On the measure of entanglement

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In this paper we present the novel qualities of entanglement of formation for general (so also infinite dimensional) quantum systems. A major benefit of our presentation is a rigorous description of entanglement of formation. In particular, we indicate how this description may be used to examine optimal decompositions. Illustrative examples showing the method of estimation of entanglement of formation are given.

I. INTRODUCTION

The problem of quantum entanglement of mixed states has attracted much attention recently and it has been widely considered in different physical contexts (cf. [2] and references therein, see also [10]). Due to recent works by Peres [1] and Horodeccy [2, 3] there exists a simple criterion allowing one to judge whether a given density matrix $\rho$ representing a $2 \times 2$ or $2 \times 3$ composite system is separable. On the other hand, the definition of a measure of entanglement for general quantum systems as well as the problem of finding operational, sufficient and necessary condition for separability in higher dimensions remain still open (cf. [2], [3], and [4] and references therein).

In this paper we are concerned with the entanglement of formation, EoF, introduced in [3]. Let us stress that the principal motivation for our generalization of the definition of EoF follows from foundations of Quantum Mechanics; a quantum system is described by infinite dimensional Hilbert space. Further, to indicate that this concept stems from mathematical structure of tensor product we develop the theory of entanglement of formation in general terms of composite systems. Moreover, we look more closely at the original definition of EoF. Namely, there is a difficulty in implementing the definition given by Bennett et al in the sense that it is not clear why the operation of taking min over the set of all decomposition of the given state into finite convex combination of pure states is well defined (for details see subsection Optimal decompositions is Section V). To overcome this problem and to have a measure with nice topological properties we shall use the theory of decomposition which is based on the theory of compact convex sets and boundary integrals. The paper is organized as follows. In Section II we set up notation and terminology, and we review some of the standard facts on the theory of decomposition. Section III contains the definition of entanglement of formation, EoF, with the proof that EoF is equal to zero if and only if a state is a separable one. In other words, EoF, leads to well established establishment of separability in section IV we review properties of EoF. Namely, we indicate how techniques based on decomposition theory can be used to study EoF. Moreover, some simple examples showing the method of estimation of EoF are given. Furthermore, the proof of convexity and detailed study of topological properties of EoF are obtained. In particular, it is shown that the family of maximally entangled states is a subset of pure states. In the a final section V, we present some other examples of explicitly calculated EoF and we clarify the relation between our and the Bennett’s et al definition. Furthermore, we provide a detailed exposition of the concept of optimal decompositions. Also, some remarks concerning uniqueness of measure of entanglement are given.

II. PRELIMINARIES

Let us consider a composite system $1 + 2$ and its Hilbert space of the pure states $\mathcal{H}_1 \otimes \mathcal{H}_2$ where $\mathcal{H}_i$ is the Hilbert space associated to subsystem $i$ ($i = 1, 2$). Let $\mathcal{B}(\mathcal{H}_i)$, denote the set of all bounded linear operators on $\mathcal{H}_i$ while $\mathcal{M}$ stands for a (unital) $C^*$-subalgebra of $\mathcal{B}(\mathcal{H}_2)$. We will assume that $\mathcal{H}_1$ is finite dimensional space (in concluding remarks, Section V, we will indicate how to dispense with that assumption). $\mathcal{H}_2$ will be an arbitrary (infinite dimensional, separable) Hilbert space. In other words, the composite system consists of small subsystem and a big heat-bath, rather typical situation for concrete physical problems. Any density matrix (normal state) on $\mathcal{H}$ determines uniquely a linear positive, normalized, functional $\omega_\rho(\cdot) \equiv \omega(\cdot) \equiv Tr{\rho \cdot}$ on $\mathcal{B}(\mathcal{H})$ which is also called a state.

We will assume the Ruelle’s separability condition for $\mathcal{M}$ (cf. [2], [3], [12]): a subset $\mathcal{F}$ of the set of all states $\mathcal{S}$ of $\mathcal{M}$ satisfies separability condition if there exists a sequence $\{M_n\}$ of sub-$C^*$-algebras of $\mathcal{M}$ such that $\bigcup_{n \geq 1} M_n$ is dense in $\mathcal{M}$, and each $M_n$ contains a closed, two-sided, separable ideal $\mathcal{I}_n$ such that

$$\mathcal{F} = \{ \omega; \omega \in \mathcal{S}, ||\omega|_{\mathcal{I}_n}|| = 1, n \geq 1 \} \quad (1)$$

This condition leads to a situation in which the subsets of states have good measurability properties. Furthermore, one can verify that this separability condition is satisfied for two important cases:
• $\mathcal{M}$ is a separable $C^*$-algebra. Then $\mathcal{S}$ is metrizable and Borel and the Baire structures on $\mathcal{S}$ coincide. We put $\mathcal{F} = \mathcal{S}$ in that case.

• $\mathcal{M} = \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and $\mathcal{F}$ is the set of all density matrices (normal states).

Thus, the separability condition covers the basic models of quantum mechanics and we will restrict our attention to models satisfying this condition. However, generalizations of our approach are possible.

We recall that the density matrix $\varrho$ (state) on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is called separable if it can be written or approximated (in the norm) by the density matrices (states) of the form:

$$\varrho = \sum p_i g_i^1 \otimes g_i^2 \quad \left( \omega(\cdot) = \sum p_i (\omega_1^i \otimes \omega_2^i)(\cdot) \right)$$

where $p_i \geq 0$, $\sum_i p_i = 1$, and $g_i^2$ are density matrices on $\mathcal{H}_\alpha$, $\alpha = 1, 2$, and $(\omega_1^i \otimes \omega_2^i)(A \otimes B) \equiv \omega_1^i(A) \cdot \omega_2^i(B) \equiv (Tr g_i^1 A) \cdot (Tr g_i^2 B) \equiv Tr (g_i^1 \otimes g_i^2 \cdot A \otimes B)$.

Now, for the convenience of the reader, we introduce some terminology and give a short summary of results from convexity and Choquet theory that we shall need in the sequel (for details see [13], [14], [19], and [12]). Let $A$ stand for a $C^*$-algebra. From now on we make the same assumption of separability for $A$ which was posed for $\mathcal{M}$.

In next sections, by a slight abuse of notation we will write $A$ for $B(\mathcal{H}_1) \otimes \mathcal{M}$. By $\mathcal{S}$ we will denote the state space of $A$, i.e., the set of linear, positive, normalized, linear functionals on $A$. We recall that $\mathcal{S}$ is a compact convex set in the $*$-weak topology. Further, we denote by $M_1(\mathcal{S})$ the set of all probability Radon measures on $\mathcal{S}$. It is well known that $M_1(\mathcal{S})$ is a compact subset of the vector space of real, regular Borel measures on $\mathcal{S}$. After these preliminaries let us recall the concept of barycenter $b(\mu)$ of a measure $\mu \in M_1(\mathcal{S})$:

$$b(\mu) = \int d\mu(\varphi)\varphi$$

where the integral is understood in the weak sense. The set $M_\omega(\mathcal{S})$ is defined as a subset of $M_1(\mathcal{S})$ with barycenter $\omega$, i.e.

$$M_\omega(\mathcal{S}) = \{ \mu \in M_1(\mathcal{S}), b(\mu) = \omega \}$$

$M_\omega(\mathcal{S})$ is a convex closed subset of $M_1(\mathcal{S})$, hence compact in the weak *-topology. Hence, it follows by the Krein-Milman theorem that there are “many” extreme points in $M_\omega(\mathcal{S})$. We say the measure $\mu$ is simplicial if $\mu$ is an extreme point in $M_\omega(\mathcal{S})$. We denote by $E_\omega(\mathcal{S})$ the set of all simplicial measures in $M_\omega(\mathcal{S})$. Finally, we will need the concept of orthogonal measures. To define that concept one introduces firstly the notion of orthogonality of positive linear functionals on $A$: given positive functionals $\phi, \psi$ on $A$ we say that $\phi$ and $\psi$ are orthogonal, in symbols, $\phi \perp \psi$, if for all positive linear functionals $\gamma$ on $A$, $\gamma \leq \phi$ and $\gamma \leq \psi$ imply that $\gamma = 0$.

Turning to measures, let $\mu$ be a regular non-negative Borel measure on $\mathcal{S}$ and let $\mu_\psi$ denote the restriction of $\mu$ to $\mathcal{V}$ for a measurable set $V$ in $\mathcal{S}$, i.e. $\mu_\psi (T) = \mu (V \cap T)$ for $T$ measurable in $\mathcal{S}$. If for all Borel sets $V$ in $\mathcal{S}$ we have

$$\int_\mathcal{S} \varphi d\mu_\psi (\varphi) \perp \int_\mathcal{S} \varphi d\mu_{\mathcal{S}\setminus \mathcal{V}} (\varphi) \quad (4)$$

we say that $\mu$ is an orthogonal measure on $\mathcal{S}$. We recall that the set of all orthogonal measures on $\mathcal{S}$ with barycenter $\omega$, $O_\omega(\mathcal{S})$, forms a subset (in general proper) of $E_\omega(\mathcal{S})$, i.e. $O_\omega(\mathcal{S}) \subset E_\omega(\mathcal{S})$.

III. ENTANGLEMENT OF FORMATION

Let us define, for a state $\omega$ on $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{M}$ the following map:

$$(r\omega)(A) \equiv \omega(A \otimes 1) \quad (5)$$

where $A \in \mathcal{B}(\mathcal{H}_1)$. Clearly, $r\omega$ is a state on $\mathcal{B}(\mathcal{H}_1)$. One has

Let $(r\omega)$ be a pure state on $\mathcal{B}(\mathcal{H}_1)$ (so a state determined by a vector from $\mathcal{H}_1$). Then $\omega$ can be written as a product state on $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{M}$.

The proof of that statement can be extracted from [10]. However, for the convenience of the reader we provide the basic idea of the proof. It is enough to consider the case with an arbitrary but fixed positive $B$ in unit ball of $\mathcal{M}$ such that $0 < \omega(1 \otimes B) < 1$. Then $(r\omega)(A)$ can be written as

$$(r\omega)(A) = \omega(1 \otimes B) \omega^I(A) + (1 - \omega(1 \otimes B))\omega^II(A) \quad (6)$$

where $\omega^I(A) = \frac{1}{\omega(1 \otimes B)} \omega(A \otimes B)$ and $\omega^II(A) = \frac{1 - \omega(1 \otimes B)}{\omega(1 \otimes (1 - B))} \omega(A \otimes (1 - B))$. Clearly $\omega^I$ and $\omega^II$ are well defined linear, positive functionals (states) on $\mathcal{B}(\mathcal{H}_1)$. Hence, the purity of $(r\omega)$ implies $\omega^I = \omega^II$. Consequently, $\omega(A \otimes B) = \omega(A \otimes 1)\omega(1 \otimes B)$. The rest is straightforward so the proof is completed. (For more details we refer the reader to [10].)

Conversely, there is another result in operator algebras saying that if $\omega$ is a state on $\mathcal{B}(\mathcal{H}_1)$ then there exists a state $\omega'$ over $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{M}$ which extends $\omega$. If $\omega$ is a pure state of $\mathcal{B}(\mathcal{H}_1)$ then $\omega'$ may be chosen to be a pure state of $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{M}$ (cf. [12]).

Now we are in position to give a modification and discuss the definition of entanglement of formation (cf. [3]).

Let $\omega$ be a state on $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{M}$. The entanglement of formation, $\text{EoF}$, is defined as

$$E(\omega) = \inf_{\mu \in M_\omega(\mathcal{S})} \int_\mathcal{S} d\mu(\varphi) S(\varphi) \quad (7)$$
where $S(\cdot)$ stands for the von Neumann entropy, i.e. $S(\varphi) = -\text{Tr}_opi \log \varphi$ where $\varphi$ is the density matrix determining the state $\varphi$.

In order to comment the above definition we recall that the map $r$ and the function $S$ are ($\ast$-weakly) continuous. At this point we want to strongly emphasize that we use the entropy function $S$ only to respect the tradition. Namely, to have a well defined concept of EoF we need a concave non-negative continuous function which vanishes on pure states (and only on pure states). In our case, with the first subsystem being finite, the von Neumann entropy meets these conditions. Thus, we define EoF as infimum of integrals evaluated on continuous function and the infimum is taken over the compact set. Therefore, the infimum is attainable, i.e. there exists a measure $\mu_0 \in M_\omega(S)$ such that

$$E(\omega) = \int_S d\mu_0(\varphi)S(r\varphi) \quad (8)$$

and

$$\omega = \int_S d\mu_0(\varphi)\varphi \quad (9)$$

Now we want to show that $F \ni \omega \mapsto E(\omega)$ is equal to 0 only for separable states (we recall that $F$ stands for the subset of states satisfying Ruelle’s condition, cf. Section II). Assume $E(\omega) = 0$. Then

$$\int_S d\mu_0(\varphi)S(r\varphi) = 0, \quad (10)$$

for some probability measure $\mu_0$. As $S(r\varphi) \geq 0$ and it is the continuous function we infer that $S(r\varphi) = 0$ for each $\varphi$ in the support of $\mu_0$. But, as the entropy is a concave function we have

$$S \circ r(\varphi) \geq \int d\xi(\nu)S(r\nu) \quad (11)$$

for any positive measure $d\xi$ on $S$ such that $\varphi = \int d\xi(\nu)\nu$. In particular, taking a measure supported on pure states (such decomposition always exists under the assumed separability condition) we infer $S(r\nu) = 0$ so $r\nu$ is a pure state and consequently $\nu$ is a product state. So $\varphi$ is a convex combination of product states. Finally, as $\omega = \int_S d\mu_0(\varphi)\varphi$ and $\mu_0$ can be well approximated by finite measures (see (10)) we infer that $\omega$ can be approximated by convex combinations of product states, so $\omega$ is a separable state.

Now, let us assume that $\omega$ is a separable state, i.e. $\omega$ can be approximated by convex combinations of product states $\omega_i^{(N)}$:

$$\omega = \lim_{N} \sum_{i=1}^{N} \lambda_i^{(N)} \omega_i^{(N)} \quad (12)$$

Define

$$\mu^N = \sum_{i=1}^{N} \lambda_i^{(N)} \delta_{\omega_i^{(N)}} \quad (13)$$

where $\delta_{\omega_i^{(N)}}$ are the Dirac measures of the point $\omega_i^{(N)}$. Considering the weak limit of $\int d\mu^N(\varphi)\varphi$ we can infer that there is a measure $\mu$ such that

$$\int d\mu(\varphi)\varphi = \omega, \quad \int d\mu(\varphi)S(r\varphi) = 0. \quad (14)$$

So we arrived to

**Theorem 1** A state $\omega \in F$ is separable if and only if EoF $E(\omega)$ is equal to 0.

### IV. Properties of EOF

**A. Relations to other decompositions**

Let us discuss some relations between decompositions used in our definition of EoF and other types of decompositions. Assume that the state $\omega$ is separable, so there is a measure $\mu_0 \in M_\omega(S)$ such that $\int d\mu_0(\varphi)S(r\varphi) = 0$. But as we consider non-negative function, and positive measures this implies that there is a simplicial measure $\mu_0^S$ (in fact there can be many such measures) such that $\int d\mu_0^S(\varphi)S(r\varphi) = 0$. In other words, the infimum is attainable on the set of simplicial measures $E_\omega(S)$ (for more detailed discussion on the role of simplicial measures in the description of EoF see subsection Optimal decompositions in Section V).

On the other hand, as $O_\omega(S) \subset E_\omega(S)$ we have

$$\inf_{\mu \in E_\omega(S)} \int d\mu(\varphi)S(r\varphi) \leq \inf_{\mu \in O_\omega(S)} \int d\mu(\varphi)S(r\varphi) \quad (15)$$

In general we can not expect the equality in (15). Namely, there are examples of simplicial measures which are not orthogonal (cf. (10)). So finding an orthogonal measure such that “inf” is attained we can infer that the state is separable but not conversely. To be more clear, let us recall some algebraic aspects of decomposition theory (cf. (12)) which are related to orthogonal measures. A finite convex decomposition of $\omega \in S$ corresponds to a finite decomposition of identity $1 = \sum_{i} T_i$, $T_i \geq 0$ within the commutant $\pi_\omega(A')$. The simplest form of such decomposition occurs when the $T_i$ are mutually orthogonal projections. This type of decomposition corresponds to that determined by orthogonal measure. So, taking the spectral resolution of density matrix $\varphi$, we obtain the very special (subcentral) orthogonal decomposition. Therefore, if we restrict ourselves to decomposition induced by spectral resolution of $\varphi$, in general, we can not expect to attain $\inf_{\mu \in M_\omega(S)} \int d\mu(\varphi)S(r\varphi)$, see also Subsection VC.
To illustrate the question of computation of EoF we start with very simple models.

1. The von Neumann entropy (for finite systems) is maximal for the state of the form $\omega_{\varrho_m}(A) = T_r \varrho_m A$ with $\varrho_m = \frac{1}{\dim \mathcal{H}} 1$ (dim stands for dimension). For such state it is equal to $\ln(\dim \mathcal{H})$ and this is the maximal value of $E$.

2. Let us consider $2 \times 2$ system with $\mathcal{H}_1 \cong \mathcal{H}_2$ (so $\dim \mathcal{H}_1 = 2 = \dim \mathcal{H}_2$) and the single state $\Psi_-$ defined as $\Psi_- = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)$. Here we adopt a notation of quantum mechanics by writing $|01\rangle \equiv e_0 \otimes e_1$ where $\{e_0, e_1\}$ is a basis in $\mathcal{H}_1$, etc. Write $\omega_{\Psi_+}(A) = Tr\{|\Psi_+\rangle \langle \Psi_+| A \otimes B\}$ where $A \in B(\mathcal{H}_1)$ while $B \in B(\mathcal{H}_2)$. Then $r\omega_{\Psi_-}(A) = Tr\{|\Psi_-\rangle \langle \Psi_-| A \otimes 1\} = Tr\{|1\rangle A\}$. So $E(\omega_{\Psi_-}) = \ln 2$.

3. Let us consider $d \times d$ system and so called maximally entangled state $|\Psi_d^+\rangle = \sqrt{d} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle$ where $\{i\rangle$ is a basis in $\mathcal{H}_1 = \mathcal{H} = \mathcal{H}_2$. Again, let us define $\omega_{\Psi_d^+}(A \otimes B) = Tr\{|\Psi_d^+\rangle \langle \Psi_d^+| A \otimes B\}$ and consider $r\omega_{\Psi_d^+}$. It is easy to note that $r\omega_{\Psi_d^+}(A) = Tr\{|1\rangle A\}$. Hence $E(\omega_{\Psi_d^+}) = \ln d$, so $E$ attains its maximal value.

The just listed results are easy to show since there is no question concerning the non-uniqueness of decomposition of the (pure) state $\omega$ into pure states.

### C. Convexity of EoF

To prove convexity of EoF let us show that the set $M_{1,\omega_1+\lambda_2\omega_2}(S)$ contains the sum of the sets $\lambda_1 M_{\omega_1}(S)$ and $\lambda_2 M_{\omega_2}(S)$ where $\lambda_1$ and $\lambda_2$ are non-negative numbers such that $\lambda_1 + \lambda_2 = 1$. To see this we recall (see e.g. [12] or [18]) that $\mu \in M_1(S)$ if and only if $\mu(f) \geq f(\omega)$ for any continuous, real-valued, convex function $f$. Thus

$$\lambda_1 \mu_1 + \lambda_2 \mu_2(f) \geq \lambda_1 f(\omega_1) + \lambda_2 f(\omega_2) \geq f(\lambda_1 \omega_1 + \lambda_2 \omega_2)$$

implies the above stated relation between sets. Hence

$$E(\lambda_1 \omega_1 + \lambda_2 \omega_2) = \inf_{\mu \in M_{1,\lambda_1 \omega_1+\lambda_2 \omega_2}(S)} \int d\mu(\varphi) S(r\varphi) \leq \lambda_1 \inf_{\mu \in M_{1,\omega_1}(S)} \int d\mu(\varphi) S(r\varphi) + \lambda_2 \inf_{\mu \in M_{1,\omega_2}(S)} \int d\mu(\varphi) S(r\varphi) = \lambda_1 E(\omega_1) + \lambda_2 E(\omega_2)$$

Consequently, the function $S \ni \omega \mapsto E(\omega)$ is the convex one.

### D. Subadditivity

Consider the tensor product of von Neumann algebras $B(\mathcal{H}_1) \otimes M \otimes B(\mathcal{H}_2) \otimes M$ and a state $\omega \otimes \omega$ over it where $\omega$ is a state on $B(\mathcal{H}_1) \otimes M$. We observe

$$E(\omega \otimes \omega) = \inf_{\mu \in M_{1,\omega_1}(S)} \int d\mu(\nu) S_{1+2}(r\nu) \leq \inf_{\mu \times \nu \in M_1(S) \times M_1(S)} \int d\mu(\nu) \int d\nu(r') S_{1+2}(r \otimes \nu \otimes \nu') \leq \inf_{\mu_1 \times \mu_2 \in M_1(S) \times M_1(S)} \int d\mu_1(\nu) \int d\mu_2(\nu')(S_1(r\nu) + S_1(r')) = 2E(\omega)$$

where $S_T$ denotes the set of all states on $B(\mathcal{H}_1) \otimes M \otimes B(\mathcal{H}_2) \otimes M$. $S_{1+2}(S_1)$ the von Neumann entropy on $B(\mathcal{H}_1) \otimes B(\mathcal{H}_2)$ (B(\mathcal{H}_1)) respectively. The last inequality follows from subadditivity of the von Neumann entropy. Consequently, EoF has also a form of subadditivity. Applying the above argument to $E(\omega \otimes \cdots \otimes \omega)$ one can consider the “density” of EoF and treat $E(\omega)$ as an extensive (thermodynamical) quantity. This feature of EoF seems to be important in quantum information (cf. [2]).

### E. Topological properties of EoF

Now we wish to examine the question of continuity of EoF. To describe that topological property we shall need some preliminaries. Let us consider $M_1(S)$ and $S$ as two compact spaces and a continuous mapping $b$ of $M_1(S)$ unto $S$ given by $M_1(S) \ni \mu \mapsto b(\mu) = \int_S d\nu(\mu, \nu)$, so $b(\mu)$ is the barycenter of the measure $\mu$. Moreover, let us consider the equivalence relation $E_0(b)$ on the set $M_1(S)$ determined by the decomposition $\{b^{-1}(\omega)\}_{\omega \in S}$. $b(\mu)$ is the quotient topology, so $q$ is a natural (quotient) topology. As $b$ is a continuous mapping of the compact (Hausdorff) space $M_1(S)$ to the compact (Hausdorff) space $S$ then the equivalence relation $E_0(b)$ is closed. We wish to represent the mapping $b : M_1(S) \rightarrow S$ as the composition $q \circ q$ of the natural mapping $q$ with the mapping $b$ of the quotient space $M_1(S)/E(b)$ onto $S$ defined by letting $b(b^{-1}(\omega)) = \omega$. It is an easy observation that the mapping $b$ is continuous. Hence we have

$$\begin{array}{ccc}
M_1(S) & \xrightarrow{b} & S \\
q \downarrow & & \downarrow q \\
M_1(S)/E(b) & \xrightarrow{\overline{b}} & S
\end{array}$$
In particular, $\bar{b}$ is one-to-one continuous mapping of $M_1(S)/E(b)$ onto $S$. We want to show that $\bar{b}$ is a homeomorphism. To prove this we observe that $q$ is the continuous mapping carrying the compact topological space $M_1(S)$ onto topological space $M_1(S)/E(b)$. Then, $M_1(S)/E(b)$ is pre-compact (not necessary Hausdorff space). But, then $\bar{b}$ is the continuous mapping of the pre-compact space $M_1(S)/E(b)$ onto compact (Hausdorff) space $S$. Thus, $\bar{b}$ is a homeomorphism. Therefore, we arrived to

**Proposition 1** $\bar{b}$ is a homeomorphism; i.e. the mapping $\bar{b}$ is quotient.

Now we wish to describe equivalence classes in $M_1(S)/E(b)$. Let $\omega_0 \in S$. We observe

$$ b^{-1}(\omega_0) = \{ \mu \in M_1(S) : \int v d\mu (v) = \omega_0 \} \tag{19} $$

In other words, $b^{-1}(\omega_0)$ is equal to the set $M_{\omega_0}(S)$ of all probabilistic measures on $S$ which represent the point $\omega_0 \in S$. We recall, see [45], that $\mu \in M_{\omega_0}(S)$ is equivalent to $\mu \sim \delta_{\omega_0}$, i.e. that $\mu$ is equivalent to the Dirac measure $\delta_{\omega_0}$ where the equivalence of (probabilistic) measures $\mu$ and $\mu'$ is defined as

$$ \int v d\mu (v) = \int v d\mu' (v) \tag{20} $$

But $\delta_{\omega_0} \sim \mu$, in turn, is equivalent to (cf. [45]) $\delta_{\omega_0} - \mu \in \mathcal{N}(S)$ where $\mathcal{N}(S)$ is the annihilator of $A(S)$ in the dual pair $< C_{BM}(S), M_{BM}(S) >$. Here, $A(S) \cap C_{BM}(S), M_{BM}(S))$ is the set of all continuous real-valued affine functions on $S$ (the vector space of all real-valued continuous functions, the vector space of all real measures on $S$ respectively). Now it should be clear that an equivalence class $[\mu]$ in $M_1(S)/E(b)$ is equal to $\{ \delta_0 + \mu, \mu \in \mathcal{N}(S), \mu (f) \geq -f(\omega_0) \}$ for any $0 \leq f \in C_{BM}(S)$ with some fixed $\omega_0 \in S$.

In order to complete our discussion of the diagram we should examine topological properties of the set-valued map $b^{-1}$.

**Proposition 2** The set-valued map $b^{-1}$ is upper semi-continuous.

**Proof:** The map $b^{-1}$ is lower (upper) semicontinuous (cf. [17]) if and only if for every closed set $K \subset M_1(S)$ the set $\mathcal{O}_K = \{ \omega : b^{-1}(\omega) \subset K \}$ (the set $\mathcal{O}_U = \{ \omega : b^{-1}(\omega) \cap K \neq \emptyset \}$) is closed in $M_1(S)$. To examine upper semicontinuity of $b^{-1}$ let us consider a net $\{ \omega_n \} \subset \mathcal{O}_U$ with a limit $\omega_0$. Furthermore, let us choose measures $\{ \mu_n \in b^{-1}(\omega_n) \cap K \}$. As $K$ is a compact subset there exists a convergent subnet $\{ \mu_\beta \}$ such that $\mu_\beta \rightarrow \mu \in K$. Since $\int_S \nu d\mu (v) = \omega_\beta$ we infer that

$$ \omega_\beta \rightarrow \omega = \int_S \nu d\mu (v) \tag{21} $$

Then, the uniqueness of the limit implies $\omega_\beta = \omega$ and the proof of upper semicontinuity of $b^{-1}$ is complete.

Having fully clarified topological relations among $S$, $M_1(S)$, and $M_1(S)/E(b)$ we wish to prove

**Proposition 3** $EoF, S \ni \omega \rightarrow E(\omega)$, is the continuous function.

**Proof:** We start with an easy proof of lower semicontinuity of $EoF$. To this end let $\{ \omega_n \}$ be a net with a limit $\omega$ and let $E(\omega_n) \leq s$, where $s$ is a real number. To show that $E(\omega) \leq s$ we take $\epsilon > 0$ and choose $\mu_n \in M_{\omega_n}(S)$ such that $\mu_n (S \circ r) < s + \epsilon$. As $M_1(S)$ is a compact set there exists a convergent subnet $\{ \mu_\beta \}$ with the limit $\mu_0$. Let $\hat{A}$ be an affine, real-valued continuous function on $S$. Then

$$ \hat{A}(\omega_\beta) = \mu_\beta (\hat{A} \rightarrow \mu_0 (\hat{A}) \tag{22} $$

and $\hat{A}(\omega) \rightarrow \hat{A}(\omega)$. Thus $\mu_0 \in M_{\omega}(S)$. Therefore, $s + \epsilon \geq \lim \mu_\beta (S \circ r) = \mu_0 (S \circ r) \geq E(\omega)$. Consequently, $E(\omega) \leq s$ and the proof of lower semicontinuity of $EoF$ is complete.

Now, let us consider the question of upper semicontinuity of $E(\omega)$. Again, let $\omega_n$ be a net with a limit point $\omega_0$. Take $s$ such that $E(\omega_n) \geq s$. We observe that for any $\mu_n \in M_{\omega_n}(S)$, $\mu_n (S \circ r) \geq s$. Again, the use of a convergent subnet $\{ \mu_\beta \}$ with a limit $\mu_0$ implies that $s \leq \lim \mu_\beta (S \circ r) = \mu_0 (S \circ r)$, where $\mu_0 \in M_{\omega_0}(S)$. Thus, to prove upper semicontinuity of $E(\omega)$ it is enough to show that any $\mu \in M_{\omega_0}(S)$ can be obtain as a limit of $\{ \mu_\beta \}$. i.e. $M_{\omega_0}(S) \ni \mu = \lim \mu_\beta$ where $\mu_\beta \in M_{\omega_0}(S)$.

In particular, we want to have

$$ \land_{\epsilon > 0} \land_{\mu \in M_{\omega_0}(S)} \lor_{\mu_\beta \in M_{\omega_0}} |\mu (f) - \mu_\beta (f)| < \epsilon \tag{23} $$

for a continuous function $f$ on $S$. Assume contrary, i.e.,

$$ \lor_{\epsilon > 0} \lor_{\mu \in M_{\omega_0}(S)} \land_{\mu_\beta \in M_{\omega_0}} |\mu (f) - \mu_\beta (f)| \geq \epsilon \tag{24} $$

for a continuous function $f$. We note (cf. [17]) that the upper semicontinuity of $b^{-1}$ implies that for every open set $U \subset M_1(S)$ the set $\{ \omega : b^{-1}(\omega) \subset U \}$ is open. Now, assuming (24) one can find the neighbourhood $U_\mu$ of $\mu$ which does not contain any $\mu_\beta$. But, $\{ \omega : b^{-1}(\omega) \subset U_\mu \}$ is a neighbourhood of $\omega_0$. The convergence $\omega_\beta \rightarrow \omega_0$ implies that each neighbourhood of $\omega_0$ should contain states of the form $\omega_\beta$. Thus, also, $\{ \omega : b^{-1}(\omega) \subset U_\mu \}$ should contains many $\omega_\beta$. Hence, one can find $M_{\omega_0}(S) \subset U_\mu$. But this contradicts (24). Therefore, (23) holds and the proof of upper continuity of $E(\omega)$ is complete. This completes the proof of Proposition.
As $S \ni \omega \mapsto E(\omega)$ is a continuous convex function, the application of the Bauer maximum principle leads to:

**Corollary 1** $E(\omega)$ attains its maximum at an extremal point of $S$, so the family of maximally entangled states is a subset of pure states.

V. EXAMPLES AND CONCLUDING REMARKS

A. Examples II

Let us begin this final section with some other illustrative examples showing the usefulness of EoF.

1. Let $\omega = \sum_k \omega_k$ be a decomposition of the state $\omega$. Then, an application of convexity would lead to the following estimation of entanglement of $\omega$: $E(\omega) \leq \max_k E(\omega_k)$.

2. In the discussion of entangled states the positive partial transposition criterion plays an important role ($\Box$). To consider that question, let us put $\mathcal{M} = \mathcal{B}(\mathcal{H}_2)$ with finite dimensional Hilbert space $\mathcal{H}_2$. We recall that the map $\alpha : \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with $\alpha = \text{id} \otimes \tau$ where $\tau$ is a transposition map of the matrix representation of an arbitrary $B \in \mathcal{B}(\mathcal{H}_2)$ in a certain fixed basis, provides essential ingredient of that criterion. Let us define $(\alpha^d \omega)(A \otimes B) \equiv \omega(\alpha(A \otimes B))$ and let us note

$$\begin{align*}
(ra^d \omega)(A) &= (a^d \omega)(A \otimes 1) = \omega(\alpha(A \otimes 1)) = \omega(A \otimes \tau(1)) = (r \omega)(A)
\end{align*}$$

Consequently, the partial transposition does not change the measure of entanglement. Therefore, the basic point of that criterion is that $id \otimes \tau$ is not a completely positive map. For further details on relations between entanglement and positive maps see [1].

3. Let us consider $d \times d$ system with the corresponding Hilbert space $\mathcal{H} \equiv \mathcal{H}_1 \otimes \mathcal{H}_2$ and let $P$ be a projector such that $PH$ does not contain product states. We recall that such projectors are related to the concept of unextendible product bases [2]. Let us define the state $\omega_P$ as $\omega_P(A \otimes B) = (TrP)^{-1}(Tr(P \cdot A \otimes B))$. We want to judge whether $\omega_P$ is a separable state. To this end let us consider an arbitrary decomposition of $\omega_P$, i.e.,

$$\omega_P(A \otimes B) = (TrP)^{-1}(TrP A \otimes B) = (26)$$

where operators $a_k \geq 0$ are defined on $\mathcal{H}$ and satisfy $\sum_k a_k = 1$ while $\omega_k$ stands for the state determined by $Pa_k$. Assume $\omega_P$ is a separable state. Then, there is a decomposition $\omega_P = \sum_k \lambda_k \omega_k$ such that $\rho_{\omega_k} \equiv \text{pure state for any } k$. But, then $\omega_k$ would be a product state, i.e.

$$\omega_k(A \otimes B) = Tr\varrho_k^1 \otimes \varrho_k^2 A \otimes B \quad (27)$$

This would imply

$$\varrho_k^1 \otimes \varrho_k^2 = \text{constant} \cdot Pa_k P \quad (28)$$

But, this is impossible as $PH$ does not contain product (vector) states. Consequently, $E(\omega_P) > 0$.

B. Comparison with the Bennett, DiVincenzo, Smolin and Wooters definition of EoF

The presented examples with explicitly calculated $E(\omega)$ suggest a close relation of our EoF with the original definition of EoF, given by Bennett et al, which will be denoted by $\text{EoF}_B$. Obviously, $EoF \leq \text{EoF}_B$. To examine the converse inequality we start with a simple observation that

$$\begin{align*}
\inf_{\mu \in \mathcal{M}_c(S)} \int d\mu(\nu)S(\rho \nu) &= \inf\{ \lambda_i S(\rho \nu_i) : \nu = \sum_{i=1}^n \lambda_i \nu_i \}\text{ (convex sum)}
\end{align*}$$

and the first infimum is attained for some $\mu \in \mathcal{M}_c(S)$.

The above observation follows from the fact that each measure $\mu$ can be (*weakly*) approximated by measures with finite support. On the other hand, measures concentrated on $S_p$, where $S_p$ is the set of all pure states, are known to be maximal with respect to the order $\mu \prec \nu$ if and only if $\mu(f) \leq \nu(f)$ for any convex, real-valued convex function $f$, cf. [13] or [15], so minimal on the set of all concave functions. It particular, such the measure is minimal on $S \circ r$. Apparently, the maximality of a measure (on all convex functions) is too strong demand and therefore to get the converse inequality between EoF and $\text{EoF}_B$ some extra argument should be given (see the next subsection).

C. Optimal decompositions

In this section we wish to examine the question of existence of the very special type of decompositions, so called optimal decompositions. A decomposition $\omega = \sum_j \lambda_j \theta_j$, where $\{\theta_j\}$ are pure states, such that the infimum in the definition of EoF is attained will be called an optimal decomposition. In other words, the infimum is attained by
a measure $\mu_0$ with finite support contained in the set of all pure states. Thus

$$E(\omega) = \inf_{\mu \in M_\omega(S)} \int_S S(rg)d\mu(\rho) = \int_S d\mu_0(\rho)S(rg)$$

with $supp\mu_0 = \{\varrho_1, \ldots, \varrho_n\}$, $n < \infty$ and $\varrho_i \in \mathcal{S}_p$. Clearly, $\mu_0 = \sum_1^n \lambda_i \delta_{\varrho_i}$, where $\delta_{\varrho_i}$ stands for the Dirac measure, $\{\varrho_i\}$ are pure states and $\omega = \sum \lambda_i \varrho_i$. We recall that each measure has maximal balayage and each state $\omega \in \mathcal{S}$ is the barycenter of a measure which is maximal for the order $\succ$ (for all necessary details see [12], [13], [18] or [19]). It is an easy observation that $\mu_0$ is a maximal measure.

In order to avoid misunderstanding we repeat: entropy is concave, maximality is defined for convex functions, so $\inf$ for "$f \mapsto -x\ln x$" means sup for "$x \mapsto x\ln x$".

We recall that maximality of the measure $\mu_0$ implies:

$$\mu_0 \text{ is supported by the set } \{\omega; \omega \in \mathcal{S}, f(\omega) = \overline{f}(\omega)\}$$

(31)

for all continuous convex functions on $\mathcal{S}$ where $\overline{f}$ is the upper envelope of $f$ i.e., $\overline{f}(\omega) = \inf \{f(\varrho); -\varrho \in \mathcal{P}(\mathcal{S}), g \geq f\}$. ($\mathcal{P}(\mathcal{S})$ stands for the set of continuous convex functions on $\mathcal{S}$). Moreover, if $\overline{f}$ is the upper envelope of a continuous function on $\mathcal{S}$ then

$$\overline{f}(\omega) = \sup_{\mu \in M_\omega(\mathcal{S})} \int \mu(d\nu)\overline{f}(\nu)$$

(32)

Denote the set of all maximal measures in $M_\omega(\mathcal{S})$ by $\mathcal{Z}_\omega(\mathcal{S})$. One can show (cf. [18]) that $\mathcal{Z}_\omega(\mathcal{S})$ is a face of $M_\omega(\mathcal{S})$. The set of all extremal measures in $\mathcal{Z}_\omega(S)$ will be denoted by $\mathcal{E}_\omega^Z(\mathcal{S})$, i.e.

$$\mathcal{E}_\omega^Z(\mathcal{S}) = \mathcal{E}_\omega(\mathcal{S}) \cap \mathcal{Z}_\omega(\mathcal{S})$$

(33)

where $\mathcal{E}_\omega(\mathcal{S})$ stands for simplicial measures (cf. Section IV). Let us consider (cf. [18])

$$F_0 = \{\mu \in M_\omega(\mathcal{S}); \mu(f) = \overline{f}(\omega)\}$$

(34)

where $f$ is a convex continuous function on $\mathcal{S}$. Obviously, by virtue of (32) the set $F_0$ is not empty. Let $\mu \in F_0$ and assume $\mu = \lambda_1 \mu_1 + (1-\lambda_1)\mu_2$ where $\mu_1, \mu_2 \in M_\omega(\mathcal{S})$ and $\lambda_1 \in (0, 1)$. Clearly, $\mu_1$ and $\mu_2$ should be in $F_0$. Thus, $F_0$ is a face. It is an easy observation that $F_0$ is the closed set with the property: $\mu \in F_0$, $\nu \in M_\omega(\mathcal{S})$ and $\mu \prec \nu$ implicate $\nu \in F_0$, i.e., $F_0$ is the hereditary upwards face. Then, the application of the Lumer existence theorem (see Proposition 1.6.4 in [18]) proves existence of a measure $\mu_0$ in $F_0 \cap \mathcal{E}_\omega^Z(\mathcal{S})$. Thus, we proved:

**Proposition 4** Maximum of the set $\{\mu(-S \circ r); \mu \in M_\omega(\mathcal{S})\}$ for a continuous convex function $-S$ is attained by a simplicial boundary measure.

Now, let us examine very special case. Namely, if the Hilbert space $\mathcal{H}$ of the composite system is of the finite dimension, say, $dim \mathcal{H} = n$ then $\mathcal{S}$ can be considered as a compact subset in the space $\mathbb{R}^{(2n)^2}$. On the other hand, a probability measure $\mu$ on a convex compact subset in $\mathbb{R}^{(2n)^2}$ is simplicial if and only if $\mu$ is supported by affinely independent set of (at most $(2n)^2 + 1$) points. Moreover, in the considered (finite dimensional) case the weak-* topology is metrizable. Hence, maximal measures are supported by extremal points in $\mathcal{S}$. Combining the just given results for the concave function $S \circ r$ we can infer the existence of optimal decomposition. Obviously, it does not exclude a possibility that infimum is attainable on some other measures. Nevertheless, for finite dimensional case we have: $EoF = EoF_B$.

Turning to others decompositions (cf. Section IV.A) we note that even in two dimensional case (so for the algebra of $2 \times 2$ matrices) one can write simplicial maximal measure that is not orthogonal (cf. [12]). Therefore, even for low dimensional case one can not say that $EoF$, $E(\omega) = \inf_{\mu \in C_\omega(S)} \mu(S \circ r)$.

Finally we want to point out that the proof existence of optimal decomposition depends only on the structure of the set $M_\omega(\mathcal{S})$.

**D. Remarks**

There is a frequently considered question of uniqueness of a measure of entanglement. We already observed, see comments following definition $EoF$ in Section III, that one can replace the von Neumann entropy by any continuous non-negative concave function which vanishes on pure states, e.g. $\varrho \mapsto \text{Tr}(\varrho(1-\varrho))$. Then, all argument can be repeated and we would arrive to a new measure of entanglement. As a matter of fact the only reason for our assumption of $dim \mathcal{H} < \infty$ was that in this case, the von Neumann entropy is continuous. Thus, to perform further generalization of $EoF$ one can replace $\varrho \mapsto -\text{Tr}(\ln \varrho)$ by another function from the just mentioned class. Clearly, changing the von Neumann entropy function, the argument leading to the subadditivity of $E(\omega)$ should be modified. The additional reason to use the von Neumann entropy throughout the paper follows from the fact that there is a nice relation (cf. [21], [22], [23]) between the entropy $H$ of subalgebra $\mathcal{M}_1$ in $\mathcal{M}_1 \otimes \mathcal{M}$ relative to the state $\omega$ $(\mathcal{M}_1, \mathcal{M}$ are von Neumann algebras), the von Neumann entropy of the restricted state and $EoF$:

$$H_{\omega, \mathcal{M}_1 \otimes \mathcal{M}}(\mathcal{M}_1) = S(\omega) - E(\omega)$$

(35)

It is clear that the main difficulty for calculating $H$ is encoded in $E(\omega)$. Therefore, our results concerning $EoF$ shed some new light on the nature of $H$.

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