A Study of Asymmetric Vibration of Polar Orthotropic Annular Plates Using Boundary Characteristic Orthonormal Polynomials

Neeraj Bhardwaj\(^1\)* and Lokendra Kumar\(^2\)

\(^1\) Department of Computer Application, BIT Meerut, India
\(^2\) Department of Mathematics, Jaypee Institute of Information Technology, Noida, India

Received: 06.02.2020 • Accepted: 19.04.2020 • Published Online: 01.06.2020

Abstract: Classical plate theory is used to study asymmetric vibration of polar orthotropic annular plates of quadratically varying thickness in the radial direction and resting on Winkler elastic foundation. Boundary characteristic orthonormal polynomials are used in Rayleigh-Ritz method. Convergence of the results is tested, and comparisons are made in particular cases with the results already reported in the existing literature. First ten frequencies for variation in orthotropy for all possible four combinations when one boundary is free are shown. Three dimensional mode shapes corresponding to the frequencies are shown.

Keywords: Annular plate, variable thickness, orthotropy, asymmetric vibration, Winkler elastic foundation, Jacobi method.

2010 Mathematics Subject Classification. 74S30, 74B20,74E10,74H15

1. Introduction

Annular circular plate is the simplest and widely used structural element in various engineering fields. The vibration of such plates has been the subject of various studies. Leissa [1-7] summarized the information in his well-known monograph and six comprehensive review articles. For the orthotropic plates, except for the few cases, no closed form solution exits, and researchers have used different approximation methods. Among them Vijaya Kumar and Ramaiah [8,9], Narita [10,11] and Gutierrez et al. [12] used Rayleigh-Ritz method, Greenberg and Stavsky [13] used finite difference method and Ginesu et al. [14] and Gorman [15] used finite element method.

A lot of information on annular circular plates having varying thickness is also available in the existing literature. Kim and Dickinson [16] have analyzed composite circular plates as a particular case of annular plates by taking inner radius very small but only few results are given on circular plates and that too for uniform thickness only. Laura et al. [17] have analysed annular circular plates having cylindrical anisotropy and non-uniform thickness using polynomial coordinates functions. Chen [18] studied lateral vibration of isotropic and orthotropic thin annular and circular plates of arbitrarily varying thickness along radius using finite element method and obtained natural frequencies and mode shapes of the axisymmetric and asymmetric modes. In these papers, variation of thickness depends only on one taper parameter and there is no mention of nodal lines and mode shapes. In recent years N. Bhardwaj and his co-researchers [19] studied asymmetric vibration of polar orthotropic annular circular plates of quadratically varying thickness with same boundary conditions

In the present paper, asymmetric vibration of annular plates of polar orthotropic material having quadratically varying thickness (along radius) and resting on Winkler elastic foundation is analysed by using boundary characteristic orthonormal polynomials in Rayleigh-Ritz method. Two taper parameters are used for quadratic thickness variation, * Correspondence: bneerajdma@gmail.com
which give more flexibility to distribute the mass economically according to thickness variation. Many thickness variations can be approximated by it by suitably choosing the values of taper constants. Convergence of frequencies at least up to five significant figures is shown. Comparison of frequencies in particular cases are made with the results already available in the literature. Mode shapes for first ten normal modes of vibrations for all possible four combinations when one of the boundaries is free are presented. Although a close agreement is found but present results are found to be better in almost all the cases.

1. ANALYSIS

A thin annular plate of outer radius $a$, inner radius $b$ variable thickness $h(r)$, made up of orthotropic material and resting on Winkler elastic foundation is considered. The plate is subjected to polar coordinates $(r, \theta)$ by taking the center of the plate as origin and the middle plane of the plate in the coordinate plane. For free flexural asymmetric vibration of the plate, let $w(r, \theta, t)$ be the deflection of the plate at any point $(r, \theta)$ at any time $t$.

For applying the Rayleigh-Ritz method, the functional

$$J(W) = \frac{\pi E_t H_a^3}{12 s_r} \int_{r_0}^{1} \left[ F^2 \left( \frac{W_{m,RR}^2}{R^2} + 2 \epsilon_r \nu_r W_{m,RR} \left( -\frac{m^2}{R^2} W_m + \frac{1}{R} W_{m,R} \right) + \epsilon_r \left( -\frac{m^2}{R^2} W_m + \frac{1}{R} W_{m,R} \right)^2 \right) + 4 s_r \left( \frac{m}{R^2} W_m - \frac{m}{R} W_{m,R} \right)^2 \right] R \, dR$$

obtained by subtracting the maximum kinetic energy from the maximum strain energy of the plate is to be minimized, where

$$R = r/a, \quad R_0 = b/a, \quad \dot{H}_a F(R) = h(r)/a, \quad \epsilon_r = E_\theta / E_r, \quad \nu_r = \nu_\theta / \nu_r, \quad g_r = G / E_r, \quad s_r = 12(1-e_r \nu_r^2), \quad w(r, \theta, t) = a W(R) \cos \omega T, \quad \kappa_f = s_r \alpha_k_f \left( \frac{E_r}{H_a^3} \right), \quad \Omega^2 = s_r \alpha^2 / H_a^2,$$

Here $E_r$ and $E_\theta$ and $\nu_r$ and $\nu_\theta$ are the Young’s moduli and Poission’s ratio’s in $r$ and $\theta$ directions, $G$ is the shear modulus, $\kappa_f$ is the foundation constant, $\rho$ is the density of the plate, $H_a$ is the thickness of the plate at the outer edge $R = 1$, $m$ is the number of nodal diameter and $\omega$ is the natural frequency of harmonic vibration. A comma followed by a suffixed variable denotes differentiation with respect to that variable.

For quadratically varying thickness of the plate in the radial direction $F(R)$ is taken as

$$F(R) = 1 - \alpha (1 - R) - \beta (1 - R)^2, \quad \alpha (1 - R_0) + \beta (1 - R_0)^2 < 1$$

where $\alpha$ and $\beta$ are the taper constants.

The $N$-term approximation of the deflection function is taken as

$$W_m(R) = \sum_{j=1}^{N} c_{m,j} \Phi_j(R),$$
where $\Phi_j$ are the orthonormal polynomials satisfying at least the geometric edge conditions of the plate. Using three terms recurrence relation given by Chihara [20], $\Phi_j$ are generated as

$$\Phi_j = \varphi_j \sqrt{\frac{\varphi_j}{\varphi_j, \varphi_j}}$$

$$\varphi_1 = 1 - R^2 p$$

(5)

$$\phi_{j+1} = \left( R - \frac{< R \phi_j, \phi_j >}{< \phi_j, \phi_j >} \right) \phi_j - \frac{< \phi_j, \phi_j >}{< \phi_j, \phi_j, \phi_j >} \phi_{j-1}, \phi_0 = 0,$$

(6)

$$< f, g > = \frac{1}{J_0} \int F(R) f(R) g(R) R dR.$$  

(7)

The values of $p$ is taken 0, 1 or 2 according as the plate is subjected to free $(F)$, simply supported $(S)$ or clamped $(C)$ edge conditions.

Substitution of $W_m(R)$ from Equation (4) into energy Equation (1) and then minimization of $J(W)$ as a function of the coefficients $c_{n,j}$ leads to the standard eigen value problem:

$$\sum_{j=1}^{N} \left( a_{ij} - \Omega^2 \delta_{ij} \right) c_{m,j} = 0 ,$$

$$i = 1 (1) N ,$$

(8)

where

$$\delta_{ij} = \begin{cases} 
1 & \text{when } i = j \\
0 & \text{when } i \neq j 
\end{cases}$$

and

$$a_{ij} = \int_{R_0}^{R} \left\{ F^3 \left( \Phi_{i,RR} \Phi_{j,RR} + e_r \nu_r \Phi_{i,R} \right) + e_r \nu_r \Phi_{i,RR} \left( \frac{m^2}{R^2} \Phi_j + \frac{1}{R} \Phi_{j,R} \right) + e_r \nu_r \Phi_{i,RR} \left( \frac{m^2}{R^2} \Phi_i + \frac{1}{R} \Phi_{i,R} \right) + e_r \nu_r \Phi_{i,RR} \left( \frac{m^2}{R^2} \Phi_j + \frac{1}{R} \Phi_{j,R} \right) + 4 s \nu_r \left\{ \frac{m^2}{R^2} \Phi_j + \frac{m^2}{R} \Phi_{j,R} \right\} \right\}$$

$$+ K_f \Phi_i \Phi_j \right] dR$$

(9)

The eigen values ($\Omega$) and the eigen vectors ($c_{m,j}$) are computed by Jacobi method. The mode shapes are computed from Equation (4).

2. RESULTS AND DISCUSSIONS

In all nine parameters $R_0, \alpha, \beta, K_f, e_r, g_r, p, \nu_r$ and $N$ are used in the analysis of this plate. The values of $\nu_r$ and $g_r$ are taken to be 0.31 and 5.0 respectively for all computation except for the Table 2, where other values are also taken for the sake of comparison to known results. The variations in thickness parameters for all possible four combinations are taken for both $\alpha$ and $\beta$ from −0.4 to 0.4 in steps of 0.1.

Table 1 shows the convergence of first ten frequencies $\Omega_{m,n}$ at least up to five significant figures for all possible four combinations C-F, S-F, F-C and F-S of edge conditions at outer and inner edges when $\alpha = \beta = 0.4, e_r = g_r = 5.0, R_0 = 0.5$ and $K_f = 500$. The suffixes $m$ and $n$ with $\Omega$ denote number of nodal diameters and number of nodal circles respectively. It can be seen that 7 terms are required to get this accuracy in all the cases.
Comparison of $\Omega_{m,n}$ for polar orthotropic annular plates of uniform thickness with Gorman[15], Narita,[10], and Kim and Dickinson[16] are given in Table 2 and with Gutierrez et al. [12] in Table 3. Our results are found to be better even for lesser number of terms in almost all the cases besides agreeing closely with their results.

Table 3 shows the variation in $\Omega_{m,n}$ with increasing $e_r$ for $C-F$, $S-F$, $F-C$ and $F-S$ plates. It is found that there is no specific pattern in $\Omega_{m,n}$ with the increase of $e_r$. For $C-F$ plate $\Omega_{0,0}$, $\Omega_{5,0}$, $\Omega_{0,1}$, $\Omega_{6,0}$ and $\Omega_{1,1}$ increase whereas $\Omega_{2,0}$, $\Omega_{3,0}$, $\Omega_{4,0}$ and $\Omega_{2,1}$ decrease. For $S-F$ plate $\Omega_{0,0}$, $\Omega_{3,0}$ and $\Omega_{4,0}$ first increase and then decrease whereas $\Omega_{1,0}$, $\Omega_{2,0}$, $\Omega_{1,1}$ and $\Omega_{2,1}$ decrease and $\Omega_{0,1}$, $\Omega_{5,0}$ and $\Omega_{6,0}$ increase. For $F-C$ plate, it is observed that $\Omega_{0,0}$ first increases and then decreases whereas $\Omega_{1,0}$, $\Omega_{2,0}$, $\Omega_{3,0}$, $\Omega_{0,1}$, $\Omega_{1,1}$ and $\Omega_{2,1}$ decrease and $\Omega_{4,0}$, $\Omega_{5,0}$ and $\Omega_{6,0}$ increase. For $F-S$ plate, $\Omega_{0,1}$ and $\Omega_{3,0}$ first increase and then decrease whereas $\Omega_{1,0}$, $\Omega_{2,0}$, $\Omega_{1,1}$ and $\Omega_{2,1}$ decrease and $\Omega_{0,0}$, $\Omega_{4,0}$, $\Omega_{5,0}$ and $\Omega_{6,0}$ increase.

Three Dimensional mode shapes for CF and FC boundary conditions are given in Figures 1-2 when when $\alpha = \beta = 0.4$, $e_r = g_r = 5.0$, $R_0 = 0.5$, $K_f = 500$.

3. CONCLUSION

It has been shown that frequencies and three dimensional mode shapes of orthotropic annular circular plate when one of the boundaries is free, with varying thickness resting and on elastic foundation can be solved efficiently by using boundary characteristic orthonormal polynomials in the Rayleigh-Ritz method. It reduces the problem into standard eigenvalue problem. Accuracy of the results can be increased by increasing the order of approximation. But the order of approximation cannot be increased arbitrarily because after a certain order the results start diverging due to accumulation of rounding off errors. This study may help the design engineers before finalizing a design in different engineering applications like aeronautical space shuttle components, machine parts, civil structures, etc. by suitably choosing the values of thickness taper parameters.

REFERENCES

[1] Leissa, A. W., “Vibration of plates. Washington: Office of Technology Utilization”, NASA. Spec. 1969, Rept. No. SP-160.

[2] Leissa, A.W., “Recent research in plate vibrations: classical theory”, The Shock Vib. Dig. 9(10), 1977, pp.13-24.

[3] Leissa, A. W., “Recent research in plate vibrations, 1973-1976: Complicating effects”, The Shock Vib. Dig. 9(11), 1977, pp. 21-35.

[4] Leissa, A. W., “Plate vibration research, 1976-1980: Classical theory”, The Shock Vib. Dig., 3(9), 1981, pp.11-22.

[5] Leissa, A. W., “Plate vibration research, 1976-1980: Complicating effects”, The Shock Vib. Dig., 13(10), 1981, pp.19-36.

[6] Leissa, A. W., “Recent studies in plate vibrations: 1981-1985 part I, Classical theory”, The Shock Vib. Dig., 19(2), 1987, pp. 11-18.

[7] Leissa, A. W., “Recent studies in plate vibrations: 1981-1985 part II, Complicated effects”, The Shock Vib. Dig. 19(3), 1987, pp.10-24.

[8] Vijaykumar, K., Ramaiah, G. K., “On the use of a coordinate transformation for analysis of axisymmetric vibration of polar orthotropic annular plates”, Journal of Sound and Vibration 24, 1972, pp.165-175.
[9] Ramiaah, G. K., Vijayakumar, K., “Natural frequencies of polar orthotropic annular plates”, Journal of Sound and Vibration 26, 1973, pp. 517-531.

[10] Narita, Y., “Natural frequencies of completely free annular and circular plates having polar orthotropy”, Journal of Sound and Vibration 92, 1984, pp. 33-38.

[11] Narita, Y., “Natural frequencies of completely free annular and circular plates having polar orthotropy”, Journal of Sound and Vibration 93, 1984, pp. 503-511.

[12] Gutierrez, R. H., Laura, P. A. A., Felix, D., Pistonesi, C., “Fundamental frequency of transverse vibration of circular, annular plates of polar orthotropy”, Journal of Sound and Vibration 230(5), 2000, pp. 1191-1195.

[13] Greenberg, J. B., Stavsky, Y., “Flexural Vibrations of certain full and annular composite orthotropic plates”, Journal of the Acoustical Society of America 66, 1979, pp. 501-508.

[14] Ginesu, F., Picasso, B., Priolo, P., “Vibration analysis of polar annular discs”, Journal of Sound and Vibration 65, 1979, pp. 97-105.

[15] Gorman, D. G., “Frequencies of polar orthotropic uniform annular plates”, Journal of Sound and Vibration 80, 1982, pp. 145-154.

[16] Kim, C. S., Dickinson, S. M., “On the lateral vibration of thin annular and circular composite plates subject to certain complicating effects”, Journal of Sound and Vibration 130(3), 1989, pp. 363-377.

[17] Laura, P. A. A., Gutierrez, R. H., Rossi, R. E., “Vibrations of circular annular plates of cylindrical anisotropy and non-uniform thickness”, Journal of Sound and Vibration 231(1), 2000, pp. 246-252.

[18] Chen, D. Y., “Axisymmetric vibration of circular and annular plates with arbitrary varying thickness”, Journal of Sound and Vibration, 206(1), 1997, pp. 114-121.

[19] N. Bhardwaj, A. P., Gupta and K. K. Choong, “Asymmetric vibration of polar orthotropic annular circular plates of quadratically varying thickness with same boundary conditions”, Shock and Vibration 15, 2008, pp. 599-617.

[20] Chihara, T. S., “An Introduction to Orthogonal Polynomials”, New York: Gordan and Breach Science Publishers, 1978.
Table 1 : Convergence of $\Omega_{m,n}$ when $\alpha = \beta = 0.4$, $e_r = g_r = 5.0$, $R_0 = 0.5$, $K_f = 500$

| Edge conditions (inner, outer) | $\Omega_{0,0}$ | $\Omega_{1,0}$ | $\Omega_{2,0}$ | $\Omega_{3,0}$ | $\Omega_{4,0}$ | $\Omega_{5,0}$ | $\Omega_{0,1}$ | $\Omega_{1,1}$ | $\Omega_{2,1}$ |
|-------------------------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $C-F$                         |                |                |                |                |                |                |                |                |                |
| 5                             | 26.122         | 27.122         | 31.050         | 40.011         | 55.348         | 77.092         | 79.424         | 89.337         | 104.38         | 114.42         |
| 6                             | 26.122         | 27.122         | 31.048         | 40.002         | 55.325         | 77.052         | 79.424         | 89.334         | 104.77         | 114.40         |
| 7                             | 26.122         | 27.122         | 31.047         | 40.000         | 55.320         | 77.044         | 79.424         | 89.334         | 104.76         | 114.40         |
| 8                             | 26.122         | 27.122         | 31.047         | 40.000         | 55.320         | 77.044         | 79.424         | 89.334         | 104.76         | 114.40         |
| $S-F$                         |                |                |                |                |                |                |                |                |                |
| 4                             | 23.725         | 24.630         | 28.444         | 37.660         | 23.498         | 60.194         | 71.909         | 75.728         | 99.841         | 103.86         |
| 5                             | 23.725         | 24.630         | 28.444         | 37.660         | 53.496         | 60.194         | 71.908         | 75.720         | 99.840         | 103.84         |
| 6                             | 23.725         | 24.630         | 28.444         | 37.660         | 53.496         | 60.194         | 71.908         | 75.719         | 99.840         | 103.83         |
| 7                             | 23.725         | 24.630         | 28.444         | 37.660         | 53.496         | 60.194         | 71.908         | 75.719         | 99.840         | 103.83         |
| $S-F$                         |                |                |                |                |                |                |                |                |                |
| 4                             | 30.980         | 41.688         | 61.097         | 85.916         | 86.360         | 104.16         | 117.13         | 141.73         | 155.19         | 184.76         |
| 5                             | 30.980         | 41.687         | 61.091         | 85.898         | 86.300         | 104.06         | 117.09         | 141.58         | 155.15         | 184.61         |
| 6                             | 30.980         | 41.686         | 61.089         | 85.890         | 86.298         | 104.06         | 117.07         | 141.58         | 155.08         | 184.59         |
| 7                             | 30.980         | 41.686         | 61.089         | 85.980         | 86.298         | 104.06         | 117.07         | 141.58         | 155.08         | 184.59         |
| $F-C$                         |                |                |                |                |                |                |                |                |                |
| 4                             | 26.620         | 35.133         | 52.671         | 64.449         | 76.765         | 83.251         | 107.55         | 120.31         | 145.14         | 162.45         |
| 5                             | 26.620         | 35.132         | 52.667         | 64.436         | 76.748         | 83.221         | 107.51         | 120.26         | 145.06         | 162.40         |
| 6                             | 26.620         | 35.132         | 52.666         | 64.435         | 76.744         | 83.219         | 107.49         | 120.25         | 145.02         | 162.38         |
| 7                             | 26.620         | 35.132         | 52.666         | 64.435         | 76.744         | 83.219         | 107.49         | 120.25         | 145.02         | 162.38         |
Table :2  Comparison of $\Omega_{m,n}$ for polar orthotropic annular plates of uniform thickness when $e_r = 5.0$, $g_r = 0.356$, $R_0 = 0.5$, $K_f = 0$

| Edge conditions (inner,outer) | Spresentce of results | $\Omega_{0,0}$ | $\Omega_{2,0}$ | $\Omega_{3,0}$ | $\Omega_{0,1}$ | $\Omega_{2,1}$ | $\Omega_{3,0}$ | $\Omega_{5,1}$ |
|------------------------------|-----------------------|----------------|---------------|---------------|---------------|---------------|---------------|---------------|
| F-C                          |                       |                |               |               |               |               |               |               |
| Gorman[15]                   | 11.30                 | 18.05          | 33.80         | 56.521        | 73.462        | 78.439        | 93.794        | 120.08        |
| Narita[10]                   | 11.30                 | 18.05          |               | 56.522        | 73.459        | 85.450        | 93.790        | 119.72        |
| Kim[16]                      | 11.305                | 18.045         | 33.795        | 56.522        | 73.459        | 85.451        | 93.790        | 119.72        |
| (Present)                    | 11.305                | 18.045         | 33.795        | 56.522        | 73.459        | 85.451        | 93.790        | 120.08        |
Table 3: Comparison of $\Omega_{m,n}$ with ref.[9] for polar orthotropic annular plates of uniform thickness when $K_f = 0, \nu_r = 0.3/e_r, g_r = 1/2(1+\nu_r)$

| Edge conditions \ (inner, outer) | $e_r$ | $R_0$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|----------------------------------|------|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|                                  |      | Ref.  |     |     |     |     |     |     |     |     |     |
| **F-S**                          |      |       |     |     |     |     |     |     |     |     |     |
| 0.50                             | [12] |       | 3.835 | 3.403 | 3.230 | 3.244 | 3.430 | 3.844 | 4.658 | 6.416 | 11.887 |
| Present                          |       | 3.6884 | 3.3692 | 3.2257 | 3.2436 | 3.4303 | 3.8443 | 4.6576 | 6.4162 | 11.887 |
| 0.75                             | [12] |       | 4.434 | 4.159 | 4.033 | 4.086 | 4.338 | 4.870 | 5.9060 | 8.139 | 15.082 |
| Present                          |       | 4.3585 | 4.1340 | 4.0292 | 4.0854 | 4.3377 | 4.8704 | 5.9060 | 8.1392 | 15.082 |
| 1.0                              | [12] |       | 4.898 | 4.737 | 4.668 | 4.765 | 5.077 | 5.711 | 6.931 | 9.555 | 17.709 |
| Present                          |       | 4.8533 | 4.7177 | 4.6641 | 4.7640 | 5.0769 | 5.7108 | 6.9310 | 9.5554 | 17.709 |
| 1.25                             | [12] |       | 5.286 | 5.213 | 5.201 | 5.344 | 5.714 | 6.439 | 7.821 | 10.787 | 19.993 |
| Present                          |       | 5.2572 | 5.1968 | 5.1972 | 5.3434 | 5.7139 | 6.4385 | 7.8209 | 10.786 | 19.993 |
| 1.50                             | [12] |       | 5.625 | 5.621 | 5.664 | 5.855 | 6.280 | 7.089 | 8.618 | 11.890 | 22.042 |
| Present                          |       | 5.6056 | 5.6075 | 5.6610 | 5.8541 | 6.2802 | 7.0886 | 8.6178 | 11.890 | 22.042 |
| **F-C**                          |      |       |     |     |     |     |     |     |     |     |     |
| 0.50                             | [12] |       | 9.083 | 9.167 | 10.333 | 12.697 | 16.966 | 25.052 | 42.619 | 92.591 | 359.969 |
| Present                          |       | 8.6657 | 9.1308 | 10.329 | 12.696 | 16.966 | 25.052 | 42.619 | 92.591 | 359.969 |
| 0.75                             | [12] |       | 9.732 | 9.858 | 10.909 | 13.164 | 17.347 | 25.366 | 42.882 | 92.813 | 360.160 |
| Present                          |       | 9.6026 | 9.8258 | 10.905 | 13.163 | 17.347 | 25.366 | 42.882 | 92.813 | 360.160 |
| 1.0                              | [12] |       | 10.244 | 10.437 | 11.428 | 13.603 | 17.715 | 25.674 | 43.142 | 93.035 | 360.350 |
| Present                          |       | 10.159 | 10.408 | 11.424 | 13.603 | 17.714 | 25.674 | 43.142 | 93.035 | 360.350 |
| 1.25                             | [12] |       | 10.577 | 10.936 | 11.901 | 14.019 | 18.070 | 25.977 | 43.400 | 93.256 | 360.541 |
| Present                          |       | 10.613 | 10.910 | 11.896 | 14.018 | 18.070 | 25.977 | 43.400 | 93.256 | 360.541 |
| 1.50                             | [12] |       | 11.044 | 11.375 | 12.334 | 14.412 | 18.415 | 26.275 | 43.656 | 93.477 | 360.731 |
| Present                          |       | 11.001 | 11.351 | 12.329 | 14.411 | 18.415 | 26.275 | 43.656 | 93.476 | 360.731 |
Fig. 1: First ten normal modes of vibration of $C-F$ plates when $\alpha = \beta = 0.4$, $e = g = 5.0$, $R_0 = 0.5$, $K_f = 500$. 
Fig. 2: First ten normal modes of vibration of $F$-$C$ plates when $\alpha = \beta = 0.4$, $\alpha = \beta = 0.4$, $R_0 = 0.5$, $K_f = 500$. 