Time of creep fracture of axisymmetrically loaded structures

I A Banshchikova¹, I V Lubashevskaya¹,²

¹ Lavrentyev Institute of Hydrodynamics SB RAS, 15 Prospekt Lavrentyeva, Novosibirsk, 630090, Russia
² Novosibirsk State University, 1 Pirogova str., Novosibirsk, 630090, Russia

E-mail: binna@ngs.ru

Abstract. A stress-strain state and a time duration up to fracture are calculated with allowance for two-stage behavior of a rotating disk under creep conditions. The duration of the stages is investigated depending on the choice of the version of the creep kinetic theory and the geometric dimensions of the disk. The first stage is the accumulation of damage and the beginning of fracture in some area of the body, where the accumulated damage reaches a critical value. The second stage is the spread of the fracture front and the complete destruction of the body. A calculation method has been developed which reduces the solution of the unsteady-state creep problem to the solution of an analogous steady-state problem.

1. Introduction

Evaluation of durability of structural elements is an actual problem. Usually, normative life of structural elements is identified by the onset of their destruction. But the time of destruction onset does not determine durability, so the determination of the additional (residual) life of the structure is an important task [1-9].

In the present work, a study of the stress-strain state and the duration of deformation of a rotating disk accounting for two-stage behavior is carried out. The solution uses an approximate calculation method based on the kinetic creep theory, taking into account the accumulation of damage [7–9]. The time of fracture onset is the time when, in some area of the structure, the accumulated damage reaches a critical value. This is the first stage of latent destruction.

The second stage is the propagation of the creep fracture front and the complete destruction of the body. The calculation method consists in the fact that the solution of the unsteady-state creep problem reduces to solving an analogous steady-state creep problem. This technique was previously developed for a ring loaded axisymmetric plates under bending [10–12] and thick-walled vessels (pipes), loaded by internal pressure [13, 14]. For a rotating disk, the duration of the stages is investigated, depending on the choice of the version of the kinetic theory of creep and the geometric dimensions of the disk.

2. Theory

2.1. Structural equations and the time of fracture onset of the axisymmetrically loaded rotating disk

Creep of a uniformly heated disk with simultaneous consideration of damage accumulation in the material is considered from the phenomenological point of view. The internal, outer and current radiuses of the disk are \( a_0 \), \( b_0 \), \( r \), \( a_0 \leq r \leq b_0 \), the thickness of the disk is \( h = h(r) \), \( h(r) = h_0 \cdot r^{-k} \), \( k \geq 0 \), \( h_0 \) is a constant (see Figure 1).
Figure 1. Diagram of a disk.

A plane stress state is assumed in the disk \((\sigma_z = 0)\), the material is incompressible \((\eta_\varphi + \eta_r + \eta_z = 0)\) and \(\eta_\varphi = dv/dr, \quad \eta_r = vr/r\); here \(v(r,t)\) is the displacement radial rate of an arbitrary point of the disk, \(\eta_\varphi, \eta_r, \eta_z\) are tangential, radial and axial components of the tensor of creep strain rates. The \(z\) axis is perpendicular to the plane of the disk, and the axes \(r, \varphi\) are directed as usual [7, 8].

The stresses \(\sigma_\varphi, \sigma_r\) at any instant of time satisfy the equation of equilibrium

\[
d(hr\sigma_r)/dr - h\sigma_\varphi + h\Omega^2 r^2 = 0
\]

and the boundary conditions \(\sigma_r(a_0) = 0, \quad \sigma_\varphi(b_0) = p\), where \(p\) is a uniform tensile pressure at the rim of the disk; \(\Omega^2 = \gamma \omega^2\), \(\gamma\) is the density of the disk material, \(\omega\) is the angular velocity of the disk. Integrating equation (1) prescribed boundary conditions and taking into account that \(\eta_\varphi(a_0) = 0\), we obtain

\[
\sigma_r(r,t) = \left[\frac{\Omega^2}{3-k} a_0^{3-k} + \int_{a_0}^r \sigma_\varphi r^{-k} dr\right] r^{k-1} + \frac{\Omega^2}{3-k} r^2 , \quad (2)
\]

\[
\int_{a_0}^{b_0} \sigma_\varphi r^{-k} dr = P, \quad P = pb_0^{3-k} + \frac{\Omega^2}{3-k} (b_0^{3-k} - a_0^{3-k}) . \quad (3)
\]

Physical relationships use Rabotnov’s kinetic theory [7, 9]

\[
\eta_{ij} = \frac{W}{\sigma_\varphi} \frac{\partial \sigma_r}{\partial \sigma_y}, \quad W = B_1 \sigma_y^{*+1}, \quad i, j = 1, 2, 3; \quad (4)
\]

\[
\frac{d\mu}{dt} = -\frac{B_g \sigma_y^{*+1}}{\phi(\mu)}, \quad \mu(r^*, t_*) = 0 , \quad (5)
\]

where \(\eta_{ij}, \sigma_{ij}\) are the components of the creep strain rates and stresses tensors; \(\sigma_\varphi, \sigma_r\) are first-order functions which are homogeneous with respect to stresses; \(W\) is the power of dissipated energy, \(W = \sigma_\varphi \eta_\varphi\). \(B_1, n, B_2, g\) are the constants of creep and long-term strength; \(\mu\) is a parameter describing the accumulation of damage during the creep of the material from the phenomenological point of view. The time of onset of body fracture \(t_*(\text{the duration of the first stage})\) is calculated from the condition for achieving the parameter of damage \(\mu\) its critical value of zero in some coordinates \(x_{r*}, \ k = 1, 2, 3\).

If \(\phi(\mu) = \phi(\mu)\), then the system (4), (5) represents the most widespread version of the kinetic theory, from which as the special cases follow the theory of short-term creep, the energy variant of creep and long-term strength, the hardening theory with its various modifications. If \(\phi(\mu) = 1\), then we get a variant of the kinetic theory of creep in the formulation of Kachanov [8]. In the following it is assumed that \(\phi(\mu) = \mu^m, \quad \phi(\mu) = \mu^m, \quad 0 \leq m \leq m\).
In (4), (5) the maximum shear stress is chosen as \( \sigma_0 \) and \( \sigma_{\alpha} \). Consider the case when the condition \( \sigma_0 > \sigma_\alpha > \sigma_r = 0 \) is satisfied at each point of the disk at any time up to destruction. Therefore \( \sigma_r = \sigma_{\alpha} = \sigma_0 / 2 \) and system (4), (5) reduced to:

\[
\begin{align*}
\eta_0 &= \frac{B_1 \sigma_0}{\mu_m}, \quad \eta_\alpha = 0, \quad \eta_r = -\frac{B_1 \sigma_0}{\mu_m}, \quad B_1 = B_z / 2^{n+1}, \\
\mu m d \mu &= -B_1 \sigma_\alpha^{(g+1)} / \mu_m, \quad \mu(r, 0) = 1, \quad \mu(r^*, t_*) = 0, \quad B_1 = B_z / 2^{n+1}.
\end{align*}
\]

(6)

Since \( \eta_r = 0 \), then \( dv/dr = 0 \) and it follows that \( v(r, t) \) is a function of time only, i.e. \( v = C(t) \). Then

\[ \eta_0 = C(t)/r, \quad \eta_\alpha = 0, \quad \eta_r = -C(t)/r \]

and taking into account (3), (6):

\[ \sigma_\alpha(r, t) = \frac{P}{J_1} \left[ \frac{\mu(r, t)}{B_1} \right]_{m/n}^{1/n} r^{-1/n} . \]

(9)

In (9) for convenience we introduce the notation:

\[ J_1 = \int_{a_0}^{t_1} r^{-(k+1/n)} dr, \quad \left[ \frac{C(t)}{B_1} \right]_{m/n}^{1/n} = \frac{P}{J_1} \left[ X(t) \right]^{-1} . \]

(10)

Substituting (9) into the equation for the parameter of damage (7) and (3), we obtain

\[
\begin{align*}
\int \mu(r, t)^{(mn-m_1(g+1))/n} d \mu &= -[(m+1) r^0(r)]^{t_1} \int_0^t [X(\tau)]^{(g+1)} d \tau , \\
\int_{a_0}^{t_1} \mu(r, t)^{m/n} r^{-(k+1/n)} dr &= J_1 X(t) ,
\end{align*}
\]

(11)

(12)

where

\[ t^0(r) = \left( (m+1) B_z (P/J_1)^{g+1} r^{-(g+1)/n} \right)^{-1} . \]

(13)

Thus, the solution of the problem of the stress-strain state calculation of the disk and the time of its fracture has been reduced to solving the system of equations (11) and (12). Integrating (11), we obtain

\[
\mu^{m/n} = \left[ 1 - \frac{\nu}{t^0(r)} \right]^{1/(m+1)} X^{-g} d \tau , \quad \beta = \frac{m_1}{n + mn - m_1(g+1)}, \quad \nu = \frac{n + mn - m_1(g+1)}{n(m+1)} .
\]

(14)

Substituting (14) into (12):

\[
\int_{a_0}^{t_1} \left[ 1 - \frac{\nu}{t^0(r)} \right]^{1/(m+1)} X^{-(g+1)} d \tau \left[ X^{-g} d \tau \right]^{\beta} r^{-(k+1/n)} dr = J_1 X(t) .
\]

(15)

Solving (15), we find \( X(t) \), then the function \( \mu(r, t) \) is determined from (14). Knowing these functions, \( \sigma_0 \) and \( \sigma_{\alpha} \), we are computed from (9) and (2), \( \eta_0 \) and \( \eta_\alpha \) are calculated from (8) using (10). The time of onset of disk fracture is determined by the condition \( \mu(r^*, t_*) = 0 \). It is easy to show that \( r^* = a_0 \), i.e. the fracture begins at the inner surface of the disk. In order that the integral on the left-hand side of equation (11) make sense in the case \( \mu = 0 \), restrictions on the constants of the material must be imposed, which are described in detail in [13].

Let \( m_1 = 0 \). Then, from (14) we get \( \beta = 0, \quad \nu = 1 \), and system (4), (5) is a variant of the kinetic theory of creep in the formulation of Kachanov. From (15) we get \( X(t) = 1 \). From (2), (8), (9), taking into account the second equation (10), it follows that the stress-strain state is stationary. From (14)

\[ \mu(r, t) = \left[ 1 - \left( \frac{t}{t^0(r)} \right)^{(g+1)/n} \right]^{1/(m+1)} , \quad t^0(r) = \left( (m+1) B_z (P/J_1)^{g+1} a_0^{-(g+1)/n} \right)^{-1} . \]

(16)
\[ t^0 = t^0(\alpha_0) \] is the time of the beginning of the disk fracture according to (13) and \( r^* = \alpha_0 \).

### 2.2. Duration of the propagation of the fracture front

The calculation of the duration of the fracture front propagation is one of the most important problems for practice. In the case of the considered problem: at the instant of time \( t = t_s \), the inner surface of the disk is destroyed, and then the fracture front, whose boundary is a circle with a radius \( r = \alpha(t) \), moves toward the outer surface of the disk \( r = \beta_0 \) (Figure 1). It is obvious, that \( \alpha_0 < \alpha(t) < \beta_0 \), \( \alpha(t_s) = \alpha_0 \), \( \alpha(t_f) = \beta_0 \), \( t_f \) is the time of disk fracture. Further relevant parameters and functions related to the fracture front will be marked by index ‘f’. In particular, at any point in time \( \tau > t_s \) will be \( J_1(\tau) = J_{1f} \), \( P(\tau) = P_f \) and they are calculated from (3), (10) and (13) with \( \alpha_0 \) replaced by \( \alpha(\tau) \); stresses \( \sigma_r \) and \( \sigma_\phi \) are still calculated from (2) and (9). Next, performing the same calculations as before, we get at the instant of time \( t \) instead of (11), (12):

\[
\int_1^{\mu(t)} [\mu(r,t)]^{m_{n-1}(g+1)/n} d\mu = -\frac{t}{t_0} [(m+1)t^0(r, \tau)]^{-1} X(\tau) \left[ X(\tau) \right]^{(g+1)} d\tau, \tag{17}
\]

\[
\int_{a(t)}^{b_0} \left[ \mu(r,t) \right]^{m_{n-1}/n} r^{-(k+1/n)} dr = J_{1f}(t) X(t). \tag{18}
\]

Integrating (17), we have instead of (14):

\[
\mu(r,t) \left[ m_{n-1}/n \right] = \left[ 1 - \frac{t}{t_0} [(m+1)t^0(r, \tau)]^{-1} X(\tau) \left[ X(\tau) \right]^{(g+1)} d\tau \right]^\beta.
\]

On the front line \( r = \alpha(t) \) we have \( \mu(\alpha(t), \tau) = 0 \), i.e. taking into account (13)

\[
1 - \frac{t}{t_0} [(m+1)t^0(r, \tau)]^{-1} X(\tau) \left[ X(\tau) \right]^{(g+1)} d\tau = 0. \tag{19}
\]

Using (19), for the damage parameter we obtain \( \mu(r,t) \left[ m_{n-1}/n \right] = \left[ 1 - (r/a(t))^{(g+1)/n} \right] \beta \) and (18) reduces to the form

\[
\int_{a(t)}^{b_0} \left[ 1 - (r/a(t))^{(g+1)/n} \right] \beta r^{-(k+1/n)} dr = J_{1f}(t) X(t). \tag{20}
\]

To obtain the equation of motion of the fracture front, we differentiate expression (19) with respect to \( t \), applying in general the rule of differentiation of the integral with respect to a parameter (here \( t \) acts as a parameter). After the simplest operations, we get:

\[
a^{1+g+1/n} d\alpha/dt = \frac{mv(m+1)B_2}{g+1} \left( \frac{P_f(t)}{J_{1f}(t)X(t)} \right)^{g+1}. \tag{21}
\]

Taking (20) into account, (21) can be reduced to the form

\[
a^{1+g+1/n} [P_f(a)^{g+1}] \left( \int_{a(t)}^{b_0} \left[ 1 - (r/a(t))^{(g+1)/n} \right] \beta r^{-(k+1/n)} dr \right)^{g+1} da = \left( \frac{mv(m+1)B_2}{g+1} \right) dt, \tag{22}
\]

where \( P_f(a) = pb_0^{\gamma} + \frac{\gamma^2}{3-k} (b_0^{\gamma} - a^{\gamma}) \). Introducing new variables \( \rho = a/a_0, 1 \leq \rho \leq \beta_1 \), \( \beta_1 = b_0/a_0, \ y = r/a, \ 1 \leq y \leq \beta_1/\rho \), and integrating (22), we obtain \( a(t) \) and the duration of the fracture front propagation \( \Delta t_s = t_{sf} - t_s \).
\[ \Delta t_s = \frac{g+1}{n\omega B_2(m+1)} \left( \beta^2 \left( \rho \beta^{1-k} + \omega \left( \rho \beta^{3-k} - \rho^3 \right) \right)^{-(g+1)} \left( \int_{1}^{\rho} y^{-(k+1/n)} \left( 1 - y^{(g+1)/n} \right) \beta \right)^{g+1} \right) d\rho, \]

Where \( d = 1 - (k-1)(g+1) \), \( \omega = a_0^2 \Omega^2 / (3-k) \).

### 3. Results and discussion

An example the rotating disk loaded on the outer contour by a tensile pressure \( p = 20 \text{ MPa} \) is considered. We estimate the influence of the choice of the variant of the creep kinetic theory and the geometric dimensions of the disk on the duration of the first and second stages. The following constant of the material corresponding to the D16 alloy are used for calculations at a temperature of 250°C \( B_1 = 3.5172 \cdot 10^{-15} \text{ (MPa}^n/\text{h}), n = 6, m = 14, g = 4.75, B_2 = 2.7563 \cdot 10^{-15} \text{ (MPa}^{4-n}/\text{h}) \) [9, 11].

Dimensions of the disk are \( a_0 = 0.025 \text{ m}, b_0 = 0.1 \text{ m}, \Omega = 70 \text{ (MPa}^{1/2}/\text{m}) \).

![Figure 2. Dependence \( \mu(r,t) \) – (a) and \( \sigma_\phi(r,t) \) – (b) for \( m_i = 10 \) and \( \sigma_\phi(r,t) \) – (c) for \( m_i = 0 \).](image)

Lines in Figure 2 a, b show the dependence \( \mu(r,t) \) and the diagrams \( \sigma_\phi(r,t) \) in the section \( a_0 < r < b_0 \) for some fixed time instants for \( m_i = 10 \): solid lines correspond to the stage of latent destruction (lines 1 for \( t = 0 \), line 2 for \( t = t_s \)); dashed lines correspond to the stage of propagation of the fracture front.

Similar lines in Figure 2 c show the diagrams \( \sigma_\phi(r,t) \) for \( m_i = 0 \) (a variant of the creep theory in Kachanov's formulation). In this case, redistribution of stresses up to the beginning of fracture is absent, i.e. the state is stationary (solid line 1 in figure 2 c). Here the function \( \mu(r,t) \) for \( m_i = 0 \) is analogous to the function \( \mu(r,t) \) shown in Figure 2 a.

Table 1 presents the results of calculating of the first stage duration \( t_s \) and \( \Delta t_s / t_s = (t_{sf} - t_s) / t_s \) for \( m_i = 0 \) (Kachanov's creep theory) and \( m_i = 10 \) (Rabotnov's kinetic theory) depending on \( a_0, k, \Omega \) ( \( p = 20 \text{ MPa}, b_0 = 0.1 \text{ m} \)).

| \( b_0/a_0 \) | \( k \) | \( \Omega \text{ (MPa}^{1/2}/\text{m}) \) | \( t_s, (10^3 \text{ h}) \) | \( \Delta t_s / t_s, (\%) \) | \( t_s, (10^3 \text{ h}) \) | \( \Delta t_s / t_s, (\%) \) |
|---|---|---|---|---|---|---|
| | | | | | | |

### Table 1. Duration \( t_s \) of the first stage and \( \Delta t_s / t_s \) for \( m_i = 0 \) and 10.
4. Conclusion

The method of calculation of stress-strain state with account of damage accumulation which used to calculate such structures as plates under bending [10–12] and pipes under internal pressure [13, 14], is developed for the analysis of a rotating disk. On the basis of the obtained results it can be concluded that the duration of the stages of front propagation of the fracture, as in the case of pipes under internal pressure, can range from a few tenths to tens of percent of the time latent fracture, depending on the choice of the variant of creep theory, fracture criterion and geometrical dimensions of the structure.

Acknowledgments

This work was partially supported by the Russian Foundation for Basic Research (project codes 15-01-07631-a and 16-08-00713-a).

References

[1] Kachanov L M 1961 Time of Fracture under Creep Conditions Problems of Continuum Mechanics (Moscow: AN SSSR) 186–201 (In Russian)
[2] Hayhurst D R 1973 Trans. ASME. J. Appl. Mech 40(4) 88-95
[3] Skrzypek J J, Ganczarski A W 1999 Modeling of Material Damage and Failure of Structures. Theory and Applications (Berlin: Springer- Verlag) p 326
[4] Lokoshchenko A M, Sokolov A V 2014 Mechanics of Solids 49(1) 49–58
[5] Lokoshchenko A M, Fomin L V 2016 J. Appl. Mech. Tech. Phy. 57(5) 792–800
[6] Altenbach H, Altenbach J, Naumenko K 1997 Mechanics of Time-Dependent Materials 1(2) 181–93
[7] Rabotnov Yu N 1969 Creep Problems in Structural Members (Amsterdam: North-Holland) p 836
[8] Kachanov L M 1960 Teorija Polzuchesti [The Theory of Creep] (Moscow, Fizmatgiz) p 455 (in Russian)
[9] Nikitenko A F 1997 Polzuchest’ i Dlitel’naya Prochnost’ Metallicheskih Materialov [Creep and Long-term Strength of Metal Materials] (Novosibirsk: Novosibirsk State University of Architecture and Civil Engineering) p 278 (In Russian)
[10] Zaev V A, Nikitenko A F 1993 J. Appl. Mech. Tech. Phy. 34(3) 423–6
[11] Banshchikova I A, Nikitenko A F 2006 J. Appl. Mech. Tech. Phy. 47(5) 747–56
[12] Banshchikova I A 2015 Computationl Continuum Mechanics 8(4) 359–68
[13] Nikitenko A F, Lyubashevskaya I V 2005 Strength Mater. 37(5) 460–70
[14] Nikitenko A F, Lyubashevskaya I V 2007 J. Appl. Mech. Tech. Phy. 48(5) 766–73