Global Well-posedness for the Generalized Navier-Stokes System

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Abstract

In this paper we investigate well-posedness of the Cauchy problem of the three dimensional generalized Navier-Stokes system. We first establish local well-posedness of the GNS system for any initial data in the Fourier-Herz space $\chi^{-1}$. Then we show that if the $\chi^{-1}$ norm of the initial data is smaller than $C\nu$ in the GNS system where $\nu$ is the viscosity coefficient, the corresponding solution exists globally in time. Moreover, we prove global well-posedness of the Navier-Stokes system without norm restrictions on the corresponding solutions provided the $\chi^{-1}$ norm of the initial data is less than $\nu$. Our obtained results cover and improve recent results in \cite{2, 11}.

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Contents

\section{Introduction}

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1 Introduction

In this paper we consider the generalized Navier-Stokes system

\[
\begin{aligned}
\partial_t u - \nu \Delta u &= Q(u, u), \quad t > 0, \quad x \in \mathbb{R}^3, \\
u u(x, 0) &= u_0,
\end{aligned}
\]

with \( \nu \) being a positive constant and the bilinear operator \( Q \) defined as

\[
Q^j(u, v) = \sum_{k, l, m} q_{k, l}^{j,m} \partial_m (u^k v^l), \quad j = 1, 2, 3,
\]

(1.1)

where \( q_{k, l}^{j,m}(a) = \sum_{n, p} a_{k, l}^{j,m,p,n} F^{-1} \left( \frac{\xi_k \xi_l}{|\xi|^2} a(\xi) \right) \), and \( a_{k, l}^{j,m,p,n} \) are real numbers.

It is obvious that the incompressible Navier-Stokes system

\[
\begin{aligned}
\partial_t u - \nu \Delta u &= \mathcal{P} \nabla (u \otimes u), \quad t > 0, \quad x \in \mathbb{R}^3, \\
\text{div } u &= 0, \\
u u(x, 0) &= u_0,
\end{aligned}
\]

(\( \text{NS}_\nu \))

is a particular case of the system (\( \text{GNS}_\nu \)). Here \( u \) stands for the velocity field of the fluid, \( \nu \) is the viscosity and \( \mathcal{P} \) is the Leray projection operator defined by the formula:

\[
\mathcal{F}(\mathcal{P} f)^j(\xi) = \mathcal{F}(f)^j(\xi) - \frac{1}{|\xi|^2} \sum_{k=1}^3 \xi_j \xi_k \mathcal{F}(f)^k(\xi), \quad j = 1, 2, 3.
\]

(1.2)

From now on we always assume that the initial data \( u_0 \) is divergence free and \( C \) denotes a generic constant.

It is well known that the space \( BMO^{-1} \) is the largest space which is included in the tempered distribution and enjoys the property of translation and scaling invariant (see [4, 6] for instance). The global well-posedness for the Navier-Stokes system in the space \( BMO^{-1} \) was studied by Koch and Tataru [6]. Many works in subspaces of the space \( BMO^{-1} \), such as \( H^{\frac{3}{2}}(\mathbb{R}^3) \), \( L^3(\mathbb{R}^3) \),
and $B_{p,\infty}^{-\frac{1}{3}+\frac{2}{3}}(\mathbb{R}^3)$ also have been down: the Navier-Stokes system is known to be globally well-posed for sufficiently small date $u_0 \in \dot{H}^{\frac{2}{3}+1}(\mathbb{R}^3)$, and locally well-posed for any $u_0 \in \dot{H}^{\frac{2}{3}+1}(\mathbb{R}^3)$, as proved by Fujita and Kato [11]: the global well-posedness for small date is due to Kato [7] in $L^{\nu}$, the system $(\parallel \cdot \parallel)$ global mild solution is unique under the condition $u \in \mathcal{M}$ for any $\mathbf{u} \in (\mathcal{M})$. Similarly, we can also get \text{reader may refer to [5, 8, 10]. Let us mention that the well-posedness for the Navier-Stokes system in those above spaces can be extended to the generalized Navier-Stokes system [3].}

Recently Lei and Lin [2] studied a new space

$$\chi^{-1} = \{f \in \mathcal{D}'(\mathbb{R}^3); \int_{\mathbb{R}^3} |\xi|^{-1} |\hat{f}|d\xi < \infty\},$$

which is contained in $BMO^{-1}$ [2] and is equivalent to the Fourier-Herz space $\mathcal{B}^{-1}_1$ [11]. They pointed out that $H^s(\mathbb{R}^3) (s > \frac{1}{2}) \subseteq \chi^{-1}$, and they also presented an example to show that $H^{\frac{2}{3}+1}(\mathbb{R}^3) \nsubseteq \chi^{-1}$. Here we give an example to show that $\chi^{-1} \nsubseteq \dot{H}^{\frac{2}{3}+1}(\mathbb{R}^3)$. In fact, let $f(x) = F^{-1}(\frac{1}{\# \xi} h(|\xi|))$, where $h(r)$ is defined by

$$h(r) = \begin{cases} 2^{\frac{2j+1}{2}}, & 1 - 2^{-j} \leq r < 1 - 2^{-(j+1)}, j = 0, 1, \cdots, \\ 0, & r \geq 1. \end{cases}$$

It is easy to deduce that

$$\int_{\mathbb{R}^3} |\xi|^{-1} |\hat{f}(\xi)|d\xi \leq C \int_0^{\infty} h(r)dr = C \sum_{j=0}^{\infty} 2^{\frac{2j+1}{2}} 2^{-(j+1)} < \infty.$$

Similarly, we can also get

$$\int_{\mathbb{R}^3} |\xi|^j |\hat{f}(\xi)|^2d\xi = C \int_0^{\infty} rh(r)^2dr \geq C \sum_{j=0}^{\infty} (1 - 2^{-j}) 2^{j+1} 2^{-(j+1)} = \infty.$$

Combing the above two inequalities yields that $f \in \chi^{-1}$, but $f \notin \dot{H}^{\frac{2}{3}+1}(\mathbb{R}^3)$. Thus, we conclude that $\chi^{-1}$ and $\dot{H}^{\frac{2}{3}+1}(\mathbb{R}^3)$ do not contain each other.

Lei and Lin [2] proved that if the initial data $u_0$ in $\chi^{-1}$ satisfying $\|u_0\|_{\chi^{-1}} < \nu$, the system (NS$\nu$) admits a global mild solution. They also proved that this global mild solution is unique under the condition $\|u\|_{L^{\infty}(\mathbb{R}^3, \chi^{-1})} < \nu$. Cannone and Wu [11] gave a global well-posedness result for small initial data in a family of critical Fourier-Herz spaces $\dot{B}^{-1}_q (q \in [1, 2])$. They also showed this global solution is unique under the condition $\|u\|_{L^{\infty}(\mathbb{R}^3, \dot{B}^{-1}_q)} \cap L^1(\mathbb{R}^3, \dot{B}^1_q) \leq \frac{3\nu}{2} \|u_0\|_{\dot{B}^{-1}_q}$.
However, it is not clear whether there exists a solution to the system \((NS_\nu)\) for large initial data. Moreover, without norm restrictions on the solutions, is the uniqueness of solutions to the system \((NS_\nu)\) still valid?

In the paper we will give definite answers to these two questions. In Section 3, we will solve the system \((GNS_\nu)\) by means of a contraction mapping argument (see Lemma 1.1 below). Thus, we can obtain a unique local mild solution to the system \((GNS_\nu)\) for any initial data in \(\chi^{-1}\) and prove the corresponding solution will be global if the initial data is sufficient small. Especially for the Navier-Stokes system, we show that if \(\|u_0\|_{\chi^{-1}} < \nu\) then the solution to the system \((NS_\nu)\) will be unique and global without norm restrictions on the solutions.

**Lemma 1.1.** \((\text{3})\) Let \(E\) be a Banach space, \(B\) a continuous bilinear map from \(E \times E \to E\), and a positive real number such that \(\alpha < \frac{1}{4\|B\|}\), with
\[
\|B\| = \sup_{\|u\| \leq 1, \|v\| \leq 1} \|B(u, v)\|.
\]
For any \(a\) in the ball \(B(0, \alpha)\) in \(E\), then there exists a unique \(x\) in \(B(0, 2\alpha)\) such that
\[
x = a + B(x, x).
\]

We will also use the spaces

\[
\chi^i = \{f \in S'(\mathbb{R}^3); \int_{\mathbb{R}^3} |\xi|^i |\hat{f}| d\xi < \infty\}, \quad i = -1, 0, 1.
\]

The norm of \(\chi^i\) is denoted by \(\|\cdot\|_{\chi^i}\). Let \(T \in (0, \infty]\). Note that the spaces \(L^2([0,T]; \chi^0)\) and \(L^\infty([0,T]; \chi^{-1}) \cap L^1([0,T]; \chi^1)\) are Banach spaces and are translation and shift invariant.

Now we are in the position to state our main results:

**Theorem 1.2.** Let \(u_0\) be in \(\chi^{-1}\). There exists a positive time \(T\) such that the system \((GNS_\nu)\) has a unique solution \(u\) in \(L^2([0,T]; \chi^0)\) which also belongs to \(C([0,T]; \chi^{-1}) \cap L^1([0,T]; \chi^1) \cap L^\infty([0,T]; \chi^{-1})\).

Let \(T_{u_0}\) denote the maximal time of existence of such a solution. Then:

(i) There exists a constant \(C\) such that if \(\|u_0\|_{\chi^{-1}} \leq \nu\), then
\[
T_{u_0} = \infty.
\]

(ii) If \(T_{u_0}\) is finite, then
\[
\int_0^{T_{u_0}} \|u(t)\|_{\chi^0}^2 dt = \infty.
\]
**Theorem 1.3.** Let $u_0$ be in $\chi^{-1}$. There exists a positive time $T$ such that the system $(NS_\nu)$ has a unique solution $u$ in $L^2([0,T];\chi^0)$ which also belongs to $C([0,T];\chi^{-1}) \cap L^1([0,T];\chi^1) \cap L^\infty([0,T];\chi^{-1})$.

Let $T_{u_0}$ denote the maximal time of existence of such a solution. Then:
(i) There exists a constant $C$ such that if $\|u_0\|_{\chi^{-1}} < \nu$, then
$$T_{u_0} = \infty.$$
(ii) If $T_{u_0}$ is finite, then
$$\int_0^{T_{u_0}} \|u(t)\|^2_{\chi^0} dt = \infty.$$

**Remark 1.4.** Although the result (i) in Theorem 1.2 is the same as Theorem (1.1) in [2], our method here is different from their method in [2]. In particular, our proof relies on the obtained local well-posedness result and the blow-up criterion (ii), but not on additional norm restrictions on the corresponding solutions.

## 2 Preliminaries

Let $B(u,v)$ be the solution to the heat equation

$$\begin{align*}
\partial_t B(u,v) - \nu \Delta B(u,v) &= Q(u,v), \\
B(u,v)|_{t=0} &= 0,
\end{align*}$$

(2.1)

with the bilinear operator $Q$ defined as in [1].

Solving $(GNS_\nu)$ amounts to finding a fixed point for the map

$$u \mapsto e^{t\nu \Delta} u_0 + B(u,u).$$

By Duhamel’s formula in Fourier space and [1], we have

$$\begin{align*}
|\widehat{B(u,v)}(t,\xi)| &= \left| \int_0^t e^{-\nu(t-s)}|\xi|^2 \widehat{Q(u,v)}(s,\xi) ds \right| \\
&\leq C \int_0^t e^{-\nu(t-s)}|\xi|^2 |\xi|(|\hat{u}| * |\hat{v}|)(s,\xi) ds.
\end{align*}$$

(2.2)

We now give two useful propositions which will be used in the sequel.
3 PROOFS OF MAIN THEOREMS

Proposition 2.1.

(2.3) \[ \|u\|_{L^2([0,T];\chi^0)}^2 \leq \|u\|_{L^\infty([0,T];\chi^{-1})} \|u\|_{L^1([0,T];\chi^1)}. \]

Proof. It is easy to check that

\[
\|u\|^2_{L^2([0,T];\chi^0)} = \int_0^T \left( \int_{\mathbb{R}^3} |\hat{u}(s,\xi)|^2 \right) dt \\
\leq \int_0^T \left( \int_{\mathbb{R}^3} \frac{1}{|\xi|^2} |\hat{u}(s,\xi)|^2 \right) \left( \int_{\mathbb{R}^3} |\hat{u}(s,\xi)| \right) dt \\
\leq \|u\|_{L^\infty([0,T];\chi^{-1})} \|u\|_{L^1([0,T];\chi^1)}. 
\]

Proposition 2.2. A constant C exists such that

(2.4) \[ \|B(u,v)\|_{L^2([0,T];\chi^0)} \leq \frac{C}{\nu^2} \|u\|_{L^2([0,T];\chi^0)} \|v\|_{L^2([0,T];\chi^0)}. \]

Proof. Thanks to the inequality (2.2) and Minkowski’s inequality, we have

\[
\|B(u,v)\|_{L^2([0,T];\chi^0)} \leq C \left\| \int_{\mathbb{R}^3} \int_0^T I_{[0,T]}(s) e^{-\nu(t-s)} |\xi|^2 |\hat{u}(s,\xi,\hat{v}(s))| ds d\xi \right\|_{L^2(0,T)} \\
\leq C \left\| \int_{\mathbb{R}^3} \int_0^T \frac{1}{\nu^2} \left( |\hat{u}| * |\hat{v}| \right)(s,\xi) ds d\xi \right\|_{L^2(0,T)} \\
\leq \frac{C}{\nu^2} \int_{\mathbb{R}^3} \|\hat{u}(s)\|_{L^1} \|\hat{v}(s)\|_{L^1} ds \\
\leq \frac{C}{\nu^2} \|u\|_{L^2([0,T];\chi^0)} \|v\|_{L^2([0,T];\chi^0)}. 
\]

3 Proofs of main theorems

To prove the first part of Theorem 1.2, we shall use Lemma 1.1. Given some \( u_0 \in \chi^{-1} \), thanks to Minkowski’s inequality, we have

(3.1) \[ \|e^{\nu t \Delta} u_0\|_{L^2([0,T];\chi^0)} \leq \left( \int_0^T \left( \int_{\mathbb{R}^3} e^{-\nu |\xi|^2} |\hat{u}_0(\xi)|^2 d\xi \right) dt \right)^{\frac{1}{2}} \\
\leq \int_{\mathbb{R}^3} \left( \int_0^T e^{-2\nu |\xi|^2} |\hat{u}_0(\xi)|^2 ds \right)^{\frac{1}{2}} d\xi. 
\]
Thus, combining Proposition 2.2 and the inequality (3.1) gives that if \( \|u_0\|_{\chi^{-1}} \leq \frac{\nu}{2\pi C_0} \), with \( C_0 > C \), then

\[
\|e^{\nu t \Delta} u_0\|_{L^2([0,T];\chi^0)} \leq \frac{1}{4} \frac{\nu}{\nu^2} \leq \frac{1}{4} \|B\|.
\]

According to Lemma 1.1, there exists a unique solution of the system \((GNS,\nu)\) in the ball with center 0 and radius \( \frac{\nu}{2\pi C_0} \) in the space \( L^2([0,T];\chi^0) \).

We now consider the case of a large initial date \( u_0 \in \chi^{-1} \). We shall split \( u_0 \) into a small part in \( \chi^{-1} \) and a large part with compactly supported Fourier transform. For that, we fix some positive real number \( \rho_{u_0} \) such that

\[
\int_{|\xi| \geq \rho_{u_0}} |\xi|^{-1} |\widehat{u_0}|(\xi)d\xi \leq \frac{\nu}{2\pi C_0}.
\]

Using the inequality (3.1) again and defining \( u_0^\flat = \mathcal{F}^{-1}(I_{B(0,\rho_{u_0})}(\xi)\widehat{u_0}(\xi)) \), we get

\[
\|e^{\nu t \Delta} u_0\|_{L^2([0,T];\chi^0)} \leq \frac{\nu}{8C_0} + \|e^{\nu t \Delta} u_0^\flat\|_{L^2([0,T];\chi^0)}.
\]

From which we can deduce that

\[
\|e^{\nu t \Delta} u_0\|_{L^2([0,T];\chi^{-1})} \leq \left( \int_0^T \left( \int_{|\xi| \leq \rho_{u_0}} e^{-\nu t|\xi|^2} |\widehat{u_0}|(\xi)d\xi \right)^2 dt \right)^{\frac{1}{2}}
\]

\[
= \left( \int_0^T \left( \int_{|\xi| \leq \rho_{u_0}} e^{-\nu t|\xi|^2} |\xi||\xi|^{-1} |\widehat{u_0}|(\xi)d\xi \right)^2 dt \right)^{\frac{1}{2}}
\]

\[
\leq \rho_{u_0} T^{\frac{1}{2}} \|u_0\|_{\chi^{-1}}.
\]

Thus if

\[
T \leq \left( \frac{\nu}{8\rho_{u_0} C_0 \|u_0\|_{\chi^{-1}}} \right)^2,
\]

then we conclude the existence of a unique solution in the ball with center 0 and radius \( \frac{\nu}{2\pi C_0} \) in the space \( L^2([0,T];\chi^0) \).
Next we claim that if $u$ is a solution of the system $(\text{GNS}_\nu)$ in $L^2([0,T];\chi^0)$, then $u$ also belongs to $C([0,T];\chi^{-1}) \cap L^1([0,T];\chi^1) \cap L^\infty([0,T];\chi^{-1})$. In fact, it is easy to deduce that

$$
\|B(u,v)(t)\|_{\chi^{-1}} \leq C \int_{\mathbb{R}^3} \int_0^t |\xi|^{-1} e^{-\nu(t-s)|\xi|^2} |\xi|(|\hat{u}| * |\hat{v}|)(s,\xi)dsd\xi
$$

$$
\leq C \int_{\mathbb{R}^3} \int_0^t e^{-\nu(t-s)|\xi|^2} (|\hat{u}| * |\hat{v}|)(s,\xi)dsd\xi
$$

$$
\leq C \int_0^t \int_{\mathbb{R}^3} (|\hat{u}| * |\hat{v}|)(s,\xi)d\xi ds
$$

$$
\leq C \|u\|_{L^2([0,T];\chi^0)} \|v\|_{L^2([0,T];\chi^0)}.
$$

Similarly, we have

$$
\|B(u,v)\|_{L^1([0,T];\chi^1)} \leq C \int_0^T \int_{\mathbb{R}^3} \int_0^T |\xi| I_{[0,t]}(s)e^{-\nu(t-s)|\xi|^2} |\xi|(|\hat{u}| * |\hat{v}|)(s,\xi)dsd\xi dt
$$

$$
\leq C \int_{\mathbb{R}^3} \int_0^T \left( \int_0^T I_{[0,t]}(s)e^{-\nu(t-s)|\xi|^2} |\xi|^2 ds \right) (|\hat{u}| * |\hat{v}|)(s,\xi)dsd\xi
$$

$$
\leq C \|u\|_{L^2([0,T];\chi^0)} \|v\|_{L^2([0,T];\chi^0)}.
$$

Combing the above two inequalities yields that

$$
B(u,u) \in L^\infty([0,T];\chi^{-1}) \cap L^1([0,T];\chi^1).
$$

Noticing the following two facts:

$$
\|e^{\nu t \Delta} u_0\|_{\chi^{-1}} \leq \int_{\mathbb{R}^3} e^{-\nu t|\xi|^2} |\xi|^{-1} |\hat{u}_0|(|\xi|)d\xi \leq \|u_0\|_{\chi^{-1}},
$$

$$
\|e^{\nu t \Delta} u_0\|_{L^1([0,T];\chi^{-1})} \leq \int_0^T \int_{\mathbb{R}^3} e^{-\nu t|\xi|^2} |\xi| |\hat{u}_0|(|\xi|)d\xi dt \leq \frac{1}{\nu} \|u_0\|_{\chi^{-1}},
$$

we have

$$
e^{\nu t \Delta} u_0 \in L^\infty([0,T];\chi^{-1}) \cap L^1([0,T];\chi^1).
$$

We can thus conclude that

$$
u \in L^\infty([0,T];\chi^{-1}) \cap L^1([0,T];\chi^1).$$
To get further the regularity of \( u(t, x) \) with respect to \( t \), we come back to the system \((GNS)\). It is obvious that \( \Delta u \) is in \( L^1([0, T]; \chi^{-1}) \). Thanks to Proposition 2.1 we have

\[
\|Q(u, v)\|_{L^1([0, T]; \chi^{-1})} \\
= \int_0^T \int_{\mathbb{R}^3} |\xi|^{-1} \hat{Q}(u, v)(s, \xi) d\xi ds \\
\leq C \int_0^T \int_{\mathbb{R}^3} (|\hat{u}| * |\hat{v}|)(s, \xi) d\xi ds \\
\leq C \|u\|_{L^2([0, T]; \chi^0)} \|v\|_{L^2([0, T]; \chi^0)} \\
\leq C \|u\|_{L^\infty([0, T]; \chi^{-1})} \|v\|_{L^2([0, T]; \chi^0)} \|v\|_{L^\frac{3}{2}([0, T]; \chi^{-1})} \|v\|_{L^2([0, T]; \chi^1)}.
\]

Thus, \( Q(u, u) \in L^1([0, T]; \chi^{-1}) \). Then we conclude that \( \partial_t u \in L^1([0, T]; \chi^{-1}) \).

We hence prove the announced properties:

\[
u \in C([0, T]; \chi^{-1}) \cap L^1([0, T]; \chi^1) \cap L^\infty([0, T]; \chi^{-1}).
\]

The uniqueness part relies on the following lemma.

**Lemma 3.1.** Let \( v \) be a solution in \( C([0, T]; S'(\mathbb{R}^3)) \) of the Cauchy problem

\[
\begin{cases}
\partial_t v - \nu \Delta v = f, \\
v|_{t=0} = v_0,
\end{cases}
\]

with \( f \in L^1([0, T]; \chi^{-1}) \) and \( v_0 \in \chi^{-1} \). Then for any \( 0 \leq t_0 \leq T \), we have

\[
\int_{\mathbb{R}^3} \sup_{0 \leq t \leq t_0} |\hat{v}(t, \xi)| \xi^{-1} d\xi \leq \|v_0\|_{\chi^{-1}} + \|f\|_{L^1([0, t_0]; \chi^{-1})},
\]

\[
\|v\|_{L^\infty([0, t_0]; \chi^{-1})} + \nu \|v\|_{L^1([0, t_0]; \chi^1)} \leq 2(\|v_0\|_{\chi^{-1}} + \|f\|_{L^1([0, t_0]; \chi^{-1})}).
\]

**Proof.** Due to Duhamel’s formula in Fourier space, we can write that

\[
\hat{v}(t, \xi) = e^{-\nu t |\xi|^2} \hat{v}_0(\xi) + \int_0^t e^{-\nu (t-s) |\xi|^2} \hat{f}(s, \xi) ds.
\]

Then we have

\[
|\hat{v}(t, \xi)| \leq e^{-\nu t |\xi|^2} |\hat{v}_0(\xi)| + \int_0^t e^{-\nu (t-s) |\xi|^2} |\hat{f}(s, \xi)| ds.
\]
For any $0 \leq t_0 \leq T$, we get
\[
\sup_{0 \leq t \leq t_0} |\hat{v}(t, \xi)| \leq |\hat{v}_0| (\xi) + \int_0^{t_0} |\hat{f}(s, \xi)| ds.
\]
Taking the $L^1$ norm with respect to $|\xi|^{-1}d\xi$ allows us to conclude that
\[
\int_{\mathbb{R}^3} \sup_{0 \leq t \leq t_0} |\hat{v}(t, \xi)|^{-1}d\xi \leq \int_{\mathbb{R}^3} |\hat{v}_0|(\xi)|\xi|^{-1}d\xi + \int_{\mathbb{R}^3} \int_0^{t_0} |\hat{f}(s, \xi)|\xi|^{-1}dsd\xi
\leq \|v_0\|_{\chi^{-1}} + \|f\|_{L^1([0,t_0];\chi^{-1})}.
\]
The first result (3.6) is thus proved.

Similarly, taking the $L^1$ norm with respect to $|\xi|^1d\xi dt$, one have,
\[
\int_0^{t_0} \int_{\mathbb{R}^3} |\hat{v}(s, \xi)||\xi|d\xi dt
\leq \int_0^{t_0} \int_{\mathbb{R}^3} e^{-\nu|\xi|^2} |\hat{v}_0|(\xi)|\xi|d\xi dt + \int_0^{t_0} \int_{\mathbb{R}^3} \int_0^{t} e^{-\nu(t-s)|\xi|^2} |\hat{f}(s, \xi)|\xi|^{-1}dsd\xi d\xi
dt
\leq \int_{\mathbb{R}^3} \left(\int_0^{t_0} e^{-\nu|\xi|^2} |\hat{v}_0|(\xi)|\xi|^{-1}d\xiight)
+ \int_0^{t_0} \int_{\mathbb{R}^3} \left(\int_s^{t_0} e^{-\nu(t-s)|\xi|^2} |\hat{f}(s, \xi)|\xi|^{-1}dsight) d\xi
\leq \frac{1}{\nu} \int_{\mathbb{R}^3} |\hat{v}_0|(\xi)|\xi|^{-1}d\xi + \|f\|_{L^1([0,t_0];\chi^{-1})}).
\]
This implies that
\[
(3.8) \quad \nu\|v\|_{L^1([0,t_0];\chi^1)} \leq \|v_0\|_{\chi^{-1}} + \|f\|_{L^1([0,t_0];\chi^{-1})}.
\]
The inequalities (3.6) and (3.8) lead to (3.7).

Now consider two solutions $u_1$ and $u_2$ with the same initial data $u_0$, and assume that
\[
u_i \in C([0, T]; \chi^{-1}) \cap L^1([0, T]; \chi^1) \cap L^\infty([0, T]; \chi^{-1}), \quad i = 1, 2.
\]
Let $w = u_1 - u_2$, we note that $w$ satisfies
\[
(3.9) \quad \begin{cases}
\partial_t w - \nu \Delta w = Q(w, u_1) + Q(u_2, w), \\
w|_{t=0} = 0.
\end{cases}
\]
Thanks to the inequality (3.4), we have that
\[
Q(w, u_1) + Q(u_2, w) \in L^1([0, T]; \chi^{-1}),
\]
and that

\[ (3.10) \quad \|Q(w, u)\|_{L^1([0,t_0]; \mathcal{X}^{-1})} \]

\[ \leq C \|w\|_{L^\infty([0,t_0]; \mathcal{X}^{-1})}^{\frac{1}{2}} (\|u_1\|_{L^\infty([0,t_0]; \mathcal{X}^{-1})} + \|u_2\|_{L^\infty([0,t_0]; \mathcal{X}^{-1})})^{\frac{1}{2}} \]

\[ \times \|w\|_{L^1([0,t_0]; \mathcal{X}^{1})} (\|u_2\|_{L^1([0,t_0]; \mathcal{X}^1)} + \|w\|_{L^1([0,t_0]; \mathcal{X}^1)})^{\frac{1}{2}}. \]

By virtue of Lemma \[3.1\] we infer that for any \(0 \leq t_0 \leq T\),

\[ (3.11) \quad \|w\|_{L^\infty([0,t_0]; \mathcal{X}^{-1})} + \nu \|w\|_{L^1([0,t_0]; \mathcal{X}^{1})} \]

\[ \leq 2 \|Q(w, u)\|_{L^1([0,t_0]; \mathcal{X}^{-1})} \]

\[ \leq C \|w\|_{L^\infty([0,t_0]; \mathcal{X}^{-1})}^{\frac{1}{2}} (\|u_1\|_{L^\infty([0,t_0]; \mathcal{X}^{-1})} + \|u_2\|_{L^\infty([0,t_0]; \mathcal{X}^{-1})})^{\frac{1}{2}} \]

\[ \times \|w\|_{L^1([0,t_0]; \mathcal{X}^{1})} (\|u_2\|_{L^1([0,t_0]; \mathcal{X}^1)} + \|w\|_{L^1([0,t_0]; \mathcal{X}^1)})^{\frac{1}{2}} \]

\[ \leq \varepsilon (\|u_1\|_{L^\infty([0,T]; \mathcal{X}^{-1})} + \|u_2\|_{L^\infty([0,T]; \mathcal{X}^{-1})}) \|w\|_{L^\infty([0,t_0]; \mathcal{X}^{-1})} \]

\[ + C_\varepsilon (\|u_1\|_{L^1([0,T]; \mathcal{X}^{1})} + \|u_2\|_{L^1([0,T]; \mathcal{X}^{1})}) \|w\|_{L^1([0,t_0]; \mathcal{X}^1)}. \]

Choosing \(\varepsilon > 0\) such that

\[ \varepsilon (\|u_1\|_{L^\infty([0,T]; \mathcal{X}^{-1})} + \|u_2\|_{L^\infty([0,T]; \mathcal{X}^{-1})}) \leq \frac{1}{2}, \]

then there exists a positive number \(\delta\) satisfying \(0 < \delta \leq T\) and

\[ C_\varepsilon (\|u_1\|_{L^1([0,\delta]; \mathcal{X}^{1})} + \|u_2\|_{L^1([0,\delta]; \mathcal{X}^{1})}) \leq \frac{\nu}{2}. \]

We then infer that

\[ \|w\|_{L^\infty([0,\delta]; \mathcal{X}^{-1})} + \nu \|w\|_{L^1([0,\delta]; \mathcal{X}^{1})} \]

\[ \leq \varepsilon (\|u_1\|_{L^\infty([0,\delta]; \mathcal{X}^{-1})} + \|u_2\|_{L^\infty([0,\delta]; \mathcal{X}^{-1})}) \|w\|_{L^\infty([0,\delta]; \mathcal{X}^{-1})} \]

\[ + C_\varepsilon (\|u_1\|_{L^1([0,\delta]; \mathcal{X}^{1})} + \|u_2\|_{L^1([0,\delta]; \mathcal{X}^{1})}) \|w\|_{L^1([0,\delta]; \mathcal{X}^1)} \]

\[ \leq \frac{1}{2} (\|w\|_{L^\infty([0,\delta]; \mathcal{X}^{-1})} + \nu \|w\|_{L^1([0,\delta]; \mathcal{X}^{1})}). \]

This implies that \(w(t) = 0, 0 \leq t \leq \delta\). Basic connective argument then yields uniqueness on \([0, T]\).

Theorem \[1.2\] is thus proved up to the blow-up criterion. Assume that we have a solution of the system \((GNS_v)\) on a time interval \([0, T]\) \((T < \infty)\) such that

\[ (3.12) \quad \int_0^T \|w\|_{\mathcal{X}^v}^2 dt < \infty. \]
We claim that the lifespan $T_{u_0}$ of $u$ is greater than $T$. Indeed, thanks to Lemma 3.1, the inequalities (3.4) and (3.12), we have
\[\int_{\mathbb{R}^3} \sup_{0 \leq t \leq T} |\hat{u}(t, \xi)| \xi^{-1} \, d\xi \leq \|u_0\|_{\chi^{-1}} + \|Q(u, u)\|_{L^1([0, T]; \chi^{-1})} \]
\[\leq \|u_0\|_{\chi^{-1}} + C\|u\|_{L^2([0, T]; \chi^0)}^2 \]
\[< \infty.\]
Thus, a positive number $\rho$ exists such that
\[\forall t \in [0, T], \int_{|\xi| \geq \rho} |\xi^{-1}| \hat{u}(t, \xi)(\xi) \, d\xi \leq \frac{\nu}{2\pi C_0}.\]
The condition (3.3) now implies that for any $t \in [0, T]$, the lifespan for a solution of $(GNS_\nu)$ with initial data $u(t)$ is bounded from below by a positive real number $C$ which is independent of $t$. Thus $T_{u_0} > T$, and the whole of Theorem 1.2 is now proved.

As the system $(NS_\nu)$ is a particular case of the system $(GNS_\nu)$, we only need to show that the Navier-Stokes system is well-posed globally in time for $\|u_0\|_{\chi^{-1}} < \nu$. This result has been obtained in [2] by mollifying initial date. Here, however, due to Theorem 1.2, for any $u_0 \in \chi^{-1}$ there already exists a unique local solution $u(x, t)$ on some interval $[0, T^*)$, thus we can directly obtain estimates of the solution $u$ instead of approximate solutions [2].

Taking the Fourier transform of the system $(NS_\nu)$, one has
\[\partial_t \hat{u}(t, \xi) + \nu|\xi|^2 \hat{u}(t, \xi)\]
\[= i \int_{\mathbb{R}^3} \hat{u}(\eta) \otimes \hat{u}(\xi - \eta) \, d\eta \cdot \xi - i \left( \frac{1}{|\xi|^2} \xi \cdot \int_{\mathbb{R}^3} \hat{u}(\eta) \otimes \hat{u}(\xi - \eta) \, d\eta \right) \xi,\]
The condition $\text{div} u = 0$ implies that $\xi \cdot \hat{u} = 0$.

Letting (3.13) $\cdot \hat{u} + \hat{u} \cdot (3.13)$, we get
\[\partial_t |\hat{u}|^2(t, \xi) + 2\nu|\xi|^2 |\hat{u}|^2(t, \xi)\]
\[= i \left[ \xi \cdot \int_{\mathbb{R}^3} \hat{u}(\eta) \otimes \hat{u}(\xi - \eta) \, d\eta \cdot \xi - \hat{u}^T \cdot \int_{\mathbb{R}^3} \hat{u}(\eta) \otimes \hat{u}(\xi - \eta) \, d\eta \cdot \xi \right].\]
For any positive \( \varepsilon \), we have

\[
\partial_t (|\hat{u}|^2 + \varepsilon)^\frac{1}{2} = \frac{\partial_t |\hat{u}|^2}{2(|\hat{u}|^2 + \varepsilon)^\frac{1}{2}}.
\]

Thus, integrating with respect to \( t \) gives

\[
(|\hat{u}|^2(t, \xi) + \varepsilon)^\frac{1}{2} + \nu \int_0^t \frac{|\hat{u}|^2(s, \xi)}{(|\hat{u}|^2(s, \xi) + \varepsilon)^\frac{1}{2}} ds 
\leq (|\hat{u}_0|^2(\xi) + \varepsilon)^\frac{1}{2} + \nu \int_0^t \int_{\mathbb{R}^3} \frac{|\hat{u}(s, \xi, \eta)| |\hat{u}(s, \eta)| |\hat{u}(s, \xi - \eta)|}{(|\hat{u}|^2(s, \xi) + \varepsilon)^\frac{1}{2}} d\eta ds.
\]

Letting \( \varepsilon \) tend to zero gives

\[
|\hat{u}(t, \xi) + \nu \int_0^t |\hat{u}|^2(s, \xi) ds | \leq |\hat{u}_0(\xi)| + \nu \int_0^t \int_{\mathbb{R}^3} |\hat{u}(s, \eta)| |\hat{u}(s, \xi - \eta)| d\eta ds.
\]

Taking the \( L^1 \) norm with respect to \( |\xi|^{-1} d\xi \), we have

\[
\int_{\mathbb{R}^3} |\hat{u}(t, \xi)| |\xi|^{-1} d\xi + \nu \int_0^t \int_{\mathbb{R}^3} |\hat{u}(s, \xi)| d\xi ds 
\leq \int_{\mathbb{R}^3} |\hat{u}_0(\xi)| |\xi|^{-1} d\xi + \|u\|_{L^1([0,t]; L^\infty)}^2 
\leq \|u_0\|_{\chi^{-1}} + \|u\|_{L^\infty([0,t]; \chi^{-1})} \|u\|_{L^1([0,t]; \chi^{-1})}.
\]

If \( \|u_0\|_{\chi^{-1}} < \nu \), then \( \|u(t)\|_{\chi^{-1}} < \nu \) at least for a very short time interval \([0, \delta]\).

Consequently, on such a time interval, we have

\[
\|u(t)\|_{\chi^{-1}} \leq \|u(0)\|_{\chi^{-1}} < \nu.
\]

The basic continuity argument yields that

\[
\|u(t)\|_{\chi^{-1}} \leq \|u(0)\|_{\chi^{-1}} < \nu,
\]

for all \( t \in [0, T^*) \) (see the inequalities (2.5)-(2.6) in [2]).

We then derive that

\[
\|u(t)\|_{\chi^{-1}} + (\nu - \|u(0)\|_{\chi^{-1}}) \int_0^t \|u(s)\|_{\chi^{-1}} ds \leq \|u_0\|_{\chi^{-1}},
\]

for all \( t \in [0, T^*) \).

Thanks to Proposition 2.1, one has

\[
\int_0^{T^*} \|u(t)\|_{\chi^{-1}} dt \leq \frac{\|u_0\|_{\chi^{-1}}^2}{\nu - \|u(0)\|_{\chi^{-1}}} < \infty.
\]
According to Theorem 1.2 again, this implies that \( T^* = \infty \), and the whole Theorem 1.3 is proved.

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**References**

[1] T. Kato and H. Fujita, *On the Navier-Stokes initial value problem I*, Arch. Ration. Mech. Anal., **16** (1964), 269-351.

[2] Z. Lei and F. Lin, *Global mild solutions of Navier-Stokes equations*, Comm. Pure Appl. Math., **64**:9 (2011), 1297C1304.

[3] H. Bahouri, J.-Y. Chemin and R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Grundlehren der Mathematischen Wissenschaften, Vol. 343. New York: Springer-Verlag (2011).

[4] J.-Y. Chemin and I. Gallagher, *Wellposedness and stability results for the Navier-Stokes equations in \( \mathbb{R}^3 \)*, Ann. I. H. Poincaré, **26**:2 (2009), 599C624.

[5] T. Kato, *Nonstationary flows of viscous and ideal fluids in \( \mathbb{R}^3 \)*, J. Funct. Anal., **9** (1972), 296-305.

[6] H. Koch and D. Tataru, *Well-posedness for the Navier-Stokes equations*, Adv. Math., **157**:1 (2001), 22C35.

[7] T. Kato, *Strong \( L^p \) solutions of the Navier-Stokes equations in \( \mathbb{R}^m \), with applications to weak solutions*, Math. Z., **187**:4 (1984), 471C480.

[8] J.-Y. Chemin, Théorèmes d’unicité pour le système de Navier-Stokes tridimensionnel, J. Anal. Math., **77** (1999), 27C50.

[9] M. Cannone, Y. Meyer and F. Planchon, *Solutions autosimilaires des équations de Navier-Stokes*, Sém. Équations aux Dérivées Partielles de l’École Polytechnique, 1993C1994.
[10] I. Gallagher, D. Iftimie and F. Planchon, *Asymptotics and stability for global solutions to the Navier-Stokes equations*, Ann. Inst. Fourier., **53** (2003), 1387–1424.

[11] M. Cannone and G. Wu, *Global well-posedness for Navier-Stokes equations in critical Fourier-Herz spaces*, Nonlinear Anal., **75** (2012), 3754–3760.