A new code for quasi-equilibrium initial data of binary neutron stars: corotating, irrotational and slowly spinning systems

Antonios Tsokaros
Institut für Theoretische Physik, Johann Wolfgang Goethe-Universität,
Max-von-Laue-Str. 1, 60438 Frankfurt am Main, Germany

Kōji Uryū
Department of Physics, University of the Ryukyus, Senbaru, Nishihara, Okinawa 903-0213, Japan

Luciano Rezzolla
Institut für Theoretische Physik, Johann Wolfgang Goethe-Universität,
Max-von-Laue-Str. 1, 60438 Frankfurt am Main, Germany
(Dated: February 20, 2015)

We present the extension of our COCAL - Compact Object CALculator - code to compute general-relativistic initial data for asymmetric binary compact-star systems. We construct quasi-equilibrium initial data for spinning binaries and multiple coordinate systems are employed. The Isenberg-Wilson-Mathews formalism is adopted and the constraint equations are solved using the representation formula with a suitable choice of a Green’s function. We validate the new code with solutions for equal-mass binaries and explore its capabilities for a wide range of compactnesses, from a white dwarf binary with compactness $\sim 10^{-4}$, up to a highly relativistic neutron-star binary with compactness $\sim 0.22$. We also present a comparison with corotating and irrotational quasi-equilibrium sequences from the spectral code LORENE [1] with different compactness, showing that the results from the two codes agree to a precision of the order of 0.05%. Finally, we present equilibria for spinning configurations with a nuclear-physics equation of state in a piecewise polytropic representation.

I. INTRODUCTION

With a compactness slightly smaller than that of a black hole, neutron stars are most probably nature’s ultimate compact matter configuration before gravitational collapse and black-hole formation. As such, they present an invaluable tool to astrophysicists in order to study a plethora of problems and test the limits of existing knowledge, from general relativity, via the emission of gravitational waves, to nuclear physics, via the input on the equation of state of nuclear matter [2–6]. For example, the leading (but not unique) candidate to explain one of the most luminous explosions in the universe, the so-called short gamma-ray bursts [7, 8] (see [9] for a recent review) is the merger of two neutron stars (or of one neutron star and one black hole) with the subsequent formation of a black hole, an accretion torus, and a jet structure of ultra-strong magnetic field [10, 11]. Yet another example has to do with the production site of the heaviest elements in the universe through the so-called, rapid neutron capture (r-process) [12–17].

Central to the processes described above [18, 19] is a binary neutron star system which in addition constitutes a prime source of gravitational waves for ground-based laser interferometric gravitational-wave detectors such as LIGO, Virgo, KAGRA, and ET [20–24]. The advanced generation of these detectors will become operational in a few years, and they will be able to observe a volume of the universe a thousand times more than their predecessors. According to present estimates [25] it may be possible to detect $\sim 1 - 100$ events per year, making the study of such systems an important step towards a practical verification of general relativity in the strong field regime, as well as an exploration of its limits. At the same time, gravitational-wave observations are expected to constrain neutron-star radius and deformability [26–32].

The broadbrush picture for the two-body problem in general relativity can be divided into three phases: the inspiral, the merger, and the ring-down, with each one having its own methods and tools of investigation. The purpose of this work lies in the interface between the first and the second phase, the so-called quasi-equilibrium stage, and the solutions presented are meant as “snapshots” at particular instants of the binary system. The purpose is twofold: on the one hand to provide initial data for the simulation of the merging phase and, on the other hand, to provide evolutionary information about the system studied by constructing quasi-equilibrium sequences. In this way, we can learn how much the star shape is deformed as the orbit shrinks, where does an instability sets in, where is the mass-shedding limit and what is the angular velocity of the system there. All of this information can be computed with a modest computational infrastructure, thus allowing for the exploration of a wide parameter space.

Because the orbital decay timescale due to gravitational wave emission is much shorter than the synchronization timescale due to the neutron star viscosity, it is unlikely that the two stars will be tidally locked before merger [33, 34]. For such slowly rotating configurations, the assumption of an irrotational flow is physically reasonable and mathematically simple to impose. An irrotational flow is also called a potential flow, since the fluid velocity is the gradient of a potential [35]. A formalism to compute initial data within a conformally flat geometry, the so-called Isenberg-Wilson-Mathews (IWM) formalism [36, 37], was presented in Ref. [38–41], and numerical implementations for a variety of physical assumptions have been discussed by several groups [1, 42–50]. Non-conformally flat formulations, have been also implemented [51–56], where the full system of Einstein equations is being solved. These more computationally expensive solutions
are expected to respect the circularity of an orbit better than the ones coming from conformally flat initial data, which in addition seem to suppress tidal effects as the compactness of the stars increases. A conformally flat geometry can still be used to produce low eccentricity initial data if one uses ideas similar to those applied to the binary black hole problem [57], as they were implemented in [58, 59].

As it was pointed in Ref. [60], the double pulsar PSR J0737-3039, has one of its stars reaching the merging epoch with a spin of \( \sim 27 \) millisecond, hence in a state that cannot be considered irrotational. Since we only know less than a dozen binary systems [61], and hundreds of millisecond pulsars, it is reasonable to simulate arbitrary spinning binary neutron stars and assess what is the impact that the stellar spin has on the gravitational-wave signal. Initial data for binary neutron stars with intermediate (and arbitrary) rotation states are more difficult to calculate, since there is no self-consistent scheme to incorporate the fluid equations with the rest of the elliptic gravitational equations. Various schemes have been proposed recently in Refs. [60, 62–65], which introduce some additional approximations, while evolution of spinning binaries have been performed in [65–67].

In this work we continue the COCAL program for computing equilibrium configurations of single [68, 69] and binary systems building on the infrastructure introduced in Refs. [70–72] for binary black holes. Here, we describe how to calculate initial data for binary stars, concentrating on neutron stars. The ability to compute configurations with a wide range of compactness was one of the goals of this work. At present, COCAL is makes use of a piecewise polytropic description to represent the equation of state (EOS), but fully tabulated EOSs can also be implemented. As in the vacuum case, we employ the Komatsu–Eriguchi–Hachisu (KEH) method [73–78] on multiple patches [79] in order to be able to treat binaries consisting of different compact objects. The gravitational equations are solved using the COCAL Poisson solvers (with appropriate Green’s functions), while for the conservation of rest mass, we follow [80] and employ the least-squared algorithm demonstrating the versatility of the methods used by our code. In this way we can calculate sequences of corotating, irrotational, as well as spinning binaries, where for the last case we use the formulation of [60], after suitably adapting it.

The paper is organized as follows: In Sec. II we discuss the equations to be solved and the assumptions made, both for the gravitational field in Sec. II A, as well as for the fluid part in Sec. II B. For the latter we present the forms used in COCAL code for corotating, irrotational, and spinning cases. Section III represents the core of this work. In Sec. III A we briefly review the gravitational multi-patch coordinate systems used in [70, 71] and discuss additional changes that are related to the neutron-star surface. In Sec. III B we describe the removal of dimensions from the equations, in conjunction with the scaling introduced in Sec. III A. Section III C describes the Green’s function used for the star patch, while in Sec. III D the least-squared method for solving a spinning configuration is introduced. Tests for our new code are presented in Sec. IV A for corotating binaries and in Sec. IV B for irrotational ones, while spinning solutions with piecewise polytropes are presented in Sec. IV C. A number of Appendices provides more technical details on several topics. More specifically, App. A reports the expressions used for the calculation of the mass and angular momentum of the binary, App. B illustrates a different approach to obtain a solution of the Tolmann–Oppenheimer–Volkoff (TOV) equations, while App. C describes in detail the full iteration scheme and App. D shows tests of COCAL in a very different regime of compactness by considering binaries of white dwarfs. Finally, App. E reports the post-Newtonian expressions for the binding energy and orbital angular momentum of a binary system in quasi-circular orbit, which are used as a reference.

Hereafter, spacetime indices will be indicated with Greek letters, while spatial indices with Latin lowercase letters. The metric has signature \(- + + +\), and we use a set of geometric units in which \( G = c = M_\odot = 1\), unless stated otherwise.

## II. QUASI-EQUILIBRIUM EQUATIONS

In this Section we review the basic equations that need to be solved to obtain binary equilibrium configurations. Details of the initial-data formalism can be found in [81–84]. Here, we only mention the points that are relevant to the COCAL’s new developments.

### A. The gravitational equations

One of the most fruitful ideas in simulating the circular motion of two bodies in general relativity was the introduction of helical-symmetry approximation [85, 86]. Solutions with such symmetry are stationary in the corotating frame and have a long history, starting from the electromagnetic two-body problem [87]. Analogous solutions in the post-Minkowski approximation have been derived in [88, 89]. Helical symmetry was also used to obtain the first law of binary star thermodynamics [90], as well as to produce equilibrium configurations of binary black holes [91, 92].

Neglecting the loss of energy due to gravitation radiation and assuming closed orbits for the binary system, results in the existence of a helical Killing vector

\[
k^\mu := t^\mu + \Omega \phi^\mu, \tag{1}
\]

such that

\[
\mathcal{L}_k g_{\alpha \beta} = 0, \tag{2}
\]

where \( \mathcal{L}_k \) is the Lie derivative along \( k \) and \( \phi^i \) is the generator of rotational symmetry. In a Cartesian coordinate system, but without loss of generality, we can assume the generator of rotational symmetry to have components

\[
\phi^i = (-y, x, 0). \tag{3}
\]

Writing the spacetime metric in 3+1 form as [35, 84, 93, 94]

\[
ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \tag{4}
\]
where $\alpha$, $\beta^i$, $\gamma_{ij}$ are respectively the lapse, the shift vector, and the three-metric on $\Sigma_t$, the generator of time translations in the rotating frame can be expressed as

$$k^\mu := \alpha n^\mu + \omega^\mu.$$  

(5)

Here, $\omega^\mu := \beta^\mu + \Omega \phi^\mu$ is the corotating shift, and $n^\mu$ the unit normal to $\Sigma_t$, $n_\mu := -\alpha \nabla_n t$. Since $\mathcal{L}_k \gamma_{ij} = 0 = \mathcal{L}_k K_{ij}$, the evolution equation of $\gamma_{ij}$ specifies the extrinsic curvature in terms of the shift and the lapse

$$K_{ij} = \frac{1}{2\alpha} (D_i \omega_j + D_j \omega_i),$$  

(6)

where $D$ the derivative operator associated with $\gamma_{ij}$. The traceless extrinsic curvature

$$A_{ij} := K_{ij} - \frac{1}{3} K_m^m \gamma_{ij} = K_{ij} - \frac{1}{3} K \gamma_{ij},$$  

(7)

can be written in terms of the longitudinal operator $\mathcal{L}$

$$A_{ij} = \frac{1}{2\alpha} \left( D_i \omega_j + D_j \omega_i - \frac{2}{3} \gamma_{ij} D_k \omega^k \right) := \frac{1}{2\alpha} (\mathcal{L} \omega)_{ij}.$$  

(8)

Assuming a maximal and conformally flat slice [36, 37]

$$\gamma_{ij} = \psi^4 \delta_{ij},$$  

(9)

the traceless extrinsic curvature Eq. (8) is written in terms of the inertial shift

$$A_{ij} = \frac{\psi^{-4}}{2\alpha} \left( \partial^i \beta^j + \partial^j \beta^i - \frac{2}{3} \delta^{ij} \partial_k \beta^k \right) = \frac{\psi^{-4}}{2\alpha} (\mathcal{L} \beta)^{ij}.$$  

(10)

The tilde symbol on the longitudinal operator $\mathcal{L}$ denotes the fact that it is related to the conformally flat geometry. In deriving Eq. (10) use has been made of the fact that $\partial_i \phi^j = 0 = \partial_i \partial_j$ [cf., Eq. (3)]. In the conformally flat geometry, the contravariant components of the shift remain the same as in the original spatial geometry (i.e., $\beta^i = \beta^i$), while this is not true for the covariant components.

With the help of Eq. (10), the constraint equations and the spatial trace of the time derivative of the extrinsic curvature result into five elliptic equations for the conformal factor $\psi$, the shift $\beta^i$, and the lapse function $\alpha$

$$\nabla^2 \psi = -\frac{\psi^5}{32\alpha^2} (\mathcal{L} \beta)^{ab} (\mathcal{L} \beta)^{ij} \delta_{ia} \delta_{jb} - 2\pi E \psi^5$$

$$:= S^g_\psi + S^f_\psi,$$  

(11)

$$\nabla^2 (\alpha \psi) = \frac{7}{32\alpha} (\mathcal{L} \beta)^{ab} (\mathcal{L} \beta)^{ij} \delta_{ia} \delta_{jb} + 2\pi \alpha \psi^5 (E + 2S)$$

$$:= S^g_\alpha + S^f_\alpha,$$  

(12)

$$\nabla^2 \beta^i = -\frac{1}{\alpha^4} \partial^i \partial_j \beta^j + \partial_j \ln \left( \frac{\alpha}{\psi^6} \right) (\mathcal{L} \beta)^{ij} + 16\pi \alpha \psi^4 \beta^i$$

$$:= S^g_\beta + S^f_\beta.$$  

(13)

We have denoted with $S^g_{\psi}$ the sources of the Poisson-type equations that come from the energy-momentum tensor, while with $S^g_{\alpha}$ are the sources that come from the nonlinear part of the Einstein tensor $G_{\alpha \beta}$. The matter sources in Eqs. (11)–(13), $S^f_{\psi}$, $S^f_{\alpha}$, $S^f_{\beta}$, are related to the corresponding projections of the energy-momentum tensor

$$E := n_\alpha n_\beta T^{\alpha \beta},$$  

(14)

$$S := \gamma_{\alpha \beta} T^{\alpha \beta},$$  

(15)

$$j^i := -\gamma_{\alpha \beta} n_\beta T^{\alpha \beta},$$  

(16)

where $E$ is the energy density as measured by a “normal” observers, that is, an observer with four-velocity $n$. Note also that since we use $G = c = M_\odot = 1$, all equations (11)–(13) are dimensionless. This is contrary to some previous works (e.g., [44]), where only $G = c = 1$ was assumed, and a procedure to remove units was applied through the use of the adiabatic constant $K$. Here the normalization scheme used is explained in detail in Sec. III.B.

The above set of equations must be supplied with conditions on the boundary of our computational region. Since we will consider only binary stars, our boundary for the gravitational equations will be only at spatial infinity, where we impose asymptotic flatness, i.e.,

$$\lim_{r \rightarrow \infty} \psi = 1, \quad \lim_{r \rightarrow \infty} \alpha = 1, \quad \lim_{r \rightarrow \infty} \beta^i = 0.$$  

(17)

We recall that a helically symmetric spacetime cannot be asymptotically flat, because a helically symmetric binary produces an infinite amount of radiation. Therefore conditions (17) seem to contradict assumption (1). In reality, the helical symmetry is only an approximation that is valid either for long times when the binary is widely separated, or only for a short time when the binary is tight. In practice, the emission of gravitational radiation reaction will lead to an inspiral, thus breaking the symmetry.

### B. The fluid equations

Let $u^\alpha$ be the four-velocity of the fluid. We consider a perfect fluid with energy-momentum tensor [35, 95]

$$T_{\alpha \beta} = (\epsilon + p) u_\alpha u_\beta + p g_{\alpha \beta} - \rho u_\alpha u_\beta + p g_{\alpha \beta},$$  

(18)

where $\rho, \epsilon, h$, and $p$ are respectively the rest-mass density, the total energy density, the specific enthalpy, and the pressure as measured in the rest frame of the fluid. The specific internal energy $e$ is related to the enthalpy through

$$h := \frac{\epsilon + p}{\rho} = 1 + e + \frac{p}{\rho}.$$  

(19)

---

1. Note that many authors use $e$ and $\epsilon$ to indicate the energy density and the specific internal energy, respectively.
The first law of thermodynamics, \(de = \rho T ds + h d\rho\), where \(s\) is the specific entropy, written in terms of the specific enthalpy \(h\), reads \(dh = T ds + dp/\rho\). For isentropic configurations, like the quasi-equilibrium solutions we are seeking for, given an EOS which relates, for example, the pressure \(p\) with the rest-mass energy density \(\rho\), we can see that the extra variables that enter our problem are \(\rho\) (or \(p\)) and the four-velocity \(u^\alpha\). For these new variables, extra equations need to be used exploiting conservation laws. In particular, from the conservation of rest mass \((\text{the "nonrotating" part of the fluid flow (i.e. the velocity in the corotating frame)}))

\[
0 = \nabla_\alpha T^{\alpha \beta} = \rho [u^\alpha \nabla_\alpha (hu^\beta) + \nabla^\beta h] + hu^\beta \nabla_\alpha (\rho u^\alpha) - \rho T \nabla^\beta s ,
\]

assuming isentropic configurations and local conservation of rest mass

\[
\nabla_\alpha (\rho u^\alpha) = 0 ,
\]

we arrive at the relativistic Euler equation

\[
u^\alpha \nabla_\alpha (hu_\beta) + \nabla_\beta h = 0 .
\]

(21)

Although we use four-dimensional indices, this is a fully spatial equation, since the projection along the fluid flow is trivially satisfied. Equations (21) and (22) provide us with four more equations for the fluid variables. If one of them is used for the determination of \(\rho\), we are left with three equations that must determine the four-velocity \(u^\alpha\), which has only three independent components.

Expressing the four-velocity as \(u^\alpha = u^i (1, v^i)\) and in analogy with a Newtonian decomposition, we can split the spatial component \(v^i\) into two parts: one that follows the orbital path \(\phi^i\), and one that represents the velocity in the corotating frame \(V^i\). Using the helical Killing vector, Eq. (1), we can write

\[
u^\alpha = u^i (k^\alpha + V^\alpha) ,
\]

(23)

where the spatial part, \(V^\alpha = (0, V^i)\) can be considered to be the "nonrotating" part of the fluid flow (i.e. the velocity in the corotating frame). The conservation of rest mass (21), and the spatial projection of the Euler equation (22), written in 3+1 form translate to

\[
\mathcal{L}_k(\rho u^i) + \frac{1}{\alpha} D_i (\alpha pu^i V^i) = 0 ,
\]

(24)

\[
\gamma_i^\alpha \mathcal{L}_k(h u_\alpha) + D_i \left( \frac{h}{u^i} + h u_j V^j \right)
+ V^j (D_j (hu_\alpha) - D_i (hu_j)) = 0 .
\]

(25)

The last term of Eq. (25) involves the relativistic vorticity tensor [35]

\[
\omega_{\alpha \beta} := \nabla_\alpha (hu_\beta) - \nabla_\beta (hu_\alpha) ,
\]

(26)

and is zero for an irrotational flow. It is not difficult to show that in the presence of a generic Killing vector field (e.g., the helical Killing field) \(k\), the following identity holds [35]

\[
\mathcal{L}_u (hu \cdot k) = 0 .
\]

(27)

In the case of a rigid corotation of the binary system, \(u = k\), so that the Lie derivative along the fluid four-velocity \(u\) in Eq. (27) can be replaced by the Lie derivative along the helical Killing vector \(k\). This yields \(\mathcal{L}_k h = 0\) and expresses that in the corotating frame the fluid properties do not change.

When the fluid four-velocity does not coincide with the helical Killing field, but the two vector fields are not too different, i.e., when \(u \approx k\), expression (27) can still be true and indeed for the flows we will consider hereafter we will assume that the following assumption

\[
\gamma_i^\alpha \mathcal{L}_k(h u_\alpha) = 0 = \mathcal{L}_k(pu^i) .
\]

(28)

While Eq. (28) is an assumption, its correctness can only be assessed a-posteriori and could indeed not represent a valid approximation if the stars are spinning very rapidly, a case we will not investigate here.

In the following we will specialize Eqs. (24) and (25) under the assumptions (28) for corotating, irrotational and slowly rotating flows. Before closing we introduce a quantity that will be used often in subsequent Sections, namely, the spatial projection of the specific enthalpy current

\[
\hat{u}_i := \gamma_i^\alpha h u_\alpha .
\]

(29)

1. Corotating binaries

The corotating case, also called of rigid rotation [96, 97], is the simplest case, since the spatial fluid velocity \(V^i\) vanishes, \(u^\alpha = u^i k^\alpha\), and thus the fluid is at rest in a corotating frame. This means that apart from the gravitational variables \(\psi, \alpha, \beta\), we have only two extra fluid variables, for example \(\rho\) and \(u^i\) once an EOS is fixed. The conservation of rest mass (24) is trivially satisfied, while the Euler equation (25) becomes a single integral equation that, together with the normalization condition \(u^\alpha u_\alpha = -1\), will determine all our fluid variables.

In particular, the specific enthalpy current becomes

\[
\hat{u}^i = hu^i \omega^i ,
\]

(30)

and from the four-velocity normalization condition we have

\[
u^i = \frac{1}{\sqrt{\alpha^2 - \omega^2}}
\]

(31)

Equation (25), on the other hand, has the first integral

\[
\frac{h}{u^i} = C ,
\]

(32)

where \(C\) is a constant to be determined. Equations (31) and (32), together with the gravitational potentials, completely determine the solution for this case. We note that in all equations to be solved (the gravitational ones included) two constant are involved. One is \(C\), the constant that comes from

\[\text{More precisely, the Euler equation is the projection orthogonal to the fluid flow of the conservation of the energy-momentum tensor and leads to three distinct spatial equations. On the other hand, the projection along the flow of } \nabla_\alpha T^{\alpha \beta} = 0 \text{ yields to a single equation expressing energy conservation [35].}\]
the Euler integral, and one is $\Omega$, the orbital angular velocity. Thus, in order to be able to achieve a solution for our system, a self-consistent scheme that involves the determination of both $C$ and $\Omega$ must be employed. As we will elaborate later on, this will be achieved in conjunction with the determination of the length scale $R_0$ of our problem.

For the corotating case the matter sources in Eqs. (11)–(13) are

\begin{align}
E &= \rho [h(\alpha u^t)^2 - q], \\
E + 2S &= \rho [h(3(\alpha u^t)^2 - 2) + 5q], \\
\rho \xi &= \rho \alpha u^i \dot{u}^i,
\end{align}

where $q := p/\rho$. As we have already mentioned, all quantities appearing above are dimensionless, while in previous studies, where geometric units were used, Eqs. (33)–(35) had units of length$^{-2}$.

2. Irrotational and spinning binaries

Irrotational configurations have $\omega_{\alpha \beta} = 0$, so that the specific enthalpy current $hu_\alpha$ can be derived from a potential $[35]$, i.e.,

\begin{equation}
hu_\alpha = \nabla_\alpha \Phi = D_\alpha \Phi + n_\alpha \nabla_\alpha \Phi,
\end{equation}

so that $\dot{u}_i = D_i \Phi$ since $\gamma \cdot n = 0$. To allow for spinning configurations we need to extend expression (36) and we do this following Ref. [64] and introducing a four-vector $s$ (not to be confused with the specific entropy $s$)

\begin{equation}
hu_\alpha = \nabla_\alpha \Phi + s_\alpha,
\end{equation}

so that $\dot{u}_i$ is decomposed as

\begin{equation}
\dot{u}_i = D_i \Phi + s^i,
\end{equation}

where the $D_i \Phi$ part corresponds to the “irrotational part” of the flow and the $s^i$ part to the “spinning part” of the flow. In what follows we will present expressions for spinning binaries ($s^i \neq 0$) and one can recover the irrotational ones by setting the spinning component $s^i$ equal to zero.

Using the decomposition (38) and the assumption (28), the Euler equation (25) can be rewritten as

\begin{equation}
\mathcal{L}_\nu s_i + D_i \left( \frac{h}{u^t} + V^j D_j \Phi \right) = 0,
\end{equation}

and hereafter we will assume

\begin{equation}
\mathcal{L}_\nu s_i = 0,
\end{equation}

which is likely to be a very good approximation in the case of slowly and uniformly rotating stars, for which $s_i$ is intrinsically small.\(^3\) Hence, the Euler equation for generic binaries (25)

\begin{equation}
\gamma_i^o [\mathcal{L}_k(hu_\alpha) + \mathcal{L}_\nu(s_\alpha)] + D_i \left( \frac{h}{u^t} + V^j D_j \Phi \right) = 0,
\end{equation}

under the assumptions (28)\(_1\) and (40), yields the reduced Euler integral

\begin{equation}
\frac{h}{u^t} + V^j D_j \Phi = C, \quad (42)
\end{equation}

where again $C$ is a constant to be determined. A few remarks should be made at this point. First, it is not difficult to obtain the following identity

\begin{equation}
\gamma_i^o \mathcal{L}_k(hu_\alpha) + \mathcal{L}_\nu(s_\alpha) = \gamma_i^o \mathcal{L}_k(\nabla_\alpha \Phi) + \mathcal{L}_\nu(\Phi^\alpha / (hu^\alpha))(s_\alpha) + \mathcal{L}_\nu(s_\alpha), \quad (43)
\end{equation}

so that our assumptions (28)\(_1\) and (40)

\begin{equation}
\gamma_i^o \mathcal{L}_k(hu_\alpha) = 0 = \gamma_i^o \mathcal{L}_\nu(s_\alpha), \quad (44)
\end{equation}

are equivalent to setting the left-hand side of Eq. (43) to zero. In turn, this implies that also the right-hand side of (43) is zero, which is true if, for instance, each of the three terms is zero, i.e., if

\begin{equation}
\gamma_i^o \mathcal{L}_k(\nabla_\alpha \Phi) = 0 = \gamma_i^o \mathcal{L}_\nu(\Phi^\alpha / (hu^\alpha))(s_\alpha) = \gamma_i^o \mathcal{L}_\nu(s_\alpha). \quad (45)
\end{equation}

The three conditions in (45) coincide with the assumptions made in [60]. Stated differently, because the conditions (44) are compatible with the conditions (45), it does not come to a surprise that we obtain the same Euler integral (42) as in [60] despite making apparently different assumptions [cf., (44) vs (45)]. Second, using the decomposition (37), it follows that

\begin{equation}
\gamma_i^o \mathcal{L}_k(hu_\alpha) = \gamma_i^o [\mathcal{L}_k(\nabla_\alpha \Phi) + \mathcal{L}_k(s_\alpha)], \quad (46)
\end{equation}

and hence that considering a flow with $\mathcal{L}_k(s_\alpha) = 0$,\(^4\) is not sufficient for having $\gamma_i^o \mathcal{L}_k(hu_\alpha) = 0$, but the additional assumption $\gamma_i^o \mathcal{L}_k(\nabla_\alpha \Phi) = 0$ is necessary. Finally, although the Euler integral has the same form for both irrotational and spinning binaries, it produces a different equation since the three-velocity $V^i$ is different in these two cases. More specifically, it is

\begin{equation}
\dot{u}_i = hu^t(\omega^i + V^i), \quad (47)
\end{equation}

so that

\begin{equation}
V^i = \frac{D_i \Phi + s^i}{hu^t} - \omega^i. \quad (48)
\end{equation}

In this case, the fluid variables are $\rho$ (or equivalently $p$ or $h$), $u^t$, and the fluid potential $\Phi$. The equations that will determine them are the normalization condition $u_\alpha u^\alpha = -1$, the Euler integral (42) [with the use of Eq. (48)], and the conservation of rest mass (24).

In particular, from the norm of $\dot{u}_i$ we get

\begin{equation}
h = \sqrt{\alpha^2 (hu^t)^2 - (D_i \Phi + s^i)(D^i \Phi + s^i)}, \quad (49)
\end{equation}

\(^3\) In practice we will consider stars with spin period down to 0.6 ms, but this is still “slowly” spinning when compared to the minimum period.

\(^4\) Note that even when the spins are aligned with the orbital angular momentum $\mathcal{L}_k(s_\alpha) \neq 0$. 
so that the relation above, the Euler integral (42) takes the following form quadratic in $hu^t$
\[\alpha^2(hu^t)^2 - \lambda(hu^t) - s_i(D^i\Phi + s^i) = 0,\]  
(50)
where $\lambda = C + \omega^i D_i\Phi$. Thus
\[hu^t = \frac{\lambda + \sqrt{\lambda^2 + 4\alpha^2s_i(D^i\Phi + s^i)}}{2\alpha^2},\]  
(51)
where we take the positive root since the negative one is incorrect at least in the limit of $s^i = 0$, when it yields $hu^t = 0$.

Having computed $\lambda$, we first calculate $hu^t$ from Eq. (51), and then $h$ from Eq. (49). For purely irrotational binaries $hu^t = \lambda/\alpha^2$ and $h = \sqrt{\lambda^2/\alpha^2 - D_i\Phi D^i\Phi}$.

Although we will not make immediate use of $u^t$ and $h$ separately, we report below their form for completeness
\[h = \sqrt{L^2 - (D^i\Phi + s_i)(D^i\Phi + s^i)},\]  
(52)
\[u^t = \frac{\sqrt{h^2 + (D^i\Phi + s_i)(D^i\Phi + s^i)}}{h\alpha},\]  
(53)
where
\[L^2 := \frac{\lambda^2 + 2\alpha^2W + \lambda\sqrt{\lambda^2 + 4\alpha^2W}}{2\alpha^2},\]  
(54)
\[W := s_i(D^i\Phi + s^i).\]  
(55)

The potential $\Phi$ will be computed from the conservation of rest mass (24), which under Eq. (48), and after expressing the spin velocity as a power law [60]
\[s^i = \psi^A \tilde{s}^i, \quad A \in \mathbb{R}\]  
(56)
will produce an extra elliptic equation
\[\nabla^2 \Phi = -\frac{2}{\psi} \partial_i \psi \partial^i \Phi + \psi^A \omega^i \partial_i (hu^t)\]
\[+ |\psi^4 hu^t \omega^i - \partial^i \Phi| \partial_i \ln \left(\frac{\alpha h}{\rho}\right)\]
\[= -\psi^A \left[ \partial_i \tilde{s}^i + \tilde{s} \partial_i \ln \left(\frac{\alpha \rho \psi^6 + A}{h}\right) \right] = S_\Phi.\]  
(57)

The boundary for the fluid is represented by the surface of the star; hence the boundary condition for Eq. (57) will be of von-Neumann type, that is, in terms of derivatives of the rest-mass density and of $\Phi$
\[\left[ \psi^4 hu^t \omega^i - \psi^4 \partial^i \Phi - \psi^A \tilde{s}^i \right] \partial_i \rho \right|_{\text{surf.}} = 0.\]  
(58)

A possible and convenient choice for the parameter $A$ that will be used in Sec. IV C is $A = -6$, as it removes the last term in Eq. (57)\(^5\). Any other value will not change the character of the equation or the boundary condition, although it will change the detailed properties of the flow velocity and therefore of the binary. We will comment on this point in Sec. IV C, where we will also illustrate the results for $A = 0$.

For the spinning case, the matter sources in Eqs. (11)–(13) are
\[E = \rho \left[ \frac{\alpha^2}{h} (hu^t)^2 - q \right],\]  
(59)
\[E + 2S = \rho \left[ \frac{3\alpha^2}{h} (hu^t)^2 - 2h + 5q \right],\]  
(60)
\[j^i = \rho \alpha \psi^A \partial_i \psi \left( \psi - 4 \partial^i \Phi + \psi^A \tilde{s}^i \right),\]  
(61)
where we have used Eq. (51) to simplify the calculations.

C. Equation of state

The EOS used in this work is represented by a sequence of polytropes called a piecewise polytrope. This is proven to be a good approximation for a great variety of models [98–101]. If $N$ is the number of such polytropes, in each piece the pressure and the rest-mass density are
\[p = K_i \rho^{\Gamma_i}, \quad i = 1, 2, \ldots, N.\]  
(62)
The order of the polytropes is $i = 1$ for the crust, and $i = N$ for the core, and Eq. (62) holds for
\[\rho_{i-1} \leq \rho < \rho_i.\]  
(63)
As we have discussed in Section II B, the first law of thermodynamics for isentropic configurations gives $dh = dp/\rho$, which can be expressed in terms of $q$ to yield
\[dh = \frac{\Gamma_i}{\Gamma_i - 1} dq,\]  
(64)
or equivalently
\[h - h_i = \frac{\Gamma_i}{\Gamma_i - 1} (q - q_i),\]  
(65)
where $h_i, q_i$ correspond to values at the right endpoint (the one closest to the core) of the $i$-th interval. In terms of $q$, we can express the rest of thermodynamic variables as
\[\rho = K_i^{1/(1-\Gamma_i)} q^{1/(\Gamma_i - 1)},\]  
(66)
\[p = K_i^{1/(1-\Gamma_i)} q^{\Gamma_i/(\Gamma_i - 1)},\]  
(67)
\[\epsilon = \rho \hbar - p.\]  
(68)
Enforcing the continuity of the pressure at the $N - 1$ interfaces of each interval constraints all adiabatic constants $K_i$ but one
\[K_i \rho_i^{\Gamma_i} = K_{i+1} \rho_{i+1}^{\Gamma_{i+1}}.\]  
(69)
As a result, the free parameters are: one adiabatic constant, $N - 1$ rest-mass densities, and $N$ adiabatic indices, a total of $2N$ parameters.

\(^5\) More precisely, $A = -6$ makes the spin have zero divergence in the three-geometry ($D_i s^i = 0$) if we choose it to have zero divergence in the conformal three-geometry ($D_i s^i = 0$).
III. NUMERICAL METHOD

The COCAL code for binary black holes has been described in detail in Refs. [70–72]. Here, we will review the most salient features of the grids used for the solution of the field equations and discuss the differences that arise from the treatment of binary stars.

When treating binary systems, COCAL employs two kinds of coordinate systems. The first kind is called COCP - from compact object coordinate patch - and has exactly two members, one centered on each star. The second kind is called ARCP - from asymptotic region coordinate patch - and can have in principle any number of members in an onion type of structure. In our computations, the ARCP patch has only one member, which is centered on the center of mass of the system. All coordinate systems use spherical coordinates \((r, \theta, \phi)\) in \([r_a, r_b] \times [0, \pi] \times [0, 2\pi]\), but the components of field variables (like the shift) are written with their Cartesian components \((\beta^x, \beta^y, \beta^z)\). The values of \(r_a, r_b\) depend on the compact object (black hole or neutron star) and the coordinate patch (COCP or ARCP). For the binary systems treated here, \(r_a\) of the COCP patch will always be zero, while for the ARCP patch \(r_a \approx \mathcal{O}(10M)\), \(M\) being the mass of the star. As for \(r_b\), the values are kept the same as in the binary black hole computation, i.e., \(\mathcal{O}(10^2M)\) for the COCP patch, and \(\mathcal{O}(10^6M)\) for the ARCP patch.

The orientation of the coordinate patches is as follows: the ARCP patch has the familiar \((x, y, z)\) orientation, the first COCP patch, which is centered on the negative \(x\)-axis of the ARCP patch, has the same orientation as the ARCP patch, while the second COCP patch, which is centered on the positive \(x\)-axis of the ARCP patch, has negative \((x, y)\) orientation, and positive \(z\) orientation with respect to the ARCP patch and to the first COCP patch. In other words the coordinate system of the second COCP patch is obtained from the first COCP patch by a rotation through an angle of \(\pi\).

The geometry of the ARCP patch (or any number of them) is that of a solid spherical shell with inner radius \(r_a\) and outer radius \(r_b\). On the other hand, the COCP patch geometry is that of a sphere of radius \(r_b\), with another sphere of radius \(r_x\) at distance \(d_s\) from the center, being removed from its interior. This second sphere whose boundary we call the excised sphere \(S_e\), is centered on the \(x\)-axis around the other compact object. For the first COCP patch, its excised sphere \(S_e\) is centered on the position of the second star, while for the second COCP patch, its excised sphere \(S_e\) is centered on the position of the first star. The size of every sphere \(S_e\) is a slightly larger than the star resolved with an opening half-angle of \(\sim \pi/3\) as seen from the origin of the COCP patch. This is done to resolve accurately the region around the other star and reduce the number of multipoles used in the Legendre expansion, and 12 multipoles are typically used in our computations. Table I summarizes the properties of the various coordinate patches used and which are illustrated schematically in Fig. 1.

| Symbol | Description |
|--------|-------------|
| \(r_a\) | Radial coordinate where the radial grids start. For the COCP patch it is \(r_a = 0\). |
| \(r_b\) | Radial coordinate where the radial grids end. |
| \(r_c\) | Center of mass point. Excised sphere is located at \(2r_c\) in the COCP patch. |
| \(r_e\) | Radius of the excised sphere. Only in the COCP patch. |
| \(r_s\) | Radius of the sphere bounding the star surface. It is \(r_s \leq 1\). Only in COCP. |
| \(N_r\) | Number of intervals \(\Delta r_i\) in \(r \in [r_a, r_b]\). |
| \(N_r^L\) | Number of intervals \(\Delta r_i\) in \(r \in [0, r_s]\) in the COCP patch. |
| \(N_r^C\) | Number of intervals \(\Delta r_i\) in \(r \in [r_a, r_a + 1]\) in the ARCP patch. |
| \(N_{\theta}\) | Number of intervals \(\Delta \theta_j\) in \(\theta \in [0, \pi]\). |
| \(N_{\phi}\) | Number of intervals \(\Delta \phi_k\) in \(\phi \in [0, 2\pi]\). |
| \(d\) | Coordinate distance between the center of \(S_e\) (\(r = 0\)) and the center of mass. |
| \(d_s\) | Coordinate distance between the center of \(S_e\) (\(r = 0\)) and the center of \(S_c\). |

TABLE I. Summary of grid the parameters used for the binary systems computed here.

A. The numerical grids

In Refs. [70, 71] we described in detail optimal numerical grids that we constructed in order to lower the error of the potentials for both close and large separations, for any kind of mass ratio. In particular, when we computed sequences, instead of keeping the radii of the black holes the same and increase their separation, we kept the separation fixed and decreased their radii. By choosing the interval separation near the black holes according to their excised radius, we were able to obtain sequences comparable to the ones produced by spectral methods.

We adopt here the same philosophy for the computation of binary stars. Contrary to previous studies [44], in order to compute sequences of binary stars we let the maximum radius of the star to be variable and we denote by \(r_s\) the infinitum of the radii of all spheres bounding the star that are centered on the origin of the COCP patch. By continuously diminishing \(r_s\) while keeping the distance between the stars constant, we can compute sequences of stars with a continuously increasing separation. In this way we can control the region around the excised sphere as described in [70, 71], while maintaining the accuracy in the area covered by the neutron star.

The COCAL radial grid for binary stars can be seen in Fig. 1, where all the radial distances are the normalized quantities discussed in Sec. III B. In that sense, they should be denoted with a hat, for example \(\hat{r}\), which is omitted here for simplicity. When comparing with Fig. 2 of [70], we can observe that there is also an important difference in the notation.
This regards the quantity $N^f_r$, which previously was used to denote the number of intervals in $[r_a, 1]$, while here is used to denote the number of intervals in $[r_a, r_s] = [0, r_s]$, with $r_s \leq 1$. The number of intervals in $[0, 1]$ is denoted by a new variable called $N^I_r$ (this plays the role of the old $N^f_r$ for grid comparisons). This change was necessary since in all previous studies [44], the surface of the star was bounded by the fixed $r = 1$ sphere, and therefore the fluid extended until that point. To compute stars at larger separation while satisfying this constraint, we would have to increase the position of the excised sphere $S_r$ and therefore expand all grid quantities analogously. To avoid such a complication, we introduce a variable $r_s$ that effectively mimics the change in separation. By varying the endpoint of the fluid (point $r_s$) we achieve the same result as varying the distance between the stars, but we maintain the good convergence properties that were established in [71] while maintaining our fluid code essentially unchanged.

As we can see from Fig. 1, there are five regions in the COCP patch of a star that are denoted by $S, I, II, III,$ and $IV$. The star is resolved by a constant grid spacing $\Delta r = r_s/N^I_r$ as region $I$, which has spacing $\Delta r_2 = 1/N^I_r$. Setting $\Delta r_i := r_i - r_{i-1}$, the grid intervals in each of them are

\[
\Delta r_i = \Delta r, \quad \text{for } i = 1, \ldots, N^I_r - 1, \quad (70)
\]

\[
\Delta r_{i+1} = h_1 \Delta r_i, \quad \text{for } i = N^I_r, \ldots, N^I_r - 1, \quad (71)
\]

\[
\Delta r_i = \Delta r_2, \quad \text{for } i = N^I_r, \ldots, N^m_r, \quad (72)
\]

\[
\Delta r_{i+1} = h_3 \Delta r_i, \quad \text{for } i = N^m_r, \ldots, N^m_r + N^I_r - 1, \quad (73)
\]

\[
\Delta r_{i+1} = h_4 \Delta r_i, \quad \text{for } i = N^m_r + N^I_r, \ldots, N_r - 1, \quad (74)
\]

which correspond to regions $S, I, II, III,$ and $IV$, respectively. The ratios $h_i(> 1)$ $(i = 1, 3, 4)$ are respectively determined from the relations

\[
1 - r_s = \Delta r \frac{h_1^2 N^I_r - N^I_r - 1}{h_1 - 1}, \quad (75)
\]

\[
2r_s = \Delta r \frac{h_3^2 h_2^2 - 1}{h_3 - 1}, \quad (76)
\]

\[
r_b - 3r_s = \Delta r \frac{h_4^2 h_2^2 - N^m_r - N^I_r - 1}{h_4 - 1}. \quad (77)
\]

For the ARCP coordinate system, there are in general two regions, one with constant grid spacing and one with increasing spacing. The grid intervals in these regions are defined by

\[
\Delta r_i = \Delta r_1, \quad \text{for } i = 1, \ldots, N^m_r, \quad (78)
\]

\[
\Delta r_{i+1} = k \Delta r_i, \quad \text{for } i = N^m_r, \ldots, N_r - 1, \quad (79)
\]
where \( \Delta r_1 = 1/N_1 \), and the ratio \( k \) is determined from
\[
rb - rc := \Delta r \frac{k(N_r - N_r^m - 1)}{k - 1}.
\tag{80}
\]

As regards the angular resolution, we keep the same grid interval in the \( \theta \) and \( \phi \) directions and therefore
\[
\Delta \theta_j = \theta_j - \theta_{j-1} = \Delta \theta = \frac{\pi}{N_\theta},
\tag{81}
\]
\[
\Delta \phi_k = \phi_k - \phi_{k-1} = \Delta \phi = \frac{2\pi}{N_\phi}.
\tag{82}
\]

One of the additional complications of having to deal with the fluid of a star, instead of a vacuum spacetime, is the need to accurately find its surface. This surface may contract or expand during the calculation, creating significant problems in close binary configurations. One very effective solution to these issues \cite{102} is the use of surface-fitted coordinates (SFC) that exist only inside each fluid and are normalized by the radius of the star. We denote this extra spherical coordinate system as \( (r_f, \theta_f, \phi_f) \), where
\[
r_f := \frac{r}{R(\theta, \phi)}, \quad \theta_f := \theta, \quad \phi_f := \phi,
\tag{83}
\]
and where the surface of the star is denoted by \( R(\theta, \phi) \).

By construction, the domain of these fluid coordinates is \([0, 1] \times [0, \pi] \times [0, 2\pi]\), and \( R(\theta_f, \phi_f) \) is a function that will be determined at the end of the self-consistent iterative method (see App. C). The advantage of SFC in the computation of derivatives on the star surface, as well as the implementation of the boundary condition Eq. \((58)\), will be discussed in Sec. III.D.

**B. Dimensionless and normalized variables**

Having removed the dimensions from our equations by using units in which \( \mathcal{G} = c = M_\odot = 1 \), we perform a normalization of all variables in order to introduce a scale in our problem that is intimately related to the variable \( r_s \), introduced in Section III.A.

We normalize variables by demanding that the intersection of the star’s surface with the positive \( x \)-axis be at \( r = r_s \). If \( R_0 \) is the scaling parameter, we impose\(^6\)
\[
\frac{R(\pi/2, 0)}{R_0} = r_s = \frac{R(\pi/2, \pi)}{R_0},
\tag{84}
\]
so that \( r_s R_0 \) is the real semi-major radius of the star. Hereafter, we will denote normalized variables with a hat and thus define
\[
\hat{x}^i := \frac{x^i}{R_0},
\tag{85}
\]
from which it follows that the normalized version of Eq. \((11)\) is\(^7\)
\[
\hat{\nabla}^2 \psi = \hat{S}_\psi^0 + R_0^2 \omega_f^i,
\tag{86}
\]
where \( \hat{\nabla} \) is the Laplacian operator associated with the variables \( \hat{x}^i \), and similarly \( \hat{S}_\psi^0 \) has all derivatives with respect to the normalized variables. Note also that
\[
\omega^i = \beta^i + \Omega \phi^i = \beta^i + \hat{\Omega} \phi^i,
\tag{87}
\]
where \( \hat{\Omega} := \Omega R_0 \). If we now define
\[
\hat{\Phi} := \frac{\Phi}{R_0},
\tag{88}
\]
we observe that all scaling factors in Eq. \((57)\) drop out. Because this is also true for the boundary condition, i.e., Eq. \((58)\), these equations are each coded in the same form but with normalized quantities replacing the original ones.

Before proceeding further with our normalization scheme, let us comment that the surface-fitted coordinates of Eq. \((83)\) are already normalized coordinates and their radial range is \([0, 1]\) irrespective of the fluid scaling profile \( r_s \). Since \( R(\theta, \phi) \leq R(\pi/2, 0) = R_0 r_s \), we have \( 0 \leq r \leq R(\theta, \phi) \leq R_0 r_s \), so that the range for \( \hat{r} \) is
\[
\hat{r} \leq \hat{R}(\theta, \phi) \leq r_s, \quad \text{or} \quad r_f = \frac{\hat{r}}{R(\theta, \phi)} \leq 1.
\tag{89}
\]

Changing the scale by modifying \( R_0 \) will affect the conformal factor and the lapse function since they scale as
\[
\psi = \hat{\psi} R_0^\alpha, \quad \alpha = \hat{\alpha} R_0^2.
\tag{90}
\]

As mentioned earlier, the system of partial differential equations that we have to solve i.e., the normalized versions of Eqs. \((11)-(13)\) and Eq. \((57)\), involve three constants: \( R_0 \), \( \hat{\Omega} \), and \( C \). To find them we will use the Euler integral evaluated at three arbitrary points to construct a nonlinear \( 3 \times 3 \) system that will be solved with a typical Newton-Raphson method. This procedure will be repeated every time we solve for any of the unknown variables \( \psi, \beta^i, \alpha, \hat{\Phi}, \) and \( q \), since any change of the them will affect the Euler integral and thus the three constants.

The arbitrary points we choose to evaluate the Euler integral are along the \( x \)-axis and in spherical coordinates are defined as
\[
r_1 = r_s, \quad \theta_1 = \pi/2, \quad \phi_1 = 0, \tag{91}
\]
\[
r_2 = 0, \quad \theta_2 = 0, \quad \phi_2 = 0, \tag{92}
\]
\[
r_3 = r_s, \quad \theta_3 = \pi/2, \quad \phi_3 = \pi. \tag{93}
\]
In the corotating case, Eqs. \((32)\) and \((31)\) will become
\[
F_c(\hat{\Omega}, R_0, C) = -\ln C + \hat{R}_0^2 \ln \hat{\alpha} + \ln h + \frac{1}{2} \ln \left[ 1 - \left( \frac{\hat{\psi}_0^2}{\hat{\alpha}} \right)^{2\hat{R}_0^2} \left( (\beta^0 + \hat{\Omega} \phi^0)^2 \right) \right] = 0,
\tag{94}
\]
\(^6\) Note that \( r_s \), but also \( r_f \) in Eq. \((83)\), are ratio of two radial coordinates and thus dimensionless for any choice of units; in this respect, they do not need to be indicated with a hat.

\(^7\) Similar normalized equations hold for Eq. \((12)\) and \((13)\).
while for the spinning case, Eqs. (42), (49), and (51) yield

\[ F_{is}(\Omega, R_0, C) = -\ln \lambda + R_0^2 \ln \alpha + \ln h \]

\[ -\ln \left[ \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{A(R_0)}{\lambda^2}} \right] + \frac{1}{2} \ln \left[ 1 + \frac{B(R_0)}{h^2} \right] = 0 \tag{95} \]

with

\[ A(R_0) := (\hat{\alpha}^2 \hat{\psi}^2 R_0^2 \delta_i \delta_j \partial_i \Phi + (\hat{\alpha}^2 \hat{\psi}^2 + 2A) R_0^2 \delta_i \delta_j \delta_k \delta_l \delta^k \delta^l) \]

\[ B(R_0) := (\hat{\psi}^2 A R_0^2 \partial_i \Phi \partial_j \Phi + 2(\hat{\psi}) A R_0^2 \delta_i \delta_j \delta_k \delta_l \delta^k \delta^l) + (\hat{\psi}^2 A R_0^2 \delta_i \delta_j \delta_k \delta_l \delta^k \delta^l) \lambda := C + (\beta y + \bar{\Omega} \bar{\phi}) \partial_\phi \Phi. \]

Evaluating either Eq. (94) or Eq. (95) at the three points given by (91)–(93) will produce a system of three equations in the three unknowns \( \Omega, R_0, \) and \( C \); the solution of the system will determine these constants.

C. Elliptic solver for the gravitational part

As discussed in detail in Refs. [70–72], Eqs. (11)–(13) are solved using the representation theorem of partial differential equations in a self-consistent way. This method, which is commonly referred to as the KEH method [78], will be suitably modified in order to account for the specific boundary conditions that exist in the new COCAL coordinate systems. For example, the conformal factor will be expressed as

\[ \psi(x) = \chi(x) + \psi_{\text{INT}}(x), \tag{96} \]

where

\[ \psi_{\text{INT}}(x) = -\frac{1}{4\pi} \int_V \frac{S^\phi_0(x') + S^\phi_\psi(x')}{|x - x'|} d^3x' \]

\[ + \frac{1}{4\pi} \int_{\partial V} \left[ \nabla_{\alpha} \psi(x') - \psi(x') \nabla_{\alpha} \frac{1}{|x - x'|} \right] dS'_a, \tag{97} \]

and

\[ \chi(x) = \frac{1}{4\pi} \int_{S_a \cup S_b} [G^{BC}(x, x') \nabla^\alpha (\psi_{\text{BC}} - \psi_{\text{INT}}(x')) - (\psi_{\text{BC}} - \psi_{\text{INT}}(x')) \nabla^\alpha G^{BC}(x, x')] dS'_a. \tag{98} \]

Note that \( G^{BC} \) is the Green’s function associated with the boundary conditions applied on the corresponding field \( \psi_{\text{BC}} \) at the boundaries \( S_a \) and \( S_b \). Formulas (96)–(98) will be applied separately on every coordinate patch. If, for example, we have one ARCP patch (as it happens in our computations) it means that the equations above will be applied three times: two for the COCP patches and one for the ARCP patch. Of course, the domains of integration vary according to the different patches considered. More specifically, if we denote by \( B(R) \) a sphere of radius \( R \) in each of the COCP patch, then the integration domain of Eq. (97) will be \( V = B(r_b) - B(r_a) \) and \( \partial V = S_a \cup S_b = \partial B(r_a) \cup \partial B(r_b) \), while that of Eq. (98) will be \( S_a \cup S_b = S_a \) since \( r_a = 0 \) for star configurations in the COCP patch. Similarly, in the ARCP patch the integration domain of Eq. (97) will be \( V = B(r_b) - B(r_a) \), and that of Eq. (98) \( \partial V = S_a \cup S_b \).

We recall that in Ref. [70] we have introduced a number of Green’s functions \( G^{BC}(x, x') \) suitable for various boundary conditions. Here we add one more Green’s function used in the COCP patch

\[ G^{SD}(x, x') := \sum_{\ell=0}^{\infty} g^{SD}_{\ell}(r, r') \sum_{m=0}^{\ell} \epsilon_m (\ell - m)!(\ell + m)! P_{\ell m}(\cos \theta) P_{\ell m}(\cos \theta') \cos[m(\phi - \phi')], \tag{99} \]

where

\[ g^{SD}_{\ell}(r, r') := r_\ell^{\epsilon_\ell} \left[ \left( \frac{r_a}{r_b} \right)^{\ell+1} - \left( \frac{r_a}{r_b} \right)^\ell \right], \tag{100} \]

and \( \epsilon_0 = 1, \epsilon_m = 2 \) for \( m \geq 1 \), while \( P_{\ell m} \) are the associated Legendre polynomials, and \( r_\ell := \sup\{r, r'\}, r_\ell := \inf\{r, r'\} \).

In the ARCP patch we use a Green’s function \( G^{DD}(x, x') \), whose radial part satisfies the Dirichlet-Dirichlet boundary conditions on \( S_a \) and \( S_b \)

\[ g^{DD}_{\ell}(r, r') := 1 - \left( \frac{r_a}{r_b} \right)^{2\ell+1} \left[ \left( \frac{r_a}{r_b} \right)^\ell - \left( \frac{r_a}{r_b} \right)^{\ell+1} \right] \left[ \left( \frac{r_\ell}{r_b} \right)^{\ell+1} - \left( \frac{r_\ell}{r_b} \right)^\ell \right]. \tag{101} \]

D. Elliptic solver for the fluid part

Next, we describe the method used to solve Eq. (57) and which is therefore valid only for the spinning binaries. The boundary condition for \( \Phi \), Eq. (58), is of von Neumann type and therefore we could apply the Poisson solver of Sec. III C to obtain a solution. Instead, and as a demonstration of the versatility of the methods employed by COCAL, we will adapt the procedure discussed in Ref. [44] and solve this boundary-value problem as an application of the least-squares algorithm.

First, we assume that the solution of Eq. (57) can be written in the form

\[ \Phi(x) = -\frac{1}{4\pi} \int_V \frac{S_{\Phi}(x')}{|x - x'|} dV + \zeta(x) = \Phi_V(x) + \zeta(x), \tag{102} \]

with \( \zeta(x) \) obeying the Laplace equation

\[ \nabla^2 \zeta(x) = 0. \tag{103} \]

Using the decomposition of Eq. (102), the boundary condition (58) is written as

\[ \psi^4 h_u^i \omega^j m_i - m^i \partial_i \Phi_V - \psi^{4+it} \delta^i m_i = m^i \partial_i \zeta, \tag{104} \]

where we used the normal to the surface unit vector \( m^i = (\hat{n}_x) \) instead of the gradient of the rest-mass density. The
equation above is evaluated on the surface of the star, \( R(\theta, \phi) \), and the velocity potential satisfies the following symmetries
\[
\Phi(r, \pi - \theta, \phi) = \Phi(r, \theta, \phi), \quad (105)
\]
\[
\Phi(r, \theta, 2\pi - \phi) = -\Phi(r, \theta, \phi), \quad (106)
\]
which in turn imply that the homogeneous solution, which is regular at the stellar center, can be expanded as
\[
\zeta(r, \theta, \phi) = \sum_{\ell=1}^{\ell} \sum_{m=1}^{m} a_{\ell m} r^{\ell} [1 + (-1)^{\ell + m}] Y_{\ell}^{m}(\cos \theta) \sin(m\phi), \quad (107)
\]
where \( Y_{\ell}^{m} \) are the spherical harmonics and \( a_{\ell m} \) coefficients to be determined.

On the other hand, if \( R(\theta, \phi) \) is the surface of the star, the spatial vector connecting any point on it with the center of coordinates is given by
\[
\vec{x}(\theta, \phi) = R(\theta, \phi)(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad (108)
\]
thus the unit normal vector will be
\[
\hat{n}_x(\theta, \phi) := \left( \frac{\partial \vec{x}}{\partial \theta} \times \frac{\partial \vec{x}}{\partial \phi} \right) \left[ \frac{\partial \vec{x}}{\partial \theta} \times \frac{\partial \vec{x}}{\partial \phi} \right]^{-1}, \quad (109)
\]
or equivalently
\[
\hat{n}_x = \frac{1}{\sqrt{h}} \left( \dot{r} - \frac{1}{R} \frac{\partial R}{\partial \theta} \dot{\theta} - \frac{1}{R \sin \theta} \frac{\partial R}{\partial \phi} \dot{\phi} \right), \quad (110)
\]
where \( \dot{r}, \dot{\theta}, \dot{\phi} \) are the spherical unit vectors and
\[
h(\theta, \phi) := 1 + \left( \frac{1}{R} \frac{\partial R}{\partial \theta} \right)^2 + \left( \frac{1}{R \sin \theta} \frac{\partial R}{\partial \phi} \right)^2. \quad (111)
\]

Using Eq. (107) and Eq. (110), the boundary condition (104) is written as
\[
\sum_{\ell=1}^{\ell} \sum_{m=1}^{m} a_{\ell m} F_{\ell m}(\theta, \phi) = H(\theta, \phi), \quad (112)
\]
where
\[
H(\theta, \phi) := \psi^4 h u^j \omega^i m_i - m^i \partial_i \Phi_V - \psi^{4+A} \hat{s}^j m_i, \quad (113)
\]
and
\[
F_{\ell m}(\theta, \phi) := [1 + (-1)^{\ell + m}] \times \left[ \ell R^{\ell - 1} Y_{\ell}^{m} \sin(m\phi) \frac{\partial R}{\partial \theta} R^{\ell - 2} \frac{\partial Y_{\ell}^{m}}{\partial \theta} \sin(m\phi) - \frac{\partial R}{\partial \phi} \frac{\partial R}{\partial \theta} \sin^2 \theta Y_{\ell}^{m} m \cos(m\phi) \right]. \quad (114)
\]
To solve for the coefficients \( a_{\ell m} \), we consider the functional
\[
\mathcal{E} := \sum_{\theta_i, \phi_j} \left[ \sum_{\ell=1}^{\ell} \sum_{m=1}^{m} a_{\ell m} F_{\ell m}(\theta_i, \phi_j) - H(\theta_i, \phi_j) \right]^2 = 0, \quad (115)
\]
of the discretized version of the boundary condition (112), and demand that for fixed indices \( p \) and \( q \)
\[
\frac{\partial \mathcal{E}}{\partial a_{pq}} = 0. \quad (116)
\]
The minimizing condition Eq. (116) then yields
\[
\sum_{\ell=1}^{\ell} \sum_{m=1}^{m} a_{\ell m} \left[ \sum_{i,j} F_{\ell m}(\theta_i, \phi_j) F_{\ell m}(\theta_i, \phi_j) \right] = \sum_{i,j} H(\theta_i, \phi_j) F_{\ell m}(\theta_i, \phi_j), \quad (117)
\]
which is a linear system in terms of the \( a_{\ell m} \) coefficients. For \( L \) even, the dimensions of the system is \( M \times M \) with \( M = L(L+1)/2 \). After determining the coefficients \( a_{\ell m} \), the solution for the velocity potential \( \Phi \) is obtained from Eq. (102) and Eq. (107).

IV. NUMERICAL RESULTS

In what follows we report tests of our new code against previous results obtained by other groups and then present some new ones. In particular, we focus on the construction of quasi-equilibrium sequences for corotating, irrotational, and spinning binaries, produce a binary white dwarf solution in order to explore the weak-field limit of our code. We compute two main global-error indicators to measure the accuracy of our converged solutions; the first one is given by the relation \( M_{K} = M_{ADM} \), where \( M_{K} \) and \( M_{ADM} \) are the Komar and Arnowitt-Deser-Misner (ADM) mass, respectively [103–107]. The second one is instead related to the first law of binary thermodynamics \( d M_{ADM} = \Omega dJ \), where \( \Omega \) is the orbital frequency and \( J \) the orbital angular momentum [90]. Explicit definitions and computational algorithms for these quantities within COCAL are presented in Appendix A.

Sequences of constant rest mass can be thought as snapshots of an evolutionary process that drives the two stars close to each other as a result of to gravitational radiation reaction. At every instant in time, the rest mass of each star is conserved; furthermore, if the flow is irrotational, the circulation of the fluid velocity on any loop is also conserved (Kelvin-Helmholtz theorem [35]). It is possible to characterize these sequences via the properties of the stars, such as the compactness or the ADM mass, when the binary has infinite separation and each star is spherical.

By varying the separation between the two stars and solving each time all the relevant equations, we obtain solutions of a given central rest-mass density, and then another loop of solutions has to be invoked in order to find the particular central rest-mass density that yields a star with the desired rest mass. Typically, this is done by a Newton-Raphson method and it takes a maximum of 10 iterations depending on the starting solution.

In this way we can monitor important quantities like the binding energy of the system, which is defined as
\[
E_b := M_{ADM} - M_{\infty}, \quad (119)
\]
FIG. 2. Quasi-equilibrium sequences for corotating binary neutron stars with EOS consisting of a single polytrope with $\Gamma = 2$. $M$ and $J$ are the total ADM mass and angular momentum of the system. The resolutions used are those given in Table II. A comparison is made with the results of Ref. [1], where similar solutions were obtained from the spectral code LORENE [112].

and represents the total energy lost in gravitational waves by the system, since $M_\infty$ is twice the ADM mass of a single isolated spherical star.

A. Corotating solutions

As mentioned in the Introduction, corotating states [108–111], i.e., states with zero angular velocity of the star with respect to a corotating observer, probably are not physically realistic due to the low viscosity of the neutron-star matter. Such solutions represent an important step in the numerical solution of the binary problem, since they provide key insights for the
numerical implementation of a stable algorithm. In particular
the surface-fitted coordinates, as well as the solution of the Euler integral, Eq. (32), can be thoroughly checked. This allows
us to perform a calibration without having to worry about the fluid flow (i.e., to solve the equation of conservation of rest mass). Corotating evolutionary configurations are known to exhibit a minimum in the mass and angular momentum versus the normalized angular velocity \( \Omega \Gamma K^{1/(2(\Gamma - 1))} \), which was taken to denote the putative innermost stable circular orbit (ISCO), beyond which the binary was thought to proceed rapidly towards a merger. In practice, fully general-relativistic simulations of inspiralling binary neutron stars do not show the existence of such an instability, revealing instead that the inspiral and merger is a smooth process [2, 3]. Nevertheless, the presence of such a minimum represents a useful test of numerical codes, as does the appearance of a familiar spike, similar to the one encountered in binary black-hole solutions, when plotting the binding energy versus the angular momentum of the binary.

In Fig. 2 we present sequences of binary neutron stars that correspond to compactness of \( C := M_{\text{ADM}}/R = 0.12 \) and 0.18, where \( M_{\text{ADM}} \) and \( R \) are the (ADM) mass and radius of each star when taken at infinite separation. The ADM and Komar mass, as well as other quantities used in COCAL, are described in detail in Appendix A. The adiabatic index is \( \Gamma = 2 \) and the polytropic constant is set to be \( K = 1 \). In the various plots a comparison is made between the results obtained with COCAL and those presented in Ref. [1], where the same initial data was computed using LORENE, a pseudo-spectral code developed by the Meudon group [1]. As we can see, the relative difference in the results between the two codes is of the order of 0.05%, even when a medium resolution is used for COCAL. The grid structure used in these calculations is the one described in Table II.

Similarly, in Fig. 3 we report the change in the central rest-mass density with respect to the one at infinity, which is \( \rho_{\infty} = 0.0922 \) for compactness \( C = 0.12 \), while it is \( \rho_{\infty} = 0.1956 \) for \( C = 0.18 \). Clearly, the central rest-mass density decreases as the binary come closer, making the onset of a instability to gravitational collapse very unlikely [113]. In addition, as a measure of accuracy of these corotating sequences, we plot in

\[
\Delta_M := \left| \frac{M_{\text{ADM}} - M_K}{M_{\text{ADM}}} \right|
\]

as a function of the binary separation \( d_s/r_s \). Note that all radii are here normalized to the scaling factor \( R_0 \) and are therefore dimensionless, so that, eg, the physical distance between the two neutron stars is \( d_s R_0 \). As we can see, even for the medium resolution used in these calculations the error is below \( 10^{-4} \). All of the quantities in the expression above have been extracted from the ARCP patch, as integrals at infinity. We note that at present COCAL does not implement a unifying mesh, and this prevents us from calculating the virial error as obtained by Friedman, Uryu, and Shibata [90], since we are using overlapping coordinate systems. We plan to revisit this issue in the future.

Finally, in Fig. 5 we report sequences of corotating binaries for increasing central rest-mass density at different separations, from \( d_s/R_0 = 4 \), down to separation in which the two stars are almost touching \( d_s/R_0 = 2.125 \). Here we use \( r_s = 1 \) therefore the radius of the neutron star is \( R_0 \). Clearly, for any given central energy density a larger mass is supported by the binary (supramassive solutions) when we move to closer con-
As anticipated in the Introduction, irrotational neutron stars have been considered as a reasonable first approximation to describe the flow in binary configurations. In such a case, the total angular momentum is less than the corresponding of a corotating binary, since in each star there is a flow in the counter-direction with respect to the orbital motion. This has two consequences. First, the inspiral of irrotational binaries is slower than that of corotating ones or, equivalently, the gravitational-wave frequency is expected to increase with a faster rate for corotating systems. Indeed, since

\[
\frac{dr}{dt} = \frac{dM_{\text{ADM}}}{dt} \left( \frac{dM_{\text{ADM}}}{dr} \right)^{-1},
\]

and the gravitational-wave luminosity \(dM_{\text{ADM}}/dt\) is approximately the same for corotating and irrotational binaries, the inspiral is determined mostly by the change of the total energy with respect to the separation between the stars, \(dM_{\text{ADM}}/dr\). This rate is smaller in the corotating system because of the inclusion of the spin kinetic energy, so that \(\left|\frac{dr}{dt}\right|_{\text{corot}} > \left|\frac{dr}{dt}\right|_{\text{irrot}}\). Second, in the light of the results obtained for binary black holes, where binaries with larger spins lead to increasingly spinning final black holes \([117, 118]\), the irrotational binary system will eventually lead to a Kerr black hole that is more slowly rotating than the corresponding one produced by the corotating binary.

As done in Sec. IV A for corotating binaries, we compare in Fig. 6 our irrotational solutions for compactness \(C = 0.12\) and 0.18 against the corresponding results presented in Ref. [1]. Although we use here a relatively small resolution, i.e., Hs2d from Table II, the relative differences with Ref. [1] is again of the order of 0.05%. Note that for binaries with compactness \(C = 0.12\), the variable \(r_s\) for ranges from \(r_s = 0.5\) to \(r_s = 0.87\), which corresponds to coordinate separations \(d_s/r_s = 2.5/0.5 = 5\) and \(d_s/r_s = 2.5/0.87 = 2.87\), respectively. On the other hand, for \(C = 0.18\), \(r_s\) varies from \(r_s = 0.5\) to \(r_s = 0.79\), which corresponds to separations from \(d_s/r_s = 5\) to \(d_s/r_s = 3.16\), respectively. Note also that the minimum in these plots marks the mass-shedding limit and the creation on the equatorial plane of a cusp in the rest-mass density.

In Fig. 7 we present the results relative to the irrotational binary solutions with \(r_s = 0.76\) and compactness \(C = 0.18\). More specifically, in the left column we report the contour plots of the conformal factor \(\psi\) from 1.0 to 1.33 with step 0.01, of the rest-mass density from 0.0 to 0.3 with step 0.01, and of the velocity potential \(\Phi\) from −0.2 to 0.2 with step 0.01, all on the \((x, y)\) plane. On the other hand, in the right

\[\text{FIG. 4. Measure of the virial error } M_{\text{vir}} = M_{\text{ADM}} \text{ for the corotating sequences in Fig. 2, as a function of separation.}\]

\[\text{FIG. 5. Corotating sequences versus central energy density for different separations. The TOV curve corresponds to infinite separation. Here we use } r_s = 1, \text{ therefore } R_0 \text{ is the radius of the star.}\]
FIG. 6. Sequences of irrotational binary neutron stars, with an EOS consisting of a single polytrope with $\Gamma = 2$. The resolution is $Hs2d$ from Table II. A comparison is made with the results presented in [1], where similar solutions were obtained from the spectral code Lorene [112].

In the column we show the shift and the fluid velocity vector fields on the $(x, y)$ plane, and a contour plot of the rest-mass density on the $(y, z)$ plane.

Similarly, in Fig. 8 we report two global error indicators computed for irrotational binaries with compactness $C = 0.18$. More specifically, the top panel shows the fractional difference in the Komar and ADM masses, while the bottom panel shows fractional error of the relation $dM_{\text{ADM}} = \Omega dJ$ relation; note that the latter is a rather stringent test and that a fractional error below $0.7\%$ gives us confidence on the accuracy of our solutions already at an intermediate resolution. Finally, in Fig. 9 we plot the relative change in the central rest-mass density as the coordinate separation between the two stars is reduced. This figure should be compared with the corresponding Fig. 3 for corotating binaries and shows that again the central rest-mass density decreases as the two stars approach, but also that this decrease is smaller, of one order of magnitude or more, than in the corotating case.
FIG. 7. Irrotational binary solution with $r_s = 0.76$ and compactness $C = 0.18$. The separation between the two neutron stars is $d_s/r_s = 2.5/0.76 = 3.29$. Left column: contour plots of the conformal factor $\psi$ from 1.0 to 1.33 with step 0.01, of the rest-mass density from 0 to 0.3 with step 0.01, and of the velocity potential $\Phi$ from $-0.2$ to 0.2 with step 0.01, all on the $(x, y)$ plane. Right column: shift and fluid velocity vector fields on the $(x, y)$ plane, and contour plot of the rest-mass density on the $(y, z)$ plane. Note that the green sphere corresponds to the excised sphere $S_e$ of COCP-1.
reports the dimensionless binding energy $E_b / M_{\infty}$ of the binary as a function of the dimensionless orbital frequency $\Omega M_{\infty}$. Considered and compared are an irrotational binary (violet solid line) and two spinning binaries, one with $\tilde{S}_0 = 0.01$ (red solid line) and another one with $\tilde{S}_0 = 0.05$ (blue solid line). All binaries are modeled with the APR1 EOS and both of the spinning binaries have velocity field with $A = 0$. Also shown for comparison is the fourth post-Newtonian (4PN) approximation (black dashed line) [119], whose explicit form is given in App. E. Clearly, the differences with respect to the irrotational binaries are very small even for these high spinning rates, but these are more evident when the looking at the right panel of Fig. 10.

C. Spinning sequences

We conclude our discussion of the results with the new CAL by presenting our first calculations of quasi-equilibrium binary systems of spinning neutron stars. The neutron-star matter is modeled using a piecewise polytrope representation of the APR1 EOS [101]. As mentioned in Sec. II C, an EOS with $N$ polytropic segments requires $2N$ parameters to be specified, which can be thought of as one adiabatic constant, $N - 1$ dividing rest-mass densities, and $N + 1$ adiabatic indices. In Ref. [98] it was found that a number of tabulated nuclear matter EOS can be modeled with three segments above nuclear density and one in the crust, thus with a total of four polytropic zones. The error in the approximation is $\sim 0.1\%$, or at worst $\sim 4\%$. A fit with a minimum error was described in [98] that had a fixed crust with $\Gamma_0 = 1.35692$, $K_0 = 3.59389 \times 10^{13}$, and three core zones with adiabatic exponents $\{\Gamma_1, \Gamma_2, \Gamma_3\}$, joining the different pieces at rest-mass densities $\rho_1 = 10^{14.7} \text{g/cm}^3$, and $\rho_2 = 10^{15} \text{g/cm}^3$. Additional information on the properties of the initial data are collected in Table IV.

The spin contribution to the fluid velocity is expressed through the spatial three-vector $\tilde{s}^i$ [cf., Eq. (56)], which we express as

$$\tilde{s}^i = \tilde{S}_0(-y, x, 0),$$

(122)

where the Cartesian coordinates $x, y$ are centered in the COCP patch, and the positive (negative) constant $\tilde{S}_0$ denotes the magnitude of corotation (counter-rotation).

In Fig. 10 we report the properties of a sequence of binary neutron stars with constant rest mass $M_0 = 1.5388 M_\odot$, corresponding to an ADM mass $M_{\text{ADM}} = 1.35 M_\odot$ when the stars are at infinite separation. The freedom in the choice of the spin velocity vector defined in Eqs. (56) and (122) has been fixed by taking $A = 0$, while for $\tilde{S}_0$ we examine two cases: $\tilde{S}_0 = 0.01$, which corresponds to a spinning period of $\sim 3\text{ ms}$, and $\tilde{S}_0 = 0.05$, which corresponds to the extreme case of a period $\sim 0.6\text{ ms}$. These choices correspond to spinning periods that are more than twice smaller than those considered in [67], where the maximum value considered was $6.7\text{ ms}$. Although it is rather unlikely that such small rotation periods are encountered in reality in binaries about to merger, it is a good consistency check for our new code and an exploration of its limits.

The left panel of Fig. 10 reports the dimensionless binding energy $E_b / M_{\infty}$ of the binary as a function of the dimensionless orbital frequency $\Omega M_{\infty}$. Considered and compared are an irrotational binary (violet solid line) and two spinning binaries, one with $\tilde{S}_0 = 0.01$ (red solid line) and another one with $\tilde{S}_0 = 0.05$ (blue solid line). All binaries are modeled with the APR1 EOS and both of the spinning binaries have velocity field with $A = 0$. Also shown for comparison is the fourth post-Newtonian (4PN) approximation (black dashed line) [119], whose explicit form is given in App. E. Clearly, the differences with respect to the irrotational binaries are very small even for these high spinning rates, but these are more evident when the looking at the right panel of Fig. 10.
which reports instead the behaviour of the dimensionless angular momentum $J/M_{\infty}^2$. In this case, the difference between the irrotational and the $S_0 = 0.05$ spinning sequence is more evident, with a relative difference of the order of $\sim 3.5\%$ at frequencies near the contact frequency. Note also that the sign of $S_0$ determines the relative position of the spinning sequence relative to the irrotational one. In particular, with a negative value for $S_0$, the angular-momentum curve for the spinning sequence would have appeared below the irrotational one.

Finally, in Fig. 11 we report an estimate of the spin contribution to the total angular momentum. More precisely, we show for the same binaries presented in Fig. 10 the relative difference between the angular momentum of the spinning binaries, $J_{sp}$, and that of an irrotational binary, $J_{ir}$. The top and bottom panels of Fig. 11 are relative to spin vector fields with $A = 0$ and $A = -6$, respectively; for each panel, red and blue lines are used to distinguish sequences with either $S_0 = 0.01$ or with $S_0 = 0.05$. Note that despite the short spin periods considered, the contribution of the spin angular momentum is at most $3.5\%$ of the total for binaries with $A = 0$, and less than $1\%$ for binaries with $A = -6$. Note that this quantity is not expected to be constant along this sequence, where only the rest mass and the spinning coefficient $S_0$ are kept constant.

If we estimate the dimensionless spin angular momentum to be

$$\chi := \frac{S}{M_{\text{ADM}}^2} := \frac{1}{2} \left( \frac{J_{sp} - J_{ir}}{M_{\text{ADM}}^2} \right),$$

then because the ADM mass of the spinning binaries is very close to the irrotational one and lies in the range $M_{\text{ADM}} \in [1.33, 1.34]$, the dimensionless spin takes values in the range $\chi \in [0.027, 0.066]$ for the case with $A = 0$ and $S_0 = 0.05$ case, and values in the range $\chi \in [0.0017, 0.0035]$ for the $A = -6$, $S_0 = 0.01$ case. In all the cases considered we have found that the error estimates discussed in the previous Sections lead quite generically to relative errors that are
is the relative difference for the same binaries presented in Fig. 5. To spin vector fields with between the angular momentum of the spinning binaries and that of an irrotational binary. The top and bottom panels refer respectively to red and blue lines are used to report sequences with \( S = 0.01 \) and \( S_0 = 0.05 \), respectively. Note that despite the short spin periods considered, the contribution of the spin angular momentum is at most 3.5% of the total.

\[ \lesssim 0.7\% \]

and that smaller errors can be obtained with grids having higher resolution.

V. CONCLUSION

We have presented the extension of the \textsc{cocal} code to treat binary configurations of compact stars within the IWM formalism of general relativity. As with the work done for binary black holes, we have used multiple coordinate patches so as to be able to treat asymmetric binaries. Also in the spirit of previous work, we have introduced a particular normalization scheme that allows us to accurately compute binary systems that have small or large separations, recovering the spherical limit at large distances. This is done by keeping the stars at fixed coordinate positions, but artificially reducing their radius. Furthermore, we have made use of surface-fitted coordinates to describe accurately the stellar shape as it varies along sequences of constant rest mass.

Also for the non-vacuum spacetimes considered here, we have employed the KEH method [73–78], in which the gravitational equations are solved using Poisson solvers with appropriate Green’s functions, while for the conservation of rest mass we employ a least-squared algorithm. The code makes use of a piecewise polytropic description to represent the EOS of stellar matter and, for the specific cases considered here, we have adopted the representation of the APR1 EOS [101].

Making use of a suitably adapted formulation described in Ref. [60], the code is able to describe fluid flows within the stars that corresponds to corotating, irrotational, but also spinning binaries. As a validation of the numerical solutions, we have constructed a number sequences of corotating and irrotational binary neutron stars having the same mass. The results for corotating and irrotational binaries have been compared with those published from the pseudo-spectral code \textsc{lorene} [1], and revealed that the relative difference in the results between the two codes is of the order of 0.05%, even when a medium resolution is used for \textsc{cocal}.

When considering spinning binaries, and although the code can handle arbitrary rotation prescriptions for the individual stars, we have concentrated here on the case of fluid flows in which the spins are parallel to the orbital angular momentum. For this class of solutions, and to explore the possible range of behaviours, we have considered sequences with stars that are either slowly spinning or that are spinning at rates that are ten times larger than those observed in binary pulsars systems. In all the cases considered, we have found that error estimates of different type leads relative errors that are \( \lesssim 0.7\% \).

A number of applications of these results and of additional developments of the code are expected to take place in the coming months. Firstly, we will explore the impacts of stellar spins in numerical simulations of binary neutron stars; more specifically, by exploiting the high convergence order of our new numerical code general-relativistic code [120], we plan to extend the work carried out in [121] for the inspiral part and the one recently published in [32, 122] for the post-merger signal. Secondly, by combining the approaches followed in the solution of binary black holes and binary neutron stars, we will extend the code to handle also binaries comprising a black hole and a neutron star of different masses and spin orientation. Third, we will explore the space of solutions in which the spins of the neutron stars are oriented arbitrarily as these are likely to correspond to the most realistic configurations. Finally, working on a parallelization of the code will allow us to obtain results with much smaller computational costs, enabling us to provide public initial data for spinning binary neutron stars under a variety of conditions.

ACKNOWLEDGMENTS

We thank John Friedman for carefully reading the manuscript and providing useful input. Partial support comes from the DFG grant SFB/Transregio 7 and by “NewCompStar”, COST Action MP1304. A.T. is supported by the
Appendix A: Mass and angular momentum

In this Appendix we review the mathematical definitions of several of the quantities that have been used to characterize the properties of the binaries. We start with the rest mass of each star, \( M_0 \), defined as an integral over the spacelike hypersurface \( \Sigma_i \) of the rest-mass density as measured by the comoving observers

\[
M_0 := \int_{\Sigma_i} \rho u^\alpha dS_{\alpha} = \int_{\Sigma_i} \rho u^\alpha \nabla_\alpha \sqrt{-g} d^3x
= \int_{\Sigma_i} \rho t^\alpha \nabla_\alpha r^2 \sin \theta d\theta d\phi . \tag{A1}
\]

In COCAL, integrals like this are computed in dimensionless form using normalized coordinates. With the help of Eq. (66), Eq. (A1) is rewritten as

\[
M_0 := R_0^3 \int_{\Sigma_i} K_i^{1/(1-\Gamma^{-1})} q^{1/(\Gamma^{-1}-1)} u^t \alpha \psi^6 r^2 \sin \theta d\theta d\phi , \tag{A2}
\]

where \( K_i \) depends on \( \hat{r} \). The integrand in Eq. (A2) is evaluated on the gravitational coordinates therefore an interpolation to the surface-fitted coordinates is needed before the integral evaluation.

Next, a measure of the total energy of the system is given by the ADM mass, \( M_{ADM} \), which is defined as a surface integral at spatial infinity as

\[
M_{ADM} := \frac{1}{16\pi} \int_{\infty} (f^{lm} f_{lm} - f^{ij} f^{mn}) \partial_j \gamma_{mn} dS_i
= -\frac{1}{2\pi} \int_{\infty} \partial t \psi dS_i = -\frac{1}{2\pi} \int_{\infty} \partial t r^2 \sin \theta d\theta d\phi , \tag{A3}
\]

and which in normalized coordinates becomes

\[
\bar{M}_{ADM} := R_0 \int_{\infty} \frac{\partial \psi}{\partial \bar{r}} \bar{r}^2 \sin \theta d\theta d\phi . \tag{A4}
\]

Note that spatial infinity in COCAL is represented by a spherical surface with radius \( r \approx 0.8r_b \) of the ARCP coordinate patch. Closely related to the ADM is the Komar mass of the binary, which is related to the timelike Killing field \( t^\alpha \) and is defined as

\[
M_K := -\frac{1}{4\pi} \int_{\infty} \nabla^\alpha t^\beta dS_{\alpha\beta} = \frac{1}{4\pi} \int_{\infty} \partial^\alpha dS_{\alpha} . \tag{A5}
\]

or in normalized form as

\[
\bar{M}_K := R_0 \int_{\infty} \frac{\partial \alpha}{\partial \bar{r}} \bar{r}^2 \sin \theta d\theta d\phi . \tag{A6}
\]

The angular momentum of the system is also calculated from a surface integral at spatial infinity

\[
\bar{J} := \frac{1}{8\pi} \int_{\infty} \bar{K}_j \bar{\psi}^i dS_i = \frac{1}{8\pi} \int_{\infty} \bar{A}_{ij} \bar{\psi}^j x^i r_{\infty} \sin \theta d\theta d\phi , \tag{A7}
\]

where \( \bar{\psi}^i \) is the generator of the orbital trajectories and we have used the maximal slicing gauge. The corresponding normalized quantity is

\[
\bar{J} := R_0^2 \frac{1}{8\pi} \int_{\infty} \bar{A}_{ij} \bar{\psi}^j x^i \bar{r}_b \sin \theta d\theta d\phi . \tag{A8}
\]

Finally, we also compute the “proper mass” of each star as the integral of the total energy density measured by the comoving observer

\[
M_p := \int_{\text{star}} \rho u^\alpha dS_{\alpha} . \tag{A9}
\]

Appendix B: Isotropic coordinates TOV solver

In this Appendix we describe our implementation for obtaining spherical solutions and the related rescaling that is used in COCAL. We can obtain the same solutions using a one-dimensional KEH solver that mimics the full three-dimensional code in a 3+1 setting. However, because most of the time the TOV equations are presented in terms of Schwarzschild coordinates while the actual calculations are performed in isotropic coordinates, in what follows we show how to transform the system of equations from Schwarzschild to isotropic coordinates without having to go through a new derivation of equations and automatically obtaining a smooth solution at the stellar surface. The results are of course identical to machine precision, at least for simple polytropes we have checked. To the best of our knowledge this approach has not been presented before in the literature.

We recall that the line element in Schwarzschild and in isotropic coordinates is given respectively by

\[
ds^2 = -A(r) dt^2 + B(r) dr^2 + r^2 d\Omega^2 , \tag{B1}
\]

\[
ds^2 = -\alpha^2(\bar{r}) dt^2 + \psi^4(\bar{r}) (d\bar{r}^2 + \bar{r}^2 d\Omega^2) , \tag{B2}
\]

with well known expressions for the functions \( A(r), B(r), \alpha(\bar{r}), \psi(\bar{r}) \) for the exterior of the star. For
the interior, instead, we need to solve the TOV equations

\[ \frac{dA}{dr} = \frac{2A}{r} \frac{dp}{dr} , \]
\[ \frac{dp}{dr} = -\frac{(\epsilon + p)(m + 4\pi r^3 p)}{r^2 - 2mr} , \]  \hspace{1cm} (B3) (B4)

where

\[ \frac{dm}{dr} = 4\pi r^2 \epsilon , \quad \text{and} \quad B(r) = \frac{1}{1 - 2m(r)/r} . \]  \hspace{1cm} (B5)

Of course it is not difficult to derive the TOV equations in the isotropic coordinates \((B2)\) and then perform a direct numerical integration in these coordinates. However, this is not necessary and it is possible to always work in Schwarzschild coordinates rescaling the radial profile of the solution so as to make the surface of the star appear at the correct position and automatically obtain a smooth solution in \([0, \infty)\) without resorting to a post-processing rescaling.

Comparing Eqs. \((B1)\) and \((B2)\) it is easy to deduce that

\[ \psi^2(\bar{r}) d\bar{r} = \sqrt{B} dr , \quad \text{and} \quad \psi^2(\bar{r}) \bar{r} = r , \]  \hspace{1cm} (B6)

which yield

\[ \frac{dr}{d\bar{r}} = \frac{r}{\bar{r}} \sqrt{1 - \frac{2m(r)}{r}} . \]  \hspace{1cm} (B7)

Using Eq. \((B7)\), we can rewrite the TOV system in terms of the isotropic radial coordinate \(\bar{r}\) as

\[ \frac{dm}{d\bar{r}} = (4\pi r^2 \epsilon) \frac{r}{\bar{r}} \sqrt{1 - \frac{2m}{r}} , \]  \hspace{1cm} (B8)
\[ \frac{dp}{d\bar{r}} = -\frac{(\epsilon + p)(m + 4\pi r^3 p)}{\sqrt{1 - 2m/r}} \frac{1}{r} , \]  \hspace{1cm} (B9)
\[ \frac{d\psi}{d\bar{r}} = \frac{\psi}{2\bar{r}} \left( \sqrt{1 - \frac{2m}{r}} - 1 \right) , \]  \hspace{1cm} (B10)
\[ \frac{d\alpha}{d\bar{r}} = \frac{\alpha m + 4\pi r^3 p}{\sqrt{1 - 2m/r}} \frac{1}{r} , \]  \hspace{1cm} (B11)

where we used Eq. \((B6)\) to derive Eqs. \((B10)\) and \((B11)\). It is possible to simply integrate the system above to obtain the star profile in isotropic coordinates. Initial values at \(\bar{r} = 0\) are needed and although this is not a problem for \(r, m,\) and \(p\) (the latter being in general a free parameter), but the values of \(\alpha, \psi\) are not available to have a smooth matching at the surface of the star \(\bar{r} = \bar{R}\), whose position is still unknown. On the other hand, one way to obtain a smooth solution across the star surface is to exploit the coordinate transformations

\[ r = \bar{r} \left( 1 + \frac{M}{2\bar{r}} \right)^2 , \]  \hspace{1cm} (B12)
\[ \bar{r} = \frac{r}{2} \left[ 1 + \sqrt{1 - \frac{2M}{r} - \frac{M}{r}} \right] , \]  \hspace{1cm} (B13)

As a result, an integration between \(r, m(r)\) leads to the following condition for the conformal \(\psi\)

\[ \psi(r) = \frac{r}{M} \left( 1 - \sqrt{1 - \frac{2M}{r}} \right) , \]  \hspace{1cm} (B14)
\[ \alpha(r) = \sqrt{1 - \frac{2M}{r}} . \]  \hspace{1cm} (B15)

Making again use of Eq. \((B7)\) and of the analytic integration of its left-hand side, Eq. \((B10)\) can be written as

\[ d\ln \psi = F(r, m(r)) dr , \]  \hspace{1cm} (B16)

where

\[ F(r, m(r)) := \frac{1}{2r} \left( 1 - \frac{1}{\sqrt{1 - 2m(r)/r}} \right) , \]  \hspace{1cm} (B17)

so that \(\psi\) (and similarly \(\alpha\)) at the star surface are guaranteed to match smoothly the exterior solution, as it can be seen in Fig. 12.

### Appendix C: Iteration scheme

The iteration procedure for binary stars is similar to the one for binary black holes described in Ref. [70], but it also contains the fluid coordinates, and a solution for the extra fluid variables in a multi-patch setting. An overall picture of this procedure is shown in Fig. 13, where the steps of the gravitational Poisson solver (essentially everything from “Compute...
The code first initializes the lapse function and the conformal factor from some initial spherical solution. For that purpose we have developed two methods. One is an isotropic TOV solver (see Appendix B), and another is a one-dimensional KEH method. This last choice reproduces the KEH approach used in three-dimensional computations, but in a one-dimensional mesh. Included in this method are all the important ingredients of the three-dimensional code, such as the renormalization of variables. Comparing the results from these two independent schemes gives us confidence about the robustness of the COCAL iterative solutions.

After a choice of the velocity fluid potential, of the orbital angular velocity, and in the case of spinning binaries, also of the rotational states of each compact object, the code proceeds to the main part of the iteration, which always starts by interpolating $q = p/\rho$, $\partial_i \Phi$, and $\tilde{s}^i$ from the surface-fitted coordinates to the gravitational coordinates. The interpolated quantities are then used in the gravitational Poisson solver, which

$S_\psi$” up to ”Invert $\nabla^2$, compute $\psi$”) have been described in detail in [70].
is executed in addition to the root-finding routine explained in Sec. III B. As discussed there, the constants related to the Euler integral, the orbital angular velocity, and the scaling of our grids \( C, \Omega, R_0 \), are calculated at this point, and the lapse function, as well as the conformal factor, are updated according to

\[
\psi^{\text{new}} = (\psi^{\text{old}})^2, \quad \alpha^{\text{new}} = (\alpha^{\text{old}})^2.
\]

When the gravitational solver ends, \( \psi, \beta^i, \alpha \) are interpolated to the surface-fitted coordinates in preparation for the fluid Poisson solver. The main steps now are the computation of the new value of \( q \), by the use of Eq. (49) or Eq. (32), and then the solution of the conservation of rest mass, Eq. (57). At this point, also the surface of the star is computed.

At each iteration step, the fluid computation is performed a few times (four times for the results presented here) since this results in a more stable final computation. A relaxation parameter \( \xi \) is used when updating a newly computed variable. If \( \Phi^{(n)}(x) \) is the \( n \)-th step value, and \( \Phi(x) \) the result of the Poisson solver, then the \( (n+1) \)-th step value will be

\[
\Phi^{(n+1)}(x) := \xi \Phi(x) + (1 - \xi) \Phi^{(n)}(x),
\]

where \( 0.1 \leq \xi \leq 0.4 \). Usually \( \xi = 0.4 \) for \( \alpha, \psi, \beta^i, q, \Phi \), while \( \xi = 0.1 \) for \( \phi \).

The criterion used by COCAL to stop the iteration is given by

\[
2 \frac{\left| \Phi^{(n)} - \Phi^{(n-1)} \right|}{\left| \Phi^{(n)} \right| + \left| \Phi^{(n-1)} \right|} < \epsilon_c,
\]

for all points of the grids, and all variables \( \alpha, \psi, \beta^i, q, \Phi \), where we used \( \epsilon_c = 10^{-6} \) in this paper. In almost all of our calculations, \( \psi \) and \( \alpha \) converge to machine precision, while the error in the fluid variables \( q \) and \( \Phi \) decreases to \( \approx 10^{-12} \) before the error in the shift reaches \( 10^{-7} \). This is due to the existence of points in the gravitational mesh where the shift has almost zero values, and convergence is much slower there. Neglecting such points can speed up a solution by a factor of at least two. Currently, COCAL is running on a serial processor and it needs around 3-4GB of RAM to produce the solutions presented in this work. With an Intel Xeon 3.60 GHz processor, about two days are needed for these computations, with the irrotational configurations taking longer than the spinning ones; this is not a surprise since convergence is faster for corotating binaries.

### Appendix D: Corotating binary white dwarfs

To test the sensitivity of our code and to prepare for future work concerning neutron star-white dwarf or black hole-white dwarf binaries, we also compute a corotating binary white dwarf solution. Here the fields are of magnitude less than the ones encountered in typical binary neutron star binaries and greater resolution is required in order to acquire smooth solutions. The resolution used is reported in Table VI where we can see that an increase in \( N_\theta, N_\phi \) by a factor of three relative to the solutions obtained in Fig. 2. Table II, was used. In Fig. 14 we show a representative binary white dwarf solution with compactness \( C = 2 \times 10^{-4} \), with centers placed at \( x = \pm 1.25 \) and unit radii. From left to right, the different panels report the contour plot of the lapse function from 0.9994 to 1.0 with step of \( 2 \times 10^{-5} \), the shift vector field, and the contour plot of the rest mass density. Note that the plots are centered at the origin of the COCP-1 patch and the green circle refers to the excised sphere \( S_e \). Values inside \( S_e \) are taken from the COCP-2 patch. The shift vector in binary white dwarfs is approximately four orders of magnitude smaller that the one typically encountered in neutron stars, while the quantity \( \left| \alpha - 1 \right| \) is about three orders of magnitude smaller. Overall we see a good convergence between the different coordinate systems even for these small values of the metric quantities.

### Appendix E: Fourth-order post-Newtonian approximation

The 4PN approximation for the binding energy and orbital angular momentum of a system of two non-spinning bodies with masses \( M_1, M_2 \) and in quasi-circular orbit has been used in Fig. 10 to compare with the numerical results of irrotational and spinning binaries. The explicit expressions for these quantities are given by [119]
TABLE VI. Grid parameters used for the white-dwarf solutions with $\Gamma = 5/3$.

| Type       | Patch | $r_a$ | $r_s$ | $r_c$ | $r_e$ | $N_a^\theta$ | $N_e^\theta$ | $N_o$ | $N_p$ | $L$ |
|------------|-------|-------|-------|-------|-------|---------------|---------------|-------|-------|-----|
| WD         | COCP 1 | 0.0 | 1.0 | 1.0 | 1.50 | 1.25 | 64 | 64 | 96 | 192 | 144 | 144 | 12 |
| WD         | COCP 2 | 0.0 | 1.0 | 1.0 | 1.50 | 1.25 | 64 | 64 | 96 | 192 | 144 | 144 | 12 |
| ARCP       |       | 5.0 | $-10^5$ | 6.25 | $-16$ | $-20$ | 20 | 192 | 144 | 144 | 12 |

FIG. 14. Binary white dwarfs solution with compactness $C = 2 \times 10^{-4}$, stellar centers placed at $x = \pm 1.25$ and unit radii. Shown from left to right are: a contour plot of the lapse function from 0.9994 to 1.0010 with step of $2 \times 10^{-5}$, the shift vector field with maximum value $8.4 \times 10^{-6}$, and a contour plot of the rest-mass density from $2 \times 10^{-8}$ to $10^{-4}$ with step $8 \times 10^{-8}$. Note that the green sphere corresponds to the excised sphere $S_e$ of COCP-1.

\[
\frac{J}{GM^2/c} = \frac{\nu}{\sqrt{x}} \left\{ 1 + \frac{3}{2} + \frac{1}{6} \nu \right\} x + \left( \frac{27}{8} - \frac{19}{8} + \frac{1}{24} \nu^2 \right) x^2 + \left[ \frac{135}{16} + \left( -\frac{6889}{144} + \frac{41}{24} \pi^2 \right) \nu + \frac{31}{24} \nu^2 + \frac{7}{1296} \nu^3 \right] x^3 + \left[ \frac{2835}{128} + \left( \frac{98869}{5760} - \frac{6455}{1536} \pi^2 - \frac{256}{3} \ln 2 - \frac{128}{3} \gamma_E \right) \nu + \left( \frac{356035}{3456} - \frac{2255}{576} \pi^2 \right) \nu^2 - \frac{215}{1728} \nu^3 - \frac{55}{31104} \nu^4 - \frac{64}{3} \nu \ln x \right\} x^4, \tag{E2}
\]

where $\gamma_E$ is the Euler constant, $M := M_1 + M_2$ is the total mass of the system, $\nu := M_1 M_2 / M^2$ is the symmetric mass ratio and $x$ the dimensionless orbital frequency

\[
x := \left( \frac{\Omega GM}{c^3} \right)^{2/3}. \tag{E3}
\]

[1] K. Taniguchi and E. Gourgoulhon, Phys. Rev. D 66, 104019 (2002).
[2] M. Shibata and K. Uryû, Phys. Rev. D 61, 064001 (2000).
[3] L. Baiotti, B. Giacomazzo, and L. Rezzolla, Phys. Rev. D 78, 084033 (2008).
[4] M. Anderson, E. W. Hirschmann, L. Lehner, S. L. Liebling, P. M. Motl, D. Neilsen, C. Palenzuela, and J. E. Tohline, Phys. Rev. D 77, 024006 (2008).
[5] Y. T. Liu, S. L. Shapiro, Z. B. Etienne, and K. Taniguchi, Phys. Rev. D 78, 024012 (2008).
[6] S. Bernuzzi, M. Thierfelder, and B. Brügmann, Phys. Rev. D 85, 104030 (2012).
[7] R. Narayan, B. Paczynski, and T. Piran, Astrophys. J. 395, L83 (1992).
[81] C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation, Freeman San Francisco 1973.
[82] J. W. York, in Sources of Gravitational Radiation, edited by L. Smarr (Cambridge University Press, Cambridge, England, 1979).
[83] G. B. Cook, Living Rev. Rel. 3, 5 (2000).
[84] E. Gourgoulhon 3 + 1 Formalism in General Relativity: Bases of Numerical Relativity (Lecture Notes in Physics vol. 846, Springer, 2012)
[85] J. K. Blackburn and S. Detweiler, Phys. Rev. D 46, 2318 (1992).
[86] S. Detweiler, Phys. Rev. D 50, 4929 (1994).
[87] A. Schild, Phys. Rev. 131, 2762 (1963).
[88] J. L. Friedman and K. Uryu, Phys. Rev. D 73, 104039 (2006).
[89] M. M. Glenz and K. Uryu, Phys. Rev. D 76, 027501 (2007).
[90] J. L. Friedman, K. Uryu, and M. Shibata, Phys. Rev. D 65, 064035 (2002).
[91] E. Gourgoulhon, P. Grandclément, and S. Bonazzola, Phys. Rev. D 65, 044020 (2002).
[92] P. Grandclément, E. Gourgoulhon, and S. Bonazzola, Phys. Rev. D 65, 044021 (2002).
[93] M. Alcubierre, Introduction to 3+1 Numerical Relativity (Oxford University Press, New York, 2008).
[94] T. W. Baumgarte and S. L. Shapiro, Numerical relativity: Solving Einstein’s equations on the computer (Cambridge University Press, Cambridge, 2010).
[95] J. L. Friedman, N. Stergioulas, Rotating Relativistic Stars Cambridge Monographs on Mathematical Physics (Cambridge University Press, 2013).
[96] R. H. Boyer, Proc. Cambridge Philos. Soc. 61, 527 (1965).
[97] J. M. Bardeen and R. W. Wagoner, Astrophys. J. 167, 359 (1971).
[98] J. S. Read, B. D. Lackey, B. J. Owen, and J. L. Friedman, Phys. Rev. D 79, 124032 (2009).
[99] J. S. Read, C. Markakis, M. Shibata, K. Uryu, J. D. E. Creighton, and J. L. Friedman, Phys. Rev. D 79, 124033 (2009).
[100] F. Douchin and P. Haensel, Astron. Astrophys. 380, 151 (2001).
[101] A. Akmal, V. R. Pandharipande, and D. G. Ravenhall, Phys. Rev. D 58, 1804 (1998).
[102] K. Uryu and Y. Eriguchi, Mon. Not. Roy. Astron. Soc. 299, 575 (1998).
[103] A. Komar, Phys. Rev. 113, 934 (1959).
[104] A. Komar, Phys. Rev. 127, 1411 (1962).
[105] R. Beig, Phys. Lett. A 69A, 153 (1978).
[106] A. Ashtekar and A. MagnonAshtekar, J. Math. Phys. 20, 793 (1979).
[107] E. Gourgoulhon and S. Bonazzola, Class. Quantum Grav. 11, 443 (1994).
[108] T. W. Baumgarte, G. B. Cook, M. A. Scheel, S. L. Shapiro, and S. A. Teukolsky, Phys. Rev. D 57, 7299 (1998).
[109] P. Marronetti, G. J. Mathews, and J. R. Wilson, Phys. Rev. D Phys. Rev. D 58, 107503 (1998).
[110] F. Usui, K. Uryu, and Y. Eriguchi, Phys. Rev. D 61, 024039 (2000).
[111] F. Usui and Y. Eriguchi, Phys. Rev. D 65, 064030 (2002).
[112] LORENE website, http://www.lorene.obspm.fr/.
[113] J. R. Wilson, G. J. Mathews and P. Marronetti, Phys. Rev. D 54, 1317 (1996).
[114] T. W. Baumgarte, G. B. Cook, M. A. Scheel, S. L. Shapiro, and S. A. Teukolsky, Phys. Rev. D 57, 6181 (1998).
[115] E. E. Flanagan, Phys. Rev. Lett. 82, 1354 (1999).
[116] G. J. Mathews and J. R. Wilson, Phys. Rev. D 61, 127304 (2000).
[117] L. Rezzolla, P. Diener, E. N. Dorband, D. Pollney, C. Reisswig, E. Schnetter, and J. Seiler, Astrophys. J. Lett., 674, L29 (2008).
[118] E. Barausse and L. Rezzolla, Astrophys. J. Lett., 704, L40 (2009).
[119] L. Blanchet, Living Rev. Rel. 17, 2 (2014).
[120] D. Radice, L. Rezzolla, and F. Galeazzi, Mon. Not. R. Astron. Soc. L. 437, L46 (2014).
[121] D. Radice, L. Rezzolla, and F. Galeazzi, Class. Quantum Grav. 31, 075012 (2014).
[122] K. Takami, L. Rezzolla, and L. Baiotti, Phys. Rev. Lett. 113, 091104 (2014).