MULTIPLICITY DISTRIBUTIONS IN STRONG INTERACTIONS: A GENERALIZED NEGATIVE BINOMIAL MODEL

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Abstract

A three-parameter discrete distribution is developed to describe the multiplicity distributions observed in total- and limited phase space volumes in different collision processes. The probability law is obtained by the Poisson transform of the KNO scaling function derived in Polyakov’s similarity hypothesis for strong interactions as well as in perturbative QCD, \( \psi(z) \propto z^\alpha \exp(-z^\beta) \). Various characteristics of the newly proposed distribution are investigated e.g. its generating function, factorial moments, factorial cumulants. Several limiting and special cases are discussed. A comparison is made to the multiplicity data available in \( e^+e^- \) annihilations at the \( Z^0 \) peak.

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1. Introduction

Over the past years much attention has been focused on the description of multiplicity distributions (MDs) for hard processes in perturbative quantum chromodynamics (pQCD), see ref. [1] for recent reviews. Already in the lowest double logarithmic approximation (DLA) it was demonstrated that QCD provides a natural explanation of the KNO scaling law

\[ P_n = \frac{1}{\langle n \rangle} \psi \left( \frac{n}{\langle n \rangle} \right) \]  

(1.1)

of the MDs [2] but the width of the scaling function \( \psi(z = n/\bar{n}) \) was overestimated. To improve on the agreement with experiments higher-order perturbative corrections should be taken into account. Mathematically this manifests itself in the appearance of a new expansion parameter \( \gamma q \), the product of the QCD multiplicity anomalous dimension \( \gamma \propto \sqrt{\alpha_s} \) and the rank \( q \) of the moments of \( P_n \). For example, the influence of recoil effects [3] on multiplicity fluctuations is seen most clearly when one considers the normalized factorial moments

\[ F_q = \frac{\langle n(n-1)\ldots(n-q+1) \rangle}{\langle n \rangle^q}. \]  

(1.2)

In the next-to-next-to-leading log approximation (NNLA) the \( F_q \) are shown [4] to be dominated for large ranks \( q \) by the \( \Gamma \)-function of the rescaled rank \( q/\mu \),

\[ F_q \propto \Gamma\left(\frac{3}{2} + q/\mu \right) D^{-q} \]  

(1.3)

where \( \mu = (1 - \gamma)^{-1} \) and \( D \) is a scale parameter depending on \( \mu \). Since \( \gamma \simeq 0.4 \) at the \( Z^0 \) peak, the strength of higher-order multiplicity fluctuations are drastically reduced by recoil effects and therefore the tail of the KNO function gets modified. Instead of the exponential fall-off predicted in the DLA, a faster than exponential decay law emerges for large \( z \) [4]

\[ \psi(z) \propto z^n \exp\left(-[Dz]^{\mu}\right) \]  

(1.4)

with \( \alpha = 3\mu/2 - 1 \) and \( \mu \simeq 5/3 \). This type of behaviour reduces the width of \( \psi(z) \) overestimated by the early DLA calculations.

Another manifestation of higher-order pQCD effects on the MDs is best seen in the \( q \)-dependence of the ratio \( H_q = K_q/F_q \) invented by Dremin [5]
where $K_q$ denotes the normalized factorial cumulant moments,

$$K_q = F_q - \sum_{i=1}^{q-1} \binom{q-1}{i} K_{q-i} F_i.$$  \hspace{1cm} (1.5)

In the DLA $H_q$ decreases monotonously as $q^{-2}$ but the inclusion of higher order perturbative corrections yields quite nontrivial $q$-dependence. In the next-to-leading log approximation (MLLA) $H_q$ acquires a negative minimum at $q \simeq 5$ and approaches for large ranks $q$ the abscissa from below. In the NNLA the negative minimum at $q \simeq 5$ is followed by sign-changing oscillations of the moment ratios [6]. Quite recently this peculiar behaviour has been experimentally confirmed in $e^+e^-$ annihilations [7].

In a historical context it is worth mentioning that some of the above predictions of pQCD were put forward already in 1970 by Polyakov formulating a similarity hypothesis for strong interactions in $e^+e^-$ annihilations [8]. He obtained asymptotic KNO scaling behaviour for the MDs (two years earlier than the original KNO-paper) with a scaling function that behaves according to

$$\psi(z) \Rightarrow \begin{cases} 
\exp(-z^{1/(1-2\delta)}) & \text{as } z \to \infty \\
0 & \text{as } z \to 0 
\end{cases}$$ \hspace{1cm} (1.6)

where $0 < \delta < 1/2$ and the small-z behaviour of the scaling function was found, with certain assumptions, to be a monomial [8]. Clearly, Polyakov arrived in his model at the same law for multiplicity fluctuations as the pQCD result (1.4). A KNO scaling function of the form of Eq. (1.4) was studied in more or less detail by others as well [9] most notably by Krasznovszky and Wagner. We call attention also to the remarks made by Koba, Nielsen and Olesen in connection with the moment problem [2] to which we shall return later.

The goal of the present paper is the development of a discrete distribution well suited to describe available multiplicity data when the popular negative binomial parametrization fails. Our guiding principle is the incorporation of the pQCD-based characteristics

\begin{enumerate}
\item[(i)] factorial moments $F_q$ dominated by the $\Gamma$-function of the rescaled rank $q/\mu$,
\item[(ii)] factorial cumulant-to-moment ratios $H_q$ displaying nontrivial sign-changing oscillations, and
\item[(iii)] distributions $P_n$ possessing squeezed high-multiplicity tail in accordance with Eq. (1.4).
\end{enumerate}

We shall see that a simple generalization of the negative binomial law with one additional parameter satisfies all these requirements.
2. The scaling function in pQCD

Let us first consider in more detail the modified KNO scaling function (1.4) obtained in pQCD with fixed coupling approximation [4]. For later convenience we use a somewhat different choice of the parameters. Introducing the shape parameter $k = (\alpha + 1)/\mu$ and denoting the scale parameter by $\lambda$ we get

$$\psi(z) = N z^{\mu k-1} \exp \left(-[\lambda z]^\mu\right)$$  \hspace{1cm} (2.1)

where $N$ is a normalization constant. The scaling function $\psi(z)$ should satisfy the two normalization conditions

$$\int_0^\infty \psi(z) \, dz = \int_0^\infty z \psi(z) \, dz = 1.$$  \hspace{1cm} (2.2)

The first condition determines the constant $N$. It can be obtained by the use of the integral [10]

$$\int_0^\infty z^{\mu k-1} \exp \left(-[\lambda z]^\mu\right) \, dz = \frac{\Gamma(k)}{\mu \lambda^{\mu k}}$$  \hspace{1cm} (2.3)

providing the reciprocal of $N$. Thus the complete form of the scaling function (at the moment without the second normalization constraint) is

$$\psi(z) = \frac{\mu}{\Gamma(k)} \lambda^{\mu k} z^{\mu k-1} \exp \left(-[\lambda z]^\mu\right).$$  \hspace{1cm} (2.4)

Eq. (2.4) is known in the mathematical literature as the generalized gamma distribution [11]. Obviously, the $\mu = 1$ special case yields the ordinary gamma distribution which corresponds in pQCD to the KNO scaling function at asymptotically high energies ($\gamma \propto \sqrt{\alpha_s} \to 0$).

The probability density function (2.4) has a long history [12]. Its first appearance in 1925 is due to L. Amoroso who analyzed the distribution of economic income. The next application is related to the grinding of materials: in 1933 Rosin, Rammler and Sperling arrived at Eq. (2.4) as the size distribution of grains produced by comminution. Nevertheless in the literature of particle size distributions the formula is named after Nukiyama and Tana-sawa who rediscovered it in 1939 studying drop size distributions in sprays. Widespread use of Eq. (2.4) by mathematicians started only in 1962 by the paper of Stacy. He introduced it as the generalized gamma distribution and
since then this name is the most frequently used one. Some of the recent applications include polymer physics [13] and the theory of multiplicative processes [14].

The moments of the scaling function (2.4) are readily obtained from the integral given by Eq. (2.3) which yields

$$\langle z^q \rangle = \int_0^\infty z^q \psi(z) \, dz = \frac{\Gamma(k + q/\mu)}{\Gamma(k)} \frac{1}{\lambda^q}.$$  \hspace{1cm} (2.5)

Eq. (2.5) with $q = 1$ and the second normalization condition in Eq. (2.2) restrict the scale parameter to

$$\lambda = \frac{\Gamma(k + 1/\mu)}{\Gamma(k)}$$ \hspace{1cm} (2.6)

which completes the analytic form of the scaling function. Let us now consider the cumulative distribution corresponding to the probability density (2.4). It is given by

$$\varphi(x) = \int_0^x \psi(z) \, dz = \frac{\gamma(k, [\lambda x]^\mu)}{\Gamma(k)}$$ \hspace{1cm} (2.7)

where $\gamma(\cdot)$ denotes the incomplete gamma function. Eq. (2.7) is particularly useful in connection with KNO-G scaling [15,16] i.e. if the discrete multiplicity distributions $P_n$ are approximated by the continuous scaling function $\psi(z)$,

$$P_n = \int_{z_n}^{z_{n+1}} \psi(z) \, dz = \varphi(z_{n+1}) - \varphi(z_n).$$ \hspace{1cm} (2.8)

Although KNO-G scaling is becoming increasingly popular in the description of MDs, with $\psi(z)$ chosen to be a shifted lognormal density, we will follow a different approach to construct $P_n$ from the scaling function.

Perhaps the most advantageous property of the generalized gamma distribution is that $\psi(z; k, \lambda, \mu)$ can be used to specify a variety of well known probability laws. Some examples are cited in Table 1 with the corresponding set of parameters. It is seen that the exponent $\mu$ can take negative values as well in which case the normalization constant in Eq. (2.4) involves $|\mu|$. Moreover, for negative $\mu$ the moments $\langle z^q \rangle$ of the distribution are finite only for ranks $q$ satisfying $q/\mu > -k$. A good example is the completely asymmetric Lévy law of index $\alpha = 1/2$ having no finite, positive-rank moments. Finally let us call special attention to the last two rows of Table 1 that display the limit distributions of $\psi(z; k, \lambda, \mu)$ involving for $\mu \to 0$ the lognormal law.
3. MDs at preasymptotic energies

Comparing the KNO function (2.4) to the pQCD result quoted in Section 1 we see that the more accurate account of recoil effects in [4] yields for \( \psi(z) \) a generalized gamma distribution with fixed shape parameter \( k = 3/2 \). Furthermore, the factorial moments given by Eq. (1.3) are the same as the ordinary moments (2.5) of \( \psi(z; \frac{3}{2}; \lambda, \mu) \) which is due to the identification of \( \langle z^q \rangle \) and \( F_q \) in ref. [4]. Their equivalence holds valid for the \( \mu = 1 \) asymptotic energy scale as well as for preasymptotic energies characterized by \( \mu > 1 \) if \( P_n \) is defined by the Poisson transform of \( \psi(z) \),

\[
P_n = \int_0^\infty \psi(z) \frac{(\bar{z})^n}{n!} e^{-\bar{z}} \, dz. \tag{3.1}
\]

For Eq. (3.1) the asymptotic scaling limit of the MDs is [17]

\[
\lim_{n \to \infty, \langle n \rangle \to \infty} P_n = \frac{1}{\langle n \rangle} \psi\left( \frac{n}{\langle n \rangle} \right). \tag{3.2}
\]

Thus the knowledge of the shape of \( P_n \) at preasymptotic energies enables one to guess the asymptotic shape of \( \psi(z) \). We mention that the scaling limit (3.2) naturally arises in Polyakov’s model [18] suggesting the use of (3.1) instead of (2.8) to construct \( P_n \) from \( \psi(z) \).

The probability generating function of \( P_n \) defined by the Poisson transform (3.1) reads as follows:

\[
G(u) = \sum_{n=0}^{\infty} (1-u)^nP_n = \int_0^\infty \psi(z) e^{-u\bar{z}} \, dz, \tag{3.3}
\]

i.e. \( G(u) \) is given by the Laplace transform of \( \psi(z) \) [17]. Unfortunately the Laplace transform of Eq. (2.4) can be expressed in terms of special functions only for a few specific values of the parameter \( \mu \). To go further we have to make use of Fox’s generalized hypergeometric function, \( H(x) \). The reader unfamiliar with the theory of Fox functions can find a compilation of the necessary formulae in the Appendix. The KNO scaling function (2.4) is expressed in terms of \( H(x) \) as the following \( H \)-function distribution:

\[
\psi(z) = \frac{\lambda}{\Gamma(k)} H_{0,1}^{1,0} \left[ \lambda \left| \begin{array}{c} \frac{1}{k-1/\mu} \frac{1}{1/\mu} \\ \end{array} \right. \right] . \tag{3.4}
\]
The above form is obtained from Eq. (A.7) using the identity (A.4) and the integral (2.3) to ensure proper normalization, see also [22]. The probability generating function of $P_n$ defined by the Poisson transform of Eq. (3.4) can be easily evaluated with the help of (A.5), the Laplace transform of $H(x)$. It yields

$$G(u) = \frac{1}{\Gamma(k)} H_{1,1}^{1,1} \left[ \frac{\lambda}{u\bar{n}} \left| \begin{array}{c} (1, 1) \\ (k, 1/\mu) \end{array} \right. \right], \quad 0 < \mu < 1$$

(3.5)

and

$$G(u) = \frac{1}{\Gamma(k)} H_{1,1}^{1,1} \left[ \frac{u\bar{n}}{\lambda} \left| \begin{array}{c} (1 - k, 1/\mu) \\ (0, 1) \end{array} \right. \right], \quad \mu > 1$$

(3.6)

where $\lambda$ is given by Eq. (2.6). The necessity of two separate expressions for $G(u)$ follows from the existence conditions of $H(x)$ discussed in the Appendix. The $\mu = 1$ case is of course the generating function of the negative binomial distribution, $G(u) = (1 + u\bar{n}/k)^{-k}$. The $\mu < 0$ case will be studied elsewhere.

The characteristic function $\phi(t)$ of the generalized gamma density (2.4) is usually written as an infinite sum [21]. In terms of the Fox function $\phi(t)$ can be expressed in a much simpler way by changing the variable in $G(u)$ given by Eqs. (3.5-6),

$$\phi(t) = \int_0^\infty \psi(z)e^{itz}dz = G(u\bar{n} \Rightarrow -it)$$

(3.7)

with, in general, unconstrained scale parameter $\lambda$. For $\mu > 1 \phi(t)$ was obtained earlier in ref. [22]. The probability generating function provides also the Poisson transform of $\psi(z)$ for $n = 0$ through $P_0 = G(1)$.

Let us now consider the Poisson transform of (3.4) for arbitrary $n$. By simple algebra, using identities (A.2-4) and the Laplace transform (A.5) of $H(x)$ one arrives at the following discrete $H$-function distributions:

$$P_n = \frac{1}{n! \Gamma(k)} H_{1,1}^{1,1} \left[ \frac{\lambda}{\bar{n}} \left| \begin{array}{c} (1 - n, 1) \\ (k, 1/\mu) \end{array} \right. \right], \quad 0 < \mu < 1$$

(3.8)

and

$$P_n = \frac{1}{n! \Gamma(k)} H_{1,1}^{1,1} \left[ \frac{\bar{n}}{\lambda} \left| \begin{array}{c} (1 - k, 1/\mu) \\ (n, 1) \end{array} \right. \right], \quad \mu > 1.$$  

(3.9)

The $\mu = 1$ case yields the Poisson transform of the ordinary gamma distribution, i.e. the negative binomial law. The splitted parameter space for $\mu$
reflects an important difference between the two expressions for $P_n$. In case of Eq. (3.8) $P_n$ is infinitely divisible, just as the negative binomial for $\mu = 1$, whereas the $\mu > 1$ case given by Eq. (3.9) violates this feature. Due to the preservation of infinite divisibility under Poisson transforms the same distinction between the two $\mu$-domains holds for the generalized gamma distribution [23].

The factorial moments of $P_n$ can be determined through the equivalence of $\langle z^q \rangle$ and $F_q$ for Poisson transforms [17]. Plugging into Eq. (2.5) with scale parameter $\lambda$ restricted by (2.6) we get

$$F_q = \frac{\Gamma(k + q/\mu)}{\Gamma(q)(k + 1/\mu)} \Gamma^{q-1}(k).$$

As intended, the factorial moments are dominated by the $\Gamma$-function of the rescaled rank $q/\mu$ for large $q$. With the help of Eq. (1.5) we have calculated the moment ratios $H_q = K_q/F_q$ for shape parameter $k = 3/2$. The behaviour of $\log |H_q|$ over the $\mu$-$q$ plane is shown in Fig. 1a. The peculiar $q$-dependence for $\mu > 1$ is due to sign-changing oscillations of $H_q$. Observe that the pattern of oscillations is nontrivial, i.e. not alternating as $q$ takes even/odd values, see Fig. 1b. Qualitatively similar sign-changing oscillations of $H_q$ occur for a different choice of the shape parameter $k$.

4. Comparison to $e^+e^-$ data

One of the advantages of $P_n$ given by Eqs. (3.8-9) is its generality: the Poisson transform of many well known probability laws (e.g. of those cited in Table 1) is a special case. The negative binomial distribution ($\mu = 1$) not rarely fails to give a reasonable description of the MDs. For example, in $e^+e^-$ annihilations the full phase-space MDs at the $Z^0$ peak show significant deviation from a negative binomial shape as reflected by the $\chi^2$/d.o.f. = 66/23 and 68/24 values reported by the Delphi and SLD collaborations [24,7]. Moreover, the sign-changing oscillations displayed by the SLD data for $H_q$ [7] can not be reproduced by a (possibly truncated) negative binomial. It is natural to ask how the $\mu \neq 1$ special cases of Eqs. (3.8-9) can describe the observed features.

To answer this question we carried out fits to the $e^+e^-$ multiplicity data at the $Z^0$ peak [7,24]. In the fitting procedure numerical evaluation of the integral (3.1) is implemented using 96-point gaussian quadrature. The scaling
function (2.4) is log-linear so that \( y = \ln z \) can be written as \( y = w/\mu - \ln \lambda \) where \( w \) has probability density \( f(w) = \exp(kw - e^w)/\Gamma(k) \). Making the parameter transformations \( p = k^{-1/2}, \sigma = p/\mu \) and \( \alpha = \ln \lambda \), further, reflecting the model about the origin \( p = 0 \) to negative \( p \) the scaling function (2.4) takes the form

\[
\psi(z) = \frac{|p|}{\Gamma(p^{-2}) \sigma z} \exp \left[ p^{-2} w - e^w \right] \quad \text{if } p \neq 0
\]  

(4.1)

with \( w = p (\ln z + \alpha)/\sigma \). The reparametrization allows the lognormal law to be mapped to the origin,

\[
\psi(z) = \frac{1}{\sqrt{2\pi} \sigma z} \exp \left[ -\frac{1}{2} \frac{(\ln z + \alpha)^2}{\sigma^2} \right] \quad \text{if } p = 0.
\]  

(4.2)

Above, the location parameter \( \alpha \) is restricted by the second normalization condition in Eq. (2.2) to

\[
\alpha = \begin{cases} 
\ln \Gamma(p^{-2} + \sigma/p) - \ln \Gamma(p^{-2}) & \text{if } p \neq 0 \\
\sigma^2/2 & \text{if } p = 0
\end{cases}
\]  

(4.3)

The distributions represented by Eqs. (4.1-2) are stochastically continuous in the parameters and include besides the lognormal \((p = 0)\) e.g. the gamma \((p = \sigma)\), exponential \((p = \sigma = 1)\) and Weibull \((p = 1)\) densities. For further details on the above transformation of generalized gamma variates the reader is referred to [25].

In the numerical evaluation of the integral (3.1) we have used Eqs. (4.1-2) for \( \psi(z) \). There is a strong reason of replacing the parametrization (2.4) with those of (4.1-2) in the fitting procedure. The original fits using the form (2.4) were able to reduce significantly the \( \chi^2 \) corresponding to the \( \mu = 1 \) negative binomial case but they resulted unexpectedly small values of \( \mu \). For example, fitting the Delphi data yields \( \chi^2/\text{d.o.f.} = 33/23, k \simeq 1000 \) and \( \mu \simeq 0.1 \) indicating that the observed \( P_n \) is a Poisson transformed lognormal distribution. Performing the fits using (4.1-2) reinforced our finding for each data set. This can be seen from Table 2 where the outcome of the fits are collected. In Fig. 2a the best-fit theoretical \( P_n \) is displayed for the SLD data.

The above result seems to be in sharp contradiction with the SLD data for the moment ratios \( H_q \) [7]. Sign-changing oscillations can occur in our model only for \( \mu > 1 \), if \( P_n \) is not infinitely divisible. Fitting the \( H_q \) data
with the help of Eqs. (3.10) and (1.5) produced unacceptable $\chi^2$. To allow the possible influence of truncation effects [26] we recalculated the factorial moments $F_q$ in terms of $P_n$ using $n_{\text{min}} = 6$ and $n_{\text{max}} = 50$ in accordance with the SLD data [24]. The resulting behaviour of $H_q$ is shown in Fig. 2b, the agreement between the theoretical curve and the data points is excellent. The corresponding $\chi^2$ and the values of the best-fit parameters are cited in the last row of Table 2. Clearly, the obtained parameters are very similar to those found in the previous fits. Although there is a small but significant deviation from $p = 0$ we can safely conclude that the multiplicity data in $e^+e^-$ annihilations at the $Z^0$ peak favour a lognormally shaped KNO function $\psi(z)$ in the asymptotic limit given by Eq. (3.2).

5. Summary and conclusions

We have developed a three-parameter discrete distribution to analyse the experimental data for MDs in different collision processes. It is obtained by the Poisson transform of the generalized gamma density (2.4) and thus it provides a generalization of the popular negative binomial distribution. The analytic form of $P_n$ is expressed in terms of $H$-functions. It is given by Eq. (3.8) for infinitely divisible $P_n$ and by Eq. (3.9) if $P_n$ does not obey this feature. The $\mu = 1$ marginal case between the two families of distributions is the negative binomial law. Violation of infinite divisibility for Eq. (3.9) allows nontrivial sign-changing oscillations of the factorial cumulant-to-moment ratios $H_q$. For $\mu > 1$ we thus have the ability to reproduce this peculiar $q$-dependence of $H_q$ data without truncation. Since the factorial moments (3.10) are dominated by the $\Gamma$-function of the rescaled rank $q/\mu$ for large $q$ we can also reproduce possible enhancement ($\mu < 1$) and suppression ($\mu > 1$) of multiplicity fluctuations with respect to the negative binomial behaviour.

Fitting the newly developed distribution to the $e^+e^-$ multiplicity data at the $Z^0$ peak we have found $\mu < 1$ departure from the negative binomial law. In fact the observed MDs are in agreement with the Poisson transform of the $\mu \to 0$ limit distribution of (2.4) which is the lognormal density. Even the sign-changing oscillations of the SLD data for $H_q$ can be fully reproduced after taking into account truncation effects. It is important to emphasize that the $\mu \to 0$ limit manifests itself through the factorial moments.
of $P_n$ being equivalent to the ordinary moments of the asymptotic $\psi(z)$. At
the $Z^0$ peak the KNO function exhibits suppressed high-multiplicity tail in
accordance with the pQCD results quoted earlier. We disagree with ref. [16]
stating that $\psi(z)$ is lognormally shaped at current energies, which became
a common belief in recent years. Fitting the scaling function (2.4) to $\psi(z)$
observed by experiments displays clear deviation from lognormality yielding
$k \simeq 5$ and $\mu \simeq 1.6$ best-fit parameters. The marked difference between the
experimental and asymptotic scaling functions warns that the ordinary and
factorial moments of $P_n$ can not be identified at current energies and sign-
changing oscillations caused by pQCD effects are more likely to occur for the
ordinary cumulant-to-moment ratios.

The Poisson transformed lognormal shape of MDs calls attention to the
moment problem considered already in ref. [2] for Eq. (2.4). According to
our results the KNO function at the asymptotics (3.2) is not determined by
its moments, a well known feature of the lognormal law. There is another
important aspect of lognormality at asymptotic energies. It suggests that the
limiting shape of $\psi(z)$ can be guessed on more elementary grounds than the
perturbative treatment of parton evolution equations. The multiplacive
nature of parton cascades in QCD and the large number of steps of the
cascade processes at very high energies cause the central limit theorem to
come into operation determining uniquely the asymptotic shape of the scaling
function.

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Appendix

Here we give a brief summary of the basic properties of Fox’s generalized hypergeometric function, $H(x)$. It heavily relies on article [19] and on the book [20] where the interested reader can find more details.

The $H$-function of Fox is defined in terms of a Mellin-Barnes type integral as follows:

$$H_{m,n}^{p,q}(x) = H_{m,n}^{p,q} \left[ x \left| \begin{array}{c} (a_p, \alpha_p) \\ (b_q, \beta_q) \end{array} \right. \right] = \frac{1}{2\pi i} \int_{L} h(s) x^{s} ds \quad (A.1)$$

where $x \neq 0$ and

$$x^{s} = \exp \{ s [ \ln |x| + i \text{arg}(x) ] \}$$

in which arg$(x)$ is not necessarily the principal value. Further,

$$h(s) = \frac{\prod_{j=1}^{p} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s) \Gamma(b_h + \nu) \Gamma(a_j - \alpha_j s)}{\prod_{j=m+1}^{p} \Gamma(1 - b_j + \beta_j s) \prod_{j=n+1}^{q} \Gamma(a_j - \alpha_j s)}$$

where $m, n, p, q$ are integers satisfying

$$0 \leq n \leq p, \quad 1 \leq m \leq q.$$ 

The parameters $(a_1, \ldots, a_p)$ and $(b_1, \ldots, b_q)$ are complex, whereas $(\alpha_1, \ldots, \alpha_p)$ and $(\beta_1, \ldots, \beta_q)$ are positive numbers. An empty product is interpreted as unity. The parameters are restricted by the condition that $\alpha_j(b_h + \nu) \neq \beta_h(a_j - 1 - \lambda)$ for $\nu, \lambda = 0, 1, \ldots; h = 1, \ldots, m; j = 1, \ldots, n$. The contour $L$ in the complex $s$ plane is such that the points $s = (b_h + \nu)/\beta_h$ and $s = (a_j - 1 - \nu)/\alpha_j$ lie to the right and left of $L$ respectively while $L$ extends from $s = \infty - ik$ to $s = \infty + ik$ where $k$ is a constant with $k > |\text{Im} \ b_h|/\beta_h$.

The $H$-function makes sense only if the following two existence conditions are satisfied:

i) \( x \neq 0 \) and $\rho > 0$ where

$$\rho = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j$$

ii) \( \rho = 0 \) and $0 < |x| < \theta^{-1}$ where

$$\theta = \prod_{j=1}^{p} \alpha_j^{-\beta_j} \prod_{j=1}^{q} \beta_j^{-\beta_j}.$$
Special cases of the $H$ associated e.g. with the Gauss- and confluent hypergeometric functions enable us to transform $H(x)$ where $c > 0$. The first identity, Eq. (A.2), is an important one because it allows us to transform $H(x)$ with $\rho < 0$ to $H(1/x)$ with $\rho > 0$ and satisfying the existence condition $i)$ given above. The Laplace transform of $H(x)$ reads as follows:

$$
\int_0^\infty H_{p,q}^{m,n}(cx) e^{-rx} dx = \frac{1}{c} H_{q,p+1}^{n+1,m} \left[ \frac{r}{c} \begin{array}{c} (1 - b_q - \beta_q, \beta_q) \\ (0, 1), (1 - a_p - \alpha_p, \alpha_p) \end{array} \right]. \tag{A.5}
$$

Elementary properties of $H(x)$ utilized in the body of the paper:

$$
H_{p,q}^{m,n} \left[ x \left| \begin{array}{c} (a_p, \alpha_p) \\ (b_q, \beta_q) \end{array} \right. \right] = H_{q,p}^{n,m} \left[ \frac{1}{x} \left| \begin{array}{c} (1 - b_q, \beta_q) \\ (1 - a_p, \alpha_p) \end{array} \right. \right] \tag{A.2}
$$

$$
x^c H_{p,q}^{m,n} \left[ x \left| \begin{array}{c} (a_p, \alpha_p) \\ (b_q, \beta_q) \end{array} \right. \right] = H_{p,q}^{m,n} \left[ x \left| \begin{array}{c} (a_p + c \alpha_p, \alpha_p) \\ (b_q + c \beta_q, \beta_q) \end{array} \right. \right] \tag{A.3}
$$

$$
x^c H_{p,q}^{m,n} \left[ x \left| \begin{array}{c} (a_p, \alpha_p) \\ (b_q, \beta_q) \end{array} \right. \right] = \frac{1}{c} H_{p,q}^{m,n} \left[ x \left| \begin{array}{c} (a_p, \alpha_p/c) \\ (b_q, \beta_q/c) \end{array} \right. \right] \tag{A.4}
$$

where $c > 0$. The first identity, Eq. (A.2), is an important one because it allows us to transform $H(x)$ with $\rho < 0$ to $H(1/x)$ with $\rho > 0$ and satisfying the existence condition $i)$ given above. The Laplace transform of $H(x)$ reads as follows:

$$
\int_0^\infty H_{p,q}^{m,n}(cx) e^{-rx} dx = \frac{1}{c} H_{q,p+1}^{n+1,m} \left[ \frac{r}{c} \begin{array}{c} (1 - b_q - \beta_q, \beta_q) \\ (0, 1), (1 - a_p - \alpha_p, \alpha_p) \end{array} \right]. \tag{A.5}
$$

Special cases of the $H$-function include such as Meijer’s $G$-function, Bessel, Legendre, Whittaker, Struve functions, the generalized hypergeometric function and several others. Meijer’s $G$-function is obtained when the $H$-function parameters $(\alpha_1, \ldots, \alpha_p)$ and $(\beta_1, \ldots, \beta_q)$ are unity. The generalized hypergeometric function $_pF_q$ is related to $H(x)$ through

$$
_{p}F_{q}(a_p, b_q; x) = \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} H_{p,q+1}^{1,p} \left[ -x \left| \begin{array}{c} (1 - a_p, 1) \\ (0, 1), (1 - b_q, 1) \end{array} \right. \right]. \tag{A.6}
$$

From this relationship one can obtain numerous distributions of statistics (associated e.g. with the Gauss- and confluent hypergeometric functions $2F_1$ and $1F_1$) in terms of $H(x)$. Two further special cases of $H(x)$ frequently encountered in statistics are

$$
x^\alpha \exp(-\lambda x^\beta) = \lambda^{-\alpha/\beta} H_{0,1}^{1,0} \left[ \lambda x^\beta \left| \begin{array}{c} - \\ (\alpha/\beta, 1) \end{array} \right. \right] \tag{A.7}
$$

and

$$
x^\alpha/(1 + \lambda x^\beta) = \lambda^{-\alpha/\beta} H_{1,1}^{1,1} \left[ \lambda x^\beta \left| \begin{array}{c} (\alpha/\beta, 1) \\ (\alpha/\beta, 1) \end{array} \right. \right]. \tag{A.8}
$$
For a rich collection of particular cases of the H-function, see ref. [20].

Let us now consider the continuous random variable \( x \in (0, \infty) \) whose probability density function is given by

\[
f(x) = \mathcal{N} H_{p, q}^{m, n} \left[ \lambda x \left| \left( \frac{a_p}{b_q}, \frac{\alpha_p}{\beta_q} \right) \right. \right]
\]  

(A.9)

with scale parameter \( \lambda > 0 \) and normalization condition \( \int_0^\infty f(x) \, dx = 1 \). The random variable \( x \) is called a H-function variate and the probability law (A.9) with \( f(x) \geq 0 \) is the so-called H-function distribution. Its parameters \((a_p, \alpha_p)\) and \((b_q, \beta_q)\) should satisfy all the restrictions given earlier in this Appendix. The characteristic function of \( f(x) \) can be obtained from Eq. (3.7) and the Laplace transform (A.5):

\[
\phi(t) = \frac{\mathcal{N}}{\lambda} H_{q, p+1}^{n+1, m} \left[ -\frac{it}{\lambda} \left| \left( 1 - b_q - \beta_q, \beta_q \right) \right. \right. \\
\left. \left. \left( 1 - a_p - \alpha_p, \alpha_p \right) \right) \right].
\]  

(A.10)

The \( q \)th moment of the H-function distribution is related to its \( r = (q + 1) \)th Mellin transform through

\[
\langle x^q \rangle = \mathcal{N} \frac{\prod_{j=1}^{m} \Gamma(b_j + r\beta_j) \prod_{j=1}^{n} \Gamma(1 - a_j - r\alpha_j)}{\prod_{j=m+1}^{p} \Gamma(1 - b_j - r\beta_j) \prod_{j=n+1}^{q} \Gamma(a_j + r\alpha_j)}.
\]  

(A.11)

Many of the classical probability densities are special cases of the H-function distribution and can be written in the form of (A.9). This was utilized in Eq. (3.4) which enabled us to express the Poisson transform of the scaling function (2.4) analytically as a discrete H-function distribution.
References

[1] I.M. Dremin, *Physics Uspekhi* 37 (1994) 715.
Yu.L. Dokshitzer, V.A. Khoze and S.I. Troyan, in *Perturbative QCD*, ed. A.H. Mueller, World Scientific, 1989.

[2] Z. Koba, H.B. Nielsen and P. Olesen, *Nucl. Phys. B* 40 (1972) 314.

[3] F. Cuypers and K. Tesima, *Z. Phys. C* 54 (1992) 87.

[4] Yu.L. Dokshitzer, *Phys. Lett. B* 305 (1993) 295.

[5] I.M. Dremin, *Mod. Phys. Lett. A* 8 (1993) 2747.

[6] I.M. Dremin and V.A. Nechitailo, *JETP Lett.* 58 (1993) 881.

[7] SLD Collab., K. Abe et al., *Phys. Lett. B* 371 (1996) 149.

[8] A.M. Polyakov, *Zh. Eksp. Teor. Fiz.* 59 (1970) 542.

[9] N.G. Antoniou et al., *Phys Rev. D* 14 (1976) 3578.
S. Krasznovszky and I. Wagner, *Nuovo Cim. A* 76 (1983) 539.
P. Carruthers, *LA-UR-84-1009*, 1984.
S. Krasznovszky and I. Wagner, *Phys. Lett. B* 306 (1993) 403.
R. Botet and M. Ploszajczak, *GANIL-P-96-07*, 1996.

[10] E. Jahnke and F. Emde, *Tables of Higher Functions*, Teubner, 1966.

[11] N.L. Johnson and S. Kotz, *Distributions in Statistics*, Vol. 2., *Continuous Univariate Distributions*, Wiley, 1970.

[12] L. Amoroso, *Ann. Mat. Pura Appl.* 21 (1925) 123.
P. Rosin et al., *Bericht C 52 des Reichskohlenrates*, Berlin, 1933.
S. Nukiyyama and Y. Tanasawa, *Trans. Soc. Mech. Japan* 5 (1939) 62.
E.W. Stacy, *Ann. Math. Statist.* 33 (1962) 1187.

[13] D.S. McKenzie, *Phys. Reports* 27 (1976) 37.

[14] A. Schenzle and H. Brand, *Phys. Rev. A* 20 (1979) 1628.
K. Lindenberg and V. Seshadri, *J. Chem. Phys.* 71 (1979) 4075.

[15] A.I. Golokhvastov, *Sov. J. Nucl. Phys.* 27 (1978) 430.

[16] R. Szwed, G. Wrochna and A.K. Wroblewski, *Mod. Phys. Lett. A* 5 (1990) 1851 and 6 (1991) 245.

[17] P. Carruthers and C.C. Shih, *Int. J. Mod. Phys. A* 2 (1987) 1447.

[18] S.J. Orfanidis and V. Rittenberg, *Phys. Rev. D* 10 (1974) 2892.

[19] B.D. Carter and M.D. Springer, *SIAM J. Appl. Math.* 33 (1977) 542.

[20] A.M. Mathai and R.K. Saxena, *The H-Function with Applications in Statistics and Other Disciplines*, Wiley Eastern, 1978.
[21] F. Oberhettinger, *Tables of Fourier Transforms and Fourier Transforms of Distributions*, Springer-Verlag, 1990.
[22] S. Krasznovszky, *Mod. Phys. Lett. A* 8 (1993) 483.
[23] L. Bondesson, *Ann. Probab.* 7 (1979) 965.
[24] Aleph Collab., D. Buskulic et al., *Z. Phys. C* 69 (1995) 15.
  Delphi Collab., P. Abreu et al., *Z. Phys. C* 50 (1991) 185.
  L3 Collab., B. Adeva et al., *Z. Phys. C* 55 (1992) 39.
  Opal Collab., P.D. Acton et al., *Z. Phys. C* 53 (1992) 539.
  SLD Collab., J. Zhou, *private communication*, 1996.
[25] R.L. Prentice, *Biometrika*, 61 (1974) 539.
  V.T. Farewell and R.L. Prentice, *Technometrics*, 19 (1977) 39.
[26] R. Ugoccioni, A. Giovannini and S. Lupia, *Phys. Lett. B* 342 (1995) 387.

FIGURE CAPTIONS

Fig. 1a: Sign-changing oscillations of \( H_q = K_q / F_q \) with \( F_q \) given by Eq. (3.10) for \( k = 3/2 \). The neighbouring bumps at a fixed \( q \) are \( H_q \)-intervals of opposite sign in \( \mu \). For \( \mu \leq 1 \) \( H_q \) is always positive. The \( \mu \)-scale is logarithmic and for clarity only the odd-rank moment ratios are shown.

Fig. 1b: Slices through Fig. 1a with \( q = 5, 20 \) (left) and with \( \mu = 1, 3/2 \) (right).

Fig. 2a-b: The best fit to the SLD data for \( P_n \) and \( H_q \) as described in the text. The fit parameters and the quality of fits are shown in the last two rows of Table 2.
### Distribution

| DISTRIBUTION          | shape par. | scale par. | exponent |
|-----------------------|------------|------------|----------|
| Generalized gamma     | $k$        | $\lambda$  | $\mu$    |
| Gamma                 | $k$        | $\lambda$  | 1        |
| Chi-square, $n$ d.o.f.| $n/2$      | $1/2$      |          |
| Exponential           | 1          | $\lambda$  |          |
| Weibull               | 1          | $\lambda$  | $\mu$    |
| Stratonovich          | $k$        | $\lambda$  | 2        |
| Rayleigh              | 1          |            |          |
| Maxwell molecular speed| 3/2       |            |          |
| Maxwell mol. velocity | 1/2        |            |          |
| Chi, $n$ d.o.f.       | $n/2$      | $1/\sqrt{2}$|          |
| Half-normal           | 1/2        |            |          |
| Circular normal       | 1          |            |          |
| Spherical normal      | 3/2        |            |          |
| Pearson type V        | $k$        | $\lambda$  | $-1$     |
| Asymmetric Lévy, $\alpha = 1/2$| 1/2 |                  |
| Lognormal             | $k \to \infty$ | $\lambda$ | $\mu \to 0$ |
| Pareto                | $k \to 0$ | $\lambda$  | $\mu \to \pm \infty$ |

Table 1. Special- and limiting cases of the generalized gamma density (2.4)

### Experiment

| EXPERIMENT | $\chi^2$/d.o.f. | $p$         | $\langle n \rangle$ | $\sigma$        |
|------------|-----------------|-------------|---------------------|-----------------|
| Aleph      | 4.2/24          | $-0.230 \pm 0.311$ | 20.970 \pm 0.407 | 0.200 \pm 0.014 |
| Delphi     | 30.2/23         | $-0.089 \pm 0.113$ | 21.311 \pm 0.140 | 0.199 \pm 0.006 |
| L3         | 14.9/20         | $-0.123 \pm 0.238$ | 20.660 \pm 0.349 | 0.205 \pm 0.016 |
| Opal       | 10.7/24         | $-0.124 \pm 0.130$ | 21.370 \pm 0.128 | 0.208 \pm 0.008 |
| SLD        | 31.2/20         | $-0.001 \pm 0.060$ | 20.892 \pm 0.068 | 0.204 \pm 0.003 |
| SLD ($H_q$) | 16.6/13        | 0.060 \pm 0.029  | 21.346 \pm 0.126 | 0.214 \pm 0.004 |

Table 2. The results of fits to the $e^+e^-$ multiplicity data at the $Z^0$ peak
Figure 1a

Figure 1b
Figure 2a

Figure 2b