On the position representation of power, net force and torque operators

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In the present article we introduce the power, net force and torque as a Hermitian linear differential operators and discuss the value of all emerging relations. In addition to the energy-frequency and momentum-wavelength relations we obtain the power-frequency and net force-wavelength analogs, respectively. Representing the power and net force operators in the position space, we determine the relevant uncertainty and describe their correspondence with the Hamiltonian and momentum operator. The position representation of the torque operator and its relation to the orbital angular momentum operator is also considered. The results are grounded by the presence of a physical constant as fundamental as the Planck’s constant that is essential to the obtained representations.

I. INTRODUCTION

Quantum mechanics is considered as one of the greatest achievements in theoretical physics [1–3]. Describing subatomic phenomena governed by principles beyond those defining the macroscopic world, it would always raise discussions on a possible interdependence with classical mechanics [4–9]. Quantum mechanics is a ground for vast number of researches accounting for atomic and subatomic processes. Some prominent examples are the photon-electron interactions [10–12], the electrons’ interactions that underpin the magnetic properties of matter [13–18] and different transport phenomena [19–24]. Furthermore, determining the time response of the electrons [25] and the time that it takes for quantum jumps to occur [26] are really fascinating and promising achievements of quantum physics nowadays. From the perspective of present technological progress, the possibility of observing and controlling quantum effects such as tunneling [27–29] and entanglement [30–34] promises a number of future applications in the field of logical devices [35–40].

Although the foundations of quantum mechanics are well established and verified in time [41–44] the existence of Hermitian linear operators representing the power, net force and torque in coordinate space remains an open question. Respectively, the impact of such representations in all fields of quantum physics is still unexplored.

In this article we report fundamental relations and operator representations for the power, net force and torque. Working in accordance to the formalism of standard quantum mechanics, we obtain the Planck–Einstein and de Broglie relations in terms of the mechanical power and net force, respectively. We introduce a quantum relativistic expression for the power of a particle in ideal gas and for a particle in rest. Further, we discuss the position representation of power, net force and torque operators and establish the commutation relations they obey. The uncertainty in power and net force is also studied. A key role within all results plays a physical constant that has a unit measure of energy. It emerges naturally in the theory and has as fundamental part in all equations as the Planck’s constant. The Planck’s constant, however, remains unique in terms of the study discussed in Refs. [42, 45, 46].

The rest of this paper is organized as follows: In Section II we briefly review the fundamental approaches and related equations used as a ground in the present research; In Section III we introduce the main equation and all originating relations; Section IV includes a summary of the obtained results.

II. GENERALITIES

A. Classical observables

The Lagrangian and Hamiltonian mechanics are among the most successfully applied approaches for studying the dynamics of an arbitrary mechanical system. Although their consideration usually includes generalized variables, henceforth, with respect to the objective to be followed, all functions and relations are given in terms of Cartesian coordinates.

For an isolated system consisting of a collection of point-like particles, the Lagrangian\(^1\) \(L = L(r_1, \dot{r}_1, r_2, \dot{r}_2, \ldots)\) where \(r_1 = (r_{11}, r_{12}, \ldots)\) are the particles position vectors and \(\dot{r}_1 = (\dot{r}_{11}, \dot{r}_{12}, \ldots)\) are the respective velocities, collects all the knowledge needed to determine how the considered system evolves over time. Respectively, predicting the particles configuration at any given moment of time, one has to solve the Euler–Lagrange equations. On the other hand, knowing the relevant momenta \(p_i = (p_{i1}, p_{i2}, \ldots)\), the evolution in time can be studied with the aid of the Hamiltonian mechanics by using the Hamiltonian \(H = H(r_i, p_i)\) and solving the Hamiltonian equations. Both, the Lagrangian and Hamiltonian approaches are related such that one can determine the total energy of the system by accounting for the relation

\[
H = \sum_i \dot{r}_i \cdot p_i - L. \tag{2.1}
\]

Although knowing the total energy is usually sufficient in determining the dynamical state of a mechanical system, studying the rate at which the transfer of energy and momentum between the interacting particles takes place with

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time may strongly enrich the study. To this end, in addition to (2.1), one has to account for the mechanical power $P$ and the respective net forces $F_i$ satisfying the relation

$$ P = \sum_i \dot{r}_i \cdot F_i - W, \quad (2.2) $$

where $W \equiv H$ gives the rate at which the system's energy changes with time. In general, $W$ is a zero function included in equation (2.2) for the sake of clarity.

Detailed study regarding the discussed approaches one can find in Refs. [47–50].

### B. Quantum observables

In quantum mechanics, the information about the dynamics of a multi particle system, with position vectors $r_i = (r_{i1}, r_{i2}, \ldots)$, is embodied in a state function $\Psi = \Psi(r_i, t)$, where $|\Psi|^2 = 1$. Each state function is related to the probability of observing the system in a particular dynamical state characterized by a number of physical quantities with a given uncertainty in their values. Satisfying the Schrodinger equation

$$ i\hbar \partial_t \Psi(r_i, t) = \hat{H} \Psi(r_i, t), \quad (2.3) $$

the explicit representation of $\Psi(r_i, t)$ is related to the Hamiltonian's representation, which in turn depends on the introduced interactions and hence the way the system is being manipulated, i.e., observed. Respectively, depending on how one interacts with the system's constituents, the latter may demonstrate either their wave-like or point-like character.

The wave-like character of the collective point-like particles' dynamics described by (2.1) become apparent on the very frontier between the classical and quantum theoretical approaches of representing observables. Within the framework of (2.3) the corresponding undulant behavior is described accounting for the relation

$$ \hat{H} \Psi(r_i, t) = \sum_i \dot{r}_i \cdot \hat{F}_i \Psi(r_i, t) - \hat{W} \Psi(r_i, t), \quad (2.4) $$

where the state function $\Psi(r_i, t)$ represents a plane wave with phase $\phi = h^{-1} \left( \sum_i \dot{r}_i - \hat{H} t \right)$ and it is such that

$$ \hat{H} \Psi(r_i, t) = H \Psi(r_i, t), \quad \hat{F}_i \Psi(r_i, t) = p_i \Psi(r_i, t), \quad \hat{W} \Psi(r_i, t) = W \Psi(r_i, t). \quad (2.5) $$

Equation (2.4) is a clear sign for the dual nature of matter and it is directly related to the de Broglie and Planck-Einstein relations that marked the origin of quantum theory. It is essential to emphasize that since $\Psi(r_i, t)$ represents a plane wave, the transformation of equation (2.4) into (2.1) holds under no semi-classical approximation. On the other hand, in contrast to the macroscopic systems in which all particles follow certain classical trajectories, on a quantum level the wave-like behavior of all objects strongly dominates. Accordingly, the plane wave case related to equation (2.4) turns out as an inadequate description for problems that include confined elementary particles. Nevertheless, equation (2.4) is of particular value for the present research since it gives the Hamiltonian of a system resembling an ideal gas in which all constituent particles have zero spin. Respectively, the state function $\Psi(r_i, t)$ is represented as a product of plane waves each with phase given by

$$ \phi_i = h^{-1} (p_i \cdot r_i - E_i t), \quad (2.6) $$

$E_i$ denotes the $i$-th particle's energy. The phase associated to the $i$-th particle is further given by

$$ \phi_i = k_i \cdot r_i - \omega_i t, \quad (2.7) $$

where $\omega_i$ and $k_i$ are the corresponding angular frequency and wave vector, respectively. We would like to point out that equations (2.5) further holds for a system composed of massless particles. In that case $\hat{L} \Psi(r_i, t) = 0$ and $|\dot{r}_i| = c$ for all $i$, where $c$ is the light speed in vacuum.

For a thoroughgoing discussion regarding the representation of different observables in quantum mechanics, we recommend the reader to consult Refs. [2, 51–57].

### III. POWER AND NET FORCE AS LINEAR OPERATORS

Relation (2.2) is of particular value, since similar to the introduction of energy, momentum and orbital angular momentum in quantum mechanics, one can find a quantum mechanical representations for the power, net force and torque. In reference to the duality principle that links equation (2.1) with (2.4) and hence the difference in representing observables in classical and quantum theories, we can see that equation (2.2) has a quantum mechanical analog. In addition to the Hamiltonian in (2.4) we have

$$ \hat{P} \Psi(r_i, t) = \sum_i \dot{r}_i \cdot \hat{F}_i \Psi(r_i, t) - \hat{W} \Psi(r_i, t), \quad (3.1) $$

such that $\Psi(r_i, t)$ represents the same plane wave with phase $\phi = \epsilon^{-1} (\sum_i F_i \cdot r_i - P t)$, where $\epsilon$ is a physical constant that has the unit measure of energy. In this case, the state function satisfies equations

$$ \hat{P} \Psi(r_i, t) = P \Psi(r_i, t), \quad \hat{F}_i \Psi(r_i, t) = F_i \Psi(r_i, t), \quad \hat{W} \Psi(r_i, t) = W \Psi(r_i, t). \quad (3.2) $$

Although on a subatomic level equation (3.1) adequately accounts for the dynamics of a system that resemble an ideal gas composed of spin-zero particles, it usually remains underestimated and infrequently used. However, analyzing it in details, we realize that it leads to valuable relations predicting an exact time duration for the processes of energy and momentum transfer between the constituent particles. Relations that have no classical analog. Thus, we can see that in addition to the representation (2.6) the phase corresponding to the $i$-th particle and satisfying (3.1) reads

$$ \phi_i = \epsilon^{-1} (F_i \cdot r_i - P_i t). \quad (3.3) $$
What makes representation (3.3) attractive is the presence of the constant $\varepsilon$ in the related phase. The latter signals for an additional to (2.3) equation and to other fundamental relations discussed hereafter.

### A. Power-frequency, force-momentum and torque-orbital momentum relations

Representing the de Broglie hypothesis, equation (3.1) leads to an additional Planck–Einstein relation. Taking into account (2.6), (2.7) and (3.3), we can obtain four fundamental relations describing the wave-like nature of an arbitrary particle in a system resembling an ideal gas. Two of them are the energy-frequency and momentum-wavevector relations

$$E = \hbar \omega, \quad p = \hbar k,$$  

(3.4)

respectively. The remaining two include the mechanical power $P$ and the net force $F$ describing how the considered particle interacts with others. They are represented as a function of the relevant angular frequency $\omega$ and wave vector $k$, respectively. Hence, we have

$$P = \varepsilon \omega, \quad F = \varepsilon k.$$  

(3.5)

The last relations show that the greater the associated frequency the greater the power that a particle may apply when transferring its energy in a process of direct interaction. Moreover, the greater the momentum the greater the force with which a particle may act in a collision processes. Notice that in a case of dispersion the power in (3.5) should vary such that $\delta P(k) = \varepsilon \delta \omega(k)$. As a result, one may find it convenient to take into account the respective average value.

Equations (3.5) lead to alternative with respect to the classical dynamics formulations for the mechanical power and net force uncover something important. Taking into account the particle's energy and momentum given in (3.4) and relations (3.5) we get

$$P = \frac{\varepsilon}{\hbar} E, \quad F = \frac{\varepsilon}{\hbar} p,$$  

(3.6)

where the physical constant $\hbar/\varepsilon$ has a unit measure of time. In contrast to the classical dynamics and the Ehrenfest's theorem, relations (3.6) give a fundamental limit for the rate at which the processes of energy and momentum transfer may happen. In other words, the theory predicts that in an ideas gas the time duration for which the constituent particles transfer energy and hence momentum is a fundamental constant equal to $\hbar/\varepsilon$. As a result, the power and net force given in (3.6) depend only on the particle's energy and momentum, respectively.

In addition to relations (3.6) we can further obtain an expression for the torque of the same particle. Denoting the its orbital angular momentum by $l$ and the corresponding torque by $\tau$, from the right hand side of equation (3.6) we have

$$\tau = \frac{\varepsilon}{\hbar} l.$$  

(3.7)

As the transfer of a momentum between particles is restricted in time, the time that it takes for the relevant orbital angular momentum to change is also a constant equals to $\hbar/\varepsilon$. We would like to point out that equations (3.6) and (3.7) suggest a zero initial or final momentum, depending on whether the particle starts form rest or transfer all of its momentum, respectively.

One then may ask is there any relation that $P$ satisfies in the case of zero momentum. Answering that question, we rewrite the phase in all three representations (2.6), (2.7) and (3.3) in terms of the four-momentum, four-wavevector and four-force, respectively. Accordingly, the force-momentum relation in (3.6) can be rewritten in Minkowski space, where the covariant four-momentum is given by $p_\mu = (E/c, -p)$.

The corresponding relativistic relation then reads

$$\mathcal{F}_\mu = \frac{\varepsilon}{\hbar} p_\mu,$$  

(3.8)

where $\mathcal{F}_\mu = (P/c, -F)$ is the respective covariant four-force. With the aid of equation (3.8) we obtain expression for the power that a particle may apply interacting with other constituents form the considered system,

$$P = \sqrt{E^2 c^2 + \frac{E^2}{\hbar^2} m^2 c^4},$$  

(3.9)

where $F = |F|$. Hence, form (3.9) follows that for a particle being at rest, $P$ will depends only on the particle's rest mass $m$. For example, when that condition is satisfied, we have

$$P = \frac{\varepsilon c^2}{\hbar} m.$$  

The last relation gives the exact amount of mechanical power that will be generated if hypothetically an elementary particle loses all of its rest mass. The existence of such fundamental limit is due to the presence of constant $\hbar/\varepsilon$ according to which relations (3.6) are defined.

### B. Time evolution and position representation

The energy and momentum transfer in a physical system is the main indication for its evolution. Since the power is an observable that accounts for such a processes in addition to the Hamiltonian of a quantum mechanical system one can rely on the respective power operator given on the left hand side in equation (3.2). Therefore, in complement to the Schrodinger equation (2.3), we have

$$i\varepsilon \partial_t \Psi(r_i, t) = \hat{\mathcal{F}} \Psi(r_i, t).$$  

(3.10)

The operator $\hat{\mathcal{F}}$ commute with the respective Hamiltonian $\hat{H}$ such that for $\hat{H} \Psi(r_i, t) = E \Psi(r_i, t)$ one obtains the relation on the left hand side in (3.6). Nevertheless, $P$ has to be treated in accordance to the problem one is solving. For example, the maximum mechanical power that a quantum harmonic oscillator with frequency $\omega$ may generate in transferring all of its energy, is given by

$$P_n = \varepsilon \omega \left(n + \frac{1}{2}\right), \quad n \in \mathbb{N}_0.$$  

(3.11)
On the other hand, the position representation of \( \hat{P} \) corresponding to a particle with rest mass \( m \) and in the case of (3.6), reads

\[
\hat{P} = -\frac{\hbar}{2m} \Delta.
\]

For \( \mathcal{K} = \{x, y, z\} \), the net force given in (3.6) is associated to the three component operator \( \hat{F} = (\hat{F}_\nu)_{\nu \in \mathcal{K}} \), that has the following position representation

\[
\hat{F} \equiv -i\epsilon \nabla.
\]

The components of the operator in (3.12) obey the commutation relation

\[
[\hat{F}, \hat{F}_\nu] = i\epsilon \delta_{\beta\nu}, \quad \beta, \nu \in \mathcal{K}
\]

and commute with each one component of the relevant momentum operator. In that respect, it is worth giving the time derivative of the force operator,

\[
d_t \hat{F} = \frac{\epsilon}{\hbar} d_t \hat{p},
\]

(3.13)

where taking into account the Ehrenfest’s theorem we get

\[
d_t \langle \hat{F}_\nu \rangle = \frac{i\epsilon}{\hbar^2} \langle [\hat{p}_\nu, \hat{H}] \rangle + \langle \partial_t \hat{F}_\nu \rangle, \quad \nu \in \mathcal{K}
\]

and

\[
d_t \langle \hat{p}_\nu \rangle = -\frac{1}{\hbar} \langle [\hat{F}_\nu, \hat{H}] \rangle + \langle \partial_t \hat{p}_\nu \rangle.
\]

The observable \( \tau \) gives the acquired orbital angular momentum of a particle due to the applied net force \( F \). It is represented by a three component operator \( \hat{\tau} = (\hat{\tau}_\nu)_{\nu \in \mathcal{K}} \) and similar to the orbital and spin angular momenta, it satisfies the algebra

\[
[\hat{\tau}_\gamma, \hat{\tau}_\beta] = i\epsilon \epsilon_{\gamma\beta\nu} \hat{\tau}_\nu, \quad \gamma, \beta, \nu \in \mathcal{K}.
\]

(3.14)

In the coordinate space the torque has the following representation

\[
\hat{\tau} = -i\epsilon (\hat{r} \times \nabla),
\]

where \( \hat{r} \) is the position operator associated with the particle’s space coordinates. As the operator in (3.12) commute with the relevant momentum operator, the following commutation relations hold

\[
[\hat{\tau}_\gamma, \hat{F}_\beta] = i\epsilon \epsilon_{\gamma\beta\nu} \hat{\tau}_\nu,
\]

where \( \hat{I} = (\hat{I}_\nu)_{\nu \in \mathcal{K}} \) is the corresponding orbital angular momentum operator. Therefore, the magnitude of the torque of a particle that has acquired an orbital momentum with quantum number \( l \), reads \( |r| = \epsilon \sqrt{\tau (\tau + 1)} \), where \( \tau \equiv l \). The equation giving the rate at which the orbital momentum changes with time is analogous to the equation (3.13). It is written as

\[
d_t \hat{\tau} = \frac{\epsilon}{\hbar} d_t \hat{I}
\]

and shows us that any variation in the orbital angular momentum corresponds to a change of the relevant torque by a constant of proportionality \( \hbar / \epsilon \).

In the relativistic case, the expression for the four-force operator follows from equation (3.8) such that one has

\[
\hat{\mathcal{F}}_\mu = i\epsilon \partial_\mu,
\]

where the derivative \( \partial_\mu = (\partial_\alpha, \nabla) \).

\[C.\] **Uncertainty relations**

With respect to the quantum mechanical representation of the power and net force, the Heisenberg uncertainty principle applies. For \( \nu \in \mathcal{K} \) one obtains the following inequalities

\[
\Delta_\nu \Delta F_\nu \geq \frac{\epsilon}{2},
\]

(3.15a)

and

\[
\Delta_\nu \Delta P_\nu \geq \frac{\epsilon}{2}.
\]

(3.15b)

Therefore, gaining a knowledge for the position of a particle leads to a lack of information for the applied net force and hence the change in its momentum. Furthermore, the greater the energy to confine a particle the greater the uncertainty in power.

\[D.\] **Electromagnetic field example**

Expressing the free-space electromagnetic field operators in terms of (3.3), in addition to the Hamiltonian and total momentum operators of the electromagnetic field, we obtain the respective power operator

\[
\hat{P} = \sum_{k, \sigma} \epsilon \omega_k \hat{n}_{k, \sigma}
\]

(3.16)

and net force operator

\[
\hat{F} = \sum_{k, \sigma} \epsilon k \hat{n}_{k, \sigma}
\]

(3.17)

where the number operator \( \hat{n}_{k, \sigma} \) gives the number of photons in a mode determined by the wave vector \( k \) and the polarization index \( \sigma \).

**IV. CONCLUSION**

Restricting our study within the standard formulation of quantum theory, we introduce a position representation of power, net force and torque operators, obtain fundamental relations they obey and discuss the essence of all results. A key role in all derived relations plays a physical constant, denoted accordingly by \( \epsilon \), that has a unit measure of energy. The named constant appears as fundamental as the Planck’s
constant in the quantum theory. In particular, for an ideal gas, $\varepsilon$ is the proportionality constant between the mechanical power of an arbitrary constituent particle generated in the processes of direct interactions with the remaining particles and the frequency of the plane wave associated to that particle, follow equation (3.5). The presence of $\varepsilon$ points out to the existence of a fundamental limit for the rate at which energy and momentum transfer take place over time, see equation (3.6). In conclusion, studying the role of $\varepsilon$ in physics more closer, we definitely have to test its uniqueness and make efforts in determining its value.

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