Combinatorial Optimization of AC Optimal Power Flow with Discrete Demands in Radial Networks

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Abstract—The AC Optimal power flow (OPF) problem is one of the most fundamental problems in power systems engineering. For the past decades, researchers have been relying on unproven heuristics to tackle OPF. The hardness of OPF stems from two issues: (1) non-convexity and (2) combinatoric constraints (e.g., discrete power ejection constraints). The recent advances in providing sufficient conditions on the exactness of convex relaxation of OPF can address the issue of non-convexity. To complete the understanding of OPF, this paper presents a polynomial-time approximation algorithm to solve the convex-relaxed OPF with discrete demands as combinatoric constraints, which has a provably small parameterized approximation ratio (also known as PTAS algorithm). Together with the sufficient conditions on the exactness of convex relaxation, we provide an efficient approximation algorithm to solve OPF with discrete demands. This paper also discusses fundamental hardness results of OPF to show that our PTAS is among the best achievable in theory. Simulations show our algorithm can produce close-to-optimal solutions in practice.

Index Terms—Optimal power flow, approximation algorithms, discrete power demands, combinatorial optimization, PTAS

 NOMENCLATURE

Electric Networks
\[ \mathcal{V}^+ \] Set of nodes in distribution network without root \( 0 \)
\[ \mathcal{E} \] Set of links in distribution network
\( \mathcal{E}_{i,j} \) Set of links of subtree from \( j \in \mathcal{V} \)
\( (i, j) \) Directed edge, \( i \) is a parent of \( j \) in a rooted tree
\( \mathcal{N} \) Set of all users
\( \mathcal{I} \) Set of users with discrete demands \( (s_k \in \{0, s_k\}) \)
\( \mathcal{F} \) Set of users with continuous demands \( (s_k \leq s_k \leq \bar{s}_k) \)
\( \mathcal{U}_{j} \) Set of users attached to node \( j \in \mathcal{V} \)
\( \mathcal{N}_j \) Set of users attached to the subtree from \( j \in \mathcal{V} \)
\( \mathcal{P}_j, \mathcal{P}_k \) Path from root 0 to node \( j \) or user \( k \)

Parameters
\( v_0 \) Voltage magnitude square at root 0
\( \bar{v}_{i,j} \) Upper limit of current magnitude sq. through \( (i, j) \)
\( \bar{S}_{i,j} \) Upper limit of power flow through \( (i, j) \)
\( \underline{v}_{j}, \bar{v}_{j} \) Upper & lower limits of voltage magnitude sq. at \( j \)
\( \underline{s}_k, \bar{s}_k \) Upper & lower limits of power demand of user \( k \)
\( f_0(\cdot) \) Cost function of total real power supply
\( f_k(\cdot) \) Objective function of real power demand of user \( k \)
\( z_{i,j} \) Impedance of link \( (i, j) \)
\( \phi \) Maximum difference between the phase angles of any pair of discrete power demands

Variables
\( s_k \) Power demand of user \( k \)
\( x_k \) Binary control variable of discrete demand of \( k \in \mathcal{I} \)

\( s_0 \) Total power supply (or injection) at root 0
\( \ell_{i,j} \) Current magnitude sq. through link \( (i, j) \)
\( v_j \) Voltage magnitude sq. at node \( j \)
\( S_{i,j} \) Power flow through link \( (i, j) \)

I. INTRODUCTION

The AC optimal power flow (OPF) problem underpins many optimization problems of power systems. However, OPF is notoriously hard to solve. The hardness of OPF stems from two issues: (1) non-convexity because of the non-convex constraints involving complex-valued entities of power systems, and (2) combinatoric constraints, for example, discrete power injection/ejection constraints. In the past, due to the lack of understanding of solvability of OPF, researchers have been relying on unproven heuristics or general numerical solvers, which suffer from the issues of excessive running-time, lack of termination guarantee, or uncertainty of how far the output solutions deviate from the true optimal solutions.

Recently, there have been advances in tackling OPF by applying convex relaxations \([1]–[4]\). These results imply that the relaxation of certain equality operating constraints to be inequality constraints can attain a more tractable convex programming problem which admits an optimal solution to the original problem, under certain mild sufficient conditions verifiable in a prior. Remarkably, these results can be applied to OPF with discrete power ejections (e.g., \([2]\)).

However, because of the lack of proper efficient algorithms to solve the convex-relaxed OPF with combinatoric constraints, most prior papers (e.g., \([1]–[4]\)) only studied OPF with continuous power ejection constraints, such that the controls of power ejections can be partially satisfied. Some papers considered OPF with discrete control variables \([5]–[7]\), but their algorithms rely mainly on heuristics and show no guarantee on optimality and running time.

In practice, there are many discrete power ejection constraints. For example, certain loads and devices can be either switched on or off, and hence, their control decision variables are binary. To tackle OPF in these settings, it is important to provide feasible solutions that satisfy the discrete power ejection constraints.

Solving combinatorial optimization problems by efficient algorithms in general is a main subject studied in theoretical computer science. Hence, we will draw on the related notions and terminology from theoretical computer science. There is a well-known class of problems, known as NP-hard problems, which are believed to be intractable to find the exact optimal solutions in polynomial-time. However, taking into consideration of approximation solutions (i.e., the solutions are within...
a certain approximation ratio over an optimal solution), it is possible to obtain efficient polynomial-time approximation algorithms for certain NP-hard problems. One efficient type of approximation algorithms is called PTAS (polynomial-time approximation scheme) [8], which allows a parametrized approximation ratio as the running time of the algorithm. Thus, one can change the desired approximation ratio, at the expense of running time. In this paper, our goal is to provide PTAS to solve OPF with discrete demands as combinatoric constraints.

This work is also related to a number of recent developments. First, the combinatorial optimization for a single-capacity power system has been studied as complex-demand knapsack problem in prior work [9]–[11]. Then, an approximation algorithm is provided for simplified DistFlow model of OPF without considering generation cost in [12]. To our best knowledge, this is the first work to present a PTAS algorithm for combinatorial optimization in a realistic OPF model.

This paper is structured as follows: First, the model of OPF and the idea of convex relaxation are reviewed in Secs. II [3]. Next, we present a PTAS algorithm to solve the convex-relaxed OPF with discrete demands in Sec. IV. Together with the sufficient conditions on the exactness of convex relaxation, we provide an efficient approximation algorithm to solve OPF with discrete demands. Furthermore, fundamental hardness results of OPF are discussed to show that our PTAS is among the best achievable in Sec. V. Lastly, simulations in Sec. VI show that the proposed algorithm can produce close-to-optimal solutions in practice.

II. PROBLEM DEFINITION AND NOTATIONS

A. Optimal Power Flow Problem on Radial Networks

As in the previous work [1], [2], this paper considers a radial (tree) electric distribution network, represented by a graph \( \mathcal{T} = (\mathcal{V}, \mathcal{E}) \). The set of nodes \( \mathcal{V} = \{0, \ldots, m\} \) denotes the electric buses, and the set of edges \( \mathcal{E} \) denotes the distribution lines. Let \( \mathcal{V}^+ \equiv \mathcal{V} \setminus \{0\} \). A substation feeder is attached to the root of the tree, denoted by node 0. We assume that root 0 is only connected to node 1 via a single edge (0,1). See an illustration in Fig. 1. Since \( \mathcal{T} \) is a tree, \( |\mathcal{V}^+| = |\mathcal{E}| = m \). Let \( \mathcal{T}_i = (\mathcal{V}_i, \mathcal{E}_i) \) be the subtree rooted at node \( i \).

Fig. 1: An illustration of a radial network.

Note that this paper adopts the flow orientation that power flows from the root (node 0) towards the leaves.\(^1\) Hence, tuple \((i, j)\) refers to a directed edge, where node \( i \) is a parent of \( j \). Denote the path from the root 0 to node \( j \) by \( \mathcal{P}_j \).

Instead of assigning a single power ejection to each node, this paper considers a general setting where a set of users are attached to each node. Each user can control his power demand individually. Let \( \mathcal{N} \) be the set of all users, where \( |\mathcal{N}| = n \). Denote the set of users attached to node \( j \) by \( \mathcal{U}_j \subseteq \mathcal{N} \). Let the set of users within subtree \( \mathcal{T}_j \) be \( \mathcal{N}_j \equiv \bigcup_{j \in \mathcal{V}_j} \mathcal{U}_j \). Denote the path from root 0 to user \( k \) by \( \mathcal{P}_k \).

The demand for user \( k \) is represented by a complex number \( s_k \in \mathbb{C} \). Among the users, some have discrete (inelastic) power demands, denoted by \( \mathcal{I} \subseteq \mathcal{N} \). A discrete demand \( s_k \), for \( k \in \mathcal{I} \), takes values from a discrete set \( \mathcal{S}_k \subseteq \mathbb{C} \). We assume \( \mathcal{S}_k \equiv \{0, \sigma_k\} \), where a demand \( s_k \) can be either completely satisfied at level \( \sigma_k \in \mathbb{C} \) or dropped, e.g., a piece of equipment that is either switched on with a fixed power rate or off. The rest of users, denoted by \( \mathcal{F} \equiv \mathcal{N} \setminus \mathcal{I} \), have continuous demands, defined by convex sets \( \mathcal{S}_k \), for \( k \in \mathcal{F} \); a typical example is a set defined by box constraints: \( \mathcal{S}_k \equiv \{s_k \in \mathbb{C} | 0 \leq s_k \leq \sigma_k\} \), for given lower and upper bounds \( \sigma_k \) and \( \sigma_k \). We consider only consumer\(^2\) users, such that Re\((s_k) \geq 0 \) (but Im\((s_k)\) may be negative) for all \( k \in \mathcal{N} \). This assumption is justified for discrete demands in Sec. V as we show fundamental difficulties of OPF when it has discrete control variables for both demand \( \text{Re}(s_k) \geq 0 \) and supply \( \text{Re}(s_k) \leq 0 \).

Let \( v_j \) and \( \ell_i,j \) be the voltage and current magnitude square at node \( j \) and edge \((i, j)\), respectively. Let \( S_{i,j} \) be the power flowing from node \( i \) towards node \( j \). Note that \( S_{i,j} \) is not symmetric, namely, \( S_{i,j} \neq S_{j,i} \).

There are several operating constraints of power systems:

- **(Power Capacity Constraints):**
  \[ |S_{i,j}| \leq \bar{S}_{i,j}, |S_{j,i}| \leq \bar{S}_{i,j}, \forall (i, j) \in \mathcal{E}. \]

- **(Current Thermal Constraints):**
  \[ \ell_{i,j} \leq \rho_{i,j}, \forall (i, j) \in \mathcal{E}. \]

- **(Voltage Constraints):**
  \[ v_j \leq v_j \leq \bar{v}_j, \forall v_j \in \mathcal{V}^+. \]

\( v_j, \sigma_j \in \mathbb{R}^+ \) are the minimum and maximum allowable voltage magnitude square at node \( j \), and \( \bar{S}_{i,j}, \rho_{i,j} \in \mathbb{R}^+ \) are the maximum allowable apparent power and current on edge \((i, j)\).

The power supply at root 0 is denoted by \( s_0 \). This paper adopts the convention to denote a power supply by a complex number with negative real part and a power demand by a complex number with positive real part (i.e., \( \text{Re}(s_0) \leq 0 \) and \( \text{Re}(s_k) \geq 0 \) for all \( k \in \mathcal{N} \)). In the following, a subscript is omitted from a variable to denote its vector form, such as \( S \equiv (S_{i,j})_{(i,j) \in \mathcal{E}}, \ell \equiv (\ell_{i,j})_{(i,j) \in \mathcal{E}}, v \equiv (v_j)_{j \in \mathcal{V}^+}, x \equiv (x_k)_{k \in \mathcal{E}}, v \equiv (v_j)_{j \in \mathcal{V}^+} \).

The goal of OPF is to find an assignment for the demand vector \( s \) and supply \( s_0 \) that minimizes a non-negative convex (cost) objective function\(^3\) \( f \):

\[
f(s_0, s) = f_0(\text{Re}(s_0)) + \sum_{k \in \mathcal{N}} f_k(\text{Re}(s_k)),
\]

\(^3\)On the other hand, when some of the continuous users are supplies (such as in distributed generation), our approximation algorithm for the discrete users with a good approximation guarantee so far cannot be generalized in a straightforward manner (see the footnote in [8]) because Lemma 8 provides a straightforward way to deal with the case when the function \( f_k(\cdot) \) depends also on the reactive power.
where $f_0$ is the non-negative and non-increasing cost for the active power supply (note that $\text{Re}(s_0) \leq 0$), and $f_k$ is the non-negative and non-increasing cost for each satisfied active power demand, such that $f_k(\text{Re}(s_k)) = 0$ (i.e., each user prefers maximum demand).

We formulate OPF using the branch flow model (BFM)\footnote{To be precise, this model is called branch flow model with angle relaxation\cite{2,3}, as it omits the phase angles of voltages and currents. But it is always possible to recover the phase angles from a feasible solution in a radial network\cite{3}.}.

The inputs are the voltage, current and transmitted power limits $\left[ v_0, (v_i, f_i) \right]_{i \in V^+}, (\tilde{S}_e, \ell_e, e_\ell)_{e \in E}, (S_k)_{k \in F}, (\tilde{S}_k)_{k \in F}$, whereas the outputs are the control decision variables and power flow states $[s_0, s, S, v, \ell]$.\footnote{As it omits the phase angles of voltages and currents.}

The OPF that utilizes BFM is given by the following mixed integer programming problem:

$$(\text{OPF}) \min_{s_0, s, S, v, \ell} f(s_0, s)$$

subject to:

$$\ell_{i,j} = \left| \frac{S_{i,j}}{v_i} \right|^2, \quad \forall (i,j) \in E, \quad (2)$$

$$S_{i,j} = \sum_{k \in I_i} s_k + \sum_{l = (i,j)} s_k + z_{i,j} \ell_{i,j}, \quad \forall (i,j) \in E, \quad (3)$$

$$S_{0,1} = -s_0, \quad (4)$$

$$v_j = v_i + \left| z_{i,j} \right|^2 \ell_{i,j} - 2\text{Re}(z_{i,j}^* S_{i,j}), \quad \forall (i,j) \in E, \quad (5)$$

$$v_j \leq \tau_j, \quad \forall j \in V^+ \quad (6)$$

$$\left| S_{i,j} \right| \leq \tilde{S}_{i,j}, \quad \left| S_{i,j} - z_{i,j} \ell_{i,j} \right| \leq \tilde{S}_{i,j}, \quad \forall (i,j) \in E, \quad (7)$$

$$\ell_{i,j} \leq \tilde{\ell}_{i,j}, \quad \forall (i,j) \in E, \quad (8)$$

$$s_k \in S_k \quad \forall k \in F, \quad (9)$$

$$s_k \in \tilde{S}_k, \quad \forall k \in I, \quad (10)$$

$$v_j \in \mathbb{R}^+, \forall j \in V^+, \quad \ell_{i,j} \in \mathbb{R}^+, \quad S_{i,j} \in \mathbb{C}, \quad \forall (i,j) \in E. \quad (11)$$

We note a few remarks:

1) BFM can also be expressed using the opposite orientation towards node 0: $S_{i,j} = \sum_{l = (i,j)} s_k + \sum_{k \in I_i} s_k$ (see e.g., [1, 2]). As shown in [3], there is a bijection between the models of the two orientations, since $S_{i,j} = -S_{i,j} + z_{i,j} \ell_{i,j}$ and $I_{j,i} = -I_{j,i}$.

2) We explicitly consider constraints (Cons. (7)) in both directions, whereas [1] implicitly considers only one direction. Our results can be applied to bi-directional capacity constraints, which are stronger than that of [1]. See Sec. III for a discussion.

3) Cons. (10) are combinatoric constraints with discrete variables. Although [2] also considers the possibility of discrete power injections, it does not solve the respective optimal solutions.

4) The non-linearity of Cons. (2) makes this program non-convex (together with the discrete Cons. (10)).

5) This paper considers a convex objective function mainly for solvability, although only a non-increasing function is required for the exactness of convex relaxation.

B. Approximation Solutions

This paper provides an efficient approximation algorithm to solve OPF with combinatoric constraints. Approximation algorithms are a well-studied subject in theoretical computer science\cite{8}. In the following, we define some standard terminology for approximation algorithms.

Consider a minimization problem $A$ with non-negative objective function $f(\cdot)$, let $F$ be a feasible solution to $A$ and $F^*$ be an optimal solution to $A$. $f(F)$ denotes the objective value of $F$. Let $\text{OPT} = f(F^*)$ be the optimal objective value of $F^*$. A common definition of approximation solution is $\alpha$-approximation, where $\alpha$ characterizes the approximation ratio between the approximation solution and an optimal solution.

**Definition 1.** For $\alpha > 1$, an $\alpha$-approximation to minimization problem $A$ is a feasible solution $F$ such that

$$f(F) \leq \alpha \cdot \text{OPT}.$$}

In particular, polynomial-time approximation scheme (PTAS) is a $(1+\epsilon)$-approximation algorithm to a minimization problem, for any $\epsilon > 0$. The running time of a PTAS is polynomial in the input size for every fixed $\epsilon$, but the exponent of the polynomial might depend on $1/\epsilon$. Namely, PTAS allows a parametrized approximation ratio as the running time.

C. Assumptions

OPF with combinatoric constraints is hard to solve. Hence, there are some assumptions to facilitate our solutions:

A0: $f_0(-s_0^R)$ is non-decreasing in $-s_0^R \in \mathbb{R}^+$

A1: $z_e \geq 0, \forall e \in E$, which naturally hold in distribution networks.

A2: $v_j < v_0 < \tau_j, \forall j \in V^+$, which is also assumed in [1]. Typically in a distribution network, $v_0 = 1$ (per unit), $\tau_j = (1.95)^2$ and $\tau_j = (1.05)^2$; in other words, 5% deviation from the nominal voltage is allowed.

A3: $\text{Re}(z_k^* \pi_k) \geq 0, \forall k \in I, e \in E$. Intuitively, A3 requires that the phase angle difference between any $z_e$ and $s_k$ for $k \in I$ is at most $\frac{\pi}{2}$. This assumption holds, if the discrete demands do not have large negative reactive power.

A4: $|\angle s_k - \angle s_k'| \leq \frac{\pi}{2}$ for any $k, l' \in N$. Intuitively, A4 requires that the demands have “similar” power factors. A4 can also be stated as $\text{Re}(s_k^I s_k') \geq 0$.

Assumptions A3 and A4 are motivated, from a theoretical point of view, by the inapproximability results which will be presented in Sec. IV (if either assumption does not hold, then the problem cannot be approximated within any polynomial factor unless P=NP). Assumption A3 also holds in reasonable practical settings, see e.g., [1]. As will be seen in Sec. IV-A by performing an axis rotation, we may assume by A4 that $s_k \geq 0$. Clearly, under this and assumption A1, the reverse power constraint in (7) is implied by the forward power constraint $|S_{i,j}| \leq \tilde{S}_{i,j}$. It will also be observed below that under assumptions A1, A2 and A3, the voltage upper bounds in (5) can be dropped.

III. REVIEW OF CONVEX RELAXATION OF OPF

This section presents a brief review of convex relaxation of OPF. The idea of relaxing OPF to a convex optimization problem can significantly improve the solvability of OPF. Convex optimization problems can be solved efficiently by a
polynomial-time algorithm. Under certain conditions, convex relaxation can be shown to obtain the optimal solutions of OPF. A second order cone programming (SOCP) relaxation of OPF is obtained by replacing Cons. (2) by \( \ell_{i,j} \geq \frac{|S_{i,j}|^2}{v_i} \). For convenience of notation, we rewrite Cons. (10) by \( s_k = \bar{s}_k x_k \), where \( x_k \in \{0, 1\} \) is a control variable. A relaxation of OPF (denoted by cOPF) is defined as follows:

\[
\text{(cOPF)} \quad \min_{s_0, s, \ell, S} f(s_0, s),
\]

s.t. \( 3, 9, 11, 12 \).

Notice that cOPF is not completely convex due to the discrete constraints in (13). In fact, the main contribution of the paper is to provide an approximation scheme for solving cOPF.

For a given \( s \in \mathbb{C}^n \), we denote by OPF[\( s \)] (resp., cOPF[\( s \)]) the restriction of OPF (resp., cOPF) where we set \( s = s \). The relaxation cOPF[\( s \)] is called (efficiently) exact, if every optimal solution \( F^* \) of cOPF[\( s \)] can be converted to an optimal solution of OPF[\( s \)], in a polynomial number of steps. This definition is adopted from [1]-[4], but also with an emphasis on efficient computation.

There are several sufficient conditions of exactness:

**C1:** There is an optimal solution \( F = (s_0, s, v, \ell, S) \) for (cOPF) such that the following linear system is feasible (in \( (\bar{S}_{i,j})_{(i,j) \in \mathcal{E}} \) and \( (\bar{v}_j)_{j \in V^+} \)):

\[
\begin{align*}
\bar{S}_{i,j} &= \sum_{k \in \mathcal{E}_j} s_k + \sum_{l:(i,l) \in \mathcal{E}} \bar{S}_{l,j}, & \forall (i,j) \in \mathcal{E}, \\
\bar{v}_i - \bar{v}_j &= 2\text{Re}(z_{i,j}^* \bar{S}_{i,j}), & \forall (i,j) \in \mathcal{E}, \\
\text{Re}(z_{i,j}^* \bar{S}_{i,j}) &\geq 0, & \forall (i,j) \in \mathcal{E}, \quad \forall (h,l) \in \mathcal{E}_j, \\
\bar{v}_j &\leq \bar{v}_j, & \forall j \in V^+.
\end{align*}
\]

**C2:** There is an optimal solution \( F = (s_0, s, v, \ell, S) \) for (cOPF) such that

\[
\sum_{k \in \mathcal{N}_j} \text{Re}(z_{i,j}^* s_k) \geq 0, \quad \forall j \in V^+, (h,l) \in \mathcal{E}_j \cup \{(i,j) \in \mathcal{E}\}
\]

where \( \mathcal{N}_j \triangleq \bigcup_{i \in \mathcal{E}} \mathcal{U}_j \) is the set of attached users within subtree \( T_j \), and \( \mathcal{E}_j \cup \{(i,j) \in \mathcal{E}\} \) is the set of edges of subtree \( T_j \) and edges that are connected to node \( j \).

In [1], it is shown that C1 is a sufficient condition for exactness of OPF considering uni-directional power capacity constraints from a leaf to the root.

In order to attain exactness of OPF with bi-directional power capacity constraints, a stronger condition ought to be considered. In addition to (16), it is also required that \( \text{Re}(z_{i,j}^* S_{i,j}) \geq 0 \) which gives (18). Note that by (18) and A2, and the recursive substitution of \( v_j \) from the root in (15), Cons. (17) is already satisfied as

\[
\bar{v}_j = v_0 - \sum_{e \in \mathcal{P}_j} \text{Re}(z_{e}^* S_e) \leq v_0 < \bar{v}_j,
\]

where \( \mathcal{P}_j \) denotes the unique path from root 0 to node \( j \). Note also that A3 implies C2 when \( N = \mathcal{I} \) (as A3 applies to only discrete demands, whereas C2 applies to all the demands and edges within a subtree).

The next theorem summarizes the sufficient condition of exactness for the relaxation of OPF.

**Theorem 1.** Let \( F'' = (s_0'', s', v', \ell'', S'') \) be a feasible solution of cOPF[\( s' \)]. Under assumptions A0, A1, A2, and C2, there is a feasible solution \( F' = (s_0', s', v', \ell', S') \) of cOPF[\( s' \)] that satisfies \( \ell'_{i,j} = \frac{|S_{i,j}|^2}{v_i} \) for all \((i,j) \in \mathcal{E} \), and \( f(F') \leq f(F'') \). Given \( F'' \), such a solution \( F' \) can be found in polynomial time.

The proof uses similar techniques as in [1], [2], but shows an additional result on the efficient conversion to an optimal solution of OPF. See the supplementary materials for a proof.

**IV. PTAS FOR OPF WITH DISCRETE POWER DEMANDS**

This section presents a \((1 + \varepsilon)\)-approximation algorithm (PTAS) for OPF. Note that we consider the number of links in the distribution network (i.e., \(|V^+| = |\mathcal{E}| = m\)) is a constant. But we allow the number of users of discrete demands \((|\mathcal{I}|)\) to be a scalable parameter to the problem. Our PTAS is polynomial in running time with respect to \(|\mathcal{I}|\) or generally \( n \).

**A. Rotational Invariance of OPF**

First, note that OPF is rotational invariant. That is, if the complex-valued parameters \((z_e)_{e \in \mathcal{E}} \) and \((s_k)_{k \in \mathcal{N}} \) are rotated by the same angle (say \( \phi \)) and the objective function \( f(s_0, s) \) is counter-rotated by \( \phi \) in \( s_0 \), then there is a bijection between the rotated OPF and the original unrotated OPF. Define the rotated objective function by \( f^\phi(s_0, s) \triangleq f(s_0 e^{-i\phi}, s) \).

Formally, rotated OPF is defined as follows:

\[
\text{(OPF}^\phi \text{)} \quad \min_{s_0, s, \ell, S} f^\phi(s_0, s),
\]

s.t. \( 2, 4, 9, 11 \).

\[
S_{i,j} = \sum_{k \in \mathcal{E}_j} s_k e^{i\phi} + \sum_{l:(i,l) \in \mathcal{E}} S_{l,j} + z_{i,j} e^{i\phi} \ell_{i,j}, \quad \forall (i,j) \in \mathcal{E}
\]

\[
v_j = v_i + |z_{i,j}|^2 \ell_{i,j} - 2\text{Re}(z_{i,j}^* e^{-i\phi} S_{i,j}), \quad \forall (i,j) \in \mathcal{E}
\]

Similarly, we define a rotated version of cOPF as cOPF^\phi.

\(^3\)It should be noted that another sufficient condition for exactness was given in [2], but we will not consider here.

\(^4\)The sufficient condition in [1] is stated in a slightly different way, because their problem formulation adopts an opposite flow orientation.
Theorem 2. There is a bijection between $\text{OPF}^\phi$ and OPF. Also, there is a bijection between $\text{COPF}^\phi$ and COPF.

Theorem 2 can be proved by showing that a feasible solution $F = (s_0, s, S, v, \ell)$ of $\text{OPF}^\phi$ can be mapped to a feasible solution $\tilde{F} = (\tilde{s}_0, \tilde{s}, S, v, \ell)$ of OPF, where $\tilde{s}_i,j = s_i,j e^{-i \phi}$, and vice versa. Similarly, it holds for $\text{COPF}^\phi$ and COPF.

Fig. 2: Angle $\phi$ is the minimum angle of rotation to rotate $(\pi_k)_{k \in I}$ into the first quadrant.

Therefore, in the rest of paper, it is more convenient to consider $\text{COPF}^\phi$ instead of COPF, where $\phi \triangleq \max \{ \max_{k \in I} \{ -\angle s_k \} , 0 \} \in [0, \pi]$. Namely, $\phi$ is the minimum angle in order to rotate all the discrete demands from the fourth quadrant to the first quadrant (see Fig. 2). Theorem 2 allows us to replace assumptions A0 and A4 by the following assumptions:

- A0': $f_0(-s_k^R \cos \phi - s_k^I \sin \phi)$ is non-decreasing in $-s_k^R, -s_k^I$.
- A4': $s_k \geq 0$ for all $k \in \mathcal{N}$. This is because all demand sets satisfying A4 are now in the first quadrant after the rotation by $\phi$.

Note that assumption A1 continues to hold for $\text{OPF}^\phi$, assuming the original OPF problem satisfies A3: $z^e e^{i \phi} \geq 0, \forall e \in \mathcal{E}$. This is because of A3, namely, $\text{Re}(z_k^e s_k) \geq 0, \forall k \in \mathcal{N}, e \in \mathcal{E}$, such that the phase angle difference between $s_x$ and $\pi_k$ is at most $\frac{\pi}{2}$. Note also that A1 and A4' already imply A3.

B. PTAS Algorithm

This section presents a PTAS for solving $\text{COPF}^\phi$. Together with Theorems 1 and 2, one can solve OPF by a PTAS. The basic steps of PTAS are illustrated in Fig. 3. After convex relaxation and rotation, we enumerate possible partial guesses for configuring the control variables of a small subset of discrete demands. For each guess, we solve the remaining subproblem by relaxing the other discrete control variables to be continuous control variables, and then rounding the continuous control variables to obtain a feasible solution. This algorithm can attain a parameterized approximation ratio by carefully adjusting the number of partial guesses and rounding.

A formal description of the PTAS algorithm (named PTAS-COPF) is presented as follows:

1) First, define a partial guess by $I_1, I_0 \subseteq \mathcal{I}$, such that $I_1 \cap I_0 = \emptyset$. For each guess, we set $x_k = 1, \forall k \in I_1$ and $x_k = 0, \forall k \in I_0$.

Fig. 3: Basic steps of PTAS for OPF.

2) Define a variant of $\text{COPF}^\phi$ with partially pre-configured and partially relaxed discrete control variables, denoted by $P1[I_0, I_1]$, as follows:

\[
\begin{align*}
(P1[I_0, I_1]) \quad & \min_{s_0, s, S, v, \ell} \quad f^\phi(s_0, s) \\
\text{s.t.} \quad & I_0 \cup I_1, \quad \sum_{k \in \mathcal{I}} \sum_{h,l \in \mathcal{P}} z^e_{h,l} s_k \leq 0, \forall e \in \mathcal{E}, \quad s_k = \pi_k x_k, \forall k \in \mathcal{I}, \\
& x_k = 1, \forall k \in I_1, \\
& x_k = 0, \forall k \in I_0, \\
& x_k \in [0,1], \forall k \in \mathcal{I} \setminus (I_0 \cup I_1)
\end{align*}
\]

Note that $P1[I_0, I_1]$ is a convex programming problem and is solvable in polynomial time. Then, obtain an optimal solution of $P1[I_0, I_1]$, denoted by $F^* = (s'_0, s', s', v', \ell')$. $F^*$ may not satisfy the discrete demand constraints (10) in $\text{COPF}^\phi$. Next, $F^*$ will be rounded to obtain a feasible solution to $\text{COPF}^\phi$.

3) Define $\mathcal{I}' \triangleq \mathcal{I} \setminus (I_0 \cup I_1)$, and $\overline{\mathcal{I}}_k \triangleq \{ f_k \} (0 \leq k \leq 5)$ for $k \in \mathcal{I}$. Define $P2[F', \mathcal{I}']$ as a sub-problem of selecting a subset of discrete control variables $(x_k)_{k \in \mathcal{I}'}$ for rounding, based on the respective objective values, as follows:

\[
\begin{align*}
(P2[F', \mathcal{I}']) \quad & \min_{x_k \in [0,1], k \in \mathcal{I}'} \quad \sum_{k \in \mathcal{I}'} \mathcal{J}_k (1 - x_k) \\
\text{s.t.} \quad & 0 \leq \sum_{k \in \mathcal{N}_j} \text{Re} \left( \sum_{(h,l) \in P_k \cap P_j} z^e_{h,l} s_k \right), \forall j \in \mathcal{V}^+ \\
& \leq \sum_{k \in \mathcal{N}_j} \text{Re} \left( \sum_{(h,l) \in P_k \cap P_j} z^e_{h,l} s_k \right), \forall j \in \mathcal{V}^+ \\
& \sum_{k \in \mathcal{N}_j} \text{Re}(s_k e^{i \phi}) \leq \sum_{k \in \mathcal{N}_j} \text{Re}(s'_k e^{i \phi}), \forall j \in \mathcal{V}^+ \\
& \sum_{k \in \mathcal{N}_j} \text{Im}(s_k e^{i \phi}) \leq \sum_{k \in \mathcal{N}_j} \text{Im}(s'_k e^{i \phi}), \forall j \in \mathcal{V}^+ \\
& s_k = \pi_k x_k, \forall k \in \mathcal{I}', \quad s_k = s'_k, \forall \mathcal{N} \setminus \mathcal{I}'
\end{align*}
\]

Note that $P2[F', \mathcal{I}']$ is a linear programming problem.
4) Suppose \((x_k')_{k \in I'}\) is an optimal solution of P2[\(F', I'\)]. Each \(x_k''\) is rounded to an integral solution such that
\[
\hat{x}_k = \begin{cases} 
\lfloor x_k' \rfloor, & \text{if } k \in I', \\
x_k', & \text{if } k \in I \setminus I'
\end{cases}
\] (30)

5) Then, obtain the corresponding \(\hat{s}_0, \hat{s}, \hat{S}, \hat{\ell}, \hat{v}\) by P3[\(\hat{x}, s'\)]:
\[
\begin{align*}
(P3[\hat{x}, s']) & \quad \min_{\hat{s}_0, \hat{s}, \hat{S}, \hat{\ell}, \hat{v}} \ f^\phi(\hat{s}_0, s) \\
\text{s.t.} & \quad \{1, 2, 6, 8, 11, 12, 19, 20\}, \ C2 \\
& \quad s_k = \hat{s}_k', \forall k \in F, \\
& \quad s_k = \hat{s}_k \hat{x}_k', \forall k \in I
\end{align*}
\] (31)
(32)

Note that P3[\(\hat{x}, s'\)] is a convex programming problem.

6) The output solution will be the one having the maximal objective value among all guesses.

The pseudo-codes of PTAS-COPF are given in Algorithm 1.

**Algorithm 1 PTAS-COPF**

**Input:** \(\epsilon, \nu, (\ell_j, \pi_j)_{j \in V^t}, (S, \ell, \ell, \nu, e)_{e \in E}, \{\bar{x}_k, \bar{s}_k\}_{k \in \Lambda}\)

**Output:** Solution \(\hat{F} = (\hat{s}_0, \hat{s}, \hat{S}, \hat{\ell}, \hat{v})\) to COPF\(^\phi\)

1: \(f_{\min} \leftarrow \infty\)
2: for each set \(I_0 \subseteq \mathcal{I}\) such that \(|I_0| \leq \frac{4m}{\epsilon}\) do
3: \(I_1 = \{k \in \mathcal{I} \setminus I_0 | \min_{k' \in I_0} \{\tilde{F}_{k'}\}\}\)
4: \(I' = \mathcal{I} \setminus (I_0 \cup I_1)\)
5: if \(P1[I_0, I_1]\) is feasible then
6: \(F' \leftarrow \text{Optimal solution of P1}[I_0, I_1]\)
7: \((x_k')_{k \in I'} \leftarrow \text{Optimal solution of P2}[F', I']\)
8: \((\hat{x}_k)_{k \in \Lambda} \leftarrow \{(x_k')_{k \in I'}(x_k')_{k \in \mathcal{I} \setminus I'}\}\)
9: \((\hat{s}_0, \hat{s}, \hat{S}, \hat{\ell}, \hat{v}) \leftarrow \text{Optimal solution of P3}[\hat{x}, s']\)
10: if \(f_{\min} > f^\phi(\hat{s}_0, \hat{s})\) then
11: \(\hat{F} \leftarrow (\hat{s}_0, \hat{s}, \hat{S}, \hat{\ell}, \hat{v}), \ f_{\min} \leftarrow f^\phi(\hat{s}_0, \hat{s})\)
end if
end if
end for
end for
end if
end for
end for
end for
return \(\hat{F}\)

**C. Proving Approximation Ratio**

In this section, the approximation ratio of PTAS will be derived as \((1 + \epsilon)\), if one sets the size of partial guesses of satisfiable discrete demands by \(|I_0| \leq \frac{4m}{\epsilon}\), where \(m\) is the number of nodes in distribution networks, and the corresponding \(I_0\) by
\[
I_0 = \{k \in \mathcal{I} \setminus I_0 | \tilde{F}_k > \min_{k' \in I_0} \{\tilde{F}_{k'}\}\}
\] (33)

Therefore, one can adjust the approximation ratio by limiting the size of \(I_0\) in partial guessing.

**Remarks:** To speed up PTAS-COPF, one can first compute the optimal objective value (denoted by \(\hat{f}\)) of P1 by taking \(I_0 = I_1 = \emptyset\), which naturally is a lower bound to that of COPF. PTAS-COPF will stop and return a solution, if the gap between the solution’s objective value and \(\hat{f}\) is sufficiently small. Hence, this may skip partial guessing, if \(\hat{f}\) is already closed to the solution of PTAS-COPF without partial guessing (which is often observed in the evaluation in Sec. VI).
By (36), (38) and non-negativity of $f_0, f_k$, it follows that

$$\sum_{k \in I} \mathcal{J}_k(1 - \hat{x}_k) \leq (1 + \epsilon) \sum_{k \in I} \mathcal{J}_k(1 - x'_k)$$

$$\leq \sum_{k \in I} \mathcal{J}_k(1 - x'_k) + \epsilon f^\phi(s'_0, s') \quad (39)$$

Finally, by (39) and Lemma 5 below, one obtains

$$f^\phi(s_0, \hat{s}) \leq (1 + \epsilon)f^\phi(s'_0, s') \leq (1 + \epsilon)f^\phi(s^*_0, s^*)$$

Hence, this completes the proof. \hfill \Box

**Lemma 4** (see [14]). Consider linear programming problem:

$$\begin{align*}
\min & \quad e^T \cdot x \\
\text{s.t.} & \quad Ax \leq b
\end{align*} \quad (40)$$

where $A$ is an $n \times r$ matrix. Then there exists an optimal solution $x^*$, such that at most $r$ components are fractional. Namely, $|\{i = 1, ..., n\} | x^*_i \in (0, 1)| \leq r$.

Proof of Lemma 4 follows from the properties of basic feasible solutions of linear programming problems.

**Lemma 5.** Let $F' = (s'_0, s', \bar{x}, S', \bar{v}, t')$ be a feasible solution of $P1[I_0, I_1]$. If $\hat{x}_k \in [0, 1], \forall k \in \mathcal{I}$ satisfies the following:

$$\sum_{k \in \mathcal{I}} \mathcal{J}_k(1 - \hat{x}_k) \leq \sum_{k \in \mathcal{I}} \mathcal{J}_k(1 - x'_k) + \epsilon f^\phi(s'_0, s'), \quad (42)$$

$$0 \leq \sum_{k \in \mathcal{N}} \text{Re} \left( \sum_{(h, l) \in \mathcal{P}_k \cap \mathcal{P}_j} z_{h,l}^* \hat{s}_k \right)$$

$$\leq \sum_{k \in \mathcal{N}} \text{Re} \left( \sum_{(h, l) \in \mathcal{P}_k \cap \mathcal{P}_j} z_{h,l}^* s'_k \right), \quad \forall j \in \mathcal{V}^+ \quad (43)$$

$$\sum_{k \in \mathcal{N}_I} \text{Re}(\hat{s}_k e^{i\phi}) \leq \sum_{k \in \mathcal{N}_I} \text{Re}(s'_k e^{i\phi}), \quad \forall j \in \mathcal{V}^+ \quad (44)$$

$$\sum_{k \in \mathcal{N}_I} \text{Im}(\hat{s}_k e^{i\phi}) \leq \sum_{k \in \mathcal{N}_I} \text{Im}(s'_k e^{i\phi}), \quad \forall j \in \mathcal{V}^+ \quad (45)$$

$$\hat{s}_k = \bar{s}_k \hat{x}_k, \forall k \in \mathcal{I}, \quad \hat{s}_k = s'_k, \forall k \in \mathcal{F} \quad (47)$$

$$\hat{x}_k = x'_k, \forall k \in \mathcal{I} \setminus \mathcal{I}' \quad (48)$$

Then, with assumptions $A0', A1, A2, A3, A4'$, there exists a feasible solution $\hat{F} = (\hat{s}_0, \hat{s}, \hat{\bar{x}}, \hat{S}, \hat{\bar{v}}, \hat{\ell}, \hat{\ell}')$ of $P3[\hat{x}, s']$, such that $f^\phi(\hat{s}_0, \hat{s}) \leq (1 + \epsilon)f^\phi(s'_0, s')$.

**Proof.** First, we aim to construct a feasible solution $\hat{F} = (\hat{s}_0, \hat{s}, \hat{\bar{x}}, \hat{S}, \hat{\bar{v}}, \hat{\ell})$ for $P3[\hat{x}, s']$. Reformulate Cons. (20) by recursively substituting $v'_j$, from the root to node $j$, and then recursively substituting $\hat{S}_{h,l}$, for each $(h, l)$ on the path from $j$ to the root:

$$v_j = v_0 - 2 \sum_{(h, l) \in \mathcal{P}_j} \text{Re}(z_{h,l}^* \phi S_{h,l}) + \sum_{(h, l) \in \mathcal{P}_j} |z_{h,l}|^2 \ell_{h,l}$$

$$= v_0 - 2 \sum_{(h, l) \in \mathcal{P}_j} \text{Re}(z_{h,l}^* \phi (\sum_{k \in \mathcal{N}_I} s_k e^{i\phi} + \sum_{e \in \mathcal{E}_I} z_e e^{i\phi} \ell_e))$$

$$+ \sum_{(h, l) \in \mathcal{P}_j} |z_{h,l}|^2 \ell_{h,l}$$

$$= v_0 - 2 \sum_{(h, l) \in \mathcal{P}_j} \text{Re} \left( \sum_{(h, l) \in \mathcal{P}_j} z_{h,l}^* \hat{s}_k \right) - 2 \sum_{(h, l) \in \mathcal{P}_j} |z_{h,l}|^2 \ell_{h,l}$$

$$- 2 \sum_{(h, l) \in \mathcal{P}_j} \text{Re} \left( z_{h,l}^* \sum_{e \in \mathcal{E}_I} z_e \ell_e \right) + \sum_{(h, l) \in \mathcal{P}_j} |z_{h,l}|^2 \ell_{h,l}$$

where the last equality follows from exchanging the summation operators, and $z_{h,l}^* \ell_e = |\ell_e|^2$. Hence, one obtains

$$v_j = v_0 - 2 \sum_{(h, l) \in \mathcal{P}_j} \text{Re} \left( \sum_{(h, l) \in \mathcal{P}_j} z_{h,l}^* \hat{s}_k \right) - 2 \sum_{(h, l) \in \mathcal{P}_j} \text{Re} \left( \sum_{e \in \mathcal{E}_I} z_e \ell_e \right) + \sum_{(h, l) \in \mathcal{P}_j} |z_{h,l}|^2 \ell_{h,l}$$

(49)

Set a feasible solution $\hat{F}$ as follows:

$$\hat{\ell}_{i,j} = \ell_{i,j}, \quad (50)$$

$$\hat{S}_i,j = \sum_{k \in \mathcal{N}} \hat{s}_k e^{i\phi} + \sum_{e \in \mathcal{E}_i} z_e e^{i\phi} \ell_e, \quad (51)$$

$$\hat{v}_j = v_0 - 2 \sum_{(h, l) \in \mathcal{P}_j} \text{Re} \left( \sum_{(h, l) \in \mathcal{P}_j} z_{h,l}^* \hat{s}_k \right) + \hat{L}_{i,j}, \quad (52)$$

where

$$\hat{L}_{i,j} = - 2 \sum_{(h, l) \in \mathcal{P}_j} \text{Re} \left( \sum_{e \in \mathcal{E}_i} z_e \ell_e \right)$$

Rewriting $S'_i,j$ recursively, one obtains

$$S'_i,j = \sum_{k \in \mathcal{N}_I} s'_k e^{i\phi} + \sum_{e \in \mathcal{E}_i} z_e e^{i\phi} \ell'_e$$

(53)

Because of (45) and (46), it follows that

$$\hat{S}_i,j \leq \sum_{k \in \mathcal{N}_I} s'_k e^{i\phi} + \sum_{e \in \mathcal{E}_i} z_e e^{i\phi} \ell_e = S'_i,j$$

(54)

The feasibility of $\hat{S}_i,j$ can be shown as follows: By $A1, A4'$, it follows that $\hat{S}_i,j \geq 0$. By (54), one obtains

$$|\hat{S}_i,j| \leq |S'_i,j| \leq S_{i,j} \quad (55)$$

The feasibility of $\hat{v}_j$ can be shown as follows: By (47), one obtains

$$\hat{v}_j \geq v_0 - 2 \sum_{k \in \mathcal{N}} \text{Re} \left( \sum_{(h, l) \in \mathcal{P}_j} z_{h,l}^* s'_k \right) + \hat{L}_{i,j} = v'_j \geq v_j$$

(56)

Note that we need here that all users are consumers.
where the last inequality follows from the feasibility of \( v_j' \). Since \( \hat{L}_{i,j} \leq 0 \) (by A1), A2 and (43), one obtains

\[
\hat{v}_j \leq v_0 - 2 \sum_{k \in N} \text{Re} \left( \sum_{(h,j) \in P_k \cap P_j} z_{h,i}^* \hat{s}_k \right) \leq v_0 - v_j.
\]

The feasibility of \( \hat{\ell}_{i,j} \) can be shown as follows: By (55), (56), one obtains \( \hat{\ell}_{i,j} = \ell_{i,j}' = \frac{|s_{i,j}^*|}{v_i} \geq |s_{i,j}| v_i^{-}, \) hence \( \hat{\ell}_{i,j} \) satisfies Cons. (12).

Finally, one obtains

\[
\hat{s}_0 = -\hat{s}_{0,1} \geq -s_{0,1}' = s_0
\]

Because A1, A4', \( e^{-i\phi} \hat{s}_0 \) and \( e^{-i\phi} s_0' \) are in the third quadrant, which is the opposite to \( s_k e^{i\phi} \) and \( z_k e^{i\phi} \) (by (26),(4)). Hence, because of (42), (48) and A0' (\( f_0(\text{Re}(\cdot)) \) is non-increasing for parameters in the third quadrant), it follows that

\[
\sum_{k \in \mathcal{I}} T_k(1 - \hat{x}_k) + \sum_{k \in \mathcal{P}} f_k(v_k) - f_0(\text{Re}(e^{-i\phi} \hat{s}_0)) \leq \sum_{k \in \mathcal{I}} T_k(1 - x_k') + \sum_{k \in \mathcal{P}} f_k(v_k') - f_0(\text{Re}(e^{-i\phi} s_0'))
\]

\[
+ \epsilon f^{\phi}(\hat{s}_0, s')
\]

\[
\Rightarrow f(\hat{s}_0, \hat{s}) \leq (1 + \epsilon) f^{\phi}(s_0', s')
\]

Hence, it completes the proof. \( \square \)

V. HARDNESS OF OPF WITH DISCRETE DEMANDS

While exact convex relaxation of OPF applies to the setting with discrete demands, the efficiency of solving OPF with discrete demands is substantially more challenging than that with only continuous demands. In prior work [9], [11], [12], [15], we show some fundamental hardness results for OPF with discrete demands. Although the results in [9], [11], [12], [15] are proven by a slightly different model of maximizing an objective function, with minor modifications to the proofs these results can also be applied to our model of minimizing an objective function.

This paper provides a PTAS for solving OPF with discrete demands. A better alternative to PTAS is a fully polynomial-time approximation scheme (FPTAS), which requires the running time to be polynomial in both input size \( n \) and \( 1/\epsilon \).

Theorem 6 (see [9]). Unless P=NP, there exists no FPTAS for OPF with discrete demands, even for a single-link distribution network \( |E| = 1 \) with \( z_c = 0 \).

By Theorems 6 a PTAS is among the best achievable efficient algorithm that can be attained for OPF with discrete demands, because FPTAS is not possible.

Next, we show that assumption A3 is necessary for PTAS.

Theorem 7 (see [12]). Unless P=NP, there does not exist an \( \alpha \)-approximation for OPF by a polynomial-time algorithm in \( n \), for any \( \alpha \) that has polynomial length in \( n \), if \( \text{Re}(z_k^* s_k) \) is allowed to be arbitrary for any \( k \in \mathcal{I} \), even for a single-link distribution network \( |E| = 1 \).

In Theorems 7 \( \alpha \) can be as large as \( 2^{P(n)} \), where \( P(n) \) is an arbitrary polynomial in \( n \). Thereom 7 shows that A3 is necessary for any approximation algorithm for OPF with discrete demands to have a practical approximation ratio.

Finally, we show that assumption A4 is necessary for PTAS. Let \( \theta \) be the maximum angle difference between any pair of demands, \( \theta \triangleq \max_{k,k' \in N} |\angle s_k - \angle s_{k'}| \). Theorem 8 shows that the approximability of OPF with discrete demands depends on \( \theta \). Hence, a PTAS requires A4.

Theorem 8 (see [11], [15]). Unless P=NP, for any \( \delta > 0 \), there is no \( \alpha \)-approximation for OPF by a polynomial-time algorithm in \( n \), for \( \theta \in \left[ \frac{\pi}{2} + \delta, \pi \right] \), where \( \alpha, \delta \) have polynomial length in \( n \) and \( \delta \) is exponentially small in \( n \), even for a single-link distribution network \( |E| = 1 \) with \( z_c = 0 \).

VI. EVALUATION STUDIES

In this section, the performance of PTAS is evaluated in simulations in terms of optimality and running time.

A. Simulation Settings

The evaluation was performed on the Bus 4 distribution system of the Roy Billinton Test System (RBTS) [1], [16], which comprises of 13 nodes, in which the generation source is attached to the sub-station node 0, the base power capacity of this network is 8MVA, and the base voltage is 11KV. The single-line diagram and line data of the RBTS 13-node network are presented in Fig. 4 and Table I respectively.

![Fig. 4: Single-line diagram of the RBTS 13-node electric network.](image-url)

| Bus | R (p.u.) | X (p.u.) | Capacity (p.u.) |
|-----|---------|---------|-----------------|
| (0,1) | 0.011636363636364 | 0.034380165289256 | 1 |
| (1,2) | 0.02644280991736 | 0.158677685950413 | 0.125 |
| (1,3) | 0.014545145454545 | 0.043636363636364 | 0.7625 |
| (3,4) | 0.02644280991736 | 0.158677685950413 | 0.25 |
| (3,5) | 0.017454545454546 | 0.042314049586777 | 0.25 |
| (5,6) | 0.02644280991736 | 0.158677685950413 | 0.25 |
| (5,7) | 0.011636363636364 | 0.0370247938843 | 0.75 |
| (7,8) | 0.02644280991736 | 0.17190826446281 | 0.25 |
| (7,9) | 0.031755571906083 | 0.18512396942141 | 0.25 |
| (7,10) | 0.014545145454545 | 0.039669421487603 | 0.75 |
| (10,11) | 0.013223140495868 | 0.161322314049587 | 0.25 |
| (10,12) | 0.029040909099090 | 0.18512396942141 | 0.25 |

TABLE I: Settings of line impedance and maximum capacity of the RBTS 13-node electric network.
The evaluation was also performed on the IEEE 123-node network, in which the generation source is attached to node 150, and the base capacity and voltage are 5MVA and 4.16KV, respectively.

The IEEE 123-node network is unbalanced three-phase networks with several devices that are not modeled in OPF. As in [2], one can modify the IEEE network by the following:

- The three phases are assumed to be decoupled into three identical single phase networks.
- Closed circuit switches are modeled as shorted lines and ignore open circuit switches.
- Transformers are modeled as lines with appropriate impedances.

All power demands are discrete and are randomly positioned at the nodes in $V^+$ uniformly. Several case studies are considered by different user types and the correlations between user demands and $f_k$:

- **User Types**:
  1) *Residential (R):* The users have small power demands ranging from 500VA to 5KVA.
  2) *Industrial (I):* The users have big demands ranging from 300KVA to 1MVA with non-negative reactive power.
  3) *Mixed (M):* The users consist of a mix of industrial and residential users, with less than 20% industrial users.

- **Cost-Demand Correlation**:
  1) *Correlated Setting (C):* The cost objective of each user is a function of his power demand as follows:

$$f_k(Re(s_k)) = (\max(|s_k| - \frac{Re(s_k)}{Re(s_k)}|s_k|) ^ 2$$

  2) *Uncorrelated Setting (U):* The cost objective of each user is independent of his/her power demand and is generated randomly from $[0, \max(|s_k|)]$. Here $\max(|s_k|)$ depends on the user type. If user $k$ is an industrial user then $\max(|s_k|) = 1MVA$, otherwise $\max(|s_k|) = 5KVA$. More precisely, given a random $r \in [\max(|s_k|)]$, and

$$f_k(Re(s_k)) = r - \frac{Re(s_k)}{Re(s_k)}$$

The case studies will be represented by the acronyms. For example, the case study named “CM” stands for the one with mixed users and correlated cost-demand setting.

In order to evaluate the performance of our algorithm PTAS, Gurobi numerical solver is used as a benchmark to obtain numerical solutions for OPF. Note that there is no guarantee that Gurobi will return an optimal solution, nor it will terminate in a reasonable time. Hence, we need to set a timeout (as 200 sec in the evaluation). Whenever Gurobi exceeds the timeout, the current best solution will be considered.

More information of the simulation settings, including parameters of RBTS and IEEE networks, can be found in [17].

The simulations were evaluated using Intel i7-3770 CPU 3.40GHz processor with 32GB of RAM. The algorithms were implemented using Python 2.7 programming language with Scipy library for scientific computation.

### Evaluation Results

1) **Optimality**: Fig 5a and 5b presents the objective values obtained by PTAS, Gurobi numerical solver, and the lower bounds to the true optimal values by fractional solutions with relaxed discrete demands (i.e., setting all $x_k \in [0, 1]$) respectively using the RBTS 13-node network (resp., IEEE 123-node network) for up to 3500 users. Each run was evaluated with over 40 random instances. PTAS will terminate, when its objective value is close to the lower bound. The objective values of PTAS are often close to the true optimal values. This is because the number of fractional components in the relaxed problem P1 is often small. Fig. 6 shows the ratio of fractional components over 4$n$m is close to 10%, which stays small when the number of users increases.

The empirical approximation ratios of PTAS for the two networks are plotted in Fig. 7 against the number of users. We observe that the empirical approximation ratio is close to 1.2 in most cases. This occurs when the optimal solution satisfies all demands, whereas PTAS (without partial guessing) rounds some fractional demands to zero, which incurs a high cost. There are few instances with a larger empirical approximation ratio, but increasing partial guessing is able to resolve this issue, which is still within polynomial running time. Although a numerical solver (e.g., Gurobi) may be able to obtain comparable solutions as PTAS, it cannot provide any theoretical guarantee on approximation ratio, unlike PTAS.

2) **Running Time**: The computation time of PTAS is compared against that of Gurobi numerical solver. Computation time is significantly important when implemented in a controller in practice, and this will have implications to the overall resilience of power grid. The running time is plotted in Fig. 8 under different case studies for IEEE 123-node network. Although the current implementation of PTAS is not fully optimized, its running time is still substantially better than that of Gurobi, and is observed to scale linearly as the number of users. On the other hand, the running time of Gurobi is much higher in many cases, which does not provide any guarantee on the termination of execution, if timeout is not set. Many instances experienced timeouts, especially for the case study UR. The actual running time of Gurobi may substantially increase if the timeout value is further increased.

### Conclusion

This paper presents a polynomial-time approximation algorithm to solve the convex relaxation of OPF with combinatoric constraints, which has a provably small parameterized approximation ratio (also known as PTAS algorithm) that can combine with the sufficient conditions on the exactness of convex relaxation to solve OPF in general. This paper also discusses fundamental hardness results of OPF to show that our PTAS is among the best achievable in theory. Further simulations show that our algorithm can produce close-to-optimal solutions in practice.

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Fig. 5: The average objective values of PTAS, Gurobi numerical solver, and fractional solutions with relaxed discrete demands (as the lower bounds to the true optimal values), against the number of users with 95% confidence intervals.

Fig. 6: The ratio of fractional components after solving P1. The ‘+’ points represent the outliers.

Fig. 7: The empirical approximation ratios of PTAS for different case studies, against the number of users.

Fig. 8: The median of running times of PTAS and Gurobi numerical solver for different case studies in IEEE 123-node network.

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A. Tree Formulation for OPF

For convenience, we give another formulation of OPF, based on the recursive “unfolding” of Eqns. (3)-(5). We start with the following simple lemma.

**Lemma 9.** Let $F \triangleq (s_0, s, v, \ell, S)$ be a vector satisfying (3)-(5). Then

$$S_{i,j} = \sum_{k \in N_j} s_k + \sum_{e \in \mathcal{E}_j \cup \{(i,j)\}} z_e \ell_e, \quad \forall (i,j) \in \mathcal{E}. \quad (60)$$

$$v_j = v_0 - 2 \sum_{(h,t) \in \mathcal{P}_j} \mathrm{Re} \left( \sum_{(i,j) \in \mathcal{E}_t \cap \mathcal{P}_j} z_{h,t}^* s_k \right) - 2 \sum_{(h,t) \in \mathcal{P}_j} \left| z_{h,t} \right|^2 \ell_{h,t} \quad (61)$$

where the last statement follows from exchanging the summation operators, and using $z_e^* z_e = |z_e|^2$.

It follows from Lemma 9 that we may equivalently formulate OPF as

$$(\text{TOPF}) \quad \min_{s_0, s, v, \ell, S} \quad f(s_0, s), \quad \text{subject to} \quad (3), (4), (5), (60) - (61).$$

We shall refer to this as the tree formulation of OPF.

**Lemma 10.** Formulations OPF and TOPF are equivalent.

**Proof.** Given a feasible solution $F \triangleq (s_0, s, v, \ell, S)$ of OPF, Lemma 9 shows that $F$ is also feasible for TOPF. Conversely, let $F' \triangleq (s_0, s, v, \ell, S')$ be a feasible solution of TOPF, we show by that $F$ satisfies (3) and (5). Consider first (3). Note by (60) that, for $(i,j) \in \mathcal{E}$,

$$\sum_{k \in N_j} s_k + \sum_{e \in \mathcal{E}_j \cup \{(i,j)\}} z_e \ell_e = \sum_{k \in N_j} s_k + \sum_{e \in \mathcal{E}_j \cup \{(i,j)\}} z_e \ell_e,$$

where the last equality follows from (60).

Consider next (5). Then (61) implies that, for $(i,j) \in \mathcal{E}$,

$$v_j - v_i = 2 \sum_{(h,t) \in \mathcal{P}_j} \mathrm{Re} \left( \sum_{(i,j) \in \mathcal{E}_t \cap \mathcal{P}_j} z_{h,t}^* s_k \right) + 2 \sum_{(h,t) \in \mathcal{P}_j} \left| z_{h,t} \right|^2 \ell_{h,t} \quad (61)$$

where the last equality follows from (60). □

B. Proof of Theorem 7

In proving the sufficient condition for exactness of the SOCP relaxation of OPF, we will make use of the following lemma.

**Lemma 11.** Let $F \triangleq (s_0, s, v, \ell, S)$ and $F' \triangleq (s'_0, s', v', \ell', S')$ be two vectors satisfying (3) (or equivalently (60)), such that $s = s'$ and $\ell \leq \ell'$ (component wise). Then under assumptions A1 and C2, $S_{i,j} \leq S'_{i,j}$ and $|S_{i,j}| \leq |S'_{i,j}|$ for all $(i,j) \in \mathcal{E}$, and $v_j \geq v'_j$ for all $j \in \mathcal{E}^+$. □
where the inequality follows by assumption A1. This implies that \( S_{i,j} \leq S'_{i,j} \). Furthermore,

\[
\Delta|S_{i,j}|^2 = |S_{i,j}|^2 - |S'_{i,j}|^2
\]

(64)

\[
= (S_{i,j}^R)^2 - (S'_{i,j})^2 + (S_{i,j}^S)^2 - (S'_{i,j}^S)^2
\]

(65)

\[
= \Delta S_{i,j}^R S_{i,j}^R + \Delta S_{i,j}^S S_{i,j}^S + \Delta S_{i,j}^R S'_{i,j}^S + \Delta S_{i,j}^S S'_{i,j}^R
\]

(66)

\[
\sum_{e \in E_j \cup \{(i,j)\}} z_e^R \Delta \ell_e (2L_{i,j}^R + L_{i,j}^I)
\]

\[
+ \sum_{e \in E_j \cup \{(i,j)\}} z_e^S \Delta \ell_e (2S_{i,j}^R + 2S_{i,j}^I + L_{i,j}^R)
\]

(67)

\[
= \sum_{e \in E_j \cup \{(i,j)\}} z_e^R \Delta \ell_e (2L_{i,j}^R + L_{i,j}^I)
\]

\[
+ \sum_{e \in E_j \cup \{(i,j)\}} z_e^S \Delta \ell_e (2S_{i,j}^R + 2S_{i,j}^I + L_{i,j}^R)
\]

(68)

where the inequality follows by A1, C2 and \( \Delta \ell_e \leq 0 \). Therefore \( |S_{i,j}| \leq |S'_{i,j}| \). Finally, using (61), we get by A1 that

\[
v_j - v'_j = - (2 \sum_{(h,t) \in P_j} \sum_{e \in E_i} \text{Re} (z_{h,t}^e s_k) - \sum_{e \in E_i} |z_{h,t}^e|^2 \Delta \ell_{h,t}^e) \geq 0.
\]

(69)

Corollary 12. Let \( F \triangleq (s_0, s, v, \ell, S) \) be a vector satisfying (3), (4) and (5) (or equivalently (60), (4) and (61)), and \( |S_{i,j}| \leq \tilde{S}_{i,j} \) for all \( (i, j) \in \mathcal{E} \). Then under assumptions A1, A2 and C2, \( F \) also satisfies \( v_j \leq \tilde{v}_j \), for all \( j \in V^p \), and \( |S_{i,j} - z_{i,j}\ell_{i,j}| \leq \tilde{S}_{i,j} \), for all \( (i, j) \in \mathcal{E} \).

Proof. The first claim is immediate from (61) and assumptions A1, A2 and C2 as

\[
v_j = v_0 - 2 \sum_{(h,t) \in P_j} \sum_{k \in N_i} \text{Re} (z_{h,t}^e s_k) - \sum_{(h,t) \in P_j} \sum_{e \in E_i} |z_{h,t}^e|^2 \Delta \ell_{h,t}^e.
\]

(70)

\[
= \sum_{e \in E_j \cup \{(i,j)\}} z_e^R \Delta \ell_e (2L_{i,j}^R + L_{i,j}^I)
\]

\[
+ \sum_{e \in E_j \cup \{(i,j)\}} z_e^S \Delta \ell_e (2S_{i,j}^R + 2S_{i,j}^I + L_{i,j}^R)
\]

(71)

The second claim follows from Lemma 11 as \( F' \triangleq (s_0, s, v, \ell', S') \) with

\[
\ell'_{i,j} \triangleq \begin{cases} 
0 & \text{if } (t,h) = (i,j) \\
\ell_{t,h} & \text{otherwise},
\end{cases}
\]

satisfies \( S'_{i,j} = S_{i,j} - z_{i,j}\ell_{i,j} \).

\[\square\]

Theorem 1. Let \( F'' = (s''_0, s', v', \ell'', S'') \) be a feasible solution of \( \text{COPF}[s'] \). Under assumptions A0, A1, A2, and C2, there is a feasible solution \( F'' = (s''_0, s', v', \ell'', S'') \) of \( \text{COPF}[s'] \) that satisfies \( \ell''_{i,j} = \frac{|S'_{i,j}|^2}{v_j} \) for all \( (i,j) \in \mathcal{E} \), and \( f(F'') \leq f(F'') \). Given \( F'' \), such a solution \( F'' \) can be found in polynomial time.

\[\square\]

Proof: The proof follows essentially the same lines as in [1-4]. We consider the following convex program

\[
(\text{COPF}[s']) \min \sum_{e \in \mathcal{E}} \ell_e,
\]

subject to (3) - (5), (11), (12)

(72)

\[\ell'_{i,j} \leq \ell''_{i,j}, \quad \forall (i,j) \in \mathcal{E} \]

(73)

\[s_k = s'_k, \quad \forall k \in N'. \]

Clearly, \( \text{COPF}[s'] \) is feasible as \( F'' \) satisfies all its constraints. Hence, it has an optimal solution \( F'' = (s''_0, s', v', \ell', S') \), which we claim to satisfy the statement of the theorem.

First, we observe that \( F' \) is an optimal solution for \( \text{COPF}[s'] \) (minimizing \( \sum_{e \in \mathcal{E}} \ell_e \) among all such solutions). Indeed, Ineq. (70) implies that

\[
\ell'_{i,j} \leq \ell''_{i,j} \leq \tilde{e}_e \quad \forall e \in \mathcal{E}.
\]

(74)

It follows by Lemma 11 and the feasibility of \( F'' \) for \( \text{COPF}[s'] \) that, for all \( (i, j) \in \mathcal{E} \),

\[
S'_{i,j} \leq S''_{i,j} \quad \text{and} \quad |S'_{i,j}| \leq |S''_{i,j}| \leq \tilde{S}_{i,j}.
\]

(75)

In particular, for \( (i, j) = (0,1) \), we obtain

\[
-s''_0 = -s''_0 = -s''_0 = -s''_0 = -s''_0
\]

(76)

implying by A0 that \( f(0,-s''_0) \leq f(0,-s''_0) \) and hence by (71),

\[
f(s_0, s') \leq f(s_0, s').
\]

(77)

Note also that, since \( F'' \) satisfies (3)-5, we have by Corollary 12 that \( |S'_{i,j} - z_{i,j}\ell_{i,j}| \leq \tilde{S}_{i,j} \), for all \( (i, j) \in \mathcal{E} \). We conclude the feasibility of \( F'' \) for \( \text{COPF}[s'] \).

Next, suppose, for the sake of contradiction, that there exists an edge \( (h,t) \) such that \( \ell'_{h,t} > |S'_{h,t}|^2 \). We construct a feasible solution \( \hat{F} = (s_0, s', \hat{v}, \hat{\ell}, \hat{S}) \) for \( \text{COPF}[s'] \) such that \( \sum_{e \in \mathcal{E}} \ell'_{e} < \sum_{e \in \mathcal{E}} \ell''_{e} \), leading to a contradiction to the optimality of \( F'' \) for \( \text{COPF}[s'] \).

To obtain \( \hat{F} = (s_0, s', \hat{v}, \hat{\ell}, \hat{S}) \) from \( F'' \), we set \( \hat{\ell}_{i,j} = \frac{|S'_{i,j}|^2}{v_j} \) and obtain \( \hat{S} \) by substituting \( \ell \leftarrow \hat{\ell} \) in Eqns. (60) and (61). To complete the proof, we show the feasibility of \( \hat{F} \).

By the way we constructed \( \bar{F} \), all equality constraints of \( \text{COPF}[s'] \) are satisfied (via Lemma 10), and by the feasibility of \( F'' \) for \( \text{COPF}[s'] \) (in particular, Ineq. (12)), we also have

\[
\hat{\ell}_{i,j} = \frac{|S'_{i,j}|^2}{v_j} \leq \ell''_{i,j} \leq \ell''_{i,j} \quad \forall (i, j) \in \mathcal{E}.
\]

(78)

It follows by Lemma 11 and the feasibility of \( F'' \) that

\[
\hat{S}_{i,j} \leq S''_{i,j} \quad \text{and} \quad |\hat{S}_{i,j}| \leq |S''_{i,j}| \leq \tilde{S}_{i,j}, \quad \forall (i,j) \in \mathcal{E},
\]

\[
\hat{v}_j \geq v_j \geq v_j \quad \forall j \in V^p.
\]

(79)

Note also that, since \( F'' \) satisfies (3)-5, we have by Corollary 12 that \( \hat{v}_j \geq v_j \) for all \( j \in V^p \). Moreover, by Ineqs. (70) and (77), \( \hat{\ell}_{i,j} = \frac{|S'_{i,j}|^2}{v_j} \geq \frac{|S_{i,j}|^2}{v_j} \), hence, \( F'' \) is feasible for \( \text{COPF}[s'] \).

8Typical convex programming solvers return a solution that is feasible within an absolute error \( \epsilon > 0 \), where the running time depends on \( \frac{1}{\epsilon} \). For simplicity, we assume that the convex program can be solved exactly.
Finally by the first inequality in (75) and the fact that $\ell'_{h,t} > \frac{|S_{h,t}|^2}{v_h} = \hat{\ell}_{h,t}$, we have $\sum_{e \in E} \tilde{\ell}_e < \sum_{e \in E} \ell'_e$, contradicting the optimality of $F'$ for cOPF'[s']. □