Canonical quantization of the Dirac oscillator field in (1+1) and (3+1) dimensions

C. J. Quimbay*† and R. A. Hernandez‡

Departamento de Física, Universidad Nacional de Colombia.
Ciudad Universitaria, Bogotá D.C., Colombia.

Y. F. Pérez§

Escuela de Física, Universidad Pedagógica y Tecnológica de Colombia, Tunja, Colombia
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Abstract

The main goal of this work is to study the Dirac oscillator as a quantum field using the canonical formalism of quantum field theory and to develop the canonical quantization procedure for this system in (1 + 1) and (3 + 1) dimensions. This is possible because the Dirac oscillator is characterized by the absence of the Klein paradox and by the completeness of its eigenfunctions. We show that the Dirac oscillator field can be seen as constituted by infinite degrees of freedom which are identified as decoupled quantum linear harmonic oscillators. We observe that while for the free Dirac field the energy quanta of the infinite harmonic oscillators are the relativistic energies of free particles, for the Dirac oscillator field the quanta are the energies of relativistic linear harmonic oscillators.

Keywords: Quantum field theory, Dirac oscillator, canonical quantization formalism, relativistic harmonic oscillators.

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* Associate researcher of Centro Internacional de Física, Bogotá D.C., Colombia.
† Electronic address: cjquimbayh@unal.edu.co
‡ Electronic address: rahernandezm@unal.edu.co
§ Electronic address: yuber.perez@uptc.edu.co
I. INTRODUCTION

Quantum states of a relativistic massive fermion are described by four-components wave functions called Dirac spinors. These wave functions, which are solutions of the Dirac equation, describe states of positive and negative energy. For the case where fermions carry the electric charge $q$, the electromagnetic interaction of fermions can be included by means of the electromagnetic four-potential $A^\mu$, which is introduced in the Dirac equation through the so called minimal substitution, changing the fourmomentum such as $p^\mu \rightarrow p^\mu - qA^\mu$. It is also possible to introduce a linear harmonic potential in the Dirac equation by substituting $\vec{p} \rightarrow \vec{p} - im\omega \beta \vec{r}$, where $m$ is the fermion mass, $\omega$ represents an oscillator frequency, $r$ is the distance of the fermion respects to the origin of the linear potential and $\beta = \gamma_0$ corresponds to the diagonal Dirac matrix.

The Dirac equation including the linear harmonic potential was initially studied by Itô et al. [1], Cook [2] and Ui et al. [3]. This system was latterly called by Moshinsky and Szczepaniak as Dirac oscillator [4], because it behaves as an harmonic oscillator with a strong spin-orbit coupling in the non-relativistic limit. As a relativistic quantum mechanical system, the Dirac oscillator has been widely studied. Several properties from this system have been considered in (1+1), (2+1), (3+1) dimensions [5]-[28]. Specifically, for the Dirac oscillator have been studied several properties as its covariance [6], its energy spectrum, its corresponding eigenfunctions and the form of the electromagnetic potential associated with its interaction in (3+1) dimensions [7], its Lie Algebra symmetries [8], the conditions for the existence of bound states [9], its connection with supersymmetric (non-relativistic) quantum mechanics [10], the absence of the Klein paradox in this system [11], its conformal invariance [12], its complete energy spectrum and its corresponding eigenfunctions in (2+1) dimensions [13], the existence of a physical picture for its interaction [14]. For this system, other aspects have been also studied as the completeness of its eigenfunctions in (1+1) and (3+1) dimensions [15], its thermodynamic properties in (1+1) dimensions [16], the characteristics of its two-point Green functions [17], its energy spectrum in the presence of the Aharonov-Bohm effect [18], the momenta representation of its exact solutions [19], the Lorenz deformed covariant algebra for the Dirac oscillator in (1+1) dimension [20], the properties of its propagator in (1+1) dimensions using the supersymmetric path integral formalism [21], its exact mapping onto a Jaynes-Cummings model [22], its nonrelativistic limit in (2+1) dimensions interpreted in terms of a Ramsey-interferometry effect [23], the existence of a chiral phase transition for this system in (2+1) dimensions in presence of a constant magnetic field [24], a new representation for its solutions using the Clifford algebra [25], its dynamics in presence of a two-component external field.
the relativistic Landau levels for this system in presence of a external magnetic field in (2+1) dimensions and its relationship with (Anti)-Jaynes-Cummings models in a (2+1) dimensional noncommutative space.

Some possible applications of the Dirac oscillator have been developed. For instance, the hadronic spectrum has been studied using the two-body Dirac oscillator and the references therein. The Dirac oscillator in (2+1) dimension has been used as a framework to study some condensed matter physical phenomena such as the study of electrons in two dimensional materials, which can be applied to study some aspects of the physics of graphene. This system has also been used in quantum optics to describe the interaction of atoms with electromagnetic fields in cavities (the Jaynes-Cummings model).

The standard point of view of the Quantum Field Theory (QFT) establishes that an excitation of one of the infinite degrees of freedom that constitute the free Dirac quantum field can be interpreted as a free relativistic massive fermion. The free Dirac quantum field which describes the quantum dynamics of a non-interacting relativistic massive fermion can be seen as constituted by infinite decoupled quantum harmonic oscillators. In this scheme a non-interacting relativistic massive fermion is described as an excitation of a degree of freedom of the free Dirac quantum field. For the free Dirac field the energy quanta of the infinite harmonic oscillators are relativistic energies of free fermions or antifermions. An analogous situation is presented in the description of a non-interacting relativistic boson as an excitation of one of the infinite degrees of freedom (harmonic oscillators) that constitute a free bosonic quantum field. For the free Boson field the energy quanta of the infinite harmonic oscillators are relativistic energies of free bosons.

On the other hand, the Dirac oscillator, which describes the interacting system constituted by a relativistic massive fermion under the action of a linear harmonic potential, has not been studied as a quantum field up to now. In this direction, we show that it is possible to study consistently the Dirac oscillator as a QFT system. The main goal of this work is to consider the Dirac oscillator as a field and perform the canonical quantization procedure of the Dirac oscillator field in (1+1) and (3+1) dimensions. We can do it, in the first place, because the Dirac oscillator is characterized by the absence of the Klein paradox, that allows to distinguish between positive and negative energy states. In the second place, the Dirac oscillator is an interacting system in which there exist bound states that are defined by means of specific quantum numbers. This aspect means that the solution of the Dirac oscillator equation leads to both a well-known energy spectrum and an eigenfunction set which satisfies the completeness and orthonormality conditions in (1+1) and (3+1) dimensions. Thus the Dirac oscillator field can be consistently treated as a quantum
field because it can be written in terms of a Fourier expansion of that eigenfunction set. We observe that after the canonical quantization procedure, the Dirac oscillator field can be seen as constituted by infinite relativistic decoupled quantum linear harmonic oscillators.

The canonical quantization procedure for the Dirac oscillator field starts from writing the Hamiltonian operator for this system in terms of annihilation and creation operators satisfying the usual anticommutation relations. This procedure allows us to obtain the Feynman propagator for the Dirac oscillator field. Thus we are able to study the differences between the Dirac oscillator field and the free Dirac field. We note that while for the free Dirac field the energy quanta of the infinite harmonic oscillators are relativistic energies of free particles, for the Dirac oscillator quantum field the quanta are energies of relativistic linear harmonic oscillators.

This work is presented as follows: First, in section 2 we introduce the notation used in this paper presenting the relevant aspects of the standard canonical quantization procedure for the free Dirac field in 3+1 dimensions. Next, in section 3 we study the Dirac oscillator system in 1+1 dimensions from a point of view of quantum relativistic mechanics; additionally we describe some aspects of the Dirac’s sea picture for this system and we develop the canonical quantization procedure for the Dirac oscillator field in 1+1 dimensions. Posteriorly, in section 4 we study the canonical quantization procedure for the Dirac oscillator field in 3+1 dimensions. Finally, in section 5 we present some conclusions of this work.

II. CANONICAL QUANTIZATION FOR THE FREE DIRAC FIELD

In this section we present the main steps of the standard canonical quantization procedure of the free Dirac field following the procedure developed in [31]. The equation of motion for a free relativistic fermion of mass $m$ is given by the Dirac equation

$$\left( i \gamma^\mu \partial_\mu - m \right) \psi(\vec{r}, t) = 0, \quad (1)$$

where we have taken $\hbar = c = 1$. In this equation $\gamma^\mu$ represents the Dirac matrices obeying the Clifford algebra $\{ \gamma^\mu, \gamma^\nu \} = 2g^{\mu\nu}$ and $\psi(\vec{r}, t)$ represents the four-component spinor wave functions. The four linearly independent solutions from the Dirac equation are plane waves of the form

$$\psi^r_p(\vec{r}, t) = (2\pi)^{-3/2} \sqrt{\frac{m}{E_p}} w_r(\vec{p}) e^{-iE_p t - i\vec{p} \cdot \vec{r}},$$

where $E_p$ is given by $E_p = +\sqrt{\vec{p}^2 + m^2}$. The two solutions of relativistic free-particle positive energy are described by $r = 1, 2$ while the two solutions of relativistic free-particle negative energy are described by $r = 3, 4$. The sign function $\epsilon_r$ takes the value $\epsilon_r = 1$, for $r = 1, 2$, and $\epsilon_r = -1$, for $r = 3, 4$. Additionally, the spinors $w_r(\vec{p})$ obey the
equation \((\gamma\mu p^\mu - \epsilon_r m) w_r (\vec{p}) = 0\).

In a quantum field theory treatment, \(\psi(\vec{r}, t)\) is the free Dirac field. The Hamiltonian associated to this field is \(H = \int d^3x \psi^\dagger (-i\alpha \cdot \nabla + m\beta) \psi\), where the matrices \(\alpha, \beta\) are defined as \(\alpha_i = \gamma^0 \gamma^i\), \(\beta = \gamma^0\). The canonical quantization procedure of the free Dirac field starts by replacing the fields \(\psi(\vec{r}, t)\) and \(\psi^\dagger(\vec{r}, t)\) with the fields operators \(\hat{\psi}(\vec{r}, t)\) and \(\hat{\psi}^\dagger(\vec{r}, t)\) which obey the usual commutation relations of the Jordan-Wigner type \([31]\). The Fourier expansion of the free Dirac field operator \(\hat{\psi}(\vec{r}, t)\), in terms of the plane waves functions, is written as

\[
\hat{\psi}(\vec{r}, t) = \sum_{r=1}^{4} \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_{\vec{p}}}} \hat{a}(\vec{p}, r) e^{-i\vec{p} \cdot \vec{r}},
\]

where \(p \cdot x \equiv p_\mu x^\mu = E_{\vec{p}} t - \vec{p} \cdot \vec{r}\) and \(\hat{a}(\vec{p}, r)\) is an operator. The operators \(\hat{a}(\vec{p}, r)\) and its conjugated \(\hat{a}^\dagger(\vec{p}, r)\) satisfy the usual anticommutation relations. The Hamiltonian operator \(\hat{H}\) associated to this system is defined in terms of the Dirac field operators \(\hat{\psi}(\vec{r}, t)\) and \(\hat{\psi}^\dagger(\vec{r}, t)\). Using the Fourier expansion of the free Dirac field operator, \(\hat{H}\) can be written as \(\hat{H} = \int d^3p E_{\vec{p}} \left[ \sum_{r=1}^{2} \hat{n}_{\vec{p}, r} + \sum_{r=3}^{4} \hat{n}_{\vec{p}, r} \right]\), where \(\hat{n}_{\vec{p}, r} = \hat{a}^\dagger(\vec{p}, r) \hat{a}(\vec{p}, r)\) is the particle-number operator and \(\hat{n}_{\vec{p}, r} = \hat{\alpha}(\vec{p}, r) \hat{\alpha}^\dagger(\vec{p}, r)\) is the antiparticle-number operator. In this Hamiltonian operator, which is positively defined, the Dirac’s see picture has been used and the zero-point energy contribution has been subtracted \([31]\).

Now it is possible to introduced the following canonical transformations over the annihilation and creation operators that allow to differentiate between operators associated to particles and antiparticles: \(\hat{c}(\vec{p}, +s) = \hat{a}(\vec{p}, 1)\), \(\hat{c}(\vec{p}, -s) = \hat{a}(\vec{p}, 2)\), \(\hat{d}(\vec{p}, +s) = \hat{a}(\vec{p}, 3)\), \(\hat{d}(\vec{p}, -s) = \hat{a}(\vec{p}, 4)\). With these transformations, the following identifications are possible: \(\hat{c}(\vec{p}, s)\) and \(\hat{c}^\dagger(\vec{p}, s)\) represent the annihilation and the creation operators of particles, respectively; \(\hat{d}(\vec{p}, s)\) and \(\hat{d}^\dagger(\vec{p}, s)\) represent the annihilation and the creation operators of antiparticles, respectively. Using the operators \(\hat{c}(\vec{p}, s)\), \(\hat{c}^\dagger(\vec{p}, s)\), \(\hat{d}(\vec{p}, s)\) and \(\hat{d}^\dagger(\vec{p}, s)\), the Hamiltonian operator of the free Dirac field can be written as \([31]\)

\[
\hat{H} = \sum_s \int d^3p E_{\vec{p}} \left( \hat{n}_{\vec{p}, s}^{(c)} + \hat{n}_{\vec{p}, s}^{(d)} \right),
\]

where \(\hat{n}_{\vec{p}, s}^{(c)} = \hat{c}^\dagger(\vec{p}, s) \hat{c}(\vec{p}, s)\) represents the fermion-number operator and \(\hat{n}_{\vec{p}, s}^{(d)} = \hat{d}^\dagger(\vec{p}, s) \hat{d}(\vec{p}, s)\) represents the antifermion-number operator. It is possible to observe that \(\hat{c}^\dagger(\vec{p}, s)\) creates a fermion of mass \(m\) having energy \(E_{\vec{p}}\), momentum \(\vec{p}\), charge \(+e\) and projection of spin \(+s\), whereas \(\hat{d}^\dagger(\vec{p}, s)\) creates an antifermion of the same mass having the same energy, the same momentum, charge \(-e\) and projection of spin \(-s\). In this scheme, a free relativistic massive fermion is described as an excitation from a degree of freedom of the free Dirac quantum field. For the free Dirac field, the energy quanta of the infinite harmonic oscillators are the relativistic energies \(E_{\vec{p}}\) of free fermions or antifermions.
Finally, the Feynman propagator $S^F_{\alpha\beta}(x-y)$ of the free Dirac field is defined as $iS^F_{\alpha\beta}(x-y) = \langle 0 | \hat{T} \left( \hat{\psi}_\alpha(x) \hat{\bar{\psi}}_\beta(y) \right) | 0 \rangle$. By using the plane wave expansion, this propagator can be re-written as $iS^F_{\alpha\beta}(x-y) = (i\gamma^\mu \partial_\mu - m)\delta_{\alpha\beta}i\Delta_F(x-y)$, where $i\Delta_F(x-y)$ represents the Feynman propagator of a scalar field.

III. DIRAC OSCILLATOR IN THE (1+1) DIMENSIONAL CASE

One of the most important and well studied systems in non-relativistic quantum mechanics is the harmonic oscillator. This system is characterized by its simplicity and usefulness. The harmonic oscillator is a simple system due to the fact that its Hamiltonian operator is quadratic both in coordinates and momenta. An analogous system in relativistic quantum mechanics is given by a linear interaction term introduced in the Dirac equation. In fact the Dirac equation corresponds to the linearization of relativistic Schrödinger equation. So the harmonic potential must be introduced in the Dirac equation as the quadratic root of the quadratic potential, i.e. the potential must be linear. The interacting system constituted by a relativistic massive fermion under the action of a linear potential is known as the Dirac oscillator [4]-[29].

In this section we will consider the Dirac oscillator in the (1+1) dimensional case. Initially we will obtain the energy spectrum and the wave functions describing the quantum states of this system. Next we develop the canonical quantization procedure for the Dirac oscillator field in 1+1 dimensions.

A. Spectrum and wave functions in (1+1) dimensions

The linear interaction over a relativistic fermion of mass $m$ that moves in one-dimension over the $z$ axis is introduced into the Dirac equation [4, 7, 14] by substituting the momentum $p_z$ as $\hat{p}_z \rightarrow \hat{p}_z - im\omega \gamma^0 \hat{z}$, where $\omega$ is the frequency associated to the oscillator. Thus, the one-dimensional Dirac oscillator equation takes the form ($\hbar = c = 1$)

$$i\frac{\partial}{\partial t} |\psi\rangle = (\alpha_3 \cdot (\hat{p}_z - im\omega \hat{z}) + \beta m) |\psi\rangle.$$  (4)

The solution of equation (4) allows to obtain both the energy spectrum and the wave functions describing the quantum states of this system. As we will show furtherly, these wave functions will be used in the canonical quantization procedure of this system. Given that the interaction does not mix positive and negative energies, it can be possible to rewrite the four-component spinor
state in eq. (4) as
\[ |\psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\chi\rangle \end{pmatrix}, \] (5)
where \(|\phi\rangle\) and \(|\chi\rangle\) are spinors. If we use the standard representation of the Dirac matrices, the time independent Hamiltonian equation takes the form
\[ \begin{pmatrix} m & \sigma_3 \cdot (\hat{p}_z + im\hat{\omega}\hat{z}) \\ \sigma_3 \cdot (\hat{p}_z - im\hat{\omega}\hat{z}) & -m \end{pmatrix} \begin{pmatrix} |\phi\rangle \\ |\chi\rangle \end{pmatrix} = E \begin{pmatrix} |\phi\rangle \\ |\chi\rangle \end{pmatrix}, \] (6)
which leads to the following coupled equations
\[ (E - m) |\phi\rangle = \sigma_3 \cdot (\hat{p}_z + im\hat{\omega}\hat{z}) |\chi\rangle, \] (7a)
\[ (E + m) |\chi\rangle = \sigma_3 \cdot (\hat{p}_z - im\hat{\omega}\hat{z}) |\phi\rangle. \] (7b)
Starting from (7a) and (7b), we obtain that
\[ (\hat{p}_z^2 + (m\omega)^2 \hat{z}^2) |\psi\rangle = \left( (E^2 - m^2) \mathbb{I} + m\omega\beta \right) |\psi\rangle, \] (8)
where the commutation relations between the operators \(\hat{p}_z\) and \(\hat{z}\) have been taken into account. In the last equation, \(\mathbb{I}\) is the identity matrix \(4 \times 4\) and \(\beta\) is the Dirac matrix \(\gamma^0\). Using the following reparametrization \[ \hat{\zeta} = (m\omega)^{\frac{1}{2}} \hat{z}, \] (9a)
\[ \hat{p}_\zeta = (m\omega)^{-\frac{1}{2}} \hat{p}_z, \] (9b)
it is possible to write the last equation as
\[ \left( \hat{p}_\zeta^2 + \zeta^2 \right) |\psi\rangle = \eta_\pm |\psi\rangle, \] (10)
where
\[ \eta_\pm = \frac{E^2 - m^2}{m\omega} \pm 1. \] (11)
We note that the equations (7a) and (7b), for the states \(|\phi\rangle\) and \(|\chi\rangle\), allow to the equation (10) that has the form of a harmonic oscillator equation. This result suggests us that we can introduce the creation and annihilation operators given by
\[ \hat{a}^\dagger = \frac{1}{\sqrt{2}} \left( \zeta - i\hat{p}_\zeta \right), \quad \hat{a} = \frac{1}{\sqrt{2}} \left( \zeta + i\hat{p}_\zeta \right), \] (12)
with the purpose to obtain the eigenvalues of the system. In this way, using these operators, the
equations (7a) and (7b) have the form

\[
|\phi\rangle = i \sqrt{\frac{2m\omega}{E-m}} \sigma_3 a^\dagger |\chi\rangle, \\
|\chi\rangle = -i \sqrt{\frac{2m\omega}{E+m}} \sigma_3 |\phi\rangle.
\]  

(13a)

(13b)

Substituting again (13a) into (13b), it is possible to find the following equation for the state $|\phi\rangle$

\[
|\phi\rangle = \frac{2m\omega}{E_n^2 - m^2} \hat{N} |\phi\rangle,
\]

(14)

while the state $|\chi\rangle$ satisfies

\[
|\chi\rangle = \frac{2m\omega}{E_{n'+1}^2 - m^2} (\hat{N} + 1) |\chi\rangle,
\]

(15)

where $\hat{N} = a^\dagger a$ is the occupation number operator of the state. Since $\hat{N}$ determines the energy
level of the state where the particle (antiparticle) is, then the energy spectrum can be deduced
from (14) and (15). Therefore we have

- From (13a), the energy spectrum for positive energy states is

\[
E_n^2 = 2nm\omega + m^2.
\]

(16)

- From (13b), the energy spectrum for negative energy states is

\[
E_{n'+1}^2 = 2(n' + 1)m\omega + m^2.
\]

(17)

These energy spectrums for fermions and antifermions can be written simultaneously if we impose
the following quantum number condition

\[
n' = n - 1, \quad \text{with } n \neq 0,
\]

(18)

thus the total spectrum can be written as

\[
E_n = \pm \sqrt{(2|n|m\omega + m^2)^2}, \quad \text{con } n \in \mathbb{Z}.
\]

(19)

The upper sign in this expression is taken for $n \geq 0$, while the lower sign is for $n < 0$, i. e. the negative quantum numbers correspond to the negative
energy states. We observe from (19), that the lower positive energy state whose energy value is $m$
corresponds to the state with $n = 0$, while the greater negative energy state whose energy value is
\[-(2m\omega + m^2)^{\frac{1}{2}}\text{ corresponds to the state } n = -1.\] Additionally, we observe that if \( \omega \ll m \), then the energy difference between the states \( \phi_0 \) and \( \chi_{-1} \) is \( \Delta E = 2m + \omega \). For \( \omega = 0 \), i.e. the harmonic potential vanishes into eq. (4), then \( \Delta E = 2m \), which is a well known result obtained from the Dirac equation in the free case.

By using the previous results, the states of the system can be written as

\[
|\psi_n\rangle = \begin{pmatrix} |\phi_n\rangle \\ |\chi_n\rangle \end{pmatrix}, \quad (20)
\]

where the quantum number \( n \), which is an integer number, can describe positive and negative energy states. Using the expression (13b), we can write that

\[
|\psi_n\rangle = \begin{pmatrix} |\phi_n\rangle \\ -i \sqrt{\frac{2m\omega}{E+m}} \sigma_3 \hat{a} |\phi_n\rangle \end{pmatrix}. \quad (21)
\]

If we apply the occupation number operator \( \hat{N} \) over the state of a system described by \( |\psi_n\rangle \), we obtain

\[
\hat{N} |\psi_n\rangle = \begin{pmatrix} |n| |\phi_n\rangle \\ (|n| - 1) |\chi_n\rangle \end{pmatrix}, \quad (22)
\]

where we have assumed that the state \( |\phi_n\rangle \) has an occupation number given by \( |n| \) and where we have used the relation (13b) and the properties of the creation and annihilation operators. For the last expression, we realize that the lowest spinor \( |\chi_n\rangle \) has associated the occupation number given by \( |n| - 1 \). Thus the states \( |\phi_n\rangle \) and \( |\chi_n\rangle \) can be written as

\[
|\phi_n\rangle = |n\rangle \xi_1^n, \quad \xi_1^n \quad (23a)
\]

\[
|\chi_n\rangle = |n-1\rangle \xi_2^n, \quad (23b)
\]

where \( \xi_1^n, \xi_2^n \) represent the two-component spinors and \( |n\rangle \) represents a state with occupation number \( |n| \). In consequence, the states of the system are rewritten as

\[
|\psi_n\rangle = \begin{pmatrix} |n\rangle \xi_1^n \\ (|n| - 1) \xi_2^n \end{pmatrix}. \quad (24)
\]

Taking into account that the creation \( \hat{a}^\dagger \) and annihilation \( \hat{a} \) operators satisfy that \( \hat{a}^\dagger |n\rangle = \sqrt{|n| + 1} \ |n+1\rangle, \hat{a} |n\rangle = \sqrt{|n|} \ |n-1\rangle \), then these operators acting on the state \( |\psi_n\rangle \) allow to
\[ \hat{a}^\dagger |\psi_n\rangle = \begin{pmatrix} \sqrt{|n|+1} & |n+1\rangle \xi_{n+1}^1 \\ \sqrt{|n|} & |n\rangle \xi_{n+1}^2 \end{pmatrix}, \text{ for } n \neq -1, \quad (25a) \]
\[ \hat{a} |\psi_n\rangle = \begin{pmatrix} \sqrt{|n|} & |n-1\rangle \xi_{n-1}^1 \\ \sqrt{|n-1|} & |n-2\rangle \xi_{n-1}^2 \end{pmatrix}, \text{ for } n \neq 0, \quad (25b) \]

whereas these operators acting on the states \( |\psi_0\rangle \) and \( |\psi_{-1}\rangle \), which have associate the occupation numbers \( n = 0, -1 \), respectively, allow to

\[ \hat{a} |\psi_0\rangle = \frac{1}{2} (1 - \beta) |\psi_{-1}\rangle, \quad (26a) \]
\[ \hat{a}^\dagger |\psi_{-1}\rangle = \sqrt{2} |\psi_0\rangle. \quad (26b) \]

In figure 1 we have schematically represented the action of the creation and annihilation operators on the positive and negative energy states. We observe that the effect of the annihilation operator on the state \( |\psi_0\rangle \), which corresponds to the lowest positive energy state, is such that it does not annihilate the state but it drives it to the state \( |\psi_{-1}\rangle \), which corresponds to the greater negative energy state. Likewise, the effect of applying the creation operator on the state \( |\psi_{-1}\rangle \) is such that it does not annihilate that state, but drives it to the state \( |\psi_0\rangle \). Therefore, we observe that the appearing of the negative energy states generates the well known problem of the Dirac theory: a minimal energy state does not exist, then it is possible to obtain an infinite energy amount from this system. In order to give a solution to this problem, it is necessary to introduce the Dirac’s sea picture for the Dirac oscillator which will be performed by means of the canonical quantization for this system.

After calculating the energy spectrum of the one-dimensional Dirac oscillator, we proceed to obtain the wave functions. We substitute (5) into (10), then we obtain the following differential equation for the wave function associated to the bispinor \( |\phi\rangle \)

\[ \left[ \frac{d^2}{d\zeta^2} + (\eta_+ - \zeta^2) \right] \phi(\zeta) = 0, \quad (27) \]

where we have used the coordinate representation of the wave function given by \( \phi(\zeta) = \langle \zeta | |\phi\rangle \).

The differential equation (27) corresponds to the one of a relativistic harmonic oscillator, whose solution is

\[ \phi_n(\zeta) = N_{|n|} H_{|n|}(\zeta)e^{-\frac{\zeta^2}{2}} \xi_{n}^1. \quad (28) \]
Figure 1: Energy spectrum of the one-dimensional Dirac oscillator and the action of the creation and annihilation operator on some system states.

Likewise the solution to the differential equation associated to the wave function \( \chi_n(\zeta) = \langle \zeta | \chi_n \rangle \) has the form

\[
\chi_n(\zeta) = N_{|n| - 1} H_{|n| - 1}(\zeta) e^{-\frac{\zeta^2}{2}} \xi_n^2.
\] (29)

In the expressions for \( \phi_n(\zeta) \) and \( \chi_n(\zeta) \), \( H_n(\zeta) \) represent the Hermite polynomials. Now, from the expression (13b) we can obtain the following relation between the spinors \( \xi_n^1 \) and \( \xi_n^2 \)

\[
\xi_n^2 = -i \sqrt{\frac{E_n - m}{E_n + m}} \sigma_3 \xi_n^1,
\] (30)

where we have used the properties of the creation and annihilation operators and the definitions given by (23). Now it is possible to write

\[
\xi_n^1 = \left( \begin{array}{c} \sqrt{\frac{E_n + m}{2E_n}} \\ 0 \end{array} \right),
\]
(31a)

\[
\xi_n^2 = \left( \begin{array}{c} -i \sqrt{\frac{E_n - m}{2E_n}} \\ 0 \end{array} \right).
\]
(31b)

We observe that the spinor \( \xi_n^2 \) is annihilated for the case \( n = 0 \), which implies \( |\chi_0\rangle \equiv 0 \). So the most general solution for the one-dimensional Dirac oscillator equation (4) is given by

\[
\psi_n(z,t) = \sqrt{m\omega} \left( \begin{array}{c} \phi_n(z) \xi_n^1 \\ \chi_n(z) \xi_n^2 \end{array} \right) e^{-iE_nt},
\] (32)
where the normalization for the wave functions has been performed and \( n = 0, \pm 1, \pm 2, \ldots \). Finally, we can write the one-dimensional Dirac oscillator equation (4) in its explicit covariant form \([7, 14]\)

\[
(i\gamma^\mu \partial_\mu - m + \sigma^{\mu\nu} F_{\mu\nu})\psi = 0,
\]

where \( \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \) and the field-strength tensor \( F_{\mu\nu} \) associated to the harmonic interaction presents in this system is given by

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,
\]

where we have defined the fourpotential associated to the interaction as

\[
A_\mu = \frac{1}{4} (2(u \cdot x)x_\mu - x^2 u_\mu).
\]

In the last expression \( u_\mu = (m\omega, \vec{0}) \) is a fourvector that depends on the reference frame. In the references \([7, 14]\), two different physical pictures for the interaction of the Dirac oscillator have been considered.

B. Canonical quantization procedure in (1+1) dimensions

By means of the previously defined quantities, we introduce the Lagrangian density of the system as

\[
\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \bar{\psi}\sigma^{\mu\nu} \psi F_{\mu\nu},
\]

which, by using the Euler-Lagrange motion equation, gives the one-dimensional Dirac oscillator equation (33). As the Lagrangian density has an explicit dependence on the position coordinate \( x \), it is not possible to determine the energy-momentum tensor of the system \([33]\). However, it can be found that the Hamiltonian density is

\[
\mathcal{H} = \bar{\psi}(\alpha_3 \cdot (\partial_z + m\omega \beta z) + \beta m)\psi.
\]

To follow the standard canonical quantization procedure \([31]\), we initially impose the following commutation relations of the Jordan-Wigner type for the fermion fields

\[
\{ \hat{\psi}_\alpha(z, t), \hat{\psi}^\dagger_\beta(z', t) \} = \delta_{\alpha\beta}\delta(z - z'),
\]

\[
\{ \hat{\psi}_\alpha(z, t), \hat{\psi}_\beta(z', t) \} = 0,
\]

\[
\{ \hat{\psi}^\dagger_\alpha(z, t), \hat{\psi}^\dagger_\beta(z', t) \} = 0.
\]
Using these relations, we obtain the Hamiltonian operator for the Dirac oscillator field operator $\hat{\psi}$ in the form

$$\hat{H} = \int dz \, \hat{\psi}^\dagger (-i\alpha_3 \cdot (\partial_z + m\omega \beta z) + \beta m) \hat{\psi}. \quad (41)$$

Considering that the Heisenberg equation for the field operator $\hat{\psi}$ is given by $i\dot{\hat{\psi}}(z,t) = -i[\hat{\psi}(z,t), \hat{H}]$, then we can obtain that the motion equation for the Dirac oscillator field operator is

$$i\dot{\hat{\psi}}(z,t) = (-i\alpha_3 \cdot (\partial_z + m\omega \beta z) + \beta m) \hat{\psi}(z,t). \quad (42)$$

Now we write the Dirac oscillator field operator using the wave functions of the Dirac oscillator (32) as the base of the expansion. These wave functions are written in terms of the Hermite polynomials which represent a complete set of orthonormal polynomials. Thus, the Fourier series expansion for the Dirac oscillator field operator can be written as

$$\hat{\psi}(z,t) = \sum_{n=-\infty}^{\infty} \hat{b}_n \psi_n(z,t) e^{-iE_n t}$$

where the positive and negative energy contributions have been separated $\hat{\psi}_\pm(z,t)$ and the spinors $u_n(z)$ and $\nu_n(z)$ have been defined as

$$u_n(z) = \begin{pmatrix} \phi_n(z) \xi_n^1 \\ \chi_n(z) \xi_n^2 \end{pmatrix}, \quad \nu_n(z) = \begin{pmatrix} \phi_{-n}(z) \xi_{-n}^1 \\ \chi_{-n}(z) \xi_{-n}^2 \end{pmatrix}. \quad (44)$$

If we make use of the anticommutation relations for the fields (38), (39) and (40), we can verify that creation $\hat{b}_n^\dagger$ and annihilation $\hat{b}_n$ operators of positive (for $n \geq 0$) and negative (for $n < 0$) energy particles satisfy the following anticommutation relations

$$\left\{ \hat{b}_n, \hat{b}_m^\dagger \right\} = \delta_{n,m}, \quad (45)$$

$$\left\{ \hat{b}_n, \hat{b}_m \right\} = 0, \quad (46)$$

$$\left\{ \hat{b}_n^\dagger, \hat{b}_m^\dagger \right\} = 0. \quad (47)$$

Using these anticommutation relations, we can find that the Hamiltonian operator is now given by

$$\hat{H} = \sum_{n=-\infty}^{\infty} E_n \hat{b}_n^\dagger \hat{b}_n$$

$$= \sum_{n=0}^{\infty} E_n \hat{b}_n^\dagger \hat{b}_n - \sum_{n=1}^{\infty} E_n \hat{b}_{-n}^\dagger \hat{b}_{-n}, \quad (48)$$
where we have also used the expansion of the field operator \((43)\) and the properties of the Hermite polynomials. We can observe that the eigenvalues of this Hamiltonian operator can take negative values without restriction at all, because it is possible the creation of negative energy particles. To solve this problem we take the Dirac’s sea picture for this system, so we impose that all the negative energy states are occupied by negative energy particles. This configuration is called the vacuum state of the Dirac oscillator field \(|0\rangle\) that is written as \([34]\)

\[
|0\rangle = \prod_{n=1}^{\infty} \hat{b}_{-n}^\dagger |0_D\rangle,
\]

where \(|0_D\rangle\) is the Dirac vacuum state which is characterized for not been occupied by fermions of positive or negative energy. We observe that if we apply the creation operator of negative energy particle \(\hat{b}_{-m}^\dagger |0\rangle\) on the vacuum state \(|0\rangle\) we obtain

\[
\hat{b}_{-m}^\dagger |0\rangle = \prod_{n=1}^{\infty} \hat{b}_{-m}^\dagger \hat{b}_{-n}^\dagger |0_D\rangle = 0,
\]

which implies that it is not possible to create a new negative energy fermion because all the negative energy states are occupied. On the other hand, it is possible that a negative energy state can be annihilated and then a hole can be originated in the Dirac’s sea. This hole represents an antiparticle with negative energy. If we use the anticommutation relations \([45], [46] and [47]\) into \((48)\), we find that the Hamiltonian operator takes the form

\[
\hat{H} = \sum_{n=0}^{\infty} E_n \hat{b}_n^\dagger \hat{b}_n + \sum_{n=1}^{\infty} E_n \left( \hat{b}_{-n} \hat{b}_{-n}^\dagger - 1 \right).
\]

If the divergent energy of the vacuum \(\sum_{n=1}^{\infty} E_n\) is subtracted, the Hamiltonian operator can be reduced to

\[
\hat{H} = \sum_{n=0}^{\infty} E_n \hat{b}_n^\dagger \hat{b}_n + \sum_{n=1}^{\infty} E_n \hat{b}_{-n} \hat{b}_{-n}^\dagger.
\]

We note that now this Hamiltonian operator does not have the problem observed before because it has been positively defined. Finally, we perform the following canonical transformations to clearly differentiate between the operators associated to particles and antiparticles \([31]\)

\[
\begin{align*}
\hat{b}_n^\dagger &= \hat{c}_n^\dagger, \\
\hat{b}_n &= \hat{c}_n, \\
\hat{b}_{-n} &= \hat{d}_n^\dagger, \\
\hat{b}_{-n}^\dagger &= \hat{d}_n,
\end{align*}
\]

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where it is possible to identify $\hat{d}_n$ and $\hat{d}^\dagger_n$ as the annihilation and the creation operators of antiparticles, respectively. These operators satisfy the anticommutation relation $\{\hat{d}_n, \hat{d}^\dagger_m\} = \delta_{n,m}$. These transformations imply that now the application of the annihilation operator of negative energy particle on the vacuum state can be understood as the application of a creation operator of antiparticle. Using this notation, the Fourier series expansion for the Dirac oscillator field (43) has now the form

$$\hat{\psi}(z,t) = \sum_{n=0}^{\infty} \hat{c}_n u_n(z) e^{-iE_n t} + \sum_{n=1}^{\infty} \hat{d}^\dagger_n v_n(z) e^{iE_n t}. \quad (54)$$

Now the Hamiltonian operator for the Dirac oscillator field can be written as

$$\hat{H} = \sum_{n=0}^{\infty} E_n \hat{c}^\dagger_n \hat{c}_n + \sum_{n=1}^{\infty} E_n \hat{d}^\dagger_n \hat{d}_n = \sum_{n=0}^{\infty} E_n \hat{n}^{(c)}_n + \sum_{n=1}^{\infty} E_n \hat{n}^{(d)}_n, \quad (55)$$

where $\hat{n}^{(c)}_n$ represents the number operator for fermions and $\hat{n}^{(d)}_n$ represents the number operator for antifermions. It is possible to observe that $\hat{c}^\dagger_n$ acting on $|0\rangle$ creates a fermion having energy $E_n$ and charge $+e$, whereas $\hat{d}^\dagger_n$ creates antifermion having energy $E_n$ and charge $-e$. In this scheme, an interacting relativistic massive fermion is described as an excitation from a degree of freedom of the Dirac oscillator quantum field.

For the Dirac oscillator field the energy quanta $E_n$ are the energies of linear harmonic oscillators. We observe that, in this sense, the Hamiltonian operator (55) is analogous to the Hamiltonian operator for the free Dirac field in three dimensions given by (3).

Using the Fourier expansion (54) it is possible to determine other relevant physical quantities. For instance, the charge operator $\hat{Q} = e \int dz \hat{\psi}^\dagger \hat{\psi}$ can be written as

$$\hat{Q} = e \sum_{n=0}^{\infty} \left( \hat{b}^\dagger_n \hat{b}_n - \hat{c}^\dagger_n \hat{c}_n \right), \quad (56)$$

where the non-observable charge of the vacuum has been removed. Likewise, the momentum operator $\hat{P} = -i \int dz \hat{\psi}^\dagger \nabla \hat{\psi}$ can be also written in an explicit form using the expansion of the Dirac oscillator field.

The Feynman propagator for the Dirac oscillator field in the coordinate space is defined as

$$iS^F_{\alpha\beta}(z - z', t - t') = \langle 0 | \hat{T} \left( \hat{\psi}_\alpha(z, t) \hat{\psi}_\beta^\dagger(z', t') \right) | 0 \rangle. \quad (57)$$

Substituting (43) into (57) and taking into account the definition of the time-ordered operator $\hat{T}$,
we obtain
\begin{align*}
    iS^F_{\alpha\beta}(z-z',t-t') = \Theta(t-t') \sum_{n=0}^{\infty} u_{\alpha,n}(z) \bar{u}_{\beta,n}(z') e^{-iE_n(t-t')}
    & - \Theta(t'-t) \sum_{n=1}^{\infty} \nu_{\alpha,n}(z) \bar{\nu}_{\beta,n}(z') e^{iE_n(t-t')}, \quad (58)
\end{align*}

Now we will obtain the Feynman propagator in the momenta space. To do it, we consider firstly that the Fourier transformation for the Hermite polynomials can be written as
\begin{align*}
    \mathcal{F}\left\{ e^{-x^2/2} H_n(x) \right\} = (-i)^n e^{-k^2/2} H_n(k), \quad (59)
\end{align*}

and secondly, that the contour integral for the energy eigenvalues is written as
\begin{align*}
    i \oint_C \frac{dp_o}{2\pi} e^{-ip_o(t-t')} = \Theta(t-t') \frac{e^{-iE_n(t-t')}}{2E_n} + \Theta(t'-t) \frac{e^{iE_n(t-t')}}{2E_n}, \quad (60)
\end{align*}

where \( p_n^2 = 2|n|m_\omega + m^2 \) and \( E_n \) is given by (19). Then we find that
\begin{align*}
    S^F_{\alpha\beta}(z-z',t-t') = \oint_C \frac{dp_o}{2\pi} \frac{dp_z}{2\pi} \frac{dp_{z'}}{2\pi} S^F_{\alpha\beta}(p_o,p_z,p_{z'}) e^{-i(p_z z + p_{z'} z')}, \quad (61)
\end{align*}

where the Feynman propagator in the momenta space for the one-dimensional Dirac oscillator field is given by
\begin{align*}
    S^F_{\alpha\beta}(p_o,p_z,p_{z'}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{p_o^2 - p_n^2} \left[ u_{\alpha,n}(p_z) \bar{u}_{\beta,n}(p_{z'}) - \nu_{\alpha,n-1}(p_z) \bar{\nu}_{\beta,n-1}(p_{z'}) \right]. \quad (62)
\end{align*}

This result is in agreement with the presented in [21], where the Feynman propagator of the one-dimensional Dirac oscillator was obtained using the path integral formalism and working with a different representation of the Dirac matrices.

IV. DIRAC OSCILLATOR IN THE (3+1) DIMENSIONAL CASE

In this section we will consider the Dirac oscillator in (3+1) dimensions. Initially we will obtain the energy spectrum and the wave functions describing the quantum states of this system. Next we develop the canonical quantization procedure for the Dirac oscillator field in the 3+1 dimensional case.

A. Spectrum and wave functions in (3+1) dimensions

In order to obtain the Dirac oscillator equation in the (3+1) dimensional case, we perform the following substitution which is analogous to the minimal substitution
\begin{align*}
    \mathbf{p} \to \mathbf{p} - i m_\omega \gamma^0 \mathbf{r}, \quad (63)
\end{align*}
where $\mathbf{\hat{p}} = (\mathbf{\hat{p}}_x, \mathbf{\hat{p}}_y, \mathbf{\hat{p}}_z)$ and $\mathbf{\hat{r}} = (\mathbf{\hat{x}}, \mathbf{\hat{y}}, \mathbf{\hat{z}})$. The Dirac oscillator equation has the form

$$i \frac{\partial}{\partial t} |\psi\rangle = (\alpha \cdot (\mathbf{\hat{p}} - im\omega \gamma_0 \mathbf{\hat{r}}) + \beta m) |\psi\rangle.$$  \hspace{1cm} (64)

The solutions of the Dirac oscillator equations are described by the spinor states $|\psi\rangle$, that can be written as

$$|\psi\rangle = \begin{pmatrix} |\varphi\rangle \\ |\chi\rangle \end{pmatrix},$$  \hspace{1cm} (65)

where $|\varphi\rangle$ and $|\chi\rangle$ are bispinors. The bispinors obey the following equations

$$\mathbf{\hat{p}}^2 + (m\omega)^2 \mathbf{\hat{r}}^2 |\varphi\rangle = (E^2 - m^2 + (3 + 4\mathbf{s} \cdot \mathbf{\hat{L}})m\omega) |\varphi\rangle,$$  \hspace{1cm} (66a)

$$\mathbf{\hat{p}}^2 + (m\omega)^2 \mathbf{\hat{r}}^2 |\chi\rangle = (E^2 - m^2 - (3 + 4\mathbf{s} \cdot \mathbf{\hat{L}})m\omega) |\chi\rangle.$$  \hspace{1cm} (66b)

From these equations, it is evident that the Dirac oscillator in the (3+1) dimensional case presents a strong spin-orbit coupling term \cite{7, 14}. It has been demonstrated that the Dirac oscillator is a system where the angular momentum is conserved \cite{7, 14}, thus the quantum numbers of full angular momentum $j$ and parity are good quantum numbers. Therefore, it is convenient to separate the energy eigenfunctions in two parts: radial and angular. The spatial coordinate representation of the states (65) allows to the spinor wave functions given by

$$\psi_{n,\kappa,g}(\mathbf{r}) = \frac{1}{r} \begin{pmatrix} F_{n,\kappa}(r) \mathcal{Y}_{\kappa,g}(\theta, \phi) \\ iG_{n,\kappa}(r) \mathcal{Y}_{-\kappa,g}(\theta, \phi) \end{pmatrix},$$  \hspace{1cm} (67)

where $n$ is the principal quantum number, $\kappa$ is a quantum number related with the angular momentum and parity, and $g$ is a quantum number related to the projection of the angular momentum in the $z$ axis. The quantum number $\kappa$ is defined as \cite{35}

$$\kappa = \mp \left(j + \frac{1}{2}\right) = \begin{cases} -(l + 1), & \text{if } j = l + \frac{1}{2}, \\ l, & \text{if } j = l - \frac{1}{2}. \end{cases} \hspace{1cm} (68)$$

The full angular momentum $j$ takes values $j = l \pm \frac{1}{2}$, where $l$ is the angular momentum and $\frac{1}{2}$ is the spin of the fermion. The angular momentum $l'$ associated to the upper and lower components is \cite{35}

$$l' = 2j - l = \begin{cases} l + 1, & \text{if } j = l + \frac{1}{2}, \\ l - 1, & \text{if } j = l - \frac{1}{2}. \end{cases} \hspace{1cm} (69)$$

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The spinorial spherical harmonics $Y_{\kappa,g}(\theta, \phi)$ and $Y_{-\kappa,g}(\theta, \phi)$ are given by \[35\]

$$
Y_{\kappa,g}(\theta, \phi) = \begin{pmatrix}
\sqrt{\frac{j+g}{2j}} Y_{l,g}^{-\frac{1}{2}}(\theta, \phi) \\
\sqrt{\frac{j-g}{2j}} Y_{l,g}^{\frac{1}{2}}(\theta, \phi)
\end{pmatrix},
$$

(70a)

$$
Y_{-\kappa,g}(\theta, \phi) = \begin{pmatrix}
-\sqrt{\frac{j-g+1}{2j+2}} Y_{l,g}^{\frac{1}{2}}(\theta, \phi) \\
\sqrt{\frac{j+g+1}{2j+2}} Y_{l,g}^{-\frac{1}{2}}(\theta, \phi)
\end{pmatrix}.
$$

(70b)

Substituting (67) into the Dirac equation (64) represented in coordinate space, we can find the following coupled equation system for the radial functions $F_{n,\kappa}(r)$ and $G_{n,\kappa}(r)$ \[15\]

$$
\left[ \frac{d}{dr} + \kappa + \frac{m\omega r^2}{r} \right] F_{n,\kappa}(r) = (E + m) G_{n,\kappa}(r),
$$

(71a)

$$
\left[ -\frac{d}{dr} + \kappa + \frac{m\omega r^2}{r} \right] G_{n,\kappa}(r) = (E - m) F_{n,\kappa}(r).
$$

(71b)

The solutions of this equation system are \[7, 14, 15\]

$$
F_{n,\kappa}(r) = A \left[ \sqrt{m\omega} \right]^{l+1} e^{-\frac{m\omega r^2}{2}} L_{n}^{l+\frac{1}{2}}(m\omega r^2),
$$

(72a)

$$
G_{n,\kappa}(r) = \pm \text{sgn}(\kappa) A' \left[ \sqrt{m\omega} \right]^{l'+1} e^{-\frac{m\omega r^2}{2}} L_{n'}^{l'+\frac{1}{2}}(m\omega r^2),
$$

(72b)

where $L_{n}^{l}(x)$ are the Laguerre associated polynomials and $A$, $A'$ are normalization constants given by

$$
A = \left[ \sqrt{m\omega} \frac{|n|!(E_{n,\kappa} + m)}{\Gamma(|n| + l + 3/2)E_{n,\kappa}} \right]^{\frac{1}{2}},
$$

(73a)

$$
A' = \left[ \sqrt{m\omega} \frac{|n'|!(E_{n,\kappa} - m)}{\Gamma(|n'| + l' + 3/2)E_{n,\kappa}} \right]^{\frac{1}{2}},
$$

(73b)

where the quantum number $|n'|$ takes the following values

$$
|n'| = \begin{cases} 
|n| - 1, & \text{for } \kappa < 0 \\
|n|, & \text{for } \kappa > 0 
\end{cases}.
$$

(74)

The radial functions $F(r)$ and $G(r)$ obtained here have the same radial structure as the one associated to the non-relativistic three-dimensional harmonic oscillator. On the other hand, the energy spectrum for this case depends on the $\kappa$ value. It is possible obtain that the energy eigenvalues are \[15\]

- For $\kappa < 0$

$$
E_{n,\kappa} = \pm \sqrt{m^2 + 4|n|m\omega},
$$

(75)

where the quantum number $n$ can take the values $n = 0, \pm 1, \pm 2, \ldots$, and the positive sign is chosen for $n \geq 0$, meanwhile the negative sign is chosen for $n < 0$. 18
\( E_{n,\kappa} = \pm \sqrt{m^2 + 4 \left( |n| + l + \frac{1}{2} \right) m\omega}, \) \hspace{1cm} (76)

where \( n = \pm 0, \pm 1, \pm 2, \ldots \) and the sign is chosen as was mentioned in the before item. Moreover, the energy spectrum satisfies the following symmetry condition \( E_{-n,\kappa} = -E_{n,\kappa}, \) except for \( n = 0 \) which means \( \kappa < 0. \) For this case, it is necessary to differentiate between the quantum numbers \( n = +0 \) and \( n = -0. \)

With the purpose of implementing the canonical quantization procedure for the Dirac oscillator in (3+1) dimension, we observe that the spinorial wave functions \( \psi_{\kappa,g} \) satisfy the following orthonormality and completeness relations \[ (77a) \]

\[
\sum_{\kappa = -\infty}^{\infty} \sum_{g = -|\kappa| + \frac{1}{2}}^{\infty} \sum_{n = -\infty}^{\infty} \psi_{n,\kappa,g}(\vec{r}) \psi_{n,\kappa,g}^\dagger(\vec{r}') = \delta^3(\vec{r} - \vec{r}') I_4,
\]

\[
\int d^3r \psi_{n,\kappa,g}^\dagger(\vec{r}) \psi_{n',\kappa',g'}(\vec{r}') = \delta_{n,n'} \delta_{\kappa,\kappa'} \delta_{g,g'},
\]

respectively. Finally, we note that the covariant Dirac oscillator equation for the (3+1) dimensional case has the same form as the equation \[ (33) \] for the (1+1) dimensional case.

**B. Canonical quantization in (3+1) dimensions**

To proceed with the canonical quantization for the Dirac oscillator field in the (3+1) dimensional case, we observe that the form of the Lagrangian density for this case is similar to the one in the (1+1) dimensional case and it is given by

\[
\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m) \psi + \bar{\psi} \sigma^{\mu\nu} \psi F_{\mu\nu}.
\] \hspace{1cm} (78)

As in the (1+1) dimensional case, the Lagrangian density depends explicitly on the spatial coordinates and then it is not possible to obtain the energy-momentum tensor of the system. However, by means of a Legendre transformation, we can obtain the following Hamiltonian density \( \mathcal{H} = \bar{\psi}(-i\alpha \cdot (\vec{\nabla} + m\omega \beta \vec{r}) + \beta m)\psi. \) Thus, the Hamiltonian of the field is

\[
H = \int d^3r \bar{\psi}(i\alpha \cdot (\vec{\nabla} + m\omega \beta \vec{r}) + \beta m)\psi.
\] \hspace{1cm} (79)
To perform the canonical quantization for this case, as is usual, the fields are considered as field operators. Then we establish the following Jordan-Wigner commutation relations

\[
\begin{aligned}
\{ \hat{\psi}_\alpha(\vec{r}, t), \hat{\psi}_\beta^\dagger(\vec{r}', t) \} &= \delta_{\alpha\beta}\delta^3(\vec{r} - \vec{r}'), \\
\{ \hat{\psi}_\alpha(\vec{r}, t), \hat{\psi}_\beta(\vec{r}', t) \} &= 0, \\
\{ \hat{\psi}_\alpha^\dagger(\vec{r}, t), \hat{\psi}_\beta^\dagger(\vec{r}', t) \} &= 0,
\end{aligned}
\]

that allow us to write the Hamiltonian operator as

\[
\hat{H} = \int d^3r \hat{\psi}^\dagger(\vec{r}, t) \left( -i\hat{\alpha} \cdot (\vec{\nabla} + m\omega\beta\vec{r}) + \beta m \right) \hat{\psi}.
\]

Now we can expand the field operator \( \hat{\psi}(\vec{r}, t) \), using the spinor wave functions \( [63] \), as follows

\[
\hat{\psi}(\vec{r}, t) = \sum_{\kappa=-\infty}^{\infty} \sum_{\kappa \neq 0} \sum_{g=-|\kappa|+\frac{1}{2}}^{\infty} \sum_{n=-\infty}^{\infty} \hat{b}_{n,\kappa,g} \psi_{n,\kappa,g}(\vec{r}) e^{-iE_{n,\kappa}t},
\]

where the operators \( \hat{b}_{n,\kappa,g} \) and \( \hat{b}_{n,\kappa,g}^\dagger \), respectively, annihilates and creates fermions in a state defined by the quantum numbers \( n, \kappa \) and \( g \). These operators obey the following anticommutation relations

\[
\begin{aligned}
\{ \hat{b}_{n,\kappa,g}, \hat{b}_{n',\kappa',g'}^\dagger \} &= \delta_{n,n'}\delta_{\kappa,\kappa'}\delta_{g,g'}, \\
\{ \hat{b}_{n,\kappa,g}, \hat{b}_{n',\kappa',g'} \} &= 0, \\
\{ \hat{b}_{n,\kappa,g}^\dagger, \hat{b}_{n',\kappa',g'}^\dagger \} &= 0,
\end{aligned}
\]

Using the properties of the energy eigenfunctions \( (77a) \) and \( (77b) \) into the Hamiltonian operator \( [84] \), we can obtain

\[
\hat{H} = \sum_{\kappa=-\infty}^{\infty} \sum_{\kappa \neq 0} \sum_{g=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} E_{n,\kappa} \hat{b}_{n,\kappa,g}^\dagger \hat{b}_{n,\kappa,g}.
\]

This operator can be rewritten by splitting the positive and negative energy contributions and taking into account the two types of spectrum depending on the \( \kappa \) value. In this way, we obtain

\[
\hat{H} = \sum_{\kappa=1}^{\infty} \sum_{g=0}^{\infty} \sum_{n=0}^{\infty} E_{n,\kappa} \hat{b}_{n,\kappa,g}^\dagger \hat{b}_{n,\kappa,g} - \sum_{\kappa=1}^{\infty} \sum_{g=0}^{\infty} \sum_{n=0}^{\infty} E_{n,\kappa} \hat{b}_{n,\kappa,g}^\dagger \hat{b}_{n,\kappa,g} + \sum_{\kappa=-\infty}^{\infty} \sum_{g=0}^{\infty} \sum_{n=0}^{\infty} E_{n,-\kappa} \hat{b}_{n,-\kappa,g}^\dagger \hat{b}_{n,-\kappa,g} - \sum_{\kappa=-\infty}^{\infty} \sum_{g=0}^{\infty} \sum_{n=0}^{\infty} E_{n,-\kappa} \hat{b}_{n,-\kappa,g}^\dagger \hat{b}_{n,-\kappa,g},
\]

where we have used the convention \( E_{n,-\kappa} \equiv E_{n,\kappa} \), for \( \kappa < 0 \), and we have omitted the limits of the sum over \( g \). It is evident that this Hamiltonian is not defined positively as happened in the \( (1+1) \)
dimensional case. Again we use the picture of the Dirac’s sea in order to solve this problem. To do it we take the vacuum state in an analogous way as it was defined in (49)

\[ |0\rangle = \prod_{\kappa=-\infty}^{\infty} \prod_{g} \prod_{n=0}^{\infty} \hat{b}_{n,\kappa,g}^\dagger |0_D\rangle. \]  

(90)

Therefore we can rewrite the Hamiltonian operator without considering the vacuum energy in the following way

\[ \hat{H}' = \sum_{\kappa=1}^{\infty} \sum_{g} \sum_{n=0}^{\infty} E_{n,\kappa} \hat{b}_{n,\kappa,g}^\dagger \hat{b}_{n,\kappa,g} + \sum_{\kappa=1}^{\infty} \sum_{g} \sum_{n=0}^{\infty} E_{n,\kappa} \hat{b}_{-n,-\kappa,g}^\dagger \hat{b}_{-n,-\kappa,g} 
+ \sum_{\kappa=1}^{\infty} \sum_{g} \sum_{n=0}^{\infty} E_{n,-\kappa} \hat{b}_{n,-\kappa,g}^\dagger \hat{b}_{n,-\kappa,g} \sum_{\kappa=1}^{\infty} \sum_{g} \sum_{n=0}^{\infty} E_{n,-\kappa} \hat{b}_{-n,-\kappa,g}^\dagger \hat{b}_{-n,-\kappa,g}. \]  

(91)

We observe that this Hamiltonian operator describes two different types of particles and antiparticles, because particles and antiparticles have different energy spectrums depending on the sign of the \( \kappa \) value. This value depends explicitly on the full angular momentum \( j \) due to the value of \( j \) is based on the spin state of the fermion. In this way, we perform the following canonical transformations

\[ \hat{b}_{n,\kappa,g}^\dagger = \hat{b}_{n,\kappa,g}, \]  

(92a)

\[ \hat{b}_{-n,-\kappa,g} = \hat{c}_{n,\kappa,g}, \]  

(92b)

\[ \hat{b}_{n,-\kappa,g}^\dagger = \hat{d}_{n,\kappa,g}, \]  

(92c)

\[ \hat{b}_{-n,-\kappa,g}^\dagger = \hat{f}_{n,\kappa,g}. \]  

(92d)

where \( \hat{b}_{n,\kappa,g}^\dagger \) is the creation operator of particles with full angular momentum \( j = l - \frac{1}{2}, \) i. e. particles with \( \kappa > 0 \) \( \{32\} \); \( \hat{c}_{n,\kappa,g} \) is the creation operator of antiparticles with full angular momentum \( j = l - \frac{1}{2}, \) i. e. antiparticles with \( \kappa > 0 \) \( \{32\} \); \( \hat{d}_{n,\kappa,g} \) is the creation operator of particles with full angular momentum \( j = l + \frac{1}{2}, \) i. e. particles with \( \kappa < 0 \) \( \{32\} \); \( \hat{f}_{n,\kappa,g} \) is the creation operator of antiparticles with full angular momentum \( j = l + \frac{1}{2}, \) i. e. antiparticles with \( \kappa < 0 \) \( \{32\} \).

Finally, we can obtain that the field operator for this case can be written by means of the expansion

\[ \hat{\psi}(\vec{r},t) = \sum_{\kappa=1}^{\infty} \sum_{g} \sum_{n=0}^{\infty} \hat{b}_{n,\kappa,g} \psi_{n,\kappa,g}(\vec{r}) e^{-iE_{n,\kappa}t} + \sum_{\kappa=1}^{\infty} \sum_{g} \sum_{n=0}^{\infty} \hat{c}_{n,\kappa,g} \psi_{-n,-\kappa,g}(\vec{r}) e^{iE_{n,\kappa}t} 
+ \sum_{\kappa=1}^{\infty} \sum_{g} \sum_{n=0}^{\infty} \hat{d}_{n,\kappa,g} \psi_{n,-\kappa,g}(\vec{r}) e^{-iE_{n,-\kappa}t} + \sum_{\kappa=1}^{\infty} \sum_{g} \sum_{n=0}^{\infty} \hat{f}_{n,\kappa,g} \psi_{-n,-\kappa,g}(\vec{r}) e^{iE_{n,-\kappa}t}. \]  

(93)

In an analogous way as was showed for the (1+1) dimensional case, starting from the expansion for the field operator \( \{93\} \), it is possible to obtain the different relevant physical quantities associated to the Dirac oscillator field in (3+1) dimensions.
V. CONCLUSIONS

In this work we have performed the canonical quantization of the Dirac oscillator field in (1+1) and (3+1) dimensions. This quantization has been possible because the solutions of the Dirac oscillator equation do not present the Klein paradox [11]. If this fact would not have satisfy, then it had not been possible to distinguish between positive and negative energy states [32], which had restricted the possibility to perform a Fourier expansion of the field operator. The Dirac oscillator field has been quantized following a similar procedure as if this field were free [31]. However, this procedure implies differences with respect to the Dirac free field quantization. For instance, the Dirac oscillator is an interacting system in which there exist bound states define with specific quantum numbers [9]. In this case the field operators create and annihilate fermions with well determined energy values in contrast to what happens in the free field case where the states have a well defined momentum [22]. Moreover, for this case, the momentum operator depends explicitly on the time thus the fermions created do not have a determined momentum. We have found that for the Dirac oscillator field it has been impossible to obtain the energy-momentum tensor due to the fact that the Lagrangian density has an explicit dependence on the spatial coordinates. Nevertheless, this problem could be solved by the introduction in the system of an additional field which describes the interaction. We have also obtained the Feynman propagator in the (1+1) dimensional case in agreement with the result obtained in the literature by using an functional procedure [21]. We note that for the Dirac oscillator field in the (3+1) dimensional case, we have found that there exist two types of particles and antiparticles because there are two possible values for the full angular momentum. Finally, we have found that while for the free Dirac field the energy quanta of the infinite harmonic oscillators are relativistic energies of free particles, for the Dirac oscillator quantum field the quanta are energies of relativistic linear harmonic oscillators. The canonical quantization procedure for the Dirac oscillator field in (2+1) dimensions is an exercise to develop explicitly. We consider that the possibility to study the Dirac oscillator as a quantum field opens the doors to future applications in different areas of the Physics.

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