BOGOMOLOV UNSTABILITY ON ARITHMETIC SURFACES

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Abstract. In this paper, we will consider an arithmetic analogue of Bogomolov unstability theorem, i.e. if $(E, h)$ is a torsion free Hermitian sheaf on an arithmetic surface $X$ and $\hat{\deg}((\text{rk } E - 1)\hat{c}_1(E, h)^2 - (2\text{rk } E)\hat{c}_2(E, h)) > 0$, then there is a non-zero saturated subsheaf $F$ of $E$ such that $\hat{c}_1(F, h|_F)/\text{rk } F - \hat{c}_1(E, h)/\text{rk } E$ lies in the positive cone of $X$.

0. Introduction

In [Bo], Bogomolov proved unstability theorem, namely, if a vector bundle $E$ on a complex projective surface $S$ satisfies an inequality $(\text{rk } E - 1)c_1(E)^2 - (2\text{rk } E)c_2(E) > 0$, then there is a saturated subsheaf $F$ of $E$ such that $c_1(F)/\text{rk } F - c_1(E)/\text{rk } E$ belongs to the positive cone of $S$. In this paper, we would like to consider an arithmetic analogue of Bogomolov unstability theorem.

Let $f : X \to \text{Spec}(O_K)$ be a regular arithmetic surface over the ring of integers of a number field $K$ with $f_*O_X = O_K$, and $\deg_K : \widehat{CH}^1(X)_\mathbb{R} \to \mathbb{R}$ the natural homomorphism given by

$$\deg_K : \widehat{CH}^1(X)_\mathbb{R} \xrightarrow{\deg} CH^1(X)_\mathbb{R} \otimes_K CH^1(X_K)_\mathbb{R} \xrightarrow{\deg} \mathbb{R}.$$ 

The positive cone $\widehat{C}_{++}(X)$ of $X$ is defined by the set of all elements $x \in \widehat{CH}^1(X)_\mathbb{R}$ with $\hat{\deg}(x^2) > 0$ and $\deg_K(x) > 0$. A torsion free Hermitian sheaf $(E, h)$ on $X$ is said to be arithmetically unstable if there is a non-zero saturated subsheaf $F$ of $E$ with

$$\frac{\hat{c}_1(F, h|_F)}{\text{rk } F} - \frac{\hat{c}_1(E, h)}{\text{rk } E} \in \widehat{C}_{++}(X),$$

where $h|_F$ is the Hermitian metric of $F$ given by the restriction of $h$ to $F$. The main theorem of this paper is the following.
Theorem A. If \((E, h)\) is a torsion free Hermitian sheaf on \(X\) and
\[
\widehat{\deg} \left((\text{rk} E - 1)\widehat{c}_1(E, h)^2 - (2 \text{rk} E)\widehat{c}_2(E, h)\right) > 0,
\]
then \((E, h)\) is arithmetically unstable.

Let \(A\) be an element of \(\widehat{\text{CH}}^1(X)_{\mathbb{R}}\) with \(\widehat{\deg}(A \cdot x) > 0\) for all \(x \in \widehat{C}_+(X)\), i.e. according to terminology in §1.1, \(A\) is an element of the weak positive cone \(\widehat{C}_+(X)\). A torsion free Hermitian sheaf \((E, h)\) on \(X\) is said to be arithmetically \(\mu\)-semistable with respect to \(A\) if, for all non-zero saturated subsheaves \(F\) of \(E\),
\[
\frac{\widehat{\deg}(\widehat{c}_1(F, h|_F) \cdot A)}{\text{rk} F} \leq \frac{\widehat{\deg}(\widehat{c}_1(E, h) \cdot A)}{\text{rk} E}.
\]
Then, we have the following corollary of Theorem A.

Corollary B. If \((E, h)\) is arithmetically \(\mu\)-semistable with respect to \(A\), then
\[
\widehat{\deg} \left((2 \text{rk} E)\widehat{c}_2(E, h) - (\text{rk} E - 1)\widehat{c}_1(E, h)^2\right) \geq 0.
\]

If \(A = \left(0, \sum_{\sigma \in K(\mathbb{C})} 1/[K : \mathbb{Q}]\right)\), then arithmetic \(\mu\)-semistability of \((E, h)\) with respect to \(A\) is nothing more than \(\mu\)-semistability of \(E_{\mathbb{Q}}\). So this corollary gives a generalization of [Mo1], [Mo2] and [So].

In §1, we will prepare several basic facts of the positive cone and Hermitian vector spaces. In §2, we will consider finiteness of saturated subsheaves in a Hermitian vector bundle, which will be crucial for the proof of Theorem A. §3 is devoted to the proof of Theorem A and Corollary B.

1. Preliminaries

1.1. Positive cone of arithmetic Chow group. Here, we consider basic properties of the positive cone of the arithmetic Chow group of codimension 1.

Let \(K\) be a number field and \(O_K\) the ring of integers of \(K\). Let \(f : X \to \text{Spec}(O_K)\) be a regular arithmetic surface with \(f_*\mathcal{O}_X = O_K\), and \(\text{deg}_K : \widehat{\text{CH}}^1(X)_{\mathbb{R}} \to \mathbb{R}\) the natural homomorphism defined by
\[
\text{deg}_K : \widehat{\text{CH}}^1(X)_{\mathbb{R}} \xrightarrow{\; z \;} \text{CH}^1(X)_{\mathbb{R}} \xrightarrow{\otimes K} \text{CH}^1(X_K)_{\mathbb{R}} \xrightarrow{\; \text{deg} \;} \mathbb{R}.
\]

We set
\[
\widehat{C}_+(X) = \left\{ x \in \widehat{\text{CH}}^1(X)_{\mathbb{R}} \mid \widehat{\deg}(x^2) > 0 \text{ and } \text{deg}_K(x) > 0 \right\} \quad \text{and}
\]
\[
\widehat{C}_+(X) = \left\{ x \in \widehat{\text{CH}}^1(X)_{\mathbb{R}} \mid \text{deg}(x \cdot y) > 0 \text{ for all } y \in \widehat{C}_+(X) \right\}.
\]

\(\widehat{C}_+(X)\) (resp. \(\widehat{C}_+(X)\)) is called the positive cone of \(X\) (resp. the weak positive cone of \(X\)). First of all, we have the following lemma.
Lemma 1.1.1. (1.1.1.1) If \( h \in \widehat{C}_+(X) \), \( x \in \widehat{CH}^1(X)_\mathbb{R} \) and \( \widehat{\deg}(x \cdot h) = 0 \), then \( \widehat{\deg}(x^2) \leq 0 \).

(1.1.1.2) If \( x \in \widehat{CH}^1(X)_\mathbb{R} \), \( \widehat{\deg}(x^2) \geq 0 \) and \( \deg_K(x) > 0 \), then \( x \in \widehat{C}_+(X) \).

(1.1.1.3) If \( h \in \widehat{C}_+(X) \), \( x \in \widehat{CH}^1(X)_\mathbb{R} \), \( \widehat{\deg}(x^2) \geq 0 \) and \( \widehat{\deg}(x \cdot h) > 0 \), then \( x \in \widehat{C}_+(X) \).

Proof. (1.1.1.1) Let \( t \) be a real number with \( \deg_K(x - th) = 0 \). Then, by Hodge index theorem (cf. [Fa], [Hr] or [Mo3]), \( \deg((x - th)^2) \leq 0 \). Thus, \( \widehat{\deg}(x^2) + t^2\deg(h^2) \leq 0 \). Therefore, \( \widehat{\deg}(x^2) \leq 0 \).

(1.1.1.2) Let \( y \in \widehat{C}_+(X) \) and \( t \) a real number with \( \deg_K(y - tx) = 0 \). Then, \( t > 0 \).

By Hodge index theorem, \( \deg((y - tx)^2) \leq 0 \). Thus, we have
\[
\widehat{\deg}(y^2) + t^2\widehat{\deg}(x^2) \leq 2t\widehat{\deg}(x \cdot y).
\]
Therefore, \( \widehat{\deg}(x \cdot y) > 0 \). Hence, \( x \in \widehat{C}_+(X) \).

(1.1.1.3) Let \( y \in \widehat{C}_+(X) \) and \( t \) a real number with \( \widehat{\deg}(y - tx \cdot h) = 0 \). Then, \( t > 0 \)
by (1.1.1.2). (1.1.1.1) implies \( \deg((y - tx)^2) \leq 0 \). Thus, by the same way as above, we have \( \widehat{\deg}(x \cdot y) > 0 \), which implies \( x \in \widehat{C}_+(X) \). \( \square \)

\( \widehat{C}_+(X) \) and \( \widehat{C}_+(X) \) have the following properties.

Proposition 1.1.2. (1.1.2.1) \( \widehat{C}_+(X) \subseteq \widehat{C}_+(X) \).

(1.1.2.2) If \( x, y \in \widehat{C}_+(X) \) and \( t > 0 \), then \( x + y, tx \in \widehat{C}_+(X) \).

(1.1.2.3) If \( x, y \in \widehat{C}_+(X) \) and \( t > 0 \), then \( x + y, tx \in \widehat{C}_+(X) \).

(1.1.2.4) \( \widehat{C}_+(X) = \left\{ x \in \widehat{CH}^1(X)_\mathbb{R} \mid \widehat{\deg}(x \cdot y) > 0 \text{ for all } y \in \widehat{C}_+(X) \right\} \).

Proof. (1.1.2.1) and (1.1.2.2) are straightforward from (1.1.1.2). (1.1.2.3) is obvious.

(1.1.2.4) Clearly, we have
\[
\widehat{C}_+(X) \subseteq \left\{ x \in \widehat{CH}^1(X)_\mathbb{R} \mid \widehat{\deg}(x \cdot y) > 0 \text{ for all } y \in \widehat{C}_+(X) \right\}.
\]
We assume that \( \widehat{\deg}(x \cdot y) > 0 \) for all \( y \in \widehat{C}_+(X) \). If we set \( B = \left( 0, \sum_{\sigma \in K(C)} 1/[K : \mathbb{Q}] \right) \), then \( B \in \widehat{C}_+(X) \). Thus, \( \deg_K(x) = \deg(x \cdot B) > 0 \). Hence, it is sufficient to show \( \widehat{\deg}(x^2) > 0 \). Here, we fix \( h \in \widehat{C}_+(X) \) with \( \widehat{\deg}(h^2) = 1 \). We set \( t = \widehat{\deg}(x \cdot h) > 0 \).

Since \( \deg(x - th \cdot h) = 0 \), by (1.1.1.1), \( \deg((x - th)^2) \leq 0 \). If \( \deg((x - th)^2) = 0 \), then
\[
\widehat{\deg}(x^2) = \widehat{\deg}((th + (x - th))^2) = t^2 > 0.
\]
Thus, we may assume that \( \widehat{\deg}((x - th)^2) < 0 \) if we set \( s = \left( -\widehat{\deg}((x - th)^2) \right)^{1/2} \) and \( l = (x - th)/s \), then \( x = th + sl \), \( \widehat{\deg}(l^2) = -1 \) and \( \widehat{\deg}(h \cdot l) = 0 \). Let us consider
Proof. Clearly, \( x \in \mathcal{C}_+(X) \). Thus, 

\[
\widehat{\text{deg}}(x^2) = t^2 - s^2 = (t + s)(t - s) > 0.
\]

Hence, \( x \in \mathcal{C}_+(X) \). \( \square \)

Finally we consider the following proposition.

**Proposition 1.1.3.** For \( z \in \overline{\mathcal{C}}^1(X) \), we set

\[
(1.1.3.1) \quad W(z) = \left\{ u \in \mathcal{C}_+(X) \mid \widehat{\text{deg}}(z \cdot u) > 0 \right\}.
\]

If \( x \not\in \mathcal{C}_+(X) \), \( y \in \mathcal{C}_+(X) \) and \( W(x) \neq \emptyset \), then \( W(x) \subsetneq W(x + y) \).

**Proof.** Clearly, \( W(x) \subseteq W(x + y) \). By virtue of (1.1.2.4), \( W(x) \subsetneq \mathcal{C}_+(X) \). Let \( u_1 \in W(x), u_2 \in \mathcal{C}_+(X) \setminus W(x) \) and \( t = -\widehat{\text{deg}}(x \cdot u_2)/\widehat{\text{deg}}(x \cdot u_1) \). Then, \( t \geq 0 \) and \( \text{deg}(x \cdot u_2 + tu_1) = 0 \). Hence, \( u_2 + tu_1 \not\in W(x) \). On the other hand, by (1.1.2.3), \( u_2 + tu_1 \in \mathcal{C}_+(X) \). Moreover, \( \widehat{\text{deg}}(y \cdot u_2 + tu_1) > 0 \). Thus, \( \text{deg}(x + y \cdot u_2 + tu_1) > 0 \). Therefore \( u_2 + tu_1 \in W(x + y) \). \( \square \)

**1.2. Hermitian vector space.** Let \( V \) be a \( \mathbb{C} \)-vector space, \( h_V \) a Hermitian metric on \( V \) and \( W \) a subvector space of \( V \). Considering the restriction of \( h \) to \( W \), the metric \( h_W \) induces a metric \( h_W \) of \( W \), which is called the submetric of \( W \) induced by \( h_V \). Let \( W^\perp \) be the orthogonal complement of \( W \). Then the natural homomorphism \( W^\perp \rightarrow V/W \) is isomorphic. Thus we have a metric \( h_{V/W} \) of \( V/W \) given by \( h|_{W^\perp} \). This metric is called the quotient metric of \( V/W \) induced by \( h_V \).

**Proposition 1.2.1.** Let \( (V, h_V) \) be a Hermitian vector space over \( \mathbb{C} \) and \( U, W \) subspaces of \( V \) with \( U \subset W \). Let \( h_W = (h_V)|_W \) and \( h_{V/U} \) the quotient metric of \( V/U \) induced by \( h_V \). We consider two Hermitian metrics of \( W/U \). Let \( h_s = (h_{V/U})|_{W/U} \) and \( h_q \) the quotient metric induced by \( h_W \). Then, we have \( h_s = h_q \).

**Proof.** Let \( V = U \oplus U^\perp \) be the orthogonal decomposition of \( V \) and \( f : U^\perp \rightarrow V/U \) the natural isomorphism. Then, for \( x, y \in W/U \), \( h_s(x, y) = h_V(f^{-1}(x), f^{-1}(y)) \).

Let \( W = U \oplus U_W^\perp \) be the orthogonal decomposition of \( W \), i.e. \( U_W^\perp \) is the orthogonal complement of \( U \) in \( W \), and \( g : U_W^\perp \rightarrow W/U \) the natural isomorphism. Then, \( h_q(x, y) = h_V(g^{-1}(x), g^{-1}(y)) \) for \( x, y \in W/U \).

On the other hand, \( U^\perp_W \subset U^\perp \) and the following diagram is commutative.

\[
\begin{array}{ccc}
U^\perp_W & \hookrightarrow & U^\perp \\
g \downarrow & & \downarrow f \\
W/U & \hookrightarrow & V/U
\end{array}
\]

Thus, we have \( h_s = h_q \). \( \square \)
2. Finiteness of saturated subsheaves

In this section, we will consider finiteness of saturated subsheaves in a Hermitian vector bundle, which will be crucial for the proof of Theorem A.

**Theorem 2.1.** Let $K$ be a number field and $O_K$ the ring of integers of $K$. Let $f : X \to \text{Spec}(O_K)$ be a regular arithmetic surface with $f_*\mathcal{O}_X = O_K$, $(E, h)$ a Hermitian vector bundle on $X$ and $(H, k)$ a Hermitian line bundle on $X$. If $H_K$ is ample, then, for constants $C_1$ and $C_2$, the set

$$F = \{ L \mid \text{$L$ is a rank-1 saturated subsheaf of $E$ with } \hat{\deg}(\hat{c}_1(L, h|_L) \cdot \hat{c}_1(H, k)) \geq C_1 \text{ and } \deg(L_K) \geq C_2 \}$$

is finite.

**Proof.** First of all, we need the following Lemma.

**Lemma 2.1.1.** Let $K$ be an infinite field, $C$ a smooth projective curve over $K$, and $E$ a vector bundle on $C$ of rank $r \geq 2$. Then, for every real number $M$, there is a rank-1 saturated subsheaf $L$ of $E$ with $\deg(L) < M$.

**Proof.** Let $H$ be an ample line bundle on $X$ and $n$ a positive integer such that $n \deg(H) > -M$ and $E \otimes H^n$ is generated by global sections. Let us consider the following closed set $\Sigma$ in $C \times \mathbb{P}(H^0(C, E \otimes H^n))$.

$$\Sigma = \{(x, s) \in C \times \mathbb{P}(H^0(C, E \otimes H^n)) \mid s(x) = 0\}.$$

Let $p : \Sigma \to C$ be the natural projection. Since $E \otimes H^n$ is generated by global sections, $\text{codim}(p^{-1}(x), \mathbb{P}(H^0(C, E \otimes H^n))) = r$. Therefore, $\text{codim}(\Sigma, C \times \mathbb{P}(H^0(C, E \otimes H^n))) = r$, which means $\dim \Sigma < \dim \mathbb{P}(H^0(C, E \otimes H^n))$. Thus, the natural projection $q : \Sigma \to \mathbb{P}(H^0(C, E \otimes H^n))$ is not surjective. Hence, since $\#(K)$ is infinite,

$$(p(\Sigma))(K) \nsubseteq \mathbb{P}(H^0(C, E \otimes H^n))(K).$$

Thus, there is a section $s \in H^0(C, E \otimes H^n)$ with $\text{div}(s) = \emptyset$, which induces an injective homomorphism $H^{-n} \to E$. Since $\text{div}(s) = \emptyset$, the image of $H^{-n} \to E$ is saturated. □

Let us start the proof of Theorem 2.1. Clearly we may assume $r = \text{rk} E \geq 2$. By Lemma 2.1.1, there is a filtration: \(\{0\} = F_0 \subset F_1 \subset \cdots \subset F_{r-1} \subset F_r = E\) such that

1. \(F_i/F_{i-1}\) is a rank-1 torsion free sheaf for every $1 \leq i \leq r$.
2. \(\deg((F_i/F_{i-1})_K) < C_2\) for $1 \leq i \leq r - 1$.

Here we claim
Claim 2.1.4. If $L$ is a line bundle on $X$ with $\deg(L_K) \geq C_2$ and $\varphi : L \to F_{r-1}$ is a homomorphism, then $\varphi = 0$.

We assume that $\varphi \neq 0$. Choose $i$ in such a way that $\varphi(L) \subseteq F_i$ and $\varphi(L) \nsubseteq F_{i-1}$. Then, we have an injective homomorphism $L \to F_i/F_{i-1}$. Since $i \leq r - 1$, $\deg(L_K) \geq C_2 > \deg((F_i/F_{i-1})_K)$. This is a contradiction. \hfill \Box

Let $Q$ be the double dual of $E/F_{r-1}$ and $h_Q$ the quotient metric of $Q$ induced by $h$ via $E \to E/F_{r-1}$, i.e. $h_Q = h|_{F_{r-1}}$. Pick up $L \in \mathcal{F}$. By Claim 2.1.4, we have the natural injection $L \to Q$. So there is an effective divisor $D_L$ on $X$ such that $L \otimes \mathcal{O}_X(D_L) \simeq Q$. Let $D_L^h$ (resp. $D_L^v$) be the horizontal part of $D_L$ (resp. the vertical part of $D_L$). Then, we have

Claim 2.1.5. If $D_L^h = D_L^v$, for $L, L' \in \mathcal{F}$, then $L = L'$.

Let us consider $M = L \otimes \mathcal{O}_X(-D_L^v)$ and $M' = L' \otimes \mathcal{O}_X(-D_L^v)$. Since $D_L^h = D_L^v$, $M$ and $M'$ has the same image in $Q$ via $E \to Q$. Therefore, we have $M + F_{r-1} = M' + F_{r-1}$.

Moreover, since $M \to Q$ and $M' \to Q$ are injective, $M \cap F_{r-1} = M' \cap F_{r-1} = \{0\}$. Hence, we have a homomorphism $M \to M' \oplus F_{r-1} \to F_{r-1}$, which must be zero by Claim 2.1.4. Thus $M \subseteq M'$. By the same way, $M' \subseteq M$. Hence, $M = M'$. So $L = L'$ because $L$ is the saturation of $M$ in $E$ and $L'$ is the saturation of $M'$ in $E$. \hfill \Box

We set $C_3 = \widehat{\deg}((\sigma_1(Q, h_Q) \cdot \sigma_1(H, k))) - C_1$ and $C_4 = \deg(Q_K) - C_2$. Let $D_L^h = \sum_i a_i \Gamma_i$ be the irreducible decomposition of $D_L^h$. Then, we have

Claim 2.1.6. $\sum_i a_i \widehat{\deg}((\sigma_1((H, k)|_{\Gamma_i}))) \leq C_3$ and $\sum_i a_i [K(\Gamma_i) : K] \leq C_4$.

Since $L \otimes \mathcal{O}_X(D_L) \simeq Q$, we have $\deg(L_K) + \deg((D_L)_K) = \deg(Q_K)$. Thus we get $\sum_i a_i [K(\Gamma_i) : K] \leq C_4$.

We choose a Hermitian metric $h_{D_L}$ of $\mathcal{O}_X(D_L)$ in such a way that $(L, h|_L) \otimes (\mathcal{O}_X(D_L), h_{D_L})$ is isometric to $(Q, h_Q)$. Let $1$ be the canonical section of $H^0(X, \mathcal{O}_X(D_L))$ with $\text{div}(1) = D_L$, and $D_L^v = \sum_j b_j l_j$ the irreducible decomposition of the vertical part of $D_L$. Then, we have

$$\widehat{\deg}(\sigma_1(\mathcal{O}_X(D_L), h_{D_L}) \cdot \sigma_1(H, k)) = \sum_i a_i \widehat{\deg}((\sigma_1((H, k)|_{\Gamma_i}))) + \sum_j b_j \deg(H|_{l_j})$$

$$- \frac{1}{2} \sum_{\sigma \in \mathcal{K}(\mathbb{C})} \int_{X_\sigma} \log(h_{D_L}(1, 1))) c_1(H_\sigma, k_\sigma).$$

Since $h_Q$ is a quotient metric of $h$, we can see $h_{D_L}(1, 1) \leq 1$ for all points of each infinite fiber $X_\sigma$. Therefore, we get

$$\sum_i a_i \widehat{\deg}((\sigma_1((H, k)|_{\Gamma_i}))) \leq \widehat{\deg}(\sigma_1(\mathcal{O}_X(D_L), h_{D_L}) \cdot \sigma_1(H, k))$$

$$= \widehat{\deg}(\sigma_1(Q, h_Q) \cdot \sigma_1(H, k)) - \widehat{\deg}(\sigma_1(L, h|_L) \cdot \sigma_1(H, k))$$

$$\leq C_3.$$
Thus, we obtain our claim. □

To complete our theorem, by Claim 2.1.5, it is sufficient to see that \( \{D^h_L \mid L \in \mathcal{F}\} \) is finite. Since \( \sum a_i[K(\Gamma_i) : K] \leq C_4 \), we have \( a_i \leq C_4 \) and \( [\mathbb{Q}(\Gamma_i) : \mathbb{Q}] \leq C_4[K : \mathbb{Q}] \). Let \( D^h_L = D^h_L(+) + D^h_L(-) \) be the decomposition of \( D^h_L \) such that

\[
D^h_L(+) = \sum_{\deg(\hat{c}_1((H,k)|_{\Gamma_i})) > 0} a_i \Gamma_i \quad \text{and} \quad D^h_L(-) = \sum_{\deg(\hat{c}_1((H,k)|_{\Gamma_i})) \leq 0} a_i \Gamma_i.
\]

By Northcott’s theorem (cf. Theorem 2.6 in Chapter 2 of [La]), the set \( \{D^h_L(-) \mid L \in \mathcal{F}\} \) is finite. Hence, there is a constant \( C_5 \) depending only on \( \mathcal{F} \) with

\[
\sum_{\deg(\hat{c}_1((H,k)|_{\Gamma_i})) > 0} a_i \deg(\hat{c}_1((H,k)|_{\Gamma_i})) \leq C_5.
\]

Thus, \( \{D^h_L(+) \mid L \in \mathcal{F}\} \) is finite. Therefore, \( \{D^h_L \mid L \in \mathcal{F}\} \) is finite. □

**Corollary 2.2.** Let \( K \) be a number field and \( O_K \) the ring of integers of \( K \). Let \( f : X \to \text{Spec}(O_K) \) be a regular arithmetic surface with \( f_*O_X = O_K \), \((E,h)\) a Hermitian vector bundle on \( X \) and \((H,k)\) a Hermitian line bundle on \( X \). If \( H_K \) is ample, then, for constants \( C_1 \) and \( C_2 \), the set

\[
\left\{ \hat{c}_1((F,h)|_{F}) \in \overline{\text{CH}}^1(X) \mid F \text{ is a non-zero saturated subsheaf of } E \text{ with } \right. \\
\hat{c}_1((F,h)|_{F}) \cdot \hat{c}_1((H,k)|_{F}) \geq C_1 \text{ and } \deg(F_K) \geq C_2 \right\}
\]

is finite.

**Proof.** Since \( \text{det } F \) is a saturated subsheaf of \( \bigwedge\text{rk } F E \), our corollary is an immediate consequence of Theorem 2.1. □

3. **Proof of Bogomolov instability (Theorem A)**

Before the proof of Theorem A, we will fix notations. Let \( K \) be a number field and \( O_K \) the ring of integers of \( K \). Let \( f : X \to \text{Spec}(O_K) \) be a regular arithmetic surface with \( f_*O_X = O_K \). Let \((F,h_F)\) and \((E,h_E)\) be torsion free Hermitian sheaves on \( X \). We set

\[
\hat{\delta}(E,h_E) = \deg\left( \frac{\text{rk } E - 1}{2 \text{rk } E} \hat{c}_1(E,h_E)^2 - \hat{c}_2(E,h_E) \right).
\]

and

\[
\hat{d}((F,h_F),(E,h_E)) = \frac{\hat{c}_1(F,h_F)}{\text{rk } F} - \frac{\hat{c}_1(E,h_E)}{\text{rk } E}.
\]

Then, we have the following formula.
Lemma 3.1. Let $0 \to (S, h_S) \to (E, h_E) \to (Q, h_Q) \to 0$ be an exact sequence of torsion free Hermitian sheaves on $X$ such that $h_S$ and $h_Q$ are the induced metric by $h_E$. Then, we have

$$\hat{\delta}(E, h_E) \leq \hat{\delta}(S, h_S) + \hat{\delta}(Q, h_Q) + \frac{(\text{rk} E)(\text{rk} S)}{2 \text{rk} Q} \text{deg}(\hat{d}((S, h_S), (E, h_E))^2)).$$

Proof. First of all, $\hat{c}_1(E, h_E) = \hat{c}_1(S, h_S) + \hat{c}_1(Q, h_Q)$. Moreover, by Proposition 7.3 of [Mo1],

$$\text{deg} (\hat{c}_2(E, h_E) - \hat{c}_2((S, h_S) \oplus (Q, h_Q))) \geq 0.$$ 

Thus, by an easy calculation, we have our lemma. $\square$

Let us start the proof of Theorem A. Let $E^\vee\vee$ be the double dual of $E$. Then,

$$\hat{c}_2(E^\vee\vee, h) = \hat{c}_2(E, h) - \log(\# (E^\vee\vee/E)).$$

So we may assume that $E$ is locally free. First we claim

Claim 3.2. There is a non-zero saturated subsheaf $F$ of $E$ such that

$$\frac{\text{deg}(F_K)}{\text{rk} F} - \frac{\text{deg}(E_K)}{\text{rk} E} > 0.$$ 

Since $\text{deg} ((\text{rk} E - 1)\hat{c}_1(E, h)^2 - (2 \text{rk} E)\hat{c}_2(E, h)) > 0$, by the main theorem of [Mo1], $E_{\overline{Q}}$ is not semistable. Let $F'$ be the maximal destabilizing sheaf of $E_{\overline{Q}}$. For $\tau \in \text{Gal}(\overline{Q}/K)$, let us consider $\tau(F')$. Then, $\tau(F') \subset E_{\overline{Q}}$, $\text{deg}(\tau(F')) = \text{deg}(F')$ and $\text{rk} \tau(F') = \text{rk} F'$, which means that $\tau(F')$ is also a maximal destabilizing sheaf of $E_{\overline{Q}}$. Thus, by the uniqueness of the maximal destabilizing sheaf, we have $\tau(F') = F'$. Therefore, $F'$ is defined over $K$. Hence, there is a saturated subsheaf $F$ of $E$ with $F_K = F'$. Thus, we have our claim. $\square$

Let $(H, k)$ be a Hermitian line bundle on $X$ such that $H_K$ is ample. Since

$$\text{deg}(\hat{d}((F, h|_F), (E, h)) \cdot \hat{c}_1(H, ck)) = \text{deg}(\hat{d}((F, h|_F), (E, h)) \cdot \hat{c}_1(H, k))$$

$$- \frac{\log(c)[K : Q]}{2} \left( \frac{\text{deg}(F_K)}{\text{rk} F} - \frac{\text{deg}(E_K)}{\text{rk} E} \right)$$

and $\text{deg}(\hat{c}_1(H, ck)^2) = \text{deg}(\hat{c}_1(H, k)^2) - \log(c)[K : Q] \text{deg}(H_K)$, we may assume that $\text{deg}(\hat{d}((F, h|_F), (E, h)) \cdot \hat{c}_1(H, k)) > 0$ and $\hat{c}_1(H, k) \in \hat{C}_{++}(X)$ if we replace $k$ by $ck$ with sufficiently small positive number $c$. Here we consider the following set.

$${\mathcal{G}} = \left\{ G \Bigg| G \text{ is a non-zero saturated subsheaf of } E \text{ with } \text{deg}(\hat{d}((G, h|_G), (E, h)) \cdot \hat{c}_1(H, k)) > 0 \text{ and } \text{deg}_K(\hat{d}((G, h|_G), (E, h))) > 0 \right\}.$$ 

Then, $F \in {\mathcal{G}}$. Moreover, by Corollary 2.2, $\left\{ \hat{d}((G, h|_G), (E, h)) \in \text{CH}^1(X)_Q \mid G \in {\mathcal{G}} \right\}$ is finite.

We will prove Theorem A by induction on $\text{rk} E$. 

Claim 3.3. If \( \text{rk} E = 2 \), then \( \hat{d}((F, h|_F), (E, h)) \in \hat{C}_+^+(X) \).

Let \( h_{E/F} \) be the quotient metric of \( E/F \), i.e. \( h_{E/F} = h|_{F^\perp} \). By Lemma 3.1,

\[
\hat{\delta}(E, h) \leq \hat{\delta}(F, h|_F) + \hat{\delta}(E/F, h_{E/F}) + \frac{(\text{rk} E)(\text{rk} F)}{2 \text{rk} E/F} \hat{\deg}(\hat{d}((F, h|_F), (E, h))^2).
\]

Since \( \text{rk} F = \text{rk} E/F = 1 \), \( \hat{\delta}(F, h|_F) \leq 0 \) and \( \hat{\delta}(E/F, h_{E/F}) \leq 0 \). Therefore,

\[
\hat{\deg}(\hat{d}((F, h|_F), (E, h))^2) > 0.
\]

Thus, \( \hat{d}((F, h|_F), (E, h)) \in \hat{C}_+^+(X) \). \( \square \)

From now on, we assume \( \text{rk} E \geq 3 \). As in (1.1.3.1), for \( x \in \hat{\text{CH}}^1(X)_R \), we set

\[
W(x) = \left\{ u \in \hat{C}_+(X) \mid \hat{\deg}(x \cdot u) > 0 \right\}.
\]

Here we claim

Claim 3.4. Under the hypothesis of induction, if \( \hat{\deg}(\hat{d}((G, h|_G), (E, h))^2) \leq 0 \) for \( G \in \mathcal{G} \), then there is \( G_1 \in \mathcal{G} \) with \( W(\hat{d}((G, h|_G), (E, h))) \subsetneq W(\hat{d}((G_1, h|_{G_1}), (E, h))) \).

We set \( h_{E/G} = h|_{G^\perp} \). First of all, by Lemma 3.1,

\[
\hat{\delta}(E, h) \leq \hat{\delta}(G, h|_G) + \hat{\delta}(E/G, h_{E/G}) + \frac{(\text{rk} E)(\text{rk} G)}{2 \text{rk} E/G} \hat{\deg}(\hat{d}((G, h|_G), (E, h))^2).
\]

Since \( \hat{\delta}(E, h) > 0 \) and \( \hat{\deg}(\hat{d}((G, h|_G), (E, h))^2) \leq 0 \), we have either \( \hat{\delta}(G, h|_G) > 0 \) or \( \hat{\delta}(E/G, h_{E/G}) > 0 \).

If \( \hat{\delta}(G, h|_G) > 0 \), then by hypothesis of induction there is a non-zero saturated subsheaf \( G_1 \) of \( G \) with \( \hat{d}((G_1, h|_{G_1}), (G, h|_G)) \in \hat{C}_+^+(X) \). Here since

\[
\hat{d}((G_1, h|_{G_1}), (E, h)) = \hat{d}((G_1, h|_{G_1}), (G, h|_G)) + \hat{d}((G, h|_G), (E, h)),$n\]

we have \( G_1 \in \mathcal{G} \). Moreover, by Proposition 1.1.3, we get

\[
W(\hat{d}((G, h|_G), (E, h))) \subsetneq W(\hat{d}((G_1, h|_{G_1}), (E, h))).
\]

If \( \hat{\delta}(E/G, h_{E/G}) > 0 \), then by hypothesis of induction there is a non-zero saturated subsheaf \( T \) of \( E/G \) such that \( \hat{d}((T, h_{E/G}|_T), (E/G, h_{E/G})) \in \hat{C}_+^+(X) \). Take a saturated subsheaf \( G_1 \) in \( E \) with \( G \subset G_1 \) and \( G_1/G = T \). Let \( h_{G_1/G} \) be the induced quotient metric
by $h_{G_1}$. By Proposition 1.2.1, we have $h_{E/G}|_T = h_{G_1/G}$. Thus, by an easy calculation, we get

\begin{align*}
\hat{d}((G_1, h|_{G_1}), (E, h)) &= \frac{\text{rk}(G_1/G)}{\text{rk}G_1} \hat{d}((T, h_{E/G}|_T), (E/G, h_{E/G})) \\
&\quad + \frac{\text{rk}G \text{rk}(E/G_1)}{\text{rk}G_1 \text{rk}(E/G)} \hat{d}((G, h|_{G}), (E, h_E)).
\end{align*}

Therefore, $G_1 \in \mathcal{G}$, and by Proposition 1.1.3,

$$W(\hat{d}((G, h|_{G}), (E, h))) \subsetneq W(\hat{d}((G_1, h|_{G_1}), (E, h))).$$

Hence we get Claim 3.4. \(\Box\)

Here we assume that $\hat{\text{deg}}(\hat{d}((G, h|_{G}), (E, h))^2) \leq 0$ for all $G \in \mathcal{G}$. Then, since $F \in \mathcal{G}$, by Claim 3.4, there is an infinite sequence $\{G_0 = F, G_1, G_2, \ldots, G_n, \ldots\}$ in $\mathcal{G}$ such that

$$W(\hat{d}((G_i, h|_{G_i}), (E, h))) \not\subset W(\hat{d}((G_j, h|_{G_j}), (E, h)))$$

for all $i < j$. In particular, $\hat{d}((G_i, h|_{G_i}), (E, h))$ gives distinct elements in $\widehat{\text{CH}}^1(X)$. On the other hand,

$$\left\{ \hat{d}((G, h|_{G}), (E, h)) \in \widehat{\text{CH}}^1(X) \mid G \in \mathcal{G} \right\}$$

is finite. This is a contradiction. So there is $G \in \mathcal{G}$ with $\hat{\text{deg}}(\hat{d}((G, h|_{G}), (E, h))^2) > 0$. Thus, we get our theorem. \(\Box\)

**3.5 Proof of Corollary B.** Finally, we give the proof of Corollary B. We assume that

$$\hat{\text{deg}}((2 \text{rk} E)\widehat{c}_2(E, h) - (\text{rk} E - 1)\widehat{c}_1(E, h)^2) < 0.$$

Then, by Theorem A, there is a non-zero saturated subsheaf $F$ of $E$ with

$$\frac{\widehat{c}_1(F, h|_F)}{\text{rk} F} - \frac{\widehat{c}_1(E, h)}{\text{rk} E} \in \widehat{C}_{++}(X).$$

Thus,

$$\frac{\hat{\text{deg}}(\widehat{c}_1(F, h|_F) \cdot A)}{\text{rk} F} - \frac{\hat{\text{deg}}(\widehat{c}_1(E, h) \cdot A)}{\text{rk} E} > 0.$$

This is a contradiction. \(\Box\)
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