Scattering of Glueballs and Mesons in Compact $QED$ in 2 + 1 Dimensions

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Abstract

We study glueball and meson scattering in compact $QED_{2+1}$ gauge theory in a Hamiltonian formulation and on a momentum lattice. We compute ground state energy and mass, and introduce a compact lattice momentum operator for the computation of dispersion relations. Using a non-perturbative time-dependent method we compute scattering cross sections for glueballs and mesons. We compare our results with strong coupling perturbation theory.

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1. Introduction. The non-perturbative computation of scattering and decay amplitudes of hadrons such as mesons and glueballs from quantum chromodynamics (QCD) is one of the most challenging problems of nuclear physics. Recently, hadron-hadron scattering lengths have been obtained from QCD\(_{3+1}\) lattice simulations by studying the finite volume behavior of the energy of a two-particle state \([1]\). The \(\pi - \pi\) and the \(N - N\) scattering length has been obtained in this way by Guagnelli et al. \([2]\). Sharpe et al. \([3]\) have computed the \(\pi - \pi\) scattering length in the \(I = 2\) channel. A method of how to obtain scattering phase shifts from the finite volume behavior of the spectrum has been worked out by Lüscher \([4]\). Meson-meson phase shifts from compact QED\(_{2+1}\) have also been obtained in lattices simulations using Lüscher’s method by Fiebig et al. \([5]\). All these studies have used the powerful numerical methods of Euclidean field theory. Scattering can also be studied using Hamiltonian field theory, such as in the work of Alessandrini et al. \([6]\) and by Hakim \([7]\), who have considered glueball scattering in strong coupling perturbation theory to lowest order in compact QED\(_{2+1}\).

At present it seems difficult to extend these Hamiltonian calculations to higher orders of perturbation theory, or to apply it to non-Abelian gauge theory. On the other hand, Lüscher’s method applied to QED features ambiguities in selecting the physical phase shifts from a determinant. These difficulties justify trying alternative methods to compute directly the scattering matrix in lattice gauge theory. Such a non-perturbative Hilbert space method to study scattering in quantum field theory on the lattice has been developed by Kröger and collaborators (see Ref. \([8]\) and references therein). It has been applied to the Schwinger model on the lattice \([9]\). Here we will present an application to QED\(_{2+1}\) and compute scattering cross sections.

This paper will be organized as follows: First, we briefly describe compact lattice QED in 2+1 dimensions. We present results for the low-lying mass spectrum and the dispersion relation for the glueball and meson and finally S-matrix elements and cross sections for glueball-glueball and meson-meson scattering.

2. Compact \(U(1)\) gauge theory in 2+1 dimensions. As shown by Polyakov \([10]\) and Banks et al. \([11]\) compact \(U(1)\) gauge theory in 2+1 dimensions has many properties of \(SU(3)\) gauge theory in 3+1 dimensions, such as field-source confinement for every non-vanishing value of the coupling constant. Ambjorn et al. \([12]\) found that the mass gap and the string tension vanish at \(g = 0\) in a non-analytic way, that is,

\[
M^2 = \frac{8\pi^2}{g^2a^3} \exp\left[\frac{-2\pi^2V(0)}{g^2a}\right],
\]

\[
\sigma = \frac{2}{\pi^2} g^2 M. \tag{1}
\]

Göpfert and Mack \([13]\) have shown rigorously that the theory has a non-zero string tension for arbitrary \(g \neq 0\) and that the value of \(\sigma\) given by eq.(1) is a rigorous lower bound.
on the string tension. An exact ground state solution for this model in a Hamiltonian lattice formulation has been obtained by Shuohong et al. [14]. Other quantities such as the mass gap and the string tension have been obtained at high orders in weak coupling perturbation theory by Morningstar [15]. This model has been investigated using weak coupling and strong coupling perturbation theory, variational methods, and Monte Carlo methods, to compute the vacuum energy, glueball masses, string tension, specific heat, etc.

We start with the usual Kogut-Susskind Hamiltonian for the compact $U(1)$ model in 2+1 dimensions, using the temporal gauge, with rescaled variables following Drell et al. [16]. To compare lattice results with strong coupling perturbation theory, we rescale $\frac{2}{g^2} H \rightarrow H[17]$ and work with the following Hamiltonian,

$$H = H_0 + H_{int} = \sum_{\text{links}} (E_{\vec{n}}^x)^2 + (E_{\vec{n}}^y)^2 - x \sum_{\text{plaq}} V_{\vec{n}} + V_{\vec{n}}^\dagger,$$

(2)

where $x = 1/g^4$ is the dimensionless coupling parameter. Starting from the vector potential $A^j_{\vec{n}}$ with $j = x, y$ and $\vec{n}$ denoting the lattice site, then a link variable is defined by $U_{\vec{n}}^j = \exp[iA^j_{\vec{n}}]$, and a plaquette variable being the product of four sequential links going around an elementary square tile is defined by $V_{\vec{n}} = U_{\vec{n}}^1 U_{\vec{n}+1}^1 [U_{\vec{n}+2}^1]^\dagger [U_{\vec{n}}^{2}]^\dagger$. The Hamiltonian is built from link and plaquette variables and it is invariant under residual, i.e., time independent gauge transformations. The Hamiltonian shows a number of symmetries: (a) The Hamiltonian can create or annihilate a single plaquette, but it cannot create or annihilate a single link. Thus an invariant subspace of Hilbert space is formed by those states which are composed of multiple plaquettes. There are other invariant subspaces built from, e.g., two links and multiple plaquettes. (b) The Hamiltonian is invariant under simultaneous change of direction of all links and plaquettes. This allows one to distinguish between symmetric and antisymmetric states. (c) The Hamiltonian conserves the total lattice wave vector. For example, if one forms a state on the $\vec{k}$-lattice by discrete Fourier transformation,

$$| \phi(\vec{k}_a, \cdots, \vec{k}_z) > = \sum_{\vec{a}, \cdots, \vec{z}} \exp \left[ ik_{\vec{a}} \cdot \vec{a} + \cdots + ik_{\vec{z}} \cdot \vec{z} \right] V_{\vec{a}} \cdots V_{\vec{m}} V_{\vec{n}}^\dagger \cdots V_{\vec{z}}^\dagger | \{0\} >, \quad (3)$$

then the Hamiltonian leaves invariant the subspace of those states with $\vec{k}_a + \cdots + \vec{k}_z = \vec{k}_{tot} = \text{const}$. Let us denote this subspace by $\mathcal{H}(\vec{k}_{tot})$. Below we will indicate the relationship between the total wave vector $\vec{k}_{tot}$ and the lattice momentum operator $\vec{P}$. We will make use of all of these symmetries when constructing glueball-like and meson-like states, and in particular scattering amplitudes, which conserve the total wave vector.

3. Hilbert space and physical states. We choose a basis of link states which diagonalizes the electrical field, $E_{\vec{n}}^i | \ell_{\vec{n}}^i > = \ell_{\vec{n}}^i | \ell_{\vec{n}}^i >$. The Hilbert space $\mathcal{H}$ is built from states
written as a tensor product of link states. The state \(| \{0\} >\) denotes an absent electric field on the lattice and is called electric vacuum.

In order to describe a meson, it has been suggested by Potvin and DeGrand \[18\] to use classical heavy [static] quarks as sources. A meson can be regarded as a state of electric flux lines between a heavy quark and its antiquark, separated by one or several units of electrical field on the lattice. To lowest order in the strong coupling expansion, a meson is given by a single link of length one between a quark and an antiquark. To higher order, fluctuations of plaquettes will occur similar to those arising from vacuum fluctuations in ordinary QED. A one-meson state obeys an eigenvalue equation of the Hamiltonian and it must satisfy Gauss’ law, with charge distribution corresponding to one quark and one antiquark.

A glueball state is a state without quark sources. It obeys the eigenvalue equation of the Hamiltonian and Gauss’ law, with a charge distribution identically zero. To lowest order in the strong coupling expansion, it is an eigenstate of the unperturbed Hamiltonian \(H_0\), it consists of just one plaquette. To higher order, more plaquettes will be added.

Similarly, the ground state of the theory obeys the eigenvalue equation plus Gauss’ law with the eigenvalue \(E_{\text{vac}}\) being the lowest eigenvalue of the spectrum. One expects the vacuum to lie in the same sector of Hilbert space as the glueball states, i.e. in the subspace \(\mathcal{H}(\text{zero-link})\), which means for the charge distribution \(\rho \equiv 0\). Moreover one expects the vacuum to have zero lattice wave vector, i.e. to lie in the subspace \(\mathcal{H}(\vec{k}_{\text{tot}} = 0)\). Both these properties have been checked in the numerical calculations and have been found to be satisfied. To lowest order in the strong coupling expansion, the vacuum is identical to the electric vacuum \(| \{0\} >\).

In order to map the Hamiltonian onto a finite dimensional matrix, we have chosen to work in the sector of Hilbert space in which states are composed of zero links and up to four plaquettes to represent a glueball and a meson is represented by one link and up to four plaquettes. This applies to the calculation of the mass spectrum. In the scattering calculations we have also worked in the sector of up to four plaquettes, i.e., glueball/meson being represented by zero/one link and up to two plaquettes each. The explicit construction and graphical representation of these state vectors is provided in Ref. \[19\]. In principle, the number of links and plaquettes limits the domain of the strong coupling regime. The actual domain of validity of our calculation will be studied by comparing with similar results obtained from strong coupling perturbation theory.

4. **Lattice momentum operator and dispersion relations.** As discussed above it is our aim to compute a scattering reaction by simulation on the lattice. As we intend to apply a Hilbert space method, we need to construct asymptotic states, which are characterized by energy, momentum and other quantum numbers. Thus in analogy to the lattice Hamiltonian, given by eq.(2), we will construct below a lattice momentum operator and
compute dispersion relations. The compact lattice Hamiltonian has been constructed by Kogut and Susskind \[20\]. We proceed to construct the compact lattice momentum operator in a similar way.

In general, the momentum operator in electrodynamics is given by the Poynting vector,

\[
\vec{P} = \int d^2 x \vec{E}(x) \times \vec{B}(x).
\]  

(4)

In two spatial dimensions, the magnetic field, or in other words the curl of the gauge field \( A \), is no longer a pseudovector but now a pseudoscalar and so we write \( B \equiv B^z \). On the lattice in non-compact form we have

\[
\vec{P} = \sum_{x_i} a^2 \left[ -E^x(x_i)B(x_i)\vec{e}_y + E^y(x_i)B(x_i)\vec{e}_x \right].
\]  

(5)

One goes over to the compact form via \( B \to \sin[ea^2B]/ea^2 \). Rescaling the variables following Drell \[19\], plus rescaling \( \vec{P} \) by \( a\vec{P} \to \vec{P} \), yields, if we write \( \vec{P} \) as Hermitian operator, the compact lattice momentum operator,

\[
\vec{P} = \frac{1}{4t} \sum_{\vec{n}} \left[ E^y_a [V_{\vec{n}} - V^\dagger_{\vec{n}}] + [V_{\vec{n}} - V^\dagger_{\vec{n}}] E^y_a \right] \vec{e}_x - \left[ E^x_a [V_{\vec{n}} - V^\dagger_{\vec{n}}] + [V_{\vec{n}} - V^\dagger_{\vec{n}}] E^x_a \right] \vec{e}_y.
\]  

(6)

The compact Hamiltonian, given by eq.(2), is gauge invariant under the residual gauge transformations compatible with temporal gauge fixing. The generator of those gauge transformations is given by [Gauss’ law] \( \nabla \cdot \vec{E} = \rho \) [see Ref. \[21\]]. The electric field \( E^a_{\vec{n}} \) and the plaquette \( V_{\vec{n}} \) commute with the generator of gauge transformations. Hence also the compact lattice momentum operator \( P^i \), given by eq.(6) is gauge invariant. Apart from gauge invariance, the compact lattice momentum operator has the same symmetries as the compact lattice Hamiltonian: (a) it creates or annihilates plaquettes but not single links, (b) it is invariant under exchange of orientation, (c) it conserves the total lattice wave vector \( \vec{k}_{tot} \).

It is obvious that the compact lattice Hamiltonian, when reversing the scaling of the variables, goes over in the classical continuum limit \( a \to 0 \) to the standard expression,

\[
H \to_{a \to 0} \frac{1}{2} \int d^2 x \vec{E}^2(x) + \vec{B}^2(x).
\]  

(7)

In the same way also the compact lattice momentum operator, when reversing the scaling of the variables, goes over in the classical continuum limit to the Poynting vector, i.e. the standard expression,

\[
\vec{P} \to_{a \to 0} \int d^2 x \vec{E}(x) \times \vec{B}(x).
\]  

(8)

The Hamiltonian and the momentum operator are part of the set of operators, which form the Lie-algebra of the Poincaré group. In the continuum case, one has \( [H_{\text{cont}}, \vec{P}_{\text{cont}}] = \)
On a finite lattice \( a \neq 0 \) the commutator for the compact lattice operators yields \( [H_{\text{latt}}, \vec{P}_{\text{latt}}] \neq 0 \). However, one has \( [H_{\text{latt}}, \vec{P}_{\text{latt}}] \to a \to 0 \).

When trying to compute numerically dispersion relations of the compact lattice Hamiltonian \( < H > \) versus the compact lattice momentum \( < \vec{P} > \), one is confronted with the problem of non-commuting operators. Thus we have chosen to diagonalize the Hamiltonian, considering a particular energy \( E \) and the corresponding eigen state \( \psi_E \) and to compute \( < \vec{P} > \) in this eigenstate. As discussed in Ref. \[19\], this eigenstate is characterized by the quantum number \( k_{\text{tot}} \) the so-called ”total wave vector” (see eq.(3)). For small momenta one can show \( < \vec{P} > = C k_{\text{tot}} + O(k_{\text{tot}}^2) \), where \( C \) is a state-dependent factor. Numerical results on this are given in Ref. \[19\]. The non-linear corrections are expected to drop off with the cut-off going to infinity, i.e. \( a \to 0 \). It is well known that on a finite lattice Lorentz symmetry is violated. It should be restored in the continuum limit. Restoration of Lorentz or Poincaré symmetry would imply that \( < M^2 > \) is invariant. It would have been interesting to compute the mass from \( < M^2 > = < H^2 - \vec{P}^2 > \). However, a meaningful evaluation of \( < \vec{P}^2 > \) would have required to work in Hilbert space with a number of plaquettes higher than four.

The results of the calculation of the dispersion relation for the anti-symmetric glueball are presented in Fig.\[1\]. The wave functions involved in such a diagonalisation are defined in the Hilbert space described above. The diagonalisation was performed numerically using IMSL. The numerical data approximate Lorentz behavior reasonably well, as is shown by a fit with a parabola. Qualitatively similar results are obtained also for the dispersion relation of the meson. The most serious source of systematic error for such an object is the finite (and small) size of the lattice. The magnitude of such an error can be assessed by interpolating the continuous curve in Fig. \[1\] to zero momentum and by comparing with the value of the mass spectrum computed from the diagonalization of the Hamiltonian using a zero-wave vector basis. The data presented here have been extended from those of Ref. \[19\] to larger values of \( x \) by using a larger number of plaquettes (up to four) in the basis.

The results of the mass spectrum calculations are shown in Figs.\[2-6\]. Fig.\[2\] shows the ground state energy density \( \omega_0/N^2 \) as a function of the coupling parameter \( x \). Using a basis containing up to four plaquettes is most accurate in the strong coupling region. As a check, these results are compared with the calculation of Irving et al. \[17\] who have used Hamiltonian diagonalisation on a space lattice. Our data are also compared with the strong coupling perturbation study of Alessandrini et al. \[6\].

We now present our results for the rest mass. These were calculated in the zero-wave vector sector, subtracting out the vacuum energy. The data of Fig.\[3\] correspond to the anti-symmetric glueball on \( 5^2 \) lattice. This particular state has been studied as a function of lattice size up to \( 11^2 \). We find a reasonably converged behavior starting at about a lattice size of \( 5^2 \). A similar behavior is found for the mass of the symmetric glueball. The mass ratio of symmetric to anti-symmetric glueball is given in Fig.\[4\]. Note that in
the strong coupling limit \( g = \infty \), i.e. \( x = 0 \), the symmetric and antisymmetric states are degenerate. Finally, in Fig.\[5\], we display the meson mass as a function of the coupling parameter. This corresponds to an anti-symmetric meson state. One observes a quite similar behavior as for the glueball. Although our lattice calculation is non-perturbative, it is based on a sector of Hilbert space. For smaller values of the coupling constant \( g \), i.e., larger coupling parameters \( x \), one needs to take into account higher excitations, e.g., electric flux between the quarks of several lattice units and more than four plaquettes. Comparing the mass spectrum extracted from the interpolation of the dispersion relation and from the direct diagonalization of the Hamiltonian in the zero-wave vector sector, we find a typical discrepancy of two percent (Figs.\[2\],\[5\]), which is an estimate of finite size effects. Those effects are thus under control in the range of the coupling parameter considered in this work.

5. Scattering on the lattice. Key elements in the computation of scattering amplitudes in a Hilbert space formulation are the generator of the time evolution, i.e., the lattice Hamiltonian, and the initial and final states. The latter are characterized by quantum numbers like energy, momentum, spin etc. However, to describe in Hilbert space a reaction like glueball+glueball \( \rightarrow \) glueball+glueball, quantum numbers alone are not sufficient; in addition one needs to construct asymptotic one-particle states. The way to do this has been proposed a long time ago in papers by Haag and Ruelle \[22\]. Haag-Ruelle’s theory specifies which operators give asymptotic one-particle states when applied on the physical vacuum. However, it does not answer how to obtain the physical vacuum state. In the truncated Hilbert space, built from link and plaquette states, we can compute the whole truncated spectrum. Thus in particular, we obtain the ground state, as discussed in sect. 4, and the next lowest eigenstate, which is a one-particle glueball state. Thus in the truncated scheme, we are able to construct a Haag-Ruelle operator, i.e., an explicit representation of an operator which maps the vacuum state onto the glueball state. An asymptotic two-particle state is obtained by applying a Haag-Ruelle operator twice on the physical vacuum. There are some tests on the construction of asymptotic one- or two-particle states. Firstly, it is quite simple to verify that an asymptotic one-particle state is an eigenstate of the Hamiltonian, but an asymptotic two-particle state is not. We have verified this for glueballs and mesons. Secondly, one can verify localisation. I.e., if \( \phi_{as} \) is an asymptotic one-particle state, then

\[
< \phi_{as} \mid C(x) \mid \phi_{as} > = < 0 \mid C(x) \mid 0 > + \epsilon(x)
\]

for a local observable \( C(x) \), where \( \epsilon(x) \) tends to zero for space-like distances \( x \rightarrow 0 \). This we have not verified in this work, because we have worked on relatively small lattices. We defer it to future studies. In this work an asymptotic one-particle state is given by the glueball/ meson state as discussed in sect.4, being the lowest lying states above the vacuum in the glueball/meson sector. Asymptotic two-particle states are product states of the one-particle states.
A time-dependent Hilbert space method to compute $S$-matrix elements has been described and reviewed in Ref. [8]. An $S$-matrix element is given by

$$S_{fi} = \lim_{t \to \infty} \langle \phi_{as}^{fi} | \exp[i E_{as} t] \exp[-i 2Ht] \exp[i E_{as} t] | \phi_{as}^{in} \rangle,$$

where $H$ denotes the Hamiltonian and $\phi_{as}^{fi}, \phi_{as}^{in}$ denote the asymptotic states. On the lattice this is replaced by

$$S_{fi}(T) = \langle \phi_{as}^{fi} | \exp[i E_{as} T] \exp[-i 2H(N)T] \exp[i E_{as} T] | \phi_{as}^{in} \rangle,$$

where $H(N)$ denotes the finite dimensional lattice Hamiltonian defined in the truncated Hilbert space. The time-evolution $\exp[-i 2H(N)T]$ is computed in the eigen representation of $H(N)$. The reason to introduce as parameter a large but finite scattering time $T$ is based on the property that eq.(11) does not have a limit when $T \to \infty$. However, study of this and other models shows [8] that there is a range of the scattering parameter $T$, such that $S_{fi}(T)$ is stable and gives physically meaningful results. The value of $T$ is determined by the physical requirement that energy violation of the scattering process measured on the lattice by

$$\Delta_{<E>}(T) = | \langle \psi_{\text{scatt}}(T) | H(N) | \psi_{\text{scatt}}(T) \rangle - E_{as} |$$

becomes minimal. The property that the position of the minimum of energy violation coincides with a region of stability of $S_{fi}(T)$ can be seen in Fig.[6] for anti-symmetric glueball scattering. A similar behavior is obtained for meson scattering.

By extracting the invariant matrix element and putting the suitable kinematical factor, we have computed the differential cross sections for scattering of two glueballs. The results are shown in Fig.[7]. Our results are compared with the predictions by Hakim [7] obtained from strong coupling perturbation theory and agree quite well. Finally, in Fig.[8] we give the cross section for meson-meson scattering which shows a quite similar behavior.

In conclusion, we have studied glueballs and mesons in compact $QED_{2+1}$ gauge theory in a Hamiltonian formulation on a momentum lattice. Mesons are treated as strings of electric field between classical quark-antiquark sources. We have computed masses and find agreement with Hamiltonian calculations on a coordinate lattice and with strong coupling perturbation theory. The new aspects of this work are: (a) In analogy to the Kogut-Susskind compact Hamiltonian, we suggest a compact lattice momentum operator and compute glueball and meson dispersion relations. (b) Using a non-perturbative time-dependent Hilbert space method we compute glueball and meson scattering cross sections. For glueballs we compare our results in the strong coupling regime with perturbation theory and find good agreement.
References

[1] H.W. Hamber, E. Marinari, G. Parisi, C. Rebbi, Nucl. Phys. B335 (1983) 475.
[2] M. Guagnelli, E. Marinari, G. Parisi, Phys. Lett. B240 (1990) 188.
[3] S.R. Sharpe, R. Gupta, G.W. Kilcup, Nucl. Phys. B393 (1992) 309.
[4] M. Lüscher, Comm. Math. Phys. 105 (1986) 153; M. Lüscher, U. Wolff, Nucl. Phys. B339 (1990) 222; M. Lüscher, Nucl. Phys. B354 (1991) 531.
[5] H.R. Fiebig, R.M. Woloshyn, A. Dominguez, preprint FIU-PHY-92-23.
[6] V. Alessandrini, A. Krzywicki, Phys.Lett. B93 (1980) 173; Phys.Rev. D24 (1981) 3237; V. Alessandrini, V. Hakim, A. Krzywicki, Nucl. Phys. B200 [FS4] (1982) 355.
[7] V. Hakim, Nucl.Phys. B180 (1981) 417.
[8] H. Kröger, Phys.Repts. 210(1992)45.
[9] J.F. Brière, H. Kröger, Phy.Rev.Lett. 63 (1989) 849; Phys.Lett. B217 (1989) 215; Phys. Rev. D41 (1990) 3197.
[10] A.M. Polyakov, Nucl. Phys. B210 (1977) 429.
[11] T. Banks, R. Myerson, J. Kogut, Nucl.Phys. B129 (1977) 493.
[12] J. Ambjorn, A.J. Hey, S. Otto, Nucl.Phys. B210 (1982) 347.
[13] M. Göpfert, G. Mack, Comm. Math.Phys 82 (1982) 545.
[14] G. Shuohong, Z. Weihong, L. Jinming, Phys.Rev. D38 (1988) 2591.
[15] C. Morningstar, Phys. Rev. D46 (1992) 824.
[16] S.D. Drell, M. Weinstein, S. Yankielowicz, Phys. Rev. D14 (1976) 478.
[17] A.C. Irving, J.F. Owens, C.J. Hamer, Phys. Rev. D28 (1983) 2059.
[18] J. Potvin, T.D. DeGrand, Phys.Rev. D30 (1984) 1285.
[19] A.M. Chaara, H. Kröger, K.J.M. Moriarty, J. Potvin, Int. J. Mod. Phys. C4(1993)919.
[20] J. Kogut, L. Susskind, Phys. Rev. D11 (1975) 395.
[21] U.M. Heller, Phys.Rev. D23 (1981) 2357.
[22] R. Haag, Phys. Rev. 112 (1958) 669; Nuov. Cim. Supp. 14 (1959) 131; D. Ruelle, Helv. Phys. Acta 35 (1962) 147.
Figure Captions

**Fig.1** Dispersion relation $E$ versus $< P_x >$ for anti-symmetric glueball state. Coupling parameter: $x = 0.26$. Lattice size: $5^2$. The curve is a fit to the numerical data in order to extract the mass.

**Fig.2** Ground state energy density versus coupling parameter $x$. Full line: diagonalisation on a coordinate lattice by Irving et al. [17]. Circle: strong coupling perturbation theory by Alessandrini et al. [6]. Triangle: this work. Lattice size: $5^2$.

**Fig.3** Mass of anti-symmetric glueball state. Otherwise same as Fig.[2].

**Fig.4** Mass ratio of symmetric to anti-symmetric glueball state. Otherwise same as Fig.[2].

**Fig.5** Meson mass versus coupling parameter $x$. Lattice size: $5^2$.

**Fig.6** Scattering of anti-symmetric glueballs as function of scattering time parameter $T$. Full line: imaginary part of S-matrix. Dashed line: violation of energy conservation. Coupling parameter: $x = 0.4$. Lattice size: $5^2$.

**Fig.7** Scattering of anti-symmetric glueballs. Total cross section versus total momentum. Full line: strong coupling perturbation theory by Hakim [7]. Triangle: this work. Coupling parameter: $x = 0.4$. Lattice size: $5^2$.

**Fig.8** Same as Fig.[7] for meson-meson scattering.
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