On the periodic Cauchy problem for a coupled system of third-order nonlinear Schrödinger equations

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Abstract
We investigate some well-posedness issues for the initial value problem (IVP) associated to the system

\[ \begin{align*}
2i \partial_t u + q \partial_x^2 u + i \gamma \partial_x^3 u &= F_1(u, w) \\
2i \partial_t w + q \partial_x^2 w + i \gamma \partial_x^3 w &= F_2(u, w),
\end{align*} \]

where \( F_1 \) and \( F_2 \) are polynomials of degree 3 involving \( u, w \) and their derivatives. This system describes the dynamics of two nonlinear short-optical pulses envelopes \( u(x, t) \) and \( w(x, t) \) in fibers (Hasegawa and Kodama in IEEE J Quantum Electron 23(5):510–524, 1987; Porsezian et al. in Phys Rev E 50:1543–1547, 1994). We prove periodic local well-posedness for the IVP with data in Sobolev spaces \( H^s(\mathbb{T}) \times H^s(\mathbb{T}) \), \( s \geq 1/2 \) and global well-posedness result in Sobolev spaces \( H^1(\mathbb{T}) \times H^1(\mathbb{T}) \).

Keywords  Coupled system of third-order nonlinear Schrödinger equations · Periodic Cauchy problem · Local and global well-posedness

Mathematics Subject Classification  35Q35 · 35Q53

1 Introduction

Consider the initial value problem (IVP)
\[
\begin{align*}
2i \partial_t u + q \partial_x^2 u + i \gamma \partial_x^3 u &= F_1(u, w), \quad x \in \mathbb{T}, \ t > 0, \\
2i \partial_t w + q \partial_x^2 w + i \gamma \partial_x^3 w &= F_2(u, w), \\
u(x, 0) = u_0, \ w(x, 0) = w_0,
\end{align*}
\] (1.1)

where \(u, w\) are complex valued functions,

\[
F_1(u, w) = -2i \beta (|u|^2 + \sigma_\beta |w|^2) \partial_x u - 2\alpha u (|u|^2 + \sigma_\alpha |w|^2) - 2i \mu u \partial_x (|u|^2 + \sigma_\mu |w|^2),
\]

\[
F_2(u, w) = F_1(w, u)\]

and \(q, \gamma, \beta, \mu, \alpha, \sigma_\alpha, \sigma_\beta\) and \(\sigma_\mu\) are real parameters.

This system describes the dynamics of two nonlinear short-optical pulses envelopes \(u(x, t)\) and \(w(x, t)\) in fibers. This model is formed by a pair of Schrödinger-Korteweg-de Vries equations coupled through nonlinear terms and it was derived by Porsezian et al. [13]. It generalizes the model (1.3) derived by Hasegawa and Kodama [8].

There is a growing interest in studying the propagation of optical soliton pulses in fiber. This is because of their potential applications in fiber-optic-based communication systems. The idea of soliton based all-optical communication systems with loss compensated by optical amplifications has provided hints of potential advantage for solitons over conventional systems. The major attraction for the soliton communication system arises from the fact that repeater spacing for this kind of system could be much larger than that required by the conventional systems.

The system above has been previously studied ([12,18]) in the particular case \(\sigma_\alpha = \sigma_\beta = \sigma_\mu = 1\) and the system of Hirota and Hirota-Satsuma studied by [1,2] respectively. Radhakrishnan and Lakshmanan [15] used the following transformation of variables in system (1.1)

\[
\begin{align*}
u(x, t) &= u_1 \left( x - \frac{q^2}{6\gamma} t, t \right) \exp i \left( \frac{q}{3\gamma} x - \frac{q^3}{27\gamma^2} t \right), \\
w(x, t) &= w_1 \left( x - \frac{q^2}{6\gamma} t, t \right) \exp i \left( \frac{q}{3\gamma} x - \frac{q^3}{27\gamma^2} t \right),
\end{align*}
\]

under the particular conditions \(\sigma_\alpha = \sigma_\beta\) and \(q \beta = 3\gamma \alpha\), to obtain the following form of coupled envelope equations corresponding to the system (1.1)

\[
\begin{align*}
2 \partial_x u_1 + \gamma \partial_x^3 u_1 + 2\beta (|u_1|^2 + \sigma_\beta |w_1|^2) \partial_x u_1 + 2 \mu u_1 \partial_x (|u_1|^2 + \sigma_\mu |w_1|^2) &= 0, \\
2 \partial_x w_1 + \gamma \partial_x^3 w_1 + 2\beta (|w_1|^2 + \sigma_\beta |u_1|^2) \partial_x w_1 + 2 \mu w_1 \partial_x (|w_1|^2 + \sigma_\mu |u_1|^2) &= 0.
\end{align*}
\] (1.2)

Then, they applied the Hirota bilinear transformation (see [9]) to (1.2) to construct dark and bright soliton solutions to (1.1) assuming further that \(\beta = \mu, q \beta = 3\gamma \alpha, \gamma \neq 0\) and \(\sigma_\alpha = \sigma_\beta = \sigma_\mu = 1\). Recently, Porsezian and Kalithasan [14] discussed the construction of new cnoidal wave solutions and found exact solutions of both bright and dark solitary wave to system (1.1).

As far as we know, the previous works in this subject do not address well-posedness issues for the system (1.1), so our aim is to fill up this gap.
Note that if the pulse $w_1 = 0$, the system (1.2) reduces to the well known modified complex KdV equation. This fact suggests that the results obtained for the periodic modified KdV equation should be the ones we expect for the system (1.1).

When $w = 0$ the system (1.1) reduces to equation

$$i \partial_t u + \frac{q}{2} \partial_x^2 u + \frac{\gamma}{2} \partial_x^3 u + \alpha u |u|^2 + i (\beta + \mu) |u|^2 \partial_x u + i \mu u^2 \partial_x \bar{u} = 0,$$

(1.3)

which describes the dynamics of one single nonlinear pulse in an optic fiber.

The initial value problem associated to Eq. (1.3) was considered by several authors ([5–7,10,16,17]) in $H^s(\mathbb{R})$, where $s \geq 1/4$ is the best result.

In the case of system (1.1), Scialom and Bragança [4] obtained local well-posedness solution in Sobolev spaces $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geq 1/4$, and global well-posedness in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > 3/5$.

In the periodic setting, Takaoka in [17], considered the IVP (1.3) and showed local well-posedness in $H^s(\mathbb{T})$, $s \geq 1/2$.

For the IVP associated to system (1.1), we are able to obtain local well-posedness for initial data in $H^s(\mathbb{T}) \times H^s(\mathbb{T})$, $s \geq 1/2$, as in the single equation case. The approach we use is similar to the one given in [17] though the presence of the coupled terms in (1.1) make the estimates more involved.

Worth to mention that our result is sharp because if $w = 0$ we reduce to (1.3) as Takaoka noted on Remark 1.2 in [17].

To describe our local result we need the following definitions.

**Definition 1.1** Let $P = C^\infty(\mathbb{T}) = \{g : \mathbb{R} \to \mathbb{C}; g \in C^\infty \text{periodic with period} 2\pi\}$. $P'$ (dual of $P$) is the collection of all linear functionals from $P$ to $\mathbb{C}$. $P'$ is periodic distributions. If $g \in P'$ denote the value of $g$ in $\varphi$ by $g(\varphi) = \langle g, \varphi \rangle$. Consider the functions $\theta_n(x) = e^{inx}, n \in \mathbb{Z}$ and $x \in \mathbb{R}$. The Fourier transform $g \in P'$ is the function $\hat{g} : \mathbb{Z} \to \mathbb{C}$ defined by $\hat{g}(n) = \langle g, \theta_n \rangle$. If $g$ is periodic of period $2\pi$, for instance, $g \in L^1(\mathbb{T})$ then

$$\hat{g}(n) = \int_{\mathbb{T}} e^{-inx} g(x) dx = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} g(x) dx,$$

where $\mathbb{T} = \mathbb{R}/2\pi \mathbb{Z}$ represents the one dimensional torus.

**Definition 1.2** Let $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{T})$ is the set of all $g \in P'$ such that

$$\|g\|_{H^s(\mathbb{T})} = \left( \frac{2\pi}{\mathbb{Z}} \sum_{n \in \mathbb{Z}} (1 + |n|^2)^s |\hat{g}(n)|^2 \right)^{\frac{1}{2}} < \infty.$$

In this work we assume that $g$ is periodic of period $2\pi$.

**Definition 1.3** We denote by $\tilde{f}$ the Fourier transform of $f$ in relation to space-time variables

$$\tilde{f}(n, \tau) = \int_{\mathbb{R}} \int_{\mathbb{T}} e^{-i(xn + t\tau)} f(x, t) dx dt.$$
The inverse Fourier transform is given by

$$f(x, t) = \sum_{n \in \mathbb{Z}} e^{inx} \int_{\mathbb{R}} e^{it \tau} \tilde{f}(n, \tau) d\tau.$$  

The Fourier transform of \(fg h\), where \(f = f(x, t), g = g(x, t)\) and \(h = h(x, t)\) are periodic functions with respect to the \(x\) variable obtained as

$$\tilde{fg h}(n, \tau) = (\tilde{f} \ast \tilde{g} \ast \tilde{h})(n, \tau) = \sum_{n_1 \in \mathbb{Z}} \sum_{n_2 \in \mathbb{Z}} \int_{\mathbb{R}^2} \tilde{f}(n_1, \tau_1) \tilde{g}(n_2, \tau_2) \tilde{h}(n - n_1 - n_2, \tau - \tau_1 - \tau_2) d\tau_1 d\tau_2.$$

**Definition 1.4** Let \(V\) be the space of functions \(f\) such that

(i) \(f : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{C}\),

(ii) \(f(x, \cdot) \in S(\mathbb{R})\) for each \(x \in \mathbb{T}\),

(iii) \(f(\cdot, t) \in C^\infty(\mathbb{T})\) for each \(t \in \mathbb{R}\).

We define the space \(X_{s,b}\) associated to operator \(\partial_t - i \frac{q}{2} \partial_x^2 + \partial_x^3 + c_0 \partial_x\) as the completion of \(V\) with respect to the following norm

$$\|f\|_{X_{s,b}} := \|f\|_{(1,s,b)} = \left\langle n \right\rangle^s \left\langle \tau - n^3 + \frac{q}{2} n^2 + c_0 n \right\rangle^b \|\tilde{f}(n, \tau)\|_{L_t^2},$$

where \(\left\langle n \right\rangle = (1 + |n|^2)^{\frac{1}{2}}\) and \(s, b, c_0 \in \mathbb{R}\).

The space \(Z_{s,b}\) is the completion of \(V\) with respect to the norm

$$\|f\|_{Z_{s,b}} := \|f\|_{(2,s,b)} = \left\langle n \right\rangle^s \left\langle \tau - n^3 + \frac{q}{2} n^2 + c_0 n \right\rangle^b \|\tilde{f}(n, \tau)\|_{L_t^1},$$

and we consider the space \(Y_s = X_{s,\frac{1}{2}} \cap Z_{s,0}\) with the norm

$$\|f\|_{Y_s} = \|f\|_{(1,s,\frac{1}{2})} + \|f\|_{(2,s,0)}.$$

**Remark 1.5** Note that for \(b > \frac{1}{2}\) we have that \(X_{s,b} \subset C(\mathbb{R}_t; H^s(\mathbb{T}))\) and for \(b = 0\), we have \(Z_{s,0} \subset C(\mathbb{R}_t; H^s(\mathbb{T}))\) and \(Y_s \subset C(\mathbb{R}_t; H^s(\mathbb{T}))\).

The space \(Y_s^T = \{ f \mid_{[-T, T]} : f \in Y_s \}\) with the norm

$$\|f\|_{Y_s^T} = \inf \left\{ \|g\|_{Y_s} : g \mid_{[-T, T]} = f \text{ and } g \in Y_s \right\},$$

satisfies \(Y_s^T \subset C([-T, T]; H^s(\mathbb{T}))\).

Now, we are in position to state the local result.
Theorem 1.6 Suppose that \( q \notin \mathbb{Z} \). If \( s \geq 1/2 \) and \( \vec{u}_0 = (u_0, w_0) \in H^s(\mathbb{T}) \times H^s(\mathbb{T}) \), then there exist \( T(\|\vec{u}_0\|_{H^s}) > 0 \) and a unique solution \( \vec{u} = (u, w) \) to the IVP (1.1) in the case \( (\beta + \mu) = \beta\sigma_\beta \) satisfying

\[
\vec{u} \in C([-T, T] : H^s(\mathbb{T}) \times H^s(\mathbb{T})), \quad \vec{u} \in Y_s \times Y_s,
\]

where \( c_0 = (\beta + \mu)\|\vec{u}_0\|_{L^2_x}^2 \) in the definition of \( Y_s \).

For each \( T' \in (0, T) \), there exists \( \epsilon > 0 \) such that the map \( \vec{v}_0 \mapsto \vec{v} \) is Lipschitz continuous from

\[
\{ \vec{v}_0 \in H^s(\mathbb{T}) \times H^s(\mathbb{T}) : \|\vec{v}_0 - \vec{u}_0\|_{H^s} < \epsilon \}
\]

to

\[
\{ \vec{v} : \|\vec{v} - \vec{u}\|_{L^\infty_{T'}H^s} + \|\Psi_{T'}(\vec{v} - \vec{u})\|_{Y_s} < \infty \}.
\]

We notice that the solution flow of (1.1) is invariant by the following quantities in the case \( \sigma_\alpha = \sigma_\beta = \sigma_\mu = 1 \).

\[
I_1(u, w) = \|u\|_{L^2(\Omega)}^2 + \|w\|_{L^2(\Omega)}^2 = I_1(u_0, w_0) \quad (1.4)
\]

and

\[
I_2(u, w) = i(-3\gamma\alpha + \beta q + 2\mu q) \int_\Omega (u\overline{\partial}_xw + w\overline{\partial}_xu) dx + \frac{3}{2}\gamma \int_\Omega (|u_x|^2 + |w_x|^2) dx
\]
\[
+ \frac{1}{2}(\beta + 2\mu) \int_\Omega (|u|^4 + |w|^4) dx + (\beta + 2\mu) \int_\Omega |u|^2|w|^2 dx
\]
\[
= I_2(u_0, w_0) \quad (1.5)
\]

for either \( \Omega = \mathbb{R} \) or \( \mathbb{T} \), see [4].

Combining the local well-posedness result with the conservation laws the following global result follows.

Theorem 1.7 Let \( \vec{u}_0 = (u_0, w_0) \in H^1(\mathbb{T}) \times H^1(\mathbb{T}) \). Then there exists a unique solution \( \vec{u} = (u, w) \) to the problem (1.1) with \( \sigma_\alpha = \sigma_\beta = \sigma_\mu = 1 \) and \( \mu = 0 \) satisfying

\[
(u, w) \in C(\mathbb{R} ; H^1(\mathbb{T}) \times H^1(\mathbb{T})).
\]

This work is organized as follows. In Sect. 2 we will list a series of estimates in the spaces defined on Definition 1.4 that will be needed in the proof of Theorem 1.6. In Sect. 3 we establish local well-posedness for the periodic initial value problem associated to (1.1) for data in \( H^s(\mathbb{T}) \times H^s(\mathbb{T}), s \geq 1/2 \) and the Sect. 4 is dedicated to global result. We finish the paper with some comments about future work.
2 Preliminary estimates

To prove our periodic results we use the spaces introduced by Bourgain [3], the contraction principle and also the properties of the solutions to the linear problem

\[
\begin{cases}
\partial_t u - i \frac{q}{2} \partial_x^2 u + \partial_x^3 u + c_0 \partial_x u = 0, & x \in \mathbb{T}, \ t > 0, \\
\partial_t w - i \frac{q}{2} \partial_x^2 w + \partial_x^3 w + c_0 \partial_x w = 0, \\
u(x, 0) = u_0(x) \quad \text{and} \quad w(x, 0) = w_0(x).
\end{cases}
\]  

(2.1)

This linear system differs from the one used in [4] because of the terms containing \(c_0 \partial_x u\) and \(c_0 \partial_x w\), where the constant \(c_0 = (\beta + \mu) \|u\|_{L^2_x}^2 + \|w\|_{L^2_x}^2\).

Note that \(\|u(t)\|_{L^2_x}^2 + \|w(t)\|_{L^2_x}^2\) is a conserved quantity, see (1.4). Therefore the constant \(c_0\) is independent of \(t\). In what follows this is the constant that appeared in the Definition (1.4). It plays important role to get the bounds we need.

Remark 2.1 The Problem (2.1) is almost the linear part of (1.1) with \(\gamma = 2\). So, as \(c_0\) is constant we assume \(\gamma = 2\) without loss of generality. It is easy to see that if \(\gamma \neq 2\) a change of variables recovers (2.1).

The solution of (2.1) is given by the unitary group \(\{W_p(t)\}_{t \in \mathbb{R}}\) in \(H^s(\mathbb{T}) \times H^s(\mathbb{T})\), defined as

\[
\overrightarrow{u}(x, t) = W_p(t) \overrightarrow{u_0} = (S_p(t) u_0, S_p(t) w_0),
\]  

(2.2)

where, the subscript \(p\) only means “periodic”, and

\[
S_p(t) u_0 = \sum_{n \in \mathbb{Z}} e^{inx} e^{it\phi(n)} \hat{u}_0(n),
\]

and \(\phi(n) = n^3 - \frac{q}{2} n^2 - c_0 n\).

Let \(q_\pm(n, \tau) = \tau - n^3 \pm \frac{q}{2} n^2 + c_0 n\), then we obtain the following equalities,

\[
\langle q_-(n, \tau) \rangle^b |\hat{u}(n, \tau)| = \langle q_-(n, -\tau) \rangle^b |\hat{u}(-n, -\tau)|,
\]

\[
\| f \|_{(1, s, b)} = \| \langle n \rangle^s \langle q_+(n, \tau) \rangle^b \hat{f}(n, \tau) \|_{L^2_x}.
\]  

(2.3)

(2.4)

The main linear estimates are the following.

Lemma 2.1 For \(s \in \mathbb{R}\) we have

\[
\| \Psi(t) W_p(t) \overrightarrow{u_0} \|_{(1, s, \frac{1}{2})} \leq c \| \overrightarrow{u_0} \|_{H^s \times H^s},
\]

\[
\| \Psi(t) W_p(t) \overrightarrow{u_0} \|_{(2, s)} \leq c \| \overrightarrow{u_0} \|_{H^s \times H^s}.
\]

Therefore

\[
\| \Psi(t) W_p(t) \overrightarrow{u_0} \|_{Y_s \times Y_s} \leq c \| \overrightarrow{u_0} \|_{H^s \times H^s}.
\]  

(2.5)
where $\Psi(t)W_p(t)\overrightarrow{u}_0$ is given by

$$\Psi(t)W_p(t)\overrightarrow{u}_0 = (\psi(t)S_p(t)u_0, \psi(t)S_p(t)w_0)$$

and $\psi$ connotes a cut-off function satisfying $\psi = 1$ in $[-1, 1]$, $\psi \in C_0^{\infty}$ and supp $\psi \subseteq (-2, 2)$.

**Proof** Taking in account the Definition 1.4 to obtain (2.5) it is enough to estimate

$$\|\psi(t)S_p(t)u_0\|_{L^2_tL^2_x}^2 = \left\langle n \right\rangle^s q_+(n, \tau) |\hat{\psi}(q_+(n, \tau))\hat{u}_0(n)|^2 d\tau$$

and

$$\|\psi(t)S_p(t)u_0\|_{L^2_tL^2_x}^2 = \sum_{n \in \mathbb{Z}} \left| \hat{u}_0(n) \right|^2 \left( \int_{\mathbb{R}} |\hat{\psi}(q_+(n, \tau))|d\tau \right)^2 \leq c\|u_0\|^2_{H^s} \|\psi\|^2_{H^{1/2}} \leq c\|u_0\|^2_{H^s}.$$ (2.6)

and

$$\|\psi(t)S_p(t)u_0\|_{L^2_tL^2_x}^2 = \sum_{n \in \mathbb{Z}} \left( \int_{\mathbb{R}} |\hat{\psi}(q_+(n, \tau))|d\tau \right)^2 \leq c\|u_0\|^2_{H^s}. \quad (2.7)$$

Details of the computations are found in [11].

\[\square\]

### 3 Proof of Theorem 1.6

The result in this section requires a new set of computations, so we will present it in more detailed setting. To simplify the notation we write (1.1) as

$$\left\{ \begin{array}{l}
\partial_t \overrightarrow{u} - \frac{q}{2} i \partial_x^2 \overrightarrow{u} + \partial_x^3 \overrightarrow{u} + c_0 \partial_x \overrightarrow{u} = G(\overrightarrow{u}), \\
\overrightarrow{u}(x, 0) = \overrightarrow{u}_0 \in H^s(\mathbb{T}) \times H^s(\mathbb{T}),
\end{array} \right. \quad (3.1)$$

where

$$G(\overrightarrow{u}) = (G_1(u, w), G_1(w, u)),$$

with

$$G_1(u, w) = (\beta + \mu) \left[ |u|^2 - \|u\|^2_{L^2_x} - \|w\|^2_{L^2_x} \right] \partial_x u + \beta \sigma_\beta |w|^2 \partial_x u$$

$$\mu|u|^2 \partial_x u + \mu \sigma_\mu u \partial_x (|w|^2) - i\alpha u (|u|^2 + \sigma_\alpha |w|^2). \quad (3.2)$$

The integral equation associated to (3.1) is

$$\Phi_{\overrightarrow{u}_0}(\overrightarrow{u}) = W_p(t)\overrightarrow{u}_0 - \int_0^t W_p(t-t') G(\overrightarrow{u})(t')dt', \quad (3.3)$$

where $W_p(t)$ is defined in (2.2).
Considering the cut-off function $\psi \in C_0^\infty$ defined on Lemma 2.1 we have the following estimate.

**Lemma 3.1** For $s \in \mathbb{R}$ we have

$$
\|\Psi(t)\int_0^t W_p(t-t')G(\overrightarrow{u})(t')dt'\|_{Y_x \times Y_x} \leq c\|G_1(u, w)\|_{1,s,-\frac{1}{2}}
$$

$$
+ c\|G_1(w, u)\|_{1,s,-\frac{1}{2}} + c \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \left(\int_{-\infty}^{+\infty} \frac{G_1(u, w)(n, \tau)}{\langle q_+(n, \tau) \rangle} d\tau\right)^2\right)^{\frac{1}{2}}
$$

$$
+ c \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \left(\int_{-\infty}^{+\infty} \frac{G_1(w, u)(n, \tau)}{\langle q_+(n, \tau) \rangle} d\tau\right)^2\right)^{\frac{1}{2}}.
$$

**Proof** To obtain (3.4) it is sufficient to estimate

$$
\|\Psi(t)\int_0^t S_p(t-t')G_1(u, w)(t')dt'\|_{Y_x} \leq c\|G_1(u, w)\|_{1,s,-\frac{1}{2}}
$$

$$
+ c \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \left(\int_{-\infty}^{+\infty} \frac{G_1(u, w)(n, \tau)}{\langle q_+(n, \tau) \rangle} d\tau\right)^2\right)^{\frac{1}{2}}.
$$

From the relation $\int_0^t h(t') dt' = \int_{-\infty}^{+\infty} \frac{e^{it\lambda} - 1}{i\lambda} \hat{h}(\lambda) d\lambda$, notation $G_1(x, t) = G_1(u, w)(x, t)$ and the definition $S_p(t-t')G_1(x, t') = \sum_{n \in \mathbb{Z}} e^{inx} e^{i(t-t')\phi(n)} \hat{G}_1(n, t')$, we obtain

$$
\psi(t)\int_0^t S_p(t-t')G_1(x, t')dt'
$$

$$
= \psi(t) \sum_{n \in \mathbb{Z}} e^{inx} \int_{-\infty}^{+\infty} \left(\frac{e^{it(q_+(n, \lambda))} - 1}{i(q_+(n, \lambda))}\right) e^{it(n^2 - \frac{q}{2}n^2 - c_0n)} \hat{G}_1(n, \lambda) d\lambda.
$$

Let $\varphi \in C_0^\infty$ be another function such that $\varphi \equiv 1$ in $[-B, B]$, $\text{supp} \varphi \subseteq (-2B, 2B)$, where $B < \frac{1}{100}$, say. Using $\varphi$, we can write (3.6) as

$$
\psi(t) \sum_{n \in \mathbb{Z}} e^{inx} \int_{-\infty}^{+\infty} \left(\frac{e^{it(q_+(n, \lambda))} - 1}{i(q_+(n, \lambda))}\right) e^{it\phi(n)} \varphi(q_+(n, \lambda)) \hat{G}_1(n, \lambda) d\lambda
$$

$$
+ \psi(t) \sum_{n \in \mathbb{Z}} e^{inx} \int_{-\infty}^{+\infty} \left(\frac{e^{it(q_+(n, \lambda))} - 1}{i(q_+(n, \lambda))}\right) e^{it\phi(n)} (1 - \varphi(q_+(n, \lambda))) \hat{G}_1(n, \lambda) d\lambda
$$

$$
= J_1(x, t) + J_2(x, t).
$$
That is,
\[
J_1(x, t) = \psi(t) \sum_{n \in \mathbb{Z}} e^{inx} e^{it\phi(n)} \times \int_{-\infty}^{+\infty} \sum_{k \geq 1} \left( \frac{(q_+ (n, \lambda))^{k-1}}{k!} i^{k-1} t^k \right) \varphi(q_+ (n, \lambda)) \tilde{G}_1(n, \lambda) d\lambda.
\]
\[
J_2(x, t) = \psi(t) \sum_{n \in \mathbb{Z}} e^{inx} e^{it\phi(n)} \int_{-\infty}^{+\infty} \left( \frac{e^{it(q_+ (n, \lambda))}}{i(q_+ (n, \lambda))} (1 - \varphi(q_+ (n, \lambda))) \tilde{G}_1(n, \lambda) d\lambda
\]
\[
- \psi(t) \sum_{n \in \mathbb{Z}} e^{inx} e^{it\phi(n)} \int_{-\infty}^{+\infty} \left( 1 - \varphi(q_+ (n, \lambda)) \right) \tilde{G}_1(n, \lambda) d\lambda
\]
\[
= J_1^1(x, t) + J_2^2(x, t).
\]
(3.7)

Therefore, \( \| \psi(t) \int_{0}^{t} S_p (t - t') G_1(u, w)(t') dt' \|_y \leq \| J_1 \|_y + \| J_2^1 \|_y + \| J_2^2 \|_y. \)

Now, we compute the norms of \( J_1, J_2^1 \) and \( J_2^2 \)

\[
\| J_1 \|_{(1, s, \frac{1}{2})} = \left\| \langle n \rangle^s \langle q_+ (n, \tau) \rangle^{\frac{1}{2}} \tilde{f}_1(n, \tau) \right\|_{L^2_t L^2_y}
\]
\[
= \left\| \langle n \rangle^s \langle q_+ (n, \tau) \rangle^{\frac{1}{2}} \sum_{k \geq 1} \frac{h_k(n)}{k!} \tilde{\psi}_k(q_+ (n, \tau)) \right\|_{L^2_t L^2_y},
\]
(3.8)

where
\[
h_k(n) = \int_{-\infty}^{+\infty} i^{k-1} (q_+ (n, \lambda))^{k-1} \varphi(q_+ (n, \lambda)) \tilde{G}_1(n, \lambda) d\lambda.
\]

and
\[
\tilde{\psi}_k(q_+ (n, \tau)) = \int_{-\infty}^{+\infty} e^{-it(q_+ (n, \tau))} \psi(t) t^k dt.
\]

Using the properties of \( \varphi \) we estimate
\[
| h_k(n) | \leq c \| \langle q_+ (n, \lambda) \rangle^{-\frac{1}{2}} \tilde{G}_1(n, \lambda) \|_{L^2_y} = L(n).
\]
(3.9)

After integration by parts, we have
\[
| \tilde{\psi}_k(q_+ (n, \tau)) | \leq \frac{1}{\| q_+ (n, \tau) \|^2} \int_{-\infty}^{+\infty} e^{-it(q_+ (n, \tau))} \left( k(k - 1) t^{k-2} \psi + 2kt^{k-1} \psi' + t^k \psi'' \right) dt
\]
\[
\leq \frac{1}{\| q_+ (n, \tau) \|^2} \left( k(k - 1) \| t^{k-2} \psi \|_{L^1} + 2k \| t^{k-1} \psi' \|_{L^1} + \| t^k \psi'' \|_{L^1} \right)
\]
\[
\leq c \frac{k^2 + k + 1}{\| q_+ (n, \tau) \|^2}.
\]
(3.10)
On the other hand, we also have

\[
|\hat{\psi}_k(q_+(n, \tau))| \leq \|t^k \psi\|_{L^1} \leq c. \tag{3.11}
\]

It follows from (3.9) to (3.11) that

\[
|\hat{\psi}_k(q_+(n, \tau))| \leq c \frac{k^2 + k + 1}{(1 + q_+(n, \tau)^2)^2}. \tag{3.12}
\]

From (3.8) to (3.12) we obtain

\[
\|J_1\|_{(1,s,\frac{1}{2})} \leq \left( \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \int_{-\infty}^{+\infty} |q_+(n, \tau)|^2 \left( \sum_{k \geq 1} \frac{|h_k(n)|}{k!} |\hat{\psi}_k(q_+(n, \tau))| \right)^2 \, d\tau \right)^{\frac{1}{2}} 
\leq c \left( \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} |L(n)|^2 \int_{-\infty}^{+\infty} \frac{1}{|q_+(n, \tau)|^3} \left( \sum_{k \geq 1} \frac{k^2 + k + 1}{k!} \right)^2 \, d\tau \right)^{\frac{1}{2}} 
\leq c \left( \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} |L(n)|^2 \right)^{\frac{1}{2}} = c \|G_1(u, w)\|_{(1,s,-\frac{1}{2})}. \tag{3.13}
\]

The following estimate follows from Young’s inequality and \(|\frac{1-\varphi(x)}{x}| \leq c_1(B)(x)^{-\frac{1}{2}}\).

\[
\|J_2^1\|_{(1,s,\frac{1}{2})} = \left\| \langle n \rangle^s \langle q_+(n, \tau) \rangle^{\frac{1}{2}} \left( \hat{\psi}(\cdot) \ast \frac{(1 - \varphi(q_+(n, \cdot)))(\tilde{G}_1(n, \cdot))}{q_+(n, \cdot)} \right)(\tau) \right\|_{L^2_t}^{L^2_t} 
\leq \left( \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \|q_+(n, \tau)\|_{L^1_t} \left\| \frac{(1 - \varphi(q_+(n, \tau)))}{q_+(n, \tau)} \tilde{G}_1(n, \tau) \right\|_{L^2_t}^2 \right)^{\frac{1}{2}} 
\leq c \left( \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \left\| \frac{(1 - \varphi(q_+(n, \tau)))}{q_+(n, \tau)} \tilde{G}_1(n, \tau) \right\|_{L^2_t}^2 \right)^{\frac{1}{2}} 
\leq c \left( \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \left\| q_+(n, \lambda) \right\|_{L^2_t}^{\frac{1}{2}} \right)^{\frac{1}{2}}. \tag{3.14}
\]

To estimate \(\|J_2^2\|_{(1,s,\frac{1}{2})}\) note that

\[
J_2^2 = -\psi(t) \sum_{n \in \mathbb{Z}} e^{inx} e^{i(n^3 - \frac{q}{2} n^2 - \text{con})} \hat{r}(n) = -\psi(t) S(t) r(x),
\]

where

\[
\hat{r}(n) = \int_{-\infty}^{+\infty} \left( 1 - \frac{\varphi(q_+(n, \lambda))}{i(q_+(n, \lambda))} \right) \tilde{G}_1.
\]
Then (2.6) and the inequality \( \left| \frac{1 - \varphi(x)}{x} \right| \leq \frac{c(B)}{(x)^{\frac{1}{2}}} \) imply

\[
\| J_{2} \|_{(1,s,\frac{1}{2})} = \| \psi(t)S_{p}(t)r \|_{(1,s,\frac{1}{2})} \leq \| r \|_{H}.
\]

Proceeding analogously to (3.14) and (3.15), we can show that

\[
\frac{\| h_{k}(n) \|}{(q_{+}(n, \lambda))} \leq \frac{\| G_{1}(n, \lambda) \|}{(q_{+}(n, \lambda)))} \leq \frac{1}{c(\lambda)}.
\]

Using the definition of \( h_{k}(n) \) it is straightforward to obtain

\[
\| h_{k}(n) \| \leq \| \chi_{[-1,1]}(q_{+}(n, \lambda)) \|_{L_{1}^{1}} \leq c \frac{\| G_{1}(n, \lambda) \|}{(q_{+}(n, \lambda))} \leq \frac{1}{c(\lambda)}.
\]

Then,

\[
\| J_{1} \|_{(2,s,0)} = \| \langle n \rangle^{s} \int_{-\infty}^{+\infty} e^{-i\tau(q_{+}(n, \tau))} \psi(t) \sum_{k \geq 1} \frac{t^{k}}{k^{2}} h_{k}(n) dt \|_{L_{1}^{1}}^{r_{L}}.
\]

Proceeding analogously to (3.14) and (3.15), we can show that

\[
\| J_{1} \|_{(2,s,0)} + \| J_{2} \|_{(2,s,0)} \leq \left( \sum_{n \in \mathbb{Z}} \| \frac{\| G_{1}(n, \lambda) \|}{(q_{+}(n, \lambda))} \|_{L_{1}^{1}}^{2} \right)^{\frac{1}{2}},
\]

which finishes the proof of the lemma. \( \square \)

Let \( n, n_{1}, n_{2} \in \mathbb{Z}, \tau, \tau_{1}, \tau_{2} \in \mathbb{R} \) and set \( n_{3} = n - n_{1} - n_{2} \) and \( \tau_{3} = \tau - \tau_{1} - \tau_{2} \), then

\[
q_{+}(n, \tau) - q_{+}(n_{1}, \tau_{1}) - q_{-}(n_{2}, \tau_{2}) - q_{+}(n_{3}, \tau_{3})
= -3(n_{1} + n_{2})(n - n_{1}) \left( n - n_{2} - \frac{q}{3} \right).
\]

Therefore,

\[
\max\{ |q_{+}(n, \tau)|, |q_{+}(n_{1}, \tau_{1})|, |q_{-}(n_{2}, \tau_{2})|, |q_{+}(n_{3}, \tau_{3})| \} \geq \frac{3}{4} |n_{1} + n_{2}| |n - n_{1}| \left| n - n_{2} - \frac{q}{3} \right|.
\]
Now, we define \( M_1(n, n_1, n_2) = M_1 \), \( L(n, n_1, n_2) = L \) and \( M_2(n, n_1, n_2) = M_2 \) as
\[
M_1(n, n_1, n_2) = \max\{|n - n_1|, |n_1 + n_2|, |n - n_2|\},
M_2(n, n_1, n_2) = \min\{|n - n_1|, |n_1 + n_2|, |n - n_2|\},
\]
and
\[
L(n, n_1, n_2) = \begin{cases} 
|n - n_1|, & \text{if } (|n - n_1| - |n_1 + n_2|)(|n - n_1| - |n - n_2|) \leq 0, \\
|n_1 + n_2|, & \text{if } (|n_1 + n_2| - |n - n_1|)(|n_1 + n_2| - |n - n_2|) \leq 0, \\
|n - n_2|, & \text{if } (|n - n_2| - |n - n_1|)(|n - n_2| - |n_1 + n_2|) \leq 0.
\end{cases}
\]

Note that \( M_2 \leq L \leq M_1 \). The following inequalities will be useful to prove the next lemma. The nonlinear term \( G_1(u, w) \) defined in (3.2) restricted to \((\beta + \mu) = \beta \sigma \beta\), writes
\[
G_1(u, w) = (\beta + \mu)[|u|^2 - \|u\|^2_{L^2_x} + |w|^2 - \|w\|^2_{L^2_x}]\partial_x u \\
\quad + \left( \mu u^2 \partial_x \tilde{u} + \mu \sigma \mu u \partial_x \tilde{w} \right) + \mu \sigma \mu u \tilde{w} \partial_x w - i \alpha u(|u|^2 + \sigma \alpha |w|^2) \\
= G_{11}(u, w) + G_{12}(u, w) + G_{13}(u, w) + G_{14}(u, w). 
\]

(3.18)

Denote by \( \tilde{G}_{1j} = G_{1j}(u, w) \), for \( j = 1, \ldots, 4 \). Next, we compute \( \tilde{G}_{1j}(n, \tau) \). To do so, note that using Parceval’s identity,
\[
|u|^2 u_x (n, \tau) = \int_{\mathbb{R}} e^{-i \tau \tau} \left( \hat{u} \ast \hat{\tilde{u}} \ast \hat{u_x} \right) d\tau = \int_{\mathbb{R}} e^{-i \tau \tau} \sum_{n_1 \in \mathbb{Z}} i n_1 \hat{\tilde{u}}(n_1) \left( \hat{u} \ast \hat{\tilde{u}} \right) (n - n_1) d\tau \\
= \int_{\mathbb{R}} e^{-i \tau \tau} \sum_{n_1 \neq n_2 \in \mathbb{Z}} i n_1 \hat{\tilde{u}}(n_1) \hat{\tilde{u}}(n_2) \tilde{u}(n_3) d\tau + \int_{\mathbb{R}} e^{-i \tau \tau} i n \hat{\tilde{u}}(n) \|u\|^2_{L^2_x} d\tau \\
= \int_{\mathbb{R}} e^{-i \tau \tau} \sum_{n_1 \neq n_2 \neq -n} i n_1 \hat{\tilde{u}}(n_1) \hat{\tilde{u}}(n_2) \tilde{u}(n_3) d\tau \\
+ \int_{\mathbb{R}} e^{-i \tau \tau} \sum_{n_1 \in \mathbb{Z}} i n_1 \hat{\tilde{u}}(n_1) \hat{\tilde{u}}(-n_1) \tilde{u}(n) d\tau - \int_{\mathbb{R}} e^{-i \tau \tau} i n \hat{\tilde{u}}(n) \hat{\tilde{u}}(-n) \tilde{u}(n) d\tau \\
+ \int_{\mathbb{R}} e^{-i \tau \tau} i n \hat{\tilde{u}}(n) \|u\|^2_{L^2_x} d\tau.
\]

Therefore,
\[
|u|^2 u_x (n, \tau) = \sum_{(n_1, n_2) \in H^4_n} i n_1 \iint_{\mathbb{R}^2} \tilde{u}(n_1, \tau_1) \hat{\tilde{u}}(n_2, \tau_2) \tilde{u}(n_3, \tau_3) d\tau_1 d\tau_2 \\
+ \sum_{n_1 \in \mathbb{Z}} i n_1 \iint_{\mathbb{R}^2} \tilde{u}(n_1, \tau_1) \hat{\tilde{u}}(-n_1, \tau_2) \tilde{u}(n, \tau_3) d\tau_1 d\tau_2
\]
\[- \int \int_{\mathbb{R}^2} in\tilde{u}(n, \tau_1)\tilde{\omega}(-n, \tau_2)\tilde{u}(n, \tau_3)d\tau_1d\tau_2 \]
\[+ \int_{\mathbb{R}} e^{-i\tau} in\tilde{u}(n)\|u\|^2_{L^2_x}dt, \quad (3.19)\]

where \(H^1_n = \{(n_1, n_2) \in \mathbb{Z}^2 : n - n_1 \neq 0, n_1 + n_2 \neq 0\}\). The term \(|\omega|^2u_x(n, \tau)\) is computed analogously.

We write \(|u|^2u_x = A + |u|^2u_{L^2_x}u_x\) and \(|\omega|^2u_x = B + |\omega|^2u_{L^2_x}u_x\). Therefore \(\tilde{G}_{11}(n, \tau)\) will be defined as \(A(n, \tau) + B(n, \tau)\). Explicitly we have that \(\tilde{G}_{11}(n, \tau)\):

\[
\tilde{G}_{11}(n, \tau) = c_1(\beta, \mu) \sum_{(n_1, n_2) \in H^1_n} in_1 \int \int_{\mathbb{R}^2} \tilde{u}(n_1, \tau_1)\tilde{\omega}(n_2, \tau_2)\tilde{u}(n_3, \tau_3)d\tau_1d\tau_2 \\
+ c_1(\beta, \mu) \sum_{(n_1, n_2) \in H^1_n} in_1 \int \int_{\mathbb{R}^2} \tilde{u}(n_1, \tau_1)\tilde{\omega}(n_2, \tau_2)\tilde{u}(n_3, \tau_3)d\tau_1d\tau_2 \\
+ c_1(\beta, \mu) \sum_{n_1 \in \mathbb{Z}} in_1 \int \int_{\mathbb{R}^2} \tilde{u}(n_1, \tau_1)\tilde{\omega}(-n_1, \tau_2)\tilde{u}(n_3, \tau_3)d\tau_1d\tau_2 \\
+ c_1(\beta, \mu) \sum_{n_1 \in \mathbb{Z}} in_1 \int \int_{\mathbb{R}^2} \tilde{u}(n_1, \tau_1)\tilde{\omega}(-n_1, \tau_2)\tilde{u}(n_3, \tau_3)d\tau_1d\tau_2 \\
- c_1(\beta, \mu) \int \int_{\mathbb{R}^2} in\tilde{u}(n, \tau_1)\tilde{\omega}(-n, \tau_2)\tilde{u}(n, \tau_3)d\tau_1d\tau_2 \\
- c_1(\beta, \mu) \int \int_{\mathbb{R}^2} in\tilde{u}(n, \tau_1)\tilde{\omega}(-n, \tau_2)\tilde{u}(n, \tau_3)d\tau_1d\tau_2 \\
= R_1(n, \tau) + R_2(n, \tau) + R_3(n, \tau) + R_4(n, \tau) + R_5(n, \tau) + R_6(n, \tau). \quad (3.20)\]

**Remark 3.1** The computation of \(\tilde{G}_{13}(n, \tau)\) and \(\tilde{G}_{12}(n, \tau)\) are similar to \(\tilde{G}_{11}(n, \tau)\).

We have for \(\tilde{G}_{14}(n, \tau)\):

\[
\tilde{G}_{14}(n, \tau) = c_4(\alpha) \sum_{(n_1, n_2) \in H^1_n} \int \int_{\mathbb{R}^2} \tilde{u}(n_1, \tau_1)\tilde{\omega}(n_2, \tau_2)\tilde{u}(n_3, \tau_3)d\tau_1d\tau_2 \\
+ c_5(\alpha, \sigma_\alpha) \sum_{(n_1, n_2) \in H^1_n} \int \int_{\mathbb{R}^2} \tilde{u}(n_1, \tau_1)\tilde{\omega}(n_2, \tau_2)\tilde{u}(n_3, \tau_3)d\tau_1d\tau_2 \\
+ c_6(\alpha) \int \int_{\mathbb{R}^2} \|\tilde{u}(\tau_1)\|_{L^2_h}^3 \|\tilde{\omega}(\tau_2)\|_{L^2_h}^3 \|\tilde{u}(n, \tau_3)\|_{L^2_h}^3 d\tau_1d\tau_2 \\
+ c_7(\alpha, \sigma_\alpha) \int \int_{\mathbb{R}^2} \|\tilde{\omega}(\tau_1)\|_{L^2_h}^3 \|\tilde{\omega}(\tau_2)\|_{L^2_h}^3 \|\tilde{u}(n, \tau_3)\|_{L^2_h}^3 d\tau_1d\tau_2 \]
Lemma 3.2 Assume \( \frac{q}{3} \) is not an integer and \( 0 < \theta < \frac{1}{12} \). For \( s \geq \frac{1}{2} \) there exists \( c > 0 \) such that

\[
\left( \sum_{n \in \mathbb{Z}} |n|^{2s} \left( \int_{-\infty}^{+\infty} \frac{G_1(u, w)(n, \tau)}{(q_+ (n, \tau))^{2(1-a)}} d\tau \right)^2 \right)^{\frac{1}{2}} \leq cf(u, w),
\]

and

\[
\|G_1(u, w)\|_{(1, s, -\frac{1}{2})} \leq cf(u, w),
\]

where

\[
f(u, w) = \left( \|u\|_{(1, s, \frac{1}{2}-\theta)}^2 + \|w\|_{(1, s, \frac{1}{2}-\theta)}^2 \right) \|u\|_{(1, s, \frac{1}{2})} + \|u\|_{(1, s, \frac{1}{2}-\theta)} \|w\|_{(1, s, \frac{1}{2}-\theta)} \|u\|_{(1, s, \frac{1}{2})} + \left( \|u\|_{(2, \frac{1}{2}, 0)}^2 + \|w\|_{(2, \frac{1}{2}, 0)}^2 \right) \|u\|_{(1, s, 0)} + \|u\|_{(2, \frac{1}{2}, 0)} \|w\|_{(2, \frac{1}{2}, 0)} \|u\|_{(1, s, 0)}.
\]

Proof The parameter \( \frac{q}{3} \) is not an integer because we need that the third factor in the right hand side of (3.17) never vanishes. Then we have \( |n - n_2 - q/3| \sim |n - n_2| \). Note also that \( n - n_1 \sim n - n_1 \) and \( n_1 + n_2 \sim n_1 + n_2 \) for \( n - n_1 \neq 0 \) and \( n_1 + n_2 \neq 0 \), respectively.

From the Cauchy–Schwarz inequality, the left hand side of (3.22) is bounded by

\[
\left( \sum_{n \in \mathbb{Z}} |n|^{2s} \int_{-\infty}^{+\infty} \frac{|G_1(u, w)(n, \tau)|^2}{(q_+ (n, \tau))^{2(1-a)}} d\tau \int_{-\infty}^{+\infty} \frac{d\tau}{(q_+ (n, \tau))^{2a}} \right)^{\frac{1}{2}},
\]

where \( a \) will be determined later. Consider first \( G_1(u, w)(n, \tau) = R_2(n, \tau) \). In this case, (3.25) is bounded by

\[
\left( \sum_n \sum_{(n_1, n_2) \in H_n^1} \left( \int_{\mathbb{R}^3} \frac{|n_1|^2}{(q_+ (n, \tau))^{2(1-a)}} |\tilde{u}(n_1, \tau_1)|^2 \right. \left. \times |\tilde{w}(n_2, \tau_2)|^2 |\tilde{w}(n_3, \tau_3)|^2 d\tau_1 d\tau_2 d\tau \right)^{\frac{1}{2}} I_a \right)
\]

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\[
\sum\left(\sum_{n} \int_{\mathbb{R}^3} \frac{|n|^2}{(n+q(n,\tau))^2(1-a)} \frac{\langle n_1 \rangle h(n_2,\tau_2)p(n_3,\tau_3)}{(n_1)^{2a}(n_2)^{2a}(n_3)^{2a}} \frac{d\tau_1 d\tau_2 d\tau}{(q_+(n_1,\tau))^1-2\theta (q_-(n_2,\tau))^1-2\theta (q_+(n_3,\tau))^1-2\theta} I_a \right)^{1/2}, \tag{3.26}
\]

where

\[I_a = \int_{-\infty}^{+\infty} \frac{d\tau}{(q_+(n,\tau))^{2a}},\]

\[g(n_1,\tau_1) = \langle n_1 \rangle^{2a} (q_+(n_1,\tau_1))^{1-2\theta} |\tilde{u}(n_1,\tau_1)|^2,\]

\[h(n_2,\tau_2) = \langle n_2 \rangle^{2a} (q_-(n_2,\tau_2))^{1-2\theta} |\tilde{w}(n_2,\tau_2)|^2\]

and

\[p(n_3,\tau_3) = \langle n_3 \rangle^{2a} (q_+(n_3,\tau_3))^{1-2\theta} |\tilde{w}(n_3,\tau_3)|^2.\]

To estimate (3.26) we divide the region of integration in four parts:

\[A_1 = \{(\tau_1, \tau_2, \tau) : |q_+(n,\tau)| \geq \max \{|q_+(n_1,\tau_1)|, |q_-(n_2,\tau_2)|, |q_+(n_3,\tau_3)|\}\},\]

\[A_2 = \{(\tau_1, \tau_2, \tau) : |q_+(n_1,\tau)| \geq \max \{|q_+(n,\tau)|, |q_-(n_2,\tau_2)|, |q_+(n_3,\tau_3)|\}\},\]

\[A_3 = \{(\tau_1, \tau_2, \tau) : |q_-(n_2,\tau_2)| \geq \max \{|q_+(n_1,\tau_1)|, |q_+(n,\tau)|, |q_+(n_3,\tau_3)|\}\},\]

\[A_4 = \{(\tau_1, \tau_2, \tau) : |q_+(n_3,\tau_3)| \geq \max \{|q_+(n_1,\tau_1)|, |q_-(n_2,\tau_2)|, |q_+(n,\tau)|\}\}.

We also consider the sum in \(n_1\) and \(n_2\) of (3.26) in the following three cases

\[M_1 \geq L \geq \frac{|n|}{5} > M_2, \tag{3.27}\]

\[M_1 \geq \frac{2|n|}{3} \geq \frac{|n|}{5} > L \geq M_2, \tag{3.28}\]

\[M_1 \geq L \geq M_2 \geq \frac{|n|}{5} \tag{3.29}\]

and \(I_a\) is bounded as

\[
\int_{-\infty}^{+\infty} \frac{(n^2 + q_+(n,\tau))^{2a} d\tau}{(n^2 + q_+(n,\tau))^{2a} (q_+(n,\tau))^{2a}} \leq \int_{-\infty}^{+\infty} \frac{c(q_+(n,\tau))^{2a d\tau}}{(n^2 + q_+(n,\tau))^{2a} (q_+(n,\tau))^{2a}} 
\leq c \int_{-\infty}^{+\infty} \frac{d\tau}{(n^2 + q_+(n,\tau))^{2a}} 
\leq \frac{c}{(n)^{4a-2}}, \text{ for } a > \frac{1}{2}, \tag{3.30}
\]
The first inequality in (3.30) follows from \(|q_+(n, \tau)| \geq cn^2(M_2) \geq cn^2\) and the last inequality is a consequence of

$$
\int_{\mathbb{R}} \frac{d\tau}{(\tau)^a(\tau - \theta)^b} \leq \frac{c}{(\theta)^d}, \quad \text{with,} \quad d = \min\{\alpha, \beta, \alpha + \beta - 1\}. \quad (3.31)
$$

Then, in the case (3.27) and in the region \(A_1\), we bound (3.26) by

$$
\sup_{n, \tau} \left[ \frac{\langle n \rangle^{s-2a+1}}{\langle q_+(n, \tau) \rangle^{(1-a)}} \left( \sum_{(n_1, n_2) \in A_{n, \tau}} \frac{|n_1|^2}{(n_1)^{2s}(n_2)^{2s}(n_3)^{2s}} H_0(n_1, n_2) \right)^{1/2} \right]^2
\times c \|u\|_{1, s, \frac{1}{2} - \theta} \|w\|_{1, s, \frac{1}{2} - \theta}^2,
$$

where

$$
A_{n, \tau} = \left\{ (n_1, n_2) : M_1 \geq \frac{|n|}{5} > M_2, n \neq n_1, n_1 \neq -n_2, \quad |q_+(n, \tau)| \geq |n_1 + n_2| \left| n - n_1 \right| \left| n - n_2 - \frac{q}{3} \right| \right\}, \quad \text{and}
$$

$$
H_0(n_1, n_2) = \int_{\mathbb{R}^2} \frac{d\tau_1 d\tau_2}{\langle q_+(n_1, \tau_1) \rangle^{1-2\theta} \langle q_-(n_2, \tau_2) \rangle^{1-2\theta} \langle q_+(n_3, \tau_3) \rangle^{1-2\theta}}.
$$

Using the identity (3.17) and inequality (3.31), we bounded \(H_0(n_1, n_2)\) by

$$
\int_{\mathbb{R}} \frac{d\tau_1}{\langle q_+(n_1, \tau_1) \rangle^{1-2\theta} \langle q_+(n_1, \tau_1) - [q_+(n, \tau) - 3(n_1 + n_2)(n - n_1)(n - n_2 - \frac{q}{3})] \rangle^{1-4\theta}}
\leq \frac{c}{\langle q_+(n, \tau) - 3(n_1 + n_2)(n - n_1)(n - n_2 - \frac{q}{3}) \rangle^{1-6\theta}}
\leq \frac{c}{\langle q_+(n, \tau) - 3(n_1 + n_2)(n - n_1)(n - n_2 - \frac{q}{3}) \rangle^{1-\varepsilon}}, \quad (3.33)
$$

for \(\theta \in (0, \frac{1}{12})\), and \(\varepsilon \in (6\theta, \frac{1}{7})\).

Then, (3.32) is estimate by

$$
c \sup_{n, \tau} \left[ \frac{\langle n \rangle^{s-2a+1}}{\langle q_+(n, \tau) \rangle^{(1-a)}} \left( \sum_{(n_1, n_2) \in A_{n, \tau}} \frac{|n_1|^2}{(n_1)^{2s}(n_2)^{2s}(n_3)^{2s}} \right)^{1/2} \right]^2
\times \|u\|_{1, s, \frac{1}{2} - \theta} \|w\|_{1, s, \frac{1}{2} - \theta}^2
= c \sup_{n, \tau} (I_1)^{1/2} \|u\|_{1, s, \frac{1}{2} - \theta} \|w\|_{1, s, \frac{1}{2} - \theta}^2 \leq c \|u\|_{1, s, \frac{1}{2}} \|w\|_{1, s, \frac{1}{2} - \theta}^2,
$$

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where
\[
I_1 = \langle n \rangle^{2s-4a+2} \left( \sum_{(n_1, n_2) \in A_{n, \tau}} \frac{|n_1|^2}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s}} \times \frac{1}{\langle q_+(n, \tau) - 3(n_1 + n_2)(n-n_1)(n-n_2) - q_3 \rangle^{1-\varepsilon}} \right).
\]

The proof that \( I_1 \leq c \) can be found in [17], Lemma 4.3. The cases (3.28) and (3.29) are analogous.

Now, we bound (3.25) in the region \( A_2 \) with \( a > \frac{1}{2} \) and \( \tilde{G}(u, w)(n, \tau) = R_2(n, \tau) \),
\[
c \left( \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \int_{\mathbb{R}} \frac{|R_2(n, \tau)|^2}{\langle q_+(n, \tau) \rangle^{2(1-a)}} d\tau \right)^{\frac{1}{2}} = c \sup_{\|h(n, \tau)\|_{L^2_2}} \left| \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{\langle n \rangle^s R_2(n, \tau) \tilde{h}(n, \tau)}{\langle q_+(n, \tau) \rangle^{(1-a)}} d\tau \right| = c \sup_{\|h(n, \tau)\|_{L^2_2}} |A(h)|. \tag{3.36}
\]

Note that
\[
|A(h)| = \left| \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{\langle n \rangle^s R_2(n, \tau) \tilde{h}(n, \tau)}{\langle q_+(n, \tau) \rangle^{(1-a)}} d\tau \right|
\leq c \sum_{n \in \mathbb{Z}} \langle n \rangle^s \left( \int_{\mathbb{R}} \sum_{(n_1, n_2) \in H_n} \frac{|n_1||\tilde{h}(n, \tau)|}{\langle q_+(n, \tau) \rangle^{(1-a)}} \int_{\mathbb{R}^2} \frac{g(n_1, \tau_1) p(n_2, \tau_2)}{\langle n_1 \rangle^s \langle n_2 \rangle^s \langle n_3 \rangle^s} d\tau_1 d\tau_2 d\tau \right)
\times \frac{r(n_3, \tau_3) d\tau_1 d\tau_2 d\tau}{\langle q_+(n_1, \tau_1) \rangle^{\frac{1}{2}} \langle q_-(n_2, \tau_2) \rangle^{\frac{1}{2}} \langle q_+(n_3, \tau_3) \rangle^{\frac{1}{2}}}
\leq c \sup_{n_1, \tau_1} \left( \sum_{n \in \mathbb{Z}} \langle n \rangle^s \left( \int_{\mathbb{R}} \sum_{n_2} \frac{|n_1||\tilde{h}(n_1, \tau_1)|}{\langle n_1 \rangle^s \langle q_+(n_1, \tau_1) \rangle^{\frac{1}{2}}} \int_{\mathbb{R}^2} \frac{p(n_2, \tau_2)}{\langle n_2 \rangle^s \langle n_3 \rangle^s} d\tau_2 d\tau \right)
\times \frac{d\tau_2 d\tau}{\langle q_+(n, \tau) \rangle^{(1-a)} \langle q_-(n_2, \tau_2) \rangle^{\frac{1}{2}} \langle q_+(n_3, \tau_3) \rangle^{\frac{1}{2}}}
\times \|g\|_{L^2_1}^2 \|r\|_{L^2_1}^2, \tag{3.37}
\]

where
\[
g(n_1, \tau_1) = \langle n_1 \rangle^{2s} \langle q_+(n_1, \tau_1) \rangle^{\frac{1}{2}} |\tilde{n}(n_1, \tau_1)|,
\]
\[
p(n_2, \tau_2) = \langle n_2 \rangle^{s} \langle q_-(n_2, \tau_2) \rangle^{\frac{1}{2}} |\tilde{w}(n_2, \tau_2)|
\]
and
\[
r(n_3, \tau_3) = \langle n_3 \rangle^{s} \langle q_+(n_3, \tau_3) \rangle^{\frac{1}{2}} |\tilde{w}(n_3, \tau_3)|.
\]
Therefore we can bound (3.39) by

\[
c \sup_{n_1, \tau_1} \left( \frac{|n_1|}{\langle n_1 \rangle^s (q_+ (n_1, \tau_1))^{\frac{1}{2}}} \left( \sum_{(n, n_2) \in D_{n_1, \tau_1}} \int_{\mathbb{R}^2} \frac{\langle n \rangle^{2s}}{(n_2)^{2s}} \langle n_3 \rangle^{2s} \right) \times \frac{d \tau_2 d \tau}{(q_+ (n, \tau))^{2(1-a)} (q_- (n_2, \tau_2))^{1-2\theta} (q_+ (n_3, \tau_3))^{1-2\theta}} \right)^{\frac{1}{2}} \times \|g\|_{n_1 L^{2}_{\tau_1}} \|r\|_{n_1 L^{2}_{\tau_1}} \|p\|_{n_2 L^{2}_{\tau_2}} \|\mathcal{T}\|_{n_2 L^{2}_{\tau_2}},
\]

where

\[
D_{n_1, \tau_1} = \left\{ (n, n_2) : n \neq n_1, n \neq -n_2, \quad |q_+ (n_1, \tau_1)| \geq |n_1 + n_2| |n - n_1| |n - n_2 - \frac{q}{3}| \right\}.
\]

It follows from (3.35)–(3.38) that (3.25) is bounded by

\[
c \sup_{n_1, \tau_1} \left( \frac{|n_1|}{\langle n_1 \rangle^s (q_+ (n_1, \tau_1))^{\frac{1}{2}}} \left( \sum_{(n, n_2) \in D_{n_1, \tau_1}} \int_{\mathbb{R}^2} \frac{\langle n \rangle^{2s}}{(n_2)^{2s}} \langle n_3 \rangle^{2s} \right) \times \frac{d \tau_2 d \tau}{(q_+ (n, \tau))^{2(1-a)} (q_- (n_2, \tau_2))^{1-2\theta} (q_+ (n_3, \tau_3))^{1-2\theta}} \right)^{\frac{1}{2}} \times \|u\|_{L^2(1,s, \tau)} \|w\|_{L^2(1,s, \tau)},
\]

where

\[
I_2 = \frac{\|n_1\|^2}{\langle n_1 \rangle^{2s} (q_+ (n_1, \tau_1))^{\frac{1}{2}}} \left( \sum_{(n, n_2) \in D_{n_1, \tau_1}} \int_{\mathbb{R}^2} \frac{\langle n \rangle^{2s}}{(n_2)^{2s}} \langle n_3 \rangle^{2s} \right) \times \frac{d \tau_2 d \tau}{(q_+ (n, \tau))^{2(1-a)} (q_- (n_2, \tau_2))^{1-2\theta} (q_+ (n_3, \tau_3))^{1-2\theta}}.
\]

The proof that \(I_2 \leq c\) is similar to Lemma 4.3 in [17]. The case \(G^1_1 (u, w) (n, \tau) = R_1 (n, \tau)\) is similar to (3.35). In the case \(G^1_1 (u, w) (n, \tau) = R_4 (n, \tau), (3.25)\) is bounded by

\[\square\]
Using Minkowski’s inequality, (3.38) is bounded by

\[
\left( \sum_{n \in \mathbb{Z}} \langle n \rangle^{2x} \int_{\mathbb{R}} \left( \sum_{n_1 \in \mathbb{Z}} \left( \int_{\mathbb{R}^2} |n_1| |\widetilde{u}(n_1, \tau_1)| \left| \widetilde{w}(-n_1, \tau_2) \right| d\tau_1 d\tau_2 \right)^2 \right) d\tau \right)^{1/2} \\
\leq c \left( \sum_{n \in \mathbb{Z}} \langle n \rangle^{2x} \int_{\mathbb{R}} |\widetilde{w}(n, \tau_3)|^2 d\tau \right)^{1/2} \left( \sum_{n_1 \in \mathbb{Z}} \left[ \int_{\mathbb{R}} \langle n_1 \rangle^{1/2} |\widetilde{u}(n_1, \tau_1)| d\tau_1 \right] \right) \\
\times \left[ \int_{\mathbb{R}} \langle n_1 \rangle^{1/2} |\widetilde{w}(-n_1, \tau_2)| d\tau_2 \right] \\
\leq c \|w\|_{(1, s, 0)} \|u\| \left( \frac{1}{2}, \frac{1}{2} \right) \|w\| \left( \frac{1}{2}, \frac{1}{2} \right).
\]

The case $G_1(u, w)(n, \tau) = R_3(n, \tau)$ is similar to $R_4(n, \tau)$, for $w = u$. In the case of $G_1(u, w)(n, \tau) = R_6(n, \tau)$ we bound (3.25) by

\[
c \left( \sum_{n \in \mathbb{Z}} \langle n \rangle^{2x} \int_{\mathbb{R}} \left( \int_{\mathbb{R}^2} |n| |\widetilde{u}(n, \tau_1)| \left| \widetilde{w}(-n, \tau_2) \right| |\widetilde{w}(n, \tau_1 - \tau_2)| d\tau_1 d\tau_2 \right)^2 d\tau \right)^{1/2} \\
\leq c \left( \sum_{n \in \mathbb{Z}} \left[ \int_{\mathbb{R}^2} |n| |\widetilde{u}(n, \tau_1)| \left| \widetilde{w}(-n, \tau_2) \right| |\widetilde{w}(n, \tau_1 - \tau_2)| d\tau_1 d\tau_2 \right]^2 \right)^{1/2} \left( \sum_{n \in \mathbb{Z}} \langle n \rangle^{2x} \int_{\mathbb{R}} |\widetilde{w}(n, \tau_3)|^2 d\tau \right)^{1/2} \\
\leq c \|w\|_{(1, s, 0)} \|u\| \left( \frac{1}{2}, \frac{1}{2} \right) \|w\| \left( \frac{1}{2}, \frac{1}{2} \right).
\]

For the cases $G_1(u, w)(n, \tau) = R_i(n, \tau)$, $i = 7, \ldots, 12$, we follow a similar argument. The proof of (3.23) follows the same lines as (3.22), choosing $a = \frac{1}{2}$ and not considering $I_a$, because
\[ \| G_1(u, w) \|_{1,s,-\frac{1}{2}} = \left( \sum_{n \in \mathbb{Z}} (n)^{2s} \int_{-\infty}^{+\infty} \frac{|G_1(u, w)(n, \tau)|^2}{(q_+(n, \tau))^2} d\tau \right)^{\frac{1}{2}}. \] (3.39)

The following lemma is found in [17], Lemma 4.6.

**Lemma 3.3** For \( s \in \mathbb{R}, 0 < \epsilon < \frac{1}{2}, T \in (0, 1) \) and \( 0 < \theta' < \theta < \frac{1}{2} \), we have the following inequalities

\[
\| \psi_T(t)f \|_{1,s,-\frac{1}{2}} \leq c(\epsilon)T^{-\epsilon} \| f \|_{1,s,-\frac{1}{2}},
\]

\[
\| \psi_T(t)f \|_{1,s,-\frac{1}{2}} \leq cT^{\theta-\theta'} \| f \|_{1,s,-\frac{1}{2}},
\]

\[
\| \psi_T(t)f \|_{2,s,0} \leq c \| f \|_{2,s,0}.
\]

Now we are able to prove the main theorem of this section.

**Proof of Theorem 1.6** Let \( s \geq \frac{1}{2} \). With all the estimates at hand, for \( T \in (0, 1) \), define

\[
Y^a_T = \{ \tilde{v} \in Y_s \times Y_s : \| \tilde{v} \|_{Y_s \times Y_s} \leq a \}.
\]

\[
\Phi(\tilde{u}) = \Psi(t) W_p(t) \overrightarrow{u_0} - \Psi(t) \int_0^t W_p(t-t')G(\Psi_T(t') \overrightarrow{u}(t')) dt' = \left( \begin{array} \psi(t)S_p(t)u_0 - \psi(t) \int_0^t S_p(t-t')G_1(\psi_T u, \psi_T w)(t') dt', \\
\psi(t)S_p(t)w_0 - \psi(t) \int_0^t S_p(t-t')G_1(\psi_T w, \psi_T u)(t') dt' \end{array} \right). \] (3.40)

From Lemma 3.2 and inequality (3.31) we have that

\[
\| \Phi(\tilde{u}) \|_{Y_s \times Y_s} \leq c\| \tilde{u}_0 \|_{H^s \times H^s} + c(\psi_T u, \psi_T w)
\]

\[
\leq c\| \tilde{u}_0 \|_{H^s \times H^s} + c \left( \| \psi_T u \|^2_{1,s,-\frac{1}{2}} + \| \psi_T w \|^2_{1,s,-\frac{1}{2}} \right) \| \psi_T u \|_{1,s,-\frac{1}{2}}
\]

\[
+ c \| \psi_T u \|^2_{1,s,-\frac{1}{2}} \| \psi_T w \|^2_{1,s,-\frac{1}{2}} \| \psi_T u \|^2_{1,s,0}
\]

\[
+ c \| \psi_T u \|^2_{1,s,0} \| \psi_T w \|^2_{1,s,0} \| \psi_T u \|^2_{1,s,0}
\]

\[
\leq c\| \tilde{u}_0 \|_{H^s \times H^s} + cT^{\theta-\theta'+\epsilon} \| \tilde{u} \|^3_{Y_s \times Y_s} \leq c\| \tilde{u}_0 \|_{H^s \times H^s} + cT^{\theta-\theta'+\epsilon} a^3,
\]

for \( \tilde{u} \in Y^a_T \) and \( 0 < \theta' - \epsilon < \theta < \frac{1}{12} \). Analogously, for \( \tilde{u}, \tilde{v} \in Y^a_T \) we have that

\[
\| \Phi(\tilde{u}) - \Phi(\tilde{v}) \|_{Y_s \times Y_s} \leq cT^{\theta-\theta'+\epsilon} a^2 \| \tilde{u} - \tilde{v} \|_{Y_s \times Y_s}. \] (3.41)
We choose \( \frac{a}{2} = c \| \mathbf{u}_0 \|_{H^s} \) and \( T \) small enough, such that \( c T^{a - a^3} \leq \frac{a}{2} \), then \( \| \Phi(\mathbf{u}) \|_{Y_s \times Y_s} \leq a \) and \( \Phi \) is a contraction. Uniqueness and continuous dependence follow in standard way.

\[ \square \]

4 Proof of Theorem 1.7

First we obtain conserved quantities for (1.1) with \( \sigma_\alpha = \sigma_\beta = \sigma_\mu = 1 \). Define

\[
H_0(u, w) = 2 \mu \text{Im} \left( \int_T \left( (u \bar{u}_x)^2 + (w \bar{w}_x)^2 \right) dx \right) + 4 \mu \text{Im} \left( \int_T \left( u \bar{u}_x \cdot w \bar{w}_x \right) dx \right),
\]

\[
H_1(u, w) = (\beta + 2\mu) \int_T \left[ |u_x|^2 \partial_x (|u|^2) + |w_x|^2 \partial_x (|w|^2) \right] dx
\]

\[
+ (\beta + 2\mu) \int_T \left[ |u_x|^2 \partial_x (|w|^2) \right] dx
\]

\[
+ (\beta + 2\mu) \int_T \left[ |w_x|^2 \partial_x (|u|^2) \right] dx + 4\alpha \text{Im} \left( \int_T (u \bar{u}_x \cdot w \bar{w}_x) dx \right)
\]

\[
+ 2\alpha \text{Im} \left( \int_T ((u \bar{u}_x)^2 + (w \bar{w}_x)^2) dx \right).
\]

\[
H_2(u, v) = \left( -\frac{1}{2} \beta + 2\mu \right) \int_T \left[ \partial_x (|u|^2) \cdot |w|^4 + \partial_x (|w|^2) \cdot |u|^4 \right] dx
\]

\[
+ q \text{Im} \int_T ((u \bar{u}_x)^2 + (w \bar{w}_x)^2) dx
\]

\[
+ \frac{3}{2} \gamma \int_T \left[ |u_x|^2 \partial_x (|u|^2) + |w_x|^2 \partial_x (|w|^2) \right] dx,
\]

\[
H_3(u, v) = \left( \frac{1}{2} \beta - 2\mu \right) \int_T \left[ \partial_x (|u|^2) \cdot |w|^4 + \partial_x (|w|^2) \cdot |u|^4 \right] dx
\]

\[
+ 2q \text{Im} \left( \int_T (u \bar{u}_x \cdot w \bar{w}_x) dx \right)
\]

\[
+ \frac{3}{2} \gamma \int_T \left[ |u_x|^2 \partial_x (|w|^2) + |w_x|^2 \partial_x (|u|^2) \right] dx.
\]

**Lemma 4.1** Let \( \mathbf{u}_0 = (u_0, w_0) \in H^s(\mathbb{T}) \times H^s(\mathbb{T}) \) with \( s' \) large enough and \( \mathbf{u} \in C([-T, T]; H^{s'}(\mathbb{T}) \times H^{s'}(\mathbb{T})) \) solutions of (1.1) with \( \sigma_\alpha = \sigma_\beta = \sigma_\mu = 1 \). Then

\[
i \partial_t \left( \int_T \left( u \bar{u}_x + w \bar{w}_x \right) dx \right) = H_0(u, w), \tag{4.1}
\]

\[
\partial_t \left( \| u_x \|_{L^2(\mathbb{T})}^2 + \| w_x \|_{L^2(\mathbb{T})}^2 \right) = -H_1(u, w), \tag{4.2}
\]

\[
\frac{1}{2} \partial_t \left( \| u_x \|_{L^4(\mathbb{T})}^4 + \| w_x \|_{L^4(\mathbb{T})}^4 \right) = -H_2(u, w). \tag{4.3}
\]
\[ \partial_t \int_T |u|^2 |w|^2 \, dx = - H_3(u, w). \] (4.4)

**Proof** Multiply the first equation in (1.1) by \( \overline{u}_x \), second equation by \( \overline{w}_x \), integrate in \( x \) and take the real part to obtain

\[
\text{Re} \left( 2i \int_T u_t \overline{u}_x \, dx \right) = i \int_T (u_t \overline{u}_x - \overline{u}_x u_x) \, dx = i \partial_t \left( \int_T u \overline{u}_x \, dx \right),
\]
\[
\text{Re} \left( 2 \mu \int_T u \partial_x (|u|^2) \overline{u}_x \, dx \right) = - \text{Im} \left( 2 \mu \int_T u \partial_x (|u|^2) \overline{u}_x \, dx \right)
= - 2 \mu \text{Im} \left( \int_T (u \overline{u}_x)^2 \, dx \right),
\]
\[
\text{Re} \left( 2 \mu \int_T u \partial_x (|w|^2) \overline{u}_x \, dx \right) + \text{Re} \left( 2 \mu \int_T w \partial_x (|u|^2) \overline{w}_x \, dx \right)
= - 4 \mu \text{Im} \left( \int_T (u \overline{u}_x w \overline{w}_x) \, dx \right),
\]
\[
\text{Re} \left( 2 \alpha \int_T u (|u|^2 + |w|^2) \overline{u}_x \, dx \right) + \text{Re} \left( 2 \alpha \int_T w (|u|^2 + |w|^2) \overline{w}_x \, dx \right)
= - \alpha \int_T \partial_x (|w|^2) |u|^2 \, dx + \alpha \int_T \partial_x (|w|^2) |u|^2 \, dx = 0.
\]

The equality (4.1) is obtained by adding the previous resulting equations. To obtain (4.2) multiply the first equation in (1.1) by \( \overline{u}_{xx} \), the second equation by \( \overline{w}_{xx} \), integrate in \( x \), take the imaginary part and add the resulting equations. The equality (4.3) is obtained multiplying the first equation in (1.1) by \( |u|^2 \overline{u} \), the second equation by \( |w|^2 \overline{w} \), integrate in \( x \), take the imaginary part and add the resulting equations. Similarly, we obtain (4.4) multiplying the first equation of (1.1) by \( |w|^2 \overline{u} \), the second equation by \( |u|^2 \overline{w} \), integrate in \( x \), take the imaginary part and sum the resulting equations. \( \square \)

The next Lemma provides the conserved quantity we need.

**Lemma 4.2** Let \( \overrightarrow{u}_0 \in H^1(\mathbb{T}) \times H^1(\mathbb{T}) \) and \( \overrightarrow{u} \in C([-T, T]; H^1(\mathbb{T}) \times H^1(\mathbb{T})) \) solution of (1.1). Then,

\[
I_1(u, w) = \|u\|^2_{L^2(\mathbb{T})} + \|w\|^2_{L^2(\mathbb{T})} = I_1(u_0, w_0),
\] (4.5)
\[
I_2(u, v) = i (-3 \gamma \alpha + \beta q + 2 \mu q) \int_T (u \overline{u}_x + w \overline{w}_x) \, dx + \frac{3}{2} \gamma \int_T (|u_x|^2 + |w_x|^2) \, dx
+ \frac{1}{2} (\beta + 2 \mu) \int_T (|u|^4 + |w|^4) \, dx + (\beta + 2 \mu) \int_T |u|^2 |w|^2 \, dx
= I_2(u_0, w_0).
\] (4.6)
Proof The following combination of (4.1)–(4.4) lead to (4.6)
\[
\left(-\frac{3\gamma\alpha + (\beta + 2\mu)q}{2\mu}\right) (4.1) + \frac{3\gamma}{2} (4.2) + (\beta + 2\mu) (4.3) + (\beta + 2\mu) (4.4).
\]

The proof of global solution in Theorem 1.7 follows from Theorem 1.6 and (4.5)–(4.6).

Final remark
An interesting mathematical problem would be to study the local well-posedness on the torus of the related higher-order nonlinear Schrödinger system
\[
\begin{align*}
2i\partial_t u + q_1 \partial_x^2 u + i\gamma_1 \partial_x^3 u &= F_1(u, w) \\
2i\partial_t w + q_2 \partial_x^2 w + i\gamma_2 \partial_x^3 w &= F_2(u, w),
\end{align*}
\]
when \(q_1 \neq q_2\) and \(\gamma_1 \neq \gamma_2\).

Note that in this case the resonant regions would be different from the ones in [3], so the proofs of [3] would need to be carefully modified.

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