$\mathcal{N} = 2$ supersymmetric sigma-models in AdS

Daniel Butter and Sergei M. Kuzenko

School of Physics M013, The University of Western Australia
35 Stirling Highway, Crawley W.A. 6009, Australia
dbutter, kuzenko@cyllene.uwa.edu.au

Abstract

We construct the most general $\mathcal{N} = 2$ supersymmetric nonlinear sigma-model in four-dimensional anti-de Sitter (AdS) space in terms of $\mathcal{N} = 1$ chiral superfields. The target space is shown to be a non-compact hyperkähler manifold restricted to possess a special Killing vector field. A remarkable property of the sigma-model constructed is that the algebra of OSp(2|4) transformations is closed off the mass shell.
1 Introduction

In 1986, Hull et al. [1] formulated, building on the earlier work of Lindström and Roček [2], general four-dimensional $\mathcal{N} = 2$ rigid supersymmetric sigma-models (without superpotentials) in terms of $\mathcal{N} = 1$ chiral superfields. In 2006, the approach of [1] was extended to include superpotentials [3]. The most general $\mathcal{N} = 2$ superconformal sigma-models have been formulated in terms of $\mathcal{N} = 1$ chiral superfields only recently in [5]. The formulation given in [1] is rather geometric, for it makes use of the geometric structures that are intrinsic to the hyperkähler target space.

In this paper, our aim is to construct the most general $\mathcal{N} = 2$ AdS supersymmetric sigma-models in terms of covariantly chiral superfields on $\mathcal{N} = 1$ AdS superspace. Achieving this goal proves to require a more involved analysis than that given in the rigid supersymmetric case [1, 3, 5], simply because the superspace geometry is curved (even if maximally symmetric). We carry out such an analysis, and its outcome turns out to be really rewarding. We prove that the $\mathcal{N} = 2$ AdS supersymmetric sigma-models constructed are off-shell, that is the algebra of the OSp(2|4) transformations closes off the mass shell. Moreover, the target space is shown to be a non-compact hyperkähler manifold restricted to possess a special Killing vector field which rotates the complex structures.

This paper is organized as follows. In section 2, we briefly review the properties of AdS nonlinear sigma-models in $\mathcal{N} = 1$ superspace. Then in section 3 we present the conditions that the $\mathcal{N} = 1$ action must obey in order to possess a second supersymmetry. In section 4, we give the component formulation of this action and the action of $\mathcal{N} = 2$ supersymmetry on its component fields. In section 5, we elaborate

1The reference [3] also considered the lift of these models to 5D $\mathcal{N} = 1$ supersymmetry. The case of 6D $\mathcal{N} = (1, 0)$ supersymmetry was further studied in [4].
2The main virtue of the $\mathcal{N} = 1$ superspace formulations [1, 3, 5] is that one of the two supersymmetries is realized off-shell. The analogous component results appeared earlier. Specifically, the rigid supersymmetric sigma-models with eight supercharges were first constructed in [6]. The construction of [3] was extended to include a superpotential in [7]. General $\mathcal{N} = 2$ rigid superconformal sigma-models were studied in [8, 9].
3General off-shell $\mathcal{N} = 2$ AdS supersymmetric sigma-models have already been formulated in the $\mathcal{N} = 2$ AdS superspace in [10], building on the projective-superspace formulation for $\mathcal{N} = 2$ supergravity-matter systems [11, 12]. Using the off-shell $\mathcal{N} = 2$ sigma-model actions of [10], one can in principle derive their reformulation in terms of $\mathcal{N} = 1$ chiral superfields upon (i) eliminating the (infinitely many) auxiliary superfields; and (ii) performing superspace duality transformations. However, these two technical procedures are quite difficult to implement explicitly in general.
4Historically, the $\mathcal{N} = 1$ AdS superspace, AdS$^{1/4} := \text{OSp}(1|4)/\text{O}(3, 1)$, was introduced in [13, 14], and the superfield approach to OSp(1|4) supersymmetry was developed by Ivanov and Sorin [15].
upon several interesting implications of our results.

2 \( \mathcal{N} = 1 \) nonlinear sigma-models in AdS

Before discussing supersymmetric nonlinear sigma-models, it is worth giving essential information about the \( \mathcal{N} = 1 \) superspace \( \text{AdS}^{4|4} \) (see [16] for more details) which is a maximally symmetric solution of old minimal supergravity with a cosmological term. The corresponding covariant derivatives

\[
\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^{\dot{\alpha}}) = E_A^M \partial_M + \frac{1}{2} \phi_A^{bc} M_{bc},
\]

obey the following (anti-)commutation relations:

\[
\{ \mathcal{D}_\alpha, \mathcal{D}_\beta \} = -4 \bar{\mu} M_{\alpha\beta}, \quad \{ \mathcal{D}_\alpha, \bar{\mathcal{D}}^{\dot{\alpha}} \} = -2i (\sigma^c)_{\alpha\dot{\beta}} \mathcal{D}_c \equiv -2i \mathcal{D}_{\alpha\dot{\beta}}, \quad [\mathcal{D}_\alpha, \mathcal{D}_\beta] = -|\mu|^2 M_{\alpha\beta} ,
\]

(2.2a)

with \( \mu \) a complex non-vanishing parameter which can be viewed as a square root of the curvature of the anti-de Sitter space. The \( \text{OSp}(1|4) \) isometries of \( \text{AdS}^{4|4} \) are generated by Killing vector fields defined as

\[
\Lambda = \lambda^a \mathcal{D}_a + \lambda^\alpha \mathcal{D}_\alpha + \bar{\lambda}^{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}}, \quad [\Lambda + \frac{1}{2} \omega^{bc} M_{bc}, \mathcal{D}_A] = 0 ,
\]

(2.3)

for some Lorentz transformation generated by \( \omega^{bc} \). As shown in [16], the equations in (2.3) are equivalent to

\[
\mathcal{D}_{(\alpha \lambda_{\beta})\dot{\beta}} = 0 , \quad \bar{\mathcal{D}}^{\dot{\beta}} \lambda_{\alpha\dot{\beta}} + 8i \lambda_\alpha = 0 , \quad \mathcal{D}_\alpha \lambda^\alpha = 0 , \quad \bar{\mathcal{D}}_{\dot{\alpha}} \lambda^{\dot{\alpha}} + \frac{i}{2} \mu \lambda_{\dot{\alpha}} = 0 , \quad \omega_{\alpha\beta} = \mathcal{D}_\alpha \lambda_\beta . \quad (2.4a, b, c)
\]

The most general nonlinear sigma-model in \( \mathcal{N} = 1 \) AdS superspace is given by

\[
S = \int d^4x \, d^4\theta \, E \mathcal{K}(\phi^a, \bar{\phi}^{\dot{a}}) ,
\]

(2.5)

where \( E^{-1} = \text{Ber} \left( E_A^M \right) \). The dynamical variables \( \phi^a \) are covariantly chiral superfields, \( \bar{\mathcal{D}}_{\dot{a}} \phi^a = 0 \), and at the same time local complex coordinates of a complex manifold \( \mathcal{M} \). Unlike in the Minkowski case, the action does not possess Kähler invariance since

\[
\int d^4x \, d^4\theta \, E F(\phi^a) = \int d^4x \, d^2\theta \, \mathcal{E} F(\phi^a) \neq 0 ,
\]

(2.6)

\[\text{We follow the notation and conventions adopted in [16], except we use lower case Roman letters for tangent-space vector indices.}\]
with $E$ the chiral density. Nevertheless, Kähler invariance naturally emerges if we represent the Lagrangian as

$$
K(\phi, \bar{\phi}) = K(\phi, \bar{\phi}) + \frac{1}{\mu} W(\phi) + \frac{1}{\bar{\mu}} \bar{W}(\bar{\phi}) ,
$$

(2.7)

for some Kähler potential $K$ and superpotential $W$. Under a Kähler transformation, these transform as

$$
K \rightarrow K + F + \bar{F}, \quad W \rightarrow W - \mu F .
$$

(2.8)

The Kähler metric defined by

$$
g_{ab} := \partial_a \partial_b K = \partial_a \partial_b K
$$

(2.9)

is obviously invariant under (2.8).

The nonlinear sigma-model (2.5) is manifestly invariant under arbitrary $\mathcal{N} = 1$ AdS isometry transformations

$$
\delta_\Lambda \phi^a = \Lambda \phi^a ,
$$

(2.10)

with the operator $\Lambda$ defined by eqs. (2.3) and (2.4).

Because of (2.9), the Lagrangian $K$ in (2.5) should be a globally defined function on the Kähler target space $\mathcal{M}$. This implies that the Kähler two-form, $\Omega = ig_{ab} \, d\phi^a \wedge d\bar{\phi}^b$, associated with (2.9), is exact and hence $\mathcal{M}$ is necessarily non-compact. We see that the sigma-model couplings in AdS are more restrictive than in the Minkowski case. The same conclusion follows from our recent analysis of AdS supercurrent multiplets [17]. In [17] we demonstrated that $\mathcal{N} = 1$ AdS supersymmetry allows the existence of just one minimal $(12 + 12)$ supercurrent, unlike the case of Poincaré supersymmetry admitting three $(12 + 12)$ supercurrents. The corresponding AdS supercurrent is associated with the old minimal supergravity and coincides with the AdS extension of the Ferrara-Zumino multiplet [18]. An immediate application of this result is that all supersymmetric sigma-models in AdS must possess a well-defined Ferrara-Zumino multiplet. The same conclusion also follows from the exactness of $\Omega$ and earlier results of Komargodski and Seiberg [19] who demonstrated that all rigid supersymmetric sigma-models with an exact Kähler two-form possess a well-defined Ferrara-Zumino multiplet. The exactness of $\Omega$ for the general $\mathcal{N} = 1$ sigma-models in AdS has independently been observed in recent publications [20] and [21] which appeared shortly after [17].
We should discuss briefly how the structure (2.5) emerges within a supergravity description. (Our discussion here is similar to that recently given in [20].) Recall that nonlinear sigma-models may be coupled to supergravity via

\[ S = -\frac{3}{\kappa^2} \int d^4x d^4\theta E e^{-\kappa^2 K / 3} + \int d^4x d^2\theta \mathcal{E} W_{\text{sugra}} + \int d^4x d^2\bar{\theta} \bar{\mathcal{E}} \bar{W}_{\text{sugra}} \]  

(2.11)

where the Kähler potential \( K \) and the superpotential \( W_{\text{sugra}} \) transform under Kähler transformations as

\[ K \to K + F + F, \quad W_{\text{sugra}} \to e^{-\kappa^2 F} W_{\text{sugra}}. \]  

(2.12)

The parameter \( \kappa \) corresponds to the inverse Planck mass. To derive an AdS model from a supergravity model, we insert a cosmological term by hand in the superpotential

\[ W_{\text{sugra}} = \frac{\mu}{\kappa^2} + W \]  

(2.13)

and consider the limit of small \( \kappa \). The terms which diverge in such a limit correspond to pure supergravity with a cosmological constant and the supergravity equations of motion may be solved to yield an AdS solution, freezing the supergravity structure. The terms which remain as \( \kappa \) tends to zero can be shown to take the form (2.5) with (2.7). The corresponding limit of (2.12) yields (2.8).

3 \textbf{ \( \mathcal{N} = 2 \) nonlinear sigma-models in AdS}

We now turn to implementing our main goal, that is to look for those restrictions on the target space geometry which guarantee that the theory (2.5) is \( \mathcal{N} = 2 \) supersymmetric.

3.1 \textbf{\( \mathcal{N} = 2 \) supersymmetry transformations}

We make the following ansatz for the action of a second supersymmetry on the chiral superfield \( \phi^a \):

\[ \delta_\epsilon \phi^a = \frac{1}{2} (\bar{D}^2 - 4\mu) (\bar{\epsilon} \Omega^a) \]  

(3.1)

The transformation law (3.1) is a generalization of that derived in [10], using manifestly \( \mathcal{N} = 2 \) supersymmetric techniques, in the case of a free \( \mathcal{N} = 2 \) hypermultiplet \( \phi^a = (\Phi, \Psi) \) for which \( \delta_\epsilon \Phi = \frac{1}{2} (\bar{D}^2 - 4\mu) (\bar{\epsilon} \Psi) \) and \( \delta_\epsilon \Psi = -\frac{1}{2} (\bar{D}^2 - 4\mu) (\bar{\epsilon} \Phi) \). The ansatz (3.1) also has a correct super-Poincaré limit [1, 3].
where $\bar{\Omega}^a$ is a function of $\phi$ and $\bar{\phi}$ which has to be determined. The parameter $\varepsilon$ is real, $\bar{\varepsilon} = \varepsilon$, and constrained to obey \[ (\bar{\mathcal{D}}^2 - 4\mu)\varepsilon = \bar{\mathcal{D}}_a\mathcal{D}_a\varepsilon = 0 \implies \bar{\mathcal{D}}_a\varepsilon = 0 . \] (3.2)

Defining $\varepsilon_a := \mathcal{D}_a\varepsilon$, the second constraint implies that $\varepsilon_a$ is chiral, $\bar{\mathcal{D}}_\dot{a}\varepsilon_a = 0$. The parameter $\varepsilon$ naturally originates within the $N = 2$ AdS superspace approach [10]. The isometries of $\mathcal{N} = 2$ AdS superspace are described by the corresponding Killing vector fields defined in [10]. Upon reduction to $\mathcal{N} = 1$ AdS superspace, any $\mathcal{N} = 2$ Killing vector produces an $\mathcal{N} = 1$ Killing vector $\Lambda$, eq. (2.3), and $\varepsilon$.

The $\theta$-dependent parameter $\varepsilon$, due to the constraints eq. (3.2), contains two components: (i) a bosonic parameter $\xi$ which is defined by $\varepsilon|_{\theta = 0} = \xi|\mu|^{-1}$ and describes the O(2) rotations; and (ii) a fermionic parameter $\varepsilon_a := \mathcal{D}_a\varepsilon|_{\theta = 0}$ along with its conjugate, which generate the second supersymmetry. Schematically, the $\varepsilon$ looks like

$$\varepsilon \sim \frac{\xi}{|\mu|} + \varepsilon^a\theta_a + \bar{\varepsilon}_\dot{a}\bar{\theta}^{\dot{a}} - \xi \left( \frac{\bar{\mu}}{|\mu|} \theta^2 + \frac{\mu}{|\mu|} \bar{\theta}^2 \right) . \quad (3.3)$$

On the mass shell, the right-hand side of (3.1) should transform as a vector field of type $(1,0)$ under reparametrizations of the target space. Due to the constraints (3.2), the transformation $\delta\phi^a$ may be rewritten in the form

$$\delta\phi^a = \bar{\varepsilon}_{\dot{a}}\bar{\mathcal{D}}^{\dot{a}}\bar{\Omega}^a + \frac{1}{2}\varepsilon\bar{\mathcal{D}}^2\bar{\Omega}^a \quad (3.4)$$

which makes clear that $\bar{\Omega}^a$ is defined only up to a holomorphic vector,

$$\bar{\Omega}^a \rightarrow \bar{\Omega}^a + H^a(\phi) . \quad (3.5)$$

### 3.2 Conditions for $\mathcal{N} = 2$ supersymmetry

We turn to discussing the conditions for the sigma-model action (2.5) to be invariant under the $\mathcal{N} = 2$ supersymmetry transformations (3.1) and (3.2).

A large amount of information can be extracted from the following requirement

$$\frac{\delta}{\delta\phi^a} \int d^4x d^4\theta E \left\{ \bar{\mathcal{K}}_b(\bar{\mathcal{D}}^2 - 4\bar{\mu})(\bar{\varepsilon}\bar{\Omega}^b) + \mathcal{K}_d(\mathcal{D}^2 - 4\mu)(\varepsilon\Omega^d) \right\} = 0 \quad (3.6)$$

which must hold if the action is invariant. This requirement is technically easier to analyse than the invariance condition $\delta S = 0$. The technical details of such an
analysis will be reported elsewhere. Here we only present the final results. As in the globally supersymmetric case \[1\], one may introduce a tensor $\omega_{\bar{a}\bar{b}}$ via

$$
\omega_{\bar{a}\bar{b}} := g_{\bar{a}c} \bar{\Omega}^c_{\bar{b}}, \quad \bar{\Omega}^c_{\bar{b}} := \partial_{\bar{b}} \bar{\Omega}^c .
$$

Eq. (3.6) implies that $\omega_{\bar{a}\bar{b}}$ is both a two-form,

$$
\omega_{\bar{a}\bar{b}} = -\omega_{\bar{b}\bar{a}} ,
$$

and covariantly constant,

$$
\nabla_c \omega_{\bar{a}\bar{b}} = 0 , \quad \nabla_{\bar{c}} \omega_{\bar{a}\bar{b}} = 0 ,
$$

and similarly for its complex conjugate $\omega_{ab}$.

These conditions imply that both $\omega_{ab}$ and $\omega^{ab} := g^{a\bar{a}} g^{b\bar{b}} \omega_{\bar{a}\bar{b}}$ are holomorphic, $\omega_{ab} = \omega_{ab}(\phi)$ and $\omega^{ab} = \omega^{ab}(\phi)$.

The conditions (3.7) and (3.8) are exactly the same as in the rigid supersymmetric case \[1\]. There is in addition one extra purely AdS condition that follows from (3.6). We find that the following equation must hold:

$$
\mu \partial_a \left( g_{\bar{a}c} \omega^{cb} K_b \right) + \bar{\mu} \partial_{\bar{a}} \left( g_a \bar{\omega}^{\bar{b}b} K_{\bar{b}} \right) = 0 .
$$

If we define the vector field

$$
V^\mu = (V^a, V^{\bar{a}}) , \quad V^a := \frac{1}{2} \mu |\omega| \omega^{ab} K_b , \quad V^{\bar{a}} := \frac{1}{2} \bar{\mu} |\omega| \omega^{\bar{a}\bar{b}} K_{\bar{b}} , \quad (3.11)
$$

then the above equation may be written as

$$
\nabla_a V_b + \nabla_b V_a = 0 .
$$

(3.12)

In addition, since $\nabla_a V_b = -\bar{\mu} \omega_{ab} / 2|\mu|$, we also have

$$
\nabla_a V_b + \nabla_b V_a = 0 .
$$

(3.13)

Together, these conditions imply that $V = V^a \partial_a + \bar{V}^{\bar{a}} \partial_{\bar{a}}$ is a Killing vector field on the Kähler manifold. By construction it also obeys

$$
V^a \partial_a K = V^{\bar{a}} \partial_{\bar{a}} K = 0 .
$$

(3.14)

This Killing vector turns out to also obey one additional critical property: it acts as a rotation on the three complex structures! One can easily show that the Lie
The derivative of the complex structures are given by

\[ \mathcal{L}_V J_1 = \frac{\text{Im} \mu}{|\mu|} J_3 = \sin \theta J_3 \]  
(3.15a)

\[ \mathcal{L}_V J_2 = -\frac{\text{Re} \mu}{|\mu|} J_3 = -\cos \theta J_3 \]  
(3.15b)

\[ \mathcal{L}_V J_3 = \frac{\text{Re} \mu}{|\mu|} J_2 - \frac{\text{Im} \mu}{|\mu|} J_1 = \cos \theta J_2 - \sin \theta J_1 \]  
(3.15c)

where \( \theta = \text{arg} \mu \). In particular, the specific linear combination \( J_1 \cos \theta + J_2 \sin \theta \) turns out to be invariant under the Lie derivative. This condition is remarkable since it implies that \( V^\mu \) is holomorphic with respect to this specific complex structure\(^8\) but not tri-holomorphic. These features have recently been observed in two papers \[24, 25\] in which supersymmetric nonlinear sigma-models in AdS\(_5\) were formulated first in terms of 4D \( \mathcal{N} = 1 \) chiral superfields \[24\] and then involving component fields \[25\]. As argued in \[24\], the AdS\(_5\) supersymmetry requires the sigma-model target space to be hyperkähler and possess a holomorphic Killing vector field.\(^9\) In that case, the Killing vector field is again holomorphic with respect to just one of the complex structures, but not tri-holomorphic.

The above properties follow solely from the requirement (3.6), without a direct analysis of the invariance condition \( \delta S = 0 \). However, taking into account the properties (3.8), (3.9) and (3.12), (3.13) it can be shown that the action is indeed invariant. We shall describe the derivation in a separate publication.

As a simple example, consider the \( \mathcal{N} = 2 \) linear sigma-model \[10\]

\[
S = \int d^4x \, d^4\theta \, E \left( \Phi \Phi + \bar{\Psi} \Psi - i \frac{\bar{\mu}}{|\mu|} \left( 1 + \frac{m}{|\mu|} \right) \Psi \Phi + i \frac{\mu}{|\mu|} \left( 1 + \frac{m}{|\mu|} \right) \bar{\Psi} \bar{\Phi} \right),
\]  
(3.16)

with \( \phi^a = (\Phi, \Psi) \) covariantly chiral superfields, and \( m \) a mass parameter (the choice \( m = -|\mu| \) corresponds to the superconformal massless hypermultiplet). Using the explicit expression for the holomorphic two-form

\[
\omega_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]  
(3.17)

it is easy to check that the vector field (3.11) given by \( V_a = (V_\phi, V_\psi) \) and \( \bar{V}_a = (\bar{V}_\phi, \bar{V}_\psi) \)

\(^8\)In other words, if one were to work in coordinates where \( J_1 \cos \theta + J_2 \sin \theta \) is diagonalized to \( \text{diag}(i \mathbb{I}_n, -i \mathbb{I}_n) \), then \( V^\mu \) would be holomorphic in the usual sense.

\(^9\)The Killing vector turns out to be holomorphic due to a certain embedding of the hypermultiplets into 4D \( \mathcal{N} = 1 \) chiral superfields.
with
\[ V_{\phi} = \frac{1}{2} \frac{\bar{\mu}}{|\mu|} \Psi + i \left( 1 + \frac{m}{|\mu|} \right) \Phi, \quad V_{\psi} = -\frac{1}{2} \frac{\bar{\mu}}{|\mu|} \Phi - i \left( 1 + \frac{m}{|\mu|} \right) \bar{\Psi} \] (3.18a)
\[ V_{\bar{\phi}} = \frac{1}{2} \frac{\mu}{|\mu|} \bar{\Psi} - i \left( 1 + \frac{m}{|\mu|} \right) \Phi, \quad V_{\bar{\psi}} = -\frac{1}{2} \frac{\mu}{|\mu|} \bar{\Phi} + i \left( 1 + \frac{m}{|\mu|} \right) \Psi \] (3.18b)
indeed obeys (3.12) and (3.13).

It should be remarked that, modulo transformations (3.5), we can choose
\[ \bar{\Omega}^a(\phi, \bar{\phi}) = \omega^{ab}(\phi) K_b(\phi, \bar{\phi}), \] (3.19)
similarly to the super-Poincaré case \[1\]. The specific feature of the AdS case is that \( K_b \) is a one-form, and thus \( \bar{\Omega}^a \) is necessarily a vector field. Comparing the expression for \( \bar{\Omega}^a \) with (3.11) shows that \( \bar{\Omega}^a \propto V^a \).

### 3.3 Closure of the supersymmetry algebra

Let us calculate the commutator of two second supersymmetry transformations (3.1). This calculation is rather short and the result is
\[ [\delta_2, \delta_1] \phi^a = -\omega^{ac} \omega_{cb} \left( -\frac{1}{2} \tilde{\lambda}^{\alpha \dot{\alpha}} D_{\alpha \dot{\alpha}} + \tilde{\lambda}^{\alpha} D_{\alpha} \right) \phi^b, \] (3.20)
where
\[ \tilde{\lambda}^{\alpha \dot{\alpha}} := 4i (\varepsilon_2^{a} \varepsilon_1^{\dot{a}} - \varepsilon_2^{\dot{a}} \varepsilon_1^{a}), \quad \tilde{\lambda}^{\alpha} := 2\mu (\varepsilon_1^{a} \varepsilon_2 - \varepsilon_1^{\dot{a}} \varepsilon_2^{\dot{a}}) \] (3.21)
are the components of the first-order operator \( \Lambda_{[\varepsilon_2, \varepsilon_1]} = -\frac{1}{2} \tilde{\lambda}^{\alpha \dot{\alpha}} D_{\alpha \dot{\alpha}} + \tilde{\lambda}^{\alpha} D_{\alpha} + \tilde{\bar{\lambda}}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}} \) which proves to be an AdS Killing vector field, see eqs. (2.3) and (2.4). If we impose
\[ \omega^{ac} \omega_{cb} = -\delta^a_b, \] (3.22)
then the above result turns into
\[ [\delta_2, \delta_1] \phi^a = \Lambda_{[\varepsilon_2, \varepsilon_1]} \phi^a. \] (3.23)

We see from (3.23) that the commutator \([\delta_2, \delta_1] \phi^a \) closes off the mass shell. This is similar to the supersymmetry structure within the Bagger-Xiong formulation \[3\] for \( \mathcal{N} = 2 \) rigid supersymmetric sigma-models. However, in the case of flat superspace, the commutator of the first and the second supersymmetries closes only on-shell.
\[ [\delta_\Lambda, \delta_\epsilon] \phi^a = -\frac{1}{2}(\bar{D}^2 - 4\mu)\big((\Lambda \epsilon)\bar{\Omega}^a\big). \quad (3.24) \]

Since \( \Lambda \) is an \( \mathcal{N} = 1 \) Killing vector field, the parameter \( \epsilon' = \Lambda \epsilon \) obeys the constraints \( (3.2) \) and hence generates a second supersymmetry transformation. We observe that commuting the \( \mathcal{N} = 1 \) AdS transformation and the second supersymmetry gives a second supersymmetry transformation,

\[ [\delta_\Lambda, \delta_\epsilon] \phi^a = -\delta_\Lambda \epsilon \phi^a. \quad (3.25) \]

As a result, the algebra of OSp(2\mid4) transformations is closed off the mass shell\(^{10}\).

Let us return to the equation (3.22). Its implications are the same as in the super-Poincaré case \([1]\). In addition to the canonical complex structure

\[ J_3 = \begin{pmatrix} i \delta_{ab} & 0 \\ 0 & -i \delta_{\bar{a}\bar{b}} \end{pmatrix}, \quad (3.26) \]

we may construct two more using \( \omega^a_{\bar{b}} \)

\[ J_1 = \begin{pmatrix} 0 & \omega^a_{\bar{b}} \\ \omega^a_{\bar{b}} & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & i \omega^a_{\bar{b}} \\ -i \omega^a_{\bar{b}} & 0 \end{pmatrix} \quad (3.27) \]

such that \( \mathcal{M} \) is Kähler with respect to each of them. The operators \( J_A = (J_1, J_2, J_3) \) obey the quaternionic algebra

\[ J_A J_B = -\delta_{AB}\mathbb{I} + \epsilon_{ABC}J_C. \quad (3.28) \]

Thus, \( \mathcal{M} \) is a hyperkähler manifold. In accordance with the discussion in section 2, this manifold is non-compact. The above analysis also shows that \( \mathcal{M} \) must possess a special Killing vector.

Using (3.22), it is easy to establish the equivalence

\[ (\bar{D}^2 - 4\mu)K_a = 0 \iff (\bar{D}^2 - 4\mu)(\omega^{ab}K_b) = 0. \quad (3.29) \]

This result implies that the following rigid symmetry of the \( \mathcal{N} = 2 \) sigma-model

\[ \delta \phi^a = \zeta(\bar{D}^2 - 4\mu)(\omega^{ab}K_b), \quad \zeta \in \mathbb{C} \quad (3.30) \]

\(^{10}\)It should be mentioned that the linearized action for all massless supermultiplets of arbitrary superspin in \( \mathcal{N} = 1 \) AdS superspace \([22]\) is also invariant under \( \mathcal{N} = 2 \) supersymmetry transformations which close off-shell.
is trivial.

It is well-known that when \( \mathcal{N} = 2 \) sigma-models are coupled to supergravity, their target spaces must be quaternionic Kähler manifolds [26]. Unlike the hyperkähler spaces which are Ricci-flat, their quaternionic Kähler cousins are Einstein spaces with a non-zero constant scalar curvature (see, e.g., [27] for a review). Since AdS is a curved geometry, one may wonder whether the target spaces of \( \mathcal{N} = 2 \) sigma-models in AdS should also be quaternionic Kähler. Yet we have shown here that within AdS, the geometry is hyperkähler just as in Minkowski space. The reason is simple. As shown in [26], the scalar curvature in the target space of locally supersymmetric sigma-models must be nonzero and proportional to \( \kappa^2 \),

\[
R = -8\kappa^2(n^2 + 2n) ,
\]

where the real dimension of the target space is \( 4n \). But AdS (or Minkowski) space can be interpreted as the \( \kappa^2 \to 0 \) limit of supergravity with (or without) a cosmological constant \( \mu \). In such a limit, we find indeed that the quaternionic Kähler requirement from supergravity reduces to a hyperkähler requirement.

### 3.4 \( \mathcal{N} = 2 \) superconformal sigma-models

Both Minkowski and AdS \( \mathcal{N} = 2 \) superspaces have the same superconformal group \( \text{SU}(2, 2|2) \). Thus all \( \mathcal{N} = 2 \) rigid superconformal sigma-models should be invariant under the \( \mathcal{N} = 2 \) AdS supergroup \( \text{OSp}(2|4) \). Here we elaborate on this point.

Target spaces for \( \mathcal{N} = 2 \) superconformal sigma-models are hyperkähler cones (see [28] and references therein). A hyperkähler cone is a hyperkähler manifold possessing a homothetic conformal Killing vector field. Let us recall the salient facts about homothetic conformal Killing vector fields (see [28, 29] for more details). By definition, a homothetic conformal Killing vector field \( \chi \) on a Kähler manifold \( (\mathcal{M}, g_{\bar{a}\bar{b}}) \),

\[
\chi = \chi^a \frac{\partial}{\partial \phi^a} + \bar{\chi}_{\bar{a}} \frac{\partial}{\partial \bar{\phi}^{\bar{a}}} \equiv \chi^\mu \frac{\partial}{\partial \phi^\mu} ,
\]

obeys the constraint

\[
\nabla_\nu \chi^\mu = \delta^\mu_\nu \quad \iff \quad \nabla_b \chi^a = \delta_b^a , \quad \nabla_{\bar{b}} \chi^{\bar{a}} = \partial_{\bar{b}} \chi^{\bar{a}} = 0 .
\]

In particular, \( \chi \) is holomorphic. Its properties include:

\[
g_{\bar{a}\bar{b}} \chi^\bar{a} \bar{\chi}^{\bar{b}} = \mathcal{K} , \quad \chi_a := g_{\bar{a}\bar{b}} \bar{\chi}^{\bar{b}} = \partial_a \mathcal{K} ,
\]
with $\mathcal{K}$ the Kähler potential. If $\mathcal{N} = 2$ superconformal sigma-models are realized in $\mathcal{N} = 1$ Minkowski superspace, the second supersymmetry is given in terms of $\chi$ [5].

We have to show that the above properties of $\chi$ imply the existence of a Killing vector field

$$V^\mu = (V^a, V^\bar{a}) = \frac{1}{2|\mu|} \left( \mu \omega^{ab} K_b, \bar{\mu} \omega^{\bar{a}\bar{b}} \bar{K}_b \right) = \frac{1}{2|\mu|} \left( \mu \omega^{ab} \chi_b, \bar{\mu} \omega^{\bar{a}\bar{b}} \bar{\chi}_b \right),$$

for any non-zero complex parameter $\mu$. By representing $2|\mu| V^a = \mu \omega_{ab} \chi^b$ and using the facts that $\omega_{ab}$ and $\chi^b$ are holomorphic, the condition (3.12) follows. The other condition, eq. (3.13), holds automatically.

It is instructive to give a slightly different proof that (3.35) is a Killing vector, which shows that $V^\mu$ belongs to the Lie algebra of the group SU(2) isometrically acting on the hyperkähler cone. As shown e.g. in [28, 29], associated with the complex structures $(J^A)_{\mu\nu}$, eqs. (3.26) and (3.27), are the three Killing vectors $X^A_\mu := (J^A)_\mu^\nu \chi^\nu$ which span the Lie algebra of SU(2). In particular, we have that $(J_1)^\mu_\nu \chi^\nu = (\omega^{ab} K_b, \omega^{\bar{a}\bar{b}} \bar{K}_b)$ and $(J_2)^\mu_\nu \chi^\nu = (i \omega^{ab} K_b, -i \omega^{\bar{a}\bar{b}} \bar{K}_b)$ are Killing vectors. The Killing vector (3.35) is simply a real combination of $(J_1)^\mu_\nu \chi^\nu$ and $(J_2)^\mu_\nu \chi^\nu$, and thus $V^\mu$ belongs the Lie algebra of SU(2).

4 $\mathcal{N} = 2$ nonlinear sigma-models in components

We turn now to the component description of $\mathcal{N} = 2$ non-linear sigma-models in AdS. The evaluation of the superspace action is straightforward, and makes use of the $\mathcal{N} = 1$ AdS reduction rule (see e.g. [30] or standard texts on $\mathcal{N} = 1$ supergravity [16, 31]) equivalent to

$$S = \int d^4 x d^4 \theta \ E \mathcal{K}$$

$$= \int d^4 x e \left\{ \frac{1}{16} \mathcal{D}^a (\mathcal{D}^2 - 4 \mu) \mathcal{D}_a \mathcal{K} - \frac{1}{4} \bar{\mu} \mathcal{D}^2 \mathcal{K} - \frac{1}{4} \mu \mathcal{D}^2 \mathcal{K} + 3 \mu \bar{\mu} \mathcal{K} \right\}$$

(4.1)

where $E^{-1} = \text{Ber}(E_A^M)$ and $e = \det(e^{a}_m)$. This form of the AdS reduction rule makes clear that $\mu$-dependent terms are the only obstruction to Kähler invariance. The first term yields the Kähler invariant kinetic and four-fermion terms while the others provide a $\mu$-dependent potential for the scalar fields and masses for the fermions. In
components, one finds

\[
S = \int d^4x \left\{ -\mathcal{D}_m \varphi^a g_{ab} \mathcal{D}^m \varphi^b - i\chi^a g_{ab} \nabla_{a\dot{a}} \bar{\chi}^{\dot{a}b} + \hat{F}^a g_{ab} \bar{\bar{F}}^b + \frac{1}{4} (\chi^a \chi^b)(\bar{\chi}^{\dot{a}} \bar{\chi}^{\dot{b}}) R_{a\dot{a}b\dot{b}} \\
- \frac{\mu}{2} (\chi^a \chi^b) \nabla_a K_b - \frac{\bar{\mu}}{2} (\bar{\chi}^{\dot{a}} \bar{\chi}^{\dot{b}}) \nabla_a K_b + \mu \bar{\hat{F}}^a K_a + \bar{\mu} \hat{F}^a K_a + 3\mu \bar{\mu} K \right\} .
\]

We have defined components in the conventional way

\[
\varphi^a := \phi^a |, \quad \chi^b := \frac{1}{\sqrt{2}} \mathcal{D}_a \phi^b |, \quad F^a := -\frac{1}{4} \mathcal{D}^2 \phi^a |
\]

and have made use of the quantity

\[
\hat{F}^a := F^a - \frac{1}{2} \Gamma^a_{bc} \chi^b \chi^c
\]

which transforms covariantly under reparametrizations. The component AdS derivative is given by the \( \theta \)-independent piece of the superspace vector derivative,

\[
\mathcal{D}_m := e_m^a \mathcal{D}_a |
\]

The reparametrization-covariant derivative acts on the fermions, for example, as

\[
\nabla_{a\dot{a}} \bar{\chi}^{\dot{a}b} := \mathcal{D}_{a\dot{a}} \bar{\chi}^{\dot{a}b} + \Gamma^b_{\dot{c}d} \mathcal{D}_{a\dot{a}} \bar{\phi}^\dot{c} \bar{\chi}^{\dot{d}b} ;
\]

their masses are given by reparametrization-covariant field derivatives of \( K \)

\[
\nabla_a K_b := \partial_a K_b - \Gamma^c_{ab} K_c , \quad K_a := \nabla_a K = \partial_a K .
\]

This action is invariant under the \( \mathcal{N} = 2 \) supersymmetry transformations

\[
\delta \varphi^a = \sqrt{2} \left( \lambda \chi^a + \omega^b_{\dot{b}} \bar{\epsilon} \chi^{\dot{b}} \right) \quad (4.8a)
\]

\[
\delta \chi^a + \Gamma^a_{bc} \delta \varphi^b \chi^c = \sqrt{2} \left( \lambda \hat{F}^a - \epsilon_{a} \omega^b_{\dot{b}} \bar{\bar{F}}^{\dot{b}} \right) + i\sqrt{2} \left( \bar{\lambda} \mathcal{D}_{a\dot{a}} \varphi^a - \bar{\epsilon} \omega^b_{\dot{b}} \mathcal{D}_{a\dot{a}} \varphi^{\dot{b}} \right) \quad (4.8b)
\]

\[
\delta \hat{F}^a + \Gamma^a_{bc} \delta \varphi^b \hat{F}^c = -\mu \sqrt{2} (\lambda \chi^a + \omega^b_{\dot{b}} \bar{\epsilon} \chi^{\dot{b}}) + i\sqrt{2} \left( \bar{\lambda} \nabla_{a\dot{a}} \chi^a + \omega^b_{\dot{b}} \epsilon_{a} \nabla_{a\dot{a}} \bar{\chi}^{\dot{a}b} \right) + \frac{1}{\sqrt{2}} \left( R_{ce\dot{a}b} \lambda \bar{\chi}^{\dot{c}} \chi^{\dot{b}} - \omega^b_{\dot{b}} R_{ce\dot{d}d} \epsilon_{a} \bar{\chi}^{\dot{d}c} \bar{\chi}^{\dot{a}b} \right) \quad (4.8c)
\]

where the spinor supersymmetry parameters \( \lambda_{a} \) and \( \epsilon_{a} \) obey the AdS Killing spinor equations

\[
\mathcal{D}_{a(\dot{a} \lambda_{\dot{\beta}})} = 0 , \quad \mathcal{D}_{a\dot{a}} \bar{\lambda}^{\dot{a}} = 2i\bar{\mu} \lambda_{a} , \quad (4.9a)
\]

\[
\mathcal{D}_{a(\dot{a} \epsilon_{\dot{\beta}})} = 0 , \quad \mathcal{D}_{a\dot{a}} \epsilon^{\dot{a}} = 2i\mu \epsilon_{a} . \quad (4.9b)
\]
In addition, the O(2) rotation of AdS acts on the fields as

\[
\delta \varphi^a = -2 \frac{\xi}{|\mu|} \omega^a_b \tilde{F}^b
\]

(4.10a)

\[
\delta \chi^a_{\alpha} + \Gamma^a_{bc} \delta \varphi^b \chi^c_{\alpha} = -2i \frac{\xi}{|\mu|} \omega^a_b \nabla_{a\dot{a}} \bar{\chi}^{\dot{a}b} + \frac{\xi}{|\mu|} \omega_{ab} R_{b\bar{c}\bar{c}} \bar{\chi}^{\bar{b}c} \chi^c_{\alpha}
\]

(4.10b)

\[
\delta \hat{F}^a + \Gamma^a_{bc} \delta \varphi^b \hat{F}^c = 6\mu \frac{\xi}{|\mu|} \omega^a_b \bar{\varphi}^b + 2i \frac{\xi}{|\mu|} \omega_{ab} D^{ab} \bar{\varphi}^c + 2i \frac{\xi}{|\mu|} \omega_{ab} R_{b\bar{c}\bar{d}} \chi^a_{\alpha} \chi^d_{\dot{\alpha}} D^{\dot{\alpha}a} \bar{\varphi}^c
\]

\[
+ \frac{\xi}{|\mu|} R^{a}_{b\bar{c}\bar{d}} \left( \omega^d_{\bar{c}} \chi^b_{\alpha} - \omega^c_{\bar{d}} \bar{\chi}^{\bar{b}c} \right) \hat{F}^b
\]

\[
- \frac{1}{2} \frac{\xi}{|\mu|} \omega^{ab} \nabla_c R_{b\bar{c}\bar{d}} \chi^d_{\alpha} \chi^e_{\dot{\alpha}} \bar{\chi}^{\bar{d}}
\]

(4.10c)

The combination of supersymmetry and O(2) transformations closes off-shell.

Integrating out the auxiliary field gives \( \hat{F}^b = -\bar{\mu} g^{bb} K_b \) and the action becomes

\[
S = \int d^4x \left\{ -D_m \varphi^a g_{ab} D^m \varphi_b - i \chi^a_{\alpha} g_{ab} \nabla_{a\dot{a}} \bar{\chi}^{\dot{a}b} + \frac{1}{4} (\chi^a_{\alpha} \chi^b_{\dot{\alpha}}) (\bar{\chi}^{\dot{a}b} \bar{\chi}^{\alpha \beta}) R_{a\dot{b}\dot{b}} - \frac{\mu}{2} (\chi^a_{\alpha} \chi^b_{\dot{\alpha}}) \nabla_a K_b - \frac{\bar{\mu}}{2} (\bar{\chi}^{\dot{a}b} \bar{\chi}^{\alpha \beta}) \nabla_a K_b - \mu \bar{\mu} g^{ab} K_a K_b + 3\mu \bar{\mu} \mathcal{K} \right\}.
\]

(4.11)

The second line can be rewritten in terms of the Killing vector \( V^a := \frac{\mu}{2|\mu|} \omega^{ab} K_b \) as

\[
S = \int d^4x \left\{ -D_m \varphi^a g_{ab} D^m \varphi_b - i \chi^a_{\alpha} g_{ab} \nabla_{a\dot{a}} \bar{\chi}^{\dot{a}b} + \frac{1}{4} (\chi^a_{\alpha} \chi^b_{\dot{\alpha}}) (\bar{\chi}^{\dot{a}b} \bar{\chi}^{\alpha \beta}) R_{a\dot{b}\dot{b}} + |\mu| (\chi^a_{\alpha} \chi^b_{\dot{\alpha}}) \omega_{\bar{c}} \nabla_a V^c + |\mu| (\bar{\chi}^{\dot{a}b} \bar{\chi}^{\alpha \beta}) \omega_{\bar{c}} \nabla_a V^c - 4\mu \bar{\mu} g_{ab} V^a V^b + 3\mu \bar{\mu} \mathcal{K} \right\}.
\]

(4.12)

Because \( \mathcal{K} \) appears explicitly in the potential, it must be a globally-defined function (up to at most a constant shift).

Using the equations of motion, the supersymmetry and O(2) transformations may be written entirely in terms of geometric quantities,

\[
\delta \varphi^a = \sqrt{2} \left( \lambda \chi^a + \omega^a_{\bar{b}} \bar{\chi}^{\bar{b}} \right) + 4\xi V^a
\]

(4.13a)

\[
\delta \chi^a_{\alpha} + \Gamma^a_{bc} \delta \varphi^b \chi^c_{\alpha} = i\sqrt{2} \left( \bar{\lambda} \bar{D}_{a\dot{a}} \varphi^a - \omega^a_{\bar{b}} \epsilon^{\dot{a}b} \bar{D}_{a\dot{a}} \varphi^b \right)
\]

\[
- 2\sqrt{2} |\mu| \left( \lambda \omega^a_{\bar{b}} \bar{V}^b - \epsilon^a \bar{V}^a \right) + 4\xi \chi^a_{\alpha} \nabla_b V^a.
\]

(4.13b)
5 Discussion

In this paper, we have constructed the most general $\mathcal{N} = 2$ supersymmetric nonlinear sigma-model in AdS in terms of $\mathcal{N} = 1$ chiral superfields. As in the rigid supersymmetric case, the target space of the sigma-model must be hyperkähler. However, the AdS supersymmetry imposes some additional geometric restrictions. The hyperkähler target space $\mathcal{M}$ must be such that (i) the Kähler two-form $\Omega = ig_{a\bar{b}}d\phi^a \wedge d\bar{\phi}^b$, which is associated with the complex structure $J_3$ used in the $\mathcal{N} = 1$ superspace formulation, is exact (and hence the target space is non-compact); (ii) $\mathcal{M}$ possesses a Killing vector defined by (3.11) which rotates the three complex structures, eq. (3.15). It should be pointed out that the exactness of $\Omega$ is a general feature of $\mathcal{N} = 1$ supersymmetric sigma-models in AdS, as demonstrated earlier in [20] and [21].

The condition that $\mathcal{M}$ must possess a certain Killing vector has in fact a simple physical explanation. As compared with the $\mathcal{N} = 2$ super-Poincaré group, its AdS counterpart OSp(2|4) includes an additional one-parameter symmetry which is the group of O(2) rotations. Invariance under this symmetry proves to require the existence of a Killing vector in the target space.

A natural question to ask is whether a given hyperkähler manifold with the properties described can be the target space of an $\mathcal{N} = 2$ sigma-model in AdS. Recall that if a hyperkähler manifold possesses a Killing vector $V^\mu$ holomorphic with respect to a complex structure, say $J_1$, then one can easily show that

$$V^\mu = \frac{1}{2} J_1^{\mu\nu} \nabla^\nu \mathcal{K}$$

(5.1)

for a real Killing potential $\mathcal{K}$. However, if in addition we make the assumption that $V^\mu$ rotates the other two complex structures, i.e.

$$\mathcal{L}_V J_1 = 0 \ , \quad \mathcal{L}_V J_2 = -J_3 \ , \quad \mathcal{L}_V J_3 = +J_1$$

(5.2)

then it is a simple exercise to show that

$$g_{\mu\nu} = \frac{1}{2}(\delta_\mu^\rho \delta_\nu^\sigma + J_3^{\mu\rho} J_3^{\nu\sigma}) \nabla_\rho \nabla_\sigma \mathcal{K}$$

(5.3)

or equivalently (in complex coordinates where $J_3$ is diagonalized)

$$g_{ab} = 0 \ , \quad g_{ab} = \partial_a \partial_b \mathcal{K} \ .$$

(5.4)

In the basis where $J_3 = \text{diag}(i \mathbb{1}_n, -i \mathbb{1}_n)$, this reduces to the usual definition of a Killing potential ([32], aside from an additional numerical factor).
In other words, the function $\mathcal{K}$ is not only the Killing potential with respect to $J_1$, but also the Kähler potential with respect to $J_3$. In fact, it is the Kähler potential with respect to any complex structure orthogonal to $J_1$.

We are thus led to the following simple prescription for generating an $\mathcal{N} = 2$ nonlinear sigma-model in AdS from a given hyperkähler manifold. If the hyperkähler manifold admits some Killing vector $V^\mu$ which rotates the complex structures (necessarily leaving one of them invariant) then one constructs a Killing potential $\mathcal{K}$ with respect to the invariant complex structure. The resulting function is the Kähler potential and, indeed, the superfield Lagrangian when written in the basis where one of the orthogonal complex structures is diagonalized. For this prescription to be consistent, the Kähler form associated with the diagonalized complex structure must be exact (hence the hyperkähler manifold must be non-compact) and the function $\mathcal{K}$ must be globally defined.

It is quite intriguing that some of the properties we have discussed have recently been independently discovered in the context of supersymmetric nonlinear sigma-models in AdS$_5$ [24, 25]. In particular, a Killing vector $V^\mu$ appears which rotates the complex structures while leaving one of them invariant. However, in these models, it is the invariant complex structure which is diagonalized, and so $V^\mu$ is holomorphic in the usual sense. This undoubtedly is related to the fact that in these models the five-dimensional space is foliated with flat four-dimensional subspaces. More precisely, the Killing vector turns out to be holomorphic due to a certain imbedding of the hypermultiplets into 4D $\mathcal{N} = 1$ chiral superfields.

The remarkable feature of our construction is that the $\mathcal{N} = 2$ supersymmetry algebra closes off the mass shell, for the most general $\mathcal{N} = 2$ supersymmetric nonlinear sigma-model in AdS realized in terms of $\mathcal{N} = 1$ chiral superfields. This is a new type of structure that has no analogue in Minkowski space. Indeed, in order to have off-shell supersymmetry for general $\mathcal{N} = 2$ nonlinear sigma-models in Minkowski space, one has to use the harmonic [33, 34] or the projective [35, 36] superspace approaches in which the off-shell hypermultiplet realizations involve an infinite number of auxiliary fields. In our construction, the hypermultiplet is described using a minimal realization of two ordinary $\mathcal{N} = 1$ chiral superfields with 8 + 8 degrees of freedom. One may wonder why the structure of supersymmetry transformations in AdS differs so drastically from that in Minkowski space. The origin of this difference can

\[ \text{[24]} \text{, which appeared on the preprint arXiv shortly after the first version of this paper, was the first to note that the Killing vector in AdS}_5 \text{ was holomorphic while [25] later noted that it rotated the complex structures. Subsequently we discovered the same features in AdS}_4 \text{, where they are more hidden.} \]
be traced back to the explicit form of the AdS superfield parameter (3.3). One can see that the leading component of \( \varepsilon \) is not analytic in the cosmological constant \( |\mu| \), which is similar to the well-known non-analyticity of the cubic interaction of massless higher spin fields in AdS \[37\]. Thus the parameter \( \varepsilon \) does not admit a smooth limit to Minkowski space. On the other hand, from the work of \[1, 3, 5\] it is known that in the case of \( \mathcal{N} = 2 \) nonlinear sigma-models in Minkowski space one has to deal with a superfield parameter of the form

\[
\varepsilon = \tau + \epsilon^\alpha \theta_\alpha + \bar{\epsilon}_\dot{\alpha} \bar{\theta}^{\dot{\alpha}}, \quad \tau = \text{const}, \quad \epsilon^\alpha = \text{const}.
\]

(5.5)

Here the bosonic parameter \( \tau \) generates a central charge transformation which can be shown to be a trivial symmetry (i.e. it coincides with the identity transformation on-shell). This transmutation of the physical \( O(2) \) symmetry, which is generated by the parameter \( \xi \) in (3.3), into a trivial \( \tau \)-symmetry is another manifestation of non-analyticity in the cosmological constant.

Off-shell supersymmetry is also characteristic of the gauge models for massless higher spin \( \mathcal{N} = 2 \) supermultiplets in AdS constructed in \[22\] using \( \mathcal{N} = 1 \) superfields. Since those theories are linearized, one may argue that their off-shell supersymmetry is not really impressive. However, now we have demonstrated that the formulation of the most general nonlinear \( \mathcal{N} = 2 \) supersymmetric sigma-models in terms of \( \mathcal{N} = 1 \) chiral superfields is also off-shell. This gives us some evidence to believe that, say, general \( \mathcal{N} = 2 \) super Yang-Mills theories in AdS possess an off-shell formulation in which the hypermultiplet is realized in terms of two chiral superfields. If this conjecture is correct, there may be nontrivial implications for quantum effective actions.

The off-shell structure of our \( \mathcal{N} = 2 \) nonlinear sigma-models in AdS implies that there should exist a manifestly \( \mathcal{N} = 2 \) supersymmetric formulation in AdS with the same finite set of auxiliary fields we have found. It would be of interest to develop such a formulation.

**Acknowledgements:**

This work is supported in part by the Australian Research Council and by a UWA Research Development Award. D.B. would like to thank Jon Bagger for correspondence.

**Note added in proof:**

After this article was accepted for publication, the authors learned that hyperkähler geometries possessing Killing vectors which rotate the complex structures were considered previously by Hitchin et al. \[38\].
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