Approximation properties for Baskakov-Kantorovich-Stancu type operators based on $q$–integers

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Abstract: In this paper, we give an interesting generalization of the Stancu type Baskakov-Kantorovich operators based on the $q$–integers and investigate their approximation properties. Also, we obtain the estimates for the rate of convergence for a sequence of them by the weighted modulus of smoothness.

Key words: $q$–integer; $q$–Baskakov-Kantorovich operators; Baskakov-Kantorovich-Stancu operators; Weighted spaces; Rate of convergence; Weighted modulus of smoothness.

2000 MSC: 41A10; 41A25; 41A36.

1 Introduction

In recent years, due to the intensive development of $q$– calculus, generalizations of some operators related to $q$– calculus have emerged (see [2, 5, 6, 10–18]). Aral and Gupta defined $q$– generalization of the Baskakov operator and investigated approximation properties of these operators in [2]. In [12], Gupta and Radu introduced the Baskakov- Kantorovich operators based on $q$–integers and investigated their weighted statistical approximation properties. They also proved some direct estimations for error using weighted modulus of smoothness in case $0 < q < 1$. In recent study Büyükyazıcı and Atakut [5] introduced a new Stancu type generalization of $q$– Baskakov operator is defined as

\[ L_{n}^{\alpha,\beta}(f;q,x) = \sum_{k=0}^{\infty} q^{-\frac{k(k-1)}{2}} \frac{D_{q}^{k}(\varphi_{n}(x))}{[k]_{q}!} (-x)^{k} f\left(\frac{1}{q^{k-1}} \frac{[k]_{q} + q^{k-1}\alpha}{[n]_{q} + \beta}\right) \]

where $0 \leq \alpha \leq \beta$, $q \in (0,1)$, $f \in C[0,\infty)$ and the following conditions are provided:

Let $\{\varphi_{n}\}$ ($n = 1, 2, \ldots$) $\varphi_{n} : \mathbb{R} \rightarrow \mathbb{R}$ be a sequence which is satisfying following conditions,
(i) $\varphi_n$ $(n = 1, 2, ...)$, $k$-times continuously $q-$ differentiable any closed interval $[0, b]$,
(ii) $\varphi_n(0) = 1$, $(n = 1, 2, ...)$,
(iii) for all $x \in [0, b]$, and $(k = 0, 1, 2, ..., n = 1, 2, ...)$, $(-1)^k D_q^k (\varphi_n(x)) \geq 0$,
(iv) there exists a positive integer $m(n)$, such that
\[
D_q^k (\varphi_n(x)) = -[n]_q D_q^{k-1} \varphi_m(n)(x)(1 + \alpha_{k,n}(x)),
\]
$k = 0, 1, ..., n = 1, 2, ...$) and $x \in [0, b]$ where $\alpha_{k,n}(x)$ converges to zero for $n \to \infty$ uniformly in $k$.
(v) $\lim_{n \to \infty} [n]_q = 1$.

Now, to explain the construction of the new $q-$ operators, we mention some basic definitions of $q-$ calculus and Lemma.

Let $q > 0$. For each nonnegative integer $n$, we define the $q-$ integer $[n]_q$ as
\[
[n]_q = \begin{cases} 
\frac{(1-q^n)}{(1-q)} & \text{if } q \neq 1 \\
n & \text{if } q = 1
\end{cases}
\]
and the $q-$ factorial $[n]_q!$ as
\[
[n]_q! = \begin{cases} 
[n]_q[n-1]_q \cdots [1]_q & \text{if } n \geq 1 \\
1 & \text{if } n = 0
\end{cases}
\]

For the integers $n$ and $k$, with $0 \leq k \leq n$, the $q-$ binomial coefficients are then defined as follows (see [14]):
\[
\begin{align*}
\left[ \begin{array}{c} n \\ k \end{array} \right]_q &= \frac{[n]_q!}{[k]_q![n-k]_q!}. \\
\end{align*}
\]
Note that the following relation is satisfied
\[
[n]_q = [n-1]_q + q^{n-1}.
\]

Definition 1 The $q-$ derivative of a function $f$ with respect to $x$ is
\[
D_q (f(x)) = \frac{f(qx) - f(x)}{qx - x}
\]
which is also known as the Jackson derivative. High $q-$ derivatives are
\[
D_q^0 (f(x)) = f(x) , \quad D_q^n (f(x)) = D_q (D_q^{n-1} (f(x))) , \quad n = 1, 2, 3, ...
\]
Note that as $q \to 1$, the $q-$ derivative approach the usual derivative.

Definition 2 The $q-$ integration is defined as
\[
\int_0^a f(t) d_q t = (1-q)a \sum_{j=0}^{\infty} f(q^j a)q^j , \quad a > 0.
\]
Over a general interval \([a, b]\), \(0 < a < b\), one defines
\[
\int_{a}^{b} f(t)q_t \, dt = \int_{a}^{b} f(t)q_t \, dt - \int_{0}^{a} f(t)q_t \, dt.
\]

**Definition 3** Let \(f(x)\) be a continuous function on some interval \([a, b]\) and \(c \in (a, b)\). Jackson’s \(q\)-Taylor formula (see [13, 14]) is given by
\[
f(x) = \sum_{k=0}^{\infty} \frac{(D_q^k f)(c)}{[k]_q!} (x - c)^k_q
\]
where \((x - c)^k_q = \prod_{i=0}^{k-1} (x - cq^i)\).

First we need the following auxiliary result. Throughout the paper, we use \(e_i(t) := t^i\) for every integer \(i \geq 0\).

**Lemma 4** (from [5]) For \(L_{n}^{\alpha, \beta}(e_i(t); q, x)\), \(i = 0, 1, 2\) the following identities hold:
\[
L_{n}^{\alpha, \beta}(e_0; q, x) = 1, \tag{3}
\]
\[
L_{n}^{\alpha, \beta}(e_1; q, x) = \frac{[n]_q}{[n]_q + \beta} x (1 + \alpha_1,n(x)) + \frac{\alpha}{[n]_q + \beta}, \tag{4}
\]
\[
L_{n}^{\alpha, \beta}(e_2; q, x) = \frac{[n]_q [m(n)]_q}{q ([n]_q + \beta)^2} x^2 (1 + \alpha_{1,m(n)}(x))(1 + \alpha_{2,n}(x)) \tag{5}
\]
\[
+ \frac{[n]_q (2\alpha + 1)}{([n]_q + \beta)^2} x (1 + \alpha_{1,n}(x)) + \frac{\alpha^2}{([n]_q + \beta)^2}.
\]

## 2 Some properties of Stancu type \(q\)-Baskakov-Kantorovich operators

In addition to the above conditions \((i)-(v)\), \(\phi_n(x)\) and \(\alpha_{k,n}(x)\) are satisfied following condition:
\[
(vi) \quad \phi_n(x)(1 + \alpha_{0,n}(x)) \leq 1, \text{ for all } x \in [0, b], (n = 1, 2, ...).
\]

In this paper, under the conditions \((i)-(vi)\), we defined a new generalization of Stancu type \(q\)-Baskakov-Kantorovich operators as following
\[ L_n^{(\alpha, \beta)}(f; q, x) = \left( [n]_q + \beta \right) \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} D^k_q \left( \varphi_\alpha(x) \right) (-x)^k \int \frac{f \left( q^{-k+1}t \right) dt}{q^{\left( \frac{[k]_q + q^{-1} \alpha}{[n]_q + \beta} \right)}} , \]

where \( x \in \mathbb{R}_+ \) , \( n \in \mathbb{N} \), \( 0 \leq \alpha \leq \beta \).

Note that, when \( q = 1 \), the operators given by (6) is reduced to the Kantorovich-Baskakov-Stancu type operators (see [3]) and if we choose \( q = 1 \), \( \varphi_\alpha(x) = (1+x)^{-n} \) and \( \alpha = \beta = 0 \), we obtain Baskakov-Kantorovich operators (see [1]).

In each of the following theorems, we assume that \( q = q_n \), where \( \{q_n\} \) is a sequence of real numbers such that \( 0 < q_n < 1 \) for all \( n \) and \( \lim_{n \to \infty} q_n = 1 \).

Now we give the following Lemmas, which are necessary to prove our theorems:

**Lemma 5** The following relations are satisfied:

\[ \int \frac{t \left( \frac{k}{[n]_q + \beta} \right)^{ \frac{k+k^{+1} \alpha}{[n]_q + \beta} } }{ q^{\left( \frac{k}{[n]_q + \beta} \right)^{ \frac{k+k^{+1} \alpha}{[n]_q + \beta} } } } dt = \frac{1}{[n]_q + \beta} \], \hspace{0.5cm} (7)

\[ \int \frac{tdt}{q^{\left( \frac{k}{[n]_q + \beta} \right)^{ \frac{k+k^{+1} \alpha}{[n]_q + \beta} } } } = \frac{[2]_q [k]_q + q^k (1+2\alpha)}{[2]_q \left( [n]_q + \beta \right)^2} \], \hspace{0.5cm} (8)

\[ \int \frac{t^2 dt}{q^{\left( \frac{k}{[n]_q + \beta} \right)^{ \frac{k+k^{+1} \alpha}{[n]_q + \beta} } } } = \frac{[3]_q [k]_q^2 + q^k [k]_q \left( (1+3\alpha) [2]_q + 1 \right) + (1+3\alpha+3\alpha^2) q^{2k}}{[3]_q \left( [n]_q + \beta \right)^3} \]. \hspace{0.5cm} (9)

**Proof.** From properties of \( q \)-analogue integration, by simple computation we obtain (7 - 9). \( \blacksquare \)

By the following Lemma Korovkin’s conditions are satisfied.

**Lemma 6** For all \( x \in \mathbb{R}_+ \), \( n \in \mathbb{N} \), \( \alpha, \beta \geq 0 \) and \( 0 < q < 1 \), we have

\[ L_n^{(\alpha, \beta)}(e_0; q, x) = 1, \] \hspace{0.5cm} (10)
\[ L_n^{(\alpha, \beta)}(e_1; q, x) = \frac{[n]_q}{[n]_q + \beta} x (1 + \alpha_1(x)) + \frac{q(1 + 2\alpha)}{[2]_q ([n]_q + \beta)}, \quad (11) \]

\[ L_n^{(\alpha, \beta)}(e_2; q, x) = \frac{[n]_q [m(n)]_q}{q ([n]_q + \beta)^2} (1 + \alpha_{1,m(n)}(x)) (1 + \alpha_{2,n}(x))^2 \]

\[ + \frac{[n]_q [3]_q + q (1 + 3\alpha) [2]_q + 1}{[3]_q ([n]_q + \beta)} (1 + \alpha_{1,n}(x)) x + q^2 (1 + 3\alpha + 3\alpha^2). \quad (12) \]

**Proof.** From definition (6) and the identities (3) and (7), we can easily obtain

\[ L_n^{(\alpha, \beta)}(e_0; q, x) = \left( [n]_q + \beta \right) \sum_{k=0}^{\infty} q^{k(k-1)} \frac{D^k_q (\varphi_n(x))}{[k]_q!} (x)^k \int_q^{(k+1)q + \alpha}_q d_q t \]

\[ = L_n^{\alpha,\beta}(e_0; q, x) = 1. \]

Now for \( e_1 \), from (3), (4) and (8) we can write

\[ L_n^{(\alpha, \beta)}(e_1; q, x) = \left( [n]_q + \beta \right) \sum_{k=0}^{\infty} q^{k(k-1)} \frac{D^k_q (\varphi_n(x))}{[k]_q!} (x)^k \int_q^{(k+1)q + \alpha}_q d_q t \]

\[ = \sum_{k=0}^{\infty} q^{k(k-1)} \frac{D^k_q (\varphi_n(x))}{[k]_q!} \sum_{k=0}^{\infty} q^{k(k-1)} \frac{[k]_q + q^{k-1} \alpha}{[n]_q + \beta} \]

\[ - \sum_{k=0}^{\infty} q^{k(k-1)} \frac{D^k_q (\varphi_n(x))}{[k]_q!} q^{k-1} \frac{[k]_q + q^{k-1} \alpha}{[n]_q + \beta} \]

\[ + q(1 + 2\alpha) \sum_{k=0}^{\infty} q^{k(k-1)} \frac{D^k_q (\varphi_n(x))}{[k]_q!} (x)^k \]

\[ = L_n^{\alpha,\beta}(e_1; q, x) - \frac{\alpha}{[n]_q + \beta} L_n^{\alpha,\beta}(e_0; q, x) + \frac{q(1 + 2\alpha)}{[2]_q ([n]_q + \beta)} L_n^{\alpha,\beta}(e_0; q, x) \]

\[ = \frac{[n]_q}{[n]_q + \beta} x (1 + \alpha_1(x)) + \frac{q(1 + 2\alpha)}{[2]_q ([n]_q + \beta)}. \]
The finally, for $e_2$, we use (3), (4), (5) and (9), one has

$$L_n(\alpha, \beta; x) = (\beta + \alpha) \sum_{k=0}^{\infty} \frac{D_k(\alpha, \beta)}{k!} \left( \frac{q, x}{1 - q, x} \right)^k.$$
This completes the proof of Lemma 6. ■

Using above Lemma, we can obtain following theorem.

**Theorem 7** Let \( f \in C[0, b] \), then
\[
\lim_{n \to \infty} L_n^{(\alpha, \beta)}(f; q, x) = f(x)
\]
uniformly on \([0, b]\).

### 3 Rate of convergence

\( B_{\rho_\gamma}(\mathbb{R}_+) \), the weighted space of real valued functions \( f \) defined on \( \mathbb{R}_+ \) with the property \(|f(x)| \leq M_f \rho_\gamma(x) \) where \( \rho_\gamma(x) = 1 + x^{\gamma+2} \) and \( M_f \) is constant depending on the function \( f \). We also consider the weighted subspace \( C_{\rho_\gamma}(\mathbb{R}_+) \) of \( B_{\rho_\gamma}(\mathbb{R}_+) \) given by
\[
C_{\rho_\gamma}(\mathbb{R}_+) := \{ f \in B_{\rho_\gamma}(\mathbb{R}_+) : f \text{ continuous on } \mathbb{R}_+ \}.
\]

The norm in \( B_{\rho_\gamma} \) is defined as
\[
\|f\|_{\rho_\gamma} = \sup_{x \in \mathbb{R}_+} \frac{|f(x)|}{\rho_\gamma(x)}.
\]

We can give some estimations of the errors \(|L_n^{(\alpha, \beta)}(f; q, x) - f(x)|, n \in \mathbb{N}, \) for unbounded functions by using a weighted modulus of smoothness associated to the space \( B_{\rho_\gamma}(\mathbb{R}_+) \).

We consider
\[
\Omega_{\rho_\gamma}(f; \delta) = \sup_{x \geq 0} \sup_{0 < h \leq \delta} \frac{|f(x + h) - f(x)|}{1 + (x + h)^{2+\gamma}}, \quad \delta > 0, \quad \gamma \geq 0. \tag{13}
\]

It is evident that for each \( f \in B_{\rho_\gamma}(\mathbb{R}_+) \), \( \Omega_{\rho_\gamma}(f; \cdot) \) is well defined and
\[
\Omega_{\rho_\gamma}(f; \delta) \leq 2 \|f\|_{\rho_\gamma}.
\]

The weighted modulus of smoothness \( \Omega_{\rho_\gamma}(f; \cdot) \) possesses the following properties.
\[
\Omega_{\rho_\gamma}(f; \lambda \delta) \leq (\lambda + 1) \Omega_{\rho_\gamma}(f; \delta), \quad \delta > 0, \quad \lambda > 0, \tag{14}
\]
\[
\Omega_{\rho_\gamma}(f; n \delta) \leq n \Omega_{\rho_\gamma}(f; \delta), \quad n \in \mathbb{N},
\]
\[
\lim_{\delta \to 0^+} \Omega_{\rho_\gamma}(f; \delta) = 0.
\]

As it is known, weighted Korovkin type theorems have been proven by Gadjiev (see [9]).
\textbf{Theorem 8} Let $q \in (0, 1)$ and $\gamma \geq 0$. For all non-decreasing $f \in B_{\rho_1}(\mathbb{R}_+)$ we have

\[ |L_n^{(\alpha, \beta)}(f; q, x) - f(x)| \leq \sqrt{L_n^{(\alpha, \beta)}(\mu_{x, \gamma}; x)} \left( 1 + \frac{1}{\delta} \sqrt{L_n^{(\alpha, \beta)}(\psi_x^2; x)} \right) \Omega_{\rho_1}(f; \delta), \]

$x \geq 0$, $\delta > 0$, $n \in N$, where $\mu_{x, \gamma}(t) := 1 + (x + |t - x|)^{2+\gamma}$, $\psi_x(t) := |t - x|$, $t \geq 0$.

\textbf{Proof.} Let $n \in N$ and $f \in B_{\rho_1}(\mathbb{R}_+)$. From (13) and (14), we can write

\[ |f(t) - f(x)| \leq \left( 1 + (x + |t - x|)^{2+\gamma} \right) \left( 1 + \frac{1}{\delta} |t - x| \right) \Omega_{\rho_1}(f; \delta). \]

Consequently, the operators $L_n^{(\alpha, \beta)}$ can be expressed as follows

\[ L_n^{(\alpha, \beta)}(f; q, x) = \left( [n]_q + \beta \right) \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{D_q^k(\varphi_n(x))}{[k]_q!} (-x)^k q^{k-1} \int f(t) d_q t. \]

By using the Cauchy-Schwartz inequality and (15), we obtain

\[ \left| L_n^{(\alpha, \beta)}(f; q, x) - f(x) \right| \]

\[ \leq \left( [n]_q + \beta \right) \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{D_q^k(\varphi_n(x))}{[k]_q!} (-x)^k q^{k-1} \int \frac{\left( L_n^{(\alpha, \beta)}(\mu_{x, \gamma}; x) + \frac{1}{\delta} L_n^{(\alpha, \beta)}(\mu_{x, \gamma} \psi_x; x) \right) \Omega_{\rho_1}(f; \delta)}{\sqrt{L_n^{(\alpha, \beta)}(\psi_x^2; x)}} \Omega_{\rho_1}(f; \delta). \]

\[ \leq \sqrt{\frac{\left( L_n^{(\alpha, \beta)}(\mu_{x, \gamma}; x) + \frac{1}{\delta} L_n^{(\alpha, \beta)}(\mu_{x, \gamma} \psi_x; x) \right) \Omega_{\rho_1}(f; \delta)}}{\sqrt{L_n^{(\alpha, \beta)}(\psi_x^2; x)}} \Omega_{\rho_1}(f; \delta). \]

\[ \Box \]
Lemma 9 For \( m \in \mathbb{N} \) and \( q \in (0, 1) \) we have

\[
L_n^{(\alpha, \beta)}(e_m; q, x) \leq A_{m,q} (1 + x^m), \quad x \in \mathbb{R}_+, \; n \in \mathbb{N},
\]

where \( A_{m,q} \) is a positive constant depending only on \( m, \alpha \) and \( q \).

**Proof.** For \( k \in \mathbb{N} \) and \( 0 < q < 1 \) the following inequality holds true

\[
1 \leq [k + 1]^q \leq 2 [k]^q.
\] (16)

Thus, for \( m \in \mathbb{N} \), from (1) and (16) we get

\[
L_n^{\alpha, \beta}(e_m; q, x) = \sum_{k=0}^{\infty} q \frac{k!}{[k]^q} \left( \frac{\varphi_n(x)}{[k]^q} \right) (-x)^k \frac{1}{q^{km-m}} \left( \frac{[k] + q^{k-1} \alpha}{[n]^q + \beta} \right)^m
\]

\[
= x \frac{[n]^q}{[n]^q + \beta} \sum_{k=0}^{\infty} q \frac{k!}{[k]^q} \left( \frac{\varphi_m(x)}{[k]^q} \right) (1 + \alpha_k n(x)) (-x)^k \frac{1}{q^{k(m-1)}} \left( \frac{[k] + q^{k-1} \alpha}{[n]^q + \beta} \right)^{m-1}
\]

\[
\leq x \frac{[n]^q}{[n]^q + \beta} \sum_{k=0}^{\infty} q \frac{k!}{[k]^q} \left( \frac{\varphi_m(x)}{[k]^q} \right) (1 + \alpha_k n(x)) (-x)^k \frac{1}{q^{k}} \left( \frac{[k] + q^k \alpha}{[n]^q + \beta} \right)^{m-1}
\]

\[
= x \frac{[n]^q}{[n]^q + \beta} \varphi_m(x) (1 + \alpha n(x)) \left( \frac{1 + \alpha}{[n]^q + \beta} \right)^{m-1}
\]

\[
+ x \frac{[n]^q}{[n]^q + \beta} \left( \frac{2}{q} \right)^m L_n^{\alpha, \beta}(e_m; q, x) + \frac{\alpha}{[n]^q + \beta} L_n^{\alpha, \beta}(e_m; q, x)
\]

\[
\leq x + \left( \frac{2}{q} \right)^m \left( \frac{1}{[n]^q + \beta} \right)^{m-1} L_n^{\alpha, \beta}(e_m; q, x)
\]

\[
\leq 2m \left( \frac{2}{q} \right)^m \left( \frac{1}{[n]^q + \beta} \right)^{m-1} \left( [n]^q (1 + x^m) + \alpha^{m-1} \right).
\]

Based on the above inequality and by using the mathematical induction over \( m \in \mathbb{N} \), we obtain

\[
L_n^{\alpha, \beta}(e_m; q, x) \leq B_{m,q} (1 + x^m),
\]

\( x \in \mathbb{R}_+, \; n \in \mathbb{N}, \) where

\[
B_{m,q} := 2m \left( \frac{2}{q} \right)^{m(m-1)} \left( 1 + \alpha^{m-1} \right).
\] (17)
On the other hand,

\[
L_n^{(\alpha, \beta)}(e_m; q, x) = \left( [n]_q + \beta \right) \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{D_q^k (\varphi_n(x))}{[k]_q!} (-x)^k \int_{q^{[k+1]_q + k\alpha}/[m+1]_q} e_m (q^{-k+1} t) \, dq \, dt
\]

\[
= \frac{[n]_q + \beta}{(m+1)[n]_q + \beta} \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{D_q^k (\varphi_n(x))}{[k]_q!} (-x)^k q^{-km+m} \left\{ (k+1)_q + q^k \alpha \right\}^{m+1} - q^{m+1} \left\{ [k]_q + q^{k-1} \alpha \right\}^{m+1}
\]

Since

\[
\left( (k+1)_q + q^k \alpha \right)^{m+1} - q^{m+1} \left( [k]_q + q^{k-1} \alpha \right)^{m+1}
\]

\[
= \left( (k+1)_q + q^k \alpha \right)^m + q \left( (k+1)_q + q^k \alpha \right)^{m-1} \left( [k]_q + q^{k-1} \alpha \right) + ... + q^m \left( [k]_q + q^{k-1} \alpha \right)^m
\]

\[
\leq (m+1) \left( [k]_q + q^k \alpha \right)^m
\]

\[
\leq (m+1) 2^m \left( [k]_q + q^k \alpha \right)^m, \quad k \in \mathbb{N},
\]

from condition (vi), we can write

\[
L_n^{(\alpha, \beta)}(e_m; q, x) \leq \frac{\varphi_n(x)(m+1)(1+\alpha)^m q^m}{([n]_q + \beta)^m [m+1]_q} + \frac{2^m(m+1)}{[m+1]_q} L_n^{(\alpha, \beta)}(e_m; q, x)
\]

\[
\leq A_{m,q} (1 + x^m),
\]

where \( A_{m,q} := \frac{(m+1)(1+\alpha)^m q^m}{[m+1]_q} + \frac{2^m(m+1)}{[m+1]_q} \) \( B_{m,q} \) and \( B_{m,q} \) is given by (17).

**Remark 10** Since any linear positive operator is monotone, from Lemma 9 we can easily see that \( L_n^{(\alpha, \beta)}(f; q, .) \in B_{\gamma, \cdot}(\mathbb{R}^+) \) for each \( f \in B_{\gamma, \cdot}(\mathbb{R}^+) \), \( \gamma \in \mathbb{N}_0 \).

**Theorem 11** Let \( f \in B_{\gamma, \cdot}(\mathbb{R}^+) \) be a non-decreasing function, then

\[
\left\| L_n^{(\alpha, \beta)}(f; q, .) - f \right\|_{\gamma, \cdot} \leq K_{\gamma, q_0} \Omega_{\gamma, \cdot}(f; \delta_n),
\]

where \( \delta_n := \frac{[n]_q + \gamma + 1}{q_0 ([n]_q + \gamma) \delta} \) and \( K_{\gamma, q_0} \) is a positive constant independent on \( f \) and \( n \).
Proof. The identities (3)-(5) imply
\[
L_{n}^{*(\alpha,\beta)}(\psi_{n}^{2}; q_{n}, x) = L_{n}^{*(\alpha,\beta)}((t - x)^{2}; q_{n}, x) \\
= \frac{[n]_{q_{n}}[m(n)]_{q_{n}}}{q_{n}} (1 + \alpha_{m(n)}(x)) (1 + \alpha_{2,n}(x)) x^{2} \\
+ \frac{[n]_{q_{n}}[3]_{q_{n}} + q_{n} \left( 1 + 3\alpha \right) [2]_{q_{n}} + 1)}{[n]_{q_{n}} + \beta}^{2} (1 + \alpha_{1,n}(x)) x \\
+ \frac{q_{n}^{2}(1 + 3\alpha + 3\alpha^{2})}{[3]_{q_{n}} \left( [n]_{q_{n}} + \beta \right)^{2}} - 2x \left\{ \frac{[n]_{q_{n}}}{[n]_{q_{n}} + \beta} x (1 + \alpha_{1,n}(x)) + \frac{q_{n}(1 + 2\alpha)}{[n]_{q_{n}} + \beta} \right\} + x^{2} \\
\leq \frac{[n]_{q_{n}} \eta_{n}(x) + 1 + \beta}{q_{n}} \left\{ \frac{[n]_{q_{n}}}{[n]_{q_{n}} + \beta} \right\} x^{2} + \frac{2(3\alpha + 3)}{q_{n}} \left\{ \frac{[n]_{q_{n}}}{[n]_{q_{n}} + \beta} \right\} x + \frac{1 + 3\alpha + 3\alpha^{2}}{q_{n}} \left\{ \frac{[n]_{q_{n}}}{[n]_{q_{n}} + \beta} \right\} \\
\leq \frac{9(1 + \beta)^{2}\rho_{0}(x)}{[n]_{q_{n}} + \beta} \left\{ \frac{[n]_{q_{n}}}{[n]_{q_{n}} + \beta} \right\} \left\{ \frac{[n]_{q_{n}} \eta_{n}(x) + 1}{[n]_{q_{n}} + \beta} \right\}
\]
where \( \eta_{n}(x) := \max \left\{ \alpha_{1,n}(x), \alpha_{m(n)}(x), \alpha_{2,n}(x) \right\} \).
Since \( \eta_{n}(x) \) converges uniformly to zero, we have \( \eta_{n} = \sup \eta_{n}(x) \) such that \( \eta_{n} \)
converges to zero as \( n \to \infty \). Let \( \gamma \in \mathbb{N}_{0} \) and \( f \in \mathcal{B}_{\rho_{\gamma}}(\mathbb{R}_{+}) \) be a fixed function.
From Theorem 8 and above inequality, we can write
\[
\left\| L_{n}^{*(\alpha,\beta)}(f; q_{n}, x) - f(x) \right\|_{\rho_{\gamma+1}(x)} \\
\leq \sqrt{\frac{L_{n}^{*(\alpha,\beta)}(\mu_{x,\gamma}^{2}; q_{n}, x)}{\rho_{\gamma+1}(x)}} \left( 1 + \frac{1}{\delta_{n}} \right) \sqrt{L_{n}^{*(\alpha,\beta)}(\psi_{n}^{2}; q_{n}, x)} \Omega_{\rho_{\gamma}}(f; \delta_{n}) \\
\leq \sqrt{\frac{L_{n}^{*(\alpha,\beta)}(\mu_{x,\gamma}^{2}; q_{n}, x)}{\rho_{\gamma+1}(x)}} \left( 1 + \frac{1}{\delta_{n}} \right) \sqrt{\frac{9(1 + \beta)^{2}\rho_{0}(x)}{[n]_{q_{n}} + \beta}} \left\{ \frac{[n]_{q_{n}} \eta_{n}(x) + 1}{[n]_{q_{n}} + \beta} \right\} \Omega_{\rho_{\gamma}}(f; \delta_{n}) \\
\leq \frac{12(1 + \beta)}{\rho_{2(\gamma+1)}(x)} \left( 1 + \frac{1}{\delta_{n}} \right) \frac{\left\{ \frac{[n]_{q_{n}} \eta_{n}(x) + 1}{[n]_{q_{n}} + \beta} \right\}}{\Omega_{\rho_{\gamma}}(f; \delta_{n})}.
\]
Since
\[
\mu_{x,\gamma}^{2}(t) = \left( 1 + (x + |t - x|)^{2+\gamma} \right)^{2} \leq 2 \left( 1 + (2x + t)^{4+2\gamma} \right) \leq 2 \left( 1 + 2^{4+2\gamma} (2x)^{4+2\gamma} + t^{4+2\gamma} \right),
\]
from Lemma 9, we get
\[
L_{n}^{*(\alpha,\beta)}(\mu_{x,\gamma}^{2}; q_{n}, x) \leq \lambda_{\gamma,q_{n}}^{2} \rho_{2(\gamma+1)}(x),
\]
where $\lambda^2_{\gamma,q_n} = 2^{5+2\gamma} \left(2^{4+2\gamma} + A_{4+2\gamma,q_n}\right)$. Choosing $\delta_n := \sqrt{\frac{\left\lfloor n\gamma/q_n \right\rfloor + 1}{n\gamma/q_n + \beta}}$ and $K_{\gamma,q_0} := 24(1+\beta)\lambda_{\gamma,q_0}$, where $q_0 := \min_{n \in \mathbb{N}} q_n$, the proof is finished. 

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