CONTINUITY OF CP-SEMIGROUPS IN THE POINT-STRONG OPERATOR TOPOLOGY

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Abstract. We prove that if \( \{ \phi_t \} \) is a CP-semigroup acting on a von Neumann algebra \( M \subseteq B(H) \), then for every \( A \in M \) and \( \xi \in H \), the map \( t \mapsto \phi_t(A)\xi \) is norm-continuous. We discuss the implications of this fact to the existence of dilations of CP-semigroups to semigroups of endomorphisms.

1. Introduction

Let \( H \) be a Hilbert space, not necessarily separable, and let \( M \subseteq B(H) \) be a von Neumann algebra. A CP-semigroup on \( M \) is a family \( \phi = \{ \phi_t : M \to M \} \) of contractive normal completely positive maps which satisfies the following properties:

1. \( \phi_0(A) = A, \forall A \in M \)
2. \( \phi_{s+t} = \phi_s \circ \phi_t, s, t \geq 0 \)
3. for all \( A \in M \) and \( \omega \in M_\ast, \lim_{t \to t_0} \omega(\phi_t(A)) = \omega(\phi_{t_0}(A)) \)

where \( M_\ast \) denotes the predual of \( M \). We shall refer to continuity condition (3) as continuity in the point-\( \sigma \)-weak topology. It is equivalent to continuity in the point-weak operator topology, i.e.

\[
\lim_{t \to t_0} \langle \phi_t(A)\xi, \eta \rangle = \langle \phi_{t_0}(A)\xi, \eta \rangle, \quad A \in M, \xi, \eta \in H.
\]

A CP-semigroup \( \phi \) is called an E-semigroup if \( \phi_t \) is a *-endomorphism for all \( t \geq 0 \).

In this note we prove that CP-semigroups satisfy a seemingly stronger continuity condition, namely

\[
\lim_{t \to t_0} \| \phi_t(A)\xi - \phi_{t_0}(A)\xi \| = 0,
\]

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for all $A \in M, \xi \in H$. A semigroup satisfying (1.1) will be said to be continuous in the point-strong operator topology. The proposition that CP-semigroups are continuous in the point-strong operator topology has appeared in the literature earlier, but the proofs that are available seem to be incomplete. In the proofs of which we are aware, only continuity from the right in the point-strong operator topology is established. By this we mean that (1.1) holds for limits taken with $t \searrow t_0$.

We consider the continuity of CP-semigroups in the point-strong operator topology to be an important property, because it plays a crucial role in the existence of dilations of CP-semigroups to E-semigroups. We are aware of five different proofs for the fact that every CP-semigroup has a dilation to an E-semigroup: Bhat [2], Selegue [7], Bhat–Skeide [4], Muhly–Solel [6] and Arveson [1] (some of the authors require some additional conditions, notably that the CP-semigroup be unital or that the Hilbert space be separable). In order to show that the minimal dilation of a CP-semigroup to an E-semigroup is continuous in the point-weak operator topology, all authors make use of continuity of the CP-semigroup in the point-strong operator topology, either implicitly or explicitly.

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2. Preliminaries

Let $M$ be a von Neumann algebra acting on a Hilbert space, which is not assumed to be separable. Let $\phi = \{\phi_t : M \to M\}_{t \geq 0}$ be a CP-semigroup acting on $M$. We denote by $M_*$ the set of $\sigma$-weakly continuous linear functionals on $M$. We shall denote by $\sigma(M_*, M)$ the pointwise convergence topology of $M_*$ as a subset of the dual space of $M$.

Let $\delta$ be the generator of $\phi$, and let $D(\delta)$ denote its domain:

$$D(\delta) = \{ A \in M : \exists \delta(A) \in M \forall \omega \in M_* \lim_{t \to 0^+} t^{-1} \omega(\phi_t(A) - A) = \omega(\delta(A)) \}.$$ 

Lemma 2.1. For every $A \in M$ and $\xi \in H$, the map $t \mapsto \phi_t(A)\xi$ is continuous from the right (in norm).

The proof of this result can be found in the literature, for example as Lemma A.1 of [3] or Proposition 4.1 item 1 in [6]. For completeness, let us present the argument from [3]. Let $A \in M$, $\xi \in H$ and $t \geq 0$. 

For all \( h > 0 \), we have, using the Schwartz inequality for completely positive maps,

\[
\|\phi_{t+h}(A)\xi - \phi_t(A)\xi\| = \\
= \langle \phi_{t+h}(A)^*\phi_{t+h}(A)\xi, \xi \rangle - 2 \text{Re}\langle \phi_{t+h}(A)\xi, \phi_t(A)\xi \rangle + \|\phi_t(A)\xi\|^2 \\
\leq \langle \phi_h(\phi_t(A)^*\phi_t(A))\xi, \xi \rangle - 2 \text{Re}\langle \phi_{t+h}(A)\xi, \phi_t(A)\xi \rangle + \|\phi_t(A)\xi\|^2 \xrightarrow{h \to 0} 0.
\]

We remark, however, that two-sided continuity does not follow directly from continuity from the right. This is in contrast with the situation of the classical theory of \( C_0 \)-semigroups on Banach spaces (see for example [5]). If \( T = \{T_t\}_{t \geq 0} \) is a semigroup of contractions on a Banach space \( X \) such that the maps

\[
t \mapsto T_t(x)
\]

are continuous from the right in norm for all \( x \in X \), then it is easy to show that these maps are also continuous from the left in norm\(^1\). In fact, when \( X \) is separable, for instance, it can be proven by measurability and integrability techniques that if the maps \( t \mapsto f(T_t(x)) \) are measurable for all \( x \in X \) and \( f \in X^* \), then the maps \( t \mapsto T_t(x) \) are continuous in norm for \( t > 0 \). In the case of CP-semigroups on von Neumann algebras, however, these techniques seem to require considerable modification. We provide here an alternative approach to the problem.

Recall that a function \( g : [0, 1] \to H \) is weakly measurable if for all \( \eta \in H \), the complex-valued function \( g_\eta(t) = \langle \eta, g(t) \rangle \) is measurable. We will say that the function \( g \) is strongly measurable if there exists a family of countably-valued functions (i.e. assuming a set of values which is at most countable) converging Lebesgue almost everywhere to \( g \). (For more details, see Definition 3.5.4, p. 72, and the surrounding discussion in [5]).

**Lemma 2.2.** For all \( \xi \in H, A \in M \), the function \( f : [0, 1] \to H \) given by \( f(t) = \phi_t(A)\xi \) is strongly measurable and Bochner integrable on the interval \([0, 1]\).

**Proof.** The function \( f \) is weakly continuous, since \( \phi \) is continuous in the point-weak operator topology. In particular, it is weakly measurable. Furthermore, by Lemma 2.1, the function \( f \) is continuous from the right in norm, hence it is separably valued (i.e., its range is contained in a separable subspace of \( H \)). By Theorem 3.5.3 of [5], the function \( f \)

\(^1\) for given \( x \in X, 0 \leq t \leq a \), \( \|T_{a-t}(x) - T_a(x)\| = \|T_{a-t}(x - T_t(x))\| \leq \|x - T_t(x)\| \).
is strongly measurable because it is is weakly measurable and separably valued.

Thanks to Theorem 3.7.4, p. 80 of [5], in order to show that $f$ is Bochner integrable it is enough to show that $f$ is strongly measurable and that

$$\int_0^1 \|f(t)\| dt < \infty.$$  

The latter condition is easy to verify, as $t \mapsto \|f(t)\|$ is a right-continuous, bounded function on $[0, 1]$.

We thank Michael Skeide for the idea to use the continuity of $f$ from the right in order to avoid making the assumption that $H$ is separable.

**Lemma 2.3.** Let $A \in B(H)$ be positive. Then there exists a sequence $A_n \in D(\delta)$ of positive operators such that $A_n \to A$ in the $\sigma$-strong* topology.

**Proof.** Recall that the sequence

$$A_n = n \int_0^{1/n} \phi_t(A) dt$$  

(integral taken in the $\sigma$-weak sense) converges in the $\sigma$-weak topology to $A$. Furthermore $A_n \in D(\delta)$ and it is a positive operator for each $n > 0$ since $\phi_t$ is a CP map for all $t$. It is easy to check that $\|A_n\| \leq \|A\|$ for all $n$ since $\phi_t$ is contractive.

Now observe that for each $\xi \in H$, the map $t \mapsto \phi_t(A)\xi$ is Bochner integrable on $[0, 1]$ (see Lemma 2.2), hence in fact we have

$$A_n\xi = n \int_0^{1/n} \phi_t(A)\xi dt$$  

where the integral is taken in the Bochner sense. The identity holds because for all $\eta \in H$, $n \in \mathbb{N}$ we have:

$$\langle A_n\xi, \eta \rangle = n \int_0^{1/n} \langle \phi_t(A)\xi, \eta \rangle dt = \langle n \int_0^{1/n} \phi_t(A)\xi dt, \eta \rangle.$$

We now show that $A_n \to A$ strongly. Let $\xi \in H$ be fixed.

$$\|A\xi - A_n\xi\| = \|n \int_0^{1/n} A\xi dt - n \int_0^{1/n} \phi_t(A)\xi dt\|$$

$$\leq n \int_0^{1/n} \|A\xi - \phi_t(A)\xi\| dt.$$  

The latter goes to zero by continuity from the right (Lemma 2.1). Since $A_n, A$ are positive operators, by considering adjoints we obtain
that $A_n \to A$ in the strong* topology. Finally, since the sequence is bounded, we have convergence in the $\sigma$-strong* topology.

Lemma 2.4. Let $A_n$ be a bounded sequence of operators in $M$ converging to $A$ in the $\sigma$-strong* topology and let $t_0 \geq 0$. Then for every sequence $t_k \to t_0$, $\xi \in H$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$\|\phi_{t_k}(A_n - A)\xi\| < \epsilon, \text{ for all } k.$$ 

Proof. Let $B_n = (A_n - A)^*(A_n - A)$, $\omega_k(X) = \langle \phi_{t_k}(X)\xi, \xi \rangle$ and $\omega(X) = \langle \phi_{t_0}(X)\xi, \xi \rangle$. Then we have that

$$\|\phi_{t_k}(A_n - A)\xi\|^2 = \langle \phi_{t_k}(A_n - A)^*\phi_{t_k}(A_n - A)\xi, \xi \rangle \leq \omega_k(B_n)$$

since $\phi_t$ is a CP map for all $t$. Since $\phi$ is a point-$\sigma$-weakly continuous semigroup, we have that $(\omega_k)$ is a sequence of $\sigma$-weakly continuous linear functionals such that $\omega_k(X) \to \omega(X)$ for all $X \in M$. Furthermore, $B_n$ is a bounded sequence converging in the $\sigma$-strong* topology to 0. The latter holds because $A_n$ is a bounded sequence converging to $A$ in the $\sigma$-strong* topology and multiplication is jointly continuous with respect to this topology in bounded sets (of course $\ast$ is also continuous). Finally, we obtain the desired conclusion by applying Lemma III.5.5, p.151 of [8], which states the following. Let $M$ be a von Neumann algebra and let $\rho_k$ be a sequence in $M_*$ converging to $\rho_0 \in M_*$ in the $\sigma(M_*, M)$ topology. If a bounded sequence $(a_n)$ converges $\sigma$-strongly* to 0, then $\lim_{n \to \infty} \rho_k(a_n) = 0$ uniformly in $k$. 

3. THE MAIN RESULT

Theorem 3.1. Let $\phi$ be a CP-semigroup acting on a von Neumann algebra $M \subseteq B(H)$. Then for all $\xi \in H$, $A \in M$ and $t_0 \geq 0$,

$$\lim_{t \to t_0} \|\phi_t(A)\xi - \phi_{t_0}(A)\xi\| = 0.$$ 

Proof. Let $\epsilon > 0$ be given, and let $(t_k)$ be a sequence converging to $t_0$. By Lemma 2.3 there is a bounded sequence $(A_n)$ of operators $A_n \in D(\delta)$ such that $A_n \to A$ in the $\sigma$-strong* topology. By Lemma 2.4 there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$\|\phi_{t_k}(A_n - A)\xi\| < \frac{\epsilon}{3}, \text{ for all } k \geq 0.$$ 

By an application of the Principle of Uniform Boundedness, if $X \in D(\delta)$ there exists $C_X > 0$ such that

$$\sup_{s \geq 0} \frac{1}{s}\|\phi_s(X) - X\| \leq C_X < \infty.$$
Now notice that $A_n \in D(\delta)$ for all $n$, and in particular $\exists C > 0$ such that

$$\sup_{s > 0} \|\phi_s(A_N) - A_N\| \leq C.$$ 

Because $\phi_s$ is a contraction for all $s$, we obtain that for all $k$,

$$\|\phi_{t_k}(A_N)\xi - \phi_{t_0}(A_N)\xi\| \leq \|\phi_{t_k}(A_N) - \phi_{t_0}(A_N)\| \|\xi\|$$

$$\leq \|\phi_{t_k - t_0}(A_N) - A_N\| \|\xi\|$$

$$\leq C\|\xi\| |t_k - t_0|.$$ 

In particular, we must have that $\|\phi_{t_k}(A_N)\xi - \phi_{t_0}(A_N)\xi\| \to 0$ as $k \to \infty$. Thus there is $K \in \mathbb{N}$ such that for $k \geq K$,

$$\|\phi_{t_k}(A_N)\xi - \phi_{t_0}(A_N)\xi\| < \frac{\epsilon}{3}.$$ 

We conclude that for $k \geq K$,

$$\|\phi_{t_k}(A)\xi - \phi_{t_0}(A)\xi\| \leq \|\phi_{t_k}(A - A_N)\xi\| +$$

$$+ \|\phi_{t_k}(A_N)\xi - \phi_{t_0}(A_N)\xi\| + \|\phi_{t_0}(A_N - A)\xi\| < \epsilon.$$

\[\square\]

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