Growth Theorems in Slice Analysis of Several Variables

Guangbin Ren and Ting Yang*

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Abstract. In this paper, we define a class of slice mappings of several variables in the quadratic cones of Clifford algebras, and the corresponding slice regular mappings. Furthermore, we establish the growth theorem for slice regular starlike or convex mappings on the unit ball of several variables in the quadratic cones of Clifford algebras, as well as on the bounded slice domains which are slice starlike and slice circular.

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1. Introduction

Ghiloni and Perotti [8] initiated the study of slice analysis on the quadratic cones of real alternative algebras; see [10,11,16] for its recent development. The theory has some distinguished models on quaternions, real Clifford algebras, and Octonions; see [2,6,7] for the pioneering works. It has important applications in the functional calculus for non-commutative operators [1,3].

Recently, the slice theory of several variables has also been studied [4,9]. In this article, we shall provide a new generalization, which enable us to extend slice analysis to higher dimensions as well as to the setting of several variables in quadratic cones of Clifford algebras.

Based on a new convex combination identity in [15], the sharp growth theorems for slice monogenic extensions of univalent functions on the unit disc $D \subset \mathbb{C}$ in the setting of Clifford algebras was established as follows:

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*Corresponding author.
Let $f$ be a slice monogenic function on the unit ball $B$ with values in the Clifford algebra $\mathbb{R}_m$, such that its restriction $f_I$ to $B_I$ is injective and such that $f(B_I) \subseteq \mathbb{C}_I$ for some $I \in \mathbb{S}_m$. If $f(0) = 0$, $f'(0) = 1$, then
\[
\frac{|x|}{(1 + |x|)^2} \leq |f(x)| \leq \frac{|x|}{(1 - |x|)^2}, \quad x \in \mathbb{B}.
\]
Moreover, equality holds for one of these two inequalities at some point $x_0 \in \mathbb{B} \setminus \{0\}$ if and only if $f$ is of the form $f(x) = x(1 - xe^{i\theta})^{-\frac{1}{2}}$, $\forall x \in \mathbb{B}$, for some $\theta \in \mathbb{R}$. (See [15] for details.)

From the classical geometric function theory in higher dimensions, it is known that the growth theorems fail for the full class of normalized univalent mappings [12]. It is Cartan who suggested to consider the subclass of starlike or convex mapping instead.

The aim of this paper is to generalize the growth results for subclasses of normalized univalent mappings on the unit ball in $\mathbb{C}^n$ to the subset of the several Clifford variables ($Q^m)_s^n$. In [15], it concluded that the maximum and minimum moduli of a slice monogenic function $f$ are actually attained on the preserved slice. Now we extend this result to higher dimensions with a different approach. As an application, we obtain the growth theorems for slice regular starlike or convex mappings on the unit ball of several variables in the quadratic cones of Clifford algebras, as well as on more general slice domains which are slice starlike and slice circular.

2. Preliminaries

The real Clifford algebra $\mathbb{R}_m$ is an universal associative algebra over $\mathbb{R}$ generated by $m$ basis elements $e_1, \ldots, e_m$, subject to the relations
\[
e_i e_j + e_j e_i = -2\delta_{ij}, \quad i, j = 1, \ldots, m.
\]
As a real vector space, $\mathbb{R}_m$ has dimension $2^m$. Each element $x$ in $\mathbb{R}_m$ can be expressed as
\[
x = \sum_{A \in \mathcal{P}(m)} x_A e_A,
\]
where
\[
\mathcal{P}(m) = \{(h_1, \ldots, h_r) \in \mathbb{N}^r \mid r = 1, \ldots, m, \ 1 \leq h_1 < \cdots < h_r \leq m\}.
\]

For each $A = \{(h_1, \ldots, h_r) \in \mathcal{P}(m)\}$, the coefficients $x_A \in \mathbb{R}$, and the products $e_A := e_{h_1} e_{h_2} \cdots e_{h_r}$ are the basis elements of the Clifford algebra $\mathbb{R}_m$. The unit of the Clifford algebra corresponds to $A = \emptyset$, and we set $e_{\emptyset} = 1$. As usual, we identify the real numbers field $\mathbb{R}$ with the subalgebra of $\mathbb{R}_m$ generated by the unit.

Let $x^c$ be the Clifford conjugate of $x \in \mathbb{R}_m$. We define
\[
t(x) = x + x^c, \quad n(x) = xx^c
\]
to be the trace and the (squared) norm of a Clifford element $x$, respectively.
The quadratic cone of the Clifford algebra $\mathbb{R}_m$ [8] is defined by

$$Q_m := \mathbb{R} \cup \{ x \in \mathbb{R}_m \mid t(x) \in \mathbb{R}, \ n(x) \in \mathbb{R}, \ 4n(x) > t(x)^2 \},$$

and the space of paravectors is defined by

$$\mathbb{R}^{(m+1)} := \{ x \in \mathbb{R}_m \mid [x]_k = 0 \text{ for every } k > 1 \},$$

where $[x]_k$ denotes the k-vector part of $x$.

We also set

$$S_m := \{ J \in Q_m \mid J^2 = -1 \}.$$

The elements of $S_m$ are called the square roots of $-1$ in the Clifford algebra $\mathbb{R}_m$.

It is known that $S_m = \{ x \in \mathbb{R}_m \mid t(x) = 0, \ n(x) = 1 \}$.

We consider the cartesian product $(\mathbb{R}_m)^n$ of the Clifford algebra $\mathbb{R}_m$. Its complexification is denoted by $(\mathbb{R}_m)^n_C := (\mathbb{R}_m)^n \otimes \mathbb{C} := (\mathbb{R}_m)^n + i(\mathbb{R}_m)^n$.

For each $x, y \in (\mathbb{R}_m)^n$, we define

$$x + iy = x - iy$$

be the complex conjugation of $x + iy$ in $(\mathbb{R}_m)^n_C$.

Note that $\mathbb{R}_0 = \mathbb{R}$ and $(\mathbb{R}_0)^n_C = \mathbb{C}^n$, then $\overline{x + iy} = x - iy$, for each $x, y \in \mathbb{R}^n$.

The following definition is a particular case of well known definitions in [5,17].

**Definition 2.1.** Let $D \subset \mathbb{C}^n$ be an open subset. A mapping $F : D \to (\mathbb{R}_m)^n_C$ is called a $\mathbb{R}_m$-stem mapping over the domain $D$ if $F$ is complex intrinsic, i.e.

$$F(\bar{z}) = \overline{F(z)}, \quad \forall \ z \in D.$$

**Remark 2.2.** (1) A mapping $F$ is a $\mathbb{R}_m$-stem mapping if and only if the $\mathbb{R}_m$-valued components $F_1, F_2$ of the $F = F_1 + iF_2$ form an even-odd pair, i.e.

$$F_1(\bar{z}) = F_1(z), \quad F_2(\bar{z}) = -F_2(z)$$

for each $z \in D$.

(2) Consider $\mathbb{R}_m$ as a $2^m$-dimensional real vector space. By means of a basis $B = \{ e_A \}_{A \in \mathcal{P}(m)}$ of $\mathbb{R}_m$, $F$ can be identify with complex intrinsic curves in $\mathbb{C}^{n \times 2^m}$.

Let $F(z) = F_1(z) + iF_2(z) = \sum_{A \in \mathcal{P}(m)} F^A(z) e_A$ with $F^A(z) \in \mathbb{C}^n$.

Then

$$\tilde{F} = \begin{pmatrix}
F_1^0 & F_1^1 & \cdots & F_1^{2^{m-1}} \\
F_2^0 & F_2^1 & \cdots & F_2^{2^{m-1}} \\
\vdots & \vdots & \ddots & \vdots \\
F_n^0 & F_n^1 & \cdots & F_n^{2^{m-1}}
\end{pmatrix} : D \to \mathbb{C}^{n \times 2^m}$$

satisfies $\tilde{F}(\bar{z}) = \overline{\tilde{F}(z)}$. Giving $\mathbb{R}_m$ the unique manifold structure as a real vector space, we get that a stem mapping $F$ is of class $C^k(k = 0, \ldots, \infty)$
or real-analytic if and only if the same property holds for $\widetilde{F}$. This notion is clearly independent of the choice of the basis of $\mathbb{R}_m$.

In the several Clifford numbers $(\mathbb{R}_m)^n$, we define

$$(Q_m)^n_s := \bigcup_{I \in \mathbb{S}_m} C^n_I,$$

where

$C^n_I := \mathbb{R}^n + I\mathbb{R}^n, \quad I \in \mathbb{S}_m.$

Let $(\mathbb{R}^{(m+1)})^n_s$ be the space of multi-paravectors. Its subspace

$$(\mathbb{R}^{(m+1)})^n_s := (\mathbb{R}^{(m+1)})^n \cap (Q_m)^n_s$$

is called the space of slice multi-paravectors.

For any $\alpha, \beta \in \mathbb{R}^n, I \in \mathbb{S}_m$ with $z = \alpha + \beta I$, we set

$[z] := \bigcup_{J \in \mathbb{S}_m} \alpha + \beta J \subseteq (Q_m)^n_s.$

Given an open subset $D$ of $\mathbb{C}^n$, let $\Omega_D$ be the subset of $(Q_m)^n_s$ obtained by the action on $D$ of the square roots of $-1$:

$$\Omega_D := \{\alpha + \beta J \in (Q_m)^n_s \mid \alpha + i\beta \in D, \alpha, \beta \in \mathbb{R}^n, J \in \mathbb{S}_m\},$$

and

$$D_I := \Omega_D \cap C^n_I, \quad I \in \mathbb{S}_m,$$

then

$$\Omega_D = \bigcup_{I \in \mathbb{S}_m} D_I, \quad I \in \mathbb{S}_m.$$ 

In particular, $\Omega_D = (Q_m)^n_s$ when $D = \mathbb{C}^n$.

**Definition 2.3.** Any stem mapping $F : D \to (\mathbb{R}_m)^n_C$ induces a (left) slice mapping

$$f = I(F) : \Omega_D \to (\mathbb{R}_m)^n.$$

If $x = \alpha + I\beta \in D_I, \forall I \in \mathbb{S}_m$, we set

$$f(x) := F_1(z) + IF_2(z) \quad (z = \alpha + i\beta).$$

The slice function $f$ is well defined, since $(F_1, F_2)$ is an even-odd pair w.r.t. $\beta$ and then $f(\alpha + (\beta)(-J)) = F_1(z) + (-J)F_2(z) = F_1(z) + JF_2(z)$.

There is an analogous definition for right slice mappings when the element $J \in \mathbb{S}_m$ is replace on the right of $F_2(z)$. In what follows, the term slice mapping will always mean left slice mappings.

We will denote the set of all (left) slice mappings on $\Omega_D$ by

$$\mathcal{S}(\Omega_D, (\mathbb{R}_m)^n) := \{f : \Omega_D \to (\mathbb{R}_m)^n \mid f = I(F), F : D \to (\mathbb{R}_m)^n_C \text{ is a } \mathbb{R}_m\text{-stem mapping}\}.$$

The distinguished property for a slice mapping is the representation formula:
Proposition 2.4. Let \( f \in \mathcal{S}(\Omega_D, (\mathbb{R}_m)^n) \) and \( J, K \in \mathbb{S}_m \) with \( J - K \) is invertible. Then
\[
f(\alpha + \beta I) = (I - K)((J - K)^{-1}f(\alpha + \beta J)) - (I - J)((J - K)^{-1}f(\alpha + \beta K))
\]
for each \( I \in \mathbb{S}_m \), \( \alpha, \beta \in \mathbb{R}^n \) with \( \alpha + i\beta \in D_I \).

Proof. For \( f \in \mathcal{S}(\Omega_D, (\mathbb{R}_m)^n) \), by definition,
\[
f(\alpha + \beta J) - f(\alpha + \beta K) = (J - K)F_2(\alpha + \beta i)
\]
Hence
\[
F_2(\alpha + \beta i) = (J - K)^{-1}(f(\alpha + \beta J) - f(\alpha + \beta K))
\]
and
\[
F_1(\alpha + \beta i) = f(\alpha + \beta J) - JF_2(\alpha + \beta i)
= f(\alpha + \beta J) - J((J - K)^{-1}(f(\alpha + \beta J) - f(\alpha + \beta K))).
\]
Therefore,
\[
f(\alpha + \beta I) = F_1(\alpha + \beta i) + IF_2(\alpha + \beta i)
= f(\alpha + \beta J) + (I - J)((J - K)^{-1}f(\alpha + \beta J) - f(\alpha + \beta K))
= (J - K + I - J)((J - K)^{-1}f(\alpha + \beta J)) - (I - J)((J - K)^{-1}f(\alpha + \beta K))
= (I - K)((J - K)^{-1}f(\alpha + \beta J)) - (I - J)((J - K)^{-1}f(\alpha + \beta K)).
\]

By setting \( K = -J \) in Proposition 2.4, we obtain the following corollary.

Corollary 2.5. Let \( f \in \mathcal{S}(\Omega_D, (\mathbb{R}_m)^n) \) and \( J \in \mathbb{S}_m \), \( \alpha, \beta \in \mathbb{R}^n \) with \( \alpha + i\beta \in \Omega_D \). Then
\[
f(\alpha + \beta I) = \frac{1}{2}(f(\alpha + \beta J) + f(\alpha - \beta J)) - \frac{I}{2}(J(f(\alpha + \beta J) - f(\alpha - \beta J)))
\]
for each \( I \in \mathbb{S}_m \).

We will denote by
\[
\mathcal{S}^1(\Omega_D, (\mathbb{R}_m)^n) := \{ f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D, (\mathbb{R}_m)^n) \mid F \in C^1(D, (\mathbb{R}_m)^n_C) \}
\]
the real vector space of slice mappings of several variables in quadratic cones of Clifford algebras with stem mapping of class \( C^1 \).

Let \( f = \mathcal{I}(F) \in \mathcal{S}^1(\Omega_D, (\mathbb{R}_m)^n) \) and \( z = \alpha + i\beta \in D \). Then the partial derivatives \( \partial F/\partial \alpha_t \) and \( i\partial F/\partial \beta_t \) are continuous \( \mathbb{R}_m \)–stem mappings on \( D \), for \( t = 1, 2, \ldots, n \). The same property holds for their linear combinations
\[
\frac{\partial F}{\partial z_t} = \frac{1}{2} \left( \frac{\partial F}{\partial \alpha_t} - i \frac{\partial F}{\partial \beta_t} \right) \quad \text{and} \quad \frac{\partial F}{\partial \bar{z}_t} = \frac{1}{2} \left( \frac{\partial F}{\partial \alpha_t} + i \frac{\partial F}{\partial \beta_t} \right),
\]
where \( z = (z_1, \ldots, z_n) \), \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( \beta = (\beta_1, \ldots, \beta_n) \), \( t = 1, 2, \ldots, n \).
Definition 2.6. Let \( f = \mathcal{I}(F) \in S^1(\Omega_D, (\mathbb{R}_m)^n) \). We set
\[
\frac{\partial f}{\partial x} := \mathcal{I} \left( \frac{\partial F}{\partial z} \right), \quad \text{and} \quad \frac{\partial f}{\partial \bar{x}} := \mathcal{I} \left( \frac{\partial F}{\partial \bar{z}} \right),
\]
i.e.
\[
\left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) = \left( \mathcal{I} \left( \frac{\partial F}{\partial z_1} \right), \ldots, \mathcal{I} \left( \frac{\partial F}{\partial z_n} \right) \right),
\]
and
\[
\left( \frac{\partial f}{\partial \bar{x}_1}, \ldots, \frac{\partial f}{\partial \bar{x}_n} \right) = \left( \mathcal{I} \left( \frac{\partial F}{\partial \bar{z}_1} \right), \ldots, \mathcal{I} \left( \frac{\partial F}{\partial \bar{z}_n} \right) \right),
\]
where
\[
\mathcal{I} \left( \frac{\partial F}{\partial z_t} \right) = \left( \mathcal{I} \left( \frac{\partial F^1}{\partial z_t} \right), \mathcal{I} \left( \frac{\partial F^2}{\partial z_t} \right), \ldots, \mathcal{I} \left( \frac{\partial F^n}{\partial z_t} \right) \right)^T,
\]
and
\[
\mathcal{I} \left( \frac{\partial F}{\partial \bar{z}_t} \right) = \left( \mathcal{I} \left( \frac{\partial F^1}{\partial \bar{z}_t} \right), \mathcal{I} \left( \frac{\partial F^2}{\partial \bar{z}_t} \right), \ldots, \mathcal{I} \left( \frac{\partial F^n}{\partial \bar{z}_t} \right) \right)^T,
\]
for \( F = (F^1, F^2, \ldots, F^n)^T \), \( t = 1, 2, \ldots, n \), and \( A^T \) stands for the transpose of the vector \( A \).

The notation \( \partial f / \partial \bar{x}_t \) is justified by the following properties:
\[
\bar{x} = (\mathcal{I}(\bar{z}_1), \ldots, \mathcal{I}(\bar{z}_n))
\]
and therefore
\[
\partial \bar{x} / \partial \bar{x}_t = (0, \ldots, 0, 1, 0, \ldots, 0), \quad \partial x / \partial \bar{x}_t = 0.
\]

Left multiplication by \( i \) defines a complex structure on \((\mathbb{R}_m)^n\). With respect to this structure, a \( C^1 \) mapping \( F = F_1 + iF_2 : D \to (\mathbb{R}_m)_C^n \) is holomorphic if and only if its components \( F_1, F_2 \) satisfy the Cauchy–Riemann equations:
\[
\frac{\partial F_1}{\partial \alpha} = \frac{\partial F_2}{\partial \beta}, \quad \frac{\partial F_1}{\partial \beta} = -\frac{\partial F_2}{\partial \alpha},
\]
i.e.
\[
\frac{\partial F}{\partial \bar{z}} = 0, \quad (z = \alpha + i\beta \in D),
\]
where
\[
\frac{\partial F_p}{\partial \alpha} = \begin{pmatrix} \frac{\partial F^1_p}{\partial \alpha_1} & \frac{\partial F^1_p}{\partial \alpha_2} & \cdots & \frac{\partial F^1_p}{\partial \alpha_n} \\ \frac{\partial F^2_p}{\partial \alpha_1} & \frac{\partial F^2_p}{\partial \alpha_2} & \cdots & \frac{\partial F^2_p}{\partial \alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F^n_p}{\partial \alpha_1} & \frac{\partial F^n_p}{\partial \alpha_2} & \cdots & \frac{\partial F^n_p}{\partial \alpha_n} \end{pmatrix}.
\]
\[
\frac{\partial F_p}{\partial \beta} = \begin{pmatrix}
\frac{\partial F_1^p}{\partial \beta_1} & \frac{\partial F_1^p}{\partial \beta_2} & \cdots & \frac{\partial F_1^p}{\partial \beta_n} \\
\frac{\partial F_2^p}{\partial \beta_1} & \frac{\partial F_2^p}{\partial \beta_2} & \cdots & \frac{\partial F_2^p}{\partial \beta_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F_n^p}{\partial \beta_1} & \frac{\partial F_n^p}{\partial \beta_2} & \cdots & \frac{\partial F_n^p}{\partial \beta_n}
\end{pmatrix},
\]

for \( F_p = (F_1^p, \ldots, F_n^p)^T, \ p = 1, 2, \) and \( \alpha = (\alpha_1, \ldots, \alpha_n), \ \beta = (\beta_1, \ldots, \beta_n). \)

This condition is equivalent to require that, for any basis \( B, \) the mapping \( \tilde{F} \) (cf. Remark 2.2) is holomorphic.

Set
\[
\frac{\partial f}{\partial x} := \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{pmatrix}, \quad \frac{\partial F}{\partial z} := \begin{pmatrix}
\frac{\partial F_1}{\partial z_1} & \frac{\partial F_1}{\partial z_2} & \cdots & \frac{\partial F_1}{\partial z_n} \\
\frac{\partial F_2}{\partial z_1} & \frac{\partial F_2}{\partial z_2} & \cdots & \frac{\partial F_2}{\partial z_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F_n}{\partial z_1} & \frac{\partial F_n}{\partial z_2} & \cdots & \frac{\partial F_n}{\partial z_n}
\end{pmatrix}.
\]

Similarly define \( \frac{\partial f}{\partial \bar{x}} \) and \( \frac{\partial F}{\partial \bar{z}}. \) Let
\[
\mathcal{H}(D) := \{ F \in C^1(D, (\mathbb{R}_m)^n_C) : \frac{\partial F}{\partial \bar{z}} = 0, \ \forall z = (z_1, \ldots, z_n) \in D \}.
\]

**Definition 2.7.** A (left) slice mapping \( f = I(F) \in S^1(\Omega_D, (\mathbb{R}_m)^n) \) is (left) slice regular if its stem mapping \( F \) is holomorphic.

We denote the vector space of slice regular functions on \( \Omega_D \) by \( \mathcal{S} \mathcal{R}(\Omega_D, (\mathbb{R}_m)^n) \). That is,
\[
\mathcal{S} \mathcal{R}(\Omega_D, (\mathbb{R}_m)^n) := \{ I(F) \in S^1(\Omega_D, (\mathbb{R}_m)^n) : F : D \rightarrow (\mathbb{R}_m)^n_C \text{ is holomorphic} \}.
\]

**Remark 2.8.** If we define \( \mathbb{S}_m^* := \{ J \in \mathbb{R}^{(m+1)} : J^2 = -1 \}, \)

and
\[
\Omega_D^* := \{ x = \alpha + \beta J \in (\mathbb{R}^{(m+1)})^n_s : \alpha + i\beta \in D, \alpha, \beta \in \mathbb{R}^n, \ J \in \mathbb{S}_m^* \},
\]

then \( \Omega_D^* \subseteq \Omega_D \) and \( f \in \mathcal{S} \mathcal{R}(\Omega_D^*, (\mathbb{R}_m)^n) \) is called slice monogenic.

**Remark 2.9.** In the case of \( m = 2, \) we call \( f \in \mathcal{S} \mathcal{R}(\Omega_D, \mathbb{H}^n) \) a slice regular mapping of several quaternionic variables.

**Proposition 2.10.** Let \( f = I(F) \in S^1(\Omega_D, (\mathbb{R}_m)^n) \). Then \( f \) is slice regular on \( \Omega_D \) if and only if the restriction
\[
f_I : D_I \rightarrow (\mathbb{R}_m)^n
\]

is holomorphic for every \( I \in \mathbb{S}_m. \)
Proof. Notice that
\[ f_I(\alpha + I\beta) = F_1(\alpha + i\beta) + IF_2(\alpha + i\beta). \]
If \( F \) is holomorphic then
\[ \frac{\partial f_I}{\partial \alpha} + I \frac{\partial f_I}{\partial \beta} = \frac{\partial F_1}{\partial \alpha} + I \frac{\partial F_2}{\partial \alpha} + I \left( \frac{\partial F_1}{\partial \beta} + I \frac{\partial F_2}{\partial \beta} \right) = 0 \]
at every point \( x = \alpha + I\beta \in D_I \).
Conversely, assume that \( f_I \) is holomorphic at every \( I \in \mathbb{S}_m \). Then
\[ 0 = \frac{\partial f_I}{\partial \alpha} + I \frac{\partial f_I}{\partial \beta} = \frac{\partial F_1}{\partial \alpha} - \frac{\partial F_2}{\partial \beta} + I \left( \frac{\partial F_1}{\partial \beta} + \frac{\partial F_2}{\partial \beta} \right) \]
at every point \( z = \alpha + i\beta \in D \). From the arbitrariness of \( I \) it follows that \( F_1, F_2 \) satisfy the Cauchy–Riemann equations. □

Proposition 2.11. Let \( f \in S(\Omega_D, (\mathbb{R}_m)^n) \). For every choice of \( I = I_1 \in \mathbb{S}_m \), let \( I_2, \ldots, I_m \) be a completion to an orthonormal basis of the algebra \( \mathbb{R}_m \). Then there exists \( 2^{m-1} \) holomorphic mappings \( F_A : D_I \to \mathbb{C}_I^n \) such that for every \( z = \alpha + i\beta \in D_I \) we have
\[ f_I(z) = \sum_{|A|=0}^{m-1} F_A(z)I_A, \quad I_A = I_{i_1} \cdots I_{i_s}. \]
Here \( A = i_1 \ldots i_s \) is a subset of \( \{2, \ldots, n\} \), with \( i_1 < \cdots < i_s \), or, when \( |A| = 0 \), \( I_\emptyset = 0 \).

3. Growth Theorems in the Unit Ball
We consider the unit ball in the set of several variables in the quadratic cones of Clifford algebras \( (\mathbb{Q}_m)^n_s \), i.e.
\[ \mathbb{B} := \left\{ x \in (\mathbb{Q}_m)^n_s \mid \|x\| = \left( \sum_{t=1}^{n} |x_t|^2 \right)^{\frac{1}{2}} < 1, \ x = (x_1, \ldots, x_n) \right\}, \]
then
\[ \mathbb{B}_I = \mathbb{B} \cap \mathbb{C}_I^n, \quad \forall \ I \in \mathbb{S}_m. \]

Lemma 3.1. Let \( f \in S(\Omega_D, (\mathbb{R}_m)^n) \), \( f(D_I) \subseteq \mathbb{C}_I^n \) for some \( I \in \mathbb{S}_m \). Then
\[ \max_{J \in \mathbb{S}_m} \|f(u + Jv)\| = \max_{J = \pm I} \|f(u + Jv)\|, \]
\[ \min_{J \in \mathbb{S}_m} \|f((u + Jv))\| = \min_{J = \pm I} \|f(u + Jv)\|, \]
for each \( u, v \in \mathbb{R}^n, \ J \in \mathbb{S}_m \) with \( u + Jv \in \Omega_D \).

Proof. Since \( f \in S(\Omega_D, (\mathbb{R}_m)^n) \), the representation formula shows that
\[ f(x) = \frac{1}{2}(f(z) + f(\bar{z})) - J \frac{I}{2}(f(z) - f(\bar{z})) \]
for each \( u, v \in \mathbb{R}^n \) and \( J \in \mathbb{S}_m \) with \( x = u + Jv \in D_J \) and \( z = u + Iv \).
Denote
\[
\alpha = \frac{1}{2}(f(z) + f(\bar{z})), \quad \beta = -\frac{I}{2}(f(z) - f(\bar{z})).
\]
By assumption, we have both \(\alpha\) and \(\beta\) in \(\mathbb{C}^n_I.\) We set
\[
\alpha = (\alpha_1, \ldots, \alpha_n), \quad \beta = (\beta_1, \ldots, \beta_n).
\]
For \(\alpha_t \neq 0,\) we set
\[
\beta_t \alpha_t^{-1} = a_t + Ib_t,
\]
where \(a_t, b_t \in \mathbb{R}\) for any \(t \in \{1, \ldots, n\}.
\]
Take \(I_2, \ldots, I_{2m-1} \in \mathbb{S}_m\) such that \(\{1, I_2, \ldots, I_{2m-1}, I, I_2 I, \ldots, I_{2m-1} I\}\) consists of an orthonormal basis of \(\mathbb{R}_m.\) For any \(J \in \mathbb{S}_m\) we can represent it under such a basis as
\[
J = wI + \sum_{l=2}^{2m-1} (\mu_l I_l + \nu_l I_l I)
\]
with real coefficients \(w, \mu_2, \mu_3, \ldots, \mu_{2m-1}, \nu_2, \nu_3, \ldots, \nu_{2m-1} \in \mathbb{R}\) such that
\[
w^2 + \sum_{l=2}^{2m-1} (\mu_l^2 + \nu_l^2) = 1.
\]
We can rewrite
\[
f(x) = \alpha + J\beta,
\]
then
\[
||f(x)||^2 = \sum_{t=1}^n |\alpha_t + J\beta_t|^2
\]
\[
= \sum_{t=1, \alpha_t \neq 0}^n |1 + J\beta_t \alpha_t^{-1}||\alpha_t|^2 + \sum_{t=1, \alpha_t = 0}^n |\beta_t|^2
\]
\[
= \sum_{t=1, \alpha_t \neq 0}^n \left| 1 + \left( wI + \sum_{l=2}^{2m-1} (\mu_l I_l + \nu_l I_l I) \right) (a_t + Ib_t) \right|^2 |\alpha_t|^2
\]
\[
+ \sum_{t=1, \alpha_t = 0}^n |\beta_t|^2
\]
\[
= \sum_{t=1, \alpha_t \neq 0}^n (1 + a_t^2 + b_t^2 - 2b_t w)|\alpha_t|^2 + \sum_{t=1, \alpha_t = 0}^n |\beta_t|^2
\]
\[
= g(w).
\]
Therefore
\[ ||f(x)||_{\text{max}} = \max_{w \in [-1,1]} g(w) = \max_{w = \pm 1} g(w), \]
\[ ||f(x)||_{\text{min}} = \min_{w \in [-1,1]} g(w) = \min_{w = \pm 1} g(w). \]

\[ \square \]

Remark 3.2. In \cite{15}, it obtained the similar result to Lemma 3.1 in the case of \( n = 1 \). Moreover, \( f \) does not need to be a slice monogenic mapping on \( \Omega_D \)(see Remark 2.8), it only need to be a slice mapping on \( \Omega_D \).

For any \( f \in SR(B, (\mathbb{R}_m)^n) \), let \( f'(0) \) be the slice derivative of a slice regular mapping \( f \) at zero, i.e.
\[ f'(0) = \frac{\partial f}{\partial x}(0). \]
(see Definition 2.6).

Theorem 3.3. Let \( f \) be a mapping in \( SR(B, (\mathbb{R}_m)^n) \) such that its restriction \( f_I \) to \( B_I \) is a starlike mapping such that \( f(B_I) \subseteq C_I^\theta \) for some \( I \in S_m \). If \( f(0) = 0, f'(0) = I_n \)(Identity matrix of order \( n \)), then
\[ \frac{||x||}{(1+||x||)^2} \leq ||f(x)|| \leq \frac{||x||}{(1-||x||)^2}, \quad \forall x \in B. \]
These estimates are sharp.

Proof. Since \( f_I : B_I \subset C_I^\theta \to C_I^\theta \) is a starlike mapping such that \( f_I(0) = 0, f_I'(0) = I_n \), we have \cite{12}
\[ \frac{||z||}{(1+||z||)^2} \leq ||f_I(z)|| \leq \frac{||z||}{(1-||z||)^2}, \quad \forall z \in B_I. \]
With \( z \) replaced by \( \bar{z} \), then
\[ \frac{||z||}{(1+||z||)^2} \leq ||f_I(\bar{z})|| \leq \frac{||z||}{(1-||z||)^2}, \quad \forall \bar{z} \in B_I. \]
These estimates are sharp.

For any \( x \in B \), there exists \( z = \alpha + I \beta \in B_I \) and \( J \in S_m \) such that \( x = \alpha + J \beta \in B \) Lemma 3.1 concludes that
\[ ||f(x)|| \geq \min\{||f_I(z)||, ||f_I(\bar{z})||\} \geq \frac{||z||}{(1+||z||)^2} = \frac{||x||}{(1+||x||)^2}. \]
This estimate is sharp.

The reverse inequality can be proved similarly. \[ \square \]

Example 3.1. We consider the function \( f : B \to (\mathbb{R}_m)^n \), defined by
\[ f(x) = (x_1(1 - x_1e^{i\theta})^{-2}, \ldots, x_n(1 - x_ne^{i\theta})^{-2}). \]
Here, for any \( x = (x_1, \ldots, x_n) \in B_I \) with \( x_t = \alpha_t + \beta_t J, \forall J \in S_m, \alpha_t, \beta_t \in \mathbb{R} \).
Let \( z = (z_1, \ldots, z_n) \) with \( z_t = \alpha_t + \beta_t I \) for some \( I \in S_m \),
\[ x_t(1 - x_te^{i\theta})^{-2} := \frac{1 + JI}{2z_t(1 - z_te^{i\theta})^{-2}} + \frac{1 - JI}{2} z_t(1 - z_te^{i\theta})^{-2}, \quad t = 1, \ldots, n. \]
It is easy to see that \( f \in \mathcal{S}(\mathbb{B}, (\mathbb{R}_m)^n) \), \( f(\mathbb{B}_I) \subseteq \mathbb{C}_I^n \) and \( f(0) = 0 \), \( f'(0) = I_n \). Since \( f_I \) is a starlike mapping on \( \mathbb{B}_I \) (see [12]), we have that \( f \) satisfies all the conditions of Theorem 3.3.

For convex mappings, the same approach yields the similar results.

**Theorem 3.4.** Let \( f \in \mathcal{S}(\mathbb{B}, (\mathbb{R}_m)^n) \) such that its restriction \( f_I \) to \( \mathbb{B}_I \) is a convex mapping and such that \( f(\mathbb{B}_I) \subseteq \mathbb{C}_I^n \) for some \( I \in \mathbb{S}_m \). If \( f(0) = 0 \), \( f'(0) = I_n \) (Identity matrix of order \( n \)), then

\[
\|x\|_{1+\|x\|} \leq \|f(x)\| \leq \|x\|_{1-\|x\|}, \quad x \in \mathbb{B}.
\]

These estimates are sharp.

**Example 3.2.** The function \( f : \mathbb{B} \rightarrow (\mathbb{R}_m)^n \), defined by

\[
f(x) = (x_1(1-x_1e^{I\theta})^{-s}, \ldots, x_n(1-x_ne^{I\theta})^{-s}),
\]

provides an example satisfying all the conditions of Theorem 3.4. Here for each \( x = (x_1, \ldots, x_n) \in \mathbb{B}_J \) with \( x_t = \alpha_t + \beta_t I \), \( \forall J \in \mathbb{S}_m \), \( \alpha_t, \beta_t \in \mathbb{R} \) and \( z_t = \alpha_t + \beta_t I \) for some \( I \in \mathbb{S}_m \),

\[
x_t(1-x_te^{I\theta})^{-s} := \frac{1+JI}{2}z_t(1-z_te^{I\theta})^{-1} + \frac{1-JI}{2}z_t(1-z_te^{I\theta})^{-1},
\]

\( t = 1, \ldots, n \).

4. Growth Theorems in the Starlike or Convex Domains

We shall now generalize the results in the last section to more general domains, other than the unit ball.

We consider the bounded slice domains which are slice starlike and slice circular.

**Definition 4.1.** A set \( \Omega \in (\mathcal{Q}_m)_I^n \) is called starlike with respect to a fixed point \( \omega_0 \in \Omega \) if the closed line segment joining \( \omega_0 \) to each point \( \omega \in \Omega \) lies entirely in \( \Omega \). Also we say that \( \Omega \) is convex if for all \( \omega_1, \omega_2 \in \Omega \), the closed line segment between \( \omega_1 \) and \( \omega_2 \) lies entirely in \( \Omega \). In other words, \( \Omega \) is convex if and only if \( \Omega \) is starlike with respect to each of its points. The term starlike will mean starlike with respect to zero.

**Definition 4.2.** A set \( \Omega \in (\mathcal{Q}_m)_I^n \) is called slice starlike if \( \Omega_I \) is starlike in \( \mathbb{C}_I^n \) for some \( I \in \mathbb{S}_m \).

**Definition 4.3.** A domain \( \Omega \in (\mathcal{Q}_m)_I^n \) is called slice circular, if for any \( I \in \mathbb{S}_m \), and \( z \in \Omega_I, \theta \in \mathbb{R} \), we have \( e^{I\theta}z \in \Omega_I \).

**Definition 4.4.** A domain \( \Omega \) is axially symmetric slice circular if and only if for some \( I \in \mathbb{S}_m \),

1. \( \Omega_I \) is circular (i.e. \( e^{I\theta}z \in \Omega_I \) if \( z \in \Omega_I \) and \( \theta \in \mathbb{R} \));
2. \( \Omega \) is axially symmetric (i.e. \( x \in \Omega \) if \( x \in [z] \) and \( z \in \Omega_I \)).
Obviously, the unit ball $B$ and polydisc $P$ are axially symmetric slice circular, where
\[ P := \{ x \in (\mathbb{Q}_m)^n_s \mid |x_t| < 1, x = (x_1, \ldots, x_n), t = 1, 2, \ldots, n \}. \]

**Definition 4.5.** A domain $\Omega \subseteq (\mathbb{Q}_m)^n_s$ is called slice domain, if
\begin{enumerate}
  \item $\Omega \cap \mathbb{R}^n \neq \emptyset$,
  \item $\Omega_I$ is a domain of $(\mathbb{Q}_m)^n_s \cap \mathbb{C}^n_I$, for any $I \in \mathbb{S}_m$.
\end{enumerate}

The bounded slice starlike slice circular and slice domain has an analytic characterization via defining functions.

**Lemma 4.6.** An axially symmetric slice domain $\Omega \subseteq (\mathbb{Q}_m)^n_s$ is bounded slice starlike and slice circular if and only if there exists a unique continuous function $\rho : (\mathbb{Q}_m)^n_s \to \mathbb{R}$, called the defining function of $\Omega$, such that
\begin{enumerate}
  \item $\rho(x) \geq 0$, $\forall x \in (\mathbb{Q}_m)^n_s$; $\rho(x) = 0 \iff x = 0$;
  \item $\rho(tx) = |t|\rho(x)$, $\forall J \in \mathbb{S}_m, \forall x \in \mathbb{C}^n_J$, $t \in \mathbb{C}_J$;
  \item $\Omega = \{ x \in (\mathbb{Q}_m)^n_s : \rho(x) < 1 \}$. \( \blacksquare \)
\end{enumerate}

**Proof.** The proof is similar to the case of several complex variables [13]. If the continuous function $\rho(x)$ satisfies (1), (2), and (3), then clearly $\Omega$ is a starlike slice circular domain. Its boundedness is also easy to prove, since if there exists a ray coming from the origin which completely falls in the starlike domain $\Omega$, then for any fixed point $x_0$ in this ray we have from (3)
\[ \rho(tx_0) < 1, \quad \forall t \in [0, \infty). \]

But we then obtain $\rho(x_0) = 0$ in terms of (2). This contradicts (1).

Conversely, if $\Omega$ is a bounded slice starlike and slice circular domain in $(\mathbb{R}^{m+1})^n_s$, then we define
\[ \rho(x) = \inf\{ c > 0 : c^{-1}x \in \Omega \}. \]

Obviously, $\rho(x)$ satisfies (2), (3), and $\rho(x) \geq 0$ for all $x$. If there exists a point $x_0 \neq 0$, such that $\rho(x_0) = 0$, then it follows from (2) and (3) that $\Omega$ includes the whole ray which comes from the origin and through the point $x_0$, hence $\Omega$ is unbounded. Thus (1) holds. Finally we prove that $\rho(x)$ is continuous. Clearly,
\[ \{ x \in (\mathbb{Q}_m)^n_s : r < \rho(x) < R \} = R\Omega \setminus \overline{r\Omega} \]
is an open set in $(\mathbb{Q}_m)^n_s$, which implies the continuity of $\rho$. \( \blacksquare \)

**Remark 4.7.** From the proof of Lemma 4.6, we have that
\[ \rho(x) = \rho(z), \quad \forall z \in [x] \subseteq \Omega, \]
where $\Omega$ is a bounded slice starlike and slice circular and slice domain.

Now we can state our main result about the growth theorem in bounded slice starlike circular and slice domain.
Theorem 4.8. Let $\Omega_D$ be a bounded slice starlike slice circular slice domain in $(\mathbb{Q}_m)^n$, its defining function $\rho(x)$ is a $C^1$ function on $\Omega_D$ except for a lower dimensional set. If $f \in \mathcal{SR}(\Omega_D, (\mathbb{R}_m)^n)$ such that its restriction $f_I$ to $D_I$ is a starlike mapping such that $f(D_I) \subseteq \mathbb{C}_I^n$. If $f(0) = 0, f'(0) = I_n$ (Identity matrix of order $n$), then for any $x \in \Omega_D$,

$$\frac{\rho(x)}{1 + \rho(x)} \leq \| f(x) \| \leq \frac{\rho(x)}{1 - \rho(x)}$$

or equivalently,

$$\frac{\| x \|}{1 + \rho(x)} \leq \| f(x) \| \leq \frac{\| x \|}{1 - \rho(x)}.$$

Proof. Since $\Omega_D$ is a bounded slice starlike slice circular domain in $(\mathbb{Q}_m)^n$, $D_I$ is bounded, complex circular in $\mathbb{C}_I^n$, and is also starlike in $\mathbb{C}_I^n$, for any $I \in \mathbb{S}_m$. By the assumption that $\rho(x)$ is a $C^1$ function on $\Omega_D$, we know that $\rho(z)$ is a $C^1$ function on $\mathbb{C}_I^n$. Notice that $f_I$ is a biholomorphic starlike mapping on $D_I$ and admits the properties that $f(D_I) \subseteq \mathbb{C}_I^n$ and $f(0) = 0, f'(0) = I_n$. Thanks to the classical results in several complex variables [13], we have

$$\frac{\rho(z)}{1 + \rho(z)} \leq \| f(z) \| \leq \frac{\rho(z)}{1 - \rho(z)}, \quad \forall z \in D_I,$$

as well as

$$\frac{\| z \|}{1 + \rho(z)} \leq \| f(z) \| \leq \frac{\| z \|}{1 - \rho(z)} \quad \forall z \in D_I.$$

By symmetry, we also have

$$\frac{\rho(\bar{z})}{1 + \rho(\bar{z})} \leq \| f(\bar{z}) \| \leq \frac{\rho(\bar{z})}{1 - \rho(\bar{z})}, \quad \forall z \in D_I.$$

Suppose $x = \alpha + J\beta \in \Omega_D$ for any $J \in \mathbb{S}_m$ and $z = \alpha + I\beta \in D_I$. By Lemma 3.1, we have

$$\| f(x) \| \geq \min\{\| f_I(z) \|, \| f_I(\bar{z}) \| \} \geq \frac{\rho(z)}{1 + \rho(z)} = \frac{\rho(x)}{1 + \rho(x)}.$$

The remaining inequality can be proved similarly.

Likewise, from the results in several complex variables [14], we can establish the growth theorem in bounded convex slice circular domain for convex mappings.

Theorem 4.9. Let $\Omega_D$ be a bounded convex slice circular slice domain in $(\mathbb{Q}_m)^n$, with defining function $\rho(x)$. If $f \in \mathcal{SR}(\Omega_D, (\mathbb{R}_m)^n)$ such that its restriction $f_I$ to $D_I$ is a convex mapping such that $f(D_I) \subseteq \mathbb{C}_I^n$. If $f(0) = 0, f'(0) = I_n$ (identity matrix of order $n$), then for any $x \in \Omega_D$,

$$\frac{\rho(x)}{1 + \rho(x)} \leq \| f(x) \| \leq \frac{\rho(x)}{1 - \rho(x)}$$
or equivalently,
\[
\frac{\| x \|}{1 + \rho(x)} \leq \| f(x) \| \leq \frac{\| x \|}{1 - \rho(x)}.
\]

5. Final Remarks

In this paper, we have proved the growth theorem for slice monogenic extensions of starlike and convex mappings on the unit ball in the subset of slice several variables in quadratic cones of Clifford algebras \((Q_m)_n\). However, the corresponding distortion theorem still deserves further investigation.

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Guangbin Ren and Ting Yang
Department of Mathematics
University of Science and Technology of China
Hefei 230026
China
e-mail: tingy@mail.ustc.edu.cn

Guangbin Ren
e-mail: rengb@ustc.edu.cn

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