Linear-Quadratic Mean Field Games

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Abstract We provide a comprehensive study of a general class of linear-quadratic mean field games. We adopt the adjoint equation approach to investigate the unique existence of their equilibrium strategies. Due to the linearity of the adjoint equations, the optimal mean field term satisfies a forward–backward ordinary differential equation. For the one-dimensional case, we establish the unique existence of the equilibrium strategy. For a dimension greater than one, by applying the Banach fixed point theorem under a suitable norm, a sufficient condition for the unique existence of the equilibrium strategy is provided, which is independent of the coefficients of controls in the underlying dynamics and is always satisfied whenever the coefficients of the mean field term are vanished, and hence, our theories include the classical linear-quadratic stochastic control problems as special cases. As a by-product, we also establish a neat

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and instructive sufficient condition, which is apparently absent in the literature and only depends on coefficients, for the unique existence of the solution for a class of non-symmetric Riccati equations. Numerical examples of nonexistence of the equilibrium strategy will also be illustrated. Finally, a similar approach has been adopted to study the linear-quadratic mean field type stochastic control problems and their comparisons with mean field games.

Keywords Mean field games · Mean field type stochastic control problems · Adjoint equations · Linear quadratic

1 Introduction

Modeling collective behaviors of individuals in account of their mutual interactions in various physical or sociological dynamical systems has been one of the major problems in the history of mankind. For instance, physicists simply applied the traditional variational methods from Lagrangian or Hamiltonian mechanics to study interacting particle system, which left a drawback of extremely high computational cost that made this microscopic approach almost mathematically intractable. To resolve this matter, a completely different macroscopic approach from statistical physics had been gradually developed, which eventually leads to the primitive notion of mean field theory. The novelty of this approach is that particles interact through a medium, namely the mean field term, aggregated by action of and reaction on each particle. Moreover, by passing the number of particles to the infinity in these macroscopic models, the mean field term will become a functional of the density function which represents the whole population of particles that leads to much less computational complexity. In biological literature, similar tools have been applied to connect human interactive motion with herding models for insects and animals. For example, the behavior that ants secrete chemical substrates for leading mates to valuable food resources resulting in a lane can be described by a mean field model (see [1] for more details).

On economics side, due to the dramatic population growth and rapid urbanization, urgent needs of in-depth understanding of collective strategic interactive behaviors of a huge group of investors is crucial for maintaining sustainable economic growth. Since the vector of good prices is determined by both demand and supply, it is natural to utilize the aggregation effect from the investors’ states as a canonical candidate of mean field term and then employs the corresponding mean field models in place of the classical equilibrium models in economics. Moreover, as the investors are usually smart in decision making (i.e., being not of zero-intelligence), it is necessary to also incorporate the theory of stochastic differential games (SDGs) in these mean field models. Over the past few decades, SDGs have been a major research topic in control theory and financial economics, especially in studying the continuous-time decision-making problem between noncooperative investors. In regard to one-dimensional setting, the theory of two-person zero-sum games is quite well developed via the notion of viscosity solutions; see, for example, [2] and [3]. Unfortunately, the most interesting SDGs are $N$-player nonzero-sum SDGs. In this direction, we mention the works of [4–6], but there are still relatively few results in the literature.
As a macroscopic equilibrium model, [7, 8] investigated stochastic differential game problems involving infinitely many players under the name “large population stochastic dynamic games.” Independently, [9–11] studied similar problems from the viewpoint of the mean field theory and termed “mean field games (MFGs).” As an combination of mean field theory and theory of stochastic differential games, MFGs provide more realistic interpretation of individual dynamics at the microscopic level, so that each player is not of zero-intelligent and will be able to strategically optimize his prescribed objectives, yet with a mathematical tractability in a macroscopic framework. To be more precise, the general theory of MFGs has been built by combining various consistent assumptions on the following modeling aspects: (1) a continuum of players; (2) homogeneity in strategic performance of players; and (3) social interactions through the impact of mean field term. The first aspect is describing the approximation of a game model with a huge number of players by one with a continuum of players but sufficient mathematical tractability. The second aspect is assuming that all players obey the same set of rules of the interactive game, which provide guidance on their own behavior that potentially leads them to ultimate success. Finally, due to the intrinsic complexity of the society in which the players participate in, the third aspect is explaining the fact that each player is so negligible that he can only affect others marginally through his own infinitesimal contribution to the society. In a MFG, the decision making of each player will base purely on his own criteria and certain summary statistics (i.e., the mean field term) about the community. In other words, in explanation of their interactions, the pair of personal and mean field characteristics of the whole population is already sufficient and exhaustive. Mathematically, each MFG will possess the following forward–backward structure: (1) a forward dynamic describes the individual strategic behavior; (2) a backward equation describes the evolution of individual optimal strategy, such as those in terms of the individual value function via the usual backward recursive techniques. For the detail of the derivation of this system of equations with forward–backward feature, one can consult the works of [8] and [9–11].

Before introducing our proposed model, we first list some relevant recent theoretical results in MFGs. (1) For problems over infinite-time horizon, [12] and [13] studied ergodic MFGs with different quadratic cost functionals and linear dynamics; [14] studied a specific MFG with a quadratic Hamiltonian and showed that the density function for the population is Gaussian; [7, 15–17] considered MFGs with quadratic cost functional; [18] extended the studies of MFGs with ergodic cost functional to Cucker–Smale flocking model; and [19] gave a bifurcation analysis of an ergodic MFG with nonlinear dynamics. (2) For problems over finite-time horizon, [20] applied a change-of-variable technique leading to separation of variables to consider MFGs with quadratic Hamiltonian; [21] and [22] extended MFGs to the setting involving two-population dynamics; [23] investigated MFGs with reflection parts and quadratic cost functional; [24] considered the risk-sensitive MFGs; and [25] adopts the MFG approach to construct a nonlinear filter. (3) Various numerical approximation scheme can also be found in [26–28]. Because of the discretization in the numerics, [29] studied discrete time MFG with finite state space directly.

MFG has been a very popular research direction. During the preparation of the final version of this paper, the following development has been noticed. For example,
Camilli and Dolcetta developed a finite difference method for approximation in [27]. Camilli and Marchi [30] investigated MFGs on a network. For linear-quadratic mean field games (LQMGs) with different settings, readers are referred to [31,32] and [33]. Huang [34] considered a MFG where a major player exists which affects the state of minor agents in a linear-quadratic setting which is later generalized by Nourian and Caines into nonlinear case in [35]. Nourian et al. [36] also applied mean field theory to solve consensus problems. Bensoussan et al. [37] studied a different setting where the dominating player guides the agent agents to follow instead of only affecting them through his state. Bensoussan et. al. investigated MFGs using a master equation approach in [38]. Lately, there are also developments in the study of mean field type forward–backward SDEs; see, for example, [39]. The above serves only as a reference and is not exhaustive. For more recent development and its applications, please also refer to the lecture notes [40], the surveys of [41,42], the book [43], and the references therein.

In this paper, we study a subclass of MFGs in which the cost functional is quadratic in all state variables, control variables, and the mean field terms, while the controlled dynamics are linear and also consist of mean field terms. These LQMGs have been previously considered in [44] by using the common Riccati equation approach. In contrast, in this paper, the stochastic maximum principle is adopted instead. Essentially, the equilibrium problem can be converted into finding a fixed point for a transformation defined by the solution of a control problem. For the part of control problem, these two approaches are certainly equivalent. However, it is not the case for the fixed point problem since a condition that is easier to be verified can be given. Indeed, in our approach, thanks to the linearity of the adjoint equations, the optimal mean field term can be expressed as the solution of a forward–backward ordinary differential equation. This method avoids solving the optimal trajectory as in the Riccati equation approach and allows generalization to higher dimension, which is a crucial setting in understanding the two-population MFGs proposed in [21] and [22]. More precisely, by choosing a suitable norm and applying the Banach fixed point theorem, we provide a more relaxed sufficient condition for the existence and uniqueness of the equilibrium strategy, which is also independent of the coefficient of control and always hold when the mean field term is zero. Under the one-dimensional setting and certain convexity assumption, we also prove that the equilibrium strategy always uniquely exists. As a by-product, we also establish a neat and instructive sufficient condition, which is apparently absent in the literature (see [45]) and only depends on coefficients, for the unique existence of the solution for a class of nontrivial nonsymmetric Riccati equations. Numerical examples showing the nonexistence of any equilibrium strategy will also be provided. Furthermore, in the Appendix, we compare explicitly our conditions with those of [44]. In summary, our present work gives a novel and totally different approach with several advantages in particular for the generalization of the classical linear-quadratic stochastic control problem in the MFG setting.

In general, the computational complexity of calibrating a Nash equilibrium of an $N$-player SDG (if it exists) is very high, especially for large values of $N$. It would be

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1 This does not mean that our condition is less restrictive. In general, both approaches cover different feasible ranges.
more convenient to find a computable approximation of this Nash equilibrium strategy. Since MFGs are obtained by setting $N \to \infty$, the equilibrium strategy serves as a natural candidate as it can be shown to be an “approximation,” or $\epsilon$-Nash equilibrium strategy for the corresponding equilibrium for $N$-player SDG. The computability of this equilibrium strategy is justifiable as it depends only on the state of the player and the mean field term, which dramatically reduces the problem dimension of the Nash equilibrium strategy of the $N$-player SDG. For more inspiring elaboration on the notion of $\epsilon$-Nash equilibrium, one can refer to, for example, [40] and [7,8,46].

If one considers a system of interacting particles under centralized control, instead of one in which every particle having the free will to choose its own control as formulated in MFGs, a stochastic control problem of mean field type would result (see [31,47] and [33]). This mean field type optimization problem shares a similar mathematical form as proposed in MFG problem, and the mean field term is now uniformly controlled by a centralizing system instead of being affected by the collective optimal trajectory. For more details about the existence and convergence rate of the related mean field backward stochastic differential equations, one can refer to [48] and [49]. By using the adjoint equation approach again, we characterize the optimal control, which exists and is unique in virtue of the convex coercive property of the underlying cost functional. Finally, we also find that, in general, this optimal control is different from the equilibrium strategy obtained in its corresponding MFG counterpart.

In Sect. 2, we will formulate a linear-quadratic $N$-player nonzero-sum stochastic differential game and demonstrate how to obtain the corresponding MFG formally. In Sect. 3, we shall employ the adjoint equation approach in order to provide a thoughtful study of the existence and uniqueness of LQMFGs. By choosing a suitable norm and applying the Banach fixed point theorem, an illuminating sufficient condition for the existence and uniqueness of the equilibrium strategy is provided. Note that this new condition is independent of the coefficients of the controls and is always satisfied whenever the coefficients of the mean field terms vanish. Relationship with nonsymmetric Riccati equations and illustrative numerical examples will also be provided. We remark that these nonsymmetric Riccati equations, appearing in the resolution of the fixed point problem, could not be found in the literature including [44], and most importantly, they are substantially different from those symmetric Riccati equation commonly arising from control theory. In Sect. 4, we shall show that the equilibrium strategy is an $\epsilon$-Nash equilibrium of the $N$-player SDG. In Sects. 5 and 6, we shall adopt a similar adjoint equation approach to solve the linear-quadratic mean field type stochastic control problem and compare its optimal control to the equilibrium strategy of the corresponding MFG counterpart. In Appendix, an example is given whose unique existence of optimum could be covered by our theory but it fails to satisfy the sufficient condition as provided in [44]. It is noticed that in some other cases, [44] may cover different possibilities from ours.

2 Problem Formulation

The present formulation of the LQMFGs will follow closely the classical linear-quadratic stochastic control problems; see, for example, [50]. Following [9–11], in
order to formulate the LQMFG, we first state the corresponding $N$-player game for $N \geq 1$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $T > 0$. Suppose that $W^1, \ldots, W^N$ are $N$ independent $n$-dimensional standard Wiener processes defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and $x^i_0, \ldots, x^N_0$ are $N$ independent, identically distributed (i.i.d.) $n$-dimensional random vectors. We also assume that $x^i_0$ is independent to $(W^1, \ldots, W^N)$ for each $i, 1 \leq i \leq N$. The dynamics of the player $i$ is modeled by

$$dx^i_t = \left( A_i x^i_t + B_t v^i_t + \tilde{A} \times \frac{1}{N-1} \sum_{j=1, j \neq i}^N x^j_t \right) dt + \sigma_t dW^i_t, \quad x^i(0) = x^i_0,$$

where $A, B, \tilde{A}$ are bounded deterministic matrix-valued functions in time of suitable sizes, $\sigma$ is a $L^2$-function in time of suitable size, and the control $v^i$ in $L^2_G(0, T; \mathbb{R}^m)$, which is the $L^2$-space of stochastic processes adapted to the filtration

$$G_t := \sigma \left( \left( x^1_0, \ldots, x^N_0 \right), \left( W^1_s, \ldots, W^N_s \right), s \leq t \right),$$

with values in $\mathbb{R}^m$. The present proposed (additive) model extends the classical linear stochastic dynamical one, as expected; the coefficient $\tilde{A}$ measures the effect brought by the state variable; and $B$ measures the impact of the control, while the new ingredient $\tilde{A}$ summarizes the symmetric influence of the rest of the players. Even though this additional term $\tilde{A}$ is natural from the modeling perspective when one attempts to investigate interactive real-time multi-player game, it causes a substantial mathematical difficulty in the discussion on the existence and calibration of the corresponding Nash equilibrium especially for large values of $N$. Although the coefficients look the same for all players, it reflects that each player obeys the same set of game rules and behaves based on the same collections of rationales. Their actual individual realized performances could be far different from each other; in particular, their single dynamics are driven by independent Wiener processes.

The cost functional for each player $i$ is assumed to be:

$$\mathcal{J}^i \left( v^1, \ldots, v^N \right) := \mathbb{E} \left[ \frac{1}{2} \int_0^T \left( x^i_T \right)^* Q_t x^i_t + \left( v^i_T \right)^* R_t v^i_t \ dt + \frac{1}{2} \left( x^i_T \right)^* Q_T x^i_T \right]$$

$$+ \mathbb{E} \left[ \frac{1}{2} \int_0^T \left( x^i_t - S_t \times \frac{\sum_{j=1, j \neq i}^N x^j_t}{N-1} \right)^* \bar{Q}_t \left( x^i_t - S_t \times \frac{\sum_{j=1, j \neq i}^N x^j_t}{N-1} \right) \ dt \right]$$

$$+ \mathbb{E} \left[ \frac{1}{2} \left( x^i_T - S_T \times \frac{\sum_{j=1, j \neq i}^N x^j_T}{N-1} \right)^* \bar{Q}_T \left( x^i_T - S_T \times \frac{\sum_{j=1, j \neq i}^N x^j_T}{N-1} \right) \right],$$
where $M^*$ denotes the transpose of a matrix $M$, and $S$ (respectively, $Q$, $\tilde{Q}$ and $R$) is bounded, deterministic (respectively, nonnegative and positive definite) matrix-valued functions in time of suitable sizes. We also suppose that $R \geq \delta I$, for some $\delta > 0$.

The reasons for the same form of the objective functional among different players are the same as in the previous discussions on the modeling of individual dynamics. The first expectation agrees with the corresponding cost functional in the classical linear-quadratic stochastic control problem; namely, it describes the sum of running expenses and the terminal costs of each player himself. The other two expectations are specific in our present model setting, they describe the extra costs incurred if a player shows deviated performance away from the average behavior of the community. These two terms truly reflect the coalescence phenomena commonly observed in the literature of socioeconomics and finance, in the sense that every agent has to pay an additional transaction cost of collecting extra profitable information if he aims to outperform from his peers. Along every direction of the deviation, the incorporated additional cost has to be nonnegative, this model constraint is ensured by the assumption of the nonnegative definiteness of $\tilde{Q}$.

The principal objective of each player is to minimize his own cost functional by properly controlling his own dynamics. In this classical nonzero-sum stochastic differential game framework, we aim to establish a Nash equilibrium $(u^1, \ldots, u^N)$ (see, for example, [4]):

\textbf{Problem 2.1} Find a Nash equilibrium $(u^1, \ldots, u^N)$ which satisfies the following comparison inequalities:

$$J^i(u^1, \ldots, u^{i-1}, v^i, u^{i+1}, \ldots, u^N) \geq J^i(u^1, \ldots, u^N),$$

for $1 \leq i \leq N$ and any admissible control $v^i$ in $L^2_G(0, T; \mathbb{R}^m)$.

In accordance with the permutation symmetry of index $i$, it suffices to consider the case for $i = 1$. In general, the computational complexity of calibration of a Nash equilibrium (if it exists) is high, especially for large values of $N$. Due to the large number of participants in most game theoretical models in practice, a convenient computable approximation of the Nash equilibrium strategy is usually demanded. By formally passing $N \to \infty$, [9–11] introduced the notion of MFGs. Now, the MFG associated with Problem 2.1 can be obtained as follows:

\textbf{Problem 2.2} Find an equilibrium strategy $u$ in $L^2_G(0, T; \mathbb{R}^m)$, with $x_0 := x^1_0$, $W := W^1$ and $\mathcal{F}_t := \sigma(x_0, W_s, s \leq t)$, which minimizes the cost functional

$$J(v) := \mathbb{E} \left[ \frac{1}{2} \int_0^T x_t^* Q_t x_t + v_t^* R_t v_t + (x_t - S_t \mathbb{E}[y_t])^* \tilde{Q}_t (x_t - S_t \mathbb{E}[y_t]) dt \right]$$

$$+ \mathbb{E} \left[ \frac{1}{2} x_T^* Q_T x_T + \frac{1}{2} (x_T - S_T \mathbb{E}[y_T])^* \tilde{Q}_T (x_T - S_T \mathbb{E}[y_T]) \right],$$

where the dynamics is given by

$$dx_t = (A_t x_t + B_t v_t + \tilde{A}_t \mathbb{E}[y_t]) dt + \sigma_t dW_t, \quad x(0) = x_0,$$
v is an admissible control in $L_{F_i}^2(0, T; \mathbb{R}^m)$, and $y$ is the trajectory corresponding to the equilibrium strategy $u$ (if it exists).

**Remark 2.1** An interesting example of our proposed one-dimensional LQMFGs has been considered earlier in [44]. In this paper, we shall provide a complete picture of the resolution of the problem by using adjoint equation approach, which results in a different sufficient condition for the unique existence of the underlying equilibrium strategy.

In comparison with the $N$-player game, in order to avoid confusion, the notations for the dynamics $x$ and $y$ stated in Problem 2.2 will be changed to $\hat{x}$ and $\hat{y}$, respectively. For if Problem 2.2 were solvable, then for each $i$, $1 \leq i \leq N$, we could obtain a strategy $u^i$ in $L_{F_i}^2(0, T; \mathbb{R}^m)$, where $F_i^t := \sigma(x_i^0, W_i^s, s \leq t)$. Since $\mathbb{E}[y_t]$ is a deterministic process, clearly $u^1, \ldots, u^N$ are i.i.d.. As the MFG is obtained from the $N$-player game, it is expected that $(u^1, \ldots, u^N)$ is an $\epsilon$-Nash equilibrium when $N \to \infty$; see, for example, [40] and [7,8,46] for more detail. We shall first present an informal description here, rigorous arguments will be provided later in Sect. 4. Again, it suffices to consider Player 1.

For any admissible control $v^1$, let $(x^1, \ldots, x^N)$ (respectively, $(y^1, \ldots, y^N)$) denote the dynamics in Problem 2.1 controlled by $(v^1, u^2, \ldots, u^N)$ (respectively, $(u^1, \ldots, u^N)$). By the definition of $\epsilon$-Nash equilibrium, $u^1$ will “approximately” minimize $\mathcal{J}^1(v^1, u^2, \ldots, u^N)$. As $N \to \infty$, for $i \neq 1$, we have

$$\frac{1}{N-1} \sum_{j=1, j \neq i}^{N} x^j_t - \frac{1}{N-1} \sum_{j=2, j \neq i}^{N} x^j_t \to 0.$$ 

By the McKean–Vlasov argument, $x^j_t \to \hat{y}^j_t$. As an application of the strong law of large numbers (SLLN), $x^j_t \to \hat{x}^j_t$ as $\hat{y}^1_t, \ldots, \hat{y}^N_t$ are i.i.d., which is a consequence of the i.i.d. nature of $u^1, \ldots, u^N$. By applying SLLN again to the cost functional, we deduce that

$$\mathcal{J}^1(v^1, u^2, \ldots, u^N) \to \mathcal{J}^1(v^1),$$

where $\mathcal{J}^1$ equals the cost functional $\mathcal{J}$ in Problem 2.2. Similarly, we also have

$$\mathcal{J}^1(u^1, u^2, \ldots, u^N) \to \mathcal{J}^1(u^1),$$

and it shows heuristically that $(u^1, \ldots, u^N)$ is an $\epsilon$-Nash equilibrium.

### 3 Solution of the Mean Field Game

To motivate for solving Problem 2.2, we first lay down some classical results in the literature of linear-quadratic stochastic control theory but from a new perspective, which aids for the development of our new methodology to tackle Problem 2.2.
Problem 3.1 Given a continuous deterministic process $z$ with values in $\mathbb{R}^n$. Find an optimal control $u$ in $L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$ which minimizes

$$J(v) := \mathbb{E} \left[ \frac{1}{2} \int_0^T x_t^* Q_t x_t + v_t^* R_t v_t + (x_t - S_t z_t)^* \bar{Q}_t (x_t - S_t z_t) \, dt \right]$$

$$+ \mathbb{E} \left[ \frac{1}{2} x_T^* Q_T x_T + \frac{1}{2} (x_T - S_T z_T)^* \bar{Q}_T (x_T - S_T z_T) \right],$$

where the dynamics is given by

$$dx_t = (A_t x_t + B_t v_t + \bar{A}_t z_t) \, dt + \sigma_t \, dW_t, \quad x(0) = x_0,$$

and $v$ is an admissible control in $L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$.

Theorem 3.1 Problem 3.1 is uniquely solvable, and the optimal control $u$ is $-R^{-1}B^* p$, where $(y, p)$ satisfy the stochastic maximum principle relation

$$dy_t = (A_t y_t - B_t R_t^{-1} B_t^* p_t + \bar{A}_t z_t) \, dt + \sigma_t \, dW_t,$$

$$y_0 = x_0,$$

$$-\frac{d\omega_t}{dt} = A_t^* \omega_t + (Q_t + \bar{Q}_t) y_t - (\bar{Q}_t S_t) z_t,$$

$$\omega_T = (Q_T + \bar{Q}_T) y_T - (\bar{Q}_T S_T) z_T,$$

such that $p_t = \mathbb{E}[\omega_t | \mathcal{F}_t]$.

Proof Clearly, Problem 3.1 is a strictly convex coercive optimization problem in the sense that $J(v) \to \infty$ as $\|v\| \to \infty$. In order to derive the stochastic maximum principle relation, we first consider the Euler equation:

$$\frac{d}{d\theta} J(u(\cdot) + \theta v(\cdot)) \bigg|_{\theta=0} = 0.$$

By linearity, under the perturbed control $u + \theta v$, the original state $y(\cdot)$ becomes $y(\cdot) + \theta \tilde{x}(\cdot)$, where

$$d\tilde{x}/dt = A_t \tilde{x}_t + B_t v_t, \quad \tilde{x}(0) = 0.$$

On the other hand, we can write $J(v(\cdot))$ as

$$J(v(\cdot)) = \mathbb{E} \left[ \int_0^T \frac{1}{2} (x_t^* (Q_t + \bar{Q}_t) x_t + v_t^* R_t v_t) - x_t^* (\bar{Q}_t S_t) z_t + \frac{1}{2} z_t^* (S_t^* \bar{Q}_t S_t) z_t \, dt \right]$$

$$+ \mathbb{E} \left[ \frac{1}{2} x_T^* (Q_T + \bar{Q}_T) x_T - x_T^* (\bar{Q}_T S_T) z_T + \frac{1}{2} z_T^* (S_T^* \bar{Q}_T S_T) z_T \right].$$
and therefore, the Euler equation becomes
\[
\mathbb{E}\left[\int_0^T \tilde{x}_t^*(Q_t + \tilde{Q}_t) y_t - \tilde{x}_t^*(\tilde{Q}_t S_t) z_t + v_t^* R_t u_t \, dt\right] \\
+ \mathbb{E}[\tilde{x}_T^*(Q_T + \tilde{Q}_T) y_T - \tilde{x}_T^*(\tilde{Q}_T S_T) z_T] = 0.
\]  
(2)

Define the adjoint process \(\omega_t\) (need not adapted to \(\mathcal{F}_t\)) by
\[
-\frac{d\omega_t}{dt} = A_t^* \omega_t + (Q_t + \tilde{Q}_t) y_t - (\tilde{Q}_t S_t ) z_t,
\]
\[\omega_T = (Q_T + \tilde{Q}_T) y_T - (\tilde{Q}_T S_T) z_T,
\]
we obtain
\[
\frac{d}{dt}(\tilde{x}_t^* \omega_t) = (\tilde{x}_t^* A_t^* + v_t^* B_t^*) \omega_t - \tilde{x}_t^*(A_t^* \omega_t + (Q_t + \tilde{Q}_t) y_t - (\tilde{Q}_t S_t) z_t)
\]
\[
= v_t^* B_t^* \omega_t - \tilde{x}_t^*((Q_t + \tilde{Q}_t) y_t - (\tilde{Q}_t S_t) z_t),
\]
and hence
\[
\mathbb{E}\left[\int_0^T v_t^* B_t^* \omega_t \, dt\right] = \mathbb{E}\left[\tilde{x}_T^*((Q_T + \tilde{Q}_T) y_T - (\tilde{Q}_T S_T) z_T)\right]
\]
\[
+ \mathbb{E}\left[\int_0^T \tilde{x}_t^*((Q_t + \tilde{Q}_t) y_t - (\tilde{Q}_t S_t) z_t) \, dt\right],
\]
and from (2), we obtain
\[
\mathbb{E}\left[\int_0^T v_t^* (B_t^* \omega_t + R_t u_t) \, dt\right] = 0.
\]  
(3)

Set \(p_t = \mathbb{E}[\omega_t | \mathcal{F}_t]\), then (3) becomes \(\mathbb{E}[\int_0^T v_t^* (p_t + R_t u_t) \, dt] = 0\). Since \(v\) is arbitrary in \(L^2_\mathcal{F}(0, T; \mathbb{R}^m)\), we deduce that \(u = -R^{-1}B^* p\). \(\Box\)

Remark 3.1 The optimal control \(u\) can be written as \(-R^{-1}B^* (\mathcal{E} y + \zeta)\), where \(\mathcal{E}\) and \(\zeta\) satisfy
\[
\frac{d\mathcal{E}_t}{dt} = \mathcal{E}_t A_t + A_t^* \mathcal{E}_t - \mathcal{E}_t (B_t R_t^{-1} B_t^*) \mathcal{E}_t + Q_t + \tilde{Q}_t = 0,
\]
\[
\mathcal{E}_T = Q_T + \tilde{Q}_T.
\]
\[
\frac{d\zeta_t}{dt} = -A_t^* \zeta_t + \mathcal{E}_t (B_t R_t^{-1} B_t^*) \zeta_t + (\tilde{Q}_t S_t - \mathcal{E}_t A_t) z_t
\]
\[
\zeta_T = -\tilde{Q}_T S_T z_T.
\]

According to Theorem 3.1, for fixed \(z\), we shall obtain the pair \((y, p)\), and the optimal control is \(u = -R^{-1}B^* p\). Hence, \(u\) is an equilibrium strategy of Problem 2.2.
if and only if there is a continuous function $z$ such that $\mathbb{E}[y] = z$. We denote the expected values $\bar{y}_t := \mathbb{E}[y_t], \bar{p}_t := \mathbb{E}[\omega_t] = \mathbb{E}[p_t]$, the stochastic maximum principle relation (1) implies that

$$\frac{d}{dt} \left( \bar{y}_t - \bar{p}_t \right) = \left( A_t - B_t R_t^{-1} B_t^* \right) \left( \bar{y}_t - \bar{p}_t \right) + \left( -\bar{A}_t z_t \right),$$

$$\bar{y}_0 = \mathbb{E}[x_0],$$

$$\bar{p}_T = (Q_T + \bar{Q}_T)\bar{y}_T - (\bar{Q}_T S_T)z_T.$$

Hence, Problem 2.2 is solvable if and only if $(\xi, \eta) = (\bar{y}, \bar{p})$ solves the following system of ordinary differential equations:

$$\frac{d}{dt} \left( \begin{array}{c} \xi_t \\ -\eta_t \end{array} \right) = \left( \begin{array}{cc} A_t + \bar{A}_t & -B_t R_t^{-1} B_t^* \\ Q_t + \bar{Q}_t \left( I - S_t \right) & A_t^* \end{array} \right) \left( \begin{array}{c} \xi_t \\ \eta_t \end{array} \right),$$

$$\xi_0 = \mathbb{E}[x_0],$$

$$\eta_T = (Q_T + \bar{Q}_T \left( I - S_T \right))\xi_T,$$

where $I$ is the identity matrix.

We next address the uniqueness issue of the Nash equilibrium. According to the previous discussion, if Eq. (4) has at most one solution, at most one possible $\mathbb{E}[y_t]$ could be found. Together with the uniqueness result in Theorem 3.1, there is at most one equilibrium strategy $u$. Conversely, suppose that there is at most one equilibrium strategy in Problem 3.1. For each solution $(\xi, \eta)$ of Eq. (4), we can associate the corresponding $y$ and $p$ with $z = \xi$. By the construction, we have $\mathbb{E}[y] = \xi$ and $\mathbb{E}[p] = \eta$, and hence, $-R^{-1}B^*p$ is an equilibrium strategy. Due to the uniqueness of equilibrium strategy, there is at most one possible choice for $-R^{-1}B^* \mathbb{E}[p]$. Hence, the solution $\xi$ is uniquely determined by

$$\frac{d\xi_t}{dt} = (A_t + \bar{A}_t)\xi_t - B_t R_t^{-1} B_t^* \mathbb{E}[p_t]. \quad \xi_0 = \mathbb{E}[x_0].$$

Similarly, $\eta$ can also be uniquely determined, and the uniqueness result follows. The following theorem summarizes the previous discussion.

**Theorem 3.2** There is a (unique) equilibrium strategy $u$ of Problem 2.2 if and only if there is a (unique) pair $(\xi, \eta)$ of the following system of ordinary differential equations (4):

$$\frac{d}{dt} \left( \begin{array}{c} \xi_t \\ -\eta_t \end{array} \right) = \left( \begin{array}{cc} A_t + \bar{A}_t & -B_t R_t^{-1} B_t^* \\ Q_t + \bar{Q}_t \left( I - S_t \right) & A_t^* \end{array} \right) \left( \begin{array}{c} \xi_t \\ \eta_t \end{array} \right),$$

$$\xi_0 = \mathbb{E}[x_0],$$

$$\eta_T = (Q_T + \bar{Q}_T \left( I - S_T \right))\xi_T.$$

Moreover, this equilibrium condition depends on $\bar{Q}$ and $S$ only through $S := \bar{Q}(I - S)$. 

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Since $S$ could be quite arbitrary, $-BR^{-1}B^*$, $Q + \tilde{Q}(I - S)$ may not often be of opposite sign, and hence, the sinusoidal functions serve as the standard example that Eq. (4) does not admit a solution if $T$ is sufficiently large. Even under the assumption of the convexity of $Q + \tilde{Q}(I - S)$, Eq. (4) is still not in a canonical form commonly found in the literature of the classical optimal control theory, and the existence of solution is not guaranteed in our present context. To overcome this hurdle, we first define

$$L := T \left( \|Q_T + S_T\|^2 + \|Q + S\|_T \right) \|BR^{-1}B^*\|_T$$

$$\times \exp \left( \left( 2\|A + \tilde{A}\|_T + 2\|A^*\|_T + \|BR^{-1}B^*\|_T + \|Q + S\|_T \right) T \right),$$

where $\|M\|_T$ denotes the supremum norm of the deterministic matrix-valued function $M$ on $[0, T]$. By applying Gronwall’s inequality and Banach fixed point theorem, we first have the following standard existence result when $T$ is sufficiently small; see, for example, [51]:

**Proposition 3.1** If $L < 1$, then there exists a unique solution of (4).

However, in general, the supremum norm of $BR^{-1}B^*$ could be large, and the condition that $L < 1$ is too restrictive. By the specific form of Eq. (4), a more relaxed condition can be provided as follows:

**Theorem 3.3** Assume that the matrix-valued function $Q_t$ is invertible. Let $\phi(t, s)$ be the fundamental solution associated with $A_t$ and

$$\|\phi\|_T := \sup_{0 \leq t \leq T} \sqrt{\|\phi^*(T, t)Q_T^{1/2}\|^2 + \int_0^T \|\phi^*(s, t)Q_s^{1/2}\|^2 ds}.$$  

Also, we define that $\|\tilde{A}\|_T := \sup_{0 < t < T} \|\tilde{A}_t Q_t^{-1/2}\|$ and $\|S\|_T := \sup_{0 \leq t \leq T} \|Q_t^{-1/2} \tilde{Q}_t (I - S_t) Q_t^{-1/2}\|$. Suppose that $\|\tilde{A}\|_T < \infty$, $\|S\|_T < \infty$

$$\sqrt{T} \|\phi\|_T \|\tilde{A}\|_T (1 + \|S\|_T) + \|S\|_T < 1. \quad (5)$$

Then, there exists a unique solution of (4).

**Proof** Let $L^2_Q(0, T; \mathbb{R}^m)$ be the Hilbert space of functions endowed with the inner product

$$\langle z, z' \rangle_Q := z_T^* Q_T z'_T + \int_0^T z_t^* Q_t z'_t dt.$$
Given a function $z$ in $L^2_Q(0, T; \mathbb{R}^m)$, and there is a pair $(\xi, \eta)$ satisfying

$$
\begin{align*}
\frac{d\xi_t}{dt} &= A_t \xi_t - B_t R_t^{-1} B_t^* \eta_t + \tilde{A}_t z_t, \\
\xi_0 &= \mathbb{E}[x_0], \\
-\frac{d\eta_t}{dt} &= A_t^* \eta_t + Q_t \xi_t + \tilde{Q}_t (I - S_t) z_t, \\
\eta_T &= Q_T \xi_T + \tilde{Q}_T (I - S_T) z_T,
\end{align*}
$$

(6)

since it corresponds to a well-defined control problem (by referring to the deterministic analog of Theorem 3.1). We remark that Eq. (6) is different from the deterministic counterpart of Eq. (1). The mapping $z \mapsto \xi$ defined in this way is affine and maps $L^2_Q(0, T; \mathbb{R}^m)$ into itself. Our objective was to show that it admits a fixed point. To apply Banach fixed point theorem, it suffices to show that the mapping $z \mapsto \xi$ is a contraction if $\mathbb{E}[x_0] = 0$. By considering the dynamics of $\xi_t^* \eta_t$, we have the following equality:

$$
\begin{align*}
\xi_t^* Q_T \xi_T + \int_0^T \xi_t^* Q_t \xi_t \, dt + \int_0^T \eta_t^* B_t R_t^{-1} B_t^* \eta_t \, dt \\
= \int_0^T \eta_t^* \tilde{A}_t z_t \, dt - \xi_T^* \tilde{Q}_T (I - S_T) z_T - \int_0^T \xi_t^* \tilde{Q}_t (I - S_t) z_t \, dt.
\end{align*}
$$

(7)

Moreover, let $\phi(t; s)$ be the fundamental solution associated with $A_t$, and we get

$$
\eta_t = \phi^*(T, t)(Q_T \xi_T + \tilde{Q}_T (I - S_T) z_T) \\
+ \int_t^T \phi^*(\tau, t)(Q_\tau \xi_\tau + \tilde{Q}_\tau (I - S_\tau) z_\tau) \, d\tau.
$$

By the Cauchy–Schwarz inequality, we have

$$
\|\eta_t\| \leq \|\phi\|_T \left(\|\xi\|_Q + \|Q^{-1/2} \tilde{Q}(I - S) z\|\right) \\
\leq \|\phi\|_T \left(\|\xi\|_Q + \|S\| \|z\|_Q\right).
$$

Therefore, from (7),

$$
\|\xi\|_Q^2 \leq \sqrt{T} \|\phi\|_T \left(\|\xi\|_Q + \|S\| \|z\|_Q\right) \|\tilde{A}\|_T \|z\|_Q + \|\xi\|_Q \|S\|_T \|z\|_Q,
$$

which shows that $z \mapsto \xi$ is a contraction under the condition (5).

**Corollary 3.1** If $S = I$, then $S = 0$, and the previous condition reduces to

$$
\sqrt{T} \|\phi\|_T \|\tilde{A}\|_T < 1.
$$
Remark 3.2 For a single-person optimization problem (i.e., the classical linear-quadratic stochastic control problem), that is, $\bar{A} = S = 0$, we recover the standard existence and uniqueness result in the literature.

Remark 3.3 Assume that $ST = 0$. Then, the nonsingularity of $QT$ is not necessary, and the norm $||S||_T$ can be weaken to $\sup_{0 < t < T} \left\| Q_t^{-1/2} \tilde{Q}_t(I - S_t)Q_t^{-1/2} \right\|$ in applying Theorem 3.3.

Remark 3.4 Suppose that $Q + S$ can be written as $Q + (Q + S - Q)$, where $Q$ is positive definite and is chosen to satisfy suitable conditions stated in Theorem 3.3. Replacing $Q, S$ by $Q$ and $Q + S - Q$, respectively, in the iterative scheme (6) in the proof, a different sufficient condition for the unique existence of the equilibrium strategy is obtained:

$$\sqrt{T} \left\| \phi \right\|_{Q,T} \left\| \bar{A} \right\|_{Q,T} \left(1 + ||S||_{Q,T}\right) + ||S||_{Q,T} < 1,$$

where

$$||\phi||_{Q,T} := \sup_{0 < t < T} \left\| \phi^*(T, t)Q_T^{1/2} \right\|^2 + \int_t^T \left\| \phi^*(s, t)Q_s^{1/2} \right\|^2 ds,$$

$$||\bar{A}||_{Q,T} := \sup_{0 < t < T} \left\| \bar{A}_tQ_t^{-1/2} \right\|,$$

$$||S||_{Q,T} := \sup_{0 < t < T} \left\| Q_t^{-1/2}(Q_t + S_t - Q_t)Q_t^{-1/2} \right\|.$$  

For example, if all the coefficients are constants and $\bar{A} = 0$, then the condition $Q + \tilde{Q}(I - S)$ is positive definite, which provides the desired unique existence by setting $Q := Q + \tilde{Q}(I - S)$.

Remark 3.5 In Appendix, an example will be constructed which illustrates that its unique existence could be covered by our theory but it fails to satisfy the sufficient condition as stated in [44].

3.1 Relationship with Nonsymmetric Riccati Equation

We can look for a solution of Eq. (4) in the form $\tilde{p}_t = \Gamma_t\tilde{y}_t$. Hence, we get the following nonsymmetric Riccati equation:

$$\frac{d\Gamma_t}{dt} + \Gamma_t(A_t + \bar{A}_t) + A_t^*\Gamma_t - \Gamma_tB_tR_t^{-1}B_t^*\Gamma_t + Q_t + S_t = 0, \quad \Gamma_T = Q_T + S_T. \quad (8)$$

If it is solvable, using Remark 3.1, we have $\Gamma_t\tilde{y}_t = \tilde{p}_t = \mathcal{L}_t\tilde{y}_t + \zeta_t$. Therefore, the optimal control $u$ is $-R^{-1}B^*(\mathcal{L}y + (\Gamma - \mathcal{L})\tilde{y})$, and the optimal trajectory $y$ satisfies
\[ \begin{align*}
d{y_t} &= \left[ \left( A_t - B_t R_t^{-1} B_t^* \Xi_t \right) y_t + \left( \tilde{A}_t - B_t R_t^{-1} B_t^* (\Gamma_t - \Xi_t) \right) \bar{y}_t \right] \, dt + \sigma_t \, dW_t, \\
y_0 &= x_0.
\end{align*} \]

However, because of the nonzero term \( \tilde{A}_t \) and \( S_t \), Equation (8) is not the standard Riccati equation. Hence, it is not always solvable, and no natural sufficient condition for the existence of the solution is known (see [45]). Moreover, \( \Gamma_t \) is not necessarily symmetric. Nevertheless, when \( n = 1 \) and \( Q + S \) is nonnegative definite, the nonsymmetric Riccati equation becomes

\[ \begin{align*}
\frac{d\Gamma_t}{dt} + \Gamma_t \left( A_t + \frac{1}{2} \tilde{A}_t \right) + \left( A_t + \frac{1}{2} \tilde{A}_t \right)^* \Gamma_t - \Gamma_t B_t R_t^{-1} B_t^* \Gamma_t + Q_t + S_t &= 0, \\
\Gamma_T &= QT + ST,
\end{align*} \]

which is of the standard form, and the existence result holds. The explicit form of the solution \( \Gamma_t \) can be established in this special case as follows. For the sake of simplicity, assume that all the coefficients are time-independent, and our Riccati equation can be simplified as:

\[ \frac{d\Gamma_t}{dt} + (2A + \tilde{A}) \Gamma_t - B^2 R^{-1} \Gamma_t^2 + Q + S = 0, \quad \Gamma_T = QT + ST. \]

1. For \( B = 0 \), we have

\[ \Gamma_t = \left( QT + ST + \frac{Q + S}{2A + \tilde{A}} \right) \exp \left( (2A + \tilde{A}) (T - t) \right) - \frac{Q + S}{2A + \tilde{A}}, \]

when \( 2A + \tilde{A} \neq 0 \) and

\[ \Gamma_t = (Q + S)(T - t) + QT + ST, \]

when \( 2A + \tilde{A} = 0 \).

2. For \( B \neq 0 \), let \( \alpha \geq 0 \) and \( -\beta \leq 0 \) be the two distinct roots of the quadratic equation

\[ Q + S + (2A + \tilde{A}) \gamma - B^2 R^{-1} \gamma^2 = 0, \]

and the solution can be explicitly written as

\[ \Gamma_t - \alpha = \frac{(Q_T + S_T - \alpha)(\alpha + \beta)}{(Q_T + S_T + \beta) \exp(B^2 R^{-1}(\alpha + \beta)(T - t)) - (Q_T + S_T - \alpha)}, \]

which is well defined.

We remark that for \( n = 1 \), \( Q + S \) is not always nonnegative, and hence, the nonsymmetric Riccati equation is not always solvable, the sufficient condition provided in Theorem 3.3 may have to be invoked. The following proposition, Radon’s lemma in [45], or a time-dependent version of Theorem 4.3 on page 48 in [51], justifies this claim.

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Proposition 3.2  Suppose that the following system of ordinary differential equations

\[ \begin{align*}
\frac{d}{dt} \begin{pmatrix} \xi_t \\ -\eta_t \end{pmatrix} &= \begin{pmatrix} A_t + \bar{A}_t & -B_t R_t^{-1} B_t^* \\ -Q_t - S_t & A_t^* \end{pmatrix} \begin{pmatrix} \xi_t \\ \eta_t \end{pmatrix}, \\
\xi_{t_0} &= 0, \\
\eta_T &= (Q_T + S_T)\xi_T.
\end{align*} \]

admits a unique solution for any \( t_0 \in [0, T] \). Then, there is a unique solution \( \Gamma_t \) of the nonsymmetric Riccati Eq. (8).

Proof  We first rewrite the system of ordinary differential equations as

\[ \frac{d}{dt} \begin{pmatrix} \xi_t \\ \eta_t \end{pmatrix} = \begin{pmatrix} A_t + \bar{A}_t & -B_t R_t^{-1} B_t^* \\ -Q_t - S_t & -A_t^* \end{pmatrix} \begin{pmatrix} \xi_t \\ \eta_t \end{pmatrix}, \]

\[ \xi_{t_0} = 0, \]

\[ \eta_T = (Q_T + S_T)\xi_T. \]

Let \( \Phi(t, s) \) be the fundamental solution of this system of forward–backward ordinary differential equations. Then, we have

\[ 0 = (Q_T + S_T, -I) \begin{pmatrix} \xi_T \\ \eta_T \end{pmatrix} = (Q_T + S_T, -I) \Phi(T, t_0) \begin{pmatrix} 0 \\ \eta_{t_0} \end{pmatrix} = (Q_T + S_T, -I) \Phi(T, t_0) \begin{pmatrix} 0 \\ I \end{pmatrix} \eta_{t_0}. \]

By the unique existence of the solution, the matrix \((Q_T + S_T, -I) \Phi(T, t_0) \begin{pmatrix} 0 \\ I \end{pmatrix}\) is invertible for any \( t_0 \in [0, T] \). By setting

\[ \Gamma_t := -\left[(Q_T + S_T, -I) \Phi(T, t) \begin{pmatrix} 0 \\ I \end{pmatrix}\right]^{-1} \left[(Q_T + S_T, -I) \Phi(T, t) \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right], \]

it can be checked that it solves for the nonsymmetric Riccati Eq. (8).

Corollary 3.2  Assume that \( \|\bar{A}\|_{T_0} < \infty \) and \( \|S\|_{T_0} < \infty \). The nonsymmetric Riccati Eq. (8) is solvable if either one of following conditions is satisfied:

1. \( \|\bar{A}\|_{T_0} = 0 \) and \( \|S\|_{T_0} < 1 \),
2. \( \|\bar{A}\|_{T_0} \neq 0 \) and

\[ T < \left(\frac{1 - \|S\|_{T_0}}{\|\phi\|_{T_0} \|\bar{A}\|_{T_0} (1 + \|S\|_{T_0})}\right)^2 \wedge T_0. \]
In particular, if all the coefficients are time-independent, then the nonsymmetric Riccati Eq. (8) is solvable if either one of following conditions is satisfied:

1. $\|\bar{A}\| = 0$ and $\|S\| < 1$,
2. $\|\bar{A}\| \neq 0$ and

\[ T < \left( \frac{1 - \|S\|}{\|\Phi\| \|\bar{A}\| (1 + \|S\|)} \right)^2. \]

### 3.2 The Case that $n = 2$

When $n = 2$, in general, the existence and uniqueness result of Eq. (4) cannot be guaranteed. For the ease of computation, in this subsection, we will assume that all coefficients are constant matrices of suitable sizes and $S = I$. The latter assumption implies that the existence and uniqueness of the equilibrium do not depend on $\bar{Q}$, and the nonsingularity of $Q_T$ is not required in Theorem 3.3.

#### 3.2.1 When $\bar{A} = -A$

Let

\[
A = \begin{pmatrix} -2.1 & -1.9 \\ -1.2 & 1.7 \end{pmatrix}, \quad R^{-1} = \begin{pmatrix} 2 & 3.1 \\ 3.1 & 4.9 \end{pmatrix}, \quad Q = \begin{pmatrix} 3.6 & -0.6 \\ -0.6 & 0.2 \end{pmatrix},
\]

$\bar{A} = -A$, $B = I$ and $Q_T = 0$. Equation (4) becomes

\[
\frac{d}{dt} \begin{pmatrix} \xi_t \\ \eta_t \end{pmatrix} = \Pi \begin{pmatrix} \xi_t \\ \eta_t \end{pmatrix},
\]

$\eta_T = 0$ and $\xi_0 = E[x_0]$, where

\[
\Pi := \begin{pmatrix} 0 & 0 & 2 & 3.1 \\ 0 & 0 & 3.1 & 4.9 \\ 3.6 & -0.6 & 2.1 & 1.2 \\ -0.6 & 0.2 & 1.9 & -1.7 \end{pmatrix}.
\]

We denote the fundamental solution of this system by

\[
\Phi_t := \begin{pmatrix} \Phi_{11}^t & \Phi_{12}^t \\ \Phi_{21}^t & \Phi_{22}^t \end{pmatrix} = \exp(\Pi \times t).
\]

This equation is (uniquely) solvable if and only if there is a (unique) $\eta_0$ such that

\[
0 = \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} \Phi_{11}^T & \Phi_{12}^T \\ \Phi_{21}^T & \Phi_{22}^T \end{pmatrix} \begin{pmatrix} E[x_0] \\ \eta_0 \end{pmatrix} = \Phi_{21}^T \times E[x_0] + \Phi_{22}^T \times \eta_0.
\]
Numerically, the determinant of \( \Phi_{0.83}^{22} \) and \( \Phi_{0.86}^{22} \) is approximately equal to 0.1244555 and -0.1295142, respectively. Since the function

\[
 t \mapsto \det \left\{ \begin{pmatrix} 0 & I \end{pmatrix} \exp(\Pi \times t) \begin{pmatrix} 0 & I \end{pmatrix} \right\} = \det \left( \Phi_T^{22} \right)
\]

is continuous, there is a scalar \( T_0 \in (0.83, 0.86) \) such that \( \det(\Phi_T^{22}) = 0 \), and hence, Eq. (4) does not have any unique solution when \( T = T_0 \). In fact, the singularity of \( \Phi_T^{22} \) shows that Eq. (9) cannot be solvable for any \( E[x_0] \) when \( T = T_0 \). Assume otherwise, that is, \( \mathcal{R}(\Phi_T^{21}) \subseteq \mathcal{R}(\Phi_T^{22}) \), where the range and the null space of a matrix \( M \) are denoted by \( \mathcal{R}(M) \) and \( \mathcal{N}(M) \), respectively. By the standard result in linear algebra, we have \( \mathcal{N}(\Phi_T^{22})^* \subseteq \mathcal{N}(\Phi_T^{21})^* \). Choose \( T = T_0 \) as defined above, and hence, \( \mathcal{N}(\Phi_T^{22})^* \) is nonempty. It shows that

\[
 \text{rank}(\Phi_{T_0}) = \text{rank} \left( \begin{pmatrix} \Phi_{T_0}^{11} & \Phi_{T_0}^{12} \\ \Phi_{T_0}^{21} & \Phi_{T_0}^{22} \end{pmatrix} \right) = \text{rank} \left( \begin{pmatrix} \Phi_{T_0}^{11} \ast \Phi_{T_0}^{21} \\ \Phi_{T_0}^{12} \ast \Phi_{T_0}^{22} \end{pmatrix} \right) \leq 2n - 1,
\]

which contradicts the invertibility of \( \Phi_{T_0} \), and the claim holds.

For the general existence issue, it suffices to show that \( \Phi_{T_0}^{21} \) is nonsingular. In fact, we can choose \( E[x_0] \) such that \( \Phi_{T_0}^{21} \times E[x_0] \) is not in the range of \( \Phi_{T_0}^{22} \), and hence, Eq. (9) is not solvable (Fig. 1).

**Fig. 1** Relationship between the time variable \( t \) and the function \( D(t) := \det(\Phi_T^{21}) \) on the interval \([0, 1]\). \( D(t) \) is strictly larger than zero in the interval \((0.83, 0.86)\) so that \( \Phi_{T_0}^{21} \) is clear nonsingular. Therefore, the general existence of the solution of Eq. (9) follows.
3.2.2 An Arbitrary Case, when $\tilde{A} \neq -A$

Let

$$A = \begin{pmatrix} -0.4 & -0.6 \\ 0.4 & -0.1 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} 1.5 & 0.7 \\ -0.1 & -0.3 \end{pmatrix},$$

$$R^{-1} = \begin{pmatrix} 1.2 & 2.2 \\ 2.2 & 4.4 \end{pmatrix}, \quad Q = \begin{pmatrix} 6.6 & -2.8 \\ -2.8 & 1.2 \end{pmatrix},$$

$$B = I$$ and $$Q_{T} = 0.$$ Equation (4) becomes

$$\frac{d}{dt} \begin{pmatrix} \xi_t \\ \eta_t \end{pmatrix} = \Pi \begin{pmatrix} \xi_t \\ \eta_t \end{pmatrix},$$

$$\eta_T = 0$$ and $$\xi_0 = \mathbb{E}[x_0],$$ where

$$\Pi := \begin{pmatrix} 1.1 & 0.1 & 1.2 & 2.2 \\ 0.3 & -0.4 & 2.2 & 4.4 \\ 6.6 & -2.8 & 0.4 & -0.4 \\ -2.8 & 1.2 & 0.6 & 0.1 \end{pmatrix}.$$  

Numerically, the determinant of $\Phi_{22}^{i}$ is approximately equal to $-0.3582768.$ Similar to the previous discussion, there is a scalar $T_0 > 0$ such that Eq. (4) does not have any unique solution, and there is no existence result for all initial points $\mathbb{E}[x_0].$

4 $\epsilon$-Nash Equilibrium

In this section, we shall show that the equilibrium strategy $u^i$ of Problem 2.2 is an $\epsilon$-Nash equilibrium of Problem 2.1. For more inspiring elaboration on the notion of $\epsilon$-Nash equilibrium, one can refer to, for example, [40] and [7, 8, 46]. Because of the permutation symmetry, it suffices to consider Player 1. In this section, $K$ will denote a generic constant, being independent of time, which may be different in line by line. In order to show that $(u^1, \ldots, u^N)$ is an $\epsilon$-Nash equilibrium, it is crucial to prove that for any $\epsilon > 0,$ there is a positive integer $N_0$ such that when $N \geq N_0,$ we have

$$J^1(v^1, u^2, \ldots, u^N) \geq J^1(u^1, \ldots, u^N) - \epsilon,$$

(10)

for any admissible control $v^1.$

To begin with, we first approximate $y^i,$ $1 \leq i \leq N,$ where

$$dy^i_t = \left( A_t y^i_t + B_t u^i_t + \tilde{A}_t \times \frac{1}{N-1} \sum_{j=1, j \neq i}^N y^j_t \right) dt + \sigma_t dW^i_t,$$

$$y^i_0 = x^i_0.$$
Proposition 4.1 As $N \to \infty$, we have
\[
\sup_{1 \leq i \leq N} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \| y^i_t - \hat{y}^i_t \|_2^2 \right] = O \left( \frac{1}{N} \right).
\]

Proof Recall that $\hat{y}$ is defined right after Problem 2.2, we first note that
\[
d(y^i_t - \hat{y}^i_t) = \left( A_t (y^i_t - \hat{y}^i_t) + \bar{A}_t \times \frac{1}{N-1} \sum_{j=1, j \neq i}^N (y^j_t - \mathbb{E}[\hat{y}^j_t]) \right) dt,
\]
\[
y^0_i - \hat{y}^0_i = 0,
\]
as $\hat{y}^1, \ldots, \hat{y}^N$ are i.i.d.. Taking square on both sides, we have
\[
\| y^i_t - \hat{y}^i_t \|_2^2 \leq K \int_0^T \left( \| y^i_s - \hat{y}^i_s \|_2^2 + \frac{1}{(N-1)^2} \left\| \sum_{j=1, j \neq i}^N (y^j_s - \mathbb{E}[\hat{y}^j_s]) \right\|_2^2 \right) ds
\]
\[
+ K \int_0^T \frac{1}{(N-1)^2} \left\| \sum_{j=1, j \neq i}^N (\hat{y}^j_s - \mathbb{E}[\hat{y}^j_s]) \right\|_2^2 ds.
\]
By first applying Jensen’s inequality to the second term in the first integrand and taking expectations on both sides, as $y^1 - \hat{y}^1, \ldots, y^N - \hat{y}^N$ are identically distributed and $\hat{y}^1, \ldots, \hat{y}^N$ are i.i.d., we have
\[
\mathbb{E} \left[ \| y^i_t - \hat{y}^i_t \|_2^2 \right] \leq K \int_0^T \left( \mathbb{E} \left[ \| y^i_s - \hat{y}^i_s \|_2^2 \right] + \frac{1}{N-1} \mathbb{E} \left[ \| \hat{y}^i_s - \mathbb{E}[\hat{y}^i_s] \|_2^2 \right] \right) ds.
\]
By the Gronwall’s inequality,
\[
\mathbb{E} \left[ \| y^i_t - \hat{y}^i_t \|_2^2 \right] \leq K \times \frac{1}{N-1} \int_0^T \mathbb{E} \left[ \| \hat{y}^i_s - \mathbb{E}[\hat{y}^i_s] \|_2^2 \right] ds,
\]
as desired, and note that the value of the last integral is independent of $i$. \qed

Note that the previous estimates are standard in the McKean–Vlasov Model; see, for example, [52]. The following corollary is crucial in proving that $(u^1, \ldots, u^N)$ is an $\epsilon$-Nash equilibrium.

Corollary 4.1 As $N \to \infty$, we have
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \| y^1_t \|_2^2 - \| \hat{y}^1_t \|_2^2 \right] = O \left( \frac{1}{\sqrt{N}} \right).
\]
Proof

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \|y_1^t\|^2 - \|\hat{y}_1^t\|^2 \right] \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} \|y_1^t - \hat{y}_1^t\|^2 \right] + 2 \sqrt{\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|\hat{y}_1^t\|^2 \right] \mathbb{E} \left[ \sup_{0 \leq t \leq T} \|y_1^t - \hat{y}_1^t\|^2 \right]}, \]

by the Cauchy–Schwarz inequality, and the result follows by applying Proposition 4.1.

\[ \square \]

Similarly, we have

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \frac{1}{N-1} \sum_{j=2}^{N} y^j_t \right|^2 - \left| \frac{1}{N-1} \sum_{j=2}^{N} \hat{y}^j_t \right|^2 \right] = O \left( \frac{1}{\sqrt{N}} \right), \]

as \( \hat{y}^1, \ldots, \hat{y}^N \) are i.i.d. Combining, we have

\[ J^1(\hat{\nu}^1, \ldots, \hat{\nu}^N) = J^1(\hat{\nu}^1) + O \left( \frac{1}{\sqrt{N}} \right). \] (11)

Because of the positive (nonnegative) definiteness of the matrix-valued parameters,

\[ J^1(\nu^1, \nu^2, \ldots, \nu^N) \geq \mathbb{E} \left[ \frac{1}{2} \int_0^T (\nu^1_t)^* R_1 \nu^1_t \, dt \right] \geq \frac{\delta}{2} \mathbb{E} \left[ \int_0^T \|\nu^1_t\|^2 \, dt \right], \]

it suffices to consider the admissible controls \( \nu^1_t \) satisfying

\[ \mathbb{E} \left[ \int_0^T \|\nu^1_t\|^2 \, dt \right] \leq \frac{2}{\delta} (J^1(\nu^1) + 1), \]

otherwise the inequality (10) holds trivially.

Having the \( L^2 \)-boundedness of the admissible controls, we are now ready to estimate \( J^1(\nu^1, \nu^2, \ldots, \nu^N) \). Recall that

\[
\begin{align*}
\, dx_1^t & = \left( A_t x_1^t + B_t \nu_1^t + \bar{A}_t \times \frac{1}{N-1} \sum_{j=2}^{N} x_j^t \right) dt + \sigma_t \, dW_1^t, \\
\, x_1^0 & = \xi^1,
\end{align*}
\]
and for $i \neq 1$,
\[
dx_i^t = \left( A_t x_i^t + B_t u_i^t + \tilde{A}_t \times \frac{1}{N-1} \left( \sum_{j=2, j \neq i}^N x_j^t + x_i^t \right) \right) dt + \sigma_i \, dW_i^t, \\
x_0^i = \xi_i.
\]

For $i \neq 1$, we claim that $x_i^t$ can be approximated by $\tilde{x}_i^t$, where
\[
dx_i^t = \left( A_t \tilde{x}_i^t + B_t u_i^t + \tilde{A}_t \times \frac{1}{N-1} \sum_{j=2, j \neq i}^N \tilde{x}_j^t \right) dt + \sigma_i \, dW_i^t, \\
x_0^i = \xi_i.
\]

The following proposition justifies this claim.

**Proposition 4.2** As $N \to \infty$, we have
\[
\sup_{2 \leq i \leq N} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \|x_i^t - \tilde{x}_i^t\|^2 \right] = O \left( \frac{1}{N^2} \right).
\]

**Proof** For each $i \neq 1$,
\[
\|x_i^t\|^2 \leq K \|\xi_i\|^2 + K \int_0^t \left( \|x_i^s\|^2 + \|u_i^s\|^2 + \frac{1}{(N-1)^2} \left\| \sum_{j=1, j \neq i}^N x_j^s \right\|^2 \right) ds \\
+ K \left\| \int_0^t \sigma_s \, dW_s^i \right\|^2,
\]
and
\[
\|x_1^t\|^2 \leq K \|\xi_1\|^2 + K \int_0^t \left( \|x_1^s\|^2 + \|v_1^s\|^2 + \frac{1}{(N-1)^2} \left\| \sum_{j=2}^N x_j^s \right\|^2 \right) ds \\
+ K \left\| \int_0^t \sigma_s \, dW_s^1 \right\|^2.
\]
Taking the summation of \( i \) from 1 to \( N \) on both sides and then using the same argument as in the proof of Proposition 4.1, we have

\[
\mathbb{E} \left[ \sum_{i=1}^{N} \|x_i^t\|^2 \right] \leq K \mathbb{E} \left[ \sum_{i=1}^{N} \|\xi^i\|^2 \right] + K \int_0^t \left( \mathbb{E} \left[ \sum_{i=1}^{N} \|x_i^s\|^2 \right] + \mathbb{E} \left[ \|v_i^s\|^2 \right] + \mathbb{E} \left[ \sum_{i=2}^{N} \mathbb{E} \left[ \|u_i^s\|^2 \right] \right] \right) ds
\]

\[
+ K \sum_{i=1}^{N} \mathbb{E} \left[ \left\| \int_0^t \sigma_s^i dW_s^i \right\|^2 \right],
\]

which shows that \( \mathbb{E} \left[ \sum_{i=1}^{N} \|x_i^t\|^2 \right] = \mathcal{O}(N) \) uniformly for all \( t \). Therefore, \( \mathbb{E} \left[ \sup_{0 \leq t \leq T} \|x_1^t\|^2 \right] \) is bounded uniformly with respect to \( N \).

Now, for \( i \neq 1 \),

\[
d(x_i^t - \tilde{x}_i^t) = A_t(x_i^t - \tilde{x}_i^t)dt + \left( \tilde{A}_t \times \sum_{j=2, j \neq i}^{N} (x_j^t - \tilde{x}_j^t) \right) dt + \tilde{A}_t \times \frac{x_1^t}{N-1} dt,
\]

\[
x_i^0 - \tilde{x}_i^0 = 0.
\]

Therefore,

\[
\|x_i^t - \tilde{x}_i^t\|^2
\]

\[
\leq K \int_0^t \|x_i^s - \tilde{x}_i^s\|^2 ds + K \int_0^t \left( \frac{\| \sum_{j=2, j \neq i}^{N} (x_j^s - \tilde{x}_j^s) \|^2}{(N-1)^2} + \frac{1}{(N-1)^2} \|x_1^s\|^2 \right) ds
\]

\[
\leq K \int_0^t \|x_i^s - \tilde{x}_i^s\|^2 ds + K \int_0^t \left( \frac{\| \sum_{j=2, j \neq i}^{N} (x_j^s - \tilde{x}_j^s) \|^2}{N-1} + \frac{1}{(N-1)^2} \|x_1^s\|^2 \right) ds.
\]

Taking summation over \( i \) from 2 to \( N \), by applying Gronwall’s inequality again, we have

\[
\mathbb{E} \left[ \sum_{i=2}^{N} \|x_i^t - \tilde{x}_i^t\|^2 \right] \leq K \times \frac{1}{N-1} \int_0^T \|x_1^s\|^2 ds.
\]

As \( x_i^t - \tilde{x}_i^t, i \neq 1 \), are identically distributed, we have \( \mathbb{E}[\|x_1^t - \tilde{x}_1^t\|^2] = \mathcal{O} \left( \frac{1}{N^2} \right) \) uniformly for all \( i \) and \( t \), as desired.

Moreover, we have

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|x_1^t - \tilde{x}_1^t\|^2 \right] = \mathcal{O} \left( \frac{1}{N} \right),
\]
and indeed,

\[
d(x^1_t - \hat{x}^1_t) = \left(A_t(x^1_t - \hat{x}^1_t) + \tilde{A}_t \times \frac{1}{N-1} \sum_{i=2}^{N} (x_i - E[\hat{y}^i_t]) \right) \, dt,
\]

\[
x^1_0 - \hat{x}^1_0 = 0,
\]

and hence,

\[
\|x^1_t - \hat{x}^1_t\|^2 \leq K \int_0^t \left( \|x^1_s - \hat{x}^1_s\|^2 + \frac{1}{N-1} \sum_{i=2}^{N} \|x^i_s - \hat{y}^i_s\|^2 \right) \, ds
\]

\[
+ K \int_0^t \frac{1}{(N-1)^2} \left\| \sum_{i=2}^{N} (\hat{y}^i_s - E[\hat{y}^i_s]) \right\|^2 \, ds.
\]

By applying Gronwall’s inequality, the claim can be deduced. In conclusion, by combining all these estimates, the main claim in this section follows.

**Theorem 4.1** \((u^1, \ldots, u^N)\) is an \(\epsilon\)-Nash equilibrium of Problem 2.1.

**Proof** First recall that we have proved

\[
J^1(u^1, \ldots, u^N) = J^1(u^1) + O \left( \frac{1}{\sqrt{N}} \right)
\]

in Eq. (11). Based on the above estimates, we have

\[
E \left[ \sup_{0 \leq t \leq T} \|x^1_t - \hat{x}^1_t\|^2 \right] = O \left( \frac{1}{N} \right),
\]

\[
E \left[ \sup_{0 \leq t \leq T} \|x^i_t - \hat{y}^i_t\|^2 \right] = O \left( \frac{1}{N} \right), \text{ for } i \neq 1.
\]

Using a similar argument as in the proof of Eq. (11), we have

\[
J^1(v^1, u^2, \ldots, u^N) = J^1(v^1) + O \left( \frac{1}{\sqrt{N}} \right)
\]

\[
\geq J^1(u^1) + O \left( \frac{1}{\sqrt{N}} \right)
\]

\[
= J^1(u^1, \ldots, u^N) + O \left( \frac{1}{\sqrt{N}} \right),
\]

as required. \(\square\)
5 Mean Field Type Linear-Quadratic Stochastic Control Problems

In this section, we will use the technique employed in the previous section to find the optimal control of the mean field type counterpart of problem 2.2.

**Problem 5.1** Let $\xi$ be a random vector that is identically distributed to $\xi^1$ and independent to $W$. The objective was to find an optimal control $u$ which minimizes

\[
J(v) := \mathbb{E}\left[\frac{1}{2} \int_0^T x_t^* Q_t x_t + v_t^* R_t v_t + (x_t - S_t \mathbb{E}[x_t])^* \tilde{Q}_t (x_t - S_t \mathbb{E}[x_t]) \, dt \right] 
\]

where the dynamics is given by

\[
dx_t = (A_t x_t + B_t v_t + \tilde{A}_t \mathbb{E}[x_t]) \, dt + \sigma_t \, dW_t, \quad x(0) = x_0,
\]

and $v$ is a control in $L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$.

In order to solve this optimization problem, we still adopt the adjoint equation approach.

**Theorem 5.1** Problem 5.1 is uniquely solvable if there is a unique solution $(y, p)$ of the following linear mean field FBSDE:

\[
d \begin{pmatrix} y_t \\ -p_t \end{pmatrix} = \begin{pmatrix} A_t & -B_t R_t^{-1} B_t^* \\ Q_t + \tilde{Q}_t & \tilde{A}_t \mathbb{E}[y_t] \end{pmatrix} \begin{pmatrix} y_t \\ p_t \end{pmatrix} \, dt + \begin{pmatrix} \sigma_t \\ q_t \end{pmatrix} \, dW_t, \\
y_0 = x_0, \\
p_T = (Q_T + \tilde{Q}_T)y_T - \tilde{Q}_T S_T \mathbb{E}[y_T] - S_T^* \tilde{Q}_T (I - S_T) \mathbb{E}[y_T].
\]

In this case, the optimal control is given by $u = -R^{-1} B^* p$.

**Proof** It is an immediate consequence of Theorem 4.1 in [47]. Note that the assumption of nonnegativity is not needed because of the linearity of the mean field term. See also our Theorem 3.1. \hfill \qed

By Theorem 3.1, Eq. (12) is uniquely solvable if and only if there is an unique pair $(\tilde{y}, \tilde{p})$ solving the following system of ordinary differential equations:

\[
d \begin{pmatrix} \tilde{y}_t \\ -\tilde{p}_t \end{pmatrix} = \begin{pmatrix} A_t + \tilde{A}_t & -B_t R_t^{-1} B_t^* \\ Q_t + (I - S_t)^* \tilde{Q}_t (I - S_t) & \tilde{A}_t^* + \tilde{A}_t^* \end{pmatrix} \begin{pmatrix} \tilde{y}_t \\ \tilde{p}_t \end{pmatrix}, \\
\tilde{y}_0 = \bar{x}_0, \\
\tilde{p}_T = (Q_T + (I - S_T)^* \tilde{Q}_T (I - S_T)) \tilde{y}_T.
\]
where \( \bar{x}_0 \) is defined to be \( \mathbb{E}[x_0] \). Unlike to Eq. 4, From the positivity of \( (Q_T + (I - S_T)^* \tilde{Q}_T (I - S_T))^* \), it is a standard result of control theory that the system has one and only one solution. Therefore, the mean field type linear-quadratic stochastic control problem is uniquely solvable.

6 Comparison of Problems 2.2 and 5.1

We will now compare the equilibrium strategy of MFG and the optimal control of mean field type stochastic control problem when \( n = 1 \) and \( S = I \). We assume that all the coefficients are constant. Since the equilibrium strategy and the optimal control are of the same form, they are different if we can show that \( \psi_1(T) \neq \psi_2(T) \), where \( (\varphi_1, \psi_1) \) satisfies

\[
\frac{d}{dt} \begin{pmatrix} \varphi_1 \\ -\psi_1 \end{pmatrix} = \begin{pmatrix} A + \tilde{A} & -BR^{-1}B^* \\ Q & A^* \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix},
\]

\[
\varphi_1(0) = \bar{x}_0,
\]

\[
\psi_1(T) = Q_T \varphi_1(T),
\]

and \( (\varphi_2, \psi_2) \) satisfies

\[
\frac{d}{dt} \begin{pmatrix} \varphi_2 \\ -\psi_2 \end{pmatrix} = \begin{pmatrix} A + \tilde{A} & -BR^{-1}B^* \\ Q & A^* + \tilde{A}^* \end{pmatrix} \begin{pmatrix} \varphi_2 \\ \psi_2 \end{pmatrix},
\]

\[
\varphi_2(0) = \bar{x}_0
\]

\[
\psi_2(T) = Q_T \varphi_2(T).
\]

To illustrate their differences, we suppose that \( Q = \tilde{Q} = 0, \tilde{A} \neq 0, R = 1, Q_T = 1 \) and \( B \neq 0 \). Solving the equations, we have \( \psi_1(t) = \psi_1(0)e^{-\tilde{A}t} \) and \( \psi_2(t) = \psi_2(0)e^{-(A+\tilde{A})t} \), and the condition that \( \psi_1(T) \neq \psi_2(T) \) is reduced to \( \varphi_1(T) \neq \varphi_2(T) \).

Solving

\[
\frac{d\psi_1}{dt} = (A + \tilde{A})\varphi_1 - B^2\varphi_1(T)e^{A(T-t)},
\]

we have

\[
\varphi_1(T) - \bar{x}_0 = -B^2\varphi_1(T)e^{AT} \frac{1}{2A + \tilde{A}} \left( 1 - e^{(2A+\tilde{A})T} \right).
\]

In a similar fashion, solving

\[
\frac{d\psi_2}{dt} = (A + \tilde{A})\varphi_2 - B^2\varphi_2(T)e^{(A+\tilde{A})(T-t)},
\]
we have
\[ e^{-(A + \tilde{A})^T} \varphi_2(T) - \tilde{x}_0 = -B^2 \varphi_2(T) e^{(A + \tilde{A})^T} \frac{1}{2A + 2\tilde{A}} \left(1 - e^{-2(A + 2\tilde{A})T}\right). \]

Therefore, the condition that \( \varphi_1(T) \neq \varphi_2(T) \) is reduced to
\[ \frac{1}{2A + \tilde{A}} \left(1 - e^{-2(A + \tilde{A})T}\right) \neq e^{\tilde{A}T} \frac{1}{2A + 2\tilde{A}} \left(1 - e^{-2(A + 2\tilde{A})T}\right). \]

It can be seen that this condition holds if we choose \( A \) sufficiently large, and we conclude that the equilibrium strategy is in general different from the optimal control.

### 7 Conclusions

In virtue of the linear structure of the adjoint equations, we show that the mean field term satisfies the forward–backward ordinary differential Eq. (4) in which, unlike the classical Riccati equation approach, our proposed argument could be much easier to be extended in the higher dimensional settings. In particular, our proposed adjoint equation approach can be freely used to facilitate more advanced studies such as those recently considered in [5, 32, 37] and [53].

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### Appendix: Comparison of the Approaches of [44] and the Present Paper

We consider the single-agent problem studied by [44] (HCM), that is, the distribution \( F(a) \) in HCM is now a Dirac distribution, to compare the approaches of HCM and the present paper (BSYY) on the same problem. More precisely, we want to find the optimal control \( u \) which minimizes the cost functional, for \( \gamma \in \mathbb{R}^{n \times n} \) and \( \eta \in \mathbb{R}^n \),
\[ J(u) = \mathbb{E} \left[ \int_0^T |z_t - \gamma(\tilde{z}_t + \eta)|^2 + ru_t^2 \, dt \right], \quad (13) \]
where
\[ dz_t = (az_t + bu_t) \, dt + \alpha \tilde{z}_t \, dt + \sigma \, dw_t, \quad z(0) = z_0, \quad (14) \]

\( \tilde{z}_t \) is fixed, deterministic, and \( z_0 \) is a random variable with zero mean, independent of the Wiener process. At the second stage, we consider a fixed point problem
\[ \bar{z}_t = \mathbb{E}[z_t], \quad (15) \]

where \( z_t \) is the optimal state of the control problem (13), (14). In the sequel, we shall describe both approaches in detail to make the comparison explicit. As in HCM, we use the notation \( z_t^* := \gamma(\bar{z}_t + \eta) \).

**HCM Approach**

For given \( \bar{z}_t \), one solves the stochastic control problem (13), (14) by the Riccati differential equation approach. The optimal control is given by
\[ u_t = -\frac{b}{r}(\Pi_t z_t + s_t), \quad (16) \]

where \( \Pi_t \) is the positive solution of the Riccati equation
\[ \frac{d\Pi_t}{dt} + 2a\Pi_t - \frac{b^2}{r}\Pi_t^2 + 1 = 0, \quad \Pi_T = 0, \quad (17) \]

and \( s_t \) solves the linear differential equation
\[ \frac{ds_t}{dt} + \left( a - \frac{b^2}{r}\Pi_t \right) s_t + \alpha \Pi_t \tilde{z}_t - z_t^* = 0, \quad s_T = 0. \quad (18) \]

Therefore, from (14), the optimal trajectory satisfies
\[ dz_t = \left( a - \frac{b^2}{r}\Pi_t \right) z_t \, dt + \left( -\frac{b^2}{r}s_t + \alpha \tilde{z}_t \right) \, dt + \sigma \, dw_t, \quad z(0) = z_0. \]

Furthermore, to satisfy (15), we must have
\[ \frac{d\bar{z}_t}{dt} = \left( a - \frac{b^2}{r}\Pi_t \right) \bar{z}_t - \frac{b^2}{r}s_t + \alpha \bar{z}_t, \quad \bar{z}(0) = 0, \quad (19) \]

and it suffices to find a solution of the deterministic system (18), (19). We now state the sufficient condition in HCM that guarantees the unique solution for the system (18), (19).
HCM Proof

Introduce

$$
\Phi(t, \tau) = \exp \left( - \int_{\tau}^{t} \left( a - \frac{b^2}{r} \Pi_{\sigma} \right) \, d\sigma \right),
$$

and note that $t$ is not necessarily greater than $\tau$. Solving (18) by the formula

$$
s_t = \int_{t}^{T} \Phi(t, \tau) \left( (\alpha \Pi_{\tau} - \gamma) \bar{z}_{\tau} - \gamma \eta \right) \, d\tau,
$$

and substituting in (19) yields

$$
\bar{z}_t = \int_{0}^{t} \Phi(\sigma, t) \left\{ \alpha \bar{z}_{\sigma} - \frac{b^2}{r} \int_{\sigma}^{T} \Phi(\sigma, \tau) \left( (\alpha \Pi_{\tau} - \gamma) \bar{z}_{\tau} - \gamma \eta \right) \, d\tau \right\} \, d\sigma. \tag{20}
$$

The problem is reduced to find a fixed point of Eq. (20), and HCM uses the contraction principle to solve for it in the space $C(0, T)$.

For simplicity, let $\eta = 0$. Consider a continuous function $\varphi$ and the linear map

$$
\Gamma \varphi(t) = \int_{0}^{t} \Phi(\sigma, t) \left\{ \alpha \varphi_{\sigma} - \frac{b^2}{r} \int_{\sigma}^{T} \Phi(\sigma, \tau) \left( (|\alpha| \Pi_{\tau} + |\gamma|) \varphi_{\tau} \right) \, d\tau \right\} \, d\sigma,
$$

then the norm $\| \Gamma \|$ must be required to be strictly less than 1. Since $\Phi$ and $\Pi$ are positive, we have

$$
|\Gamma \varphi(t)| \leq \| \varphi \| \int_{0}^{t} \Phi(\sigma, t) \left\{ |\alpha| + \frac{b^2}{r} \int_{\sigma}^{T} \Phi(\sigma, \tau) (|\alpha| \Pi_{\tau} + |\gamma|) \, d\tau \right\} \, d\sigma,
$$

and thus, the assumption

$$
\| \Gamma \| \leq \sup_{0 < t < T} \int_{0}^{t} \Phi(\sigma, t) \left\{ |\alpha| + \frac{b^2}{r} \int_{\sigma}^{T} \Phi(\sigma, \tau) (|\alpha| \Pi_{\tau} + |\gamma|) \, d\tau \right\} \, d\sigma < 1, \tag{21}
$$

guarantees the contraction property.

Remark 7.1 There are three typos in the statement of the condition in HCM. On Page 170 (although the previous one is corrected), $\frac{b^2}{r}$ appears as a multiplicative factor for the whole right-hand side of (21), which should not be. Also the integral sign $\int_{0}^{T}$ is written as $\int_{T}^{0}$. On Page 171, the integral sign $\int_{\sigma}^{T}$ is written as $\int_{0}^{T}$.

Correcting these typos, this is the HCM result in the present framework.
The present paper uses the stochastic maximum principle to solve (13), (14). The optimal control is

\[ u_t = -\frac{b}{r} \rho_t, \]  

(22)

where \( \rho_t = \mathbb{E}[\omega_t | \mathcal{F}_t] \), \( \mathcal{F}_t \) is the filtration generated by \( z_0 \), and the Wiener process up to time \( t \) and \( \omega_t \) solves the adjoint equation

\[ -\frac{d\omega_t}{dt} = a\omega_t + z_t - z_t^*, \quad \omega_T = 0. \]

Combining the following system of necessary conditions holds:

\[ dz_t = \left( a z_t - \frac{b^2}{r} \rho_t + \alpha \bar{z}_t \right) dt + \sigma dw_t, \]

\[ z(0) = z_0, \]

\[ -\frac{d\omega_t}{dt} = a\omega_t + z_t - z_t^*, \]

\[ \omega_T = 0, \]

(23)

where \( \rho_t = \mathbb{E}[\omega_t | \mathcal{F}_t] \). It is well known that \( \rho_t = \prod_t z_t + s_t \) and (16), (22) coincide. The difference concerns the fixed point argument.

In our approach, we define \( \bar{z}_t = \mathbb{E}[z_t] \) and \( \bar{\rho}_t = \mathbb{E}[\rho_t] = \mathbb{E}[\omega_t] \). Therefore, (23) becomes

\[ \frac{d\bar{z}_t}{dt} = (a + \alpha)\bar{z}_t - \frac{b^2}{r} \bar{\rho}_t, \]

\[ \bar{z}(0) = 0, \]

\[ -\frac{d\bar{\rho}_t}{dt} = a\bar{\rho}_t + (1 - \gamma)\bar{z}_t - \gamma \eta, \]

\[ \bar{\rho}_T = 0, \]

(24)

which we shall solve. Again, \( \bar{\rho}_t = \Pi_t \bar{z}_t + s_t \), and the systems (24) and (18), (19) are equivalent.

**BSYY Proof and Nonsymmetric Riccati Equation**

The trouble with the relation \( \bar{\rho}_t = \Pi_t \bar{z}_t + s_t \) is that it does not express the adjoint variable \( \bar{\rho}_t \) as an affine function of \( \bar{z}_t \) alone. In fact, we need both \( \bar{z}_t \) and \( s_t \), which is a coupled system, to be solved first, as done in HCM. Our approach is to express \( \bar{\rho}_t \) as an affine function on \( \bar{z}_t \) only. We write
\[ \bar{p}_t = P_t \bar{z}_t + \rho_t, \quad (25) \]

and by identification, we obtain:

\[
\begin{align*}
\frac{dP_t}{dt} &= -(2a + \alpha)P_t + \frac{b^2}{r} P_t^2 - 1 + \gamma, \quad P_T = 0, \\
\frac{d\rho_t}{dt} &= -\left( a - \frac{b^2}{r} P_t \right) \rho_t + \gamma \eta, \quad \rho_T = 0. \quad (26)
\end{align*}
\]

The Riccati Eq. (26) is different from (17), and it is called a nonsymmetric Riccati equation because in dimension larger than 1, it leads to (nonstandard) nonsymmetric Riccati equations. Solving (24) amounts to solve the Riccati equation (26), since using (25) in (24), \( \bar{z}_t \) is a solution of linear equation.

If we assume that

\[ \gamma \leq 1 \quad (27) \]

(which is independent of the choice of \( b \)), then the second-order equation

\[ -\frac{b^2}{r} \varsigma^2 + (2a + \alpha)\varsigma + 1 - \gamma = 0 \]

has two roots \( \varsigma_1 \geq 0, \varsigma_2 \leq 0 \), and the solution of the Riccati equation is

\[ P_t = \frac{(1 - \gamma)r}{b^2} \exp \left( \frac{(\varsigma_1 - \varsigma_2) b^2}{r} (T - t) \right) - 1 \]

\[ \varsigma_1 \varsigma_2 \exp \left( \frac{(\varsigma_1 - \varsigma_2) b^2}{r} (T - t) \right). \]

We can compare assumption (27) with respect to assumption (21), in obtaining the fixed point property. For instance, take \( a = 0, \alpha = 0 \) and \( r = 1 \), Condition (21) means that

\[ \sup_{0 \leq t \leq T} b^2 |\gamma| \int_0^t \Phi(\sigma, t) \left( \int_\sigma^T \Phi(\sigma, \tau) d\tau \right) d\sigma < 1 \quad (28) \]

where

\[ \Phi(t, \tau) = \exp \left( b^2 \int_\tau^t \Pi_\sigma d\sigma \right) \]

and

\[ \frac{d\Pi_t}{dt} - b^2 \Pi_t^2 + 1 = 0, \quad \Pi_T = 0. \]

Thus,

\[ b \Pi_t + 1 = \frac{2}{1 + e^{-2b(T-t)}}, \]

and from (28),

\[ \text{Springer} \]
\[ \sup_{0 < t < T} b^2 |\gamma| \int_0^t \exp \left( \int_t^\sigma b^2 \Pi_\lambda \, d\lambda \right) \left[ \int_\sigma^T \exp \left( \int_{\sigma}^{\tau} b^2 \Pi_\mu \, d\mu \right) \, d\tau \right] \, d\sigma < 1, \]

which means that
\[ \sup_{0 < t < T} b^2 |\gamma| \int_0^t \exp \left( - \int_\sigma^t b^2 \Pi_\lambda \, d\lambda \right) \left[ \int_\sigma^T \exp \left( - \int_\sigma^{\tau} b^2 \Pi_\mu \, d\mu \right) \, d\tau \right] \, d\sigma < 1. \] (29)

Since
\[ b \Pi_t = \frac{1 - e^{-2b(T-t)}}{1 + e^{-2b(T-t)}} < 1, \]

from (29), we obtain
\[
1 > \sup_{0 < t < T} b^2 |\gamma| \int_0^t e^{-b(t-\sigma)} \left\{ \int_\sigma^T e^{-b(\tau-\sigma)} \, d\tau \right\} \, d\sigma \\
= \sup_{0 < t < T} b |\gamma| \int_0^t e^{-b(t-\sigma)} (1 - e^{-b(T-\sigma)}) \, d\sigma \\
= |\gamma| \sup_{0 < t < T} \left( 1 - e^{-bt} - \frac{1}{2} e^{-b(T-t)} + \frac{1}{2} e^{-b(T+t)} \right),
\]

which amounts to
\[ |\gamma| (1 - e^{-bT}) < 1. \] (30)

Obviously, (27) and (30) are not equivalent. We remark that in order to guarantee the existence uniformly for arbitrarily choice of \( b \), (30) only holds when \( |\gamma| \leq 1 \).

**Generalization to High Dimensions**

Both approaches can be considered in \( n \) dimension. However, the condition for the existence and uniqueness of the fixed point in the HCM approach becomes extremely difficult to be checked, since it involves the solution of Riccati equations. Our approach leads to conditions which are much easier to be verified and also introduces an interesting and new direction for solvable nonsymmetric Riccati equations that do not correspond to any usual control problems at all. The contribution of BSYY provides a complement to HCM theory, with insights which deserve to be known.

**References**

1. Kirman, A.: Ants, rationality, and recruitment. Q. J. Econ. **108**(1), 137–156 (1993)
2. Elliott, R.J., Kalton, N.J., Markus, L.: Saddle points for linear differential games. SIAM J. Control **11**, 100–112 (1973)
3. Fleming, W.H., Souganidis, P.E.: On the existence of value functions of two player, zero sum stochastic differential games. Indiana Univ. Math. J. 38, 293–314 (1989)
4. Bensoussan, A., Frehse, J.: Stochastic games for N players. J. Optim. Theory Appl. 105(3), 543–565 (2000)
5. Bensoussan, A., Frehse, J.: On diagonal elliptic and parabolic systems with super-quadratic Hamiltonians. Commun. Pure Appl. Anal. 8, 83–94 (2009)
6. Bensoussan, A., Frehse, J., Vogelgesang, J.: Systems of Bellman equations to stochastic differential games with non-compact coupling. Discret. Contin. Dyn. Syst. 27(4), 1375–1389 (2010)
7. Huang, M., Caines, P.E., Malhamé, R.P.: Individual and mass behaviour in large population stochastic wireless power control problems: centralized and Nash equilibrium solutions. In: Proceedings of the 42nd IEEE Conference on Decision and Control, Maui, Hawaii, December 2003, pp. 98–103 (2003)
8. Huang, M., Malhamé, R.P., Caines, P.E.: Large population stochastic dynamic games: closed-loop McKean–Vlasov systems and the Nash certainty equivalence principle. Commun. Inf. Syst. 6(3), 221–252 (2006)
9. Lasry, J.-M., Lions, P.-L.: Jeux á champ moyen I—Le cas stationnaire. C. R. Acad. Sci. Ser. I 343, 619–625 (2006a)
10. Lasry, J.-M., Lions, P.-L.: Jeux á champ moyen II. Horizon fini et contrôle optimal. C. R. Acad. Sci. Ser. I 343, 679–684 (2006b)
11. Lasry, J.M., Lions, P.L.: Mean field games equation with quadratic hamiltonian: a specific approach. Math. Models Methods Appl. Sci. 22(9), 1250022 (2012)
12. Tembine, H., Zhu, Q., Başar, T.: Risk-sensitive mean-field stochastic differential games. In: Proceedings of the 18th IFAC World Congress, Milan, August 2011, pp. 3222–3227 (2011)
13. Achdou, Y., Capuzzo-Dolcetta, I.: Mean field games: numerical methods. SIAM J. Control Optim. 51(5), 3598–3620 (2013)
30. Camilli, F., Marchi, C.: Stationary mean field games systems defined on networks. Preprint, arXiv:1505.04953 (2015)
31. Bardi, M., Priuli, F.S.: Linear-quadratic N-person and mean-field games with ergodic cost. SIAM J. Control Optim. 52(5), 3022–3052 (2014)
32. Bensoussan, A., Sung, K.C.J., Yam, S.C.P.: Linear-quadratic time-inconsistent mean field games. Dyn. Games Appl. 3(4), 537–552 (2013)
33. Priuli, F.S.: Linear-quadratic N-person and mean-field games: infinite horizon games with discounted cost and singular limits. Dyn. Games Appl. 5(3), 397–419 (2014)
34. Huang, M.: Large-population LQG games involving a major player: the Nash certainty equivalence principle. SIAM J. Control Optim. 48(5), 3318–3353 (2010)
35. Nourian, M., Caines, P.E.: ε-Nash mean field game theory for nonlinear stochastic dynamical systems with major and minor agents. SIAM J. Control Optim. 51(4), 3302–3331 (2013)
36. Nourian, M., Caines, P.E., Malhamé, R.P., Huang, M.: Nash, social and centralized solutions to consensus problems via mean field control theory. IEEE Trans. Automat. Control 58(3), 639–653 (2013)
37. Bensoussan, A., Chau, M.H.M., Yam, S.C.P.: Mean-field Stackelberg games: aggregation of delayed instructions. SIAM J. Control Optim. 53(4), 2237–2266 (2015)
38. Bensoussan, A., Frehse, J., Yam, S.C.P.: The master equation in mean field theory. J. Math. Pures Appl. 103(6), 1441–1474 (2015)
39. Bensoussan, A., Yam, S.C.P., Zhang, Z.: Well-posedness of mean-field type forward–backward stochastic differential equations. Stoch. Process. Their Appl. 125(9), 3327–3354 (2015)
40. Cardaliaguet, P.: Notes on mean field games (2013)
41. Guéant, O.: Mean field games and applications to economics. Ph.D. Thesis, Université Paris-Dauphine (2009)
42. Guéant, O., Lasry, J.M., Lions, P.L.: Mean field games and applications. Paris-Princeton Lectures on Mathematical Finance, Springer, Berlin Heidelberg, pp. 205–266 (2011)
43. Bensoussan, A., Frehse, J., Yam, S.C.P.: Mean field games and mean field type control theory. Springer, Berlin (2013)
44. Huang, M., Caines, P.E., Malhamé, R.P.: An invariance principle in large population stochastic dynamic games. J. Syst. Sci. Complex. 20(2), 162–172 (2007a)
45. Freiling, G.: A survey of nonsymmetric Riccati equations. Linear Algebra Appl. 351, 243–270 (2002)
46. Huang, M., Caines, P.E., Malhamé, R.P.: Large-population cost-coupled LQG problems: generalizations to non-uniform individuals. In: Proceedings of the 43rd IEEE Conference on Decision and Control, Atlantis, Paradise Island, Bahamas, December 2004, pp. 3453–3458 (2004)
47. Andersson, D., Djehiche, B.: A maximum principle for SDEs of mean-field type. Appl. Math. Optim. 63(3), 341–356 (2010)
48. Buckdahn, R., Djehiche, B., Li, J., Peng, S.: Mean-field backward stochastic differential equations: a limit approach. Ann. Probab. 37(4), 1524–1565 (2009)
49. Buckdahn, R., Li, J., Peng, S.: Mean-field backward stochastic differential equations and related partial differential equations. Stoch. Process. Appl. 119(10), 3133–3154 (2007)
50. Bensoussan, A.: Stochastic control of partially observable systems. Cambridge University Press, Cambridge (1992)
51. Ma, J., Yong, J.: Forward–backward stochastic differential equations and their applications. Lecture Notes in Mathematics, vol. 1702, Springer, Berlin (1999)
52. Sznitman, A.S.: Topics in Propagation of Chaos. In: École de Probabilités de Saint Flour, XIX-1989, Lecture Notes in Mathematics, vol. 1464, pp. 165–251 (1989)
53. Bensoussan, A., Chau, M.H.M., Yam, S.C.P.: Mean field games with a dominating player. Appl. Math. Optim. (2015, forthcoming)