Effect of intensity-dependent couplings and the atomic dissipation on the interaction of two three-level atoms and two modes of radiation field

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Abstract
The effect of intensity-dependent coupling and the atomic dissipation on the interaction of a pair of three-level atoms in the \( \Xi \) configuration with a bi-mode cavity field are studied. The solution of the proposed model is obtained in the resonance case. Some statistical quantities are discussed to discover the characteristics of the model. With different nonlinearity functions, the population inversion and the photon number dynamics, which display the phenomenon of collapses and revivals are discussed. The non-classical effects by examining the Mandel Q-parameter are considered. The amount of correlation between the two atoms and the field is estimated by analyzing the results of the fidelity. The entanglement periods between the parts of the quantum system, the collapse and revival periods were affected by the choice of coupling intensity, where the collapse periods improved with increasing the effect of the Rabi frequency and decreased when the effect of the Rabi frequency decreased.

Keywords  Trapped ions · Two-mode cavity field · Two three-level atom · Q-parameter · Fidelity

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1 Introduction

The Jaynes-Cummings model (JCM) is a fully quantum mechanical and exactly solvable model which displays the description of the most basic and important interaction between light (a single mode quantized field) and matter (a two-level atom) in the rotating wave approximation (RWA) (Jaynes and Cummings 1963). Based on this model and also its various generalizations different studies have been reported (Nielsen and Chuang 2000). This is not only due to its mathematical solvability but also to its richness in showing many phenomena such as the Rabi-oscillation and the collapses-revivals phenomenon etc. Eberly et al. (1980). Thus, much attention has been paid to extend and generalize this model. For example, it has been generalized and extended to include the effects of cavity mode decay, as well as black-body radiation fields (Puri and Agarwal 1987). Generalizations include applying to Tavies Cummings multi-atoms (Tavis and Cummings 1968) and multiphoton processes (Naim et al. 2019; Abdel-Hafez et al. 1987). Collapses and revivals of oscillations in the photon number distribution were investigated (Buzek et al. 1992). General formalism of interaction of a two-level atom with cavity field in arbitrary forms of nonlinearities was introduced and studied (Haifa and Khalil 2018).

One can find the trapping ions (atoms) to be in a strong connection with (JCM) (atom-field interaction). This is due to a single-trapped and the laser-irradiated ion exhibits a strongly nonlinear multiquantum Jaynes-Cummings dynamics (Gerry and Knight 2005; Abdel-Aty and Abdel-khalek 2008; Vogel and de Matos Filho 1995). These models possess two types of internal and external degrees of freedom. Their internal degrees relate to their internal ionic levels and their external ones refer to their center-of-masses vibrational motion (Yazdanpanah and Tavassoly 2017; Obada et al. 2003). In the meantime the system of trapped ions has been employed, to demonstrate experimentally, the generation and measurement of nonclassical states of the ion’s center-of-mass motion (Meekhof et al. 1996; Leibfried et al. 1996). Furthermore trapped ions have been used to implement quantum logic gates (Meekhof et al. 1996; Cirac and Zoller 1995). This in fact encouraged us to handle the problem of ion trap instead of JCM problem. In recent years advances in the trapping of ions have led to a situation in which the center-of-mass motion of trapped ions has to be treated quantum mechanically (Diedrich et al. 1989). The three-level system was studied including generalizations to multiphoton interaction, field-dependent coupling (Yoo and Eberly 1985; Abdel-Hafez et al. 1989) and for different types of transitions (V-type, Λ-type and Ξ-type) in the presence of nonlinearities (Abdel-Wahab 2007; Abdalla et al. 2012). Dynamics of a two two-level atoms system has been investigated (Tavassoly and Hekmatara 2015; Abdalla et al. 2013) with constant and intensity-dependent coupling regimes. Also, different studies have been carried out on the interaction between two three-level atoms with two mode fields (Baghshahi and Tavassoly 2015; Rasim and Abdel-Khalek 2011; Abdel-Khalek 2014, 2015). It has been stated that, for instance, intensity-dependent coupling seems to be a more realistic approach to the problem of atom-field interaction, especially in the strong coupling regimes where the rotating wave approximation does not work well (Naderi 2011; Klimov and Chumakov 2009).

Previous researchers interested in the interaction of a three-level atom with a single-mode and multi-photon field studied the effect of the nonlinear field on the entanglement periods between the field and the atom. They also studied the phenomenon of collapse and revival and the relationship between them and between entanglement periods (Debray and Das 2011). While in our study a pair of three-level atoms with a multimode field is studied in the presence of nonlinear coupling that depends on the number of photons operator. The main target here
is to examine the influences of the intensity-dependent matter-field coupling and the atomic dissipation on the collapses-revivals phenomenon, the photon number operators, Mandel Q-parameter and the fidelity for a two three-level systems interacting with two mode field.

The rest of the manuscript is arranged as follows: in Sect. 2, we introduce the Hamiltonian of the model and calculate the wave function. Section 3 contains the atomic population inversion. Section 4 is devoted to calculate the analytical expression of the time evolution of photon number operators for each mode in two cases (two phonons and four phonons process). Section 5 is devoted to display the Mandel Q-parameter. Section 6 is devoted to display the fidelity. Finally, in Sect. 7 conclusion is displayed.

2 The model and its solution

The interactions of a single-mode, multi-mode or multi-photon field with a single atom or several atoms are becoming increasingly important because of its applications in quantum information (Naim et al. 2019; Abdel-Hafez et al. 1987). The investigations of these processes are of theoretical value in themselves and may have some potential applications in the near future. On the other hand, this increased interest is partly due to the fact that such nonlinear processes have been experimentally realized in the field of ion trapping (Leibfried et al. 1996).

The Hamiltonian which describes the system of two three-level particles in the Ξ configuration interacting with non-linearity two laser fields in the following form ($\hbar = 1$):

$$\hat{H} = \hat{H}_0 + \hat{H}_I$$

with

$$\hat{H}_0 = \Omega_1 \hat{a}_1^\dagger \hat{a}_1 + \Omega_2 \hat{a}_2^\dagger \hat{a}_2 + \sum_{j=1}^{2} (\alpha_{e}^{(j)} \sigma_{ee}^{(j)} + \omega_{i}^{(j)} \sigma_{ii}^{(j)} + \omega_{g}^{(j)} \sigma_{gg}^{(j)}),$$

$$\hat{H}_I = \sum_{j=1}^{2} (\sigma^{(j)}_{e} f_1(X_j) + h.c) + \sum_{j=1}^{2} (\sigma^{(j)}_{g} f_2(Y_j) + h.c)$$

where

$$f_1(X_j) = \lambda_1^{(j)} E_1^{(j)} = e_1^{(j)} \lambda_1^{(j)} \exp[i(K_j X_j - \omega_1 t)],$$

$$f_2(Y_j) = \lambda_2^{(j)} E_2^{(j)} = e_2^{(j)} \lambda_2^{(j)} \exp[i(K_j Y_j - \omega_2 t)]$$

where $E_1^{(j)}$ and $e_1^{(j)}$, $(l = 1, 2)$ are the classical laser fields at the particles position and the modulated amplitude of the irradiating field, respectively. We can express the center of mass positions in terms of annihilation and creation operators of the two dimensional trap for every particle, namely $X_j, Y_j$ (Leibfried et al. 1996).

Suppose that the two particles are irradiated by the same field, therefore

$$X = \sqrt{\frac{1}{2 M \Omega_1}} (\hat{a}_1^\dagger + \hat{a}_1) \quad Y = \sqrt{\frac{1}{2 M \Omega_2}} (\hat{a}_2^\dagger + \hat{a}_2)$$
where $\Omega_j, (j = 1, 2)$ is the center-of-mass motion frequency, while $\sigma^{(j)}_{AB} = |A\rangle\langle B|$. 

Using the special form of the Baker-Hausdorff theorem (Louisell 1973), the operator $e^{i\eta(\hat{a}_j + \hat{a}^+_j)}$ may be written as a product of two operators as

$$e^{i\eta(\hat{a}_j + \hat{a}^+_j)} = e^{-\frac{\eta^2}{2}} \sum_{l=0}^{\infty} \frac{(i\eta)^l}{l!} \sum_{m=0}^{\infty} \frac{(i\eta)^m}{m!} (\hat{a}_j^+)^m (\hat{a}_j)^l$$

(6)

where $\eta_j = K_j \sqrt{\frac{1}{2M\Omega_j}}$ is Lamb-Dick parameter. In the case of $l = m + k$, when the particle is in the resolved sideband limit and the laser is irradiated resonantly to the $k$th vibrational sidebands, $\omega_1 = (\omega_e^{(j)} - \omega_i^{(j)}) - k_1 \Omega_1$ and $\omega_2 = (\omega_i^{(j)} - \omega_g^{(j)}) - k_2 \Omega_2$ we can write the Hamiltonian $\hat{H}_\text{int.}$ in the interaction picture as follows,

$$\hat{H}_\text{int.} = U^+_0 \hat{H} U_0, \quad U_0(t) = e^{-i\hat{H}_\text{int}t}$$

(7)

The Lamb-Dicke regime is $\eta \ll 1$. In this limit we can expand the Hamiltonian of Eq. (3) up to the requested order in $\eta_j$. Then the Hamiltonian can be simplified to

$$\hat{H}_\text{int.} = \sum_{j=1}^{2} (\sigma^{(j)}_{ei} f_1(n_j) \hat{a}^+_1 \hat{a}^+_1 + \hat{a}_2^+ \hat{a}_2^+ f_1(n_j) \sigma^{(j)}_{ie}) + \sum_{j=1}^{2} (\sigma^{(j)}_{ig} f_2(n_j) \hat{a}^+_2 \hat{a}^+_2 + \hat{a}_2^+ \hat{a}_2^+ f_2(n_j) \sigma^{(j)}_{gi})$$

(8)

where

$$f_j(n_j) = \lambda_j \epsilon_j \exp \left(-\frac{n_j^2}{2}\right) \sum_{m=0}^{\infty} \frac{(i\eta)^{2m+k_j}}{m!(m+k_j)!} (\hat{a}_j^+)^m (\hat{a}_j)^m, \quad (l, 1, 2)$$

since $f_j(n_j)$ are the couplings between the trapped ions and the fields which supposed to be intensity-dependent. However, these function are be engineered at well by applying a number of laser fields (Di Fidio et al. 2001). The environment surrounding of an open quantum systems naturally affects the decoherence of these systems. Therefore, the process of decoherence can be studied using the following master equation (Louisell 1973).

$$\frac{\partial \hat{\rho}(t)}{\partial t} = -i\hbar [\hat{H}, \hat{\rho}(t)] + \frac{\gamma}{2} \sum_{j=1,2} \left(2\sigma^{(j)}_{ei} \hat{\rho}(t) \sigma^{(j)}_{ie} - \sigma^{(j)}_{ee} \hat{\rho}(t) - \hat{\rho}(t) \sigma^{(j)}_{ee} \right)$$

$$+ \frac{\gamma}{2} \sum_{j=1,2} \left(2\sigma^{(j)}_{ig} \hat{\rho}(t) \sigma^{(j)}_{gi} - \sigma^{(j)}_{ii} \hat{\rho}(t) - \hat{\rho}(t) \sigma^{(j)}_{ii} \right)$$

(9)

where $\gamma$ is the atomic dissipation constant. Now, consider the weak dissipations condition, where the off-diagonal terms $2\sigma^{(j)}_{ei} \hat{\rho}(t) \sigma^{(j)}_{ie}$ and $2\sigma^{(j)}_{ig} \hat{\rho}(t) \sigma^{(j)}_{gi}$ can be neglected (Abd-Rabou et al. 2019; Obada et al. 2021). Therefore, the master Eq. (9) becomes,
\[ \frac{i}{\hbar} \frac{\partial \hat{\rho}(t)}{\partial t} = \left\{ \hat{H} - \frac{i}{2} \sum_{j=1,2} (\sigma_{e_j}^{(j)} + \sigma_{i_j}^{(j)}) \right\} \hat{\rho}(t) \]

\[ - \hat{\rho}(t) \left\{ \hat{H} + \frac{i}{2} \sum_{j=1,2} (\sigma_{e_j}^{(j)} + \sigma_{i_j}^{(j)}) \right\}, \]

therefore, the system can be described by using the non-Hermitian operator \( H_{\text{non}} \) to be given by,

\[ H_{\text{non}} = \hat{H} - \frac{i}{2} \sum_{j=1,2} (\sigma_{e_j}^{(j)} + \sigma_{i_j}^{(j)}). \]

Assume that the two particles and the field are initially prepared in their excited states of the atom and uncorrelated two coherent states. In this case the wave function of the system at \( t = 0 \) can be written as,

\[ |\Psi(0)\rangle = |\Psi(0)\rangle_{\text{atoms}} \otimes |\Psi(0)\rangle_{\text{field}} \]

\[ = |e_1, e_2\rangle \otimes \sum_{n_1, n_2}^{\infty} q_{n_1} q_{n_2} |n_1, n_2\rangle \]

where

\[ q_{n_1} q_{n_2} = \exp \left[ -\frac{1}{2}(|\alpha_1|^2 + |\alpha_2|^2) \right] \frac{\alpha_1^{n_1} \alpha_2^{n_2}}{\sqrt{n_1! n_2!}} \]

where the mean photon number is \( \bar{n}_j = |\alpha_j|^2, (j = 1, 2) \).

The wave function \( |\Psi(t)\rangle \) at \( t > 0 \) is supposed to take the following form,

\[ |\Psi(t)\rangle = \sum_{n_1, n_2}^{\infty} [x_1(n_1, n_2, t)|e_1, e_2, n_1, n_2\rangle + x_2(n_1, n_2, t)|e_1, i_2, n_1 + k_1, n_2\rangle 
+ x_3(n_1, n_2, t)|i_1, e_2, n_1 + k_1, n_2 + k_2\rangle + x_4(n_1, n_2, t)|i_1, i_2, n_1 + k_1, n_2\rangle 
+ x_5(n_1, n_2, t)|i_1, i_2, n_1 + 2k_1, n_2\rangle + x_6(n_1, n_2, t)|i_1, g_2, n_1 + 2k_1, n_2 + k_2\rangle 
+ x_7(n_1, n_2, t)|g_1, e_2, n_1 + k_1, n_2 + k_2\rangle + x_8(n_1, n_2, t)|g_1, i_2, n_1 + 2k_1, n_2 + k_2\rangle 
+ x_9(n_1, n_2, t)|g_1, g_2, n_1 + 2k_1, n_2 + 2k_2\rangle] \]

with condition,

\[ \sum_{i=1}^{9} |x_i(n_1, n_2, t)|^2 = 1 \]

The Shrödinger equation of motion for the wave function in the interaction picture is given by,

\[ i \frac{\partial}{\partial t} |\Psi\rangle = H_{\text{non}} |\Psi\rangle \]

Thus we have the following system of equation in the matrix form,
\[
\frac{dx(n_1, n_2, t)}{dt} = Ax(n_1, n_2, t)
\]  

where

\[
A = \begin{pmatrix}
-i\frac{\gamma}{2} & \lambda_1^{(2)}R_1 & 0 & \lambda_1^{(1)}R_1 & 0 & 0 & 0 & 0 & 0 \\
\lambda_1^{(2)}R_1 & -i\frac{\gamma}{2} & \lambda_2^{(2)}R_2 & 0 & \lambda_1^{(1)}R_1 & 0 & 0 & 0 & 0 \\
0 & \lambda_2^{(2)}R_2 & -i\frac{\gamma}{2} & \lambda_1^{(1)}R_1 & 0 & \lambda_2^{(1)}R_2 & 0 & 0 & 0 \\
\lambda_1^{(1)}R_1 & 0 & 0 & -i\frac{\gamma}{2} & \lambda_1^{(2)}R_1 & 0 & \lambda_2^{(1)}R_2 & 0 & 0 \\
0 & \lambda_1^{(1)}R_1 & 0 & \lambda_1^{(2)}R_1 & -i\frac{\gamma}{2} & \lambda_2^{(2)}R_2 & 0 & \lambda_2^{(1)}R_2 & 0 \\
0 & 0 & \lambda_1^{(1)}R_3 & \lambda_2^{(2)}R_2 & -i\frac{\gamma}{2} & 0 & 0 & \lambda_2^{(1)}R_4 & 0 \\
0 & 0 & 0 & \lambda_2^{(1)}R_2 & 0 & \lambda_2^{(2)}R_4 & -i\frac{\gamma}{2} & \lambda_2^{(1)}R_4 & 0 \\
0 & 0 & 0 & 0 & \lambda_2^{(1)}R_4 & 0 & \lambda_2^{(2)}R_4 & -i\frac{\gamma}{2} & \lambda_2^{(1)}R_4 & 0 \\
\end{pmatrix}
\]  

(18)

Where

\[
R_j = v_j(n_j)f_j(n_j), \quad R_{j+2} = v_{j+2}(n_j + k_j)f_j(n_j + k_j)
\]  

(19)

with

\[
v_j(n_j) = \sqrt{\frac{n_j + k_j}{n_j!}}, \quad v_{j+2}(n_j) = v_j(n_j + k_j) \quad j = 1, 2
\]  

(20)

and

\[
x(n_1, n_2, t) = [x_1(n_1, n_2, t), x_2(n_1, n_2, t), ..., x_9(n_1, n_2, t)]^T
\]  

(21)

Therefore, by taking Laplace transform we obtain

\[
x(s)(n_1, n_2, s) - x(0) = -iAx(n_1, n_2, s) \\
(sI + iA)x(n_1, n_2, s) = x(n_1, n_2, 0)
\]  

(22)

Then taking inverse Laplace transform,

\[
x(n_1, n_2, t) = L^{-1}[(sI + iA)^{-1}]x(n_1, n_2, 0)
\]

Let

\[
Q(n_1, n_2, s) = (sI + iA)^{-1}
\]  

(23)

then

\[
Q(n_1, n_2, t) = L^{-1}[Q(n_1, n_2, s)]
\]  

(24)

The $ij$ element of the matrix has the form,

\[
Q(n_1, n_2, s)_{ij} = \frac{\text{Cof}(sI + iA)_{ij}}{\text{Det}(sI + iA)}
\]  

(25)

then
\[ x_i(n_1, n_2, t) = \sum_{j=1}^{9} \sum_{n_1, n_2} Q_{ij}(n_1, n_2, t) x_j(n_1, n_2, 0) \]  

(26)

According to the initial condition as stated in Eq. (12), that the two particles in their excited states, i.e., only \( x_1(n_1, n_2, 0) = q_{n_1} q_{n_2} \); otherwise \( x_i(n_1, n_2, 0) = 0 \), therefore

\[ x_i(n_1, n_2, t) = \sum_{j=1}^{9} Q_{ij}(n_1, n_2, t) x_j(n_1, n_2, 0). \]  

(27)

We recourse to numerical investigations in what follows to consider the influence of the dissipation terms on atomic population inversion, field dynamics, the Mandel Q-parameter and the fidelity.

3 Population inversion

The population inversion is one of the important atomic variables of the system. This would give information about the state of the particle during the interaction period, which leads one to observe when the particle is in its excited or ground states. The population inversion for the ion \( j \), \( (j = 1, 2) \) is given by,

\[ W^{(j)}(t) = \left\langle \sigma_{ee}^{(j)}(t) \right\rangle - \left\langle \sigma_{gg}^{(j)}(t) \right\rangle \]  

(28)

The expectation values for the atomic occupation operators [the probability of finding the particle \((j)\) in the state \(l\)] are given by using (14) as follows,

\[ \left\langle \sigma_{ll}^{(j)}(t) \right\rangle = \left\langle \Psi(t) \right| \sigma_{ll}^{(j)} \left| \Psi(t) \right\rangle \quad (j = 1, 2, \quad l = e, i, g) \]  

(29)

We display the evolution of the population inversion for the particle \( W^{(1)}(t) \) against the scaled time \( \lambda t \). When excluding dissipation \( \gamma = 0 \), for the initial atomic states \( |e_1, e_2\rangle \), the mean photon number for the coherent states are \( \bar{n}_1 = \bar{n}_2 = 9 \) and \( k_1 = 1 = k_2 \). Where \( f_1(n_1) = 1, f_2(n_2) = 1 \) as shown in Fig. 1a the revivals and a collapses period are clearly apparent in the range of time considered. The oscillations are around the horizontal axis, the behavior of the function \( W^{(1)}(t) \) looks different from the two-level atom with single-photon (Scully and Zubairy 1997). There is a revival period around 0.2, but with small amplitude compared with the revival period around zero with the amplitudes are larger. After considering the dissipation \( (\gamma = 0.05) \), the previous oscillations quickly collapse and the function \( W^{(1)}(t) \) reaches the smallest values after a short period of time leading to a superposition state. For \( f_1 = 1, f_2 = \sqrt{\frac{n_2!}{(n_2+k_2)!}} \) that makes the Rabi oscillation in the lower transition constant, note that the middle revival period appears with larger amplitudes corresponding to the earlier period shown in Fig. 1b. The oscillations of the function \( W^{(1)}(t) \) are centered around the excited state and never reach to the ground state. Moreover, it is shifted upwards and the phenomena of collapse and revivals are more pronounced. After taking into account the dissipation, the maximum values gradually decrease and it takes a long time to reach the minimum values compared to the previous case in Fig. 1a. For the case of \( f_1 = \sqrt{\frac{n_1!}{(n_1+k_1)!}}, f_2 = 1 \), we note that the first
ion \(W(1)(t)\) has a first revival period with small amplitude and the particle loses energy for \(\gamma = 0\) for blue (solid) curve while \(\gamma = 0.05\) for red (dashed) curve. (a) \(f_1 = f_2\), (b) \(f_1 = 1, f_2 = \sqrt{\frac{n_1}{(n_1+k_1)!}}\), (c) \(f_1 = \sqrt{\frac{n_1}{(n_1+k_1)!}}; f_2 = 1\), (d) \(f_1 = \sqrt{\frac{n_1}{(n_1+k_1)!}}; f_2 = \sqrt{\frac{n_1}{(n_1+k_1)!}}\).

Fig. 1 Evaluation of the atomic population inversion for the first ion \(\langle W(1)(t) \rangle\) against the scaled time \(\lambda t\) for \(k_1 = 1 = k_2\), for initial atomic state \(|e_1, e_2\rangle\) and coherent state with photon number \(\bar{n}_1 = \bar{n}_2 = 9\), where \(\gamma = 0\) for blue (solid) curve while \(\gamma = 0.05\) for red (dashed) curve. (a) \(f_1 = f_2\), (b) \(f_1 = 1, f_2 = \sqrt{\frac{n_1}{(n_1+k_1)!}}\), (c) \(f_1 = \sqrt{\frac{n_1}{(n_1+k_1)!}}; f_2 = 1\), (d) \(f_1 = \sqrt{\frac{n_1}{(n_1+k_1)!}}; f_2 = \sqrt{\frac{n_1}{(n_1+k_1)!}}\).

ion \(W(1)(t)\) has a first revival period with small amplitude and the particle loses energy for \(\gamma = 0.05\) and its mean value reduce to a value around zero as time increases. Moreover, the upward shift of \(W(1)(t)\) is more pronounced. The function \(W(1)(t)\) decays quickly to minimum values after taking the dissipation into account as shown in Fig. 1c. Finally, when take \(f_1 = \sqrt{\frac{n_1}{(n_1+k_1)!}}; f_2 = \sqrt{\frac{n_2}{(n_2+k_2)!}}\) which means that \(R_1 = 1 = R_2\) and here both Rabi frequencies for the two transitions are constant. The function \(W(1)(t)\) oscillates regularly, the collapse and revivals phenomenon disappear. After the dissipation is taken into account, the amplitude of the oscillations decreases gradually as seen in Fig. 1d.
The case of two-photon transition i.e. $k_1 = 2, k_2 = 2$ is shown in Fig. 2, the same previous conditions are considered. In the first case $f_1 = 1, f_2 = 1$, both the collapse period and the amplitude of the oscillations are significantly reduced as in Fig. 2a. The maximum values of the function $W^{(1)}(t)$ decrease to zero leading to superposition state ($W^{(1)}(t) = 0$) after a short period of time for $\gamma = 0.05$. In the second case $f_1 = 1, f_2 = \sqrt{\frac{n_2^{\dagger}}{(n_1 + k_1)^2}}$, collapse and revival phenomenon improves, collapse periods are regularly period. The particle gains energy from the field, which causes the axis of symmetry of $W^{(1)}(t)$ to rise upwards, as shown in Fig. 2b. The axis of symmetry incline downward and amplitude of the oscillations decrease after the dissipation is inserted into the interaction. For the third case $f_1 = \sqrt{\frac{n_1^{\dagger}}{(n_1 + k_1)^2}}$, $f_2 = 1$, the particle stores more energy, so we notice the function $W^{(1)}(t)$ is very close to one. Although $W^{(1)}(t)$ oscillates only in the excited state, nevertheless it

Fig. 2 Evaluation of the atomic inversion for the first particle $\langle W^{(1)}(t) \rangle$ against the scaled time $\lambda t$ for $k_1 = 2 = k_2$, for initial atomic state $|e_1, e_2 \rangle$ and coherent state with photon number $\tilde{n}_1 = \tilde{n}_2 = 9$, where $\gamma = 0$ for blue curve while $\gamma = 0.05$ for red (dashed) curve and other parameters such as Fig. 1
collapses rapidly to the limit $W^1(t) \to 0$ after the inclusion of the dissipation, see Fig. 2c. While the last case is like the previous case Figs. 1d, 2d, because the frequency of Rabi is a fixed amount, but the envelope of the oscillations has a slightly waves behavior.

## 4 Photon numbers dynamics

Consider here, the collapses-revivals phenomenon via the dynamics of the photon numbers. The detection of this phenomenon appears during the course of interaction between the field and the particles within the cavity. This phenomenon is a pure quantum effect and has its origin in the granular structure of the photon-number distribution of the initial field (Scully and Zubairy 1997). In order to show these phenomenon we would like to calculate the expectation value for $\hat{n}_1^r \hat{n}_2^r$,

$$\langle \hat{n}_1^r \hat{n}_2^r \rangle = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left[ \hat{n}_1^r (\hat{n}_2^r) \left( |x_1(m_1, m_2, t)|^2 \right) \right]$$

$$+ (n_1 + k_1)^r (n_2)^r \left( |x_2(m_1, m_2, t)|^2 + |x_4(m_1, m_2, t)|^2 \right)$$

$$+ (n_1 + k_1)^r (n_2 + k_2)^r \left( |x_3(m_1, m_2, t)|^2 + |x_7(m_1, m_2, t)|^2 \right)$$

$$+ (n_1 + 2k_1)^r (n_2)^r \left( |x_5(m_1, m_2, t)|^2 \right)$$

$$+ (n_1 + 2k_1)^r (n_2 + k_2)^r \left( |x_6(m_1, m_2, t)|^2 + |x_8(m_1, m_2, t)|^2 \right)$$

$$+ (n_1 + 2k_1)^r (n_2 + 2k_2)^r \left( |x_9(m_1, m_2, t)|^2 \right)$$

Using the above expression to calculate $\langle n_1(t) \rangle$ and $\langle n_2(t) \rangle$, we put $(r_1 = 1, r_2 = 0)$ and $(r_1 = 0, r_2 = 1)$.

$$\langle n_2(t) \rangle - \bar{n}_2 = k_2 \sum_{n_1,n_2} \left[ |x_3(n_1, n_2, t)|^2 + |x_6(n_1, n_2, t)|^2 \right]$$

$$+ |x_7(n_1, n_2, t)|^2 + |x_8(n_1, n_2, t)|^2 + 2|x_9(n_1, n_2, t)|^2$$

Generally, there is a monotonic relation between the population inversion and the number of photons. In Figs. 3-4 we plot $\langle \hat{n}_i(t) - \bar{n}_i \rangle$, $(i = 1, 2)$ for one photon process $k_1 = 1 = k_2$ with equal mean value of the two modes $\bar{n}_2 = 9$, and the atom in the $|e_1, e_2\rangle$ state. We look for the influence of the intensity-dependent coupling functions on the time evolution of the photon number to display the phenomenon of the collapses and revivals. When the function $f_1(n_i) = 1$, the evolution of the operator $\langle \hat{n}_i(t) - \bar{n}_i \rangle$ shows collapses and revivals and the function oscillates around 1.2 Fig. 3a with small amplitude for the first mode and around 0.6 Fig. 4a with small amplitude for the second mode. The oscillations decay quickly and the function $\langle \hat{n}_i(t) - \bar{n}_i \rangle$ returns to zero after taking the dissipation into account as seen in Figs. 3a, 4a. In Figs. 3b, 4b, for $f_1(n_1) = 1, f_2(n_2) = \sqrt{\frac{n_2!}{(n_1+k_2)!}}$, it leads to difference in the shape of the oscillations. When the dissipation is included in the interaction cavity, the function $\langle n(t) \rangle - \bar{n}$ needs more time to reach a steady state. In Figs. 3c, 4c, for $f_1(n_1) = \sqrt{\frac{n_1!}{(n_1+k_1)!}}, f_2(n_2) = 1$, more improvement in the revival times, the amplitude of
fluctuations increased significantly especially in the second mode. The axis of symmetry drops down roughly 0.4. The oscillations quickly collapse until they reach a steady state ($\langle n(t) \rangle - \bar{n} \approx 1$) after a short period. In Figs. 3d, 4d, we set $f_1(n_1) = \sqrt{\frac{n_1!}{(n_1+k_1)!}}$, $f_2(n_2) = \sqrt{\frac{n_2!}{(n_2+k_2)!}}$. As a result of the Rabi frequency not being dependent on the number of photons, regular and repeated oscillations are generated. These fluctuations are quickly erased when the dissipation is taken into account.

In the case of the two photon transition ($k_1 = 2 = k_2$), we plot Figs. 5, 6 for the photon number functions $\langle \hat{n}_i(t) - \bar{n}_i \rangle$, ($i = 1, 2$). In the first case, oscillations of small amplitude are formed. The collapse appears in the first period, while the revival phenomenon recurs in multiple periods. These oscillations are gradually eliminated until the function $\langle \hat{n}_i(t) - \bar{n}_i \rangle$ reaches its steady state (for the first mode equal 2, for the second mode equal zero) as
shown in Figs. 5a, 6a. Both the collapses period and amplitude of oscillations improve and appear periodically for the 1st mode after $f_1(n_1) = 1$ and $f_2(n_2) = \sqrt{\frac{n_1}{(n_1+k_1)!}}$ are taken into account. The oscillations decay rapidly when the dissipation is inserted into the interaction, as observed in Figs. 5b, 6b. When $f_1(n_1) = \sqrt{\frac{n_1}{(n_1+k_1)!}}$, $f_2(n_2) = 1$, in the first mode the axis of symmetry is shifted downward remarkably the regular oscillation for the 2nd are in amplitude than the first mode, the collapse periods improve. The curve of the function $(\hat{n}_1(t) - \bar{n}_{1})$ returns to stability quickly after the dissipation joins the interaction, see Figs. 5c, 6c. While for $f_1(n_1) = \sqrt{\frac{n_1}{(n_1+k_1)!}}$, $f_2(n_2) = \sqrt{\frac{n_1}{(n_1+k_2)!}}$ the behaviour of the photon numbers is similar to the previous case (two photon case).
5 The Mandel Q-parameter

Mandel parameter is a useful criterion to check the non-classical statistics of any state. It is defined as Scully and Zubairy (1997), Mandel (1979), Paul (1982),

\[
Q_i(t) = \frac{\langle \hat{n}_i(t)^2 \rangle - \langle \hat{n}_i(t) \rangle^2}{\langle \hat{n}_i(t) \rangle}, \quad i = 1, 2
\]  

(32)

This quantity attains 1, > 1 or < 1 values when the state is standard coherent state (Poissonian distribution), classical state (super-Poissonian) or nonclassical state (sub-Poissonian), respectively. To proceed further, the expectation values are easily calculated when we use

Fig. 5 Evaluation of the photon number $\langle \hat{a}_i^\dagger \hat{a}_i \rangle - \bar{n}_i$ against the scaled time $\lambda t$ for $k_1 = 2 = k_2$, for initial atomic state $|e_1, e_2\rangle$ and coherent state with photon number $\bar{n}_1 = \bar{n}_2 = 9$, where $\gamma = 0$ for blue curve while $\gamma = 0.05$ for green (dashed) curve and other parameters such as Fig. 1
the expression (30) as follows, when we take $r_2 = 0$ we get the expectation values belonging to the first mode, getting $\langle \hat{n}_1 \rangle$ at $r_1 = 1$ and $\langle \hat{n}_2^2 \rangle$ at $r_1 = 2$. Otherwise if we take $r_1 = 0$, $r_2 = 1$ we get $\langle \hat{n}_2 \rangle$, while getting $\langle \hat{n}_2^2 \rangle$ for $r_1 = 0$, $r_2 = 2$.

We plot Mandel $Q$-parameter $Q_1(t)$ for the first mode against the scaled time $\lambda t$ in Fig. 7 for $k_1 = 2$, $k_2 = 1$ taking into account $|e_1, e_2 \rangle$ as the initial state of the particles and $\bar{n}_1 = \bar{n}_2 = 9$ for the coherent state as in the previous cases. In the first case, for $f_1 = 1$, $f_2 = 1$ the state begins with Poisson distribution, followed by the distribution oscillates between sub-Poissonian and super-Poissonian. The distribution stabilizes at Poisson during the period of collapse, where during the revival period the distribution oscillates between the classical and the non-classical. After adding the dissipation, a classical and the non-classical distribution are formed for a short period, followed by a pure classical
distribution as seen in Fig. 7a. In the second case \( f_1 = 1, f_2 = \sqrt{\frac{n_2!}{(n_2+k_2)!}} \), the non-classical behavior improves significantly. The state passes to the classical behavior after inserting the dissipation into the interaction as observed in Fig. 7b. In the third case \( f_1 = \sqrt{\frac{n_1!}{(n_1+k_1)!}} \), \( f_2 = 1 \), the Poisson distribution appears for most of the interaction periods. The Poisson distribution shifts to super-Poissonian sharply when the dissipation is taken into account, as seen in Fig. 7c. In the fourth case \( f_1(n_1) = \sqrt{\frac{n_1!}{(n_1+k_1)!}} \), \( f_2(n_2) = \sqrt{\frac{n_2!}{(n_2+k_2)!}} \), a sub-Poissonian distribution is formed over most of the interaction periods. The non classical behavior quickly becomes classical after the dissipation is placed into the interaction cavity as shown in Fig. 7d.
6 The fidelity

The process of measuring the entanglement between parts of a quantum system is of paramount importance in quantum information. Fidelity is a measure of the distance between two quantum states (Nielsen and Chuang 2000). The fidelity is not a metric for density operators, but we will see that it yields a useful metric. The fidelity values in this frame range [0,1], the maximum entangled state when the fidelity equal zero and the non-entanglement state at the fidelity equal one, between these two limits, partial entanglement is formed. Therefore, consider the fidelity (Nielsen and Chuang 2000; Obada and Khalil 2010; Abdalla et al. 2017) to indicate the amount of entanglement between the particle and the cavity field. The definition of fidelity is given by

$$F(t) = |\langle \psi(0)|\psi(t)\rangle|$$

(33)

The same conditions mentioned in the population inversion is used. For the process $k_1 = 1 = k_2$, in the first case $f_1 = 1 = f_2$ after excluding the dissipation, the fidelity oscillates around 0.5. The fidelity is thus moderate during the interaction time, as seen in Fig. 8a. Gradually, the fidelity decreases until it reaches a stable state after adding the dissipation to the interaction. In the second case $f_1 = 1, f_2 = \sqrt{\frac{n_2^2}{(n_2+k_2)^2}}$, the fidelity oscillates around 0.4 the amplitude of the oscillations increases. The effect of the dissipation on the entanglement is reduced, so the function $F(t)$ needs a long time to reach a stable state, as shown in Fig. 8b. In the third case $f_1 = \sqrt{\frac{n_1^2}{(n_1+k_1)^2}}, f_2 = 1$, the fidelity attains higher value around 0.8 in addition to the decrease in the intensity of the oscillations. The function $F(t)$ starts from a the value ($F(t) = 1$) followed by generating partial fidelity around 0.45. The function $F(t)$ reaches the maximum values at mid-revival period. While it improves in the middle of the collapse. A mixed state is generated and fidelity is gradually degraded after the dissipation is taken into account as shown in Fig. 9a. In the second case, the fidelity decreases slightly and it is reaches maximum values uniformly regularly. After inclusion of the dissipation, a mixed state is generated and the function $F(t)$ takes longer time to reach the steady state as observed in Fig. 9b. In the third case, a higher value for $F(t) \approx 0.95$ attained i.e higher fidelity. When the dissipation is taken into account, the function $F(t)$ reaches a steady state quickly, see Fig. 9c. In Fig. 9d, as a result of the Rabi frequency being independent on the number of photons, the behaviour of the final case is similar to Fig. 8d.
The interaction between two three-level particles in $\Xi$ configuration and a two-mode field in presence of the atomic dissipation terms is studied. The mathematical form of the corresponding wave function is obtained, considering the two particles initially in an uppermost state and the field in uncorrelated two coherent states. The time-dependent Schrodinger equation is solved by using Laplace transformations for the coupled system of differential equations in the resonance case. The effects of the intensity dependent coupling functions and the atomic dissipation on some non-classical statistical aspects are examined. We have studied the population inversion and noted that as we use the nonlinear function of the photon number operators, it is found that the first particle

![Graphs showing the fidelity $F(t)$ against the scaled time $\lambda t$ for different cases.]

**Fig. 8** The fidelity $F(t)$ against the scaled time $\lambda t$ for $k_1 = k_2$, for initial atomic state $|e_1, g_2\rangle$ and coherent state with photon number $\bar{n}_1 = \bar{n}_2 = 9$, where $\gamma = 0$ for blue (solid) curve while $\gamma = 0.05$ for red (dashed) curve and other parameters such as Fig. 1.

### 7 Conclusion

The interaction between two three-level particles in $\Xi$ configuration and a two-mode field in presence of the atomic dissipation terms is studied. The mathematical form of the corresponding wave function is obtained, considering the two particles initially in an uppermost state and the field in uncorrelated two coherent states. The time-dependent Schrodinger equation is solved by using Laplace transformations for the coupled system of differential equations in the resonance case. The effects of the intensity dependent coupling functions and the atomic dissipation on some non-classical statistical aspects are examined. We have studied the population inversion and noted that as we use the nonlinear function of the photon number operators, it is found that the first particle...
again energy and moves from its state to exited state and never reach the ground state. The intensity dependent coupling functions are strongly influencing the behavior of atomic population. The oscillations are quickly eliminated and the population reaches a steady state after taking the dissipation into account. We have studied the variation in photon number under the effect of the intensity-dependent couplings showing collapses and revivals in the case of $k_1 = 2 = k_2$ photons and the maximum value of the function increase for the $k_1 = 2 = k_2$ photons process. Also, we have discussed the Q-Mandel-parameter for classical and non-classical effects. The results confirm that the dissipation parameter significantly affects the photons number and the Q-Mandel-parameter. Fidelity was studied to estimate the amount of correlation between the final state and the initial state. A mixed state is formed after adding the dissipation and the fidelity is completely erased with increasing the time of interaction.

**Fig. 9** The fidelity $F(t)$ against the scaled time $\lambda \tau$ for $k_1 = 2 = k_2$, for initial atomic state $|e_1, e_2\rangle$ and coherent state with photon number $\bar{n}_1 = \bar{n}_2 = 9$, where $\gamma = 0$ for blue (solid) curve while $\gamma = 0.05$ for red (dashed) curve and other parameters such as Fig. 1.
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