THE WEYL ALGEBRA IS COHERENT

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Abstract. In this paper, we show that the Weyl algebra $A_n(k)$ is a coherent ring for any field $k$ and any integer $n \geq 1$.

1. Introduction

It is well-known that the ring $D = D(A)$ of $k$-linear differential operators on an affine $k$-algebra $A$ is not Noetherian in general. For instance, $D$ is not Noetherian when $A = k[x_1, x_2, \ldots, x_n]$ and $\text{char}(k) = p > 0$ or when $A = k[x, y, z]/(x^3 + y^3 + z^3)$ and $\text{char}(k) = 0$. In this paper, we prove that in the first of these cases, the ring $D = A_n(k)$, called the Weyl algebra, is a coherent ring. It is, as far as we know, an open question if $D$ is coherent in the second case.

Over a coherent ring, the category of finitely presented (or coherent) modules is well-behaved. For instance, any finitely presented module has a free resolution $(L_\bullet, d_\bullet)$ such that $L_i$ has finite rank for all $i$. This suggests that it is possible to do homological algebra for coherent modules over coherent rings.

When $D$ is a ring of differential operators, any (finite) system of linear differential operators corresponds to a finitely presented left $D$-module. This seems to suggest that finitely presented $D$-modules are interesting to study.

This work was inspired by the recent paper Bavula [1], where the author studies the Weyl algebra $D = A_n(k)$ in prime characteristic $p > 0$ and its holonomic modules, and (among other things) obtains an explicit classification of the finitely presented simple $D$-modules. Unfortunately, it turns out that these modules are not very interesting considered as differential systems. In fact, the finitely presented holonomic $D$-modules are the finitely presented $D$-modules of finite length, and it follows from Bavula’s results that these $D$-modules correspond to systems of linear differential equations of order zero.

2. The Weyl algebra is a coherent ring

We recall that a ring $D$ is left (right) coherent if any of its finitely generated left (right) ideals is finitely presented, and that $D$ is coherent if it is left and right coherent. Clearly, any ring that is either Noetherian or semi-hereditary is coherent. We refer to Bourbaki [2] and Glaz [4] for details on coherent rings.

Let $k$ be any field, and let $A = k[x_1, x_2, \ldots, x_n]$ be the polynomial algebra in $n$ variables over $k$ for an integer $n \geq 1$. We define the $n$’th Weyl algebra over $k$ to be the ring of $k$-linear differential operators on $A$ in the sense of Grothendieck [5], and denote it by $A_n(k)$. 
We consider the partial derivations \( \partial_i = \partial/\partial x_i \in \text{Der}_k(A) \) for \( 1 \leq i \leq n \), and define their \textit{divided powers} \( \partial_i^{[r]} : A \to A \) to be the \( k \)-linear operators given by

\[
\partial_i^{[r]}(x^m) = \binom{r}{m} x^{m - r},
\]

for all multi-indices \( m = (m_1, m_2, \ldots, m_n) \in \mathbb{N}_0^n \) and for all integers \( r \geq 0 \), where we use multi-index notation

\[
x^m = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}
\]

and write \( m - re_i = (m_1, \ldots, m_i - r, \ldots, m_n) \). Notice that the binomial coefficients in \( k \) are the canonical images of the usual integer-valued binomial coefficients. The name divided powers come from the fact that \( r! \partial_i^{[r]} = \partial_i^r \).

**Lemma 1.** The Weyl algebra \( A_n(k) \) is the \( k \)-subalgebra of \( \text{End}_k(A) \) generated by \( \{x_1, x_2, \ldots, x_n\} \cup \{\partial_i^{[r]} : 1 \leq i \leq n, r \geq 1\} \), with relations

1. \( [x_i, x_j] = 0 \)
2. \( [\partial_i^{[r]}, \partial_j^{[s]}] = 0 \)
3. \( \partial_i^{[r]} \partial_j^{[s]} = (r+s) \partial_i^{[r+s]} \)
4. \( [\partial_i^{[r]}, x_j] = \delta_{ij} \partial_i^{[r-1]} \)

for all \( 1 \leq i, j \leq n \) and all \( r, s \geq 1 \).

**Proof.** This result is well-known; see for instance Bavula [1]. \( \square \)

If \( \text{char}(k) = 0 \), then the Weyl algebra \( A_n(k) \) is a finitely generated Noetherian ring, generated by \( \{x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\} \). We will assume that \( \text{char}(k) = p > 0 \) in the rest of this paper.

We shall use the following construction, introduced in Section 3 of Chase [3]. Let \( A_r \subseteq A = k[x_1, \ldots, x_n] \) be the \( k \)-subalgebra generated by \( \{a^{p^r} : a \in A\} \) for all \( r \geq 0 \). This implies that \( A_r = k[x_1^{p^r}, x_2^{p^r}, \ldots, x_n^{p^r}] \); see Lemma 2.6 in Smith [6]. We obtain a descending chain

\[
A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_r \supseteq \cdots
\]

of \( k \)-algebras such that \( A_i \) is a free \( A_j \)-module of finite rank for all \( i \leq j \). We define \( D_r = \text{End}_{A_r}(A) \subseteq \text{End}_k(A) \) for \( r \geq 0 \), and identify \( D_0 = \text{End}_A(A) \) with \( A \). By Lemma 3.3 in Chase [3], it follows that

\[
A = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_r \subseteq \cdots \subseteq D = A_n(k)
\]

is an ascending chain of subalgebras of \( D = A_n(k) \) such that \( D = \bigcup_{r \geq 0} D_r \). In fact, we may identify \( D_r \) with the matrix ring \( M_q(A_r) \), where \( q = \text{rk}_{A_r}(A) \).

**Theorem 2.** Let \( k \) be a field of characteristic \( p > 0 \), and let \( n \geq 1 \) be an integer. Then the Weyl algebra \( A_n(k) \) is a coherent ring.

**Proof.** Using the construction and notation from Chase [3] mentioned above, we have that \( D = A_n(k) \) is a direct limit

\[
D = \lim_{r \to \infty} D_r
\]

We note that \( A_r \) is a finitely generated commutative \( k \)-algebra, and therefore a Noetherian ring. Since \( D_r \cong M_q(A_r) \) is a matrix ring with \( q = \text{rk}_{A_r}(A) \), it is clear that \( D_r \) is Morita equivalent to \( A_r \) and that \( D_r \) is a flat (left and right) \( D_s \)-module.
for all $r \geq s$; see also the proof of Lemma 3.4 in Chase [3] and Proposition 3.2 in Smith [6]. In particular, $D_r$ is Noetherian and therefore coherent for all $r \geq 0$. Hence the results follows from the well-known fact that a flat direct limit of coherent rings is coherent; see for instance Exercise I.2.12 in Bourbaki [2].

\[\square\]

**Remark 1.** It is known that first Weyl algebra $A_1(k)$ has global dimension one when $k$ is a field of characteristic $p > 0$, and this implies that $A_1(k)$ is semi-hereditary and therefore a coherent ring. However, when $n \geq 2$, the Weyl algebra $A_n(k)$ is not semi-hereditary.

**References**

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