AN ALGEBRAIC APPROACH TO HOUGH TRANSFORMS

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Abstract. The main purpose of this paper is to lay the foundations of a general theory which encompasses the features of the classical Hough transform and extend them to general algebraic objects such as affine schemes. The main motivation comes from problems of detection of special shapes in medical and astronomical images. The classical Hough transform has been used mainly to detect simple curves such as lines and circles. We generalize this notion using reduced Gröbner bases of flat families of affine schemes. To this end we introduce and develop the theory of Hough regularity. The theory is highly effective and we give some examples computed with CoCoA (see [2]).

1. Introduction

The Hough transform (or transformation) is a technique mainly used in image analysis and digital image processing. It was introduced by P.V.C. Hough in 1962 in the form of a patent (see [5]). Its original application was in physics for detection of lines and arcs in the photographs obtained in particle detectors, and many extensions and refinements of this method have been investigated since. In principle it can detect arbitrary shapes in images, given a parametrized description of the shape in question.

The main tool to achieve this result is a voting procedure in the parameter space. Roughly speaking, for instance to detect line segments it works in the following way. We can analytically describe a straight line in a number of forms. Although in practice the most convenient representation is the polar equation $x \cos \theta + y \sin \theta = \rho$, suppose that instead we use the cartesian equation $y = mx + n$. For each point $(a_1, a_2)$ in the source image, a potential line passing through it has parameters $m, n$ which necessarily satisfy the equation $a_2 = ma_1 + n$. This is the equation of a straight line in the parameter space with coordinates $m, n$. If many points in the source image lie on the straight line $y = ma_0x + n_0$, many lines in the parameter space will pass through the same point $(m_0, n_0)$. A discretization of the parameter space into small cells and an accumulator matrix whose entries count the number of lines passing through the cell will tell us that there is a maximum value corresponding to the cell containing $(m_0, n_0)$. More details can be found for instance in the paper [3].

A key winning factor for this strategy is that in the plane, a straight line which is not parallel to the $y$-axis has a well-defined unique representation of the type $y = mx + n$. Therefore, if a local maximum is obtained in the cell corresponding to $(m_0, n_0)$, the line of equation $y = m_0x + n_0$ is detected in the source image. Moreover if two local maxima are obtained in different cells, they correspond to
two different lines. This is a fundamental property which we call Hough regularity and which must be kept in every generalization.

In recent years, in particular in problems of recognition of special shapes in medical and astronomical images, much effort has been made to apply the above described procedure to the detection of more complicated objects, in particular special algebraic plane and space curves.

In this paper we want to lay the foundations of a general theory which encompasses the features of the classical Hough transform and extends them to general algebraic objects such as affine schemes.

To this end, a family of algebraic schemes is required to have the property that its fibers are irreducible and share the most essential properties such as the degree. So, we restrict ourselves to free families over an affine space and to achieve this property we use the notion of reduced Gröbner basis (see Section 2).

Moreover, the uniqueness of the reduced Gröbner basis of an ideal, given a term ordering $\sigma$ (see [6], Section 2.4.C), implies the possibility of associating a well-defined set of coefficients to every fiber of the family.

We define the Hough transform (Definition 3.2) of a point, and prove Theorem 3.10 which describes the interplay between algebraic objects in the source space and in the parameter space. The main consequence is that we are in the position of defining the key notion of Hough $\sigma$-regularity (see Definition 3.11). It describes the situation where equality of fibers implies equality of the corresponding parameters. In algebro-geometric terminology it means the following. If we consider the unirational variety $V$ defined parametrically by the set of coefficients of the reduced Gröbner bases of the generic fiber, the given parametrization represents $V$ as a rational variety.

Our main result is a criterion for detecting Hough $\sigma$-regularity which is embodied in Theorem 4.6 which rests on Theorem 4.3. It is then extended to the generic Hough regularity (see Threm 4.10), while the special case where the fibers of our family are hypersurfaces is described in Remark 4.7.

Another important feature of our presentation is that Hough $\sigma$-regularity and generic Hough regularity are computable. The last section is devoted to the illustration of several examples computed with the help of CoCoA (see [2]).

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2. Families of Schemes

We start the section by recalling some definitions. The notation is borrowed from [6] and [7], in particular we let $x_1, \ldots, x_n$ be indeterminates and let $\mathbb{T}^n$ be the monoid of the power products in the symbols $x_1, \ldots, x_n$. Most of the time, for simplicity we use the notation $\mathbf{x} = x_1, \ldots, x_n$. If $K$ is a field, the multivariate polynomial ring $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ is denoted by $P$, and if $f_1(\mathbf{x}), \ldots, f_k(\mathbf{x})$ are polynomials in $P$, the set $\{f_1(\mathbf{x}), \ldots, f_k(\mathbf{x})\}$ is denoted by $\mathbf{f}(\mathbf{x})$ (or simply by $\mathbf{f}$). Finally, we denote the polynomial system associated to $\mathbf{f}(\mathbf{x})$ by $\mathbf{f}(\mathbf{x}) = 0$, and we say that the system is $d$-dimensional if the ideal generated by $\mathbf{f}(\mathbf{x})$ is $d$-dimensional (see [6], Section 3.7).
Let $n$ be a positive integer, let $P = K[x_1, \ldots, x_n]$, let $f(x) = \{f_1(x), \ldots, f_n(x)\}$ define a $d$-dimensional irreducible, reduced scheme, and let $I$ be the ideal of $P$ generated by $f(x)$. We let $m$ be a positive integer and let $a = (a_1, \ldots, a_m)$ be an $m$-tuple of indeterminates which will play the role of parameters. If $F_1(a, x), \ldots, F_n(a, x)$ are polynomials in $K[a, x]$ we let $F(a, x) = 0$ be the corresponding family of systems of equations parametrized by $a$, and the ideal generated by $F(a, x)$ in $K[a, x]$ is denoted by $I(a, x)$. If the scheme of the $a$-parameters is denote by $S$, then there exists a $K$-algebra homomorphism $\phi : K[a] \rightarrow K[a, x]/I(a, x)$ or, equivalently, a morphism of schemes $\Phi : F \rightarrow S$ where $F = \text{Spec}(K[a, x]/I(a, x))$.

Although it is not strictly necessary for the theory, for our applications it suffices to consider independent parameters. Here is the formal definition.

**Definition 2.1.** If $S = K^m_R$ and $I(a, x) \cap K[a] = (0)$, then the parameters $a$ are said to be independent with respect to $F(a, x)$, or simply independent if the context is clear.

A theorem called generic flatness (see [4], Theorem 14.4) prescribes the existence of a non-empty Zariski-open subscheme $U$ of $S$ over which the morphism of schemes $\Phi^{-1}(U) \rightarrow U$ is flat. In particular, it is possible to explicitly compute a subscheme over which the morphism is free. To do this, Gröbner bases reveal themselves as a fundamental tool.

**Definition 2.2.** Let $F(a, x)$ be a family which contains a scheme defined by $f(x)$. Let $S = K^m_R$ be the scheme of the independent $a$-parameters and let

$$\Phi : \text{Spec}(K[a, x]/I(a, x)) \rightarrow S$$

be the associated morphism of schemes. A dense Zariski-open subscheme $U$ of $S$ such that $\Phi^{-1}(U) \rightarrow U$ is free (flat, faithfully flat), is said to be an $I$-free ($I$-flat, $I$-faithfully flat) subscheme of $S$ or simply an $I$-free ($I$-flat, $I$-faithfully flat) scheme.

**Proposition 2.3.** With the above assumptions and notation, let $I(a, x)$ be the ideal generated by $F(a, x)$ in $K[a, x]$, let $\sigma$ be a term ordering on $T^n$, let $G(a, x)$ be the reduced $\sigma$-Gröbner basis of the ideal $I(a, x)K(a)[x]$, let $d(a)$ be the least common multiple of all the denominators of the coefficients of the polynomials in $G(a, x)$, and let $T = T^n \setminus \text{LT}_\sigma(I(a, x)K(a)[x])$.

Then the open subscheme $U$ of $K^m_R$ defined by $d(a) \neq 0$ is $I$-free.

**Proof.** We consider the morphism $\text{Spec}(K(a)[x]/I(a, x)K(a)[x]) \rightarrow \text{Spec}(K(a))$. A standard result in Gröbner basis theory (see for instance [4], Theorem 1.5.7) shows that the residue classes of the elements in $T$ form a $K(a)$-basis of this vector space. We denote by $U$ the open subscheme of $K^m_R$ defined by $d(a) \neq 0$. For every point in $U$, the given reduced Gröbner basis evaluates to the reduced Gröbner basis of the corresponding ideal. Therefore the leading term ideal is the same for all these fibers, and so is its complement $T$. If we denote by $K[a]_{d(a)}$ the localization of $K[a]$ at the element $d(a)$ and by $I(a, x)^c$ the extension of the ideal $I(a, x)$ to the ring $K[a]_{d(a)}$, then $K[a]_{d(a)}[x]/I(a, x)^c$ turns out to be a free $K[a]_{d(a)}$-module. This consideration concludes the proof.

**Remark 2.4.** We observe that the term ordering $\sigma$ can be chosen arbitrarily.
Example 2.5. We let $P = \mathbb{C}[x]$, the univariate polynomial ring, and embed the ideal $I$ generated by the polynomial $x^2 - 3x + 2$ into the generically zero-dimensional family $F(a, x) = \{a_1x^2 - a_2x + a_3\}$. Such family is given by the canonical $K$-algebra homomorphism
\[
\varphi : \mathbb{C}[a] \twoheadrightarrow \mathbb{C}[a, x]/(a_1x^2 - a_2x + a_3)
\]
It is a zero dimensional complete intersection for
\[
\{(a_1, a_2, a_3) \in \mathbb{A}^3_\mathbb{C} \mid a_1 \neq 0\} \cup \{(a_1, a_2, a_3) \in \mathbb{A}^3_\mathbb{C} \mid a_1 = 0, a_2 \neq 0\}.
\]

3. The Hough Transform

We are not assuming that $K$ is algebraically closed, hence we must distinguish between maximal ideals and maximal linear ideals. The last ones correspond to $K$-rational points.

Suppose that $\Phi : \mathcal{F} \rightarrow \mathbb{A}^n_K$ represents a dominant family of sub-schemes of $\mathbb{A}^n_K$ parametrized by $\mathbb{A}^n_K$ which corresponds to a $K$-algebra homomorphism $\varphi : K[a] \twoheadrightarrow K[a, x]/I(a, x)$. The dominance implies that the $a$-parameters are independent, therefore $\varphi$ is injective. If we fix $\alpha = (a_1, \ldots, a_m)$, a rational “parameter point” in $\mathbb{A}^n_K$, we get a special fiber of $\Phi$, namely $\text{Spec}(K[a, x]/I(\alpha, x))$, hence a special member of the family. Clearly we have $K[\alpha, x] = K[x]$ so that $I(\alpha, x)$ can be seen as an ideal in $K[x]$. With this convention we denote the scheme $\text{Spec}(K[x]/I(\alpha, x))$ by $\Gamma_{\alpha, x}$.

On the other hand, there exists another morphism $\Psi : \mathcal{F} \rightarrow \mathbb{A}^n_K$ which corresponds to the $K$-algebra homomorphism $\psi : K[x] \twoheadrightarrow K[a, x]/I(a, x)$. If we fix a rational point $p = (\xi_1, \ldots, \xi_n) \in \mathbb{A}_K^n$, we get a special fiber of the morphism $\Psi$, namely $\text{Spec}(K[\alpha, p]/I(a, p))$. Clearly we have $K[\alpha, p] = K[a]$ so that $I(\alpha, p)$ can be seen as an ideal in $K[a]$. With this convention we denote the scheme $\text{Spec}(K[a]/I(\alpha, p))$ by $\Gamma_{\alpha, p}$. If $p \in C_{\alpha, x}$ then the pair $(C_{\alpha, x}, p)$ will be called a pointed fiber of $\Phi$.

Remark 3.1. We observe that a rational zero of $I(\alpha, p)$ is an $m$-tuple $\alpha$ such that $f(\alpha, p) = 0$ for every $f(\alpha, p) \in I(\alpha, p)$. Therefore $I(\alpha, p)$ corresponds to the sub-scheme of $\mathbb{A}^n_K$ which parametrizes the fibers of $\Phi$ which contain the rational point $p$.

Definition 3.2. (The Hough Transform)
We use the notation introduced above, we let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{A}^n_K$ and let $p = (\xi_1, \ldots, \xi_n) \in \mathbb{A}_K^n$. Then the scheme $\Gamma_{\alpha, p}$ is said to be the Hough transform of the point $p$ with respect to the family $\Phi$. If it is clear from the context, we simply say that the scheme $\Gamma_{\alpha, p}$ is the Hough transform of the point $p$.

Example 3.3. Let $a = (a, b)$, let $x = (x, y)$, and let $\mathcal{F} = \text{Spec}(K[\alpha, x]/F_{\alpha, x})$, where
\[
F_{\alpha, x} = y(x - ay)^2 - b(x^4 + y^4)
\]
Then let $\alpha = (1, 1)$. The corresponding fiber in $\mathbb{A}_K^2$ is $C_{(1,1), x}$ which is defined by the polynomial $F_{(1,1), x} = y(x - y)^2 - (x^4 + y^4)$. The point $p = (0, 1)$ belongs to $C_{(1,1), x}$ and it corresponds to the curve $\Gamma_{\alpha, (0,1)}$ which is defined by the polynomial $F_{\alpha, (0,1)} = a^2 - b$. The curve $\Gamma_{\alpha, (0,1)}$ is the Hough transform of the point $(0, 1)$.
Next, we are going to describe some properties of the Hough transforms.

**Proposition 3.4.** Let $\Phi : \mathcal{F} \to \mathbb{A}^n_K$ be a dominant family of sub-schemes of $\mathbb{A}^n_K$ parametrized by $\mathbb{A}^m_K$ and let $\alpha, p, C_{\alpha,x}, \Gamma_{\alpha,p}$ be as described above.

(a) We have $\alpha \in \cap_{p \in C_{\alpha,x}} \Gamma_{\alpha,p}$.

(b) If $\eta \in \cap_{p \in C_{\alpha,x}} \Gamma_{\alpha,p}$, then $C_{\alpha,x} \subseteq C_{\eta,x}$.

**Proof.** To prove (a) we observe that by definition $(\alpha,p)$ is a rational point of $\mathcal{F}$ for every $p \in C_{\alpha,x}$. This statement can be rephrased by saying that $\alpha \in \Gamma_{\alpha,p}$ for every $p \in C_{\alpha,x}$. Now we prove (b). To say that $\eta \in \cap_{p \in C_{\alpha,x}} \Gamma_{\alpha,p}$ is equivalent to saying that $(\eta,p)$ is a rational point of $\mathcal{F}$ for all the points $p$ such that $(\alpha,p)$ is a rational point of $\mathcal{F}$. This fact can be rephrased by saying that $C_{\alpha,x} \subseteq C_{\eta,x}$. \hfill $\square$

**Example 3.5.** Let $a = (a,b)$, let $x = (x,y)$, and let $\mathcal{F} = \text{Spec}(\mathbb{R}[a,x]/F_{a,x})$, where

$$F_{a,x} = x^2 + ay^2 - b$$

Then let $\alpha = (1,0)$. The corresponding fiber in $\mathbb{A}^2_K$ is $C_{(1,0),x}$ which is defined by the polynomial $F_{(1,0),x} = x^2 + y^2$. There is only one rational point on it, i.e. $p = (0,0)$. It corresponds to the curve $\Gamma_{(a,0,0)}$ which is defined by the polynomial $F_{a,(0,0)} = b$. Therefore $\cap_{p \in C_{(1,0),x}} \Gamma_{\alpha,p} = \Gamma_{(a,0,0)}$. We observe that $\eta = (-1,0)$ belongs to $\Gamma_{(a,0,0)}$ and that $C_{(-1,0),x}$ is defined by the polynomial $F_{(-1,0),x} = x^2 - y^2$. It consists of two lines which pass through the origin. Consequently $C_{(1,0),x} \subseteq C_{(-1,0),x}$.

While Example 3.5 shows that in general $C_{\alpha,x} \subseteq C_{\eta,x}$ (see statement (b) of the above proposition), we have a better result if $K$ is algebraically closed, but we have to rule out some extremal cases, as the following example shows.

**Example 3.6.** Let $a = a$, let $x = x$, and let $\mathcal{F} = \text{Spec}(\mathbb{C}[a,x]/F_{a,x})$, where

$$F_{a,x} = ax^2 + x$$

Then let $\alpha = 0$. The corresponding fiber in $\mathbb{A}^1_C(x)$ is $C_{0,x}$ which is defined by the polynomial $F_{0,x} = x$. There is only one rational point on it, i.e. $p = 0$. It corresponds to the entire line $\mathbb{A}^1_C(a)$ which is defined by the polynomial $F_{a,0} = 0$. Therefore $\cap_{p \in C_{0,x}} \Gamma_{\alpha,0} = \mathbb{A}^1_C(a)$. We observe that $\eta = 1$ belongs to $\mathbb{A}^1_C(a)$ and that $C_{1,x}$ is defined by the polynomial $F_{1,x} = x^2 + x$, so that $C_{1,x} = \{0,1\}$. Consequently $C_{0,x} \subseteq C_{1,x}$.

As previously announced, we need to rule out extremal cases. Since the application that we have in mind deal with continuous families, we resort to Proposition 3.4. Moreover, instead of seeking the broadest generality, we mainly consider irreducible fibers of $\Phi$. Finally, we use degree-compatible term orderings, the reason being that the fibers in our families must share the affine Hilbert function. In particular, if the fibers are hypersurfaces we want that they share the degree.

We start our investigation by recalling a straightforward result.

**Lemma 3.7.** Let $K[x]$ be a polynomial ring over a field $K$ and let $I$, $J$ be ideals in $P$ such that $\dim(P/I) = \dim(P/J)$, the radicals of $I$ and $J$ are prime ideals, and $\sqrt{J} \subseteq \sqrt{I}$. Then $\sqrt{J} = \sqrt{I}$.

**Proof.** Let $d = \dim(P/I)$, let $p = \sqrt{J}$, and let $\mathfrak{p} = \sqrt{I}$. We have $p \subseteq \mathfrak{p}$. On the other hand, we have $\dim(P/p) = \dim(P/\mathfrak{p})$, hence $p = \mathfrak{p}$. \hfill $\square$
Remark 3.8. The example with $I = (x^2, y)$, $J = (x, y^2)$ excludes the conclusion that $I = J$ in the previous lemma. The example $I = (x)$, $J = (x^2 - xy)$ excludes the possibility of assuming $I$ and $J$ to be equidimensional even if they belong to the flat family defined by $x^2 + axy = 0$.

Definition 3.9. Let $\Phi : F \longrightarrow \mathbb{A}_K^n$ be a dominant family of sub-schemes of $\mathbb{A}_K^n$ parametrized by $\mathbb{A}_K^n$. Then let $\sigma$ be a degree-compatible term ordering on $\mathbb{T}^n$, let $G(a, x)$ be the reduced $\sigma$-Gröbner basis of the ideal $I(a, x)K(a)[x]$, and let $d_\sigma(a)$ be the least common multiple of all the denominators of the coefficients of the polynomials in $G(a, x)$. We say that $d_\sigma(a)$ is the $\sigma$-denominator of $\Phi$. Moreover, let $U_\sigma = A_K^n \setminus \{d_\sigma(a) = 0\}$ and let $\Phi_{d_\sigma(a)} : \Phi^{-1}(U) \longrightarrow U_\sigma$ be the corresponding restriction of $\Phi$. We say that $\Phi_{d_\sigma(a)}$ is the $\sigma$-flat restriction of $\Phi$.

Theorem 3.10. Let $K$ be algebraically closed, let $\Phi : F \longrightarrow \mathbb{A}_K^n$ be a dominant family of sub-schemes of $\mathbb{A}_K^n$ parametrized by $\mathbb{A}_K^n$, let $\sigma$ be a degree-compatible term ordering on $\mathbb{T}^n$, let $d_\sigma(a)$ be a $\sigma$-denominator of $\Phi$, and $\Phi_{d_\sigma(a)}$ be the corresponding $\sigma$-flat restriction. If we restrict to the family $\Phi_{d_\sigma(a)}$, let $\alpha, p, C_{\alpha, x}$, $\Gamma_{a, p}$ be as described before, let $\eta \in \cap_{p \in C_{\alpha, x}} \Gamma_{a, p}$, and assume that $C_{\alpha, x}$ and $C_{\eta, x}$ are irreducible, then we have $C_{\alpha, x} = C_{\eta, x}$.

Proof. We know that the morphism $\Phi_{d_\sigma(a)}$ is flat and that $G(a, x)$ specializes to the reduced $\sigma$-Gröbner basis of $I(\alpha, x)$ for every $\alpha \in U_\sigma$ (see for instance the proof of Proposition 2.3 in [3]). By assumption, the term ordering $\sigma$ is degree-compatible, hence all the fibers of $\Phi_{d_\sigma(a)}$ share the same affine Hilbert function (see Chapter 5 of [7]), hence they share the same dimension. On the other hand, we know that $C_{\alpha, x} \subseteq C_{\eta, x}$ by Proposition 3.3 and hence the assumption that $K$ is algebraically closed implies that $\sqrt{I(\eta, x)} \subseteq \sqrt{I(\alpha, x)}$. The assumption that $C_{\alpha, x}$ and $C_{\eta, x}$ are irreducible implies that these two radical ideals are prime, hence we conclude the proof using Lemma 3.7.

Definition 3.11. With the assumptions as in Theorem 3.10, let $\cap_{p \in C_{\alpha, x}} \Gamma_{a, p} = \{\alpha\}$ for all $\alpha \in U_\sigma$. Then $\Phi_{d_\sigma(a)}$ is said to be Hough $\sigma$-regular.

An immediate consequence of the theorem is the following corollary.

Corollary 3.12. With the same assumptions as in the theorem, the following conditions are equivalent

(a) For all $\alpha, \eta \in U$, the equality $C_{\alpha, x} = C_{\eta, x}$ implies $\alpha = \eta$.
(b) The morphism $\Phi_{d_\sigma(a)}$ is Hough $\sigma$-regular.

Proof. The theorem states that if $\eta \in \cap_{p \in C_{\alpha, x}} \Gamma_{a, p}$ then $C_{\alpha, x} = C_{\eta, x}$, hence the conclusion follows immediately.

The meaning of this result is that under the assumption that $C_{\alpha, x} = C_{\eta, x}$ implies $\alpha = \eta$, the intersection of the Hough transforms of the pointed fibers $(C_{\alpha, x}, p)$, when $p$ ranges through the points of $C_{\alpha, x}$, turns out to be exactly $\{\alpha\}$, so that such intersection identifies the fiber $C_{\alpha, x}$. Therefore it is of great importance to detect situations where $\Phi_{d_\sigma(a)}$ is Hough $\sigma$-regular. For example, Hough regularity has been used as the conceptual basis for a pattern recognition algorithm (see [1]). Hough $\sigma$-regularity does not hold in general as the following easy example shows.
Example 3.13. Let \( a = a \), let \( x = x \), and let \( \mathcal{F} = \text{Spec}(\mathbb{C}[a,x]/F_{a,x}) \), where 
\[
F_{a,x} = a^2x^2 + x
\]
Then \( d(a) = a \) (there is only one term ordering, so that we do not need to write \( d_\sigma(a) \)). If we let \( \alpha = 1 \), the corresponding fiber in \( \mathbb{A}^1_\mathbb{C}(x) \) is \( C_{1,x} \) which is defined by the polynomial \( F_{0,x} = x^2 + x \). Therefore \( C_{1,x} = \{0, -1\} \) and \( \cap_{p \in C_{1,x}} \Gamma_{a,p} = \Gamma_{a,0} \cap \Gamma_{a,-1} \) is defined by the polynomial \( F_a = a^2 - 1 \). We observe that \( \eta = -1 \) is a zero of \( F_a \) and that \( C_{-1,x} = C_{1,x} \).

4. Detecting Hough-regularity

After Corollary 8.12 the problem of finding Hough \( \sigma \)-regular families rests on the ability of detecting families where \( C_{x,x} = C_{y,x} \) implies \( \alpha = \eta \). The schematic meaning of \( C_{x,x} = C_{y,x} \) is \( \sqrt{I(\alpha,x)} = \sqrt{I(\eta,x)} \), but we look for a more restricted condition, namely we consider the following algebraic problem. When does the equality \( I(\alpha,x) = I(\eta,x) \) imply \( \alpha = \eta \)? To answer this question, we address a seemingly unrelated problem and prove some algebraic facts. To do that, we make the following assumptions.

Let \( K \) be an algebraically closed field and \( m, s \) be two positive integers, let \( a = (a_1, \ldots, a_m), \ y = (y_1, \ldots, y_s) \), let \( p_1(a), \ldots, p_s(a), d_1(a), \ldots, d_s(a) \) be polynomials in \( K[a] \), and let \( d(a) = \text{lcm}(d_1(a), \ldots, d_s(a)) \). Then let \( C \) be the affine sub-scheme of \( \mathbb{A}^s \) given parametrically by \( y_i = \frac{p_i(a)}{d(a)} \), let \( D = \mathbb{A}^m \setminus \{d(a) = 0\} \), and denote the parametrization by \( \mathcal{P} \).

Definition 4.1. With these assumptions, we say that \( \mathcal{P} \) is injective if the corresponding morphism of schemes \( D \to C \) is injective. In other words, \( \mathcal{P} \) is injective if \( (\frac{p_1(a)}{d(a)}, \ldots, \frac{p_s(a)}{d(a)}, \frac{p_1(\eta)}{d(\eta)}, \ldots, \frac{p_s(\eta)}{d(\eta)}) \) implies \( \alpha = \eta \). If this is the case, \( \mathcal{P} \) is a rational representation of \( C \).

Definition 4.2. With the same assumptions, we double the set of indeterminates \( a = (a_1, \ldots, a_m) \) by considering a new set of indeterminates \( e = (e_1, \ldots, e_m) \). Then we define two ideals in \( K[a, e] \), the ideal \( I(\text{Doub}) \) generated by 
\[
\{p_1(a)d_1(e) - p_1(e)d_1(a), \ldots, p_s(a)d_s(e) - p_s(e)d_s(a)\},
\]
called the ideal of doubling coefficients of \( \mathcal{P} \), and the ideal \( I(\Delta) \) generated by \( \{a_1 - e_1, \ldots, a_m - e_m\} \), called the diagonal ideal.

The next theorem provides conditions under which a parametrization of the type described above is injective.

Theorem 4.3. Let \( K \) be an algebraically closed field and \( m, t \) be two positive integers, let \( a = (a_1, \ldots, a_m), \ y = (y_1, \ldots, y_s) \), let \( p_1(a), \ldots, p_s(a), d_1(a), \ldots, d_s(a) \) be polynomials in \( K[a] \), and let \( d(a) = \text{lcm}(d_1(a), \ldots, d_s(a)) \). Let \( C \) be an affine rational sub-scheme of \( \mathbb{A}^s \) defined by the parametrization \( \mathcal{P} \) given by \( y_i = \frac{p_i(a)}{d(a)} \), let \( D = \mathbb{A}^m \setminus \{d = 0\} \), and let \( I(\text{Doub}) \) and \( I(\Delta) \) be as described in Definition 4.2. Finally, let \( S(\Delta) \) be the saturation of \( I(\text{Doub}) \) with respect to \( I(\Delta) \). Then the following conditions are equivalent.

(a) The parametrization \( \mathcal{P} \) is injective.
(b) The ideal \( I(\Delta) \) is contained in the radical of the ideal \( I(\text{Doub}) \).
(c) The ideal \( I(\Delta) \) coincides with the radical of the ideal \( I(\text{Doub}) \).
(d) We have $S(\Delta) = (1)$.

Proof. Clearly (b) and (d) are equivalent. We observe that $I(Doub) \subseteq I(\Delta)$ and that $I(\Delta)$ is a prime ideal. Therefore if (b) is satisfied then we have the inclusions $I(Doub) \subseteq I(\Delta) \subseteq \sqrt{I(Doub)}$. Passing to the radicals we get the chain of inclusions $\sqrt{I(Doub)} \subseteq I(\Delta) \subseteq \sqrt{I(Doub)}$ which proves that (b) implies (c) while the implication (c) $\Rightarrow$ (b) is obvious. It is clear that $(\alpha_1, \ldots, \alpha_m)$ and $(\eta_1, \ldots, \eta_m)$ yield the same image point in $C$ if and only if $(\alpha_1, \ldots, \alpha_m, \eta_1, \ldots, \eta_m)$ is a zero of the ideal $I(Doub)$. Consequently, condition (a) is equivalent to the zero set of $I(Doub)$ being contained in the zero set of $I(\Delta)$. Since $K$ is algebraically closed, the Nullstellensatz implies that this condition is equivalent to (b), and the proof is complete. \hfill \Box

Now we are ready to use these facts to construct a method for detecting Hough-regularity.

**Definition 4.4.** Let $\sigma$ be a term ordering on $\mathbb{T}^n$ and let $H$ be a tuple of polynomials in $K(a)[x]$. If they are listed with $\sigma$-increasing leading terms, we get a well-defined list of non constant coefficients which is denoted by $NCC_H$ and called the **non constant coefficient list** of $H$.

For example, if $\sigma = \text{DegLex}$ and $H = (x_1x_2 - \frac{a_2^2 - 1}{a_1 - a_2}x_2, \ x_1^3 - \frac{a_1^2}{a_1 - a_2}x_1x_2 + \frac{a_2^3}{a_1})$, we have $NCC_H = [-\frac{a_2^2 - 1}{a_1 - a_2}, -\frac{a_1^2}{a_1 - a_2}, \frac{a_2^3}{a_1}]$.

In the following definition we use the terminology of Definition 3.9.

**Definition 4.5.** Let $\sigma$ be a degree compatible term ordering on $\mathbb{T}^n$ and let $G$ be the reduced $\sigma$-Gröbner basis of $I(a,x)K(a)[x]$, listed with $\sigma$-increasing leading terms. Let $NCC_G = (\frac{p_1(a)}{d_1(a)}, \ldots, \frac{p_m(a)}{d_m(a)})$ be the non constant coefficient list of $G$ and let $d_\sigma(a) = \text{lcm}(d_1(a), \ldots, d_m(a))$ be the $\sigma$-denominator of $G$. Let $e_1, \ldots, e_m$ be $m$ new indeterminates, let $e = (e_1, \ldots, e_m)$, and consider the following two ideals in the localization $K[a,e]_{d_\sigma(a)-d_\sigma(e)}$. The first ideal is generated by the $s$ polynomials

\[\{p_1(a)d_1(e) - p_1(e)d_1(a), \ldots, p_s(a)d_s(e) - p_s(e)d_s(a)\},\]

is denoted by $I(\text{DC}_G)$, and called the **ideal of doubling coefficients** of $G$. The second ideal is generated by the $m$ polynomials $\{a_1 - e_1, \ldots, a_m - e_m\}$, is denoted by $I(\Delta)$ and called the **diagonal ideal**.

**Theorem 4.6.** (Hough $\sigma$-regularity)

Let $K$ be algebraically closed, let $\Phi : \mathcal{F} \rightarrow \mathbb{K}_K^m$ be a dominant family of subschemes of $\mathbb{K}_K^n$ parametrized by $\mathbb{K}_K^m$, let $\sigma$ be a degree compatible term ordering on $\mathbb{T}^n$ and let $G$ be the reduced $\sigma$-Gröbner basis of $I(a,x)K(a)[x]$ listed with $\sigma$-increasing leading terms. Then let $d_\sigma(a)$ be the $\sigma$-denominator of $G$, let $I(\text{DC}_G)$ be the ideal of doubling coefficients of $G$, let $I(\Delta)$ be the diagonal ideal, and let $S(\Delta)$ be the saturation of $I(\text{DC}_G)$ with respect to $I(\Delta)$. Then the following conditions are equivalent.

(a) The morphism $\Phi_{d_\sigma(a)}$ is Hough $\sigma$-regular.

(b) The ideal $I(\Delta)$ is contained in the radical of the ideal $I(\text{DC}_G)$.

(c) The ideal $I(\Delta)$ coincides with the radical of the ideal $I(\text{DC}_G)$.

(d) We have $S(\Delta) = (1)$.
Proof. The equivalence of (b), (c), (d) follows as a special case of Theorem \[4.3\]. After Corollary \[3.12\] we observed that condition (a) can be expressed by saying that the equality \(I(\alpha, x) = I(\eta, x)\) implies \(\alpha = \eta\). On the other hand, to say that \(I(\alpha, x) = I(\eta, x)\) is equivalent to saying that the reduced Gröbner bases of \(I(\alpha, x)\) and \(I(\eta, x)\) are identical. As we have already observed in the proof of Theorem \[4.10\], outside the hypersurface defined by \(d^r(\alpha) = 0\) the reduced \(\sigma\)-Gröbner basis of \(I(\alpha, x)K(\alpha)|x|\) specializes to the reduced Gröbner bases of the fibers. Therefore the equality of the defining ideals of two fibers is equivalent to the equality of the corresponding coefficients in the reduced Gröbner bases. Hence \((\alpha_1, \ldots, \alpha_m)\) yields the same reduced Gröbner basis as \((\eta_1, \ldots, \eta_m)\) if and only if \((\alpha_1, \ldots, \alpha_m, \eta_1, \ldots, \eta_m)\) is a zero of the ideal \(I(DC_G)\). Consequently, condition (a) is equivalent to the zero set of \(I(DC_G)\) being contained in the zero set of \(I(\Delta)\). Again the conclusion follows directly from Theorem \[4.3\].

Remark 4.7. (Hypersurfaces)
Clearly, if the ideal \(I(\alpha, x)\) is principal all the matter is simplified. Let \(F = F(\alpha, x)\) be a generator of \(I(\alpha, x)\). In the case that all the coefficients of the leading form of \(F\) contain parameters, we have to choose \(\sigma\) and then invert the leading term of \(F\) to produce a monic polynomial \(\overline{F}\). Then \(\{\overline{F}\}\) is the reduced \(\sigma\)-Gröbner basis of \(I(\alpha, x)K(\alpha)|x|\) and we can use the above theorem. On the other hand, if one of the coefficients of the leading form of \(F\) is constant, then the family is flat over the parameter space \(K^m\). In this case we do not need to invert anything, hence if \(\{p_1(\alpha), \ldots, p_s(\alpha)\}\) is the list of non constant coefficients, we may consider the ideal generated by \(\{p_1(\alpha) - p_1(\epsilon), \ldots, p_s(\alpha) - p_s(\epsilon)\}\) as the ideal of doubling coefficients, and then use the theorem.

Theorem \[4.6\] yields a nice criterion to detect Hough \(\sigma\)-regularity. It depends on the term ordering chosen, hence it refers to a specific open sub-scheme of \(K^m\). However, it turns out to be of particular importance the detection of cases where the regularity is achieved generically, i.e. over a possibly different Zariski-open subschemes of the parameter space \(K^m\).

Definition 4.8. Let \(\Phi : F \rightarrow K^m\) be a dominant family of sub-schemes of \(K^m\) parametrized by \(K\), and let \(\alpha, p, C_{\alpha, x}, \Gamma_{\alpha, p}\) be as described at the beginning of the section.

(a) We say that the morphism \(\Phi\) is generically Hough regular if there exists a non-empty open subscheme \(U\) of \(K^m\) such that \(\cap_{p \in C_{\alpha, x}} \Gamma_{\alpha, p} = \{\alpha\}\) for all \(\alpha \in U\). In this case we can say that \(\Phi\) is Hough \(U\)-regular.

(b) We say that the morphism \(\Phi\) is Hough regular if it is generically Hough regular and \(U = K^m\).

Corollary 4.9. With the same assumptions as in Theorem \[4.6\] we let \(0 \neq h(\alpha)\) be an element of \(K[\alpha]\), let \(d = d_s(\alpha) \cdot h(\alpha) \cdot d_r(\epsilon) \cdot h(\epsilon)\), and consider \(I(\Delta), I(DC_G), S(\Delta)\) as ideals of \(K[\alpha, e]|a\). Then the following conditions are equivalent.

(a) The morphism \(\Phi\) is Hough \(U\)-regular where \(U = K^m \setminus \{d_s(\alpha) \cdot h(\alpha) = 0\}\).

(b) The ideal \(I(\Delta)\) is contained in the radical of the ideal \(I(DC_G)\).

(c) The ideal \(I(\Delta)\) coincides with the radical of the ideal \(I(DC_G)\).

(d) We have \(S(\Delta) = (1)\).

Proof. We observed that outside the hypersurface of \(K^m\) defined by the vanishing of the \(\sigma\)-denominator \(d^r(\alpha)\) of \(\Phi\), the reduced \(\sigma\)-Gröbner basis of \(I(\alpha, x)K(\alpha)|x|\)
specializes to the reduced $\sigma$-Gröbner bases of the fibers (see the proof of Theorem 3.10). Consequently, the equivalences in Theorem 4.6 extend, with the same proof, to the complement of every hypersurface of type $d_\sigma(a) \cdot h(a)$ and the corresponding localization $K[a]_d$. □

By comparing this definition with Definition 3.11, we observe that the notion of Hough $\sigma$-regularity is a special instance of the notion of generic Hough regularity. Example 5.6 in the next section illustrates this relation. Now the question is: how can we detect generic Hough regularity? Even more, if we discover generic Hough regularity, can we find $U$ explicitly? The next theorem provides an answer to both questions.

**Theorem 4.10. (Generic Hough regularity)**

With the same assumptions as in Theorem 4.6, suppose that the morphism $\Phi$ is not Hough $\sigma$-regular and that the ideal $S(\Delta) \cap K[a]$ is different from zero. Then we have the following facts.

(a) The morphism $\Phi$ is generically Hough regular.

(b) If $h(a)$ is a non-zero element of $S(\Delta) \cap K[a]$ and $U = K[a]_d \{d_\sigma(a) \cdot h(a) = 0\}$ then $\Phi$ is Hough $U$-regular.

**Proof.** Let $h(a)$ be a non-zero element of $S(\Delta) \cap K[a]$. We observe that $I(DC_G)$ and $I(\Delta)$ are invariant under the action of switching $a_1, \ldots, a_m$ with $e_1, \ldots, e_m$. Therefore $S(\Delta)$ enjoys the same property which implies that $h(e) \in S(\Delta)$. Consequently, if we let $d = d_\sigma(a) \cdot h(a) \cdot d_\sigma(e) \cdot h(e)$ we deduce that $S(\Delta) K[a, e]_d = (1)$

Therefore the two claims of the theorem follow directly from Corollary 4.9. □

**Question:** Is it true that condition (a) implies that $S(\Delta) \cap K[a]$ is different from zero?

---

5. Examples and Code

The computation in the following examples uses CoCoA-5. Here we see the basic functions which were written by Anna Bigatti.

```
Define NonConstCoefficients(X)
If Type(X) = RINGELEM Then
  Return [C In Coefficients(X) | deg(den(C))<>0 Or deg(num(C))<>0];
Else If Type(X) = LIST Then
  Return flatten([NonConstCoefficients(F) | F In X], 1);
EndIf;
Error("Unknown type");
EndDefine; -- NonConstCoefficients

Define IdealOfDoublingCoefficients(S, L, a_Name, e_Name, t_Name)
If L = [] Then Error("list is empty"); EndIf;
R := RingOf(num(L[1]));
phia := PolyAlgebraHom(R, S, IndetsCalled(a_Name, S));
phie := PolyAlgebraHom(R, S, IndetsCalled(e_Name, S));
DC := [phia(num(F))*phie(den(F))-phie(num(F))*phia(den(F)) | F In L];
```
Example 5.1. (A space line)
Let $\Phi : F \rightarrow \mathbb{A}^4_{k}$ be defined by the ideal $I(a, x) = (x - a_1 y - a_2 z, x - a_3 y - a_4 z)$.
If $\sigma = \text{DegRevLex}$, the reduced $\sigma$-Gröbner basis of $I(a, x)[a][x]$ is
$$G = (y + \frac{a_2 - a_4}{a_1 - a_3} z, x + \frac{a_2 a_3 - a_1 a_4}{a_1 - a_3} z)$$
Consequently $I(DC)$ is generated by the set $\{(a_2 - a_4)(e_1 - e_3) - (e_2 - e_4)(a_1 - a_3), (a_2 a_3 - a_1 a_4)(e_1 - e_3) - (e_2 e_3 - e_1 e_4)(a_1 - a_3), (a_1 - a_3)(e_1 - e_3) t - 1\}$. The ideal $I(\Delta)$ is generated by $\{a_1 - e_1, a_2 - e_2, a_3 - e_3, a_4 - e_4\}$. We ask CoCoA-5 to check if $I(\Delta)$ is contained in the radical of $I(DC_G)$ and the answer is negative. Here we see the CoCoA-code.

```plaintext
N := 4; R := QQ[a[1..N]]; S := QQ[a[1..N], e[1..N], t]; K := NewFractionField(R); Use P := K[x,y,z]; I := Ideal(x-a[1]*y-a[2]*z, x-a[3]*y-a[4]*z); RGB := ReducedGBasis(I); NCC := NonConstCoefficients(RGB); Use S; IDElta := ideal([a[i]-e[i] | i In 1..N]); IDC := IdealOfDoublingCoefficients(S, NCC, "a", "e", "t"); IsInRadical(IDElta, IDC); --false
```

We conclude that the morphism $\Phi$ is not Hough $\sigma$-regular. A bit of further easy investigation shows that indeed we get the same line for instance if we assign the values $(0, 1, 2, 3)$ and $(0, 1, 1, 2)$ to $(a_1, a_2, a_3, a_4)$.

Example 5.2. (A canonical space line)
A completely different situation happens when the space line is presented in canonical form. Let $\Phi : F \rightarrow \mathbb{A}^4_{k}$ be defined by $I(a, x) = (x - a_1 z - a_2, y - a_3 z - a_4)$. If $\sigma = \text{DegRevLex}$, the reduced $\sigma$-Gröbner basis of $I(a, x)[a][x]$ is clearly
$$G = (x - a_1 z - a_2, y - a_3 z - a_4)$$
The trivial conclusion is that the morphism $\Phi$ is Hough $\sigma$-regular. Since there are no denominators, $\Phi$ is Hough regular.

Example 5.3. (First conic)
Let $\Phi : F \rightarrow \mathbb{A}^4_{k}$ be defined by the ideal $I(a, x) = (x^2 - a^2 y - a^3)$. If $\sigma = \text{DegRevLex}$, the reduced $\sigma$-Gröbner basis of $I(a, x)[a][x]$ is $G = (x^2 - a^2 y - a^3)$. There is no denominator, hence $U = \mathbb{A}^4_{k}$. Therefore $NCC_G = [a^2, a^3]$, and we do not need to invert anything, so that $I(DC_G) = (a^2 - e^2, a^3 - e^3)$. We have $I(\Delta) = (a - e)$, we ask CoCoA-5 to check if $I(\Delta)$ is contained in the radical of $I(DC_G)$ and the answer is positive. Here we see the CoCoA-code.
\[ R ::= \mathbb{Q}[a]; \]
\[ S ::= \mathbb{Q}[a,e,t]; \]
\[ K := \text{NewFractionField}(R); \]
\[ \text{Use } P ::= K[x,y]; \]
\[ I := \text{Ideal}(x^2-a^2y-a^4); \]
\[ \text{RGB} ::= \text{ReducedGBasis}(I); \]
\[ \text{NCC} := \text{NonConstCoefficients}(\text{RGB}); \]
\[ \text{Use } S; \]
\[ \text{IDelta} ::= \text{ideal}(a-e); \]
\[ \text{IDC} ::= \text{IdealOfDoublingCoefficients}(S, \text{NCC}, "a", "e", "t"); \]
\[ \text{IsInRadical}(\text{IDelta}, \text{IDC}); \]
\[ --true \]

Therefore the morphism \( \Phi \) is Hough regular.

**Example 5.4. (Second conic)**

Let \( \Phi : \mathbb{F} \rightarrow \mathbb{A}^{1}_{\mathbb{C}} \) be defined by the ideal \( I(a, x) = (x^2-a^2y-a^4) \). If \( \sigma = \text{DegRevLex} \), the reduced \( \sigma \)-Gröbner basis of \( I(a, x)\mathbb{C}(a)[x] \) is \( G = (x^2-a^2y-a^4) \). There is no denominator, hence \( \mathcal{U} = \mathbb{A}^{1}_{\mathbb{C}} \). Therefore \( \text{NCC}_G = [a^2, a^4] \), and we do not need to invert anything, so that \( I(\text{DC}_G) = (a^2 - e^2, a^4 - e^4) \). We have \( I(\Delta) = (a - e) \), we ask GoGA-5 to check if \( I(\Delta) \) is contained in the radical of \( I(\text{DC}_G) \) and the answer is negative. Here we see the GoGA-code.

**Example 5.5. (A quartic curve)**

Let \( \Phi : \mathbb{F} \rightarrow \mathbb{A}^{2}_{\mathbb{C}} \) be defined by \( I(a, x) = (x^2 + y^2 + z^2 - 1, a_1xy - a_2y^2 - z) \). If \( \sigma = \text{DegRevLex} \), the reduced \( \sigma \)-Gröbner basis of \( I(a, x)\mathbb{C}(a)[x] \) is
\[
G = (xy - \frac{a_2}{a_1}y^2 - \frac{1}{a_1}z, \ x^2 + y^2 + z^2 - 1, \ y^3 + \frac{a_1^2}{h}yz^2 + \frac{a_1}{h}xz + \frac{a_2}{h}yz - \frac{a^2}{h}y)
\]
where \( h = a_1^2 + a_2^2 \). Then \( \mathcal{U} = \mathbb{A}^{2}_{\mathbb{C}} \setminus \{a_1h = 0\} \). We ask GoGA-5 to check if \( I(\Delta) \) is contained in the radical of \( I(\text{DC}_G) \) and the answer is positive. Here we see the GoGA-code.
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\[ N := 2; \quad R ::= \mathbb{QQ}[a[1..N]]; \]
\[ S ::= \mathbb{QQ}[a[1..N], e[1..N], t]; \]
\[ K ::= \text{NewFractionField}(R); \]
\[ \text{Use } P ::= K[x,y,z]; \]
\[ I ::= \text{Ideal}(x^2+y^2+z^2-1, a[1]*x*y-a[2]*y^2-z); \]
\[ \text{RGB} ::= \text{ReducedGBasis}(I); \]
\[ \text{NCC} ::= \text{NonConstCoefficients}(	ext{RGB}); \]
\[ \text{Use } S; \]
\[ \text{IDelta} ::= \text{ideal}([a[i]-e[i] \mid i \in 1..N]); \]
\[ \text{IDC} ::= \text{IdealOfDoublingCoefficients}(S, \text{NCC}, "a", "e", "t"); \]
\[ \text{IsInRadical}(\text{IDelta}, \text{IDC}); \]
\[ \text{true} \]

Therefore the morphism \( \Phi \) is Hough \( \sigma \)-regular. If we had removed the last generator of \( I(DC_G) \), i.e. the generator which imposes the invertibility of \( d(a) \) and \( d(e) \), the answer would have been false.

**Example 5.6. (A family of Viviani curves)**

Let \( \Phi : F \rightarrow \mathbb{A}^2_K \) be defined by \( I(a,x) = (a_2(z-a_1)^2+y^2-a_2^2z^2, x^2+y^2+z^2-4a_1^2) \). Classical Vivian curves are obtained for \( a_2 = 1 \) (see [9], p. 461). If \( \sigma = \text{DegRevLex} \), the reduced \( \sigma \)-Gröbner basis of \( I(a,x) \mathbb{C}[a][x] \) is

\[
G = (y^2 + a_2z^2 - 2a_1a_2z, \quad x^2 + (1 - a_2)z^2 + 2a_1a_2z - 4a_1^2)
\]

There are no denominators in the coefficients, hence the family is flat all over \( \mathbb{A}^2_K \). However the family is not Hough regular, as we check with CoCasa.

Here we see the CoCasa-code.

\[ N := 2; \quad R ::= \mathbb{QQ}[a[1..N]]; \]
\[ S ::= \mathbb{QQ}[t, e[1..N], a[1..N]]; \]
\[ K ::= \text{NewFractionField}(R); \]
\[ \text{Use } P ::= K[x,y,z,t]; \]
\[ \text{ID} ::= \text{Ideal}(a[2]*(z-a[1])^2 + y^2 - a[2]*a[1]^2, \quad x^2 + y^2 + z^2 - 4*a[1]^2); \]
\[ \text{RGB} ::= \text{ReducedGBasis}(\text{ID}); \]
\[ \text{NCC} ::= \text{NonConstCoefficients}(\text{RGB}); \]
\[ \text{Use } S; \]
\[ \text{IDelta} ::= \text{ideal}([a[i]-e[i] \mid i \in 1..N]); \]
\[ \text{IDC} ::= \text{IdealOfDoublingCoefficients}(S, \text{NCC}, "a", "e", "t"); \]
\[ \text{IsInRadical}('\text{IDelta}', '\text{IDC}); \]
\[ \text{false} \]

At this point we compute the saturation of \( I(DC_G) \) with respect to \( I(\Delta) \) and its intersection with \( \mathbb{C}[a_1,a_2] \).

\[
\text{Sat} ::= \text{Saturation}(\text{IDC}, \text{IDelta});
\]
\[ \text{ideal}(a[2], e[2], e[1]*a[1], t-1) \]
\[ \text{Elim}([e[1], e[2], t], \text{Sat}); \]
\[ \text{ideal}(a[2]) \]

We get the ideal generated by \( a_2 \) which means that \( \Phi \) is Hough \( U \)-regular where \( U = \mathbb{A}^2_K \setminus \{a_2 = 0\} \) (see Theorem 4.10).
Example 5.7. (A monomial curve)

Let \( \Phi : F \rightarrow \mathbb{A}^2_K \) be defined parametrically by

\[
x_1 = a_1 u^3, \quad x_2 = a_2 u^4, \quad x_3 = u^5
\]

By eliminating \( u \) we get generators of the ideal \( I(a, x) \). If \( \sigma = \text{DegRevLex} \), the reduced \( \sigma \)-Gröbner basis of \( I(a, x) \) is

\[
G = (x_2^2 - \frac{a_2}{a_1} x_1 x_3, \quad x_1 x_2 - \frac{a_1 a_2}{a_2} x_2^2, \quad x_3 - \frac{a_3}{a_2} x_2 x_3)
\]

The family is not Hough-regular as the following CoCoA-code shows.

Here we see the CoCoA-code.

\[
\begin{align*}
N := 2; & \quad R := \mathbb{Q}[a[1..N]]; \\
S := \mathbb{Q}[t, a[1..N], e[1..N]]; & \\
K := \text{NewFractionField}(R); & \\
Use P := K[x[1..3],u]; & \\
L := [3,4,5]; & \\
ID := \text{Ideal}(x[1]-a[1]*u(L[1]), x[2]-a[2]*u(L[2]), x[3]-u(L[3])); & \\
E := \text{Elim}([u], ID); & \\
RGB := \text{ReducedGBasis}(E); & \\
NCC := \text{NonConstCoefficients}(RGB); & \\
Use S; & \\
IDelta := \text{ideal}(a[i] - e[i] \mid i \text{ In } 1..N); & \\
IDC := \text{IdealOfDoublingCoefficients}(S, NCC, "a", "e", "t"); & \\
\text{IsInRadical}(IDelta, IDC); & \\
\text{--false}
\end{align*}
\]

At this point we check the reduced Gröbner basis of \( E \) and get the following set

\[
\{a_1^2 - e_1, a_2^2 - e_2, a_2^3 - e_1 e_2, a_1 a_2 e_3 - e_1^2 e_2, a_1 a_2^2 - e_1^3 e_2, ta_1 a_2 e_1 e_2 - 1, ta_1^3 e_3 - a_2, te_1^3 e_2 - a_1, te_1^2 e_3 - a_2^2, ta_1^2 e_2^2 - e_1^3, ta_1 e_1 e_2^2 - a_4^3\}.
\]

Just looking at the first two polynomials, we see that if we restrict our check to real numbers, we get

\[
a_1 = e_1, \quad a_2 = e_2,
\]

hence we may conclude that the family is real Hough regular.

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