NEW UNIVERSAL DEFORMATION FORMULAS FOR DEFORMATION QUANTIZATION

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ABSTRACT. Universal Deformation Formulas (UDFs) for the deformation of associative algebras play a key role in deformation quantization. Here we present examples for certain classes of infinitesimals. A basic representable 2-cocycle $F$ of an associative algebra $A$ is one for which there exist commuting derivations $D_1, \ldots, D_n$ of $A$ such that $F = \sum_{ij} a_{ij} D_i \circ D_j$, where the $a_{ij}$ are central elements of $A$. When $A$ is defined over the rationals, there is a natural definition of the exponential of such a cocycle. With this $\exp \hbar F$ defines a formal one-parameter family of deformations of $A$, where $\hbar$ is a deformation parameter. The rational quantization of smooth functions on a smooth manifold using a bivector field as an infinitesimal deformation is a special case.

1. INTRODUCTION

Quantization arises naturally within the context of the deformation theory of algebras, a subject created in the author’s papers of 1963 and 1964, [5], [6]. The question implicitly raised in [6] is, given an algebra $A$ in some category, how can we create formal one-parameter families of deformations of $A$ with multiplication of the form

$$a \star b = ab + \hbar F_1(a, b) + \hbar^2 F_2(a, b) + \cdots .$$

This was initially studied in the associative case in [6], where it was observed that the Lie case proceeds similarly. The extension of the basic idea to algebras defined over an operad is evident.

The $F_i$ in (1) are 2-cochains of $A$ with coefficients in itself, tacitly extended to be defined over $A[[\hbar]]$, where $\hbar$ is a deformation parameter. The original multiplication in $A$ will be denoted by $F_0$.

Recall, for the special case needed here, the definition of the composition products $\circ_1$ and $\circ$ introduced in [5]: If $F, G$ are 2-cochains of $A$ then $F \circ_1 G$, $F \circ_2 G$, $F \circ G$ are the 3-cochains defined by setting, respectively,

$$(F \circ_1 G)(a, b, c) = F(G(a, b), c), \quad (F \circ_2 G)(a, b, c) = F(a, G(b, c)),$$
and

$$(F \circ G)(a, b, c) = (F \circ_1 G - F \circ_2 G)(a, b, c) = F(G(a, b), c) - F(a, G(b, c)).$$

Associativity of the multiplication defined by (1), in the form $a \star (b \star c) = (a \star b) \star c$, is now expressed by the fundamental equations

$$\sum_{i=0}^{n} F_i \circ_2 F_{n-i} = \sum_{i=0}^{n} F_i \circ_1 F_{n-i}, \text{ for all } n.$$

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There is a homological reformulation of (2). Separating out the terms where either \( i = 0 \) or \( j = 0 \), (2) can be rewritten as

\[
\sum_{i+j=n, i,j>0} F_i \circ F_j = -\delta F_n,
\]

where \( \delta \) is the Hochschild coboundary operator. When \( n = 1 \) the left side vanishes, so \( F_1 \) must be a 2-cocycle, usually called the \textit{infinitesimal} of the deformation. When a 2-cocycle \( F_1 \) is given, the question of whether one can construct a formal deformation of \( A \) as in (1) depends only on the cohomology class of \( F_1 \). This was shown in [6] by considering the effect of one-parameter families of automorphisms of the algebra \( A \) \textit{as a linear space}; these are now often called gauge transformations.

When \( F_1 \) is a cocycle then so is \( F_1 \circ F_1 \). Its cohomology class in \( H^3(A,A) \), which depends only on the class of \( F_1 \), is its \textit{primary obstruction}. When it vanishes, one can choose an \( F_2 \) with \( -\delta F_2 = F_1 \circ F_1 \) and one can ask if an \( F_3 \) exists so that one can continue building the series. However, (3) with \( n = 3 \) shows that one may encounter another obstruction in \( H^3(A,A) \), and so on indefinitely. If, with a given 2-cocycle \( F_1 \), we are able to construct a series such as that in (1) with the given \( F_1 \), then that series is said to be an \textit{integral} of the infinitesimal \( F_1 \) and to \textit{quantize} \( A \). In principle, one may encounter an infinite sequence of obstructions, [6]. Moreover, an integral, if it exists, is generally not unique. The homological formulation is useful when one knows in advance that all obstructions vanish, e.g., when \( H^3(A,A) \) vanishes identically, as for a polynomial ring in two variables by the Hochschild-Kostant-Rosenberg Theorem, or when one knows that there are only a limited number and these can be overcome by calculation, cf [2]. A \textit{Universal Deformation Formula} (UDF), by contrast, exhibits an explicit integral for some class of cocycles. Our main theorem exhibits a family of UDFs generalizing the classical first UDF of the next section.

Note in the following that if \( a \) is a central element of \( A \) and \( F \) a cocycle of any dimension, then \( aF \) is again a cocycle, and if \( D \) is a derivation, then \( Da \) is again central.

2. The first UDF

When \( f, g \) are 1-cochains of an associative algebra \( A \), one can attempt to define a ‘naive’ multiplication on 2-cochains of the form \( f \sim g \) by setting \((f_1 \sim g_1)(f_2 \sim g_2) = f_1f_2 \sim g_1g_2 \). However, this is generally not well defined. For if \( a \) is a central element of \( A \), then as 2-cochains one has \( a(f \sim g) = af \sim ag = f \sim ag \) but such changes in representation will usually change the naive product. Suppose now that we have a set of commuting derivations \( \{D_i\} \) of \( A \), a choice of which we will later call a base of derivations. When \( I = \{i_1, \ldots, i_r\} \) is an unordered set of indices of the \( D_i \), set \( D_I = D_{i_1} \cdots D_{i_r} \). Then the foregoing problem does not arise when the naive multiplication is restricted to 2-cochains of the form \( D_I \sim D_J \).

(When the set \( I \) is empty, interpret \( D_I \) as the identity map \( A \); when \( I \) and \( J \) are both empty then \( D_I \sim D_J \) is the multiplication map.) In particular, if \( D, \overline{D} \) are commuting derivations of \( A \) then \((D \sim \overline{D})^n = D^n \sim \overline{D}^n \) is well defined, as is \( \exp(D \sim \overline{D}) = \sum_{n=0}^{\infty} \frac{1}{n!}(D \sim \overline{D})^n \) when \( A \) is defined over \( \mathbb{Q} \), which we henceforth assume.
Theorem 1 (First UDF). If $D, \overline{D}$ are commuting derivations of $A$, then setting
\[ a \star b = \exp(\hbar(D \sim \overline{D}))(a, b) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} D^n a \cdot \overline{D}^n b \]
defines a formal one-parameter family of deformations of $A$. \(\square\)

Proof. Setting $F_i = D^i \sim \overline{D}^i$ in (2), what must be shown is that for all $n$ one has
\[ \sum_{i=0}^{n} \frac{1}{i!(n-i)!} D^i \sim \overline{D}^i (D^{n-i} \sim \overline{D}^{n-i}) = \sum_{i=0}^{n} \frac{1}{i!(n-i)!} D^i (D^{n-i} \sim \overline{D}^{n-i}) \sim \overline{D}^i. \]
By Leibniz’ rule, the left of (4) can be written as
\[ \sum_{i,j=0}^{n} \frac{1}{(n-i)!j!(i-j)!} D^i \sim \overline{D}^j D^{n-i} \sim \overline{D}^{n-j} \overline{D}^i. \]
The right side of (6) can be written as
\[ \sum_{i,j=0}^{n} \frac{1}{(n-i)!j!(i-j)!} D^{n-j} \sim \overline{D}^j D^{n-i} \sim \overline{D}^{n-j}. \]
On the right side of (6) we can replace now replace the dummy variable $i$ by $n-j$ and $j$ by $n-i$. The sum remains formally over the same set of indices (as it must) since $i \geq j$ if and only if $n-j \geq n-i$. As $D$ and $\overline{D}$ commute, the right side of (6) then becomes identical to the right side of (5), proving the assertion. \(\square\)

This theorem, although first explicitly stated (in a more general context) in [7, Lemma 1, p.17, 1968, was already implicit in two important papers on quantum mechanics, Groenewold, 1946, [8], and Moyal [10], 1949. The algebra being deformed in both [8] and [10] is the algebra of observables in phase space, where before deformation the position variable $q$ and momentum variable $p$ commute but after deformation satisfy $qp - pq = i\hbar$. Here $\hbar$ is the reduced Planck’s constant, $\hbar/2\pi$, where $\hbar \approx 6.626176 \times 10^{-34}$ Joule-seconds is Planck’s original constant. Subsequently, using [8], Bayen, Flato, Frønsdal, Lichnerowitz, and Sternheimer, [1], developed deformation quantization. They showed that quantum theory, and in particular, the spectrum of the hydrogen atom, could be deduced without the use of Schrödinger’s equation, directly from deformation theory. For a summary of this and some later developments, cf. [11].

Theorem 1 has the following immediate generalization, of which we will give two proofs.

Theorem 2. If $D_i, \overline{D}_i, i = 1, \ldots, n$ are commuting derivations of $A$, then
\[ \exp(\sum_{i=1}^{n} D_i \sim \overline{D}_i) \text{ quantizes } A. \] \(\square\)
First Proof of Theorem 2. By induction. Any derivation which commutes with all $D_i$, remains a derivation after the deformation which they induce. A pair of such therefore induces a further deformation. As the derivations commute, the product of the exponentials involved is the exponential of the sum of the exponents.

It follows, in particular, that if $D_1, \ldots, D_n$ are commuting derivations of $\mathcal{A}$ and $a_{ij}$, $i,j = 1, \ldots, n$ are constants for all $D_i$ (i.e., annihilated by all), then $\sum_{ij} a_{ij} D_i \sim D_j$ quantizes $\mathcal{A}$. If $D,D'$ are derivations, which need not commute, then $\delta(DD') = -(D \circ D' + D' \circ D)$. It follows that the 2-cocycle $F = \sum_{ij} a_{ij} D_i \sim D_j$ is cohomologous to that obtained by replacing the matrix $(a_{ij})$ by its skew-symmetric part. The resulting deformation will then be equivalent by a gauge transformation to the original. Further, cocycles $F = \sum_{ij} a_{ij} D_i \sim D_j$ and $F' = \sum_{ij} a'_{ij} D_i \sim D_j$ are cohomologous if and only if $(a_{ij})$ and $(a'_{ij})$ have the same skew part. Theorem 2 therefore has the following refinement.

**Theorem 3.** If $D_i$, $i = 1, \ldots, n$ are commuting derivations of $\mathcal{A}$ and $a_{ij}$ are constants then $\exp(\sum a_{ij} D_i \sim D_j)$ quantizes $\mathcal{A}$. Every such quantization is gauge equivalent to one in which the matrix $(a_{ij})$ is skew. □

The simple argument used in the proof of Theorem 2 fails in the non-commutative case. To overcome this, we will need an approach yielding Theorem 2 directly without relying on Theorem 1.

3. Polarization and depolarization

Polarization (in mathematics) is a known operation in which a homogeneous identity of degree $n$ in one variable is replaced by a multilinear identity in $n$ independent variables. When $I = (i_1, \ldots, i_n)$ is an $n$-tuple of non-negative integers, and $\lambda_1, \ldots, \lambda_n$ are commuting variables, set $\lambda^I = \lambda_1^{i_1} \cdots \lambda_n^{i_n}$. Suppose now that in a unital, not necessarily commutative, ring $\mathcal{R}$ that we have a homogeneous form $F$ of degree $n$ in one variable $x$, i.e., a sum of $k$ monomial terms,

$$F(x) = \sum_{j=1}^k a_{0,j} x a_{1,j} x \cdots a_{n-1,j} x a_{n,j}.$$  

Here $x$ is not treated as central, since it represents an element of $\mathcal{R}$. Taking $n$ variables $x_1, \ldots, x_n$, which like $x$ are not central, and treating the $\lambda_i$ as central in $\mathcal{R}$, we can write

$$F(\lambda_1 x_1 + \cdots + \lambda_n x_n) = \sum_I \lambda^I F_I (x_1, \ldots, x_n)$$

for certain functions $F_I$ of the $n$ variables $x_1, \ldots, x_n$. Here $I$ can only run through those $n$-tuples with $i_1 + \cdots + i_n = n$.

Set $F_{(1,\ldots,1)} = F$ and call this the weak polarization of $F$. Then

$$F(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \sum_{j=1}^k a_{0,j} x_{\sigma 1} a_{1,j} x_{\sigma 2} \cdots a_{n-1,j} x_{\sigma n} a_{n,j},$$

where $S_n$ is the permutation group on $n$ elements. It follows from (8) that setting all $x_i$ equal to the one variable $x$, one has

$$F(x, \ldots, x) = n! F(x).$$
Here \( n! \) is a unit; we define \( (1/n!) F \) to be the polarization, or polar form, of \( F \).

**Theorem 4 (Polarization).** With the foregoing notation, if a homogeneous form \( F \) vanishes identically then its polarization as well as all \( F_I \) vanish identically for all \( I \). \( \square \)

Depolarization may be new. Observe that in the foregoing, all the \( F_I \) are specializations of \( F \) obtained by setting the elements in various subsets of \( x_1, \ldots, x_n \) equal to each other. It follows that if the polarization of \( F \) vanishes identically then so do all the \( F_I \), so the left side of (7) must also vanish. This yields the following

**Theorem 5 (Depolarization).** Suppose that we have a ring \( R \) with an additive subgroup \( L \) spanned by certain generators \( x_1, x_2, \ldots \) and we know that the polarized form 
\[
\frac{1}{n!} \sum_{\sigma \in S_n} F(x_{\sigma_1}, \ldots, x_{\sigma_n}) = 0
\]
of some homogeneous identity \( F(x) = 0 \) of degree \( n \) holds whenever the variables are among these generators. Then it must hold for all elements of the linear space \( L \) spanned by these generators. \( \square \)

Analogous assertions hold for systems with several binary operations, as long as they are all bilinear.

### 4. Second proof of theorem 2

In this section, \( D_1, D_2, \ldots, D_n, \overline{D}_1, \overline{D}_2, \ldots, \overline{D}_n \) will denote commuting derivations, \( I, J, \ldots \) will denote subsets of \( \{1, \ldots, n\} \), possibly empty, \( I', J', \ldots \) will denote their complements, and \( \sqcup \) will denote a union of disjoint sets.

**Second Proof of Theorem 2.** Multiply the basic equation (2) by \( n! \). Taking any pair of commuting derivations \( D \) and \( \overline{D} \), and setting \( F = D \sim \overline{D} \), the basic identities (2), which we must prove, become

\[
\sum_{r=0}^{n} \binom{n}{r} (D \sim \overline{D})^r \circ_2 (D \sim \overline{D})^{n-r} = \sum_{r=0}^{n} \binom{n}{r} (D \sim \overline{D})^r \circ_1 (D \sim \overline{D})^{n-r}.
\]

The polarized form of the identities (9) is then

\[
\sum_{I \subseteq \{1, \ldots, n\}} (D_I \sim \overline{D}_I) \circ_2 (D_{I'} \sim \overline{D}_{I'}) = \sum_{I \subseteq \{1, \ldots, n\}} (D_I \sim \overline{D}_I) \circ_1 (D_{I'} \sim \overline{D}_{I'}).
\]

The binomial coefficients do not appear explicitly, but this form reduces to (9) when all \( D_i \) are identified with a single \( D \) and all \( \overline{D}_i \) with a single \( \overline{D} \). The left side of (10) is

\[
\sum_{I \subseteq \{1, \ldots, n\}} D_I \sim \overline{D}_I (D_{I'} \sim \overline{D}_{I'})
\]

\[
= \sum_{I \subseteq \{1, \ldots, n\}} D_I \sim \sum_{J \sqcup K = I} (\overline{D}_J D_{I'} \sim \overline{D}_K \overline{D}_{I'})
\]

\[
= \sum_{I \subseteq \{1, \ldots, n\}} D_I \sim \sum_{J \sqcup K = I} (\overline{D}_J D_{I'} \sim \overline{D}_{K \sqcup I'}).\]
which can be written as

$$\sum_{I, J \subseteq \{1, \ldots, n\}, I \supseteq J} D_I \sim D_J D_{I^c} \sim D_{J^c},$$

since $K \cup I^c$ is the complement to $J$, which is a subset of $I$.

The right side of (10) is

$$\sum_{I \subseteq \{1, \ldots, n\}} D_I (D_{I^c} \sim D_{J^c}) \sim D_i$$

$$= \sum_{I \subseteq \{1, \ldots, n\}} \sum_{J \cup K = I} D_J D_{I^c} \sim D_K D_{I^c} \sim D_I$$

$$= \sum_{I \subseteq \{1, \ldots, n\}} \sum_{J \cup K = I} D_J D_{I^c} \sim D_K \sim D_J.$$

In the last expression note that $K$ is the complement to $J \cup I^c$. We may therefore make the following changes in notation. Replace $J \cup I^c$ by $I$, $K$ by $I^c$, and $I$ by $J^c$. Since $J = I \setminus K$, this implies that $J$ is replaced by $J^c \setminus I^c$. The final double sum in (13) then becomes formally identical with the the sum in (12), so the last expression in (13) then becomes

$$\sum_{I, J \subseteq \{1, \ldots, n\}, I \supseteq J} D_I \sim D_{I^c} D_J \sim D_{J^c}.$$

Since $D_J$ and $D_{I^c}$ commute, the last expressions in (11) and (13) are indeed equal, so the polarized form, (11), of the basic identity in its form used here, (9), holds. By depolarization, it follows that (9) must continue to hold when $D \sim D$ is replaced by any sum of the form $\sum (D_i \sim D_i)$. This proves the assertion. \qed

While we showed that the final sum in (13) can be given the same formal expression as that in (11), that was not a necessary part of the proof; the renaming of indices can not change the set over which the summation actually takes place.

The Second Proof of Theorem 4 does not use Theorem 1, which has become a special case of a stronger theorem obtained directly by depolarization.

5. AN INTRINSIC PRODUCT ON REPRESENTABLE COCHAINS

A set of commuting derivations of $A, \{D_i\}$, will be called a base of derivations, or simply a base. As before, we will write $D_I$ for a product $D_{i_1} \cdots D_{i_r}$ of base elements, where $I = (i_1, \ldots, i_r)$. An $n$-cochain $F$ of $A$ which can be written in the form $F = a_1 D_{i_1} \sim \cdots \sim a_n D_{i_n}$, where $a_1, \ldots, a_n$ are central elements of $A$, will be called representable relative to the base $\{D_i\}$, and the given expression will be called a representation of $F$. Those $n$-cochains which are representable relative to a given base form a left module over the enter of $A$. A basic representable $n$-cochain will be one of the form $a_1 D_{i_1} \sim \cdots \sim a_n D_{i_n}$ where the $a_i$ are central and $D_{i_1}, \ldots, D_{i_n}$ are elements of the base of commuting derivations. Being cup products of derivations, these are cocycles; when $n = 2$ they are infinitesimal deformations.

Fixing $n$, one can define an intrinsic product on $n$-cochains representable relative to a given base. We do this explicitly only for 2-cochains, but the definition for arbitrary $n$ will be evident. The multiplication is required to be linear in the
first factor over the center of $A$, so we only need to define products of the form $(D_I \sim D_J)(aD_K \sim D_L)$. With $\sqcup$ again denoting disjoint union, set

$$ (D_I \sim D_J)(aD_K \sim D_L) = \sum_{I', J', J'' = I, \sqcup J' \sqcup J'' = J} (D_{I'} D_{J'} a) D_{I''} D_K \sim D_{J''} D_L, $$

where the sum runs over all complementary subsequences $I', I''$ of $I$ and $J', J''$ of $J$, the empty subsequence and entire sequence being allowed. When a sequence is empty, the corresponding operator is the identity.

**Theorem 6.** The multiplication of 2-cochains all of which are representable with respect to one fixed base is associative (but not commutative). It does not depend on the choice of base; if the cochains multiplied are also all representable with respect to a second base, then the product is unchanged.

**Proof.** Leibniz’ rule (which is applicable even to a product of non-commuting derivations if one is careful about their order), asserts that

$$ D_I(ab) = \sum_{I', I'' = I} D_{I'} a D_{I''} b, \quad \text{whence} \quad D_I a D_J = \sum_{I', I'' = I} D_{I'} a D_{I''} D_J. $$

For the case of 2-cochains, consider two copies of the set of derivations $D_i, \ldots, D_n$, a left copy which will be left unmarked, and a right copy marked by overbars, all commuting. Instead of $D_I \sim D_J$ write $D_I \overline{D}_J$. Then what is asserted here just becomes the preceding assertion about derivations. $\square$

The intrinsic multiplication is analogous to multiplication of ordinary integers, where one chooses a base for the number system in order to calculate a product, but the product does not depend on the choice of base.

Two $n$-cochains might each be representable relative to some base but possibly not relative to a common base, in which case the foregoing can not define a product. However, we can always define the powers of a representable 2-cocycle $F$, as well as its formal exponential $\exp \hbar F$. Neither will depend on the choice of representation. It is evident that the analog of the foregoing holds for all dimensions but we do not know its significance for higher dimensions.

### 6. First UDF with Non-Commuting Derivations

The first UDF with non-commuting derivations, [3], is important as a significant antecedent to our main theorem.

**Theorem 7.** Suppose that $\phi, \psi$ are derivations $A$ with $[\phi, \psi] = \phi$. Then

$$ \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \phi^n \sim \psi(\psi - 1) \cdots (\psi - n + 1) $$

quantizes $A$.

As noted in [3], an expression of the form $x(x - 1) \cdots (x - n + 1)$ is often called a descending factorial and denoted by $[x]_n$. The coefficient of $x^i$ in $[x]_n$ is the $i$th Stirling number of the first kind; it is equal to the number of elements in the symmetric group on $n$ elements which are expressible as a product of exactly $i$ disjoint cycles. The following shows that (16) is basically an exponential.
Theorem 8. Let $D, \overline{D}$ be commuting derivations and $a$ be a central element of $A$ such that $Da = -a$. Then

$$\exp \hbar (aD \sim \overline{D}) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} (aD) ^n \sim \overline{D} (\overline{D} - 1) \cdots (\overline{D} - n + 1).$$

Proof using the notation introduced for the descending exponential, we must show that $(aD \sim \overline{D})^n = (aD)^n \sim [\overline{D}]_n$. The proof is by induction, the case $n = 1$ being tautologous. Since $D$ and $\overline{D}$ commute, one has $\overline{D}D_i a = -D_i a$ for all $i$. It follows that if $b$ is a product of $n$ factors of the form $D_i a$ with varying $i$ then $\overline{D} b = -nb$. With the intrinsic product on separable 2-cochains, one has

$$\sum_{r=0}^{n} \binom{n}{r} \left( aD \sim \overline{D} \right)^{n-r} = \sum_{r=0}^{n} \binom{n}{r} (aD \sim \overline{D})^r \circ_1 (aD \sim \overline{D})^{n-r}.$$

The assertion follows. $\blacksquare$

The proof of Theorem 7 will be given in the next section.

7. A new UDF for deformation quantization

Let $D, \overline{D}$ be commuting derivations and $a$ be a central element of $A$. Set $F = aD \sim \overline{D}$. After multiplying by $n!$, the basic equations (2) become

$$\sum_{r=0}^{n} \binom{n}{r} (aD \sim \overline{D})^r \circ_2 (aD \sim \overline{D})^{n-r} = \sum_{r=0}^{n} \binom{n}{r} (aD \sim \overline{D})^r \circ_1 (aD \sim \overline{D})^{n-r}.$$

We shall need the polarization of $F^n = (aD \sim \overline{D})^n$. To this end, consider first the simpler polarized form of $(aD)^n$, where $D$ is a derivation and $a$ central. This is

$$\sum_{\sigma \in S_n} a_{\sigma n} D_{\sigma n} a_{\sigma(n-1)} D_{\sigma(n-1)} \cdots a_{\sigma 2} D_{\sigma 2} a_{\sigma 1} D_{\sigma 1}.$$

When $D$ is a derivation and $a$ an element of $A$, then as an operator, $Da$ is a sum of two terms, $Da = (Da) + aD$, for $D(ab) = (Da)b + aDb$. Therefore, when (19) is expanded, each $D_{\sigma i}$ will operate on all terms generated in $a_{\sigma(i-1)} D_{\sigma(i-1)} \cdots a_{\sigma 1} D_{\sigma 1}$, and from each of these $D_{\sigma i}$ will generate a sum of terms in each of which it either appears as an operator on exactly one of the $a_{\sigma j}$ with $j < i$ or on none of them, in which case it appears in that term as an operator in front of all of them. If the product (19) is not empty, then each term in it will therefore appear as some product of $n$ central elements of $A$ on each of which certain of the $D_i$ have operated, preceded as an operator (written on the right) by the product of those $D_i$ which have not operated on any of the central elements. The terms in the expanded form of (19) can now be collected according to the various products of the $D_i$ appearing on the right of the various products of $n$ central elements. Since at least one $D_i$ will always appear to the right, and since the $D_i$ commute, (19) can therefore be written as

$$\sum_{I \subseteq \{1, \ldots, n\}} a_I D_{I^c}.$$
Considering now both $D$ and $\overline{D}$, the polarization of $(aD - \overline{D})^n$ can be written similarly as
\[
\sum_{I,J \subseteq \{1, \ldots, n\}} a_{IJ} D_I^e - \overline{D}_J^e.
\]

**Theorem 9.** Let $F = \sum_{i=1}^n (a_i D^i - \overline{D}^i)$ be a basic representable 2-cocycle of an algebra $A$ defined over $\mathbb{Q}$, where the $a_i$ are central elements and $D_1, \ldots, D_n, \overline{D}_1, \ldots, \overline{D}_n$ are commuting derivations $A$. Then $\exp hF$ quantizes $A$, i.e., it defines a formal one-parameter family of deformations of $A$.

**Proof.** We must show that the basic identities (2) hold for all $n$. As in the Second Proof of Theorem 2, it will be sufficient to show that their polarized form holds with $(a_1 D^1 - \overline{D}^1), \ldots, (a_n D^n - \overline{D}^n)$ inserted as the variables. Fixing $n$ and multiplying by $n!$, the polarized form is then expressible as
\[
\sum (a_{IJ} D_I^e - \overline{D}_J^e) = \sum (a_{IJ} D_I^e - \overline{D}_J^e) = 
\]
Here $I \cup J^e = J \cup J^e$, $K \cup K^e = L \cup L^e$, and $I \cup I^e \cup J^e = J \cup J^e \cup L \cup L^e = \{1, \ldots, n\}$. (Think of $I, J$ as subsets of a common set $\mathcal{I}$, of $K, L$ as subsets of its complement $\mathcal{I}^e$ inside $\{1, \ldots, n\}$, the sets $I^e$ and $J^e$ as the complements of $I, J$, respectively within $\mathcal{I}$, and $K^e, L^e$ as the respective complements of $K, L$ within $\mathcal{I}^e$.)

With this, the left side of the basic equation (22) is
\[
\sum a_{IJ} D_I^e - \overline{D}_J^e (a_{KL} D_K^e - \overline{D}_L^e).
\]
Applying Leibniz rule applied to the operation of $D_I^e$ on the three factors $a_{KL}, D_K^e$ and $\overline{D}_L^e$, the expression in (22) becomes
\[
\sum a_{IJ} (D_I^e a_{KL}) D_I^e \sim D_{I_1} D_K^e \sim D_{I_1} \overline{D}_{L}^e,
\]
where $J_1 \cup J_2 \cup J_3 = J^e$. Since $(D_I^e a_{KL}) = a_{KL, I_1} J_1$, this reduces to
\[
\sum a_{IJ} a_{KL, I_1} D_{I_1} D_K^e \sim D_{I_1} \overline{D}_{L}^e,
\]
where $I \cup K \cup I^e = J \cup (L \cup J_1^c) \cup J_2^c \cup (J_3^c \cup L^c) = \{1, \ldots, n\}$. The right side of (22) is
\[
\sum a_{IJ} D_I^e (a_{KL} D_K^e - \overline{D}_L^e) \sim D_J^e.
\]
Applying Leibniz rule applied to the operation of $D_I^e$ on the three factors $a_{KL}, D_K^e$ and $\overline{D}_L^e$, the expression in (25) becomes
\[
\sum a_{IJ} (D_I^e a_{KL}) D_I^e D_K^e \sim D_{I_1} \overline{D}_{I_2} \overline{D}_{L}^e \sim D_J^e,
\]
where $I_1 \cup I_2 \cup I_3 = \mathcal{I}^e$. Since $(D_I^e a_{KL}) = a_{KL, I_1} J_1$, this reduces to
\[
\sum a_{IJ} a_{KL, I_1} D_{I_1} D_K^e \sim D_{I_1} \overline{D}_{I_2} \overline{D}_{L}^e \sim D_J^e.
\]
Substituting $I_1 \cup I_2, I_3 \cup I_2, I_3 \cup L^c, J^c$ for $K, L \cup J_1, I^e, K^c, J_2^c, J_3^c \cup L^c$, respectively, (24) becomes identical with (26) since $D_{I_1}^c$ and $\overline{D}_{I_2}^c$ commute. Depolarizing, it follows that (18) must hold when $aD - \overline{D}$ is replaced by any linear combination $\sum (a_i D_i - \overline{D}_i)$. This proves the assertion. \square

Theorem 9 has the following refinement, generalizing Theorem 6.
Theorem 10. If $D_i$, $i = 1, \ldots, n$ are commuting derivations of $A$ and $a_{ij}$ are central elements, then $\exp(\sum a_{ij} D_i \lrcorner D_j)$ quantizes $A$. Every such quantization is gauge equivalent to exactly one in which the matrix $(a_{ij})$ is skew. □

We come finally to the following.

Proof of Theorem 7. This is the special case of Theorem 9 where $F$ is just $aD \lrcorner D$ and $Da = -a$. □

8. Quantization of manifolds and concluding remarks

A bivector $\pi$ on a smooth manifold $\mathcal{M}$ can be viewed as a 2-cocycle of the algebra $C^\infty(\mathcal{M})$, for one can define $\pi(f, g) = \pi(df, dg)$. The problem of quantizing a manifold is to determine whether one can construct a formal one-parameter family of deformations of $C^\infty(\mathcal{M})$ whose infinitesimal is $\pi$. Its solution is a special case of Theorem 9, since if the dimension of $\mathcal{M}$ is $d$, then locally one can choose coordinates $x_1, \ldots, x_d$; the bivector $\pi$ is then expressible in the form $\sum a_{ij} \partial x_i \lrcorner \partial x_j$ for certain functions $a_{ij}$. This is a basic representable 2-cocycle and the desired quantization is simply given by $\exp(\hbar \pi)$.

This solution to the quantization problem has the evident property that all cochains involved are constructed by rational processes from the infinitesimal of the deformation. This could not be shown by the methods used in the solutions by Kontsevich, 9, and Tamarkin 12. (Dolgushev, 4, has more recently shown, however, that if a solution exists, then so does a rational one.) We conjecture that for the special case of quantization of manifolds, Kontsevich’s solution can be reduced to that yielded by Theorem 9. If so, then the integrals over configuration spaces employed in his solution must all have rational values, something of number-theoretic interest.

Another property of the present solution is that the problem of extending a local solution to a global one does not arise, since the infinitesimal, $\pi$, is already a global object; its exponential does not depend on a choice of coordinate system, which is, in effect, just a choice of representation. By contrast, Kontsevich and Tamarkin first prove the possibility of quantization for $\mathbb{R}^d$. A separate argument is then needed to pass to the global case.

Attempts to solve the quantization problem have all previously been concerned with manifolds, while the UDFs constructed here are applicable not only to non-commutative algebras, but in the commutative case, also to singular varieties.

We conclude with some conjectures. The cochains of an algebra which are representable relative to a fixed base of derivations form a subcomplex of the Hochschild complex which is closed under the composition products; it is a Gerstenhaber sub-algebra. We conjecture that its cohomology is formal. If so, then the formality theorem of Kontsevich would be a special case. Finally, we conjecture that the exponentials of basic separable cocycles of dimension 3 can be related to Drinfeld’s associators.

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