The signs of the Stieltjes constants associated with the Dedekind zeta function

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Abstract: The Stieltjes constants \( \gamma_n(K) \) of a number field \( K \) are the coefficients of the Laurent expansion of the Dedekind zeta function \( \zeta_K(s) \) at its pole \( s = 1 \). In this paper, we establish a similar expression of \( \gamma_n(K) \) as Stieltjes obtained in 1885 for \( \gamma_n(\mathbb{Q}) \). We also study the signs of \( \gamma_n(K) \).

Key words: Stieltjes constants; Riemann zeta function; Dedekind zeta function.

1. Introduction. Let \( K \) be a number field and \( \mathcal{O}_K \) be its ring of integers. Define for \( \Re s > 1 \) the Dedekind zeta function

\[
\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N\mathfrak{a}^s} = \prod_p \frac{1}{1-Np^{-s}},
\]

where \( \mathfrak{a} \) runs over non-zero ideals in \( \mathcal{O}_K \), \( p \) runs over the prime ideals in \( \mathcal{O}_K \) and \( Na \) is the norm of \( \mathfrak{a} \). It is known that \( \zeta_K(s) \) can be analytically continued to \( C - \{1\} \), and that at \( s = 1 \) it has a simple pole, with residue \( \gamma_{-1}(K) \), given by the analytic class number formula:

\[
\gamma_{-1}(K) = \frac{2^{r_1} (2\pi)^2 h(K) R(K)}{\omega(K) \sqrt{|D(K)|}},
\]

where \( r_1 \) denotes the number of real embeddings of \( K \), \( r_2 \) is the number of complex embeddings of \( K \), \( h(K) \) is the class number of \( K \), \( R(K) \) is the regulator of \( K \), \( \omega(K) \) is the number of roots of unity contained in \( K \) and \( D(K) \) is the discriminant of the extension \( K/\mathbb{Q} \). The Laurent expansion of \( \zeta_K(s) \) at \( s = 1 \) is

\[
(1) \quad \zeta_K(s) = \frac{\gamma_{-1}(K)}{s-1} + \frac{\gamma_0(K)}{2} + \gamma_1(K)(s-1) + \gamma_2(K)(s-1)^2 + \cdots.
\]

The constants \( \gamma_n(K) \) are sometimes called the Stieltjes constants associated with the Dedekind zeta function. In [6] they are called by higher Euler’s constants of \( K \). While the constant \( \gamma_K = \gamma_0(K)/\gamma_{-1}(K) \) is called the Euler-Kronecker constant in [7] and [16].

In case \( K = \mathbb{Q} \), the Laurent expansion of the Riemann zeta function \( \zeta(s) \) at its pole \( s = 1 \) is given by

\[
(2) \quad \gamma_n = (-1)^n \lim_{x \to \infty} \frac{n!}{x^n} \left( \sum_{m=1}^{n} \frac{(\log x)^{m}}{m} - \frac{(\log x)^{n+1}}{n+1} \right).
\]

Stieltjes in 1885 was the first to propose this definition of \( \gamma_n \) for this reason these constants are today called by his name. The asymptotic behaviour of \( \gamma_n \), as \( n \to \infty \), has been widely studied by many authors (for instance: Briggs [3], Mitrović [12], Isailov [8], Matsuoka [11] and more recently Coffey [4] and [5], Knessl and Coffey [9], Adell [2], Adell and Lekuona [1] and Saad Eddin [14]). Their main interest is focused on the growth, the sign changes of the sequence \( \gamma_n \) and on giving explicit upper estimates for \( |\gamma_n| \). Moreover, they obtained relations between this sequence and the zeros of \( \zeta(s) \) (see [11], [15]). In this paper we are interested in the Stieltjes coefficients \( \gamma_n(K) \) for the Dedekind zeta function. We first give the following formula of \( \gamma_n(K) \) which is similar to Stieltjes’s formula given by Eq. (2).

**Theorem 1.** For any \( n \geq 1 \), we have

\[
\gamma_n(K) = (-1)^n \lim_{x \to \infty} \frac{n!}{x^n} \sum_{Na \leq x} \frac{\log Na}{Na} - \gamma_{-1}(K) \log x \left( \frac{\log x}{n+1} \right),
\]

and

\[
\gamma_0(K) = \lim_{x \to \infty} \left( \sum_{Na \leq x} \frac{1}{Na} - \gamma_{-1}(K) \log x \right) + \gamma_{-1}(K).
\]
This result seems similar to another one obtained by Hashimoto et al. [6] for the higher Euler-Selberg constants. Despite a considerable effort the author has not been able to find Theorem 1 in the literature.

In 1962, Mitrović [12] studied the sign changes of the constants \( \gamma_n \) and prove that; each of the inequalities
\[
\gamma_{2n} > 0, \quad \gamma_{2n} < 0, \quad \gamma_{2n-1} > 0, \quad \gamma_{2n-1} < 0,
\]
holds for infinitely many \( n \). In [11], Matsuoka gave precise conditions for the sign of \( \gamma_n \). By the same techniques used in [12], we prove that

**Theorem 2.** For the coefficients in the expansion (1), each of the inequalities
\[
\gamma_{2n}(K) > 0, \quad \gamma_{2n}(K) < 0, \\
\gamma_{2n-1}(K) > 0, \quad \gamma_{2n-1}(K) < 0,
\]
holds for infinitely many \( n \).

It immediately follows that

**Corollary 1.** Infinitely many \( \gamma_n(K) \) are positive and infinitely many are negative.

2. **Proofs.**

Proof of Theorem 1. By Eq. (1), we note that
\[
(3) \quad \zeta_K(s) - \left( \frac{\gamma_{-1}(K)}{s} - 1 \right) \zeta_K(s) - \frac{\gamma_{-1}(K)}{s} - \frac{\gamma_{-1}(K)}{s} = \sum_{n \geq 0} \alpha_n(K)(s - 1)^n,
\]
where \( \alpha_0(K) = \gamma_0(K) - \gamma_{-1}(K) \) and \( \alpha_n(K) = \gamma_n(K) \) for \( n \geq 1 \). By the definition of \( \zeta_K(s) \), we write
\[
\zeta_K(s) = \int_{1-}^{+\infty} \frac{dN_K(t)}{t^s} = s \int_{1-}^{+\infty} \frac{N_K(t)}{t^{s+1}} dt,
\]
where
\[
N_K(t) = \sum_{N \leq t} 1.
\]
Then, we get
\[
(4) \quad \zeta_K(s) - \left( \frac{\gamma_{-1}(K)}{s} - 1 \right) = s \int_{1-}^{+\infty} \frac{N_K(t) - \gamma_{-1}(K)t}{t^{s+1}} dt.
\]
Put \( \sum_{n \geq 0} \alpha_n(K)(s - 1)^n = h(s) \). From Eqs. (3) and (4), we have
\[
h(s) = s \int_{1-}^{+\infty} \frac{N_K(t) - \gamma_{-1}(K)t}{t^{s+1}} dt.
\]
From [10, Satz 210] we have \( N_K(t) = \gamma_{-1}(K)t + O(t^{-1/m}) \), where \( m \) is the degree of \( K \) and \( Q \). For \( \Re s > 1 - 1/m \), it is easily seen that the \( n \)-th derivative of \( h(s) \) at \( s = 1 \) is
\[
(5) \quad h^{(n)}(1) = n! \alpha_n(K) = (-1)^n (I_1 - I_2),
\]
where
\[
I_1 = \int_{1-}^{+\infty} N_K(t) \left( \frac{\log^n t - n(\log t)^{n-1}}{t^2} \right) dt,
\]
and
\[
I_2 = \gamma_{-1}(K) \int_{1-}^{+\infty} \frac{\log^n t - n(\log t)^{n-1}}{t^2} dt.
\]
On the other hand, we have
\[
\sum_{N \leq x} \frac{(\log Na)^n}{Na} = \int_{1-}^{x} \frac{\log^n t}{t^2} dN_K(t) = N_K(x) \frac{\log^n x}{x} + \int_{1-}^{x} N_K(t) \left( \frac{\log^n t - n(\log t)^{n-1}}{t^2} \right) dt.
\]
Thus, we get
\[
\int_{1-}^{x} N_K(t) \left( \frac{\log^n t - n(\log t)^{n-1}}{t^2} \right) dt = \sum_{N \leq x} \frac{(\log Na)^n}{Na} - N_K(x) \frac{\log^n x}{x}.
\]
Again using the fact that \( N_K(t) = \gamma_{-1}(K)t + O(t^{-1/m}) \), we find that
\[
\int_{1-}^{x} N_K(t) \left( \frac{\log^n t - n(\log t)^{n-1}}{t^2} \right) dt = \sum_{N \leq x} \frac{(\log Na)^n}{Na} - \gamma_{-1}(K) \log^n x + O\left( \frac{\log^n x}{x^{1/m}} \right).
\]
Taking \( x \to +\infty \), the above becomes
\[
\lim_{x \to +\infty} \sum_{N \leq x} \frac{(\log Na)^n}{Na} - \gamma_{-1}(K) \log^n x = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}
\]
Now, notice that
\[
(7) \quad I_2 = \lim_{x \to +\infty} \left[ \gamma_{-1}(K) \frac{(\log x)^{n+1}}{n+1} - \gamma_{-1}(K) \log^n x \right].
\]
From Eqs. (5), (6) and (7), we conclude that, for \( n \geq 1 \),
\[
\gamma_n(K) = \alpha_n(K) = \frac{(-1)^n}{n!} \lim_{x \to +\infty} \left[ \sum_{N \leq x} \frac{(\log Na)^n}{Na} - \gamma_{-1}(K) \frac{(\log x)^{n+1}}{n+1} \right]
\]
and \( \gamma_0(K) = \alpha_0(K) + \gamma_{-1}(K) \). This completes the proof. \( \square \)
Proof of Theorem 2. To prove Theorem 2, we apply the same technique used in [12]. Let $C$ be the set of all positive integers $n$ such that $\gamma_n(K) \neq 0$. Define

$$C_1 = \{ n : \gamma_n(K) \neq 0 \text{ and } (-1)^n = 1 \}$$

$$C_1^* = \{ n : \gamma_n(K) < 0 \text{ and } (-1)^n = 1 \}$$

$$C_1^+ = \{ n : \gamma_n(K) > 0 \text{ and } (-1)^n = 1 \}$$

and

$$C_2 = \{ n : \gamma_n(K) \neq 0 \text{ and } (-1)^n = -1 \}$$

$$C_2^* = \{ n : \gamma_n(K) < 0 \text{ and } (-1)^n = -1 \}$$

$$C_2^+ = \{ n : \gamma_n(K) > 0 \text{ and } (-1)^n = -1 \}$$

From [13], we have

$$\zeta_K(s) = \frac{\gamma_{-1}(K)}{s - 1}$$

is an entire transcendental function. So the cardinal number of the set $C$ is equal to the cardinal number of the set of all positive integers $\aleph_0$. Then, we can write

$$\zeta_K(s) = \frac{\gamma_{-1}(K)}{s - 1}$$

$$= \left( \sum_{n \in C_1} + \sum_{n \in C_1^*} + \sum_{n \in C_1^+} + \sum_{n \in C_2} \right) \gamma_n(K)(s - 1)^n.$$ 

Replacing $s$ by $t + 1$ and then by $-t + 1$ in the above. Adding and then subtracting the results, we find that

$$(8) \quad \zeta_K(t + 1) + \zeta_K(-t + 1)$$

$$= 2 \left( \sum_{n \in C_1} + \sum_{n \in C_1^*} \right) \gamma_n(K)t^n,$$

and

$$(9) \quad \zeta_K(t + 1) - \zeta_K(-t + 1) = \frac{2 \gamma_{-1}(K)}{t}$$

$$= 2 \left( \sum_{n \in C_1} + \sum_{n \in C_1^*} \right) \gamma_n(K)t^n.$$ 

Taking $t = 2m + 1$ with $m > 0$ and using the fact that the $\zeta_K(s)$ vanishes at all negative even integers. We find the left-hand side of Eq. (8) approaches to 1 when $m \to +\infty$. It follows that the right-hand side of this equation can’t be polynomial. That means the cardinal of the set $C_1$ is $\aleph_0$. On the other hand, if we assume that the cardinal of the set $C_1^*$ is less than $\aleph_0$. Then the right-hand side of Eq. (8) approaches $+\infty$. Similarly, if the cardinal of the set $C_1^+$ is less than $\aleph_0$. Then the right-hand side of Eq. (8) approaches $-\infty$, this leads to a contradiction. We thus conclude that the cardinal of the sets $C_1^-$ and $C_1^+$ are $\aleph_0$. By a similar argument, we show that the cardinal of the sets $C_2^-$ and $C_2^+$ in Eq. (9) are $\aleph_0$. That completes the proof. 

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