Dynamics and uncertainty for maximally entangled bipartite system constrained on a helicoid

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Abstract  The classical and quantum dynamics of particles constrained on a right helicoid is discussed via Dirac approach. We show how the uncertainty in measurement for observables of maximally entangled system is affected by the total number of constrained particles \( \sigma \), external magnetic field \( \vec{B} \); as well as by geometric parameters like the pitch \( \rho \) and the radial position of particles. In doing so we also highlight numeric bounds on the external field strengths to tune both intraparticle and bipartite entanglement and we remark that the bipartite entanglement is more robust to changes in the fields than the intraparticle entanglement, in this framework. We also highlight specific parameter regimes which lead the uncertainty (in measurement) to achieve respective parameter independence and, for a particular subset of commutation relations, the system remains confined in the quantum regime even in the limit \( \sigma \to \infty \). It is observed that the uncertainties are strongly influenced by the geometric parameters e.g., \( \rho \), and the strength of bipartite as well as intraparticle entanglement might be controllable through \( \rho \). The energy equation for this setup is obtained and the additional terms are discussed which arise due to quantum correlations, orbit–orbit interaction and the normal Zeeman effect, which leads to the splitting of the energy level into 11 non-degenerate levels. Finally we comment that, a linkage of this phenomenology with Aharonov–Bohm like effect might be possible by strictly confining \( \vec{B} \) along the central axis of the helicoid.

1 Introduction

From a classical physics perspective it is possible to specify all the degrees of freedom of a system simultaneously with arbitrary precision, but uncertainty relation is fundamental in quantum mechanics and is one of the chief features that underlines a difference between the classical and quantum physics. In the seminal EPR paper it was shown that if two systems are correlated then the uncertainty of measurement between a conjugate pair of observables of these systems reduces [1]. The extended uncertainty relation (by Robertson) sets a limit on the precision of simultaneous measurement of any pair of observables in terms of a lower bound, which gives a quantitative measure of the uncertainty of measurement [2].

The uncertainty relations in more recent literature are usually expressed in the well known entropic form e.g., see Ref. [3], in which the lower bounds quantifying uncertainty of measurement reduce if the two parties are maximally entangled, implying that quantum entanglement has a significant effect on the product of dispersion. In [4] it was reported that the lower bound deviates from the traditional Heisenberg Uncertainty Relation (HUR) in such a way that the precision of simultaneous measurement increases for maximally entangled bipartite and tripartite system.

One can also consider various types of entanglement where quantum correlations can exist between distinct degrees of freedom of a single particle i.e., intraparticle entanglement [5], or two particles (bipartite entanglement), as well as various background geometries. Dirac approach for constrained Hamiltonian dynamics provides a modified symplectic structure to obtain the constrained dynamics of the system [6]. To study the quantum dynamics, one must work in the Hamiltonian formalism.
However, for singular Lagrangians i.e.,
\[ \left\| \frac{\partial^2 L}{\partial \dot{x}_a \dot{x}_b} \right\| = 0 \]  
(1)
one cannot express all \( \dot{x}_a \) in terms of \( p_a \) and the Hamiltonian dynamics can not be obtained by the usual procedures. Dirac proposed a generic technique to handle singular Lagrangians in which the constrained dynamics can be obtained by Dirac brackets instead of Poisson brackets [6,7]. Dirac brackets can be used to assign geometric meaning to Hamiltonian formulation in the presence of second class constraints. A similar study in this direction was done in Ref. [8], and a different approach to find geometric duality for quantum equations in the pilot-wave limit was presented in Refs. [9–11]. Studies on Dirac approach for particle constrained on a helicoid and for a free particle on \( S^3 \) were carried out in [12,13]. Dirac quantization on curved spaces leads to extra terms in the energy equation which can be linked with the curvature of space or shift in the energy spectrum leading to novel physics. It is therefore also of interest to discuss such energy terms for constrained quantum correlated particles on curved spaces [14], which was done in [15] for a maximally entangled system constrained on \( S^1 \).

In view of the usefulness of generic constrained quantum systems for modern avenues in quantum research, e.g., quantum matter, it is instructive to explore some fundamental aspects of such systems e.g., the effect of entanglement on product of dispersion for quantum systems constrained on certain geometries. We address this question in this paper for a maximally entangled bipartite system on a right helicoid, which is a minimal surface. We will develop uncertainty relations and study the effect of various parameters on the product of dispersion. For this, we choose a pure bipartite state [16,17],
\[ |\Psi\rangle = \sum_{ij} \Lambda_{ij} |\psi^i_a\rangle \otimes |\psi^j_b\rangle \]
(2)and compute the lower bounds using Robertson relation to quantify the uncertainty of measurement for various observables acting on this bipartite state. Note that throughout this paper, the statements \textit{Uncertainty of (simultaneous) measurement/product of dispersion/precision of simultaneous measurement} will be used interchangeably, as appropriated by context.

The paper is organized as: In Sect. 2, we present the constrained classical dynamics on a right helicoid for \( \sigma = 1 \) and \( \sigma = 2 \). In Sect. 3, we quantize the classical system via Dirac quantization rule for particles constituting a maximally entangled bipartite system. In Sect. 4, we discuss the results of the effect of \( \sigma, \tilde{A}, \rho \) and radial position on the uncertainty of measurement. In Sect. 5, we find the energy equation and address the significance of additional terms arising in the equation. Section 6 is the summary of results.

### 2 Constrained classical dynamics on a right helicoid

#### 2.1 Single particle case (\( \sigma = 1 \))

We consider a free particle (\( \sigma = 1 \)) constrained on a right helicoid. The right helicoid is parameterized by the following equation [18]
\[ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \eta \cos(\theta) \\ \eta \sin(\theta) \\ \rho \theta \end{pmatrix} \]
(3)
where \( \eta \) is the radial distance from \( z \)-axis (hereafter called \( x^c \)-axis); \( \theta \) and \( \rho \) is the angle of the twist and pitch of the helicoid, respectively. The Lagrangian of this system can be written as
\[ L = \frac{1}{2} m \dot{\eta}^2 - eV(\eta, t). \]
(4)
Since the particle is constrained on the helicoid, this constraint is given by [12]
\[ f(x^a, x^b, x^c) = x^c - \rho \tan^{-1}\left(\frac{x^b}{x^a}\right) \]
(5)
where \( x^a = x, x^b = y, x^c = z \). In order to get the full dynamics of the system, we must incorporate this constraint in Eq. (5) into the Lagrangian in Eq. (4).
\[ L = \frac{1}{2} m \dot{\eta}^2 - eV(\eta, t) - \omega \left( x^c - \rho \tan^{-1}\left(\frac{x^b}{x^a}\right) \right) \]
(6)
where \( \omega \) is the Lagrange multiplier which serves as a degree of freedom. We have assumed the particle to be constrained on helicoid under the effect of external magnetic field and no additional force acts on the particle (i.e., \( V = 0 \)). The Hamiltonian of the system is
\[ H = \frac{1}{2m} p_\eta^2 + p_\rho \dot{\rho} + \omega \left( x^c - \rho \tan^{-1}\left(\frac{x^b}{x^a}\right) \right) \]
(7)
As the Hamiltonian in Eq. (7) is not independent of velocities, therefore we need to follow the Dirac’s approach to obtain the dynamics of the system. In such cases one needs to consider the relations which prevents the Hamiltonian to be independent of \( \dot{\eta} \). These relations which emerge directly from the Lagrangian are the primary constraints \( \xi_\lambda \approx 0 \). One can obtain a series of second class constraints from such relations via the consistency condition [6]:
\[ \dot{\xi}_\lambda = [\xi_\lambda, H_T] \approx 0 \]
(8)where \( H_T \) is the total Hamiltonian which is obtained by multiplying a function of time (\( \kappa \)), with the primary constraint and including this term into the Hamiltonian Eq. (7).
\[ H_T = \frac{1}{2m} p_\eta^2 + \omega \left( x^c - \rho \tan^{-1}\left(\frac{x^b}{x^a}\right) \right) + \kappa p_\rho \]
(9)
where $\dot{\omega}$ is absorbed into $\kappa$. The constraint emerging directly from the Lagrangian is given by $p_\omega = \frac{\partial L}{\partial \omega} = 0$. We will call this the primary constraint and denote it as $\xi_1 = p_\omega \approx 0$. The secondary constraints are obtained by means of the consistency condition $\xi_1 = [\xi_1, H_T] \approx 0$. These constraints are listed below,

$$\xi_2 = -x^c + \rho \tan^{-1} \left( \frac{x^b}{x^a} \right) \approx 0$$  \hspace{1cm} (10)

$$\xi_3 = -\left[ p_c \Pi^2 - \rho p_b x^a + \rho p_a x^b \right] \approx 0$$  \hspace{1cm} (11)

where, $\Pi^2 = (x^a)^2 + (y^b)^2$ and $\Upsilon = \{\rho^2 + (x^a)^2 + (x^b)^2\}

$$\xi_4 = \omega \Upsilon - 2\rho \left[ -p_a^2 x^a x^b + p_{\rho x} (x^a)^2 - p_{\rho y} (x^b)^2 + p_{\rho e} (x^c)^2 \right] \approx 0.$$  \hspace{1cm} (12)

In such cases, the dynamics of the system can not be obtained by the usual Poisson brackets but by Dirac brackets. For any two physical quantities $U$ and $V$ the Dirac bracket is given by

$$[U, V]_D = [U, V]_P - \sum_{\mu=1}^\lambda \left[ U, \xi_\mu \right] \Lambda_{\mu \nu} \left[ v_\nu, V \right]$$  \hspace{1cm} (13)

where, $\Lambda_{\mu \nu} = (\Omega^{-1})_{\mu \nu}$ is the inverse matrix of second class constraints with $\Omega_{\mu \nu} = [\xi_\mu, \xi_\nu]$ [6]. In case of singular Lagrangians, we need to include the constraints into the Lagrangian, Hamiltonian as well as the brackets to obtain the correct constrained dynamics of the system. The inverse matrix of second class constraints is given by

$$\Lambda_{\mu \nu} = \left( \begin{array}{cccc} 0 & \frac{4\rho^2 (p_a x^c - p_c x^a)}{2m^2} & \frac{2\rho^2 (p_a x^c - p_c x^a)}{2m^2} & \frac{4\rho^2 (p_a x^c - p_c x^a)}{2m^2} \\ \frac{2\rho^2 (p_a x^c - p_c x^a)}{2m^2} & 0 & \frac{2\rho^2 (p_a x^c - p_c x^a)}{2m^2} & \frac{2\rho^2 (p_a x^c - p_c x^a)}{2m^2} \\ \frac{2\rho^2 (p_a x^c - p_c x^a)}{2m^2} & \frac{2\rho^2 (p_a x^c - p_c x^a)}{2m^2} & 0 & \frac{2\rho^2 (p_a x^c - p_c x^a)}{2m^2} \\ \frac{2\rho^2 (p_a x^c - p_c x^a)}{2m^2} & \frac{2\rho^2 (p_a x^c - p_c x^a)}{2m^2} & \frac{2\rho^2 (p_a x^c - p_c x^a)}{2m^2} & 0 \end{array} \right).$$  \hspace{1cm} (14)

Using Dirac's approach, the classical dynamics is found to be

$$[x^a, p_b] = \frac{\rho^2 x^b x^a}{\Pi^2 \Upsilon}, \quad [x^b, p_a] = \frac{\rho^2 x^a x^b}{\Pi^2 \Upsilon}$$  \hspace{1cm} (15)

$$[x^a, p_c] = \frac{\rho x^b}{\Upsilon}, \quad [x^c, p_a] = \frac{-\rho x^b}{\Upsilon}$$  \hspace{1cm} (16)

$$[x^b, p_c] = \frac{\rho x^a}{\Upsilon}, \quad [x^c, p_b] = \frac{-\rho x^a}{\Upsilon}$$  \hspace{1cm} (17)

$$[p_a, p_b] = \frac{\rho^2 (x^a p_b - x^b p_a) - 2\rho p_a \Pi^2}{2m^2 \Upsilon},$$  \hspace{1cm} (18)

$$[p_a, p_c] = \frac{-\rho p_b + 2x^c p_a}{\Upsilon}$$  \hspace{1cm} (19)

$$[p_b, p_c] = \frac{\rho p_a + 2x^b p_c}{\Upsilon}.$$

The corresponding quantum dynamics of a free particle constrained on a helicoid can be obtained by the Dirac quantization rule: $\frac{[L]}{h} \rightarrow [.]$.

2.2 Two particle case ($\sigma = 2$)

We consider two particle case under the influence of scalar and vector potential. The Lagrangian is given by

$$L = \sum_{\alpha} \frac{1}{2} m \dot{\eta}_\alpha^2 - eV(x, t) + e \dot{\eta}_\alpha \cdot \vec{A}_{\eta\alpha}(x, t)$$  \hspace{1cm} (20)

where $\alpha$ is the particle index such that $\alpha = 1, \ldots, \sigma$ and

$$\eta_\alpha^2 = \dot{x}_\alpha^2 + \dot{y}_\alpha^2 + \dot{z}_\alpha^2 = x_{\alpha 1}^2 + y_{\alpha 1}^2 + x_{\alpha 2}^2 + y_{\alpha 2}^2 + x_{\alpha 1}^2 + y_{\alpha 2}^2.$$

The particles are assumed to keep their motion on the surface of the helicoid under the action of a uniform magnetic field $\vec{B}$ directed along $x^c$-axis. Applying this constraint on the motion of these particles leads to

$$L = \frac{1}{2} m \dot{\eta}_\alpha^2 - eV(x, t) + e \dot{\eta}_\alpha \cdot \vec{A}_{\eta\alpha}(x, t)$$  \hspace{1cm} (21)

$$- \omega \left( x^c_\alpha - \rho \tan^{-1} \left( \frac{x^b_\alpha}{x^a_\alpha} \right) \right).$$  \hspace{1cm} (22)

The Hamiltonian is:

$$H = p_\omega \dot{\omega} + \frac{1}{2m} (p_{\rho x} - e \vec{A}_{\rho x})^2 + eV(x, t)$$

$$+ \omega \left( x^c_\alpha - \rho \tan^{-1} \left( \frac{x^b_\alpha}{x^a_\alpha} \right) \right).$$  \hspace{1cm} (23)

This is the primary constraint since it is derived directly from the Lagrangian. The total Hamiltonian is thus given by:

$$H_T = \frac{(p_{\rho x} - e \vec{A}_{\rho x})^2}{2m} + \omega \left[ x^c_\alpha - \rho \tan^{-1} \left( \frac{x^b_\alpha}{x^a_\alpha} \right) \right] + \kappa p_\omega$$  \hspace{1cm} (24)

where $\dot{\omega}$ is absorbed in $\kappa$.

$$\xi_1 = p_\omega \approx 0.$$  \hspace{1cm} (25)

If a constraint is initially zero, it must be zero for all times. The resulting equation is taken as another constraint. Following the consistency condition, one obtains a series of constraints as follows:

$$\xi_2 = -\left( x^c_\alpha - \rho \tan^{-1} (x^b_\alpha/x^a_\alpha) \right) \approx 0$$  \hspace{1cm} (26)

$$\xi_3 = \frac{-\rho x^b (p_{\rho x} - e \vec{A}_{\rho x}) + \rho x^c (p_{\rho y} - e \vec{A}_{\rho y}) - (p_{\rho x} - e \vec{A}_{\rho x}) \Pi^2_a}{m \Pi^2_a} \approx 0.$$  \hspace{1cm} (27)

$$\xi_4 = \frac{-\rho x^b (p_{\rho x} - e \vec{A}_{\rho x}) + \rho x^c (p_{\rho y} - e \vec{A}_{\rho y}) - (p_{\rho x} - e \vec{A}_{\rho x}) \Pi^2_a}{m \Pi^2_a} \approx 0.$$  \hspace{1cm} (28)

$$-2\rho \left( -x^2_{\alpha 1} + 2x_{\alpha 1} x_{\alpha 2} + x^2_{\alpha 2} - 2x_{\alpha 1} x_{\alpha 2} - p^2_{\rho x} + p^2_{\rho y} \right) x^b_{\alpha 1}$$

$$+ (e^2 \vec{A}_{\rho x} \vec{A}_{\rho y} - e \vec{A}_{\rho x} p_{\rho y} - e \vec{A}_{\rho y} p_{\rho x} + p_{\rho x} p_{\rho y}) (x^c_{\alpha 1})^2$$

$$+ (-\vec{A}_{\rho x} \vec{A}_{\rho y} + e \vec{A}_{\rho x} p_{\rho y} + e \vec{A}_{\rho y} p_{\rho x} - p_{\rho x} p_{\rho y}) (x^c_{\alpha 2})^2 \right] \approx 0.$$

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Using $\zeta_1 = \zeta_2 = 0$, the total Hamilton (24) is

$$\mathcal{H}_T = \frac{1}{2}(p_{b\alpha} - e A_{b\alpha})^2.$$ 

(29)

To find the Dirac brackets for this system, we follow (13), where $\zeta$ represents the constraint evaluated above, and $\Lambda_{\mu\nu}$ is a matrix such that $\Lambda_{\mu\nu} = (\Omega^{-1})_{\mu\nu}$ and $\Omega_{\mu\nu} = [\zeta_\mu, \zeta_\nu]$. The inverse matrix is computed and is given below

$$\Lambda_{\mu\nu} = \left(\begin{array}{cc}
I & J \\
N & O
\end{array}\right)$$

(30)

with,

$$I = \begin{bmatrix}
0 & \frac{4\rho^2 (p_{b\alpha} x_\alpha^b - p_{b\alpha} x_\beta^b -e x_\alpha^b A_0 + e x_\beta^b A_0)^2}{m\Pi_{\alpha}^2 \Pi_{\beta}^2} \\
-4\rho^2 (p_{b\alpha} x_\alpha^b - p_{b\alpha} x_\beta^b -e x_\alpha^b A_0 + e x_\beta^b A_0)^2 & 0
\end{bmatrix}$$

$$J = \begin{bmatrix}
\frac{2\rho^2 (p_{b\alpha} x_\alpha^b - p_{b\alpha} x_\beta^b -e x_\alpha^b A_0 + e x_\beta^b A_0)}{m\Pi_{\alpha}^2 \Pi_{\beta}^2} & \frac{m\Pi_{\alpha}^2}{\Pi_{\beta}^2} \\
-\frac{m\Pi_{\alpha}^2}{\Pi_{\beta}^2} & 0
\end{bmatrix}$$

$$N = \begin{bmatrix}
\frac{2\rho^2 (p_{b\alpha} x_\alpha^b - p_{b\alpha} x_\beta^b -e x_\alpha^b A_0 + e x_\beta^b A_0)}{m\Pi_{\alpha}^2 \Pi_{\beta}^2} & \frac{m\Pi_{\alpha}^2}{\Pi_{\beta}^2} \\
-\frac{m\Pi_{\alpha}^2}{\Pi_{\beta}^2} & 0
\end{bmatrix}$$

$$O = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}.$$ 

The classical dynamics obtained via the Dirac’s approach is given in the Appendix. Following Dirac’s approach we achieved four sets of equations (listed in Appendix) which we have labelled as Type-I, Type-II, Type-III and Type-IV equations.

Type-I This set of equations reflects how the particle’s motion deviates from a linear trajectory along a particular axis. According to classical mechanics, the trajectory of a particle in a particular direction is encapsulated in the Poisson brackets. Type-I Dirac brackets differ from Poisson brackets due to the presence of additional terms $(\frac{\rho^2 (p_{b\alpha} x_\alpha^b - p_{b\alpha} x_\beta^b -e x_\alpha^b A_0 + e x_\beta^b A_0)^2}{m\Pi_{\alpha}^2 \Pi_{\beta}^2})$, see Eqs. (45)–(47) in Appendix for details) originating from the constraints. The particle’s dynamics is deviated from their original axis by these additional factors which include constituents ($\rho$, $x^\alpha$, $x^\beta$) underpinning the fact that underlying geometry is a Helicoid.

Type-II All canonical brackets of type-II vanish in classical mechanics; on the contrary, if a particle is moving along $x^\alpha$ axis then $p_{b\beta}$ can generate its translation along $x^\beta$ axis due to the constraints incorporated into the dynamics of the system. Therefore, these brackets are non-zero in this framework.

Type-III This is the set of equations where Dirac brackets coincide with the standard canonical brackets of classical mechanics. This is because none of the $\zeta_\mu$ is function of $p$ only, thus these brackets are not affected by the constraints.

Type-IV These equations contain Dirac brackets between various momentum components of both particles. These equations contain $\gamma_\alpha = \rho^2 + (x_\alpha^b)^2 + (x_\beta^b)^2 = \rho^2 + \sigma (x_\alpha^b)^2 + (x_\beta^b)^2$ where, the quantities $\Pi = \sqrt{(x_\alpha^b)^2 + (x_\beta^b)^2}$ and $\rho$ define the radius and pitch of the helicoid. Thus, these brackets specify the trajectory of particles on the surface of right helicoid.

3 Quantum dynamics

We consider bipartitioning the system in such a way that $\tau(\sigma) = \tau(\alpha) + \tau(\beta)$ (i.e., the system with $\sigma$ number of particles is partitioned into two subsystems containing $\alpha$ and $\beta$ number of particles) The bipartitioning of the classical system discussed above treats two particles as two parties (A and B) which constitute a global system, whose Hilbert space can be written as

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B.$$ 

(35)

The basis sets spanning $\mathcal{H}_A$ and $\mathcal{H}_B$ are taken to be $|\psi_\alpha\rangle_{\alpha=1}$ and $|\phi_\beta\rangle_{\beta=1}$ respectively. We assume these subsystems to be maximally entangled having unit concurrence. The maximally entangled pure state with respect to this partitioning is given by an equation of the following form

$$\Psi(x_1, x_2) = \epsilon \psi_1(x_1)\phi_1(x_2) + \xi \psi_1(x_1)\phi_2(x_2)$$

$$+ \eta \psi_2(x_1)\phi_1(x_2) + \delta \psi_2(x_1)\phi_2(x_2)$$

(36)

where $C = 2|\epsilon \delta - \xi \eta| = 1$ for a maximally entangled state [19]. The state of a single free (unentangled) particle on helicoid is defined as: $\Psi(\eta, \theta) = \exp(i l \theta)\psi(\eta)$, where $l$ is angular momentum [20]. The maximally entangled pure bipartite state for two particles constrained on a helicoid is
given by an equation of the following form [15],
\[ \Psi(\eta, \vartheta) = e^{i(\vartheta_1 + \vartheta_2 + \vartheta_1 \vartheta_2)} \left[ \psi_1(\eta_1)\phi_1(\eta_2) + \psi_2(\eta_1)\phi_2(\eta_2) \right]. \]
(37)

To quantize the Dirac brackets we use Dirac quantization rule. The commutators in Eq. (67) contain \( x^c \)-component of angular momentum. The operator is defined as [7]:
\[ L_c = xa p_a - xb p_b = \left( -i\hbar \frac{\partial}{\partial \vartheta} - \varrho \right). \]
(38)

It is important to note that the operators acting on an entangled state lead to a reduced uncertainty of measurement in view of the single free (unentangled) particle discussed in Sect. 2.1. This deviation in uncertainty leads to an enhanced precision in simultaneous measurement of observables. The Robertson relation serves as a measure of this deviation by computing the numerical lower bounds for a pure bipartite state of the form given in [2] and in Eq. (37). The lower bounds are computed under the assumption that \( \psi(\eta) \) is a constant function. These lower bounds are affected by a series of parameters including: the number of particles \( (\sigma) \), external fields, pitch of the helicoid \( (\rho) \) and the radial position of particles on the helicoid. The lower bounds are computed numerically using Robertson relation and a few of these are given in the Appendix. These lower bounds will be plotted as a function of some parameters in the following section.

4 Results

4.1 Uncertainty of measurement as a function of \( \sigma \)

The plots in Figs. 1 and 2 present the product of dispersion as a function of number of particles constrained on the right helicoid. In Figs. 1 and 2, the lower bounds of Type-I and Type-IV equations are evaluated at \( x^i = 2, x^j = 3.1, x^k = 2.45, x^l = 0.65, A = 0.5, l = 5, h = 1, \varrho = 4, \rho = 10, e = 1 \) and the angular position (in radians) of the particles (constrained on the helicoid) is: \( \vartheta_a = \tan^{-1}\left( \frac{\theta_a}{a} \right) \).

Figure 1 represents uncertainty of measurement of Eqs. (45)–(47) as a function of \( \sigma \). These plots essentially tell us how the lower bounds of the subsystem of two particles are affected by varying the total number of particles in the global system. The precision of simultaneous measurement of position and momentum decreases for both particles but this trend is not as pronounced for \( x^c \) and \( x^d \) component of particle 2 as in the case of \( x^b \) component. However, for particle 1, \( x^a \) and \( x^b \) manifest a sharp deviation with increasing \( \sigma \). The lower bounds of Fig. 1 become independent of \( \sigma \) as \( \sigma \rightarrow \infty \), i.e., for a very large number of particles, the system achieves \( \sigma \)-independence. Moreover, the uncertainty increases with increasing \( \sigma \) (for observables of the same particle) and never touches zero even in the limit \( \sigma \rightarrow \infty \), implying that the system remains in the quantum regime for an arbitrarily large number of identical constrained entangled particles, a similar conclusion was given for a different setup in [21].

Figure 2 illustrates that the uncertainty decreases exponentially for all commutation relations implying that the system tends to exhibit a classical-like behaviour for \( \sigma \rightarrow \infty \). It is evident from Figs. 1 and 2 that the product of dispersion between the observables of the same particle increases and that between the observables of different particles decreases in view of the single particle case. This indicates that the entanglement between the two parties increases (in accordance with Ref. [3] which suggests that increased entanglement leads to smaller uncertainty), however no intraparticle entanglement exists between position and momentum observables of either particle.

4.2 Uncertainty of measurement as a function of external field

The left and right panel plots in Fig. 3 present the lower bounds of Eqs. (67) and (73) as a function of \( \mathcal{A} \), respectively.

In the left panel plot, red (squares) and green (circles) curves represent the lower bound of Eq. (67) for particle 1 and 2, respectively, and in the right panel, red and green curves represent the lower bound of Eq. (73) i.e., \( [p_{a1}, p_{b2}] \) and \( [p_{a2}, p_{b1}] \), respectively. The (dashed) black line in both plots corresponds to the single free particle case discussed in Sect. 2.1. The left plot depicts that the lower bounds quantifying uncertainty of simultaneous measurement between various momentum degrees of freedom of each particle (i.e., \( [p_{a1}, p_{b1}] \) and \( [p_{a2}, p_{b2}] \)) initially decrease as a function of \( \mathcal{A} \) and hit minimum at \( \mathcal{A}_1 = -0.4 \) (for subsystem 1) and \( \mathcal{A}_2 = 2.3 \) (for subsystem 2) which, for our choice of parameters correspond to the maximum intraparticle entanglement that can be achieved for subsystem 1 and 2, respectively. The inset reveals that in view of the single particle case the intraparticle entanglement for both subsystems exists for lower bounds below \( 2 \times 10^{-3} \); the intraparticle entanglement is zero beyond this cutoff. Region-I \((-0.8 \leq A \leq 0) \) and region-II \((-0.2 \leq A \leq 4.7) \) identified in the inset of the left plot correspond to the entanglement regions for subsystem 1 and subsystem 2, respectively. The range of vector potential that favors intraparticle entanglement in the total system (subsystem 1 and 2) is thus \(-0.8 \leq A \leq 4.7 \).

Lower bounds quantifying the uncertainty of measurement between momentum degrees of freedom of distinct particles, as presented in the right panel plot of Fig. 3, also illustrates an initially decreasing trend for negative values of \( \mathcal{A} \). These bounds are zero for red (squares) and green (circles) curves at \( \mathcal{A}_{12} = 0 \) and \( \mathcal{A}_{21} = 0 \), respectively. The bipartite entanglement exists (for both red and green curves) for
Fig. 1 Product of dispersion for Type-I equations as a function of $\sigma$. The left and right panel plots represent curves quantifying intraparticle entanglement as a function of $\sigma$ for particle 1 and 2, respectively. The red (dashed) curve corresponds to the lower bound of $[x^a, p_a]$, the green curve (circles) corresponds to $[x_b^b, p_b]$ and the magenta (diamonds) corresponds to the lower bound of $[x_c^c, p_c]$.

Fig. 2 Product of dispersion for Type-IV equations as a function of $\sigma$, (all other parameters as before). In the left plot, the red (dashed) and green (circles) represent the lower bounds of Eq. (67) (quantifying intraparticle entanglement) for particle 1 and 2, respectively. In the right panel, the red (dashed) and green (circles) represent the lower bound of Eq. (73) (quantifying bipartite entanglement) i.e., $[p_{a1}, p_{b2}]$ and $[p_{a2}, p_{b1}]$, respectively.

lower bounds smaller than approx. $2.1 \times 10^{-3}$ and is maximum at $A_{12} = A_{21} = 0$ for any choice of parameters. The bipartite entanglement is nonzero for red (squares) curve in region-I identified by $-1.2 \leq A \leq 1.2$ and for green (circles) curve in region-II, i.e., $-5.1 \leq A \leq 5.1$. For $A \geq 0$ both curves exhibit an increasing trend and at $A = 1.2$, the bipartite entanglement between $p_{a1}$ and $p_{b2}$ degrees of freedom is lost. Further increasing the vector potential to $A = 5.1$ renders disentanglement between all momentum degrees of freedom of the global system.

It is important to note, from both plots in Fig. 3, that at $A = 0$, the intraparticle entanglement for subsystem 1 is zero whereas the bipartite entanglement is maximum which suggests that, for a physical setup like ours, the intraparticle entanglement can only be achieved in the presence of external field $\mathbf{B}$. However, the bipartite entanglement can be achieved and appropriately tuned (in the absence of external magnetic field) for maximum strength in regions of the parameter space respecting the corresponding cutoff values, as discussed above.

It is possible to calculate the minimum/maximum strength of the external field $\mathbf{B}$ that enforces intraparticle and bipartite entanglement in the system, from vector potential. Since the external field is assumed to be uniform and directed along $x^c$-axis, the vector potential corresponding to this field rotates about $x^c$-axis and is: $A = nB\Pi/2$, where $n$ is the number of twists of the helicoid. We choose a right helicoid with $n = 2$, the minimum and maximum strength of external magnetic field required for intraparticle entanglement is $B_{i}^{c} = 0$ and $B_{i}^{c} = 0.47$, respectively. The minimum/maximum strength of the field required for bipartite entanglement is $B_{ii}^{bi} = 0.12$ and $B_{ii}^{bi} = 0.51$, respectively. Thus it suggests that the bipartite entanglement is more robust against small fluctuations in $\mathbf{B}$ as compared to the intraparticle entanglement.
Fig. 3  Product of dispersion for Type-IV equations as a function of external field at $A = 0.1$, and all other parameters as before. In the left plot, the red (dotted) and green (circles) represent the lower bounds of Eq. (67) (quantifying intraparticle entanglement) for particle 1 and 2, respectively. In the right panel, the red (dotted) and green (circles) represent the lower bound of Eq. (73) (quantifying bipartite entanglement) i.e., $[p_{a1}, p_{b2}]$ and $[p_{a2}, p_{b1}]$, respectively. The dashed black line corresponds to single free particle case discussed in Sect. 2.1, i.e., $(\Delta p_a)^2(\Delta p_b)^2 = 2.065 \times 10^{-5}$.

Fig. 4  Product of dispersion for Type-I equations as a function of pitch. Left: Uncertainty of measurement as a function of $\rho$ for particle 1. Right: Uncertainty of measurement as a function of $\rho$ for particle 2. All parameters as defined earlier.

4.3 Uncertainty of measurement as a function of $\rho$

The left and right panel plot in Fig. 4 show the uncertainty as a function of $\rho$ for particle 1 and 2, respectively. The red (solid), green (circles) and magenta (dashed) curves correspond to the lower bounds of $[x^a, p_a], [x^b, p_b]$ and $[x^c, p_c]$, respectively. The left plot in Fig. 4 infers that the uncertainty of measurement in the case of $[x^a, p_a], [x^b, p_b]$ (i.e., red and green curves) drops sharply for $\rho \leq 10$ and stabilizes for higher $\rho > 10$; a phenomenon which is reversed for the case of $[x^c, p_c]$ (magenta) curve where it rises sharply for small $\rho$ regime. The plot in the right panel of Fig. 4 also portrays a similar scenario but with a much smaller lower bound for green (circles) $[x^b, p_b]$ that tend to zero for $\rho > 10$ approximately, and a higher bound for red (solid) curve $[x^a, p_a]$. Figure 4 illustrates that uncertainty of measurement is not stable against variations in the pitch for $\rho < 10$; with the parameter values utilized. It however achieves robustness in parameter regimes with large $\rho$. Thus, the lower bounds (and hence uncertainty of measurement) can achieve $\rho$-independence for particles constrained on helicoidal geometries with large $\rho$. The left plot in Fig. 4 depicts that the lower bounds increase with increasing $\rho$, and reach a somewhat stable point for regimes in parameter ranges $\rho > 10$. Besides, there is no intraparticle entanglement between momentum degrees of freedom (of each particle) in the region $1 \leq \rho \leq 3$. For $\rho > 3$, intraparticle entanglement between momentum degrees of freedom for subsystem 2 can be achieved. The inset in right panel plot tells that initially the bounds increase and then decrease as a function of $\rho$, for both $[p_{a1}, p_{b2}]$ and $[p_{a2}, p_{b1}]$. It is interesting to note (from right panel plot) that both cases lie below the black (dashed) curve which corresponds to single particle bounds. Thus, one can conclude from Fig. 5 that bipartite entanglement between the
momentum degrees of freedom exists for all $\rho$, with a tunable strength controllable by the value of $\rho$.

4.4 Uncertainty of measurement as a function of radial position

The colour plots in Fig. 6 present product of dispersion as a function of radial position of particles. The upper left plot illustrates that the lower bound of Eq. (45) $((\Delta x_a^a)^2(\Delta p_{a\alpha}))^2$ is minimum at $(x_a^a, x_b^b) = (0.10649, 1.1714)$ with a numerical value of 0.003002. The plot also shows an asymmetrically located minima at $(-x_a^a, -x_b^b)$ having the same magnitude but opposite direction. The middle plot (in upper panel) gives the same result for the lower bound of Eq. (46) $((\Delta x_a^b)^2(\Delta p_{b\alpha}))^2$, only the plot is flipped by $\frac{\pi}{2}$ for observables depicting the location of particles at right angles to $x^a$-axis. The parameter regions in bright yellow colour in Fig. 6 correspond to relatively higher uncertainty ranges, and it is easy to see that the top left and middle panel plots hint at an increased uncertainty away from the origin for both cases. Increasing and decreasing $(x_a^a, x_b^b)$ imply that the particles move towards the edges or towards the central axis $(x^a$-axis) of the helicoid, respectively. The top right plot for the $x^b$-component represents the opposite feature with the uncertainty increasing with decrease in $(x_a^a, x_b^b)$ and attaining the maximum value of 0.25 at the origin $(0, 0)$. The uncertainty of measurement increases with the decrease in the pitch $\rho$ and it is evident from the left and middle panel plots in the bottom row of Fig. 6 that the uncertainty of measurement increases for significantly large parameter regimes. Although, the lower bounds reduce with increase in $\rho$, no reduction in uncertainty is obtained in view of the single particle case and no intraparticle entanglement exists between position and momentum observables.

The left and right panel plots in Fig. 7 give the lower bounds of Eq. (73) $(\Delta p_{a\alpha})^2(\Delta p_{b\alpha})^2$ and $(\Delta p_{a\alpha})^2(\Delta p_{b\alpha})^2$ respectively, as a function of $(x_a^a, x_b^b)$. The left plot shows a small faint peak and one relatively large and broad peak leading to the global maxima value 0.093403. In the right plot there are two peaks located at $(-6.28, -0.01)$ and $(6.28, -0.01)$. Figure 7 also exhibits the sensitive dependence of uncertainty on the radial coordinates; the uncertainty decreases quickly as one deviates from the peaks.

5 Energy equation

The Dirac quantization leads to some additional terms in the energy equation which can be of great physical significance, in connection to curved spaces [7,14,15] and to the external fields. In this section, we obtain the energy equation for our setup and study the effect of various parameters i.e., $\vec{B}, \vec{A}$ etc., on these terms. By utilizing Eq. (38), the total Hamiltonian in Eq. (29) can be written as

$$H_T = \frac{E_2^{(a)}}{2m(\Pi_2^a + \rho^2)} + \frac{(x_a^a)^2 + \rho^2}{2m(\Pi_2^a + \rho^2)} + \frac{(x_b^b)^2 + \rho^2}{2m(\Pi_2^b + \rho^2)} + \frac{2x_a^a x_b^b}{2m(\Pi_2^a + \rho^2)} + \frac{2x_a^b x_b^a}{2m} + \frac{e^2 A_{1\alpha}}{2m} - \frac{ep_{1\alpha}}{m} A_{1\alpha}$$

where $\Pi^2 = (x_a^a)^2 + (x_b^b)^2$ is the radius of right helicoid (assuming the full helicoid to perfectly fit inside a cylinder of radius $\approx \Pi^2$). The solution of the Schrödinger equation for maximally entangled bipartite state in Eq. (37) leads to the following energy equation,
Fig. 6 Color plots of lower bounds of Eqs. (45)–(47) (quantifying intraparticle entanglement) as a function of radial position. The upper and lower horizontal panels correspond to $\rho = 10$ and $\rho = 1$ respectively. The left, middle and right plot represent the lower bounds of Eqs. (45)–(47) respectively, as a function of $(x_a^\sigma, x_b^\sigma)$. The lower bounds are evaluated for parameter values same as earlier. The angular position of particles: $\vartheta_\sigma = \tan^{-1}(x_b^\sigma/x_a^\sigma)$.

Fig. 7 Color plots of the lower bounds of Eq. (73) (i.e., $[p_{a1}, p_{b2}]$ in the left plot and $[p_{a2}, p_{b1}]$ in the right plot), as a function of $(x_a^\sigma, x_b^\sigma)$. The plots correspond to $\rho = 10$. All other parameter values same as before.

\[
\mathcal{E}\psi + (Q\mathcal{E})(Q\psi) = \frac{1}{2m(\Pi^2_\sigma + \rho^2)} \left[ \left( \frac{l^2}{2} - \left( \frac{l}{2} - \tilde{\varrho} \right)^2 \right) - \partial l(2\vartheta_a + 1) + \left( \vartheta_a^2 - \frac{\varrho^2}{1 + \varrho^2} \right)^2 \right] \Psi \\
- \frac{1}{2m(\Pi^2_\sigma + \rho^2)} \left[ ((x_a^\psi)^2 + \rho^2)Q_{\psi\psi}^{1}\psi + ((x_b^\psi)^2 + \rho^2)Q_{\psi\psi}^{1}\psi + ((x_a^\psi)^2 + \rho^2)Q_{\psi\psi}^{1}\psi + ((x_b^\psi)^2 + \rho^2)Q_{\psi\psi}^{1}\psi + (\Pi^2_\sigma + \rho^2)((Q_{\psi\psi}^{1}\psi + Q_{\psi\psi}^{1}\psi + e^2A_{\psi\psi}) + 2ieA_{\psi\psi}(Q_{\psi\psi}^{1}\psi + Q_{\psi\psi}^{1}\psi))] \right.
\]

where, $\tilde{\varrho} = \varrho - 1$, $\hbar = 1$ and $Q_{\psi\psi}^{1} = \frac{e_i^{\psi(\psi_1 + \psi_2 + \eta_1 \eta_2)}}{\sqrt{2}} [\frac{\partial \psi_1(\eta_1)}{\partial x_a^\psi} \Phi_1(\eta_2) + \frac{\partial \psi_2(\eta_1)}{\partial x_b^\psi} \Phi_2(\eta_2)]$ indicates the variation in state and $Q\mathcal{E}$ represents the additional energy terms associated with $Q\psi$. Keeping in view the definition of $Q$, the indices in the above equation can be interpreted as: the index $a, b, c, n$ in the subscript refers to the derivative of $\psi(\phi)$ with respect to $x^a, x^b, x^c, \eta$ respectively, and the subscript $(ab)$ refers to the second order derivative with respect to $x^a$ and $x^b$. The superscript represents the particle number and the index $\psi(\phi)$ in the subscript refers to the derivative of $\psi(\phi)$. The first term in the energy equation has the form $\mathcal{E} = \frac{E}{2m\varrho^2}$.
with \( r \rightarrow (\Pi_\alpha^2 + \rho^2) \); \( \Pi_\alpha^2 \) and \( \rho \) being the radius and pitch of the helicoid respectively. Hence this term characterizes the energy of particles (constrained to retain their motion on a right helicoid) as a function of angular position \( \vartheta_\alpha \). One can see that: \( E = E(1, 2) \neq \bar{E}_1 + \bar{E}_2 \) i.e., this is the case of interacting particles, where it is not possible to specify each particle independently of the other (particle(s)) making up the system. The particles are maximally entangled and the energy of each particle is affected by the simultaneous angular position \( \vartheta_\alpha \) of the other particle. It is also to be noted that the energy equation contains additional terms in view of the equation for particles under the effect of a vector potential (but free from any additional constraints on its trajectory), as given by the following equation

\[
\mathcal{H} = \left( \hat{p}_{\vartheta_\alpha} - e\eta_\alpha \hat{A}_{\vartheta_\alpha} \right)^2 \frac{2m}{(\Pi_\alpha^2 + \rho^2)}. \tag{41}
\]

The energy equation for free particle \( f(x^a, x^b, x^c) = V = 0 \) and \( \hbar = 1 \) takes the following form

\[
\mathcal{E} \Psi = \frac{1}{2m(\Pi_\alpha^2 + \rho^2)} \left[ \left( l + \frac{1}{2} \right)^2 \right] \Psi - \frac{e\eta_\alpha \hat{A}_{\vartheta_\alpha}}{m(\Pi_\alpha^2 + \rho^2)} \Psi + \frac{e^2 \Pi_\alpha^2 \hat{A}_{\vartheta_\alpha}}{2m(\Pi_\alpha^2 + \rho^2)} \Psi. \tag{42}
\]

The additional energy term (second term) in Eq. (40) has non-constant terms, which involve derivatives of the function characterizing the radial distance of particles from the central axis of the helicoid. Also note that, since we choose the particles with \( l = 5 \), the interaction between the external field \( \vec{B} \) and the field generated by the orbital motion of the particles splits the energy level into 11 different energy level (i.e., normal Zeeman effect leads to 11 split levels with slightly different energies). In the absence of spin–spin interaction, the total angular momentum vector (obtained as a result of orbit–orbit coupling of the two particles) precess about the magnetic field and the projection of orbital angular momentum vector of these particles along \( x^c \)-axis is given by the following equation

\[
\mathcal{L}_c^1 = l(1 + \vartheta_2) - \varphi, \quad \mathcal{L}_c^2 = l(1 + \vartheta_1) - \varphi. \tag{43}
\]

The total \( x^c \)-component of angular momentum is given by

\[
\mathcal{L}_c = \sum_\alpha \mathcal{L}_c^{(\alpha)} \text{ i.e.}, \quad \mathcal{L}_c = 2l + \vartheta_\alpha - 2\varphi. \tag{44}
\]

From the above discussion one can see that, the additional terms arising in Eq. (40) represent the energies of split levels which are a consequence of a perturbation collectively caused by quantum correlations, orbit–orbit interaction and the normal Zeeman effect.

Lastly it is interesting to note that, if the external field \( \vec{B} \) is strictly confined to only the region along the central axis of the helicoid (along \( x^c \)-axis), then it is very likely to achieve a situation where no field penetrates the surface of the helicoid which may lead to a possible linkage of this geometric scheme to the Aharonov–Bohm effect. The energy equation Eq. (40) gives the dependence of energy spectrum on the vector potential instead of \( \vec{B} \) and hence on magnetic flux through \( A = \frac{\Phi}{2\pi H} \), in accordance with Aharonov–Bohm effect.

### 6 Summary

In this work, the uncertainty of simultaneous measurement for various observables is quantified by computing their respective lower bounds using Robertson relation [2]. The classical dynamics of two constrained particle system under the influence of an external magnetic field is handled via Dirac formalism, the classical system is then quantized by Dirac quantization rule assuming the two particles constitute a maximally entangled bipartite system. We have shown how these uncertainties, in a constrained maximally entangled system on a helicoid, are affected under various geometric and non-geometric system parameters.

- **Effect of \( \sigma \):** It was shown that the uncertainty between various observables of the two particles reduces as a function of \( \sigma \). However, no reduction in uncertainty is possible for position and momentum observables of a single particle i.e., there is no intraparticle entanglement between these degrees of freedom. In view of a quantum to classical transition perspective, our results reveal that for Type-I equations the quantum regime persists for a system with a significantly large number of identical entangled particles constrained on helical geometries. Besides, the lower bounds of Type-I equations achieve \( \sigma \)-independence at the expense of a very large number of constrained particles i.e., in the limit \( \sigma \rightarrow \infty \).

- **Effect of \( \vec{B} \):** It was shown that the external field \( \vec{B} \) plays an important role in preserving quantum correlations and suitable strengths of \( \vec{A} \) were revealed for tuning intraparticle as well as bipartite entanglement. Numerical bounds on the strength of \( \vec{B} \) required for intraparticle and bipartite entanglement were given. We also show that for this system, it is possible to bring the uncertainty between momentum observables (of the different particles) to zero at \( \vec{A} = 0 \), which corresponds to maximal bipartite entanglement. However, intraparticle entanglement can only be tuned in the presence of non-zero external fields.
- **Effect of $\vec{\rho}$**: We showed that the product of dispersion (between position and momentum observables of the same particle) is susceptible to variations in the pitch for $\rho \leq 10$, and achieves $\rho$-independence for large values of $\rho$. Moreover, it is observed that the bipartite entanglement exists for all $\rho$ and is tunable by adjusting $\rho$.

- **Effect of radial position of particles**: It has also been highlighted that the lower bounds are sensitive to changes in radial coordinates. The lower bounds are also affected by the pitch $\rho$, in such a way that increasing $\rho$ reduces the lower bounds between position and momentum observables of the same particles. However, this reduction does not imply existence of intraparticle entanglement in view of the single particle case.

It has been observed that the bipartite entanglement is more robust to fluctuations in the fields as compared with the intraparticle entanglement. The energy equation is obtained and few additional energy terms arise in comparison to Eq. (42). These new energy terms arise due to coupling between the angular momentum of two particles, quantum correlations and the normal Zeeman effect which leads to the splitting of energy level into 11 levels. Finally, we remark that a suitable linkage might be possible between this geometric setup and the Aharonov–Bohm effect under the $\vec{A}$ dependence of energy, which might be an interesting avenue to explore further.

**Data Availability Statement** This manuscript has no associated data or the data will not be deposited. [Authors’ comment: The work is purely mathematical and all the techniques are standard. The simulation reported is also simple equation plotting which is straightforward. Thus there is no data required to be deposited.]

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**Appendix**

In all these equations, $\sigma = \beta = 1, 2$, but $\sigma \neq \beta$ in a given bracket.

### A. Type-I equations

\[
[x^b_\sigma, p_{a\sigma}]_D = (1 - \frac{\rho^2(x^b_\sigma)^2}{\Pi_\sigma^2 Y_\sigma})
\]  
(45)

\[
[x^a_\sigma, p_{b\sigma}]_D = (1 - \frac{\rho^2(x^a_\sigma)^2}{\Pi_\sigma^2 Y_\sigma})
\]  
(46)

\[
[x^c_\sigma, p_{c\sigma}]_D = (1 - \frac{(x^c_\sigma)^2 + (x^b_\sigma)^2}{Y_\sigma})
\]  
(47)

### B. Type-II equations

\[
[x^b_\sigma, p_{b\sigma}]_D = \frac{\rho^2 x^b_\sigma x^a_\sigma}{\Pi_\sigma^2 Y_\sigma}
\]  
(48)

\[
[x^a_\sigma, p_{c\sigma}]_D = -\frac{\rho x^b_\sigma}{Y_\sigma}
\]  
(49)

\[
[x^b_\sigma, p_{a\sigma}]_D = \frac{\rho^2 x^a_\sigma x^b_\sigma}{\Pi_\sigma^2 Y_\sigma}
\]  
(50)

\[
[x^c_\sigma, p_{a\sigma}]_D = -\frac{\rho x^b_\sigma}{Y_\sigma}
\]  
(52)

\[
[x^c_\sigma, p_{b\sigma}]_D = \frac{\rho x^a_\sigma}{Y_\sigma}
\]  
(53)

\[
[x^b_\sigma, p_{a\beta}]_D = -\frac{\rho^2 x^b_\sigma x^b_\beta}{\Pi_\beta^2 Y_\alpha}
\]  
(54)

\[
[x^c_\sigma, p_{c\beta}]_D = -\frac{\rho x^a_\sigma}{Y_\alpha}
\]  
(55)

\[
[x^c_\sigma, p_{b\beta}]_D = \frac{\rho^2 x^a_\beta x^b_\beta}{\Pi_\beta^2 Y_\alpha}
\]  
(56)

\[
[x^b_\sigma, p_{b\beta}]_D = -\frac{\rho^2 x^b_\sigma x^b_\beta}{\Pi_\beta^2 Y_\alpha}
\]  
(57)

\[
[x^b_\sigma, p_{a\beta}]_D = \frac{\rho^2 x^a_\beta x^b_\beta}{\Pi_\beta^2 Y_\alpha}
\]  
(58)

\[
[x^c_\sigma, p_{c\beta}]_D = \frac{\rho x^a_\sigma}{Y_\alpha}
\]  
(59)

\[
[x^c_\sigma, p_{a\beta}]_D = -\frac{\rho x^b_\beta ((x^a_\sigma)^2 + (x^b_\sigma)^2)}{\Pi_\beta^2 Y_\alpha}
\]  
(60)

\[
[x^c_\sigma, p_{b\beta}]_D = \frac{\rho x^a_\beta ((x^a_\sigma)^2 + (x^b_\sigma)^2)}{\Pi_\beta^2 Y_\alpha}
\]  
(61)

\[
[x^c_\sigma, p_{c\beta}]_D = -\frac{(x^c_\sigma)^2 + (x^b_\sigma)^2}{Y_\alpha}
\]  
(62)

### C. Type-III equations

\[
[x^a_\sigma, x^a_\beta]_D = [x^b_\sigma, x^b_\beta]_D = [x^c_\sigma, x^b_\beta]_D = 0
\]  
(63)

\[
[x^a_\sigma, x^a_\beta]_D = [x^b_\sigma, x^b_\beta]_D = [x^c_\sigma, x^b_\beta]_D = 0
\]  
(64)
[x_a^b, x_a^b]_D = [x_a^b, x_a^b]_D = [x_a^b, x_a^b]_D = 0 
(65)

[x_a^b, x_b^b]_D = [x_a^b, x_b^b]_D = [x_b^b, x_b^b]_D = 0. 
(66)

D. Type-IV equations

Let \( \chi_a^\alpha = (p_a - eA_a), \chi_b^\beta = (p_b - eA_b), \chi_c^\gamma = (p_c - eA_c) \)

\[
[p_a, p_b] = \frac{\rho^2(\chi_a^\alpha \chi_b^\beta - \chi_a^\alpha \chi_b^\beta) - 2\rho((\chi_a^\alpha)^2 + (\chi_b^\beta)^2)\chi_c^\gamma}{\Pi_2^2 \gamma_a} \quad (67)
\]

\[
[p_a, p_c] = -\frac{\rho \chi_a^\alpha + 2\chi_a^\alpha \chi_b^\beta}{\gamma_a} \quad (68)
\]

\[
[p_b, p_c] = \frac{\rho \chi_b^\beta + 2\chi_b^\beta \chi_c^\gamma}{\gamma_b} \quad (69)
\]

\[
[p_a, p_b] = \frac{\rho x_a^b \rho \chi_b^\beta + 2\chi_b^\beta \chi_b^\beta}{\gamma_b} - \frac{(-\rho \chi_b^\beta + 2\chi_b^\beta \chi_b^\beta)\rho x_b^a}{\gamma_a} \quad (70)
\]

\[
[p_b, p_c] = \rho x_b^a \rho \chi_b^\beta + 2\chi_b^\beta \chi_b^\beta \quad (71)
\]

\[
[p_c, p_b] = 0 \quad (72)
\]

\[
[p_a, p_b] = -\frac{\rho x_a^b \rho \chi_b^\beta + 2\chi_b^\beta \chi_b^\beta}{\gamma_a} + \frac{(-\rho \chi_b^\beta + 2\chi_b^\beta \chi_b^\beta)\rho x_b^a}{\gamma_b} \quad (73)
\]

\[
[p_a, p_c] = -\frac{\rho x_a^c \rho \chi_a^\beta + 2\chi_a^\beta \chi_a^\beta}{\gamma_a} \quad (74)
\]

\[
[p_b, p_c] = \frac{\rho x_c^b \rho \chi_a^\gamma + 2\chi_a^\gamma \chi_a^\gamma}{\gamma_a} \quad (75)
\]

E. Lower bounds

The lower bounds are computed numerically using Robertson relation [2] as:

\[
(\Delta x_a^\alpha)^2(\Delta p_a^\alpha)^2 \geq \frac{\hbar^2}{4} - \frac{\rho^2(\chi_a^\alpha)^2}{\Pi_2^2 \gamma_a} \quad (76)
\]

\[
(\Delta p_a^\alpha)^2(\Delta p_a^\alpha)^2 = \frac{1}{4} \times \frac{|\hbar \rho^2((\hbar - \phi) + \theta_2) - [\rho^2 \chi_a^\alpha - \rho^2 \chi_a^\alpha - 2\rho \pi \Pi_2^2 |A|]^2}{\Pi_2^2 \gamma_a} \quad (77)
\]

\[
(\Delta p_a^\alpha)^2(\Delta p_a^\alpha)^2 = \frac{1}{4} \times \frac{|\hbar \rho^2((\hbar - \phi) + \theta_1) - [\rho^2 \chi_a^\alpha - \rho^2 \chi_a^\alpha - 2\rho \pi \Pi_2^2 |A|]^2}{\Pi_2^2 \gamma_a} \quad (78)
\]

We computed the lower bounds by assuming that, \( \eta \) is fixed, so that the wavefunction is independent of \( \eta \) and the corresponding derivatives w.r.t \( \eta \) are zero.

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