CONTRACTION OF GRAPHS AND SPANNING K-END TREES

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Abstract. A tree with at most $k$ leaves is called $k$-ended tree, and a tree with exactly $k$ leaves is called $k$-end tree, where a leaf is a vertex of degree one. Contraction of a graph $G$ along the edge $e$ means deleting the edge $e$ and identifying its end vertices and deleting all edges between every two vertex except one edge to gain again a simple graph. Contraction of edge $e$ on graph $G$ is denoted by $G/e$. In this paper we prove some theorems related to a graph and its contraction. For example we prove the following theorem. If $G$ is a connected graph that has a spanning $k$-end tree and $|V(G)| > K + 1$ then there exist an edge $e$ such $G/e$ has a spanning $k$-end tree.

1. Introduction

In this paper all graphs are simple. Vertex set and edge set of graph $G$ are denoted by $V(G)$ and $E(G)$ successively, degree of vertex $v$ in graph $G$ is denoted by $\text{deg}_G(v)$. If $v$ is a vertex of graph $G$, $N_G(v)$ is set of all vertices adjacent to $v$ in $G$. If $T$ is a tree then the unique path between very two vertices $v$ and $u$ is denoted by $uTv$ and $v^-$ is the vertex adjacent to $u$ in this path, and also $v^-$. We also denote the edge $e$ with $uv$ or $vu$ where $u$ and $v$ are end vertices of $e$. A Hamiltonian path in graph $G$ is a path that contains all vertices of graph. Subdividing the edge $e$ with end vertices $u$ and $v$ in graph $G$ is an operation that produces a new graph whose vertex set is $V(G) \cup \{w\}$ and edge set is $(E(G) - \{e\}) \cup \{e', e''\}$ where $w$ is a new vertex, $e' = uw$ and $e'' = vw$(figure 1.1). A graph that is obtained of finite sequence of subdivisions of edges of graph $G$ is called a subdivision of $G$.

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Now we presents some famous theorems about $k$-ended trees. First one is Ore’s theorem.

**Theorem 1.1.** Suppose $G$ be a graph with $|V(G)| \geq 3$, if for every two non-adjacent vertices $v$ and $w$ of $G$ we have $\deg v + \deg w \geq |V(G)| - 1$ then $G$ has a spanning 2-ended tree or a Hamiltonian path.

A graph is called $K_{1,4}$-free if doesn’t contain $K_{1,4}$ as a induced sub graph where $K_{1,4}$ is complete bipartite graph illustrate in figure 1.2. If $k$ is a positive integer then $\sigma_k(G) = \min \{ \deg(U) : U$ is an independent set of $G$ with $|U| = k \}$, where $\deg(U) = \sum_{u \in U} \deg(u)$

![Figure 1.1: right side graph is a subdivision of left side](image)

**Theorem 1.2.** Every connected $K_{1,4}$-free graph $G$ with $\sigma_4 \geq |G| - 1$ contains a spanning tree with at most $k$ leaves.

2. Main results

If $e$ is an edge of graph $G$ with end vertices $u$ and $v$ then we define $N_e(u) = \{ x \in V(G); xu \in E(G), x \neq v \}$ and so $N_e(v) = \{ x \in V(G); xv \in E(G), x \neq u \}$.

**Theorem 2.1.** Suppose $G$ is a connected graph and has a spanning $k$-end tree ($k \in \mathbb{N}, k \geq 2$), and $|V(G)| > k + 1$. Then there exist an edge $e$ such $G/e$ has a spanning $k$-end tree.

At theorem 2.1 we cannot choose an arbitrary edge and make contraction on graph with that edge to get the theorem 2.1 result. For example, if we consider the graph in figure 2.1 and $e=py$ then it has a
spanning 3-end tree but $G/e$ has just a spanning 4-end tree.

\[ G : \]

\[ x \quad y \quad z \quad w \]

\[ p \]

\[ v \quad u \]

\[ \text{Figure 2.1} \]

Proof. Suppose $T$ is a spanning $k$-end tree of $G$, put:
\[ A = \{ x \in V(G); \deg_T(x) \neq 1 \} \] if $|A| \leq 1$ then because $|V(G)| > k+1$ so the number of vertices with degree one in $T$ is greater than $k$, and this is contradiction. So $|A|>1$, now choose two different vertices $u, v \in A$, in the unique path $uTv$ if consider $uu^-$ then $T/uu^-$ is a spanning tree of $G/uu^-$ with $k$ leaves.

Theorem 2.2. Consider a connected graph $G$ that has a spanning $k$-end tree with $k \geq 3$ then there exist a sequence $e_1, e_2, \ldots, e_m$ of edges in $G(m \in \mathbb{N})$ such, if put $G_1 = G/e_1$ and $G_i = G_{i-1}/e_i (i = 2, \ldots, m)$ then $G_m$ has a spanning $k-1$-end tree.

Proof. Suppose $T$ is a spanning $k$-end tree of $G$. We choose a vertex $v$ of degree one in $T$, because $k \geq 3$ there exist vertex or vertices with degree greater than 2 in $T$, now we choose one of them with minimum distance from $v$ and call it $w$. Consider $vv_1v_2\ldots v_{m-1}w$ as the unique path from $v$ to $w$ in $T$ and put $e_1 = vv_1, e_2 = v_1v_2, \ldots, e_m = v_{m-1}w$.

It is obvious every cycle $C_n(n \geq 2)$ has a spanning 2-end tree and if in cycle $C_n$ with $n \geq 3$ we contract an edge then we have $C_{n-1}$. It is interesting to know for which edges of a graph $G$ if we contract each of them, for example $e$, then for the minimum $k$ such that $G/e$ has a spanning $k$-end tree, $G$ also has a spanning $k$-end tree. In cycle $C_n$ for every edge $e = uv$ have $|N_e(u) - N_e(v)| \leq 1$ and $|N_e(v) - N_e(u)| \leq 1$.

At the following theorem we prove that, if these two inequalities hold in a graph then we can conclude our above ideal result.

Theorem 2.3. Suppose in a connected graph $G$ for an edge $e$ with end vertices $u$ and $v$ we have $|N_e(u) - N_e(v)| \leq 1$ and $|N_e(v) - N_e(u)| \leq 1$. If $G/e$ has a spanning $k$-end tree then $G$ has a spanning $k$-end tree.
Proof. Suppose $T$ is a spanning $k$-end tree of $G/e$ and $w$ is the vertex of $G/e$ that produced by contracting edge $e$ by identifying vertices $v$ and $u$. If $\deg_T w = 1$ and $wy \in E(T)$ then $vy \in E(G)$ or $uy \in E(G)$, if $vy \in E(G)$ then we make a subdivision of $T$ with a new vertex (called $v$) on edge $wy$ and rename $w$ to $u$, then this subdivision is a spanning $k$-end tree of $G$, and if $uy \in E(G)$ do similar.

If $\deg_T w > 1$ then at least one of it’s $|N_T(w)|$ adjacent vertices is adjacent to $v$ in $G$ and at least one of them is adjacent to $u$ in $G$. Now we replace the $w$ in $T$ with the edge $e$ and consider $N_T(w)$, we can choose $x_1, x_2 \in N_T(w)$ such $x_1 v \in E(G)$ and $x_2 u \in E(G)$ and draw these two edges and for other vertices in $N_T(w)$ we connect each one to just one of $u$ and $v$ such that they are adjacent in $G$. Now we have a spanning $k$-end tree of $G$.

\[\square\]

Corollary 2.4. \textit{Suppose a graph $G$ has a spanning $k$-end tree such that $k$ is as minimal as possible, then there is no edge $e$ such $G/e$ has a spanning $p$-ended tree, where $p < k - 1$.}

Proof. If $T$ is a spanning $p$-ended tree of $G/e$ where $p < k - 1$ then like proof of Theorem 2.3 if $w$ is the vertex of $G/e$ produced by contraction on edge $e$ with end vertices $u$ and $v$, if we replace $w$ with edge $e$ and each vertex in $N_T(w)$ connect to just one of $u$ and $v$ such they are adjacent in $G$ the the new graph is a spanning $p + 1$-ended tree of $G$ and this contradicts with minimality of $k$. \[\square\]

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