Towards a sufficient criterion for collapse in 3D Euler equations

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Abstract

A sufficient integral criterion for a blow-up solution of the Hopf equations (the Euler equations with zero pressure) is found. This criterion shows that a certain positive integral quantity blows up in a finite time under specific initial conditions. Blow-up of this quantity means that solution of the Hopf equation in 3D can not be continued in the Sobolev space $H^2(\mathbb{R}^3)$ for infinite time.

1 Introduction

In 1984 Beale, Kato and Maida \cite{1} showed that sufficient and necessary condition for a smooth solution to the 3D Euler equations for ideal incompressible fluids on the time interval $[0, t_0]$ is the finiteness of the integral,

$$\int_0^{t_0} \sup_r |\omega| \, dt < \infty.$$  \hspace{1cm} (1)

Here $\omega$ is the vorticity, connected with the velocity field $v$ by the standard relation:

$$\omega = \text{curl} \, v.$$

If the integral (1) is divergent, then the vorticity blows up (or collapses) in a finite time.

Proof of this criterion is based on the local existence theorem \cite{2}. According to this theorem a smooth solution to the 3D incompressible Euler equations exists if the initial conditions $v(0)$ belong to the Sobolev space $H^q(\mathbb{R}^3)$ for $q \geq 3$ and the solution itself is from the class,

$$v \in C([0, t_0]; H^q) \cap C^1([0, t_0]; H^{q-1}),$$

where the norm of $H^q$ is defined as follows,

$$\|v\|_{H^q(\mathbb{R}^3)} = \left( \sum_{\alpha \leq q} \int \left( \nabla^\alpha v \right)^2 \, dr \right)^{1/2}, \hspace{1cm} (2)$$

with $\alpha$ being a multi-index. In accordance with \cite{1}, the violation of this property leads to divergence of the integral (1) and to collapse for vorticity.

However, in order to use the criterion (1), effectively one needs to have an explicit solution to the Euler equation, that is practically very difficult or impossible. Therefore, the main task is to find sufficient conditions which guarantee blow-up of the vorticity.

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The first step in this direction would be to construct such criteria for models more simple than the incompressible Euler equations which is one of the main aims of this paper.

We consider the simplest variant – the Euler equations for a compressible fluid without pressure (the hydrodynamics of dust) which are sometimes called the Hopf equations:

\[
\frac{\partial \rho}{\partial t} + \text{div } \rho \mathbf{v} = 0, \tag{3}
\]

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = 0. \tag{4}
\]

These equations can be successfully solved by means of the Lagrangian description. In terms of Lagrangian variables the equation (4) describes the motion of free fluid particles. For this reason breaking is possible in this model: it happens when trajectories of fluid particles intersect. In terms of the mapping, describing the change of variables from the Eulerian description to the Lagrangian one, the process of breaking corresponds to a vanishing Jacobian, \( J \), of the mapping. This process can be considered as a collapse in this system: the density \( \rho \) as well as the velocity gradient become infinite at a finite time in the points where the Jacobian vanishes. By virtue of this the breaking is sometimes called gradient catastrophe.

In this paper we construct sufficient integral criteria for breaking within the model (3), (4) and verify what the functional conditions of the theorem [1] correspond to those for the given system. Our main conjecture is the following: breaking in the 3D case within the system yields a violation of Sobolev space already for \( q = 2 \) instead of \( q \geq 3 \) in the theorem [1].

To complete the introduction we would like to note that the model (3), (4) has a lot of astrophysical applications. According to the pioneering idea of Ya.B. Zeldovich [3], the formation of proto-galaxies due to breaking of stellar dust can be described by the system (3), (4) (see also the review [4]). It should be noticed also that recently [5, 6] the mechanism of breaking was used to explain collapse in incompressible fluids when, instead of the breaking of fluid particles, the breaking of vortex lines can happen which results in a blow-up of vorticity. Therefore we believe that the results presented in this paper can also be useful from the point of view of collapse in incompressible fluids.

\section{One-dimensional analysis}

We begin with a one-dimensional calculations where Eq. (4) has the form

\[
v_t + vv_x = 0. \tag{5}
\]

Consider the integrals

\[
I_n = \int v^n x \, dx
\]

with integer \( n \), where integration is assumed to be from \( -\infty \) to \( +\infty \).

It is easy to show that the evolution of these integrals is defined by the relations:

\[
\frac{dI_n}{dt} = -(n - 1) I_{n+1}, \tag{6}
\]

which can be obtained by means of (5) and integration by parts.
The relations for \( n = 2, 3 \) read:

\[
\frac{d}{dt} \int v_x^2 \, dx = - \int v_x^3 \, dx ,
\]

\[
\frac{d}{dt} \int v_x^3 \, dx = -2 \int v_x^4 \, dx .
\]

These differential relations allow one to find a closed differential inequality for the integral \( I_2 = \int v_x^2 \, dx \). Applying the Cauchy-Bunyakovsky inequality to (7) gives the estimation

\[
\frac{dI_2}{dt} \leq \left( \int v_x^4 \, dx \right)^{1/2} I_2^{1/2}.
\]

Substituting,

\[
I_4 = \int v_x^4 \, dx = \frac{1}{2} \frac{d^2I_2}{dt^2} ,
\]

into (8) we arrive at the closed differential inequality for \( I_2 \):

\[
I_2 \cdot \frac{d^2I_2}{dt^2} - 2 \left( \frac{dI_2}{dt} \right)^2 \geq 0.
\]

This inequality is solved by means of change of variables

\[
I_2 = X^\alpha > 0,
\]

with unknown exponent \( \alpha \). We will define the exponent by requiring the absence in (10) of the terms \( \sim X^{\alpha-2} X_t^2 \). Hence \( \alpha = -1 \) and the inequality becomes,

\[
X_t \leq 0.
\]

Using a mechanical interpretation of \( X \) as the coordinate of a particle and taking into account that the particle acceleration is negative, we can immediately conclude that the particle can reach origin \( X = 0 \) in a finite time if the initial particle velocity \( X_t(0) \) is negative. For \( I_2 \) this means a blow-up:

\[
I_2 = \frac{1}{X} \to \infty.
\]

Hence, by elementary calculation, we can estimate the collapse time \( t_0 \):

\[
t_0 < \frac{X(0)}{|X_t(0)|} \equiv \frac{I_2(0)}{|I_{2t}(0)|}.
\]

After multiplying (11) by \( X_t < 0 \) and integrating the result over time, we arrive at the estimation from above for \( I_2 \):

\[
I_2(t) \leq \frac{I_2^2(0)}{|I_{2t}(0)| (t_0 - t)}.
\]

Thus, the blow-up of \( I_2 \) takes place if the initial velocity \( X_t(0) \) is negative which is equivalent to the initial condition,

\[
I_{2t}(0) > 0
\]
or,
\[ I_3(0) = \int v_x^3 dx < 0. \]
If the initial distribution of \( v_0(x) \) is symmetric with respect to \( x \), then \( I_3(0) \equiv 0 \). However, as follows from (7), \( I_3 \) becomes negative already at \( t = +0 \) that results in a blow-up of the integral \( I_2 \).

To complete this section, let us compare the estimate (12) with the exact time dependence of \( I_2(t) \) near the breaking point.

In order to define this dependence, we differentiate (5) with respect to \( x \) that results in the equation,
\[ \frac{dv_x}{dt} = -v_x^2, \quad \left( \frac{d}{dt} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right). \] (13)
After integration we have
\[ v_x = \frac{v_x'(a)}{1 + v_x'(a)t}, \] (14)
where \( a \) is the initial coordinate of the fluid particle at \( t = 0 \), \( v_x(a) \) is the initial velocity, the prime denotes a derivative. The denominator in (14), \( 1 + v_x(a)t \), represents the Jacobian \( J \) of the mapping,
\[ x = a + v(a)t. \]
The Jacobian tends to zero for the first time at \( t = t_0 \), defined by
\[ t_0 = \min_a [-v_x^{-1}(a)] > 0. \] (15)
Near the singular point, \( a = a_0 \), corresponding to the minimum, \( (15) \) \( J \) can be approximated by the expression,
\[ J \approx \alpha(t_0 - t) + \gamma(a - a_0)^2. \]
Here
\[ \alpha = \frac{\partial J}{\partial t} \bigg|_{t_0,a_0} = 1/t_0, \quad 2\gamma = \frac{\partial^2 J}{\partial a^2} \bigg|_{t_0,a_0} > 0. \]

As a result, \( v_x \) takes a singularity as \( \tau = t_0 - t \to 0 \):
\[ v_x = \frac{v_x'(a_0)}{\alpha \tau + \gamma(a - a_0)^2}. \] (16)

Hence, one can see that the contribution from the singularity to the integral \( I_2 \),
\[ I_2 \approx \int \frac{v_x'(a_0)^2}{\alpha \tau + \gamma(a - a_0)^2} da \sim \frac{1}{\tau^{1/2}}, \] (17)
diverges as \( \tau \to 0 \) and satisfies the inequality (11).

3 Multi-dimensional breaking

In this section we generalize the above analysis to multi-dimensions.

In the multi-dimensional case, instead of \( v_x \) it is convenient to introduce the matrix \( U \) with matrix elements
\[ U_{ij} = \frac{\partial v_j}{\partial x_i}. \]
The equations of motion for this matrix have a form analogous to (12):

\[
\frac{dU}{dt} = -U^2
\]  

(18)

where

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla).
\]

Our aim is now to find the inequality corresponding to (9). Consider two scalar characteristics of the matrix \(U\): its trace, \(\text{tr } U \equiv \text{div } \mathbf{v}\) and determinant, \(\text{det } U\). From (18) we derive the following equations for these two quantities:

\[
\frac{d}{dt} \text{tr } U = -\text{tr } (U^2),
\]  

(19)

\[
\frac{d}{dt} \text{det } U = -\text{tr } U \cdot \text{det } U.
\]  

(20)

Now we introduce the positive definite integral,

\[
I = \int (\text{det } U)^2 \, dr,
\]

which, in the 1D case, coincides with \(I_2\). Due to (20), we have

\[
\frac{dI}{dt} = -\int \text{tr } U (\text{det } U)^2 dr.
\]  

(21)

This equation generalizes the equation (7) to the multi-dimensional case. The second derivative of \(I\) will be given by the expression,

\[
\frac{d^2 I}{dt^2} = \int [(\text{tr } U)^2 + \text{tr}(U^2)](\text{det } U)^2 \, dr.
\]  

(22)

Applying the Cauchy-Bunyakovsky inequality to the r.h.s. of (20) yields

\[
\frac{dI}{dt} \leq I^{1/2} \cdot \left( \int (\text{tr } U^2) (\text{det } U)^2 \, dr \right)^{1/2}.
\]  

(23)

From (22) we write the integral

\[
\int (\text{tr } U^2)(\text{det } U)^2 \, dr = \frac{d^2 I}{dt^2} - \int \text{tr}(U^2)(\text{det } U)^2 \, dr.
\]  

(24)

Now, we shall estimate the integral on the r.h.s. of this equation. We shall assume that all eigenvalues of the matrix \(U\) are real. Such an assumption means that the matrix \(U\) is close to its symmetric part, \(S = 1/2(U + U^T)\), the so-called stress tensor (here \(T\) denotes transpose). In this case the antisymmetric part of the matrix \(U\), the vorticity tensor, \(\Omega = 1/2(U - U^T)\) is small compared to \(S\). In particular, if \(\Omega = 0\) the matrix \(U\) coincides with \(S\), representing the Hessian of the velocity potential \(\Phi\):

\[
U_{ij} = S_{ij} = \partial^2 \Phi / \partial x_i \partial x_j.
\]

\[\text{The vorticity tensor } \Omega \text{ is connected with the vorticity } \omega \text{ by the relation } \Omega_{ij} = \frac{1}{2} \varepsilon_{ijk} \omega_k.\]
Under this assumption the trace of the matrix \( U \), \( \text{tr}(U^2) = \sum_{i=1}^{D} \lambda_i^2 \), where \( \lambda_i \) are eigenvalues of the \( U \) and \( D \) dimension, becomes positive. In this case the following relation between traces of \( U \) and \( U^2 \) can be easily proven:

\[
\sum_{i=1}^{D} \lambda_i^2 \geq \frac{1}{D} \left( \sum_{i=1}^{D} \lambda_i \right)^2.
\]

This inequality generates the following estimate between integrals:

\[
\int (\det U)^2 dr \int \text{tr} (U^2)(\det U)^2 dr \geq \frac{1}{D} \left( \int \text{tr} U \cdot (\det U)^2 dr \right)^2 \equiv \frac{1}{D} \left( \frac{dI}{dt} \right)^2.
\]  \( \text{(25)} \)

Substitution (24) and (25) into (23) gives the desired differential inequality,

\[
I \frac{d^2I}{dt^2} - \left( 1 + \frac{1}{D} \right) \left( \frac{dI}{dt} \right)^2 \geq 0.
\]  \( \text{(26)} \)

Its solution is sought in power form as before: \( I = x^\alpha \). Excluding terms proportional to \( X^{\alpha-2}X_t^2 \) we find that \( \alpha = -D \) and

\[
X = \frac{1}{ID}.
\]

For \( X \) this results in the same inequality as (10): \( X_{tt} < 0 \). The criterion for attaining the origin \( X = 0 \) will be also analogous:

\[
X_t(0) < 0 \text{ or } I_t(0) > 0.
\]

Almost the same form will have the estimate for the collapse time

\[
t_0 < \frac{DI(0)}{I_t(0)} = -\frac{1}{\langle \lambda(0) \rangle},
\]  \( \text{(27)} \)

where \( \langle \lambda \rangle \) is a mean eigenvalue defined in accordance with (20):

\[
\langle \lambda \rangle = \frac{1}{D} \sum_i \bar{\lambda}_i = \frac{1}{DI} \int \text{tr} U (\det U)^2 dr.
\]

For arbitrary \( D \), instead of (11), the following estimation appears for \( I \):

\[
I(t) \leq \frac{I^{D+1}(0)}{(DI_t(0)(t_0 - t))^D}.
\]  \( \text{(28)} \)

4 Comparison with exact solution

In order to compare the estimation (28) with the exact dependence of \( I \) we have to solve equation (18). This solution is

\[
U = U_0(a)(1 + U_0(a)t)^{-1}.
\]  \( \text{(29)} \)

Here \( a \) is the initial coordinates of a fluid particle and \( U_0(a) \) is the initial value of the matrix \( U \). By introducing the projectors \( P^{(k)} \) of the matrix \( U_0(a) \) \( (P^{(k)})^2 = P^{(k)} \) corresponding to
each of the eigenvalues \( \lambda_{0k}(a) \)), this expression can be rewritten in the form of a spectral expansion:

\[
U = \sum_{k=1}^{D} \frac{\lambda_{0k}}{1 + \lambda_{0k}t} P^{(k)}.
\]  

(30)

The projector \( P^{(k)} \), being a matrix function of \( a \), is expressed through the eigenvectors for the direct \( (U_0(a)\psi = \lambda_0\psi) \) and conjugated \( (\phi U_0(a) = \phi \lambda_0) \) spectral problems for the matrix \( U_0(a) \):

\[
P^{(k)}_{ij} = \psi^{(k)}_i \phi^{(k)}_j.
\]

where the vectors \( \psi^{(n)} \) and \( \phi^{(m)} \) with different \( n \) and \( m \) are mutually orthogonal:

\[
\psi^{(m)}_i \phi^{(n)}_i = \delta_{mn}.
\]

Hence, the determinant of the matrix \( U \) is defined by the product,

\[
\det U = \prod_{k=1}^{D} \frac{\lambda_{0k}}{1 + \lambda_{0k}t}.
\]

From (30) it follows also that singularity in \( U \) first time appears at \( t = t_0 \), defined from the condition [4, 7]:

\[
t_0 = \min_{k,a} [-1/\lambda_{0k}(a)],
\]

(31)

(compare this with (27)).

From (30), one can see that near the singular point only one term in the sum (30) survives,

\[
U \approx -\frac{P^{(n)}}{\tau + \gamma_{\alpha\beta} \Delta a_\alpha \Delta a_\beta},
\]

(32)

where the projector \( P^{(n)} \) is evaluated at the point \( a = a_0 \) and \( k = n \), corresponding to the minimum (31), \( \tau = t_0 - t, \Delta a = a - a_0 \), and

\[
2\gamma_{\alpha\beta} = -\frac{\partial^2 \lambda_{0n}^{-1}}{\partial a_\alpha \partial a_\beta} \bigg|_{a = a_0}
\]

is a positive definite matrix.

The remarkable formula (32) demonstrates that i) the matrix \( U \) tends to the degenerate one as \( t \to t_0 \) and ii) both parts of the matrix \( U \) in this limit, i.e. the stress tensor \( S \) and the vorticity tensor \( \Omega \), become simultaneously infinite (compare with [8]). It is interesting to note that at near singular time the ratio between both parts is fixed and governs by two relations following from the definition of the projector \( P \):

\[
P_S = P_S^2 + P_A^2, \quad P_A = P_S P_A + P_A P_S
\]

where \( P_S \) and \( P_A \) are respectively symmetric ("potential") and antisymmetric (vortical) parts of the projector \( P \). In particular, the second relation provides the collapsing solution for the equation for vorticity

\[
\frac{\partial \omega}{\partial t} = \text{curl} [\mathbf{v} \times \omega]
\]

which has the same form for both compressible and incompressible cases. It is also interesting to note that in the sense of the criterion [4], the collapsing solution (32) represents the marginal solution.
The asymptotic solution (32), far from the collapsing point should be matched with a "regular" solution. The corresponding matching scale can be estimated as $l_0 \approx \gamma^{2/3}$. This scale $l_0$ alone can be taken as the size of the collapsing region for (32). This remark now allows one to calculate the contribution from the breaking area to the integral $I$.

Substituting (29) into $I$ and using a change of variables, from $r$ to $a$, one can get the expression for this contribution,

$$I = C \int_V \frac{d^D a}{\tau + \gamma_{\alpha\beta}a_\alpha a_\beta},$$

(33)

where the constant is

$$C = \lambda_{0n} \det U_0 \prod_{k \neq n} \frac{\lambda_{0k}}{1 + \lambda_{0k}l_0} \bigg|_{a = a_0}.$$  

The integral is taken over the spherical volume $V$ with coordinate center at $a = a_0$ and size $\sim l_0$. Introducing a self-similar variable $\xi = a\tau^{-1/2}$, one can see that the contribution depends significantly on the dimension $D$. At $D = 1$ this integral behaves like $\tau^{-1/2}$ in full correspondence with (17). In this case, the integral over $\xi$ is convergent at large $\xi$ and is not sensitive to the cut-off size $l_0$. In the two-dimensional geometry, however, the integral (33) has a power dependence on $\tau$, but a logarithmic dependence on $l_0$ arising from integration on $\xi$:

$$I \sim \log \frac{l_0}{\tau^{1/2}}$$

that satisfies the inequality (28).

In the three-dimensional case, the integral (33) diverges at large scales as the first power of $\xi$, becomes proportional to the size of collapsing area $l_0$ as $\tau \to 0$:

$$I \sim l_0.$$  

This result for $D = 3$ formally contradicts to the blow-up sufficient condition found above. This contradiction indicates only that in the three-dimensional case the blow-up of the integral $I$ has no universal behavior near the singular time which should be expected following to the universal asymptotics (32).

5 Concluding remarks

Thus, the initial condition,

$$\frac{dI}{dt}(0) > 0,$$

represents a sufficient integral criterion for the collapse if the vorticity matrix $U$ is small in comparison to the stress tensor. Under this condition, the integral $I = \int (\det U)^2 d\tau$ becomes infinite in a finite time.

In turn, divergence of $I$ means that a solution can not be continued in the corresponding functional space. In the 1D case this is the Sobolev space $H^1(\mathbb{R})$. For $D = 2$, from the inequality,

$$I \leq \int |U|^4 d^2 r \quad (|U|^2 \equiv U_{ik}^2),$$

represents a sufficient integral criterion for the collapse if the vorticity matrix $U$ is small in comparison to the stress tensor. Under this condition, the integral $I = \int (\det U)^2 d\tau$ becomes infinite in a finite time.
together with the embedding Sobolev inequality,

$$
\left( \int |U|^4 d^2r \right)^{1/4} \leq C \|U\|_{H^1(\mathbb{R}^2)}
$$

it follows that $\|U\|_{H^1(\mathbb{R}^2)} \to \infty$ as $I \to \infty$. In terms of the velocity, this means that the solution is not continued in the Sobolev space $H^2(\mathbb{R}^2)$.

In the 3D case it is possible to write the following set of inequalities,

$$
I^{1/6} \leq \left( \int |U|^6 d^3r \right)^{1/6} \leq C \|U\|_{H^1(\mathbb{R}^3)},
$$

where the second inequality represents the partial case of the Sobolev embedding inequality [10].

Hence, one can see that $I \to \infty$ is equivalent to the divergence of the norm (2) for the Sobolev space $H^2(\mathbb{R}^3)$. Thus, the requirements for strong solutions in the hydrodynamic model (3), (4) are different from those for the 3D Euler equation for incompressible fluids. At the moment it is hard to say whether the results presented in this paper contradict to the theorem [2] (see also [9]) or not. In any case, it is a very interesting question. It should be added that collapse in incompressible fluids might happen through breaking of vortex lines (there are some arguments both analytical and numerical [3, 6] in a favor of such point of view). In this case the corresponding norms would also blow up for the same Sobolev space $H^2(\mathbb{R}^3)$.

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