On boundary super algebras

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Abstract

We examine the symmetry breaking of super algebras due to the presence of appropriate integrable boundary conditions. We investigate the boundary breaking symmetry associated to both reflection algebras and twisted super Yangians. We extract the generators of the resulting boundary symmetry as well as we provide explicit expressions of the associated Casimir operators.
1 Introduction

Symmetry breaking processes are of the most fundamental concepts in physics. It was shown in a series of earlier works (see e.g. [1]–[9]), within the context of quantum integrability, that the presence of suitable boundary conditions may break a symmetry down without spoiling the integrability of the system. It was also shown [10] that the breaking symmetry mechanism due to the presence of integrable boundary conditions may be utilized to provide certain centrally extended algebras. Here, we shall investigate in detail the resulting boundary symmetries in the context of quantum integrable models associated to various super-algebras.

More precisely, the main aim of the present work is the study of the algebraic structures underlying quantum integrable systems associated to certain super-algebras, \(\mathcal{Y}(gl(m|n))\) and \(U_q(gl(m|n))\), once non-trivial boundary conditions are implemented. In this investigation we shall focus on the relevant algebraic content, and our primary objectives will be the study of the related exact symmetries as well as the construction of the relevant Casimir operators. The existence of centrally extended super algebras emerging from these boundary algebras is one of the main motivations for the present investigation, and will be discussed in full detail in a forthcoming work given that is a separate significant topic.
We shall focus here on the super Yangian $\mathcal{Y}(gl(m|n))$ and its $q$-deformed counterpart the $U_q(gl(m|n))$ algebra. It is necessary to first introduce some useful notation associated to super algebras. Consider the $m+n$ dimensional column vectors $\hat{e}_i$, with 1 at position $i$ and zero everywhere else, and the $(m+n) \times (m+n)$ $e_{ij}$ matrices: $(e_{ij})_{kl} = \delta_{ik} \delta_{jl}$. Then define the grades:

$$[\hat{e}_i] = [i], \quad [e_{ij}] = [i] + [j].$$

The tensor product is also graded as:

$$(A_{ij} \otimes A_{kl})(A_{mn} \otimes A_{pq}) = (-1)^{([k]+[l])([m]+[n])}A_{ij}A_{mn} \otimes A_{kl}A_{pq}.$$

Define also the transposition

$$A^T = \sum_{i,j=1}^{m+n} (-1)^{[i][j]+[j]}e_{ji} \otimes A_{ij}, \quad A = \sum_{i,j=1}^{m+n} e_{ij} \otimes A_{ij},$$

and the super-trace as:

$$strA = \sum_i (-1)^{[i]}A_{ii}.$$ 

It will also be convenient for our purposes here to define the super-transposition as:

$$A^t = V^{-1}A^T V$$

where the matrix $V$ will be defined later in the text when appropriate. Also it is convenient for what follows to introduce the distinguished and symmetric grading, corresponding apparently to the distinguished and symmetric Dynkin diagrams. In the distinguished grading we define:

$$[i] = \begin{cases} 0, & 1 \leq i \leq m, \\ 1, & m + 1 \leq i \leq m + n. \end{cases}$$

In the $gl(m|2k)$ we also define the symmetric grading as:

$$[i] = \begin{cases} 0, & 1 \leq i \leq k, \\ m + k + 1 \leq i \leq m + 2k, \\ 1, & k + 1 \leq i \leq m + k. \end{cases}$$

2 The super Yangian $\mathcal{Y}(gl(m|n))$

Let us first introduce the basic algebraic objects associated to the Yangian $\mathcal{Y}(gl(m|n))$. The $R$ matrix solution of the Yang-Baxter equation [11] associated to $\mathcal{Y}(gl(m|n))$ is [12, 13, 14, 15]:

$$R(\lambda) = \lambda + iP$$

(2.1)
where $P$ is the super-permutation operator defined as:

$$P = \sum_{i,j} (-1)^{[j]} e_{ij} \otimes e_{ji}.$$  \hspace{1cm} (2.2)

Also define

$$\bar{R}_{12}(\lambda) := R_{12}^1(\lambda - i\rho), \quad \bar{R}_{21}(\lambda) := R_{12}^2(-\lambda - i\rho) \quad \text{and} \quad \bar{R}_{12}(\lambda) = \bar{R}_{21}(\lambda)$$

$$\bar{\lambda} = -\lambda - i\rho, \quad \text{and} \quad \rho = \frac{n - m}{2}.$$ \hspace{1cm} (2.3)

The $\bar{R}$ matrix may be written as

$$\bar{R}_{12}(\lambda) = \bar{\lambda} + iQ_{12}$$ \hspace{1cm} (2.4)

where $Q$ is a projector satisfying

$$Q^2 = 2\rho Q, \quad P Q = Q P = Q.$$ \hspace{1cm} (2.5)

Consider also the $L$-operator expressed as

$$L(\lambda) = \lambda + iP, \quad P = \sum_{a,b} e_{ab} \otimes P_{ab}$$ \hspace{1cm} (2.6)

with $P_{ab} \in gl(m|n)$. $L$ is a solution of the equation:

$$R_{12}(\lambda_1 - \lambda_2) L_1(\lambda_1) L_2(\lambda_2) = L_2(\lambda_2) L_1(\lambda_1) R_{12}(\lambda_1 - \lambda_2)$$ \hspace{1cm} (2.7)

with $R$ being the matrix above (2.1). The algebra defined by (2.7) is equipped with a co product: let $L(\lambda) = \sum_{i,j} e_{ij} \otimes l_{ij}(\lambda)$

$$\Delta(L(\lambda)) = L_{02}(\lambda) L_{01}(\lambda) \Rightarrow \Delta(l_{ij}(\lambda)) = \sum_j l_{ij}(\lambda) \otimes l_{ij}(\lambda).$$ \hspace{1cm} (2.8)

Define also the opposite co product. Let $\Pi$ be the ‘shift operator’ $\Pi : V_1 \otimes V_2 \leftrightarrow V_2 \otimes V_1$

$$\Delta' = \Pi \circ \Delta.$$ \hspace{1cm} (2.9)

in particular

$$\Delta'(L(\lambda)) = L_{01}(\lambda)L_{02}(\lambda) \Rightarrow \Delta'(l_{ij}(\lambda)) = \sum_j (-1)^{[i][j]} l_{ij}(\lambda) \otimes l_{ij}(\lambda).$$ \hspace{1cm} (2.10)

The $L$ co-products are derived by iteration as:

$$\Delta^{(L)} = (\text{id} \otimes \Delta^{(L-1)}) \Delta, \quad \Delta'^{(L)} = (\text{id} \otimes \Delta^{(L-1)}) \Delta'.$$ \hspace{1cm} (2.11)
Let us now define the super commutator as

\[
\left\{ A, \ B \right\} = AB - (-1)^{[A][B]} AB.
\]  
(2.12)

It is easy to show from (2.7) that \( P_{ab} \) satisfy the \( gl(m|n) \) algebra, which reads as

\[
\begin{align*}
\left\{ P_{ij}, \ P_{kl} \right\} &= 0, \quad k \neq j, \quad i \neq l \\
\left\{ P_{ij}, \ P_{kj} \right\} &= (-1)^{[i][j]} P_{ki}, \quad k \neq j \\
\left\{ P_{ij}, \ P_{jl} \right\} &= -(1)^{[i][j]+[i][l]} P_{il} \quad i \neq l \\
\left\{ P_{ij}, \ P_{ji} \right\} &= (-1)^{[i]}(P_{jj} - P_{ii}).
\end{align*}
\]  
(2.13)

### 2.1 The reflection algebra

This section serves mostly as a warm up, although some alternative proofs for the symmetries are provided, and explicit expressions of quadratic Casimir operators are also given. Consider now the situation of a boundary integrable system described by the reflection equation [16, 17], which also provides the exchange relations of the underlying algebra, i.e. the reflection algebra

\[
R_{12}(\lambda_1 - \lambda_2) \ K_1(\lambda_1) \ R_{21}(\lambda_1 + \lambda_2) \ K_2(\lambda_2) = K_2(\lambda_2) \ R_{12}(\lambda_1 + \lambda_2) \ K_1(\lambda_1) \ R_{21}(\lambda_1 - \lambda_2)
\]  
(2.14)

As shown in [17] a tensorial type representation of the reflection algebra is given by:

\[
T(\lambda) = T(\lambda) \ K(\lambda) \ \hat{T}(\lambda)
\]  
(2.15)

where we define

\[
\hat{T}(\lambda) = T^{-1}(-\lambda), \text{ and } T(\lambda) = \Delta^{(N+1)}(L) = L_0 N(\lambda) \ldots L_0 1(\lambda)
\]  
(2.16)

\( K \) is a \( c \)-number solution of the reflection equation.

The associated transfer matrix is defined as

\[
t(\lambda) = str \{ K^+(\lambda) \ \mathbb{T}(\lambda) \}
\]  
(2.17)

\( K^+ \) is also a solution of the reflection equation, and

\[
\left[ t(\lambda), \ t(\lambda') \right] = 0.
\]  
(2.18)
In the special case where $K^+ = K = I$ the transfer matrix enjoys the full $gl(m|n)$ symmetry. Here we shall provide an explicit proof based on the linear relations satisfied by the algebra co products and the matrix $T$. The proof of this statement goes as follows: let us first recall the co-product of the $gl(m|n)$ elements

$$
\Delta(\mathbb{P}_{ab}) = I \otimes \mathbb{P}_{ab} + \mathbb{P}_{ab} \otimes I.
$$

Define also the following representation $\pi : gl(m|n) \hookrightarrow \operatorname{End}(\mathbb{C}^{n+m})$ such that $\pi(\mathbb{P}_{ab}) = P_{ab}$.

The $N + 1$ co-product satisfies the following commutation relations with the monodromy matrix

$$
(\pi \otimes \text{id}^{\otimes N})\Delta^{(N+1)}(\mathbb{P}_{ab})\ T(\lambda) = T(\lambda) \ (\pi \otimes \text{id}^{\otimes N})\Delta^{(N+1)}(\mathbb{P}_{ab}).
$$

The later relations may be written in a more straightforward form as:

$$
\left(P_{ab} \otimes I + I \otimes \Delta^{(N)}(\mathbb{P}_{ab})\right)\ T(\lambda) = T(\lambda) \left(P_{ab} \otimes I + I \otimes \Delta^{(N)}(\mathbb{P}_{ab})\right).
$$

If we now express $T(\lambda) = \sum_{i,j} e_{ij} \otimes T_{ij}(\lambda)$ then the latter relations become

$$
\sum_j (-1)^{[b]} e_{aj} \otimes T_{bj}(\lambda) + \sum_i (-1)^{([a]+[b])} e_{ij} \otimes \Delta^{(N)}(\mathbb{P}_{ab})T_{ij}(\lambda)
$$

$$
= \sum_i (-1)^{[b]+([a]+[b])} e_{ib} \otimes T_{ia}(\lambda) + \sum_{i,j} e_{ij} \otimes T_{ij}(\lambda)\Delta^{(N)}(\mathbb{P}_{ab}).
$$

We are however interested in the super-trace over the auxiliary space, so we are dealing basically with the diagonal terms of the above equation, hence we obtain the following exchange relations:

$$
\left[T_{ii}(\lambda), \Delta^{(N)}(\mathbb{P}_{ab})\right] = 0, \quad i \neq a, \ i \neq b
$$

$$
\left[T_{aa}(\lambda), \Delta^{(N)}(\mathbb{P}_{ab})\right] = (-1)^{[a]} T_{ba}(\lambda), \quad \left[T_{bb}(\lambda), \Delta^{(N)}(\mathbb{P}_{ab})\right] = -(-1)^{[a]} T_{ba}(\lambda).
$$

By taking now the super-trace (we are considering $K^+ = I$) we have

$$
\sum_i (-1)^{[i]} T_{ii}(\lambda), \Delta^{(N)}(\mathbb{P}_{ab}) = \left[(-1)^{[a]} T_{aa}(\lambda) + (-1)^{[b]} T_{bb}, \Delta^{(N)}(\mathbb{P}_{ab})\right] = \ldots = 0
$$

\Rightarrow \left[t(\lambda), \Delta^{(N)}(\mathbb{P}_{ab})\right]= 0

$$
(2.25)
$$
and consequently:

\[
\begin{bmatrix}
t(\lambda), & gl(m|n)
\end{bmatrix} = 0. \tag{2.26}
\]

Recall that here we are focusing on the distinguished Dynkin diagram (1.6). Consider now the non-trivial situation where the \(K\)-matrix has the following diagonal form (see also [4]).

\[
K(\lambda) = \text{diag}(1, \ldots, 1, -1, \ldots, -1, 1, \ldots, 1) \tag{2.27}
\]

such that \(m = m_1 + m_2, \ n = n_1 + n_2\). In general, any solution [4] may be written in the form \(K(\lambda) = i\xi + \lambda \mathcal{E}, \ \mathcal{E}^2 = \mathbb{I}\). \(\mathcal{E}\) may be diagonalized into (2.27) and that is why we make this convenient choice for the \(K\) matrix (2.27), we also chose for simplicity \(\xi = 0\).

Let us first extract the non-local charges for any generic \(K\)-matrix of the form:

\[
K(\lambda) = K + \frac{1}{\lambda} \xi_1 + \frac{1}{\lambda^2} \xi_2 + \mathcal{O}(\frac{1}{\lambda^3}), \tag{2.28}
\]

(see also [18] for a brief discussion on the symmetry). From the asymptotic behavior of the dynamical \(T\) we then obtain:

\[
T(\lambda \to \infty) \sim K + \frac{i}{\lambda} Q^{(0)} - \frac{1}{\lambda^2} Q^{(1)} + \ldots \tag{2.29}
\]

The first order quantity provides the generators of the remaining boundary symmetry,

\[
Q^{(0)} = \Delta^{(N)}(\mathbb{P}) K + K \Delta^{(N)}(\mathbb{P}) + \xi_1 \tag{2.30}
\]

and for the special choice of \(K\)-matrix (2.27) we conclude

\[
Q^{(0)} = \sum_{i,j=1}^{m_1} e_{ij} \otimes \Delta^{(N)}(\mathbb{P}_{ij}) + \sum_{i,j=m+n_2}^{m+n} e_{ij} \otimes \Delta^{(N)}(\mathbb{P}_{ij}) - \sum_{i,j=1}^{m+n_2} e_{ij} \otimes \Delta^{(N)}(\mathbb{P}_{ij}) + \sum_{i=1}^{m_1} \sum_{j=m+n_2+1}^{m+n} e_{ij} \otimes \Delta^{(N)}(\mathbb{P}_{ij}) + \sum_{i=m+n_2+1}^{m+n} \sum_{j=1}^{m_1} e_{ij} \otimes \Delta^{(N)}(\mathbb{P}_{ij}). \tag{2.31}
\]

More precisely, the elements:

\[
\mathbb{P}_{ij}, \ i, j \in (1, m_1) \cup (m + n_2 + 1, m + n) \quad \text{form the} \quad gl(m_1|n_1)
\]

\[
\mathbb{P}_{ij}, \ i, j \in (m_1 + 1, m + n_2) \quad \text{form the} \quad gl(m_2|n_2). \tag{2.32}
\]
These are exactly the generators that, in the fundamental representation, commute with the $K$ matrix (2.27).

That is the $gl(m|n)$ symmetry breaks down to $gl(m_1|n_1) \otimes gl(m_2|n_2)$. Since the $K$-matrix commutes with all the above generators following the procedure above we can show relations (2.21) but only with the generators (2.32) and finally:

$$\left[ t(\lambda), \; gl(m_1|n_1) \otimes gl(m_2|n_2) \right] = 0.$$  \hfill (2.33)

We shall focus now for simplicity on the ‘one-particle’ representation $N = 1$. The trace of the second order quantity provides the quadratic Casimir associated to the $gl(m|n)$ in the case where $K = 1$. When $K$ is of the diagonal form (2.27) the Casimir (see also [19, 20]) is associated to $gl(m_1|n_1) \otimes gl(m_2|n_2)$.

More specifically, for $K \propto I$ (set $N = 1$):

$$Q^{(1)} = 2P^2 \quad \text{and} \quad C = str Q^{(1)} = 2 \sum_{i,j=1}^{m+n} (-1)^j P_{ij} P_{ji} \quad (2.34)$$

and for $K$ given by the diagonal matrix (2.28)

$$Q^{(1)} = PKP + K^2 P^2 - iP_1 - i \xi_1 - P_2 \quad \text{and} \quad C = str Q^{(1)}. \quad (2.35)$$

For the special choice of $K$-matrix (2.27) we have:

$$C = \sum_{i=1}^{m_1} \left( \sum_{j=1}^{m_1} (-1)^j P_{ij} P_{ji} \right) \sum_{i=m+n_2}^{m+n} (-1)^j P_{ij} P_{ji} \right) + \sum_{j=1}^{m_1} \left( \sum_{i=1}^{m_1} (-1)^j P_{ij} P_{ji} \right) + \sum_{i=m+n_2}^{m+n} (-1)^j P_{ij} P_{ji} \right) - \sum_{i,j=m_1+1}^{m+n_2} (-1)^j P_{ij} P_{ji}. \quad (2.36)$$

Higher Casimir operators may be extracted by considering the higher order terms in the expansion of the transfer matrix in powers of $\frac{1}{\lambda}$. More precisely, let us focus on the $N = 1$ case and see more precisely how one obtains the higher Casimir operators from the expansion of the transfer matrix $t(\lambda) = \sum_{k=1}^{2N} \frac{i^{(k-1)}}{\lambda^k}$. Recall the $N = 1$ representation of the reflection algebra

$$T(\lambda) = L(\lambda) \cdot \hat{L}(\lambda) = (1 + \frac{i}{\lambda} P) \cdot (1 + \frac{i}{\lambda} P - \frac{1}{\lambda^2} P^2 - \frac{i}{\lambda^3} P^3 + \frac{1}{\lambda^4} P^4 \ldots)

= \hat{K} + \frac{i}{\lambda} (P \hat{K} + \hat{K} P) - \frac{1}{\lambda^2} (P \hat{K} P + k P^2) - \frac{1}{\lambda^3} (P \hat{K} P^2 + k P^3) \ldots \quad (2.37)$$
where $k$ is diagonal then

$$
t^{(k-1)} \propto \sum_{a,b} (P_{ab} P_{ba}^{k-1} + k_{aa} P_{aa}^k)
$$

(2.38)

All $t^{(k)}$ are the higher Casimir quantities and by construction they commute with each other and they commute as shown earlier with the exact symmetry of the system. Depending on the rank of the considered algebra the expansion of $t(\lambda)$ should truncate at some point; note that expressions (2.37), (2.38) are generic and hold for any $gl(m|n)$. The spectra of all Casimir operators associated to a specific algebra may be derived via the Bethe ansatz methodology. In particular, the spectrum for generic representations of super symmetric algebras is known (see e.g. [21]). By appropriately expanding the eigenvalues in powers of $\frac{1}{\lambda}$ we may identify the spectrum of each one of the relevant Casimir operators.

### 2.2 The twisted super Yangian

Note that we focus here in the $gl(m|2k)$ case and the symmetric Dynkin diagramm (1.7). Let us first define some basic notation useful or our purposes here. Consider the matrix

$$V = \sum_i f_i e_{\bar{i}}, \quad \text{where,} \quad \bar{i} = m + 2k - i + 1.
$$

(2.39)

More precisely we shall consider here the following anti-diagonal matrix:

$$V = \text{antidiag}(1, \ldots, 1, -1, \ldots, -1).$$

(2.40)

The twisted super Yangian defined by [22, 23, 24] (for more details on the physical meaning of reflection algebra and twisted Yangian see [25, 4]):

$$R_{12}(\lambda_1 - \lambda_2) K_1(\lambda_1) \bar{R}_{12}(\lambda_1 - \lambda_2) \bar{K}_2(\lambda_2) = \bar{K}_2(\lambda_2) \bar{R}_{12}(\lambda_1 - \lambda_2) K_1(\lambda_1) R_{12}(\lambda_1 - \lambda_2)$$

(2.41)

the matrix $\bar{R}_{12}$ is defined in [223].

Define also

$$\hat{L}_{0n}(\lambda) = L_{0n}^t(-\lambda - i\rho) \propto 1 + \frac{i}{\lambda} \hat{P}_{0n}, \quad \text{where} \quad \hat{P}_{0n} = \rho - P_{0n}^t.
$$

(2.42)

Consider now the generic tensorial representation of the twisted super Yangian:

$$\hat{T}(\lambda) = T(\lambda) K(\lambda) T^t(-\lambda - i\rho).
$$

(2.43)
$K$ is a $c$-number solution of the twisted Yangian (2.41).

As in the previous section express the above tensor representation in powers of $\frac{1}{\lambda}$:

$$\mathbb{T}(\lambda) = 1 + \frac{i}{\lambda} (\bar{Q}^{(0)} + N\rho) - \frac{1}{\lambda^2} \bar{Q}^{(1)} + \ldots$$  \hspace{1cm} (2.44)

where

$$\bar{Q}^{(0)} = \Delta^{(N)}(P) - \Delta^{(N)}(P^{t_0}).$$  \hspace{1cm} (2.45)

It is clear that the elements $\bar{Q}_{ab}$ form the $osp(m|2n)$ algebra, and this corresponds essentially to a folding of the $gl(m|2n)$ to $osp(m|2n)$. Such a folding occurs in the corresponding symmetric Dynkin diagram. Henceforth, we shall consider the simplest solution $K \propto \mathbb{I}$, although a full classification is presented in [4]. Based on the same logic as in the previous paragraph we may extract the corresponding exact symmetry. We have in this case:

$$(\pi \otimes \text{id}^{\otimes N}) \Delta^{(N+1)}(\bar{Q}_{ab}^{(0)}) \mathbb{T}(\lambda) = \mathbb{T}(\lambda) (\pi \otimes \text{id}^{\otimes N}) \Delta^{(N+1)}(\bar{Q}_{ab}^{(0)}),$$  \hspace{1cm} (2.46)

which leads to

$$\left[ t(\lambda), \text{osp}(m|2n) \right] = 0,$$  \hspace{1cm} (2.47)

so the exact symmetry of the considered transfer matrix is indeed $osp(m|2n)$.

The quadratic Casimir operator associated to $osp(m|n)$ emerges from the super-trace of $\bar{Q}^{(1)}$ ($N = 1$):

$$\bar{Q}^{(1)} = P \hat{P} \quad C = \text{str} \bar{Q}^{(1)} = \sum_{i,j} (-1)^{|j|} P_{ij} \hat{P}_{ji}. $$  \hspace{1cm} (2.48)

### 3 The $U_q(gl(m|n))$ algebra

We come now to the $q$ deformed situation. The $R$-matrix associated to the $U_q(gl(m|n))$ algebra is given by the following expressions [26]:

$$R(\lambda) = \sum_{i=1}^{m+n} a_i(\lambda) e_{ii} \otimes e_{ii} + b(\lambda) \sum_{i \neq j=1}^{m+n} e_{ii} \otimes e_{jj} + \sum_{i \neq j=1}^{m+n} c_{ij}(\lambda) e_{ij} \otimes e_{ji},$$  \hspace{1cm} (3.1)

where we define

$$a_j(\lambda) = \sinh(\lambda + i\mu - 2i\mu[j]), \quad b(\lambda) = \sinh \lambda, \quad c_{ij}(\lambda) = \sinh(i\mu)e^{\text{sign}(j-i)\lambda}(-1)^{|j|}. $$  \hspace{1cm} (3.2)
Let us now introduce the supersymmetric Lax operator associated to $U_q(gl(m|n))$

$$L(\lambda) = e^\lambda L^+ - e^{-\lambda} L^-$$  \hspace{1cm} (3.3)

and $L$ satisfies the fundamental algebraic relation (2.7) with the $R$ matrix given in (3.1). The elements $L^\pm$ satisfy [27, 28]

$$R_{12}^\pm L^+_1 L^+_2 = L^+_2 L^+_1 R_{12}^+, \quad R_{12}^\pm L^-_1 L^-_2 = L^-_2 L^-_1 R_{12}^+, \quad R_{12}^\pm L^\pm_1 L^\pm_2 = L^\pm_2 L^\pm_1 R_{12}^+$$  \hspace{1cm} (3.4)

$L^\pm$ are expressed as:

$$L^+ = \sum_{i \leq j} e_{ij} \otimes l^+_{ij}, \quad L^- = \sum_{i \geq j} e_{ij} \otimes l^-_{ij}$$  \hspace{1cm} (3.5)

definitions of $l^\pm_{ij}$, $\tilde{l}^\pm_{ij}$ in terms of the $U_q(gl(m|n))$ algebra generators in the Chevalley-Serre basis are given in Appendix A. The equations above (3.4) provide all the exchange relations of the $U_q(gl(m|n))$ algebra. This is in fact the so called FRT realization of the $U_q(gl(m|n))$ algebra (see e.g. [27]).

### 3.1 The reflection algebra

Our main objective here is to extract the exact symmetry of the open transfer matrix associated to $U_q(gl(m|n))$. We shall focus in this section on the distinguished Dynkin diagram. The open transfer matrix is given by (2.17), and from now on we consider for our purposes here the left boundary to be the trivial solution

$$K^+ = M = \sum_{k=1}^{m+n} q^{n+m-2k+1} q^{-2[k]+4 \sum_{i=1}^{k} i} e_{kk}.$$  \hspace{1cm} (3.6)

The elements one extracts from the asymptotic expansion of $T$ by keeping the leading contribution, $T^\pm_{ab}$ are the boundary non-local charges, which form the boundary super algebra with exchange relations dictated by:

$$R_{12}^+ T_1^+ R_{21}^+ T_2^+ = T_2^+ R_{21}^+ T_1^+ R_{12}^+.$$  \hspace{1cm} (3.7)
In general, it may be shown that the boundary super algebra is an exact symmetry of the double row transfer matrix. Indeed by introducing the element

$$\tau^{\pm} = \text{str}(e_{ab} T^{\pm}) = (-1)^{[a]} T^{\pm}_{ba}$$

(3.8)

it is quite straightforward to show along the lines described in [8] that

$$[\tau^{\pm}, t(\lambda)] = 0 \Rightarrow [T^{\pm}_{ab}, t(\lambda)] = 0.$$  

(3.9)

Let

$$T^{\pm} = \sum_{ab} e_{ab} \otimes T^{\pm}_{ab}, \quad \hat{T}^{\pm} = \sum_{ab} e_{ab} \otimes \hat{T}^{\pm}_{ab}$$

and

$$T^{\pm}_{ab} = \Delta^{(N)}(l^{\pm}_{ab}), \quad \hat{T}^{\pm}_{ab} = \Delta^{(N)}(\hat{l}^{\pm}_{ab}),$$

(3.10)

see also Appendix A for definitions of $l^{\pm}_{ij}$, $\hat{l}^{\pm}_{ij}$.

The boundary non-local charges emanating from $T^{\pm}$ are of the explicit form:

$$T^{\pm}_{ad} = \sum_{b,c} (-1)^{([a]+[b])([b]+[d])} T^{\pm}_{ab} K^{\pm}_{bc} \hat{T}^{\pm}_{cd}.$$  

(3.11)

It is clear that different choice of $K$-matrix leads to different non-local charges and consequently to different symmetry. Also, the quadratic Casimir operators are obtained by

$$C^{\pm} = \text{str}_0(\mathcal{M}_0 T^{\pm}_0) = \sum_{a,b,c} (-1)^{[b]} M_{aa} T^{\pm}_{ab} K^{\pm}_{bc} \hat{T}^{\pm}_{ca}.$$  

(3.12)

Explicit expressions for the quadratic Casimir operators will be given below for particular simple examples. Note that higher Casimir operators may be extracted from the higher order terms in the expansion of the transfer matrix in powers of $e^{\pm 2\lambda}$.

### 3.1.1 Diagonal reflection matrices

We shall distinguish two main cases of diagonal matrices, and we shall examine the corresponding exact symmetry. First consider the simplest boundary conditions described by $K \propto \mathbb{I}$, we shall explicitly show that the associated symmetry is the $U_q(gl(m|n))$. One could just notice that the resulted $T^{\pm}$ form essentially the $U_q(gl(m|n))$, but this is not quite obvious. Then via (3.9) one can show the exact symmetry of the transfer matrix. Let us however here follow a more straightforward and clean approach regarding the symmetry,
that is we shall show that the open transfer matrix commutes with each one of the algebra
generators. Our proofs hold for generic non-trivial integrable boundary conditions as will be
more transparent subsequently. In fact, in this and the previous section we prepare some-
how the general algebraic setting so that we may extract the symmetry for generic integrable
boundary conditions.

Let us now set the basic algebraic machinery necessary for the proofs that follow. Our
aim now is to show that the double row transfer matrix with trivial bou
ndary conditions
that is
\[ K = I, \quad K^{(L)} = M \]
enjoys the full \( U_q(gl(m|n)) \) symmetry. We shall show in particular
that the open transfer matrix commutes with each one of the generators of \( U_q(gl(m|n)) \). Let
us outline the proof; first it is quite easy to show from the form of the co product for \( \epsilon_i \) and
following the logic described in the previous section see equation (2.21) that
\[ [\Delta^{(N)}(\epsilon_i), t(\lambda)] = 0. \quad (3.13) \]

We may now show the commutation between the transfer matrix and the other elements of
the super algebra. Introduce the representation \( \pi : U_q(gl(m|n)) \rightarrow \text{End}(\mathbb{C}^N) \)
\[ \pi(e_i) = \varepsilon^i, \quad \pi(f_i) = \phi^i, \quad \pi(q^{h_i^+}) = k^i. \quad (3.14) \]

Consider also the following more convenient notation
\[ \Delta^{(N)}(q^{\pm h_i^+}) \equiv (K_N^i)^{\pm 1}, \quad \Delta^{(N)}(e_i) \equiv E_N^i, \quad \Delta^{(N)}(f_i) \equiv F_N^i. \quad (3.15) \]

We shall now make use of the generic relations, which clearly \( T \) satisfies due to the particular
choice of boundary conditions (see also [7]):
\[ (\pi \otimes \text{id}^\otimes N)\Delta^{(N+1)}(Y) \ T(\lambda) = T(\lambda) \ (\pi \otimes \text{id}^\otimes N)\Delta^{(N+1)}(Y), \quad Y \in U_q(gl(m|n)). \quad (3.16) \]

Let us restrict our proof to \( e_i \) although the same logic follows for proving the commutation
of the transfer matrix with \( f_i \). In addition to the algebra exchange relations, presented in
Appendix A, we shall need for our proof the following relations:
\[ M e_i = q^{a_{ii}} e_i M, \quad M f_i = q^{-a_{ii}} f_i M \]
\[ (k_0^i K_N^i)^{\pm 1} \ T_0(\lambda) = T_0(\lambda) \ (k_0^i K_N^i)^{\pm 1}. \quad (3.17) \]
It is convenient to rewrite the relation above for $e_i$ as follows:

$$(k^i_0 E^i_N + \varepsilon^i_0 (K^i_N)^{-1}) T_0(\lambda) = T_0(\lambda) (k^i_0 E^i_N + \varepsilon^i_0 (K^i_N)^{-1}) \ldots \Rightarrow$$

$$E^i_N M_0 T_0(\lambda) + q \frac{2i}{2} \varepsilon^i_0 M_0 T_0(\lambda) (k^i_0)^{-1} (K^i_N)^{-1} = (k^i_0)^{-1} M_0 T_0(\lambda) k^i_0 E^i_N + (k^i_0)^{-1} M_0 T_0(\lambda) \varepsilon^i_0 (K^i_N)^{-1},$$

(3.18)

where the subscript 0 denotes the auxiliary space, whereas the $N$ co products leave exclusively on the quantum space, recall also that $\varepsilon^i \propto e_{ii+1}$. By focusing only on the diagonal contributions over the auxiliary space, after some algebraic manipulations, and bearing in mind (3.18), we end up to the following exchange relations (recall $M$ is diagonal):

$$[E^i_N, M_{aa} T_{aa}(\lambda)] = 0, \quad a \neq i, \ i + 1$$

$$[E^i_N, M_{ii} T_{ii}(\lambda)] = -\rho_i M_{i+1i+1} T_{i+1i+1}(\lambda) (K^i_N)^{-1}$$

$$[E^i_N, M_{i+1i+1} T_{i+1i+1}(\lambda)] = (-1)^{[i][i+1]} \rho_i M_{i+1i+1} T_{i+1i+1}(\lambda) (K^i_N)^{-1}$$

(3.19)

$\rho_i$ is a scalar depending on $a_{ii}$ and the grading. Then based on the relations above we have

$$[E^i_N, \sum_{a=1}^{N} (-1)^{[a]} M_{aa} T_{aa}(\lambda)] = 0 \quad \Rightarrow \quad [E^i_N, t(\lambda)] = 0.$$  

(3.20)

Similarly one may show the commutativity of the transfer matrix with $F^i_N$, hence

$$[\Delta^{(N)}(x), t(\lambda)] = 0, \quad x \in U_q(gl(m|n)),$$

(3.21)

and this concludes our proof on the exact symmetry of the double row transfer matrix for the particular choice of boundary conditions. It is clear that the proof above can be easily applied to the usual non super symmetric deformed algebra. We have not seen, as far as we know, such an explicit and elegant proof elsewhere not even in the non super symmetric case. In [7] the proof is explicit, but rather tedious, whereas in [2] one has to realize that the emanating non local charges form the $U_q(gl(N))$, and this is not quite obvious. Note that $T^+_{ab}$ are quadratic combinations of the algebra generators, and in fact the corresponding super-trace provides the associated Casimir, which again gives a hint about the associated exact symmetry. In the case of trivial boundary conditions ($K \propto I$) there is a discussion on the symmetry in [29], but is restricted only in this particular case, whereas our proof holds for generic non-trivial integrable boundary condition (see also next section).
Let us now give explicit expressions of the quadratic $q$-Casimir for the simplest case (see also \[30\]), that is the $U^q(gl(1|1))$ situation. In general, the quadratic Casimir operator is given by (3.12), but now $K^\pm_{ab} = \delta_{ab}$

\[
\begin{align*}
C^+ &\propto \Delta^{(N)}(q^{2\epsilon_1}) - \Delta^{(N)}(q^{2\epsilon_2}) - (q - q^{-1})^2 \Delta^{(N)}(q^{\frac{\epsilon_1 + \epsilon_2}{2}}) \Delta^{(N)}(f_1) \Delta^{(N)}(q^{\frac{\epsilon_1 + \epsilon_2}{2}}) \Delta^{(N)}(e_1) \\
C^- &\propto \Delta^{(N)}(q^{-\epsilon_1}) - \Delta^{(N)}(-q^{2\epsilon_2}) + (q - q^{-1})^2 \Delta^{(N)}(e_1) \Delta^{(N)}(q^{-\frac{\epsilon_1 + \epsilon_2}{2}}) \Delta^{(N)}(f_1) \Delta^{(N)}(q^{-\frac{\epsilon_1 + \epsilon_2}{2}}).
\end{align*}
\]

(3.22)

Consider diagonal non-trivial solutions of the reflection equation (see also a brief discussion on the symmetry in \[21\]):

\[
K(\lambda) = \text{diag}(a(\lambda), \ldots, a(\lambda), b(\lambda), \ldots, b(\lambda)).
\]

(3.23)

In this case it is clear that the $K$ matrix satisfies (recall (3.14))

\[
[\pi(x), K(\lambda)] = 0 \quad x \in U_q(gl(m - \alpha|n)) \otimes U_q(gl(\alpha)), \quad \text{if } \alpha \text{ bosonic}
\]

\[
x \in U_q(gl(m|\tilde{\alpha})) \otimes U_q(gl(n - \tilde{\alpha})) \quad \text{if } \alpha = m + \tilde{\alpha} \text{ fermionic}
\]

(3.24)

and consequently,

\[
(\pi \otimes \text{id}^{\otimes N})\Delta^{(N+1)}(x) T(\lambda) = T(\lambda) (\pi \otimes \text{id}^{\otimes N})\Delta^{(N+1)}(x)
\]

\[
x \in U_q(gl(m - \alpha|n)) \otimes U_q(gl(\alpha)), \quad \text{if } \alpha \text{ bosonic}
\]

\[
x \in U_q(gl(m|\tilde{\alpha})) \otimes U_q(gl(n - \tilde{\alpha})) \quad \text{if } \alpha = m + \tilde{\alpha} \text{ fermionic}
\]

(3.25)

Thus following the logic described in the case where $K \propto \mathbb{I}$ we show that:

\[
\left[ t(\lambda), U_q(gl(\alpha|n)) \otimes U_q(gl(m - \alpha)) \right], \quad \text{if } \alpha \text{ bosonic}
\]

\[
\left[ t(\lambda), U_q(gl(m|\tilde{\alpha})) \otimes U_q(gl(n - \tilde{\alpha})) \right] \quad \text{if } \alpha = m + \tilde{\alpha} \text{ fermionic}
\]

(3.26)

Consider the simplest non-trivial cases, that is the $U_q(gl(2|1))$ and $U_q(gl(2|2))$ algebras with diagonal $K$ matrices \[3.23\] with $\alpha = 2$. According to the preceding discussion the
\( U_q(gl(2|1)) \) and \( U_q(gl(2|2)) \) symmetries break down to \( U_q(gl(2)) \otimes u(1) \) and \( U_q(gl(2)) \otimes U_q(gl(2)) \) respectively. It is worth presenting the associated Casimir operators for the two particular examples. Consider first the \( U_q(gl(2|1)) \) case, then from the \( \lambda \to \pm \infty \) asymptotic behavior of the open transfer matrix we obtain:

\[
\begin{align*}
C^+ &= q^2 \Delta^{(N)}(q^{\epsilon_1+\epsilon_2}) \left( q \Delta^{(N)}(q^{\epsilon_1-\epsilon_2}) + q^{-1} \Delta^{(N)}(q^{-\epsilon_1+\epsilon_2}) + (q - q^{-1})^2 \Delta^{(N)}(f_1) \Delta^{(N)}(e_1) \right) \\
C^- &= q^2 \Delta^{(N)}(q^{-2\epsilon_3}).
\end{align*}
\]

(3.27)

Notice that all \( \epsilon_i, i = 1, 2, 3 \) elements commute with the transfer matrix and also the parenthesis in the first line of (3.27) is essentially the typical \( U_q(sl_2) \) quadratic Casimir. \( C^- \) is basically a \( u(1) \) type quantity. It is thus clear that \( C^+ \) is the quadratic Casimir associated to \( U_q(gl(2)) \); indeed in this case the \( U_q(gl(2|1)) \) symmetry breaks down to \( U_q(gl(2)) \otimes u(1) \). In general the implementation of boundary conditions described by the diagonal matrix (3.23) with \( \alpha = m \) obviously breaks the super symmetry to \( U_q(gl(m)) \otimes U_q(gl(n)) \), so in fact the super algebra reduces to two non super symmetric quantum algebras. Also, \( C^+ \) is the Casimir associated \( U_q(gl(m)) \) to whereas \( C^- \) is the Casimir associated to \( U_q(gl(n)) \).

This will become more transparent when examining the \( U_q(gl(2|2)) \) case with boundary conditions described by \( K \) (3.23) and \( \alpha = 2 \). The associated Casimir operators are then given by

\[
\begin{align*}
C^+ &= q^2 \Delta^{(N)}(q^{\epsilon_1+\epsilon_2}) \left( q \Delta^{(N)}(q^{\epsilon_1-\epsilon_2}) + q^{-1} \Delta^{(N)}(q^{-\epsilon_1+\epsilon_2}) + (q - q^{-1})^2 \Delta^{(N)}(f_1) \Delta^{(N)}(e_1) \right) \\
C^- &= q^2 \Delta^{(N)}(q^{-2\epsilon_3}) \left( q \Delta^{(N)}(q^{\epsilon_4-\epsilon_3}) + q^{-1} \Delta^{(N)}(q^{-\epsilon_4+\epsilon_3}) + (q - q^{-1})^2 \Delta^{(N)}(f_3) \Delta^{(N)}(e_3) \right).
\end{align*}
\]

(3.28)

As expected in this case, since now the symmetry is broken to \( U_q(gl_2) \otimes U_q(gl_2) \), the \( C^+ \) Casimir is associated to the one \( U_q(gl_2) \) symmetry whereas the \( C^- \) is associated to the other \( U_q(gl_2) \).

### 3.1.2 Non-diagonal reflection matrices

Let us finally consider non-diagonal reflection matrices. A new class of non-diagonal reflection matrices associated to \( U_q(gl m|n) \) was recently derived in [31]. Specifically, first define
the conjugate index \( \bar{a} \) such that: \([a] = [\bar{a}]\), and more specifically:
\[
\bar{a} = 2k + m + 1 - a; \quad \text{Symmetric diagram}
\]
\[
\bar{a} = m + 1 - a, \quad a \text{ bosonic; } \bar{a} = 2m + n + 1 - a, \quad a \text{ fermionic; } \text{Distinguished diagram.}
\]

Then the non diagonal matrices read as:

\[\text{A. Symmetric Dynkin diagram:}\]
\[
K_{\bar{a}a}(\lambda) = e^{2\lambda} \cosh i\mu m - \cosh 2i\mu \zeta, \quad K_{\bar{a}\bar{a}}(\lambda) = e^{-2\lambda} \cosh i\mu m - \cosh 2i\mu \zeta,
\]
\[
K_{a\bar{a}}(\lambda) = ic_a \sinh 2\lambda, \quad K_{\bar{a}a}(\lambda) = ic_a \sinh 2\lambda, \quad 1 \leq a \leq L
\]
\[
K_{aa}(\lambda) = K_{\bar{a}\bar{a}}(\lambda) = \cosh(2\lambda + i\mu \tilde{\zeta}) - \cosh 2i\mu \zeta, \quad K_{a\bar{a}}(\lambda) = K_{\bar{a}a}(\lambda) = 0, \quad L < a \leq \frac{m + 2k}{2};
\]
\[
1 \leq L \leq \frac{m + 2k}{2}
\]
\[
K_{AA} = \cosh(2\lambda + i\mu \tilde{\zeta}) - \cosh 2i\mu \zeta, \quad A = \frac{m + 2k + 1}{2} \quad \text{if } m \text{ odd.} \tag{3.30}
\]

\[\text{B. Distinguished Dynkin diagram:}\]

bosonic:
\[
K_{aa}(\lambda) = e^{2\lambda} \cosh i\mu m - \cosh 2i\mu \zeta, \quad K_{\bar{a}\bar{a}}(\lambda) = e^{-2\lambda} \cosh i\mu m - \cosh 2i\mu \zeta,
\]
\[
K_{a\bar{a}}(\lambda) = ic_a \sinh 2\lambda, \quad K_{\bar{a}a}(\lambda) = ic_a \sinh 2\lambda, \quad 1 \leq a \leq L
\]
\[
K_{aa}(\lambda) = K_{\bar{a}\bar{a}}(\lambda) = \cosh(2\lambda + i\mu \tilde{\zeta}) - \cosh 2i\mu \zeta, \quad K_{a\bar{a}}(\lambda) = K_{\bar{a}a}(\lambda) = 0; \quad L < a \leq \frac{m}{2}
\]
\[
1 \leq L \leq \frac{m}{2}
\]
\[
K_{AA} = \cosh(2\lambda + i\mu \tilde{\zeta}) - \cosh 2i\mu \zeta, \quad A = \frac{m + 1}{2} \quad \text{if } m \text{ odd,}
\]

and
\[
K_{aa}(\lambda) = K_{\bar{a}\bar{a}}(\lambda) = \cosh(2\lambda + i\mu \tilde{\zeta}) - \cosh 2i\mu \zeta, \quad K_{a\bar{a}}(\lambda) = K_{\bar{a}a}(\lambda) = 0, \quad a > m,
\]

\[
(3.31)
\]
fermionic:

\[ K_{aa}(\lambda) = e^{2\lambda} \cosh i\mu m - \cosh 2i\mu \zeta, \quad K_{\bar{a}a}(\lambda) = e^{-2\lambda} \cosh i\mu m - \cosh 2i\mu \zeta, \]
\[ K_{\bar{a}\bar{a}}(\lambda) = i c_a \sinh 2\lambda, \quad K_{a\bar{a}}(\lambda) = ic_a \sinh 2\lambda, \]

\[ m + 1 \leq a \leq L \]

\[ K_{AA} = \cosh(2\lambda + im\mu) - \cosh 2i\mu \zeta, \quad A = m + \frac{n + 1}{2} \quad \text{if } n \text{ odd}, \]

and \[ K_{aa}(\lambda) = K_{\bar{a}a}(\lambda) = \cosh(2\lambda + im\mu) - \cosh 2i\mu \zeta, \quad K_{\bar{a}\bar{a}}(\lambda) = K_{a\bar{a}}(\lambda) = 0, \quad a \leq m. \] (3.32)

\( m, \zeta \) are free boundary parameters.

Note that for the distinguished Dynkin diagram we have purely bosonic or purely fermionic reflection matrices and not mixed. Let us first focus on the solution with the minimal number of non-zero entries, and let us consider the symmetric case. The elements \( T_{ij}^\pm, i, j \in \{2, \ldots, 2k + m - 1\} \) form essentially the \( U_q(gl(m|2(k - 1))) \) algebra, and it can be shown, based on the logic described in the previous section, that the transfer matrix commutes with \( U_q(gl(m|2(k - 1))) \) plus the elements \( T_{ij}^\pm \cup T_{ji}^\pm \cup T_{Xj}^\pm \cup T_{jX}^\pm \) \( (N = m + 2k) \). Let us now deal with the generic situation described by the solutions presented above (3.31)–(3.32). First define: \( U = T_{Aj}^\pm \cup T_{jA}^\pm \cup T_{Aj}^\pm \cup T_{jA}^\pm = 0, \quad A \in \{1, \ldots, L\} \). Then it is shown for the generic non-diagonal \( K \)-matrix:

**Symmetric Dynkin diagram**

\[
\begin{bmatrix}
 t(\lambda), & U_q(gl(m|2(k - L))) \otimes U
\end{bmatrix} = 0 \quad \text{bosonic solution,}
\]

\[
\begin{bmatrix}
 t(\lambda), & U_q(gl(m - 2l)) \otimes U
\end{bmatrix} = 0 \quad L = k + l, \quad \text{fermionic solution.} \quad (3.33)
\]

**Distinguished Dynkin diagram**

\[
\begin{bmatrix}
 t(\lambda), & U_q(gl(m - 2L)) \otimes U_q(gl(n)) \otimes U
\end{bmatrix} = 0, \quad \text{bosonic solution,}
\]

\[
\begin{bmatrix}
 t(\lambda), & U_q(gl(m)) \otimes U_q(gl(n - 2l)) \otimes U
\end{bmatrix} = 0, \quad L = m + l, \quad \text{fermionic solution}. \quad (3.34)
\]

The generic Casimir is given by (3.12), where now the only non-zero entries are given by: \( K_{aa}, K_{\bar{a}a} \)
It is clear that different choices of non-diagonal reflection matrices lead to distinct preserving symmetries. This may perhaps be utilized together with the contraction process presented in [10] to offer an algebraic description regarding of the underlying algebra emerging in the AdS/CFT context. Recall that this type of contractions leads to centrally extended algebras, and as in known in the context of AdS/CFT we deal basically with a centrally extended $gl(2|2)$ algebra [32].

The explicit form of the boundary nonlocal charges $T_{ab}^{(3.11)}$ in addition to the existence of some familiar symmetry is essential, and may be for instance utilized for deriving reflection matrices associated to higher representations of $U_q(gl(m|n))$ (see e.g. [33]). In fact, the logic we follow here is rather two-fold: on the one hand we try to extract a familiar symmetry algebra if any, and following the process of section 3.1.1 to derive the exact symmetry. On the other hand for generic $K$ that may break all familiar symmetries we show via the reflection equation that the boundary non-local charges form an algebra, and via (3.9) we show that provide an extra symmetry for the open transfer matrix. Moreover, the knowledge of the explicit form of the boundary non-local charges is of great significance given that they may be used, as already mentioned, for deriving reflection matrices for arbitrary representations [33].

3.2 The q twisted super Yangian

To complete our analysis on the boundary super symmetric algebras we shall now briefly discuss the $q$ twisted Yangian. A more detailed analysis together with the classification of the corresponding $c$-number solutions will be pursued elsewhere. As in the case of the twisted Yangian we focus on the symmetric Dynkin diagram (1.7) and introduce

\[ V = \sum_i f_i e_{\bar{i}} : \quad V^T V = M \]  \hspace{1cm} (3.35)

and define the super transposition as in the rational case. Also define the following matrices:

\[ \tilde{R}_{12}(\lambda) := R_{21}^{(1)}(-\lambda - i\rho), \quad \tilde{R}_{21}(\lambda) := R_{12}^{(1)}(-\lambda - i\rho). \]  \hspace{1cm} (3.36)

Recall that in the isotropic case $R_{12} = R_{21}$.

Then the $q$-Twisted Yangian is defined by:

\[ R_{12}(\lambda_1 - \lambda_2)K_1(\lambda_1)\tilde{R}_{21}(\lambda_1 + \lambda_2)K_2(\lambda_2) = K_2(\lambda_2)\tilde{R}_{12}(\lambda_1 + \lambda_2)K_1(\lambda_1)R_{21}(\lambda_1 - \lambda_2). \]  \hspace{1cm} (3.37)
The non-local charges are derived from the asymptotic behavior of the tensor representation of the $q$-twisted Yangian:

$$T_{ab}^{\pm} = \sum_k (-1)^{([a]+[k])([k]+[b])} T_{ak}^\pm T_{kb}^\mp.$$  \hspace{1cm} (3.38)

As in the non super symmetric case the non local charges derived above satisfy exchange relations of the type (see also e.g. [9])

$$R_{12}^\pm T_{12}^\pm = T_{21}^\pm R_{21}^\pm$$ \hspace{1cm} (3.39)

it turns out that they do not provide an exact symmetry of the transfer matrix [9], as opposed to the isotropic limit described in section 2.2. And although in the context of discrete integrable models described by the double row transfer matrix the boundary non-local charges do not form an exact symmetry one may show that in the corresponding field theoretical context they do provide exact symmetries (see e.g. [3]). Such investigations regarding super symmetric field theories will be left however for future investigations. Note finally that the classification of solutions of the $q$ twisted Yangian for the $U_q(gl(m|n))$ is still an open question, which we hope to address in a forthcoming publication.

**A Appendix**

We shall recall here some details regarding the $U_q(gl(m|n))$ algebra. The algebra is defined by generators $\epsilon_i, e_j f_j, i = 1, \ldots, \mathcal{N}, \ j = 1, \ldots, \mathcal{N} - 1$, and the exchange relations of the $U_q(gl(m|n))$ algebra are given below:

$$q^{\epsilon_i} q^{-\epsilon_i} = q^{-\epsilon_i} q^{\epsilon_i} = 1$$

$$q^{\epsilon_i} e_j = q^{(-1)^{[j]\delta_{ij}^{(-1)^{[j+1]}\delta_{i+1}}}} e_j q^{\epsilon_i}$$

$$q^{\epsilon_i} f_j = q^{(-1)^{[j]\delta_{ij}^{(-1)^{[j+1]}\delta_{i+1}}}} f_j q^{\epsilon_i}$$

$$e_i f_j - (-1)^{([i]+[i+1])([j]+[j+1])} f_j e_i = \delta_{ij} \frac{q^{\epsilon_i - \epsilon_i+1} - q^{-\epsilon_i+\epsilon_i+1}}{q - q^{-1}}$$

$$x_i x_j = (-1)^{([i]+[i+1])([j]+[j+1])} x_j x_i, \quad x_i \in \{\epsilon_i, f_i\}, \hspace{1cm} (A.1)$$

and $q$ Chevallay-Serre relations:

$$x_i^2 x_{i+1} - (q + q^{-1}) x_i x_{i+1} x_i + x_{i+1} x_i^2 = 0, \quad x_i \in \{\epsilon_i, f_i\} \quad i \neq m. \hspace{1cm} (A.2)$$
Now set \( h_i = \epsilon_i \) then the \( U_q(sl(m|n)) \) algebra is defined by generators \( e_i, f_i, h_i \). Let \( a_{ij} \) the elements of the related Cartan \( \mathcal{N} \times \mathcal{N} \) matrix, which for instance for the distinguished Dynkin diagram is:

\[
a = \begin{pmatrix}
2 & -1 & & & \\
-1 & 2 & -1 & & \\
 & & \ddots & & \\
 & & -1 & 0 & 1 \\
 & & & \ddots & \\
1 & -2 & 1 & & \\
1 & & & & -2
\end{pmatrix}
\] (A.3)

the zero diagonal element occurs in the \( m \) position. Define also:

\[
[h]_q = \frac{q^h - q^{-h}}{q - q^{-1}}
\] (A.4)

then the \( U_q(sl(m|n)) \) super algebra for the distinguished Dynkin diagram reads as:

\[
[e_i, f_i] = [h]_q, \quad i < m, \quad \{e_m, f_m\} = -[h_m]_q, \quad [e_i, f_i] = -[h_i]_q, \quad i > m
\]

\[
[h_j, h_k] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j
\] (A.5)

plus the Chevalley-Serre relations (A.2). The algebra above is equipped with a non-trivial co-product:

\[
\Delta(\epsilon_i) = \epsilon_i \otimes I + I \otimes \epsilon_i
\]

\[
\Delta(x) = q^{-h x} \otimes x + x \otimes q^{h x}, \quad x \in \{e_i, f_i\}.
\] (A.6)

There is also an isomorphism between the FRT representation of the algebra and the Chevalley-Serre basis. Recall that

\[
L(\lambda) = L^+ - e^{-2\lambda} L^-,
\]

\[
\hat{L}(\lambda) = \hat{L}^+ - O(e^{-2\lambda}), \quad \lambda \to \infty
\]

\[
\hat{L}(\lambda) = \hat{L}^- - O(e^{2\lambda}), \quad \lambda \to -\infty.
\] (A.7)

Recall for the reflection algebra \( \hat{L}(\lambda) = L^{-1}(-\lambda) \) and also

\[
L^\pm = \sum_{i,j} e_{ij} \otimes l_{ij}^\pm, \quad \hat{L}^\pm = \sum_{i,j} e_{ij} \otimes \hat{l}_{ij}^\pm.
\] (A.8)
Then we have the following identifications \[27, 34\]:

\[
\begin{align*}
l_{ii}^+ &= (-1)^{[i]} q^e_i, \quad l_{ii}^{+1} = (-1)^{[i]} (q - q^{-1}) q^{\frac{e_i + e_{i+1}}{2}} f_i, \quad l_{i+1i}^+ = 0 \\
l_{ii}^- &= (-1)^{[i]} q^{-e_i}, \quad l_{i+1i}^- = (-1)^{[i]} (q - q^{-1}) e_i q^{-\frac{e_{i+1}}{2}}, \quad l_{ii+1}^- = 0 \\
\hat{l}_{ii}^+ &= (-1)^{[i]} q^e_i, \quad \hat{l}_{i+1i}^+ = (-1)^{[i+1]} (q - q^{-1}) q^{-\frac{2e_i}{2}} q^{-\frac{e_i + e_{i+1}}{2}} e_i, \quad \hat{l}_{i+1i}^+ = 0 \\
\hat{l}_{ii}^- &= (-1)^{[i]} q^{-e_i}, \quad \hat{l}_{i+1i}^- = (-1)^{[i]} (q - q^{-1}) q^{\frac{2e_i}{2}} f_i q^{\frac{e_i + e_{i+1}}{2}}, \quad \hat{l}_{i+1i}^- = 0 \quad (A.9)
\end{align*}
\]

\(l_{ij}^+, \hat{l}_{ij}, \quad i < j\) and \(l_{ij}^-, \hat{l}_{ij}^+, \quad i > j\) are also non zero, and are expressed as combinations of the \(U_q(gl(m|n))\) generators, but are omitted here for brevity, see e.g expressions in \[35, 7\] for the \(U_q(gl(n))\) case.

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