A NOTE ON LOG-ALGEBRAICITY ON ELLIPTIC CURVES

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Abstract. We analyze log-algebraic power series identities for formal groups of elliptic curves over $\mathbb{Q}$ which arise from modular parametrizations. We further investigate applications to special values of elliptic curve $L$-functions.

1. Introduction

The notion of log-algebraicity was termed by Anderson [1], [2], to describe certain power series identities in the context of exponential functions of Drinfeld modules and twisted harmonic series over global function fields. The basic motivating example from characteristic 0 is the familiar formal power series identity for the multiplicative group,

$$\exp \left( - \sum_{n=1}^{\infty} \frac{t^n}{n} \right) = 1 - t.$$ 

If we take $\beta = \sum_{k=-d}^{d} m_k u^k \in \mathbb{Z}[u, u^{-1}],$ then only slightly more complicated is that

$$\exp \left( - \sum_{n=1}^{\infty} \frac{\beta(u^n)}{n} t^n \right) = \prod_{k=-d}^{d} \left( 1 - u^k t \right)^{m_k}.$$ 

These inner harmonic sums are thus "log-algebraic," as one obtains polynomial, rational, or algebraic power series upon exponentiation. Specializations of these identities can be used to recover classical formulas for Dirichlet $L$-functions at $s = 1,$ as in [29, Thm. 4.9].

Anderson extended these types of identities to sign-normalized Drinfeld modules over function fields in positive characteristic, based on special identities for Carlitz-Goss zeta values due to Thakur [27]. For example, Anderson [2, Thm. 3] showed for the Carlitz module $C$ over $A = \mathbb{F}_q[\theta], \ K = \mathbb{F}_q(\theta),$ that in $K[u][t]$ we have

$$\exp_C \left( \sum_{a \in A_+} \frac{\beta(C_a(u))}{a} t^{\deg a} \right) \in A[u, t],$$

where $\exp_C$ is the Carlitz exponential, $A_+$ denotes the monic elements of $A$, $\beta \in A[u]$ is fixed, and $C_a(u) \in A[u]$ represents multiplication by $a$ on $C$. Anderson used these identities to express special values of Goss $L$-series of Dirichlet type in terms of Carlitz logarithms of special points, which themselves arise from a theory of circular units for $C$. Log-algebraic identities have been widely studied in function field arithmetic in recent years, extending Anderson’s results to analogues of Stark units and abelian $L$-series, Drinfeld modules of arbitrary ranks, and certain Anderson $t$-modules (e.g., see [3]–[6],

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These identities are also closely connected to Taelman’s work on special values of Goss $L$-series for Drinfeld modules [25]. Although log-algebraic identities were constructed in the theory of function fields, the purpose of the present note is to investigate how they occur in characteristic 0 on other algebraic groups, particularly on the formal groups of elliptic curves over $\mathbb{Q}$. Unlike for Drinfeld modules and Anderson $t$-modules over $A$ or for the multiplicative group, where we exponentiate twisted harmonic power series, we find that log-algebraic formulas for an elliptic curve over $\mathbb{Q}$ arise most naturally through the curve’s modular parametrization.

Our main results in these directions are in §2 (see Corollary 2.6 and Theorem 2.8). For example, we show for an elliptic curve $E/\mathbb{Q}$ and $\beta = \sum_{d=0}^{m} m_k u^k \in \mathbb{Z}[u]$, that in $\mathbb{Q}[u][[t]]$,

$$\exp_{\mathcal{E}} \left( \sum_{n=1}^{\infty} \frac{a_n \beta(u^n)}{n} t^n \right) = \sum_{k=0}^{d} [m_k]_{\mathcal{E}} (\Phi(u^k t)).$$

Here $\exp_{\mathcal{E}}(t)$ denotes the exponential on the formal group of $E$, the sequence $\{a_n\}$ provides the Fourier coefficients of the newform $f \in S_2(\Gamma_0(N))$ attached to $E$, $\Phi(t) \in \mathbb{Q}[t]$ is induced by the modular parametrization, and the sum “$\sum_{\mathcal{E}}$” on the right is taken in the formal group $\mathcal{E}$. The reason to term this identity as “log-algebraic” is that the series $\Phi(t)$ formally represents the algebraic map $X_0(N) \to E$, and so the right-hand side represents a formal sum of algebraic points on $E$. Thus the interior sum on the left is a formal logarithm of this sum of algebraic points. This identity is also closely related to the formal group of Honda [17], [18], obtained from $f$.

We present examples of applications to $L$-functions of elliptic curves in §3 by demonstrating how Corollary 2.6 and Theorem 2.8 can be used to determine exactly the values of $L(E, 1)$ and $L(E, \chi, 1)$ for a Dirichlet character $\chi$. For example, if $E_0 = X_0(11)$ and $\psi$ is a given cubic Dirichlet character modulo 7, we show in Example 3.3 that

$$L(E_0/\mathbb{Q}, \psi, 1) = \frac{5}{14} (1 + \sqrt{-3}) g(\psi) \Omega,$$

where $g(\psi)$ is a Gauss sum and $\Omega$ is the positive real period of $E_0$. To be sure, these special value formulas can be obtained using previous methods of modular symbols that are not far from our considerations, e.g., see [11, §2.8–2.12] and [22], but one underlying goal of this note is to investigate how special $L$-values are interpolated by power series identities.

2. Formal groups and log-algebraic identities

Suppose we have an elliptic curve,

$$E : y^2 = x^3 - \frac{g_2}{4} x - \frac{g_3}{4}, \quad g_2, \ g_3 \in \mathbb{C},$$

and let $\Lambda \subseteq \mathbb{C}$ be its associated lattice so that $g_i = g_i(\Lambda)$ for $i = 2, 3$. Let $\varphi(z) = \varphi_{\Lambda}(z)$ be its associated Weierstrass $\varphi$-function so that $(\varphi(z), \frac{1}{2} \varphi'(z))$ represents a point on $E(\mathbb{C})$. Recall by [24, Thm. VI.3.5] that the Laurent series expansion of $\varphi(z)$ at $z = 0$ is

$$\varphi(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k + 1) G_{2k+2}(\Lambda) z^{2k},$$

where $G_{2k}(\Lambda)$ is the weight $2k$ Eisenstein series for $\Lambda$. 

Let $E$ denote the formal group of $E$ over $\mathbb{Q}[g_2, g_3]$, and by abuse of notation we let $E(t_1, t_2) \in \mathbb{Q}[g_2, g_3][t_1, t_2]$ be the power series defining this formal group law as in [24 Ch. IV]. In particular we have the formal $x$- and $y$-coordinates in $\mathbb{Q}[g_2, g_3](t)$,

\begin{align}
(5) \quad x(t) &= \frac{1}{t^2} + \frac{g_2}{4} t^2 + \frac{g_3}{4} t^4 - \frac{g_2^2}{16} t^6 + \cdots, \\
y(t) &= -\frac{1}{t^3} - \frac{g_2}{4} t^4 - \frac{g_3}{4} t^6 + \frac{g_2^2}{16} t^8 + \cdots,
\end{align}

and formal invariant differential,

$$\omega_E(t) = \left(1 - \frac{g_2}{2} t^4 - \frac{3g_3}{4} t^6 + \frac{3g_2^2}{8} t^8 + \cdots\right) dt.$$ 

We collect a couple results about formal groups. This first follows from [24 §VII.2].

**Lemma 2.1.** The maps

$$\alpha(t) \mapsto (x(\alpha(t)), y(\alpha(t))), \quad -\frac{x_0(t)}{y_0(t)} \longleftarrow (x_0(t), y_0(t)),$$

induce mutually inverse isomorphisms of abelian groups,

$$E(t \mathbb{Q}(g_2, g_3)[t]) \cong \{(x_0(t), y_0(t)) \in E(\mathbb{Q}(g_2, g_3)(t)) : \text{ord}_t(x_0(t)/y_0(t)) \geq 1\} \cup \{O\}.$$

This next follows from the fact that the formal exponential and logarithms for a formal group over a field of characteristic 0 are isomorphisms with the formal additive group $\widehat{\mathbb{G}}_a$ (see [16 §5.4] or [24 §IV.5])

**Lemma 2.2.** Let $K$ be a field of characteristic 0, and let $\mathcal{F}(t_1, t_2), \mathcal{G}(t_1, t_2) \in K[t_1, t_2]$ be formal groups over $K$. For a power series $m(t) \in K[t]$, the following are equivalent.

(a) $m : \mathcal{F} \to \mathcal{G}$ is a morphism of formal groups,

(b) $m \circ \exp_{\mathcal{F}}(t) = \exp_{\mathcal{G}}(m'(0)t)$,

(c) $\log_{\mathcal{G}} \circ m(t) = m'(0) \cdot \log_{\mathcal{F}}(t)$,

(d) $\omega_{\mathcal{G}} \circ m(t) = m'(0) \cdot \omega_{\mathcal{F}}(t)$.

We then obtain the following result for the formal exponential of $E$.

**Proposition 2.3.** Considering $\varphi(z) \in \mathbb{Q}[g_2, g_3](z)$ as a formal Laurent series in $z$,

$$\exp_{\mathcal{E}}(z) = -\frac{2\varphi(z)}{\varphi'(z)} \in \mathbb{Q}[g_2, g_3][z].$$

**Proof.** The invariant differential $\omega_E$ on $E$ satisfies $\omega_E = dx/2y = d\varphi(z)/\varphi'(z) = dz$. On the other hand, letting $t = -x/y$ be the formal parameter on $E$, the invariant differential formally satisfies $\omega_E(t) = dx(t)/2y(t)$. By definition the formal logarithm $\log_{\mathcal{E}}(t)$ satisfies

$$\log_{\mathcal{E}}(t) = \int \omega_E(t) = \int dz = z,$$

and so

$$\exp_{\mathcal{E}}(z) = t = -\frac{x(z)}{y(z)} = -\frac{2\varphi(z)}{\varphi'(z)}. \quad \square$$

Now let $E_0/\mathbb{Q}$ be an elliptic curve of conductor $N$,

$$E_0 : y^2 + e_1 xy + e_3 y = x^3 + e_2 x^2 + e_4 x + e_6, \quad e_i \in \mathbb{Z},$$
which is modular by [8], [26], [30]. Assume further that the modular parametrization
\[ \mu : X_0(N) \to E_0 \] is optimal, i.e., \( E_0 \) is a strong Weil curve. Let \( f \in S_2(\Gamma_0(N)) \) be the
unique normalized newform associated to \( E_0 \) with Fourier expansion
\[ f(\tau) = \sum_{n=1}^{\infty} a_n q^n, \quad q = e^{2\pi i \tau}, \quad a_n \in \mathbb{Z}, \quad a_1 = 1. \]
Then the pullback of the invariant differential \( \omega_{E_0} \) satisfies
\[ \mu^* \omega_{E_0} = c \cdot 2\pi i f(\tau) d\tau = c \cdot \sum_{n=1}^{\infty} a_n q^{n-1} dq, \]
where \( c \) is the Manin constant, which henceforth we assume to be 1. Replacing \( q \) by a
formal parameter \( t \), we set
\[ \lambda(t) := \int \mu^* \omega_{E_0} = \sum_{n=1}^{\infty} \frac{a_n}{n} t^n, \]
and recall a theorem of Honda.

**Theorem 2.4** (Honda [17, Thm. 5], [18]). There is a formal group \( \mathcal{L} \) over \( \mathbb{Z} \) so that
\[ \log_{\mathcal{L}}(t) = \lambda(t), \]
and in particular, \( \mathcal{L}(t_1, t_2) = \lambda^{-1}(\lambda(t_1) + \lambda(t_2)) \in \mathbb{Z}[t_1, t_2] \). Furthermore, \( \mathcal{L} \) is strongly
isomorphic over \( \mathbb{Z} \) to the formal group of \( E_0 \).

We also recall that (6) is closely related to the Eichler integral
\[ 2\pi i \int_{z}^{i\infty} f(\tau) d\tau = -\sum_{n=1}^{\infty} \frac{a_n}{n} e^{2\pi i n z}, \]
for \( z \in \mathbb{C} \) with \( \text{Im}(z) > 0 \), and as such \( \lambda(t) \) is a formal Eichler integral in the sense of [9].

Defining invariants \( b_i, c_i \in \mathbb{Q} \) as in [24, §III.1], we change coordinates on \( E_0 \) by
\[ x \leftarrow x - \frac{b_2}{12}, \quad y \leftarrow y - \frac{c_1}{2} x + \frac{e_1 b_2}{24} - \frac{e_3}{2}, \]
and obtain the \( \mathbb{Q} \)-isomorphism \( \psi : E_0 \to E \), where
\[ E : y^2 = x^3 - \frac{c_4}{48} x - \frac{c_6}{864}. \]
Setting \( g_2 := c_4/12 \) and \( g_3 := c_6/216 \), we obtain \( \Lambda, \psi(z) = \varphi_\Lambda(z), \log_\mathcal{E}(z), \) and \( \exp_\mathcal{E}(z) \),
all associated to \( E \), as in the beginning of this section. Thus we can define \( \phi := \psi \circ \mu, \)
\[ \phi : X_0(N) \xrightarrow{\mu} E_0 \xrightarrow{\psi} E, \]
from which we see that
\[ \phi^* \omega_E = \mu^* \omega_{E_0} = \sum_{n=1}^{\infty} a_n q^{n-1} dq = d\lambda(q). \]
Moreover, we can set \( X(q) := \phi^*(x) \) and \( Y(q) := \phi^*(y) \) in \( \mathbb{Q}(X_0(N)) \). Then as \( X(q), \)
\( Y(q) \) satisfy the defining equation for \( E \) and
\[ q \cdot dX/dq = 2Y \cdot f, \]
it follows recursively (see [11, §3]) that $X, Y$ have Laurent series expansions in $q$,

$$X(q) = \frac{1}{q^2} - \frac{a_2}{4} q + \frac{3a_2^2}{3} + \cdots, \quad Y(q) = -\frac{1}{q^3} + \frac{3a_2}{q^2} - \frac{3a_2^2 - 2a_3}{2q} + \cdots.$$  

We let $X(t), Y(t) \in \mathbb{Q}[[t]]$ be the formal series obtained by replacing $q$ with $t$, and we set

$$\Phi(t) := \frac{-X(t)}{Y(t)} = t + \frac{a_2}{2} t^2 + \frac{a_3}{3} t^3 + \frac{a_4}{4} t^4 + \left( \frac{c_4}{120} + \frac{a_5}{5} \right) t^5 + \cdots \in \mathbb{Q}[[t]].$$  

The following proposition underlies our log-algebraic identities.

**Proposition 2.5.** The power series $\Phi(t) \in \mathbb{Q}[[t]]$ is an isomorphism $\Phi : \mathcal{L} \to \mathcal{E}$ of formal groups over $\mathbb{Q}$.

**Proof.** Combining Lemma 2.1 and (10), we see that

$$X(t) = x(\Phi(t)), \quad Y(t) = y(\Phi(t)).$$  

It then follows that

$$\omega_{\mathcal{E}} \circ \Phi = \frac{d(x \circ \Phi(t))}{2y \circ \Phi(t)} = \frac{dX(t)}{2Y(t)} = \sum_{n=1}^{\infty} a_n t^{n-1} dt,$$

where the last equality follows from (5). Then by Theorem 2.4 we conclude that $\omega_{\mathcal{E}} \circ \Phi = \Phi'(0) \omega_{\mathcal{L}}$ (since $\Phi'(0) = 1$), and the result follows from Lemma 2.2. \(\square\)

We then obtain the following log-algebraic identity on power series in $\mathbb{Q}[[t]]$ together with a specialization relating it to a special $L$-value. This expression is “log-algebraic” in that $\Phi(t)$ formally represents the algebraic map $\phi : X_0(N) \to E$.

**Corollary 2.6.** For $E_0/\mathbb{Q}$ a strong Weil curve of conductor $N$, let $E/\mathbb{Q}$, $\exp_{\mathcal{E}}(t)$, and $\Phi(t)$ be chosen as above.

(a) We have the identity of formal power series in $\mathbb{Q}[[t]]$,

$$\exp_{\mathcal{E}} \left( \sum_{n=1}^{\infty} \frac{a_n}{n} t^n \right) = \Phi(t).$$

(b) Suppose that the sign of the functional equation of $L(E_0/\mathbb{Q}, s) = L(f, s)$ is $\varepsilon = +1$. Then

$$\varphi \left( \frac{1}{2} L(E_0/\mathbb{Q}, 1) \right) = X(e^{-2\pi/\sqrt{N}}),$$

when both sides converge.

**Proof.** For (a) we combine Theorem 2.4 and Proposition 2.5 (again using $\Phi'(0) = 1$). For (b), in general if $\varepsilon = \pm 1$ is the sign of the functional equation, then by [11, Prop. 2.11.1] we have the rapidly converging formula

$$L(E_0/\mathbb{Q}, 1) = (1 + \varepsilon) \sum_{n=1}^{\infty} \frac{a_n}{n} e^{-2\pi n/\sqrt{N}}.$$  

When $\varepsilon = +1$, we could obtain $\exp_{\mathcal{E}} \left( \frac{1}{2} L(E_0/\mathbb{Q}, 1) \right) = \Phi(\exp(-2\pi/\sqrt{N}))$ by specializing into part (a), and this can converge for some elliptic curves $E_0$ and $E$ (see Example 3.1), but it runs into issues near zeros of $Y(t)$ since $\Phi(t) = -X(t)/Y(t)$. Instead, using
Lemma [2.1] we see that we have identities in \( \mathbb{Q}(t) \): \( x(\exp_E(t)) = \varphi(t) \) and \( y(\exp_E(t)) = \frac{1}{2} \varphi'(t) \). Combining part (a) with [11], we obtain formal identities in \( \mathbb{Q}(t) \),

\[
\varphi \left( \sum_{n=1}^{\infty} \frac{a_n t^n}{n} \right) = X(t), \quad \frac{1}{2} \varphi' \left( \sum_{n=1}^{\infty} \frac{a_n t^n}{n} \right) = Y(t).
\]

Our desired identity follows from (12) by substituting \( t = e^{-2\pi t/\sqrt{N}} \). We note that this is the same as letting \( t = e^{2\pi i \tau} \) with \( \tau = i/\sqrt{N} \) from the upper half-plane.

**Remark 2.7.** Even when convergence is an issue in Corollary 2.6(b), one finds that

\[
\varphi \left( \frac{1}{2} L(E_0/Q, 1) \right) = x(\phi(i/\sqrt{N})),
\]

where as usual \( \varphi \) is extended to a meromorphic function on \( \mathbb{C} \).

Our main result constitutes the following power series identities.

**Theorem 2.8.** For \( E_0/Q \) a strong Weil curve of conductor \( N \), let \( E/Q, \varphi, \mathcal{E}/Q[t_1, t_2] \), \( \exp_E(t) \), and \( \Phi(t) \) be chosen as above. Let \( \beta = \sum_{k=0}^{d} m_k u^k \in \mathbb{Z}[u] \).

(a) We have the identity of formal power series in \( \mathbb{Q}[u][t] \),

\[
\exp_{E} \left( \sum_{n=1}^{\infty} \frac{a_n \beta(u^n)}{n} t^n \right) = \sum_{k=0}^{d} \left[ m_k \right]_{E} (\Phi(u^k t)),
\]

where \( \sum_{E} \) indicates that the sum is taken with respect to the formal group law \( \mathcal{E} \).

(b) Let \( P(t) := \left( X(t), Y(t) \right) \in E(\mathbb{Q}(t)) \). Then in \( \mathbb{Q}(u)(t) \),

\[
\varphi \left( \sum_{n=1}^{\infty} \frac{a_n \beta(u^n)}{n} t^n \right) = x \left( \sum_{k=0}^{d} \left[ m_k \right]_{E} (P(u^k t)) \right),
\]

where \( \sum_{E} \) indicates that the sum is taken with respect to the group law on \( E \).

**Proof.** Part (a) is a consequence of Corollary 2.6. We recall that since \( \exp_{E} : \hat{G}_{\mathcal{E}} \rightarrow \mathcal{E} \) is a morphism of formal groups, for \( g, h \in t \cdot \mathbb{Q}[u][t] \), we have \( \exp_{E}(g + h) = \mathcal{E}(g, h) \) and likewise for \( m \in \mathbb{Z} \), we have \( \exp_{E}(mg) = [m]_{E}(\exp_{E}(g)) \). We then observe that

\[
\sum_{n=1}^{\infty} \frac{a_n \beta(u^n)}{n} t^n = \sum_{k=0}^{d} m_k \sum_{n=1}^{\infty} \frac{a_n}{n} (u^k t)^n.
\]

By applying \( \exp_{E} \) to both sides and using Corollary 2.6 we obtain (a). For part (b), we take the \( x \)-coordinate of both sides of (a) and use (13). If \( g \in t \cdot \mathbb{Q}[u][t] \), then \( x(\Phi(g)) = X(g) \) by (11), which leads to the expression on the right side of (b). \( \square \)

### 3. Special L-values

As in [2], the identities in Corollary 2.6 and Theorem 2.8 can be used to recover information about \( L(E_0/Q, 1) \) or \( L(E_0/Q, \chi, 1) \), for a Dirichlet character \( \chi \), when these \( L \)-values are non-vanishing, by specializing at certain values of \( u \) and \( t \) and for judicious choices of \( \beta \). As mentioned in [11] the special value identities we obtain are closely related to those obtained previously (e.g., see [11] §2.8–2.12, App. Ex. 1), but here our goal is to highlight how special \( L \)-value formulas can be reflected in formal power series identities.
We continue with the notation of the previous section, and let $\mathbb{H}$ be the upper half-plane in $\mathbb{C}$. We assume that the sign of the functional equation of $L(E_0/\mathbb{Q}, s)$ is $\varepsilon = +1$. Recall that $\phi : X_0(N) \to E$ is the modular parametrization of $E$. Since $\varepsilon = +1$, it is well-known using the Atkin-Lehner $w_N$ operator that for $\tau \in \mathbb{H}$,

$$\phi(w_N(\tau)) = \phi\left(-\frac{1}{N\tau}\right) = [-1]_E(\phi(\tau)) + \phi(0),$$

where the image $\phi(0)$ of the cusp 0 is necessarily a torsion point on $E$ (e.g., see [7, §2]).

Since $\tau = i/\sqrt{N}$ is fixed by $w_N$, we have

$$[2]_E \circ \phi\left(\frac{i}{\sqrt{N}}\right) = \phi(0) \in E(\mathbb{Q})_{\text{tor}}.$$

By Corollary 2.6(b), we see that $\wp(E_0(1))$ represents the $x$-coordinate of $\phi(i/\sqrt{N})$ and thus a torsion point. Letting $d$ be its order, we see that

$$L(E_0/\mathbb{Q}, 1) \in \frac{2\Omega}{d} \cdot \mathbb{Z},$$

where $\Omega$ is the positive real period of $E$. We can pin down the exact value by computing the value of $L(E_0/\mathbb{Q}, 1)$ to enough precision so that its value can be determined from the fact that $2\Omega/d \cdot \mathbb{Z}$ is a discrete subset of $\mathbb{R}$. See Example 3.1.

In a similar manner, and by adapting the approach of [2], we can evaluate twists $L(E_0/\mathbb{Q}, \chi, 1)$ by applying Theorem 2.8. It is somewhat more complicated, relying on calculations with Heegner points, which we exhibit in Examples 3.2 and 3.3.

The computations of the exact values of $L(E_0/\mathbb{Q}, 1)$ and $L(E_0/\mathbb{Q}, \chi, 1)$ in Examples 3.1 and 3.2 are essentially the same as what can be found in [11, §2.8–2.11]. They are warm-ups for Example 3.3, which requires additional considerations and where the point of view of Theorem 2.8 is most useful. Calculations were performed using PARI [21].

Example 3.1. Let $E_0 = X_0(11) : y^2 + y = x^3 - x^2 - 10x - 20$ be the strong Weil curve of conductor 11, with Hecke eigenform in $S_2(\Gamma_0(11))$,

$$f = \prod_{m=1}^{\infty} (1-q^m)^2(1-q^{11m})^2 = q - 2q^2 - q^3 + 2q^4 + q^5 + \cdots = \sum_{n=1}^{\infty} a_n q^n.$$

The sign of the functional equation of $L(E_0/\mathbb{Q}, s) = L(f, s)$ is $\varepsilon = +1$. Applying the change of coordinates in (7), we obtain

$$E : y^2 = x^3 - \frac{31}{3}x - \frac{2501}{108}.$$

As formal series, we have

$$\wp(z) = \frac{1}{z^2} + \frac{31}{15}z^2 + \frac{2501}{756}z^4 + \frac{961}{675}z^6 + \frac{77531}{41580}z^8 \cdots,$$

and by Proposition 2.3

$$\exp_{\varepsilon}(z) = \frac{2\wp(z)}{\wp'(z)} = z + \frac{62}{15}z^5 + \frac{2501}{252}z^7 + \frac{1922}{135}z^9 + \cdots.$$

We compute

$$X(t) = \frac{1}{t^2} + \frac{2}{t} + \frac{11}{3} + 5t + 8t^2 + t^3 + 7t^4 + \cdots,$$
\[
Y(t) = -\frac{1}{t^3} - \frac{3}{t^2} - \frac{7}{t} - \frac{25}{2} - 17t - 26t^2 - 19t^3 + \cdots,
\]
and so
\[
\Phi(t) = -\frac{X(t)}{Y(t)} = t - t^2 - \frac{1}{3}t^3 + \frac{13}{3}t^5 - \frac{61}{3}t^6 + \frac{529}{12}t^7 + \cdots.
\]
We find that
\[
\Phi(e^{-2\pi/\sqrt{11}}) \approx 0.1270624598..., 
\]
which corresponds to the point
\[
P = (X(e^{-2\pi/\sqrt{11}}), Y(e^{-2\pi/\sqrt{11}})) \approx (62.111554..., -488.826947...).
\]
We calculate \(2P \approx (15.666666..., -60.499999...),\) and since by (15),
\[
(17) \quad 2P = \phi(0) \in E(\mathbb{Q})_{\text{tor}} = \{O, (14/3, \pm11/2), (47/3, \pm121/2)\},
\]
we find
\[
(18) \quad 2P = (47/3, -121/2).
\]
Since \(P\) itself does not approximate any of the elements of \(E(\mathbb{Q})_{\text{tor}}\), we see that \(P\) has order 10. By (12) we obtain \(L(E_0/\mathbb{Q}, 1) \approx 0.2538418608...\), and we can approximate that the minimal positive real period of \(E\) is \(\Omega \approx 1.2692093042...\). By (16), we see that \(L(E_0/\mathbb{Q}, 1) \in (2\Omega/10)\mathbb{Z},\) and since this is a discrete set, by comparing approximations we find that
\[
(19) \quad L(E_0/\mathbb{Q}, 1) = \frac{\Omega}{5}.
\]

**Example 3.2.** We continue with the notation of Example 3.1 Let \(\chi : \mathbb{Z} \to \{0, \pm1\}\) be the quadratic Dirichlet character for \(\mathbb{Q}(\sqrt{-3})\), and let
\[
L(E_0/\mathbb{Q}, \chi, s) = \sum_{n=1}^{\infty} \chi(n) a_n n^{-s},
\]
be the corresponding quadratic twist of \(L(E_0/\mathbb{Q}, s)\). By [11] Prop. 2.11.2,
\[
(20) \quad L(E_0/\mathbb{Q}, \chi, 1) = 2 \sum_{n=1}^{\infty} \frac{\chi(n) a_n}{n} e^{-2\pi n/3\sqrt{11}},
\]
from which we obtain \(L(E_0/\mathbb{Q}, \chi, 1) \approx 1.6844963329...\)

Now let \(\beta = u - u^2\), and let \(\rho = e^{2\pi i/3}\). It is a quick calculation to determine that
\[
\beta(\rho^n) = \rho^n - \rho^{-n} = \chi(n) \cdot \sqrt{-3}.
\]
We note that together with (20) this implies,
\[
(21) \quad \sum_{n=1}^{\infty} \frac{a_n \beta(u^n)}{n} t^n \bigg|_{u=\rho, t=\exp(-2\pi/3\sqrt{11})} = \frac{\sqrt{-3}}{2} L(E_0/\mathbb{Q}, \chi, 1).
\]
From Theorem 2.8 (b), letting \(P(t) = (X(t), Y(t)) \in E(\mathbb{Q}(t))\) we see that
\[
\varphi \left( \sum_{n=1}^{\infty} \frac{a_n \beta(u^n)}{n} t^n \right) = x \left( P(ut) - P(u^2t) \right),
\]
where the difference on the right is calculated in $E(\mathbb{Q}(u)((t)))$. From this we obtain
\begin{equation}
(22) \quad \varphi \left( \frac{\sqrt{-3}}{2} L(E_0/\mathbb{Q}, \chi, 1) \right) = x \left( P(\rho \cdot e^{-2\pi/3\sqrt{11}}) - P(\rho^{-1} \cdot e^{-2\pi/3\sqrt{11}}) \right).
\end{equation}

Let
\[ Q := P(\rho \cdot e^{-2\pi/3\sqrt{11}}) \approx (-2.055777... + i \cdot 1.071828..., -0.336526... - i \cdot 1.905429...) \]
and note $Q = P(\rho^{-1} \cdot e^{-2\pi/3\sqrt{11}})$, where $Q$ is the complex conjugate of $Q$ in $E(\mathbb{C})$. Now
\[ \rho \cdot \exp \left( -\frac{2\pi}{3\sqrt{11}} \right) = \exp \left( 2\pi i \left( \frac{1}{3} + \sqrt{-11} \right) \right), \]
and so $\tau = 1/3 + \sqrt{-11}/33 \in \mathbb{H}$ represents the Heegner point $Q := P(\rho \cdot e^{-2\pi/3\sqrt{11}}) \in E_0(\mathbb{Q})$. As $\tau$ is a root of $33x^2 - 22x + 4$, this is a Heegner point of discriminant $-44$. Moreover, in the notation of [15],
\[ Q = \phi([Z[\sqrt{-11}], (\sqrt{-11}), [(11 + \sqrt{-11}, 33)]], \]
where $Z[\sqrt{-11}]$ is the order of discriminant $-44$, $(\sqrt{-11}) \subseteq Z[\sqrt{-11}]$ is an ideal of norm $11$, and $[(11 + \sqrt{-11}, 33)]$ is an ideal class in the class group of $Z[\sqrt{-11}]$. What is important to note here is that by [15, Eq. (5.2)], $Q$ is fixed under the induced Atkin-Lehner involution on $E$:
\[ w_{11}(Q) = \phi([Z[\sqrt{-11}], (-\sqrt{-11}), [(11 + \sqrt{-11}, 33) \cdot (\sqrt{-11})^{-1}])] = Q. \]

Therefore, as in [14], [17], and [18], we obtain
\[ 2Q = \phi(0) = (47/3, -121/2). \]

For $\overline{Q}$ we start with $\tau = -1/3 + \sqrt{-11}/33$, and in the same manner find that $2\overline{Q} = \phi(0)$ as well. Therefore, $Q - \overline{Q}$ is a 2-torsion point, and since $Q \neq \overline{Q}$, its order is 2. By [23, Thm. V.2.3] (and its proof), since $E(\mathbb{R})$ has a single component, we can write the period lattice of $E$ as $Z\Omega + Z\Omega'$, where $\Omega$ is the minimal positive real period and $\text{Re}(\Omega') = \frac{1}{2} \Omega$, $\text{Im}(\Omega') \approx -1.4588166169...$. From (22), as $(\sqrt{-3}/2)L(E_0/\mathbb{Q}, \chi, 1)$ is purely imaginary, we then have
\[ \frac{\sqrt{-3}}{2} L(E_0/\mathbb{Q}, \chi, 1) \in \frac{\Omega - 2\Omega'}{2} \cdot Z, \]
and by comparing approximations we find exactly
\begin{equation}
(23) \quad L(E_0/\mathbb{Q}, \chi, 1) = \frac{\Omega - 2\Omega'}{\sqrt{-3}}.
\end{equation}

A few comments are in order. (1) The specializations on the right-hand side of (22) converge in $\mathbb{C}$, and the right-hand side is well-defined because $\varphi$ is meromorphic on all of $\mathbb{C}$. However, the value $(\sqrt{-3}/2)L(E_0/\mathbb{Q}, \chi, 1)$ is outside of the radius of convergence of the power series for $\varphi(z)$ centered at $z = 0$. On the other hand, $(\sqrt{-3}/2)L(E_0/\mathbb{Q}, \chi, 1) + \Omega'$ is within the radius of convergence, so by shifting the series for $\varphi(z)$ by $\Omega'$, the identity in (22) holds.

(2) By taking the quadratic twist of $E_0$ by $-3$, we arrive at the strong Weil curve $F_0 : y^2 + y = x^3 - 3x - 5$ of conductor 99. Then $L(E_0/\mathbb{Q}, \chi, s) = L(F_0/\mathbb{Q}, s)$, and these same calculations can proceed as in Example 3.1. Indeed $(\Omega - 2\Omega')/\sqrt{-3}$ turns out to be
the real period associated to $F_0$. In Example 3.3 we consider the case of a cubic character where this method is not available and where we can generalize the present example.

**Example 3.3.** We continue with the notation of the previous two examples, and now let $\psi$ be the cubic Dirichlet character modulo 7, satisfying for $\rho = e^{2\pi i / 3}$,

$$
\psi(1) = 1, \quad \psi(2) = \rho^2, \quad \psi(3) = \rho, \ldots
$$

For $\zeta = e^{2\pi i / 7}$, we let $g(\psi) = \sum_{j=1}^6 \psi(j)\zeta^j$ be the associated Gauss sum, and we set

$$
C_\psi := \psi(-11) \frac{g(\psi)}{g(\overline{\psi})} = \rho \cdot \frac{g(\psi)}{g(\overline{\psi})}, \quad C_{\overline{\psi}} := \overline{\psi}(-11) \frac{g(\psi)}{g(\overline{\psi})} = \rho^2 \cdot \frac{g(\psi)}{g(\overline{\psi})}
$$

Then $C_\psi$ is the sign of the functional equation of $L(f, \psi, s)$ by [19, Thm. 7.6], so that if $\Lambda(E_0/\mathbb{Q}, \psi, s) = (7\sqrt{11}/2\pi)^s \Gamma(s)L(E_0/\mathbb{Q}, \psi, s)$, then

$$
\Lambda(E_0/\mathbb{Q}, \psi, s) = C_\psi \Lambda(E_0/\mathbb{Q}, \overline{\psi}, 2 - s).
$$

In a similar manner to the derivation of (20) in [11, Prop. 2.11.2], one finds

$$
L(E_0/\mathbb{Q}, \psi, 1) = S_\psi + C_\psi S_{\overline{\psi}},
$$

where

$$
S_\psi := \sum_{n=1}^\infty \frac{\psi(n)a_n}{n} e^{-2\pi n/7\sqrt{11}}, \quad S_{\overline{\psi}} := \sum_{n=1}^\infty \frac{\overline{\psi}(n)a_n}{n} e^{-2\pi n/7\sqrt{11}}.
$$

This leads to the approximation $L(E_0/\mathbb{Q}, \psi, 1) \approx 1.997106... + i \cdot 1.328439...$. Setting $\gamma = \sum_{j=1}^6 \psi(j)u^j$ and $\overline{\gamma} = \sum_{j=1}^6 \overline{\psi}(j)u^j$, the theory of Gauss sums implies that

$$
\gamma(\zeta^k) = \overline{\psi}(k)g(\psi), \quad \overline{\gamma}(\zeta^k) = \psi(k)g(\overline{\psi}).
$$

We form polynomials in $\mathbb{Z}[u]$,

$$
\beta_1 = \gamma + \overline{\gamma} = 2u - u^2 - u^3 - u^4 - u^5 + 2u^6,
$$

$$
\beta_2 = \frac{1}{\sqrt{-3}}(\gamma - \overline{\gamma}) = -u^2 + u^3 + u^4 - u^5,
$$

and by setting

$$
T_i := \sum_{n=1}^\infty \frac{a_n\beta_i(u^n)}{n} t^n \bigg|_{u = \zeta, t = \exp(-2\pi i/7\sqrt{11})} \quad i = 1, 2,
$$

we find that

$$
T_1 = g(\overline{\psi})S_\psi + g(\psi)S_{\overline{\psi}}, \quad T_2 = -\frac{g(\psi)}{\sqrt{-3}}S_\psi + \frac{g(\overline{\psi})}{\sqrt{-3}}S_{\overline{\psi}}.
$$

Both $T_1$ and $T_2$ are in $\mathbb{R}$, and by comparing with (24), we obtain

$$
(1 - \sqrt{-3})g(\overline{\psi})L(E_0/\mathbb{Q}, \psi, 1) = T_1 - 3T_2.
$$

With this in mind, we define

$$
\beta := \beta_1 - 3\beta_2 = 2u + 2u^2 - 4u^3 - 4u^4 + 2u^5 + 2u^6,
$$

and

$$
T := T_1 - 3T_2 = \sum_{n=1}^\infty \frac{a_n\beta(u^n)}{n} t^n \bigg|_{u = \zeta, t = \exp(-2\pi i/7\sqrt{11})} \in \mathbb{R}.
$$
Letting \( P_k = P(\zeta^k \cdot e^{-2\pi i/\sqrt{11}}) \in E(\mathbb{Q}) \), we see that
\[
P_k = \phi\left( \frac{k}{7} + \frac{\sqrt{-11}}{77} \right), \quad 1 \leq k \leq 6.
\]
For \( k = 1, \ldots, 6 \), the number \( \tau_k = k/7 + \sqrt{-11}/77 \) is a root of \( 539x^2 - 154kx + 11k^2 + 1 \), and so each \( \tau_k \) represents a Heegner point of discriminant \(-2156\). In the notation of [15],
\[
P_k = \phi\left( \mathbb{Z}[\sqrt{11}], (11, \sqrt{-11}), [(77k + 7\sqrt{-11}, 539)] \right),
\]
where \( \mathbb{Z}[\sqrt{11}] \) is the order of discriminant \(-2156\), \( \mathfrak{n} := (11, \sqrt{-11}) \) is an ideal in this order of norm \( 11 \), and \( [\mathfrak{a}_k] \) represents the ideal class of \( \mathfrak{a}_k := (77k + 7\sqrt{-11}, 539) \). Through computations using PARI [21], the ideal class \([\mathfrak{a}_1] \in \text{Cl}(\mathbb{Z}[\sqrt{11}])\) has order 8, and furthermore,
\[
[a_1]^2 = [a_4], \quad [a_1]^3 = [a_2], \quad [a_1]^4 = [\mathfrak{n}], \quad [a_1]^5 = [a_5], \quad [a_1]^6 = [a_3], \quad [a_1]^7 = [a_6].
\]
By [15] Eq. (5.2), the induced action of the Atkin-Lehner operator on \( P_k \) is
\[
w_{11}(P_k) = \phi\left( \mathbb{Z}[\sqrt{11}], [\mathfrak{a}_k\mathfrak{n}] \right),
\]
and so
\[
w_{11}(P_1) = P_5, \quad w_{11}(P_2) = P_6, \quad w_{11}(P_3) = P_4.
\]
By (27) and (28), when we apply Theorem 2.8(b) to \( \beta \), we have as in Example 3.2
\[
\varphi(T) = x(2P_1 + 2P_2 - 4P_3 - 4P_4 + 2P_5 + 2P_6)
\quad = x(2P_1 + 2w_{11}(P_1) + 2P_2 + 2w_{11}(P_2) - 4P_3 - 4w_{11}(P_3)),
\]
and so \((\varphi(T), \frac{1}{2}\varphi'(T)) = O\) by (14). Therefore, by (26) and the fact that \( T \in \mathbb{R} \), we see
\[
(1 - \sqrt{-3})g(\psi)L(E_0/\mathbb{Q}, \psi, 1) \in \Omega \cdot \mathbb{Z},
\]
and by comparing approximations, we find that the multiple on the left-hand side is \( 10\Omega \). After some rearrangement we obtain the identity
\[
L(E_0/\mathbb{Q}, \psi, 1) = \frac{5}{14}(1 + \sqrt{-3})g(\psi)\Omega,
\]
which aligns with [22] Thm. 1 but provides the precise algebraic multiple of \( \Omega \).

With appropriate modifications to these examples, one could in principle determine the special value \( L(E_0/\mathbb{Q}, \psi, 1) \) exactly for an arbitrary elliptic curve and Dirichlet character, e.g., using the techniques of [2] §4.7. This would be interesting to carry out, but without more specific information about the particular curve and its Heegner points, these log-algebraic methods are limited and determining the values beyond what is already known qualitatively in [22] Thm. 1 would be difficult.

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