Abstract

This work presents an operational and geometric approach to logic. It starts from the multilinear elective decomposition of Boolean functions in the original form introduced by George Boole. It is then shown that this algebraic polynomial formulation can be naturally extended to operators in finite vector spaces. Logical operators will appear as commuting projectors and the truth values, which take the binary values \{0, 1\}, are the respective eigenvalues. In this view the solution of a logical proposition resulting from the operation on a combination of arguments will appear as a selection where the outcome can only be one of the eigenvalues. In this way propositional logic can be formalized in linear algebra by using elective developments which correspond here to combinations of tensored elementary projectors.

1 Introduction

This year 2015 celebrates the 200\textsuperscript{th} anniversary of the birth of George Boole (1815-1864). His visionary approach to logic has led to the formalization in simple mathematical language what was prior to him a language and philosophy oriented discipline. His initial motivation as it appears clear in his first work on logic in 1847: “Mathematical Analysis of Logic” [1], was to propose an algebraic formulation which could generate all the possible logical propositions.

It has to be pointed out that George Boole was already an outstanding mathematician, he was awarded the Gold Medal of the Royal Society in 1844 for his memoir “On a General Method in Analysis”. He was an expert in the resolution of nonlinear differential equations and introduced many new methods using symbolic algebra as stated by Maria Panteki [2].

His approach can be viewed as operational, this characteristic is rarely considered nowadays as pointed out by Theodeore Halperin [3]. George Boole (see [1] p.16) uses \(X, Y, Z\ldots\) to represent the individual members of classes. He then introduces the symbol \(x\), which he named “elective symbol”, operating upon any object comprehending individuals or classes by selecting all the \(X\)’s which it contains. It follows that the product of elective symbols “\(xy\)” will represent, in succession, the selection of the class \(Y\), and the selection from the class \(Y\) of such objects of the class \(X\) that are contained in it, the result being the class common to both \(X\)’s and \(Y\)’s”. In logical language this is the operation of conjunction, \(\text{AND}\).

An expression in which the elective symbols, \(x, y, z,\ldots\), are involved becomes an elective function only if it can be considered “interpretable” in logic. This is the case when the expression sums up to the two possible values 0 and 1. In logic the numbers 0 and 1 correspond to false and true respectively. So, according to George Boole, all the quantities become interpretable when they take the values 0 and 1. This means that all arguments and functions in logic can be considered as elective symbols.
His interpretation of logic using symbolic algebra was different and new because he was convinced that logic had not only to do with “quantity” but should possess a “deeper system of relations” that had to do with the activity of “deductive reasoning”. Now with these premises he was able to use all the common operations of ordinary algebra but introducing a special condition on the symbols: the idempotence law. This law can only be satisfied by the numbers 0 and 1 and was by him considered as the peculiar law for logic.

The aspect of Boole’s method which has been much discussed was his interpretation given to these two special numbers. The number 1 represented for him the class of all conceivable objects i.e. the entire universe, and the number 0 the empty class. The latter being acceptable but the former gives a metaphysical flavor which has not helped to the acceptance of his method.

But as was outlined in [3] the elective symbols and functions denote operators and it will be emphasized in this work that the algebra of elective symbols can also be interpreted as an algebra of commuting projection operators and used for developing propositional logic in a linear algebra framework.

2 Elective functions

Here are briefly presented the basic concepts underlying the elective decomposition method, starting from the very first intuition of George Boole regarding his “digitization” of logic.

Elective symbols obey the following laws, these are sufficient to build an algebra.

Law 1 says that elective symbols are distributive. This means, according to Boole, that “the result of an act of election is independent of the grouping or classification of the subject”.

\[ x(u + v) = xu + xv \]  (1)

Law 2 says that elective symbols commute, this because: “it is indifferent in what order two successive acts of election are performed”.

\[ xy = yx \]  (2)

Law 3 called “index law” by George Boole represents the idempotence of an elective symbol, he states: “that the result of a given act of election performed twice or any number of times in succession is the result of the same act performed once”.

\[ x^n = x \]  (3)

As a consequence of this law George Boole formulated the two following equivalent equations.

\[ x^2 = x \]
\[ x(1 - x) = 0 \]  (4)

Equation 4 explicitly shows that the numbers 0 and 1 are the only possible ones. It also states the orthogonality between the elective symbol \( x \) and \( (1 - x) \), which represents the complement or negation of \( x \). Also:

\[ x + (1 - x) = 1 \]  (5)

this equation shows that the symbol \( x \) and its complement \( (1 - x) \) form the whole class.

Now with these laws and symbols elective functions can be calculated. It is interesting to show how George Boole came to a general expression of an elective function using the Mac Laurin development of a function \( f(x) \) around the number 0. Because of the idempotence law [3] the symbol \( x \) becomes a factor of the series starting from the second term in the Mac Laurin development, this gives:
\[ f(x) = f(0) + x[f'(0) + \frac{1}{2!}f''(0) + \frac{1}{3!}f'''(0) + ...] \]

Then by calculating the function at the value 1, \( f(x = 1) \), using equation 6 one finds a substitute expression of the series. By substituting this expression back in equation 6 one finally gets:

\[
\begin{align*}
    f(x) &= f(0) + x(f(1) - f(0)) \\
    &= f(0)(1 - x) + f(1)x
\end{align*}
\]

Equation 7 shows that the elective function can be developed using the two orthogonal elective symbols \( x \) and \( (1 - x) \). Now if the function is to be “interpretable” it can take only the values 0 and 1, and this means that both numbers \( f(0) \) and \( f(1) \) take the values 0 or 1. As will be shown hereafter these coefficients represent the “truth values” of the logical functions.

How many possibilities, or stated in logical language, how many different logical functions can we build using \( n \) arguments? The possible combinations are \( 2^n \). So considering a unique symbol one obtains 4 functions. These are shown on table 4.

A similar procedure can be used for elective functions of two arguments \( f(x, y) \), this gives the following development using 4 orthogonal and idempotent polynomials:

\[
\begin{align*}
    f(x, y) &= f(0, 0)(1 - x)(1 - y) + f(0, 1)(1 - x)y + f(1, 0)x(1 - y) + f(1, 1)xy
\end{align*}
\]

And so on for increasing \( n \). For \( n = 2 \) we have \( 2^{2n} = 16 \) different elective functions and for \( n = 3 \), \( 2^{3n} = 256 \). All obtained elective functions are idempotent: \( f^2_{el} = f_{el} \).

Equation 8 represents the canonical elective development of a two argument elective function and has the same structure as the “minterm” disjunction canonical form in Boolean algebra [5] which represents the disjunction of mutually exclusive conjunctions.

From equation 5 one sees that all logical functions can be expressed as a combination of degree 1 multilinear polynomials. It can be shown that this decomposition is unique.

George Boole has also developed a method of resolution of what he called “elective equations” where for example the question is: for what values an elective function is true? (for the developments see [1] p. 70).

A very simple method used for resolving elective equations uses the orthogonality of the elective polynomials multiplying the respective coefficients \( f^m(a, b, c, ...) \) which are named \( \pi^m_{a,b,c,...} \). This gives the following equation for selecting the individual coefficients for an \( n \) symbol elective function:

\[
f^m(x, y, z, ...) \cdot \pi^m_{a,b,c,...} = f^m(a, b, c, ...) \pi^m_{a,b,c,...} \]

Equation 8 can be used whatever the number of symbols and also when the functions are not explicitly put in the canonical form. For example if one wants to select the coefficient \( f(0, 1) \) out of \( f(x, y) \) in equation 4 one simply multiplies the function by the corresponding orthogonal polynomial \( (1 - x)y \).

Even though this procedure is simple and straightforward it is not in the habits of logic to use these algebraic expressions, and the reason why is not so clear. One explanation, in my opinion, could be because of technology driven habits: the development of computers using logical gates as building blocks, has generalized what is called “Boolean algebra”, which is not exactly the original George Boole’s algebra [3]. For example addition is considered in Boolean algebra as a modulo 1 sum giving: \( x + x = x \). For a Boolean ring we have even a different rule: \( x + x = 0 \). Whereas the elective calculation employs normal arithmetic addition and subtraction as will be discussed hereafter.

It seems that, historically, only John Venn explicitly used the original reasoning of George Boole in order to build his logical graphic diagrams [3]. He used surfaces on a 2 dimensional space which represented the different logical propositions and more precisely intersection and union corresponding to conjunction and disjunction. Doing this he had, in some cases, to subtract portions of surfaces in order to get the correct
surface measure. For example considering two overlapping surfaces, the surface representing disjunction, \( OR \), is obtained by the sum of the two surfaces minus their intersecting surface (without this subtraction one would count twice the intersecting surface), also for exclusive disjunction, \( XOR \), one has to subtract twice the intersecting surface.

### 3 Elective symbolic logic

In this section the link of elective functions with ordinary propositional logic is presented. Functions and symbols will only take the two binary values 0 and 1 representing respectively the false (\( F \)) and true (\( T \)) character of a given proposition. Logical functions are classified according to their truth tables.

Starting from the very simple propositions derived from the single elective symbol \( x \), according to the function development in equation 7, one sees that there are 4 possible functions depending on the values taken by \( f(0) \) and \( f(1) \) respectively. This is shown on table 1:

In this case the two non trivial propositions are projection \( A \) and its negation \( \bar{A} \). The other two give constant outcomes: false \( F \) and true \( T \) whatever the value of the argument.

On table 2 are shown the 16 elective functions corresponding to \( n = 2 \) arguments. The corresponding elective polynomials can be straightforwardly obtained by substituting the truth value in front of the four polynomial terms in equation 8. According to the standard classification, given for example by Donald Knuth, logical functions are ordered with increasing binary number in the truth table (counting order goes from left to right: the lower digit is on the left). The representation used here corresponds to what is often called the “truth vector” of the function \( f \).

\( f_0^{[2]} \) has the truth values \((0,0,0,0)\) and represents contradiction, \( f_1^{[2]} \) is NOR with truth values \((1,0,0,0)\) and so on... For example conjunction (\( AND \), \( \wedge \)) is \( f_8^{[2]} \) with \((0,0,0,1)\), disjunction (\( OR \), \( \lor \)) is \( f_{14}^{[2]} \) with \((0,1,1,1)\) and exclusive disjunction (\( XOR \), \( \oplus \)) is \( f_6^{[2]} \) with \((0,1,1,0)\).

In table 2 are also shown the canonical polynomial forms issued directly from eq. 8 and the respective simplified polynomial expressions.

Some precisions on other logical propositions: the expression \( A \Rightarrow B \) signifies “\( A \) implies \( B \)”, and the converse \( A \Leftarrow B \) signifies “\( B \) implies \( A \)” the symbol \( \Rightarrow \) signifies non-implication. The expression for \( NAND \) which is “not \( AND \)” is given according to the De Morgan’s law by \( A \lor B \). The same for \( NOR \), “not \( OR \)” given by \( A \land B \).

Negation which is complementation is obtained by subtracting from the number 1 the function. Conjunction \( AND \) corresponds to the following elective function:

\[
f_8^{[2]}(x, y) = f_{AND}^{[2]}(x, y) = xy
\]

and its negation \( NAND \) is simply:

\[
f_7^{[2]}(x, y) = 1 - xy = 1 - f_{AND}^{[2]}(x, y) = f_{NAND}^{[2]}(x, y)
\]
Table 2: The sixteen two argument logical elective functions

By complementing the input symbols i.e. by replacing the symbols $x$ and $y$ by $1-x$ and $1-y$ respectively one gets other logical functions. For example considering:

$$f_1^{[2]}(x, y) = (1 - x)(1 - y) = 1 - x - y - xy = 1 - (x + y - xy)$$
$$= 1 - f_{14}^{[2]}(x, y) = 1 - f_{OR}^{[2]}(x, y) = f_{NOR}^{[2]}(x, y)$$  (12)

this is the complement of the disjunction $OR$ named $NOR$. This result corresponds to De Morgan’s law \[5\] that states that the conjunction $AND$ of the complements is the complement of the disjunction $OR$.

Remark that the expression of the disjunction $OR$ is given by a polynomial expression containing the minus sign “$-”:

$$f_1^{[2]}(x, y) = f_{OR}^{[2]}(x, y) = x + y - xy$$  (13)

this is specific to elective functions, and it must be this way in order that the functions be “interpretable”.

The expression for the exclusive disjunction $XOR$ is given by:

$$f_6^{[2]}(x, y) = f_{XOR}^{[2]}(x, y) = x + y - 2xy$$  (14)

this expression differs from what is usually used in logic where the last term is omitted due to the fact that the addition operation is considered a modulo 1 sum in Boolean algebra. By the way this function is also the parity function giving 1 when the total number of 1's of the arguments is odd.

Implication can also be obtained by the same method, the function corresponding to $A \Rightarrow B$ will be $f_{\Rightarrow}^{[2]}$ and the converse $f_{\Leftarrow}^{[2]}$. According to table 2

$$f_{\Rightarrow}^{[2]}(x, y) = f_{11}^{[2]}(x, y) = 1 - x + xy$$
$$f_{\Leftarrow}^{[2]}(x, y) = f_{13}^{[2]}(x, y) = 1 - y + xy$$  (15)
By De Morgan’s law, by complementing the arguments, it is easy to verify that \( f^{[2]}_{\Sigma} \) transforms into \( f^{[2]}_{\Sigma} \).

The non-implication cases will be respectively \( f^{[2]}_{\Sigma} \) and \( f^{[2]}_{\Sigma} \) and are given by:

\[
f^{[2]}_{\Sigma}(x, y) = f^{[2]}_1(x, y) = x - xy = 1 - f^{[2]}_2 \quad \quad f^{[2]}_{\Sigma}(x, y) = f^{[2]}_2(x, y) = y - xy = 1 - f^{[2]}_1
\]  

One can of course go on by increasing the number of arguments \( n \) in a straightforward way. Let’s consider the cases for \( n = 3 \). The conjunction becomes:

\[
f^{[3]}_{\text{AND}}(x, y, z) = xyz
\]  

The expression of disjunction is obtained in the same way as in equation \( \text{[5]} \) but with three elective symbols \( x, y \) and \( z \). Doing straightforward calculation using the 8 truth values \( (0, 1, 1, 0, 0, 1, 0, 1) \) gives:

\[
f^{[3]}_{\text{OR}}(x, y, z) = x + y + z - xy - xz - yz + xyz
\]  

this expression represents the well-known inclusion-exclusion rule. This formula can be extended to any \( n \) by recurrence.

For the XOR function with \( n = 3 \) one gets, using the truth values \( (0, 1, 1, 0, 0, 1, 0, 1) \):

\[
f^{[3]}_{\text{XOR}}(x, y, z) = x + y + z - 2xy - 2xz - 2yz + 4xyz
\]  

Another very popular function for \( n = 3 \) arguments is the majority \( \text{MAJ3} \) which gives the value 1 when there is a majority of 1’s in the arguments. The function is obtained using the truth values \( (0, 0, 0, 1, 0, 1, 1, 1) \):

\[
f^{[3]}_{\text{MAJ3}}(x, y, z) = xy + xz + yz - 2xyz
\]  

These two last three argument logical functions are used together in digital electronics to build a binary full-adder using logical gates, the three input XOR gives the binary sum and the three input MAJ3 gives the carry out.

So it can be seen that this method is completely general and can be straightforwardly applied to all logical functions whatever the number arguments.

An important remark must be made about the use of the two different polynomial developments named respectively “canonical form” and “polynomial form” shown in the two last columns of table \( \text{[2]} \). The canonical form corresponds to what is named in modern digital logic the canonical minterm decomposition. The minterms correspond here to the products of the elective polynomials. For example for \( n = 2 \) arguments the minterms are the 4 orthogonal polynomials in equation \( \text{[5]} \) in logical language each minterm is one of the possible 4 conjunctions obtained by complementing none, one or two arguments.

One can always put whatever logical function in the canonical form SOP (Sum Of Products), also named the “full conjunctive normal form” \( \text{[5]} \) which is a sum of minterms. A minterm being formed by all input arguments, in a given combination complemented or not, connected by conjunction \( \land \) and the “sum” corresponding here to disjunction \( \lor \) (also exclusive disjunction \( \oplus \) , as discussed earlier). Another canonical decomposition is POS (Product Of Sums) of “maxterms”. A maxterm being all input arguments connected by disjunction \( \lor \) , in a given combination complemented or not, and the “product” corresponding here to conjunction \( \land \), this form is also named the “disjunctive normal form”.

An example of a SOP with four input arguments is:

\[
F^{[4]}_{\Sigma m(5,7,10,15)}(A, B, C, D) = (\neg A \land B \land \neg C \land D) \lor (\neg A \land B \land C \land D) \lor (A \land B \land \neg C \land D) \lor (A \land B \land C \land D)
\]  

The expression \( \Sigma m(5,7,10,15) \) is the standard minterm notation where the numbers correspond to the binary order count in the minterm order. In the minterm form one can easily verify that only one among all minterms can be true at a time, this means that each disjunction \( \lor \) becomes an exclusive disjunction \( \oplus \).
This last point is important regarding the discussion at the end of the preceding section \(2\). One sees that in the minterm SOP decomposition, because all the terms are orthogonal, disjunction and exclusive disjunction play the same role. But in general this is not the case when the logical inputs are not mutually exclusive, as is the case in general (see discussion about Venn’s diagrams in section \(2\)). For example for \(n = 2\) arguments, the two logical projectors \(A\) and \(B\) are not orthogonal.

One can express the expression given in equation \(21\) using the formalism presented in this paper by writing directly the elective decomposition:

\[
\sum_{m(5, 7, 10, 15)} f(x, y, z, r) = (1 - x)y(1 - z)r + (1 - x)yzr + xy(1 - z)r + xyzr
\]

then one can transform this expression into other polynomial forms in order to get a simpler expression. Significant simplifications are obtained when one can factor an argument and its complement, for example \(x\) and \((1 - x)\). The simplest case being the logical projectors themselves such as \(A\) in table \(2\) where the canonical form \(x(1 - y) + xy\) reduces to \(x\). This last argument is essentially what is used to reduce logical functions by using Karnaugh’s diagrams \([5]\).

4 Elective projector logic

This section presents the real new part of this work. It will be shown that the results given above can be applied within the framework of the following formalism.

If one goes back to the motivation of George Boole’s elective symbols, one sees that he applies them as selecting operators on classes of objects. But in some way he was limited by his interpretation of the true value represented by the number 1 which he considered as the unique class \(U\) representing the whole universe. Because of this, without going into all the details, see for example \([2, 3]\), he did not apply his method to subclasses of the universal class \(U\). It must also be stated that at the time of George Boole modern methods in matrix linear algebra were not used.

In this work the elective symbols will represent operators acting on a given class of objects (a subclass of the universe class \(U\)). In this way the elective operator represented by the number 1 will simply become the identity operator for the considered subclass. Using the framework of linear algebra, operators will be defined on a vector space whose dimension depends on the number of arguments in the propositional system.

So what operator can represent the selection of elements out of a class? The straightforward answer in linear algebra are the projection operators which have the property of idempotence. Considering the case of objects belonging to one single class, the corresponding projection operator \(\Pi\) of this class will act on vectors \(\vec{a}\). Now what are the expected outcomes when applying this projector? If a vector \(\vec{a}\) corresponds exactly to elements of the class, the following matrix equations will be verified:

\[
\Pi(1) \cdot \vec{a} = 1 \cdot \vec{a} \quad \Pi(0) \cdot \vec{a} = 0 \cdot \vec{a}
\]

The values 0 and 1 are actually the two eigenvalues of the two projectors associated with the eigenvector \(\vec{a}\). As before, if interpretable results are to be considered in logic, the only possible numbers for these eigenvalues are 0 and 1. 1 will be obtained for objects belonging to the considered class and 0 for objects not belonging to it. In the second case one can also define the complement vector \(\vec{\overline{a}}\).

When these properties are expressed in matrix form: vectors \(\vec{a}\) and \(\vec{\overline{a}}\) will be 2 dimensional orthonormal column vectors and the projection operators \(2 \times 2\) square matrices:

\[
\Pi(1) = \Pi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \Pi(0) = I_2 - \Pi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
\vec{a} = (1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \vec{\overline{a}} = (0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]
The choice of the position of the value 1 in the column vector is arbitrary, here it is in agreement with computer science conventions for a “bit-1” corresponding here to \( \overrightarrow{a} = (1) \).

The two projectors given in equation (24) are complementary and idempotent, this last condition is written:

\[
\Pi \cdot \Pi = \Pi^2 = \Pi
\]  

(26)

One can then construct the 4 logical operators corresponding to the 4 elective functions given in table [1] corresponding to the single argument case \( n = 1 \). Capital bold letters are used here to represent operators.

\[
A = \Pi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \bar{A} = I_2 - \Pi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
T = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad F = 0_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]  

(27)

\( A \) is the logical projector and \( \bar{A} \) its complement. \( T \) is “True” and corresponds here to the identity operator in 2 dimensions \( I_2 \). \( F \) is “False” and corresponds here to the nil operator \( 0_2 \).

Remark that \( I_2 \) and \( 0_2 \) are also projectors (idempotent).

The extensions to more arguments can be obtained by increasing the dimension, this is done by using the Kronecker product \( \otimes \). In the following, as before for the elective logical functions, superscripts are used in order to indicate how many arguments are used in the propositional system.

Projectors considered here are idempotent (equation 26) and commuting pairwise but they need not to be orthogonal. One can verify that in equation (27) all the four projectors are effectively idempotent and commuting. The correspondence of the elective symbol \( x \) with the elementary “seed” projector \( \Pi \) will be used in the following to build the logical operators.

In the case \( n = 2 \) one needs 4 commuting orthogonal rank 1 projectors in order to express the elective decomposition in the same way as shown in equation [8]

Some properties of the Kronecker product on projectors have to be reminded.

(i) The Kronecker product of two projectors is also a projector.

(ii) If projectors are rank 1 projectors (a single eigenvalue is 1 all the others are 0) then their Kronecker product is also a rank 1 projector.

Using these two properties, the 4 commuting orthogonal rank 1 projectors spanning the 4 dimensional vector space can be calculated in a straightforward way:

\[
\Pi^{[2]}_{(0,0)} = (I_2 - \Pi) \otimes (I_2 - \Pi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \Pi^{[2]}_{(0,1)} = (I_2 - \Pi) \otimes \Pi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\Pi^{[2]}_{(1,0)} = \Pi \otimes (I_2 - \Pi) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \Pi^{[2]}_{(1,1)} = \Pi \otimes \Pi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]  

(28)

The logical operators can be obtained in a similar way as before, the notation will be here: \( F_i^{[n]} \).
By the same procedure as in equation (28) one can write the operators for \( n = 2 \) arguments using the projectors given in equation (25):

\[
F^{[2]}_i = f_i^{[2]}(0, 0) \Pi^{[2]}_{(0,0)} + f_i^{[2]}(0, 1) \Pi^{[2]}_{(0,1)} + f_i^{[2]}(1, 0) \Pi^{[2]}_{(1,0)} + f_i^{[2]}(1, 1) \Pi^{[2]}_{(1,1)}
\]  

(29)

The coefficients are the logical function’s truth values given on table 2.

This method can be extended to whatever number of arguments \( n \) using the “seed” projector \( \Pi \) and its complement \((I_2 - \Pi)\).

For \( n = 3 \) arguments one can generate 8 orthogonal 8 dimensional rank 1 projectors, for example two of these are given, by:

\[
\Pi^{[3]}_{(1,1,1)} = \Pi \otimes \Pi \otimes \Pi \quad \Pi^{[3]}_{(0,1,0)} = (I_2 - \Pi) \otimes \Pi \otimes (I_2 - \Pi)
\]  

(30)

For \( n = 2 \) the polynomial expressions have already been calculated in table 2, so one can write down directly the corresponding operator. One has to express the logical projectors corresponding to the two functions \( f^{[2]}_{12} \) and \( f^{[2]}_{10} \), these operators are:

\[
A^{[2]} = F^{[2]}_{12} = 1 \cdot \Pi^{[2]}_{(1,0)} + 1 \cdot \Pi^{[2]}_{(1,1)} = \Pi \otimes (I_2 - \Pi) + \Pi \otimes \Pi = \Pi \otimes I_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]  

(31)

\[
B^{[2]} = F^{[2]}_{10} = 1 \cdot \Pi^{[2]}_{(0,1)} + 1 \cdot \Pi^{[2]}_{(1,1)} = (I_2 - \Pi) \otimes \Pi + \Pi \otimes \Pi = I_2 \otimes \Pi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]  

(32)

This procedure can be generalized to higher dimensions for example in the case \( n = 3 \) one has directly:

\[
A^{[3]} = \Pi \otimes I_2 \otimes I_2 \quad B^{[3]} = I_2 \otimes \Pi \otimes I_2 \quad C^{[3]} = I_2 \otimes I_2 \otimes \Pi
\]  

(33)

Here are some examples: the conjunction operator for \( n = 2 \) will simply be the product of the two logical projectors:

\[
F^{[2]}_{\text{AND}} = A^{[2]} \cdot B^{[2]} = (\Pi \otimes I_2) \cdot (I_2 \otimes \Pi) = \Pi \otimes \Pi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]  

(34)

where the following property of the Kronecker product was used. If \( P, Q, R \) and \( S \) are operators then:

\[
(P \otimes Q) \cdot (R \otimes S) = (P \cdot R) \otimes (Q \cdot S)
\]  

(35)

The disjunction operator can be directly written, using equation (34):

\[
F^{[2]}_{\text{OR}} = A^{[2]} + B^{[2]} - A^{[2]} \cdot B^{[2]} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]  

(36)

The exclusive disjunction can also be directly written, using equation (34):
\[ F_{\text{XOR}}^{[2]} = A^{[2]} + B^{[2]} - 2A^{[2]} \cdot B^{[2]} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \] (37)

Negation is obtained by subtracting from the identity operator (complementation) giving for \( n \) arguments:
\[ \overline{A}^{[n]} = I_{2^n} - A^{[n]} \] (38)

This equation can be used to obtain the NAND operator:
\[ F_{\text{NAND}}^{[2]} = I_4 - F_{\text{AND}}^{[2]} = I_4 - A^{[2]} \cdot B^{[2]} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \] (39)

Using De Morgan’s law:
\[ F_{\text{NOR}}^{[2]} = (I_4 - A^{[2]}) \cdot (I_4 - B^{[2]}) = I_4 - A^{[2]} - B^{[2]} + A^{[2]} \cdot B^{[2]} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = (I_2 - \Pi) \otimes (I_2 - \Pi) \] (40)

For \( n = 3 \) the majority \( \text{MAJ} \) operator will be a \( 8 \times 8 \) matrix, its expression can written directly using equation 20 and equation 33:
\[ F_{\text{MAJ}}^{[3]} = A^{[3]} \cdot B^{[3]} + A^{[3]} \cdot C^{[3]} + B^{[3]} \cdot C^{[3]} - 2A^{[3]} \cdot B^{[3]} \cdot C^{[3]} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \] (41)

So “mechanically” one finds the complete correspondence between elective developments and what we will name from now on “Eigenlogic”.

## 5 Eigenvectors, eigenvalues and truth values

The logical operators given above will act on the vector space, for example when \( n = 2 \) the vectors will have the dimension \( 2^n = 2 = 4 \).

Only operations on the eigenvectors of the corresponding logical projector family are considered in this paper. For \( n = 2 \) a complete family of 16 commuting projectors represents all possible logical propositions and will be “interpretable” when applied on the four possible orthonormal eigenvectors of this family. These vectors form the complete canonical basis.

Vectors \((a, b)\), where the arguments \( a, b \) take the values \( \{0, 1\} \) represent one of the four possible cases shown here:
When applying the logical operators on these vectors the resulting eigenvalue will correspond to the truth value of the corresponding logical proposition.

The method for selecting eigenvalues is similar to the one for elective functions given in equation 9. Because the projectors of the type \( \Pi_{(a,b,c,...)}^{[n]} \) are rank 1 projectors, the product (matrix product) with whatever other projector (for example the logical operator \( F_{i}^{[n]} \)) will also give a rank 1 projector and more precisely this will be the same projector multiplied by the eigenvalue. So for whatever logical operator \( F_{i}^{[n]} \) of the considered family:

\[
F_{i}^{[n]} \cdot \Pi_{(a,b,c,...)}^{[n]} = f_{i}^{[n]}(a, b, c,...) \Pi_{(a,b,c,...)}^{[n]}
\]  

On the right of equation 43 the truth value is multiplied by the corresponding rank 1 projector.

To get explicitly the eigenvalue one can take the trace of the product of the two operators on the left of equation 43. In this way one obtains the truth value \( f_{i}^{[n]}(a, b, c,...) \) corresponding to a case given by a fixed combination of the values \( (a, b, c,...)^{[n]} \) of the logical arguments.

6 Properties of Eigenlogic

To summarize, all the logical operators of a given family (same number of arguments \( n \)) have the following properties in Eigenlogic.

1. The dimension of the vector space spanned by the logical operators is \( d_{n} = 2^{n} \). All logical operators of the same family are \( d_{n} \times d_{n} \) square matrices.

2. All logical operators are idempotent projectors (see eq. 26). This means that in the logical eigenbasis of the family the matrices are diagonal with eigenvalues either 0 or 1.

3. All the logical operators of a given family are commutative pairwise. This means that all the respective matrices are diagonal on the logical eigenbasis of the family.

4. The logical operators are not necessarily orthogonal. This means that the matrix product of two logical operators is not necessarily the nil operator.

5. The number of different logical operators of a given family is \( 2^{2^{n}} \), representing a complete system of logical propositions. This number corresponds to the number of different diagonal matrices obtained for all the combinations of 0’s and 1’s on the diagonal of the matrices.

6. For each family there are \( 2^{n} \) orthogonal rank 1 projectors spanning the entire vector space. The corresponding matrices will have a single eigenvalue of value 1, the other eigenvalues being 0.

7. Every logical operator can be expressed as an elective decomposition using the \( 2^{n} \) orthogonal rank 1 projectors, where the coefficients of the decomposition can only take the values 0 or 1 (see eq. 29 for \( n = 2 \)).
8. Every orthogonal rank 1 projector can be obtained by the means of the Kronecker product, the seed projector \( \Pi \) and its complement \( (I_2 - \Pi) \) (see eq. 24, eq. 28 and eq. 30).

9. The negation of a logical operator, which is its complement, is obtained by subtracting the operator from the identity operator (see eq. 38).

10. The eigenvectors of the family of commuting logical operators form an orthonormal complete basis of dimension \( d_n = 2^n \). This basis is the canonical basis and each eigenvector corresponds to a certain combination of logical arguments (named an “atomic proposition”) for the logical propositional system.

11. The eigenvalues of the logical operators are the truth values of the respective logical proposition and each eigenvalue is associated to a given eigenvector corresponding to a case of the logical proposition.

12. The truth value of a given logical operator for a given case of values of the \( n \) arguments can be obtained using equation 43.

7 Discussion and Conclusion

This work presents a formulation of logic, named here “Eigenlogic”, which uses logical operators in linear algebra and which is similar to the formulation of the elective symbolic algebra of George Boole in [1]. This similarity is striking and is more than just an analogy, at the heart of this is the idempotence property of logical operators. These operators belong to families of commuting projectors. The interesting feature is that the eigenvalues of these operators are the truth values of the corresponding logical functions.

Attempts to link linear algebra and more generally geometry to logic are very numerous and date back to the first efforts to formalize logic. The methods developed by John Venn [4] were already mentioned. In the following are briefly quoted some recent works which came up during this research and which support this approach.

Starting with “Matrix Logic” developed by August Stern [6] which gives directly a matrix formulation for logical operators, by putting the truth values as the matrix coefficients, in the way of Karnaugh diagrams. So for example a two argument logical function becomes a \( 2 \times 2 \) matrix, this is a fundamental difference when compared with the method given here above where \( 4 \times 4 \) matrices are used. Using scalar products on vectors and mean values on operators “Matrix Logic” gives a method to resolve logical equations. This formalism also allows to enlarge the alphabet of the truth-values with negative logic antivalues.

A breakthrough has been undoubtedly made by “Vector Logic” developed by Eduardo Mizraji [7]. This approach vectorizes logic where the truth values map on orthonormal vectors. Technically this approach is different from the one presented here because the resulting operators for 2 arguments are represented by \( 2 \times 4 \) matrices and do not represent projectors. “Vector Logic” can also deal with three-valued logic. Applications have been proposed for neural networks.

A very pertinent development, which is close to the approach in this paper, was done by Vannet Aggarwal and Robert Caldebrabnk [8] in the framework of coding theory, their work was also justified by the “Projection Logic” formulation of David Cohen [9]. In their method they connect Boolean logic to projection operators derived initially from the Heisenberg-Weyl group. They associate the dimension of the considered projector with the Hamming weight (number of 1’s in the truth table) of the corresponding Boolean function. The logical operators they obtain are commuting projectors, as in the work presented here. So their work can be considered as the first formalization of these kind of logical operators in linear algebra.

In the formulation given here, I think that a more general method is proposed, enabling the construction of logical projectors from a single “seed” projector using the Kronecker product. It gives also a simpler formulation because the elective interpretation of logic shows that the idempotence property (eq. 4 and eq. 11) in association with distributivity (eq. 11) and commutativity (eq. 2) permit to identify directly commuting projectors with logical functions.
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