Quantum Dynamics in Regions of Quaternionic Curvature

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Abstract

The complex unit appearing in the equations of quantum mechanics is generalised to a quaternionic structure on spacetime, leading to the consideration of complex quantum mechanical particles whose dynamical behaviour is governed by inhomogeneous Dirac and Schrödinger equations. Mixing of hyper-complex components of wavefunctions occurs through their interaction with potentials dissipative into the extra quaternionic degrees of freedom. An interferometric experiment is analysed to illustrate the effect.

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I. INTRODUCTION

The foundations of quaternionic quantum mechanics (QQM) were laid by Finkelstein, Jauch, Schiminovich and Speiser [1]. They sought to generalise standard complex quantum mechanics (CQM) by introducing additional geometrical concepts. Guiding them in their enterprise was the example of Einstein’s geometrisation of gravity. Thus, they were led to proposing a quantum theory that was locally identical to CQM, but with a generalised global structure requiring the introduction of a connection on the spacetime manifold, called the Q–connection, in order to relate complex algebras, and hence states and measurements, at different points. A nontrivial global structure was described by a nonvanishing Q–curvature operator, defined to be the commutator of the Q–covariant derivative at each point. The theory is complicated by ambiguity over the construction of tensor products, [2,3], and questions of how complex analytic techniques (e.g. harmonic analysis) are to be carried over to QQM’s nontrivial bundle of complex state-spaces over spacetime.

Given that investigations of quantum gravity have suggested that deBroglie–wave scattering off regions of gravitational curvature leads to effective particle creation, (e.g. Hawking radiation [4]), we should allow the possibility that regions of nonzero Q–curvature can act as sources and sinks of quantum probability.

More recently, [5–7], the zero Q–curvature (i.e. Q–flat) limit of the theory has been investigated. Coupling of the wavefunction components only occurs in the presence of quaternionic potentials. A common feature of these investigations is the adoption of conventions regarding the ordering of quaternionic factors in the dynamical equations, resulting in exponentially decaying hyper-complex components. Thus, the Q–flat limit introduces physics that is in principle difficult to observe.

We instead consider particle dynamics in the intermediate case termed the “electromagnetic” limit by [1]. As in the Q–flat limit, the presence of potentials with quaternionic elements causes mixing of the wavefunction components.

Our approach to QQM is guided by an analogy with the importance of Killing vector
fields in general relativity. The compatibility of the metric tensor with the covariant derivative is a very strong constraint on the connection, simplifying the analysis of the system’s consequent dynamical behaviour. In the QQM case, imposing an analogous constraint on the Q–connection singles out a field of unit, pure imaginary quaternions which we identify with the $i$ of CQM and with which we generalise complex analysis. Further, we adopt a convention regarding the ordering of factors in the dynamical equations which has major implications for the experimental verification of the theory. We use Green functions to analyse an interferometric experiment in the presence of weak quaternionic potentials. Multiple barrier potentials are then tractable using this technique in the weak potential limit. Our results differ qualitatively from those of previous investigators [7,8].

II. GENERAL RESULTS

We generalise from the CQM formalism by introducing the correspondence

$$
\hat{E}^{CQM} \Psi = (\partial_t \Psi)i\hbar \leftrightarrow \hat{E}^{QQM} \Psi = (D_t \Psi)\eta\hbar,
$$

where the new (Lorentz scalar) field on spacetime $\eta(x^\mu)$ is defined formally by

$$
\eta^* = -\eta, \quad \eta^2 = -1,
$$

and now, because the algebra of quaternions $\mathbb{H}$ is noncommutative, the order of the factors is crucial.

That is, the canonical 4–momentum operator of the quaternionic theory acts on a state $\Psi$ according to the prescription

$$
P_\mu \Psi = (D_\mu \Psi)\eta = (\partial_\mu \Psi + \frac{1}{2}[Q_\mu, \Psi])\eta,
$$

where $D_\mu$ is the Q–covariant derivative and $Q_j$ is the Q–connection.

By imposing the compatibility condition that $D_\mu \eta \equiv 0$, we can implement the programme of canonical quantisation, with $\hbar = 1$ and $c = 1$ in the appropriate units,
\[ [X^j, X^k] = 0, \quad [P_j, P_k] = 0, \quad [X^j, P_k] = \eta \delta^j_k. \] (4)

The algebra of observables generated from the fundamental operators \( X^j, P_k, E = P_o, \eta \mathbb{1} \), is formally identical to the algebra of operators of CQM. The possibility remains that there exist operators with additional quaternionic components. We use the symplectic decomposition of any operator at a point,

\[ O = O^\eta + \zeta O^\zeta, \] (5)

where \( O^\eta, O^\zeta \) are \( \eta \)-complex, (i.e. a real linear combination of the unit operator, \( \mathbb{1} \), and the operator \( \eta \mathbb{1} \)). Here we have introduced the new quaternionic field \( \zeta = \zeta(x^\mu) \) with properties

\[ \zeta^2 = -1, \quad \{ \eta(x^\mu), \zeta(x^\mu) \} = 0. \] (6)

Note that \( D_\mu \zeta \neq 0 \) in general, and that \( O^\zeta \) is not necessarily zero. The dynamical evolution of fully quaternionic operators is expected to be complicated.

Also, there is no \textit{a priori} reason why the wavefunction (considering single particle quantum systems) has to be restricted to be \( \eta \)-complex. Instead, in the spirit of Finkelstein \textit{et al.}, we decompose the wavefunction into its natural symplectic components at each point of spacetime

\[ \Psi = \Psi_\eta + \zeta \Psi_\zeta. \] (7)

We interpret the probability measure associated with the first symplectic component of the quaternionic wavefunction as corresponding to the usual probability distribution of CQM. Our convention regarding the order of \( Q \)-covariant differentiation and multiplication by \( \eta \) ensures that the second symplectic component of the quaternionic wavefunction will be oscillatory in free space, and, hence, asymptotically relevant.

In the relativistic regime, the full \( Q \)-curvature of the system is given by \([D_\mu, D_\nu]\), where

\[ [D_\mu, D_\nu] \Psi \equiv \frac{1}{2} [\mathcal{K}_{\mu\nu}, \Psi], \] (8a)

\[ \mathcal{K}_{\mu\nu} = Q_{\nu,\mu} - Q_{\mu,\nu} + \frac{1}{2} [Q_\mu, Q_\nu]. \] (8b)
The compatibility condition on the covariant derivative of $\eta$ implies that $[K_{\mu\nu}, \eta] = 0$, but in general $[K_{\mu\nu}, \zeta]$ will not vanish.

This convention also allows us to treat the components of the Q–wavefunction under EM–gauge transformation in a way formally identical to the CQM case:

$$D_\mu \Psi \rightarrow D^A_\mu \Psi = D_\mu \Psi + e \Psi A_\mu \eta, \quad P_\mu \Psi \rightarrow (D^A_\mu \Psi) \eta,$$

where the components of the EM–field $A_\mu$ are real. Then under any EM–gauge transformation, where $\Theta(x^\mu)$ is a real function,

$$\Psi \rightarrow \Psi^A = \Psi \exp[-e \Theta(x^\mu)] \eta, \quad A_\mu \rightarrow A_\mu + \Theta_\mu .$$

The Aharonov–Bohm effect can be analysed in the quaternionic case in an analogous fashion [10].

The most general linear combination of the quaternion generators that satisfies the modulus constraint on $\eta$ is

$$\eta(x^\mu) = \sin \theta \cos \phi i_1 + \sin \theta \sin \phi i_2 + \cos \theta i_3 ,$$

where $\theta = \theta(x^\mu)$ and $\phi = \phi(x^\mu)$ are real functions on spacetime. Then the most general $\zeta$ field that anticommutes with $\eta$ and has unit modulus is

$$\zeta(x^\mu) = \cos \theta \cos \phi i_1 + \cos \theta \sin \phi i_2 - \sin \theta i_3 .$$

The third generator of the quaternionic algebra at $x^\mu$ is

$$\xi(x^\mu) \equiv \frac{1}{2} [\eta, \zeta] = - \sin \phi i_1 + \cos \phi i_2 + 0i_3 .$$

Invoking Liebniz’s rule for the application of covariant derivatives to products of quaternions, Eqs. (2,3) imply

$$\{ \eta, D\eta \} = 0, \quad \{ \zeta, D\zeta \} = 0, \quad \{ \xi, D\xi \} = 0 .$$

Therefore, we define
\[ D_\mu \zeta = a_\mu \eta + b_\mu \zeta \eta, \]  
\[ (14) \]

where \( a_\mu, b_\mu \) are real functions.

Then \( D_\mu (D_\nu \zeta) = A_{\mu\nu} + \zeta B_{\mu\nu} \), where

\[ A_{\mu\nu} = a_{\nu,\mu} \eta - a_\mu b_\nu, \quad B_{\mu\nu} = b_{\nu,\mu} \eta - b_\mu b_\nu. \]  
\[ (15) \]

If \( a_\mu \) (respectively \( b_\mu \)) vanishes, then so does \( A_{\mu\nu} \) (respectively \( B_{\mu\nu} \)). The ramifications of these circumstances will be explored below.

Now the additional postulate \( D\eta \equiv 0 \) implies for integer \( k \),

\[ \{ \eta, D^k \zeta \} = 0, \quad \{ \eta, D^k \xi \} = 0. \]  
\[ (16) \]

Hence, \( D\zeta \propto \xi \) and \( D\xi \propto \zeta \). That is, the compatibility condition is sufficient to force \( a = 0 \), decoupling the symplectic components of the wavefunction in the absence of fully quaternionic potentials.

Explicitly,

\[ \partial_\mu \eta = \theta_{\mu}(\cos \theta \cos \phi i_1 + \cos \theta \sin \phi i_2 - \sin \theta i_3) \]
\[ + \phi_{\mu}(\sin \theta \sin \phi i_1 + \sin \theta \cos \phi i_2) , \]  
\[ (17) \]

\[ D_\mu \eta = \theta_{\mu} \zeta + \phi_{\mu} \sin \theta \xi + \frac{1}{2} [Q^\eta_{\mu} \eta + Q^\zeta_{\mu} \zeta + Q^\xi_{\mu} \xi, \eta] \equiv 0. \]  
\[ (18) \]

Therefore \( Q^\eta_{\mu} \) is a free real parameter, \( Q^\zeta_{\mu} = \phi_{\mu} \sin \theta \), and \( Q^\xi_{\mu} = -\theta_{\mu} \). Hence

\[ D_\mu \zeta = -\theta_{\mu} \eta + \phi_{\mu} \cos \theta \zeta + \frac{1}{2} [Q^\eta_{\mu} \eta + Q^\zeta_{\mu} \zeta, \zeta] , \]
\[ = (Q^\eta_{\mu} + \phi_{\mu} \cos \theta) \zeta , \]  
\[ (19) \]

\[ D_\mu \xi = \phi_{\mu} (-\cos \phi i_1 - \sin \phi i_2) + \frac{1}{2} [Q^\eta_{\mu} \eta + Q^\zeta_{\mu} \zeta, \xi] , \]
\[ = -(Q^\eta_{\mu} + \phi_{\mu} \cos \theta) \xi , \]  
\[ (20) \]

so we have

\[ a_\mu = 0, \quad b_\mu = -(Q^\eta_{\mu} + \phi_{\mu} \cos \theta) . \]  
\[ (21) \]

We see that the imposition of the compatibility condition results in the decoupling of the symplectic components of the quaternionic wavefunction.
III. Q–POTENTIALS

In the nonrelativistic limit, we have the quaternionic analogue of the Schrödinger equation

\[(D_t\Psi)\eta + \frac{1}{2m} \mathbf{D} \cdot \mathbf{D} \Psi = 0, \quad (22)\]
equivalent to the pair of equations

\[(i\partial_t + \frac{1}{2m} \Delta)\psi = 0, \quad (23a)\]

\[(i\partial_t + \frac{1}{2m} \Delta)\varphi = (b_t + \frac{i}{m} \mathbf{b} \cdot \mathbf{\partial} - \frac{1}{2m} \mathbf{B}_k^k)\varphi. \quad (23b)\]

That is, we carry out a local, quaternionic gauge transformation. The \(b\)–field contains the remaining degrees of freedom associated with the choice of a \(\zeta\)–field at each point of spacetime. We do not consider here the physical meaning which might be ascribed to restrictions on this procedure arising from the existence of topological defects in the spacetime manifold.

These dynamical equations are independent of intrinsic spin, and so an ensemble of spin states (correlated or statistical mixture) will retain its structure until a measurement of spin is made. This would imply that Bell experiments will be fundamentally unchanged by progressing to QQM, but that the expectation values will be different to the CQM case due to the new dependence of the Pauli spin operators on spacetime [11].

In the presence of a potential \(V = V^n + \zeta V^\zeta, V^\zeta \neq 0\), in the nonrelativistic limit, and, as in Eq.(9a), postulating the right-multiplication of the \(\eta\)–complex \(V\)–components on the wavefunction, we have the pair of coupled dynamical equations

\[(i\partial_t + \frac{1}{2m} \Delta - V^n)\psi = -V^\zeta \varphi, \quad (24a)\]

\[(i\partial_t + \frac{1}{2m} \Delta - V^n - b_t - \frac{i}{m} \mathbf{b} \cdot \mathbf{\partial} + \frac{1}{2m} \mathbf{B}_k^k)\varphi = V^\zeta \psi. \quad (24b)\]

Note that this differs from [7] only by an irrelevant choice of signs for the quaternionic components of the external potential.
Introducing Green functions $G_j(x|x')$, the most general solutions to these P.D.E.’s are

$$\psi = \psi_o - \int dx' G_1(x|x')V^\zeta(x')\varphi(x'),$$  \hspace{1cm} (25a)  

$$\varphi = \varphi_o + \int dx' G_2(x|x')V^\zeta(x')\psi(x'),$$  \hspace{1cm} (25b)  

where $\psi_o, \varphi_o$ are solutions to their respective homogeneous P.D.E.’s, and we assume homogeneous boundary conditions.

Thus we have the formal, iterative solution for the symplectic components

$$\psi = \psi_o - \int dx' G_1(x|x')V^\zeta(x')\{\varphi_o(x')$$  

$$+ \int dx'' G_2(x'|x'')V^\zeta(x'')\psi(x'')\}\, , \text{ etc.}$$  \hspace{1cm} (26)  

Hence, for small $V^\zeta$ we have

$$\psi = \psi_o - \int dx' G_1(x|x')V^\zeta(x')\varphi_o(x') + O(|V^\zeta|^2),$$  \hspace{1cm} (27)  

and we can see that QQM with weak Q–potentials leads to inhomogeneous dynamical equations. Spontaneous creation and dematerialisation of particles is a consequence of this situation.

**IV. INTERFEROMETRY EXPERIMENT**

To illustrate, we consider an interferometry experiment consisting of a deBroglie–wave, say of slow neutrons, split into two coherent beams. One beam is passed through a constant, quaternionic potential of bounded support, and then they are recombined to interfere [12].

The final intensity pattern

$$I(x, t) \propto |\psi + [S\psi - V^\zeta I_{[0,L]}(\varphi_o) + O(|V^\zeta|^2)]|^2,$$  \hspace{1cm} (28)  

where $\psi$ is the reference wave, $|S|^2$ is the transmission coefficient of the single barrier potential $V$ which is nonzero constant on the interval $[0, L]$ and vanishes elsewhere, $\varphi_o$ is
the time-dependent solution of the corresponding hyper-complex component’s homogeneous
dynamical equation, and

\[ I_{[0,L]}(\varphi_0) = \int_0^L dx' \mathcal{G}_1(x|x') \varphi_0(x', t). \] (29)

The Green function for the 1 dimensional, finite barrier potential is, from [13],

\[ \mathcal{G}_1(x|x') = -2\pi i(k_1)^{-1} e^{ik_1|x-x'|}, \] (30)

where \( k_1 = (\omega_1 - V^\eta)^{1/2} \), and \( i\partial_t \psi = \omega_1 \psi \). Assuming an effective constant potential, \( V_Q \), due to the presence of Q–curvature in the interval \([0, L]\), and that \( i\partial_t \varphi = \omega_2 \varphi \), we have in the region \( x > L \),

\[ I_{[0,L]}(\varphi_0) = -2\pi i(k_1)^{-1} N_\varphi e^{ik_1 x-i\omega_2 t} \]
\[ \times \int_0^L dx'(e^{i(k_2-k_1)x'} + Re^{-i(k_2+k_1)x'}), \] (31)

where \( N_\varphi \) is a normalisation factor, \( R > 0 \) due to reflection within the barrier, and \( k_2 = (\omega_2 - V^\eta - V_Q)^{1/2} \). This is not translationally invariant, which is intuitively obvious as this is a source problem. Note that the condition for the intensity to be time independent is that \( \omega_1 = \omega_2 \).

In the case of a series of spatially bounded and nonintersecting potential barriers, the order of traversal is critical (as previously suggested [5], but now for new reasons). That is, given a set of weakly quaternionic potential barriers \( V_{(n)} \) of width \( L_{(n)} \), at positions \( x_{(n)} \), to first order in the hyper-complex barrier components we have as the QQM contribution to the particle amplitude at the detector

\[ -\sum_n V_{(n)}^\zeta I_{[x_{(n)},x_{(n)}+L_{(n)}]}(\varphi_0) + O(\{|V_{(n)}^\zeta|^2\}). \] (32)

The Q–curvature will affect the \( I \)–integrals through the \( \varphi_0 \) field, and so a permutation of the order of the barriers will in general result in a new intensity pattern being produced, which cannot be explained merely in terms of a re-assignment of complex phases.
Thus, our quaternionic expectation value departs from the predictions of CQM in a way qualitatively different to previous investigations of QQM in the Q-flat limit. In particular, QQM with nonvanishing Q-curvature manifests itself through “external source” effects, rather than through noncommuting phase factors.

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REFERENCES

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[1] D. Finkelstein, J. M. Jauch, S. Schiminovich, and D. Speiser, J. Math. Phys. 3, 207 (1962); 4, 788 (1963).

[2] C. G. Nash and G. C. Joshi, Int. J. Theor. Phys. 31, 965 (1992); J. Math. Phys. 28, 2883 (1987); 28, 2886 (1987);

[3] A. Razon and L. P. Horwitz, Acta Appl. Math. 24, 141 (1991); 24, 179 (1991).

[4] S. W. Hawking, Commun. Math. Phys. 87, 395 (1982).

[5] S. L. Adler, Phys. Rev. D 37, 3654 (1988); Phys. Rev. Lett. 57, 167 (1986); Commun. Math. Phys. 104, 611 (1986).

[6] L. P. Horwitz and L. C. Biedenharn, Ann. Phys. 157, 432 (1984).

[7] A. J. Davies and B. H. J. McKellar, Phys. Rev. A 40, 4209 (1989); 46, 3671 (1992); A. J. Davies, Phys. Rev. D 41, 2628 (1990).

[8] A. Peres, Phys. Rev. Lett. 42, 683 (1979).

[9] The algebra of quaternions is the real span of a set of abstract units \( \{1, i_1, i_2, i_3\} \) with defining multiplication rules: \( 1i_j = i_j1 = i_j, \ i_ji_k = -\delta_{jk}1 + \sum_l \epsilon_{jkl}i_l \), where \( j, k, l = 1, 2, 3 \).

[10] S. P. Brumby and G. C. Joshi, work in progress.

[11] S. P. Brumby, G. C. Joshi, and R. Anderson, University of Melbourne preprint UM–P–94/54 ; RCHEP–94/15.

[12] A. G. Klein and S. A. Werner, Rep. Prog. Phys. 46, 259 (1983).
[13] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw–Hill, New York, 1953), Vol. 1.