ZETA INTEGRALS ON ARITHMETIC SURFACES

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Abstract. Let $C$ be a smooth, projective, geometrically connected curve over a global field $k$, with $L$-function $L_C$. Standard conjectural properties of the Hasse–Weil $L$-function $L_C$ correspond to analogous properties of zeta functions $\zeta_S$ of proper regular models $S$ of $C$. By taking into account the additional contribution of finitely many horizontal curves on $S$, we are lead to study a modified zeta function $Z_S$. When $C$ has simple reduction properties, we show that $Z_S$ is an integral over certain two-dimensional “analytic adeles”. We conclude by understanding a mean-periodicity condition on a transform $Z_S$, comparable to automorphy of $L_C$, in terms of adelic integrals. These results are an extension of those of Fesenko in the case where $C = E$ is an elliptic curve, the primary difference being a renormalising factor whose arithmetic interpretation is a power of $\zeta_{\mathbb{P}^1(O_k)}$.

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1. Introduction

Let $S$ be a scheme of finite type over $\mathbb{Z}$. The zeta function of $S$ is defined on $\Re(s) > \dim(S)$ by the Euler product

$$\zeta(S, s) = \prod_{x \in |S|} \frac{1}{1 - |k(x)|^{-s}},$$

where $|S|$ denotes the atomisation of $S$, that is, its set of closed points, and $k(x)$ is the residue field at a closed point $x$. The question of meromorphic continuation of such zeta functions remains open, along with the conjectural functional equation with respect to $s \mapsto \dim(S) - s$.

A basic case is when $S$ is a proper, regular model of a smooth projective curve $C$ over a number field $k$. Associated to each étale cohomology group of $C$, one has a Hasse–Weil $L$-function $L(H^i(C), s)$ defined as the Euler product of characteristic polynomials of the action of Frobenius. The cases $i = 0, 2$ reduce to Dedekind zeta functions of the ground field $k$, and so the unknown quantity is $L(C, s) := L(H^1(C), s)$. The meromorphic continuation and functional equation of $\zeta(S, s)$ is equivalent to that of $L(C, s)$. The precise relationship between $\zeta$ and $L$ takes the following form

$$\zeta(S, s) = n(S, s) \frac{\zeta(k_i(s)) \zeta(s - 1)}{L(C, s)}.$$

This can essentially be viewed as the definition of $n(S, s)$, which is easily seen as a product of explicit functions rational in a variable of the form $q^{-s}$. This product is finite, including only bad primes. That it has the required functional equation with respect to $s \mapsto 2 - s$ is an exercise in étale cohomology.

By taking into account the additional contribution of finitely many horizontal curves on $S$, in this paper we study a modified zeta function

$$Z(S, \{k_i\}, s) = \zeta(S, s) \prod_{i=1}^n \zeta(k_i, s/2),$$

where the number fields $k_i/k$ are determined by the horizontal curves. Up to sign, the functional equation of $L(C, s)$ is equivalent to

$$Z(S, \{k_i\}, s) = Z(S, \{k_i\}, 2 - s)^2,$$

where $Z$ is not quite the product of the completions:

$$Z(S, \{k_i\}, s) = A(S)^{(1-s)/2} \Gamma(S, s) \zeta(S, s) \prod_{i=1}^n \xi(k_i, s/2).$$

In the above expression $A(S)$ denotes the conductor of $S$ and $\Gamma(S, s)$ denotes the gamma factor, both of which will be defined in the main body.

For number fields, or curves over finite fields, it is well understood that the analytic properties of zeta functions can be obtained through harmonic analysis on a commutative adelic group - this is reviewed in section 2. These techniques have long since been extended to various non-commutative algebraic groups. Our goal is to develop this idea on certain two-dimensional adelic groups. We will review the theory of two dimensional local fields in section 4.3, and two-dimensional analytic adeles in 4.4. The most fundamental issue is that these groups are not locally compact, and so what we mean by “harmonic analysis” has to be somewhat modified. The development of harmonic analysis on more general topological
groups is of the utmost importance, as mentioned as far back as Weil [29, Foreword]. Following a sketch given in [9, Section 57], section 6 introduces zeta integrals extending those of Fesenko in the case where \( C = E \) is an elliptic curve, the primary difference being a renormalising factor whose arithmetic interpretation is a power of \( \zeta(P_{O_k}) \). Fesenko’s original zeta integrals for elliptic curves diverge for higher genus curves, due to a certain incompatibility of the additive and multiplicative measures, which is rectified by the renormalizing factor. Moreover, there is a simple connection between this factor and the gamma factor of the completed zeta function, as will be explained.

When \( C \) has simple reduction properties (which can always be obtained after base change), we will show that \( Z(S, \{ k_i \}, s)^2 \) is an integral over certain two dimensional “analytic adeles”, up to the square of a rational function \( Q(s) \) of the following form

\[
Q(s) = \frac{C}{(1-s)^m},
\]

for constants \( C \in \mathbb{C} \) and \( m \in \mathbb{N} \), each of which depend on the base field \( k \). Therefore \( Q(s)^2 \) is invariant with respect to \( s \mapsto 2-s \). The required reduction properties for the integral expressions are explained in [3] and are broadly comparable to semistability. When \( C \) possesses these reduction properties, the conductor \( A(S) \) arises from counting singularities on bad fibres.

The fact that it is the square of the modified completed zeta function appearing in the integral expressions is due to the fact we integrate over two-copies of the multiplicative group of the analytic adeles. Of course, one might expect that integrating over a single copy would give rise to the completed zeta function itself - this is not true at finitely many factors. There are two further reasons for considering the square of the zeta function. Firstly, in this way we avoid issues with the sign of the functional equation. The second reason concerns a compatibility with two-dimensional class field theory, which will be considered again in the final chapter where we consider the foundations of a \( \text{GL}_1(A(S)) \)-theory.

The Hasse–Weil \( L \)-function is expected to be automorphic. This expectation is not held for the zeta function \( \zeta(S, s) \). One possible replacement is the notion of mean-periodicity, as studied extensively in [5]. We will state the relevant conjecture in 7.2 and conclude this paper with an adelic interpretation of the mean-periodicity condition.

2. Tate’s Thesis

We will begin by summarizing the content of Tate’s thesis for Dedekind zeta functions [28]. Let \( k \) be a number field with ring of integers \( O_k \). The Dedekind zeta function of \( O_k \) is then the zeta function of the arithmetic scheme \( S = \text{Spec}(O_k) \):

\[
\zeta(S, s) = \zeta(k, s).
\]

Up to a constant factor, the locally compact topological group of ideles \( A_k^\times \) of \( k \) has a unique Haar measure \( \mu \). Each Haar measure defines module map

\[
| | : A_k^\times \to \mathbb{R}_+^\times,
\]

which we will normalize so that, for \( \alpha \in k^\times \hookrightarrow A_k^\times \) (diagonal embedding):

\[
|\alpha| = 1.
\]
Let \( f \in S(\mathbb{A}_k) \) be defined as follows

\[
f = \bigotimes_v f_v
\]

\[
f_v(\alpha) =
\begin{cases}
\text{char}(\mathcal{O}_v)(\alpha), & v \nmid \infty \\
\exp(-\pi \alpha^2), & v \text{ real.} \\
\exp(-2\pi |\alpha|^2), & v \text{ complex.}
\end{cases}
\]

For all \( s > 1 \), the following integral absolutely converges

\[
\zeta(f, s) := \int_{\mathbb{A}_k^\times} f(x)|x|^s d\mu(x) = \xi(k, s).
\]

We will call integrals of this form “one-dimensional zeta integrals.” Such integrals were known and studied by Artin, Weil, Iwasawa, Tate and many other mathematicians.

In order to proceed, one applies basic techniques of integration (the Fubini property) and harmonic analysis on the locally compact multiplicative group of ideles. Adelic duality, whose incarnation is the theta formula and Riemann-Roch theorem, then implies the analytic continuation and functional equation of the zeta-function. More precisely, one shows that there exists an entire function \( \eta_f(s) \) such that

\[
\xi(k, s) = \eta_f(s) + \eta_f(1 - s) + \omega_f(s),
\]

where \( \hat{f} \) is the Fourier transform of \( f \), and \( \omega_f \) is the Laplace transform of a rational function:

\[
\omega_f(s) = \int_0^1 h_f(x)x^s \frac{dx}{x},
\]

\[
h_f(x) = -\mu(\mathbb{A}_k^1/x)(f(0) - x^{-1}\hat{f}(0)).
\]

The explicit form of \( h_f(x) \) clearly implies the meromorphic continuation and functional equation of \( \xi(k, s) \). The function \( h_f(x) \) is closely related to an integral over the weak topological boundary of a global subspace of the adeles. More precisely, if \( \mathbb{A}_k^1 \) denotes the set of ideles of norm 1, then

\[
h_f(x) := -\int_{\gamma \in \mathbb{A}_k^1/k^\times} \int_{\beta \in \partial k^\times} (f(x\gamma\beta) - x^{-1}\hat{f}(x^{-1}\gamma\beta))d\mu(\beta)d\mu(\gamma).
\]

The boundary \( \partial k^\times \) is with respect to the weak (or “initial” topology - see \([2, 1, 2.3]\)) on \( \mathbb{A}_k \), which is simply \( k \backslash k^\times = \{0\} \).

More generally, one can consider one-dimensional zeta integrals where \( |\cdot|^s \) is replaced by an arbitrary quasi-character \( \chi \) of the multiplicative group of ideles. In this setting one deduces the basic analytic properties of Hecke \( \Lambda \)-functions.

Fesenko attempted to extend these ideas to dimension 2 as follows \([9, 8]\). Let \( E \) be a proper, regular model of an elliptic curve \( E \) over a number field \( k \), then \([9, \text{Section 3}]\) shows that there is an entire function \( \eta_E \) such that

\[
A(E)^{1-s}\zeta(E, s)^2 = \eta_E(s) + \eta_E(2 - s) + \omega_E(s),
\]

where \( \omega_E(s) \) is defined for \( \Re(s) > 2 \). Generalizing the embedding \( k \hookrightarrow \mathbb{A}_k \), there is a semiglobal ring of adeles \( \mathbb{B}(E) \hookrightarrow \mathbb{A}(E) \) such that, with respect to an inductive limit of weak topologies for a given family of characters, \( \omega_E(s) \) is closely related to an integral over the boundary of \( \mathbb{B}(E) \). We will develop these ideas for higher genus curves.
3. Choice of Model for Adelic Analysis

We now specify a model $\mathcal{S}$ of $C$ simplistic enough for application of two dimensional adelic analysis in its current form. A further development of the theory of lifted harmonic analysis should allow for application to a more general class of arithmetic surfaces. If one is willing to base-change, no restrictions are required. The final section considers peripherally the problem of descent to the ground field, invoking integral representations of certain twisted zeta functions.

Let $\mathcal{B}$ be a Dedekind scheme of dimension 1, and let $\pi : \mathcal{S} \to \mathcal{B}$ be a regular, integral, projective, flat $\mathcal{B}$-scheme. We will call such an $\mathcal{S}$ an arithmetic surface. Closed, irreducible curves on $\mathcal{S}$ are either horizontal or vertical. More precisely, such curves are either an irreducible component of a special fiber or the closure of a closed point of the generic fiber, the latter being finite and surjective onto the base $\mathcal{B}$.

Since $\mathcal{S}$ is regular, the special fibre $\mathcal{S}_b$ over a closed point $b \in \mathcal{B}$ is the Cartier divisor $\pi^*b$. If a given special fiber $\mathcal{S}_b$ contains $r$ irreducible components $\mathcal{S}_{b,i}$, with multiplicity $d_i$, then, as Weil divisors

$$\mathcal{S}_b = \sum_{1 \leq i \leq r} d_i \mathcal{S}_{b,i}.$$ 

An effective divisor $D$ on a regular Noetherian scheme $X$ is said to have normal crossings if, at each point $x \in X$, there exist a system of parameters $f_1, \ldots, f_n$ of $X$ at $x$ such that, for some positive integer $m \leq n$, there are integers $r_1, \ldots, r_m$ such that $\mathcal{O}_X(-D)_x$ is generated by $f_1^{r_1} \ldots f_m^{r_m}$. If $D = \mathcal{S}_b$ is the fibre over $b \in \mathcal{B}$, below we will ask for this property to be true over the residue field $k(b)$, in short, we will be asking for split singularities.

The zeta function depends only on the atomization of $\mathcal{S}$, in particular the zeta function agrees with that of the reduced part $\mathcal{S}_{\text{red}}$. With that in mind, for the purposes of adelic analysis we will only work with the reduced part of each fiber, tacitly using the same notation:

$$\mathcal{S}_b := \sum_{1 \leq i \leq r} \mathcal{S}_{b,i}.$$ 

On finitely many reduced fibres $\mathcal{S}_b$, there may well be non-smooth points. In this chapter we will only work with ordinary double points. Moreover, we need $\mathcal{S}_b$ to be a normal crossing divisor over $k(b)$, so we will assume the ordinary double points are split. To summarize, we will assume that the reduced part of each fiber on $\mathcal{S}$ has only split ordinary double points.

From now on, $\mathcal{B}$ will be $\text{Spec}(\mathcal{O}_k)$, where $k$ is a number field. Let $C$ be a smooth, projective geometrically irreducible curve of genus $g$ over $k$, such that $C$ has good reduction in all residual characteristics less than $2g + 1$. This ensures that the Swan character is trivial and the conductor of $\mathcal{S}$ can be computed by counting singularities as in [26].

**Remark 3.1.** Some authors describe an arithmetic surface $\mathcal{S} \to \text{Spec}(\mathcal{O}_k)$ as semistable if $\mathcal{S}$ is a regular $\text{Spec}(\mathcal{O}_k)$-curve with smooth generic fiber and all closed fibers are reduced normal crossing divisors.

One could work with a more general class of curves by incorporating extensions of the base field. More precisely, let $C$ be a smooth, projective geometrically connected curve of genus $\geq 2$ over the function field $K$ of a one-dimensional Dedekind scheme $\mathcal{B}$. By the Deligne–Mumford theorem ([3], [19, Theorem 10.4.3]), there exists a Dedekind scheme $\mathcal{B}'$ with function field $K'$ such that the extension $C_{K'}$ has a unique stable model over $\mathcal{S'}$. One can
take the extension $K'/K$ to be separable. The extension of base field required by the Deligne–Mumford theorem is not intractable. Let $G := \text{Gal}(K'/K)$, which has a natural action on $\mathcal{S}$, lifting its natural action on $\text{Spec}(\mathcal{O}_{K'})$. The stable reduction, along with its natural $G$-action determines the local factors of the $L$-function (for example, see [3, Theorem 1.1]). Throughout, the function field of $\mathcal{S}$ is denoted by $K$. Closed points of $\mathcal{S}$ are denoted $x$, and $y$ will denote an irreducible fibre or horizontal curve. When $y$ is an irreducible component of a fibre, its genus is denoted $g_y$ and function field $k(y)$. The maximal finite subfield of $k(y)$ has cardinality denoted $q(y)$. The set of components of a fibre $S_p$ is denoted by $\text{comp}(S_p)$.

If $x$ is a singular point on a fibre $S_p$, then $S_p(x) = \bigcup_{y \in \text{comp}(S_p)} y(x)$, where $y(x)$ denotes the set of local branches of $y$ at $x$.

### 4. Two-Dimensional Local Fields

Let $\mathcal{S}$ be a two dimensional, irreducible, Noetherian scheme and let $x \in y \subset \mathcal{S}$ be a complete flag of irreducible closed subschemes. If $m$ is a local equation for $x$ and $p$ is a local equation for $y$, then let $\mathcal{O} = \hat{\mathcal{O}}_{\mathcal{S},x}$ and

$$K_{x,y} = \text{Frac}(\hat{\mathcal{O}}_{\mathcal{S},x}) / \mathcal{O},$$

see, for example, [1], [2], [3], [10], [23] Sections 6, 7]. If $x$ is a smooth point of $y$, then $K_{x,y}$ is an example of a two-dimensional local field; it is a complete discrete valuation field whose residue field is a one-dimensional local field. If $x$ is a singular point on $y$, the same construction yields a direct product of two-dimensional local fields. Recall that $y(x)$ is the set of local branches of $y$ at $x$, then

$$K_{x,y} = \prod_{z \in y(x)} K_{x,z},$$

where $K_{x,z}$ is the two-dimensional local field associated to $x$ and the minimal prime $z$. The residue field of $K_{x,z}$ will always be denoted $E_{x,z}$. A lift of a local parameter from $E_{x,z}$ to $K_{x,z}$ will be denoted $t_{1,x,z}$, and the cardinality of the residue field of $E_{x,z}$ (which is the second residue field of $K_{x,z}$) is denoted $q(x,z)\). Let $F$ be a two-dimensional local field. As a complete discrete valuation field, it has the discrete valuation

$$v_2 : F \to \mathbb{Z}.$$ 

For this valuation we fix a local parameter $t_2$. We denote the ring of integers with respect to this valuation $\mathcal{O}_F$. On the residue field $\overline{F}$ we have the discrete valuation

$$v_1 : \overline{F} \to \mathbb{Z}.$$ 

Together, $v_1$ and $v_2$ induce a “rank 2” valuation on $F$, which depends on $t_2$:

$$\underline{v} : F \to \mathbb{Z}^2
\alpha \mapsto (v_1(\alpha t_2^{-v_2(\alpha)}), v_2(\alpha)),$$

where $\mathbb{Z}^2$ is given the lexicographic ordering. Let $O_F$ denote the ring of integers with respect to $v$, we have:

$$O_F = \{x \in \mathcal{O}_F : \pi \in \mathcal{O}_F\}.$$
Unlike the classical situation, there are infinitely many different rank 2 discrete valuations on $F$, however, the ring of integers and maximal ideal do not depend on this choice. When $F = K_{x,y}$ we use the notations:

$$\mathcal{O}_F = \mathcal{O}_{x,y},$$

$$O_F = O_{x,y},$$

and when $y = S_p$ is the fibre of $S$ over $p \in \text{Spec}(\mathcal{O}_k)$ we will write

$$\mathcal{O}_{x,p} := \mathcal{O}_{x,S_p},$$

$$O_{x,p} := O_{x,S_p}.$$

It is well known that complete discrete valuation fields have a nontrivial $\mathbb{R}$-valued Haar measure only when their residue field is finite. In particular, there is no $\mathbb{R}$-valued Haar measure on higher dimensional local fields. A lifted $\mathbb{R}((X))$-valued Haar measure and integration theory appeared in \cite{7, 9}. In these papers Fesenko develops two approaches to the theory of higher Haar measure on higher local fields, taking values in formal power series over $\mathbb{R}$. A third, lifting approach, suggested in \cite{9} was further developed by Morrow in \cite{21}. All these approaches give essentially the same translation invariant measure on a class of measurable subsets of $F$. There is also a model-theoretic approach of Hrushovski-Kazdhan \cite{12}.

**Example 4.1.** Let $F$ be a two-dimensional local field, with a fixed local parameter $t_2$ and residue field $K$. On the locally compact field $K$ we have a Haar measure $\mu_K$, normalized so that $\mu_K(\mathcal{O}_K) = 1$. Let $\mathcal{A}$ be the minimal ring of sets generated by $\alpha + t_2^j p^{-1}(S)$, where $S$ is $\mu_K$-measurable, the “measure” of a generator of $\mathcal{A}$ is $X^i \mu_K(S) \in \mathbb{R}((X))$. For example $\mu_F(\mathcal{O}_F) = 1$, where $\mathcal{O}_F$ is the rank two ring of integers. This measure extends to a well-defined additive function on $\mathcal{A}$, which is moreover, countably additive in a certain refined sense, \cite{6, Part 6}, \cite{7}, \cite{9}.

We observe the following:

1. Essential role was played by a choice of a local parameter $t_2$, i.e. the splitting of the residue map. An analogous statement will be true in the adelic counterpart of example \cite{7, 2}.

2. In the mixed characteristic case there are nonlinear changes of variables for which the Fubini property of the measure does not hold \cite{20}. This could be considered as an example of the non-commutativity inherent in studying $L$-functions of curves over global fields. In this chapter, such considerations will not cause a problem.

5. **Analytic Adeles**

Let $X$ be a Noetherian scheme, let $M$ be a quasi-coherent sheaf on $X$, and let $T$ be a set of reduced chains on $X$. To such a triple $(X, M, T)$, one can associate an abelian group $A(X, M, T)$ of adeles. We will call these groups “geometric adeles” and recommend the following references for details \cite{24}, \cite{25}, \cite{11}, \cite{13}, \cite{10} and \cite{23, Section 8}.

The adelic group $A(X, M, T)$ can be interpreted as a restricted product over $T$ of local factors, which are obtained by localising and completing along each flag. Often, one takes $T$ to be the set of all reduced chains on $X$, and we denote the resulting group by $A(X, M)$. $A(X, M)$ has more structure than that of an abelian group - it admits a semi-cosimplicial structure whose cohomology is that of $M$. 

7
Let \( y \) be an irreducible curve on \( S \). If \( T \) is the set of all reduced chains formed by closed points on \( y \), then we will denote \( A(S, \mathcal{O}_S, T) \) by \( A(y) \). Later (remark 5.3) we will see that this space is "too big" for integration, which motivates us to introduce the smaller spaces of "analytic" adeles \( \mathbb{A}(y) \), following the constructions of [9, Section 1].

As mentioned in 3, there are two types of irreducible curves on \( S \) - irreducible components of fibres and horizontal curves. Whilst there is no real difference in the construction of \( A(y) \), we will treat the two cases separately so as to emphasize some important aspects in each setting. In particular, the fibres may well be singular, and the horizontal curves contain archimedean information.

5.1. Fibres. Let \( y \) be an irreducible component of the fibre \( S_p \) over \( p \in \text{Spec}(\mathcal{O}_k) \). For any \( n \geq 0 \) and any point \( x \in y \), one can define local lifting maps

\[
l^n_{x,y} : E^{\oplus n}_{x,y} \to \begin{cases} \mathcal{O}_{x,y}, & \text{if } K_{x,y} \text{ is of equal characteristic;} \\ \mathcal{O}_{x,y}/t^n \mathcal{O}_{x,y}, & \text{otherwise,} \end{cases}
\]

and, subsequently, adelic lifting maps

\[
L^n_{y} : \mathbb{A}(k(y))^{\oplus n} \to \begin{cases} (K_{x,y})_{x \in y} \\ (\mathcal{O}_{x,y}/t^n \mathcal{O}_{x,y})_{x \in y} \end{cases}.
\]

For details of these constructions, the reader is referred to [9, Section 1.1]. The \( y \)-component of the analytic adeles is the following ring:

\[
\mathbb{A}(y) = \{(a_{x,y})_{x \in y} : a_{x,y} \in K_{x,y}, \forall n \geq 0, (a_{x,y}) + t^n \mathcal{O}_y \in \text{im}(L^n_{y}) \}.
\]

Recall that, when \( x \) is a singular point on \( y \), \( K_{x,y} \) is in fact \( \prod_{z \in y(x)} K_{x,z} \)

For \( a_{x,y} \in \mathcal{O}_{x,y} = \prod_{z \in y(x)} \mathcal{O}_{x,z} \), let \( \overline{a}_{x,y} = (\overline{a}_{x,z})_{z \in y(x)} \) denote the image of \( a_{x,y} \) under the residue map to \( \prod E_{x,z} \). We thus have

\[
p_y : \mathbb{A}(y) \to \mathbb{A}(k(y))
\]

\[
(a_{x,y}) \mapsto (\overline{a}_{x,y}).
\]

**Definition 5.1.** Let \( S_p \) denote the fibre of \( S \) over \( p \). The \( S_p \)-component \( \mathbb{A}(S_p) \) of the analytic adeles is

\[
\mathbb{A}(S_p) = \prod_{y \in \text{comp}(S_p)} \mathbb{A}(y).
\]

We have a residue map

\[
p_p = (p_y) : \mathbb{A}(S_p) \to \prod_{y \in \text{comp}(S_p)} \mathbb{A}(k(y)).
\]

The following example gives a concrete interpretation of analytic adelic spaces and the subsequent remark explains why their measure theory cannot be extended to geometric adeles.

---

1The geometric adeles \( \mathbb{A}(S) \) exist for arbitrary Noetherian schemes \( S \). One can consider what the general definition of the analytic space \( \mathbb{A}(S) \) is and what role it plays in algebraic geometry.
Example 5.2. Let $S$ be a surface over a finite field and let $y$ be a nonsingular irreducible curve on $S$, with function field $k(y)$. Associated to $y$ we have the complete discrete valuation field

$$K_y = \text{Frac}(\hat{O}_y).$$

We fix a local parameter and denote it by $t_y$, it can be taken as a second local parameter for all two-dimensional local fields associated to closed points $x$ on $y$. We will refer to it as a local parameter of $y$. The ring $\mathbb{A}(k(y))$ is locally compact and has a Haar measure $\mu_{\mathbb{A}(k(y))}$.

We have a non-canonical isomorphism

$$\mathbb{A}(y) \cong \mathbb{A}(k(y))((t_y)).$$

Let $p$ be the map to $\mathbb{A}(k(y))$ sending a power series to its free coefficient. As in example 4.1, one can construct an $\mathbb{R}((X))$-measure $\mu_y$ on $\mathbb{A}(y)$. Let $S$ be a measurable subset of $\mathbb{A}(k(y))$, then

$$\mu_y(t^i_y p^{-1}(S)) = X^i \mu_{\mathbb{A}(k(y))}(S).$$

Remark 5.3. As in the above situation, let $y$ be an irreducible curve on $S$. If $A(y)$ is the group of geometric adeles associated to $y$ on $S$, $M = \mathcal{O}_S$ and $T$ is the set of all reduced chains of the form $x \in y \subset S$, then

$$A(y) = \bigcup_{r \in \mathbb{Z}} t^r_y A(y).$$

$A(y)$ can be understood as a restricted direct product of $A(y)$ in which almost all components lie in $A(y)$. Since the measure of $A(k(y))$ is infinite, the measure of $A(y)$ in the previous example is infinite. The geometric adeles are therefore a restricted product with respect to a set of infinite measure, and so we cannot extend the measure to $A$.

5.2. Horizontal Curves. Horizontal curves on $S$ will play a crucial role in this chapter. We will begin by explaining their archimedean content - roughly, each horizontal curve intersects the archimedean fibers of the surface, as we now explain in more detail (see also [22, Section 5]).

By an archimedean fibre, we mean the fiber product

$$S_\sigma = S \times_{\text{Spec}(\mathcal{O}_k)} k_\sigma.$$

where $\sigma$ is an archimedean place of the base field $k$, with corresponding completion $k_\sigma$. There is a natural morphism from an archimedean fiber to the generic fiber

$$S_\sigma \to S \times_{\mathcal{O}_k} k = S_\eta \cong C.$$

The fibre over any closed point on the generic fibre $C \cong S_\eta$ is a finite reduced scheme. A horizontal curve $y$ on $S$ is the closure $\overline{\{z\}}$ of a unique closed point $z \in C$, which has residue field $k(z)$. There are only finitely many points on $S_\sigma$ which map to $z$, and they are the primes of $k_\sigma \otimes_k k(z)$, which correspond to the infinite places of $k(z)$ extending $\sigma$ on $k$.

At a closed point $\omega$ on $S_\sigma$ we have a two-dimensional local field

$$K_{\omega,\sigma} = \text{Frac}(\mathcal{O}_{S_\omega,\omega}).$$

The residue field of $K_{\omega,\sigma}$ is denoted $k_\sigma(\omega)$ and is either $\mathbb{R}$ or $\mathbb{C}$. We have, respectively

$$K_{\omega,\sigma} \cong \begin{cases} \mathbb{R}((t)) \\ \mathbb{C}((t)) \end{cases}$$
Let \( y \) be a horizontal curve on \( S \), and let \( \sigma \) be an archimedean place of \( k \). By the correspondence just described, we have an archimedean place \( \omega \) of \( k(y) \) and a two-dimensional local field
\[
K_{\omega,y} = k(y)_{\omega}((t_{\omega})),
\]
where \( k(y)_{\omega} \) is the completion of \( k(y) \) at \( \omega \).

Repeating the construction from 5.1, we obtain a lifting map
\[
l_{\omega,y}^{n} : k(y)^{\otimes n}_{\omega} \to \mathcal{O}_{\omega,y} \cong \begin{cases} \mathbb{R}[t], \\ \mathbb{C}[t]. \end{cases}
\]

Also, at a closed point \( x \in y \), we have a local lifting
\[
l^{n}_{x,y} : E_{x,y} \to \mathcal{O}_{x,y}.
\]

Altogether, we have an adelic map:
\[
L_{y} : \oplus_{x \in y} \oplus_{\omega} l_{x,y}^{n} : \mathbb{A}(k(y))^{\otimes n} \to \prod_{x \in y} K_{x,y} \prod_{\omega} K_{\omega,y}.
\]

**Definition 5.4.** Let \( y \) be a horizontal curve on \( S \). The \( y \)-component of the analytic adelic space is:
\[
\mathbb{A}(y) = \{(a_{x,y})_{x \in y}, (a_{\omega,y})_{\omega} \in \prod_{x \in y} K_{x,y} \prod_{\omega} K_{\omega,y} : \forall n \geq 1, \ (a_{x,y})_{x \in y}, (a_{\omega,y})_{\omega} \in \text{im}(L_{y}^{n})\}.
\]

The residue maps \( \mathcal{O}_{x,z} \to E_{x,z} \) and \( \mathcal{O}_{\omega,y} \to k(y)_{\omega} \) induce
\[
p_{y} : \mathbb{A}(y) \to \mathbb{A}(k(y)).
\]

**5.3. Additive Normalization.** We want to extend example 5.2 to \( \mathbb{A}(S_{p}) \) and \( \mathbb{A}(y) \), where \( p \in \text{Spec}(\mathcal{O}_{k}) \) and \( y \) is a horizontal curve on \( S \).

If \( y \) is a fiber or horizontal curve, the additive group of the ring \( \mathbb{A}(k(y)) \) is a locally compact abelian group. It thus has a Haar measure, which is unique up to scalar multiplication. The \( \mathbb{R}((X)) \)-measure on \( \mathbb{A}(y) \) will depend on a choice of normalization of the Haar measure on \( \mathbb{A}(k(y)) \).

Let \( F \) be a two-dimensional local field. If \( F \) is nonarchimedean and \( \psi_{F} : F \to \mathbb{C}^{\times} \) is a character, then we will refer to the orthogonal complement of \( O_{F} \) as the conductor of \( \psi_{F} \). If \( F \) is an archimedean two-dimensional local field, the conductor is the orthogonal complement of \( O_{F} \).

When \( F = K_{x,z} \) (resp. \( K_{\omega,y} \)), we denote \( \psi_{F} \) by \( \psi_{x,z} \) (resp. \( \psi_{\omega,y} \)). The aim is to define the normalization of the measure on \( \mathbb{A}(k(y)) \) through the characters \( \psi_{x,z} \).

**Lemma 5.5.** For any closed point \( x \) on the fibre \( S_{p} \) of \( S \), let \( z \) be a branch of an irreducible component of \( S_{p} \) at \( x \). There are characters \( \psi_{x,z} \) of the two-dimensional local fields \( K_{x,z} \) such that if
\[
\psi_{x,p} = \otimes_{z \in S_{p}(x)} \psi_{x,z},
\]
then the following is defined on \( \mathbb{A}(S_{p}) \):
\[
\psi_{p} = \otimes_{x \in S_{p}} \psi_{x,p}.
\]

Moreover, the conductor \( A_{x,p} \) is commensurable with \( O_{x,p} \), with equality at almost all \( x \in S_{p} \), including the singular points. There is a nontrivial
\[
\varphi_{p} : \mathbb{A}(k(S_{p})) \to \mathbb{C}^{\times}
\]
such that
\[ \psi_p = \varphi_p p_p. \]
Similarly, if \( y \) is a horizontal curve, for all points \( x \in y \) and archimedean places \( \omega \) of \( k(y) \), there are local characters \( \psi_{x,y} \) and \( \psi_{\omega,y} \) such that
\[ \psi_y = \otimes_{x \in y} \psi_{x,y} \otimes_\omega \psi_{\omega,y} \]
is defined on \( \mathbb{A}(y) \) with the same properties.

Proof. [9, Proposition 27]. \( \square \)

From now on we fix such \( \psi_{x,p} \) (resp. \( \psi_{x,y} \)) for all closed points on fibres \( S_p \) (resp. horizontal curves \( y \)).

Definition 5.6. Let \( y \) be an irreducible component of a fibre. If \( x \) is a nonsingular point on \( y \), define \( d(x,y) \) by
\[ A_{x,y} = t_1^{d(x,y)} O_{x,y}, \]
where \( A_{x,y} \) is the conductor of \( \psi_{x,y} \) and \( t_1 \) denotes the local parameter of \( K_{x,y} \). If \( x \) is a split ordinary double point on \( y \), and \( z, z' \) are local branches of \( y \) at \( x \), we will write \( d(x,z) = d(x,z') = -1 \).

Lemma 5.7. Let \( S_p \) be a smooth fibre on \( S \), then
\[ \prod_{x \in S_p} q_{x,p}^{d(x,p)} = 1 \]

Proof. Let \( k(S_p) \) denote the function field of \( S_p \) and let \( v \) be a place of this global field. The residue field at \( v \) has cardinality \( q_v \). The lemma then follows from the representation of the canonical divisor \( C \) on \( z \) as
\[ C = \sum_v m_v v, \]
where \( P_{v}^{m_v} \) is the \( v \)-component of the conductor of the standard character on \( \mathbb{A}(k(z)) \). We know:
\[ N(P_{v}^{m_v}) = q_v^{m_v} = q^{\deg(v) m_v} \]
So
\[ \prod_v N(P_{v}^{m_v}) = q^{-\deg(C)}, \]
and formula follows from the fact that \( \deg(C) = 2g - 2 \). \( \square \)

Let \( \mu_{\mathbb{A}(k(y))} \) be the Haar measure on \( \mathbb{A}(k(y)) \) which is self dual with respect to the character \( \varphi_y \) from lemma [5.5]. If \( y \) is an irreducible curve on \( S \), let \( L \) denote a measurable subset of \( \mathbb{A}(k(y)) \) with respect to the above Haar measure. Consider the lifted measure \( M_{\mathbb{A}(y)} \) on \( \mathbb{A}(y) \) such that
\[ M_{\mathbb{A}(y)}(t_1^{i} p_y^{-1}(L)) = X^i \mu_{\mathbb{A}(k(y))}(L). \]
For example, let \( S_p \) be a smooth fibre of \( S \) over \( p \). Consider the subset
\[ O_{\mathbb{A}}(S_p) = \prod_{x \in S_p} O_{x,p}, \]
then
\[
M_{\mathbb{A}(S_p)}(O\mathbb{A}(S_p)) = \mu_{\mathbb{A}(k(S_p))}(\prod_{x \in S_p} O_x),
\]
\[
= \prod_{x \in S_p} \mu_{k(S_p)}(O_x)
\]
\[
= \prod_{x \in S_p} q^{d(x,y)/2}
\]
\[
= q^{1-g(S_p)},
\]
where \(g(S_p) = g\) denotes the genus of the special fibre \(S_p\).

**Definition 5.8.** Let \(g\) be the genus of \(C\) and \(S_p\) be a smooth fibre, define:
\[
\mu_{\mathbb{A}(S_p)} = q(S_p)^{g-1} M_{\mathbb{A}(y)}.
\]
If \(S_p\) is a singular fiber, then
\[
\mu_{\mathbb{A}(S_p)} = M_{\mathbb{A}(S_p)}.
\]
If \(y\) is a horizontal curve, let \(\mu_{\mathbb{A}(y)} = M_{\mathbb{A}(y)}\).

So, for all smooth fibres and horizontal curves \(\mu_{\mathbb{A}(y)}(O\mathbb{A}(y)) = 1\).

In this paper we will only integrate linear combinations of characteristic functions of measurable sets. For a more general theory, see [9, 1.3].

**6. Zeta Integrals on \(S\)**

Two-dimensional zeta integrals were first studied by Fesenko for proper regular models of elliptic curves [8, 9]. We extend his results to a model \(S\) as in 3, following the sketch in [8, Part 57].

**6.1. Multiplicative Normalization.** First, we recall the relationship between the measure on the additive and multiplicative group of one-dimensional local fields and adeles.

**Example 6.1.** Let \(k\) be a number field. At each non-archimedean prime \(p\) we have a normalized Haar measure \(d\mu\) on the locally compact additive abelian group \(k_p\), which has finite residue field of cardinality \(q(p)\). One then integrates on the multiplicative group \(k_p^\times\) with the measure
\[
(1 - q(p)^{-1}) \frac{d\mu}{|k_p|}.
\]
In turn, one integrates over the idele group \(\mathbb{A}_k^\times\) with the tensor product of these measures.

Similarly, on an arithmetic surface, we need a measure compatible with the multiplicative structure of a two-dimensional local field.

Let \(F\) be a two-dimensional local field with local parameter \(t_2\) and let \(t_1\) be a lift of the local parameter of the residue field of \(F\). Let \(U\) denote the group of principal units. Let \(q\) denote the cardinality of the final (finite) residue field. One can decompose the multiplicative group \(F^\times\) as follows:
\[
F^\times = < t_1 >^x < t_2 >^x U.
\]
Using this decomposition we define the $\mathbb{R}((X))$-valued module $\mid F$ by

$$|t^j u|^F = q^{-j} X^i.$$  

When $F = K_{x,y}$, we use the notation $\mid F = \mid x,y$. Let $z$ denote a local branch of an irreducible curve on $S$ at a point $x$. Motivated by example 6.1 we will use the following measure on $K_{x,z}$:

$$M_{K_{x,z}} = \frac{M_{K_{x,z}}}{(1-q(x,z)^{-1})\mid x,z}.$$  

**Example 6.2.** Let $S_p$ be a smooth fibre of $S$ and consider the following measurable function for each closed $x \in y$

$$f_{x,S_p} = \mid x,S_p \text{ char}(O_{x,p}),$$

and define $f_{S_p} = \otimes_{x \in S_p} f_{x,S_p}$. Then, $f_{S_p}$ is integrable and

$$\int_{A(S_p)} f_{S_p} d\mu_{A(S_p)} = \zeta(S_p, s) \prod_{x \in S_p} q(x,S_p)^{d(x,S_p)(1-s)} = \zeta(S_p, s) \prod_{x \in S_p} q(x,S_p)^{(1-g)(1-s)}.$$  

In our case the special fibres have at worst split ordinary double singularities and we will use an ad hoc variant of the function in example 6.2 to recover the corresponding factor of the zeta-function - for a more complete approach see [9, 36, Remark 1, 37]. When $x$ is a singular point of $S_p$, define

$$M_{K_{x,S_p}} = \otimes_{x \in S_p(x)} M_{K_{x,S_p}}.$$  

**6.2. Zeta Integrals on the Projective Line.** We would like to take the product over all the fibres in order to obtain the non-archi-medean part of the zeta function of $S$, including the conductor. Unfortunately, the product diverges due to the additional factors appearing in examples 6.2 and 6.5.

To resolve this, we begin by observing something complementary that happens when we apply the adelic analysis on the scheme $P := \mathbb{P}^1(O_k)$. At a non-archimedean place $p$ of the base field $k$, the fibre $P_p = \mathbb{P}^1(k(p))$. At a closed point $x \in P_p$, define

$$g_{x,p} = \text{ char}(O_{x,p}),$$

and subsequently,

$$g_p = \otimes_{x \in P_p} g_{x,p}.$$  

Then

$$\int_{A(P_p)} g_p \mid s_p d\mu_{A(P_p)} = \zeta(P_p, s) \prod_{x \in P_p} q_s^{(1-s)}.$$  

Combining this computation with examples 6.2 and 6.5 we see that:

$$\prod_{p \in \text{Spec}(O_k)} \int_{A(S_p)} f_p \mid s_p d\mu_{A(S_p)} \cdot \left(\int_{A(P_p)} g_p \mid s_p d\mu_{A(p)}\right)^{g-1} = \zeta(\mathbb{P}^1(O_k), s)^{1-g} \zeta(S, s) A(S)^{(1-s)/2}.$$  

This is essentially the non-archimedean part of the zeta integral in 6.5 below. The $(1-g)$th power of the zeta integral on $\mathbb{P}^1(O_k)$ conveniently cancels the divergent part of the zeta integral over $S$. But that is not all, as by a completion process for the zeta function
of \( \mathbb{P}^1(\mathcal{O}_k)^{1-g} \), we can recover the gamma factor of \( S \) up to an \([s \mapsto 2 - s]\)–invariant rational function. We will now make this idea precise.

6.3. **The Gamma Factor.** The gamma factor for the zeta function of \( S \) is the quotient of the gamma factors of its Hasse–Weil decomposition, ie.

\[
\Gamma(S, s) = \frac{\Gamma(k, s)\Gamma(k, s-1)}{\Gamma(C, s)}.
\]

The renormalizing factor in fact induces the gamma factor in a very natural way. The zeta function of \( \mathbb{P}^1(\mathcal{O}_k) \) is very simple:

\[
\zeta(\mathbb{P}^1(\mathcal{O}_k), s) = \zeta(k, s)\zeta(k, s-1),
\]

and so its gamma factor is

\[
\Gamma(\mathbb{P}^1(\mathcal{O}_k), s) = \frac{1}{Q(s)\Gamma(S, s)},
\]

where

\[
Q(s) = \frac{\pi^{-(r_1+r_2)(g-1)}(s-1)^{(r_1+r_2)(g-1)}}{R(s)}\Gamma(S, s),
\]

so

\[
Q(2-s) = \pm Q(s).
\]

We thus see that completing the normalizing factor gives us the transcendental part of the gamma factor of \( S \).

6.4. **Integration on Horizontal Curves.** We will remind ourselves of the Haar measure on \( \mathbb{R} \) and \( \mathbb{C} \).

\[
\mu_{k_\sigma}(\omega) = \begin{cases} 
\text{Lebesgue measure}, & dx \\
\text{twice Lebesgue measure}, & 2dz
\end{cases} \quad k_\sigma(\omega) = \begin{cases} 
\mathbb{R} \\
\mathbb{C}
\end{cases}
\]

One then integrates on the multiplicative group \( \mathbb{R}^\times \) (resp. \( \mathbb{C}^\times \)) with the measure \( \frac{dx}{|x|^2} \) (resp. \( \frac{2dz}{|z|^2} \)).
Example 6.3. We have the well-known identities:

\[ \int_{\mathbb{R}^\times} e^{-\pi x^2} |x|^s \frac{dx}{x} = \Gamma(\mathbb{R}, s), \]
\[ \int_{\mathbb{C}^\times} e^{-2\pi|z|^2} |z|^s \frac{2dz}{|z|^2} = 2\pi \Gamma(\mathbb{C}, s). \]

These are precisely the Gamma factors required for the Dedekind-zeta function of a number field at a real (respectively complex) place.

We will integrate on \( K_{\omega,y} \) with the lifted measure from \( k(y) \). At all closed points \( x \) of \( y \) we have the two-dimensional local field \( K_{x,y} \) and the natural higher Haar measure. Altogether, we have a measure on \( A(y) = \prod_{x \in y} A(y) \) for a horizontal curve \( y \).

On a horizontal curve \( y \), we redefine \( |_{y} \) to be \( \prod_{x \in y} |_{x,y} \). On a set \( S \) of fibers and finitely many horizontal curves:

\[ |_S = \prod_{y \in S} |_{y}, \]
where

\[ |_{y} = \prod_{x \in y} |_{x,y}. \]

On a horizontal curve \( y \), we redefine \( A(y)^\times \) to be a maximal subgroup such that the image of \( |_{y} \) is equal to that of \( |_{y}^2 \).

Example 6.4. At a nonsingular \( x \in y \), let \( f_{x,y} = \text{char}(O_x) \). At an archimedean place \( \omega \) of \( y \), let \( f_{\omega,y}(\alpha) = \text{char}(O_{\omega,y})(\alpha)\exp(\text{Tr}_{k(y)/\mathbb{R}}(1) \text{res}_{\omega}(\alpha)) \). Then

\[ \int_{A(y)^\times} f |_{y}^s d\mu_{A(y)^\times} = \zeta(k, f, |_{k}^{|s/2|}, \)

where \( \zeta(k, g, \chi) \) is a classical Iwasawa–Tate zeta integral and

\[ \tilde{f} = \otimes_v \tilde{f}_v \]

\[ \tilde{f}_v(\alpha_v) = \begin{cases} \text{char}(O_v)(\alpha_v), & v \text{ archimedean} \\ e^{-\pi \alpha_v^2}, & v \text{ real} \\ e^{-2\pi |\alpha_v|^2}, & v \text{ complex} \end{cases} \]

by the well-known theory of Iwasawa–Tate this integral defines a meromorphic function on \( \mathbb{C} \) and satisfies a functional equation with respect to \( s \mapsto 2 - s \).

6.5. Zeta Integrals. We are missing the factor at a bad prime \( p \). At a split ordinary double singularity on the fibre \( S_p \) we have two local branches, so that integrating over multiplicative group of the two-dimensional analytic adelic space for \( S_p \) gives us an additional factor that is not present in the zeta function. One way of treating singular and smooth fibres \( y \) in a regular way is by integrating over \( A(y)^\times \times A(y)^\times \), which we give the product measure.

\[ ^2 \text{This potentially confusing notation will be used throughout without much further comment} \]
Example 6.5. Let $y = S_p$ be a fibre over $p$ with singular point $x$. Let $z$ be a branch of $y$ at $x$, then define $f_{x,y}$ on $O_{x,y} \times O_{x,y}$ as follows:

$$f_{x,y} = q_x^{-1} \text{char}(O_{x,z}, t_{1,x,z}^{-1}O_{x,z}).$$

For nonsingular points $x \in y$ put

$$f_{x,y} = \text{char}(O_{x,y}, O_{x,y}).$$

Combining, put $f_y = \otimes_{x \in y} f_{x,y}$, then

$$\int f_y d\mu_{\mathbb{A}(y) \times \mathbb{A}(y)} = A_p(S)^{(1-s)} \zeta(y, s)^2 \prod_{z \in S_p} q_z^{2(1-g_z)(1-s)}.$$

All that remains is to put everything together as an integral over the whole adelic space $\mathbb{A}(S)^\times$, where $S$ contains all fibres of $S$ and finitely many horizontal curves.

Definition 6.6. Combining the previous examples, let

$$f = \otimes_{y \in S} f_y,$$

where $y$ runs over all curves in $S$. $f_y$ is defined as follows:

1. Let $y$ be a nonsingular fiber and $x \in y$ be a closed point, put

$$f_{x,y} = \text{char}((O_{x,y}, O_{x,y})).$$

2. Let $y$ be a fiber with singular point $x$. Choose branches $z, z' \in y(x)$ and put

$$f_{x,y} = q_x^{-1} \text{char}(O_{x,z}, t_{1,x,z}O_{x,z}).$$

3. Let $y$ be a (nonsingular) horizontal curve, for nonarchimedean places of $k$ define $f_{x,y}$ as in point (1). At archimedean places $\omega$ take

$$f_{\omega,y}(\alpha) = \text{char}(O_{\omega,y})(\alpha) \exp(\text{Tr}_{k(\omega)_/\mathbb{R}}(1)|\text{res}_{\omega}(\alpha)|).$$

We introduce the following abbreviated notation:

Definition 6.7. Let $k$ be a number field and $\mathcal{P} = \mathbb{P}^1(\mathcal{O}_k)$. Let $S$ denote a set of curves on $S$, consisting of all fibres and a finite set $S_h$ of horizontal curves on $S$. If $f$ is an integrable function on $\mathbb{A}(S)^\times \times \mathbb{A}(S)^\times$ and $h$ is an integrable function on $\mathbb{A}(\mathcal{P})^\times \times \mathbb{A}(\mathcal{P})^\times$, then the zeta integral $\zeta_{S_2}^{(2)}(f, h, s)$ is defined to be the following product:

$$\prod_{p \in \text{Spec} \mathcal{O}_k} \left( \int_{\mathbb{A}(\mathcal{P}_p)^\times \times \mathbb{A}(\mathcal{P}_p)^\times} h_p |_p \right) \left| s_p^{\mathbb{A}(\mathcal{P}_p)} \right|^{g-1} \int_{\mathbb{A}(S_p)^\times \times \mathbb{A}(S_p)^\times} f_{S_p} |_p \right| s_p^{\mathbb{A}(S_p)} \times$$

$$\prod_{y \in S_h} \left( \int_{\mathbb{A}(y)^\times} f_y |_y \right) \left| s_y^{\mathbb{A}(y)} \right| \times \left( \xi(\mathbb{P}^1(\mathcal{O}_k), s) \right)^{1-g}. $$

---

3 This is the image of $\text{Char}(O_{x,y})$ under Fesenko’s “diamond operator”.
Remark 6.8. This can be viewed as a “renormalized” integral over the adelic spaces $A(S, S)^x$ and $A(P, S)^x$ [9, Part 57]. In the next chapter we will consider this as an integral over the analytic adeles of the non-connected arithmetic scheme

$$S \prod_{i=1}^{g-1} P$$

Remark 6.9. In this chapter we have only specified one integrable function, for a more complete theory see [9, Section 1.3]. In general, integrable functions will only differ at finitely many components.

At a closed point $x \in P_p$, define $h_{x,P_p} = \text{char}(O_{x,P_p}, O_{x,P_p})$, and let $h_{P_p} = \otimes_{x \in P_p} h_{x,P_p}$. Convergence of the preliminary zeta integral in some specified half plane will be a corollary (6.11) of the following computation.

Theorem 6.10. Let $S$ be a set of curves consisting of all fibers and finitely many horizontal curves $\{y_i\}$, each of function field $k(y_i)$. If $f$ is as in definition 6.6 and $h$ is as above, then

$$\zeta^{(2)}(f, h, s) = Q(s)^2 \Gamma(S, s)^2 A(S)^{(1-s)/2} \prod_i \xi(k(y_i), s/2)^2,$$

where $Q(s)$ is a rational function such that

$$Q(s) = \pm Q(2-s),$$

and $\xi(k(y_i), s)$ is the completed Dedekind zeta function of the finite extension $k(y_i)/k$ which satisfies the functional equation:

$$\xi(k(y_i))\left(\frac{s}{2}\right) = \xi(k(y_i))\left(\frac{2-s}{2}\right).$$

Proof. This follows from combining examples 6.2, 6.5, 6.3 and 6.4.

Corollary 6.11. Assume that the integral in the above definition is defined at $f, h$, then it converges for $s \in \{\Re(s) > 2\}$.

Proof. In the case of $f, h$ as in the theorem, the convergence of the zeta integral follows from the well known properties of $\zeta(S, s)$ which are described in [27]. For arbitrary $f, g$ such that the zeta integral is defined, the zeta integral will differ by only finitely many factors.

We will introduce the notation

$$\mathcal{Z}(S, \{y_i\}, s) = \zeta(S, s)A(S)^{(1-s)/2} \Gamma(S, s)Q(s) \prod_i \xi(k(y_i), s/2).$$

Clearly, the zeta function verifies conjecture 22 if and only if $\mathcal{Z}(S, \{y_i\}, s)$ does.

Let $T$ be a two-dimensional arithmetic scheme over $\text{Spec}(O_k)$. For $p \in \text{Spec}(O_k)$, define $|n|_{T_p}$ on $(A(T_p)^x)^x$ by $|(a_1, \ldots, a_n)|^{(n)} = |a_1| \ldots |a_n|$. We will use the product measure on $(A(T_p)^x)^x$. Let $f^{(n)} = (f, \ldots, f)$ and $g^{(n)} = (g, \ldots, g)$, and define $\zeta^{(n)}(S, f, h, s)$ as the following product:

$$\prod_{p \in \text{Spec}(O_k)} \left( \int_{(A(T_p)^x)^x} f^{(n)}_{T_p}(|\cdot|^{(n)}_{T_p})^s d\mu(A(T_p)^x)^x \right)^{g-1} \int_{(A(S_p)^x)^x} f^{(n)}_{S_p}(|\cdot|^{(n)}_{S_p})^s d\mu(A(S_p)^x)^x.$$
Corollary 6.12. For each positive integer $m$, we have
\[ \zeta^{(2m)}(S, f, h, s) = Z(S, \{y_i\}, s)^{2m}. \]

Proof. This follows from Theorem 6.10 and the definitions of measures above. \qed

When the genus of $C$ is 1 and $2m = 2$, we recover Fesenko’s zeta integrals for elliptic curves and the formula in corollary 6.12 agrees with his first calculation. This motivates the following definition:

**Definition 6.13.** Let $S$ be a set of curves on $S$, for integrable functions on
\[ f : (A(S, S)^{\times})^{\times} \to \mathbb{C} \]
and $h$ on $(A(\mathbb{P}^1(O_k))^\times)^{\times}$, the “two-dimensional unramified zeta integral” is
\[ \zeta(S, S, f, h, | |^s) := \zeta^{(2)}(S, f, h, s). \]

We make the following conjecture, extending that of [9, Section 4]:

**Conjecture 6.14.** Provided the set $S$ of curves on $S$ contains finitely many horizontal curves, the zeta integral $\zeta(S, S, f, h, | |^s)$ meromorphically extends to the complex plane and satisfies the following functional equation
\[ \zeta(S, S, f, h, | |^s) = \zeta(S, S, f, h, | |^{2-s}). \]

**Remark 6.15.** Let $S$ contain all fibres of $S$ and finitely many horizontal curves, then
\[ \zeta(S, S, f, h, | |^s) = \zeta(S, S, f, h, | |^{2-s}) \iff \xi(S, s)^2 = \xi(S, 2 - s)^2 \iff \xi(S, s) = \pm \xi(S, 2 - s) \iff \Lambda(C, s) = \pm \Lambda(C, 2 - s). \]

We have integrated over two copies of the multiplicative group of the ring of analytic adeles so as to get the correct factor of the zeta function at split ordinary double points. This is not the only motivation for doing so. In fact, there is a certain compatibility with two-dimensional class field theory that allows us to define “twisted” zeta integrals whose evaluation is an analogue of Hecke $L$-functions for arithmetic surfaces. This will be the subject of a later chapter. Before then, we will formulate the mean periodicity correspondence in terms of this “two dimensional adelic analysis” on $S$.

### 7. Adelic Duality and Filtrations

In dimension two, there are three “levels” to the adeles. On the purely local level, one has the products of fields associated to closed points on irreducible curves. The other extreme is the global object, ie. the function field of the surface. In between one has the local-global complete discrete valuation fields associated to irreducible curves, or closed points on the surface. In combining these spaces, one obtains a filtration on the adeles, and, moreover, semi-cosimplicial complexes which compute the cohomology of quasi-coherent sheaves. The additive duality of the adeles and associated quotients can then be used to deduce the Riemann–Roch theorem.
Results in class field theory can be viewed in terms of the duality of multiplicative, and $K$-theoretic, adelic structures. In studying zeta integrals, we use these ideas implicitly. In this section we derive a harmono-analytic expression of a mixture of additive and multiplicative duality and apply this to the zeta integrals of the previous chapter. This will allow us to understand an important component of the boundary function as an integral over the boundary of a local-global adelic space. More precisely, we will write the boundary function as a sum of two adelic integrals, one of which is an integral over the boundary, in such a way that the meromorphic continuation, functional equation, and even poles, of the zeta integrals are all reduced to the analogous properties of the boundary integral.

The formula we deduce is the so-called “two-dimensional theta formula.” This terminology is by analogy to the classical theta formula, expressing the functional equation of the theta function. This classical result can be verified through Poisson summation on the adeles and is used in the Iwasawa–Tate method for the functional equation of Hecke $L$-functions. The two-dimensional theta formula for elliptic surfaces was first proved by Fesenko in [9, §3.6].

First we must construct integrals on the local-global adelic spaces. The measures required do not factorize as a product of local factors, so the ad hoc method of renormalizing in the previous chapter is not sufficient. Instead, we will consider convergent integrals on certain non-connected arithmetic schemes.

7.1. **A Second Calculation of the Zeta Integral.** As always, let $S$ be a proper, regular model of a smooth, projective, geometrically connected curve $C$ over a number field $k$, and let $P$ denote the relative projective line $\mathbb{P}^1(O_k)$. Due to the renormalizing factors of the previous chapter, we are interested in the zeta function of the disjoint union

$$\mathcal{X} = S \bigcup_{i=1}^{g-1} P.$$

Given any disjoint union $X = \bigcup X_i$ of schemes of finite type over $\mathbb{Z}$, one has

$$\zeta(X, s) = \prod \zeta(X_i, s),$$

so that

$$\zeta(\mathcal{X}, s) = \zeta(S, s)\zeta(P, s)^{g-1}.$$

Let $S(S)$ denote a set of curves on $S$, and $S(P)$ denote a set of curves on $P$. We will assume throughout that $S(S)$ contains at least one horizontal curve and each set contains all fibres. Let $S(\mathcal{X})$ denote the union:

$$S(\mathcal{X}) = S(S) \cup S(P).$$

We will define an analytic adelic space on $\mathcal{X}$ as the following product

$$\mathbb{A}(\mathcal{X}, S(\mathcal{X})) = \mathbb{A}(S, S(S)) \times \mathbb{A}(P, S(P)) \times \cdots \times \mathbb{A}(P, S(P)).$$

To avoid cumbersome notation, for an arithmetic surface $\mathcal{A}$ and a set $S$ of curves on $\mathcal{A}$ we will use the notation

$$T(\mathcal{A}, S) = (\mathbb{A}(\mathcal{A}, S) \times \mathbb{A}(\mathcal{A}, S))^x.$$
Note that if $S$ contains only finitely many horizontal curves on $S$ then the following integral converges for $s > 1$:

$$\int_{T(A)} (f \prod h)(\alpha)|\alpha|^s d\mu(\alpha),$$

where the measure on $T(A)$ is simply the product measure on the multiplicative adelic groups. Indeed, in the notation of the previous chapter, this is equal to the zeta integral

$$\zeta(S, f, h, |^s).$$

Often, we will omit the set of curves from the notation, and simply use $T(A, S)$ and

$$\zeta(f, h, s) := \zeta(S, S, f, h, |^s).$$

Due to the presence of a horizontal curve in $S$, we have a surjective module on $T(A)$,

$$|_S : T(A) \to \mathbb{R}_+^\times,$$

given as the product of modules on $S$ and $P$, which are modified at horizontal curves as in the previous chapter. $T_1(A)$ denotes the kernel of this module, namely

$$T_1(A) = \{x \in T(A) : |x| = 1\}.$$

We may choose a splitting

$$T(A) \cong \mathbb{R}_+^\times \times T_1(A).$$

The aim is to integrate over $T_1(A)$. In order to do so, we must first consider a finite subset $S^0 \subset S(A)$ containing at least one horizontal curve. $S^0$ can be decomposed into a union

$$S^0(S) \cup S^0(P),$$

where the first set contains only curves on $S$ and the second only curves on $P$. For such an $S^0$, its complement will be denoted $S_0 = S(A) - S^0$. We define

$$T_{S^0}(A) = \prod_{y \in S^0(S)} (A(S, y) \times A(S, y))^x \prod_{y \in S^0(P)} \prod_{i=1}^{g-1} (A(P, y) \times A(P, y))^x.$$

Again, we have a surjective map

$$|_{S^0} : T_{S^0}(A) \to \mathbb{R}_+^\times,$$

defined in the obvious manner. Its kernel is denoted

$$T_{S^0,1}(A),$$

and we have a splitting

$$T_{S^0}(A) \cong \mathbb{R}_+^\times \times T_{S^0,1}(A).$$

Let $p(S^0)$ denote the product of projections to one-dimensional adelic spaces as introduced in the previous chapter. We will fix a Haar measure on $p(T_{S^0,1}(A))$ such that the Haar measure on $p(T_{S^0}(A))$ is the product of this Haar measure and that on $\mathbb{R}_+^\times$, and let $\mu(T_{S^0,1})$ denote the lift of this Haar measure. For an integrable $\mathcal{F}$ on $T(A)$, for example $\mathcal{F} = f \prod h$ defined by the functions in the previous chapter, let

$$\int_{T_1(A)} \mathcal{F} = \int_{T_{S^0}(A)} \int_{T_{S^0,1}(A)} \mathcal{F}(\alpha^0 \gamma) d\mu(T_{S^0,1}) d\mu(T_{S^0}).$$
where $\alpha^0 \in T_{S^0}$ is such that

$$|\alpha^0|_{S^0} = |\alpha|_{S^0}^{-1}.$$ 

The integral does not depend on the choice of $S^0$, and we have the following lemma as a consequence of [9, Lemma 43].

**Lemma 7.1.** For an integrable function $F$ on $T(X)$, we have the following

$$\int_{T(X)} F \, d\mu = \int_{\mathbb{R}_+^\times} \int_{T_1(X)} F(x\alpha) d\mu(\alpha) \frac{dx}{x}.$$ 

In particular, we can decompose the zeta integrals of the previous section as

$$\zeta(f, h, s) = \int_{\mathbb{R}_+^\times} \zeta_x(f \coprod h, s) \frac{dx}{x} \cdot \xi(P, s)^{1-g},$$

where

$$\zeta_x(f \coprod h, s) = \int_{T_1(X)} (f \coprod h)(m_x \alpha) |m_x \alpha|^{-s} d\mu(\alpha).$$

This decomposition is the key to our second calculation of the zeta integral.

**Proposition 7.2.** Let $f, h$ be as in the previous chapter, then we may decompose the zeta function as a sum of the form

$$\zeta(f, h, s) = \eta(s) + \eta(2 - s) + \omega(s),$$

where $\eta(s)$ absolutely converges for all $s$, and so extends to an entire function on $\mathbb{C}$.

**Proof.** We decompose the multiplicative group $M = \mathbb{R}_+^\times$ of positive real numbers as $M = M^+ \cup M^-$, where

$$M^\pm = \{m \in M : \pm(|m| - 1) \geq 0\}.$$ 

We give these spaces the measure

$$\mu_{M^\pm} = \begin{cases} 
\mu_M & \text{on } M - M \cap T_1 \\
\frac{1}{2} \mu_M & \text{on } M \cap T_1.
\end{cases}$$

The result then follows directly from

$$\zeta(f, h, s) = \int_{M^+} \zeta_m(f, h, s) d\mu_{M^+}(m) + \int_{M^-} \zeta_m(f, h, s) d\mu_{M^-}(m),$$

and

$$\omega_m(s) = \zeta_m(f, h, s) - |m|^{-2} \zeta_{m^{-1}}(f, h, s),$$

by writing

$$\eta(s) = \int_{M^+} \zeta_m(f, s) d\mu_{M^+}(m),$$

$$\omega(s) = \int_{M^-} \omega_m(s) d\mu_{M^-}(m).$$

$\square$
Let \( \{y_i\} \) denote the complete set of horizontal curves in \( S(S) \). We will define the adelic boundary function \( h(S, \{y_i\}) : \mathbb{R}_+^\times \to \mathbb{C} \) as follows

\[
h(S, \{y_i\})(x) = \int_{T_1(x)} (x^2 f(m_x \gamma) - f(m_x^{-1} \gamma)) d\mu(\gamma),
\]

where \( m_x \in M \subset T(X) \) is a choice of representative of \( x \in \mathbb{R}_+^\times \).

From the above proposition we deduce the following:

**Corollary 7.3.** Let \( f(S, \{y_i\}) \) be the inverse Mellin transform of \( Z(S, \{y_i\}) \), then

\[
h(S, \{y_i\})(x) = f(S, \{y_i\})(x)x^2 - f(S, \{y_i\})(x^{-1}).
\]

In this way, we understand the boundary function \( h \) of the mean-periodicity correspondence as an adelic integral. The next step is to understand the role of local-global adelic boundaries. In another work, the mean-periodicity correspondence will be compared to automorphy conjectures for the generic fibre of \( S \).

### 7.2. The Adelic Boundary Term

The second goal of the chapter is to understand the boundary function as a boundary integral (thus motivating the terminology used throughout). This is the first step towards a verification of the mean-periodicity conjecture stated in [5] through two-dimensional adelic duality. The role of mean-periodicity in zeta functions is considered independently of adelic analysis in [5].

We require the notion of “weak” or “initial” topology, and “final” topology, see, for example [2, I, 2.3, 2.4]. If \( G \) is a topological group, then the weak topology is the weakest topology with respect to which every character of \( G \) is continuous.

**Example 7.4.** Let \( k \) be a number field, to verify the meromorphic continuation and functional equation of \( \zeta(k, s) \), Tate utilizes the following decomposition of his zeta integrals:

\[
\zeta(f, \chi) = \int_0^\infty \int_{A_k/k^\times} f(t\alpha)\chi(t\alpha)dt\,d\alpha,
\]

see [28, Main Theorem 4.4.1]. Let \( A_k \) have the weak topology, then the boundary \( \partial k^\times \) of \( k^\times \) in \( A_k \) is \( \{0\} \). After applying the theta formula, the functional equation of the zeta integrals is equivalent to that of

\[
\int_0^1 h_f(x)x^s dx,
\]

where the boundary function \( h_f \) has the following integral representation

\[
h_f(x) := -\int_{\gamma \in A_k/k^\times} \int_{\beta \in \partial k^\times} (f(x\gamma/\beta) - x^{-1}\hat{f}(x^{-1} \gamma/\beta)) d\mu(\beta) d\mu(\gamma).
\]

Explicitly, this is the following rational function:

\[-\mu(A_k/k^\times)(f(0) - x^{-1}\hat{f}(0)).\]

We will need to use a two-dimensional analogue of the inclusion \( k^\times \hookrightarrow A_k^\times \).

Let \( \mathcal{A} \) be an arithmetic surface with function field \( K \). If \( y \) is a curve on \( \mathcal{A} \), the field \( K_y = \text{Frac}(\mathcal{O}_y) \) is a complete discrete valuation field whose residue field is the global field \( k(y) \). It is therefore neither truly local, nor truly global in nature. For all closed points \( x \in y \), we have an embedding

\[
K_y \hookrightarrow K_{x,y}.
\]
which together induce a diagonal embedding

\[ K_y \hookrightarrow \prod_{x \in y} K_{x,y}. \]

For a curve \( y \) on \( \mathcal{A} \), let \( \mathbb{B}(\mathcal{A}, y) \) denote the intersection of the image of this embedding with \( \mathbb{A}(\mathcal{A}, y) \). The counting measure on \( k(y) \) lifts to an \( \mathbb{R}((X)) \)-valued measure on \( \mathbb{B}(\mathcal{A}, y) \).

Let \( S^0 \) be a finite set of curves on \( \mathcal{A} \), and define

\[ \mathbb{B}(\mathcal{A}, S^0) = (\prod_{y \in S^0} \mathbb{B}(\mathcal{A}, y)) \cap \mathbb{A}(\mathcal{A}, S^0). \]

We integrate on \( \mathbb{B}(\mathcal{A}, S^0) \) with the measure induced from the product measure on each \( \mathbb{B}(\mathcal{A}, y) \), for \( y \in S^0 \). We then take the product measure on \( \mathbb{B}(\mathcal{A}, S^0) \times \mathbb{B}(\mathcal{A}, S^0) \).

Let \( F \) be a two-dimensional local field, and let \( \psi \) be a choice of character such that all continuous characters of \( F \) are of the form \( \psi_a : \alpha \mapsto \psi(a\alpha) \), for \( a \in F \). For an integrable function \( f \) on \( F \), the Fourier transform \( \mathcal{F}(f) \) with respect to \( \psi \) is defined by

\[ \mathcal{F}(f)(\beta) = \int_F f(\alpha)\psi(\alpha\beta)d\alpha. \]

In particular, this applies to fields of the form \( K_{x,y} \) and we denote the Fourier transform on these fields by \( \mathcal{F}_{x,y} \). For any integrable function \( f_y \) on \( \mathbb{A}(\mathcal{A}, y) \), we may define

\[ \mathcal{F}_y(f_y) = \otimes_{x \in y} \mathcal{F}_{x,y}(f_{x,y}). \]

By [9, Proposition 32], we have a “summation formula”\(^4\).

**Proposition 7.5.** Let \( S^0 \) be a finite set of curves on an arithmetic surface \( \mathcal{A} \), and \( f \) be an integrable function on \( \mathbb{B}(\mathcal{A}, S^0) \) then

\[ \int f(\alpha\beta)d\mu_{\mathbb{B} \times \mathbb{B}}(\beta) = \frac{1}{|\alpha|} \int \mathcal{F}(f)(\alpha^{-1}\beta)d\mu_{\mathbb{B} \times \mathbb{B}}(\beta). \]

Let \( y \) be a curve on \( \mathcal{A} \), we introduce the notation

\[ T_0(\mathcal{A}, y) = \mathbb{B}(\mathcal{A}, y)^\times \times \mathbb{B}(\mathcal{A}, y)^\times \subset T(\mathcal{A}, y). \]

We may take the product measure on \( \mathbb{B}(\mathcal{A}, y) \times \mathbb{B}(\mathcal{A}, y) \), which induces the subspace measure \( M_y \) on \( T_0(\mathcal{A}, y) \). For a finite subset \( S^0 \) of curves on \( \mathcal{A} \), let

\[ T_0(\mathcal{A}, S^0) = \prod_{y \in S^0} T_0(\mathcal{A}, y) \subset T(\mathcal{A}, S^0). \]

On \( T_0(\mathcal{A}, S^0) \), we introduce the measure

\[ \mu(T_0, S^0) = \prod_{y \in S^0} (q_y - 1)^{-2} M_y. \]

Finally, let \( S \) be a set containing all fibres and finitely many horizontal curves, we define

\[ T_0(\mathcal{A}, S) = \prod_{y \in S} T_0(\mathcal{A}, y). \]

---

\(^4\) The semi-global adelic object \( \mathbb{B} \) is discrete in \( \mathbb{A} \).
On this space, we integrate using the following rule
\[ \int_{T_0(A,S)} \mathcal{F} = \lim_{S_0 \subseteq S} \int_{T_0(A,S^0)} \mathcal{F}. \]

We have a filtration
\[ T_0(A,S) \subset T_1(A,S) \subset T(A,S). \]

In [9, Section 3.5], Fesenko introduces a measure on the quotient such that, for an integrable function \( g \) on \( T(A,S) \):
\[ \int_{T_1(A,S)} g = \int_{T_1(A,S)/T_0(A,S)} \int_{T_0(A,S)} g(\gamma/\beta) \, d\mu(\beta) \, d\mu(\gamma). \]

Let \( S^0 \subseteq S \) be a finite subset. We endow \( \mathbb{A}(A,S^0) \times \mathbb{A}(A,S^0) \) with the weakest topology such that each character lifted by \( p \) is continuous. With respect to this topology, we will call the boundary \( \partial T_0(A,S^0) \) of \( T_0(A,S^0) \subseteq \mathbb{A}(A,S^0) \times \mathbb{A}(A,S^0) \) the “weak boundary”. We are interested in an inductive limit of these weak boundaries:
\[ \partial T_0(A,S) = \bigcup_{S_0 \subseteq S} T_0(A,S^0), \]
where the union runs over finite subsets \( S^0 \subseteq S \). If \( g \) is an integrable function on \( \mathbb{A}(A,S) \times \mathbb{A}(A,S) \), then one defines
\[ \int_{\partial T_0(A,S)} g = \lim_{S_0 \subseteq S} \int_{\partial T_0(A,S^0)} g, \]
where
\[ \int_{\partial T_0(A,S^0)} g = d(S^0) \int_{\partial(B(A,S^0) \times B(A,S^0))} g \, d\mu_{B(A,S^0) \times B(A,S^0)}, \]
\[ d(S^0) = \prod_{y \in S^0} (q_y - 1)^{-2}. \]

The limit expressing the integral is only finite in exceptional circumstances. One such case is expressed by the so-called “two-dimensional theta formula”.

**Theorem 7.6.** Let \( A \) be an arithmetic surface, \( S \) denote a set of curves on \( A \), and \( f \) be an integrable function on \( \mathbb{A}(A,S) \times \mathbb{A}(A,S) \), then
\[ \int_{T_0(A)} (f(\alpha\beta) - \mathcal{F}(f_\alpha)(\beta)) \, d\mu(\beta) = \int_{\partial T_0(A)} (\mathcal{F}(f_\alpha(\beta)) - f(\alpha\beta)) \, d\mu(\beta). \]

One applies this result to \( A = X \) and \( g = f \prod h \). Consequently, one obtains the boundary integral contribution to the boundary term. It transpires that this boundary integral knows much about the analytic properties of zeta – this will be discussed in chapter 7.

**Corollary 7.7.** Let \( \{y_i\} \) be the set of horizontal curves in \( S \). We may decompose the boundary integral as follows
\[ h(S,\{y_i\})(x) = h_1(S,\{y_i\})(x) + h_2(S,\{y_i\})(x), \]
where
\[ h_1(x) = \int_{T_1(x,S)} (|x^{-1} - 1| \cdot f(m_x^{-1} x^{-1}) \, d\mu(x) \]


\[ h_2(x) = x^2 \int_{[T_1/T_0](X,S)} \int_{\partial T_0(X,S)} \left( |m_x|^{-1} f(m_x^{-1} \nu^{-1} \beta) - f(m_x \gamma \beta) \right) d\mu(\beta) d\mu(\gamma), \]

and \( m_x \) are lifts of \( x \in \mathbb{R}_+^\times \) to \( T(X) \), and \( \nu \) is as in [9, Section 3.4].

For elliptic curves, this result was first deduced by Fesenko. An analogous decomposition, not involving adelic integrals, appeared in [5, Remark 5.11].

8. Twisted Zeta Integrals; \( \text{GL}_1(\mathbb{A}(S)) \)-Theory

We will now discuss some notions from a prospective “higher dimensional Langlands program”, over two-dimensional local and global fields. For a notion of Satake isomorphism for two-dimensional local fields, the reader is referred to [17]. Below we will discuss the rudiments of integral representations of \( \text{GL}_1(\mathbb{A}(S)) \)-automorphic representations. Much remains to be done even in this most basic case. What we consider are “two-dimensional” twisted zeta functions, which are analogues of Artin L-functions for characters of two-dimensional global fields. When the character factors through a Galois group on the residue level, we recover the usual Artin L-functions. This factorization occurs when the character has suitable ramification properties. For sufficiently unramified, abelian characters, we will interpret these new twisted zeta functions as integrals over two-dimensional adelic spaces.

The function field \( K \) of an arithmetic surface \( S \) is a two-dimensional global field, and abelian aspects of its arithmetic can be studied through two-dimensional class field theory [18], [14], [15], [16] and [9, Theorem 34]. The first pair of papers use higher \( K \)-groups, whereas the second pair do not. The final reference describes explicit adelic class field theory.

Let \( F \) be a two-dimensional local field. According to [11, Section 10], there is an injective map with dense image:

\[ K_2^{\text{top}}(F) \to \text{Gal}(F^{\text{ab}}/F), \]

where the topological (Milnor) \( K_2 \) group is defined in [11, Section 6]. \( K_2^{\text{top}}(F) \) was introduced because the reciprocity map \( \psi_F : K_2(F) \to \text{Gal}(F^{\text{ab}}/F) \) is not injective. In fact, the topological \( K_2 \) group is equal to the quotient \( K_2(F)/\ker(\psi_F) \). On the other hand, \( \ker(\psi_F) \) is equal to the intersection of all neighbourhoods of 0 in \( K_2(F) \) with respect to a certain topology.

The boundary map \( \partial \) from \( K \)-theory induces a map, also denoted by \( \partial \):

\[ K_2^{\text{top}}(F) \to K_1(F) = F^\times, \]

where \( F \) is the residue field of \( F \). Let \( \mathbb{F}_q \) be the second residue field of \( F \). Upon applying \( \partial \) again, we obtain a map

\[ v_F : K_2^{\text{top}}(F) \to K_0(\mathbb{F}_q) = \mathbb{Z}. \]

We will refer to an element mapped to 1 under \( v \) as a prime of \( K_2^{\text{top}}(F) \). \( v \) induces a module

\[ |_2,F : K_2^{\text{top}}(F) \to \mathbb{R}_+^\times \]

\[ \alpha \mapsto q^{-v(\alpha)}. \]

This map is continuous in the topologies of \( \mathbb{R} \). The module \( |_2,F \) has a certain compatibility with \( |_F \), see equation \( \ref{eq:compatibility} \).

Let \( z \) be a branch at a point \( x \) of an irreducible curve \( y \) on \( S \), and let \( F = K_{x,z} \). We will use the notations:

\[ v_F = v_{x,z}, \]
An adelic analogue of $K_{\text{top}}(F)$ was constructed in \cite[Section 2]{9}, by restricting with respect to the rank 2 structure:

$$J_S = \prod_{y} \prod_{x \in y} \mathcal{K}^\text{top}(K_{x,y}) \times \prod_{\sigma \in S_\sigma} \mathcal{K}^\text{top}(K_{\omega,\sigma}),$$

where $y$ runs over all irreducible curves on $S$ and $\sigma$ over all archimedean places of $k$. The restricted product over closed points $\omega$ on the archimedean fibre $S_\sigma$ means that all but finitely many components lie in $K^\text{top}_2(O_{\omega,\sigma}).$ The first pair of restricted products mean that for almost all $(x, y)$, the $(x, y)$-component of an element in $J_S$ lies in $K^\text{top}_2(O_{x,y}).$

Let $x$ be a closed point on an irreducible curve $y$ and let $\sigma$ be a closed point on the archimedean fibre $S_\omega$, where $\omega$ is an archimedean place of $k$. We have field embeddings

$$K_x \hookrightarrow K_{x,y},$$

$$K_y \hookrightarrow K_{x,y},$$

$$K_\sigma \hookrightarrow K_{\omega,\sigma}.$$

These embeddings induce diagonal maps

$$\Delta : \begin{cases} K_2(K_y) \to K^\text{top}_2(K_{x,y}) \\ K_2(K_x) \to K^\text{top}_2(K_{x,y}) \\ K_2(K_\sigma) \to K^\text{top}_2(K_{\omega,\sigma}) \end{cases}.$$

The product of all $\Delta$ gives a map to $J_S$, which we use to define the semi-global adelic objects:

$$P_S = \Delta \prod_{y} \mathcal{K}^*(K_y) + \Delta \prod_{x} \mathcal{K}^*(K_x) + \Delta \prod_{\sigma} \mathcal{K}^*(K_\sigma),$$

where the restricted direct product is short hand for intersection with $\Delta^{-1}(J_S)$.

The product of all $| |_{2,x,y}$ induces a module on $J_S$. Let $J^1_S$ denote those elements of module 1. Note that

$$P_S \subset J^1_S.$$

We will use the topology on $J_S$ as in \cite[theorem 34]{9}. Then, by two-dimensional class field theory, finite order characters of the Galois group $\text{Gal}(K^{ab}/K)$ correspond to characters of $J_S$ which are trivial on $P_S$.

If $x$ is a closed point on the local branch $z$ of an irreducible fibre $y$, and $\chi : J_S \to \mathbb{C}^\times$ is a character, let $\chi_{x,z}$ denote the composite

$$K^\text{top}_2(K_{x,z}) \hookrightarrow J_S \to \mathbb{C}^\times.$$

If $\omega$ is an archimedean point on a horizontal curve $y$, let $\chi_{\omega,y}$ denote the composite

$$K^\text{top}_2(K_{\omega,y}) \to K^\text{top}_2(K_{\omega,\sigma}) \hookrightarrow J_S \to \mathbb{C}^\times,$$

where the first map uses the isomorphism $K_{\omega,y} \cong K_{\omega,\sigma},$ for an archimedean fibre $S_\sigma$.

We are interested in finite order characters $\chi$ of the group $J_S/P_S$. We will assume the following:
Assumption 8.1. there is a finite order character $\chi_0$, lifting a character of $J^1_S/P_S$ of the same order, such that

$$\chi(\alpha) = \chi_0(\alpha)|\alpha|_2^s,$$

where $\chi_0$ is of finite order, lifting a character of $J^1_S/P_S$ of the same order.

In this case, both $\chi_0$ and $s$ are both uniquely determined.

Bearing in mind that, for a two-dimensional local field $F$, $K^\text{top}_2(F)$ is the Galois group of the maximal abelian extension $F^{ab}/F$, we will say that $\chi_{x,z}$ is unramified if it is trivial on the subgroup of $K^\text{top}_2(K_{x,z})$ generated by $K_1(O_{x,z})$.

Let $x$ be a (possibly singular) point on a fibre $S_p$. On each branch $S_p(x)$, let $\pi_{x,z}$ be a prime with respect to $v_{x,z}$. When $\chi$ is unramified, let $t = q_x^{-s}$, where $\chi = \chi_0|_2^s$. We have polynomials

$$f_{x,z}(t) = 1 - \chi_{x,z}(\pi_{x,z}) \in \mathbb{C}[t].$$

We define

$$L(S, \chi_0, s) = L(S, \chi) = \prod_{p \in \text{Spec}(O_k)} \prod_{x \in S_p} (\gcd\{f_{x,z}\}_{z \in S_p(x)})^{-1}.$$

In particular, if $x \in S_p =: y$ is nonsingular, then the $(x, y)$-factor is simple

$$(1 - \chi_{x,y}(t))^{-1},$$

where $t$ is a prime of $K_{x,y}$.

This is a two dimensional analogue the Hecke $L$-function of a character of the idele class group of a number field, and first appeared in [9, Section 3.2]. We will discuss its convergence after our main theorem.

Let $F$ be a two-dimensional local field. An element $\alpha \in O_F^\times$ can be written in the form $t_1^i u$, where $u \in O_F^\times$, for a choice of local parameter $t_1$. We have a map

$$(2) \quad t : O_F^\times \times O_F^\times \to K^\text{top}_2(F)$$

$$(t_1^i u_1, t_2^j u_2) = (j + l)\{t_1, t_2\} + \{t_1, u_1\} + \{u_2, t_2\}.$$

The compatibility of the modules $|$ and $|_2$ is described through this map:

$$(3) \quad |t(\alpha_1, \alpha_2)|_2 = |\alpha_1| |\alpha_2|.$$

For a character $\chi$ of $K^\text{top}_2(F)$, put

$$\chi_t : F^\times \times F^\times \to \mathbb{C}^\times$$

$$\chi_t(\alpha) = \begin{cases} \chi \circ t(\alpha), & \alpha \in O_F^\times \times O_F^\times, \\ 0, & \text{otherwise.} \end{cases}$$

When $F = K_{x,z}$, we will use the notation $\chi_{t,x,z}$. We will make a second assumption about the character $\chi_{x,z}$.

**Assumption 8.2.** $\forall x \in z$, $\chi_{0,t,x,z}$ factors through the residue map:

$$T \to \overline{F}^\times \times \overline{F}^\times.$$ 

This assumption concerns the ramification of $\chi_{0,x,z}$, as illustrated in the following examples.
Example 8.3. Let $F$ be a two-dimensional local field, and let $L$ be a finite abelian extension of $F$. A character of $\text{Gal}(L/F)$ corresponds to a character of $K^{\text{top}}_2(F)$ which vanishes on $N_{L/F}K^{\text{top}}_2(F)$ by two-dimensional class field theory. Consider two possibilities:
- $L/F$ is an unramified extension of complete discrete valuation fields. So, $L/F$ is separable and $|L:F| = |L:F|$.
- $L/F$ is a totally unramified extension in the second parameter. In this case $L = F$, $p \nmid |L:F|$ and $t_2 \in N_{L/F}L^\times$.

In either case, the induced character $\chi_t$ of $T$ factors through the residue map according to [21] Examples 7.11, 7.12.

Under assumption 8.2, we can write
\[ \chi_t(a,b) = \omega_1(a)\omega_2(b), \]
where $\omega_1$ and $\omega_2$ are quasi-characters of the local field $F$, the residue field of $F$. When $F = K_{x,z}$, we will write $\omega_1(x,z)$ and $\omega_2(x,z)$.

Integration against an unramified character see [6, Section 1], [9, Section 1] and Theorem 8.4.

Theorem 8.4. Let $\chi$ be a character of $J_S/P_S$, satisfying assumptions 8.1 and 8.2. Then, for all $f,h$ such that the integrals are defined, there is an entire meromorphic function $\Phi_{\chi,f,h}(s)$ such that
\[
\zeta_{\text{na}}^{(2)}(S, f, h, \chi) = \Phi_{\chi,f,h}(s) A(S, \chi)^{1-s} L(S, s, \chi_0)^2 Q(s)^2 \cdot \prod_{y \in S_h} \Lambda(k(y), \omega_1, s) \Lambda(k(y), \omega_2, s),
\]

where $f$ and $g$ are the functions from chapter 4, $\Lambda(k(y), \omega, s)$ is a completed Hecke $L$-function for the Hecke character $\omega$ on the number field $k(y)$, $S_h$ is a finite set of horizontal curves, and $\zeta_{\text{na}}^{(2)}(S, f, g, \chi)$ is the following product
\[
\prod_{p \in \text{Spec}(O_k)} \int_{(A(S_p) \times)^2} f_p((\chi_0| |_{\mathbb{P}^1})^t)(\alpha)d\mu(\alpha) \int_{(A(S_p) \times)^2} g_p(|\alpha|^s d\mu(\alpha))^{g-1} 
\]
\[
\cdot \prod_{y \in S_h} \int_{(A(y) \times)^2} f_y(\alpha)|\chi_0(\alpha)|^s d\mu(\alpha).
\]

Proof. We have
\[ L(S, \chi) = \prod_p L_p(S, \chi), \]
where
\[ L_p(S, \chi) = \prod_{x \in S_p} (\gcd\{f_{x,z}\}_{z \in S_p(z)})^{-1}. \]

Due to assumption 8.2 if $S_p$ is a smooth fibre, then
\[ L_p(S, \chi) = L(k(S_p), \omega_{1,p}, s)L(k(S_p), \omega_{2,p}, s), \]
where, for \( i = 1, 2 \), \( \omega_{i,p} = \otimes_{x \in S_p} \omega_{1,x,S_p} \), for some characters on the local field \( k(S_p) \). By [21 Proposition 7.3], \( L_p(S, \chi) \) is equal to

\[
\int_{(A(S_p) \times \mathbb{A})^{\times 2}} f_p(\alpha)((\chi_0|_{|2,p})\chi)(\alpha)d\mu(\alpha)(\int_{(A(P_p) \times \mathbb{A})^{\times 2}} h_p(\alpha)|\alpha|^s d\mu(\alpha))^{g-1}.
\]

Finally, let

\[
\Phi_{\chi,f,h}(s) = \varepsilon(S, \chi)A(S, \chi)^{1-s} \prod_{p \text{ bad}} \frac{1}{\zeta(S, f_p, h_p, \chi)}.
\]

The result follows from taking the product over all primes \( p \) of the integrals. \( \square \)

The above computation does not incorporate an archimedean contribution - indeed, it is not clear how to do this in general. However, if \( K/k \) is a finite Galois extension of number fields and \( \chi \) is a character of \( \text{Gal}(K/k) \), we can apply the above theorem to the character \( \psi \) lifted to \( \mathbb{A}(S, S) \) and \( \mathbb{A}(P, S) \). Observe that

\[
L(P_K, \chi, s) = L(K/k, \chi, s)L(K/k, \chi, s),
\]

and

\[
\Gamma(P, \chi, s)^{1-g} = Q(\chi, s)\Gamma(S, \chi, s),
\]

where \( Q(\chi, s) \in \mathbb{C}(s) \) is invariant with respect to \( s \mapsto 2 - s \). Using this, and applying the previous theorem, one obtains

**Corollary 8.5.** With the conventions above

\[
\xi(P^1(O_k), \psi, s)^{1-g} \zeta_{na}(S, f, h, \chi) = \Phi_{\chi,f,h}(s)A(S, \psi)^{1-s}L(S, s, \psi_0)^2 Q(s)^2 \Gamma(S, \psi, s)
\]

\[
\cdot \prod_{y \in S_h} \Lambda(k(y), \omega_{1,y}, s) \Lambda(k(y), \omega_{2,y}, s).
\]

Unfortunately, this integral description is nowhere near detailed enough to investigate the functional equation, however we deduce for such a restricted class of characters that \( L(S, \chi, s) \) is a meromorphic function on the half plane \( \Re(s) > 2 \) due to the convergence property of the zeta integrals, and \( L(S, \chi, s) \) has meromorphic continuation if and only if the zeta integral does.

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