SOME APPLICATIONS OF THE ADM FORMALISM*

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The ADM Formalism is discussed in the context of 2 + 1-dimensional gravity, uniting two areas of relativity theory in which Stanley Deser has been particularly active. For spacetimes with topology $\mathbb{R} \times T^2$ the partially reduced and fully reduced ADM formalism are related and quantized, and the role of "large diffeomorphisms" (the modular group) in the quantum theory is illustrated.

1. Introduction

Over forty years ago Arnowitt, Deser and Misner (ADM) studied the 3+1-decomposition of general relativity, its initial value problem, the dynamical structure of the field equations and calculated the Hamiltonian. This extraordinary piece of work has become a fundamental ingredient of modern relativity theory. It is now regularly taught as an integral part of relativity courses, and usually occupies at least a chapter in relativity textbooks. The problem had actually been considered previously by Dirac who applied his theory of constrained systems to the gravitational field. But Dirac’s treatment was incomplete and in a particular gauge.

In this Section I briefly summarise the ADM results, and in Section 2 discuss the main differences between the 3 and 4 dimensional theories. In Section 3 the second-order, partially reduced, ADM formalism, for spacetimes of topology $\mathbb{R} \times T^2$ is reviewed, and I show how in principle the system can be quantized. In Section 4 the first-order fully reduced holonomy approach is presented. In Section 5 the two approaches are related, both classically and quantum mechanically, using the action of the modular group, or "large diffeomorphisms" - those that remain after ADM reduction.

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The $3 + 1$–decomposition of the Einstein–Hilbert action calculated by ADM is

\[ I_{\text{Ein}} = \int \sqrt{-g} \left( \frac{\partial I}{\partial \dot{g}_{ij}} \frac{\partial I}{\partial g_{ij}} \right) \]

where spacetime is of the form $I \times \Sigma$, and time runs along $I$. In (1) the metric has been decomposed as

\[ ds^2 = N^2 dt^2 - g_{ij} (dx^i + N^i dt)(dx^j + N^j dt) \quad i, j = 1, 2, 3 \]

and $\pi^{ij} = \sqrt{g} (K^{ij} - g^{ij} K)$, where $K^{ij}$ is the extrinsic curvature of the surface $\Sigma$ labeled by $t = \text{const}$. In (2) the lapse $N^i$ and shift $N^j$ functions are related to the non-dynamical components of $g_{ij}$ and their variation in (1) leads to the supermomentum and super-Hamiltonian constraints on $g_{ij}$ and $\pi^{kl}$.

\[ \mathcal{H}_i = -2 \nabla_j \pi^{ji} = 0, \quad \mathcal{H} = \frac{1}{\sqrt{g}} g_{ij} g_{kl} \left( \pi^{ik} \pi^{jl} - \frac{1}{2} \pi^{ij} \pi^{kl} \right) - \sqrt{g} R = 0 \]

where $\nabla_j$ is the covariant derivative for the connection compatible with $g_{ij}$, and indices are now raised and lowered with $g_{ij}$. The $\mathcal{H}_i, \mathcal{H}$ in (3) are non-polynomial in $g_{ij}$ and $\pi^{kl}$ and involve $(g_{ij})^{-1}$. They are directly proportional to the components $G^\mu^\nu$ of the Einstein tensor defined by

\[ G^\mu^\nu = \frac{\delta I_{\text{Ein}}}{\delta g_{\mu^\nu}} = R^\mu^\nu - \frac{1}{2} R g^\mu^\nu \]

so finding a solution to (3) would correspond to finding a general solution of Einstein’s equations (the other components of (4) are zero by the Bianchi identities $G^\mu^\nu|_\nu = 0$).

The constraints (3) generate, through the Poisson brackets obtained from (1)

\[ \{g_{ij}(x), \pi^{kl}(y)\} = \frac{1}{2} (\delta^k_j \delta^l_i + \delta^k_i \delta^l_j) \delta^3(x - y) \]

three-dimensional diffeomorphisms in $\Sigma$, and the time development of the variables $g_{ij}(x), \pi^{kl}(y)$.

One can ask what effect the constraints (3) would have when applied on wave functions $\psi(y)$. If the brackets (5) are represented by letting the momenta $\pi^{ij}$ act by differentiation

\[ \pi^{ij}(x) \sim \frac{\delta}{\delta g_{ij}(x)} \]

This is standard ADM notation: $g_{ij}$ and $R$ refer to the induced metric and scalar curvature of a time slice, while the spacetime metric and curvature are denoted $^{(4)}g_{\mu^\nu}$ and $^{(4)}R$.\]
and the metric components $g_{ij}(x)$ by multiplication, the supermomentum constraint $\mathcal{H}_i \psi(g) = 0$ is easy to interpret, since

$$\int_\Sigma d^3x (N^i \mathcal{H}_i) \Psi(g) = \frac{\delta \Psi(g)}{\delta g_{ij}(x)} (\nabla_i N_j + \nabla_j N_i)$$

(7)

and implies that one should identify wave functions of metrics $g_{ij}$ and $\tilde{g}_{ij}$ when they differ as

$$\tilde{g}_{ij} = g_{ij} + \nabla_i N_j + \nabla_j N_i.$$  

(8)

But (8) is a Lie derivative, or coordinate transformation, in the spatial surface $\Sigma$, so the supermomentum constraint reflects the freedom to choose the 3 spatial coordinates on $\Sigma$. The space of metrics (8) with $\tilde{g}_{ij}$ identified with $g_{ij}$ was named superspace in 1963 by Wheeler.

The super-Hamiltonian constraint $\mathcal{H} \psi(g) = 0$ (also known as the Wheeler DeWitt equation) is much harder to interpret, and alone does not generate the dynamics, or time reparametrization invariance, of wave functions $\psi(g)$. Instead one needs to use the full Hamiltonian, namely the combination

$$\int_\Sigma d^3x (N^i \mathcal{H}_i + N \mathcal{H}).$$

(9)

The gravitational field in 4 spacetime dimensions has correctly (for a massless field) 2 independent degrees of freedom per spacetime point. This is most easily seen by noting that the induced metric of a time slice $g_{ij}$ has 6 independent components, and there are the 4 constraints (3).

### 2. 2 + 1–Dimensional ADM Decomposition

In 2 + 1 spacetime dimensions the description of Section (1) is essentially identical, apart from a factor of $\frac{1}{2}$ in the super-Hamiltonian (3). The counting of degrees of freedom is, however, quite different. There are in fact zero degrees of freedom, and this can be seen in several ways. The simplest is perhaps to note that now the induced metric $g_{ij}, i, j = 1, 2$ has only 3 independent components, but there are 3 constraints $\mathcal{H} = 0, \mathcal{H}_i = 0$ analogous to (3). Alternatively, since the Weyl tensor vanishes in 3 dimensions (but not in 4, see 6), it follows that the full Riemann curvature tensor $R_{\alpha\beta\mu\nu}$ can be decomposed uniquely in terms of only the Ricci tensor $R_{\mu\nu}$ the scalar curvature $R$ and the metric tensor $g_{\mu\nu}$ itself.

$$R_{\lambda\mu\nu k} = g_{\lambda\nu} R_{\mu k} - g_{\mu\nu} R_{\lambda k} - g_{\lambda k} R_{\mu\nu} + g_{\mu k} R_{\lambda\nu}$$

$$+ \frac{1}{2} R (g_{\mu\nu} g_{\lambda k} - g_{\lambda\nu} g_{\mu k})$$

(10)
In fact in $d$ dimensions $R_{\alpha\beta\mu\nu}$ has $\frac{d(d^2-1)}{12}$ independent degrees of freedom and $R_{\mu\nu}$ has $\frac{d(d+1)}{2}$. These coincide when $d = 3$. In terms of the Einstein tensor $G^{\alpha\beta}$ (equation (4)) the decomposition (10) is

$$R_{\lambda\mu\nu\kappa} = \epsilon_{\lambda\mu\beta} \epsilon_{\nu\kappa\alpha} G^{\alpha\beta}$$

(11)

so that when Einstein’s vacuum equations $G^{\alpha\beta} = 0$ are satisfied, the full curvature tensor (all components) are zero, i.e. $R_{\lambda\mu\nu\kappa} = 0$ and spacetime is flat. Thus vacuum solutions of Einstein’s equations correspond to flat spacetimes, and there are no local degrees of freedom.

It is possible, however, to solve the field equations and introduce some dynamics, in several ways. The first - developed extensively by Deser et al. and others, is to add sources, or matter, thus creating local degrees of freedom. When Einstein’s equations read

$$G^{\alpha\beta} = T^{\alpha\beta}$$

(12)

where $T^{\alpha\beta}$ is the stress-energy tensor of the sources, the curvature (11) is no longer zero, but is proportional, from (12), to $T^{\alpha\beta}$.

The second creates propagating massive gravitational modes by adding a topological term to the action, always possible in an odd number of dimensions. For gravity in 3 dimensions, this is the Chern-Simons form

$$\int (\omega^{ab} \wedge d\omega_{ab} + \frac{2}{3} \omega^{ac} \wedge \omega^d_c \wedge \omega_{da})$$

(13)

where the components of the spin connection $\omega_{\mu}^{ab}$ are to be considered as functionals of the triads $e^a_{\mu}$ by solving the torsion equation.

$$R^a = de^a - \omega^{ab} \wedge e_b = 0$$

with $\epsilon^a_{\mu} \epsilon^b_{\nu} \eta_{ab} = g_{\mu\nu}$. Variation of (13) with respect to the metric tensor $g_{\mu\nu}$ gives the Cotton tensor

$$C^{\mu\nu} = g^{-\frac{1}{2}} \epsilon^{\mu\lambda\beta} D_\lambda \left( R^\nu_\beta - \frac{1}{4} \delta^\nu_\beta R \right)$$

which is symmetric, traceless, conserved, and vanishes if the theory is conformally invariant. Therefore, adding the Chern-Simons term (13) to the three-dimensional scalar curvature action (1) with a constant factor $\frac{1}{\mu}$ leads to the field equations

$$G^{\mu\nu} + \frac{1}{\mu} C^{\mu\nu} = 0$$
which can be transformed into
\[
\left( \Box + \mu^2 \right) R_{\mu\nu} = \text{terms in } \left( R_{\mu\nu} \right)^2
\]  
(14)

In the linearized limit the R.H.S. of (14) vanishes and it is shown in \(^6\) that the solutions of (14) correspond to massive, spin ±2, particles.

A way to introduce global rather than local degrees of freedom in flat spacetime is to consider non-trivial topologies. Recall that curvature is defined by commutators of covariant derivatives, or, by parallel transport around non-collapsible curves i.e. curves which are not homotopic to the identity. The change effected by parallel transport around closed curves of this type is often called holonomy - and is used to characterise flat space-times. A simple example is when the spatial surfaces are tori, i.e. \(\Sigma = T^2\) - then the meridian and parallel are clearly non-collapsible. This will be discussed explicitly in Sections (3) and (4).

3. Second–Order, Partially Reduced ADM Formalism

Here I summarise work by Moncrief \(^8\) and Hosoya and Nakao \(^9\), adding a cosmological constant \(\Lambda\). It is known that any two-metric \(g_{ij}\) on \(\Sigma_g\), where \(\Sigma_g\) is a Riemann surface of genus \(g\), is conformal (up to a diffeomorphism) to a finite-dimensional family of constant curvature metrics \(\bar{g}_{ij}(m_\alpha)\),

\[
g_{ij} = e^{2\lambda} \bar{g}_{ij}(m_\alpha),
\]
(15)

labelled by a set of moduli \(m_\alpha, \alpha = 1 \ldots 6g - 6\) (see Abikoff \(^10\)), and

\[
\begin{align*}
1 & \quad g = 0 \\
R(\bar{g}) &= 0 \quad g = 1 \\
-1 & \quad g > 1
\end{align*}
\]
(16)

A similar decomposition of the momenta \(\pi^{ij}\) gives

\[
\pi^{ij} = e^{-2\lambda} \sqrt{\bar{g}} \left( p^{ij} + \frac{1}{2} \bar{g}^{ij} \pi / \sqrt{\bar{g}} + \nabla^i Y^j + \nabla^j Y^i - \bar{g}^{ij} \nabla_k Y^k \right)
\]
(17)

where \(\nabla_i\) is the covariant derivative for \(\bar{g}_{ij}\), indices are now raised and lowered with \(\bar{g}_{ij}\), and \(p^{ij}\) - the momentum conjugate to \(\bar{g}_{ij}\) - is transverse traceless with respect to \(\nabla_i\), i.e., \(\nabla_i p^{ij} = 0\).

This decomposition uses York time \(^11\), the mean (extrinsic) curvature \(K = \pi / \sqrt{\bar{g}} = T\), shown to be a good global coordinate choice in \(^8\).

The supermomentum constraints now imply that \(Y^i = 0\), while the super-Hamiltonian constraint,

\[
\mathcal{H} = - \frac{1}{2} \sqrt{\bar{g}} e^{2\lambda} (T^2 - 4\Lambda) + \sqrt{\bar{g}} e^{-2\lambda} p^{ij} p_{ij} + 2 \sqrt{\bar{g}} \left[ \Delta \lambda - \frac{1}{2} \bar{R} \right] = 0,
\]
(18)
reduces to a differential equation for the conformal factor $\lambda$ as a function of $g_{ij}, p^{ij}$ and $T$. For $g > 1$ a solution of (18) always exists and the three-dimensional Einstein–Hilbert action is

$$J_{\text{Ein}} = \int dT \left( p^{\alpha} \frac{dm_{\alpha}}{dT} - H(m, p, T) \right)$$  \hspace{1cm} (19)$$

where $p^\alpha$ are the momenta conjugate to the moduli $m_{\alpha}$ defined by

$$p^{\alpha} = \int_{\Sigma} d^2x \pi^{ij} \frac{\partial}{\partial m_{\alpha}} g_{ij}.$$  \hspace{1cm} (20)$$

and $H(m, p, T)$ is an effective, or reduced, ADM Hamiltonian

$$H(m, p, T) = \int_{\Sigma} \sqrt{\bar{g}} d^2x = \int_{\Sigma} e^{2\lambda(m, p, T)} \sqrt{g} d^2x$$  \hspace{1cm} (21)$$

which represents the surface area at time $T$, with $\lambda(m, p, T)$ determined by (18). The reduced ADM Hamiltonian (21) generates the $T = K$ or time development of $m_{\alpha}, p^\beta$ through the Poisson brackets

$$\{m_{\alpha}, p^\beta\} = \delta^\beta_{\alpha}.$$  \hspace{1cm} (22)$$

For $g = 1$ the modulus is the complex number $m = m_1 + im_2$ (with $m_2 > 0$), with momenta $p = p^1 + ip^2$ satisfying the Poisson brackets

$$\{m, \bar{p}\} = \{\bar{m}, p\} = 2, \quad \{m, p\} = \{\bar{m}, \bar{p}\} = 0$$  \hspace{1cm} (23)$$

and

$$d\sigma^2 = m_2^{-1} |dx + m dy|^2,$$  \hspace{1cm} (24)$$

is the spatial metric for a given $m$ where $x$ and $y$ each have period 1. The surface curvature (16) is zero and (18) is explicitly solved. The reduced ADM Hamiltonian (21) becomes

$$H(m, p, T) = (T^2 - 4\Lambda)^{-1/2} \left[ m_2 p^1 \bar{p} \right]^{1/2}.$$  \hspace{1cm} (25)$$

One can recognise in (25) the square of the momentum with respect to the Poincaré (constant negative curvature) metric on the torus moduli space

$$m_2^{-2} |d\bar{m}^2|.$$  \hspace{1cm} (26)$$

Hamilton’s equations for the motion of $m, p$ on the hyperbolic upper half plane (Teichmüller space) using the reduced Hamiltonian (25) can be solved exactly $^{12,13}$ and correspond to motion on a semicircle, a geodesic with respect to the metric (26).
This reduced phase space can, in principle, be quantized by replacing the Poisson brackets (22) with commutators,
\[
\hat{m}_\alpha, \hat{\rho}^\beta = i\hbar \delta^\beta_\alpha
\]  
representing the momenta as derivatives,
\[
\hat{\rho}^\alpha = \hbar \frac{\partial}{i \partial m_\alpha},
\]
and imposing the Schrödinger equation
\[
\frac{i\hbar}{\partial T} \partial \psi(m, T) = \hat{H}\psi(m, T),
\]
where the Hamiltonian \(\hat{H}\) is obtained from (25) by some suitable operator ordering. With the ordering of (25), the Hamiltonian is
\[
\hat{H} = \frac{\hbar}{\sqrt{T^2 - 4\Lambda}} \Delta_0^{1/2},
\]
where \(\Delta_0\) is the scalar Laplacian for the constant negative curvature moduli space with metric (26). Other orderings exist, but all consist of replacing \(\Delta_0\) in (30) by \(\Delta_n\), the weight \(n\) Maass Laplacian (see e.g. Carlip 14).

This approach also depends on the arbitrary, albeit good, choice of \(K = \pi/\sqrt{g} = T\) as a time variable. It is not at all clear that a different choice would lead to the same quantum theory.

4. First–Order Fully Reduced, ADM Formalism

The first-order, connection approach to (2+1)-dimensional gravity, in which the triad one-form \(e^a = e^a_\mu dx^\mu\) and the spin connection \(\omega^{ab} = \omega^{ab}_\mu dx^\mu\) are treated as independent variables was inspired by Witten 15 (see also 16) and developed by Nelson, Regge and Zertuche 17,18,19,20. The three dimensional Einstein-Hilbert action is
\[
I_{\text{Ein}} = \int (d\omega^{ab} - \omega^{a}_d \wedge \omega^{db} + \frac{\Lambda}{3} e^a \wedge e^b) \wedge e^c \epsilon_{abc}, \quad a, b, c = 0, 1, 2.
\]

For \(\Lambda < 0\) this action can be written (up to a total derivative) as
\[
I_{\text{CS}} = -\frac{\alpha}{4} \int (d\Omega^{AB} - \frac{2}{3} \Omega^A_E \wedge \Omega^{EB} \wedge \Omega^{CD} \epsilon_{ABCD},
\]
\[b\text{For } \Lambda \geq 0 \text{ see e.g. the discussion in 21.}\]
where $A, B, C, \ldots = 0, 1, 2, 3, \epsilon_abc = -\epsilon_{abc}$, the tangent space metric is $\eta_{AB} = (-1, 1, 1, -1)$ and the (anti-)de Sitter $SO(2, 2)$ spin connection $\Omega^{AB}$ is

$$\Omega^{A}_{B} = \begin{pmatrix} \omega^{a}_{b} - \frac{\epsilon^{a}}{\alpha} \\ -\frac{\epsilon_{a}}{\alpha} \\ 0 \end{pmatrix},$$

with $\Lambda = -\alpha^{-2}$. The canonical $2 + 1$-decomposition of (32) is

$$I_{E=0} = \int d^{3}x (\Omega^{i}_{A} \bar{\Omega}^{j}_{B} \epsilon_{ABCD} - \Omega^{0}_{A} \bar{R}^{ij}_{AB}) \epsilon^{ij}.$$  

(34)

In (34) the curvature two-form $R^{AB} = d \Omega^{AB} - \Omega^{AC} \wedge \Omega_{C}^{B}$ has components $R^{ab} + \Lambda e^{a} \wedge e^{b}$ (proportional to the constraints (3)) and $R^{a} = \frac{1}{\Lambda} R^{a}$ (proportional to a rotation constraint $J^{a}ab$ on the triads), where

$$R^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega_{c}^{b}, \quad R^{a} = de^{a} - \omega^{ab} \wedge e_{b}$$

are the $(2+1)$-dimensional curvature and torsion. The field equations (constraints) derived from the action (34) are simply $R^{AB} = 0$, and imply that the $SO(2, 2)$ connection $\Omega^{AB}$ is flat, or, equivalently, from (35) that the torsion vanishes everywhere and that the curvature $R^{ab}$ is constant. They generate, through the Poisson brackets

$$\{\Omega^{AB}(x), \Omega^{CD}(y)\} = \frac{1}{2\alpha} \epsilon_{ij} \epsilon^{ABCD} \delta^{2}(x - y).$$

(36)

infinitesimal gauge and coordinate transformations $\delta \Omega^{AB} = Du^{AB}$ on the connections $\Omega^{AB}$.

Since the connection $\Omega^{AB}$ is flat, it can be written locally in terms of an $SO(2, 2)$-valued zero-form $\psi^{AB}$ as $dG^{AB} = \Omega^{AC} G_{C}^{B}$. This sets to zero all the constraints $R^{AB} = 0$ and is therefore a fully reduced ADM formalism in which the Hamiltonian is identically zero. However, some global degrees of freedom remain, as can be seen by now taking into account the non-trivial topology of the Riemann surface. For each path $\sigma$ on $\Sigma$ define the holonomy (Wilson loop)

$$G_{\sigma}^{AB} = \exp P \int_{\sigma} \Omega^{AB}$$

(37)

where $P$ denotes path-ordered, and $G_{\sigma}$ depends on the base (starting) point and the homotopy class $\{\sigma\}$ of $\sigma$, and satisfies $G_{\sigma \rho} = G_{\sigma} G_{\rho}$. Integrating the brackets (36) along paths $\rho, \sigma$ with non-zero intersection gives

$$\{G_{\rho}, G_{\sigma}\} \neq 0$$

(38)

and this is the starting point for holonomy quantization.
It is actually more convenient to use the spinor groups $\text{SL}(2, \mathbb{R}) \otimes \text{SL}(2, \mathbb{R})$ for $\text{SO}(2, 2)$ (see \cite{18} for details). For each path $\sigma$ we have
\begin{equation}
G^{AB}_\sigma \gamma_B = S^{-1}(\sigma) \gamma^A S(\sigma) \tag{39}
\end{equation}
where $\gamma^A$ are the Dirac matrices and $S = S^+ \otimes S^-, S^\pm \in \text{SL}(2, \mathbb{R})$. Explicitly, if the paths $\rho, \sigma$ have a single intersection then
\begin{align*}
\{ S^\pm (\rho)^\alpha_\beta, S^\pm (\sigma)^\gamma_\delta \} &= \pm s ( - S^\pm (\rho)^\alpha_\beta S^\pm (\sigma)^\gamma_\delta + 2 S^\pm (\rho^\alpha_\beta \sigma^\gamma_\delta) \sigma^3_1 \rho^3_1 ) \\
\{ S^+ (\rho)^\alpha_\beta, S^- (\sigma)^\gamma_\delta \} &= 0 \quad \alpha, \beta, \ldots = 1, 2. \tag{40}
\end{align*}
where $s$ is the intersection number (now set to 1) and $\sigma_1, \rho_1$ (resp. $\sigma_3, \rho_3$) are the segments of paths before (resp. after) the intersection. The gauge invariance can be implemented by taking traces since, if $\delta$ is an open path
\begin{equation}
R^\pm (\sigma) = tr S^\pm (\sigma) = tr S^\pm (\delta^{-1} \sigma \delta) = R^\pm (\delta^{-1} \sigma \delta) \tag{41}
\end{equation}
where now $\delta^{-1} \sigma \delta$ is closed. For $g = 1$ it is enough to have just six traces $R^\pm_i, i = 1, 2, 3$, corresponding to the three paths $\gamma_1, \gamma_2, \gamma_3 = \gamma_1 \cdot \gamma_2$. From (40) they satisfy the non–linear cyclical Poisson bracket algebra $\cite{18}$
\begin{align*}
\{ R^\pm_i, R^\pm_j \} &= \frac{\sqrt{-\Lambda}}{4} ( \epsilon_{ij} k R^\pm_k - R^\pm_i R^\pm_j ), \quad \epsilon_{123} = 1 \tag{42}
\end{align*}
and the cubic Casimir
\begin{equation}
1 - (R^\pm_1)^2 - (R^\pm_2)^2 - (R^\pm_3)^2 + 2 R^\pm_1 R^\pm_2 R^\pm_3 = 0. \tag{43}
\end{equation}
The traces (holonomies) of (42) can be represented classically as
\begin{equation}
R^\pm_1 = \cosh r^\pm_1, \quad R^\pm_2 = \cosh r^\pm_2, \quad R^\pm_3 = \cosh (r^\pm_1 + r^\pm_2), \tag{44}
\end{equation}
where $r^\pm_{1,2}$ are real, global, time-independent (but undetermined) parameters which, from (42) satisfy the Poisson brackets$^c$
\begin{align*}
\{ r^\pm_1, r^\pm_2 \} &= \pm \frac{\sqrt{-\Lambda}}{4}, \quad \{ r^\pm_{1,2}, r^\pm_{1,2} \} = 0. \tag{45}
\end{align*}
The above fully reduced system can be easily quantized either by replacing the Poisson brackets (45) by the commutators
\begin{align*}
[i^\pm_1, i^\pm_2] &= \pm \frac{i \hbar \sqrt{-\Lambda}}{4}, \quad [i^\pm_{1,2}, i^\pm_{1,2}] = 0. \tag{46}
\end{align*}
$^c$The parameters $r^\pm_{1,2}$ used here have been scaled by a factor of $\frac{1}{2}$ with respect to previous articles.
or by directly quantizing the algebra (42). This gives for the (+) algebra
\[ q^\frac{1}{2} \hat{R}_1^+ \hat{R}_2^+ - q^{-\frac{1}{2}} \hat{R}_2^+ \hat{R}_1^+ = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \hat{R}_3^+ \]
(47)
where \( q = \exp 2i\theta \) and \( \tan \theta = -\frac{\hbar \sqrt{-\Lambda}}{8} \). The algebra (47) is related to the Lie algebra of the quantum group \( SU(2)_q \), and can be represented (up to rescalings of \( O(\hbar) \)) by e.g. \( \hat{R}_i = \frac{1}{2}(A_i + A_i^{-1}) \), \( i = 1, 2, 3 \) where the \( A_i \) satisfy
\[ A_1 A_2 = q A_2 A_1, \quad A_1 A_2 A_3 = q^{\frac{1}{2}} \]
(48)
The first of (48) is called either a \( q \)-commutator, or a quantum plane relation, or it is said that \( A_1, A_2 \) form a Weyl pair. Relations (48) can be satisfied by the assignments \( A_1 = e^{\hat{r}_1}, A_2 = e^{\hat{r}_2}, A_3 = e^{-(\hat{r}_1 + \hat{r}_2)} \) with \( \hat{r}_1, \hat{r}_2 \) satisfying (46).

5. Classical and Quantum Equivalence

5.1. Classical equivalence

The classical solution of Section (3) can be related to the parameters \( r_{1,2}^\pm \) of Section (4) as follows \( 2 \).

The ADM reduced actions (19) and (34) are related by
\[ I = \int dt \int d^2 x \pi^{ij} \dot{g}_{ij} = \int d^3 x \ \Omega_i^{ab} \hat{Y}^C_{ij} \mathcal{E}_{ABCD} \]
\[ = \int \frac{1}{2}(\hat{p}dm + p\hat{m}) - HdT - d(p^1 m_1 + p^2 m_2) \]
\[ = \int \alpha(r_{1}^- dr_{2}^- - r_{1}^+ dr_{2}^+) \]
(49)
and show that with the time coordinate \( t \) determined by \( T = -\frac{2}{\alpha} \cot \frac{2t}{\alpha} \) the parameters \( r_{1,2}^\pm \) are related to the complex modulus \( m \) and momentum \( p \) through a (time-dependent) canonical transformation. Explicitly, with \( r_a(t) = r_a^- e^{\hat{r}_a^+} + r_a^+ e^{-\hat{r}_a^+}, \quad a = 1, 2 \) and the \( r_a^\pm \) satisfying (45), then
\[ m = r_2^{-1}(t)r_1(t), \quad \text{and} \quad p = -i \frac{\sqrt{T^2 - 4\Lambda}}{4\Lambda} r_2^2(t) \]
(50)
will satisfy the Poisson brackets (23).

The Hamiltonian (25) is now
\[ H = \frac{1}{\sqrt{T^2 - 4\Lambda}}(r_{1}^- r_{2}^+ - r_{1}^+ r_{2}^-) \]
(51)
\( \text{The (-) algebra is the same as (47) but uses } q^{-1} \text{ rather than } q \)
and generates the development of the modulus and momentum (50) as functions of the parameters $r_a^\pm$ and time $T$ through

$$\frac{dp}{dT} = \{p, H\}, \quad \frac{dm}{dT} = \{m, H\} \quad (52)$$

### 5.2. Large Diffeomorphisms

The reduction to the modulus $m$ and momenta $p$ means there are no more “small diffeomorphisms”- coordinate transformations (the constraints which generate them are all identically zero). But there remain “large diffeomorphisms” due to the topology. These are transformations that are not connected to the identity, cannot be built up from infinitesimal transformations and are generated by “Dehn twists”, i.e. by the operation of cutting open a handle, twisting one end by $2\pi$, and regluing the cut edges. For $g > 1$ the set of equivalence classes of such large diffeomorphisms (modulo diffeomorphisms that can be deformed to the identity) is known as the mapping class group. For $g = 1$ it is also called the modular group, and the Dehn twists of the two independent circumferences $\gamma_1$ and $\gamma_2$ (which have intersection number +1) act by

$$S : \gamma_1 \to \gamma_2^{-1}, \quad \gamma_2 \to \gamma_1 \quad \text{and} \quad T : \gamma_1 \to \gamma_1 \cdot \gamma_2, \quad \gamma_2 \to \gamma_2. \quad (53)$$

These transformations induce the modular transformations

$$S : m \to -m^{-1} \quad p \to \bar{m}^2 p$$
$$T : m \to m + 1 \quad p \to p. \quad (54)$$

which preserve the Poincaré metric (26), the Hamiltonian (25) and the Poisson brackets (23). The figure illustrates this group action on the modulus configuration space, with the invariant semicircle representing the geodesic motion of the modulus $m$. 
Classically, one could ask that observables be invariant under all space-time diffeomorphisms, including those in the modular group. Since equation (54) shows that the modular group is well behaved on configuration space, invariant functions of $m$ exist (see $^{14}$). So the reduced ADM approach of Section 3 looks like a standard “Schrödinger picture” quantum theory, with time-dependent states $\psi(m,T)$ whose evolution is determined by the Hamiltonian operator (30).

On the traces of holonomies the transformations (53) induce the following

\begin{align*}
S & : R^+_1 \to R^+_-2, \quad R^+_2 \to R^+^+_1, \quad R^+_3 \to 2R^+_1 R^+_2 - R^+_3 \\
T & : R^+_1 \to R^+^+_3, \quad R^+_2 \to R^+^+_2, \quad R^+_3 \to 2R^+_3 R^+_2 - R^+_1.
\end{align*}

which preserve the algebra (42). The corresponding transformations on the holonomy parameters preserve their Poisson brackets (45)

\begin{align*}
S & : r^+_1 \to r^+^+_2, \quad r^+_2 \to -r^+_1 \\
T & : r^+_1 \to r^+_1 + r^+_2, \quad r^+_2 \to r^+_2.
\end{align*}

In this approach the modular group action (56) on the parameters is not well behaved since it mixes $r_1$ and $r_2$, and quantization normally requires a polarization. So the quantum theory of Section 4 resembles a “Heisenberg picture” quantum theory, with time-independent states $\psi(r)$, and, for some ordering, time–dependent operators (50).

### 5.3. The Quantum Modular Group

Here I present work in collaboration with Carlip $^{22}$. It is useful to note that the modular transformations can also be implemented quantum-mechanically, by conjugation with the unitary operators $^{19,22}$

\begin{align*}
\hat{T} &= \hat{T}^+ \hat{T}^- = \exp\left\{ \frac{i}{2\hbar}(\hat{p} + \hat{p}^\dagger) \right\} \\
\hat{S} &= \hat{S}^+ \hat{S}^- = \exp\left\{ \frac{i\pi}{8\hbar} [2(\hat{p}^\dagger \hat{p}) + \hat{m}^\dagger (\hat{m} \hat{p} + \hat{p} \hat{m}^\dagger) + (\hat{m} \hat{p}^\dagger + \hat{p}^\dagger \hat{m})] \right\}
\end{align*}

The $S$ transformation for $p$ differs from its classical version (54)

\begin{equation}
S : \hat{p} \to \frac{\hat{m}^\dagger}{2}(\hat{m} \hat{p} + \hat{p} \hat{m}^\dagger),
\end{equation}

by terms of order $\hbar$. In terms of the holonomy parameters these are

\begin{align*}
\hat{T}^\pm &= \exp\left\{ \pm \frac{i\alpha}{2\hbar} (r^\pm_2)^2 \right\}, \quad \hat{S}^\pm &= \exp\left\{ \pm \frac{i\pi\alpha}{4\hbar} \left[ (r^\pm_1)^2 + (r^\pm_2)^2 \right] \right\}
\end{align*}

\begin{equation}
(58)
\end{equation}
Using the above construction the two representations, classically equivalent as shown in Section 5.1 can be related as follows. Start by diagonalizing the commuting moduli operators \( \hat{m} \) and \( \hat{m}^\dagger \), considered as functions of time and initial data \( r_{1,2}^t \) through (50)). Now if the \( r_2(t) \) are “coordinates” \( u(t) \) and \( r_1(t) \) their “momenta” then \( \hat{m} \) and \( \hat{p} \) act as

\[
\hat{m} \sim u^{-1} \frac{\partial}{\partial u} \quad \hat{p} \sim \bar{u}^2
\]  

The normalized eigenstates of \( \hat{m} \) with eigenvalues \( m \) (and \( \bar{m} \) for \( \hat{m}^\dagger \)) are

\[
K(m, \bar{m}, t | u, \bar{u}) = \frac{\alpha m^2}{2\pi \hbar} \bar{u} \exp \left\{ -\frac{\alpha}{4\hbar} mu^2 + \frac{\alpha}{4\hbar} \bar{m} \bar{u}^2 \right\}.
\]  

So candidates for “Schrödinger picture” wave functions are the superpositions

\[
\tilde{\psi}(m, \bar{m}, t) = \int du_1 du_2 K^\ast(m, \bar{m}, t | u, \bar{u}) \psi(u, \bar{u})
\]  

of the “Heisenberg picture” wave functions \( \psi(u, \bar{u}) \). Inverting (61) gives

\[
\psi(u, \bar{u}) = \int F \frac{d^2 m}{m_2^2} K(m, \bar{m}, t | u, \bar{u}) \tilde{\psi}(m, \bar{m}, t).
\]  

where \( F \) is a fundamental region for the modular group. Now apply the \( T \) transformation (54) to (62)

\[
\hat{T}\psi(u, \bar{u}) = \hat{T} \int F \frac{d^2 m}{m_2^2} K(m, \bar{m}, t | u, \bar{u}) \tilde{\psi}(m, \bar{m}, t)
\]

\[
= \int F \frac{d^2 m}{m_2^2} K(m + 1, \bar{m} + 1, t | u, \bar{u}) \tilde{\psi}(m + 1, \bar{m} + 1, t)
\]

\[
= \int_{T^{-1}F} F \frac{d^2 m}{m_2^2} K(m, \bar{m}, t | u, \bar{u}) \tilde{\psi}(m, \bar{m}, t),
\]  

where \( T^{-1}F \) is the new fundamental region obtained from \( F \) by a \( T^{-1} \) transformation, and in (63) \( \tilde{\psi}(m, \bar{m}, t) \) and the integration measure are modular invariant. A similar argument holds for the \( S \) transformation, and shows that there are no invariant “Heisenberg picture” wave functions \( \psi(u, \bar{u}) \), since the integration regions in (62) and (63) are disjoint except on a set of measure zero. Further, \( \psi(u, \bar{u}) \) and \( \hat{T}\psi(u, \bar{u}) \) are orthogonal since

\[
\langle \psi | \hat{T}\psi \rangle = \int_{T^{-1}F} \frac{d^2 m}{m_2^2} \int F \frac{d^2 m'}{m_2^2} m_2^2 \delta^2(m - m') \tilde{\psi}(m, \bar{m}, t) \tilde{\psi}^\ast(m', \bar{m}', t) = 0,
\]  

and similarly for \( S \).
Equation (64) shows that the modular group splits the Hilbert space of square-integrable functions of \((u_1, u_2)\) into an infinite set of orthonormal fundamental subspaces consisting of wave functions of the form (62) for a fixed fundamental region \(\mathcal{F}\). It is shown in \(^{22}\) that they are physically equivalent, because matrix elements of invariant operators can be computed in any of these subspaces, and each one is equivalent to the ADM Hilbert space.

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