Virtual head waves in ocean ambient noise: Theory and modeling

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ABSTRACT:
The Green’s function retrieval in media with horizontal boundaries usually only considers the extraction of direct and reflected waves but ignores the virtual head waves, which have been observed experimentally from ocean ambient noise and used to invert for geometric and environmental parameters. This paper derives the extraction of virtual head waves from ocean ambient noise using a vertically spaced sensor pair in a Pekeris waveguide. Ocean ambient noise in the water column is a superposition of direct, reflected, and head waves. The virtual head waves are produced by the cross-correlations between head waves and either reflected waves or other head waves. The locations of sources that contribute to the virtual head waves are derived based on the method of stationary phase. It is the integration over time of contributions from these sources that makes the virtual head waves observable. The estimation of seabed sound speed with virtual head waves using a vertical line array is also demonstrated. The slope of the virtual head waves is different from that of direct and reflected waves in the virtual source gather; it is therefore possible to constructively stack the virtual head waves. The predictions are verified with simulations.

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I. INTRODUCTION

The Green’s function estimate between two receivers in both open and closed systems can be extracted by cross-correlating the field measured from sources that surround the receivers; this is referred to as Green’s function retrieval1,2 or seismic interferometry.3,4,6,32 Experimental evidence of this result has been presented in helioseismology,7 ultrasonics,8 underwater acoustics,9,10 and seismology.11

In an environment with boundaries, the emergence of a Green’s function estimate from correlation of ambient noise recordings was demonstrated theoretically. Lobkis and Weaver8 demonstrated this principle using a normal-mode formulation in a reverberant ultrasonic cavity. However, the equilibration of normal modes is not a necessary condition. By cross-correlating multiple scattered waves in a free-space medium with embedded scatterers uniformly distributed, Snieder12 proposed a stationary phase derivation to explain the extraction of the ballistic Green’s function in seismology. The method of stationary phase was often combined with ray theory to explain the extraction of Green’s function estimates in a medium with horizontal reflectors. For example, in a homogeneous medium with one horizontal reflector and without a free surface, Snieder et al.13 showed the correlation of waves recorded by two receivers correctly yielded the Green’s function (direct plus single-reflected wave) between the two receivers. In an ocean waveguide bounded above by a pressure release surface and below by a seabed with attenuation, Sabra et al.9 formulated the time domain Green’s function for time-averaged surface generated ambient noise cross-correlation, where the sources were modeled as point sources evenly distributed on a horizontal plane at a constant depth. Brooks and Gerstoft14 described the relation between the stacked cross-correlations from a line of vertical sources, located in the same vertical plane as two receivers, and the Green’s function between the receivers.

Recently, the virtual head waves have been observed from ocean surface generated noise both in simulation and experiment using vertical15,17 and horizontal15,18 arrays. The processing used is a generalization of the passive fathometer19–23 and produces cross-beam correlations,15–17 and it is equivalent to seismic interferometry techniques for delay and sum beamforming (but is not for adaptive beamforming).15 The virtual head waves have the same phase speed as the real, acoustic head waves,24–27 but the travel time is offset due to the cross-correlation method that the paths in common disappear and only the difference remains, thus the term virtual. The virtual head waves, also called spurious multiples,13,14,28,29 are non-physical energy in Green’s function estimates.30 However, they are useful since...
the travel time and angle of arrival of the virtual head waves can be used to invert for geometric and environmental properties.\textsuperscript{15,17}

Most research on Green’s function retrieval in media with horizontal boundaries only mentions the extraction of direct and reflected waves between receivers\textsuperscript{9,13,14} but not head waves or virtual head waves. The goal of this paper is to investigate the extraction of virtual head waves from ocean ambient noise in the simple case of a Pekeris waveguide with the method of stationary phase and ray theory. The differences between this work and literature mentioned above\textsuperscript{9,13,14} are as follows: (1) besides direct and reflected waves, head waves are also considered between the noise sources and receivers and (2) instead of two receivers that are arbitrarily positioned in the \(x-z\) plane, the focus is on two vertically placed receivers in the water column, and it is extended later to the case of a vertical line array. Under these conditions, nine terms can be obtained by cross-correlating ocean surface noise recorded at two receivers. The method of stationary phase is used to analyze each term, and it is proved that the virtual head waves are produced by three terms.

In the following, Sec. II proposes a simple model of the sea surface generated ambient noise and explains the extraction of virtual head waves for a vertically spaced sensor pair. Section III presents the details of seabed sound speed estimation using the extracted virtual head waves from a vertical line array. Section IV demonstrates the theory above with simulations. Section V contains the summary and conclusion.

II. VIRTUAL HEAD WAVES IN OCEAN AMBIENT NOISE

A. Ambient noise cross-correlation function

The model geometry is depicted in Fig. 1. The water column is bounded above by a pressure-release surface and below by a semi-infinite bottom layer. The density and sound speed of the water and the bottom are given by \(\rho_1, v_1\) and \(\rho_2, v_2\), respectively. Consider an infinite plane parallel to the surface and located below the surface at depth \(z_s\).\textsuperscript{31} In this plane, let each noise source strength be \(S(r_s, z_s)\), \(r_s = (r_s \cos \varphi_s, r_s \sin \varphi_s, z_s)\), and \(\varphi_s \in [0, 2\pi]\). The two receivers \(V_1\) and \(V_2\) are on the \(z\)-axis with coordinates \((r_1, z_1)\) and \((r_2, z_2)\), \(r_1 = r_2 = (0, 0), 0 < z_1 \leq z_2 < Z\). The field then becomes independent of azimuthal angle \(\varphi\). The pressure from noise source \(S\) to receiver \(V_1\) is

\[
P(V_1, z_1; \omega) = \int S(r_s, z_s; \omega)G(r_1, z_1, r_s, z_s; \omega)d^2r_s,
\]

where \(G(r_1, z_1, r_s, z_s; \omega)\) is the Green’s function between \(S\) and \(V_1\). In this section, frequency dependence \(\omega\) is suppressed since the derivation is in the frequency domain. Based on ray theory, the full Green’s function consists of three terms,

\[
G(r_1, z_1, r_s, z_s) = G_D(r_1, z_1, r_s, z_s) + G_R(r_1, z_1, r_s, z_s) + G_H(r_1, z_1, r_s, z_s),
\]

where \(G_D, G_R,\) and \(G_H\) are direct, reflected, and head waves between \(S\) and \(V_1\). Assuming that sources are close to the surface \((z_s \approx 0)\), there are only down-going waves from the source, but both down-going \((\theta \in (0, \pi/2))\) and up-going \((\theta \in [-\pi/2, 0])\) waves at receivers. The grazing angle \(\theta\) shown in Fig. 1 is down-going. Since \(G_D\) and \(G_H\) have up- and down-going contributions they can be expanded to

\[
G_D = G_{D\uparrow} + G_{D\downarrow}, G_R = G_{R\downarrow}, G_H = G_{H\downarrow} + G_{H\uparrow},
\]

the superscripts \(-\) and ++ represent up- and down-going waves, and

\[
G_{D\downarrow}(r_1, z_1, r_s, z_s) = AD(r_1, z_1)e^{-i\omega\sqrt{r_1^2 + (z_1 - z_s)^2}/v_1},
\]

\[
G_{R\downarrow}(r_1, z_1, r_s, z_s) = \sum_{m=1}^{\infty} A_{R\downarrow}(m, r_s, z_s, \gamma_{nb})e^{-i\omega\sqrt{r_1^2 + [mZ - z_s]^2}/v_1},
\]

\[
G_{H\downarrow}(r_1, z_1, r_s, z_s) = \sum_{m=0}^{\infty} A_{H\downarrow}(m, r_s, z_s, \gamma_{nb})e^{-i\omega(r_1/v_2 + [mZ - z_s] \sin \theta/v_1)},
\]

where \(AD\) is the amplitude of the direct wave down-going from \(S\) and down-going to \(V_1\), \(AR\downarrow\), and \(AH\downarrow\) are those of the reflected and head waves down-going from \(S\) and up- \((-\) or down-going \((+)\) to \(V_1\), \(z_{ls} = [z_1, z_s]\) is the depth of receiver and source, \(\gamma_{nb} = [Z, \rho_1, \rho_2, v_1, v_2]\) contains parameters for the water column and sea bottom, \(\theta = \arccos(v_1/v_2)\) is the critical grazing angle. Due to the ocean surface and bottom, the reflected and head waves can bounce several times in the waveguide thus have various ray paths from \(S\) to \(V_1\). Defining \(m \in \mathbb{N}\) as the bounce number from the bottom interface at \(z = Z\) that occur between \(S\) and \(V_1\), \(AR\downarrow\), and \(AH\downarrow\) are expressed as the superposition of wavefields from each ray path. For the reflected waves, \(m \in [1, \infty]\), while for the head waves, \(m \in [0, l_z]\), where \(m = 0\) corresponds to the case when the horizontal range between \(S\) and \(V_1\) is less than the critical offset defined as

\[
X_{\text{c}}(m) = (2mZ - z_s \pm z_1) \cot \theta,\textsuperscript{17}
\]

with \(AH\downarrow(m = 0) = 0\). This leads to the largest number of head waves bounces that are up- or down-going from \(S\) to \(V_1\) as

\[
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\]
Note, this paper is aimed at discussing the location of sources that contribute to virtual head waves, therefore the amplitudes are not expressed accurately. The expressions of \( A_D, A_R, \) and \( A_H \) can be found in other references.\(^9\,\)\(^{14}\,\)\(^{26}\) The cross-correlation between noise sources at receivers \( V_1 \) and \( V_2 \) is

\[
C(r_2, z_2, r_1, z_1) = P(r_1, z_1) P^\ast(r_2, z_2) \]

\[
= \int \int S(r_3, z_3) S^\ast(r_3', z_3' G(r_1, z_1, r_3, z_3') \times G^\ast(r_2, z_2, r_3', z_3') dr_3 dz_3 dr_3' dz_3',
\]

(4)

where Eq. (1) is used. Inserting Eq. (2) into Eq. (4),

\[
C = C_{DD} + C_{DR} + C_{DH} + C_{RD} + C_{RR} + C_{RH} + C_{HR} + C_{HH},
\]

(5)

where \( C_{ab} (a = \{ D, R, H \}, b = \{ D, R, H \}) \) is the cross-correlation between the three types of waves and

\[
C_{ab} = \int \int S(r_3, z_3) S^\ast(r_3', z_3') G_a(r_1, z_1, r_3, z_3) \times G^\ast_b(r_2, z_2, r_3', z_3') dr_3 dz_3 dr_3' dz_3'.
\]

(6)

For uncorrelated noise, assume that \( \langle S(r_3, z_3) S^\ast(r_3', z_3') \rangle = Q(r_3) \delta(r_3 - r_3') \), where \( \langle \cdot \rangle \) is the ensemble average, \( Q(r_3) \) the noise power spectrum density, and \( \delta \) the Dirac delta function. For uniformly distributed noise, \( Q(r_3) = \bar{Q} \) is a constant. In the high-frequency regime, the spatial integration in Eq. (6) over the distribution of noise sources can be estimated using a stationary phase approximation where the phase of \( C_{ab} \) is \( \phi_{ab} = \omega(t_3 - t_3') \). The spatial integration is estimated by finding the stationary phase points, where \( \phi_{ab} \) has vanishing derivatives, \( d\phi_{ab}/dr_3 = 0 \). Since the two receivers are on the z-axis, \( \phi_{ab} \in [0, 2\pi] \).

For \( a = H \) or \( b = H \), the horizontal range between the noise source \( S \) and receivers \( V_1 \) and \( V_2 \), \( |r_3 - r_1| = |r_3 - r_2| = r_3 \), should be greater than the critical offset \( X_0^\ast(m) = (2mZ - z_\pm z_\pm) \cot \theta_e \) or \( X_0^\ast(n) = (2nZ - z_\pm z_\pm) \cot \theta_e \),\(^{17}\) to produce head waves between them, where \( n \in \mathbb{N}^+ \) is the number of bounces from the bottom interface at depth \( Z \) that occur between \( S \) and \( V_2 \).

### B. Evaluation of cross-correlation functions

Equation (5) shows that, by decomposing the noise field on each receiver into direct, reflected, and head waves, nine terms are obtained after the cross-correlation processing. In fact, since the reflected and head waves have both up- and down-going propagation to the receiver \( (G_R = G_{R^+} + G_{R^-}, \ G_{HR} = G_{HR^+} + G_{HR^-}) \), the nine terms can be further separated into 25 terms. Specifically, the 25 terms are the result of expanding \( C_{DR}, C_{DH}, C_{RD}, \) and \( C_{HR} \), where each consists of two terms (e.g., \( C_{DR} = C_{DR^+} + C_{DR^-} \)), and expanding \( C_{RR} \).

The cross-correlation terms \( C_{DD}, C_{DR}, C_{RD}, \) and \( C_{RR} \) have been analyzed in an ocean waveguide for two arbitrarily spaced receivers in the \( x-z \) domain.

In the following, only the results of these four terms are shown, the analysis will be ignored. However, the cross-correlations between head waves and direct, reflected, and head waves, \( C_{DH}, C_{HR}, C_{HR}, \) and \( C_{HH} \), are discussed in detail in Secs. \( \text{II B 2} - \text{II B 5} \).

#### 1. \( C_{DD}, C_{DR}, C_{RD}, \) and \( C_{RR} \)

Based on Eqs. (3) and (5), the cross-correlations \( C_{DD}, C_{DR}, C_{RD}, \) and \( C_{RR} \) are expressed as

\[
C_{DD} = Q_{DD} e^{-i\omega(z_1 - z_2)/\tau},
\]

\[
C_{DR} = \sum \sum \sum a_{DR} \cdot a_{DR^\ast} \cdot e^{-i\omega(2nZ - z_\pm z_\pm)/\tau},
\]

\[
C_{RD} = \sum \sum \sum a_{RD} \cdot a_{RD^\ast} \cdot e^{-i\omega(2nZ - z_\pm z_\pm)/\tau},
\]

\[
C_{RR} = \sum \sum \sum a_{RR} \cdot a_{RR^\ast} \cdot e^{-i\omega(2nZ - z_\pm z_\pm)/\tau},
\]

(7)

where

\[
\begin{aligned}
A_{DD} &= A_D(z_1) A_D^\ast(z_2), \\
A_{DR} &= A_D(z_1) A_{R^+}(z_2), \\
A_{RD} &= A_R(z_1) A_{R^+}(z_2), \\
A_{RR} &= A_R(z_1) A_{R^+}(z_2),
\end{aligned}
\]

\[
A_{RR^\ast} = \begin{cases} 
\sum \sum a_{RR^\ast} \cdot (m, z_{15}, \gamma_{w\beta}) \cdot (m, z_{25}, \gamma_{w\beta}) \\
\sum \sum a_{RR^\ast} \cdot (m, z_{15}, \gamma_{w\beta}) \cdot (m, z_{25}, \gamma_{w\beta}) \
\end{cases}
\]

\[
= \begin{cases} 
\Delta_m \geq 0, \\
\Delta_m \leq -1,
\end{cases}
\]

\( \Delta_m = m - n \) is the difference of seabed bounces of rays from noise source to two receivers.

#### 2. \( C_{DH} \) and \( C_{HR} \)

Equation (6) represents the cross-correlation between direct and head waves when \( a = D, b = H \). Inserting \( G_D \) and \( G_{HR^-} \) in Eq. (3) into Eq. (6), one obtains
\[ C_{DH} = \sum_{n=0}^{\infty} C_{DH}^{n} \]
\[ = 2\pi Q \sum_{n=0}^{\infty} \sum_{m=1}^{l_n} \phi_{DH}^{(n,m,r_s)} e^{-i\phi_{DH}^{(n,m,r_s)}} dr_s, \]

where 2\pi is due to the integration of \( \phi_s \) over 0 to 2\pi, \( \phi_{DH}^{(n,m,r_s)} = A_D(r_s, \gamma_{n+1}) A_H^{(n,m,r_s)}, \)
\[ \phi_{DH}^{(n,m,r_s)} = \alpha \left( \sqrt{r_s^2 + (z_1 - z_s)^2}/v_1 - r_s/v_2 \right. \]
\[ - \left. (2nZ - z_s + z_2) \sin \theta_v / v_1 \right). \]

To find the location of stationary sources of \( C_{DH} \), let \( d\phi_{DH}^{(n,m,r_s)}/dr_s = 0 \), the solution is \( r_s = r_{DH} = (z_1 - z_s)\cot \theta_v \), where \( r_{DH} \) is the horizontal range of the stationary point. However, \( r_{DH} \) does not satisfy the condition for head waves as the horizontal distance between source and receiver \( |x_r - x_2| = r_{DH} \) is less than the critical range \( |X_0^\pm(n,\alpha)| \), thus there are no head waves between two receivers, \( C_{DH} = 0 \). Similarly, \( C_{IDH} = 0 \).

3. \( C_{RH} \)

Equation (6) represents the cross-correlation between reflected and head waves when \( a = R, b = H \). Inserting \( G_{R^H} \) and \( G_{H^H} \) in Eq. (3) into Eq. (6), we find
\[ C_{RH} = \sum_{n=0}^{\infty} C_{RH}^{n} \]
\[ = 2\pi Q \sum_{n=0}^{\infty} \sum_{m=1}^{l_n} \phi_{RH}^{(n,m,r_s)} e^{-i\phi_{RH}^{(n,m,r_s)}} dr_s, \]

where \( \phi_{RH}^{(n,m,r_s)} = A_R(m, r_s, z_1, z_2, \gamma_{n+1}) A_H^{(n,m,r_s)}, \)
\[ \phi_{RH}^{(n,m,r_s)} = \alpha \left( \sqrt{r_s^2 + (2mZ - z_s + z_1)^2}/v_1 \right) \]
\[ - r_s/v_2 - (2nZ - z_s + z_2) \sin \theta_v / v_1 \right). \]

Let \( d\phi_{RH}^{(n,m,r_s)}/dr_s = 0 \), the horizontal range for the stationary point is \( r_s = r_{RH}^{(m)} = X_0^\pm(n,\alpha) \). Here, \( C_{RH} \) satisfies \( |r_s - r_2| > X_0^\pm(n,\alpha) \) when \( \Delta_m \geq 0 \), while the condition is met for \( C_{RH} \) when \( \Delta_m \geq 1 \). Therefore, \( C_{RH} \) has contributions from discrete sources on concentric circles with coordinates \( S_{RH}^{(m)} = X_0^\pm(n,\alpha) \) \( \cos \varphi_s, \)
\[ r_{RH}^{(m)}(\varphi_s, z_s) = X_0^\pm(n,\alpha), \varphi_s \in [0, 2\pi] \]. In detail, we have
\[ r_{RH}^{(m)}(\varphi_s, z_s) = \begin{cases} X_0^+(n,\alpha), & m \geq 1, \\ X_0^-(n,\alpha), & m \geq 2. \end{cases} \]

The location of stationary sources at the ocean surface is depicted in Fig. 2(a). For each cross-correlation term \( C_{RH} \), only stationary sources at the closest circle \( S_{RH}^{(m=1)} \) and \( S_{RH}^{(m=2)} \) are shown in Fig. 2(a). The corresponding ray geometry for sources in panel (a) in the \( x-z \) domain are shown in panel (b). The expression of \( C_{RH} \) is simplified as
\[ C_{RH} \approx 2\pi Q \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \phi_{RH}^{(n,m,r_s)} e^{-i\phi_{RH}^{(n,m,r_s)}} dr_s, \]

where \( \kappa = 0 \) for \( \phi_{RH}^+, \kappa = 1 \) for \( \phi_{RH}^- \).
\[ \phi_{RH}^{(n,m,r_s)} = \sum_{m=1}^{\infty} A_R^{(n,m,r_s)} A_H^{(n,m,r_s)}, \]
\[ \times A_R^{(n,m,r_s)}(\Delta_m, r_{RH}^{(m)}, z_1, z_2, \gamma_{n+1}). \]

**FIG. 2.** (Color online) (a) The location of stationary sources \( S_{RH}^{(m=1)} \) and \( S_{RH}^{(m=2)} \) at the ocean surface (circle), the contents in the brackets are ignored. (b) The corresponding ray geometry of labeled sources in (a) in the \( x-z \) domain. After the cross-correlation processing, the common ray paths between \( S_{RH}^{(m=1)} \) and \( V_1 \) and \( V_2 \) disappear, only ray path I (in the water) and II (along the seabed) are left.
The expressions of \( \varphi_{H^+} (\Delta_m) \), \( \varphi_{H^-} (\Delta_m) \), and \( \varphi_{R^+} (\Delta_m) \) are similar to \( \varphi_{R^-} (\Delta_m) \), and not shown here. From Eq. (13), by cross-correlating up- and down-going reflected and head waves, one can obtain four types of virtual head waves with different amplitudes \( 2\pi Q_{R^+} (\Delta_m) \) and travel times,

\[
\begin{align*}
t_{R^+} (\Delta_m) &= (2\Delta_m Z \pm z_1 + z_2) \sin \theta_t / v_1, \quad \Delta_m \geq 0, \\
t_{R^-} (\Delta_m) &= (2\Delta_m Z \pm z_1 - z_2) \sin \theta_t / v_1, \quad \Delta_m \geq 1.
\end{align*}
\]

The virtual head wave travel times \( t_{R^+} (\Delta_m = 0) \) and \( t_{R^-} (\Delta_m = 1) \) are equal to the travel time difference of sound waves on path I and path II (\( t_{1} - t_{2} \)) in Fig. 2(b).

The virtual head waves have the same phase speed as the head waves, \( v_1 / \sin \theta_t \), but the travel time is offset due to the cross-correlation processing, thus the term virtual.\(^{17}\) From Eq. (15), the virtual head wave travel times are a function of \( \Delta_m \). Similarly, the stationary source ranges \( r_{R^+} (\Delta_m) \) are also dependent on \( \Delta_m \) by using \( m = \Delta_m + n \),

\[
\begin{align*}
r_{R^+} (\Delta_m, n) &= X_1^+ (\Delta_m + n), \quad \Delta_m \geq 0, \\
r_{R^-} (\Delta_m, n) &= X_1^- (\Delta_m + n), \quad \Delta_m \geq 1.
\end{align*}
\]

Therefore, the virtual head wave travel times \( t_{R^+} (\Delta_m) \) and stationary source ranges \( r_{R^+} (\Delta_m, n) \) are related through \( \Delta_m \) [see Fig. 3(a)] for \( Z = 150 \) m, \( z_1 = 0 \) m, \( z_2 = 20 \) m, \( z_1 = 86 \) m, \( v_1 = 1500 \) m/s, \( v_2 = 1600 \) m/s. As predicted by Eq. (16), the figure shows that \( \Delta_m < 0 \) does not exist for \( C_{R^+} \), and \( \Delta_m < 1 \) does not exist for \( C_{R^-} \). The virtual head waves produced by \( C_{R^+} \) at a fixed \( \Delta_m \) are contributed by multiple stationary sources \( r_{R^+} (\Delta_m = \text{constant}, n), \ n \in \mathbb{N}^+ \) simultaneously. For example, for \( C_{R^+} \), the virtual head waves at \( \Delta_m = 0 \) (0.015 s) have contributions from stationary sources at three discrete ranges 754, 1563, and 2371 m due to \( n = 1, 2, 3 \). Besides, for a fixed \( n \), the stationary source ranges \( r_{R^+} (\Delta_m, n = \text{constant}) \) increase with \( \Delta_m \) [for \( C_{R^+} \) \( (\Delta_m, n = 1) \), from 754 m at \( \Delta_m = 0 \) to 2371 m at \( \Delta_m = 2 \)].

4. \( C_{HR} \)

Equation (6) represents the cross-correlation between head waves and reflected waves when \( a = H, b = R \). As with \( C_{HR} \), the horizontal range of the stationary point is \( r_s = r_{HR^+} (n) = X_0^+ (n) \) by letting \( d \phi_{HR^+} (m, n, r) / dr_s = 0 \). For \( C_{HR^+} \), it satisfies \( |r_s - r_1| > X_0^+ (m) \) when \( \Delta_m \leq -1 \), while for \( C_{HR^-} \), the condition is met when \( \Delta_m \leq 0 \). Therefore, \( C_{HR^+} \) has contributions from sources on concentric circles with coordinates \( S_{HR^+} (n) = S_{HR^+} (r_{HR^+} (n) \cos \varphi_s, r_{HR^+} (n) \sin \varphi_s, z_1) \), \( \varphi_s \in [0, 2\pi] \). In detail, we have

\[
\begin{align*}
r_{HR^+} (n) &= X_0^+ (n), \quad n \geq 2, \\
r_{HR^-} (n) &= X_0^+ (n), \quad n \geq 1.
\end{align*}
\]

The distribution of stationary noise sources and the corresponding ray geometry are shown in Fig. 4. The expression of \( C_{HR} \) is

\[
C_{HR} \approx 2\pi Q \sum_{n = 1}^{\infty} \sum_{m = -\infty}^{\infty} \varphi_{HR^+} (\Delta_m) e^{-i\omega (2\Delta_m Z \pm z_1 \mp z_2) \sin \theta_t / v_1},
\]

where \( \kappa = -1 \) for \( C_{HR^+} \) and \( \kappa = 0 \) for \( C_{HR^-} \). Similar to \( C_{RH} \), there are four types of virtual head waves with amplitudes \( 2\pi Q_{2HR^+} (\Delta_m) \) and travel times,

\[
\begin{align*}
t_{HR^+} (\Delta_m) &= (2\Delta_m Z \pm z_1 \pm z_2) \sin \theta_t / v_1, \quad \Delta_m \leq -1, \\
t_{HR^-} (\Delta_m) &= (2\Delta_m Z \pm z_1 \mp z_2) \sin \theta_t / v_1, \quad \Delta_m \leq 0.
\end{align*}
\]

The virtual head wave travel times \( t_{HR^+} (\Delta_m = 1) \) and \( t_{HR^-} (\Delta_m = 0) \) are equal to the travel time difference of sound waves on path II and path I (\( t_1 - t_2 \)) in Fig. 4(b).

The expressions for \( t_{HR^+} \), not shown here, are similar to \( t_{R^+} \). The stationary source ranges \( r_{HR^+} (n) \) are dependent on \( \Delta_m \) by using \( n = m - \Delta_m \),

\[
\begin{align*}
r_{HR^+} (\Delta_m, m) &= X_0^+ (m - \Delta_m), \quad \Delta_m \leq -1, \\
r_{HR^-} (\Delta_m, m) &= X_0^+ (m - \Delta_m), \quad \Delta_m \leq 0.
\end{align*}
\]

Figure 3(b) shows the relation between virtual head wave travel times \( t_{HR^+} (\Delta_m) \) and stationary source ranges \( r_{HR^+} (\Delta_m, m) \). As predicted by Eq. (20), \( \Delta_m > -1 \) does not exist for \( C_{HR^+} \), and \( \Delta_m > 0 \) does not exist for \( C_{HR^+} \). The virtual head waves produced by \( C_{HR^+} \) at a fixed \( \Delta_m \) have contributions from multiple stationary sources \( r_{HR^+} (\Delta_m = \text{constant}, m), \ m \in \mathbb{N}^+ \) simultaneously. For example, for \( C_{HR^-} \), the virtual head waves at \( \Delta_m = -1 \) (–0.05 s) have contributions from stationary sources at two discrete ranges.
1385 and 2193 m due to $m = 1, 2$. Besides, for a fixed $m$, the stationary source ranges $r_{H^{+}R^{-}}(\Delta_m, m = \text{constant})$ increase with $|\Delta_m|$ [for $C_{H^{-}R^{-}}(\Delta_m, m = 1)$, from 1385 m at $\Delta_m = -1$ to 2193 m at $\Delta_m = -2$].

5. $C_{HH}$

Equation (6) represents the cross-correlation between head waves and head waves when $a = H$, $b = H$. Inserting $G_{HH}$ in Eqs. (3) into (6), one obtains

$$C_{HH} = \sum_{\pm x} C_{H^{+}H^{-}}$$

$$= 2\pi Q \sum_{\pm x} \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} z_{H^{+}H^{-}}(m, n, r_s) e^{-i\phi_{H^{+}H^{-}}(\Delta_m)} dr_s,$$

where $z_{H^{+}H^{-}}(m, n, r_s) = A_{H^{+}}(m, r_s, z_{1s}, \gamma_{\text{shb}}) A_{H^{-}}(n, r_s, z_{2s}, \gamma_{\text{shb}})$,

$$\phi_{H^{+}H^{-}}(\Delta_m) = \omega(2\Delta_m Z + z_1 + z_2) \sin \theta_v / v_1. \quad (22)$$

Therefore, $C_{HH}$ can be integrated without having to account for the phase term. To satisfy $r_s = r_{H^{+}H^{-}}(m) > X_1^+(m)$ and $r_s = r_{H^{+}H^{-}}(n) > X_2^+(n)$ at the same time, $C_{HH}$ has contributions from sources on several annulus $\{S_{H^{+}H^{-}}(m) \cup S_{H^{+}H^{-}}(n)\}$, where $S_{H^{+}H^{-}}(m) = S_{H^{+}H^{-}}(r_{H^{+}H^{-}}(m) \cos \varphi_s,$

$$r_{H^{+}H^{-}}(m) \sin \varphi_s, z_s), \quad S_{H^{+}H^{-}}(n) = S_{H^{+}H^{-}}(r_{H^{+}H^{-}}(n) \cos \varphi_s,$$

$$r_{H^{+}H^{-}}(n) \sin \varphi_s, z_s), \varphi_s \in [0, 2\pi]$. In detail, we have

$$r_{H^{+}H^{-}}(m) > X_1^+(m), \quad m \geq n \geq 1,$$

$$r_{H^{+}H^{-}}(n) > X_2^+(n), \quad n > m \geq 1,$$

$$r_{H^{+}H^{-}}(m) > X_1^+(n), \quad m > n \geq 1,$$

$$r_{H^{+}H^{-}}(n) > X_2^+(n), \quad n \geq m \geq 1. \quad (23)$$

Note, the sources in the inner diameter of the annuli, $X_1^+(m)$ or $X_2^+(n)$, do not contribute to $C_{HH}$. Besides, the outer diameter of each annulus is not infinite, but related to attenuation and not discussed here. The sources on the three annuli with smallest inner diameter $S_{H^{+}H^{-}}(m = 1)$, $S_{H^{+}H^{-}}(m = 1)$, and $S_{H^{+}H^{-}}(n = 1)$ are shown in Fig. 5(a), and the coordinates in the brackets are ignored. The corresponding ray geometry for panel (a) is shown in panel (b). Sources at the critical offsets $X_1^+(m = 1)$, $X_1^+(m = 1)$, $X_1^+(n = 1)$, and $X_2^+(n = 1)$ (hollow star) are shown in the top-to-bottom panels. The expression for $C_{HH}$ is

$$C_{HH} = 2\pi Q \sum_{\pm x} \sum_{\Delta_m=-\infty}^{\infty} z_{H^{+}H^{-}}(\Delta_m) e^{-i\omega(2\Delta_m Z + z_1 + z_2) \sin \theta_v / v_1}, \quad (24)$$

where

$$z_{H^{+}H^{-}}(\Delta_m) = \begin{cases} \sum_{m=-\Delta_m+1}^{\infty} A_{H^{+}}(m, r_s, z_{1s}, \gamma_{\text{shb}}) A_{H^{-}}(m - \Delta_m, r_s, z_{2s}, \gamma_{\text{shb}}) dr_s, & \Delta_m \geq 0, \\ \sum_{m=1}^{\infty} A_{H^{-}}(m, r_s, z_{1s}, \gamma_{\text{shb}}) A_{H^{+}}(m - \Delta_m, r_s, z_{2s}, \gamma_{\text{shb}}) dr_s, & \Delta_m \leq -1, \end{cases} \quad (25)$$

FIG. 4. (Color online) (a) The location of stationary sources $S_{H^{+}H^{-}}(n = 1)$ and $S_{H^{+}H^{-}}(n = 2)$ at the ocean surface (circle), the contents in the brackets are ignored. (b) Similar to Fig. 2b, but for $C_{RR}$. 

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there are four types of virtual head waves with amplitudes \( D \). For example, for \( CH \), contributions from multiple sources 
\[
\frac{t_{H}^{CH}}{C_0} = n \quad \text{and} \quad r_{H}^{CH} = \frac{m}{C_0} + \frac{n}{C_0}
\]
are equal to \( t_{II} \) in the top two panels of Fig. 5(b), while \( t_{II} \) is in the bottom two panels.

\[
t_{H}^{H+H^\pm}(\Delta_m) = (2\Delta_m Z \pm z_1 \mp z_2) \sin \theta_i / v_1, \quad -\infty \leq \Delta_m \leq +\infty.
\]

(26)

The virtual head wave travel times \( t_{H}^{H+H^\pm}(\Delta_m = 0) \) are equal to \( t_{II} - t_{II} \) in the top two panels of Fig. 5(b), while \( t_{H}^{H+H^\pm}(\Delta_m = 0) \) are equal to \( t_{II} - t_{II} \) in the bottom two panels.

The source ranges \( r_{H}^{H+H^\pm}(m) \) and \( r_{H}^{H+H^\pm}(n) \) are expressed in terms of \( \Delta_m \) by using \( m = \Delta_m + n \) and \( n = m - \Delta_m \),

\[
\begin{align*}
 r_{H}^{H+H^\pm}(\Delta_m, n) &> X_1^\pm(\Delta_m + n), \quad \Delta_m \geq 0, \\
 r_{H}^{H+H^\pm}(\Delta_m, m) &> X_2^\pm(\Delta_m - m), \quad \Delta_m \leq -1, \\
 r_{H}^{H+H^\pm}(\Delta_m, n) &> X_1^\pm(\Delta_m + n), \quad \Delta_m \geq 1, \\
 r_{H}^{H+H^\pm}(\Delta_m, m) &> X_2^\pm(\Delta_m - m), \quad \Delta_m \leq 0.
\end{align*}
\]

(27)

Figure 6(a) shows the relationship between virtual head wave travel times \( t_{H}^{H+H^\pm}(\Delta_m) \) and source ranges \( r_{H}^{H+H^\pm}(\Delta_m, n = 1) \) and \( r_{H}^{H+H^\pm}(\Delta_m, m = 1) \). Different from \( C_{RH} \) and \( C_{IH} \), \( \Delta_m \) can take all the values (here \( \Delta_m = \{-2, \ldots, 2\} \)) for each type of virtual head waves. The virtual head waves produced by \( C_{H}^{H+H^\pm} \) at a fixed \( \Delta_m \) have contributions from multiple sources \( r_{H}^{H+H^\pm}(\Delta_m = \text{constant}, n = 1) \) or \( r_{H}^{H+H^\pm}(\Delta_m = \text{constant}, m = 1) \) simultaneously. For example, for \( C_{H}^{H+H^\pm} \), the virtual head waves at \( \Delta_m = 0 \) (0.015 s) have contributions from all the sources greater than the critical offset \( X_1^\pm(1) = 754 \) m. Besides, for a fixed \( m \) or \( n \), the source ranges \( r_{H}^{H+H^\pm}(\Delta_m = \text{constant}) \) or \( r_{H}^{H+H^\pm}(\Delta_m = \text{constant}) \) increase with \( |\Delta_m| \) [for \( C_{H}^{H+H^\pm}(\Delta_m, n = 1) \), from 754 m at \( \Delta_m = 0 \) to 2371 m at \( \Delta_m = 2 \), for \( C_{H}^{H+H^\pm}(\Delta_m, m = 1) \), from 1385 m at \( \Delta_m = -1 \) to 2193 m at \( \Delta_m = -2 \)].

6. The sum of five terms

Considering (2–5), for a vertical sensor pair, \( C_{DH} = 0 \), \( C_{HD} = 0 \). The other three terms produce the virtual head waves,
\[ C_{RH} + C_{HR} + C_{HH} \]
\[ \approx 2\pi Q \sum_{\pm} \sum_{\Delta m = -\infty}^{\infty} \alpha_{\pm}(-\Delta m) e^{-i\omega t(2\Delta m z_1 + z_2)} \sin \theta/v_1, \]  
(28)

where

\[ \alpha_{\pm}(-\Delta m) = \begin{cases} 
\alpha_{H+H^-}(\Delta m) + \alpha_{R+H^-}(\Delta m), & \Delta m \geq 0, \\
\alpha_{H+R^-}(\Delta m) + \alpha_{R+R^-}(\Delta m), & \Delta m \leq -1, \\
\alpha_{H+R^-}(\Delta m) + \alpha_{R+H^-}(\Delta m), & \Delta m \geq 1, \\
\alpha_{H+H^-}(\Delta m) + \alpha_{R+R^-}(\Delta m), & \Delta m \leq 0.
\end{cases} \]

From Eq. (28), each type of virtual head waves are produced by \( C_{RH}, C_{HR}, \) and \( C_{HH} \) simultaneously. Therefore, the virtual head wave travel times \( t_{\pm}(-\Delta m) = \{ r_{H+H^-}(\Delta m) \cup r_{R+H^-}(\Delta m) \} = t_{H+H^-}(\Delta m), \) and the sources that contribute to the virtual head waves are as ranges

\[ r_{\pm}(-\Delta m, n) = \{ r_{H+H^-}(\Delta m, n) \cup r_{R+H^-}(\Delta m, n) \}, \Delta m \geq 0, \]
\[ r_{\pm}(-\Delta m, m) = \{ r_{H+H^-}(\Delta m, m) \cup r_{R+H^-}(\Delta m, m) \}, \Delta m \leq -1, \]
\[ r_{\pm}(-\Delta m, n) = \{ r_{H+H^-}(\Delta m, n) \cup r_{R+H^-}(\Delta m, n) \}, \Delta m \geq 1, \]
\[ r_{\pm}(-\Delta m, m) = \{ r_{H+H^-}(\Delta m, m) \cup r_{R+H^-}(\Delta m, m) \}, \Delta m \leq 0. \]
(30)

In detail, we find

\[ t_{\pm}(-\Delta m) = (2\Delta m z_1 + z_2) \sin \theta/v_1, \quad -\infty \leq \Delta m \leq +\infty. \]  
(29)

Combining Figs. 3 and 6(a), the final relation between virtual head wave travel times \( t_{\pm}(-\Delta m) \) and source ranges \( r_{\pm}(-\Delta m, m = 1) \) and \( r_{\pm}(-\Delta m, n = 1) \) are obtained, see Fig. 6(b). All the sources greater than or equal to the critical offsets \( X_1^+(-\Delta m + 1) \) and \( X_2^+(1 - \Delta m) \) [on the right hand side of Eq. (30)] have contributions to the virtual head waves. Although the virtual head waves produced by a single noise source are weak, they are observable by averaging over time of multiple noise sources.

7. The sum of nine terms

Based on Eqs. (7) and (28), the cross-correlation result for a vertical sensor pair is the sum of nine terms,

\[ C(r_2, z_2, r_1, z_1) = P(r_1, z_1)P^*(r_2, z_2) \]
\[ \approx Q \left[ \alpha_{DD} e^{-i\omega(z_1 - z_2)/v_1} \right. \]
\[ + \sum_{n=1}^{\infty} \sum_{\Delta m = -\infty}^{\infty} \alpha_{DR^+}(n) e^{-i\omega(2mZ - z_1 + z_2)/v_1} \]
\[ + \sum_{m=1}^{\infty} \sum_{\Delta m = -\infty}^{\infty} \alpha_{R^+D}(m) e^{-i\omega(2mZ - z_1 - z_2)/v_1} \]
\[ + \sum_{\Delta m = -\infty}^{\infty} \alpha_{H+H^-}(-\Delta m) e^{-i\omega(2\Delta m Z + z_1 + z_2) \sin \theta/v_1} \]  
(31)

The first four terms on the right hand side of Eq. (31) are from the cross-correlation of direct and reflected waves \( C_{DD}, C_{DR^+}, C_{RD} \), and \( C_{RR} \), producing physical waves including direct, surface reflected, bottom reflected, and surface-bottom reflected waves between \( V_1 \) and \( V_2 \). The last term is from the cross-correlation between reflected and head waves \( C_{HR} \), and between head and head waves \( C_{HH} \), producing virtual head waves that do not propagate between two receivers.

III. SEABED SOUND SPEED ESTIMATION WITH VIRTUAL HEAD WAVES

It has been shown that the time domain cross-correlation function can be extracted through beamforming or seismic interferometry, and the seabed sound speed can be estimated by stacking the virtual head waves in the time domain cross-correlation function.15 In this section, we will show theoretically that the time domain cross-correlation function of vertical sensor pairs consists of direct, reflected, and virtual head waves based on the analysis in Sec. III. The slope of the direct and reflected waves is different from that of virtual head waves, therefore it is possible to stack the virtual head waves while other waves are added destructively.

Assuming a vertical array with \( N_R \) hydrophones, \( V_1 \) and \( V_k \) are two array elements, \( j, k \in \{1, \ldots, N_R\} \). Then, the time domain cross-correlation function \( c_{jk}(\tau) \), also called virtual source gather with the virtual source at \( V_k \),15 is obtained by transforming Eq. (31) to the time domain, while the subscripts 1 and 2 are replaced by \( j \) and \( k \),

\[ c_{jk}(\tau) = \mathcal{F}^{-1}(C(r_k, z_k; r_j, z_j, \omega)) \]
\[ \approx s(\tau - (z_2 - z_k)/v_1) \]
\[ + \sum_{n=1}^{\infty} \sum_{\Delta m = -\infty}^{\infty} \alpha_{DR}(n) s(\tau + (2mZ - z_1 + z_2)/v_1) \]
\[ + \sum_{m=1}^{\infty} \sum_{\Delta m = -\infty}^{\infty} \alpha_{R^+D}(m) s(\tau - (2mZ - z_1 - z_2)/v_1) \]
\[ + \sum_{\Delta m = -\infty}^{\infty} \alpha_{H+H^-}(-\Delta m) s(\tau - (2\Delta m Z + z_1 + z_2) \sin \theta/v_1). \]  
(32)
From Eq. (32), the slope of the direct and reflected waves is $|\partial(z_j)/\partial t| = v_1$, while that of virtual head waves is $|\partial(z_j)/\partial t| = v_1/\sin \theta_c = 1/\sqrt{v_1^2 - v_2^2}$. Therefore, it is possible to estimate the seabed sound speed $v_2$ by stacking the virtual head waves, while the direct and reflected waves add destructively through this processing. The virtual source gather can be transformed to the $r$-$s_r$ domain by summing over virtual sources $V_k$ and receivers $V_j$ using difference or sum of hydrophone depth, $\pm \Delta z_k = \pm (z_j - z_k)$ and $\pm \sum z_k = \pm (z_j + z_k)$,

$$c(\tau, s_r, \pm \Delta z_k) = \sum_{j=1}^{N_j} \sum_{k=1}^{N_k} h_j h_k c_{jk} \left( \tau \pm \sum z_k s_r \right),$$

$$c(\tau, s_r, \pm \sum z_k) = \sum_{j=1}^{N_j} \sum_{k=1}^{N_k} h_j h_k c_{jk} \left( \tau \pm \sum z_k s_r \right),$$

where $h_j$ and $h_k$ are spatial shading windows (e.g., Hanning windows), $s_r = \sqrt{v_1^2 - v_2^2}$. In fact, the seismic interferometric processing described in Eq. (33) is identical to delay-and-sum, cross-correlating, and auto-correlating beams. In detail, for $c(\tau, s_r, -\sum z_k)$,

$$c(\tau, s_r, -\sum z_k) = \sum_{j=1}^{N_j} \sum_{k=1}^{N_k} h_j h_k z_{jk} (\Delta m) \times s(\tau - (2\Delta m^2 z_{jk} + z_k) s_r - \sum z_k s_r)$$

$$= \sum_{j=1}^{N_j} \sum_{k=1}^{N_k} h_j h_k z_{jk} (\Delta m) \times s(\tau - \tau_{v2} (\Delta m) + \sum z_k s_r),$$

Equations (34) and (35) show that when stacking according to $-\sum z_k$, the direct and reflected waves are added destructively, and the virtual head waves appear at $\tau = \tau_{v2} (\Delta m)$ and $s_r = s_{v2}$ in the $r$-$s_r$ domain. However, if $k$ is a constant,

$$c(\tau, s_r, -\sum z_k) = \sum_{j=1}^{N_j} \sum_{k=1}^{N_k} h_j h_k z_{jk} (\Delta m) \times s(\tau - (2\Delta m^2 z_{jk} + z_k) s_r - \sum z_k s_r)$$

$$= \sum_{j=1}^{N_j} \sum_{k=1}^{N_k} h_j [z_{jk} (\Delta m) s(\tau - \tau_{v2} (\Delta m) + \sum z_k s_r)$$

$$+ z_{jk} (\Delta m) s(\tau - \tau_{v2} (\Delta m) + \sum z_k s_r)]$$

$$= \sum_{j=1}^{N_j} \sum_{k=1}^{N_k} h_j [z_{jk} (\Delta m) s(\tau - \tau_{v2} (\Delta m) + \sum z_k s_r)$$

$$+ z_{jk} (\Delta m) s(\tau - \tau_{v2} (\Delta m) + \sum z_k s_r)]$$

$$+ \sum_{j=1}^{N_j} \sum_{k=1}^{N_k} h_j [z_{jk} (\Delta m) s(\tau - \tau_{v2} (\Delta m) + \sum z_k s_r)$$

$$+ z_{jk} (\Delta m) s(\tau - \tau_{v2} (\Delta m) + \sum z_k s_r)]$$

where $\sum_{j=1}^{N_j} \sum_{k=1}^{N_k} h_j [z_{jk} (\Delta m) s(\tau - \tau_{v2} (\Delta m) + \sum z_k s_r)$$

$$+ z_{jk} (\Delta m) s(\tau - \tau_{v2} (\Delta m) + \sum z_k s_r)]$$

Assuming $v = v_2$, then $s_r = s_{v2}$, $\Delta s_r = 0$, Eq. (36) is simplified as

$$c(\tau, s_{v2}, -\sum z_k)$$

$$= A \sum_{j=1}^{N_j} \sum_{k=1}^{N_k} h_j h_k [z_{jk} (\Delta m) s(\tau - \tau_{v2} (\Delta m) - \sum z_k s_r)$$

$$+ z_{jk} (\Delta m) s(\tau - \tau_{v2} (\Delta m) - \sum z_k s_r)]$$

$$= A \sum_{j=1}^{N_j} \sum_{k=1}^{N_k} h_j h_k [z_{jk} (\Delta m) s(\tau - \tau_{v2} (\Delta m) + \sum z_k s_r)$$

$$+ z_{jk} (\Delta m) s(\tau - \tau_{v2} (\Delta m) + \sum z_k s_r)]$$

Equation (37) shows that both geometries are able to retrieve virtual head waves, but the processing techniques are different.
to the controlled, impulsive point sources near the surface. For both simulations, the frequencies are computed every 0.25 Hz from 400 to 1000 Hz.

Panel (a)–(d) are the corresponding stacking results of virtual head waves, the first four types of waves (D, SR, BR, and BSR) are produced by the first four terms on the right hand side of Eq. (32), while the virtual head waves are observed at the same time from noise cross-correlations. By stacking the time domain cross-correlation result in Figs. 8(a)–8(d), where $c_{jk}(\tau)$ is defined as

$$c_{jk}(\tau) = \mathcal{F}^{-1}\left\{P(r_j, z_j; \omega)P^*(r_k, z_k; \omega)\right\},$$

(41)

where $j, k \in [1, \ldots, 200]$.

However, for the active correlation, the processing involves measuring the acoustic field at $V_1$ through $V_{200}$ due to the controlled, impulsive point sources near the surface. This is followed by cross-correlating over sources $N_S$ ($1 \leq N_S \leq 2401$) and summing to create a virtual source at $V_{\hat{k}}$.

$$c_{jk}^v(\tau) = \sum_{j=1}^{N_S} h_j \mathcal{F}^{-1}\left\{P(r_j, z_j; \omega)P^*(r_k, z_k; \omega)\right\}.$$  

(42)

Here, the passive case demonstrates the theory in Sec. III that five types of waves including direct, surface reflected, bottom reflected, surface-bottom reflected waves between two receivers and virtual head waves can be extracted at the same time from noise cross-correlations. By stacking the virtual head waves, it is possible to estimate the seabed sound speed. Active sources at different ranges are simulated to validate the relation between virtual head wave travel times and location of sources that contribute to the virtual head waves. For both simulations, the frequencies are computed every 0.25 Hz from 400 to 1000 Hz.

Figure 8 shows the envelope of virtual source gather $c_{jk}(\tau)$, $k = 1, 67, 133, 199$. That is, this plot is similar to what would be observed if the receiver ($V_1$) were a source and the time-domain response is plotted for all $j$ receivers $V_1-V_{200}$ in rows of the plot. The dark and gray areas indicate the arrival of wavefronts. These lines with different slopes and travel times indicate five types of waves: the direct wave (D), surface reflected wave (SR), bottom reflected wave (BR), bottom-surface reflected wave (BSR), and virtual head waves. The first four types of waves (D, SR, BR, and BSR) are produced by the first four terms on the right hand side of Eq. (32), while the virtual head waves are produced by the last term. For the virtual head waves, the $\Lambda_m$ along the top of Fig. 8 refers to difference of seabed bounces of rays from sources to two receivers $V_j$ and $V_k$, and five groups of virtual head waves with $\Lambda_m = [-2, \ldots, 2]$ are observed.

By stacking the time domain cross-correlation result $c_{jk}(\tau)$ in Fig. 8 according to $c(\tau, s_{\tau} - \sum_{j \neq k} z_{jk})$, the virtual head waves are expected to be observed, while other waves (D, SR, BR, and BSR) are added destructively, see Fig. 9. Panels (a)–(d) are the corresponding stacking results of $c_{jk}(\tau)$ in Figs. 8(a)–8(d), where $k$ is a constant. As predicted by Eq. (37), the virtual head waves are expected to be observed at $\tau = \tau_{v2}(\Lambda_m), s_{\tau} = s_{v2}$, and $\tau = \tau_{v1}(\Lambda_m) + 2z_{ks_{v2}}, s_{\tau} = s_k$ in the $\tau$-s$_{\tau}$ domain. Let $v = \sqrt{1/(v_1^2 - z_k^2)}$, the virtual head waves are therefore observed at $\tau = \tau_{v2}(\Lambda_m) = 0.069\Lambda_m$ s, $v = v_2 = 1600$ m/s (circles), and $\tau = \tau_{v1}(\Lambda_m) + 2z_k s_{v2} = 0.069\Lambda_m + 4.6398 \times 10^{-4} z_k$ s, $v = v_2 = 1600$ m/s.
(right triangles) in the $\tau-v$ domain. After summing over all the virtual sources $V_k, k \in [1, \ldots, 200]$, as predicted by Eq. (35), the virtual head waves are only observed at $\tau = \tau_{zj}(\Delta_m)$ and $v = v_2$, while other virtual head waves at $\tau = \tau_{zj}(\Delta_m) + 2z_k\Delta_z$ and $v = v_2$ are added destructively, see panel (e).

In the following, active sources at different ranges are summed to validate the relation between virtual head wave travel times $\left\{ t_{\pm \Delta_m} \right\}$ and source ranges $\left\{ r_{\pm \Delta_m, m} \right\}$. Noise sources are not considered because these are already summed in $P(r, z, \omega)$ and $P(r_k, z_k, \omega)$ before the correlation. Figure 10 shows the virtual head waves at different vertically spaced receiver pairs from sources at different ranges. The y axis of panels (a)–(d) represents sources from ranges $[600: 1: 3000]$ m (corresponding to $N = SR - 599$ in Eq. (42), where SR $= 700: 50: 3000$). The reason for not using sources with ranges less than 600 m is to avoid the appearance of direct, surface reflected, bottom reflected, and surface-bottom reflected waves.

In panel (a), virtual head waves at different $\Delta_m$ are observed, however, for short distances $[1600, 1500]$ m, only virtual head waves at $\Delta_m = 0$ are observed. With increasing range $[1500, 2500]$ m, virtual head waves at $\Delta_m = -1$ and $\Delta_m = 1$ begin to appear. As range increases to 2500 m, virtual head waves at $\Delta_m = -2$ and $\Delta_m = 2$ are weakly observed due to attenuation. Similar phenomena are observed in panels (b)–(d), but the virtual head waves shift in time and range due to different receiver depth. As explained in Sec. III, the virtual head waves are produced by three terms, $C_{RRH}, C_{CHR}$, and $C_{CHR}$, and they have contributions from all the sources greater than or equal to the critical offsets $X_{\pm \Delta_m, 1}$ and $X_{\pm \Delta_m, m}$, see Fig. 6(b). Here, only sources at the critical offsets $X_{\pm \Delta_m, n}$ and $X_{\pm \Delta_m, m}$ are plotted to simplify. The theoretical predictions in each panel match well with the simulation result.

V. DISCUSSION AND CONCLUSION

This study derives the extraction of virtual head waves from sea surface generated ambient noise in a Pekeris waveguide using a vertically spaced sensor pair. Based on ray theory, it is assumed that noise recorded at each receiver consists of direct, reflected, and head waves. After cross-correlating noise measured on two receivers, nine terms are obtained. Analyzing these nine terms with the method of stationary phase, it is shown that four types of physical waves (direct, surface reflected, bottom reflected, and surface-bottom reflected waves) between two receivers and virtual
head waves can be extracted from ocean ambient noise. The virtual head waves are produced from cross-correlations of reflected waves and head waves, and head waves and head waves. They have contributions from sources located on an annulus, where the inner radius is the critical offset, and the external radius is the farthest source-receiver horizontal distance that makes the head waves detectable. These sources’ contributions are integrated over time, making it possible to observe the virtual head waves.

The slope of the virtual head waves is different from that of physical waves in the virtual source gather, making it possible to constructively stack the virtual head waves, while the other waves are summed destructively through this processing. The virtual head waves are therefore observed at the seabed sound speed.

The simulation with noise sources confirms the theoretically predicted five types of waves and the estimation of seabed sound speed by stacking the virtual head waves. The controlled, active sources at different ranges are simulated to validate the relation between the virtual head wave travel times and ranges of sources that contribute to the virtual head waves.

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