On a degenerate singular elliptic problem

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Abstract

In this article we provide existence, uniqueness and regularity results of a degenerate singular elliptic boundary value problem whose prototype is given by

\[
\begin{cases}
-\text{div} \left( w(x) |\nabla u|^{p-2} \nabla u \right) = \frac{f(x)}{u^2} \quad \text{in} \quad \Omega, \\
u > 0 \quad \text{in} \quad \Omega, \\
u = 0 \quad \text{on} \quad \partial \Omega,
\end{cases}
\]

where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^N \) with \( N \geq 2 \), \( w \) belongs to the Muckenhoupt class \( A_p \) for some \( 1 < p < \infty \), \( f \) is a nonnegative function belonging to some Lebesgue space and \( \delta > 0 \).

KEYWORDS

degenerate elliptic equation, Muckenhoupt weight, singular nonlinearity, weighted Sobolev space

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1 INTRODUCTION

In this article, we establish existence, uniqueness and regularity results to the following degenerate singular elliptic boundary value problem:

\[
\begin{cases}
-\text{div}(\mathcal{A}(x, \nabla u)) = \frac{f(x)}{u^2} \quad \text{in} \quad \Omega, \\
u > 0 \quad \text{in} \quad \Omega, \\
u = 0 \quad \text{on} \quad \partial \Omega,
\end{cases}
\]

where \( \delta > 0 \), \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^N \) with \( N \geq 2 \) and \( f \) is a nonnegative function belonging to some Lebesgue space but not identically zero. The function \( \mathcal{A} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N \) is Carathéodory, that is,

- the function \( \mathcal{A}(\cdot, s) \) is measurable on \( \Omega \) for every \( s \in \mathbb{R}^N \), and
- the function \( \mathcal{A}(x, \cdot) \) is continuous on \( \mathbb{R}^N \) for a.e. \( x \in \Omega \).
Moreover, the following additional hypothesis on the function $A$ will be imposed throughout the paper.

(H1) Every $w$ belongs to the Muckenhoupt class $A_p$ (defined in Section 2),

(H2) (Growth) $|A(x, \zeta)| \leq |\zeta|^{p-1} w(x)$, for a.e. $x \in \Omega$, $\forall \zeta \in \mathbb{R}^N$.

(H3) (Degeneracy) $A(x, \zeta) : \zeta \leq |\zeta|^p w(x)$, for a.e. $x \in \Omega$, $\forall \zeta \in \mathbb{R}^N$.

(H4) (Homogeneity) $A(x, t\zeta) = t |t|^{p-2} A(x, \zeta)$, for $t \in \mathbb{R}$, $t \neq 0$.

(H5) (Strong Monotonicity) For $\gamma = \max\{p, 2\}$,

$$\langle A(x, \zeta_1) - A(x, \zeta_2), \zeta_1 - \zeta_2 \rangle \geq c \left| \zeta_1 - \zeta_2 \right|^{1-\gamma/p} \left\{ \bar{A}(x, \zeta_1, \zeta_2) \right\}^{\gamma} w(x),$$

for some positive constant $c$ where $\bar{A}$ is defined by

$$\bar{A}(x, \zeta_1, \zeta_2) := \frac{1}{w(x)} \left( \langle A(x, \zeta_1), \zeta_1 \rangle + \langle A(x, \zeta_2), \zeta_2 \rangle \right).$$

A prototype of Equation (1.1) is given by the following boundary value problem

$$\begin{cases}
Lu := -\text{div}(M(x)|\nabla u|^{p-2}\nabla u) = \frac{f(x)}{u^\delta} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

(1.2)

where $M(\cdot)$ is a continuous function with values in the set of $N \times N$ symmetric matrix satisfying

$|M(x)\zeta| \leq w(x)|\zeta|$, $M(x)\zeta \cdot \zeta \geq w(x)|\zeta|^2$, for a.e. $x \in \Omega$, for all $\zeta \in \mathbb{R}^N$.

In case of $M(x) = w(x)I$, where $I$ is the $N \times N$ identity matrix, the operator $L$ reduces to the weighted $p$-Laplace operator $\Delta_{p,w}$ defined by

$$\Delta_{p,w} u := \text{div}(w(x)|\nabla u|^{p-2}\nabla u).$$

We observe that for $w = 1$, $\Delta_{p,w} u = \Delta p u$, which is the standard $p$-Laplace operator. For the constant weight $w$, singular problems of type (1.2) has been widely studied in the last three decades, see [2–5, 7–10, 20–23, 30] and the references therein. We would like to point out some historical developments made in this direction which are closely related to the problem (1.2).

The case of $M = I$ and $p = 2$ with Dirichlet boundary condition is settled in the pioneering work of Crandall et al. [13], where the existence of a unique classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to Equation (1.2) is proved for any $\delta > 0$. This solution $u \in W^{1,2}_0(\Omega)$ if and only if $\delta < 3$, and for $\delta > 1$, $u$ does not belong to $C^1(\overline{\Omega})$ is proved by Lazer–McKenna [28] for a positive Hölder continuous data $f$.

Boccardo–Orsina [7] studied the semilinear case $p = 2$ for a constant weight function $w$ and nonnegative (not identically zero) data $f$ in some Lebesgue space to obtain existence and regularity results for any $\delta > 0$ to the problem (1.2). De Cave [14] generalised these results in the quasilinear case $1 < p < N$. Further Canino et al. [9] proved existence in addition to uniqueness results to the problem (1.2) in the full range $1 < \delta < \infty$ for $M = I$. In summary, depending on $p$ and the nonlinearity $f$, the authors in [7, 9, 14] proved existence of a solution $u \in W^{1,p}_0(\Omega)$ if $0 < \delta < 1$ and $u \in W^{1,p}_{loc}(\Omega)$ such that $u^{(\delta+1)/p} \in W^{1,p}_0(\Omega)$ (this was the meaning of $u = 0$ on $\partial \Omega$) if $\delta \geq 1$ to the problem (1.2). Moreover, we emphasize that when $0 < \delta < 1$ under the assumption $f \in L^1(\Omega)$, the authors in [7] (for $p = 2$) proved existence result in a larger Sobolev space than $W^{1,2}_0(\Omega)$, whereas if $2 - \delta + (\delta - 1)/N \leq p < N$, the author in [14] proved existence result in a larger Sobolev space than $W^{1,p}_0(\Omega)$ to the problem (1.2). When $f$ is a Radon measure, existence results to singular $p$-Laplace equations has also been investigated in the recent past and we refer the reader to De Cave et al. [15], De Cave–Oliva [16] and the references therein.
In contrast to [7, 9, 14], a natural question can be posed to say what happens to Equation (1.2) in the presence of a nonconstant weight function \( w \)? Indeed, our main motive in this paper is to answer this question affirmatively by providing a certain class of weight function (which may vanish or blow up near the origin) to ensure existence, uniqueness and regularity results analogous to [7, 9, 14] for the more general weighted singular problem (1.1).

We have started with choosing the weight function \( w \) in the class of Muckenhoupt weight \( A_p \) whose theory is well developed, see [11, 17, 19, 24, 26, 31, 33]. Such class of weights was firstly introduced by Muckenhoupt [31], where the author proved these are the only class of weights such that the Hardy–Littlewood maximal operator is bounded from the weighted Lebesgue space into itself and thus plays a very significant role in harmonic analysis.

Due to the presence of the weight function, solutions of (1.1) are investigated in a weighted Sobolev space (see Section 2 for definition). We mainly adapt the approximation approach introduced by the authors in [7] along [9, 14] although there are some difficulties we will face in our setting. To be more precise, by regularizing the right hand size of (1.1) we prove existence of a uniform positive and bounded solutions to the approximated problem (3.2). But in contrast to [7], weak convergence is not enough to pass the limit in the approximated problem (3.2). In this concern a gradient convergence theorem was proved by the De Cave in [14] which allows to pass the limit (see also [9]). Here, we establish a counterpart of gradient convergence theorem in our setting (see Theorem 2.12) by following the idea from Boccardo–Murat [6] in order to pass the limit in Equation (3.2) and obtain our existence results. The availability of embedding results in the classical Sobolev space \( W^{1, p}(\Omega) \) (see [1, 18]) is one of the main ingredients in [7, 9, 14]. Such embeddings are not readily available in our setting which we establish here for a subclass of \( A_p \) (see Theorem 2.7). Then following the idea from [7, 14] choosing suitable test functions into Equation (3.2) along with an application of our embedding theorem we obtain regularity results depending on the summability of \( f \). Finally, to obtain uniqueness results, we establish a variational inequality (see Lemma 2.14) further avoiding the use of boundary continuity of solutions to the regularized \( p \)-Laplace equations (see, e.g., [29, 32, 34, 35]) as implemented by the authors in [9].

**Notations:** Throughout the paper, the following notations will be used:

- \( X := W^{1, p}_0(\Omega, w) \)
- \( X^* := \text{Dual space of } X \)
- \( ||x||_X := ||x||_{1, p, w} \)
- \( c, c_i, i \in \mathbb{N} \), will denote constants whose values may vary depending on the situation from line to line or even in the same line.
- \( |S| := \text{Lebesgue measure of a set } S \)
- \( T_\eta(s) := \min\{\eta, s\} \) for \( \eta > 0, s \geq 0 \)
- \( B(x, r) := \text{Ball of radius } r \text{ with center } x \)

This paper is organized as follows: In Section 2, we present some preliminary results. In Section 3, existence and regularity results and in Section 4, uniqueness results are proved.

2 | PRELIMINARIES

In this section, we present some basic properties of \( A_p \) weights and a brief literature of the corresponding weighted Sobolev space. For a more general theory we refer the reader to the nice surveys by Drábek et al. [17], Fabes et al. [19], Heinonen et al. [24] and Kilpeläinen [26].

2.1 | Muckenhoupt weight

**Definition 2.1.** Let \( w \) be a locally integrable function in \( \mathbb{R}^N \) such that \( 0 < w < \infty \) a.e. in \( \mathbb{R}^N \). Then for \( 1 < p < \infty \), we say that \( w \) belongs to the Muckenhoupt class \( A_p \) or \( w \) is an \( A_p \)-weight, if there exists a positive constant \( c_{p, w} \) (called the \( A_p \) constant of \( w \)) depending only on \( p \) and \( w \) such that for all balls \( B \) in \( \mathbb{R}^N \),

\[
\left( \frac{1}{|B|} \int_B w \, dx \right) \left( \frac{1}{|B|} \int_B w^{-1/(p-1)} \, dx \right)^{p-1} \leq c_{p, w}.
\]
Example 2.2.

- \( w(x) = |x|^{\alpha} \in A_p \) if and only if \(-N < \alpha < N(p-1)\), see [24, 26].

**Definition 2.3.** (Weighted Sobolev Space) For any \( w \in A_p \), define the weighted Sobolev space \( W^{1,p}(\Omega, w) \) by

\[
W^{1,p}(\Omega, w) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable} : ||u||_{1,p,w} < \infty \right\},
\]

where

\[
||u||_{1,p,w} = \left( \int_{\Omega} |u(x)|^p w(x) \, dx \right)^{1/p} + \left( \int_{\Omega} |\nabla u|^p w(x) \, dx \right)^{1/p}.
\]

(2.1)

- Observe that if \( 0 < c \leq w \leq d \) for some constants \( c \) and \( d \), the weighted Sobolev space \( W^{1,p}(\Omega, w) \) becomes the classical Sobolev space \( W^{1,p}(\Omega) \).
- The fact \( w \in A_p \) implies \( w \in L^1_{\text{loc}}(\Omega) \) and hence \( C_c^\infty(\Omega) \subset W^{1,p}(\Omega, w) \). Therefore we can introduce the space

\[
W_0^{1,p}(\Omega, w) = (C_c^\infty(\Omega), ||.||_{1,p,w}).
\]

- Both the spaces \( W^{1,p}(\Omega, w) \) and \( W_0^{1,p}(\Omega, w) \) are uniformly convex Banach spaces with respect to the norm \( ||.||_{1,p,w} \), see [24].

**Definition 2.4.** We say that \( u \in W^{1,p}_{\text{loc}}(\Omega, w) \) if and only if \( u \in W^{1,p}(\Omega', w) \) for every \( \Omega' \Subset \Omega \).

**Theorem 2.5** (Poincaré inequality [24]). For any \( w \in A_p \), we have

\[
\int_{\Omega} |\phi|^p w(x) \, dx \leq c \int_{\Omega} |\nabla \phi|^p w(x) \, dx \text{ for all } \phi \in C_c^\infty(\Omega),
\]

for some positive constant \( c \) independent of \( \phi \).

By using Theorem 2.5, an equivalent norm to (2.1) on the space \( W_0^{1,p}(\Omega, w) \) can be defined by

\[
||u||_{1,p,w} = \left( \int_{\Omega} |\nabla u(x)|^p w(x) \, dx \right)^{1/p}.
\]

(2.2)

2.2 | Embedding theorems

The following compactness result follows from Chua et al. [11].

**Theorem 2.6** (Theorem 2.2, [11]). Let \( w \in A_p \) with \( 1 < p < \infty \), then the inclusion map

\[
W^{1,p}(\Omega, w) \hookrightarrow L^p(\Omega, w)
\]

is compact.

For the rest of the paper, we assume that the weight function \( w \in A_s \), unless otherwise stated, where \( A_s \) is a subclass of \( A_p \) given by

\[
A_s := \left\{ w \in A_p : w^{-s} \in L^1(\Omega) \text{ for some } s \in \left[ \frac{1}{p-1}, \infty \right) \cap \left( \frac{N}{p}, \infty \right) \right\}.
\]
For example, $w(x) = |x|^a$ with $-N/s < a < N/s$ belong to $A_s$ for any $s \in \left[1/(p-1), \infty \right) \cap \left(N/p, \infty \right)$, provided $1 < p < N$. This subclass allows one to shift from the weighted Sobolev space into the classical Sobolev space using the idea of [17]. Indeed, we prove the following embedding theorem.

**Theorem 2.7** (Embedding from weighted to classical Sobolev space).

- For any $w \in A_s$, we have the following continuous inclusion map

$$W^{1,p}(\Omega, w) \hookrightarrow W^{1,p_s}(\Omega) \hookrightarrow \begin{cases} L^q(\Omega), & \text{for } p_s \leq q \leq p_s^*, \text{ in case of } 1 \leq p_s < N, \\ L^q(\Omega), & \text{for } 1 \leq q < \infty, \text{ in case of } p_s = N, \\ C(\Omega), & \text{in case of } p_s > N, \end{cases}$$

where $p_s = (ps)/(s + 1) \in [1, p)$.

- Moreover, the above embeddings are compact except for $q = p_s^*$ in case of $1 \leq p_s < N$.

- The same result holds for the space $W^{1,p}_0(\Omega, w)$.

**Proof.** Let $u \in W^{1,p}(\Omega, w)$. Since $p/p_s > 1$, by using the Hölder inequality with exponents $p/p_s$ and $(p/p_s)' = s + 1$, we obtain

$$\int_\Omega |u(x)|^{p_s} \, dx = \int_\Omega |u(x)|^{p_s} w(x)^{p/p} w(x)^{-p/p} \, dx \leq \left( \int_\Omega |u(x)|^p w(x) \, dx \right)^{p_s/p} \left( \int_\Omega w(x)^{-s} \, dx \right)^{1/(s+1)},$$

which implies

$$||u||_{L^{p_s}(\Omega)} \leq \left( \int_\Omega w(x)^{-s} \, dx \right)^{1/p_s} \left( \int_\Omega |u(x)|^p w(x) \, dx \right)^{1/p}.$$  \hspace{1cm} (2.3)

Replacing $u$ by $\nabla u$, similarly we obtain

$$||\nabla u||_{L^{p_s}(\Omega)} \leq \left( \int_\Omega w(x)^{-s} \, dx \right)^{1/p_s} \left( \int_\Omega |\nabla u|^p w(x) \, dx \right)^{1/p}.$$  \hspace{1cm} (2.4)

Adding (2.3) and (2.4) we have

$$||u||_{W^{1,p_s}(\Omega)} \leq ||w^{-s}||_{L^1(\Omega)}^{1/p_s} ||u||_{1,p,w}.$$  

Hence the embedding

$$W^{1,p}(\Omega, w) \hookrightarrow W^{1,p_s}(\Omega)$$

is continuous. The rest of the proof follows from the classical Sobolev embedding theorem (Theorem 1.1.3 of Ambrosetti–Arcoya [1]). \qed

**Remark 2.8.** Observe that the fact $s \in \left[1/(p-1), \infty \right) \cap \left(N/p, \infty \right)$ implies that $p_s^* > p$. Therefore, by Theorem 2.7 there exists a constant $q > p$ such that the inclusion

$$W^{1,p}(\Omega, w) \hookrightarrow L^q(\Omega)$$

is continuous. The existence of such $q$ is an important tool to prove some a priori estimates later, see [17] for more applications.
Now we state two important theorems on \(A\) superharmonic functions, for the definition of such functions we refer the reader to [24].

**Theorem 2.9** (Theorem 7.12, [24]). A nonconstant \(A\) superharmonic function cannot attain its infimum in \(\Omega\).

**Theorem 2.10** (Corollary 7.18, [24]). If \(u \in W^{1,p}_\text{loc}(\Omega, w)\) is a weak supersolution of the equation
\[- \text{div} \ A(x, \nabla u) = 0\]
in \(\Omega\), i.e.,
\[\int_\Omega A(x, \nabla u) \cdot \nabla \phi \, dx \geq 0\]
whenever \(\phi \in C_c^\infty(\Omega)\) is nonnegative, then there exists \(A\) superharmonic function \(v\) such that \(v = u\) a.e.

**Theorem 2.11.** Let \(u \in W^{1,p}_\text{loc}(\Omega, w)\) be positive a.e. in \(\Omega\) and let \(\alpha \geq 1\) be such that \(u^\alpha \in X\). Then for every \(\epsilon > 0\), we have \((u - \epsilon)^+ \in X\).

**Proof.** Since \(u^\alpha \in X\), there exists a sequence of nonnegative functions \(\{\phi_n\} \in C_c^\infty(\Omega)\) such that \(\phi_n\) converges to \(u^\alpha\) in the norm of \(X\). Set
\[v_n := \left(\frac{1}{\alpha} - \epsilon \right)^+\]
Observe that, since \(\alpha \geq 1\), one has
\[\|v_n\|_X^p = \int_\Omega w(x)|\nabla v_n|^p \, dx \leq \int_{\{\phi_n > \epsilon \alpha\}} w(x)\epsilon^{\alpha(1/\alpha - 1)}|\nabla \phi_n|^\alpha \, dx \leq M,
\]
where \(M\) is a constant independent of \(n\), since \(\|\phi_n\|_X \leq c\) for some positive constant \(c\) independent of \(n\). Therefore, the sequence \(v_n\) is uniformly bounded in \(X\) and by the reflexivity of \(X\), it follows that \((u - \epsilon)^+ \in X\).  

**Theorem 2.12** (Gradient Convergence Theorem). Given \(n \in \mathbb{N}\) and \(w \in A_s\), consider the following equation
\[- \text{div} \ (A(x, \nabla u_n)) = G_n \text{ in } \Omega.\]

Assume that \(u_n \rightharpoonup u\) weakly in \(W^{1,p}(\Omega, w)\). In addition, suppose \(G_n\) satisfies
\[|\langle G_n, \phi \rangle| \leq C_K \|\phi\|_{L^\infty(\Omega)},\]
for all \(\phi \in C_c^\infty(\Omega)\) with support \(\phi \subset K\), where \(C_K\) is a constant depending on \(K\). Then, up to a subsequence \(\nabla u_n \rightharpoonup \nabla u\) pointwise a.e. in \(\Omega\).

**Proof.** In the unweighted case this theorem was proved in Theorem 2.1 of [6] and following the same arguments we present the proof in the weighted case as follows.

**Step 1.** Fix a compact set \(K \subset \Omega\) and a function \(\phi_K \in C_c^\infty(\Omega)\) such that \(0 \leq \phi_K \leq 1\) and \(\phi_K \equiv 1\) on \(K\). Define the truncated function
\[L_\mu(s) := \begin{cases} \frac{s}{\mu} & \text{for } |s| \leq \mu, \\ \frac{1 - s}{|s|} & \text{for } |s| > \mu. \end{cases}\]
Then \( v_n = \phi_K L_\mu(u_n - u) \in W^{1,p}_0(\Omega, w) \) with compact support.

\[
\int_\Omega \phi_K \{ A(x, \nabla u_n) - A(x, \nabla u) \} \cdot \nabla L_\mu(u_n - u) \, dx
= \langle G_n, v_n \rangle - \int_\Omega L_\mu(u_n - u) A(x, \nabla u_n) \cdot \nabla \phi_K \, dx
- \int_\Omega \phi_K A(x, \nabla u) \cdot \nabla L_\mu(u_n - u) \, dx.
\]

Now,

\[
I_n := \left| \int_\Omega L_\mu(u_n - u) A(x, \nabla u_n) \cdot \nabla \phi_K \, dx \right|
\leq \| \nabla \phi_K \|_{L^\infty(\Omega)} \int_K w|u_n - u|\|\nabla u_n\|^{p-1} \, dx
\leq \| \nabla \phi_K \|_{L^\infty(\Omega)} \| u_n - u \|_{L^p(\Omega, w)} \| u_n \|_{W^{1,p}(\Omega, w)}^{p-1}.
\]

Since \( u_n \to u \) weakly in \( W^{1,p}(\Omega, w) \), by Theorem 2.6 the sequence \( I_n \) converges to 0 as \( n \to \infty \). Moreover, since the sequence \( L_\mu(u_n - u) \to 0 \) weakly in \( W^{1,p}(\Omega, w) \) as \( n \to \infty \), it follows that the sequence

\[
J_n := \int_\Omega \phi_K A(x, \nabla u) \cdot \nabla L_\mu(u_n - u) \, dx
\]

converges to 0 as \( n \to \infty \). Now, by the given condition we have \( |\langle G_n, v_n \rangle| \leq c_K \mu \).

**Step 2.** Fix \( \theta \in (0,1) \) and define the sequence of function

\[
e_n(x) = \{ A(x, \nabla u_n) - A(x, \nabla u) \} \cdot \nabla (u_n - u)(x).
\]

We denote by

\[
S_n^\mu = \{ x \in K : |u_n(x) - u(x)| \leq \mu \}, \quad G_n^\mu = \{ x \in K : |u_n(x) - u(x)| > \mu \}.
\]

Therefore

\[
\int_K e_n^\theta \, dx = \int_{S_n^\mu} e_n^\theta \, dx + \int_{G_n^\mu} e_n^\theta \, dx \leq \left( \int_{S_n^\mu} e_n^1 \, dx \right)^{\theta} \left| S_n^\mu \right|^{1-\theta} + \left( \int_{G_n^\mu} e_n^1 \, dx \right)^{\theta} \left| G_n^\mu \right|^{1-\theta}.
\]

By Theorem 2.7, we have \( u_n \to u \) strongly in \( L^p(\Omega) \). Therefore \( |G_n^\mu| \to 0 \) as \( n \to \infty \). Also, the sequence \( \{ e_n \} \) is bounded in \( L^1(\Omega) \), since

\[
\int_\Omega |e_n| \, dx = \int_\Omega |A(x, \nabla u_n) \cdot \nabla u_n - A(x, \nabla u) \cdot \nabla u_n - A(x, \nabla u_n) \cdot \nabla u + A(x, \nabla u) \cdot \nabla u| \, dx
\leq \int_\Omega w \left( |\nabla u_n|^p + |\nabla u_n||\nabla u|^p + |\nabla u_n|^{p-1} + |\nabla u|^{p-1} + |\nabla u|^p \right) \, dx
\leq M,
\]

for some constant \( M \) independent of \( n \). By Step 1 and the fact \( \phi_K \equiv 1 \) on \( K \), we obtain

\[
\limsup_{n \to \infty} \int_K e_n^\theta \, dx \leq (c_K \mu)^\theta |\Omega|^{1-\theta}.
\]

Letting \( \mu \to 0 \), we have \( e_n^\theta \to 0 \) in \( L^1(K) \). Therefore up to a subsequence \( e_n(x) \to 0 \) a.e. in \( \Omega \) and by using the hypothesis (H5), we obtain up to a subsequence \( \nabla u_n \to \nabla u \) pointwise a.e. in \( \Omega \). 

\[ \square \]
Moreover, we will use the following three important results, see Ciarlet [12] for Theorem 2.13–Theorem 2.14 and Kinderlehrer–Stampacchia [27] for Theorem 2.15 respectively.

**Theorem 2.13** (Theorem 9.14, [12]). Let $V$ be a real reflexive Banach space and let $A : V \to V^*$ be a coercive and demi-continuous monotone operator. Then $A$ is surjective, i.e., given any $f \in V^*$ there exists $u \in V$ such that $A(u) = f$. If $A$ is strictly monotone, then $A$ is also injective.

**Theorem 2.14** ([12]). Let $U$ be a nonempty closed and convex subset of a real separable reflexive Banach space and let $A : V \to V^*$ be a coercive and demi-continuous monotone operator. Then for every $f \in V^*$ there exists $u \in U$ such that

$$\langle A(u), v - u \rangle \geq \langle f, v - u \rangle \text{ for all } v \in U.$$ 

Moreover, if $A$ is strictly monotone, then $u$ is unique.

**Theorem 2.15** (Lemma B.1, [27]). Let $\phi(t), k_0 \leq t < \infty$, be nonnegative and nonincreasing such that

$$\phi(h) \leq \left[ \frac{c}{(h - k)^l} \right]^m_k h > k > k_0,$$

where $c, l, m$ are positive constants with $m > 1$. Then $\phi(k_0 + d) = 0$, where

$$d^l = c \left[ \phi(k_0) \right]^{m-1} 2^{(m)/(m-1)}.$$

# 3 EXISTENCE AND REGULARITY RESULTS

**Definition 3.1.** A function $u \in W^{1,p}_{\text{loc}}(\Omega, w)$ is said to be a weak solution of the problem (1.1), if for every $K \Subset \Omega$ there exists a positive constant $c_K$ such that $u \geq c_K > 0$ in $K$ and for all $\phi \in C^1_c(\Omega)$, one has

$$\begin{cases}
\int_{\Omega} A(x, \nabla u(x)) \cdot \nabla \phi(x) \, dx = \int_{\Omega} \frac{f(x)}{u^2} \phi(x) \, dx, \\
u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,
\end{cases} \tag{3.1}$$

where by $u = 0$ on $\partial \Omega$, we mean that for some $\alpha \geq 1$, the function $u^\alpha \in X$.

Our main existence and regularity results in this paper reads as follows:

## 3.1 The case $0 < \delta < 1$

**Theorem 3.2.** For any $0 < \delta < 1$, the problem (1.1) has at least one weak solution in $X$, if

(a) $f \in L^m(\Omega), m = \left( \frac{p_\delta}{p_\delta - \delta} \right)\prime$, provided $1 \leq p_\delta < N$, or
(b) $f \in L^m(\Omega)$ for some $m > 1$, provided $p_\delta = N$, or
(c) $f \in L^1(\Omega)$ for $p_\delta > N$.

**Theorem 3.3.** Let $0 < \delta < 1$, then the solution $u$ given by Theorem 3.2 satisfies the following properties:

(a) For $1 \leq p_\delta < N$,
   (i) if $f \in L^m(\Omega)$ for some $m \in \left( \left( \frac{p_\delta}{p_\delta - \delta} \right)\prime, \frac{p_\delta}{p_\delta - \delta} \right)$, then $u \in L^t(\Omega), t = \frac{p_\delta}{p_\delta - \delta} \gamma$ where $\gamma = \frac{(\delta + p - 1)m'}{(pm' - p)}$.
   (ii) if $f \in L^m(\Omega)$ for some $m \geq \frac{p_\delta}{p_\delta - \delta} - \delta$, then $u \in L^\infty(\Omega)$. 

(b) Let \( p_s = N \) and assume \( q > p \). Then if \( f \in L^m(\Omega) \) for some \( m \in \left( \frac{q}{1-\delta}, \frac{q}{q-p} \right) \), we have \( u \in L^t(\Omega) \), \( t = p\gamma \) where
\[
\gamma = \frac{pm'}{pm'-q}.
\]
(c) For \( p_s > N \) and \( f \in L^1(\Omega) \), we have \( u \in L^\infty(\Omega) \).

### 3.2 The case \( \delta = 1 \)

**Theorem 3.4.** For \( \delta = 1 \) with any \( p_s \), the problem (1.1) has at least one weak solution in \( X \), provided \( f \in L^1(\Omega) \).

**Theorem 3.5.** Let \( \delta = 1 \), then the solution \( u \) given by Theorem 3.4 satisfies the following properties:

(a) For \( 1 \leq p_s < N \),

(i) if \( f \in L^m(\Omega) \) for some \( m \in \left( 1, \frac{p^*_s}{(p^*_s - p)} \right) \), then \( u \in L^t(\Omega) \), \( t = p^*_s \gamma \), where \( \gamma = \frac{pm'}{pm'-p^*_s} \).

(ii) if \( f \in L^m(\Omega) \) for some \( m > \frac{p^*_s}{(p^*_s - p)} \), then \( u \in L^\infty(\Omega) \).

(b) Let \( p_s = N \) and assume \( q > p \). Then if \( f \in L^m(\Omega) \) for some \( m \in \left( 1, \frac{q}{q-p} \right) \), we have \( u \in L^t(\Omega) \), \( t = q \gamma \), where \( \gamma = \frac{pm'}{pm'-q} \).

(c) For \( p_s > N \) and \( f \in L^1(\Omega) \), we have \( u \in L^\infty(\Omega) \).

### 3.3 The case \( \delta > 1 \)

**Theorem 3.6.** For \( \delta > 1 \) with any \( p_s \), the problem (1.1) has at least one weak solution, say \( u \in W^{1,p}_{\text{loc}}(\Omega, w) \) such that \( u^{(\delta+p-1)/p} \in X \), provided \( f \in L^1(\Omega) \).

**Theorem 3.7.** Let \( \delta > 1 \), then the solution \( u \) given by Theorem 3.6 satisfies the following properties:

(a) For \( 1 \leq p_s < N \),

(i) if \( f \in L^m(\Omega) \) for some \( m \in \left( 1, \frac{p^*_s}{(p^*_s - p)} \right) \), then \( u \in L^t(\Omega) \) where \( t = p^*_s \gamma \), where \( \gamma = \frac{(\delta+p-1)m'}{pm'-p^*_s} \).

(ii) if \( f \in L^m(\Omega) \) for some \( m > \frac{p^*_s}{(p^*_s - p)} \), then \( u \in L^\infty(\Omega) \).

(b) Let \( p_s = N \) and assume \( q > p \). Then if \( f \in L^m(\Omega) \) for some \( m \in \left( 1, \frac{q}{q-p} \right) \), we have \( u \in L^t(\Omega) \), \( t = q \gamma \), where \( \gamma = \frac{(\delta+p-1)m'}{pm'-q} \).

(c) For \( p_s > N \) and \( f \in L^1(\Omega) \), we have \( u \in L^\infty(\Omega) \).

### 3.4 Preliminaries

For \( n \in \mathbb{N} \), define \( f_n(x) := \min \{ f(x), n \} \) and consider for \( \delta > 0 \), the approximated problem

\[
\begin{aligned}
-\text{div} \left( A(x, \nabla u) \right) &= \frac{f_n(x)}{(u + \frac{1}{n})^{\frac{\delta}{\delta}}} \quad \text{in } \Omega, \\
u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega.
\end{aligned}
\]

**Definition 3.8.** A function \( u \in X \) is said to be a weak solution of the problem (3.2) if \( u > 0 \) in \( \Omega \) and for all \( \phi \in X \), one has

\[
\int_\Omega A(x, \nabla u) \cdot \nabla \phi(x) \, dx = \int_\Omega \frac{f_n(x)}{(u + \frac{1}{n})^{\frac{\delta}{\delta}}} \phi(x) \, dx.
\]
Define the operator \( J : X \to X^* \) by
\[
\langle J(u), \phi \rangle := \int_{\Omega} A(x, \nabla u) \cdot \nabla \phi \, dx, \text{ for all } \phi, u \in X.
\]

**Lemma 3.9.** \( J \) is a surjective and strictly monotone operator.

**Proof.** The proof follows applying Theorem 2.13, since

1. **Boundedness:** By using Hölder’s inequality and hypothesis (H2) we obtain
   \[
   ||J(u)||_{X^*} = \sup_{||\phi||_X \leq 1} |\langle J(u), \phi \rangle| \\
   \leq \sup_{||\phi||_X \leq 1} \left| \int_{\Omega} A(x, \nabla u) \cdot \nabla \phi \, dx \right| \\
   \leq \sup_{||\phi||_X \leq 1} \left| \int_{\Omega} (w^{1/p'} |\nabla u|^{p-1}) (w^{1/p} |\nabla \phi|) \, dx \right| \\
   \leq ||u||_{X}^{p-1}.
   \]
   Hence \( J \) is bounded.

2. **Demicontinuity:** Let \( u_n \to u \) in the norm of \( X \), then \( w^{1/p} \nabla u_n \to w^{1/p} \nabla u \) in \( L^p(\Omega) \). Therefore up to a subsequence \( u_{n_k} \) of \( u_n \), we have \( \nabla u_{n_k}(x) \to \nabla u(x) \) pointwise for a.e. \( x \in \Omega \). Since the function \( A(x, \cdot) \) is continuous in the second variable, we have
   \[
   w(x)^{-1/p} A(x, \nabla u_{n_k}(x)) \to w(x)^{-1/p} A(x, \nabla u(x))
   \]
   pointwise for a.e. \( x \in \Omega \). Now using the growth condition (H2), we obtain
   \[
   \left\| w^{-1/p} A(x, \nabla u_{n_k}) \right\|_{L^{p/(p-1)}(\Omega)}^{p/(p-1)} = \int_{\Omega} w^{-1/(p-1)}(x) A(x, \nabla u_{n_k}(x)) \right|^{p/(p-1)} \, dx \\
   \leq \int_{\Omega} w^{-1/(p-1)}(x) w^{p/(p-1)}(x) |\nabla u_{n_k}(x)|^p \, dx \\
   \leq \left\| u_{n_k} \right\|_X^p \\
   \leq c^p
   \]
   where \( \left\| u_{n_k} \right\|_X \leq c \). Therefore since the sequence \( w^{-1/p} A(x, \nabla u_{n_k}(x)) \) is uniformly bounded in \( L^{p/(p-1)}(\Omega) \), we have \( w^{-1/p} A(x, \nabla u_{n_k}(x)) \to w^{-1/p} A(x, \nabla u(x)) \) weakly in \( L^{p/(p-1)}(\Omega) \), see Jakszto [25]. Since the weak limit is independent of the choice of the subsequence \( u_{n_k} \), it follows that
   \[
   w^{-1/p} A(x, \nabla u_{n}(x)) \to w^{-1/p} A(x, \nabla u(x))
   \]
   weakly. Now \( \phi \in X \) implies the function \( w^{1/p} \nabla \phi \in L^p(\Omega) \) and therefore by the weak convergence, we obtain
   \[
   \langle J(u_n), \phi \rangle \to \langle J(u), \phi \rangle
   \]
as \( n \to \infty \) and hence \( J \) is demicontinuous.
3. **Coercivity:** By using (H3), we have the inequality
\[ \langle J(u), u \rangle = \int_{\Omega} A(x, \nabla u) \cdot \nabla u \, dx \geq \int_{\Omega} w|\nabla u|^p \, dx = ||u||_X^p. \]
Therefore \( J \) is coercive.

4. **Strict monotonicity:** By using the strong monotonicity condition (H5), for all \( u \neq v \in X \), we have
\[ \langle J(u) - J(v), u - v \rangle = \int_{\Omega} \{ A(x, \nabla u(x)) - A(x, \nabla v(x)) \} \cdot \nabla (u(x) - v(x)) \, dx > 0. \]

**Lemma 3.10.** The operator \( J^{-1} : X^* \to X \) is bounded and continuous.

**Proof.** By using H"older’s inequality for all \( u, v \in X \), we have the estimate
\[ \langle J(v) - J(u), v - u \rangle \geq \left( ||v||_X^{p-1} - ||u||_X^{p-1} \right) \left( ||v||_X - ||u||_X \right), \]
which implies that the operator \( J^{-1} \) is bounded. Suppose by contradiction \( J^{-1} \) is not continuous, then there exists \( g_k \to g \) strongly in \( X^* \) such that \( ||J^{-1}(g_k) - J^{-1}(g)||_X \geq \gamma \) for some \( \gamma > 0 \). Denote by \( u_k = J^{-1}(g_k) \) and \( u = J^{-1}(g) \). Therefore, by using (H3) we have
\[ ||u_k||_X^p \leq \int_{\Omega} A(x, \nabla u_k(x)) \cdot \nabla u_k(x) \, dx = \langle J(u_k), u_k \rangle = \langle g_k, u_k \rangle \leq ||g_k||_{X^*} ||u_k||_X, \]
which implies
\[ ||u_k||_X^{p-1} \leq ||g_k||_{X^*}. \]
Since \( g_k \to g \) strongly in \( X^* \), we have that the sequence \( \{ u_k \} \) is uniformly bounded in \( X \). Therefore up to a subsequence there exists \( u^1 \in X \) such that \( u_k \to u^1 \) weakly in \( X \). Now
\[ \langle J(u_k) - J(u^1), u_k - u^1 \rangle = \langle J(u_k) - J(u) + J(u) - J(u^1), u_k - u^1 \rangle = \langle J(u_k) - J(u), u_k - u^1 \rangle + \langle J(u) - J(u^1), u_k - u^1 \rangle. \]
Since \( J(u_k) \to J(u) \) in \( X^* \) and \( u_k \to u^1 \) weakly in \( X \), both the terms
\[ \langle J(u_k) - J(u), u_k - u^1 \rangle \text{ and } \langle J(u) - J(u^1), u_k - u^1 \rangle \]
converges to 0 as \( k \to \infty \). Therefore,
\[ \langle J(u_k) - J(u^1), u_k - u^1 \rangle \to 0 \text{ as } k \to \infty. \]
Putting \( v = u_k \) and \( u = u^1 \) in the inequality (3.4) we obtain \( ||u_k||_X \to ||u^1||_X \). Therefore by the uniform convexity of \( X \), it follows that \( u_k \to u^1 \) in \( X \) which together with the convergence \( J(u_k) \to J(u) \) in \( X^* \) implies that \( J(u^1) = J(u) \). Now the injectivity of \( J \) implies \( u = u^1 \), a contradiction to our assumption. Hence \( J^{-1} \) is continuous. \( \square \)
Lemma 3.11. Let $\zeta_k, \zeta \in X$ satisfies
\[
\langle J(\zeta_k), \phi \rangle = \langle h_k, \phi \rangle,
\]
\[
\langle J(\zeta), \phi \rangle = \langle h, \phi \rangle,
\]
for all $\phi \in X$ where $\langle , \rangle$ denotes the dual product between $X^*$ and $X$. If $h_k \to h$ in $X^*$, then we have $\zeta_k \to \zeta$ in $X$.

Proof. By the strict monotonicity of $J$, we have $J(\zeta) = h$ and $J(\zeta_k) = h_k$. Therefore applying Lemma 3.10, if $h_k \to h$ in $X^*$ then $J^{-1}(h_k) \to J^{-1}(h)$, i.e., $\zeta_k \to \zeta$ as $k \to \infty$. Hence the proof. □

By using Lemma 3.9, we can define the operator $A: L^{p_s}(\Omega) \to X$ by $A(v) = u$ where $u \in X$ is the unique weak solution of the problem
\[
-\text{div}(A(x, \nabla u)) = \frac{f_n(x)}{|v| + \frac{1}{n}} \delta \text{ in } \Omega, \tag{3.5}
\]
i.e., for all $\phi \in X$,
\[
\int_{\Omega} A(x, \nabla u(x)) \cdot \nabla \phi(x) \, dx = \int_{\Omega} \frac{f_n(x)}{|v(x)| + \frac{1}{n}} \delta \phi(x) \, dx.
\]

Lemma 3.12. The map $A: L^{p_s}(\Omega) \to X$ is continuous as defined above.

Proof. Let $v_k \to v$ in $L^{p_s}(\Omega)$. Suppose $A(v_k) = \zeta_k$ and $A(v) = \zeta$. Then for every fixed $n \in \mathbb{N}$ and for all $\phi \in X$, we have
\[
\int_{\Omega} A(x, \nabla \zeta_k(x)) \cdot \nabla \phi(x) \, dx = \int_{\Omega} \frac{f_n(x)}{|v_k(x)| + \frac{1}{n}} \delta \phi(x) \, dx,
\]
\[
\int_{\Omega} A(x, \nabla \zeta(x)) \cdot \nabla \phi(x) \, dx = \int_{\Omega} \frac{f_n(x)}{|v(x)| + \frac{1}{n}} \delta \phi(x) \, dx.
\]
Denote by
\[
g_k(x) = \frac{f_n(x)}{|v_k(x)| + \frac{1}{n}} \delta \text{ and } g(x) = \frac{f_n(x)}{|v(x)| + \frac{1}{n}} \delta.
\]
Now, by Theorem 2.7, one has
\[
\|g_k - g\|_{X^*} = \sup_{\|\phi\|_{X} \leq 1} \left| \int_{\Omega} f_n \left\{ \left( |v_k| + \frac{1}{n} \right)^{-\delta} - \left( |v| + \frac{1}{n} \right)^{-\delta} \right\} \phi \, dx \right| 
\leq n \|\phi\|_{L^{p_s}(\Omega)} \left\| \left( |v_k| + \frac{1}{n} \right)^{-\delta} - \left( |v| + \frac{1}{n} \right)^{-\delta} \right\|_{L^{p_s'}(\Omega)}.
\]
Now since \(|v_k| + \frac{1}{n}\)^{−δ} − \(|v| + \frac{1}{n}\)^{−δ} \leq 2n^{\delta+1} and \(v_k \to v\) in \(L^{p_\delta}(\Omega)\), up to a subsequence \(v_{k_l} \to v\) pointwise a.e. in \(\Omega\).

As a consequence of the Lebesgue dominated theorem, we obtain

\[
\left\|\left(|v_{k_l}| + \frac{1}{n}\right)^{−\delta} − \left(|v| + \frac{1}{n}\right)^{−\delta}\right\|_{L^{p_\delta}(\Omega)} \to 0 \quad \text{as} \quad k_l \to \infty.
\]

Since the limit is independent of the choice of the subsequence, we have

\[
\left\|\left(|v_{k}| + \frac{1}{n}\right)^{−\delta} − \left(|v| + \frac{1}{n}\right)^{−\delta}\right\|_{L^{p_\delta}(\Omega)} \to 0 \quad \text{as} \quad k \to \infty.
\]

Therefore by Lemma 3.12, we have \(\zeta_{k_l} \to \zeta\) as \(k \to \infty\). Hence \(A : L^{p_\delta}(\Omega) \to X\) is a continuous map.

**Theorem 3.13.** For any \(p_\delta \geq 1\) the following holds:

1. The problem (3.2) has a unique weak solution, say \(u_n\) in \(X \cap L^\infty(\Omega)\) for every fixed \(n \in \mathbb{N}\),
2. \(u_{n+1} \geq u_n\) for every \(n \in \mathbb{N}\),
3. For every \(K \Subset \Omega\) there exists a positive constant \(C_K\) (independent of \(n\)) such that \(u_n \geq C_K > 0\) in \(K\).

**Proof.**

1. **Existence:** Define

\[
S := \{ v \in L^{p_\delta}(\Omega) : \lambda A(v) = v, \ 0 \leq \lambda \leq 1 \}.
\]

Let \(v_i \in S\) and \(A(v_i) = u_i\) for \(i = 1, 2\). Using \(u_i\) as test function in (3.5) we obtain

\[
\|u_i\|_X \leq c(n),
\]

where \(c(n)\) is a constant depending on \(n\) but not on \(u_i, i = 1, 2\). Therefore, by Lemma 3.12 and the compactness of the inclusion

\[
X \hookrightarrow L^{p_\delta}(\Omega)
\]

together with the inequality (3.6), it follows that the map

\[
A : L^{p_\delta}(\Omega) \to L^{p_\delta}(\Omega)
\]

is both continuous and compact.

Observe that

\[
\|v_1 - v_2\|_{L^{p_\delta}(\Omega)} = \lambda \|A(v_1) - A(v_2)\|_{L^{p_\delta}(\Omega)}
\]

\[
= \lambda \|u_1 - u_2\|_X
\]

\[
\leq 2\lambda c(n) < \infty.
\]

Hence the set \(S\) is bounded in \(L^{p_\delta}(\Omega)\). By Schaefer’s fixed point theorem, there exists a fixed point of the map \(A\), say \(u_n\) i.e., \(A(u_n) = u_n\), and hence \(u_n \in X\) is a solution of (3.2).

**\(L^\infty\)-estimate:** For any \(k > 1\), define the set

\[
A(k) := \{ x \in \Omega : u_n(x) \geq k \text{ a.e. in } \Omega \}.
\]

Choosing

\[
\phi_k(x) := \begin{cases} u_n(x) - k, & \text{if } x \in A(k), \\ 0, & \text{otherwise}, \end{cases}
\]

as a test function in (3.3) together with the Hölder inequality and Remark 2.8, we obtain

\[
\int_{\Omega} |\nabla \phi_k|^p w(x) \, dx \leq n^{\delta+1} \int_{A(k)} |u_n(x) - k| \, dx \leq c n^{\delta+1} |A(k)|^{(q-1)/q} \|\phi_k\|_X.
\]
Therefore we get

\[ \| \phi_k \|_{X}^{p-1} \leq c |A(k)|^{(q-1)/q}, \]

where \( c \) depends on \( n \). Now for \( 1 < k < h \), by the Remark 2.8, we obtain

\[ (h-k)^p |A(h)|^{p/q} \leq \left( \int_{A(h)} (u_n(x) - k)^q \, dx \right)^{p/q} \]

\[ \leq \left( \int_{A(k)} (u_n(x) - k)^q \, dx \right)^{p/q} \]

\[ \leq \int_{\Omega} |\nabla \phi_k|^p w(x) \, dx \]

\[ \leq c |A(k)|^{p'/q'}. \]

Hence we obtain the inequality

\[ |A(h)| \leq \frac{c}{(h-k)^q} |A(k)|^{(p'q)/(pq')} \]

Now \( q > p \) implies \((p'q)/(pq') > 1\), therefore by Theorem 2.15, we obtain

\[ \| u_n \|_{L^\infty(\Omega)} \leq c, \]

where \( c \) is a constant dependent on \( n \).

2. **Monotonicity:** Let \( u_n \) and \( u_{n+1} \) satisfies the equations

\[ \int_{\Omega} A(x, \nabla u_n(x)) \cdot \nabla \phi(x) \, dx = \int_{\Omega} \frac{f_n(x)}{(u_n + \frac{1}{n})^\delta} \phi(x) \, dx \tag{3.7} \]

and

\[ \int_{\Omega} A(x, \nabla u_{n+1}(x)) \cdot \nabla \phi(x) \, dx = \int_{\Omega} \frac{f_{n+1}(x)}{(u_{n+1} + \frac{1}{n+1})^\delta} \phi(x) \, dx \tag{3.8} \]

respectively for all \( \phi \in X \). Choosing \( \phi = (u_n - u_{n+1})^+ \in X \) and using the inequality \( f_n(x) \leq f_{n+1}(x) \) we obtain after subtracting Equation (3.7) from (3.8)

\[ I := \int_{\Omega} \left\{ A(x, \nabla u_n(x)) - A(x, \nabla u_{n+1}(x)) \right\} \cdot \nabla \left( u_n - u_{n+1} \right)^+(x) \, dx \]

\[ = \int_{\Omega} \left\{ \frac{f_n(x)}{(u_n + \frac{1}{n})^\delta} - \frac{f_{n+1}(x)}{(u_{n+1} + \frac{1}{n+1})^\delta} \right\} \left( u_n - u_{n+1} \right)^+(x) \, dx \]

\[ \leq \int_{\Omega} f_{n+1}(x) \left\{ \frac{1}{(u_n + \frac{1}{n})^\delta} - \frac{1}{(u_{n+1} + \frac{1}{n+1})^\delta} \right\} \left( u_n - u_{n+1} \right)^+(x) \, dx \]

\[ \leq 0. \]
Again by using the strong monotonicity condition (H5), we have

- for \( p \geq 2, \)
  \[
  0 \leq \left\| (u_n - u_{n+1})^+ \right\|_X^p \leq I \leq 0,
  \]

- and for \( 1 < p < 2, \)
  \[
  0 \leq \int_{\Omega} w(x) \left\| \nabla (u_n - u_{n+1})^+ \right\|^2 \left\{ \left\| \nabla u_n \right\|^p + \left\| \nabla u_{n+1} \right\|^p \right\}^{1-(2/p)} \leq I \leq 0,
  \]

which gives \( u_{n+1} \geq u_n. \)

**Uniqueness:** The uniqueness of \( u_n \) follows by arguing similarly as in monotonicity.

3. Choosing \( \phi = \min \{ u_n, 0 \} \) as a test function in Equation (3.3) we get \( u_n \geq 0 \) in \( \Omega. \) Applying Theorem 2.9 we get \( u_1 > 0 \) in \( \Omega. \) Hence by the monotonicity and Theorem 2.10 there exists \( C_K > 0 \) (independent of \( n \)) such that \( u_n \geq C_K > 0 \) for every \( K \subset \Omega. \)

\[ \square \]

### 3.5 Proof of the existence and regularity results

#### 3.6 The case \( 0 < \delta < 1 \)

*Proof of Theorem 3.2.* Let \( 0 < \delta < 1. \)

(a) Let \( 1 \leq p_s < N. \) Choosing \( \phi = u_n \in X \) as a test function in Equation (3.3) and by using Hölder’s inequality together with the continuous embedding \( X \hookrightarrow L^{p^*}(\Omega) \) we obtain

\[
\left\| u_n \right\|_X^p \leq \int_{\Omega} \left| f \right| u_n^{1-\delta} \, dx
\]

\[
\leq \left\| f \right\|_{L^m(\Omega)} \left( \int_{\Omega} \left| u_n \right|^{(1-\delta)m'} \, dx \right)^{1/m'}
\]

\[
\leq c \left\| f \right\|_{L^m(\Omega)} \left\| u_n \right\|_X^{1-\delta}.
\]

Since \( \delta + p - 1 > 0, \) we have \( \left\| u_n \right\|_X \leq c, \) where \( c \) is a constant independent of \( n. \) Therefore one can apply Theorem 2.12 to conclude that up to a subsequence \( \nabla u_{n_k} \rightarrow \nabla u \) pointwise a.e. in \( \Omega. \) Since the function \( A(x, \cdot) \) is continuous, we have \( w^{-1/p}(x) A(x, \nabla u_{n_k}(x)) \rightarrow w^{-1/p}(x) A(x, \nabla u(x)) \) pointwise for a.e. \( x \in \Omega. \) Now we observe that

\[
\left\| w^{-1/p} A(x, \nabla u_{n_k}) \right\|_{L^{p/(p-1)}(\Omega)}^{p/(p-1)} = \int_{\Omega} w^{-1/(p-1)}(x) A(x, \nabla u_{n_k}(x))^{p/(p-1)} \, dx
\]

\[
\leq \left\| u_{n_k} \right\|_X^p \leq c^p.
\]

Since the sequence \( w^{-1/p} A(x, \nabla u_{n_k}) \) is uniformly bounded in \( L^{p/(p-1)}(\Omega), \) the sequence

\[
w^{-1/p} A(x, \nabla u_{n_k}(x)) \rightarrow w^{-1/p} A(x, \nabla u(x))
\]

weakly in \( L^{p/(p-1)}(\Omega). \) As the weak limit is independent of the choice of the subsequence \( u_{n_k}, \) it follows that \( w^{-1/p} A(x, \nabla u(x)) \rightarrow w^{-1/p} A(x, \nabla u(x)) \) weakly. Now \( \phi \in X \) implies the function \( w^{1/p} \nabla \phi \in L^p(\Omega) \) and hence by the weak convergence, we obtain

\[
\lim_{n \rightarrow \infty} \int_{\Omega} A(x, \nabla u_n(x)) \cdot \nabla \phi(x) \, dx = \int_{\Omega} A(x, \nabla u(x)) \cdot \nabla \phi(x) \, dx.
\]
Moreover, by Theorem 3.13 we have \( u \geq u_n \geq c_K > 0 \) for every \( K \Subset \Omega \). Since for \( \phi \in C^1_c(\Omega) \), one has

\[
\left| \frac{f_n \phi}{(u_n + \frac{1}{n})^\delta} \right| \leq \frac{||\phi||_\infty}{c_K^\delta} f \in L^1(\Omega),
\]

and \( \frac{f_n}{(u_n + \frac{1}{n})^\delta} \phi \rightarrow \frac{f}{u^\delta} \phi \) pointwise a.e. in \( \Omega \) as \( n \to \infty \), by the Lebesgue dominated convergence theorem we obtain

\[
\lim_{n \to \infty} \int_\Omega \frac{f_n}{(u_n + \frac{1}{n})^\delta} \phi \, dx = \int_\Omega \frac{f}{u^\delta} \phi \, dx.
\]

Therefore we have for all \( \phi \in C^1_c(\Omega) \),

\[
\int_\Omega A(x, \nabla u(x)) \cdot \nabla \phi(x) \, dx = \int_\Omega \frac{f}{u^\delta} \phi \, dx
\]

and hence \( u \in X \) is a weak solution of (1.1).

(b) Let \( p_s = N \). Choosing \( \phi = u_n \in X \) as a test function in (3.3) and by using the Hölder inequality together with the continuous embedding \( X \hookrightarrow L^q(\Omega), \; q \in [1, \infty) \), we obtain

\[
\|u_n\|_{X}^p \leq \int_\Omega |f||u_n|^{1-\delta} \, dx
\]

\[
\leq ||f||_{L^m(\Omega)} \left( \int_\Omega |u_n|^{(1-\delta)m'} \, dx \right)^{1/m'}
\]

\[
\leq c ||f||_{L^m(\Omega)} \left( \int_\Omega |u_n|^{m'} \, dx \right)^{(1-\delta)/m'}
\]

\[
\leq c ||f||_{L^m(\Omega)} \|u_n\|_{X}^{1-\delta},
\]

where \( c \) is a constant independent of \( n \). Since \( \delta + p - 1 > 0 \) we have that the sequence \( \{u_n\} \) is uniformly bounded in \( X \). Now arguing similarly as in case (a) we obtain the required result.

(c) Let \( p_s > N \). Choosing \( \phi = u_n \in X \) as a test function in (3.3) and by using the Hölder inequality together with the continuous embedding \( X \hookrightarrow L^\infty(\Omega) \) we obtain

\[
\|u_n\|_{X}^p \leq \int_\Omega |f||u_n|^{1-\delta} \, dx
\]

\[
\leq ||f||_{L^1(\Omega)} \|u_n\|_{L^\infty(\Omega)}^{1-\delta}
\]

\[
\leq c ||f||_{L^1(\Omega)} \|u_n\|_{X}^{1-\delta}.
\]

Since \( \delta + p - 1 > 0 \), we have \( \|u_n\|_{X} \leq c \), where \( c \) is a constant independent of \( n \). Therefore the sequence \( \{u_n\} \) is uniformly bounded in \( X \). Arguing similarly as in (a) we obtain the required result.

\[ \square \]

Proof of Theorem 3.3.

(a) Let \( 1 \leq p_s < N \), then \( p_s^* > p \).
(i) We observe that

- for $m = \left( \frac{p^*_s}{(1 - \delta)} \right)'$, i.e., $(1 - \delta)m' = p^*_s$, we have $\gamma = \frac{(\delta + p - 1)m'}{(p^*_s - p)} = 1$ and

- $m \in \left( \frac{p^*_s}{1 - \delta}, \frac{p^*_s}{p^*_s - p} \right)$ implies $\gamma = \frac{(\delta + p - 1)m'}{(p^*_s - p)} > 1$.

Note that $(p\gamma - p + 1 - \delta)m' = p^*_s \gamma$ and choosing $\phi = u^* \gamma^{-p+1} \in X$ as a test function in (3.3) we obtain

$$
\|u^*_n\|^p_X \leq \|f\|_{L^m(\Omega)} \left( \int_{\Omega} |u_n|^p \gamma \ dx \right)^{1/m'}.
$$

Now using the continuous embedding $X \hookrightarrow L^{p^*_s}(\Omega)$ and the fact $\frac{p}{p^*_s} - \frac{1}{m'} > 0$ we obtain $\|u^*_n\|_{L^{p^*_s}(\Omega)} \leq c$, where $c$ is independent of $n$, which implies that the sequence $\{u^*_n\}$ is uniformly bounded in $L^t(\Omega)$ where $t = p^*_s \gamma$.

Therefore the pointwise limit $u$ belongs to $L^t(\Omega)$, e.g., see [25].

(ii) Let $m > \frac{p^*_s}{(p^*_s - p)}$ and for $k > 1$. Choosing $\phi_k = (u_n - k)^{+} \in X$ as a test function in (3.3) we obtain from Hölder’s and Young’s inequalities with $\epsilon \in (0, 1)$

$$
\int_{\Omega} w|\nabla \phi_k|^p \ dx \leq c \int_{A(k)} |f||u_n - k| \ dx
$$

$$
\leq c \left( \int_{A(k)} |f|^\frac{p^*_s}{p^*_s + p'} \ dx \right)^{1/\left(\frac{p^*_s}{p^*_s + p'}\right)} \left( \int_{A(k)} |u_n - k|^{p^*_s} \ dx \right)^{1/\left(\frac{p^*_s}{p^*_s + p'}\right)}
$$

$$
\leq c \left( \int_{A(k)} |f|^\frac{p^*_s}{p^*_s + p'} \ dx \right)^{1/\left(\frac{p^*_s}{p^*_s + p'}\right)} \left( \int_{\Omega} w|\nabla \phi_k|^p \ dx \right)^{1/p}
$$

$$
\leq c \epsilon \left( \int_{A(k)} |f|^\frac{p^*_s}{p^*_s + p'} \ dx \right)^{\frac{p^*_s}{p^*_s + p'} - 1} + c \left( \int_{\Omega} w|\nabla \phi_k|^p \ dx \right)^{1/p}
$$

where $A(k) = \{ x \in \Omega : u_n \geq k \text{ a.e. in } \Omega \}$. Since $m > \frac{p^*_s}{(p^*_s - p)}$, we have $m > p^*_s'$. By using Hölder’s inequality in the above estimate we obtain

$$
\int_{\Omega} w|\nabla \phi_k|^p \ dx \leq c \|f\|_{L^m(\Omega)}^{\frac{p^*_s}{p^*_s + p'}} \left( \frac{m}{p^*_s} \right)^{\frac{1}{p^*_s}}
$$

where $c$ is a constant independent of $n$. Now using the continuous embedding $X \hookrightarrow L^{p^*_s}(\Omega)$ we obtain for $1 < k < h$,

$$
(h - k)^p |A(h)|^{p/p^*_s} \leq \left( \int_{A(h)} (u_n - k)^{p^*_s} \ dx \right)^{p/p^*_s}
$$

$$
\leq \left( \int_{A(h)} (u_n - k)^{p^*_s} \ dx \right)^{p/p^*_s}
$$

$$
\leq c \int_{\Omega} w|\nabla \phi_k|^p \ dx
$$

$$
\leq c \|f\|_{L^m(\Omega)}^{\frac{p^*_s}{p^*_s + p'}} \left( \frac{m}{p^*_s} \right)^{\frac{1}{p^*_s}}.
$$
Therefore

\[ |A(h)| \leq \frac{c|f|^p_{L^m(\Omega)}}{(h-k)^{p_{s}^*}} |A(k)| \left( \frac{p_{s}^*}{m} \right)^{1/p_{s}^*} \left( \frac{m}{p_{s}^*} \right)^{1/m} \].

Since \( p_{s}^* > 1 \), by Theorem 2.15, we have \( \|u_n\|_{L^{\infty}(\Omega)} \leq c \), where \( c \) is a constant independent of \( n \). Therefore we have \( u \in L^{\infty}(\Omega) \).

(b) Let \( p_{s} = N \) and \( q > p \). Observe that

- for \( m \in (q/(1-\delta), \infty) \), i.e., \( (1-\delta)m' = q \), we have \( \gamma = \frac{(\delta+p-1)m'}{(pm'-q)} = 1 \) and
  - \( m \in \left[ \frac{q}{1-\delta}, \frac{pm'}{pm'-q} \right) \) implies \( \gamma = \frac{(\delta+p-1)m'}{pm'-q} > 1 \).

Note that \( (p_{s}^* - p + 1 - \delta)m' = q \gamma \) and choosing \( \phi = u_n^{p_{s}^* - p + 1} \in X \) as a test function in (3.3) we obtain

\[ \|u_n\|_{L^{m}(\Omega)}^{p_{s}^*} \leq \|f\|_{L^{m}(\Omega)} \left( \int_{\Omega} |u_n|^q \right)^{1/m}. \]

Now using the continuous embedding \( X \hookrightarrow L^q(\Omega) \) and the fact \( p/q - 1/m' > 0 \) we obtain \( \|u_n\|_{L^q(\Omega)} \leq c \), where \( c \) is independent of \( n \), which implies that the sequence \( \{u_n\} \) is uniformly bounded in \( L^t(\Omega) \) where \( t = q \gamma \). Therefore \( u \) belong to \( L^t(\Omega) \).

(c) Follows from Theorem 3.2 using the continuous embedding \( X \hookrightarrow L^{\infty}(\Omega) \).

3.7 | The case \( \delta = 1 \)

Proof of Theorem 3.4. Let \( \delta = 1 \) and \( f \in L^1(\Omega) \). Then choosing \( \phi = u_n \in X \) as a test function in (3.3) for any \( p_{s} \geq 1 \), we obtain \( \|u_n\|_{X} \leq \|f\|_{L^1(\Omega)} \). Now arguing similarly as in Theorem 3.2 we obtain the existence of weak solution \( u \in X \) of (1.1).

Proof of Theorem 3.5.

(a) Let \( 1 \leq p_{s} < N \), then \( p_{s}^* > p \).

(i) Observe that \( m \in (1, p_{s}^*/(p_{s}^* - p)) \) implies \( \gamma = \frac{pm'}{(pm'-p_{s}^*)} > 1 \). Now choosing \( \phi = u_n^{p_{s}^* - p + 1} \in X \) as a test function in (3.3) together with the continuous embedding \( X \hookrightarrow L^{p_{s}^*}(\Omega) \) and arguing similarly as in part (i) of Theorem 3.3 we obtain the required result.

(ii) Follows arguing similarly as in part (ii) of Theorem 3.3.

(b) Let \( p_{s} = N \) and \( q > p \). Observe that \( m \in (1, q/(q - p)) \) implies \( \gamma = \frac{pm'}{pm'-q} > 1 \). Choosing \( \phi = u_n^{p_{s}^* - p + 1} \in X \) as a test function in (3.3) together with the continuous embedding \( X \hookrightarrow L^q(\Omega) \) and proceeding similarly as in part (b) of Theorem 3.3 we obtain the required result.

(c) Follows from Theorem 3.4 using the continuous embedding \( X \hookrightarrow L^{\infty}(\Omega) \).

3.8 | The case \( \delta > 1 \)

Proof of Theorem 3.6. Let \( \delta > 1 \) and \( f \in L^1(\Omega) \) with \( p_{s} \geq 1 \). By Theorem 3.13 for every fixed \( n \in \mathbb{N} \) we have \( u_n \in L^{\infty}(\Omega) \) (the bound may depend on \( n \)). Choosing \( \phi = u_n^{\delta} \in X \) as a test function in (3.3) (which is admissible since \( \delta > 1 \) and \( u_n \in L^{\infty}(\Omega) \) by Theorem 3.13) we obtain

\[ \int_{\Omega} \delta u_n^{\delta-1} |\nabla u_n|^p w(x) dx \leq \int_{\Omega} \delta u_n^{\delta-1} A(x, \nabla u_n) \cdot \nabla u_n dx \leq \int_{\Omega} |f(x)| dx, \]
which implies
\[ \int_{\Omega} w \left\| \nabla \left( u_n \left( \frac{\delta + p - 1}{p} \right) \right) \right|^p dx \leq c \| f \|_{L^1(\Omega)}, \]

where \( c \) is independent of \( n \). Therefore the sequence \( \{ u_n \left( \frac{\delta + p - 1}{p} \right) \} \) is uniformly bounded in \( X \). Let \( \phi \in C_c^\infty(\Omega) \) and consider \( v_n = \phi u_n \in X \). We observe that
\[ \int_{\Omega} A(x, \nabla u_n) \cdot \nabla (\phi u_n) dx = p \int_{\Omega} \phi^{p-1} u_n A(x, \nabla u_n) \cdot \nabla \phi dx + \int_{\Omega} \phi^p A(x, \nabla u_n) \cdot \nabla u_n dx, \tag{3.9} \]

and by using Young’s inequality for \( \varepsilon \in (0, 1) \), we obtain for some positive constant \( c_\varepsilon \),
\[ \left| p \int_{\Omega} \phi^{p-1} u_n A(x, \nabla u_n) \cdot \nabla \phi dx \right| \leq \varepsilon \int_{\Omega} w |\phi|^p |\nabla u_n|^p dx + c_\varepsilon \int_{\Omega} w |u_n|^p |\nabla \phi|^p dx. \tag{3.10} \]

Now choosing \( \phi = v_n \in X \) as a test function in (3.3) and using the estimates (3.9), (3.10), we obtain
\[ \int_{\Omega} \phi^p |\nabla u_n|^p w(x) dx \leq \int_{\Omega} \phi^p A(x, \nabla u_n) \cdot \nabla u_n dx \]
\[ = \int_{\Omega} \frac{f_n}{u_n + \frac{1}{n} \delta} \phi^p u_n dx - p \int_{\Omega} \phi^{p-1} u_n A(x, \nabla u_n) \cdot \nabla \phi dx \]
\[ \leq \int_{K} \frac{f_n}{u_n} \phi^p dx + \varepsilon \int_{\Omega} |\phi|^p |\nabla u_n|^p w(x) dx + c_\varepsilon \int_{\Omega} |u_n|^p |\nabla \phi|^p w(x) dx \]
\[ \leq \frac{\| \phi \|_{L^\infty(\Omega)} \| f \|_{L^1(\Omega)} + \varepsilon \int_{\Omega} |\phi|^p |\nabla u_n|^p w(x) dx + c_\varepsilon \| \nabla \phi \|_{L^\infty(\Omega)} }{c_\delta K} \int_{K} \frac{1}{u_n^{\delta-1}} |u_n^{(\delta+p-1)/p}|^p dx \]
\[ \leq c_\delta \| f \|_{L^1(\Omega)} + \varepsilon \int_{\Omega} |\phi|^p |\nabla u_n|^p w(x) dx + c_\phi \| u_n^{(\delta+p-1)/p} \|_{X'}, \]

where \( K \) is the support of \( \phi \) and \( c_\phi \) is a constant depending on \( \phi \). Therefore we have
\[ (1 - \varepsilon) \int_{\Omega} \phi^p |\nabla u_n|^p w(x) dx \leq c_\phi \left\{ \| f \|_{L^1(\Omega)} + \| u_n^{(\delta+p-1)/p} \|_{X'} \right\}. \]

Now since the sequence \( \{ u_n^{(\delta+p-1)/p} \} \) is uniformly bounded in \( X \) we have the sequence \( \{ u_n \} \) is uniformly bounded in \( W^{1,p}_{\text{loc}}(\Omega, w) \). Now arguing similarly as in Theorem 3.2, we obtain \( u \in W^{1,p}_{\text{loc}}(\Omega, w) \) is a weak solution of (1.1). The fact that \( u^{(\delta+p-1)/p} \in X \) follows from the uniform boundedness of the sequence \( \{ u_n^{(\delta+p-1)/p} \} \) in \( X \). \qed

**Proof of Theorem 3.7.**

(a) Let \( 1 \leq p_s < N \), then \( p_s^* > p \).
   (i) Observe that \( m \in (1, p_s^*/(p - p_s)) \) implies \( \gamma = \frac{(\delta+p-1)m'}{pmt'-p} > \frac{\delta+p-1}{p} > 1 \), since \( \delta > 1 \). Now choosing \( \phi = u_n^{p_{s'}^{p-1}} \in X \) as a test function in (3.3) together with the continuous embedding \( X \hookrightarrow L^{p_{s'}}(\Omega) \) and arguing similarly as in part (i) of Theorem 3.3 the result follows.
   (ii) Follows by arguing similarly as in part (ii) of Theorem 3.3.

(b) Let \( p_s = N \) and \( q > p \). Observe that \( \delta > 1, m \in (1, q/(q - p)) \) implies \( \gamma = \frac{(\delta+p-1)m'}{pmt'-q} > 1 \). Choosing \( \phi = u_n^{p_{s'}^{p-1}} \in X \) as a test function in (3.3) together with the continuous embedding \( X \hookrightarrow L^{p}(\Omega) \) and proceeding similarly as in part (b) of Theorem 3.3 we obtain the required result.

(c) Follows from Theorem 3.6 using the continuous embedding \( X \hookrightarrow L^{\infty}(\Omega) \). \qed
4 | UNIQUENESS RESULTS

In this section we state and prove our main uniqueness results.

4.1 | The case $0 < \delta \leq 1$

**Theorem 4.1.** For any $0 < \delta \leq 1$ and $w \in A_p$, the problem (1.1) admits at most one weak solution in $W_0^{1,p}(\Omega, w)$ for any nonnegative $f \in L^1(\Omega)$.

**Proof.** Let $0 < \delta \leq 1$, let $w \in A_p$ be arbitrary and let $u_1, u_2 \in X$ be two solutions of Equation (1.1). The fact $(u_1 - u_2)^+ \in X$ allows us to choose $\{\varphi_n\} \in C_c^\infty(\Omega)$ converging to $(u_1 - u_2)^+$ in $|| \cdot ||_X$. Now setting,

$$
\psi_n := \min \left\{ (u_1 - u_2)^+, \varphi_n^+ \right\} \in X \cap L_\infty^c(\Omega)
$$

as a test function in (1.1) we get

$$
\int_\Omega (A(x, \nabla u_1) - A(x, \nabla u_2)) \cdot \nabla \psi_n \, dx \leq \int_\Omega \left( \frac{1}{u_1^\delta} - \frac{1}{u_2^\delta} \right) \psi_n \, dx \leq 0.
$$

Passing to the limit and using the strong monotonicity condition (H5), $(u_1 - u_2)^+ = 0$ a.e. in $\Omega$ which implies $u_1 \leq u_2$. Similarly changing the role of $u_1$ and $u_2$, we get $u_2 \leq u_1$. Therefore, $u_1 \equiv u_2$. \hfill \Box

4.2 | The case $\delta > 1$

**Theorem 4.2.** Let $\delta > 1$ and $w \in A_p$. Then the problem (1.1) has at most one weak solution in $W^{1,p}_{\text{loc}}(\Omega, w)$ if

1. $f \in L^m(\Omega)$ for some $m = (p_s^\ast)'$, provided $1 \leq p_s < N$, or
2. $f \in L^m(\Omega)$ for some $m > 1$, provided $p_s = N$, or
3. $f \in L^1(\Omega)$ for $p_s > N$.

**Remark 4.3.** In case $w \equiv 1$ our main results in this paper will hold by replacing $p_s$ by $p$. Moreover, since $(p^\ast)' < N/p$, Theorem 4.2 improves the range of $f$ in Theorem 1.5 of [9] to get the uniqueness provided $1 < p < N$.

**Preliminaries:** Define for $k > 0$ and $\delta > 1$ the truncated function

$$
g_k(s) := \begin{cases} 
\min \{s^{-\delta}, k\}, & \text{for } s > 0, \\
k, & \text{for } s \leq 0.
\end{cases}
$$

**Definition 4.4.** We say that $v(> 0) \in W^{1,p}_{\text{loc}}(\Omega, w)$ is a super-solution of the problem (1.1), if for every $K \Subset \Omega$ there exists a positive constant $c_K$ such that $v \geq c_K > 0$ in $K$ and for every nonnegative $\phi \in C_c^\infty(\Omega)$ one has

$$
\int_\Omega A(x, \nabla u) \cdot \nabla \phi \, dx \geq \int_\Omega \frac{f(x)}{v^\delta} \phi \, dx.
$$

**Definition 4.5.** We say that $v(> 0) \in W^{1,p}_{\text{loc}}(\Omega, w)$ is a sub-solution of the problem (1.1), if for every $K \Subset \Omega$ there exists a positive constant $c_K$ such that $v \geq c_K > 0$ in $K$ and for every nonnegative $\phi \in C_c^\infty(\Omega)$ one has

$$
\int_\Omega A(x, \nabla u) \cdot \nabla \phi \, dx \leq \int_\Omega \frac{f(x)}{v^\delta} \phi \, dx.
$$
For a fixed super-solution $v$ of (1.1), consider the following non-empty closed and convex set

$$ \mathcal{K} := \{ \phi \in X : 0 \leq \phi \leq v \text{ a.e. in } \Omega \}. $$

**Lemma 4.6.** There exists $z \in \mathcal{K}$ such that for every nonnegative $\phi \in C^1_c(\Omega)$ one has

$$ \int_{\Omega} A(x, \nabla z) \cdot \nabla \phi \, dx \geq \int_{\Omega} f(x) g_k(z) \phi \, dx. \quad (4.3) $$

**Proof.** Under the assumptions on $f$ and applying Theorem 2.7 one can define the operator $J_k : X \to X^*$ for every $u, \psi \in X$ by

$$ \langle J_k(u), \psi \rangle := \int_{\Omega} A(x, \nabla u) \cdot \nabla \psi \, dx - \int_{\Omega} f g_k(u) \psi \, dx. $$

Following the same arguments as in Lemma 3.9, it follows that $J_k$ is demicontinuous, coercive and strictly monotone. As a consequence of Theorem 2.14, there exists a unique $z \in \mathcal{K}$ such that for every $\psi \in \mathcal{K}$, one has

$$ \int_{\Omega} A(x, \nabla z) \cdot \nabla (\psi - z) \, dx \geq \int_{\Omega} f(x) g_k(z)(\psi - z) \, dx. \quad (4.4) $$

Let us consider a real valued function $g \in C^\infty_c(\mathbb{R})$ such that $0 \leq g \leq 1$, $g \equiv 1$ in $[-1, 1]$ and $g \equiv 0$ in $(-\infty, -2] \cup [2, \infty)$. Define the function

$$ \phi_h := g \left( \frac{z}{h} \right) \phi $$

and

$$ \phi_{h,t} := \min \{ z + t \phi_h, v \} $$

with $h \geq 1$ and $t > 0$

for a given nonnegative $\phi \in C^1_c(\Omega)$. Then by the inequality (4.4), we have

$$ \int_{\Omega} A(x, \nabla z) \cdot \nabla (\phi_{h,t} - z) \, dx \geq \int_{\Omega} f(x) g_k(z)(\phi_{h,t} - z) \, dx. \quad (4.5) $$

By (H5), we have

$$ I = c \int_{\Omega} | \nabla (\phi_{h,t} - z) |^\gamma \left\{ A(x, \nabla \phi_{h,t}, \nabla z) \right\}^{1-\gamma/p} w(x) \, dx $$

$$ \leq \int_{\Omega} \{ A(x, \nabla \phi_{h,t}) - A(x, \nabla z) \} \cdot \nabla (\phi_{h,t} - z) \, dx $$

$$ = \int_{\Omega} A(x, \nabla \phi_{h,t}) \cdot \nabla (\phi_{h,t} - z) \, dx - \int_{\Omega} A(x, \nabla z) \cdot \nabla (\phi_{h,t} - z) \, dx $$

$$ \leq \int_{\Omega} A(x, \nabla \phi_{h,t}) \cdot \nabla (\phi_{h,t} - z) \, dx - \int_{\Omega} f(x) g_k(z)(\phi_{h,t} - z) \, dx \text{ (using (4.5)).} $$

Therefore,

$$ I - \int_{\Omega} f(x)(g_k(\phi_{h,t}) - g_k(z))(\phi_{h,t} - z) \, dx $$

$$ \leq \int_{\Omega} A(x, \nabla \phi_{h,t}) \cdot \nabla (\phi_{h,t} - z) \, dx - \int_{\Omega} f(x) g_k(\phi_{h,t})(\phi_{h,t} - z) \, dx $$

$$ = \int_{\Omega} g(x) \, dx - \int_{\Omega} f(x) g_k(\phi_{h,t})(\phi_{h,t} - z - t \phi_h) \, dx + t \int_{\Omega} A(x, \nabla \phi_{h,t}) \cdot \nabla \phi_h \, dx - t \int_{\Omega} f(x) g_k(\phi_{h,t}) \phi_h \, dx, $$

$$ \quad (4.6) $$
where
\[ g(x) := A(x, \nabla \phi_h) \cdot \nabla (\phi_{h,t} - z - \phi_h). \]

Let us denote by
\[ g_v(x) := A(x, \nabla v) \cdot \nabla (\phi_{h,t} - z - \phi_h). \]

Set \( \Omega = S_v \cup S_v^c \), where \( S_v := \{ x \in \Omega : \phi_h(x) = v(x) \} \) and \( S_v^c := \Omega \setminus S_v \). Observe that \( g(x) = g_v(x) = 0 \) on \( S_v^c \) and \( g(x) = g_v(x) \) on \( S_v \). This gives from (4.6),
\[
I - \int_\Omega f(x)(g_k(\phi_{h,t}) - g_k(z))(\phi_{h,t} - z) \, dx
= \int_\Omega g_v(x) \, dx - \int_\Omega f(x)g_k(\phi_{h,t})(\phi_{h,t} - z - t\phi_h) \, dx + t \int_\Omega A(x, \nabla \phi_{h,t}) \cdot \nabla \phi_h \, dx - t \int_\Omega f(x)g_k(\phi_{h,t})\phi_h \, dx.
\]
(4.7)

Since \( v \) is a super-solution of (1.1), choosing \((z + t\phi_h - \phi_{h,t})\) as a test function in (4.1) and using the fact that \( \phi_{h,t} = v \) on \( S_v \), we obtain
\[
\int_\Omega g_v(x) \, dx - \int_\Omega f(x)g_k(\phi_{h,t})(\phi_{h,t} - z - t\phi_h) \, dx \leq 0.
\]

Since \( I \geq 0 \) and \( \phi_{h,t} - z \leq t\phi_h \), using the inequality (4.7), we get
\[
\int_\Omega A(x, \nabla \phi_{h,t}) \cdot \nabla \phi_h \, dx - \int_\Omega f(x)g_k(\phi_{h,t})\phi_h \, dx \geq - \int_\Omega f|g_k(\phi_{h,t}) - g_k(z)|\phi_h \, dx.
\]

Therefore letting \( t \to 0 \), we obtain
\[
\int_\Omega A(x, \nabla z) \cdot \nabla \phi_h \, dx - \int_\Omega f(x)g_k(z)\phi_h \, dx \geq 0.
\]

As \( h \to \infty \), we obtain
\[
\int_\Omega A(x, \nabla z) \cdot \nabla \phi \, dx \geq \int_\Omega f(x)g_k(z)\phi \, dx.
\]

Hence the proof. \( \square \)

**Proof of Theorem 4.2.** Suppose \( u, v \in W^{1,p}_{loc}(\Omega, w) \) both are solutions of the problem (1.1). Then, we can assume that \( u \) is a sub-solution and \( v \) is a super-solution of (1.1). By the given condition on \( f \), one can use Lemma 4.6 to get the existence of \( z \in \mathbb{K} \) satisfying the inequality (4.3). Let \( \varepsilon = 2k^{-1/\delta} \) for \( k > 0 \). Since \( u = 0 \) on \( \partial \Omega \), one can use Theorem 2.11 to obtain \((u - z - \varepsilon)^+ \in X \). Applying Lemma 4.6, for any \( \eta > 0 \), by standard density arguments one has
\[
\int_\Omega A(x, \nabla z) \cdot \nabla T_\eta((u - z - \varepsilon)^+) \, dx \geq \int_\Omega f(x)g_k(z)T_\eta((u - z - \varepsilon)^+) \, dx.
\]
(4.8)

Since \((u - z - \varepsilon)^+ \in X \), there exists a sequence \( \phi_n \in C^\infty_c(\Omega) \) such that \( \phi_n \to (u - z - \varepsilon)^+ \) in \( || \cdot ||_X \). Denote by
\[
\psi_{n,\eta} := T_\eta(\min\{(u - z - \varepsilon)^+, \phi_n^+\}) \in X \cap L^\infty_c(\Omega),
\]
and since \( u \) is a sub-solution of (1.1), we obtain
\[
\int_\Omega A(x, \nabla u) \cdot \nabla \psi_{n,\varepsilon} \, dx \leq \int_\Omega \frac{f}{u^\delta}\psi_{n,\varepsilon} \, dx.
\]
Since \( w|\nabla u|^p \) is integrable on the support of \((u - z - \epsilon)^+\), one can pass to the limit as \( n \to \infty \) and obtain

\[
\int_{\Omega} A(x, \nabla u) \cdot \nabla \eta \ ((u - z - \epsilon)^+) \, dx \leq \int_{\Omega} \frac{f}{w^\frac{p}{2}} \eta \ ((u - z - \epsilon)^+) \, dx.
\] (4.9)

By using (4.8), (4.9), the fact \( \epsilon > k^{-1/\delta} \) together with (H5), we obtain for \( \gamma := \max \{ p, 2 \} \),

\[
\int_{\Omega} |\nabla \eta \ ((u - z - \epsilon)^+) |^{\gamma/\psi} (|\nabla u|^p + |\nabla z|^p)^{1-\gamma/p} w(x) \, dx \leq \int_{\Omega} \left\{ A(x, \nabla u) - A(x, \nabla z) \right\} \cdot \nabla \eta \ ((u - z - \epsilon)^+) \, dx
\]

\[
\leq \int_{\Omega} f(x) \left( \frac{1}{w^\frac{p}{2}} - g_k \ (z) \right) \eta \ ((u - z - \epsilon)^+) \, dx
\]

\[
\leq \int_{\Omega} f(x) (g_k (u) - g_k (z)) \eta \ ((u - z - \epsilon)^+) \, dx \leq 0.
\]

Since \( \eta > 0 \) is arbitrary, we have \( u \leq z + 2k^{-1/\delta} \leq v + 2k^{-1/\delta} \). Letting \( k \to \infty \), we get \( u \leq v \) a.e. in \( \Omega \). Arguing similarly we obtain \( v \leq u \) a.e. in \( \Omega \). Hence \( u \equiv v \). □

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