\textbf{$\mathfrak{sl}_N$-WEB CATEGORIES}

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Abstract. In this paper we show how the colored Khovanov-Rozansky $\mathfrak{sl}_N$-matrix factorizations, due to Wu \cite{39} and Y.Y \cite{40,41}, can be used to categorify the quantum skew Howe duality defined by Cautis, Kamnitzer and Morrison in \cite{10}. In particular, we define web categories and 2-representations of Khovanov and Lauda’s categorical quantum $\mathfrak{sl}_m$ on them. We also show that this implies that each such web category is equivalent to the category of finite-dimensional graded projective modules over a certain level-$N$ cyclotomic KLR-algebra.

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1. Introduction

Recently Cautis, Kamnitzer and Morrison [10] found a complete set of relations on colored $\mathfrak{sl}_N$-webs. We recall that these webs represent intertwiners between tensor products of fundamental $U_q(\mathfrak{sl}_N)$-representations. The main ingredient in [10] was a diagrammatic version of quantum skew Howe duality, which shows that $U_q(\mathfrak{sl}_m)$ acts on $\mathfrak{sl}_N$-web spaces, where $m$ and $N$ are distinct non-negative integers.

In this paper we show how the colored $\mathfrak{sl}_N$-matrix factorizations can be used to categorify Cautis, Kamnitzer and Morrison’s results. These matrix factorizations are due to Wu [39] and Y.Y. [40, 41] and generalize Khovanov and Rozansky’s [20] matrix factorizations in their groundbreaking work on $\mathfrak{sl}_N$-link homologies.

To be a bit more precise, let $N \geq 2$ and $m, d \geq 0$ be arbitrary integers. We first define a 2-functor $\Gamma_{m,d,N}$ from Khovanov and Lauda’s categorified quantum $\mathfrak{sl}_m$, defined in [18] and denoted $U_Q(\mathfrak{sl}_m)$ in this paper, to a certain 2-category of colored $\mathfrak{sl}_N$-matrix factorizations, denoted $\text{HMF}_{m,d,N}$. This 2-functor is similar to Khovanov and Lauda’s 2-functor $\Gamma_d$ from $U_Q(\mathfrak{sl}_m)$ to a 2-category built from the cohomology rings of partial flag varieties (of flags in $C^d$). However, they are not equivalent and do not categorify the same maps, as we will explain.

Now assume that $d = N\ell$, for some $\ell \in \mathbb{N}_{\geq 1}$, and that $m \geq d$. Denote by $\omega_\ell$ the $\ell$-th fundamental $\mathfrak{sl}_m$-weight and take $\Lambda = N\omega_\ell$. We define an additive graded $\mathfrak{sl}_N$-web category $\mathcal{W}_\Lambda^\circ$ and use $\Gamma_{m,d,N}$ to define a 2-representation of categorified quantum $\mathfrak{sl}_m$ on it.

We prove that this implies that $\mathcal{W}_\Lambda^\circ$, the Karoubi envelope of $\mathcal{W}_\Lambda^\circ$, is equivalent to the category of finite-dimensional graded projective $R_\Lambda$-modules, where $R_\Lambda$ is the level-$N$ cyclotomic KLR-algebra of highest weight $\Lambda$. In particular, this implies that the split Grothendieck group of $\mathcal{W}_\Lambda^\circ$ is isomorphic to the corresponding web space.

The category $\mathcal{W}_\Lambda^\circ$ decomposes into blocks, as we will show. Each of these blocks is equivalent to the category of finite-dimensional graded projective modules over a certain finite-dimensional algebra, called the $\mathfrak{sl}_N$-web algebra, which is studied in a sequel to the present paper [25].

For $N = 2$ these algebras were introduced by Khovanov [14], who called them arc algebras. Huerfano and Khovanov [13] categorified certain level-two irreducible $U_q(\mathfrak{sl}_m)$-representations using arc algebras. In those days categorified quantum groups and cyclotomic KLR algebras had not been invented yet, so they did not work out the full $\mathfrak{sl}_m$ 2-representations. But other than that our results can be seen as the level-$N$ generalization of theirs.

The representation theory of the arc algebras and its relation to the geometry of 2-block Springer varieties was studied in detail in [4, 5, 6, 7, 8, 12, 14, 15, 35, 36]. For $N = 3$ the
web algebras were introduced and studied by Pan, Tubbenhauer and M.M. in [26]. The categorified quantum skew Howe duality was proved in that paper. In general less is known about the web algebras for \( N = 3 \) than for \( N = 2 \).

For \( N = 2 \) and \( N = 3 \) the web category can be defined using cobordisms or foams respectively. The reason we do not use foams in this paper, is that they have not yet been defined for \( \mathfrak{sl}_N \) in general. For \( N = 2 \) and \( N = 3 \) it is known that the space of \( \mathfrak{sl}_N \)-foams between two webs is isomorphic to the EXT-group of the corresponding matrix factorizations [20, 29]. For \( N \geq 4 \), \( \mathfrak{sl}_N \)-foams were defined and studied in [27], but only for the colors 1,2 and 3. To use \( \mathfrak{sl}_N \)-foams for the categorification of \( \mathfrak{sl}_N \)-webs in general, one would have to define \( \mathfrak{sl}_N \)-foams for all colors and find a consistent and complete set of relations on them. Perhaps our categorification of quantum skew Howe duality in this paper can help to achieve that goal, which in our view would be a proper and complete categorification of the results in [10].

The results in this paper might also help to find a proof that the Khovanov-Rozansky \( \mathfrak{sl}_N \)-link homology is isomorphic to Webster’s \( \mathfrak{sl}_N \)-link homology [37, 38], for which he used generalizations of the cyclotomic KLR algebras. For \( N = 2 \) and \( N = 3 \) a first step in that direction has already been taken by Lauda, Queffelec and Rose in [22]. They used categorified quantum skew Howe duality to prove that Khovanov’s (and therefore Khovanov and Rozansky’s) \( \mathfrak{sl}_N \) link homologies are isomorphic to link homologies obtained from the so called Chuang-Rouquier complexes over level-\( N \) cyclotomic KLR algebras, for \( N = 2,3 \). Their result probably generalizes to arbitrary \( N \geq 2 \), using categorified skew Howe duality and matrix factorizations as in this paper.

Webster [38] showed that his link homologies are isomorphic to Mazorchuk and Stroppel’s representation theoretic link homologies [30], and also to the Koszul dual version of these homologies independently obtained by Sussan [34]. Mazorchuk and Stroppel proved that their link homologies are isomorphic to Khovanov’s and Khovanov and Rozansky’s link homologies for \( N = 2,3 \), and conjectured that to be true for \( N \geq 4 \) also. Thus we know that all the link homologies mentioned in this introduction are isomorphic for \( N = 2,3 \). By the remarks above, our work might help to prove the same result for arbitrary \( N \geq 2 \).

We should also remark that there is an algebro-geometric categorification of quantum skew Howe duality, due to Cautis, Kamnitzer and Licata [9]. It would be interesting to understand the precise relation with the results in this paper.

The paper is organized as follows:

- After fixing some notation and conventions in Section 2, we briefly recall \( \mathcal{U}_q(\mathfrak{sl}_m) \) and its fundamental representations in Section 3. After this section, we will always use the parameters \( m \) and \( N \) instead of \( n \) in order to distinguish the two sides which occur in Howe duality.
- In Section 4 we recall the necessary material on \( \mathfrak{sl}_N \) webs and quantum skew Howe duality.

1The idea to relate Khovanov-Lauda diagrams to foams was first suggested by Khovanov to M. M. in 2008 and worked out in an unpublished preprint [24] for \( \mathfrak{sl}_3 \) foams over \( \mathbb{Z}/2\mathbb{Z} \). In [26] and [22] the sign problem in that preprint got fixed; by “brute force” in the first case and by the introduction of a Blanchet-like version of \( \mathfrak{sl}_3 \) foams in the second case.
• In Section 5 we recall some material on matrix factorizations, which can be found in more detail in [20, 39, 40, 41]. These matrix factorizations categorify colored $\mathfrak{sl}_N$ webs.

• In Section 6 we briefly recall Khovanov and Lauda’s [18] diagrammatic categorification of quantum $\hat{U}_q(\mathfrak{sl}_m)$, denoted $\hat{U}_q(\mathfrak{sl}_m)$ in this paper, and the cyclotomic KLR algebras, denoted $R_\Lambda$. We also recall Brundan and Kleshchev’s [2] categorification theorem and Rouquier’s [33] additive universality theorem for $R_\Lambda$.

• In Sections 7 and 8 we prove all the technical results about matrix factorizations that are needed in Section 9.

• In Section 9 we first give the 2-functor $\Gamma^*_{m,d,N} : \mathcal{U}_Q(\mathfrak{sl}_m)^* \to \mathcal{HMF}^*_{m,d,N}$ and prove that is well-defined in Theorem 9.2. The meaning of $*$ is explained in Section 2.

Then we define the web categories using the matrix factorizations from Section 5 and prove the aforementioned relation with the level-$N$ cyclotomic KLR algebras in Theorem 9.7. As a consequence we see that the web categories categorify the web spaces in Corollary 9.8.

2. Notation and conventions

In this sections fix some notations and explain some conventions.

Let $\mathcal{C}^*$ be a $\mathbb{Z}$-graded $\mathbb{C}$-linear additive category which admits translation (for a precise definition, see [21] for example). Then $\{t\}$ denotes a positive translation/shift of $t$ units. For any Laurent polynomial $f(q) = \sum a_i q^i \in \mathbb{N}[q, q^{-1}]$, we define

$$X \oplus f(q) := \bigoplus_i (X\{i\})^{\oplus a_i}.$$

Let $\mathcal{C}$ be the subcategory with the same objects as $\mathcal{C}^*$ but only degree-zero morphisms. For any pair of objects $X, Y \in \mathcal{C}$, let $\text{Hom}(X, Y)$ be the usual hom-space in $\mathcal{C}$. Then the hom-space in $\mathcal{C}^*$ is given by

$$\text{HOM}(X, Y) := \bigoplus_{t \in \mathbb{Z}} \text{Hom}(X\{t\}, Y).$$

For simplicity, assume that $\text{HOM}(X, Y)$ is finite-dimensional. Then we define the $q$-dimension of $\text{HOM}(X, Y)$ by

$$\dim_q \text{HOM}(X, Y) := \sum_{t \in \mathbb{Z}} q^t \dim \text{Hom}(X\{t\}, Y).$$

Assume that $\mathcal{C}$ is Krull-Schmidt. The split Grothendieck group $K_0(\mathcal{C})$ is by definition the Abelian group generated by the isomorphism classes of the objects in $\mathcal{C}$ modulo the relation

$$[X \oplus Y] = [X] + [Y],$$

for any objects $X, Y \in \mathcal{C}$. It becomes a $\mathbb{Z}[q, q^{-1}]$-module, by defining

$$q[X] = [X\{1\}],$$
for any object $X \in \mathcal{C}$. For any Laurent polynomial $f(q) = \sum a_i q^i \in \mathbb{N}[q, q^{-1}]$, we get
\[ f(q)[X] = [X^{\otimes f(q)}]. \]

Assume that $S = \{X_1, \ldots, X_s\}$ is a set of indecomposable objects in $\mathcal{C}$ such that
- any indecomposable object in $\mathcal{C}$ is isomorphic to $X_i\{t\}$ for a certain $i \in \{1, \ldots, s\}$ and $t \in \mathbb{Z}$;
- for all $i \neq j \in \{1, \ldots, s\}$ and all $t \in \mathbb{Z}$ we have $X_i \not\cong X_j\{t\}$. Then it is well-known that $K_0(\mathcal{C})$ is freely generated by $S$.

In this paper we will mostly tensor $K_0(\mathcal{C})$ with $\mathbb{C}(q)$, so we define
\[ K_0^q(\mathcal{C}) := K_0(\mathcal{C}) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{C}(q). \]

A $q$-sesquilinear form on $K_0^q(\mathcal{C})$ is by definition a form
\[ \langle \cdot, \cdot \rangle : K_0(\mathcal{C}) \times K_0(\mathcal{C}) \to \mathbb{C}(q) \]
satisfying
\[ \langle f(q)[X], [Y] \rangle = f(q^{-1}) \langle [X], [Y] \rangle \]
\[ \langle [X], f(q)[Y] \rangle = f(q) \langle [X], [Y] \rangle, \]
for any $f(q) \in \mathbb{C}(q)$. There exists a well-known $q$-sesquilinear form on $K_0^q(\mathcal{C})$ called the Euler form. It is defined by
\[ \langle [X], [Y] \rangle := \dim_q \text{HOM}(X, Y), \]
for any objects $X, Y \in \mathcal{C}$. Note that the Euler form takes values in $\mathbb{N}[q, q^{-1}]$ if the HOM-spaces are finite-dimensional.

3. The special linear quantum algebra and its fundamental representations

We briefly recall the special linear quantum algebra and the pivotal category of its fundamental representations.

3.1. The special linear quantum algebra. Let $n \in \mathbb{N}_{>1}$ be arbitrary. We write
\[ \alpha_i := (0, \ldots, 1, -1, \ldots, 0) \in \mathbb{Z}^n \]
with 1 on the $i$-th position, for $i = 1, \ldots, n - 1$. We denote the Euclidean inner product on $\mathbb{Z}^n$ by $(\cdot, \cdot)$.

**Definition 3.1.** The quantum special linear algebra $U_q(\mathfrak{sl}_n)$ is the associative unital $\mathbb{C}(q)$-algebra generated by $K_i^{\pm 1}, E_{\pm i}$, for $i = 1, \ldots, n - 1$, subject to the relations
\[ K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \]
\[ E_{-i} E_j - E_{-j} E_{+i} = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \]
\[ K_i E_{\pm j} = q^{\pm (\alpha_i, \alpha_j)} E_{\pm j} K_i, \]
\[ E_{\pm i} E_{\pm j} - (q + q^{-1}) E_{\pm i} E_{\pm j} E_{\pm i} + E_{\pm j} E_{\pm i}^2 = 0, \quad \text{if } |i - j| = 1, \]
\[ E_{\pm j} E_{\pm i} - E_{\pm i} E_{\pm j} = 0, \quad \text{else} \]
Recall that $\mathbf{U}_q(\mathfrak{sl}_n)$ is a Hopf algebra with coproduct given by
$$\Delta(E_{+i}) = E_{+i} \otimes K_i + 1 \otimes E_{+i}, \quad \Delta(E_{-i}) = E_{-i} \otimes K_i^{-1} \otimes E_{-i}, \quad \Delta(K_i) = K_i \otimes K_i$$
and antipode by
$$S(E_{+i}) = -E_{+i} K_i^{-1}, \quad S(E_{-i}) = -K_i E_{-i}, \quad S(K_i) = K_i^{-1}.$$ 

The counit is given by $\epsilon(E_{\pm i}) = 0, \epsilon(K_i) = 1$.

The Hopf algebra structure is used to define $\mathbf{U}_q(\mathfrak{sl}_n)$ actions on tensor products and duals of $\mathbf{U}_q(\mathfrak{sl}_n)$-modules.

Recall that the $\mathbf{U}_q(\mathfrak{sl}_n)$-weight lattice is isomorphic to $\mathbb{Z}^{n-1}$. For any $i = 1, \ldots, n-1$, the element $K_i$ acts as $q^\lambda_i$ on the $\lambda$-weight space of any weight representation.

Although we have not recalled the definition of $\mathbf{U}_q(\mathfrak{gl}_n)$, it is sometimes convenient to use $\mathbf{U}_q(\mathfrak{gl}_n)$-weights. Recall that the $\mathbf{U}_q(\mathfrak{gl}_n)$-weight lattice is isomorphic to $\mathbb{Z}^n$ and that any $\vec{k} = (k_1, \ldots, k_N) \in \mathbb{Z}^n$ determines a unique $\mathbf{U}_q(\mathfrak{sl}_n)$-weight
$$\lambda = (k_1 - k_2, \ldots, k_{n-1} - k_n) \in \mathbb{Z}^{n-1}.$$ 

In this way, we get an isomorphism
$$\mathbb{Z}^n / \langle (1^n) \rangle \cong \mathbb{Z}^{n-1}.$$ 

Conversely, given a $\mathbf{U}_q(\mathfrak{sl}_n)$-weight $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$, there is not a unique choice of $\mathbf{U}_q(\mathfrak{gl}_n)$-weight. We first have to fix the sum of the entries: for any $d \in \mathbb{Z}$, the equations
$$k_i - k_{i+1} = \lambda_i,$$
$$\sum_{i=1}^n k_i = d$$
determine $\vec{k} = (k_1, \ldots, k_n)$ uniquely, if there exists a solution to (2) and (3) at all.

**Remark 3.2.** Since the $\mathbf{U}_q(\mathfrak{sl}_n)$ and $\mathbf{U}_q(\mathfrak{gl}_n)$ weights and weight lattices are equal those of the corresponding classical algebras, we will often refer to them as the $\mathfrak{sl}_n$ and $\mathfrak{gl}_n$-weights and weight lattices.

For weight representations, we can also use the idempotent version of $\mathbf{U}_q(\mathfrak{sl}_n)$, denoted $\mathbf{U}_q(\mathfrak{sl}_n)$, due to Beilinson, Lusztig and MacPherson [1]. For $n = 2$, define $i' = (2)$. For $n > 2$, define
$$i' := \begin{cases} (2, -1, 0, \ldots, 0), & \text{for } i = 1; \\ (0, \ldots, -1, 2, -1, \ldots, 0), & \text{for } 2 \leq i \leq n-2; \\ (0, \ldots, 0, -1, 2), & \text{for } i = n-1. \end{cases}$$

Adjoin an idempotent $1_\lambda$ for each $\lambda \in \mathbb{Z}^{n-1}$ and add the relations
$$1_\lambda 1_\mu = \delta_{\lambda,\mu} 1_\lambda,$$
$$E_{\pm i} 1_\lambda = 1_{\lambda \pm i'} E_i,$$
$$K_i 1_\lambda = q^{\lambda_i} 1_\lambda.$$
Definition 3.3. The idempotented quantum special linear algebra is defined by

\[ \dot{U}_q(\mathfrak{sl}_n) = \bigoplus_{\lambda, \mu \in \mathbb{Z}^{n-1}} 1_\lambda U_q(\mathfrak{sl}_n)1_\mu. \]

The following remark is useful for Proposition 4.5.

Remark 3.4. Sometimes we will consider \( U_q(\mathfrak{sl}_n) \) as a \( \mathbb{C}((q)) \)-linear category rather than an algebra. The objects are the idempotents \( 1_\lambda \), for \( \lambda \in \mathbb{Z}^{n-1} \), and

\[ \text{Hom}(1_\lambda, 1_\mu) := 1_\lambda U_q(\mathfrak{sl}_n)1_\mu. \]

3.2. Fundamental representations. As already remarked in the introduction, from now on we distinguish the two sides which occur in Howe duality by using the parameters \( m \) and \( N \) instead of \( n \) for the general linear quantum groups.

In this section we recall the fundamental \( U_q(\mathfrak{sl}_N) \) representations, following [10, 31]. The basic \( U_q(\mathfrak{sl}_N) \)-representation is denoted \( C^N_q \). It has a standard basis \( \{x_1, \ldots, x_N\} \) on which the action is given by

\[ E_{+i}(x_j) = \begin{cases} x_i, & \text{if } j = i + 1; \\ 0, & \text{else.} \end{cases}, \quad E_{-i}(x_j) = \begin{cases} x_{i+1}, & \text{if } j = i; \\ 0, & \text{else.} \end{cases}, \quad K_i(x_j) = \begin{cases} qx_i, & \text{if } j = i; \\ q^{-1}x_{i+1}, & \text{if } j = i + 1; \\ x_j, & \text{else.} \end{cases} \]

Using the basic representation, one can define all fundamental \( U_q(\mathfrak{sl}_N) \)-representations. Define the quantum exterior algebra

\[ \Lambda^\bullet_q(C^N_q) := T C^N_q / \langle \{ x_i \otimes x_i, x_i \otimes x_j + qx_j \otimes x_i \mid 1 \leq i < j \leq N \} \rangle. \]

We denote the equivalence class of \( x \otimes y \) by \( x \wedge_q y \). Note that

\[ \Lambda^\bullet_q(C^N_q) = \bigoplus_{k=0}^N \Lambda^k(C^N_q) \]

is a graded algebra. For each \( 0 \leq k \leq N \), the homogeneous direct summand \( \Lambda^k_q(C^N_q) \) is an irreducible \( U_q(\mathfrak{sl}_N) \)-representation. For \( k = 0, N \) it is the trivial representation and for \( 1 \leq k \leq N \) it is called the \( k \)-th fundamental representation. It is well-known that the dual of the \( k \)-th fundamental representation is isomorphic to the \((N-k)\)-th fundamental representation.

In this paper we will use tensor products of fundamental representations and their duals, which are also \( U_q(\mathfrak{sl}_N) \)-representation by the Hopf algebra structure on \( U_q(\mathfrak{sl}_N) \).

Definition 3.5. Let \( \text{Rep}(\text{SL}_N) \) be the pivotal category whose objects are tensor products of fundamental \( U_q(\mathfrak{sl}_N) \)-representations and their duals and whose morphisms are intertwiners.

Morrison (Theorem 3.5.8 in [31]) defined a generating set of intertwiners in \( \text{Rep}(\text{SL}_N) \), the precise definition of which is not relevant here. In Section 4 we recall Cautis, Kamnitzer and Morrison’s diagrammatic presentation of \( \text{Rep}(\text{SL}_N) \) in [10], which will be used in the rest of this paper.
4. **SL\(_N\) webs**

The morphisms in \(\text{Rep}(\text{SL}\_N)\) can be represented graphically by webs. These are certain trivalent graphs, whose edges are colored by integers belonging to \(\{0,\ldots,N\}\). Webs can be seen as morphisms in a pivotal category, which in the literature is called a *spider* or *spider category*, denoted \(\mathcal{S}p(\text{SL}_N)\).

4.1. **The SL\(_N\) spider.** Recently, Cautis, Kamnitzer and Morrison [10] gave a presentation of \(\mathcal{S}p(\text{SL}_N)\) in terms of generating webs and relations.

**Definition 4.1** (Cautis-Kamnitzer-Morrison). The objects of \(\mathcal{S}p(\text{SL}_N)\) are finite sequences \(\vec{k}\) of elements in \(\{0^\pm,\ldots,(N)^\pm\}\).

The hom-space \(\text{Hom}(\vec{k},\vec{l})\) is the \(\mathbb{C}(q)\) vector space freely generated by all diagrams, with lower and top boundary labeled by \(\vec{k}\) and \(\vec{l}\) respectively, which can be obtained by glueing and juxtaposing labeled cups and caps and the following elementary webs, together with the ones obtained by mirror reflections and arrow reversals:

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with all labels between 0 and N, modded out by planar isotopies (i.e. zig-zag relations for cups and caps) and the following relations:

\[
\begin{align*}
(4) & \quad \frac{N-a}{a} = (-1)^{a(N-a)} \frac{N-a}{a} \\
(5) & \quad \frac{b+a}{a} = \left[ \frac{a+b}{a} \right]_q \frac{b+a}{a} \\
(6) & \quad \frac{b+a}{a} = \left[ \frac{N-a}{b} \right]_q \frac{a}{a} \\
(7) & \quad \frac{a+b+c}{a} = \frac{a+b+c}{a} \\
(8) & \quad \frac{a-s-t}{a} = \left[ \frac{s+t}{t} \right]_q \frac{a-s-t}{a}
together with the analogous relations obtained by mirror reflections and arrow reversals.

Note that with tags one can invert the orientation of cups and caps, so one only needs the above cups and caps as generators.

The following result can be found in [10] (Theorems 3.2.1 and 3.3.1):

**Theorem 4.2** (Cautis-Kamnitzer-Morrison). There exists an equivalence of pivotal categories $\gamma_N : \text{Sp}(\text{SL}_N) \to \text{Rep}(\text{SL}_N)$, which on objects is defined by

$$\vec{k} = (k^1, \ldots, k^m) \mapsto \Lambda^\vec{k}_q(C^N_q) = (\Lambda^k_q(C^N_q))^{e_1} \otimes \cdots \otimes (\Lambda^k_q(C^N_q))^{e_m},$$

where $V^1 := V$ and $V^{-1} := V^\ast$.

**Remark 4.3.** There is a slight discrepancy with the setup in [10]. We allow the colors 0 and $N$ too, whereas Cautis, Kamnitzer and Morrison do not. Of course $\Lambda^0_q(C^N_q)$ and $\Lambda^N_q(C^N_q)$ are both isomorphic to the trivial $U_q(\mathfrak{sl}_N)$-representation and one can decide to not draw edges labeled with them. However, for our purposes they are actually useful.

4.2. **Quantum skew Howe duality.** Let us briefly recall the instance of quantum skew Howe duality which we will categorify. For more details, see [10] and the references therein.

For the rest of this paper, let $N \geq 2$ and $m, d \geq 0$ be arbitrary integers.

Below we are only interested in $\mathfrak{gl}_m$-weights with entries between 0 and $N$. Following [10], we call these weights $N$-bounded. Let $\Lambda(m, d)_N$ denote the set of $N$-bounded $\mathfrak{gl}_m$-weights whose entries sum up to $d$, i.e.

$$\Lambda(m, d)_N := \{ \vec{k} \in \{0, \ldots, N\}^m | k_1 + \cdots + k_m = d \}.$$

**Definition 4.4.** We define the map

$$\phi_{m,d,N} : \mathbb{Z}^{m-1} \to \Lambda(m, d)_N \cup \{\ast\}$$

by

$$\phi_{m,d,N}(\lambda) = \begin{cases} \vec{k}, & \text{if (2) and (3) have a solution in } \Lambda(m, d)_N; \\ \ast, & \text{otherwise.} \end{cases}$$

The following Proposition is due to Cautis, Kamnitzer and Morrison and follows from Propositions 5.1.2 and 5.2.1 in [10]. In this proposition we consider $\check{U}_q(\mathfrak{sl}_m)$ as a $\mathbb{C}(q)$-category rather than an idempotented algebra (see Remark 3.4).

**Proposition 4.5** (Cautis-Kamnitzer-Morrison). The functor

$$\gamma_{m,d,N} : \check{U}_q(\mathfrak{sl}_m) \to \text{Sp}(\text{SL}_N)$$
detected by

\[ 1_\lambda \mapsto \begin{cases} \vec{k} & \text{if } \phi_{m,d,N}(\lambda) = \vec{k}; \\ 0, & \text{if } \phi_{m,N,d}(\lambda) = *. \end{cases} \]

\[ E_{+i}1_\lambda \mapsto \begin{array}{c} \cdots \\downarrow k_{i+2} \\downarrow k_{i+1} \\downarrow 1 \\downarrow k_{i} \\downarrow k_{i-1} \\downarrow k_{1} \\ \cdots \\uparrow k_{i} \\uparrow k_{i+1} \\uparrow k_{i+2} \end{array} \]

\[ E_{-i}1_\lambda \mapsto \begin{array}{c} \cdots \\downarrow k_{i+2} \\downarrow k_{i+1} \\downarrow 1 \\downarrow k_{i} \\downarrow k_{i-1} \\downarrow k_{1} \\ \cdots \\uparrow k_{i} \\uparrow k_{i+1} \\uparrow k_{i+2} \end{array} \]

is well-defined.

**Remark 4.6.** We label the vertical edges with the entries of \( \vec{k} \) in the reverse order. This differs from the convention in [10], but their results hold true with our convention too. The reason that we opted for this “opposite” convention, is that it permits us very easily to compare our 2-representation \( \Gamma_{m,N} \) in Theorem 9.2 to the 2-representations in [18] and [28]. Finding a consistent set of signs for the definition of such 2-representations is very tricky, so we prefer to stick to the conventions in those two papers.

Note that by (5) and (7), it is easy to determine the images of the divided powers

\[ E_{+i}^{(a)} := E_{+i}^a/[a]! \text{ and } E_{-i}^{(a)} = E_{-i}^a/[a]! \]

\[ E_{+i}^{(a)}1_\lambda \mapsto \begin{array}{c} \cdots \\downarrow k_{i+2} \\downarrow k_{i+1} \\downarrow k_{i} \\downarrow k_{i-1} \\downarrow k_{1} \\ \cdots \\uparrow k_{i} \\uparrow k_{i+1} \\uparrow k_{i+2} \end{array} \]

\[ E_{-i}^{(a)}1_\lambda \mapsto \begin{array}{c} \cdots \\downarrow k_{i+2} \\downarrow k_{i+1} \\downarrow k_{i} \\downarrow k_{i-1} \\downarrow k_{1} \\ \cdots \\uparrow k_{i} \\uparrow k_{i+1} \\uparrow k_{i+2} \end{array} \]
Proposition 4.5 singles out a special class of web diagrams, called ladders, which Cautis, Kamnitzer and Morrison defined in their Section 5.

**Definition 4.7 (Cautis-Kamnitzer-Morrison).** An $N$-ladder with $m$ uprights is a rectangular $\mathfrak{sl}_N$-web diagram without tags,
- whose vertical edges are all oriented upwards and lie on $m$ parallel vertical lines running from bottom to top;
- which contains a certain number of horizontal oriented rungs connecting adjacent uprights.

Since ladders do not have tags, at each trivalent vertex the sum of the labels of the incoming edges has to be equal to the sum of the labels of the outgoing edges.

It is clear that any $N$-ladder with $m$ uprights whose labels add up to $d$, at any generic level between the rungs, is the image under $\gamma_{m,d,N}$ of a product of divided powers in $\hat{U}_q(\mathfrak{sl}_m)$ by Proposition 4.5.

**Remark 4.8.** In [10] the authors are very careful in distinguishing between a ladder, which for them is just a diagram in the “free spider”, and its image in $\mathcal{S}p(SL_N)$. In this paper that distinction is irrelevant and we use the term “ladder” more loosely to designate any web which can be represented by a ladder diagram in $\mathcal{S}p(SL_N)$.

Now suppose that $d = N\ell$, for some integer $\ell \geq 0$, and that $m \geq d$. Let $\Lambda = N \omega_\ell$ be $N$ times the $\ell$-th fundamental $\mathfrak{sl}_m$-weight. We denote by $P_{\Lambda}$ the set of $\mathfrak{sl}_m$-weights in the irreducible $\hat{U}_q(\mathfrak{sl}_m)$-module $V_{\Lambda}$. Note that $\phi_{m,d,N}(P_{\Lambda}) = \Lambda(m,d)_N$. In particular, we have $\phi_{m,d,N}(\Lambda) = (N^\ell) \in \Lambda(m,d)_N$.

**Definition 4.9.** Define the $\hat{U}_q(\mathfrak{sl}_m)$-web module with highest weight $\Lambda$ by

$$W_{\Lambda} := \bigoplus_{\vec{k} \in \Lambda(m,d)_N} W(\vec{k}, N),$$

where $W(\vec{k}, N)$ is the web space defined by

$$W(\vec{k}, N) := \text{Hom}((N^\ell), \vec{k}).$$

Let $\vec{k}, \vec{k}' \in \Lambda(m,d)_N$. Any web in $\text{Hom}(\vec{k}, \vec{k}')$ defines a linear map $W(\vec{k}, N) \to W(\vec{k}', N)$ by gluing it on top of the webs in $W(\vec{k}, N)$. Therefore, the homomorphism $\gamma_{m,d,N}$ induces a well-defined action of $\hat{U}_q(\mathfrak{sl}_m)$ on $W_{\Lambda}$. We are going to show that $W_{\Lambda}$ is an irreducible $\hat{U}_q(\mathfrak{sl}_m)$-representation.

For any $u \in W(\vec{k}, N)$, let

$$u^* \in \text{Hom}(\vec{k}, (N^\ell))$$

be the web obtained via reflexion in the $x$-axis and reorientation. Note that $u$ and $u^*$ can be glued together such that $u^* u \in \text{End}((N^\ell))$.

Let

$$\text{ev} : \text{End}((N^\ell)) \to \mathbb{C}(q)$$

denote the isomorphism given by the evaluation of closed webs (forgetting about the edges labeled $N$). We can define a $q$-sesquilinear form on $W(\vec{k}, N)$, which we call the $q$-sesquilinear
By Theorem 3.3.1 in [10], we see that
\[ \dim W(\vec{k}, N) = \dim \text{Inv}(V(\vec{k}, N)) \]
for any \( \vec{k} \in \phi_{m,d,N}(P_\Lambda) \). By Theorem 4.2.1(3) in the same paper, this implies that
\[ \dim W_\Lambda = \dim V_\Lambda. \]

The first statement in the corollary now follows by the uniqueness up to isomorphism of irreducible highest weight \( \dot{U}_q(\mathfrak{sl}_m) \)-modules.

The web form clearly satisfies \( \langle w_\Lambda, w_\Lambda \rangle = 1 \). Note that from the definition of the action it follows immediately that
\[ (E_{i+1}1_\lambda u)^*v = u^*(1_\lambda E_{-i}v), \]
for any \( i = 1, \ldots, N - 1 \) and any \( \lambda \in \mathbb{Z}^{N-1} \). Let us assume that \( \phi_{m,d,N}(\lambda) = \vec{k} \) (otherwise there is nothing to prove) and \( \phi_{m,d,N}(\lambda + i') = \vec{k}' \). Then
\[ \vec{k}' = (k_1, \ldots, k_i + 1, k_i - 1, \ldots, k_m), \]
so
\[ \sum_{j=1}^m k_j'(k_j' - 1) = \sum_{j=1}^m k_j(k_j - 1) + 2 + 2(k_i - k_{i+1}), \]
which implies
\[ d(\vec{k}') = d(\vec{k}) - 1 - (k_i - k_{i+1}). \]
The normalization of the web form then gives
\[ \langle E_{i+1}^+ \lambda, v \rangle = q^{d(\vec{k})} \text{ev}((E_{i+1}^+ \lambda u)^* v) \]
and
\[ \langle u, \tau(E_{i+1}^+ \lambda)v \rangle = q^{-1-(k_i-k_{i+1})} \langle u, 1^+ \lambda E_{-i} v \rangle = q^{d(\vec{k})-1-(k_i-k_{i+1})} \text{ev}(u^* (1^+ \lambda E_{-i} v)). \]
This shows that
\[ \langle E_{i+1}^+ \lambda u, v \rangle = \langle u, \tau(E_{i+1}^+ \lambda)v \rangle. \]
Similarly, one can check that
\[ \langle E_{-i}^+ \lambda u, v \rangle = \langle u, \tau(E_{-i}^+ \lambda)v \rangle. \]
Since any element in $\hat{U}_q(\mathfrak{sl}_m)$ is a linear combination of products of $E_{i+1}^+$’s and $E_{-i}^+$’s, this shows that the web form satisfies the defining axioms of the $q$-Shapovalov form. □

Corollary 4.10 shows that any monomial web in $W_\Lambda$ is equal to a linear combination of $N$-ladders with $m$ uprights. The following corollary shows that ladders are special.

**Corollary 4.11.** Let $u \in W(\Lambda, N)$ be any $N$-ladder with $m$ uprights. Then there exists an $\alpha \in \mathbb{N}[q, q^{-1}]$ such that
\[ u = \alpha w_\Lambda. \]

**Proof.** Since $u$ is a ladder, Proposition 4.5 shows that it can be obtained from $w_\Lambda$ by applying a product of divided powers of $E_{i+1}^+$’s. The result now follows from Theorem 22.1.7 in [23] and the fact that $w_\Lambda$ is the unique canonical basis element of weight $\Lambda$. □

**Remark 4.12.** A good question is whether any monomial web $w \in W(\vec{k}, N)$ can be represented as an $N$-ladder with $m$ uprights. For $N = 2$ and $N = 3$ the answer is affirmative. For $N = 2$ this is immediate and for $N = 3$ this was proved in Lemma 5.2.10 in [26]. For $N > 3$ we do not know the answer. Theorem 3.5.1 in [10] cannot be applied, at least not immediately, because it only shows that $w$ is isomorphic to the image of an $N$-ladder with a “certain number” of uprights after removing the edges labeled 0 and $N$. The algorithm in the proof of that theorem will give a ladder with more than $m$ uprights in general.

## 5. The Categorification of Webs

Wu [39] and independently Y. Y [40, 41] defined matrix factorizations associated to colored $\mathfrak{sl}_N$ webs, generalizing the groundbreaking work of Khovanov and Rozansky [20]. In this section we recall these matrix factorizations, which we will use to define our web categories in Definition 9.3.
5.1. Partially symmetric polynomials. Let \( k \) and \( r \) be two integers. We define
\[
T^k_r := \{t_{1,r}, \ldots, t_{|k|,r}\},
\]
where the \( t_{i,r} \) are certain formal variables, which are graded by putting \( \deg(t_{i,r}) = 2 \), for all \( i = 1, \ldots, k \). The integer \( r \) just serves as an index, which will be useful later.

Denote the elementary symmetric and the complete symmetric functions in \( T^k_r \) by
\[
e_{0,r}, e_{1,r}, \ldots, e_{k,r} \quad \text{and} \quad h_{0,r}, h_{1,r}, \ldots, h_{k,r}, \ldots
\]
respectively.

Write
\[
s(k) = \begin{cases} 1 & k \geq 0 \\ -1 & k < 0 \end{cases}.
\]

Let \( m \) be a non-negative integer and let \( \vec{k} = (k_1, k_2, \ldots, k_m) \) and \( \vec{r} = (r_1, \ldots, r_m) \) be \( m \)-tuples of integers and define \( |\vec{k}| := \sum_{a=1}^{m} |k_a| \). We write
\[
T^{\vec{k}}_{\vec{r}} := T^{k_1}_{r_1} \cup \cdots \cup T^{k_m}_{r_m},
\]
which is the ring of partially symmetric polynomials which are symmetric in each \( T^{k_a}_{r_a} \) separately for \( a = 1, \ldots, m \).

We define the rational function \( X^{\vec{k}}_{\vec{r}} \) in the alphabet \( T^{\vec{k}}_{\vec{r}} \) by
\[
X^{\vec{k}}_{\vec{r}} := \prod_{a=0}^{m} \left( \frac{|k_a|}{\sum_{b=0}^{a} e_{b,r_a}} \right)^{s(k_a)}.
\]

Expand \( X^{\vec{k}}_{\vec{r}} \) as a power series and let \( X^{\vec{k}}_{j,\vec{r}} \) \((j \in \mathbb{Z}_{\geq 0})\) be the polynomial given by the homogeneous summand of \( X^{\vec{k}}_{\vec{r}} \) of degree \( 2j \). Note that
\[
X^{\vec{k}}_{j,\vec{r}} \in R^{\vec{k}}_{\vec{r}}
\]
for all \( j \in \mathbb{Z}_{\geq 0} \).

**Example 5.1.** Let \( m = 1 \) and \( k \geq 0 \) \((r \text{ is just an index, which is not important here})\), then
\[
X^{k}_{j,r} = e_{j,r} \quad \text{and} \quad X^{-k}_{j,r} = (-1)^j h_{j,r}
\]
for \( j = 1, \ldots, k \).

Define
\[
X^{\vec{k}}_{\vec{r}} := \left\{ X^{\vec{k}}_{j,\vec{r}} \mid 0 \leq j \leq |\vec{k}| \right\},
\]
which we treat as an alphabet in its own right.

Fix \( N \geq 2 \) and let
\[
p_{N+1}(T^k_r) := t_{1,r}^{N+1} + t_{2,r}^{N+1} + \cdots + t_{k,r}^{N+1}.
\]

Define \( P_{N+1}(X^{\vec{k}}_{\vec{r}}) \) by
\[
P_{N+1}(X^{\vec{k}}_{\vec{r}}) := P_{N+1}(X^{k_1}_{r_1}) + P_{N+1}(X^{k_2}_{r_2}) + \cdots + P_{N+1}(X^{k_m}_{r_m})
\]
where
\[ P_{N+1}(X_{r_a}^k) := p_{N+1}(T_{r_a}^k) \]
for each \( a = 1, \ldots, m \).

**Example 5.2.** Let \( m = 1 \) and \( k, N = 2 \). Then
\[ p_3(t_1, t_2) = t_1^3 + t_2^3 = (t_1 + t_2)^3 - 3(t_1 + t_2)t_1t_2, \]
so
\[ P_3(e_1, e_2) = e_1^3 - 3e_1e_2. \]

### 5.2. Graded matrix factorizations

Let \( R = \mathbb{C}[X_1, \ldots, X_k] \) and put a grading on \( R \) by taking \( \deg(X_i) \) to be an even positive integer for each \( i = 1, \ldots, k \). Let \( P \) be a polynomial in \( R \).

**Definition 5.3.** A graded matrix factorization with potential \( P \) is a 4-tuple
\[ \hat{M} = (M_0, M_1, d_{M_0}, d_{M_1}) \]
such that
\[ M_0 \xrightarrow{d_{M_0}} M_1 \xrightarrow{d_{M_1}} M_0, \]
is a 2-chain of free graded \( R \)-modules (possibly of infinite rank) such that
\[ \deg(d_{M_0}) = \deg(d_{M_1}) = \frac{1}{2}\deg(P) \]
and
\[ d_{M_1}d_{M_0} = P\text{Id}_{M_0} \quad \text{and} \quad d_{M_0}d_{M_1} = P\text{Id}_{M_1}. \]

**Definition 5.4.** A matrix factorization \( \hat{M} = (M_0, M_1, d_{M_0}, d_{M_1}) \) is finite if as an \( R \)-module rank \( (M_0) = \text{rank}(M_1) < \infty \).

We define a grading shift \( \{ m \} \) \( (m \in \mathbb{Z}) \) and a translation \( \langle 1 \rangle \) on \( \hat{M} = (M_0, M_1, d_{M_0}, d_{M_1}) \) by
\[ \hat{M}\{ m \} = (M_0\{ m \}, M_1\{ m \}, d_{M_0}, d_{M_1}), \]
\[ \hat{M}\langle 1 \rangle = (M_1, M_0, -d_{M_1}, -d_{M_0}). \]

A morphism \( f : \hat{M} \to \hat{N} \) of matrix factorizations is a pair of degree preserving \( R \)-module morphisms \( f_0 : M_0 \to N_0 \) and \( f_1 : M_1 \to N_1 \) such that
\[ d_{N_0}f_0 = f_1d_{M_0}, \quad d_{N_1}f_1 = f_0d_{M_1}. \]

A morphism \( f : \hat{M} \to \hat{N} \) of matrix factorizations is null-homotopic if there exists a pair of \( R \)-module morphisms \( h_0 : M_0 \to N_1 \) and \( h_1 : M_1 \to N_0 \) such that
\[ f_0 = d_{N_0}h_0 + h_1d_{M_0}, \quad f_1 = d_{N_0}h_1 + h_0d_{M_1}. \]

Two such morphisms \( f, g \) are homotopic if \( f - g \) is null-homotopic.

Let \( \text{HMF}_R(P) \) be the homotopy category of matrix factorizations with potential \( P \). This is an additive Krull-Schmidt category (see Propositions 24 and 25 in [20]). Recall that \( \text{HMF}^*_R(P) \), which was defined in Section 2, contains all homogeneous morphisms.
Let $X$ and $Y$ be two sets of variables and put $U = X \cap Y$ and $V = X \cup Y$. Take $R = \mathbb{C}[X]$, $R' = \mathbb{C}[Y]$ and $S = \mathbb{C}[U]$ and $Q = \mathbb{C}[V]$. Note that

$$Q = R \otimes_S R'.$$

For $\widehat{M} = (M_0, M_1, d_{M_0}, d_{M_1})$ in $\text{HMF}_R(P)$ and $\widehat{N} = (N_0, N_1, d_{N_0}, d_{N_1})$ in $\text{HMF}_{R'}(P')$, we define the tensor product $\widehat{M} \boxtimes \widehat{N}$ in $\text{HMF}_Q(P + P')$ by

$$\widehat{M} \boxtimes \widehat{N} := \left( \begin{array}{ccc} M_0 \otimes N_0 \\ M_1 \otimes N_1 \end{array} \right), \left( \begin{array}{ccc} M_0 \otimes N_0 \\ M_1 \otimes N_1 \end{array} \right), \left( \begin{array}{ccc} d_{M_0} - d_{N_1} \\ d_{N_0} \\ d_{M_1} \end{array} \right), \left( \begin{array}{ccc} d_{M_1} \\ -d_{N_0} \\ d_{M_0} \end{array} \right).$$

If no confusion is possible, we will write $\widehat{M} \boxtimes \widehat{N}$.

**Example 5.5.** Let $p$ and $q$ be two homogeneous polynomials in a graded polynomial ring $R$ and let $M$ be a free graded $R$-module. We define the matrix factorization $K(p; q)_M$ with potential $pq$ by

$$K(p; q)_M := (M, M\{\frac{1}{2}(\deg(q) - \deg(p))\}, p, q).$$

More generally, for sequences $p = (p_1, p_2, \ldots, p_r)$, $q = (q_1, q_2, \ldots, q_r)$ of homogeneous polynomials in $R$, we define the matrix factorization $K(p; q)_M$ with potential $\sum_{i=1}^r p_i q_i$ by

$$K(p; q)_M := \boxtimes_{R}^{r} K(p_i; q_i)_R \boxtimes_{R}^{r} (M, 0, 0, 0).$$

These matrix factorizations are called Koszul matrix factorizations [20].

5.3. **The 2-complex** $\text{HOM}_R(\widehat{M}, \widehat{N})$. We define the structure of a 2-complex on $\text{HOM}_R(\widehat{M}, \widehat{N})$ by

$$\text{HOM}_R^0(\widehat{M}, \widehat{N}) \xrightarrow{d_0} \text{HOM}_R^1(\widehat{M}, \widehat{N}) \xrightarrow{d_1} \text{HOM}_R^0(\widehat{M}, \widehat{N}),$$

where

$$\text{HOM}_R^0(\widehat{M}, \widehat{N}) = \text{HOM}_R(M_0, N_0) \oplus \text{HOM}_R(M_1, N_1),$$

$$\text{HOM}_R^1(\widehat{M}, \widehat{N}) = \text{HOM}_R(M_0, N_1) \oplus \text{HOM}_R(M_0, N_1),$$

and

$$d_i(f) = d_{N} f + (-1)^i f d_{M} \quad (i = 0, 1).$$

The cohomology of this complex is denoted by

$$\text{EXT}(\widehat{M}, \widehat{N}) = \text{EXT}^0(\widehat{M}, \widehat{N}) \oplus \text{EXT}^1(\widehat{M}, \widehat{N}).$$

By definition, we have the following proposition.

**Proposition 5.6.** We have

$$\text{EXT}^0(\widehat{M}, \widehat{N}) \simeq \text{HOM}_{\text{HMF}}(\widehat{M}, \widehat{N}),$$

$$\text{EXT}^1(\widehat{M}, \widehat{N}) \simeq \text{HOM}_{\text{HMF}}(\widehat{M}, \widehat{N}(1)).$$

We also recall the following result, which can be found in Proposition 12 and Corollary 6 [20]. Given a matrix factorization $\widehat{N} = (N_0, N_1, d_{N_0}, d_{N_1})$, one can define its dual by $\widehat{N}_* = (N_0^*, N_1^*, -d_{N_1}^*, d_{N_0}^*)$, where $N^* = \text{HOM}_R(N, R)$. 
Lemma 5.7. If $M$ is finite, we have an isomorphism
\[ \text{EXT}(\hat{M}, \hat{N}) \cong H(\hat{M}, R \hat{N}) \]
which preserves the $q$-degree.

5.4. Matrix factorizations associated to webs. For a given web $\Gamma$, we will always denote the corresponding matrix factorization by $\hat{\Gamma}$.

We define a matrix factorization for the following web with formal indices associated to its boundaries. We assume that $k \geq 1$.

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (0,-1) {2};
  \draw[-stealth] (1) -- (2) node [midway, above] {\textcolor{red}{$k$}};
\end{tikzpicture}
\caption{$L^{[k]}_{(1;2)}$}
\end{figure}

Definition 5.8. We define the matrix factorization
\[ \hat{L}^{[k]}_{(1;2)} := \bigotimes_{a=1}^{k} K \left( P_{a,(1;2)}^{[k]} ; X_{a,(1)}^{(k)} - X_{a,(2)}^{(k)} \right)_{k,(1;2)} \]
where
\[ P_{a,(1;2)}^{[k]} = P_{N+1}(X_{1,(2)}^{(k)}, \ldots, X_{a-1,(2)}^{(k)}, X_{a,(1)}^{(k)}, \ldots, X_{k,(1)}^{(k)}) - P_{N+1}(X_{1,(2)}^{(k)}, \ldots, X_{a-1,(2)}^{(k)}, X_{a+1,(1)}^{(k)}, \ldots, X_{k,(1)}^{(k)}). \]

Proposition 5.9. We have the following results:
\begin{enumerate}
  \item $\hat{L}^{[k]}_{(1;2)}$ is homotopy equivalent to the zero matrix factorization if $k \geq N + 1$;
  \item $\hat{L}^{[k]}_{(1;2)}$ ($1 \leq k \leq N$) is indecomposable.
\end{enumerate}

Proof. (1) Expressing the $N + 1$-th power sum in terms of the elementary symmetric polynomials, we see that
\[ P_{N+1}^{[k]} \in \mathbb{C} \]
Therefore, $K(P_{N+1}^{[k]} ; X_{N,(1)}^{(k)} - X_{N,(2)}^{(k)})$ is homotopy equivalent to the zero matrix factorization, which implies that $\hat{L}^{[k]}_{(1;2)}$ is homotopy equivalent to the zero matrix factorization.

(2) We have
\[ \text{EXT}(\hat{L}^{[k]}_{(1;2)} ; \hat{L}^{[k]}_{(1;2)}) \cong H^*(G_k, \mathbb{C}), \]
where the latter is the cohomology ring of the Grassmannian of $k$-dimensional complex planes in $\mathbb{C}^N$. This shows that $\text{EXT}(\hat{L}^{[k]}_{(1;2)} ; \hat{L}^{[k]}_{(1;2)})$ has dimension one in degree zero. Therefore, the identity is a primitive idempotent, which means that $\hat{L}^{[k]}_{(1;2)}$ is indecomposable. \qed
We define the matrix factorizations

\[ \Lambda_{(3,1,2)}^{[k_1,k_2]} := \bigotimes_{a=1}^{k_3} K \left( P_{a,(3;1,2)}^{[k_1,k_2]} \cdot X_{a,(3)}^{(k_3)} - X_{a,(1,2)}^{(k_1,k_2)} \right) \]

where

\[ P_{a,(3;1,2)}^{[k_1,k_2]} = P_{N+1}(\ldots, X_{a-1,(1,2)}^{(k_1,k_2)}, X_{a,(3)}^{(k_3)}, X_{a,(1,2)}^{(k_1,k_2)}, \ldots) - P_{N+1}(\ldots, X_{a-1,(1,2)}^{(k_1,k_2)}, X_{a,(1,3)}^{(k_1,k_2)}, X_{a,(1,2)}^{(k_1,k_2)}), \]

and

\[ \widehat{V}_{(1,2;3)}^{[k_1,k_2]} := \bigotimes_{a=1}^{k_3} K \left( P_{a,(1;2;3)}^{[k_1,k_2]} \cdot X_{a,(1,2)}^{(k_3)} - X_{a,(3)}^{(k_3)} \right) \]

where

\[ P_{a,(1;2;3)}^{[k_1,k_2]} = P_{N+1}(\ldots, X_{a-1,(3)}^{(k_3)}, X_{a,(1,2)}^{(k_1,k_2)}, X_{a,(1,3)}^{(k_3)}, X_{a,(1,2)}^{(k_1,k_2)}, \ldots) - P_{N+1}(\ldots, X_{a-1,(3)}^{(k_3)}, X_{a,(1,2)}^{(k_3)}, X_{a,(1,2)}^{(k_1,k_2)}). \]

The proof of the following proposition is analogous to the one of Proposition 5.9.

**Proposition 5.11.** We have the following results:

1. \( \Lambda_{(3,1,2)}^{[k_1,k_2]} \) and \( \widehat{V}_{(1,2;3)}^{[k_1,k_2]} \) are homotopy equivalent to the zero matrix factorization if \( k_3 \geq N + 1 \);

2. \( \Lambda_{(3,1,2)}^{[k_1,k_2]} \) and \( \widehat{V}_{(1,2;3)}^{[k_1,k_2]} \) are indecomposable for \( 0 \leq k_3 \leq N \).

The following proposition can be proved by direct computation.

**Proposition 5.12.** We have the following isomorphisms:

\[ \Lambda_{(3,1,2)}^{[0,k_2]} \sim \widehat{L}_{(3;2)}^{[k_2]}, \quad \Lambda_{(3,1,2)}^{[k_1,0]} \sim \widehat{L}_{(3;1)}^{[k_1]}, \]

\[ \widehat{V}_{(1,2;3)}^{[0,k_2]} \sim \widehat{L}_{(2;3)}^{[k_2]}, \quad \Lambda_{(1,2;3)}^{[k_1,0]} \sim \widehat{L}_{(1;3)}^{[k_1]}, \]
Tensoring the matrix factorizations above, we can associate a matrix factorizations \( \hat{u} \) to any monomial \( \mathfrak{sl}_N \)-web from \( \vec{k} \) to \( \vec{k}' \) without tags (but with oriented \( N \)-colored edges). Note that for cups and caps we use the same matrix factorization as for \( L^{[\vec{k}]}_{(1,2)} \). We also have

\[
\hat{u} \cdot \cong \hat{u}^* \{ d(\vec{k}) \}\langle 1 \rangle,
\]

for any monomial web \( u \in W(\vec{k}, N) \). By Lemma 5.7, this implies that

\[
\text{EXT}(\hat{u}, \hat{v}) \cong H(\hat{u} \otimes_{R^{\vec{k}}} \hat{v}) \cong H(\hat{u}^*v)\{d(\vec{k})\}\langle 1 \rangle
\]

for any \( u, v \in W(\vec{k}, N) \).

For the proof of the following theorem, we refer to Sections 6 through 11 in [39] and Section 3 in [40].

**Theorem 5.13** (Wu, Yonezawa). The matrix factorizations associated to webs without tags satisfy all relations in Definition 4.1, except the first one, up to homotopy equivalence. These equivalences are \( q \)-degree preserving, but might involve homological degree shifts.

### 6. CATEGORIZED QUANTUM \( \mathfrak{sl}_m \) AND 2-REPRESENTATIONS

#### 6.1. CATEGORIZED \( \hat{U}_q(\mathfrak{sl}_m) \)

Khovanov and Lauda introduced diagrammatic 2-categories \( \mathcal{U}(g) \) which categorify the integral version of the corresponding idempotented quantum groups [18]. Independently, Rouquier also introduced similar 2-categories [33]. Subsequently, Cautis and Lauda [11] defined diagrammatic 2-categories \( \mathcal{U}_Q(g) \) with implicit scalars \( Q \) consisting of \( t_{ij} \), \( r_i \) and \( s_{ij}^{pq} \) which determine certain signs in the definition of the categorized quantum groups.

In this section, we recall \( \mathcal{U}_Q(\mathfrak{sl}_m) \) briefly and choose the implicit scalars \( Q \) to be given by \( t_{ij} = -1 \) if \( j = i + 1 \), \( t_{ij} = 1 \) otherwise, \( r_i = 1 \) and \( s_{ij}^{pq} = 0 \). This corresponds precisely to the signed version in [18, 19]. The other conventions here are the same as those in Section 3.

**Definition 6.1** (Khovanov-Lauda). The 2-category \( \mathcal{U}_Q(\mathfrak{sl}_m) \) is defined as follows:

- The objects in \( \mathcal{U}_Q(\mathfrak{sl}_m) \) are the weights \( \lambda \in \mathbb{Z}^{m-1} \).

For any pair of objects \( \lambda \) and \( \lambda' \) in \( \mathcal{U}_Q(\mathfrak{sl}_m) \), the hom category \( \mathcal{U}_Q(\mathfrak{sl}_m)(\lambda, \lambda') \) is the graded additive \( \mathbb{C} \)-linear category consisting of:

- objects (1-morphisms in \( \mathcal{U}_Q(\mathfrak{sl}_m) \)), which are finite formal sums of the form \( \mathcal{E}_{\vec{t}} I_\lambda \{ t \} \) where \( t \in \mathbb{Z} \) is the grading shift and \( \vec{t} \) is a signed sequence such that \( \lambda' = \lambda + \sum_{a=1}^{\vec{t}} \epsilon_a t'_a \).
- morphisms from \( \mathcal{E}_{\vec{t}} I_\lambda \{ t \} \) to \( \mathcal{E}_{\vec{t}'} I_{\lambda'} \{ t' \} \) in \( \mathcal{U}_Q(\mathfrak{sl}_m)(\lambda, \lambda') \) (2-morphisms in \( \mathcal{U}_Q(\mathfrak{sl}_m) \)) are \( \mathbb{C} \)-linear combinations of diagrams with degree \( t' - t \) spanned by composites of the following diagrams:
As already remarked, the relations on the 2-morphisms are those of the signed version in [18, 19], which we do not recall here because we do not need them explicitly in this paper.

We recall Khovanov and Lauda’s Proposition 1.4 in [18].

**Theorem 6.2 (Khovanov-Lauda).** The linear map

\[ \bar{U}_q(\mathfrak{sl}_m) \rightarrow K_q^0(U_Q(\mathfrak{sl}_m)) \]

defined by

\[ q^t E_n 1 \lambda \rightarrow \mathcal{E}_n 1 \lambda \{ t \} \]

is an isomorphism of algebras.

### 6.2. Cyclotomic KLR algebras and 2-representations

Let \( \Lambda \) be a dominant \( \mathfrak{sl}_m \)-weight, \( V_\Lambda \) the irreducible \( \bar{U}_q(\mathfrak{sl}_m) \)-module of highest weight \( \Lambda \) and \( P_\Lambda \) the set of weights in \( V_\Lambda \).

**Definition 6.3 (Khovanov-Lauda, Rouquier).** The cyclotomic KLR algebra \( R_\Lambda \) is the subquotient of \( U_Q(\mathfrak{sl}_m) \) defined by the subalgebra of all diagrams with only downward oriented strands and right-most region labeled \( \Lambda \) modded out by the ideal generated by diagrams of the form

\[ \mathcal{E}_{(i, i+1)}^\Lambda \rightarrow \mathcal{E}_{(i, i+1)}^\Lambda \{ -a \} \]

Note that

\[ R_\Lambda = \bigoplus_{\mu \in P_\Lambda} R_\Lambda(\mu), \]

where \( R_\Lambda(\mu) \) is the subalgebra generated by all diagrams whose left-most region is labeled \( \mu \). Brundan and Kleshchev proved that \( R_\Lambda \) is finite-dimensional in Corollary 2.2 in [3]. We also define

\[ V_\Lambda^p := R_\Lambda - \text{pmod}_{gr}. \]

Below we will use Cautis and Lauda’s language of strong \( \mathfrak{sl}_m \) 2-representations (see Definition 1.2 in [11]). For a comparison with Rouquier’s [33] definition of a Kac-Moody 2-representation, see Cautis and Lauda’s remark (1) below their Definition 1.2. Since we
always use the same choice of \( Q \) in this paper, which we specified above, we call the 2-representations below simply strong 2-representations.

In Section 4.4 in [2] Brundan and Kleshchev defined a strong \( \mathfrak{sl}_m \) 2-representation on \( \mathcal{V}_\Lambda \), which can be restricted to \( \mathcal{V}_\Lambda^p \).

Brundan and Kleshchev proved the following theorem (Proposition 4.16 and Theorem 4.18 in [2]), which was conjectured by Khovanov and Lauda [17].

**Theorem 6.4** (Brundan-Kleshchev). There exists an isomorphism

\[
\delta : \mathcal{V}_\Lambda \to K_0^q(\mathcal{V}_\Lambda^p)
\]

of \( \mathcal{U}_q(\mathfrak{sl}_m) \)-modules.

Moreover, this isomorphism maps intertwines the \( q \)-Shapovalov form and the Euler form.

Rouquier proved that \( R_\Lambda \) is the universal categorification of \( \mathcal{V}_\Lambda \) in the following sense. For each \( \mu \in P_\Lambda \), let \( C(\mu) \) be a graded Krull-Schmidt \( \mathbb{C} \)-linear category with finite-dimensional hom-spaces. Take

\[
C_\Lambda := \bigoplus_{\mu \in P_\Lambda} C(\mu).
\]

For a proof of the following result, see Lemma 5.4, Proposition 5.6 and Corollary 5.7 in [33].

**Proposition 6.5** (Rouquier’s Universality Proposition, additive version). Suppose that

- \( C_\Lambda \) is a strong 2-representation of \( \mathfrak{sl}_m \) by \( \mathbb{C} \)-linear functors;
- There exists an indecomposable object \( V(\Lambda) \in C(\Lambda) \) such that \( \mathcal{E}_i V(\Lambda) = 0 \), for any \( i = 1, \ldots, m - 1 \), and \( \text{End}(V(\Lambda)) \cong \mathbb{C} \);
- any object in \( C_\Lambda \) is a direct summand of \( XV(\Lambda) \), for some 1-morphism \( X \in \mathcal{U}_Q(\mathfrak{sl}_m) \).

Then there exists an equivalence

\[
\mathcal{V}_\Lambda^p \to C_\Lambda
\]

of additive strong \( \mathfrak{sl}_m \) 2-representations.

In particular, we have

\[
\mathcal{V}_\Lambda \cong K_0^q(R_\Lambda) \cong K_0^q(C_\Lambda)
\]

as \( \mathcal{U}_q(\mathfrak{sl}_m) \)-modules.

7. Some morphisms in HMF

In this section, we recall some useful morphisms between matrix factorizations.

As before, let \( R \) be a graded polynomial ring and \( p_a, q_a \ (a = 1, \ldots, l) \) be polynomials in \( R \). Consider the Koszul matrix factorization

\[
\bigotimes_{a=1}^l K(p_a; q_a)_R.
\]

Recall the following result which we use frequently in the following sections. The proof follows directly from the definitions.

**Proposition 7.1.** Multiplication by \( p_a \) or \( q_a \) defines an endomorphism of \( \bigotimes_{a=1}^l K(p_a; q_a)_R \) which is homotopic to 0.
7.1. Morphism in HMF (1). We consider the following diagrams:

\[ \begin{array}{c}
\Gamma_1 \\
\downarrow \Gamma_1 \quad \Gamma_2 \\
L_{(1;2)}^{[k+1]} \\
\end{array} \]

The matrix factorization \( \hat{\Gamma}_1 \) is isomorphic to

\[ \hat{\Lambda}_{(1,3;4)}^{[k]} \otimes_{R_{(3,4)}} \hat{\Lambda}_{(3,4;2)}^{[k]} \]

\[ \hat{L}_{(1;2)}^{[k+1]} \otimes_{R_{(1,2)}} R_{(1,2)}^{(k+1,k+1,k+1)} / J_{(3,4)}^{(1,k)} \{ -k \} \cong \hat{L}_{(1;2)}^{[k+1]} \oplus \hat{L}_{(1;2)}^{[k+1]} \]

where \( J_{(3,4)}^{(1,k)} = \langle X_{1,3;4}, \ldots, X_{k+1,3;4} \rangle \). We have two \( R_{(1,2)}^{(k,k)} \)-module morphisms of degree \(-k\)

\[ I_{(4,3)} : \]

\[ \xymatrix{ \ar[rr]^{R_{(1,2)}^{(k+1+k+1,k+1)}} & & \ar[r]_{J_{(3,4)}^{(1,k)}} & \ar[r]_{\{ -k \}} & \}

\[ \downarrow \]

\[ \ar[r]_{\{ -k \}} & \]

\[ D_{(4,3)} : \]

\[ \xymatrix{ \ar[rr]^{R_{(1,2,3,4)}^{(k+1+k+1+k+1)}} & & \ar[r]_{J_{(3,4)}^{(1,k)}} & \ar[r]_{\{ -k \}} & \}

\[ \downarrow \]

\[ \ar[r]_{\{ -k \}} & \]

\[ \partial_{t_{1},t_{1},3} \partial_{t_{2},t_{1},3} \ldots \partial_{t_{k},t_{1},3} f, \]

where \( \partial_{t_{j},t_{1},3} g(t_{1},t_{1},t_{3},t_{j},t_{4},\ldots,t_{k},t_{4}) \) is defined by

\[ g(t_{1},t_{3},\ldots,t_{j},t_{4},\ldots,t_{k},t_{4}) - g(t_{j},t_{j},t_{1},t_{3},\ldots,t_{k}) \]

\[ t_{j} - t_{1},t_{3} \]

Using (17), we can extend the maps \( I_{(4,3)} \) and \( D_{(4,3)} \) to morphisms of matrix factorizations of degree \(-k\)

\[ \hat{I}_{(1,k)} : \hat{L}_{(1;2)}^{[k+1]} \rightarrow \hat{\Gamma}_1, \]

\[ \hat{D}_{(1,k)} : \hat{\Gamma}_1 \rightarrow \hat{L}_{(1;2)}^{[k+1]} \]

Since \( \Gamma_2 \) is symmetric to \( \Gamma_1 \), we also have

\[ \hat{I}_{(k,1)} : \hat{L}_{(1;2)}^{[k+1]} \rightarrow \hat{\Gamma}_2, \]

\[ \hat{D}_{(k,1)} : \hat{\Gamma}_2 \rightarrow \hat{L}_{(1;2)}^{[k+1]} \]

7.2. Morphisms in HMF (2). We consider the following diagrams:

\[ \Gamma_3 \\
\downarrow \Gamma_3 \\
L_{(1;3)}^{[1]} \sqcup L_{(2;4)}^{[1]} \]

\[ \Gamma_4 \\
\downarrow \Gamma_4 \\
L_{(1;3)}^{[k]} \sqcup L_{(2;4)}^{[1]} \]
We have

\begin{equation}
\hat{\Gamma}_3 = \hat{\Lambda}_{(5,3)}^{[1,k]} \bigotimes_{R(5)}^{(k+1)} \hat{V}_{(1,2;5)}^{[1,k]} \simeq \\
\hat{S}_{(1,2;3,4)} \bigotimes_{R(1,2;3,4)}^{(1,k,1,k)} K(p_{k+1}; (t_{1,1} - t_{1,3})X_{k,(2,3)}^{(k,-1)})_{R^{(1,k,1,k)}_{(1,2;3,4)}} \{ -k \}
\end{equation}

and

\begin{equation}
\hat{L}^{[1]}_{(1;3)} \bigotimes L^{[k]}_{(2;4)} \simeq \hat{S}_{(1,2;3,4)} \bigotimes_{R(1,2;3,4)}^{(1,k,1,k)} K(p_{k+1}X_{k,(2,3)}^{(k,-1)}; (t_{1,1} - t_{1,3}))_{R^{(1,k,1,k)}_{(1,2;3,4)}}
\end{equation}

where

\[ \hat{S}_{(1,2;3,4)} = \bigotimes_{a=1}^{k+1} K(p_a; (t_{1,1} - t_{1,3})X_{a-1,(2,3)}^{(k,-1)} + x_{a,2} - x_{a,3})_{R^{(1,k,1,k)}_{(1,2;3,4)}} \]

with

\[
p_a = \frac{P_{N+1}(..., X_{a-1,(3,4)}^{(1,k)}, X_{a,(1,2)}, ...) - P_{N+1}(..., X_{a,(3,4)}, X_{1,3+1,(3,4)}, ...) - X_{a,(1,2)} + X_{a,(3,4)}}{X_{a+1,(1,2)} - X_{a,(3,4)}} + t_{1,3} \frac{P_{N+1}(..., X_{a-1,(3,4)}^{(1,k)}, X_{a+1,(1,2)}, ...) - P_{N+1}(..., X_{a+1,(3,4)}, X_{a+2,(1,2)}, ...) - X_{a+1,(1,2)} + X_{a,(3,4)}}{X_{a+1,(1,2)} - X_{a,(3,4)}}
\]

for \( 1 \leq a \leq k \), and

\[ p_{k+1} = \frac{P_{N+1}(..., X_{k,(3,4)}^{(1,k)}, X_{k+1,(1,2)}, ...) - P_{N+1}(..., X_{k,(3,4)}, X_{k+1,(1,2)}, ...) - X_{k+1,(1,2)} + X_{k,(3,4)}}{X_{k+1,(1,2)} - X_{k,(3,4)}}. \]

By (23) and (24), the morphisms of degree \( k \)

\[ (1, X_{k,(2,3)}^{(k,-1)} ) : K(p_{k+1}; (t_{1,1} - t_{1,3})X_{k,(2,3)}^{(k,-1)})_{R^{(1,k,1,k)}_{(1,2;3,4)}} \]

\[ \rightarrow K(p_{k+1}X_{k,(2,3)}^{(k,-1)}; (t_{1,1} - t_{1,3}))_{R^{(1,k,1,k)}_{(1,2;3,4)}} \]

\[ (X_{k,(2,3)}^{(k,-1)}, 1) : K(p_{k+1}X_{k,(2,3)}^{(k,-1)}; (t_{1,1} - t_{1,3}))_{R^{(1,k,1,k)}_{(1,2;3,4)}} \]

\[ \rightarrow K(p_{k+1}; (t_{1,1} - t_{1,3})X_{k,(2,3)}^{(k,-1)})_{R^{(1,k,1,k)}_{(1,2;3,4)}} \]

induce morphisms of degree \( k \) between \( \hat{\Gamma}_3 \) and \( \hat{\Gamma}^{[1]}_{(1;3)} \bigotimes \hat{L}^{[k]}_{(2;4)} \)

\[ \hat{U}_{(k,1)} : \hat{\Gamma}_3 \xrightarrow{\text{Id}_C \otimes X_{k,(2,3)}^{(k,-1)}} \hat{L}^{[1]}_{(1;3)} \bigotimes \hat{L}^{[k]}_{(2;4)} \]

\[ \hat{Z}_{(k,1)} : \hat{L}^{[1]}_{(1;3)} \bigotimes \hat{L}^{[k]}_{(2;4)} \xrightarrow{\text{Id}_C \otimes X_{k,(2,3)}^{(k,-1)}} \hat{\Gamma}_3. \]

Using Proposition 5.6, we get

**Proposition 7.2.**

1. The morphism \( \hat{U}_{(k,1)} \) induces the \( R^{(1,k,1,k)}_{(1,2;3,4)} \)-linear map in

\[ \text{EXT}^0(\hat{\Gamma}_3, \hat{L}^{[1]}_{(1;3)} \bigotimes \hat{L}^{[k]}_{(2;4)}) \]

_determined by sending 1 to 1._
(2) The morphism $\hat{Z}_{(k,1)}$ induces the $R_{(1,2,3,4)}^{(1,k,1,k)}$-linear map in
\[
\text{EXT}^0(\hat{L}^{[1]}_{(1;3)} \boxtimes \hat{L}^{[k]}_{(2;4)}, \hat{\Gamma}_3)
\]
determined by multiplying with $X_{k(2,3)}^{(k-1)}$.

Similarly, we get degree $k$ morphisms
\[
\hat{U}_{(1,k)}: \hat{\Gamma}_4 \xrightarrow{\text{Id}_{\hat{\mathfrak{g}} \boxtimes (1, X_{k(1,4)}^{(k-1)})}} \hat{L}^{[k]}_{(1;3)} \boxtimes \hat{L}^{[1]}_{(2;4)}
\]
\[
\hat{Z}_{(k,1)}: \hat{L}^{[k]}_{(1;3)} \boxtimes \hat{L}^{[1]}_{(2;4)} \xrightarrow{\text{Id}_{\hat{\mathfrak{g}} \boxtimes (X_{k(1,4)}^{(k-1)}, 1)}} \hat{\Gamma}_4.
\]

7.3. Morphism in HMF (3). We consider the following diagrams:

\[
\begin{array}{c}
\begin{array}{c}
1 \\
\downarrow^k \\
3 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
2 \\
\downarrow^k \\
4 \\
\end{array}
\end{array}
\}
\Gamma_5
\]
\[
\begin{array}{c}
\begin{array}{c}
1 \\
\downarrow^k \\
3 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
2 \\
\downarrow^k \\
4 \\
\end{array}
\end{array}
\}
\Gamma_6
\]

We have
\[
\hat{\Gamma}_5 = \hat{\Lambda}_{(5,3,4)}^{[1,1]} \boxtimes R_{(5)}^{(1,k,1)} \hat{V}_{(1,2,5)}^{[k,1]} \simeq \hat{T}_{(1,2,3,4)} \boxtimes R_{(1,2,3,4)}^{(k,1,k,1)} K(q_{k+1}; (t_{1,1} - t_{1,4}) X_{k(1,3)}^{(-1,k)}),
\]
\[
\hat{\Gamma}_6 = \hat{\Lambda}_{(2,5,4)}^{[k-1,1]} \boxtimes R_{(5)}^{(k,1)} \hat{V}_{(1,5,3)}^{[1,k-1]} \simeq \hat{T}_{(1,2,3,4)} \boxtimes R_{(1,2,3,4)}^{(k,1,k,1)} K(q_{k+1} (t_{1,1} - t_{1,4}); X_{k(1,3)}^{(-1,k)}),
\]

where
\[
\hat{T}_{(1,2,3,4)} = \bigotimes_{a=1}^k K(q_a; (t_{1,1} - t_{1,4}) X_{a-1,(1,3)}^{(-1,k)}) + x_{a,2} - x_{a,3}) R_{(1,2,3,4)}^{(1,k,1)}
\]
\[
q_a = \frac{P_{N+1}(..., X_{a-1,(3,4)}^{(k,1)}, X_{a,(1,2)}^{(1,k)}, ...) - P_{N+1}(..., X_{a,(3,4)}^{(k,1)}, X_{a+1,(1,2)}^{(1,k)}, ...)}{X_{a,(1,2)}^{(1,k)} - X_{a,(3,4)}^{(k,1)}} + t_{1,1} \frac{P_{N+1}(..., X_{a,(3,4)}^{(k,1)}, X_{a+1,(1,2)}^{(1,k)}, ...) - P_{N+1}(..., X_{a+1,(3,4)}^{(k,1)}, X_{a+2,(1,2)}^{(1,k)}, ...)}{X_{a+1,(1,2)}^{(1,k)} - X_{a+1,(3,4)}^{(k,1)}}
\]
\[
q_{k+1} = \frac{P_{N+1}(..., X_{k,(3,4)}^{(k,1)}, X_{k+1,(1,2)}^{(1,k)}, ...) - P_{N+1}(..., X_{k,(3,4)}^{(k,1)}, X_{k+1,(3,4)}^{(k,1)}, ...)}{X_{k+1,(1,2)}^{(1,k)} - X_{k+1,(3,4)}^{(k,1)}}.
\]
By the isomorphisms in (25) and (26), we see that the degree 1 morphisms
\[(1, t_{1,1} - t_{1,4}) : K(q_{k+1}; (t_{1,1} - t_{1,4})X^{(-1,k)}_{k,(1,3)})_{R(1,2,3,4)} \]
\[\rightarrow K(q_{k+1}(t_{1,1} - t_{1,4}); X^{(-1,k)}_{k,(1,3)})_{R(1,2,3,4)} \{1\}\]
\[(t_{1,1} - t_{1,4}, 1) : K(q_{k+1}(t_{1,1} - t_{1,4}); X^{(-1,k)}_{k,(1,3)})_{R(1,2,3,4)} \{1\}\]
induce morphisms
\[\overline{TU}_{(k,1)} : \Gamma_5 \rightarrow \Gamma_6\]
\[\overline{TZ}_{(k,1)} : \Gamma_6 \rightarrow \Gamma_5.\]

**Proposition 7.3.**
(1) \(\overline{TU}_{(k,1)}\) corresponds to the \(R(1,k,k,1)\)-linear map in \(\text{EXT}^0(\Gamma_5, \Gamma_6)\) determined by sending 1 to 1.
(2) \(\overline{TZ}_{(k,1)}\) corresponds to the \(R(1,k,k,1)\)-linear map in \(\text{EXT}^0(\Gamma_6, \Gamma_5)\) determined by multiplying with \(t_{1,1} - t_{1,4}\).

8. The 2-category \(HMF_{m,d,N}\)

8.1. The matrix factorizations \(\hat{E}_{\pm 1}\)

In this case, we have \(\alpha = (1, -1)\). For \(1 \leq a \leq k_2\), we define the matrix factorization
\[(27) \quad \hat{E}^{(a)}_{+, [\vec{k}]} := \hat{V}^{[k_2-a,a]}_{(2', j; 2)} \hat{\Lambda}^{[a,k_1]}_{(1', j; 1;)} \in \text{HOM}_{HMF_{m,N}}(\vec{k}, \vec{k} + a\alpha)\]
associated to the diagram in Figure 3.

\[\begin{array}{c}
2' \\
\downarrow k_2-a \\
\downarrow j \\
2 \\
\end{array} \quad \begin{array}{c}
1' \\
\downarrow k_1+a \\
\downarrow k_1 \\
1 \\
\end{array} \quad \begin{array}{c}
\uparrow a \\
\end{array} \]

**Figure 3.** \(E^{(a)}_{+, [\vec{k}]}\)

Similarly, for \(1 \leq a \leq k_1\), we define the matrix factorization
\[(28) \quad \hat{E}^{(a)}_{-, [\vec{k}]} := \hat{V}^{[k_2,a]}_{(2'; 2; j)} \hat{\Lambda}^{[a,k_1-a]}_{(j, 1'; j; 1)} \in \text{HOM}_{HMF_{m,N}}(\vec{k}, \vec{k} - a\alpha)\]
associated to the diagram in Figure 4.
The proof of the following proposition is analogous to that of Proposition 5.9(2).

**Proposition 8.1.** The matrix factorizations \(\hat{E}^{(a)}_{+, [\vec{k}]}\) and \(\hat{E}^{(a)}_{-, [\vec{k}]}\) are indecomposable.
The general case $\check{k} = (k_1, \ldots, k_m)$:

In this case we have $\alpha_i = \epsilon_i - \epsilon_{i+1}$, for $i = 1, \ldots, m - 1$. We define

$$
\hat{\mathbf{1}}_{[\check{k}]} := \hat{L}^{[k_1]} \times \hat{L}^{[k_2]} \times \cdots \times \hat{L}^{[k_m]} \in \text{HOM}_{\text{HMF}_{m,N}}(\check{k}, \check{k})
$$

distributed to the diagram in Figure 5.

For $i = 1, \ldots, m - 1$ and $1 \leq a \leq k_{i+1}$, we define the matrix factorization

$$
\hat{E}_+^{(a)}_{i,[\check{k}]} := \hat{1}_{[(k_{i+2}, \ldots, k_m)]} \times \hat{E}_+^{(a)}_{i,[(k_i, k_{i+1})]} \in \text{HOM}_{\text{HMF}_{m,N}}(\check{k}, \check{k} + a\alpha_i)
$$

distributed to the diagram in Figure 6.

and, for $1 \leq a \leq k_i$, the matrix factorization

$$
\hat{E}_-^{(a)}_{-i,[\check{k}]} := \hat{1}_{[(k_{i+2}, \ldots, k_m)]} \times \hat{E}_-^{(a)}_{-i,[(k_i, k_{i+1})]} \in \text{HOM}_{\text{HMF}_{m,N}}(\check{k}, \check{k} - a\alpha_i)
$$

distributed to the diagram in Figure 7.

The proof of the following proposition follows from Propositions 5.9(2) and 8.1.

Proposition 8.2. $\hat{E}_+^{(a)}_{i,[\check{k}]}$ and $\hat{E}_-^{(a)}_{-i,[\check{k}]}$ are indecomposable.
Next we have to explain in which order we glue these webs and tensor the corresponding matrix factorizations. The logic is determined by the fact that these webs determine an action of $\hat{U}_q(\mathfrak{sl}_m)$ on $W_\Lambda$ by Proposition 4.5, which the corresponding matrix factorizations categorify, as we will show in Theorem 9.2 and Section 9.

For any signed sequence $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l)$ and any sequence of non-negative integers $a = (a_1, \ldots, a_l)$, let

$$ (\vec{k}, a, \varepsilon)_j := \vec{k} + \sum_{s=j+1}^l \varepsilon_s a_s \alpha_s $$

for $j = 1, \ldots, l - 1$. By convention, we put

$$ (\vec{k}, a, \varepsilon)_l := \vec{k}. $$

As before, let $N \geq 2$ and $m, d \geq 0$ be arbitrary integers.

**Definition 8.4.** We define the 2-category $\text{HMF}_{m,d,N}$ as follows:

- The set of objects is $\Lambda(m, d)_N$.
- For any pair of objects $\vec{k}, \vec{k}' \in \Lambda(m, d)_N$, we define the hom-category $\text{Hom}_{\text{HMF}_{m,d,N}}(\vec{k}, \vec{k}')$ to be the full subcategory of $\text{HMF}_{R^N}(P_{N+1}(X^{\vec{k}}) - P_{N+1}(X^{\vec{k}'})$ whose set of objects is

$$ \left\{ \hat{E}^{(\varepsilon)}_{\vec{k}, \vec{k}'} \mid (\vec{k}, a, \varepsilon) = \vec{k}' \right\}. $$

Horizontal composition is defined by tensoring and vertical composition by composing morphisms between matrix factorizations.
In the rest of this section we define certain 1 and 2-morphisms in $HMF^*_{m,N}$ which are crucial for Section 9. We always assume that $\vec{k} \in \Lambda(m,d)_N$.

8.2. Some useful 2-morphisms. In this section we define some useful 2-morphisms in $HMF_{m,N}$.

Definition 8.5. We define the endomorphisms of $\hat{E}_{+,[k]}$ and $\hat{E}_{-,[k]}$ of degree $2b$, denoted $\hat{t}^b_{1,j}$, by

$$\hat{t}^b_{1,j} := (t^b_{1,j}, t^b_{1,j}) \in \text{End}_{HMF_{m,N}}(\hat{E}_{+,[k]})$$

For the proof of the following proposition see [39, 40, 41].

Proposition 8.6. If $b \geq N$, then $\hat{t}^b_{1,j} = 0$.

For the following definition consult Figure 8. The map $\hat{\phi}_{j_1,j_2}$ is induced by the homotopy equivalences

$$\hat{\Gamma}_2 \simeq \hat{\Gamma}_2/\langle(t_{1,j_1} - t_{1,j_2}) \rangle \simeq \hat{\Gamma}_2^\prime$$

and $\hat{\psi}_{j_1,j_2}$ is its inverse.

Definition 8.7. We define the morphisms

\begin{align*}
\overline{CU}_{+,[\vec{k}]} & : \quad \hat{1}_{[\vec{k}]} \quad \rightarrow \quad \hat{1}_{[\vec{k}]} \\
\overline{CU}_{-,[\vec{k}]} & : \quad \hat{1}_{[\vec{k}]} \quad \rightarrow \quad \hat{1}_{[\vec{k}]}
\end{align*}

by

$$\overline{CU}_{+,[\vec{k}]} := (-1)^{k_i} \tilde{Z}_{(1,k_i)} \hat{\phi}_{j_1,j_2} \hat{I}_{(k_{i+1}-1,1)};$$

$$\overline{CU}_{-,[\vec{k}]} := \tilde{Z}_{(k_{i+1},1)} \hat{\phi}_{j_1,j_2} \hat{I}_{(1,k_i-1)}.$$

Their degree is:

$$\deg(\overline{CU}_{+,[\vec{k}]}) = k_i - k_{i+1} + 1,$$

$$\deg(\overline{CU}_{-,[\vec{k}]}) = -k_i + k_{i+1} + 1.$$

The following lemma follows directly from the definitions and its proof is left to the reader.
Lemma 8.8. $\widehat{CU}_{+i,[k]}$ is the $R^k$-linear map in $\text{EXT}^0(\widehat{1}_{[k]}, \widehat{E}_{(-i,+i),[k]})$ determined by multiplying with $(-1)^{k_i}X_{k_i(i,j_2)}^{(k_i-1)}$.

Similarly, $\widehat{CU}_{-i,[k]}$ is the $R^k$-linear map in $\text{EXT}^0(\widehat{1}_{[k]}, \widehat{E}_{(+i,-i),[k]})$ determined by multiplying with $X_{k_{i+1}(i+1,j_2)}^{(k_{i+1}-1)}$.

The following definition is similar.

Definition 8.9. We define

$\widehat{CA}_{+i,[k]} :$ \hspace{1cm} $\widehat{CA}_{-i,[k]} :$

by

$\widehat{CA}_{+i,[k]} := \widehat{D}(k_{i+1}-1,1)\widehat{\psi}_{j_1,j_2}\widehat{U}(1,k_i)$;

$\widehat{CA}_{-i,[k]} := (-1)^{k_i}\widehat{D}(1,k_i-1)\widehat{\psi}_{j_1,j_2}\widehat{U}(k_{i+1},1)$.

Their degree is:

$\deg(\widehat{CA}_{+i,[k]}) = k_i - k_{i+1} + 1$,

$\deg(\widehat{CA}_{-i,[k]}) = -k_i + k_{i+1} + 1$.

Lemma 8.10. $\widehat{CA}_{+i,[k]}$ corresponds to the $R^k$-linear map in $\text{EXT}^0(\widehat{E}_{(-i,+i),[k]}, \widehat{1}_{[k]})$ given by

$f(t_1, \ldots, t_{k_{i+1}-1}, t_{1,j_1}, t_{1,j_2}) \mapsto \partial_{t_1} t_{1,j_1} \partial_{t_{2} t_{1,j_1}} \ldots \partial_{t_{k_{i+1}-1} t_{1,j_1}} f_{j_1=j_2}$.

Here $t_1, \ldots, t_{k_{i+1}-1}$ are the edge variables for the left vertical edge inside the square and $f_{j_1=j_2}$ is the polynomial obtained by putting $t_{1,j_1}$ equal to $t_{1,j_2}$ in $f$.

$\widehat{CA}_{-i,[k]}$ corresponds to the $R^k$-linear map in $\text{EXT}^0(\widehat{E}_{(+i,-i),[k]}, \widehat{1}_{[k]})$ given by

$f(t_1, \ldots, t_{k_i-1}, t_{1,j_1}, t_{1,j_2}) \mapsto \partial_{t_{1,j_1}} t_{1,j_1} \partial_{t_{1,j_1} t_{2}} \ldots \partial_{t_{1,j_1} t_{k_i-1}} f_{j_1=j_2}$.

Here $t_1, \ldots, t_{k_i-1}$ are the edge variables for the right vertical edge inside the square.

For the following definition, consider Figure 9. We have a canonical isomorphism

$\widehat{\Phi}_1 : \widehat{E}_{(-i,-i),[k]} \rightarrow \widehat{1}$

and the morphism

$\widehat{D}_{(j_1,j_2)} : \widehat{1} \rightarrow \widehat{E}^{(2)}_{-i,[k]}$.

We have a similar Figure for $\widehat{E}_{(+i,+i),[k]}$ and $\widehat{E}^{(2)}_{+i,[k]}$ and corresponding morphisms, for which we use the same notation.
Definition 8.11. We define the morphisms

\[ \hat{D}_{+i(j_1,j_2)} : \]

\[ \hat{D}_{-i(j_1,j_2)} : \]

by

\[ \hat{D}_{+i(j_1,j_2)} := \hat{D}_{(1,1)} \hat{\Phi}_1 \]
\[ \hat{D}_{-i(j_1,j_2)} := -\hat{D}_{(1,1)} \hat{\Phi}_1. \]

Their degree is:

\[ \deg(\hat{D}_{+i(j_1,j_2)}) = \deg(\hat{D}_{-i(j_1,j_2)}) = -1. \]

Let

\[ \hat{\Phi}_2 : \hat{\Gamma}_1 \to \hat{E}_{(-i,-i),[\vec{k}]} \]

be the inverse of \( \hat{\Phi}_1 \). Recall also the morphism

\[ \hat{I}_{(j_1,j_2)} : \hat{E}_{-i,[\vec{k}]}^{(2)} \to \hat{\Gamma}_1. \]

Definition 8.12. We define the morphisms

\[ \hat{I}_{+i(j_1,j_2)} : \]

\[ \hat{I}_{-i(j_1,j_2)} : \]

by

\[ \hat{I}_{+i(j_1,j_2)} := \hat{\Phi}_2 \hat{I}_{(1,1)} \]
\[ \hat{I}_{-i(j_1,j_2)} := \hat{\Phi}_2 \hat{I}_{(1,1)}. \]
Their degree is:

\[ \text{deg}(\hat{r}_{i+j_1,j_2}) = \text{deg}(\hat{r}_{-(i+j_1,j_2)}) = -1. \]

Consider the following figure:

\[ \hat{E}_{(-i,-(i+1),[k])} \quad \hat{E}'_{(-(i+1),-i),[k]} \quad \hat{E}_{(-(i+1),-i),[k]} \]

**Figure 10. 2-morphisms III**

In

\[ \Gamma'_3 \quad \Gamma'_4 \quad \Gamma'_5, \]

which is part of Figure 10, we have the morphism

\[ \hat{TZ}(k_{i+1,1}) : \hat{\Gamma}'_3 \to \hat{\Gamma}'_4 \]

from Section 7.3 and the isomorphism

\[ \hat{s}_{j_1,j_2} : \hat{\Gamma}'_4 \to \hat{\Gamma}'_5 \]

which swaps the variables \( t_{1,j_1} \) and \( t_{1,j_2} \).

Of course, there also exists an analogous figure for \( \hat{E}_{(i,i+1),[k]} \) and \( \hat{E}_{(i+1,i),[k]} \).

**Definition 8.13.** We define the morphisms

\[ \hat{CR}_{+(i+1),i} : \quad \hat{CR}_{-(i,-(i+1))} : \]

by

\[ \hat{CR}_{+(i+1),i} := \hat{s}_{j_1,j_2} (\hat{\text{Id}}_{[k_1,k_i]} \otimes \hat{TZ}(1,k_{i+1}) \otimes \hat{\text{Id}}_{[1,k_{i+2}-1]}_{(j_1,k_{i+2}^{'},k_{i+2}^{'})}); \]

\[ \hat{CR}_{-(i,-(i+1))} := -\hat{s}_{j_1,j_2} (\hat{\text{Id}}_{[k_1,k_i]} \otimes \hat{TZ}(k_{i+1,1}) \otimes \hat{\text{Id}}_{[1,k_{i+2}-1]}_{(j_1,k_{i+2}^{'},k_{i+2}^{'})}). \]
Their degree is:
\[
\deg(\widehat{CR}_{(+i,+(i+1))}) = \deg(\widehat{CR}_{(-i,-(i+1))}) = 1.
\]

**Lemma 8.14.** \(\widehat{CR}_{(+i,+(i+1))}^{[k]}\) corresponds to the \(R^k\)-linear map in
\[
\text{EXT}^0(\widehat{E}_{(+i,+(i+1))}[\overline{k}], \widehat{E}_{(+i,+(i+1))}[\overline{k}])
\]
determined by
\[
f \mapsto \hat{s}_{j_1,j_2}(t_{1,j_2} - t_{1,j_1})f.
\]

\(\widehat{CR}_{(-i,-(i+1))}^{[k]}\) corresponds to the \(R^k\)-linear map in
\[
\text{EXT}^0(\widehat{E}_{(-i,-(i+1))}[\overline{k}], \widehat{E}_{(-i,-(i+1))}[\overline{k}])
\]
determined by
\[
f \mapsto -\hat{s}_{j_1,j_2}(t_{1,j_2} - t_{1,j_1})f.
\]

Still with respect to Figure 10 and the subsequent Figure, we also have the morphism
\[
\widehat{TU}_{(k_{i+1,1})} : \widehat{\Gamma}'_4 \rightarrow \widehat{\Gamma}'_3
\]
from Section 7.3.

**Definition 8.15.** We define the morphisms

\[
\widehat{CR}_{(+i,+(i+1))} : \begin{array}{c}
\begin{array}{c}
1 \\
1
\end{array}
\end{array}
\]

\[
\widehat{CR}_{(-i,-(i+1))} : \begin{array}{c}
\begin{array}{c}
1 \\
1
\end{array}
\end{array}
\]

by
\[
\widehat{CR}_{(+i,+(i+1))} := \hat{s}_{j_1,j_2}(\text{Id}_{\Lambda_{(k_i,k_{i+1})}} \boxtimes \widehat{TU}_{(1,k_{i+1})} \boxtimes \text{Id}_{\Lambda_{(k_i+k_{i+1},k_{i+1})}});
\]

\[
\widehat{CR}_{(-i,-(i+1))} := \hat{s}_{j_1,j_2}(\text{Id}_{\Lambda_{(k_{i+1},k_i)}} \boxtimes \widehat{TU}_{(k_{i+1},1)} \boxtimes \text{Id}_{\Lambda_{(k_{i+1},k_i+k_{i+1})}}).
\]

Their degree is:
\[
\deg(\widehat{CR}_{(+i,+(i+1))}) = \deg(\widehat{CR}_{(-i,-(i+1))}) = 1.
\]

**Lemma 8.16.** \(\widehat{CR}_{(+i,+(i+1))}^{[k]}\) corresponds to the \(R^k\)-linear map in
\[
\text{EXT}^0(\widehat{E}_{(+i,+(i+1))}^{(1,1)[k]}, \widehat{E}_{(+i,+(i+1))}^{(1,1)[k]})
\]
determined by
\[
1 \mapsto 1.
\]
\( \overline{CR}^{[k]}_{(-i+1), -i} \) corresponds to the \( R^\k \)-linear map in
\[
 \text{EXT}^0(\widehat{E}_{(-i+1), -i}^{(1, 1)[k]}, \overline{E}_{(-i, -(i+1))}^{(1, 1)[k]})
\]
determined by
\[
 1 \mapsto 1.
\]

9. Categorified skew Howe duality

In this section we categorify the quantum skew Howe duality which was explained in Section 4.2. We first define a 2-functor \( \Gamma_{m,d,N} : U_Q(sl_m)^* \to HMF_{m,d,N}^* \) in Subsection 9.1 and then use it to categorify the results in Theorem 4.2 and Corollary 4.10 in Subsection 9.2.

9.1. Definition of the 2-representation. Let \( m, d, N \geq 2 \) be arbitrary integers, until further notice.

**Definition 9.1.** We define a 2-functor
\[
 \Gamma_{m,d,N} : U_Q(sl_m)^* \to HMF_{m,d,N}^*
\]
which sends:

- **the objects** \( \lambda = (\lambda_1, ..., \lambda_{m-1}) \in \mathbb{Z}^{m-1} \) to \( \vec{k} \), if \( \phi_{m,d,N}(\lambda) = \vec{k} \in \Lambda(m, d)_N \), or else to zero.
- **the 1-morphisms to matrix factorizations:**

\[
\begin{align*}
1_\lambda \{t\} &\xrightarrow{\widetilde{E}_+ \{k\}} \begin{cases} 
1_{[k]} \{t\} & \text{if } \phi_{m,d,N}(\lambda) \in \Lambda(m, d)_N \\
0 & \text{otherwise.}
\end{cases} \\
\mathcal{E}_+ 1_\lambda \{t\} &\xrightarrow{\widehat{E}_+ \{k\} \{t + k - k_{i+1} + 1\}} \begin{cases} 
\widehat{E}_+ \{k\} \{t + k - k_{i+1} + 1\} & \text{if } \phi_{m,d,N}(\lambda) \in \Lambda(m, d)_N \\
0 & \text{otherwise.}
\end{cases} \\
\mathcal{E}_- 1_\lambda \{t\} &\xrightarrow{\widehat{E}_- \{k\} \{t - k + k_{i+1} + 1\}} \begin{cases} 
\widehat{E}_- \{k\} \{t - k + k_{i+1} + 1\} & \text{if } \phi_{m,d,N}(\lambda) \in \Lambda(m, d)_N \\
0 & \text{otherwise.}
\end{cases}
\end{align*}
\]

In general, the 1-morphism \( 1_\lambda \mathcal{E}_\lambda 1_\lambda \{t\} \) is mapped to

\[
\begin{cases} 
\widehat{E}_{\lambda} \{k\} \{t + d(\vec{k}) - d(\vec{k}')\} & \text{if } \phi_{m,d,N}(\lambda) \in \Lambda(m, d)_N \\
0 & \text{otherwise.}
\end{cases}
\]

with \( \vec{k} = \phi_{m,d,N}(\lambda) \) and \( \vec{k}' = \phi_{m,d,N}(\lambda') \).

In order to avoid cluttering of notation, we will write
\[
\widehat{E}_{\lambda}': \overline{E}_{\lambda} \{t + d(\vec{k}) - d(\vec{k}')\} = \Gamma_{m,d,N}(\mathcal{E}_\lambda 1_\lambda).
\]

Of course, we have
\[
1'_{[k]} = 1_{[\vec{k}]}
\]
for any \( \vec{k} \in \Lambda(m, d)_N \).
• the generating 2-morphisms in $\mathcal{U}_Q(\mathfrak{sl}_m)^*$ to the following 2-morphisms in $\text{HMF}_{m,d,N}^*$:
Theorem 9.2. $\Gamma_{m,d,N}: \mathcal{U}_q(\mathfrak{sl}_m)^* \to \text{HMF}^*_{m,d,N}$ is a well-defined 2-functor.

Proof. Our $\Gamma_{m,d,N}$ is very similar to Khovanov and Lauda’s 2-representation

$$\Gamma_d: \mathcal{U}_q(\mathfrak{sl}_m)^* \to \text{Flag}_d^*$$

in Section 6 in [18]. The proof that $\Gamma_{m,d,N}$ is well-defined is completely analogous to their proof of Theorem 6.9, as can be seen by comparing the images of the generating 2-morphisms in $\mathcal{U}_q(\mathfrak{sl}_m)^*$ for $\Gamma_{m,d,N}$ and $\Gamma_d$. We will therefore omit the actual calculations here. Note that our sign conventions in the definition of $\Gamma_{m,d,N}$ correspond to the ones used in [28]. □

Although $\Gamma_{m,d,N}$ and $\Gamma_d$ are similar and the proof of their well-definedness relies on the same calculations, they are not equivalent. To explain the difference, let us compare the decategorification of both 2-functors. First let us explain the statement in Theorem 6.14 in [18].

As explained in [28], the 2-functor $\Gamma_d$ actually categorifies the surjective homomorphism of $\mathcal{U}_q(\mathfrak{sl}_m)$ onto

$$\prod_{\phi_{m,d,N}(\lambda) \in \Lambda(m,d)^+} \text{End}_{C_q}(V_{\lambda}),$$

which is isomorphic to the quantum Schur algebra $S_q(m,d)$. Here

$$\Lambda(m,d) := \{\mu \in \mathbb{N}^m \mid \mu_1 + \cdots + \mu_m = d\}$$

is the set of $m$-part compositions of $d$ and $\Lambda(m,d)^+$ is its subset of partitions, i.e. those $\mu \in \Lambda(m,d)$ such that $\mu_1 \geq \mu_2 \geq \cdots \mu_m$. Thus $\Gamma_d$ descends to a quotient of $\mathcal{U}_q(\mathfrak{sl}_m)$ which categorifies $S_q(m,d)$, as was proved in [28].

The irreducible $\mathcal{U}_q(\mathfrak{sl}_m)$-module $V_d$ with highest weight $(d,0,\ldots,0)$ is isomorphic to the left ideal $S_q(m,d)1_{(d,0,\ldots,0)} \lhd S_q(m,d)$ and is categorified by Khovanov and Lauda’s category

$$\bigoplus_k H_k^\text{gmod}.$$ 

So the statement of Theorem 6.14 in [18] should be interpreted as meaning that the category in (34) categorifies the underlying vector space $V_d$ and that $\Gamma_d$ categorifies the action of $\mathcal{U}_q(\mathfrak{sl}_m)$ on $V_d$. For more details on the quantum Schur algebra and its categorification see [28] and references therein.

Our $\Gamma_{m,d,N}$ categorifies the surjective homomorphism of $\mathcal{U}_q(\mathfrak{sl}_m)$ onto the quotient of the algebra in (33) by the ideal generated by those $\mu \in \Lambda(m,d)^+$ which are not $N$-bounded.

For $m \geq d = N\ell$ and $\Lambda = N\omega_\ell$, the irreducible $\mathcal{U}_q(\mathfrak{sl}_m)$-representation $V_\Lambda$ can be obtained as (sub)quotient of $S_q(m,d)$, but a different one. We have

$$V_\Lambda \cong S_q(m,d)1_{\Lambda}/[\mu > \Lambda],$$

where the ideal $[\mu > \Lambda]$ is generated by all $1_{\mu}$ with $\mu > \Lambda$.

Recall that $V_\Lambda \cong W_\Lambda$, by Corollary 4.10. As we will show in Theorem 9.7, our $\Gamma_{m,d,N}$ defines an additive strong $\mathfrak{sl}_m$ 2-representation on $W_\Lambda^3$, which categorifies the $\mathcal{U}_q(\mathfrak{sl}_m)$-representation on $W_\Lambda$. 
9.2. **The graded web category.** Let \( m \geq d = N\ell \) and \( \Lambda = N\omega\ell \). In this section we define a \( \mathbb{C} \)-linear category \( \mathcal{W}^0(\vec{k}, N) \).

**Definition 9.3.** The objects of \( \mathcal{W}^0(\vec{k}, N) \) are by definition all matrix factorizations which are homotopy equivalent to direct sums of matrix factorizations of the form \( \hat{u}\{t\} \), where \( u \) is an \( N \)-ladder with \( m \) uprights in \( W(\vec{k}, N) \) and \( t \in \mathbb{Z} \).

For any pair of objects \( X, Y \in \mathcal{W}^0(\vec{k}, N) \), we define the hom-space between them as
\[
\mathcal{W}^0(X, Y) := \text{Ext}(X, Y).
\]
Composition in \( \mathcal{W}^0(\vec{k}, N) \) is induced by the composition of homomorphisms between matrix factorizations.

Note that \( \mathcal{W}^0(\vec{k}, N)^* \) is a \( \mathbb{Z} \)-graded \( \mathbb{C} \)-linear additive category which admits translation and has finite-dimensional hom-spaces.

By definition, \( \mathcal{W}^0(\vec{k}, N) \) is a full subcategory of the homotopy category of matrix factorizations with fixed potential determined by \( \vec{k} \) and \( N \). The latter category is Krull-Schmidt by Propositions 24 and 25 in [20]. Therefore, we can take the Karoubi envelope or idempotent completion of \( \mathcal{W}^0(\vec{k}, N) \), denoted \( \hat{\mathcal{W}}^0(\vec{k}, N) \), which is also Krull-Schmidt. The point here is that a matrix factorization associated to a monomial web might have direct summands in the homotopy category which are not associated to any monomial webs, so we have to include these in our web category by taking the Karoubi envelope. For \( N \geq 3 \) it is very hard to determine the indecomposable summands of webs in general, which is why surjectivity in Corollary 9.8 would be hard to prove directly. For more information for \( N = 3 \) see [16, 26, 32] and for general \( N \) see the sequel to this paper [25].

For any monomial web \( u \in W(\vec{k}, N) \), we write
\[
\hat{u}' := \hat{u}\{-d(\vec{k})\}.
\]
This is consistent with our notation in Definition 9.1.

**Definition 9.4.** We define a linear map
\[
\delta^0_{\vec{k}, N} : W(\vec{k}, N) \to K^0_\delta(W^0(\vec{k}, N))
\]
by
\[
u \mapsto [\hat{u}'] = q^{-d(\vec{k})}[\hat{u}],
\]
for any \( N \)-ladder with \( m \)-uprights \( u \in W(\vec{k}, N) \).

In Corollary 9.8, we will show that \( \delta^0_{\vec{k}, N} \) is an isomorphism. For now, we can only show injectivity.

**Lemma 9.5.** The map \( \delta^0_{\vec{k}, N} \) is injective.

**Proof.** Recall that the Euler form
\[
\langle [P], [Q] \rangle = \dim_q \text{HOM}(P, Q)
\]
is a non-degenerate \( q \)-sesquilinear form on \( K^0_\delta(W^0(\vec{k}, N)) \). Furthermore, the sesquilinear web form gives a non-degenerate \( q \)-sesquilinear form on \( W(\vec{k}, N) \).
The map $\delta^\circ_{\bar{k},N}$ is an isometry w.r.t. these two forms because we have
\[
\dim_q(\mathcal{W}^\circ(\hat{u}', \hat{v}')) = \dim_q(\Ext(\hat{u}', \hat{v}')) = q^{d(\hat{k})}\dim_q(H(\hat{u}^*v)) = q^{d(\hat{k})}\ev(u^*v) = \langle u, v \rangle.
\]
for any $N$-ladders with $m$ uprights $u, v \in \hat{W}(\bar{k}, N)$, by Theorem 5.13.

Since isometries for non-degenerate forms are always injective, this proves the lemma. □

**Definition 9.6.** Define
\[
\mathcal{W}_\Lambda^\circ := \bigoplus_{\bar{k} \in \Lambda(m, d)_N} \mathcal{W}^\circ(\bar{k}, N).
\]

We will now show that $\Gamma_{m, N}$ induces a strong $\mathfrak{sl}_m$ 2-representation on $\mathcal{W}_\Lambda^\circ$. The idea is quite simple. Given an object $1_X^m \mathcal{E}_1^m 1_\lambda$ in $\mathcal{U}_Q(\mathfrak{sl}_m)$ and an object $\hat{w}' \in \mathcal{W}^\circ(\bar{k}, N)$, such that $\phi_{m,d,N}(\lambda) = \bar{k}$ and $\phi_{m,d,N}(\lambda') = \bar{k}'$, we define the action of $1_X^m \mathcal{E}_1^m 1_\lambda$ on $\hat{w}'$ by gluing the ladder associated to $E_i 1_\lambda$ (see Proposition 4.5) on top of $w$, which gives a ladder again, and taking the corresponding matrix factorization
\[
\hat{E}'_{\overrightarrow{\bar{k}}} \boxtimes \hat{w}' \simeq \hat{E}'_{\overrightarrow{\bar{k}}} \hat{w}' \in \mathcal{W}^\circ(\bar{k}', N).
\]
This defines the 2-representation on objects.

We continue to assume that $\phi_{m,d,N}(\lambda) = \bar{k} \in \Lambda(m, d)_N$. Given any 2-morphism $f \in \Hom(\mathcal{E}_1^m 1_\lambda, \mathcal{E}_1^m 1_{\lambda'})$ in $\mathcal{U}_Q(\mathfrak{sl}_m)$, any pair of objects $\hat{u}', \hat{v}' \in \mathcal{W}^\circ(\bar{k}, N)$ and any morphism $\phi \in \mathcal{W}^\circ(\hat{u}', \hat{v}') = \Ext(\hat{u}', \hat{v}')$, we define the action of $f$ on $\phi$ by
\[
\hat{f} \boxtimes \phi \in \Ext(\hat{E}'_{\overrightarrow{\bar{k}}} \boxtimes \hat{u}', \hat{E}'_{\overrightarrow{\bar{k}}} \hat{v}').
\]

By Theorem 9.2, the definition above gives a well-defined 2-action on $\mathcal{W}_\Lambda^\circ$, which extends to a well-defined strong $\mathfrak{sl}_m$ 2-representation on the Karoubi envelope $\mathcal{W}_\Lambda^\circ$.

Let us now show that we can apply the additive version of Rouquier’s universality proposition, reproduced in Proposition 6.5, to this strong $\mathfrak{sl}_m$ 2-representation.

**Theorem 9.7.** With the definitions above, $\mathcal{W}_\Lambda^\circ$ is a strong $\mathfrak{sl}_m$ 2-representation equivalent to $\mathcal{V}_\Lambda^\circ$.

**Proof.** We have to show that the last three conditions in Proposition 6.5 are also fulfilled.

Take $w_\Lambda$ to be the web consisting of $\ell$ vertical upward oriented $N$-edges in $W(\Lambda, N)$. By Corollary 4.11, any indecomposable object in $\hat{W}^\circ(\Lambda, N)$ is isomorphic to $\hat{w}_\Lambda$ up to a grading shift.

Clearly we have
\[
\End(\hat{w}_\Lambda) \cong \mathbb{C}.
\]

And finally, any $N$-ladder with $m$ uprights is the image of a product of divided powers by Proposition 4.5, so we see that the third condition of Proposition 6.5 is fulfilled. □

Let
\[
\delta^\circ: W_\Lambda \rightarrow K_0^q(\mathcal{W}_\Lambda^\circ)
\]
be the direct sum of the maps $\delta^\circ_{k,N}$ from Definition 9.4 over all $\bar{k} \in \Lambda(m, d)_N$. 

Corollary 9.8. The map \( \delta^0 \) is an isomorphism of \( \tilde{U}_q(\mathfrak{sl}_m) \)-representations. In particular, \( \delta^0_{k,N} \) is an isomorphism for all \( k \in \Lambda(m,d)_N \).

Proof. By Theorem 9.7, we have
\[
\dim_q K^q_0(\tilde{\mathcal{W}}^o_\Lambda) = \dim_q K^q_0(V_\Lambda) = \dim_q V_\Lambda.
\]
Since \( \dim_q V_\Lambda = \dim_q V_\Lambda \) by Corollary 4.10, we see that
\[
\dim_q K^q_0(\tilde{\mathcal{W}}^o_\Lambda) = \dim_q W_\Lambda,
\]
which proves this corollary. \( \square \)

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