REDUCTION OF THE WAVEPACKET: HOW LONG DOES IT TAKE?*

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Abstract

We show that the “reduction of the wavepacket” caused by the interaction with the environment occurs on a timescale which is typically many orders of magnitude shorter than the relaxation timescale $\tau$. In particular, we show that in a system interacting with a “canonical” heat bath of harmonic oscillators decorrelation timescale of two pieces of the wave-packet separated by $N$ thermal de Broglie wavelengths is approximately $\tau/N^2$. Therefore, in the classical limit $\hbar \to 0$ dynamical reversibility ($\tau \to \infty$) is compatible with “instantaneous” coherence loss.

INTRODUCTION

It is sometimes argued that observables of macroscopic objects which obey, to a good approximation, reversible classical dynamics – i.e. their relaxation timescale $t$ is, for all practical purposes, infinite – could not have

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lost coherence and become “classical” due to the interaction with the environment through environment-induced superselection.\textsuperscript{1–5} For, the reasoning goes, relaxation rate is the measure of the strength of the coupling with the environment. In particular, when $\tau \to \infty$ one can neglect dissipation of energy. Consequently, one should be equally justified in neglecting any influence of the environment. We show that this argument is fallacious in an example of a free particle interacting with the environment of quantum oscillators in the high-temperature weak coupling limit. In particular, we show that the coherence between two pieces of the wave-packet $\Delta x$ apart is lost on a decorrelation timescale $\theta$ which is typically

$$\theta = \tau \left[ \left( \frac{\hbar}{\sqrt{4mkT}} / \Delta x \right) \right]^2 . \quad (1)$$

Here, $m$ is the mass of the particle, $k$ is Boltzmann’s constant, and $T$ is temperature. For “canonical” classical systems ($m \sim 1g$, $T \sim 300^oK$) and standard “macroscopic” separations $\Delta x \sim 1cm$, $\theta / \tau \sim 10^{-40}$. Moreover, in the classical limit, $\hbar \to 0$, $\theta / \tau \to 0$. This enormous disparity between the two timescales can be regarded as the explanation of the apparent “instantaneous” collapse of the state vector of macroscopic objects, including distinguishable (i.e. separated by many de Broglie wavelength\textsuperscript{*} $\lambda_{dB} = h / \sqrt{4mkT}$) outcomes of measurements performed by a classical apparatus on a quantum system.

**DECORRELATION OF A “FREE PARTICLE”**

Consider an otherwise free particle of mass $m$ interacting with the environment of many harmonic oscillators via the Hamiltonian:

$$H_{INT} = x \sum c_i q_i . \quad (2)$$

Above, $x$ is the coordinate of the free particle while $q_i$ are the coordinates of harmonic oscillators. This interaction Hamiltonian was used extensively in many earlier discussions of relaxation,\textsuperscript{7,8} and, more recently it is being used in calculations of dephasing in a harmonic oscillator.\textsuperscript{9}

\textsuperscript{*The more popular definition of thermal de Broglie wavelength is $\lambda_{TB}^2 = h^2 / 2\pi m kT$. It differs by a factor $\sqrt{\pi / 2}$ ($\lambda_{dB}^2 = (2/\pi)\lambda_{TB}^2$) from the de Broglie wavelength $\lambda_{dB}$ we shall use here.}
We note that $H_{INT}$, Eq. (1), commutes with the position observable of the free particle:

$$\left[ H_{INT}, x \right] = 0 \quad .$$

Therefore, position can be regarded as pointer observable,$^{1,3,5}$ measured continuously by the environment of harmonic oscillators. In the absence of the self-Hamiltonian:

$$H_0 = -\left( \frac{\hbar^2}{2m} \right) \left( \partial^2 / \partial x^2 \right)$$

$x$ would be a constant of motion. One would then expect combined system-environment state vector to evolve from an initial, uncorrelated state

$$|\phi_0\rangle = |\psi\rangle|\epsilon\rangle$$

into the time dependent, correlated state:

$$|\phi_t\rangle \propto \int dx |\psi(x)\rangle|\epsilon\rangle \quad .$$

Tracing out an environment after it has performed idealized, perfect “measurement” – i.e. after states of the environment are correlated with different positions become orthogonal, $\langle \epsilon_x | \epsilon_y \rangle \sim \delta(x - y)$ – yields, for the system, the density matrix:

$$\rho \propto \int dx |\psi(x)|^2 |x\rangle\langle x|$$

This density matrix is diagonal in the pointer basis $\{|x\rangle\}$.

In the more realistic case of finite $H_0$ the density matrix $\rho$ will not achieve perfect diagonalization, Eq. (5). Rather, it will have a finite correlation length $\sim \lambda_{dB}$. Moreover, the distribution will become uniform, $\langle x | \rho | x \rangle = \text{const.}$, on a relaxation timescale. The estimate of the timescales of these two processes can be obtained from the effective master equation for the free particle. We shall use it in the form given by Caldeira and Leggett$^7$. Its three consecutive terms correspond to the von Neumann’s equation for the density matrix of a free particle, to the dissipation with viscosity $\eta = 2m\gamma$, and to the fluctuating force responsible for Brownian motion:

$$\dot{\rho} = \left\{ \left( \frac{i\hbar}{2m} \right) \left( \partial^2 / \partial x^2 - \partial^2 / \partial y^2 \right) - \gamma(x - y) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) - \left( \frac{2m\pi kT}{\hbar^2} \right) (x - y)^2 \right\} \rho$$

(6)
To compare relaxation and decorrelation timescales we consider an initial wavepacket of half-width $\delta$. As we shall argue in the next section, this half-width will be typically of the order of the de Broglie wavelength. We now suppose that the initial wavepacket has been “split,” coherently, into two pieces, $|\alpha\rangle$ and $|\beta\rangle$, so that the free particle is described by the wave function:

$$|\psi\rangle = (|\alpha\rangle + |\beta\rangle)/\sqrt{2} .$$  (7)

Here we assume for simplicity:

$$\langle x|\alpha\rangle = (2\pi\delta^2)^{-1/4} \exp \left[ -(x - \Delta x/2)^2/4\delta^2 \right] .$$  (8a)

$$\langle x|\beta\rangle = (2\pi\delta^2)^{-1/4} \exp \left[ -(x - \Delta x/2)^2/4\delta^2 \right] .$$  (8b)

The resulting initial density matrix $\rho = |\psi\rangle\langle\psi|$ has, in the position representation, four extremes. Two of them occur on the diagonal: (1) $x = y = \Delta x/2$; (2) $x = y = -\Delta x/2$. They are the maxima of $|\langle x|\alpha\rangle|^2$ and $|\langle x|\beta\rangle|^2$. In addition, there are off-diagonal maxima of $\langle x|\alpha\rangle\langle x|\beta\rangle$ and of its Hermitian conjugate which lie at: (3) $x = -y = \Delta x/2$; (4) $x = -y = -\Delta x/2$. The size of these off-diagonal maxima provides a measure of the coherence between $|\alpha\rangle$ and $|\beta\rangle$.

The rate of change of the diagonal terms due to the interaction with the environment can be estimated by calculating, from Eq. (6),

$$\tau^{-1} = \langle \alpha_t|\dot{\rho}|\alpha_t\rangle \simeq -\left( \gamma/2 \right) \langle \alpha_t| (x - y)^2 |\alpha_t\rangle \left( 1/\delta^2 + 1/\lambda_{dB}^2 \right)$$  (10a)

Here $|\alpha_t\rangle = \exp(-iH_0 t/\hbar)|\alpha\rangle$ was used to separate out the evolution due to the environment from the evolution induced by the self-Hamiltonian $H_0$. Similarly, the rate of change of the off-diagonal term is:

$$\theta^{-1} = \langle \alpha_t|\dot{\rho}|\beta_t\rangle \simeq -\left( \gamma/2 \right) \langle \alpha_t| (x - y)^2 |\beta_t\rangle \left( 1/\delta^2 + 1/\delta_{dB}^2 \right)$$  (10b)

The key and only difference between the two rates is then the size of the factor $\langle (x - y)^2 \rangle$. For the diagonal terms it is given by

$$\langle \alpha_t| (x - y)^2 |\alpha_t\rangle = \delta^2 \sim \lambda_{dB}^2 .$$  (11a)
For the off-diagonal elements, it is, on the other hand
\[ \langle \alpha_t | (x - y)^2 | \beta_t \rangle = (\Delta x)^2 . \] (11b)

Therefore, the ratio of the two rates is
\[ \tau/\theta = (\Delta x/\delta)^2 \sim (\Delta x/\lambda_{dB})^2 . \] (12)
in accord with Eq. (1). For “macroscopic” values of \( \Delta x, m, \) and \( T, \) this ratio is enormous and enforces environment-induced superselection. It is worth pointing out that qualitative conclusions of our discussion are in accord with more elaborate path integral treatment of the harmonic oscillator, given recently by Caldeira and Leggett.\(^9\)

**DISCUSSION: A CLASSICAL LIMIT?**

In the previous section we have shown that when \( \delta \sim \lambda_{dB}, \) decorrelation is much more rapid than relaxation. The purpose of this section is to justify why, in the practical context, the assumption \( \delta \sim \lambda_{dB} \) is natural. Moreover, we shall briefly point out consequences of the disparity between the two timescales for the interpretation of quantum mechanics.

Let us first consider a classic example of measurement, patterned after the one discussed by von Neumann.\(^10\) We couple the measured system, initially in a state \( |\phi\rangle \), with the free particle measuring apparatus, so that their total Hamiltonian is
\[ H = H_{SYSTEM} + H_0 - i\hbar \Delta x \delta(t - t_0) P(\partial/\partial x) . \] (13)

Here \( P \) is the measured operator which we assume has 0 and 1 as the eigenvalues, while \( x \) is the position of the free particle which will record the outcome of the measurement.

Just before the observation the state of the apparatus must be determined with the accuracy better than \( \Delta x \). If the free particle apparatus is already in contact with the heat bath of temperature \( T, \) as discussed previously, then the measurement of its position with some accuracy \( \sigma, \Delta x \gg \sigma \gg \lambda_{dB}, \) will be a typical, sufficient preparation. Therefore, the apparatus will be left in an incoherent mixture of \( n = \sigma/\lambda_{dB} \) wavelets. Such inexhaustive measurements may be not only “realistic,” but also advantageous, as the resulting mixture will spread slower than the pure wavepacket of comparable
In the course of the interaction at $t = t_0$, each of the de Broglie-sized wavepackets will be split into an “unmoved” $|\alpha\rangle$ portion, and into the shifted one: $\exp(-i\Delta x \partial/\partial x)|\alpha\rangle = |\beta\rangle$. Therefore, immediately after the observation, the state of the combination (system - free particle apparatus) is, in effect, described by a mixture of terms of the form:

$$|\Upsilon\rangle = |\alpha\rangle(1 - P)|\phi\rangle + |\beta\rangle P|\phi\rangle$$

(14)

with all the $|\alpha\rangle$ contained within $\sigma$. Now the analysis of the decay of the pure state $|T\rangle$ into the density matrix of the form

$$\rho = |\alpha\rangle\langle\alpha|(1 - P)|\phi\rangle\langle\phi|(1 - P) + |\beta\rangle\langle\beta|P|\phi\rangle\langle\phi|P$$

(15)

can be conducted in accord with the discussion of the previous section. In particular, $\delta \sim \lambda_{dB}$ will apply as long as the resolution $\sigma$ of the measurement which prepares free-particle apparatus is worse than $\lambda_{dB}$. Moreover, even if $\sigma < \lambda_{dB}$, our qualitative conclusions still hold, as in that case decorrelation will be even more rapid.

The most intriguing corollary of our discussion is, perhaps, the possibility that in the classical limit of $\hbar \to 0$ the relaxation timescale may approach infinity,

$$\tau \to \infty$$

(16)

which allows the system to follow reversible, Newtonian dynamics, and yet the decorrelation timescale will remain arbitrarily short, or, indeed, it may approach zero:

$$\theta \to 0.$$  

(17)

We regard this limit as a true classical limit: Not only does it allow classical Newton’s equations of motion, but it also prevents long-range quantum correlations, by imposing the environment-induced superselection.$^{1,2}$

It is worth stressing that the loss of coherence and the accompanying “irreversibility” is a consequence of the deliberate tracing out of the environment, which disposes of the mutual information,$^5$ and not of the approximations involved in the derivation of Eq. (6). This is particularly clearly demonstrated by the analogous results obtained by path-integral methods for harmonic oscillators.$^9$

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