Singularly perturbed vector fields

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Abstract. A geometrically invariant concept of singularly perturbed systems of ordinary
differential equations (singularly perturbed vector fields) is proposed in this paper. Singularly
perturbed vector fields can be represented locally as singularly perturbed systems (for
Corresponding coordinate system choice. The paper focuses on possible ways of fast and slow
directions/manifolds evaluations. A special algorithm for the evaluation is proposed. The
algorithm is called as a global quasi-linearization procedure. A practical application of the
proposed algorithm for numerical simulations is the main issue of the paper.

1. Introduction

The decomposition of complex systems into simpler subsystems using different rates of change
(time scales) for different subsystems is widely used in physical and engineering models. The
main difficulty for complex realistic models is the “hidden”, implicit nature of the time scales
of the original systems of governing equations. This gap between realistic models and the
Corresponding mathematical theory of singularly perturbed systems (SPS) represents a major
problem for application of the mathematical theory based on an explicit knowledge about a
“fast-slow” subdivision. For relatively simple models such a subdivision can be found by using
various non-dimensionalization procedures and/or a priori physical or engineering knowledge
about the processes.

For complicated models it might be reasonable to develop mathematical and/or algorithmic
tools that can permit us to evaluate a possible “fast-slow” subdivision and this is a main
issue of the current work. There are a number of numerical methods developed to deal with
an implicit fast-slow structure [8, 11, 3, 10, 1]. These methods are based on a division of
eigenvalues of the corresponding Jacobian matrix according to the values of their real parts
into two groups: large and small. If such a division does exist then the first group of “large”
eigenvectors are declared as the fast directions and the second group is declared as the slow
directions. This decomposition permits us to evaluate slow invariant manifolds (see, for example,
[1, 10, 13, 12, 8, 9]). Nevertheless, the procedure produces a number of so-called “ghost” manifolds
that have no relation to the true invariant manifolds and the true system dynamics. From our
previous research we can state that the main reason for the appearance of “ghost” manifolds is
an incorrect evaluation of fast directions through the Jacobi matrix eigenvalues.

In this paper we focus mainly on analytical and/or numerical algorithms that permit us
to evaluate correctly the fast directions. Since we need a formal mathematical basis for such
procedures we will start with the introduction and notion of singularly perturbed vector fields.
To the best of our knowledge this notion has not been discussed previously in any analytical framework.

Roughly speaking a singularly perturbed vector field (SPVF) $F(x, \varepsilon)$ is a vector field defined in a domain $G$ of Euclidian space $\mathbb{R}^n$ that depends on a small parameter $\varepsilon \geq 0$ such that, for any point $x$, $F(x, 0)$ belongs to an a priori fixed fast subspace $L_f(x)$ of smaller dimension - $\dim L_f(x) < n$. Moreover, the dimension of $L_f(x)$ does not depend on the choice of the point $x$. Thus, the vector field $F(x, \varepsilon)$ can be decomposed into a fast subfield that belongs to the fast subspace $L_f(x)$ and its complement represents a slow subfield. Of course this is not a formal description, which is more sophisticated. Note, if $L_f(x)$ does not depend on $x$ then the vector field $F(x, \varepsilon)$ represents (by definition) a linearly decomposed singularly perturbed vector field. Accordingly, the notion of the linearly decomposed singularly perturbed vector field is a geometrical analog of a singularly perturbed system of ordinary differential equations.

A formal concept (a theory of SPVFs) can be useful for practical applications if it is supported by an identification algorithm for fast subfields. In this paper we construct an algorithm for linearly decomposed singularly perturbed vector fields. This algorithm is based on a global linear interpolation procedure for an original vector field that we call a global quasi-linearization (GQL) (see e.g. [23]). As a reader will see, the procedure is flexible and permits us to adopt an appropriate version of the global quasi-linearization for different applications. A number of applications are discussed at the end of the paper. Some of these are purely illustrative and demonstrate the main properties of the proposed framework.

2. General Notion of Singularly Perturbed Vector Fields

In this section we introduce an abstract general notation for a singularly perturbed vector field based on the concept of vector bundles. On the one hand this general construction will be useful for our future investigations, on the other hand it permits us to understand better the corresponding simplest possible “linear” situation.

2.1. Motivation

Let us consider a standard singularly perturbed system (SPS). It can be written in two different equivalent forms [19]

$$
\varepsilon \frac{dx}{dt} = F(x, y, \varepsilon) \quad (1)
$$

$$
\frac{dy}{dt} = G(x, y, \varepsilon) \quad (2)
$$

for the so called fast time $t$ or

$$
\frac{dx}{d\tau} = F(x, y, \varepsilon) \quad (3)
$$

$$
\frac{dy}{d\tau} = \varepsilon G(x, y, \varepsilon) \quad (4)
$$

for the so called slow time ($\tau : \varepsilon d\tau = dt$).

Here $x \in \mathbb{R}^{n_f}$, $y \in \mathbb{R}^{n_s}$, $n_f + n_s = n$, smooth vector functions $F(x, y, \varepsilon)$, $G(x, y, \varepsilon)$ are defined in a domain $V \subset \mathbb{R}^n$ and $F(x, y, \varepsilon)$, $G(x, y, \varepsilon) \sim O(1)$. The vector field corresponding to the system (1)-(2) is

$$
\Phi_\varepsilon(x, y) = \left(\frac{1}{\varepsilon} F(x, y, \varepsilon), G(x, y, \varepsilon)\right) \quad (5)
$$
the vector field corresponding to the system (3)-(4) is

$$\Phi^\varepsilon(x, y) = (F(x, y, \varepsilon), \varepsilon G(x, y, \varepsilon)).$$

Both vector fields have the same directions but different asymptotic behavior for the limit \(\varepsilon \to 0\):

1) \(\Phi_x(x, y) \to (\infty, G(x, y, 0))\) for any point \((x, y) \in V\) such that \(F(x, y, 0) \neq 0\), and \(\Phi^\varepsilon(x, y) = (0, G(x, y, 0))\) for any point \((x, y)\) such that \(F(x, y, 0) = 0\), i.e. for slow manifolds;

2) \(\Phi^\varepsilon(x, y) \to (F(x, y, 0), 0)\) for any point \((x, y) \in V\).

Hence the first asymptotic representation is more convenient for describing the slow motion \(G(x, y, \varepsilon)\) and the second one is better for the fast motion \(F(x, y, \varepsilon)\).

In this work we focus on the fast motion’s subspace \(R^{n_f}\) and, therefore, the second asymptotic representation is more important for our study. Using the language of geometric invariance the second asymptotic representation can be described in the following way: \(\lim_{\varepsilon \to 0} \Phi^\varepsilon(x, y) \subset R^{n_f}\) for any \((x, y) \in V\). This remark is a motivation for the definition of asymptotic singularly perturbed vector fields.

2.2. Asymptotic singularly perturbed vector fields

We will start from a standard definition of vector bundles and will use vector bundles as a formal substitute for so-called nonlinear coordinate systems.

**Definition 2.1** A vector bundle \(\xi\) over a connected manifold \(N \subset R^n\) consists of a set \(E \subset R^n\) (the total set), a smooth map \(p : E \to N\) (the projection) which is onto, and each fiber \(F_x^\xi = p^{-1}(x)\) is a finite dimensional affine subspace. These objects are required to satisfy the following condition: for each \(x \in N\), there is a neighborhood \(U\) of \(x\) in \(N\), an integer \(k\) and a diffeomorphism \(\varphi : p^{-1}(U) \to U \times R^k\) such that is an isomorphism of vector spaces on each fiber \(\varphi\).

All fibers have the same dimension \(k\).

**Definition 2.2** Call a domain \(V \subset R^n\) a structured domain (or a domain structured by a vector bundle) if there exists a vector bundle \(\xi\) and a diffeomorphism \(\psi : V \to U\) onto an open subset \(U \subset E\), where \(E\) is the total set of \(\xi\).

Fix a parametric family of smooth vector fields \(\Phi(z, \delta)\) defined in a domain \(V \subset R^n\) for any \(0 < \delta < \delta_0\). Here \(\delta_0\) is a fixed positive number and \(\delta\) is a small positive parameter, an explicit form of the small parameter in the system is needed at least in initial stage.

A corresponding system of ODEs is

$$\frac{dz}{dt} = \Phi(z, \delta).$$

**Definition 2.3** Suppose that \(V\) is a domain structured by a vector bundle \(\xi\) and a diffeomorphism \(\psi\). For any point \(z \in G\) call \(M_z := \psi^{-1}(p^{-1}(\psi(z) \cap U))\) a fast manifold associated with the point \(z\). Call the set of all fast manifolds \(M_z\) a family of fast manifolds of \(V\).

By construction any point \(z \in G\) belongs to only one fast manifold. If \(z \neq z_1\) either \(M_z \cap M_{z_1} = \emptyset\) or \(M_z = M_{z_1}\). The dimension of any manifold \(M_z\) is the same. Denote this dimension by \(n_f\) and call it the fast dimension of \(G\). For instance, a singularly perturbed system (1-2) has \(M_z = R^{n_f}\) and \(\dim M_z = n_f\).

**Definition 2.4** A family of fast manifolds \(M_z\) is linear if there exists a linear subspace \(L_f\) of \(R^n\) such that \(M_z = \{z\} + L_f\) for any \(z \subset V\).
Call $L_f$ a fast subspace.

This is the simplest possible “linear” situation. Using a corresponding linear change of variables it is possible to move $L_f$ to a coordinate subspace like in the case (1-2).

Denote by $TM_z$ a tangent space to $M_z$ at the point $z$.

**Definition 2.5** A parametric family $\Phi(z, \delta) : V \to \mathbb{R}^n$ of vector fields defined in a domain $V$ structured by a vector bundle $\xi$ and a diffeomorphism $\psi$ is an asymptotic singularly perturbed vector field if $\lim_{\delta \to 0} \Phi(z, \delta) \in TM_z$ for any $z \in V$ and the structure of the domain $G$ is minimal for the vector field $\Phi(z, \delta) : G \to \mathbb{R}^n$ in the following sense.

There is no proper vector subbundle $\xi_1$ of the vector bundle $\xi$ such that $\Phi(z, \delta) : V \to \mathbb{R}^n$ is an asymptotic singularly perturbed vector field in a domain $V$ structured by the vector subbundle $\xi_1$ and the same diffeomorphism $\psi$.

**Remark 1** This property of minimality means that it is not possible to reduce the dimension of fast manifolds $\{M_z\}$ using subbundles.

From this point onwards, without loss of generality, we suppose that a family of fast manifolds $\{M_z\}$ associated with a singularly perturbed vector field $\Phi(z, \delta)$ is minimal.

For a linear family of fast manifolds associated with a singularly perturbed vector field $\Phi(z, \delta)$ the property of minimality can be written in a rather simple way. If $L_f$ is a minimal fast linear subspace associated with a singularly perturbed vector field $\Phi(z, \delta)$ then dimension $n_f = \dim L_f$ cannot be reduced.

Call this minimal subspace $L_f$ a linear subspace of fast motions of $\Phi(z, \delta)$.

### 2.3. Fast-slow decomposition of singularly perturbed vector fields.

Fix an asymptotic singularly perturbed vector field $\Phi(z, \delta)$. Suppose $\{M_z\}$ is a fast family associated with $\Phi(z, \delta)$ and the fast dimension of $\{M_z\}$ is $n_f$. Then the vector field $\Phi(z, \delta)$ is a sum of two vector fields $\Phi_f(z, \delta) := \Pr_f \Phi(z, \delta)$ and $\Phi_s(z, \delta) := \Phi(z, \delta) - \Pr_f \Phi(z, \delta)$. Here $\Pr_f \Phi(z, \delta)$ is a projection of $\Phi(z, \delta)$ onto the tangent space $TM_z$ of the fast manifold $M_z$, and $\Phi_s(z, \delta)$ is a projection of $\Phi(z, \delta)$ onto the linear subspace $TM^s$ of slow motions that is orthogonal to $TM_z$.

Call an asymptotic singularly perturbed vector field $\Phi(z, \delta)$ a uniformly asymptotic singularly perturbed vector field (or simply a uniform vector field) if

$$\lim_{\delta \to 0} \sup_{z \in V} |\Pr_s \Phi(z, \delta)| = 0.$$  

Define $\varepsilon := \sup_{z \in V} |\Pr_s \Phi(z, \delta)|$ which is a new small parameter, $\varepsilon < \varepsilon_0 = \sup_{z \in V} |\Pr_s \Phi(z, \delta_0)|$; $F(z, \delta) := \Pr_f \Phi(z, \delta)$ is the fast subfield and $G(z, \delta) := \frac{\sup_{z \in V} |\Pr_s \Phi(z, \delta)|}{\sup_{z \in V} |\Pr_s \Phi(z, \delta)|}$ is a slow subfield of $\Phi(z, \delta)$. Then the vector field $\Phi(z, \delta)$ is a linear combination of its fast and slow subfields i.e.

$$\Phi(z, \delta) = F(z, \delta) + \varepsilon G(z, \delta).$$  

(8)

**Remark:** the small parameter $\varepsilon$ is a function of the small parameter $\delta$. If $\delta \to 0$ then $\varepsilon \to 0$.

For any practical implementation of the proposed construction of singularly perturbed vector fields we have to find a way to determine the fast manifolds. For the moment we have this for the linear case i.e. for the case where all fast manifolds are parallel to a fixed linear subspace $L_f$. 

2.4. Analytical example of a singularly perturbed vector field - Polar coordinates

This example was constructed by V.A Sobolev and it demonstrates that a fast subfield can depend on a point even in very simple cases.

Consider a simple SPS in polar coordinates

$$\begin{align*}
\varepsilon \frac{dr}{dt} &= (1 - r) \\
\frac{d\theta}{dt} &= 1
\end{align*}$$

with initial conditions $r(t = 0) = r_0, \theta(t = 0) = \theta_0$. Now, we change the polar coordinates to Cartesian

$$\begin{align*}
x &= r \cos(\theta) \\
y &= r \sin(\theta)
\end{align*}$$

The system (9) is singularly perturbed and its dynamics in both $(r, \theta)$ and $(x, y)$ planes are schematically presented in Figure 1.

After the substitution (10) the system (9) has the following representation

$$\begin{align*}
\frac{dx}{dt} &= -\frac{y}{x^2+y^2} + \frac{1}{\varepsilon} \left(1 - \sqrt{x^2+y^2}\right) \\
\frac{dy}{dt} &= \frac{x}{x^2+y^2} + \frac{1}{\varepsilon} \left(1 - \sqrt{x^2+y^2}\right)
\end{align*}$$

with initial conditions $x_0 = r_0 \cos(\theta_0), y_0 = r_0 \sin(\theta_0)$. Assume that the original form (9) of the system is unknown. Then a natural question is how to find (redefine) new global coordinates similar to (9) in such a way that the system goes into the standard form of SPS. Remark: despite the existence a small parameter and the multi-scale (singularly perturbed) nature of the system, system (11) is not written as SPS in Cartesian coordinates.

The proposed framework allows us treat and redefine the global transformation. An initial step in this direction is a decomposition of the given vector field (11) and it might be performed according to

$$\Phi = F + G,$$

where the fast subfield $F$ and the slow subfield $G$ can easily be found due to the explicit appearance of the small parameter $\varepsilon$:
Nevertheless, the fast subfield (subsystem) - the original system (11), it is clear that to find the solution in any form is not a simple task. (it reduces the system dimension and leads to simple subsystems). For example, if we consider propagation through an inert porous media filled with a gaseous combustible mixture. Such framework to a ‘hot’ combustion problem. Namely, we consider the phenomenon of flame Let us now present another example representing an application of the suggested theoretical 2.5. Pressure driven flames (model system)

\[
F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \frac{1}{\varepsilon} \begin{pmatrix} x \left(1 - \sqrt{x^2 + y^2} \right) \\ \frac{y \left(1 - \sqrt{x^2 + y^2} \right)}{\sqrt{x^2 + y^2}} \end{pmatrix} \sim O\left(\frac{1}{\varepsilon}\right),
\]

\[
G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \begin{pmatrix} -\frac{y}{\sqrt{x^2 + y^2}} \\ \frac{-x}{\sqrt{x^2 + y^2}} \end{pmatrix} \sim O(1).
\]

Now, in order to represent (11) as SPS we can analyze the fast and slow subfields separately. This approach represents a considerable simplification which is important in an analytical study (it reduces the system dimension and leads to simple subsystems). For example, if we consider the original system (11), it is clear that to find the solution in any form is not a simple task. Nevertheless, the fast subfield (subsystem) - \( \frac{dx}{dt} = F \) can be integrated separately and it has the general solution in implicit form - \( u = \ln\left(\frac{x}{y}\right) + C_u = 0 \). This means that the fast motion is performed in the \((x, y)\) plane along straight lines \( y = Cx \), which represents the fast manifold \( \{M_{(x, y)}\} \) of the singularly perturbed vector field \( F \). Moreover, it would be natural to expect that the new variable \( u \) is slow. In fact, the variable \( u \) describes the slow system dynamics of the original system

\[
\frac{du}{dt} = u_x \frac{dx}{dt} + u_y \frac{dy}{dt} = u_x(F_1 + G_1) + u_y(F_2 + G_2) = u_xG_1 + u_yG_2 \sim O(1).
\]

At the same time the slow subsystem \( \frac{dv}{dt} = G \) has the general solution \( v = x^2 + y^2 + C_v = 0 \) and it is expected to be fast according to

\[
\frac{dv}{dt} = v_x \frac{dx}{dt} + v_y \frac{dy}{dt} = v_x(F_1 + G_1) + v_y(F_2 + G_2) = v_xF_1 + v_yF_2 \sim O\left(\frac{1}{\varepsilon}\right).
\]

Hence, the standard SPS form of the system (11) becomes

\[
\frac{du}{dt} = -\frac{y}{x} \frac{1}{\sqrt{x^2 + y^2}} - \frac{x}{y} \frac{1}{\sqrt{x^2 + y^2}} = -\frac{(\exp(-u) + \exp(u))}{\varepsilon
\frac{dv}{dt} = \frac{1}{\varepsilon} \left( \frac{2x^2 \left(1 - \sqrt{x^2 + y^2}\right)}{(x^2 + y^2)^{3/2}} + \frac{2y^2 \left(1 - \sqrt{x^2 + y^2}\right)}{(x^2 + y^2)^{3/2}} \right) = 2(1 - \sqrt{v}) \varepsilon \sqrt{v}.
\]

Here the constants - \( C_u, C_v \) were chosen equal to zero. This finalizes the task of the redefinition of a global coordinate system. Future simplifications and redefinition of the polar coordinates can be done by taking \( \theta = \arctan\left(\frac{y}{x}\right) = \arctan(\exp(u)) \) as the slow variable instead of \( u \), because, as was mentioned during the fast stage, the angle between the \( x \)-axis and direction of the fast motion is constant. Additionally one can use \( r = \sqrt{v} = \sqrt{x^2 + y^2} \) instead of \( v \).

It should be noted that this type of the analysis might be considered as a preliminary step (converting to the standard SPS form (1)-(2)) of asymptotic analysis and it is possible only if a decomposition of a singularly perturbed vector field is known and/or a small system parameter is explicitly specified as in the example considered.

2.5. Pressure driven flames (model system)

Let us now present another example representing an application of the suggested theoretical framework to a ‘hot’ combustion problem. Namely, we consider the phenomenon of flame propagation through an inert porous media filled with a gaseous combustible mixture. Such
flame waves are often referred to as pressure driven or barodiffusion flames [25]. It should be mentioned that our attempts to use a conventional approach (the method of inner and outer asymptotic expansions, well accepted in the field of the mathematical theory of combustion) to solve the problem under consideration failed. In the authors’ opinion this failure was caused by the failure of the standard assumption of a constant pressure which is obviously not correct for pressure driven flames.

The machinery based on the SPVF approach permits us to overcome the main problems of conventional asymptotics involved in the analysis of combustion problems, and allows us to investigate the flame structure and to obtain an asymptotic estimation of the flame propagation velocity.

The main physical assumptions required to construct this simplified model are the following: the heat transfer mechanism is considered negligible (to single out an impact of the local pressure elevation), the inertia effects are negligible and the friction force is proportional to the gas velocity (Darcy law) (see e.g. [26]). Non-dimensionalization and suitable integration allow us to re-write the original system in the form of the three ODEs [5, 6]

\[
\frac{d\theta}{d\xi} = \Lambda \left( \Pi - \theta \right) + \varepsilon_1 \eta \frac{1 + \beta \Pi}{1 + \beta \theta} \exp \left( \frac{\theta}{1 + \beta \theta} \right), \quad (14)
\]

\[
\sigma \frac{d\Pi}{d\xi} = \Lambda \left( \Pi - \theta \right), \quad (15)
\]

\[
\frac{d\eta}{d\xi} = -\varepsilon_2 \eta \frac{1 + \beta \Pi}{1 + \beta \theta} \exp \left( \frac{\theta}{1 + \beta \theta} \right). \quad (16)
\]

The system is written in the moving frame coordinate system using the so called auto-model variable $\xi$. Here $\Pi, \theta, \eta$ are dimensionless quantities of pressure, temperature and deficient reactant concentration and $\Lambda$ is a dimensionless flame velocity that corresponds to the quasi-stationary approach.

To demonstrate the essence of the new approach, let us additionally apply the Frank-Kamenetskii approximation and put $\beta = 0$ in the set of the governing equations (14)-(16). The simplified model system reads

\[
\frac{d\theta}{d\xi} = \Lambda (\Pi - \theta) + \varepsilon_1 \eta \exp (\theta), \quad (17)
\]

\[
\sigma \frac{d\Pi}{d\xi} = \Lambda (\Pi - \theta), \quad (18)
\]

\[
\frac{d\eta}{d\xi} = -\varepsilon_2 \eta \exp (\theta). \quad (19)
\]

Further simplifications are possible due to the fact that the systems (17)-(19) and (14)-(16) have an energy integral (the adiabatic approach was used)

\[
\eta - 1 + \frac{\varepsilon_2}{\varepsilon_1} (\theta - \sigma \Pi) = 0, \quad (20)
\]

which permits us to determine the parameters of the burnt mixture (behind the flame front):

\[
\eta_b = 0; \ \theta_b = \Pi_b = \frac{\varepsilon_1}{\varepsilon_2 (1 - \sigma)}. \quad (21)
\]

In addition, the existence of the energy integral allows to reduce a number of governing equations by one. The reduced system reads
Figure 2. Profiles of quasi-stationary solution (left) and phase plane of reduced system with solution trajectory (right figure). Dashed line - approximate analytical solution, solid line - numerical solution.

\[
\frac{d\theta}{d\xi} = \Lambda (\Pi - \theta) + \varepsilon_1 \left(1 - \frac{\varepsilon_2}{\varepsilon_1} (\theta - \sigma \Pi)\right) \exp(\theta),
\]

and is subject to the initial conditions

\[
\theta(0) = \varepsilon \ll 1; \quad \Pi(0) = \frac{\varepsilon}{\sigma} \ll 1,
\]

whose choice in the form (24) eliminates the so-called cold-boundary problem for initial data (a well known problem for initial data in combustion models). The boundary conditions (24) and (21) permit us to solve the problem of the evaluation of the wave velocity \(\Lambda\) since the system is over-determined. It becomes a boundary value problem.

Before starting the asymptotic investigation, let us mention typical values of the parameters for combustion systems: \(0.1 \leq \sigma \leq 0.3, 10^{-6} \leq \varepsilon_2 \ll \varepsilon_1 \leq 10^{-4}\). Accordingly, a characteristic value of the adiabatic temperature and pressure has the order of magnitude: \(O(10^2)\).

Additionally, the singular/adiabatic point of the system (22)-(23) is of saddle type. It means in particular, that this point is reachable only along the unique direction (separatrix) which is found from

\[
\frac{d\theta}{d\Pi} = \sigma \left(\frac{1}{\sigma} + \frac{\varepsilon_2}{\Lambda} \exp(\theta_b)\right)
\]

corresponding to the negative eigenvalue \(\lambda = -\varepsilon_2 \exp(\theta_b)\) and, therefore, if a trajectory of the system reaches this singular point it must do so along this direction.

Now, consider the system (22)-(23) and construct an approximate solution in the vicinity of the singular (adiabatic) saddle point \(P_{fin}\) of the system (see Fig. 2, where profiles and the trajectories of the system in the phase plane are presented). Then using the analysis of the approximate solution we transform the system into standard SPS form and resolve the boundary value problem for the system eigenvalue \(\Lambda\).

The behavior of the trajectory of the system in the phase plane \((\theta, \Pi)\) is determined by the solution of the following equation (plane system)

\[
\frac{d\theta}{d\Pi} = \sigma \left(1 + \frac{\varepsilon_1 (1 - \frac{\varepsilon_2}{\varepsilon_1} (\theta - \sigma \Pi))}{\Lambda (\Pi - \theta)}\right) \exp(\theta).
\]
In order to find an approximate solution let us simplify this equation using the information regarding the direction of the system trajectory in the vicinity of the singular point. This direction can be readily found from (26)

\[
\frac{d\theta}{d\Pi} \bigg|_{\theta \to \theta_b} = \sigma \left( 1 + A \exp(\theta_b) \right),
\]

(27)

where the limit of the pre-exponential term \( A \) is evaluated using (25) in the following way (an exponentially small term (i.e. \( \exp(-\theta_b) \ll 1, \theta_b = O(10^2) \)) is omitted)

\[
\sigma \left( \frac{1}{2} + \frac{\varepsilon_2}{\Lambda} \exp(\theta_b) \right) \equiv \sigma \left( 1 + A \exp(\theta_b) \right),
\]

\[
A = \left( \frac{1}{2} - 1 \right) \exp(-\theta_b) + \frac{\varepsilon_2}{\Lambda} \approx \frac{\varepsilon_2}{\Lambda}.
\]

(28)

Thus, the simplified solution, which defines the fast motion zero order approximation as well as the transformation into SPS form, can be easily found from (26)

\[
\int \frac{d\theta}{\left( 1 + \frac{\varepsilon_2}{\Lambda} \exp(\theta) \right)} = \sigma \int d\Pi,
\]

(29)

which, after integration, yields

\[
\theta - \ln \left( 1 + \frac{\varepsilon_2}{\Lambda} \exp(\theta) \right) = \sigma \Pi + C.
\]

(30)

This corresponds to the solution of the fast subsystem where the decomposition of the type (8), (12) is made by using the special system behavior near the singular point (21). Since the fast motion is realized along the integral curve (30) (at least near the stationary point) a candidate for the slow variable can be extracted easily as

\[
u = \sigma \Pi - \theta + \ln \left( 1 + \frac{\varepsilon_2}{\Lambda} \exp(\theta) \right).
\]

(31)

We now transform the model system considered to the standard SPS form. Following the suggested framework we apply a new coordinate system \((\nu, \theta)\) in the form

\[
\frac{d\theta}{d\xi} = \Lambda \left( \Pi(u, \theta) - \theta \right) + \varepsilon_1 \left( 1 - \frac{\varepsilon_2}{\varepsilon_1} (\theta - \sigma \Pi(u, \theta)) \right) \exp(\theta),
\]

(32)

\[
\frac{du}{d\xi} = \varepsilon_1 \left( 1 - \frac{\varepsilon_2}{\varepsilon_1} (1 - \sigma) \Pi(u, \theta) \right) \exp(\theta) \left( 1 + \frac{\varepsilon_2}{\Lambda} \exp(\theta) \right),
\]

(33)

where \( \Pi(u, \theta) \) is given by (31). We remark that the RHS of (33) is \( O(\varepsilon_1) \) everywhere in the phase plane, at the initial point as well as near the stationary point. Therefore \( \theta \) is the fast variable. The slow variable \( u \) does not change much along the system solution and has to be close to a constant defined at the stationary point \( u = u_b \).

To find the zero order approximation for \( \varepsilon_1 \ll 1 \) we have to find the constant of integration \( C \) in (30) which is determined by the values at the stationary point. After substitution we have

\[
\theta - \theta_b - \ln \left( \frac{1 + \frac{\varepsilon_2}{\Lambda} \exp(\theta)}{1 + \frac{\varepsilon_2}{\Lambda} \exp(\theta_b)} \right) = \sigma (\Pi - \Pi_b).
\]

(34)

Now, expand the approximate solution of the fast subsystem up to the initial point taking the limit \( \theta(0) = \varepsilon \to 0 \).
\[- \ln \left( \frac{1 + \frac{\Lambda}{\theta_b}}{1 + \frac{\Lambda}{\theta_b} \exp (\theta_b)} \right) = \theta_b (1 - \sigma). \tag{35} \]

An estimate of the parameter \( \Lambda \) is obtained as the solution of this equation

\[ \Lambda = \varepsilon_2 \exp (\sigma \theta_b) \left( \frac{1 - \exp (-\sigma \theta_b)}{1 - \exp(-(1 - \sigma) \theta_b)} \right) \approx \varepsilon_2 \exp (\sigma \theta_b). \tag{36} \]

It has to be mentioned here that further improvements and increasing accuracy is possible by the standard SPS analysis of the decomposed system (32)-(33). Comparison with previous results [5, 7, 23] shows that these estimates coincide with exponentially small order of magnitude \((\theta_b \gg 1)\)

\[ \Lambda = \frac{\varepsilon_1}{(1 - \sigma) \theta_b} \exp (\sigma \theta_b) = \varepsilon_2 \exp (\sigma \theta_b). \tag{37} \]

The formula (37) finalizes the solution of the problem (17)-(19). Thus, the well-known property of the saddle point together with the proposed framework allowed us to construct the required decomposition in the whole phase plane. Moreover, it justified our confidence that there are asymptotic tools which are able to provide us with more accurate asymptotic solutions of the combustion problems than the existing ones.

3. Slow Invariant Manifolds
A few words about slow manifolds. Suppose \( \Phi(z, \delta) = F(z, \delta) + \varepsilon G(z, \delta) \) is a singularly perturbed vector field. The equation \( F(z, \delta) = 0 \) represents a fast isocline of the vector field \( \Phi(z, \delta) \). If its solution is a manifold \( S_0 \) with tangent spaces transverse to fast manifolds then this manifold has dimension \( n_s \) and using local charts of the corresponding vector bundle and a standard local representation of manifold \( S_0 \) as a graph of a vector function it is possible to prove the existence of the slow invariant manifold \( S \). Similarly to the standard theory of singularly perturbed systems, \( S_0 \) is a zero approximation (for \( \delta = 0 \)) of the slow invariant manifold \( S \).

Of course this restriction on \( S_0 \) does not hold for all points of \( S_0 \) in any realistic situation. Even if \( S_0 \) is a manifold it can happen that for some submanifold its tangent space is not transverse to the fast manifolds. This submanifold represents a manifold of turning (jumping) points of \( S_0 \) where slow dynamics change to fast ones (for realistic situations). In this paper we will avoid any discussion about these delicate submanifolds and will leave it for future work.

Dynamics on \( S_0 \) is dynamics of the slow subfield \( G(z, \delta) \) which belongs to the tangent bundle of \( S_0 \). To obtain the first approximation of the slow invariant manifold let us differentiate the fast vector field \( F(z, \delta) \) w.r.t. time (using the dynamical system \( dz/dt = \Phi(z, \delta) \)). Because \( \Phi(z, \delta) = F(z, \delta) + \varepsilon G(z, \delta) \) we obtain an implicit expression

\[ F_z(z, \delta)(F(z, \delta) + \varepsilon G(z, \delta)) = 0 \]
as the first approximation, where terms of order \( o(\delta) \) are neglected. By repeated differentiation, approximations of high orders can be obtained. This procedure coincides with the evaluation of implicit invariant slow manifolds (see e.g. [15], [16]). Since this paper is focused on the evaluation of fast manifolds, detailed analysis of slow manifolds will be given in future works.

4. Singularly Perturbed Vector Fields with Linear Fast Subspace
For any realistic complex model the small parameter \( \delta \) is unknown and this fact restricts possible applications of the proposed asymptotic theory. In this section we will try to explain how to adapt the asymptotic theory developed for a practical problem in the simplest possible case of linear fast manifolds.
Thus any fast manifold at any point $z$ is parallel to a linear subspace $L_f(z)$ with fixed dimension $n_f$. Note that for many applications an assumption that $L_f$ does not depend on $z$ is natural.

4.1. Fast-slow decomposition

Fix a uniformly asymptotic singularly perturbed vector field $\Phi(z, \delta)$. Suppose that the fast subspace $L_f$ does not depend on $z$ and $\dim L_f = n_f$. The vector field $\Phi(z, \delta)$ is a sum of two vector fields $\Phi_f(z, \delta) := Pr_f \Phi(z, \delta)$ and $\Phi_s(z, \delta) := \Phi(z, \delta) - Pr_f \Phi(z, \delta)$. Here $Pr_f \Phi(z, \delta)$ is a projection of $\Phi(z, \delta)$ onto the fast subspace $L_f$, and $\Phi_s(z, \delta)$ is a projection of $\Phi(z, \delta)$ onto the linear slow subspace $L_s$ that is orthogonal to $L_f$.

The uniformity condition permits us to represent a uniformly singularly perturbed vector field and a corresponding dynamical system (7) as a standard singularly perturbed system (SPS). A corresponding construction is now possible. Suppose $x := Pr_f z$ and $y := Pr_s z$ are fast and slow variables that represent a new coordinate system with $n_f$ fast variables $x$ and $n_s = n - n_f$ slow variables $y$; $\varepsilon := \sup_{z \in V} |Pr_s \Phi(z, \delta)|$ is a small parameter, $\varepsilon < \varepsilon_0 = \sup_{z \in V} |Pr_s \Phi(z, \delta_0)|$; $F(x, y, \delta)$ is a representation of $F(z, \delta) := Pr_f \Phi(z, \delta)$ in the new coordinate system $(x, y)$ and $G(x, y, \delta)$ is a representation of $G(z, \delta) := \frac{Pr_s \Phi(z, \delta)}{\sup_{z \in V} |Pr_s \Phi(z)|}$ in the new coordinate system $(x, y)$.

Hence the system (7) has the standard SPS form

$$\frac{dx}{d\tau} = F(x, y, \delta)$$

$$\frac{dy}{d\tau} = \varepsilon G(x, y, \delta).$$

in the new coordinate system $(x, y)$.

Remember that the small parameter $\varepsilon$ is a function of the small parameter $\delta$. If $\delta \to 0$ then $\varepsilon \to 0$.

4.2. Singularly perturbed vector fields: non asymptotic definition.

In many practical situations the previous definition of an asymptotic singularly perturbed vector field $\Phi(z, \delta)$ cannot be useful because the small parameter $\delta$ is unknown. Meanwhile the main geometrical idea is still useful if some previous knowledge about a scaling is known. This means that some “small” number $\varepsilon_0$ is fixed for corresponding processes (models) and any parameter $\varepsilon < \varepsilon_0$ can be considered as a small parameter.

Suppose a smooth vector field $\Phi(z)$ is defined in a structured domain $V \subset \mathbb{R}^n$ and $L_f$ is the fast subspace. Moreover $\sup_{z \in V} |Pr_s \Phi(z)| < \varepsilon_0$.

Suppose as well, as in the previous subsection, that $x := Pr_f z$ and $y := Pr_s z$ are fast and slow variables that represent a new coordinate system with $n_f$ fast variables $x$ and $n_s = n - n_f$ slow variables $y$; $\varepsilon := \sup_{z \in V} |Pr_s \Phi(z)|$ is a small system parameter; $F(x, y)$ is a representation of $Pr_f \Phi(z)$ and $G(x, y)$ is a representation of $\frac{Pr_s \Phi(z)}{\sup_{z \in V} |Pr_s \Phi(z)|}$ in the new coordinate system $(x, y)$.

Hence the system (7) has a standard SPS form

$$\frac{dx}{d\tau} = F(x, y)$$

(38)

$$\frac{dy}{d\tau} = \varepsilon G(x, y).$$

(39)

in the new coordinate system $(x, y)$ and $\Psi(x, y, \varepsilon) = F(x, y) + \varepsilon G(x, y)$ is a singularly perturbed vector field.
5. Evaluation of fast subspaces

In this section some procedures for the evaluation of the dimension and structure of fast subspaces is proposed. We call such a type of procedure quasi-linearizations of vector fields. Different versions of quasi-linearization were discussed in our previous paper [23]. In what follows we present a general construction with the help of the approach developed here.

5.1. Asymptotic global quasi-linearization procedure

Suppose \( \Phi(z, \delta) \) is a singularly perturbed vector field defined in a structured domain \( V \). This means that \( \Phi(z, \delta) \) can be represented as a sum \( \Phi(z, \delta) := F(z, \delta) + \varepsilon G(z, \delta) \) where \( F(z, \delta) \) is a fast part of the vector field \( \Phi(z, \delta) \) and \( \varepsilon G(z, \delta) \) is a slow part of the vector field \( \Phi(z, \delta) \). Exact descriptions of \( F(z, \delta), G(z, \delta) \) and \( \varepsilon \) can be found as in the previous section.

Suppose there exist such \( n \) points \( z_1, ..., z_n \in V \) such that vectors \( \Phi(z_1, \delta), ..., \Phi(z_n, \delta) \) are linearly independent and its fast parts \( F(z_1, \delta), ..., F(z_n, \delta) \) represent a basis of \( L_f \) for any \( 0 \leq \delta \leq \delta_0 \). Call an \( n \times n \)-matrix \( T := [\Phi(z_1, \delta), ..., \Phi(z_n, \delta)] \) a quasi-linearization matrix of the vector field \( \Phi(z, \delta) \) associated with points \( z_1, ..., z_n \). As usual, \( \Phi(z_1, \delta), ..., \Phi(z_n, \delta) \) are columns of the matrix \( T \). Using the representation \( \Phi(z, \delta) := F(z, \delta) + \varepsilon G(z, \delta) \), the matrix \( T \) can be represented as a sum

\[
T = T_f + \varepsilon T_s = [F(z_1, \delta), ..., F(z_n, \delta)] + \varepsilon [G(z_1, \delta), ..., G(z_n, \delta)]
\]

of the fast and the slow matrices \( T_f \) and \( \varepsilon T_s \).

Denote by \( A^* \) a matrix conjugate to a matrix \( A \). For the evaluation of the fast subspace \( L_f \) we use the matrix \( TT^* \), because the eigenvalue problem (possible complex eigenvalues, non stable numerics etc.) for \( T \) that can lead to non coincidence of the “fast” subspace of \( T \) with \( L_f \) ([4], [24]). Recall that all eigenvalues of the matrix \( TT^* \) are nonnegative (even positive) and the corresponding eigenspaces are orthogonal. By the construction of matrices \( T_f \) and \( T_s \), the matrix \( TT^* = (T_f + \varepsilon T_s) (T_f + \varepsilon T_s)^* = T_f T_f^* + \varepsilon (T_s T_f^* + T_f T_s^*) + \varepsilon^2 T_s T_s^* \) has no more then \( n_f \) positive eigenvalues. Additionally all these eigenvalues are positive (non negative) for small enough \( \varepsilon \) and for \( \varepsilon \to 0 \) another \( n_s \) eigenvalues tend to zero. Eigenvectors that correspond to \( n_f \) positive eigenvalues belong to \( L_f \) and form a basis of \( L_f \).

Therefore for small enough \( \varepsilon \) (asymptotically) the matrix \( TT^* \) has two groups of eigenvalues: \( n_f \) “large” eigenvalues, whose eigenvectors asymptotically form a basis of the fast subspace \( L_f \) and \( n_s \) “small” eigenvalues.

5.2. Practical implementation

Here we will discuss a variation of the global quasi-linearization procedure (GQL) that was motivated by numerical simulations. A reason for the modification is that in many practical situations a system has dimensional form and therefore cannot be used without proper non-dimensionalization beforehand. Otherwise the system hierarchy can be perturbed by different scales of the system variables. The suggested modification overcomes this problem and can be applied to the system in dimensional form as well.

Suppose now \( \Phi(z) \) is a vector field and our assumption is that this vector field is singularly perturbed and it depends on a “hidden” small parameter \( \delta \). We will not even try to look for \( \delta \) because only the parameter \( \varepsilon = \varepsilon(\delta) \) has practical importance.

The proposed modification of GQL is the following procedure. Choose \( n \) linearly independent points \( z_1, ..., z_n \) in such a way that vectors \( \Phi(z_1), ..., \Phi(z_n) \) are also linearly independent. It is
impossible to check linear independence for any relatively small \( \delta \) because the small parameter \( \delta \) is unknown. Using \( z_1, ..., z_n \) and \( \Phi(z_1), ..., \Phi(z_n) \) as columns for matrices we define a new matrix \( T \) as follows

\[
T := [\Phi(z_1), ..., \Phi(z_n)] [z_1, ..., z_n]^{-1}.
\]

This matrix has a simple geometrical interpretation. It is the matrix of the linear mapping that transforms points \( z_1, ..., z_n \) to \( \Phi(z_1), ..., \Phi(z_n) \). Of course the efficiency of this procedure depends essentially on the choice of \( z_1, ..., z_n \). Practical recommendations for the choice are the following. Points \( z_1, ..., z_n \) cannot be close to another because it can lead to a degeneracy of the matrix \([z_1, ..., z_n]^{-1}\); also the vector field \( \Phi(z) \) should have essentially “different” behavior for different points \( z_1, ..., z_n \). It is obvious that this is a crucial point of the algorithm and must be adopted in every particular model.

For the evaluation of the fast subspace \( L_f \) we will use the matrix \( TT^* \) in the previous subsection.

6. Examples

Let us finally demonstrate the global quasi-linearization procedure on the basis of comparatively simple examples.

6.1. Linear example

The idea to use principal subspaces of matrix \( TT^* \) instead of the principal subspaces of \( T \) is illustrated by a simple linear system

\[
\begin{align*}
\frac{du}{dt} &= au + bv \\
\frac{dv}{dt} &= cv
\end{align*}
\]

Because this system is linear any quasi-linearization procedure gives the same system. Suppose that \( |b| \gg |c|, |b| \gg |a| \) and \(|c| \sim |a|\). Using the previous procedure for the evaluation of the fast direction it is necessary to calculate first the eigenvalues of \( TT^* \); where \( T = \begin{bmatrix} a & b \\ o & c \end{bmatrix} \) is the matrix of the system (40) (it is the Jacobian matrix of the system). It is obvious that any method based on the eigenvectors of \((J = T)\) will fail to produce any decomposition. In contrast consider the matrix \( TT^* \), which has two positive eigenvalues

\[
\mu_{1,2} = \frac{a^2 + b^2 + c^2}{2} \pm \sqrt{\left(\frac{a^2 + b^2 + c^2}{2} \right)^2 - a^2c^2}.
\]

We can conclude that \( \mu_1 \approx a^2 + b^2 + c^2 \) is a “big” eigenvalue and \( \mu_2 \) (which is close to zero) is a “small” eigenvalue because \( |b| \gg |a|, |b| \gg |c| \). A “fast” eigenvector is \( e_f = \left(1, \frac{\mu_1-a^2-b^2}{2} \right) \approx (1, \frac{b}{c})\). Therefore we obtain a reasonably good fast direction whose deviation from \( e_1 = (1, 0) \) is of order of \( c/b \).

6.2. Numerical example - Rotation of the original coordinates

The second type of example illustrates an application of the linear version of the singularly perturbed vector field framework and the global quasi-linearization procedure. It has been developed to treat a system having a relatively simple inner hidden hierarchy. To construct a model example let us take the following systems in the standard SPS form

\[
\begin{align*}
\epsilon \frac{dx}{dt} &= x^3 - 2xy + y - 1 \\
\frac{dy}{dt} &= -(x + y - 1)
\end{align*}
\]
Figure 3. The left figure shows the system dynamics of (41) in the \((x, y)\) state plane of original variables with \(\varepsilon = 0.3\); the right figure - the same as in the left, but in the new system coordinates \((u, v)\) after rotation with \(\theta = -\pi/4\).

Figure 4. The left figure shows the system dynamics of (42) in the \((x, y)\) state plane of the original variables with \(\varepsilon = 0.1\); the right figure - the same as in the left, but in the new system coordinates \((u, v)\) after rotation with \(\theta = -\pi/4\).

and

\[
\varepsilon \frac{dx}{dt} = -(x + \sin(x) + \sin(y)) \\
\frac{dy}{dt} = -y
\]

(42)

Rotate the system coordinates according to

\[
\begin{aligned}
&u = \cos(\theta)x + \sin(\theta)y \\
v = -\sin(\theta)x + \cos(\theta)y
\end{aligned}
\]

and apply now the suggested global quasi-linear approximation (GQL) to redefine the decomposition in the \((x, y)\) phase plane. Reference points were chosen uniformly distributed in the cube \((0, x_0) \times (0, y_0)\) where \((x_0, y_0) = P_0\) is an initial point. Results for \(\varepsilon = 0.3\) and \(0.1, \theta = -\pi/4\) are shown in figures 3 and 4, where the slow curves and the system trajectories are drawn in both \((x, y)\) and \((u, v)\) phase planes. Solid curves \(E\) represent the slow curves of the original system which are based on standard SPS theory and defined by the RHS of the fast subsystem (first equations) in (41) and (42). The solid line \(AB\) shows the direction of the fast motion redefined by the GQL approximation, while the dashed curves - \(E'\) are slow curves obtained with the help of the approximation (GQL). Dashed black lines are system
trajectories starting at the same $P_0$, but in different phase planes. It is not hard to see that the proposed method produces a very good approximation for the decomposition, especially for the fast manifolds.

6.3. Numerical example - Michaelis-Menten enzyme model

The last example is the Michaelis-Menten model [17, 18]. A simplified mathematical model is the following

\[
\begin{align*}
\frac{dx}{dt} &= -x (1 - y) + L_1 y \\
\frac{dy}{dt} &= -\frac{1}{L_2} (x (1 - y) - y)
\end{align*}
\]

This example has been extensively studied in [17] to show, in particular, that the standard method of quasi-stationary concentrations leads to a significant inaccuracy in defining the slow system manifold. It was shown that the zero order approximation for the slow invariant manifold given by the RHS of the second equation in (43), when the value of the system parameter $L_2$ is small, is not good enough (see Fig. 5 line $E$).

![Figure 5. State plane with two system trajectories - dashed black lines, fast directions ($AB$, $CD$) - solid blue lines and the slow curve $E$ - solid green line. The first order approximation - $F$ is shown by solid red curve. The system parameters equal $L_1 = 25, L_2 = 0.2$.](image)

Indeed, for relatively large but still finite values of the system parameter, one gets only a rough approximation at least to zero order.

Let us demonstrate now how the suggested method (GQL) overcomes this problem. Figure 5 shows typical trajectories of system solutions (starting at some initial points $P_1$ and $P_2$) by dashed black lines. Straight lines $AB$ and $CD$ represent the linear approximation to the fast system manifold.

Curve $F$ corresponds to the first order approximation of the system slow invariant manifold based on the new SPS form generated by GQL and the corrected direction of the fast motion $AB$. It has to be emphasized that the zero order approximation (slow curve) based on GQL is also not so good and actually close to $E$ since the assumption about the asymptotic smallness
of $L_2$ is not fully satisfied. Nevertheless, the first order approximation, even for this value of the system’s small parameter, approximates quite well (line $F$) the slow invariant manifold of the system.

7. Conclusions

The theoretical framework of Singular Perturbed Vector Fields (SPVF) has been proposed in this work. To some extent it can be considered as a generalization of a standard approach of the SPS. It was shown that the proposed approach can be successfully applied in both analytical and numerical versions. While the original system of ODEs is treated analytically, SPVF provides us with an accurate description of the fast and slow system manifolds and allows to transform the original system into standard SPS form so that a powerful method of singular perturbations can be applied afterward. Additionally, within the framework of SPVF a numerical technique for the construction of a linear approximation of the system’s fast manifold(s) has been suggested.

A number of examples were considered. They clearly illustrate that the proposed method allows us to overcome a number of difficulties, which other analyzes face. It was demonstrated that the proposed theoretical framework of SPVF has potential as a possible theoretical background for the construction of numerical schemes to automatically identify the decomposition of the system into the “slow” and “fast” subsystems.

The proposed technique can be useful for researchers interested in analytical or numerical evaluation of a ‘hidden’ decomposition of the system dynamics into so called “slow” and “fast” motions.

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