From Kernel Machines to Ensemble Learning

Chunhua Shen, Fayao Liu

Abstract—Ensemble methods such as boosting combine multiple learners to obtain better prediction than could be obtained from any individual learner. Here we propose a principled framework for directly constructing ensemble learning methods from kernel methods. Unlike previous studies showing the equivalence between boosting and support vector machines (SVMs), which needs a translation procedure, we show that it is possible to design boosting-like procedure to solve the SVM optimization problems. In other words, it is possible to design ensemble methods directly from SVM without any middle procedure. This finding not only enables us to design new ensemble learning methods directly from kernel methods, but also makes it possible to take advantage of those highly-optimized fast linear SVM solvers for ensemble learning. We exemplify this framework for designing binary ensemble learning as well as a new multi-class ensemble learning methods. Experimental results demonstrate the flexibility and usefulness of the proposed framework.

Index Terms—Kernel, Support vector machines, Ensemble learning, Column generation, Multi-class classification.

I. INTRODUCTION

Ensemble learning methods, with a typical example being boosting [1]–[5], have been successfully applied to many machine learning and computer vision applications. Its excellent performance and fast evaluation have made ensemble learning one of the most widely-used learning methods, together with kernel machines like SVMs. In the literature, the general connection between boosting and SVM has been shown by Schapire et al. [2], and Rätsch et al. [6]. In particular, Rätsch et al. [6] developed a mechanism to convert SVM algorithms to boosting-like algorithms by translating the quadratic programs (QP) of SVMs into linear programs (LP) of boosting (similar to LPBoost [7]). A one-class boosting method was then designed by converting one-class SVM into an LP. Following this vein, a direct approach to multi-class boosting was developed in [8] by using the loss function in Crammer and Singer’s multi-class SVM [9]. The recipe to transfer algorithms is essentially [6]: “The SV-kernel is replaced by an appropriately constructed hypothesis space for leveraging where the optimization of an analogous mathematical program is done using \( \ell_1 \) instead of \( \ell_2 \)-norm.” This transfer is indirect in the sense that one has to design a different \( \ell_1 \) norm regularized mathematical program. We suspect that this is due to the widely-adopted belief that boosting methods need the sparsity-inducing \( \ell_1 \)-norm regularization so that the final ensemble model only relies on a subset of weak learners [3]. In this work, we show that it is possible to design ensemble learning methods by directly solving standard SVM optimization problems. Unlike [6, 8], no mathematical transform is needed. The only optimization technique that our framework relies on is column generation. With the proposed framework, the advantages that we can think of are: 1) Many kernel methods can directly have an equivalent ensemble model; 2) As conventional boosting methods, our ensemble models are iteratively learned. At each iteration, compared with the \( \ell_1 \) optimization involved in the indirect approach [6, 6], our optimization problems are much simpler. For the first time, we enable the use of fast linear SVM optimization software for ensemble learning. 3) As the fully-corrective boosting methods in [6, 8], our ensemble learning procedure is also fully corrective. Therefore the convergence speed is often much faster than stage-wise boosting. 4) Kernel SVMs usually offer promising classification accuracies at the price of high usage of memory and evaluation time, especially when the size of training data is large. Recall that the number of support vectors is linearly proportional to the number of training data [11]. Ensemble models, on the other hand, are often much faster to evaluate. Ensemble learning is also more flexible in that the user can determine the number of weak learners used. Typically an ensemble model uses less than a few thousand weak learners. Ensemble learning can also select features by using decision stumps or trees as weak learners, while nonlinear kernels are defined on the entire feature space. The developed framework tries to enjoy the best of both worlds of kernel machines and ensemble learning. Additional contributions of this work include: 1) To exemplify the usefulness of this proposed framework, we introduce a new multi-class boosting method based on the recent multi-class SVM [12]. The new multi-class boosting is effective in performance and can be efficiently learned since a closed-form solution exists at each iteration. 2) We introduce Fourier features as weak learners for learning the strong classifier. Fourier features approximate the radial basis function (RBF) Gaussian kernel. Our experiments demonstrate that Fourier weak learners usually outperforms decision stumps and linear perceptrons. 3) We also show that multiple kernel learning is made much easier with the proposed framework.

II. RELATED WORK

The general connection between SVM and boosting has been discussed by a few researchers [3, 6] at a high level. To our knowledge, the work here is the first one that attempts to build ensemble models by solving SVM’s optimization problem. We review some closest work next. Boosting has been extensively studied in the past decade [1]–[3], [7], [8]. Our methods are close to [3, 7] in that we also use column generation (CG) to select weak learners and fully-correctively update weak learners’ coefficients. Because we are solving the SVM problem, instead of the \( \ell_1 \) regularized boosting problem, conventional CG cannot be directly applied. We use CG in a novel way—instead of looking at dual constraints, we rely on the KKT conditions.

If one uses an infinitely many weak learners in boosting [13] (or hidden units in neural networks [14]), the model is equivalent to SVM with a certain kernel. In particular, it shows that when the feature mapping function \( \Phi(\cdot) \) contains infinitely many randomly distributed decision stumps, the kernel function \( k(x, x') = \langle \Phi(x), \Phi(x') \rangle \) is the stump kernel of the form \( \Delta = \|x - x'\|_1 \). Here \( \Delta \) is a constant, which has no impact on the SVM training. Moreover, when \( \Phi(x) = \text{sign}(\theta^\top x - \kappa) \), i.e., a perceptron, the corresponding kernel is called the perceptron kernel \( k(x, x') = \Delta - \|x - x'\|_2 \).

Loosely speaking, boosting can be seen as explicitly computing the kernel mapping functions because, as pointed out in [6], a kernel constructed by the inner product of weak learners’ outputs satisfies the Mercer’s condition. Random Fourier features (RFF) [15] have been applied to large-scale kernel methods. RFF is designed by using the fact that a shift-invariant kernel is the Fourier transform of a non-negative measure. Yang et al. show that RFF does not perform well due to its data-independent sampling strategy when there is a large gap in the eigen-spectrum of the kernel matrix [16]. In [17], [18], it shows that for homogeneous additive kernels, the kernel mapping function can be exactly computed. When RFF is used as weak learners in the proposed framework here, the greedy CG based RFF selection can be viewed as data-dependent feature selection. Indeed, our experiments demonstrate that our method performs much better than random sampling.
III. KERNEL METHODS AND ENSEMBLE MODELS

We first review some fundamental concepts in SVMs and boosting. Then we show the connection between these two methods and show how to design column generation based ensemble learning methods that directly solve the optimization problems in kernel methods.

Let us consider binary classification for the time being. Assume that the input data points are \((x_i, y_i) \in X \times \{-1, 1\}\), with \(i = 1 \cdots m\). For SVMs, it is well known that the original data \(x\) are implicitly mapped to a feature space through a mapping function \(\Phi : X \rightarrow \mathcal{F}\). The function \(\Phi\) is implicitly defined by a kernel function \(k(x, x') = \langle \Phi(x), \Phi(x') \rangle\), which computes the inner product in \(\mathcal{F}\). SVM finds a hyperplane that best separates the data by solving:

\[
\min_{w, b, \xi \geq 0} \frac{1}{2} \|w\|^2 + C \sum_i \xi_i, \quad \text{subject to} \quad y_i (\langle w, \Phi(x_i) \rangle + b) \geq 1 - \xi_i, \quad \forall i.
\]

The Lagrange dual can be easily derived:

\[
\max \sum_i \alpha_i - \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j y_i y_j k(x_i, x_j), \quad \text{s.t.} \quad 0 \leq \alpha_i \leq C, \ y_i \alpha_i = 0.
\]

Here \(C\) is the trade-off parameter; \(K\) is the kernel matrix with \(K_{ij} = k(x_i, x_j)\); and \(\circ\) denotes element-wise matrix multiplication, i.e., Hadamard product. \(y y^\top\) is the label matrix with \(y = [y_1, \ldots, y_m]\). Note that in the case of linear SVMs, i.e., \(k(x, x') = \langle x, x' \rangle\), there are fast and scalable algorithms for training linear SVMs, e.g., LIBLINEAR [19].

Ensemble learning methods, with boosting being the typical example, usually learns a strong classifier/model by linearly combining a finite set of weak learners. Formally the learned model is \(F(x) = w^\top \Phi(x)\) with

\[
\Phi(x) = [h_1(x), \ldots, h_J(x)]^\top.
\]

Therefore, in the case of boosting, the feature mapping function is explicitly learned: \(\Phi : x \rightarrow [h_1(x), \ldots, h_J(x)]^\top\), where \(h(\cdot) \in \mathcal{F}\) is the weak learner. It is easy to see that a kernel induced by the weak learner set \(k(x, x') = \sum_j h_j(x) h_j(x')\) is a valid one and its corresponding kernel matrix must be positive semidefinite. Next let us take LPBoost as an example to see how CG is used to explicitly learn weak learners, which is the core of most boosting methods [3, 7].

The primal program of LPBoost can be written as

\[
\min_{w \geq 0, \xi \geq 0} f_p = \|w\|_1 + C 1^\top \xi,
\]

subject to the margin constraint \(y_i (\langle w, \Phi(x_i) \rangle + b) \geq 1 - \xi_i, \quad \forall i\), with \(\Phi(\cdot)\) defined in [3]. The dual of (4) is

\[
\max \sum_i \xi_i - \frac{1}{2} \sum_{i,j} \xi_i \xi_j y_i y_j \langle h_i, h_j \rangle, \quad \text{s.t.} \quad 0 \leq \xi_i \leq C, \ y_i \xi_i = 0.
\]

Note that the last constraint in the dual is a set of \(J\) constraints. Often, the number of possible weak learners can be infinitely large. In this case it is intractable to solve either the primal or dual. In this case, CG can be used to solve the problem. These original problems are referred to as the master problems. The CG method solves these problems by incrementally selecting a subset of columns (variables in the primal and constraints in the dual) and optimizing the restricted problem on the subset of variables. So the basic idea of CG is to add one constraint at a time to the dual problem until an optimal solution is identified. In terms of the primal problem, CG solves the problem on a subset of variables, which corresponds to a subset of constraints in the dual. If a constraint absent from the dual problem is violated by the solution to the restricted problem, this constraint needs to be included in the dual problem to further restrict its feasible region. To speed convergence we would like to find the one with maximum deviation (most violated dual constraint), that is, the base learning algorithm must deliver a function \(\hat{h}(\cdot)\) such that

\[
\hat{h}(\cdot) = \arg\max_{h \in \mathcal{H}} \sum_i y_i \alpha_i h(x_i).
\]

If there is no weak learner \(h(\cdot)\) for which the dual constraint \(\sum_i y_i \alpha_i h(x_i) \leq 1\) is violated, then the current combined hypothesis is the optimal solution over all combinations of weak learners. That is the main idea of LPBoost [7] and its extension [3]. It has been believed that here two components have played an essential role in this procedure of deriving this meaningful dual such that CG can be applied. 1) This derivation relies on the \(\ell_1\) norm regularization in the primal objective of \(f_p\). 2) The constraint of nonnegative \(w\) lead to the dual inequality constraint. Without this nonnegative constraint, the last dual constraint becomes an equality: \(\sum_i y_i \alpha_i \Phi(x_i) = 1\). In terms of optimization, the constraint \(w \geq 0\) causes difficulties. We will show the remedies for these difficulties in the next section.

IV. FROM SVM TO ENSEMBLE LEARNING

We show how to derive ensemble learning directly from kernel methods like SVM. Our goal is to explicitly solve (1) without using the kernel trick. In other words, similar to boosting, we iteratively solve (1) by explicitly learning the kernel mapping function \(\Phi(\cdot)\). At the first glance, it is unclear how to use the idea of CG to derive a boosting-like procedure similar to LPBoost, as discussed above. In order to add a weak learner \(h(\cdot)\) into \(\Phi(\cdot)\) by finding the most violated dual constraint—as a starting point—we must have a dual constraint containing \(\Phi(\cdot)\). From the dual problem of SVM (3), the main difficulty here is that the dual constraints are two types of simple linear constraints on the dual variable \(\alpha\). The dual constraints do not have \(\Phi(\cdot)\) at all. A condition for applying CG is that the duality gap between the primal and dual problems is zero (strong duality). Generally, the primal problem must be convex [8] and both the primal and dual are feasible, so the Slater condition holds. In such a case, the KKT conditions are necessary conditions for a solution to be optimal. One such condition in deriving the dual (3) from (1) is

\[
w = \sum_{i=1}^m y_i \alpha_i \Phi(x_i).
\]

This KKT condition is the root of the representer theorem in kernel methods, which states that a minimizer of a regularized empirical risk function defined over a reproducing kernel Hilbert space can be represented as a finite linear combination of kernel products evaluated on the input points.

We can verify the optimality by checking the dual feasibility and KKT conditions. At optimality, (7) must hold for all \(j\), i.e., \(w_j = \sum_i y_i \alpha_i h_j(x_i)\) must hold for all \(j\). For the columns/weak learners in the current working set, i.e., \(j = 1 \cdots J\), the corresponding condition in (7) is satisfied by the current solution. For the weak learners that are not selected yet, they do not appear in the current restricted optimization problem and the corresponding \(w_j = 0\). It is easy to see that if \(\sum_j y_i \alpha_i h_j(x_i) = 0\) for any \(h(\cdot)\) that is not in the current working set, then current solution is already the globally optimal one. So, our base learning strategy to check the optimality as well as to select the best weak learner \(\hat{h}(\cdot)\) is:

\[
\hat{h}(\cdot) = \arg\max_{h \in \mathcal{H}} \sum_i y_i \alpha_i h(x_i).
\]

Different from (6), here we select the weak learner with the score \(\sum_i y_i \alpha_i h(x_i)\) farthest from 0, which can be negative. Now we show that using (3) to choose a weak learner is not heuristic in terms of solving the SVM problem of (1).

A Lagrange dual problem is always convex.
Algorithm 1 CGENS: Column generation for learning ensembles

Input: Training data \((x_i, y_i), i = 1 \cdots m\); termination threshold \(\epsilon > 0\); regularization parameter \(C > 0\); (optional) maximum iteration \(J_{\text{max}}\).

1 Initialize: \(J = 0\); \(w = 0\); \(\alpha_i = \text{const} (0 < \text{const} < C)\).

2 while true do

3 \quad . Find a new weak learner \(\tilde{h}(\cdot)\) by solving (6).

4 \quad . Check the termination condition: if \(\sum_j y_j \alpha_i \tilde{h}(x_i) < \epsilon\), then terminate (problem solved).

5 \quad . Add \(\tilde{h}(\cdot)\) to the restricted master problem;

6 \quad . Solve either the primal \(P\) or dual \(D\), and update \(\alpha, w, b\).

7 \quad . \(J = J + 1\); (optional) if \(J > J_{\text{max}}\), then terminate.

Output: The final learned model:

\[ F(x) = \sum_j w_j h_j(x) + b. \]

Claim 4.1: At iteration \(J + 1\), the weak learner selected using (6) decreases the duality gap for the most promising solution obtained at iteration \(J\), in terms of solving the SVM primal problem \((P)\) or dual \((D)\).

To prove the result, let us check the dual objective in (6).

We denote the current working set (corresponding to current selected weak learners) by \(W\) and the rest by \(\overline{W}\). The dual objective in (6) is

\[ 1^T \alpha = -\frac{1}{2} \alpha^T (K^W \odot \mathbf{y} y^T) \alpha = -\frac{1}{2} \alpha^T (K^W \odot \mathbf{y} y^T) \alpha. \] (9)

Here the \((s, t)\) entry of \(K^W\) is

\[ (\Phi^W(x_s), \Phi^W(x_t)) = \sum_{j \in W} h_j(x_s) h_j(x_t), \]

and likewise, \(K_{ij}^\overline{W} = \sum_{j \in \overline{W}} h_j(x_s) h_j(x_t). \)

Clearly the sum of first terms in (9) equals to the objective value of the primal problem with the current solution: \(\frac{1}{2} \|w\|_2^2 + C^T \mathbf{y} \xi\). Therefore the duality gap is the last term of (9): \(-\frac{1}{2} \alpha^T (K^W \odot \mathbf{y} y^T) \alpha = -\frac{1}{2} \sum_{j \in W} \sum_{i} y_i \alpha_i h(x_i)\). Clearly minimalization of this duality gap leads to the base learning rule (8). Next, we show that it can be equivalent between (9) and (8).

Claim 4.2: Let us assume that the weak learn set \(\Psi\) is negation complete; i.e., if \(\tilde{h}(\cdot) \in \Psi\), then \(-\tilde{h}(\cdot) \in \Psi\); and vice versa. Here \([-\tilde{h}(\cdot)]\) means the function \([-\tilde{h}(\cdot)] = -(\tilde{h}(\cdot))\). Then to solve \((8)\), we only need to solve \((9)\).

This result is straightforward. Because \(\Psi\) is negation complete, if a maximizer of \((8)\), \(\tilde{h}(\cdot)\), leads to \(\sum_i y_i \alpha_i \tilde{h}(x_i) < 0\), then \([-\tilde{h}(\cdot)] \in \Psi\) is also a maximizer of \((9)\) such that \(\sum_i y_i \alpha_i \tilde{h}(x_i) > 0\). Therefore we can always solve \((8)\) to obtain the maximizers of \((9)\).

At this point, we are ready to design CG based ensemble learning for solving the SVM problem, analogue to boosting, e.g., LPBoost. The proposed ensemble learning method termed CGENS, is summarized in Algorithm 1. Note that at Line 6, we can use very efficient linear SVM solvers to solve either the primal or dual. In our experiments, we have used LIBLINEAR [19].

Having shown how to solve the standard SVM problem using CG, we provide another example application of the proposed framework by developing a new multi-class ensemble method using the idea of simplex coding [12].

V. MULTI-CLASS ENSEMBLE LEARNING

As most real-world problems are inherently multi-class, multi-class learning is becoming increasingly important. Coding matrix based boosting methods are one of the popular boosting approaches to multi-class classification. Methods in this category include AdaBoost.MO [20], AdaBoost.OC and AdaBoost.ECC [21]. Shen and Hao proposed a direct approach to multi-class boosting in [8].

\footnote{In order not to confuse the terms, we use “ensemble learning” instead of “boosting” for our boosting-like algorithms.}

Algorithm 2 CGENS-SLS: Simplex coding multi-class ensembles

Input: Training data \((x_i, y_i), i = 1 \cdots m\); termination threshold \(\epsilon > 0\); regularization parameter \(C > 0\); (optional) maximum iteration \(J_{\text{max}}\).

1 Initialize: \(J = 0\); \(L_0 = \text{spx}(y_1 y^T)\); Assign a positive constant to each element of \(U_{tr}, w_r = 0, b_r = 0, r = 1 \cdots l\).

2 while true do

3 \quad . Find a new weak learner \(\tilde{h}(\cdot)\) by solving (13).

4 \quad . Check the termination condition: if \(\max_r \sum_r \tilde{h}(x_i) U_{tr} < \epsilon\), then terminate (problem solved);

5 \quad . Add \(\tilde{h}(\cdot)\) to the restricted master problem;

6 \quad . Update \(H, U, w_r\) using (11) (solving (10));

7 \quad . \(J = J + 1\); (optional) if \(J > J_{\text{max}}\), then terminate.

Output: The learned \(l\) classifiers:

\[ F_r(x) = \sum_i w_i \Phi(x_i) + b_r, \]

\(r = 1 \cdots l\), with \(\Phi(x) = [\Phi_1(x), \cdots, \Phi_m(x)]^T\).

Here we proffer a new multi-class ensemble learning method based on the simplex least-squares SVM (SLS-SVM) introduced in [12]. SLS-SVM can be seen as a generalization of the binary LS-SVM (least-squares SVM). For binary classification, LS-SVM fits the decision function output to the label: \(\min \frac{1}{2} \|w\|_2^2 + \frac{C}{2} \sum_i \xi_i^2\) with \(\xi_i = w^T \Phi(x_i) + b - y_i\). In the case of multi-class classification, label \(y_i \in \{1, 2, \ldots, l\}\) here. We have \(l + 1\) classes. Simplex coding maps each class label to \(l + 1\) most separated vectors \(\{c_1, c_2, \ldots, c_{l+1}\}\) on the unit hypersphere in \(\mathbb{R}^m\). So we need to learn \(l\) classifiers. Let us assume that the simplex label coding function is \(\text{spx} : \{1, \ldots, l\} \rightarrow \{c_1, \ldots, c_l\}\) (see Appendix). The label matrix \(L \in \mathbb{R}^{m \times l}\) collects training data’s coded labels such that each row \(L_l = \text{spx}(y_1 y^T)\).

SLS-SVM trains the \(l\) classifiers simultaneously by minimizing the following regularized problem:

\[
\min \frac{1}{2} \sum_{i} \|w_r\|_2^2 + \frac{C}{2} \sum_{i} \sum_{r=1}^{m} \xi_i^2 \mathbf{O}_{ir}, \quad \text{s.t. } \mathbf{O}_{ir} = w_r \Phi(x_i) + b_r.
\] (10)

Here the model parameters to optimize are \(\mathbf{w}_r \in \mathbb{R}^l, b_r \in \mathbb{R}\), for \(r = 1 \cdots l\). Problem (10) can be solved by deriving its dual (see Appendix) and the solutions are:

\[
\mathbf{b} = \left( \frac{1}{\mathbf{S}^{-1}} \mathbf{L}_\tau \right)^T, \quad \mathbf{U} = \mathbf{S}^{-1} (\mathbf{L} - 1 \mathbf{b}^T), \quad \mathbf{w}_r = \mathbf{H} \mathbf{U}_{\tau r}.
\] (11)

Here \(\mathbf{H} \in \mathbb{R}^{r \times m}\) denotes the learned weak classifiers’ response on the whole training data such that each column \(H \in \mathbb{R}^{r \times l} = \Phi(x_i); \mathbf{S} = \mathbf{H}^T \mathbf{L}_\tau \mathbf{H}, \mathbf{L}_\tau \) is the \(m \times m\) identity matrix; \(\mathbf{U} \in \mathbb{R}^{m \times l}\) is the dual Lagrange multiplier associated with the equality constraints of \(\mathbf{O}\). Note that, the inverse of \(\mathbf{S}\) can be computed efficiently incrementally (see the supplementary document).

Here one of the KKT conditions that CG relies on is (last equation of (11)):

\[ \mathbf{w}_r = \sum_i U_{ir} \Phi(x_i). \] (12)

Similar to the binary case, the subproblem for generating weak learners is

\[ \tilde{h}(\cdot) = \arg \max_{r = 1 \cdots l, h \in \Psi} \sum_i U_{ir} h(x_i). \] (13)

For the same reason as in Result 5.2 we have removed the absolute operation without changing the essential problem. A subtle difference from the binary classification is that we pick the best weak learner across all the \(l\) classifiers.

We summarize our multi-class ensemble method in Algorithm 2. The output is the \(l\) classifiers:

\[ F(x) = \left[ w_1 \Phi(x) + b_1, \cdots, w_l \Phi(x) + b_l \right]. \] (14)
The classification rule assigns the label

$$\arg\max_{y=1,\ldots,f+1} \langle F(x), c_y \rangle$$

(15)
to the test datum $x$.

VI. EXPERIMENTS

We run experiments on binary and multi-class classification problems, and compare our methods against classical boosting methods.

A. Binary classification

We conduct experiments on synthetic as well as real datasets, namely, 2D toy, 13 UCI benchmark datasets [7] and then on several vision tasks such as digits recognition, pedestrian detection etc..

2D toy data The data are generated by sampling points from a 2D Gaussian distribution. All points within a certain radius belong to one class and all outside belong to the other class. The decision boundary obtained by CGENS with decision stumps is plotted in Fig. 1. As can be seen, our method converges faster than AdaBoost because CGENS is fully corrective.

UCI benchmark For the UCI experiment, we use three different weak learners, namely, decision stumps, perceptrons and Fourier weak learners, with each compared with the corresponding kernel SVMs and other boosting methods.

We use decision stumps (stump kernel for SVM) in this experiment. We compared our method with AdaBoost, LPBoost, all using decision stumps as the weak learner. All the parameters are chosen using 5-fold cross validation. The maximum iteration for AdaBoost, LPBoost [7] and our CGENS are searched from $\{25, 50, 100, 250, 500\}$. Results of SVMs with the stump kernel are also reported [13]. Results are the average of 5 random splits on each dataset. From Table I, we can see that overall, all the methods achieve comparable accuracy, with CGENS being marginally the best, and SVM the second best.

In the second experiment, we compare our method (using 500 weak learners) against several other methods such as SVM using (1) perceptrons $\text{sign}(\theta^\top x - \kappa)$ as weak learners and the perceptron kernel for SVM, and (2) Fourier cosine functions $\cos(\theta^\top x - \kappa)$ as weak learners and Gaussian RBF kernel for SVM. We did not optimize the weak learner’s parameters $\{\theta, \kappa\}$. Instead, we sample 2000 pairs of $\{\theta, \kappa\}$ according to their distributions as described in [13] and [15], and then pick the one that maximizes the weak learner selection criterion in Equ. (5).

In the case of the perceptron kernel, same as the decision stump kernel, it is parameter free. In the case of Gaussian RBF kernel (corresponding Fourier weak learners), there is Gaussian bandwidth parameter $\sigma$. Here $\theta$ in Fourier is sampled from a Gaussian distribution with the same bandwidth. We cross validate this bandwidth parameter $\sigma$ with the SVM and use the same $\sigma$ for sampling Fourier weak learners for use in CGENS. Ideally one can cross validate $\sigma$ with CGENS, which needs extra computation overhead. This might be the reason why RBF SVM is slightly better CGENS with Fourier weak learners as shown in Table I because CGENS uses the optimal $\sigma$ of SVM. While in the case of perceptrons, CGENS performs on par with SVM. Note that as expected, in general, CGENS and SVM again outperform AdaBoost and LPBoost.

We have also compared our CGENS with RFF [15]. Although CGENS uses 500 features (weak learners) and RFF uses 2000 features, CGENS still slightly outperforms RFF.

Computation efficiency of CGENS We evaluate the computation efficiency of the proposed CGENS. As mentioned, at each CG iteration of CGENS, we need to solve a linear SVM and therefore we can take advantage of off-the-shelf fast linear SVM solvers. Here we have used LIBLINEAR. We compare CGENS against LPBoost because both are fully-corrective boosting. At each iteration, LPBoost needs to solve a linear program. We use the state-of-the-art commercial solver Mosek to solve the dual problem [5]. The dual problem has less number of variables than the primal [4]. Thus it more efficient for Mosek to solve the dual problem.

We run experiments on a standard desktop using the MNIST data

|                       | AdaBoost | LPBoost | SVM     | CGENS     |
|-----------------------|----------|---------|---------|-----------|
| banana                | 28.38±0.55 | 27.28±0.25 | 27.16±1.70 | 26.97±1.35 |
| b-cancer              | 28.83±6.12 | 29.90±1.84 | 29.61±1.69 | 30.13±2.13 |
| diabetes              | 25.13±1.48 | 25.66±2.41 | 25.53±1.32 | 25.07±2.35 |
| f-solar               | 32.85±1.44 | 31.98±1.69 | 32.30±2.45 | 31.80±1.70 |
| german                | 26.27±2.14 | 22.88±1.74 | 22.87±1.74 | 22.87±2.39 |
| heart                 | 18.80±4.60 | 18.60±4.45 | 17.00±3.39 | 18.40±4.51 |
| image                 | 2.55±0.67  | 2.44±0.63  | 3.62±0.58  | 2.85±0.45  |
| ringnorm              | 8.72±0.66  | 8.98±0.88  | 3.93±1.66  | 5.71±0.42  |
| splice                | 8.48±0.57  | 8.30±0.87  | 7.56±0.24  | 6.23±0.47  |
| thyroid               | 5.87±4.58  | 6.94±5.56  | 5.87±3.60  | 5.60±3.45  |
| Titanic               | 24.53±2.86 | 22.38±0.33 | 22.30±0.53 | 22.30±0.53 |
| tennessee             | 4.66±0.34  | 5.32±0.73  | 3.07±0.54  | 4.40±0.46  |
| waveform              | 12.86±0.46 | 12.96±0.59 | 12.84±2.32 | 12.63±0.67 |

TABLE I: Mean and standard deviation of test errors (%) on 13 UCI datasets using stumps (stump kernel for SVM).
B. Multi-class classification

To demonstrate the potential effectiveness of the proposed ensemble learning method in multi-class classification task, we test the proposed CGENS-SLS algorithm both on UCI and image benchmark datasets. For fair comparison, we focus on the multi-class algorithms using binary weak learners, including AdaBoost.ECC [21] , AdaBoost.MH [20] and MultiBoost [8] using the exponential loss.

The proposed CGENS-SLS method is more related to MultiBoost in the sense that both use the fully-corrective boosting framework, yet it employs an LS regression-type formulation of multi-class classifier and a closed-form solution exists for the sub optimization problem during each iteration. For all boosting algorithms, decision stumps are chose as the weak learners due to its simplicity and the controlled complexity. Similar to the binary classification experiments, the maximum number of iteration is set to 500. The regularization parameters in our CGENS-SLS and MultiBoost [8] are both determined by 5-fold cross validation. We first test the proposed CGENS-SLS on 7 UCI datasets, and then on several vision tasks. The results are summarized in Table III.

**UCI datasets** For each of the 7 dataset, all samples are randomly divided into 75% for training and 25% for test, regardless of the existence of a pre-specified split. Each algorithm is evaluated 10 times and the average results are reported.

**Handwritten digit recognition** Three handwritten digit datasets are evaluated here, namely, MNIST, USPS and PENDIGITS. For MNIST, we randomly sample 1000 examples from each class for training and use the original test set of 10000 examples for test. For USPS, we randomly select 75% for training and the rest for test.

**Image classification** We then apply the proposed CGENS-SLS for image classification on several datasets: PASCAL07 [23], LabelMe [24] and CIFAR10. For PASCAL07, we use 5 types of features provided in [25]. For LabelMe, we use the LabelMe-12-504 subset and generate the GIST [26] features. Images which have more than one class labels are excluded for these two datasets. We use 70% of the data for training, and the rest for test. As for CIFAR10, we also use the GIST [26] features and use the provided test and training sets.

**Scene recognition** The Scene15 dataset consists of 4,485 images of 9 outdoor scenes and 6 indoor scenes. We randomly select 100 images per class for training and the rest for test. Each image is divided into 31 sub-windows, each of which is represented as a histogram of 200

---

Table II: Test errors (%) on the 13 UCI benchmarks using Fourier weak learners (corresponding to Gaussian RBF kernels for SVM) and perceptrons weak learners.

| Fourier weak learner/RBF kernel | Perceptron/Perceptron kernel |
|---------------------------------|------------------------------|
| RBF [18] SVM CGENS AdaBoost LPBoost SVM CGENS |
| banana | 11.41±0.84 | 11.06±0.66 | 11.74±0.37 | 11.40±0.46 | 12.52±1.13 | 11.72±0.78 | 10.88±0.28 |
| b-cancer | 28.83±4.35 | 28.31±4.63 | 27.79±3.26 | 30.91±5.91 | 30.68±4.27 | 29.87±6.30 | 28.83±4.81 |
| diabetes | 24.27±0.68 | 24.33±1.51 | 23.87±2.34 | 26.87±3.01 | 25.54±1.58 | 24.07±1.40 | 24.73±1.55 |
| f-solar | 31.85±1.66 | 31.85±1.66 | 32.15±1.45 | 34.80±1.57 | 33.96±1.50 | 33.10±2.02 | 32.05±1.59 |
| german | 23.53±2.58 | 22.80±2.06 | 24.13±2.35 | 26.00±1.05 | 24.04±2.59 | 24.33±3.79 | 21.60±2.70 |
| heart | 17.00±4.53 | 17.60±4.67 | 17.80±4.82 | 20.60±6.43 | 20.06±4.83 | 18.60±6.07 | 17.80±2.95 |
| image | 3.80±1.68 | 2.99±0.49 | 3.90±0.98 | 2.75±0.50 | 3.06±0.51 | 3.19±0.44 | 2.75±0.50 |
| ringnorm | 2.03±0.13 | 1.66±0.08 | 2.47±0.21 | 4.99±0.50 | 7.96±0.45 | 2.15±0.23 | 3.02±0.21 |
| splice | 13.14±0.89 | 13.12±0.77 | 12.02±0.40 | 13.87±0.84 | 15.28±0.64 | 12.46±1.28 | 13.51±1.08 |
| thyroid | 4.27±2.19 | 0.71±0.46 | 0.73±2.56 | 2.47±1.19 | 7.46±1.59 | 6.93±3.04 | 4.80±3.21 |
| titanic | 22.42±0.66 | 22.42±0.66 | 22.37±0.73 | 24.05±1.89 | 22.56±1.89 | 22.56±0.75 | 22.09±1.83 |
| twonorm | 3.71±1.59 | 2.81±0.27 | 3.00±0.61 | 3.27±0.36 | 3.42±0.48 | 2.45±0.13 | 3.19±0.49 |
| waveform | 10.80±0.37 | 10.96±0.32 | 10.47±0.23 | 11.20±0.23 | 11.98±0.55 | 11.96±0.63 | 10.80±0.33 |

---

5http://www.ais.uni-bonn.de/download/datasets.html
6http://www.cs.toronto.edu/~kriz/cifar.html
TABLE III: Comparison of test errors (%) on 7 UCI and 7 vision datasets. The results of the UCI datasets (marked with *) are averaged over 10 different runs while all the others are averaged over 5 tests. Decision stumps are used as weak learners here. The best results are bold faced.

| Algorithm | Error rate (%) |
|-----------|----------------|
| AdaBoost  | 6.32±0.66     |
| Ours      | 5.80±0.46     |

TABLE IV: Error rates (%) of AdaBoost and the proposed method on spam dataset using decision stump.

| Algorithm | Error rate (%) |
|-----------|----------------|
| AdaBoost  |                |
| Ours      |                |

and \(O_{tr}\) to zeros:

\[
\begin{align*}
\frac{\partial L}{\partial w_r} &= w_r - \sum_{i=1}^{m} U_{ir} \Phi(x_i) = 0, \\
\frac{\partial L}{\partial b_r} &= -\sum_{i=1}^{m} U_{ir} = 0, \\
\frac{\partial L}{\partial O_{tr}} &= CO_{tr} - U_{tr} = 0, \\
\frac{\partial L}{\partial U_{tr}} &= w_{tr}^\top \Phi(x_i) + b_r - L_{tr} + O_{tr} = 0.
\end{align*}
\]

Use \(H \in \mathbb{R}^{J \times m}\) to denote the weak classifiers’ response on the whole training data such that each column \(H_i = \Phi(x_i)\). The previous conditions can be rewritten to the following form:

\[
\begin{align*}
\begin{align*}
L_{tr} &= (H^\top H + C^{-1}I_m)U_{tr} + b_r1, \\
\Rightarrow S^{-1}L_{tr} &= S^{-1}SU_{tr} + b_rS^{-1}1 \\
\Rightarrow S^{-1}L_{tr} &= S^{-1}SU_{tr} + b_rS^{-1}1 \\
&= 0, \text{ due to (18b)} \\
\Rightarrow 1^\top S^{-1}L_{tr} &= 1^\top U_{tr} + b_r1^\top S^{-1}1 \\
\Rightarrow b_r &= 1^\top S^{-1}L_{tr}, \quad U_{tr} = S^{-1}(L_{tr} - b_r1) \\
\Rightarrow b &= 1^\top S^{-1}L_{tr}, \quad U = S^{-1}(L - 1b^\top).
\end{align*}
\end{align*}
\]

The inverse for \(S\) can be computed efficiently as follows. Suppose \(H_{(j)}, S_{(j)}, S_{(j)}\) and \(H_{(j+1)}, S_{(j+1)}, S_{(j)}^{-1}\) are matrices in the \(j\)th and \((j + 1)\)th iteration, respectively. We have \(H_{(j+1)} = [H_{(j)}^\top h_{j+1}]^\top\), where \(h_{j+1} = [h_{j+1}(x_1), h_{j+1}(x_2), \ldots, h_{j+1}(x_n)]^\top\). It is easy to see that \(S_{(j)}, S_{(j)}, S_{(j+1)}\) and \(S_{(j)}^{-1}\) are symmetric matrices. So,

\[
S_{(j+1)}^{-1} = \left(C^{-1}I_m + H_{(j+1)}^\top H_{(j+1)}\right)^{-1} = \left(C^{-1}I_m + H_{(j)}^\top H_{(j)} + h_{j+1}h_{j+1}^\top\right)^{-1} = \left(S_{(j)} + h_{j+1}h_{j+1}^\top\right)^{-1} = S_{(j)}^{-1} - S_{(j)}^{-1}h_{j+1} \left(1 + h_{j+1}S_{(j)}^{-1}h_{j+1}^\top\right)^{-1} h_{j+1}S_{(j)}^{-1}. \\
\]

Let \(s_{j+1} = S_{(j+1)}^{-1}h_{j+1}\), the update process finally is

\[
S_{(j+1)}^{-1} = S_{(j)}^{-1} - \frac{s_{(j+1)}h_{j+1}}{1 + h_{j+1}s_{(j+1)}}. \\
\]

VII. CONCLUSION

Kernel methods are popular in domains even outside of the computer science community largely because they are easy to use and there are highly optimized software available. On the other hand, ensemble learning is being developed in a separate direction and there are highly optimized software available. On the other hand, ensemble learning is being developed in a separate direction and there are highly optimized software available. On the other hand, ensemble learning is being developed in a separate direction and there are highly optimized software available. On the other hand, ensemble learning is being developed in a separate direction and there are highly optimized software available. On the other hand, ensemble learning is being developed in a separate direction and there are highly optimized software available. On the other hand, ensemble learning is being developed in a separate direction and there are highly optimized software available. On the other hand, ensemble learning is being developed in a separate direction and there are highly optimized software available.

VIII. APPENDIX

A. Solutions of the multi-class SLS-SVM and ensemble learning

The Lagrangian of problem \([10]\) is

\[
\begin{align*}
L(w_1, \ldots, w_m, b, O, U) &= \frac{1}{2} \sum_{j=1}^{J} \|w_j\|_2^2 + \frac{C}{2} \sum_{i=1}^{m} \sum_{j=1}^{J} O_{ij}^2 - \sum_{i=1}^{m} \sum_{j=1}^{J} \sum_{r=1}^{t} \lambda_{ijr} \Phi(x_i) + b_r - L_{ir} + O_{ir},
\end{align*}
\]

where \(U \in \mathbb{R}^{m \times 1}\) is the collection of Lagrange variables corresponded to \(O\). The optimization problem can be solved by setting its first order partial derivative with respect to the parameters \(w_r, b_r, L_{ir}\)
decision stump as weak learner. The task is to differentiate spam emails according to word frequencies. We use AdaBoost as a baseline. The maximum iterations are both set to 60 due to fast convergence and no overfitting observed hereafter. We use 5-fold cross validation to choose the best hyper parameter $C$ in CGENS. The results, shown in Table [IV] are reported over 20 different runs with training/testing ratio being 3 : 2. Fig. [V] plots the average frequencies over the 20 rounds. As can be observed, both algorithms select important features such as “free” (feature #16), “hp” (25) and “!” (52) with high frequencies. As for the other features, the two methods showed different inclinations. CGENS tends to select features like “remove” (7), “you” (19), “$” (53) which are intuitively meaningful for the classification. On the contrary, the favorite ones of AdaBoost are “george” (27), “meeting” (42) and “edu” (46) which are more irrelevant for spam email detection. This explains why our method slightly outperformed AdaBoost in test accuracy.

REFERENCES

[1] Y. Freund and R. E. Schapire, “A decision-theoretic generalization of on-line learning and an application to boosting,” J. Comp. & Syst. Sci., vol. 55, no. 1, pp. 119–139, 1997.

[2] R. E. Schapire, Y. Freund, P. Bartlett, and W. S. Lee, “Boosting the margin: a new explanation for the effectiveness of voting methods,” Annals of Statistics, vol. 26, pp. 322–330, 1998.
[3] C. Shen and H. Li, “On the dual formulation of boosting algorithms,” *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 32, 2010.

[4] S. Paisitkriangkrai, C. Shen, Q. Shi, and A. van den Hengel, “RandomBoost: Simplified multi-class boosting through randomization,” *IEEE Transactions on Neural Networks and Learning Systems*, 2014.

[5] C. Shen and H. Li, “Boosting through optimization of margin distributions,” *IEEE Transactions on Neural Networks*, vol. 21, no. 4, pp. 659–666, 2010.

[6] G. Rätsch, S. Mika, B. Schölkopf, and K.-R. Müller, “Constructing boosting algorithms from SVMs: An application to one-class classification,” *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 24, no. 9, 2002.

[7] A. Demiriz, K. P. Bennett, and J. Shawe-Taylor, “Linear programming boosting via column generation,” *Mach. Learn.*, vol. 46, no. 1-3, pp. 225–254, 2002.

[8] C. Shen and Z. Hao, “A direct formulation for totally-corrective multi-class boosting,” in *Proc. IEEE Conf. Comp. Vis. Patt. Recogn.*, 2011, pp. 2585–2592.

[9] K. Crammer and Y. Singer, “On the algorithmic implementation of multiclass kernel-based vector machines,” *J. Mach. Learn. Res.*, vol. 2, pp. 265–292, 2001.

[10] J. Zhu, S. Rosset, T. Hastie, and R. Tibshirani, “l1-norm support vector machines,” in *Proc. Adv. Neural Inf. Process. Syst.*, 2003.

[11] I. Steinwart, “Sparseness of support vector machines,” *J. Mach. Learn. Res.*, vol. 4, pp. 1071–1105, 2003.

[12] Y. Mroueh, T. Poggio, L. Rosasco, and J. J. Slotine, “Multi-class learning with simplex coding,” in *Proc. Adv. Neural Inf. Process. Syst.*, 2012, [http://arxiv.org/abs/1209.1360](http://arxiv.org/abs/1209.1360).

[13] H.-T. Lin and L. Li, “Support vector machinery for infinite ensemble learning,” *J. Mach. Learn. Res.*, vol. 9, pp. 285–312, 2008.

[14] N. Le Roux and Y. Bengio, “Continuous neural networks,” in *Int. Conf. Artificial Intell. & Stat.*, 2007.

[15] A. Rahimi and B. Recht, “Random features for large-scale kernel machines,” in *Proc. Adv. Neural Inf. Process. Syst.*, 2007.

[16] T. Yang, Y.-F. Li, M. Mahdavi, R. Jin, and Z.-H. Zhou, “Nyström method vs. random Fourier features: A theoretical and empirical comparison,” in *Proc. Adv. Neural Inf. Process. Syst.*, 2012.

[17] S. Maji, A. C. Berg, and J. Malik, “Efficient classification for additive kernel SVMs,” *IEEE Trans. Pattern Anal. Mach. Intell.*, 2012.

[18] A. Vedaldi and A. Zisserman, “Efficient additive kernels via explicit feature maps,” *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 34, no. 3, 2012.

[19] R.-E. Fan, K.-W. Chang, C.-J. Hsieh, X.-R. Wang, and C.-J. Lin, “LIBLINEAR: A library for large linear classification,” *J. Mach. Learn. Res.*, vol. 9, 2008.

[20] R. E. Schapire and Y. Singer, “Improved boosting algorithms using confidence-rated predictions,” *Mach. Learn.*, vol. 37, no. 3, pp. 297–336, 1999.

[21] V. Guruswami and A. Sahai, “Multiclass learning, boosting, and error-correcting codes,” in *Proc. Ann. Conf. Computat. Learn. Theory*, 1999.

[22] Mosek, “The MOSEK interior point optimizer,” [http://www.mosek.com](http://www.mosek.com).

[23] M. Everingham, L. Van Gool, C. Williams, J. Winn, and A. Zisserman, “The PASCAL visual object classes challenge 2007,” in *2th PASCAL Challenge Workshop*, 2007.

[24] B. Russell, A. Torralba, K. Murphy, and W. Freeman, “LabelMe: A database and web-based tool for image annotation,” *Int. J. Comp. Vis.*, vol. 77, no. 1, pp. 157–173, 2008.

[25] M. Guillaumin, J. Verbeek, and C. Schmid, “Multimodal semi-supervised learning for image classification,” in *Proc. IEEE Conf. Comp. Vis. Patt. Recogn.*, 2010, pp. 902–909.

[26] A. Oliva and A. Torralba, “Modeling the shape of the scene: A holistic representation of the spatial envelope,” *Int. J. Comp. Vis.*, vol. 42, no. 3, pp. 145–175, 2001.

[27] J. Xiao, J. Hays, K. Ehinger, A. Oliva, and A. Torralba, “Sun database: Large-scale scene recognition from abbey to zoo,” in *Proc. IEEE Conf. Comp. Vis. Patt. Recogn.*, 2010, pp. 3485–3492.