Solving the fully-connected spherical $p$-spin model with the cavity method:
equivalence with the replica results

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The spherical $p$-spin is a fundamental model for glassy physics, thanks to its analytic solution achievable via the replica method. Unfortunately the replica method has some drawbacks: it is very hard to apply to diluted models and the assumptions beyond it are not immediately clear. Both drawbacks can be overcome by the use of the cavity method, which, however, needs to be applied with care to spherical models.

Here we show how to write the cavity equations for spherical $p$-spin models on complete graphs, both in the Replica Symmetric (RS) ansatz (corresponding to Belief Propagation) and in the 1-step Replica Symmetry Breaking (1RSB) ansatz (corresponding to Survey Propagation). The cavity equations can be solved by a Gaussian (RS) and multivariate Gaussian (1RSB) ansatz for the distribution of the cavity fields. We compute the free energy in both ansatzes and check that the results are identical to the replica computation, predicting a phase transition to a 1RSB phase at low temperatures.

The advantages of solving the model with the cavity method are many. The physical meaning of any ansatz for the cavity marginals is very clear. The cavity method works directly with the distribution of local quantities, which allows to generalize the method to diluted graphs. What we are presenting here is the first step towards the solution of the diluted version of the spherical $p$-spin model, which is a fundamental model in the theory of random lasers and interesting *per se* as an easier-to-simulate version of the classical fully-connected $p$-spin model.
I. INTRODUCTION

Spherical models are made of $N$ real variables $\sigma_i \in \mathbb{R}$ satisfying the global constraint $\sum_i \sigma_i^2 = N$. They play a key role among solvable models in statistical physics, because they usually allow for closed and compact algebraic solutions \[1, 2\]. Moreover, being the variables reals, the space of configurations is continuous and differentiable, thus allowing one to study in these models several kind of dynamics (e.g. Langevin dynamics or gradient descent like relaxations). At variance, models whose variables satisfy local constraints pose more problems. For example, in Ising and Potts models the variables take discrete values and so the space of configuration is not continuous; while in $O(n)$ models (e.g. with XY or Heisenberg spins) each variable is continuous, but needs to satisfy a local constraint of unit norm in an $n$-dimensional space, and this in turn makes the analytic solution much more complicated, see for instance \[3–5\].

The success of spherical models is well witnessed by the fully-connected spherical $p$-spin model. For $p \geq 3$ this model is the most used mean-field model for the glassy dynamics. We learned a lot from it exactly because both the thermodynamics and the dynamics can be easily solved \[6–9\]. The thermodynamic solution has been obtained via the replica method, and it has a compact analytic form thanks to the spherical constraint: the solution predicts a random first-order transition from a high-temperature paramagnetic phase to a low temperature spin glass phase. The equilibrium and out-of-equilibrium dynamics have been solved via the generating functional formalism, and it is exact thanks to the mean-field nature of the model and the spherical constraint \[9, 10\].

Notwithstanding the success of fully-connected spherical models, we are well aware they have several unrealistic features: fully-connectedness is unlikely to happen in any realistic phenomenon and the spherical constraint is just a global surrogate for the actual constraint each variable should satisfy locally. In other words, in realistic models each variable is somehow bounded and one uses the single global spherical constraint to make computations easier. Although this approximation is extremely useful, it has some drawbacks. For example, when the interactions are diluted, a condensation phenomenon may take place \[11, 12\].
The diluted and sparse versions of a model are particularly interesting, because moving away from the fully connected limit is needed in order to study more realistic phenomena \cite{14,15}. We reserve the word sparse for graphs with a mean degree $O(1)$, i.e., not growing with $N$, while we use the term diluted for a graph which is not fully-connected, but whose mean degree still grows with $N$. In sparse models the couplings do not vanish in the large $N$ limit and this implies the solution is deeply non-perturbative. The cavity method has been developed exactly to solve sparse models \cite{19}. Diluted models are much less studied in the literature with respect to fully-connected and sparse models. Nonetheless they are very interesting for several aspects. They can be used in numerical simulations as a proxy for fully-connected models which are very demanding in terms of computing resources. They appear in models of random lasers where dilution is induced by the selection rules for the coupling of light modes in random media \cite{20–22}. Depending on the level of dilution, they allow for heterogeneities and local fluctuations in models that can still be solved similarly to the fully-connected version, that is exploiting that couplings are weak and the graph mean degree diverges. We believe it is worth dedicating more efforts in studying the realm of diluted models. In the present contribution we would like to set up the framework that would allow us to study diluted models via the cavity method. We are specially interested in spherical models, because they are models whose solution turns out to be particularly simple and compact. However, spherical models may undergo a condensation transition when the interaction graph is diluted. How the condensation transition can be avoided in a $p$-spin model by just modifying the spherical constraint is another open problem which we are currently investigating and which will be discussed elsewhere \cite{23}.

The study of whether condensation takes place is a delicate matter: this depends on a competition between the functional form of the global constraint, which can even be non-spherical, and the strength of the interactions, the latter depending on both the order of the non-linearity and the amount of dilution in the graph. Working with Hamiltonian models where variables interact via $p$-body terms and calling $M = O(N^\delta)$ the number of interaction terms, one would like to single out the threshold exponent $\delta_c$ such that for $\delta > \delta_c$ at finite temperature there is no condensation while for $\delta < \delta_c$ at any temperature the system is in the condensed phase. So far the situation is clear only for the two boundaries of the interval of possible values for $\delta$. For $\delta = p$, which represents the complete graph, condensation is never found at finite temperature, while the sparse graph, i.e., $\delta = 0$, is always in the condensed phase provided that interactions are non-linear, i.e., $p > 2$. The situation for intermediate values of $\delta$ is under current investigation, and we expect the present work to be an important footprint in this direction. For the moment we focus on the dilution regime where such a condensation phenomenon does not take place.

In the following we present the zero-th order step of the above program by showing how to use the cavity method to solve the fully-connected version of spherical spin glass models. Although the cavity method is well known \cite{24}, its use in spherical models did not appear before in the literature (to the best of our knowledge). The application of the cavity method to spherical models is not straightforward, because one has to decide how to convert a global constraint in a set of local ones. We will discuss this aspect explicitly and propose a standardized solution. Once the cavity equations are written, their solution requires some Ansatz for the distribution of local fields. This is one of the advantage of the cavity method with respect to the replica method: all assumptions made in the derivation have a clear and direct physical meaning. By using a Gaussian Ansatz for the distribution of local fields in the spin glass phase where the replica symmetry spontaneously breaks down we are able to obtain the exact solution to the spherical $p$-spin spin glass model, that was previously derived via the replica method. We dedicate the main text to the derivation of the saddle point equations, to the illustration of the Ansatz for the local field distributions, to the discussion on how to implement the spherical constraint and to report the resulting free-energies. More technical and lengthy derivations, as the explicit calculations of the free-energy, are postponed to the Appendices.

More in detail: in Sec. \textbf{II} we explain why a Gaussian ansatz for the cavity marginals is correct in the large degree limit and how to use it to obtain a closure of the Belief Propagation equations. In particular, in Sec. \textbf{III} we discuss the two possible choices to implement the spherical constraint in the Belief Propagation equations, which are equivalent only in the large degree limit. Sec. \textbf{III} is dedicated to the study of Survey Propagation equations, i.e., the generalization of Belief Propagation equations in the case of a one-step-replica-symmetry-breaking scenario. In Sec. \textbf{III}A we present the multivariate Gaussian ansatz needed for the Survey Propagation equations and in Sec. \textbf{III}B how their explicit closure is obtained by means of this ansatz. While the 1RSB expression of the free energy is reported in Sec. \textbf{III}C its explicit derivation in full detail can be found in the Appendices.
II. CAVITY EQUATIONS WITH SPHERICAL CONSTRAINT

A. Spherical models

We consider models with $N$ real variables $\sigma_i \in \mathbb{R}$ constrained to satisfy the condition

$$A[\sigma] = \sum_{i=1}^{N} \sigma_i^2 = N$$  \hspace{1cm} (1)

and interacting via $p$-body interactions

$$\mathcal{H} = -\sum_{a=1}^{M} J_a \prod_{i \in \partial a} \sigma_i,$$ \hspace{1cm} (2)

where $\partial a$ is the set of variables entering the $a$-th interaction and we fix $|\partial a| = p$. If the interaction graph is fully-connected then $M = \binom{N}{p}$ and the sum runs over all possible $p$-uples; otherwise, in diluted models, the $M$ interactions are randomly chosen among the $\binom{N}{p}$ possible $p$-uples. The fully-connected versions have been solved via the replica method.

For $p = 2$ the model is particularly simple because the energy function has only two minima and the free-energy can be computed from the spectrum of the interaction matrix $J$. The model possesses a spin glass phase at low temperatures, but the replica symmetry never breaks down and a replica symmetric (RS) ansatz provides the exact solution \[25\]. In this case the spherical constraint, although efficient in keeping variables bounded, changes drastically the low energy physics with respect to models with e.g. Ising variables: indeed the Sherrington-Kirkpatrick model \[26\] has a spin glass phase with spontaneous breaking of the replica symmetry \[27, 28\].

For $p \geq 3$ the spherical model is much more interesting since it undergoes a phase transition to a spin glass phase where the replica symmetry is broken just once (1RSB phase) \[6\] as in the analogous model with Ising variables \[29\]. More importantly the thermodynamic phase transition is preceded by a dynamical phase transition \[7\] which has been connected to the structural glass transition \[30, 31\] and to the mode coupling theory \[32\]. The spherical $p$-spin model with $p \geq 3$ represents now the most used mean-field model for the random first order transition \[33\].

B. Self-consistent cavity equations for the local marginals

The replica method allows to fully characterize the static properties of the spherical $p$-spin model, as was firstly done in Ref. \[6\]. Our purpose is to study spherical $p$-spin models on complete graphs, showing that the cavity method is equivalent to replicas. A complete hypergraph can be seen as a bipartite graph made of function nodes, representing the interaction $p$-uplets, and variable nodes, representing the $N$ spins $\sigma_i$'s. We will indicate the set of links between function and variable nodes as edges $E$. A complete graph has $M = \binom{N}{p} = O(N^p)$ function nodes, each of which is linked to $p$ variable nodes. On the other hand, each variable node is linked to $K = \binom{N-1}{p-1} = O(N^{p-1})$ function nodes.

In order to ensure the extensivity of the energy, not only the $N$ real variables must satisfy the spherical constraint in Eq. \[1\], but the couplings $\{J_a\}$, which are independent and identically distributed quenched random variables, must be properly normalized: in the case of symmetric couplings we have

$$\langle J \rangle = 0 \quad , \quad \langle J^2 \rangle = \frac{p! J_2}{2^{Np-1}},$$ \hspace{1cm} (3)

with $J_2 = O(1)$ to ensure an extensive energy. Since we have in mind to extend the results of the present study to the case of increasing dilution of the hypergraph, let us start from the statistical ensemble where the partition function of the model, and hence the corresponding thermodynamic potentials, is always well defined, i.e., the microcanonical ensemble.

In presence of the spherical constraint written in Eq. \[1\] the partition function of the model reads thus

$$\Omega_A(E, N) = \int d\sigma \delta(E - \mathcal{H}[\sigma]) \delta(A - A[\sigma]).$$  \hspace{1cm} (4)

The first, very important, assumption of the present derivation is the equivalence between the ensemble with hard
constraints on both $A$ and $E$, i.e. the partition function written in Eq. (1), and the one where the same spherical constraints are realized via a Lagrange multiplier. This means that the study of the partition function in Eq. (1) is fully equivalent to that of its Laplace transform:

$$
Z_{\lambda}(\beta, N) = \int_0^\infty dA \ e^{-\lambda A} \int_{-\infty}^{+\infty} dE \ e^{-\beta E} \ Omega_A(E, N) = \int \prod_{i=1}^N d\sigma_i \ \exp \left\{ \ -\lambda \sum_{i=1}^N \sigma_i^2 + \beta \sum_{a=1}^M \sum_{i \in \partial a} \sigma_i \right\}, \quad (5)
$$

For a given choice of values $A$ and $E$ the ensembles are equivalent if and only if it is possible to find real values of the Lagrange multipliers $\lambda$ and $\beta$ such that

$$
A = -\frac{\partial}{\partial \lambda} \log [Z_{\lambda}(\beta, N)]
$$

$$
E = -\frac{\partial}{\partial \beta} \log [Z_{\lambda}(\beta, N)] \quad (6)
$$

In this paper we will consider only choices of $E \propto A \propto N$ such that it is possible to find real positive values of $\lambda$ and $\beta$ which solve the equations in Eq. (6). It is nevertheless important to keep in mind that there are situations where Eq. (6) does not have a solution in terms of either a real $\lambda$ or a real $\beta$: this is the situation where the equivalence of ensembles breaks down and we expect it to happen in sparse hypergraphs, where condensation takes place. See for instance the recent discussion in [13].

Let us now introduce the cavity approach to solve the analyzed problem. We will introduce two kinds of cavity messages: with $\eta_{i \rightarrow a}(\sigma_i)$ we will indicate the variable-to-function cavity message, that indicates the probability that the spin on the $i$ node assumes the value $\sigma_i$ in the absence of the link between the variable node $i$ and the function node $a$. Analogously with $\hat{\eta}_{b \rightarrow i}(\sigma_i)$ we indicate the function-to-variable cavity message. In the general case $\eta_{i \rightarrow a}(\sigma_i)$ will depend on all the messages $\hat{\eta}_{b \rightarrow i}(\sigma_i)$, with $b \in \partial i \setminus a$, that are correlated random variables. However for tree-like graphs they are independent, due to the absence of loops. Loops are negligible at the leading order also on the Bethe lattice, which is locally tree-like (there are loops of size $\log(N)$). A complete graph is not at all locally tree-like, since each spin participates in $O(N^p)$ interactions, and there are always short loops. Nevertheless, due to the vanishing intensity of coupling constants $J_a$, i.e. $(J^2) \sim 1/N^p-1$, $\hat{\eta}_{b \rightarrow i}(\sigma_i)$, with $b \in \partial i \setminus a$, behave as independent random variables even on complete graphs.

This allows us to introduce the following cavity equations:

$$
\eta_{i \rightarrow a}(\sigma_i) = \frac{1}{Z_{i \rightarrow a}} \prod_{b \in \partial i \setminus a} \hat{\eta}_{b \rightarrow i}(\sigma_i) \quad (7)
$$

$$
\hat{\eta}_{a \rightarrow i}(\sigma_i) = \frac{1}{Z_{a \rightarrow i}} e^{-\frac{\lambda \sigma_i^2}{2K}} \prod_{j \in \partial a \setminus i} \hat{\eta}_{j \rightarrow a}(\sigma_j) \ \exp \left\{ \ \beta J_a \sigma_i \prod_{j \in \partial a \setminus i} \sigma_j \right\}, \quad (8)
$$

with $Z_{i \rightarrow a}$ and $\hat{Z}_{a \rightarrow i}$ that are normalization constants to ensure that the messages are normalized:

$$
\int_{-\infty}^{\infty} d\sigma_i \eta_{i \rightarrow a}(\sigma_i) = \int_{-\infty}^{\infty} d\sigma_i \hat{\eta}_{i \rightarrow a}(\sigma_i) = 1.
$$

Let us spend few words on the way we have transformed the global spherical constraint into the local terms $\exp\left(\frac{\lambda \sigma_i^2}{2K}\right)$ appearing in the equations for the cavity marginals $\hat{\eta}_{a \rightarrow i}(\sigma_i)$. The factor $1/2$ is just convenient for the definition of the Gaussian distributions (Lagrange multiplier can be changed by a multiplicative factor without changing the physics). Although the most natural place to insert the spherical constraint would be as an external field in the equation for the cavity marginal $\eta_{i \rightarrow a}(\sigma_i)$, our choice turn out to simplify the computations and we prove in Sec. II F to be equivalent to the other one. We notice that the idea of moving the external field from the variables to the interactions is not new. It is used, for example, in the real space renormalization group.

Once Eqs. (7,8) are solved (e.g. in an iterative way as in the Belief Propagation algorithm), the local marginals for each spin $a$ are given by

$$
\eta_i(\sigma_i) = \frac{1}{Z_i} \prod_{b \in \partial i} \hat{\eta}_{b \rightarrow i}(\sigma_i) \quad (9)
$$

with $Z_i$ a new normalization constant.
C. The Gaussian Ansatz in the large degree limit

In the fully-connected model, but also in diluted models, the mean degree grows and diverges in the large $N$ limit. At the same time the coupling intensities decrease as $N^{-(p-1)/2}$ to ensure well defined local fields. In this limit we can close the cavity equations with the following Gaussian Ansatz for the cavity marginal distribution

$$
\eta_{i \to a}(\sigma_i) = \frac{1}{\sqrt{2\pi v_{i \to a}}} \exp \left[ -\frac{(\sigma_i - m_{i \to a})^2}{2v_{i \to a}} \right] \propto \exp \left[ \frac{m_{i \to a}}{v_{i \to a}} \sigma_i - \frac{1}{2v_{i \to a}} \sigma_i^2 \right] \tag{10}
$$

Since $\langle J^2 \rangle \sim 1/N^{p-1}$ the large $N$ limit is equivalent to a small $J$ or high-temperature expansion, known as the Plefka/Georges-Yedidia expansion \cite{34,35}. Expanding to second order in $J$, and inserting the Ansatz Eq. (10) we get

$$
\hat{\eta}_{a \to i}(\sigma_i) = \frac{1}{Z_{a \to i}} e^{-\frac{\lambda^2}{2\pi}} \int \prod_{j \in \partial a \setminus i} d\sigma_j \eta_{j \to a}(\sigma_j) \exp \left\{ \beta J_a \sigma_i \prod_{j \in \partial a \setminus i} \sigma_j \right\} = \frac{1}{Z_{a \to i}} e^{-\frac{\lambda^2}{2\pi}} \left[ 1 + \beta J_a \sigma_i \prod_{j \in \partial a \setminus i} m_{j \to a} + \frac{\beta^2 J_a^2}{2} \prod_{j \in \partial a \setminus i} \left( m_{j \to a}^2 + v_{j \to a} \right) \right] = \frac{1}{Z_{a \to i}} e^{-\frac{\lambda^2}{2\pi}} \exp \left\{ \beta J_a \sigma_i \prod_{j \in \partial a \setminus i} m_{j \to a} + \frac{\beta^2 J_a^2}{2} \prod_{j \in \partial a \setminus i} \left( m_{j \to a}^2 + v_{j \to a} \right) - \prod_{j \in \partial a \setminus i} m_{j \to a}^2 \right\} \tag{11}
$$

and

$$
\eta_{i \to a}(\sigma_i) = \frac{1}{Z_{i \to a}} \prod_{b \in \partial a \setminus \{i\}} \hat{\eta}_{b \to i}(\sigma_i) = e^{-\frac{\lambda^2}{2\pi}} \exp \left\{ \beta \sigma_i \sum_{b \in \partial a \setminus \{i\}} J_b \prod_{j \in \partial b \setminus i} m_{j \to b} + \frac{\beta^2}{2} \sum_{b \in \partial a \setminus \{i\}} J_b^2 \left( \prod_{j \in \partial b \setminus i} \left( m_{j \to b}^2 + v_{j \to b} \right) - \prod_{j \in \partial b \setminus i} m_{j \to b}^2 \right) \right\} \tag{12}
$$

Comparing Eq. (11) and Eq. (12), one obtains the following self consistency equations for the means and the variances of the Gaussian marginals:

$$
\frac{m_{i \to a}}{v_{i \to a}} = \beta \sum_{b \in \partial a \setminus \{i\}} J_b \prod_{j \in \partial b \setminus i} m_{j \to b}
$$

$$
\frac{1}{v_{i \to a}} = \lambda - \beta^2 \sum_{b \in \partial a \setminus \{i\}} J_b^2 \left( \prod_{j \in \partial b \setminus i} \left( m_{j \to b}^2 + v_{j \to b} \right) - \prod_{j \in \partial b \setminus i} m_{j \to b}^2 \right) \tag{13}
$$

The $\lambda$ parameter has to be fixed in order to satisfy the spherical constraint $\sum_i (\sigma_i^2) = N$, where the average is taken over the marginals defined in Eq. (11). However, given that we are in a dense system, cavity marginal and full marginals differ by just terms of order $O(1/N)$, so we can impose the spherical constraint using cavity marginals. These are the replica symmetric cavity equations for dense (fully-connected or diluted) spherical $p$-spin models.

D. The homogeneous solution to the cavity equations

Let us now consider the homogeneous solution for the replica symmetric case: $m_{i \to a} = m$, $v_{i \to a} = v$ and $m^2 = g$, where $g$ is called the self overlap. This solution is the correct one if the graph is not too heterogeneous (e.g. a complete graph) and the replica symmetry is not broken. In this case the Gaussian ansatz in Eq. (11) simply reduces to

$$
\eta_{i \to a}(\sigma_i) = \frac{1}{\sqrt{2\pi v}} \exp \left[ -\frac{(\sigma_i - m)^2}{2v} \right] \tag{14}
$$
By imposing the spherical constraint, \( \sum_i \langle \sigma_i^2 \rangle = N \), one gets the identity \( v = 1 - m^2 = 1 - q \), from which we have
\[
v = 1 - m^2 = 1 - q, \quad (15)
\]
and the function-to-variable cavity message (11) assumes the simple form:
\[
\hat{\eta}_{a \rightarrow i}(\sigma_i) = \frac{1}{Z_{a \rightarrow i}} \exp \left\{ -\frac{\lambda}{2K} \sigma_i^2 + \beta J_a m^{p-1} \sigma_i + \frac{1}{2} \beta^2 J_a^2 (1 - q^{p-1}) \sigma_i^2 \right\}. \quad (16)
\]
The self-consistent conditions in Eq (13) simply reduce to:
\[
mv = \beta m^{p-1} \sum_{a=1}^{K-1} J_a, \quad (17)
\]
\[
\frac{1}{v} = \lambda - \beta^2 \left[ (v + m^2)^{p-1} - m^{2(p-1)} \right] \sum_{a=1}^{K-1} J_a^2, \quad (18)
\]
We notice that in the dense limit \( K \) diverges with \( N \) and the two summations in the saddle point equations can be replaced by the first moments of the coupling distribution. From Eq. (18) we rewrite the Lagrange multiplier \( \lambda \) as
\[
\lambda = \frac{1}{1 - q} + \beta^2 (1 - q^{p-1}) \sum_{a=1}^{K-1} J_a^2. \quad (19)
\]
It can be checked by using the last expression of \( \lambda \) that the normalization of messages \( \hat{\eta}_{a \rightarrow i}(\sigma_i) \) is always well defined in the limit of large \( N \).

**E. The replica symmetric free energy**

We have now all the pieces we need to compute the replica symmetric free energy of the model, which is defined as [24]:
\[
-\beta F \equiv \beta \left( \sum_{a=1}^{M} F_a + \sum_{i=1}^{N} F_i - \sum_{(ai) \in E} F_{ai} \right) \equiv \sum_{a=1}^{M} \log(Z_a) + \sum_{i=1}^{N} \log(Z_i) - \sum_{(ai) \in E} \log(Z_{(ai)}), \quad (20)
\]
where we have respectively
\[
Z_a = \int_{-\infty}^{\infty} \prod_{i \in \partial a} d\sigma_i \, \hat{\eta}_{i \rightarrow a}(\sigma_i) \, e^{\beta J_a} \prod_{\ell \in \partial a} \sigma_i, \quad (21)
\]
\[
Z_i = \int_{-\infty}^{\infty} d\sigma_i \, \prod_{a \in \partial i} \hat{\eta}_{a \rightarrow i}(\sigma_i) \quad (22)
\]
\[
Z_{(ai)} = \int_{-\infty}^{\infty} d\sigma_i \, \hat{\eta}_{a \rightarrow i}(\sigma_i) \, \eta_{i \rightarrow a}(\sigma_i). \quad (23)
\]
The computation of these three terms is reported in the Appendix A. Here we just report the final result:
\[
-\beta F_{RS} = \frac{N}{2} \left[ \beta^2 (1 - q^p) J_2 + \log(1 - q) + \frac{q}{(1 - q)} \right], \quad (24)
\]
The free energy written in Eq. (24) is identical to that of the spherical \( p \)-spin computed with replicas in the replica symmetric case, see Eq. (4.4) of [6]. From now on we will set \( J_2 = 1 \).
F. Alternatives for the spherical constraint: equivalence in the large-$N$ limit.

The experienced reader will have probably noticed that the way we have introduced the spherical constraint in the cavity equations is not, perhaps, the most natural one, that would correspond to an external field of intensity $\lambda$ acting on every spin. As such, we should have put

$$\eta_{i \to a}(\sigma_i) \propto e^{-\frac{1}{2} \lambda^2 \sigma_i^2},$$

(25)

rather than

$$\hat{\eta}_{a \to i}(\sigma_i) \propto e^{-\frac{1}{2\lambda} \sigma_i^2},$$

(26)

as we have done in the equations for the cavity marginals, Eq. (17) and Eq. (18). In what follows we show that the choice of where to put the spherical constraint is arbitrary in the large-$N$ limit. In practice we are going to show that either we let the constraint act as an external field in the variable-to-function message $\eta_{i \to a}(\sigma_i)$, as in Eq. (25), or inside the function-to-variable marginal $\hat{\eta}_{a \to i}(\sigma_i)$, as in Eq. (26), in both cases we obtain the same expression for the free energy to the leading order in $N$. The reader must therefore bear in mind that the two ways to put the constraint in the cavity equations might not be equivalent in the case of a graph with finite connectivity.

After a trial and error procedure we realized that the choice in Eq. (26) makes all calculations simpler, so that we opted for this one. We have already shown that by doing so we obtain, at high temperature, a free energy which is identical to the one obtained from mean-field replica calculations, Eq. (24). We now want to show explicitly that, term by term and beside any further assumption as the one of homogeneity, the free energy in the high temperature ergodic phase is identical for the two choices [Eq. (25) and Eq. (26)] to introduce the constraint.

Let us term $\eta^{(A)}_{i \to a}(\sigma_i)$ and $\hat{\eta}^{(A)}_{a \to i}(\sigma_i)$ the local cavity marginals corresponding to the case where the field $\lambda$ acts directly on the spin:

$$\eta^{(A)}_{i \to a}(\sigma_i) = \frac{1}{Z^{(A)}_{i \to a}} e^{-\lambda \sigma_i^2} \prod_{b \in \partial i \setminus a} \hat{\eta}^{(A)}_{b \to i}(\sigma_i)$$

(27)

$$\hat{\eta}^{(A)}_{a \to i}(\sigma_i) = \frac{1}{Z^{(A)}_{a \to i}} \int_{-\infty}^{\infty} \prod_{i \in \partial a \setminus i} d\sigma_j \eta^{(A)}_{j \to a}(\sigma_j) \exp \left\{ \beta J a \sigma_i \prod_{j \in \partial a \setminus i} \sigma_j \right\} .$$

(28)

Accordingly, since in the function-to-variable messages there is now no trace of the external field, one has to consider the following modified definition of the entropic term in the local partition functions:

$$Z^{(A)}_{a} = \int_{-\infty}^{\infty} \prod_{i \in \partial a} d\sigma_i \eta^{(A)}_{i \to a}(\sigma_i) e^{\beta J a} \prod_{i \in \partial a} \sigma_i$$

(29)

$$Z^{(A)}_{i} = \int_{-\infty}^{\infty} d\sigma_i e^{-\lambda \sigma_i^2 / 2} \prod_{a \in \partial i} \hat{\eta}^{(A)}_{a \to i}(\sigma_i)$$

(30)

$$Z^{(A)}_{(ai)} = \int_{-\infty}^{\infty} d\sigma_i \hat{\eta}^{(A)}_{a \to i}(\sigma_i) \eta^{(A)}_{i \to a}(\sigma_i).$$

(31)

Our task is now to show that:

$$\sum_{a=1}^{M} \log(Z_a) + \sum_{i=1}^{N} \log(Z_i) - \sum_{(ai) \in E} \log(Z_{(ai)}) = \sum_{a=1}^{M} \log(Z^{(A)}_{a}) + \sum_{i=1}^{N} \log(Z^{(A)}_{i}) - \sum_{(ai) \in E} \log(Z^{(A)}_{(ai)}).$$

(32)

The key observation is that, in order to have overall consistency, the Gaussian ansatz for the variable-to-function message must be the same in both cases, that is:

$$\eta_{i \to a}(\sigma_i) = \frac{1}{\sqrt{2\pi \nu_{i \to a}}} \exp \left[ -\frac{(\sigma_i - m_{i \to a})^2}{2 \nu_{i \to a}} \right] = \eta^{(A)}_{i \to a}(\sigma_i).$$

(33)

The assumption of Eq. (33) allows us to conclude immediately that $Z^{(A)}_{a} = Z_{a}$, so that the identity we need to prove
reduces to:

\[ \sum_{i=1}^N \log(Z_i) - \sum_{(a_i) \in E} \log(Z_{(a_i)}) = \sum_{i=1}^N \log(Z_i^{(\lambda)}) - \sum_{(a_i) \in E} \log(Z_{(a_i)}^{(\lambda)}) \]  

(34)

By exploiting Eq. (33) once again we obtain

\[ \dot{\eta}_{a ightarrow i}(\sigma_i) = \frac{1}{Z_{a ightarrow i}^{(\lambda)}} \int_{-\infty}^{\infty} d\sigma_j \sum_{j \in \partial a \setminus i} \eta_{j ightarrow a}(\sigma_j) \exp \left\{ \beta J_a \sigma_i \prod_{j \in \partial a \setminus i} \sigma_j \right\}, \]  

(35)

that, by comparison with the definition Eq. (8) leads to

\[ \dot{\eta}_{a ightarrow i}(\sigma_i) \dot{Z}_{a ightarrow i}^{(\lambda)} = \dot{\eta}_{a ightarrow i}(\sigma_i) \dot{Z}_{a ightarrow i}^{(\lambda)} e^{\frac{\lambda \sigma_i^2}{2}}, \]  

(36)

so that

\[ \dot{\eta}_{a ightarrow i}(\sigma_i) \dot{Z}_{a ightarrow i}^{(\lambda)} = \dot{\eta}_{a ightarrow i}(\sigma_i) \dot{Z}_{a ightarrow i}^{(\lambda)} e^{\frac{\lambda \sigma_i^2}{2}}. \]  

(37)

By inserting Eq. (37) in the definition of \( Z_i^{(\lambda)} \) in Eq. (30) one finds:

\[ Z_i^{(\lambda)} = \int_{-\infty}^{\infty} d\sigma_i e^{-\gamma_0 \sigma_i^2/2} \prod_{a \in \partial i} \dot{\eta}_{a ightarrow i}(\sigma_i) \]  

\[ = \prod_{a \in \partial i} \left( \frac{\dot{Z}_{a ightarrow i}}{Z_{a ightarrow i}^{(\lambda)}} \right) \int_{-\infty}^{\infty} d\sigma_i \prod_{a \in \partial i} \dot{\eta}_{a ightarrow i}(\sigma_i) \]  

\[ = \prod_{a \in \partial i} \left( \frac{\dot{Z}_{a ightarrow i}}{Z_{a ightarrow i}^{(\lambda)}} \right) Z_i, \]  

(38)

so that the identity that we want prove is further simplified in

\[ \sum_{(a_i) \in E} \log(Z_{(a_i)}) = \sum_{(a_i) \in E} \log(Z_{(a_i)}^{(\lambda)}) - \sum_{i=1}^N \sum_{a \in \partial i} \log \left( \frac{\dot{Z}_{a ightarrow i}}{Z_{a ightarrow i}^{(\lambda)}} \right). \]  

(39)

Using, once again, Eq. (33) we can write

\[ Z_{(a_i)}^{(\lambda)} = \int_{-\infty}^{\infty} d\sigma_i \dot{\eta}_{a ightarrow i}(\sigma_i) \eta_{i ightarrow a}(\sigma_i) \]  

\[ = \int_{-\infty}^{\infty} d\sigma_i \dot{\eta}_{a ightarrow i}(\sigma_i) \eta_{i ightarrow a}(\sigma_i) \]  

\[ = \frac{\dot{Z}_{a ightarrow i}}{Z_{a ightarrow i}^{(\lambda)}} \int_{-\infty}^{\infty} d\sigma_i e^{\frac{\lambda \sigma_i^2}{2}} \dot{\eta}_{a ightarrow i}(\sigma_i) \eta_{i ightarrow a}(\sigma_i) \]  

\[ \approx \frac{\dot{Z}_{a ightarrow i}}{Z_{a ightarrow i}^{(\lambda)}} Z_{(a_i)}, \]  

(40)

where the last line equality holds for large \( N \) (see Eqns. (A12), (A14) and (A15)). The \( N \rightarrow \infty \) limit is equivalent to the \( K \rightarrow \infty \) limit, since \( K \sim N^{p-1} \). Finally, by plugging the result of Eq. (40) into Eq. (39) we can conclude that the identity in Eq. (39) is true in the limit \( N \rightarrow \infty \). We have thus demonstrated that in the large-\( N \) limit it is equivalent, and thus just a matter of convenience, to write down explicitly the spherical constraint inside the definition of the function-to-variable message \( \hat{\eta}_{a ightarrow i}(\sigma) \), as we have done, or inside the definition of the variable-to-function one, \( \hat{\eta}_{i ightarrow a}(\sigma) \).
In the previous sections we have reviewed the replica symmetric solution that is the stable one for high temperatures. In this phase we have written closed cavity equations for the marginal distributions of the variables, relying on the assumption that the joint distribution of the cavity variables is factorized as in a single pure state.

However, lowering the temperature, it is known from the replica solution [6], that several metastable glassy states arise on top of the paramagnetic state. Their number being exponential in N with a rate Σ called complexity. The function Σ(f) is in general an increasing function of the state free-energy f, with a downwards curvature (for stability reasons as for the entropy).

Comparing the total free-energy of the glassy states computed using Σ(f) and the paramagnetic free-energy [36], one can derive the dynamical critical temperature \( T_d \) where the ergodicity breaks down and the thermodynamic critical temperature, also called Kauzmann temperature \( T_K \), where a phase transition to a replica symmetry breaking phase takes place.

Below \( T_d \) the dynamics of the model is dominated by the states of larger free-energy, so-called threshold states, which are the most abundant and always exponentially many in N (although a more refined picture has been recently presented in [37]).

For \( T < T_d \) the Gibbs measure is split over many different states, such that two different equilibrium configurations can be in the same (metastable) state or in different states. Defining the overlap between two different configurations as how much they are close to each other, the 1RSB phase is characterized by an overlap \( q_1 \) between configurations inside the same pure state (independently of the pure state) and an overlap \( q_0 < q_1 \) between configurations in two different states.

In formulas, the presence of many metastable pure states yields an additional contribution to the free-energy. The complexity \( \Sigma(f) \), that counts the number of “states” (disjoint ergodic components of the phase-space) with the same free-energy \( f \) can be written as

\[
\Sigma(f) = \frac{1}{N} \log \left[ \sum_{\eta=1}^{N} \delta(f - f_{\eta}) \right],
\]

where \( N \) is the total number of metastable glassy states (formally they can be defined as the non-paramagnetic stationary points of the TAP free-energy [38]) and \( f_{\eta} \) is the free energy of the glassy state \( \eta \). Please notice that expression in Eq. (41) is identical to the standard microcanonical definition of entropy, with the only difference that now we measure the number of phase-space regions with the same free-energy rather than the volume of phase space with the same energy. The total free-energy is thus given by:

\[
\mathcal{F} = -\frac{1}{\beta N} \log Z = -\frac{1}{\beta N} \log \left( \sum_{\eta} e^{-\beta N f_{\eta}} \right) = -\frac{1}{\beta N} \log \int df \sum_{\eta} \delta(f - f_{\eta}) e^{-\beta N f} = -\frac{1}{\beta N} \log \int df e^{-N(\beta f - \Sigma)}
\]  

(42)

The problem is that we do not know how to characterize the different states and how to count them to obtain \( \Sigma \): we are still not able to compute \( \mathcal{F} \). In the following we will solve this problem applying the method of real coupled replicas introduced by Monasson in Ref. [39] (see also [22] for a rigorous derivation and [36] for a pedagogical review). This method was applied to the spherical \( p \)-spin in Ref. [41] to compute the 1RSB free-energy with a replica computation. The idea of Ref. [39] is to introduce \( x \) real clones, that we will call replicas, on a single realization of a graph. These replicas will be infinitesimally coupled together in such a way that, even when the coupling between them goes to zero, they will all fall in the same pure state below \( T_d \): this cloning method is a way to select a state equivalent to what is usually done in ferromagnetic systems to select a state adding an infinitesimal magnetic field.

The free energy \( \Phi(x) \) of \( x \) replicas in the same state is:

\[
\Phi(x) = -\frac{1}{\beta N} \log \left( \sum_{\eta} e^{-\beta N x f_{\eta}} \right) = -\frac{1}{\beta N} \log \int df e^{-N(\beta x f - \Sigma)} = -\frac{1}{\beta} \max_{f}(\beta x f - \Sigma(f)).
\]

(43)

The complexity in this way simply results in the Legendre transform of the free energy of the replicated system. The total free-energy in the 1RSB phase is derived passing to the analytical continuation of \( x \) to real values and turns out to be: \( \mathcal{F} = \min_{x} \frac{\Phi(x)}{x} \).

In the following we will use this cloning method to write 1RSB closed cavity equations for the spherical \( p \)-spin, in a way analogous to what has been done for the planted SK model in Ref. [41]. In a situation with many pure states, the factorization of the distribution of the cavity variables is valid only inside a single pure state: we can thus still write
some cavity closed equations considering the coupled replicas in a same pure state. Then, we will compute the 1RSB free energy in a cavity approach below $T_d$, obtaining exactly the same expression found with replica computations in Refs. \[40\].

### A. The ansatz for the distribution of $x$ coupled replicas

For the RS phase, in the dense case, we have written a Gaussian ansatz for the marginal probability of the spin on a given site in Eq. (14). In the 1RSB phase, we will consider the joint probability distribution of $x$ coupled replicas that are all in a same pure state. We will comment in the next sections on the choice and the physical meaning of $x$. In order to lighten the notation, let us indicate as $\sigma_i = \{\sigma_i^\alpha\}$, $\alpha = 1, \ldots, x$, the vector of all $x$ replicas on site $i$. The 1RSB form of the ansatz for the marginal probability $\eta_{i \rightarrow a}(\sigma_i)$ amounts to

$$\eta_{i \rightarrow a}(\sigma_i) = \int_{-\infty}^{\infty} dm_{i \rightarrow a} \frac{1}{\sqrt{2\pi} \Delta_i^{(0)}(a)} \exp \left(-\frac{(m_{i \rightarrow a} - h_{i \rightarrow a})^2}{2\Delta_i^{(0)}(a)}\right) \prod_{\alpha=1}^{x} \exp \left(-\frac{\sum_{i=1}^{\alpha} (\sigma_i^\alpha - m_{i \rightarrow a})^2}{2\Delta_i^{(1)}(a)}\right)$$  (44)

By shortening the integration measure for the joint probability distribution $\sigma_i$ with the symbol

$$\int D\sigma_i = \int_{-\infty}^{\infty} \prod_{\alpha=1}^{x} d\sigma_i^\alpha,$$

and defining the distribution

$$Q_{i \rightarrow a}(m_{i \rightarrow a}) \equiv \frac{1}{\sqrt{2\pi} \Delta_i^{(0)}(a)} \exp \left(-\frac{(m_{i \rightarrow a} - h_{i \rightarrow a})^2}{2\Delta_i^{(0)}(a)}\right),$$  (46)

the first diagonal and second moments of the cavity marginal are simply computed as:

$$\langle \sigma_i^\alpha \rangle = \int D\sigma_i \sigma_i^\alpha \eta_{i \rightarrow a}(\sigma_i) = \int_{-\infty}^{\infty} dm_{i \rightarrow a} m_{i \rightarrow a} Q_{i \rightarrow a}(m_{i \rightarrow a}) = h_{i \rightarrow a}$$

$$\langle (\sigma_i^\alpha)^2 \rangle = \int D\sigma_i (\sigma_i^\alpha)^2 \eta_{i \rightarrow a}(\sigma_i) = \int_{-\infty}^{\infty} dm_{i \rightarrow a} \left(\Delta_i^{(1)}(a) + m_i^2\right) Q_{i \rightarrow a}(m_{i \rightarrow a}) = \Delta_i^{(1)}(a) + \Delta_i^{(0)}(a) + h_{i \rightarrow a}^2$$

$$\langle \sigma_i^\alpha \sigma_i^\beta \rangle = \int D\sigma_i \sigma_i^\alpha \sigma_i^\beta \eta_{i \rightarrow a}(\sigma_i) = \int_{-\infty}^{\infty} dm_{i \rightarrow a} m_{i \rightarrow a}^2 Q_{i \rightarrow a}(m_{i \rightarrow a}) = \Delta_i^{(0)}(a) + h_{i \rightarrow a}^2$$  (47)

Let us comment briefly on the form of the Ansatz. The marginal probability of a single replica in a given state is still a Gaussian, being on a dense graph. If the real replicas are coupled, they will fall in the same state. The only effect of the infinitesimal coupling between the replicas will be that the configurations of the real replicas will be independent variables extracted from the same distribution in each state, once average $m$ and variance $\Delta^{(1)}$ are given:

$$\eta_{i \rightarrow a}(\sigma_i^\alpha) \equiv \frac{1}{\sqrt{2\pi} \Delta_i^{(1)}(a)} \exp \left(-\frac{(\sigma_i^\alpha - m_{i \rightarrow a})^2}{2\Delta_i^{(1)}(a)}\right).$$  (48)

In the same way, the average magnetizations in different states will be independent variables extracted from the same distribution $Q_{i \rightarrow a}(m_{i \rightarrow a})$, that will depend on $\Delta^{(0)}$ and $h$, see, e.g., ref. \[42\].

With this simple scenario in mind, we can give a simple physical interpretation to the parameters of the distribution in Eq. (14) rewriting them as:

$$h_{i \rightarrow a} = \langle \sigma_i^\alpha \rangle$$

$$\Delta_i^{(1)}(a) = \langle (\sigma_i^\alpha)^2 \rangle - \langle \sigma_i^\alpha \sigma_i^\beta \rangle = 1 - q_{i \rightarrow a}^{\alpha \beta}$$

$$\Delta_i^{(0)}(a) = \langle \sigma_i^\alpha \sigma_i^\beta \rangle - \langle \sigma_i^\alpha \rangle^2 = q_{i \rightarrow a}^{\alpha \beta} - q_{i \rightarrow a}^{\alpha \alpha},$$  (49)

where the average is taken with respect to the probability distribution in Eq. (14). $q_{i \rightarrow a}^{\alpha \beta}$ and $q_{i \rightarrow a}^{\alpha \alpha}$ are the local overlap (in absence of the link from $i$ to $a$) inside a state and between states that we mentioned at the beginning of
this Section. Obviously on a complete graph, they will be independent of \(i\) and \(a\), as for the only parameter in the RS case (the magnetization) in the homogeneous case. However, here we prefer to write explicitly the dependence on \(i\) and \(a\), because in this way the equations we will obtain could be easily applied to non-complete graphs.

### B. 1RSB cavity equations

We now write the replicated cavity equations for the 1RSB Ansatz introduced in the previous section:

\[
\eta_{i \rightarrow a}(\sigma_i) \propto \prod_{k \in \partial a} \hat{\eta}_{k \rightarrow i}(\sigma_i)
\]

\[
\hat{\eta}_{a \rightarrow i}(\sigma_i) \propto e^{-\frac{1}{2\beta} \sum_{\alpha=1}^{x} (\sigma_{i}^{\alpha})^{2}} \int_{-\infty}^{\infty} \prod_{k \in \partial a \setminus i} \mathcal{D}\sigma_k \ \eta_{k \rightarrow a}(\sigma_k) \ \exp \left\{ \beta J_a \sum_{\alpha=1}^{x} \sigma_{i}^{\alpha} \prod_{k \in \partial a \setminus i} \sigma_{k}^{\alpha} \right\}.
\]

We have omitted the normalization factors that are irrelevant in the subsequent computations. As we did for the RS case, in the dense limit we take the leading term in a small \(J_a\) expansion (valid in the large \(N\) limit for dense graphs) and in this setting we will close the equations on the parameters of the multivariate Gaussian. That is, we write:

\[
\int_{-\infty}^{\infty} \prod_{k \in \partial a \setminus i} \mathcal{D}\sigma_k \ \eta_{k \rightarrow a}(\sigma_k) \ \exp \left\{ \beta J_a \sum_{\alpha=1}^{x} \sigma_{i}^{\alpha} \prod_{k \in \partial a \setminus i} \sigma_{k}^{\alpha} \right\} \approx \int_{-\infty}^{\infty} \prod_{k \in \partial a \setminus i} \mathcal{D}\sigma_k \ \eta_{k \rightarrow a}(\sigma_k) \ \left[ 1 + \beta J_a \sum_{\alpha=1}^{x} \sigma_{i}^{\alpha} \prod_{k \in \partial a \setminus i} \sigma_{k}^{\alpha} + \frac{1}{2} \beta^2 J_a^2 \left( \sum_{\alpha=1}^{x} \sigma_{i}^{\alpha} \prod_{k \in \partial a \setminus i} \sigma_{k}^{\alpha} \right)^2 \right] =
\]

\[
= 1 + \beta J_a \left( \prod_{k \in \partial a \setminus i} h_{k \rightarrow a} \right) \sum_{\alpha=1}^{x} \sigma_{i}^{\alpha} + \frac{1}{2} \beta^2 J_a^2 \left[ \prod_{k \in \partial a \setminus i} \left( \Delta_{k \rightarrow a}^{(1)} + \Delta_{k \rightarrow a}^{(0)} + h_{k \rightarrow a}^2 \right) \right] \sum_{\alpha=1}^{x} \sigma_{i}^{\alpha} \sum_{\alpha \neq \beta} \sigma_{i}^{\beta} \approx \exp \left\{ \hat{A}_{a \rightarrow i} \sum_{\alpha=1}^{x} \sigma_{i}^{\alpha} - \frac{1}{2} \hat{B}_{a \rightarrow i}^{(d)} \sum_{\alpha=1}^{x} (\sigma_{i}^{\alpha})^2 + \frac{1}{2} \hat{B}_{a \rightarrow i}^{(nd)} \sum_{\alpha \neq \beta} \sigma_{i}^{\alpha} \sigma_{i}^{\beta} \right\},
\]

where the three coefficients are respectively

\[
\hat{A}_{a \rightarrow i} = \beta J_a \prod_{k \in \partial a \setminus i} h_{k \rightarrow a}
\]

\[
\hat{B}_{a \rightarrow i}^{(d)} = \beta^2 J_a^2 \left[ \prod_{k \in \partial a \setminus i} h_{k \rightarrow a}^2 - \prod_{k \in \partial a \setminus i} \left( \Delta_{k \rightarrow a}^{(1)} + \Delta_{k \rightarrow a}^{(0)} + h_{k \rightarrow a}^2 \right) \right]
\]

\[
\hat{B}_{a \rightarrow i}^{(nd)} = \beta^2 J_a^2 \left[ \prod_{k \in \partial a \setminus i} \left( \Delta_{k \rightarrow a}^{(0)} + h_{k \rightarrow a}^2 \right) - \prod_{k \in \partial a \setminus i} h_{k \rightarrow a}^2 \right].
\]

The function-to-variable message, expressed by Eq. 51, reads therefore as:

\[
\hat{\eta}_{a \rightarrow i}(\sigma_i) \propto \exp \left\{ \hat{A}_{a \rightarrow i} \sum_{\alpha=1}^{x} \sigma_{i}^{\alpha} - \frac{1}{2} \left( \hat{B}_{a \rightarrow i}^{(d)} + \frac{\lambda}{K} \right) \sum_{\alpha=1}^{x} (\sigma_{i}^{\alpha})^2 + \frac{1}{2} \hat{B}_{a \rightarrow i}^{(nd)} \sum_{\alpha \neq \beta} \sigma_{i}^{\alpha} \sigma_{i}^{\beta} \right\}.
\]
while from Eq. (50) we have that the variable-to-function message reads as

$$
\eta_{i \rightarrow a}(\sigma_i) \propto \exp \left\{ \left( \sum_{b \in \partial i \setminus a} \hat{A}_{b \rightarrow i} \right) \sum_{\alpha=1}^{x} \sigma_i^\alpha - \frac{1}{2} \left( \sum_{b \in \partial i \setminus a} \hat{B}_{b \rightarrow i}^{(d)} + \lambda \right) \sum_{\alpha=1}^{x} (\sigma_i^\alpha)^2 + \frac{1}{2} \left( \sum_{b \in \partial i \setminus a} \hat{B}_{b \rightarrow i}^{(nd)} \right) \sum_{\alpha \neq \beta}^{x} \sigma_i^\alpha \sigma_i^\beta \right\}.
$$

(55)

In order to keep the notation simple let us define:

$$
A_{i \rightarrow a} = \sum_{b \in \partial i \setminus a} \hat{A}_{b \rightarrow i} = \sum_{b \in \partial i \setminus a} \beta J_b \prod_{k \in \partial b \setminus i} h_{k \rightarrow b}
$$

$$
B_{i \rightarrow a}^{(d)} = \lambda + \sum_{b \in \partial i \setminus a} \hat{B}_{b \rightarrow i}^{(d)} = \lambda - \sum_{b \in \partial i \setminus a} \beta^2 J_b^2 \left[ \prod_{k \in \partial b \setminus i} \left( \Delta_{k \rightarrow b}^{(1)} + \Delta_{k \rightarrow b}^{(0)} + h_{k \rightarrow b}^2 \right) - \prod_{k \in \partial b \setminus i} h_{k \rightarrow b}^2 \right]
$$

$$
B_{i \rightarrow a}^{(nd)} = \sum_{b \in \partial i \setminus a} \hat{B}_{b \rightarrow i}^{(nd)} = \sum_{b \in \partial i \setminus a} \beta^2 J_b^2 \left[ \prod_{k \in \partial b \setminus i} \left( \Delta_{k \rightarrow b}^{(0)} + h_{k \rightarrow b}^2 \right) - \prod_{k \in \partial b \setminus i} h_{k \rightarrow b}^2 \right],
$$

(56)

so that Eq. (50) can be rewritten in the more compact form as:

$$
\eta_{i \rightarrow a}(\sigma_i) \propto \exp \left\{ A_{i \rightarrow a} \sum_{\alpha=1}^{x} \sigma_i^\alpha - \frac{1}{2} B_{i \rightarrow a}^{(d)} \sum_{\alpha=1}^{x} (\sigma_i^\alpha)^2 + \frac{1}{2} B_{i \rightarrow a}^{(nd)} \sum_{\alpha \neq \beta}^{x} \sigma_i^\alpha \sigma_i^\beta \right\}.
$$

(57)

The expression above can be further simplified by introducing the matrix $\mathcal{M}_{\alpha\beta}$ and the vector $u_\alpha$ such that

$$
u_\alpha = \frac{A_{i \rightarrow a}}{B_{i \rightarrow a}^{(d)} - (x - 1)B_{i \rightarrow a}^{(nd)}} \quad \forall \alpha,
$$

$$
\mathcal{M}_{\alpha\beta} = \delta_{\alpha\beta} B_{i \rightarrow a}^{(d)} + (1 - \delta_{\alpha\beta}) (-B_{i \rightarrow a}^{(nd)}),
$$

(58)

and the normalized distribution, written in the standard form for a multivariate Gaussian, reads

$$
\eta_{i \rightarrow a}(\sigma_i) = \sqrt{\det \mathcal{M}} \frac{1}{(2\pi)^{rac{x}{2}}} \exp \left\{ -\frac{1}{2} (\sigma_i - u)^T \mathcal{M} (\sigma_i - u) \right\}.
$$

(59)

The closed cavity equations, which in the 1RSB case are three rather than two, are simply obtained by taking the averages in Eq. (13) with respect to the marginal distribution $\eta_{i \rightarrow a}(\sigma_i)$:

$$
h_{i \rightarrow a} = \langle \sigma_i^\alpha \rangle = u_\alpha
$$

$$
\Delta_{i \rightarrow a}^{(0)} = \langle \sigma_i^\alpha \sigma_i^\beta \rangle - \langle \sigma_i^\alpha \rangle \langle \sigma_i^\beta \rangle = \mathcal{M}_{\alpha\beta}^{-1}
$$

$$
\Delta_{i \rightarrow a}^{(1)} = \langle \sigma_i^\alpha \rangle \langle \sigma_i^\alpha \rangle - \langle \sigma_i^\alpha \sigma_i^\alpha \rangle = \mathcal{M}_{\alpha\alpha}^{-1} - \mathcal{M}_{\alpha\beta}^{-1},
$$

where the general expression of the inverse matrix element is:

$$
\mathcal{M}_{\alpha\beta}^{-1} = \frac{1}{B_{i \rightarrow a}^{(d)} + B_{i \rightarrow a}^{(nd)}} \delta_{\alpha\beta} + \frac{B_{i \rightarrow a}^{(nd)}}{B_{i \rightarrow a}^{(d)} + B_{i \rightarrow a}^{(nd)}} \frac{B_{i \rightarrow a}^{(nd)}}{B_{i \rightarrow a}^{(d)} + (1 - x)B_{i \rightarrow a}^{(nd)}}.
$$

(60)
For the ease of the reader willing to implement them in a code, let us write explicitly the closed cavity equations:

\[
\begin{align*}
    h_{i \rightarrow a} &= \frac{\beta \sum_{b \in \partial i \setminus a} J_b \prod_{k \in \partial b \setminus i} h_{k \rightarrow b}}{D_{i \rightarrow a}^{(1)} - x D_{i \rightarrow a}^{(0)}} \\
    \Delta_{i \rightarrow a}^{(0)} &= \frac{D_{i \rightarrow a}^{(0)} - x D_{i \rightarrow a}^{(0)}}{D_{i \rightarrow a}^{(1)}(D_{i \rightarrow a}^{(1)} - x D_{i \rightarrow a}^{(0)})} \\
    \Delta_{i \rightarrow a}^{(1)} &= \frac{1}{D_{i \rightarrow a}^{(1)}}
\end{align*}
\]  

(61)

with

\[
\begin{align*}
    D_{i \rightarrow a}^{(1)} &= \lambda - \beta^2 \sum_{b \in \partial i \setminus a} J_b^2 \left[ \prod_{k \in \partial b \setminus i} \left( \Delta_{k \rightarrow b}^{(0)} + \Delta_{k \rightarrow b}^{(1)} + h_{k \rightarrow b}^2 \right) - \prod_{k \in \partial b \setminus i} \left( \Delta_{k \rightarrow b}^{(0)} + h_{k \rightarrow b}^2 \right) \right] \\
    D_{i \rightarrow a}^{(0)} &= \beta^2 \sum_{b \in \partial i \setminus a} J_b^2 \left[ \prod_{k \in \partial b \setminus i} \left( \Delta_{k \rightarrow b}^{(0)} + h_{k \rightarrow b}^2 \right) - \prod_{k \in \partial b \setminus i} h_{k \rightarrow b}^2 \right]
\end{align*}
\]

While the parameter \( \lambda \) is fixed by the normalization condition, the parameter \( x \) is a variational one and has to be choosen in order to extremize the free-energy, a quantity that is computed explicitly in the next subsection in the case of a complete graph. The population dynamics version of the above equations, where one takes the average on the graph statistics reads:

\[
\begin{align*}
    h_i &= \frac{\beta(K-1)\langle J \rangle \prod_{k=1}^{p-1} h_k}{D^{(1)} - x D^{(0)}} \\
    \Delta_i^{(0)} &= \frac{D^{(0)}}{D^{(1)}(D^{(1)} - x D^{(0)})} \\
    \Delta_i^{(1)} &= \frac{1}{D^{(1)}}
\end{align*}
\]  

(62)

where

\[
\begin{align*}
    D^{(1)} &= \lambda - \beta^2(K-1)\langle J^2 \rangle \left[ \prod_{k=1}^{p-1} \left( \Delta_k^{(1)} + \Delta_k^{(0)} + h_k^2 \right) - \prod_{k=1}^{p-1} \left( \Delta_k^{(0)} + h_k^2 \right) \right] \\
    D^{(0)} &= \beta^2(K-1)\langle J^2 \rangle \left[ \prod_{k=1}^{p-1} \left( \Delta_k^{(0)} + h_k^2 \right) - \prod_{k=1}^{p-1} h_k^2 \right]
\end{align*}
\]

To practically implement the above equations, one creates a population of triplets \( \{h, \Delta^{(0)}, \Delta^{(1)}\} \) which is updated as follows: at each time step \( p-1 \) triplets randomly extracted from the population are used to compute the right hand sides of Eq. (62), that provide a new triplet, which is then inserted back in the population in a random position. The process is repeat until the population becomes stationary in time. From the fixed point distribution physical observables are computed.

A further step of simplification of the population dynamics equation can be taken if one assumes that there is a homogeneous solution and looking at it as we did in the RS ansatz. In this case the population dynamics equations turn into simple iterative equations for the three parameters \( \Delta^{(1)}, \Delta^{(0)} \) and \( h \):

\[
\begin{align*}
    h &= \frac{\beta(K-1)\langle J \rangle h^{p-1}}{D_1 - x D_0} ; \quad \Delta^{(0)} = \frac{D_0}{D_1(D_1 - x D_0)} ; \quad \Delta^{(1)} = \frac{1}{D_1} , \\
    D_1 &= \lambda - \beta^2\langle J^2 \rangle(K-1) \left[ \left( \Delta^{(1)} + \Delta^{(0)} + h^2 \right)^{p-1} - \left( \Delta^{(0)} + h^2 \right)^{p-1} \right] \\
    D_0 &= \beta^2\langle J^2 \rangle(K-1) \left[ \left( \Delta^{(0)} + h^2 \right)^{p-1} - h^{2(p-1)} \right]
\end{align*}
\]  

(63)
Using Eqs. (33) and (49) that we recall for clarity (taking $J_2 = 1$):

\begin{align}
\langle J \rangle &= 0 \quad \langle J^2 \rangle = \frac{p!}{2N^{p-1}}, \\
h^2 &= q_0 \quad \Delta^{(0)} = q_1 - q_0 \quad \Delta^{(1)} = 1 - q_1,
\end{align}

we are left with the following two closed equations:

\begin{align}
q_1 - q_0 &= \frac{p\beta^2}{2} \left( q_1^{p-1} - q_0^{p-1} \right) \\
1 - q_1 &= \frac{1}{\lambda - p\beta^2 \left( 1 - q_1^{p-1} \right)}
\end{align}

Let us notice that Eq. (66) allows us to easily re-express the spherical constraint parameter $\lambda$ as a function of $q_1$ and $\beta$, i.e.

\begin{equation}
\lambda = \frac{1}{1 - q_1} + \frac{p\beta^2}{2} \left( 1 - q_1^{p-1} \right),
\end{equation}

which will be useful later on. Comparing the expression of the Langrange multiplier with the one obtained in the RS case, Eq. (19), we see that they are the same with the substitution $q \rightarrow q_1$: in the 1RSB phase the Lagrange multiplier is enforcing the spherical constraint inside each pure state.

C. 1RSB free energy

We now want to compute the free-energy $F(x)$ in the presence of a one-step replica symmetric Ansatz. The free-energy for the replicated system is [24, 36]:

\begin{equation}
\Phi(x) = - \left( \sum_{a=1}^{M} F_a^{\text{RSB}}(x) + \sum_{i=1}^{N} F_i^{\text{RSB}}(x) - \sum_{(ai) \in E} F_{ai}^{\text{RSB}}(x) \right).
\end{equation}

The total free energy of the system is just the free energy of the $x$ coupled replicas, divided by $x$ (and extremized over $x$). In principle the three contributions, respectively representing the energetic and the entropic contributions and a normalization, read as:

\begin{align}
\beta F_a^{\text{RSB}}(x) &= \log \left\{ \prod_{i \in \partial a} \left[ dm_i \rightarrow a Q_{i \rightarrow a}(m_i \rightarrow a) \right] e^{\beta F_a attempt{\{m_i \rightarrow a\}}} \right\} \\
\beta F_i^{\text{RSB}}(x) &= \log \left\{ \prod_{a \in \partial i} \left[ dm_a \rightarrow i \hat{Q}_{a \rightarrow i}(m_a \rightarrow i) \right] e^{\beta F_i attempt{\{m_a \rightarrow i\}}} \right\} \\
\beta F_{ai}^{\text{RSB}}(x) &= \log \left\{ \int dm_a \rightarrow i \int dm_i \rightarrow a \hat{Q}_{a \rightarrow i}(m_a \rightarrow i) Q_{i \rightarrow a}(m_i \rightarrow a) e^{\beta F_{ai} attempt{\{m_i \rightarrow a, m_a \rightarrow i\}}} \right\}
\end{align}

where $F_a$, $F_i$ and $F_{ai}$ are the RS free energy parts in Eq. (20), $Q_{i \rightarrow a}(m_i \rightarrow a)$ is the distribution of the cavity marginals that in the case of the dense Gaussian 1RSB ansatz has been defined in Eq. (66), Sec. III A and the distribution $\hat{Q}_{a \rightarrow i}(m_a \rightarrow i)$ should be the one of the function-to-variable fields.

The detailed computation is reported in the Appendix [B2] here we just write the result that is the same found
with the replica approach [6]:

\[
-x\beta F(x) = \sum_{a=1}^{M} \beta F_{RSB}^{a} + \sum_{i=1}^{N} \beta F_{RSB}^{i} - \sum_{(ai)\in E} \beta F_{RSB}^{ai}
\]

\[
= xN\left\{ \frac{\beta^2}{2} \left[ 1 - (1 - x)q_1^p - xq_0^p \right] + \frac{1}{1 - xq_0 - (1 - x)q_1} + \frac{x - 1}{x} \log (1 - q_1) + \frac{1}{x} \log [1 - xq_0 - (1 - x)q_1] \right\}
\]

(72)

IV. CONCLUSIONS

In this paper we have derived the cavity equations for solving diluted spherical $p$-spin models. Such a cavity-based derivation makes evident the underlying assumption that reflects itself in the distribution of local fields: in the RS ansatz replicas are uncorrelated and have Gaussian local fields, while in the 1RSB ansatz replicas have correlated Gaussian local fields whose covariance matrix depends on whether the replicas are in the same state or not.

We have derived the cavity equations exploiting the same high-temperature expansion that leads to mean-field approximations [43].

We have solved the cavity equations in the fully-connected case. In this case the solution is homogeneous, depends on very few parameters and can be written explicitly, leading to the same expression that was obtained via the replica method.

The approach based on the cavity method has several advantages:

- it makes clear the underlying assumptions;
- it holds also for the diluted version of the model (provided the solution does not condensate);
- it can be converted in message-passing algorithms, the RS Belief Propagation and the 1RSB Survey Propagation;
- it allows to study heterogeneous solutions in diluted models, until the condensation transition.

Our work, besides providing the first complete reference on the equivalence between the replica and the cavity methods for spherical disordered models, paves the way to a more systematic study of inhomogeneous glassy phases in diluted mean-field models and represents a reference point for the systematic development of algorithms for combinatorial optimization and inference problems characterized by continuous variables [41, 44–46].

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Appendix A: Replica Symmetric free energy

In this section we compute the replica-symmetric free energy of the model, which is defined as [24]:

\[
-\beta F \equiv \beta \left( \sum_{a=1}^{M} F_{a} + \sum_{i=1}^{N} F_{i} - \sum_{(ai)\in E} F_{ai} \right) \equiv \sum_{a=1}^{M} \log(Z_{a}) + \sum_{i=1}^{N} \log(Z_{i}) - \sum_{(ai)\in E} \log(Z_{(ai)}),
\]

(A1)

where we have respectively

\[
Z_{a} = \int_{-\infty}^{\infty} \prod_{i \in \partial a} d\sigma_{i} \eta_{i\rightarrow a}(\sigma_{i}) e^{\beta J_{a} \prod_{i \in \partial a} \sigma_{i}}
\]

(A2)

\[
Z_{i} = \int_{-\infty}^{\infty} d\sigma_{i} \prod_{a \in \partial i} \hat{\eta}_{a\rightarrow i}(\sigma_{i})
\]

(A3)

\[
Z_{(ai)} = \int_{-\infty}^{\infty} d\sigma_{i} \eta_{a\rightarrow i}(\sigma_{i}) \eta_{i\rightarrow a}(\sigma_{i})
\]

(A4)
Let us now compute the three contributions to the free-energy, starting from the simplest, which is the energy per functional node \( \log(Z_a) \).

By expanding the Boltzmann weight in Eq. (A2) we get:

\[
Z_a \simeq \int \left( \prod_{i \in \partial a} d\sigma_i \right) \exp \left( -\frac{\lambda}{2K} \sigma_i^2 + \beta J_a \sigma_i + \frac{\beta^2}{2} J_a^2 (1 - q^p - 1) \sigma_i^2 \right)
\]

\[
= \left[ 1 + \beta J_a m^p + \frac{\beta^2}{2} J_a^2 \right] \simeq \exp \left( \beta J_a m^p + \frac{\beta^2}{2} J_a^2 (1 - q^p) \right),
\]

where we used the spherical constraint to write \((m^2 + v)^p = 1\). We have, therefore, that the sum over all the \( M = N^p \) interaction nodes reads as

\[
\sum_{a=1}^M \log(Z_a) = \beta m^p \left( \sum_{a=1}^M J_a \right) + \frac{\beta^2}{2} (1 - q^p) \left( \sum_{a=1}^M J_a^2 \right) = N \frac{\beta^2}{4} (1 - q^p) J_2,
\]

since we have assumed \( \sum_{a=1}^M J_a \simeq 0 \) and \( \sum_{a=1}^M J_a^2 \simeq M J_2^2 = \frac{M J_2^2}{2N^p} \simeq N J_2/2 \) for large \( N \).

Let us now compute the entropy per spin \( \log(Z_i) \). From Eq. (22) we have that:

\[
Z_i = \int_{-\infty}^{\infty} d\sigma_i \left[ 1 \right] \sum_{a=1}^N \exp \left\{ -\frac{\lambda}{2K} \sigma_i^2 + \beta m^p - 1 J_a \sigma_i + \frac{\beta^2}{2} J_a^2 (1 - q^p - 1) \sigma_i^2 \right\}
\]

\[
= \prod_{a \in \partial i} \frac{1}{\tilde{Z}_{a \rightarrow i}} \int_{-\infty}^{\infty} d\sigma_i \exp \left\{ -\frac{\lambda}{2\sigma_i^2} + \beta m^p - 1 \left( \sum_{a=1}^{K-1} J_a \right) \sigma_i + \frac{\beta^2}{2} (1 - q^p - 1) \left( \sum_{a=1}^{K-1} J_a^2 \right) \sigma_i^2 \right\}.
\]

If we now plug the expression of the Lagrange multiplier \( \lambda \) of Eq. (19) inside Eq. (A7) we find out that there are two terms proportional to \( \beta^2 (1 - q^p - 1) \) which cancel out. We are left in the end with

\[
Z_i = \prod_{a \in \partial i} \frac{1}{\tilde{Z}_{a \rightarrow i}} \int_{-\infty}^{\infty} d\sigma_i \exp \left\{ -\frac{1}{2(1 - q)} \sigma_i^2 + \beta m^p - 1 \left( \sum_{a=1}^{K-1} J_a \right) \sigma_i \right\}
\]

\[
Z_i = \prod_{a \in \partial i} \frac{1}{\tilde{Z}_{a \rightarrow i}} \sqrt{2\pi} (1 - q)^{1/2} \exp \left\{ \frac{1}{2} \beta^2 m^{2(p-1)} \left( \sum_{a=1}^{K-1} J_a \right) \sigma_i^2 \right\}.
\]

At this stage we can use the following identity, obtained by taking the square of Eq. (18):

\[
m^{2(p-1)} \beta^2 \left( \sum_{a=1}^{K-1} J_a \right)^2 = \left( \frac{m^2}{v} \right)^2.
\]

By plugging the expression of Eq. (A9) into Eq. (A8) and recalling that \( q = m^2 \) we get

\[
Z_i = \prod_{a \in \partial i} \frac{1}{\tilde{Z}_{a \rightarrow i}} \sqrt{2\pi} (1 - q)^{1/2} \exp \left\{ \frac{1}{2} \frac{q}{(1 - q)} \right\}.
\]

From the expression of Eq. (A10), summing over all spins (and neglecting constant terms) one finds:

\[
\sum_{i=1}^N \log(Z_i) = \frac{N}{2} \left[ \log(1 - q) + \frac{q}{(1 - q)} \right] - \sum_{i=1}^N \sum_{a \in \partial i} \log(\tilde{Z}_{a \rightarrow i}).
\]
Finally, the contribution to the free-energy coming from the edges reads as:

\[
Z_{(ai)} = \int_{-\infty}^{\infty} d\sigma_i \; \hat{\eta}_{a\rightarrow i}(\sigma_i) \; \eta_{i\rightarrow a}^{(0)}(\sigma_i) \\
= \frac{1}{\sqrt{2\pi v}} \frac{1}{Z_{a\rightarrow i}} \int_{-\infty}^{\infty} d\sigma_i \exp \left\{ -\frac{(\sigma_i - m)^2}{2v} - \frac{\lambda}{2K} \sigma_i^2 + \beta J_a m^{p-1} \sigma_i + \frac{1}{2} \beta^2 J_a^2 (1 - q^{p-1}) \sigma_i^2 \right\}
\]  

(A12)

By looking at Eq. (18), one can easily see that:

\[
\frac{m}{v} = \beta m^{p-1} \sum_a J_a \gg \beta J_a m^{p-1}
\]  

(A13)

\[
\frac{1}{v} = \lambda - \beta^2 (1 - q^{p-1}) \sum_a J_a^2 \gg \frac{\lambda}{K} - \beta^2 (1 - q^{p-1}) J_a^2,
\]  

(A14)

so that in the limit of large \( N \) all terms but the one proportional to \( 1/v \) can be neglected in the integral of Eq. (A12) and one is left with

\[
Z_{(ai)} \approx \frac{1}{\sqrt{2\pi v}} \frac{1}{Z_{a\rightarrow i}} \int_{-\infty}^{\infty} d\sigma_i \exp \left\{ -\frac{(\sigma_i - m)^2}{2v} \right\} = \frac{1}{Z_{a\rightarrow i}}.
\]  

(A15)

Finally, summing over all the edges we get:

\[
\sum_{(ai)\in E} \log(Z_{(ai)}) = - \sum_{(ai)\in E} \log(\hat{\rho}_{a\rightarrow i}) = - \sum_{i=1}^{N} \sum_{a\in \partial i} \log(\hat{\rho}_{a\rightarrow i}).
\]  

(A16)

By plugging our results for \( Z_a, Z_i \) and \( Z_{(ai)} \) within the expression for the free-energy in Eq. (20) we get:

\[
-\beta F_{RS} = \frac{N}{2} \left[ \frac{\beta^2}{2} (1 - q^p) J_2 + \log(1 - q) + \frac{q}{1 - q} \right],
\]  

(A17)

obtained assuming \( J_2 = 1 \) and (all terms \( \log(\hat{\rho}_{a\rightarrow i}) \) are identical) using the fact that \( N \sum_{a\in \partial i} 1 = \sum_{(ai)\in E} 1 \).

Appendix B: One-step Replica Symmetry Breaking solution

1. 1RSB Cavity Equations

We derive here some useful identities which can be obtained from the 1RSB cavity equations. One has just to equate the coefficients of the diagonal, i.e. \( \sum_{a=1}^{N}(\alpha^a)^2 \), and off-diagonal, i.e. \( \sum_{a \neq b}\alpha^a\beta^b \) on the left and on the right of the first of the cavity equations in Eq. (31). Before doing this let us just write the local 1RSB marginal \( \eta(\sigma_i) \) explicitly, i.e. we compute explicitly the integral over \( m_{i\rightarrow a} \) in Eq. (11):

\[
\eta_{i\rightarrow a}(\sigma_i) = \int_{-\infty}^{\infty} dm_{i\rightarrow a} \frac{1}{\sqrt{2\pi \Delta_{i\rightarrow a}^{(0)}}} \exp \left\{ -\frac{(m_{i\rightarrow a} - h_{i\rightarrow a})^2}{2\Delta_{i\rightarrow a}^{(0)}} \right\}
\]

\[
\exp \left\{ -\frac{\sigma_i^2}{2\Delta_{i\rightarrow a}^{(0)}} \right\} \exp \left\{ -\sum_{a=1}^{x} \frac{(\sigma_i^a - m_{i\rightarrow a})^2}{2\Delta_{i\rightarrow a}^{(1)}} \right\}
\]

\[
\approx \exp \left\{ -\frac{1}{2\Delta_{i\rightarrow a}^{(1)}} \sum_{a=1}^{x} \sigma_i^a - h_{i\rightarrow a} \right\} \int_{-\infty}^{\infty} dm_{i\rightarrow a} \exp \left\{ -\frac{1}{2} \left( \frac{\Delta_{i\rightarrow a}^{(1)}}{\Delta_{i\rightarrow a}^{(0)} + x\Delta_{i\rightarrow a}^{(0)}} \right) \right\}
\]

\[
\approx \exp \left\{ -\frac{1}{2\Delta_{i\rightarrow a}^{(1)}} \sum_{a=1}^{x} \sigma_i^a - h_{i\rightarrow a} \right\} \approx \exp \left\{ -\frac{1}{2\Delta_{i\rightarrow a}^{(1)}} \sum_{a=1}^{x} \sigma_i^a - h_{i\rightarrow a} \right\}
\]

\[
\approx \exp \left\{ -\frac{1}{2\Delta_{i\rightarrow a}^{(1)}} \left( 1 - \frac{\Delta_{i\rightarrow a}^{(0)}}{\Delta_{i\rightarrow a}^{(1)} + x\Delta_{i\rightarrow a}^{(0)}} \right) \sum_{a=1}^{x} \sigma_i^a \right\}
\]

(B1)
where, in the last line of Eq. [B1], we have retained only terms at least linear in \( \sigma_i^n \). Thus, matching the coefficients of the linear and quadratic in \( \sigma_i^n \), both diagonal and non-diagonal, appearing in the expression of \( \eta(\sigma_i^n) \) in Eq. [B1] above here and in Eq. [B7] we get the set of three equations:

\[
\begin{align*}
\frac{h_{i \rightarrow a}}{\Delta_{i \rightarrow a}^{(1)} + x \Delta_{i \rightarrow a}^{(0)}} &= \beta \sum_{b \in \partial_i \setminus \{ a \}} J_b \prod_{k \in \partial_b \setminus i} h_{k \rightarrow b} = A \\
\frac{1}{\Delta_{i \rightarrow a}^{(1)} \left( 1 - \frac{\Delta_{i \rightarrow a}^{(0)}}{\Delta_{i \rightarrow a}^{(1)} + x \Delta_{i \rightarrow a}^{(0)}} \right)} &= \lambda + \sum_{b \in \partial_i \setminus \{ a \}} \beta^2 J_b^2 \left[ \prod_{k \in \partial_b \setminus i} h_{k \rightarrow b}^2 - \prod_{k \in \partial_b \setminus i} \left( \Delta_{k \rightarrow b}^{(1)} + \Delta_{k \rightarrow b}^{(0)} + h_{k \rightarrow b}^2 \right) \right] = B^{(d)} \\
\frac{1}{\Delta_{i \rightarrow a}^{(1)} \Delta_{i \rightarrow a}^{(0)} \left( \Delta_{i \rightarrow a}^{(1)} + x \Delta_{i \rightarrow a}^{(0)} \right)} &= \sum_{b \in \partial_i \setminus \{ a \}} \beta^2 J_b^2 \left[ \prod_{k \in \partial_b \setminus i} \left( \Delta_{k \rightarrow b}^{(0)} + h_{k \rightarrow b}^2 \right) - \prod_{k \in \partial_b \setminus i} h_{k \rightarrow b}^2 \right] = B^{(nd)}
\end{align*}
\]

(B2)

The equations above [Eq. [B2]] are particularly useful to derive identities between the parameters of homogeneous solution of the cavity equations. In particular, we can do the following replacements:

\[
\begin{align*}
h_{i \rightarrow a} &= h \\
h_{i \rightarrow a}^2 &= q_0 \\
\Delta_{i \rightarrow a}^{(0)} &= q_1 - q_0 \\
\Delta_{i \rightarrow a}^{(1)} &= 1 - q_1,
\end{align*}
\]

so that from Eq. [B2] we are left with the following couple of equations:

\[
\begin{align*}
\frac{h}{1 - x q_0 - (1 - x) q_1} &= \beta \left( \sum_{b \in \partial_i \setminus \{ a \}} J_b \right) h^{p-1} = A \\
\frac{1}{1 - q_1} \left[ 1 - \frac{q_1 - q_0}{1 - x q_0 - (1 - x) q_1} \right] &= \lambda - \beta^2 \sum_{b \in \partial_i \setminus \{ a \}} J_b^2 \left( 1 - q_0^{p-1} \right) = B^{(d)} \\
\frac{q_1 - q_0}{1 - q_1} \frac{1}{1 - x q_0 - (1 - x) q_1} &= \beta^2 \sum_{b \in \partial_i \setminus \{ a \}} J_b^2 \left( q_1^{p-1} - q_0^{p-1} \right) = B^{(nd)}
\end{align*}
\]

(B4)

2. 1RSB Free energy

Let us reproduce here the exact formulas that we need to compute \( \mathcal{F}(x) \) in the presence of a one-step replica symmetry Ansatz. The free-energy for the replicated system is [24, 34]:

\[
\Phi(x) = - \left( \sum_{a=1}^{M} \mathcal{F}^{RSB}_a(x) + \sum_{i=1}^{N} \mathcal{F}^{RSB}_i(x) - \sum_{(ai) \in E} \mathcal{F}^{RSB}_{ai}(x) \right).
\]

(B5)

In principle the three contributions, respectively representing the energetic and the entropic contributions and a normalization, read as:

\[
\begin{align*}
\beta \mathcal{F}^{RSB}_a(x) &= \log \left\{ \int \prod_{i \in \partial a} [d\sigma_i \rightarrow a Q_i \rightarrow a(\sigma_i \rightarrow a)] \ e^{x \beta \mathcal{F}_a(\{\sigma_i \rightarrow a\})} \right\} \\
\beta \mathcal{F}^{RSB}_i(x) &= \log \left\{ \int \prod_{a \in \partial i} [d\hat{\sigma}_a \rightarrow i Q_i \rightarrow a(\hat{\sigma}_a \rightarrow i)] \ e^{x \beta \mathcal{F}_i(\{\hat{\sigma}_a \rightarrow i\})} \right\}
\end{align*}
\]

(B6) (B7)
\[ \beta E_{RSB}^{RSB}(x) = \log \left\{ \int dm_{a\rightarrow i} \ dm_{i\rightarrow a} \ \hat{Q}_{a\rightarrow i}(m_{a\rightarrow i}) \ Q_{i\rightarrow a}(m_{i\rightarrow a}) \ e^{x \beta F_{a\rightarrow i}(m_{i\rightarrow a}, m_{a\rightarrow i})} \right\} \]  

(B8)

where \( F_{a}, F_{i}, \) and \( F_{a\rightarrow i} \) are the RS free energy parts in Eq. (20), \( Q_{i\rightarrow a}(m_{i\rightarrow a}) \) is the distribution of the cavity marginals that in the case of the dense Gaussian 1RSB ansatz has been defined in Eq. (46), Sec. III A and the distribution \( \hat{Q}_{a\rightarrow i}(m_{a\rightarrow i}) \) should be the one of the function-to-variable fields. Since we are going to write everything in terms of the “variable to function” fields \( m_{i\rightarrow a} \), it is convenient to write the probability distribution \( \hat{Q}_{a\rightarrow i}(m_{a\rightarrow i}) \) of “function to variable” fields \( m_{a\rightarrow i} \) in terms of \( Q_{i\rightarrow a}(m_{i\rightarrow a}) \) and \( m_{i\rightarrow a} \). This can be done taking advantage of the identity (24):

\[ \hat{Q}_{a\rightarrow i}(m_{a\rightarrow i}) = \int \prod_{k \in \partial a \setminus i} \left[ dm_{k\rightarrow a} Q_{k\rightarrow a}(m_{k\rightarrow a}) \right] \hat{Z}_{a\rightarrow i}^{x}(\{m_{k\rightarrow a}\}) \delta \left[ m_{a\rightarrow i} - f(\{m_{k\rightarrow a}\}) \right], \]  

(B9)

where \( f(x) \) is a shorthand to refer to the cavity equations and \( \hat{Z}_{a\rightarrow i}^{x} \) is the normalization factor introduced in the RS equation [3]. By exploiting the last identity we can rewrite the expression of the 1RSB entropy per site as

\[ \beta E_{RSB}^{RSB}(x) = \log \left\{ \int \prod_{a \in \partial i} \prod_{k \in \partial a \setminus i} dm_{k\rightarrow a} Q_{k\rightarrow a}(m_{k\rightarrow a}) \hat{Z}_{a\rightarrow i}^{x}(\{m_{k\rightarrow a}\}) e^{x \beta F_{a}(\{f(\{m_{k\rightarrow a}\})\})} \right\} \]  

(B10)

3. 1RSB Energy

We compute in what follows the expression of the energetic part of the free-energy. By plugging into the same expression the definition of the energetic contribution to the local free energy in the RS case, see Eq. (21), and the 1RSB weighted average over the local messages written in Eq. (69) we get:

\[ \beta E_{RSB}^{RSB}(x) = \log \left\{ \int \prod_{i \in \partial a} \left[ dm_{i\rightarrow a} Q_{i\rightarrow a}(m_{i\rightarrow a}) \right] \left( \int_{-\infty}^{\infty} \prod_{i \in \partial a} d\sigma_{i} \eta_{i\rightarrow a}^{\alpha}(\sigma_{i}) e^{x \beta J_{a} \sum_{i \in \partial a} \sigma_{i}} \right)^{x} \right\} = \log \left\{ \int \prod_{i \in \partial a} \left[ dm_{i\rightarrow a} Q_{i\rightarrow a}(m_{i\rightarrow a}) \right] \prod_{i \in \partial a} \prod_{\alpha=1}^{x} d\sigma_{i}^{\alpha} \eta_{i\rightarrow a}^{\alpha}(\sigma_{i}^{\alpha}) e^{x \beta J_{a} \sum_{\alpha=1}^{x} \sum_{i \in \partial a} \sigma_{i}^{\alpha}} \right\} \]  

(B11)

The expression in Eq. (B11) is formally correct but completely useless unless an explicit ansatz for the distribution \( Q_{i\rightarrow a}(m_{i\rightarrow a}) \) is given. We consider the ansatz in Eqs. (46,48), which assumes three parameters, the local magnetization \( h_{i\rightarrow a} \) and the two parameters \( \Delta^{(0)}_{i\rightarrow a} \) and \( \Delta^{(1)}_{i\rightarrow a} \) relative to the coupling between replicas:

\[ \int dm_{i\rightarrow a} Q_{i\rightarrow a}(m_{i\rightarrow a}) \prod_{\alpha=1}^{x} \eta_{i\rightarrow a}^{\alpha}(\sigma_{i}^{\alpha}) = \int_{-\infty}^{\infty} dm_{i\rightarrow a} e^{-(m_{i\rightarrow a}-h_{i\rightarrow a})^{2}/(2\Delta^{(0)}_{i\rightarrow a})} \prod_{\alpha=1}^{x} e^{-(\sigma_{i}^{\alpha}-m_{i\rightarrow a})^{2}/(2\Delta^{(1)}_{i\rightarrow a})} \]  

(B12)

In particular we have that \( h_{i\rightarrow a} \) and \( \Delta^{(0)}_{i\rightarrow a} \) are variational parameter for the probability distribution of the fields \( m_{i\rightarrow a} \) on a given edge, while \( \Delta^{(1)}_{i\rightarrow a} \) is a variational parameter for the coupling between replicas.

In order to concretely carry on the calculation we need first to expand to the leading order the interaction term in the partition function:

\[ \exp \left\{ \beta J_{a} \sum_{\alpha=1}^{x} \sum_{i \in \partial a} \sigma_{i}^{\alpha} \right\} \simeq \left( 1 + \beta J_{a} \sum_{\alpha=1}^{x} \sum_{i \in \partial a} \sigma_{i}^{\alpha} + \frac{1}{2} \beta^{2} J_{a}^{2} \sum_{\alpha,\beta=1}^{x} \sum_{i \in \partial a} \sigma_{i}^{\alpha} \sigma_{i}^{\beta} \right) \]  

(B13)

By then indicating

\[ \int dm_{i\rightarrow a} Q_{i\rightarrow a}(m_{i\rightarrow a}) \prod_{\alpha=1}^{x} \eta_{i\rightarrow a}(\sigma_{i}^{\alpha}, m_{i\rightarrow a}) \mathcal{G}(\sigma_{i}^{\alpha}) = \langle \mathcal{G}(\sigma_{i}^{\alpha}) \rangle, \]  

(B14)
we have

\[
\beta F_a^{\text{RSB}}(x) = \log \left\{ e^{\beta J_a \sum_a^{x=1} \prod_{i \in \partial a} \sigma_i^a} \right\} \simeq \\
\simeq \log \left\{ 1 + \beta J_a \sum_{a=1}^{x} \prod_{i \in \partial a} \langle \sigma_i^a \rangle + \frac{1}{2} \beta^2 J_a^2 \sum_{a=1}^{x} \prod_{i \in \partial a} \langle \sigma_i^a \sigma_i^a \rangle \right\} \simeq \\
\simeq \beta J_a \sum_{a=1}^{x} \prod_{i \in \partial a} \langle \sigma_i^a \rangle + \frac{1}{2} \beta^2 J_a^2 \sum_{a=1}^{x} \prod_{i \in \partial a} \langle \sigma_i^a \sigma_i^a \rangle - \frac{1}{2} \beta^2 J_a^2 \sum_{a=1}^{x} \prod_{i \in \partial a} \langle \sigma_i^a \sigma_i^a \rangle 
\]

\[
= \beta J_a \sum_{a=1}^{x} \prod_{i \in \partial a} \langle \sigma_i^a \rangle + \frac{1}{2} \beta^2 J_a^2 \sum_{a=1}^{x} \prod_{i \in \partial a} \langle \sigma_i^a \sigma_i^a \rangle - \frac{1}{2} \beta^2 J_a^2 \sum_{a \neq \beta i \in \partial a} \langle \sigma_i^a \sigma_i^a \rangle 
\]

Finally, by making use of the definition of the moments \( \langle \sigma_i^a \rangle \), \( \langle \sigma_i^a \sigma_i^a \rangle \) and \( \langle \sigma_i^a \sigma_i^a \rangle \) given in Eq. (47) and taking the homogeneous ansatz, \( \Delta^{(1)}_{i \to a} = 1 - q_1 \), \( \Delta^{(0)}_{i \to a} = q_1 - q_0 \), \( h_{i \to a} = q_0 \) and \( h_{i \to a} = h \), we get

\[
\beta F_a^{\text{RSB}}(x) = \beta J_a x h + \frac{1}{2} x \beta^2 J_a^2 + \frac{1}{2} \beta^2 J_a^2 x(x - 1)q_0^P - \frac{1}{2} x \beta^2 J_a^2 q_0^P - \frac{1}{2} x(x - 1) \beta^2 J_a^2 q_0^P = \\
= \beta J_a x h + \frac{1}{2} x \beta^2 J_a^2 [1 - (1 - x) q_0^P - x q_0^P] 
\]

The energetic part of the free-energy summed over all the \( M \) interactions, substituting \( \sum_{a=1}^{M} J_a = 0 \) and \( \sum_{a=1}^{M} J_a^2 = M \frac{\beta^2}{2N^P} \simeq N/2 \), eventually reads:

\[
\sum_{a=1}^{M} \beta F_a^{\text{RSB}}(x) = xN \frac{\beta^2}{4} [1 - (1 - x) q_0^P - x q_0^P] 
\]

### 4. 1RSB Entropy

The 1RSB entropy is defined in eq. (B10) and we rewrite here for the ease of the reader:

\[
\beta F_1^{\text{RSB}}(x) = \log \left\{ \int \prod_{a \in \partial i} \prod_{k \in \partial a \setminus i} \ dm_{k \to a} Q_{k \to a}(m_{k \to a}) \ Z_a^{-1}(\{m_{k \to a}\}_{k \in \partial a \setminus i}) e^{x \beta F_1(\{m_{k \to a}\}_{a \in \partial i})} \right\} 
\]

where the local free-energy \( F_1(\{m_{k \to a}\}_{k \in \partial a \setminus i}) \) is the RS one:

\[
\exp (\beta F_1) = \int_{-\infty}^{\infty} \ d\sigma_i e^{-\lambda \sigma_i^2/2} \prod_{a \in \partial i} \ Z_a^{-1}(\{m_{k \to a}\}_{k \in \partial a \setminus i}) \int \prod_{k \in \partial a \setminus i} \ d\sigma_k \eta_k^{m_{k \to a}(\sigma_k)} e^{\beta J_a \sigma_i \sum_{k \in \partial a \setminus i} \sigma_k} 
\]

By then assuming that \( x \) is integer (the analytic continuation to real values will be taken afterwards) it is straightforward to write:

\[
\prod_{a \in \partial i} Z_a^{-1}(\{m_{k \to a}\}_{k \in \partial a \setminus i}) e^{x \beta F_1(\{m_{k \to a}\}_{a \in \partial i})} = \\
= \int_{-\infty}^{\infty} \ d\sigma_i e^{-\lambda \sum_{a=1}^{x} (\sigma_i^a)^2/2} \prod_{a \in \partial i} \int \prod_{\alpha=1}^{x} \ d\sigma_k \eta_k^{m_{k \to a}(\sigma_k)} e^{\beta J_a \sum_{k \in \partial a \setminus i} \sigma_k^a} 
\]

The implementation of the 1RSB ansatz comes at this stage very natural, we just have to insert for each link \((ka)\) the ansatz in Eqs. (10), (15). Thus, if we take the average over local fields before taking the one over local variables—as
for any replica calculation, but locally—it is convenient to define
\[
\eta_{k \to a}(\sigma_k) = \int dm_{k \to a} \, Q_{k \to a}(m_{k \to a}) \prod_{\alpha=1}^{x} \eta_{k \to a}^{\alpha}(\sigma_k^{\alpha}),
\]  
which is precisely the quantity defined in Eq. (A11). The above steps allow us to rewrite the 1RSB entropy simply as:
\[
\beta F_{1}^{\text{RSB}} = \log \left( \int_{-\infty}^{\infty} \prod_{\alpha=1}^{x} d\sigma_i^{\alpha} \, e^{-\lambda \sum_{\alpha=1}^{x} \sigma_i^{\alpha}^2/2} \prod_{\alpha \in \partial i} \left[ \int_{-\infty}^{\infty} \prod_{k \in \partial \alpha \setminus i} d\sigma_k \, \eta_{k \to a}(\sigma_k) \right] \right)
\]

where the quantities \( \hat{A}_{a \to i}, \hat{B}_{a \to i}^{(d)} \) and \( \hat{B}_{a \to i}^{(nd)} \) are those defined in Eq. (A3). Since at the leading order we have
\[
A_{i \to a} = \sum_{a \in \partial i} \hat{A}_{a \to i}
\]
\[
B_{i \to a}^{(d)} = \lambda + \sum_{a \in \partial i} \hat{B}_{a \to i}^{(d)}
\]
\[
B_{i \to a}^{(nd)} = \sum_{a \in \partial i} \hat{B}_{a \to i}^{(nd)},
\]
and we are interested in the free-entropy of the homogeneous solution, we can drop the subscript \( i \to a \). Using the results of Appendix B.1 we can use the self-consistent solution of the 1RSB cavity equation to write:
\[
A = \sum_{a \in \partial i} \hat{A}_{a \to i} = \frac{h}{1 - xq_0 - (1 - x)q_1}
\]
\[
B^{(d)} = \lambda + \sum_{a \in \partial i} \hat{B}_{a \to i}^{(d)} = \frac{1}{1 - q_1} \left[ 1 - \frac{1}{1 - xq_0 - (1 - x)q_1} \right]
\]
\[
B^{(nd)} = \sum_{a \in \partial i} \hat{B}_{a \to i}^{(nd)} = \frac{q_1 - q_0}{1 - q_1} \cdot \frac{1}{1 - xq_0 - (1 - x)q_1}
\]

exactly as we did in the RS case when passing from eq. (A7) to eq. (A10). We can, therefore, define a vector \( \mathbf{A} \) and a matrix \( \mathbf{M} \) (already introduced in eq. (58)) as:
\[
A_\alpha = A, \quad \forall \alpha
\]
\[
\mathbf{M}_{\alpha \beta} = \delta_{\alpha \beta}B^{(d)} + (1 - \delta_{\alpha \beta})(-B^{(nd)}).
\]
\[
\alpha, \beta = 1, \ldots, x.
\]

The 1RSB entropy can be easily written as:
\[
\beta F_{1}^{\text{RSB}} = \log \left( \int \mathcal{D}\sigma \exp \left\{ \sum_{\alpha=1}^{x} A_\alpha \sigma_\alpha - \frac{1}{2} \sum_{\alpha \beta} \sigma_\alpha \mathbf{M}_{\alpha \beta} \sigma_\beta \right\} \right)
\]
\[
= \sqrt{\frac{2\pi}{\det \mathbf{M}}} \exp \left( \frac{1}{2} A^2 \sum_{\alpha \beta} \mathbf{M}_{\alpha \beta}^{-1} \right)
\]  

where, making use of the definition of the inverse \( \mathbf{M}^{-1} \) given in Eq. (60) and with the help of a little algebra we get:
\[
\sum_{\alpha \beta} \mathbf{M}_{\alpha \beta}^{-1} = \frac{x}{B^{(d)} + (1 - x)B^{(nd)}} = \frac{x}{1/q_1 - (1 - q_1)/(1 - xq_0 - (1 - x)q_1)} = x(1 - xq_0 - (1 - x)q_1)
\]

```
We thus have
\[
\frac{1}{2} A^2 \sum_{\alpha \beta} \mathcal{M}_{\alpha \beta}^{-1} = \frac{1}{2} \frac{h^2}{(1 - x q_0 - (1 - x) q_1)^2} \left( \sum_{\alpha \beta} \mathcal{M}_{\alpha \beta}^{-1} \right) = \frac{x q_0}{1 - x q_0 - (1 - x) q_1}
\]  
(B27)

We also need to compute \( \det \mathcal{M} \), that is the determinant of a symmetric \( x \times x \) matrix of the form:

\[
\mathcal{M} = \begin{pmatrix}
    a_1 & b & b & b & b \\
    b & a_2 & b & b & b \\
    b & b & a_3 & b & b \\
    b & b & b & a_4 & b \\
    b & b & b & b & a_5,
\end{pmatrix}
= \begin{pmatrix}
    a_1 - b & 0 & 0 & 0 & 0 \\
    0 & a_2 - b & 0 & 0 & 0 \\
    0 & 0 & a_3 - b & 0 & 0 \\
    0 & 0 & 0 & a_4 - b & 0 \\
    0 & 0 & 0 & 0 & a_5 - b,
\end{pmatrix} + \begin{pmatrix}
    b & b & b & b \\
    b & b & b & b \\
    b & b & b & b \\
    b & b & b & b \\
    b & b & b & b,
\end{pmatrix}
\]

The general formula for such a determinant is:

\[
\det \left[ \text{Diag} (a_1 - b, \ldots, a_x - b) + b \cdot \mathbf{1}_x^T \otimes \mathbf{1}_x \right] = \prod_{i=1}^{x} (a_i - b) + b \sum_{i=1}^{x} \sum_{j=1 \atop j \neq i}^{x} (a_i - b).
\]  
(B28)

In the case of the rank-\( x \) symmetric matrix \( \mathcal{M} \) defined in Eq. (B24), where in addition all the elements on the diagonal are identical, from Eq. (B28) we have that:

\[
\det (\mathcal{M}) = \left( B^{(d)} + B^{(nd)} \right)^x - x B^{(nd)} \left( B^{(d)} + B^{(nd)} \right)^{x-1}.
\]  
(B29)

By exploiting then the definition of \( B^{(d)} \) and \( B^{(nd)} \) in terms of \( x, q_0 \) and \( q_1 \) written in Eq. (B23) one gets, after a very simple algebra:

\[
\det (\mathcal{M}) = \frac{1}{(1 - q_1)^{x-1}} \cdot \frac{1}{(1 - x q_0 - (1 - x) q_1)},
\]  
(B30)

so that

\[
\frac{1}{\sqrt{\det (\mathcal{M})}} = \exp \left\{ \frac{1}{2} (x - 1) \log (1 - q_1) + \frac{1}{2} \log (1 - x q_0 - (1 - x) q_1) \right\}
\]  
(B31)

At this point it is immediate to write (neglecting constant terms) the 1RSB local free entropy term:

\[
\sum_{i=1}^{N} \beta F_{\text{RSB}}^{i} = N \left\{ \frac{x}{2} \frac{q_0}{[1 - x q_0 - (1 - x) q_1]} + \frac{x - 1}{2} \log (1 - q_1) + \frac{1}{2} \log [1 - x q_0 - (1 - x) q_1] \right\}
\]  
(B32)

5. Vanishing of the normalization terms \( \beta F_{\text{RSB}}^{i} \)

Let us now show how the terms \( \beta F_{\text{RSB}}^{i} \) vanish in the limit \( N \rightarrow \infty \), provided that normalized distributions are used everywhere. The first step is to plug the definition of \( \hat{Q}_{a \rightarrow i} (\hat{m}_{a \rightarrow i}) \) of Eq. (59) into the definition of \( \beta F_{\text{RSB}}^{i} \) in Eq. (71), thus obtaining:

\[
\exp \left[ \beta F_{\text{RSB}}^{i} (x) \right] = \int \prod_{k \in \partial a \setminus i} dm_{k \rightarrow a} Q_{k \rightarrow a} (m_{k \rightarrow a}) \prod_{i \rightarrow a} dm_{i \rightarrow a} \hat{Z}_{a \rightarrow i}^{x} (\{m_{k \rightarrow a}\}) Q_{i \rightarrow a} (m_{i \rightarrow a}) e^{x \beta F_{\text{RSB}}^{i} (m_{i \rightarrow a}, \{m_{k \rightarrow a}\})}. \]  
(B33)
From the RS equations Eqns. (8) and (23) we then have that:

\[ e^{x\beta f_{ai}(m_{i\rightarrow a}, \{m_{k\rightarrow a}\})} = Z_{(ai)}^x \]

\[ = \hat{Z}_{a\rightarrow i}(\{m_{k\rightarrow a}\}) \left[ \int_{-\infty}^{\infty} d\sigma_i \; \eta_{i\rightarrow a}(\sigma_i) \; e^{\frac{\lambda x^2}{2N} \int_{-\infty}^{\infty} \prod_{k \in \partial a \setminus i} d\sigma_k \; \eta_{k\rightarrow a}(\sigma_k) \; e^{\beta J_a \sigma_i \prod_{k \in \partial a \setminus i} \sigma_k} } \right]^{x} \]

\[ = \hat{Z}_{a\rightarrow i}^x \left[ 1 + \beta J_a \prod_{k \in \partial a} m_{k\rightarrow a} + O(1/K) \right]^{x}. \]

(B34)

Then, since we have \( J_a \sim 1/N^{(p-1)/2} \) and \( 1/K \sim 1/N^{p-1} \), to the leading order in \( N \) we can simply write

\[ x\beta f_{ai}(m_{i\rightarrow a}, \{m_{k\rightarrow a}\}) = -x \log \left[ \hat{Z}_{a\rightarrow i}(\{m_{k\rightarrow a}\}) \right] + O(1/N^{(p-1)/2}), \]

(B35)

so that inside the integral of Eq. (B33) we have

\[ \hat{Z}_{a\rightarrow i}^x(\{m_{k\rightarrow a}\}) \; e^{x\beta f_{ai}(m_{i\rightarrow a}, \{m_{k\rightarrow a}\})} \simeq 1. \]

(B36)

In conclusion, to the leading order in \( N \), we can write

\[ \exp \left[ \beta f_{RSB}^{ai}(x) \right] = \int \prod_{k \in \partial a} dm_{k\rightarrow a} \; Q_{k\rightarrow a}(m_{k\rightarrow a}) = 1, \]

(B37)

by the closure condition over the probability density \( Q \). We have therefore shown that, to the leading order in \( N \), one has \( \beta f_{RSB}^{ai}(x) = 0 \) for each edge \((ai)\).

6. 1RSB total free energy

Putting together the results of Sec. [B3] and Sec. [B4] and remembering that the the total free energy of the system is just the free energy of the \( x \) coupled replicas that we computed, divided by \( x \) (and extremized over \( x \)), we obtain:

\[ -x\beta \mathcal{F}(x) = \sum_{a=1}^{M} \beta f_{RSB}^{ai} + \sum_{i=1}^{N} \beta f_{RSB}^{ai} - \sum_{(ai) \in E} \beta f_{RSB}^{ai} \]

\[ = \frac{xN}{2} \left\{ \frac{\beta^2}{2} \left[ 1 - (1 - x)q_0^p - xq_0^p \right] + \frac{q_0}{[1 - xq_0 - (1 - x)q_1]} \right\} + \frac{x - 1}{x} \log (1 - q_1) + \frac{1}{x} \log [1 - xq_0 - (1 - x)q_1] \}

(B38)

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