On geometry and mechanics

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Abstract

Our purpose in this article is first, following [14], to find the topological upper limits of projections of secant planes to $C^1$ surfaces and the topological upper limits of projections of secant hyperplanes to $C^1$ hypersurfaces and second to prove that every $C^2$ space curve can be the solution of the principle of stationary action.

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The definition of a manifold tangent vector for dimensions 5, 6, 7, ... (see [17]) makes use of the Axiom of Choice and hence is a theorem of at least ZFC - Axiom of Foundation, which implies the Banach - Tarski paradox (see [18]), since for any point $P$ and for any tangent vector $v$ at $P$ one has to choose a path $c$ such that $v = c'(0)$, while the Banach - Tarski paradox contradicts the Ideal Gas Law (see [4] or [19]). Since in dimensions 2, 3, 4 the projection of the intersection of the secant hyperplane with the hypersurface is in dimensions 1, 2, 3, one has the classical notion of tangent vector. So our purpose in this article is to prove in ZF - Axiom of Foundation + Axiom of Countable Choice two theorems for the property of being tangent and an exclusion principle for the potential in Lagrangian mechanics.

* Dedicated to the memory of my grandparents Nikolaos and Alexandra, and Konstantinos and Eleni.

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1 Two theorems on geometry

1.1. Definition. If \( X \) is any compact Polish space, then we denote by \( K(X) \) the compact Polish space of compact subsets of \( X \) equipped with the Hausdorff metric and for any sequence \( (K_n)_{n \in \mathbb{N}} \) of compact subsets of \( X \), we denote by \( \limsup_{n \to \infty} K_n \) the topological upper limit of \( (K_n)_{n \in \mathbb{N}} \), i.e., the set of all \( x \in X \) with the property that for any \( n \in \mathbb{N} \), there exists \( x_n \in K_n \) such that \( x \) is a limit point of the sequence \( (x_n)_{n \in \mathbb{N}} \). (See, for example, Section 4.F on pages 24-28 of [12] and page 18 of [2] as it is cited on page 341 of [10].)

1.2. Definition. Let \( \Omega \in K(\mathbb{R}^2) \) be such that \( \Omega^o \neq \emptyset \) and let \( f : \Omega \to \mathbb{R} \) be \( C^1 \), while \( (x_0, y_0) \in \Omega^o \). If \( A, B \) are any real numbers and

\[
P(A, B) = \{ (x, y, z) \in \mathbb{R}^3 : z - f(x_0, y_0) = A(x - x_0) + B(y - y_0) \}
\]

is any plane in \( \mathbb{R}^3 \) that passes through \( (x_0, y_0, f(x_0, y_0)) \), then \( P(A, B) \) intersects the surface \( \text{Graph}(f) \) whose equation is \( z = f(x, y) \) exactly at the points \( (x, y, z) \) in \( \mathbb{R}^3 \) that satisfy the equation \( f(x, y) - f(x_0, y_0) = A(x - x_0) + B(y - y_0) \) (see, for example, [9] and Paragraphs a and b of Section 8.4.6 on pages 470-471 of [16] and pages 495-497 of [11] and pages 326-327 of [3]) and let

\[
C(A, B) = \{ (x, y) \in \Omega : f(x, y) - f(x_0, y_0) = A(x - x_0) + B(y - y_0) \}.
\]

1.3. Theorem. If

\[
(A_n, B_n) \to \left( \frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right)
\]

in \( \mathbb{R}^2 \) as \( n \to \infty \), then

\[
\limsup_{n \to \infty} C(A_n, B_n) \subseteq C \left( \frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right)
\]
Proof. If \((x^*_0, y^*_0) \in \limsup_{n \to \infty} C(A_n, B_n)\), then there exists a subsequence \((A_{n_k}, B_{n_k})\) of the sequence \((A_n, B_n)\) such that for any \(k \in \mathbb{N}\), there exists \((x_{n_k}, y_{n_k}) \in C(A_{n_k}, B_{n_k})\) with the property that
\[
\lim_{k \to \infty} (x_{n_k}, y_{n_k}) = (x^*_0, y^*_0),
\]
so
\[
\begin{align*}
f(x_{n_k}, y_{n_k}) - f(x_0, y_0) &= A_{n_k} (x_{n_k} - x_0) + B_{n_k} (y_{n_k} - y_0) \\
&\text{for every } k \in \mathbb{N} \text{ and by passing to the limit as } k \to \infty 
\end{align*}
\]
follows that
\[
f(x^*_0, y^*_0) - f(x_0, y_0) = A(x^*_0 - x_0) + B(y^*_0 - y_0),
\]
so \((x^*_0, y^*_0) \in C\left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0)\right)\). \(\triangle\)

1.4. Remark. If \(\phi(x) = \begin{cases} e^{-\frac{1}{x}} &\text{if } x > 0 \\ 0 &\text{if } x \leq 0 \end{cases}\) and \(\Omega = [-1, 1]^2\), while \(f(x, y) = \phi(x)\), whenever \((x, y) \in \Omega\), then, since \(\phi\) is \(C^1\), it follows that \(f\) is \(C^1\) and the tangent plane at \((0, 0)\) is the one with equation \(z = 0\), since \(f(0, 0) = \phi(0) = 0\), \(\frac{\partial f}{\partial x}(0, 0) = \phi'(0) = 0\) and \(\frac{\partial f}{\partial y}(0, 0) = 0\). So
\[
\begin{align*}
C(0, 0) &= \{(x, y) \in [-1, 1]^2 : f(x, y) - f(0, 0) = 0 \cdot (x - 0) + 0 \cdot (y - 0)\} \\
&= \{(x, y) \in [-1, 1]^2 : \phi(x) = 0\} \\
&= [-1, 0] \times [-1, 1],
\end{align*}
\]
while
\[
\begin{align*}
C\left(\frac{1}{n}, 0\right) &= \{(x, y) \in [-1, 1]^2 : f(x, y) - f(0, 0) = \frac{1}{n} \cdot (x - 0) + 0 \cdot (y - 0)\} \\
&= \{(x, y) \in [-1, 1]^2 : \phi(x) = \frac{1}{n} x\} \\
&= \{(x, y) \in [-1, 0] \times [-1, 1] : 0 = \frac{1}{n} x\}
\end{align*}
\]

3
\[ \cup \{(x, y) \in \{0\} \times [-1, 1] : 0 = \frac{1}{n} \cdot 0\} \]

\[ \cup \{(x, y) \in (0, 1] \times [-1, 1] : e^{-\frac{x}{n}} = \frac{1}{n}x\} \]

\[ = \{0\} \times [-1, 1] \]

\[ \cup \{(x, y) \in (0, 1] \times [-1, 1] : e^{-\frac{x}{n}} = \frac{1}{n}x\} \]

and consequently \( \limsup_{n \to \infty} C \left( \frac{1}{n}, 0 \right) \) is a proper subset of \( C(0, 0) \).

1.5. Remark. So, it follows that the tangent plane is the limiting position of the normal vector of a secant plane.

1.6. Definition. Let \( \Omega \in K(\mathbb{R}^3) \) be such that \( \Omega^c \neq \emptyset \) and let \( f : \Omega \to \mathbb{R} \) be \( C^1 \), while \((x_0, y_0, z_0) \in \Omega^c \). If \( A, B, \Gamma \) are any real numbers and

\[ P(A, B, \Gamma) = \{(x, y, z, t) \in \mathbb{R}^4 : t - f(x_0, y_0, z_0) = A(x - x_0) + B(y - y_0) + \Gamma(z - z_0)\} \]

is any hyperplane in \( \mathbb{R}^4 \) that passes through \((x_0, y_0, z_0, f(x_0, y_0, z_0))\) (see, for example, page 192 of [7] and page 295 of [8] and page 27 of [13]), then \( P(A, B, \Gamma) \) intersects the hypersurface \( \text{Graph}(f) \) whose equation is

\[ t = f(x, y, z) \]

exactly at the points \((x, y, z, t)\) in \( \mathbb{R}^4 \) that satisfy the equation

\[ f(x, y, z) - f(x_0, y_0, z_0) = A(x - x_0) + B(y - y_0) + \Gamma(z - z_0) \]

and let

\[ H(A, B, \Gamma) = \{(x, y, z) \in \Omega : f(x, y, z) - f(x_0, y_0, z_0) = A(x - x_0) + B(y - y_0) + \Gamma(z - z_0)\} \].

It is not difficult to see that since \( \Omega \) is compact and \( f \) is \( C^1 \), and so continuous, it follows that

\[ f(x, y, z) - f(x_0, y_0, z_0) = A(x - x_0) + B(y - y_0) + \Gamma(z - z_0) \]
is a closed condition and consequently $H(A, B, \Gamma)$ is a closed subset of $\Omega$, which implies in its turn that $H(A, B, \Gamma) \in K(\Omega)$. (See, for example, (3.15.1) and (3.17.3) of [6].)

1.7. Theorem. If
\[(A_n, B_n, \Gamma_n) \rightarrow (\partial f/\partial x(x_0, y_0, z_0), \partial f/\partial y(x_0, y_0, z_0), \partial f/\partial z(x_0, y_0, z_0))\]
in $\mathbb{R}^3$ as $n \to \infty$, then
\[\lim \sup \, H(A_n, B_n, \Gamma_n) \subseteq \lim \sup \, H(\partial f/\partial x(x_0, y_0, z_0), \partial f/\partial y(x_0, y_0, z_0), \partial f/\partial z(x_0, y_0, z_0))\]
in $K(\Omega)$.

Proof. If $(x^*_0, y^*_0, z^*_0) \in \lim \sup \, H(A_n, B_n, \Gamma_n)$, then there exists a subsequence $(y_n, \Gamma_n)_{k \in \mathbb{N}}$ of the sequence $(y_n, \Gamma_n)_{n \in \mathbb{N}}$ such that for any $k \in \mathbb{N}$, there exists $(x_n, y_n, z_n) \in H(A_n, B_n, \Gamma_n)$ with the property that
\[\lim \, (x_n, y_n, z_n) = (x^*_0, y^*_0, z^*_0),\]
so
\[f(x_n, y_n, z_n) - f(x_0, y_0, z_0) = A_n (x_n - x_0) + B_n (y_n - y_0) + \Gamma_n (z_n - z_0)\]
for every $k \in \mathbb{N}$ and by passing to the limit as $k \to \infty$ it follows that
\[f(x^*_0, y^*_0, z^*_0) - f(x_0, y_0, z_0) = A(x^*_0 - x_0) + B(y^*_0 - y_0) + \Gamma(z^*_0 - z_0),\]
so $(x^*_0, y^*_0, z^*_0) \in H(\partial f/\partial x(x_0, y_0, z_0), \partial f/\partial y(x_0, y_0, z_0), \partial f/\partial z(x_0, y_0, z_0)).$ \[\triangle\]

1.8. Remark. If $\phi(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$ and $\Omega = [-1, 1]^3$, while $f(x, y, z) = \phi(x)$, whenever $(x, y, z) \in \Omega$, then, since $\phi$ is $C^1$, it follows that $f$ is $C^1$ and the tangent hyperplane at $(0,0,0)$ is the one with equation $t = 0$, since $f(0,0,0) = \phi(0) = 0$, $\partial f/\partial x(0,0,0) = \phi'(0) = 0$, $\partial f/\partial y(0,0,0) = 0$ and $\partial f/\partial z(0,0,0) = 0$. So
\[ H(0, 0, 0) = \{(x, y, z) \in [-1, 1]^3 : \]
\[ f(x, y, z) - f(0, 0, 0) = 0 \cdot (x - 0) + 0 \cdot (y - 0) + 0 \cdot (z - 0) \} = \{(x, y, z) \in [-1, 1]^3 : \phi(x) = 0\} = [-1, 0] \times [-1, 1]^2, \]

while
\[ H \left( \frac{1}{n}, 0, 0 \right) = \{(x, y, z) \in [-1, 1]^3 : \]
\[ f(x, y, z) - f(0, 0, 0) = \frac{1}{n} \cdot (x - 0) + 0 \cdot (y - 0) + 0 \cdot (z - 0) \} = \{(x, y, z) \in [-1, 1]^3 : \phi(x) = \frac{1}{n} x\} = \{(x, y, z) \in [-1, 0] \times [-1, 1]^2 : 0 = \frac{1}{n} x\} \cup \{(x, y, z) \in \{0\} \times [-1, 1]^2 : 0 = \frac{1}{n} x\} \cup \{(x, y, z) \in (0, 1] \times [-1, 1]^2 : e^{-\frac{1}{n}} = \frac{1}{n} x\} = \{0\} \times [-1, 1]^2 \cup \{(x, y, z) \in (0, 1] \times [-1, 1]^2 : e^{-\frac{1}{n}} = \frac{1}{n} x\} \]

and consequently \( \limsup_{n \to \infty} H \left( \frac{1}{n}, 0, 0 \right) \) is a proper subset of \( H(0, 0, 0) \).

1.9. **Remark.** So, it follows that the tangent hyperplane is the limiting position of the normal vector of a secant hyperplane.

## 2 One theorem in mechanics

Our purpose in this section is to prove that every \( C^2 \) space curve can be the solution of the principle of stationary action. See Section 13 on page 59 of [1], where the principle is erroneously called of least action. To this end, it is enough to prove the following:
2.1. Theorem. If

\[ L(t, x, y, z, x', y', z') = P(t, x, y, z) + Q_1(t, x)x' + Q_2(t, y)y' + Q_3(t, z)z', \]

whenever \((t, x, y, z, x', y', z') \in \mathbb{R}^7\), where

\[
\begin{align*}
\frac{\partial P}{\partial x} &= \frac{\partial Q_1}{\partial t} \\
\frac{\partial P}{\partial y} &= \frac{\partial Q_2}{\partial t} \\
\frac{\partial P}{\partial z} &= \frac{\partial Q_3}{\partial t}
\end{align*}
\]

then given \(-\infty < a < b < \infty\) and given any \(C^2\) function

\[ r : (a, b) \ni t \mapsto r(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3, \]

the function \(r\) is a solution of the principle of stationary action with Lagrangian \(L\), i.e., \(r\) is a stationary point of the action integral

\[ I(r) = \int_a^b L(t, r(t), r'(t)) \, dt. \]

**Proof.** By virtue of D.4 on page 152 of [15], it follows that \(r\) must solve the system of equations

\[
\begin{align*}
\frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial x'} \right) &= 0 \\
\frac{\partial L}{\partial y} - \frac{d}{dt} \left( \frac{\partial L}{\partial y'} \right) &= 0 \\
\frac{\partial L}{\partial z} - \frac{d}{dt} \left( \frac{\partial L}{\partial z'} \right) &= 0
\end{align*}
\]

So, since

\[ \frac{\partial L}{\partial x} = \frac{\partial P}{\partial x} + \frac{\partial Q_1}{\partial x} x' \]

and

\[ \frac{\partial L}{\partial x'} = Q_1(t, x) \Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial x'} \right) = \frac{\partial Q_1}{\partial t} + \frac{\partial Q_1}{\partial x} x', \]

due to

\[ \frac{\partial P}{\partial x} = \frac{\partial Q_1}{\partial t}, \]
it follows that
\[ \frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial x'} \right) = 0. \]
In addition, since
\[ \frac{\partial L}{\partial y} = \frac{\partial P}{\partial y} + \frac{\partial Q_2}{\partial y} y', \]
and
\[ \frac{\partial L}{\partial y'} = Q_2(t, y) \Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial y'} \right) = \frac{\partial Q_2}{\partial t} + \frac{\partial Q_2}{\partial y} y', \]
due to
\[ \frac{\partial P}{\partial y} = \frac{\partial Q_2}{\partial t}, \]
it follows that
\[ \frac{\partial L}{\partial y} - \frac{d}{dt} \left( \frac{\partial L}{\partial y'} \right) = 0. \]
And once more, since
\[ \frac{\partial L}{\partial z} = \frac{\partial P}{\partial z} + \frac{\partial Q_3}{\partial z} z', \]
and
\[ \frac{\partial L}{\partial z'} = Q_3(t, z) \Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial z'} \right) = \frac{\partial Q_3}{\partial t} + \frac{\partial Q_3}{\partial z} z', \]
due to
\[ \frac{\partial P}{\partial z} = \frac{\partial Q_3}{\partial t}, \]
it follows that
\[ \frac{\partial L}{\partial z} - \frac{d}{dt} \left( \frac{\partial L}{\partial z'} \right) = 0. \]

2.2. Theorem. Keeping the notation as in 2.1., if
\[ P(t, x, y, z) = P_1(t, x) + P_2(t, y) + P_3(t, z), \]
whenever \((t, x, y, z) \in \mathbb{R}^4\), then the system of equations
\[
\begin{align*}
\frac{\partial P}{\partial x} &= \frac{\partial Q_1}{\partial t} \\
\frac{\partial P}{\partial y} &= \frac{\partial Q_2}{\partial t} \\
\frac{\partial P}{\partial z} &= \frac{\partial Q_3}{\partial t}
\end{align*}
\]
is equivalent to the system of equations
\[
\begin{align*}
\frac{\partial P_1}{\partial x} &= \frac{\partial Q_1}{\partial t} \\
\frac{\partial P_2}{\partial y} &= \frac{\partial Q_2}{\partial t} \\
\frac{\partial P_3}{\partial z} &= \frac{\partial Q_3}{\partial t}
\end{align*}
\]

**Proof.** It is enough to notice that
\[
\frac{\partial P}{\partial x} = \frac{\partial P_1}{\partial x}
\]
and
\[
\frac{\partial P}{\partial y} = \frac{\partial P_2}{\partial y}
\]
and
\[
\frac{\partial P}{\partial z} = \frac{\partial P_3}{\partial z}
\]
\(\triangle\)

2.3. **Theorem.** Keeping the notation as in 2.2., one can solve easily
\[
\begin{align*}
\frac{\partial P_1}{\partial x} &= \frac{\partial Q_1}{\partial t} \\
\frac{\partial P_2}{\partial y} &= \frac{\partial Q_2}{\partial t} \\
\frac{\partial P_3}{\partial z} &= \frac{\partial Q_3}{\partial t}
\end{align*}
\]

**Proof.** By virtue of 4.1.5 on page 147 of [5], since
\[
\frac{\partial P_1}{\partial x} = \frac{\partial Q_1}{\partial t},
\]
the differential equation

\[ P_1(t, x) dt + Q_1(t, x) dx = 0 \]

has as solution the equation

\[ u_1(t, x) = C, \]

where \( C \in \mathbb{R} \), while

\[ u_1(t, x) = \int_{t_1}^{t} P_1(t, x) dt + \int_{x_1}^{x} Q_1(t_1, x) dx, \]

so it is enough for \( P_1, Q_1 \) to be \( C^1 \). In addition, by virtue of 4.1.5 on page 147 of [5], since

\[ \frac{\partial P_2}{\partial y} = \frac{\partial Q_2}{\partial t}, \]

the differential equation

\[ P_2(t, y) dt + Q_2(t, y) dy = 0 \]

has as solution the equation

\[ u_2(t, y) = C, \]

where \( C \in \mathbb{R} \), while

\[ u_2(t, y) = \int_{t_2}^{t} P_2(t, y) dt + \int_{y_2}^{y} Q_2(t_2, y) dy, \]

so it is enough for \( P_2, Q_2 \) to be \( C^1 \). And once more, by virtue of 4.1.5 on page 147 of [5], since

\[ \frac{\partial P_3}{\partial z} = \frac{\partial Q_3}{\partial t}, \]

the differential equation

\[ P_3(t, z) dt + Q_3(t, z) dz = 0 \]

has as solution the equation

\[ u_3(t, z) = C, \]
where $C \in \mathbb{R}$, while

$$u_3(t, z) = \int_{t_3}^t P_3(t, z) \, dt + \int_{z_3}^z Q_3(t, z) \, dz,$$

so it is enough for $P_3, Q_3$ to be $C^1$. $\triangle$

2.4. Remark. In order to do infinitely many examples of $L$’s, do the following: Take three $C^2$ functions $f(t, x), g(t, y), h(t, z)$ and set

$$L = \frac{\partial f}{\partial t} + \frac{\partial g}{\partial t} + \frac{\partial h}{\partial t} + \frac{\partial f}{\partial x} x' + \frac{\partial g}{\partial y} y' + \frac{\partial h}{\partial z} z'.$$

For example,

$$L = (2t + 3x^2 t) + (2ty) + (-2z) + (3t^2 x) x' + (t^2 - y^2) y' + (z - 2t) z'.$$

Since for any such $L$, the equations of motion in Lagrangian mechanics give as solution every $C^2$ space curve and since $L = T - U$, where $T$ is kinetic energy and $U$ is potential energy, when kinetic energy $T$ is given, it follows that potentials $U = T - L$ with $L$ as above do not exist. See Section 13 on page 59 of [1].

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