Relativistic wave–particle duality for spinors

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We propose that relativistic wave–particle duality can be embodied in a relation
\[ \langle u^i \rangle = \bar{\psi} \gamma^i \psi / \bar{\psi} \psi, \]
which determines the mean four-velocity of the fermion particle associated with a Dirac wave function. We use the Einstein–Cartan theory of gravity with torsion, which incorporates the spin-orbit interaction in curved spacetime. This relation is satisfied by a spinor plane wave and it is consistent with the energy-momentum tensor for particles. We suggest that spacetime guides the evolution of a wave function, that in turn guides the mean motion of the associated particle. Consequently, spacetime guides the motion of particles. The exact motion is limited by the uncertainty principle.

I. SPINORS IN EINSTEIN-CARTAN GRAVITY

The torsion tensor \( S^i_{jk} \) is the antisymmetric part of the affine connection \( \Gamma^i_{jk} \):
\[ S^i_{jk} = \Gamma^i_{jk}. \] (1)

The general theory of relativity (GR) assumes that this tensor vanishes [2, 3]. In this theory, the orbital angular momentum is conserved [3, 4]. However, the Dirac equation of relativistic quantum mechanics gives the conservation law for the total (orbital and spin) angular momentum which allows the exchange between the orbital and spin parts that are not separately conserved [6]. This conservation requires that the torsion tensor is not constrained to zero [7]. The simplest theory of gravity that extends GR by including torsion is the Einstein–Cartan–Sciama–Kibble (EC) theory [7, 8, 9]. In this theory, the Lagrangian density for the gravitational field is proportional to the Ricci tensor, as in GR. The Ricci tensor is composed from the affine connection which contains torsion:
\[ \tilde{R}^i_{jk} = \Gamma^i_{jk,l} - \Gamma^i_{jkl} + \Gamma^i_{jl,k} - \Gamma^i_{jl,k}, \] where the comma denotes a coordinate partial derivative.

Torsion is determined by the field equations obtained from varying the action for the gravitational field and matter with respect to the torsion tensor [4, 5, 6, 7]. This variation leads to the Cartan equations:
\[ S^i_{jk} - S_i \delta^j_k + S_k \delta^i_j = -\frac{\kappa}{2} S^i_{jk}, \] (2)

where \( S^i_{jk} = S_i \) is the torsion vector, \( \delta^j_k \) is the Kronecker tensor, and \( \kappa = 8\pi G \) (in the natural units \( c = 1 \)). These algebraic equations show that torsion is proportional to the spin tensor, defined as a functional derivative:
\[ s^i_{ab} = \frac{2}{\sqrt{-g}} \frac{\delta L_m}{\delta \omega^i_{ab}} = \frac{2}{\sqrt{-g}} \frac{\delta L_m}{\delta C_{ij}^k}. \] (3)

where \( g \) is the determinant of the metric tensor \( g_{ik} \), \( L_m \) is the Lagrangian density for matter, and \( C^k_{ij} = 2 S^k_{(ij)} + S^k_{ij} \) is the contortion tensor. The spin connection is given by [4, 5, 6, 7]
\[ \omega^a_{bi} = e^a_k (e^k_b,i + \Gamma^k_{j} e^j_b), \] (4)

where \( e^a_k \) is the tetrad and \( e^a_k \) is the inverse tetrad.

The affine connection can be decomposed into the Christoffel symbols (the Levi-Civita connection) \( \{ j^i_k \} = (1/2) g^{il} (g_{lk,j} + g_{lj,k} - g_{jk,l}) \) and the contortion tensor:
\[ \Gamma^i_{jk} = \{ j^i_k \} + C^i_{jk}. \] (5)

Accordingly, the Ricci tensor \( R_{ik} \) can be decomposed into the Riemannian part \( R_{ik} \) (the Ricci tensor of the Levi-Civita connection) that depends only on the metric tensor and its partial derivatives and a part that contains the torsion.

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tensor. Using the Cartan equations, the part with the torsion tensor can be written in terms of the spin tensor and regarded as a modification of the Lagrangian density for matter. Consequently, EC can be rewritten as GR in which the energy-momentum tensor of matter acquires additional terms that are quadratic in the spin tensor. The variation of the corresponding action for the gravitational field and matter with respect to the metric tensor gives the Einstein equations:

$$R_{ik} - \frac{1}{2}R g_{ik} = \kappa(T_{ik} + U_{ik}),$$

(6)

where $R$ is the Ricci scalar of the Levi-Civita connection, $T_{ik}$ is the energy-momentum tensor:

$$T_{ik} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta g^{ik}},$$

(7)

and $U_{ik}$ is its modification arising from the spin tensor $\mathcal{E}_{ijkl}$:

$$U_{ik} = \kappa \left( -s^j_{[i} \gamma^{kl]} - \frac{1}{2} s^j_{[ls} s_{k]} + \frac{1}{4} s^{j[i} s_{k]} + \frac{1}{8} g^{ik} (-4 s_{j[l} s^{jm]} + s^{jm} s_{jm}) \right).$$

(8)

Consequently, the spin of matter affects the curvature of spacetime through terms that are quadratic in the spin density and proportional to $\kappa^2$, making them negligible except at extremely high densities that exist only in black holes or in the very early Universe. Accordingly, EC passes all current tests of GR.

The Lagrangian for matter represented by a spinor field $\psi$ (in the natural units $\hbar = 1$) is given by

$$\mathcal{L}_m = \frac{i}{2} (\bar{\psi} \omega^a \gamma^i \psi - \bar{\psi} \gamma^i \psi) - m \bar{\psi} \psi,$$

(9)

where $\bar{\psi}$ is the adjoint spinor and the Dirac matrices $\gamma^i$ with a coordinate index are related to the Dirac matrices $\gamma^a$ with the Lorentz index by $\gamma^i = e^a_i \gamma^a$. The semicolon denotes the EC covariant derivative with respect to the affine connection, so $\bar{\psi}_i = \bar{\psi}, - \Gamma_i \psi$ is the covariant derivative of the spinor, $\bar{\psi}_i = \bar{\psi}, + \psi \Gamma_i$ is the covariant derivative of the adjoint spinor, and $\Gamma_i$ is the Fock–Ivanenko spinor connection:

$$\Gamma_i = -\frac{1}{4} \omega_{ab} \gamma^a \gamma^b.$$

(10)

The variation of the corresponding action with respect to $\bar{\psi}$ and $\psi$ gives the Dirac equation for the spinor and the adjoint spinor, respectively:

$$i \gamma^k \psi_{;k} = m \psi, \quad -i \bar{\psi}_{;k} \gamma^k = m \bar{\psi}.$$

(11)

Substituting the covariant derivatives and the spinor connection to the Dirac follows and using the decomposition of the affine connection gives

$$i \gamma^k \psi_{;k} + \frac{3\kappa}{8} (\bar{\psi} \gamma^k \gamma^5 \psi) \bar{\gamma}^k \gamma^5 \psi = m \psi, \quad -i \bar{\psi}_{;k} \gamma^k + \frac{3\kappa}{8} (\psi \gamma^k \gamma^5) \bar{\psi} \gamma^k \gamma^5 \psi = m \bar{\psi},$$

(12)

where the colon denotes the GR covariant derivative with respect to the Levi-Civita connection. The relations are cubic in the spin field. This nonlinearity restricts the normalization of a spinor wave function.

The spin tensor for a Dirac spinor field is given by

$$s^{ijkl} = i \bar{\psi} \gamma^{[i} \gamma^{j} \gamma^{k]} \psi.$$

(13)

Because this tensor is completely antisymmetric, it has the corresponding dual spin pseudovector $s^i$ that is related by

$$s^{ijkl} = -e^{ijkl} s_l,$$

where $e^{ijkl}$ is the Levi-Civita pseudotensor. This pseudovector is

$$s^i = \frac{1}{2} \bar{\psi} \gamma^i \gamma^5 \psi.$$

(14)

Accordingly, the torsion tensor is also completely antisymmetric. The combined energy-momentum tensor for a Dirac spinor field is given by

$$T_{ik} + U_{ik} = i \left( \bar{\psi} \delta^{ij}_i \gamma_k \psi_{;j} - \bar{\psi}_{;j} \delta^{ij}_i \gamma_k \psi \right) + \frac{3\kappa}{4} s^i s_l g_{ik}.$$

(15)
If GR is used instead of EC, then the relations (12) and (15) would not have the terms with $\kappa$.

At extremely high densities that exists in black holes and in the very early Universe, torsion acts like gravitational repulsion and prevents the formation of singularities [10]. Consequently, a black hole may create a new universes on the other side of its event horizon [11]. Moreover, torsion may be a source of the positive cosmological constant [12]. In addition, it may give spatial extension of elementary particles and regularize Feynman diagrams in quantum field theory [13].

II. PLANE-WAVE SOLUTION OF DIRAC EQUATION WITH TORSION

In the local Minkowski spacetime, the coordinate and Lorentz indices coincide and the Dirac equation (12) can be written in the Hamiltonian form:

$$i \frac{\partial \psi}{\partial t} = -i \alpha \cdot \nabla \psi + m \beta \psi - \frac{3\kappa}{8} (\bar{\psi} \gamma^5 \psi) \gamma^5 \psi,$$

where $\alpha$ is the vector formed from the matrices $\alpha^i = \beta \gamma^i$ and $\beta = \gamma^0$. A free particle is represented by a plane wave. Therefore, the corresponding Dirac spinor has a form $\psi \sim \exp(-ip_k x^k)$, where $p_k$ is the four-momentum and $x^k$ are the spacetime coordinates. Consequently, (17) becomes

$$E \psi = p \cdot \alpha \psi + m \beta \psi - \frac{3\kappa}{8} (\bar{\psi} \gamma^5 \psi) \gamma^5 \psi,$$

where $E$ is the energy and $p$ is the momentum of the particle. We use the Dirac representation:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix}, \quad \alpha^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

where $I$ is the two-dimensional unit matrix, $\sigma^\mu$ are the Pauli matrices, and $\mu = 1, 2, 3$. The adjoint spinor is given by $\bar{\psi} = \psi^\dagger \gamma^0$, where $\psi^\dagger$ is the Hermitian conjugated spinor. In the rest frame of reference ($p = 0$), the normalized ($\bar{\psi} \psi = 1$), spin-up, positive-energy solution of the Dirac equation (17) is [6]

$$\psi(r, t) = \begin{pmatrix} \xi \\ \eta \end{pmatrix} e^{-iEt},$$

where $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\eta = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. For this solution, the last term in (17) is composed from

$$\bar{\psi} \gamma^0 \gamma^5 \psi = \psi^\dagger \gamma^5 \psi = 0, \quad \bar{\psi} \gamma^\mu \gamma^5 \psi = \psi^\dagger \alpha^\mu \gamma^5 \psi = \xi^\dagger \sigma^\mu \xi = \delta^\mu_5,$$

$$\beta \gamma^0 \gamma^5 \psi = \begin{pmatrix} \eta \\ -\eta \end{pmatrix} e^{-iEt}, \quad \beta \gamma^\mu \gamma^5 \psi = \begin{pmatrix} \sigma^\mu \xi \\ \sigma^\mu \eta \end{pmatrix} e^{-iEt} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \delta^\mu_5 e^{-iEt},$$

giving

$$\frac{3\kappa}{8} (\bar{\psi} \gamma_k \gamma^5 \psi) \beta \gamma^5 \psi = -\frac{3\kappa}{8} \begin{pmatrix} \xi \\ \eta \end{pmatrix} e^{-iEt}.$$

Consequently, (17) determines the energy in the rest frame:

$$E = m + \frac{3\kappa}{8} = M.$$

In the frame of reference in which the particle has momentum $p$, the corresponding spinor wave function has a form of a plane wave proportional to $\exp(-ip_k x^k)$:

$$\psi(r, t) = \frac{1}{\sqrt{2M(E + M)}} \begin{pmatrix} E + M \\ \sigma \cdot p \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} e^{i(p \cdot r - Et)},$$

where $\sigma$ is the vector formed from the Pauli matrices and the square matrix in the above equation is the spinor representation of the boost from rest to velocity $v = p/E$ [4, 5, 14]. The energy and momentum of the particle are related to one another by

$$E^2 = p^2 + M^2.$$
If the normalization constant is \( \bar{\psi}\psi = N \), then \[20\]

\[ M = m + \frac{3\kappa}{8}N. \] \(25\)

Consequently, the normalization of a spinor wave function is related to the mass of the particle associated with the spinor. Similar calculations can be carried out for spin-down and negative-energy solutions.

### III. FOUR-VELOCITY OF A SPINOR

We propose that relativistic wave–particle duality can be embodied in a relation

\[ \langle u^i \rangle = \frac{\bar{\psi}\gamma^i\psi}{\psi\psi}. \] \(26\)

According to this relation, a wave represented by a spinor wave function \( \psi \) can be regarded as a fermion particle with the mean four-velocity \( u^i \) \[16\]. The relation \(26\) shows that the four-velocity of a particle is proportional to the Dirac four-current \( j^i = \bar{\psi}\gamma^i\psi \), which is covariantly conserved \[4, 5, 14\]: \( j^i_j = 0 \), where the colon denotes the covariant derivative with respect to the Levi-Civita connection.

The mean value is taken over the region occupied by the wave function and signifies that the particle satisfies the Heisenberg uncertainty principle which establishes an intrinsic uncertainty in the measurement of its momentum. It can indicate that a particle is not a point but, instead, an extended body that is represented by a wave function. The measurement interacts with the wave function and its size and shape assume a form of a point-like particle. However, the uncertainty principle is necessary to derive the quantization of angular momentum that determines its eigenvalues and eigenfunctions \[18\] that are observed \[19\]. Consequently, the particle guided by its own wave \[17\]. However, the uncertainty principle is necessary to derive the quantization of angular momentum that determines its eigenvalues and eigenfunctions \[18\] that are observed \[19\]. Consequently, the particle guided by its own wave.

In the de Broglie–Bohm pilot-wave interpretation of quantum mechanics, the relation \(26\) could describe a point particle guided by its own wave \[17\]. However, the uncertainty principle is necessary to derive the quantization of angular momentum that determines its eigenvalues and eigenfunctions \[18\] that are observed \[19\]. Consequently, the relation \(26\) cannot define the exact value of the four-velocity of a particle (and the same conclusion applies to the nonrelativistic limit \(20\) ) but, instead, it determines its mean value.

It is straightforward to demonstrate that the relation \(26\) is satisfied for a free particle associated with a plane spinor wave \[23\]. Using the Dirac representation gives

\[ \bar{\psi}\psi = \psi^\dagger\gamma^0\psi = \frac{1}{2M(E + M)}[(E + M)^2 - (1, 0)(\sigma \cdot p)(\sigma \cdot p) \begin{pmatrix} 1 \\ 0 \end{pmatrix}] = \frac{(E + M)^2 - p^2}{2M(E + M)} = 1. \] \(27\)

Similarly, the time component gives, using \( p^i = Mu^i \),

\[ \bar{\psi}\gamma^0\psi = \psi^\dagger\psi = \frac{1}{2M(E + M)}[(E + M)^2 + (1, 0)(\sigma \cdot p)(\sigma \cdot p) \begin{pmatrix} 1 \\ 0 \end{pmatrix}] = \frac{(E + M)^2 + p^2}{2M(E + M)} = \frac{E}{M} = u^0, \] \(28\)

and the space components give, using \( \sigma^\mu\sigma^\nu + \sigma^\nu\sigma^\mu = 2\delta^{\mu\nu}I \),

\[ \bar{\psi}\gamma^\mu\psi = \psi^\dagger\sigma^\mu\psi = \frac{1}{2M(E + M)}[(E + M)(1, 0)\sigma^\mu(\sigma \cdot p) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (1, 0)(\sigma \cdot p)\sigma^\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix}] \begin{pmatrix} E + M \end{pmatrix} \]

\[ = \frac{2(E + M)p^\mu}{2M(E + M)} = u^\mu. \] \(29\)

Equations \(28\) and \(29\) prove the relation \(26\). The normalization of a spinor wave function \( \psi \) affects the effective mass \( M \) \[26\] but it does not affect the mean four-velocity \(26\) of the particle associated with this wave.

The four-velocity of a particle is normalized: \( u^iu_i = 1 \). The mean four-velocity \(26\) for a plane spinor wave is also normalized. This condition, however, does not need to be satisfied for a general spinor wave function because \( \langle u^i \rangle \langle u_i \rangle \neq \langle u^iu_i \rangle \).

For a plane spinor wave function that represents a free particle, the energy and momentum are constant. The functions \( \bar{\psi}\gamma^i\psi \) and \( \bar{\psi}\psi \) do not depend on the coordinates. Consequently, \( u^i \) in \(26\) is constant and identically satisfies the geodesic equation of motion in the local Minkowski spacetime. From the principle of general covariance it follows that the four-velocity satisfies the mean geodesic equation also in curved spacetime. This equation also follows from \(26\) and the Dirac equation, as the Newton second law of motion follows from the Schrödinger equation in the pilot-wave quantum mechanics \[17\].
IV. ENERGY-MOMENTUM TENSOR FOR A SPINOR

The multipole expansion of the conservation law for the energy-momentum tensor \[ T_{ik}(r) = \frac{\delta(r - r_0)}{\sqrt{-g}} \mu_{u_i u_k}, \] where all the quantities on the right-hand side are evaluated at \( r_0 \). It is straightforward to demonstrate that the combined energy-momentum tensor (15) for a spinor wave function (23) has a similar form. In the local Minkowski spacetime, this tensor is, using (26):

\[ T_{ik} + U_{ik} = \frac{i}{2} \left( \bar{\psi} \delta_{i(i')k} \gamma_{i'} \psi_j - \bar{\psi}_j \delta_{i(i')k} \gamma_{i'} \psi \right) + \frac{3\kappa}{4} s^i s g_{ik} = \bar{\psi} \psi u_{(i} p_{k)} - \frac{3\kappa}{16} (\bar{\psi} \psi)^2 g_{ik} = M \bar{\psi} \psi u_{i} u_{k} - \frac{3\kappa}{16} (\bar{\psi} \psi)^2 g_{ik}. \]

Following (25), it becomes

\[ T_{ik} + U_{ik} = m \bar{\psi} \psi u_{i} u_{k} + \frac{3\kappa}{16} (\bar{\psi} \psi)^2 (2u_{i} u_{k} - g_{ik}). \] The first term on the right-hand side of (32) is analogous to (20), giving an interpretation of \( \bar{\psi} \psi \). When a spinor wave function is measured as a particle at a position \( r_0 \), \( \bar{\psi} \psi \) as a function of \( r \) assumes a form of \( \delta(r - r_0)/(\sqrt{-g} u^0) \).

The second term on the right-hand side of (32) manifests itself as an additional source of the gravitational field. It may explain the effects that are attributed to dark matter. The covariant conservation (\( T_{ik} + U_{ik} \)) determines the motion of the particle associated with the wave function. If GR is used instead of EC, then the right-hand side of (32) would have only the first term that is analogous to (30).

The Einstein equations for the metric tensor that curves spacetime with a spinor field as a source are given by (6), (14), and (15):

\[ R_{ik} - \frac{1}{2} R g_{ik} = \kappa \left( \frac{i}{2} \left( \bar{\psi} \delta_{i(i')k} \gamma_{i'} \psi_j - \bar{\psi}_j \delta_{i(i')k} \gamma_{i'} \psi \right) + \frac{3\kappa}{16} (\bar{\psi} \gamma^5 \psi)(\bar{\psi} \gamma^5 \psi) g_{ik} \right). \]

The Einstein equations relate the curvature to the energy and momentum of the wave function and the Cartan equations relate the torsion to the spin of the wave function. We think that spacetime is continuous. Matter waves tell spacetime how to curve. The Dirac equation for the spinor wave function in curved spacetime is given by (12). Spacetime tells matter waves how to move. It governs deterministically how the wave function evolves. We think that an elementary particle is a wave function itself; it can change size and shape through interaction with other bodies. This view lies between the Bohr Copenhagen interpretation of quantum mechanics, in which the wave function is not real, and the pilot-wave interpretation, in which the wave function determines the exact velocity of a guided point particle. Regarding matter waves as particles whose mean velocity is given by (20) turns the Einstein equations into those with a particle source and turns the Dirac equation into the relativistic equation of motion. Einstein’s view of spacetime guiding the motion of a particle and Bohm’s view of the wave function guiding the motion of the particle may therefore be unified. The continuous spacetime guides the evolution of the wave function, that in turn guides the mean motion of the particle associated with this wave function. Measuring the motion of particles is still restricted by the uncertainty principle and the wave function determines, through the Born rule (18), the probabilities of various measurements. The physical origin of this determination is still unknown.

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