Quantum Tera-Hertz electrodynamics in Layered Superconductors

Sergey Savel’ev,1,2 A.L. Rakhmanov,1,3 and Franco Nori1,4

1Frontier Research System, The Institute of Physical and Chemical Research (RIKEN), Wako-shi, Saitama, 351-0198, Japan
2Department of Physics, Loughborough University, Loughborough LE11 3TU, United Kingdom
3Institute for Theoretical and Applied Electrodynamics RAS, 125412 Moscow, Russia
4Department of Physics and MCTP, University of Michigan, Ann Arbor, MI 48109-1040, USA

(Dated: March 29, 2022)

In close analogy to quantum electrodynamics, we derive a quantum field theory of Josephson plasma waves (JPWs) in layered superconductors (LSCs), which describes two types of interacting JPW bosonic quanta: one massive and the other almost-massless. We also calculate the amplitude of their decay and scattering. We propose a mechanism of enhancement of macroscopic quantum tunneling (MQT) in stacks of intrinsic Josephson-junctions (SIJJs). Due to the long-range interactions between many junctions in the LSCs, the calculated MQT escape rate $\Gamma$ has a very nonlinear dependence on the number of junctions in the stack. This allows to quantitatively describe striking recent experiments in Bi2212 stacks.

PACS numbers: 74.72.Hs, 74.78.Fk

The recent surge of interest on Stacks of Intrinsic Josephson Junctions (SIJJs) is partly motivated by the desire to develop THz devices, including emitters, filters, detectors, and nonlinear devices. Macroscopic quantum tunnelling (MQT) has been, until recently, considered to be negligible in high-$T_c$ superconductors due to the d-wave symmetry of the order parameter. Recent unexpected experimental evidence could open a new avenue for the applicability of SIJJs in quantum electronics. This requires a quantum theory for SIJJs capable of describing quantitatively this new stream of remarkable experimental data. In contrast to a single Josephson junction, SIJJs are strongly coupled along the direction perpendicular to the layers. This is because the thickness of these layers is of the order of a few nm, which is much smaller than the magnetic penetration length. This results in a profoundly nonlocal electrodynamics that strongly affects the quantum fluctuations in SIJJs.

Using a general Lagrangian approach, we derive the quantum electrodynamics of JPWs, which describes two interacting quantum fields. We analyze the first-order quantum electrodynamics of JPWs, which describes two interacting quantum fields. We also calculate the amplitude of their decay and scattering. We propose a mechanism of enhancement of macroscopic quantum tunneling (MQT) in stacks of intrinsic Josephson-junctions (SIJJs). Due to the long-range interactions between many junctions in the LSCs, the calculated MQT escape rate $\Gamma$ has a very nonlinear dependence on the number of junctions in the stack. This allows to quantitatively describe striking recent experiments in Bi2212 stacks.

Quantum theory for layered superconductors.— The electrodynamics of SIJJs can be described by the Lagrangian:

$$
\mathcal{L} = \sum_n \int dx \left( \frac{1}{2} \varphi_n'^2 + \frac{1}{2\gamma^2} p_n^2 - \frac{1}{2} (\partial_x \varphi_n)^2 - \frac{1}{2} (\partial_y p_n)^2 \right) \frac{1}{2} (\partial_x p_n + \partial_y \varphi_n + \partial_y p_n \partial_x \varphi_n). \tag{1}
$$

where $\varphi_n \equiv \chi_{n+1} - \chi_n - 2 \pi s A_{y}^{(n)}/\Phi_0$ is the gauge-invariant interlayer phase difference, and $p_n \equiv (s/\lambda_{ab}) \partial_{x} \chi_n - 2 \gamma s A_{y}^{(n)}/\Phi_0$ is the normalized superconducting momentum in the $n$th layer. Here, we introduce the phase $\chi_n$, the interlayer parameter, the interlayer distance $s$, the in-plane $\lambda_{ab}$ and out-of-plane $\lambda_{c}$ penetration depths, the anisotropy parameter $\gamma = \lambda_{c}/\lambda_{ab}$, flux quantum $\Phi_0$, and vector potential $\vec{A}$. The in-plane coordinate $x$ is normalized by $\lambda_{c}$; the time $t$ is normalized by $1/\omega_j$, where the plasma frequency is $\omega_j$; also, $\partial_x = \partial/\partial x$, $\partial_y f_n = \lambda_{ab} (f_{n+1} - f_n)/s$, and $\partial = \partial/\partial t$. We choose the $z$ axis pointed along the magnetic field. Varying the action $S = \int dt \mathcal{L}$ produces the dynamical equations

$$
\begin{align*}
\dot{\varphi}_n - \gamma^2 \varphi_n + \sin \varphi_n + \partial_x \partial_y p_n &= 0, \\
\dot{p}_n - \partial_x^2 p_n + p_n + \partial_x \partial_y \varphi_n &= 0, \tag{2}
\end{align*}
$$

which reduces to the usual coupled sine-Gordon equations for $\gamma^2 > 1$. Note that a Lagrangian approach for SIJJs can be formulated only for two interacting fields $\varphi$ and $p$, but not for $\varphi$ alone. This because of the 2D nature of the vector potential in SIJJs. So particles with two types of polarization can propagate. For a 1D Josephson junction, only one polarization is enough.

Linearizing Eqs. (2) results in the spectrum $\omega^2 = 1 + k_x^2/(1 + k_y^4)$ of the classical JPWs in the continuous limit (i.e., $k_y s \ll 1$) and $\gamma^2 \gg 1$. Here, $k_x$ and $k_y$ are the wave vectors (momentums in the quantum description; here, $\hbar = 1$) of the JPWs. In or-
order to quantize the JPWs we introduce the Hamiltonian, \( \mathcal{H} = \sum_n \int dx \{ \Pi_n \varphi_n + \Pi_n p_n - \mathcal{L} \} \), with the momenta \( \Pi_n \) and \( p_n \), of the \( \varphi_n \) and \( p_n \) fields, and require the standard commutation relations \( [\varphi_n^\dagger(x), \varphi_n(x')] = -i \delta(x - x') \delta_{n,n'} \), \( [\Pi_n(x), p_n(x')] = -i \delta(x - x') \delta_{n,n'} \) (all other commutators are zero), where \( \delta \) is either a delta function or Kronecker symbol. Expanding \( \varphi_n = 1 - \varphi_n^2/2 + \varphi_n^4/24 - \ldots \), we can write \( \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{an} \), where we include terms up to \( \varphi_n^4 \) in \( \mathcal{H}_0 \), and the anharmonic terms in \( \mathcal{H}_{an} \). Diagonalizing \( \mathcal{H}_0 \), we obtain the Hamiltonian for the Bosonic free fields \( a \) and \( b \): \( \mathcal{H}_0 = \sum_{k_y} \int (dk_x/2\pi) \left( \varepsilon_a(k) a^a + \varepsilon_b(k) b^b \right) \), where the energy of the quasiparticles are

\[
\varepsilon_a(k) = \left( 1 + \frac{k_x^2}{1 + k_y^2} \right)^{1/2}, \quad \varepsilon_b(k) = \frac{1}{\gamma} \frac{|k_x k_y|}{\sqrt{k_y^2 + 1}} (3)
\]

up to \( 1/\gamma^2 \), for \( \gamma \gg 1 \). The energy \( \varepsilon_a(k) \) coincides with the frequency \( \omega(k_x, k_y) \) for classical JPWs, while the quantum bosonic field \( b \) corresponds to the gapless branch of the excitations. The original fields \( \varphi_n, p_n \) in Eq. (1) are related to the free Bosonic fields \( a \) and \( b \) by

\[
\varphi \approx \frac{a^+ + a}{\sqrt{2} \varepsilon_a} + \frac{2^2 b^+ + b}{\sqrt{2} \varepsilon_b}, \quad p \approx 2 \left( \frac{a^+ + a}{\sqrt{2} \varepsilon_a} - \frac{b^+ + b}{\sqrt{2} \varepsilon_b} \right) (4)
\]

where \( Z = k_x k_y/(k_y^2 + 1) \). The case of a single Josephson junction corresponds to \( k_y = 0 \) resulting in a single field, \( a \).

Thermodynamics of quantum JPWs.— Finite temperatures, \( T \), excite both \( a \) and \( b \) quasiparticles providing contributions of the JPWs to the thermodynamical quantities. The thermal equilibrium internal energy \( E(T) = \sum_{k_y} \int (dk_x/2\pi) \varepsilon_a n_a + \varepsilon_b n_b \) of the system can be calculated using the usual Bosonic distributions \( n_{a,b} = 1/\exp(\varepsilon_{a,b}/T) - 1 \). The calculated dependence of \( E(T) \) for SIJJ's and, for comparison, for an “equivalent” stack of non-interacting Josephson junctions is shown in Fig. 1a. This clearly shows that the thermodynamic energies are significantly different for these systems, especially at low temperatures. Thus, finite temperatures easily thermally excite JPWs in layered superconductors, compared with the case of non-interacting junctions. The main origin of this enhancement is the suppression of both excitation energies \( \varepsilon_{a,b}(k_x, k_y) \) when increasing \( k_y \), which is associated with a stronger interlayer interaction. Other thermodynamic quantities, e.g., heat capacity, can be easily calculated using the standard expressions.

Interaction of Bosonic fields.— The interaction between the \( a \) and \( b \) fields, including the self-interaction, occurs due to the anharmonic terms in \( \mathcal{H}_{an} \approx (1/24) \sum_n \int dx \varphi_n^4 + \ldots \), where \( \varphi \) is given in (1). Here we consider the dominant first-order perturbation terms. Using the interaction representation, we obtain the amplitude \( S_{I} = 2\pi i \langle f | \mathcal{H}_{an} | i \rangle \delta(\varepsilon_i - \varepsilon_f) \) for the transition from the initial state \( |i\rangle \) to a final state \( |f\rangle \), where \( \varepsilon_i, f \) are the energies of the initial and final states. To first-order approximation in \( 1/\gamma^2 \), a decay (see Fig. 1b) of an \( a \)-JPW can occur in two channels: either \( 3a \) or \( 2a + b \). The amplitude, \( S_{\text{decay}} \), of the decay of the quantum \( a \)-JPW propagating along the \( x \)-axis, i.e., along the layers, \( k_1 = (k_x, 0) \), is determined by

\[
S_{\text{decay}} = \int \frac{i}{2\pi^3} \frac{d^2 k_1}{\gamma} \left\{ \frac{\delta(\varepsilon_a(k_1) - \varepsilon_a(k_1'))}{\sqrt{\varepsilon_a(k_1) \varepsilon_a(k_1')}} \right\} \delta(k_1 - k_1') (5)
\]

where the sums and products are performed over the final states with momenta \( k_1 \). Eq. (5) predicts the probability \( |S_{\text{decay}}|^2 \) to create JPWs propagating perpendicular to the layers by an \( a \)-JPW quantum propagating along the layers. Using Eqs. (3) and (5), one can conclude that the amplitude \( S_{\text{decay}} \) diverges for large \( k_y \), when resonance conditions \( |\varepsilon_a(k_1)| = 2 \) are fulfilled. For the former case (in dimensional units, \( \varepsilon_a(k_1) = 2\hbar \omega_J \)) the \( a \)-JPWs create \( a \)-JPW pairs, while \( \varepsilon_a = 3\hbar \omega_J \) the \( a \)-excitations diverge. Indeed, due to the \( \varphi^4 \) nonlinear interaction, a particle can only create two more additional particles, which could be either \( 2a \) or \( a + b \). The first process has a threshold \( 2\hbar \omega_J \) (similar to the \( 2m^2c^2 \) rest energy threshold for \( e^- + e^- \) pair creation in QED), while the second one has a \( \hbar \omega_J \) energy threshold due to the gapless nature of the \( b \) particles. Figure 1d shows the calculated probability, \( |S_{\text{decay}}|^2 \), of decay of a JPW-a-quantum versus the energy \( \varepsilon_a \) of the initial a-quantum. Both resonance peaks are clearly seen.

We can similarly analyze the scattering of \( a \)-JPWs. The diagrams in Fig. 1c show two input particles as the initial state \( |i\rangle \), corresponding to particles “1” and “2”, while the final state \( |f\rangle \) contains free particles “3” and “4”. These diagrams do not diverge for any input particle momentum. However, the scattering probability enormously increases for large transverse momentum transfer \( (k_y^2 - k_y^2) \), if the energies of the initial particles are close to \( \hbar \omega_J \). This can occur either for low \( k_x \) or large \( k_y \) of the particles 1 and 2. The decay and scattering resonances occur due to the unusual anisotropic spectrum of the JPWs, i.e., \( \varepsilon_a(k_x, k_y \to \infty) = 1 \) and \( \varepsilon_b(k_x, k_y \to \infty) = k_x/\gamma \).

Enhancement of macroscopic quantum tunneling.— Now we apply our theory to interpret very recent experiments [7] on MQT in Bi2212. To observe MQT, an external current \( J \), close to the critical value \( J_c \), was applied [7]. This produces an additional contribution \( J \varphi_n \) in the Lagrangian (1). When tunneling occurs, the phase difference in a junction changes from 0 to \( 2\pi \), which can be interpreted as the tunnelling of a fluxon through the contact. This process can be safely described within a semiclassical approximation and we use
the approach developed in Refs. 7, 10 to calculate the escape rate \( \Gamma = (\omega_p/2\pi)\sqrt{2\pi\hbar B}\exp(-B) \) of a fluxon through the potential barrier. Here, \( \omega_p \) is the oscillation frequency of a fluxon near the effective potential minimum, and \( B = \int_{-\infty}^{\infty} \alpha \tau \mathcal{L} \) is described by the Lagrangian Eq. 4 with the classical fields determined by Eqs. 2, if we add the term \( j = J/J_c \) in the right-hand-side of the first equation. In the limit \( \gamma^2 \gg 1 \), the equation for \( \varphi \) is reduced to standard sine-Gordon equations 8, which in the continuous limit, \( k_s \lambda_b \ll 1 \) and \( y = n_s/\lambda_b \), reads \( (1 - \partial^2/\partial y^2) [\varphi + \sin \varphi] - \partial^2 \varphi/\partial x^2 = j \). We seek a solution of the last equation in the form \( \varphi = \psi(x, y, t) + \arcsin(j) \), where the field \( \psi \) obeys

\[
\left(1 - \frac{\partial^2}{\partial y^2}\right) \left[\ddot{\psi} - j(1 - \cos \psi) + \sqrt{1 - j^2} \sin \psi \right] - \frac{\partial^2 \psi}{\partial x^2} = 0.
\]

Following the experimental setup 2, here we consider the SIJJs having the size \( L \gg s \) along the \( y \) direction, i.e., the total number of contacts \( N = L/s \gg 1 \), and the size of the SIJJs in the \( x \) direction, 2\( d \), is smaller than the Josephson length, \( \lambda_J = \gamma \sqrt{s\lambda_b}/2 \).

We can linearize Eq. 3 in all junctions except one, where the fluxon tunnels. The linearized equation can be solved by using the Fourier transformation, \( \psi = \sum_m \exp(-i\omega t) \cos(k_{xm}m) \psi_m(y, \omega) d\omega/2\pi \), where \( k_{xm} = \lambda_c \pi (2m + 1)/2d \). Since in the experiment 2 the sample connects two bulk superconductors, we can choose the phase difference to be zero at the top \( (y = L_1) \) and bottom \( (y = L_1 - L) \) layers of the sample, and \( y = 0 \) corresponds to the position of the fluxon tunneling. As a result, we derive the solution of the linearized equations in the form \( \psi_m(y) = \psi_m(0) \sinh[q_m(L_1 - y)]/\sinh[q_mL_1] \), for \( y > 0 \), and \( \psi_m(y) = \psi_m(0) \sinh[q_m(L_1 - y)]/\sinh[q_m(L - L_1)] \) for \( y < 0 \). Here, \( q_m = (k_{xm} + \sqrt{1 - j^2} - \omega^2)/\sqrt{1 - j^2} - \omega^2 \). Following the method described in Ref. 2 and requiring the continuity of both \( \psi \) and the current flowing through the central \( (y = 0) \) contact, we obtain the nonlinear equation for the phase difference in the junction with \( y = 0 \):

\[
\dot{\psi} - j(1 - \cos \psi) + \sqrt{1 - j^2} \sin \psi \sinh(q_mL_1) \sinh(q_m(L - L_1)) - \frac{\lambda^2}{\lambda_c^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \sum_m k_{xm}^2 \sinh(q_mL_1) \sinh(q_m(L - L_1)) / \sinh(q_mL) \cos(k_{xm}m) \psi_m(\omega).
\]

For an infinite \( (L, d \to \infty) \) sample, this equation coincides with the nonlocal equation for the Josephson vortex in the SIJJs 2. Eq. 7 can be used to describe the tunneling of a fluxon through a SIJJs with any width \( d \) and any number of layers \( N \). However, such a treatment can only be done numerically.

Now, we adopt Eq. 7 for the short \( (d/\lambda_J \ll 1) \) SIJJs used in 7, where \( d/\lambda_J \approx 2 \mu m/5 \mu m = 0.4 \). In this case, the phase difference \( \varphi \) changes slowly with \( x \) and the main contribution to the sum in the right-hand-side of 7 comes from the first harmonic \( k_{xm} = \pi \lambda_c/2d \). Neglecting contributions to the tunnelling process arising from higher-frequencies, \( \omega \gg \omega_J(1 - j^2)^{1/4} \), and integrating Eq. 7 over \( dx \), we derive for the phase difference \( \ddot{\psi} \), averaged over \( x \), the equation: \( d^2\dot{\psi}/dx^2 = -\partial V/\partial \psi \).

Here the effective potential \( V(\ddot{\psi}) \) can be written as

\[
V(\ddot{\psi}) = j(\sin \ddot{\psi} - \sqrt{1 - j^2} \cos \ddot{\psi} - 1) - g_n(j) \frac{\ddot{\psi}^2}{2},
\]

where \( n = L_1 \lambda_a/s \) labels the contact through which the
Fluxon tunnels, 

\[ g_n(j) = \frac{2(1 - j^2)^{1/2}}{\pi} \frac{Q}{\sinh(Qn)\sinh(Q(N-n))}, \]

and \( Q(j) = \pi \gamma s \left( 1 - j^2 \right)^{1/2} / 2d \). For an applied current \( J \) close to \( J_c \), where tunnelling was observed, we can expand both \( \cos \psi \) and \( \sin \psi \) and finally, derive \( V(\psi) = -\dot{\psi}^2(\psi - \psi_1)/6 \), where \( \psi_1(j) = 3j^2 \sqrt{1 - j^2 - g_n(j)} \). Using a semiclassical approach \( \Gamma_0 \) (i.e., \( B = k_0 \sum N \left[ 2V(\psi) \right]^{1/2} d\psi \)), and taking into account that the fluxon can tunnel through any junction of the SIJJs, we derive (now in dimensional units) 

\[ \Gamma_0 = \frac{\sum N (1 - g_n)^{5/4}}{\exp \left( -\frac{36 U_0}{5 h \omega_p} \left[ (1 - g_n)^{5/2} - 1 \right] \right)}, \]

where the summation is taken over all \( N \) contacts. Here, the effective Josephson frequency is \( \omega_p(j) = \omega_J(1 - j^2)^{1/4} \), the height of the potential barrier \( U_0 = 2E_J(1 - j^2)^{3/2}/3 \), the Josephson energy \( E_J = \Phi_0 J_c/2\pi c \), and the escape rate \( \Gamma_0(j) \) for a single Josephson junction (see, e.g., \[ \Gamma_0 \] is given by 

\[ \Gamma_0(j) = \frac{6 \omega_p(j)}{\pi} \frac{\sqrt{6\pi U_0(j)}}{h \omega_p(j)} \exp \left( -\frac{36 U_0(j)}{5 h \omega_p(j)} \right), \]

Figure 2 shows \( \Gamma(j) \), which very well describes experimental results in \[ \Gamma_0 \]. Some deviation between the experimental data and the theoretical prediction at high currents is due to a significant lowering of the potential barrier resulting in a decrease of the accuracy of the semiclassical approximation. The dependence of \( \Gamma \) on the number \( N \) of junctions is nonlinear due to the long-range interaction between different junctions, described by the last term in the expression \( \Gamma_0 \) for the effective potential. This nonlinearity is strong for relatively small \( N \lesssim N_c = d/\gamma L \) and the escape rate becomes proportional to \( N \) when the SIJJs thickness \( L \) exceeds the effective interaction length \( d/\gamma \). Very different types of MQT models in SIJJs, with no quantitative comparison with experimental data, are also being studied in \[ \Gamma_0 \]. For instance, here we consider the inductive coupling among layers, which is known to be strong, instead of the weak capacitive coupling among layers used in \[ \Gamma_0 \].

Conclusions.— We analyze the quantum effects in SIJJs. We develop a model for quantum excitations in SIJJs using two Bosonic fields. We also describe the interactions and thermodynamics of these fields. Moreover, we suggest a semiclassical theory of the fluxon quantum tunneling in SIJJs, which is in good agreement with recent remarkable experimental observations. The obtained results might be potentially useful for future designs of quantum THz devices.

We acknowledge partial support from the NSA, LPS, ARO, NSF grant No. EIA-0130383, JSPS-RFBR 06-02-91200, RFBR 06-02-16691, MEXT Grant-in-Aid for Young Scientists No 18740224, and an EPSRC Advanced Research Fellowship.

\[1\] R. Kleiner et al., Phys. Rev. Lett. 68, 2394 (1992); G. Hechtfischer et al., Phys. Rev. Lett. 79, 1365 (1997); E. Golobin et al., Phys. Rev. B 57, 130 (1998); J. Zitzmann et al., Phys. Rev. B 66, 064527 (2002); M. Tachiki et al., Phys. Rev. B 71, 134515 (2005); A. A. Abdumalikov et al., Phys. Rev. B 72, 144526 (2005).

\[2\] S. Sav'ev et al., Phys. Rev. B 72, 205 (2005); Physica C 437-438 281 (2006).

\[3\] S. Sav'ev et al., Phys. Rev. Lett. 94, 157004 (2005); H. Susanto et al., Phys. Rev. B 71, 174510 (2005).

\[4\] S. Sav'ev et al., Phys. Rev. Lett. 95, 187002 (2005).

\[5\] S. Sav'ev et al., Nature Physics 2, No 8, in press (2006).

\[6\] T. Bauch et al., Phys. Rev. Lett. 94, 087003 (2005); T. Bauch et al., Science 311, 57 (2006).

\[7\] X. Y. Jin et al., Phys. Rev. Lett. 96, 177003 (2006).

\[8\] J.Q. You, F. Nori, Phys. Today 58 (11), 42 (2005); R. McDermott et al., Science 307, 1299 (2005); A. J. Berkley et al., Science 300, 1548 (2003).

\[9\] L. N. Bulaevskii et al., Phys. Rev. B 50, 12831 (1994).

\[10\] S. Coleman, Phys. Rev. D 15, 2929 (1977).

\[11\] H. Grabert, P. Olschowski, and U. Weiss, Phys. Rev. B 36, 1931 (1987).

\[12\] M. Machida, T. Koyama, cond-mat/0605404 preprint.
M.V. Fistul, cond-mat/0606751