STOCHASTIC-LIKE BEHAVIOUR
IN NONUNIFORMLY EXPANDING MAPS

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1. INTRODUCTION

1.1. Determinism versus randomness. A feature of many real-life phenomena in areas as diverse as physics, biology, finance, economics, and many others, is the random-like behaviour of processes which nevertheless are clearly deterministic. On the level of applications this dual aspect has proved very problematic. Specific mathematical models tend to be developed either on the basis that the process is deterministic, in which case sophisticated numerical techniques can be used to attempt to understand and predict the evolution, or that it is random, in which case probability theory is used to model the process. Both approaches lose sight of what is probably the most important and significant characteristic of the system which is precisely that it is deterministic and has random-like behaviour. The theory of Dynamical Systems has contributed a phenomenal amount of work showing that it is perfectly natural for completely deterministic systems to behave in a very random-like way and achieving a quite remarkable understanding of the mechanisms by which this occurs. The purpose of these notes is to survey some of this research.

1.2. Nonuniform expansivity. We shall assume that the state space can be represented by a compact Riemannian manifold $M$ and that the evolution of the process is given by a map $f : M \to M$ which is piecewise differentiable. Following an approach which goes back at least to the first half of the 20th century, we shall discuss how certain statistical properties can be deduced from geometrical assumptions on $f$ formulated explicitly in terms of assumptions on the derivative map $Df$ of $f$. The basic strategy is to construct certain geometrical structures which then imply some statistical/probabilistic properties of the dynamics (a striking and pioneering example of this is the work of Hopf on the ergodicity of geodesic flows on manifolds of negative curvature [Hop39]). Research work following this basic line of reasoning goes under the heading of Hyperbolic Dynamics and/or Smooth Ergodic Theory.

The main focus of these notes will be on maps which satisfy an asymptotic expansivity condition.

**Definition 1.** We say that $f : M \to M$ is (nonuniformly) expanding if there exists $\lambda > 0$ such that

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df_{f^i(x)}^{-1}\|^{-1} > \lambda$$

for almost every $x \in M$. Equivalently, for almost every $x \in M$ there exists a constant $C_x > 0$ such that

$$\prod_{i=0}^{n-1} \|Df_{f^i(x)}^{-1}\|^{-1} \geq C_x e^{\lambda n}$$

for every $n \geq 1$. 
The definition and the corresponding results can be generalized to the case in which the expansivity condition holds only on an invariant set of positive measure instead of on the entire manifold $M$. See also [Alv03] for a detailed treatment of the theory of nonuniformly expanding maps. Notice that the condition $\log \| Df^{-1} \|^{-1} > 0$, which is equivalent to $\| Df^{-1} \|^{-1} > 1$ and which in turn is equivalent to $\| Df^{-1} \| < 1$, implies that all vectors in all directions are contracted by the inverse of $Df$ and thus that all vectors in all directions are expanded by $Df$; the intuitively more obvious condition $\log \| Df \| > 0$, which is equivalent to $\| Df \| > 1$, implies only that there is at least one direction in which vectors are expanded by $Df$. Thus a map is nonuniformly expanding if every vector is asymptotically expanded at a uniform exponential rate. The constant $C_x$ can in principle be arbitrarily small and indicates that an arbitrarily large number of iterates may be needed before this exponential growth becomes apparent.

In the special case in which condition $(\ast)$ holds at every point $x$ and the constant $C$ can be chosen uniformly positive independent of $x$ we say that $f$ is uniformly expanding. Thus uniformly expanding is a special case of nonuniformly expanding. The terminology is slightly awkward for historical reasons: uniformly expanding maps have traditionally been referred to simply as expanding maps whereas this term should more appropriately refer to the more general (i.e. possibly nonuniformly) expanding case. We shall generally say that $f$ is strictly nonuniformly expanding if $f$ satisfies condition $(\ast)$ but is strictly not uniformly expanding. A basic theme of these notes is to discuss the difference between uniformly and nonuniformly expanding maps: how the nonuniformity affects the results and the ideas and techniques used in the proofs and how different degrees of nonuniformity can be made precise.

Nonuniform expansivity is a special case of nonuniform hyperbolicity. This concept was first formulated and studied by Pesin [Pes76, Pes77] and has since become one of the main areas of research in dynamical systems, see [Bunetal89, You95h, BarPes02] and [BarPes04] by Barreira and Pesin in this volume, for extensive and in-depth surveys. The formal definition is in terms of non-zero Lyapunov exponents which means that the tangent bundle can be decomposed into subbundles in which vectors either contract or expand at an asymptotically exponential rate. Nonuniform expansivity corresponds to the case in which all the Lyapunov exponents are positive and therefore all vectors expand asymptotically at an exponential rate. The natural setting for this situation is that of (non-invertible) local diffeomorphisms whereas the theory of nonuniform hyperbolicity has been developed mainly for diffeomorphisms (however see also [Rue82] and [BarPes04, Section 5.8]). For greater generality, and also because this has great importance for applications, we shall also allow various kinds of critical and/or singular points for $f$ or its derivative.

1.3. General overview of the notes. We first review the basic notions of invariant measure, ergodicity, mixing, and decay of correlations in order to fix the notation and to motivate the results and techniques. In section 2 we discuss the key idea of a Markov Structure and sketch some of the arguments used to study systems which admit such a structure. In section 3, 4, and 5 we give a historical and technical survey of many classes of systems for which results are known, giving references to the original proofs whenever possible, and sketching in varying amounts of details the construction of Markov structures in such systems. In section 6 we present some recent abstract results which go towards a general theory of nonuniformly expanding maps. In section 7 we discuss the important problem of verifying the geometric nonuniform expansivity assumptions in specific classes of maps. Finally, in section 8 we make some concluding remarks and present some open questions and conjectures.

The focus on Markov structures is partly a matter of personal preference; in some cases the results can be proved and/or were first proved using completely different arguments and techniques.
Of particular importance is the so-called Functional-Analytic approach in which the problems are reformulated and reduced to questions about the spectrum of a certain linear operator on some functional space. There are several excellent survey texts focussing on this approach, see [Bal01, Liv04, Via]. In any case, it is hard to see how the study of systems in which the hyperbolicity or expansivity is nonuniform can be carried out without constructing or defining some kind of subdivision into subsets on which relevant estimates satisfy uniform bounds. The Markov structures to be described below provide one very useful way in which this can be done and give some concrete geometrical structure. It seems very likely that these structures will prove useful in studying many other features of nonuniformly hyperbolic or expanding systems such as their stability, persistence, and even existence in particular settings. Another quite different way to partition a set satisfying nonuniform hyperbolicity conditions is with so-called Pesin or regular sets, see [BarPes04, Section 4.5]. These sets play a very useful role in the general theory of nonuniform hyperbolicity for diffeomorphisms, for example in the construction of the stable and unstable foliations.

We shall always assume that $M$ is a smooth, compact, Riemannian manifold of dimension $d \geq 1$. For simplicity we shall call the Riemannian volume Lebesgue measure, denote it by $m$ or $| \cdot |$ and assume that it is normalized so that $m(M) = |M| = 1$. We let $f : M \to M$ denote a Lebesgue-measurable map. In practice we shall always assume significantly more regularity on $f$, e.g. that $f$ is $C^2$ or at least piecewise $C^2$, but the main definitions apply in the more general case of $f$ measurable. All measures on $M$ will be assumed to be defined on the Borel $\sigma$-algebra of $M$.

1.4. Invariant measures. For a set $A \in M$ and a map $f : M \to M$ we define $f^{-1}(A) = \{ x : f(x) \in A \}$.

**Definition 2.** We say that a probability measure $\mu$ on $M$ is invariant under $f$ if

$$\mu(f^{-1}(A)) = \mu(A)$$

for every $\mu$-measurable set $A \subset M$.

A given measure can be invariant for many different maps. For example Lebesgue measure on the circle $M = S^1$ is invariant for the identity map $f(\theta) = \theta$, the rotation $f(\theta) = \theta + \alpha$ for any $\alpha \in \mathbb{R}$, and the covering map $f(\theta) = \kappa \theta$ for any $\kappa \in \mathbb{N}$. Similarly, for a given point $p \in M$, the Dirac-$\delta$ measure $\delta_p$ defined by

$$\delta_p(A) = \begin{cases} 1 & p \in A \\ 0 & p \not\in A \end{cases}$$

is invariant for any map $f$ for which $f(p) = p$. On the other hand, a given map $f$ can admit many invariant measures. For example any probability measure is invariant for the identity map $f(x) = x$ and, more generally, any map which admits multiple fixed or periodic points admits as invariant measures the Dirac-$\delta$ measures supported on such fixed points or their natural generalizations distributed along the orbit of the periodic points. There exist also maps that do not admit any invariant probability measures. However some mild conditions, e.g. continuity of $f$, do guarantee that there exists at least one.

A first step in the application of the theory and methods of ergodic theory is to introduce some ways of distinguishing between the various invariant measures. We do this by introducing various properties which such measures may or may not satisfy. Unless we specify otherwise we shall use $\mu$ to denote a generic invariant probability measure for a given unspecified map $f : M \to M$.

1.5. Ergodicity.
Definition 3. We say that $\mu$ is ergodic if there does not exist a measurable set $A$ with

$$f^{-1}(A) = A \quad \text{and} \quad \mu(A) \in (0, 1)$$

In other words, any fully invariant set $A$, i.e. a set for which $f^{-1}(A) = A$, has either zero or full measure. This is a kind of indecomposability property of the measure. If such a set existed, its complement $B = A^c$ would also be fully invariant and, in particular, both $A$ and $B$ would be also forward invariant: $f(A) = A$ and $f(B) = B$. Thus no point originating in $A$ could ever intersect $B$ and vice-versa and we essentially have two independent dynamical systems.

Simple examples such as the Dirac-$\delta_p$ measure on a fixed point $p$ are easily shown to be ergodic, but in general this is a highly non-trivial property to prove. A lot of the techniques and methods to be described below are fundamentally motivated by the basic question of whether some relevant invariant measures are ergodic. It is known that Lebesgue measure is ergodic for circle rotations $f(\theta) = \theta + \alpha$ when $\alpha$ is irrational and for covering maps $f(\theta) = \kappa \theta$ when $\kappa \in \mathbb{N}$ is $\geq 2$ (the proof of ergodicity for the latter case will be sketched below). Irrational circle rotations are very special because they do not admit any other invariant measures besides Lebesgue measure. On the other hand covering maps have infinitely many periodic points and thus admit infinitely many invariant measures. It is sometimes easier to show that certain examples are not ergodic. This is clearly true for example for Lebesgue measure and the identity map since any subset is fully invariant. A less trivial example is the map $f : [0, 1] \rightarrow [0, 1]$ given by

$$f(x) = \begin{cases} 
2x & \text{if } 0 \leq x \leq 1/4 \\
-2x + 1 & \text{if } 1/4 \leq x < 1/2 \\
2x - 1/2 & \text{if } 1/2 \leq x \leq 3/4 \\
-2x + 5/2 & \text{if } 3/4 \leq x \leq 1.
\end{cases}$$

Lebesgue measure is invariant, but the intervals $[0, 1/2)$ and $[1/2, 1]$ are both backward (and forward) invariant. This example can easily be generalized by defining two different Lebesgue measure preserving transformations mapping each of the two subintervals $[0, 1/2)$ and $[1/2, 1]$ into themselves.

The fundamental role played by the notion of ergodicity is given by the well known and classical Birkhoff Ergodic Theorem. We give here only a special case of this result.

Theorem ([Br31][Br42]). Let $f : M \rightarrow M$ be a measurable map and let $\mu$ be an ergodic invariant probability measure for $f$. Then, for any function $\varphi : M \rightarrow \mathbb{R}$ in $L^1(\mu)$, i.e. such that $\int \varphi d\mu < \infty$, and for $\mu$ almost every $x$ we have

$$\frac{1}{n} \sum_{i=1}^{n} \varphi(f^i(x)) \rightarrow \int \varphi d\mu$$

In particular, for any measurable set $A \subset M$, letting $\varphi = 1_A$ be the characteristic function of $A$, we have for $\mu$ almost every $x \in M$,

$$\frac{\# \{1 \leq j \leq n : f^j(x) \in A \}}{n} \rightarrow \mu(A).$$

(1)

Here $\# \{1 \leq j \leq n : f^j(x) \in A \}$ denotes the cardinality of the set of indices $j$ for which $f^j(x) \in A$. Thus the average proportion of time which the orbit of a typical point spends in $A$ converges precisely to the $\mu$-measure of $A$. Notice that the convergence of this proportion as $n \rightarrow \infty$ is in itself an extremely remarkable and non-intuitive result. The fact that the limit is given $a \ priori$ by $\mu(A)$ means in particular that this limit is independent of the specific initial condition $x$. Thus $\mu$-almost every initial condition has the same statistical distribution in space and this distribution depends
only on \(\mu\) and not even on the map \(f\), except implicitly for the fact that \(\mu\) is ergodic and invariant for \(f\).

1.6. Absolute continuity. Some care needs to be taken when applying Birkhoff’s ergodic theorem to maps which admit several ergodic invariant measures. Consider for example the circle map \(f(\theta) = 10\theta\). This maps preserves Lebesgue measure and also has several fixed points, e.g. \(p = 0.2222\ldots\), on which we can consider the Dirac-\(\delta_p\) measure. Both these measures are ergodic. Thus an application of Birkhoff’s theorem says that “almost every” point spends an average proportion of time converging to \(m(A)\) in the set \(A\) but also that “almost every” point spends an average proportion of time converging to \(\delta_p(A)\) in the set \(A\). If \(m(A) \neq \delta_p(A)\) this may appear to generate a contradiction.

The crucial observation here is that the notion of *almost every* point is always understood with respect to a particular measure. Thus Birkhoff’s ergodic theorem asserts that for a given measure \(\mu\) there exists a set \(\tilde{M} \subset M\) with \(\mu(\tilde{M}) = 1\) such that the convergence property holds for every \(x \in \tilde{M}\) and in general it may not be possible to identify \(\tilde{M}\) explicitly. Conversely, if \(X \in M\) satisfies \(\mu(X) = 0\) then no conclusion can be drawn about whether \(|\square|\) holds for any point of \(X\). Returning to the example given above we have the following situation: the convergence \(|\square|\) of the time averages to \(\delta_p(A)\) can be guaranteed only for points belonging to a minimal set of full measure. But in this case this set reduces to the single point \(p\) for which \(|\square|\) clearly holds. On the other hand the single point \(p\) clearly has zero Lebesgue measure and thus the convergence \(|\square|\) to \(m(A)\) is not guaranteed by Birkhoff’s Theorem. Thus there is no contradiction.

An important point therefore is that the information provided by Birkhoff’s ergodic theorem depends on the measure \(\mu\) under consideration. Based on the premise that Lebesgue is the given “physical” measure and that we consider a satisfactory description of the dynamics one which accounts for a sufficiently large set of points from the point of view of Lebesgue measure, it is clear that if \(\mu\) is a Dirac-\(\delta\) measure on a fixed point it gives essentially no useful information. On the other hand, if \(\mu\) is Lebesgue measure itself then we do get a convergence result that holds for Lebesgue almost every starting condition. The invariance of Lebesgue measure is a very special property but much more generally we can ask about the existence of ergodic invariant measures \(\mu\) which are *absolutely continuous* with respect to \(m\).

**Definition 4.** \(\mu\) is absolutely continuous with respect to \(m\) if

\[
m(A) = 0 \quad \text{implies} \quad \mu(A) = 0
\]

for every measurable set \(A \subset M\).

In this case, Birkhoff’s theorem implies that \(|\square|\) holds for all points belonging to a set \(\tilde{M} \subset M\) with \(\mu(\tilde{M}) = 1\) and the absolute continuity of \(\mu\) with respect to \(m\) therefore implies that \(m(\tilde{M}) > 0\). Thus the existence of an ergodic *absolutely continuous invariant probability (acip)* \(\mu\) implies some control over the asymptotic distribution of at least a set of positive Lebesgue measure. It also implies that such points tend to have a dynamics which is non-trivial in the sense that it is distributed over some relatively large subset of the space as opposed to converging for example to some attracting fixed point. Thus it indicates that there is a minimum amount of inherent *complexity* as well as *structure*. This motivates the basic question:

1) Under what conditions does \(f\) admit an ergodic acip?

This question is already addressed explicitly by Hopf ([Hop32]) for invertible transformations. Interestingly he formulates some conditions in terms of the existence of what are essentially some induced
transformations, similar in some respects to the Markov structures to be defined below.\(^1\) In these notes we shall discuss what is effectively a generalization of this basic approach.

1.7. Mixing. Birkhoff’s ergodic theorem is very powerful but it is easy to see that the asymptotic space distribution given by (1) does not necessarily tell the whole story about the dynamics of a given map \(f\). Indeed these conclusions depend not on \(f\) but simply on the fact that Lebesgue measure is invariant and ergodic. Thus from this point of view the dynamics of an irrational circle rotation \(f(\theta) = \theta + \alpha\) and of the map \(f(\theta) = 2\theta\) are indistinguishable. However it is clear that they give rise to very different kinds of dynamics. In one case for example, nearby points remain nearby for all time, whereas in the other they tend to move apart at an exponential speed. This creates a kind of unpredictability in one case which is not present in the other.

**Definition 5.** We say that an invariant probability measure \(\mu\) is **mixing** if

\[
|\mu(A \cap f^{-n}(B)) - \mu(A)\mu(B)| \to 0
\]

as \(n \to \infty\), for all measurable sets \(A, B \subseteq M\).

Notice that mixing implies ergodicity and is therefore a stronger property. Thus a natural follow up to question 1 is the following. Suppose that \(f\) admits an ergodic acip \(\mu\).

2) Under what conditions is \(\mu\) mixing?\(^2\)

Early work in ergodic theory in the 1940’s considered the question of the genericity of the mixing property is in various spaces of systems [Hal44, Roh48, Ros56, Hol57, KacKes58] but, as with ergodicity, in specific classes of systems it is generally easier to show that a system is not mixing rather than that it is mixing. For example it is immediate that irrational circle rotations are not mixing. On the other hand it is non-trivial that maps of the form \(f(\theta) = \kappa\theta\) for integers \(\kappa \geq 2\) are mixing.

To develop an intuition for the concept of mixing, notice that mixing is equivalent to the condition

\[
\left| \frac{\mu(A \cap f^{-n}(B))}{\mu(B)} - \mu(A) \right| \to 0
\]

as \(n \to \infty\), for all measurable sets \(A, B \subseteq M\), with \(\mu(B) \neq 0\). In this form there are two natural interpretations of mixing, one geometrical and one probabilistic. From a geometrical point of view, recall that \(\mu(f^{-n}(B)) = \mu(B)\) by the invariance of the measure. Then one can think of \(f^{-n}(B)\) as a “redistribution of mass” and the mixing condition says that for large \(n\) the proportion of \(f^{-n}(B)\) which intersects \(A\) is just proportional to the measure of \(A\). In other words \(f^{-n}(B)\) is spreading itself uniformly with respect to the measure \(\mu\). A more probabilistic point of view is to think of \(\mu(A \cap f^{-n}(B))/\mu(B)\) as the conditional probability of having \(x \in A\) given that \(f^n(x) \in B\), i.e. the probability that the occurrence of the event \(B\) today is a consequence of the occurrence of the event \(A\) \(n\) steps in the past. The mixing condition then says that this probability converges to the probability of \(A\), i.e., asymptotically, there is no causal relation between the two events. This is why we say that a mixing system exhibits stochastic-like or random-like behaviour.

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\(^1\)Hopf’s result is the following: suppose that for every measurable partition \(\mathcal{P}\) of the manifold \(M\) and every stopping time function \(p\) such that the images \(f^{p(n)}(\omega)\) for \(\omega \in \mathcal{P}\) are all disjoint, the union of all images has full measure. Then \(f\) admits an absolutely continuous invariant probability measure.
1.8. Decay of correlations. It turns out that mixing is indeed a quite generic property under certain assumptions which will generally hold in the examples we shall be interested in. Thus apparently very different systems admit mixing *acip’s* and become, in some sense, statistically indistinguishable at this level of description. Thus it is natural to want to dig deeper in an attempt relate finer statistical properties with specific geometric characteristics of systems under considerations. One way to do this is to try to distinguish systems which mix at different speeds. To formalize this idea we need to generalize the definition of mixing. Notice first of all that the original definition can be written in integral form as

$$\left| \int 1_{A \cap f^{-n}(B)} d\mu - \int 1_A d\mu \int 1_B d\mu \right| \to 0$$

where $1_X$ denotes the characteristic function of the set $X$. This can be written in the equivalent form

$$\left| \int 1_A(1_B \circ f^n) d\mu - \int 1_A d\mu \int 1_B d\mu \right| \to 0$$

and this last formulation now admits a natural generalization by replacing the characteristic functions with arbitrary measurable functions.

**Definition 6.** For real valued measurable functions $\varphi, \psi : M \to \mathbb{R}$ we define the correlation function $C_n(\varphi, \psi) = \left| \int \psi(\varphi \circ f^n) d\mu - \int \psi d\mu \int \varphi d\mu \right|$.

In this context, the functions $\varphi$ and $\psi$ are often called *observables*. If $\mu$ is mixing, the correlation function decays to zero whenever the observables $\varphi, \psi$ are characteristic functions. It is possible to show that indeed it decays also for many other classes of functions. We then have the following very natural question. Suppose that the measure $\mu$ is mixing, fix two observables $\varphi, \psi$, and let $C_n = C_n(\varphi, \psi)$.

3) Does $C_n$ decay at a specific rate depending only on $f$? The idea behind this question is that a system may have an intrinsic *rate of mixing* which reflects some characteristic geometrical structures. It turns out that an intrinsic rate does sometimes exist and is in some cases possible to determine, but only by restricting to a suitable class of observables. Indeed, a classical result says that even in the “best” cases it is possible to choose subsets $A, B$ such that the correlation function $C_n(1_A, 1_B)$ of the corresponding characteristic functions decays at an arbitrarily slow rate. Instead positive results exist in many cases by restricting to, for example, the space of observables of bounded variation, or Hölder continuous, or even continuous with non-Hölder modulus of continuity. Once the space $\mathcal{H}$ of observables has been fixed, the goal is to show that there exists a sequence $\gamma_n \to 0$ (e.g. $\gamma_n = e^{-\alpha n}$ or $\gamma_n = n^{-\alpha}$ for some $\alpha > 0$) depending only on $f$ and $\mathcal{H}$, such that for any two $\varphi, \psi \in \mathcal{H}$ there exists a constant $C = C(\varphi, \psi)$ (generally depending on the observables $\varphi, \psi$) such that

$$C_n \leq C\gamma_n$$

for all $n \geq 1$. Ideally we would like to show that $C_n$ actually decays like $\gamma_n$, i.e. to have both lower and upper bounds, but this is known only in some very particular cases. Most known results at present

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The derivation of the correlation function from the definition of mixing as given here does not perhaps correspond to the historical development. I believe that the notion of decay of correlation arose in the context of statistical mechanics and was not directly linked to abstract dynamical systems framework until the work of Bowen, Lebowitz, Ruelle and Sinai in the 1960’s and 1970’s. [Sin68, Bow70, Sin72, PenLeb74, Bow75]
are upper bounds and thus we say that the correlation functions decays at a certain rate we will usually mean that it decays at least at that rate. Also, most known results deal with Hölder continuous observables and thus, to simplify the presentation, we shall assume assume that we are dealing with this class unless we mention otherwise.

We shall discuss below several examples of systems whose correlations decay at different rates, for example exponential, polynomial or even logarithmic, and a basic theme of these notes will be gain some understanding about how and why such differences occur and what this tells us about the system.

2. Markov Structures

Definition 7. \( f : M \to M \) is (or admits) a Markov map if there exists a finite or countable partition \( P \) (mod 0) of \( M \) into open sets with smooth boundaries such that \( f(\omega) = M \) for every partition element \( \omega \in P \) and \( f|_\omega \) is a continuous non-singular bijection.

We recall that a partition mod 0 of \( M \) means that Lebesgue almost every point belongs to the interior of some partition elements. Also, \( f|_\omega \) is non-singular if \( |A| > 0 \) implies \( |f(A)| > 0 \) for every (measurable) \( A \subset \omega \). These two conditions together immediately imply that the full forward orbit of almost every point always lies in the interior of some partition element. The condition \( f(\omega) = M \) is a particularly strong version of what is generally referred to as the Markov property where it is only required that the image of each \( \omega \) be a continuous non-singular bijection onto some union of partition elements and not necessarily all of \( M \). The stronger requirement we use here is sometimes called a Bernoulli property. A significant generalization of this definition allows the partition element to be just measurable sets and not necessarily open; the general results to be given below apply in this case also. However we shall not need this for any of the applications which we shall discuss.

A natural but extremely far-reaching generalization of the notion of a Markov map is the following.

Definition 8. \( f : M \to M \) admits an induced Markov map if there exists an open set \( \Delta \subset M \), a partition \( P \) (mod 0) of \( \Delta \) and a return time function \( R : \Delta \to \mathbb{N} \), piecewise constant on each element of \( P \), such that the induced map \( F : \Delta \to \Delta \) defined by \( F(x) = f^{R(x)} \) is a Markov map.

Again, the condition that \( \Delta \) is open is not strictly necessary. For the rest of this section we shall suppose that \( F : \Delta \to \Delta \) is an induced Markov map associated to some map \( f : M \to M \). We call \( P \) the Markov partition associated to the Markov map \( F \). Clearly if \( f \) is a Markov map to begin with, it trivially admits an induced Markov map with \( \Delta = M \) and \( R \equiv 1 \). Since \( P \) is assumed to be countable, we can define an indexing set \( \omega = \{0, 1, 2, \ldots \} \) of the Markov partition \( P \). Then, for any finite sequence \( a_0 a_1 a_2 \ldots, a_n \) with \( a_i \in I \), we can define the cylinder set of order \( n \) by

\[
\omega_{a_0 a_1 \ldots a_n}^{(n)} \{ x : F^i(x) \in \omega_{a_i} \text{ for } 0 \leq i \leq n \}.
\]

Inductively, given \( \omega_{a_0 a_1 \ldots a_{n-1}} \), then \( \omega_{a_0 a_1 \ldots a_n} \) is the part of \( \omega_{a_0 a_1 \ldots a_{n-1}} \) mapped to \( \omega_{a_n} \) by \( F^n \). The cylinder sets define refinements of the partition \( P \). We let \( \omega_0 \) denote generic elements of \( P_0 = P \) and \( \omega^{(n)} \) denote generic elements of \( P^{(n)} \). Notice that by the non-singularity of the map \( F \) on each partition element and the fact that \( P \) is a partition mod 0, it follows that each \( P^{(n)} \) is also a partition mod 0 and that Lebesgue almost every point in \( \Delta \) falls in the interior of some partition element of \( P \) for all future iterates. In particular almost every \( x \in \Delta \) has an associated infinite symbolic sequence \( a(x) \) determined by the future iterates of \( x \) in relation to the partition \( P \). To get the much more sophisticated results on the statistical properties of \( f \) we need first of all the following two additional conditions.
Definition 9. \( F : \Delta \to \Delta \) has integrable or summable return times if
\[
\int_{\Delta} R(x) dx = \sum_{\omega \in P} |\omega| R(\omega) < \infty.
\]

Definition 10. \( F : \Delta \to \Delta \) has the geometric self-similarity property (or, more prosaically, the (volume) bounded distortion property) if there exists a constant \( D > 0 \) such that for all \( n \geq 1 \) and any measurable subset \( \tilde{\omega}^{(n)} \subset \omega^{(n)} \in \mathcal{P}^{(n)} \) we have
\[
\frac{1}{D} |\tilde{\omega}^{(n)}| \leq |f^n(\tilde{\omega}^{(n)})| \leq D |\tilde{\omega}^{(n)}|.
\]

This means that the relative measure of subsets of a cylinder set of any level \( n \) are preserved up to some factor \( D \) under iteration by \( f^n \). A crucial observation here is that the constant \( D \) is independent of \( n \). Thus in some sense the geometrical structure of any subset of \( \Delta \) re-occurs at every scale inside each partition element of \( \mathcal{P}^{(n)} \) up to some bounded distortion factor. This is in principle a very strong condition but we shall see below that it is possible to verify it in many situations. We shall discuss in the next section some techniques for verifying this condition in practice. First of all we state the first result of this section.

Theorem 1. Suppose that \( f : M \to M \) admits an induced Markov map satisfying the geometric self-similarity property and having summable return times. Then it admits an ergodic absolutely continuous invariant probability measure \( \mu \).

This result goes back to the 1950’s and is often referred to as the Folklore Theorem of dynamics. We will sketch below the main ideas of the proof. First however we state a much more recent result which applies in the same setting but takes the conclusions much further in the direction of mixing and rates of decay of correlations. First of all we shall assume without loss of generality that the greatest common divisor of all values taken by the return time function \( R \) is 1. If this were not the case all return times would be multiples of some integer \( k \geq 2 \) and the measure \( \mu \) given by the Theorem stated above would clearly not be mixing. If this is the case however, we could just consider the map \( \tilde{f} = f^k \) and the results to be stated below will apply to \( \tilde{f} \) instead of \( f \). We define the tail of the return times as the measure of the set
\[
R_n = \{ x \in \Delta : R(x) > n \}
\]
of points whose return times is strictly larger than \( n \). The integrability condition implies that \( R(x) < \infty \) for almost every point and thus
\[
|R_n| \to 0
\]
as \( n \to \infty \). However there is a range of possible rates of decay of \( |R_n| \) all of which are compatible with the integrability condition. L.-S. Young observed and proved that a bound on the decay of correlations for Hölder continuous observables can be obtained from bounds on the rate of decay of the tail of the return times.

Theorem 2 (You98 You99). Suppose that \( f : M \to M \) admits an induced Markov map satisfying the geometric self-similarity property and having summable return times. Then it admits an ergodic (and mixing) absolutely continuous invariant probability measure. Moreover the correlation function for Hölder continuous observables satisfies the following bounds:

**Exponential tail:** If \( \exists \alpha > 0 \) such that \( |R_n| = \mathcal{O}(e^{-\alpha n}) \), then \( \exists \alpha > 0 \) such that \( \mathcal{C}_n = \mathcal{O}(e^{-\tilde{\alpha} n}) \).

**Polynomial tail:** If \( \exists \alpha > 1 \) such that \( |R_n| = \mathcal{O}(n^\alpha) \), then \( \mathcal{C}_n = \mathcal{O}(n^{-\alpha+1}) \).
Other papers have also addressed the question of the decay of correlations for similar setups mainly using spectral operator methods \cite{You98,Bre99,Mau01a,BreFerGal99,Mau01b}. We remark that the results about the rates of decay of correlations generally require an a priori slightly stronger form of bounded distortion than that given in (2). The proof in \cite{You99} uses a very geometrical/probabilistic coupling argument which appears to be quite versatile and flexible. Variations of the argument have been applied to prove the following generalizations which apply in the same setting as above (in both cases we state only a particular case of the theorems proved in the cited papers).

The first one extends Young’s result to arbitrarily slow rates of decay. We say that $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is slowly varying \cite{Aar97} if for all $y > 0$ we have $\lim_{x \to \infty} \rho(xy)/\rho(x) = 0$. A simple example of a slowly varying function is the function $\rho(x) = e^{(\log x)/(\log \log x)}$. Let $\hat{R}_n = \sum_{n \geq n} R_n$.

**Theorem 3** (Hol04). The correlation function for H"older continuous observables satisfies the following bound.

--- Slowly varying tail: If $\hat{R}_n = \mathcal{O}(\rho(n))$ where $\rho$ is a monotonically decreasing to zero, slowly varying, $C^\infty$ function, then $\mathcal{C}_n = \mathcal{O}(\rho(n))$.

The second extends Young’s result to observables with very weak, non-H"older, modulus of continuity. We say that $\psi : I \to \mathbb{R}$ has a logarithmic modulus of continuity $\gamma$ if there exists $C > 0$ such that for all $x, y \in I$ we have

$$|\psi(x) - \psi(y)| \leq C|\log |x - y||^{-\gamma}.$$  

For both the exponential and polynomial tail situations we have the following

**Theorem 4** (Lyn04). There exists $\alpha > 0$ such that for all $\gamma$ sufficiently large and observables with logarithmic modulus of continuity $\gamma$, we have $\mathcal{C}_n = \mathcal{O}(n^{-\alpha})$.

These general results indicate that the rate of decay of correlations is linked to what is in effect the geometrical structure of $f$ as reflected in the tail of the return times for the induced map $F$. From a technical point of view they shift the problem of the statistical properties of $f$ to the problem of the geometrical structure of $f$ and thus to the (still highly non-trivial) problem of showing that $f$ admits an induced Markov map and of estimating the tail of the return times of this map. The construction of an induced map in certain examples is relatively straightforward and essentially canonical but the most interesting constructions require statistical arguments to even show that such a map exists and to estimate the tail of the return times. In these cases the construction is not canonical and it is usually not completely clear to what extent the estimates might depend on the construction.

We now give a sketch of the proof of Theorem 1. The proofs of Theorems 2, 3, and 4 are in a similar spirit and we refer the interested reader to the original papers. We assume throughout the next few sections that $F : \Delta \to \Delta$ is the Markov induced map associated to $f : I \to I$ and $\mathcal{P}^{(n)}$ are the family of cylinder sets generated by the Markov partition $\mathcal{P} = \mathcal{P}^{(0)}$ of $\Delta$. We first define a measure $\nu$ on $\Delta$ and show that it is $F$-invariant, ergodic, and absolutely continuous with respect to Lebesgue. Then we define the measure $\mu$ on $I$ in terms of $\nu$ and show that it is $f$-invariant, ergodic, and absolutely continuous.

### 2.1. The invariant measure for $F$.\footnote{We start with a preliminary result which is a consequence of the bounded distortion property.}

#### 2.1.1. The measure of cylinder sets.\footnote{A straightforward but remarkable consequence of the bounded distortion property is that the measure of cylinder sets tends to zero uniformly.}
Lemma 2.1.

\[ \max \{ |\omega^{(n)}|; \omega^{(n)} \in P^{(n)} \} \to 0 \quad \text{as} \quad n \to 0 \]

Notice that in the one-dimensional case, the measure of an interval coincides with its diameter and so this implies in particular that the diameter of cylinder sets tends to zero, implying the essential uniqueness of the symbolic representation of itineraries.

Proof. It is sufficient to show that there exists a constant \( \tau \in (0, 1) \) such that for every \( n \geq 0 \) and every \( \omega^{(n)} \subset \omega^{(n-1)} \) we have

\[ |\omega^{(n)}|/|\omega^{(n-1)}| \leq \tau. \]  

Applying this inequality recursively then implies \( |\omega^{(n)}| \leq \tau|\omega^{(n-1)}| \leq \tau^2|\omega^{(n-2)}| \leq \cdots \leq \tau^n|\omega^0| \leq \tau^n|\Delta| \). To verify (4) we shall show that

\[ 1 - \frac{|\omega^{(n)}|}{|\omega^{(n-1)}|} = \frac{|\omega^{(n-1)}| - |\omega^{(n)}|}{|\omega^{(n)}|} = \frac{|\omega^{(n-1)} \setminus \omega^{(n)}|}{|\omega^{(n)}|} \geq 1 - \tau. \]

To prove (4) let first of all \( \delta = \max_{\omega \in P} |\omega| < |\Delta| \). Then, from the definition of cylinder sets we have that \( F^n(\omega^{(n-1)}) = \Delta \) and that \( F^n(\omega^{(n)}) \in P = P^{(0)} \), and therefore \( |F^n(\omega^{(n)})| \leq \delta \) or, equivalently, \( |F^n(\omega^{(n-1)} \setminus \omega^{(n)})| \geq |\Delta| - \delta > 0 \). Thus, using the bounded distortion property we have

\[ \frac{|\omega^{(n-1)} \setminus \omega^{(n)}|}{|\omega^{(n)}|} \geq \frac{1}{D} \frac{|F^n(\omega^{(n-1)} \setminus \omega^{(n)})|}{|F^n(\omega^{(n)})|} \geq \frac{|\Delta| - \delta}{|\Delta|D} \]

and (4) follows choosing \( \tau = 1 - ((|\Delta| - \delta)/|\Delta|D) \). \( \square \)

The next property actually follows only from the conclusions of Lemma 2.1 rather than from the bounded distortion property itself. It essentially says that it is possible to “zoom in” to any given set of positive measure.

Lemma 2.2. For any \( \varepsilon > 0 \) and any Borel set \( A \) with \( |A| > 0 \) there exists \( n \geq 1 \) and \( \omega^{(n)} \in P^{(n)} \) such that

\[ |A \cap \omega^{(n)}| \geq (1 - \varepsilon)|\omega^{(n)}|. \]

Proof. Fix some \( \varepsilon > 0 \). Suppose first of all that \( A \) is compact. Then, using the properties of Lebesgue measure it is possible to show that for any \( \eta > 0 \) there exists an integer \( n \geq 1 \) and a collection \( I_\eta = \{ \omega^{(n)} \} \subset P^{(n)} \) such that \( A \subset \cup_{\omega^{(n)} \in I_\eta} \omega^{(n)} \) and \( |\omega^{(n)}| \leq |A| + \eta \). Now suppose by contradiction that \( |\omega^{(n)} \cap A| \leq (1 - \varepsilon)|\omega^{(n)}| \) for every \( \omega^{(n)} \in \omega_\eta \) for any given \( \eta > 0 \). Using that fact that the \( \omega^{(n)} \in \omega_\eta \) are disjoint and thus \( \sum |\omega^{(n)}| = |\omega_\eta| \), this implies that

\[ |A| = \sum_{\omega^{(n)} \in \omega_\eta} |\omega^{(n)} \cap A| \leq (1 - \varepsilon) \sum_{\omega^{(n)} \in \omega_\eta} |\omega^{(n)}| \leq (1 - \varepsilon) (|A| + \eta). \]

Since \( \eta \) can be chosen arbitrarily small after fixing \( \varepsilon \) this gives a contradiction. If \( A \) is not compact we can approximate it from below in measure by compact sets and repeat essentially the same argument. \( \square \)
2.1.2. Absolute continuity. The following estimate also follows immediately from the bounded distortion property. It says that the absolute continuity property of \( F \) on partition elements is preserved up to arbitrary scale with uniform bounds.

**Lemma 2.3.** Let \( A \subset \Delta \) and \( n \geq 1 \). Then

\[
|F^{-n}(A)| \leq D|A|.
\]

**Proof.** The Markov property implies that \( F^{-n}(A) \) is a union of disjoint sets each contained in the interior of some element \( \omega^{(n)} \in \mathcal{P}(n) \). Moreover each \( \omega^{(n)} \) is mapped by \( F^n \) to \( \Delta \) with uniformly bounded distortion, thus we have \( |F^{-n}(A) \cap \omega^{(n)}|/|\omega^{(n)}| \leq D|A|/|\Delta| \) or, equivalently, \( |F^{-n}(A) \cap \omega^{(n)}| \leq D|A||\omega^{(n)}|/|\Delta| \). Therefore

\[
|F^{-n}(A)| = \sum_{\omega^{(n)} \in \mathcal{P}(n)} |F^{-n}(A) \cap \omega^{(n)}| \leq \frac{D|A|}{|\Delta|} \sum_{\omega^{(n)} \in \mathcal{P}(n)} |\omega^{(n)}| = D|A|
\]

\( \Box \)

2.1.3. The pull-back of a measure. For any \( n \geq 1 \) and Borel \( A \subseteq \Delta \), let

\[
\nu_n(A) = \frac{1}{n} \sum_{i=0}^{n-1} |F^{-i}(A)|.
\]

It is easy to see that \( \nu_n \) is a probability measure on \( \Delta \) and absolutely continuous with respect to Lebesgue. Moreover, lemma 2.3 implies that the absolute continuity property is uniform in \( n \) and \( A \) in the sense that \( \nu_n(A) \leq D|A| \) for any \( A \) and for any \( n \geq 1 \). By some standard results of functional analysis, this implies the following

**Lemma 2.4.** There exists a probability measure \( \nu \) and a subsequence \( \{\nu_{n_k}\} \) such that, for every measurable set \( A \),

\[
(5) \quad \nu_{n_k}(A) \to \nu(A) \leq D|A|.
\]

In particular \( A \) is absolutely continuous with respect to Lebesgue.

2.1.4. Invariance. To show that \( \nu \) is \( F \)-invariant, let \( A \subset \Delta \) be a measurable set. Then, by (5) we have

\[
\nu(F^{-1}(A)) = \lim_{k \to \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} |F^{-(i+1)}(A)|
\]

\[
= \lim_{k \to \infty} \left[ \frac{1}{n_k} \sum_{i=0}^{n_k-1} |F^{-1}(A)| - \frac{|A|}{n_k} + \frac{|F^{-n_k}(A)|}{n_k} \right]
\]

Since \( |A| \) and \( |F^{-n}(A)| \) are both uniformly bounded by 1, we have \( |A|/n_k \to 0 \) and \( |F^{-n_k}(A)|/n_k \to 0 \) as \( k \to 0 \). Therefore

\[
\nu(F^{-1}(A)) = \lim_{k \to \infty} \left[ \frac{1}{n_k} \sum_{i=0}^{n_k-1} |F^{-1}(A)| - \frac{|A|}{n_k} + \frac{|F^{-n_k}(A)|}{n_k} \right]
\]

\[
= \lim_{k \to \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} |F^{-1}(A)| = \nu(A).
\]

Therefore \( \nu \) is \( F \)-invariant.
2.1.5. Ergodicity and uniqueness. Let $A \subset I$ be a measurable set with $F^{-1}(A) = A$ and $\mu(A) > 0$. We shall show that $\nu(A) = |A| = 1$. This implies both ergodicity and uniqueness of $\nu$. Indeed, if $\tilde{\nu}$ were another such measure invariant absolutely continuous measure, there would be have to be a set $B$ with $F^{-1}(B) = B$ and $\tilde{\nu}(B) = 1$. But in this case we would have also $|B| = 1$ and thus $A = B \mod 0$. This is impossible since two absolutely continuous invariant measures must have disjoint support.

To prove that $|A| = 1$, let $A^c = \Delta \setminus A$ denote the complement of $A$. Notice that $x \in A^c$ if and only if $F(x) \in A^c$ and therefore $F(A^c) = A^c$. By Lemma 2.2, for any $\varepsilon > 0$ there exists some $n \geq 1$ and $\omega^{(n)} \in \mathcal{P}^{(n)}$ such that $|A \cap \omega^{(n)}| \geq (1 - \varepsilon)|\omega^{(n)}|$ and therefore

$$|A^c \cap \omega^{(n)}| \leq \varepsilon|\omega^{(n)}|.$$

Using that fact that $F^n(\omega^{(n)}) = I$ and the invariance of $A^c$ have $F^n(\omega^{(n)} \cap A^c) = A^c$. The bounded distortion property then gives

$$|A^c| = \frac{|F^n(\omega^{(n)} \cap A^c)|}{|F^n(\omega^{(n)})|} \leq D|\omega^{(n)} \cap A^c| / |\omega^{(n)}| \leq D\varepsilon.$$

Since $\varepsilon$ is arbitrary this implies $|A^c| = 0$ and thus $|A| = 1$.

2.2. The invariant measure for $f$. We now show how to define a probability measure $\mu$ which is invariant for the original map $f$ and satisfies all the required properties.

2.2.1. The probability measure $\mu$. We let $\nu_\omega$ denote the restriction of $\nu$ to the partition element $\omega \in \mathcal{P}$, i.e. for any measurable set $A \subset \Delta$ we have $\nu_\omega(A) = \nu(A \cap \omega)$. Then $\nu(A) = \sum_{\omega \in \mathcal{P}} \nu_\omega(A)$.

Then, for any measurable set $A \subseteq M$ (we no longer restrict our attention to $\Delta$) we define

$$\hat{\mu}(A) = \sum_{\omega \in \mathcal{P}} \sum_{j=0}^{R(\omega)-1} \nu_\omega(f^{-j}(A)).$$

Notice that this is a sum of non-negative terms and is uniformly bounded since

$$\hat{\mu}(A) \leq \hat{\mu}(M) = \sum_{\omega \in \mathcal{P}} \sum_{j=0}^{R(\omega)-1} \nu_\omega(f^{-j}(M))$$

$$= \sum_{\omega \in \mathcal{P}} \sum_{j=0}^{R(\omega)-1} \nu_\omega(M) = \sum_{\omega \in \mathcal{P}} R(\omega) \nu(\omega) < \infty$$

by the assumption on the summability of the return times. Thus it defines a finite measure on $M$ and from this we define a probability measure by normalizing to get

$$\mu(A) = \hat{\mu}(A) / \hat{\mu}(M).$$

2.2.2. Absolute continuity. The absolute continuity of $\mu$ is an almost immediate consequence of the definition and the absolute continuity of $\mu$. Indeed, $|A| = 0$ implies $\nu(A) = 0$ which implies $\nu_\omega(A) = 0$ for all $I\omega \in \mathcal{P}$, which therefore implies that we have

$$\sum_{j=0}^{R(\omega)-1} \nu_\omega(f^{-j}(A)) = 0$$

and therefore $\mu(A) = 0$. 

2.2.3. Invariance. Recall first of all that by definition \( f^R(\omega) = \Delta \) for any \( \omega \in \mathcal{P} \). Therefore, for any \( A \subset M \) we have
\[
f^{-R(\omega)}(A) \cap \omega = F^{-1}_{\omega}(A) \cap \omega,
\]
where \( F^{-1}_{\omega} \) denotes the inverse of the restriction \( F|_{\omega} \) of \( F \) to \( \omega \) (notice that \( f^{-R(\omega)}(A) \cap \omega = \emptyset \) if \( A \cap \Delta = \emptyset \)). In particular, using the invariance of \( \nu \) under \( F \), this gives
\[
\sum_{\omega \in \mathcal{P}} \nu(f^{-R(\omega)}(A) \cap \omega) = \sum_{\omega \in \mathcal{P}} \nu(F^{-1}_{\omega}(A) \cap \omega) = \nu(F^{-1}(A)) = \nu(A).
\]
Using this equality we get, for any measurable set \( A \subseteq I \),
\[
\mu(f^{-1}(A)) = \sum_{\omega \in \mathcal{P}} \sum_{j=0}^{R(\omega)-1} \nu_{\omega}(f^{-j}(A))
= \sum_{\omega \in \mathcal{P}} \sum_{j=0}^{R(\omega)-1} \nu(f^{-j+1}(A) \cap \omega)
= \sum_{\omega \in \mathcal{P}} \nu\left((f^{-1}(A) \cap \omega) + \cdots + (f^{-R(\omega)}(A) \cap \omega)\right)
= \sum_{\omega \in \mathcal{P}} \sum_{j=1}^{R(\omega)-1} \nu(f^{-j}(A) \cap \omega) + \sum_{\omega \in \mathcal{P}} (f^{-R(\omega)}(A) \cap \omega)
= \sum_{\omega \in \mathcal{P}} \sum_{j=1}^{R(\omega)-1} \nu(f^{-j}(A) \cap \omega) + \nu(A)
= \sum_{\omega \in \mathcal{P}} \sum_{j=0}^{R(\omega)-1} \nu(f^{-j}(A) \cap \omega)
= \mu(A).
\]

2.2.4. Ergodicity and uniqueness. Ergodicity of \( \mu \) follows immediately from the ergodicity of \( \nu \) since every fully invariant set for of positive measure must intersect the image of some partition element \( \omega \) and therefore must have positive (and therefore full) measure for \( \nu \) and therefore must have full measure for \( \mu \). Notice however that we can only claim a limited form of uniqueness for the measure \( \mu \). Indeed, the support of \( \mu \) is given by
\[
\text{supp}(\mu) = \bigcup_{\omega \in \mathcal{P}} \bigcup_{j=0}^{R_k-1} f^j(I_k)
\]
which is the union of all the images of all partition elements. Then \( \mu \) is indeed the unique ergodic absolutely continuous invariant measure on this set. However in a completely abstract setting there is no way of saying that \( \text{supp}(\mu) = M \) nor that there may not be other relevant measures in \( M \setminus \text{supp}(\nu) \).

2.3. Expansion and distortion estimates. The application of the abstract results discussed above to specific examples involves three main steps:
- Combinatorial construction of the induced map;
- Verification of the bounded distortion property;
• Estimation of the tail of the return times function and verification of the integrability of the return times.

We shall discuss some of these step in some detail in relation to some of the specific case as we go through them below. Here we just make a few remarks concerning the bounded distortion property and in particular the crucial role played by regularity and derivative conditions in these calculations.

We begin with a quite general observation which relates the geometric self-similarity condition to a property involving the derivative of $F$. Let $F : \Delta \to \Delta$ be a Markov map which is continuously differentiable on each element of the partition $P$. We let $\det DF^n$ denote the determinant of the derivative of the map $F^n$.

**Definition 11.** We say that $F$ has uniformly bounded derivative distortion if there exists a constant $D > 0$ such that for all $n \geq 1$ and $\omega \in P^{(n)}$ we have

$$\text{Dist}(f^n, \omega) := \max_{x,y \in f^n(\omega)} \log \frac{\det DF^n(x)}{\det DF^n(y)} \leq D$$

Notice that this is just the infinitesimal version of the self-similarity bounded distortion property and indeed it is possible to show that this condition implies the geometric self-similarity property. In the one-dimensional setting and assuming $J \subset \omega$ to be an open set, this implication follows immediately from the Mean Value Theorem. Indeed, in one dimension the determinant of the derivative is just the derivative itself. Thus, the Mean Value Theorem implies that there exists $x \in I\omega$ such that $|Df^n(x)| = |DF^n(\omega)|/|\omega|$ and $y \in J$ such that $|DF(y)| = |DF^n(J)|/|J|$. Therefore

$$\frac{|\omega| |F^n(J)|}{|J| |F^n(\omega)|} \leq \frac{|DF^n(\omega)|}{|DF^n(x)|} \leq D.$$  

To verify (6) we use the chain rule to write

$$\log \left| \frac{\det DF^n(x)}{\det DF^n(y)} \right| = \log \prod_{i=0}^{n-1} \left| \frac{\det DF(F^i(x))}{\det DF(F^i(y))} \right| = \sum_{i=0}^{n-1} \log \left| \frac{\det DF(F^i(x))}{\det DF(F^i(y))} \right|.$$  

Now adding and subtracting $\log |\det DF(F^i(y))|/|\det DF(F^i(y))|$ and using that fact that $\log(1+x) < x$ for $x > 0$ gives

$$\log \left| \frac{\det DF(F^i(x))}{\det DF(F^i(y))} \right| \leq \log \left( \frac{|\det DF(F^i(x)) - \det DF(F^i(y))|}{|\det DF(F^i(y))|} + 1 \right) \leq \frac{|\det DF(F^i(x)) - \det DF(F^i(y))|}{|\det DF(F^i(y))|}.$$  

Therefore we have

$$\log \left| \frac{\det DF^n(x)}{\det DF^n(y)} \right| \leq \sum_{i=0}^{n-1} \frac{|\det DF(F^i(x)) - \det DF(F^i(y))|}{|\det DF(F^i(y))|}.$$  

The inequality (8) gives us the basic tool for verifying the required distortion properties in particular examples.

3. **Uniformly Expanding Maps**

In this section we discuss maps which are uniformly expanding.
3.1. The smooth/Markov case. We say that $f$ is uniformly expanding if there exist constants $C, \lambda > 0$ such that for all $x \in M$, all $v \in T_x M$, and all $n \geq 0$, we have
$$\|Df^n_x(v)\| \geq Ce^{\lambda n}\|v\|.$$ 
We remark once again that this is a special case of the nonuniform expansivity condition.

**Theorem 5.** Let $f : M \to M$ be $C^2$ uniformly expanding. Then there exists a unique acip $\mu$. If $f$ is uniformly expanding, then $\mu$ is mixing and the correlation function decays exponentially fast. [Sin72, Per74, Bow75, Rue76].

The references given here use a variety of arguments some of which use the remarkable observation that uniformly expanding maps are intrinsically Markov in the strong sense given above, with $\Delta = M$, a finite number of partition elements and return time $R \equiv 1$ (this is particularly easy to see in the case of one-dimensional circle maps $f : \mathbb{S}^1 \to \mathbb{S}^1$). Thus the main issue here is the verification of the distortion condition.

One way to show this is to show that there is a uniform upper bound independent of $n$ for the sum in (8) above. Indeed, notice first of all that the expansivity condition implies in particular that $|\det DF(F^i(y))| \geq Ce^{\lambda i} \geq C > 0$ for every $y$, and the $C^2$ regularity condition implies that $\det Df$ is Lipschitz: there exists $L > 0$ such that $|\det Df(F^i(x)) - \det Df(F^i(y))| \leq L|F^i(x) - F^i(y)|$, for all $x, y \in M$. Substituting these inequalities into (8) we get
$$\sum_{i=0}^{n-1} \frac{|\det Df(F^i(x)) - \det Df(F^i(y))|}{|\det Df(F^i(y))|} \leq \frac{L}{C} \sum_{i=0}^{n-1} |F^i(x) - F^i(y)|$$

The next step, and final, step uses the expansivity condition as well as, implicitly, the Markov property in a crucial way. Indeed, let $\text{diam} M$ denote the diameter of $M$, i.e. the maximum distance between any two points in $M$. The definition of $\mathcal{P}^{(n)}$ implies that $\omega$ is mapped diffeomorphically to $M$ by $F^n$ and thus
$$|\text{diam} M| \geq |F^n(x) - F^n(y)| \geq Ce^{\lambda(n-1)}|F^i(x) - F^i(y)|$$
for every $i = 0, \ldots, n - 1$. Therefore
$$\sum_{i=0}^{n-1} |F^i(x) - F^i(y)| \leq \frac{\text{diam} M}{C} \sum_{i=0}^{n-1} e^{-\lambda(n-i)} \leq \frac{\text{diam} M}{C} \sum_{i=0}^{\infty} e^{-\lambda i}$$
Substituting back into (9) and (8) gives a bound for the distortion which is independent of $n$.

The regularity condition on $\det Df$ can be weakened somewhat but not completely. There exist examples of one-dimensional circle maps $f : \mathbb{S}^1 \to \mathbb{S}^1$ which are $C^1$ uniformly expanding (and thus Markov as above) but for which the uniqueness of the absolutely continuous invariant measure fails [Qua96, CamQua01], essentially due to the failure of the bounded distortion calculation. On the other hand, the distortion calculation above goes through with minor modifications as long as $\det Df$ is just Hölder continuous. In some situations, such as the one-dimensional Gauss map $f(x) = x^{-1} \mod 1$ which is Markov but for which the derivative $Df$ is not even Hölder continuous, one can compensate by taking advantage of the large derivative. Then it is possible to show directly that the right hand side of (8) is uniformly bounded, even though (9) does not hold.

3.2. The non-Markov case. The general (non-Markov) piecewise expanding case is significantly more complicated and even the existence of an absolutely continuous invariant measure is no longer guaranteed [LasYor73, GorSch89, Qua99, Tsu00a, Buz01a]. One possible problem is that the images of the discontinuity set can be very badly distributed and cause havoc with any kind of structure. In the Markov case this does not happen because the set of discontinuities gets mapped to itself by definition.
Also the possibility of components being translated in different directions can destroy on a global level the local expansiveness given by the derivative. Moreover, where results exist for rates of decay of correlations, they do not always apply to the case of Hölder continuous observables, as technical reasons sometimes require that different functions spaces be considered which are more compatible with the discontinuous nature of the maps. We shall not explicitly comment on the particular classes of observables considered in each case.

In the one-dimensional case these problems are somewhat more controllable and relatively simple conditions guaranteeing the existence of an ergodic invariant probability measure can be formulated even in the case of a countable number of domains of smoothness of the map. These essentially require that the size of the image of all domains on which the map is $C^2$ be strictly positive and that certain conditions on the second derivative are satisfied \cite{LasYor73, Adl73, Bow77, Bow79}. In the higher dimensional case, the situation is considerably more complicated and there are a variety of possible conditions which can be assumed on the discontinuities. The conditions of \cite{LasYor73} were generalized to the two-dimensional context in \cite{Kel79} and then to arbitrary dimensions in \cite{GorBoy89, Buz00a, Tsu01b}. There are also several other papers which prove similar results under various conditions, we mention \cite{Alv00, BuzKel01, BuzPacSch01, Buz01c, BuSa03}. In \cite{Buz99a, Cow02} it is shown that conditions sufficient for the existence of a measure are generic in a certain sense within the class of piecewise expanding maps.

Estimates for the decay of correlations have been proved for non-Markov piecewise smooth maps, although again the techniques have had to be considerably generalized. In terms of setting up the basic arguments and techniques, a similar role to that played by \cite{LasYor73} for the existence of absolutely continuous invariant measures can be attributed to \cite{Kel80, HofKel82, Ryc83} for the problem of decay of correlations in the one-dimensional context. More recently, alternative approaches have been proposed and implemented in \cite{Liv95a, Liv95, You98}. The approach of \cite{You98} has proved particularly suitable for handling some higher dimensional cases such as \cite{BuzMau02} in which assumptions on the discontinuity set are formulated in terms of topological pressure and \cite{ AlvLuzPin, Gou04} in which they are formulated as geometrical non-degeneracy assumptions and dynamical assumptions on the rate of recurrence of typical points to the discontinuities. The construction of an induced Markov map is combined in \cite{Dia04} with the Theorem 4 to obtain estimates for the decay of correlations of non-Hölder observables for Lorenz-like expanding maps. We remark also that the results of \cite{ AlvLuzPin, Gou04} apply to more general piecewise nonuniformly expanding maps, see section 6. It would be interesting to understand the relation between the assumptions of \cite{ AlvLuzPin, Gou04} and those of \cite{BuzMau02}.

4. Almost Uniformly Expanding Maps

Perhaps the simplest way to relax the uniform expansivity condition is to allow some fixed (or periodic) point $p$ to have a neutral eigenvalue, e.g in the one-dimensional setting $|Df(p)| = 1$, while still requiring all other vectors in all directions over the tangent spaces of all points to be strictly expanded by the action of the derivative (though of course not uniformly since the expansion must degenerate near the point $p$). Remarkably this can have extremely dramatic consequences on the dynamics.

There are some recent results for higher-dimensional systems \cite{PolYur01a, Hu01, Gou03} but a more complete picture is available the one-dimensional setting and thus we concentrate on this case. An initial motivation for these kinds of examples arose from the concept of intermittency in fluid dynamics. A class of one-dimensional maps expanding everywhere except at a fixed point was introduced by Maneville and Pomeau in \cite{ManPom80} as a model of intermittency since numerical studies showed
that orbits tend to spend a long time *trapped* in a neighbourhood of the fixed point with relatively short *bursts of chaotic activity* outside this neighbourhood. Recent work shows that indeed, these long periods of inactivity near the fixed point are a key to slowing down the mixing process and obtaining examples of systems with subexponential decay of correlations.

We shall consider interval maps \( f \) which are piecewise \( C^2 \) with a \( C^1 \) extension to the boundaries of the \( C^2 \) domains and for which the derivative is strictly greater than 1 everywhere except at a fixed point \( p \) (which for simplicity we can assume lies at the origin) where \( Df(p) = 1 \). For definiteness, let us suppose that on a small neighbourhood of 0 the map takes the form

\[
f(x) \approx x + x^2 \phi(x)
\]

where \( \approx \) means that the terms on the two sides of the expression as well as their first and second order derivatives converge as \( x \to 0 \). We assume moreover that \( \phi \) is \( C^\infty \) for \( x \neq 0 \); the precise form of \( \phi \) determines the precise degree of *neutrality* of the fixed point, and in particular affects the second derivative \( D^2 f \). It turn out that it plays a crucial role in determining the mixing properties and even the very existence of an absolutely continuous invariant measure. For the moment we assume also a strong Markov property: each domain of regularity of \( f \) is mapped bijectively to the whole interval.

The following result shows that the situation can be drastically different from the uniformly expanding case.

**Theorem 6.** (Pia80) If \( f \) is \( C^2 \) at \( p \) (e.g. \( \phi(x) \equiv 1 \)) then \( f \) does not admit any acip.

Note that \( f \) has the same topological behaviour as a uniformly expanding map, typical orbits continue to wander densely on the whole interval, but the proportion of time which they spend in various regions tends to concentrate on the fixed point, so that, asymptotically, typical orbits spend all their time near 0. It turns out that in this situation there exists an infinite (\( \sigma \)-finite) absolutely continuous invariant measure which gives finite mass to any set not containing the fixed point and infinite mass to any neighbourhood of \( p \) (Tha83).

The situation changes if we relax the condition that \( f \) be \( C^2 \) at \( p \) and allow the second derivative \( D^2 f(x) \) to diverge to infinity as \( x \to p \). This means that the derivative increases quickly as one moves away from \( p \) and thus nearby points are repelled at a faster rate. This is a very subtle change but it makes all the difference.

**Theorem 7.** If \( \phi \) is of the form \( \phi(x) = x^{-\alpha} \) for some \( \alpha \in (0, 1) \), then (Pia80, HuYou95) \( f \) admits an ergodic acip \( \mu \) and (Iso99, LivSauVai98, You99, PolYur01a, Sar02, Hu04, Gou04a) \( \mu \) is mixing with decay of correlations

\[
C_n = O(n^{1 - \frac{1}{\alpha}}).
\]

If \( \phi(1/x) = \log x \log(2) x \ldots \log(r-1) x (\log(r) x)^{1-\alpha} \) for some \( r \geq 1, \alpha \in (-1, \infty) \) where \( \log(r) \) \( = \log \log \ldots \log \) repeated \( r \) times, then (Hol04) \( f \) admits a mixing acip \( \mu \) with decay of correlations

\[
C_n = O((\log(r)^n)^{-\alpha}).
\]

Thus, the existence of an absolutely continuous invariant measure as in the uniformly expanding case has been recovered, but the exponential rate of decay of correlation has not. We can think of the indifferent fixed point as having the effect of *slowing down* this process by trapping nearby points for disproportionately long time. The estimates in (Sar02, Hu04, Gou04a) include lower bounds as well as upper bounds. The approach in (You99) using Markov induced maps applies also to non-Markov cases and (Hol04) can also be generalized to these cases.

The proofs of Theorem 7 do not use directly the fact that \( f \) is nonuniformly expanding. Indeed the fact that \( f \) is nonuniformly expanding does not follow automatically from the fact that the map is
expanding away from the fixed point \( p \). However we can use the existence of the acip to show that this condition is satisfied. Indeed by Birkhoff’s Ergodic Theorem, typical points spend a large proportion of time near \( p \) but also a positive proportion of time in the remaining part of the space. More formally, by a simple application of Birkhoff’s Ergodic Theorem to the function \( \log |Df(x)| \), we have that, for \( \mu \)-almost every \( x \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |Df(f^i(x))| \to \int \log |Df| d\mu > 0.
\]

The fact that \( \int \log |Df| d\mu > 0 \) follows from the simple observation that \( \mu \) is absolutely continuous, finite, and that \( \log |Df| > 0 \) except at the neutral fixed point.

5. One-Dimensional Maps with Critical Points

We now consider another class of systems which can also exhibit various rates of decay of correlations, but where the mechanism for producing these different rates is significantly more subtle. The most general set-up is that of a piecewise smooth one-dimensional map \( f : I \to I \) with some finite set \( C \) of critical/singular points at which \( Df = 0 \) or \( Df = \pm \infty \) and/or at which \( f \) may be discontinuous. There are at least two ways to quantify the “uniformity” of the expansivity of \( f \) in ways that get reflected in different rates of decay of correlations:

- To consider the rate of growth of the derivatives along the orbits of the critical points;
- To consider the average rate of growth of the derivative along typical orbits.

In this section we will concentrate on the first, somewhat more concrete, approach and describe the main results which have been obtained over the last 20/25 years. We shall focus specifically on the smooth case since this is where most results have been obtained. Some partial generalization to the piecewise smooth case can be found in [DiaHolLuz04]. The second approach is somewhat more abstract but also more general since it extends naturally to the higher dimensional context where critical points are not so well defined and/or cannot play such a fundamental role. The main results in this direction will be described in Section 6 below in the framework of a general theory of non-uniformly expanding maps.

5.1. Unimodal maps. We consider the class of \( C^3 \) interval maps \( f : I \to I \) with some finite set \( C \) of non-flat critical points. We recall that \( c \) is a critical point if \( Df(c) = 0 \); the critical point is non-flat if there exists an \( 0 < \ell < \infty \) called the order of the critical point, such that \( |Df(x)| \approx |x - c|^{\ell - 1} \) for \( x \) near \( c \); \( f \) is unimodal if it has only one critical point, and multimodal if it has more than one. Several results to be mentioned below have been proved under a standard technical negative Schwarzian derivative condition which is a kind of convexity assumption on the derivative of \( f \), see [MelStr88] for details. Recent results [Koz00] indicate that this condition is often superfluous and thus we will not mention it explicitly.

The first result on the statistical properties of such maps goes back to Ulam and von Neumann [UlaNeu47] who showed that the top unimodal quadratic map, \( f(x) = x^2 - 2 \) has an acip. Notice that this map is actually a Markov map but does not satisfy the bounded distortion condition due to the presence of the critical point. It is possible to construct an induced Markov map for \( f \) which does satisfy this condition and gives the result, but Ulam and von Neumann used a very direct approach, observing that \( f \) is \( C^1 \) conjugate to a piecewise-linear uniformly expanding Markov map for which Lebesgue measure is invariant and ergodic. This implies that the pull-back of the Lebesgue measure by the conjugacy is an acip for \( f \). However, the existence of a smooth conjugacy is extremely rare and such an approach is not particularly effective in general.
More general and more powerful approaches have allowed the existence of an acip to be proved under increasingly general assumptions on the behaviour of the critical point. Let
\[ D_n(c) = |Df^n(f(c))|. \]
Notice that the derivative along the critical orbit needs to be calculated starting from the critical value and not from the critical point itself for otherwise it would be identically 0.

**Theorem 8.** Let \( f : I \to I \) be a unimodal map. Then \( f \) admits an ergodic acip if the following conditions hold (each condition is implied by the preceding ones):

- The critical point is pre-periodic [Rue77];
- The critical point is non-recurrent [Mis81];
- \( D_n \to \infty \) exponentially fast [ColEck83, NowStr88];
- \( D_n \to \infty \) sufficiently fast so that \( \sum_n D_n^{1/\ell} < \infty \) [NowStr91];
- \( D_n \to \infty \) [BruSheStr03].

If \( D_n \to \infty \) exponentially fast then some power of \( f \) is mixing and exhibits exponential decay of correlations [KelNow92] (and [You92] with additional bounded recurrence assumptions on the critical point).

Notice that the condition of [BruSheStr03] is extremely weak. In fact they show that it is sufficient for \( D_n \) to be eventually bounded below by some constant depending only on the order of the critical point. However even this condition is not optimal as there are examples of maps for which \( \lim \inf D_n = 0 \) but which still admit an ergodic acip. It would be interesting to know whether an optimal condition is even theoretically possible: it is conceivable that a complete characterization of maps admitting acip’s in terms of the behaviour of the critical point is not possible because other subtleties come into play.

5.2. **Multimodal maps.** Given the number of people and research papers in one-dimensional dynamics it is remarkable that until very recently there were essentially no results at all on the existence of acip’s for multimodal maps. A significant breakthrough was achieved by implementing the strategy of constructing induced Markov maps and estimating the rate of decay of the tail. This strategy yields also estimates for various rates of decay in the unimodal case and extends very naturally to the multimodal case.

**Theorem 9** ([BruLuzStr03]). Let \( f \) by a multimodal map with a finite set of critical points of order \( \ell \) and suppose that
\[ \sum_n D_n^{-1/(2\ell-1)} < \infty \]
for each critical point \( c \). Then there exists an ergodic acip \( \mu \) for \( f \). Moreover some power of \( f \) is mixing and the correlation function decays at the following rates:

**Polynomial case:** If there exists \( C > 0, \tau > 2\ell - 1 \) such that
\[ D_n(c) \geq Cn^\tau, \]
for all \( c \in \mathcal{C} \) and \( n \geq 1 \), then, for any \( \hat{\tau} < \frac{\tau}{\ell - 1} - 1 \), we have
\[ C_n = O(n^{-\hat{\tau}}) \]

**Exponential case:** If there exist \( C, \beta > 0 \) such that
\[ D_n(c) \geq Ce^{\beta n} \]
for all \( c \in C \) and \( n \geq 1 \), then there exist \( \tilde{\beta} > 0 \) such that
\[
C_n = \mathcal{O}(e^{-\tilde{\beta}n}).
\]

These results gives previously unknown estimates for the decay of correlations even for unimodal maps in the quadratic family. For example they imply that the so-called Fibonacci maps \([\text{LyuMil93}]\) exhibit decay of correlation at rates which are faster than any polynomial. It seems likely that these estimates are essentially optimal although the argument only provides upper bounds. The general framework of \([\text{Lyn04}]\) applies to these cases to provide estimates for the decay of correlation for observable which satisfy weaker than Hölder conditions on the modulus of continuity. The condition for the existence of an acip have recently been weakened to the summability condition \( \sum D_n^{-1/\ell} < \infty \) and to allow the possibility of critical points of different orders \([\text{BruStr01}]\). Some improved technical expansion estimates have been also obtained in \([\text{Ced04}]\) which allow the results on the decay of correlations to apply to maps with critical points of different orders.

Based on these results, the conceptual picture of the causes of slow rates of decay of correlations appears much more similar to the case of maps with indifferent fixed points than would appear at first sight: we can think of the case in which the rate of growth of \( D_n \) is subexponential as a situation in which the critical orbit is neutral or indifferent and points which land close to the critical point tend to remain close to (“trapped” by) its orbit for a particularly long time. During this time orbits are behaving “non-generically” and are not distributing themselves over the whole space as uniformly as they should. Thus the mixing process is delayed and the rate of decay of correlations is correspondingly slower. When \( D_n \) grows exponentially, the critical orbit can be thought of (and indeed is) a non-periodic hyperbolic repelling orbit and nearby points are pushed away exponentially fast. Thus there is no significant loss in the rate of mixing, and the decay of correlations is not significantly slowed down notwithstanding the presence of a critical point.

5.3. Benedicks-Carleson maps. We give here a sketch of the construction of the induced Markov map for a class of unimodal maps. We shall try to give a conceptually clear description of the main steps and ingredients required in the construction. The details of the argument are unfortunately particularly technical and a lot of notation and calculations are carried out only to formally verify statements which are intuitively obvious. It is very difficult therefore to be at one and the same time conceptually clear and technically honest. We shall therefore concentrate here on the former approach and make some remarks about the technical details which we omit or present in a simplified form.

Let
\[
f_a(x) = x^2 - a
\]
for \( x \in I = [-2, 2] \) and
\[
a \in \Omega_\varepsilon = [2 - \varepsilon, 2]
\]
for some \( \varepsilon > 0 \) sufficiently small. The assumptions and the details of the proof require the introduction of several additional constants, some intrinsic to the maps under consideration and some auxiliary for the purposes of the argument. In particular we suppose that there is a \( \lambda \in (0, \log 2) \) and constants
\[
\lambda \gg \alpha \gg \delta \gg \delta > 0
\]
where \( x \gg y \) means that \( y \) must be sufficiently small relative to \( x \). Finally, to simplify the notation we also let \( \beta = \alpha/\lambda \). We restrict ourselves to parameter values \( a \in \Omega_\varepsilon \) which satisfy the Benedicks-Carleson conditions:

Hyperbolicity: There exist \( C > 0 \) such that
\[
D_n \geq Ce^{\lambda n} \quad \forall \ n \geq 1;
\]
Slow recurrence:

\[ |c_n| \geq e^{-\alpha n} \quad \forall \, n \geq 1. \]

In section 7 on page 31 we sketch a proof of the fact that these conditions are satisfied for a positive measure set of parameters in \( \Omega_\varepsilon \) (for any \( \lambda \in (0, \log 2) \) and any \( \alpha > 0 \)). They are therefore reasonably generic conditions. Assuming them here will allow us to present in a compact form an almost complete proof. During the discussion we shall make some comments about how the argument can be modified to deal with slower rates of growth of \( D_n \) and arbitrary recurrence patterns of the critical orbit.

We remark that the overall strategy as well as several details of the construction in the two arguments (one proving that the hyperbolicity and slow recurrence conditions occur with positive probability and the other proving that they imply the existence of an \( \text{acip} \)) are remarkably similar. This suggests a deeper, yet to be fully understood and exploited, relationship between the structure of dynamical space and that of parameter space.

We let \( \Delta = (\delta, \delta) \subset (-\hat{\delta}, \hat{\delta}) = \hat{\Delta} \) denote \( \delta \) and \( \hat{\delta} \) neighbourhood of this critical point \( c \). The aim is to construct a Markov induced map \( F : \Delta \to \Delta \).

We shall do this in three steps. We first define an induced map \( f^p : \Delta \to I \) which is essentially based on the time during which points in \( \Delta \) shadow the critical orbit. The shadowing time \( p \) is piecewise constant on a countable partition of \( \Delta \) but the images of partition elements can be arbitrarily small. Then we define an induced map \( f^E : \Delta \to I \) which is still not Markov but has the property that the images of partition elements are uniformly large. Finally we define the Markov induced map \( F^R : \Delta \to \Delta \) as required.

5.4. Expansion outside \( \Delta \). Before starting the construction of the induced maps, we state a lemma which gives some derivative expansion estimates outside the critical neighbourhood \( \Delta \).

**Lemma 5.1.** There exists a constant \( C > 0 \) independent of \( \delta \) such that for \( \varepsilon > 0 \) sufficiently small, all \( a \in \Omega_\varepsilon, f = f_a, x \in I \) and \( n \geq 1 \) such that \( x, f(x), \ldots, f^{n-1}(x) \notin \Delta \) we have

\[ |Df^n(x)| \geq \delta e^{\lambda n} \]

and if, moreover, \( f^n(x) \in \hat{\Delta} \) and/or \( x \in f(\hat{\Delta}) \) then

\[ |Df^n(x)| \geq C e^{\lambda n} \]

Notice that the constant \( C \) and the exponent \( \lambda \) do not depend on \( \delta \) or \( \hat{\delta} \). This allows us to choose \( \hat{\delta} \) and \( \delta \) small in the following argument without worrying about this affecting the expansivity estimates given here. In general of course both the constants \( C \) and \( \lambda \) depend on the size of this neighbourhood and it is an extremely useful feature of this particular range of parameter values that they do not. In the context of the quadratic family these estimates can be proved directly using the smooth conjugacy of the top map \( f_2 \) with the piecewise affine tent map, see [UhlNeu47, Luz00]. However there are general theorems in one-dimensional dynamics to the effect that one has uniform expansivity outside an arbitrary neighbourhood of the critical point under extremely mild conditions [Man85] and this is sufficient to treat the general case in [BruLuzStr03].
5.5. **Shadowing the critical orbit.** We start by defining a partition of the critical neighbourhoods $\Delta$ and $\hat{\Delta}$. For any integer $r \geq 1$ let $I_r = [e^{-r}, e^{-r+1})$ and $I_{-r} = (-e^{-r+1}, -e^{-r})$ and, for each $r \geq r_\delta + 1$, let $I_r = I_{r-1} \cup I_r \cup I_{r+1}$. We can suppose without loss of generality that $r_\delta = \log \delta^{-1}$ and $r_\delta = \log \delta^{-1}$ are integers Then

$$\Delta = \{0\} \cup \bigcup_{|r| \geq r_\delta + 1} I_r, \quad \text{and} \quad \hat{\Delta} = \{0\} \cup \bigcup_{|r| \geq r_\delta + 1} I_r.$$  

This is one of the minor technical points of which we do not give a completely accurate description. Strictly speaking, the distortion estimates to be given below require a further subdivision of each $I_r$ into $\delta^2$ subintervals of equal length. This does not affect significantly any of the other estimates. A similar partition is defined in Section 7.1.2 on page 34 in somewhat more detail. We remark also that the need for the two neighbourhoods $\Delta$ and $\hat{\Delta}$ will not become apparent in the following sketch of the argument. We mention it however because it is a crucial technical detail: the region $\hat{\Delta} \setminus \Delta$ acts as a buffer zone in which we can choose to apply the derivative estimates of Lemma 5.1 or the shadowing argument of Lemma 5.2 according to which one is more convenient in a particular situation.

Now let

$$p(r) = \max\{k : |f^{j+1}(x) - f^{j+1}(c)| \leq e^{-2\alpha j} \forall x \in \hat{I}_r, \forall j < k\}$$  

This definition was essentially first formulated in [BenCar85] and [BenCar91]. The key characteristic is that it guarantees a bounded distortion property which in turn allows us to make several estimates based on information about the derivative growth along the critical orbit. Notice that the definition in terms of $\alpha$ is based crucially on the fact that the critical orbit satisfies the slow recurrence condition. We mention below how this definition can be generalized.

**Lemma 5.2.** For all points $x \in \hat{I}_r$ and $p = p(r)$ we have

$$|Df^{p+1}(x)| = |Df(x)| \cdot |Df^p(x_0)| \geq e^{(1-\beta)r}$$

Recall that $\beta = \alpha / \lambda$ can be chosen arbitrarily small.

**Proof.** First of all, using the bounded recurrence condition, the definition of the binding period, and arguing as in the distortion estimates for the uniformly expanding maps above, it is not difficult to show that there is a constant $D_1$, depending on $\alpha$ but independent of $r$ and $\delta$, such that for all $x_0, y_0 \in f(\hat{I}_r)$ and $1 \leq k \leq p$,

$$\frac{|Df^k(x_0)|}{|Df^k(y_0)|} \leq D_1. \quad (11)$$

Using the definition of $p$ this implies

$$e^{-2\alpha(p-1)} \geq |x_{p-1} - c_{p-1}| \geq D_1^{-1} |Df^{p-1}(c_0)| \geq D_1^{-1} \lambda^{p-1} e^{-2r}$$
and thus $D_1 e^{-2\alpha p} e^{2\alpha} \geq e^{\lambda p} e^{-\lambda} e^{-2r}$. Rearranging gives

$$p + 1 \leq \frac{\log D_1 + 2\alpha + 2\lambda + 2r}{\lambda + 2\alpha} \leq \frac{3r}{\lambda} \quad (12)$$

as long as we choose $\delta$ so that $r_\delta$ is sufficiently large in comparison to the other constants, none of which depend on $\delta$. Moreover

$$D e^{-2r} |Df^p(x_0)| \geq D |x_0 - c_0| \cdot |Df^p(x_0)| \geq |x_p - c_p| \geq e^{-2\alpha p}$$
and therefore, using (12), we have $|Df^p(x_0)| \geq D^{-1} e^{2r} e^{-2\alpha p} \geq D^{-1} e^{(2-\frac{\lambda}{\lambda})r}$. Since $x \in \hat{I}_r$ we have $|Df(x)| = 2|x - c| \geq 2e^{-(r+2)}$ and therefore $|Df^{p+1}(x)| = |Df^p(x_0)| |Df(x)| \geq \ldots$
Thus we have a first induced map \( F_p : \Delta \to I \) given by \( F_p(x) = f^{p(x)}(x) \) where \( p(x) = p(r) \) for \( x \in I_{\pm r} \) which is uniformly expanding. Indeed notice that \( Df^{p(x)}(x) \to \infty \) as \( x \to c \). However there is no reason for which this map should satisfy the Markov property and indeed, an easy calculation shows that the images of the partition elements are \( \sim e^{-7\beta r} \to 0 \) and thus not even of uniform size.

The notion of shadowing can be generalized without any assumptions on the recurrence of the critical orbit in the following way, see [BruLuzStr03]: let \( \{\gamma_n\} \) be a monotonically decreasing sequence with \( 1 > \gamma_n > 0 \) and \( \sum \gamma_n < \infty \). Then for \( x \in \Delta \), let

\[
p(x) := \max\{p : |f^k(x) - f^k(c)| \leq \gamma_k|f^k(c) - c| ~ \forall k \leq p - 1\}.
\]

A simple variation of the distortion calculation used above shows that the summability of \( \gamma_n \) implies that (1) holds with this definition also. Analogous bounds on \( D_n \) will reflect the rate growth of the derivative along the critical orbit. If the growth of \( D_n \) is subexponential, the binding period will last much longer because the interval \( |f^k(x) - f^k(c)| \) is growing at a slower rate. The generality of the definition means that it is more natural to define a partition \( I_p \) as the “level sets” of the function \( p(x) \). The drawback is that we have much less control over the precise size of these intervals and their distance from the critical point. Some estimates of the tail \( \{x > p\} \) can be obtained and it turns out these are closely related to the rate of growth of \( D_n \) and to those of the return time function for the final induced Markov map. This is because the additional two steps, the escape time and the return time occur exponentially fast. Thus the only bottleneck is the delay caused by the long shadowing of the critical orbit.

5.6. The escape partition. Now let \( J \subset I \) be an arbitrary interval (which could also be \( \Delta \) itself). We want to construct a partition \( \mathcal{P} \) of \( J \) and a stopping time \( E : J \to \mathbb{N} \) constant on elements of \( \mathcal{P} \) with the property that for each \( \omega \in \mathcal{P} \), \( f^{E(\omega)}(\omega) \approx \delta \). We think of \( \delta \) as being our definition of large scale; we call \( E(\omega) \) the escape time of \( \omega \), we call the interval \( f^{E(\omega)}(\omega) \) and escape interval, and call \( \mathcal{P} \) the escape time partition of \( J \).

The construction is carried out inductively in the following way. Let \( k \geq 1 \) and suppose that the intervals with \( E < k \) have already been defined. Let \( \omega \) be a connected component of the complement of the set \( \{E < k\} \subset J \). We consider the various cases depending on the position of \( f^k(\omega) \). If \( f^k(\omega) \) contains \( \Delta \cup I_r \cup I_{-r} \) then we subdivide \( \omega \) into three subintervals satisfying the required properties, and let \( E = k \) on each of them. If \( f^k(\omega) \cap \Delta = \emptyset \) we do nothing. If \( f^k(\omega) \cap \Delta \neq \emptyset \) but \( f^k(\omega) \) does not intersect more than two adjacent \( I_r \)’s then we say that \( k \) is an inessential return and define the corresponding return depth by \( r = \max\{|r| : f^k(\omega) \cap I_r \neq \emptyset\} \). If \( f^k(\omega) \cap \Delta \neq \emptyset \) and \( f^k(\omega) \) intersects at least three elements of \( \mathcal{I} \), then we simply subdivide \( \omega \) into subintervals \( \omega_r \) in such a way that each \( \omega_r \) satisfies \( I_r \subset f^k(\omega_r) \subset I_r \). For \( r > r_0 \) we say that \( \omega_r \) has an essential return at time \( k \) with an associated return depth \( r \). For all other \( r \), the \( \omega_r \) are escape intervals, and for these intervals we set \( E = k \). Finally we consider one more important case. If \( f^k(\omega) \) contains \( \Delta \) as well as (at least) the two adjacent partition elements then we subdivide as described above except for the fact that we keep together that portion of \( \omega \) which maps exactly to \( \Delta \). We let this belong to the escape partition \( \mathcal{P} \) but, as we shall see, we also let it belong to the final partition associated to the full induced Markov map.

This purely combinatorial algorithm is designed to achieve two things, neither one of which follows immediately from the construction:

1. Guarantee uniformly bounded distortion on each partition element up to the escape time;
(2) Guarantee that almost every point eventually belongs to the interior of a partition element \( \omega \in \mathcal{P} \).

We shall not enter into the details of the distortion estimates here but discuss the strategy for showing that \( \mathcal{P} \) is a partition mod 0 of \( J \). Indeed this follows from a much stronger estimate concerning the tail of the escape time function: there exists a constant \( \gamma > 0 \) such that for any interval \( J \) with \( |J| \geq \delta \) we have

\[
|\{ x \in J : E(x) \geq n \}| = \sum_{\omega' \in \mathcal{P}} |\omega'| \lesssim e^{-\gamma n} |J|.
\]

The argument for proving (13) revolves fundamentally around the combinatorial information defined in the construction. More specifically, for \( \omega \in \mathcal{P} \) let \( r_1, r_2, \ldots, r_s \) denote the sequence of return depths associated to essential return times occurring before \( E(\omega') \), and let \( E(\omega) = \sum_{i=0}^{s} r_s \). Notice that this sequence may be empty if \( \omega \) escapes without intersecting \( \Delta \), in this case we set \( E(\omega) = 0 \). We now split the proof into three steps:

1) Relation between escape time and return depths. The first observation is that the escape time is bound by a constant multiple of the sum of the return depths: there exists a \( \kappa \) depending only on \( \lambda \) such that

\[
E(\omega) \lesssim \kappa E(\omega).
\]

Notice that a constant \( T_0 \) should also be added to take care of the case in which \( E(\omega) = 0 \), corresponding to the situation in which \( \omega \) has an escape the first time that iterates of \( \omega \) intersect \( \Delta \). Since \( \omega \) is an escape, it has a minimum size and the exponential growth outside \( \Delta \) gives a uniform bound for the maximum number of iterates within which such a return must occur. Since this constant is uniform it does not play a significant role and we do not add it explicitly to simplify the notation. For the situation in which \( E(\omega) > 0 \) it is sufficient to show that each essential return with return depth \( r \) has the next essential return or escape within at most \( \kappa r \) iterations. Again this follows from the observation that the derivative is growing exponentially on average during all these iterations: we have exponential growth outside \( \Delta \) and also exponential growth on average during each complete inessential binding period. This implies (14). From (14) we then have

\[
\sum_{\omega \in \mathcal{P}(\omega)} |\omega| \lesssim \sum_{\omega \in \mathcal{P}(\omega)} |\omega| \quad \text{for} \quad E(\omega) \geq n / \kappa.
\]

Thus it is enough to estimate the right hand side of (15) which is saying that there is an exponentially small probability of having a large total accumulated return depth before escaping, i.e. most intervals escape after relatively few and shallow return depths. The strategy is perfectly naive and consists of showing that the size of interval with a certain return depth \( \mathcal{E} \) is exponentially small in \( \mathcal{E} \) and that there cannot be too many so that their total sum is still exponentially small.

2) Relation between size of \( \omega \) and return depth. The size of each partition element can be estimated in terms of the essential return depths in a very coarse, i.e. non-sharp, way which is nevertheless sufficient for your purposes. The argument relies on the following observation. Every return depth corresponds to a return which is followed by a binding period. During this binding period there is a certain overall growth of the derivative. During the remaining iterates there is also derivative growth, either from being outside \( \Delta \) or from the binding period associated to some inessential return.
Therefore a simple application of the Mean Value Theorem gives
\begin{equation}
|\omega| \lesssim e^{-\frac{1}{4}\lambda_E(\omega)}.
\end{equation}

3) The cardinality of the $\omega$ with a certain return depth. It therefore remains only to estimate the cardinality of the set of elements $\omega$ which can have the same value of $E$. To do this, notice first of all that we have a bounded multiplicity of elements of $P(\omega)$ which can share exactly the same sequence of return depths. More precisely this corresponds to the number of escaping intervals which can arise at any given time from the subdivision procedure described above, and is therefore less than $r_\delta$. Moreover, every return depth is bigger than $r_\delta$ and therefore for a given sequence $r_1, \ldots, r_s$, we must have $s \leq E/r_\delta$, Therefore letting $\eta = r_\delta^{-1}$, choosing $\delta$ sufficiently small the result follows from the following fact: Let $N_{k,s}$ denote the number of sequences $(t_1, \ldots, t_s)$, $t_i \geq 1$ for all $i$, $1 \leq i \leq s$, such that $\sum_{i=1}^s t_i = k$. Then, for all $\eta > 0$, there exists $\eta > 0$ such that for any integers $s, k$ with $s < \eta k$ we have
\begin{equation}
N_{k,s} \leq e^{\eta k}.
\end{equation}

Indeed, applying (17) we get that the total number of possible sequences is $N_k = \sum_{s=1}^{\eta E} N_{E,s} \leq \eta E e^{\eta E} \leq e^{2\eta E}$. Taking into account the multiplicity of the number of elements sharing the same sequence we get the bound on this quantity as $\lesssim r_\delta^{-1} e^{2\eta E} \leq e^{3\eta E}$. Multiplying this by (16) and substituting into (15) gives the result.

To prove (17), notice first of all that $N_{k,s}$ can be bounded above by the number of ways to choose $s$ balls from a row of $k + s$ balls, thus partitioning the remaining $k$ balls into at most $s + 1$ disjoint subsets. Notice also that this expression is monotonically increasing in $s$, and therefore
\[ N_{k,s} \leq \binom{k + s}{s} = \binom{k + s}{k} \leq \binom{(1 + \eta)k}{\eta k} \leq \frac{[(1 + \eta)k]!(\eta k)!}{(\eta k)!k!}. \]

Using Stirling’s approximation formula $k! \in [1, 1 + \frac{1}{2k}]\sqrt{2\pi k}ke^{-k}$, we have $N_{k,s} \leq \frac{[(1 + \eta)k]!(\eta k)!}{(\eta k)!k!} \leq (1 + \eta)^{1+\eta}k^{\eta} - \eta k^{\eta} \leq \exp\{((1 + \eta)k \log(1 + \eta) - \eta k \log \eta) \leq \exp\{((1 + \eta)\eta - \eta \log \eta)k\}$. Clearly $(1 + \eta)\eta - \eta \log \eta \to 0$ as $\eta \to 0$. This completes the proof of (13).

5.7. The return partition. Finally we need to construct the full induced Markov map. To do this we simply start with $\Delta$ and construct the escape partition $P$ of $\Delta$. Notice that this is a refinement of the binding partition into intervals $I_j$. Notice also that the definition of this escape partition allows as a special case the possibility that $f^E(\omega)(\omega) = \Delta$. In this case of course $\omega$ satisfies exactly the required properties and we let it belong by definition to the partition $Q$ and define the return time of $\omega$ as $R(\omega) = E(\omega)$. Otherwise we consider each escape interval $J = f^E(\omega)(\omega)$ and use it as a starting interval for constructing and escape partition and escape time function. Again some of the partition elements constructed in this way will actually have returns to $\Delta$. These we define to belong to $Q$ and let their return time be the sum of the two escape times, i.e. the total number of iterations since they left $\Delta$, so that $f^{R(\omega)}(\omega) = \Delta$. For those that don’t return to $\Delta$ we repeat the procedure. We claim that almost every point of $\Delta$ eventually belongs to an element which returns to $\Delta$ in a good (Markov) way at some point and that the tail estimates for the return time function are not significantly affected, i.e. they are still exponential.

The final calculation to support this claim is based on the following fairly intuitive observation. Once an interval $\omega$ has an escape, it has reached large scale and therefore it will certainly cover $\Delta$ after some uniformly bounded number of iterations. In particular it contains some subinterval $\tilde{\omega} \subset \omega$ which has a return to $\delta$ with at most this uniformly bounded number of iterates after the escape.
Moreover, and crucially, the proportion of $\tilde{\omega}$ in $\omega$ is uniformly bounded below, i.e. there exists a constant $\xi > 0$ independent of $\omega$ such that

(18) \[ |\tilde{\omega}| \geq \xi |\omega|. \]

Using this fact we are now ready to estimate the tail of return times, $|\{\omega \in Q \mid R(\omega) > n\}|$. The argument is again based on taking into account some combinatorial information related to the itinerary of elements of the final partition $Q$. In particular we shall keep track of the *number of escape times* which occur before time $n$ for all elements whose return is greater than $n$. First of all we let

(19) \[ Q^{(n)} = \{\omega \in Q \mid R(\omega) \geq n\}. \]

Then, for each $1 \leq i \leq n$ we let

(20) \[ Q^{(n)}_i = \{\omega \in Q^{(n)} \mid E_{i-1}(\omega) \leq n < E_i(\omega)\} \]

be the set of partition elements in $Q^{(n)}$ who have exactly $i$ escapes. Amongst those we distinguish those with a specific escape combinatorics. More precisely, for $(t_1, \ldots, t_i)$ such that $t_j \geq 1$ and $\sum t_j = n$, let

(21) \[ Q^{(n)}_i(t_1, \ldots, t_i) = \left\{ \omega \in Q^{(n)} \mid \sum_{j=1}^{k} t_j = E_k(\omega), 1 \leq k \leq i - 1 \right\}. \]

We then fix some small $\eta > 0$ to be determined below and write

(22) \[ |\{\omega \in Q \mid R(\omega) \geq n\}| = \sum_{i \leq n} |Q^{(n)}_i| = \sum_{i \leq \eta n} |Q^{(n)}_i| + \sum_{\eta n < i \leq n} |Q^{(n)}_i|. \]

By (18) we have $|Q^{(n)}_i| \lesssim (1 - \xi)^i$, which gives

(23) \[ \sum_{\eta n < i \leq n} |Q^{(n)}_i| \lesssim \sum_{\eta n < i \leq n} (1 - \xi)^i \lesssim (1 - \xi)^\eta n \approx e^{-\gamma n} \]

for some $\gamma_\xi > 0$. Now let $\omega \subset \tilde{\omega} \in P_{E_i}$ be one of the non-returning parts of an interval $\tilde{\omega}$ that had its $i$th escape at time $E_i$. Note that $f^{E_i}(P_{E_{i+1}}(\omega)) = P(f^{E_i}(\omega))$.

Therefore

(24) \[ \sum_{\omega' \subset \tilde{\omega}, E_{i+1}(\omega') \geq E_{i} + n} |\omega'| \lesssim e^{-\gamma n} |\omega|, \]

where $\gamma$ is as in (13). Let $Q^{(n)}_i$ denote the set of intervals $\omega \in Q$ that have precisely $i$ escapes before time $n$ then

(25) \[ \sum_{\omega \in Q^{(n)}(E_1, \ldots, E_i)} |\omega| \lesssim e^{-\gamma n} |\Delta|. \]

Therefore using again the combinatorial counting argument and the inequality (17) we get

(26) \[ \sum_{i \leq \eta n} |Q^{(n)}_i| \approx \sum_{i \leq \eta n} \sum_{(t_1, \ldots, t_i)} |Q^{(n)}_i(t_1, \ldots, t_i)| \lesssim \sum_{i \leq \eta n} N_n e^{-\gamma n} |\Delta| \lesssim e^{\eta n} e^{-\gamma n}. \]
Recall that by (17) \( \hat{\eta} \) can be chosen arbitrarily small by choosing \( \eta \) small. Thus, combining (23) and (26) and substituting into (22) we get
\[
|\{ \omega \in Q | R(\omega) \geq n \}| \lesssim e^{-\gamma \epsilon n} + e^{(\hat{\eta} - \gamma)n} \lesssim e^{-cn}.
\]

6. General Theory of Nonuniformly Expanding Maps

The intuitive picture which emerges from the examples discussed above is that of a default exponential mixing rate for uniformly expanding systems and Hölder continuous observables. However it is clear that general nonuniformly expanding systems can exhibit a variety of rates of decay. Sometimes these rates can be linked to properties of specific neutral orbits which can slow down the mixing process. However it is natural to ask whether there is some intrinsic information related to the very definition of nonuniform expansivity which determines the rate of decay of correlation. We recall that \( f \) is nonuniformly expanding if there exists \( \lambda > 0 \) such that for almost every \( x \in M \)
\[
(*) \quad \liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \| Df^{-1}_{f^i(x)} \|^{-1} > \lambda.
\]
Although the constant \( \lambda > 0 \) is uniform for Lebesgue almost every point, the convergence to the \( \lim \inf \) is not generally uniform.

A measure of nonuniformity has been proposed in [AlvLuzPin03] based precisely on the idea of quantifying the rate of convergence. The measure has been shown to be directly linked to the rate of decay of correlations in [AlvLuzPindim1] in the one-dimensional setting and in [AlvLuzPin] in arbitrary dimensions, in the case of polynomial rates of decay. Recently the theory has been very extended to cover the exponential case as well [Gou04]. We give here the precise statements.

6.1. Measuring the degree of nonuniformity.

6.1.1. The critical set. Let \( f : M \to M \) be a (piecewise) \( C^2 \) map. For \( x \in M \) we let \( Df_x \) denote the derivative of \( f \) at \( x \) and define \( \| Df_x \| = \max \{ \| Df_x(v) \| : v \in T_x M, \| v \| = 1 \} \). We suppose that \( f \) fails to be a local diffeomorphism on some zero measure critical set \( C \) at which \( f \) may be discontinuous and/or \( Df \) may be discontinuous and/or singular and/or blow up to infinity. Remarkably, all these cases can be treated in a unified way as problematic points as will be seen below. In particular we can define a natural generalization of the non-degeneracy (non-flatness) condition for critical points of one-dimensional maps.

**Definition 12.** The critical set \( C \subset M \) is non-degenerate if \( m(C) = 0 \) and there is a constant \( \beta > 0 \) such that for every \( x \in M \setminus C \) we have \( \text{dist}(x, C)^\beta \lesssim \| Df_x v \| / \| v \| \lesssim \text{dist}(x, C)^{-\beta} \) for all \( v \in T_x M \), and the functions \( \log \det Df \) and \( \log \| Df^{-1} \| \) are locally Lipschitz with Lipschitz constant \( \lesssim \text{dist}(x, C)^{-\beta} \).

From now on we shall always assume these non-degeneracy conditions. We remark that the results to be stated below are non-trivial even when the critical set \( C \) is empty and \( f \) is a local diffeomorphism everywhere. For simplicity we suppose also that \( f \) is topologically transitive, i.e. there exists a point \( x \) whose orbit is dense in \( M \). Without the topological transitivity condition we would just get that the measure \( \mu \) admits a finite number of ergodic components and the results to be given below would then apply to each of its components.
6.1.2. Expansion and recurrence time functions. Since we have no geometrical information about $f$ we want to show that the statistical properties such as the rate of decay of correlations somehow depends on abstract information related to the non-uniform expansivity condition only. Thus we make the following

**Definition 13.** For $x \in M$, we define the expansion time function

$$
\mathcal{E}(x) = \min \left\{ N : \frac{1}{n} \sum_{i=0}^{n-1} \log \left\| Df^{-1}_{f^i(x)} \right\|^{-1} \geq \frac{\lambda}{2} \forall n \geq N \right\}.
$$

By condition (*) this function is defined and finite almost everywhere. It measures the amount of time one has to wait before the uniform exponential growth of the derivative kicks in. If $\mathcal{E}(x)$ was uniformly bounded, we would essentially be in the uniformly expanding case. In general it will take on arbitrarily large values and not be defined everywhere. If $\mathcal{E}(x)$ is large only on a small set of points, then it makes sense to think of the map as being not very non-uniform, whereas, if it is large on a large set of points it is in some sense, very non-uniform. We remark that the choice of $\lambda/2$ in the definition of the expansion time function $\mathcal{E}(x)$ is fairly arbitrary and does not affect the asymptotic rate estimates. Any positive number smaller than $\lambda$ would yield the same results.

We also need to assume some dynamical conditions concerning the rate of recurrence of typical points near the critical set. We let $d_\delta(x, C)$ denote the $\delta$-truncated distance from $x$ to $C$ defined as $d_\delta(x, C) = d(x, C)$ if $d(x, C) \leq \delta$ and $d_\delta(x, C) = 1$ otherwise.

**Definition 14.** We say that $f$ satisfies the property of subexponential recurrence to the critical set if for any $\epsilon > 0$ there exists $\delta > 0$ such that for Lebesgue almost every $x \in M$

$$
(\ast\ast) \quad \limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} - \log d_\delta(f^j(x), C) \leq \epsilon.
$$

We remark that although condition (\ast\ast) might appear to be a very technical condition, it is actually quite natural and in fact almost necessary. Indeed, suppose that an absolutely continuous invariant measure $\mu$ did exist for $f$. Then, a simple application of Birkhoff’s Ergodic theorem implies that condition (\ast\ast) is equivalent to the integrability condition

$$
\int_M |\log d_\delta(x, S)| d\mu < \infty
$$

which is simply saying that the invariant measure does not give too much weight to a neighbourhood of the discontinuity set.

Again, we want to differentiate between different degrees of recurrence in a similar way to the way we differentiated between different degrees of non-uniformity of the expansion.

**Definition 15.** For $x \in M$, we define the recurrence time function

$$
\mathcal{R}(x) = \min \left\{ N \geq 1 : \frac{1}{n} \sum_{j=0}^{n-1} - \log d_\delta(f^j(x), C) \leq 2\epsilon, \forall n \geq N \right\}
$$

Then, for a map satisfying both conditions (*) and (\ast\ast) we let

$$
\Gamma_n = \{ x : \mathcal{E}(x) > n \text{ or } \mathcal{R}(x) > n \}
$$

Notice that $\mathcal{E}(x)$ and $\mathcal{R}(x)$ are finite almost everywhere and thus $\Gamma_n \to 0$. It turns out that the rate of decay of $|\Gamma_n|$ is closely related to the rate of decay of correlations. In the statement of the theorem we let $C_n$ denote the correlation function for Hölder continuous observables.
**Theorem 10.** Let \( f : M \rightarrow M \) be a transitive \( C^2 \) local diffeomorphism outside a non-degenerate critical set \( \mathcal{C} \), satisfying conditions (\#) and (\###). Then

1. \[ \text{[AlvBonVia00]} \] \( f \) admits an acip \( \mu \). Some power of \( f \) is mixing
2. \[ \text{[AlvLucPin03, AlvLucPin, AlvLucPindim1]} \] Suppose that there exists \( \gamma > 0 \) such that
   \[ |\Gamma_n| = O(n^{-\gamma}). \]
   Then
   \[ C_n = O(n^{-\gamma+1}). \]
3. \[ \text{[Gou04]} \] Suppose that there exists \( \gamma > 0 \) such that
   \[ |\Gamma_n| = O(e^{-\gamma n}). \]
   Then there exists \( \gamma' > 0 \) such that
   \[ C_n = O(e^{-\gamma'n}). \]

**6.2. Viana maps.** A main application of the general results described above are a class of maps known as Viana or Alves-Viana maps. Viana maps were introduced in \[ \text{[Via97]} \] as an example of a class of higher dimensional systems which are strictly not uniformly expanding but for which the non-uniform expansivity condition is satisfied and, most remarkably, is persistent under small \( C^3 \) perturbations, which is not the case for any of the examples discussed above. These maps are defined as skew-products on a two dimensional cylinder of the form \( f : S^1 \times \mathbb{R} \rightarrow S^1 \times \mathbb{R} \)

\[ f(\theta, x) = (\kappa \theta, x^2 + a + \varepsilon \sin 2\pi \theta) \]

where \( \varepsilon \) is assumed sufficiently small and \( a \) is chosen so that the one-dimensional quadratic map \( x \mapsto x^2 + a \) for which the critical point lands after a finite number of iterates onto a hyperbolic repelling periodic orbit (and thus is a good parameter value and satisfies the non-uniform expansivity conditions as mentioned above). The map \( \kappa \theta \) is taken modulo \( 2\pi \), and the constant \( \kappa \) is a positive integer which was required to be \( \geq 16 \) in \[ \text{[Via97]} \] although it was later shown in \[ \text{[BuzSesTsu03]} \] that any integer \( \geq 2 \) will work. The \( \sin \) function in the skew product can also be replaced by more general Morse functions.

**Theorem 11.** Viana maps

- \[ \text{[Via97]} \] satisfy (\#) and (\###). In particular they are nonuniformly expanding;
- \[ \text{[Alv00, AlvVia02]} \] are topologically mixing and have a unique ergodic acip (with respect to two-dimensional Lebesgue measure);
- \[ \text{[AlvLucPin]} \] have super-polynomial decay of correlations: for any \( \gamma > 0 \) we have
  \[ C_n = O(n^{-\gamma}); \]
- \[ \text{[BaiGou03, Gou04]} \] have stretched exponential decay of correlations: there exists \( \gamma > 0 \) such that
  \[ C_n = O(e^{-\gamma\sqrt{n}}). \]

**7. Existence of Nonuniformly Expanding Maps**

An important point which we have not yet discussed is the fact that the verification of the nonuniform expansivity assumptions is a highly non-trivial problem. For example, the verification that Viana maps are nonuniformly expanding is one of the main results of \[ \text{[Via97]} \]. Only in some special cases can the required assumptions be verified directly and easily. The definition of nonuniform expansivity is in terms of asymptotic properties of the map which are therefore intrinsically not checkable in any given finite number of steps. The same is true also for the derivative growth assumptions on the critical
orbits of one-dimensional maps as in Theorems 8 on page 21 and 9 on page 21. A perfectly legitimate question is therefore whether these conditions actually do occur for any map at all. Moreover, recent results suggest that this situation is at best extremely rare in the sense that the set of one-dimensional maps which have attracting periodic orbits, and in particular do not have an $acip$, is open and dense in the space of all one-dimensional maps [GraSwi97, Lyu97, Koz03, She, KozSheStr03]. However, this topological point of view is only one way of defining “genericity” and it turns out that for general one-parameter families of one-dimensional maps, the set of parameters for which an $acip$ does exist can have positive Lebesgue measure (even though it may be topologically nowhere dense).

We give here a fairly complete sketch (!) of the argument in a special case, giving the complete description of the combinatorial construction and just brief overview of how the analytic estimates are obtained. For definiteness and simplicity we focus on the family

\[ f_a(x) = x^2 - a \]

for \( x \in I = [-2, 2] \) and \( a \in \Omega_\varepsilon = [2 - \varepsilon, 2] \) for some \( \varepsilon > 0 \).

**Theorem 12** ([Jak81]). For every \( \eta > 0 \) there exists an \( \varepsilon > 0 \) and a set \( \Omega^* \subset \Omega_\varepsilon \) such that for all \( a \in \Omega^* \), \( f_a \) admits an ergodic absolutely continuous invariant probability measure, and such that

\[ |\Omega^*| > (1 - \eta)|\Omega_\varepsilon| > 0. \]

There exists several generalizations of this result for families of smooth maps [Jak81, BenCar85, Ryc88, ThiTreYou94, MelStr93, Tsu93a, Tsu93] and even to families with completely degenerate (flat) critical points [Thu99] and to piecewise smooth maps with critical points [LuzVia00, LuzTuc99]. The arguments in the proofs are all fundamentally of a probabilistic nature and the conclusions depend on the fact that if \( f \) is nonuniformly expanding for a large number of iterates \( n \) then it has a “high probability” of being nonuniformly expanding for \( n + 1 \) iterates. Thus, by successively deleting those parameters which fail to be nonuniformly expanding up to some finite number of iterates has to delete smaller and smaller proportions. Therefore a positive proportion survives all exclusions.

In section 7.1 we give the formal inductive construction of the set \( \Omega^* \). In sections 7.2 and 7.3 we prove the two main technical lemmas which give expansion estimates for orbit starting respectively outside and inside some critical neighbourhood. In section 7.4 we prove the inductive step in the definition of \( \Omega^* \) and in section 7.5 we obtain the lower bound on the size of \( |\Omega^*| \).

The proof involves several constants, some intrinsic to the family under consideration and some auxiliary for the purposes of the construction. The relationships between these constants and the order in which they are chosen is quite subtle and also crucial to the argument. However this subtlety cannot easily be made explicit in such a sketch as we shall give here. We just mention therefore that there are essentially only two intrinsic constants: \( \lambda \) which is the expansivity exponent outside some (in fact any) critical neighbourhood, and \( \varepsilon \) which is the size of the parameter interval \( \Omega_\varepsilon \), \( \lambda \) can be chosen first and is essentially arbitrary as long as \( \lambda \in (0, \log 2) \); \( \varepsilon \) needs to be chosen last to guarantee that the auxiliary constants can be chosen sufficiently small. The main auxiliary constants are \( \lambda_0 \) which can be chosen arbitrarily in \((0, \lambda)\) and which gives the target Lyapunov exponent of the critical orbit for good parameters, and

\[ \lambda \gg \alpha \gg \delta = \delta^* \gg \delta > 0 \]

which are chosen in the order given and sufficiently small with respect to the previous ones. During the proof we will introduce also some “second order” auxiliary constants which depend on these.
Finally we shall use the constant $C > 0$ to denote a generic constant whose specific value can in different formulae.

7.1. The definition of $\Omega^s$. We let $c_0 = c_0(a) = f_a(0)$ denote the critical value of $f_a$ and for $i \geq 0$, $c_i = c_i(a) = f^i(c_0)$. For $n \geq 0$ and $\omega \subseteq \Omega$ let $\omega_n = \{c_n(a); a \in \omega\} \subseteq I$. Notice that for $a = 2$ the critical value maps to a fixed point. Therefore iterates of the critical point for parameter values sufficiently close to 2 remain in an arbitrarily small neighbourhood of this fixed point for an arbitrarily long time. In particular it is easy to see that all the inductive assumptions to be formulated below hold for all $k \leq N$ where $N$ can be taken arbitrarily large if $\varepsilon$ is small enough. This observation will play an important role in the very last step of the proof.

7.1.1. Inductive assumptions. Let $\Omega^{(0)} = \Omega$ and $\mathcal{P}^{(0)} = \{\Omega^{(0)}\}$ denote the trivial partition of $\Omega$. Given $n \geq 1$ suppose that for each $k \leq n - 1$ there exists a set $\Omega^{(k)} \subseteq \Omega$ satisfying the following properties.

Combinatorics: For the moment we describe the combinatorial structure as abstract data, the geometrical meaning of this data will become clear in the next section. There exists a partition $\mathcal{P}^{(k)}$ of $\Omega^{(k)}$ into intervals such that each $\omega \in \mathcal{P}^{(k)}$ has an associated itinerary constituted by the following information To each $\omega \in \mathcal{P}^{(k)}$ is associated a sequence $0 = \theta_0 < \theta_1 < \cdots < \theta_r \leq k, r = r(\omega) \geq 0$ of escape times. Escape times are divided into three categories, i.e. substantial, essential, and inessential. Inessential escapes possess no combinatorial feature and are only relevant to the analytic bounded distortion argument to be developed later. Substantial and essential escapes play a role in splitting itineraries into segments in the following sense. Let $0 = \eta_0 < \eta_1 < \cdots < \eta_s < k, s = s(\omega) \geq 0$ be the maximal sequence of substantial and essential escape times. Between any of the two $\eta_{i-1}$ and $\eta_i$ (and between $\eta_s$ and $k$) there is a sequence $\eta_{i-1} < \nu_1 < \cdots < \nu_t < \eta_i, t = t(\omega, i) \geq 0$ of essential return times (or essential returns) and between any two essential returns $\nu_{j-1}$ and $\nu_j$ (and between $\nu_t$ and $\eta_i$ there is a sequence $\nu_{j-1} < \mu_1 < \cdots < \mu_u < \nu_j, u = u(\omega, i, j) \geq 0$ of inessential return times (or inessential returns). Following essential and inessential return (resp. escape) there is a time interval $[\nu_j + 1, \nu_j + p_j]$ (resp. $[\mu_j + 1, \mu_j + p_j]$) with $p_j > 0$ called the binding period. A binding period cannot contain any return and escape times. Finally, associated to each essential and inessential return time (resp. escape) is a positive integer $r$ called the return depth (resp. escape depth).

Bounded Recurrence: We define the function $E^{(k)} : \Omega^{(k)} \to \mathbb{N}$ which associates to each $a \in \Omega^{(k)}$ the total sum of all essential return depths of the element $\omega \in \mathcal{P}^{(k)}$ containing $a$ in its itinerary up to and including time $k$. Notice that $E^{(k)}$ is constant on elements of $\mathcal{P}^{(k)}$ by construction. Then, for all $a \in \Omega^{(k)}$

$$(BR)_k E^{(k)}(a) \leq \alpha k$$

Slow Recurrence: For all $a \in \Omega^{(k)}$ and all $i \leq k$ we have

$$(SR)_k |c_i(a)| \geq \varepsilon^{-\alpha i}$$

Notice that $\alpha$ can be chosen arbitrarily small as long as $\varepsilon$ is small in order for this to hold for all $i \leq N$.

Hyperbolicity: For all $a \in \Omega^{(k)}$

$$(EG)_k |(f_a^{k+1})'(c_0)| \geq Ce^{\lambda_0(k+1)}.$$
**Bounded Distortion:** Critical orbits with the same combinatorics satisfy uniformly comparable derivative estimates: For every \( \omega \in \mathcal{P}^{(k)} \), every pair of parameter values \( a, b \in \omega \) and every \( j \leq \nu + p + 1 \) where \( \nu \) is the last return or escape before or equal to time \( k \) and \( p \) is the associated binding period, we have

\[
(BD)_k = \left| \frac{(f_j')'(c_0)}{(f_j')'(c_0)} \right| \leq D \quad \text{and} \quad \left| \frac{c_j'(a)}{c_j'(b)} \right| \leq D
\]

Moreover if \( k \) is a substantial escape a similar distortion estimate holds for all \( j \leq l \) (\( l \) is the next chopping time) replacing \( D \) by \( D \) and \( \omega \) by any subinterval \( \omega' \subseteq \omega \) which satisfies \( \omega'_l \subseteq \Delta^+ \). In particular for \( j \leq k \), the map \( c_j : \omega \to \omega_j = \{ c_j(a) : a \in \omega \} \) is a bijection.

### 7.1.2. Definition of \( \Omega^{(n)} \) and \( \mathcal{P}^{(n)} \)

For \( r \in \mathbb{N} \), let \( I_r = [e^{-r}, e^{-r+1}] \), \( I_{-r} = -I_r \) and define

\[
\Delta^+ = \{ 0 \} \cup \bigcup_{|r| \geq r_\delta + 1} I_r \quad \text{and} \quad \Delta = \{ 0 \} \cup \bigcup_{|r| \geq r_\delta + 1} I_r,
\]

where \( r_\delta = \log \delta^{-1}, r_{\delta^+} = \log \delta^{-1} \). We can suppose without loss of generality that \( r_\delta, r_{\delta^+} \in \mathbb{N} \). For technical reasons related to the distortion calculation we also need to subdivide each \( I_r \) into \( r^2 \) subintervals of equal length. This defines partitions \( \mathcal{I}, \mathcal{I}^+ \) of \( \Delta^+ \) with \( \mathcal{I} = \mathcal{I}^+ | \Delta \). An interval belonging to either one of these partitions is of the form \( I_{r,m} \) with \( m \in [1, r^2] \). Let \( I_{r,m}^f \) and \( I_{r,m}^p \) denote the elements of \( \mathcal{I}^+ \) adjacent to \( I_{r,m} \) and let \( \hat{I}_{r,m} = I_{r,m}^f \cup I_{r,m}^p \cup I_{r,m}^b \). If \( I_{r,m}^b \) happens to be one of the extreme subintervals of \( \mathcal{I}^+ \) then let \( I_{r,m}^f \) or \( I_{r,m}^p \), depending on whether \( I_{r,m}^b \) is a left or right extreme, denote the intervals \( (-\delta^0, -\delta^0) \) or \( (\delta^0, \delta^0) \) respectively. We now use this partition to define a refinement \( \mathcal{P}^{(n)} \) of \( \mathcal{P}^{(n-1)} \). Let \( \omega \in \mathcal{P}^{(n-1)} \). We distinguish two different cases.

**Non-chopping times:** We say that \( n \) is a non-chopping time for \( \omega \in \mathcal{P}^{(n-1)} \) if one (or more) of the following situations occur: (1) \( \omega_n \cap \Delta^+ = \emptyset \); (2) \( n \) belongs to the binding period associated to some return or escape time \( \nu < n \) of \( \omega \); (3) \( \omega_n \cap \Delta^+ \neq \emptyset \) but \( \omega_n \) does not intersect more than two elements of the partition \( \mathcal{I}^+ \). In all three cases we let \( \omega \in \mathcal{P}^{(n)} \). In cases (1) and (2) no additional combinatorial information is added to the itinerary of \( \omega \). In case (3), if \( \omega_n \cap (\Delta \cup I_{\pm r_\delta}) \neq \emptyset \) (resp. \( \omega_n \subset \Delta^+ \setminus (\Delta \cup I_{\pm r_\delta}) \)), we say that \( n \) is an inessential return time (resp. inessential escape time) for \( \omega \in \mathcal{P}^{(n)} \). We define the corresponding depth by \( r = \max \{|r| : \omega_n \cap I_r \neq \emptyset \} \).

**Chopping times:** In all remaining cases, i.e. if \( \omega_n \cap \Delta^+ \neq \emptyset \) and \( \omega_n \) intersects at least three elements of \( \mathcal{I}^+ \), we say that \( n \) is a chopping time for \( \omega \in \mathcal{P}^{(n-1)} \). We define a natural subdivision

\[
\omega = \omega^f \cup \bigcup_{(r,m)} \omega^{(r,m)} \cup \omega^p,
\]

so that each \( \omega_n^{(r,m)} \) fully contains a unique element of \( \mathcal{I}_+ \) (though possibly extending to intersect adjacent elements) and \( \omega_n^f \) and \( \omega_n^p \) are components of \( \omega_n \setminus (\Delta^+ \cup \omega_n) \) with \( |\omega_n^f| \geq \delta^0/(\log \delta^{-1})^2 \) and \( |\omega_n^p| \geq \delta^0/(\log \delta^{-1})^2 \). If the connected components of \( \omega_n \setminus (\Delta^+ \cup \omega_n) \) fail to satisfy the above condition on their length we just glue them to the adjacent interval of the form \( \omega_n^{(r,m)} \). By definition we let each of the resulting subintervals of \( \omega_n \) be elements of \( \mathcal{P}^{(n)} \). The intervals \( \omega^f, \omega^p \) and \( \omega^{(r,m)} \) with \( |r| < r_\delta \) are called escape components and are said to have a substantial escape and essential escape respectively at time \( n \). The corresponding values of \( |r| < r_\delta \) are the associated essential escape depths. All other intervals are said to
have an essential return at time $n$ and the corresponding values of $|r|$ are the associated essential return depths. We remark that partition elements $I_{\pm r_3}$ do not belong to $\Delta$ but we still say that the associated intervals $\omega^{(\pm r_3, m)}$ have a return rather than an escape.

This completes the definition of the partition $\hat{P}^{(n)}$ of $\Omega^{(n-1)}$ and of the function $E^{(n)}$ on $\Omega^{(n-1)}$. We define

$$\Omega^{(n)} = \{ a \in \Omega^{(n-1)} : E^{(n)}(a) \leq \alpha n \}$$

Notice that $E^{(n)}$ is constant on elements of $\hat{P}^{(n)}$. Thus $\Omega^{(n)}$ is the union of elements of $\hat{P}^{(n)}$ and we can define

$$P^{(n)} = \hat{P}^{(n)}|_{\Omega^{(n)}}.$$ 

Notice that the combinatorics and the recurrence condition $(BR)_n$ are satisfied for every $a \in \Omega^{(n)}$ by construction. In Section 7.4 we shall prove that conditions $(EG)_n, (SR)_n, (BD)_n$ all hold for $\Omega^{(n)}$. Then we define

$$\Omega^* = \bigcap_{n \geq 0} \Omega^{(n)}.$$ 

In particular, for every $a \in \Omega^*$, the map $f_a$ has an exponentially growing derivative along the critical orbit and thus, in particular, by Lemma 8 admits an ergodic acip. In Section 7.5 we prove that $|\Omega^*| > 0$.

We recall that a sketch of the proof of the existence of an induced Markov map under precisely the hyperbolicity and slow recurrence assumptions given here is carried out in section 5.3. As mentioned there, the strategy for construction of the induced Markov map is remarkably similar to the strategy for the construction carried out here for estimating the probability that such conditions hold. The deeper meaning of this similarity is not clear.

7.2. Expansion outside the critical neighbourhood. On some deep level, the statement in the Theorem depends essentially on the following result which we have already used in section 5.3.

**Lemma 7.1.** There exists a constant $C > 0$ independent of $\delta$ such that for $\varepsilon > 0$ sufficiently small, all $a \in \Omega_{\varepsilon}, f = f_a, x \in I$ and $n \geq 1$ such that $x, f(x), \ldots, f^{n-1}(x) \notin \Delta$ we have

$$|Df^n(x)| \geq \delta e^{\lambda n}$$

and if, moreover, $f^n(x) \in \Delta^+ \text{ and/or } x \in f(\Delta^+)$ then

$$|Df^n(x)| \geq Ce^{\lambda n}$$

In the proof of the theorem we will use some other features of the quadratic family and of the specific parameter interval $\Omega_{\varepsilon}$ but it is arguable that they are inessential and that the statement of Lemma 7.1 are to a certain extent sufficient conditions for the argument. It would be very interesting to try to prove the main theorem using only the properties stated in Lemma 7.1. On a general “philosophical” level, the idea, as I believe originally articulated explicitly by L.-S. Young, is that

**Uniform Hyperbolicity for all parameters in most of the state space implies**

**Nonuniform Hyperbolicity in all the state space for most parameters.**
7.3. The binding period. Next we make precise the definition of the binding period which is part of the combinatorial information given above. Let $k \leq n - 1$, $\omega \in \mathcal{P}^{(k)}$ and suppose that $k$ is an essential or inessential return or escape time for $\omega$ with return depth $r$. Then we define the binding period of $\omega_k$ as

$$p(\omega_k) = \min_{a \in \omega} \{ p(c_k(a)) \}$$

where

$$p(c_k(a)) = \min \{ i : |c_{k+1+i}(a) - c_i(a)| \geq e^{-2\alpha i} \}.$$  

This is the time for which the future orbit of $c_k(a)$ can be thought of as shadowing or being bound to the orbit of the critical point (that is, in some sense, the number of iterations for which the orbit of $c(a)$ repeats its early history after the $k$’th iterate). We will obtain some estimates concerning the length of this binding period and the overall derivative growth during this time.

**Lemma 7.2.** There exist constants $\tau_0 > 0$ and $\gamma_1 \in (0, 1)$ such that the following holds. Let $k \leq n - 1$, $\omega \in \mathcal{P}^{(k)}$ and suppose that $k$ is an essential or inessential return or escape time for $\omega$ with return depth $r$. Let $p = p(\omega_k)$. Then for every $a \in \omega$ we have

$$p \leq \tau_0 \log |c_k(a)|^{-1} < k$$

and

$$|Df^{p+1}(c_k(a))| \geq Ce^{r(1-\gamma_1)} \geq Ce^{\frac{1}{\alpha_0}(p+1)}$$

and, if $k$ is an essential return or an essential escape, then

$$|\omega_{k+p+1}| \geq Ce^{-\gamma_1 r}.$$ 

To simplify the notation we write $x = c_k(a)$ and $x_0 = c_{k+1}(a)$ and omit the dependence on the parameter $a$ where there is no risk of confusion. The first step in the proof is to obtain a bounded distortion estimate during binding periods: there exists a constant $D_1(\alpha_0, \alpha_1)$ independent of $x$, such that for all $a \in \omega$, all $y_0, z_0 \in [x_0, c_0]$ and all $0 \leq j \leq \min\{p - 1, k\}$ we have

$$\left| \frac{(f^j)'(z_0)}{(f^j)'(y_0)} \right| < D_1.$$ 

This follows from the standard distortion calculations as in (8) on page 16 using the upper bound $e^{-2\alpha i}$ from the definition of binding in the numerator and the lower bound $e^{-\alpha i}$ from the bounded recurrence condition $(SR)_k$ in the denominator. Notice that for this reason the distortion bound is formally calculated for iterates $j \leq \min\{p - 1, k\}$ (the bounded recurrence condition cannot be guaranteed for iterates larger than $k$). The next step however gives an estimate for the duration of the binding period and implies that $p < k$ and therefore the distortion estimates do indeed hold throughout the duration of the binding period. The basic idea for the upper bound on $p$ is simple. The length of the interval $[x_0, c_0]$ is determined by the length of the interval $[x, c]$ which is $c_k(a)$. The exponential growth of the derivative along the critical orbit and the bounded distortion imply that this interval is growing exponentially fast. The condition which determines the end of the binding period is shrinking exponentially fast. Some standard mean value theorem estimates using these two facts give the result. Finally the average derivative growth during the binding is given by the combined effect of the small derivative of order $c_k(a)$ at the return to the critical neighbourhood and the exponential growth during the binding period. The result then intuitively boils down to showing that the binding period is long enough to (over) compensate the small derivative at the return.
The final statement in the lemma requires some control over the way that the derivatives with respect to the parameter are related to the standard derivatives with respect to a point. This is a fairly important point which will be used again and therefore we give a more formal statement.

**Lemma 7.3.** There exists a constant $\mathcal{D}_2 > 0$ such that for any $1 \leq k \leq n - 1$, $\omega \in \mathcal{P}^{(k-1)}$ and $a \in \omega$ we have

$$\mathcal{D}_2 \geq \frac{|c_k'(a)|}{|Df_{\tilde{a}}^k(c_0)|} \geq \mathcal{D}_2$$

and, for all $1 \leq i < j \leq k + 1$, there exists $\bar{a} \in \omega$ such that

$$\frac{1}{\mathcal{D}_2}|Df_{\bar{a}}^{j-i}(c_i(\bar{a}))| \leq \frac{\omega_j}{\omega_i} \leq \mathcal{D}_2|Df_{\bar{a}}^{j-i}(c_i(\bar{a}))|$$

**Proof.** The second statement is a sort of parameter mean value theorem and follows immediately from the first one and the standard mean value theorem. To prove the first one let $F : \Omega \times I \rightarrow I$ be the function of two variables defined inductively by $F(a, x) = f_a(x)$ and $F^k(a, x) = F(F_{a, F}^{k-1}(a, x))$. Then, for $x = c_0$, we have

$$c_k'(a) = \partial_{\tilde{a}}F_k(a, c_0) = \partial_{\tilde{a}}F(a, f_{\tilde{a}}^{k-1}c_0) = -1 + f_{\tilde{a}}'(c_{k-1})c_k'(a).$$

Iterating this expression gives

$$-c_k'(a) = 1 + f_{\tilde{a}}'(c_{k-1}) + f_{\tilde{a}}'(c_{k-2})f_{\tilde{a}}'(c_{k-1}) + \ldots + f_{\tilde{a}}'(c_1)f_{\tilde{a}}'(c_2)\ldots f_{\tilde{a}}'(c_1)f_{\tilde{a}}'(c_0)$$

and dividing both sides by $(f_{\tilde{a}}^k)'(c_0) = f_{\tilde{a}}'(c_{k-1})f_{\tilde{a}}'(c_{k-2})\ldots f_{\tilde{a}}'(c_1)f_{\tilde{a}}'(c_0)$ gives

$$\frac{c_k'(a)}{(f_{\tilde{a}}^k)'(c_0)} = 1 + \sum_{i=1}^{k} \frac{1}{(f_{\tilde{a}}^i)'(c_0)}.$$ 

The result then depends on making sure that the sum on the right hand side is bounded away from -1. Since the critical point spends an arbitrarily large number $N$ of iterates in an arbitrarily small neighbourhood of a fixed point at which the derivative is $-4$ we can bound an arbitrarily long initial part of this sum by $-1/2$. By the exponential growth condition the tail of the sum is still geometric and by taking $N$ large enough we can make sure that this tail is less than $1/2$ in absolute value. \qed

Returning to the proof of Lemma 7.2 we can use the parameter/space derivative bound to extend the derivative expansion result to the entire interval $\omega_k$ and therefore to estimate the growth of this interval during the binding period.

**7.4. Positive exponents in dynamical space.** Using a combination of the expansivity estimates outside $\Delta$ and the binding period estimates for returns to $\Delta$ it is possible to prove the inductive step stated above.

The slow recurrence condition is essentially an immediate consequence of the parameter exclusion condition.

The exponential growth condition relies on the following crucial and non-trivial observation: the overall proportion of bound iterates is small. This follows from the parameter exclusion condition which bounds the total sum of return depths (an estimate is required to show that inessential return do not contribute significantly to the total) and the binding period estimates which show that the length of the binding period is bounded by a fraction of the return depth. This implies that the overall derivative growth is essentially built up from the free iterates outside $\Delta$ and this gives an overall derivative growth at an exponential rate independent of $n$. 

The bounded distortion estimates again starts with the basic estimate as in (8) on page 16. By Lemma 7.3, it is sufficient to prove the estimate for the space derivatives $Df^k$; intuitively this is saying that critical orbits with the same combinatorics satisfy the same derivative estimates. The difficulty here is that although the images of parameter intervals $\omega$ are growing exponentially, they do not satisfy a uniform backward exponential bound as required to carry out the step leading to (9) on page 17 also images of $\omega$ can come arbitrarily close to the critical point and thus the denominator does not admit any uniform bounds. The calculation therefore is technically quite involved and we refer the reader to published proofs such as [Luz04] for the details. Here we just mention that the argument involves decomposing the sum into “pieces” corresponding to free and bound iterates and estimating each one independently, and taking advantage of the subdivision of the critical neighbourhood into interval $I_r$ each of which is crucially further subdivided into further $r^2$ subintervals of equal length. This implies that the contribution to the distortion of each return is at most of the order of $1/r^2$ instead of order 1 and allows us to obtain the desired conclusion using the fact that $1/r^2$ is summable in $r$.

7.5. Positive measure in parameter space. Recall that $\hat{P}^{(n)}$ is the partition of $\Omega^{(n-1)}$ which takes into account the dynamics at time $n$ and which restricts to the partition $\mathcal{P}^{(n)}$ of $\Omega^{(n)}$ after the exclusion of a certain elements of $\hat{P}^{(n)}$. Our aim here is to develop some combinatorial and metric estimates which will allow us to estimate the measure of parameters to be excluded at time $n$.

The first step is to take a fresh look at the combinatorial structure and “re-formulate it” in a way which is more appropriate. To each $\omega \in \hat{P}^{(n)}$ is associated a sequence $0 = \eta_0 < \eta_1 < \cdots < \eta_n \leq n$, $s = s(\omega) \geq 0$ of escape times and a corresponding sequence of escaping components $\omega \subseteq \omega^{(\eta_0)} \subseteq \cdots \subseteq \omega^{(\eta_n)}$ with $\omega^{(\eta_i)} \subseteq \Omega^{(\eta_i)}$ and $\omega^{(\eta_i)} \in \hat{P}^{(n)}$. To simplify the formalism we also define some “fake” escapes by letting $\omega^{(\eta_i)} = \omega$ for all $s + 1 \leq i \leq n$. In this way we have a well defined parameter interval $\omega^{(\eta_i)}$ associated to $\omega \in \hat{P}^{(n)}$ for each $0 \leq i \leq n$. Notice that for two intervals $\omega, \tilde{\omega} \in \hat{P}^{(n)}$ and any $0 \leq i \leq n$, the corresponding intervals $\omega^{(\eta_i)}$ and $\tilde{\omega}^{(\eta_i)}$ are either disjoint or coincide. Then we define

$$Q^{(i)} = \bigcup_{\omega \in \hat{P}^{(n)}} \omega^{(\eta_i)}$$

and let $Q^{(i)} = \{\omega^{(\eta_i)}\}$ denote the natural partition of $Q^{(i)}$ into intervals of the form $\omega^{(\eta_i)}$. Notice that $\Omega^{(n-1)} = Q^{(n)} \subseteq \cdots \subseteq Q^{(0)} = \Omega^{(0)}$ and $Q^{(n)} = \hat{P}^{(n)}$ since the number $s$ of escape times is always strictly less than $n$ and therefore in particular $\omega^{(\eta_n)} = \omega$ for all $\omega \in \hat{P}^{(n)}$. For a given $\omega = \omega^{(\eta_i)} \in Q^{(i)}$, $0 \leq i \leq n - 1$ we let

$$Q^{(i+1)}(\omega) = \{\omega' = \omega^{(\eta_{i+1})} \in Q^{(i+1)} : \omega' \subseteq \omega\}$$

denote all the elements of $Q^{(i+1)}$ which are contained in $\omega$ and let $Q^{(i+1)}(\omega)$ denote the corresponding partition. Then we define a function $\Delta\mathcal{E}^{(i)} : Q^{(i+1)}(\omega) \to \mathbb{N}$ by

$$\Delta\mathcal{E}^{(i)}(a) = \mathcal{E}^{(\eta_{i+1})}(a) - \mathcal{E}^{(\eta_i)}(a).$$

This gives the total sum of all essential return depths associated to the itinerary of the element $\omega' \in Q^{(i+1)}(\omega)$ containing $a$, between the escape at time $\eta_i$ and the escape at time $\eta_{i+1}$. Clearly $\Delta\mathcal{E}^{(i)}(a)$ is constant on elements of $Q^{(i+1)}(\omega)$. Finally we let

$$Q^{(i+1)}(\omega, R) = \{\omega' \in Q^{(i+1)} : \omega' \subseteq \omega, \Delta\mathcal{E}^{(i)}(\omega') = R\}.$$

Notice that the entire construction given here depends on $n$. The main motivation for this construction and is the following
Lemma 7.4. There exists a constant $\gamma_0 \in (0, 1 - \gamma_1)$ such that the following holds. For all $i \leq n - 1$, $\omega \in Q^{(i)}$ and $R \geq 0$ we have

$$\sum_{\tilde{\omega} \in Q^{(i+1)}(\omega, R)} |\tilde{\omega}| \leq e^{(\gamma_1 + \gamma_0 - 1)R} |\omega|.$$  

(34)

This says essentially that the probability of accumulating a large total return depth between one escape and the next is exponentially small. The strategy for proving this result is straightforward. We show first of all that for $0 \leq i \leq n - 1$, $\omega \in Q^{(i)}$, $R \geq 0$ and $\tilde{\omega} \in Q^{(i+1)}(\omega, R)$ we have

$$|\tilde{\omega}| \leq e^{(\gamma_1 - 1)R} |\omega|.$$  

(35)

The proof is not completely straightforward but depends on the intuitively obvious fact that an interval which has a deep return must necessarily be very small (since it is only allowed to contain at most three adjacent partition elements at the return). Notice moreover that this statement on its own is not sufficient to imply (34) as there could be many small intervals which together add up to a lot of intervals having large return. However we can control to some extent the multiplicity of these intervals and show that we can choose an arbitrarily small $\gamma_0$ (by choosing the critical neighbourhood $\Delta$ sufficiently small) so that for all $0 \leq i \leq n - 1$, $\omega \in Q^{(i)}$ and $R \geq r_\delta$, we have

$$\# Q^{(i+1)}(\omega, R) \leq e^{\gamma_0 R}.$$  

(36)

This depends on the observation that each $\omega$ has an essentially unique (uniformly bounded multiplicity) sequence of return depths. Thus the estimate can be approached via purely combinatorial arguments very similar to those used in relation to equation (17). Choosing $\delta$ small means the sequences of return depths have terms bounded below by $r_\delta$ which can be chosen large, and this allows the exponential rate of increase of the combinatorially distinct sequences with $R$ to be taken small.

Combining (35) and (36) immediately gives (34).

Now choose some $\gamma_2 \in (0, 1 - \gamma_0 - \gamma_1)$ and let

$$\gamma = \gamma_0 + \gamma_1 + \gamma_2 > 0.$$  

For $0 \leq i \leq n - 1$, and $\omega \in Q^{(i)}$ write

$$\sum_{\omega' \in Q^{(i+1)}(\omega)} e^{\gamma_2 \Delta \mathcal{E}^{(i)}(\omega')} |\omega'| = \sum_{\omega' \in Q^{(i+1)}(\omega, 0)} |\omega'| + \sum_{R \geq \gamma_2} e^{\gamma_2 R} \sum_{\omega' \in Q^{(i+1)}(\omega, R)} |\omega'|.$$  

By (34) we then have

$$\sum_{\omega' \in Q^{(i+1)}(\omega, 0)} |\omega'| + \sum_{R \geq \gamma_2} e^{\gamma_2 R} \sum_{\omega' \in Q^{(i+1)}(\omega, R)} |\omega'| \leq \left(1 + \sum_{R \geq \gamma_2} e^{(\gamma_0 + \gamma_1 + \gamma_2 - 1)R}\right) |\omega|.$$  

(37)

Since $\mathcal{E}^{(n)} = \Delta \mathcal{E}^{(0)} + \cdots + \Delta \mathcal{E}^{(n-1)}$ and $\Delta \mathcal{E}^{(i)}$ is constant on elements of $Q^{(i)}$ we can write

$$\sum_{\omega \in Q^{(n)}} e^{\gamma_2 \mathcal{E}^{(n)}(\omega)} |\omega| = \sum_{\omega^{(1)} \in Q^{(1)}(\omega^{(2)})} e^{\gamma_2 \Delta \mathcal{E}^{(1)}(\omega^{(1)})} \sum_{\omega^{(2)} \in Q^{(2)}(\omega^{(3)})} e^{\gamma_2 \Delta \mathcal{E}^{(2)}(\omega^{(2)})} \cdots \sum_{\omega^{(n-1)} \in Q^{(n-1)}(\omega^{(n)})} e^{\gamma_2 \Delta \mathcal{E}^{(n-1)}(\omega^{(n-1)})} \sum_{\omega^{(n)} \in Q^{(n)}} e^{\gamma_2 \Delta \mathcal{E}^{(n)}(\omega^{(n)})} |\omega|.$$  

Notice the nested nature of the expression. Applying (37) repeatedly gives

$$\int_{\Omega^{(n-1)}} e^{\gamma_2 \mathcal{E}^{(n)}(\omega)} |\omega| \leq \left(1 + \sum_{R \geq \gamma_2} e^{(\gamma_0 - 1)R}\right)^n |\Omega|.$$  

(38)
The definition of $\Omega^{(n)}$ gives
$$|\Omega^{(n-1)}| - |\Omega^{(n)}| = |\Omega^{(n-1)} \setminus \Omega^{(n)}| = |\{\omega \in \hat{P}^{(n)} : e^{\gamma_2 \mathcal{E}(n)} \geq e^{\gamma_2 \alpha n}\}|$$
and therefore using Chebyshev’s inequality and (38) we have
$$|\Omega^{(n-1)}| - |\Omega^{(n)}| \leq e^{-\gamma_2 \alpha n} \int_{\Omega^{(n-1)}} e^{\gamma_2 \mathcal{E}(n)} \leq \left[ e^{-\gamma_2 \alpha} \left( 1 + \sum_{R \geq \tau_4} e^{(\gamma-1)R} \right) \right]^n |\Omega|$$
which implies
$$|\Omega^{(n)}| \geq |\Omega^{(n-1)}| - \left[ e^{-\gamma_2 \alpha} \left( 1 + \sum_{R \geq \tau_4} e^{(\gamma-1)R} \right) \right]^n |\Omega|$$
and thus
$$|\Omega^*| \geq \left( 1 - \sum_{j=N}^{\infty} \left[ e^{-\gamma_2 \alpha} \left( 1 + \sum_{R \geq \tau_4} e^{(\gamma-1)R} \right) \right]^j \right) |\Omega|.$$  

Choosing $N$ sufficiently large, by taking $\varepsilon$ sufficiently small, guarantees that the right hand side is positive.

8. Conclusion

In this final section we make some concluding remarks and present some questions and open problems.

8.1. What causes slow decay of correlations? The general theory described in section 6 is based on a certain way of quantifying the intrinsic nonuniformity of $f$ which does not rely on identifying particular critical and/or neutral orbits. However, the conceptual picture according to which slow rates of decay are caused by a slowing down process due to the presence of neutral orbits can also be generalized. Indeed, the abstract formulation of the concept of a neutral orbit is naturally that of an orbit with a zero Lyapunov exponent. The definition of nonuniform expansivity implies that almost all orbits have uniformly positive Lyapunov exponents but this does not exclude the possibility of some other point having a zero Lyapunov exponent. It seems reasonable to imagine that a point with a zero Lyapunov exponent could slow down the overall mixing process in a way which is completely analogous to the specific examples mentioned above. Therefore we present here, in a heuristic form, a natural conjecture.

Conjecture 1. Suppose $f$ is non-uniformly expanding. Then $f$ has exponential decay of correlations if and only it has no orbits with zero Lyapunov exponent.

An attempt to state this conjecture in a precise way reveal several subtle points which need to be considered. We discuss some of these briefly. Let $\mathcal{M}$ denote the space of all probability $f$-invariant measures $\mu$ on $M$ which satisfy the integrability condition $\int \log ||Df_x|| d\mu < \infty$. Then by standard theory, see also [BarPes04, Section 5.8], we can apply a version of Oseledet’s Theorem for non-invertible maps which says that there exist constants $\lambda_1, \ldots, \lambda_k$ with $k \leq d$, and a measurable decomposition $T_x M = E^1_x \oplus \cdots \oplus E^d_x$ of the tangent bundle over $M$ such that the decomposition is invariant by the derivative and such that for all $j = 1, \ldots, k$ and for all non zero vectors $v^{(j)}(x) \in E^j_x$ we have
$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log ||Df^n_x(v^{(j)})|| = \lambda_j.$$
The constants $\lambda_1, \ldots, \lambda_k$ are called the *Lyapunov exponents* associated to the measure $\mu$. The definition of nonuniform expansivity implies that all Lyapunov exponents associated to the $acip$ $\mu$ are $\geq \lambda$ and thus uniformly positive, but it certainly does not exclude the possibility that there exist some other (singular with respect to Lebesgue) invariant probability measure with some zero Lyapunov exponent. This is the case for example for the maps of Section 3 for which the Dirac measure on the indifferent fixed point has a zero Lyapunov exponent.

Thus one way to state precisely the above conjecture is to claim that $f$ has exponential decay of correlations if and only if all Lyapunov exponents associated to all invariant probability measure in $\mathcal{M}$ are uniformly positive. Of course, a priori, there may also be some exceptional points, not typical for any measure in $\mathcal{M}$, along whose orbit the derivative expands subexponentially and which therefore might similarly have a slowing down effect. Also it may be that one zero Lyapunov exponent along one specific direction may not have a significant effect whereas a measure for which all Lyapunov exponents were zero would. Positive results in the direction of this conjecture include the remarkable observation that local diffeomorphisms for which all Lyapunov exponents for all measures are positive, must actually be uniformly expanding [AlvAraSau03, Cao03, CaoLuzRioHyp] and thus in particular have exponential decay of correlations. Moreover, in the context of one-dimensional smooth maps with critical points it is known that in the unimodal case exponential growth of the derivative along the critical orbit (the Collet-Eckmann condition) implies uniform hyperbolicity on periodic orbits [Now88] which in turn implies that all Lyapunov exponents of all measures as positive [BruKel98] and the converse is also true [NowSan98]. Thus conjecture 1 is true in the one-dimensional unimodal setting.

We remark that the assumption of non-uniform expansion is crucial here. There are several examples of systems which have exponential decay of correlations but clearly have invariant measures with zero Lyapunov exponents, e.g. partially hyperbolic maps or maps obtained as time-1 maps of certain flows [Dol98a, Dol98b, Dol00]. These examples however are not non-uniformly expanding, and are generally partially hyperbolic which means that there are two continuous subbundles such that the derivative restricted to one subbundle has very good expanding properties or contracting properties and the other subbundle has the zero Lyapunov exponents. For reasons which are not at all clear, this might be better from the point of view of decay of correlations than a situation in which all the Lyapunov exponents of the absolutely continuous measure are positive but there is some embedded singular measure with zero Lyapunov exponent slowing down the mixing process. Certainly there is still a lot to be understood on this topic.

8.2. **Stability.** The results on the existence of nonuniformly expanding maps for open sets or positive measure sets of parameters are partly *stability* results. They say that certain properties of a system, e.g. being nonuniformly expanding, are stable in a certain sense. We mention here two other forms of stability which can be investigated.

8.2.1. **Topological rigidity.** The notion of (nonuniform) expansivity is, *a priori*, completely metric: it depends on the differentiable structure of $f$ and most constructions and estimates related to nonuniform expansivity require delicate metric distortion bounds. However the statistical properties we deduce (the existence of an $acip$, the rate of decay of correlation) are objects and quantities which make sense in a much more general setting. A natural question therefore is whether the metric properties are really necessary or just very useful conditions and to what extent the statistical properties might depend only on the underlying topological structure of $f$. We recall that two maps $f : M \to M$ and $g : N \to N$ are topologically conjugate, $f \sim g$, if there exists a homeomorphism $h : M \to N$ such that $h \circ f = g \circ h$. We say that a property of $f$ is topological or depends only on the topological structure of $f$ if it holds for all maps in the topological conjugacy class of $f$. 
The existence of an absolutely continuous invariant measure is clearly not a topological invariant in general: if \( \mu_f \) is an acip for \( f \) then we can define \( \mu_g = h^* \mu_f \) by \( \mu_g(A) = \mu_f(h^{-1}(A)) \) which gives an invariant probability measure but not absolutely continuous unless the conjugating homeomorphism \( h \) is itself absolutely continuous. For example the map of Theorem 6 has no acip even though it is topologically conjugate to a uniformly expanding Markov map. However it turns out, quite remarkably, that there are many situations in the setting of one-dimensional maps with critical points in which the existence of an acip is indeed a topological property (although there are also examples in which it is not [Bru98a]). Topological conditions which imply the existence of an acip for unimodal maps were given in [Bru94, San95, Bru98b]. In [NowPrz98] (bringing together results of [NowSan98, PrzRoh98]) it was shown that the exponential growth condition along the critical orbit for unimodal maps (which in particular implies the existence of an acip, see Theorem 8) is a topological property. A counterexample to this result in the multimodal case was obtained in [PrzRivSmi03]. However it was shown in [LuzWan03] that in the general multimodal case, if all critical points are generic with respect to the acip, then the existence of an acip still holds for all maps in the same conjugacy class (although not necessarily the genericity of the critical points).

We emphasize that all these results do not rely on showing that all conjugacies in question are absolutely continuous. Rather they depend on the existence of some topological property which forces the existence of an acip in each map in the conjugacy class. These acip’s are generally not mapped to each other by the conjugacy.

8.2.2. Stochastic stability. Stochastic stability is one way to formalize the idea that the statistical properties of a dynamical systems are stable under small random perturbations. There are several positive results on stochastic stability for uniformly expanding [You86b, BalYou93, Cow00] and nonuniformly expanding maps in dimension 1 [BalVia96, AraLuzVia] and higher [Ara01, AlvVia02]. See [Alv03] for a comprehensive treatment of the results.

8.3. Nonuniform hyperbolicity and induced Markov maps. The definition of nonuniform hyperbolicity in terms of conditions (\( * \)) and (\( ** \)) given above are quite natural as they are assumptions which do not a priori require the existence of an invariant measure. However they do imply the existence of an acip \( \mu \) which has all positive Lyapunov exponents. Thus the system \((f, \mu)\) is also nonuniformly expanding in the more abstract sense of Pesin theory, see [BarPes93]. The systematic construction of induced Markov maps in many examples and under quite general assumptions, as described above, naturally leads to the question of whether such a construction is always possible in this abstract setting. Since the existence of an induced Markov map implies nonuniform expansivity this would essentially give an equivalent characterization of nonuniform expansivity. A general result in this direction has been given for smooth one-dimensional maps in [San03]. It would be interesting to extend this to arbitrary dimension. A generalization to nonuniformly hyperbolic surface diffeomorphisms is work in progress [LuzSan].

It seems reasonable to believe that the scope of application of induced Markov towers may go well beyond the statistical properties of a map \( f \). The construction of the induced Markov map in [San03] for example is primarily motivated by the study if the Hausdorff dimension of certain sets. A particularly interesting application would be a generalization of the existence (parameter exclusion) argument sketched in section 7. Even in a very general setting, with no information about the map \( f \) except perhaps the existence of an induced Markov map, it is natural to ask about the possible existence of induced Markov maps for small perturbations of \( f \). If, moreover, the existence of an induced Markov map were essentially equivalent to nonuniform expansivity then this would be a question about the persistence of nonuniform expansivity under small perturbations.
**Conjecture 2.** Suppose that $f$ is nonuniformly expanding. Then sufficiently small perturbations of $f$ have positive probability of also being nonuniformly expanding.

Using the Markov induced maps one could define, even in a very abstract setting, a critical region $\Delta$ formed by those points that have very large return time. Then outside $\Delta$ one would have essentially uniform expansivity and these, as well as the Markov structure, would essentially persist under small perturbations. One could then perturb $f$ and, up to parameter exclusions, try to show that the Markov structure can be extended once again to the whole of $\Delta$ for some nearby map $g$.

8.4. **Verifying nonuniform expansivity.** The verification of the conditions of nonuniform hyperbolicity are a big problem on both a theoretical and a practical level. As mentioned in Section 7 for the important class of one-dimensional maps with critical point, nonuniform expansivity occurs with positive probability but for sets of parameters which are topologically negligible and thus essentially impossible to pinpoint exactly. The best we can hope for is to show they occur with “very high” probability in some given small range of parameter values.

However even this is generally impossible with the available techniques. Indeed, all existing argument rely on choosing a sufficiently small parameter interval centred on some sufficiently good parameter value. The closeness to this parameter value is then used to obtain the various conditions which are required to start the induction. However the problem then reduces to showing that such a good parameter value exists in the particular parameter interval of interest, and this is again both practically and theoretically impossible in general. Moreover, even if such a parameter value was determined (as in the special case of the “top” quadratic map) existing estimates do not control the size of the neighbourhood in which the good parameters are obtained nor the relative proportion of good parameters. For example there are no explicit bounds for the actual measure of the set of parameters in the quadratic family which have an acip. A standard coffee-break joke directed towards authors of the papers on the existence of such maps is that so much work has gone into proving the existence of a set of parameters which as far as we know might be infinitesimal. Moreover, there just does not seem to be any even heuristic argument for believing that such a set is or isn’t very small. Thus, for no particular reason other than a reaction to these coffee-break jokers (!), we formulate the following

**Conjecture 3.** The set of parameters in the quadratic family which admit an acip is “large”.

For definiteness let us say that “large” means at least 50% but it seems perfectly reasonable to expect even 80% or 90%, and of course we mean here those parameters between the Feigenbaum period doubling limit and the top map. An obvious strategy for proving (or disproving) this conjecture would be to develop a technique for estimating the proportion of maps having an acip in any given small one-parameter family of maps. The large parameter interval of the quadratic family could then be subdivided into small intervals each and the contribution of each of these small intervals could then be added up.

A general technique of this kind would also be interesting in a much broader context of applications. As mentioned in the introduction, many real-life systems appear to have a combination of deterministic and random-like behaviour which suggests that some form of expansivity and/or hyperbolicity might underly the basic driving mechanisms. In modelling such a system it seems likely that one may obtain a parametrized family and be interested in a possibly narrow range of parameter values. It would be desirable therefore to be able to obtain a rigorous prove of the existence of stochastic like behaviour such as mixing with exponential decay of correlations in this family and to be able to estimate the probability of such behaviour occurring. An extremely promising strategy has been proposed recently by K. Mischakow. The idea is to combine non-trivial numerical estimates with the geometric and probabilistic parameter exclusion argument discussed above. Indeed the parameter
exclusion argument, see section [7] relies fundamentally on an induction which shows that the probability of being excluded at time \( n \) are exponentially small in \( n \). The implementation of this argument however also requires several delicate relations between different system constants to be satisfied and in particular no exclusions to be required before some sufficiently large \( N \) so that the exponentially small exclusions occurring for \( n > N \) cannot cumulatively add up to the full measure of the parameter interval under consideration. The assumption of the existence of a particularly good parameter value \( \alpha^* \) and the assumption that the parameter interval is a sufficiently small neighbourhood of \( \alpha^* \) are used in all existing proofs to make sure that certain constants can be chosen arbitrarily small or arbitrarily large thus guaranteeing that the necessary relations are satisfied. Mischaikow’s suggestion is to reformulate the induction argument in such a way that the inductive assumptions can, at least in principle, be explicitly verified computationally. This requires the dependence of all the constants in the argument to be made completely explicit in such a way that the inductive assumptions boil down to a finite set of open conditions on the family of maps which can be verified with finite precision in finite time.

Besides the interest of the argument in this particular setting this could perhaps develop into an extremely fruitful interaction between the “numerical” and the “geometric/probabilistic” approach to Dynamical Systems, and contribute significantly to the applicability of the powerful methods of Dynamical Systems to the solution and understanding of real-life phenomena.

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