Time-Dependent Perpendicular Transport of Energetic Particles

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Abstract. The motion of energetic particles in magnetic turbulence across a mean magnetic field can be explored analytically. The approach discussed in this paper allows for a full time-dependent description of the transport, including compound sub-diffusion. For the first time it is shown systematically that as soon as there is transverse structure of the turbulence, diffusion is restored even if no Coulomb collisions are invoked. Compared to other non-linear theories the new approach has the advantage that a diffusion approximation is no longer part of that theory. Criteria for sub-diffusion and normal Markovian diffusion are provided as well. A comparison with test-particle simulations is also discussed.

1. Introduction
An important problem in space science is the understanding of energetic particles moving through a magnetized plasma. In the interplanetary space one finds a magnetic field configuration of the form

\[ \vec{B}(\vec{x}, t) = B_0 \hat{z} + \delta \vec{B}(\vec{x}, t) \]  

where we assumed that the mean magnetic field \( B_0 \) points into the \( z \)-direction and is constant. This mean field deflects energetic and electrically charged particles forcing them to follow a helical path. Furthermore, there is a complicated turbulent component \( \delta \vec{B} \) of the magnetic field. This component is, in general, time-dependent and strongly depends on the different directions of space. Some simplification can be achieved by considering the incompressible case \( \delta B_z = 0 \) and/or the static case where we neglect the time-dependence of the turbulent field.

The turbulent magnetic field scatters the energetic particles and leads to a diffusive motion. In this case the transport of particles can be described by Parker’s famous transport equation (see Parker (1965))

\[ \frac{\partial f}{\partial t} + \sum_i \vec{u}_i \frac{\partial f}{\partial \vec{x}_i} = \sum_{i,j} \frac{\partial}{\partial \vec{x}_i} \left( \kappa_{ij} \frac{\partial f}{\partial \vec{x}_j} \right) + \frac{1}{3} \sum_i \left( \frac{\partial \vec{u}_i}{\partial \vec{x}_i} \right) \frac{\partial f}{\partial \ln p} + S. \]  

Here we have used the plasma bulk velocity \( \vec{u} \) and the components of the spatial diffusion tensor \( \kappa_{ij} \). It should be noted, that more general versions of this equation can be found in the literature (see, e.g., Schlickeiser (2002)) involving further terms such as momentum diffusion. Furthermore, non-diffusive transport, also known as anomalous diffusion, was discussed in several papers such as Perri & Zimbardo (2009a) and Perri & Zimbardo (2009b). Assuming normal or Markovian...
diffusion, and axi-symmetry with respect to the mean magnetic field allows us to use a diffusion tensor which has, in this case, the following form

$$\kappa = \begin{pmatrix} \kappa_\perp & \kappa_A & 0 \\ -\kappa_A & \kappa_\perp & 0 \\ 0 & 0 & \kappa_\parallel \end{pmatrix}$$

(3)

where we have used the parallel diffusion coefficient \(\kappa_\parallel\), the perpendicular diffusion coefficient \(\kappa_\perp\), and the drift coefficient \(\kappa_A\). In the current paper we focus on the perpendicular diffusion coefficient and treat the parallel diffusion coefficient as a variable entering non-linear theories for perpendicular transport. The drift coefficient and its importance in solar modulation studies were explored before in the literature (see, e.g., Engelbrecht & Burger (2015) and Engelbrecht et al. (2017)) but is not further discussed in the current paper.

The theoretical exploration of perpendicular transport has a long history starting with the pioneering paper of Jokipii (1966). In the latter article the first time a perpendicular diffusion coefficient was calculated based on quasi-linear theory (QLT). In this case perpendicular diffusion is entirely caused by particles following magnetic field lines which themselves behave diffusively. Further important steps leading to an improved understanding of perpendicular transport which are reviewed in the current paper are:

- The compound-diffusion model describes particles which are tied to magnetic field lines while they move diffusively in the parallel direction (see, e.g., Kóta & Jokipii (2000) and Webb et al. (2006)). As a consequence perpendicular transport behaves sub-diffusively.
- The non-linear guiding center (NLGC) theory of Matthaeus et al. (2003) was the first systematic theory for perpendicular transport. Furthermore, the theory showed agreement with several simulations and solar wind observations if combined with a two-component turbulence model (see Bieber et al. (2004)).
- The extended non-linear guiding center (ENLGC) theory was developed in Shalchi (2006) for describing perpendicular diffusion in two-component turbulence. While the NLGC theory does not provide a sub-diffusive result for slab turbulence, the ENLGC theory describes perpendicular transport due to slab modes sub-diffusively while the two-dimensional contribution is as in NLGC theory.
- The unified non-linear transport (UNLT) theory of Shalchi (2010) was developed with the aim to obtain a diffusion theory which works for all cases ranging from slab to full three-dimensional turbulence. Furthermore, UNLT theory contains the Matthaeus et al. (1995) theory of field line random walk as well as other theories and limits.
- The time-dependent UNLT theory was developed in Shalchi (2017) and Lasuik & Shalchi (2017). The theory does not longer rely on a diffusion approximation and can describe the initial ballistic regime, sub-diffusion, as well as the turnover to the diffusive regime. Furthermore, this theory provides a condition which needs to be satisfied in order to obtain normal diffusion.

The analytical description of perpendicular transport in not just important in order to improve our fundamental understanding of processes in nature. There are several applications of analytical forms of particle diffusion coefficients. Examples are the acceleration of particles at shock waves (see, e.g., Zank et al. (2004), Dosch & Shalchi (2010), Li et al. (2012), Ferrand et al. (2014), and Hu et al. (2017)) and solar modulation studies (see, e.g. Hitge & Burger (2010), Engelbrecht & Burger (2010), Wawrzynczak & Alania (2010), Alania et al. (2011), Potgieter & Ndanganeni (2013), Engelbrecht & Burger (2014), Manuel et al. (2014), Potgieter et al. (2014), Ahluwalia & Ygbuhay (2015), and Shen & Qin (2018)).
It is the purpose of the current article to review the analytical description of the perpendicular transport of energetic particles. The organization is as follows. In Sect. 2 we discuss simple models and previous theories for perpendicular diffusion. In Sect. 3 and 4, diffusive and time-dependent UNLT theories are discussed, respectively. In Sect. 5 a comparison between analytical results and test-particle simulations are shown as an example. In Sect. 6 we summarize and conclude.

2. Simple Models and Previous Theories for Perpendicular Diffusion
Before more advanced theories for perpendicular diffusion are discussed, we briefly review three previous approaches, namely the field line random walk (FLRW) limit, compound sub-diffusion, and the non-linear guiding center (NLGC) theory.

2.1. The FLRW Limit
One of the first descriptions of perpendicular transport was entirely based on the effect of FLRW and was presented in Jokipii (1966). This approach is simple but does not work in the general case as shown later. As an approximation we assume that field line random walk is diffusive for all distances meaning that we neglect the initial free-streaming regime entirely. Thus, we can employ

$$\langle (\Delta x)^2 \rangle_{FL} = 2\kappa_{FL} |z| \quad \forall \ z$$

(4)

with the diffusion coefficient of FLRW $\kappa_{FL}$. As a simple model for the perpendicular transport of energetic particles we assume that the particle follows a single magnetic field line while it moves with constant velocity in the parallel direction meaning that

$$z_p = v\mu t$$

(5)

where we have used the pitch-angle cosine $\mu$ and particle speed $v$. Furthermore, if particles follow field lines, the perpendicular mean square displacements of particle trajectories and field lines are the same. Therefore, we can combine Eqs. (4) and (5) to find

$$\langle (\Delta x)^2 \rangle_p = 2vt\kappa_{FL} |\mu|.$$  

(6)

After averaging over all $-1 \leq \mu \leq +1$, this becomes

$$\langle (\Delta x)^2 \rangle_p = vt\kappa_{FL}$$

(7)

corresponding to normal diffusion with the perpendicular diffusion coefficient of the energetic particle

$$\kappa_\perp = \frac{v}{2}\kappa_{FL}.$$  

(8)

We can clearly see that the perpendicular diffusion coefficient of the energetic particles is directly proportional to the diffusion coefficient of random walking magnetic field lines. The only property of the particle entering this simple equation is the particle speed $v$. Alternatively, one can compute the perpendicular mean free path

$$\lambda_\perp = \frac{3}{v}\kappa_\perp = \frac{3}{2}\kappa_{FL}.$$  

(9)

Obviously the latter quantity does not depend on any particle properties. This means that the perpendicular mean free path does even not depend on particle energy, momentum, or rigidity. Later in this review paper we will argue that this behavior can typically be found for high particle energies.
2.2. Compound Sub-diffusion

Real particles do not perform an unperturbed motion in the parallel direction as assumed in the previous paragraph. Therefore, we now assume a diffusive parallel motion. However, we still assume that the particle is tied to a single magnetic field line as before. A very comprehensive description of compound sub-diffusion was presented in Webb et al. (2006). The latter authors employed an approach based on the so-called Chapman-Kolmogorov equation (see, e.g., Gardiner (1985))

\[ f_{\perp}(x, y; t) = \int_{-\infty}^{+\infty} dz \, f_{FL}(x, y; z) f_{\parallel}(z; t) \]  

(10)

with the particle distributions in the parallel direction \( f_{\parallel}(z; t) \), in the perpendicular direction \( f_{\perp}(x, y; t) \), and the field line distribution function \( f_{FL}(x, y; z) \). Whereas Webb et al. (2006) computed the function \( f_{\perp}(x, y; t) \) by solving Eq. (10), we follow Shalchi & Kourakis (2007) and only consider the second moment of \( f_{\perp}(x, y; t) \) leading to

\[ \langle (\Delta x(t))^2 \rangle_P = \int_{-\infty}^{+\infty} dz \, \langle (\Delta x(z))^2 \rangle_{FL} f_{\parallel}(z; t) \]  

(11)

giving us a relation between particle and field line mean square displacements. In the following we employ a Gaussian particle distribution and assume that parallel transport is diffusive for all times. Thus, the parallel distribution function has the form

\[ f_{\parallel}(z, t) = \frac{1}{\sqrt{4\pi \kappa_{\parallel} t}} e^{-\frac{z^2}{4\kappa_{\parallel} t}}. \]  

(12)

Using this and Eq. (4) in Eq. (11) leads to the following integral

\[ \langle (\Delta x)^2 \rangle_P = \frac{\kappa_{FL}}{\sqrt{\pi \kappa_{\parallel} t}} \int_{-\infty}^{+\infty} dz \, |z| e^{-\frac{z^2}{4\kappa_{\parallel} t}}. \]  

(13)

This integral can easily be solved yielding

\[ \langle (\Delta x)^2 \rangle_P = 4\kappa_{FL} \sqrt{\frac{\kappa_{\parallel} t}{\pi}} \]  

(14)

corresponding to sub-diffusive transport. Webb et al. (2006) computed not just the second moment but also the perpendicular distribution function \( f_{\perp}(x, y; t) \). They found a so-called Fox function which is more sharply peaked at the center compared to the usual Gaussian found in diffusive cases. Arendt & Shalchi (2018) found this type of distribution via test-particle simulations confirming the work of Webb et al. (2006). A velocity correlation function based approach to describe compound sub-diffusion was presented in Kóta & Jokipii (2000).

Compound sub-diffusion is highly relevant for particle transport in slab turbulence. According to the so-called theorem on reduced dimensionality presented in Jokipii et al. (1993) and Jones et al. (1998), particles are tied to magnetic field lines if the turbulence has reduced dimensionality. Undoubtedly this is the case for slab turbulence. If parallel transport is diffusive, this means that for slab turbulence we find compound sub-diffusion. Numerically it was confirmed via simulations by Qin & Shalchi (2015) that for slab turbulence particles follow indeed field lines.

2.3. The Non-linear Guiding Center Theory

Matthaeus et al. (2003) developed a more systematic approach in order to describe perpendicular diffusion called the non-linear guiding center (NLGC) theory. To derive a simple equation
of motion for perpendicular transport, we combine the Newton-Lorentz equation for a purely magnetic system

$$\frac{d\vec{p}}{dt} = \frac{q}{c} \vec{v} \times \vec{B}(\vec{x})$$

(15)

with the guiding center transformation (see, e.g., Schlickeiser (2002))

$$\vec{X} = \vec{x} + \frac{1}{\Omega} (\vec{v} \times \hat{z})$$

(16)

The gyrofrequency is given by

$$\Omega = \frac{qB_0}{mc\gamma}$$

(17)

where we have used the Lorentz factor $\gamma$, the particle’s electric charge $q$ and rest mass $m$, and the speed of light $c$. After some straightforward algebra, we find the following equation of motion

$$V_x = v_z \frac{\delta B_x(\vec{x})}{B_0}$$

(18)

for the guiding center velocity component $V_x = \dot{X}$. A similar equation can be obtained for $V_y$ but this is not needed due to the assumption of axi-symmetry. If Eq. (18) is combined with the TGK formula (see Taylor (1922), Green (1951), and Kubo (1957))

$$\kappa_\perp = \int_0^\infty dt \langle v_z(t)v_z(0)\rangle \langle \delta B_x(t)\delta B_x(0) \rangle$$

(19)

one obtains

$$\kappa_\perp = \frac{a^2}{B_0^2} \int_0^\infty dt \langle v_z(t)v_z(0)\rangle \langle \delta B_x(t)\delta B_x(0) \rangle$$

(20)

Here we have used an additional parameter $a^2$. In the original paper by Matthaeus et al. (2003), the parameter $a$ was introduced directly in Eq. (18) but it was shown in Qin & Shalchi (2016) that Eq. (18) is correct as it is without containing any type of additional parameter. It is still unclear what the exact value of this parameter is, nor is it known what the physics of $a^2$ is (see, however, the work of Qin & Zhang (2014)). In the following we just keep this parameter in our equations and assume that this parameter is there in order to balance out some inaccuracies of the theory related to some of the used approximations (e.g., the parameter $a^2$ can balance out the inaccuracy of the Corrsin approximation used below).

To proceed, Matthaeus et al. (2003) assumed that the fourth-order correlation in Eq. (20) can be replaced by a product of two second-order correlations

$$\langle v_z(t)v_z(0)\delta B_x(t)\delta B_x^*(0) \rangle \approx \langle v_z(t)v_z(0)\rangle \langle \delta B_x(t)\delta B_x^*(0) \rangle$$

(21)

leading to

$$\kappa_\perp = \frac{a^2}{B_0^2} \int_0^\infty dt \langle v_z(t)v_z(0)\rangle \langle \delta B_x(t)\delta B_x^*(0) \rangle$$

(22)

Approximation (21) is problematic and does not work in the general case as shown in the numerical work of Qin & Shalchi (2016). Therefore, this approximation is dropped in the next section.

Furthermore, Matthaeus et al. (2003) modeled the parallel velocity correlation function by the isotropic exponential model

$$\langle v_z(t)v_z(0)\rangle = \frac{a^2}{3} e^{-vt/\lambda_p}$$

(23)
where we used the parallel mean free path $\lambda_\parallel$. It was shown in Shalchi (2011a) that Eq. (23) is only exact for an isotropic pitch-angle Fokker-Planck coefficient. However, for other forms of $D_{\mu\mu}$, Eq. (23) should still provide an accurate approximation. To proceed, the magnetic correlation function \[ \langle \delta B_x(k) \delta B_x^*(\vec{k}) e^{i\vec{k} \cdot \vec{x}} \rangle \langle e^{-vt/\lambda_\parallel} \] (24)

where we used $\vec{x}(t = 0) = 0$. Now we employ Corrsin’s independence hypothesis (see Corrsin (1959))

\[ \langle \delta B_x(k) \delta B_x^*(\vec{k}) e^{i\vec{k} \cdot \vec{x}} \rangle \approx \langle \delta B_x(\vec{k}) \delta B_x^*(\vec{k}) \rangle \langle e^{i\vec{k} \cdot \vec{x}} \rangle \] (25)

and assume homogeneous turbulence

\[ \langle \delta B_n(\vec{k}) \delta B_m^*(\vec{k}) \rangle = P_{nm}(\vec{k}) \delta(\vec{k} - \vec{k}') \] (26)

to derive

\[ \kappa_\perp = \frac{a^2 v^2}{3 B_0^2} \int d^3 k \int_0^\infty dt \int e^{i\vec{k} \cdot \vec{x}} \langle e^{-vt/\lambda_\parallel} \rangle \] (27)

Next, the characteristic function $\langle e^{i\vec{k} \cdot \vec{x}} \rangle$ must be approximated. Matthaeus et al. assumed a Gaussian distribution of the particles and that the particle motion is diffusive for all times. Therefore, one can use

\[ \langle e^{i\vec{k} \cdot \vec{x}} \rangle = e^{-\kappa_\perp k_\perp^2 t - \kappa_\parallel k_\parallel^2 t} \] (28)

corresponding to the characteristic function of a usual diffusion equation for the axi-symmetric case. Combining Eq. (27) with Eq. (28) and evaluating the time-integral, yields

\[ \kappa_\perp = \frac{a^2 v^2}{3 B_0^2} \int d^3 k \frac{P_{xx}(\vec{k})}{v/\lambda_\parallel + \kappa_\perp k_\perp^2 + \kappa_\parallel k_\parallel^2} \] (29)

This is the non-linear integral equation of the NLGC theory as originally derived by Matthaeus et al. (2003). This integral equation can be used for turbulence models with purely magnetic fluctuations, where $\delta B_z = 0$ corresponding to the incompressible case. Furthermore, we assumed static turbulence here but the theory can easily be generalized to allow for dynamical turbulence (see Matthaeus et al. (2003) and Shalchi et al. (2004)).

As shown in Matthaeus et al. (2003) and Bieber et al. (2004), Eq. (29) agrees with some simulations as well as solar wind observations as long as a two-component turbulence model is used and one sets $a^2 = 1/3$. This was undoubtedly a breakthrough in the theory of perpendicular diffusion. However, Eq. (29) cannot be seen as the final solution to this problem. One of the major problems is that NLGC theory does not work for slab turbulence where perpendicular transport should be sub-diffusive.

### 3. Diffusive UNLT Theory

After NLGC theory was developed it was pointed out quickly that the theory has some weaknesses. In Shalchi (2006) a modification was presented called the extended non-linear guiding center (ENLGC) theory. The latter theory was developed specifically for slab/2D turbulence and describes the slab contribution as sub-diffusive whereas the 2D contribution is as in NLGC theory. Historically the next step was the formulation of the unified non-linear transport (UNLT) theory (see Shalchi (2010)) which is now called diffusive UNLT theory.
In order to avoid using Eq. (21) one has to be able to compute higher order correlations analytically. In Shalchi (2010) this was done based on the pitch-angle dependent Fokker-Planck equation (see, e.g., Schlickeiser (2002) for a more complete version of this equation)

\[
\frac{\partial f}{\partial t} + v \mu \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu} \left( D_{\mu \mu} \frac{\partial f}{\partial \mu} \right) + D_\perp \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right). \tag{30}
\]

In principle the solution of this equation \( f = f(\vec{x}, \mu, t) \) allows one to determine higher order correlations. After using a Fourier representation in Eq. (20) and after employing Corrsin’s approximation, one ends up with a correlation of the form

\[
\langle v_z(t)v_z(0)e^{i\vec{k} \cdot \vec{x}} \rangle = \frac{v^2}{4} \int d^3 x \int_{-1}^{1} d\mu_0 \int_{-1}^{1} d\mu_0 \mu_0 e^{i\vec{k} \cdot \vec{x}} f(\vec{x}, \mu, t) \tag{31}
\]

where \( f \) is the solution of Eq. (30) and \( \mu_0 \) is the initial pitch-angle cosine. This idea was used in Shalchi (2010) in order to find an improved theory for perpendicular diffusion. However, the derivation is lengthy and not straightforward. Therefore, we only provide the result, namely

\[
\kappa_\perp = \frac{a^2 v^2}{3 B_0^2} \int d^3 k \frac{P_{xx}(\vec{k})}{F(k_\|, k_\perp) + (4/3)\kappa_\perp k_\perp^2 + \nu/\lambda_\|} \tag{32}
\]

with the function

\[
F(k_\|, k_\perp) = (v k_\|)^2/(3 \kappa_\perp k_\perp^2). \tag{33}
\]

As mentioned above, the original derivation of this result is not easy. However, a simpler derivation from time-dependent UNLT theory developed by Shalchi (2017) and Lasuik & Shalchi (2017) can be found in Sect. 4.4 of this paper.

Although Eq. (32) has some similarity with Eq. (29), it is not equal nor does it provide the same solution in the general case. For pure slab turbulence, for instance, Eq. (32) provides the solution \( \kappa_\perp = 0 \), corresponding to sub-diffusion, whereas NLGC theory provides a finite result, corresponding to normal diffusion. One can easily generalize Eq. (32) to allow for dynamical turbulence (see, e.g., Shalchi (2011b)).

In the following we discuss asymptotic limits which are contained in this theory. This will link Eq. (32) to previously derived and discussed theories such as the FLRW limit. It will be shown that the form of the solution of Eq. (32) depends on two parameters, namely the Kubo number (Kubo (1963))

\[
K = \frac{\ell_\|}{\ell_\perp} \frac{\delta B_x}{B_0} \tag{34}
\]

as well as the ratio \( \lambda_\|/\ell_\| \). Here we have used the correlation lengths\(^1\) in the different directions of space \( \ell_\| \) and \( \ell_\perp \), respectively.

3.1. The FLRW-Limit and the Matthaeus et al. (1995) Theory

First we consider the limit \( \lambda_\|/\ell_\|= \infty \) corresponding to the case that pitch-angle scattering and therewith parallel diffusion are suppressed. In this case we can set \( v/\lambda_\| = 0 \) in Eq. (32) and we find the solution

\[
\kappa_\perp = \frac{v}{2} \kappa_{FL} \tag{35}
\]

\(^1\) It has to be noted that magnetic turbulence depends on several characteristic lengths scales. Besides the bendover scales denoting the turnover from the energy to the inertial range in the spectrum, there are also integral scales, the ultra-scale, and the Kolmogorov scale. Therefore, special care is required if those scales are used and how the Kubo number is defined.
where the field line diffusion coefficient $\kappa_{FL}$ is given by

$$
\kappa_{FL} = \frac{1}{B_0^2} \int d^3k \, P_{xx}(\vec{k}) \frac{\kappa_{FL} k_\perp^2}{k^2 + (\kappa_{FL} k_\perp^2)^2}
$$

(36)

where we have used the effective mean field $\tilde{B}_0 = B_0/a$. Eq. (35) agrees with Eq. (8) corresponding to the FLRW limit. Eq. (36), on the other hand, is in perfect agreement with the Matthaeus et al. (1995) theory for field line wandering. Obviously the latter theory is contained in Eq. (32). Therefore, UNLT theory can be seen as a unified transport theory for magnetic field lines and energetic particles, hence the name unified non-linear transport theory.

### 3.2. The Quasi-Linear Regime

The quasi-linear limit can be obtained if we additionally consider $\kappa_{FL} \to 0$ at the right hand side of Eq. (36). Using the relation (Zwillinger (2012))

$$
\lim_{\xi \to 0} \frac{\xi}{\xi^2 + x^2} = \pi \delta(x)
$$

(37)

in Eq. (36) yields

$$
\kappa_{FL} = \frac{\pi}{B_0^2} \int d^3k \, P_{xx}(\vec{k}) \delta(k_\parallel)
$$

(38)

corresponding to the well-known quasi-linear result. As shown here, QLT for perpendicular diffusion can be obtained in the limit of long parallel mean free paths and small field line diffusion coefficients corresponding to small Kubo numbers. It can be shown (see Shalchi (2015)) that for most turbulence models Eq. (35) with (38) leads to the scaling law

$$
\lambda_\perp \propto \ell_\perp \frac{\delta B^2}{B_0^2}
$$

(39)

where we used the parallel correlation scale of the turbulence $\ell_\parallel$. As discussed above, this result should be valid for small Kubo numbers but also for long parallel mean free paths. It is usually assumed that the parallel mean free path increases with increasing magnetic rigidity. Therefore, the quasi-linear scaling should become relevant if high energy particles propagating through small Kubo number turbulence is considered.

### 3.3. The Kadomtsev & Pogutse Scaling

Here we still consider the case of $\lambda_\parallel/\ell_\parallel \to \infty$ and, thus, Eqs. (35) and (36) are still valid. However, we now assume a large field line diffusion coefficient corresponding to large values of the Kubo number. Considering $\kappa_{FL} \to \infty$ at the right hand side of Eq. (36), we derive (see again Shalchi (2015) for the mathematical details)

$$
\kappa_{FL} = \frac{1}{B_0} \left[ \int d^3k \, P_{xx}(\vec{k}) k_\perp^{-2} \right]^{1/2}
$$

(40)

The latter limit is sometimes called the non-linear regime or the Bohm limit of field line diffusion and was originally obtained by Kadomtsev & Pogutse (1978). In this case particles still follow magnetic field lines but the field lines are highly non-linear. The more experienced reader can easily see the occurrence of the so-called ultra-scale in Eq. (40). For most spectral tensors Eq. (40) leads to the scaling law

$$
\lambda_\perp \propto \ell_\perp \frac{\delta B_x}{B_0}
$$

(41)

where we used the perpendicular correlation scale of the turbulence $\ell_\perp$. This is very different compared to the quasi-linear scaling given by Eq. (39). This result should be valid for high particle energies and large Kubo number turbulence.
3.4. The Owens-Zybin-Istomin Scaling

We now consider the limit \( \lambda_\parallel / \ell_\parallel \to 0 \) corresponding to strong pitch-angle scattering. In this case Eq. (32) becomes

\[
\kappa_\perp = \frac{a^2 v^2}{3B_0^2} \int d^3k \frac{P_{xx}(\vec{k})}{F(k_\parallel, k_\perp) + v/\lambda_\parallel}.
\]

Here we kept the function \( F(k_\parallel, k_\perp) \) in the equation because for a small parallel diffusion coefficient, we also expect that the perpendicular diffusion coefficient is small. This means that we can neglect the term directly proportional to \( \kappa_\perp k_2 \) in Eq. (32) but not the term \( F(k_\parallel, k_\perp) \) given by Eq. (33). Replacing the parallel mean free path via \( \lambda_\parallel = 3\kappa_\parallel / v \) and using Eq. (33) allows us to write

\[
\kappa_\perp = \frac{a^2}{B_0^2} \kappa_\parallel \int d^3k P_{xx}(\vec{k}) \frac{\kappa_\perp k_2^2}{k_\parallel^2 + \kappa_\parallel k_\perp^2}.
\]

In order to simplify this further, we consider two cases. First we assume that the ratio \( \kappa_\perp / \kappa_\parallel \) is large leading to

\[
\kappa_\perp = \frac{a^2}{B_0^2} \kappa_\parallel \kappa_\perp \int d^3k P_{xx}(\vec{k}) = a^2 \kappa_\parallel \delta B_x^2 / B_0^2.
\]

Important here is to note that this result does not depend on the details of turbulence. Only the ratio of the turbulent field with respect to the mean field enters the latter equation. The result obtained here was derived before by Owens (1974) as well as Zybin & Istomin (1985).

3.5. The Collisionless Rechester & Rosenbluth Scaling

Here we still consider the case \( \lambda_\parallel / \ell_\parallel \to 0 \) and, thus, Eq. (43) is still valid. First we write Eq. (43) as

\[
\frac{\kappa_\perp}{\kappa_\parallel} = \frac{a^2}{B_0^2} \int d^3k P_{xx}(\vec{k}) \frac{\kappa_\perp k_2}{k_\parallel^2 + \kappa_\parallel k_\perp^2} \sqrt{\kappa_\perp k_\perp^2}.\]

Now we assume that the ratio \( \kappa_\perp / \kappa_\parallel \) is small. Using Eq. (37) therein yields

\[
\sqrt{\frac{\kappa_\perp}{\kappa_\parallel}} = \pi \frac{a^2}{B_0^2} \int d^3k P_{xx}(\vec{k}) k_\perp \delta(k_\parallel).
\]

The latter formula can easily be combined with any turbulence model as long as the occurring integrals are convergent. For most spectral tensors we now find the scaling law

\[
\lambda_\perp \propto \lambda_\parallel \delta B_x^4 / B_0^2.
\]

which sensitively depends on the scale ratio \( \ell_\parallel / \ell_\perp \) as well as the magnetic field ratio \( \delta B_x / B_0 \). This result should be valid for short parallel mean free paths corresponding to strong pitch-angle scattering and small Kubo numbers. Characteristic here is that the ratio \( \kappa_\perp / \kappa_\parallel \) does not depend on magnetic rigidity. Furthermore, we expect to find small values of that ratio in this limit (see next subsection for an example).

Spatschek (2008) discussed different transport regimes as well. Besides quasi-linear and Kadomtsev & Pogutse scalings, the famous work of Rechester & Rosenbluth (1978) was also discussed. Rechester & Rosenbluth derived a formula for the perpendicular diffusion coefficient for the case that deviations from the magnetic field lines occur. Assuming that the so-called Kolmogorov length \( L_K \) is the characteristic scale in the parallel direction for which this deviation occurs, one can estimate the corresponding mean square displacement of the field lines via
\[ \langle (\Delta x)^2 \rangle \propto \kappa_{FL} L_K. \]  

The parallel motion is assumed to be diffusive so that \( L_K^2 \propto \kappa_{\parallel} t_c \) where \( t_c \) is the time the particles need to travel the distance \( L_K \) in the parallel direction. The corresponding perpendicular diffusion coefficient of the particle is then given by

\[
\kappa_{\perp} = \frac{\langle (\Delta x)^2 \rangle}{2t_c} \propto \frac{\kappa_{FL} L_K}{t_c} \propto \frac{\kappa_{FL} \kappa_{\parallel}}{L_K}. \tag{48}
\]

This formula was also derived in Shalchi (2019) by using arguments based on field line separation theory. According to Neuer & Spatschek (2006, equation (46)) the Kolmogorov scale is given by

\[
L_K \propto \frac{\ell_{\perp}^2 B_0^2}{B_2^x}. \tag{49}
\]

Using this in Eq. (48) yields

\[
\kappa_{\perp} \propto \frac{\ell_{\perp}^2}{\ell_{\perp}^2 \delta B_{2x}^x \kappa_{\parallel}} \tag{50}
\]

in perfect agreement with the scaling obtained in Eq. (47) from diffusive UNLT theory. Therefore, we refer to the limit (47) as the collisionless Rechester & Rosenbluth (CLRR) scaling. As shown here and originally in Shalchi (2015) this limit can be derived in a very systematic way from diffusive UNLT theory.

It also has to be emphasized that the case considered here (strong pitch-angle scattering and small Kubo number turbulence) includes the case of pure slab turbulence. However, for slab turbulence, we have by definition \( \ell_{\perp} = \infty \). Using this in Eq. (47) yields \( \kappa_{\perp} = 0 \) indicating that perpendicular transport in slab turbulence is sub-diffusive. Therefore, diffusive UNLT theory correctly describes transport in slab turbulence in the sense that \( \kappa_{\perp} = 0 \). Later we shall see that time-dependent UNLT theory agrees perfectly with compound sub-diffusion for this case.

### 3.6. An Illustrative Example

Three out of the four limits derived above from diffusive UNLT theory can easily be understood in terms of FLRW and parallel diffusion. Eq. (46), however, is an exception and one can assume that this type of diffusion requires the particles to get scattered away from the original magnetic field line (see Shalchi (2019) for some ideas supporting this assumption). Therefore, we focus more on the CLRR limit and consider an example. This example is the noisy slab model defined via

\[
F_{nm}^{\text{ns}}(\vec{k}) = g_{\text{slab}}(k_{||}) \frac{2\ell_{\perp}}{k_{\perp}} \Theta(1 - k_{\perp} \ell_{\perp}) \left( \delta_{nm} - \frac{k_n k_m}{k_{\perp}^2} \right). \tag{51}
\]

This model was used in Shalchi (2015) and is useful to study field line and particle transport in turbulence with non-vanishing but weak transverse complexity. Because of the Heaviside step function in Eq. (51) the usual slab model is broadened so that the wave vectors are no longer aligned perfectly parallel with respect to the mean field. In this case Eq. (46) becomes

\[
\sqrt{\frac{\kappa_{\perp}}{\kappa_{\parallel}}} = \frac{\pi^2 a^2}{\ell_{\perp} B_2^x} g_{\text{slab}}(k_{||} = 0). \tag{52}
\]

For \( g_{\text{slab}}(k_{||}) \) we employ the model spectrum given by Bieber et al. (1994)

\[
g_{\text{slab}}(k_{||}) = \frac{1}{2\pi} C(s) \delta B_{slab}^2 \ell_{\parallel} \left[ 1 + \left( k_{\parallel} \ell_{\parallel} \right)^2 \right]^{-s/2}. \tag{53}
\]
with the normalization function

\[ C(s) = \frac{\Gamma\left(\frac{s}{2}\right)}{2\sqrt{\pi} \Gamma\left(\frac{s-1}{2}\right)}. \]  

(54)

In the model spectrum we have used the turbulent magnetic field \( \delta B_{\text{slab}} \), the bendover scale \( \ell_\parallel \) denoting the turnover from the energy to the inertial range, and the inertial range spectral index \( s \). Using this spectrum in Eq. (52) leads to

\[ \frac{\kappa_\perp}{\kappa_\parallel} = \frac{\pi}{2} C(s) a_\delta \frac{\ell_\parallel}{\ell_\perp} \frac{\delta B^2}{B_0^2} \]  

(55)

which is clearly a special case of the scaling law given by Eq. (47). In order to estimate a number for the ratio \( \kappa_\perp/\kappa_\parallel \) in the limit considered here, we use \( \delta B^2/B_0^2 \approx 0.5, \ell_\parallel/\ell_\perp \approx 0.75, \) and \( a^2 \approx 1 \). For \( s = 5/3 \) (Kolmogorov (1941)) we find \( C(s = 5/3) \approx 0.12 \) and, thus, Eq. (55) provides \( \kappa_\perp/\kappa_\parallel \approx 0.005 \). Perpendicular transport in noisy slab turbulence is discussed in more detail in the next section. However, we can already see that we found a finite result for \( \kappa_\perp \) corresponding to normal diffusion. Furthermore, the ratio \( \kappa_\perp/\kappa_\parallel \) is small and energy independent.

4. Time-Dependent Perpendicular Transport

For certain applications, a time-dependent description of perpendicular transport could be needed. Perpendicular transport is described by the auto-correlation function \( \langle V_z(t)V_z(0) \rangle \) where the guiding center velocity is given by Eq. (18). It is useful to write the latter equation as

\[ V_z(t) = \frac{1}{B_0} \int d^3k \ \delta B_z(\vec{k}, t)v_z(t)e^{izk_\parallel + i\vec{x}_\perp \cdot \vec{k}_\perp} \]  

(56)

where we have used the notation \( \vec{x}_\perp \vec{k} = zk_\perp + \vec{x}_\perp \cdot \vec{k}_\perp \) with the two-dimensional vectors \( \vec{x}_\perp = (x, y) \) and \( \vec{k}_\perp = (k_x, k_y) \). As before, we employ a Fourier representation to find for the velocity correlation function

\[ \langle V_z(t)V_z(0) \rangle = \frac{1}{B_0^3} \int d^3k \int d^3k' \ \delta B_z(\vec{k}, t)\delta B_z(\vec{k}', 0)v_z(t)v_z(0)e^{iz(t)k_\parallel}e^{i\vec{x}_\perp(t) \cdot \vec{k}_\perp} \]  

(57)

where we have set \( \vec{x}(t = 0) = 0 \). The central idea of time-dependent UNLT theory is to use the following approximation

\[ \langle v_z(t)v_z(0)\delta B_z(\vec{k})\delta B_z^*(\vec{k}')e^{iz(t)k_\parallel} \rangle \approx \langle v_z(t)v_z(0)e^{ik_\parallel z(t)} \rangle \langle \delta B_z(\vec{k})\delta B_z^*(\vec{k}') \rangle \langle e^{i\vec{x}_\perp(t) \cdot \vec{k}_\perp} \rangle \]  

(58)

meaning that we group together magnetic fields, all particle properties associated with their parallel motion, and their perpendicular motion. This can be understood as an extension of the Corrsin approximation used before in this review. Clearly this is different compared to approximation (21) used during the derivation of NLGC theory. Using this in Eq. (57) and employing Eq. (26) allows us to write the velocity auto-correlation function as

\[ \langle V_z(t)V_z(0) \rangle = \frac{1}{B_0^3} \int d^3k \ P_{zz}(\vec{k}, t)\xi(k_\parallel, t)\langle e^{i\vec{x}_\perp(t) \cdot \vec{k}_\perp} \rangle \]  

(59)

where we have used the parallel correlation function

\[ \xi(k_\parallel, t) = \langle v_z(t)v_z(0)e^{iz(t)k_\parallel} \rangle. \]  

(60)
For the perpendicular characteristic function in Eq. (59) we set
\[
\langle e^{i\vec{x}_\perp \cdot \vec{k}_\perp} \rangle = e^{-\langle (\Delta x)^2 \rangle k^2_\perp / 2}
\] (61)
corresponding to a Gaussian distribution with vanishing mean. In Lasuki & Shalchi (2018) other forms of distribution functions, such as kappa distributions, were employed but it was shown that the assumed statistics has only a minor influence on the resulting perpendicular diffusion parameter.

After combining Eqs. (59) and (61), we can write the velocity correlation function as
\[
\langle V_x(t)V_x(0) \rangle = \frac{1}{B_0^2} \int d^3k P_{xx}(\vec{k}, t) \xi \left( k_\parallel, t \right) e^{-\frac{1}{2} \langle (\Delta x)^2 \rangle k^2_\perp} \] (62)
and after employing the relation
\[
\frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle = 2 \langle v_x(t)v_x(0) \rangle
\] (63)
we find
\[
\frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle = 2 \frac{v^2}{B_0^2} \int d^3k P_{xx}(\vec{k}, t) \xi \left( k_\parallel, t \right) e^{-\frac{1}{2} \langle (\Delta x)^2 \rangle k^2_\perp}.
\] (64)
This ordinary differential equation can be evaluated for any given turbulence model described by the spectral tensor \( P_{nm} \) as long as the parallel correlation function \( \xi(k_\parallel, t) \) is specified as well. It has to be emphasized that at no point have we assumed that perpendicular transport is diffusive. Within a two-dimensional sub-space approximation, it was derived in Shalchi et al. (2011) as well as Lasuki & Shalchi (2019) that \( \xi(k_\parallel, t) \) is given by\(^2\)
\[
\xi \left( k_\parallel, t \right) = \frac{v^2}{3} \frac{1}{\omega_+ - \omega_-} \left[ \omega_+ e^{\omega_+ t} - \omega_- e^{\omega_- t} \right]
\] (65)
with the parameters
\[
\omega_\pm = \frac{v}{2k_\parallel} \pm \sqrt{\left( \frac{v}{2\lambda_\parallel} \right)^2 - \frac{1}{3} \left( vk_\parallel \right)^2}.
\] (66)
Since the function \( \xi(k_\parallel, t) \) is known, the ordinary differential equation (64) can be evaluated numerically for any given turbulence model including dynamical turbulence. In the following we consider some limits and special cases.

4.1. The Initial Free-Streaming Regime

First we focus on early times. From Eq. (65) it follows for \( t \to 0 \) that
\[
\xi \left( k_\parallel, t \to 0 \right) = \frac{v^2}{3}.
\] (67)
Also assuming \( \langle (\Delta x)^2 \rangle \to 0 \) allows us to approximate Eq. (64) by
\[
\frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle = \frac{2v^2}{3B_0^2} \int d^3k P_{xx}(\vec{k}, t) = \frac{2}{3} \frac{v^2 \delta B^2}{B_0^2}.
\] (68)
Integrating this over time twice yields
\[
\langle (\Delta x)^2 \rangle = \frac{v^2}{3} \frac{\delta B^2}{B_0^2} t^2
\] (69)
corresponding to ballistic perpendicular transport at early times.

\(^2\) Please note that Eq. (65) is the correct solution of the problem considered here. Eq. (98) of Lasuki & Shalchi (2019), where this result was also derived, has the wrong sign.
4.2. Pure Slab Turbulence

For pure slab turbulence we expect that particles follow magnetic field lines and, thus, compound sub-diffusion should occur. This has to come out of time-dependent UNLT theory. For slab turbulence, defined via

\[ P_{nm}^{slab}(k) = g_{slab}(k_\parallel) \delta(k_\perp) \delta_{nm}, \] (70)

Eq. (62) becomes

\[ \langle V_x(t)V_x(0) \rangle = \frac{4\pi}{B_0^2} \int_0^\infty dk_\parallel g_{slab}(k_\parallel) \xi(k_\parallel, t). \] (71)

Using the time-dependent version of the TGK formula (19)

\[ d_\perp(t) = \int_0^t d\tau \langle v_x(\tau)v_x(0) \rangle \] (72)

and employing Eq. (65) yields for the running perpendicular diffusion coefficient

\[ d_\perp(t) = \frac{4\pi v_2^2}{3B_0^2} \int_0^\infty dk_\parallel g_{slab}(k_\parallel) \frac{1}{\omega_+ - \omega_-} \left( e^{\omega_+ t} - e^{\omega_- t} \right). \] (73)

In Fig. 1 we show a comparison of the latter formula with test-particle simulations showing an almost perfect agreement between time-dependent UNLT theory and the simulations. In order to understand Eq. (73) better, we consider the late time limit \( t \to \infty \). One can easily see by considering the exponentials in Eq. (73) together with Eq. (66), that the main contribution to the \( k_\parallel \)-integral comes from very small wave numbers. Therefore, we find

\[ d_\perp(t) \approx \frac{4\pi k_\parallel}{B_0^2} g_{slab}(k_\parallel = 0) \int_0^\infty dk_\parallel e^{-\kappa_\parallel k_\parallel^2 t}. \] (74)

For the turbulence spectrum we employ Eq. (53) and this becomes

\[ d_\perp(t) = C(s) \ell_\parallel \delta B_{slab}^2 B_0^2 \sqrt{\frac{\pi \kappa_\parallel}{t}} \] (75)

corresponding to compound sub-diffusion discussed above. Clearly we can see how compound sub-diffusion is correctly described by time-dependent UNLT theory.

4.3. Diffusive Perpendicular Transport in Noisy Slab Turbulence

As demonstrated above, time-dependent UNLT theory provides compound sub-diffusion for slab turbulence. In the following we try to explore what ingredients are needed in order to restore normal diffusion. Therefore, we employ a turbulence model which is very close to the slab model but contains some transverse structure. This model is the noisy slab model defined via Eq. (51). Combining this with Eq. (64) yields

\[ \frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle = \frac{8\pi \ell_\perp^2}{B_0^2} \int_0^{1/\ell_\perp} dk_\perp e^{-\frac{1}{2}(\Delta x)^2} k_\perp^2 \int_0^\infty dk_\parallel g_{slab}(k_\parallel) \xi(k_\parallel, t). \] (76)

The \( k_\perp \)-integral can be expressed by an error function

\[ \int_0^{1/\ell_\perp} dk_\perp e^{-\frac{1}{2}(\Delta x)^2} k_\perp^2 = \sqrt{\frac{\pi}{2(\Delta x)^2}} \text{Erf} \left( \sqrt{\frac{(\Delta x)^2}{2\ell_\perp^2}} \right) \] (77)
so that
\[
\frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle = \frac{8\pi\ell_{\perp}}{B_0^2} \int_0^\infty dk_\parallel g_{\text{slab}}(k_\parallel) \xi(k_\parallel, t). \tag{78}
\]

The error function has the following asymptotic properties (see, e.g., Abramowitz & Stegun (1974))
\[
\text{Erf} \approx \begin{cases} 
\frac{2\sqrt{x}}{\sqrt{\pi}} & \text{for } x \ll 1 \\
1 & \text{for } x \gg 1.
\end{cases} \tag{79}
\]

Therewith, it follows from Eq. (78) that we find compound sub-diffusion as long as the condition \( \langle (\Delta x)^2 \rangle \ll 2\ell_{\perp}^2 \) is satisfied. A numerical solution of Eq. (78) is visualized in Shalchi (2017) where one can see that diffusion is restored for later times. From Eq. (78) we can learn that in order to recover normal diffusion, we need to satisfy the condition \( \langle (\Delta x)^2 \rangle \gg 2\ell_{\perp}^2 \). For slab turbulence we have \( \ell_{\perp} = \infty \) and we never satisfy this. As soon as there is transverse complexity, corresponding to a finite \( \ell_{\perp} \), diffusion is restored as soon as the perpendicular particle mean square displacement exceeds the perpendicular bendover scale squared. Everything which is needed in order to get Markovian diffusion in the perpendicular direction is transverse structure in magnetic turbulence and the condition \( \langle (\Delta x)^2 \rangle \gg 2\ell_{\perp}^2 \) needs to be satisfied.

4.4. Diffusive UNLT Theory

Of course the question arises how diffusive and time-dependent UNLT theories are related to each other. In terms of equations, this means one has to explore how Eqs. (32) and (64) are related. Diffusive UNLT theory is based on the diffusion approximation meaning that \( \langle (\Delta x)^2 \rangle = 2\kappa_{\perp} t \).
for all \( t \). Using this at the right hand side of Eq. (64) and integrating over all times yields for the perpendicular diffusion coefficient

\[
\kappa_\perp = \frac{a^2 v^2}{3 B_0^2} \int d^3 k \frac{1}{P_{xx}(\vec{k})} \left( \frac{\omega_+ - \omega_-}{\omega_+ - \omega_-} \right) e^{(\omega_- - \omega_-)(\kappa_\perp k^2_\perp)t} - \omega_+ e^{(\omega_+ - \omega_-)(\kappa_\perp k^2_\perp)t},
\]

\[
= \frac{v^2}{3 B_0^2} \int d^3 k \frac{\kappa_\perp k^2_\perp}{P_{xx}(\vec{k})} \left( \frac{\omega_+ - \omega_-}{(\omega_+ + \omega_-) \kappa_\perp k^2_\perp + \nu / \lambda_{\parallel}} \right)
\]

(80)
in the case of magnetostatic turbulence. Using Eq. (66) in order to replace the parameters \( \omega_+ \) and \( \omega_- \) therein, and after performing some straightforward algebra we can derive

\[
\kappa_\perp = \frac{a^2 v^2}{3 B_0^2} \int d^3 k \frac{P_{xx}(\vec{k})}{F(k_{\parallel}, k_{\perp}) + \kappa_\perp k^2_\perp + v / \lambda_{\parallel}}
\]

(81)

where \( F(k_{\parallel}, k_{\perp}) \) is still given by Eq. (33). This result agrees with Eq. (32) apart from a factor 4/3. The derivation presented here has the advantage that one can see it as a special case of the time-dependent theory. The only difference between Eqs. (81) and (32) is the factor 4/3 in the denominator. Although this factor is a minor difference, strictly speaking Eq. (32) should be more accurate than Eq. (81).

5. An Example for Three-dimensional Turbulence: The Goldreich-Sridhar Model

In the recent years spectral tensors based on the Goldreich & Sridhar (1995) model became more popular and it was used in test-particle simulations in the paper by Sun & Jokipii (2011). For our analytical considerations, we use the following spectral tensor

\[
P_{nm}^{GS}(\vec{k}) = g^{GS}(k_{\parallel}, k_{\perp}) \left( \delta_{nm} - \frac{k_n k_m}{k^2} \right),
\]

(82)

It needs to be emphasized that this corresponds to the compressible case \( \delta B_z \neq 0 \) but the original work of Goldreich & Sridhar is based on incompressible MHD. For the spectrum Shalchi (2013) suggested the following form

\[
g^{GS}(k_{\parallel}, k_{\perp}) = \frac{D(s, q)}{2\pi} \ell^3 \delta B^2 \frac{(k_{\parallel} \ell)^{q-s}}{1 + (k_{\parallel} \ell)^2} e^{-\ell^2-s|k_{\parallel}|k_{\perp}^{-s}}
\]

(83)

which is only valid for \( s = 5/3 \). The exponential function therein ensures that the critical balance condition of Goldreich & Sridhar \( |k_{\parallel}| \propto k_{\perp}^{2/3} \ell^{-1/3} \) is satisfied. Furthermore, the spectrum used here contains an energy range where the spectral index is given by \( q \).

Sun & Jokipii (2011) used a model spectrum which corresponds to the one given by Eq. (83) for the case \( q = 0 \). Diffusive UNLT theory was combined with this spectral tensor and spectrum to test the theory via a comparison with the aforementioned simulations in Shalchi & Hussein (2015). The comparison is visualized in Fig. 2. As shown, we find good agreement between analytical theory and the simulations. Interesting here is that we find good agreement for \( a^2 = 1 \) meaning that the correction factor \( a^2 \) is not needed at all. This was also found for slab as well as noisy slab turbulence but not for two-component turbulence where we need to set \( a^2 = 1/3 \) (see, e.g., Arendt & Shalchi (2018)). This suggests that UNLT theory works well for small and intermediate Kubo numbers and only for large Kubo numbers one needs the correction factor \( a^2 \).

Furthermore, we can clearly see by considering Fig. 2 that \( \lambda_{\perp} \propto \lambda_{\parallel} \) for small \( \lambda_{\parallel} \) whereas the perpendicular mean free path becomes independent of the parallel mean free path for large \( \lambda_{\parallel} \).
Figure 2. The perpendicular mean free path $\lambda_\perp$ versus the parallel mean free path $\lambda_\parallel$ for Goldreich-Sridhar turbulence. Both parameters are normalized with respect to the bendover scale $\ell$. Shown are the mean free paths obtained from the simulations (dots) performed by Sun & Jokipii (2011) and the results derived from diffusive UNLT theory (solid line). Also the quasi-linear perpendicular mean free path (see Eq. (3) from Sun & Jokipii (2011)) is shown (dotted line).

This behavior can also be seen for the noisy slab model and two-component turbulence (see, e.g., Hussein et al. (2015)). Thus there is some universality in the transport of particles across the mean magnetic field (see, e.g., Shalchi (2014) and Hussein et al. (2015)). However, this universal behavior of the perpendicular mean free path requires that all fundamental turbulence scales such as the integral scales and the ultra-scale are finite (see Matthaeus et al. (2007) for scales in two-dimensional turbulence and Shalchi (2014) for scales in different three-dimensional models).

6. Summary and Conclusion
The problem of perpendicular transport has a long history going back to the famous paper of Jokipii (1966) where a quasi-linear theory has been developed. A more advanced non-linear tool is provided by the diffusive unified non-linear transport theory and its time-dependent generalization. One important feature of this theory is that it contains several known limits as special cases. Those are listed in the following:

- In the limit of long parallel mean free paths, the unified non-linear transport equation (32) turns into the field line random walk limit and one obtains automatically the non-linear field line diffusion theory of Matthaeus et al. (1995). Therefore, the latter theory is contained in the non-linear integral equation provided by diffusive UNLT theory.
- For magnetostatic slab turbulence time-dependent UNLT theory provides compound sub-
diffusion.

- For long parallel mean free path and small Kubo numbers, UNLT theory is identical compared to quasi-linear theory.

- For long parallel mean free path and large Kubo numbers, on the other hand, UNLT theory provides the limit known as Kadomtsev & Pogutse (1978) limit.

- For short parallel mean free paths and small Kubo numbers, one can derive a Rechester & Rosenbluth (1978) type of scaling from UNLT theory. Since this process does not require to incorporate collisions, this limit can be called the collisionless Rechester & Rosenbluth scaling. This result is interesting because it provides a ratio $\kappa_\perp/\kappa_\parallel$ which is small and does not depend on particle energy.

- For short parallel mean free paths and large Kubo numbers one finds $\kappa_\perp/\kappa_\parallel \approx \delta B_2^2/B_0^2$ which could be explained by assuming that particles diffusive along ballistic magnetic field lines.

Furthermore, UNLT theory agrees well with most test-particle simulations. This is in particular the case for full three-dimensional turbulence and slab turbulence. For instance, UNLT theory agree almost perfectly with simulations performed for Goldreich-Sridhar turbulence.

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