Extending Partial Representations of Circular-Arc Graphs

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Abstract

The partial representation extension problem generalizes the recognition problem for classes of graphs defined in terms of vertex representations. We exhibit circular-arc graphs as the first example of a graph class where the recognition is polynomially solvable while the representation extension problem is \( \text{NP}\)-complete. In this setting, several arcs are predrawn and we ask whether this partial representation can be completed.

We complement this hardness argument with tractability results of the representation extension problem on various subclasses of circular-arc graphs, most notably on all variants of Helly circular-arc graphs. In particular, we give linear-time algorithms for extending normal proper Helly and proper Helly representations. For normal Helly circular-arc representations we give an \( O(n^3) \)-time algorithm.

Surprisingly, for Helly representations, the complexity hinges on the seemingly irrelevant detail of whether the predrawn arcs have distinct or non-distinct endpoints: In the former case the previous algorithm can be extended, whereas the latter case turns out to be \( \text{NP}\)-complete. We also prove that representation extension problem of unit circular-arc graphs is \( \text{NP}\)-complete.

1 Introduction

An intersection representation \( \mathcal{R} \) of a graph \( G \) is a collection of sets \( \{R(v) : v \in V(G)\} \) such that \( R(u) \cap R(v) \neq \emptyset \) if and only if \( uv \in E(G) \). Important classes of graphs are obtained by restricting the sets \( R(v) \) to some specific geometric objects. In an interval representation of a graph, each set \( R(v) \) is a closed interval of the real line; and in a circular-arc representation, the sets \( R(v) \) are closed arcs of a circle; see Fig. 1. A graph is an interval graph if it admit an interval representation and it is a circular-arc graph if it admits a circular-arc representation. We also denote the corresponding classes of graphs by INT and CA, respectively.

In many cases, the availability of a geometric representation makes computational problems tractable that are otherwise \( \text{NP}\)-complete. For example, maximum clique can be solved in polynomial time for both interval graphs and circular-arc graphs. Another example is the coloring problem, which can be solved in polynomial time for interval graphs but remains \( \text{NP}\)-complete for circular-arc graphs [10].

A key problem in the study of geometric intersection graphs is the recognition problem, which asks whether a given graph has a specific type of intersection representation. It is a classic result

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that interval graphs can be recognized in linear time. For circular-arc graphs the first polynomial-time recognition algorithm was given by Tucker [33]. McConnell gave a linear-time recognition algorithm [27].

In this paper, we are interested in a generalization of the recognition problem. For a class \( \mathcal{X} \) of intersection representations, the partial representation extension problem for \( \mathcal{X} \) (RepExt(\( \mathcal{X} \)) for short) is defined as follows. In addition to a graph \( G \), the input consists of a partial representation \( \mathcal{R}' \) that is a representation of an induced subgraph \( G' \) of \( G \). The question is whether there exists a representation \( \mathcal{R} \in \mathcal{X} \) of \( G \) that extends \( \mathcal{R}' \) in the sense that \( R(u) = R'(u) \) for all \( u \in V(G') \).

The recognition problem is the special case where the partial representation is empty. The partial representation extension problem has been recently studied for many different classes of intersection graphs, e.g., interval graphs [21], proper/unit interval graphs [18], function and permutation graphs [16], circle graphs [4], chordal graphs [19], and trapezoid graphs [22]. Related extension problems have also been considered, e.g., for planar topological [1, 14] and straight-line [28] drawings, for contact representations [3], and rectangular duals [5].

In many cases, the key to solving the partial representation extension problem is to understand the structure of all possible representations. For interval representations, the basis for this is the characterization of Fulkerson and Gross [7], which establishes a bijection between the combinatorially distinct interval representations of a graph \( G \) on the one hand and the linear orderings \( \preceq \) of the maximal cliques of \( G \) where for each vertex \( v \) the cliques containing \( v \) appear consecutively in \( \preceq \) on the other hand. This not only forms the basis for the linear-time algorithm using PQ-trees by Booth and Lueker [2], but also shows that a PQ-tree can compactly store the set of all possible interval representations of a graph. The partial representation problem for interval graphs can be solved efficiently by searching this set for one that is compatible with the given partial representation.

Despite the fact that circular-arc graphs straightforwardly generalize interval graphs, the structure of their representations is much less understood. It is not clear whether there exists a way to compactly represent the structure of all representations of a circular-arc graph. There are two structural obstructions to this aim. First, in contrast to interval graphs, it may happen that two arcs have disconnected intersection, namely in the case when their union covers the entire circle. Secondly, intervals of the real line satisfy the Helly property: if any pair of sets in a set system intersects, then the intersection of the entire set system is non-empty. Consequently, the maximal cliques of interval graphs can be associated to distinct points of the line and also the number of maximal cliques in an interval graph is linear in the number of its vertices. In contrary, arcs of a circle do not necessarily satisfy the Helly property and indeed the number of maximal cliques can be exponential. The complement of a perfect matching \( nK_2 \) is an example of this phenomenon, see Fig. 1b.

To capture the above properties, that may have substantial impact on explorations of circular-

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**Figure 1**: (a) The graph \( 3K_2 \) and (b) its circular-arc representation. (c) A non-Helly representation of \( K_4 \).
arc graphs, the following specific subclasses of circular-arc graphs have been defined and intensively studied [31, 11, 26, 24]:

- **Normal circular-arc graphs (NCA)** are circular-arc graphs that have an intersection representation in which the intersection of any two arcs is either empty or connected.
- **Helly circular-arc graphs (HCA)** have an intersection representation that satisfies the Helly property, i.e., there are no \( k \geq 3 \) pairwise intersecting arcs without a point in common.
- **Proper circular-arc graphs (PCA)** are circular-arc graphs that have an intersection representation in which no arc properly contains another.
- **Unit circular-arc graphs (UCA)** are circular-arc graphs with an intersection representation in which every arc has a unit length.

The above properties can be combined together in the sense that a single representation shall satisfy more properties simultaneously, e.g., **Proper Helly circular-arc graphs (PHCA)** are circular-arc graphs with an intersection representation that is both proper and Helly [23]. This is stronger than requiring that a graph is a proper circular-arc graph as well as a Helly circular-arc graph (with each property guaranteed by a different representation), i.e., \( \text{PHCA} \subseteq \text{PCA} \cap \text{HCA} \).

Analogously, since \( C_4 \) has a unique representation, the wheel \( W_4 \) is a graph with a Helly representation (the universal vertex covers all four clique points) or a normal representation (it covers three clique points) but not normal Helly representation. Thus also \( \text{NHCA} \subseteq \text{NCA} \cap \text{HCA} \).

Moreover, Tucker [32] proved that every representation of a proper (Helly) circular-arc graph that is not normal can be transformed into a normal representation. Hence, the following graph classes coincide \( \text{PCA} = \text{NPCA} \) and \( \text{PHCA} = \text{NPHCA} \). Fig. 2a shows inclusions between the defined graph classes.

We use an analogous notation for the classes of possible representations, i.e., for \( X \subseteq \{N,P,H\} \) the symbol \( X\text{CAR} \) for the class of all \( X\text{CA} \) representations, see Fig. 2b. We note that whether a graph \( G \) with a partial representation \( \mathcal{R}' \) admits an extension depends crucially on the class of allowed representations, as illustrated by the example of \( W_4 \) above.

Our results. While for many classes efficient algorithms for the representation extension problem have been found, the problem has been open for circular arc graphs for nine years [17]. We prove that \( \text{RepExt}(\text{CAR}) \) is NP-hard. To the best of our knowledge, it is the first known representation class for which the extension problem is NP-hard while the recognition problem is in P. Our reduction also works for \( \text{RepExt}(\text{HCA}) \).

**Theorem 1.** The problems \( \text{RepExt}(\text{HCA}) \) and \( \text{RepExt}(\text{CAR}) \) are NP-hard. \( \text{RepExt}(\text{CAR}) \) is also NP-hard if the predrawn arcs have pairwise distinct endpoints.

We complement this result by showing tractability for several subclasses, including all Helly variants; see Figure 2b. Linear-time algorithms for recognizing Helly circular-arc graphs [25, 15] use McConnell’s [27] algorithm to construct a circular-arc representation and transform it to a Helly circular-arc representation. This cannot be exploited in the case of partial representation extension.

Deng et al. [6] and Lin et al. [25] characterize proper and proper Helly circular-arc representations in terms of vertex orderings of the graph. They show that these orderings are unique under certain conditions. Building on these results, we prove the following two theorems.

**Theorem 2.** The problem \( \text{RepExt}(\text{NPHCA}) \) can be solved in linear time.

**Theorem 3.** The problem \( \text{RepExt}(\text{PHCA}) \) can be solved in linear time.
Recall that in the case of interval graphs, PQ-trees can be used to capture all plausible linear orderings of the maximal cliques. Klavík et al. [18] use this to solve RepExt for interval representation by determining an order that is represented by the PQ-tree and that extends a partial order that is derived from the partial representation.

The fact that Gavril [12] shows that a graph $G$ is a Helly circular-arc graph if and only if there exists a cyclic ordering $\prec$ of its maximal cliques such that for every vertex $v$, the maximal cliques containing $v$ appear consecutively in $\prec$ and that Hsu and McConnell [13] use PC-trees to capture all plausible cyclic orderings of the maximal cliques of a Helly circular-arc graph makes it tempting to simply apply the same techniques to generalize the algorithm of Klavík et al. However, this cannot be straightforwardly applied for two reasons. First, the clique ordering carries little information about whether a representation is normal or not, and, even more severely, extending a partial cyclic ordering is $\mathbf{NP}$-complete, even without requiring that the order be additionally represented by some given PC-tree [8]. We overcome this by working with suitably linearized partial orders to show the following results.

**Theorem 4.** The problem RepExt(NHCA) can be solved in $O(n^3)$ time.

**Theorem 5.** The problem RepExt(HCAR) can be solved in $O(n^3)$ time if the partial representation consists of arcs with pairwise distinct endpoints.

It follows that Helly representations used in our reduction essentially involve arcs that share endpoints. This is surprising since non-degeneracy assumptions like this are often made without much consideration of the impact on the problem when working with graph representations.

The bottleneck of our NHCA-algorithms is the testing of the consecutivity constraints for all universal pairs of vertices. A closer exploration of the structure of the set of universal pairs may yield improvements of the running time upper bound.

Finally, we show that involving the most tight constraints on arc lengths, the problem becomes again computationally difficult.

**Theorem 6.** The problem RepExt(UCAR) is $\mathbf{NP}$-complete.

The $\mathbf{NP}$-hardness of Theorems 1 and 6 follows by a reduction from the 3-Partition problem [9]. For the unit case, the membership in $\mathbf{NP}$ can be seen by a linear programming argument.
2 Preliminaries

Cyclic order. Let \( \leq = v_0, \ldots, v_{n-1} \) and \( \leq' = u_0, \ldots, u_{n-1} \) be two linear orders on a finite set \( S \). We say that \( \leq \) and \( \leq' \) are cyclically equivalent if there is \( k \in \{0, \ldots, n-1\} \) such that \( v_i = u_{i+k} \), where the addition is modulo \( n \). Clearly, this is an equivalence relation on the set of all linear orders on \( S \). A cyclic order \( \leq \) on \( S \) is an equivalence class of this relation. For a linear order \( \leq \), we denote the corresponding cyclic ordering by \( [\leq] \).

Every linear order \( \leq \) on \( S \) induces a linear order \( \leq' \) on a subset \( S' \subseteq S \) by omitting all ordered pairs in which the elements of \( S \setminus S' \) occur. In this case we say that \( \leq \) extends \( \leq' \) and similarly that the cyclic order \( [\leq] \) extends \( [\leq'] \).

Circular-arc representations. For any circular-arc representation \( \mathcal{R} \) and each connected component \( C \) of a graph \( G \) the set \( \bigcup_{v \in V(C)} R(v) \) is a connected subset of the circle. Therefore, if \( G \) is a disconnected circular-arc graph, then each connected component of \( G \) has to be an interval graph. These cases can be treated with the corresponding algorithms for interval graphs of \[18, 21\]. Hence without loss of generality we restrict ourselves to connected graphs in this paper.

Let \( \mathcal{R} \) be a representation of a circular-arc graph \( G \). For a vertex \( v \) of \( G \), we call the tail \( R(v)_t \) and the head \( R(v)_h \) the two endpoints of \( R(v) \). We use the convention of traversing the arc from the tail to the head in the clockwise direction along the circle. We denote such an arc as \( R(v) = [R(v)_t, R(v)_h] \), and its complement \( (R(v)_h, R(v)_t) \) as \( R(v)^c \).

Let \( \mathcal{R} \) be a Helly representation of a circular-arc graph \( G \). Denote by \( C \) the set of maximal cliques of \( G \). We assign every maximal clique \( C \in C \) a unique point \( cp(C) \in \bigcap_{v \in C} R(v) \) and call it the clique-point of \( C \).

Lemma 2.1 (Gavril [12]). A graph \( G \) is a Helly circular-arc graph if and only if there exists a cyclic ordering \( \leq \) of its maximal cliques such that for every vertex \( v \), the maximal cliques containing \( v \) appear consecutively in \( \leq \).

Note that if we distribute clique points on the circle according to a cyclic ordering \( \leq \) of Lemma 2.1, then a representation \( \mathcal{R} \) of \( G \) can be obtained by choosing for each vertex \( v \) an arc \( R(v) \) that covers exactly the clique-points \( v \) belongs to.

PC-Trees and the Reordering Problem. A PC-tree \( T \) on a set \( L \) of leaves is a tree whose inner nodes have one of two types: P-nodes and C-nodes. The neighbors around a P-node can be permuted arbitrarily, whereas the order of the neighbors of a C-node is fixed up to reversal. In this way a PC-tree represents a set of cyclic orderings of its leaf set \( L \). The usefulness of PC-trees derives from the fact that they can represent cyclic orderings subject to consecutivity constraints. Namely, given a set \( L \) and sets \( X_1, \ldots, X_r \subseteq L \), a PC-tree that represents precisely those cyclic orderings of \( L \) where each of the subsets \( X_1, \ldots, X_r \) is consecutive can be computed in \( O(|L| + \sum_{i=1}^{r} |X_i|) \) time [14]. In our setting, in the spirit of Lemma 2.1 the leaf set \( L \) will always be the set of all maximal cliques of a Helly circular-arc graph.

We make use of the following reordering problem, which can be solved analogously to the topological sorting of PQ-trees [18]. The input consists of a PC-tree \( T \), a leaf \( u \) of \( T \), and a partial ordering (not cyclic) \( \leq \) of the remaining leaves \( L' = L \setminus \{u\} \) of \( T \). The question is whether \( T \) represents a cyclic order \( \leq' \) which induces a linear extension of \( \leq \) on \( L' \). If this is the case, \( T \) is called compatible with \( \leq \) with respect to \( u \). We denote an instance of this problem by \textsc{Reorder}(\( T, u, \leq \)).

Lemma 2.2. An instance \textsc{Reorder}(\( T, u, \leq \)) can be solved in time \( O(\ell + c) \), where \( \ell \) is the number of leaves of \( T \) and \( c \) is the number of comparable pairs in the partial ordering \( \leq \).

For a detailed proof see Section 6.
3 Complexity

Sketch of proof for Theorem 1. We first reduce the strongly NP-complete problem 3-PARTITION to RepExt(HCAR). Figure 3 illustrates the proof. Figure 3a shows a representation with four universal vertices (blue). The green dots indicate the endpoints that are shared by these arcs. The red blocker vertex covers all other points of the circle shared by all universal vertices. The key insight is that each vertex that is not adjacent to the blocker vertex must not be contained in the complement of a universal vertex and thus contain at least one of the green points. Figure 3b shows an instance resulting from our reduction, and Figure 3c shows a corresponding solution from the instance \{1, 1, 2, 2, 3, 3, 4\}.

For RepExt(CAR), the same construction works. However, we can avoid shared endpoints in the partial representation with a simple modification. Namely, we slightly shorten the arc for each universal vertex; see Figure 3d. Then we have for each former green dot a green area between the corresponding arc ends. Each vertex that is not adjacent to the blocker vertex must now contain a green area. Note that in every solution each leaf of a star in G must contain exactly one green area. It thus violates the Helly property with the two universal arcs ending there.

For details see Section 7.1.

4 Normal Proper Helly Circular-Arc Graphs

We show how to extend partial representations of normal proper Helly circular-arc graphs in linear time. To do this, we give a characterization of all partial representations that are extendable. This generalizes the characterization of the extendable partial representations of proper interval graphs [18, Lemma 2.4]. We first simplify the possible instances by reducing the number of vertices with the same neighborhood as follows.

Twin vertices. Recall that vertices \( u, v \in V(G) \) are called twins if \( N[u] = N[v] \). It is possible to find the equivalence classes of twin vertices in linear time [30]. If vertices \( u \) and \( v \) are twins and if either \( u \) is not predrawn or if both \( u \) and \( v \) are predrawn with the same arc, then we may remove \( u \) and in the final representation we can set \( R(u) = R(v) \). This allows us to assume that each twin class either consists of a single vertex, which is not predrawn, or it consists only of predrawn vertices that are represented by distinct arcs.
Consecutive orderings of vertices. Let \( \prec = [v_0, \ldots, v_{n-1}] \) be a cyclic ordering of the vertices of some graph \( G \). We say that \( \prec \) is consecutive if the closed neighborhood \( N[v] \) of each \( v \in V(G) \) is consecutive in \( \prec \). Note that \( N[v_i] \) is consecutive in \( \prec \) if there exist positive integers \( a \) and \( b \) such that \( a + b \leq n \) and
\[
N[v_i] = \{v_{i-a}, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_{i+b}\},
\]
where the addition is performed modulo \( n \).

Roberts characterized connected proper interval graphs in terms of consecutive linear orderings [29] that are unique up to permuting twin vertices and a complete reversal. The following lemma, proved in slightly different terminology by Deng et al. as Corollaries 2.7 and 2.9. in [6] give a characterization of proper circular-arc graphs by cyclic ordering:

**Lemma 4.1.** A graph \( G \) belongs to PCA if and only if \( V(G) \) has a consecutive cyclic ordering.

They also showed that this ordering is unique up to permuting twin vertices and a complete reversal only in the case when the complement is connected or nonbipartite. For example, the complement of a perfect matching \( K_{2,2,\ldots,2} \) on at least 6 vertices allows several distinct consecutive orderings, like \([1,2,3,4,5,6]\) and \([1,3,2,4,6,5]\) for the graph depicted in Fig. [1]. For normal proper Helly circular-arc graphs, the following strengthening of Lemma 4.1 follows from the results summarized in Lin et al. [24]:

**Lemma 4.2.** Every NPHCA graph \( G \) has a unique consecutive cyclic ordering of \( V(G) \), up to permuting twins and a reversal. Such an ordering can be obtained in \( O(m + n) \) time, where \( n \) and \( m \) stand for the number of vertices and edges, respectively.

Note that each such consecutive cyclic ordering of vertices corresponds to the cyclic order on tails of the arcs in some normal proper Helly representation.

Characterization of extendable instances. Let \( G \in \text{NPHCA} \) and let \( R' \) be a partial representation of \( G \). By the discussion above, we may assume that if \( u \) and \( v \) are twins, then both are predrawn with \( R(u) \neq R(v) \).

A constraint specific to the representation extension of proper circular-arc graphs is imposed by any pair of touching arcs, for which the tail of one coincides with the head of the other, i.e., \( R'(v_i)_h = R'(v_j)_t \). In this situation, it is impossible to place another tail or head between \( R'(v_i)_h \) and \( R'(v_j)_t \) – in fact on this point of the circle – as the resulting representation would not be proper.

We use this fact as well as the uniqueness of the ordering of Lemma 4.2 to characterize all instances that allow a representation extension; see Section 8.

**Lemma 4.3.** A partial normal proper Helly circular-arc representation \( R' \) of a connected graph \( G \), where all twins are predrawn by distinct arcs, is extendable if and only if \( V(G) \) has a consecutive cyclic ordering \( \prec = [v_0, \ldots, v_{n-1}] \) satisfying:

1. The ordering \( \prec \) extends the cyclic ordering \( \prec' \) of the predrawn vertices corresponding to the clockwise cyclic ordering of the tails of their predrawn arcs.
2. If \( R'(v_i)_h = R'(v_j)_t \), then \( N[v_i] = \{v_a, \ldots, v_i, \ldots, v_b\} \) and \( N[v_j] = \{v_i, \ldots, v_j, \ldots, v_b\} \), for some non-negative integers \( a, b \).

We are ready to prove that \( \text{RepExt}(\text{NPHCAR}) \) can be solved in time \( O(n + m) \), where \( n \) is the number of vertices and \( m \) is the number of edges of the given graph \( G \).

**Theorem 2.** The problem \( \text{RepExt}(\text{NPHCAR}) \) can be solved in linear time.
Data: A graph $G$ and a partial representation $R'$.

Result: A NPHCA representation $R$ of $G$ extending $R'$ or a message that it does not exist.

1 begin
2 Determine the twin classes and prune not predrawn twins;
3 Find the ordering $<$ by Lemma 4.2;
4 Transform $<$ to $\prec$ by reversal and permutation of predrawn twins;
5 forall arcs $R'(v_i)$ and $R'(v_j)$ such that $R'(v_i)_h = R'(v_j)_t$ do
6 $\text{if } N[v_i] \cap N[v_j] \neq \{v_i, \ldots, v_j\}$ then return $R'$ has no extension.;
7 Construct the cyclic order $(A, \prec)$ from the proof of Lemma 4.3;
8 Convert $(A, \prec)$ into the representation $R$ as described in Lemma 4.3;
9 Replicate the pruned twins into $R$ from the non-pruned ones;
10 return $R$

Algorithm 1: The algorithm for the RepExt(NPHCAR) problem.

Proof. The procedure is given as Algorithm 1. Its correctness follows directly from Lemma 4.3.

For the computational complexity note that:
- Step 3 could be achieved in $O(n + m)$ time by Lemma 4.2.
- Step 4. To check whether reversal is necessary it suffices to check two consecutive vertices belonging to distinct twin classes. The correct ordering of the predrawn twins can be achieved by a single traversal of $<$.\footnote{Note: The proof details are not shown here.}
- The test at line 6 can be done in constant time since it suffices to check whether $v_i - 1 v_j \in E(G)$ or $v_i v_{j+1} \in E(G)$.
- Step 8. Following the proof of Lemma 4.3, we perform a single traversal of the initial order $[t_0, \ldots, t_{n-1}]$ and insert the elements $h_0, \ldots, h_{n-1}$ appropriately according to $\prec$.
- For step 9, the positions of heads and tails of non-predrawn arcs can be determined in $O(n)$ by a single traversal of $(A, \prec)$.
- At line 10, each pruned twin requires only constant time to be replicated.

For Theorem 3, which deals with RepExt(PHCA), note that even though all proper Helly circular-arc graphs allow also a normal proper representation, we cannot reduce RepExt(PHCAR) to RepExt(NPHCAR) directly, as the given partial representation need not to be normal.

However, the existence of a proper Helly representation extending a pair of arcs in a not-normal position imposes strong conditions on the structure of $G$: after pruning all universal vertices only two disjoint cliques remain. Such instances can be solved in linear time. For details see Section 9.

5 Normal Helly Circular-Arc Graphs

With the following lemma, we can avoid universal vertices since they allow considering instances of RepExt(NHCAR) as instances for the interval case.

Lemma 5.1. Let $G$ be a graph with a universal vertex $u$. Then for every normal Helly circular-arc representation of $G$ there exists a point on the circle that is contained in no arc.

Proof. Let $R$ be a normal Helly circular-arc representation of $G$. Assume that every point of the circle is contained in some arc. Without loss of generality, we may assume that $R(u)$ is not strictly
Definition 7. For a maximal clique $C \subseteq G$, we define sets

$$\text{Reg}^+(C) = \bigcap_{w \in \text{Pre}(C)} R'(w) \quad \text{and} \quad \text{Reg}^-(C) = \bigcup_{w \in C'} R'(w).$$

The region of $C$ is the set $\text{Reg}(C) = \text{Reg}^+(C) \setminus \text{Reg}^-(C)$.

We assume for the rest of this section that our graph contains no universal vertices.

Let $G = (V, E)$ be a graph. Then two adjacent vertices $u, v \in V$ form a universal pair if each vertex $w \in V$ is adjacent either to $u$ or to $v$.

Lemma 5.2. A graph $G$ without a universal vertex is a normal Helly circular-arc graph if and only if there exists a cyclic ordering $<$ of its maximal cliques such that

(i) for every vertex $v$, the maximal cliques containing $v$ appear consecutively in $<$.

(ii) for every universal pair $u, w$, the maximal cliques containing $u$ and $w$ appear consecutively in $<$.

Proof. If $G$ has a normal Helly circular-arc representation, then, due to the Helly property, we have for each maximal clique $C$ a clique point $cp(C)$ where the arcs of all vertices in $C$ intersect. For each vertex $v$ and for each universal pair $u, w$ the corresponding cliques are consecutive in $<$ since $R(v)$ and $R(u) \cap R(w)$ each are connected (the latter due to the normal property).

Next assume that we have a cyclic ordering $<$ of the maximal cliques of $G$ with properties (i) and (ii). We obtain a normal Helly circular-arc representation as follows; see Fig. 9. We first arrange the maximal clique points on the circle according to $<$. Then, for each vertex $v$, we define the arc $R(v)$ as the smallest arc that contains exactly the clique points of the maximal cliques that contain $v$. Note that $R(v)$ is well-defined since $v$ is not universal. This defines a circular-arc representation $R$ of $G$ since any two intersecting arcs share a clique point. Moreover, the existence of the clique points shows the Helly property. It remains to show that $R$ is normal.

Assume that there are two nodes $u, w$ such that $R(u) \cap R(w)$ is not connected. Then $u, w$ is a universal pair. Thus the cliques containing $u$ and $w$ are consecutive in $<$ which contradicts $R(u) \cap R(w)$ being not connected. Hence, the representation is normal. □

Extendable Partial Representations. We characterize all partial representations of a normal Helly circular-arc graph $G$ that are extendable. Let $R'$ be a partial representation of $G$ and let $C$ be a maximal clique of $G$. We define $\text{Pre}(C) = \{ R'(v) : v \in C \cap V(G') \}$ to be the predrawn arcs corresponding to the vertices in $C$.

The region of $C$ is the set $\text{Reg}(C) = \text{Reg}^+(C) \setminus \text{Reg}^-(C)$.
The set $\text{Reg}^+(C)$ means the set of possible locations of $\text{cp}(C)$ with respect to predrawn arcs, while $\text{Reg}^-(C)$ means the forbidden locations. Thus if the region $\text{Reg}(C)$ is empty for some clique $C \in C$, then the given partial representation is not extendable. We thus assume in the following that no region is empty. The following lemmas give some useful properties that hold for the regions of maximal cliques.

**Lemma 5.3.** For maximal cliques $C$ and $D$, we have either $\text{Reg}(C) \cap \text{Reg}(D) = \emptyset$, or $\text{Reg}(C) = \text{Reg}(D)$.

**Proof.** If $\text{Pre}(C) = \text{Pre}(D)$, then clearly $\text{Reg}(C) = \text{Reg}(D)$. So, we assume that $\text{Pre}(C) \neq \text{Pre}(D)$. We can further assume that there exists an arc $R' \in \text{Pre}(C) \setminus \text{Pre}(D)$. Since the clique point $\text{cp}(C)$ must be placed on the point of $R'$ and the clique point $\text{cp}(D)$ cannot be placed on any point of $R'$, we have that $\text{Reg}(C)$ and $\text{Reg}(D)$ are disjoint. \hfill \square

For every maximal clique $C$, we call the connected components of $\text{Reg}(C)$ *islands* and the connected components of its complement $\text{Reg}^c(C)$ we call *gaps* of $C$. We say an island and a gap are neighboring, if they share an endpoint (where one end is open and the other is closed.) Note that every island has two neighboring gaps and every gap has two neighboring islands.

Observe that if two maximal cliques $C, D \in C$ satisfy $\text{Pre}(C) = \text{Pre}(D)$ then $\text{Reg}(C) = \text{Reg}(D)$ by definition. In the other case we obtain the following relationship:

**Lemma 5.4.** Let $C$ and $D$ be two maximal cliques with $\text{Pre}(C) \neq \text{Pre}(D)$. Then $D$ has a gap $J$ with $\text{Reg}(C) \subseteq J$.

**Proof.** From $\text{Pre}(C) \neq \text{Pre}(D)$ we obtain $\text{Reg}(C) \subseteq \text{Reg}^c(D)$. Assume that there are two gaps $J_1, J_2$ of $D$ that contain points $j_1, j_2 \in \text{Reg}(C)$. Let $I_1, I_2$ be the islands of $D$ neighboring $J_1$. Let $i_1 \in I_1, i_2 \in I_2$. For $v \in \text{Pre}(C)$ it holds that $j_1, j_2 \in R'(v)$ and thus $i_1 \in R'(v)$ or $i_2 \in R'(v)$ since $i_1, i_2$ separate $j_1, j_2$ on the circle; see Fig. 10. This implies $\text{Pre}(C) \subseteq \{v \in V(G') : i_1 \in R'(v) \lor i_2 \in R'(v)\} = \{v \in V(G') : i_1 \in R'(v)\} \cup \{v \in V(G') : i_2 \in R'(v)\} = \text{Pre}(D)$ as both sets in the union are equal to $\text{Pre}(D)$. Likewise, we obtain $\text{Pre}(D) \subseteq \text{Pre}(C)$. This contradicts $\text{Pre}(C) \neq \text{Pre}(D)$. Hence, $\text{Reg}(C)$ is contained in a single gap of $D$. \hfill \square

This means that for any two maximal cliques $C$ and $D$, the clique point $\text{cp}(C)$ must be placed in a given gap of $D$. We obtain additional consecutivity constraints. For every gap $J$, we define the set $S_J = \{C \in C \mid \text{Reg}(C) \cap J \neq \emptyset\} = \{C \in C \mid \text{Reg}(C) \subseteq J\}$. Recall that we assumed that no region is empty.

**Lemma 5.5.** Let $\mathcal{R}$ be a NHCA extension of $\mathcal{R}'$. Let $J$ be a gap of some maximal clique $D$. Let $\leq$ be the clique ordering derived from $\mathcal{R}$. Then $\leq$ is the set $S_J$ is consecutive.

**Proof.** Direct consequence of Lemma 5.4 since all clique points of that set must be placed on $J$ and all other clique points must be placed on its complement $J^c$. \hfill \square

**Lemma 5.6.** There exists a maximal clique $D$ with a single island.

**Proof.** Let $D$ be a maximal clique with $|\text{Pre}(D)|$ maximal. Then $\text{Reg}^+(D)$ and $\text{Reg}^-(D)$ are disjoint as otherwise there would exist a point $p$ in $\text{Reg}^+(D) \cap \text{Reg}^-(D)$ with $\text{Pre}(D) \subseteq \{v \in V(G') : p \in R'(v)\}$. This implies the existence of a (maximal) clique $C$ in $G$ with $\text{Pre}(D) \subseteq \text{Pre}(C)$ in contradiction to the choice of $D$.

Since $\mathcal{R}'$ is NHCA, this yields that $\text{Reg}(D) = \text{Reg}^+(D)$ is connected. In other words, $D$ has a single island. \hfill \square
We obtain properties 2, 3 and 4. Property 1 is necessary, since no two clique points can be placed at the same point. We define a partial order $\prec$ on the maximal cliques $\mathcal{C}$ such that for each $C \in \mathcal{C} \setminus \{D\}$ we have $D \prec C$ and for each $C, C' \in \mathcal{C} \setminus \{D\}$ satisfying $\forall p \in \text{Reg}(C), q \in \text{Reg}(C')$: $p < q$ we set $C \prec C'$. Note that every linearization of a clique ordering of an extension of $\mathcal{R}'$ that starts with $D$ extends $\prec$. For any vertex $v$ let $M_v$ denote the set of maximal cliques containing $v$.

**Theorem 8.** Let $G$ be a graph without universal vertices and let $\mathcal{R}'$ be a partial normal Helly circular-arc representation of $G$. There exists a normal Helly circular-arc representation of $G$ that extends $\mathcal{R}'$ if and only if there exists a linear extension $\prec$ of $\prec$ such that

1. For any pair of distinct maximal cliques $C \neq C'$ with $\text{Reg}(C) = \text{Reg}(C')$, the region $\text{Reg}(C)$ is not a single point.
2. For every vertex $v$, the set $M_v$ is consecutive in $[\prec]$.
3. For every universal pair $u, w$, the set $M_u \cap M_w$ is consecutive in $[\prec]$.
4. For every gap $J$ of some $C \in \mathcal{C}$, the set $S_J$ is consecutive in $[\prec]$.

**Proof.** We first show that, if there is an NHCA-extension of $\mathcal{R}'$, then these properties are satisfied. We obtain $\prec$ as the linearization of a clique ordering of an extension of $\mathcal{R}'$ starting with $D$ where the clique point of $D$ is $p_D$. By the construction of $\prec$, the order $\prec$ is a linear extension of $\prec$. By Lemmas 5.2 and 5.5, we obtain properties 2, 3 and 4. Property 1 is necessary, since no two clique points can be placed at the same point.

For the opposite implication, let $\leq = C_1, C_2, \ldots, C_k$ be a linear extension of $\prec$ such that properties 1, 2, 3 and 4 are satisfied. We show that each $C_i \in \mathcal{C}$ can be assigned its clique point $\text{cp}(C_i) \in \text{Reg}(C_i)$ such that $\text{cp}(C_j) < \text{cp}(C_i)$ whenever $j < i$.

Let $\varepsilon > 0$ be the $\frac{1}{2n+1}$-fraction of the length of the shortest nontrivial island. This allows to draw all new endpoints at distance at least $\varepsilon$ but still within any chosen island or side of island $\text{Reg}(D)$. For $C_1 = D$ we place $\text{cp}(C_1)$ on $p_D$. In a greedy way, when the location of the clique points $\text{cp}(C_1), \ldots, \text{cp}(C_{i-1})$ is settled, we determine the set $P$ of feasible points for $\text{cp}(C_i)$ that is $P = \text{Reg}(C_i) \cap \{p: p > \text{cp}(C_{i-1}) + \varepsilon\}$. If $P$ has minimum, we place $\text{cp}(C_i)$ there, otherwise we put $\text{cp}(C_i)$ at $\inf(P) + \varepsilon$. We argue that such choice always exists.

Assume for a contradiction that $C_i$ is the first maximal clique in the order $\prec$ whose clique point $\text{cp}(C_i)$ cannot be properly placed. Note that a clique $C \neq C_i$ can only have an island $I_C$ consisting of a single point if we have $I_C = \text{Reg}(C)$, since $\text{Reg}^+$ is a closed arc and all islands separated by gaps within $\text{Reg}^+$ have an open end. Hence, by the choice of $\varepsilon$, we only place a clique $C$ at the very last point of $\text{Reg}(C)$, if $\text{Reg}(C)$ consists of a single point. With property 4 this can not happen if $\text{Reg}(C) = \text{Reg}(C_i)$. Therefore, one clique point must be placed to the right of $\text{Reg}(C_i)$ before placing $\text{cp}(C_i)$. We identify the first maximal clique $C_j, j < i$ that is placed to the right of all points in $\text{Reg}(C_i)$. Since $\text{cp}(C_j) \notin \text{Reg}(C_i)$, we have that $\text{Pre}(C_i) \neq \text{Pre}(C_j)$. By Lemma 5.4 the maximal clique $C_j$ has a gap $J$ with $\text{Reg}(C_i) \subseteq J$, see Fig. 4.
Data: A graph $G$ and a partial representation $\mathcal{R}'$.  
Result: A NHCA representation $\mathcal{R}$ of $G$ extending $\mathcal{R}'$ or a message that none exists.

begin
1. if $G$ has a universal vertex then resolve $\operatorname{RepExt}(\text{INTR})$ instead;
   
   else
2. Determine the set of maximal cliques $C$ of $G$;
3. foreach $C \in C$ do
4.   Determine $\operatorname{Reg}(C)$;
5.   if $\operatorname{Reg}(C) = \emptyset$ then return $\mathcal{R}'$ has no extension;
6.   $\varepsilon := \frac{1}{2m+1} \min \{|I|, I \text{ is a non-trivial island}|$;
7.   Find $C_1 \in C$ with a single island;
8.   if such $C_1$ does not exist then return $\mathcal{R}'$ has no extension;
9.   Set $\operatorname{cp}(C_1)$ to the middle of $\operatorname{Reg}(C_1)$;
10. Determine the partial order $<$;
11. Build a PC tree $T$ on $C$ capturing the constraints stated in Theorem 8;
12. if such $T$ does not exist then return $\mathcal{R}'$ has no extension;
13. Solve $\operatorname{Reorder}(T, C_1, <)$ to get the order $C_1 < C_2 < \cdots < C_k$;
14. if such order $<$ does not exist then return $\mathcal{R}'$ has no extension;
15. for $i = 2$ to $k$ do
16.   $P := \operatorname{Reg}(C_i) \cap \{p : p > \operatorname{cp}(C_{i-1}) + \varepsilon\}$;
17.   if $\min P$ exists then $\operatorname{cp}(C_i) := \min P$;
18.   else $\operatorname{cp}(C_i) := \inf(P) + \varepsilon$;
19.   foreach $u \notin V(G')$ do draw $R(u)$ to cover exactly $\{\operatorname{cp}(C) : C \in M_u\}$;
20. return $\mathcal{R}$

Algorithm 2: The algorithm for the $\operatorname{RepExt}(\text{NHCAR})$ problem.

Consider the neighboring island $I$ of $C_j$ to the left of $J$. Since $\operatorname{cp}(C_j)$ was not placed on $I$, the clique point $\operatorname{cp}(C_{j-1})$ has been placed to the right of $I$. By the choice of $C_j$, we have that $\operatorname{cp}(C_{j-1})$ is not placed to the right of $J$ and thus $\operatorname{cp}(C_{j-1}) \in J$. Since $\operatorname{cp}(C_j)$ has been placed to the right of $\operatorname{Reg}(C_i)$ and thus to the right of $J$, we have $p_D \notin J$ and thus $D \notin S_J$. With $D < C_{j-1} < C_j < C_i$, where $C_{j-1}, C_i \in S_J$ and $D, C_j \notin S_J$ we get a contradiction with the property 4, since $S_J$ is not consecutive in $\langle \rangle$.

From the placement of clique points, we obtain a NHCA-extension of $\mathcal{R}'$ so that for every not yet represented vertex $u \notin V(G')$ we choose $R(u)$ to be the minimal arc containing exactly the clique points of the maximal cliques from $M_u$.

By the proof of Lemma 5.2 we obtain from properties 3, 4 that this results in an NHCA representation of $G$ extending $\mathcal{R}'$, since the maximal clique points are placed correctly with regards to both predrawn arcs and new arcs, and moreover, two arcs intersect if and only if they are predrawn or share a maximal clique.

Theorem 4. The problem $\operatorname{RepExt}(\text{NHCAR})$ can be solved in $O(n^3)$ time.

Proof. The correctness of Algorithm 2 follows from the already mentioned arguments. For the computational complexity of the more complex steps note that:

- Line 2: $\operatorname{RepExt}(\text{INTR})$ can be solved in linear time [20].
• Line 4: We run the linear time recognition algorithm for Helly circular-arc graphs \[25\] on \(G\) and read its at most \(n\) maximal cliques from any of its representation.

• Lines 5–8: The regions can be obtained in time \(O(n)\) by traversing the circle once. During the traversal two consecutive predrawn endpoints specify possible islands which can be assigned to appropriate maximal cliques.

• Line 12: The comparable pairs of the partial order \(\prec\) can also be determined during the traversal in Steps 5–8.

• Line 13: The \(O(n^2)\) constraints can be computed in \(O(n^3)\) time. The construction of the PC-tree from \[13\] also runs in \(O(n^3)\) time.

• Line 16: The \(\text{REORDER}(T, C_1, \prec)\) problem can be solved in \(O(n^2)\) time by Lemma 2.2.

In light of the hardness result from Theorem 1, it is unlikely that this result can be generalized to \(\text{REPEXT}(\text{HCAR})\). However, the hardness result for Helly representations crucially relies on predrawn arcs sharing endpoints. Indeed, if all predrawn arcs have distinct endpoints, the problem can be solved in a similar fashion.

**Theorem 5.** The problem \(\text{REPEXT}(\text{HCAR})\) can be solved in \(O(n^3)\) time if the partial representation consists of arcs with pairwise distinct endpoints.

**Proof sketch.** We characterize extendable HCA instances as in Theorem 8 (see Theorem 9) with Lemma 2.1 instead of Lemma 5.2. Since the predrawn arcs have pairwise distinct endpoints, we have no islands consisting of a single point. Note that Lemma 5.6 no longer applies and thus the placement of \(p_D\) cannot be chosen freely. Instead, observe that every maximal clique \(D\) in a Helly circular-arc representation of \(R'\) has a clique point \(\text{cp}(D)\) that can be chosen as \(p_D\). Thus, we choose an arbitrary clique as \(C_1\) and apply the remaining procedure for every island \(I\) of \(C_1\), choosing \(\text{cp}(C_1) \in I\). In contrast to our method for \(\text{REPEXT}(\text{NHCAR})\) we have no special procedure for universal vertices.

For details see Section 11.

### 6 Details for PC-trees and The Reordering Problem

**Lemma 2.2.** An instance \(\text{REORDER}(T, u, \prec)\) can be solved in time \(O(\ell + c)\), where \(\ell\) is the number of leaves of \(T\) and \(c\) is the number of comparable pairs in the partial ordering \(\prec\).

**Proof.** We root \(T\) by \(u\). We represent the ordering \(\prec\) by a digraph \(D\) having \(c\) edges. The algorithm reorders the nodes from the bottom to the root and modifies \(D\) by contractions. Once we finish reordering a subtree, it is never modified later. After reordering a subtree, the corresponding vertices in \(D\) are contracted. We process a node of \(T\) when all its subtrees are finished and the corresponding digraphs are contracted to single vertices. Note that since the tree \(T\) is now rooted, the equivalent transformations (i) and (ii) correspond to permuting the children of a P-node and reversing the children of a C-node.

For a P-node, we then permute its children according to any topological sort of the subdigraph \(D'\) induced by the vertices corresponding to the children of the P-node. Note that \(D'\) is acyclic since \(\prec\) is a partial ordering. For a C-node, there are two possible orderings of its children and we check whether one of them is feasible. The resulting PC-tree \(T'\) is compatible with \(\prec\).
7 Detailed Hardness Proofs

7.1 RepExt(CAR) and RepExt(HCAR)

Theorem 1. The problems RepExt(HCAR) and RepExt(CAR) are NP-hard. RepExt(CAR) is also NP-hard if the predrawn arcs have pairwise distinct endpoints.

Proof. We reduce the 3-Partition problem \([9]\). Let \(S = \{s_1, \ldots, s_{3n}\}\) be its instance, i.e., a collection of \(3n\) integers summing up to \(nt\) for some positive integer \(t\) such that each satisfies \(\frac{1}{3} < s_i < \frac{2}{3}\), where the goal is to partition \(S\) into \(n\) disjoint subsets whose sum is always \(t\). Note that the size constraints on \(s_i\)'s ensure that every subset, which sums exactly to \(t\), is a suitable triple.

From \(S\) and \(t\) we construct a graph \(G\) as follows: For the set of vertices we choose \(V(G) = \{u_1, \ldots, u_{(t+1)n}, v_1, \ldots, v_n, w_1, \ldots, w_{tn}, z_1, \ldots, z_{3n}\}\), where \(u_1, \ldots, u_{(t+1)n}\) are universal vertices; vertices \(v_1, \ldots, v_n, w_1, \ldots, w_{tn}\) form an independent set; and each \(z_i\) is connected to all of \(u_1, \ldots, u_{(t+1)n}\) and some private vertices among \(w_1, \ldots, w_{tn}\).

Observe that this graph has exactly \((t+1)n\) maximal cliques, each identified by a vertex from \(v_1, \ldots, v_n, w_1, \ldots, w_{tn}\). Moreover, each \(z_i\) belongs to exactly \(s_i\) maximal cliques.

Let \(p_1, \ldots, p_{(t+1)n}\) be distinct points of a circle ordered in a clockwise direction. For each \(u_i\) let \(R'(u_i) = [p_i, p_{i-1}]\). For each \(v_i\), let \(R'(v_i) = [p_{(t+1)i}, p_{(t+1)i}]\), (alternatively these degenerate intervals could be extended to nondegenerate by expanding them by \(\frac{1}{3}\) distance of two nearest points \(p_i, p_{i+1}\)). The partial representation \(R'\) is the just described representation of the subgraph induced by \(\{u_1, \ldots, u_{(t+1)n}, v_1, \ldots, v_n\}\), see Fig. [3] for an example.

When \(G\) allows a representation \(R\) extending \(R'\), then each maximal clique with some \(w_i\) will be placed on some point \(p_j\) where \(j \not\equiv 0 \pmod{t+1}\). Moreover clique points corresponding to some \(z_i\) must be consecutive and do not correspond to any other \(z_i\) or any \(v_j\).

Therefore the set system \(\{\{s_i : R(z_i) \cap \{p_{(t+1)i}, \ldots, p_{(t+1)i+t}\} \neq \emptyset : j \in \{1, \ldots, n\}\}\) yields the desired partition of \(S\). A construction of the representation from the set system is straightforward.

To argue that this reduction works also for NP-hardness of RepExt(CAR) observe that we did not put any condition on the representation to enforce the Helly property, it is guaranteed by the partial representation and the construction itself.

This NP-hardness reduction for RepExt(CAR) can be altered to the situation when the predrawn arcs are required to have pairwise distinct endpoints just by shrinking each arc \(R'(u_i)\) by moving its tail by \(\frac{1}{t}\) distance of the two nearest points \(p_i, p_{i+1}\). The argument is the same, only to guarantee that each \(w_i\) is adjacent to all universal vertices \(u_1, \ldots, u_{(t+1)n}\), the corresponding arc \(R(w_i)\) would have to cover not only a single point \(p_j\) for some \(j\), but instead the whole arc to the nearest tail in the clockwise direction, formally the arc \([p_j, R'(u_j)]\). Observe that by this modification we loose the Helly property as the universal vertices share no common point.

\[\square\]

7.2 RepExt(UCAR)

We first argue the NP-hardness of Theorem 6 by showing a reduction from the 3-Partition problem defined already in the proof of Theorem 1.

Then we provide a linear programming argument for the membership in the class NP.

Theorem 6. The problem RepExt(UCAR) is NP-complete.

Proof. For a given instance of 3-Partition, we construct a unit circular-arc graph \(G\) and its partial representation \(R'\). For technical reasons, we assume that \(t \geq 8\).
Let $P_{2\ell}$ be a path of length $2\ell$. There exists a unit circular-arc representation $P_{2\ell}$ such that it spans $\ell + \varepsilon$ units, for some $\varepsilon > 0$. To see this, note that $P_{2\ell}$ has two independent sets of size $\ell$ and each of this independent sets needs at least $\ell + \varepsilon$. Let $a, b, c$ be positive integers such that $a + b + c = t$. It follows that the disjoint union of $P_{2a}, P_{2b},$ and $P_{2c}$ has a representation such that it spans $t + \varepsilon$ units, for some $\varepsilon > 0$, and therefore, it can be fit into $t + 1$ units.

Let $x_0, \ldots, x_{n(t+2)-1}$ be points of the circle that divide it into $n(t+2)$ equal parts, i.e., vertices of a regular $n(t+2)$-gon. The graph $G$ is a disconnected graph consisting of $4n$ connected components. For each $s_i$, we take the path $P_{2s_i}$. We further add an isolated vertex $v_j$, for $j = 0, \ldots, n-1$. The partial representation $R'$ is the collection $\{R(v_j) : j = 0, \ldots, n-1\}$, where $R(v_j)$ is the arc of the circle from $x_{j(t+2)}$ to $x_{j(t+2)+1}$ in the clockwise direction.

The predrawn arcs $R(v_0), \ldots, R(v_{n-1})$ split the circle into $n$ gaps, where each gap has exactly $t + 1$ units. By the discussion above, if the $s_i$’s can be partitioned into $n$ triples such each triple sums to $t$, then a representation of the disjoint union of the paths corresponding to a triple can be placed in one of the $n$ gaps, see Fig. 5. If the partial representation $R'$ can be extended, then we have a partition of the $s_i$’s into $n$ triples such that each triple sums to $t$.

The certificate for the membership in $\text{NP}$ of an instance $G$, $R'$ is a linear order of $\prec$ on $V_G$ corresponding to the cyclic ordering of the intervals together with a subset of edges $E' \subseteq E_G$. Without loss of generality we assume that the predrawn arcs are $R(v_1), \ldots, R(v_k)$.

The following linear program, where for each $i \in \{1, \ldots, n\}$ the variable $x_i$ corresponds to the tail coordinate $R(v_i)_t$, has a feasible solution if and only if $R'$ can be extended to a unit interval representation $R$ where its cyclic ordering of intervals (as the equivalence class) contains $\prec$, and where the set $E'$ specifies pairs of adjacent vertices with tail coordinates at least $\ell - 1$ apart.

\[
\begin{align*}
  x_i &= R(v_i)_t & \text{for all } i \in \{1, \ldots, k\} \\
  x_i &\geq 0 & \text{for all } i \in \{k + 1, \ldots, n\} \\
  x_i &< \ell & \text{for all } i \in \{k + 1, \ldots, n\} \\
  x_j - x_i &\geq 0 & \text{if } i < j \text{ and } (v_i, v_j) \in E_G \setminus E' \\
  x_j - x_i &\leq 1 & \text{if } i < j \text{ and } (v_i, v_j) \in E_G \setminus E' \\
  x_j - x_i &\geq \ell - 1 & \text{if } i < j \text{ and } (v_i, v_j) \in E' \\
  x_j - x_i &> 1 & \text{if } i < j \text{ and } (v_i, v_j) \notin E_G \\
  x_j - x_i &< \ell - 1 & \text{if } i < j \text{ and } (v_i, v_j) \notin E_G 
\end{align*}
\]

\[\square\]

8 Additional Details for Normal Proper Helly Circular Arc Graphs

Lemma 4.3. A partial normal proper Helly circular-arc representation $R'$ of a connected graph $G$, where all twins are predrawn by distinct arcs, is extendable if and only if $V(G)$ has a consecutive cyclic ordering $\prec = [v_0, \ldots, v_{n-1}]$ satisfying:

\[\square\]
(1) The ordering $\prec$ extends the cyclic ordering $\prec'$ of the predrawn vertices corresponding to the clockwise cyclic ordering of the tails of their predrawn arcs.

(2) If $R'(v_i)$ and $R'(v_j)$, are distinct touching predrawn arcs such that $R'(v_i)_t = R'(v_j)_t$, then $N[v_i] = \{v_a, \ldots, v_i, \ldots, v_j\}$ and $N[v_j] = \{v_i, \ldots, v_j, \ldots, v_b\}$, for some non-negative integers $a, b$.

Proof. Suppose that there is a normal proper Helly representation $R$ extending $R'$. Let $\prec$ be the cyclic ordering $[v_0, \ldots, v_{n-1}]$ of $V(G)$ induced by the clockwise cyclic ordering $[t_0, \ldots, t_{n-1}]$ of tails $t_i = R(v_i)_t$. For convenience, we also write $h_i$ for $R(v_i)_h$. The ordering $\prec$ extends $\prec'$ since $R'$ is contained in $R$, so the condition (1) is satisfied.

To verify condition (2), suppose that there are two predrawn arcs $R(v_i)$ and $R(v_j)$ such that $h_i = t_j$. If $v_i$ is a universal vertex (adjacent to all other vertices), then the condition on $N[v_i]$ holds trivially. Otherwise, since $N[v_i] \neq V(G)$ and $N[v_i]$ is consecutive, $\prec$ induces a linear order on $N[v_i]$, where the minimum and maximum elements $v_a$ and $v_b$ correspond to the first and the last neighbor of $v_i$ in clockwise direction. Since $h_i = t_j$, it must be that $b = j$. An analogous argument holds for $v_j$.

For the opposite implication, let $\prec = [v_0, \ldots, v_{n-1}]$ be a consecutive cyclic ordering of $V(G)$ satisfying all three conditions. Without loss of generality we may assume that $G$ is not a complete graph, as otherwise the problem is trivial – we pick any predrawn arc and replicate it for every non-predrawn vertex. To construct the representation $R = \{R(v_0), \ldots, R(v_{n-1})\}$ of $G$, we need to determine the position of the endpoints of $R(v_i)$, for every $v_i$ that is not predrawn.

First, we construct a cyclic order $\prec$ on the auxiliary set $A = \{t_0, \ldots, t_{n-1}, h_0, \ldots, h_{n-1}\}$.

We construct $\prec$ in several steps. Initially, we set $\prec = [t_0, \ldots, t_{n-1}]$. Next, we insert $h_i$ into $\prec$, for $i = 0, \ldots, n-1$, in the following way. For each $i = 0, \ldots, n-1$ we distinguish two cases, depending on whether $v_i$ is universal or not, see Fig. 6.

(a) If $v_i$ is not universal, then we determine the last neighbor $v_b$ of $v_i$, i.e., the vertex satisfying $N[v_i] = \{v_a, \ldots, v_i, \ldots, v_b\}$ and insert $h_i$ into $\prec$ immediately before $t_{b+1}$.

(b) Otherwise, i.e., when $v_i$ is universal, we determine the first non-universal vertex $v_a$ for which $N[v_a] = \{\ldots, v_a, \ldots, v_i, \ldots\}$, i.e., the arc $R(v_a)$ shall contain the tail $R(v_i)_t$, and insert $h_i$ into $\prec$ immediately before $t_a$. The phrase the first means that all other non-universal vertices $v_{a'}$ which have $v_{a'}$ before $v_i$ in their neighborhood $N[v_{a'}]$ are between $v_a$ and $v_i$ in $\prec$.

We compute the order $\prec$ on the set $A$ from the cyclic vertex order $\prec$ and from adjacencies between vertices of the graph $G$. As described, the order $\prec$ depends uniquely on $\prec$ and $G$.

Figure 6: (a) Position of the head $h_i$ in $\prec$ for a non-universal $v_i$, and (b) for a universal $v_i$. 
The representation $R'$ imposes a cyclic ordering on the points representing the tails and heads of the predrawn arcs. We extend this cyclic ordering to the set of tails and heads of the predrawn arcs, i.e. on $\{t_i, h_i : R'(v_i) \in R'\}$. In the case when a single point of the circle is the tail and the head of touching arcs, i.e., when $R'(v_i)_b = R'(v_j)_t$, we insert $h_i$ immediately after $t_j$. Note that as $G$ is a proper circular-arc graph, only a single pair of arcs may touch on a single point. Denote the resulting cyclic ordering by $\prec'$.

Since $R'$ represents an induced subgraph of $G$ and as the cyclic ordering $\prec'$ on tails is a sub-ordering of $\prec$ due to (1), it follows that $\prec'$ is a sub-ordering of $\prec$. In particular, it could be obtained from $\prec'$ and $G'$ by the same process as $\prec$ was constructed from $\prec$ and $G$.

Now, we use the cyclic order $\prec$ to construct the representation $R$. We identify every $t_i \in A$ with $R(v_i)_t$ and $h_i \in A$ with $R(v_i)_h$ and with a slight abuse of notation we view the elements of $A$ as the points of the circle.

Consider now two predrawn elements $e$ and $e'$ that are consecutive in $\prec'$ and that corresponds to distinct points of the circle — in other words, the interior of the arc $[e, e']$ contains no other predrawn head or tail. Especially we exclude here the case when $e$ and $e'$ represent the head and the tail of touching arcs. For any such pair $e$ and $e'$, let $e_0, \ldots, e_{\ell-1}$ be the linear ordering of the elements in $\prec$ that are between $e$ and $e'$ (if any). We place $e_0, \ldots, e_{\ell-1}$ equidistantly between $e$ and $e'$ on the circle. This way we obtain the representation $R$. By the same argument we have used for $R'$, the cyclic ordering on the points representing the tails and heads can be extended to $\prec$ by putting heads of touching arcs immediately after the matching tails.

Since we do not modify the endpoints of predrawn arcs, $R$ extends $R'$.

Consider now any pair of vertices $v_i, v_j \in V(G)$. When $v_iv_j \in E(G)$, we distinguish the following cases:

- If both $v_i$ and $v_j$ are not universal, then we may without loss of generality assume that $v_j$ appears after $v_i$ in $N[v_i] = \{v_a, \ldots, v_i, \ldots, v_j, \ldots, v_b\}$. In this case the head $h_i$ has been inserted in $\prec$ immediately before the tail $t_{b+1}$. As the tail $t_b$ is between $t_i$ and $t_{b+1}$ and $t_j = t_b$ or $t_j$ is between $t_i$ and $t_b$, it follows that the tail $t_j$ is inside the arc $R(v_i)$, see Fig. 6.
- If $v_i$ in universal but $v_j$ not, then either $v_i$ appears before $v_j$ or after it in $N[v_j]$. When $v_i$ is after $v_j$ in $N[v_j]$ we deduce that $t_i$ is inside the arc $R(v_j)$ as in the previous case. Otherwise the head $h_i$ has been inserted in $\prec$ immediately before $t_a$, the first non-universal vertex which has in $N[v_a]$ the vertex $v_j$ positioned after $v_a$. Therefore $h_j$ is between $h_i$ and $t_j$ and the tail $t_j$ is inside the arc $R(v_j)$, see Fig. 6.
- Finally, when both $v_i$ and $v_j$ are universal, then they are twins and hence are predrawn and intersecting already in $R'$.

In all three cases, the arcs $R(v_i)$ and $R(v_j)$ intersect.

On the other hand, when $v_iv_j \notin E(G)$, then none of these vertices is universal. In the moment of insertion of $t_i$ in $\prec$, i.e., immediately before $t_{b+1}$ for $N[v_i] = \{v_a, \ldots, v_i, \ldots, v_b\}$, the tail $t_i$ appeared between $h_i$ and $h_j$ as otherwise $N[v_i]$ would not be consecutive. Analogously, the tail $t_j$ was inserted between $h_j$ and $h_i$. Therefore, the arcs $R(v_i)$ and $R(v_j)$ are disjoint.

## 9 Proper Helly Circular-Arc Graphs

Note that even though all proper Helly circular-arc graphs allow also a normal proper Helly circular-arc representation [24], we cannot use Algorithm 1 directly, as the given partial representation may not admit a normal extension. We can, however, still use the same machinery.

**Lemma 9.1** ([24] Lemma 1). Let $R$ be a proper circular-arc representation of a graph $G$. Any two vertices whose arcs are in non-normal position are both universal, i.e. adjacent to all vertices.
of $G$.

Proof. Let $u$, $v$ be two vertices with arcs in non-normal position. Then a non-neighbor of $u$ would have to be represented as a proper sub-arc of $R(v)$ (and vice-versa) in contradiction to $\mathcal{R}$ being proper.

Lemma 9.2. A yes-instance of $\text{RepExt}(\text{PHCAR})$ is a yes-instance of $\text{RepExt}(\text{NPHCAR})$ if and only if no two predrawn arcs are in non-normal position.

Proof. Clearly, if the partial representation $\mathcal{R}'$ contains two predrawn arcs in non-normal position, then there is no normal extension of $\mathcal{R}'$.

Conversely, assume that there is no such pair and consider a PHCAR-extension $\mathcal{R}$ of $\mathcal{R}'$. By Lemma 9.1 only universal vertices can have arcs in non-normal position. If $\mathcal{R}'$ prescribes no universal vertex, we modify $\mathcal{R}$ so that all universal vertices are represented by the same arc $R(v)$ for some arbitrary universal vertex $v$. If $\mathcal{R}'$ prescribes a universal vertex $v$, we modify $\mathcal{R}$ by representing all unprescribed universal vertices by the same arc $R(v)$.

After this modification, if there still exists a non-normal pair of arcs, there also exists a non-normal pair of prescribed arcs. Since such a pair does not exist by assumption, it follows that the modified representation is an NPHCAR-extension of $\mathcal{R}'$.

Theorem 3. The problem $\text{RepExt}(\text{PHCAR})$ can be solved in linear time.

Proof. Without loss of generality we assume that $G$ is not complete as otherwise an extension can always be obtained by duplication of any predrawn arc.

Let $\mathcal{R}'$ be a partial representation of a graph $G \in \text{PHCA}$. Without loss of generality we also assume that predrawn arcs are distinct as otherwise the identical arcs must correspond to twins and it suffices to keep only one.

If there is no pair of prescribed arcs in non-normal position, Lemma 9.2 implies that the problem can be solved with Theorem 2. Hence assume that there exists a pair $R'(u), R'(v)$ of prescribed arcs in non-normal position that by Lemma 9.1 must correspond to universal vertices.

Denote by $A$ and $B$ the two disjoint circular arcs whose union is the intersection of $R'(u)$ and $R'(v)$ and by $C$ and $D$, resp., the arcs formed by the points of the circle that belong only to $R'(u)$ but not $R'(v)$ and vice-versa. The two closed arcs $A, B$ together with the two open $C$ and $D$ cover the whole circle, see Fig. 7a.

In order to characterize all graphs that allow a representation extension, consider now any such graph $G$ together with its proper Helly representation $\mathcal{R}$.
Figure 8: Position of $R(w_A)_t$ and $R(w_B)_h$ with respect to universal arcs containig $D$.

Since $\mathcal{R}$ is proper, any arc $R(w) \in \mathcal{R}$ distinct from $R(v)$ must intersect $C$ to guarantee $R(w) \not\subset R(v)$. With an analogous condition on $D$ we get that either $A \subset R(w)$ or $B \subset R(w)$.

Then let $V_A = \{w_A \in V(G) : A \subseteq R(w_A)\}$ and $V_B = \{w_B \in V(G) : B \subseteq R(w_B)\}$ be the sets of vertices whose arcs contain $A$ and $B$, respectively. By the definition $V_A$ and $V_B$ are two cliques covering $V(G)$. Since $G$ is not complete, both $V_A$ and $V_B$ contain a non-universal vertex. Any arc intersecting $A$ and $B$ contains either $C$ or $D$, thus due to proper and distinct positions of arcs we have $V_A \cap V_B = \{u, v\}$.

Observe that when the arc representing some $w_A \in V_A$ intersects $B$ (or vice versa), then $w_A$ is universal — it would intersect all vertices from $V_B$ on $R(w_A) \cap B$. On the other hand, we claim that every arc of a universal vertex intersects both $A$ and $B$. Assume by a contradiction that such arc $R(w_A)$ would intersect only $A$. By Lemma 9.1, it is in normal position with respect to the arc $R(w_B)$ of some non-universal vertex $w_B \in V_B$. Without loss of generality the intersection of these two arcs is in $C$. Then as depicted in Fig. 7b the arcs of $w_A, w_B$ and $v$ violate the Helly property, a contradiction. In summary, each universal arc includes one of $A$ or $B$, one of $C$ or $D$ and intersects the remaining two.

By identical arguments in slightly altered context we show that after pruning all universal vertices from $G$ the resulting graph $G'$ is disjoint union of two cliques $V'_A = V_A \setminus V_B$ and $V'_B = V_B \setminus V_A$. Assume for contradiction that $G'$ contains two adjacent vertices $w_A \in V'_A$ and $w_B \in V'_B$. Since $w_A, w_B$ are not universal, $R(w_A), R(w_B)$ must be in normal position by Lemma 9.1. Therefore, the intersection $R(w_A) \cap R(w_B)$ is fully contained either in $C$ or in $D$. In the first case we get a contradiction as the arcs $R(v), R(w_A), R(w_B)$ violate the Helly property, see again Fig. 7b. In the other case, the arcs $R(u), R(w_A)$ and $R(w_B)$ also violate the Helly property.

By Lemma 9.1 any arc $R(w_A)$ for a non universal vertex $w_A \in V'_A$ contains exactly one endpoint of the each arc $R(x)$ representing a universal vertex. The other endpoint of $R(x)$ must then belong to $R(w_B)$ of each $w_B \in V'_B$. This way we get a partition of the endpoints of universal arcs into two consecutive sets. We indeed describe this partition more precisely: The arc $R(w_A)$ contains $\{h_i : C \subset R(u_i)\} \cup \{t_i : D \subset R(u_i)\}$, while $\{t_i : C \subset R(u_i)\} \cup \{h_i : D \subset R(u_i)\} \subset R(w_B)$. When $C \subset R(u_i)$ then $h_i$ belongs either to $A$ or $D$. The first case is enforced immediately and in the latter $t_i \in B$. Hence we also have to put $t_i$ into $R(w_A)$.

This necessary condition is also sufficient, namely a partial representation $\mathcal{R}'$ of a non-complete graph proper-Helly circular graph $G$ with arcs $R(w), R(v)$ in not-normal position allows an extension if and only if the sets $\{h_i : C \subset R'(u_i)\} \cup \{t_i : D \subset R'(u_i)\}$, and $\{t_i : C \subset R'(u_i)\} \cup \{h_i : D \subset R'(u_i)\}$ are consecutive — if no $w_A \in V'_A$ is pre-drawn, we choose $R(w_A)$ to be the shortest arc containing the first of these two sets within $A \cup C \cup D$ and analogously for any $w_B \in V'_B$, see Fig. 8. The
remaining arcs could be obtained by replication.

Our arguments convert directly to the following algorithm: Accept if $G$ is complete. If the partial proper Helly representation has no pair of arcs in non-normal position, use Algorithm $[\mathbb{1}]$. Otherwise check whether $G'$ is isomorphic to the union of two disjoint cliques; if not, reject. Finally, check whether the endpoints of the predrawn arcs of universal vertices could be split into two consecutive sets described above and if the predrawn arcs non-universal ones cover these sets appropriately. If not, reject.

In the affirmative case, a representation could be constructed by replication of arcs of three representatives: a universal vertex, one vertex from $V'_{A}$ and one from $V'_{B}$. A minor simplification could applied when a representative from $V'_{A}$ or $V'_{B}$ is already pre-drawn and recognized.

10 Illustrations for the Normal Helly Case

Figure 9: An example of a construction of a normal Helly circular-arc representation.

Figure 10: The region of a maximal clique cannot intersect two gaps of another.

11 Helly Circular Arc Graphs

Surprisingly, when the Helly property is required, we can solve $\text{RepExt}(\text{HCAR})$ similarly to $\text{RepExt}(\text{NHCAR})$ when the predrawn arcs have pairwise distinct endpoints. The most notable difference is that Lemma $5.6$ no longer applies and we have to test every island of clique $D$ for placing $p_{D}$.

**Theorem 9.** Let $R'$ be a partial HCA representation of $G$ where the arcs have pairwise distinct endpoints and let $D$ be a maximal clique of $G$. There exists an HCA representation of $G$ that extends
if and only if for some point \( p_D \in \operatorname{Reg}(D) \) with the corresponding partial order \( \prec \) as defined in Section 5 there exists a linear extension \( \prec \) of \( \prec \) such that:

1. For every vertex \( v \), \( M_v \) is consecutive in \([<]\).
2. For every gap \( J \) of a \( C \in \mathcal{C} \), the set \( S_J \) is consecutive in \([<]\).

Proof. We first show that, if there is such an HCA-extension of \( \mathcal{R}' \), then these properties are satisfied.

We obtain \( \prec \) as the linearization of a clique ordering of an extension of \( \mathcal{R}' \) starting with \( D \) where the clique point of \( D \) is \( p_D \). By the construction of \( \prec \), the order \( \prec \) is a linear extension of \( \prec \). By Lemmas 2.1 and 5.5 we obtain properties 1 and 4.

For the opposite implication, let \( p_D \in \operatorname{Reg}(D) \) be on an island \( I_1 \) of \( D \) and let \( \leq C_1, C_2, \ldots, C_k \) be a linear extension of \( \prec \) such that properties 1 and 2 are satisfied. We show that each \( C_i \in \mathcal{C} \) can be assigned its clique point \( \operatorname{cp}(C_i) \in \operatorname{Reg}(C_i) \) such that \( \operatorname{cp}(C_j) < \operatorname{cp}(C_i) \) whenever \( j < i \). We relocate \( p_D \) to the middle of island \( I_1 \). Note that the properties 1 and 2 remain satisfied for the new ordering \( \prec \) since the only possible change is that relations to cliques with island \( I_1 \) may get lost.

Let \( \varepsilon > 0 \) be the \( \frac{1}{n+1} \)-fraction of the length of the shortest nontrivial island. This choice allows us to draw all new endpoints at distance at least \( \varepsilon \) but still within any chosen island or side of island \( I_1 \).

For \( C_1 = D \) we place \( \operatorname{cp}(C_1) \) on \( p_D \). In a greedy way, when the location of the clique points \( \operatorname{cp}(C_1), \ldots, \operatorname{cp}(C_{i-1}) \) is settled, we determine the set \( P \) of feasible points for \( \operatorname{cp}(C_i) \) that is \( P = \operatorname{Reg}(C_i) \cap \{ p : p > \operatorname{cp}(C_{i-1}) + \varepsilon \} \). If \( P \) has minimum, we place \( \operatorname{cp}(C_i) \) there, otherwise we put \( \operatorname{cp}(C_i) \) at \( \inf(P) + \varepsilon \). We argue that such choice always exists.

Assume for a contradiction that \( C_i \) is the first maximal clique in the order \( \prec \) whose clique point \( \operatorname{cp}(C_i) \) cannot be properly placed. Note that a maximal clique \( C \neq C_i \) can never have an island \( I_C \) consisting of a single point since this can only occur as intersection of two predrawn arcs ending at that point. Hence, by the choice of \( \varepsilon \), we never place a clique \( C \) at the very last point of \( \operatorname{Reg}(C) \). Therefore, one clique point must be placed to the right of \( \operatorname{Reg}(C_i) \) before placing \( \operatorname{cp}(C_i) \).

We identify the first maximal clique \( C_j, j < i \) that is placed to the right of all points in \( \operatorname{Reg}(C_i) \). Since \( \operatorname{cp}(C_j) \notin \operatorname{Reg}(C_i) \), we have that \( \operatorname{Pre}(C_i) \neq \operatorname{Pre}(C_j) \). By Lemma 5.4 the maximal clique \( C_j \) has a gap \( J \) with \( \operatorname{Reg}(C_i) \subseteq J \), see Fig. 4.

Consider the neighboring island \( I \) of \( C_j \) to the left of \( J \). Since \( \operatorname{cp}(C_j) \) was not placed on \( I \), the clique point \( \operatorname{cp}(C_{j-1}) \) has been placed to the right of \( I \). By the choice of \( C_j \), we have that \( \operatorname{cp}(C_{j-1}) \) is not placed to the right of \( J \) and thus \( \operatorname{cp}(C_{j-1}) \in J \). Since \( \operatorname{cp}(C_j) \) has been placed to the right of \( \operatorname{Reg}(C_i) \) and thus to the right of \( J \), we have \( p_D \notin J \) and thus \( D \notin S_J \). With \( D < C_{j-1} < C_j < C_i \), where \( C_{j-1}, C_i \in S_J \) and \( D, C_j \notin S_J \) we get a contradiction with the property 2 since \( S_J \) is not consecutive in \([<]\).

From the placement of clique points, we obtain an HCA-extension of \( \mathcal{R}' \) so that for every not yet represented vertex \( u \notin V(G') \) we choose \( R(u) \) to be a minimal arc containing exactly the clique points of the maximal cliques from \( M_u \).

By the Lemma 2.1 we obtain from properties 1, 2 that this results in an HCA representation of \( G \) extending \( \mathcal{R}' \), since the maximal clique points are placed correctly with regards to both predrawn arcs and new arcs, and moreover, two arcs intersect if and only if they are predrawn or share a maximal clique. Note that we can perturbate the endpoints of the non-predrawn arcs to avoid shared endpoints.

Theorem 5. The problem \( \text{REPExt}(\text{HCA}) \) can be solved in \( \mathcal{O}(n^3) \) time if the partial representation consists of arcs with pairwise distinct endpoints.
Data: A graph $G$ and a partial representation $\mathcal{R}'$ consisting of arcs with pairwise distinct ends.

Result: An HCA representation $\mathcal{R}$ of $G$ extending $\mathcal{R}'$ or a message that it does not exist.

1 begin
2 \hspace{.5cm} Determine the set of maximal cliques $\mathcal{C}$ of $G$;
3 \hspace{.5cm} foreach $C \in \mathcal{C}$ do
4 \hspace{1cm} Determine $\text{Reg}(C)$;
5 \hspace{1cm} if $\text{Reg}(C) = \emptyset$ then return $\mathcal{R}'$ has no extension;
6 \hspace{.5cm} $\varepsilon := \frac{1}{2n+1} \min \{|I|, I \text{ is an island}|$;
7 \hspace{.5cm} Choose any $C_1 \in \mathcal{C}$;
8 \hspace{.5cm} foreach island $I$ of $C_1$ do
9 \hspace{1cm} Set $\text{cp}(C_1)$ to the middle of $I$;
10 \hspace{1cm} Determine the partial order $\prec$;
11 \hspace{1cm} Build a PC tree $T$ on $\mathcal{C}$ capturing the constraints stated in Theorem 9;
12 \hspace{1cm} if such $T$ does not exist then continue;
13 \hspace{1cm} Solve $\text{REORDER}(T, C_1, \prec)$ to get the order $C_1 < C_2 < \cdots < C_k$;
14 \hspace{1cm} if such order $\prec$ exists then for $i = 2$ to $k$ do
15 \hspace{1.5cm} $P := \text{Reg}(C_i) \cap \{p : p > \text{cp}(C_{i-1}) + \varepsilon\}$;
16 \hspace{1.5cm} if $\min P$ exists then $\text{cp}(C_i) := \min P$;
17 \hspace{1.5cm} else $\text{cp}(C_i) := \inf(P) + \varepsilon$;
18 \hspace{1cm} foreach $u \notin V(G')$ do draw $R(u)$ to cover exactly $\{\text{cp}(C) : C \in M_u\}$;
19 \hspace{.5cm} return $\mathcal{R}$;
20 return $\mathcal{R}'$ has no extension;

Algorithm 3: The algorithm for the RepExt(HCAR) problem.

\textbf{Proof.} The correctness of Algorithm 3 follows from the already mentioned arguments. For the computational complexity of the more complex steps note that:

- Line 2: We run the linear time recognition algorithm for Helly circular-arc graphs \[25\] on $G$ and read its at most $n$ maximal cliques from any of its representation.
- Lines 3–5: The regions can be obtained in time $\mathcal{O}(n)$ by traversing the circle once. During the traversal two consecutive predrawn endpoints specify possible islands which can be assigned to appropriate maximal cliques.
- Line 8: There are at most $\mathcal{O}(n)$ islands for $C_1$.
- Line 10: The comparable pairs of the partial order $\prec$ can also be determined by a traversal as in Steps 3–5.
- Line 11: The construction of the PC-tree follows the standard approach from the recognition of Helly circular-arc graphs \[13\]. Each arc is at the left start of at most one gap and each of the $O(n)$ maximal cliques has at most one gap without an arc inside. Hence, there are $O(n)$ gaps and thus $O(n)$ constraints. $T$ can thus be constructed in $O(n^2)$ time \[13\].
- Line 13: The $\text{REORDER}(T, C_1, \prec)$ problem can be solved in $\mathcal{O}(n^2)$ time by Lemma 2.2.

12 Conclusions and Open Problems

Our study of the RepExt problem has been restricted in two ways:
First, we have considered mostly representations satisfying Helly property as this allows us to consider the clique points of maximal cliques. For representations that do not have this property one would involve a completely different approach.

Secondly, for the recognition problem it is irrelevant whether arcs are closed or open, but this is not the case for the representation extension. Observe that touching intervals in RepExt(NPHCAR) in Lemma 4.3 imply constraints on the ordering. For the sake of completeness it might be worth to check whether use of open or semi-open intervals would yield significant impact on the computational complexity.

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