Self-dual codes over $\mathbb{F}_5$ and $s$-extremal unimodular lattices

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Abstract

New $s$-extremal extremal unimodular lattices in dimensions 38, 40, 42 and 44 are constructed from self-dual codes over $\mathbb{F}_5$ by Construction A. In the process of constructing these codes, we obtain a self-dual $[44, 22, 14]$ code over $\mathbb{F}_5$. In addition, the code implies a $[43, 22, 13]$ code over $\mathbb{F}_5$. These codes have larger minimum weights than the previously known $[44, 22]$ codes and $[43, 22]$ codes, respectively.

1 Introduction

A (linear) $[n, k]$ code $C$ over $\mathbb{F}_5$ is a $k$-dimensional subspace of $\mathbb{F}_5^n$, where $\mathbb{F}_5$ is the finite field of order 5. The dual code $C^\perp$ of $C$ is defined as $\{x \in \mathbb{F}_5^n \mid x \cdot y = 0 \text{ for all } y \in C\}$, where $x \cdot y$ is the standard inner product. A code $C$ is called self-dual if $C = C^\perp$. Self-dual codes are one of the most interesting classes of codes. This interest is justified by many combinatorial objects and algebraic objects related to self-dual codes. Unimodular lattices are one of the objects related to self-dual codes. In addition, there are many similarities between self-dual codes and unimodular lattices (see [4]). Construction A, which is the most important construction of unimodular lattices from self-dual codes, gives some similarities.

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Extremal unimodular lattices are unimodular lattices meeting a certain upper bound on minimum norms, and s-extremal unimodular lattices are odd unimodular lattices meeting a certain upper bound on minimum norms of the shadows. In this paper, new s-extremal extremal unimodular lattices in dimensions 38, 40, 42 and 44 are constructed from self-dual codes over \( \mathbb{F}_5 \) by Construction A. In the process of constructing the above self-dual codes, we obtain a self-dual \([44, 22, 14]\) code over \( \mathbb{F}_5 \). This code has larger minimum weight than the previously known self-dual \([44, 22]\) codes. It is a fundamental problem to determine the largest minimum weight \( d_5(n) \) among all self-dual \([n, n/2]\) codes over \( \mathbb{F}_5 \) for a given \( n \). Much work has been done concerning this fundamental problem (see e.g. \([9], [10], [12], [14], [16], [17] \) and \([23]\)). In addition, the new self-dual code implies a \([43, 22, 13]\) code over \( \mathbb{F}_5 \), which has larger minimum weight than the previously known \([43, 22]\) codes.

This paper is organized as follows. In Section 2, we give some definitions, notations and basic results used in this paper. Two methods for constructing self-dual codes are given, namely, quasi-twisted codes and four-negacirculant codes. In Section 3, new s-extremal extremal unimodular lattices in dimension 44 are constructed from four-negacirculant self-dual codes over \( \mathbb{F}_5 \) by Construction A. In the process of constructing these codes, we obtain a self-dual \([44, 22, 14]\) code \( C_{44} \) over \( \mathbb{F}_5 \). This code has larger minimum weight than the previously known self-dual codes. We emphasize that \( C_{44} \) is the first example as not only self-dual \([44, 22, 14]\) codes but also (linear) \([44, 22, 14]\) codes. In addition, \( C_{44} \) implies a \([43, 22, 13]\) code over \( \mathbb{F}_5 \), which has larger minimum weight than the previously known \([43, 22]\) codes. Section 3 also lists the current information on the largest minimum weight among self-dual \([n, n/2]\) codes over \( \mathbb{F}_5 \) for \( 22 \leq n \leq 72 \). In Section 4, new s-extremal extremal unimodular lattices in dimensions 38, 40 and 42 are constructed from self-dual codes over \( \mathbb{F}_5 \) by Construction A. These self-dual codes are constructed as quasi-twisted codes or four-negacirculant codes. As a summary, we list the current information on the existence of non-isomorphic s-extremal unimodular lattices with minimum norm 4.

All computer calculations in this paper were done by programs in MAGMA [1].

2 Preliminaries

In this section, we give definitions, notations and basic results used in this paper.
2.1 Unimodular lattices

A (Euclidean) lattice $L \subset \mathbb{R}^n$ in dimension $n$ is unimodular if $L = L^*$, where the dual lattice $L^*$ of $L$ is defined as $\{ x \in \mathbb{R}^n \mid x \cdot y \in \mathbb{Z} \text{ for all } y \in L \}$ under the standard inner product $x \cdot y$. A unimodular lattice $L$ is even if the norm $x \cdot x$ of every vector $x$ of $L$ is even, and odd otherwise. An even unimodular lattice in dimension $n$ exists if and only if $n$ is divisible by eight, while an odd unimodular lattice exists for every dimension. The minimum norm $\min(L)$ of a unimodular lattice $L$ is the smallest norm among all nonzero vectors of $L$. The kissing number of $L$ is the number of vectors of minimum norm in $L$. Two lattices $L$ and $L'$ are isomorphic, denoted $L \cong L'$, if there is an orthogonal matrix $A$ with $L' = \{ xA \mid x \in L \}$.

Let $L$ be an odd unimodular lattice in dimension $n$. The shadow $S(L)$ of $L$ is defined to be $S(L) = L_0 \setminus L$, where $L_0$ denotes the even sublattice of $L$. Shadows of odd unimodular lattices appeared in [3] (see also [4, p. 440]), and shadows play an important role in the study of odd unimodular lattices. The theta series of an odd unimodular lattice $L$ and its shadow $S(L)$ are the formal power series $\theta_L(q) = \sum_{x \in L} q^{x \cdot x}$ and $\theta_{S(L)}(q) = \sum_{x \in S(L)} q^{x \cdot x}$, respectively. Conway and Sloane [3] showed that when the theta series of an odd unimodular lattice $L$ in dimension $n$ is written as:

$$\sum_{j=0}^{[n/8]} a_j \theta_3(q)^{n-8j} \Delta_8(q)^j,$$  \hspace{1cm} (1)

the theta series of the shadow $S(L)$ is written as:

$$\sum_{j=0}^{[n/8]} \frac{(-1)^j}{16^j} a_j \theta_2(q)^{n-8j} \theta_4(q^2)^{8j},$$  \hspace{1cm} (2)

where $\Delta_8(q) = q \prod_{m=1}^{\infty} (1 - q^{2m-1})^8(1 - q^{4m})^8$, and $\theta_2(q), \theta_3(q), \theta_4(q)$ are the Jacobi theta series [4].

2.2 Extremal unimodular lattices and $s$-extremal unimodular lattices

By considering the theta series of odd unimodular lattices and their shadows, Rains and Sloane [25] showed that the minimum norm $\min(L)$ of an odd
unimodular lattice $L$ in dimension $n$ is bounded by
\[
\min(L) \leq \begin{cases} 
2\left\lfloor \frac{n}{24} \right\rfloor + 2 & \text{if } n \neq 23, \\
3 & \text{if } n = 23.
\end{cases} \tag{3}
\]

A unimodular lattice meeting the bound (3) with equality is called \textit{extremal}. In addition, it was shown in \cite{8} that
\[
4 \min(S(L)) = 15 \quad \text{if } n = 23 \text{ and } \min(L) = 3,
\]
\[
8 \min(L) + 4 \min(S(L)) \leq 8 + n \quad \text{otherwise}, \tag{4}
\]
where $\min(S(L))$ denotes the minimum norm of $S(L)$, that is, the smallest norm among all vectors of $S(L)$. An odd unimodular lattice meeting the bound (4) with equality is called \textit{s-extremal}. If an s-extremal unimodular lattice $L$ in dimension $n$ having $\min(L) = 4$ exists, then $n \in \{32, 36, 37, \ldots, 47\}$ \cite{8, p. 148}.

### 2.3 Invariants $\text{inv}(L)_i$

For a given unimodular lattice $L$ in dimension $n$ having $\min(L) = 4$, let $L(4)$ denote the set of vectors of norm 4 in $L$. There is a subset $L(4)^+$ of $L(4)$ such that
\[
L(4) = L(4)^+ \cup L(4)^- \text{ and } L(4)^+ \cap L(4)^- = \emptyset,
\]
where $L(4)^- = \{-x \mid x \in L(4)^+\}$. For a given unimodular lattice $L$ and a nonnegative integer $i$, we define
\[
\text{inv}(L)_i = \left| \{ (x, y) \in L(4)^+ \times L(4)^+ \mid x \cdot y \in \{i, -i\} \} \right|.
\]

It is trivial that $\text{inv}(L)_i = \text{inv}(L')_i$ if $L \cong L'$ for a nonnegative integer $i$.

**Lemma 1.** $\text{inv}(L)_3 = 0$, $\text{inv}(L)_4 = |L^+(4)|$ and $\text{inv}(L)_0 + \text{inv}(L)_1 + \text{inv}(L)_2 = |L^+(4)|^2 - |L^+(4)|$.

**Proof.** Let $x$ and $y$ be vectors of $L(4)$. By considering $(x + y) \cdot (x + y)$ and $(x-y) \cdot (x-y)$, it follows that $-2 \leq x \cdot y \leq 2$ if $x \not\in \{y, -y\}$, and it follows that $x \in \{y, -y\}$ if and only if $x \cdot y \in \{-4, 4\}$. The last assertion is trivial.  

Hence, it is sufficient to consider only $\text{inv}(L)_0$ and $\text{inv}(L)_1$. 


2.4 Self-dual codes and Construction A

Let $\mathbb{F}_5$ be the finite field of order 5. Throughout this paper, we take the elements of $\mathbb{F}_5$ to be either 0, 1, 2, 3, 4 or 0, ±1, ±2, using whichever form is more convenient. A (linear) $[n, k]$ code $C$ over $\mathbb{F}_5$ is a $k$-dimensional subspace of $\mathbb{F}_5^n$. All codes in this paper mean linear codes and we omit the term linear. The weight $\text{wt}(x)$ of a vector $x$ of $\mathbb{F}_5^n$ is the number of non-zero components of $x$. A vector of $C$ is called a codeword. The minimum non-zero weight of all codewords in $C$ is called the minimum weight of $C$. An $[n, k]$ code with minimum weight $d$ is called an $[n, k, d]$ code. The weight enumerator of $C$ is $\sum_{c \in C} y^{\text{wt}(c)}$.

The dual code $C^\perp$ of an $[n, k]$ code $C$ over $\mathbb{F}_5$ is defined as

$$C^\perp = \{ x \in \mathbb{F}_5^n \mid x \cdot y = 0 \text{ for all } y \in C \},$$

where $x \cdot y$ is the standard inner product. A code $C$ is called self-dual if $C = C^\perp$. If an $[n, k]$ code is self-dual, then it is trivial that $k = n/2$. It is a fundamental problem to determine the largest minimum weight $d_5(n)$ among all self-dual $[n, n/2]$ codes over $\mathbb{F}_5$ for a given $n$.

Let $C$ be a self-dual $[n, n/2]$ code over $\mathbb{F}_5$. Then the following lattice

$$A_5(C) = \frac{1}{\sqrt{5}} \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid (x_1 \mod 5, \ldots, x_n \mod 5) \in C \}$$

is a unimodular lattice in dimension $n$. This construction of lattices is well known as Construction A.

2.5 Self-dual codes constructed from negacirculant matrices

An $n \times n$ negacirculant matrix has the following form

$$\begin{pmatrix}
  r_0 & r_1 & r_2 & \cdots & r_{n-1} \\
  -r_{n-1} & r_0 & r_1 & \cdots & r_{n-2} \\
  -r_{n-2} & -r_{n-1} & r_0 & \cdots & r_{n-3} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  -r_1 & -r_2 & -r_3 & \cdots & r_0
\end{pmatrix}$$

Throughout this paper, let $I_n$ denote the identity matrix of order $n$ and let $A^T$ denote the transpose of a matrix $A$. 5
Let $A$ be an $n \times n$ negacirculant matrix. A $[2n, n]$ code over $\mathbb{F}_5$ having the following generator matrix

$$
( I_n \ A )
$$

is called a *quasi-twisted* code or a double twistulant code. Let $A$ and $B$ be $n \times n$ negacirculant matrices. A $[4n, 2n]$ code over $\mathbb{F}_5$ having the following generator matrix

$$
\begin{pmatrix}
I_{2n} & A & B \\
-A^T & B & A^T
\end{pmatrix}
$$

is called a *four-negacirculant* code. Many quasi-twisted self-dual codes and four-negacirculant self-dual codes with large minimum weights are known (see e.g. [16], [17], [21] and [22]).

3 New $s$-extremal extremal unimodular lattices in dimension 44 and new self-dual codes over $\mathbb{F}_5$

In this section, new $s$-extremal extremal unimodular lattices in dimension 44 are constructed from four-negacirculant self-dual codes over $\mathbb{F}_5$ by Construction A. In the process of constructing these codes, we obtain a self-dual $[44, 22, 14]$ code over $\mathbb{F}_5$. This code has larger minimum weight than the previously known $[44, 22]$ codes. In addition, the new self-dual code implies a $[43, 22, 13]$ code over $\mathbb{F}_5$, which has larger minimum weight than the previously known $[43, 22]$ codes.

3.1 New $s$-extremal extremal unimodular lattices in dimension 44

Let $L_{44}$ be an odd unimodular lattice in dimension 44 having minimum norm 4. By using [1] and [2], the possible theta series of $L_{44}$ and its shadow $S(L_{44})$ are determined as follows:

$$
\theta_{L_{44}}(q) = 1 + (6600 + 16\alpha)q^4 + (811008 - 128\alpha - 65536\beta)q^5 + \cdots \quad \text{and}
$$

$$
\theta_{S(L_{44})}(q) = \beta q^3 + (\alpha - 76\beta)q^3 + (1622016 - 52\alpha + 2806\beta)q^5 + \cdots ,
$$
respectively, where \( \alpha \) and \( \beta \) are nonnegative integers \([19]\). It turns out that \( L_{44} \) has kissing number 6600 if and only if \( L_{44} \) is \( s \)-extremal. Two \( s \)-extremal extremal unimodular lattices in dimension 44 are previously known. More precisely, \( A_5(C_{44}) \) in \([19]\) and \( A_5(C_{44,5}(D_{22})) \) in \([21]\) are the known lattices, and we denote the lattices by \( L_{44,1} \) and \( L_{44,2} \), respectively. We remark that the two lattices are constructed from some self-dual codes over \( \mathbb{F}_5 \) by Construction A.

We calculated by MAGMA the invariants \( \text{inv}(L)_i \) \( (i = 0, 1) \) given in Section 2.3 for \( L = L_{44,1} \) and \( L_{44,2} \), where the results are listed in Table 1. This was done using the MAGMA function \texttt{ShortVectors}. From the table, it holds that the two lattices \( L_{44,1} \) and \( L_{44,2} \) are non-isomorphic.

| \( L \) | \( \text{inv}(L)_0 \) | \( \text{inv}(L)_1 \) |
|---|---|---|
| \( L_{44,1} \) | 7097112 | 3750384 |
| \( L_{44,2} \) | 7089192 | 3760944 |

In order to construct new \( s \)-extremal extremal odd unimodular lattices, we tried to find four-negacirculant self-dual \([44,22]\) codes over \( \mathbb{F}_5 \). Then we found 50 codes \( C_{44,i} \) satisfying the condition that \( A_5(C_{44,i}) \) have minimum norm 4 and kissing number 6600 and the condition that

\[
|\{(\text{inv}(L)_0, \text{inv}(L)_1) \mid L \in \mathcal{L}\}| = 52,
\]

where \( \mathcal{L} = \{L_{44,1}, L_{44,2}\} \cup \{A_5(C_{44,i}) \mid i \in \{1, 2, \ldots, 50\}\} \). The self-duality was verified by the MAGMA function \texttt{IsSelfDual}. The minimum norm and the kissing number were calculated by the MAGMA functions \texttt{Minimum} and \texttt{KissingNumber}, respectively. For \( L = A_5(C_{44,i}) \), the results \( \text{inv}(L)_j \) \( (j = 0, 1) \) are listed in Table 2. From Tables 1 and 2 we have the following:

**Lemma 2.** The 52 lattices \( L_{44,1}, L_{44,2} \) and \( A_5(C_{44,i}) \) \( (i = 1, 2, \ldots, 50) \) are non-isomorphic.

For the 50 codes \( C_{44,i} \), the first rows \( r_A(C_{44,i}) \) and \( r_B(C_{44,i}) \) of the negacirculant matrices \( A \) and \( B \) in the generator matrices of form (6) are listed in Table 3.

For an odd unimodular lattice \( L \), there are cosets \( L_1, L_2, L_3 \) of \( L_0 \) such that \( L_0 = L_0 \cup L_1 \cup L_2 \cup L_3 \), where \( L = L_0 \cup L_2 \) and \( S(L) = L_1 \cup L_3 \). Two
Table 2: \( \text{inv}(L)_j \) \((j = 0, 1)\) for \( L = A_5(C_{44,i}) \)

| \( i \) | \( \text{inv}(L)_0 \) | \( \text{inv}(L)_1 \) | \( i \) | \( \text{inv}(L)_0 \) | \( \text{inv}(L)_1 \) | \( i \) | \( \text{inv}(L)_0 \) | \( \text{inv}(L)_1 \) |
|---|---|---|---|---|---|---|---|---|
| 1 | 7068600 | 3788400 | 18 | 7083252 | 3768864 | 35 | 7092756 | 3756192 |
| 2 | 7071372 | 3784704 | 19 | 7083648 | 3768336 | 36 | 7093152 | 3755664 |
| 3 | 7074144 | 3781008 | 20 | 7084440 | 3767280 | 37 | 7093548 | 3755136 |
| 4 | 7074540 | 3780480 | 21 | 7085232 | 3766224 | 38 | 7093944 | 3754608 |
| 5 | 7074936 | 3779952 | 22 | 7085628 | 3765696 | 39 | 7094340 | 3754080 |
| 6 | 7075332 | 3779424 | 23 | 7086420 | 3764640 | 40 | 7094736 | 3753552 |
| 7 | 7076520 | 3777840 | 24 | 7086816 | 3764112 | 41 | 7095132 | 3753024 |
| 8 | 7077312 | 3776784 | 25 | 7087212 | 3763584 | 42 | 7095528 | 3752496 |
| 9 | 7078104 | 3775728 | 26 | 7087608 | 3763056 | 43 | 7096320 | 3751440 |
| 10 | 7078500 | 3775200 | 27 | 7088004 | 3762528 | 44 | 7097904 | 3749328 |
| 11 | 7078896 | 3774672 | 28 | 7089588 | 3760416 | 45 | 7098300 | 3748800 |
| 12 | 7079292 | 3774144 | 29 | 7099984 | 3759888 | 46 | 7100676 | 3745632 |
| 13 | 7079688 | 3773616 | 30 | 7099380 | 3759360 | 47 | 7103448 | 3741936 |
| 14 | 7081272 | 3771504 | 31 | 7091172 | 3758304 | 48 | 7105824 | 3738768 |
| 15 | 7081668 | 3770976 | 32 | 7091568 | 3757776 | 49 | 7107012 | 3737184 |
| 16 | 7082460 | 3769920 | 33 | 7091964 | 3757248 | 50 | 7107804 | 3736128 |
| 17 | 7082856 | 3769392 | 34 | 7092360 | 3756720 |

Lattices \( L \) and \( L' \) are neighbors if both lattices contain a sublattice of index 2 in common. If \( L \) is an odd unimodular lattice in dimension \( n \) and \( n \) is a multiple of four, then there are two unimodular lattices containing \( L_0 \), which are rather than \( L \), namely, \( L_0 \cup L_1 \) and \( L_0 \cup L_3 \) (see [5]). Note that the two neighbors are even if \( n \) is a multiple of eight. We denote the two unimodular neighbors by

\[ N_1(L) = L_0 \cup L_1 \text{ and } N_2(L) = L_0 \cup L_3. \]

**Lemma 3.** Let \( L \) be an \( s \)-extremal extremal unimodular lattice in dimension 44. Then \( N_1(L) \) and \( N_2(L) \) are also \( s \)-extremal extremal unimodular lattices.

**Proof.** Since \( L \) is an \( s \)-extremal extremal unimodular lattice in dimension 44, the minimum norm of the shadow \( S(L) \) is 5. Thus, \( N_1(L) \) and \( N_2(L) \) have minimum norm 4. In addition, the shadows of \( N_1(L) \) and \( N_2(L) \) are \( L_2 \cup L_3 \) and \( L_2 \cup L_1 \), respectively. Hence, \( \min(S(N_1(L))) = \min(S(N_2(L))) = 5 \). The result follows. \[ \square \]

By the above lemma, more \( s \)-extremal extremal unimodular lattices in dimension 44 are constructed as \( N_j(L_{44,i}) \), \( N_j(L_{44,2}) \) and \( N_j(A_5(C_{44,i})) \) (\( i = 1, 2, \ldots, 50 \)) (\( j = 1, 2 \)).
Lemma 4. Let $L$ and $L'$ be $s$-extremal extremal unimodular lattices in dimension 44. If $\text{inv}(L)_0 \neq \text{inv}(L')_0$ or $\text{inv}(L)_1 \neq \text{inv}(L')_1$, then $L \not\cong N_1(L')$ and $L \not\cong N_2(L')$.

Proof. Since the minimum norm of the shadow of $L'$ is 5, the two sets of vectors of norm 4 in $L'$ and $M$ are identical for $M = N_1(L')$ and $N_2(L')$. The result follows.

In addition, we verified that the three lattices $L, N_1(L)$ and $N_2(L)$ are non-isomorphic for each lattice $L = L_{44,1}, L_{44,2}$ and $A_5(C_{44,i}) (i = 1, 2, \ldots, 50)$. The neighbors $N_1(L)$ and $N_2(L)$ were constructed using the MAGMA functions EvenSublattice and Dual. In addition, the non-isomorphisms were verified by the MAGMA function IsIsomorphic. By Lemma 4, we have the following:

**Proposition 5.** There are at least 156 non-isomorphic $s$-extremal extremal unimodular lattices in dimension 44.

We stopped our search after finding the 50 self-dual codes $C_{44,i}$, which give 150 lattices $A_5(C_{44,i}), N_1(A_5(C_{44,i}))$ and $N_2(A_5(C_{44,i}))$. Our feeling is that the number of non-isomorphic $s$-extremal extremal unimodular lattices in dimension 44 might be even bigger.

### 3.2 A self-dual $[44, 22, 14]$ code over $\mathbb{F}_5$ and its related codes

We verified by MAGMA that $C_{44,50}$ has minimum weight 14, $C_{44,29}$ has minimum weight 13 and $C_{44,i}$ ($i = 1, 2, \ldots, 28, 30, \ldots, 49$) have minimum weight 12. The minimum weight was calculated by the MAGMA function MinimumWeight.

**Proposition 6.** There is a self-dual $[44, 22, 14]$ code over $\mathbb{F}_5$.

The code $C_{44,50}$ has generator matrix of form (6), where the negacirculant matrices $A$ and $B$ have the first rows

$$(1003210404) \text{ and } (1224113344),$$

respectively. The first few terms of the weight enumerator of $C_{44,50}$ are given by

$$1 + 12056y^{14} + 95920y^{15} + 807312y^{16} + 4677728y^{17} + \cdots.$$
This was calculated by the MAGMA function `NumberOfWord`.

Let $d_5(n)$ denote the largest minimum weight among all self-dual $[n, n/2]$ codes over $\mathbb{F}_5$. It was known that $13 \leq d_5(44) \leq 19$ [16, Table III]. Hence, $C_{44,50}$ improves the previously known lower bound on the largest minimum weight $d_5(44)$. As a summary, we list the current information on the largest minimum weight $d_5(n)$ in Table 3 along with references for $22 \leq n \leq 72$. The table updates [16, Table III].

| $n$  | $d_5(n)$ | References | $n$  | $d_5(n)$ | References |
|------|----------|------------|------|----------|------------|
| 22   | 8–10     | [23]       | 48   | 14–20    | [6]        |
| 24   | 9–10     | [23]       | 50   | 14–20    | [9]        |
| 26   | 9–11     | [9]        | 52   | 15–21    | [9]        |
| 28   | 10–12    | [9]        | 54   | 16–22    | [9]        |
| 30   | 10–12    | [9]        | 56   | 16–23    | [9]        |
| 32   | 11–13    | [17]       | 58   | 16–24    | [6]        |
| 34   | 11–14    | [9]        | 60   | 18–24    | [15]       |
| 36   | 12–15    | [9]        | 62   | 17–25    | [9]        |
| 38   | 12–16    | [6]        | 64   | 18–26    | [15]       |
| 40   | 13–17    | [6]        | 66   | 18–27    | [12]       |
| 42   | 13–18    | [16]       | 68   | 18–28    | [9]        |
| 44   | 14–19    | $C_{44,50}$| 70   | 20–29    | [12]       |
| 46   | 14–20    | [6]        | 72   | 22–29    | [12]       |

It is a main problem in coding theory to determine the largest minimum weights $d_q(n,k)$ among all $[n,k]$ codes over a finite field of order $q$ for a given $(q,n,k)$. The current information on $d_5(n,k)$ can be found in [11]. For example, it was known that $12 \leq d_5(43,22) \leq 18$ and $13 \leq d_5(44,22) \leq 19$. We emphasize that $C_{44,50}$ is the first example as not only self-dual [44, 22, 14] codes but also (linear) [44, 22, 14] codes. We verified that all punctured codes of $C_{44,50}$ are [43, 22, 13] codes over $\mathbb{F}_5$. The punctured codes were constructed by the MAGMA functions `PunctureCode`.

**Proposition 7.** There is a [43, 22, 13] code over $\mathbb{F}_5$.

The self-dual [44, 22, 14] code $C_{44,50}$ and the punctured codes improve the previously known lower bounds on $d_5(43,22)$ and $d_5(44,22)$.

**Corollary 8.** $13 \leq d_5(43,22) \leq 18$ and $14 \leq d_5(44,22) \leq 19$. 

10
4 Construction of $s$-extremal extremal unimodular lattices in dimensions 38, 40 and 42

In this section, we construct new $s$-extremal extremal unimodular lattices in dimensions 38, 40 and 42 from self-dual codes over $\mathbb{F}_5$ by Construction A. These self-dual codes are constructed as four-negacirculant self-dual codes or quasi-twisted self-dual codes.

4.1 New $s$-extremal extremal odd unimodular lattices in dimension 40

By using (1) and (2), the possible theta series of an extremal odd unimodular lattice $L_{40}$ in dimension 40 and its shadow $S(L_{40})$ are determined as follows:

$$\theta_{L_{40}}(q) = 1 + (19120 + 256\alpha)q^4 + (1376256 - 4096\alpha)q^5 + \cdots$$
$$\theta_{S(L_{40})}(q) = \alpha q^2 + (40960 - 56\alpha)q^4 + (87818240 + 1500\alpha)q^6 + \cdots,$$

respectively, where $\alpha$ is an even integer with $0 \leq \alpha \leq 80$ [2]. It is trivial that $L_{40}$ has kissing number 19120 if and only if $L_{40}$ is $s$-extremal. An $s$-extremal extremal odd unimodular lattice in dimension 40 was explicitly constructed in [2] and three non-isomorphic $s$-extremal extremal odd unimodular lattices were explicitly constructed in [21]. We calculated by MAGMA $\text{inv}(L)_i (i = 0, 1)$ for the four lattices, where the results are listed in Table 4.

| $L$          | $\text{inv}(L)_0$ | $\text{inv}(L)_1$ |
|-------------|------------------|------------------|
| [2]         | 56589480         | 34257920         |
| $A_{13}(C_{13,40})$ in [21] | 56644200 | 34184960 |
| $A_9(C_{9,40})$ in [21] | 565549160 | 34311680 |
| $A_{19}(C_{19,40})$ in [21] | 56553480 | 34305920 |

In order to construct new $s$-extremal extremal odd unimodular lattices, we tried to find self-dual [40, 20] codes $C$ over $\mathbb{F}_5$ by considering four-negacirculant codes such that $A_5(C)$ have minimum norm 4 and kissing number 19120. Then we found 50 self-dual [40, 20] codes $C_{40,i}$ over $\mathbb{F}_5$ such that their lattices $A_5(C_{40,i})$ are $s$-extremal extremal odd unimodular lattices and

$$|\{(\text{inv}(L)_0, \text{inv}(L)_1) \mid L \in \{A_5(C_{40,i}) \mid i \in \{1, 2, \ldots, 50\}\}| = 50.$$
The results $\text{inv}(A_5(C_{40,i}))_j$ ($j = 0, 1$) are listed in Table 5. From Tables 4 and 5 we have the following:

**Proposition 9.** There are at least 54 non-isomorphic $s$-extremal extremal odd unimodular lattices in dimension 40.

Table 5: $\text{inv}(L)_j$ ($j = 0, 1$) for $L = A_5(C_{40,i})$

| $i$ | $\text{inv}(L)_0$ | $\text{inv}(L)_1$ | $i$ | $\text{inv}(L)_0$ | $\text{inv}(L)_1$ |
|-----|-------------------|-------------------|-----|-------------------|-------------------|
| 1   | 56523240          | 34346240          | 18  | 56593800          | 34252160          |
| 2   | 56536200          | 34328960          | 19  | 56596680          | 34248320          |
| 3   | 56554920          | 34304000          | 20  | 56598120          | 34246400          |
| 4   | 56559240          | 34298240          | 21  | 56601000          | 34242560          |
| 5   | 56562120          | 34294400          | 22  | 56605320          | 34236800          |
| 6   | 56563560          | 34292480          | 23  | 56606760          | 34234880          |
| 7   | 56566440          | 34288640          | 24  | 56609640          | 34231040          |
| 8   | 56570760          | 34282880          | 25  | 56611080          | 34229120          |
| 9   | 56572200          | 34280960          | 26  | 56613960          | 34225280          |
| 10  | 56575080          | 34277120          | 27  | 56618280          | 34219520          |
| 11  | 56576520          | 34275200          | 28  | 56619720          | 34217600          |
| 12  | 56579400          | 34271360          | 29  | 56622600          | 34213760          |
| 13  | 56580840          | 34269440          | 30  | 56624040          | 34211840          |
| 14  | 56583720          | 34265600          | 31  | 56626920          | 34208000          |
| 15  | 56585160          | 34263680          | 32  | 56628360          | 34206080          |
| 16  | 56588040          | 34259840          | 33  | 56631240          | 34202240          |
| 17  | 56592360          | 34254080          | 34  | 56632680          | 34200320          |

For the 50 codes $C_{40,i}$, the first rows $r_A(C_{40,i})$ and $r_B(C_{40,i})$ of the negacirculant matrices $A$ and $B$ in the generator matrices of form (6) are listed in Table 10. We verified by Magma that the 50 codes $C_{40,i}$ have minimum weight 12. Although these codes have minimum weights less than $d_5(40)$, these codes are useful for constructing $s$-extremal extremal odd unimodular lattices.

We stopped our search after finding the new 50 lattices. Our feeling is that the number of non-isomorphic $s$-extremal extremal unimodular lattices in dimension 40 might be even bigger.

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4.2 New s-extremal extremal odd unimodular lattices in dimension 38

Let $L_{38}$ be an extremal odd unimodular lattice in dimension 38. By using (1) and (2), one can determine the possible theta series $\theta_{L_{38}}(q)$ and $\theta_{S(L_{38})}(q)$ as follows. Since the minimum norm of $L_{38}$ is 4, we have that

$$a_0 = 1, a_1 = -76, a_2 = 1140 \text{ and } a_3 = -1520,$$

in (1). By considering the coefficient of $q^\frac{3}{2}$ in (2), $a_4$ is written as:

$$a_4 = 2^{10} \alpha,$$

by using an integer $\alpha$. Hence, we have that

$$\theta_{L_{38}}(q) = 1 + (29260 + 1024 \alpha)q + (1668352 - 20480 \alpha)q^5 + \cdots$$

and

$$\theta_{S(L_{38})}(q) = \alpha q^\frac{3}{2} + (6080 - 58 \alpha)q^\frac{7}{2} + (18471040 + 1615 \alpha)q^\frac{11}{2} + \cdots.$$

It is trivial that $L_{38}$ has kissing number 29260 if and only if $L_{38}$ is s-extremal.

An s-extremal extremal unimodular lattice in dimension 38 is previously known and this lattice is denoted by $G_{38}$ in [7]. We calculated by Magma

$$\text{inv}(G_{38})_0 = 129060350 \text{ and } \text{inv}(G_{38})_1 = 83320320. \quad (7)$$

For $n \equiv 2 \pmod{4}$, in order to construct new s-extremal extremal odd unimodular lattices in dimension $n$, we consider quasi-twisted self-dual $[n, n/2]$ codes over $\mathbb{F}_5$. Then we found 15 self-dual [38, 19] codes $C_{38,i}$ over $\mathbb{F}_5$ such that $A_5(C_{38,i})$ have minimum norm 4 and kissing number 29260. This means that $A_5(C_{38,i})$ are s-extremal extremal odd unimodular lattices in dimension 38. For $L = A_5(C_{38,i})$, we calculated by Magma $\text{inv}(L)_j$ ($j = 0, 1$), where the results are listed in Table 6. From (7) and the table, we know that

$$(\text{inv}(G_{38})_0, \text{inv}(G_{38})_1) = (\text{inv}(A_5(C_{38,i}))_0, \text{inv}(A_5(C_{38,i}))_1),$$

however, we verified by Magma that these lattices are non-isomorphic. In addition, there are pairs $(i_1, i_2)$ such that

$$(\text{inv}(A_5(C_{38,i_1}))_0, \text{inv}(A_5(C_{38,i_1}))_1) = (\text{inv}(A_5(C_{38,i_2}))_0, \text{inv}(A_5(C_{38,i_2}))_1).$$

For the pairs, we verified by Magma that $A_5(C_{38,i_1})$ and $A_5(C_{38,i_2})$ are non-isomorphic. Therefore, we have the following:
Table 6: \( r(C) \) and \( \text{inv}(A_5(C))_j \) \((j = 0, 1)\) for \( C = C_{38,i} \)

| \(i\) | \( r(C) \) | \( \text{inv}(A_5(C))_0 \) | \( \text{inv}(A_5(C))_1 \) |
|-----|-------|----------------|----------------|
| 1   | (1, 4, 0, 1, 2, 3, 3, 1, 3, 4, 0, 4, 0, 1, 2, 1, 1, 2) | 128961854 | 83451648 |
| 2   | (1, 0, 0, 0, 4, 2, 2, 1, 0, 3, 4, 2, 0, 3, 1, 2, 1, 0) | 129027518 | 83364096 |
| 3   | (1, 0, 4, 3, 0, 3, 2, 1, 3, 3, 0, 1, 3, 4, 0, 0, 3, 4) | 129060350 | 83320320 |
| 4   | (1, 0, 0, 2, 1, 1, 1, 1, 1, 0, 0, 2, 0, 2, 0, 2, 0, 1) | 129093182 | 83276544 |
| 5   | (1, 0, 0, 2, 2, 0, 4, 0, 0, 2, 3, 1, 0, 0, 2, 1, 0, 1, 3) | 129126014 | 83232768 |
| 6   | (1, 0, 0, 3, 3, 3, 4, 3, 1, 0, 3, 2, 0, 4, 4, 4, 2, 3) | 129126014 | 83232768 |
| 7   | (1, 0, 2, 1, 1, 3, 3, 2, 1, 0, 4, 2, 3, 0, 0, 3, 1, 4, 2) | 129126014 | 83232768 |
| 8   | (1, 0, 0, 4, 0, 0, 2, 3, 4, 2, 1, 0, 0, 3, 1, 4, 1, 0, 4) | 129158846 | 83189992 |
| 9   | (1, 0, 3, 2, 0, 1, 1, 4, 3, 1, 0, 1, 3, 4, 1, 2, 4, 3, 4) | 129158846 | 83189992 |
| 10  | (1, 0, 3, 2, 2, 2, 3, 2, 0, 1, 3, 3, 3, 0, 4, 2, 1, 4, 3) | 129191678 | 83154216 |
| 11  | (1, 0, 0, 2, 1, 2, 0, 0, 2, 1, 0, 3, 3, 0, 2, 1, 4, 2, 1) | 129224510 | 83101440 |
| 12  | (1, 0, 3, 3, 3, 2, 2, 1, 3, 2, 4, 1, 3, 1, 3, 3, 4, 1) | 129224510 | 83101440 |
| 13  | (1, 0, 0, 1, 3, 3, 2, 3, 0, 2, 1, 1, 3, 1, 1, 3, 1, 1, 1) | 129257342 | 83057664 |
| 14  | (1, 0, 1, 4, 4, 4, 4, 1, 2, 0, 3, 3, 4, 2, 2, 0, 4, 1, 3) | 129257342 | 83057664 |
| 15  | (1, 0, 2, 2, 3, 2, 4, 2, 3, 3, 1, 2, 4, 4, 1, 0, 0, 1, 0) | 129257342 | 83057664 |

**Proposition 10.** There are at least 16 non-isomorphic \( s \)-extremal extremal odd unimodular lattices in dimension 38.

For the 15 codes \( C_{38,i} \), the first rows \( r(C_{38,i}) \) of the negacirculant matrices \( A \) in the generator matrices of form (4) are listed in Table 6. We verified by Magma that the codes \( C_{38,i} \) have minimum weight 10 if \( i \in \{1, 3, 4, 5, 8, 9, 12, 13, 15\} \) and the other codes have minimum weight 11. Although these codes have minimum weights less than \( d_5(38) \), these codes are useful for constructing \( s \)-extremal extremal odd unimodular lattices.

### 4.3 New \( s \)-extremal extremal odd unimodular lattices in dimension 42

Let \( L_{42} \) be an extremal odd unimodular lattice in dimension 42. By using (1) and (2), one can determine the possible theta series \( \theta_{L_{42}}(q) \) and \( \theta_{S(L_{42})}(q) \) as follows. Since the minimum norm of \( L_{42} \) is 4, we have that

\[
a_0 = 1, \quad a_1 = -84, \quad a_2 = 1596 \quad \text{and} \quad a_3 = -4144,
\]

in (1). By considering the coefficients of \( q^{\frac{1}{2}} \) and \( q^{\frac{5}{2}} \) in (2), \( a_4 \) and \( a_5 \) are written as:

\[
a_4 = 2^6 \alpha \quad \text{and} \quad a_5 = -2^{18} \beta,
\]
by using integers $\alpha$ and $\beta$. Hence, we have that
\[
\theta_{L_{42}}(q) = 1 + (11844 + 64\alpha)q^4 + (1080576 - 768\alpha - 262144\beta)q^5 + \cdots 
\]
and
\[
\theta_{S(L_{42})}(q) = \beta q^{\frac{1}{2}} + (\alpha - 78\beta)q^{\frac{5}{2}} + (265216 - 54\alpha + 2961\beta)q^9 + \cdots .
\]

An $s$-extremal extremal unimodular lattice in dimension 42 is previously known and this lattice is denoted by $G_{42}$ in [7]. We calculated by MAGMA

\[
\text{inv}(G_{42})_0 = 22272390 \quad \text{and} \quad \text{inv}(G_{42})_1 = 12633936 .
\]

**Lemma 11.** Let $L_{42}$ be an extremal odd unimodular lattice in dimension 42. Then $L_{42}$ has kissing number 11844 if and only if $L_{42}$ is $s$-extremal.

**Proof.** It is sufficient to show that if $L_{42}$ has kissing number 11844 then $L_{42}$ is $s$-extremal. Suppose that $L_{42}$ has kissing number 11844. By considering the coefficients of $q^4$ in $\theta_{L_{42}}(q)$, we have $\alpha = 0$. By considering the coefficients of $q^{\frac{1}{2}}$ and $q^{\frac{5}{2}}$ in $\theta_{S(L_{42})}(q)$, we have $\beta = 0$. The result follows. $\Box$

By considering quasi-twisted codes, we found 30 self-dual $[42, 21]$ codes $C_{42,i}$ over $\mathbb{F}_5$ such that $A_5(C_{42,i})$ have minimum norm 4 and kissing number 11844. This means that $A_5(C_{42,i})$ are $s$-extremal extremal odd unimodular lattices in dimension 42. For $L = A_5(C_{42,i})$, we calculated by MAGMA $\text{inv}(L)_j \quad (j = 0, 1)$, where the results are listed in Table 7. There are pairs $(i_1, i_2)$ such that

\[
(\text{inv}(A_5(C_{42,i_1})))_0, \text{inv}(A_5(C_{42,i_1})))_1 = (\text{inv}(A_5(C_{42,i_2})))_0, \text{inv}(A_5(C_{42,i_2})))_1 .
\]

For the pairs, we verified by MAGMA that $A_5(C_{42,i_1})$ and $A_5(C_{42,i_2})$ are non-isomorphic. Therefore, we have the following:

**Proposition 12.** There are at least 31 non-isomorphic $s$-extremal extremal odd unimodular lattices in dimension 42.

For the 30 codes $C_{42,i}$, the first rows $r(C_{42,i})$ of the negacirculant matrices $A$ in the generator matrices of form (5) are listed in Table 4. We verified by MAGMA that the codes $C_{42,i}$ have minimum weight 10 if $i \in \{3, 5, 6, 22, 26, 28, 29\}$, the code $C_{42,19}$ have minimum weight 11 and the other codes have minimum weight 12. Although these codes have minimum weights less than $d_5(42)$, these codes are useful for constructing $s$-extremal extremal odd unimodular lattices.
Table 7: \(r(C)\) and \(\text{inv}(A_5(C))_j\) \((j = 0, 1)\) for \(C = C_{42,i}\)

| \(i\) | \(r(C)\) | \(\text{inv}(A_5(C))_0\) | \(\text{inv}(A_5(C))_1\) |
|-----|----------|----------------|----------------|
| 1   | (1, 3, 2, 1, 1, 2, 3, 2, 2, 2, 4, 1, 0, 4, 0, 4, 3, 2, 1, 1) | 22228542 | 12692400 |
| 2   | (1, 0, 3, 0, 2, 0, 4, 4, 0, 1, 4, 2, 0, 3, 4, 3, 2, 2, 2, 0) | 22242150 | 12674256 |
| 3   | (1, 0, 2, 0, 1, 3, 1, 2, 1, 4, 0, 0, 4, 2, 3, 2, 3, 0, 0, 0, 0) | 22247946 | 12666528 |
| 4   | (1, 0, 4, 0, 3, 1, 0, 0, 1, 1, 2, 2, 1, 0, 1, 1, 1, 0, 1, 1) | 22258026 | 12653088 |
| 5   | (1, 0, 1, 2, 3, 0, 4, 4, 0, 1, 0, 2, 2, 3, 2, 4, 0, 2, 0, 3, 1) | 22261806 | 12648048 |
| 6   | (1, 0, 4, 1, 2, 2, 1, 0, 4, 3, 4, 3, 0, 1, 1, 1, 1, 4, 0, 1) | 22261806 | 12648048 |
| 7   | (1, 0, 0, 1, 1, 1, 4, 1, 1, 1, 4, 1, 3, 1, 1, 2, 1, 4, 4, 1, 0) | 22262562 | 12647040 |
| 8   | (1, 3, 1, 4, 4, 0, 4, 0, 3, 2, 2, 1, 2, 0, 1, 4, 2, 0, 4, 1) | 22263318 | 12646032 |
| 9   | (1, 2, 2, 0, 2, 2, 1, 4, 0, 1, 1, 3, 1, 3, 2, 3, 0, 4, 2) | 22264830 | 12644016 |
| 10  | (1, 0, 1, 3, 0, 0, 1, 4, 2, 3, 2, 0, 0, 3, 0, 3, 0, 1, 4, 3) | 22267602 | 12640320 |
| 11  | (1, 4, 3, 3, 1, 3, 4, 4, 4, 4, 4, 2, 0, 2, 1, 2, 0, 1, 3, 4) | 22270626 | 12636288 |
| 12  | (1, 3, 2, 0, 3, 0, 3, 2, 2, 1, 1, 2, 4, 2, 3, 4, 2, 4, 3, 2) | 22271634 | 12634944 |
| 13  | (1, 3, 4, 2, 2, 4, 4, 0, 1, 3, 2, 0, 3, 0, 0, 1, 0, 3, 0, 1, 2) | 22272138 | 12634272 |
| 14  | (1, 0, 4, 1, 4, 0, 2, 0, 3, 2, 2, 1, 2, 0, 4, 1, 0, 1, 0, 1) | 22273902 | 12631920 |
| 15  | (1, 0, 2, 4, 2, 2, 3, 2, 1, 3, 0, 0, 1, 0, 3, 0, 1, 2, 3, 3) | 22273902 | 12631920 |
| 16  | (1, 0, 4, 0, 2, 1, 4, 4, 0, 3, 2, 0, 4, 3, 4, 1, 3, 1, 2, 0, 4) | 22274658 | 12630912 |
| 17  | (1, 0, 0, 1, 1, 1, 3, 1, 1, 4, 1, 1, 2, 1, 2, 0, 0, 4, 0, 0) | 22275918 | 12629232 |
| 18  | (1, 1, 0, 0, 0, 2, 0, 2, 0, 2, 0, 2, 1, 0, 4, 2, 4, 1, 1, 0, 4) | 22278438 | 12625872 |
| 19  | (1, 3, 3, 4, 2, 4, 0, 4, 2, 4, 1, 1, 3, 3, 3, 3, 0, 3, 0, 0, 1) | 22280706 | 12622848 |
| 20  | (1, 4, 3, 3, 0, 0, 2, 2, 1, 0, 0, 0, 3, 4, 0, 4, 4, 3, 0, 2) | 22284486 | 12617808 |
| 21  | (1, 2, 2, 1, 2, 2, 1, 1, 3, 2, 3, 0, 0, 1, 4, 2, 0, 1, 0, 2, 1) | 22285242 | 12616800 |
| 22  | (1, 0, 3, 0, 2, 0, 4, 0, 1, 4, 0, 0, 0, 4, 4, 3, 3, 3, 1) | 22286754 | 12614784 |
| 23  | (1, 1, 1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 1, 4, 3, 1, 3, 1) | 22300866 | 12595968 |
| 24  | (1, 0, 2, 0, 3, 4, 3, 0, 3, 0, 3, 2, 3, 2, 3, 4, 2, 3, 0, 3, 1) | 22304142 | 12591600 |
| 25  | (1, 0, 3, 3, 3, 4, 3, 0, 2, 1, 4, 0, 2, 1, 0, 3, 0, 2, 3, 2, 3) | 22311450 | 12581856 |
| 26  | (1, 1, 2, 1, 4, 1, 1, 2, 3, 1, 4, 2, 1, 0, 2, 2, 2, 4, 4, 1, 3) | 22320774 | 12569424 |
| 27  | (1, 2, 3, 3, 2, 1, 3, 0, 1, 3, 0, 4, 2, 2, 3, 2, 3, 0, 4, 4) | 22338162 | 12546240 |
| 28  | (1, 0, 2, 1, 0, 4, 4, 1, 2, 1, 0, 0, 4, 3, 3, 1, 3, 1, 2, 1, 0) | 22339674 | 12544224 |
| 29  | (1, 0, 1, 0, 1, 3, 0, 3, 1, 2, 0, 1, 1, 0, 2, 4, 3, 4, 2, 4) | 22365378 | 12509952 |
| 30  | (1, 0, 2, 1, 4, 2, 4, 0, 3, 1, 2, 0, 1, 1, 0, 1, 0, 1, 0, 3, 0, 3) | 22391586 | 12475008 |
4.4 Summary of the existence of $s$-extremal unimodular lattices with minimum norm 4

In Section 3 and this section, we constructed 15, 50, 30 and 153 new non-isomorphic $s$-extremal extremal unimodular lattices in dimensions 38, 40, 42 and 44, respectively. As a summary, we list in Table 8 the current information on the number $N(n)$ of non-isomorphic $s$-extremal unimodular lattices $L$ in dimension $n$ with $\min(L) = 4$. The table updates the table given in [8, p. 148] and Table 2 in [19].

Table 8: Existence of $s$-extremal unimodular lattices $L$ with $\min(L) = 4$

| $n$  | $N(n)$ | References | $n$  | $N(n)$ | References |
|------|--------|------------|------|--------|------------|
| 32   | 5      | [3]        | 42   | $\geq 31$ | Proposition [12] |
| 36   | $\geq 19$ | [20]      | 43   | $\geq 1$  | [19]       |
| 37   | $?$    |            | 44   | $\geq 156$ | Proposition [5] |
| 38   | $\geq 16$ | Proposition [10] | 45   | $?$    |            |
| 39   | $\geq 1$ | [13]       | 46   | $\geq 1$  | [18]       |
| 40   | $\geq 51$ | Proposition [9] | 47   | $\geq 1$  | [18]       |
| 41   | $\geq 1$ | [19]       |      |         |            |

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| $i$ | $r_A(C_{44,i})$ | $r_B(C_{44,i})$ | $i$ | $r_A(C_{44,i})$ | $r_B(C_{44,i})$ |
|-----|-----------------|-----------------|-----|-----------------|-----------------|
| 1   | $1,0,0,4,4,4,0,2,3,1,1$ | $2,0,1,0,2,0,0,4,3,4,0$ | 26  | $1,0,0,3,2,4,3,2,0,3,1$ | $2,0,1,4,4,1,3,2,3,0$ |
| 2   | $1,0,0,1,2,2,2,3,2,0,4$ | $4,0,3,4,1,2,0,1,1,2,2$ | 27  | $1,0,0,1,2,1,3,0,4,2,1$ | $1,1,3,2,3,4,3,2,3,4$ |
| 3   | $1,0,0,1,1,3,4,1,1,4,0$ | $1,3,3,2,2,0,2,2,0,3,2$ | 28  | $1,0,0,4,0,0,1,1,1,0,2$ | $3,4,3,3,2,4,1,2,2,2$ |
| 4   | $1,0,0,4,0,2,1,4,0,1,2$ | $3,1,0,4,4,1,3,2,0,4,2$ | 29  | $1,0,0,0,4,4,1,1,0,2,4$ | $1,0,0,3,0,3,0,2,4,4,2$ |
| 5   | $1,0,0,0,3,1,1,4,1,0,1$ | $(1,4,3,2,4,2,0,4,2,0,2)$ | 30  | $1,0,0,1,2,1,4,2,1,4,3$ | $(3,3,0,1,1,4,1,4,1,4)$ |
| 6   | $1,0,0,1,1,0,4,1,2,2,3$ | $(2,1,0,1,0,3,1,0,2,4,4)$ | 31  | $1,0,0,0,2,3,3,4,2,0,4$ | $(4,0,3,4,1,2,0,1,4,1,4)$ |
| 7   | $1,0,0,3,3,0,3,1,1,4,0$ | $(1,3,3,2,2,0,2,0,1,0,4)$ | 32  | $1,0,0,3,1,3,3,1,1,4,0$ | $(1,3,3,2,2,0,2,0,4,0,0)$ |
| 8   | $1,0,0,4,1,1,4,1,0,1,3$ | $(1,4,1,0,0,2,1,4,3,4,3)$ | 33  | $1,0,0,2,4,1,2,3,4,4,2$ | $(3,2,3,4,2,2,0,3,0,3)$ |
| 9   | $1,0,0,0,1,4,4,3,4,0,4$ | $(4,0,1,0,2,1,3,1,2,2,2)$ | 34  | $1,0,0,0,1,3,2,0,4,4,2$ | $(1,4,1,1,3,1,0,0,1,2,2)$ |
| 10  | $1,0,0,0,2,3,3,3,2,4,2$ | $(2,1,0,1,3,0,4,2,1,4,1)$ | 35  | $1,0,0,4,1,1,2,2,2,4,2$ | $(1,3,1,3,2,0,4,2,2,4,2)$ |
| 11  | $1,0,0,3,3,2,4,1,1,4,1$ | $(2,2,1,0,0,1,0,1,1,3,0)$ | 36  | $1,0,0,2,3,4,4,1,4,2,2$ | $(3,3,1,4,3,3,3,0,3,4)$ |
| 12  | $1,0,0,3,0,3,0,1,0,0,3$ | $(0,4,2,1,0,1,1,3,3,0,3)$ | 37  | $1,0,0,1,4,0,4,2,0,4$ | $(4,0,3,4,1,2,0,2,3,4,3)$ |
| 13  | $1,0,0,4,3,4,0,4,2,2,3$ | $(2,3,0,4,2,3,1,4,4,2,0)$ | 38  | $1,0,0,2,3,3,3,0,3,0,2$ | $(4,1,2,2,4,4,0,0,1,4,0)$ |
| 14  | $1,0,0,1,2,1,3,1,1,4,0$ | $(1,3,3,2,2,0,2,0,3,1,2)$ | 39  | $1,0,0,4,0,1,3,1,3,0,1$ | $(1,4,1,1,1,0,0,0,2,1,1)$ |
| 15  | $1,0,0,2,2,1,1,4,0,2,2$ | $(3,0,4,0,0,0,4,4,3,3,3)$ | 40  | $1,0,0,1,1,0,3,0,1,4,1$ | $(2,2,1,0,0,1,0,0,3,2,4)$ |
| 16  | $1,0,0,0,4,3,1,1,1,4,1$ | $(2,2,1,0,0,1,0,3,3,4,2)$ | 41  | $1,0,0,0,4,0,3,3,3,2,4$ | $(3,2,0,1,3,3,1,0,3,2,2)$ |
| 17  | $1,0,0,2,2,0,0,2,2,0,2$ | $(0,0,1,2,2,3,4,3,2,1,0)$ | 42  | $1,0,0,4,3,1,2,4,0,2,2$ | $(3,0,4,0,0,0,4,3,2,0,0)$ |
| 18  | $1,0,0,2,3,0,0,3,2,0,4$ | $(4,0,3,4,1,2,0,2,4,4,2)$ | 43  | $1,0,0,4,2,2,1,1,1,0,2$ | $(4,3,3,3,3,3,4,0,4,0,2)$ |
| 19  | $1,0,0,4,2,4,0,1,0,4,0$ | $(3,2,4,3,3,4,4,4,4,0,2)$ | 44  | $1,0,0,4,1,0,4,4,2,2,4$ | $(4,3,0,3,4,2,2,1,1,0)$ |
| 20  | $1,0,0,3,3,3,2,4,1,2,3$ | $(0,0,4,2,2,0,0,2,2,3,4)$ | 45  | $1,0,0,4,4,0,4,3,4,3,2$ | $(2,2,1,2,1,3,1,2,3,3,1)$ |
| 21  | $1,0,0,4,4,1,2,1,0,4,3$ | $(0,3,2,2,4,3,1,2,3,2,0)$ | 46  | $1,0,0,0,4,4,3,0,1,1,4$ | $(3,3,3,1,4,1,2,4,3,4,3)$ |
| 22  | $1,0,0,0,0,3,0,3,4,0,4$ | $(4,0,1,0,2,1,3,2,1,4,1)$ | 47  | $1,0,0,0,4,0,2,1,4,2,2$ | $(3,3,1,4,3,3,0,4,4,2)$ |
| 23  | $1,0,0,2,3,1,4,4,1,2,3$ | $(0,0,4,2,2,0,0,0,2,4,3)$ | 48  | $1,0,0,1,4,0,4,2,4,2,2$ | $(3,3,1,4,3,3,1,0,0,3)$ |
| 24  | $1,0,0,0,2,3,0,2,1,0,3$ | $(1,0,4,3,0,2,3,2,4,4,4)$ | 49  | $1,0,0,4,2,0,4,1,1,3,0$ | $(4,4,3,4,3,2,2,0,0,1,1)$ |
| 25  | $1,0,0,4,2,0,4,0,3,0,2$ | $(4,1,2,2,4,4,0,1,1,2,4)$ | 50  | $1,0,0,3,3,2,1,0,4,0,4$ | $(1,2,2,4,1,4,1,3,3,4,4)$ |
Table 10: Self-dual \([40, 20, 12]\) codes \(C_{40,i}\) over \(\mathbb{F}_5\)

| \(i\) | \(r_A(C_{40,i})\) | \(r_B(C_{40,i})\) | \(i\) | \(r_A(C_{40,i})\) | \(r_B(C_{40,i})\) |
|------|------------------|------------------|------|------------------|------------------|
| 1    | \((1,0,0,3,0,3,2,3,3,1)\) | \((4,3,0,3,2,2,3,3,1,1)\) | 26   | \((1,0,0,3,3,2,1,3,1,1)\) | \((2,3,2,2,2,4,2,1,1,2)\) |
| 2    | \((1,0,0,3,2,3,3,3,4,1)\) | \((4,3,0,3,2,2,1,2,2,0)\) | 27   | \((1,0,0,4,0,0,0,1,1,3)\) | \((2,3,2,2,2,4,2,0,0,1)\) |
| 3    | \((1,0,0,2,4,0,0,3,2,3)\) | \((2,3,2,2,2,4,4,4,2,3)\) | 28   | \((1,0,0,3,3,1,3,3,1,3)\) | \((2,3,2,2,2,4,2,0,4,0)\) |
| 4    | \((1,0,0,0,3,1,0,2,1,3)\) | \((2,3,2,2,2,4,2,3,3,4)\) | 29   | \((1,0,0,4,3,4,3,0,1,3)\) | \((2,3,2,2,2,4,2,1,4,1)\) |
| 5    | \((1,0,0,2,4,0,2,3,1,3)\) | \((4,3,0,3,2,2,2,4,1,0)\) | 30   | \((1,0,0,2,2,1,2,0,2,3)\) | \((2,3,2,2,2,4,4,0,2,4)\) |
| 6    | \((1,0,0,2,4,3,3,2,1,3)\) | \((2,3,2,2,2,4,4,4,2,3)\) | 31   | \((1,0,0,4,1,4,4,4,3,1)\) | \((4,3,0,3,2,2,0,1,2,4)\) |
| 7    | \((1,0,0,2,0,2,4,3,3,1)\) | \((4,3,0,3,2,2,3,3,4,2)\) | 32   | \((1,0,0,2,2,4,1,1,3,1)\) | \((4,3,0,3,2,2,1,2,1,2)\) |
| 8    | \((1,0,0,0,4,1,4,3,4,1)\) | \((4,3,0,3,2,2,1,1,2,1)\) | 33   | \((1,0,0,0,4,3,4,3,2,3)\) | \((2,3,2,2,2,4,1,2,0,2)\) |
| 9    | \((1,0,0,3,2,1,3,1,2,3)\) | \((2,3,2,2,2,4,2,4,2,4,1)\) | 34   | \((1,0,0,3,2,4,2,1,2,3)\) | \((2,3,2,2,2,4,0,1,3,0)\) |
| 10   | \((1,0,0,4,3,0,3,4,1,3)\) | \((2,3,2,2,2,4,0,0,1,1)\) | 35   | \((1,0,0,4,3,3,3,1,4,1)\) | \((4,3,0,3,2,2,1,3,2,4)\) |
| 11   | \((1,0,0,0,4,0,3,3,2,3)\) | \((2,3,2,2,2,4,0,1,0,3)\) | 36   | \((1,0,0,0,1,0,1,3,1,3)\) | \((2,3,2,2,2,4,4,2,1,0)\) |
| 12   | \((1,0,0,0,0,1,2,2,4,1)\) | \((1,3,1,0,1,3,1,0,4,2)\) | 37   | \((1,0,0,3,4,0,3,2,1,3)\) | \((2,3,2,2,2,4,1,2,0,2)\) |
| 13   | \((1,0,0,0,4,3,4,1,3,1)\) | \((4,3,0,3,2,2,3,2,4,0)\) | 38   | \((1,0,0,0,1,1,4,2,2,3)\) | \((2,3,2,2,2,4,3,3,2,0)\) |
| 14   | \((1,0,0,4,0,0,2,2,4,1)\) | \((4,3,0,3,2,2,4,3,2,4)\) | 39   | \((1,0,0,3,0,1,1,1,1,3)\) | \((2,3,2,2,2,4,4,2,0,0)\) |
| 15   | \((1,0,0,2,0,2,4,0,3,1)\) | \((4,3,0,3,2,2,3,3,0,3)\) | 40   | \((1,0,0,1,3,4,1,0,2,3)\) | \((2,3,2,2,2,4,3,3,3,0)\) |
| 16   | \((1,0,0,4,0,0,0,2,3,1)\) | \((4,3,0,3,2,2,3,1,0,1)\) | 41   | \((1,0,0,1,3,0,0,2,2,3)\) | \((2,3,2,2,2,4,0,2,0,4)\) |
| 17   | \((1,0,0,4,1,3,3,1,1,3)\) | \((2,3,2,2,2,4,0,3,1,1)\) | 42   | \((1,0,0,3,1,3,1,0,2,3)\) | \((2,3,2,2,2,4,0,4,3,3)\) |
| 18   | \((1,0,0,0,1,2,3,3,3,1)\) | \((4,3,0,3,2,2,3,3,0,0)\) | 43   | \((1,0,0,2,0,0,4,1,3,1)\) | \((4,3,0,3,2,2,1,3,1,3)\) |
| 19   | \((1,0,0,2,3,4,2,4,3,1)\) | \((4,3,0,3,2,2,1,1,3,1)\) | 44   | \((1,0,0,4,1,4,1,0,1,3)\) | \((2,3,2,2,2,4,4,3,3,3)\) |
| 20   | \((1,0,0,0,3,3,2,3,3,1)\) | \((2,3,2,2,2,4,3,1,0,1)\) | 45   | \((1,0,0,2,4,3,1,0,2,3)\) | \((2,3,2,2,2,4,3,2,4)\) |
| 21   | \((1,0,0,3,4,1,4,4,1,3)\) | \((2,3,2,2,2,4,4,4,1,4)\) | 46   | \((1,0,0,2,0,0,4,1,3,1)\) | \((4,3,0,3,2,2,0,0,1,2)\) |
| 22   | \((1,0,0,1,0,1,2,1,1,3)\) | \((2,3,2,2,2,4,1,4,3,2)\) | 47   | \((1,0,0,1,1,2,1,0,1,3)\) | \((2,3,2,2,2,4,3,0,4,0)\) |
| 23   | \((1,0,0,3,1,4,3,1,1,3)\) | \((2,3,2,2,2,4,0,3,4,1)\) | 48   | \((1,0,0,4,1,3,0,1,1,3)\) | \((2,3,2,2,2,4,3,1,3,1)\) |
| 24   | \((1,0,0,3,2,3,0,3,1,3)\) | \((2,3,2,2,2,4,4,0,4,3)\) | 49   | \((1,0,0,2,3,2,4,3,1,3)\) | \((2,3,2,2,2,4,0,2,0,4)\) |
| 25   | \((1,0,0,2,1,1,3,3,3,1)\) | \((4,3,0,3,2,2,3,0,2,2)\) | 50   | \((1,0,0,4,4,3,4,2,2,3)\) | \((2,3,2,2,2,4,1,4,1,0)\) |