THE INDEX FORMULA FOR FAMILIES OF DIRAC TYPE OPERATORS ON PSEUDOMANIFOLDS

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Abstract. We study families of Dirac-type operators, with compatible perturbations, associated to wedge metrics on stratified spaces. We define a closed domain and, under an assumption of invertible boundary families, prove that the operators are self-adjoint and Fredholm with compact resolvents and trace-class heat kernels. We establish a formula for the Chern character of their index.

Introduction

In this article we establish an index formula for families of Dirac-type operators on stratified spaces endowed with metrics of iterated conic-type singularities. In the process we construct the resolvent and heat kernel of these operators by extending the b-calculus of Melrose and the edge calculus of Mazzeo to manifolds with corners and fibered boundaries. We establish refined asymptotic expansions of these operators of relevance to the study of analytic and spectral constructions and their connections to topology. In the present paper we use them to carry out the heat equation proof of the index theorem.

Our index theorem seems to be the first example of an index theorem for singular metrics on stratified spaces of arbitrary depth since the Gauss-Bonnet and signature theorems for piecewise flat admissible metrics established by Cheeger [Che83]. Since this seminal paper there has been much interest and progress in index theory on spaces with isolated conic singularities and, more recently, non-isolated conic singularities which we will review below.

Most singular spaces arising from smooth objects, such as zero sets of polynomials, orbits of group actions, and many moduli spaces, are Thom-Mather stratified spaces. They can be written as the union of smooth manifolds known as strata. One stratum is dense and referred to as the regular part, while the others make up the singular part. Each singular stratum has a tubular neighborhood in the stratified space that fibers over it, the fiber is itself a stratified space and is known as the link of that stratum. The depth of a stratified space is the length of the longest chain of inclusions among the closures of singular strata. (See, e.g., [Alb17, Klo09] and references therein for more on stratified spaces.)

Every Thom-Mather stratified space \( \hat{X} \) can be ‘resolved’ to a manifold with corners \( X \) by a procedure that goes back to Thom [Tho69] and was recently reformulated by Melrose (see [ALMP12, AM11, Alb17]). The boundary hypersurfaces of \( X \) are partitioned into collective (i.e. disjoint unions of) boundary hypersurfaces, \( \mathcal{B}_Y \), one per singular stratum \( Y^\circ \) of \( \hat{X} \), which participate in smooth fiber bundles of manifolds with corners,

\[
Z \rightarrow \mathcal{B}_Y \xrightarrow{\phi_Y} Y,
\]

in which \( Y \) is the resolution of the closure of \( Y^\circ \) in \( \hat{X} \) and \( Z \) is the resolution of the link of \( Y^\circ \) in \( \hat{X} \). These fiber bundles satisfy compatibility conditions at intersections of collective
boundary hypersurfaces, see Definition 1.1. We refer to this structure on $X$ as an iterated fibration structure and emphasize that it is equivalent to the Thom-Mather structure on $\hat{X}$.

There is a map $\beta: X \rightarrow \hat{X}$ relating a stratified space and its resolution that restricts to a diffeomorphism between the interior of $X$ and the regular part of $\hat{X}$, $\hat{X}^{\text{reg}}$. This relation allows us to define the analogues of smooth objects on $\hat{X}$. For example, we can define a smooth function on $\hat{X}$ to be a continuous function on $\hat{X}$ whose restriction to $\hat{X}^{\text{reg}} = X^{\circ}$ extends to a smooth function on $X$. We can study these functions on $X$ without reference to $\hat{X}$.

We define $C^\infty_\Phi(X) = \{ f \in C^\infty(X) : f|_Y \in \phi_Y^* C^\infty(Y) \text{ for all } Y \}$.

The differentials of these functions locally span a vector bundle, known as the wedge cotangent bundle, $^wT^*X \rightarrow X$, described in more detail below.

Following the paradigm of [Me93], we can think of the present paper as carrying out geometric analysis in the ‘wedge category’ where, e.g., the role of the contangent bundle is supplanted by the wedge cotangent bundle. For example, our metrics will be Riemannian metrics on the interior that extend to bundle metrics on the wedge (co)tangent bundle, and our Clifford bundles will have Clifford actions over the interior that extend to actions of the Clifford algebra of the wedge cotangent bundle.

In contrast to the situation in [Me93], the dual bundle $^wTX$, known as the wedge tangent bundle, does not have a natural Lie bracket; this leads us to define ‘wedge differential operators’ in terms of ‘edge differential operators’. A related issue is that wedge differential operators generally have multiple extensions from smooth sections of compact support to closed operators on $L^2$-spaces. Indeed, formally self-adjoint operators will often have closed extensions that are not self-adjoint.

Thus our first task in studying Dirac-type wedge operators is to identify a well-behaved closed extension. We carry this out under the assumption that certain model operators analogous to the ‘boundary families’ in [BC90a, BC90b] are invertible; however as in [MP97a, MP97b] we include appropriate pseudodifferential perturbations and we will show in a forthcoming paper that as in loc. cit. this is a ‘minimal’ assumption in that it corresponds to the vanishing of a topological obstruction (see Remark 4.2).

**Theorem 1.** Let $X$ be a manifold with corners and an iterated fibration structure endowed with a totally geodesic wedge metric, let $(E, g_E, \nabla^E, \mathcal{D})$ be a wedge Clifford module with associated Dirac-type operator $\delta_X$ and let $Q_X$ be a compatible perturbation. If $\delta_{X,Q} = \delta_X + Q_X$ satisfies the Witt assumption (i.e., has invertible boundary families, see Section 2 below, in particular Definition 2.4) then $\delta_{X,Q}$ with its vertical APS domain

$$D_{\text{VAPS}}(\delta_{X,Q}) = \text{graph closure of } \{ u \in H^1_X(X; E) : \delta_{X,Q}u \in L^2(X; E) \},$$

is a closed operator on $L^2(X; E)$ that is self-adjoint and Fredholm with compact resolvent.

The heat kernel of $\delta_{X,Q}^2$ with the induced domain is trace-class and has a short-time asymptotic expansion of the form

$$\text{Tr}(e^{-t\delta_{X,Q}^2}) \sim t^{-(\dim X)/2} \sum_{j=0}^{\infty} \sum_{k=0}^{\text{depth}(X)} a_{j,k} t^{j/2} (\log t)^{k/2}.$$
We refer the reader to the text for precise definitions and statements. For example the trace of the heat kernel has fewer powers of log $t$ for small values of $j$, see Corollary 5.7. The nomenclature ‘Witt assumption’ stems from the case of the signature operator which satisfies this assumption if and only if $X$ is a Witt space in the sense of [Sie83, Che79], see, e.g., [ALMP12]. As explained in a forthcoming companion paper, allowing for a compatible perturbation in our theorems means in particular that they apply to the signature operator on the more general class of ‘Cheeger spaces’ studied in [ALMP13, ALMP17] (originally introduced by Banagl [Ban02] and known as L-spaces, see [ABL+]15). Below we also discuss the ‘geometric Witt assumption’ assumed in most of these references and analyzed in the recent preprint [HLV].

There are immediate consequences to the spectral theory of such $\partial_{X,Q}$, for example:

**Corollary 2.** The spectrum of $(\partial_{X,Q}^\sharp, D_{VAPS})$ is a discrete subset of $\mathbb{R}^+$ that satisfies Weyl’s law

$$\#\{\text{eigenvalues, with multiplicity, less than } \Lambda\} \sim \frac{\text{Vol}(X)}{(4\pi)^{\frac{1}{2}\dim X}} \Lambda^{\frac{1}{2}\dim X}.$$ 

The zeta function of $\partial_{X,Q}^\sharp$, $\zeta(s) = \text{Tr}((\partial_{X,Q}^\sharp|_{\ker(\partial_{X,Q}^\sharp)})^{-2s})$, is holomorphic on $\{s \in \mathbb{C} : \text{Re}(s) > \frac{1}{2}\dim X\}$ and extends to a meromorphic function on $\mathbb{C}$ with poles of order at most $1 + \text{depth}(X)$.

In light of the theorem above, a family of wedge Dirac-type operators $\partial_{M/B}$ on the fibers of a fiber bundle, $X \to M \to B$, of manifolds with corners and iterated fibration structures determines an index in $K^{\dim X}(B)$. Our second main result is a formula for the Chern character of this index.

Recall that Bismut and Cheeger, in their study of the families index theorem for manifolds with boundary [BC90a, BC90b, BC91], established a formula for the Chern character of the index of a family of spin Dirac operators on even-dimensional spaces with isolated conic singularities and invertible boundary families. Namely, if $M \to B$ is the resolution of these spaces to manifolds with boundary,

$$\text{Ch}_{\text{even}}(\text{Ind}(\partial_{M/B})) = \int_{M/B} \hat{A}(M/B) \text{Ch}'(E) - \mathcal{J}(\partial M/B) \text{ in } H^{\text{even}}(B),$$

where $\mathcal{J}$ is a differential form depending globally on the geometry of the boundary fibration $\partial M \to B$. Our theorem involves Bismut-Cheeger $\mathcal{J}$-forms and $\eta$-forms extended to allow for Dirac operators on singular spaces and compatible smoothing perturbations.

**Theorem 3.** Let $M \to B$ be a fiber bundle of manifolds with corners and iterated fibration structures, $E \to M$ a wedge Clifford bundle with associated Dirac-type operator $\partial_{M/B}$ and $Q$ an admissible perturbation. If $\partial_{M/B,Q}$ with its vertical APS domain satisfies the Witt assumption, then

$$\text{Ch}_{\dim(M/B)}(\text{Ind}(\partial_{M/B,Q}), \nabla^{\text{Ind}}) = \int_{M/B} \hat{A}(M/B) \text{Ch}'(E) - \sum_{N \in \mathcal{S}(M)} \int_{N/B} \hat{A}(N/B) \mathcal{J}_{Q}(\mathfrak{M}_{N}/N) + d\eta_{Q}(M/B),$$

where $\mathcal{J}_{Q}$ is a differential form depending on the geometry of the boundary fibration $\partial M \to B$. Our theorem involves Bismut-Cheeger $\mathcal{J}_{Q}$-forms and $\eta$-forms extended to allow for Dirac operators on singular spaces and compatible smoothing perturbations.
where $\text{Ch}_j$ denotes the even or odd Chern character in accordance with the parity of $j$, and $\dim(M/B) = \dim X$. The sum is taken over the set $S_\psi(M)$ of boundary hypersurfaces of $M$ transverse to the map $\psi: M \to B$, i.e. those which also fibre over $B$. (In the even dimensional case, the left hand side is the Chern character obtained after stabilizing the null space of $\partial_{M/B,Q}$ to a bundle over $B$ by compressing the Bismut superconnection; in the odd case, the left hand side is the differential form obtained by suspension and integration.)

Bismut and Cheeger obtained the index formula for a family of spin Dirac operators on manifolds with boundary with invertible boundary families by deforming a neighborhood of the conic singularities to a cylindrical end. In the process the $J$-form was shown to converge to the Bismut-Cheeger $\eta$-form, introduced in [BC89]. Melrose and Piazza [MP97a, MP97b] used the b-calculus to establish the families index theorem for arbitrary families of Dirac-type operators by allowing appropriate pseudodifferential perturbations. The boundary contributions correspondingly depend on the perturbation.

We discuss the definition of the Bismut-Cheeger $\eta$ and $J$ forms for families of compatibly perturbed Dirac-type operators on stratified spaces in §6.2. This extends the definition of $\eta$-invariants for spaces with conic singularities in [Che87, §8] and for spaces with non-isolated conic singularities (i.e., stratified spaces of depth one) in [PV]. Heuristically, given a fiber bundle $\tilde{X} \to \tilde{M}$, a vertical family of Dirac-type operators $\tilde{\partial}_{\tilde{M}/\tilde{B}}$, and a connection for $\tilde{\psi}$, the $\eta$-forms and $J$-forms are both related to the heat kernel of a family of Dirac-type operators on $\tilde{M} \times \mathbb{R}^+$, but with different extensions of $\tilde{\psi}$. Indeed,

$$
\eta \longleftrightarrow \left( \begin{array}{c}
\tilde{X} \\
\tilde{B}
\end{array} \right) \left( \begin{array}{c}
\tilde{M} \\
\tilde{B}
\end{array} \right),
\quad
J \longleftrightarrow \left( \begin{array}{c}
\tilde{X} \times \mathbb{R}^+ \\
\tilde{B} \times \mathbb{R}^+
\end{array} \right) \left( \begin{array}{c}
\tilde{M} \times \mathbb{R}^+ \\
\tilde{B}
\end{array} \right)
$$

where in the former case the $\mathbb{R}^+$ factor results in an ‘auxiliary Grassman variable’ but does not change the fiber, while in the latter the fiber $\tilde{X} \times \mathbb{R}^+$ is endowed with an exact conic metric $dr^2 + r^2 g_{\tilde{X}}$.

In order to find the relation between the $\eta$ and $J$ forms associated to $\tilde{M} \to \tilde{B}$, we attach the cone over $\tilde{M}$ to the boundary of a half-cylinder over $\tilde{M}$. That is, we form a ‘b-c suspension’ (where b-c refers to b-metric and conic-metric) of $\tilde{M}$ of the form

$$
(\tilde{M} \times [0, 1], g_{(\tilde{M} \times [0, 1])/(\tilde{M} \times [0, 1])})
$$

\[ g_{(\tilde{M} \times [0, 1])/(\tilde{M} \times [0, 1])} = \begin{cases} 
\frac{ds^2 + s^2 g_{\tilde{M}/\tilde{B}}}{(1-s)^2} + g_{\tilde{M}/\tilde{B}} & \text{near } s = 0 \\
\frac{ds^2}{(1-s)^2} + g_{\tilde{M}/\tilde{B}} & \text{near } s = 1 
\end{cases}
\]

we extend $\tilde{\partial}_{\tilde{M}/\tilde{B}}$ to a family of Dirac-type operators acting on the fibers of $\tilde{M} \times [0, 1] \to \tilde{B}$ and then, using an extended families index formula together with some characteristic form computations, conclude the following in Theorem 7.4. See Section 7 for a detailed description of the transgression forms herein.
Theorem 4. For a fiber bundle $\tilde{M} \to \tilde{B}$ as above, the relation between the $\eta$ and $J$-forms for wedge Dirac-type operators with a compatible perturbation $Q$ is

$$J_Q(\tilde{M}/\tilde{B}) - \tilde{\eta}_Q(\tilde{M}/\tilde{B}) = \int_{\tilde{M}/\tilde{B}} T\hat{A}(\tilde{M}/\tilde{B}) \text{Ch}'(E) + \sum_{\tilde{N} \in S(\tilde{M})} \int_{\tilde{N}/\tilde{B}} T\hat{A}(\tilde{N}/\tilde{B}) J_Q(\mathfrak{B}_{\tilde{N}}/\tilde{N}) + d\eta_{b-w,Q},$$

where, for any family $L \to \tilde{B}$ as above, $T\hat{A}(L/\tilde{B})$ is the restriction to $L \times \{0\}$ of a transgression between the $\hat{A}$-forms on $L \times \mathbb{R}^+$ corresponding to a cylindrical metric and a conic metric, and $\eta_{b-w,Q}$ denotes a ‘b-wedge $\eta$-form’.

To prove Theorem 4 we extend Theorem 3 to manifolds with corners with metrics that are of ‘wedge-type’ at all collective boundary hypersurfaces but one, where they are of ‘b-type’ (i.e., asymptotically cylindrical, albeit with a singular cross-section).

The formula for the exterior derivative of the $\eta$-forms is already in Theorem 3. Together with Theorem 4 this implies a formula for the exterior derivative of the $J$-forms, namely

$$dJ_Q(\tilde{M}/\tilde{B}) = \int_{\tilde{M}/\tilde{B}} \hat{A}_c(\tilde{M}/\tilde{B}) \text{Ch}'(E) + \sum_{\tilde{N} \in S(\tilde{M})} \int_{\tilde{N}/\tilde{B}} \hat{A}_c(\tilde{N}/\tilde{B}) J_Q(\mathfrak{B}_{\tilde{N}}/\tilde{N}).$$

Here, for any $L$, $\hat{A}_c(L/\tilde{B})$ is the restriction to $L \times \{0\}$ of the $\hat{A}$-form on $L \times \mathbb{R}^+$ corresponding to a conic metric.

Previous results. The index theorem of Atiyah-Singer for closed manifolds [AS63] was soon generalized to operators on manifolds with boundary admitting local elliptic boundary conditions by Atiyah and Bott [AB64] and later extended to operators admitting global elliptic boundary conditions by Atiyah, Patodi, and Singer [APS75]. The resulting formula for a Dirac-type operator $\partial$ on a Clifford bundle $E \to M$ takes the form

$$\text{ind}(\partial) = \int_M \hat{A}(M) \text{Ch}'(E) - \eta(\partial_{\partial M})$$

where $\hat{A}$ is the A-hat genus of $M$, $\text{Ch}'(E)$ the twisting curvature of the Clifford bundle, and $\eta(\partial_{\partial M})$ the $\eta$-invariant of the induced Dirac-type operator on the boundary. The boundary conditions in this paper involve the projection onto the sum of the eigenspaces corresponding to positive eigenvalues of an induced boundary operator and are now known as Atiyah-Patodi-Singer, or APS, boundary conditions.

Already in [APS75] it was pointed out that the resulting domain had a natural interpretation as the domain of an operator on a complete non-compact manifold obtained by attaching a cylinder to the boundary. Melrose [Mel93] showed that in fact one can consider the APS index theorem as the index theorem in the ‘category’ of manifolds with asymptotically cylindrical end. The present project is analogous to Melrose’s treatment, we obtain the index theorem in the ‘category’ of manifolds with corners and iterated fibration structures endowed with wedge metrics.

Cheeger’s proof of the Ray-Singer conjecture connecting analytic and Reidemeister torsion [Che77, Che79a] (established independently by Müller [M78]), led him to develop analysis on spaces with singularities, particularly what we refer to as stratified spaces with wedge metrics. Cheeger realized that the Atiyah-Patodi-Singer index formula can be obtained as
the natural index theorem in the context of manifolds with conic singularities, see [Che79b, Che80, Che83] for the signature and Gauss-Bonnet theorem both for isolated singularities and for piecewise flat metrics on stratified spaces. Chou [Cho85, Cho89] showed for isolated conic singularities that the same was true for the Dirac operator.

For a fiber bundle of closed operators $M \xrightarrow{\psi} B$, Atiyah and Singer [AS71] showed that the index theorem generalizes to families of operators on the fibers of $\psi$. They showed that such a family has an index in the form of a virtual bundle over $B$ and, among other things, they computed the Chern character of this index bundle. Bismut [Bis86] used heat equation methods to establish the formula for the Chern character of the index bundle.

In [Wit85], Witten derived a formula for the $\eta$ invariant of a manifold fibering over a circle. This formula was established rigorously by Bismut and Freed [BF86a, BF86b] for spin Dirac operators and Cheeger [Che87] for the signature operator. This was generalized by Bismut and Cheeger [BC89] and Dai [Dai91] to fiber bundles of arbitrary closed manifolds. They considered the behavior of the $\eta$ invariant as the metric on the total space of the fiber bundle undergoes an ‘adiabatic limit’ in which the fibers are collapsed to a point. The limit involves a ‘higher’ version of the $\eta$-invariant, known as the Bismut-Cheeger $\eta$-forms. These forms are of even degree if the dimension of the fiber is odd and of odd degree if the dimension of the fiber is even.

The rôle played by the $\eta$-invariant in the Atiyah-Patodi-Singer index theorem for manifolds with boundary is played by the $\eta$-forms in the formula for the Chern character of the index bundle of a family of Dirac-type operators on a manifold with boundary. This was established by Bismut and Cheeger for spin Dirac operators on even dimensional manifolds with boundary [BC90a, BC90b, BC91] assuming invertibility of an induced boundary family of Dirac operators (referred to as the ‘Witt assumption’ below). Melrose and Piazza [MP97a, MP97b] proved a formula conjectured in [BC91] for the case of odd dimensional manifolds with boundary, extended the Bismut-Cheeger result to Dirac-type operators, and removed the assumption of invertible boundary families. They introduced the notion of a ‘spectral section’ of the boundary family and proved an index theorem for each spectral section.

(In the present paper we make the assumption of invertible boundary families as in the work of Bismut-Cheeger mentioned above, but we allow perturbations as in the approach of Melrose-Piazza. In a subsequent paper we will characterize the existence of these perturbations in terms of spectral sections.)

The approach adopted by Bismut and Cheeger to establish the families index theorem for Dirac operators on manifolds with boundary was to attach a cone to the boundary and consider a family of metrics, parametrized by $\varepsilon \in [0, 1]$, interpolating between the conic singularity and an infinite cylindrical end. An intermediate result is an index theorem for families of Dirac operators on spaces with isolated conic singularities. One effect of the $\varepsilon$ degeneration is to ‘scale away’ the small eigenvalues of the boundary family of Dirac operators so that the Dirac operators on the spaces with conical singularities are essentially self-adjoint and there is no need to choose a domain. The Chern character of the index bundle of the family of Dirac operators on spaces with isolated conic singularities involves another differential form invariant, the Bismut-Cheeger $\mathcal{J}$-forms. Bismut-Cheeger showed that the $\varepsilon$-limit of the $\mathcal{J}$-form in this case is the $\eta$ form.
Index theory is now a vast field. Among the many ways in which singular spaces arise in index theory there are spaces arising from foliations (see, e.g., [CS84, BKR10]) and group actions (see, e.g., [Ati74]). Index theorems on complete metrics on manifolds with possibly fibered boundary include [Car01, LM05, MR04, MR06, Vai01, MR11, AM09a, AM10, AM09b, AR09a, LMP06, Hun07, Pia93, MN]. Index theory on manifolds with corners endowed with complete metrics have been studied in, e.g., [MP92, MN98, LM02, Loy05, Bun09, MN12, Ste89, M96, HMM97].

There is a powerful groupoid approach to index problems on the interior of manifolds with corners, see e.g., [ALN07, DLR15, CN14, CRLM14, BS] for complete metrics, [DLN09] for isolated conic singularities, and [DS] for a Boutet de Monvel type calculus.

An approach of Nazaikinskii, Yu, Savin, Sternin, and Schulze, see [NkSSS05, NkSSS06, SS10] in the stratified setting proceeds by decomposing the index of a pseudodifferential operator on a stratified space into a sum of contributions from each stratum with the property that each is a homotopy invariant of the symbol. These papers make use of the analytic tools developed by Schulze and his collaborators, see e.g., [Sch07, SS95]. Our treatment benefits from recent advances in parallel analytic tools in, e.g., [ARS, MV14, MV12, KM16, GRS15, MW17].

Already in [Ati71, Sin71] Atiyah and Singer called for the development of index theory on stratified spaces such as algebraic varieties. Index theory on spaces with isolated conic singularities is now very well understood see, e.g., [Les97, FH95, BL96, BS88]. However, there are few explicit index formulæ associated to singular metrics on stratified spaces beyond the case of isolated conic singularities. For a stratified space with a single singular stratum and a wedge metric there are index theorems for the signature operator by Chan, Hunsicker-Mazzeo, Brüning, and Cheeger-Dai [Cha97, HM05, Bru09, CD09]. The $\eta$ and $\rho$ invariants have been studied by Piazza and Vertman [PV]. Atiyah and LeBrun [AL13] obtain an index theorem on a smooth four dimensional manifold endowed with a singular wedge metric. Lock and Viaclovsky [LV13] prove an index theorem for anti-self-dual orbifold-cone metrics, again in four dimensions. In previous work, the authors [AGR16] proved an index theorem for spin Dirac operators satisfying the geometric Witt condition.

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Contents

| Section | Page |
|---------|------|
| Introduction | 1 |
| Previous results | 5 |
| 1. Families of Dirac-type wedge operators | 8 |
| 1.1. Iterated fibration structures and wedge geometry on manifolds with corners | 8 |
| 1.2. Totally geodesic wedge metrics | 13 |
| 1.3. Dirac-type wedge operators | 18 |
| 1.4. Bismut superconnection | 19 |
| 2. Witt condition | 22 |
| 2.1. Boundary families | 22 |
1. Families of Dirac-type wedge operators

1.1. Iterated fibration structures and wedge geometry on manifolds with corners. Let $X$ be an $n$ dimensional manifold with corners, by which we mean an $n$ dimensional topological manifold with boundary with a smooth atlas modeled on $(\mathbb{R}^+)^n$ whose boundary hypersurfaces are embedded. We denote the set of boundary hypersurfaces of $X$ by $\mathcal{M}_1(X)$.

A collective boundary hypersurface refers to a finite union of non-intersecting boundary hypersurfaces.

**Definition 1.1** (Melrose [AM11, ALMP12, Alb17]). An iterated fibration structure on a manifold with corners $X$ consists of a collection of fiber bundles

$$Z_Y \to \mathcal{B}_Y \xrightarrow{\phi_Y} Y$$
where $\mathcal{B}_Y$ is a collective boundary hypersurface of $X$ with base and fiber manifolds with corners such that:

i) Each boundary hypersurface of $X$ occurs in exactly one collective boundary hypersurface $\mathcal{B}_Y$.

ii) If $\mathcal{B}_Y$ and $\mathcal{B}_{\tilde{Y}}$ intersect, then $\dim Y \neq \dim \tilde{Y}$, and we write

$$Y < \tilde{Y} \text{ if } \dim Y < \dim \tilde{Y}.$$  

iii) If $Y < \tilde{Y}$ then $\tilde{Y}$ has a collective boundary hypersurface $\mathcal{B}_{Y\tilde{Y}}$ participating in a fiber bundle $\phi_{Y\tilde{Y}} : \mathcal{B}_{Y\tilde{Y}} \to Y$ such that the diagram

(1.1) \[
\begin{array}{ccc}
\mathcal{B}_Y \cap \mathcal{B}_{\tilde{Y}} & \xrightarrow{\phi_{\tilde{Y}}} & \mathcal{B}_{Y\tilde{Y}} \\
\phi_Y \downarrow & & \phi_{Y\tilde{Y}} \\
Y & \xrightarrow{\phi_Y} & \tilde{Y}
\end{array}
\]

commutes.

Unless stated otherwise, we will assume that $X$ is compact and that $\dim Z_Y > 0$ for all $Y$.

There is no real loss of generality in assuming that the bases are connected, but the fibers of the boundary fibrations will generally be disconnected.

There is a functorial equivalence between Thom-Mather stratified spaces and manifolds with corners and iterated fibration structures see, e.g., [ALMP12], [Alb17, Theorem 6.3]. Under this equivalence, the bases of the boundary fibrations correspond to the different strata. We will denote this set by

$$\mathcal{S}(X) = \{Y : Y \text{ is the base of a boundary fibration of } X\}.$$  

Both the bases and fibers of the boundary fiber bundles themselves are manifolds with corners and iterated fibration structures, see e.g., [AM11, Lemma 3.4]. The assumption $\dim Z_Y > 0$ corresponds to the category of pseudomanifolds within the larger category of stratified spaces.

The partial order on $\mathcal{S}(X)$ gives us a notion of depth

$$\text{depth}_X(Y) = \max\{n \in \mathbb{N}_0 : \exists Y_i \in \mathcal{S}(X) \text{ s.t. } Y = Y_0 < Y_1 < \ldots < Y_n\}.$$  

The depth of $X$ is then the maximum of the integers $\text{depth}_X(Y)$ over $Y \in \mathcal{S}(X)$.

If $H$ is a boundary hypersurface then, because it is assumed embedded, there is a non-negative function $\rho_H$ such that

$$\rho_H^{-1}(0) = H \text{ and } |d\rho_H| \neq 0 \text{ on } H,$$

where $| \cdot |$ denote the norm with respect to some smooth background metric on $M$; we call any such function a **boundary defining function** for $H$. It is always possible (see, e.g., [AM11, Proposition 1.2]) to choose: a boundary defining function $\rho_H$ for each $H \in \mathcal{M}_1(X)$, an open neighborhood $\mathcal{U}_H \subseteq X$ of each $H$, and a smooth vector field $V_H$ defined in $\mathcal{U}_H$ such
that
\[
V_H \rho_K = \begin{cases} 
1 & \text{in } U_H \\
0 & \text{in } U_H \cap U_K
\end{cases}
\]
if \( K = H \) or \( K \neq H \),

\([V_H, V_K] = 0 \text{ in } U_H \cap U_K\)
for all \( H, K \in \mathcal{M}_1(X) \). We refer to these choices as a **boundary product structure**, and will always assume that our boundary defining functions are chosen this way.

For each \( Y \in \mathcal{S}(X) \) we denote a collective boundary defining function by

\[ \rho_Y = \prod_{H \in \mathcal{B}_Y} \rho_H, \]
we also use the notation

\[ \rho_X = \prod_{H \in \mathcal{M}_1(X)} \rho_H \]
for a **total boundary defining function**.

A boundary product structure allows us to extend an iterated fibration structure to a **collared iterated fibration structure**. Indeed, let us assume for simplicity of notation that the neighborhoods \( U_H \) coincide with \( \rho_H^{-1}([0,1]) \), so that \( U_H \cong [0,1)_{\rho_H} \times H \). For each \( Y \in \mathcal{S}(X) \), we write

\[ \mathcal{C}(\mathcal{B}_Y) = \bigcup_{H \in \mathcal{B}_Y} U_H \cong [0,1)_{\rho_Y} \times \mathcal{B}_Y, \quad Y^+ = [0,1)_{\rho_Y} \times Y \]

and we denote the natural extension of \( \phi_Y \) by

\[ \phi_{Y^+} : \mathcal{C}(\mathcal{B}_Y) \to Y^+. \]

Choosing compatible boundary product structures on each \( Y \in \mathcal{S}(X) \) (existence is checked by a simple induction on the depth of \( X \)), the extended boundary fibrations participate in commutative diagrams,

\[
\begin{array}{ccc}
\mathcal{C}(\mathcal{B}_Y) \cap \mathcal{C}(\mathcal{B}_{\bar{Y}}) & \xrightarrow{\phi_{Y^+}} & \mathcal{C}(\mathcal{B}_{Y\bar{Y}}) \subseteq \bar{Y}^+ \\
\phi_{Y^+} \downarrow & & \downarrow \phi_{(Y\bar{Y})^+} \\
Y^+ & & Y^+
\end{array}
\]
whenever \( Y < \bar{Y} \).

This structure will be useful when we discuss Getzler rescaling below \((\S 5)\). In that setting, we will have a filtration of a vector bundle defined on collective boundary hypersurfaces and we will need to extend it into a neighborhood of the boundary consistently; a collared iterated fibration structure makes this easy to do.

Various differential geometric objects have natural analogues that take the iterated fibration structure into account. For example, we define

\[
C^\infty_{\phi}(X) = \{ f \in C^\infty(X) : f\big|_{\mathcal{B}_Y} \in \phi_Y^* C^\infty(Y) \text{ for all } Y \in \mathcal{S}(X) \}\]
(This corresponds to the smooth functions on \( X \) that are continuous on the underlying stratified space. If an open cover of \( X \) is the lift of a cover of the underlying stratified space, then there is a compatible partition of unity in \( C^\infty_{\phi}(X) \) see, e.g., \([ABL+15\) Lemma 5.2\).
The edge vector fields on $X$ \cite{Maz91} are
\[ V_e = \{ V \in \mathcal{C}^\infty(X; TX) : V|_{\partial Y} \text{ is tangent to the fibers of } \phi_Y \text{ for all } Y \in \mathcal{S}(X) \}, \]
or, equivalently, they are the b-vector fields (vector fields tangent to the boundary) that
when applied to $\mathcal{C}^\infty_\phi(X)$ yield functions that vanish at the boundary of $X$.

There is a vector bundle, the edge tangent bundle, $^{e}TX$, together with a natural vector
bundle map $i_e : ^eTX \to TX$ that is an isomorphism over the interior and satisfies
\[ (i_e)_* \mathcal{C}^\infty(X; ^eTX) = V_e. \]

In local coordinates near a point in $\mathcal{B}_Y$, $(x, y, z)$, where $x$ is a bdf, $y$ denotes coordinates
along $Y$, and $z$ denotes coordinates along $Z$, a local frame for $^{e}TX$ is given by
\[ x \partial_x, \quad x \partial_y, \quad \partial_z. \]

Note that the vector fields $x \partial_x$ and $x \partial_y$ are degenerate as sections of $TX$ but not as sections
of $^{e}TX$. If $Y = \{ \text{pt} \}$ then the edge tangent bundle coincides with the b-tangent bundle
discussed in \S3.1.

The universal enveloping algebra of $V_e$ is the ring $\text{Diff}_e^*(X)$ of edge differential operators
\cite{Maz91} \S2. Thus these are the differential operators on $X$ that can be expressed locally as
finite sums of products of elements of $V_e$. They have the usual notion of degree and extension
to sections of vector bundles, as well as an edge symbol map defined on the edge cotangent
bundle, see \cite{Maz91, ALMP12, ALMP13}.

**Remark 1.2.** In \cite{ALMP12, ALMP13, ALMP17} the edge tangent bundle was referred to as
the ‘iterated edge tangent bundle’. We prefer to think of it as the edge tangent bundle of the
iterated fibration structure. Similarly, the wedge tangent bundle, defined below, was referred
to in loc. cit. as the ‘iterated incomplete edge tangent bundle’.

Among the metrics most closely associated to these spaces are metrics that degenerate
conically reflecting the conic degeneration of the space. We will define these formally in
Section 1.2 but roughly speaking their are the geometric object arising from iterating the
process of taking the geometric cones of spaces and local bundles of such cones. Metrics of
this form, wedge metrics, are singular at the boundary of $X$. However, they can be seen as
smooth (or more generally $\mathcal{I}$-smooth or polyhomogeneous) sections of a rescaled bundle of
symmetric tensors.

Formally, we proceed as follows. Let $X$ be a manifold with corners and iterated fibration
structure. Consider the ‘wedge one-forms’
\[ \mathcal{V}_w^* = \{ \omega \in \mathcal{C}^\infty(X; T^*X) : \text{ for each } Y \in \mathcal{S}(X), \ i_{\mathcal{B}_Y}^* \omega(V) = 0 \text{ for all } V \in \ker D\phi_Y \}. \]

Just as we have done with the edge tangent bundle, we can identify $\mathcal{V}_w^*$ with the space of
sections of a vector bundle. That is, there exists a vector bundle $^{w}T^*X$, the wedge cotangent
bundle of $X$, together with a bundle map
\[ i_w : ^wT^*X \to T^*X \]
that is an isomorphism over the interior of $X$ and such that
\[ (i_w)_* \mathcal{C}^\infty(X; ^wT^*X) = \mathcal{V}_w^* \subseteq \mathcal{C}^\infty(X; T^*X). \]

In particular, in local coordinates near the collective boundary hypersurface $\mathcal{B}_Y$ the wedge
cotangent bundle is spanned by
\[ dx, \quad xdz, \quad dy \]
where $x$ is a boundary defining function for $\mathcal{B}_Y$, $dz$ represents covectors along the fibers and $dy$ covectors along the base.

The dual bundle to the wedge cotangent bundle is the \textit{wedge tangent bundle}, $wTX$. It is locally spanned by

$$\partial_x, \frac{1}{x} \partial_z, \partial_y.$$ 

A \textit{wedge metric} is simply a bundle metric on the wedge tangent bundle. Below we will make further assumptions on the metric, see Section 1.2.

Wedge differential operators are defined in terms of edge differential operators: $P$ is a wedge differential operator of order $k$ acting on sections of a vector bundle $E$ if $\rho^k_X P$ is an edge differential operator of order $k$ acting on sections of $E$,

\begin{equation}
\text{Diff}_w^k(X; E) = \rho^{-k}_X \text{Diff}_e^k(X; E).
\end{equation}

See, e.g., [MV14, GKM13, ALMP12, ALMP13].

By a smooth family of manifolds with corners and iterated fibration structures we will mean first of all a fiber bundle $X \xrightarrow{\psi} M \xrightarrow{\psi} B$ where $X, M,$ and $B$ are manifolds with corners. Since $M$ is locally diffeomorphic to $X \times U$, $U \subseteq B$, every boundary hypersurface of $M$ corresponds to either a boundary hypersurface of $B$ or a boundary hypersurface of $X$. The latter are the boundary hypersurfaces that are transverse to $\psi$. We want the fibers of $\psi$ to have iterated fibration structures that themselves vary smoothly. We formalize this as follows.

\textbf{Definition 1.3.} A \textit{locally trivial family of manifolds with corners and iterated fibration structures} over $B$ is:

- a fiber bundle of manifolds with corners $X \xrightarrow{\psi} M \xrightarrow{\psi} B$,
- a partition of the boundary hypersurfaces of $M$ transverse to $\psi$, which we denote by $S_\psi(M) \subseteq S(M)$, into collective boundary hypersurfaces $\{\mathcal{B}_N : N \in S_\psi(M)\}$, where each $N$ is a manifold with corners endowed with a fiber bundle map $N \xrightarrow{\psi_N} B$,
- a collection of fiber bundles $Z_N \xrightarrow{\phi_N} \mathcal{B}_N \xrightarrow{\phi_N} N$ satisfying Definition 1.1(ii-iii),

satisfying that, for all $N \in S_\psi(M)$, the diagram

\begin{equation}
\begin{array}{ccc}
\mathcal{B}_N & \xrightarrow{\phi_N} & M \\
\downarrow \psi_N & & \downarrow \psi \\
N & \xrightarrow{\psi} & B \\
\end{array}
\end{equation}

commutes.

For each $b \in B$ the fiber of $\psi : M \rightarrow B$, $X = \psi^{-1}(b)$ has a iterated fibration structure with

$$S(X) = \{Y_b = \psi^{-1}_N(b) : N \in S_\psi(M)\}$$
and boundary fiber bundles determined by the diagrams, one for each $N \in S_\psi(M)$,

![Diagram](image)

In particular, $\mathcal{B}_Y$ is the typical fiber of $\phi_N \circ \psi_N$. We will always use $\mathcal{B}_N$ to denote collective boundary hypersurfaces of $M$ and $\mathcal{B}_Y$ to denote collective boundary hypersurfaces of $X$ and hope the similar notation does not cause confusion.

Analogously to what we have done before, the $\psi$-wedge one forms are the covectors on $M$ that vanish on vertical vectors at all boundary hypersurfaces transverse to $\psi$,

$$V^*_w(\psi) = \{\omega \in C^\infty(M; T^*M) : \text{ for each } N \in S_\psi(M), \, \tilde{\iota}_N^*\omega(V) = 0 \text{ for all } V \in \ker D\phi_N\},$$

and can be identified with the sections of a vector bundle, $w(\psi)T^*M$. We refer to this as the ‘$\psi$-wedge cotangent bundle’ and to the dual bundle $w(\psi)TM$, as the ‘$\psi$-wedge tangent bundle’. The latter has a sub-bundle determined by its sections,

$$C^\infty(M; w^*TM/B) = \{V \in C^\infty(M; w(\psi)TM) : (\rho_X V)(\psi_B^* f) = 0 \text{ for all } f \in C^\infty(B)\},$$

where $\rho_XV \in C^\infty(M; TM)$ acts by differentiation, which we will call the vertical wedge tangent bundle. The vertical wedge cotangent bundle is the dual bundle $w(T^*M/B) \rightarrow M$. A choice of connection for $M \rightarrow B$ induces splittings

$$w(\psi)TM = w^*TM/B \oplus \psi^*T^*B, \quad w(\psi)T^*M = w(T^*M/B \oplus \psi^*T^*B).$$

1.2. Totally geodesic wedge metrics. Let $M \rightarrow B$ be a family of manifolds with corners and iterated fibration structures. A vertical wedge metric on $M$ will refer to a bundle metric on $w^*TM/B$. For simplicity we will work with a subset of these metrics which we call ‘totally geodesic vertical wedge metrics’. For simplicity of notation, in this section we discuss these metrics on a fixed fiber $X = \psi^{-1}(b)$ of $\psi$.

We define totally geodesic wedge metrics inductively by the depth of the space. If $X$ has depth zero, so is a smooth manifold, a wedge metric is simply a Riemannian metric. Assuming we have defined totally geodesic wedge metrics at spaces of depth less than $k$, let $X$ have depth $k$. A wedge metric $g_w$ on $X$ is a totally geodesic wedge metric if, for every $Y \in S(X)$ of depth $k$ there is a collar neighborhood $\mathcal{C}(\mathcal{B}_Y) = [0,1)_x \times \mathcal{B}_Y$ of $\mathcal{B}_Y$ in $X$, a metric $g_{w,pt}$ of the form

$$g_{w,pt} = dx^2 + x^2 g_{\mathcal{B}_Y/Y} + \phi^*_Y g_Y$$

where $g_Y$ is a totally geodesic wedge metric on $Y$, $g_{Z_Y} + \phi^* g_Y$ is a submersion metric for $\mathcal{B}_Y \rightarrow Y$, and $g_{Z_Y}$ restricts to each fiber of $\phi_Y$ to be a totally geodesic wedge metric on $Z_Y$, and

$$g_w = g_{w,pt} \in x^2 C^\infty(\mathcal{C}(\mathcal{B}_Y; S^2(w^*X))).$$

Off of these collar neighborhoods the form of the metric is fixed by the induction.
If at every step \( g_w = g_{w,pt} \) we say that \( g_w \) is a rigid or product-type wedge metric. If at every step \( g_w - g_{w,pt} = \mathcal{O}(x) \) as a symmetric two-tensor on the wedge tangent bundle, we say that \( g_w \) is an exact wedge metric. We will always work with totally geodesic wedge metrics.

It is worthwhile describing a product-type metric near boundary faces of arbitrary depth. As a simple example, the metric on the cone over a cone has the form

\[
ds^2 + s^2(d\tau^2 + r^2g_Z).\]

This has the form above near \( \{s = 0\} \), but not near \( \{r = 0\} \). However, from [1.1], every point \( \zeta \in \partial X \) lies over the interior of a unique \( Y \in \mathcal{S}(X) \). We can choose a boundary defining function \( x \) for \( \mathcal{B}_Y \), and identify a neighborhood of \( \zeta \) in \( X \) with the form \([0, 1)_x \times Z \times \mathcal{U} \), where \( \mathcal{U} \) is a subset of \( Y \) with closure contained in the interior of \( Y \), and then the metric takes the form

\[
dx^2 + x^2g_Z + \phi^*_Y g_Y\]

in this neighborhood. This is consistent with the description above since the boundary hypersurface of greatest depth in this neighborhood is \( \mathcal{B}_Y \). A common theme throughout this work is to work over the interior of each \( Y \in \mathcal{S}(X) \), and make use of the fact that this exhausts all of \( \partial X \).

Let us describe the asymptotics of the Levi-Civita connection of a totally geodesic wedge metric at \( \mathcal{B}_Y \) for \( Y \in \mathcal{S}(X) \). First let us start by recalling the behavior of the Levi-Civita connection of a submersion metric. Endow \( \mathcal{B}_Y \) with a submersion metric of the form \( g_{\mathcal{B}_Y} = g_{\mathcal{B}_Y/Y} + \phi^*_Y g_Y \). We denote the associated splitting of the tangent bundle \( T\partial X \) by

\[
T\mathcal{B}_Y = T\mathcal{B}_Y/Y \oplus \phi^*_Y TY
\]

and the orthogonal projections onto each summand by

\[
h : T\mathcal{B}_Y \rightarrow \phi^* TY, \quad v : T\mathcal{B}_Y \rightarrow T\mathcal{B}_Y/Y.
\]

Given a vector field \( U \) on \( Y \), we denote its horizontal lift to \( \mathcal{B}_Y \) by \( \tilde{U} \). The Levi-Civita connection of \((\mathcal{B}_Y, g_{\mathcal{B}_Y})\), \( \nabla^{\mathcal{B}_Y} \), can be written in terms of the Levi-Civita connections \( \nabla^Y \) on the base and the connections \( \nabla^{\mathcal{B}_Y/Y} \) on the fibers using two tensors: 1) the second fundamental form of the fibers, defined by

\[
S^{\phi_Y} : T\mathcal{B}_Y/Y \times T\mathcal{B}_Y/Y \rightarrow \phi^* TY, \quad S^{\phi_Y}(V_1, V_2) = h(V_{\mathcal{B}_Y/Y})V_1 V_2
\]

and, 2) the curvature of the fibration, defined by

\[
R^{\phi_Y} : \phi^* TY \times \phi^* TY \rightarrow T\mathcal{B}_Y/Y, \quad R^{\phi_Y}(\tilde{U}_1, \tilde{U}_2) = v([\tilde{U}_1, \tilde{U}_2]).
\]

The behavior of the Levi-Civita connection (cf. [HHM04, Proposition 13]) is then summed up in the table:

| \( g_{\mathcal{B}_Y} \left( V_{\mathcal{B}_Y/Y} W_1, W_2 \right) \) | \( V_0 \) | \( \tilde{U}_0 \) |
| --- | --- | --- |
| \( \nabla^{\mathcal{B}_Y}_{V_1} V_2 \) | \( g_{\partial X/Y}(V_{\mathcal{B}_Y/Y}, V_0) \) | \( \phi^*_Y g_Y(S^{\phi_Y}(V_1, V_2), \tilde{U}_0) \) |
| \( \nabla^{\mathcal{B}_Y}_{\tilde{U}_1} V \) | \( g_{\mathcal{B}_Y/Y}(\tilde{U}, V_0) - \phi^*_Y g_Y(S^{\phi_Y}(V, V_0), \tilde{U}) \) | \( -\frac{1}{2} g_{\mathcal{B}_Y/Y}(R^{\phi_Y}(\tilde{U}, \tilde{U}_0), V) \) |
| \( \nabla^{\mathcal{B}_Y}_{\tilde{U}_1} U \) | \( -\phi^*_Y g_Y(S^{\phi_Y}(V, V_0), \tilde{U}) \) | \( \frac{1}{2} g_{\mathcal{B}_Y/Y}(R^{\phi_Y}(\tilde{U}_1, \tilde{U}_2), V) \) |
| \( \nabla^{\mathcal{B}_Y}_{\tilde{U}_1} U_2 \) | \( \frac{1}{2} g_{\mathcal{B}_Y/Y}(R^{\phi_Y}(\tilde{U}_1, \tilde{U}_2), V_0) \) | \( g_Y(\nabla^{\mathcal{B}_Y}_{\tilde{U}_1} U_2, U_0) \) |
We want a similar description of the Levi-Civita connection of a totally geodesic wedge metric. We define an operator $\nabla$ on sections of the wedge tangent bundle through the usual Koszul formula

$$2g_w(\nabla_{W_0}W_1, W_2) = W_0g_w(W_1, W_2) + W_1g_w(W_0, W_2) - W_2g_w(W_0, W_1) + g_w([W_0, W_1], W_2) - g_w([W_0, W_2], W_1) - g_w([W_1, W_2], W_0).$$

We will consider wedge metrics $g_w$ and $g'_w$ that differ by $g_w - g'_w = O_w(x^2)$, where the $O_w(x^p)$ notation reminds the reader that these are tensors on wedge vector fields. In particular, for any such pair, if $W_i \in C^\infty(X; wTX)$, $i = 0, 1, 2$, then $g_w(W_i, W_j) - g'_w(W_i, W_j) = O(x^2)$, and by the Koszul formula and the fact, seen below, that $x[W_i, W_j]$ is a smooth wedge vector field,

$$(1.9) \quad g_w(\nabla_{W_0}W_1, W_2) - g'_w(\nabla'_{W_0}W_1, W_2) = O(x).$$

Thus to understand the leading asymptotics of the Levi-Civita connection of a totally geodesic wedge metric, it suffices to understand the leading asymptotics of a product-type wedge metric.

Fix a product-type wedge metric $g_{w,pt} = dx^2 + x^2g_{\mathfrak{B}Y/Y} + \phi^* g_Y$ such that $g_w - g_{w,pt} = x^2 \tilde{g}$, the splitting of the tangent bundle of $X$ associated to $g_{\mathfrak{B}Y/Y} + \phi^* g_Y$ extends to a splitting of the wedge tangent bundle of $\mathcal{C}(\mathfrak{B}_Y)$ and hence induces a splitting

$$(1.10) \quad wT\mathcal{C}(\mathfrak{B}_Y) = \langle \partial_x \rangle \oplus \frac{1}{x}wT\mathfrak{B}_Y/Y \oplus \phi_Y^* TY.$$

In terms of which a convenient choice of vector fields is

$$\partial_x, \quad \frac{1}{x} V, \quad \tilde{U}$$

where $V$ denotes a $\phi_Y$-vertical wedge vector field at $\{x = 0\}$ extended trivially to $\mathcal{C}(\mathfrak{B}_Y)$ and $\tilde{U}$ denotes a wedge vector field on $Y$, lifted horizontally to $\mathfrak{B}_Y$ and then extended trivially to $\mathcal{C}(\mathfrak{B}_Y)$. Note that, with respect to $g_{w,pt}$, these three types of vector fields are orthogonal, and that their commutators satisfy

$$[\partial_x, \frac{1}{x} V] = -\frac{1}{x^2} V \in x^{-1}C^\infty(\mathcal{C}(\mathfrak{B}_Y), \frac{1}{x}wT\mathfrak{B}_Y/Y), \quad [\partial_x, \tilde{U}] = 0,$$

$$[\frac{1}{x} V, \frac{1}{x} V_2] = \frac{1}{x^2} [V, V_2] \in x^{-1}C^\infty(\mathcal{C}(\mathfrak{B}_Y), \frac{1}{x}wT\mathfrak{B}_Y/Y),$$

$$[\frac{1}{x} V, \tilde{U}] = \frac{1}{x} \left[ V, \tilde{U} \right] \in C^\infty(\mathcal{C}(\mathfrak{B}_Y), \frac{1}{x}wT\mathfrak{B}_Y/Y),$$

$$[\tilde{U}_1, \tilde{U}_2] \in xC^\infty(\mathcal{C}(\mathfrak{B}_Y), \frac{1}{x}wT\mathfrak{B}_Y/Y) + C^\infty(\mathcal{C}(\mathfrak{B}_Y), \phi_Y^* TY).$$

**Remark 1.4.** It is important to understand that the inductive definition of product type and totally geodesic wedge vector fields does not imply that for a fixed $g_w$, every $\mathfrak{B}_Y$ has a collar neighborhood with a metric that of the canonical form above. Indeed, such a decomposition can only be assumed on the complement of those $\mathfrak{B}_Y$, with $Y' < Y$. Here we compute the asymptotics of the connection and curvature on a neighborhood where the decomposition holds, and when we work at a corner, i.e. an intersection of $\mathfrak{B}_Y \cap \mathfrak{B}_{Y'}$, we assume the decomposition holds only on a neighborhood of the deeper one.

Below we will work with a local frame of wedge vector fields, orthogonal with respect to $g_{w,pt}$,

$$\partial_x, \quad \frac{1}{x} V_\alpha, \quad \tilde{U}_i, \quad \alpha = 1, \ldots, v_Y = \dim Z_Y, \quad i = 1, \ldots, h_Y = \dim Y,$$

$$\partial_x, \frac{1}{x} V_\alpha, \tilde{U}_i, \alpha = 1, \ldots, v_Y = \dim Z_Y, i = 1, \ldots, h_Y = \dim Y,$$
where the $V_\alpha$ are a a local frame of $w^wT\mathcal{B}_Y$, the $\tilde{U}_i$ are the horizontal lifts of a local frame $U_i$ of $w^TY$.

If $W_1 \in \{\partial_x, V_\alpha, \tilde{U}_i\}$ and $W_2, W_3 \in \{\partial_x, \frac{1}{x}V_\alpha, \tilde{U}_i\}$ then we find
\[ g_{w,pt}(\nabla_{W_1}W_2, W_3) = 0 \text{ if } \partial_x \in \{W_1, W_2, W_3\} \]

except for $g_{w,pt}(\nabla_{V_1}\partial_x, \frac{1}{x}V_2) = -g_{w,pt}(\nabla_{V_1}\frac{1}{x}V_2, \partial_x) = g_{\mathcal{B}_Y/Y}(V_1, V_2)$
and otherwise

| $g_{w,pt}(\nabla_{W_1}W_2, W_3)$ | $\frac{1}{x}V_3$ | $U_3$ |
|-----------------------------|----------------|-------|
| $\nabla_{V_1}\frac{1}{x}V_2$ | $g_{\mathcal{B}_Y/Y}(\nabla^2_{V_1}^\mathcal{B}_Y V_2, V_3)$ | $x\phi^*_Y g_Y(S^\phi_Y (V_1, V_2), \tilde{U}_3)$ |
| $\nabla_{\tilde{U}_i}V_2$ | $g_{\mathcal{B}_Y/Y}(\tilde{U}_i, V_3)$ | $-\frac{1}{2}g_{\mathcal{B}_Y/Y}(\mathcal{R}^\phi_Y (\tilde{U}_i, \tilde{U}_3), V)$ |
| $\nabla_{\tilde{U}_i}U_2$ | $\frac{1}{2}g_{\mathcal{B}_Y/Y}(\mathcal{R}^\phi_Y (\tilde{U}_i, U_2), V)$ | $g_Y(\mathcal{V}_{\mathcal{B}_Y} U_3)$ |

We point out a few consequences of these computations, valid for an arbitrary totally geodesic wedge metric. First note that the operator
\[ \nabla : C^\infty(X; w^wTX) \to C^\infty(X; T^*X \otimes w^wTX) \]
defines a connection on the wedge tangent bundle. Thus, in particular, the curvature tensor $R_w$ of $\nabla$ is a well-defined 2-form on all of $X$ with values in endomorphisms of $w^wTX$. Also note that this connection asymptotically preserves the splitting of $w^wT\mathcal{E}(\mathcal{B}_Y)$ into two bundles
\[ w^wT\mathcal{E}(\mathcal{B}_Y) = [(\partial_x) \oplus \frac{1}{x}w^wT\mathcal{B}_Y/Y] \oplus \phi^*_Y TY \]
in that if $W_1, W_2 \in \mathcal{V}_w$ are sections of the two different summands then
\[ g_w(\nabla_{W_0}W_1, W_2) = O(x) \text{ for all } W_0 \in C^\infty(X; TX). \]

Let us denote by
\[ v_+ : w^wT\mathcal{E}(\mathcal{B}_Y) \to (\partial_x) \oplus \frac{1}{x}w^wT\mathcal{B}_Y/Y, \quad h_+ : w^wT\mathcal{E}(\mathcal{B}_Y) \to (\partial_x) \oplus \phi^*TY, \]
the orthogonal projections onto their images, while $v$ and $h$ continue to denote projection onto $w^wT\mathcal{B}_Y/Y$ and $\phi^*TY$, and define connections
\[ \nabla^v = v_+ \circ \nabla \circ v_+ : C^\infty(\mathcal{E}(\mathcal{B}_Y); (\partial_x) \oplus \frac{1}{x}w^wT\mathcal{B}_Y/Y) \to C^\infty(\mathcal{E}(\mathcal{B}_Y); (\partial_x) \oplus \frac{1}{x}w^wT\mathcal{B}_Y/Y), \]
\[ \nabla^h = \phi^*\nabla^Y : C^\infty(\mathcal{E}(\mathcal{B}_Y); \phi^*_Y TY) \to C^\infty(\mathcal{E}; T^*\mathcal{E}(\mathcal{B}_Y) \otimes \phi^*_Y TY). \]

Denote by $j_\varepsilon : \{x = \varepsilon\} \hookrightarrow \mathcal{E}(\mathcal{B}_Y)$ the inclusion, and identify $\{x = \varepsilon\}$ with $\mathcal{B}_Y = \{x = 0\}$, note that the pull-back connections $j^*_\varepsilon \nabla^v$ and $j^*_\varepsilon \nabla^h$ are independent of $\varepsilon$ and
\[ j^*_0 \nabla = j^*_0 \nabla^v + j^*_0 \nabla^h. \]

Let us describe the asymptotics of the curvature.

**Proposition 1.5.** Let $(X, g_w)$ be a manifold with corners and an iterated fibration structure endowed with a totally geodesic wedge metric. Let $Y \in \mathcal{S}(X)$ and let $x$ be a bdf for $\mathcal{B}_Y$ in which the canonical metric form decomposition \[1.7]-\[1.8\] holds.

1. If $W_1, W_2$ are vector fields tangent to $\mathcal{B}_Y$ then $R_w(W_1, W_2)$ asymptotically preserves the splitting
\[ w^wT\mathcal{E}(\mathcal{B}_Y) = [(\partial_x) \oplus \frac{1}{x}w^wT\mathcal{B}_Y/Y] \oplus \phi^*_Y TY \]
(2) For \( N = N(x) \partial_x, W_0 \in C^\infty(X; TX) \), with \( W_0 \) tangent to the fibers of \( \phi_Y \), \( W_1, W_2 \in C^\infty(X; w^*TX) \), we have
\[
g_w(R_w(N, W_0)W_1, W_2) = g_w(R_w(N, vW_0)vW_1, vW_2) - N(x) \left( \phi^*_Y g_Y(S^\phi_Y(vW_0, vW_2), hW_1) - \phi^*_Y g_Y(S^\phi_Y(vW_0, vW_1), hW_2) \right) + O(x).
\]

(3) For \( N = N(x) \partial_x \) and \( W_i \) as in part (2),
\[
g_w(\nabla_N(R_w(N, W_0))h_iW_1, h_iW_2) = (N(x))^2 g_{2Y}(R_{2Y}(hW_2, hW_1), vW_0) + O(x).
\]

Proof. If a connection asymptotically preserves a splitting of the bundle, then its curvature evaluated in vector fields tangent to the boundary hypersurface will also asymptotically preserve that splitting. By (1.9) and (1.12), the \( g_w \) connection preserves the splitting, so statement (1) above is correct.

Now consider \( W_0 \) tangent to the fibers of \( \phi_Y \) and \( W_1, W_2 \in C^\infty(X; w^*TX) \). Since \( R_w \) is a tensor, its value at \( \partial X \) only depends on the values of the vector fields at the boundary, so if \( \tilde{W}_0 \) agrees with \( W_0 \) at \( x = 0 \) as elements of \( C^\infty(X; TX) \), and \( \tilde{W}_1, \tilde{W}_2 \) agree with \( W_1, W_2 \) at \( x = 0 \) as elements of \( C^\infty(X; w^*TX) \), then
\[
g_w(R_w(N, W_0)W_1, W_2) = g_w(R_w(N, \tilde{W}_0)\tilde{W}_1, \tilde{W}_2) + O(x).
\]

Thus we may assume that \( W_0 = V_0 \in \{ \xi_\alpha \} \) (since \( R_w(N, x\partial_x) \) and \( R_w(N, x\tilde{U}) \) are \( O(x) \) and \( W_1, W_2 \in \{ \partial_x, 1/2 V_1, \tilde{U}_1 \} \) extended to \( C(\mathfrak{B}_Y) \) as above. The advantage being that \( \nabla_{\partial_x} W_i \in xC^\infty(X; w^*TX) \).

Thus consider \( N \in \{ \partial_x \} \), \( N = N(x) \partial_x \), and \( W_0 = vW_0 \) vertical. Using \( \partial_x, W_0 = 0 \),
\[
g_w(R_w(N, vW_0)W_1, W_2) = N(x) \partial_x(g_w(\nabla_{vW_0}W_1, W_2)) + O(x).
\]

If \( W_1 = vW_1, W_2 = vW_2 \), then we get the first term in part (2). If either \( W_1, W_2 = \partial_x \), or
\[
g_w(-R_w(N, vW_0)vW_1, vW_2) = -N(x)\phi^*_Y g_Y(S^\phi_Y(vW_0, vW_1), vW_2) + O(x).
\]

Next consider \( g_w(\nabla_N(R_w(N, W_0))h_iW_1, h_iW_2) \). Here \( R_w(N, W_0) \) is a section of \( \text{hom}(w^*TX) \) and correspondingly
\[
\nabla_N(R_w(N, W_0))h_iW_1 = \nabla_N(R_w(N, W_0)h_iW_1) - R_w(N, W_0)\nabla_N(h_iW_1).
\]

The second term in \( O(x) \), so we have
\[
g_w(\nabla_N(R_w(N, W_0))h_iW_1, h_iW_2) = N(g_w(R_w(N, W_0)h_iW_1, h_iW_2)) + O(x).
\]

Since \( W_0 \) is an edge vector field we may write it as \( x(h_+W_0) + vW_0 \). For the first summand we have
\[
N(g_w(R_w(N, xh_+W_0)h_+W_1, h_+W_2)) = N(x)g_w(R_w(N, h_+W_0)h_+W_1, h_+W_2) + O(x).
\]

For the second summand we have
\[
N(g_w(R_w(N, vW_0)h_+W_1, h_+W_2)) = Ng_w(-R_w(vW_0, h_+W_1)N - R_w(h_+W_1, N)vW_0, h_+W_2) = N \left( R_w(N, h_+W_1) + g_w(R_w(N, h_+W_1)vW_0, h_+W_2) \right) = (N(x) \left( -g_w(R_w(N, h_+W_1)1/2 vW_0, h_+W_1) + g_w(R_w(N, h_+W_1)1/2 vW_0, h_+W_2) \right))
\]
Now
\[ g_w(R_w(N, h_+ W_2) \frac{1}{2} v W_0, h_+ W_1) = N(g_w(\nabla_{h_+ W_2} \frac{1}{2} v W_0, h_+ W_1)) \]
and so altogether
\[ g_w((\nabla_N R_w)(N, v W_0) h_+ W_1, h_+ W_2) = \frac{1}{2} (N x)^2 (g_{\mathcal{B}_Y/Y}(\mathcal{R}^{\phi_Y}(h W_2, h W_1), v W_0) - g_{\mathcal{B}_Y/Y}(\mathcal{R}^{\phi_Y}(h W_1, h W_2), v W_0)) + \mathcal{O}(x) \]
\[ = (N x)^2 g_{\mathcal{B}_Y/Y}(\mathcal{R}^{\phi_Y}(h W_2, h W_1), v W_0) + \mathcal{O}(x). \]

This establishes the asymptotics of the curvature at each boundary hypersurface. Let us consider the implications at a corner. Suppose \( Y, \tilde{Y} \in \mathcal{S}(X) \) and \( Y < \tilde{Y} \) so that the boundary fiber bundles participate in (1.1). Near \( \mathcal{B}_Y \cap \mathcal{B}_{\tilde{Y}} \) a ‘boundary product structure’ as in (1.1) yields a collar neighborhood of the form
\[ [0, 1)_x \times [0, 1)_r \times (\mathcal{B}_Y \cap \mathcal{B}_{\tilde{Y}}) \]
with \( x \) a boundary defining function for \( \mathcal{B}_Y \) and \( r \) a boundary defining function for \( \mathcal{B}_{\tilde{Y}} \). In this collar, the vector field \( \partial_x \) is \( \phi_{\tilde{Y}} \)-horizontal, the vector field \( \frac{1}{2} \partial_r \) is \( \phi_Y \)-vertical, and any wedge vector field that is vertical with respect to \( \phi_{\tilde{Y}} \) is also vertical with respect to \( \phi_Y \). We will eventually carry out a Getzler rescaling argument where we rescale in the horizontal directions at each boundary hypersurface, so the interesting expressions at the corner are the ones of the form
\[ g_w(R_w(\partial_r, W_0)v_{\tilde{Y}} W_1, \partial_x), \quad g_w(\nabla_{\partial_r} R_w(\partial_r, W_0) h_{\tilde{Y}} W_1, \partial_x), \]
with \( W_0 \) a vector field tangent to the fibers of \( \phi_{\tilde{Y}} \) and (hence) \( \phi_Y \) and \( W_1 \) a wedge vector field. The first expression is equal to
\[ g_w(R_w(v_{\tilde{Y}} W_1, \partial_x) \partial_r, W_0) = x g_w(R_w(x r v_{\tilde{Y}} W_1, \partial_x) \frac{1}{2} \partial_r, \frac{1}{x r} W_0) = \mathcal{O}(x), \]
while the second expression has leading term at the corner involving
\[ \mathcal{R}^{\phi_{\tilde{Y}}}(h_{\tilde{Y}} W_1, \partial_x) = v_{\tilde{Y}}([h_{\tilde{Y}} W_1, \partial_x]) = 0. \]
The upshot is the vanishing at the corner of every term in these asymptotics in which \( \partial_x \) occurs as a horizontal vector field.

1.3. **Dirac-type wedge operators.** Let \( X \xrightarrow{\psi} B \) be a fiber bundle of manifolds with corners and iterated fibration structures in the sense of Definition 1.3 and fix a choice of splitting as in (1.6).

**Definition 1.6.** Let \( g_{M/B} \) be a totally geodesic vertical wedge metric on \( M \). A **wedge Clifford module** along the fibers of \( \psi \) consists of

1. a complex vector bundle \( E \rightarrow M \)
2. a Hermitian bundle metric \( g_E \)
3. a connection \( \nabla^E \) on \( E \) compatible with \( g_E \)
4. a bundle homomorphism from the ‘vertical wedge Clifford algebra’ into the endomorphism bundle of \( E \),
\[ \mathcal{C} : \mathcal{C}_w(M/B) = \mathbb{C} \otimes \mathcal{C}(^{*}T^*M/B, g_{M/B}) \rightarrow \text{End}(E) \]
compatible with the metric and connection in that, for all \( \theta \in \mathcal{C}^\infty(M;^{*}T^*M/B) \),
\[ g_E(\mathcal{C}(\theta)\cdot, \cdot) = -g_E(\cdot, \mathcal{C}(\theta)\cdot) \].
• \( \nabla^E_W \alpha(\theta) = \alpha(\theta) \nabla^E_W + \alpha(\nabla^M/B \theta) \) as endomorphisms of \( E \), for all \( W \in T M \).

This information determines a smooth family of wedge Dirac-type operators by

\[
D_{M/B} : C_c^\infty(M^\circ; E) \xrightarrow{\nabla^E} C_c^\infty(M^\circ; T^* M \otimes E) \xrightarrow{\alpha} C_c^\infty(M^\circ; E)
\]

where we have used that \( T^* M \) and \( {}^wT^* M \) are canonically isomorphic over the interior of \( M \).

If the fibers of \( \psi \) are even dimensional we will want for \( E \) to admit a \( \mathbb{Z}_2 \) grading

\[
E = E^+ \oplus E^-
\]

compatible in that it is orthogonal with respect to \( g_E \), parallel with respect to \( \nabla^E \), and odd with respect to \( \alpha \).

In local coordinates, we can write

\[
D_{M/B} = \sum_{i=1}^n \alpha(\theta^i) \nabla^E_{(\theta^i)x}
\]

where \( \theta^i \) runs over a \( g_w \)-orthonormal frame of \( T^* M/B \). If we restrict to a fiber \( X \) of \( \psi \) and then further to a collar neighborhood of \( \mathcal{B}_Y, Y \in S(X) \), this takes the form

\[
(1.15) \quad \alpha(dx) \nabla^E_{\partial_x} + \alpha(x \, dz) \nabla^E_{\frac{1}{x} \partial_z} + \alpha(dy^j) \nabla^E_{\partial_y^j} = \alpha(dx) \nabla^E_{\partial_x} + \alpha(x \, dz) \frac{1}{x} \nabla^E_{\partial_x} + \alpha(dy^j) \nabla^E_{\partial_y^j}
\]

plus an error in \( \text{Diff}_1^\circ(X; E) \). Here \( x \) is a boundary defining function for \( \mathcal{B}_Y \), and we recognize \( (1.15) \) as a wedge differential operator of order one.

We are interested in this operator acting on the natural family of vertical \( L^2 \) spaces associated to the wedge metric \( g_w \) and the Hermitian metric \( g_E \), which we denote \( L^2_w(M/B; E) \). However it is convenient to work with \( L^2 \) spaces with respect to a non-degenerate density, so let us define a multiweight on \( M \),

\[
b(H) = \frac{1}{2} (\text{dim } \mathcal{B}_N/N) \text{ for all } H \subseteq \mathcal{B}_N \text{ and } N \in \mathcal{S}_\psi(M)
\]

so that

\[
(1.17) \quad L^2_w(M/B; E) = \rho_M \rho^b_M L^2(M/B; E).
\]

(On each fiber \( X \) of \( \psi \) we have \( b(H) = \frac{1}{2} (\text{dim } \mathcal{B}_Y/Y) \) for all \( H \subseteq \mathcal{B}_Y \) and \( Y \in S(X) \).)

Define the unitarily equivalent family of operators \( \partial_{M/B} \) by

\[
(1.18) \quad \partial_{M/B} = \rho_M D_{M/B} \rho_M = D_{M/B} - \sum_{N \in \mathcal{S}_\psi(M)} \left( \frac{\text{dim } \mathcal{B}_N/N}{2 \rho_N} \right) \alpha(d\rho_N)
\]

Then \( \partial_{M/B} \) is also a vertical family of wedge differential operators of order one, and studying \( \partial_{M/B} \) as operators on \( L^2(M/B; E) \) is equivalent to studying \( D_{M/B} \) as a family of operators on \( L^2_w(M/B; E) \).

1.4. Bismut superconnection. We briefly recall the construction of the Bismut superconnection and refer to [BGV04 Chapter 9-10], [MP97a], [AR09a] for more details.

Let \( M \xrightarrow{\psi} B \) be a family of manifolds with corners and iterated fibration structures as in Definition 1.3, endowed with a splitting

\[
{}^w(T M) = wTM/B \oplus \psi^*TB
\]
as in (1.6) and a vertical wedge metric \( g_{M/B} \). Denote the projections onto the summand of the splitting by \( v_\psi \) (left) and \( h_\psi \) (right). These data determine a connection on the bundle \( wTM/B, \nabla^{M/B} \), as follows. We choose a Riemannian metric \( g_B \) on \( B \), and obtain

\[
g_M = g_{M/B} \oplus \psi^*g_B,
\]

a wedge metric on \( M \). As in §1.2 the Koszul formula defines a Levi-Civita connection on \( w^{(\psi)}TM \) which we denote \( \nabla^M \) and use to define

\[
\nabla^{M/B} = v_\psi \nabla^M v_\psi.
\]

Just as for families of closed manifolds, this defines a connection on \( wTM/B \) that is independent of the choice of the metric \( g_B \).

We embed \( g_M \) in a one-parameter family of wedge metrics on \( M \),

\[
g_{M,\epsilon} = g_{M/B} + \frac{1}{\epsilon} \psi^*g_B,
\]

limiting to the degenerate metric on \( w^{(\psi)}TM \),

\[
g_{M,0}(V,W) = g_{M/B}(v_\psi V, v_\psi W),
\]

in that the dual metrics on the wedge cotangent bundle converge.

Consider the connection

\[
\nabla^\oplus = v_\psi \nabla^M v_\psi \oplus h_\psi \nabla^M h_\psi.
\]

Following Bismut, we can describe the difference between this connection and \( \nabla^M \) in terms of the fundamental tensors of \( \psi \).

Define

\[
\begin{align*}
S^\psi & \in C^\infty(M; w^{*}T^*M/B \otimes w^{*}T^*M/B \otimes \psi^*TB), \\
\hat{R}^\psi & \in C^\infty(M; \psi^*TB \otimes \psi^*TB \otimes w^{*}T^*M/B)
\end{align*}
\]

by the equations

\[
\begin{align*}
S^\psi(W_1,W_2)(A) &= g_{M/B}(\nabla_{A/B}W_1 - [A,W_1],W_2) \\
\hat{R}^\psi(A_1,A_2)(W) &= g_{M/B}([A_1,A_2],W).
\end{align*}
\]

Using the splitting, we extend these trivially to \( \otimes^3(w^{(\psi)}TM) \), and then define

\[
\omega^\psi \in C^\infty(M; w^{(\psi)}T^*M \otimes \Lambda^2(w^{(\psi)}T^*M)),
\]

\[
\begin{align*}
\omega^\psi(X)(Y,Z) &= S^\psi(X,Z)(Y) - S^\psi(X,Y)(Z) \\
&+ \frac{1}{2} \hat{R}^\psi(X,Z)(Y) - \frac{1}{2} \hat{R}^\psi(X,Y)(Z) + \frac{1}{2} \hat{R}^\psi(Y,Z)(X).
\end{align*}
\]

This tensor is isomorphic to \( \nabla^M - \nabla^\oplus \) via \( g_M \) by [BGV04, Prop 10.6] and allows us to define

\[
\nabla^0 = \nabla^\oplus + \frac{1}{2} \tau(\omega^\psi)
\]

where

\[
\tau : \Lambda^2(w^{(\psi)}T^*M) \longrightarrow \text{hom}(w^{(\psi)}T^*M), \quad \tau(\alpha \wedge \beta)\delta = 2(g_{M,0}(\alpha, \delta)\beta - g_{M,0}(\beta, \delta)\alpha).
\]

As the notation is meant to indicate, this is the limiting connection of the Levi-Civita connections of the metrics \( g_{M,\epsilon} \) as \( \epsilon \to 0 \).

Given a wedge Clifford module \( E \longrightarrow M \), we can extend it to the bundle

\[
E = \psi^*\Lambda^*B \otimes E.
\]
This has a natural Clifford action
\[\alpha_0 : \mathfrak{Cl}_0(\wedge^i T^* M) = \mathbb{C} \otimes \mathfrak{Cl}(\psi^* T^* B \oplus \wedge^i T^* M/B, g_{M,0}) \longrightarrow \text{End}(\mathbb{E})\]
\[\alpha_0(\alpha) = \epsilon(h_\psi \alpha) + \alpha(v_\psi \alpha) \text{ for } \alpha \in C^\infty(M; \wedge^i T^* M)\]
where \(\epsilon\) is exterior multiplication.

The square of the wedge Bismut superconnection satisfies a Lichnerowicz with the Clifford action and is known as the ‘twisting curvature’ of \(\nabla\).

\[\nabla^{E,0} = \psi^* \nabla^B \oplus \nabla^E + \frac{1}{2} \alpha_0(\omega^\psi)\]
that is compatible with \(\nabla^0\) in that
\[\nabla^{E,0}_W \alpha_0(\theta) = \alpha_0(\theta) \nabla^{E,0}_W + \alpha_0(\nabla^0_W \theta), \text{ for all } W \in C^\infty(M; TM)\].

We will need to know about the curvature of \(\nabla^{E,0}\). It is easy to see that, for any \(U, V \in C^\infty(M; TM)\),
\[v_\psi R^{M,\varepsilon}(U, V) = R^{M/B}(U, V), \quad h_\psi R^{M,\varepsilon}(U, V) h_\psi = \psi^* R^B(U, V)\].

It follows from Proposition 10.9 of [BGV04] that
\[R^{M,0}(T_1, T_2)(T_3, T_4) = \lim_{\varepsilon \to 0} R^{M,\varepsilon}(T_1, T_2)(T_3, T_4)\]
as long as \(h_\psi T_i = 0\) for some \(i\) (which will hold in all cases we need to consider). In particular, applying Proposition 1.5 to \(R^{M/B,\varepsilon}\) lets us conclude that the corresponding asymptotics hold for \(R^{M,0}\).

The Dirac-type operator corresponding to \(\mathbb{E}\) with Clifford action \(\alpha_0\) and connection \(\nabla^{E,0}\) is known as the Bismut superconnection and denoted \(A_{M/B}\). As a map \(C^\infty(M; E) \longrightarrow C^\infty(M; \mathbb{E})\) it is given by
\[A_{M/B} = A_{M/B, [0]} + A_{M/B, [1]} + A_{M/B, [2]}\]
where \(A_{\psi, [j]} : C^\infty(M; E) \longrightarrow C^\infty(M; \psi^* \Lambda^j B \otimes E)\). Explicitly, in terms of local orthonormal frames \(\{f_\alpha\}\) for \(TB\) and \(\{e_i\}\) for \(\wedge^i TM/B\), with dual coframes \(\{f^\alpha\}\), \(\{e^i\}\), we have
\[A_{M/B} = \alpha(e^i) \nabla^E_{e^i} + \alpha^a (\nabla^E_{f_\alpha} + \frac{1}{2} k_\psi(f_\alpha)) - \frac{1}{4} \sum_{\alpha < \beta} \tilde{R}^\psi(f_\alpha, f_\beta)(e_\iota) e^\alpha e^\beta \alpha(e^i),\]
where \(k_\psi\) is the trace of \(S^\psi\). Note that \(A_{M/B, [0]}\) is \(D_{M/B}\), the Dirac-type operator associated to \(E\).

As \(E\) is a wedge Clifford bundle, its homomorphism bundle has a decomposition
\[\text{hom}(E) \cong \mathfrak{Cl}(\wedge^i T^* M/B) \otimes \text{hom}_{\mathfrak{Cl}(\wedge^i T^* M/B)}(E)\]
where \(\text{hom}_{\mathfrak{Cl}(\wedge^i T^* M/B)}(E)\) denotes homomorphisms that commute with \(\mathfrak{Cl}(\wedge^i T^* M/B)\), see e.g., [Vai01, Lemma 5.1]. The curvature of \(\nabla^E\) decomposes as \(\frac{1}{4} \alpha'(R^M) + K_E^0\), where \(K_E^0\) commutes with the Clifford action and is known as the ‘twisting curvature’ of \(\nabla^E\), see e.g., [Mel93, Lemma 8.33]. The square of the wedge Bismut superconnection satisfies a Lichnerowicz formula [BGV04, Theorem 3.52],
\[A_{M/B}^2 = \Delta^{M/B,0} + \frac{1}{4} \text{scal}(g_w) - \frac{1}{2} \sum_{a,b} K_E^0(e_a, e_b) \alpha_0(e^a) \alpha_0(e^b)\]
where the sum is over both horizontal and vertical tangent vectors and \(\Delta^{M/B,0}\) is the vertical family of operators which at \(M_b\) is the Bochner Laplacian corresponding to \(\nabla^{E,0}|_{M_b}\).
For each $N \in \mathcal{S}_\psi(M)$, we have three related fiber bundles at the corresponding collective boundary hypersurface

$$
\begin{array}{ccc}
Z_Y & \longrightarrow & B_N \\
\phi_N \downarrow & & \psi|_{B_N} \\
Y \longrightarrow & N & \psi_N \downarrow \\
\psi_N \downarrow & & B
\end{array}
$$

and from the asymptotics of wedge connections we see that

$$
(1.23) \quad S^\psi|_{B_N} = S^{\psi N}, \quad \hat{R}^\psi|_{B_N} = \hat{R}^{\psi N}.
$$

We will see that the contribution of $\phi_N$ to these tensors can be recovered by passing to ‘rescaled normal operators’.

2. Witt condition

2.1. Boundary families. Let $M \xrightarrow{\psi} B$ be a fiber bundle of manifolds with corners and iterated fibration structures and $g_{M/B}$ a totally geodesic vertical wedge metric on $M$. Given a $\psi$-vertical wedge Clifford module over $M$, $E \hookrightarrow M$, we will explain how at each $N \in \mathcal{S}_\psi(M)$ there is an induced $\phi_N$-vertical wedge Clifford module on $B_N$.

This has a Clifford action not just by $\omega^*_{w^*B_N/N}$ but by all of $\omega^*_{w^*M/B|_{B_N}}$; we encode this as a $\mathbb{C}l(F)$-wedge Clifford bundle for an appropriate bundle $F \rightarrow B$. We refer to the corresponding family of vertical Dirac-type operators as the boundary family of $D_{M/B}$ at $N$ and denote it $D_{B_N/N}$.

Let $N \in \mathcal{S}_\psi(M)$ with corresponding collective boundary hypersurface $\mathcal{B}_N$ and fix a choice of boundary defining function $x$. Choose a collar neighborhood $C(B_N) \cong [0,1) \times \mathcal{B}_N$ and identify

$$
{w^*T^*M/B}|_{\mathcal{B}_N} = N^*_M\mathcal{B}_N \oplus x{w^*T^*\mathcal{B}_N/N} \oplus \phi^*_N T^*N,
$$

where $N^*_M\mathcal{B}_N$ is the (rank one) conormal bundle of $\mathcal{B}_N$, and then further identify

$$
N^*_M\mathcal{B}_N = \langle dx \rangle, \quad x{w^*T^*\mathcal{B}_N/N} \cong {w^*T^*\mathcal{B}_N/N}.
$$

The $\psi$-vertical wedge metrics $g_{M/B}$ on ${w^*T^*M/B}$ induce a $\phi_N$-vertical wedge metric $g_{\mathcal{B}_N/N}$ on ${w^*T^*\mathcal{B}_N/N}$. Choose a metric $g_B$ on $B$ and let $g_M = g_{M/B} \oplus \psi^*g_B$, so that $\nabla_{M/B}$ is given, as above, in terms of the Levi-Civita connection $\nabla^M$ of $g_M$ by

$$
\nabla_{M/B} = v_\psi \circ \nabla^M \circ v_\psi.
$$

The metric $g_M$ is a totally geodesic wedge metric on $M$ and thus in particular there is a corresponding wedge metric on $\mathcal{B}_N$, with vertical connection

$$
\nabla^{\mathcal{B}_N/N} = v_{\phi_N} \circ \nabla^{\mathcal{B}_N} \circ v_{\phi_N}.
$$

In order to relate $\nabla^{\mathcal{B}_N/N}$ with the restriction of $\nabla_{M/B}$ to $\mathcal{B}_N$, recall from §1.2 that the restriction of $\nabla^M$ to $\mathcal{B}_N$ will respect the splitting

$$
(2.1) \quad w^*\mathcal{E}(\mathcal{B}_N) = [\langle \partial_x \rangle \oplus x^*{w^*\mathcal{B}_N/N}] \oplus \phi_N^*T^*N.
$$
so that \( j_0^\ast \nabla^M = \mathbf{v}^+_{\phi_N} \circ \nabla^M \circ \mathbf{v}^+_{\phi_N} \oplus \mathbf{h}_{\phi_N} \circ \nabla^M \circ \mathbf{h}_{\phi_N} \). Let us denote the fully diagonal connection by

\[
\nabla^\oplus = \frac{\partial}{\partial x} dx \oplus \nabla^\mathfrak{B}_{N/N} \oplus \mathbf{h}_{\phi_N} \nabla^M \mathbf{h}_{\phi_N}.
\]

The difference between this and a direct sum connection with respect to the splitting \( (2.1) \) comes from the fact that in a frame like \( (1.11) \), letting the connection act on differential forms,

\[
\nabla^M_{V_1} (\frac{1}{x} V_2)^\flat (\partial_x) = -g_{\mathfrak{B}_{N/N}} (V_1, V_2), \quad \nabla^M_{\frac{1}{x} V_1} dx = (\frac{1}{x} V_1)^\flat.
\]

Thus we have

\[
\nabla^\oplus_{V_1} \theta = j_0^\ast \nabla^M_{V_1} \theta - (g(dx, \theta)(\frac{1}{x} v W)^\flat - g((\frac{1}{x} v W)^\flat, \theta) dx).
\]

The Clifford connection \( \nabla^E \) which is by definition compatible with \( \nabla^M \) can be modified in a standard way to obtain a Clifford connection compatible with \( \nabla^\oplus \), namely, writing \( S = \nabla^M - \nabla^\oplus \), following [BC90a] page 375,

\[
\nabla^E = \nabla^M - \frac{1}{4} (S(W)e_i, e_j) g_M d\mathbf{e}(e^i) d\mathbf{e}(e^j)
\]

for orthonormal frame \( e_i \) and dual frame \( e^i \). (Indeed, from the fact that \( S(W) \) is anti-symmetric, it follows that \( \frac{1}{4} (S(W)e_i, e_j) g_M d\mathbf{e}(e^i) d\mathbf{e}(e^j)X = d\mathbf{e}(S(X)). \) Restricting to \( \mathfrak{B}_N \) via \( j_0 \) from \( (1.14) \), we let \( \nabla^E \mid \mathfrak{B}_N \) be given by

\[
\nabla^E \mid \mathfrak{B}_N = j_0^\ast \nabla^M \mid \mathfrak{B}_N - \frac{1}{2} d(dx) d((\frac{1}{x} v W)^\flat)
\]

and this is compatible with the restriction of \( \nabla^\oplus \) to \( \mathfrak{B}_N \).

After identifying \( x^w T^w \mathfrak{B}_N \mid \mathfrak{B}_N \) with \( T^w \mathfrak{B}_N \mid \mathfrak{B}_N \) we see that \( \nabla^E \mid \mathfrak{B}_N \) is a metric connection with respect to the restriction of \( g_E \) which restricts to the fibers of \( \mathfrak{B}_N \mid \mathfrak{B}_N \) to be compatible with \( g_{\mathfrak{B}_N/N} \). Thus altogether we obtain a wedge Clifford module for the induced vertical wedge metric \( g_{\mathfrak{B}_N/N} \) on \( \mathfrak{B}_N \rightarrow N \) that moreover is compatible with the Clifford action of the base and normal covectors i.e. the Clifford action by sections of the bundle \( (dx) \oplus \phi_N^w T^* N/B \).

As the bundle \( N_M^* \mathfrak{B}_N = \langle dx \rangle \rightarrow \mathfrak{B}_N \) is trivial there is no loss, and some convenience, in treating it as the pull-back of a trivial bundle over \( N \). We introduce the notation \( T^w N^+ / B \) for the direct sum of \( T^w N / B \) and a trivial bundle over \( N \) formally generated by \( dx \), so that \( N_M^* \mathfrak{B}_N \oplus \phi_N^w T^* N / B = \phi_N^w T^* N^+ / B \).

**Definition 2.1.** Let \( X \rightarrow M \xrightarrow{\psi} B \) be a fiber bundle of manifolds with corners and iterated fibration structures in the sense of Definition \( 1.3 \) with a vertical wedge metric \( g_{M/B} \). If \( F \rightarrow B \) is a real vector bundle with bundle metric \( g_F \), a wedge Clifford module \( (E, g_E, \nabla^E, d\mathbf{e}) \) is a \( \Cl(F) \)-wedge Clifford module if there is a bundle homomorphism (also denoted \( d\mathbf{e} \))

\[
d\mathbf{e} : \Cl(\psi^* F) = \mathbb{C} \otimes \Cl(\psi^* F) \rightarrow \text{End}(E)
\]

satisfying, for all \( \eta \in \mathcal{C}^\infty(M; \psi^* F) \),

\[
g_E(d\mathbf{e}(\eta)s_1, s_2) = -g_E(s_1, d\mathbf{e}(\eta)s_2) \quad \text{for all } s_i \in \mathcal{C}^\infty(M; E)
\]

\[
\nabla^E(d\mathbf{e}(\eta)s) = d\mathbf{e}(\eta) \nabla^E s, \quad \text{for all } s \in \mathcal{C}^\infty(M; E)
\]

\[
d\mathbf{e}(\eta)d\mathbf{e}(\theta) + d\mathbf{e}(\theta)d\mathbf{e}(\eta) = 0 \quad \text{for all } \theta \in \mathcal{C}^\infty(M; \psi^* T^* M / B). \]
Note that if $F$ has rank zero, a $\mathbb{C}l(F)$-wedge Clifford module is just a wedge Clifford module.

Clearly if $D$ is the Dirac-type operator corresponding to (the underlying wedge Clifford module $(E, g_E, \nabla^E, \alpha)$ of) a $\mathbb{C}l(F)$-wedge module, and $\theta \in \mathcal{C}^\infty(M; F)$ then $D\alpha(\theta) = -\alpha(\theta)D$.

We have seen that a wedge Clifford bundle $(E, g_E, \nabla^E, \alpha)$ along the fibers of $M \xrightarrow{\psi} B$ induces a $\mathbb{C}l(T^*N^+ / B)$-wedge Clifford module, $(E|_N, g_E|_N, \nabla^E|_N, \alpha)$ along the fibers of $\mathcal{B}_N \xrightarrow{\phi_N} N$. Let us finally consider the relation between the corresponding Dirac-type operators.

To each $N \in \mathcal{S}_\psi(M)$ we can associate a $\phi_N$-vertical family of operators

$$\rho_N D_{M/B}|_{\mathcal{B}_N}.$$ 

From the local expression (1.15) we see, letting $V_i$ be an orthonormal frame of vertical vectors and $V^i$ the dual frame, that

$$\rho_N D_{M/B}|_{\mathcal{B}_N} = \alpha(x V^i) J^0 \nabla^E V_i$$

where $V_i$ runs over a local frame for the vertical wedge tangent bundle associated to $\mathcal{B}_N \xrightarrow{\phi_N} N$. Replacing the connection $J^0 \nabla^E$ by $\nabla^E|_N$ yields

$$\alpha(x dz^i)(\nabla^E|_N + \frac{1}{2} \alpha(dx) \alpha(x dz^i)) = D_{\mathcal{B}_N/N} + \frac{v}{2} \alpha(dx)$$

where we recall that $v = \dim \mathcal{B}_N/N$. Thus we can conclude:

**Lemma 2.2.** A wedge Clifford module along the fibers of $\psi : M \rightarrow B$ induces, for each $N \in \mathcal{S}(N)$, a $\mathbb{C}l(T^*N^+ / B)$-wedge Clifford module along the fibers of $\phi_N : \mathcal{B}_N \rightarrow N$. The vertical operator of a family of wedge Dirac-type operators $D_{M/B}$ defined by the former is equal to the family of wedge Dirac-type operators defined by the latter, which we denote $D_{\mathcal{B}_N/N}$, plus a zero-th order term

$$\rho_N D_{M/B}|_{\mathcal{B}_N} = D_{\mathcal{B}_N/N} + \frac{v}{2} \alpha(dx).$$

We will denote the restriction of $D_{\mathcal{B}_N/N}$ to the fiber over $y \in N$ by

$$D_{\mathcal{B}_N/N}|_{Z_y} = D_{Z_y}$$

when it is clear from context. Note that Clifford multiplication by the global section $dx$ of $T^*N^+$ satisfies

$$D_{\mathcal{B}_N/N} \circ \alpha(dx) = -\alpha(dx) \circ D_{\mathcal{B}_N/N}.$$

Hence for any choice of closed domain for $D_{Z_y}$, invariant under multiplication by $\alpha(dx)$, we have

$$s \in \lambda\text{-eigenspace of } D_{Z_y} \iff \alpha(dx)s \in (-\lambda)\text{-eigenspace of } D_{Z_y}.$$ 

and so the eigenvalues of $D_{Z_y}, -D_{Z_y}$, and $\alpha(dx)D_{Z_y}$ coincide, including multiplicity.

**Lemma 2.2** highlights one advantage of working with $\bar{\alpha}_{M/B}$ from (1.18) since the induced vertical family is precisely the boundary family of Dirac operators,

$$\rho_N \bar{\alpha}_{M/B}|_{\mathcal{B}_N} = D_{\mathcal{B}_N/N}.$$
Note that since we are interested in $\partial_{M/B}$ on the space $L^2(M/B; E)$, we are interested in the boundary family $D_{\mathcal{B}_N/N}$ as an operator on $L^2(\mathcal{B}_N/N; E_N)$.

### 2.2. Witt assumption and vertical APS domain.

Let $\psi: M \longrightarrow B$ be a fibration of manifolds with corners with iterated fibration structure as in Definition 1.3 (so $B$ is closed) with typical fiber $X$, let $g_{M/B}$ be a vertical wedge metric, and $E \longrightarrow M$ a wedge Clifford module as in Definition 1.6. Let $D_{M/B}$ be the corresponding family of vertical wedge Dirac-type operators and $D_X$ the restriction of $D_{M/B}$ to a fixed fiber $X$. As an unbounded operator on $L^2_w(X; E)$, for an arbitrary totally geodesic wedge metric $g_{M/B}|_X$, $D_X$ generally has many closed extensions. As discussed in, e.g., [ALMP13], the two canonical closed domains,

\[
\mathcal{D}_{\text{min}}(D_X) = \{ u \in L^2_w(X; E) : \exists (u_n) \subseteq C^\infty_c(X; E) \text{ s.t. } u_n \rightarrow u \text{ and } (D_X u_n) \text{ is } L^2_w\text{-Cauchy} \},
\]

\[
\mathcal{D}_{\text{max}}(D_X) = \{ u \in L^2_w(X; E) : D_X u \in L^2_w(X; E) \},
\]

where in the latter $D_X u$ is computed distributionally, satisfy

\[
\rho_X H^1_e(X; E) \subseteq \mathcal{D}_{\text{min}}(D_X) \subseteq \mathcal{D}_{\text{max}}(D_X) \subseteq H^1_e(X; E).
\]

Here

\[
H^1_e(X; E) = \{ u \in L^2(X; E) : Vu \in L^2(X; E) \text{ for all } V \in C^\infty(X; \mathcal{E}TX) \}
\]

is the edge Sobolev space introduced in [Maz91]. We consider the following domain:

**Definition 2.3.** The **vertical APS domain** of $D_X$ is the graph closure of $\rho_X^{1/2}H^1_e(X; E) \cap \mathcal{D}_{\text{max}}(D_X)$,

\[
\mathcal{D}_{\text{VAPS}}(D_X) = \{ u \in L^2_w(X; E) : \exists (u_n) \subseteq \rho_X^{1/2}H^1_e(X; E) \cap \mathcal{D}_{\text{max}}(D_X) \text{ s.t. } u_n \rightarrow u \text{ and } (D_X u_n) \text{ is } L^2_w\text{-Cauchy} \}.
\]

As in [ALMP13], this domain induces a domain for each vertical family $D_{\mathcal{B}_Y/Y}$, namely the corresponding vertical APS domain. Note that this domain is invariant under multiplication by $\mathcal{G}(dx)$ so that the spectrum of each $D_{Z_y}$ has the symmetries mentioned at the end of §2.1.

**Definition 2.4.** The operator $(D_X, \mathcal{D}_{\text{VAPS}})$ satisfies the **geometric Witt condition** if

\[
Y \in \mathcal{S}(X), y \in Y \implies \text{Spec}(D_{Z_y}) \cap (-\frac{1}{2}, \frac{1}{2}) = \emptyset.
\]

If, instead, we only require

\[
Y \in \mathcal{S}(X), y \in Y \implies \text{Spec}(D_{Z_y}) \cap \{0\} = \emptyset
\]

then we say that $(D_X, \mathcal{D}_{\text{VAPS}})$ satisfies the **Witt condition.** (Tacitly, we take the domain of the links to be the VAPS domains. Thus the definition could be stated without reference to a domain on the whole of $X$, specifically only with reference to the spectrum of $\text{Spec}(D_{Z_y}, \mathcal{D}_{\text{VAPS}}(D_{Z_y}))$. Since this notation is cumbersome we speak only of the VAPS domain on $X$ and think of the domains on the fibres as induced.)

The analysis in [ALMP13] can be used to show that the geometric Witt condition

\[
Y \in \mathcal{S}(X), y \in Y \implies \text{Spec}(D_{Z_y}) \cap (-\frac{1}{2}, \frac{1}{2}) = \emptyset
\]

implies $\mathcal{D}_{\text{min}}(D_X) = \mathcal{D}_{\text{max}}(D_X)$ so that $D_X$ is essentially self-adjoint.
Remark 2.5. We use the nomenclature ‘vertical APS domain’ because the different local ideal boundary conditions for $D_X$ involve the spectrum of $D_{Z_y}$ in the interval $(-\frac{1}{2}, \frac{1}{2})$. The vertical APS domain corresponds to projecting off of the negative half of this interval, analogous to the Atiyah-Patodi-Singer boundary conditions [APS75].

2.3. Normal operator. As in [27], let $D_X$ be the restriction of $D_{M/B}$ to a fiber of $\psi : M \to B$. At every $Y \in \mathcal{S}(X)$, $y \in Y^\circ$, there is a normal operator of $D_X$, modeling its behavior on a model wedge,

$$\mathbb{R}_s^+ \times \mathbb{R}_u^h \times Z_y,$$

where $h = \dim Y$, acting on sections of the bundle $E|_{Z_y}$ pulled back along the natural projection. This operator is given by

$$N_y(D_X) = \mathcal{D}(dx)\partial_s + \frac{1}{s}(D_{Z_y} + \frac{\nu}{2}\mathcal{D}(dx)) + \sum \mathcal{D}(H_j^\nu)H_j = \mathcal{D}(dx)\partial_s + \frac{1}{s}(D_{Z_y} + \frac{\nu}{2}\mathcal{D}(dx)) + D_{\mathbb{R}_u^h}$$

where the sum ranges over an orthonormal frame for $T_yY$, and throughout this section we use the notation

$$v = \dim Z_y.$$

The normal operator of $\partial_X$ is then

$$N_y(\partial_X) = N_y(D_X) - s^{-\frac{\nu}{2}}\mathcal{D}(dx).$$

The vertical APS domain for $D_X$ induces domains for both $D_{Z_y}$ and $N_y(D_X)$, which are easily identified as the corresponding vertical APS domains. The induced domain for the normal operator can be described in terms of $t$-smooth (i.e., polyhomogeneous) asymptotic expansions.

Indeed, the operator $D_{Z_y}$ is independent of $s$, and so we are in the ‘constant indicial root’ situation studied in, e.g., [ALMP13]. Note that if $f$ is a section of $E|_{Z_y}$ over the model wedge and is $\mathcal{O}(s^0)$ as $s \to 0$ then for almost all $\sigma \in \mathbb{C}$, $sN_y(D_X)(s^\sigma f) = \mathcal{O}(s^\sigma)$. We say that $\sigma$ is an indicial root if there is an $f$ such that $sN_y(D_X)(s^\sigma f) = \mathcal{O}(s^{\sigma - 1})$.

Let us describe the induced domain for the normal operator in more detail. Analogously to [ALMP13, §1], we can consider an intermediate domain where we have imposed the vertical APS ‘boundary condition’ at all of the $\mathcal{B}_Y$, with $Y < Y'$,

$$\mathcal{D}_{\max,Y <}((\partial_X) = \text{graph closure of } \mathcal{D}_{\max}(\partial_X) \cap \left( \prod_{Y < Y'} \rho_{Y'}^{1/2} \right) L^2(X; E).$$

As explained in loc. cit., elements in the induced domain for the normal operator have a partial asymptotic expansion. (In the setting of [ALMP13], under an assumption of constant indicial roots, domains were defined by an inductive process. Elements of the maximal domain have partial asymptotic expansions with distributional coefficients at strata of depth one on which we can impose ideal boundary conditions. Elements of the resulting domain have partial asymptotic expansions with distributional coefficients at strata of depth two and so on.)

Lemma 2.6. The indicial roots of $N_y(D_X)$ are the eigenvalues of $D_{Z_y}$ shifted by $-\frac{\nu}{2}$. Every $f$ in

$$\mathcal{D}_{\max,Y <}(N_y(D_X)) = \text{graph closure of } \mathcal{D}_{\max}(N_y(D_X)) \cap \left( \prod_{Y < Y'} \rho_{Y'}^{1/2} \right) L^2_{\mathbb{R}_s^+ \times \mathbb{R}_u^h \times Z_y; E|_{Z_y}}(\mathbb{R}_s^+ \times \mathbb{R}_u^h \times Z_y; E|_{Z_y}),$$
has a partial asymptotic expansion as $s \to 0$,

$$f \sim \sum_{\lambda \in \text{Spec}(D_{Z_y})} f_\lambda s^{-\frac{u}{2} + \lambda} + \tilde{f}$$

in which each $f_\lambda$ is a distributional element of the $\lambda$ eigenspace of $\alpha(dx)D_{Z_y}$ and $\tilde{f} \in x^1 - \overline{H}^{-1}(\mathbb{R}^+ \times \mathbb{R}_+^h \times Z_y)$. The vertical APS domain of $N_u(D_X)$ consists of those $f$ such that $f_\lambda = 0$ whenever $\lambda \leq 0$.

Proof. Applying $N_u(D_X)$ to $s^\sigma f$ yields

$$sN_u(D_X)(s^\sigma f) = s^\sigma((\sigma + \frac{u}{2})\alpha(dx) + D_{Z_y})f + \mathcal{O}(s^{\sigma+1})$$

so $\sigma$ will be an indicial root when $(\sigma + \frac{u}{2} - \alpha(dx)D_{Z_y})$ is not invertible, i.e.,

$$\sigma \in -\frac{u}{2} + \text{Spec}(\alpha(dx)D_{Z_y}) = -\frac{u}{2} + \text{Spec}(D_{Z_y}).$$

The existence of the asymptotic expansion is established in [ALMP13, Lemma 2.2].

Note that the translation invariance of $N_u(D_X)$ in $\mathbb{R}^h$ allows us to Fourier transform in $u$ and obtain the family of model operators,

$$Y \times \mathbb{R}^h \ni (y, \eta) \mapsto N_u(y, \eta)(D_X) = \alpha(dx)(\partial_s + \frac{u}{2}) + \frac{1}{s}D_{Z_y} + i\alpha(\eta).$$

By a standard computation (cf. [AGR16, Lemma 2.10], [ALMP12, Lemma 5.5], [Les93, Proposition 4.1], [Cho85, Proposition 2.25]) we can establish injectivity and self-adjointness as follows.

**Proposition 2.7.** For each $(y, \eta) \in Y \times \mathbb{R}^h$, the operator $N_{(y, \eta)}(D_X)$, together with the corresponding domain $D_{\max,Y<}(N_{(y, \eta)}(D_X))$, is injective if $\eta = 0$ and otherwise has null space spanned by

$$\bigcup_{\lambda \in \text{Spec}(D_{Z_y}) \cap [0,1/2]} \left\{ s^{1/2}K_{\lambda - 1/2}(s|\eta|)\phi_\lambda - \alpha(\frac{\eta}{|\eta|})s^{1/2}K_{\lambda+1/2}(s|\eta|)\alpha(dx)\phi_\lambda : \alpha(dx)D_{Z_y}\phi_\lambda = \lambda\phi_\lambda \right\}.$$

It follows that, if $D_X$ satisfies the Witt condition at $Y$, $N_y$ is injective and self-adjoint with its vertical APS domain.

Proof. Since $L^2(s^v ds) = s^{-v/2}L^2(ds)$, the operator $N_{(y, \eta)}(D_X)$ acting on $L^2_w$ is equivalent to the operator

$$N_{(y, \eta)}(\overline{\alpha}) = s^{v/2}N_{(y, \eta)}(D_X)s^{-v/2} = \alpha(dx)\partial_s + \frac{1}{s}D_{Z_y} + i\alpha(\eta)$$

acting on $L^2(ds)$.

First note that, with $A_y = \alpha(dx)D_{Z_y}$, we have

$$N_{(y, \eta)}(\overline{\alpha})^2 = -\partial_s^2 + \frac{1}{s^2}(D_{Z_y} - \alpha(dx)D_{Z_y}) + |\eta|^2 = -\partial_s^2 + \frac{1}{s^2}(A_y^2 - A_y) + |\eta|^2.$$

It will be useful to recall that the null space of

$$-\partial_s^2 + \frac{1}{s^2}(\lambda^2 - \lambda) + |\eta|^2 = s^{1/2}(-\partial_s^2 - \frac{1}{s}\partial_s + \frac{1}{s^2}(\lambda - \frac{1}{2})^2 + |\eta|^2)s^{-1/2}$$

$$= -s^{1/2 - 2}(s^2\partial_s^2 + s\partial_s - ((\lambda - \frac{1}{2})^2 + s^2|\eta|^2))s^{-1/2}$$
is spanned by $s^\lambda$ and $s^{1-\lambda}$ if $\eta = 0$, and otherwise by $s^{1/2}I_{\lambda - \frac{1}{2}}(s|\eta|)$ and $s^{1/2}K_{\lambda - \frac{1}{2}}(s|\eta|)$. In view of the asymptotics

$$I_\alpha(z) = O(z^{|\alpha|}) \text{ as } z \to 0, \quad I_\alpha(z) = O\left(\frac{1}{z}\right) e^z \text{ as } z \to \infty$$

$$K_\alpha(z) = O(z^{-|\alpha|}) \text{ as } z \to 0 (\alpha \neq 0), \quad K_\alpha(z) = O\left(\frac{1}{z}\right) e^{-z} \text{ as } z \to \infty,$$

none of these are in $L^2(ds)$ except for $s^{1/2}K_{\lambda - \frac{1}{2}}(s|\eta|)$ when $|\lambda - \frac{1}{2}| < 1$.

Now since

$$\varphi(dx)D_{Z_y} = -D_{Z_y}\varphi(dx), \quad \varphi(\eta)D_{Z_y} = -D_{Z_y}\varphi(\eta)$$

$$\implies \varphi(dx)A_y = -\varphi(dx)D_{Z_y}\varphi(dx) = -A_y\varphi(dx), \quad \varphi(\eta)A_y = A_y\varphi(\eta),$$

the operator $\mathcal{N}_{(y,\eta)}(\partial_X) = \varphi(dx)(\partial_s - \frac{1}{s}A_y) + i\varphi(\eta)$ preserves the space $F_\lambda = E_\lambda(A_y) \oplus E_{-\lambda}(A_y)$, on which it acts by

$$\mathcal{N}_{(y,\eta,\lambda)} = \begin{pmatrix}
    i\varphi(\eta) & \varphi(dx)(\partial_s + \frac{1}{s}\lambda)
    \\
    \varphi(dx)(\partial_s - \frac{1}{s}\lambda) & \varphi(\eta)
\end{pmatrix}.$$

If we further identify

$$F_\lambda \longrightarrow E^2_\lambda = E_\lambda(A_y)^2$$

$$(a, b) \longrightarrow (a, \varphi(dx)b)$$

we end up with the map

$$\tilde{\mathcal{N}}_{y,\eta,\lambda} = \begin{pmatrix}
    i\varphi(\eta) & \varphi(dx)(\partial_s + \frac{1}{s}\lambda) \varphi(dx)
    \\
    \varphi(dx)(\partial_s - \frac{1}{s}\lambda) & \varphi(\eta)
\end{pmatrix} = \begin{pmatrix}
    i\varphi(\eta) & \partial_s + \frac{1}{s}\lambda
    \\
    -(\partial_s - \frac{1}{s}\lambda) & \varphi(\eta)
\end{pmatrix}$$

satisfying $\tilde{\mathcal{N}}^2_{y,\eta,\lambda} = \begin{pmatrix}
    |\eta|^2 - \partial_s^2 + \frac{1}{s^2}(\lambda^2 - \lambda)
    \\
    0 & |\eta|^2 - \partial_s^2 + \frac{1}{s^2}(\lambda^2 + \lambda)
\end{pmatrix}$.

Thus any element of the $L^2$-null space of $\tilde{\mathcal{N}}^2_{y,\eta,\lambda}$ has the form

$$(as^{1/2}K_{\lambda - \frac{1}{2}}(s|\eta|), bs^{1/2}K_{\lambda + \frac{1}{2}}(s|\eta|)).$$

Recall that $[AS64]$ (9.6.26]

$$\partial_s K_\nu(z) = -K_{\nu - 1}(z) - \frac{\nu}{z}K_\nu(z) = -K_{\nu + 1}(z) + \frac{\nu}{z}K_\nu(z)$$

hence applying $\tilde{\mathcal{N}}_{y,\eta,\lambda}$ to the putative element of the null space results in

$$\begin{pmatrix}
    (\varphi(\eta)a - b|\eta|)s^{1/2}K_{\lambda - \frac{1}{2}}(s|\eta|),
    (a|\eta| - \varphi(\eta)b)s^{1/2}K_{\lambda + \frac{1}{2}}(s|\eta|)
\end{pmatrix} = 0 \iff \varphi(\eta)a = |\eta|b,$$

and significantly $a = 0 \iff b = 0$. This means that to get an element in $L^2$ we need both $|\lambda - \frac{1}{2}| < 1$ and $|\lambda + \frac{1}{2}| < 1$, i.e., $|\lambda| < \frac{1}{2}$. This establishes the first part of the proposition.

Recall that, for $\nu \in \mathbb{R}^*$, $K_\nu(z) \sim C_\nu z^{-|\nu|}$ as $z \to 0$. Thus the elements of the null space of $\mathcal{N}_y$ are spanned by elements with non-trivial asymptotics at both exponents $\lambda$ and $-\lambda$, for $\lambda \in \text{Spec}(A_y) \cap [0, 1/2]$. There are no such elements in the vertical APS domain.

Finally we show that the vertical APS domain is self-adjoint by showing the vanishing of its deficiency indices by a similar argument. Indeed, the null space of $\mathcal{N}_y \pm i$ is contained
in the null space of $\mathcal{N}_y^2 + 1$. Analyzing this as above shows that solutions are built up from elements of the form
\[
(as^{1/2}K_{\lambda-\frac{1}{2}}(s\langle\eta\rangle), bs^{1/2}K_{\lambda+\frac{1}{2}}(s\langle\eta\rangle)).
\]
where $\langle\eta\rangle = \sqrt{\|\eta\|^2 + 1}$ and solutions coming from the null space of $\mathcal{N}_y \pm i$ further satisfy that $a = 0 \iff b = 0$. Since this requires non-trivial asymptotics at exponents $\lambda$ and $-\lambda$, there are no such solutions in the vertical APS domain.

3. Edge calculus with bounds and wedge heat calculus

3.1. Conormal distributions on manifolds with corners. We briefly recall some of the results of [Mel92] that we will use in our constructions and refer the reader to loc. cit. for details. (See also, e.g., [Gri01], [Maz91, §2A].)

Recall that we use the notation
\[
\rho_X = \prod_{H \in \mathcal{M}_1(X)} \rho_H
\]
for a ‘total boundary defining function’. A multiweight for $X$ is a map
\[
s : \mathcal{M}_1(X) \rightarrow \mathbb{R} \cup \{\infty\}
\]
and we denote the corresponding product of boundary defining functions by
\[
\rho^s_X = \prod_{H \in \mathcal{M}_1(X)} \rho^s_H.
\]
We write $s \leq s'$ if $s(H) \leq s'(H)$ for all $H \in \mathcal{M}_1(X)$.

A smooth map between manifolds with corners $f : X \rightarrow Y$ is a b-map if, for each $H \in \mathcal{M}_1(Y)$, and some choice of boundary defining functions, we have
\[
f^*\rho_H = \prod_{H \in \mathcal{M}_1(X)} e_f(H,G) \rho^G
\]
where $e_f(H,G)$ is a non-negative integer. (These are called ‘interior b-maps’ in [Mel92] because they map the interior of the domain into the interior of the target.) The map
\[
e_f : \mathcal{M}_1(X) \times \mathcal{M}_1(Y) \rightarrow \mathbb{N}_0
\]
is known as the exponent matrix of the b-map $f$ and we write
\[
\ker(e_f) = \{H \in \mathcal{M}_1(X) : e_f(H,G) = 0 \text{ for all } G \in \mathcal{M}_1(Y)\}.
\]
The vector fields tangent to the boundary hypersurfaces of $X$ are known as the $b$-vector fields and are denoted
\[
\mathcal{V}_b = \{V \in C^\infty(X; TX) : V \text{ is tangent to each } H \in \mathcal{M}_1(X)\}.
\]
There is a vector bundle, the $b$-vector bundle, $\mathcal{V}_b$, together with a natural vector bundle map $i_b : \mathcal{V}_b \rightarrow TX$ that is an isomorphism over the interior of $X$ and satisfies
\[
(i_b)_* C^\infty(X; \mathcal{V}_b) = \mathcal{V}_b.
\]
Thus, for example, if $x$ is a boundary defining function for a boundary hypersurface $H$ of $X$ then, near $H$, the vector field $x\partial_x$ is non-degenerate at $H$ as a section of $\mathcal{V}_b$. Indeed, it
does not vanish at $H$ because it is not an element of $xV_b$. We refer to any such vector field as a radial vector field for $H$. It is determined up to an element of $xV_b$, and its restriction to the boundary generates a canonical trivialization of the null space of $i_b$ over $H$, known as the $b$-normal bundle, $\mathcal{b}NH$.

The differential of a $b$-map $f : X \to Y$ extends to a bundle map between $b$-tangent bundles and $b$-normal bundles. If both of these induced maps are surjective, $f$ is a $b$-fibration.

**Conormal functions.** Let $\mu$ denote a positive section of the density bundle $\Omega(X)$. Denote $L^2(X) = L^2(X, \mu) = \{ u : X \to \mathbb{C} \text{ measurable} : \int_X |u|^2 \mu < \infty \}$ and, for $n \in \mathbb{N}_0$ and $s$ a multiweight, the weighted $b$-Sobolev spaces corresponding to $\mu$ are $\rho_X^s H^n_b(X, \mu) = \{ u : X \to \mathbb{C} \text{ measurable} : \mathcal{V}_b(X)^n(\rho_X^{-s} u) \subseteq L^2(X) \}$.

The $L^2$-based conormal spaces are $\rho_X^s H^\infty_b(X) = \bigcap_{n \in \mathbb{N}_0} \rho_X^s H^n_b(X)$

though we shall usually use $A^s_-(X) = \bigcap_{s' < s} \rho_X^s H^\infty_b(X)$.

We refer to these as conormal functions with multiweight $s - \cdot$. We denote the union over all multiweights by $A^s_+(X) = \bigcup_s \rho_X^s H^\infty_b(X)$.

By Sobolev embedding, any function in $A^s_+(X)$ is smooth in the interior of $X$, and indeed the individual $\rho_X^s H^\infty_b(X)$ are preserved by the action of $\mathcal{V}_b(X)$. They are also $\mathcal{C}^\infty(X)$-modules, so it makes sense to talk about conormal sections of a vector bundle $E \to X$, e.g., $A^s_-(X; E) = A^s_-(X) \otimes_{\mathcal{C}^\infty(X)} \mathcal{C}^\infty(X; E)$.

**$I$-smooth (or polyhomogeneous) expansions.** Regularity at the boundary hypersurfaces is often manifest in an asymptotic expansion reminiscent of the Taylor series but with exponents that are not necessarily integers and with the presence of powers of logarithms,

$$u \sim \sum u_{s_j, p} x^{s_j} (\log x)^p$$

with coefficients $u_{s_j, p}$ themselves conormal functions. We keep track of the allowed exponents in index sets and refer to this class of functions as $I$-smooth ($I$ for index set) or as polyhomogeneous.

An index set $E$ is a discrete subset of $\mathbb{C} \times \mathbb{N}_0$ such that

$$\{(s_j, p_j)\} \subseteq E, |(s_j, p_j)| \to \infty \implies \Re s_j \to \infty.$$ 

To ensure independence from the choice of $b$df $x$ we also require

$$(z, p) \in E, p \geq 1 \implies (z, p - 1) \in E$$

$$(z, p) \in E \implies (z + k, p) \in E \text{ for all } k \in \mathbb{N}.$$
THE INDEX FORMULA FOR FAMILIES OF DIRAC TYPE OPERATORS ON PSEUDOMANIFOLDS

We often denote the index set \( \{(\alpha + n, 0) \in \mathbb{C} \times \mathbb{N}_0 : n \in \mathbb{N}_0 \} \) simply as \( \alpha \). The extended union of two index sets is

\[ E \cup F = E \cup F \cup \{(z, p) \in \mathbb{C} \times \mathbb{N}_0 : (z, q) \in E \text{ and } (z, p - q - 1) \in F \text{ for some } q \in \mathbb{N}_0 \}. \]

Given an index set \( E \) we define

\[ \text{Re } E = \{ \text{Re}(z) : (z, 0) \in E \}, \quad \inf E = \min \text{Re } E. \]

We allow the empty set as an index set and define \( \inf \emptyset = \infty \).

To each index set \( E \) and \( w \in \mathbb{R} \) we assign the polynomial

\[ b(E, w; \lambda) = \prod_{(z, p) \in E} (\lambda - z). \]

Note that if \( r_H \) is a radial vector field for \( H \) then the null space of the differential operator \( b(E, s; r_H) \) is spanned by

\[ \{ x^z (\log x)^p : (z, p) \in E, \text{Re } z < w \}. \]

An index family \( E \) on a manifold with corners is an assignment of an index set \( E(H) \) to each boundary hypersurface \( H \). To each index family \( E \) we associate a multiweight \( \inf E \).

Given an index family, a multiweight \( w \), a choice of radial vector field \( r_H \) for each boundary hypersurface, and an ordering of the boundary hypersurfaces we define the differential operator

\[ b(E, w) = \prod_{H \in \mathcal{M}_1(X)} b(E(H), w(H); r_H) \]

and the spaces of partially \( \mathcal{I} \)-smooth conormal functions by

\[ (3.1) \quad \mathcal{B}^{E/w}_{\text{phg}} \mathcal{A}^\mathcal{I}(X) = \{ u \in \mathcal{A}^\mathcal{I}(X) : b(E, r)u \in \mathcal{A}^\mathcal{I}(X) \text{ for all } s \leq r \leq w \}. \]

Thus these are conormal functions with multiweight \( s \) — that have an asymptotic expansion at each boundary hypersurface \( H \) with exponents in \( E(H) \) with real part less than \( w(H) \) and with remainder a conormal function with multiweight \( w \) —. The space of (totally) \( \mathcal{I} \)-smooth conormal functions with index family \( E \) is

\[ \mathcal{A}^E_{\text{phg}}(X) = \bigcap_w \mathcal{B}^{E/w}_{\text{phg}} \mathcal{A}^\mathcal{I}(X) \]

where \( s \) is any multiweight satisfying \( s < \inf E \).

When the empty set is used as an index set we interpret \( \mathcal{B}^{0/w}_{\text{phg}} \mathcal{A}^\mathcal{I}(X) = \mathcal{A}^\mathcal{I}(X) \) whenever \( s \leq w \).

Pull back and push forward. If \( f : X \to Y \) is a b-map, then to each multiweight \( r \) on \( Y \) we associate a multiweight on \( X \),

\[ \mathcal{M}_1(X) \ni H \mapsto f^* r(H) = \sum_{G \in \mathcal{M}_1(Y)} e_f(H, G) r(G). \]

Note that \( f^* r(H) = 0 \) for any \( H \in \ker(e_f) \). Let \( n_f \) be the multiweight on \( X \),

\[ n_f(H) = \begin{cases} \infty & \text{if } H \in \ker(e_f) \\ 0 & \text{else} \end{cases} \]
To an index family $\mathcal{F}$ on $Y$ we associate an index family on $X$,

$$\mathcal{M}_1(X) \setminus \ker(e_f) \ni H \mapsto f^*\mathcal{F}(H) = \left\{ (S, P) : \exists (s_G, p_G) \in \mathcal{F}(G) : e_f(H, G) \neq 0 \right\}$$

s.t. $S = \sum e_f(H, G)s_G, P = \sum p_G$

and $f^*\mathcal{F}(H) = 0$ for all $H \in \ker(e_f)$. For any multiweights $r, r'$ and index family $\mathcal{F}$ on $Y$, pull-back along $f$ gives a map [Mel92 Theorem 3]

$$f^*: \mathcal{A}^1/(f^1r + n_r) \mathcal{A}^{f^1r}(X) \rightarrow \mathcal{A}^1/(f^1r + n_r) \mathcal{A}^{f^1r}(X).$$

Similarly, if $f: X \rightarrow Y$ is a b-fibration (defined above) we can associate to each multi-weight $s$ on $X$ a multiweight on $Y$,

$$\mathcal{M}_1(Y) \ni G \mapsto f_gG = \min\{s(H)/e_f(H, G) : H \in \mathcal{M}_1(X), e_f(H, G) \neq 0\},$$

and to each index family $\mathcal{E}$ on $X$ an index family on $Y$,

$$\mathcal{M}_1(Y) \ni G \mapsto f_g\mathcal{E}(G) = \bigcup_{H \in \mathcal{M}_1(X), e_f(H, G) \neq 0} \{(z/e_f(H, G), p) : (z, p) \in \mathcal{E}(H)\}.$$

For any multiweights $s, s'$ and index family $\mathcal{E}$ on $X$ satisfying

$$H \in \ker(e_f) \implies \inf \mathcal{E}(H) > 0,$$

push-forward along $f$ gives a map [Mel92 Theorem 5]

$$f_*: \mathcal{A}^1/(s' \mathcal{A}^1) \mathcal{A}^{s'}(X; \rho^{-1}_X \Omega) \rightarrow \mathcal{A}^1/(s' \mathcal{A}^1) \mathcal{A}^{s'}(Y; \rho^{-1}_Y \Omega).$$

These theorems hold with functions replaced by sections of a vector bundle with only notational differences. Another useful extension is to sections that are also conormal with respect to an interior p-submanifold. A submanifold $W \subseteq X$ is a p-submanifold if every point in $W$ has a neighborhood $U$ in $X$ such that

$$X \cap U = X' \times X'', \text{ where } \partial X'' = \emptyset,$$

$$W \cap U = X' \times \{p''\} \text{ for some } p'' \in X''.$$  (3.2)

We will not detail this extension but refer the reader to e.g., [EMM91 Appendix B].

3.2. **Edge double space.** Given a manifold with corners and an iterated fibration structure $X$, we follow [Maz91, MW17] and define edge pseudodifferential operators by describing their integral kernels on a replacement of $X^2$ that takes the iterated fibration structure into account.

Recall that the radial blow-up of a manifold with corners $X$ along a p-submanifold $W$ (as in (3.2)) is the manifold with corners $[X; W]$ obtained by replacing $W$ with the inward-pointing part of its spherical normal bundle, see e.g., [Mel93 §4.2], [MM95 §2.2], [Mel Chapter 5].

Recall that there is a partial order on $\mathcal{S}(X)$, $Y < Y'$ iff $\mathcal{B}_Y \cap \mathcal{B}_{Y'} = \emptyset$ and dim $Y < \dim Y'$. The edge double space associated to $X$ is obtained from $X^2$ by blowing-up certain p-submanifolds. For each $Y \in \mathcal{S}(X)$ we denote the the fiber diagonal of $\phi_Y$ in $X^2$ by

$$\mathcal{B}_Y \times_{\phi_Y} \mathcal{B}_Y = \{(\zeta, \zeta') \in (\mathcal{B}_Y)^2 : \phi_Y(\zeta) = \phi_Y(\zeta')\}.$$
Definition 3.1. Let $X$ be a manifold with corners and an iterated fibration structure. Let $\mathcal{S}(X) = \{Y_1, Y_2, \ldots, Y_t\}$ be a listing of $\mathcal{S}(X)$ such that $Y_i < Y_j \implies i < j$, i.e., such that the list is non-decreasing in depth. The edge double space of $X$ is

\[(3.3) \quad X_e^2 = [X^2; \mathcal{B}_{Y_1} \times \phi_{Y_1} \mathcal{B}_{Y_1}; \mathcal{B}_{Y_2} \times \phi_{Y_2} \mathcal{B}_{Y_2}; \ldots; \mathcal{B}_{Y_{t}} \times \phi_{Y_{t}} \mathcal{B}_{Y_{t}}].\]

As in, e.g., [DM12, MP97a], there is an analogous construction for families of manifolds with corners.

Definition 3.2. Given a fiber bundle $M \xrightarrow{\psi} B$ of manifolds with corners and iterated fibration structures as in Definition 1.3, fix a non-decreasing list of $\mathcal{S}_\psi(M)$, $\{N_1, N_2, \ldots, N_t\}$, let the families edge double space be

\[(M/B)_e^2 = [M \times \psi M; \mathcal{B}_{N_1} \times \phi_{N_1} \mathcal{B}_{N_1}; \ldots; \mathcal{B}_{N_t} \times \phi_{N_t} \mathcal{B}_{N_t}].\]

The map $\psi$ induces a fiber bundle

\[X_e^2 \xrightarrow{(M/B)_e^2} (M/B)_e^2 \xrightarrow{\psi(2)} B.\]

Let us check, as is implicit in Definition 3.1, that after performing the first $k-1$ blow-ups, the lift of $\mathcal{B}_{Y_k} \times \phi_{Y_k} \mathcal{B}_{Y_k}$ to the blown-up space is a p-submanifold. Local coordinates near $\mathcal{B}_Y$, say $x, y, z$ where $x$ is a bdf for $\mathcal{B}_Y$, $y$ are coordinates along $Y$ and $z$ coordinates along the fiber $Z$ of $\phi_Y$, induce coordinates $x, y, z, x', y', z'$ near $\mathcal{B}_Y \times \mathcal{B}_Y$, in which

\[\mathcal{B}_Y \times \phi_Y \mathcal{B}_Y = \{x = x' = 0, y = y'\}\]

and so this is a p-submanifold whenever $Y$ is a closed manifold, e.g., for $Y_1$.

If $Y < \bar{Y}$, so that $\bar{Y}$ has a collective boundary hypersurface $\mathcal{B}_{Y\bar{Y}}$ as in (1.1), let us label the fibers of these fiber bundles,

\[(3.4) \quad Z \supseteq \mathcal{B}_{Y\bar{Y}} \xrightarrow{\phi_{Y\bar{Y}}} W \quad \bar{Z} \xrightarrow{\bar{Y}} \mathcal{B}_Y \cap \mathcal{B}_{\bar{Y}} \xrightarrow{\phi_{\bar{Y}}} \mathcal{B}_{Y\bar{Y}} \subseteq \bar{Y} \xrightarrow{\phi_{\bar{Y}}} Y \xrightarrow{\phi_Y} \bar{Y}\]

and choose coordinates near $\mathcal{B}_Y \cap \mathcal{B}_{\bar{Y}}$ of the form

\[(3.5) \quad x, \ y, \ w, \ r, \ \bar{z}, \]

in which $x$ is a bdf for $Y$ and $r$ is a bdf for $\bar{Y}$, $y$ are coordinates along $Y$, $w$ coordinates along $W$ and $\bar{z}$ coordinates along $\bar{Z}$, so that $(x, y, w)$ are coordinates along $\bar{Y}$ and $(w, r, \bar{z})$ are coordinates along $Z$. In the induced coordinates $x, y, w, r, \bar{z}, x', y', w', r', \bar{z}'$, we have

\[\mathcal{B}_Y \times \phi_Y \mathcal{B}_Y = \{x = x' = 0, y = y'\}, \quad \mathcal{B}_{\bar{Y}} \times \phi_{\bar{Y}} \mathcal{B}_{\bar{Y}} = \{r = r' = 0, (x, y, w) = (x', y', w')\}\]

which shows that the latter is not a p-submanifold. After blowing-up the former, projective coordinates with respect to $x'$ are given by

\[(3.6) \quad s = \frac{x}{x'}, \quad u = \frac{y - y'}{x'}, \quad w, \ r, \ \bar{z}, \ x', \ y', \ w', \ r', \ \bar{z}'\]
and the interior lift of $\mathcal{B}_Y \times_{\phi} \mathcal{B}_Y$ is given by

\begin{equation}
\{ r = r' = 0, \ s = 1, \ u = 0, \ w = w' \},
\end{equation}

which is a $p$-submanifold.

Thus the manifolds blown-up in (3.3) are $p$-submanifolds. We denote the blow-down map by

$$\beta_{(2)} : X^2_e \longrightarrow X^2$$

and the compositions with the projections onto the left and right factors by

$$\beta_{(2),L} : X^2_e \longrightarrow X^2 \longrightarrow X,$$

$$\beta_{(2),R} : X^2_e \longrightarrow X^2 \longrightarrow X.$$ 

Since the collective boundary hypersurfaces may contain more than one connected component, this is an ‘overblown’ version of the double space in [Maz91, §2] and an edge version of the double space in [MP92, Appendix].

The edge double space has collective boundary hypersurfaces, for each $Y \in \mathcal{S}(X)$,

$$\mathcal{B}_Y^{(1)} \times X \leftrightarrow \mathcal{B}_Y^{(2)}(Y), \quad X \times \mathcal{B}_Y^{(1)} \leftrightarrow \mathcal{B}_Y^{(2)}(Y), \quad \mathcal{B}_Y^{(1)} \times_{\phi_Y} \mathcal{B}_Y^{(1)} \leftrightarrow \mathcal{B}_Y^{(2)}(Y),$$

where the notation indicates that, e.g., the interior lift of $\mathcal{B}_Y^{(1)} \times X$ is the boundary hypersurface $\mathcal{B}_Y^{(2)}(Y)$ of $X^2_e$. We denote the family of collective boundary hypersurfaces produced by the blow-ups by $\text{ff}(X^2_e)$ (the ‘front faces’) and the other collective boundary hypersurfaces by $\text{sf}(X^2_e)$ (the ‘side faces’), thus

\begin{align}
\text{ff}(X^2_e) = \{ \mathcal{B}_Y^{(2)}(Y) : Y \in \mathcal{S}(X) \}, \\
\text{sf}(X^2_e) = \{ \mathcal{B}_{10}^{(2)}(Y), \mathcal{B}_{01}^{(2)}(Y) : Y \in \mathcal{S}(X) \}.
\end{align}

We use similar notations for the family edge double space $(M/B)^2_e$, e.g., $\mathcal{B}_{10}^{(2)}(N)$.

It will be useful to describe the structure of these collective hypersurfaces in more detail. If $\mathcal{S}(X) = \{ Y \}$, the case treated in [Maz91], then the restriction of the blow-down map

$$\mathcal{B}_Y^{(2)}(Y) \longrightarrow \mathcal{B}_Y \times_{\phi_Y} \mathcal{B}_Y$$

is the fiber bundle map of the inward pointing spherical normal bundle of $\mathcal{B}_Y \times_{\phi_Y} \mathcal{B}_Y$ in $X^2$. The fiber is a quarter sphere $\mathbb{S}^{\dim Y-1}$, where $h = \dim Y$. Invariantly the spherical normal bundle at a point $q \in \mathcal{B}_Y \times_{\phi_Y} \mathcal{B}_Y$ is obtained from $T_qX^2$ by moding out by tangent vectors to $\mathcal{B}_Y \times_{\phi_Y} \mathcal{B}_Y$, removing zero, and taking the $\mathbb{R}^+$-orbit space of the dilation action,

$$\mathcal{S}(N_{X^2}(\mathcal{B}_Y \times_{\phi_Y} \mathcal{B}_Y)_q) = \mathbb{R}^+ \backslash \{(T_qX^2/T_q(\mathcal{B}_Y \times_{\phi_Y} \mathcal{B}_Y)) \setminus \{0\}\}.$$

and so every vector field on $X^2$ transverse to $\mathcal{B}_Y \times_{\phi_Y} \mathcal{B}_Y$ (i.e. not in the image of its tangent space via the inclusion) defines a section of the spherical normal bundle, and every inward-pointing vector field defines a section of the fiber bundle $\mathcal{B}_{10}^{(2)}(Y) \longrightarrow \mathcal{B}_Y \times_{\phi_Y} \mathcal{B}_Y$. As pointed out in [MM87, (3.10)], there is a canonical section: let $\nu'$ be any vector field on $X$ that is inward pointing at $\mathcal{B}_Y$, denote the corresponding vector fields acting on the left, respectively
right, factor of $X$ in $X^2$ by $\nu'_L$, respectively $\nu'_R$, set $\nu = \nu'_L + \nu'_R$ and let $[\nu_{\phi_Y}]$ be the induced section of $\mathcal{B}^{(2)}_{\phi}(Y) \longrightarrow \mathcal{B}_Y \times_{\phi_Y} \mathcal{B}_Y$,

$$[\nu_{\phi_Y}] : \mathcal{B}_Y \times_{\phi_Y} \mathcal{B}_Y \longrightarrow \mathcal{B}^{(2)}_{\phi}(Y).$$

A different choice of $\nu'$ would change the value $\nu'(y, z)$ at a point $(y, z) \in \mathcal{B}_Y$ by multiplication by a positive constant and addition of a vector tangent to $T_{(y, z)} \mathcal{B}_Y$, and correspondingly change $\nu(y, z, z')$ by multiplication by a positive constant and addition of a vector in $T_{(y, z, z')}(\mathcal{B}_Y \times_{\phi_Y} \mathcal{B}_Y)$, and hence would not change $[\nu](y, z, z')$. We denote the image of $[\nu]$ by

$$\nu_{\phi_Y}(\mathcal{B}_Y) = [\nu_{\phi_Y}] (\mathcal{B}_Y \times_{\phi_Y} \mathcal{B}_Y) \subseteq \mathcal{B}^{(2)}_{\phi}(Y).$$

For reasons described below, this will be referred to as the identity section of the $S^{h+1}$-bundle, in analogy to the zero section of a vector bundle.

A choice of connection for $\mathcal{B}_Y \times_{\phi_Y} \mathcal{B}_Y \longrightarrow Y$ lets us identify the normal bundle to $\mathcal{B}_Y \times_{\phi_Y} \mathcal{B}_Y$ in $X^2$ with the pull-back of $TY$ (as the normal bundle to the diagonal of $Y$ in $Y^2$), times two copies of $N^+_X \mathcal{B}_Y$, the inward-pointing normal bundle to $\mathcal{B}_Y$ in $X$, one for each factor of $X^2$,

$$S(TY \times (N^+_X \mathcal{B}_Y)^2) = \mathcal{B}^{(2)}_{\phi}(Y) \longrightarrow \mathcal{B}_Y \times_{\phi_Y} \mathcal{B}_Y.$$

With this identification, the identity section $\nu_{\phi_Y}(\mathcal{B}_Y)$ is the subset $\pi(\{0\} \times \{N_L\} \times \{N_R\})$ where $\pi$ is the projection from $TY \times (N^+_X \mathcal{B}_Y)^2$ minus its zero section onto its sphere bundle, and $N_L, N_R$, denote the pull-back along the left and right of an inward-pointing vector field transverse to $\mathcal{B}_Y$ in $X$. We can compose the blow-down map with the fiber bundle map $\mathcal{B}_Y \times_{\phi_Y} \mathcal{B}_Y \longrightarrow Y$ to obtain a fiber bundle

$$S^{h+1} \times Z^2 \longrightarrow \mathcal{B}^{(2)}_{\phi}(Y) \longrightarrow Y.$$

Thus near its front face, $X^2_e$ is locally diffeomorphic to $\mathbb{R}^+ \times S^{h+1} \times Z^2 \times \mathcal{U}_Y$ with $\mathcal{U}_Y$ an open set in $Y$ over which $\phi_Y$ is trivial.

Finally, in local projective coordinates analogous to (3.6),

$$s = \frac{x}{x'}, \quad u = \frac{y - y'}{x'}, \quad z, \quad x', \quad y', \quad z',$$

$X^2_e$ is locally diffeomorphic to

$$\mathbb{R}_+^+ \times \mathbb{R}_u^h \times [0, 1]_{x'} \times \mathcal{U}_Y \times Z^2.$$

The submanifold $\nu_{\phi_Y}(\mathcal{B}_Y)$ in these coordinates is $\{s = 1, u = 0\}$. Notice that if we view $\mathbb{R}_+^+ \times \mathbb{R}_u^h$ as the ‘ax+b’ group, $\mathbb{R}^+ \times \mathbb{R}^h$, with product

$$(s, u) \cdot (s', u') = (ss', s'u + u'),$$

then $(1, 0)$ is the identity element of the group. This is the reason why we refer to $\nu_{\phi_Y}(\mathcal{B}_Y)$ as the identity section. The action of edge pseudodifferential operators is by convolution with respect to this group (see, e.g., [Maz91 (3.5)]).

As we discuss now, this structure persists in a modified way in the setting of manifolds with iterated fibration structures.
Remark 3.3. It may be useful to consider a ‘toy case’ with underlying stratified space
\[ \hat{X} = Y \times C_{[0,1]}(W \times C_{[0,1]}(Z)) , \]
where \(C_{[0,1]}\) denotes the truncated cone, so that
\[ X = Y \times [0,1)_x \times W \times [0,1)_r \times \tilde{Z} , \quad \mathcal{S}(X) = \{ Y, \ Y = Y \times [0,1)_x \times W \} \]
with collective boundary hypersurfaces
\[ \mathcal{B}_Y = \{ x = 0 \} = Y \times W \times [0,1)_r \times \tilde{Z} , \quad \mathcal{B}_{Y^*} = \{ r = 0 \} = Y \times [0,1)_x \times W \times \tilde{Z} \]
participating in the (trivial) fiber bundles
\[ Z = W \times [0,1)_r \times \tilde{Z} \to \mathcal{B}_Y \xrightarrow{\phi_Y} Y , \quad \tilde{Z} \to \mathcal{B}_{Y^*} \xrightarrow{\phi_{Y^*}} \tilde{Y} \]
whose compatibility diagram takes the form
\[
\begin{align*}
\tilde{Z} & \quad \mathcal{B}_Y \cap \mathcal{B}_{Y^*} = Y \times W \times \tilde{Z} \xrightarrow{\phi_Y} \mathcal{B}_{Y^*} = Y \times W \subseteq \tilde{Y} \quad \phi_{Y^*} \\
\phi_{Y} & \quad W \\
\end{align*}
\]
To construct \( X^2 \), we start with \( X^2 \) and blow-up
\[ \mathcal{B}_Y \times_{\phi_Y} \mathcal{B}_{Y^*} = \{ x = x' = 0, y = y' \} = \text{diag}(Y^2) \times \{ x = x' = 0 \} \times W^2 \times [0,1)_r^2 \times \tilde{Z}^2 . \]
Near the resulting front face the blow-up space is locally diffeomorphic to
\[ \mathbb{R}^+ \times \text{S}^{h_Y+1 \times Y} \times W^2 \times [0,1)_r^2 \times \tilde{Z}^2 \]
where \( h_Y \) is the dimension of \( Y \) and \( R \) is a defining function for the front face. In this local description, the interior lift of the submanifold
\[ \mathcal{B}_{\mathcal{Y}^*} \times_{\phi_{Y^*}} \mathcal{B}_{Y^*} = \{ r = r' = 0, (y, x, w) = (y', x', w') \} = \text{diag}(Y^2 \times [0,1)_x^2 \times W^2) \times \{ r = r' = 0 \} \times \tilde{Z}^2 \]
is equal to (cf. (3.7))
\[
\mathbb{R}^+_R \times \{ (1,0) \} \times Y \times \text{diag}(W^2) \times \{ r = r' = 0 \} \times \tilde{Z}^2 \\
= \mathbb{R}^+_R \times \left( \nu_{\phi_Y}(\mathcal{B}_{Y^*}) \cap \text{diag}(W^2) \times \{ r = r' = 0 \} \times \tilde{Z}^2 \right) ,
\]
where \( \{ (1,0) \} \) denotes the identity element of the ‘ax+b group’, as discussed above, and \( \nu_{\phi_Y}(\mathcal{B}_{Y^*}) \) is the identity section of \( \text{S}(TY \times \mathbb{R}^+_R) \).

Blowing-up the interior lift of \( \mathcal{B}_{\mathcal{Y}^*} \times_{\phi_{Y^*}} \mathcal{B}_{Y^*} \) produces the manifold \( X^2 \). The normal bundle of the interior lift of \( \mathcal{B}_{\mathcal{Y}^*} \times_{\phi_{Y^*}} \mathcal{B}_{Y^*} \) fibres over \( [0,1)_R \times Y \times W \) with fiber \( \mathbb{R}^{h_Y + h_W + 1} \) where \( h_Y, h_W \) are the dimensions of \( Y \) and \( W \), respectively; indeed the normal bundle is exactly \( \mathbb{R}^+ \times \mathbb{R} \times TY \times TW \). So near the intersection of the front faces this space is locally diffeomorphic to
\[ \mathbb{R}^+_R \times \mathbb{R}^+_R \times \text{S}(\mathbb{R} \times TY \times TW \times \mathbb{R}^+_R) \times \tilde{Z}^2 , \]
where \( R, \tilde{R} \) are defining functions for \( \mathcal{B}_{\mathcal{Y}^*}^{(2)}(Y) \) and \( \mathcal{B}_{\mathcal{Y}^*}^{(2)}(\tilde{Y}) \), respectively. The third term is a fibre bundle \( \text{S}^{h_Y + h_W + 2} \rightarrow \mathbb{R} \times TY \times TW \times \mathbb{R}^+_R \rightarrow [0,1)_R \times Y \times W \simeq \tilde{Y} . \)
With an eye to the case of non-trivial fiber bundles, note that the normal bundle of $\mathcal{B}_{\tilde{Y}^2} \times_{\phi_{\tilde{Y}}}$ $\mathcal{B}_{\tilde{Y}^2}$ is naturally isomorphic to $eT\tilde{Y} \times \mathbb{R}_+^2$, where $eT\tilde{Y}$ is the edge tangent bundle, generated by the vector fields $x\partial_x, x\partial_y, \partial_u$, as can be seen by lifting these vector fields to $X^2$ from the left projection and restricting to the interior lift of $\mathcal{B}_{\tilde{Y}^2} \times_{\phi_{\tilde{Y}}} \mathcal{B}_{\tilde{Y}}$. So in fact this neighborhood can be expressed as

$$\mathbb{R}_R^+ \times S(eT\tilde{Y} \times \mathbb{R}_+^2) \times \tilde{Z}^2,$$

where $S_{++}^{h_v + h_w + 2} \to S(eT\tilde{Y} \times \mathbb{R}_+^2) \to \tilde{Y}$.

In particular, $\mathcal{B}_{\phi_{\tilde{Y}}}(Y)$ is locally diffeomorphic to

$$\mathbb{R}_R^+ \times S_{++}^{h_v + h_w + 2} \times Y \times W \times \tilde{Z}^2.$$

To compare this to

$$Z^2_e = [Z^2; \mathcal{B}_{\tilde{Y}^2} \times_{\phi_{\tilde{Y}}} \mathcal{B}_{\tilde{Y}^2}] = ([W \times [0, 1], r \times Z]^2; \text{diag}(W^2) \times \{r = r' = 0\} \times \tilde{Z}^2]$$

note that this space has a similar description near its front face, namely

$$\mathbb{R}_R^+ \times S_{++}^{h_v + 2} \times Y \times W \times \tilde{Z}^2.$$

Thus the local description of $\mathcal{B}_{\phi_{\tilde{Y}}}(Y)$ fibers over $Y$ and we can think of the fiber as a ‘suspended’ version of the edge double space of $Z$,

$$[S_{++}^{h_v + 1} \times Z^2; \nu_{\phi_{\tilde{Y}}}(\mathcal{B}_Y) \cap (\mathcal{B}_{\tilde{Y}^2} \times_{\phi_{\tilde{Y}}} \mathcal{B}_{\tilde{Y}^2})].$$

The front faces of the edge double space of $X$ are related to the edge double spaces of the fibers of its boundary fibrations, but as pointed out in the remark, they are ‘suspended’ versions. To define this structure in general we momentarily replace a boundary fiber bundle $Z \to \mathcal{B}_Y \to Y$ with an arbitrary fiber bundle of manifolds with corners with iterated fibration structures:

**Definition 3.4.** Let $\tilde{X} \to \tilde{M} \xrightarrow{\psi} \tilde{B}$ be a fiber bundle of manifolds with corners and iterated fibration structures and let $S_{\psi}(\tilde{M}) = \{N_1, \ldots, N_\ell\}$ be a non-decreasing listing of $S_{\psi}(\tilde{M})$. Let $\pi: S_{++}(\tilde{M} \times_{\psi} \tilde{M}) \to \tilde{M} \times_{\psi} \tilde{M}$ be the pull-back of the fiber bundle $S(eT\tilde{B} \times \mathbb{R}_+^2)$ from $B$ to $\tilde{M} \times_{\psi} \tilde{M}$ and let $\nu_{\psi}(\tilde{M})$ denote the identity section. The **suspended edge double space** of $\tilde{M}/\tilde{B}$ is

$$(\tilde{M}/\tilde{B})_{\text{Sus}(e)}^2 = [S_{++}(\tilde{M} \times_{\psi} \tilde{M}); \nu_{\psi}(\tilde{M}) \cap \pi^{-1}(\mathcal{B}_{N_1} \times_{\phi_{N_1}} \mathcal{B}_{N_1}); \ldots; \nu_{\psi}(\tilde{M}) \cap \pi^{-1}(\mathcal{B}_{N_\ell} \times_{\phi_{N_\ell}} \mathcal{B}_{N_\ell})].$$

This fibers over $\tilde{B}$ and we denote the typical fiber by $\tilde{X}_{\text{Sus}(e)}^2; \tilde{X}^2_{\text{Sus}(e)} \to (\tilde{M}/\tilde{B})_{\text{Sus}(e)}^2 \to \tilde{B}$.

**Proposition 3.5** (Structure of the front faces of $X^2_e$). Let $X$ be a manifold with corners and an iterated fibration structure. For each $Y \in S(X)$, let

$$\phi_{Y}^{(2)}: \mathcal{B}_{\phi_{\tilde{Y}}}(Y) \to Y.$$
denote the composition of $\beta(2) : X^2_e \longrightarrow X^2$ with the fibration $\mathfrak{B}_Y \times_{\phi_Y} \mathfrak{B}_Y \longrightarrow Y$, restricted to $\mathfrak{B}^{(2)}_{\phi_Y}(Y)$. This map participates in a fiber bundle with fiber the suspended edge double space of $Z$.

\[
Z^2_{\text{Sus}_Y(e)} = \mathfrak{B}^{(2)}_{\phi_Y}(Y) = (\mathfrak{B}_Y/Y)^2_{\text{Sus}_Y(e)} \longrightarrow Y.
\]

Note that if $Y$ is a maximal element of $\mathcal{S}(X)$ (and hence the fiber $Z$ of $\mathfrak{B}_Y \longrightarrow Y$ is a closed manifold), then the fiber bundle in the proposition over $Y$ is

\[
\mathcal{S}(\mathbb{R}^{\dim Y} \times \mathbb{R}^2_+) \times Z^2 \longrightarrow \mathfrak{B}^{(2)}_{\phi_Y}(Y) \longrightarrow Y,
\]

just as in [Maz91].

Proof. If $Y$ has depth $\chi(Y) = k$ then the fiber $Z$ of $\phi_Y : \mathfrak{B}_Y \longrightarrow Y$ is a manifold with corners and an iterated fibration structure of depth equal to $k - 1$. $Z$ has one collective boundary hypersurface for each $\tilde{Y} \in \mathcal{S}(X)$ such that $Y < \tilde{Y}$. Indeed following diagram (3.4), this boundary hypersurface, which we denote $\mathfrak{B}_{\tilde{Y}Z}$, is the fiber of the restriction of $\phi_{\tilde{Y}}$ to $\mathfrak{B}_{Y} \cap \mathfrak{B}_{\tilde{Y}}$ and its boundary fibration is the restriction of $\phi_{\tilde{Y}}$. Consequently we have

\[
(3.9) \quad Z^2 \supseteq \mathfrak{B}_{\tilde{Y}Z} \times_{\phi_{\tilde{Y}Z}} \mathfrak{B}_{\tilde{Y}Z} \longrightarrow W
\]

\[
\tilde{Z}^2 \longrightarrow (\mathfrak{B}_Y \cap \mathfrak{B}_{\tilde{Y}}) \times_{\phi_{\tilde{Y}}} (\mathfrak{B}_Y \cap \mathfrak{B}_{\tilde{Y}}) \longrightarrow \mathfrak{B}_{Y_{\tilde{Y}}}
\]

\[
\mathfrak{B}_{\tilde{Y}Z} \times_{\phi_{\tilde{Y}}} \mathfrak{B}_{\tilde{Y}Z} \longrightarrow W,
\]

Let $X^2_e(k+1)$ denote the blow up in $X^2$ of all the fibre diagonals of strata of depth greater than or equal to $k + 1$. Let $\mathcal{N}_+(Y)$ denote the inward pointing normal bundle of the interior lift of $\mathfrak{B}_Y \times_{\phi_Y} \mathfrak{B}_Y$ to $X^2_e(k + 1)$. Composing the projection down to $\mathfrak{B}_Y \times_{\phi_Y} \mathfrak{B}_Y$ with the projection of this space down to $Y$, we see, e.g. by using (3.6) and lifting vector fields from the left projection and restricting to the interior lift, that we have a diagram of fiber bundles

\[
\begin{align*}
Z^2 & \longrightarrow \mathcal{N}_+(Y) \\
\mathbb{R}^{h_Y} & \longrightarrow eTY \times \mathbb{R}^2_+ \\
Y & \longrightarrow Y
\end{align*}
\]

where $eTY$ is the edge tangent bundle.

Thus when we blow up the interior lift of $\mathfrak{B}_Y \times_{\phi_Y} \mathfrak{B}_Y$ to $X^2_e(k + 1)$, we produce a boundary hypersurface $\mathfrak{B}^{(2)}_{\phi_Y}(Y) = \mathcal{S}(\mathcal{N}_+(Y))$ which we can identify with the pull-back to $\mathcal{S}(eTY \times \mathbb{R}^2_+) \longrightarrow Y$ of two copies of the $Z$ bundle over $Y$. In particular we have a fiber bundle map

\[
\mathcal{S}^{h+1}_+ \times Z^2 \longrightarrow \mathfrak{B}^{(2)}_{\phi_Y}(Y) \longrightarrow Y.
\]
If $\tilde{Y} > Y$, the interior lift of $B_{\tilde{Y}} \times_{\phi_{\tilde{Y}}} B_{\tilde{Y}}$ intersects $B_{\phi_{\phi}}(Y)$ at the identity section of $\mathcal{S}(\pi TY \times \mathbb{R}^2)$ over the submanifold $B_{\phi_{\phi}}(Y) \times_{\phi_{\phi}} B_{\phi_{\phi}}(Y)$ of $Z$ (e.g., by the computation (3.7)).

The restriction of $\phi_{\phi}(2)$ fibers over $Y$,

\[(0, 1, 1) \times B_{\phi_{\phi}}(Y) \to B_{\phi_{\phi}}(Y) \cap B_{\phi_{\phi}}(Y) \to B_{\phi_{\phi}}(Y) \to Y.\]

Hence blowing these submanifolds up in the appropriate order produces $(\mathcal{B}_Y/Y)^2_{\text{Sus(e)}}$ as required.

We point out that, just as when $S(X) = \{Y\}$, the maps

\[
\begin{array}{ccc}
\beta_{(2),L} & \beta_{(2),R} \\
X_e^2 & \downarrow & \downarrow \\
X & X & X
\end{array}
\]

obtained by blowing-down and then projecting onto the left or right factor of $X$, are b-fibrations.

The double edge space has a distinguished submanifold, the interior lift of the diagonal, which is known as the **edge diagonal** and denoted $\text{diag}_e$. It is a $p$-submanifold of $X_e^2$ and its normal bundle is canonically identified with the edge tangent bundle.

### 3.3. Edge pseudodifferential operators

Let $X$ be a manifold with an iterated fibration structure. We define the edge pseudodifferential operators as the natural analogue of the operators defined by Mazzeo in [Maz91] by specifying the structure of their integral kernels. These will be conormal distributions as in §3.1 on the manifold with corners $X_e^2$ defined in §3.2. We will first define the ‘small calculus’ which includes edge differential operators and then the ‘large calculus’ which can be shown to include the inverse of invertible edge differential operators when they have constant indicial roots. Elements in this calculus are very well behaved but since the hypothesis of constant indicial roots is very restrictive, we also define a ‘calculus with bounds.’

Our convention is that the integral kernels of operators acting on functions will be weighted sections of the density bundle of $X$, pulled back along the projection onto the second factor of $X^2$. We introduce the multiweight

\[
\varnothing : \mathcal{M}_1(X_e^2) \to \mathbb{R}, \quad \varnothing(H) = \begin{cases} 
-(\dim(Y) + 1) & \text{if } H \subseteq \mathcal{B}_{\phi_{\phi}}(Y) \text{ for some } Y \in \mathcal{S}(X) \\
0 & \text{otherwise}
\end{cases}
\]

and the weighted right density bundle over $X_e^2$,

\[
\Omega_{\varnothing,R} = \rho_{X_e^2}\beta_{(2),R}^\beta \Omega(X).
\]

Following [Maz91] Definition 3.3] in the simple edge case, the **small edge calculus of a manifold with corners and an iterated fibration structure** is the filtrated algebra of pseudodifferential operators consisting of the union, over $r \in \mathbb{R}$, of the operators defined by the integral kernels:

\[
\Psi^r_e(X) = \rho_{\text{sfl}(X_e^2)}^{\infty}I^r(X_e^2, \text{diag}_e; \Omega_{\varnothing,R}).
\]
where \( \rho_{sf}(X^2_e) \) is a total boundary defining function for the ‘side faces’ defined in \( (3.8) \), i.e.

\[
\rho_{sf}(X^2_e) = \prod_{H \in \mathfrak{d}(X^2_e)} \rho_H
\]

(We use classical, one-step distributions conormal to the diagonal, see loc. cit.) Note that our convention is that the integral kernels are right-densities so that they will map functions to functions.

If \( E \) and \( F \) are vector bundles over \( X \), we define the vector bundle \( \text{Hom}(E, F) \) over \( X^2_e \) by

\[
\text{Hom}(E, F) = \beta^*_{(2),L} F \otimes \beta^*_{(2),R} E'
\]

where \( E' \) denotes the dual bundle to \( E \), and then the edge pseudodifferential operators acting between sections of \( E \) and \( F \) are given by

\[
\Psi_e^r(X; E, F) = \rho_{sf}(X^2_e)^r(X^2_e, \text{diag}_e; \text{Hom}(E, F) \otimes \Omega_{0,R})
\]

for each \( r \in \mathbb{R} \). We abbreviate \( \Psi_e^r(X; E) = \Psi_e^r(X; E, E) \).

The edge smoothing operators in the small calculus are

\[
\Psi_e^{-\infty}(X; E, F) = \bigcap_{r \in \mathbb{R}} \Psi_e^r(X; E, F) = \rho_{sf}(X^2_e)^\infty(X^2_e; \text{Hom}(E, F) \otimes \Omega_{0,R}).
\]

The integral kernels of edge differential operators lifted to \( X^2_e \) are supported on the edge diagonal and identifying the operators with their kernels (and multiplying by a section of the weighted density bundle, on which the operators act trivially) we have

\[
\text{Diff}^k_e(X, E, F) \subseteq \Psi_e^k(X; E, F), \text{ for all } k \in \mathbb{N}_0.
\]

The conormal singularity at the diagonal means [Hör07, Definition 18.2.6] that elements of the small calculus have a symbol map defined on the conormal bundle to the diagonal, i.e., the edge cotangent bundle,

\[
\sigma_r : \Psi_e^r(X; E, F) \longrightarrow \rho_{RC}^r C^\infty(RC(\ast^T X), \pi^* \text{hom}(E, F))
\]

where \( RC(\ast^T X) \) denotes the radial compactification of the edge cotangent bundle, \( \pi : RC(\ast^T X) \longrightarrow X \) denotes the projection, and \( \rho_{RC} \) denotes a boundary defining for the boundary at radial infinity. Multiplying by \( \rho_{RC}^r \) and restricting to the boundary maps into \( C^\infty(\ast^0 X, \pi^* \text{hom}(E, F)) \), and we denote the resulting map by \( \sigma_r \). The symbol fits into a short exact sequence,

\[
0 \longrightarrow \Psi^{-1}_e(X; E, F) \longrightarrow \Psi^r_e(X; E, F) \overset{\sigma_r}{\longrightarrow} C^\infty(\ast^0 X, \pi^* \text{hom}(E, F)) \longrightarrow 0.
\]

In Appendix A we construct a triple edge space \( X^3_e \) such that composition of edge pseudodifferential operators is given by pull-back, multiplication, and push-forward along fibrations; the behavior of distributions conormal along the lifted diagonal is essentially the same as in, e.g., [Maz91], and hence, for any \( r_A, r_B \in \mathbb{R} \),

\[
A \in \Psi^{r_A}_e(X; G, F), \quad B \in \Psi^{r_B}_e(X; E, G)
\]

\[
\implies A \circ B \in \Psi^{r_A+r_B}_e(X; E, F) \quad \text{and} \quad \sigma_{r_A+r_B}(A \circ B) = \sigma_{r_A}(A) \circ \sigma_{r_B}(B).
\]

If \( A \in \Psi^r_e(X; E, F) \) has invertible symbol, we say that \( A \) is elliptic (or edge elliptic). If \( A \) is elliptic then we can find \( B \in \Psi^{-r}_e(X; F, E) \) satisfying

\[
\sigma_{-r}(B) = \sigma_r(A)^{-1}
\]
and any such is known as a *symbolic parametrix* of $A$. These satisfy
\[
A \circ B - \text{Id} \in \Psi^{-1}_e(X; F), \quad B \circ A - \text{Id} \in \Psi^{-1}_e(X; E).
\]

The **large edge calculus of a manifold with corners and an iterated fibration structure** consists of, for any $r \in \mathbb{R}$, and $E$ an index family for $X_e^2$,
\[
\Psi_{e, phg}^{r, E}(X; E, F) = \Psi_{e}^{r}(X; E, F) + \mathcal{A}_{phg}(X_e^2; \text{Hom}(E, F) \otimes \Omega_{3, R}).
\]

**Definition 3.6.** Let $\mathcal{E}_{ff}$ be the index set for $X_e^2$ given by
\[
\mathcal{E}_{ff}(\mathfrak{B}^{(2)}(Y)) = \mathbb{N}_0, \quad \mathcal{E}_{ff}(\mathfrak{B}^{(2)}_{10}(Y)) = \emptyset, \quad \text{for all } Y \in \mathcal{S}(X).
\]

The **edge calculus with bounds of a manifold with corners and an iterated fibration structure** consists of, for any $r \in \mathbb{R}$, *multiweight* $\mathfrak{w}$ for $X_e^2$,
\[
\Psi_{e}^{-\infty, \mathfrak{w}}(X; E, F) = \mathcal{B}_{phg}^{\mathcal{E}_{ff}/\mathfrak{w}} \mathcal{A}_{-N}^{-}(X; \text{Hom}(E, F) \otimes \Omega_{3, R}),
\]
\[
\Psi_{e}^{r, \mathfrak{w}}(X; E, F) = \Psi_{e}^{r}(X; E, F) + \Psi_{e}^{-\infty, \mathfrak{w}}(X; E, F).
\]

with notation as in \((3.1)\) and $-N < \min(0, \mathfrak{w})$.

(We will always implicitly assume that the multiweight used in the edge calculus with bounds, $\mathfrak{w}$ above, is positive on $\mathcal{f}(X_e^2)$ so that one can restrict to $\mathcal{f}(X_e^2)$.)

As in [Maz91 §5], for each $Y \in \mathcal{S}(X)$ we have a restriction map (with $\text{sf}(\mathfrak{B}^{(2)}(Y))$) the collective boundary hypersurface given by intersections of $\mathfrak{B}^{(2)}(Y)$ with the side faces of $X_e^2$)
\[
(3.12) \quad \mathcal{E}Y : \Psi_{e}^{r, \mathfrak{w}}(X; E) \longrightarrow \Psi_{\text{nsus}(TY^+)}^{r, \mathfrak{w}}(\mathfrak{B}_Y; Y; E), \quad \text{where}
\]
\[
\Psi_{\text{nsus}(TY^+)}^{r, \mathfrak{w}}(\mathfrak{B}_Y; Y; E) = \rho_{\text{sf}(\mathfrak{B}^{(2)}(Y))}^{\infty} I'(\mathfrak{B}^{(2)}(Y), \text{diag} \cap \mathfrak{B}^{(2)}(Y) ; \text{Hom}(E) \otimes \Omega_{3, R})
\]
\[
+ \mathcal{B}_{phg}^{(\mathcal{E}_{ff}/\mathfrak{w})} \mathcal{A}_{-N}^{-}(\mathfrak{B}^{(2)}(Y); \text{Hom}(E) \otimes \Omega_{3, R}).
\]

The notation indicates that the latter space is a ‘non-commutative suspension’ (cf. [AM10 §1], [MM98 §4]) which refers to the following. We can identify the fibers of the bundle $^eTY^+$, over $Y$, with the Lie group $G = \mathbb{R}^+ \ltimes \mathbb{R}^h$, i.e.,
\[
(s, u) \cdot (s', u') = (ss', u + su'),
\]
and the composition of edge pseudodifferential operators induces convolution with respect to this action for the normal operators.

Moreover, $\Psi_{\text{nsus}(^eTY^+)}^{r, \mathfrak{w}}(\mathfrak{B}_Y; Y; E)$ is naturally a bundle of operators over $Y$, and the normal operators of differential operator form a special sub-bundle, namely the product lie algebra $U(\mathfrak{g}) \times \text{Diff}_e^+(Z)$ where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, the Lie algebra of the Lie Group $G$ above. This goes also for zeroeth order operators, including the identity operator, in particular
\[
(3.13) \quad \mathcal{E}Y(Id) = \delta_{\nu_{\phi_Y}(\mathfrak{B}_Y)}Id_Z,
\]
with $\nu_{\phi_Y}(\mathfrak{B}_Y)$ the identity section.
With $\rho_X$ a total boundary defining function for $X$, for $Y \in S(X)$ we now have a composite map $\bar{\rho} \in \text{Diff}_p(X)$ to $^cN_Y(\rho_X \bar{\rho})$, which is related to the ‘wedge’ normal operator above by
\begin{equation}
(3.14) \quad ^cN_Y(\rho_X \bar{\rho}) = (\rho_X / \prod_{\gamma \leq Y} \rho_Y^\#)N_Y(\bar{\rho}),
\end{equation}
where $\rho_Y^\#$ is the pullback of $\rho_Y$ to $X^2_\varepsilon$ to the right factor. $N_Y(\bar{\rho})$ is not the normal operator of an edge operator; it is a wedge operator on $\mathfrak{B}^{(2)}_\phi(Y)$.

Consider the normal operators $\{^cN_Y(A)\}_{Y \in S(X)}$ of an edge pseudodifferential operator with bounds $A \in \Psi_{r,m}^{\phi \phi}(X; E, F)$. Since these are defined by restriction of the integral kernel to the front faces of $X^2_\varepsilon$, they automatically agree on the intersections of the front faces $\mathfrak{B}^{(2)}_\phi(Y)$, but it will be useful in the parametrix construction below to have a concrete understanding of these intersections. Let $Y < \tilde{Y}$, so we have a diagram (3.9). The intersection of $\mathfrak{B}^{(2)}_\phi(Y)$ with $\phi^{(2)}_Y : \mathfrak{B}^{(2)}_\phi(\tilde{Y}) \to \tilde{Y}$ takes place ‘in the base’ of $\phi_Y^{(2)}$, i.e. exactly over $\mathfrak{B}_\phi \tilde{Y} \in \mathcal{M}_1(\tilde{Y})$. Thus
\begin{equation}
(3.15) \quad ^cN_{\tilde{Y}}(A)|_{\mathfrak{B}^{(2)}_\phi(Y) \cap \mathfrak{B}^{(2)}_\phi(\tilde{Y})} \in \Psi_{r,m}^{\phi \phi}((\mathfrak{B}_{\tilde{Y}}/\tilde{Y}; E)|_Y,
\end{equation}
where the restriction on the right hand side comes from fibration of $\Psi_{r,m}^{\phi \phi}((\mathfrak{B}_{\tilde{Y}}/\tilde{Y}; E)$ over $\tilde{Y}$ discussed in the previous paragraph. On the other hand, the intersection $\mathfrak{B}^{(2)}_\phi(Y) \cap \mathfrak{B}^{(2)}_\phi(\tilde{Y})$ is the front face obtained from blow up of $\pi^2(\mathfrak{B}_\tilde{Y} \times \phi_{\tilde{Y}} \mathfrak{B}_\phi) \subset \mathfrak{B}^{(2)}_\phi(Y)$ (see (3.10)), and is trivially equal to $(\phi_Y^{(2)})^{-1}(\mathfrak{B}_\phi \tilde{Y})$. Thus restriction of $^cN_Y(A)$ to this front face is really taking the normal operator of an (albeit suspended) edge pseudodifferential operator. This is summarized in the diagram
\begin{equation}
(3.16) \quad \Psi_{r,m}^{\phi \phi}(X; E) \xrightarrow{\quad ^cN_{\tilde{Y}} \quad} \Psi_{r,m}^{\phi \phi}((\mathfrak{B}_{\tilde{Y}}/\tilde{Y}; E)
\end{equation}

$\Psi_{r,m}^{\phi \phi}(X/Y; E) \xrightarrow{\quad \text{res} \quad} \Psi_{r,m}^{\phi \phi}((\mathfrak{B}_{\tilde{Y}}/\tilde{Y}; E)|_Y$

where res means ‘restriction’.

To each index family $\mathcal{E}$ we assign a multiweight $\mathfrak{w}(\mathcal{E})$ such that
\begin{equation}
\Psi_{r,m}^{\phi \phi}(X; E, F) \subseteq \Psi_{r,m}^{\phi \phi}(X; E, F)
\end{equation}
by defining, for each $Y \in S(X),$
\begin{align*}
\mathfrak{w}(\mathcal{E})(\mathfrak{B}_{10}^{(2)}(Y)) &= \inf \mathcal{E}(\mathfrak{B}_{10}^{(2)}(Y)), \\
\mathfrak{w}(\mathcal{E})(\mathfrak{B}_{01}^{(2)}(Y)) &= \inf \mathcal{E}(\mathfrak{B}_{01}^{(2)}(Y)), \\
\mathfrak{w}(\mathcal{E})(\mathfrak{B}_{\phi \phi}^{(2)}(Y)) &= \inf(\text{Re} \mathcal{E}(\mathfrak{B}_{\phi \phi}^{(2)}(Y) \setminus N_0)
\end{align*}
with the convention that the infimum of the empty set is $\infty$.

These operators act on Sobolev sections of vector bundles. In view of the inclusion of the large calculus into the calculus with bounds, it suffices to describe the mapping properties of the latter. We prove the following theorem in Appendix A, where as usual we identify operators with their integral kernels, so for a multiweight $\mathfrak{g}$, $\rho_X^\# \Psi_{r,m}^{\phi \phi}(X; E, F)$ is the space of operators with integral kernel in $\rho_X^\#(\Psi_{r,m}^{\phi \phi}(X; E, F) + \Psi_{r,m}^{\phi \phi}(X; E, F))$. Moreover we will use
multiweights for the front faces, specifically multiweights $\mathfrak{f}: \mathbb{H}(X_e^2) \to \mathbb{R}$, notation as in (3.8), and corresponding weight functions

$$\rho^{\mathfrak{f}}_{\mathbb{H}(X_e^2)} = \prod_{H \in \mathbb{H}(X_e^2)} \rho^{(H)}_{H}$$

**Theorem 3.7** (Action on edge Sobolev spaces). Let $\mathfrak{f}$ be a multiweight for $\mathbb{H}(X_e^2)$. Any $A \in \rho^{\mathfrak{f}}_{\mathbb{H}(X_e^2)} \Psi_{e,\theta}^{r,e}(X;E,F)$ defines a bounded map, for any $t \in \mathbb{R}$,

$$(3.17) \quad \rho^s H^t_e(X;E) \to \rho^{s'} H^{t'}_e(X;F)$$

as long as $t \geq t' + r$ and, for each $Y \in \mathcal{S}(X)$,

$$(3.18) \quad g(\mathfrak{B}^{(2)}_{01}(Y)) + \mathfrak{s}(Y) > -\frac{1}{2}$$

Essentially by Arzela-Ascoli we can see that the inclusion

$$\rho^s H^t_e(X;E) \to \rho^{s'} H^{t'}_e(X;F)$$

is compact if (and only if) $\mathfrak{s} > \mathfrak{s}'$ and $t > t'$. Combining with the mapping properties, we can identify the edge pseudodifferential operators that act as compact operators.

**Corollary 3.8.** If $A$ is as in Theorem 3.7 then the operator (3.17) is compact if and only if $t > t' + r$ and the inequalities in (3.18) are strict.

In Appendix A we study the composition of these pseudodifferential operators at the level of their integral kernels. One advantage of studying composition at this level is that one can then deduce composition results for functions spaces (Sobolev spaces, Hölder spaces, etc.) see, e.g., [Maz91].

**Theorem 3.9** (Composition of edge pseudodifferential operators).

1. Let $r_A, r_B \in \mathbb{R}$ and let $\mathcal{E}_A, \mathcal{E}_B$ be index families for $X_e^2$ such that

$$\text{Re}(\mathcal{E}_A(\mathfrak{B}^{(2)}_{01}(Y))) + \text{Re}(\mathcal{E}_B(\mathfrak{B}^{(2)}_{10}(Y))) > -1 \text{ for all } Y \in \mathcal{S}(X).$$

If $A \in \Psi_{e,\theta}^{r_A,\mathcal{E}_A}(X_e^2;G,F)$ and $B \in \Psi_{e,\theta}^{r_B,\mathcal{E}_B}(X_e^2;E,G)$ then

$$C = A \circ B \in \Psi_{e,\theta}^{r_A + r_B,\mathcal{E}_C}(X_e^2;E,F)$$

where $\mathcal{E}_C$ is the index family on $X_e^2$ given by, for each $Y \in \mathcal{S}(X)$,

$$\mathcal{E}_C(\mathfrak{B}^{(2)}_{10}(Y)) = \mathcal{E}_A(\mathfrak{B}^{(2)}_{10}(Y)) \cup \left( \mathcal{E}_A(\mathfrak{B}^{(2)}_{\phi\theta}(Y)) + \mathcal{E}_B(\mathfrak{B}^{(2)}_{10}(Y)) \right),$$

$$\mathcal{E}_C(\mathfrak{B}^{(2)}_{01}(Y)) = \mathcal{E}_B(\mathfrak{B}^{(2)}_{01}(Y)) \cup \left( \mathcal{E}_A(\mathfrak{B}^{(2)}_{01}(Y)) + \mathcal{E}_B(\mathfrak{B}^{(2)}_{\phi\theta}(Y)) \right),$$

$$\mathcal{E}_C(\mathfrak{B}^{(2)}_{\phi\theta}(Y)) = \left( \mathcal{E}_A(\mathfrak{B}^{(2)}_{10}(Y)) + \mathcal{E}_B(\mathfrak{B}^{(2)}_{01}(Y)) + \text{dim}(Y) + 1 \right) \cup \left( \mathcal{E}_A(\mathfrak{B}^{(2)}_{\phi\theta}(Y)) + \mathcal{E}_B(\mathfrak{B}^{(2)}_{\phi\theta}(Y)) \right).$$
Let \( r_A, r_B \in \mathbb{R} \) and let \( g_A, g_B \) be multiweights for \( X_e^2 \) such that
\[
g_A(\mathfrak{B}_{10}^{(2)}(Y)) + g_B(\mathfrak{B}_{10}^{(2)}(Y)) > -1 \quad \text{for all } Y \in \mathcal{S}(X).
\]
If \( A \in \Psi_{1}^{r_A;G}(X_e^2; G, F) \) and \( B \in \Psi_{1}^{r_B;G}(X_e^2; E, G) \) then
\[
C = A \circ B \in \Psi_{1}^{r_A+r_B;G}(X_e^2; E, F)
\]
where \( g_C \) is the multiweight on \( X_e^2 \) given by, for each \( Y \in \mathcal{S}(X) \),
\[
\begin{align*}
g_C(\mathfrak{B}_{10}^{(2)}(Y)) &= \min \left( g_A(\mathfrak{B}_{10}^{(2)}(Y)), g_B(\mathfrak{B}_{10}^{(2)}(Y)) \right), \\
g_C(\mathfrak{B}_{01}^{(2)}(Y)) &= \min \left( g_B(\mathfrak{B}_{01}^{(2)}(Y)), g_A(\mathfrak{B}_{01}^{(2)}(Y)) \right), \\
g_C(\mathfrak{B}_{\phi \phi}^{(2)}(Y)) &= \min \left( g_A(\mathfrak{B}_{\phi \phi}^{(2)}(Y)) + g_B(\mathfrak{B}_{\phi \phi}^{(2)}(Y)) + \dim(Y) + 1, g_A(\mathfrak{B}_{\phi \phi}^{(2)}(Y)), g_B(\mathfrak{B}_{\phi \phi}^{(2)}(Y)) \right)
\end{align*}
\]
Proof. The proof of (1) is carried out in Appendix A following [Maz91] by constructing a ‘triple edge space’ and analyzing the integral kernel of the composite geometrically via the push-forward and pull-back theorems. As explained in §3.1, these same theorems apply to partially polyhomogenous distributions with conormal errors. Once we recall that the multiweight \( g_C \) denotes the order of the conormal error, we can deduce the behavior of the multiweights in (2) from the behavior of the index sets in (1).

We formalize the notion of smooth family of edge operators using the space \((M/B)^2\), e.g.,
\[
\begin{align*}
\Psi_e^\infty(M/B; E, F) &= \rho_{\text{st}}^\infty(\Omega^2) \Gamma((M/B)^2; \text{diag}_M; \text{Hom}(E, F) \otimes \Omega_{0,R}), \\
\Psi_e^{\infty,a}(M/B; E, F) &= \mathcal{A}_{\text{phg}}^{\infty,a} \mathcal{A}_{\text{phg}}^{-1}(M/B; \text{Hom}(E, F) \otimes \Omega_{0,R}).
\end{align*}
\]
The composition results in Appendix A are established in the setting of families.

3.4. Bi-ideal. As in, e.g., [Mel93] Proposition 5.38, [MM95], §4.12, we point out a useful bi-ideal property of some edge pseudodifferential operators.

For each \( a \in \mathbb{R}^+ \), define the residual edge pseudodifferential operators of weight \( a \) to be
\[
\Psi_{e,\text{res}}^{-\infty,a}(X; E, F) = \rho_{\text{phg}}^a \Psi_e^{-\infty,a}(X; E, F).
\]

**Theorem 3.10.** For \( a \in \mathbb{R}^+ \)
\[
\Psi_{e,\text{res}}^{-\infty,a}(X; G, H) \circ \mathcal{B}(L^2(X; F, G)) \circ \Psi_{e,\text{res}}^{-\infty,a}(X; E, F) \subseteq \Psi_{e,\text{res}}^{-\infty,a}(X; E, H),
\]
where \( \mathcal{B}(L^2(X; F, G)) \) is the space of bounded operators on \( L^2(X; F, G) \).

**Proof.** This result is by now standard, see e.g. [MM95], §4.12, but we sketch a proof for the convenience of the reader. For simplicity of notation, let us assume that \( E, F, G, H \) are trivial line bundles. Let \( A, C \in \Psi_{e,\text{res}}^{-\infty,a}(X) \) and \( B \in \mathcal{B}(L^2(X)) \). In terms of their distributional kernels on \( X^2 \), the composition is given by
\[
\mathcal{K}_{ABC}(\zeta, \zeta') = \int \int \mathcal{K}_A(\zeta, \zeta'') \mathcal{K}_B(\zeta'', \zeta''') \mathcal{K}_C(\zeta''', \zeta') \, d\zeta'' \, d\zeta''',
\]
and so smoothness of \( \mathcal{K}_{ABC} \) in \( \zeta \) is inherited from smoothness of \( \mathcal{K}_A \) in \( \zeta \), while smoothness of \( \mathcal{K}_{ABC} \) in \( \zeta' \) is inherited from the corresponding smoothness of \( \mathcal{K}_C \).

Next we lift this smooth function from \( X^2 \) to \( X^2_e \) and check that it is conormal to the boundary hypersurfaces. Indeed, the \( b \)-vector fields on \( X^2_e \) are spanned by the lifts of the \( b \)-vector fields on \( X \) along \( \beta_{(2),L} \) and \( \beta_{(2),R} \). The kernel \( \beta_{(2)}^* \mathcal{K}_{ABC} \) has stable regularity with
3.5. Wedge heat space. Recall that the edge double space was defined in (3.2) as

\[ X_e = \left[ X^2; \mathcal{B}_{Y_1}^{(1)} \times_{\phi_{Y_1}} \mathcal{B}_{Y_1}; \ldots; \mathcal{B}_{Y_L}^{(1)} \times_{\phi_{Y_L}} \mathcal{B}_{Y_L} \right], \]

where \( \{Y_1, \ldots, Y_L\} \) is a non-decreasing listing of \( S(X) \), and has collective boundary hypersurfaces, for each \( Y \in S(X) \),

\[ \mathcal{B}_{Y}^{(1)} \times X \leftrightarrow \mathcal{B}_{Y}^{(2)}(Y), \quad X \times \mathcal{B}_{Y}^{(1)} \leftrightarrow \mathcal{B}_{Y}^{(2)}(Y), \quad \mathcal{B}_{Y}^{(1)} \times_{\phi_{Y}} \mathcal{B}_{Y}^{(1)} \leftrightarrow \mathcal{B}_{\phi_{Y}}^{(2)}(Y). \]

Now we construct the wedge heat space. Starting with the space \( X^2 \times \mathbb{R}_t^+ \) we blow-up \( \{t = 0\} \) parabolically so that \( \tau = \sqrt{t} \) is a smooth function. We will not include this blow-up explicitly but simply change the notation to \( X^2 \times \mathbb{R}_t^+ \).

**Definition 3.11.** Let \( X \) be a manifold with corners and an iterated fibration structure and \( \{Y_1, \ldots, Y_L\} \) a non-decreasing listing of \( S(X) \). The **wedge heat space** of \( X \) is the space \( HX_w \) defined by

\[ HX_{w,0} = \left[ X^2 \times \mathbb{R}_t^+; \mathcal{B}_{Y_1} \times_{\phi_{Y_1}} \mathcal{B}_{Y_1} \times \{0\}; \ldots; \mathcal{B}_{Y_L} \times_{\phi_{Y_L}} \mathcal{B}_{Y_L} \times \{0\}; \text{diag}_X \times \{0\} \right]. \]

(3.19)

\[ HX_w = \left[ HX_{w,0}; \mathcal{B}_{Y_1} \times_{\phi_{Y_1}} \mathcal{B}_{Y_1} \times \mathbb{R}_t^+; \ldots; \mathcal{B}_{Y_L} \times_{\phi_{Y_L}} \mathcal{B}_{Y_L} \times \mathbb{R}_t^+ \right]. \]

(3.20)

**Remark 3.12.** In order to describe the heat kernel of \( \tilde{\partial}_X^2 \) as a conormal distribution with bounds the intermediate space \( HX_{w,0} \) would suffice, see, e.g., [MV12]. However below we will allow for perturbations of \( \tilde{\partial}_X \) by smoothing edge pseudodifferential operators and this requires the slightly more complicated space \( HX_w \).

To deal smoothly with a family of wedge heat operators we construct a families wedge heat space to carry their integral kernels.

**Definition 3.13.** Given a fiber bundle \( M \overset{\psi}{\longrightarrow} B \) of manifolds with corners and iterated fibration structures as in Definition 1.3, fix a non-decreasing list of \( S(\psi(M)), \{N_1, N_2, \ldots, N_L\} \), and let the families wedge heat space be

\[ H(M/B)_w = \left[ M \times_{\psi} M \times \mathbb{R}_t^+; \mathcal{B}_{N_1} \times_{\phi_{N_1}} \mathcal{B}_{N_1} \times \{0\}; \ldots; \mathcal{B}_{N_L} \times_{\phi_{N_L}} \mathcal{B}_{N_L} \times \{0\}; \mathcal{B}_{N_1} \times_{\phi_{N_1}} \mathcal{B}_{N_1} \times \mathbb{R}_t^+; \ldots; \mathcal{B}_{N_L} \times_{\phi_{N_L}} \mathcal{B}_{N_L} \times \mathbb{R}_t^+; \text{diag}_M \times \{0\} \right]. \]

The map \( \psi \) induces a fiber bundle

\[ HX_w \longrightarrow H(M/B)_w \overset{\psi(H)}{\longrightarrow} B. \]

As with the double space, implicit in the definition of \( HX_w \) is the fact that for \( \tilde{Y} \in S(X) \) of depth \( k \), the interior lift of \( \mathcal{B}_{\tilde{Y}} \times_{\phi_{\tilde{Y}}} \mathcal{B}_{\tilde{Y}} \times \{0\} \) to the space in which the \( \mathcal{B}_{Y} \times_{\phi_{Y}} \mathcal{B}_{Y} \times \{0\} \) have been blown up for all \( Y < \tilde{Y} \) is a \( p \)-submanifold. It is helpful to see this explicitly. If \( Y < \tilde{Y} \), we have a diagram as in (3.4) and attendant coordinates \( x, y, w, r, z \), together with their primed versions on the right factor on \( X^2 \). Working in the interior of \( Y \), after blowing-up \( \mathcal{B}_Y \times_{\phi_Y} \mathcal{B}_Y \times \{\tau = 0\} \), projective coordinates with respect to \( x' \) are given by
\[ T = \tau/x' \text{ and the other coordinates in } (3.0), \text{ in which the interior lift of } \mathcal{B}_Y \times_\phi \mathcal{B}_Y \times \{ \tau = 0 \} \text{ is given by} \]

\[(3.21) \quad \{T = 0, \ r = r' = 0, \ s = 1, \ u = 0, \ w = w'\}, \]

again a p-submanifold.

We denote the blow-down map by

\[ \beta_{(H)} : HX_w \longrightarrow X^2 \times \mathbb{R}^+ \]

and its composition with the projections onto the left or right factor of \( X \) by \( \beta_{(H),L}, \beta_{(H),R} \) respectively. There are boundary hypersurfaces

\[ X^2 \times \{0\} \leftrightarrow \mathcal{B}_{00,1}^{(H)}, \quad \text{diag}_X \times \{0\} \leftrightarrow \mathcal{B}_{dd,1}^{(H)} \]

and collective boundary hypersurfaces, one for each \( Y \in \mathcal{S}(X) \),

\[ \mathcal{B}_Y \times X \times \mathbb{R}^+ \leftrightarrow \mathcal{B}_{10,0}^{(H)}(Y), \quad X \times \mathcal{B}_Y \times \mathbb{R}^+ \leftrightarrow \mathcal{B}_{01,0}^{(H)}(Y), \]

\[ \mathcal{B}_Y \times_{\phi_Y} \mathcal{B}_Y \times \{0\} \leftrightarrow \mathcal{B}_{00,0}^{(H)}(Y), \quad \mathcal{B}_Y \times_{\phi_Y} \mathcal{B}_Y \times \mathbb{R}^+ \leftrightarrow \mathcal{B}_{\phi,0}^{(H)}(Y). \]

We denote the collective boundary hypersurfaces of \( H(M/B)_w \) analogously to those of \( HX_w \), e.g., \( \mathcal{B}_{\phi,0}^{(H)}(N) \).

We introduce the abbreviations

\[ \text{ff}(HX_w) = \bigcup_{Y \in \mathcal{S}(X)} \mathcal{B}_{\phi,0,1}^{(H)}(Y), \quad \text{lf}(HX_w) = \bigcup_{Y \in \mathcal{S}(X)} \mathcal{B}_{10,0}^{(H)}(Y), \]

\[ \text{rf}(HX_w) = \bigcup_{Y \in \mathcal{S}(X)} \mathcal{B}_{01,0}^{(H)}(Y), \quad \text{ef}(HX_w) = \bigcup_{Y \in \mathcal{S}(X)} \mathcal{B}_{\phi,0}^{(H)}(Y). \]

so that, e.g., \( \rho_{\text{ff}(HX_w)} \) refers to the product of boundary defining functions over all \( \mathcal{B}_{10,0}^{(H)}(Y) \) for \( Y \in \mathcal{S}(X) \). The 'edge faces' making up \( \text{ef}(HX_w) \) do not intersect the lower depth front faces, nor the \( \tau = 0 \) diagonal

\[ \text{diag}_X \times \{0\} \cap \text{ef}(HX_w) = \emptyset = \mathcal{B}_{\phi,0,1}^{(H)}(Y) \cap \mathcal{B}_{\phi,0,0}^{(H)}(Y), \quad Y < \tilde{Y}. \]

Also as with the double space, the faces created by the blow ups are fibre bundles whose fibers are suspended versions of wedge heat spaces. This will be a wedge heat space where the time \([0, \infty)\) is compactified along with other normal directions, and we need analogues of the identity section above. Given a fiber bundle \( \tilde{M} \longrightarrow \tilde{B} \) and a vector bundle \( E \longrightarrow \tilde{B} \), the pull-back of \( \mathcal{S}(E \times \mathbb{R}^3) = (E \times \mathbb{R}^3)/\mathbb{R}^3 \) to \( \tilde{M} \) has two subbundles, \( \tilde{\nu}_0 \) and \( \tilde{\nu}_t \), given, respectively, by the inclusion of \( \mathbb{R}^3_+ \hookrightarrow \mathbb{R}^3_+ \times \mathbb{R}^3_+ \) into the right factor and the projection \( \mathbb{R}^3_+ \times \mathbb{R}^3_+ \longrightarrow \mathbb{R}^3_+ \) off the right factor, of the identity section. Concretely, \( \tilde{\nu}_0 \) is given by the lift of the subbundle \([0] \times (1, 1, 0) \subset \mathcal{S}(E \times \mathbb{R}^3_+) \) to \( \tilde{M} \), and \( \tilde{\nu}_t \) is given by the lift of \([0] \times (x, x, \sqrt{1 + x^2})\). For trivial fibrations \( \tilde{M} = \text{pt} = \tilde{B} \) with \( E = \mathbb{R}^h \) we denote these by

\[ \tilde{\nu}_0(\mathbb{S}^{h+2}) = [(0, 1, 1, 0)] \in (\mathbb{R}^h \times \mathbb{R}^3_+ \setminus (0, 0, 0, 0))/\mathbb{R}^3_+, \]

\[ \tilde{\nu}_t(\mathbb{S}^{h+2}) = [(0, x, x, \sqrt{1 + x^2})] \in (\mathbb{R}^h \times \mathbb{R}^3_+ \setminus (0, 0, 0, 0))/\mathbb{R}^3_+, \]

the \( \mathbb{R}^3_+ \) acting by dilation.
Definition 3.14. Let \( ar{X} \rightarrow \bar{M} \to \bar{B} \) be a fiber bundle of manifolds with corners and iterated fibration structures and let \( \{ \bar{N}_1, \ldots, \bar{N}_r \} \) be a non-decreasing listing of \( S_{\psi}(\bar{M}) \). Let \( S_{+++}(\bar{M} \times \psi \bar{M}) \) be the pull-back of the fiber bundle \( S(T\bar{B} \times \mathbb{R}^3_+) \) from \( \bar{M} \times \psi \bar{M} \) and let \( \tilde{\nu}_{\psi,0}(\bar{M}) \) denote the \( \tau = 0 \) identity section. The \textit{intermediate suspended wedge heat space} \( H(\bar{M}/\bar{B})_{\text{Sus}(w),0} \) is

\[
H(\bar{M}/\bar{B})_{\text{Sus}(w),0} = \left[ S_{+++}(\bar{M} \times \psi \bar{M}); \nu_{\psi}^{\bar{M}}(\bar{M}) \cap \pi^{-1}(\mathcal{B}_{\bar{N}_1} \times \phi_{\bar{N}_1} \mathcal{B}_{\bar{N}_1}); \ldots; \right]
\]

This fibers over \( \bar{B} \) and we denote the typical fiber by \( H\bar{X}_{\text{Sus}(w),0} \) so that

\[
H\bar{X}_{\text{Sus}(w),0} \to H(\bar{M}/\bar{B})_{\text{Sus}(w),0} \to \bar{B}.
\]

The \textit{suspended wedge heat space} \( H(\bar{M}/\bar{B})_{\text{Sus}(w)} \) is

\[
H(\bar{M}/\bar{B})_{\text{Sus}(w)} = \left[ H(\bar{M}/\bar{B})_{\text{Sus}(w),0}; \tilde{\nu}_{\psi,t}(\bar{M}) \cap \pi^{-1}(\mathcal{B}_{\bar{N}_1} \times \phi_{\bar{N}_1} \mathcal{B}_{\bar{N}_1}); \ldots; \right]
\]

and participates in the fiber bundle

\[
H\bar{X}_{\text{Sus}(w)} \to H(\bar{M}/\bar{B})_{\text{Sus}(w)} \to \bar{B}.
\]

As anticipated, the suspended wedge heat spaces describe the structure of the front faces of the wedge heat space.

\textbf{Proposition 3.15 (Structure of the front faces of \( HX_w \)).} Let \( X \) be a manifold with corners and an iterated fibration structure.

For each \( Y \in \mathcal{S}(X) \), let

\[
\phi_Y^{(H)} : \mathcal{B}_{\phi,Y}^{(H)}(Y) \to Y
\]

denote the composition of \( \beta : HX_w \to X^2 \times \mathbb{R}^3_{++} \) with the fibration \( \mathcal{B}_Y \times \phi_Y \mathcal{B}_Y \times \{0\} \to Y \). Then \( \mathcal{B}_{\phi,Y}^{(H)}(Y) = H(\mathcal{B}_Y/Y)_{\text{Sus}(w)} \) and \( \phi_Y^{(H)} \) is the fiber bundle map

\[
HZ_{\text{Sus}(w)} \to \mathcal{B}_{\phi,Y}^{(H)}(Y) \xrightarrow{\phi_Y^{(H)}} Y.
\]

For the intermediate heat space, \( HX_{w,0} \), the corresponding front face is the total space of the fiber bundle

\[
HZ_{\text{Sus}(w),0} \to H(\mathcal{B}_Y/Y)_{\text{Sus}(w),0} \xrightarrow{\phi_Y^{(H)}} Y.
\]

The edge face corresponding to \( Y, \mathcal{B}_{\phi,Y}^{(H)}(Y) \), participates in a fiber bundle with typical fiber the suspended edge double space of \( Z \),

\[
Z_{\text{Sus}(w,e)}^2 \to \mathcal{B}_{\phi,Y}^{(H)}(Y) \to (Y \times \mathbb{R}^3_{\text{res}})
\]

and base given by

\[
(Y \times \mathbb{R}^3_{\text{res}}) = [Y \times \mathbb{R}^3; \mathcal{B}_{Y_1,Y} \times \{0\}; \ldots; \mathcal{B}_{Y_r,Y} \times \{0\}]
\]

where \( \{Y'_1, \ldots, Y'_r\} \) are the strata with \( Y'_i < Y \) indexed in non-increasing order of depth.
Finally, the interior of the boundary hypersurface $\mathfrak{B}_{dd,1}^{(H)}$ is naturally identified with the edge tangent bundle $^eTX$.

Proof. Let $HX_{w,0}(k+1)$ be the intermediate space obtained by blow-up of all the interior lifts of the $\mathfrak{B}_Y \times \phi_Y$, $\mathfrak{B}_Y \times \{\tau = 0\}$ with $Y'$ of depth not less than $k+1$, and let $Y \in S(X)$ have depth $k$. Then the normal bundle of $\mathfrak{B}_Y \times \phi_Y$ $\mathfrak{B}_Y \times \{0\}$ fiber over $^eTY \times (\mathbb{R}_+)^3$ with fiber $Z^2$. Thus blow up of the interior lift of $\mathfrak{B}_Y \times \phi_Y$ $\mathfrak{B}_Y \times \{0\}$ gives a front face ff that fibres

$$S^{hy+2}_{++} \times Z^2 \rightarrow ff \rightarrow Y.$$  

The section $\tilde{\nu}_0(ff)$ is the subbundle of ff over $Y$ given by $\tilde{\nu}_0(S^{hy+2}_{++}) \times Z^2$. From this we see that for $\tilde{Y} > Y$, the interior lift of $\mathfrak{B}_Y \times \phi_Y$ $\mathfrak{B}_Y \times \{0\}$ intersects ff exactly in the fibres at $\tilde{\nu}_0(S^{hy+2}_{++}) \times \mathfrak{B}_Y \times \phi_Y \mathfrak{B}_Y \times \{0\}$, and the diagonal intersects it at $\tilde{\nu}_0(S^{hy+2}_{++}) \times Z^2$. This yields the structure of the front faces of the intermediate wedge heat space. For the wedge heat space it suffices to note that the blow-ups of the $\mathfrak{B}_Y \times \phi_Y$ $\mathfrak{B}_Y \times \mathbb{R}_+$ intersect the intermediate front faces exactly at the $\tilde{\nu}_t$.

To see that the statement for the $\mathfrak{B}_{\phi_0,0}(Y)$ holds, note that the lifts of $\mathfrak{B}_Y \times \phi_Y$, $\mathfrak{B}_Y \times \{0\}$ into $Y' \times \{0\}$ for $Y' < Y$ intersect the blowdown, $\mathfrak{B}_Y \times \phi_Y$ $\mathfrak{B}_Y \times \mathbb{R}_+ \rightarrow Y \times \mathbb{R}_+$ exactly in the bases over the $\mathfrak{B}_Y \times \{0\} \subseteq Y \times \mathbb{R}_+$

Finally the boundary hypersurface $\mathfrak{B}_{dd,1}^{(H)}$ is the inward-pointing part of the spherical normal bundle to $diag_e \times \{0\}$ and, as the normal bundle to the edge diagonal in the edge double space is the edge tangent bundle, $\mathfrak{B}_{dd,1}^{(H)}$ is its radial compactification. \hfill \Box

3.6. Wedge heat operators. Let us specify the weighted density bundle we will use for operators. Define a multi-weight for $HX_w$ by

$$\eta : M_1(HX_w) \rightarrow \mathbb{R},$$

$$\eta(J) = \begin{cases} -(\dim Y + 3) & \text{if } J \subseteq \mathfrak{B}_{\phi_0,0}^{(H)}(Y) \\ -(\dim Y + 1) & \text{if } J \subseteq \mathfrak{B}_{\phi_0,0}^{(H)}(Y) \\ -(\dim X + 2) & \text{if } J \subseteq \mathfrak{B}_{dd,1}^{(H)} \\ 0 & \text{otherwise} \end{cases}$$

and then

$$(3.22) \quad \Omega_{\eta,R} = \rho^b \beta_{(H),R}^* \Omega(X).$$

We will often denote a nowhere-vanishing section of $\beta_{(H),R}^* \Omega(X)$ by $\mu_R$.

By a wedge heat operator we will mean an element of

$$\mathcal{R}_{phg}^{\mathcal{E}/\mathcal{W}}_{\eta^{-m-1}}(HX_w; \text{Hom}(E) \otimes \Omega_{0,R})$$

where $\mathcal{E}$ and $\mathcal{W}$ are, respectively, an index set and multi-weight for $HX_w$.

Recall that, e.g., on a smooth manifold $L$ the composition of two heat operators is given by the formula

$$\mathcal{K}_{AB}(\zeta, \zeta', t) = \int_0^t \int_L \mathcal{K}_A(\zeta, \zeta'', t - t') \mathcal{K}_B(\zeta'', \zeta', t') \, d\zeta'' \, dt'.$$
In Appendix B we define the composition of two wedge heat operators by a version of this formula and then analyze it using the geometric microlocal approach of Melrose, cf. [MP97a, Appendix].

**Theorem 3.16** (Composition of wedge heat operators).

1. Let \( E_A, E_B \) be index families for \( HX_w \) such that
   \[
   \text{Re}(E_A(B_{dd,1}^{(H)})) > 0, \quad \text{Re}(E_B(B_{dd,1}^{(H)})) > 0, \quad \text{and}
   \]
   \[
   \text{Re}(E_A(B_{10,0}^{(H)}(Y))) + \text{Re}(E_B(B_{10,0}^{(H)}(Y))) + 1 > 0 \quad \text{for all} \ Y \in S(X),
   \]
   and let
   \[
   A \in \mathcal{A}_{phg}^E(HX_w; \text{Hom}(E) \otimes \Omega_{h,R}), \quad B \in \mathcal{A}_{phg}^E(HX_w; \text{Hom}(E) \otimes \Omega_{h,R}),
   \]
   then the composition is defined and satisfies
   \[
   C = A \circ B \in \mathcal{A}_{phg}^E(HX_w; \text{Hom}(E) \otimes \Omega_{h,R}),
   \]
   with \( E_C(B_{dd,1}^{(H)}) = E_A(B_{dd,1}^{(H)}) + E_B(B_{dd,1}^{(H)}) \) and, for each \( Y \in S(X) \),
   \[
   E_C(B_{10,0}^{(H)}(Y)) = E_A(B_{10,0}^{(H)}(Y)) \cup (E_A(B_{\phi,1}^{(H)}(Y)) + E_B(B_{\phi,1}^{(H)}(Y)))
   \]
   \[
   E_C(B_{01,0}^{(H)}(Y)) = E_B(B_{01,0}^{(H)}(Y)) \cup (E_A(B_{01,0}^{(H)}(Y)) + E_B(B_{\phi,1}^{(H)}(Y)))
   \]
   \[
   E_C(B_{\phi,1}^{(H)}(Y)) = E_A(B_{\phi,1}^{(H)}(Y)) + E_B(B_{\phi,1}^{(H)}(Y))
   \]
   \[
   E_C(B_{\phi,0}^{(H)}(Y)) = (E_A(B_{\phi,0}^{(H)}(Y)) + E_B(B_{\phi,0}^{(H)}(Y)))
   \]
   \[
   \cup (E_A(B_{10,0}^{(H)}(Y)) + E_B(B_{01,0}^{(H)}(Y)) + \text{dim} \ Y + 1).
   \]

2. Let \( w_A, w_B \) be multiweights for \( HX_w \) such that
   \[
   \{w(B_{dd,1}^{(H)}) \cup \{w(B_{\phi,1}^{(H)}(Y)) : Y \in S(X)\} \subseteq (0, \infty) \cup \{\infty\},
   \]
   and let \( E_A \) and \( E_B \) be index sets as above. If we have
   \[
   w_A(B_{dd,1}^{(H)}(Y)) + w_B(B_{10,0}^{(H)}(Y)) + 1 > 0 \quad \text{for all} \ Y \in S(X),
   \]
   then for any
   \[
   A \in \mathcal{B}_{phg}^{E_A/w_A} \mathcal{A}_{m-1}^{-}(HX_w; \text{Hom}(E) \otimes \Omega_{h,R}),
   \]
   \[
   B \in \mathcal{B}_{phg}^{E_B/w_B} \mathcal{A}_{m-1}^{-}(HX_w; \text{Hom}(E) \otimes \Omega_{h,R}),
   \]
   the composition is defined and satisfies
   \[
   C = A \circ B \in \mathcal{B}_{phg}^{E_C/w_C} \mathcal{A}_{m-1}^{-}(HX_w; \text{Hom}(E) \otimes \Omega_{h,R}),
   \]
   where \( E_C \) is as above and \( w_C \) is the multiweight on \( HX_w \) given by
   \[
   w_C(B_{dd,1}^{(H)}) = \min(w(E_A)(B_{dd,1}^{(H)})) + w_B(B_{dd,1}^{(H)}), \quad w_A(B_{dd,1}^{(H)}) + w(E_B)(B_{dd,1}^{(H)})),
   \]
and, for each $Y \in \mathcal{S}(X)$,

$$w_C(\mathcal{B}_{10,0}^{(H)}(Y)) = \min(w_A(\mathcal{B}_{10,0}^{(H)}(Y)), w_B(\mathcal{B}_{10,0}^{(H)}(Y)), w_A(\mathcal{B}_{\phi,1}^{(H)}(Y)) + w(\mathcal{E}_A)(\mathcal{B}_{\phi,1}^{(H)}(Y))),$$

$$w_C(\mathcal{B}_{01,0}^{(H)}(Y)) = \min(w_B(\mathcal{B}_{01,0}^{(H)}(Y)), w_A(\mathcal{B}_{01,0}^{(H)}(Y)) + w_B(\mathcal{B}_{\phi,1}^{(H)}(Y)), w_A(\mathcal{B}_{\phi,1}^{(H)}(Y)) + w(\mathcal{E}_B)(\mathcal{B}_{\phi,1}^{(H)}(Y))),$$

$$w_C(\mathcal{B}_{\phi,1}^{(H)}(Y)) = \min(\min(w_A(\mathcal{B}_{\phi,1}^{(H)}(Y)) + w_B(\mathcal{B}_{\phi,1}^{(H)}(Y)), w_A(\mathcal{B}_{\phi,1}^{(H)}(Y)) + w(\mathcal{E}_B)(\mathcal{B}_{\phi,1}^{(H)}(Y))),$$

$$w_C(\mathcal{B}_{01,0}^{(H)}(Y)) = \min(\dim Y + 1 + w_A(\mathcal{B}_{10,0}^{(H)}(Y)) + w(\mathcal{E}_B)(\mathcal{B}_{01,0}^{(H)}(Y)), \dim Y + 1 + w_A(\mathcal{B}_{10,0}^{(H)}(Y)) + w_B(\mathcal{B}_{01,0}^{(H)}(Y)), w_A(\mathcal{B}_{\phi,1}^{(H)}(Y)) + w_B(\mathcal{B}_{\phi,1}^{(H)}(Y)), w_A(\mathcal{B}_{\phi,0}^{(H)}(Y)) + w(\mathcal{E}_B)(\mathcal{B}_{\phi,0}^{(H)}(Y)))).$$

The restriction on the multiweights in the second part of the theorem, which holds for all of the multiweights that we will make use of, is made only to simplify the statement of the theorem.

**Proof.** The proof of (1) is carried out in Appendix B following Melrose’s geometric microlocal approach, see, e.g., [MP97a, DM12, Alb07, MM95]. As explained in §3.1, the same pull-back and push-forward theorems used to prove (1) establish (2). \qed

We formalize the notion of smooth family of wedge heat operators using the space $H(M/B)_w$, as elements of

$$\mathcal{B}^{\mathcal{E} \cap w} \mathcal{A}_{-m}^{-1}(H(M/B)_w) \cdot \text{Hom}(E) \otimes \Omega_{h,R}$$

where $\mathcal{E}$ and $w$ are, respectively, an index set and multiweight for $H(M/B)_w$, and $h$ is the multiweight above extended to $H(M/B)_w$, i.e.,

$$h : M_1(H(M/B)_w) \longrightarrow \mathbb{R},$$

$$h(H) = \begin{cases} 
-(\dim (N/B) + 3) & \text{if } H \subseteq \mathcal{B}_{\phi,1}^{(H)}(N) \text{ for some } N \in \mathcal{S}_\psi(M) \\
-(\dim (N/B) + 1) & \text{if } H \subseteq \mathcal{B}_{\phi,0}^{(H)}(N) \text{ for some } N \in \mathcal{S}_\psi(M) \\
-(\dim (M/B) + 2) & \text{if } H = \mathcal{B}_{d,1}^{(H)} \\
\infty & \text{if } H = \mathcal{B}_{0,1}^{(H)} \\
0 & \text{otherwise}
\end{cases}$$

(3.23)

The composition results in Appendix B are established in the setting of families of operators.

### 4. Resolvent and heat kernel of $\overline{\partial}_{M/B}$

Let us return to our usual setting with $M \xrightarrow{\psi} B$ a locally trivial family of manifolds with corners and iterated fibration structures with a totally geodesic vertical wedge metric $g_{M/B}$, and $E \longrightarrow M$ wedge Clifford module along the fibers of $\psi$. Recall that $D_{M/B}$ denotes the associated Dirac-type operator acting on $L^2(M/B; E)$, the wedge $L^2$-space and $\overline{\partial}_{M/B}$ denotes the unitarily equivalent operator acting on $L^2(M/B; E)$, see (1.17). We assigned a ‘vertical
APS domain' to this operator, $D_{VAPS}(\partial_{M/B})$, and in this section we will describe the structure of the resolvent and heat kernel of $\partial_{M/B}$ under the Witt assumption from Definition 2.4.

4.1. Compatible perturbations. Before carrying out these constructions, we will generalize the operators under consideration by allowing certain perturbations by smoothing operators. The perturbations we will use in the main result of this paper will be compactly supported in the interior of $(M/B)^2$, but we allow more general perturbations that share sufficiently many properties of $\partial_{M/B}$ so as to not seriously affect the analysis. In a future publication we will make use of these more general perturbations.

Let

$$Q \in \rho_{\mathbb{R}((M/B)_0)}^{-1} \Psi_c^{-\infty}(M/B; E)$$

so that $\partial_{M/B} + Q$ has a model operator at every $N \in S_\psi(M)$, $y \in N^\circ$, modeling its behavior on the model wedge $\mathbb{R}^+_s \times \mathbb{R}^h_u \times \mathbb{R}^n_y$, acting on sections of the pull-back of $E|_{\mathbb{R}^n_y}$, given by

$$\mathcal{N}_y(\partial_{M/B} + Q) = \alpha(dx)\partial_s + \frac{1}{s}(D_{Z_y} + \mathcal{N}_y(\rho_y Q)) + D_{\mathbb{R}^h}.$$ 

Here $\mathcal{N}_y(\rho_y Q) = Q_{Z_y}$ is the restriction of the integral kernel of $\rho_y Q$ to the fiber of $\mathfrak{B}^{(2)}(\phi,y,N)$ over $y \in N^\circ$. Thus, as in (3.12), $Q_{Z_y}$ is a non-commutative suspension operator, translation invariant with respect to the Lie group $\mathbb{R}^+ \times \mathbb{R}^h$.

Our analysis of $\mathcal{N}_y(\partial_{M/B})$ in (2.3) made use of two convenient facts, first that $D_{Z_y}$ is independent of the variables $(s,u)$, and secondly that it anti-commutes with Clifford multiplication by covectors in $T^*(N/B)^+$. Our methods are insensitive to perturbations that maintain these two properties.

**Definition 4.1.** Let $M \to B$ be a family of manifolds with corners and iterated fibration structures with a vertical wedge metric $g_{M/B}$ and $E \to M$ a wedge Clifford module along the fibers of $\psi$. By a compatible perturbation (of the associated $\partial_{M/B}$) we will mean a self-adjoint family of operators $Q = Q_{M/B}$ satisfying two properties:

i) The integral kernel of $Q_{M/B}$ is an element of

$$\rho_{\mathbb{R}((M/B)_0)}^{-1} \beta^{(2)}(\phi,y_0) C^\infty(M \times \psi M; \text{Hom}(E)),$$

ii) At every $N \in S_\psi(M)$, $y \in N$,

$$\alpha(\theta)Q_{Z_y} + Q_{Z_y} \alpha(\theta) = 0,$$

for every covector $\theta$ in $T^*_y(N/B)^+$, where $Q_{Z_y}$ is the operator on $Z_y$ whose integral kernel is the restriction of $\rho_N Q_{M/B}$ to the fiber of $\mathfrak{B}_N \times \phi_N \mathfrak{B}_N \subseteq M \times \psi M$ over $y$.

**Remark 4.2.** In a subsequent paper we will study the existence of compatible perturbations. For the purpose of this paper we restrict ourselves to an example of how these will arise.

Consider a single manifold with boundary, $X$, whose boundary $\mathfrak{B}_Y$ participates in a fiber bundle of closed manifolds

$$Z \to \mathfrak{B}_Y \to Y,$$

together with a wedge Clifford module $(E, g_E, \nabla^E, \alpha)$ and associated Dirac-type operator $\partial_X$. The boundary family $D_{\mathfrak{B}_Y/Y}$ is a family of Dirac-type operators that anti-commute with Clifford multiplication in the $T^*Y^+$-directions and so determine an index class in the $C^*-K$-theory group $K_*(\mathbb{C}(T^*Y^+))$. This index vanishes if and only if there is a family of smoothing
operators $Q_{\mathcal{B}_Y/Y} \in \Psi^{-\infty}(\mathcal{B}_Y/Y; E)$ such that $D_{\mathcal{B}_Y/Y} + Q_{\mathcal{B}_Y/Y}$ is a family of invertible operators with the same anti-commutation property. If $q$ is any smooth function on $X^2$ that is equal to the Schwartz kernel of $Q_{\mathcal{B}_Y/Y}$ on $\text{diag}_Y \times Z^2 \subseteq \{ x = x^\prime = 0 \} \subseteq X^2$, and we set $Q_X = \rho_{\mathcal{B}_{\phi(Y)}}^{-1}(Z_{\phi})(q)$, then $Q_X$ is a compatible perturbation of $\bar{\partial}X$.

We will use the notation

$$\bar{\partial}_{M/B,Q} = \bar{\partial}_{M/B} + Q_{M/B}, \quad D_{Z,y,Q} = D_{Z,y} + Q_{Z,y}, \quad \text{etc.},$$

with the understanding that $Q_{M/B}$ is a compatible perturbation. We define the vertical APS domain of $\bar{\partial}_{M/B,Q}$ as the graph closure of $\rho_{X}^{1/2}H^1_e(X; E) \cap D_{\text{max}}(\bar{\partial}_{M/B,Q})$ and say that the Witt condition is satisfied if

$$0 \notin \text{Spec}(D_{Z,y,Q}),$$

where the spectrum refers to $D_{Z,y} + Q_{Z,y}$ with its vertical APS domain in $L^2(Z_y; E|_{Z_y})$. The compatibility conditions are chosen so that Proposition 2.7 holds after replacing $\bar{\partial}_{M/B}$ with $\bar{\partial}_{M/B,Q}$ with the same proof.

From §2.3 we know that the indicial roots of $\bar{\partial}_{M/B,Q}$ at $y \in N, N \in S_{\psi}(M)$ are equal to the positive eigenvalues of the induced Dirac-type operator $D_{Z,y,Q}$, acting on $L^2(\mathcal{B}_N/N; E|_N)$ with its vertical APS domain. Define an ‘indicial multiweight’, $\mathcal{I}$, for $M$ by

$$(4.1) \quad \mathcal{I}(\mathcal{B}_N) = \min \{ \lambda \in \text{Spec}(D_{Z,y,Q}) \cap \mathbb{R}^+: y \in N \} \text{ for all } N \in S_{\psi}(M)$$

and a corresponding multiweight $\mathcal{J}^{(2)}$ for $(M/B)^2_{e}$ by

$$\mathcal{J}^{(2)}(\mathcal{B}^{(2)}_{10}(N)) = \mathcal{J}^{(2)}(\mathcal{B}^{(2)}_{01}(N)) = \mathcal{I}(\mathcal{B}_N)$$

$$\mathcal{J}^{(2)}(\mathcal{B}^{(2)}_{\phi}(N)) = 2\mathcal{I}(\mathcal{B}_N) + \dim(N/B) + 1 \text{ for all } N \in S_{\psi}(M).$$

The weight at the front faces $\mathcal{B}^{(2)}_{\phi}(N)$ is explained by the composition formula for edge pseudodifferential operators: composing an operator with the given weights at the side faces produces this weight at the front faces. We use the same notation for the indicial multweights of $X$ and $X^2$.

**Theorem 4.3.** Let $\bar{\partial}_{M/B,Q}$ be a family of compatibly perturbed Dirac-type wedge operators endowed with its vertical APS domain and satisfying the Witt assumption. Then $(\bar{\partial}_{M/B,Q}, D_{\text{VAPS}})$ is a family of self-adjoint, Fredholm operators with compact resolvent. The generalized inverse of $\bar{\partial}_{M/B,Q}$ is a family valued in $\rho_{\Pi((M/B)^2_{e})}(\mathcal{B}_Y/Y)\Psi^{-1,3(2)}(M/B; E)$.

For each fiber $X$ of $\psi$, the eigenfunctions of $\bar{\partial}_{X,Q}$ are elements of $\rho_{X}^{2}H^\infty_e(X; E)$, the resolvent is a meromorphic function on $\mathbb{C}$ with values in the edge calculus with bounds,

$$(\bar{\partial}_{X,Q} - \lambda)^{-1} \in \rho_{\Pi(X^2)}\Psi^{-1,3(2)}(X; E),$$

and the projection onto the $\lambda$-eigenspace of $\bar{\partial}_{X,Q}$ satisfies

$$(4.2) \quad \Pi_{\lambda} \in \rho_{\Pi(X^2)}\Psi^{-\infty,3(2)}(X; E).$$
Define the indicial multiweight for the heat space, in terms of \([4.1]\), by
\[
\mathcal{J}^{(H)}(\mathcal{B}_{10,0}^{(N)}) = \mathcal{J}^{(H)}(\mathcal{B}_{01,0}^{(N)}) = \mathcal{J}(\mathcal{B}_N),
\]
(4.3) \[
\mathcal{J}^{(H)}(\mathcal{B}_{\phi,0}^{(N)}) = 2\mathcal{J}(\mathcal{B}_N) + \dim(N/B) + 1, \quad \mathcal{J}^{(H)}(\mathcal{B}_{\phi,1}^{(N)}) = \infty \ \forall N \in \mathcal{S}_N(M),
\]
and \(\mathcal{J}^{(H)}(\mathcal{B}_{00,1}^{(N)}) = \mathcal{J}(\mathcal{B}_{dd,1}^{(N)}) = \infty\).

We also define an index set for the heat space by
\[
\mathcal{H}(\mathcal{B}_{dd,1}^{(N)}) = 2, \quad \mathcal{H}(\mathcal{B}_{00,1}^{(N)}) = 0, \quad \text{and}
\]
(4.4) \[
\mathcal{H}(\mathcal{B}_{10,0}^{(N)}) = \mathcal{H}(\mathcal{B}_{01,0}^{(N)}) = \mathcal{H}(\mathcal{B}_{\phi,0}^{(N)}) = 0, \quad \mathcal{H}(\mathcal{B}_{\phi,1}^{(N)}) = 2 \ \forall N \in \mathcal{S}_N(M).
\]

**Theorem 4.4.** Let \(\partial_{M/B,Q}\) be a compatibly perturbed family of Dirac-type wedge operators endowed with its vertical APS domain and satisfying the Witt assumption. The heat kernel of \(\partial_{M/B,Q}^2\) satisfies
\[
e^{-\Omega_{M/B,Q}} \in \mathcal{A}^{m-1}_{H^\phi}(H(M/B); \text{Hom}(E) \otimes \Omega_{h,R})
\]
where \(\mathcal{J}^{(H)}\) and \(\mathcal{H}\) are given by (4.3), (4.4) and \(\Omega_{h,R}\) is the density bundle from (3.22).

The rest of this section consists of a proof of Theorems 4.3 and 4.4 by induction on the depth of \(M\). Our base case consists of closed manifolds, for which these theorems are well-known, even with a smoothing perturbation (e.g., [BGV04, Proposition 9.46], [MP97a, Appendix], [AR09a, AR09b, AR13]). Thus we now assume that these theorems are known for all spaces of depth less than \(k\), and that \(X\) has depth \(k\).

### 4.2. The model wedge

In the situation above, choose \(N \in \mathcal{S}_N(M), \ y \in N^\circ\), let \(Z = \phi_N^{-1}(y)\) and let
\[
\mathcal{N}_y(\partial_{M/B,Q}) = d(dx)\partial_s + \frac{1}{s} D_{Z,Q} + D_{R^h} = \mathcal{N}_y(D_{M/B,Q}) - d(dx)\frac{\dim Z}{2s}
\]
be the normal operator of \(\partial_{M/B,Q}\) on the model wedge at \(y, \mathbb{R}_s^+ \times \mathbb{R}^h \times Z\), from \(\S 2.3\). In this section, we make use of the inductive hypothesis that Theorems 4.3 and 4.4 hold for \(D_{Z,Q}\) to describe the Green’s function and heat kernel of \(\mathcal{N}_y(\partial_{M/B,Q})\).

Our assumptions on the perturbation and inductive hypothesis on the link invite us to analyze \(\mathcal{N}_y(\partial_{M/B,Q})\) by using the Fourier transform on \(\mathbb{R}^h\) and the Hankel transform on each eigenspace in a spectral decomposition on \(Z\) (as is done in, e.g., [Che79b], [CT82], [Che83], [Cho85], [Les97], \S 2.3, [Tay11], \S 8.8, [MV12] \S 3.2]).

Thus we consider
\[
(\mathcal{N}_y(D,Q))^2 = -\partial_s^2 + \frac{1}{s^2}((d(dx)D_{Z,Q} - \frac{1}{2})^2 - \frac{1}{4}) + \Delta_{R^h}
\]
as an operator on \(L^2(ds \, dq \, dz)\) and using the inductive hypothesis of discrete spectrum of \(D_{Z,Q}\) with its vertical APS domain, denote the eigenvalues of the self-adjoint operator \((d(dx)D_{Z,Q} - 1/2)^2\) by \(\{\ell_i\}_{i=1}^\infty\) and the corresponding eigesections by \(\{\phi_i\}\). As in [Tay11] \S 8.8, by writing
\[
F(s, z) = \sum f_i(s)\phi_i(z)
\]
for appropriate coefficients \(f_i\), we have
\[
(-\partial_s^2 + \frac{1}{s^2}((d(dx)D_{Z,Q} - \frac{1}{2})^2 - \frac{1}{4})) = \sum (-\partial_s^2 + \ell_i \frac{1}{s^2})(f_i(s)\phi_i(z))
\]
and this will equal $\mu^2 F(s, z)$ if we take

$$f_i(s) = \sqrt{s} J_{\nu_i}(\mu s)$$

with $\nu_i^2 = \ell_i$.

Note that there is a potential sign ambiguity in $\nu_i$. As $s \to 0^+$, $f_i(s) = \mathcal{O}(s^{\frac{1}{2} + \nu_i})$ will be in $L^2_{\text{loc}}(ds)$ for $\nu_i > -1$, so the ambiguity is only for the small eigenvalues $\ell_i < 1$. The choice of square root corresponds to different domains for $\mathcal{N}_y(\partial_{M/B,Q})$. In terms of the eigenvalues $\{\lambda_i\}$ of $d(dx)D_{Z,Q}$, we have $\lambda_i = (\lambda_i - \frac{1}{2})^2$ so this ambiguity corresponds to $\lambda$ in $(-\frac{1}{2}, \frac{1}{2})$. If we restrict attention to domains of $\mathcal{N}_y(\partial_{M/B,Q})^2$ induced from domains of $\mathcal{N}_y(\partial_{M/B,Q})$ then, arguing as in the proof of Proposition 2.7, the ambiguity corresponds to $\lambda$ in $(-\frac{1}{2}, \frac{3}{2}) \cap (-\frac{3}{2}, \frac{1}{2}) = (-\frac{1}{2}, \frac{1}{2})$. (Thus, as is well-known, these are the small eigenvalues that distinguish domains, see for example the discussion in [Cho85, top of page 37] in terms of which we are taking $\Delta_b$.)

In particular, as we are interested in the vertical APS domain for $\mathcal{N}_y(\partial_{M/B,Q})$, which induces

$$\mathcal{D}_{\text{VAPS}}(\mathcal{N}_y(\partial_{M/B,Q})^2) = \{ F \in \mathcal{D}_{\text{VAPS}}(\mathcal{N}_y(\partial_{M/B,Q})) : \mathcal{N}_y(\partial_{M/B,Q})(F) \in \mathcal{D}_{\text{VAPS}}(\mathcal{N}_y(\partial_{M/B,Q})) \},$$

we define $\nu_{\text{APS}}$ by

$$\nu_{\text{APS}}(\lambda) = \begin{cases} -|\lambda - \frac{1}{2}| & \text{if } \lambda \in \text{Spec}(D_{Z,y,Q}) \cap (0, \frac{1}{2}) \\ |\lambda - \frac{1}{2}| & \text{if } \lambda \in \text{Spec}(D_{Z,y,Q}) \setminus (0, \frac{1}{2}) \end{cases}$$

Then we can, as in [Tay11, Sec. 8.8], diagonalize $\mathcal{N}_y(\partial_{M/B,Q})^2$ by combining first the map

$$\mathcal{H}(g) = \bigoplus_{\lambda \in \text{Spec}(D_{Z,y,Q})} (H_{\nu_{\text{APS}}(\lambda)}(s^{-1/2} g_\lambda)),$$

where $H_{\nu_{\text{APS}}(\lambda)}$ denotes the Hankel transform, $g_\lambda$ denotes the projection of $g$ onto the corresponding (i.e., $(\lambda - \frac{1}{2})^2 - \frac{1}{4}$) eigenspace of $(d(dx)D_{Z,Q} - 1/2)^2 - 1/4$, and then the Fourier transform in $\mathbb{R}^h$. This yields a unitary map onto $L^2(\lambda \, d\lambda \, d\xi, \ell^2)$ which replaces $\mathcal{N}_y(\partial_{M/B,Q}^2)$ with multiplication by $\lambda^2 + |\xi|^2$.

The operator $\mathcal{N}_y(\partial_{M/B,Q})^2$ is injective on its vertical APS domain and has an (unbounded) inverse $\mathcal{G}$, determined by multiplication by $\mathcal{H}(\mathcal{G}g) = (\lambda^2 + |\xi|^2)^{-1} \mathcal{H}(g)$, and satisfying

$$\mathcal{N}_y(\partial_{M/B,Q})^2 \mathcal{G} = \text{Id}.$$ 

Below we will analyze the integral kernels of the heat kernel and of $\mathcal{G}$, in particular we will use the integral kernel of the unbounded operator

$$G = \mathcal{N}_y(\partial_{M/B,Q}) \mathcal{G},$$

the Green’s function for $\mathcal{N}_y(\partial_{M/B,Q})$, to construct a parametrix for $\partial_{M/B,Q}$.

**The heat kernel on the model wedge.** The heat kernel of a product is the product of the heat kernels, so to begin with let us disregard the factor of $\mathbb{R}^h$ and focus instead on the exact Riemannian cone

$$Z^+ = \mathbb{R}_{+}^2 \times Z, \quad g_{Z^+} = ds^2 + s^2 g_Z.$$

We denote $E$ pulled-back to $\mathbb{R}_{+}^2 \times Z$ by the same symbol, and the corresponding Dirac-type operator by

$$\partial_{Z^+,Q} = d(dx)\partial_s + \frac{1}{s} D_{Z,Q}.$$
It follows from Proposition 2.7 that $\partial_{Z^+,Q}$ is injective and self-adjoint with its vertical APS domain. Thus the corresponding domain for

$$\partial_{Z^+,Q}^2 = -\partial_z^2 + \frac{1}{s^2}((d^2(dx)D_{Z,Q} - \frac{1}{2})^2 - \frac{1}{4}),$$

namely

$$\mathcal{D}_{\text{VAPS}}(\partial_{Z^+,Q}^2) = \{ u \in \mathcal{D}_{\text{VAPS}}(\partial_{Z^+,Q}) : \partial_{Z^+,Q} u \in \mathcal{D}_{\text{APS}}(\partial_{Z^+,Q}) \},$$

is also a self-adjoint domain.

Let $e^{-\partial_{Z^+,Q}^2}$ be the heat kernel of $(\partial_{Z^+,Q}, \mathcal{D}_{\text{APS}}(\partial_{Z^+,Q}))$ considered as a density,

$$e^{-\partial_{Z^+,Q}^2} = \mathcal{K} \mu_R.$$

From the spectral theorem we know that $\mathcal{K}$ is a distribution on the space

$$(Z^+)^2 \times \mathbb{R}_t^+$$

such that:

- $\lim_{t \to 0} e^{-\partial_{Z^+,Q}^2} = \text{Id},$
- For every $t > 0$, the map

$$\mathcal{D}_{\text{VAPS}}(\partial_{Z^+,Q}^2) \ni s \mapsto e^{-\partial_{Z^+,Q}^2} s \in L^2(Z^+; E)$$

is valued in $\mathcal{D}_{\text{VAPS}}(\partial_{Z^+,Q}^2) = \bigcap \mathcal{D}_{\text{VAPS}}(\partial_{Z^+,Q})$. In particular, $\mathcal{K}(r, z, r', z', t)$ is smooth in all of its variables in the interior of $(Z^+)^2 \times \mathbb{R}_t^+$,

- For each $t$, $e^{-\partial_{Z^+,Q}^2}$ is a self-adjoint operator, and hence $\mathcal{K}(r, z, r', z', t) = \mathcal{K}(r', z', r, z, t)$.

We will improve these properties by showing that $e^{-\partial_{Z^+,Q}^2}$, viewed as a distribution on a different compactification of the interior of $(Z^+)^2 \times \mathbb{R}_t^+$, extends nicely to the boundary.

Recall, e.g., from [Les97, Proposition 2.3.9], that given $a > -1$ and $f \in \mathcal{C}_\infty(\mathbb{R}^+)$, the solution to

$$\begin{cases}
(\partial_t + (-\partial_z^2 + s^2(a^2 - \frac{1}{4})))u(s, t) = 0 \\
\lim_{t \to 0} u(s, t) = f(s)
\end{cases}$$

is given by

$$u(s, t) = \int_0^\infty \frac{\sqrt{s^2}}{2\sigma} I_{p(a^2)} \left( \frac{s^2}{2\sigma} \right) \exp \left( -\frac{s^2 + s'^2}{4\sigma} \right) f(s') ds',$$

where $I_{p(a^2)}$ denotes the modified Bessel function of the first kind, $p(a^2) = a$ if $a \geq 1$ and otherwise satisfies $p(a^2) \in \{ \pm a \}$ with different choices corresponding to different domains of $(-\partial_z^2 + s^2(a^2 - \frac{1}{4}))$ as an unbounded operator on $L^2(\mathbb{R}^+)$.

In projective coordinates as above, this shows that this heat kernel is a right density times the function

$$\frac{\sqrt{s^2}}{2\sigma} I_{p(a^2)} \left( \frac{s}{2\sigma^2} \right) \exp \left( -\frac{s^2 + s'^2}{4\sigma^2} \right).$$

Hence we can write the heat kernel on the exact cone as

$$(4.9) \sum_{\lambda \in \text{Spec}(D_{Z^Q})} \frac{\sqrt{s^2}}{2\sigma} I_{\nu_{\text{APS}}(\lambda)} \left( \frac{s^2 + (s')^2}{4\sigma^2} \right) \Phi_\lambda(z, \bar{z}).$$

where $\nu_{\text{APS}}$ is given by (4.5) and $\Phi_\lambda(z, \bar{z})$ is the projection onto the $\lambda$-eigenspace of $d^2(dx)D_{Z^Q}$. Convergence of this sum in the space of polyhomogeneous conormal distributions is used in [MVT12, Proposition 3.2]. See also [Che83, Example 3.1].
To establish the asymptotics of this kernel our strategy, following Mooers [Moo99] and others, e.g., [Che83, §2], [Les97, §2.2], is to exploit the homogeneity of the cone. For each $c > 0$, we set
\[ \Upsilon_c : (Z^+) \times \mathbb{R}^+ \longrightarrow (Z^+) \times \mathbb{R}^+, \quad \Upsilon_c(s, z, s', z', t) = (cs, z, cs', z', c^2t); \]
we use the same symbol to denote the corresponding scalings on $(Z^+) \times \mathbb{R}^+$ and trust that this will not lead to confusion.

As $E$ is pulled-back from $Z$, it makes sense to pull-back a section of $E$ over $Z^+$ along $\Upsilon_c$ and it is easy to see that
\[ \Upsilon_c^* : C^\infty_0((Z^+) \circ E) \longrightarrow C^\infty_0((Z^+) \circ E) \]
extends to a bounded map on $L^2(Z^+; E)$ and satisfies
\[ \Upsilon_c^* \circ r \partial_r = r \partial_r \circ \Upsilon_c^*, \quad \Upsilon_c^* \circ \partial_z = \partial_z \circ \Upsilon_c^*. \]
It follows that $\Upsilon_c^*$ preserves $\mathcal{D}_{VAPS}(\bar{\partial}_{Z^+, Q})$ for any $\ell \in \mathbb{N}$ and satisfies
\[ \Upsilon_c^* \circ \bar{\partial}_{Z^+, Q} = c^{-1} \bar{\partial}_{Z^+, Q} \circ \Upsilon_c^*, \quad \Upsilon_c^* \circ (t \partial_t + t \bar{\partial}_{Z^+, Q}) = (t \partial_t + t \bar{\partial}_{Z^+, Q}) \circ \Upsilon_c^*. \]
In particular, if $u$ is a solution of the heat equation with initial data $f$, then $\Upsilon_c^* u$ solves the heat equation with initial data $\Upsilon_c^* f$. However,
\[ \Upsilon_c^* u(\zeta, t) = \int_0^t \int_{Z^+} K(cs, z, s', z', c^2t) f(s', z') \, ds' \, dz' \, dt \]
\[ \quad \overset{s' = c r}{\longrightarrow} c \int_0^t \int_{Z^+} K(cs, z, cr, z', c^2t) f(cr, z') \, dr' \, dz' \, dt, \]
so uniqueness for the heat equation shows that $\Upsilon_c^* K = c^{-1} K$ and hence, as a right density, the heat kernel is invariant under this dilation,
\[ (4.10) \quad \Upsilon_c^* e^{-\bar{\partial}_{Z^+, Q}} = e^{-\bar{\partial}_{Z^+, Q}}. \]
This is the dilation invariance that we will exploit.

To do so, we first blow-up $\{ t = 0 \}$ parabolically so that $\tau = \sqrt{t}$ is a generator of the smooth structure. Secondly we blow-up the cone-tip at time zero, to obtain the space
\[ H_1 Z^+ = [\mathbb{R}_s^+ \times Z \times \mathbb{R}_s^+ \times Z' \times \mathbb{R}_z^+] \cong Z \times Z' \times \mathbb{S}_+^2 \times \mathbb{R}_R^+ \]
where $\mathbb{S}_+^2 = \{ (\omega_s, \omega_s', \omega_r) \in [0, \infty)^3 : \omega_s^2 + \omega_s'^2 + \omega_r^2 = 1 \}$. The blow-down map is
\[ H_1 Z^+ \overset{\beta}{\longrightarrow} Z^+ \times Z^+ \times \mathbb{R}_t^+ \]
\[ (z, z', (\omega_s, \omega_s', \omega_r), R) \longmapsto ((R \omega_s, z), (R \omega_s', z'), R^2 \tau^2) \]
and we note that $\Upsilon_c$ lifts to be simply $R \mapsto c R$. We denote $\{ R = 0 \}$ by $\mathfrak{B}_R(H_1 Z^+)$. Instead of using polar coordinates, we can use projective coordinates
\[ r = \frac{s}{s'}, \quad z, \quad s', \quad z', \quad \sigma = \frac{t}{s'} \]
valid away from $\omega_s' = 0$, in which $s'$ is a boundary defining function for $\mathfrak{B}_R(H_1 Z^+)$ and $\Upsilon_c$ is $s' \mapsto c s'$.
Let \( \pi_L : Z^+ \times Z^+ \times \mathbb{R}_t^+ \rightarrow Z^+ \) be the projection onto the left factor of \( Z^+ \), and let \( \beta_L = \pi_L \circ \beta \). We have
\[
\beta_L^* D_{Z,+}^{\beta} = (s')^{-2} \left( -\partial_s^2 + \frac{1}{s}((dZ)D_{Z,Q} - \frac{1}{2})^2 - \frac{1}{4} \right).
\]
The plan is to identify the heat kernel at \( s' = 1 \) and then use dilation invariance.

To this end, let \( \chi : \mathbb{R} \rightarrow \mathbb{R}^+ \) satisfy
\[
\chi(r) = \begin{cases} 
1 & \text{if } |r - 1| < 1/4 \\
0 & \text{if } |r - 1| > 1/2
\end{cases}
\]
and define the operator
\[
\tilde{D} = d'(dx)\partial_s + ((1 - \chi(s)) + \frac{\chi(s)}{s})D_{Z,Q},
\]
which we can interpret as an operator on \( S^1 \times Z \) where \( S^1 = [0, 2]/0 \sim 2 \) and coincides (thought of as an operator on \( \mathbb{R}_+ \times Z \)) with \( D_{Z,+},Q \) on \( 3/4 \leq r \leq 5/4 \).

Let \( e^{-t\tilde{D}^2} \) denote the heat kernel of \( \tilde{D}^2 \) on \( S^1 \times Z \) endowed with its vertical APS domain and note that the inductive hypothesis applies to it. We consider \( e^{-t\tilde{D}^2} \) as a right density and denote it as \( \tilde{K}\mu_R \).

We will multiply the model wedge heat kernel \( K \) by cut-off functions to obtain a kernel on \( S^1 \times Z \) that we can compare to \( e^{-t\tilde{D}^2} = \tilde{K}\mu_R \). Let \( \tilde{\chi} : \mathbb{R} \rightarrow \mathbb{R}_+ \) satisfy \( \chi\tilde{\chi} = \tilde{\chi} \) and \( \tilde{\chi}(r) = 1 \) for \( |r - 1| < \delta \) for some \( \delta > 0 \). Set
\[
F_1(r, z, r', z', \tau) = \tilde{\chi}(r)\tilde{\chi}(r')K(r, z, r', z', \tau)
\]
so that
\[
(\partial_t + \tilde{D}^2)F_1(r, z, r', z', \tau) = E(r, z, r', z', \tau)
\]
Note that \( E \equiv 0 \) for \( |r - 1| < \delta \), indeed
\[
E(r, z, r', z, \tau) = (-\tilde{\chi''}(r)K(r, z, r', z', \tau) - 2\tilde{\chi'}(r)\partial_r(K(r, z, r', z', \tau)))\tilde{\chi}(r').
\]
Letting \( F_2(r, z, r', z', \tau) = \tilde{K} \circ E \), so \( (\partial_t + \tilde{D}^2)(F_1 - F_2) = 0 \), we have

- \( F_2(r, z, r', z', \tau) = O(r^\infty) \) for \( r \in [1 - \delta, 1 + \delta] \). Indeed, in
\[
\int_0^t \int_{S^1 \times Z} \tilde{K}(r, z, r'', z'', \tau - s) \circ E(r'', z'', r', z', s) ds dr'' dz'',
\]
where \( r'' \in [0, 2] \) is the variable on \( S^1 \), we can take the spatial integral over the set \( V = U^c \) where \( U = \{ r'' \in [1 - \delta, 1 + \delta] \} \) since on \( U \), \( E \equiv 0 \), but on \( V \), if the left variable \( r \in U \) we have \( \tilde{K} = O(r^\infty) \) by our inductive hypothesis.

- The restriction of \( F_1 - F_2 \) to \( r'' = 1 \) satisfies
\[
\lim_{\tau \rightarrow 0}(F_1 - F_2)(r, z, 1, z', \tau) = \tilde{\chi}(r, z)\delta(r, z)(1, z') = \delta(r, z)(1, z').
\]

By uniqueness, it follows that
\[
\tilde{K}(r, z, 1, z', \tau) = (F_1 - F_2)(r, z, 1, z', \tau), \quad \tilde{K}(r, z, s', z', \tau) = (s')^{-1}(F_1 - F_2)(r, z, 1, z', \tau),
\]
in particular
\[
\tilde{K} - K = O(\tau^\infty) \text{ for } r'' = 1, \ r \in [1 - \delta, 1 + \delta],
\]
and this goes most of the way toward proving:
Proposition 4.5. The heat kernel of the normal operator, $e^{-tN_y(\mathcal{B}_{M/B,Q})}$, is a conormal distribution in the space

$$\mathcal{B}_{phg}^{(H)} \mathcal{A}^{-m-1}(\mathcal{B}_{\phi,1}(N); \text{Hom}(E) \otimes \rho_{dd,1}^{-2\beta(H),R} \Omega(M/B))$$

where $\mathcal{B}_N^{(H)}$ is the restriction of $\mathcal{B}_N^{(H)}$ to $\mathcal{B}_{\phi,1}(N)$, and

$$\mathcal{R}_N(\mathcal{B}_{10,0}^{(H)} \cap \mathcal{B}_{\phi,1}^{(H)}) = \mathcal{R}_N(\mathcal{B}_{01,0}^{(H)} \cap \mathcal{B}_{\phi,1}^{(H)}) = \mathcal{R}_N(\mathcal{B}_{00,0}^{(H)} \cap \mathcal{B}_{\phi,1}^{(H)}) = \mathcal{R}_N(\mathcal{B}_{0,0,0}^{(H)} \cap \mathcal{B}_{\phi,1}^{(H)}) = 0,$$

that inverts the vertical heat operator $\frac{1}{2} \sigma \bar{\partial}_s + \sigma^2(N_y(\mathcal{B}_{M/B,Q}))^2$.

Proof of Proposition 4.5 under inductive hypothesis. Without loss of generality we assume that $B = pt$, so $X = M$ is a manifold with corners with iterated fibration structure of depth $k$ carrying a single wedge Dirac type operator; thus the base of a boundary hypersurface $\mathcal{B}_Y$ corresponds to a heat front face $\mathcal{B}_{\phi,1}^{(H)}(Y)$ that is a bundle with fibre $HZ_{Sus}(w)$ with $Z$ of depth less than $k$, and again without loss of generality we assume that $Y$ is a point since the heat kernel is product type, so $\mathcal{B}_{\phi,1}^{(H)}(Y) = HZ_{Sus}(w)$. By our inductive hypothesis, Theorem 4.4 applies to $\tilde{K}$ above, and thus $\tilde{K}$ lies in $\mathcal{B}_{phg}^{(H)} \mathcal{A}^{-m-1}(H(S^1 \times Z)_w; \text{Hom}(E) \otimes \Omega_{b,R})$. The lift of the set $s' = 1$ to $H(S^1 \times Z)_w$ is a suspended heat space itself, and a neighborhood of its diagonal, say $s \in [1 - \delta, 1 + \delta]$, may be identified with the same neighborhood in $HZ_{Sus}(w)$.

From the expression for the heat kernel in (4.9), the high order asymptotics of the modified Bessel functions, and the inductive hypothesis, we see that for any $\varepsilon > 0$, with $s \in [0, 1 - \varepsilon)$, $s' = 1$, the heat kernel has the appropriate asymptotics at the side faces. This together with (4.12) gives the proposition. □

Green’s function on the model wedge. We now prove the analogous statement above for the right inverse $G(N)$ of $N_y(\mathcal{B}_{M/B,Q})$.

Proposition 4.6. Let $N \in S_\nu(M)$ and denote the Green’s function of $N_y(\mathcal{B}_{M/B,Q})$ constructed above by $G(N)$. The integral kernel of $G(N)$ is an element of

$$\Psi_{N_{sus}(T(N/B)^+)}^{-1}(\mathcal{B}_N/N; E)$$

where $\mathcal{B}_N^{(2)}$ denotes the restriction of the indicial multiweight to $\mathcal{B}_N^{(2)}(N)$.

Proof of Proposition 4.6 under inductive hypothesis. The proof is directly analogous to that of Proposition 4.5 and we use the notation from that proof. First we study $\tilde{G}$, the Green’s function of the self-adjoint extension of $N_y(\mathcal{B}_{M/B,Q})^2 = \sigma_{Z,+}^2 + \Delta_{\mathbb{R}^h}$ from (4.6). Letting $\tilde{D}$ be as in (4.11), the operator $\tilde{L} := \tilde{D}^2 + \Delta_{\mathbb{R}^h}$ on sections of $E$ over $S^1 \times Z \times \mathbb{R}^h$ is the square of a Dirac-type operator on a lower depth space, and thus the inductive hypothesis applies to it. Writing $F_1(s, z, y, s', z', y') = \tilde{\chi}(s)\tilde{G}\tilde{\chi}(s')$ and thinking of this as the Schwartz kernel of an operator on the $S^1 \times Z \times \mathbb{R}^h$, we have $\tilde{L}F_1 - \tilde{\chi} = E$ is a smooth distribution, in fact vanishing for $|s - 1|, |s' - 1| < \delta$. Writing $\tilde{G} \circ E = F_2$ and noting that all operators here can be taken symmetric, using $\tilde{G}\tilde{L} = I - \Pi$, we see that in the region $|s - 1|, |s' - 1| < \delta$, $F_1$ differs from $\tilde{G}$ by a smooth function, and thus restricted to $s' = 1$ is a polyhomogeneous conormal distribution on the suspended double space. Homogeneity in $s, s'$ then gives the result. □
Remark 4.7. Recall from our discussion of the normal operators above that $\mathfrak{B}_{\phi \phi}^{(2)}(\bar{Y}) \cap \mathfrak{B}_{\phi \phi}^{(2)}(\bar{Y})$ is a front face of the resolved (suspended) double space on which the normal operators’ Schwartz kernels live. It follows from the proof and the inductive hypothesis that for $N' < N$, $G(N)|_{\mathfrak{B}_{\phi \phi}^{(2)}(N) \cap \mathfrak{B}_{\phi \phi}^{(2)}(N')}$ is equal to the Green’s function for the normal operator on that face.

Remark 4.8. The computation of the null space of the normal operator in Proposition 2.7 can be used to write down integral kernels of the inverses of these operators as in, e.g., [DS88, §XIII.3, Theorem 16]. For the vertical APS domain we have, see e.g., [BS88, Lemma 4.1]

\[
(-\partial_s^2 + \frac{1}{x^2}((\partial'(dx)D_{Z,Q} - \frac{1}{2})^2 - \frac{1}{4}) + z^2)^{-1}(r, \bar{r})
\]

\[
= \bigoplus_{\lambda \in \text{Spec}(A_{\nu})} \sqrt{r\bar{r}}I_{\nu_{APS}}(\lambda)(rz)K_{\nu_{APS}}(\lambda)(\bar{r}z)\Pi_{\lambda}, \quad \text{for } r < \bar{r}
\]

(and interchanging $r$ with $\bar{r}$ for $\bar{r} < r$) where $\nu_{APS}$ is defined in (4.5) and $\text{Im}(z^2) \neq 0$. It is a bit delicate to establish the asymptotic behavior of the inverse from this explicit expression.

4.3. Resolvent of $\bar{\partial}_{M/B,Q}$. We have described, for each $N \in \mathcal{S}_\psi(M), y \in N^\circ$, the inverse of the normal operator $\mathcal{N}_y(\bar{\partial}_{M/B,Q})$. Putting these together, we specified the integral kernel

\[G(N) = \{G_y(N) : N \in \mathcal{S}_\psi(M), y \in N\}.
\]

In fact this also determines the integral kernel over the boundary of each $N \in \mathcal{S}_\psi(M)$. Indeed, recall that the structure of $\mathfrak{B}_{\phi \phi}^{(2)}(N)$ is described in Proposition 3.5. At a boundary hypersurface $\mathfrak{B}_{N,N}$ of $N$, fibered over $N' < N$, the fibers of $\mathfrak{B}_{\phi \phi}^{(2)}(N)$ comprise one of the front faces of $(Z')^2_e$. Namely, they comprise the front face corresponding to the boundary hypersurface $\mathfrak{B}_{N,N'}$ of $Z'$ in the notation of (3.4). Since the integral kernel $G(N')$ has been specified over $N'$, taking its normal operator over the front face corresponding to $\mathfrak{B}_{N,N'}$, yields the extension of $G(N)$ to points over the boundary hypersurface $\mathfrak{B}_{N,N}$ of $N$. By Remark 4.7, these normal operators fit together smoothly since at each intersection $\mathfrak{B}_{\phi \phi}^{(2)}(N) \cap \mathfrak{B}_{\phi \phi}^{(2)}(N')$ the restriction of the integral kernel is the Green’s function of the model operator induced by $\bar{\partial}_{M/B,Q}$ with its vertical APS domain.

We now proceed as in [ALMP13, §4] and obtain a parametrix for $\bar{\partial}_{M/B,Q}$ from the integral kernels $G(N)$.

Proposition 4.9. Let $\bar{\partial}_{M/B,Q}$ be a Dirac-type wedge operator endowed with its vertical APS domain and satisfying the Witt assumption. There is an edge pseudodifferential operator in the calculus with bounds

\[G(M) \in \rho_{H_e((M/B)^2)}\Psi^{-1,3}_{e}(M/B; E),\]

such that $\mathcal{N}_y(G(M))\bar{\partial}_{M/B,Q} = \text{Id}$ for each $N \in \mathcal{S}_\psi(M), y \in N^\circ$, and hence

\[G(M)\bar{\partial}_{M/B,Q} - \text{Id} = \rho_{H_e((M/B)^2)}\Psi^{-1,3}_{e}(M/B; E).
\]

Proof. At each $N \in \mathcal{S}_\psi(M)$, recall that $\mathcal{N}_y(\bar{\partial}_{M/B,Q})$ is not the restriction to $\mathfrak{B}_{\phi \phi}^{(2)}(N)$ of the lift of $\bar{\partial}_{M/B,Q}$ to $(M/B)^2_e$, because this lift is not tangent to that boundary hypersurface. Instead we have

\[\mathcal{N}_y(\bar{\partial}_{M/B,Q}) = \frac{1}{s}\mathcal{N}_y(\rho_N\bar{\partial}_{M/B,Q}).\]
forcing equality of these three spaces. So

g are bounded operators, as are

\( G \) for \( \rho \).

Proof of Theorem 4.3. To simplify notation we assume that

\[ \text{Id} \] and \( G \) are adjoints and specifying domains when necessary for clarity,

\( \Box \).

We proceed as in [Mel93, Proof of Proposition 5.43] to choose an extension of these kernels

\( \tilde{e} \).

This deals with the first operator in (4.14), for the second we point out that

\( \text{Id} \) and

\( \text{Id} \) tend

\( e \) 60 PIERRE ALBIN AND JESSE GELL-REDMAN

\( \text{Id} \) and

\( \text{Id} \) tend

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\( \text{Id} \) and

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\( \text{Id} \) and

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\( \text{Id} \) and

\( \text{Id} \) tend

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Next note that the mapping properties in Theorem 3.7 show that

\( G(X), R = \partial_{X,Q} \circ G(X) - \text{Id}: L^2_w = H^1_\text{c}(X; E) \rightarrow \rho^{1/2} H^1_\text{c}(X; E) \),

are bounded operators, as are \( G(X)^* \) and \( R^* \) (since as is well-known their integral kernels are obtained from those of \( G(X) \) and \( R \) by interchanging the two factors of \( X^2 \)). Taking adjoints and specifying domains when necessary for clarity,

\( (\partial_{X,Q}, \mathcal{D}_{\text{VAPS}}) \circ G(X) = \text{Id} - R \iff (\partial_{X,Q} \circ G(X))^* = \text{Id} - R^* \),

and since \( G(X)^* \circ (\partial_{X,Q}, \mathcal{D}_{\text{VAPS}})^* \subseteq (\partial_{X,Q} \circ G(X))^* \), we have that

\( u \in \mathcal{D} ((\partial_{X,Q}, \mathcal{D}_{\text{VAPS}})^*) \)

\( \Rightarrow u = G(X)^* \partial_{X,Q} u + R^* u \in D_{\text{max}}(\partial_{X,Q}) \cap \rho^{1/2} H^1_\text{c}(X; E) \subseteq \mathcal{D}_{\text{VAPS}}(\partial_{X,Q}) \)

so

\( \mathcal{D}_{\text{VAPS}}(\partial_{X,Q})^* \subset D_{\text{max}}(\partial_{X,Q}) \cap \rho^{1/2} H^1_\text{c}(X; E) \subset \mathcal{D}_{\text{VAPS}}(\partial_{X,Q}) \),

forcing equality of these three spaces.

Since \( \rho^{1/2} H^1_\text{c}(X; E) \) is compactly contained in \( L^2_w \), the resolvent and the errors \( G(X)\partial_{X,Q} - \text{Id} \) and \( \partial_{X,Q}G(X) - \text{Id} \) are compact on \( L^2_w \), so the Fredholm and discrete spectrum properties follow.
Now note that, since $G(X)$ is a compact operator, it is simultaneously a parametrix for $(\tilde{\partial}_{X,Q} - \lambda)$ for all $\lambda$. It follows that the space of eigensections of a given eigenvalue is an element of $\rho_{\text{III}}(X^2_\mathbb{C})\Psi^{-1,3(2)}_e(X; E)$, and hence eigensections are in $\rho_X^*H^\infty_e(X; E)$.

For $\lambda \in \mathbb{C}$ define $R_i(\lambda)$ by

$$G(X)(\tilde{\partial}_{X,Q} - \lambda) = \text{Id} - R_1(\lambda), \quad (\tilde{\partial}_{X,Q} - \lambda)G(X) = \text{Id} - R_2(\lambda)$$

and note that $R_i(\lambda) \in \rho_{\text{III}}(X^2_\mathbb{C})\Psi^{-1,3(2)}_e(X; E)$. These errors can be improved to $R_i(\lambda) \in \rho_{\text{III}}(X^2_\mathbb{C})\Psi^{-\infty,3(2)}_e(X; E)$ by the standard symbolic construction. For each $\lambda \in \mathbb{C}\setminus \text{Spec}(\tilde{\partial}_X, \mathcal{D}_{\text{VAPS}})$ we have (cf. [Maz91] (4.25)]

$$(\tilde{\partial}_{X,Q} - \lambda)^{-1} = G(X) + R_1(\lambda)(\tilde{\partial}_{X,Q} - \lambda)^{-1}R_2(\lambda) + R_1(\lambda)G(X)$$

which, by virtue of Theorems 3.9 and 3.10, is an element of $\rho_{\text{III}}(X^2_\mathbb{C})\Psi^{-1,3(2)}_e(X; E)$. (As in the proof of [Maz91] Theorem 4.20, the weights at the various faces of $X^2_\mathbb{C}$ follow from the fact that this inverts $(\tilde{\partial}_{X,Q} - \lambda)$.) This formula also shows that the resolvent is holomorphic as a map from $\mathbb{C}\setminus \text{Spec}(\tilde{\partial}_{X,Q}, \mathcal{D}_{\text{VAPS}})$ into $\rho_{\text{III}}(X^2_\mathbb{C})\Psi^{-1,3(2)}_e(X; E)$ and analytic Fredholm theory shows that it extends to a meromorphic function on $\mathbb{C}$.

Writing the projection onto the $\lambda$-eigenspace as a contour integral, together with elliptic regularity, then proves (4.2) and completes the induction and the proof of Theorem 4.3. □

**Remark 4.10.** It is easy to see from this construction that every family of wedge Dirac-type operators whose vertical APS domain satisfies the Witt condition can be connected smoothly through wedge Dirac-type operators to a family whose vertical APS domains satisfy the geometric Witt condition.

Indeed recall, e.g., from [Hit74] §1.4, [Val01] §A.2, that for a conformal change of metric $g_\omega = \omega^2 g$ there is a Clifford bundle adapted to this metric with Dirac-type operator $D_\omega = \omega^{-1}(\omega^{-(n-1)/2}D\omega^{(n-1)/2})$. Thus if we scale each of the metrics $g_Z$ we can vary the operators, through Dirac-type operators, and push away any small indicial roots while maintaining the Witt condition. We lose special structures, e.g., this variation will take the signature operator through Dirac-type operators not equal to the signature operators of the varying metrics. This yields a family over $B \times [0,1]$ with the original family at $B \times \{0\}$, and such that the family over $B \times \{1\}$ satisfies the geometric Witt condition. The wedge Dirac-type operators over $B \times [0,1]$ with their vertical APS domains form a smooth family of Fredholm operators. (For a discussion of smoothness of this family of operators, see [MP97a], the vertical APS domain corresponds to the spectral section coming from a spectral gap at zero.)

### 4.4. Heat kernel of $\tilde{\partial}_{M/B,Q}^2$.

The construction of the heat kernel proceeds by solving model problems at the critical boundary hypersurfaces of the wedge heat space. In Proposition 4.5 we have described the solution of the model problem at each $\mathfrak{B}^{(H)}_{\phi \psi,1}(N), N \in \mathcal{S}_\psi(M)$.

We can similarly solve the model problem at $\mathfrak{B}^{(H)}_{dd,1}$. This proceeds exactly as in, e.g., [Mel93] Chapter 7, [AM]. The result is naturally compatible with $e^{-t\mathcal{N}_\lambda(\tilde{\partial}_{M/B,Q}^2)}$ at $\mathfrak{B}^{(H)}_{\phi \psi,1}(N) \cap \mathfrak{B}^{(H)}_{dd,1}$ and combining these we find a conormal density $H_{1,1}$ with Schwartz kernel

$$H_{1,1} \in \mathcal{D}_{phg}^{H/3(H)} \mathcal{A}^{-m-1}(H(M/B)_w; \text{Hom}(E) \otimes \Omega_{h,R})$$
with $\mathcal{H}$ the index set from (4.4) and

$$\beta^*_{(H),L}(t(\partial_t + \partial^2_{M/B,Q})) H_{1,1}$$

\[ \in \rho^1_{H(\mathcal{H})} \rho_{H(\mathcal{H})} \rho_{H(\mathcal{H})} \rho_{dd,1} \mathcal{B}^{H(\mathcal{H})}_{\phi\phi} \mathcal{A}_{-m-1}(H(M/B)_w; \text{Hom}(E) \otimes \Omega_{b,R}). \]

Indeed, the vanishing at $\text{ff}(H(M/B)_w)$ and $\mathcal{B}^{(H)}_{dd,1}$ comes from solving the model problems at these faces, while the singular power of $\rho(H(M/B)_w)$ comes from the singularity of $\mathcal{B}_{M/B,Q}$ in $x$ as $x \to 0$, at each boundary hypersurface. This singular term would \textit{a priori} be of order $-2$, but proceeding as in [Mel93] Proof of Proposition 5.43 the extension can be carried out so that the leading term vanishes.

We can improve this parametrix by removing the Taylor expansion at $\mathcal{B}^{(H)}_{dd,1}$ exactly as in [Mel93] Chapter 7. This results in

$$H_{1,\infty} \in \mathcal{B}^{H(\mathcal{H})}_{\phi\phi} \mathcal{A}_{-m-1}(H(M/B)_w; \text{Hom}(E) \otimes \Omega_{b,R}),$$

$$\beta^*_{(H),L}(t(\partial_t + \partial^2_{M/B,Q})) H_{1,\infty}$$

\[ \in \rho^1_{H(\mathcal{H})} \rho_{H(\mathcal{H})} \rho_{H(\mathcal{H})} \rho_{dd,1} \mathcal{B}^{H(\mathcal{H})}_{\phi\phi} \mathcal{A}_{-m-1}(H(M/B)_w; \text{Hom}(E) \otimes \Omega_{b,R}). \]

Similarly, we can solve away the expansion at $\text{ff}(H(M/B)_w)$ by proceeding as in Proposition 7.28 of [Mel93] which in this context takes the following form.

**Proposition 4.11.** For each $N \in \mathcal{S}_\phi(M)$, let $\rho(H < N) = 1$ if $N$ is minimal in $\mathcal{S}_\phi(M)$ and otherwise

$$\rho(H < N) = \prod_{N' \in \mathcal{S}_\phi(M)} \rho_{\mathcal{B}^{H(\mathcal{H})}_{\phi\phi,1}(N')}.$$ Given

$$f \in \rho_{dd,1} \rho_{H(\mathcal{H})} \rho_{\mathcal{B}^{H(\mathcal{H})}_{\phi\phi,1}(N)} \mathcal{A}_{-m-1}(\mathcal{B}^{(H)}_{\phi\phi,1}(N); \text{Hom}(E) \otimes \rho_{dd,1} \beta^*_{(H),R} \Omega(M/B))$$

the equation

$$\beta^*_{(H),L}(t(\partial_t + \partial^2_{M/B,Q})) \mathcal{B}^{(H)}_{\phi\phi,1}(N) u = f$$

has a unique solution

$$u \in \rho_{dd,1} \rho_{H(\mathcal{H})} \rho_{\mathcal{B}^{H(\mathcal{H})}_{\phi\phi,1}(N)} \mathcal{A}_{-m-1}(\mathcal{B}^{(H)}_{\phi\phi,1}(N); \text{Hom}(E) \otimes \rho_{dd,1} \beta^*_{(H),R} \Omega(M/B)).$$

**Proof.** As usual, the solution to the equation is given by

$$u(t, \zeta, \zeta') = \int_0^t \int_M e^{-\mathcal{B}^{(H)}_{\phi\phi,1}(N)(\partial^2_{M/B,Q})} (t-s, \zeta, \zeta') f(s, \zeta'', \zeta') ds d\zeta''$$

but in order to understand the structure of $u$ it is best to express this in terms of pull-back and push-forward. The asymptotics of the first factor are given by Proposition 1.5 and the composition is a particular case of Proposition B.1 (e.g., by extending off of $\mathcal{B}^{(H)}_{\phi\phi,1}(N)$, composing, and then restricting back), which gives the asymptotics of the result. □

We can use this proposition to solve away the expansion at $\text{ff}(H(M/B)_w)$, one face at a time. Let $\{N_1, \ldots, N_\ell\}$ be the list of $\mathcal{S}_\psi(M)$ used in the construction of the heat space.
Using this proposition to solve away successive terms at $\mathfrak{B}^{(H)}_{\phi,1}(N_1)$ we can construct, for any $\ell \geq 1$, an improved parametrization

$$H_{\ell,\infty}^{N_1} \in \mathfrak{B}^{(H/\gamma)}_{\rho_{\text{phg}}} \mathscr{A}^{-\ell-1}(H(M/B)_w; \text{Hom}(E) \otimes \Omega_{h,R}),$$

$$\beta_{(H),L}^*(t(\partial_t + \bar{\partial}_M^{2}(B,Q)))H_{\ell,\infty}^{N_1} \in \rho_{\text{H}(H(M/B)_w)}^{-1} \rho_{\text{ef}(H(M/B)_w)}^{-1} \rho_{\text{H}(H(M/B)_w)}^{-1} \mathfrak{B}^{(H/\gamma)}_{\text{phg}} \mathscr{A}^{-\ell-1}(H(M/B)_w; \text{Hom}(E) \otimes \Omega_{h,R}).$$

Asymptotically summing successive differences we can remove the error at $\mathfrak{B}^{(H)}_{\phi,1}(N)$ altogether,

$$H_{\infty,\infty}^{N_1} \in \mathfrak{B}^{(H/\gamma)}_{\rho_{\text{phg}}} \mathscr{A}^{-m-1}(H(M/B)_w; \text{Hom}(E) \otimes \Omega_{h,R}),$$

$$\beta_{(H),L}^*(t(\partial_t + \bar{\partial}_M^{2}(B,Q)))H_{\infty,\infty}^{N_1} \in \rho_{\text{H}(H(M/B)_w)}^{-1} \rho_{\text{ef}(H(M/B)_w)}^{-1} \rho_{\text{H}(H(M/B)_w)}^{-1} \mathfrak{B}^{(H/\gamma)}_{\text{phg}} \mathscr{A}^{-m-1}(H(M/B)_w; \text{Hom}(E) \otimes \Omega_{h,R}).$$

Relabeling $H_{\infty,\infty}^N = H_{\infty}^N$, we now proceed in the same way at $\mathfrak{B}^{(H)}_{\phi,1}(N_2)$. After carrying this out at $\mathfrak{B}^{(H)}_{\phi,1}(N_1), \ldots, \mathfrak{B}^{(H)}_{\phi,1}(N_{\ell})$, we end up with

$$H_{\infty,\infty}^{N_1,\ldots,N_\ell} \in \mathfrak{B}^{(H/\gamma)}_{\rho_{\text{phg}}} \mathscr{A}^{-m-1}(H(M/B)_w; \text{Hom}(E) \otimes \Omega_{h,R}),$$

$$\beta_{(H),L}^*(t(\partial_t + \bar{\partial}_M^{2}(B,Q)))H_{\infty,\infty}^{N_1,\ldots,N_\ell} \in \rho_{\text{H}(H(M/B)_w)}^{-1} \rho_{\text{ef}(H(M/B)_w)}^{-1} \rho_{\text{H}(H(M/B)_w)}^{-1} \mathfrak{B}^{(H/\gamma)}_{\text{phg}} \mathscr{A}^{-m-1}(H(M/B)_w; \text{Hom}(E) \otimes \Omega_{h,R}).$$

Note that the error now vanishes to infinite order at all boundary hypersurfaces lying over $\{t = 0\}$, so we can just as well view it as a distribution on a simpler space, $\mathbb{R}^+ \times (M/B)_{\epsilon}$,

$$\pi_L^*(t \partial_t + \bar{\partial}_M^{2}(B,Q))H_{\infty,\infty} \in \rho_{\text{H}(H(M/B)_w)}^{-1} \rho_{\text{ef}(H(M/B)_w)}^{-1} \mathfrak{B}^{(H/\gamma)}_{\text{phg}} \mathscr{A}^{-m-1}(\mathbb{R}^+ \times (M/B)_{\epsilon}; \text{Hom}(E) \otimes \pi_R^* \Omega(M/B)).$$

A natural next step (as in [Mel93, Proposition 7.17]) is to interpret the heat kernel as an operator with respect to convolution in $t$, so that the error term satisfies

$$\pi_L^*(t \partial_t + \bar{\partial}_M^{2}(B,Q))H_{\infty,\infty} = \text{Id} - A$$

with $A \in \rho_{\text{H}(H(M/B)_w)}^{-1} \rho_{\text{ef}(H(M/B)_w)}^{-1} \mathfrak{B}^{(H/\gamma)}_{\text{phg}} \mathscr{A}^{-m-1}(\mathbb{R}^+ \times (M/B)_{\epsilon}; \text{Hom}(E) \otimes \pi_R^* \Omega(M/B))$ and use a Volterra series to invert $\text{Id} - A$.

**Proposition 4.12.** Let $A$ be the operator above, then

$$(\text{Id} - A)^{-1} = \text{Id} + S,$$

with $S \in \rho_{\text{H}(H(M/B)_w)}^{-1} \rho_{\text{ef}(H(M/B)_w)}^{-1} \mathfrak{B}^{(H/\gamma)}_{\text{phg}} \mathscr{A}^{-m-1}(\mathbb{R}^+ \times (M/B)^2_{\epsilon}; \text{Hom}(E) \otimes \pi_R^* \Omega(M/B))$.

**Proof.** Fix $t_0 > 0$. Let us write the Schwartz kernel of $A$ as $\mathcal{K}_A(\rho_{\text{H}(H(M/B)_w)} \rho_{\text{ef}(H(M/B)_w)})^{-1+\varepsilon \mu_R}$ for any fixed

$$\varepsilon \in (0, \min_{N \in S_v(M)} \mathfrak{I}(\mathfrak{B}_N^2)),$$

and point out that $|\mathcal{K}_A|_{t \leq t_0}$ is uniformly bounded on $(M/B)_\epsilon^2$, say by $C$, and that

$$V_\varepsilon = \max_B \int_{(M/B)_\epsilon^2} \rho_M^{-1+\varepsilon \mu} < \infty.$$
During this proof, let us write $\rho_\natural(H(M/B)_w)\rho_\natural(H(M/B)_w)$ as $x$.

If we similarly write the Schwartz kernel of $A^k$ as $K_{A^k}x^{-1+\varepsilon}\mu_R$ then assuming that $|K_{A^k}|_{t \leq t_0}$ is bounded by $C_k t_0^k/k!$ we see from

$$K_{A^{k+1}}(t,\zeta,\zeta''')x^{-1+\varepsilon}\mu_{\zeta'''} = \int_0^t \int_{\zeta''\in\psi^{-1}(\psi(\zeta))}(K_{A^k}(s, \zeta, \zeta')(x^{-1+\varepsilon}\mu_{\zeta'})(K_A(t-s, \zeta', \zeta''')(x')^{-1+\varepsilon}\mu_{\zeta''}')) \, ds$$

that

$$|K_{A^{k+1}}|_{t \leq t_0} \leq \int_0^{t_0} \int_{\zeta'''\in\psi^{-1}(\psi(\zeta))}(Cs^k/k!)C(x')^{-1+\varepsilon}\mu_{\zeta'''} \, ds \leq CC_k V_\varepsilon t_0^{k+1}/(k+1)!$$

Hence we see that the Volterra series $\sum A^k$ converges uniformly for $t \leq t_0$ and arbitrary $t_0$. We may run the same argument after differentiating by any vector field on $(M/B)^2 \times \mathbb{R}^+$ that is tangent to the boundary hypersurfaces, and so we can conclude that the Volterra series converges in the space of conormal sections of $\text{Hom}(E)$.

**Theorem 4.13.** Let $\delta_{M/B,Q}$ be a family of compatibly perturbed Dirac-type wedge operators acting on a Clifford bundle $E$ on a family of manifolds with corners and iterated fibration structures, $M \rightarrow \psi$ $B$. The heat kernel of $\delta_{M/B,Q}^2$ satisfies

$$e^{-\delta_{M/B,Q}^2} \in \mathcal{R}_{phg}^{H/\mathcal{H}(H)} \mathcal{I}_{-m-1}(H(M/B)_w; \text{Hom}(E) \otimes \Omega_{b,R})$$

where $\mathcal{I}(H)$ and $\mathcal{H}$ are given by (4.3), (4.4) and $\Omega_{b,R}$ is the density bundle from (3.22). The leading terms at $\mathcal{B}_{ady,1}^H$ and each $\mathcal{B}_{\phi,1}^H(N)$ are given by

$$\mathcal{N}_{\mathcal{B}_{ady,1}^H}(e^{-\delta_{M/B,Q}^2}) = e^{-\Delta_{T(M/B)}}$$

$$\mathcal{N}_{\mathcal{B}_{\phi,1}^H}(e^{-\delta_{M/B,Q}^2}) = e^{-\sigma^2\Delta_{T(N/B)}} e^{-\sigma^2D_{C(\phi_N)}}$$

where $e^{-\Delta_{T(M/B)}}$ denotes the Euclidean heat kernel on the fibers of $T(M/B)$ (at time one) and similarly $e^{-\sigma^2\Delta_{T(N/B)}}$, $C(\phi_N)$ denotes the mapping cylinder of $\phi_N$.

(4.15) $\mathcal{Z}^+ \rightarrow C(\phi_N) \xrightarrow{\phi_N^*} N$

and $D_{C(\phi_N)}$ is the family of operators $N \ni y \mapsto D_{Z_y^+}$ from (4.7).

**Proof.** The operator $H_{\infty,\infty}(\text{Id} - S)$ satisfies the wedge surgery heat equation with initial condition given by the (lift of the) identity since

$$\beta_{H,L}(\partial_t + \delta_{M/B,Q}^2)(G_{\infty}(\text{Id} - S)) = (\text{Id} - A)(\text{Id} - S) = \text{Id}.$$  

The composition result, Proposition [B.1], yields the asymptotics of this composition. The leading terms are immediate from the construction above.  

5. **Getzler rescaling and the trace of the heat kernel**

5.1. **Getzler rescaling.** The heat kernel construction above did not significantly use that $\delta_{M/B}^2$ is the square of a Dirac-type operator rather than an arbitrary Laplace-type operator. We now refine the construction to take advantage of the Clifford action on $E$ and its compatibility with $\delta_{M/B,Q}$. Specifically, we proceed as in [Mel93, Chapter 8] to carry out Getzler
rescaling geometrically (see also [DM12, Va01, AR09a]) by constructing ‘rescaled homomorphism bundles’ on the resolved heat space which capture the relationship between the heat kernel and the Clifford action. The second property in Definition 4.1 ensures that compatible perturbations will not affect the discussion. After carrying out the rescaling for the family of Dirac-type operators we carry out the analogous rescaling for the Bismut superconnection.

Recall the decomposition of the homomorphism bundle of $E$ from (1.21),

$$\text{Hom}(E) \cong \mathcal{C}l(\omega T^* M/B) \otimes \text{hom}'_{\mathcal{C}l(\omega T^* M/B)}(E).$$

The heat kernel is a section of $\text{Hom}(E) \rightarrow H(M/B)_w$, the lift of $\pi_L^* E^* \otimes \pi_R^* E$ from $M \times \psi M$ to $H(M/B)_w$. The restriction of this bundle to $\mathcal{B}_M$ is (canonically isomorphic to) $\text{hom}(E)$ and hence inherits the decomposition above. In this way we see that $\text{Hom}(E) \rightarrow H(M/B)_w$ has compatible filtrations at $\mathcal{B}_{dd,1}$ and each $\mathcal{B}_{\phi,1}^H(N)$,

$$\text{Hom}(E)_{\mathcal{B}_{dd,1}^H} = \mathcal{C}l^*(\omega T^* M/B) \otimes \text{Hom}'_{\mathcal{C}l(\omega T^* M/B)}(E),$$

$$\text{Hom}(E)_{\mathcal{B}_{\phi,1}^H(N)} = \mathcal{C}l^*(\omega T^* N/B) \otimes \text{Hom}'_{\mathcal{C}l(\omega T^* N/B)}(E),$$

where $\text{Hom}'_{\mathcal{C}l(\omega T^* M/B)}(E)$ denotes the elements of $\text{Hom}(E)$ that commute with $\mathcal{C}l(\omega T^* M/B)$, and similarly $\text{Hom}'_{\mathcal{C}l(\omega T^* N/B)}(E)$. In fact we can extend the filtration of $\text{Hom}(E)_{\mathcal{B}_{\phi,1}^H(N)}$ even further by including Clifford multiplication by $d \rho_{\mathcal{B}_N}$, but it will be convenient not to do so. It is easy to see from (1.1) that these filtrations are compatible.

We define a connection on $\text{Hom}(E) \rightarrow H(M/B)_w$ by

$$\nabla_{\text{Hom}(E)} = \partial_t dt \otimes \beta_L^* \nabla^* \otimes \beta_R^* \nabla$$

and then choose vector fields $\nu$ and, for each $N \in \mathcal{S}_\psi(M)$, $\nu_N$ transverse to $\mathcal{B}_{dd,1}^H$ and $\mathcal{B}_{\phi,1}^H(N)$, respectively, and tangent to all other boundary hypersurfaces (e.g., by modifying the vector fields in a boundary product structure as in (1.1)). We define the space of rescaled sections of $\text{Hom}(E)$ by

$$\Gamma = \{ s \in C^\infty(H(M/B)_w; \text{Hom}(E)) :$$

$$\text{for } j \in \{0, \ldots, \dim M/B \}, \quad (\nabla_{\nu}^\text{Hom(E)})^j s|_{\mathcal{B}_{dd,1}^H} \in \mathcal{C}l^j(\omega T^* M/B) \otimes \text{Hom}'_{\mathcal{C}l(\omega T^* M/B)}(E),$$

$$\text{for each } N \in \mathcal{S}_\psi(M) \text{ and } k \in \{0, \ldots, \dim N/B\},$$

$$\text{(}\nabla_{\nu_N}^\text{Hom(E)})^k s|_{\mathcal{B}_{\phi,1}^H(N)} \in \mathcal{C}l^k(\omega T^* N/B) \otimes \text{Hom}'_{\mathcal{C}l(\omega T^* N/B)}(E)\}.$$
to increase filtration degree by at most one (where $\tilde{v}$ is the appropriate transverse vector field), and for $(\nabla^{\text{Hom}(E)}_v)(K^{\text{Hom}(E)}(\tilde{v}, W))$ to increase filtration degree by at most two for all $j \geq 1$.

**Lemma 5.1.** If $W \in C^\infty(H(M/B)_w; T(H(M/B)_w))$ is tangent to the fibers of the fiber bundles

$$\mathfrak{B}^{(H)}_{dd,1} \to \text{diag}_M \quad \text{and} \quad \mathfrak{B}^{(H)}_{\phi,1}(N) \to N \quad \text{for all} \ N \in \mathcal{S}_\psi(M),$$

then $\nabla^{\text{Hom}(E)}_W$ acts on sections of $\text{Hom}_G(E)$.

In particular, if $W'$ is any vector field on $\mathbb{R}^+ \times M \times \psi M$, vertical with respect to the projection onto $B$, then $\nabla^{\text{Hom}(E)}_{\beta'}(\sqrt{T}W')$ acts on sections of $\text{Hom}_G(E)$.

**Proof.** Since $\nabla^E$ is a Clifford connection, it satisfies

$$[\nabla^E, \mathcal{L}(\theta)] = \mathcal{L}(\nabla^E \theta).$$

Hence it is immediate that $\nabla^{\text{Hom}(E)}$ preserves the filtration at $\mathfrak{B}^{(H)}_{dd,1}$, while checking that it preserves the filtration at $\mathfrak{B}^{(H)}_{\phi,1}(N)$ comes down to checking that in a local frame as in (1.11)

$$g(\nabla_{W_1}W_2, \tilde{U}) = \mathcal{O}(x) \quad \text{for all} \ W_1 \in \mathcal{V}, W_2 \in \{\partial_x, \frac{1}{x}V\},$$

which follows from (1.14). (Incidentally, this is why we do not rescale at $\mathfrak{B}^{(H)}_{\phi,1}(N)$ by $\mathcal{C}(T^*(N/B)^+)$, as the connection would not preserve this filtration.)

Next recall that

$$K^{\text{Hom}(E)}(W_1, W_2) = K^E((\beta_L)_*, W_1, (\beta_L)_*, W_2) \circ \cdots \circ K^E((\beta_R)_*, W_1, (\beta_R)_*, W_2)$$

and that, since $\nabla^E$ is a Clifford connection,

$$K^E(S_1, S_2) = \frac{1}{4} \mathcal{L}(R(S_1, S_2)) + K^{E'}(S_1, S_2), \quad \text{with} \ K^{E'}(S_1, S_2) \in C^\infty(M, \text{hom}_{\mathcal{C}(\mathcal{L}(T^*M/B))}(E)).$$

Thus the covariant derivatives of the curvature of $\text{Hom}(E)$ involve at most two Clifford multiplications and hence can move the filtrations at $\mathfrak{B}^{(H)}_{dd,1}$ and each $\mathfrak{B}^{(H)}_{\phi,1}(N)$ by at most two.

It follows that $\nabla^{\text{Hom}(E)}_W$ will act on $\mathcal{D}$ whenever

$$R((\beta)_*, \nu)|_{\mathfrak{B}^{(H)}_{dd,1}(N)} = 0, \quad \text{and} \quad R((\beta)_*, \nu)|_{\mathfrak{B}^{(H)}_{\phi,1}(N)} = 0 \quad \text{in} \ C^\infty(\mathfrak{B}_N, \Lambda^1(T^*N/B) \wedge \Lambda^1(dx) \otimes xT^*\mathfrak{B}_N/N).$$

where $\beta$ can be either $\beta_L$ or $\beta_R$. If $W|_{\mathfrak{B}^{(H)}_{dd,1}}$ is tangent to the fibers of $\mathfrak{B}^{(H)}_{dd,1} \to \text{diag}_M$ then $(\beta)_*W|_{\mathfrak{B}^{(H)}_{dd,1}} = 0$, so the first condition holds. From Proposition 1.5 we know that the second condition will hold as long as $(\beta)_*W|_{\mathfrak{B}^{(H)}_{\phi,1}(N)}$ is tangent to the fibers over $N^\circ$. \hfill \Box

Now that we know that the connection acts on $\mathcal{D}$, we can use the Lichnerowicz formula (see, e.g., [Mel93, §8.8])

$$\bar{\Delta}^2_{M/B} = \Delta^{M/B} + \frac{1}{4} \text{scal}_M + \frac{1}{2} \sum_{a,b} K^E(e_a, e_b)\mathcal{L}(e^a) \mathcal{L}(e^b)$$

in which $\Delta^{M/B}$ is the Bochner Laplacian of $\nabla^E$ and the sum runs over an orthonormal frame of $TM/B$, to see that $t\bar{\Delta}^2_{M/B}$ acts on sections of $\text{Hom}_G(E)$.
For vector fields as in (1.11), we can read off from [Mel93, (8.36)] and Proposition 1.5 the rescaled normal operators, for each $N \in \mathcal{S}_\psi(M)$ we have

$$\mathcal{N}^G_{\mathfrak{B}_{\phi_{N}}(N)}(\nabla^E_{\tau \partial_y}) = \sigma \partial_y, \quad \mathcal{N}^G_{\mathfrak{B}_{\phi_{N}}(N)}(\nabla^E_{\tau \partial_y}) = \sigma(\partial_y + \frac{1}{4} \epsilon (R^{N/B}(\mathcal{R}, \partial_y)),$$

where $\mathcal{R}$ denotes the radial vector field in $e^T N/B$ and the appearance of the curvature $R^{N/B}$ can be traced back to Proposition 1.5 (1), and

$$\tau_{\mathfrak{B}_{\phi_{N}}(N)}(\nabla^E_{\tau \partial_y}) = \sigma \left( \nabla^E_{\tau \partial_y} + \frac{1}{4} \sum_{i=1}^{2} g^{N/B}(\mathfrak{S}^\phi_{N}(V, V'), \tilde{U}) \epsilon ((\frac{1}{2} V')^i) \epsilon (\tilde{U}^i) + \frac{1}{2} \sum_{i=1}^{2} g^{N/B}(\mathfrak{R}^\phi_{N}(\tilde{U}, V), \mathcal{E}(\tilde{U}^i \wedge (\tilde{U}^i)') \right)$$

where the appearance of $\mathfrak{S}^\phi_{N}$, $\mathfrak{R}^\phi_{N}$ traces back to (2)-(3) in Proposition 1.5. We have similar behavior at $\mathfrak{B}_{\phi_{N}}(N)$, save that all vector fields are horizontal.

Combining this with the Lichnerowicz formula we obtain their rescaled normal operator.

**Lemma 5.2.** Let $\mathfrak{B}_{M/B}$ be a family of Dirac-type wedge operators on the fibers of $M \to B$. The rescaled normal operators of $\tau^2 \mathfrak{B}_{M/B}$ are

$$\mathcal{N}^G_{\mathfrak{B}_{\phi_{N}}(N)}(\tau^2 \mathfrak{B}_{M/B}) = - \sum_{(\partial_y) = e^{TN/B}} (\partial_y) \epsilon (R(\partial_y, \mathcal{R}_{e^{TN/B}})) + \epsilon (K_E)$$

$$\mathcal{N}^G_{\mathfrak{B}_{\phi_{N}}(N)}(\tau^2 \mathfrak{B}_{M/B}) = - \sigma^2 \sum_{(\partial_y) = e^{TN/B}} (\partial_y) \epsilon (R(\partial_y, \mathcal{R}_{e^{TN/B}})) + \sigma^2 \mathfrak{B}_{\mathfrak{C}(\phi_{N})/N}$$

where $\mathfrak{B}_{\mathfrak{C}(\phi_{N})/N}$ assigns to each $b \in B$ the Bismut superconnection on the mapping cylinder of $\phi_{N_b}(= \phi_{N_{\mathfrak{f}^{-1}N_b}})$, $Z^+ : C(\phi_{N_b}) \to N_b$.

(Noe that the rescaled normal operator of $\tau^2 \mathfrak{B}_{M/B}$ corresponding to $N \in \mathcal{S}_\psi(M)$ is a family of superconnections; for the rescaled superconnection of $\tau^2 \mathfrak{A}_{M/B}$ we will instead obtain the superconnection of a family.)

**Proof.** For the rescaled normal operator at $\mathfrak{B}_{\phi_{N}}(N)$ for some $N \in \mathcal{S}_\psi(M)$, we start by considering $\tau^2$ times the Bochner Laplacian in (5.1),

$$\tau^2 \Delta_{M/B} = - \sum_{e^a}(\nabla^E_{\tau e^a})^* (\nabla^E_{\tau e^a})$$

where the sum runs over an orthonormal frame of $TM/B$. We can write this as a sum over a frame of $e^{TN/B}$ plus a sum over a frame of the orthogonal complement; the former has rescaled normal operator equal to the harmonic oscillator $-\sigma^2 \sum_{(\partial_y) = e^{TN/B}} (\partial_y) + \frac{1}{2} \epsilon (R(\partial_y, \mathcal{R}_{e^{TN/B}}))^2$, while the latter has rescaled normal operator equal to $\sigma^2$ times the Bochner Laplacian term in the Lichnerowicz formula (1.22) for the Bismut superconnection of $C(\phi_{N})/N$. The twisting curvature term in (5.1) gives rise to the twisting curvature term in (1.22) and similarly for the scalar curvature terms, since, at each $\mathfrak{B}_N$, $\text{scal}(X, g_w) \sim \rho_N^2 \text{scal}(C(Z), d\rho_N^2 + \rho_N^2 g_Z) + \mathcal{O}(\rho_N^2)$.

The rescaled normal operator at $\mathfrak{B}_{\phi_{N}}(N)$ is similar but simpler. 

\[\square\]
This same analysis applies to the Bismut superconnection $\mathcal{A}_{M/B}$ from Section 1.4. Indeed, one can either repeat the analysis above or consider the parameter $\varepsilon$ from section 1.4 and note that these results for $\varepsilon > 0$ imply the analogous results for $\varepsilon = 0$. First, if $Q$ is a compatible perturbation of $\partial M/B$ then we define

$$\mathcal{A}_{M/B,Q} = \mathcal{A}_{M/B} + Q_{M/B}, \quad \mathcal{A}_{C(\phi_N)}/N,Q = \mathcal{A}_{C(\phi_N)/N} + Q_{B/N}$$

where $Q_{B/N}$ is the family $N \ni y \mapsto Q_{Z_y}$.

In this case the bundle $E$ is replaced by $E = \psi^* \Lambda^* T^* B \otimes E$, the connection $\nabla^E$ by $\nabla^{E,0}$ and the Clifford action by $\mathcal{C}_0$. For the bundle $E$, we set

$$\text{Hom}(E) = \psi^* \Lambda^* T^* B \otimes \text{Hom}(E)$$

and

$$\mathcal{C}_0^{(wT^* M)} = C \otimes \mathcal{C}(\psi^* T^* B \oplus T^* M/B, g_M,0)$$

so that

$$\text{Hom}(E)|_{\mathcal{B}^{(H)}_{dd,1}} = \mathcal{C}_0^{(wT^* M)} \otimes \text{Hom}'_{\mathcal{C}(\psi T^* M/B)}(E),$$

$$\text{Hom}(E)|_{\mathcal{B}^{(H)}_{\psi,1}(N)} = \mathcal{C}_0^{(wT^* N)} \otimes \text{Hom}'_{\mathcal{C}(\psi T^* N/B)}(E),$$

for each $N \in \mathcal{S}_\psi(M)$.

We define a connection on $\text{Hom}(E) \to H(M/B)_w$ by

$$\nabla^\text{Hom}(E) = \partial_t dt \otimes \beta^*_E \nabla^{E,0} \otimes \beta_R \nabla^{E*,0}$$

and the space of rescaled sections of Hom$(E)$ by

$$\mathcal{D} = \{ s \in C^\infty(H(M/B)_w; \text{Hom}(E)) :$$

for $j \in \{0, \ldots, \text{dim } M\}$, \( (\nabla^\text{Hom}(E))^j s|_{\mathcal{B}^{(H)}_{dd,1}} \in \mathcal{C}_0^{l_j(wT^* M)} \otimes \text{Hom}'_{\mathcal{C}(\psi T^* M/B)}(E), \)

for each $N \in \mathcal{S}_\psi(M)$ and $k \in \{0, \ldots, \text{dim } N\}$,

$$\left(\nabla^\text{Hom}(E)^k s|_{\mathcal{B}^{(H)}_{\psi,1}(N)} \in \mathcal{C}_0^{l_k(wT^* N)} \otimes \text{Hom}'_{\mathcal{C}(\psi T^* N/B)}(E)\right).$$

The corresponding rescaled bundle is denoted $\text{Hom}_\mathcal{C}(E)$.

The rescaled normal operators of $\mathcal{A}_{M/B}^2$ are similar to those of $\mathcal{A}_{M/B}^2$ but valued in differential forms in $M$ instead of differential forms in $M/B$. Indeed, the Lichnerowicz formula (1.22) combined with the above yields the following.

**Lemma 5.3.** Let $\mathcal{A}_{M/B}$ be a family of Dirac-type wedge operators on the fibers of $M \xrightarrow{\psi} B$, and let $\mathcal{A}_{M/B}$ be a Bismut superconnection extending $\partial M/B$. The rescaled normal operators of $\tau^2 \mathcal{A}_{M/B}^2$ are

$$\mathcal{N}^{G}_{\mathcal{B}^{(H)}_{dd,1}}(\tau^2 \mathcal{A}_{M/B}^2) = - \sum_{\langle \partial_j \rangle = e_{TM/B}} (\partial_j + \frac{1}{4} \mathcal{C}(R(\partial_j, \mathcal{R}_{e_{TM/B}})))^2 + \mathcal{C}(K'_E) = \mathcal{H}_{M/B}$$

$$\mathcal{N}^{G}_{\mathcal{B}^{(H)}_{\psi,1}(N)}(\tau^2 \mathcal{A}_{M/B}^2) = - \sigma^2 \sum_{\langle \partial_j \rangle = e_{TN/B}} (\partial_j + \frac{1}{4} \mathcal{C}(R(\partial_j, \mathcal{R}_{e_{TN/B}})))^2 + \sigma^2 \mathcal{A}_{C(\phi_N)/N}^2$$

$$= \sigma^2 (\mathcal{H}_{N/B} + \mathcal{A}_{C(\phi_N)/N}^2)$$

where $\mathcal{A}_{C(\phi_N)/N}$ is the induced superconnection for the family of cones given by the mapping cylinder of $\phi_N$, $C(Z) \xrightarrow{\psi} C(\phi_N) \to N$ from (1.15).
Remark 5.4. Note from \cite{[1,23]} that the unrescaled normal operators of $\tau^2 A^2_{M/B}$ would involve the tensors of $\psi_N$ but not the tensors of $\phi_N$. The rescaling makes explicit the contribution of the tensors of $\phi_N$ to the expansion at $\mathcal{B}_{\phi,1}^{(H)}(N)$.

Theorem 5.5. Let $A_{M/B}$ be the Bismut superconnection associated to a family of perturbed Dirac-type wedge operators acting on a Clifford bundle $E$ on a family of manifolds with corners and iterated fiber bundle structures, $M \rightarrow B$, and let $Q$ be a compatible perturbation. The heat kernel of $A^2_{M/B,Q}$ satisfies
\begin{equation}
e^{-tA^2_{M/B,Q}} \in \mathcal{B}^{(H)}(E) \otimes \mathcal{F}_{M/B}^{-1}(H(M/B)_w, \text{Hom}_G(E) \otimes \Omega_{b,R}).
\end{equation}
where $\mathcal{I}(H)$ and $\mathcal{H}$ are given by \cite{[3,4]}, \cite{[4.4]} and $\Omega_{b,R}$ is the density bundle from \cite{[3.22]}. The leading terms at $\mathcal{B}_{dd,1}$ and each $\mathcal{B}_{\phi,1}^{(H)}(N)$ are given by
\begin{align*}
N_{\mathcal{S}_{\phi,1}^{dd}}(e^{-tA^2_{M/B,Q}}) &= e^{-\mathcal{H}^2_{M/B}} \mathcal{N}_{\mathcal{S}_{\phi,1}^{dd}}(N) = e^{-2\mathcal{H}^2_{M/B}} e^{-\phi_{N,B}/N,Q}
\end{align*}

Proof. We proceed as in \cite{[11]}. Secondly we recall (e.g., \cite{[Mel93], [BGV04], Chapter 9 Appendix]) that in a situation like ours where $\mathcal{F} = H + \mathcal{F}_{[+]}$ with $\mathcal{F}_{[+]}$ nilpotent we have
\begin{equation}
\exp(-t\mathcal{F}) = \exp(-tH) + \sum (-1)^k I_k,
\end{equation}
with the nilpotence of $\mathcal{F}_{[+]}$ guaranteeing that the $I_k$ are eventually zero.

We apply this at $\mathcal{B}_{\phi,1}^{(H)}(N)$ to see that heat kernel of the rescaled normal operator is a section with the same asymptotics as those of the heat kernel of the normal operator. (Since $\mathcal{F}_{[+]}$ is a tensor and so its integral kernel is supported on the diagonal.) Mehler’s formula directly yields a solution of the model heat problem at $\mathcal{B}_{dd,1}^{(H)}$. These models are compatible at the corner because the restriction to the corner solves the corresponding model problem.

The rest of the construction proceeds as in \cite{[4.4]}. □

5.2. Trace of the heat kernel. We discuss the trace of integral kernels on the heat space (extending \cite{[MV12], Theorem 4.2]) and then specialize to the case of the heat kernel.

The trace of an operator is intimately related to the integral of its Schwartz kernel along the diagonal (see \cite{[Bri88]} for a general discussion). The interior lift of the diagonal of $M$ from $M \times \psi M \times \mathbb{R}^+_t$ to $H(M/B)_w$ can be identified with
\begin{equation}
\text{diag}^{(H)}_w(M) = \left[ M \times \mathbb{R}^+; \mathcal{B}_{N_1} \times \{0\}; \cdots ; \mathcal{B}_{N_\ell} \times \{0\} \right],
\end{equation}
where $\tau = t^{1/2}$ and $S_\psi(M) = \{N_1, \ldots, N_\ell\}$ is listed in a non-decreasing order. The fiber bundle $X \rightarrow M \rightarrow B$ induces a fiber bundle
\begin{equation}
\text{diag}^{(H)}_w(X) \rightarrow \text{diag}^{(H)}_w(M) \rightarrow B,
\end{equation}
which we continue to denote $\psi$.

We will denote the blow-down map by

$$\beta_\Delta : \text{diag}_w^{(H)}(M) \longrightarrow M \times \mathbb{R}_+^\tau,$$

and the collective boundary hypersurfaces of $\text{diag}_w^{(H)}(M)$ by

$$M \times \{0\} \leftrightarrow \mathcal{B}_{0,1}^{(\Delta)}$$

and, for each $N \in \mathcal{S}_\psi(M)$,

$$\mathcal{B}_N \times \mathbb{R}_+^\tau \leftrightarrow \mathcal{B}_{1,0}^{(\Delta)}(N), \quad \mathcal{B}_N \times \{0\} \leftrightarrow \mathcal{B}_{1,1}^{(\Delta)}(N).$$

Assume that the kernel of $A$ has the form $K_A \rho^h \mu_R$ with $K_A \in \mathcal{A}_{phg}^E(H(M/B)_w; \text{Hom}(E))$ and $h$ the multiweight from (3.23). Ultimately we are interested in kernels that are merely conormal with bounds acting on sections of a vector bundle, but the corresponding trace result will follow easily from this one.

For appropriate index sets, this kernel is trace-class and its trace is the integral of its restriction to the diagonal. In terms of the composition of the blow-down map with the projection onto $B \times \mathbb{R}_+^\tau$,

$$\beta_\Delta,t : \text{diag}_w^{(H)}(M) \longrightarrow M \times \mathbb{R}_+^\tau \longrightarrow B \times \mathbb{R}_+^\tau,$$

this means that

$$\text{Tr}(A) = (\beta_\Delta,t)_* \left(K_A \rho^h \mu_R \rvert_{\text{diag}_w^{(H)}(M)}\right).$$

**Theorem 5.6.** If $A$ has integral kernel $K_A \rho^h \mu_R$ with $K_A \in \mathcal{A}_{phg}^E(H(M/B)_w; \text{Hom}(E))$ and

$$\text{Re}(E_A(\mathcal{B}^{(H)}_{\phi,0}(N))) - \dim(N/B) > -1$$

for all $N \in \mathcal{S}_\psi(M)$ then it is trace-class with trace polyhomogeneous in $\tau$ satisfying

$$\text{Tr} A \in \mathcal{A}_{phg}^{E_A,\tau}(\mathbb{R}_+^\tau; C^\infty(B)),$$

$$E_{A,\tau} = (E_A(\mathcal{B}^{(H)}_{\text{diag},1}) - \dim(M/B) - 2) \bigcup_{N \in \mathcal{S}_\psi(M)} (E_A(\mathcal{B}^{(H)}_{\phi,1}(N))) - \dim(N/B) - 2)$$

*Proof.* The operator is trace-class because its integral kernel is a bounded smooth function times a measure of finite volume.

Next let us discuss the effect of restricting the kernel to the diagonal. Let $\mu_\Delta = \beta_\Delta^* \mu(M/B)$. The restriction of the weight (3.23) to the diagonal is

$$h^\Delta : M_1(\text{diag}_w^{(H)}) \longrightarrow \mathbb{R},$$

$$h^\Delta(H) = \begin{cases} -(\dim(N/B) + 3) & \text{if } H \subseteq \mathcal{B}_{1,1}^{(\Delta)}(N) \text{ for some } N \in \mathcal{S}_\psi(M) \\ -(\dim(N/B) + 1) & \text{if } H \subseteq \mathcal{B}_{1,0}^{(\Delta)}(N) \text{ for some } N \in \mathcal{S}_\psi(M) \\ -(\dim(M/B) + 2) & \text{if } H = \mathcal{B}_{0,1}^{(\Delta)}(N) \\ 0 & \text{otherwise} \end{cases}$$

Let $E_A^\Delta$ be the index sets given by

$$E_A^\Delta(\mathcal{B}_{0,1}^{(\Delta)}) = E_A(\mathcal{B}^{(H)}_{dd,1})$$

and, for each $N \in \mathcal{S}_\psi(M)$,

$$E_A^\Delta(\mathcal{B}_{1,0}^{(\Delta)}(N)) = E_A(\mathcal{B}^{(H)}_{\phi,0}(N)), \quad E_A^\Delta(\mathcal{B}_{1,1}^{(\Delta)}(N)) = E_A(\mathcal{B}^{(H)}_{\phi,1}(N)).$$
Then we have
\[ K_A \rho^h \mu_R \big|_{\text{diag}(H)} = K_A \rho^h \mu_\Delta \text{ with } K_A \in \mathcal{A}_{\text{phg}}^E \left( \text{diag}(H) \right) \]

Next note that the map \( \beta_{(\Delta),t} \) is a b-fibration which sends \( \{ \mathfrak{B}_{1,0}^{(\Delta)}(N) : N \in \mathcal{S}_\psi(M) \} \) to the interior of \( \mathbb{R}_+^\times \) and the other boundary hypersurfaces to \( \{ \tau = 0 \} \), so we can apply the push-forward theorem once we pass to b-densities. In this setting, we start with
\[ (\beta_{(\Delta),t})_*(K_A \rho^h \mu_\Delta) = (\text{Tr} A), \]
and multiply both sides by \( \frac{d\tau}{\tau} \) to obtain
\[ (\beta_{(\Delta),t})_*(K_A \rho^h \beta_{(\Delta),t}^*(\frac{1}{\tau} \mu(M/B \times \mathbb{R}^+))) = (\text{Tr} A) \frac{d\tau}{\tau}. \]

Now,
\[ \beta_{(\Delta),t}^*(\mu(M/B \times \mathbb{R}^+)) = \prod_{N \in \mathcal{S}_\psi(M)} \rho_{\mathfrak{B}_{1,0}^{(\Delta)}}(N) \mu(\text{diag}_w(H)(M)/B) \]
\[ = \rho_{\mathfrak{B}_{0,1}^{(\Delta)}} \prod_{N \in \mathcal{S}_\psi(M)} \rho_{\mathfrak{B}_{1,0}^{(\Delta)}}(N) \rho_{\mathfrak{B}_{1,1}^{(\Delta)}}(N) \mu_b(\text{diag}_w(H)(M)/B) \]
\[ \beta_{(\Delta),t}^*(\tau^{-1}) = \rho_{\mathfrak{B}_{0,1}^{(\Delta)}}^{-1} \prod_{N \in \mathcal{S}_\psi(M)} \rho_{\mathfrak{B}_{1,1}^{(\Delta)}}^{-1}(N), \]
so that we need to push-forward
\[ K_A \rho^h \prod_{N \in \mathcal{S}_\psi(M)} \rho_{\mathfrak{B}_{1,0}^{(\Delta)}}(N) \rho_{\mathfrak{B}_{1,1}^{(\Delta)}}(N) \mu_b(\text{diag}_w(H)(M)/B). \]

This is a polyhomogeneous b-density with index sets
\[ \mathcal{E}_A(\mathfrak{B}_{dd,1}^{(H)}) = \dim(M/B) - 2 \text{ at } \mathfrak{B}_{0,1}^{(\Delta)} \]
and, for each \( N \in \mathcal{S}_\psi(M), \)
\[ \mathcal{E}_A(\mathfrak{B}_{\phi,0}^{(H)}(N)) + 1 - \dim(N/B) - 1 \text{ at } \mathfrak{B}_{1,0}^{(\Delta)}(N) \]
\[ \mathcal{E}_A(\mathfrak{B}_{\phi,1}^{(H)}(N)) + 1 - \dim(N/B) - 3 \text{ at } \mathfrak{B}_{1,1}^{(\Delta)}(N). \]

Applying the push-forward theorem yields the index sets for \( (\text{Tr} A) \frac{d\tau}{\tau} \), and finally we cancel the factor of \( \frac{d\tau}{\tau} \).

\[ \square \]

**Corollary 5.7.** Let \( M \xrightarrow{\psi} B \) be a family of even dimensional manifolds with iterated fibration structures and let \( E \rightarrow M \) be a \( \mathbb{Z}_2 \)-graded wedge vertical Clifford module. If \( A \) has integral kernel satisfying \( K_A \rho^h \mu_R \) with
\[ K_A \in \mathfrak{B}_{\text{phg}}^{H/3} \mathcal{I}_{-m-1}^\tau \left( H(M/B) ; \text{Hom}_G(E) \otimes \Omega_{b,R} \right), \]
where \( \mathcal{I}^{(H)} \) and \( H \) are given by \( \{4.3\}, \{4.4\} \) and \( \Omega_{b,R} \) is the density bundle from \( \{3.22\} \) then \( A \) is trace-class and
\[ \text{Tr}(A) \in \mathcal{A}_{\text{phg}}^{H/\tau}(\mathbb{R}_+^\times ; C^\infty(B; \Lambda^* T^* B)), \]
\[ \mathcal{H}_\tau = (\mathbb{N}_0 - \dim M/B) \bigcup \bigcup_{N \in \mathcal{S}_\psi(M)} (\mathbb{N}_0 - \dim N/B), \]
and
\[ \text{Str}(A) \in \mathcal{A}_{\text{phg}}^{H_0, \ldots, H_0}(\mathbb{R}_+^\times ; C^\infty(B; \Lambda^* T^* B)), \]


where $N_0$ is repeated $1 + \text{depth}(M)$ times. That is, the short-time expansion of $\text{Str}(A)$ has the form

$$\text{Str}(A) \sim \sum_{j \geq 0} \sum_{k=0}^{\text{depth}(M)} \alpha_{j,k} \tau^j (\log \tau)^k,$$

where the coefficients $\alpha_{j,k}$ are smooth differential forms on $B$.

For $e^{-t \mathcal{A}^2_{M/B,Q}}$ we have the improvement

$$\text{Str}(e^{-t \mathcal{A}^2_{M/B,Q}}) \in \mathcal{A}^{N_0 \cup \ldots \cup N\{0\}}_{\text{phg}}(\mathbb{R}^+; C^\infty(B; \Lambda^* T^* B)),$$

where the index set is the extended union of one copy of $N_0$ and $\text{depth}(M)$ copies of $\mathbb{N}$. That is, the short-time expansion of $\text{Str}(e^{-t \mathcal{A}^2_{M/B,Q}})$ has the form

$$\text{(5.3) } \text{Str}(e^{-t \mathcal{A}^2_{M/B,Q}}) \sim \alpha_{0,0} + \sum_{j \geq 1} \sum_{k=0}^{\text{depth}(M)} \alpha_{j,k} \tau^j (\log \tau)^k,$$

where the coefficients $\alpha_{j,k}$ are smooth differential forms on $B$.

**Proof.** The statement about the trace of $A$ follows directly from Theorem 5.6. For the statement about the supertrace, recall Patodi’s observation that the supertrace on the Clifford algebra vanishes on homomorphisms whose Clifford degree is less than the maximum Clifford degree. Thus the index set of $\mathcal{K}_A|_{\text{diag}(M)}$ at $\mathcal{B}^{(1)}_{1,1}(N)$, for each $N \in \mathcal{S}_{\psi}(M)$, is shifted by $\dim(N/B)$. The result is that each of these index sets contributes an $N_0$ to the asymptotics as $\tau \to 0$.

In principle the index set of $\text{Str}(A)$ is then $N_0 \cup \ldots \cup N_0$ with $N_0$ repeated as many times as there are boundary hypersurfaces of $H(M/B)_w$ over $\{\tau = 0\}$. However only actual intersections of boundary hypersurfaces produce accidental multiplicities and so it suffices to take the extended product over $1 + \text{depth}(M)$ copies of $N_0$.

Finally, for $A = e^{-t \mathcal{A}^2_{M/B,Q}}$, the improvement is that $\alpha_{0,k} = 0$ for all $k > 0$. To establish this, it suffices to show that the pointwise supertrace of the heat kernel vanishes at corners (cf. [Vai01, Lemma 5.27]). The discussion at the end of §1.2 shows that the supertrace vanishes at any intersection $\mathcal{B}^{(1)}(N) \cap \mathcal{B}^{(2)}_{1,1}(\tilde{N})$ since there can not be a term of full Clifford degree. Similarly at an intersection of the form $\mathcal{B}^{(1)}(N) \cap \mathcal{B}^{(2)}_{1,1}$ the supertrace vanishes as the can not be a term of full Clifford degree; in this case this follows from the rescaled normal operator in Lemma 5.3 at $\mathcal{B}^{(2)}_{1,1}$. Indeed, the curvature of the Levi-Civita connection is evaluated on edge vector fields, so the vector field $\rho_N \partial_{\rho_N}$ does not occur without the $\rho_N$ factor, and hence any term with $e(d\rho_N)$ will vanish at $\mathcal{B}^{(2)}_{1,1}(N)$.

\[\square\]

### 6. Families index formula

Let $M \xrightarrow{\psi} B$ be a fiber bundle of manifolds with corners and iterated fibration structures, let $E \to M$ a wedge Clifford module with compatibly perturbed Dirac-type operator $\partial_{M/B,Q}$ equipped with its vertical APS domain satisfying the Witt condition, and let $\mathcal{A}_{M/B,Q}$ be the perturbed Bismut superconnection.
6.1. The finite time limit. Given an arbitrary superconnection on \( M \rightarrow B \),
\[
\mathbb{A} = \mathbb{A}_0 + \mathbb{A}_1 + \ldots + \mathbb{A}_k,
\]
recall that the rescaled superconnection is
\[
\mathbb{A}^t = t^{1/2} \left( \mathbb{A}_0 + t^{-1/2} \mathbb{A}_1 + \ldots + t^{-k/2} \mathbb{A}_k \right) = \tau \delta^B_t \mathbb{A} (\delta^B_t)^{-1}
\]
where \( \delta^B_t \) multiplies forms on \( B \) of degree \( k \) by \( \tau^{-k/2} \).

Let us recall the notion of twisted supertrace. If \( \alpha_E \) is a grading operator on \( E \), i.e., the operator which is identity on even degree sections and multiplication by \(-1\) on odd degree sections, so that
\[
\text{str}_E(\cdot) = \text{tr}_E(\alpha_E \cdot),
\]
then since \( \alpha_E \) is a section of
\[
(6.1) \quad \text{hom}(E) = \mathbb{C}l(\mathbb{w}T^*M/B) \otimes \text{hom}_{\mathbb{C}l(\mathbb{w}T^*M/B)}(E),
\]
we have
\[
\alpha_E = \alpha_{M/B} \otimes \alpha'_E
\]
where \( \alpha_{M/B} \) is the grading operator on \( \mathbb{C}l(\mathbb{w}T^*M/B) \) and \( \alpha'_E \) commutes with Clifford multiplication and squares to the identity. The supertrace functional decomposes with respect to (6.1) into the product of two supertrace functionals,
\[
\text{str}(A \otimes A') = \text{tr}_{\mathbb{C}l(\mathbb{w}T^*M/B)}(\alpha_{M/B} A) \text{tr}(\alpha'_E A') = \text{str}_{\mathbb{C}l(M/B)}(A) \text{str}'_{\mathbb{C}l(M/B)}(A')
\]
for all \( A \in \mathbb{C}l(\mathbb{w}T^*M/B) \), \( A' \in \text{hom}_{\mathbb{C}l(\mathbb{w}T^*M/B)}(E) \), and we refer to \( \text{str}'_{\mathbb{C}l(M/B)} \), defined by this equation, as the twisted supertrace.

We have similar decompositions at each \( \mathfrak{B}^{(H)}_{\phi \phi',1}(N) \) of \( H(M/B)_w \); indeed, we have seen that
\[
\text{Hom}(E)|_{\mathfrak{B}^{(H)}_{\phi \phi',1}(N)} = \mathbb{C}l(\mathbb{w}T^*N/B) \otimes \text{hom}_{\mathbb{C}l(\mathbb{w}T^*N/B)}(E)
\]
and consequently the supertrace functional decomposes as \( \text{str}_{\mathbb{C}l(N/B)} \otimes \text{str}'_{\mathbb{C}l(N/B)} \).

Let us introduce the notation for the terms appearing in the short-time limit of the supertrace of the heat kernel. Let
\[
\widehat{A}(M/B) = \det^{1/2} \left( \frac{R^{M/B}/4\pi}{\sinh(R^{M/B}/4\pi)} \right) \in C^\infty(M; \Lambda^*T^*M)
\]
and similarly for \( \widehat{A}(N/B) \), with \( N \in \mathcal{S}_\psi(M) \), and denote the twisted Chern character by
\[
\text{Ch}'(E) = \text{str}'_{\mathbb{C}l(M/B)}(\exp(-K'_E/2\pi)) \in C^\infty(M; \Lambda^*T^*M)
\]
where \( K'_E \) is the twisting curvature from [1.22].

Define, for each \( N \in \mathcal{S}_\psi(M) \), the Bismut-Cheeger \( J \)-form,
\[
J_Q(\mathfrak{B}_{N/N}) = \int_0^R \int_{\mathfrak{B}_{N/N}} \text{str}'_{\mathbb{C}l(N/B)} \left( \exp(-\langle \mathbb{A}_C(\phi_N)/N,Q \rangle^2) \right) \bigg|_{s=1} \frac{dt}{2t} \in C^\infty(N; \Lambda^*T^*N)
\]
where \( s \) is the radial variable along the cone. Here \( \int_0^\infty \) denotes the renormalized integral (also known as the b-integral see, e.g., [Mel93, §4.19], [Alb09], [HMM95, §2.3]) we will see below that this integral is convergent, so that no renormalization is necessary.
Proposition 6.1. Under the Witt assumption,
\[ \lim_{t \to 0} \text{Str}(e^{-(\Delta^2_{M/B,Q})^2}) = \int_{M/B} \hat{A}(M/B) \text{Ch}'(E) - \sum_{N \in \mathcal{S}_\psi(M)} \int_{N/B} \hat{A}(N/B) \mathcal{J}_Q(\mathcal{B}_N/N) \]

Proof. We proceed as in [AR09a, BGV04, MP97a]. Let 
\[ \mathcal{K} = e^{-t\Delta^2_{M/B,Q}} \bigg|_{\text{diag}_w(M)} \]
where the lifted diagonal, \( \text{diag}_w^H(M) \), is described in [5.2]. Corollary 5.7 established the existence of the small time limit of the supertrace and that it is given by 
\[ \int_{\mathcal{B}_{\text{dd},1}^H \cap \text{diag}_M} \text{str}(\mathcal{K})|_{\mathcal{B}_{\text{dd},1}^H} + \sum_{N \in \mathcal{S}_\psi(M)} \int_{\mathcal{B}_{\phi,1}^H(N) \cap \text{diag}_M} \text{str}(\mathcal{K})|_{\mathcal{B}_{\phi,1}^H(N)} \]
Recall from, e.g., [BGV04, Lemma 10.22]
\[ \text{str}(e^{-(\Delta^2_{M/B,Q})^2}) = \delta_t^B \left( \text{str}(e^{-t\Delta^2_{M/B,Q}}) \right) \]
and Patodi’s observation that the supertrace in \( \mathcal{C}l(V) \) only depends on terms of top Clifford degree. Thus if at \( \mathcal{B}_{\text{dd},1}^H \) we have an expansion 
\[ \mathcal{K}|_{\text{diag}_M} \sim t^{-(\dim M/B)/2} \sum_{\ell \in \mathbb{N}_0} U_\ell t^{\ell/2}, \]
with each term \( U_\ell \) of Clifford degree at most \( \ell \) then, just as in [BGV04, MP97a], this implies that 
\[ \text{str}(\mathcal{K}|_{\text{diag}_w})|_{\mathcal{B}_{\text{dd},1}^H} = (-2i)^{\dim M/B/2} \text{Ev}_{M/B} \left[ \text{str} \mathcal{N}_{\text{dd},1}^G \left( e^{-t\Delta^2_{M/B,Q}} \right) \right]|_{\text{diag}_M} \]
\[ = (-2i)^{\dim M/B/2} \text{Ev}_{M/B} \left[ \text{str} e^{-\mathcal{H}_{M/B}(\zeta)} \bigg|_{\zeta=0} \right] = \text{Ev}_{M/B} \left( \hat{A}(M/B) \text{Ch}'(E) \right) \]
where \( \text{Ev}_{M/B} \) is the projection of differential forms on \( M \) onto those with top \( \psi \)-vertical degree. Thus the contribution from this face is 
\[ \int_{\mathcal{B}_{\text{dd},1}^H \cap \text{diag}_M} \text{str}(\mathcal{K})|_{\mathcal{B}_{\text{dd},1}^H} = \int_{M/B} \hat{A}(M/B) \text{Ch}'(E). \]

We now consider the situation at a face \( \mathcal{B}_{\phi,1}^H(N) \) for some \( N \in \mathcal{S}_\psi(M) \). Let \( x \) denote a boundary defining function for this face, \( \sigma = \frac{x}{2} \) a rescaled time parameter, and note that the rescaling operator \( \delta_B^x \) becomes \( \delta_B^x \delta_B^2 \). Since the Taylor expansion of \( \mathcal{K} \) at \( \mathcal{B}_{\phi,1}^H(N) \) is in powers of \( x \), the rescaling operator \( \delta_B^x \) plays the same rôle at this face as \( \delta_B^2 \) plays at \( \mathcal{B}_{\text{dd},1}^H \), it mediates between the degree in \( \mathcal{C}l(w^TN/B) \) and that in \( \mathcal{C}l(w^TN/B) \otimes \psi^* \Lambda^* B \). Proceeding as in [BC89, (1.23),(1.24)],
\[ \text{str}(\mathcal{K})|_{\mathcal{B}_{\phi,1}^H(N)} = (-2i)^{(1+\dim N/B)/2} \text{Ev}_{N/B} \left[ \text{str} \mathcal{N}_{\phi,1}^G(N) \left( e^{-t\Delta^2_{M/B,Q}} \right) \right]|_{\text{diag}_M} \]
\[ = (-2i)^{(1+\dim N/B)/2} \text{Ev}_{N/B} \left[ \delta_B^{\sigma^2} \hat{A}^2 \right] \left( \text{str} e^{-\sigma^2 \mathcal{H}_{N/B}(\zeta)} e^{-\sigma^2 \mathcal{H}_{\psi,1}(\zeta)} \right) \bigg|_{\zeta=0,s=1} \]
\[ = \text{Ev}_{N/B} \left[ \delta_B^{\sigma^2} \hat{A}^2 \right] \left( \text{str} e^{-\sigma^2 \mathcal{H}_{\psi,1}(\zeta)} \right) \bigg|_{\zeta=0,s=1} \]
\[ = \text{Ev}_{N/B} \left[ \hat{A}(N/B) \right] \left( \text{str} e^{-\sigma^2 \mathcal{H}_{\psi,1}(\zeta)} \right) \bigg|_{\zeta=0,s=1} \]


Thus the contribution from this face is

\[
\int_{B} (H) \phi_{N} \hat{A}(N/B) R \int_{0}^{\infty} \int_{\mathfrak{B}_{N/N}} \text{str}'_{\text{Cl}(N/B)} (\exp(-(\hat{A}_{C}(\phi_{N}/N,Q)^{2})) \bigg |_{s=1} \frac{d\tau}{\tau}
\]

\[
= \int_{N/B} (N/B) J_{Q}(\mathfrak{B}_{N/N})
\]

as required. \[\square\]

6.2. Bismut-Cheeger \(\eta\) and \(J\) forms. Bismut and Cheeger \cite{BC89, BC90a, BC90b, BC91} defined differential forms on \(B\), \(\eta\) and \(J\) in the setting of closed manifold fibers. The former were also defined on spaces of depth one by Cheeger \cite{Che87, §8}, for isolated conic singularities, and by Piazza-Vertman \cite{PV} in general. Melrose and Piazza \cite{MP97a, MP97b} introduced \(\eta\)-forms for perturbed Dirac-type operators. In this section we generalize their construction to compatibly perturbed families of wedge Dirac-type operators with vertical APS domain satisfying the Witt condition.

We define these forms for a family of manifolds with iterated fibration structures without assuming that they come from a boundary fibration. To differentiate the fiber bundle used in this subsection with the one in the main body of the text, we will adopt the notation

\[
\hat{X} \rightarrow \hat{M} \psi \rightarrow B.
\]

As we will review, both the \(\eta\)-forms and \(J\)-forms can be thought of as arising from \(\hat{M} \times \mathbb{R}^{+}\). In the former case the factor of \(\mathbb{R}^{+}\) is added to the base, in the latter it is added to the fiber. Analogously to Definition 2.1, we will use the following notation; for a bundle \(\hat{X} \rightarrow \hat{M} \psi \rightarrow B\) with vertical metric \(g_{\hat{M}/B}\), a \(\text{Cl}(1)\) bundle is a bundle \(E \rightarrow \hat{M}\) with an action of \(\text{Cl}(T^{*}\hat{M}/B \oplus \mathbb{R})\) where the additional \(\mathbb{R}\) factor is orthogonal. (In particular the fibre is the complexified Clifford algebra on \(\mathbb{R}^{\dim \hat{X}+1}\).) We will denote the additional generator of this Clifford algebra by \(\gamma\).

**Definition 6.2.** Let \(E \rightarrow \hat{M}\) be a \(\psi\)-vertical wedge Clifford bundle if \(\dim \hat{M}/B\) is even dimensional and a \(\text{Cl}(1)\)-bundle if \(\dim \hat{M}/B\) is odd and \(Q\) a compatible perturbation such that the associated family of Dirac-type operators \(\partial_{\hat{M}/B,Q}\), endowed with its vertical APS domain, satisfies the Witt assumption and is such that \(\ker \partial_{\hat{M}/B,Q}\) forms a vector bundle over \(B\).

The Bismut-Cheeger \(\eta\)-form of \(\partial_{\hat{M}/B,Q}\),

\[
\eta_{Q}(\hat{M}/B) = \eta(\partial_{\hat{M}/B,Q}) \in \mathcal{C}^{\infty}(B; \Lambda^{*}T^{*}B)
\]
is given by
\[
\eta_Q(\hat{M}/B) = \begin{cases} 
\frac{1}{2\sqrt{\pi}} \int_0^\infty \text{Str} \left( \frac{\partial (A^*_M/B,Q)}{\partial t} e^{-\frac{(A^*_M/B,Q)^2}{2}} \right) dt & \text{if dim}(\hat{M}/B) \text{ even} \\
\frac{1}{2\sqrt{\pi}} \int_0^\infty \text{Str}_{C(1)} \left( \frac{\partial (A^*_M/B,Q)}{\partial t} e^{-\frac{(A^*_M/B,Q)^2}{2}} \right) dt & \text{if dim}(\hat{M}/B) \text{ odd}
\end{cases}
\]

The normalized \( \eta \) form, \( \bar{\eta}_Q(\hat{M}/B) \) is the form obtained from \( \eta_Q(\hat{M}/B) \) by multiplying the forms of degree \( \ell \) by \((2\pi i)^{-[\ell/2]}\).

The form \( \bar{\eta}_Q(\hat{M}/B) \) has even degree if \( \dim \hat{X} \) is odd, and odd degree if \( \dim \hat{X} \) is even.

Implicit in this definition is the fact that the integral converges. As pointed out in, e.g., \cite[Theorem 2.10]{BF86b}, \cite[§3]{BC89}, \cite[Theorem 2.11]{BGV04}, this can be established by considering the heat kernel of the Bismut superconnection in one dimension higher. Following \cite[(117), Remark A.15]{Vai01}, we can treat the even and odd cases uniformly by extending the fiber bundle \( \hat{\psi} \) to
\[
\hat{X} = M^+ = \hat{M} \times \mathbb{R}_s^+ \xrightarrow{\hat{\psi} \times \text{id}} B^+ = B \times \mathbb{R}_s^+,
\]
and noting that, in terms of the extended Bismut superconnection \( A_{(M^+)/(B^+)} \) corresponding to the natural extensions of the vertical covariant derivative and choice of horizontal tangent bundle,
\[
\bar{\eta}(\partial_M/B) = \int_0^\infty \nu(\partial_s) \text{Str}' \left( e^{-\frac{(A^*_M/B)^2}{2}} \right) dt
\]
where \( \text{Str}' \) denotes the appropriate supertrace on \( \Lambda^*T^*(B \times \mathbb{R}^+) \otimes E \) corresponding to the parity of \( \dim \hat{M}/B \). The short-time asymptotic expansion of the supertrace of the heat kernel together with the long-time limit implies the convergence of this integral and so the well-definedness of the \( \eta \)-forms.

Next we define the Bismut-Cheeger \( J \)-forms in our context. Suppose \( E \to \hat{M} \) is a \( C(1) \)-bundle so we have an action of \( d'(\gamma) \) on \( E \) (see above) and that \( E \) is \( \mathbb{Z}_2 \)-graded if \( \dim \hat{M}/B \) is odd. Consider the extension of \( \hat{\psi} \) to
\[
\hat{X}^+ = \hat{X} \times \mathbb{R}^+ = M^+ = \hat{M} \times \mathbb{R}^+ \xrightarrow{\hat{\psi}^+} B,
\]
and the warped product metric
\[
g^+ = g_{\hat{M}^+/B} = ds^2 + s^2 g_{\hat{M}/B}.
\]

Recall from \eqref{eq:Levi-Civita} that the Levi-Civita connection of \( g^+ \) on the wedge cotangent bundle differs from the product Levi-Civita connection by
\[
\nabla_{W^+}^g \theta = \nabla_{W^+}^\oplus + \left( g^+(dx,\theta) \left( \frac{1}{x} vW \right)^\gamma - g^+((\frac{1}{x} vW)^\gamma, \theta)dx \right).
\]

Let \( E^+ \), denote \( E \) pulled-back to \( \hat{M}^+ \) via \( \pi : \hat{M}^+ \to \hat{M} \) and \( g_{E^+} \) the pull-back metric of \( g_E \). Over \( \hat{M}^+ \) we have a bundle isomorphism
\[
^wT\hat{M}^+/B \cong \langle ds \rangle \oplus s T^*\hat{M}/B \xrightarrow{\varphi} \langle \gamma \rangle \oplus T^*\hat{M}/B, \quad a_0 ds + a_i s^i \theta^i \mapsto a_0 \gamma + a_i \theta^i
\]
which we use to define a Clifford action of $\text{w} T^* \hat{M}^+/B$ on $E$,

$$\epsilon^+ (\theta) = \epsilon(\Xi(\theta)).$$

This Clifford action is compatible with the metric $g_{E^+}$. We modify the connection on $E$ to get a connection on $E^+$ following (2.3),

$$\nabla^E_{w^+} = \pi^* \nabla^E_w + \frac{1}{2} \epsilon^+(dx) \epsilon^+(\frac{1}{2} x W^b).$$

This is a $g_{E^+}$ metric connection compatible with the extended Clifford action $\epsilon^+$ and the Levi-Civita connection of $g^+$. Hence $E^+$ is a $\hat{\psi}^+$-vertical wedge Clifford module.

The families of Dirac operators on $\hat{M}^+/B$ produced by the preceding construction do not constitute arbitrary families; indeed, those produced here have in a sense $s$-independent twisting bundle (though the twisting bundle is only defined globally in the spin case). In fact the twisting curvature of the connection $\nabla^E_{w^+}$ is $s$ and $d s$-independent:

$$K_E' (\partial_s, \cdot) \equiv 0 \equiv \partial_s K_E'.$$

Indeed, locally in the bulk of $\hat{M}$, the Clifford action induces a splitting $E \simeq S(T^* \hat{M}^+/B) \otimes W$ with $S(T^* \hat{M}^+/B)$ the (locally defined) bundle of spinors, with Clifford multiplication acting on the left and the connection decomposing as a tensor product connection $s \nabla \otimes id + id \otimes W \nabla$. The pullback construction above changes the connection only by adding a zeroeth order Clifford multiplication term, i.e. only by modifying the connection on the spinor bundle part. The twisting curvature depends only on $W \nabla$ and is therefore (6.5) holds.

A choice of connection for $\hat{\psi}$,

$$\text{w} T \hat{M} \cong \text{w} T \hat{M}/B \oplus \hat{\psi}^* TB,$$

readily extends to a choice of connection for $\hat{\psi}^+$,

$$\text{w} T \hat{M}^+ \cong \text{w} T \hat{M}^+/B \oplus (\hat{\psi}^+)^* TB.$$

The fundamental tensors (1.19) of $\hat{\psi}$ and $\hat{\psi}^+$ are essentially unchanged,

$$S^{\hat{\psi}^+} (W_1, W_2) (A) = S^{\hat{\psi}} (W_1, W_2) (A), \quad \hat{R}^{\hat{\psi}^+} (A_1, A_2) (W) = \hat{R}^{\hat{\psi}} (A_1, A_2) (W),$$

where the latter tensors are understood to vanish if any of the vertical vector fields is $\partial_s$. Thus the Bismut superconnection on $\hat{M}^+$ is given by (cf. Lemma 2.2 and [AGR16 Lemma 1.2])

$$\hat{A}_{\hat{M}^+/B} = \left( \frac{1}{s} \hat{\partial}_{\hat{M}/B} + \epsilon (d s) \partial_s + \frac{\text{dim} \hat{M}/B}{2} \epsilon (d s) \right) + (\hat{A}_{\hat{M}/B})^{[1]} + (\hat{A}_{\hat{M}/B})^{[2]},$$

and the square of the rescaled Bismut superconnection satisfies [BC90b, (6.37)]

$$(\hat{A}_{\hat{M}^+/B})^2 = -t (\partial_s + \frac{\text{dim} \hat{M}/B}{2s})^2 + (\hat{A}_{\hat{M}/B})^{[2]} + \frac{1}{s} \epsilon (d s) \hat{A}_{\hat{M}/B, [0]} + \frac{1}{4} \epsilon (d s) \hat{A}_{\hat{M}/B, [2]}.$$

**Definition 6.3.** With notation as above, the Bismut-Cheeger $J$-**form** of $\hat{\partial}_{\hat{M}/B,Q}$

$$\mathcal{J}_Q (\hat{M}/B) = \mathcal{J}(\hat{\partial}_{\hat{M}/B,Q}) \in C^\infty (B; \Lambda^* T^* B)$$

is given by

$$\mathcal{J}_Q (\hat{M}/B) = \begin{cases} - \int_0^\infty \int_{\hat{M}/B} \text{str} \left( e^{- \left( \hat{A}_{\hat{M}^+/B}^t \right)^2} \right)_{\text{diag} \hat{g}^{\hat{M}^+,s=1}} \frac{dt}{2t} & \text{if } \text{dim}(\hat{M}^+/B) \text{ even} \\ - \int_0^\infty \int_{\hat{M}/B} \text{str}_{\mathcal{L}(1)} \left( e^{- \left( \hat{A}_{\hat{M}^+/B}^t \right)^2} \right)_{\text{diag} \hat{g}^{\hat{M}^+,s=1}} \frac{dt}{2t} & \text{if } \text{dim}(\hat{M}^+/B) \text{ odd} \end{cases}$$
The convergence of the integral follows as in \[BC90b, \S VI(a)\] from the dilation invariance property (4.10). Indeed, the short time asymptotic expansion of the supertrace from (5.3) and Proposition 6.1 guarantees the convergence as \( t \to 0 \), and the vanishing of the null space established in Proposition 2.7 together with the resulting decay of the heat kernel, establishes the convergence as \( t \to \infty \).

6.3. Even dimensional fibers. Let \( M \xrightarrow{\psi} B \) be a fiber bundle of manifolds with corners and iterated fibration structures such that \( \text{dim}(M/B) \) is even. Let \( E \to M \) be a wedge Clifford bundle, with associated Dirac-type operator \( \bar{\partial}_{M/B} \) and \( Q \) a compatible perturbation such that \( \bar{\partial}_{M/B,Q} \) with its vertical APS domain satisfies the Witt assumption.

We may, as in \[MR06, \text{Lemma 1.1}\], perturb \( \bar{\partial}_{M/B,Q} \) by smoothing operators compactly supported in the interior of \( M \) without changing the families index in K-theory and so that the null spaces form a vector bundle over \( B \). Since this perturbation is supported in the interior it will not change the boundary families of \( \bar{\partial}_{M/B,Q} \) and the arguments in \[BGV04, \text{Proposition 9.46}\] apply to show that the effect on the short-time asymptotic expansion of the trace of the heat kernel is \( O(t) \). By incorporating this perturbation into \( Q \), we will assume that \( \ker \bar{\partial}_{M/B,Q} \) forms a smooth vector bundle over \( B \).

Given an arbitrary superconnection on \( M \to B \), \( A \), the Chern character of \( A \) is

\[
\text{Ch}(A) = \text{Str}(e^{-A^2}).
\]

For the Bismut superconnection and a compatible perturbation, \( A_{M/B,Q} \), the arguments in \[BGV04, \S 9.3\] apply directly and show that

\[
\frac{\partial}{\partial t} \text{Ch}(A^t_{M/B,Q}) = -d_B \text{Str} \left( \frac{\partial A^t_{M/B,Q}}{\partial t} e^{-(A^t_{M/B,Q})^2} \right)
\]

and

\[
\lim_{t \to \infty} \text{Ch}(A^t_{M/B,Q}) = \text{Ch}(\text{Ind}(\bar{\partial}_{M/B,Q}), \nabla^{\text{Ind}})
\]

where \( \text{Ind}(\bar{\partial}_{M/B,Q}) \) is the virtual index bundle of \( \bar{\partial}_{M/B,Q} \) and \( \nabla^{\text{Ind}} \) is the contraction,

\[
\nabla^{\text{Ind}} = \mathcal{P}_{\text{Ind}}(A^t_{M/B,Q}_{|1})|\mathcal{P}_{\text{Ind}}.
\]

Proposition 6.1 and integration in \( t \) yield the families index formula.

**Theorem 6.4.** Let \( M \xrightarrow{\psi} B \) be a fiber bundle of manifolds with corners and iterated fibration structures with even-dimensional fibers, \( E \to M \) a \( \mathbb{Z}_2 \)-graded wedge Clifford bundle with associated Dirac-type operator \( \bar{\partial}_{M/B} \) and let \( Q \) be a compatible perturbation. If \( \bar{\partial}_{M/B,Q} \) with its vertical APS domain satisfies the Witt assumption, then

\[
\text{Ch}_{\text{even}}(\text{Ind}(\bar{\partial}_{M/B}), \nabla^{\text{Ind}}) = \int_{M/B} \tilde{A}(M/B) \text{Ch}'(E) - \sum_{N \in \mathcal{S}_\psi(M)} \int_{N/B} \tilde{A}(N/B) J_Q(\mathfrak{B}_N/N) + d\eta_Q(M/B)
\]
6.4. **Odd dimensional fibers.** A standard argument going back to [AS69] reduces the families index for odd dimensional fibers to the families index for even dimensional fibers by suspension. This was carried out for Dirac operators on closed manifolds in [BF86b]. We will follow the treatment of Melrose and Piazza [MP97b], though note that our Clifford multiplication conventions differ by a factor of \( i \).

Lemma 2 of [MP97b, §5] says that: If \( L_1 \) and \( L_2 \) are Clifford modules, with Clifford actions \( \text{cl}_1 \) and \( \text{cl}_2 \), over Riemannian manifolds \( X_1, X_2 \), respectively, then the bundle
\[
L = L_1 \otimes L_2, C^2 \to X_1 \times X_2
\]
has a Clifford action compatible with the product metric on \( X = X_1 \times X_2 \) given by
\[
\text{cl}(\alpha) = \text{cl}_1(\alpha) \otimes \text{Id} \otimes \Gamma_1 \text{ for all } \alpha \in C^\infty(X_1; T^*X_1)
\]
\[
\text{cl}(\beta) = \text{Id} \otimes \text{cl}_2(\beta) \otimes \Gamma_2 \text{ for all } \beta \in C^\infty(X_2; T^*X_2)
\]
for any choice of \( \Gamma_i \in \mathcal{M}_2(\mathbb{C}) \) satisfying
\[
\Gamma_1 \Gamma_2 + \Gamma_2 \Gamma_1 = 0, \quad \Gamma_1^2 = \Gamma_2^2 = -1.
\]

For example we can take
\[
\Gamma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]
If \( X_1 \times X_2 \) is even-dimensional then \( L \) can be taken to be \( \mathbb{Z}_2 \)-graded as follows: If \( X_1 \) and \( X_2 \) are both even-dimensional, so that \( L_1 \) and \( L_2 \) are \( \mathbb{Z}_2 \)-graded, then we take the product grading of \( L_1 \otimes L_2 \) and then tensor with \( C^2 \).

If \( X_1 \) and \( X_2 \) are both odd-dimensional, so that \( L_1 \) and \( L_2 \) are ungraded, then we put a grading on \( C^2 \) by
\[
(C^2)^+ = \mathbb{C} \oplus \{0\}, \quad (C^2)^- = \{0\} \oplus \mathbb{C},
\]
and then tensor with \( L_1 \otimes L_2 \). Endowing \( C^2 \) with the trivial metric and connection, \( L \) has the structure of a Clifford module over \( X_1 \times X_2 \).

Set \( T = S^1_\theta \times S^1_\xi \), where we parametrize the first circle by \( \theta \in [0, 2\pi] \) and the second by \( \xi \in [0, \pi] \), and let \( L_T \) be the Hermitian line bundle over \( T \) given by identifying the points \((\theta, 0, v)\) with \((\theta, \pi, e^{-i\theta} v)\) endowed with the Hermitian connection
\[
\nabla_{L_T} = d - \frac{2\xi - 1}{2\pi} d\theta.
\]
These data define a family of Dirac operators on the fibers of
\[
T \xrightarrow{\psi_T} S^1_\xi
\]
given by \( \delta_{T/S^1} = \frac{1}{i} \partial_\theta + \frac{2\xi - 1}{2\pi} \) with spectral flow equal to one.

We replace the original fiber bundle \( M \xrightarrow{\psi} B \) with
\[
X \times S^1_\theta \rightarrow S^2 M = M \times T \xrightarrow{\psi_T} SB = B \times S^1_\xi,
\]
and each boundary fiber bundle \( B_N \xrightarrow{\phi_N} N \) with
\[
S^2 B_N = B_N \times T \xrightarrow{\phi_N \times \psi_T} SN = N \times S^1_\xi,
\]
replace \( E \to M \) with \( F = E \otimes L \otimes \mathbb{C}^2 \), and then the extension of the Clifford structure described above yields the family of Dirac-type wedge operators

\[
\delta_{S^2 M/\mathbb{S}^B} = \begin{pmatrix} 0 & \delta_{M/B} \otimes \text{Id} + \text{Id} \otimes \delta_{\mathbb{T}/\mathbb{S}^1} \\ \delta_{M/B} \otimes \text{Id} - \text{Id} \otimes \delta_{\mathbb{T}/\mathbb{S}^1} & 0 \end{pmatrix}.
\]

The invertibility of \( \delta_{\mathbb{T}/\mathbb{S}^1} \) on each fiber of \( \psi_{\mathbb{T}} \), together with the invertibility of the boundary families of \( \delta_{M/B,Q} \) easily yields the invertibility of the boundary families of \( \delta_{S^2 M/\mathbb{S}^B,Q} \), as well as the invertibility of \( \delta_{\mathbb{T}/\mathbb{S}^1} \) on the fibers lying over some neighborhood of the point \( \{ t = 0 \} \in \mathbb{S}^1 \), as in [MP97b, Lemma 3].

This implies that the index class of \( \delta_{M/B,Q} \) in the odd K-theory of \( B \) is mapped by suspension into the index class of \( \delta_{S^2 M/\mathbb{S}^B,Q} \) [MP97b, Proposition 6]. Thus the odd Chern character of the index class of \( \delta_{M/B,Q} \) satisfies

\[
\text{Ch}_{\text{odd}}(\text{Ind}(\delta_{M/B,Q})) = \frac{i}{2\pi} \int_{\mathbb{S}^1} \text{Ch}_{\text{even}}(\text{Ind}(\delta_{S^2 M/\mathbb{S}^B,Q}))
\]

and we obtain a formula for the odd Chern character by integrating our formula for the even dimensional Chern character over the circle.

This leaves us to consider the effect of suspension on the \( \mathcal{J}_Q \)-forms. Let us start by recalling [MP97b, §8] the effect of suspension on the Bismut superconnection. The Bismut superconnection depends on the choice of a vertical metric and connection, in this case we take

\[
T(S^2 M/\mathbb{S}^B) = TM/B \oplus T\mathbb{S}^1, \quad g_{S^2 M/\mathbb{S}^B} = g_{M/B} \oplus dt^2,
\]

where we leave implicit the pull-back maps. The corresponding Bismut superconnection satisfies

\[
\mathbb{A}_{S^2 M/\mathbb{S}^B,Q} = (\delta_{M/B,Q} \otimes \Gamma_2 + \delta_{\mathbb{T}/\mathbb{S}^1} \otimes \Gamma_1) + (\mathbb{A}_{[1]} + \mathbb{A}_{[2]} \otimes \Gamma_2)
\]

\[
\mathbb{A}_{S^2 M/\mathbb{S}^B,Q} = \mathbb{A}_{M/B,Q} + \delta_{\mathbb{T}/\mathbb{S}^1} + \frac{1}{\pi^2} \mathbb{A}_{[2]} \otimes \Gamma_2
\]

and hence its square can be written

\[
(\mathbb{A}_{S^2 \mathbb{B}_N/SN, Q}^t)^2 = \begin{aligned}
&= -t(\delta_s + \frac{\dim S^2 \mathbb{B}_N/SN}{2} \mathbb{A}_{S^2 \mathbb{B}_N/SN, Q})^2 + \frac{t}{\pi^2} \mathbb{A}_{M/B,Q}^t \mathbb{A}_{S^2 \mathbb{B}_N/SN, Q, [0]} \\
&= -t(\delta_s + \frac{\dim S^2 \mathbb{B}_N/SN}{2} \mathbb{A}_{M/B,Q}^t \mathbb{A}_{S^2 \mathbb{B}_N/SN, Q})^2 + \frac{t}{\pi^2} \mathbb{A}_{M/B,Q}^t \mathbb{A}_{S^2 \mathbb{B}_N/SN, Q}^t
\end{aligned}
\]

\[
= \begin{aligned}
&= -t(\delta_s + \frac{\dim S^2 \mathbb{B}_N/SN}{2} \mathbb{A}_{M/B,Q}^t \mathbb{A}_{S^2 \mathbb{B}_N/SN, Q})^2 + \frac{t}{\pi^2} \mathbb{A}_{M/B,Q}^t \mathbb{A}_{S^2 \mathbb{B}_N/SN, Q}^t + \frac{\sqrt{t}}{\pi^2} \mathbb{A}_{M/B,Q}^t \mathbb{A}_{S^2 \mathbb{B}_N/SN, Q}^t
\end{aligned}
\]

\[
\begin{aligned}
&= -t(\delta_s + \frac{\dim S^2 \mathbb{B}_N/SN}{2} \mathbb{A}_{M/B,Q}^t \mathbb{A}_{S^2 \mathbb{B}_N/SN, Q})^2 + \frac{t}{\pi^2} \mathbb{A}_{M/B,Q}^t \mathbb{A}_{S^2 \mathbb{B}_N/SN, Q}^t + \frac{\sqrt{t}}{\pi^2} \mathbb{A}_{M/B,Q}^t \mathbb{A}_{S^2 \mathbb{B}_N/SN, Q}^t
\end{aligned}
\]

\[
\begin{aligned}
&= \frac{\sqrt{t}}{\pi^2} \mathbb{A}_{M/B,Q}^t \mathbb{A}_{S^2 \mathbb{B}_N/SN, Q}^t + \frac{t}{\pi^2} \mathbb{A}_{M/B,Q}^t \mathbb{A}_{S^2 \mathbb{B}_N/SN, Q}^t
\end{aligned}
\]
Note that in the final formula the three summands commute, and so we have
\[
\exp\left(-\left(\mathcal{A}_{S^2B_N/SN,Q}^t + \mathcal{A}(d\theta) \otimes \Gamma_1\right) + \mathcal{A}(d\theta) (\partial_{T/S1} \otimes \Gamma_1)\right) = \exp\left(-s^{1/2}(\mathcal{A}_{S^2B_N/SN,Q}^t)^2 s^{-1/2}\right) .
\]
Now, from the odd families index theorem of Bismut-Freed, we have
\[
\frac{i}{2\pi} \int_{S^1} \text{Tr} \left(\left(1 + \frac{\sqrt{T}}{s\pi} e(d\xi) (d\theta) \otimes \Gamma_1\right) + \mathcal{A}(d\theta) (\partial_{T/S1} \otimes \Gamma_1)\right) = \frac{i}{2\pi} \int_{S^1} \text{Tr} \left(\frac{\sqrt{T}}{s\pi} e(d\theta) (\partial_{T/S1} + \frac{1}{2})^2\right) = \text{spectral flow}(\mathcal{A}(d\theta) (\partial_{T/S1} + 1/2)) = 1
\]
and hence the \(J\)-forms satisfy
\[
\frac{i}{2\pi} \int_{S^1} J_Q(S^2B_N/SN) = J_Q(\mathcal{B}_N/N) .
\]

**Theorem 6.5.** Let \(M \xrightarrow{\psi} B\) be a fiber bundle of manifolds with corners and iterated fibration structures with odd-dimensional fibers, \(E \rightarrow M\) a wedge Clifford bundle with associated Dirac-type operator \(\partial_{M/B}\) and compatible perturbation \(Q\). If \(\partial_{M/B,Q}\) with its vertical APS domain satisfies the Witt assumption, then
\[
\text{Ch}_{\text{odd}}(\text{Ind}(\partial_{M/B,Q}), \nabla^{\text{Ind}}) = \int_{M/B} \hat{A}(M/B) \text{Ch'}(E) - \sum_{N \in S_\psi(M)} \int_{N/B} \hat{A}(N/B) J_Q(\mathcal{B}_N/N) + d\eta_Q(M/B).
\]

**7. An extended index formula and the relation between \(J\) and \(\eta\)**

In §6 we have found a formula for the Chern character of the index of a family of wedge Dirac-type operators in terms of the Bismut-Cheeger \(J\)-forms. In this section we establish the relation between the \(J\)-forms and the \(\eta\) forms. In the process we establish a families index theorem on manifolds with corners and an iterated fibration structure endowed with a metric that is of wedge ‘type’ at all boundary hypersurfaces save one, where it is of ‘b’ or asymptotically cylindrical type.

Before we start, we give an example to show that \(J\) and \(\hat{\eta}\) do not coincide in general. Consider an embedded surface, \(Y\), in a closed spin 4-manifold \(L\) and let \(X = [L; Y]\), endowed with a wedge metric with constant cone angle \(2\pi\beta\),
\[
dx^2 + x^2 \beta^2 d\theta^2 + \phi_Y^* g_Y,
\]
where \(\phi_Y\) is the boundary fiber bundle \(S^1 \rightarrow \partial X \xrightarrow{\psi} Y\). It follows from [AGR16, Corollary 1.2, §6.1] that, if \(\beta \leq 1\),
\[
\int_Y \hat{A}(Y) J(\partial X/Y) = \frac{1}{24}(\beta^2 - 1)[Y]^2, \quad \int_Y \hat{A}(Y) \eta(\partial X/Y) = \frac{1}{24}[Y]^2,
\]
where \([Y]^2\) denotes the self-intersection number of \(Y\) in \(L\).
7.1. **b-c suspension.** The ultimate aim of this section is to determine the relation between the $\mathcal{J}$ and $\eta$ forms. We do not assume that the fiber bundles treated here arise as boundary fiber bundle so to distinguish this setting from that above we will use $\tilde{M}$ instead of $M$ or $\mathfrak{B}_N$. Starting with a fiber bundle

$$
\tilde{X} \rightarrow \tilde{M} \xrightarrow{\tilde{\psi}} \tilde{B},
$$

we consider $\mathbb{R}_s^+ \times \tilde{M}$ together with a vertical metric that is of wedge type near $\{s = 0\}$ and cylindrical away from $\{s = 0\}$, which we refer to as a ‘b,wedge metric’. In this subsection we show that a wedge Clifford bundle on $\tilde{M}$ with a compatible perturbation induces a b,wedge Clifford bundle on $\mathbb{R}^+ \times \tilde{M}$ with vanishing index. In the following subsection we will find an explicit formula for the Chern character of this index involving both $\eta(\tilde{M}/\tilde{B})$ and $\mathcal{J}(\tilde{M}/\tilde{B})$.

Let

$$
\tilde{X}^+ \rightarrow \mathbb{R}_s^+ \times \tilde{X} \rightarrow \mathbb{R}_s^+ \times \tilde{M} \xrightarrow{\tilde{\psi}^+} \tilde{B}
$$

be the extended fiber bundle. Given a $\tilde{\psi}$-vertical wedge metric $g_{\tilde{M}/\tilde{B}}$, let

$$
g_h^+ = ds^2 + h(s)^2 g_{\tilde{M}/\tilde{B}}
$$

where $h$ is a smooth function satisfying

$$
h(s) = \begin{cases} 
  s & \text{for } s < 1, \\
  1 & \text{for } s > 2,
\end{cases} \quad h(s) > 0 \text{ for } s > 0.
$$

Just as in §6.2, given a $\mathcal{C}l(1)$-wedge Clifford bundle $E \rightarrow \tilde{M}$, we can endow the pull-back bundle $E \rightarrow \tilde{M}^+$ with an action of $\mathcal{C}l(\mathbb{R}^+ \times \tilde{M}^+/\tilde{B})$, a compatible Hermitian metric $g_E$ and connection $\nabla_E$. In particular the connection $\nabla_E^+$ in (6.4) and the Clifford action (6.3) have obvious analogies when the warping factor $s^2$ is replaced by an arbitrary warping factor $k(s)^2$.

Given a compatible perturbation $Q$ over $\tilde{M}$, let us also use $Q$ to denote the trivial extension of $Q$ to $\tilde{M}^+$.

Consider a fiber $\tilde{X}$ of $\tilde{\psi}$. Given a frame $\{V_1, \ldots, V_m\}$ for $\tilde{X}$ over $\mathcal{U} \subseteq \tilde{X}$, consider the frame $\{\partial_s, \tilde{V}_i\}$ with $\tilde{V}_i = \frac{1}{h} V_i$, over $\mathbb{R}^+ \times \mathcal{U}$. Using the Koszul formula we can express the Levi-Civita connection of $g_h^+$ in terms of $h$ and the Levi-Civita connection of $g_\tilde{X}$,

| $g_h^+(\nabla_{W_1}^+ W_2)$ | $\partial_s$ | $\tilde{V}_i$ |
|-------------------|-------------|-------------|
| $\nabla_{\partial_s}^+ \partial_s$ | 0 | 0 |
| $\nabla_{\partial_s}^+ V_0$ | 0 | 0 |
| $\nabla_{\tilde{V}_0}^+ \partial_s$ | 0 | $\frac{h'}{h} g_\tilde{X}(V_0, V_2)$ |
| $\nabla_{\tilde{V}_0}^+ V_1$ | $-\frac{h'}{h} g_\tilde{X}(V_0, V_1)$ | $\frac{h'}{h} g_\tilde{X}(\nabla_{\tilde{V}_0} V_1, V_2)$ |

Thus we have

$$
\nabla_{\partial_s}^+ = 0, \quad \nabla_{\tilde{V}_0}^+ \partial_s = \frac{h'}{h} \tilde{V}_0, \quad \nabla_{\tilde{V}_0}^+ \tilde{V}_1 = -\frac{h'}{h} g_\tilde{X}(V_0, V_1) \partial_s + \frac{1}{h} \nabla_{\tilde{V}_0} \tilde{V}_1
$$
or, equivalently,
\[ \nabla^\flat_{\partial_s} = 0, \quad \nabla^\flat_{\partial_s} \partial_s = h' \nabla_0, \quad \nabla^\flat_{\partial_s} \nabla_1 = -h' g_{\tilde{X}}(V_0, V_1) \partial_s + \nabla_{V_0} \nabla_1. \]

This can be interpreted as saying that the Levi-Civita connection induces a connection on the ‘rescaled tangent bundle’, locally spanned by \{\partial_s, \nabla_1\}.

Let \{\theta^i\} be the dual coframe to \{V_i\} on \tilde{X}, so that \{ds, h \theta^i\} is a coframe on \tilde{X}⁺, and the Dirac-type operator on \(E\) is
\[ D_{\tilde{X}⁺} = \alpha'(ds) \nabla^E_{\partial_s} + \sum \alpha'(h \theta^i) \nabla^E_{\nabla_1} = \alpha'(ds) \partial_s + \sum \alpha'(h \theta^i) \nabla^E_{\nabla_1}. \]

We can use \(\alpha'(ds)\) to split \(E\),
\[ E = E_i \oplus E_{-i}, \quad \alpha'(ds)|_{E_{\pm i}} = \pm i, \]
and we have natural projections onto each summand,
\[ \frac{1}{2}(1 \pm \frac{1}{i} \alpha'(ds)) : E \to E_{\pm i}. \]

Since \(\nabla^E\) is a Clifford connection we have, for \(\varepsilon \in \{\pm 1\}, \sigma \in \mathcal{C}^\infty(\tilde{X}⁺; E)\), and \(W\) a vector field satisfying \(ds(W) = 0\),
\[ \nabla^E_W \left( \frac{1}{2} (1 + \frac{i}{\varepsilon} \alpha'(ds)) \sigma \right) = \frac{1}{2} (1 + \frac{i}{\varepsilon} \alpha'(ds)) \nabla^E_W \sigma + \frac{\varepsilon}{2i} \alpha'(\nabla_W ds) \sigma \]
\[ = \frac{1}{2} (1 + \frac{i}{\varepsilon} \alpha'(ds)) \nabla^E_W \sigma + \frac{\varepsilon}{2i} \frac{h'}{\hbar} \alpha'(W^\flat) \sigma, \]
or, in terms of the splitting of \(E\),
\[ \nabla^E_W = \begin{pmatrix} \nabla^E_W & -\frac{1}{2i} \frac{h'}{\hbar} \alpha'(W^\flat) \\ \frac{1}{2i} \frac{h'}{\hbar} \alpha'(W^\flat) & \nabla^E_W \end{pmatrix}. \]

Hence the associated Dirac-type operator satisfies
\[ D_{\tilde{X}⁺} = \alpha'(ds) \partial_s + \sum \left( \begin{array}{cc} 0 & \alpha'(h \theta^i) \\ \alpha'(h \theta^i) & 0 \end{array} \right) \left( \begin{array}{cc} \nabla^E_{\nabla_1} & -\frac{1}{2i} \frac{h'}{\hbar} \alpha'(\nabla_1^\flat) \\ \frac{1}{2i} \frac{h'}{\hbar} \alpha'(\nabla_1^\flat) & \nabla^E_{\nabla_1} \end{array} \right) \]
\[ = \left( i(\partial_s + \frac{m}{2} \frac{h'}{\hbar}) - \frac{i}{\hbar} \tilde{\partial}_{\tilde{X}} \right) \left( \begin{array}{cc} 0 & \alpha'(\tilde{\partial}_{\tilde{X}}) \\ \alpha'(\tilde{\partial}_{\tilde{X}}) & 0 \end{array} \right) \]
\[ = \left( i(\partial_s + \frac{m}{2} \frac{h'}{\hbar}) - \frac{i}{\hbar} \tilde{\partial}_{\tilde{X}} \right) \left( \begin{array}{cc} 0 & \alpha'(\tilde{\partial}_{\tilde{X}, Q}) \\ \alpha'(\tilde{\partial}_{\tilde{X}, Q}) & 0 \end{array} \right) \]
with \(\tilde{\partial}_{\tilde{X}} = \sum \alpha'(\theta^i) \nabla^E_{\nabla_1} \). Thus
\[ h\tilde{\partial}_{\tilde{X}, Q} = h^{m/2}(h(D_{\tilde{X}⁺} + Q))h^{-m/2} = \left( \begin{array}{cc} i h \partial_s & \tilde{\partial}_{\tilde{X}, Q} \\ \tilde{\partial}_{\tilde{X}, Q} & -i h \partial_s \end{array} \right) = \bigoplus_{\mu \in \text{Spec}(\tilde{\partial}_{\tilde{X}, Q})} \left( \begin{array}{cc} i h \partial_s & \mu \\ \mu & -i h \partial_s \end{array} \right), \]
where we have used that \(\partial_s\) and \(\tilde{\partial}_{\tilde{X}, Q}\) commute. Define
\[ R(s) = \int_1^s dt \frac{dt}{h(t)} \]
so that \(\partial_R = h(s) \partial_s\). If \(\begin{pmatrix} a \\ b \end{pmatrix}\) is in the null space of \(D_{\tilde{X}⁺} + Q\) and the \(\mu\) eigenspace of \(\tilde{\partial}_{\tilde{X}, Q}\), \(\mu \neq 0\), we have
\[ a''(R) = \mu^2 a(R), \quad b''(R) = \mu^2 b(R) \implies \begin{pmatrix} a(R, \mu) \\ b(R, \mu) \end{pmatrix} = a_1(\mu) \left( \begin{array}{cc} 1 \\ -i \end{array} \right) e^{\mu R} + a_2(\mu) \left( \begin{array}{cc} 1 \\ i \end{array} \right) e^{-\mu R} \]
Lemma 7.1. If $\mathcal{D}_{\text{VAPS}}(\partial_{\tilde{X}^+,Q})$ is the graph closure of $\mathcal{D}_{\text{max}}(\partial_{\tilde{X}^+,Q}) \cap (h(s)^{1/2}H^1(\tilde{X}^+; E))$ then

$$\langle \partial_{\tilde{X}^+,Q}, \mathcal{D}_{\text{VAPS}} \rangle$$

is self-adjoint and invertible with bounded inverse.

Proof. The conditions (7.2) on $h$ imply that $R(s) = \log s$ for all $s < 1$ and $R(s) = s + C$ for some constant $C$ and $s \gg 0$. Since $R(s) = \log s$ for all $s < 1$, the elements of the null space of $\partial_{\tilde{X}^+,Q}$ that are in the $\mu$-eigenspace of $\partial_{\tilde{X}^+,Q}$ are of the form

$$a_1(\mu) \left( \begin{array}{c} 1 \\ -i \end{array} \right) s^\mu + a_2(\mu) \left( \begin{array}{c} 1 \\ i \end{array} \right) s^{-\mu}$$

Now

$$\int_0^1 s^k \, ds < \infty \iff k > -1$$

shows that for a solution to be in $L^2$ for $s < 1$ it must be of the form $e^{\mu R}$ for $\mu > -\frac{1}{2}$.

On the other hand, since $R(s) = C + s$ for $s \gg 0$ and

$$\int_1^\infty e^{ks} \, ds < \infty \iff k < 0$$

shows that for a solution to be in $L^2$ for $s > 1$ it must be of the form $e^{\mu R}$ for $\mu < 0$.

In particular there are no elements in the null space of $\partial_{\tilde{X}^+}$ with the domain $\mathcal{D}_{\text{VAPS}}(\partial_{\tilde{X}^+})$.

Consider the operator $I_b(\partial_{\tilde{X}^+,Q}) = c'(ds)\partial_s + \partial_{\tilde{X}^+,Q}$ over $\mathbb{R}_s \times \tilde{X}$. Since $\partial_{\tilde{X}^+}$ is self-adjoint on $\tilde{X}$, $I_b(\partial_{\tilde{X}^+,Q})$ is self-adjoint on $\mathbb{R} \times \tilde{X}$. Note that the square of $I_b(\partial_{\tilde{X}^+,Q})$ is $-\partial_s^2 + (\partial_{\tilde{X}^+,Q})^2$ and is bounded below by the smallest eigenvalue of $(\partial_{\tilde{X}^+})^2$, which is positive by the Witt assumption.

Let $v \in \mathcal{D}_{\text{max}}(\partial_{\tilde{X}^+})$ be such that

$$\langle \partial_{\tilde{X}^+} u, v \rangle = \langle u, \partial_{\tilde{X}^+} v \rangle$$

for all $u \in \mathcal{D}_{\text{VAPS}}(\partial_{\tilde{X}^+})$. By choosing $u$ with support in $\{ s \leq C \}$, and using the self-adjointness of wedge Dirac-type operators with compatible perturbations on manifolds with corners and iterated fibration structures, we see that $v$ is in the vertical APS domain of $\partial_{\tilde{X}^+}$ at any boundary hypersurface of $\tilde{X}$. By choosing $u$ with support in $\{ s \geq C > 1 \}$, and using the self-adjointness of $I_b(\partial_{\tilde{X}^+})$, we see that $v \in \mathcal{D}_{\text{VAPS}}(\partial_{\tilde{X}^+})$ and hence this is a self-adjoint domain.

Similarly, the fact that wedge Dirac-type operators with compatible perturbations on manifolds with corners and iterated fibration structures have closed range and the lower bound for $I_b(\partial_{\tilde{X}^+})$ combine to show that $(\partial_{\tilde{X}^+,Q}, \mathcal{D}_{\text{VAPS}})$ has closed range. As we have already shown that this operator is injective, it follows that it has a bounded inverse. \hfill $\square$

Since we have shown that the individual operators in the family $\partial_{\tilde{M}^+/\tilde{B},Q}$ are invertible, its families index is identically zero.
7.2. **Extended families index formula.** To exploit the vanishing of the families index of $\partial_{M'/B, Q}$ we will work out an extension of the discussion of the families index above. To distinguish this setting from that above we will use $M'$ instead of $M$, etc. For simplicity we only consider the case of even dimensional fibers; the odd dimensional case can be established by suspension as above. For the most part the constructions above extend easily to the case we will consider here, in which case we will simply indicate the changes necessary.

Let $M' \to B'$ be a locally trivial family of manifolds with corners and iterated fibration structures over $B'$ as in Definition 1.3. Assume that a minimal element $N'_0 \in S(M')$ is such that $\dim N'_0/B' = 0$ and let $\rho_{N'_0}$ be a boundary defining function for $B_{N'_0}$. By a **b-wedge metric** on $M'$ (with respect to $N'_0$) we mean a metric conformally related to a totally geodesic wedge metric on $M'$,

$$g_{M'/B', b-w} = \rho_{N'_0}^{-2}g_{M'/B'}.$$  

In particular this is a metric on $TM'/B$ that near $B_{N'_0}$ takes the form

$$\frac{d\rho_{N'_0}^2}{\rho_{N'_0}^2} + g_{B_{N'_0}/N'_0}$$

with $g_{B_{N'_0}/N'_0}$ a vertical family of wedge metrics, and, for any other $N' \in S(M')$, near $B_N$ takes the form

$$dx^2 + x^2g_{B_{N'}/N'} + \phi_{N'}^*g_{N'/B'}$$

with $g_{B_{N'}/N'}$ a vertical wedge metric, while $g_{N'/B'}$ is a family of b-wedge metrics if $N'_0 < N'$ and a family of wedge metrics if $B_{N'_0} \cap B_{N'} = \emptyset$. This is best understood as a non-degenerate bundle metric on the bundle

$$b,wTM'/B' = \rho_{N'_0}^2wTM'/B'.$$

A Clifford b-wedge bundle (with respect to $N'_0$) is defined just as in Definition 1.6 but with an action of $\mathbb{C} \otimes \mathfrak{Cl}(b,wTM'/B', b,w_{TM'/B'})$. We denote the corresponding Dirac-type operator by $D_{M'/B', Q}$ and, if $Q$ is a compatible perturbation, $D_{M'/B', Q} + Q$ will be denoted $D_{M'/B', Q}$. We assume that the perturbation has stabilized the index, so that ker $D_{M'/B', Q}$ is a vector bundle over $B'$.

Define a multiweight on $M'$ by $b'(H) = 0$ if $H \subseteq N'_0$ and $b'(H) = b(H)$ otherwise (where $b$ is defined in (1.16)). Let

$$L^2(M'/B', E) = \rho_{M'}^{-b'}L_{b,w}^2(M'/B'; E)$$

where the latter is defined using the b-wedge metric on $M'/B'$ and the Hermitian metric on $E$. Define $\partial_{M'/B', Q}$ to be the operator $\frac{d}{d\rho_{M'}}D_{M'/B', Q}\rho_{M'}^{-b}$, so that $\partial_{M'/B', Q}$ acting on $L^2(M'/B'; E)$ is unitarily equivalent to $D_{M'/B', Q}$ acting on $L_{b,w}^2(M'/B'; E)$. We define the vertical APS domain to be the graph closure of

$$D_{\max}(\partial_{M'/B', Q}) \cap \left( \prod_{N' \in S_o(M) \setminus \{N'_0\}} \rho_{N'}^{1/2}H_e^{1}(M'/B'; E) \right).$$

For each $N' \in S(M')$, there is a boundary operator of $\partial_{M'/B', Q}$ given by

$$D_{N'_0/B', Q} = \partial_{N'_0/B', Q}|_{B_{N'_0}}, \quad D_{N'/B'} = \rho_{N'}\partial_{N'/B', Q}|_{B_{N'}}$$

if $N' \neq N'_0$. 


all of which are families of wedge Dirac-type operators. The Witt assumption in this case is that each of these boundary operators are invertible. Just as in §4.3 under the Witt condition we can construct a generalized inverse of $\partial_{M'/B',Q}$ with compact errors within the edge calculus. However in this case, the proof of Proposition 4.9 should be modified at $\mathcal{B}_{\phi_\partial}^{(2)}(N'_0)$ because $D_{N'_0/B',Q}$ is simply the restriction to $\mathcal{B}_{N'_0}$ without having to multiply by the boundary defining function. For this reason the generalized inverse has order zero at $\mathcal{B}_{\phi_\partial}^{(2)}(N'_0)$ while having order one at $\mathcal{B}_{\phi_\partial}^{(2)}(N')$ for $N' \neq N'_0$. The upshot is that the generalized inverse is not compact and so does not guarantee discrete spectrum. Indeed, one can argue as in [Mel93] and see that the spectrum will not be discrete. Nevertheless, the Witt assumption does guarantee that $\partial_{M'/B',Q}$ is a smooth family of self-adjoint Fredholm operators.

The heat kernel construction is, as we now briefly describe, an easy amalgamation of the heat kernel construction in [Mel93, Chapter 7] for the ‘b’-face corresponding to $\mathcal{B}_{N'_0}$ and the heat kernel construction of §3.5 at the other boundary hypersurfaces.

The b-wedge heat space is given by

$$H(M'/B')_{b,w} = \left[ M' \times \phi M' \times \mathbb{R}^+_s; \mathcal{B}_{N'_0} \times \phi_{N'_0} \mathcal{B}_{N'_0} \times \mathbb{R}^+_s; \cdots \mathcal{B}_{N'_1} \times \phi_{N'_1} \mathcal{B}_{N'_1} \times \mathbb{R}^+_s; \mathcal{B}_{N'_\ell} \times \phi_{N'_\ell} \mathcal{B}_{N'_\ell} \times \{0\}; \cdots \mathcal{B}_{N'_1} \times \phi_{N'_1} \mathcal{B}_{N'_1} \times \{0\} \right],$$

where $\{N'_0, N'_1, \ldots, N'_\ell\}$ is a non-decreasing list of $\mathcal{S}(M')$. (The difference between this heat space and the wedge heat space in §3.5 is that there is no boundary hypersurface corresponding to $\mathcal{B}_{N'_0}$ at time zero, as is to be expected from [Mel93, Chapter 7].) The composition heat space is described in Appendix C below, where a composition result is established.

Blowing-up $\mathcal{B}_{N'_0} \times \phi_{N'_0} \mathcal{B}_{N'_0} \times \mathbb{R}^+_s$ results in a collective boundary hypersurface $\mathcal{B}_{11,0}^{(H)}(N'_0)$ which we can identify with $\mathbb{R}^+_s \times H(\mathcal{B}_{N'_0}/B)_w$ and at which the model heat operator is

$$\partial_t - (-\partial_s^2 + D_{N'_0/B',Q}^2).$$

Hence the model heat kernel at this face is $e^{-t(-\partial_s)} e^{-t D_{N'_0/B',Q}^2}$. The other blow-ups produce model problems that are identical to the ones in §4.4. Once the model problems are solved we can solve away the expansion at each face using the composition result from Appendix C and obtain the heat kernel as an element of

$$e^{-t\partial_{M'/B',Q}} \in \mathcal{B}_{\mathcal{H}_{\phi_\partial}}^{H,\mathcal{J}(H)} \mathcal{A}_{-m-1}(H(M'/B')_{b,w}; \text{Hom}(E) \otimes \Omega_{b,R})$$

where the index set $\mathcal{H}$ and multiweights $\mathcal{J}(H)$, $\mathcal{J}$, are defined as before for $\mathcal{B}_{10,0}^{(H)}(N'_0), \mathcal{B}_{01,0}^{(H)}(N'_0)$ with $N' \neq N'_0$, and are given by

$$\mathcal{H}(\mathcal{B}_{11,0}^{(H)}(N'_0)) = N'_0, \quad \mathcal{H}(\mathcal{B}_{10,0}^{(H)}(N'_0)) = \mathcal{H}(\mathcal{B}_{01,0}^{(H)}(N'_0)) = 0,$$

$$\mathcal{J}(\mathcal{B}_{11,0}^{(H)}(N'_0)) = \mathcal{J}(\mathcal{B}_{10,0}^{(H)}(N'_0)) = \mathcal{J}(\mathcal{B}_{01,0}^{(H)}(N'_0)) = \infty,$$

$$\mathcal{J}(\mathcal{B}_{11,0}^{(H)}(N'_0)) = -1, \quad \mathcal{J}(\mathcal{B}_{10,0}^{(H)}(N'_0)) = \mathcal{J}(\mathcal{B}_{01,0}^{(H)}(N'_0)) = 0,$$

at the collective boundary hypersurfaces associated to $N'_0$.

Just as in [Mel93, Chapter 7], the heat kernel is not trace-class because at $\mathcal{B}_{11,0}^{(H)}(N'_0)$ it is $\mathcal{O}(\rho_{N'_0}^{-1})$ times a non-degenerate density. However the renormalized (fibrewise) trace of the
heat kernel,
\[
R \, \text{Tr} \left( e^{-\partial_{M'/B',\varrho}} \right) = \left. R \int_{M'/B'} \text{tr}(e^{-\partial_{M'/B',\varrho}}) \right|_{\text{diag}_{M'}} = \text{FP} \int_{M'/B'} \rho_{N_0}^\varrho \left. \text{tr}(e^{-\partial_{M'/B',\varrho}}) \right|_{\text{diag}_{M'}}
\]
will stand in for the trace as it does in, e.g. [Mel93, MP97a]. (For more on these renormalizations, see [Alb09] and [Alb07] for another application to an index theorem.) In particular, the renormalized trace converges as \( t \to \infty \) to the dimensions of the null spaces of \( \partial_{M'/B',Q} \), and the corresponding renormalized supertrace converges to the index, while as \( t \to 0 \), the renormalized trace has short-time asymptotics as before.

Thus the renormalized supertrace mediates between the index and the short-time asymptotic expansion of the heat kernel but crucially the renormalized supertrace does depend on \( t \). This dependence can be computed via the trace-defect formula,
\[
R \, \text{Tr} \left( [A, B] \right) = \text{FP} \int_{z=0} \text{Tr}(\rho_{N_0}^z A B) = \text{FP} \int_{z=0} \text{Tr}(\rho_{N_0}^z A B)
\]
where, e.g., \( I_{N_0}^0(A; \sigma) \) is the restriction of \( \rho_{N_0}^z A \rho_{N_0}^z \) to \( \rho_{N_0}^z = 0 \), and gives rise to the \( \eta \) invariant.

The next step in obtaining the families index theorem is to fix a connection for \( \psi' \), and define the Bismut superconnection just as in 5.1.4, \( A_{M'/B',Q} \). The construction of the heat kernel of \( A_{M'/B',Q} \) follows quickly from that of \( \partial_{M'/B',Q} \) and the composition results from Appendix C as in Theorem 5.5. (The rescaling only takes place at collective boundary hypersurfaces \( \mathcal{B}^{(H)}_{\phi\psi}(N') \) with \( N' \neq N_0' \), and so proceeds exactly as before.) If the null spaces of \( \partial_{M'/B',Q} \) do not form a vector bundle over \( B \) we find a smoothing perturbation as in [MR06, Lemma 1.1] that is compactly supported in the interior of \( M' \) (and hence does not affect any of our other arguments) and incorporate this perturbation into \( Q \) without further comment.

**Theorem 7.2.** Let \( M' \xrightarrow{\psi'} B' \) be a fiber bundle of manifolds with corners and iterated fibration structures, such that \( \dim M'/B' \) is even, with a minimal element \( N_0' \) such that \( \dim N_0'/B' = 0 \), \( E \to M' \) a \( \mathbb{Z}_2 \)-graded b, wedge Clifford bundle with associated Dirac-type operator \( \partial_{M'/B',Q} \). If \( \partial_{M'/B',Q} \) with its vertical APS domain satisfies the Witt assumption, then

\[
\text{Cheven}(\text{Ind}(\partial_{M'/B',Q}), \nabla^\text{Ind}) = \int_{M'/B'} \tilde{A}(M'/B') \text{Ch}'(E) \eta_Q(\tilde{\mathcal{B}}_{N_0'}/B')
\]

\[
- \sum_{N' \in S_{\psi'}(M') \setminus \{N_0'\}} \int_{N'/B'} \tilde{A}(N'/B') R^Q(\mathcal{B}_{N'/N'}) + d \int_0^\infty R \text{Str} \left( \frac{\partial A^t_{M'/B'}}{\partial t} e^{-A^t_{M'/B'}} \right) dt
\]

where \( \eta_Q(\tilde{\mathcal{B}}_{N_0'}/B') \) is the normalized Bismut-Cheeger \( \eta \)-form.

7.3. **b-c suspension families index formula.** Let us return to the context of the family \( \tilde{\partial}_{M'+\bar{B},Q} \) from 7.1. where we showed that this operators families index vanishes.
We compactify $\tilde{M}^+ = \mathbb{R}_+^+ \times \tilde{M}$ to
\[ [0, 1]_\sigma \times \tilde{M} \]
using the logarithm so that the warped product metric $g^+$ is a b-metric near $\{1\} \times \tilde{M}$. We continue to denote $[0, 1]_\sigma \times \tilde{M}$ by $\tilde{M}^+$. Note that $\tilde{M}^+$ is naturally a locally trivial fiber bundle of manifolds with corners and iterated fibration structures, with each of $\{0\} \times \tilde{M}$, $\{1\} \times \tilde{M}$ a collective boundary hypersurface over $\tilde{B}$, and in sum
\[ S(\tilde{M}^+) = \{\tilde{B}\} \cup \{[0, 1]_\sigma \times \tilde{N} = \tilde{N}^+ : \tilde{N} \in S_Q(\tilde{M})\}. \]

By the Theorem 7.2 above we have
\[ (7.3) \quad \mathcal{J}_Q(\tilde{M}/\tilde{B}) - \eta_Q(\tilde{M}/\tilde{B}) \]
\[ = \int_{\tilde{M}^+/\tilde{B}} \hat{A}(\tilde{M}^+/\tilde{B}) \text{Ch}'(E) + \sum_{\tilde{N} \in S_Q(\tilde{M})} \int_{\tilde{N}^+/\tilde{B}} \hat{A}(\tilde{N}^+/\tilde{B}) \mathcal{J}_Q(\mathfrak{B}_{\tilde{N}}^+/\tilde{N}^+) + d\eta_{b-w,q}. \]

We will simplify this formula by carrying out the integrals over $[0, 1]_\sigma$.

More generally for any connection on $T\tilde{M}^+ / \tilde{B}$, $\nabla$, and any polynomial $f$, let
\[ A_f(\nabla) = \text{Tr}(f(\nabla^2)) \in \Omega^*(\tilde{M}^+) \]
where $\nabla^2$ denotes the curvature of $\nabla$. Following, e.g., [BGV04, Proposition 1.41], given two connections $\nabla$, $\nabla'$, we fix a transgression form $TA_f(\nabla, \nabla')$ satisfying $dTA_f(\nabla, \nabla') = A_f(\nabla') - A_f(\nabla)$ by the formula
\[ TA_f(\nabla, \nabla') = \int_0^1 \text{Tr}(\frac{\partial \nabla_t}{\partial t} f'(\nabla^2_t)) \, dt \]
where $\nabla_t = (1 - t)\nabla + t\nabla'$.

As we have fixed a connection for $\tilde{M}^+ \rightarrow \tilde{B}$, we get a connection on $T\tilde{M}^+/\tilde{B}$, for each choice of vertical metric. We are particularly interested in the connections
\[ \nabla^h \leftrightarrow ds^2 + h(s)^2 g_{\tilde{M}/\tilde{B}}, \quad \nabla^\text{con} \leftrightarrow ds^2 + \hat{s}^2 g_{\tilde{M}/\tilde{B}}, \quad \hat{\nabla} \leftrightarrow ds^2 + g_{\tilde{M}/\tilde{B}}. \]

Let $\pi : \tilde{M}^+ \rightarrow \tilde{M}$ be the natural projection and $j : \tilde{M} \rightarrow \tilde{M}^+$ the inclusion of the left endpoint.

**Proposition 7.3.** For any polynomial $f$, we have
\[ \pi_*(A_f(\nabla)) = j^*(TA_f(\nabla, \nabla^h)) = TA_f(\nabla, \nabla^h)|_{s=0} = TA_f(\nabla, \nabla^\text{con})|_{s=0}. \]

**Proof.** The middle equality holds by definition and the final holds because $h \equiv s$ near $s = 0$, so we focus on the first equality.

Let $\{\tilde{e}_i, \tilde{f}_\alpha\}$ be a local frame for $T\tilde{M}$, in which $\{\tilde{e}_i\}$ constitute an orthonormal frame for $T\tilde{M}/\tilde{B}$, and let $\{\tilde{e}^i, \tilde{f}^\alpha\}$ be the dual coframe. Denote their lifts/pull-backs to $\tilde{M}^+$ by the same symbols. Let
\[ \{V_a\} = \{\partial_s, \frac{1}{h} \tilde{e}_i\}, \quad \{V^a\} = \{ds, h \tilde{e}^i\} \]
be the corresponding frames for $T\tilde{M}^+/\tilde{B}$ on $\tilde{M}^+$. (We will use $a, b, c$ for indices that begin at 0 and $i, j, \ell$ for indices corresponding to $T\tilde{M}/\tilde{B}$.)
Let $\omega$ and $\overline{\omega}$ denote the one-form matrices corresponding to these frames and the connections $\nabla, \overline{\nabla}$, by
\[ \nabla V_a = \omega^b_a V_b, \quad \overline{\nabla} V_a = \overline{\omega}^b_a V_b. \]

Let $d_{\overline{M}^+}$ denote the exterior derivative on $\overline{M}^+$, $d_{\overline{M}^+ / \overline{B}}$ the part of the exterior derivative that raises the $\psi^+$-vertical degree by one, and $\hat{\nabla}$ the difference between these two, so that
\[ d_{\overline{M}^+} = d_{\overline{M}^+ / \overline{B}} + \hat{\nabla}. \]

(See, e.g., [BGV04, Proposition 10.1], [HHM04, Proposition 14], for descriptions of $\hat{\nabla}$.) It is easy to see that the forms $\omega^b_a$ satisfy
\[ d_{\overline{M}^+ / \overline{B}} V^a = V^b \wedge \omega^a_b \]
and hence
\[ \omega^j_0 = -\omega^0_j = h'(s) \overline{\nu}^j, \quad \omega^j_i = \overline{\omega}^j_i. \]

Then $\theta = \omega - \overline{\omega}$ satisfies
\[ \theta^j_0 = -\theta^0_j = h'(s) \overline{\nu}^j, \quad \theta^j_i = 0 \text{ otherwise}. \]

Let $\nabla(t) = (1-t)\nabla + t \overline{\nabla}$ so that its connection one-form is $\omega(t) = \overline{\omega} + tgt$ and its curvature $\Omega(t) = d\omega(t) - \omega(t) \wedge \omega(t)$ is given by
\[ \Omega^j_0(t) = -\Omega^0_j(t) = d(th'(s) \overline{\nu}^j) - (th'(s) \overline{\nu}^j) \wedge (\overline{\omega}^j_t) = th''(s) \, ds \wedge \overline{\nu}^j + th'(s) \, \hat{\nabla}_W \overline{\nu}^j \]
\[ \Omega^j_i(t) = \overline{\Omega}^j_i + t^2(h'(s))^2 \overline{\nu}^i \wedge \overline{\nu}^j \]
where $\overline{\Omega}$ denotes the curvature of $\overline{\nabla}$.

We can write these expressions succinctly in terms of the one-form valued matrix given by
\[ \gamma^j_i = \gamma^0_i = -\overline{\nu}^i, \quad \gamma^j_i = 0 \text{ for all } i, j. \]

Indeed,
\[ (\gamma^2)^j_i = \gamma^0_i \wedge \gamma^j_i = -\overline{\nu}^i \wedge \overline{\nu}^j, \quad (\gamma^2)^b_a = 0 \text{ otherwise} \]
and hence
\[ \theta = h'(s) \gamma, \quad \Omega(t) = \overline{\Omega} + th''(s) \, ds \wedge \gamma + th'(s) \, \hat{\nabla}_W \gamma - t^2(h'(s))^2 \gamma^2. \]

Now let $f$ be as in the statement of the proposition and note that, since $h'(0) = 1$,
\[ j^*(TA_f) = j^* \left( \int_0^1 h'(t) \mathrm{Tr}(\gamma f'(\Omega(t))) \, dt \right) = \int_0^1 \mathrm{Tr}(\gamma f'(\overline{\Omega} + t \, \hat{\nabla}_W \gamma - t^2 \gamma^2)) \, dt. \]

On the other hand, if we denote $\Omega = \overline{\Omega} + h''(s) \, ds \wedge \gamma$, then as in [BGV04, pg.48-49] we have
\[ \mathrm{Tr}(f(\Omega)) = \mathrm{Tr}(f(\overline{\Omega})) + h''(s) \, ds \wedge \mathrm{Tr}(\gamma f'(\overline{\Omega})), \]
and so
\[ \pi_*(A_f) = \int_0^1 h''(s) \, \mathrm{Tr}(\gamma f'(\overline{\Omega})) \, ds = \int_0^1 h''(s) \, \mathrm{Tr}(\gamma f'((\overline{\Omega} + h'(s) \, \hat{\nabla}_W \gamma - (h'(s))^2 \gamma^2)) \, ds \]
\[ \overset{t=h'(s)}{\longrightarrow} \int_0^1 \mathrm{Tr}(\gamma f'(\overline{\Omega} + t \, \hat{\nabla}_W \gamma - t^2 \gamma^2)) \, dt \]
which coincides with $j^*(TA_f)$ as required. \[ \square \]
To apply this to simplify formula (7.3), note that
\[ \partial_s(J_Q(B_N^+/N^+)) = 0 \] and \[ \partial_s \text{Ch}'(E) = 0 = \iota_{\partial_s} \text{Ch}'(E). \]

For the twisted Chern character this follows from the fact that the twisted curvature is independent of \( s \). On the other hand, the \( J \) forms only depend on the vertical metric, and are unchanged by rescaling the metric. The warping factor \( h \), as it only depends on \( s \), has the effect of rescaling the vertical metric at each \( s \).

Applying the proposition to simplify the index formula (7.3) yields the first part of the following theorem. Taking the exterior derivative and applying the expression for \( d\tilde{\eta} \) yields the second part. We introduce the abbreviation
\[ \hat{A}_c(\tilde{M}/\tilde{B}) = \hat{A}(\nabla^{|s|=0}), \quad \hat{T}\tilde{A}(\tilde{M}/\tilde{B}) = TA_f(\nabla^\text{cpl}, \nabla^\text{con})|_{s=0}. \]

and similarly for \( \tilde{N}/\tilde{B} \).

**Theorem 7.4.**

\[ J_Q(\tilde{M}/\tilde{B}) - \pi Q(\tilde{M}/\tilde{B}) = \int_{\tilde{M}/\tilde{B}} T\hat{A}(\tilde{M}/\tilde{B}) \text{Ch}'(E) + \sum_{\tilde{N} \in S_q(M)} \int_{\tilde{N}/\tilde{B}} T\hat{A}(\tilde{N}/\tilde{B}) J_Q(\mathfrak{B}_{\tilde{N}}/\tilde{N}) + d\eta_{b-w,q}. \]

Moreover
\[ dJ_Q(\tilde{M}/\tilde{B}) = \int_{\tilde{M}/\tilde{B}} \hat{A}_c(\tilde{M}/\tilde{B}) \text{Ch}'(E) + \sum_{\tilde{N} \in S_q(M)} \int_{\tilde{N}/\tilde{B}} \hat{A}_c(\tilde{N}/\tilde{B}) J_Q(\mathfrak{B}_{\tilde{N}}/\tilde{N}). \]

**Remark 7.5.** In the case \( \tilde{B} \) is a point, the deduction of the first part of Theorem [7.4 from (7.3)] follows from [AGR16, Section 5]. Since this follows from the preceding proof also we only briefly sketch the argument. To evaluate the term \( \int_{\tilde{X}^+} \hat{A}(\tilde{X}^+) \text{Ch}'(E) \) in (7.3), using that \( \text{Ch}'(E) \) is closed and that \( \hat{A}(\nabla^h) - \hat{A}(\nabla^\text{cpl}) = dT\hat{A}(\nabla^h, \nabla^\text{cpl}) \). Since both \( A(\nabla^\text{cpl}) \) and \( \text{Ch}'(E) \) have no \( ds \) component, the integral is \( \int_{\tilde{X}^+} dT\hat{A}(\nabla^h, \nabla^\text{cpl}) \text{Ch}'(E) \). From loc. cit., at the interior of each boundary hypersurface, the transgression splits into according to whether the base or vertical connections change; at the \( s = 0 \) only the vertical metric changes and there \( T\hat{A}(\nabla^h, \nabla^\text{cpl})|_{s=0} = T\hat{A}(\nabla^\text{con}, \nabla^\text{cpl}) \) while at the other boundary faces (the \( \tilde{Y}^+ \)) the base metric changes from \( ds^2 + g_Y \) to \( ds^2 + h(s)^2 g_Y \) and as discussed the transgression of \( \hat{A} \) for such metrics is zero.

When the fibers of \( \tilde{M} \to B \) are closed manifolds this reads
\[ J_Q(\tilde{M}/\tilde{B}) - \pi Q(\tilde{M}/\tilde{B}) = \int_{\tilde{M}/\tilde{B}} T\hat{A}(\tilde{M}/\tilde{B}) \text{Ch}'(E) + d\eta_{b-w,q} \]

which is consistent with [AGR16, Main Theorem]. Note that if \( \tilde{B} \) is a point then by conformal invariance \( T\hat{A}(\nabla^\text{con}, \nabla^\text{cpl}) = 0 \), so \( J_Q(\tilde{M}) = \eta_Q(\tilde{M}) \) However as noted above this is generally not the case.
THE INDEX FORMULA FOR FAMILIES OF DIRAC TYPE OPERATORS ON PSEUDOMANIFOLDS 91

APPENDIX A. COMPOSITION OF EDGE PSEUDODIFFERENTIAL OPERATORS

Let \( M \xrightarrow{\psi} B \) be a family of manifolds with corners and iterated fibration structures as in Definition 1.3. In this section we prove the composition formula for families of edge pseudodifferential operators acting on the fibers of \( \psi \).

**Edge triple space.** Our construction of the triple space is the natural combination of [MP92, Appendix] and [Maz91, §3]. Let

\[
\begin{align*}
M_\psi^2 &= M \times_\psi M = \{ (\zeta, \zeta') \in M^2 : \psi(\zeta) = \psi(\zeta') \}, \\
M_\psi^3 &= \{ (\zeta, \zeta', \zeta'') \in M^3 : \psi(\zeta) = \psi(\zeta') = \psi(\zeta'') \}.
\end{align*}
\]

We start with the three natural projections,

\[
\begin{array}{ccc}
M_\psi^3 & \xrightarrow{\pi_{LM}} & M_\psi^2 \\
& & \xrightarrow{\pi_{LR}} \\
M_\psi^2 & \xrightarrow{\pi_{LR}} & M_\psi^2
\end{array}
\]

and we will modify \( M_\psi^3 \) so as to end up with \( b \)-fibrations down to \( (M/B)_e^3 \).

For each \( N \in \mathcal{S}_\psi(M) \), let

\[(A.1)\]

\[
T(N) = \{ (\zeta, \zeta', \zeta'') \in \mathcal{B}_N^3 : \phi_N(\zeta) = \phi_N(\zeta') = \phi_N(\zeta'') \}, \quad S_{LM}(N) = \pi_{LM}^{-1}(\mathcal{B}_N \times_{\phi_N} \mathcal{B}_N), \quad S_{LR}(N) = \pi_{LR}^{-1}(\mathcal{B}_N \times_{\phi_N} \mathcal{B}_N), \quad S_{MR}(N) = \pi_{MR}^{-1}(\mathcal{B}_N \times_{\phi_N} \mathcal{B}_N).
\]

Let \( \mathcal{S}_\psi(M) = \{ N_1, N_2, \ldots, N_\ell \} \) be a listing of \( \mathcal{S}_\psi(M) \) with non-decreasing depth.

We construct \( (M/B)_e^3 \) in two steps. In the first step, we blow-up the submanifolds \( T(N_i) \) in order,

\[
M_{\psi,T}^3 = \left[ M_\psi^3 ; T(N_1) ; T(N_2) ; \ldots ; T(N_\ell) \right].
\]

As in §3.2, \( T(N_i) \) is a p-submanifold of \( M_\psi^3 \) and each \( T(N_i) \) is a p-submanifold once we have blown-up \( T(N_j) \) for all \( N_j < N_i \).

The second step is to blow-up the submanifolds \( \{ S_{\bullet}(N) \} \in \{ \mathcal{L}_{LM,LR,MR} \} \) as the \( N \) range over \( \mathcal{S}_\psi(M) \). As we will see below, these are separated by the blow-ups performed in the first step. Thus,

\[
(M/B)_e^3 = \left[ M_{\psi,T}^3 ; (S_{LM}(N_1) \cup S_{LR}(N_1) \cup S_{MR}(N_1)) ; \ldots ; (S_{LM}(N_\ell) \cup S_{LR}(N_\ell) \cup S_{MR}(N_\ell)) \right].
\]

We denote the blow-down map by \( \beta_3 : (M/B)_e^3 \longrightarrow M_\psi^3 \).

We denote the collective boundary hypersurfaces obtained from these blow-ups by

\[
T(N) \leftrightarrow \mathcal{B}_{\phi_{\psi\phi}}^3(N), \quad S_{LM}(N) \leftrightarrow \mathcal{B}_{\phi_{\psi\phi}}^3(N), \quad S_{LR}(N) \leftrightarrow \mathcal{B}_{\phi_{\psi\phi}}^3(N), \quad S_{MR}(N) \leftrightarrow \mathcal{B}_{\phi_{\psi\phi}}^3(N).
\]

We denote the other collective boundary hypersurfaces by

\[
\mathcal{B}_N^{(1)} \times_\psi M_\psi^2 \leftrightarrow \mathcal{B}_{100}^3(N), \quad M \times_\psi \mathcal{B}_N^{(1)} \times_\psi M \leftrightarrow \mathcal{B}_{010}^3(N), \quad M_\psi^2 \times_\psi \mathcal{B}_N^{(1)} \leftrightarrow \mathcal{B}_{001}^3(N).
\]
Lemma A.2. Let $\pi^0 : M_1 \to M_2$ be a $b$-fibration of manifolds with corners, $N_i \subset M_i$ $p$-submanifolds, and assume $N_1 \subset \pi^{-1}(N_2)$ such that $\pi^0|_{N_1} : N_1 \to N_2$ is a diffeomorphism. Then $\pi^0$ extends to a $b$-fibration

$$
\pi : [M_1; N_1; (\pi^0)^{-1}(N_2)] \to [M_2; N_2].
$$

If $ff$ is the introduced front face of $[M_2; N_2]$ and $ff_1, ff_2$ are the two introduced front faces of $[M_1; N_1; (\pi^0)^{-1}(N_2)]$, then the exponent matrix $e$ for $\pi$ satisfies

$$
e(ff_1', ff) = e(ff_2', ff) = 1.
$$

Proof. This is straightforward from the local description of radial blow up. Indeed, by assumption, there are coordinates $(x, y, z_1, z_2)$ in which the map $\pi^0$ has action $(x, y, z_1, z_2) \to (x, y)$ and such that $N_1 = \{x = 0 = z_1\}$ while $(\pi^0)^{-1}(N_2) = \{x = 0\}$. Blowing up $N_1$ in $M_1$ gives polar coordinates $\rho = (|x|^2 + |z_1|^2)^{1/2}$, $\phi = (x, z_1)/\rho$ together with $y$ and $z$. The intersection with $(\pi^0)^{-1}(N_2)$ is contained in $\phi = (0, z_1/|z_1|)$ so nearby this intersection one can use coordinates $\xi = x/|z_1|$, $y$, $|z_1|$, $\zeta_1 = z_1/|z_1|z_2$, and blowing up $\xi = 0$ gives coordinates $r = |\xi|$, $\hat{r} = \xi/|\xi| = x/|x|$, $y$, $|z_1|$, $\hat{\zeta}_1, z_2$. The map to $[M_2; N_2]$ is locally (in polar coordinates) $(r, \hat{r}, y, |z_1|, \hat{\zeta}_1, z_2) \to (\tilde{r}, \hat{x}, y)$ where $\tilde{r} = |x|$, $\hat{x} = x/|x|$. From this one can deduce that the map is a $b$-fibration with this given exponent matrix.

Proof of Proposition A.1. Since $B$ enters only parametrically we assume that $B = pt$, so $X = M$ and all the strata $N$ are in fact the strata $Y$ of $X$. Letting $X^3_{e,T}(k+1) = [X^3; T(Y_1), \ldots, T(Y_j)]$ where the $Y_1, \ldots, Y_j$ are the strata of depth greater than $k$, let $Y$ be a stratum of depth $k$. In the interior $Y^0$ and near the intersection of $Y$ with a lower depth stratum $\tilde{Y}$ we have a diagram as in (3.4), and work locally in the interior of $Y^0$ and $W$ in coordinates $(x, y, r, w, \tilde{z})$ on $X$ as in (3.5). Recall that near the intersection $\mathcal{B}_{e,\phi}^{(2)}(Y) \cap \mathcal{B}_{e,\tilde{\phi}}^{(2)}(\tilde{Y})$ we have a local diffeomorphism to

$$
[0, 1)_{x'} \times [0, 1)_{\tilde{R}} \times S^2_{+}^{4+\dim{Y}+\dim{W}} \times \tilde{Z}^2
$$

with

$$
\tilde{R} = (r^2 + (r')^2 + ((x/x') - 1)^2 + |(y - y')/x'|^2 + |w - w'|^2)^{1/2}
$$

and $S^2_{+}^{4+\dim{Y}+\dim{W}}$ parametrized by

$$
\tilde{\phi} = (r, r', (x/x') - 1, (y - y')/x', w - w')/\tilde{R}.
$$

The front face $ff$ of $[X^3_{e,T}(k+1); T(Y)]$ fibers $\pi : ff \to Y$, and the inverse image $\pi^{-1}(U)$ of an open neighborhood $U \subset Y^0$ intersected with a neighborhood of the interior lift of $T(\tilde{Y})$ is diffeomorphic to

$$
[0, 1)_{\rho} \times S^2_{++}^{4+2\dim{Y}} \times U \times (\mathcal{B}_{Y^0} \times [0, 1)_{x'})^3.
$$
where \( \tilde{Z} - \mathfrak{D}_{\tilde{Y}} \rightarrow W \). Here we can take \( \rho = (x_1^2 + x_2^2 + x_3^2 + |y_1 - y_3|^2 + |y_2 - y_3|^2)^{1/2} \) and parametrize \( S^2_{+}^{2+2 \dim Y} \) with \( \phi = (x_1, x_2, x_3, y_1 - y_3, y_2 - y_3) / \rho \), where \( x_1 \) is the pullback of \( x \) from projection of \( X^3 \) onto the left factor, etc. From this it is easy to check that the interior lifts of the \( S_\bullet(Y) \) are pairwise disjoint, since they are contained, respectively, in \( \phi \in \{(x_1, 0, 0, y_1 - y_3, 0) / \rho \}, \phi \in \{(0, x_2, 0, 0, y_2 - y_3) / \rho \}, \phi \in \{(0, 0, x_3, y_1 - y_3, y_1 - y_3) / \rho \}. \) Moreover, the inverse image of \( T(\tilde{Y}) \) lies in

\[
\{ \phi = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}, 0, 0) \}
\]

so the interior lifts of the \( S_\bullet(Y) \) and \( T(\tilde{Y}) \) are disjoint p-submanifolds in \([X^3_{x,T}(k + 1); T(Y)]\).

The proposition then follows from application of Lemma A.2 and induction. Indeed, \( T(Y) \) and \( S_\bullet(Y) \) satisfy the hypotheses of Lemma A.2 with \( N_1 = T(Y) \) and \( N_2 = S_\bullet(Y) \), and since for varying \( \bullet \in \{LM, LR, MR\} \) these become disjoint in the lift and the b-fibration property is local, the proposition follows. \( \square \)

Inspection of the proof and the lemma above show that the exponent matrices have only zeros and ones, so we specify them by listing the preimages of the collective boundary hypersurfaces. Recall that the boundary hypersurfaces of \( (M/B)^2 \) are collective boundary hypersurfaces \( \mathfrak{D}_{10}^{(2)}(N), \mathfrak{D}_{10}^{(2)}(N) \), and the front face over \( Y \), \( \mathfrak{D}_{\phi\phi}^{(2)}(N) \). We have, for each \( \{N \in \mathcal{S}_\phi(M)\}, \)

\[
\beta^{\ast}_{LM} \mathfrak{D}_{10}^{(2)}(N) = \{ \mathfrak{D}_{100}^{(3)}(N), \mathfrak{D}_{\phi\phi}^{(3)}(N) \}, \quad \beta^{\ast}_{LM} \mathfrak{D}_{01}^{(2)}(N) = \{ \mathfrak{D}_{010}^{(3)}(N), \mathfrak{D}_{\phi\phi}^{(3)}(N) \},
\]

\[
\beta^{\ast}_{LM} \mathfrak{D}_{20}^{(2)}(N) = \{ \mathfrak{D}_{200}^{(3)}(N), \mathfrak{D}_{\phi\phi}^{(3)}(N) \},
\]

\[
\beta^{\ast}_{LR} \mathfrak{D}_{10}^{(2)}(N) = \{ \mathfrak{D}_{100}^{(3)}(N), \mathfrak{D}_{\phi\phi}^{(3)}(N) \}, \quad \beta^{\ast}_{LR} \mathfrak{D}_{01}^{(2)}(N) = \{ \mathfrak{D}_{010}^{(3)}(N), \mathfrak{D}_{\phi\phi}^{(3)}(N) \},
\]

\[
\beta^{\ast}_{LR} \mathfrak{D}_{20}^{(2)}(N) = \{ \mathfrak{D}_{200}^{(3)}(N), \mathfrak{D}_{\phi\phi}^{(3)}(N) \},
\]

\[
\beta^{\ast}_{MR} \mathfrak{D}_{10}^{(2)}(N) = \{ \mathfrak{D}_{100}^{(3)}(N), \mathfrak{D}_{\phi\phi}^{(3)}(N) \}, \quad \beta^{\ast}_{MR} \mathfrak{D}_{01}^{(2)}(N) = \{ \mathfrak{D}_{010}^{(3)}(N), \mathfrak{D}_{\phi\phi}^{(3)}(N) \},
\]

\[
\beta^{\ast}_{MR} \mathfrak{D}_{20}^{(2)}(N) = \{ \mathfrak{D}_{200}^{(3)}(N), \mathfrak{D}_{\phi\phi}^{(3)}(N) \}.
\]

Applying Melrose's push-forward and pull-back theorems this leads to a composition result for the large edge calculus. The behavior with respect to the conormal singularity at the diagonal is standard, so we will focus on operators of order \(-\infty\). We will also simplify notation by not including vector bundles.

Thus, from (3.11) and Definition 3.6 we will establish composition results for conormal distributions in

\[
\mathcal{A}_{\rho\phi}^{\mathcal{E}}((M/B)^2; \text{Hom}(E, F) \otimes \Omega_{\partial, R})
\]

where we recall that \( \Omega_{\partial, R} = \rho_{(M/B)^2}^{\partial} \beta^{\ast}_{(2), R} \Omega(M/B) \), with \( \partial : \mathcal{M}_{1}((M/B)^2) \rightarrow \mathbb{R} \),

\[
\partial(H) = \begin{cases} -\text{dim}(N/B) + 1 & \text{if } H \subseteq \mathfrak{D}_{\phi\phi}^{(2)}(N) \text{ for some } N \in \mathcal{S}_\phi(M) \\ 0 & \text{otherwise} \end{cases}
\]

For an operator \( A \), let us write its integral kernel as

\[
K_{A} \rho_{(M/B)^2}^{\partial} \mu_{R}.
\]
Then if the composition $C = A \circ B$ is defined, its integral kernel is given by

$$\mathcal{K}_C \rho^3 \mu_R = (\beta_{LR})^* (\beta_{LM}^*(\mathcal{K}_A \rho^3 \mu_R) \cdot \beta_{MR}^*(\mathcal{K}_B \rho^3 \mu_R)).$$

**Theorem A.3.** If $\mathcal{K}_A \in \mathcal{A}^{E_A}_\phi(M/B)^2_e$ and $\mathcal{K}_B \in \mathcal{A}^{E_B}_\phi(M/B)^2_e$ where the index sets satisfy

\[
\text{Re}(\mathcal{E}_A(\mathfrak{B}^{(2)}_{01}(N))) + \text{Re}(\mathcal{E}_B(\mathfrak{B}^{(2)}_{10}(N))) > -1 \quad \text{for all } N \in \mathcal{S}_\phi(M)
\]

then $\mathcal{K}_C \in \mathcal{A}^{E_C}_\phi(M/B)^2_e$ with, for each $N \in \mathcal{S}_\phi(M)$,

\[
\mathcal{E}_C(\mathfrak{B}^{(2)}_{01}(N)) = \mathcal{E}_A(\mathfrak{B}^{(2)}_{01}(N)) \cup \left( \mathcal{E}_A(\mathfrak{B}^{(2)}_{00}(N)) + \mathcal{E}_B(\mathfrak{B}^{(2)}_{01}(N)) \right),
\]

\[
\mathcal{E}_C(\mathfrak{B}^{(2)}_{00}(N)) = \mathcal{E}_B(\mathfrak{B}^{(2)}_{01}(N)) \cup \left( \mathcal{E}_A(\mathfrak{B}^{(2)}_{00}(N)) + \mathcal{E}_B(\mathfrak{B}^{(2)}_{00}(N)) \right),
\]

\[
\mathcal{E}_C(\mathfrak{B}^{(2)}_{0\phi}(N)) = \left( \mathcal{E}_A(\mathfrak{B}^{(2)}_{0\phi}(N)) + \mathcal{E}_B(\mathfrak{B}^{(2)}_{0\phi}(N)) \right)
\]

\[
\cup \left( \mathcal{E}_A(\mathfrak{B}^{(2)}_{10}(N)) + \mathcal{E}_B(\mathfrak{B}^{(2)}_{01}(N)) + \dim(N/B) + 1 \right)
\]

**Proof.** Let $\mu(W)$ denote a nowhere vanishing section of $\Omega(W)$ and $\mu_b(W)$ a nowhere vanishing section of $\Omega_b(W)$.

The behavior of the densities under pull-back is given by

\[
(\beta^{(2)})^*(\mu(M^2_{\psi}/B)) = \prod_{N \in \mathcal{S}_\phi(M)} \rho^{(2)}_{\mathfrak{B}^{(2)}_{0\phi}(N)} \mu((M/B)^2_e/B)
\]

\[
(\beta^{(3)})^*(\mu(M^3_{\psi}/B)) = \prod_{N \in \mathcal{S}_\phi(M)} (\rho^{(3)}_{\mathfrak{B}^{(3)}_{0\phi}(N)} \rho^{(3)}_{\mathfrak{B}^{(3)}_{0\phi}(N)} \rho^{(3)}_{\mathfrak{B}^{(3)}_{0\phi}(N)})^{\dim(M/B) + 1} \prod_{N \in \mathcal{S}_\phi(M)} (\rho^{(3)}_{\mathfrak{B}^{(3)}_{0\phi}(N)})^{\dim(M/B) + 1} \mu((M/B)^3_e/B).
\]

Multiplying the integral kernel of $C$ by $\mu_L = (\beta^{(2)})^* \mu$ yields

\[
\mathcal{K}_C \rho^3 (\beta^{(2)})^* \mu(M^2_{\psi}/B) = (\beta_{LR})^* (\beta^{(2)}_{LM}^*(\mathcal{K}_A \rho^3 \mu) \cdot \beta^{(2)}_{MR}^*(\mathcal{K}_B \rho^3 \mu) \cdot \beta^{(2)}_{LM} \mu_L \cdot \beta^{(2)}_{MR} \mu_R)
\]

\[
= (\beta_{LR})^* (\beta^{(2)}_{LM}^*(\mathcal{K}_A \rho^3) \cdot \beta^{(2)}_{MR}^*(\mathcal{K}_B \rho^3) \cdot (\beta^{(3)})^* \mu(M^3_{\psi}/B)),
\]

hence

\[
\mathcal{K}_C \mu((M/B)^2_e/B) = (\beta_{LR})^* \left( \beta^{(2)}_{LM}^*(\mathcal{K}_A) \cdot \beta^{(2)}_{MR}^*(\mathcal{K}_B) \cdot \prod_{N \in \mathcal{S}_\phi(M)} \rho^{(2)}_{\mathfrak{B}^{(2)}_{0\phi}(N)} \mu((M/B)^2_e/B) \right).
\]

Now we write this in terms of $b$-densities

\[
\mathcal{K}_C \mu_b((M/B)^2_e/B) = (\beta_{LR})^* \left( \beta^{(2)}_{LM}^*(\mathcal{K}_A) \cdot \beta^{(2)}_{MR}^*(\mathcal{K}_B) \cdot \prod_{N \in \mathcal{S}_\phi(M)} \rho^{(2)}_{\mathfrak{B}^{(2)}_{0\phi}(N)} \mu_b((M/B)^2_e/B) \right)
\]

and we can apply the pull-back and push-forward theorems.

The action on polyhomogeneous functions is also easy to write down. Given $A$ as above and $f \in \mathcal{A}^{E_f}_\phi(X)$ we define

\[
A f = (\beta_{L})^*(\mathcal{K}_A \rho^3 \mu_R \cdot \beta^{(2)}_{R} f).
\]

**Proposition A.4.** If $\mathcal{K}_A \in \mathcal{A}^{E_A}_\phi(M/B)^2_e$ and $f \in \mathcal{A}^{G_f}_{\phi}(M/B)$ where the index sets satisfy

\[
\text{Re}(\mathcal{E}_A(\mathfrak{B}^{(2)}_{01}(N))) + \text{Re}(\mathcal{E}_B(\mathfrak{B}^{(1)}_{N})) > -1 \quad \text{for all } N \in \mathcal{S}_\phi(M)
\]
then \( Af \in \mathcal{A}^{\mathcal{G}_{Af}}(N) \) with, for each \( N \in \mathcal{S}_\psi(M) \),

\[
\mathcal{G}_{Af}(\mathcal{B}^{(1)}_N) = \mathcal{E}_A(\mathcal{B}^{(2)}_{10}(N)) \cup \left( \mathcal{E}_A(\mathcal{B}^{(2)}_{\partial\phi}(N)) + \mathcal{G}_f(\mathcal{B}^{(1)}_N) \right)
\]

**Proof.** Multiplying \( Af \) by \( \mu(M/B) \) yields

\[
Af\mu(M/B) = (\beta_L)_* \left( \mathcal{K}_A \rho^0 \cdot \beta^*_R f \cdot (\beta^{(2)}_\psi)^* \mu(M^2_\psi) \right) = (\beta_L)_* \left( \mathcal{K}_A \cdot \beta^*_R f \mu((M/B)^2_e/B) \right).
\]

and passing to \( b \)-densities

\[
Af\mu_b(M/B) = (\beta_L)_* \left( \mathcal{K}_A \cdot \beta^*_R f \prod_{N \in \mathcal{S}_\psi(M)} \rho_{\mathcal{B}^{(2)}_{01}(N)} \mu_b((M/B)^2_e/B) \right).
\]

and we can apply the pull-back and push-forward theorems.

\[\square\]

**Corollary A.5.** If the integral kernel of \( A \) satisfies \( \mathcal{K}_A \in \mathcal{A}^{\mathcal{E}_A}((M/B)^2_e) \) then \( A \) defines a bounded map, for any \( t \in \mathbb{R} \),

\[
\rho^t H^t_e(M/B) \longrightarrow \rho^t H^\infty_e(M/B)
\]

as long as

\[
\text{Re}(\mathcal{E}_A(\mathcal{B}^{(2)}_{01}(N))) + \mathfrak{s}(N) > -\frac{1}{2}
\]

\[
\text{Re}(\mathcal{E}_A(\mathcal{B}^{(2)}_{10}(N))) > \mathfrak{s}'(N) - \frac{1}{2}
\]

\[
\text{Re}(\mathcal{E}_A(\mathcal{B}^{(2)}_{\partial\phi}(N))) + \mathfrak{s}(N) \geq \mathfrak{s}'(N)
\]

**Appendix B. Composition of wedge heat operators**

The **wedge heat composition space**. Composition is through convolution in time,

\[
f(t) \ dt = \int_0^t g(t - t') h(t') \ dt' dt = \int_{t'' + t' = t}^t (g(t'') \ dt'')(h(t') \ dt').
\]

We prefer to work with \( \tau = \sqrt{t} \) instead of \( t \), so this becomes

\[
f(\tau) \ d\tau = \frac{2}{\tau} \int \frac{(s\bar{s}) (g(s) \ ds) (h(\bar{s}) \ d\bar{s})}{\sqrt{s^2 + \bar{s}^2}}.
\]

In terms of the maps

\[
\begin{array}{ccc}
\mathbb{R}_+^+ \times \mathbb{R}_+^+ & \xrightarrow{\pi_L} & \mathbb{R}_+^+ \\
\pi_{\tau} & \xrightarrow{\pi_C} & \pi_{\tau} \\
\mathbb{R}_+^+ & \xrightarrow{\pi_R} & \mathbb{R}_+^+ \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathbb{R}_+^+ \times \mathbb{R}_+^+ & \xrightarrow{\pi_{\tau}} & \mathbb{R}_+^+ \\
\pi_{\tau} & \xrightarrow{\pi_C} & \pi_{\tau} \\
\mathbb{R}_+^+ & \xrightarrow{\pi_R} & \mathbb{R}_+^+ \\
\end{array}
\]

the composition is given by

\[
f(\tau) \ d\tau = \frac{2}{\tau} (\pi_{\tau})_* (\pi_{\tau}^* (g(\tau) \ d\tau) \pi_{\tau}^* (h(\tau) \ d\tau))
\]

This push-forward is not well-behaved because \( \pi_C \) is not a \( b \)-fibration but we can fix this by replacing \( (\mathbb{R}_+^+)^2 \) with

\[
\mathcal{J}_0^2 = [\mathbb{R}_+^+ \times \mathbb{R}_+^+; \{(0,0)\}]
\]
and then the composition formula is well-behaved. We denote the boundary hypersurfaces of this space by

\[ \{0\} \times \mathbb{R}^+ \leftrightarrow B_{10}^{(\mathcal{F})}, \quad \mathbb{R}^+ \times \{0\} \leftrightarrow B_0^{(\mathcal{F})}, \quad \{0\} \times \{0\} \leftrightarrow B_{11}^{(\mathcal{F})}. \]

Now bringing in the spatial variables, we want to construct a space \( H(M/B)^3_w \) so that the maps

\[
\begin{align*}
M^3_\psi \times (\mathbb{R}^+)^2 & \quad \xrightarrow{\pi_{LM,L}} \quad M^2_\psi \times \mathbb{R}^+ & \quad \xrightarrow{\pi_{LR,C}} \quad (\zeta, \zeta', \zeta'', s, \tilde{s}) \\
M^2_\psi \times \mathbb{R}^+ & \quad \xrightarrow{\pi_{MR,R}} \quad M^2_\psi \times \mathbb{R}^+ & \quad \xrightarrow{\pi_{LR,C}} \quad (\zeta, \zeta', s) & \quad \xrightarrow{\pi_{LM,L}} \quad (\zeta, \zeta'', \sqrt{s^2 + \tilde{s}^2}) & \quad \xrightarrow{\pi_{MR,R}} \quad (\zeta', \zeta'', \tilde{s})
\end{align*}
\]

(where we are using the notation \( M^2_\psi, M^3_\psi \) from Appendix A) lift to b-maps

\[
\begin{align*}
H(M/B)^3_w & \quad \xrightarrow{\beta_{LM,L}} \quad H(M/B)_w & \quad \xrightarrow{\beta_{LR,C}} \quad H(M/B)_w & \quad \xrightarrow{\beta_{MR,R}} \quad H(M/B)_w
\end{align*}
\]

We construct the space \( H(M/B)^3_w \) in steps starting from

\[ H_0(M/B)^3_w = M^3_\psi \times \mathcal{F}_0^2. \]

Let \( S_\psi(M) = \{N_1, \ldots, N_\ell\} \) be a non-decreasing list and recall the notation \( T(N), S_{LM}(N), \) etc. from \( A.1 \). Inductively, for \( 1 \leq i \leq \ell \), let

\[
H_i(M/B)^3_w = \left[ H_{i-1}(M/B)^3_w; T(N_i) \times B_{11}^{(\mathcal{F})}; (S_{LM}(N_i) \cup S_{LR}(N_i) \cup S_{MR}(N_i)) \times B_{11}^{(\mathcal{F})}; \right.
\]

\[
S_{LM}(N_i) \times B_{10}^{(\mathcal{F})}; S_{MR}(N_i) \times B_{01}^{(\mathcal{F})}; T(N_i); (S_{LM}(N_i) \cup S_{LR}(N_i) \cup S_{MR}(N_i)) \left. \right].
\]

Finally, we need to blow-up the (interior) lifts of the partial diagonals. So let

\[
\text{diag}_{LM} = \pi_{LM}^{-1}(\text{diag}_M), \quad \text{diag}_{LR} = \pi_{LR}^{-1}(\text{diag}_M), \quad \text{diag}_{MR} = \pi_{MR}^{-1}(\text{diag}_M),
\]

and let \( \text{diag}_{LMR} \) be their intersection and then define

\[
H(M/B)^3_w = \left[ H_\ell(M/B)^3_w; \text{diag}_{LMR} \times B_{11}^{(\mathcal{F})}; \right.
\]

\[
(\text{diag}_{LM} \cup \text{diag}_{LR} \cup \text{diag}_{MR}) \times B_{11}^{(\mathcal{F})}; \text{diag}_{LM} \times B_{10}^{(\mathcal{F})}; \text{diag}_{MR} \times B_{01}^{(\mathcal{F})} \left. \right].
\]

Our notation for the (collective) boundary hypersurfaces of \( H(M/B)^3_w \) is as follows. First we have,

\[
\begin{align*}
M^3_\psi \times B_{10}^{(\mathcal{F})} & \leftrightarrow B_{000,10}^{(\mathcal{F})}, \quad M^3_\psi \times B_{01}^{(\mathcal{F})} \leftrightarrow B_{000,01}^{(\mathcal{F})}, \quad M^3_\psi \times B_{11}^{(\mathcal{F})} \leftrightarrow B_{000,11}^{(\mathcal{F})}, \\
\text{diag}_{LMR} \times B_{11}^{(\mathcal{F})} & \leftrightarrow B_{0dd,11}^{(\mathcal{F})}, \quad \text{diag}_{LM} \times B_{11}^{(\mathcal{F})} \leftrightarrow B_{0dd,01}^{(\mathcal{F})}, \quad \text{diag}_{LR} \times B_{11}^{(\mathcal{F})} \leftrightarrow B_{0dd,11}^{(\mathcal{F})}, \\
\text{diag}_{MR} \times B_{11}^{(\mathcal{F})} & \leftrightarrow B_{0dd,11}^{(\mathcal{F})}, \quad \text{diag}_{LM} \times B_{10}^{(\mathcal{F})} \leftrightarrow B_{0dd,01}^{(\mathcal{F})}, \quad \text{diag}_{MR} \times B_{01}^{(\mathcal{F})} \leftrightarrow B_{0dd,01}^{(\mathcal{F})}.
\end{align*}
\]
and then, for each $N \in \mathcal{S}_\psi(M)$,
\[
\mathcal{B}_{N}^{(i)} \times_\psi M^2 \times \mathcal{T}_0^2 \leftrightarrow \mathcal{B}_{100,00}^{(C)}(N), \quad M \times_\psi \mathcal{B}_{N}^{(i)} \times_\psi M \times \mathcal{T}_0^2 \leftrightarrow \mathcal{B}_{010,00}^{(C)}(N), \\
M^2 \times_\psi \mathcal{B}_{N}^{(i)} \times \mathcal{T}_0^2 \leftrightarrow \mathcal{B}_{001,00}^{(C)}(N), \quad T(N) \times \mathcal{B}_{11}^{(T)} \leftrightarrow \mathcal{B}_{\phi\phi,11}^{(C)}(N), \\
S_{LM}(N) \times \mathcal{B}_{11}^{(T)} \leftrightarrow \mathcal{B}_{\phi\phi,01}^{(C)}(N), \quad S_{LM}(N) \times \mathcal{B}_{11}^{(T)} \leftrightarrow \mathcal{B}_{\phi\phi,11}^{(C)}(N), \\
S_{MR}(N) \times \mathcal{B}_{11}^{(T)} \leftrightarrow \mathcal{B}_{\phi\phi,01}^{(C)}(N), \quad S_{LM}(N) \times \mathcal{B}_{10}^{(T)} \leftrightarrow \mathcal{B}_{\phi\phi,00}^{(C)}(N), \\
S_{MR}(N) \times \mathcal{B}_{10}^{(T)} \leftrightarrow \mathcal{B}_{\phi\phi,00}^{(C)}(N), \quad S_{LM}(N) \times \mathcal{T}_0^2 \leftrightarrow \mathcal{B}_{\phi\phi,00}^{(C)}(N), \\
S_{MR}(N) \times \mathcal{T}_0^2 \leftrightarrow \mathcal{B}_{\phi\phi,00}^{(C)}(N). 
\]

**The exponent matrices.** We specify the exponent matrix of the maps $\beta_{\ldots}$ by specifying the pull-back of the collective boundary hypersurfaces. Thus $\beta_{LM,L}^{*}$ maps $\mathcal{B}_{000,01}^{(C)}$, $\mathcal{B}_{000,01}^{(C)}$ and, for each $N \in \mathcal{S}_\psi(M)$, $\mathcal{B}_{001,00}^{(C)}(N)$, into the interior of $H(M/B)_w$; otherwise
\[
\beta_{LM,L}^{*} \mathcal{B}_{00,1}^{(H)} = \{ \mathcal{B}_{000,10}^{(C)}, \mathcal{B}_{000,11}^{(C)}, \mathcal{B}_{00d,11}^{(C)}, \mathcal{B}_{00d,11}^{(C)} \} \cup \bigcup_{N \in \mathcal{S}_\psi(M)} \{ \mathcal{B}_{00\phi,11}^{(C)}(N), \mathcal{B}_{00\phi,11}^{(C)}(N) \}, \\
\beta_{LM,L}^{*} \mathcal{B}_{0d,1}^{(H)} = \{ \mathcal{B}_{0dd,11}^{(C)}, \mathcal{B}_{0dd,11}^{(C)} \}.
\]
and, for each $N \in \mathcal{S}_\psi(M)$,
\[
\beta_{LM,L}^{*} \mathcal{B}_{10,0}^{(H)}(N) = \{ \mathcal{B}_{100,00}^{(C)}(N), \mathcal{B}_{\phi\phi,11}^{(C)}(N), \mathcal{B}_{\phi\phi,00}^{(C)}(N) \} \\
\beta_{LM,L}^{*} \mathcal{B}_{01,0}^{(H)}(N) = \{ \mathcal{B}_{010,00}^{(C)}(N), \mathcal{B}_{\phi\phi,11}^{(C)}(N), \mathcal{B}_{\phi\phi,00}^{(C)}(N), \mathcal{B}_{\phi\phi,00}^{(C)}(N) \} \\
\beta_{LM,L}^{*} \mathcal{B}_{\phi,1}^{(H)}(N) = \{ \mathcal{B}_{\phi\phi,11}^{(C)}(N), \mathcal{B}_{\phi\phi,11}^{(C)}(N), \mathcal{B}_{\phi\phi,00}^{(C)}(N) \}, \\
\beta_{LM,L}^{*} \mathcal{B}_{\phi,0}^{(H)}(N) = \{ \mathcal{B}_{\phi\phi,00}^{(C)}(N), \mathcal{B}_{\phi\phi,00}^{(C)}(N) \}.
\]
where we note that $\mathcal{B}_{\phi\phi,11}^{(C)}(N)$ and $\mathcal{B}_{\phi\phi,11}^{(C)}(N)$ are repeated.

Similarly, the map $\beta_{LRC}^{*}$ maps $\mathcal{B}_{000,10}^{(C)}$, $\mathcal{B}_{000,11}^{(C)}$, $\mathcal{B}_{0dd,10}^{(C)}$, $\mathcal{B}_{0dd,11}^{(C)}$ and, for each $N \in \mathcal{S}_\psi(M)$, $\mathcal{B}_{010,00}^{(C)}(N)$, into the interior of $H(M/B)_w$; otherwise
\[
\beta_{LRC}^{*} \mathcal{B}_{00,1}^{(H)} = \{ \mathcal{B}_{000,11}^{(C)}, \mathcal{B}_{0dd,11}^{(C)}, \mathcal{B}_{0dd,11}^{(C)} \} \cup \bigcup_{N \in \mathcal{S}_\psi(M)} \{ \mathcal{B}_{\phi\phi,11}^{(C)}(N), \mathcal{B}_{\phi\phi,11}^{(C)}(N) \}, \\
\beta_{LRC}^{*} \mathcal{B}_{0d,1}^{(H)} = \{ \mathcal{B}_{0dd,11}^{(C)}, \mathcal{B}_{0dd,11}^{(C)} \}.
\]
and, for each $N \in \mathcal{S}_\psi(M)$,
\[
\beta_{LRC}^{*} \mathcal{B}_{10,0}^{(H)}(N) = \{ \mathcal{B}_{100,00}^{(C)}(N), \mathcal{B}_{\phi\phi,11}^{(C)}(N), \mathcal{B}_{\phi\phi,10}^{(C)}(N), \mathcal{B}_{\phi\phi,00}^{(C)}(N) \} \\
\beta_{LRC}^{*} \mathcal{B}_{01,0}^{(H)}(N) = \{ \mathcal{B}_{010,00}^{(C)}(N), \mathcal{B}_{\phi\phi,11}^{(C)}(N), \mathcal{B}_{\phi\phi,00}^{(C)}(N) \} \\
\beta_{LRC}^{*} \mathcal{B}_{\phi,1}^{(H)}(N) = \{ \mathcal{B}_{\phi\phi,11}^{(C)}(N), \mathcal{B}_{\phi\phi,11}^{(C)}(N) \}, \\
\beta_{LRC}^{*} \mathcal{B}_{\phi,0}^{(H)}(N) = \{ \mathcal{B}_{\phi\phi,00}^{(C)}(N), \mathcal{B}_{\phi\phi,00}^{(C)}(N) \}
\]
where we note that $\mathcal{B}_{\phi\phi,11}^{(C)}(N)$ and $\mathcal{B}_{\phi\phi,11}^{(C)}(N)$ are repeated.
Finally, the map $\beta_{MR,R}$ maps $\vB_{00,10}^{(C)}, \vB_{00,01}^{(C)}$ and, for each $N \in \vS_{\psi}(M), \vB_{10,00}^{(C)}(N)$, into the interior of $H(M/B)_{w}$; otherwise

$$
\beta_{MR,R}^{*(H)} = \{ \vB_{00,01}^{(C)}, \vB_{00,11}^{(C)}, \vB_{01,01}^{(C)} \} \cup \bigcup_{N \in \vS_{\psi}(M)} \{ \vB_{00,01}^{(C)(N)}, \vB_{00,11}^{(C)(N)} \}
$$

and, for each $N \in \vS_{\psi}(M),$

$$
\beta_{MR,R}^{*(H)(0)} = \{ \vB_{00,01}^{(C)(N)}, \vB_{00,11}^{(C)(N)}, \vB_{00,10}^{(C)(N)} \}
$$

$$
\beta_{MR,R}^{*(H)(1)} = \{ \vB_{00,01}^{(C)(N)}, \vB_{00,11}^{(C)(N)}, \vB_{00,00}^{(C)(N)} \}
$$

where we note that $\vB_{00,01}^{(C)(N)}$ and $\vB_{00,11}^{(C)(N)}$ are repeated.

**Composition.** Now let us discuss the composition law. As before we are interested in integral kernels that are sections of a weighted density bundle. Let us start by recalling the weight from (3.23), namely

$$
\mathfrak{h} : M_{1}(H(M/B)_{w}) \longrightarrow \mathbb{R},
$$

$$
\mathfrak{h}(H) = \begin{cases} 
-(\dim(N/B) + 3) & \text{if } H \subseteq \vB_{00,01}^{(H)(N)} \text{ for some } N \in \vS_{\psi}(M) \\
-(\dim(N/B) + 1) & \text{if } H \subseteq \vB_{00,00}^{(H)(N)} \text{ for some } N \in \vS_{\psi}(M) \\
-(\dim(M/B) + 2) & \text{if } H = \vB_{dd,1}^{(H)} \\
\infty & \text{if } H = \vB_{00,1}^{(H)} \\
0 & \text{otherwise}
\end{cases}
$$

and let $\mu_{R} = \beta_{(H),R}^{*}(M/B)$. We will determine the behavior under composition for integral kernels of the form $\vK_{A}\rho^{b}_{\mu_{R}}$ with $\vK_{A} \in \vA_{\rho_{by}}^{(A)}(H(M/B)_{w})$. Ultimately we are interested in kernels that are merely conormal with bounds acting on sections of a vector bundle, but the corresponding composition result follows easily from this one.

**Proposition B.1.** Let $A$ have kernel $\vK_{A}\rho^{b}_{\mu_{R}}$ with $\vK_{A} \in \vA_{\rho_{by}}^{(A)}(H(M/B)_{w})$ and $B$ have kernel $\vK_{B}\rho^{b}_{\mu_{R}}$ with $\vK_{B} \in \vA_{\rho_{by}}^{(B)}(H(M/B)_{w})$. If

$$
\text{Re}(\vE_{A}(\vB_{dd,1}^{(H)(N)})) > 0, \quad \text{Re}(\vE_{B}(\vB_{dd,1}^{(H)})) > 0, \quad \text{and}
$$

$$
\text{Re}(\vE_{A}(\vB_{00,1}^{(H)(N)})) + \text{Re}(\vE_{B}(\vB_{00,1}^{(H)(N)})) + 1 > 0 \text{ for all } N \in \vS_{\psi}(M),
$$

then we may define their composition $C = A \circ B$ by the formula

$$
\tilde{\vK}_{C} \mu_{R} \beta_{(H)}^{*}(\tau d\tau) = (\beta_{LM,R}^{*}(\vK_{A}\rho^{b}_{\mu_{R}}(\tau d\tau))) \cdot \beta_{MR,R}^{*}(\vK_{B}\rho^{b}_{\mu_{R}}(\tau d\tau))
$$

and we have $\tilde{\vK}_{C} \in \rho^{b}_{\vA_{\rho_{by}}^{(C)}(H(M/B)_{w})}$ with

$$
\vE_{C}(\vB_{dd,1}^{(H)}) = \vE_{A}(\vB_{dd,1}^{(H)}) + \vE_{B}(\vB_{dd,1}^{(H)})
$$
and, for each $N \in S_\psi(M)$,

$$
\begin{align*}
\mathcal{E}_C(\mathfrak{B}_{10,0}(N)) &= \mathcal{E}_A(\mathfrak{B}_{10,0}(N)) \cup \mathcal{E}_A(\mathfrak{B}_{\phi,1}(N)) + \mathcal{E}_B(\mathfrak{B}_{10,0}(N)), \\
\mathcal{E}_C(\mathfrak{B}_{01,0}(N)) &= \mathcal{E}_B(\mathfrak{B}_{01,0}(N)) \cup \mathcal{E}_A(\mathfrak{B}_{01,0}(N)) + \mathcal{E}_B(\mathfrak{B}_{01,0}(N)), \\
\mathcal{E}_C(\mathfrak{B}_{\phi,1}(N)) &= \mathcal{E}_A(\mathfrak{B}_{\phi,1}(N)) + \mathcal{E}_B(\mathfrak{B}_{\phi,1}(N)), \\
\mathcal{E}_C(\mathfrak{B}_{\phi,0,0}(N)) &= \mathcal{E}_A(\mathfrak{B}_{\phi,0,0}(N)) + \mathcal{E}_B(\mathfrak{B}_{\phi,0,0}(N)) + \dim(N/B) + 1
\end{align*}
$$

Proof. Let us write, during the proof,

$$
\mathfrak{h} : \mathcal{M}_1(H(M/B)_w) \rightarrow \mathbb{R},
$$

$$
\mathfrak{h}(H) = \begin{cases}
-(\dim(N/B) + a) & \text{if } H \subseteq \mathfrak{B}_{\phi,1}(N) \text{ for some } N \in S_\psi(M) \\
-(\dim(N/B) + a') & \text{if } H \subseteq \mathfrak{B}_{\phi,0,0}(N) \text{ for some } N \in S_\psi(M) \\
-(\dim(M/B) + b) & \text{if } H = \mathfrak{B}_{0,1}^{dd} \\
\infty & \text{if } H = \mathfrak{B}_{00,1}^{dd} \\
0 & \text{otherwise}
\end{cases}
$$

to motivate the choice ($a = 3$, $a' = 1$, $b = 2$) made above.

Note that $\beta_{LR,C}$ is not a b-fibration so that push-forward along it does not preserve polyhomogeneous functions. However, the weight $\mathfrak{h}$ is such that the problem faces can be blown-down which is why we will end up with a polyhomogeneous function. Indeed, the product $\beta_{LM,L}^*(\rho^b) \cdot \beta_{MR,R}^*(\rho^b)$ vanishes to infinite order at every face in

$$
\beta_{LM,L}^*\mathfrak{B}_{00,1}^{H} \cup \beta_{MR,R}^*\mathfrak{B}_{00,1}^{H} = \left\{ \mathfrak{B}_{00,1}^{(C)}, \mathfrak{B}_{00,01}^{(C)}, \mathfrak{B}_{00,11}^{(C)}, \mathfrak{B}_{dd,01}^{(C)}, \mathfrak{B}_{dd,11}^{(C)}, \mathfrak{B}_{dd,01}^{(C)} \right\} \\
\cup \bigcup_{N \in S_\psi(M)} \left\{ \mathfrak{B}_{\phi,0,0,1}^{(C)}, \mathfrak{B}_{\phi,0,1,1}^{(C)}, \mathfrak{B}_{\phi,1,1}^{(C)} \right\},
$$

so we see that the push-forward along $\beta_{LR,C}$ will be polyhomogeneous and will vanish at $\mathfrak{B}_{00,1}$ to infinite order.

Thus let us write $\tilde{K}_C = K_C \rho^b$ for some $K_C$ polyhomogeneous, which after multiplying both sides of the formula for $C$ by $\mu_L = \beta_{(H),L}^*(M/B)$, satisfies

$$
K_C \rho^b \beta_{(H)}^*(\tau \cdot \mu(M^2_B / B \times \mathbb{R}^+)) = (\beta_{LR,C})^*(\beta_{LM,L}^*(\mathcal{K}_A \rho^b \beta_{(H)}^*)(\tau)) \cdot \beta_{MR,R}^*(\mathcal{K}_B \rho^b \beta_{(H)}^*(\tau)) \cdot \beta_{(C)}^*(\mu(M^2_B / (\mathbb{R}^+)^2))).
$$

We need to work out the density weight factors. Start by noting that

$$
\beta_{(H)}^* \tau = \rho_{\mathfrak{B}_{00,1}^{H}} \rho_{\mathfrak{B}_{00,1}^{H}} \prod_{N \in S_\psi(M)} \rho_{\mathfrak{B}_{\phi,1}^{H}}(N)
$$
and then, ignoring the faces where we have seen infinite order decay,
\[
\beta^*_{LR,C}(\rho^b \beta^*_{\mathcal{H}})^{-1} \beta^*_{\mathcal{L},M,L}(\rho^b \beta^*_{\mathcal{H}}) \beta^*_{M,R,R}(\rho^b \beta^*_{\mathcal{H}})
\]
\[
= (\rho_{\alpha \beta}^{\mathfrak{C}_{\mathcal{H}}} \rho_{\gamma \delta}^{\mathfrak{C}_{\mathcal{H}}} \rho_{\epsilon \zeta}^{\mathfrak{C}_{\mathcal{H}}})^{-b+1} \prod_{N \in S_p(M)} (\rho_{\alpha \beta}^{\mathfrak{C}_{\mathcal{H}}} \rho_{\gamma \delta}^{\mathfrak{C}_{\mathcal{H}}}(N) \rho_{\epsilon \zeta}^{\mathfrak{C}_{\mathcal{H}}}(N))^{-\dim(N/B) - a} \\
\prod_{N \in S_p(M)} (\rho_{\alpha \beta}^{\mathfrak{C}_{\mathcal{H}}} \rho_{\gamma \delta}^{\mathfrak{C}_{\mathcal{H}}}(N) \rho_{\epsilon \zeta}^{\mathfrak{C}_{\mathcal{H}}}(N) \rho_{\alpha \beta}^{\mathfrak{C}_{\mathcal{H}}}(N))^{-\dim(N/B) - a'}
\]

Next, the lifts of the densities (continuing to ignore faces where we have infinite decay) are
\[
\beta^*_{\mathcal{H}} \mu(M^2_{\psi}/B \times \mathbb{R}^+) = \left[ \rho_{\alpha \beta}^{\mathfrak{C}_{\mathcal{H}}} \prod_{N \in S_p(M)} \rho_{\gamma \delta}^{\mathfrak{C}_{\mathcal{H}}}(N) \right] \mu(H(M/B)^w) \\
= \rho_{\alpha \beta}^{\mathfrak{C}_{\mathcal{H}}} \mu(H(M/B)^w/B)
\]
\[
\beta^*_{\mathcal{H}} \mu(M^3_{\psi}/B \times (\mathbb{R}^+)^2) = \left[ \rho_{\alpha \beta}^{\mathfrak{C}_{\mathcal{H}}} \prod_{N \in S_p(M)} \rho_{\gamma \delta}^{\mathfrak{C}_{\mathcal{H}}}(N) \right] \mu(H(M/B)^3) \\
= \rho_{\alpha \beta}^{\mathfrak{C}_{\mathcal{H}}} \mu(H(M/B)^3/B)
\]

where \( \rho_{\mathcal{H}} \) and \( \rho_{\mathcal{C}} \) are, respectively, total boundary defining functions for \( H(M/B)^w \) and \( H(M/B)^3_w \). From the exponent matrices of \( \beta_{LR,C} \) we have
\[
(\beta^*_{LR,C} \rho_{\mathcal{H}})^{-1} \rho_{\mathcal{C}} = (\rho_{\alpha \beta}^{\mathfrak{C}_{\mathcal{H}}} \rho_{\gamma \delta}^{\mathfrak{C}_{\mathcal{H}}} \rho_{\epsilon \zeta}^{\mathfrak{C}_{\mathcal{H}}})^{-b+1} \\
\prod_{N \in S_p(M)} (\rho_{\alpha \beta}^{\mathfrak{C}_{\mathcal{H}}} \rho_{\gamma \delta}^{\mathfrak{C}_{\mathcal{H}}}(N) \rho_{\epsilon \zeta}^{\mathfrak{C}_{\mathcal{H}}}(N))^{-\dim(N/B) + 2} \\
\prod_{N \in S_p(M)} (\rho_{\alpha \beta}^{\mathfrak{C}_{\mathcal{H}}} \rho_{\gamma \delta}^{\mathfrak{C}_{\mathcal{H}}}(N) \rho_{\epsilon \zeta}^{\mathfrak{C}_{\mathcal{H}}}(N) \rho_{\alpha \beta}^{\mathfrak{C}_{\mathcal{H}}}(N))^{\dim(N/B) + 1} \rho_{\alpha \beta}^{\mathfrak{C}_{\mathcal{H}}}(N)
\]

So altogether
\[
\mathcal{K}_{C} \mu_0(H(M/B)^3/B) = (\beta_{LR,C})^* (\beta^*_{\mathcal{L},M,L}(\mathcal{K}_{A}) \cdot \beta^*_{M,R,R}(\mathcal{K}_{B}) \rho^{a_3} \mu_0(H(M/B)^3/B)).
\]

where, ignoring faces with infinite decay, \( \rho^{a_3}_{\mathcal{C}} \) is equal to
\[
(\rho_{\alpha \beta}^{\mathfrak{C}_{\mathcal{H}}} \rho_{\gamma \delta}^{\mathfrak{C}_{\mathcal{H}}} \rho_{\epsilon \zeta}^{\mathfrak{C}_{\mathcal{H}}})^{-b+2} \prod_{N \in S_p(M)} (\rho_{\alpha \beta}^{\mathfrak{C}_{\mathcal{H}}} \rho_{\gamma \delta}^{\mathfrak{C}_{\mathcal{H}}}(N) \rho_{\epsilon \zeta}^{\mathfrak{C}_{\mathcal{H}}}(N))^{-a+3} \\
(\rho_{\alpha \beta}^{\mathfrak{C}_{\mathcal{H}}}(N) \rho_{\gamma \delta}^{\mathfrak{C}_{\mathcal{H}}}(N) \rho_{\epsilon \zeta}^{\mathfrak{C}_{\mathcal{H}}}(N))^{-a'+\dim(N/B)} \\
\rho_{\alpha \beta}^{\mathfrak{C}_{\mathcal{H}}}(N) \rho_{\epsilon \zeta}^{\mathfrak{C}_{\mathcal{H}}}(N).
\]

Now we can apply the pull-back and push-forward theorems to get the result.
Appendix C. Composition of b, wedge heat kernels

In section [7.2], we prove a families index theorem on manifolds with iterated fibration structures endowed with b-wedge metrics. The locally trivial family is denoted

\[ X' \rightarrow M' \overset{\psi'}{\rightarrow} B' \]

in order to distinguish it from the fiber bundle used in the bulk of the text. There is a minimal \( N_0' \in \mathcal{S}(M') \) such that \( \dim(N_0'/B') = 0 \), and the metric is of b-type near \( \mathfrak{B}_{N_0} \) and of wedge type at all other \( N' \in \mathcal{S}(M') \).

In this section we establish a composition result for b-wedge heat operators.

**The b, wedge heat composition space.** Recall that the b,wedge heat space is given by

\[ H(M'/B')_{b,w} = \left[ M' \times \psi' \times M' \times \mathbb{R}_+^\ell; \mathfrak{B}_{N_0} \times \mathfrak{B}_{N_1} \times \cdots \times \mathfrak{B}_{N_{\ell}} \times \mathbb{R}_+^\ell; \right. \]

\[ \left. \mathfrak{B}_{N_1} \times \mathfrak{B}_{N_2} \times \cdots \times \{0\} \right], \]

where \( \{N_0', N_1', \ldots, N_\ell'\} \) is a non-decreasing list of \( \mathcal{S}(M') \). Thus this space is constructed by treating \( N_0' \) as in the construction of the b-heat space [Mc93, Chapter 7], and treating the other \( N' \) as in the construction of the wedge heat space in [3.5]. We will follow the same pattern below.

We use the space \( \mathcal{T}_0^2 \) defined in Appendix [B] as

\[ \mathcal{T}_0^2 = [\mathbb{R}_+^s \times \mathbb{R}_+^r; \{(0,0)\}] \]

together with the notation for its boundary hypersurfaces \( \mathfrak{B}_{10}^{(\mathcal{T})}, \mathfrak{B}_{01}^{(\mathcal{T})}, \mathfrak{B}_{11}^{(\mathcal{T})} \). We define the b, wedge composition space using the notation from (A.1) starting from

\[ H_0(M'/B')_{b,w}^3 = [(M')_{\psi'}^3 \times \mathcal{T}_0^2; T(N_0'); (S_{LM}(N_0') \cup S_{LR}(N_0') \cup S_{MR}(N_0'))]. \]

Inductively, for \( 1 \leq i \leq \ell \), let

\[ H_i(M'/B')_{b,w}^3 = \left[ H_{i-1}(M'/B')_{b,w}^3; T(N_i') \times \mathfrak{B}_{11}^{(\mathcal{T})}; (S_{LM}(N_i') \cup S_{LR}(N_i') \cup S_{MR}(N_i')) \right. \]

\[ \left. \times \mathfrak{B}_{11}^{(\mathcal{T})} \right]. \]

Finally, we need to blow-up the (interior) lifts of the partial diagonals. So let

\[ \text{diag}_{LM} = \pi_{LM}^{-1}(\text{diag}_M), \quad \text{diag}_{LR} = \pi_{LR}^{-1}(\text{diag}_M), \quad \text{diag}_{MR} = \pi_{MR}^{-1}(\text{diag}_M), \]

and let \( \text{diag}_{LMR} \) be their intersection and then define

\[ H(M'/B')_{b,w}^3 = \left[ H_i(M'/B')_{b,w}^3; \text{diag}_{LMR} \times \mathfrak{B}_{11}^{(\mathcal{T})} \right. \]

\[ \left. (\text{diag}_{LM} \cup \text{diag}_{LR} \cup \text{diag}_{MR}) \times \mathfrak{B}_{11}^{(\mathcal{T})}; \text{diag}_{LM} \times \mathfrak{B}_{11}^{(\mathcal{T})}; \text{diag}_{MR} \times \mathfrak{B}_{11}^{(\mathcal{T})} \right]. \]

As anticipated, \( N_0' \) gives rise to the same blow-ups as in the composition space for the b-calculus [Alb07] and the other \( N' \) give rise to the same blow-ups as in the wedge heat composition space in Appendix [B].
Our notation for the (collective) boundary hypersurfaces of $H(M'/B')_{b,w}^3$ is as follows. First we have,

\[
(M')_{\psi}^3 \times B_{10} \leftrightarrow B_{00,10}^{(C)}, \quad (M')_{\psi}^3 \times B_{01} \leftrightarrow B_{00,01}^{(C)}, \quad (M')_{\psi}^3 \times B_{11} \leftrightarrow B_{00,01}^{(C)},
\]

\[
\text{diag}_{LMR} \times B_{11} \leftrightarrow B_{odd,11}^{(C)}, \quad \text{diag}_{LM} \times B_{11} \leftrightarrow B_{odd,11}^{(C)}, \quad \text{diag}_{LR} \times B_{11} \leftrightarrow B_{odd,11}^{(C)},
\]

\[
\text{diag}_{MR} \times B_{11} \leftrightarrow B_{odd,11}^{(C)}, \quad \text{diag}_{LM} \times B_{10} \leftrightarrow B_{00,10}^{(C)}, \quad \text{diag}_{MR} \times B_{01} \leftrightarrow B_{00,01}^{(C)}
\]

and then, for each $N' \in S(M')$,

\[
B_{N'}^{(1)} \times \psi \ M_{\psi}^2 \times \mathcal{F}_0^2 \leftrightarrow B_{100,00}^{(C)}(N'), \quad M \times \psi \ B_{N'}^{(1)} \times \psi \ M \times \mathcal{F}_0^2 \leftrightarrow B_{010,00}^{(C)}(N'),
\]

\[
M_{\psi}^2 \times \psi \ B_{N'}^{(1)} \times \mathcal{F}_0^2 \leftrightarrow B_{001,00}^{(C)}(N'), \quad T(N') \times \mathcal{F}_0^2 \leftrightarrow B_{00,00}^{(C)}(N'),
\]

\[
S_{LM}(N') \times \mathcal{F}_0^2 \leftrightarrow B_{odd,11}^{(C)}(N'), \quad S_{LR}(N') \times \mathcal{F}_0^2 \leftrightarrow B_{odd,11}^{(C)}(N'),
\]

\[
S_{MR}(N') \times \mathcal{F}_0^2 \leftrightarrow B_{00,00}^{(C)}(N'),
\]

and, for each $N' \in S(M') \setminus \{N'_0\}$,

\[
T(N') \times B_{11}^{(F)} \leftrightarrow B_{00,11}^{(C)}(N'), \quad S_{LM}(N') \times B_{11}^{(F)} \leftrightarrow B_{00,11}^{(C)}(N'),
\]

\[
S_{LR}(N') \times B_{11}^{(F)} \leftrightarrow B_{00,11}^{(C)}(N'), \quad S_{MR}(N') \times B_{11}^{(F)} \leftrightarrow B_{00,11}^{(C)}(N'),
\]

\[
S_{LM}(N') \times B_{10}^{(F)} \leftrightarrow B_{00,10}^{(C)}(N'), \quad S_{MR}(N') \times B_{01}^{(F)} \leftrightarrow B_{00,11}^{(C)}(N').
\]

The exponent matrices. As in Appendix B we have b-maps

\[
\begin{array}{ccc}
H(M'/B')_{b,w}^3 & \text{by} & H(M'/B')_{b,w}^3 \\
\beta_{LM,L} & \text{by} & \beta_{L,R,C} \quad \beta_{MR,R}
\end{array}
\]

and we now specify their behavior with respect to the boundary hypersurfaces.

First, $\beta_{LM,L}$ maps $B_{00,01}^{(C)}$, $B_{00,01}^{(C)}$ and, for each $N' \in S(M')$, $B_{00,10}^{(C)}(N')$, into the interior of $H(M'/B')_{b,w}$; otherwise

\[
\beta_{LM,L}^* B_{00,01}^{(H)} = \{B_{00,01}^{(C)}(N'), B_{00,11}^{(C)}(N'), B_{odd,11}^{(C)}(N'), B_{odd,11}^{(C)}(N')\} \cup \bigcup_{N' \in S(M') \setminus \{N'_0\}} \{B_{00,11}^{(C)}(N'), B_{00,11}^{(C)}(N')\},
\]

\[
\beta_{LM,L}^* B_{00,11}^{(H)} = \{B_{00,11}^{(C)}(N'), B_{odd,11}^{(C)}(N'), B_{odd,11}^{(C)}(N'), B_{odd,11}^{(C)}(N')\},
\]

for $N'_0$,

\[
\beta_{LM,L}^* B_{10,0}^{(H)}(N'_0) = \{B_{10,00}^{(C)}(N'_0), B_{00,00}^{(C)}(N'_0)\}
\]

\[
\beta_{LM,L}^* B_{01,0}^{(H)}(N'_0) = \{B_{01,00}^{(C)}(N'_0), B_{00,00}^{(C)}(N'_0)\}
\]

\[
\beta_{LM,L}^* B_{11,0}^{(H)}(N'_0) = \{B_{00,11}^{(C)}(N'_0), B_{00,11}^{(C)}(N'_0)\},
\]
and, for each $N' \in \mathcal{S}(M') \setminus \{N_0'\}$,

$$\beta^*_{LM,L} \mathcal{B}^{(H)}_{10,0}(N') = \{ \mathcal{B}^{(C)}_{100,00}(N'), \mathcal{B}^{(C)}_{\phi\phi,11}(N'), \mathcal{B}^{(C)}_{\phi\phi,00}(N') \}$$

$$\beta^*_{LM,L} \mathcal{B}^{(H)}_{01,0}(N') = \{ \mathcal{B}^{(C)}_{010,00}(N'), \mathcal{B}^{(C)}_{\phi\phi,11}(N'), \mathcal{B}^{(C)}_{0\phi\phi,01}(N'), \mathcal{B}^{(C)}_{0\phi\phi,00}(N') \}$$

$$\beta^*_{LM,L} \mathcal{B}^{(H)}_{\phi\phi,0}(N') = \{ \mathcal{B}^{(C)}_{\phi\phi,11}(N'), \mathcal{B}^{(C)}_{\phi\phi,11}(N'), \mathcal{B}^{(C)}_{\phi\phi,10}(N') \}$$

$$\beta^*_{LM,L} \mathcal{B}^{(H)}_{\phi\phi,1}(N') = \{ \mathcal{B}^{(C)}_{\phi\phi,00}(N'), \mathcal{B}^{(C)}_{\phi\phi,00}(N') \}$$

We note that $\mathcal{B}^{(C)}_{\phi\phi,11}(N')$ and $\mathcal{B}^{(C)}_{0\phi\phi,11}(N')$ are repeated, for $N' \in \mathcal{S}(M') \setminus \{N_0'\}$.

Similarly, the map $\beta^*_{LR,C}$ maps $\mathcal{B}^{(C)}_{00,01,0}, \mathcal{B}^{(C)}_{00,01,0}, \mathcal{B}^{(C)}_{00,01,0}$ and, for each $N' \in \mathcal{S}(M')$, $\mathcal{B}^{(C)}_{00,01,0}(N')$, into the interior of $H(M'/B')_{b,w}$; otherwise

$$\beta^*_{LR,C} \mathcal{B}^{(H)}_{00,1}(N') = \{ \mathcal{B}^{(C)}_{000,11,11}, \mathcal{B}^{(C)}_{000,11,11}, \mathcal{B}^{(C)}_{000,11,11} \} \cup \bigcup_{N' \in \mathcal{S}(M') \setminus \{N_0'\}} \{ \mathcal{B}^{(C)}_{\phi\phi,01,11}(N'), \mathcal{B}^{(C)}_{\phi\phi,01,11}(N') \}$$

$$\beta^*_{LR,C} \mathcal{B}^{(H)}_{dd,0}(N') = \{ \mathcal{B}^{(C)}_{dd0,11,11}, \mathcal{B}^{(C)}_{dd0,11,11} \}$$

for $N_0'$,

$$\beta^*_{LR,C} \mathcal{B}^{(H)}_{10,0}(N_0') = \{ \mathcal{B}^{(C)}_{100,00}(N_0'), \mathcal{B}^{(C)}_{\phi\phi,00,00}(N_0') \}$$

$$\beta^*_{LR,C} \mathcal{B}^{(H)}_{01,0}(N_0') = \{ \mathcal{B}^{(C)}_{010,00}(N_0'), \mathcal{B}^{(C)}_{\phi\phi,00,00}(N_0') \}$$

$$\beta^*_{LR,C} \mathcal{B}^{(H)}_{\phi\phi,0}(N_0') = \{ \mathcal{B}^{(C)}_{\phi\phi,00,00}(N_0'), \mathcal{B}^{(C)}_{\phi\phi,00,00}(N_0') \}$$

and, for each $N' \in \mathcal{S}(M') \setminus \{N_0'\}$,

$$\beta^*_{LR,C} \mathcal{B}^{(H)}_{10,0}(N') = \{ \mathcal{B}^{(C)}_{100,00}(N'), \mathcal{B}^{(C)}_{\phi\phi,11,11}(N'), \mathcal{B}^{(C)}_{\phi\phi,10,10}(N'), \mathcal{B}^{(C)}_{\phi\phi,00,00}(N') \}$$

$$\beta^*_{LR,C} \mathcal{B}^{(H)}_{01,0}(N') = \{ \mathcal{B}^{(C)}_{001,00}(N'), \mathcal{B}^{(C)}_{\phi\phi,11,11}(N'), \mathcal{B}^{(C)}_{\phi\phi,00,01}(N'), \mathcal{B}^{(C)}_{\phi\phi,00,00}(N') \}$$

$$\beta^*_{LR,C} \mathcal{B}^{(H)}_{\phi\phi,0}(N') = \{ \mathcal{B}^{(C)}_{\phi\phi,11,11}(N'), \mathcal{B}^{(C)}_{\phi\phi,11,11}(N') \}$$

$$\beta^*_{LR,C} \mathcal{B}^{(H)}_{\phi\phi,1}(N') = \{ \mathcal{B}^{(C)}_{\phi\phi,00,00}(N'), \mathcal{B}^{(C)}_{\phi\phi,00,00}(N') \}$$

We note that $\mathcal{B}^{(C)}_{\phi\phi,11,11}(N')$ and $\mathcal{B}^{(C)}_{\phi\phi,00,00}(N')$ are repeated for $N' \in \mathcal{S}(M') \setminus \{N_0'\}$.

Finally, the map $\beta^*_{MR,R}$ maps $\mathcal{B}^{(C)}_{00,01,0}, \mathcal{B}^{(C)}_{dd0,10}$ and, for each $N' \in \mathcal{S}(M')$, $\mathcal{B}^{(C)}_{100,00}(N')$, into the interior of $H(M'/B')_{b,w}$; otherwise

$$\beta^*_{MR,R} \mathcal{B}^{(H)}_{00,1} = \{ \mathcal{B}^{(C)}_{000,11,11}, \mathcal{B}^{(C)}_{000,11,11}, \mathcal{B}^{(C)}_{000,11,11} \} \cup \bigcup_{N' \in \mathcal{S}(M') \setminus \{N_0'\}} \{ \mathcal{B}^{(C)}_{\phi\phi,01,11}(N'), \mathcal{B}^{(C)}_{\phi\phi,01,11}(N') \}$$

$$\beta^*_{MR,R} \mathcal{B}^{(H)}_{dd,1} = \{ \mathcal{B}^{(C)}_{dd0,11,11}, \mathcal{B}^{(C)}_{dd0,11,11}, \mathcal{B}^{(C)}_{dd0,11,11} \}$$

for $N_0'$,

$$\beta^*_{MR,R} \mathcal{B}^{(H)}_{10,0}(N_0') = \{ \mathcal{B}^{(C)}_{100,00}(N_0'), \mathcal{B}^{(C)}_{\phi\phi,00,00}(N_0') \}$$

$$\beta^*_{MR,R} \mathcal{B}^{(H)}_{01,0}(N_0') = \{ \mathcal{B}^{(C)}_{010,00}(N_0'), \mathcal{B}^{(C)}_{\phi\phi,00,00}(N_0') \}$$

$$\beta^*_{MR,R} \mathcal{B}^{(H)}_{\phi\phi,0}(N_0') = \{ \mathcal{B}^{(C)}_{\phi\phi,00,00}(N_0'), \mathcal{B}^{(C)}_{\phi\phi,00,00}(N_0') \}$$
and, for each $N' \in \mathcal{S}(M') \setminus \{N'_0\}$,

$$
\beta_{\mathcal{M},R}^*(\mathfrak{B}_{00,00}(N')) = \{ \mathfrak{B}_{00,00}(N'), \mathfrak{B}_{00,11}(N'), \mathfrak{B}_{00,01}(N'), \mathfrak{B}_{00,00}(N')\}
$$

and $\beta_{\mathcal{M},R}^*(\mathfrak{B}_{00,00}(N'))$.

We note that $\mathfrak{B}_{00,11}(N')$ and $\mathfrak{B}_{00,01}(N')$ are repeated, for $N' \in \mathcal{S}(M') \setminus \{N'_0\}$.

**Composition.** Now let us discuss the composition law. As before we are interested in integral kernels that are sections of a weighted density bundle. Let us start by recalling the weight from (7.2) namely

$$
\mathfrak{h} : \mathcal{M}_1(H(M'/B')_{b,w}) \to \mathbb{R},
$$

\[
\mathfrak{h}(H) = \begin{cases} 
-(\dim(N'/B') + 3) & \text{if } H \subseteq \mathfrak{B}_{\phi_1}(N') \text{ for some } N' \in \mathcal{S}(M') \setminus \{N'_0\} \\
-(\dim(N'/B') + 1) & \text{if } H \subseteq \mathfrak{B}_{\phi_00}(N') \text{ for some } N' \in \mathcal{S}(M') \\
-(\dim(M'/B') + 2) & \text{if } H = \mathfrak{B}_{dd,1} \\
\infty & \text{if } H = \mathfrak{B}_{00,1} \\
0 & \text{otherwise}
\end{cases}
\]

and let $\mu_R = \beta_{(H),R}^*(\mu_R)$. We will determine the behavior under composition for integral kernels of the form $\mathcal{K}_{A\rho^h} \mu_R$ with $\mathcal{K}_A \in \mathcal{A}_{\text{phg}}^E(H(M'/B')_{b,w})$. Ultimately we are interested in kernels that are merely conormal with bounds acting on sections of a vector bundle, but the corresponding composition result follows easily from this one.

**Proposition C.1.** Let $A$ have kernel $\mathcal{K}_{A\rho^h} \mu_R$ with $\mathcal{K}_A \in \mathcal{A}_{\text{phg}}^E(H(M'/B')_{b,w})$ and $B$ have kernel $\mathcal{K}_{B\rho^h} \mu_R$ with $\mathcal{K}_B \in \mathcal{A}_{\text{phg}}^E(H(M'/B')_{b,w})$. If

$$
\text{Re}(\mathcal{E}_A(\mathfrak{B}^{(H)}_{dd,1})) > 0, \quad \text{Re}(\mathcal{E}_B(\mathfrak{B}^{(H)}_{dd,1})) > 0,
$$

and

$$
\text{Re}(\mathcal{E}_A(\mathfrak{B}^{(H)}_{01,0}(N')) + \mathcal{E}_B(\mathfrak{B}^{(H)}_{10,0}(N')) + 1 > 0 \quad \text{for all } N' \in \mathcal{S}(M'),
$$

then we may define their composition $C = A \circ B$ by the formula

$$
\tilde{\mathcal{K}}_{C\rho^h} \beta_{(H)}^*(\tau d\tau) = (\beta_{LR,C}^*(\beta_{LM,L}^*(\mathcal{K}_{A\rho^h} \mu_R \beta_{(H)}^*(\tau d\tau)) \cdot \beta_{M,R,R}^*(\mathcal{K}_{B\rho^h} \mu_R \beta_{(H)}^*(\tau d\tau)))
$$

and we have $\tilde{\mathcal{K}}_{C} \in \rho^h \mathcal{A}_{\text{phg}}^E(H(M'/B')_{b,w})$ with

$$
\mathcal{E}_C(\mathfrak{B}^{(H)}_{dd,1}) = \mathcal{E}_A(\mathfrak{B}^{(H)}_{dd,1}) + \mathcal{E}_B(\mathfrak{B}^{(H)}_{dd,1})
$$

and, for $N'_0$,

$$
\mathcal{E}_C(\mathfrak{B}^{(H)}_{01,0}(N'_0)) = \mathcal{E}_A(\mathfrak{B}^{(H)}_{01,0}(N'_0)) \cup \mathcal{E}_B(\mathfrak{B}^{(H)}_{01,0}(N'_0))
$$

$$
\mathcal{E}_C(\mathfrak{B}^{(H)}_{10,0}(N'_0)) = \mathcal{E}_B(\mathfrak{B}^{(H)}_{10,0}(N'_0)) \cup \mathcal{E}_A(\mathfrak{B}^{(H)}_{01,0}(N'_0))
$$

$$
\mathcal{E}_C(\mathfrak{B}^{(H)}_{\phi_0,0}(N'_0)) = (\mathcal{E}_A(\mathfrak{B}^{(H)}_{\phi_0,0}(N'_0)) + \mathcal{E}_B(\mathfrak{B}^{(H)}_{\phi_0,0}(N'_0)))
$$

$$
\cup (\mathcal{E}_A(\mathfrak{B}^{(H)}_{10,0}(N'_0)) + \mathcal{E}_B(\mathfrak{B}^{(H)}_{01,0}(N'_0)) + \dim(N'_0/B) + 1)
and, for each \( N' \in \mathcal{S}(M') \setminus \{ N'_0 \}, \)

\[
\mathcal{E}_C(\mathfrak{B}_{10,0}^{(H)}(N')) = \mathcal{E}_A(\mathfrak{B}_{10,0}^{(H)}(N')) \cup (\mathcal{E}_A(\mathfrak{B}_{\phi,1}^{(H)}(N')) + \mathcal{E}_B(\mathfrak{B}_{10,0}^{(H)}(N')))
\]

\[
\mathcal{E}_C(\mathfrak{B}_{01,0}^{(H)}(N')) = \mathcal{E}_B(\mathfrak{B}_{01,0}(N')) \cup (\mathcal{E}_A(\mathfrak{B}_{01,0}^{(H)}(N')) + \mathcal{E}_B(\mathfrak{B}_{\phi,1}^{(H)}(N')))
\]

\[
\mathcal{E}_C(\mathfrak{B}_{\phi,1}^{(H)}(N')) = \mathcal{E}_A(\mathfrak{B}_{\phi,1}^{(H)}(N')) + \mathcal{E}_B(\mathfrak{B}_{\phi,1}^{(H)}(N'))
\]

\[
\mathcal{E}_C(\mathfrak{B}_{\phi,0}^{(H)}(N')) = (\mathcal{E}_A(\mathfrak{B}_{\phi,0}^{(H)}(N')) + \mathcal{E}_B(\mathfrak{B}_{\phi,0}^{(H)}(N'))) \\
\cup (\mathcal{E}_A(\mathfrak{B}_{10,0}^{(H)}(N')) + \mathcal{E}_B(\mathfrak{B}_{01,0}^{(H)}(N')) + \dim(N'/B') + 1).
\]

**Proof.** The proof is essentially the same as that of Proposition B.1 using the exponent matrices computed above. 
\[\square\]
Index of Notation

\( \mathcal{A}_{phg}^{I}(X) \) The space of (totally) \( I \)-smooth conormal functions with index family \( \mathcal{E} \), page 31.

\( \mathcal{B}_{phg}^{I}(X) \) The partially \( I \)-smooth conormal functions, page 31.

\( \mathcal{D}_{VAPS}(\partial_{X,Q}) \) The vertical APS domain of a Dirac-type operator, page 2.

\( H_{c}^{I}(X;E) \) The \( L^{2} \)-based edge Sobolev space, page 25.

\( S_{\psi}(M) \) The boundary hyperfaces of a manifold with corners \( M \) that are transverse to a fibration \( \psi: M \to B \), page 12.

\( wT^{\ast}X \) The wedge cotangent bundles of \( X \), a manifold with corners with iterated fibration structure. Similarly \( wTX \) denotes the wedge tangent bundle, page 11.

References

[AS64] Milton Abramowitz and Irene A. Stegun. Handbook of mathematical functions with formulas, graphs, and mathematical tables, volume 55 of National Bureau of Standards Applied Mathematics Series. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[Alb07] Pierre Albin. A renormalized index theorem for some complete asymptotically regular metrics: the Gauss-Bonnet theorem. Adv. Math., 213(1):1–52, 2007.

[Alb09] Pierre Albin. Renormalizing curvature integrals on Poincaré-Einstein manifolds. Adv. Math., 221(1):140–169, 2009.

[Alb17] Pierre Albin. On the Hodge theory of stratified spaces. In Hodge theory and \( L^{2} \)-analysis, volume 39 of Advanced lectures in mathematics, pages 1–78. Higher Education Press and International Press, Beijing-Boston, 2017.

[ABL+15] Pierre Albin, Markus Banagl, Eric Leichtnam, Rafe Mazzeo, and Paolo Piazza. Refined intersection homology on non-Witt spaces. J. Topol. Anal., 7(1):105–133, 2015.

[AGR16] Pierre Albin and Jesse Gell-Redman. The index of Dirac operators on incomplete edge spaces. SIGMA Symmetry Integrability Geom. Methods Appl., 12:Paper No. 089, 45, 2016.

[ALMP12] Pierre Albin, Eric Leichtnam, Rafe Mazzeo, and Paolo Piazza. The signature package on Witt spaces. Ann. Sci. Éc. Norm. Supér. (4), 45(2):241–310, 2012.

[ALMP13] Pierre Albin, Eric Leichtnam, Rafe Mazzeo, and Paolo Piazza. Hodge theory on Cheeger spaces. Available online at [http://arxiv.org/abs/1307.5473.v2](http://arxiv.org/abs/1307.5473.v2) to appear in Crelle’s journal, 2013.

[ALMP17] Pierre Albin, Eric Leichtnam, Rafe Mazzeo, and Paolo Piazza. The Novikov conjecture on Cheeger spaces. J. Noncommut. Geom., 11(2):451–506, 2017.

[AM] Pierre Albin and Rafe Mazzeo. Geometric constructions of heat kernels: a user’s guide. to appear.

[AM09a] Pierre Albin and Richard Melrose. Fredholm realizations of elliptic symbols on manifolds with boundary. J. Reine Angew. Math., 627:155–181, 2009.

[AM09b] Pierre Albin and Richard Melrose. Relative Chern character, boundaries and index formulas. J. Topol. Anal., 1(3):207–250, 2009.

[AM10] Pierre Albin and Richard Melrose. Fredholm realizations of elliptic symbols on manifolds with boundary II: fibered boundary. In Motives, quantum field theory, and pseudodifferential operators, volume 12 of Clay Math. Proc., pages 99–117. Amer. Math. Soc., Providence, RI, 2010.

[AM11] Pierre Albin and Richard Melrose. Resolution of smooth group actions. In Spectral theory and geometric analysis, volume 535 of Contemp. Math., pages 1–26. Amer. Math. Soc., Providence, RI, 2011.

[AR09a] Pierre Albin and Frédéric Rochon. Families index for manifolds with hyperbolic cusp singularities. Int. Math. Res. Not. IMRN, (4):625–697, 2009.

[AR09b] Pierre Albin and Frédéric Rochon. A local families index formula for d-bar operators on punctured Riemann surfaces. Comm. Math. Phys., 289(2):483–527, 2009.

[AR13] Pierre Albin and Frédéric Rochon. Some index formulae on the moduli space of stable parabolic vector bundles. J. Aust. Math. Soc., 94:1–37, 2013.

[ARS] Pierre Albin, Frédéric Rochon, and David Sher. A Cheeger-Müller theorem for manifolds with wedge singularities. forthcoming.
[Bun09] Ulrich Bunke. Index theory, eta forms, and Deligne cohomology. Mem. Amer. Math. Soc., 198(928):vi+120, 2009.

[CRLM14] Paulo Carrillo Rouse, Jean-Marie Lesure, and Bertrand Monthubert. A cohomological formula for the Atiyah-Patodi-Singer index on manifolds with boundary. J. Topol. Anal., 6(1):27–74, 2014.

[Car01] Gilles Carron. Théorèmes de l’indice sur les variétés non-compactes. J. Reine Angew. Math., 541:81–115, 2001.

[CN14] Catarina Carvalho and Victor Nistor. An index formula for perturbed Dirac operators on Lie manifolds. J. Geom. Anal., 24(4):1808–1843, 2014.

[Cha97] Shing-Wai Chan. $L$-classes on pseudomanifolds with one singular stratum. Proc. Amer. Math. Soc., 125(7):1955–1968, 1997.

[Che77] Jeff Cheeger. Analytic torsion and Reidemeister torsion. Proc. Nat. Acad. Sci. U.S.A., 74(7):2651–2654, 1977.

[Che79a] Jeff Cheeger. Analytic torsion and the heat equation. Ann. of Math. (2), 109(2):259–322, 1979.

[Che79b] Jeff Cheeger. On the spectral geometry of spaces with cone-like singularities. Proc. Nat. Acad. Sci. U.S.A., 76(5):2103–2106, 1979.

[Che80] Jeff Cheeger. On the Hodge theory of Riemannian pseudomanifolds. In Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., XXXVI, pages 91–146. Amer. Math. Soc., Providence, R.I., 1980.

[Che83] Jeff Cheeger. Spectral geometry of singular Riemannian spaces. J. Differential Geom., 18(4):575–657 (1984), 1983.

[Che87] Jeff Cheeger. $\eta$-invariants, the adiabatic approximation and conical singularities. I. The adiabatic approximation. J. Differential Geom., 26(1):175–221, 1987.

[CD09] Jeff Cheeger and Xianzhe Dai. $L^2$-cohomology of spaces with nonisolated conical singularities and nonmultiplicativity of the signature. In Riemannian topology and geometric structures on manifolds, volume 271 of Progr. Math., pages 1–24. Birkhäuser Boston, Boston, MA, 2009.

[CT82] Jeff Cheeger and Michael Taylor. On the diffraction of waves by conical singularities. I. Comm. Pure Appl. Math., 35(3):275–331, 1982.

[Cho85] Arthur W. Chou. The Dirac operator on spaces with conical singularities and positive scalar curvatures. Trans. Amer. Math. Soc., 289(1):1–40, 1985.

[Cho89] Arthur W. Chou. Criteria for selfadjointness of the Dirac operator on pseudomanifolds. Proc. Amer. Math. Soc., 106(4):1107–1116, 1989.

[CS84] Alain Connes and Georges Skandalis. The longitudinal index theorem for foliations. Publ. Res. Inst. Math. Sci., 20(6):1139–1183, 1984.

[Dai91] Xianzhe Dai. Adiabatic limits, nonmultiplicativity of signature, and Leray spectral sequence. J. Amer. Math. Soc., 4:265 – 321, 1991.

[DM12] Xianzhe Dai and Richard B. Melrose. Adiabatic limit, heat kernel and analytic torsion. In Metric and differential geometry, volume 297 of Progr. Math., pages 233–298. Birkhäuser/Springer, Basel, 2012.

[DLN09] Claire Debord, Jean-Marie Lesure, and Victor Nistor. Groupoids and an index theorem for conical pseudo-manifolds. J. Reine Angew. Math., 628:1–35, 2009.

[DLR15] Claire Debord, Jean-Marie Lesure, and Frédéric Rochon. Pseudodifferential operators on manifolds with fibred corners. Ann. Inst. Fourier (Grenoble), 65(4):1799–1880, 2015.

[DS] Claire Debord and Georges Skandalis. Blowup constructions for Lie groupoids and a Boutet de Monvel type calculus. available online at [arXiv:1705.09588]

[DS88] Nelson Dunford and Jacob T. Schwartz. Linear operators. Part II. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1988. Spectral theory. Selfadjoint operators in Hilbert space, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1963 original, A Wiley-Interscience Publication.

[EMM91] Charles L. Epstein, Richard B. Melrose, and Gerardo A. Mendoza. Resolvent of the Laplacian on strictly pseudoconvex domains. Acta Math., 167(1-2):1–106, 1991.

[FH95] Jeffrey Fox and Peter Haskell. Index theory for perturbed Dirac operators on manifolds with conical singularities. Proc. Amer. Math. Soc., 123(7):2265–2273, 1995.
THE INDEX FORMULA FOR FAMILIES OF DIRAC TYPE OPERATORS ON PSEUDOMANIFOLDS

[GRS15] Jesse Gell-Redman and Jan Swoboda. Spectral and Hodge theory of ‘Witt’ incomplete cusp edge spaces, 2015. available online at [arXiv:1509.06359](http://arxiv.org/abs/1509.06359).

[GKM13] Juan B. Gil, Thomas Krainer, and Gerardo A. Mendoza. On the closure of elliptic wedge operators. *J. Geom. Anal.*, 23(4):2035–2062, 2013.

[Gri01] Daniel Grieser. Basics of the $b$-calculus. In *Approaches to singular analysis (Berlin, 1999)*, volume 125 of *Oper. Theory Adv. Appl.*, pages 30–84. Birkhäuser, Basel, 2001.

[HLV] Luiz Hartmann, Matthias Lesch, and Boris Vertman. On the domain of a Dirac operator on stratified spaces. Available online at [arXiv:1709.01636](http://arxiv.org/abs/1709.01636).

[HMM95] Andrew Hassell, Rafe Mazzeo, and Richard B. Melrose. Analytic surgery and the accumulation of eigenvalues. *Comm. Anal. Geom.*, 3(1-2):115–222, 1995.

[HMM97] Andrew Hassell, Rafe Mazzeo, and Richard B. Melrose. A signature formula for manifolds with corners of codimension two. *Topology*, 36(5):1055–1075, 1997.

[HMM04] Tamás Hausel, Eugénie Hunsicker, and Rafe Mazzeo. Hodge cohomology of gravitational instantons. *Duke Math. J.*, 122(3):485–548, 2004.

[Hit74] Nigel Hitchin. Harmonic spinors. *Advances in Math.*, 14:1–55, 1974.

[HM05] Eugénie Hunsicker and Rafe Mazzeo. Harmonic forms on manifolds with edges. *Int. Math. Res. Not.*, (52):3229–3272, 2005.

[Hör07] Lars Hörmander. *The analysis of linear partial differential operators. III*. Classics in Mathematics. Springer-Verlag, Berlin, 2007. Pseudo-differential Operators, Reprint of the 1994 edition.

[Hun07] Eugénie Hunsicker. Hodge and signature theorems for a family of manifolds with fibre bundle boundary. *Geom. Topol.*, 11:1581–1622, 2007.

[Klo09] Benoît Kloeckner. Quelques notions d’espaces stratifiés. In *Actes du Séminaire de Théorie Spectrale et Géométrie. Vol. 26. Année 2007–2008*, volume 26 of *Sémin. Théor. Spectr. Géom.*, pages 13–28. Univ. Grenoble I, Saint, 2009.

[KM16] Thomas Krainer and Gerardo A. Mendoza. Boundary value problems for first order elliptic wedge operators. *Amer. J. Math.*, 138(3):585–656, 2016.

[Les93] Matthias Lesch. Deficiency indices for symmetric Dirac operators on manifolds with conic singularities. *Topology*, 32(3):611–623, 1993.

[Les97] Matthias Lesch. *Operators of Fuchs type, conical singularities, and asymptotic methods*, volume 136 of *Teubner-Texte auf Math.* B.G. Teubner, Stuttgart, Leipzig, 1997.

[LV13] Michael T. Lock and Jeff A. Viaclovsky. An index theorem for anti-self-dual orbifold-cone metrics. *Adv. Math.*, 248:698–716, 2013.

[Loy05] Paul Loya. The index of $b$-pseudodifferential operators on manifolds with corners. *Ann. Global Anal. Geom.*, 27(2):101–133, 2005.

[Maz91] Rafe Mazzeo. Elliptic theory of differential edge operators. I. *Comm. Partial Differential Equations*, 16(10):1615–1664, 1991.

[MM87] Rafe R. Mazzeo and Richard B. Melrose. Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature. *J. Funct. Anal.*, 75(2):260–310, 1987.

[MM95] Rafe Mazzeo and Richard B. Melrose. Analytic surgery and the eta invariant. *Geom. Funct. Anal.*, 5(1):14–75, 1995.

[MM98] Rafe Mazzeo and Richard B. Melrose. Pseudodifferential operators on manifolds with fibred boundaries. *Asian J. Math.*, 2(4):833–866, 1998. Mikio Sato: a great Japanese mathematician of the twentieth century.

[MV12] Rafe Mazzeo and Boris Vertman. Analytic torsion on manifolds with edges. *Adv. Math.*, 231(2):1000–1040, 2012.

[MV14] Rafe Mazzeo and Boris Vertman. Elliptic theory of differential edge operators, II: Boundary value problems. *Indiana Univ. Math. J.*, 63(6):1911–1955, 2014.
[MW17] Rafe Mazzeo and Edward Witten. The KW Equations and the Nahm Pole Boundary Condition with Knots. available online at [arXiv:1712.00835], 2017.

[Mel] Richard B. Melrose. *Differential Analysis on Manifolds with Corners*. in preparation. Available online at [http://www-math.mit.edu/~rbm/book.html].

[Mel92] Richard B. Melrose. Calculus of conormal distributions on manifolds with corners. *Int Math Res Notices*, 1992:51–61, 1992.

[Mel93] Richard B. Melrose. The Atiyah-Patodi-Singer index theorem, volume 4 of *Research Notes in Mathematics*. A K Peters Ltd., Wellesley, MA, 1993.

[MN] Richard Melrose and Victor Nistor. Homology of pseudodifferential operators I. Manifolds with boundary. available online at: [arXiv:funct-an/9606005].

[MN98] Richard Melrose and Victor Nistor. K-theory of C*-algebras of b-pseudodifferential operators. *Geom. Funct. Anal.*, 8(1):88–122, 1998.

[MP92] Richard B. Melrose and Paolo Piazza. Analytic K-theory on manifolds with corners. *Adv. Math.*, 92(1):1–26, 1992.

[MP97a] Richard B. Melrose and Paolo Piazza. Families of Dirac operators, boundaries and the b-calculus. *J. Differential Geom.*, 46(1):99–180, 1997.

[MP97b] Richard B. Melrose and Paolo Piazza. An index theorem for families of Dirac operators on odd-dimensional manifolds with boundary. *J. Differential Geom.*, 46(2):287–334, 1997.

[MR04] Richard Melrose and Frédéric Rochon. Families index for pseudodifferential operators on manifolds with boundary. *Int. Math. Res. Not.*, (22):1115–1141, 2004.

[MR06] Richard Melrose and Frédéric Rochon. Index in K-theory for families of fibred cusp operators. *K-Theory*, 37(1-2):25–104, 2006.

[MR11] Richard Melrose and Frédéric Rochon. Eta forms and the odd pseudodifferential families index. In *Surveys in differential geometry. Volume XV. Perspectives in mathematics and physics*, volume 15 of *Surv. Differ. Geom.* pages 279–322. Int. Press, Somerville, MA, 2011.

[MN12] Bertrand Monthubert and Victor Nistor. A topological index theorem for manifolds with corners. *Compos. Math.*, 148(2):640–668, 2012.

[Moo99] Edith A. Mooers. Heat kernel asymptotics on manifolds with conic singularities. *J. Anal. Math.*, 78:1–36, 1999.

[M78] Werner Müller. Analytic torsion and R-torsion of Riemannian manifolds. *Adv. in Math.*, 28(3):233–305, 1978.

[M96] Werner Müller. On the L2-index of Dirac operators on manifolds with corners of codimension two. I. *J. Differential Geom.*, 44(1):97–177, 1996.

[NkSSS05] Vladimir E. Nazaikinski, Anton Yu. Savin, Bert-Wolfgang Schulze, and Boris Yu. Sternin. On the index of elliptic operators on manifolds with edges. *Mat. Sb.*, 196(9):23–58, 2005.

[NkSSS06] Vladimir E. Nazaikinski, Anton Yu. Savin, Bert-Wolfgang Schulze, and Boris Yu. Sternin. Elliptic theory on singular manifolds, volume 7 of *Differential and Integral Equations and Their Applications*. Chapman & Hall/CRC, Boca Raton, FL, 2006.

[Pia93] Paolo Piazza. On the index of elliptic operators on manifolds with boundary. *J. Funct. Anal.*, 117(2):308–359, 1993.

[PV] Paolo Piazza and Boris Vertman. Eta and rho invariants on manifolds with edges. available online at [arXiv:1604.07420].

[SS10] Anton Yu. Savin and Boris Yu. Sternin. Index formulas for stratified manifolds. *Differ. Uram.*, 46(8):1135–1146, 2010.

[SS95] Elmar Schrohe and Bert-Wolfgang Schulze. Boundary Value Problems in Boutet de Monvel's Algebra for Manifolds with Conical Singularities II. In *Boundary Value Problems, Schrödinger Operators, Deformation Quantization*, volume 2 of *Advances in Partial Differential Equations*, pages 70–205. Akademie Verlag, Berlin, 1995.

[Sch07] Bert-Wolfgang Schulze. Pseudo-differential calculus on manifolds with geometric singularities. In *Pseudo-differential operators: partial differential equations and time-frequency analysis*, volume 52 of *Fields Inst. Commun.*, pages 37–83. Amer. Math. Soc., Providence, RI, 2007.

[Sie83] Paul Siegel. Witt spaces: A geometric cycle theory for KO-homology theory at odd primes. *Amer. J. Math.*, 105:1067–1105, 1983.
[Sin71] Isadore Singer. Future extensions of index theory and elliptic operators. In Prospects in Mathematics, volume 70 of Annals of Mathematics Studies in Mathematics, pages 171–185. 1971.

[Ste89] Mark Stern. $L^2$-index theorems on locally symmetric spaces. Invent. Math., 96(2):231–282, 1989.

[Tay11] Michael E. Taylor. Partial differential equations II. Qualitative studies of linear equations, volume 116 of Applied Mathematical Sciences. Springer, New York, second edition, 2011.

[Tho69] René Thom. Ensembles et morphismes stratifiés. Bull. Amer. Math. Soc., 75:240–284, 1969.

[Vai01] Boris Vaillant. Index- and spectral theory for manifolds with generalized fibred cusps, volume 344 of Bonner Mathematische Schriften [Bonn Mathematical Publications]. Universität Bonn, Mathematisches Institut, Bonn, 2001. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 2001.

[Wit85] Edward Witten. Global gravitational anomalies. Comm. Math. Phys., 100(2):197–229, 1985.

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