Generic Newton polygon for exponential sums in $n$ variables with parallelootope base

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GENERIC NEWTON POLYGON FOR EXPONENTIAL SUMS IN $n$ VARIABLES WITH PARALLELOTOPE BASE

By Rufei Ren

Abstract. Let $p$ be a prime number. Every $n$-variable polynomial $f(x)$ over a finite field of characteristic $p$ defines an Artin-Schreier-Witt tower of varieties whose Galois group is isomorphic to $\mathbb{Z}_p$. Our goal of this paper is to study the Newton polygon of the $L$-function associated to a nontrivial finite character of $\mathbb{Z}_p$ and a generic polynomial whose convex hull is an $n$-dimensional paralleltope $\Delta$. We denote this polygon by $\text{GNP}(\Delta)$. We prove a lower bound of $\text{GNP}(\Delta)$, which is called the improved Hodge polygon $\text{IHP}(\Delta)$. We show that $\text{IHP}(\Delta)$ lies above the usual Hodge polygon $\text{HP}(\Delta)$ at certain infinitely many points, and when $p$ is larger than a fixed number determined by $\Delta$, it coincides with $\text{GNP}(\Delta)$ at these points. As a corollary, we roughly determine the distribution of the slopes of $\text{GNP}(\Delta)$.

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1. Introduction.
2. Dwork’s trace formula.
3. The improved Hodge polygon.
4. The generic Newton polygon.
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References.

1. Introduction. We shall state our main results and their motivation after recalling the notion of $L$-functions for Witt coverings. Let $p$ be a prime number, and

$$f(x) := \sum_{P \in \mathbb{Z}_p^n \geq 0} a_P x^P$$

be an $n$-variable polynomial in $\mathbb{F}_p[x_1, \ldots, x_n]$. We denote by $\mathbb{F}_p(f)$ the coefficient field of $f(x)$ and set $m(f) := [\mathbb{F}_p(f) : \mathbb{F}_p]$. Then we put $\hat{a}_P \in \mathbb{Z}_p^{m(f)}$ to be the Teichmüller lift of $a_P$, where $\mathbb{Z}_p^{m(f)}$ is the unramified extension of $\mathbb{Z}_p$ of degree $m(f)$, and call

$$\hat{f}(x) := \sum_{P \in \mathbb{Z}_p^n \geq 0} \hat{a}_P x^P$$

the Teichmüller lift of $f(x)$. 
Viewing $\mathbb{Z}^n$ as the lattice points in $\mathbb{R}^n$ with origin denoted by $O$, we call the convex hull of $O \cup \{P \mid a_P \neq 0\}$ the polytope of $f$ and denote it by $\Delta_f$.

The Artin-Schreier-Witt tower associated to $f$ is the sequence of varieties $\mathcal{V}_i$ over $\mathbb{F}_{p^m(f)}$ defined by the following equations:

$$\mathcal{V}_i : y_i^F - y_i = \sum_{P \in \Delta_f} (a_P x_P^P, 0, 0, \ldots)_i,$$

where $y_i = (y_i^{(1)}, y_i^{(2)}, \ldots, y_i^{(i)})$ are Witt vectors of length $i$, and $\bullet^F$ means raising each Witt coordinate to the $p$th power. The Artin-Schreier-Witt tower $\cdots \to \mathcal{V}_i \to \cdots \to \mathcal{V}_0 := \mathbb{A}^n$ is a tower of Galois covers of $\mathbb{A}^n$ with total Galois group $\mathbb{Z}_p$, and consequently the study of zeta function of the tower can be reduced to the study of the $L$-functions associated to (finite) characters of the Galois group $\mathbb{Z}_p$. For more details we refer the readers to [DWX, Section 1].

Let $(\mathbb{G}_m)^n$ be the $n$-dimensional torus over $\mathbb{F}_{p^m(f)}$. The main subject of our study is the $L$-function associated to a finite character $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$ of conductor $p^m\chi$ which is given by

$$L^*_f(\chi, s) := \prod_{x \in [(\mathbb{G}_m)^n]} \frac{1}{1 - \chi(\text{Tr}_{\mathbb{Q}_p^{p^m(f)}(\hat{x})}/\mathbb{Q}_p(\hat{\mathbb{A}^n}))} s^{\deg(x)},$$

where $[(\mathbb{G}_m)^n]$ is the set of closed points of $(\mathbb{G}_m)^n$, $\hat{x}$ is the Teichmuller lift of a geometric point at $x$, and $\deg(x)$ stands for the degree of $x$ over $\mathbb{F}_{p^m(f)}$.

It is proved in [LWei, Theorem 1.3] that when $f$ is a non-degenerate polynomial with convex hull $\Delta_f$, the function

$$L^*_f(\chi, s)(-1)^{-n-1} = \sum_{i=0}^{n!p^{n(m-1)}\text{Vol}(\Delta_f)} v_i s^i \in \mathbb{Z}_p[\zeta_{p^m\chi}][s]$$

is a polynomial of degree $n!p^{n(m-1)}\text{Vol}(\Delta_f)$, where $\zeta_{p^m\chi}$ is a primitive $p^m\chi$th root of unity.

In this paper, we confine ourselves to studying these $f$ whose polytopes are of the simpler shape. Let $\Delta$ be an $n$-dimensional parallelotope generated by linearly independent vectors $\overrightarrow{OV}_1, \overrightarrow{OV}_2, \ldots, \overrightarrow{OV}_n$, where $V_1, V_2, \ldots, V_n$ are integral points.

**Hypothesis 1.1.** We assume that $p$ is a prime such that $p \nmid (n!\text{Vol}(\Delta))$ and $p > (n+4)D$, where $D$ is a positive integer depending only on $\Delta$ (See Definition 2.2).

In particular, when $\Delta$ is an $n$-dimensional cube with side length $d$, the integer $D = d$. 

**Notation 1.2.** For every integer \( k \geq 1 \), let \( k\Delta \) denote a scaling of \( \Delta \). We write
\[
\Delta^+_k := k\Delta \cap \mathbb{Z}^n
\]
for the set consisting of the lattice points in \( k\Delta \) and put
\[
\# \Delta^+_k := \# \Delta^+_k.
\]
Let \( \Delta^\circ \) denote the parallelotope \( \Delta \) with all faces not containing \( O \) removed. We put
\[
\Delta^-_k := k\Delta^\circ \cap \mathbb{Z}^n \quad \text{and} \quad \# \Delta^-_k := \# \Delta^-_k.
\]
For simplicity, we write \( \Delta^\pm \) for \( \Delta^\pm_1 \) when no confusion can rise.

**Notation 1.3.** One may naturally identify the set of polynomials with polytope \( \Delta \) as an open subscheme of \( \mathbb{F}_p^{\Delta^\pm} \) by recording the coefficients \( a_P \) of \( f \).

**Definition 1.4.** If \( f \) satisfies the non-degenerate condition in [LWei], we call the lower convex hull of the set of points \( \left( i, p^{m_{x-1}}(p-1)\text{val}_{p^m(f)}(v_i) \right) \) the normalized Newton polygon of \( L^*_f(\chi,s)(-1)^{n-1} \) which is denoted by \( \text{NP}(f,\chi)_L \). Here, \( \text{val}_{p^m(f)}(-) \) is the \( p \)-adic valuation normalized so that \( \text{val}_{p^m(f)}(p^m(f)) = 1 \).

The following Theorems 1.5 and 1.8 are the main results of this paper.

**Theorem 1.5.** Assume Hypothesis 1.1. Let \( \overline{E}(\Delta) \) denote the set of all non-degenerate polynomials \( f(x) = \sum_{P \in \Delta^+_x} a_P x^P \in \mathbb{F}_p[x] \) with \( \Delta_f = \Delta \). Then there is a Zariski open subset \( \mathcal{O}_{\text{Zar}} \subset \overline{E}(\Delta) \) such that for any \( f \in \mathcal{O}_{\text{Zar}} \) and any finite character \( \chi : \mathbb{Z}_p \to \mathbb{C}_p^\times \) of conductor \( p^{m_{x-1}} \), if we put \( \{\alpha_1, \ldots, \alpha_{p^{m_{x-1}}n!\text{Vol}(\Delta)}\} \) to be the set of \( p^{m_{x-1}} \)-adic Newton slopes of \( L^*_f(\chi,s)(-1)^{n-1} \), it has the following distribution.

For every \( 0 \leq i_1 \leq n-1 \) and every \( 0 \leq i_2 \leq p^{m_{x-1}}-1 \) we have
\[
\begin{align*}
\# \left\{ \alpha_j \mid \alpha_j \in \left( i_1 + \frac{i_2}{p^{m_{x-1}}}, i_1 + \frac{i_2 + 1}{p^{m_{x-1}}} \right) \right\} \\
= \sum_{t=0}^{i_1} (-1)^t \binom{n}{t} \left( \frac{\# \Delta^-_k}{(i_1-t)p^{m_{x-1}}+i_2+1} - \frac{\# \Delta^+_k}{(i_1-t)p^{m_{x-1}}+i_2} \right),
\end{align*}
\]
\[
\begin{align*}
\# \left\{ \alpha_j \mid \alpha_j = i_1 + \frac{i_2}{p^{m_{x-1}}} \right\} \\
= \sum_{t=0}^{i_1} (-1)^t \binom{n}{t} \left( \frac{\# \Delta^+_k}{(i_1-t)p^{m_{x-1}}+i_2} - \frac{\# \Delta^-_k}{(i_1-t)p^{m_{x-1}}+i_2} \right),
\end{align*}
\]
\[ C^*_f(\chi, s) := \left( \prod_{j=0}^{\infty} L^*_f(\chi, p^m(f)^j s)^{\binom{n+j-1}{n-1}} \right)^{(-1)^{n-1}}, \]

whose normalized $p$-adic Newton polygon is denoted by $NP(f, \chi)_C$.

**Definition 1.7.** The generic Newton polygon of $\Delta$ is defined by

\[ GNP(\Delta) := \inf_{\chi: \mathbb{Z}_p/p^m \chi \mathbb{Z}_p \to \mathbb{C}_p^\times, \Delta_f=\Delta} (NP(f, \chi)_C), \]

where $\chi: \mathbb{Z}_p \to \mathbb{C}_p^\times$ runs over all nontrivial finite characters, and $f$ runs over all non-degenerate polynomials in $\mathbb{F}_p[x]$ such that $\Delta_f = \Delta$.

**Theorem 1.8.** The generic Newton polygon $GNP(\Delta)$ passes through the points $(\pi_k^\pm(\Delta), h(\Delta_k^\pm))$ for any $k \geq 0$, where $\pi_k^\pm$ and $h(\Delta_k^\pm)$ are defined in Notation 2.1 and Definition 3.11 respectively.

In [DWX], Davis, Wan, and Xiao studied the $p$-adic Newton slopes of $L^*_f(\chi, s)$ and $C^*_f(\chi, s)$ when $f$ is a one-variable polynomial whose degree $d$ is coprime to $p$. They concluded that, for each character $\chi: \mathbb{Z}_p \to \mathbb{C}_p^\times$ of a relatively large conductor, the $p$-adic Newton slopes of $L^*_f(\chi, s)$ are in a finite union of arithmetic progressions. Their proof strongly inspired the proof of spectral halo conjecture by Liu, Wan, and Xiao in [LWX]; we refer to [RWXY, Section 1.5] for the discussion on the analogy of the two proofs.

Motivated by the attempt of extending spectral halo type results beyond the case of modular forms, it is natural to ask whether one can generalize the main results of [DWX] to more general cases:

1. changing the tower to $\mathbb{Z}_p^\ell$ for $\ell \geq 2$, and
2. making the base to higher dimensional.

The first case is examined in a joint work with Wan, Xiao, and Yu see [RWXY]. The goal of this current paper is to investigate the second case.

From the Iwasawa theory point of view, it is important to have access to the Newton polygon $NP(f, \chi)_C$ associated to this Artin-Schreier-Witt tower. When $p$ is “ordinary”, this polygon was explicitly computed by Adolphson-Sperber [AS], Berndt-Evans [BE], and Wan [Wan] in many special cases, and by Liu-Wan [LWan] in the general case (and in the $T$-adic setup).
Going beyond the ordinary case, there has been many researches on understanding the generic Newton polygon of $L_f(\chi, s)$ when $f$ is a polynomial of a single variable. Here is an incomplete list.

- In [DWX], Davis, Wan, and Xiao proved that the Newton slopes of $L_f(\chi, s)$ form a finite union of arithmetic progressions, when $f$ is a one-variable polynomial and $\chi$ is a finite character of a relatively large conductor.
- When $p$ is large enough, Zhu [Zhu1] and Scholten-Zhu [SZ] showed that for a non-degenerate one-variable polynomial $f$ and a finite character $\chi_0$ of conductor $p$, the Newton polygon $NP(f,\chi_0)_L$ coincides $GNP(\Delta)$.
- Later, Blache, Ferard, and Zhu in [BFZ] proved a lower bound for the Newton polygon of $f(x) \in \mathbb{F}_q[x, \frac{1}{x}]$ of degree $(d_1, d_2)$, which is called a Hodge-Stickelberger polygon. They also showed that when $p$ approaches to infinity, the Newton polygon $NP(\chi, f)_L$ coincides with the Hodge-Stickelberger polygon.
- In [BF], Blache and Ferard worked on the generic Newton polygon associated to characters of large conductors.
- In [OY], Ouyang and Yang studied the one-variable polynomial $f(x) = x^d + a_1 x$. A similar result can be found in [OZ], where Ouyang and Zhang studied the family of polynomials of the form $f(x) = x^d + a_{d-1} x^{d-1}$.
- In [KW], Koster and Wan studied a more general $\mathbb{Z}_p$-tower, and they proved the genus stability of all such $\mathbb{Z}_p$-tower.

However, for technical reasons, it is difficult to prove that the slopes of the $L$-function form a union of arithmetic progression when $f$ is a multi-variable polynomial. Zhu in [Zhu2] shows that $GNP(\Delta)$ and $IHP(\Delta)$ coincide for characters of $\mathbb{Z}_p$ of conductor $p$ when $\Delta$ is a rectangular and $p$ is large enough. A similar result is obtained by the author in [Ren] when the polytope of $f$ is an isosceles right triangle.

In this paper, we focus on the generic Newton polygon of an $n$-dimensional parallelotope $\Delta$. Our main contribution in this paper is to prove the distribution of slopes of the generic Newton polygon $GNP(\Delta)$, when $p$ is not necessary to be ordinary with respect to $\Delta$. We refer reader to Theorem 1.5 for the statement.

Now we list the key steps of the proof of our main theorems.

Step 1. Instead of working with the $L$-function itself, it is more convenient (from the point of view of Dwork trace formula) to work with the characteristic power series $C_f^*(\chi, s)$, which recovers the $L$-function by

$$L_f^*(\chi, s) = \prod_{j=0}^{n-1} C_f^*(\chi, p^{m(f)} s^{\frac{1}{p^n}})^{-1}.$$ (1.2)

The power series $C_f^*(\chi, s)$ is genuinely the characteristic power series of a nuclear operator (or equivalently an infinite matrix $N$ with respect to some canonical basis). Moreover, we can do this for the universal character (as opposed to just finite characters) of the Galois group of the tower $\mathbb{Z}_p$. 
Step 2. We construct the improved Hodge polygon $\text{IHP}(\Delta)$ for $\Delta$ in Definition 3.1, and prove that it is a lower bound of $\text{NP}(f, \chi)_C$ for every nontrivial finite character $\chi$. The polygon $\text{IHP}(\Delta)$ lies above the usual Hodge polygon at $x = \pm x_k$ for every $k \geq 1$. In fact, we show in Proposition 3.3 the condition that $\text{IHP}(\Delta)$ lies strictly above the usual Hodge polygon at $x = \pm x_k$.

The key point of this step lies in: the usual way of obtaining Hodge polygon is to conjugate $N$ by an appropriate diagonal matrix, and observe that each row is entirely divisible by a certain power of $p$. In this paper, we dig into the definition of characteristic power series as the sum over permutations, which allows us to slightly but crucially improve the usual Hodge polygon.

Step 3. We show in Proposition 4.3 that for every polynomial $f \in \mathcal{F}(\Delta)$, if there is a finite character $\chi_0$ of conductor $p$ such that $\text{NP}(f, \chi_0)_C$ coincides with $\text{IHP}(\Delta)$ at

$$x = \pm x_k \quad \text{for } 1 \leq k \leq n + 2,$$

then for every nontrivial finite character $\chi$, $\text{NP}(f, \chi)_C$ and $\text{IHP}(\Delta)$ coincide at

$$x = \pm x_k \quad \text{for all } k \geq 0.$$

This proposition reduces the problem to show that all the polynomials $f \in \mathcal{F}(\Delta)$ such that $\text{NP}(f, \chi_0)_C$ and $\text{IHP}(\Delta)$ coincide at $x = \pm x_k$ for $1 \leq k \leq n + 2$ form a Zariski open dense subset of $\mathcal{F}(\Delta)$.

For this, one may consider the characteristic power series for a “universal” polynomial $\tilde{f}$, namely all coefficients of $\tilde{f}$ are viewed as variables. We need to show that when we write $\tilde{u}_{x_k} = \tilde{u}_{x_k}^{\pm} \cdot h(\Delta_k^\pm) + O(T^h(\Delta_k^\pm) + 1)$, the coefficients

$$\tilde{u}_{x_k}^{\pm} \cdot h(\Delta_k^\pm) \not\equiv 0 \pmod{p} \quad \text{for every } 1 \leq k \leq n + 2. \quad (1.3)$$

Step 4. We show (1.3) in Section 5. The technical core of this paper lies in proving (1.3). Roughly speaking, the key is to show that for each $0 \leq k \leq n + 2$, a certain monomial of $\tilde{u}_{x_k}$ is nonzero. Tracing back to the definition of $\tilde{u}_{x_k}$, we see the contribution to such leading monomial must come from a unique special permutation that appears in the definition of the characteristic power series. Computing explicitly the contribution of this special permutation to the leading term, which itself was subdivided into simpler cases, allows us to prove (1.3).

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2. **Dwork’s trace formula.** In this section, we will introduce Dwork trace formula to express $C_f^T(\chi, s)$ as the characteristic power series of some infinite matrix and deduce a natural lower bound of $\text{NP}(T, f)_C$ called the improved Hodge polygon.

Recall from introduction that $\Delta$ is an $n$-dimensional paralletope generated by linearly independent vectors $\overrightarrow{OV}_1, \overrightarrow{OV}_2, \ldots, \overrightarrow{OV}_n$.

**Notation 2.1.** We denote the cone of $\Delta$ by

$$\text{Cone}(\Delta) := \{ Q \in \mathbb{R}^n | kQ \in \Delta \text{ for some } k \in \mathbb{R}_{>0} \}$$

and put

$$\mathbb{M}(\Delta) := \text{Cone}(\Delta) \cap \mathbb{Z}^n$$

to be the set of integer point points in $\text{Cone}(\Delta)$.

**Definition 2.2.** Let $D$ be the smallest positive integer such that $\mathbb{M}(\Delta) \subset \{ z_1 \overrightarrow{V}_1 + z_2 \overrightarrow{V}_2 + \cdots + z_n \overrightarrow{V}_n | z_i \in \frac{1}{D} \mathbb{Z}_{\geq 0} \}$. (2.1)

In particular, when $\Delta$ is an $n$-cube with side length $d$, we have $D = d$.

**Notation 2.3.** Let

$$\Lambda_{\Delta} := \bigoplus_{i=1}^{n} \mathbb{Z}_{\geq 0} \cdot \overrightarrow{V}_i \subset \mathbb{M}(\Delta).$$

From now on, let $p$ be a prime satisfying Hypothesis 1.1, and let $\mathbb{E}(\Delta)$ denote the set of all non-degenerate polynomials $f(x) = \sum_{P \in \Delta^+} a_P x^P \in \mathbb{F}_p[x]$ with $\Delta_f = \Delta$. We denote by $\mathbb{F}_p(f)$ the coefficient field of $f$ which is the finite field generated by the coefficients of $f$.

We will fix such a polynomial $f \in \mathbb{E}(\Delta)$ in Sections 2 and 3. We set $\mathbb{F}_q := \mathbb{F}_p(f)$ and $m = [\mathbb{F}_q : \mathbb{F}_p]$. Let $\hat{a}_P \in \mathbb{Z}_q$ be the Teichmüller lift of $a_P$. We call $\hat{f}(x) := \sum_{P \in \Delta^+} \hat{a}_P x^P$ the Teichmüller lift of $f(x)$.

**2.1. $T$-adic exponential sums.**

**Notation 2.4.** We recall that the Artin-Hasse exponential series is defined by

$$E(\pi) := \sum_{i=0}^{\infty} c_i \pi^i = \exp \left( \sum_{i=0}^{\infty} \frac{\pi^p}{p^i} \right) \in 1 + \pi + \pi^2 \mathbb{Z}_p[[\pi]].$$

Setting $E(\pi) = 1 + T$ gives an isomorphism $\mathbb{Z}_p[[\pi]] \cong \mathbb{Z}_p[[T]]$.

**Definition 2.5.** For a ring $R$ and a power series $g \in R[T]$, we define its $T$-adic valuation, denoted by $\text{val}_T(g)$, as the largest integer $k$ such that $g \in T^k R[T]$. 
**Definition 2.6.** For every \( k \geq 1 \), the \( T\)-adic exponential sum of \( f \) over \( \mathbb{F}_{q^k}^\times \) is

\[
S^*_f(k, T) := \sum_{x \in (\mathbb{F}_{q^k}^\times)^n} (1 + T)^{\text{Tr}_{q^k/q}f(\hat{x})} \in \mathbb{Z}_p[[T]].
\]

**Definition 2.7.** The \( T\)-adic characteristic power series of \( f \) is defined by

\[
C^*_f(T, s) := \exp \left( \sum_{k=1}^{\infty} -(q^k - 1)^{-n} S^*_f(k, T) \frac{s^k}{k} \right) = \sum_{k=0}^{\infty} u_k(T)s^k \in \mathbb{Z}_p[[T, s]].
\]

It is not difficult to check that every nontrivial finite character \( \chi : \mathbb{Z}_p \to \mathbb{C}_p^\times \) satisfies

\[
C^*_f(\chi, s) = C^*_f(T, s) \big|_{T=\chi(1)-1},
\]

where \( C^*_f(\chi, s) \) is defined in Section 1. We refer the readers to [DWX, Section 2] for the proof.

**Notation 2.8.** We put

\[
E_f(x) := \prod_{P \in \Delta^+} E(\tilde{a}_P \pi x^P) \in \mathbb{Z}_q[[T]][x]
\]

and

\[
\prod_{P \in \Delta^+ \setminus \{O\}} E(\tilde{a}_P \pi x^P) := \sum_{Q \in \mathbb{Z}_q^n} e_Q(T)x^Q \in \mathbb{Z}_q[[T]][x].
\]

We shall later in Sections 4 and 5 need a version of \( E_f(x) \) for a universal polynomial \( \tilde{f}(x) \). Namely, we consider the universal polynomial

\[
\tilde{f}(x) = \sum_{P \in \Delta^+} \tilde{a}_P x^P \in \mathbb{F}_p[[\tilde{a}_P; P \in \Delta^+]][x],
\]

where \( \tilde{a}_P \) are treated as variables. Then we put

\[
\prod_{P \in \Delta^+ \setminus \{O\}} E(\tilde{a}_P \pi x^P) := \sum_{Q \in \mathbb{Z}_q^n} \tilde{e}_Q(T)x^Q \in \mathbb{Z}_p[[\tilde{a}_P; P \in \Delta^+ \setminus \{O\}]][T][x].
\]

2.2. **Dwork’s trace formula.**

**Definition 2.9.** We fix a \( D \)th root \( T^{1/D} \) of \( T \). Define

\[
\mathcal{B} = \left\{ \sum_{Q \in \mathcal{M}(\Delta)} b_Q x^Q \mid b_Q \in \mathbb{Z}_q[T^{1/D}], \text{ val}_T(b_Q) \longrightarrow +\infty, \text{ when } Q \longrightarrow \infty \right\}.
\]
Let $\psi_p$ denote the operator on $\mathcal{B}$ such that

$$\psi_p\left(\sum_{Q \in \mathcal{M}(\Delta)} b_Q x^Q\right) := \sum_{Q \in \mathcal{M}(\Delta)} b_p Q x^Q.$$ 

**Definition 2.10.** Define

$$\psi := \sigma^{-1}_{\text{Frob}} \circ \psi_p \circ E_f(x) : \mathcal{B} \rightarrow \mathcal{B},$$

where $\sigma_{\text{Frob}}$ represents the arithmetic Frobenius acting on the coefficients, and $E_f(x)(g) := E_f(x) \cdot g$ for every $g \in \mathcal{B}$.

Note that its $k$th iterate satisfies

$$\psi^k = \sigma^{-k}_{\text{Frob}} \circ \psi^k_p \circ \prod_{i=0}^{k-1} E_f^k(x_1^{p^i}, x_2^{p^i}, \ldots, x_n^{p^i}).$$

**Lemma 2.11.** Let $N$ be the matrix of $\psi$ acting on $\mathcal{B}$ with respect to the basis $\{x^Q\}$, then the entries

$$N_{Q', Q} = E(\hat{a}_{\mathcal{O}} \pi) e_{pQ'-Q}(T),$$

where $e_{pQ'-Q}(T)$ is defined in (2.6).

**Proof.** From

$$\psi_p \circ E_f(x)(x^Q) = \psi_p\left(E(\hat{a}_{\mathcal{O}} \pi) \sum_{Q'' \in \mathcal{M}(\Delta)} e_{Q''}(T)x^{Q''+Q}\right)$$

$$= E(\hat{a}_{\mathcal{O}} \pi) \sum_{Q'' \in \mathcal{M}(\Delta)} e_{Q''}(T)x^{Q'}$$

$$= \sum_{Q'' \in \mathcal{M}(\Delta)} E(\hat{a}_{\mathcal{O}} \pi) e_{pQ'-Q}(T)x^{Q'},$$

we complete the proof. \(\square\)

Recall that $m = [\mathbb{F}_q : \mathbb{F}_p]$ and $C_f^*(T, s) = \sum_{k=0}^{\infty} u_k(T)s^k \in \mathbb{Z}_p[[T, s]]$.

**Theorem 2.12.** (Dwork Trace Formula) For every integer $k \geq 1$, we have

$$(-1)^{n-1}(q^k - 1)^n S_f^*(k, \pi) = \text{Tr}_{\mathcal{B}/\mathbb{Z}_q[[\pi]]}(\psi^{mk}).$$

**Proof.** See [LWan, Lemma 4.7]. \(\square\)

We refer the reader to [Wan] for a thorough treatment of the universal Dwork trace formula.
THEOREM 2.13. (Analytic trace formula) The theorem above has an equivalent multiplicative form:

\[ C_f^*(T, s) = \det \left( I - s\psi^m | B/\mathbb{Z}_q[\pi] \right). \]  

\[ (2.9) \]

Proof. It follows from Dwork trace formula. For proof, see [LWan, Theorem 4.8]. \[\square\]

Definition 2.14. The normalized Newton polygon of \( C_f^*(T, s) \), denoted by \( \text{NP}(f, T)_C \), is the lower convex hull of the set of points \( \left\{ \left( i, \frac{\text{val}_T(u_i(T))}{m} \right) \right\} \).

Now we recall the weight function and the usual Hodge polygon.

Definition 2.15. For an integer point \( Q \) in \( \mathbb{Z}^n \), assume that the line \( OQ \) intersects some surface of \( \Delta \) at a point \( Q' \). Then we define the weight of \( Q \) by the dilation \( d \in Q \) such that \( OQ = d \cdot OQ' \), and denote it by \( w(Q) \). Note that \( w(Q) \) is negative if \( OQ \) and \( OQ' \) have opposite directions.

Definition 2.16. Let \( \mathcal{W}_\ell \) denote a set consisting of \( \ell \) elements of \( \mathbb{M}(\Delta) \) with minimal weights. The usual Hodge polygon, denoted by \( \text{HP}(\Delta) \), is the lower convex hull of

\[ \left\{ \left( \ell, \sum_{Q \in \mathcal{W}_\ell} (p - 1)w(Q) \right) \right\}. \]

PROPOSITION 2.17. The Newton polygon \( \text{NP}(f, T)_C \) lies on or above \( \text{HP}(\Delta) \).

Proof. See [LWei, Theorem 1.3]. \[\square\]

Definition 2.18. We call \( p \) ordinary with respect to \( \Delta \) if \( p \equiv 1 \) (mod \( D \)).

3. The improved Hodge polygon. As we have explained in Section 2 that for \( f \in \mathcal{F}(\Delta) \), the Newton polygon \( \text{NP}(f, T)_C \) lies on or above the Hodge polygon introduced in Definition 2.16. However, unless \( p \) is ordinary with respect to \( \Delta \), this Hodge polygon is not expected to be sharp. In this section, we introduce an improved Hodge polygon which is again a lower bound of \( \text{NP}(f, T)_C \), and we prove that for a generic \( f \), our improved Hodge polygon coincides with \( \text{NP}(f, T)_C \) at infinitely many points.

Definition 3.1. The improved Hodge polygon of \( \Delta \), denoted by \( \text{IHP}(\Delta) \), is the lower convex hull of the set of points

\[ \left\{ \left( \ell, \sum_{Q \in \mathcal{W}_\ell} \left( \lfloor w(pQ) \rfloor - \lfloor w(Q) \rfloor \right) \right) \right\}, \]

\[ (3.1) \]

where \( \mathcal{W}_\ell \) consists of \( \ell \) elements of \( \mathbb{M}(\Delta) \) with minimal weights as in Definition 2.16.
We will later give a simplified expression of this polygon at some particular points.

**Remark 3.2.** Each point \( Q \in \mathbb{M}(\Delta) \) can be written as a rational linear combination \( \sum_{i=1}^{n} z_i V_i \) of the basis vectors. It is straightforward to see that

\[
(3.2) \quad w(Q) = \max_{1 \leq i \leq n} \{ z_i \}.
\]

In particular, since each \( z_i \in \frac{1}{D} \mathbb{Z} \), we have \( w(Q) \in \frac{1}{D} \mathbb{Z} \) for every \( Q \in \mathbb{M}(\Delta) \).

Moreover, the weight function is subadditive, namely, for every two points \( Q_1, Q_2 \in \mathbb{M}(\Delta) \), we have

\[
(3.3) \quad w(Q_1) + w(Q_2) \geq w(Q_1 + Q_2).
\]

**Proposition 3.3.** (1) The improved Hodge polygon \( IHP(\Delta) \) lies on or above \( HP(\Delta) \) at \( x = \frac{\pm}{D} \mathbb{Z} \) for every \( k \geq 1 \).

(2) If there is a point \( P_0 = \sum_{i=1}^{n} r_i V_i \in \Delta^- \) and \( j_1, j_2 \in \{1, 2, \ldots, n\} \) such that

(a) \( r_{j_1} < r_{j_2} \), and

(b) \( pr_{j_1} - \lfloor pr_{j_1} \rfloor > pr_{j_2} - \lfloor pr_{j_2} \rfloor \),

then \( IHP(\Delta) \) lies strictly above \( HP(\Delta) \) at \( x = \frac{\pm}{D} \mathbb{Z} \) for every \( k \geq 2 \).

(3) If there is a point \( P_0 = \sum_{i=1}^{n} r_i V_i \in \Delta^- \) such that

\[
\left\{ j \mid r_j = \max_{1 \leq i \leq n} (r_i) \right\} \cap \left\{ j \mid pr_j - \lfloor pr_j \rfloor = \max_{1 \leq i \leq n} (pr_i - \lfloor pr_i \rfloor) \right\} = \emptyset,
\]

then \( IHP(\Delta) \) lies strictly above \( HP(\Delta) \) at \( x = \frac{\pm}{D} \mathbb{Z} \) for every \( k \geq 1 \).

**Notation 3.4.** For a point \( Q \) in \( \mathbb{M}(\Delta) \) we write \( Q\% \) for its residue in \( \Delta^- \) modulo \( \Lambda_\Delta \), and set

\[
\eta: \Delta^- \longrightarrow \Delta^- \quad Q \longmapsto (pQ)\%.
\]

be a permutation of \( \Delta^- \).

**Proof of Proposition 3.3.** (1) Since any point \( Q \in \Delta_k^+ \setminus \Delta_k^- \) has integer weight, we get

\[
\lceil w(pQ) \rceil - \lfloor w(Q) \rfloor = w(pQ) - w(Q).
\]

Therefore, we only need to prove this proposition for \( x = \frac{\pm}{k} \).

Since \( \Delta_k^- \) can be decomposed into a disjoint union of shifts of \( \Delta^- \) by points in \( \Lambda_\Delta \), we reduce the question to show

\[
(3.4) \quad \sum_{Q-Q_1 \in \Delta^-} \left( \lceil w(pQ) \rceil - \lfloor w(Q) \rfloor - w(pQ) + w(Q) \right) \geq 0 \text{ for every } Q_1 = \sum_{i=1}^{n} m_i V_i \in \Lambda_\Delta.
\]
Let \( Q - Q_1 = \sum_{i=1}^{n} r_i V_i \in \Delta^- \). We set
\[
m_{\text{max}} := \max_{1 \leq j \leq n} (m_j) \quad \text{and} \quad S =: \{ 1 \leq i \leq n \mid m_i = m_{\text{max}} \}.
\]

Take \( j \in S \) such that \( r_j = \max_{i \in S} (r_i) \). Then we have
\[
(3.5) \quad \lfloor w(Q) \rfloor + w(pQ) = m_j + pr_j + pm_j = w(pQ + Q_1).
\]

Since \( \eta \) is a permutation of \( \Delta^- \), we know that
\[
(3.6) \quad \sum_{Q - Q_1 \in \Delta^-} w(Q) = \sum_{Q - Q_1 \in \Delta^-} w(Q_1 + \eta(Q - Q_1)).
\]

Since \( pQ \equiv \eta(Q - Q_1) \pmod{\Lambda_\Delta} \), we know
\[
(3.7) \quad w(pQ - \eta(Q - Q_1)) = (pr_j + pm_j) - (pr_j - \lfloor pr_j \rfloor)
= pm_j + \lfloor pr_j \rfloor = \lfloor w(pQ) \rfloor.
\]

Combining (3.3), (3.5), (3.6), and (3.7), we get
\[
\sum_{Q - Q_1 \in \Delta^-} \left( \lfloor w(pQ) \rfloor - \lfloor w(Q) \rfloor - w(pQ) + w(Q) \right)
\overset{(3.6)}{=} \sum_{Q - Q_1 \in \Delta^-} \left( \lfloor w(pQ) \rfloor - \lfloor w(Q) \rfloor - w(pQ) + w(Q_1 + \eta(Q - Q_1)) \right)
\overset{(3.7)}{=} \sum_{Q - Q_1 \in \Delta^-} \left( w(pQ - \eta(Q - Q_1)) - \lfloor w(Q) \rfloor - w(pQ) + w(Q_1 + \eta(Q - Q_1)) \right)
\overset{(3.5)}{=} \sum_{Q - Q_1 \in \Delta^-} \left( w(pQ - \eta(Q - Q_1)) + w(Q_1 + \eta(Q - Q_1)) - w(pQ + Q_1) \right)
\overset{(3.3)}{=} 0.
\]

(2) We put \( Q_1 = V_{j_1} + V_{j_2} \) and \( Q = Q_1 + P_0 \). From the assumptions (a) and (b), we have
\[
w(pQ - \eta(Q - Q_1)) + w(Q_1 + \eta(Q - Q_1)) - w(pQ + Q_1)
= p + \lfloor pr_{j_1} \rfloor + (1 + pr_{j_1} - \lfloor pr_{j_1} \rfloor) - (p + pr_2 + 1)
= pr_{j_1} - \lfloor pr_{j_1} \rfloor - (pr_{j_2} - \lfloor pr_{j_2} \rfloor) > 0,
\]
which completes the proof.

(3) The proof of (3) is similar to (2). \( \square \)
**Example 3.5.** Let \( p = 29, V_1 = (2, 0) \) and \( V_2 = (0, 3) \). Then we have

\[
\mathbb{W}_6 = \Delta^-_1 = \{ (0, 0), (0, 1), (1, 0), (1, 1), (0, 2), (1, 2) \},
\]

and the chart of weight of points in \( \Delta^- \):

| \( P_0 \) | (0, 0) | (0, 1) | (1, 0) | (1, 1) | (0, 2) | (1, 2) |
|-----------|--------|--------|--------|--------|--------|--------|
| \( w(P_0) \) | 0      | \( \frac{1}{3} \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{2}{3} \) | \( \frac{2}{3} \) |
| \( w(pP_0) \) | 0      | \( \frac{29}{3} \) | \( \frac{29}{2} \) | \( \frac{29}{2} \) | \( \frac{58}{3} \) | \( \frac{58}{3} \) |

Computing the left-hand side of (3.4) for \( Q_1 = (0, 0) \), we have

\[
\sum_{P_0 \in \Delta^-} \left( \lfloor w(pP_0) \rfloor - \lfloor w(P_0) \rfloor - w(pP_0) + w(P_0) \right) = \frac{1}{3} > 0.
\]

We next give an estimate of the \( T \)-adic valuation of each entry of the matrix \( N \). For this, we need to control the \( T \)-adic valuation of each \( e_Q(T) \).

**Lemma 3.6.** (1) Recall that \( \prod_{P \in \Delta^+ \backslash \{O\}} E(\hat{a}_P \pi P) \) expands as \( \sum_{Q \in \mathcal{M}(\Delta)} e_Q(T)x^Q \). Then we have

\[
e_O(T) = 1 \quad \text{and} \quad \text{val}_T(e_Q(T)) \geq \lfloor w(Q) \rfloor \quad \text{for all } Q \in \mathcal{M}(\Delta).
\]

(2) Recall that \( \prod_{P \in \Delta^+ \backslash \{O\}} E(\tilde{a}_P \pi P) \) expands as \( \sum_{Q \in \mathcal{M}(\Delta)} \tilde{e}_Q(T)x^Q \). Then we have

\[
\tilde{e}_O(T) = 1 \quad \text{and} \quad \text{val}_T(\tilde{e}_Q(T)) \geq \lfloor w(Q) \rfloor \quad \text{for all } Q \in \mathcal{M}(\Delta).
\]

**Proof.** (1) We will only prove (1) since the proof of (2) is similar.

It follows from Definition 2.6 that \( e_O(T) = 1 \).

For each \( P \in \Delta^+ \backslash \{O\} = \Delta \cap \mathbb{Z}^n \backslash \{O\} \), we expand \( E(\hat{a}_P \pi P) \) to a power series in variables \( x_1, x_2, \ldots, x_n \), and get

\[
\prod_{P \in \Delta^+ \backslash \{O\}} E(\hat{a}_P \pi P) = \sum_{\tilde{f}(x) \in Z_p} \left( \prod_{P \in \Delta^+ \backslash \{O\}} c_{jP}(\hat{a}_P \pi P)^{jp} \right),
\]

where \( \{ \hat{a}_P \} \) is the set of coefficients of \( \tilde{f}(x) \) and \( c_i \in Z_p \) is the \( \pi^i \) coefficient of \( E(\pi) \).
Expanding this product and the sum, we deduce

\[
e_Q(T) = \sum_{\{j \in \mathbb{Z}_{\geq 0} \mid \sum_{P \in \Delta^+ \setminus \{\mathcal{O}\}} j_P P = Q\}} \left( \prod_{P \in \Delta^+ \setminus \{\mathcal{O}\}} (c_{j_P} (\hat{a}_P)^{j_P}) \right)
\]

\[
= \sum_{\{j \in \mathbb{Z}_{\geq 0} \mid \sum_{P \in \Delta^+ \setminus \{\mathcal{O}\}} j_P P = Q\}} \left( \prod_{P \in \Delta^+ \setminus \{\mathcal{O}\}} (c_{j_P} (\hat{a}_P)^{j_P}) \pi \sum_{P \in \Delta^+ \setminus \{\mathcal{O}\}} j_P \right).
\]

Since each point in \(\Delta^+ \setminus \{\mathcal{O}\}\) has weight less or equal to 1, for each \(\vec{j}\) such that \(\sum_{P \in \Delta^+ \setminus \{\mathcal{O}\}} j_P P = Q\), we have

\[
\text{val}_T \left( \prod_{P \in \Delta^+ \setminus \{\mathcal{O}\}} c_{j_P} (\hat{a}_P)^{j_P} \pi \sum_{P \in \Delta^+ \setminus \{\mathcal{O}\}} j_P \right) \geq \sum_{P \in \Delta^+ \setminus \{\mathcal{O}\}} j_P w(P) \geq w(Q).
\]

Note that \(\text{val}_T(\pi) = 1\). Therefore, we immediately obtain \(\text{val}_T(e_Q(T)) \geq w(Q)\). Since \(\text{val}_T(e_Q(T))\) is an integer, we have

\[
\text{val}_T(e_Q(T)) \geq \left\lfloor w(Q) \right\rfloor.
\]

\[\square\]

**Corollary 3.7.** If both \(Q\) and \(Q'\) belong to \(\mathbb{M}(\Delta)\), then

\[
(3.8) \quad \text{val}_T(N_{Q',Q}) = \text{val}_T(e_{pQ'-Q}) \geq \left\lfloor w(pQ') - w(Q) \right\rfloor,
\]

where \(N_{Q',Q}\) is the entry of matrix \(N\) as in Lemma 2.11.

**Proof.** Since \(\text{val}_T(E(\hat{a}_\mathcal{O})) = 0\), we have

\[
\text{val}_T(N_{Q',Q}) = \text{val}_T(E(\hat{a}_\mathcal{O}) \cdot e_{pQ'-Q}) = \text{val}_T(e_{pQ'-Q}).
\]

We assume that \(pQ' - Q \in \mathbb{M}(\Delta)\), for otherwise \(\text{val}(e_{pQ'-Q}) = \infty\), which leads to (3.8) directly.

By Lemma 3.6, we have

\[
\text{val}(e_{pQ'-Q}) \geq \left\lfloor w(pQ') - w(Q) \right\rfloor = \left\lfloor w(pQ') - (\left\lfloor w(pQ') \right\rfloor - \left\lfloor w(Q) \right\rfloor) \right\rfloor + \left\lfloor w(pQ') \right\rfloor - \left\lfloor w(Q) \right\rfloor \geq \left\lfloor w(pQ') \right\rfloor - \left\lfloor w(Q) \right\rfloor.
\]

\[\square\]

**Notation 3.8.** (1) Let \(S_1\) and \(S_2\) be two sets of the same cardinality \(\ell\). We denote by \(\text{Iso}(S_1,S_2)\) the set of bijections from \(S_1\) to \(S_2\). If the elements in these two sets
are labeled as
\[ S_1 := \{ Q_1, Q_2, \ldots, Q_\ell \} \quad \text{and} \quad S_2 := \{ Q'_1, Q'_2, \ldots, Q'_\ell \} , \]
for a bijection \( \tau \in \text{Iso}(S_1, S_2) \) such that \( \tau(Q_i) = Q'_j \), we put
\[ \text{sgn}(\tau) = \text{sgn}(j_1, j_2, \ldots, j_\ell) . \]

(2) For a function \( G \) on \( S_1 \times S_2 \), we put
\[ \det(G(Q, Q'))_{Q, Q' \in S_1 \times S_2} := \pm \sum_{\tau \in \text{Iso}(S_1, S_2)} \text{sgn}(\tau) \prod_{Q \in S_1} G(Q, \tau(Q)) , \]
which is independent to the order of elements in \( S_1 \) and \( S_2 \) up to a sign.

**Notation 3.9.** (1) For the rest of this section, we shall consider multisets, i.e., sets of possibly repeating elements. They are marked with a superscript star to be distinguished from usual sets.

(2) The disjoint union of two multisets \( S^* \) and \( S'^* \) is denoted by \( S^* \cup S'^* \) as a multiset.

(3) For a set \( S \), we write \( S^*m \) (resp. \( S^*\infty \)) for the union of \( m \) (resp. countably infinite) copies of \( S \) as a multiset.

**Notation 3.10.** For two multisets \( S_1^* \) and \( S_2^* \) of the same cardinality, we denote by \( \text{Iso}(S_1^*, S_2^*) \) the set of bijections (as multisets) from \( S_1^* \) to \( S_2^* \). When \( S_1^* = S_2^* = S^* \), we simply set \( \text{Iso}(S^*) := \text{Iso}(S^*, S^*) \).

**Definition 3.11.** Let \( S^* \) be a subset of \( M(\Delta)^*\infty \). We define
\[ h(S^*) := \sum_{Q \in S^*} \left[ w(pQ) \right] - \left[ w(Q) \right] . \tag{3.9} \]
The expression on the right-hand side will be related to the estimate in Corollary 3.7, and to the expression (3.1).

If \( S^* \) belongs to \( M(\Delta) \), we suppress the star from the notation.

**Notation 3.12.** We denote \( M_\ell(k) \) to be the set consisting of all sub-multisets of \( M(\Delta)^*k \) of cardinality \( k\ell \). For simplicity, we put \( M_\ell := M_\ell(1) \).

**Remark 3.13.** It is clear that \( \text{IHP}(\Delta) \) is the lower convex hull of the set of points \( \{ (\ell, \min_{S \in M_\ell} h(S)) \} \).

Now we gather more information of \( \text{IHP}(\Delta) \). Recall that \( D \) is the smallest positive integer that satisfies (2.1).

**Lemma 3.14.** Let \( S_1, \ldots, S_m \in M_\ell \) so that their disjoint union \( S^* = \bigcup_{j=0}^{m-1} S_j \in M_\ell(m) \). Then the following two statements are equivalent.

(1) The minimum of \( h(S'^*) \) over all \( S'^* \in M_\ell(m) \) is achieved by \( S^* \).
(2) For every $1 \leq i \leq m$, the sum of weights $\sum_{Q \in S_i} w(Q)$ achieves the minimum of $\sum_{Q \in S'} w(Q)$ over all $S' \in \mathcal{M}_\ell$.

Proof. It is obvious that $h$ is additive. Hence, without loss of generality, we assume $m = 1$ in the proof.

First, we claim that for $Q$ and $Q'$ two points in $\mathcal{M}(\Delta)$, if $w(Q) > w(Q')$, then

$$|w(pQ)| - |w(Q)| - (|w(pQ')| - |w(Q')|) > 0.$$  

Indeed, by Definition 2.2, if $w(Q) > w(Q')$, then

$$w(Q) \geq w(Q') + \frac{1}{D}.$$ 

Since we assume $p > D(n + 4)$ in Hypothesis 1.1, we have

$$|w(pQ)| - |w(Q)| - (|w(pQ')| - |w(Q')|) > w(pQ) - w(pQ') - w(Q) + w(Q') - 2 \geq \frac{p-1}{D} - 2 > 0.$$ 

Therefore, if $S' \in \mathcal{M}_\ell$ such that $h(S')$ is minimal over all $S \in \mathcal{M}_\ell$, then $\sum_{Q \in S'} w(Q)$ takes the minimum of $\sum_{Q \in S} w(Q)$ for all $S \in \mathcal{M}_\ell$.

Let $S'$ and $S''$ be two subsets of $\mathcal{M}_\ell$ such that $\sum_{Q \in S'} w(Q)$ and $\sum_{Q \in S''} w(Q)$ reach the minimum of $\sum_{Q \in S} w(Q)$ over all $S \in \mathcal{M}_\ell$. Clearly, the two multisets $\{w(Q) \mid Q \in S'\}$ and $\{w(Q) \mid Q \in S''\}$ are the same, which implies $h(S') = h(S'')$ and this lemma.  

PROPOSITION 3.15. For every integer $\ell \geq 1$, let $\mathcal{W}_\ell$ be a set of $\ell$ elements in $\mathcal{M}(\Delta)$ with minimal weights. Then we have

1. $\min_{S^* \in \mathcal{M}_\ell(m)} h(S^*) = mh(\mathcal{W}_\ell)$.

2. Order the elements in $\mathcal{M}(\Delta)$ by their weights (in the non-decreasing order): $P_1, P_2, \ldots$. Suppose that $n_1, n_2, \ldots$ are exactly the indices such that $w(P_{n_i+1}) > w(P_{n_i})$. Then IHP($\Delta$) has vertices $(n_i, h(\mathcal{W}_{n_i}))$ for every $i \geq 1$.

Proof. (1) From the proof of Lemma 3.14, we know that the minimum on the left is achieved when $S^* = \left\{ y^m_{j=0} S_j \right\}$ with $S_j = \mathcal{W}_\ell$. Since $h$ is additive, we have

$$h \left( \sum_{j=0}^{m-1} S_j \right) = \sum_{j=0}^{m-1} h(S_j) = mh(\mathcal{W}_\ell).$$

(2) Since $\{w(P_n)\}$ is non-decreasing, by Lemma 3.14, the improved Hodge polygon IHP($\Delta$) passes through the point $(\ell, \sum_{i=1}^\ell (|w(pP_i)| - |w(P_i)|))$ for every $\ell \geq 1$. Since $w(P_{n_i+1}) > w(P_{n_i})$, by (3.10), we have

$$|w(pP_{n+1})| - |w(P_{n+1})| - (|w(pP_{n})| - |w(P_{n})|) > 0.$$ 

Therefore, $(n_i, h(\mathcal{W}_{n_i}))$ is a vertex of IHP($\Delta$) for every $i \geq 1$.  

COROLLARY 3.16. (1) The height of the improved Hodge polygon at $x = x_k^\pm$ satisfies
\[ \min_{S \in \#_k^\pm} h(S) = h(\Delta_k^\pm). \]

(2) Every Newton slope of IHP($\Delta$) before point $x = x_k^-$ is strictly less than $k(p - 1)$.

(3) Every slope after the point $x = x_k^+$ is strictly greater than $k(p - 1)$.

(4) Every Newton slope of IHP($\Delta$) between points $x = x_k^-$ and $x = x_k^+$ is equal to $k(p - 1)$.

(5) For every $k \geq 1$, the point $(x_k^\pm, h(\Delta_k^\pm))$ is a vertex of IHP($\Delta$).

Proof. They are straightforward corollaries of Proposition 3.15. $\square$

Notation 3.17. For a subset $I \subseteq \{1, 2, \ldots, n\}$ we put
\[ \Delta^-(I) = \left\{ \sum_{i=1}^{n} r_i \mathbf{V}_i \in \Delta^- \mid r_i = 0 \text{ if } i \in I; r_i \in (0, 1) \text{ otherwise} \right\} \subseteq \Delta^- . \]

Lemma 3.18. For every integer $k \geq 1$ we have
1. $x_k^- = k^n \text{Vol}(\Delta)$, and
2. $x_k^+ = \sum_{I \subseteq \{1, 2, \ldots, n\}} \#\Delta^-(I) k^{|I|} (k + 1)^{n-|I|}$.

Proof. (1) Every point in $Q \in \Delta_k^-$ can be uniquely written as a sum of a point $P_0 \in \Delta^-$ and a point $Q_1 = \sum_{i=1}^{n} m_i \mathbf{V}_i \in \Lambda_\Delta$ for an $n$-dimensional vector $m \in \{0, \ldots, k - 1\}^n$. It implies $x_k^\pm = \#(\Delta^k) k^n$. Since $\Delta$ is a parallelootope, we have $\#\Delta^- = \text{Vol}(\Delta)$, and hence $x_k^- = k^n \text{Vol}(\Delta)$.

(2) Let $I \subseteq \{1, 2, \ldots, n\}$, $P_0 \in \Delta^-(I)$ and $Q \in \mathbb{M}(\Delta)$ such that $Q \equiv P_0 \pmod{\Lambda_\Delta}$. Write $Q = P_0 + Q_1$ for some $Q_1 = \sum_{i=1}^{n} m_i \mathbf{V}_i \in \Lambda_\Delta$. Then $Q$ belongs to $\Delta_k^\pm$ if and only if $0 \leq m_i \leq k$ for every $i \in I$ and $0 \leq m_i \leq k - 1$ for every $i \notin I$. Therefore, there are $k^{|I|} (k + 1)^{n-|I|}$ points in $\Delta^\pm_k$ with the residue $P_0$ module $\Lambda_\Delta$. It completes the proof.

Lemma 3.19. The functions $h(\Delta^\pm_k)$ and $h(\Delta^\pm_k)$ are both polynomials in $k$ of degree $n + 1$, i.e., for every $k \geq 1$ we have
\[ h(\Delta^\pm_k) = \sum_{i=0}^{n+1} A_i^\pm k^i, \]
where $A_i^\pm$ are integers which depend only on $\Delta$.

Proof. By Lemma 3.18, we have
\[
\begin{align*}
\text{h}(\Delta_k^+) &= \text{h}(\Delta_k^-) + (p - 1)k(x_k^+ - x_k^-) \\
&= \text{h}(\Delta_k^-) + (p - 1)k \sum_{I \subseteq \{1, 2, \ldots, n\}} (\#\Delta^-(I) k^{|I|} (k + 1)^{n-|I|} - \#\Delta^-(I) k^n).
\end{align*}
\]
Hence, it is enough to prove the statement for \( h(\Delta_k^-) \). For every nonempty subset \( I \subseteq \{1, 2, \ldots, n\} \) and \( k \geq 1 \), we put
\[
U(k; I) = \left\{ \sum_{i=1}^{n} m_i V_i \in \Lambda_{\Delta} \mid \begin{array}{l}
m_i = k \text{ if } i \in I \\
m_i < k \text{ otherwise}
\end{array} \right\}.
\]

Clearly, a point \( Q \in \mathbb{M}(\Delta) \) belongs to \( \Delta_k^- \setminus \Delta^- \) if and only if there exist \( P_0 \in \Delta^- \), \( \ell \leq k - 1 \), and a nonempty subset \( I \subseteq \{1, 2, \ldots, n\} \) such that \( Q \in P_0 + U(\ell; I) \).

Therefore, we have the following decomposition:
\[
\Delta_k^- = \Delta^- \cup \bigcup_{P_0 \in \Delta^-} \bigcup_{I \subseteq \{1, 2, \ldots, n\}} \bigcup_{I \neq \emptyset} (P_0 + U(\ell; I)).
\]

Since \( h \) is additive, we obtain
\[
h(\Delta_k^-) = h(\Delta^-) + \sum_{P_0 \in \Delta^-} \sum_{I \subseteq \{1, 2, \ldots, n\}} \sum_{I \neq \emptyset} h(P_0 + U(\ell; I)).
\]

It is enough to show that for a fixed \( P_0 \) and a nonempty subset \( I \subseteq I(P_0) \), the sum \( \sum_{\ell=1}^{k-1} h(P_0 + U(\ell; I)) \) is a polynomial in \( k \) of degree \( \leq n + 1 \).

Now we prove that every pair of points \( P_1 = P_0 + \sum_{i=1}^{n} m_{1,i} V_i \in P_0 + U(k_1; I) \) and \( P_2 = P_0 + \sum_{i=1}^{n} m_{2,i} V_i \in P_0 + U(k_2; I) \) satisfies
\[
h(P_1) = h(P_2) + (k_1 - k_2)(p - 1), \tag{3.11}
\]
where \( h(P) := [w(pP)] - [w(P)] \).

Let \( P_0 = \sum_{i=1}^{n} r_i V_i \) and \( r_s = \max_{i \in I} \{r_i\} \) for some \( s \in I \). By (3.2), we have
\[
\left[ w \left( p \left( P_0 + \sum_{i=1}^{n} m_{j,i} V_i \right) \right) \right] - \left[ w \left( P_0 + \sum_{i=1}^{n} m_{j,i} V_i \right) \right] = \lfloor pr_s \rfloor + pk_j - k_j
\]
for \( j = 1, 2 \), which implies (3.11).

We choose a representative \( P' \in U(1; I) \) and put
\[
h(P_0; I) := h(P_0 + P').
\]

By (3.11), \( h(P_0; I) \) is well defined and for every point \( P \in P_0 + U(\ell; I) \) we have
\[
h(P) = h(P_0; I) + (\ell - 1)(p - 1).
\]
Therefore, we have

\begin{equation}
(3.12)
\end{equation}

\begin{align*}
h(P_0 + U(\ell; I)) &= \#U(\ell; I)\left(h(P_0; I) + (\ell - 1)(p - 1)\right) \\
&= \ell^{n-\#I}\left(h(P_0; I) + (\ell - 1)(p - 1)\right).
\end{align*}

For every $\ell \geq 1$ and $k \geq 0$ we put $G_{k,\ell} := \sum_{i=1}^\ell i^k$, which is well known as a polynomial in $\ell$ of degree $k + 1$. Therefore, the function

\begin{align*}
\sum_{\ell=1}^{k-1} \ell^{n-\#I}\left(h(P_0; I) + (\ell - 1)(p - 1)\right) &= G_{n-\#I,k-1}\left(h(P_0; I) - p + 1\right) \\
&\quad+ (p - 1)G_{n+1-\#I,k-1}
\end{align*}

is a polynomial in $k$ of degree in $n + 1$, so is $h(\Delta_{\ell-1})$ when combined with (3.12).

\[\square\]

**Notation 3.20.** We denote by

\[
\begin{bmatrix}
P_0 & P_1 & \cdots & P_{\ell-1} \\
Q_0 & Q_1 & \cdots & Q_{\ell-1}
\end{bmatrix}_M
\]

the $\ell \times \ell$-submatrix formed by elements of a matrix $M$ whose row indices belong to $\{P_0, P_1, \ldots, P_{\ell-1}\}$ and whose column indices belong to $\{Q_0, Q_1, \ldots, Q_{\ell-1}\}$.

Recall that we defined the improved Hodge polygon $\text{IHP}(\Delta)$ in Definition 3.1.

**Proposition 3.21.** For every $f \in \mathbb{E}(\Delta)$, the normalized Newton polygon $\text{NP}(f, T)_{\mathcal{C}}$ lies above $\text{IHP}(\Delta)$.

**Proof.** We first recall the definition of $u_\ell$ in (2.3), and write $N$ for the standard matrix of $\psi_p \circ E_f$ corresponding to the basis $\{x^Q\}_{Q \in \mathcal{M}(\Delta)}$ of the Banach space $\mathbf{B}$. By [RWXY, Corollary 3.9], we know that the standard matrix of $\psi^n$ corresponding to the same basis is equal to $\sigma^{m-1}_{\text{Frob}}(N) \circ \sigma^{m-2}_{\text{Frob}}(N) \circ \cdots \circ N$. By [RWXY, Proposition 4.6], for every $\ell \in \mathbb{N}$ we have

\begin{equation}
(3.13)
\end{equation}

\[u_\ell(T) = \sum_{\{Q_0,0, Q_0,1, \ldots, Q_0,\ell-1\} \in \mathcal{M}_\ell} \det\left(\prod_{j=0}^{m-1} \begin{bmatrix}
Q_{j+1,0} & Q_{j+1,1} & \cdots & Q_{j+1,\ell-1} \\
Q_j,0 & Q_j,1 & \cdots & Q_j,\ell-1
\end{bmatrix}\sigma_{\text{Frob}}^j(N)\right) ,
\]

where $Q_{m,i} := Q_{0,i}$ for each $0 \leq i \leq \ell - 1$. 

We set $S_j = \{Q_{j,0}, Q_{j,1}, \ldots, Q_{j,\ell-1}\}$, where $S_m = S_0$, then

$$\text{val}_T \left( \det \left( \prod_{j=0}^{m-1} \begin{bmatrix} Q_{j,1,0} & Q_{j,1,1} & \cdots & Q_{j,1,\ell-1} \\ Q_{j,0} & Q_{j,1} & \cdots & Q_{j,\ell-1} \end{bmatrix} \sigma_j^j \text{Frob}(N) \right) \right)$$

$$= \text{val}_T \left( \prod_{j=0}^{m-1} \det \left( \sigma_j^j \text{Frob}(e_pQ' - Q) \right) \right)_{Q,Q' \in S_{j+1} \times S_j}$$

(3.14) $$\geq \sum_{j=0}^{m-1} \min_{\tau_j \in \text{Iso}(S_{j+1}, S_j)} \left\{ \sum_{Q \in S_{j+1}} \text{val}_T \left( \sigma_j^j \text{Frob}(e_p\tau_j(Q) - Q) \right) \right\}$$

$$\geq \sum_{j=0}^{m-1} \min_{\tau_j \in \text{Iso}(S_{j+1}, S_j)} \left\{ \sum_{Q \in S_{j+1}} \left[ w(\tau_j(Q)) \right] - \left[ w(Q) \right] \right\}$$

$$= \sum_{Q \in \bigcup_{j=0}^{m-1} S_j} \left( \left\lfloor w(pQ) \right\rfloor - \left\lfloor w(Q) \right\rfloor \right) = h \left( \bigcup_{j=0}^{m-1} S_j \right).$$

Therefore, we deduce

$$\frac{\text{val}_T(u_{\ell}(T))}{m} \geq \frac{\min_{S_0, \ldots, S_{m-1} \in \mathcal{M}_\ell} h \left( \bigcup_{j=0}^{m-1} S_j \right)}{m} = \min_{S \in \mathcal{M}_\ell} h(S) = h(\mathbb{W}_\ell),$$

which implies that $\text{NP}(T,f)_C$ lies on or above $\text{IHP}(\Delta)$. \hfill \qed

**Corollary 3.22.** Let $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$ be a finite character of conductor $p^m$. The improved Hodge polygon is a lower bound of the normalized Newton polygon $\text{NP}(f,\chi)_C$.

**Proof.** Note that specializing $T$ with $\chi(1) - 1$ does not lower the polygon. Therefore, for every nontrivial finite character $\chi$, the Newton polygon $\text{NP}(f,\chi)_C$ lies on or above $\text{NP}(f,T)_C$. Hence, $\text{NP}(f,\chi)_C$ lies on or above $\text{IHP}(\Delta)$, when combined with Proposition 3.21. \hfill \qed

**Corollary 3.23.** For every $k \geq 1$ we have

$$u_{\pm_k} \equiv \prod_{j=0}^{m-1} \sigma_j^j \left( \text{det} \left( e_pQ' - Q \right)_{Q,Q' \in \Delta_k^\pm} \right) \pmod{T^{mh(\Delta_k^\pm) + 1}}.$$

**Proof.** By Proposition 3.15, when $\ell = \pm_k$, the equalities hold in (3.14) if and only if

$$S_j = \Delta_k^\pm \quad \text{for every } 0 \leq j \leq m - 1. \hfill \qed$$
4. The generic Newton polygon. In this section, we will prove Theorem 1.8, which is given by a sequence of sufficient statements.

Recall that \( \tilde{e}_Q(T) \) is defined in (2.7), and satisfies \( \text{val}_T(\tilde{e}_Q(T)) \geq \lfloor w(pQ) \rfloor - \lfloor w(Q) \rfloor \) for all \( Q \in \mathbb{M}(\Delta) \) (see Lemma 3.6 (2)).

_Notation 4.1._ We put

\[
\det \left( I - \tilde{p}Q' - Qs \right)_{Q, Q' \in \mathbb{M}(\Delta)} = \sum_{i=0}^{\infty} \tilde{u}_i(T)s^i \in \mathbb{Z}_p[\tilde{a}; P \in \Delta^+ \setminus \{O\}]/[s].
\]

(4.1)

Similar to (3.15), we have

\[
\text{val}_T(\tilde{u}_{x_k^\pm}(T)) \geq h(\Delta_k^\pm),
\]

(4.2)

which allows us to put

\[
\tilde{u}_{x_k^\pm}(T) := \sum_{i=h(\Delta_k^\pm)}^{\infty} \tilde{u}_{x_k^\pm,i} T^i \quad \text{for} \quad \tilde{u}_{x_k^\pm,i} \in \mathbb{Z}_p[\tilde{a}; P \in \Delta^+ \setminus \{O\}]
\]

(4.3)

with an explicit lower bound of the summation.

**Proposition 4.2.** For every integer \( 1 \leq k \leq n + 2 \), we have

\[
\tilde{u}_{x_k^\pm, h(\Delta_k^\pm)} \not\equiv 0 \pmod{p}.
\]

(4.4)

We will give its proof in Section 5.

**Proposition 4.3.** Proposition 4.2 implies Theorems 1.8 and 1.5.

We give its proof after several lemmas.

**Lemma 4.4.** Assume Proposition 4.2. Then there exists a Zariski open subset \( O_{\text{Zar}} \subseteq \mathbb{E}(\Delta) \) such that for every \( f \in O_{\text{Zar}} \) and every finite character \( \chi_0 \) of conductor \( p \), the Newton polygon \( \text{NP}(f, \chi_0)_C \) passes through the point \( (x_k^{\pm, h(\Delta_k^\pm)}) \) for every \( 1 \leq k \leq n + 2 \).

_Proof._ Since \( \tilde{u}_{x_k^{\pm, h(\Delta_k^\pm)}} \not\equiv 0 \pmod{p} \) for every \( 1 \leq k \leq n + 2 \), the subset

\[
O_{\text{Zar}}(\Delta) := \left\{ f(x) = \sum_{P \in \Delta^+} a_P x^P \in \mathbb{E}(\Delta) : \tilde{u}_{x_k^{\pm, h(\Delta_k^\pm)}}(x)|_{a_P = \tilde{a}_P} \not\equiv 0 \pmod{p} \right\}
\]

for every \( 1 \leq k \leq n + 2 \)

(4.5)

is a nonempty open subset of \( \mathbb{E}(\Delta) \).
Similar to Corollary 3.23, we have

\[(4.6) \quad \det(\tilde{e}_p Q' - Q)_{Q,Q' \in \Delta^+_k} \equiv \tilde{u}_{x_k^\pm, h(\Delta^+_k)} T^{h(\Delta^+_k)} \mod (p, T^{h(\Delta^+_k)+1}).\]

Hence, for every \( f(x) = \sum_{P \in \Delta^+} a_P x^P \in O_{\text{Zar}}(\Delta) \) and every \( 1 \leq k \leq n + 2 \) we have

\[(4.7) \quad u_{x_k^\pm} = \prod_{j=0}^{m(f)-1} \sigma_j^f(\det(\tilde{e}_p Q' - Q)_{Q,Q' \in \Delta^+_k})^j \equiv \prod_{j=0}^{m(f)-1} \sigma_j^f(\tilde{u}_{x_k^\pm, h(\Delta^+_k)} |_{\tilde{a}_p = \hat{a}_P})^j \equiv \prod_{j=0}^{m(f)-1} \sigma_j^f(\tilde{u}_{x_k^\pm, h(\Delta^+_k)} |_{\tilde{a}_p = \hat{a}_P}) T^{m(f) h(\Delta^+_k)} \not\equiv 0 \pmod{p, T^{m(f) h(\Delta^+_k)+1}},\]

where \( m(f) = [F_p(f) : \mathbb{F}_p] \).

Combining (2.4) with Corollary 3.22, we get

\[C^*_f(\chi_0, s) = C^*_f(T, s) |_{T = \chi_0(1)-1} \quad \text{and} \quad u_{x_k^\pm} = \sum_{\ell = T^{m h(\Delta^+_k)}} \infty u_{x_k^\pm, \ell} T^\ell.\]

Together with (4.7), these two equalities imply that for every \( 1 \leq k \leq n + 2 \) the \( p \)-adic valuation of the coefficient of \( s^\pm_{x_k^\pm} \) in \( C^*_f(\chi_0, s) \) satisfies

\[\text{val}_p \left( u_{x_k^\pm} |_{T = \chi_0(1)-1} \right) = m \text{val}_p (\xi_p - 1) h(\Delta^\pm) = \frac{m h(\Delta^\pm)}{p-1}.\]

It implies that NP\((f, \chi_0)_C\) lies below \((x_k^\pm, h(\Delta^\pm))\) for every \( 1 \leq k \leq n + 2 \).

Since \((x_k^\pm, h(\Delta^\pm))\) are vertices of IHP\((\Delta)\) which, by Corollary 3.22, is a lower bound for NP\((f, \chi_0)_C\), we conclude that NP\((f, \chi_0)_C\) must pass though the points \((x_k^\pm, h(\Delta^\pm))\) for every \( 1 \leq k \leq n + 2 \)\(\square\).

**Lemma 4.5.** For a polynomial \( f \in \mathcal{F}(\Delta) \) and a finite character \( \chi_0 \) of conductor \( p \), if NP\((f, \chi_0)_C\) passes though \((x_k^\pm, h(\Delta^\pm))\) for some \( 1 \leq k \leq n + 2 \), then it passes \((x_k^\pm, h(\Delta^\pm))\) for all \( k \geq 1 \).

**Proof.** Since for every \( 1 \leq k \leq n + 2 \) the Newton polygon NP\((f, \chi_0)_C\) passes though the points \((x_k^-, h(\Delta^-_k))\) and \((x_k^+, h(\Delta^+_k))\), which are vertices of its lower bound IHP\((\Delta)\), we have the following.

(a) The points \((x_k^-, h(\Delta^-_k))\) and \((x_k^+, h(\Delta^+_k))\) are vertices of NP\((f, \chi_0)_C\), and the segment connecting them is contained in NP\((f, \chi_0)_C\).

(b) There are \( x_k^- \) slopes of NP\((f, \chi_0)_C\) strictly less than \( k(p-1) \).
(c) There are $x_k^+$ slopes of $NP(f, \chi_0)_C$ less than or equal to $k(p - 1)$.
We denote by $L^*: = \{a_1, a_2, \ldots, a_n!Vol(\Delta)\}$ the Newton slopes of $NP(f, \chi_0)_L$ in a non-descending order. By Weil conjecture, $\alpha \in [0, n(p - 1)]$ for every $\alpha \in L^*$. We put
\[
\tau_i = \# \{ \alpha \in L^* \mid \alpha < i(p - 1) \} \quad \text{for } 0 \leq i \leq n.
\]
Set
\[
X_i := \sum_{j=1}^{\tau_i} \alpha_j \quad \text{and} \quad C_f^*(\chi_0, s) := \sum_{i=0}^{\infty} u_{\chi_0, i}s^i \in \mathbb{Z}_p[\bar{s}].
\]
Consider the relation between $C_f^*(\chi_0, s)$ and $L_f^*(\chi_0, s)$:
\[
(4.8) \quad C_f^*(\chi_0, s) = \left( \prod_{j=0}^{\infty} L_f^*(\chi_0, q^j) \left( \frac{n+j-1}{n-1} \right)^{-1} \right)^{(n-1)}
\]
We obtain that for every $\ell \geq 1$ there are
\[
\sum_{j=\ell-n}^{\ell-1} \binom{n+j-1}{n-1} (\tau_{\ell-j+1} - \tau_{\ell-j})
\]
slopes of $NP(f, \chi_0)_C$ which are contained in $[(\ell-1)(p-1), \ell(p-1)]$, where $\tau_{<0} = 0$. Therefore, the slopes of $NP(f, \chi_0)_C$ strictly less than $k(p - 1)$ is equal to
\[
\sum_{\ell=1}^{k} \sum_{j=\ell-n}^{\ell-1} \binom{n+j-1}{n-1} (\tau_{\ell-j+1} - \tau_{\ell-j})
\]
and $NP(f, \chi_0)_C$ passes through the points
\[
\left( \sum_{\ell=1}^{k} \sum_{j=\ell-n}^{\ell-1} \binom{n+j-1}{n-1} (\tau_{\ell-j+1} - \tau_{\ell-j}) \right),
\]
\[
\left( \sum_{\ell=1}^{k} \sum_{j=\ell-n}^{\ell-1} \binom{n+j-1}{n-1} (X_{\ell-j+1} - X_{\ell-j} + j(p-1)(\tau_{\ell-j+1} - \tau_{\ell-j})) \right)
\]
for all $k \geq 1$.
Combining it with (b), for every $1 \leq k \leq n + 2$ we have
\[
(4.9) \quad \sum_{\ell=1}^{k} \sum_{j=\ell-n}^{\ell-1} \binom{n+j-1}{n-1} (\tau_{\ell-j+1} - \tau_{\ell-j}) = x_k^- = k^nVol(\Delta).
\]
Note that the left-hand side of (4.9) is equal to
\[
\sum_{i=1}^{n} \left( \prod_{i=1}^{k} \sum_{\ell=1}^{n} \left( \frac{n+\ell-i-1}{n-1} \right) \right),
\]
which is a polynomial of degree \( n \). Since the both sides of (4.9) are polynomials in \( k \) of degree \( n \) and they have the same values for every \( 1 \leq k \leq n+2 \), they must be identical as polynomials. Namely, the equality (4.9) holds for all \( k \geq 1 \).

On the other hand, combining (4.9) with (a), we get
\[
\sum_{\ell=1}^{k} \sum_{j=\ell-n}^{\ell-1} \left( \frac{n+j-1}{n-1} \right) (X_{\ell-j+1} - X_{\ell-j} + j(p-1) (\#_{\ell-j+1} - \#_{\ell-j}) = h(\Delta_{-k})
\]
for every \( 1 \leq k \leq n+2 \).

Clearly, the left-hand side of (4.10) is a polynomial in \( k \) of degree \( n+1 \). By Lemma 3.19, the right-hand side of (4.10) is also a polynomial in \( k \) of degree \( n+1 \). Running the similar argument as above, we conclude that the equality (4.10) holds for every \( k \geq 1 \). Therefore, the Newton polygon \( NP(f,\chi_0)_C \) passes through the point \( (x^-_{-k}, h(\Delta^-_{-k})) \) for every \( k \geq 1 \).

A similar argument shows that \( NP(f,\chi_0)_C \) passes through the point \( (x^+_{-k}, h(\Delta^+_{-k})) \) for every \( k \geq 1 \).

\[\Box\]

**Lemma 4.6.** For a polynomial \( f \in F(\Delta) \) with \( m = [F_p(f) : F_p] \), if there exists a finite character \( \chi_0 \) of conductor \( p \) such that \( NP(f,\chi_0)_C \) passes through \( (x^-_{-k}, h(\Delta^-_{-k})) \) for all integer \( k \geq 1 \), then for every nontrivial finite character \( \chi \) the Newton polygon \( NP(f,\chi)_C \) passes through the points \( (x^\pm_{-k}, h(\Delta^\pm_{-k})) \) for all \( k \geq 0 \).

**Proof.** Recall that \( C_1^*(T,s) = \sum_{i=0}^{\infty} u_i s^i \). Since
\[
u_{x^{-}_0} = 1 \quad \text{and} \quad u_{x^+_0} = -\text{Norm}_{F_p(f)/F_p}(E(\hat{\alpha}_{\mathcal{O}p})) \equiv -1 \pmod{T},
\]
the Newton polygon \( NP(f,\chi)_C \) passes through the points \( (x^-_{-k}, h(\Delta^-_{-k})) \).

We next show that for every integer \( k \geq 1 \) we have
\[
u_{x^{-}_k} \equiv 0 \pmod{p, T^m h(\Delta^{-}_{-k})+1}.
\]

Suppose it is false. Without loss of generality, we may assume that there exists \( k_0 \geq 1 \) such that
\[
u_{x^{-}_{k_0}} \equiv 0 \pmod{p, T^m h(\Delta^{-}_{-k_0})+1}.
\]
This congruence equality implies that the $p$-adic valuation of the coefficient of $s^{\chi_{k_0}}$ in $C_f^*(\chi_0, s)$ satisfies
\[
\text{val}_p \left( u_{\chi_{k_0}} \big|_{T=\chi_0(1)-1} \right) > \frac{mh(\Delta_{k_0}^-)}{p-1},
\]
and hence the Newton polygon $\text{NP}(f, \chi_0)_C$ can never pass through the point $(x_{k_0}^-, h(\Delta_{k_0}^-))$. However, $(x_{k_0}^-, h(\Delta_{k_0}^-))$ is a vertex of the lower bound $\text{IHP}(\Delta)$ of $\text{NP}(f, \chi_0)_C$, a contradiction to the assumption in this lemma.

Therefore, we prove the inequality (4.11) and hence
\[
\text{val}_p \left( u_{x_k^\pm} \big|_{T=\chi(1)-1} \right) = \frac{mh(\Delta_k^\pm)}{(p-1)p^{m\chi-1}}.
\]
The last equality implies that $\text{NP}(f, \chi)_C$ passes through $(x_k^\pm, h(\Delta_k^\pm))$ for every $k \geq 1$ when combined with Corollary 3.22.

**Proof of Theorem 1.8.** We know already from Corollary 3.22 that $\text{IHP}(\Delta)$ is a lower bound for $\text{GNP}(\Delta)$. Now we show that it is sharp at the point $(x_k^\pm, h(\Delta_k^\pm))$ for every $k \geq 0$.

If we choose a polynomial $f \in O_{Zar}$, by Lemmas 4.4 and 4.5, for every finite character $\chi_0$ of conductor $p^{m\chi_0}$ and every integer $k \geq 0$, the Newton polygon $\text{NP}(f, \chi_0)_C$ passes through the point $(x_k^\pm, h(\Delta_k^\pm))$.

**Proof of Theorem 1.5.** Let $\chi$ be a nontrivial finite character and $f \in O_{Zar}$, where $O_{Zar}$ is defined as in Lemma 4.4. By Lemmas 4.4, 4.5 and 4.6, the Newton polygon $\text{NP}(f, \chi)_C$ passes through the point $(x_k^\pm, h(\Delta_k^\pm))$ for every $k \geq 0$. Since these points are vertices of $\text{IHP}(\Delta)$ which is a lower bound of $\text{NP}(f, \chi)_C$, they must also be vertices of $\text{NP}(f, \chi)_C$.

Therefore, by (1.2), for every $0 \leq i_1 \leq n - 1$ and every $0 \leq i_2 \leq p^{m\chi-1} - 1$ there are
\[
\sum_{t=0}^{i_1} (-1)^t \binom{n}{i_1-t} x_{(i_1-t)p^{m\chi-1}+i_2} \quad \text{resp.} \quad \sum_{t=0}^{i_1} (-1)^t \binom{n}{i_2} x_{(i_1-t)p^{m\chi-1}+i_2}
\]
slopes of $\text{NP}(f, \chi)_L$ less than or equal to (resp. strictly less than) $i_1(p-1)p^{m\chi-1} + (p-1)i_2$. Since the slopes of $\text{NP}(f, \chi)_L$ divided by $(p-1)p^{m\chi}$ are the $p^{m(f)}$-adic Newton slopes of $L_f^*(\chi, s)$, we complete the proof.

5. **Nonvanishing of leading terms in universal coefficients.** To show (4.4) for $1 \leq k \leq n + 2$, it is enough to show the coefficient of one special term in $\tilde{u}_{x_k^\pm}$ is not zero. Computing explicitly the contribution of this permutation of $\Delta_k^\pm$ to this special term, which itself is subdivided into simpler cases, allows us to prove (4.4).
Proof of Proposition 4.2 with assuming Propositions 5.13 and 5.14.

Recall that in Notation 3.4, we define the function \( \eta \) which maps \( Q \in \Delta^- \) to the residue of \( pQ \) module \( \Lambda_{\Delta} \) in \( \Delta^- \).

Notation 5.1. (1) For every nonempty subset \( S \subseteq \{1,2,\ldots,n\} \), we set \( V_S := \sum_{i \in S} V_i \), and denote by

\[
\text{Ver}(\Delta) := \{ V_S \mid S \subseteq \{1,2,\ldots,n\} \text{ and } S \neq \emptyset \}
\]

the set consisting of all vertices of \( \Delta \) except the origin \( O \).

(2) For a polynomial \( g(\tilde{a}_P) \in \mathbb{Z}_p[\tilde{a}_P, \ P \in \Delta^+ \setminus \{O\}] \), we denote by \( g_{\text{res}} \) the polynomial in \( \mathbb{Z}_p[\zeta_1, \zeta_2, \ldots, \zeta_n] \) obtained by

- specializing \( \tilde{a}_P \) with 0 if \( P \notin \text{Ver}(\Delta) \), and
- replacing \( \tilde{a}_P \) by \( \zeta_{\#S} \) if \( P = V_S \).

(3) For a point \( P_0 \in \Delta^- \), we put

\[
\Delta_\pm^k(P_0) := \{ Q \in \Delta_\pm^k \mid Q \equiv P_0 \pmod{\Lambda_{\Delta}} \}.
\]

Theorem 5.2. For every \( 1 \leq k \leq n + 2 \), we have

\[
\det(\tilde{e}_{pQ'-Q})_{Q,Q' \in \Delta_\pm^k} \neq 0 \pmod{p,T^{h(\Delta_\pm^k)+1}}.
\]

The rest of the paper is devoted to the proof of this theorem.

Lemma 5.3. Theorem 5.2 implies Proposition 4.2.

Proof. From (5.1) we have

\[
\det(\tilde{e}_{pQ'-Q})_{Q,Q' \in \Delta_\pm^k} \neq 0 \pmod{p,T^{h(\Delta_\pm^k)+1}}
\]

for every \( 1 \leq k \leq n + 2 \). Combined with (4.6), these congruence inequalities imply that

\[
\tilde{u}_{x_\pm^k,h(\Delta_\pm^k)} \neq 0 \pmod{p}
\]

for every \( 1 \leq k \leq n + 2 \). □

Lemma 5.4. For every \( \tau \in \text{Iso}(\Delta_\pm^k) \), if there exist \( P_0 \in \Delta^- \) and \( Q_0 \in \Delta_\pm^k(P_0) \) such that \( \tau(Q_0) \notin \Delta_\pm^k(\eta(P_0)) \), then

\[
\prod_{Q \in \Delta_\pm^k(P_0)} \tilde{e}_{pQ'-\tau(Q)} = 0.
\]
Proof. Recall that for every $Q \in \mathbb{M}(\Delta)$, we have

\begin{equation}
\tilde{e}_Q(T) = \sum_{\{j \in \mathbb{Z}_{\geq 0}^n \mid \sum_{P \in \Delta^+ \setminus \{O\}} j_P P = Q\}} \left( \prod_{P \in \Delta^+ \setminus \{O\}} c_{j_P} (\tilde{a}_P \pi)^{j_P} \right).
\end{equation}

(5.3)

If $\tau(Q_0) \notin \Delta^\pm_k(\eta(P_0))$, then $pQ_0 - \tau(Q_0) \not\equiv O \mod \Lambda_\Delta$. Therefore, every linear combination

$$pQ_0 - \tau(Q_0) = \sum_{P \in \Delta^+ \setminus \{O\}} j_P P$$

that contains $P \notin \text{Ver}(\Delta)$ must have $j_P \neq 0$, which implies

$$\tilde{e}_{pQ_0 - \tau(Q_0)} = 0.$$ 

Hence, to compute $\det(\tilde{e}_{pQ-Q'} pQ, q' \in \Delta^\pm)$, it suffices to take sum over all the permutations $\tau$ such that $\tau(Q) \equiv pQ \pmod{\Lambda_\Delta}$ for every $Q \in \Delta^\pm_k$.

Notation 5.5. For a point $Q = \sum_{i=1}^n z_i V_i$, we set

$$I(Q) := \{ i \mid z_i = 0 \}.$$ 

For a subset $I \subseteq \{1, 2, \ldots, n\}$, we generalize Notations 3.17 and 5.1(3) to

- $\Delta^\pm_k(I) := \left\{ Q = \sum_{i=1}^n z_i V_i \in \Delta^\pm \mid \begin{array}{l} z_i = 0 \text{ if } i \in I \\ z_i > 0 \text{ otherwise} \end{array} \right\},$
- $\Delta^\pm_k(I, P_0) := \Delta^\pm_k(I) \cap \Delta^\pm_k(P_0)$ and $\Delta^-_k(I, P_0) := \Delta^-_k(I) \cap \Delta^-_k(P_0)$.

Lemma 5.6. (1) For each $P_0 \in \Delta^-$, if $I \not\subseteq I(P_0)$, then $\Delta^\pm_k(I, P_0) = \emptyset$.

(2) The set $\Delta^\pm_k$ is a disjoint union of $\Delta^\pm_k(P_0, I)$ for all points $P_0 \in \Delta^-$ and all subsets $I \subseteq I(P_0)$.

Proof. Both (1) and (2) are straightforward.

Lemma 5.7. For every $\tau \in \text{Iso}(\Delta^\pm_k)$, if there exists a subset $I \subseteq \{1, 2, \ldots, n\}$ such that $\tau(\Delta^\pm_k(I)) \neq \Delta^\pm_k(I)$, then

\begin{equation}
\prod_{Q \in \Delta^\pm_k(P_0)} \tilde{e}_{pQ - \tau(Q)} = 0.
\end{equation}

(5.4)

Proof. We put $I_0 \subseteq \{1, 2, \ldots, n\}$ to be one of the smallest subset such that

$$\tau(\Delta^\pm_k(I_0)) \neq \Delta^\pm_k(I_0).$$

Namely, every set $I$ that contains fewer elements than $I_0$ satisfies

$$\tau(\Delta^\pm_k(I)) = \Delta^\pm_k(I).$$
It implies a point \( Q_0 \in \Delta_k^\pm(I_0) \) such that \( pQ_0 - \tau(Q_0) \notin \text{Cone}(\Delta) \). Hence, we obtain \( \bar{e}_{pQ_0-\tau(Q_0)} = 0 \) and consequently (5.4).

**Proposition 5.8.** We have the following equality

\[
\det(\bar{e}_{pQ - Q}^\text{res})_{Q, Q' \in \Delta_k^\pm} = \prod_{P_0 \in \Delta^{-}} \prod_{I \leq I(P_0)} \det(\bar{e}_{P_0 - \eta(P_0) + pQ - Q}^\text{res})_{Q, Q' \in \Delta_k^\pm(I, P_0)}.
\]

**Proof.** By Lemmas 5.4 and 5.7, for every \( \tau \in \text{Iso}(\Delta_k^\pm) \) if

\[ \prod_{Q \in \Delta^+(P_0)} \bar{e}_{pQ - \tau(Q)}^\text{res} \neq 0, \]

then \( \tau \) must map \( \Delta_k^\pm(I, P_0) \) onto \( \Delta_k^\pm(I, \eta(P_0)) \) for every \( P_0 \in \Delta^{-} \) and every integer \( I \in I(P_0) \). Note that

\[
\Delta_k^\pm(I, \eta(P_0)) = \Delta_k^\pm(I, P_0) - P_0 + \eta(P_0).
\]

Then we have

\[
\det(\bar{e}_{pQ - Q}^\text{res})_{Q, Q' \in \Delta_k^\pm} = \prod_{P_0 \in \Delta^{-}} \prod_{I \leq I(P_0)} \det(\bar{e}_{P_0 - \eta(P_0) + pQ - Q}^\text{res})_{Q, Q' \in \Delta_k^\pm(I, P_0)} \quad \square
\]

**Definition 5.9.** (1) The partial degree of a monomial \( \prod_{i=1}^{n} \zeta_i^{t_i} \), denoted by \( \text{Deg}(\prod_{i=1}^{n} \zeta_i^{t_i}) \), is defined to be the vector \( (t_n, t_{n-1}, \ldots, t_1) \in \mathbb{Z}_0^n \), where \( \mathbb{Z}_0^n \) equips with a reverse lexicographic order. Namely, for two vectors \( \vec{v} = (v_1, v_2, \ldots, v_n) \) and \( \vec{u} = (u_1, u_2, \ldots, u_n) \) in \( \mathbb{R}_0^n \), we call \( \vec{v} \) strictly greater than \( \vec{u} \), denoted by \( \vec{v} \succ \vec{u} \), if there is \( 1 \leq k \leq n \) such that

- \( v_i = u_i \) for every \( 1 \leq i \leq k - 1 \), and
- \( v_k > u_k \).

Note that the partial degree of every nonzero constant is just \( (0, 0, \ldots, 0) \).

(2) The partial degree of a polynomial \( g \in \mathbb{Z}_p[T][\zeta_1, \zeta_2, \ldots, \zeta_n] \), denoted by \( \text{Deg}(g) \), is the maximal partial degree of nonzero monomials in \( g \).

(3) For a point \( Q = \sum_{j=1}^{\ell} z_j V_{S_j} \) such that \( 0 = z_0 < z_1 < \cdots < z_\ell \), where \( \prod_{i=1}^{\ell} S_i \subseteq \{1, 2, \ldots, n\} \), the degree of \( Q \), denoted by \( \text{Deg}(Q) \), is the \( n \)-dimensional vector \( (v_1, v_2, \ldots, v_n) \) such that

\[
v_{n+1 - \sum_{j=i}^{\ell} \#S_j} = z_i - z_{i-1} \quad \text{for every} \ 1 \leq i \leq \ell,
\]

and all other components are zero.
(4) The leading term of \( g \), denoted by \( \text{LD}(g) \), is the sum of monomials in \( g \) of the maximal partial degree.

**Property 5.10.** (1) Every nonzero \( g \in \mathbb{Z}_p[[T]][\zeta_1, \zeta_2, \ldots, \zeta_n] \) satisfies

\[
\text{Deg}(g - \text{LD}(g)) \prec \text{Deg}(g).
\]

(2) For every two polynomials \( g_1, g_2 \in \mathbb{Z}_p[[T]][\zeta_1, \zeta_2, \ldots, \zeta_n] \), we have

(I) \( \text{Deg}(g_1 - g_2) \preceq \max(\text{Deg}(g_1), \text{Deg}(g_2)) \).

(II) \( \text{Deg}(g_1g_2) = \text{Deg}(g_1) + \text{Deg}(g_2) \).

(III) \( \text{LD}(g_1g_2) = \text{LD}(g_1)\text{LD}(g_2) \).

(IV) \( \text{LD}(g_1) = \text{LD}(g_2) \) if and only if \( \text{Deg}(g_1) \succ \text{Deg}(g_1 - g_2) \).

**Lemma 5.11.** Let \( \prod_{i=1}^\ell S_i \subseteq \{1, 2, \ldots, n'\} \) for some \( n' \leq n \) and \( \sum_{i=1}^\ell m_i V_{S_i} \in \Lambda_\Delta \) such that \( 0 = m_0 < m_1 < \cdots < m_\ell \). Then

(1) the leading term

\[
\text{LD}(\widetilde{e}_{Q_1}^{\text{res}}) = \prod_{i=1}^\ell c_{m_i-m_{i-1}} \left( \zeta^{\sum_{j=1}^n \#S_j \pi} \right)^{m_i-m_{i-1}},
\]

where \( \{c_i\} \) is defined in (2.2).

(2) \( \text{Deg}(\widetilde{e}_{Q_1}^{\text{res}}) = \text{Deg}(Q_1) \).

**Proof.** (1) By (5.3) and Notation 5.1(3), the polynomial \( \widetilde{e}_{Q_1} \) can be written explicitly as

\[
\widetilde{e}_{Q_1}^{\text{res}} = \sum_{\substack{S \subseteq \{1, 2, \ldots, n\} \setminus S \neq \emptyset \atop jS \subseteq Q_1}} \left( \prod_{S \subseteq \{1, 2, \ldots, n\}} c_{js} (\zeta^{S_j \pi})^{jS} \right).
\]

The monomial

\[
\prod_{i=1}^n c_{m_i-m_{i-1}} \left( \zeta^{\sum_{j=1}^n \#S_j \pi} \right)^{m_i-m_{i-1}}
\]

is the unique term in \( \widetilde{e}_{Q_1}^{\text{res}} \) of the maximal degree, which completes the proof.

(2) It follows directly from (1). \(\square\)

**Notation 5.12.** (1) Let \( P_0 \in \Delta^- \) and \( I \subseteq I(P_0) \). By relabeling indices, we may assume that \( I = \{n' + 1, n' + 2, \ldots, n\} \). Let \( S_1, S_2, \ldots, S_\ell \) be an ordered disjoint subsets of \( \{1, 2, \ldots, n'\} \) such that \( \prod_{i=1}^\ell S_i \subseteq \{1, 2, \ldots, n'\} \).
We set

\[ \Delta_k^\pm(I, P_0, S_1, \ldots, S_\ell) := \left\{ \sum_{j=1}^\ell z_j V_{S_j} \in \Delta_k^\pm(I, P_0) \mid 0 < z_1 < \cdots < z_\ell \right\}. \]

(2) For \(1 \leq \ell \leq n'\), we set

\[ \mathcal{J}_\ell(\Delta_k^\pm(I, P_0)) := \{ \Delta_k^\pm(I, P_0, S_1, \ldots, S_\ell) \}, \]

where \(S_1, \ldots, S_\ell\) runs over all ordered disjoint subsets of \(\{1, 2, \ldots, n'\}\).

(3) Let

\[ \mathcal{J}(\Delta_k^\pm(I, P_0)) := \prod_{\ell=0}^{n'} \mathcal{J}_\ell(\Delta_k^\pm(I, P_0)). \]

Clearly, we have

\[ \Delta_k^\pm(I, P_0) = \prod_{S \in \mathcal{J}(\Delta_k^\pm(I, P_0))} S. \]

We next to show that Theorem 5.2 follows from Propositions 5.13 and 5.14 whose proofs are given later in Sections 5.2 and 5.3 respectively.

**Proposition 5.13.** For every subset \(I \subseteq I(P_0)\), we have

\[ \text{LD} \left( \begin{vmatrix} \tilde{e}_{\text{res}}^{P_0} & p Q - Q' \end{vmatrix} Q, Q' \in \Delta_k^\pm(I, P_0) \right) = \prod_{S \in \mathcal{J}(\Delta_k^\pm(I, P_0))} \text{det} \left( \text{LD} \left( \tilde{e}_{\text{res}}^{P_0} - \eta(P_0) + p Q - Q' \right) \right)_{Q, Q' \in S}. \]

**Proposition 5.14.** For every point \(P_0 \in \Delta^-\) and every subset \(S \in \mathcal{J}(\Delta_k^\pm(I, P_0))\) we have

\[ \begin{vmatrix} \text{det} \left( \tilde{e}_{\text{res}}^{P_0} - \eta(P_0) + p Q - Q' \right) \end{vmatrix}_{Q, Q' \in S} = b_{P_0, S} \times g_{P_0, S}(\zeta), \]

where \(b_{P_0, S}\) is a \(p\)-adic unit in \(\mathbb{Z}_p\) and \(g_{P_0, S}(\zeta)\) is a monomial in \(\mathbb{Z}_p[\zeta_1, \zeta_2, \ldots, \zeta_n]\).
Proof of Theorem 5.2. (Assuming Propositions 5.13 and 5.14) By Property 5.10, Propositions 5.8, 5.13, and 5.14, we have

\[ \text{LD} \left( \det \left( e^{\text{res}}_{P^0 - \eta(P^0) + p Q - Q'} \right) \right)_{Q, Q' \in \Delta^\pm_k} = \prod_{P^0 \in \Delta^-} \prod_{I \subseteq I(P^0)} \prod_{S \in \mathcal{S}(\Delta_k^+(I, P^0))} \text{LD} \left( \det \left( e^{\text{res}}_{P^0 - \eta(P^0) + p Q - Q'} \right) \right)_{Q, Q' \in S} \]

(Prop 5.8)

\[ \prod_{P^0 \in \Delta^-} \prod_{I \subseteq I(P^0)} \prod_{S \in \mathcal{S}(\Delta_k^+(I, P^0))} \text{LD} \left( \det \left( e^{\text{res}}_{P^0 - \eta(P^0) + p Q - Q'} \right) \right)_{Q, Q' \in S} = \prod_{P^0 \in \Delta^-} \prod_{I \subseteq I(P^0)} \prod_{S \in \mathcal{S}(\Delta_k^+(I, P^0))} b_{P^0, S} \times g_{P^0, S}(\zeta) \]

(Prop 5.13)

\[ h(\Delta_k^\pm) = \prod_{P^0 \in \Delta^-} \prod_{I \subseteq I(P^0)} \prod_{S \in \mathcal{S}(\Delta_k^+(I, P^0))} \left\{ Q - P^0 + \eta(P^0) \mid Q \in S \right\}. \]

Now we prove Propositions 5.13 and 5.14.

5.2. Proof of Proposition 5.13 assuming Proposition 5.14. The essence of the proof is the only terms that contribute to the left-hand side of (5.8) come from the determinant of the leading terms on its right-hand side.

Lemma 5.15. Let

\[ \prod_{j=1}^{\ell'} S'_j \subseteq \prod_{i=1}^{\ell} S_i \subseteq \{1, 2, \ldots, n'\}. \]

For every two points \( Q = \sum_{i=1}^{\ell} z_i V_{S_i} \) and \( Q' = \sum_{j=1}^{\ell'} z_j' V_{S_j'} \) in \( \Lambda_\Delta \) such that

1. \( 0 < z_1 < z_2 < \cdots < z_\ell \) and \( 0 < z_1' < z_2' < \cdots < z_{\ell'}' \);
2. \( z_i - z_{i-1} > z_{i'}' - z_1' \) for every \( 1 \leq i \leq \ell \),

we have

\[ \text{Deg}(Q - Q') \leq \text{Deg}(Q) - \text{Deg}(Q') \]

(5.10)
with the equality if and only if for every $1 \leq i \leq \ell$, there is $1 \leq j_i \leq \ell'$ such that
(I) $S_i \subseteq S_{j_i}'$, and
(II) $j_{i_1} \leq j_{i_2}$ for every $1 \leq i_1 < i_2 \leq \ell$.

Proof. When $\ell = 1$, if $\ell' \leq 1$, it is easy to show that
\[
\text{Deg}(Q - Q') = \text{Deg}(Q) - \text{Deg}(Q').
\]

Now we assume that $\ell' > 1$. By condition (2) and Definition 5.9(3), we obtain
\[
\text{Deg}(Q - Q') = \left( n - \sum_{j=1}^{\ell} \# S_j, \sum_{j=1}^{\ell} \# S_{j-1}, 0, \ldots, 0, (z_1 - z_{\ell'}'), \ldots \right) \quad (5.11)
\]
and
\[
\text{Deg}(Q) - \text{Deg}(Q') = \left( n - \sum_{j=1}^{\ell} \# S_j, \sum_{j=1}^{\ell} \# S_{j-1}, 0, \ldots, 0, (z_1 - z_1'), \ldots \right), \quad (5.12)
\]
which directly imply $\text{Deg}(Q - Q') \prec \text{Deg}(Q) - \text{Deg}(Q')$.

Assume (5.10) and its equality condition holds for $\ell > 1$; we will prove them for $\ell + 1$.

We set $j' := \max\{ j \mid S_1 \cap S_j' \neq \emptyset \}$. By condition (2) and Definition 5.9(3) again, we have
\[
\text{Deg}(Q - Q') = \left( 0, \ldots, 0, (z_1 - z_{j'}'), \ldots \right) \quad (5.13)
\]
and
\[
\text{Deg}(Q) - \text{Deg}(Q') = \left( 0, \ldots, 0, (z_1 - z_1'), \ldots \right), \quad (5.14)
\]
If $j' > 1$, then
\[
\text{Deg}(Q - Q') \prec \text{Deg}(Q) - \text{Deg}(Q').
\]
Then we show that in this case (I) and (II) cannot be both satisfied. Otherwise, from (I), we have $S_1 \subseteq S_{j'}'$. Since $j' > 1$, there exists $1 < i \leq \ell$ such that $S_i \subseteq S_{j'}'$. Therefore, we obtain $j_1 = j' > 1 = j_i$, which contradicts to (II).

If $j' = 1$, we have $S_1 \subseteq S_{j_1}'$, hence
\[
\prod_{j=2}^{\ell'} S_j' \subseteq \prod_{i=2}^{\ell} S_i \subseteq \{1, 2, \ldots, n'\}.
\]
We set
\[ Q_0 := \sum_{i=2}^{\ell+1} (z_i - z_1) V_{S_i} \quad \text{and} \quad Q'_0 := \sum_{j=2}^{\ell'} (z'_j - z'_1) V_{S'_j}. \]

It is easy to check that \( Q_0 \) and \( Q'_0 \) also satisfy the conditions (1) and (2). By condition (2) and Lemma 5.11, we get
\[ \deg(Q) - \deg(Q') - \deg(Q - Q') = \deg(Q_0) - \deg(Q'_0) - \deg(Q_0 - Q'_0). \]

Therefore, we are left to show that this result holds for \( Q_0 \) and \( Q'_0 \), which follows directly from the induction. \(\square\)

**Lemma 5.16.** Let \( \bigcup_{i=1}^\ell S_i \subseteq \{1,2,\ldots,n'\} \) and \( Q = \sum_{i=1}^\ell z_i V_{S_i} \in \text{Cone}(\Delta) \) such that \( 0 < z_1 < \cdots < z_\ell \). The point \((p-1)Q - \eta(Q\%)\) is a linear combination of \( \{V_{S_i}\}_{i=1}^\ell \) with integer coefficients, i.e.,
\[ (p-1)Q - \eta(Q\%) = \sum_{i=1}^\ell t_i V_{S_i}, \]
and \( t_i - t_{i-1} > n + 2 \) for every \( 1 \leq i \leq \ell \), where \( t_0 = 0 \).

**Proof.** Taking \( t_i = \lfloor pz_i \rfloor - z_i \) we prove
\[ (p-1)Q - \eta(Q\%) = \sum_{i=1}^\ell t_i V_{S_i}. \]

Combined with our assumption that \( p > D(n+4) \), this implies that
\[ t_{i+1} - t_i \geq (p-1)(z_{i+1} - z_i) - 1 \geq \frac{p-1}{D} - 1 > n + 2 \geq 0 \]
for every \( 1 \leq i \leq \ell \). \(\square\)

**Lemma 5.17.** Let \( 1 \leq k \leq n + 2, \bigcup_{i=1}^\ell S_i \subseteq \{1,2,\ldots,n'\}, \) and \( \bigcup_{j=1}^{\ell'} S'_j \subseteq \{1,2,\ldots,n'\}. \) For every pair of points \( Q \in \Delta_k^\pm(I,P_0;S_1,\ldots,S_\ell) \) and \( Q' \in \Delta_k^\pm(I,P_0;S'_1,S'_2,\ldots,S'_{\ell'}) \), we have
\[ \deg(pQ - \eta(P_0) + P_0 - Q') \leq \deg(pQ - \eta(P_0) + P_0) - \deg(Q'), \]
with the equality if and only if \( S_i \)'s and \( S'_j \)'s satisfy that
- for every \( 1 \leq i \leq \ell' \) there exists \( 1 \leq j_i \leq \ell \) such that \( S_i \subseteq S'_j \), and
- for every \( 1 \leq i_1 \leq \ell' \) and \( 1 \leq i_2 \leq \ell' \) if \( i_1 < i_2 \), then \( j_{i_1} \leq j_{i_2} \).
Proof. Let
\[ Q = \sum_{i=1}^\ell z_i V_{S_i}, \quad pQ - \eta(P_0) + P_0 = \sum_{i=1}^\ell t_i V_{S_i} \quad \text{and} \quad Q' = \sum_{j=1}^{\ell'} z'_j V_{S'_j} \]
such that \( 0 < z_1 < \cdots < z_\ell \) and \( 0 < z'_1 < \cdots < z'_{\ell'} \).

By Lemma 5.16, we have
\[ t_i - t_{i-1} > n + 2 \geq z'_\ell - z'_1, \tag{5.15} \]
for every \( 1 \leq i \leq \ell \). Then it follows directly from Lemma 5.15. \( \square \)

Notation 5.18. We set
\[ K := \min \{ i \mid \mathcal{S}_i(\Delta_k(\pm I, P_0)) \neq \emptyset \}. \]

Corollary 5.19. Let \( \mathcal{S} \in \mathcal{S}_\ell(\Delta_k(\pm I, P_0)). \) If \( Q \in \mathcal{S} \) and \( Q' \in \Delta_k(I, P_0) \) satisfy
\[ \text{Deg}(pQ - \eta(P_0) + P_0 - Q') = \text{Deg}(pQ - \eta(P_0) + P_0) - \text{Deg}(Q'), \tag{5.16} \]
then
\[ Q' \in \prod_{j=K}^{\ell} \prod_{S' \in \mathcal{S}_j(\Delta_k(I, P_0))} S'. \tag{5.17} \]
Moreover, if
\[ Q' \in \prod_{S' \in \mathcal{S}_\ell(\pm \Delta_k(I, P_0))} S', \]
then \( Q' \in \mathcal{S} \).

Proof. It follows directly from Lemma 5.17. \( \square \)

Proposition 5.20. For a permutation \( \tau \in \text{Iso}(\pm \Delta_k(I, P_0)) \), the equality
\[ \text{Deg}(pQ - \eta(P_0) + P_0 - \tau(Q)) = \text{Deg}(pQ - \eta(P_0) + P_0) - \text{Deg}(\tau(Q)) \tag{5.18} \]
holds for every \( Q \in \Delta_k(I, P_0) \) if and only if
\[ \tau(\mathcal{S}) = \mathcal{S} \]
for every \( \mathcal{S} \in \mathcal{S}(\pm \Delta_k(I, P_0)). \)
Proof. The “if part” follows from Lemma 5.17.

Now we prove the “only if part”. Assume $\tau$ is a permutation of $\mathcal{S}(\Delta_k^\pm(I,P_0))$ such that every $Q \in \Delta_k^\pm(I,P_0)$ satisfies (5.18). Combining two statements in Corollary 5.19 gives that $\tau(S_K) = S_K$.

Assume that any $K \leq \ell' < \ell$ and any $S_{\ell'} \in \mathcal{S}(\Delta_k^\pm(I,P_0))$ satisfies $\tau(S_{\ell'}) = S_{\ell'}$. Let $S_\ell$ be any subset of $\Delta_k^\pm(I,P_0)$ in $\mathcal{S}(\Delta_k^\pm(I,P_0))$. Combining the induction assumption with Corollary 5.19, we know

$$\tau(S_\ell) \subseteq \prod_{S' \in \mathcal{S}(\Delta_k^\pm(I,P_0))} S',$$

and hence $\tau(S_\ell) = S_\ell$. \qed

Notation 5.21. Let

$$\text{Iso}^{\text{sp}}(\Delta_k^\pm(I,P_0)) := \{ \tau \in \text{Iso}(\Delta_k^\pm(I,P_0)) \mid \tau(S) = S \text{ for every } S \in \mathcal{S}(\Delta_k^\pm(I,P_0)) \}.$$ 

Lemma 5.22. Let $\tau$ be a permutation of $\Delta_k^\pm(I,P_0)$. We have

$$\text{Deg} \left( \prod_{Q \in \Delta_k^\pm(I,P_0)} e_{pQ - \eta(P_0) + P_0 - \tau(Q)} \right)$$

$$= \max_{\tau' \in \text{Iso}(\Delta_k^\pm(I,P_0))} \left( \text{Deg} \left( \prod_{Q \in \Delta_k^\pm(I,P_0)} e_{pQ - \eta(P_0) + P_0 - \tau'(Q)} \right) \right)$$

if and only if $\tau \in \text{Iso}^{\text{sp}}(\Delta_k^\pm(I,P_0))$.

Proof. By Property 5.10(2)(II), Lemmas 5.11(2) and 5.17, we know that

$$\text{Deg} \left( \prod_{Q \in \Delta_k^\pm(I,P_0)} e_{pQ - \eta(P_0) + P_0 - \tau(Q)} \right)$$

(Property 5.10(2)(II))

$$= \sum_{Q \in \Delta_k^\pm(I,P_0)} \text{Deg} \left( e_{pQ - \eta(P_0) + P_0 - \tau(Q)} \right)$$

(Lemma 5.11(2))

$$= \sum_{Q \in \Delta_k^\pm(I,P_0)} \text{Deg} \left( pQ - \eta(P_0) + P_0 - \tau(Q) \right)$$

(Lemma 5.17)

$$\leq \sum_{Q \in \Delta_k^\pm(I,P_0)} \left( \text{Deg}(pQ - \eta(P_0) + P_0) - \text{Deg}(\tau(Q)) \right)$$

$$= \sum_{Q \in \Delta_k^\pm(I,P_0)} \left( \text{Deg}(pQ - \eta(P_0) + P_0) - \text{Deg}(Q) \right).$$
By Proposition 5.20, the equality holds in the above inequality if and only if

\[
\tau \in \text{Iso}^\text{sp}(\Delta_k^\pm(I, P_0)). \quad \square
\]

**Lemma 5.23.** Let \( J \) be an index set and \( \{ g_j \mid j \in J \} \) be a set of polynomials in \( \mathbb{Z}_p[T][\zeta_1, \zeta_2, \ldots, \zeta_n] \). Denote by \( J_0 \) the subset of \( J \) such that for every \( j \in J \), we have

\[
\text{Deg}(g_j) = \max_{i \in J} \big( \text{Deg}(g_i) \big) \quad \text{if and only if} \quad j \in J_0.
\]

Then we have either

1. \( \sum_{j \in J_0} \text{LD}(g_j) = \text{LD}(\sum_{j \in J} g_j) \) or
2. \( \sum_{j \in J_0} \text{LD}(g_j) = 0. \)

**Proof.** It is straightforward. \( \square \)

We prove Proposition 5.13 assuming Proposition 5.14.

**Proof of Proposition 5.13.** By Property 5.10(2)(III) and the definition of \( \text{Iso}^\text{sp}(\Delta_k^\pm(I, P_0)) \), we have

\[
\sum_{\tau \in \text{Iso}^\text{sp}(\Delta_k^\pm(I, P_0))} \text{LD} \left( \left( \text{sgn}(\tau) \prod_{P \in \Delta_k^\pm(I, P_0)} \tilde{e}_{P_0 - \eta(P_0) + pQ - \tau(Q)} \right) \right) = \prod_{S \in \mathcal{S}(\Delta_k^\pm(I, P_0))} \det \left( \text{LD} \left( \tilde{e}_{P_0 - \eta(P_0) + pQ - Q'} \right) \right)_{Q, Q' \in S}. \tag{5.19}
\]

Assuming Proposition 5.14, we know that

\[
\prod_{S \in \mathcal{S}(\Delta_k^\pm(I, P_0))} \det \left( \text{LD} \left( \tilde{e}_{P_0 - \eta(P_0) + pQ - Q'} \right) \right)_{Q, Q' \in S} \neq 0. \tag{5.20}
\]

Combining (5.20) with Lemmas 5.22 and 5.23, we obtain (5.8). \( \square \)

### 5.3. Proof of Proposition 5.14.

**Notation 5.24.** We set

\[
Y_k(S_1, \ldots, S_\ell) := \left\{ \sum_{i=1}^\ell m_i V_{S_i} \mid m_i \in \mathbb{Z} \text{ and } 0 \leq m_1 \leq m_2 \leq \cdots \leq m_\ell \leq k \right\}.
\]

As in Notation 5.12, we assume that \( I = \{ n' + 1, n' + 2, \ldots, n \} \).

**Lemma 5.25.** For every nonempty set \( \Delta_k^\pm(I, P_0; S_1, \ldots, S_\ell) \), there exists

\[
Q_{\min} = \sum_{i=1}^\ell z_{\min, i} V_{S_i} \in \Delta_k^\pm(I, P_0; S_1, \ldots, S_\ell),
\]
where \(0 < z_{\min,i} - z_{\min,i-1} \leq 1\) for all \(1 \leq i \leq \ell\), and integers \(K^\pm\) such that

\[
\Delta_k^\pm(I, P_0; S_1, \ldots, S_\ell) = Q_{\min} + Y_{K^\pm}(S_1, \ldots, S_\ell).
\]

**Proof.** Let \(Q_{\min}\) denote the point in \(\Delta_k^\pm(I, P_0; S_1, \ldots, S_\ell)\) with the minimal degree. Now we show that \(0 \leq z_{\min,i} - z_{\min,i-1} \leq 1\) for all \(1 \leq i \leq \ell\). Suppose that it is false, then there exists an integer \(j\) such that \(z_{\min,j} - z_{\min,j-1} > 1\). It is easy to check that \(\sum_{i=1}^\ell z_{\min,i} V_{S_i} - V_{S_j}\) is a point in \(\Delta_k^\pm(I, P_0; S_1, \ldots, S_\ell)\) of a smaller degree than \(Q_{\min}\), a contradiction.

The rest of this lemma is obvious. We show it by an example. \(\Box\)

**Example 5.26.** When \(\Delta\) is a cube generated by \((3,0,0),(0,3,0),(0,0,3), p = 29,\) and \(P_0 = (1,0,0)\).

(1) For \(\Delta_3^\pm(\{2\}, P_0; \{1\}, \{3\})\), it is easy to show that

- \(\Delta_3^-(\{2\}, P_0; \{1\}, \{3\}) = \{(1,0,3), (1,0,6), (4,0,6)\}\) and
- \(\Delta_3^+(\{2\}, P_0; \{1\}, \{3\}) = \{(1,0,3), (1,0,6), (4,0,6), (1,0,9), (4,0,9), (7,0,9)\}\),

hence \(Q_{\min} = (1,0,3)\).

It is easy to check that

\[
Q_{\min} + Y_1(\{1\}, \{3\}) = \Delta_3^-(\{2\}, P_0; \{1\}, \{3\})
\]

and

\[
Q_{\min} + Y_2(\{1\}, \{3\}) = \Delta_3^+(\{2\}, P_0; \{1\}, \{3\}).
\]

(2) For \(\Delta_3^\pm(\{2\}, P_0; \{3\}, \{1\})\), we get

\[
\Delta_3^\pm(\{2\}, P_0; \{3\}, \{1\}) = \{(4,0,3), (7,0,3), (7,0,6)\},
\]

hence \(Q_{\min} = (4,0,3)\).

It is easy to check that

\[
Q_{\min} + Y_1(\{3\}, \{1\}) = \Delta_3^+(\{2\}, P_0; \{3\}, \{1\}).
\]

Recall that in Notation 2.4, we denote by \(c_i \in \mathbb{Z}_p\) the coefficients of \(\pi^i\) in \(E(\pi)\).

**Notation 5.27.** Let \(Q_1 = \sum_{i=1}^\ell m_i V_{S_i} \in \Lambda_\Delta\) such that \(0 \leq m_1 \leq \cdots \leq m_\ell\). We put

\[
\gamma(Q_1) := \pi^{m_\ell} \prod_{i=1}^\ell \left(\zeta_{\sum_{j=i}^\ell \#S_j(Q_{i+1})}^{m_j - m_{j-1}}\right).
\]

**Remark 5.28.** Although the method of writing \(Q_1\) as a sum like that is not unique, \(\gamma(Q_1)\) is well defined.
Lemma 5.29. Let \( k \geq 1 \) be an integer, and let \( Q = \sum_{i=1}^{\ell} z_i V_{S_i} \) such that
\[
z_i - z_{i-1} > k \quad \text{for every } 1 \leq i \leq \ell.
\]

For every \( Q_1 = \sum_{i=1}^{\ell} m_i V_{S_i} \in Y_k(S_1, \ldots, S_\ell) \) and every \( Q_2 = \sum_{i=1}^{\ell} m'_i V_{S_i} \in Y_k(S_1, \ldots, S_\ell) \), the leading term
\[
LD(\tilde{e}_{Q+Q_1-Q_2}) = \gamma(Q) \gamma(pQ_1) \prod_{i=1}^{\ell} c_{z_i + pm_i - m'_i}.
\]

Proof. By Lemma 5.11, we have
\[
LD(\tilde{e}_{Q+Q_1-Q_2}) = \gamma(Q + pQ_1 - Q_2) \prod_{i=1}^{\ell} c_{z_i + pm_i - m'_i}.
\]

By Notation 5.27, we know that
\[
\gamma(Q + pQ_1 - Q_2) = \frac{\gamma(Q) \gamma(pQ_1)}{\gamma(Q_2)}.
\]

Combining them, we complete the proof. \( \square \)

Notation 5.30. (1) For a vector \( \vec{w} = (w_1, w_2, \ldots, w_\ell) \in \mathbb{Z}_\ell \) such that \( w_0 < w_1 < \cdots < w_\ell \), we denote
\[
\xi(\vec{w}) := \prod_{i=1}^{\ell-1} c_{w_{i+1} - w_i}.
\]

(2) Let
\[
V_{\ell,k} := \{(z_1, z_2, \ldots, z_\ell) \in \mathbb{Z}_\ell \mid 0 \leq z_1 \leq \cdots \leq z_\ell \leq k\}.
\]

Rearranging the vectors in \( V_{\ell,k} \) with increasing partial order as defined in Definition 5.9(1), we obtain a sequence of vectors as \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{(\ell+k)} \). For a vector \( \vec{w} = (w_1, w_2, \ldots, w_\ell) \in \mathbb{Z}_\ell \) such that \( k < w_i - w_{i-1} \) for every \( 1 \leq i \leq \ell \), we set
\[
M(\vec{w}, k) := (\xi(\vec{w} + p\vec{v}_i - \vec{v}_j))_{1 \leq i,j \leq (\ell+k)}.
\]

Proposition 5.31. Let \( k \in \mathbb{Z}_{>0} \) and \( Q = \sum_{i=1}^{\ell} z_i V_{S_i} \) such that \( 0 = z_0 < z_1 < \cdots < z_\ell \). If
\[
z_i - z_{i-1} > k \quad \text{for every } 1 \leq i \leq \ell,
\]
where \( \text{diag} \) (Lemma 5.29)

\[ (5.23) \]

\[ \ldots \]

\[ (5.24) \]

**Proof.** By Lemma 5.29, we simplify \( \det \left( \text{LD}\left( e^{Q+1}Q - Q_2 \right) \right)_{Q_1, Q_2 \in Y_k(S_1, \ldots, S_\ell)} \) as follows:

\[
\begin{align*}
\det \left( \text{LD}\left( e^{Q+1}Q - Q_2 \right) \right)_{Q_1, Q_2 \in Y_k(S_1, \ldots, S_\ell)} &= \\
&= \det \left( \text{diag} \left( \gamma(Q) \right) \right)_{Q_1 \in Y_k(S_1, \ldots, S_\ell)} \\
&\quad \times \left( \text{LD}\left( e^{Q+1}Q - Q_2 \right) \right)_{Q_1, Q_2 \in Y_k(S_1, \ldots, S_\ell)} \\
&\quad \times \text{diag} \left( \gamma(Q_2) \right)_{Q_2 \in Y_k(S_1, \ldots, S_\ell)} \\
&= \det \left( \text{LD}\left( e^{Q+1}Q - Q_2 \gamma^{-1}(Q_1) \right) \right)_{Q_1, Q_2 \in Y_k(S_1, \ldots, S_\ell)} \\
\text{(Lemma 5.29)} \\
&= \det \left( \text{diag} \left( \gamma(Q+pQ_1 - Q_1) \right) \right)_{Q_1 \in Y_k(S_1, \ldots, S_\ell)} M\left( (z_1, \ldots, z_\ell), k ) \right) \\
&= \det \left( M\left( (z_1, \ldots, z_\ell), k \right) \right) \prod_{Q_1 \in Y_k(S_1, \ldots, S_\ell)} \left( \gamma(Q+pQ_1 - Q_1) \right),
\end{align*}
\]

where \( \text{diag} \left( \gamma(Q) \right)_{Q_1 \in Y_k(S_1, \ldots, S_\ell)} \) is a diagonal matrix whose rows and columns are indexed by the points in \( Y_k(S_1, \ldots, S_\ell) \). \( \square \)

**Lemma 5.32.** Let \( \vec{w} = (w_1, w_2, \ldots, w_\ell) \) be an \( \ell \)-dimensional vector. If

\[ (5.24) \]

\[ k \leq w_i - w_{i-1} \leq p - k \]

for every \( 1 \leq i \leq \ell \), then

\[ \det \left( M(\vec{w}, k) \right) \neq 0 \pmod{p}. \]

**Proof.** We prove it by induction. When \( \ell = 1 \), we have \( \vec{w} = (w_1) \). For every \( k \geq 1 \) we write the determinant explicitly as

\[
\begin{vmatrix}
    c_{w_1} & c_{w_1 + p} & \cdots & c_{w_1 + kp} \\
    c_{w_1 - 1} & c_{w_1 + p - 1} & \cdots & c_{w_1 + kp - 1} \\
    c_{w_1 - 2} & c_{w_1 + p - 2} & \cdots & c_{w_1 + kp - 2} \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{w_1 - k} & c_{w_1 + p - k} & \cdots & c_{w_1 + kp - k}
\end{vmatrix}
\]

\[ (5.25) \]

\[ \det \left( M(\vec{w}, k) \right) = \det \left( \begin{vmatrix} 
    c_{w_1} & c_{w_1 + p} & \cdots & c_{w_1 + kp} \\
    c_{w_1 - 1} & c_{w_1 + p - 1} & \cdots & c_{w_1 + kp - 1} \\
    c_{w_1 - 2} & c_{w_1 + p - 2} & \cdots & c_{w_1 + kp - 2} \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{w_1 - k} & c_{w_1 + p - k} & \cdots & c_{w_1 + kp - k}
\end{vmatrix} \right). \]
By [RWXY, Lemma 5.2] and (5.24), the congruence relation

\[(w_1 - j)c_{w_1 + ip - j} - c_{w_1 + ip - j - 1} \equiv c_{w_1 + (i-1)p - j} \pmod{p}\]

holds for every \(0 \leq i \leq k\) and \(0 \leq j \leq k - 1\), where \(c_s = 0\) for all \(s < 0\).

Therefore, by (5.26), the equality (5.25) can be simplified as

\[(5.27)\]

\[\det(M(\vec{w}, k)) = \det\begin{pmatrix}
  c_{w_1} & c_{w_1 + p} & \cdots & c_{w_1 + kp} \\
  c_{w_1 - w_1 - 1} & c_{w_1 + p - w_1 - 1} & c_{w_1 + kp - w_1 - 1} & \cdots & c_{w_1 + kp - w_1 - 1} \\
  c_{w_1 - 2(w_1 - 1)c_{w_1 - 1}} & c_{w_1 + p - 2(w_1 - 1)c_{w_1 - 1}} & c_{w_1 + kp - 2(w_1 - 1)c_{w_1 - 1}} & \cdots & c_{w_1 + kp - 2(w_1 - 1)c_{w_1 - 1}} \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  c_{w_1 - (w_1 - k + 1)c_{w_1 - k + 1}} & c_{w_1 + p - (w_1 - k + 1)c_{w_1 - k + 1}} & c_{w_1 + kp - (w_1 - k + 1)c_{w_1 - k + 1}} & \cdots & c_{w_1 + kp - (w_1 - k + 1)c_{w_1 - k + 1}}
\end{pmatrix}
\equiv (1)^k c_{w_1} \det(M(\vec{w}, k - 1)) \pmod{p}.

Since \(k - 1 \leq w_i + 1 - w_i \leq p - k + 1\) for every \(0 \leq i \leq k - 1\), by simply taking induction on \(k\), we know that

\[(5.28)\]

\[\det(M(\vec{w}, k)) \equiv (-1)^{\frac{k(k+1)}{2}} c_{w_1}^{k-1} \pmod{p}.
\]

Condition (5.24) shows

\[(5.29)\]

\[c_{w_1} = \frac{1}{w_1!} \not\equiv 0 \pmod{p}.
\]

Combining (5.29) and (5.28), we complete the proof of the case when \(\ell = 1\).

For \(\ell = 2\), by definition, we have

\[M(\vec{w}, k) = \begin{pmatrix}
  c_{w_1} M_{00} & c_{w_1 + p} M_{01} & \cdots & c_{w_1 + kp} M_{0k} \\
  c_{w_1 - 1} M_{10} & c_{w_1 - 1 + p} M_{11} & \cdots & c_{w_1 - 1 + kp} M_{1k} \\
  c_{w_1 - 2} M_{20} & c_{w_1 - 2 + p} M_{21} & \cdots & c_{w_1 - 2 + kp} M_{2k} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{w_1 - k} M_{k0} & c_{w_1 - k + p} M_{k1} & \cdots & c_{w_1 - k + kp} M_{kk}
\end{pmatrix},
\]
where for every $0 \leq i \leq k$ and every $0 \leq j \leq k$,

$$M_{ij} = \begin{pmatrix}
    c_{w_2 - w_1} & c_{w_2 - w_1 + p} & \cdots & c_{w_2 - w_1 + jp - 1} \\
    c_{w_2 - w_1 - 1} & c_{w_2 - w_1 + p - 1} & \cdots & c_{w_2 - w_1 + jp - 2} \\
    c_{w_2 - w_1 - 2} & c_{w_2 - w_1 + p - 2} & \cdots & c_{w_2 - w_1 + jp - 3} \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{w_2 - w_1 - i} & c_{w_2 - w_1 + p - i} & \cdots & c_{w_2 - w_1 + jp - i - 1}
\end{pmatrix}.$$  

Let $A_i = [I_{(n-i)\times(n-i)}, 0_{(n-i)\times1}]$ for $1 \leq i \leq k$. It is easy to see that $M_{i,j} = A_i M_{i-1,j}$ for every $0 \leq j \leq k$ and every $1 \leq i \leq k$. Therefore, imitating the row operations used in (5.27), we modify $M(\vec{w}, k)$ by “block row operations” as follows:

$$\begin{pmatrix}
    I_0 \\
    -w_1 A_1 \\
    \vdots \\
    I_k
\end{pmatrix}
\begin{pmatrix}
    I_0 \\
    I_1 \\
    \vdots \\
    I_k
\end{pmatrix}
\begin{pmatrix}
    I_0 \\
    -(w_1 - 1) A_2 \\
    \vdots \\
    I_k
\end{pmatrix}$$

$$\times
\begin{pmatrix}
    I_0 \\
    I_1 \\
    \vdots \\
    I_{k-1} \\
    -(w_1 - k + 1) A_k \\
    I_k
\end{pmatrix}
M(\vec{w}, k)$$

$$= \begin{pmatrix}
    c_{w_1} M_{00} & c_{w_1 + p} M_{01} & \cdots & c_{w_1 + kp} M_{0k} \\
    c_{w_1 - 1 - w_1 c_{w_1}} M_{10} & c_{w_1 - 1 - w_1 c_{w_1} + p} M_{11} & \cdots & c_{w_1 - 1 - w_1 c_{w_1} + kp} M_{1k} \\
    (c_{w_1 - 2} - (w_1 - 1)) M_{20} & (c_{w_1 - 2} - (w_1 - 1) + c_{w_1 - 1}) M_{21} & \cdots & (c_{w_1 - 2} - (w_1 - 1) + c_{w_1 - 1} + kp) M_{2k} \\
    \vdots & \vdots & \ddots & \vdots \\
    (c_{w_1 - k} - (w_1 - k + 1)) M_{k0} & (c_{w_1 - k} - (w_1 - k + 1) + c_{w_1 - k+1}) M_{k1} & \cdots & (c_{w_1 - k} - (w_1 - k + 1) + c_{w_1 - k+1} + kp) M_{kk}
\end{pmatrix}$$

$$\equiv \begin{pmatrix}
    c_{w_1} M_{00} & c_{w_1 + p} M_{01} & \cdots & -c_{w_1 + kp} M_{0k} \\
    0 & -c_{w_1} M_{11} & \cdots & -c_{w_1 + (k-1)p} M_{1k} \\
    0 & -c_{w_1 - 1} M_{21} & \cdots & -c_{w_1 + (k-1)p - 1} M_{2k} \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & -c_{w_1 - k+1} M_{k1} & \cdots & -c_{w_1 + (k-1)p - k+1} M_{kk}
\end{pmatrix} \pmod{p}.$$  

It is not hard to show that

$$M(\vec{w}, k - 1) = \begin{pmatrix}
    c_{w_1} M_{11} & \cdots & c_{w_1 + (k-1)p} M_{1k} \\
    c_{w_1 - 1} M_{21} & \cdots & c_{w_1 + (k-1)p - 1} M_{2k} \\
    \vdots & \ddots & \vdots \\
    c_{w_1 - k+1} M_{k1} & \cdots & c_{w_1 + (k-1)(p-1)} M_{kk}
\end{pmatrix}.$$
Combining it with (5.30) gives

$$\det(M(\vec{w}, k)) \equiv \det\left(\begin{array}{cc}
c_{w_1}M(\vec{w}', k) & * \\
0 & -M(\vec{w}, k - 1)
\end{array}\right) \pmod{p},$$

where $\vec{w}' = (w_1)$ and $*$ represents a $(k + \ell - 1 \choose k)$ matrix.

Since $\vec{w}'$ is a one-dimensional vector, as the argument above, the determinant

$$\det(M(\vec{w}', k)) \not\equiv 0 \pmod{p}.$$

Combining it with (5.29) shows that

$$\det(M(\vec{w}, k)) \not\equiv 0 \pmod{p} \quad \text{if and only if} \quad \det(M(\vec{w}, k - 1)) \not\equiv 0 \pmod{p}.$$

Since

$$M(\vec{w}, 0) = c_{w_2 - w_1} = \frac{1}{(w_2 - w_1)!} \not\equiv 0 \pmod{p},$$

we show that

$$\det(M(\vec{w}, k)) \not\equiv 0 \pmod{p}.$$

Assume this statement holds for all $t \leq \ell - 1$; we will prove it for $\ell$.

We first put

$$M(\vec{w}, k) = \begin{pmatrix}
c_{w_1}M_{00} & c_{w_1 + p}M_{01} & \cdots & c_{w_1 + kp}M_{0k} \\
c_{w_1 - 1}M_{10} & c_{w_1 - 1 + p}M_{11} & \cdots & c_{w_1 - 1 + kp}M_{1k} \\
c_{w_1 - 2}M_{20} & c_{w_1 - 2 + p}M_{21} & \cdots & c_{w_1 - 2 + kp}M_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
c_{w_1 - k}M_{k0} & c_{w_1 - k + p}M_{k1} & \cdots & c_{w_1 - k + kp}M_{kk}
\end{pmatrix},$$

where $M_{ij}$ is a $(k - 1 + \ell - i \choose k - 1) \times (k - 1 + \ell - j \choose k - 1)$ matrix; and

(5.32) $M_{00} = M(\vec{w}', k)$

for $\vec{w}' = (w_2 - w_1, w_3 - w_1, \ldots, w_\ell - w_1)$.

Similar to the case $\ell = 2$ there exists a set of matrices $\{A_i\}_{1 \leq i \leq k}$, such that for every $1 \leq i \leq k$ and $0 \leq j \leq k$, $A_i$ is a $(k - 1 + \ell - i \choose k - 1) \times (k - 1 + \ell - i + j \choose k - 1)$ reduced echelon matrix and

$$M_{i,j} = A_i M_{i-1,j}.$$
Similar to (5.30) and (5.31), we have
\[
\begin{pmatrix}
I_0 & I_1 & \ldots & I_k \\
-w_1 A_1 & I_1 & \ldots & I_k \\
\vdots & \ddots & \ddots & \vdots \\
I_0 & \ldots & & I_k \\
-1 & I_1 & \ldots & I_k \\
\end{pmatrix}
\begin{pmatrix}
I_0 \\
I_1 \\
\vdots \\
I_k \\
\end{pmatrix}
\begin{pmatrix}
I_0 & I_1 & \ldots & I_k \\
-(w_1 - 1) A_2 & I_2 & \ldots & I_k \\
\vdots & \ddots & \ddots & \vdots \\
I_0 & \ldots & & I_k \\
-1 & I_1 & \ldots & I_k \\
\end{pmatrix}
\begin{pmatrix}
I_0 \\
I_1 \\
\vdots \\
I_k \\
\end{pmatrix}
\]
(5.33)

\[
\begin{pmatrix}
c_{w_1} M_{00} & c_{w_1} M_{01} & \ldots & c_{w_1+k_p M_{0k}} \\
0 & -c_{w_1} M_{11} & \ldots & -c_{w_1+(k-1)p} M_{1k} \\
0 & -c_{w_1-1} M_{21} & \ldots & -c_{w_1+(k-1)p-1} M_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
0 & -c_{w_1-k+1} M_{k1} & \ldots & -c_{w_1+(k-1)(p-1)} M_{kk} \\
\end{pmatrix}
\equiv
\begin{pmatrix}
c_{w_1} M (\bar{w}', k) & * \\
0 & -M (\bar{w}, k-1) \\
\end{pmatrix}
\pmod{p},
\]
where * represents a \( \binom{k+\ell-1}{k} \) \times \( \binom{k+\ell-1}{k-1} \) matrix, and
\[
M (\bar{w}, k-1) =
\begin{pmatrix}
c_{w_1} M_{11} & \ldots & c_{w_1+(k-1)p} M_{1k} \\
c_{w_1-1} M_{21} & \ldots & c_{w_1+(k-1)p-1} M_{2k} \\
\vdots & \ddots & \vdots \\
c_{w_1-k+1} M_{k1} & \ldots & c_{w_1+(k-1)(p-1)} M_{kk} \\
\end{pmatrix}.
\]

By induction and (5.29), the determinant
\[
\det (c_{w_1} M (\bar{w}', k)) \not\equiv 0 \pmod{p},
\]
which implies
\[
\det (M (\bar{w}, k)) \not\equiv 0 \pmod{p} \quad \text{if and only if} \quad \det (M (\bar{w}, k-1)) \not\equiv 0 \pmod{p}.
\]

Since
\[
M (\bar{w}, 0) = \prod_{i=0}^{\ell} c_{w_{i+1} - w_1} = \prod_{i=0}^{\ell} \frac{1}{(w_{i+1} - w_1)!} \not\equiv 0 \pmod{p},
\]
we show that
\[
\det (M (\bar{w}, k)) \not\equiv 0 \pmod{p}.
\]
\[\square\]
Proof of Proposition 5.14. Let $S = \Delta^\pm_k(P_0, I; S_1, S_2, \ldots, S_\ell)$. By Lemma 5.25, there exists

$$Q_{\min} = \sum_{i=1}^{\ell} z_{\min,i} V_{S_i} \in \Delta^\pm_k(I, P_0; S_1, \ldots, S_\ell),$$

where $0 < z_{\min,i} - z_{\min,i-1} \leq 1$ for all $1 \leq i \leq \ell$, and integers $K^\pm$ such that

$$\Delta^\pm_k(I, P_0; S_1, \ldots, S_\ell) = Q_{\min} + Y_{K^\pm}(S_1, \ldots, S_\ell).$$

(5.34)

It is easy to see that we can put

$$Q := (p-1)Q_{\min} - \eta(P_0) + P_0 = \sum_{i=1}^{\ell} z_i V_{S_i} \in \Lambda_\Delta.$$

Since $P_0$ and $\eta(P_0)$ are both in $\Delta^-$ and $z_i \in \mathbb{Z}$ for every $1 \leq i \leq \ell$, we know that

$$|z_i - z_i - (p-1)(z_{\min,i} - z_{\min,i-1})| \leq 1.$$

Therefore, by Hypothesis 1.1, we get

$$n + 2 < z_i - z_{i-1} < p - (n + 2)$$

for every $1 \leq i \leq \ell$.

Since $K^\pm < n + 2$, we know that $Q$ and $Y_{K^\pm}(S_1, S_2, \ldots, S_\ell)$ satisfy the condition in Proposition 5.31, hence

$$\det \left( \text{LD} \left( \Gamma_{\text{res}}^{\pm}(P_0 - \eta(P_0) + p\eta - Q) \right) \right)_{P, Q \in S} = \det \left( \text{LD} \left( \Gamma_{\text{res}}^{\pm}(Q - \eta(Q) + p\eta - Q') \right) \right)_{Q_1, Q_2 \in \Lambda_\Delta(S_1, \ldots, S_\ell)}$$

$$= \det \left( M \left( (z_1, \ldots, z_\ell), \pm \right) \right) \times \prod_{Q_1 \in \Lambda_\Delta(S_1, \ldots, S_\ell)} (\gamma(Q + pQ_1 - Q_1))$$

(5.35)

$$= \det \left( M \left( (z_1, \ldots, z_\ell), \pm \right) \right) \times g_{P_0, S} \left( \sum_{Q_1 \in \Lambda_\Delta(S_1, \ldots, S_\ell)} w(Q + pQ_1 - Q_1) \right)$$

$$= \det \left( M \left( (z_1, \ldots, z_\ell), \pm \right) \right) \times g_{P_0, S} \left( \sum_{Q' \in S} w(P_0 - \eta(P_0) + p\eta' - Q') \right)$$

$$= \det \left( M \left( (z_1, \ldots, z_\ell), \pm \right) \right) \times g_{P_0, S} \left( \sum_{Q' \in S} \left[ w(Q') \right] - \left[ w(\eta(P_0) - P_0 + Q') \right] \right).$$

(5.36)

By (5.35), the vector $(z_1, z_2, \ldots, z_\ell)$ satisfies the conditions in Lemma 5.32. Therefore, we obtain

$$\det \left( M \left( (z_1, \ldots, z_\ell), \pm \right) \right) \neq 0 \pmod{p},$$

which completes the proof. □
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