Geometric Aspects of Young Integral: Decomposition of Flows

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Abstract. In this paper we study geometric aspects of dynamics generated by Young differential equations (YDE) driven by \( \alpha \)-Hölder trajectories with \( \alpha \in (1/2, 1) \). We present a number of properties and geometrical constructions in this context: Young Itô geometrical formula, horizontal lift in principal fibre bundles, parallel transport, among others. Our main application here is a geometrical decomposition of flows generated by YDEs according to diffeomorphisms generated by complementary distributions (integrable or not). The proof of existence of this decomposition is based on an Itô-Wentzel type formula for Young integration along \( \alpha \)-Hölder paths proved by Castrequini and Catuogno (Chaos Solitons Fractals, 2022).

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1. Introduction

In this paper we study geometric aspects of dynamics generated by Young differential equations (YDE) driven by \( \alpha \)-Hölder trajectories with \( \alpha \in (1/2, 1) \). More precisely, given a smooth manifold \( M \), we focus on geometrical properties of equations of the type:

\[ dx_t = X(x_t) \, dZ_t, \tag{1} \]

with initial condition \( x_0 \in M \) at \( t = 0 \), where \( x \to X(x) \in \mathcal{L}(\mathbb{R}^d, T_x M) \) is a smooth assignment of \( d \) vector fields on \( M \) and \( Z \in C^\alpha([0,T],\mathbb{R}^d) \) is an \( \alpha \)-Hölder continuous trajectory in \( \mathbb{R}^d \). We say that a path \( x : [0,T] \to M \) is a solution of equation (1) if for all test function \( f \in C^\infty(M;\mathbb{R}) \) we have that

\[ f(x_t) = f(x_0) + \int_0^t Xf(x_s) \, dZ_s, \tag{2} \]

where \( Xf \) is a short term for \( \sum Df(x)X(x)e_i \), with \( e_i \)'s the elements of the canonical basis of \( \mathbb{R}^d \). The last term of equation (2) is an integral in the
Young sense, see e.g. the classical [33], or more recent Hairer and Friz [14], Gubinelli et al. [15], Lyons [21], Castrequini and Russo [7], Castrequini and Catuogno [8], Cong [10], Ruzmaikina [30], among many others.

Here, in a scenario of low regularity of trajectories, the Itô type formula in the context of Young integration, Theorem 2.3 opens the possibility for many basic geometric constructions on this dynamics. These topics are exploited in the next section, where we prove the existence of horizontal lifts in principal fibre bundles with an affine connection. In particular, considering a Riemannian manifold and its orthonormal bundle, parallel transport and covariant derivatives can be established along \( \alpha \)-Hölder trajectories. Development and anti-development can also be constructed.

Motivated by the fact that in many kinds of dynamical systems, in order to obtain local or asymptotic parameters of the dynamics, one performs a best-fitting decomposition of the associated flow, our main example of application of this low regularity techniques in manifolds concerns a decomposition of the associated Young flow. In fact, many examples of this kind of decomposition appear in the literature related to distinct geometrical or analytical contexts. We mention few of them: given a system in a semi-simple Lie group, we get a lot of information if we decompose the system into each component of the Iwasawa decomposition (see, e.g. in the stochastic context Malliavin and Malliavin [25]); given a stochastic flow in a Riemannian manifold, one can write this flow (up to some conditions) as a Markovian process in the group of isometries of the manifold composed with a process in the Lie group of diffeomorphisms which fix the initial condition and has derivatives at this point given by an upper triangular matrix, see Ming Liao [23], [22]. Also, given a flow in an \( m \)-dimensional manifold with a pair of complementary foliation (i.e. locally, the manifold and foliations are diffeomorphic to \( \mathbb{R}^k \times \mathbb{R}^{m-k} \)) then, locally in time and space, a stochastic flow can be written as a composition of diffeomorphisms whose components preserve the leaves of each of these foliations, see [26], [27]. We are going to make this last example more precise and explore its potential in the Young integral context.

The decomposition of Young flows is allowed thanks to an Itô-Wentzel type formula in this Young integration context, Theorem 3.4 due to Castrequini and Catuogno [8]. The framework where we apply this formula is a pair of geometrical distributions. The main result in Section 3 establishes the local decomposition of an Young flow of diffeomorphisms as one component given by a diffeomorphism generated by vector fields in one distribution and another component given by diffeomorphism generated by the other distribution. Precise definition are given in Section 3. Many previous works have treated the geometrical aspects of low regularity of paths on manifolds for \( \alpha < 1/2 \) (rough path) see e.g. Cass, Litterer and Lyons [5], Armstrong [3], Duc [12] among others; the approach in these articles depends on an enhanced \( \alpha \)-Hölder trajectory, i.e. it relies on additional information given by a second (or higher) order function. Besides, an Itô- Wentzel type formula for composition of flows in the rough path context is much more demanding in terms of geometrical and analytical hypothesis. Our approach here relies only on the geometry and the trajectory itself.
The article is organized as follows: in the next section we recall basic properties of the Young integral and prove the relevant geometric results we use latter on. In Section 3 we prove the decomposition of flows given complementary distributions. In Section 4 we present examples. Initially linear systems are treated with a pair of foliations given by affine parallel hyper-planes: we present conditions to the existence of global decomposition for any time in this context. More precisely, if dimension \( n > 2 \), then there exist a pair of parallel foliations \( \mathcal{F}(E_1), \mathcal{F}(E_2) \) generated by complementary subspaces \( E_1 \) and \( E_2 \) such that the decomposition of the flow of the linear equation exists for all time \( t \in [0,T] \), i.e. there is no explosion time of the decomposition. Dimension of \( E_1 \) can be chosen as a number of the form as established in Proposition 4.1. The last example provides explicit calculations in the case of fibre bundles over homogeneous space \( G \to M = G/H \) where \( G \) is a Lie group and \( H < G \) is a closed subgroup.

2. Geometric Set Up

2.1. Young Differential Equation on Manifolds

We recall that for a general metric space \((M,d)\), a curve \( \sigma : [0,T] \to M \) is \( \alpha \)-Hölder continuous, with \( \alpha > 0 \) if there exists a constant \( C > 0 \), such that

\[
d(\sigma(t),\sigma(s)) \leq C|t - s|^{\alpha},
\]

for all \( s, t \in [0,T] \). This concept extends naturally to a Riemannian manifold, since it carries the well known induced metric \( d(x,y) \) given by

\[
d(x,y) = \inf \left\{ \int_0^1 ||\gamma'(t)||dt; \gamma : [0,1] \to M \text{ differentiable such that } \gamma(0) = x, \gamma(1) = y \right\}.
\]

See e.g. [11] among many other classical books. Hence, naturally, \( \alpha \)-Hölder paths are also well defined in Riemannian manifolds. Most of the classical analytic results on this regularity theory also holds for \( \alpha \)-Hölder paths in a Riemannian manifold. For instance, composition of a differentiable function with an \( \alpha \)-Hölder trajectory is also an \( \alpha \)-Hölder path. Particularly, in a geometrical context, for readers’ convenience we prove the following

**Proposition 2.1.** Let \( M \) and \( N \) be Riemannian manifolds, \( \dim N \geq 1 \). A path \( \sigma : [0,T] \to M \) is \( \alpha \)-Hölder continuous on \( M \) if and only if, for all differentiable map \( f : M \to N \), the path \( f(\sigma(t)) \) is \( \alpha \)-Hölder continuous on \( N \).

**Proof.** There are many interesting ways to prove this result. Here, we use an embedding argument. Initially consider that \( N \) is an Euclidean space \( \mathbb{R}^n \) and take \( \sigma(t) \) an \( \alpha \)-Hölder trajectory on \( M \). There exists an isometric embedding \( i : M \to \mathbb{R}^d \) for a sufficiently large integer \( d \) (Nash theorem). For sake of notation we write \( \sigma_t := \sigma(t) \).
Since \( \|i(x) - i(y)\|_{\mathbb{R}^d} \leq d_M(x, y) \) for all \( x, y \in M \), we have the following inequalities:
\[
\|i(\sigma_t) - i(\sigma_s)\|_{\mathbb{R}^d} \leq d_M(\sigma_t, \sigma_s) \leq C|t - s|^\alpha,
\]
which implies that \( i(\sigma_t) \) is \( \alpha \)-Hölder in \( \mathbb{R}^d \). Now, for any differentiable function \( f : M \to \mathbb{R}^n \), use the fact that it can be extended to a differentiable function \( \tilde{f} : U \to \mathbb{R}^n \) defined in a tubular neighbourhood \( U \) of \( i \circ \sigma([0, T]) \) in \( \mathbb{R}^d \). Hence, \( f(\sigma_t) = \tilde{f}(i(\sigma_t)) \). Since Hölder regularity is preserved by differentiable functions on Euclidean spaces, \( f(\sigma_t) \) is \( \alpha \)-Hölder continuous in \( \mathbb{R}^n \). Conversely if \( \tilde{f}(i(\sigma_t)) \) is \( \alpha \)-Hölder for all differentiable function \( \tilde{f} \), in particular taking convenient projections shows that \( i(\sigma_t) \) is \( \alpha \)-Hölder in \( \mathbb{R}^d \). Mind that, in fact, in the compact set \( i(\sigma_t) \) the metrics \( d_M \), and \( \ell_2 \) in \( \mathbb{R}^d \) are uniformly equivalents, see Lemma 2.2 [19]. Hence, \( \ell_2 \)-norm Hölder regularity in \( \mathbb{R}^d \) is equivalent to Hölder regularity on \((M, d_M)\).

For a general Riemannian manifold \( N \) and a differentiable map \( f : M \to N \), consider another isometric embedding \( i' : N \to \mathbb{R}^{d'} \) for an integer \( d' \) sufficiently large. Then, the last paragraph shows that \( i' \circ f(\sigma) \) is \( \alpha \)-Hölder in \( \mathbb{R}^{d'} \). From Lemma 2.2 [19] we have that there exists a positive constant \( C_1 \) such that
\[
d_N(f(\sigma_t), f(\sigma_s)) \leq C_1\|i(f(\sigma_t)) - i(f(\sigma_t))\|_{\mathbb{R}^{d'}} \leq C_2|t - s|^\alpha,
\]
for a positive constant \( C_2 \), which shows that \( f(\sigma_t) \) is \( \alpha \)-Hölder continuous in \( N \).

Conversely, suppose that \( f(\sigma_t) \in N \) is \( \alpha \)-Hölder for all differentiable function \( f : M \to N \). Denote the projections of \( i(\sigma_t) \in \mathbb{R}^d \) by \( \sigma^j_t := p_j \circ i(\sigma_t) \) for each \( 1 \leq j \leq d \). Let \( \varphi : V \to W \subset N \) be a local parametrization for \( N \), with \( V \) an open set in an Euclidean space. There exist another local parametrization obtained from the previous one, just enlarging the domain by homothety, if necessary, which we call again by \( \varphi : \tilde{V} \to W \subset N \) such that the set \( \{(x, 0, \ldots, 0); x = \sigma^j(t) \text{ for some } t \in [0, T]\} \subset \tilde{V} \) for all \( 1 \leq j \leq d \). Consider the differentiable functions \( f_j : M \to N \) given by \( f_j(x) := \varphi(p_j(i(x)), 0, \ldots, 0) \). Then \( f_j(\sigma_t) := \varphi(\sigma^j_t, 0, \ldots, 0) \) is \( \alpha \)-Hölder by hypothesis. Since \( \varphi \) is a diffeomorphism and by metric equivalence in compact sets in the domain of the local parametrization, we have that \( \sigma^j_t \) is \( \alpha \)-Hölder for all \( 1 \leq j \leq d \). We conclude that \( \sigma(t) \in M \) is \( \alpha \)-Hölder continuous on \( M \).

Before we show conditions for existence and uniqueness of solutions for equation (1), we state the main geometric theorem that is a version of Itô’s formula for \( \alpha \)-Hölder continuous paths. We start with the definition of the Young integral of a real 1-form:

**Definition 2.2.** (Integration of real 1-forms) Let \( N \) be an \( n \)-dimensional differentiable manifold with \( \bigwedge^1(N) \) the space of real differentiable 1-forms. Consider \( \beta \in \bigwedge^1(N) \) and a chart \((U, (y_1, \ldots, y_n)) \) in \( N \) such that
\[
\beta = \sum_{i=1}^n \beta_i dy^i.
\]
The integral of $\beta$ along an $\alpha$-Hölder path $x : [0, T] \to N$ is defined by
\[
\int_0^T \beta \, dx_t := \sum_{i=1}^n \int_0^T \beta_i(x_t) \, dx^i_t,
\]
where the above integrals are the Young integrals of $\beta_i$ with respect to the $i$-th coordinate of the path $x_t$. Among others properties, this integration is independent of the local chart, see e.g., Abraham, Marsden and Ratiu [1], Ikeda and Manabe [16] and Warner [32].

The integration above allows one to integrate vector valued 1-forms in a manifold. In particular, if $F : M \to \mathbb{R}^d$ is a smooth function, the integration
\[
\int_0^t DF(x_s) \, dx_s
\]
makes sense, looking at each coordinate of $\mathbb{R}^d$. Next theorem is a basic property of the $\alpha$-Hölder calculus on manifolds. It corresponds to an Itô type formula for Young integration.

**Theorem 2.3.** Let $M$ and $N$ be Riemannian manifolds. Consider $x \in C^\alpha([0, T], M)$ and a smooth function $F : M \to N$. Then
\[
dF(x_t) = DF(x_s) \, dx_s.
\]

**Remark:** We highlight that formula (6) above means that if $\beta$ is a 1-form in $N$ then
\[
\int_0^t \beta \, dF(x_s) = \int_0^t (dF(x_s))^* \beta \, dx_s.
\]
In particular, if $N$ is an Euclidean space:
\[
F(x_t) = F(x_0) + \int_0^t DF(x_s)dx_s.
\]

**Proof.** Initially we prove the result for an Euclidean space $N = \mathbb{R}^d$. We use again the embedding argument from Nash’s theorem: there exists a sufficiently large $p \in \mathbb{N}$ such that $M$ can be isometrically embedded into $\mathbb{R}^{m+p}$. Abusing notation, we have $x \in C^\alpha([0, T], \mathbb{R}^{m+p})$, $F$ is defined in a tubular neighbourhood of the image of $M$ and $DF(x) \in L(\mathbb{R}^{m+p}, \mathbb{R}^d)$. By Taylor’s formula in Euclidean space,
\[
F(x_t) - F(x_s) = DF(x_s) \cdot (x_t - x_s) + R(x_s, x_t),
\]
with
\[
R(x_s, x_t) = \int_0^1 (1 - u) \text{Hess} (F)(x_s + u(x_t - x_s))(x_t - x_s, x_t - x_s) \, du.
\]
Since $F$ is smooth,
\[
\|R(x_s, x_t)\| \leq C\|x_t - x_s\|^2 \leq C'|t - s|^{2\alpha}.
\]
Let $\pi = \{s_i\}$ be a partition of $[0, T]$. Then
\[
F(x_t) - F(x_0) = \sum_i \left[ F(x_{s_{i+1}}) - F(x_{s_i}) \right] = \sum_i DF(x_{s_i}) \cdot (x_{s_{i+1}} - x_{s_i})
+ \sum_i R(x_{s_i}, x_{s_{i+1}}).
\] (9)

We have that
\[
\sum_i \|R(x_{s_i}, x_{s_{i+1}})\| \leq C' \sum_i |s_{i+1} - s_i|^{2\alpha} \leq C'T \sup_i |s_{i+1} - s_i|^{2\alpha - 1}.
\]

Thus, since $\alpha > 1/2$ we have that
\[
\lim_{|\pi| \to 0} \sum_i \|R(x_{s_i}, x_{s_{i+1}})\| = 0.
\]

Take the limit $|\pi| \to 0$ in equation (9) and the definition of Stieltjes (Young) integral to finish the proof in this context.

In general, when $N$ is a Riemannian manifold, consider a local chart $\phi$. The previous calculations hold with $\phi \circ F$ whose integration is independent of the coordinate system.

Note that multidimensional versions of Itô type formula above, integration by parts and other formulae can be obtained from formula (6) considering the manifold $M$ above as an appropriate product space. We proceed to prove a theorem of existence and uniqueness of solution for equation (1).

**Theorem 2.4.** Given an initial condition $x_0 \in M$, there exists a unique maximal solution of the Young differential equation (1) such that $x(0) = x_0$. Moreover, there exists a flow of (local) diffeomorphisms associated to the solutions.

**Proof.** A simple way to prove the result for local solutions is based on the existence and uniqueness results in the Euclidean space. In fact, given the initial condition $x_0$, let $(U, \Psi)$ be a chart on $M$ with $x_0 \in U$. Let $\tilde{X} := D\Psi(X(\Psi^{-1}(p)))$ be the induced vector field in the image of $\Psi$. The Young differential equation $dy_t = \tilde{X}(y_t) dZ_t$ has a unique solution local solution $y_t$ with $y_0 = \Psi(x_0)$. See e.g. Lejay [20], Caruana, Lyons and Thierry [24], Li and Lyons [21], Friz and Hairer [14] and references therein. Take $x_t = \Psi^{-1}(y_t) \subset U$. We claim that $x_t$ is a solution of equation (1). In fact, consider a test function $f \in C^\infty(M)$. By Theorem 2.3, it follows that
\[
f(\Psi^{-1}(y_s)) = f(\Psi^{-1}(y_0)) + \int_0^t D(f \circ \Psi^{-1})(y_s) dy_s
= f(\Psi^{-1}(y_0)) + \int_0^t Df \Psi^{-1}\Psi_s X(y_s) dZ_s
= f(x_0) + \int_0^t Xf(x_s) dZ_s.
\]
Moreover, the solution $x_t$ does not depend on the choice of local coordinate. In fact, let $(V, \Phi)$ be another chart on $M$, with $x_t \in U \cap V$ and let $z_t$ be the solution of the Young differential equation $dz_t = \Phi^*X(z_t)dZ_t$. Then

\[ dy_t = \Psi^*X(y_t)dZ_t = \Psi^*\Phi^{-1}_*\Phi^*X(z_t)dZ_t = \Psi^*\Phi^{-1}_*dz_t. \]

Hence $y_t = \Psi\Phi^{-1}(z_t)$ and therefore $\Phi^{-1}(z_t) = \Psi^{-1}(y_t) = x_t$. A maximal solution is obtained in the classical way by extending a local solution up to its explosion time. The existence of local flow of (local) diffeomorphisms is also concluded from the Euclidean case using the same local chart argument.

\[ \square \]

### 2.2. Horizontal Lifts

Let $\{P, M, G, \pi\}$ be a principal fibre bundle with base $M$, structure group $G$ and total space $P$. In this case $M$ is a smooth, connected and paracompact manifold. The projection $\pi$ is taken as $\pi : P \rightarrow M$. The group $G$ acts freely on $P$ on the right by the action $R_g : P \rightarrow P$ defined by $R_g(u) = ug$, for $u \in P$ and $g \in G$. Let $\mathfrak{g}$ be the Lie algebra of $G$, then an element $A \in \mathfrak{g}$ generates the exponential $\{\exp tA, \ t \in \mathbb{R}\}$, which induces a vector field on $P$ by

\[ A^*u = \frac{d}{dt}\exp (tA)u \bigg|_{t=0}, \]

If $\Gamma^\infty(TP)$ is the section of all smooth vector fields on $P$, then the map $A \rightarrow A^*$, from $\mathfrak{g}$ into $\Gamma^\infty(TP)$ is a Lie algebra homomorphism. For more details, see e.g. Shigekawa [31], the classical Kobayashi and Nomizu [17] among many others. The tangent space $TP$ has a naturally defined subspace called the vertical tangent bundle $VT_P$ given by $VT_uP := \ker d\pi u$ for all $u \in P$. Note that $A^*u \in VT_uP$ for all $A \in \mathfrak{g}$.

A connection in the principal fibre bundle is an assignment of a horizontal subspace $HT_uP$ of $T_uP$ which is the kernel of a $\mathfrak{g}$-valued 1-form $\omega$ in $P$ with the following properties:

(i) (well-behaved vertically) $\omega dR_g = \text{Ad}(g^{-1})\omega$, for all $g \in G$. Here the linear map $\text{Ad}(g^{-1}) : \mathfrak{g} \rightarrow \mathfrak{g}$ is the derivative at the identity of the adjoint $\text{Ad}(g^{-1}) : G \rightarrow G$ defined by $\text{Ad}(g^{-1})a = g^{-1}ag$.

(ii) (vertical calibration) $\omega(A^*) = A$, where $A^*$ is a vector field on $VT_P$.

Such 1-form $\omega$ is called a connection form in the principal fibre bundle $\{P, M, G, \pi\}$. Moreover, $\omega$ defines the horizontal tangent bundle $HT_P$ given by $HT_uP = \ker \omega_u$. Hence, for all $u \in P$, the tangent space $T_uP$ splits into $HT_uP \oplus VT_uP$ and $dR_g(HT_uP) = HT_{ug}P$.

Now we have the geometric set up to define the horizontal lift of $\alpha$-Hölder continuous paths. It is worth mentioning the next definitions and results give a natural geometric bases for $\alpha$-Hölder geometry ($\alpha \in (\frac{1}{2}, 1)$), see e.g. [21].

**Definition 2.5.** Let $x : [0, T] \rightarrow M$ be an $\alpha$-Hölder continuous path. Consider $u \in P$, with $\pi(u) = x_0$. The horizontal lift of $x_t$ starting at $u$ is a path $\tilde{x} : [0, T] \rightarrow P$ such that:
(i) $\tilde{x}_0 = u$.
(ii) $\pi(\tilde{x}_t) = x_t$ for all $t \in [0, T]$.
(iii) $\int_0^t \omega \, d\tilde{x}_s = 0$ for all $t \in [0, T]$.

Next result shows the existence and uniqueness of the horizontal lift for an $\alpha$-Hölder continuous paths in manifolds. In the proof we apply the same technique used in Kobayashi and Nomizu [17] and in Shigekawa [31] where the existence and uniqueness of horizontal lift were proved in the context of $C^1$ paths and semimartingales respectively.

**Theorem 2.6.** Given an $\alpha$-Hölder continuous path $x : [0, T] \to M$ and an element $u$ in the fibre $\pi^{-1}(x_0)$, there exists (up to an explosion time) a unique horizontal lift $\tilde{x} : [0, T] \to P$ with $\tilde{x}_0 = u$.

**Proof.** Consider a local trivialization $\phi : \pi^{-1}(U) \to U \times G$ with $x_0 \in U$ and take the $\alpha$-Hölder path $\nu_t = \phi^{-1}(x_t, e)$. If the horizontal lift of $x_t$ exists at all, it has to be of the form $\tilde{x}_t = \nu_t a_t$, where $a_t \in G$ is an appropriate path which makes $\tilde{x}_t$ horizontal and $\nu_0 a_0 = u$.

Let $\Psi : P \times G \to P$ be the right free action of $G$ on $P$. Then, by Theorem 2.3 we have that

$$
\int_0^t \omega \, d\tilde{x}_t = \int_0^t (\partial_1 \Psi(\nu_t, a_t))^* \omega \, d\nu_t + \int_0^t (\partial_2 \Psi(\nu_t, a_t))^* \omega \, da_t.
$$

Hence:

$$
\int_0^t \omega \, d\tilde{x}_t = \int_0^t R_{a_t} \omega \, d\nu_t + \int_0^t \theta \, da_t,
$$

by the vertical calibration of the connection $\omega$, where $\theta$ is the canonical Cartan 1-form given by $\theta_g(dR_g A) = A$ for all $g \in G$ and $A \in \mathfrak{g}$. The lift $\tilde{x}_t$ is horizontal if and only if equation (10) vanishes for all $t \in [0, T]$, i.e. if and only if $\text{Ad}(a_t^{-1}) \omega \, d\nu_t = -\theta \, da_t$. Let $F_1, \ldots, F_n$ be a basis of the right invariant Lie algebra $\mathfrak{g}$. For all $t \in [0, T]$, there exist $\alpha$-Hölder continuous real functions $\alpha^1_t, \ldots, \alpha^n_t$, such that:

$$
\int_0^t \omega \, d\nu_s = \sum_{i=1}^n F_i \alpha^i_t.
$$

Using this notation we have that a necessary and sufficient condition such that equation (10) vanishes is that

$$
\int_0^t \omega \, d\tilde{x}_t = \sum_{i=1}^n \int_0^t \text{Ad}(a_t^{-1}) F_i \, d\alpha^i_t + \int_0^t dR_{a_t^{-1}} \, da_t = 0,
$$

for all $t \in [0, T]$, i.e. trajectory $a_t$ has to satisfy

$$
da_t = -\sum_{i=1}^n dR_{a_t} \, \text{Ad}(a_t^{-1}) F_i \, d\alpha^i_t,
$$
with initial condition $a_0$. There exists a unique solution by Theorem 2.4, hence there exists a unique horizontal lift $\tilde{x}_t$ up to an explosion. Mind that at the border of the local trivialization, one can extend further the solution by applying again the same construction as above. The maximal solution covers the whole interval $[0,T]$ (by compactness) if there is no explosion in the fibre.

Note that for initial element in the fibre $a_0g$, the horizontal lift is given by $a_tg$.

□

Besides the dynamics and the principal fibre bundle approach presented so far (which are basic to the next Sections), this low regularity Young calculus of Theorem 2.3 allows one to develop further geometrical properties. We mention the following three classical geometric aspects:

A. Parallel Transport and covariant derivative: Given a smooth manifold $M$, consider the frame bundle $BM \to M$ of basis $u : \mathbb{R}^n \to T_pM$, with $p \in M$, with the structure group $G = Gl(n, \mathbb{R})$. Last Theorem applied in this context establishes a parallel transport along $\alpha$-Hölder path $x_t \in M$. In fact, given a horizontal lift $u_t$, the parallel transport of a vector $v \in T_{x_0}M$ is obtained by

$$\parallel_t v = u_t \circ u^{-1}_0(v) \in T_{x(t)}M.$$  

It does not depend on the choice of the horizontal lift. Moreover, if we take the orthonormal frame bundle $OM \to M$ of basis orthonormal basis given by linear isometries $u : \mathbb{R}^n \to T_xM$, with $x \in M$, with the structure group $G = O(n, \mathbb{R})$, the parallel transport is also an isometry.

Covariant derivative can now be defined along an $\alpha$-Hölder continuous path $x_t \in M$.

Given a differentiable vector field $Y$, we have that its covariant derivative along $x(t)$ is given by:

$$D_Y(x_t) = \parallel_t d \parallel_t^{-1} Y(x_t).$$

where the differentials are interpreted in the sense of Young (Definition 2.2).

B. Development and anti-development: Let $M$ be an $m$-dimensional Riemannian manifold, and consider an $\alpha$-Hölder continuous path $x : [0,T] \to \mathbb{R}^m$. Take the horizontal operator $H : OM \times \mathbb{R}^m \to HTOM$ where $H(u, v)$ is the horizontal lift of $u(v) \in T_{\pi(u)}M$ up to $HT_uOM$. The development of $x_t$ on $M$ with initial orthonormal frame $u_0$ is obtained from $u_t$, the solution of the YDE:

$$du_t = H(u_t, dx_t),$$

i.e. $\pi(u_t)$ is the development of $x(t)$ on $M$ (rolling without slipping, with initial “contact plane” given by $u_0$). On the other hand, the anti-development of an $\alpha$-Hölder continuous path $x : [0,T] \to M$ is described using its horizontal lift $\tilde{x}_t$ (Theorem 2.6) with initial condition $\tilde{x}_0$:

$$y_t = \int_0^t \tilde{x}_s^{-1} ds.$$
Note that, as expected, $y_t$ depends on the choice of $\tilde{x}_0$. Compare this approach with the classical Brownian motion approach by Eells and Elworthy [13], and the isotropic Lévy processes approach in Applebaum and Estrade [2], among many others.

**C. Continuous $\alpha$-Hölder paths in $M$ are solutions of Young differential equations:** As established before, solutions of Young equations driven by $\alpha$-Hölder paths on a manifold are also $\alpha$-Hölder continuous paths. Reciprocally, every $\alpha$-Hölder continuous path on $M$ is a solution of a Young differential equation (YDE) driven by an $\alpha$-Hölder function. In fact, take an embedding $i : M \to \mathbb{R}^{m+p}$ of $M$ into a sufficiently large dimensional Euclidean space. Let $U$ be a tubular neighbourhood with $\pi : U \to i(M)$ a projection of $U$ into $i(M)$. Given an $\alpha$-Hölder path $y_t$ on $M$, let $z_t = i(y_t)$. Then $z_t$ is an $\alpha$-Hölder trajectory in $\mathbb{R}^{m+p}$. Consider the YDE in $i(M)$:

$$dx_t = D\pi(x_t) \, dz_t,$$

Then $z_t$ is the solution of this YDE with initial condition $x_0 = z_0$: just check that the YDE is the differential version of the identity $z_t = \pi(z_t)$, according to formula of theorem 2.3. If the projection $\pi$ is orthogonal, as in Elworthy [13] then the vector fields are gradients of the embedding. In general, the dynamics of other trajectories starting at $x_0 \neq y_0$ depends on the embedding and on the projection. This is an interesting topic to be studied further.

**3. Decomposition of Flow Generated by Young Differential Equation**

Let $\text{Diff}(M)$ be the infinite dimensional Lie group of smooth diffeomorphisms of a compact connected manifold $M$. The Lie algebra associated to $\text{Diff}(M)$ is the infinite dimensional space of smooth vector fields on $M$, see e.g. Neeb [28], Omori [29], among others. The exponential map $\exp\{tY\} \in \text{Diff}(M)$ is the associated flow of diffeomorphisms generated by the smooth vector field $Y$. In this context, given an element $\varphi \in \text{Diff}(M)$ the derivative of the right translation is given by $R_{\varphi^*}Y = Y(\varphi)$ for any smooth vector $Y$. The derivative of left translation $L_{\varphi^*}Y = D\varphi(Y)$, and $\text{Ad}(\varphi)Y = \varphi_*(Y(\varphi^{-1}))$.

In this Lie group notation, a solution flow $\varphi_t$ of an YDE is written as the solution of a right invariant Young differential equation in the Lie group of diffeomorphisms $\text{Diff}(M)$:

$$d\varphi_t = R_{\varphi_t^*} X \, dZ_t.$$  \hspace{1cm} (12)

Here we abuse notation in the sense that (using the same notation as in equation 1) one can write

$$X \, dZ_t = \sum_{i=1}^{d} X_i \, dZ^i_t,$$
where $X_j = X(e_j)$ with $e_j$ the elements of the canonical basis. Hence, equation (12) has to be interpreted as

$$d\varphi_t = \sum_{i=1}^d R_{\varphi_t} X_i \ dZ^i_t.$$  

Interesting problems arise when one decomposes a (flow of) diffeomorphism $\varphi \in \text{Diff}(M)$, into composition of convenient prescribed components. This kind of decomposition appears in the literature, for example, in Bismut [4], Kunita [18] and many others. In particular, it is also relevant when each component of the decomposition belongs to prescribed subgroups of Diff($M$), see e.g Melo et al [27], Catuogno et al [8], Iwasawa and non-linear Iwasawa decomposition [9], Ming Liao [23] among many others.

In this section, we explore the Young calculus to prove the existence of a geometrical decomposition of flows generated by $\alpha$-Hölder systems $\varphi_t$ given by equation (1). Suppose that locally $M$ is endowed with a pair of regular differentiable distributions: i.e. every point $x \in M$ has a neighbourhood $U$ and differentiable mappings $\Delta^1 : U \to Gr_p(M)$ and $\Delta^2 : U \to Gr_{m-k}(M)$ respectively, where

$$Gr_p(M) = \bigcup_{x \in M} Gr_p(T_x M)$$

is the Grasmannian bundle of $p$-dimensional subspaces over $M$, with $1 \leq p \leq m$. We assume that $\Delta^1$ and $\Delta^2$ are complementary in the sense that $\Delta^1(x) \oplus \Delta^2(x) = T_x M$, for all $x \in U$. With this notation we define the subgroup of Diff($M$) which is generated by a certain distribution $\Delta$ by:

$$\text{Diff}(\Delta, M) = \text{cl}\{\exp(t_1 X_1) \ldots \exp(t_n X_n), \text{ with } X_i \in \Delta, t_i \in \mathbb{R}, \forall n \in \mathbb{N}\}.$$  

Note that if a distribution $\Delta$ is involutive, then each element of the group $\text{Diff}(\Delta, M)$ preserves the leaves of the corresponding foliation.

In particular, in this section we focus on the subgroups $\text{Diff}(\Delta^1, M)$ and $\text{Diff}(\Delta^2, M)$. The main result of this paper (Theorem 3.4) establishes a local decomposition of the solution flow $\varphi_t$ into two components: a curve (solution of an autonomous YDE) in $\text{Diff}(\Delta^1, M)$ composed with a non-autonomous path in $\text{Diff}(\Delta^2, M)$.

**Definition 3.1.** We say that an element $\eta \in \text{Diff}(M)$ preserves transversality of $\Delta^1$ and $\Delta^2$ in a neighbourhood $U \subset M$ if $\eta_* \Delta^2 (\eta^{-1}(p)) \cap \Delta^1 (p) = \{0\}$, for all $p \in U$.

By continuity, for any pair of complementary distributions, there always exists a neighbourhood of the identity $Id \in \text{Diff}(M)$ where all elements in this neighbourhood preserve transversality. Moreover, if the distribution $\Delta^1$ is involutive then all elements in $\text{Diff}(\Delta^1, M)$ preserves transversality of $\Delta^1$ and $\Delta^2$: in fact, the derivative $\eta_*$ above is a linear isomorphism which sends tangent spaces of the associated foliation to tangent spaces in the same leaf. In the sequence, we state an extended scope of the Itô-Wentzel type formula (see [18]) in the geometrical Young calculus.
Theorem 3.2. (Itô-Wentzel type formula) Let $X, Y \in C^2(M, \mathcal{L}(\mathbb{R}^d, TM))$ and $Z \in C^\alpha([0, T], \mathbb{R}^d)$ and suppose that $\eta_t$ and $\psi_t$ are solutions maps associated to the Young differential equations $d\eta_t = X(\eta_t)dZ_t$ and $d\psi_t = Y(\psi_t)dZ_t$ respectively. Then, $\varphi_t = \eta_t \circ \psi_t$ is the solution map associated with the Young differential equation

$$d\varphi_t = X(\varphi_t)dZ_t + \text{Ad}(\eta_t)Y(\varphi_t)dZ_t.$$  \hspace{1cm} (13)

For a proof in this low regularity context, see Castrequini and Catuogno [6, Thm. 4.1]. Next Corollary shows that the inverse of the solution flow of an YDE is also $\alpha$-Hölder continuous.

Corollary 3.3. If $\eta_t$ is the solution flow the Young differential equation on $M$

$$dx_t = X(x_t)dZ_t,$$  \hspace{1cm} (14)

then, the inverse map $\eta_t^{-1}$ is the solution of the Young differential equation on $M$

$$dz_t = -D\eta_t^{-1}(z_t)X(\eta_t(z_t))dZ_t.$$  \hspace{1cm} (15)

Proof. In fact, just apply expressions (14) and (15) into equation (13). \hfill \square

For a constructive proof of last Corollary see [6, Thm. 4.2]. Next theorem states the main result of this section:

Theorem 3.4. (Decomposition of flows of YDE) Up to a life time $\tau \in [0, T]$, the solution flow $\varphi_t$ can be locally decomposed as

$$\varphi_t = \eta_t \circ \psi_t,$$

where $\eta_t$ is solution of an (autonomous) Young differential equation in $\text{Diff}(\Delta^1, M)$ and $\psi_t$ is a path in $\text{Diff}(\Delta^2, M)$.

Proof. Given $p \in M$, take $\eta \in \text{Diff}(\Delta^1, M)$ sufficiently close to the identity such that it preserves transversality, i.e. $\text{Ad}(\eta_t)\Delta^2$ and $\Delta^1$ are complementary. The tangent vector(s) $X(p)$ can be decomposed uniquely as

$$X(p) = h(p) + V(\eta_t, p),$$  \hspace{1cm} (16)

where $h(p) \in \Delta^1(p)$ and $V(\eta_t, p) \in \text{Ad}(\eta_t)\Delta^2(p)$, for all $p \in M$. We take the first component $\eta_t$ as the solution map of the following Young differential equation in $\text{Diff}(\Delta^1, M)$:

$$d\eta_t = R_{\eta_t} \cdot h dZ_t,$$  \hspace{1cm} (17)

with initial condition $\eta_0 = \text{Id}$, the identity. Even though the equation above is described in terms of a right translation, it is not a right invariant equation since $h$ in general depends on $\eta_t$. We obtain the second component of decomposition of $\varphi_t$ using that $\psi_t = \eta_t^{-1} \circ \varphi_t$. Applying Corollary 3.3, it follows that:

$$d\eta_t^{-1} = -L_{\eta_t^{-1}} h dZ_t,$$
where $L_{\eta^{-1}}$ is the derivative of the left translation at the identity by $\eta^{-1}$. Finally, we find an equation for $\psi_t$ by applying Theorem 3.2.

$$d\psi_t = (\eta_t^{-1} h \eta_t \psi_t - \eta_t^{-1} X \eta_t \psi_t) d\zeta_t$$

$$= \text{Ad}(\eta_t^{-1}) V(\eta_t) d\zeta_t.$$  \hspace{1cm} (18)

Note that $V(\eta,p)$ does not necessarily belong to $\Delta^2$. Still, $d\psi_t \in \Delta^2$ since $d\psi_t \in \text{Ad}(\eta_t^{-1}) \text{Ad}(\eta) \Delta^2 = \Delta^2$. Then $\psi_t$ is the $\Delta^2$-component of $\varphi_t$.

\[\square\]

**Corollary 3.5.** If the distributions $\Delta^1$ and $\Delta^2$ are integrable, then the decomposition of Theorem 3.4 is unique.

**Proof.** In fact, in this case $\text{Diff}(\Delta^1, M) \cap \text{Diff}(\Delta^2, M) = \{\text{Id}\}$.

\[\square\]

### 4. Examples of Decomposition of Flows of YDE

In this section we consider the same geometric structure used in section 2, i.e., a principal fibre bundle $\{P, M, G, \pi\}$, with base $M$, structure group $G$ and total space $P$. Our main goal is to apply the decomposition which was proposed in theorem 3.4 in general fibre bundles. Important notions such as 1-forms connection on fibre bundles, horizontal and vertical tangent bundles and others were discussed briefly on section 2, more details can be found for example in Kobayashi and Nomizu [17].

#### 4.1. Linear Systems

Consider an Euclidean space $\mathbb{R}^n$, with a pair of complementary foliations given by the trivial Cartesian product $\mathbb{R}^k \times \mathbb{R}^\ell$, with $k + \ell = n$. More precisely, the horizontal foliation $\mathcal{F}_H$ is given by parallel leaves generated by affine translations $x + (\mathbb{R}^k \times \{0\})$, with $x \in \mathbb{R}^n$. Analogously, the vertical foliation $\mathcal{F}_V$ is given by parallel vertical leaves $x + (\{0\} \times \mathbb{R}^\ell)$, for all $x \in \mathbb{R}^n$. We consider the linear Young differential equation:

$$dx_t = A x_t \, d\zeta_t,$$  \hspace{1cm} (19)

with $x_0 \in \mathbb{R}^n$ and $\zeta_t$ an $\alpha$-Hölder continuous trajectory in the real line. The Young calculus presented in the previous section shows that the fundamental linear solution flow of (19) is the exponential

$$F_t = \exp \{A(\zeta_t - \zeta_0)\}.$$  \hspace{1cm} (20)

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}_{k \times \ell}$$

The decomposition we are interested here is

$$F_t = \eta_t \circ \psi_t.$$
such that $\eta_t \in \text{Diff}(\mathcal{F}_H, \mathbb{R}^n)$ and $\psi_t \in \text{Diff}(\mathcal{F}_V, \mathbb{R}^n)$. In general $\eta_t$ and $\psi_t$ does not have to be linear, even in quite symmetric situations. For example, if the pair of foliations in $\mathbb{R}^n \setminus \{0\}$ are given by radial and spherical coordinates, the components of the decomposition are not necessarily linear: in fact, the linear radial diffeomorphisms is reduced to a one dimensional group of uniform contractions and expansions $\lambda Id$, with $\lambda > 0$, which, obviously, is not big enough to perform the decomposition. For the Cartesian pair of foliation $\mathbb{R}^k \times \mathbb{R}^\ell$ considered in this section, we do have that $\eta_t$ and $\psi_t$ are linear. In fact, in coordinates, write

$$F_t = \begin{pmatrix} (F_1(t))_{k \times k} & (F_2(t))_{k \times \ell} \\ (F_3(t))_{\ell \times k} & (F_4(t))_{\ell \times \ell} \end{pmatrix}.$$ 

Since $\eta_t$ does not change the last $\ell$ coordinates the diffeomorphisms $\psi_t$ must satisfies

$$\psi_t = \begin{pmatrix} (Id)_{k \times k} & 0 \\ F_3(t) & F_4(t) \end{pmatrix}.$$ 

Hence diffeomorphisms $\psi_t$ and $\eta_t$, when exist, are global and linear.

**A simple example:** A system which illustrates not only these formulae, but also the lifetime of the decomposition is the pure rotation in $\mathbb{R}^2$ given by

$$dx_t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x_t dZ_t,$$

whose decomposition of flow can be easily calculated as:

$$\begin{pmatrix} \cos Z_t - \sin Z_t \\ \sin Z_t \cos Z_t \end{pmatrix} = \begin{pmatrix} \sec Z_t - \tan Z_t \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sin Z_t \cos Z_t \end{pmatrix}.$$ 

(21)

Note that if $Z_t \in \{\pi/2 + k\pi, k \in \mathbb{Z}\}$, then the decomposition (21) no longer exists at the corresponding time $t$, i.e. we have explosion of the solutions of equations (17) or (18). 

□

Back to the general linear case, the components of the decomposition in fact lie in the Lie group:

$$\psi_t \in G_V = \left\{ g \in Gl(n, \mathbb{R}); g = \begin{pmatrix} (Id)_{k \times k} \\ g_3 \\ g_4 \end{pmatrix}_{\ell \times \ell} \right\}$$

whose Lie algebra is given by the vector space generated by

$$\begin{pmatrix} (0)_{k \times k} & 0 \\ (\ast) & (\ast)_{\ell \times \ell} \end{pmatrix},$$
where (*) means nonzero matrices of the appropriate dimension. Analogously for the horizontal component:

\[ \eta_t \in G_H = \left\{ g \in \text{Gl}(n, \mathbb{R}); g = \begin{pmatrix} g_1 & g_2 \\ 0 & \text{Id}_{\ell \times \ell} \end{pmatrix} \right\} \]

whose Lie algebra is given by the vector space generated by

\[ \left( \begin{array}{cc} * & * \\ 0 & 0 \end{array} \right) \]

Using the properties of the Young integral, we find the differential equations for the constituents submatrices \( g_1, g_2 \) and \( g_3, g_4 \) of \( \eta_t \) and \( \psi_t \) respectively. Let \( \pi_2 : \mathbb{R}^k \times \mathbb{R}^\ell \to \mathbb{R}^\ell \) be the projection on the second subspace. From formula (16) we have that

\[ V(\eta, \cdot) = \eta \circ \pi_2 \circ A(\cdot). \]

In fact, it is enough to check that \( V(\eta, \cdot) \) is in the image of the vertical component by \( \eta \) and that \( \pi_2 V(\eta, \cdot) = \pi_2 A(\cdot) \). From this formula, equations (17) and (18) we find the autonomous equation:

\[ d\eta_t = (\text{Id} - \eta_t \circ \pi_2) A \eta_t \ dZ_t, \]

and the well expected nonautonomous vertical diffeomorphisms:

\[ d\psi_t = \pi_2 A \eta_t \circ \psi_t \ dZ_t. \]

Rewriting each constituent submatrices we find:

\[ \begin{align*}
  dg_1(t) &= [A_1 g_1(t) - g_2(t) A_3 g_1(t)] \ dZ_t, \\
  dg_2(t) &= [A_1 g_2(t) + A_2 - g_2(t) A_4 - g_2 A_3 g_2(t)] \ dZ_t, \\
  dg_3(t) &= [A_3 g_1 + A_3 g_2 g_3 + A_4 g_3] \ dZ_t, \\
  dg_4(t) &= [A_3 g_2 g_4 + A_4 g_4] \ dZ_t.
\end{align*} \tag{22} \]

Explosion in the solutions of the equations of \( g_1 \) and \( g_2 \) can appear if \( A_3 \) is not zero (see example of equation (21), where \( A_3 = [1] \)). Otherwise, if \( A_3 = 0 \) then there exists the decomposition for all time \( t \geq 0 \). Using this feature, and the Jordan canonical form we can extend the scope of the decomposition in the next Proposition. Before that let us fix a notation. Given two complementary subspaces \( E_1 \oplus E_2 = \mathbb{R}^n \), let us denote by \( \mathcal{F}(E_1) \) and \( \mathcal{F}(E_2) \) the corresponding pair of complementary parallel foliations in \( \mathbb{R}^n \).

**Proposition 4.1.** Consider a Young linear system in \( \mathbb{R}^n \)

\[ dx_t = A x_t \ dZ_t. \tag{23} \]

If dimension \( n > 2 \), then there exist a pair of parallel foliations \( \mathcal{F}(E_1), \mathcal{F}(E_2) \) generated by complementary subspaces \( E_1 \) and \( E_2 \) such that the decomposition of the flow of equation (23) exists for all time \( t \in [0, T] \), i.e. there is no explosion time of the decomposition. Dimension of \( E_1 \) can be chosen as a number of the form \( a+2b \) where \( a = 0, 1, \ldots, r = \# \{ \text{real eigenvalues with multiplicities} \} \), and \( b = 0, 1, \ldots, (n - r)/2 \).
Proof. Let $A = PJP^{-1}$ be the canonical real Jordan form of $A$, with the choice of bases $P$ such that the nilpotent component has, if necessary, 1’s and identities $I_2$’s above the diagonal. The change of coordinates $y = Px$ establishes the conjugate Young system:

$$dy_t = Jy_t \, dZ_t.$$ 

If $n > 2$, it is possible to write

$$J = \begin{pmatrix}
(J_1)_{k \times k} & (J_2)_{k \times \ell} \\
(J_3)_{\ell \times k} & (J_4)_{\ell \times \ell}
\end{pmatrix}$$

with $k = a + 2b$ and its complementary $\ell = n - k$, such that the submatrix $(J_3)_{\ell \times k} = 0$. The number $a$ represents the number of real eigenvalues in the block $J_3$ and $b$ represents the number of pairs of conjugate nonreal eigenvalues in this block. Hence, equations (22) guarantee the there is no explosion in the decomposition of $y_t$. By conjugacy, there is also no explosion in the decomposition of the linear fundamental solution $F_t$ of (23) along the foliations generated by $E_1 = P(\mathbb{R}^k \times \{0\})$ and $E_2 = P(\{0\} \times \mathbb{R}^\ell)$. This proves the Proposition. \[\square\]

Using the notation in the proof of last Proposition, the decomposition of $F_t = \eta_t \circ \psi_t$ above are such that $\eta_t$ lies in the group $PG_H P^{-1}$ and $\psi_t$ lies in $PG_V P^{-1}$.

4.2. Principal Fibre Bundles Over Homogeneous Spaces

Let $G$ be a connected Lie group with a closed subgroup $H$ and denote by $\mathfrak{g}$ and $\mathfrak{h}$ their Lie algebras of right invariant vector fields, respectively. The group $G$ acts on $H$ by left translation $gH$, for all $g \in G$ and the orbits generate the homogeneous space $M := G/H$, see e.g. [17]. We have a principal fibre bundle given by the canonical projection $\pi : G \rightarrow M$. Given an element $A \in \mathfrak{g}$ consider the right invariant YDE:

$$dg_t = Ag_t \, dZ_t. \tag{24}$$

As it has done in Section 2.2, here, we consider a connection $\omega$ in the principal fibre bundle $\pi : G \rightarrow M$. In this example we construct our decomposition of flow according to the vertical subspaces (involutive) and the horizontal subspace established by this connection. The solution flow (global in $G$ up to lifetime of $Z_t$) is given by left action:

$$\varphi_t(x) = g_t x,$$

where $g_t = \exp\{AZ_t\}$. In this example the distributions $\Delta^1$ and $\Delta^2$ in the tangent space $TG$ are given by the horizontal subspaces with respect to the connection $\omega$ and the tangent to the fibres $gH$ (involutive). In order to decompose the flow $\varphi_t$ as in Theorem 3.4, one has to identify the vector fields $V$ and $h$ as in equation (16) in the proof of the theorem, i.e.:

$$Ax := h + V(\eta, x)$$
Elements \( \eta \in \text{Diff}(\Delta^1, G) \) can be written pointwise (with respect to \( x \in G \)) as a left action of elements of \( G \) at \( x \). This action preserves the vertical component, i.e. \( g_* \Delta^2 = \Delta^2 \) for all \( g \in G \). Hence, vector field \( V \) above is independent of \( \eta \) and one can easily calculate:

\[
V(x) = (Ax)^* \quad \text{and} \quad h = Ag - \omega(Ax)^*.
\]

By equations (17) and (18) we have that each component of the decomposition \( \varphi_t(\cdot) = \eta_t \circ \psi_t(\cdot) \) are given by:

\[
d\eta_t = R_{\eta_t}(A\eta_t(\cdot) - \omega(A\eta_t(\cdot)))^* \quad \text{(25)}
\]

and

\[
d\psi_t = \text{Ad}(\eta_t) \omega(A\eta_t(\cdot))^*. \quad \text{(26)}
\]

Proposition 4.2. Consider the decomposition \( \varphi_t(\cdot) = \eta_t \circ \psi_t(\cdot) \) of the solution flow of equation (24) according to horizontal and vertical distribution of the fibre bundle in the sense of Theorem 3.4. Then, at each point \( x \in G \), the first component can be written as the left action:

\[
\eta_t(x) = g_t^{H,x} x,
\]

and the second component can be written as the right action:

\[
\psi_t(x) = x h_t
\]

where \( h_t = x^{-1} (g_t^{H,x})^{-1} g_t x \).

Proof. The proof of the first equation follows straightforward when one applies equations (25) at a fixed initial condition \( x \in G \): it is the horizontal lift of \( \pi(g_t x) \), cf. Theorem 2.6, using Itô formula 2.3. Regarding the second equation of the statement, one sees that \( (x^{-1} (g_t^{H,x})^{-1} g_t x) \in H \) by definition of the horizontal lift: \( g_t^{H,x} x = g_t x v_t \) for some \( v_t \in H \). With this notation, fixing the action at a point \( x \in G \), the equations above reduce to well known finite dimensional equations (in \( G \)). This is the content of the following

Trivial fibre bundles: As a particular case, consider a trivial principal fibre bundle \( \pi : G \times H \rightarrow H \) with structural group \( H \), where \( G \) and \( H \) are connected Lie groups. The trivial connection is given by \( \omega(x, y)(g_t', h_t') = y^{-1} h_t' \in \mathfrak{h} \). Consider a right invariant YDE in \( G \times H \):

\[
d(x_t, y_t) = (A \times B) (x_t, y_t) dZ_t
\]

where \( A \in \mathfrak{g} \) and \( B \in \mathfrak{h} \), the Lie algebras of \( G \) and \( H \) respectively, with an initial condition \( (x_0, y_0) \). Since the connection in this case is invariant by left action of \( G \times \{Id\} \), the factor \( g_t^{H,x} \in G \times H \) of Proposition 4.2 does not depend on \( (x, y) \). One recovers the trivial components of the decomposition.
In fact we get a global decomposition where the first component is given by the left action:
\[ \eta_t(\cdot, \cdot) = (\exp(AZ_t), Id)(\cdot, \cdot). \]

And the second (vertical) component is given in terms of the right action:
\[ \psi_t(\cdot, \cdot) = (\cdot, \cdot)(Id, h_t) \]

where \( h_t = y^{-1} \exp(BZ_t) y \), according to Proposition 4.2.

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