Quantum Gravity Hamiltonian for
Manifolds with Boundary

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Abstract

In canonical quantum gravity, when space is a compact manifold with boundary there is a Hamiltonian given by an integral over the boundary. Here we compute the action of this ‘boundary Hamiltonian’ on observables corresponding to open Wilson lines in the new variables formulation of quantum gravity. In cases where the boundary conditions fix the metric on the boundary (e.g., in the asymptotically Minkowskian case) one can obtain a finite result, given by a ‘shift operator’ generating translations of the Wilson line in the direction of its tangent vector. A similar shift operator serves as the Hamiltonian constraint in Morales-Técotl and Rovelli’s work on quantum gravity coupled to Weyl spinors. This suggests the appearance of an induced field theory of Weyl spinors on the boundary, analogous to that considered in Carlip’s work on the statistical mechanics of the 2+1-dimensional black hole.
1 Introduction

On globally hyperbolic, spatially compact spacetimes it is a characteristic feature of general relativity that the Hamiltonian vanishes when Einstein’s equations hold. Suppose that spacetime is of the form $\mathbb{R} \times S$ with $S$ compact without boundary. Then in the metric representation of general relativity without matter, the Hamiltonian density $\mathcal{H}$ is, up to a total divergence, given by a linear combination of the components of the Einstein tensor. The vacuum Einstein equations therefore imply the vanishing of the Hamiltonian

$$H_S = \int_S d^3x \, N \mathcal{H},$$

which for this reason is usually called the Hamiltonian constraint. (Here $N$ is an arbitrary densitized lapse function, and for simplicity we consider only the case of vanishing shift.) In canonical quantum gravity one thus expects physical states to satisfy the Wheeler-DeWitt equation $\hat{H}_S \Psi = 0$. This leads to the ‘problem of time’: the usual recipe for time evolution in quantum mechanics

$$\Psi \mapsto e^{-i\hat{H}_S} \Psi,$$

does not capture the dynamics of quantum gravity. Conceptually, since the state $\Psi$ is diffeomorphism-invariant, it makes no sense to ‘evolve $\Psi$ in time’.

Many approaches to this problem have been proposed \cite{1}. A rather obvious strategy is to introduce a nonzero Hamiltonian on physical states. Doing so essentially amounts to choosing a notion of time evolution applicable to physical states. For example, one can try to gauge-fix Einstein’s equation by using one degree of freedom of the gravitational field as the time coordinate, so that the canonically conjugate variable serves as a Hamiltonian generating evolution with respect to this choice of

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time. Alternatively, one can try to introduce a ‘clock field’: a matter field whose value serves as as time coordinate, and whose canonically conjugate field serves as a Hamiltonian density.

Here we consider another way to introduce a nonzero Hamiltonian on physical states. Suppose that we take as space a compact manifold $\Sigma$ with boundary [2]. Classically, given initial data on $\Sigma$, Einstein’s equations need not have a unique solution, even locally and up to diffeomorphism, unless we impose some boundary conditions. Moreover, to obtain Hamilton’s equations, the Hamiltonian must be functionally differentiable with respect to the fields on which it depends, at least with respect to variations preserving the boundary conditions. Since functional differentiation usually involves an integration by parts, to obtain a differentiable Hamiltonian one must add a surface term:

$$H_\Sigma = \int_\Sigma d^3x N \mathcal{H} + \int_{\partial\Sigma} d^2S_a N \mathcal{K}^a. \tag{2}$$

The quantity $H_\Sigma$ need not vanish when Einstein’s equations and the boundary conditions hold. Thus, at least in principle, we can hope to quantize the theory and obtain a space of states on which there is a nonzero Hamiltonian.

To what choice of time evolution does the Hamiltonian $H_\Sigma$ correspond? The detailed answer, of course, depends on the choice of boundary conditions and the surface term $\mathcal{K}$. But in general we may say this: physical states need not be invariant under diffeomorphisms of spacetime that are not the identity on $\mathbb{R} \times \partial\Sigma$, and the Hamiltonian $H_\Sigma$ generates time evolution corresponding to diffeomorphisms pushing the surface $\{0\} \times \Sigma$ in the direction $Nv$, where $v$ is the unit timelike vector normal to this surface.

In what follows we shall work using Ashtekar’s ‘new variables’ [3], namely a densi-
tized complex triad field $E^a_i$ and a chiral spin connection $A_b = A^i_b \tau_j$, where $a, b, c, \ldots$ are spacelike indices and $i, j, k, \ldots$ are internal indices. (Here we omit the tilde sometimes written over the $E$ field to indicate that it is densitized.) The Hamiltonian density is then

$$\mathcal{H} = -\frac{1}{2} \varepsilon^{ijk} F_{abk} E^a_i E^b_j,$$

and we take as our boundary term

$$\mathcal{K}^a = \varepsilon^{ijk} A_{bk} E^a_i E^b_j.$$

The resulting Hamiltonian $H_\Sigma$ is compatible with a variety of boundary conditions. Smolin [4], for example, obtains essentially this Hamiltonian (but with an additional cosmological constant term) in his study of quantum gravity with ‘self-dual’ boundary conditions. However, these conditions have not been thoroughly studied yet at the classical level.

Asymptotically Minkowskian boundary conditions are much better understood, at least classically [3]. Here $\Sigma$ is a ball of coordinate radius $r$, but one is really interested in the limit as $r \to \infty$. The boundary $\partial \Sigma = S^2$ then represents spacelike infinity, where there is a fixed Euclidean 3-metric. In this limit, the triad field $E^i_a$ is constant on $\partial \Sigma$, and the 3-metric at spacelike infinity is given by $q_{ab} = \delta_{ij} E^i_a E^j_b$. When we set $\tilde{N} = 1$, the Hamiltonian $H_\Sigma$ then generates time evolution with respect to the standard Minkowski time coordinate at spacelike infinity. One might hope, therefore, that the corresponding quantum Hamiltonian $\hat{H}_\Sigma$ will generate nontrivial time evolution for asymptotically Minkowskian states of quantum gravity. Of course the very notion of an asymptotically flat state of quantum gravity is problematic. However, the extent of the problems can only be understood by investigation.
In most of what follows we will not need a specific choice of boundary conditions; all we will need is boundary conditions for which $E_i^a$ and $A_j^b$ have the usual Poisson brackets \( \{ E_i^a(x), A_j^b(y) \} = -i\delta_i^a\delta_j^b\delta^3(x,y) \). This is not true for self-dual boundary conditions, where there are extra boundary terms for the Poisson brackets, but it is true in the asymptotically flat case. Given these Poisson brackets, one then expects the corresponding quantum operators to be

\[
\hat{H} = -\frac{1}{2} \epsilon^{ijk} F_{abk} \frac{\delta}{\delta A_i^a} \frac{\delta}{\delta A_j^b},
\]

\[
\hat{K}^a = \epsilon^{ijk} A_{bk} \frac{\delta}{\delta A_i^a} \frac{\delta}{\delta A_j^b},
\]

in units where $\hbar = 1$.

Here we should note that two operator orderings for the Hamiltonian constraint have been widely studied, the ‘FEE’ and ‘EEF’ orderings, and the former turns out to be the appropriate one when we think of the constraint as acting on wavefunctions on the space of connections (rather than dually on measures on the space of connections). Briefly, the FEE ordering is the one for which the constraint may be described as a ‘shift operator’ [5], and also the one for which the relation between quantum gravity with cosmological constant and Chern-Simons theory becomes apparent [6]. Thus we adopt this ordering and the compatible ordering for the boundary term.

Crucial to Rovelli and Smolin’s [5] original paper on the loop representation of quantum gravity was their calculation of the action of the Hamiltonian constraint on Wilson loops. Surprisingly, they obtained a 3-dimensional geometrical interpretation of the Hamiltonian constraint, which allowed them to find — at least heuristically — a large set of solutions of the Wheeler-DeWitt equation.
Recall that in this work, Wilson loops play a dual role. In the Heisenberg picture we may think of them as multiplication operators of the form $\text{tr}(U[1,0])$, where

$$U[t',t] = \mathcal{P} \exp \int_t^{t'} ds \dot{\gamma}^c(s) A^k_c(\gamma(s)) \tau_k$$

is the holonomy from $t$ to $t'$ and $\gamma:[0,1] \to S$ is a loop. These form a maximal commuting set of gauge-invariant kinematical observables, on which the diffeomorphism constraint acts in a simple way. Alternatively, in the Schrödinger picture we may think of Wilson loops as kinematical states, that is, wavefunctions of the form $\Psi(A) = \text{tr}(U[1,0])$ on the space of connections. This permits the construction of a loop representation in which states solving the diffeomorphism constraint are described by (diffeomorphism equivalence classes of) collections of loops.

In the Heisenberg picture the action of the Hamiltonian constraint on Wilson loops is given, as usual, by a commutator

$$\left[\hat{H}_S, \text{tr}(U[1,0])\right],$$

while in the Schrödinger picture it is given by

$$\hat{H}_S \Psi.$$

In either case, a regularization procedure is needed to compute the action. Initially, Rovelli and Smolin used a simple point-splitting regularization, and obtained an integral involving $F_{ab}^i \dot{\gamma}^a \dot{\gamma}^b$. Since $F_{ab}^i$ is antisymmetric in the indices $a, b$ this vanishes when $\gamma$ is smooth and without self-intersections. They concluded that the Wilson loop states associated to such loops were solutions of the Wheeler-DeWitt equation, and thus physical states.

Later work made it clearer that the regularization issues are quite delicate \[7, 8, 9, 10, 11\]. In fact, they remain controversial, and mathematically rigorous work on
the loop representation is just nearing the point of being able to definitively deal with them \cite{3,12}. We will not address these issues in the present work. Instead, we will work at a level of rigor similar to that of Rovelli and Smolin’s original work, and concentrate on the new features that arise when space is a manifold with boundary.

2 Hamiltonian Action on Wilson Lines

When space has no boundary, physical states of quantum gravity in terms of the new variables are invariant under $\text{SL}(2,\mathbb{C})$ gauge transformations. When space is a compact manifold $\Sigma$ with boundary, the boundary conditions may break gauge-invariance at the boundary. For example, in the asymptotically Minkowskian case the condition that $E^a_i$ be constant at the boundary is not gauge-invariant. In such cases, Wilson loops will not suffice as a complete set of kinematical states in the Schrödinger picture, since loop (or multi-loop) states are gauge-invariant. Similarly, in the Heisenberg picture the Wilson loops will not form a complete (i.e., maximal commuting) set of kinematical observables.

For this reason it is important to consider the action of the Hamiltonian not only on Wilson loops but also on ‘Wilson lines’ starting and ending on the boundary. Let $\gamma$ be a smooth path for which the initial and final points $\gamma_0 = \gamma(0)$ and $\gamma_1 = \gamma(1)$ lie on $\partial \Sigma$. Then in the Heisenberg picture, the associated Wilson line is the matrix-valued observable $U[1,0]$. Now recall that $A_b$ is a chiral spin connection, so the holonomy $U[1,0]$ describes the parallel transport of Weyl spinors along the curve $\gamma$. If we fix Weyl spinors $\psi$ and $\psi'$ at the endpoints of $\gamma$, then in the Schrödinger picture we may also define a Wilson line state by

$$\Psi(A) = \psi'_{A} U[1,0]^{A}_{B} \psi^{B}. \quad (3)$$
These Wilson lines are invariant under gauge transformations that are the identity on the boundary. Moreover, by using Wilson lines in addition to Wilson loops, one can obtain a complete set of kinematical observables (or states) that are invariant under gauge transformations that are the identity on the boundary \[6, 12\].

To the same degree of rigor as in the case without boundary, Rovelli and Smolin’s argument shows that the action of the Hamiltonian on smooth Wilson loops without self-intersections is identically zero. The simplest case exhibiting the effect of the boundary term is that of a smooth Wilson line \(\gamma\) intersecting the boundary transversally at its endpoints \(\gamma_0, \gamma_1\). Since

\[
\left[ \int_\Sigma d^3x \mathcal{N} \hat{H}, U[1,0] \right] = 0
\]

by Rovelli and Smolin’s original computation, it follows that

\[
\left[ \hat{H}_\Sigma, U[1,0] \right] = \left[ \hat{H}_{\partial \Sigma}, U[1,0] \right].
\]

Using the fact that

\[
\left[ \frac{\delta}{\delta A^i_a(x)} \frac{\delta}{\delta A^j_b(x)}, U[1,0] \right] = \frac{\delta^2 U[1,0]}{\delta A^i_a(x) \delta A^j_b(x)} + \frac{\delta U[1,0]}{\delta A^i_a(x)} \frac{\delta}{\delta A^j_b(x)} + \frac{\delta U[1,0]}{\delta A^j_b(x)} \frac{\delta}{\delta A^i_a(x)},
\]

it follows that

\[
\left[ \hat{H}_\Sigma, U[1,0] \right] = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3
\]

where, letting \(A^{ij}_b = \epsilon^{ijk} A_{bk}\), we have

\[
\mathcal{C}_1 = \int_{\partial \Sigma} d^2 S_a(x) N(x) A^{ij}_b(x) \frac{\delta}{\delta A^i_a(x)} \frac{\delta}{\delta A^j_b(x)} U[1,0],
\]

\[
\mathcal{C}_2 = \int_{\partial \Sigma} d^2 S_a(x) N(x) A^{ij}_b(x) \frac{\delta U[1,0]}{\delta A^i_a(x)} \frac{\delta}{\delta A^j_b(x)},
\]

\[
\mathcal{C}_3 = \int_{\partial \Sigma} d^2 S_a(x) N(x) A^{ij}_b(x) \frac{\delta U[1,0]}{\delta A^j_b(x)} \frac{\delta}{\delta A^i_a(x)}.
\]
If instead we work in the Schrödinger picture and define a Wilson line state by eq. (3), we have

$$\hat{H}_\Sigma \Psi = \psi' A \left[ \hat{H}_\Sigma, U[1,0] \right] \psi^B = \psi'_A C_A^B \psi^B. \quad (4)$$

In what follows we compute $C_1$, $C_2$ and $C_3$, with the results appearing in eqs. (5), (7) and (8), respectively. We begin by evaluating $C_1$, which can be written as an integral over $\Sigma$ of a total divergence:

$$C_1 = \int_\Sigma d^3 x \partial^x_a \left( \int_\Sigma d^3 y z_\epsilon(x, y) N(x) A^i_j(y) \frac{\delta}{\delta A^i_a(x)} \frac{\delta}{\delta A^j_b(x)} U[1,0] \right).$$

Introducing a point-splitting by letting $z_\epsilon(x, y)$ be a function that tends to $\delta^3(x,y)$ as $\epsilon \downarrow 0$, we have

$$C_1 = \int_\Sigma d^3 y \int_\Sigma d^3 x \partial^x_a \left( \int_\Sigma d^3 y \frac{\delta}{\delta A^j_b(x)} \frac{\delta}{\delta A^i_a(y)} U[1,0] \right).$$

Note that the reason we rewrite $C_1$ as an integral over $\Sigma$ is precisely to carry out this point-splitting.

Since

$$\frac{\delta}{\delta A^i_a(x)} \frac{\delta}{\delta A^j_b(y)} U[1,0] = \int_0^1 ds \int_0^t dt \left( \delta(x, \gamma(s)) \delta(y, \gamma(t)) \theta(t-s) U[1,t] \tau_i U[t,s] \tau_j U[s,0] + \theta(s-t) U[1,s] \tau_j U[s,t] \tau_i U[t,0] \right),$$

we obtain

$$C_1 = \int_0^1 ds \int_0^t dt \int d^3 x \int d^3 y \partial^x_a \left( z_\epsilon(x, y) N(x) A^i_j(y) \frac{\delta}{\delta A^i_a(x)} \frac{\delta}{\delta A^j_b(x)} \right) T_{ij}(t,s),$$

where

$$T_{ij}(t,s) = \theta(t-s) U[1,t] \tau_i U[t,s] \tau_j U[s,0].$$

Turning this back into a surface integral and doing the integral over $y$, this gives
\[ C_1 = \int_0^1 ds \int_0^1 dt \int_{\partial \Sigma} d^2 S_a(x) N(x) \left( z_e(x, \gamma(t)) A^{ij}_b(\gamma(t)) \dot{\gamma}^a(s) \dot{\gamma}^b(t) \delta(x, \gamma(s)) - z_e(x, \gamma(s)) A^{ij}_b(\gamma(s)) \dot{\gamma}^a(t) \dot{\gamma}^b(s) \delta(x, \gamma(t)) \right) T_{ij}(t, s). \]

Using the fact that for \( x \in \partial \Sigma, \)
\[ \int_0^1 ds \dot{\gamma}^a(s) \int d^2 S_a(x) \delta(x, \gamma(s)) f(x) = f(\gamma_0) + f(\gamma_1), \]
where \( d^2 S_a(x) = d^2 x n_a(x), \) we obtain
\[ C_1 = \int_0^1 dt \ N(\gamma_0) z_e(\gamma_0, \gamma(t)) A^{ij}_b(\gamma(t)) \dot{\gamma}^b(t) T_{ij}(t, 0) - \int_0^1 dt \ N(\gamma_1) z_e(\gamma_1, \gamma(t)) A^{ij}_b(\gamma(t)) \dot{\gamma}^b(t) T_{ij}(1, t). \]

Taking the limit as \( \epsilon \to 0, \) and using the fact that \( \gamma_0 \neq \gamma_1, \) we have
\[ \lim_{\epsilon \to 0} z_e(\gamma_0, \gamma(t)) = \delta^{(2)}(\gamma_0, \gamma_0) \frac{\delta(t)}{n_c(\gamma_0) \dot{\gamma}^c(0)}, \]
and a similar result for \( z_e(\gamma_1, \gamma(t)) \). Thus we obtain
\[ C_1 = -N(\gamma_0) \delta^{(2)}(\gamma_0, \gamma_0) \frac{A^{ij}_b(\gamma_0) \dot{\gamma}^b(0)}{n_c(\gamma_0) \dot{\gamma}^c(0)} U[1, 0] \tau_i \tau_j - N(\gamma_0) \delta^{(2)}(\gamma_0, \gamma_0) \frac{A^{ij}_b(\gamma_1) \dot{\gamma}^b(1)}{n_c(\gamma_1) \dot{\gamma}^c(1)} \tau_i \tau_j U[1, 0]. \]

Using the identity \( \epsilon^{ijk} \tau_i \tau_j = 2i \tau^k, \) we obtain the final result:
\[ C_1 = 2i N(\gamma_0) \delta^{(2)}(\gamma_0, \gamma_0) U[1, 0] \frac{A_b(\gamma_0) \dot{\gamma}^b(0)}{n_c(\gamma_0) \dot{\gamma}^c(0)} - 2i N(\gamma_1) \delta^{(2)}(\gamma_1, \gamma_1) \frac{A_b(\gamma_1) \dot{\gamma}^b(1)}{n_c(\gamma_1) \dot{\gamma}^c(1)} U[1, 0], \]

where we have defined \( A_b = A^k_b \tau_k. \) Note that this result is purely formal, as \( \delta^{(2)}(\gamma_0, \gamma_0) \) is infinite. It should be regarded as a precise description of the behavior of \( C_1 \) in the
limit as the point-splitting function $z_e(x, y)$ converges to $\delta(x, y)$. (At the end of this section we discuss how to ‘renormalize’ the Hamiltonian to obtain a finite result.)

In a similar manner we can evaluate the second term of $[H_{\partial \Sigma}, U[1, 0]]$, obtaining

$$C_2 = \int_{\Sigma} d^3x \delta_a \left( N(x) A^{ij}_b(x) \frac{\delta U[1, 0]}{\delta A^a_b(x)} \frac{\delta}{\delta A^b_a(x)} \right)$$

$$= \int_0^1 dt \int_{\Sigma} d^3x \delta_a \left( \gamma^a(t) \ N(x) A^{ij}_b(x) \delta(x, \gamma(t)) U[1, t] \tau_i U[t, 0] \frac{\delta}{\delta A^b_a(x)} \right)$$

$$= \int_0^1 dt \gamma^a(t) \int_{\partial \Sigma} d^2S(x) \delta(x, \gamma(t)) \ N(x) A^{ij}_b(x) U[1, t] \tau_i U[t, 0] \frac{\delta}{\delta A^b_a(x)}.$$  

By (5), we obtain:

$$C_2 = \int_0^1 dt \gamma^a(t) \int_{\partial \Sigma} d^2S(x) \delta(x, \gamma(t)) \ N(x) A^{ij}_b(x) U[1, t] \tau_i U[t, 0] \frac{\delta}{\delta A^b_a(x)}.$$  

With the help of some Pauli matrix identities, and setting

$$\frac{\delta}{\delta A^a} = \tau_i \frac{\delta}{\delta A^a},$$

the final result can be written as

$$C_2 \psi = i \ N(\gamma_0) U[1, 0] \left[ A_b(\gamma_0), \frac{\delta \psi}{\delta A_b(\gamma_0)} \right] + i \ N(\gamma_1) \left[ A_b(\gamma_1), \frac{\delta \psi}{\delta A_b(\gamma_1)} \right] U[1, 0].$$

Proceeding in the same way as before we can evaluate the third term in the commutator as

$$C_3 \psi = i \ N(\gamma_0) U[1, 0] \left[ n_a(\gamma_0), \frac{\delta \psi}{\delta A_a(\gamma_0)} \right] + A_b(\gamma_0) \frac{\delta^b(0)}{\delta A_a(\gamma_0)} + i \ N(\gamma_1) \left[ n_a(\gamma_1), \frac{\delta \psi}{\delta A_a(\gamma_1)} \right] + A_b(\gamma_1) \frac{\delta^b(1)}{\delta A_a(\gamma_1)} U[1, 0].$$

It is unfortunate, but not unexpected, that the term $C_1$ which determines the action of the Hamiltonian on a Wilson line state is singular. In fact, the action of the
Hamiltonian constraint on Wilson loop states is singular in a very similar way. Rovelli and Smolin dealt with this problem by point-splitting the Hamiltonian constraint and then ‘renormalizing’ it, that is, multiplying it by $\epsilon$ before taking the limit as $\epsilon \to 0$. We can do the same sort of thing when our boundary conditions are such that the 3-metric is fixed on $\partial \Sigma$ — for example, in the asymptotically flat case. Namely, we pick any metric $g_{ij}$ on $\Sigma$ extending the fixed metric on $\partial \Sigma$, and using this metric choose the regulator to be

$$z_\epsilon(x, y) = \sqrt{q/(\pi \epsilon)^3} e^{-d(x, y)^2/\epsilon},$$

where $d(x, y)^2 \sim q_{ij}(x - y)^i(x - y)^j$ is the square of the distance from $x$ to $y$, and $q$ stands for the determinant of the metric. We then renormalize the Hamiltonian $\mathcal{H}_\Sigma(\epsilon)$ by writing the boundary term as an integral of a total divergence, performing a point-splitting of both terms, and introducing a factor of $\epsilon$ before taking the $\epsilon \to 0$ limit:

$$\hat{\mathcal{H}}_\Sigma^{\text{ren}} = \lim_{\epsilon \to 0} \epsilon \{ - \frac{1}{2} \int_{\Sigma} d^3x \ z_\epsilon(x, y) \ N(x) \ e^{ijk} F_{ahlk}(x) \frac{\delta}{\delta A_i^a(x)} \frac{\delta}{\delta A_j^b(y)} + \int_{\Sigma} d^3x \ \partial^a x \left( \int_{\Sigma} d^3y \ z_\epsilon(x, y) \ N(x) A_{ij}^b(y) \frac{\delta}{\delta A_i^a(x)} \frac{\delta}{\delta A_j^b(y)} \right) \}. $$

If we apply this to the Wilson line state given in eq. (3), Rovelli and Smolin’s argument shows that the first term is zero. Following our earlier computation of $C_1$ but using the formula

$$\lim_{\epsilon \to 0} \epsilon z_\epsilon(\gamma_0, \gamma(t)) = \frac{\delta(t)}{n_c(\gamma_0) \dot{\gamma}^c(0)},$$

we find that the second term gives

$$\hat{\mathcal{H}}_\Sigma^{\text{ren}} \Psi = 2i \psi(\xi(\gamma_0)) U[1, 0] \frac{A_b(\gamma_0) \dot{\gamma}^b(0)}{n_c(\gamma_0) \dot{\gamma}^c(0)} - \mathcal{N}(\gamma_1) \frac{A_b(\gamma_1) \dot{\gamma}^b(1)}{n_c(\gamma_1) \dot{\gamma}^c(1)} U[1, 0] \psi,$$

(9)
Now one might worry that since our regulator depended on a choice of metric $q_{ij}$ on $\Sigma$, this renormalization prescription spoils the diffeomorphism-invariance of the problem. However, we are assuming that the boundary conditions are such that the metric $q_{ij}$ is fixed on $\partial \Sigma$, so the symmetry group has been reduced to the group of diffeomorphisms that preserve this metric on $\partial \Sigma$. Note that the final result in eq. (11) only depends on the metric via the unit normal vector $n_c$. It thus depends only on the value of $q_{ij}$ on $\partial \Sigma$. Thus $\mathcal{H}_{\Sigma}^{ren}$ is preserved, as it should be, by all diffeomorphisms of $\Sigma$ preserving the metric on $\partial \Sigma$.

3 Conclusions

In addition to its conceptual subtleties, canonical quantum gravity presents many technical problems, so the regularization procedure leading to our final result should be carefully checked. It is encouraging, however, that eq. (11) has a strikingly simple geometrical interpretation. The Hamiltonian $\hat{H}_{\Sigma}^{ren}$ acts on the Wilson line state $\Psi$ to give a difference of two terms. In the first, $U[1,0]$ is multiplied on the right by a term proportional to the component of the connection $A$ in the direction $\dot{\gamma}(0)$. In the second, $U[1,0]$ is multiplied on the left by a term proportional to the component of $A$ in the direction $\dot{\gamma}(1)$. In short, the Hamiltonian acts as a kind of ‘shift operator’ corresponding to an infinitesimal displacement of the Wilson line in the direction of its tangent vector.

This ‘shift operator’ interpretation is closely related to two earlier results in the loop representation of quantum gravity. First, Rovelli and Smolin [5] have already shown that in the case of a space without boundary, the Hamiltonian constraint acts on Wilson loops as a shift operator. This 3-dimensional geometrical interpretation
of the Hamiltonian constraint is what enabled them (and others) to find solutions of the Wheeler-DeWitt equation.

Second, a formula very similar to ours also arises in the work by Morales-Técotl and Rovelli on quantum gravity coupled to massless chiral fermions, that is, a Weyl spinor field \[ \psi^A \]. They consider a space without boundary, so there is no Hamiltonian, only a Hamiltonian constraint. Moreover, while for us the spinors \( \psi^A \) appearing at the ends of Wilson lines are only a device to extract numbers out of the matrix-valued observable \( U[1,0] \), for them the spinors are dynamical fields appearing in the Lagrangian. Nonetheless, when they switch to the Schrödinger representation, construct states as in eq. (3) and compute the action of the Hamiltonian constraint on these states, the answer is given by a shift operator as in our work.

Why should the Hamiltonian for quantum gravity on a space with boundary have the same action on Wilson lines as the Hamiltonian constraint for quantum gravity coupled to Weyl spinors? A clue is provided by Carlip’s computation [14] of black hole entropy in 2+1-dimensional quantum gravity. He treats the event horizon as a boundary and notes, as we do above, that the boundary conditions break diffeomorphism invariance at the boundary. Due to this reduction of symmetry, modes of the gravitational field that would otherwise be treated as ‘gauge’ manifest themselves as physical degrees of freedom. It is these degrees of freedom that account for the black hole entropy in his computation.

Technically speaking, Carlip proceeds via Witten’s description of 2+1 quantum gravity (with cosmological term) as a Chern-Simons theory, and uses the fact that Chern-Simons theory on a manifold with boundary induces a field theory on the boundary, namely a WZW model. As noted by Balachandran et al. [15], the math-
matics involved here is the same as that which describes the fractional quantum Hall effect. In the fractional quantum Hall effect the induced WZW model is known to describe a chiral fermion field on the boundary.

Balachandran et al have suggested that in 3+1-dimensional quantum gravity as well, boundary conditions breaking diffeomorphism invariance should give rise to an induced field theory on the boundary, whose ‘edge states’ account for the black hole entropy. Our work suggests that when the boundary conditions fix the metric at the boundary, this induced field theory describes a Weyl spinor field coupled to the connection $A_b$ by means of the Hamiltonian given in eq. (9). In this interpretation the endpoints of Wilson lines act as chiral fermions living on $\partial \Sigma$. Trying to work out the black hole entropy by this method is especially tempting, because the area operator in the loop representation essentially counts the number points where Wilson lines intersect a surface [10]. Smolin has already begun work towards proving area-entropy relations for black holes by exploiting this fact [4]. It is also interesting to compare the work of Hawking and Horowitz [17].

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