HARDY AND HARDY PDO TYPE
INEQUALITIES IN DOMAINS
PART I

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ABSTRACT. Here is given mainly an extensive treatment of Hardy inequalities in domains in $\mathbb{R}^N$. Part II will continue on this theme and furthermore also treat Hardy PDO inequalities. The latter are Hardy inequalities involving Partial Differential Operators instead of gradients in the inequalities. General subsets of the function space are also treated. Examples are nonnegative cones and are given a special treatment.

A part of the material is given in an encyclopedic fashion. This in order to get better overview and also be helpful in cases of outside applications.

These papers are a continuation of [WAN5], which should be regarded a prerequisite.

Part II will contain more about Hardy inequalities and will treat Hardy PDO type inequalities in domains.

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0. SOME INITIAL DEFINITIONS

Let $u$ be a function defined on a subset of $\mathbb{R}^N$ and let $\nabla^m u$ denote the $m$-th gradient of $u$. This is the set of partial derivatives in some sense of $u$ of order $m$. This set written in the multiindex notations is $\{D^\alpha u\}_{|\alpha|=m}$. The $L^p$-norm of this $m$-th gradient can be given in various ways, usually equivalent. We give the following

\[(0.0) \quad ||\nabla^m u||_{L^p} = \left( \int |\nabla^m u|^p dx \right)^{\frac{1}{p}} = \left( \sum_{|\alpha|=m} \int |D^\alpha u|^p dx \right)^{\frac{1}{p}} \]
and

\[ \sum_{|\alpha|=m} (\int |D^\alpha u|^p dx)^{\frac{1}{p}}. \]

Which definition to use is usually only of interest when determining e.g. the constants in the inequalities. The definition is then chosen to be correct with respect to the application.

The formula (0.1) emphasizes the point of view of the norm as a sum of seminorms.

Two versions of Sobolev spaces are the most common.

**Definition 0.0.** Let \( \Omega \) be open in \( \mathbb{R}^N \). Let \( W^{m,p}(\Omega) \) be defined as the completion of \( C^m(\Omega) \) in the Sobolev norm \( \| \cdot \|_{W^{m,p}(\Omega)} \), which is defined as

\[ \|v\|_{W^{m,p}(\Omega)} = \sum_{k=0}^m \|\nabla^k v\|_{L^p(\Omega)}. \]

If the completion is taken with respect to \( C_0^\infty(\Omega) \) instead then the resulting Sobolev space is denoted \( W_0^{m,p}(\Omega) \).

These Sobolev space functions are quasicontinuous. This property means that if the indices for the Sobolev space are \( m, p \), then such a function is continuous except for an open set of capacity \( \epsilon \) any \( \epsilon > 0 \). The capacity is the Sobolev space capacity, see [WAN5], here written \( C_{m,p} \). It is equivalent to Bessel capacity \( B_{m,p} \) if \( p > 1 \). The Sobolev functions are defined up to capacity zero. The capacity \( C_{0,p}(M) \) is Lebesgue measure for every \( p > 0 \).

1. **On the history of this paper**

The author’s research in the area began with a suggestion by L.I. Hedberg to study the note [ANC1] by Ancona. Ancona studied the following question, when is a Sobolev space of type \( W_0^{m,p}(\Omega) \) generated by its nonnegative cone, i.e. when is every function in the space equal to the difference of two nonnegative functions in the space?

The aim was first to improve on the results by Ancona. However it was found that sometimes his results were the best possible!

This question is connected to Hardy inequalities in domains.

Hence the study turned into a deeper study of these. The choice was also motivated by that these inequalities in many cases are very useful tools and hence progress here will influence other areas, such as parts of analysis and mathematical physics through their influence on the study of PDE:s, eigenvalues, analysis on manifolds, etc.

There is also a related question on the compactness and the weights/domains that make the corresponding imbedding compact. This is not touched upon here. Certainly it is kind of a twin problem though.
The study of Poincaré inequalities for functions in cubes and polynomial capacities has been used as a part of this work. We refer to [WAN5] for this. The study of these polynomial capacities has also other applications to Sobolev space theory. Some aspects of these are treated in [WAN5] as well.

In order to illustrate the discussion below on Hardy inequalities for domains we give a simple formulation of the Hardy inequality in a domain $\Omega$ of $\mathbb{R}^N$ namely

\begin{equation}
\int_{\Omega} |u|^p d_{\partial \Omega}(x)^{s-mp} dx \leq A_0 \int_{\Omega} |\nabla^m u|^p d_{\partial \Omega}(x)^{s} dx.
\end{equation}

Here $d_{\partial \Omega}(x)$ is the usual distance function from a point $x \in \Omega$ to the boundary, $\partial \Omega$.

The most standard question concerning this inequality is that given $m, p, s$, which sufficient (necessary) conditions are there on $\Omega$ for inequality (1.0) to hold say for all $u \in W^{m,p}_0(\Omega)$ with some constant $A_0$. Furthermore if (1.0) holds then it is also of great interest to get information about the best possible constant $A_0$, since it gives one of the eigenvalues to a corresponding PDO, e.g. the Laplacian.

This last question is not treated here. The methods given here are constructive and a value of $A_0$ can be estimated.

– It should be observed that the particular inequality (1.0) is scale invariant and this makes it stand out among Hardy inequalities.

In the case of more general weights it should be said that the most interesting ones in terms of applications are weights that are functions of the distance to the boundary. The treatment here is much adjusted to this.

The results given here, say for the simple situation (1.0), is given in terms of a polynomial capacity. This way both largeness and shape of $\Omega^c$ is measured locally and at all scales. The bigger this capacity uniformly, the smaller the value of $A_0$.

This gives a formulation of a sufficient condition that holds in the case (1.0) if $s < s_0$. Here $s_0 > 0$ is calculable.

If $s \geq s_0$ this condition is not enough and a more involved condition is used. The result is that there is a cost, a lesser good polynomial capacity has to be used. How much so is regulated by the badness of $\partial \Omega$. This badness is measured by the value of a dimension of $\partial \Omega$. The dimension is designed to reflect local properties uniformly at all scales.

This dimension is denoted $\text{dim}_{\text{loc}}$. It was studied by the author in the mid 80:ies. (There is a typed manuscript by the author from about that time treating Hardy inequalities.)

The polynomial capacities were originally given by Maz'ya, see [MAZ1], and also treated in his book [MAZ2]. In [WAN5] we gave a more extensive treatment and also constructed a different polynomial capacity. In [WAN5] also the setup is generalized.

Since the polynomial capacities can estimated from below by Bessel capacities the same sufficient conditions holds with Bessel capacities.

Generally this estimate give fewer cases of possible $\Omega$:s then the polynomial capacity condition. However these conditions coincide is when $m = 1$ and $p > 1$.  

To begin with my technique for these problems only worked for \( s < 0, \) \((m, p \text{ general})\).

However Ancona then visited Sweden (Uppsala) and then I asked him the question about the case \( s = 0 \) for inequality (1.0) with \( m = 1 \) and \( p > 1 \) and with this uniform Bessel capacity condition as discussed above.

Since the question of possible \( \Omega \)s in (0.1) is in a way is harder when \( s \) is increasing, this was the first simple but general case unknown to me.

Ancona later in [ANC2] answered this question affirmatively and gave a theorem covering this question just as we discussed. He restricted himself to \( p = 2 \) and proved the converse in two dimensions.

Then Lewis [LEW] got interested in this part of Ancona’s paper. He devoted a paper to (1.0) above and were able to generalize the statements by Ancona to \( m = 1, p > 1 \) and \( s \leq 0 \). He also proved a converse statement for the cases with \( s = 0 \) and \( p = N \), (here \( m = 1 \)).

The present paper as well as e.g. [WAN2] or [WAN1] contains the results of Ancona and Lewis and much more. The converse statements by Ancona-Lewis are not treated though.

However the author made a kind of such announcement in [WAN4].

The methods of the author, Ancona and Lewis are all different, but the formulation used was given and used first by the author, then communicated to Ancona and then indirectly to Lewis through [ANC2].

There have been several manuscripts by the author with this kind of material from the mid 80:ies and on. The names of these have been different but the contents have been similar but expanding.

– These manuscripts have been circulated.

The version [WAN1], which was not at all the first one, has been much circulated and dates from (Febr) 1992. Though also this manuscript has been somewhat expanded later and circulated as well.

The present paper follows much the manuscript [WAN1] though the part on polynomial capacities and Poincaré inequalities from [WAN1] has already been taken up in [WAN5].

The thesis [NYS] by Nyström contains some on these matters. However his only contribution to the area is an estimate of the Maz’ya polynomial capacity in the case of domains with boundaries that have the so called Markov property. (This will be discussed elsewhere.)

This property is used (by the Umeå group) as a means to get the same properties other fractal sets as selfsimilarity gives this offer.

2. Somewhat on Polynomial Capacities

The first polynomial capacities were invented by Maz’ya. In [WAN5], (and in [WAN1]) a deeper treatment is made, which included a generalization and further properties. Also another and different kind of polynomial capacity was constructed.
It is the decided opinion of the author that these polynomial capacities have an important role to play in the theory related to Sobolev spaces. The work by the author have so far given a some corroboration.

However we leave this issue to future research to tell.

Since it is rather complicated to go through the matter of definitions, properties etc. of these two polynomial capacities we refer the reader to [WAN5]. This paper is thus a necessary prerequisite for reading the present paper. (However the reader is refered first to the first part of Section 6 in order to get advice how to get a simpler way to a first understanding of these two papers.)

Anyway we now repeat some of the background of polynomial capacities in a simplified manner for the benefit of the reader.

These polynomial capacities are based on functions in a unit cube. The choice of a cube is merely a practical matter. Say that someone wants to improve the constants in the Hardy inequalities that follows from calculations based on the proofs given here. Then there can be a point to for instance choose a unit ball instead. However in order to have a relationship to the Poincaré inequalities, which is the at issue here, you need that what in [WAN5] is called weak Poincaré inequalities should hold for the domain.

If one leaves this main road then of course it is possible to make estimates in some cases. But if one treat domains in general and make conditions in terms of corresponding polynomial capacities then the condition may be without meaning.

– The problem of statements (theorems) and their meaning/contents or lack of this is a common problem in the area of Sobolev space theory and requires a constant attention.

We return to the setting of polynomial capacities in the framework of a unit cube, $Q$. We study the simple inequality

\[(2.0) \quad ||u||_{L^p(Q)} \leq C_0||\nabla^m u||_{L^p(Q)},\]

where $u \in \mathcal{A}$ and $\mathcal{A}$ is a subset of $W^{m,p}(Q)$. We assume that the constant $C_0$ is the best possible. The treatment of general subsets $\mathcal{A}$ given here comes from [WAN1].

Then $C_0^{-p}$ is equivalent to a certain polynomial capacity when $C_0$ is big enough or as well if the polynomial capacity is small enough.

In this case the polynomial capacity can be chosen as the original polynomial capacity by Maz’ya. It is here denoted by $\Gamma(\mathcal{A})$ with suitable indices added.

However also the polynomial capacity given by the author also gives a correct answer. It is denoted by $\Theta(\mathcal{A})$. Hence these two types of polynomial capacities are here equivalent in the case $(2.0)$ when $\mathcal{A}$ is varied. These polynomial capacities with their respective indices written out are in this case $(2.0)$ $\Gamma_{m,m-1,p}(\mathcal{A})$ and $\Theta_{m,m-1,p}(\mathcal{A})$. Here $m$ is the order of the gradient, $m - 1$ is the degree of the polynomials in the zero space of the RHS and $p$ is exponent in the norm of the highest order term in the RHS. (Finally $\alpha$ is a kind of dummy parameter needed only for some proof procedures.)

Next we study two somewhat more complicated inequalities,
\begin{equation}
\|u\|_{L^p(Q)} \leq C_1(\|\nabla^{k+1} u\|_{L^p(Q)} + \|\nabla^m u\|_{L^p(Q)})
\end{equation}
and

\begin{equation}
\|u\|_{L^p(Q)} \leq A_0\|\nabla^{k+1} u\|_{L^p(Q)} + C_2\|\nabla^m u\|_{L^p(Q)}.
\end{equation}

In the last case the constant $A_0$ is assumed to be fixed and big enough.

Just as before the quantity $C_i^{-p}$ is studied.

The inequalities hold generally when the respective polynomial capacity is used and is nonnegative. As before there is the same relationship between best possible constant $C_i$ and the respective polynomial capacity.

In (2.1) the polynomial capacity is denoted $\Gamma_{m,k,p}(A)$.

In (2.2) the polynomial capacity is denoted $\Theta_{m,k,p}(A)$.

The latter polynomial capacity was (as said before) constructed by the author, see [WAN5] (or [WAN1]).

In both cases the index $k$ denotes that the zero space of RHS is the polynomials of degree less than or equal to $k$.

A type of inequality of the kind (2.2) was first studied by Hedberg [HED]. He gave a Bessel capacity estimate of the constant which gives a sufficient condition. With the polynomial capacity $\Theta$ there is a necessary and sufficient formulation instead. It is important to emphasize the very different nature of the polynomial capacities. Even in a standard setting. There are key geometrical dependences that does not exist at all in ordinary (say Bessel) capacities.

– Anyhow Hedberg managed to get what he wanted, using his estimate.

If a situation is hand that makes both corresponding polynomial capacities equivalent, then of course the inequality (2.2) is better than (2.1).

– However there is reason! to pose a question, which is not natural because it seems contraintuitive.

Anyway experience give some hope for an affirmative answer anyway.

**Open question 2.1.**

– Are these two types of polynomials capacities equivalent?

– Refrased, does (2.1) imply (2.2)? Maybe with new fixed constants.

**Comment on vocabulary.**

From our point of view the word polynomial in the expression polynomial capacity here should be regarded as derived from that the set of polynomials of degree less or equal to $m - 1$ is the zero space of the seminorm $\|\nabla^m u\|_{L^p(Q)}$.

The use of the word capacity should not be seen as derived from the word capacity as used in any ordinary sense. Instead it should be seen as the fact that the polynomial capacities are intimately related to say Sobolev (or Bessel potential) space capacities in force of the actual formulas.

– Formally, i.e. in a logical sense, the concept of polynomial capacity is entirely different from that of capacity.
3. HARDY AND HARDY PDO TYPE INEQUALITIES IN DOMAINS – BACKGROUND

The first person to generalize Hardy inequalities to more general domains was Nečas. He took the standard case in one dimension to the corresponding case for Lipschitz domains.

Then Kufner used Nečas’ ideas and generalized to Hölder domains. He wrote a book [KUF] with this as the main message, together with treatment of applications etc.

It is a thin volume but has anyway been influential. In fact since his results were not contested Kufner thought that maybe his parameter settings might be optimal, until he got to know our work in the field.

The results in [KUF] on Hardy inequalities in domains and imbeddings are special cases of the single theorem in [WAN3], part of author’s thesis 1991. However Kufner gives also another setting (variant) of inequalities that we do not discuss.

Again we return to the following aspect.

–In my opinion there is a big problem in this area of Hardy inequalities and also other areas. Namely sometimes there are results given that are posing as very good ones – for instances necessary and sufficient, but instead these results are dubious since they do not lead very far when you want a specific answer.

This can be said about the result in [MAZ2] p.113, i.e. if it is regarded as a Hardy inequality. He gives a very general statement about first order Hardy inequalities with general domains and general measures as weights in the LHS and an expression for the RHS that can be very involved and can be used for a defining a capacity of Choquet type.

The problem is that he makes a comparison between the values of the measure and the capacity every compact in the domain with respect to the full domain. The infimum you get is the best constant in the inequality.

The problem is that these numbers usually cannot be calculated.

Maz’ya discusses the problem in [NIK] p.153 and that shows that he is aware the problem.

Anyway his result in the context of the book [MAZ2] is a very important lemma, that gives sharp constants in unweighted inequalities like the Sobolev inequality and variants, see also Stredulinsky [STR] for a different short proof. It is based on another technique – introduced by Maz’ya – “capacitary integrals”.

Then we have the many results in [GUR-OPI]. They do not give examples and it seems hard/impossible? to get interesting ones. This makes gives their results a dubious ring.

Horiuchi in [HOR] gives a useful condition for a Hölder boundary such that the boundary has dimension less than the dimension of the domain minus one. Here it ought to be worthwhile to generalize along the lines of [WAN3].

3.1 HARDY PDO TYPE INEQUALITIES

We make a somewhat brief and uncomplete history.

These inequalities appeared in [WAN1] and shortly thereafter also in mathematical journals.
The general idea here have been to use as condition that if we take a point in $\Omega$ with a distance $r$ from the boundary then there should exist a ball with radius $\lambda r$ for some fixed $\lambda$ such that it within this ball exist a ball in the complement of $\Omega$ with radius $\mu r$ with $\mu$ fixed. Then the situation should about the same as for the corresponding Hardy inequality.

In fact in [WAN1] there has much more results than those that appeared in journals. However this part of [WAN1] needs some reorganization.

Furthermore the papers [SHA] and [H-K-S] contain results which seem to be very much related the question of Hardy PDO type inequalities, though the connection has not been worked out yet. They have many applications of their results.

A treatment of Hardy PDO type inequalities will be included in Part II.

4. On Applications

The Hardy inequalities for domains and the accompanying problem of compactness of the corresponding inequality (which is not treated here) are heavily linked to many areas of mathematical physics and more generally the study of PDE:s as well as their eigenvalue problems. However the potential use is even greater.

– Another aspect is that these Hardy inequalities can be seen as models for other kinds of inequalities.

One main aspect is that the Dirichlet problem generalizing Poission’s equation can be treated for very bad domains using a variation of Lax-Milgram’s lemma. For this see the relevant chapter in [KUF]. In fact the results there are extremely general.

The theory given there can be said to be just waiting for better Hardy inequalities in order to be upgraded, also the book [KUF-SÅN] by Kufner and Sändig can be recommended as source of many examples of applications.

– Ancona’s paper [ANC2] is quite helpful and with an alternative point of view to that in the Kufner book [KUF]. He treats the classical Dirichlet problem more classically.

Nyström in [NYS] discusses some aspects of the Dirichlet problem.

Horiochi have motivated his study of Hardy inequalities for domains by the fact that they are suited for treating Dirichlet problems with perturbed ellipticity. This aspect is also treated in [KUF-SÅN].

A field in physics where these Hardy inequalities are much present is General Relativity.

Generally when PDE:s are treated in physics there is often a possible use for Hardy inequalities.

Also generally speaking when there is progress on Hardy inequalities in domains in $\mathbb{R}^N$, then this can be translated to manifolds with/(without) boundary.

A special question here, where Hardy inequalities in domains are of interest is the so called Yamabe problem, see Maz’ya’s opinion on this, [NIK].

5. Overview of the Results

I.
The first result of the paper is a summation lemma, Lemma 6.4. It is used to go from local information in the sense of local inequalities to a global one. It involves sums of integrals over enlarged Whitney cubes and concerns general functions. Since it does not involve Sobolev functions or special kind of functions it can be used in many contexts.

II.

We define a dimension here denoted dim_{loc}, see Definition 6.8. It appears naturally by the use of local Hölder inequalities.

We also define another dimension dim_{mc,loc} using a similar definition which is based on Minkowski content, see Definition 6.9. The dimensions can be proved to be equal for compacts but the proof is not included. Also the same can be asked about a similar constructed dimension based on Hausdorff measure instead.

III.

We make a minor variation of the definition of Hölder quotient in order to suit the summation process, see Definition 6.12.

IV.

The main result on Hardy inequalities is collected in Theorem 6.17. This theorem should be regarded as a kind of look-up table, since there are too many possibilities squeezed into one theorem to make any nice version.

The results are given with the sum of two integrals in the LHS since this sometimes is advantageous. Such situations will be treated in Part II.

The two different types of polynomial capacities make a difference when the two-integral formulation is used.

The reader is advised to look through the passage “Advice for the first reading” given in the beginning of section 6 before more careful reading.

The contents of Theorem 6.17 follows in more detail from the knowledge of the respective capacities, i.e. see [WAN5].

V.

Corollary 6.19 is a rewriting of the main theorem in certain situations for the benefit of the reader. Here also [WAN5] is used. We observe for instance that if \( m = 2 \) then nonnegative functions often have better \( \Theta \)-capacities and that way they get better Hardy inequalities as well.

Corollary 6.19 contains a long list but is a good place to look for adequate information.

VI.

Theorem 6.20 gives sufficient conditions for when certain Sobolev spaces equal the difference formed from its nonnegative cone. Let lower index + denote the non-negative cone. Then this is written

\[
W_0^{m,p}(\Omega, d_{\partial \Omega}(x)^s) = W_0^{m,p}(\Omega, d_{\partial \Omega}(x)^s)_+ - W_0^{m,p}(\Omega, d_{\partial \Omega}(x)^s)_+. \tag{5.0}
\]
i.e. also weighted Sobolev spaces are covered. Actually the conditions given make substantial improvement to the corresponding theorem by Ancona in [ANC1]. He studies the cases with \( s = 0, p > 1 \) and \( \Omega \) Lipschitz, or \( p > N \).

Theorem 6.23 is just an upgrading of his result with essentially his original proof. Theorem 6.21 treats a similar problem and depends on Theorem 6.23 together with our Hardy inequality results.

With one of several given conditions given (listed) we have that \( u \in W^{2,p}_0(\Omega, d\partial\Omega(x)^s) \) implies

\[
(5.1) \quad u \in W^{2,p}_0(\Omega, d\partial\Omega(x)^s) - W^{2,p}_0(\Omega, d\partial\Omega(x)^s)
\]

if and only if

\[
(5.2) \quad \int |u|^p d\partial\Omega(x)^{-2p+s} dx < \infty.
\]

VII.

We formulate a rather spectacular conjecture.

**Conjecture 5.0.**

*Let \( m \) be odd, \( p > 1 \) then

\[
(5.3) \quad W^{m,p}_0(\Omega) = W^{m,p}_0(\Omega) + W^{m,p}_0(\Omega)
\]

holds for all \( \Omega \) open in \( \mathbb{R}^N \).

Let \( m \) be even and positive then there always exist some \( N, p, \Omega \), such that (5.3) does not hold.*

**Remark 5.1.**

It be conjectured that the weighted version of Conjecture 5.0 would hold.

This is not done since then features are introduced that makes the answer possibly out of reach.

– For ever!

6. The main body of results and proofs

**Advice for a first reading.**

The nature of the information given here and in [WAN5] certainly motivates some advice. The actual formulas for the main results are given as inequalities in formulas (6.24) and (6.25). Hence LHS and RHS below refer to these.

(i) The first term in the RHS, which involves \( p_1 \), usually is used only in special situations. If this term is skipped it simplifies. To begin with there is no longer any
difference then between the \( \Gamma \)- and the \( \Theta \)-capacities and this reduces the number of cases. Furthermore Lemma 6.14 can be postponed.

(ii) The key lemma is Lemma 6.4. – It should be read. The main theorem is Theorem 6.17.

(iii) Temporarily the information on polynomial capacities given in [WAN5] can wait. – Instead they can for the moment be seen as a representation of the constant in the Poincaré inequality in a cube, see Section 2.

(iv) Put \( q = p \) and get some simplification.

(v) Now only Case A – (6.25) and Case E are left. Some part of the proof is given in Case A – (6.24). Hence it is needed to read this too. Choose one of the two suggested cases.

(vi) To understand the outcome. Some understanding of the polynomial capacities is needed. To faciliate go to Corollary 6.19 first. Then choose a special case of interest and in accordance to the reduction made. Then return to [WAN5], look for the fact and its explanation.

**Observe.** *When constants \( A \) are used several times their values may change as the steps in a proof proceed.*

To begin with we first introduce the Whitney cube concept. It is in fact a way to look upon the open sets (with non-empty complement) as combinatorial objects.

**Definition 6.0.**

*Given an open set \( \Omega \in \mathbb{R}^N \) and a set of cubes \( F_\Omega \). This set is called the Whitney cubes of \( \Omega \) or are said to form a Whitney decomposition of \( \Omega \), if the following conditions hold.*

*The cubes in \( F_\Omega \) are open, dyadic and disjoint. Furthermore*

\[
\Omega = \bigcup_{Q \in F_\Omega} \bar{Q},
\]

\[
\text{diam } Q \leq \text{dist}(\Omega^c, Q) \leq 4 \text{diam } Q
\]

*and, if \( \bar{Q} \cap \bar{Q}' \neq \emptyset \), then*

\[
\frac{1}{4} \leq \frac{\text{diam } Q}{\text{diam } Q'} \leq 4.
\]

**Theorem 6.1. (Whitney.)**

*A set of Whitney cubes as defined above exist (non-uniquely) for every open subset of \( \mathbb{R}^N \) with a non-empty complement.*

Proof. See e.g. the book by Stein, [STE].

We now give a very useful Summation Lemma involving integrals and weights. – This lemma gives a generic method for many situations. First some notation.
**Definition 6.2.**

Let $\Omega$ be open $\mathbb{R}^N$. Let $\mathcal{F}_\Omega$ be a Whitney decomposition of $\Omega$. For $Q \in \mathcal{F}_\Omega$ define a new cube as follows. Take any point $x_0$ on $\partial \Omega$ that has smallest distance to $Q$. Then let $x_0$ be centre of a cube $R_Q$ which has the smallest side length and covers $Q$.

**Notation 6.3.**

Fix $Q \in \mathcal{F}_\Omega$. Denote with $\cdot$ for the scaling which has the property that $R_Q$ is mapped on a unit cube. This cube is denoted $\tilde{R}_Q$. By the same scaling $\Omega$ is mapped on $\tilde{\Omega}$ etc. When $\cdot$ is used it should be clear what $Q$ is refered to.

**Lemma 6.4.**

Let $\Omega$ be an open set in $\mathbb{R}^N$ with a Whitney decomposition $\mathcal{F}_\Omega$. Let $f$ be a nonnegative function on $\mathbb{R}^N$ with $f|_{\Omega^c} = 0$ and let $s > 0$. Then

\[ \sum_{Q \in \mathcal{F}_\Omega} (\text{diam } Q)^{-s} \int_{R_Q} f d\partial \Omega(x)^s dx \leq \frac{A(N)}{1 - 2^{-s}} \int_{\Omega} f dx. \]

**Proof.**

It follows from the properties of Whitney cubes that $d(x, \partial \Omega) \leq 5\text{diam } Q$ for $x \in Q$. This together with a change of order of summation gives that

\[ \sum_{Q \in \mathcal{F}} (\text{diam } Q)^{-s} \int_{R_Q} f d\partial \Omega(x)^s dx \leq \sum_{Q \in \mathcal{F}} \sum_{Q' \in \mathcal{F}} (\text{diam } Q)^{-s} (5\text{diam } Q')^s \int_{Q'} f dx \]

\[ = \sum_{Q' \in \mathcal{F}} \sum_{Q \in \mathcal{F}} 5^s (\text{diam } Q)^{-s} (\text{diam } Q')^s \int_{Q'} f dx. \]

We want to evaluate the inner sum of the RHS of (6.4) for fixed $Q'$. We have to estimate some of the entities involved. First we estimate diam $Q'$, when $Q'$ intersects $R_Q$.

To this end we use a sublemma.

**Sublemma 6.5.**

*With the situation as in Lemma 6.4, there is a constant $a$ such that*

\[ \frac{\text{diam } Q'}{\text{diam } Q} > a \]

implies that $Q' \cap R_Q$ is empty. The value of $a$ can be taken as $a = 5\sqrt{N}$. 

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Proof of Sublemma 6.5.  
First we determine how large diam $R_Q$ can be.  
The quotient diam $R_Q/diam Q$ takes its largest value if $Q$ is situated exactly at the middle of a face of $R_Q$ and $4diam Q = dist(Q, \partial \Omega)$, since if we move $Q$ about in $R_Q$ following a face we get an equal or longer distance otherwise and then $Q$ has to be made larger in order to satisfy the Whitney cube condition $4diam Q \geq dist(Q, \partial \Omega)$.  

Hence, the side of $R_Q$ has length at most $10diam Q$ and the diameter of $R_Q$ has length at most $10\sqrt{N}diam Q$. Now according to the Whitney cube property $diam Q' \leq 5\sqrt{N}diam Q$.

End of proof of sublemma 6.5.

The proof of Lemma 6.4 continues.

The diameters of the Whitney cubes are dyadic and there is no restriction to take $diam Q = 2^{-n}$ and $diam Q' = 2^{-k}$, with $k$ and $n$ integers. By Sublemma 6.5 we have that $k + 2\log(5\sqrt{N}) \geq n$ implies that $Q' \cap R_Q = \emptyset$.

It follows from a volume consideration that there are at most only a fixed number $A(N)$ of $Q$'s with $n$ fixed, such that $R_Q$ intersects $Q'$. If $Q' \cap R_Q \neq \emptyset$, then the cube $Q$ lies in a ball with centre in the centre of $Q'$ and

\[ \text{radius of ball} = diam R_Q + (1/2)diam Q' = 10\sqrt{N}diam Q + (5/2)\sqrt{N}diam Q. \]

The conclusion follows.

Now we can evaluate the inner sum in (6.4)

\[
\sum_{Q \in F} 5^s(diam Q)^{-s}(diam Q')^s \leq A(N) \sup_k \left\{ \sum_{n=-\infty}^{k+\lfloor 2\log(5\sqrt{N}) \rfloor} 2^{(n-k)s} \right\} \leq \frac{A(N)}{1 - 2^{-s}}.
\]

Lemma 6.4 now follows from (6.5) and (6.6).

End of proof of Lemma 6.4.

**Observe:**
It holds that for $s$ small

\[
\frac{1}{1 - 2^{-s}} \sim \frac{1}{s}.
\]

**Definition 6.6.**

Let $\Omega$ be open in $\mathbb{R}^N$ and $s$ be real, then define

\[
G_s(\Omega) = \sup_{Q \in F} (diam Q)^{s-1} \int_{R_Q \cap \Omega} d\partial \Omega(x)^{-s} dx,
\]

where $F_\Omega$ is a Whitney decomposition of $\Omega$.  

Observation 6.7.

$G_s(\Omega)$ is invariant when $\Omega$ and $\mathcal{F}_\Omega$ are dilated.

We will define a useful related dimension concept.

Definition 6.8.

Let

\begin{equation}
\dim_{\text{loc}}(\partial \Omega, \Omega) = N - s_0,
\end{equation}

where

\begin{equation}
s_0 = \sup \{ s : G_s(\Omega) < \infty \}.
\end{equation}

This concept goes back to our work on the problem of Hardy inequalities in the mid 80-ties. (Documented in a typed manuscript from that time.)

We discuss this dimension concept only as a remark. The definitions of Hausdorff measure, $h_d$, and Hausdorff dimension, $\dim_h$, as well the definitions of Minkowski content $mc_d$ and Minkowski dimension $\dim_{mc}$ are assumed to be known.

Definition 6.9.

Define

\begin{equation}
\dim_{mc,\text{loc}}(\partial \Omega, \Omega) = \inf_d \{ d : MC_d < \infty \}
\end{equation}

with

\begin{equation}
MC_d = \sup_{Q \in \mathcal{F}_\Omega} (mc_d(\tilde{R}_Q \cap \tilde{\partial} \Omega))
\end{equation}

and define in the same way

\begin{equation}
\dim_{h,\text{loc}}(\partial \Omega, \Omega)
\end{equation}

Theorem 6.10.

It holds for closed sets that

\begin{equation}
\dim_{\text{loc}} = \dim_{mc,\text{loc}}
\end{equation}

Proof. Only in the typed notes from the 80:ies.

Question 6.11.

Does it hold that

\begin{equation}
\dim_{\text{loc}} = \dim_{h,\text{loc}}.
\end{equation}

For some cases in the main theorem we want a variant of Hölder quotients that performs well under summation.
Definition 6.12.

Let $\Omega$ be open in $\mathbb{R}^N$, $0 < \lambda \leq 1$ and $h$ be nonnegative integer. The pointwise H"older quotient is defined by

\begin{equation}
||\nabla^h u||_{H^\lambda,pnt(\Omega)} = \sup_{x \in \Omega} \sup_{|\alpha| = h} \limsup_{y \to x} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^{\lambda}}.
\end{equation}

In weighted formulation with weight function $\kappa$

\begin{equation}
||\nabla^h u||_{H^\lambda,pnt(\Omega,\kappa)} = \sup_{x \in \Omega} \sup_{|\alpha| = h} \limsup_{y \to x} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^{\lambda}} \cdot \kappa(x).
\end{equation}

The following definition is made for examples later.

– It is not generally agreed on one definition of selfsimilarity. Here is a definition suitable for the discussion here.

Definition 6.13.

Let $K$ be a closed set in $\mathbb{R}^N$ and let $F_{\Omega}$ be a Whitney decomposition of $K^c$.

Then $K$ is selfsimilar if for every $Q \in F_{\Omega}$, $\tilde{R}_Q$ contains a ball $\tilde{B}_Q$ such that diam $\tilde{B}_Q$ is uniformly bounded off zero and every $\tilde{B}_Q \cap \tilde{K}$ has the property that there is a similarity transformation taking one to the other.

The following lemma is a consequence of a theorem by Sobolev on equivalent norms in Sobolev space.

Lemma 6.14.

Given conditions

(i) Let $Q_0$ be unit cube, $Q$ cube with $Q \subset Q_0$ and sidelength $l(Q) = a$,

(ii) let $p \geq 1$ and let $m, k$ with $m > k + 1$ be positive integers,

(iii) let $p_1$ with $0 < p_1 \leq \frac{Np}{N-(m-k-1)p}$ for $N > (m-k-1)p$,

(iv) let $0 < p_1 < \infty$ for $N = (m-k-1)p$,

(v) let $0 < p_1 \leq \infty$ for $N < (m-k-1)p$,

(vi) let $\nabla^m u$ etc. be the vectors of all weak derivatives of order $m$ of $u$ that are functions a.e.

Then it holds that there exist a constant $A = A(N,m,p,p_1,a)$ independent of the position of $Q$ in $Q_0$ such that

\begin{equation}
(\int_{Q_0} |\nabla^{k+1} u|^{p_1} dx)^{\frac{1}{p_1}} \leq A((\int_Q |\nabla^{k+1} u|^{p_1} dx)^{\frac{1}{p_1}} + (\int_{Q_0} |\nabla^m u|^{p} dx)^{\frac{1}{p}}).
\end{equation}

Proof. It is enough to prove

\begin{equation}
(\int_{Q_0} |D^\alpha u|^{p_1} dx)^{\frac{1}{p_1}} \leq A((\int_Q |D^\alpha u|^{p_1} dx)^{\frac{1}{p_1}} + (\int_{Q_0} |\nabla^{m-k-1} D^\alpha u|^{p} dx)^{\frac{1}{p}})
\end{equation}
for all $\alpha$, $|\alpha| = k + 1$.

Hence define

$$(6.20) \quad F(D^\alpha u) = \min_{Q \subset Q_0 \atop l(Q) \geq a} \left( \int_Q |D^\alpha u|^{p_1} dx \right)^{\frac{1}{p_1}}. $$

The functional $F$ is obviously continuous in $W^{m-k-1,p}(Q_0)$. Furthermore if $P \in \mathcal{P}_{m-k-2}$ and $P \neq 0$, then $F(P) \neq 0$. These conditions gives that the result follows from 1.1.15 in [MAZ2].

End of proof.

**THE MAIN RESULTS ON HARDY INEQUALITIES**

The main Theorem and other results are in order to avoid endless repetetivity given as

**Statements structured as**

– A Look-up Table With Inputs.

Thus minimizing the formulation of the theorem to about two pages only!

Later on some of the information in [WAN5] on polynomial and usual capacities are used to make lists that are less general but easier to grasp and overview.

**Discussion 6.15.**

The results from now on can be said to be organized in a somewhat peculiar manner. Furthermore not all situations that can covered by the methods here are treated. – A selection has been made. This selection is done to show different possibilities when using different extra ideas and also to show their effects, i.e. the outcome for the possible Hardy inequalities.

– By a consistent use of power type weight with respect to distance to the boundary (in the RHS) a concentration to the in applications most important situations is achieved. – Certainly the readability suffers anyway and certainly this anyway is in nature of the subject itself.

In order to simplify the exposition we give some special notation.

**Notation 6.16.** (For Theorem 6.17.)

Let $m, k$ be integers and $p \geq 1$ and $p_1 > 0$.

Let $\Omega$ be open proper subset in $\mathbb{R}^N$, with Whitney decomposition $\mathcal{F}_\Omega$.

Let $A$ be a subset of $W^{m,p}_{loc}$ with $A|_{\Omega^c} = 0$ q.e.

Let $\delta_{\partial \Omega}(x)$ be the regularized distance function, $\delta_{\partial \Omega}(x) \sim d_{\partial \Omega}(x)$ and $\delta_{\partial \Omega}(x) \in C^\infty(\Omega)$, see [STE].

As shorthand denote

$$ \sum_Q = \sum_{Q \in \mathcal{F}_\Omega} Q. $$
Denote \([p, p_1] = \max \{p, p_1\}\).
Define for \(x \in Q \in \mathcal{F}_\Omega\)
\[(6.21)\]
\[
\begin{align*}
\Gamma_{m,k+1,p}\mathcal{A}(x) &= \Gamma_{m,k+1,p}(\mathcal{A}|_{R_Q \cap \tilde{\Omega}^c}), \\
\Theta_{m,k+1,p}\mathcal{A}(x) &= \Theta_{m,k+1,p}(\mathcal{A}|_{R_Q \cap \tilde{\Omega}^c}).
\end{align*}
\]
Let
\[
f_{p,p_1} : \mathcal{F}_\Omega \to [0,1].
\]
Let \(s\) be given by
\[
s' = s/(s - 1) \text{ and } s' = \frac{\max \{p, p_1\}}{q}
\]
and then let \(f_{p,p_1} \in l^s\) with norm 1.
Denote \(f(x) = f_{p,p_1}(Q)\).

Theorem 6.17 is organized as follows. First some general conditions are given that do not depend on which of the inequalities (6.24) or (6.25) is at hand nor the “Cases”. Then two different the two types of Hardy inequalities (6.24) and (6.25) are given and since they are different there are also given general conditions to each of them. After this the different “Cases” are given. They include more conditions needed as well as “inputs” to the inequalities (6.24) and (6.25).

This system of presentation is also used later on.

**Theorem 6.17.**

The results are given in notation 6.16.
Let \(u \in \mathcal{A}\).
In both (6.23) and (6.24) let
\[(6.22)\]
\[
s_1 = -(m - k - 1)p_1 - N + \frac{p_1}{p}(s + N).
\]

Preconditions for (6.23).
(i) \((m - h)p > N > (m - h - 1)p,\)
(ii) \(0 < \lambda \leq m - h - \frac{N}{p},\)
(iii) \(0 < \lambda < 1,\)
(iv) \(t = m - h - \lambda - \frac{s+N}{p}.\)

Preconditions for (6.24).
(i) \(0 < q \leq \frac{pN}{N - mp},\)
(ii) \(N > mp,\)
(iii) $0 < q < \infty$, when $N = mp$
and $q \leq \infty$ when $N < mp$,
(iv) $t = mq - (\frac{q}{p} - 1)N - \frac{2}{p} s$.

\[
||\nabla^h u||_{H^{\lambda, pm, t}(\Omega, \Lambda(x)^{\frac{1}{p}} d\partial\Omega(x)^{-t})} \leq A((\int_{\Omega} |\nabla^{k+1} u|^{p_1} \Lambda_1(x)^{\frac{1}{p_1}} d\partial\Omega(x)^{s_1} dx)^{\frac{1}{p_1}}
+ (\int_{\Omega} |\nabla^{m} u|^{p} d\partial\Omega(x)^{s} dx)^{\frac{1}{p}}).
\]

\[
(\int_{\Omega} \frac{|u|^q \Lambda(x)^{\frac{q}{p}}}{d\partial\Omega(x)^{t}} dx)^{\frac{1}{q}} \leq A((\int_{\Omega} |\nabla^{k+1} u|^{p_1} \Lambda_1(x)^{p_1} d\partial\Omega(x)^{s_1} dx)^{\frac{1}{p_1}}
+ (\int_{\Omega} |\nabla^{m} u|^{p} d\partial\Omega(x)^{s} dx)^{\frac{1}{p}}).
\]

Then (6.23) and (6.24) holds respectively when the general condition (6.22), the
preconditions as well as the conditions in Case A-E are satisfied in the form the
respective inputs give.

**Case A.**
(i) Let $A$ be arbitrary,
(ii) let $s < 0$, $p \geq 1$ and $\Lambda_1(x) = 1$,
(iii) in (6.23) let $\Lambda(x) = \Gamma_{m, k, p}(x)$,
(iv) in (6.24) let $\Lambda(x) = \Gamma_{m, k, p}(x)$ if $q \geq \max\{p, p_1\}$,

and if $0 < q < \max\{p, p_1\}$ then $\Lambda(x) = \Gamma_{m, k, p}(x) f(x)^{\frac{p}{q}}$.

Here $A = A(N, m, p, p_1)(1 - 2^{-s})^{\frac{1}{p}} \sim A(N, m, p, p_1)s^{-\frac{1}{p}}$ for small $s$.

**Case B.**
(i) Let $A$ be arbitrary,
(ii) let $\Lambda_1(x) = 1$,
(iii) let $1 \leq p_0 < p$,
(iii) let $\dim_{\text{loc}}(\partial\Omega, \Omega) < N$,
(iv) let $s < \frac{p-p_0}{p_0} (N - \dim_{\text{loc}}(\partial\Omega, \Omega))$,
(v) in (6.23) let $\Lambda(x) = \Gamma_{m, k, p_0}(x)$,
(vi) in (6.24) let $\Lambda(x) = \Gamma_{m, k, p_0}(x)$, if $q \geq \max\{p, p_1\}$ and $q \leq p_0^*$ with $p_0^*$
(Sobolev exponent)

but if $0 < q < \max\{p, p_1\}$, then $\Lambda(x) = \Gamma_{m, k, p_0}(x) f(x)^{\frac{p}{q}}$.

Here $A = A(N, m, p, p_1, s, \Omega)$

**Comment.** The upper bound on $q$ in (vi) is not made optimal.
CASE C.
(i) Let $A$ be arbitrary,
(ii) let $s < 0$,
(iii) let $p \geq 1$,
(iv) let $\Lambda_1(x) = \Theta_{m, k, p}^{\alpha}(x)$,
(v) in (6.23) let $\Lambda(x) = \Theta_{m, k, p}^{\alpha}(x)$,
(vi) in (6.24) let $\Lambda(x)$ be the same if $q \geq \max\{p, p_1\}$
and if $0 < q < \max\{p, p_1\}$, then let $\Lambda(x) = \Theta_{m, k, p}^{\alpha}(x)f(x)^{\frac{q}{p}}$.

Here $A = A(N, m, p, p_1)(1 - 2^{-s})^{-\frac{s}{p}}$.

CASE D

(i) Let $A$ be arbitrary,
(ii) let $1 \leq p_0 < p$ and let $\dim_{\text{loc}}(\partial \Omega, \Omega) < N$,
(iii) let $s < \frac{p - p_0}{p_0}(N - \dim_{\text{loc}}(\partial \Omega, \Omega))$,
(iv) let $\Lambda_1(x) = \Theta_{m, k, p_0}^{\alpha}(x)$,
(v) in (6.23) let $\Lambda(x) = \Theta_{m, k, p_0}^{\alpha}(x)$,
(vi) in (6.24) let $\Lambda(x)$ be the same if $q \geq \max\{p, p_1\}$
and if $0 < q < \max\{p, p_1\}$, then let $\Lambda(x) = \Theta_{m, k, p_0}^{\alpha}(x)f(x)^{\frac{q}{p}}$.

Here $A = A(N, m, p, p_1, s, \Omega)$.

CASE E – (6.24).

(i) Let $A \subset C_0^\infty(\Omega)$,
(ii) let $\Gamma_{m, m - 1, p, A}(x) \geq b > 0$ for all $Q \in F$,
(\text{where the index } m - 1 \text{ gives the one-integral case in the RHS}),
(iii) let $s_0 > 0$ be a constant that can be calculated, let $s < s_0$,
let $\Lambda(x) = 1$ and $q \geq p$.

Here $A = A(N, m, p, b)$ and $s_0 = s_0(N, m, p, b)$.

Comment on Case E.

The situations in the previous Cases with with $p_1$-term, $\Theta$-, $\Gamma$-capacities, $\dim_{\text{loc}}$, $f$-case, (6.23) and (6.24) have already been treated and can be adjusted to the Case E as well. Hence one situation only is treated, also with the simplification that $A$ is subset of $C_0^\infty(\Omega)$.

PROOFS:

PROOF OF CASE A – (6.23)

Obviously

\begin{equation}
(6.25) \quad ||\nabla^h u||_{H^{\lambda, p_{n^*}}(\tilde{Q})} \leq ||\nabla^h u||_{H^{\lambda, p_{n^*}}(\tilde{R}_Q)} \leq ||\nabla^h u||_{H^{\lambda}(\tilde{R}_Q)}.
\end{equation}
By a Poincaré inequality for a cube with H"older seminorms in the LHS, see [WAN5], and (6.25)

\[ ||\nabla^h u||_{H^{\lambda,p_{nt}(\hat{Q})}} \leq \]

\[ \frac{A}{\Gamma_{m,k,p,A}(x)^{\frac{1}{p}}} \left((\int_{R_Q} |\nabla^{k+1} u|^{p_1} dx)^{\frac{1}{p_1}} + (\int_{\hat{R}_Q} |\nabla^m u|^{p} dx)^{\frac{1}{p}}\right) \]

\[ \leq \frac{A}{\Gamma_{m,k,p,A}(x)^{\frac{1}{p}}} \left((\int_{Q} |\nabla^{k+1} u|^{p_1} dx)^{\frac{1}{p_1}} + (\int_{R_Q} |\nabla^m u|^{p} dx)^{\frac{1}{p}}\right). \]

Dilate back. The dilation is a change of the independent variable with a factor \( \gamma = \frac{\text{diam } R_Q}{\text{diam } \hat{R}_Q} \) involved. In this construction \( \text{diam } \hat{R}_Q = 1 \).

**Dilation table** – Dilation factor \( \gamma \)

\[ \nabla^k \tilde{u} \rightarrow \gamma^k \cdot \nabla^k u \]
\[ d_{\partial \tilde{\Omega}}(x) \rightarrow \gamma^{-1} \cdot d_{\partial \Omega}(x) \]
\[ dx \rightarrow \gamma^{-N} \cdot dx \]

Next multiply with \( \Gamma_{m,k,p,A}(x)^{\frac{1}{p}} \). Observe that \( \text{diam } R_Q \sim \text{diam } Q \). Then multiply both sides with the same factor so that \( (\text{diam } Q)^{-t} \) is the scale factor left for the LHS. Then

\[ ||\nabla^h u||_{H^{\lambda,p_{nt}(Q)} \cdot (\text{diam } Q)^{-t} \cdot \Gamma_{m,k,p,A}(x)^{\frac{1}{p}}} \leq A \left((\int_{Q} |\nabla^{k+1} u|^{p_1} dx)^{\frac{1}{p_1}} \cdot (\text{diam } Q)^{-t}) + (\int_{\hat{R}_Q} |\nabla^m u|^{p} dx)^{\frac{1}{p}} \cdot (\text{diam } Q)^{-t}\right). \]

Since \( d_{\partial \Omega}(x)^{-t} \sim (\text{diam } Q)^{-t} \) in the cube \( Q \) we have simply

\[ ||\nabla^h u||_{H^{\lambda,p_{nt}(Q,\Lambda(x)^{\frac{1}{p}} d_{\partial \Omega}(x)^{-t})} \leq A \cdot ||\nabla^h u||_{H^{\lambda,p_{nt}(Q) \cdot \Gamma_{m,k,p,A}(x)^{\frac{1}{p}} (\text{diam } Q)^{-t}}. \]

Now raise (6.28) and (6.29) to power \( \max \{p, p_1\} \). Combine and sum over \( Q \).

Then it holds

\[ ||\nabla^h u||_{H^{\lambda,p_{nt}(\tilde{\Omega},\Lambda(x)^{\frac{1}{p}} d_{\partial \Omega}(x)^{-t})} \leq \sum_Q ||\chi_Q \nabla^h u||_{H^{\lambda,p_{nt}(\tilde{\Omega},\Lambda(x)^{\frac{1}{p}} d_{\partial \Omega}(x)^{-t})}. \]
This follows since the pointwise Hölder quotient has local properties and the contribution in the LHS of (6.30) is (roughly) coming from one cube only and this contribution is found in (roughly) one of cubes of the RHS of (6.30).

Next we we calculate the new RHS after these operations have been done.

Though first some (basic) inequalities are pointed out.

Note that by equivalent norms and/or quasinorms in finite dimensions we have for $a, b, r > 0$ that

\[(a^r + b^r)^{\frac{1}{r}} \leq A(r)(a + b)\]

There are also two well-known elementary inequalities for non-negative numbers.

**Lemma 6.18.**

\[(6.32a, b) \quad \text{For } r \geq 1 \sum_{n=1}^{\infty} |a_n|^r \leq (\sum_{n=1}^{\infty} |a_n|)^r; \quad r \leq 1 \sum_{n=1}^{\infty} |a_n|^r \geq (\sum_{n=1}^{\infty} |a_n|)^r.\]

Proof. Exercise.

We begin with (6.28), follow up the calculations and make use of Lemma 6.18. Then

\[(6.33) \quad \sum_{Q} \left( \int_{Q} |\nabla^{k+1}u|^{p_1} d_{\partial \Omega}(x)^{s_1} dx \right)^{\frac{[p,p_1]}{r_1}} \leq \left( \int_{\Omega} |\nabla^{k+1}u|^{p_1} d_{\partial \Omega}(x)^{s_1} dx \right)^{\frac{[p,p_1]}{r_1}}.\]

In the same way the second term of the same RHS becomes

\[
\sum_{Q} \left( \int_{R_Q} |\nabla^m u|^{p} d x \right)^{\frac{[p,p_1]}{p}} \leq \left( \sum_{Q} \int_{R_Q} |\nabla^m u|^{p} d_{\partial \Omega}(x)^{s} d_{\partial \Omega}(x)^{-s} dx \cdot (diam \ Q)^{s} \right)^{\frac{[p,p_1]}{p}}.
\]

The last expression is estimated from above with Lemma 6.4. Then it holds

\[(6.35) \quad \sum_{Q} \int_{R_Q} |\nabla^m u|^{p} d_{\partial \Omega}(x)^{s} d_{\partial \Omega}(x)^{-s} dx \cdot (diam \ Q)^{s} \leq \frac{A}{1 - 2^{-s}} \int_{\Omega} |\nabla^m u|^{p} d_{\partial \Omega}(x)^{s} d x.\]
The results are collected as

\[
\|\nabla^h u\|_{H^\lambda,pnt(\Omega,\Lambda(x)^{-t})}^{[p,p_1]} \leq A((\int_{\Omega} |\nabla^{k+1} u|^{p_1} d\partial\Omega(x)^{s_1} dx)^{\frac{1}{p_1}}
+ \frac{1}{(1 - 2^{-s})^{\frac{1}{p}}} (\int_{\Omega} |\nabla^m u|^p d\partial\Omega(x)^{s} dx)^{\frac{1}{p}}).
\]

(6.36)

But by (6.31) we can take a root of these terms and finally obtain

\[
\|\nabla^h u\|_{H^\lambda,pnt(\Omega,\Lambda(x)^{-t})}^{[p,p_1]} \leq A((\int_{\Omega} |\nabla^{k+1} u|^{p_1} d\partial\Omega(x)^{s_1} dx)^{\frac{1}{p_1}}
+ \frac{1}{(1 - 2^{-s})^{\frac{1}{p}}} (\int_{\Omega} |\nabla^m u|^p d\partial\Omega(x)^{s} dx)^{\frac{1}{p}}).
\]

(6.37)

End of proof Case A – (6.23).

**Comment.**

*Note that the constants for the two terms in the RHS differs in dependency on s.*

Proof Case A – (6.24)

The starting point is again a Poincaré inequality, see [WAN5], but now with another LHS. This makes no major difference. We conclude that

\[
(\int_{Q} |u|^q dx)^{\frac{1}{q}} \leq A_{\Gamma_m,k,p,A(x)^q} ((\int_{Q} |\nabla^{k+1} u|^{p_1} dx)^{\frac{1}{p_1}} + (\int_{Q} |\nabla^m u|^p dx)^{\frac{1}{p}}).
\]

(6.38)

Then the RHS of the wanted (6.24) is got in exactly the same way as in the earlier proof of (6.23), i.e. dilate and rearrange like in the beginning of that proof. Then sum in the same way as in the proof of (6.23).

The result is the RHS of (6.24).

However this procedure gives another LHS

\[
\sum_{Q} (\int_{Q} |u|^q \cdot \Gamma_{m,k,p}(x)^{\frac{q}{p}} dx)^{\frac{[p,p_1]}{q}}.
\]

(6.39)

If \( q \geq [p,p_1] \), then the desired result follows from Lemma 6.18.

Hence the first part of the statement in Case A – (6.24) is proved.

Let instead \( 0 < q < [p,p_1] \). Then the procedure is much the same. Something else is needed instead of Lemma 6.18 though. Here the tool is the Hölder inequality for sums.

Since \( s' \) is defined as \( \frac{[p,p_1]}{q} \) and \( f \) is defined with \( \|f\|_{L^s} = 1 \), we observe that
\[
\left( \sum_Q f(Q)^s \right)^\frac{1}{s} \left( \sum_Q \left( \int_Q |u|^q \cdot \Gamma_{m,k,p}(x)^\frac{q}{p} dx \right)^{s'} \right)^\frac{1}{s'} \\
\geq \sum_Q f(Q) \int_Q |u|^q \cdot \Gamma_{m,k,p}(x)^\frac{q}{p} dx \\
= \int_\Omega |u|^q f(x) \Gamma_{m,k,p}(x)^\frac{q}{p} dx.
\]

and then raise both sides to power \( s' \). We obtain

\[
\left( \sum_Q f(Q)^s \right)^\frac{1}{s} \sum_Q \left( \int_Q |u|^q \Gamma_{m,k,p}(x)^\frac{q}{p} dx \right)^{s'} \geq \left( \int_\Omega |u|^q f(x) \Gamma_{m,k,p}(x)^\frac{q}{p} dx \right)^{s'} \\
= \left( \int_\Omega |u|^q f(x) \Gamma_{m,k,p}(x)^\frac{q}{p} dx \right)^{\frac{1}{s}} [p,p_1].
\]

as before the result follows by taking a root of each term and use (6.31).
End of proof of Case A – (6.24).
End of proof of Case A.

Proof of Case B
The proof is almost the same as the one for Case A. The difference is in the beginning of the argument.
We want to prove (6.23). We begin with (6.26) with exponent \( p_0 \) instead of \( p \),

\[
||\nabla^h u||_{H^{\lambda,p_0} (Q)} \leq \frac{A}{\Gamma_{m,k,p_0,A}(x)^\frac{1}{p_0}} \left( \int_{\tilde{Q}} |\nabla^{k+1} u|^{p_1} \ dx \right)^\frac{1}{p_1} \\
+ \left( \int_{\tilde{R}_Q} |\nabla^m u|^{p_0} \ dx \right)^\frac{1}{p_0} \leq \ \\
\leq \frac{A}{\Gamma_{m,k,p_0}(x)^\frac{1}{p_0}} \left( \int_{\tilde{Q}} |\nabla^{k+1} u|^{p_1} \ dx \right)^\frac{1}{p_1} \\
+ \left( \int_{\tilde{R}_Q} |\nabla^m u|^{p_0} d\partial^\Omega(x)^{s+a} \ dx \right)^\frac{1}{p_0} \left( \int_{\tilde{R}_Q} d\partial^\Omega(x)^{-\frac{s+a}{p_0}} \ dx \right)^\frac{p-p_0}{p_0 p_0} \ \\
\text{for } a > 0.
\]
Now the last integral has a uniform bound independent of \( Q \) if there is a positive \( a \) with

\[
(6.44) \quad s + a < \frac{p - p_0}{p_0} (N - \operatorname{dim}_{\text{loc}}(\partial \Omega, \Omega)).
\]

This follows from the given condition on \( s \) and the definition of \( \operatorname{dim}_{\text{loc}}. \)

Then the proof is completed in the same way as for Case A.

End of proof of Case B – (6.23)

The Case B – (6.24) is similar.

End of proof of Case B.

Proof of Case C and D

The proofs proceed in the same way as for Case A and B above, but with the difference that the \( \Theta \)-capacity is used instead of the \( \Gamma \)-capacity.

End of proof of Case C and D.

Proof of Case E

It follows from Case A that for \( \beta \) small and positive

\[
(6.45) \quad \sum_{k=0}^{m-1} \left( \frac{\int |\nabla^k u|^{p} \delta_{\partial \Omega}(x)^{\frac{-\beta - \beta'}{p}} \delta_{\partial \Omega}(x)^{(m-k)p} \, dx}{\delta_{\partial \Omega}(x)^{(m-k)p}} \right)^{\frac{1}{p}} \leq A \left( \frac{\int |\nabla^m u|^{p} \delta_{\partial \Omega}(x)^{\frac{-\beta}{p}} \, dx}{\delta_{\partial \Omega}(x)^{(m-k)p}} \right)^{\frac{1}{p}}.
\]

Make a change of the dependent variable as follows

\[
(6.46) \quad u' = u \cdot \delta_{\partial \Omega}(x)^{\frac{\alpha' - \beta}{p}}.
\]

This transformation is clearly a set-isomorphism of \( C_0^\infty(\Omega) \) to itself.

This change of variable is evaluated for the following expression. The triangle inequality has also been used.

\[
(6.47) \quad \leq A \left( \frac{\int |\nabla^k u'|^{p} \delta_{\partial \Omega}(x)^{\frac{-\beta}{p}} \, dx}{\delta_{\partial \Omega}(x)^{(m-k)p}} \right)^{\frac{1}{p}} + A \frac{\beta' - \beta}{p} \sum_{k=0}^{m-1} \left( \frac{\int |\nabla^r u|^{p} \delta_{\partial \Omega}(x)^{\frac{-\beta}{p}} \, dx}{\delta_{\partial \Omega}(x)^{(k-r)p}} \right)^{\frac{1}{p}}.
\]

Then sum over \( k \). We obtain

\[
(6.48) \quad \sum_{k=0}^{m-1} \left( \frac{\int |\nabla^k u'|^{p} \delta_{\partial \Omega}(x)^{\frac{-\beta'}{p}} \delta_{\partial \Omega}(x)^{(m-k)p} \, dx}{\delta_{\partial \Omega}(x)^{(m-k)p}} \right)^{\frac{1}{p}} \leq A \sum_{k=0}^{m-1} \left( \frac{\int |\nabla^k u|^{p} \delta_{\partial \Omega}(x)^{\frac{-\beta}{p}} \, dx}{\delta_{\partial \Omega}(x)^{(m-k)p}} \right)^{\frac{1}{p}}.
\]
Next repeat (6.47) but with $m$ instead and use of the triangle inequality

\[
\left( \int \frac{|\nabla^m u|^p \delta_{\partial_\Omega(x)}^{\beta'} \delta_{\partial_\Omega(x)}^{(m-k)p} dx \right)^\frac{1}{p} \geq A' \left( \int \frac{|\nabla^m u|^p \delta_{\partial_\Omega(x)}^{\beta} \delta_{\partial_\Omega(x)}^{(m-k)p} dx \right)^\frac{1}{p}
\]

\[
- A'' \frac{\beta' - \beta}{p} \sum_{k=0}^{m-1} \left( \int \frac{|\nabla^k u|^p \delta_{\partial_\Omega(x)}^{\beta} \delta_{\partial_\Omega(x)}^{(m-k)p} dx \right)^\frac{1}{p}.
\]

(6.49)

However if say

\[
\frac{A'}{\beta^\frac{1}{p}} \geq 2 \frac{A'' |\beta' - \beta|}{p},
\]

(6.50)

then the first term of the RHS of (6.49) dominates and the inequality (6.24) with only the higher order term in RHS holds with $s = \beta'$ say.

It remains to calculate the possible $\beta'$:s. We make the choice $\beta' = -\beta$ in order to simplify. Then $\beta$ has to be chosen so that

\[
c \geq \beta^{1-\frac{1}{p}},
\]

(6.51)

for some constant $c > 0$, which can be calculated. This is possible to do if $p > 1$ and $\beta$ small enough.

End of proof of Case E.

End of proof of Theorem 6.17.

We will give examples in Part II of how the two-integral RHS in Theorem 6.17 in certain cases can be used to get a better one-integral in the RHS Hardy inequality than the one-integral formulation taken directly from Theorem 6.17.

– This is the motivation for the two-integral RHS formulation in Theorem 6.17.

Next we give a corollary to Theorem 6.17.

It consists of a list of cases where the one-integral RHS of 6.17 have been used. The weights are specified to give a dilation invariant formulation. Hence the weights are similar to the original one dimensional Hardy inequality.

We include the $f$-cases with $q < p$ since they can be stated simultaneously.

**Corollary 6.19 to Theorem 6.17.** *For notation see 6.16.*

Let $u \in A$

In the cases (i)-(x) it follows from [WAN5] that there is a $b > 0$ with

\[
\Gamma_{m,m-1,A}(x) \geq b > 0.
\]

(6.52)

Preconditions for (6.55).
\begin{align*}
\begin{cases}
(m-h)p > N > (m-h-1)p, \\
0 < l \leq m-h - \frac{N}{p}, \\
t = m-h - l - \frac{s}{p} - \frac{N}{p}.
\end{cases}
\end{align*}

Preconditions for (6.56).

\begin{align*}
\begin{cases}
t = mq - \left(\frac{a}{p} - 1\right)N - \frac{as}{p} \\
N > mp & \text{with } 0 < q \leq \frac{pN}{N-mp} \\
or \quad N \leq mp & \text{with } 0 < q \leq \infty.
\end{cases}
\end{align*}

\begin{align*}
||\nabla^h u||_{H^{\lambda,p,n-t}(\Omega,d_{d\Omega}(x)^{-t})} & \leq A \left( \int |\nabla^m u|^p d_{d\Omega}(x)^s dx \right)^{\frac{1}{p}}. \\
(6.55)
\end{align*}

\begin{align*}
\left( \int \frac{|u|^q F(x)}{d_{d\Omega}(x)^t} dx \right)^{\frac{1}{q}} & \leq A \left( \int |\nabla^m u|^p d_{d\Omega}(x)^s dx \right)^{\frac{1}{p}}. \\
(6.56)
\end{align*}

Here put \( F(x) = 1 \) if \( q \geq p \) and \( F(x) = f(x) \) if \( q < p \).

Then (6.55) respectively (6.56) holds in cases (i)-(x).

To be added:

For (ii) and (vi) let \( 1 \leq p_0 < p \),

for (iv), (viii) and (x) let \( 1 < p_0 < p \),

for (v)-(viii), let there exist an \( r, 0 \leq r \leq N \), such that for all \( Q \in \mathcal{F}_\Omega \) there is a set of orthogonal projections onto hyperplanes, \( \{ S_i \}_{i=1}^{N-r} \), with \( r\)-dim cube \( Q' \)

\[
Q' \subset \bigcap_{i=1}^{N-r} (\tilde{R}_Q \cap \tilde{\Omega}^c)
\]

and \( \text{diam } Q' \geq b > 0 \) for all \( Q \in \mathcal{F}_\Omega \).

Below \( Q \in \mathcal{F}_\Omega \) and \( A, s_0 \) are positive constants independent of \( u \).

(i) Let \( \mathcal{A} = W^{m,p}_0(\Omega) \) and let \( C_{1,p}(\tilde{\Omega}^c \cap \tilde{R}_Q) \geq \text{const.} > 0 \),

If \( s < 0 \), then \( p \geq 1 \) and if \( 0 \leq s < s_0 \), then \( p > 1 \).

(ii) Let \( \mathcal{A} = W^{m,p}_0(\Omega) \) and let \( C_{1,p_0}(\tilde{\Omega}^c \cap \tilde{R}_Q) \geq \text{const.} > 0 \). Let

\[
s < (\frac{p}{p_0} - 1)(N - \dim_{\text{loc}}(\partial\Omega, \Omega)).
\]

(iii) Let \( \mathcal{A} = W^{2,p}_0(\Omega) \) for \( p \geq 1 \). Let \( C_{2,p}(\tilde{\Omega}^c \cap \tilde{R}_Q) \geq \text{const.} > 0 \).

If \( s < 0 \), then \( p \geq 1 \), and if \( 0 \leq s < s_0 \), then \( p > 1 \).
(iv) \( \mathcal{A} = W_0^{2,p}(\Omega)_+ \) and \( p > 1 \). Let \( C_{2,p_0}(\Omega^c \cap \tilde{R}_Q) \geq \text{const.} > 0 \). Let \( s < \left( \frac{p}{p_0} - 1 \right)(N - \dim_{\text{loc}}(\partial \Omega, \Omega)). \)

(v) \( \mathcal{A} = W_{0}^{m,p}(\Omega) \) and let \( p > N - r \). Let \( s < s_0 \) and let \( p > 1 \).

(vi) \( \mathcal{A} = W_{0}^{m,p}(\Omega), \) let \( p_0 > N - r \) and let \( s < \left( \frac{p}{p_0} - 1 \right)(N - \dim_{\text{loc}}(\partial \Omega, \Omega)). \)

(vii) \( \mathcal{A} = W_0^{2,p}(\Omega)_+ \) and let \( 2p > N - r \). Let \( s < s_0 \) and let \( p > 1 \).

(viii) \( \mathcal{A} = W_0^{2,p}(\Omega)_+, \) let \( 2p_0 > N - r \), and let \( s < \left( \frac{p}{p_0} - 1 \right)(N - \dim_{\text{loc}}(\partial \Omega, \Omega)). \)

(ix) \( \mathcal{A} = W_{0}^{m,p}(\Omega), \) let \( s < 0 \), let \( p > 1 \), let \( C_{m,p}(\Omega^c) \neq 0 \), let \( \Omega^c \) be selfsimilar and let \( \partial \Omega \) not be a subset of a hyperplane.

(x) \( \mathcal{A} = W_{0}^{m,p}(\Omega), \) let \( s < \left( \frac{p}{p_0} - 1 \right)(N - \dim_{\text{loc}}(\partial \Omega, \Omega)), \) let \( C_{m,p_0}(\Omega^c) \neq 0 \), let \( \Omega^c \) be selfsimilar and let \( \partial \Omega \) not be a subset of a hyperplane.

Some earlier results.

Ancona has treated the case \( q = p, \) \( p > N \) and \( \Omega \) bounded in [ANC1] and the case (i) in Corollary 6.19 for \( m = 1, \) \( p = 2, \) \( s = 0 \) in [ANC2]. Lewis has treated the case (i) with \( m = 1, \) general \( p, \) \( s \leq 0 \) in [LEW].

In [WAN2] it is given a deliberately short version, with a different proof. (Here is also reference made to more extensive unpublished work material).

The results of Ancona in [ANC1], [ANC2] and Lewis in [LEW] on Hardy inequalities are special cases of the result in [WAN2], except their necessity condition.

This have been discussed in the introductory part.

Observation.

In Theorem 6.17 the constant \( A \) does not depend on \( q \) in (6.25). This property has been the subject of a paper by Kavian, see [KAV], whose result was used by Brezis and Turner on a nonlinear partial differential equation and the corresponding eigenvalue problem, see [BRE-TUR].

The paper [BRE-BRO] by Brezis and Browder got the following problem into focus. The question is, when holds

\[
W_0^{m,p}(\Omega) = W_0^{m,p}(\Omega)_+ - W_0^{m,p}(\Omega)_+.
\]

Ancona gave some answer \( (p > 1) \), see [ANC1], and he proved that, if either \( \Omega \) was bounded Lipschitz or if \( \Omega \) was bounded and \( p > N, \) then (6.58) holds.

The following result is given here
**Theorem 6.20.** Let one of the conditions (i), (ii), (v), (vi), (ix) and (x) of Corollary 6.19 hold together with eventual extra conditions given in the Corollary. Let $q = p$. Then

\[ W^{m,p}_0(\Omega, d_{\partial \Omega}(x)^s) = W^{m,p}_0(\Omega, d_{\partial \Omega}(x)^s)_+ - W^{m,p}_0(\Omega, d_{\partial \Omega}(x)^s)_{++}, \]

also the $W^{m,p}(\Omega, d_{\partial \Omega}(x)^s)$-norms for the nonnegative functions are less than a factor times the norm of the original function. (– A feature also in the Ancona result.)

**Proof.**

Follows from Theorem 6.17 and Theorem 6.23.

There is another interesting result.

**Theorem 6.21.** Let $u \in W^{2,p}_0(\Omega, d_{\partial \Omega}(x)^s)$ and $1 < p$. If one of the conditions (i)–(iv) below is satisfied, then

\[ u \in W^{2,p}_0(\Omega, d_{\partial \Omega}(x)^s)_+ - W^{2,p}_0(\Omega, d_{\partial \Omega}(x)^s)_+ \]

holds, if and only if

\[ \int |u|^p d_{\partial \Omega}(x)^{-2p+s} dx < \infty. \]

**Additional comments:**

In (iv) let there exist an $r$, $0 \leq r \leq N$, such that to all $Q \in F_\Omega$ there is set of orthogonal projections onto hyperplanes, $\{S_i\}_{i=1}^{N-r}$, with $r$-dim cube $Q'$

\[ Q' \subset (\prod_{i=1}^{N-r} S_i) \left( \tilde{R}_Q \cap \tilde{\Omega}^c \right) \]

and $\text{diam } Q' \geq b > 0$ for all $Q \in F_\Omega$.

Below $Q \in F_\Omega$ and $A$, $s_0$ are positive constants independent of $u$.)

(i) Let $C_{2,p}(\tilde{\Omega}^c \cap \tilde{R}_Q) \geq \text{const} > 0$ for all $Q \in F_\Omega$. If $s < 0$, then $p \geq 1$, and if $s < s_0$, then $p > 1$.

(ii) Let $C_{2,p_0}(\tilde{\Omega}^c \cap \tilde{R}_Q) \geq \text{const} > 0$ for all $Q \in F_\Omega$. Let $s < \left(\frac{p}{p_0} - 1\right)(N - \text{dim}_{loc}(\partial \Omega, \Omega))$. Let $1 \leq p_0 < p$.

(iii) Let $2p > N - r$, let $s < s_0$ and let $p > 1$.

(iv) Let $2p_0 > N - r$ and let $s < \left(\frac{p}{p_0} - 1\right)(N - \text{dim}_{loc}(\partial \Omega, \Omega))$. Let $1 < p_0 < p$. 28
Proof.

The sufficiency follows from Theorem 6.23. The necessity follows from Corollary 6.19.

The following theorem can be read out from the proof of the first theorem by Ancona in [ANC1]. We give a different theorem though. Since Ancona’s account is brief we repeat the arguments andz more generously.

– This version is more general.

**Theorem 6.22.** Let $1 < p$ and let $s$ be a real number. Let $\Omega$ be open subset of $\mathbb{R}^N$, with non-empty complement. Let

$$u \in W^{m,p}_0(\Omega, d_{\partial \Omega}(x)^s).$$

Then

$$\int_{\Omega} |u|^p d_{\partial \Omega}(x)^{-mp+s} dx < \infty$$

implies that there exist

$$u_i \in W^{m,p}_0(\Omega, d_{\partial \Omega}(x)^s)_+, \ i = 1, 2,$$

with $u = u_1 - u_2$.

Proof.

Let $u_Q \in W^{m,p}_0(\alpha Q)$, with say $\alpha = 4/3$.

If there exists $v_Q \in W^{m,p}_0(\beta Q)$, with say $\beta = 4^2/3^2$, such that $v_Q \geq u_Q$ and $v_Q \geq 0$ and if furthermore it holds that

$$||v_Q||_{W^{m,p}} \leq A_0 ||u_Q||_{W^{m,p}},$$

then the problem is solved locally in a sense.

The procedure now is first to show that this implies that there is a global solution. After that we return and solve the local problem.

Let $Q_0$ be a unit cube. Take an $\eta \in C_0^\infty \left(\frac{4}{3}Q_0\right)_+, \eta|Q_0 = 1$. Let $Q$ be any cube, centre $x_Q$. Define

$$\eta_Q = \eta\left(\frac{x - x_Q}{l(Q)}\right)$$

(6.62)

Let $Q \in F_\Omega$, a set of Whitney cubes for $\Omega$. Let $u \in W^{m,p}_0(\Omega, d_{\partial \Omega}(x)^s)$. Define $u_Q = \eta_Q u$, then by assumption there is an $v_Q$ with properties as above. Define

$$v = \sum_{Q \in F_\Omega} v_Q.$$
Clearly $v \geq 0$ and $v \geq u$.

In this global situation it only remains to make the norm estimate,

$$\|v\|_{W^{m,p}(\Omega, d_{\partial \Omega}(x)^s)}^p \leq A \sum_{k=0}^{m} \|\nabla^k v\|_{L^p(\Omega, d_{\partial \Omega}(x)^s)}^p$$

(6.64)

$$= A \sum_{k=0}^{m} \sum_{Q \in \mathcal{F}_\Omega} \|\nabla^k v_Q\|_{L^p(\Omega, d_{\partial \Omega}(x)^s)}^p \leq$$

Next we use the fact that the enlarged cubes only have at most a fixed number of overlap. This means that in this situation the integrals can be decomposed at the cost of a constant factor only,

$$\leq A \sum_{k=0}^{m} \sum_{Q \in \mathcal{F}_\Omega} \|\nabla^k v_Q\|_{L^p(\Omega, d_{\partial \Omega}(x)^s)}^p \leq$$

(6.65)

$$\leq A \sum_{Q \in \mathcal{F}_\Omega} \|v_Q\|_{W^{m,p}(\Omega)}^p l(Q)^s$$

Next step is a standard interpolation of norm and the scale factor involved is e.g. the result of a dilation of a unit dimension case

$$\leq A \sum_{Q \in \mathcal{F}_\Omega} \|u_Q\|_{W^{m,p}(\Omega)}^p l(Q)^s \leq$$

(6.66)

$$\leq A \sum_{Q \in \mathcal{F}_\Omega} \|u_Q\|_{L^p(\Omega, d_{\partial \Omega}(x)^s)}^p l(Q)^s + \|\nabla^m u\|_{L^p(\Omega, d_{\partial \Omega}(x)^s)}^p.$$ 

The last step is a consequence of that the $p$-powers make norms into integrals and then this inverse kind of inequality becomes natural.

In (6.66) it is clear that the first term in the last expression is finite and that then $u \in W^{m,p}(\Omega, d_{\partial \Omega}(x)^s)$ makes the whole expression finite. Hence the wanted function $v$ exist with finite norm. This step-wise build-up of $v$ also ensures that $v \in W^{m,p}_0(\Omega, d_{\partial \Omega}(x)^s)$ as wanted.

Hence it remains only to prove the local cube-wise property.

Let $u \in W^{m,p}(\Omega, d_{\partial \Omega}(x)^s)$ and $u_Q = \eta_Q u$. The Bessel kernel $G_m$ together with an $f \in L^p$ solves

30
\begin{equation}
G_m * f = u_Q.
\end{equation}

This is $u_Q$ written as a Bessel potential, $(1 < p)$, and the norm $||f||_{L^p}$ is equivalent to the Sobolev space norm. Further $G_m$ acts positively, i.e. if $0 \leq g$ and $g \in L^p$ then $0 \leq G_m * g$. In the situation above take $f_+$ instead of $f$ and then $G_m * f \leq G_m * f_+$ and of course $||f_+||_{L^p} \leq ||f||_{L^p}$.

Note that as usual the Sobolev space $W^{m,p}$ is defined by the quasicontinuous functions only.

To ensure the short range of support of $v_Q$ we define

\begin{equation}
v_Q = \eta(4^2/3^2)(G_m * f_+).
\end{equation}

By the standard Poincaré inequality and the construction of the Bessel potentials it holds that

\begin{equation}
||v_Q||_{W^{m,p}} \leq ||f||_{L^p} = ||G_m * f||_{W^{m,p}} = ||u_Q||_{W^{m,p}}.
\end{equation}

Hence the local problem is solved.

The proof is complete.

REFERENCES

[ANC1] A. Ancona, Une propriété des espaces de Sobolev, C.R. Acad. Sc. Paris \textbf{292} (1981), 477–480.

[ANC2] A. Ancona, On strong barriers and an inequality of Hardy for domains in $\mathbb{R}^n$, J. London Math. Soc. (2) \textbf{34} (1986), 274–290.

[BRE-BRO] H. Brezis and F.E. Browder, Some properties of higher order Sobolev spaces, J. Math. pure et appl. \textbf{61} (1982), 245–259.

[BRE-TUR] H. Brezis and R.E.L. Turner, On a class of superlinear elliptic problems, Comm. Partial Differential Equations \textbf{2} (1977), 601-614.

[COF-GRO] C.V. Coffman and C.L. Grover, Obtuse cones and applications to partial differential equations, J. Funct. Anal. \textbf{35} (1980), 369–396.

[GUR-OPI] P. Gurka and B. Opic, Continuous and compact imbeddings of weighted Sobolev space $I, II, III$, I and II, Czech. Math. J., III preprint \textbf{38 (113)}, \textbf{39 (114)} (1988 1989).

[HED] L.I. Hedberg, Spectral synthesis in Sobolev spaces, and uniqueness of solutions of the Dirichlet problem, Acta Mathematica \textbf{147} (1981), 237–264.

[HOR] T. Horiuchi, The imbedding theorems for weighted Sobolev spaces, preprint, Institute Mittag-Leffler \textbf{9} (1987).

[JON] P.W. Jones, Quasiconformal mappings and extendability of functions in Sobolev space, Acta Math. (1981), 71–88.

[H-K-S] W.K. Hayman, L. Karp, H.S. Shapiro, Newtonian capacity and Quasi-Balaylage, preprint, Royal Inst. of Tech. Stockholm, Sweden (1998), 1-30.

[KAV] O. Kavian, In égalité de Hardy-Sobolev, C.R. Acad. Sc. Paris \textbf{t.286 (8 mai)} (1978), Série A 779–781.

[KUF] A. Kufner, Weighted Sobolev Spaces, Teubner-texte, 1980.
[KUF-SAN] A. Kufner and A-M Sandig, Some Applications of Weighted Sobolev Spaces, Teubner-texte, 1987.

[LEW] John L. Lewis, Uniformly fat sets, Trans. Amer. Math. Soc. 308 (1988), 177–196.

[MAZ1] V. G. Maz’ja, On (p,l)-capacity, imbedding theorems, and the spectrum of a selfadjoint elliptic operator, Math. USSR Izvestija 7. No. 2 (1973), 357–387.

[MAZ2] V.G. Maz’ja, Sobolev Spaces, Springer, 1985.

[NIK] S.M. Nikol’ii, Encyclopedia of Mathematical Sciences, vol. 26, Springer, 1988.

[NYS] K. Nyström, Smoothness properties of solutions to Dirichlet problems in domains with a fractal boundary, Doctoral Thesis No 7, Dept. Math., Univ. Umeå (1994).

[SHA] Harold. S. Shapiro, Multivariate Approximation, Akademie Verlag Berlin, 1997, pp. 203-254.

[STE] E.M. Stein, Singular integrals and differentiability properties of functions, Princeton University Press, 1970.

[STR] E.W. Stredulinsky, Lecture Notes in Math. 1074, Weighted inequalities and degenerated elliptic partial differential equations, Springer-Verlag, 1984.

[WAN1] A. Wannebo, Aspects of Sobolev theory I, Circulated work material (1991), 1-63.

[WAN2] A. Wannebo, Hardy inequalities, Proc. AMS 109 (1990), 85-95.

[WAN3] A. Wannebo, Hardy inequalities and imbeddings in domains generalizing $C^{0,\lambda}$ domains, Part of Thesis 1991, Proc. Amer. Math. Soc. 122 (1994).

[WAN4] A. Wannebo, Equivalent norms for the Sobolev space $W_0^{m,p}(\Omega)$, Part of Thesis 1991, Ark. Mat. 32 (1994).

[WAN5] A. Wannebo, Polynomial Capacities, Poincaré type Inequalities and Spectral Synthesis in Sobolev Space, Preprint, Dept. Math. Royal Institute of Technology (1998), 1-33.