The Reflection Principle in the Control Problem of the Heat Equation

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Abstract
We consider the control problem for the generalized heat equation for a Schrödinger operator on a domain with a reflection symmetry with respect to a hyperplane. We show that if this system is null-controllable, then so is the system on its respective parts and the corresponding control cost does not exceed the one on the whole domain. As an application, we obtain null-controllability results for the heat equation on half-spaces, orthants, and sectors of angle $\pi/2^n$. As a byproduct, we also obtain explicit control cost bounds for the heat equation on certain triangles and corresponding prisms in terms of geometric parameters of the control set.

Keywords Heat equation · Reflection principle · Null-controllability · Observability · Thick set · Equidistributed set

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1 Introduction and Main Result
Let $\Omega \subset \mathbb{R}^d$ be open, $d \in \mathbb{N}$, and let $A : \Omega \to \mathbb{R}^{d \times d}$ be measurable with $A(x)$ a symmetric matrix for almost every $x \in \Omega$. Suppose, in addition, that there exist $\theta_1, \theta_2 > 0$ such that

$$\theta_1 \|\xi\|^2_{C^d} \leq \langle A(x)\xi, \xi \rangle_{C^d} \leq \theta_2 \|\xi\|^2_{C^d} \quad \forall \xi \in \mathbb{C}^d, \text{ a.e. } x \in \Omega, \quad (1.1)$$

and let $V \in L^\infty(\Omega)$ be real-valued. We denote by $H^D_\Omega = H^D_\Omega(A, V)$ and $H^N_\Omega = H^N_\Omega(A, V)$ the Dirichlet and Neumann realizations of the differential expression

$$-\nabla \cdot (A \nabla) + V$$

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as self-adjoint lower semibounded operators on $L^2(\Omega)$ defined via their quadratic forms with form domain $H^1_0(\Omega)$ and $H^1(\Omega)$, respectively. For details of this construction, we refer the reader to the discussion in Section 4 below.

Let $T > 0$, $\omega \subset \Omega$ be a measurable subset, and $\bullet \in \{D, N\}$. We consider the heat-like system

$$\partial_t u(t) + H^\omega_{\Omega} u(t) = \chi_\omega v(t) \quad \text{for } 0 < t < T, \quad u(0) = u_0,$$

(1.2)

with $u_0 \in L^2(\Omega)$ and $v \in L^2((0, T), L^2(\Omega))$. Here, $\chi_\omega$ denotes the characteristic function of $\omega$, and we call $\omega$ a control set for the system (1.2).

System (1.2) is said to be null-controllable in time $T > 0$ if for every initial data $u_0 \in L^2(\Omega)$ there exists a control function $v \in L^2((0, T), L^2(\Omega))$ such that the mild solution of (1.2) satisfies $u(T) = 0$, that is,

$$e^{-TH^\omega_{\Omega} u_0} + \int_0^T e^{-(T-s)H^\omega_{\Omega} } \chi_\omega v(s) \, ds = 0 \quad \text{in } L^2(\Omega).$$

(1.3)

In this case, the quantity

$$C_T := \sup_{\|u_0\|_{L^2(\Omega)} = 1} \inf \{\|v\|_{L^2((0, T), L^2(\omega))} : v \text{ satisfies (1.3)}\}$$

(1.4)

is called control cost.

For bounded domains $\Omega$, there is already a rich literature on null-controllability results for system (1.2) with $H^\omega_{\Omega}$ being the Laplace or a Schrödinger operator with an open or measurable control set $\omega$, see, e.g., [2, 9, 15, 19, 27]. On the other hand, null-controllability of these systems on unbounded domains has been an issue of growing interest only recently, see for example [7, 9, 19] and the references therein. As far as bounds on the control cost are concerned, one classically asks for the type of time dependency, which has been largely explored in [5, 10, 11, 16, 18, 19, 21, 27], see also the references therein. However, the dependency of the control cost on geometric parameters of $\Omega$ and $\omega$ has only recently been addressed in [6, 7, 9, 19], see also [17] for previous results. Especially the work [19] focused on these dependencies, which have been exploited in several asymptotic regimes there in order to discuss homogenization.

The aim of the present paper is to prove null-controllability results for the above situation on some new unbounded domains, such as sectors of certain angles, half-spaces, and orthants, with bounds on the associated control cost that are fully or partially consistent with the known ones on cubes and the full space. As a byproduct, we also obtain explicit control cost bounds in terms of only geometric parameters of the control set for the corresponding system on some bounded domains with flat boundary that were not accessible before. We believe that the lack of the control cost’s dependency on the domain may be an effect of the flatness of its boundary. However, it is not our current scope to explore this matter in depth, but merely to add examples to support this conjecture. All these results are discussed in detail in Section 2 below as applications of a reflection principle which we now describe.

The key idea in our considerations is to relate system (1.2) for certain domains $\Omega$ to a corresponding symmetrized system on a larger domain, where null-controllability results are available. In order to make this precise, let $M : \mathbb{R}^d \to \mathbb{R}^d$ be the reflection with respect to the first coordinate, that is, $M(x_1, \ldots, x_d) = (-x_1, x_2, \ldots, x_d)$. We suppose that $\Omega$ is contained in the half-space $(0, +\infty) \times \mathbb{R}^{d-1}$ such that $\Omega = \hat{\Omega} \cap ((0, +\infty) \times \mathbb{R}^{d-1})$ for some open set $\hat{\Omega} \subset \mathbb{R}^d$ with $\Gamma := \hat{\Omega} \cap ((0) \times \mathbb{R}^{d-1}) \neq \emptyset$ and which is symmetric with respect to the reflection $M$, that is, $M(\hat{\Omega}) = \hat{\Omega}$. In particular, we have $\hat{\Omega} = \Omega \cup \Gamma \cup M(\Omega)$, cf. Figure 1.
Let $\tilde{A}: \tilde{\Omega} \to \mathbb{R}^{d \times d}$ and $\tilde{V}: \tilde{\Omega} \to \mathbb{R}$ be measurable with
\[
\tilde{A}(x) = \begin{cases} \lambda(x), & x \in \Omega \\ U(A \circ M)(x)U, & x \in M(\Omega) \end{cases}, \quad U := \text{diag}(-1, 1, \ldots, 1),
\]
and
\[
\tilde{V}(x) = \begin{cases} V(x), & x \in \Omega \\ (V \circ M)(x), & x \in M(\Omega) \end{cases}.
\]
Observe that by construction $\tilde{A}(x)$ is for almost every $x \in \tilde{\Omega}$ a symmetric matrix satisfying (1.1) with the same constants $\theta_1, \theta_2$.

Let the self-adjoint operators $H^*_{\tilde{\Omega}} = H^*_{\tilde{\Omega}}(\tilde{A}, \tilde{V})$ on $L^2(\tilde{\Omega})$ associated with the differential expression
\[
-\nabla \cdot (\tilde{A}\nabla) + \tilde{V}
\]
be defined analogously to $H^*_{\Omega}(A, V)$ above. Set $\tilde{\omega} := \omega \cup M(\omega) \subset \tilde{\Omega}$, and consider the corresponding symmetrized system
\[
\partial_t \tilde{u}(t) + H^*_{\tilde{\Omega}}\tilde{u}(t) = \chi_{\tilde{\omega}}\tilde{v}(t) \quad \text{for} \quad 0 < t < T, \quad \tilde{u}(0) = \tilde{u}_0,
\]
with $\tilde{u}_0 \in L^2(\tilde{\Omega})$, $\tilde{v} \in L^2((0, T), L^2(\tilde{\Omega}))$, and control set $\tilde{\omega}$. The notions of null-controllability and control cost for this system carry over verbatim.

The main result of the present paper now reads as follows.

**Theorem 1.1** If the system (1.5) is null-controllable in time $T > 0$ with control cost $\tilde{C}_T$, then also system (1.2) is null-controllable in time $T > 0$ with control cost $C_T \leq \tilde{C}_T$.

The restriction to the reflection symmetry with respect to the hyperplane $\{0\} \times \mathbb{R}^{d-1}$ in Theorem 1.1 is not essential. Indeed, by rotating the whole system, we can deal with reflection symmetries with respect to any hyperplane in $\mathbb{R}^d$. This way, the above theorem allows us to infer null-controllability on the respective parts of a domain with a reflection symmetry if the system on the whole domain is null-controllable.

Theorem 1.1 is, in fact, an instance of a much more general result, which is formulated in Theorem 3.1 below. The proof of this abstract result heavily relies on an extension relation, which in the situation of Theorem 1.1 takes the form
\[
(X^*)^*H^*_{\tilde{\Omega}} \subset H^*_{\Omega}(X^*),
\]
where $X^*: L^2(\Omega) \to L^2(\bar{\Omega})$ corresponds to the extension of functions by symmetric or antisymmetric reflection, respectively. Since $X^*$ here is a multiple of an isometry, this essentially means that the subspace $\text{Ran} X^*$ reduces the operator $H^*_\Omega$, cf. Remark 3.5 below. With this in mind, the extension relation (1.6) transfers to operators related to $H^*_\Omega$ and $H^*_\bar{\Omega}$ by functional calculus, such as fractional Laplacians, yielding an analogue of Theorem 1.1 also for such operators; cf. Lemma 3.3 and Remark 4.2 (a) below. More generally, an extension relation of the form (1.6) can be formulated with respect to every reducing subspace of $H^*_\Omega$, opening the way to null-controllability of subsystems corresponding to reducing subspaces, even in a general abstract framework, see Remark 3.5 below. In this regard, our abstract result provides a flexible and convenient way to derive new results from existing ones independently of how the existing results have originally been proved.

The rest of the paper is organized as follows. In Section 2, we discuss the null-controllability results and control cost bounds announced above for the case where $\Lambda(x)$ is the identity matrix as consequences of Theorem 1.1. Here, we will make explicit use of recent results from [9, 19] for cubes and the full space. The assumption on the essential boundedness of the potential $V$ is tailored towards these applications but it is not essential for the general argument. In Section 3, we prove the abstract result discussed above, which is the core of the proof of Theorem 1.1 and yields applications in broader settings, see, e.g., Remarks 2.11, 4.2, and 4.3 below. Section 4 deals with the proof of Theorem 1.1 based on the abstract result by establishing the extension relation (1.6). Finally, Appendix A provides an integration by parts formula used in Section 4.

1.1 Addendum

After completion of the present work, the current authors obtained a family of so-called spectral inequalities in the recent article [8], which together with [19, Theorem 2.8] can be used to reproduce the results from Section 2.1.

2 Applications

We here discuss the applications of Theorem 1.1 mentioned in the previous section for the particular case where each $\Lambda(x)$ is the identity matrix, that is, $H^*_\Omega = -\Delta^*_\Omega + V$. These applications draw upon null-controllability results from the works [9, 19] on cubes and the full space. As considered there, we also take control sets of the form $\omega = \Omega \cap S$, where $S \subset \mathbb{R}^d$ is some measurable set with certain geometric properties, namely a thick set or an equidistributed set. We treat the two types of sets in the subsections below separately.

The general strategy for our applications is to build from the given set $S$ a symmetric set $\bar{S}$ with respect to the reflection such that $\Omega \cap \bar{S} = \bar{\Omega} \cap S$. After having verified corresponding geometric properties of $\bar{S}$, this new set yields a suitable control set $\bar{\omega} = \bar{\Omega} \cap \bar{S}$ for the symmetrized system on $\bar{\Omega}$, where a null-controllability result is available. Applying Theorem 1.1 then gives the desired null-controllability result for the original system on $\Omega$. In summary, our main task here is to establish the geometric properties of the symmetrized set $\bar{S}$ from those of the given set $S$.

Recall that by rotation of the whole system Theorem 1.1 can be applied for a reflection with respect to any hyperplane in $\mathbb{R}^d$. Since the Laplacian is rotation invariant, only the potential $V$ then gets rotated, but remains essentially bounded. In this context, we remind the
reader that $M : \mathbb{R}^d \to \mathbb{R}^d$ denotes the reflection with respect to the hyperplane $\{0\} \times \mathbb{R}^{d-1}$. Furthermore, for the rest of this section, we use the following notation:

- $H_\theta := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_2 < (\tan \theta)x_1\}$ for $0 \leq \theta < \pi/2$;
- $M_\theta : \mathbb{R}^d \to \mathbb{R}^d$ denotes the reflection with respect to $\partial H_\theta$;
- $\Lambda^d_L := (0, L)^d$ for $L > 0$.

### 2.1 Null-controllability from Thick Sets

Let $V = 0$, and consider the system

$$\partial_t u(t) - \Delta u(t) = \chi_{\omega} v(t) \quad \text{for} \quad 0 < t < T, \quad u(0) = u_0, \quad (2.1)$$

with $u_0 \in L^2(\Omega)$ and $v \in L^2((0, T), L^2(\Omega))$. Moreover, assume that the control set $\omega$ is of the form $\omega = \Omega \cap S$ with some $S \subset \mathbb{R}^d$ that is $(\gamma, a)$-thick in the sense of the following definition.

**Definition 2.1** A measurable set $S \subset \mathbb{R}^d$ is called *thick* if there exist $\gamma \in (0, 1]$ and $a = (a_1, \ldots, a_d) \in (0, +\infty)^d$ such that

$$|S \cap (x + [0, a_1] \times \cdots \times [0, a_d])| \geq \gamma \prod_{j=1}^d a_j \quad \forall x \in \mathbb{R}^d,$$

where $| \cdot |$ denotes the Lebesgue measure. In this case, $S$ is also referred to as $(\gamma, a)$-thick to emphasise the parameters.

The above definition has played a crucial role in a recent development on the null-controllability of the heat equation on $\mathbb{R}^d$. In [9], see also [28], it is shown that thickness of $S$ is a necessary and sufficient condition for the heat equation (2.1) on $\Omega = \mathbb{R}^d$ to be null-controllable. Moreover, in [9] the authors give an explicit estimate of the control cost in terms of the thickness parameters of the set $S$ and the time $T$. In the same paper, the authors also consider the heat equation (2.1) on the cube $\Omega = \Lambda^d_1$ with control set $\omega = \Omega \cap S$, where $S$ is $(\gamma, a)$-thick with $a_j \leq L$ for all $j = 1, \ldots, d$. In this case, they show that null-controllability holds in any time $T > 0$ with a bound on the control cost of the same form as for the full space case and independent of the scale $L$.

In [19], these bounds have been strengthened in both time and geometric parameters dependency and the authors have shown that they are close to optimality in certain asymptotic regimes, see [19, Section 5]. For both cases $\Omega \in \{\mathbb{R}^d, \Lambda^d_L\}$, they can be written as

$$C_T \leq \frac{1}{\sqrt{T}} \left( \frac{K^d}{\gamma} \right)^{Kd/2} \exp \left( \frac{K\|a\|^2}{2} \ln^2 \left( \frac{K^d}{\gamma} \right) \right),$$

where $K > 0$ is a universal constant and $\|a\|_1 = a_1 + \cdots + a_d$, see [19, Theorem 4.9].

We show below that system (2.1) on half-spaces, positive orthants, and sectors of angle $\pi/2^n$, $n \geq 2$, is null-controllable with a control cost bound fully or partially consistent with (2.2); as a consequence, certain asymptotic properties of (2.2) are inherited, see Remark 2.10 below. Moreover, we obtain null-controllability and explicit control cost bounds for the same system on isosceles right-angled triangles and corresponding prisms. It is worth to note that although the results for the listed unbounded domains are new$^1$, the

$^1$Note, however, the addendum on page 4.
heat equation on half-spaces and positive orthants with Dirichlet boundary conditions could also be treated with the techniques from [26] (see also Remark 2.6 below), while the novelty of the result on triangles and corresponding prisms is the explicitness of the control cost bound on the model parameters.

In order to enter the setting of the main theorem, let \( \tilde{\Omega} \in \{\mathbb{R}^d, \Lambda^d_L\} \) and \( \tilde{\omega} = \tilde{\Omega} \cap \tilde{S} \) with some \((\tilde{\gamma}, \tilde{a})\)-thick set \( \tilde{S} \subset \mathbb{R}^d \), and consider the system

\[
\partial_t \tilde{u}(t) - \Delta^*_\omega \tilde{u}(t) = \chi_{\tilde{\omega}} \tilde{v}(t) \quad \text{for} \quad 0 < t < T, \quad \tilde{u}(0) = \tilde{u}_0,
\]

with \( \tilde{u}_0 \in L^2(\tilde{\Omega}) \) and \( \tilde{v} \in L^2((0, T), L^2(\tilde{\Omega})) \).

We need the following easy geometric lemma, parts of which are already contained in [9].

**Lemma 2.2** Let \( S \subset \mathbb{R}^d \) be \((\gamma, a)\)-thick.

(a) Let \( S' = S \cap ((0, +\infty) \times \mathbb{R}^{d-1}) \). Then, the set \( \tilde{S} = S' \cup M(S') \) is \((\tilde{\gamma}, \tilde{a})\)-thick with \( \tilde{\gamma} = \gamma / 2 \) and \( \tilde{a} = (2a_1, a_2, \ldots, a_d) \).

(b) The set \( \tilde{S} = \{ (x_1, \ldots, x_d) : (|x_1|, \ldots, |x_d|) \in S \} \) is \((\gamma / 2^d, 2a)\)-thick, where \( 2a = (2a_1, \ldots, 2a_d) \).

(c) Let \( S' = S \cap H_0 \). Then, \( \tilde{S} = S' \cup M_0(S') \) is a \((\tilde{\gamma}, \tilde{a})\)-thick set with parameters \( \tilde{\gamma} = \frac{\gamma a_0}{4(a_1^2 + a_2^2)} \) and \( \tilde{a} = (2\sqrt{a_1^2 + a_2^2}, 2\sqrt{a_2^2 + a_3^2}, \ldots, a_d) \).

**Proof** We abbreviate \( Q_a := [0, a_1] \times \cdots \times [0, a_d], \quad a = (a_1, \ldots, a_d) \subset (0, +\infty)^d \).

For part (a), we need to show that

\[
|\tilde{S} \cap (x + Q_\tilde{a})| \geq \frac{\gamma}{2} 2a_1 \prod_{j=2}^{d} a_j = \tilde{\gamma} |Q_\tilde{a}| \quad \forall x \in \mathbb{R}^d.
\]

Let \( x \in \mathbb{R}^d \). There is \( y \in \mathbb{R}^d \) such that \( y + Q_a \subset (x + Q_\tilde{a}) \cap ([0, +\infty) \times \mathbb{R}^{d-1}) \) or \( M(y + Q_a) \subset (x + Q_\tilde{a}) \cap ((-\infty, 0]) \times \mathbb{R}^{d-1} \), cf. Figure 2(a). In the first case, we have

\[
|\tilde{S} \cap (x + Q_\tilde{a})| \geq |S' \cap (y + Q_a)| = |S \cap (y + Q_a)| \geq \gamma \prod_{j=1}^{d} a_j = \tilde{\gamma} |Q_\tilde{a}|.
\]

In the second case, we have

\[
|\tilde{S} \cap (x + Q_\tilde{a})| \geq |M(S') \cap M(y + Q_a)| = |S \cap (y + Q_a)| \geq \tilde{\gamma} |Q_\tilde{a}|,
\]

which completes the proof of part (a).

For part (b), we observe that \( \tilde{S} \) can be obtained from \( S \cap [0, +\infty)^d \) by successive reflection with respect to all coordinate axes. In this regard, part (b) follows by analogous arguments as in part (a), cf. Figure 2(b); for more details, see also the proof of [9, Theorem 4].

Finally, we prove part (c). Let \( x \in \mathbb{R}^d \). Then, there exists \( y \in \mathbb{R}^d \) such that \( y + Q_a \subset (x + Q_\tilde{a}) \cap H_0 \) or \( M_0(y + Q_a) \subset (x + Q_\tilde{a}) \cap (\mathbb{R}^d \setminus H_0) \). This follows from the fact that \( x + Q_\tilde{a} \) contains the cylinder \( x' + (B_2(0, \sqrt{a_1^2 + a_2^2}) \times \mathbb{R}^{d-2} \cap [0, a_1]) \) for some \( x' \in \mathbb{R}^d \), where \( B_2(0, r) \) stands for the Euclidean 2-dimensional ball centred at zero with radius \( r \), and such a cylinder contains a parallelepiped of type \( y + Q_a \subset H_0 \) or a reflection \( M_0(y + Q_a) \subset \mathbb{R}^d \setminus H_0 \), or both, cf. Figure 3(c). The claim now follows by analogous calculations as in part (a), taking into account that \( \gamma |Q_\tilde{a}| = \frac{\gamma a_1 a_2}{4(a_1^2 + a_2^2)} |Q_\tilde{a}| = \tilde{\gamma} |Q_\tilde{a}| \).  \( \Box \)
Fig. 2  Positions of parallelepipeds in dimension $d = 2$: (a) $y + Q_\alpha \subset x + Q_\alpha$ or $M(y + Q_\alpha \subset x + Q_\alpha$ is for some $y \in \mathbb{R}^d$ contained in one of the two half spaces. (b) A parallelepiped $y + Q_\alpha \subset x + Q_\alpha$ is contained in one of the orthants. (c) The ball of radius $\sqrt{a_1^2 + a_2^2}$ at the origin contains both a parallelepiped of sides $a_1$ and $a_2$ and its reflection with respect to $M_\theta$

**Remark 2.3** The choice of the parameters $\tilde{y}$ and $\tilde{a}$ in part (c) of Lemma 2.2 takes care of every possible orientation of reflected parallelepipeds, which are in general not parallel to coordinate axes (with the exception of $\theta = \pi/4$). In this regard, the angle $\theta = \pi/8$ constitutes a kind of worst case, whereas especially the angle $\theta = \pi/4$ could have been handled in a slightly more efficient way. We refrained from doing so for the sake of simplicity.

**Proposition 2.4** (The heat equation on half-spaces) The system (2.1) on $\Omega = (0, +\infty) \times \mathbb{R}^{d-1}$ is null-controllable in any time $T > 0$ with control cost

$$C_T \leq \frac{1}{\sqrt{T}} \left( \frac{2K^d}{\gamma} \right)^{K_0/2} \exp \left( \frac{K(2a_1 + a_2 + \ldots + a_d)^2}{2T} \ln^2(2K^d/\gamma) \right), \quad (2.4)$$

where $K > 0$ is the universal constant from (2.2).

**Proof** Choose the $(\gamma/2, (2a_1, a_2, \ldots, a_d))$-thick set $\tilde{S}$ as in Lemma 2.2 (a). Then, the heat equation (2.3) on $\tilde{\Omega} = \mathbb{R}^d$ is null-controllable in any time $T > 0$ from the control set $\tilde{\omega} = \tilde{S}$. The associated bound on the control cost $\tilde{C}_T$ from (2.2) now reads as in (2.4).

Since $\tilde{\Omega} = \Omega \cup M(\Omega) \cup \Gamma$ with $\Gamma = \{0\} \times \mathbb{R}^{d-1}$ and $\tilde{\omega} = \omega \cup M(\omega)$, the claim follows by Theorem 1.1. □
Proposition 2.5 (Heat equation on positive orthants) The system (2.1) on $\Omega = (0, +\infty)^d$ is null-controllable in any time $T > 0$ with control cost

$$C_T \leq \frac{1}{\sqrt{T}} \left( \frac{(2K)^d}{\gamma} \right)^{Kd/2} \exp \left( \frac{4K\|a\|_1^2 \ln^2((2K)^d/\gamma)}{2T} \right), \quad (2.5)$$

where $K > 0$ is the universal constant from (2.2).

Proof Choose the $(\gamma/2^d, 2a)$-thick set $\hat{S}$ as in Lemma 2.2 (b). Then, the heat equation (2.3) on $\hat{\Omega} = \mathbb{R}^d$ is null-controllable in any time $T > 0$ from the control set $\hat{\omega} = \hat{S}$. The associated bound on the control cost $\hat{C}_T$ now reads as in (2.5).

Since the set $\hat{S}$ is symmetric with respect to every coordinate axis, the claim now follows by successively applying Theorem 1.1 for the reflections with respect to the coordinate axes. This step by step leads to null-controllability of the system (2.1) on $\Omega = (0, +\infty)^j \times \mathbb{R}^{d-j}$, $j = 1, \ldots, d$, with control set $\omega = ((0, +\infty)^j \times \mathbb{R}^{d-j}) \cap \hat{S}$. Each control cost does not exceed the one from the previous step and, thus, the corresponding bound reads as in (2.5).

Taking into account that $(0, +\infty)^d \cap \hat{S} = (0, +\infty)^d \cap \hat{S}$, the final step $j = d$ then proves the claim. \qed

Remark 2.6 Control cost estimates on the half-space and the positive orthant can alternatively be obtained by means of the exhaustion approach studied in [26]. In this case, the corresponding bounds will not incorporate any change in the thickness parameters as the above bounds do. However, the considerations from [26] currently allow to apply this approach only in the case of Dirichlet boundary conditions.

Proposition 2.7 (Heat equation on a sector of angle $\pi/2^n$, $n \geq 2$) The system (2.1) on the sector $\Omega = \hat{S}(\{x, y\} \in (0, +\infty)^2 : y < (\tan(\pi/2^n))x\}$ of angle $\pi/2^n$, $n \geq 2$, is null-controllable in any time $T > 0$ with control cost

$$C_T \leq \frac{1}{\sqrt{T}} \left( \frac{\gamma^{3n-4} K^2}{\hat{\gamma}} \right)^{K} \exp \left( \frac{\gamma^{3n-4} K \|a\|_1^2 \ln^2(\gamma^{3n-4} K^2/\hat{\gamma})}{2T} \right), \quad (2.6)$$

where

$$\hat{\gamma} = \frac{\gamma a_1 a_2}{4(a_1^2 + a_2^2)}, \quad \hat{a} = (2\sqrt{a_1^2 + a_2^2}, 2\sqrt{a_1^2 + a_2^2}).$$

and $K > 0$ is the universal constant from (2.2).

Proof We prove the claim by induction on $n$. For $n = 2$, we choose the $(\hat{\gamma}, \hat{a})$-thick set $\hat{S}$ as in Lemma 2.2 (c) with $d = 2$ and $\theta = \pi/4$. Then, the heat equation (2.3) on the orthant $\hat{\Omega} = (0, +\infty)^2$ is null-controllable in any time $T > 0$, and the bound on the associated control cost from Proposition 2.5 reads as in (2.6) for $n = 2$. The claim for $n = 2$ then follows by applying Theorem 1.1 with respect to the reflection $M_{\pi/4}$.

Now, suppose that the claim holds for some $n \geq 2$. Choose the $(\hat{\gamma}, \hat{a})$-thick set $\hat{S}$ as in Lemma 2.2 (c) with $d = 2$ and $\theta = \pi/2^{n+1}$. Since $\hat{a} = (\hat{a}_1, \hat{a}_2)$ is a multiple of $(1, 1)$, we observe that

$$\frac{\gamma^{3n-4} K}{4(\hat{a}_1^3 + \hat{a}_2^3)} = \frac{\hat{\gamma}}{8} \quad \text{and} \quad (2\sqrt{\hat{a}_1^2 + \hat{a}_2^2}, 2\sqrt{\hat{a}_1^2 + \hat{a}_2^2}) = 2\sqrt{2}\hat{a}.$$

Thus, by the induction hypothesis, the system (2.3) with $\hat{\Omega}$ being the sector of angle $\pi/2^n$ is null-controllable in any time $T$, and the corresponding bound on the control cost reads as in (2.6), but with $\hat{\gamma}$ and $\hat{a}$ replaced by $\hat{\gamma}/8$ and $2\sqrt{2}\hat{a}$, respectively. The claim for $\Omega$ being
the sector of angle $\pi/2^{n+1}$ therefore follows by applying Theorem 1.1 with respect to the reflection $M_{\pi/2^{n+1}}$. This concludes the proof.

For the last proposition of this subsection, we suppose that the thickness parameter $a = (a_1, \ldots, a_d)$ additionally satisfies $2\sqrt{a_1^2 + a_2^2} \leq L$ and $a_j \leq L$ for $j \in \{3, \ldots, d\}$.

**Proposition 2.8** (Heat equation on triangles and triangular prisms) Let

\[ T_L := \{(x, y) \in \Lambda^2_L : y < x\} \quad \text{and} \quad \mathcal{P}_L := T_L \times (0, L). \]

Then, the system (2.1) on $\Omega \in \{T_L, \mathcal{P}_L\}$ is null-controllable in any time $T > 0$ with control cost

\[ C_T \leq \frac{1}{\sqrt{T}} \left( \frac{K}{\tilde{\gamma}} \right)^{Kd/2} \exp \left( \frac{K\|\tilde{a}\|_1^2 \ln^2(Kd/\tilde{\gamma})}{2T} \right), \]

where $\tilde{\gamma}$ and $\tilde{a}$ are as in part (c) of Lemma 2.2 and $K > 0$ is the universal constant from (2.2). Here, $d = 2$ if $\Omega = T_L$ and $d = 3$ if $\Omega = \mathcal{P}_L$.

**Proof** This is proved analogously to the case $n = 2$ in Proposition 2.7 with $\tilde{\Omega} = \Lambda^d_L$ and the corresponding control cost bound from (2.2).

**Remark 2.9** In the recent work [6], it has been shown that the system (2.1) on the strip $\Omega = \Lambda^d_L \times \mathbb{R}$ is null-controllable in any time $T > 0$ if and only if $\mathcal{S}$ is a thick set (which can be arbitrarily changed outside the strip), and an explicit control cost bound has been provided.

With similar arguments as in the propositions above, one can infer null-controllability results for system (2.1) on the product spaces $\Lambda^d_L \times (0, +\infty)$, $T_L \times \mathbb{R}$, and $T_L \times (0, +\infty)$ by means of Theorem 1.1.

**Remark 2.10** For the situations discussed above, [19, Theorem 2.13] tells us that $C_T \geq 1/\sqrt{T}$, and equality is attained if, for instance, $\mathcal{S}$ agrees with $\mathbb{R}^d$, which corresponds to the case of full control on $\Omega$. The upper bounds on $C_T$ obtained above can be compared to this lower bound in the asymptotic regime where $\alpha \to 0$ while $\gamma$ remains fixed. This means that $\omega = \Omega \cap \mathcal{S}$ becomes more and more “well-distributed” within $\Omega$ and is sometimes also called the homogenization limit, see, e.g., [19, Section 5] and the references cited therein. More precisely, for simplicity, consider $a = \alpha(1, \ldots, 1) \in (0, \infty)^d$ with $\alpha > 0$ and fixed $\gamma$. In the limit as $\alpha \to 0$, the obtained upper bounds on $C_T$ have the form

\[ \frac{1}{\sqrt{T}} \left( \frac{CKd}{\gamma} \right)^{Kd/2} \quad \text{with} \quad C = C(d, \Omega) > 0, \]

which, up to constants, agrees with the lower bound on $C_T$ in terms of time dependence.

On the other hand, in the de-homogenization limit where $\alpha \to \infty$, the simultaneous choice of $T \sim \|a\|_1^2$ prevents the blow-up of the control cost as $\alpha \to \infty$. Here, the choice $T \sim \|a\|_1^2$ recovers the relation between time and space derivatives in (2.1), cf. [19, Example 5.4].

**Remark 2.11** (a) Null-controllability results and corresponding bounds on the control cost for fractional heat equations, that is, system (2.3) with $(-\Delta^s)\gamma$, $s > \frac{1}{2}$, instead
of $-\Delta^{\cdot}_{\Omega}$, have also been investigated in [19, Theorem 4.10] on $\tilde{\Omega} \in \{\mathbb{R}^d, \Lambda_L^d\}$ with a control cost bound of the form (2.2) but with $\|a\|^2_1 \ln^2(K^d/\gamma)/T$ replaced by $(\|a\|^2_1 \ln(K^d/\gamma))^{2s/(2s-1)}/T^{1/(2s-1)}$.

Although fractional Laplacians are formally not of divergence form as the operators considered in Theorem 1.1, the more general considerations in Section 3 below allow us to obtain corresponding results on the domains considered in Propositions 2.4–2.8, cf. Remark 4.2 (a) below.

(b) Recent developments on null-controllability for parabolic equations in Banach spaces [3, 12] together with the abstract considerations in Section 3 also open the way towards similar results for certain strongly elliptic differential operators with constant coefficients on $L^p$, $1 < p < \infty$; a bound on the associated control cost on $\mathbb{R}^d$ of a form close to (2.2) is given in [3, Theorem 2.3], see also [12, Corollary 4.6]. We briefly show how to infer a corresponding result for the $L^p$-Laplacian on the half-space $(0, +\infty) \times \mathbb{R}^{d-1}$ in Remark 4.2 (b) below.

2.2 Null-controllability from Equidistributed Sets

Let $V \in L^\infty(\Omega)$ be real-valued, and consider the heat-like system

$$\partial_t u(t) + (-\Delta^{\cdot}_{\Omega} + V)u(t) = \chi_\Omega v(t) \quad \text{for} \quad 0 < t < T, \quad u(0) = u_0,$$  \tag{2.8}$$

with $u_0 \in L^2(\Omega)$ and $v \in L^2((0, T), L^2(\Omega))$. Moreover, let the control set $\omega$ be of the form $\omega = \Omega \cap S$ with some $S \subset \mathbb{R}^d$ that is $(G, \delta)$-equidistributed in the sense of the following definition.

**Definition 2.12** Let $G > 0$ and $\delta \in (0, G/2)$. A measurable set $S \subset \mathbb{R}^d$ is called $(G, \delta)$-equidistributed if

$$S \supset \bigcup_{j \in \mathbb{Z}^d} B(z_j, \delta)$$

for some sequence $(z_j)_{j \in \mathbb{Z}^d} \subset \mathbb{R}^d$ with $B(z_j, \delta) \subset (Gj + \Lambda_G^d)$ for all $j \in \mathbb{Z}^d$, where $B(z_j, \delta)$ denotes the open ball in $\mathbb{R}^d$ of radius $\delta$ around $z_j$.

It is shown in [19] that system (2.8) is null-controllable in any time $T > 0$ on domains of the form $\Omega = \times_{j=1}^d (a_j, b_j)$ with $a_j, b_j \in \mathbb{R} \cup \{\pm \infty\}$, $a_j < b_j$, such that $\Lambda_G \subset \Omega$. A corresponding bound on the control cost is given by

$$C_T \leq \left(\frac{\delta}{G}\right)^{-D/(1+G^{4/3} \|V\|_{\infty}^{2/3})} \frac{D}{\sqrt{T}} \exp \left(\frac{DG^2 \ln^2(\delta/G)}{2T} + \|V_\cdot\|_{\infty}T\right),$$  \tag{2.9}$$

where $V_\cdot = \max(-V, 0)$ denotes the negative part of $V$ and $D = D(d)$ depends only on the dimension, cf. [19, Theorem 4.11] and [26, Proposition 2.4 and Corollary 2.5].

Below we extend the above result to sectors of angle $\pi/2^n$, $n \geq 2$, isosceles right-angled triangles, and corresponding prisms. A treatment of some related product spaces would also be possible, but we will not do this here for simplicity. In the framework of the main theorem, let $\tilde{\Omega} \in \{\Lambda^d_L, (0, +\infty)^d\}$, $L \geq 2G$, and $\tilde{\omega} = \tilde{\Omega} \cap \tilde{S}$ with some $(2G, \delta)$-
equidistributed set for $\mathcal{S}$. As in the previous subsection, we use the above-mentioned results for the system
\[
\partial_t \tilde{u}(t) + (-\Delta_\Omega^\ast + \tilde{V})\tilde{u}(t) = \chi_0 \tilde{v}(t) \quad \text{for} \quad 0 < t < T, \quad \tilde{u}(0) = \tilde{u}_0,
\]
with $\tilde{u}_0 \in L^2(\tilde{\Omega})$ and $\tilde{v} \in L^2((0, T), L^2(\tilde{\Omega}))$ to infer null-controllability and related control cost bounds on the desired domains.

Analogously to Lemma 2.2 in the case of thick sets, we first need the following geometric consideration.

**Lemma 2.13** Let $S \subset \mathbb{R}^d$ be $(G, \delta)$-equidistributed, and let $S' = S \cap H_\theta$. Then, the set $\mathcal{S} = S' \cup M_\theta(S')$ is $(4G, \delta)$-equidistributed.

**Proof** We need to show that for every cube $\Lambda_j := 4Gj + \Lambda^d_{4G}$, $j \in \mathbb{Z}^d$, there exists $j' \in \mathbb{Z}^d$ such that $Gj' + \Lambda^d_{4G} \subset \Lambda_j \cap \mathcal{H}_{\theta}$ or $M_\theta(Gj' + \Lambda^d_{4G}) \subset \Lambda_j \cap (\mathbb{R}^d \setminus H_\theta)$.

To this end, we observe that for every $j \in \mathbb{Z}^d$ there exists $k \in \mathbb{Z}^d$ such that $Gk + \Lambda^d_G \subset \Lambda_j \cap \mathcal{H}_{\theta}$ or $3Gk + \Lambda^d_{3G} \subset \Lambda_j \cap (\mathbb{R}^d \setminus H_\theta)$, cf. Figure 3(a). In the first case, we are done with $j' = k$. In the second case, the cube $3Gk + \Lambda^d_{3G}$ contains a ball of radius $\sqrt{2}G$, which, in turn, contains the reflection of a cube $Q \subset H_\theta$ with sides of length $2G$ parallel to coordinate axes, cf. Figure 3(b). This cube $Q$ must contain at least one cube of the form $Gj' + \Lambda^d_G$ with $j' \in \mathbb{Z}^d$, for which $M_\theta(Gj' + \Lambda^d_{G}) \subset 3Gk + \Lambda^d_{3G} \subset \Lambda_j \cap (\mathbb{R}^d \setminus H_\theta)$. This concludes the proof. \hfill $\square$

**Remark 2.14** If $\theta = \pi/4$, it is possible to slightly strengthen the above lemma. Indeed, in this case the resulting set $\mathcal{S}$ is $(2G, \delta)$-equidistributed since the reflection $M_{\pi/4}$ maps cells of the lattice $(G\mathbb{Z})^d$ in $\mathcal{H}_{\pi/4}$ to cells of the lattice in $\mathbb{R}^d \setminus H_{\pi/4}$ and vice versa, and each cube $2Gj + \Lambda^d_{2G}$, $j \in \mathbb{Z}^d$, contains at least one cell of the lattice $(G\mathbb{Z})^d$ that belongs either to $\mathcal{H}_{\pi/4}$ or to $\mathbb{R}^d \setminus H_{\pi/4}$. However, for the sake of simplicity we opted for a unified statement valid for every reflection $M_\theta$.

![Fig. 3](image-url)

(a) For every reflection hyperplane cutting a $4G$-cell, there is a $G$-subcell in the lower or a $3G$-subcell in the upper half-space. (b) The blue $3G$-subcell contains the reflection of the red cube $Q$. 

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Proposition 2.15 (Heat-like equation on a sector of angle $\pi/2^n$, $n \geq 2$) The system (2.8) on the sector $\Omega = \{(x, y) \in (0, +\infty)^2 : y < (\tan(\pi/2^n))x\}$ of angle $\pi/2^n$, $n \geq 2$, is null-controllable in any time $T > 0$ with control cost

$$C_T \leq \left(\frac{\delta}{G}\right)^{-D(1+\tilde{G}^{4/3}\|V\|^2_{L^\infty})} \frac{D}{\sqrt{T}} \exp\left(\frac{DG^2 \ln^2(\delta/\tilde{G})}{2T} + \|V\|_{L^\infty} T\right),$$

(2.11)

where $\tilde{G} = 4^{n-1} G$ and $D = D(2)$ is the constant from (2.9) for $d = 2$.

Proof This is proved by induction on $n$ analogously to Proposition 2.7, the only difference being the use of Lemma 2.13 instead of Lemma 2.2 (c). \qed

Proposition 2.16 (Heat-like equation on triangles and triangular prisms) Let $L \geq 2G$, and let $\mathcal{T}_L$ and $\mathcal{P}_L$ as in Proposition 2.8. Then, the system (2.8) on $\Omega \in \{\mathcal{T}_L, \mathcal{P}_L\}$ is null-controllable in any time $T > 0$ with control cost

$$C_T \leq \left(\frac{\delta}{2G}\right)^{-R(1+2G)^{4/3}\|V\|^2_{L^\infty}} \frac{R}{\sqrt{T}} \exp\left(\frac{R(2G)^2 \ln^2(\delta/(2G))}{2T} + \|V\|_{L^\infty} T\right),$$

where $R = \max\{D(2), D(3)\}$ with $D(d)$ from (2.9).

Proof Taking into account Remark 2.14, this is proved analogously to the case $n = 2$ in Proposition 2.15 with $\tilde{\Omega} = \Lambda^d_L$. \qed

Remark 2.17 In the situation of equidistributed sets, similarly as for thick sets in Remark 2.10, the homogenization limit corresponds to the limit as $\delta, G \to 0$ while the quotient $\nu := G/\delta$ remains constant. In this regime, the obtained upper bounds on $C_T$ have the form

$$(C\nu)^{-R} \cdot \frac{R}{\sqrt{T}} \exp(\|V\|_{L^\infty} T) \quad \text{with} \quad C = C(\Omega) > 0, \quad R = \max\{D(2), D(3)\},$$

which has the same asymptotic behaviour for $T \to 0$ as the lower bound $C_T \geq 1/\sqrt{T}$.

3 Abstract Result

In this section, we prove a general version of Theorem 1.1 for more abstract control systems, which in principle allows one to apply this result, for instance, also to other types of differential operators than discussed here so far, such as fractional Laplacians, second order elliptic operators, and magnetic Schrödinger operators.

Let $\mathcal{H}$ and $\mathcal{U}$ be Banach spaces, $-H$ an infinitesimal generator of a $C_0$-semigroup $(S(t))_{t \geq 0}$ on $\mathcal{H}$, $B : \mathcal{U} \to \mathcal{H}$ a bounded linear operator, and $T > 0$. The abstract Cauchy problem

$$\partial_t u(t) + Hu(t) = Bv(t) \quad \text{for} \quad 0 < t < T, \quad u(0) = u_0,$$

(3.1)

with $u_0 \in \mathcal{H}$ and $v \in L^1((0, T), \mathcal{U})$ is said to be null-controllable in time $T > 0$ with respect to $L^p((0, T), \mathcal{U})$, $p \in [1, \infty]$, if for every initial datum $u_0 \in \mathcal{H}$ there is a function $v \in L^p((0, T), \mathcal{U})$ with

$$S(T)u_0 + \int_0^T S(T-s)Bv(s) \, ds = 0,$$

(3.2)
that is, if the mild solution to (3.1) vanishes at time $T$; see, e.g., [20] for the notion of a mild solution to abstract Cauchy problems. The associated control cost in time $T > 0$ is then defined as

$$C_{T, p} := \sup_{\|u_0\|_H = 1} \inf_{v} \{\|v\|_{L^p((0, T), \mathcal{U})} : v \text{ satisfies (3.2)}\} < \infty.$$ 

The main result of this section is as follows.

**Theorem 3.1** Let $\mathcal{H}, \tilde{\mathcal{H}}, \mathcal{U}, \tilde{\mathcal{U}}$ be Banach spaces, $-\mathcal{H}$ and $-\tilde{\mathcal{H}}$ infinitesimal generators of $C_0$-semigroups on $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively, and let $B : \mathcal{U} \to \mathcal{H}$ and $\tilde{B} : \tilde{\mathcal{U}} \to \tilde{\mathcal{H}}$ be bounded linear operators.

Suppose that there is a bounded linear operator $Y : \tilde{\mathcal{H}} \to \mathcal{H}$ with a bounded right inverse $\tilde{Y} : \mathcal{H} \to \tilde{\mathcal{H}}$ such that

$$Y \tilde{\mathcal{H}} \subset HY,$$

that is, $Yg \in \mathcal{D}(H)$ and $Y \tilde{\mathcal{H}} g = HYg$ for all $g \in \mathcal{D}(\tilde{\mathcal{H}})$. Furthermore, suppose that for some bounded linear operator $Z : \mathcal{U} \to \tilde{\mathcal{U}}$ one has

$$Y \tilde{\mathcal{H}} = BZ.$$

If the system

$$\partial_t \tilde{u}(t) + \tilde{\mathcal{H}} \tilde{u}(t) = \tilde{B} \tilde{v}(t) \quad \text{for} \quad 0 < t < T, \quad \tilde{u}(0) = \tilde{u}_0,$$

with $\tilde{u}_0 \in \tilde{\mathcal{H}}$ and $\tilde{v} \in L^1((0, T), \tilde{\mathcal{U}})$ is null-controllable in time $T > 0$ with respect to $L^p((0, T), \mathcal{U})$ with control cost $C_{T, p} > 0$, then also the system (3.1) is null-controllable in time $T > 0$ with respect to $L^p((0, T), \mathcal{U})$ with control cost $C_{T, p} > 0$ satisfying

$$C_{T, p} \leq \|\tilde{Y}\|_{\mathcal{H} \to \tilde{\mathcal{H}}} \|Z\|_{\mathcal{U} \to \tilde{\mathcal{U}}} \cdot C_{T, p}.$$ 

**Proof** Let $(S(t))_{t \geq 0}$ and $(\tilde{S}(t))_{t \geq 0}$ be the $C_0$-semigroups associated to $-\mathcal{H}$ and $-\tilde{\mathcal{H}}$, respectively. We first show that

$$Y \tilde{S}(t) = S(t)Y \quad \text{for all} \quad t > 0.$$ 

To this end, let $\tilde{u}_0 \in \mathcal{D}(\tilde{\mathcal{H}})$, and define the function $w : [0, \infty) \to \mathcal{H}$ by $w(t) := Y \tilde{S}(t) \tilde{u}_0$. Since $\tilde{S}(t)\tilde{u}_0 \in \mathcal{D}(\tilde{\mathcal{H}})$ holds for all $t \geq 0$, we then have $w(t) \in \mathcal{D}(H)$ and, in particular, $w(0) = Y\tilde{u}_0 \in \mathcal{D}(H)$. Moreover,

$$\partial_t w(t) = -Y \tilde{\mathcal{H}} \tilde{S}(t) \tilde{u}_0 = -HY \tilde{S}(t) \tilde{u}_0 = -H w(t)$$

for $t > 0$. By uniqueness of the solution, see, e.g., [20, Theorem 4.1.3], we conclude that $w(t) = S(t)w(0)$, that is, $Y \tilde{S}(t) \tilde{u}_0 = S(t)Y \tilde{u}_0$ for all $t \geq 0$. Since $\mathcal{D}(\tilde{H})$ is dense in $\tilde{\mathcal{H}}$ and $S(t)$ and $\tilde{S}(t)$ are bounded operators, this proves (3.6).

Let now $u_0 \in \mathcal{H}$, and set $\tilde{u}_0 := \tilde{Y} u_0 \in \tilde{\mathcal{H}}$. Then, $Y \tilde{u}_0 = u_0$, and by hypothesis there is $\tilde{v} \in L^p((0, T), \mathcal{U})$ such that

$$\tilde{S}(T)\tilde{u}_0 + \int_0^T \tilde{S}(T-s)\tilde{B}\tilde{v}(s) ds = 0.$$

Define $v \in L^p((0, T), \mathcal{U})$ by $v(s) := Z \tilde{v}(s)$. By (3.4) and (3.6), we then conclude that

$$S(T)u_0 + \int_0^T S(T-s)B v(s) \ ds = S(T)Y \tilde{u}_0 + \int_0^T S(T-s)BZ \tilde{v}(s) \ ds$$

$$= Y \left(\tilde{S}(T)\tilde{u}_0 + \int_0^T \tilde{S}(T-s)\tilde{B}\tilde{v}(s) ds\right) = 0,$$

so that system (3.1) is indeed null-controllable in time $T > 0$ with respect to $L^p((0, T), \mathcal{U})$. 

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Finally, using \( \|v(s)\|_{\mathcal{U}} \leq \|Z\|_{\mathcal{U}}\rightarrow\mathcal{U}\|v(s)\|_{\mathcal{U}} \) for \( s \in (0, T) \), we observe that
\[ \|v\|_{L^p((0,T),\mathcal{U})} \leq \|Z\|_{\mathcal{U}}\rightarrow\mathcal{U}\|v\|_{L^p((0,T),\mathcal{U})}. \]
Since also \( \|u_0\|_{\mathcal{H}} \leq \|v\|_{\mathcal{U}}\rightarrow\mathcal{U}\|u_0\|_{\mathcal{H}} \), the claimed estimate for the control cost \( C_{T,p} \) is immediate. This completes the proof.

**Remark 3.2** Since the extension relation (3.3) is only used to establish the semigroup relation (3.6), the conclusion of Theorem 3.1 still holds if (3.6) is directly taken as an assumption.

We refrained from doing so because the extension relation between the operators gives a nice understanding of the interplay between the systems (3.1) and (3.5).

In the particular case where \( H \) and \( \widetilde{H} \) are lower semibounded self-adjoint operators on Hilbert spaces \( \mathcal{H} \) and \( \widetilde{\mathcal{H}} \), respectively, relation (3.6) for the semigroups can be extended to far more general functions of the operators. The following result formulates this explicitly. It is of particular interest, for instance, when considering fractional Laplacians of the form \((-\Delta)^s\) with \( s > 1/2 \), see Remark 4.2 (a) below. The result itself is probably well known and can be proved, for instance, with a standard reasoning using Stone’s formula, which relates the spectral family of a self-adjoint operator with its resolvents. We give a brief variant of this reasoning below for the convenience of the reader.

**Lemma 3.3** Let \( H \) and \( \widetilde{H} \) be lower semibounded self-adjoint operators on Hilbert spaces \( \mathcal{H} \) and \( \widetilde{\mathcal{H}} \), respectively, and let \( Y : \widetilde{\mathcal{H}} \rightarrow \mathcal{H} \) be a bounded linear operator such that \( Y\widetilde{H} \subseteq HY \). Then, the spectral families \( E_{\mathcal{H}} \) and \( E_{\widetilde{\mathcal{H}}} \) for \( H \) and \( \widetilde{H} \), respectively, satisfy
\[ YE_{\widetilde{\mathcal{H}}} = E_{\mathcal{H}} Y \]
for all \( \lambda \in \mathbb{R} \).

Moreover, for every Borel measurable function \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) the relation
\[ Y\phi(\widetilde{H}) \subseteq \phi(H)Y \]
holds, where \( \phi(H) \) and \( \phi(\widetilde{H}) \) are defined by functional calculus.

**Proof** It suffices to consider the case where \( \mathcal{H} \) and \( \widetilde{\mathcal{H}} \) are complex Hilbert spaces. The case of real Hilbert spaces can be obtained from this by complexification.

In view of the relation \( Y\widetilde{H} \subseteq HY \), we have \( Y(\widetilde{H} - s \pm i\varepsilon) \subseteq (H - s \pm i\varepsilon)Y \) for every \( s \in \mathbb{R} \) and \( \varepsilon > 0 \), so that \( Y(\widetilde{H} - s \pm i\varepsilon)^{-1}g = (H - s \pm i\varepsilon)^{-1}Yg \) for all \( g \in \widetilde{\mathcal{H}} \) and, hence,
\[ \langle (\widetilde{H} - s \pm i\varepsilon)^{-1}g, Y^*f \rangle_{\widetilde{\mathcal{H}}} = \langle (H - s \pm i\varepsilon)^{-1}Yg, f \rangle_{\mathcal{H}} \quad \forall f \in \mathcal{H}, \quad \forall g \in \widetilde{\mathcal{H}}. \]
Stone’s formula for the spectral families (see, e.g., [29, Satz 8.11]) then implies that
\[ \langle Y\mathcal{E}_{\widetilde{\mathcal{H}}} \rangle_{H} = \langle \mathcal{E}_{H} \rangle_{\mathcal{H}} \quad \forall f \in \mathcal{H}, \quad \forall \lambda \in \mathbb{R}. \]
for all \( f \in \mathcal{H} \) and \( \lambda \in \mathbb{R} \), that is, \( YE_{\widetilde{\mathcal{H}}} = E_{\mathcal{H}} Y \) for all \( \lambda \in \mathbb{R} \), which proves the first claim of the lemma.

Now, let \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) be a Borel measurable function and \( g \in \mathcal{D}(\phi(\widetilde{H})) \). The above then implies that \( \langle \mathcal{E}_{H}(Yg, g) \rangle_{\mathcal{H}} \leq \langle Y^2 \mathcal{E}_{\widetilde{\mathcal{H}}} \rangle_{\widetilde{\mathcal{H}}} \) for all \( \lambda \in \mathbb{R} \), so that by functional calculus we conclude that \( Yg \in \mathcal{D}(\phi(H)) \) and
\[ \langle Y\phi(\widetilde{H})g, f \rangle_{\mathcal{H}} = \langle \phi(\widetilde{H})g, Y^*f \rangle_{\widetilde{\mathcal{H}}} = \langle \phi(H)Yg, f \rangle_{\mathcal{H}} \quad \forall f \in \mathcal{H}. \]
This proves the second claim and, hence, completes the proof.

**Remark 3.4** In the case of complex Hilbert spaces in Lemma 3.3, also complex-valued Borel measurable functions \( \phi \) can be considered, obviously, without any change to the proof.
Remark 3.5 Given a lower semibounded operator $\tilde{H}$ on $\tilde{\mathcal{H}}$, an extension relation of the form (3.3) can clearly be formulated with respect to every subspace $\mathcal{M} \subset \tilde{\mathcal{H}}$ that reduces $\tilde{H}$, in which case $\hat{Y}$ can be chosen as the embedding of $\mathcal{H} = \mathcal{M}$ into $\tilde{\mathcal{H}}$ and $H$ is the part of $\tilde{H}$ corresponding to $\mathcal{M}$. More generally, $\hat{Y}$ can be any isometry from a Hilbert space $\mathcal{H}$ to $\tilde{\mathcal{H}}$ with $\text{Ran}\hat{Y} = \mathcal{M}$ and then $H = \hat{Y}^* \tilde{H} \hat{Y}$. Indeed, in this case, with $Y = \hat{Y}^*$ the orthogonal projection $P_M$ in $\tilde{\mathcal{H}}$ onto $\mathcal{M}$ agrees with $\hat{Y}Y$ and we have $YP_M = Y$. Since for $g \in D(\tilde{H})$ we have $P_M g \in D(\tilde{H})$ with $\tilde{H}P_M g = P_M \tilde{H}g$, this gives $Y g \in D(H)$ and

$$HYg = Y \tilde{H} \hat{Y} Y g = Y \tilde{H} P_M g = Y P_M \tilde{H} g = Y \tilde{H}g,$$

which proves (3.3). Upon a suitable choice of the operators $B$ and $\tilde{B}$, this can be used to infer null-controllability of subsystems of null-controllable systems corresponding to reducing subspaces.

Conversely, if $\hat{Y}$ is an isometry such that $Y = \hat{Y}^*$ satisfies (3.3), then the subspace $\text{Ran}\hat{Y}$ reduces $\tilde{H}$. Indeed, by taking adjoints, it follows from (3.3) that $\tilde{H} \hat{Y} = \hat{Y}^* (\tilde{H} \hat{Y})^* \supset \hat{Y} H$ and, therefore, $P \tilde{H} \subset \tilde{H} P$ with the orthogonal projection $P = \hat{Y} \hat{Y}^* = Y^* Y$. Up to a constant multiple in the definition of $\hat{Y}$, this is the situation we encounter in Theorem 1.1, see Section 4 below.

We close this section with a remark on a general lower bound on the control cost in the Hilbert space setting.

Remark 3.6 Let $H$ and $\tilde{H}$ be as in Lemma 3.3, and let $\mathcal{U}$ and $\tilde{\mathcal{U}}$ be Hilbert spaces. In this case, for system (3.5) we obtain from [19, Theorem 2.13] the sharp lower bound

$$\tilde{C}_T := \tilde{C}_{T,2} \geq \|B\|_{\tilde{\mathcal{U}} \to \tilde{\mathcal{H}}}^{-1} \|\hat{Z}\|_{\tilde{\mathcal{H}} \to \tilde{\mathcal{U}}} \left( \frac{1}{\sqrt{\kappa}}, \frac{2\kappa}{\exp(2\kappa T) - 1} \right)^{\frac{1}{2}}, \quad \tilde{\kappa} = 0,$$

on the control cost, where $\tilde{\kappa} = \min \sigma(\tilde{H})$. Here, equality is attained if, for instance, $\tilde{\mathcal{U}} = \tilde{\mathcal{H}}$ and $\tilde{B}$ is (a multiple of) the identity, see [19, Remark 2.14]. The analogous bound holds for the control cost $C_T := C_{T,2}$ corresponding to system (3.1), with $B$ and $\kappa$ replaced by $\tilde{B}$ and $\tilde{\kappa}$, respectively.

Now, if the operator $Z$ in Theorem 3.1 has a bounded right inverse $\hat{Z}$, then the relation (3.4) gives $\|B\|_{\mathcal{U} \to \mathcal{H}} \leq \|Y\|_{\tilde{\mathcal{H}} \to \tilde{\mathcal{U}}} \|\hat{Z}\|_{\tilde{\mathcal{H}} \to \tilde{\mathcal{U}}} \|\tilde{B}\|_{\tilde{\mathcal{U}} \to \tilde{\mathcal{H}}}$, and the lower bound on $C_T$ implies that

$$C_T \geq (\|Y\|_{\tilde{\mathcal{H}} \to \tilde{\mathcal{U}}} \|\hat{Z}\|_{\tilde{\mathcal{H}} \to \tilde{\mathcal{U}}} \|\tilde{B}\|_{\tilde{\mathcal{U}} \to \tilde{\mathcal{H}}})^{-1} \left( \frac{1}{\sqrt{\kappa}}, \frac{2\kappa}{\exp(2\kappa T) - 1} \right)^{\frac{1}{2}}, \quad \kappa = 0,$$

$$\geq \left( \frac{1}{\sqrt{\kappa}}, \frac{2\kappa}{\exp(2\kappa T) - 1} \right)^{\frac{1}{2}}, \quad \kappa \neq 0.$$

Up to a relation between $\kappa$ and $\tilde{\kappa}$, this allows to interpret the lower bound for $C_T$ in terms of the lower bound for $\tilde{C}_T$. This gets particularly interesting when $\tilde{\mathcal{U}} = \tilde{\mathcal{H}}$, $\mathcal{U} = \mathcal{H}$, and $Z = Y$, a situation which we encounter in Theorem 1.1, see Section 4 below. In this case, the factor $\|Y\|_{\tilde{\mathcal{H}} \to \tilde{\mathcal{U}}} \|\hat{Z}\|_{\tilde{\mathcal{H}} \to \tilde{\mathcal{U}}} \tilde{B}$ agrees with the factor $\|\hat{Y}\|_{\tilde{\mathcal{H}} \to \tilde{\mathcal{U}}} \tilde{B}$ obtained in the bound from Theorem 3.1.

4 Proof of the Main Theorem

We deduce Theorem 1.1 from the abstract result Theorem 3.1 with the choices $\mathcal{H} = \mathcal{U} = L^2(\Omega)$, $\tilde{\mathcal{H}} = \tilde{\mathcal{U}} = L^2(\tilde{\Omega})$, $H = H_{\tilde{\Omega}}$, $\tilde{H} = H_{\Omega}^*$, $B = \chi_\omega$, $\tilde{B} = \chi_{\tilde{\omega}}$, and $p = 2$. Set $\lambda^N := 1$.
and \( \lambda^D := -1 \), and for \( \bullet \in \{ D, N \} \) define

\[
X^\bullet : L^2(\Omega) \to L^2(\tilde{\Omega}), \quad X^\bullet f := f \oplus (\lambda^\bullet J f),
\]

with respect to the orthogonal decomposition \( L^2(\tilde{\Omega}) = L^2(\Omega) \oplus L^2(M(\tilde{\Omega})) \), where \( J : L^2(\Omega) \to L^2(M(\tilde{\Omega})) \) is defined as \( Jf := f \circ M \).

The adjoint \((X^\bullet)^*\) of \( X^\bullet \) acts as

\[
(X^\bullet)^* g = (g + \lambda^\bullet g \circ M)|_\Omega
\]

for all \( g \in L^2(\tilde{\Omega}) \). Indeed, for \( f \in L^2(\Omega) \) and \( g \in L^2(\tilde{\Omega}) \) we have

\[
\int_{M(\tilde{\Omega})} (Jf)(x)g(x)\, dx = \int_{M(\tilde{\Omega})} f(M(x))g(x)\, dx = \int_{\Omega} f(x)g(M(x))\, dx
\]

by change of variables, so that

\[
\langle X^\bullet f, g \rangle_{L^2(\tilde{\Omega})} = \int_{\Omega} f(x)(g(x) + \lambda^\bullet g(M(x)))\, dx.
\]

It is then easy to see that

\[
(X^\bullet)^* X^\bullet f = 2f, \quad \forall f \in L^2(\Omega).
\]

Moreover, since \( \tilde{\omega} = \omega \cup M(\omega) \) by construction, we have \( \chi_{\tilde{\omega}} \circ M = \chi_\omega \) and, hence,

\[
(X^\bullet)^* \chi_\omega g = (\chi_\omega(g + \lambda^\bullet g \circ M)|_\Omega = \chi_\omega (X^\bullet)^* g, \quad \forall g \in L^2(\tilde{\Omega}).
\]

Hence, with the choice \( Y = Z = (X^\bullet)^* \), condition (3.4) in Theorem 3.1 is satisfied, and we may take \( \tilde{Y} = X^\bullet/2 \) as a bounded right inverse of \( Y \) with \( \| \tilde{Y} \| \| Z \| = 1 \).

It remains to verify the extension relation (3.3) for the operators \( H^\bullet_\Omega \) and \( H^\bullet_\tilde{\Omega} \), that is, \((X^\bullet)^* H^\bullet_\Omega \subset H^\bullet_\tilde{\Omega} (X^\bullet)^* \). To this end, let us first briefly recall how the operators \( H^\bullet_\Omega \) and \( H^\bullet_\tilde{\Omega} \) are constructed. Since it is in general hard to define them directly by their differential expression, especially if the matrix function \( A \) has discontinuities, the established approach to do that is via their quadratic forms:

Consider the form \( h^\bullet_\Omega = a^\bullet_\Omega + V \) with

\[
h^\bullet_\Omega[f, g] := a^\bullet_\Omega[f, g] + \langle Vf, g \rangle_{L^2(\Omega)}, \quad f, g \in \mathcal{D}(a^\bullet_\Omega) := \mathcal{D}(a^\bullet_\Omega),
\]

where

\[
a^N_\Omega[f, g] := \int_{\Omega} (A(x)\nabla f(x), \nabla g(x))_{C^d}\, dx, \quad f, g \in \mathcal{D}(a^N_\Omega) := H^1(\Omega),
\]

and

\[
a^D_\Omega[f, g] := a^N_\Omega[f, g], \quad f, g \in \mathcal{D}(a^D_\Omega) := H^1_0(\Omega).
\]

The ellipticity condition (1.1) guarantees that this form \( h^\bullet_\Omega \) is densely defined, lower semibounded, and closed, cf. [25, Example 4.19]. Hence, there is a self-adjoint operator \( H^\bullet_\Omega = H^\bullet_\Omega(A, V) \) on \( L^2(\Omega) \) given by

\[
\mathcal{D}(H^\bullet_\Omega) = \{ f \in \mathcal{D}(h^\bullet_\Omega) : \exists h \in L^2(\Omega) \text{ s.t.} \}
\]

\[
b^\bullet_\Omega[f, g] = \langle h, g \rangle_{L^2(\Omega)} \quad \forall g \in \mathcal{D}(h^\bullet_\Omega)
\]

and

\[
h^\bullet_\Omega[f, g] = \langle H^\bullet_\Omega f, g \rangle_{L^2(\Omega)}, \quad f \in \mathcal{D}(H^\bullet_\Omega), \quad g \in \mathcal{D}(h^\bullet_\Omega),
\]

see, e.g., [14, Theorem VI.2.6], [24, Theorem 10.7], or [23, Theorem VIII.15]. The operator \( H^\bullet_\tilde{\Omega} \) on \( L^2(\tilde{\Omega}) \) is defined completely analogous via the form \( h^\bullet_\tilde{\Omega} = a^\bullet_\tilde{\Omega} + V \) with

\[
a^N_\tilde{\Omega}[f, g] := \int_{\tilde{\Omega}} (\tilde{A}(x)\nabla f(x), \nabla g(x))_{C^d}\, dx, \quad f, g \in \mathcal{D}(a^N_\tilde{\Omega}) := H^1(\tilde{\Omega}),
\]

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and
\[ a^D_Ω[f, g] := a^N_Ω[f, g], \quad f, g \in \mathcal{D}(a^D_Ω) := H^1_0(Ω). \]

The desired extension relation \((X^*)^* H^*_{Ω} \subset H^*_{Ω}(X^*)^*\) can now be formulated in terms of the quadratic forms as
\[ h^*_Ω[g, X^* f] = h^*_Ω[(X^*)^* g, f], \quad f \in \mathcal{D}(h^*_Ω), \quad g \in \mathcal{D}(h^*_Ω), \tag{4.6} \]
provided that \(X^* f \in \mathcal{D}(h^*_Ω) = \mathcal{D}(a^*_Ω)\) and \((X^*)^* g \in \mathcal{D}(h^*_Ω) = \mathcal{D}(a^*_Ω);\) see the formal proof below.

In order to verify (4.6), we first take care of the mapping properties of \(X^*\) and \((X^*)^*\).

**Lemma 4.1**

(a) \((X^*)^*\) maps \(\mathcal{D}(a^*_Ω)\) into \(\mathcal{D}(a^*_Ω)\).

(b) \(X^*\) maps \(\mathcal{D}(a^*_Ω)\) into \(\mathcal{D}(a^*_Ω)\), and one has
\[ (\nabla (X^* f))|Ω = \nabla f \quad \text{and} \quad (\nabla (X^* f))|M(Ω) = \lambda^* U(\nabla f) \circ M \]
for \(f \in \mathcal{D}(a^*_Ω)\) with \(U = \text{diag}(-1, 1, \ldots, 1).\)

**Proof**

(a). The case \(\bullet = N\) is clear by (4.1). For \(\bullet = D\) let first \(g \in C^∞_c(Ω)\). Then, \(g - g \circ M \in C^∞_c(Ω)\) and \(g - g \circ M = 0\) on \(Γ\), and part (c) of Lemma A.1 in the appendix yields \((X^D)^* g = (g - g \circ M)|Ω \in H^1_0(Ω)\). The case of general \(g \in H^1_0(Ω)\) follows from this by approximation.

(b). Let \(f \in \mathcal{D}(a^*_Ω)\) and \(φ \in C^∞_c(Ω)\), and abbreviate \(α_1 := -1\) and \(α_k := 1\) for \(k \geq 2\). We then have to show that
\[ -\int_Ω (X^* f)(x)(∂_k φ)(x) \, dx = \int_Ω \left(∂_k f \oplus \lambda^* α_k (∂_k f) \circ M\right)(x) φ(x) \, dx \tag{4.7} \]
for \(k = 1, \ldots, d\). To this end, observe that \((∂_k φ) \circ M = α_k ∂_k φ \circ M\), so that
\[ \int_Ω (X^* f)(x)(∂_k φ)(x) \, dx = \int_Ω f(x) ∂_k (φ + λ^* α_k φ \circ M) \, dx \tag{4.8} \]
by change of variables. Now, for each \(k \in \{1, \ldots, d\}\) and \(\bullet \in \{N, D\}\), parts (a) and (b) of Lemma A.1 allow to perform integration by parts without boundary terms on the right-hand side of (4.8). We therefore obtain
\[ -\int_Ω f(x) ∂_k (φ + λ^* α_k φ \circ M)(x) \, dx = \int_Ω (φ + λ^* α_k φ \circ M)(x)(∂_k f)(x) \, dx \]
\[ = \int_Ω (∂_k f \oplus λ^* α_k (∂_k f) \circ M)(x) φ(x) \, dx, \]
where the last equality follows again by a change of variables. This equality together with (4.8) proves (4.7) and, hence, completes the proof.

We are finally in position to prove our main result.

**Proof of Theorem 1.1**

We need to verify identity (4.6). To this end, let \(f \in \mathcal{D}(h^*_Ω) = \mathcal{D}(a^*_Ω)\) and \(g \in \mathcal{D}(h^*_Ω) = \mathcal{D}(a^*_Ω)\). Since \(U(\nabla g) \circ M = \nabla(g \circ M)\) and, by definition, \(A(x) = U A(M(x)) U\) for \(x \in M(Ω)\), we obtain from part (b) of Lemma 4.1 with change of variables that
\[ \int_{M(Ω)} \langle A(x) \nabla (g \circ M)(x), \nabla f(x) \rangle_{C_{Ω}^d} \, dx = \lambda^* \int_Ω \langle A(x) \nabla (g \circ M)(x), \nabla f(x) \rangle_{C_{Ω}^d} \, dx. \]
Thus,
\[
\alpha^{*}_{\Omega}[g, X^{*}f] = \int_{\Omega} (A(x)\nabla(g + \lambda^{*}g \circ M)(x), (\nabla f)(x))_{C^{0'}} \, dx
\]  
(4.9)
where we have taken into account \((4.1)\) and part (a) of Lemma 4.1. Since also \(\tilde{V} \circ M = \tilde{V}\) and, therefore, \((X^{*})^*\tilde{V} = V(X^{*})^*\), this proves \((4.6)\). In turn, for \(g \in D(H^{*}_{\Omega}) \subset D(h^{*}_{\Omega})\) we conclude that
\[
b^{*}_{\Omega}(X^{*})^*g, f) = \langle H^{*}_{\Omega}g, X^{*}f \rangle_{L^{2}(\Omega)} = \langle (X^{*})^*H^{*}_{\Omega}g, f \rangle_{L^{2}(\tilde{\Omega})}
\]
for all \(f \in D(h^{*}_{\Omega})\), so that \((X^{*})^*g \in D(H^{*}_{\Omega})\) and \(H^{*}_{\Omega}(X^{*})^*g = (X^{*})^*H^{*}_{\Omega}g\), which proves the desired extension relation \((3.3)\). Since also \((3.4)\) is satisfied by \((4.3)\), applying Theorem 3.1 with \(\tilde{Y} = X^{*}/2\) completes the proof.

Remark 4.2 (a) The above proof in particular shows the extension relation \((X^{*})^*(-\Delta^{*}_{\Omega}) \subset (-\Delta^{*}_{\Omega})(X^{*})^*\). By Lemma 3.3 with the particular choice \(\phi = (\cdot)^s\mathbb{1}_{[0,\infty)}\), \(s > 1/2\), we then also have \((X^{*})^*(-\Delta^{*}_{\Omega})f \subset (-\Delta^{*}_{\Omega})f(X^{*})^*\). This allows us to apply Theorem 3.1 also in this situation, which gives an analogue of Theorem 1.1 for such fractional Laplacians, as claimed in Remark 2.11 (a).

(b) A modification of the above allows to consider also certain operators on \(L^{p}_{\Omega}\), \(p \in (1, \infty)\). We prove this briefly for the pure Laplacian on \(L^{p}(\Omega), \Omega = (0, \infty) \times \mathbb{R}^{d-1}\): Set \(\tilde{\Omega} = \mathbb{R}^{d}\) and let \(-\Delta^{*}_{\Omega,p}\) and \(-\Delta_{\tilde{\Omega},2}\) be the realizations on the differential expression \(-\Delta\) on \(L^{p}(\Omega)\) and \(L^{p}(\tilde{\Omega})\), respectively, as closed sectorial operators, see, e.g., [4, Theorem 5.4 and Corollary 6.11]. These agree with \(-\Delta^{*}_{\Omega} = -\Delta^{*}_{\Omega,2}\) and \(-\Delta_{\tilde{\Omega}} = -\Delta_{\tilde{\Omega},2}\), respectively, on common elements of their domains. Consider now the bounded operator \(Y^{*}_{p}: L^{p}(\tilde{\Omega}) \to L^{p}(\Omega), Y^{*}_{p} g = (g + \lambda^{*}g \circ M)\mathbb{1}_{\Omega}\), with \(\|Y^{*}_{p}\|_{L^{p} \to L^{p}} = 2^{1/q}, 1/p + 1/q = 1\), a right inverse of which is given by \(\hat{Y}^{*}_{p}: L^{p}(\Omega) \to L^{p}(\tilde{\Omega}), \hat{Y}^{*}_{p} f = 2^{-1}(f \oplus \lambda^{*}f \circ M)\), with \(\|\hat{Y}^{*}_{p}\|_{L^{p} \to L^{p}} = 2^{1/p-1}\). Let \(g \in \mathcal{S}(\mathbb{R}^{d})\) be a Schwartz function. We then have \(Y^{*}_{p} g = (X^{*})^*g \in D(-\Delta^{*}_{\Omega,p}) \cap D(-\Delta^{*}_{\tilde{\Omega},2})\) and
\[
Y^{*}_{p}(-\Delta_{\tilde{\Omega},p})g = (X^{*})^*(-\Delta_{\tilde{\Omega},2})g = (-\Delta^{*}_{\tilde{\Omega},2})(X^{*})^*g = (-\Delta_{\tilde{\Omega},p})Y^{*}_{p} g.
\]
The extension relation \(Y^{*}_{p}(-\Delta_{\tilde{\Omega},p}) \subset (-\Delta_{\tilde{\Omega},p})Y^{*}_{p}\) is then obtained by approximation since \(\mathcal{S}(\mathbb{R}^{d})\) is an operator core for \(-\Delta_{\tilde{\Omega},p}\). Theorem 3.1 can then be applied with \(Y = Z = Y^{*}_{p}\) and \(\hat{Y} = \hat{Y}^{*}_{p}\).

Remark 4.3 In the above considerations, the operator \(X^{*}\) models the extension of a function in \(L^{2}(\Omega)\) to a function in \(L^{2}(\tilde{\Omega})\) by reflection with respect to one hyperplane. However, in view of the general form of Theorem 3.1, also other forms of extensions are feasible. For instance, one could consider to prolong functions on a sector of angle \(\pi/n\) in the plane to functions on the whole plane by successive reflections with respect to different lines. In a similar way, in the work [22] the author considers the prolongation of functions on the hemiequilateral triangle with corners \((0, 0), (0, 1), (1/\sqrt{3}, 0)\) to functions on the rectangle \((0, \sqrt{3}) \times (0, 1)\). One then has to prove results analogous to Lemma 4.1 and relation \((4.9)\), allowing to infer null-controllability results on such sectors and the (hemi-)equilateral triangle from those on the whole plane and the rectangle, respectively. In fact, in case of the hemiequilateral triangle, these analogous results follow to some extent from the considerations in [22] already.
One may also consider the rescaling that maps a given sector of arbitrary angle $0 < \theta < \pi/2$ to the sector with angle $\pi/4$. This transforms the pure Laplacian on the given sector to a divergence-type operator on the sector of angle $\pi/4$ with constant diagonal matrix and, therefore, fits into the framework of our abstract Theorem 3.1. However, there are currently no results on null-controllability of the form discussed in Section 2 available for such operators on this specific sector, so that Theorem 3.1 cannot be applied yet.

**Appendix A. An integration by parts formula**

The following result is probably folklore, but in lack of a suitable reference we give an elementary proof using only integration by parts on hypercubes. We emphasize that we are not assuming any boundary regularity for the set $\Omega$.

**Lemma A.1** Let $\tilde{\Omega} \subset \mathbb{R}^d$ be an open set with $\Gamma := \tilde{\Omega} \cap \{(0) \times \mathbb{R}^{d-1}\} \neq \emptyset$, and set $\Omega := \tilde{\Omega} \cap (0, +\infty) \times \mathbb{R}^{d-1}$). Let $f \in H^1(\Omega)$ and $\phi \in C_c^\infty(\Omega)$.

(a) One has
\[
\int_{\Omega} \phi(x)(\partial_k f)(x) \, dx = -\int_{\Omega} f(x)(\partial_k \phi)(x) \, dx
\]
for $k \in \{2, \ldots, d\}$.

(b) If, in addition, $f \in H^1_0(\Omega)$ or $\phi|_{\Gamma} = 0$, then
\[
\int_{\Omega} \phi(x)(\partial_1 f)(x) \, dx = -\int_{\Omega} f(x)(\partial_1 \phi)(x) \, dx.
\]

(c) If $\phi|_{\Gamma} = 0$, then $\phi|_{\Omega} \in H^1_0(\Omega)$.

**Proof** Since $\text{supp} \, \phi \subset \tilde{\Omega}$ is compact, there is $\varepsilon > 0$ with
\[
2\varepsilon < \text{dist}_{\infty}(\text{supp} \, \phi, \partial \tilde{\Omega}),
\]
where the distance is taken with respect to the maximum norm. We now cover $\text{supp} \, \phi \cap (\{0\} \times \mathbb{R}^{d-1})$ with suitable cubes from an equidistant lattice of mesh size $\varepsilon$. More precisely, we choose a finite subset $J \subset \mathbb{N} \times \mathbb{Z}^{d-1}$ with $\mathbb{N} = \{1, 2, \ldots\}$ such that the compact set
\[
Q := \bigcup_{j \in J} \Lambda_\varepsilon(j), \quad \Lambda_\varepsilon(j) := \varepsilon j + \left( \frac{-\varepsilon}{2}, \frac{\varepsilon}{2} \right)^d,
\]
is a subset of $\tilde{\Omega} \cap (\{0\} \times \mathbb{R}^{d-1})$ containing $\text{supp} \, \phi \cap (\{0\} \times \mathbb{R}^{d-1})$ and such that $\text{supp} \, \phi \cap (\{0\} \times \mathbb{R}^{d-1})$ lies in the interior $Q^\circ$ of $Q$. Clearly, we have $Q \setminus (\{0\} \times \mathbb{R}^{d-1}) \subset \Omega$.

An integration by parts on each cube $\Lambda = \Lambda_\varepsilon(j) \subset \Omega$, $j \in J$, yields
\[
\int_{\Lambda} \phi(x)(\partial_k f)(x) \, dx = \int_{\Gamma_k^+} f(x)\phi(x) \, d\sigma(x) - \int_{\Gamma_k^-} f(x)\phi(x) \, d\sigma(x)
\]
\[
- \int_{\Lambda} f(x)(\partial_k \phi)(x) \, dx,
\]
where $\Gamma_k^+$ and $\Gamma_k^-$ denote the two opposite faces of the cube $\Lambda$ (relatively open in the corresponding hyperplane) with outward unit normal in positive and negative direction with respect to the $k$-th coordinate axis, respectively. Now, each of the faces $\Gamma_k^+$ and $\Gamma_k^-$ either
corresponds to the face of exactly one other cube $\omega_{\varepsilon}(l)$, $l \in J \setminus \{j\}$, but then with the opposite direction of the corresponding outward unit normal, or the face belongs to the boundary of $Q$. In the latter case, the integral over the face vanishes unless the face belongs to the hyperplane \( \{0\} \times \mathbb{R}^{d-1} \), since \( \text{supp } \phi \cap ((0, +\infty) \times \mathbb{R}^{d-1}) \) lies in the interior of $Q$. This means that after summing over all cubes $\omega_{\varepsilon}(j)$, $j \in J$, in (A.1) only boundary integrals for faces in the hyperplane \( \{0\} \times \mathbb{R}^{d-1} \) can survive, and these faces have an outward unit normal in the negative direction in the first coordinate ($k = 1$).

Thus, after summing in (A.1) over all cubes $\omega_{\varepsilon}(j)$, $j \in J$, in both cases (a) and (b) we have no remaining boundary integrals, so that

$$\int_Q \phi(x)(\partial_k f)(x) \, dx = -\int_Q f(x)(\partial_k \phi)(x) \, dx,$$

where for $k = 1$ we may approximate $f \in H^1_0(\Omega)$ with functions from $C_c^\infty(\Omega)$ and then take the limit. Since \( \text{supp } \phi \cap ((0, +\infty) \times \mathbb{R}^{d-1}) \subset Q^\circ \subset \Omega \), this proves (a) and (b).

It remains to show that $\phi\vert_{A'} \in H^1_0(A')$ if $\phi\vert_{A} = 0$. For that, we choose an open cube $\tilde{\Lambda} \subset \mathbb{R}^d$ with \( \text{supp } \phi \subset \tilde{\Lambda} \) and $\Lambda' := \tilde{\Lambda} \cap ((0, +\infty) \times \mathbb{R}^{d-1}) \neq \emptyset$. Extending $\phi$ by zero on $\tilde{\Lambda} \setminus \tilde{\Omega}$, and taking into account that $\phi\vert_{\tilde{\Omega}} = 0$ by hypothesis, we have $\phi \in C_c^\infty(\tilde{\Lambda})$ and $\phi = 0$ on $\partial A'$.

Since $\Lambda'$ is convex and therefore has a Lipschitz boundary, see, e.g., [13, Corollary 1.2.2.3], this implies that $\phi\vert_{\Lambda'} \in H^1_0(\Lambda')$, see, e.g., [13, Corollary 1.5.1.6] or [1, Lemma A6.10]. Thus, there exists a sequence $(\varphi_k)$ in $C_c^\infty(\Lambda')$ such that $\varphi_k \to \phi\vert_{\Lambda'}$ in $H^1(\Lambda')$ as $k \to \infty$. We choose a smooth cutoff function $\eta \in C_c^\infty(\tilde{\Omega})$ with $\eta = 1$ on supp $\phi$. Since $(\eta \varphi_k)\vert_{\Omega} = \phi\vert_{\Omega}$, it is straightforward to verify that $(\eta \varphi_k)\vert_{\Omega} \to \phi\vert_{\Omega}$ in $H^1(\Omega)$ as $k \to \infty$. Since also $(\eta \varphi_k)\vert_{\Omega} \in C_c^\infty(\Omega)$, we conclude that $\phi\vert_{\Omega} \in H^1_0(\Omega)$. This proves (c) and, hence, completes the proof of the lemma.

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