Abstract

I compute the deconfinement order parameter for the $SU(2)$ lattice gauge theory, the Polyakov loop, using the fixed scale approach for two different scales and show how one can obtain a physical, renormalized, order parameter. The generalization to other gauge theories, including quenched or full QCD, is straightforward.

Keywords: Polyakov loop, Deconfinement Renormalization Lattice QCD

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Since the pioneering work of Hagedorn [1] on the limiting temperature for hadrons, it has been widely expected that the strongly interacting matter will show unusual features at high enough temperatures. With the advent of quantum chromodynamics (QCD) as the theory of strong interaction with quarks and gluons as its basic constituents, quark-gluon plasma was identified as this new phase with a possible phase transition in between. Using the lattice formulation of QCD, such a transition was shown to be akin to spin models. The Polyakov loop, $L$, defined as the product of the timelike gauge...
links at a given site, is the order parameter for this deconfinement transition \[2\]. On an Euclidean \(N^3_\sigma \times N_\tau\) lattice \(L(\vec{x})\) is defined at a site \(\vec{x}\) as

\[
L(\vec{x}) = \frac{1}{N_c} \text{Tr} \prod_{x_0=1}^{N_\tau} U^4(\vec{x}, x_0),
\]

where \(N_c\) is the number of colours, being three for QCD, and \(U^\mu(x)\) are the gauge variables associated with the directed links in the \(\mu\)th direction, \(\mu = 1,4\). It is convenient to define its average over the spatial volume, \(\bar{L} = \sum_{\vec{x}} L(\vec{x})/N^3_\sigma\). \(\langle |\bar{L}| \rangle\) was used to establish a second order deconfinement transition in numerical simulations of the \(SU(2)\) pure gauge theory. Since then it has been used for similar studies of the deconfinement phase transitions for a variety of \(N_c\) \[3\], for establishing the universality \[4\] of the continuum limit, as well as for theories with dynamical quarks \[5\]. Indeed, one hopes to be able to construct effective actions \[6\] for it in a Wilsonian RG approach, which will be similar to the spin models in the same universality class but with possibly additional interaction terms. A large number of models of quark-hadron transitions use the Polyakov loop as the order parameter for the deconfinement transition as well.

An order parameter should be physical, i.e., independent of the lattice spacing \(a\) in the continuum limit. Furthermore, it must be so in both the phases it seeks to distinguish. As is the case for any bare Wilson loop, the Polyakov loop, needs to be renormalized for this to be true. Since the bare Polyakov loop is further known to decrease progressively with \(N_\tau\), suggesting it to be zero in the continuum limit in the high temperature phase, renormalized \(L\) is even more desirable to have.

The physical interpretation of the order parameter as a measure of the free energy of a single quark, \(\langle \bar{L}(T) \rangle = \exp(-F_Q(T)/T)\) provides a straightforward clue for renormalization. Since many years it is known that the single quark free energy contains divergent contribution in the continuum limit, as shown by the computations employing lattice perturbation theory \[7\]. Subtracting these off was the first step \[7\] towards the renormalized Polyakov loop. Kaczmarek et al. \[8\] proposed to use the heavy quark-antiquark free energy, as determined from the Polyakov loop correlations at short distances to define the renormalized the Polyakov loop and showed it to then become \(N_\tau\)-independent. Subsequently, fits to \(\langle \bar{L} \rangle\) on \(N_\tau\)-grids \[9\] and an iterative direct renormalization procedure \[10\] for \(\langle \bar{L} \rangle\) were used to extract the renormalize Polyakov loop, and shown to yield similar results.
Figure 1: The average Polyakov loop as a function of $T/T_c$ for two different scales. The lattice sizes are as indicated in the key.

Figure 2: The heavy quark free energy $F$ a function of $T/T_c$ for two different scales. The lattice sizes are as indicated in the key.
While these are clearly nice results, it would be more satisfying to have a better definition of the renormalized order parameter for the following reasons. The definition of Ref. 8 needs heavy quark potential at short distances. The lattice artifacts are at their worse when one is at the shortest distance, with maximal violation of the rotational invariance. Finite volume of the lattice also enters in defining the maximum distance between the heavy quarks, or Polyakov loops. Similarly the iterative procedure used in Ref. 10 to obtain the renormalization constants needs large lattices in both spatial and temporal directions, and should ideally be further tested on lattice of another temporal size to check whether the same renormalization constants apply. Physically perhaps an undesirable aspect of the definition of Ref. 10 is that it works only on the plasma side, i.e., for \( T \geq T_c \), where \( T_c \) is the position of the peak in the Polyakov loop susceptibility. The other definition 8 has so far been employed only in the \( T \geq T_c \) for pure gauge theories for which \( L \) is an order parameter. Furthermore, it would be nice if the renormalization procedure is applicable to the usually employed \( \langle |\bar{L}| \rangle \), which is used as an order parameter on finite volumes.

In this letter I show that a renormalized Polyakov loop which is valid for both the phases below and above \( T_c \) can be defined, and it becomes the true order parameter in the infinite spatial volume limit. Of course, it is also physical, i.e., \( N_\tau \)-independent on finite volumes as well. I use the fixed scale approach 11 to do so. It was introduced to minimize the computational costs for the zero temperature simulations needed to subtract the vacuum contribution in thermodynamic quantities such as the pressure and to isolate pure thermal effects in computation of \( T_c \) 12. Furthermore, its advantage is that all the simulations stay on the line of constant physics in a straightforward way. What I argue is that it is indeed this advantage which also permits an easy renormalization of the Polyakov loop. Although these considerations are general, and apply to any \( SU(N) \) gauge theory, I shall consider below the simplest case of the \( SU(2) \) lattice gauge theory to illustrate how and why it works.

Recall that the temperature \( T \) is varied in this approach by varying \( N_\tau \), holding the lattice spacing \( a \), or equivalently the gauge coupling \( \beta = 2N_c/g^2 \) fixed. The single quark free energy \( F_b(N_\tau, a) \) is then obtained from the \( \bar{L} \) by the canonical relation,

\[
\ln \langle |\bar{L}| \rangle = -aN_\tau F_b(N_\tau, a) .
\]

The subscript \( b \) reminds us that one obtains the bare free energy this way.
If the chosen coupling is $\beta_c$, corresponding to the position of the peak of the $|L|$-susceptibility in the usual fixed $N_\tau$ approach, and it lies in the scaling region, then the physical deconfinement temperature $T_c = 1/N_{\tau,c}a_c$, and in the fixed scale approach then $T/T_c = N_{\tau,c}/N_\tau$, with the free energy given by $F_b(T/T_c, a_c)$. Writing it as a sum of a divergent and a regular contribution, one has $a_c F_b(T/T_c, a_c) = a_c F(T/T_c, a_c) - a_c A(a_c)$, where $A$ is the divergent free energy in physical units. Clearly, the divergent contribution will be same at all temperatures in the fixed scale approach since it depends only on $a_c$.

Since $\beta_c$ or $a_c$ is known precisely for the Wilson action of the $SU(2)$ theory for many different $N_\tau$, I chose $\beta_{c1} = 2.4265$, and $\beta_{c2} = 2.5104$ corresponding to the known transition temperatures on $N_\tau = 6$ [13] and 8 [14] respectively. Note that $T/T_c$ is given simply by $6/N_\tau$ and $8/N_\tau$ respectively. Employing then $N_\tau = 3$ to 12, I varied the temperature in the range $2 \geq T/T_c \geq 0.6$. I used two different spatial lattice sizes for the smaller $\beta$, $20^3$ and $28^3$, while a $26^3$ lattice was used for the larger one. Note that fixed $a_c$ means that the spatial volume was constant in physical units in each case, being $37.03$ and $101.63$ in the units of $T_0^3$ for $\beta_{c1}$ and $34.44$ for $\beta_{c2}$. Note that in contrast to these simulations, the spatial volume varies with $T$ in the usual fixed $N_\tau$ approach.

Figure 1 shows the results for the thermal expectation value of $\bar{L}$ as a function of the temperature in the units of $T_c$. In most cases, I used both a random and an ordered start. The errors are corrected for autocorrelations. The agreement in the data for the two starts suggest the statistics of 200K iterations to be sufficient. As expected, the two different scales lead to two different curves for the order parameter. Figure 2 displays the behaviour of the corresponding bare free energy, obtained by using the eq.(2). It reinforces the expectation of the effect of the divergent free energy, since it increases with the decrease in the lattice cut-off $a_c$.

The two different scales, $a_{c1}$ and $a_{c2}$ have their respective divergent contributions, $a_{c1} A(a_{c1})$ and $a_{c2} A(a_{c2})$. Multiplying eq.(2) by $N_j$, for $j = 1$ and 2 corresponding to the critical $N_\tau$ for the scale choices above, i.e, 6 and 8, one obtains

$$\frac{T}{T_c} \ln \langle |\bar{L}| \rangle = -\frac{F_b(T/T_c, a_{cj})}{T_c},$$

(3)

where $F_b(T/T_c, a_{cj})/T_c = F(T/T_c, a_{cj})/T_c - A(a_{cj})/T_c$. Thus the free energies at the same temperatures but two different scales are related by a mere constant, $[A(a_{c1}) - A(a_{c2})]/T_c$. Figure 3 shows the same results as Figure 2 but with a constant shift of -0.55 in the free energy for the higher $\beta$. A
Figure 3: The same as Figure 2 but with a constant shift, as explained in the text.

Figure 4: Same as Figure 1 but with the shifted free energy for the upper curve, as explained in the text.
universal curve for the free energy seems to result as a result for a wide range of $T/T_c$, covering both the phases. The finite free energy in the confined phase should not surprise us. In the infinite volume limit, the free energy should increase to infinity in the confined phase whereas it should essentially remain constant in the deconfined phase. Such an expectation is indeed borne out by my results for larger spatial volume, $28^3$ and $\beta_{c1}$, shown in the same Figure.

One clearly sees a substantial increase in the free energy in the confined phase due to the 2.74 fold increase in spatial volume without affecting the high temperature phase in any significant way. The points at the lowest $T/T_c$ seem to suggest a drop in the free energy in all cases, which I believe is a finite volume effect. This is also evident from the results for these points for the $28^3$ lattice in the same figure.

Finally, it should now be clear how one can obtain a universal curve for the order parameter from the universal free energy curve. The $\langle|\bar{L}|\rangle$ obtained at the two different scales will lie on a universal curve by simply multiplying the results for the scale corresponding to $\beta_{c2}$ by the factor $\exp(N\tau[A(a_{c1})/T_c - A(a_{c2})/T_c])$. This is exhibited in Figure 4. This can be continued to as many scales as one wishes, and the same universal order parameter should result in both below and above $T_c$. Furthermore, each new scale introduces only one unknown constant which can be fixed by free energy difference at any $T > T_c$. The entire low and high temperature region of the order parameter is then uniquely fixed, and has to be universal. As in case of any renormalized quantity, it depends on the scale chosen to define the scheme. Here it is the inclusion of a constant free energy $A(a_c)/T_c$ in the free energy for the chosen scale $a_c$ which defines the choice. The details of the shape of the physical order parameter are therefore scale-dependent in the plasma phase but it is universal none the less once a choice is made. Moreover, any further change of scale leads to a computable change in the shape.

In conclusion, I showed that the fixed scale approach leads to a natural definition of a physical, $N_c$-independent, order parameter which is defined in both the confined and the deconfined phases. The definition itself does not depend on any lattice artifacts or the lattice size in the deconfined phase. Moreover, it displays the expected behaviour in the confined phase as the physical volume is increased, suggesting that the so determined physical free energy of a single quark in the confined phase, $F$, goes to infinity in the infinite volume limit. It is straightforward to generalize this idea to $SU(N_c)$ gauge theories and QCD as well as to sources in higher representations.

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