Abstract

The RG expansions for renormalized coupling constants $g_6$ and $g_8$ of the 3D $n$-vector model are calculated in the 4-loop and 3-loop approximations respectively. Resummation of the RG series for $g_6$ by the Padé-Borel-Leroy technique results in the estimates for its universal critical values $g_6^*(n)$ which are arqued to deviate from the exact numbers by less than 0.3 %. The RG expansion for $g_8$ demonstrates stronger divergence being much less suitable for getting proper numerical estimates.

Recently, the free energy (effective action) and, in particular, higher-order renormalized coupling constants $g_{2k}$ for the basic models of phase transitions became the target of intensive theoretical studies (see, e. g. [1, 2] and references therein). These constants enter the scaling equation of state and thus play a key role at criticality. Along with critical exponents, they are universal, i.e. possess, under $T \to T_c$, numerical values which depend only on the space dimensionality and the symmetry of the order parameter. Calculation of the universal critical values of $g_6$, $g_8$, etc. for the 3D Ising model by different methods showed that the field-theoretical RG approach in fixed dimensions yields the most accurate numerical estimates. It is a consequence of a rapid convergence of the iteration schemes originating from RG expansions [1, 3, 4]. It is natural, therefore, to use the field theory for calculation of renormalized higher-order coupling constants for the 3D $n$-vector model. In the report, the 3D RG expansion for the renormalized coupling constants $g_6$ and $g_8$ will be found and the universal critical values of $g_6$ will be estimated.
The 3D $O(n)$-symmetric model is described at criticality by Euclidean scalar
field theory with the Hamiltonian

$$H = \int d^3 x \left[ \frac{1}{2} \left( m_0^2 \varphi^2 + (\nabla \varphi)^2 \right) + \lambda (\varphi^2)^2 \right],$$

where $m_0^2$ is proportional to $T - T_c^{(0)}$, $T_c^{(0)}$ being the phase transition temperature in
the absence of the order parameter fluctuations. The fluctuations give rise to many-
point correlations $\langle \varphi(x_1) \varphi(x_2) ... \varphi(x_{2k}) \rangle$ and, correspondingly, to higher-order
terms in the expansion of the free energy in powers of the magnetization $M$:

$$F(M, m) = F(0, m) + \sum_{k=1}^{\infty} g_{2k} m^{3-k(1+\eta)} M^{2k},$$

where $m$ is a renormalized mass, $\eta$ is a Fisher exponent, and $g_{2k}$ are dimensionless
 coupling constants. Let, as usually, $g_2 = 1/2$. Then $g_4, g_6, g_8, ...$ will acquire, under
$T \to T_c$, the universal values.

The asymptotic critical values of $g_4$, $g_4^*(n)$, determining critical exponents and
other universal quantities, have been found from the 6-loop expansion for RG $\beta$-
function [1, 3, 8]; they are known with rather high accuracy. On the contrary, the
information about the universal numbers $g_6^*, g_8^*$, etc. for $O(n)$-symmetric model
is very poor today. To estimate them, we calculate corresponding RG series and
perform their resummation by means of the Pade-Borel-Leroy technique. The RG
series for $g_6$ and $g_8$ are obtained from conventional Feynman graph expansions for the
6-point and 8-point vertices in terms of the bare coupling constant $\lambda$. In its turn, $\lambda$
is expressed perturbatively via the renormalized coupling constant $g_4$. Substituting
then the series for $\lambda$ into the "bare" expansions, we obtain the RG expansions for
$g_6$ and $g_8$.

As was shown [3, 8], the 1-, 2-, 3-, and 4-loop contributions to $g_6$ are formed by
1, 3, 16, and 94 one-particle irreducible Feynman graphs, respectively. In this case,
the calculations just described give:

$$g_6 = \frac{9}{\pi} g_4^3 \left[ \frac{n + 26}{27} - \frac{17 n + 226}{81 \pi} g_4 + (0.000999164 n^2 + 0.14768927 n + 1.24127452) g_4^2 \\
- (-0.0000094 g_4^3 + 0.00783129 n^2 + 0.34565683 n + 2.14825455) g_4^3 \right].$$

In the case of $g_8$, the 1-, 2-, and 3-loop contributions are given by 1, 5, and 36
graphs respectively [3]. Corresponding RG expansion is found to be:

$$g_8 = \frac{81}{2 \pi} g_4^3 \left[ \frac{n + 80}{81} - \frac{81 n^2 + 7114 n + 134960}{13122 \pi} g_4 \\
+ (0.00943497 n^2 + 0.60941312 n + 7.15615323) g_4^2 \right].$$
Being a field-theoretical perturbative expansions these series are divergent (asymptotic). To get reasonable numerical estimates for \( g^*_6 \) and \( g^*_8 \) some procedure making them convergent should be applied. The Borel-Leroy transformation

\[
f(x) = \sum_{i=0}^{\infty} c_i x^i = \int_0^b e^{-t} F(xt) dt, \quad F(y) = \sum_{i=0}^{\infty} \frac{c_i}{(a+b)!} y^i.
\]

(5)
can play a role of such a procedure. Since the RG series considered turns out to be alternating the analytical continuation of the Borel-Leroy transform may be then performed by using Padé approximants \([L/M]\).

For \( g_6 \) we have the 4-loop RG expansion and can construct, in principle, three different Padé approximants: \([2/1]\), \([1/2]\), and \([0/3]\). To obtain proper approximation schemes, however, only diagonal \([L/L]\) and near-diagonal Padé approximants should be employed. That’s why further we limit ourselves with approximants \([2/1]\) and \([1/2]\). Moreover, the diagonal Padé approximant \([1/1]\) will be also dealt with although this corresponds to the usage of the lower-order, 3-loop approximation.

The algorithm of estimating \( g^*_6 \) we use here is as follows. Since the Taylor expansion for the effective action contains as coefficients the ratios \( R_{2k} = g_{2k}/g_{k-1}^4 \) we work with the RG series for \( R_6 \). It is resummed in three different ways based on the Padé approximants just mentioned. The Borel-Leroy integral is evaluated as a function of the parameter \( b \) under \( g_4 = g_4^*(n) \). For the fixed point coordinate \( g_4^*(n) \) the values extracted from the six-loop RG expansion are adopted \([1, 5]\). The optimal value of \( b \) providing the fastest convergence of the iteration scheme is then determined. It is deduced from the condition that the Padé approximants employed should give, for \( b = b_{opt} \), the values of \( R^*_6 \) which are as close as possible to each other. Finally, the average over three estimates for \( R^*_6 \) is found and claimed to be a numerical value of this universal ratio.

The results of our calculations are presented in Table 1. It contains numerical estimates for \( g^*_6 \) resulting from the 4-loop RG expansion (column 3) and their analogs given by the Padé-Borel resummed 3-loop RG series \([3]\) (column 4). As is seen, with increasing \( n \) the difference between the 4-loop and 3-loop estimates rapidly diminishes: being small (0.9 %) even for \( n = 1 \), it becomes negligible at \( n = 10 \) and practically disappears for \( n \geq 14 \). Such a behaviour is quite natural since with increasing \( n \) the approximating properties of RG series for \( g_6 \) are improving \([3]\).

How close to the exact values of \( g^*_6 \)? To clear up this point, let us compare our 4-loop estimate for \( R^*_6 \) at \( n = 1 \) with those obtained recently by an analysis of the 5-loop scaling equation of state for the 3D Ising model \([4, 9]\). R. Guida and J. Zinn-Justin have obtained \( R^*_6 = 1.644 \) and, taking into account some additional information, \( R^*_6 = 1.643 \), while our estimate is \( R^*_6 = 1.648 \). Keeping in mind that the exact value of \( R^*_6 \) should lie between the 4-loop and 5-loop estimates (the RG series is alternating), our estimate can differ from the exact number by no more than 0.3 %. Since for \( n > 1 \) the RG expansion \([3]\) should provide better numerical estimates than in the Ising case, this value (0.3 %) represents an
upper bound for the deviation of the numbers in column 3 of Table 1 from their exact counterparts.

It is interesting to compare our estimates with those obtained by other methods. Since 1994, the universal values of the sextic coupling constant for the 3D $O(n)$-symmetric model were estimated by solving the exact RG equations [10], by lattice calculations [11], and by a constrained analysis of the $\epsilon$–expansion [2]; corresponding results are collected in columns 5, 6, and 7 of Table 1 respectively. As is seen, they are, in general, in accord with ours. For large $n$, our estimates are consistent also with those given by the $1/n$-expansion which are presented in column 8.

The RG expansion for the octic coupling constant $g_8$ turns out to be much worse than the series (3) from the point of view of their summability. Indeed, the series (3) diverges considerably stronger and is one term shorter than that for $g_6$. It implies that the only Pade approximant – $[1/1]$ – may be really used in a course of the resummation of this series. In the Ising case $n = 1$, such a simple Pade-Borel procedure, when applied to the 3-loop RG expansion for $g_8$, was found to lead to rather crude numerical estimates [3]. As our analysis shows, with increasing $n$ the situation becomes better but, nevertheless, the RG estimates for $g_8(n)$ remain much less accurate than those obtained for the sextic coupling constant. Corresponding numerical results will be published elsewhere.

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Table 1: Our estimates of universal critical values of the renormalized sextic coupling constant for the 3D $n$-vector model (column 3). The fixed point coordinates $g^*$ are taken from [5] ($1 \leq n \leq 3$) and [1] ($4 \leq n \leq 40$). The $g_6^*$ estimates extracted from the Pade-Borel resummed 3-loop RG expansion (column 4), from the exact RG equations (column 5), obtained by the lattice calculations (column 6), resulting from a constrained analysis of the $\epsilon$-expansions (column 7), and given by the $1/n$-expansion (column 8) are presented for comparison.

| $n$ | $g^*$ | $g_6^*$ | $g_6^{[4]}$ | $g_6^{[10]}$ | $g_6^{[1]}$ | $g_6^{[2]}$ | $g_6^* (1/n)$ |
|-----|-------|---------|-------------|-------------|-------------|-------------|----------------|
| 2   | 1.415 | 1.608   | 1.622       | 1.52        | 1.92(24)    | 1.609(9)    |                 |
| 3   | 1.406 | 1.228   | 1.236       | 1.14        | 1.27(25)    | 1.21(7)     |                 |
| 4   | 1.392 | 0.951   | 0.956       | 0.88        | 0.93(20)    | 0.931(46)   |                 |
| 5   | 1.3745| 0.747   | 0.751       | 0.68        | 0.62(15)    | 0.725(29)   | 1.6449          |
| 6   | 1.3565| 0.596   | 0.599       |             |             |             | 1.0528          |
| 7   | 1.3385| 0.483   | 0.485       |             |             |             | 0.7311          |
| 8   | 1.321 | 0.396   | 0.398       |             |             |             | 0.5371          |
| 9   | 1.3045| 0.329   | 0.331       |             |             |             | 0.319(4)        |
| 10  | 1.289 | 0.277   | 0.278       |             |             |             | 0.3249          |
| 12  | 1.2487| 0.174   | 0.175       |             |             |             | 0.2632          |
| 14  | 1.2266| 0.134   | 0.134       |             |             |             | 0.1828          |
| 16  | 1.2077| 0.105   | 0.105       |             |             |             | 0.1343          |
| 18  | 1.1914| 0.0845  | 0.0847      |             |             |             | 0.0812          |
| 20  | 1.1773| 0.0693  | 0.0694      |             |             |             | 0.0658          |
| 24  | 1.1542| 0.0487  | 0.0488      |             |             |             | 0.0457          |
| 28  | 1.1361| 0.0360  | 0.0361      |             |             |             | 0.0336          |
| 32  | 1.1218| 0.0276  | 0.0276      |             |             |             | 0.0275(1)       |
| 36  | 1.1099| 0.0218  | 0.0218      |             |             |             | 0.0203          |
| 40  | 1.1003| 0.0176  | 0.0176      |             |             |             | 0.0164          |