Determination of the Pythagorean scale using a half wavelength theory of standing-wave

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Abstract. We have derived the formula to determine the number parameter \( p = 0.943874313 \) instead of the Pythagorean scale parameter \( \Pi = 17.817154 \) which is the basic parameter in determining the length of the string to produce various tones. Both parameters are connected through the formula \( \Pi = 1/(1-p) \). The formula is derived through a method built from vibration produced by half the wavelength of the string bound at both ends with the assumption that from the basic tone to the first high note there are 12 tones, namely E, F, Fis, G, Gis, A, Bes, B, C, Cis, D, and Dis.

1. Introduction

A musical instrument is one tool that produces sound (source of sound) with regular frequency, which is generally called "tone". In music theory, natural tone symbols are generally written in letters C, D, E, F, G, A, B, c, d, e, f, g, a, b, and so on. The tones are produced by a sound source that vibrates with a certain frequency range. In other words, if we determine one particular frequency for the first tone, say C, then the determination of the next tones must meet a certain ratio referring to that first tone frequency[1].

In a string-based musical instrument, for example, a guitar, sound is produced from the vibrations of the strings with both ends tied. Therefore the vibration that produces sound satisfies the theory of vibration or wave on the string with both ends bound (standing waves with both ends fixed). The tones produced by the string vibrations depend on the length and tension (strong pull) of the string. The stronger the string tension and the shorter the string, the higher the tone (the frequency increases), and vice versa.

On the guitar, the tension on a string is fixed, but to get a variety of tones, the length of the string being vibrated varies, which is by pressing the string slightly on the edge of the metal stick (fret) (Figure 1). In other words, the variety of tones that appear on a guitar string is caused by differences in length, for example, strings with length \( l_0 \) will produce an E note, then if the length \( l_1 \) will produce a note F, and so on[2]. The determination of the length of the string to produce various tones turns out to have to satisfy a certain constant, which is commonly referred to as the Pythagorean scale.
In determining the distance between frets on a guitar, the Pythagorean scale $\bar{\Pi} = 17,817,154$ is an important scale that seems to appear just like that. The problem is that the scale is inseparable from the determination of the vibrating frequency that produces various tones [3][4]. However, it is difficult to trace where the scale numbers appear. Since this problem is inseparable from the problem of sound waves produced by the vibrations of the strings with both ends bound, then, physically, the appearance of that number should be traceable through the standing wave theory for strings which are bound at both ends.

In this paper, a general theoretical parameter will be derived as a standard scale (Pythagorean scale) to determine the fret length for a certain length of string based on standing wave theory.

2. Determination of the fret scale using a half wavelength theory of standing-wave

The string vibration on the guitar satisfies the rule of $1/2 \lambda$. In other words, wherever its position on the string is picked, the string guitar only has $1/2 \lambda$. The frequency $f$ of the vibrations satisfies the relationship

$$ f = \frac{v}{\lambda} \quad \text{with} \quad \lambda = 2l$$

(1)

where $v$, $\lambda$, and $l$ are speed of wave propagation, wavelength, and string length, respectively [5]. If $f_0$ is the base tone frequency of the string, and $f'$ is the first high-tone frequency in the string, it must be satisfied

$$ f' = 2f_0.$$ 

(2)

If the frequency of the basic tone and the first high tone is

$$ f_0 = \frac{v_0}{2l_0} \quad \text{and} \quad f' = \frac{v'}{2l'}$$

then equation (2) and $l' = 1/2l_0$ produce $2f_0 = v'/l_0$. Because $v = \sqrt{F/\mu}$ with $F$ (string tension) and $\mu$ (mass per unit length) are constant, then $v' = v_0$, so that $f_0 = v_0/(2l_0)$ which is equation (1) for the basic level.

Guitar (for example) has a different tone caused by a long change in the string. Generally, there are 24 fret points, which means there are 24 different tones for each string. If the base note in a string is a string tone picked at the "string-free" condition (string is not pressed on the fret board) then the first high note is the string tone picked at the string condition pressed at the 12th fret (see figure2).
Figure 2. The base tone has a string length of $l_0$, the first high-note (with the frequency $f' = 2f_0$ having the length of string $\frac{1}{2}l_0$).

One octave of tone, for example from E to E', there are 8 general tones, namely E-F-G-A-B-C-D-E'. In the guitar one octave of the tone is divided into 12 frets (for example, giving the notes E, F, Fis, G, Gis, A, Bes, B, C, Cis, D, Dis) as shown in figure 3.

If $l_0 - l_1 = \Delta l_1$, $l_1 - l_2 = \Delta l_2$, and $l_2 - l_3 = \Delta l_3$, than it can be formulated

$$\Delta l_n = l_{n-1} - l_n \text{ where } n = 1, 2, 3, 4, \ldots$$

(3)

For example, $f_0$ is a frequency of a base tone and $f_1$ is a frequency of the first fret,

$$f_0 : f_1 = \frac{v_0}{2l_0} : \frac{v_1}{2l_1} = \frac{2l_1 v_0}{2l_0 v_1}.$$  

Since $v_1 = v_0$ than

$$\frac{f_0}{f_1} = \frac{l_1}{l_0}$$

(4)

In the same way, it can be written

$$\frac{f_0}{f_1} = \frac{l_1}{l_0}, \ldots, \frac{f_{n-1}}{f_n} = \frac{l_n}{l_{n-1}}.$$  

(5)

If the ratio between $f_{n-1}$ and $f_n$ is equal to a constant for all $n$, then, from the equation (5), it is obtained

$$\frac{l_n}{l_{n-1}} = p \quad \text{or} \quad l_n = p.l_{n-1}$$

(6)

where $p$ is the constant to be searched, from the equation (6) a comparative relationship between the length $l_n$ and $l_0$ can be obtained with the appointment as follows:

$$l_1 = p. l_0; \quad l_2 = p. l_1 = p. p. l_0 = p^2 l_0; \quad l_3 = p. l_2 = p. p^2. l_0 = p^3 l_0; \ldots \text{etc.}$$

(7)

or it can be written recursively as

$$l_n = p^n. l_0.$$  

(8)
Furthermore, the first high tone frequency level, \( f' \), is 2 times the base tone, \( f_0 \).

\[ f' = 2f_0. \]

Since from the basic note to the first high note divided by 12 frets, it can be written

\[ f' = f_{12} = 2f_0. \]  \( \text{(9)} \)

Since \( 2f_0 \) is obtained when the string length is \( \frac{1}{2}l_0 \) then

\[ l_{12} = \frac{1}{2}l_0 \]  \( \text{(10)} \)

By considering equations (8) and (10), the \( p \) value is obtained as follows:

\[ l_{12} = p^{12}l_0; \frac{1}{2}l_0 = p^{12}l_0; \quad p^{12} = \frac{1}{2} \]

which means

\[ p = \sqrt[12]{\frac{1}{2}} = 0.943874313. \]  \( \text{(11)} \)

From equations (3) and (8) are obtained

\[ \Delta l_n = l_{n-1} - l_n = p^{n-1}l_0 - p^n l_0 = (1 - p)p^{n-1}l_0 \]  \( \text{(12)} \)

which is the distance between frets with the constant \( p \) given by equation (11).

3. Fret Distance Determination Using the Pythagorean Theory

Calculation of fret distance by using Pythagorean parameter (\( \Pi \)) is done assuming that the distance between frets is proportional to the fret distance from the saddle divided by the same number (\( \Pi \)), as illustrated in figure (4).

\[ \Delta l_n = \frac{l_{n-1}}{\Pi} \]

\[ \Delta l_1 = \frac{l_0}{\Pi} \]

\[ \Delta l_2 = \frac{l_1}{\Pi} \]

\[ \Delta l_3 = \frac{l_2}{\Pi} \]

\[ \Delta l_n = \frac{l_{n-1}}{\Pi} \]

**Figure 4.** Illustration of distance between frets correspond to the Pythagorean parameter (\( \Pi \))

Mathematically, the distance between the frets is expressed as

\[ \Delta l_n = \frac{l_{n-1}}{\Pi}. \]  \( \text{(13)} \)

The \( n^{th} \) fret distance from the saddle can be obtained through equations (3) and (13) [6][7], i.e.

\[ l_n = l_{n-1} - \Delta l_n = l_{n-1} - \frac{l_{n-1}}{\Pi} = l_{n-1} \left( 1 - \frac{1}{\Pi} \right), \]  \( \text{(14)} \)

while the relationship between \( l_n \) and \( l_0 \) can be obtained through the translation of equation (14), i.e.

\[ l_1 = l_0 \]

\[ l_2 = l_1 \left( 1 - \frac{1}{\Pi} \right) = l_0 \left( 1 - \frac{1}{\Pi} \right) \left( 1 - \frac{1}{\Pi} \right) = l_0 \left( 1 - \frac{1}{\Pi} \right)^2 \]

\[ l_3 = l_2 \left( 1 - \frac{1}{\Pi} \right) = l_0 \left( 1 - \frac{1}{\Pi} \right) \left( 1 - \frac{1}{\Pi} \right) \left( 1 - \frac{1}{\Pi} \right) = l_0 \left( 1 - \frac{1}{\Pi} \right)^3 \]

\[ . \]

\[ . \]

\[ l_n = l_0 \left( 1 - \frac{1}{\Pi} \right) \ldots \left( 1 - \frac{1}{\Pi} \right) = l_0 \left( 1 - \frac{1}{\Pi} \right)^n. \]  \( \text{(15)} \)
The parameter $\Pi$ (referred to as the Pythagorean parameter) can be searched from equation (15) with assumptions (as in equation (10)) $l_{12} = l_0 \left(1 - \frac{1}{\Pi}\right)^{12}$; that is:

$$l_{12} = l_0 \left(1 - \frac{1}{\Pi}\right)^{12} \frac{1}{2} = \left(1 - \frac{1}{\Pi}\right)^{12} \sqrt[12]{0.5} = \left(1 - \frac{1}{\Pi}\right)^{12} \frac{1}{2} = 1 - \frac{1}{2} \sqrt[12]{0.5}$$

or

$$\Pi = \left(1 - \frac{1}{2} \sqrt[12]{0.5}\right)^{-1}. \quad (16)$$

From the equations (11) and (16) can be established the relationship between the parameters $p$ and $\Pi$ as follows

$$p = 1 - \frac{1}{\Pi} \quad \text{or} \quad \Pi = (1 - p)^{-1}. \quad (17)$$

4. Conclusion

A fundamental parameter in determining the distance between frets on string-based musical instruments (e.g., guitars) can be derived through a method constructed from vibrations produced by half the wavelength of a string tied at both ends. From the derivation, the parameter $p = 0.943874313$ is generated. The Pythagorean scale $\Pi = 17.817154$ which has been a "sacred" number in determining the distance between frets associated with parameter $p$ through the equation $\Pi = 1/(1-p)$.

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