A geometrical method towards first integrals for dynamical systems

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Abstract

We develop a method, based on Darboux’ and Liouville’s works, to find first integrals and/or invariant manifolds for a physically relevant class of dynamical systems, without making any assumption on these elements’ form. We apply it to three dynamical systems: Lotka–Volterra, Lorenz and Rikitake.

I. HISTORICAL OVERVIEW.

In 1,2, Roger Liouville and A. Tresse developed a method for deciding whether two differential equations of the form

$$\frac{d^2y}{dx^2} + a_1(x, y) \left( \frac{dy}{dx} \right)^3 + 3 a_2(x, y) \left( \frac{dy}{dx} \right)^2 + 3 a_3(x, y) \frac{dy}{dx} + a_4(x, y) = 0 \quad (1)$$

where the $a_i$ are arbitrary functions of the real or complex variables $x$ and $y$, are geometrically equivalent, i.e. can be transformed into each other by the most general dependent and independent variable change

$$x' = \varphi(x, y), \quad y' = \psi(x, y) \quad (2)$$

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First integrals from geometrical equivalence

This method was based on the construction of a “relative invariant” function called $\nu_5$ of the $a_i$ and of their derivatives, such that in any transformation (2) it becomes $\nu'_5 = J(x, y)^{-5} \nu_5$ where $J(x, y)$ is the Jacobian of the transformation. In the general case, two equations such that their $\nu_5$ are non-zero and proportional to each other are indeed equivalent. If $\nu_5 = 0$ for both, however, one cannot conclude at first, and other invariants, involving higher derivatives of the $a_i$, must be calculated in order to decide. As an application, Liouville proposed the effective reduction of Equation (1) into its simplest canonical form, which in most cases leads to an explicit integration.

Here we will adopt another point of view. We have derived from these theories a method for finding out first integrals for a wide and physically important class of dynamical systems without having to make any ansatz on their functional form. In the rest of this section, we shall recall some mathematical results of Darboux, Liouville and Tresse. Then we explain our method in Section II, and apply it in Section III to three well-known dynamical systems. Finally, Section IV discusses our results summarised in Table I.

A. Essentials of Liouville theory.

Consider a differential equation like (1). Liouville defined the following functions — which are seen as functions of $(x, y)$, forgetting the supposed relation between those variables.

\[
L_2 = \frac{\partial}{\partial x} \left( \frac{\partial a_1}{\partial x} - 3a_1a_3 \right) + \frac{\partial}{\partial y} \left( \frac{\partial a_3}{\partial y} - 2 \frac{\partial a_2}{\partial x} + a_1a_4 \right) - 3a_2 \left( \frac{\partial a_3}{\partial y} - 2 \frac{\partial a_2}{\partial x} + a_1a_4 \right) + a_1 \left( \frac{\partial a_4}{\partial y} + 3a_2a_4 \right) \tag{3}
\]

\[
L_1 = \frac{\partial}{\partial y} \left( \frac{\partial a_4}{\partial y} + 3a_2a_4 \right) - \frac{\partial}{\partial x} \left( 2 \frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial x} + a_1a_4 \right) - 3a_3 \left( 2 \frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial x} + a_1a_4 \right) - a_4 \left( \frac{\partial a_1}{\partial x} - 3a_1a_3 \right) \tag{4}
\]

\[
\nu_5 = L_2 \left( L_1 \frac{\partial L_2}{\partial x} - L_2 \frac{\partial L_1}{\partial x} \right) + L_1 \left( L_2 \frac{\partial L_1}{\partial y} - L_1 \frac{\partial L_2}{\partial y} \right) - a_1L_1^3 + 3a_2L_1^2L_2 - 3a_3L_1L_2^2 + a_4L_2^3 \tag{5}
\]

The equation $\nu_5 = 0$ means that
defines a particular solution of Equation (6). We shall call (6) a subequation of Equation (6), i.e. (6) is a differential consequence of (6). Notice that in the case \( L_2 \equiv 0 \) the solution is not \( L_1 \equiv 0 \) — which would mean an unexpected lowering of this equation’s differential order — but \( dx = 0 \), an absurdity as \( x \) is seen as the independent variable. Similarly if \( L_1 \equiv 0 \), the solution is \( dy = 0 \), a solution possibly present in Equation (6) but not very interesting.

Now suppose neither \( L_1 \) nor \( L_2 \) vanish identically and define \( \alpha = -L_2/L_1 \). Then

\[
\nu_5 = L_1^3 \left( \alpha \frac{\partial \alpha}{\partial y} + \frac{\partial \alpha}{\partial x} + a_1 \alpha^3 + 3 a_2 \alpha^2 + 3 a_3 \alpha + a_4 \right) \tag{7}
\]

and Equation (6) can be rewritten \( dy/dx = \alpha(x,y) \). Conversely suppose there is a first-order subequation to Equation (6), namely \( dy/dx = A(x,y) \). Then \( A \) is a solution to the first-order non-linear PDE

\[
A \frac{\partial A}{\partial y} + \frac{\partial A}{\partial x} + a_1 A^3 + 3 a_2 A^2 + 3 a_3 A + a_4 = 0 \tag{8}
\]

we shall discuss later. Liouville theory is a finite effort tool for finding particular solutions to Equation (8).

B. Darboux polynomials and first integrals for polynomial dynamical systems.

Consider an autonomous polynomial dynamical system

\[
\dot{x}_i = V_i(x), \quad i = 1 \ldots n \tag{9}
\]

We say that a polynomial \( f(x_1, \ldots, x_n) \) is a Darboux polynomial of (6) if there exists a polynomial “eigenvalue” \( p \) such that

\[
\frac{df}{dt} \equiv \sum_{i=1}^{n} V_i \frac{\partial f}{\partial x_i} = pf
\]

In other words, there is an algebraic variety, defined by \( f(x_1, \ldots, x_n) = 0 \), which is invariant by the flow of \( V \). In this respect, this notion is a neighbour of the notion of subequation we have seen.
Darboux polynomials are tools for finding out\cite{1}, but also proving the non-existence (cf.\cite{2} for an example) of first integrals to polynomial dynamical systems. We shall not enter into the details. Let us just notice that a polynomial first integral is simply a Darboux polynomial with eigenvalue 0; and that a Darboux polynomial $f$ with constant eigenvalue $\alpha$ gives rise to the time-dependent first integral $f e^{-\alpha t}$. More rational and algebraic first integrals can be built with the “basic blocks” of Darboux polynomials\cite{3,4}; and, conversely, a theorem of Bruns\cite{5} says there cannot be an algebraic first integral of (8) unless there is a rational one, which in turn implies the existence of Darboux polynomials.

In brief, the problems of existence of first integrals and Darboux polynomials for a polynomial dynamical system are very tightly linked. Notice also that all these objects, like ordinary eigenvectors and -values of linear endomorphisms, naturally live in $\mathbb{C}$.

II. PRINCIPLE OF THE METHOD.

We shall draw our interest to autonomous three-dimensional polynomial dynamical systems which are of first degree in one of their three variables, e.g. $z$. Their general form is thus:

$$
\begin{align*}
\dot{x} &= V_x^0(x, y) + z V_x^1(x, y) \\
\dot{y} &= V_y^0(x, y) + z V_y^1(x, y) \\
\dot{z} &= V_z^0(x, y) + z V_z^1(x, y)
\end{align*}
$$

(10)

which we may abbreviate as $\dot{X} = V(X)$. Dynamical systems of this kind are frequently met in physics: well-known examples are the Lorenz model, or the various three-wave interaction problems (Rabinovich etc.). Very often they are indeed of first degree in all their variables. We can use this feature for harvesting more information — an example is given in the paragraph about the Lorenz model.

We assume to have found out and studied all solutions with $x =$ cst. Assuming $x$ nonconstant, we shall transform the system (10) into a non-autonomous second-order differential equation linking $y$ and $x$ which will turn out to be of type (1).
Now we settle in a region of space where \( \dot{x} \neq 0 \) and take \( x \) as the independent variable, parametrising the integral curves of (10). The relation

\[
\left(V^0_x(x, y) + z V^1_x(x, y)\right) \frac{dy}{dx} = V^0_y(x, y) + z V^1_y(x, y)
\]

is satisfied along all integral curves. Hence, writing \( p = \frac{dy}{dx} \),

\[
z \left(V^1_x p - V^1_y \right) = V^0_y - p V^0_x
\]

These equations define the mappings \( \phi: (x, y, z) \mapsto (x, y, p) \) and \( \phi^*: (x, y, p) \mapsto (x, y, z) \), which are homographic and hence:

1. They are one-to-one wherever they are defined and their determinant \( C(x, y) = V^0_x V^1_y - V^0_y V^1_x \) is non-zero; as it involves only the variables \( x \) and \( y \) the surface \( \Sigma = \{C(x, y) = 0\} \) can be seen either as a submanifold in the \((x, y, z)\) space or in the \((x, y, p)\) space.

2. The surfaces \( S_1 = \{V^0_x(x, y) + z V^1_x(x, y) = 0\} \) in the \((x, y, z)\) space and \( S_2 = \{V^1_x(x, y) p - V^1_y(x, y) = 0\} \) in the \((x, y, p)\) space are singular. Any point on \( S_1 \setminus \Sigma \) is sent to \( p = \infty \): this happens when \( \dot{x} = 0 \), and the tangent to the integral curve is orthogonal to the \( x \)-axis, i.e. “vertical” in \((x, y)\) representation (\( dy/dx = \infty \)). Similarly any point on \( S_2 \setminus \Sigma \) is sent to \( z = \infty \).

3. On \( \Sigma \), \( \phi \) and \( \phi^* \) are “constant along fibres”, i.e. two points of \( \Sigma \) having different \( z \) (or \( p \)) are sent to the same image, having the same \((x, y)\) as the original point, hence lying on \( \Sigma \). Thus it is always possible to get it as the image of a point on \( \Sigma \setminus S_1 \) and calculating it that way shows that \( \phi(\Sigma) = \Sigma \cap S_2 \); and similarly \( \phi^*(\Sigma) = \Sigma \cap S_1 \).

Since we are concerned with a differential problem, we have to study what the vector field \( V(X) \) becomes under the action of the tangent map \( T_X\phi \). And, indeed, points \( X \) on \( \Sigma \) differing only in the \( z \) coordinate have the same image by \( \phi \), but different \( V(X) \) such that the corresponding \( T_X\phi(V(X)) \) also generally differ. As all these vectors are attached to the common image of the points \( X \), this can cause a loss of information, leading, as we shall see, to important practical difficulties.
Similarly, we find that
\[
\left( V^0_x(x, y) + z V^1_x(x, y) \right) \frac{dz}{dx} = V^0_y(x, y) + z V^1_y(x, y)
\]
We calculate \( dz/dx \) by differentiating (12) with respect to \( x \), putting the result into the previous formula, and then replacing \( z \) itself with its value in function of \( p \) given by Equation (12). This leads to a differential equation in \( p \) which, in Cauchy form, reads
\[
\frac{dp}{dx} = \frac{N(x, y, p)}{C(x, y)^2}
\]
where \( N \) is polynomial in \((x, y, p)\) and of degree three in \( p \). Interpreting \( p \) as \( dy/dx \), we see Equation (13) as a differential equation of Liouville type like (4). There are two essential facts in these computations. One is that the denominator of (13) is exactly the square of the determinant \( C(x, y) \) of \( \phi \), so Equation (13) will not set any further problem as long as its construction is valid. The other one is that Equations (10) are of degree one in \( z \): it ensures not only the good behaviour of the \( z \leftrightarrow p \) correspondence but also the Liouville form of the differential equation (13).

We intend to apply Liouville theory to Equation (13) in order to obtain subequations for it. Now, we must take care of their possible relationships with the forbidden surfaces. If a subequation defines a curve in the \((x, y, p)\) space which is not contained in \( \Sigma \) or \( S_2 \), there is no problem: it will be pulled back into the \((x, y, z)\) space by the \( \phi^* \) map, which coincides then with the reciprocal of \( \phi \).

But in the computation of Equation (13), we have used the \( z = \phi^*(x, y, p) \) map, and then suppressed the denominator \( V^1_x p - V^1_y \). Thus, the singular manifold at \( S_2 \) has disappeared in (13). But consider a curve plotted on \( S_2 \) (i.e. \( V^1_x p - V^1_y \equiv 0 \)) which is, moreover, a jet (i.e. \( p \equiv dy/dx \)). Then, identically
\[
\frac{dy}{dx} = A(x, y) = \frac{V^1_y(x, y)}{V^1_x(x, y)}
\]
We can check that this \( A \) is always a solution to Equation (8), whatever the vector field \( V \) may be. Hence any jet plotted on \( S_2 \) is a subequation of Equation (13). However, this jet
cannot yield an invariant manifold in the \((x, y, z)\) space unless it is made of images by \(\phi\) of points in this space. Now, it is easy to show that the only points on \(S_2\) that can be written as \(\phi(x, y, z)\) are those on \(\Sigma \cap S_2\); therefore, as we have seen, they are images of points also lying on \(\Sigma\).

Yet we know that the pullback of the vector field \(V\) is not necessarily well-behaved on \(\Sigma\). Thus, the system (11) and the differential equation (13) can behave quite differently on \(\Sigma\).

If we find as Equation (6) the equation of a jet on \(S_2\), we have to check independently whether \(\Sigma\) is an invariant manifold for \(V\) or not. In dynamical systems containing parameters, this can be rephrased as: find at what condition on the system’s parameters the equation denominator \(C(x, y)\) is a Darboux polynomial for the system (11).

### III. RESULTS FOR SEVERAL DYNAMICAL SYSTEMS.

We have applied the method exposed in Section II to three different dynamical systems of type (11) depending on real parameters: Lotka–Volterra, Lorenz, Rikitake. We shall discuss the results obtained in the rest of this section.

#### A. The \((a, b, c)\) Lotka–Volterra system.

This remarkably symmetric system

\[
\begin{aligned}
\dot{x} &= x(cy + z) \\
\dot{y} &= y(az + x) \\
\dot{z} &= z(bx + y)
\end{aligned}
\]  

appeared first as a model for three-species competition, yet has been found later in plasma physics. A considerable amount of research has been done on it, using many techniques. Here we shall follow the process exposed in Section II.

Since Equations (14) are invariant by simultaneous circular permutations of \((x, y, z)\) and \((a, b, c)\), it is equivalent to perform the method with any couple of variables. Once this is
done, more results can be got by the above symmetry. There is also a symmetry in taking $x = x'/b$, $y = z'/c$, $z = y'/a$ and $a = 1/c'$, $b = 1/b'$, $c = 1/a'$, which will appear in the distribution of the $\nu_5 = 0$ cases.

So, we take $z$ as the independent variable and eliminate $x$, and find

$$L_1 = \frac{(b - 1)(1 + a b c) Q_{abc}^1(y, z)}{z (y - a b z)^4}$$

(15)

and

$$L_2 = \frac{(b - 1)(1 + a b c) Q_{abc}^2(y, z)}{z (y - a b z)^4}$$

(16)

and

$$\nu_5 = \frac{(b - 1)^3 (1 + a b c)^3 P_{abc}(y, z)}{z^2 y^2 (y - a b z)^10}$$

(17)

where $P_{abc}$, $Q_{abc}^1$, $Q_{abc}^2$ are polynomials whose coefficients depend on $(a, b, c)$ and which we do not write down for the sake of brevity. The cases $1 + a b c = 0$ and $b = 1$ are known: the first one is the full integrability of the system, with, in particular, the first integral $a b x + y - a z$; in the second one we have the Darboux polynomial $y - a z$ whose eigenvalue is $x$. We remark this Darboux polynomial is exactly the denominator... Notice also that in those two cases, $L_1$ and $L_2$ vanish together with $\nu_5$ so that Equation (14) is an identity and cannot be used for finding out Darboux polynomials.

Now the cases where all coefficients of $P_{abc}$ are zero are listed below in Table [I]. We notice the presence of the symmetry $a = 1/c'$, $b = 1/b'$, $c = 1/a'$ in this list; one case $(a = 1, b = 1, c = 1)$ is self-symmetric. We shall handle in some detail one of these “exotic” cases, viz. $(a = 1/4, b = 2, c = -5)$. The subequation (14) reads:

$$y \left(-3 z^2 + 16 z y - 32 y^2\right) + 4 z \frac{dy}{dz} \left(z^2 - 7 z y + 16 y^2\right) = 0$$

(18)

Reverting to the original variables $(x, y, z)$ changes this equation in

$$y(z - 2 y) (16 y^2 - 2 x z - 8 y z + z^2) = 0$$

(19)
The expression $z - 2y$ is proportional to the equation denominator; its presence here is an artefact due, as we have seen, to a former suppression of denominator. It should not be taken in account since it corresponds to $b = 1$. On the other hand, the other two factors are Darboux polynomials, since their derivatives with respect to the system (14) are

$$\dot{y} = y\left(\frac{z}{4} + x\right)$$

and

$$\frac{d}{dt}\left(16y^2 - 2xz - 8yz + z^2\right) = 2x\left(16y^2 - 2xz - 8yz + z^2\right)$$

This illustrates the validity of the method in the general case. The results for all cases are summarised in Table I. We get no information for the self-symmetric case since it is a specialisation of $b = 1$, so our method cannot be applied.

B. The Lorenz model.

Another well-known and intensively studied dynamical system, the Lorenz model

$$\begin{cases} \dot{x} &= \sigma (y - x) \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= xy - bz \end{cases}$$

originally thought as a simple model for atmospheric turbulence, was the first example of a low-dimensional chaotic deterministic dynamical system. All known first integrals have been obtained or reobtained by Kus, using the non-decisive procedure of Carleman embedding. Here we shall recover some of them methodically by the means of Liouville theory.

Since there is no symmetry among variables here, we can proceed three times to the calculation of the $L_{1,2}$ and $\nu_5$ functions, eliminating each time one of the three variables. Indeed, the Liouville equation like (13) contains no more information than the dynamical system like (10) does, but it has more singularities; yet, as we have seen, these singularities often contain useful matter about the dynamical system’s invariant manifold structure.
Eliminating $z$ and choosing $x$ as the independent variable yields $C(x, y) = \sigma x (y - x)$. Neither $x = 0$ nor $y - x = 0$ can be interesting invariant manifolds, since both imply $\dot{x} = 0$, so $x = \text{cst}$ and $y = \text{cst}$. This, in turn, also implies that $z$ is constant, and the manifold reduces to a fixed point.

The functions $L_2$ and $\nu_5$ are (cf. 14)

$$L_2 = \frac{1 + b + \sigma}{\sigma (y - x)^3}$$

and

$$\nu_5 = \frac{(1 + b + \sigma) P_{b\sigma \varrho}(x, y)}{\sigma^5 x^2 (y - x)^{10}}$$

In the obvious case $1 + b + \sigma = 0$, we also have $L_2 = 0$. Thus, as explained in Section I, our choice of variables was a bad one. Looking for other cases, we only get three points in the $(b, \sigma, \varrho)$ space, viz. $(b = 0, \sigma = -1)$, $(b = 2/3, \sigma = 1/3)$ and $(b = -16/5, \sigma = -1/5, \varrho = -7/5)$. They are specialisations either of known case, or of the $1 + b + \sigma = 0$ case. In those cases, we get subequations that do not give rise to Darboux polynomials, i.e. equations that represent jets plotted on the surface $S_2$.

Now, if we eliminate $y$ and choose $x$ as the independent variable, we get as equation denominator $C(x, z) = bz - x^2$. Its derivative with respect to the system (20) is

$$\frac{d}{dt} (bz - x^2) = (b - 2 \sigma) x y - (b^2 z - 2 \sigma x^2)
= -2 \sigma (bz - x^2) + (b - 2 \sigma) (xy - bz)$$

The remainder is of degree one in $x$. Thus $C$ is a Darboux polynomial iff $b = 2 \sigma$; then the eigenvalue is $-2 \sigma$, so $I = (x^2 - 2 \sigma z) e^{2 \sigma t}$ is a first integral.

We have calculated $\nu_5$ and found

$$\nu_5 = \frac{(b - 2 \sigma)(1 + b + \sigma) P_{b\sigma \varrho}}{\sigma^5 (bz - x^2)^{10}}$$

$P_{b\sigma \varrho}$ being such that its coefficients never simultaneously vanish. In the case $b = 2 \sigma$, Equation (3) yields $x - \sigma dz/dx = 0$. This is the equation of a jet on $S_2$, and since we are in the good case, we find $\Sigma$ as the invariant surface.
When \(1 + b + \sigma = 0\), Equation (6) still yields \(x - \sigma \frac{dz}{dx} = 0\). But in this case, \(\Sigma\) is not invariant and we do not have a Darboux polynomial.

Finally, we have taken \(z\) as the independent variable and eliminated \(x\). We find
\[
C(y, z) = b \varrho z - b z^2 - y^2
\]
and
\[
\frac{dC}{dt} = x y \left((b - 2) \varrho + 2 (1 - b) z\right) + 2 y^2 + 2 b^2 z^2 - b^2 \varrho z
\]
(25)
the remainder \(R\) being of first degree in \(y\); hence \(C\) is a Darboux polynomial iff \(b = 1\) and \(\varrho = 0\). Then, \(\frac{dC}{dt} = -2C\) and we get that way the first integral \(I = (z^2 + y^2) e^{2t}\).

As for \(\nu_5\), it is equal to
\[
\nu_5 = \frac{(1 + b + \sigma) y P_{b\varrho}(y, z)}{(b \varrho z - b z^2 - y^2)^{10}}
\]
(26)
where \(P_{b\varrho}(y, z) \equiv 0\) iff \(b = 1\) and \(\varrho = 0\). Let us handle first the latter case. In that case, \(L_1 = -\sigma z y / (z^2 + y^2)^2\) and \(L_2 = -\sigma y^2 / (z^2 + y^2)^2\), so Equation (3) simplifies as
\(z \frac{dz}{dt} + y \frac{dy}{dt} = 0\). This is the equation of a jet on \(S_2\), and we get \(\Sigma\) as invariant manifold.

Now, in the case \(1 + b + \sigma = 0\), there is another simplification in Equation (3), namely \(y L_1 \equiv (z - \varrho) L_2\) and hence \((z - \varrho) \frac{dz}{dt} + y \frac{dy}{dt} = 0\). But this is the jet on \(S_2\), so we get no information in this case.

C. The Rikitake dynamo.

This dynamical system
\[
\begin{align*}
\dot{x} &= -\mu x + y (z + \beta) \\
\dot{y} &= -\mu y + x (z - \beta) \\
\dot{z} &= \alpha - xy
\end{align*}
\]
models the variation of the earth’s magnetic field with time.

Let us take \(x\) as the privileged variable and eliminate \(z\). The denominator in Equation (13) is
\[
C(x, y) = \mu (y^2 - x^2) + 2 \beta x y,
\]
and
\[ \nu_5 = \frac{\beta^2 \mu^2 P_{\alpha\beta\mu}(x, y)}{(\mu (y^2 - x^2) + 2 \beta xy)^{10}} \]  

(28)

where the coefficients of \( P_{\alpha\beta\mu} \) cannot vanish together unless \( \beta = 0 \) or \( \mu = 0 \).

The derivative of \( C(x, y) \) with respect to the system (27) is

\[ \frac{dC}{dt} = (y^2 - x^2)(2 \beta^2 - 2 \mu^2) - 2 \beta xy + 2 \beta z(x + y) \]  

(29)

Assume \( C \) is a Darboux polynomial of eigenvalue \( P(x, y, z) = A(x, y) + z B(x, y) \). Then the identification of the \( z \) terms in Equation (29) gives \( BC = 2 \beta (x + y) \). Since \( C \) is of second degree, this is impossible unless \( \beta = 0 \).

When \( \beta = 0 \), Equation (29) reads \( \frac{dC}{dt} = -2 \mu C \), so \( C \) is a Darboux polynomial of this system, which gives the first integral \( (y^2 - x^2) e^{2 \mu t} \). On the other hand,

\[ L_1 = \frac{4 x^2 y}{\mu^2 (x^2 - y^2)^2}, \quad L_2 = \frac{-4 x y^2}{\mu^2 (x^2 - y^2)^2} \]

hence Equation (3) becomes \(-x dx + y dy = 0\). This is a jet on \( S_2 \), but we are in the “good” case, and we recover the Darboux polynomial \( y^2 - x^2 \).

If now \( \mu = 0 \) then

\[ L_1 = \frac{-\alpha (3 x^4 + 2 x^2 y^2 + 3 y^4)}{4 \beta^2 x^3 y^4}, \quad L_2 = \frac{\alpha (3 x^4 + 2 x^2 y^2 + 3 y^4)}{4 \beta^2 x^4 y^3} \]

so the subequation is once more \(-x dx + y dy = 0\), the jet on \( S_2 \). Hence we do not obtain any Darboux polynomial unless \( \beta = 0 \).

We have also performed the computations with the other two couples of variables. They have not given more information than the previous ones.

**IV. CONCLUSION.**

We have obtained, *by a methodic procedure*, numerous cases of Darboux polynomials for the Lotka–Volterra system. This “Darboux polynomial searcher” can be seen as an input to algorithms which need Darboux polynomials, such as the Prelle–Singer procedure. Up
to now, that procedure began with a systematic search, which obliged to set an *a priori* limit on the polynomial’s degree in all its variables. Some new results allow to refine the search by restricting the choice of the possible highest-degree homogeneous components of the tentative Darboux polynomials, while speeding it up when the system’s coefficients are rational numbers. Yet they are valid for dynamical systems of dimension 2, and until now have no counterpart in dimension 3.

Our method has also reobtained some known first integrals for the Lorenz and Rikitake systems, though all cases have not been found, and despite the “divergence enigma” we now explain.

In both Lorenz and Rikitake systems, the divergence is a constant, respectively $-1-b-\sigma$ and $-2\mu$. In both cases, its vanishing triggers the vanishing of $\nu_5$, but also the reduction of Equation (6) to a singularity from which no information can be extracted. However, the Rikitake dynamo possesses when $\mu = 0$ a time-dependent first integral $I = x^2 - y^2 + 4\beta z - 4\alpha \beta t$; this kind of first integral cannot be detected by our method, since it does not arise from a Darboux polynomial, but from a polynomial $f(x, y, z)$ such that $df/dt = \text{cst.}$ Such a first integral may exist only for dynamical systems having a constant term, so there is no chance to find any for the Lorenz model. But there may be a first integral of some special kind — indeed, numerical experiments exhibit a regular behaviour when $1+b+\sigma = 0$.

Table summarises the results obtained by our method. Abbreviations are: DS for “dynamical system”, FI for “first integral”, and DE for “Darboux element”, i.e. a couple of polynomials $(f, p)$ such that $df/dt = pf$. 
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### TABLES

| (1) DS | $\nu_5$ | Denominator | Parameters | Information obtained |
|--------|---------|-------------|------------|----------------------|
| Lotka  | $(y, z):[17]$ | $y - a b z$ | $(0, 2, c)$ | DE: $(y; x)$ |
|        |         |             | $(1/(2 b), b, 1)$ | DE: $(b x - z; y)$ |
|        |         |             | $(1, b, 2/b)$ | DE: $(x - c y; z)$ |
|        |         |             | $(1/4, 2, -5)$ | DE: $(16 y^2 - 2 x z - 8 y z + z^2; 2 x)$ |
|        |         |             | $(-1/5, 1/2, 4)$ | DE: $(100 y^2 - 25 x y + 40 y z + 4 z^2; x)$ |
|        |         |             | $(1, 2, -2)$ | DE: $(y^2 - x z - y z; 2 x + z)$ |
|        |         |             | $(-1/2, 1/2, 1)$ | DE: $(-2 x y + 2 y z + z^2; x + y)$ |
|        |         |             | $(1, 1, 1)$ | none |
|        |         |             | $(-1/2, 0, 1)$ | none |
|        |         |             | $(0, 2/3, 1)$ | DE: $(3 z - 2 x; y)$ |
|        |         |             | cyclic permutation of the above results | |
|        | $(x, y):[22]$ | $x - y$ | $(-1 - \sigma, \sigma, g)$ | none |
|        |         |             | $(2/3, 1/3, g)$ | none |
|        |         |             | $(-16/5, -1/5, -7/5)$ | none |
| Lorenz | $(x, z):[24]$ | $b z - x^2$ | $(2 \sigma, \sigma, g)$ | FI: $(x^2 - 2 \sigma z) e^{2 \sigma t}$ |
|        |         |             | $(-1 - \sigma, \sigma, g)$ | none |
|        | $(z, y):[2] | b z (g - z) - y^2 | $(1, \sigma, 0)$ | FI: $(z^2 + y^2) e^{2 t}$ |
|        |         |             | $(-1 - \sigma, \sigma, g)$ | none |
| Rikitake | $(x, y):[28]$ | $\mu (y^2 - x^2)$ | $(\alpha, 0, \mu)$ | FI: $(y^2 - x^2) e^{2 \mu t}$ |
|        |         |             | $+2 \beta x y$ | none |
|        |         |             | $(\alpha, \beta, 0)$ | |
|        | other | ... | | nothing more |

**TABLE I.** Results obtained by our method