Rigidity results with curvature conditions from Lichnerowicz Laplacian and applications

Gunhee Cho, Nguyen Thac Dung, and Tran Quang Huy

Abstract

In this paper, we show several rigidity results of harmonic tensors in terms of Lichnerowicz Laplacians. These are obtained mainly from the method of controlling the curvature term of Lichnerowicz Laplacians due to P. Petersen and M. Wink in [PW21a]. Geometric applications on several classes of Riemannian manifolds including Einstein manifolds and immersed submanifolds are provided.

1 Introduction

In a recent work, motivated by the Hodge theory and the results due to Meyer, Gallot-Meyer, and Gallot (see [Mey71, GM75, Gal81]), in [PW21a], Petersen and Wink introduced a new curvature conditions for Bochner technique. Recall that given an \( n \)-dimensional Riemannian manifold \((M^n, g)\), the curvature operator of \( M \) is said to be \( m \)-positive (non-negative) if the sum of its lowest \( m \) eigenvalues is positive (non-negative), where \( 1 \leq m \leq n - 1 \). Petersen and Wink proved that if \( M \) is a closed Riemannian manifold of dimension \( n \geq 3 \) and the curvature operator is \((n - \ell)\)-positive, for some \( 1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor \) then the Betti numbers \( b_1(M), \ldots, b_\ell(M) \) vanish. It is worth to note that there is a corresponding rigidity result which states that if \( M \) is a closed Riemannian manifold of dimension \( n \geq 3 \) and the curvature operator is \((n - \ell)\)-positive, for some \( 1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor \) then every harmonic \( \ell \)-form is parallel. Moreover, Petersen and Wink also obtained a new rigidity result for harmonic \( \ell \)-forms on a closed \( n \)-dimensional Riemannian manifold with \((n - \ell)\)-nonnegative curvature operator. Furthermore, they introduced a beautiful application of their methods to obtain a generalization of a theorem due to Tachibana which states that if \( M \) is a connected closed \( n \)-dimensional Einstein manifold and the curvature is \( \lfloor \frac{n-1}{2} \rfloor \)-positive then \( M \) has a constant sectional curvature. We also would like to mention that there is a version of these results developed by Petersen and Wink [PW20] in the setting on complete non-compact weighted manifolds. The authors in [PW20] study vanishing properties for Betti numbers and for weighted harmonic \( \ell \)-forms with finite weighted \( L^2 \) energy. We emphasize that the theory of \( L^2 \) (weighted) harmonic \( \ell \)-forms has a very close relationship
with the so-called reduced $L^2$-cohomology. For further discussion on these topics, we refer the interested readers to [Car02, PW20] and the references therein.

Motivated by the investigation of Petersen and Wink in their series of papers [PW20, PW21a], in this paper, we are interested in $L^Q$ harmonic $\ell$-harmonic tensors on complete non-compact Riemannian manifolds. Let $\mathcal{R}$ be the curvature tensor associated with the Riemannian curvature tensor. Our first result is stated as follows.

**THEOREM 1.1.** Let $(M, g)$ be a complete, non-compact $n$-dimensional Riemannian manifold. Denote $\Delta_L = \nabla^* \nabla + c \text{Ric}$ as Lichnerowicz Laplacian for $c > 0$. Assume that the curvature tensor is $\left[\frac{n}{2}\right]$-nonnegative. Then every harmonic tensor $T$ (with respect to the Lichnerowicz Laplacian) is vanishing if $|T| \in L^Q(M)$ for some $Q \geq 2$. Here $\hat{T}$ is the associated tensor defined in Definition 2.2.

For general $\kappa \geq 0$, if we assume that $g(\mathcal{R}(\hat{T}), \hat{T}) \geq -\kappa |T|^2$, we can also prove an analogous result. However, we need some more assumption that the weighted Poincaré inequality holds ([LW06, Def 0.1]): we say that a complete Riemannian manifold $M$ satisfies a weighted Poincaré inequality with a nonnegative weight function $\rho \geq 0$ on $M$, if the inequality

$$\int_M \rho(x) \phi^2(x) dV \leq \int_M |\nabla \phi|^2 dV$$

is valid for all compactly supported smooth function $\phi \in C_0^\infty(M)$. We put two additional hypotheses on $\rho$, one is

(1.1) \quad \lim \inf_{x \to +\infty} \rho(x) > 0,

(1.2) \quad M \text{ is nonparabolic},

i.e., there exists a symmetric positive Green’s function $G(x, y)$ for the Laplacian on $L^2$ functions (otherwise, we say $M$ is parabolic). All assumptions on a weight function $\rho$ can be regarded as the generalization of the positivity condition of the first Dirichlet eigenvalue $\lambda_1(M)$ [LW06].

Our second result is formulated as follows.

**THEOREM 1.2.** Let $(M, g)$ be a connected complete non-compact Riemannian manifold. Denote $\Delta_L = \nabla^* \nabla + c \text{Ric}$ as Lichnerowicz Laplacian for $c > 0$. Assume that $M$ satisfies a weighted Poincaré inequality with a nonnegative weight function $\rho$ with (1.1) and (1.2), and also $g(\mathcal{R}(\hat{T}), \hat{T}) \geq -\kappa \rho |T|^2$ for all $(0, k)$-tensors $T$, where $\kappa \geq 0$ is given. Then every harmonic tensor $T$ (with respect to the Lichnerowicz Laplacian) vanishes provided that $|T| \in L^Q(M), Q \geq 2$ and $0 \leq \kappa < \frac{4(Q-1)}{cQ^2}$.

As an application, with the fact that the Riemannian curvature tensor is harmonic with respect to the Lichnerowicz Laplacian (namely, $c = \frac{1}{2}$) on $n$-dimensional Ricci-flat manifolds, we
obtain the following rigidity Theorem. This theorem is a consequence of more general rigidity result for Weyl tensor on complete, non-compact Riemannian manifolds with \( \lfloor \frac{n-1}{2} \rfloor \)-positive curvature in Theorem 4.3.

**Theorem 1.3.** For \( n \geq 3 \), let \((M, g)\) be a complete connected non-compact \( n \)-dimensional Ricci-flat manifold. Assume that the Riemannian curvature tensor \( R \) satisfies \( |R| \in L^Q(M) \) for some \( Q \geq 2 \). If the curvature operator is \( \lfloor \frac{n-1}{2} \rfloor \)-nonnegative, then the curvature tensor is vanishing. Consequently, \((M, g)\) must be flat.

For compact Riemannian manifolds, the relationships between Ricci-flatness and flatness have been studied. By using Bochner technique, Fischer and Wolf in [FW75] showed that a compact manifold cannot have both flat Riemannian structures and nonflat Ricci-flat Riemannian structures. Moreover, if a compact connected Riemannian manifold with positive semi-definite Ricci curvature is homotopy-equivalent to a generalized nilmanifold, then it must be Riemannian flat. In particular, if the manifold is homotopy-equivalent to a euclidean space form, then it is Riemannian flat. It is also well-known that a manifold with Schwarzschild metric is Ricci-flat but is not Riemannian flat. For further discussion on Ricci-flat manifolds we refer the reader to [AKW19] and [FW75].

Now, we give some applications to study submanifolds. For notations, let \( M \) be an immersed submanifold in a space form of constant sectional curvature \( K \) and \( b_p(M) \) the \( p \)-th Betti number of \( M \).

**Definition 1.4.** Let \( A = (h_{ij}) \) be the second fundamental form of \( M \) and \( \{ \lambda_i : i = 1, \ldots, n \} \) its principle curvatures. Suppose that \( \mu_1 \leq \mu_2 \leq \ldots \leq \mu_{\lfloor \frac{n}{2} \rfloor} \), where \( \mu_k \) is of form \( \lambda_i \lambda_j \) for any \( 1 \leq k \leq \binom{n}{2} \) and \( 1 \leq i < j \leq n \). We say that the mean curvature of second kind of the hypersurface \( M \) is \( m \)-positive (non-negative) if \( \sum_{k=1}^{m} \mu_k \) is positive (non-negative).

**Theorem 1.5.** Let \( n \geq 3 \), \( 1 \leq p \leq \lfloor \frac{n}{2} \rfloor \) and \( M \) be an immersed hypersurface in a space form of constant sectional curvature \( K \). If \( M \) is closed and
\[
\mu_1 + \ldots + \mu_{n-p} > -(n-p)K
\]
then \( b_1(M) = \ldots = b_p(M) = 0 \) and \( b_{n-p}(M) = \ldots = b_{n-1}(M) = 0 \).

Consequently, we have the following result.

**Corollary 1.6.** Let \( n \geq 3 \) and \( M \) be an immersed hypersurface in a space form of constant sectional curvature \( K \). If \( M \) is closed and
\[
\mu_1 + \ldots + \mu_{\lfloor \frac{n}{2} \rfloor} > -\left( n - \left\lfloor \frac{n}{2} \right\rfloor \right) K
\]
then \( b_p(M) = 0 \) for all \( 0 < p < n \).
We also note that there is a similar phenomenon in higher dimension if $M$ is totally umbilical.

The paper is organized as follows. In Section 2, we introduce some basic results on Lichnerowicz Laplacian and estimation of algebraic curvature tensors. Then we prove Theorem 1.1 and 1.2 in Section 3. We derive several applications of these results when harmonic tensors are harmonic forms. Finally, we use Section 4 to show geometric applications, in particular, we will prove Theorem 1.3 and rigidity properties for Weyl tensors in complete, non-compact Riemannian manifolds. The new Bochner techniques to study Riemannian submanifolds are investigated in Section 5.

2 Preliminaries

Let $(M, g)$ be an $n$-dimensional Riemannian manifold and denote $R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z$ as $(1,3)$-Riemannian curvature tensor. We denote by $T^{(0,k)}(M)$ the vector bundle of $(0,k)$-tensors on $M$. Recall that the Weitzenböck curvature operator on a tensor $T \in T^{(0,k)}(M)$ is defined by

$$\text{Ric}(T) (X_1, \ldots, X_k) = \sum_{i=1}^{k} \sum_{j=1}^{n} (R(X_i, e_j) T) (X_1, \ldots, e_j, \ldots, X_k).$$

For $c > 0$ the Lichnerowicz Laplacian is given by

$$\Delta_L = \nabla^* \nabla + c \text{Ric}.$$

Here, we use the convention $\nabla^* T(X_2, \ldots, X_k) = - (\nabla_{E_i} T)(E_i, X_2, \ldots, X_k)$ on a tensor $T \in T^{(0,k)}(M)$. A tensor $T \in T^{(0,k)}(M)$ is called harmonic if $\Delta_L T = 0$.

**Example 2.1.** There are some important examples of Lichnerowicz Laplacian for different $c > 0$.

(a) The Hodge Laplacian is a Lichnerowicz Laplacian on any forms for $c = 1$.

(b) For $c = \frac{1}{2}$ the Riemannian curvature tensor $Rm$ is harmonic if it is divergence free. This follows from the formula

$$(\nabla^* \nabla Rm + \frac{1}{2} \text{Ric}(Rm))(X, Y, Z, W) = \frac{1}{2}(\nabla_X \nabla^* Rm)(Y, Z, W) - \frac{1}{2}(\nabla_Y \nabla^* Rm)(X, Z, W) + \frac{1}{2}(\nabla_Z \nabla^* Rm)(W, X, Y) - \frac{1}{2}(\nabla_W \nabla^* Rm)(Z, X, Y).$$

The symmetric $(0,2)$ tensor $h$ is called a Codazzi tensor if $d^\nabla h(X, Y, Z) := (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) = 0$, and harmonic if in addition it is divergence free. The fact is that $Rm$
is divergence free if and only if its Ricci tensor is a Codazzi tensor, in this case its scalar curvature is constant. In particular, the contracted Bianchi identity implies the Ricci tensor is harmonic if and only if \( Rm \) is harmonic. If the manifold is Einstein, its Ricci tensor is always Codazzi. Therefore, the curvature tensors of Einstein manifolds are harmonic.

Associated with a fixed \( T \in T^{(0,k)}(M) \), in [Peter16], we have the following tensor.

**Definition 2.2.** For \( T \in T^{(0,k)}(M) \), we define \( \hat{T} \in \Lambda^2 \otimes T^{(0,k)}(M) \) implicitly by

\[
g(L, \hat{T}(X_1, \ldots, X_k)) = (LT)(X_1, \ldots, X_k)
\]

for all \( L \in \mathfrak{so}(M) = \Lambda^2(M) \). Here for any \( k \) smooth vector fields \( X_1, \ldots, X_k \) on \( M \), we have

\[
LT(X_1, \ldots, X_k) = -\sum_{i=1}^{k} T(X_1, \ldots, LX_i, \ldots, X_k).
\]

Note that \( \hat{T} \) encodes the information on how all \( L \in \mathfrak{so}(M) \) interact with \( T \). The following proposition give a relation between \( \text{Ric}(T) \) and \( \hat{T} \) established in [Peter16].

**Proposition 2.3.** If \( S, T \in T^{(0,k)}(M) \), then

\[
g(\text{Ric}(S), T) = g(\mathfrak{R}(\hat{S}), \hat{T}).
\]

Here \( \mathfrak{R} : \Lambda^2(TM) \to \Lambda^2(TM) \) is the curvature operator defined by

\[
g(\mathfrak{R}(X \wedge Y), Z \wedge W) = Rm(X, Y, Z, W)
\]

for any smooth vector fields \( X, Y, Z, W \). In particular, \( \text{Ric} \) is self-adjoint.

Recall the Bochner formula for a tensor \( T \in T^{(0,k)}(M) \)

\[
\Delta \frac{1}{2}|T|^2 = |\nabla T|^2 - g(\nabla^* \nabla T, T).
\]

Therefore, if \( T \) is harmonic, then \( \nabla^* \nabla T = -c \text{Ric} T \). Hence, this togeher with Proposition 2.3 implies

\[
\Delta \frac{1}{2}|T|^2 = |\nabla T|^2 + c \cdot g(\mathfrak{R}(\hat{T}), \hat{T}).
\]

The following lemmas due to Petersen and Wink in [PW21a] provide a framework to control the curvature term of the Lichnerowicz Laplacian on tensors. To give a precisely definition, let us recall some notations. Suppose that \((V, g)\) is an \( n \)-dimensional Euclidean vector space and \( T^{(0,k)}(V) \) stands for the space of \((0,k)\)-tensors. We denote the vector space of symmetric \((0,2)\)-tensor by \( \text{Sym}^2(V) \). It is well known that there is an orthogonal decomposition

\[
\text{Sym}^2(\Lambda^2 V) = \text{Sym}^2_B(\Lambda^2 V) \oplus \Lambda^4 V,
\]
where the vector space $\text{Sym}^2_B(\Lambda^2 V)$ consists of all tensors $R \in \text{Sym}^2(\Lambda^2 V)$ satisfying the first Bianchi identity. An $R \in \text{Sym}^2_B(\Lambda^2 V)$ is called an algebraic curvature tensor. The associated algebraic curvature $(0,4)$-tensor $Rm$ is defined by

$$Rm(x, y, z, w) = R(x \wedge y, z \wedge w), \forall x, y, z, w \in V.$$  

For $S, T \in \text{Sym}^2(V)$, the Kulkarni-Nomizu product of $S, T$ is given by

$$(S \otimes T)(x, y, z, w) = S(x, z)T(y, w) - S(x, w)T(y, z) + S(y, w)T(x, z) - S(y, z)T(x, w).$$

Recall that every algebraic $(0,4)$-curvature tensor $Rm$ satisfies the orthogonal decomposition

$$Rm = \frac{\text{Scal}}{2(n-1)n}g \otimes g + \frac{1}{n-2}g \otimes R^o \text{Ric} + W,$$

where $R^o \text{Ric} = \text{Ric} - \frac{\text{Scal}}{n}g$ is traceless Ricci tensor and $W$ denotes the Weyl part.

**Lemma 2.4.** Let $\mathcal{R} : \Lambda^2 V \to \Lambda^2 V$ be an algebraic curvature operator with eigenvalues $\mu_1 \leq \ldots \leq \mu_{(\frac{n}{2})}$ and let $T \in T^{(0,k)}(V)$. Suppose there is $C \geq 1$ such that

$$|LT|^2 \leq \frac{1}{C} |\hat{T}|^2 |L|^2$$

for all $L \in \mathfrak{so}(V)$. Let $\kappa \leq 0$.

1. If $\frac{1}{|e_j|} (\mu_1 + \ldots + \mu_{|e_j|}) \geq \kappa$, then $g(\mathcal{R}(\hat{T}), \hat{T}) \geq \kappa|\hat{T}|^2$.

2. If $\mu_1 + \ldots + \mu_{|e_j|} > 0$, then $g(\mathcal{R}(\hat{T}), \hat{T}) > 0$ unless $\hat{T} = 0$.

**Proof.** See [PW21a, Lemma 2.1].

The next lemma allows us to estimate $|LT|^2$ for various types of tensors.

**Lemma 2.5.** Let $(V, g)$ be an $n$-dimensional Euclidean vector space and $L \in \mathfrak{so}(V)$. The followings hold:

(a) Every $T \in T^{(0,k)}(V)$ satisfies

$$|LT|^2 \leq k^2 |T|^2 |L|^2.$$  

(b) Every $\ell$-form $\omega$

$$|L\omega|^2 \leq \min\{\ell, n - \ell\}|\omega|^2 |L|^2.$$
(c) Every algebraic curvature $(0, 4)$-tensor $R_m$ satisfies

$$|LR_m|^2 \leq 8|R_m|^2|L|^2,$$

where $R_m$ is traceless algebraic curvature $(0, 4)$-tensor given by

$$R_m = R_m - \frac{\text{Scal}}{2(n-1)n} g \otimes g.$$

**Proof.** See [PW21a, Lemma 2.2].

Finally, we need the following result about the relation between $|\hat{T}|$ and $|T|$ for several types of tensors above.

**Proposition 2.6.** Let $(V, g)$ be an $n$-dimensional Euclidean vector space. The followings hold:

(a) Every $\ell$-form $\omega$ satisfies

$$|\hat{\omega}|^2 = \ell(n-\ell)|\omega|^2.$$

(b) Every algebraic $(0, 4)$-curvature tensor $R_m$ and every $R \in \text{Sym}^2_B(\Lambda^2 V)$ satisfies

$$|\hat{R}_m|^2 = |\hat{\circ}R_m|^2 = 4(n-1)|\circ R_m|^2 - 8|\circ \text{Ric}|^2.$$

In particular $\hat{R}_m = 0$ if and only if $R_m = \frac{\kappa}{2} g \otimes g$ for some $\kappa \in \mathbb{R}$.

**Proof.** See [PW21a, Proposition 2.5].

3 Rigidity theorems for harmonic tensors

**Proof of Theorem 1.1.** For $T$ is a harmonic tensor, recall that the Bochner formula for harmonic tensor $T$ implies

$$\Delta \frac{1}{2}|T|^2 = |\nabla T|^2 + c \cdot g(\Re(\hat{T}), \hat{T}).$$

Since $g(\Re(\hat{T}), \hat{T}) \geq 0$, the Kato inequality $|\nabla T|^2 \geq |\nabla |T||^2$ implies

$$\frac{1}{2}\Delta |T|^2 \geq |\nabla |T||^2.$$  

(3.1)

Take $q \in \mathbb{R}^+$ arbitrarily and a smooth nonnegative function $\varphi$ with compact support. Multiplying inequality (3.1) side by side by $\varphi^2 |T|^q$ and integrating over $M$, we have

$$\frac{1}{2} \int_M \varphi^2 |T|^q \Delta |T|^2 \geq \int_M \varphi^2 |T|^q |\nabla |T||^2$$
Using the integration by part, we infer

\begin{equation}
- \int_M \phi^2 |T|^q |\nabla|T|^2| \leq \frac{1}{2} \int_M \langle \nabla (\phi^2 |T|^q), \nabla |T|^2 \rangle \\
= 2 \int_M \phi |T|^{q+1} (\nabla \phi, \nabla |T|) + q \int_M \phi^2 |T||\nabla|T|^2|.
\end{equation}

By Cauchy-Schwarz inequality, we have

\[ 2 |\phi|T|^{q+1} (\nabla \phi, \nabla |T|) \leq \varepsilon \phi^2 |T|^q |\nabla|T|^2| + \frac{1}{\varepsilon} |\nabla \phi|^2 |T|^{q+2}, \varepsilon > 0. \]

Using this and (3.2), we obtain that

\[(q + 1 - \varepsilon) \int_M \phi^2 |T|^q |\nabla|T|^2| \leq \frac{1}{\varepsilon} \int_M |\nabla \phi|^2 |T|^{q+2}. \]

For $1 + q > 0$, we can choose $\varepsilon > 0$ sufficiently small so that $1 + q - \varepsilon > 0$. Thus,

\begin{equation}
\int_M \phi^2 |T|^q |\nabla|T|^2| \leq C \int_M |\nabla \phi|^2 |T|^{q+2},
\end{equation}

where $C = C(q, \varepsilon) > 0$.

Now, for $Q \geq 2$, we can choose $q \geq 0$ such that $Q = q + 2$ and also choose $\phi$ satisfying

\[ \phi = \begin{cases} 1 & \text{on } B(R) \\ 0 & \text{on } M \setminus B(2R) \end{cases} \]

and $|\nabla \phi| \leq \frac{2}{R}$. The inequality (3.3) implies

\[ \int_M \phi^2 |T|^{Q-2} |\nabla|T|^2| \leq \frac{4C}{R^2} \int_M |T|^Q. \]

Let $R \to \infty$ and since $|T| \in L^Q(M)$, then $|T|$ is constant on each connected component of $M$. Now, since the Ricci curvature is bounded from below by the sum of the lowest $(n - 1)$ eigenvalues of the curvature operator, the curvature assumption implies $\text{Ric} \geq 0$ (see also Remark 1.10 in [PW20]). Hence, due to non-compactness of $M$, a lower bound estimate in [Yau76] yields the volume of the geodesic ball $B_o(R)$ has at least linear growth. Consequently, $M$ is of infinite volume. Since $|T|$ is constant and $|T| \in L^Q(M)$, we conclude that $T$ must be vanishing. The proof is complete. \(\square\)

We note the if we only assume $g(\mathfrak{R}(\hat{T}), \hat{T}) \geq 0$ for every harmonic tensor $T$ instead of the $\lfloor \frac{n}{2} \rfloor$-non-negativity of the curvature operator then we still obtain that $T$ is parallel provided $T \in L^Q(M)$. Moreover, $T$ must be vanishing if $g(\mathfrak{R}(\hat{T}), \hat{T}) > 0$ unless $T = 0$.

Now, we give a proof of Theorem 1.2.
Proof of Theorem 1.2. Using the similar argument as in the proof of Theorem 1.1 and the assumption \( g(\tilde{\mathcal{R}}(\hat{T}), \hat{T}) \geq -\kappa \rho |T|^2 \), we obtain that
\[
ck \int_M \rho \varphi^2 |T|^{q+2} \geq 2 \int_M \varphi |T|^{q+1} \langle \nabla \varphi, \nabla |T| \rangle + (q + 1) \int_M \varphi^2 |T|^q |\nabla |T||^2.
\]
By the weighted Poincaré inequality, we have the following estimation
\[
\int_M \rho \varphi^2 |T|^{q+2} \leq \int_M \left| \nabla \left( \varphi |T|^\frac{q+2}{2} \right) \right|^2 \leq (1 + \varepsilon) \left( \frac{q + 2}{2} \right)^2 \int_M |T|^q |\nabla |T||^2 \varphi^2 + \left( 1 + \frac{1}{\varepsilon} \right) \int_M |T|^{q+2} |\nabla \varphi|^2.
\]
(3.4)
By Cauchy-Schwarz inequality, we have
\[
2 \left| \varphi |T|^{q+1} \langle \nabla \varphi, \nabla |T| \rangle \right| \leq \varepsilon \varphi^2 |T|^q |\nabla |T||^2 + \frac{1}{\varepsilon} |\nabla \varphi| |T|^{q+2}, \varepsilon > 0.
\]
Combining this and (3.4), we obtain
\[
\left( c\kappa \left( 1 + \frac{1}{\varepsilon} \right) + \frac{1}{\varepsilon} \right) \int_M |T|^{q+2} |\nabla \varphi|^2 \geq \left( q + 1 - \varepsilon - c\kappa \left( 1 + \varepsilon \right) \left( \frac{q + 2}{2} \right)^2 \right) \int_M |T|^q |\nabla |T||^2 \varphi^2.
\]
(3.5)
On the other hand, if we assume that \( q + 1 - c\kappa \left( \frac{q+2}{2} \right)^2 > 0 \) or equivalently, \( \kappa < \frac{4(q+1)}{c(q+2)^2} \), we can find \( \varepsilon > 0 \) sufficiently small for which
\[
q + 1 - \varepsilon - c\kappa \left( 1 + \varepsilon \right) \left( \frac{q + 2}{2} \right)^2 > 0.
\]
For such \( \varepsilon \), we have
\[
\int_M |T|^q |\nabla |T||^2 \varphi^2 \leq C \int_M |T|^{q+2} |\nabla \varphi|^2,
\]
where \( C = C(\varepsilon, q) > 0 \). Now, it is easy to see that for any \( Q \geq 2 \), we can find \( q \geq 0 \) and \( Q = q + 2 \). Then, for any \( \kappa < \frac{4(Q-1)}{cQ^2} \), it holds that
\[
\int_M |T|^{Q-2} |\nabla |T||^2 \varphi^2 \leq C \int_M |T|^Q |\nabla \varphi|^2.
\]
Using the cut-off function \( \varphi \) and also the same argument as in the Theorem 1.1, we obtain that \( |T| \) is constant on \( M \) due to the fact that \( M \) is connected. From the hypothesis (1.2), the volume of \( M \) is infinite [LW06, Corollary 3.2]. By \( |T| \in L^Q(M) \), it follows that \( |T| = 0 \). Therefore, \( T \equiv 0 \). The proof is complete.

Remark 3.1. We note that if \( T \) satisfies a refine Kato inequality, namely there exists a constant \( a \geq 0 \) such that
\[
|\nabla T|^2 \geq (1 + a)|\nabla |T||^2,
\]
then the equation (3.5) becomes

$$\left(-c\kappa \left(1 + \frac{1}{\varepsilon}\right) + \frac{1}{\varepsilon}\right) \int_M |T|q+2|\nabla \varphi|^2 \geq \left(q + 1 + a - \varepsilon + c\kappa \left(1 + \varepsilon\right) \left(\frac{q + 2}{2}\right)^2\right) \int_M |T|^q|\nabla T|^2 \varphi^2.$$ 

Hence, we can improve the upper bound of $\kappa$ to be

$$\kappa \leq \frac{4(Q - 1 + a)}{cQ^2}.$$ 

As applications, when we consider the harmonic tensors $T$ to be harmonic $\ell$-forms, we obtain the following vanishing results.

**Theorem 3.2.** Let $n \geq 3$ and $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$. If $(M, g)$ is a complete, non-compact $n$-dimensional Riemannian manifold with $(n - p)$-nonnegative curvature operator. Then every harmonic $\ell$-form $\omega$, for all $1 \leq \ell \leq p$ and $n - p \leq \ell \leq n - 1$, is vanishing if $|\omega| \in L^Q(M)$ for some $Q \geq 2$.

**Proof.** Let $\omega$ is a harmonic $\ell$-form or $(n - \ell)$-form with $|\omega| \in L^Q(M)$ for some $Q \geq 2$ and $1 \leq \ell \leq p$. Applying Lemma 2.5 and Proposition 2.6, we obtain that

$$|L\omega|^2 \leq \ell |\omega|^2 |L|^2 = \frac{1}{n - \ell} |\hat{\omega}|^2 |L|^2$$

for all $L \in \mathfrak{so}(TM)$.

If the curvature tensor is $(n - p)$-nonnegative, then it is also $(n - \ell)$-nonnegative. Hence Lemma 2.4 implies

$$g(\Re(\hat{\omega}), \hat{\omega}) \geq 0.$$ 

An application of Theorem 1.1 to Hodge Laplacian yields that $\omega$ is parallel. Moreover, $|\omega|$ is constant. Now, we can follow the argument as in Theorem 1.1 to complete the proof. 

Due to [Car02], the above result has a reduced $L^2$ cohomology interpretation as follows. Let $H^\ell(M)$ be the space of $L^2$ harmonic $\ell$-forms, saying $H^\ell(M) = \{\omega \in L^2(\Lambda^\ell T^* M) : d\omega = \delta \omega = 0\}$, where $\delta$ is the dual of the differential operator $d$ and $Z^\ell_2(M)$ the kernel of the unbounded operator $d$ acting on $L^2(\Lambda^\ell T^* M)$, or equivalently

$$Z^\ell_2(M) = \{\omega \in L^2(\Lambda^\ell T^* M) : d\omega = 0\}.$$ 

The space $H^\ell(M)$ can be used to characterize the reduced $L^2$ cohomology group as follows

$$H^\ell(M) = Z^\ell_2(M) / dC^\infty_0(L^2(\Lambda^{\ell-1} T^* M),$$

where the closure is taken with respect to the $L^2$ topology. It is worth to note that the finiteness of $\dim H^\ell(M)$ depends only on the geometry of ends ([Lott97]). Theorem 3.2 leads immediately to the following result:
**Corollary 3.3.** Let $n \geq 3$ and let $(M, g)$ be a complete non-compact $n$-dimensional Riemannian manifold. Then every harmonic $\ell$-form $\omega$ with $|\omega| \in L^Q(M)$ for some $Q \geq 2$ is vanishing if the curvature tensor is $\lceil \frac{n}{2} \rceil$-nonnegative. In particular, every harmonic $\ell$-form $\omega$ with $|\omega| \in L^2(M)$ is vanishing, consequently, every reduced $L^2$ cohomology groups are trivial.

We note that by [CGH00], there is a refined Kato inequality for harmonic $\ell$-forms stated as follows

$$|\nabla \omega|^2 \geq \left(1 + \frac{1}{\max\{\ell, n-\ell\}}\right)|\nabla|\omega|^2.$$  

The next results with a general curvature condition is a direct consequence of Theorem 1.2 and the above Kato inequality.

**Theorem 3.4.** Let $(M, g)$ be a complete non-compact $n$-dimensional manifold, $n \geq 3$. Assume that $M$ satisfies a weighted Poincaré inequality with a nonnegative weight function $\rho$ with (1.1) and (1.2). Denote $\mu_1 \leq \ldots \leq \mu_{\frac{n}{\ell}}$ eigenvalues of the curvature operator of $(M, g)$. For $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$ and $\kappa \geq 0$, if

$$\frac{\mu_1 + \ldots + \mu_{n-p}}{n-p} \geq -\kappa \rho$$

then every harmonic $\ell$-form $\omega$, for all $1 \leq \ell \leq p$ and $n-p \leq \ell \leq n-1$, vanishes provided that $|\omega| \in L^Q(M)$ for some $Q \geq 2$ satisfying

$$\kappa < \frac{4(Q-1) + \frac{1}{\max\{\ell, n-\ell\}}}{\ell(n-\ell)Q^2}.$$  

**Proof.** For $\omega$ is a harmonic $\ell$-form or $(n-\ell)$-form, using the estimate (3.6) and Lemma 2.4, we obtain that

$$g(\Re(\hat{\omega}), \hat{\omega}) \geq -\kappa \rho |\hat{\omega}|^2 = -\kappa \rho (n-\ell)|\omega|^2.$$  

Following the proof of Theorem 1.2 and Remark 3.1, we complete the proof.

Finally, the above theorem infers following vanishing result for reduced $L^2$ cohomology groups.

**Corollary 3.5.** Let $(M, g)$ be a complete non-compact $n$-dimensional manifold, $n \geq 3$. Assume that $M$ satisfies a weighted Poincaré inequality with a nonnegative weight function $\rho$ with (1.1) and (1.2). Denote $\mu_1 \leq \ldots \leq \mu_{\frac{n}{\ell}}$ eigenvalues of the curvature operator of $(M, g)$. If

$$\frac{\mu_1 + \ldots + \mu_{n-\lfloor \frac{n}{2} \rfloor}}{n-\lfloor \frac{n}{2} \rfloor} \geq -\kappa \rho$$

then every harmonic $\ell$-form $\omega$, for all $1 \leq \ell \leq p$ and $n-p \leq \ell \leq n-1$, vanishes provided that $|\omega| \in L^Q(M)$ for some $Q \geq 2$ satisfying

$$\kappa < \frac{4(Q-1) + \frac{1}{\max\{\ell, n-\ell\}}}{\ell(n-\ell)Q^2}.$$  

**Proof.** For $\omega$ is a harmonic $\ell$-form or $(n-\ell)$-form, using the estimate (3.6) and Lemma 2.4, we obtain that

$$g(\Re(\hat{\omega}), \hat{\omega}) \geq -\kappa \rho |\hat{\omega}|^2 = -\kappa \rho (n-\ell)|\omega|^2.$$  

Following the proof of Theorem 1.2 and Remark 3.1, we complete the proof.
then every harmonic $\ell$-form $\omega$, for all $1 \leq \ell \leq n - 1$ vanishes provided that $|\omega| \in L^2(M)$ for some $\kappa$ satisfying

$$\kappa < \frac{1 + \frac{1}{\max\{\ell, n - \ell\}}}{\ell(n - \ell)}.$$  

Consequently, every reduced $L^2$-cohomology groups are trivial.

4 Geometric Applications

In this section, we apply the vanishing results in the section 3 for indicated types of tensors. First, we now a rigidity property for Weyl tensor. To begin with, let us recall the following result (see [PW21b, Proposition 2.2]).

**Proposition 4.1.** Let $(M, g)$ be a Riemannian manifold. If the Weyl tensor $W$ is divergence free, then $W$ satisfies the second Bianchi identity and

$$\nabla^* \nabla W + \frac{1}{2} \text{Ric}(W) = 0.$$  

Hence, if the Weyl tensor is divergence free, we have $\nabla^* \nabla W = -\frac{1}{2} \text{Ric}(W)$. Then by the Bochner formula, it yields

$$\Delta \frac{1}{2} |W|^2 = |\nabla W|^2 + \frac{1}{2} g(\text{Ric}(W), W) = |\nabla W|^2 + \frac{1}{2} g(\mathfrak{R}(\hat{W}), \hat{W}).$$  

**Remark 4.2.** For $n \geq 4$, $(M, g)$ is an $n$-dimensional Riemannian manifold. It is well known that $(M, g)$ is locally conformally flat if and only if the Weyl tensor vanishes.

With the help of this Bochner type formula, we have the following theorem on complete non-compact manifolds which is a non-compact version of Petersen and Wink’s rigidity for the Weyl tensor in [PW21b].

**Theorem 4.3.** For $n \geq 4$, let $(M, g)$ be a complete non-compact $n$-dimensional Riemannian manifold. Suppose that the Weyl tensor satisfies $\nabla^* W = 0$. If curvature operator is $\lfloor \frac{n-1}{2} \rfloor$-nonnegative, then the Weyl tensor $W$ is vanishing provided that $|W| \in L^Q(M)$, $Q \geq 2$. Consequently, $(M, g)$ is locally conformally flat.

**Proof.** Applying Lemma 2.5 and Proposition 2.6 for $Rm$ replaced by the Weyl tensor $W$ with the note that the Weyl tensor is traceless and its associated Ricci tensor vanishes, we obtain that

$$|LW|^2 \leq 8|W|^2|L|^2 = \frac{2}{n-1} |\hat{W}|^2|L|^2$$  

(4.2)
for all $L \in \mathfrak{so}(TM)$. By the assumption that the curvature operator is $\lfloor \frac{n-1}{2} \rfloor$-nonnegative and Lemma 2.4 implies
\[ g(\mathbb{R}(\hat{W}), \hat{W}) \geq 0. \]
Applying Theorem 1.1, we obtain that $\hat{W}$ is vanishing, so is $W$. Therefore, $M$ is locally conformally flat.

Now we give a proof of Theorem 1.3.

**Proof.** Since $M$ is Ricci flat, it must be Einstein. Moreover, the Riemannian curvature tensor is divergence free. Note that because of Ricci-flatness, the Riemannian curvature tensor and the Weyl tensor $W$ coincide. Consequently, $|W| \in L^Q(M)$ and $W$ is divergence free. Hence, Theorem 4.3 implies $M$ is locally conformally flat. Since $M$ is Ricci-flat, it must be Riemannian flat.

We note that there is a more general statement for the Weyl tensor. In fact, if $M$ is Einstein, then the Riemannian tensor is also divergence free. Therefore, by Lemma 1 in [Der88], the Weyl tensor is also divergence free. For general $\kappa \geq 0$, we have the following result.

**Theorem 4.4.** For $n \geq 4$, let $(M, g)$ be a connected complete non-compact $n$-dimensional Riemannian manifold. Suppose that the Weyl tensor is divergence free and assume that $M$ satisfies a weighted Poincaré inequality with a nonnegative weight function $\rho$. If the eigenvalues $\mu_1 \leq \ldots \leq \mu_{\lfloor \frac{n}{2} \rfloor}$ of the curvature operator satisfies
\[
\frac{\mu_1 + \ldots + \mu_{\lfloor \frac{n-1}{2} \rfloor}}{\lfloor \frac{n-1}{2} \rfloor} \geq -\kappa \rho, \kappa \geq 0
\]
and the Weyl tensor satisfies $|W| \in L^Q(M)$ such that $\kappa < \frac{2(Q-1)}{(n-1)Q^2}$, then $(M, g)$ is locally conformally flat.

**Proof.** Since the estimate (4.2) and Lemma 2.4, we obtain that
\[ g(\mathbb{R}(\hat{W}), \hat{W}) \geq -\kappa \rho |\hat{W}|^2 = -4\kappa (n-1) \rho |W|^2. \]
Applying Theorem 1.2, we obtain that $W \equiv 0$. This implies $(M, g)$ is locally conformally flat.

Furthermore, if we assume that $M$ is Einstein, we can improve the upper bound of $\kappa$ to obtain the following rigidity.

**Theorem 4.5.** For $n \geq 4$, let $(M, g)$ be a connected complete non-compact $n$-dimensional Einstein manifold. Suppose that the Weyl tensor is divergence free and assume that $M$ satisfies
a weighted Poincaré inequality with a nonnegative weight function $\rho$. If the eigenvalues $\mu_1 \leq \ldots \leq \mu_{\binom{n}{2}}$ of the curvature operator satisfies

$$\frac{\mu_1 + \ldots + \mu_{\binom{n}{2}}}{\binom{n}{2}} \geq -\kappa \rho, \kappa \geq 0$$

and the Weyl tensor satisfies $|W| \in L^Q(M)$ such that $\kappa < \frac{2(Q-1 + \frac{2}{n-1})}{(n-1)Q^2}$, then $(M, g)$ has constant sectional curvature.

**Proof.** Since $M$ is Einstein, an improved Kato inequality in [BKN89] (see also [CGH00]), we have

$$|\nabla W|^2 \geq \left(1 + \frac{2}{n-1}\right) |\nabla W|^2.$$ 

Following the proof of Theorem 1.2 and using Remark 3.1, we conclude that if

$$\kappa \leq \frac{2 \left(Q - 1 + \frac{2}{n-1}\right)}{(n-1)Q^2}$$

then $W = 0$. Since $M$ is Einstein, this implies that $M$ has constant sectional curvature. \qed

Before coming to next result, we recall a volume comparison theorem due to Bishop-Gromov:

**Theorem 4.6 (Bishop-Gromov).** If $(M, g)$ is a complete $n$-dimensional Riemannian manifold with $\text{Ric} \geq (n - 1)k$, and $p \in M$ is an arbitrary point. Then the function

$$r \mapsto \frac{\text{Vol}(B_r(p))}{\text{Vol}(B^k_r)}$$

is a non-increasing function which tends to $1$ as $r$ goes to $0$, where $B^k_r$ is a geodesic ball of radius $r$ in the space form $M^k_n$. In particular, $\text{Vol}(B_r(p)) \leq \text{Vol}(B^k_r)$.

**Theorem 4.7.** For $n \geq 3$, let $(M, g)$ be a complete, connected, non-compact $n$-dimensional Einstein manifold. Let $\mu_1 \leq \ldots \leq \mu_{\binom{n}{2}}$ be eigenvalues of curvature operator of $(M, g)$. For $\kappa \geq 0$, if $M$ satisfies a weighted Poincaré inequality with a nonnegative weight function $\rho$ with (1.1) and (1.2) and

$$\frac{\mu_1 + \ldots + \mu_{\binom{n}{2}}}{\binom{n}{2}} \geq -\kappa \rho,$$

then for any $\kappa < \frac{2(Q-1)}{(n-1)Q^2}$, we have

$$\int_M |\text{Rm}|^Q = \infty.$$
PROOF. In contradiction, we assume that \( \liminf_{x \to \infty} \rho(x) = l > 0 \), \( M \) satisfies a weighted Poincaré inequality with a nonnegative weight function \( \rho \), and \( \| Rm \| \in L^Q(M) \) for some \( Q \geq 2 \) and \( 0 \leq \kappa < \frac{2(Q-1)}{(n-1)Q} \). Since \( \text{Ric} = 0 \), Proposition 2.6 follows that \( |\hat{Rm}|^2 = 4(n-1)|\hat{Rm}|^2 \) and thus Lemma 2.5 implies

\[
|LRm|^2 \leq 8 |Rm|^2 |L|^2 = \frac{2}{n-1} |\hat{Rm}|^2 |L|^2.
\]

for all \( L \in \mathfrak{so}(TM) \). Using the estimate (4.3) and Lemma 2.4, we have

\[
g(\hat{R}(\hat{Rm}), \hat{Rm}) \geq -\kappa \rho |\hat{Rm}|^2 = -4\kappa(n-1)\rho |\hat{Rm}|^2 \geq -4\kappa(n-1)\rho |Rm|^2
\]

Applying Theorem 1.2, we obtain that \( Rm \equiv 0 \). This implies \( \text{Ric} = 0 \). By Theorem 4.6, we obtain

\[
\frac{\text{Vol}(B(p, 2R))}{\text{Vol}(B(p, R))} \leq \frac{\text{Vol}(B_0(p_0, 2R))}{\text{Vol}(B_0(p_0, R))} = 2^n,
\]

where \( B_0(p_0, R) \) denote the ball of radius \( R \) around the point \( p_0 \) in the \( n \)-dimensional Euclidean space \( \mathbb{E}^n \). Choose a family of nonnegative smooth functions \( \varphi_R \) satisfying

\[
\varphi_R = \begin{cases} 
 1 & \text{on } B(R) \\
 0 & \text{on } M \setminus B(2R)
\end{cases}
\]

and \( 0 \leq \varphi_R \leq 1, |\nabla \varphi_R| \leq \frac{2}{R} \). From \( \liminf_{x \to \infty} \rho(x) = l > 0 \), there exists \( \delta > 0 \) such that

\[
\rho(x) > l - \epsilon > 0
\]

on the complement of \( B(p, \delta) \) for some \( \epsilon > 0 \). Then \( R > \delta \),

\[
(l- \epsilon) \int_{M \setminus B(p, \delta)} |\varphi_R|^2 \leq \int_M |\nabla \varphi_R|^2.
\]

It is easy to see that for \( R \geq \delta \),

\[
l - \epsilon \leq \frac{\int_M |\nabla \varphi_R|^2}{\int_{M \setminus B(p, \delta)} |\varphi_R|^2} \leq \frac{4}{R^2} \frac{\text{Vol}(B(0, 2R))}{\text{Vol}(B(0, R) - \text{Vol}(B(0, \delta))} \leq \frac{4}{R^2} 2^n \frac{1}{1 - \frac{\text{Vol}(B(0, \delta))}{\text{Vol}(B(0, R))}} \to 0
\]

as \( R \to \infty \). This implies \( l - \epsilon = 0 \). This contradiction completes the proof.

Finally, we want to note that if the manifold is not Einstein, but has zero scalar curvature, we have the following rigidity.

**Theorem 4.8.** For \( n \geq 3 \), let \( (M, g) \) be a complete, connected, non-compact \( n \)-dimensional Riemannian manifold with zero scalar curvature. Let \( \mu_1 \leq \ldots \leq \mu_{\binom{n}{2}} \) be eigenvalues of curvature operator of \( (M, g) \). For \( \kappa \geq 0 \), suppose that \( M \) satisfies a weighted Poincaré inequality with a nonnegative weight function \( \rho \) with (1.1) and (1.2) and

\[
\frac{\mu_1 + \ldots + \mu_{\binom{n}{2}} - \frac{n-1}{2}}{\binom{n}{2}} \geq -\kappa \rho,
\]
for some $\kappa < \frac{2 \left( \frac{Q - 1}{2} \right)}{(n-1)Q^2}$. If $\text{Rm}$ is divergence free then,

$$\int_M |\text{Rm}|^Q = \infty.$$ 

**Proof.** We recall that if $\text{Rm}$ is divergence free and $M$ has zero scalar curvature, then there is a refined Kato inequality proved by [TV05] that

$$|\nabla \text{Rm}|^2 \geq \frac{3}{2} |\nabla |\text{Rm}||^2.$$ 

Therefore, using the argument in the proof of Theorem 4.7 and Remark 3.1, we are done.

**Remark 4.9.** We would like to mention that the assumption $M$ has zero scalar curvature is quite natural. This is because of a result by Derdzinski in [Der88] which stated that if the Riemannian tensor is harmonic then the scalar curvature must be constant.

## 5 Geometry of immersed submanifolds

Now, let $M^n$ be an immersed hypersurface in a Riemannian manifold $\overline{M}$. Denote $\text{Rm}, \overline{\text{Rm}}$ the Riemannian curvature tensors on $M$ and $\overline{M}$, respectively. For any unit tangent vectors $X, Y, Z, W$ in $T_p M$, where $p \in M$, the Gaussian equation implies

$$\text{Rm}(X, Y, Z, W) = \overline{\text{Rm}}(X, Y, Z, W) - h(X, W)h(Y, Z) + h(X, Z)h(Y, W),$$ 

where $A = (h_{ij})$ is the second fundamental form of $M$. Recall that the algebraic curvature operator can be given by

$$g(\mathcal{R}(X \wedge Y), Z \wedge W) = \text{Rm}(X, Y, Z, W).$$ 

Assume that $\overline{M}$ is a space form with constant section curvature $K$, namely

$$\overline{\text{Rm}}(X, Y, Z, W) = K(g(X, Z)g(Y, W) - g(x, W)g(Y, Z)).$$

Suppose that $\{e_1, \ldots, e_n\}$ is an orthonormal basic of $T_p M$ such that $\mathcal{A}e_i = \lambda_i e_i$, where $\lambda_i$’s are principal curvatures. Using the Gaussian equation, we have

$$g(\mathcal{R}(e_i \wedge e_j, e_k \wedge e_\ell)) = K(g_{ik}g_{j\ell} - g_{i\ell}g_{jk}) + h_{ik}h_{j\ell} - h_{i\ell}h_{jk}. $$
Since \( \{e_1, \ldots, e_n\} \) is an orthonormal basis, \( g_{ij} = \delta_{ij} \). Moreover, \( h_{ij} = \lambda_i \delta_{ik} \). It is easy to see that the algebraic curvature operator is presented by the following matrix

\[
\begin{pmatrix}
K + \lambda_1 \lambda_2 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & K + \lambda_1 \lambda_n & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & K + \lambda_2 \lambda_3 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & K + \lambda_2 \lambda_n & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & K + \lambda_3 \lambda_4 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & K + \lambda_{n-1} \lambda_n
\end{pmatrix}
\]  

This implies that all eigenvalues of \([R]\) are of form \( K + \lambda_i \lambda_j, 1 \leq i < j \leq n \). We now give a proof of Theorem 1.5.

**Proof of Theorem 1.5.** Since \( \mu_1 + \ldots + \mu_{n-p} > -(n-p)K \), the curvature operator of \( M \) is \((n-p)\)-positive. Hence, the proof follows by Theorem A in [PW21a]. \(\square\)

Now, we consider the case that \( \mu_1 + \ldots + \mu_{n-p} \) is non-negative. This is directly consequence of Theorem B in [PW21a].

**Theorem 5.1.** Let \( n \geq 3, 1 \leq p \leq \left\lfloor \frac{n}{2} \right\rfloor \) and \( M \) be an immersed Riemannian submanifold in a space form of constant sectional curvature \( K \). If \( M \) is close and

\[
\mu_1 + \ldots + \mu_{n-p} \geq -(n-p)K
\]

then every harmonic \( p \)-form is parallel. Similarly, every harmonic \((n-p)\)-form is parallel.

Next, we study complete non-compact immersed submanifolds. First, we note that if the first eigenvalue \( \lambda_1(M) \) of \( M \) is positive then \( M \) satisfies a Poincaré inequality with weight function \( \rho \equiv \lambda_1(M) \). As a consequence of Theorem 3.4, we have

**Theorem 5.2.** Let \( n \geq 3, 1 \leq p \leq \left\lfloor \frac{n}{2} \right\rfloor \) and \( M \) be an immersed Riemannian submanifold in a space form of constant sectional curvature \( K \). If \( M \) is complete non-compact and

\[
\frac{(n-p) + \mu_1 + \ldots + \mu_{n-p}}{n-p} \geq -\kappa \lambda_1(M)
\]

then every harmonic \( \ell \)-form \( \omega \), for all \( 1 \leq \ell \leq p \) and \( n-p \leq \ell \leq n-1 \), vanishes provided that \( |\omega| \in L^Q(M), Q \geq 2 \) and

\[
0 \leq k < \frac{4 \left( Q - 1 + \frac{1}{\max\{\ell, n-\ell\}} \right)}{\ell(n-\ell)Q^2}.
\]
In the rest of this section, we study totally umbilical submanifolds. We assume that $M^n$ is an immersed submanifold in a Riemannian manifolds $\overline{M}^m$. Furthermore, $M$ is assumed to be totally umbilical, namely, 
\[ h(X, Y) = H g(X, Y) \]
for any unit tangent vector fields $X, Y$, where again $h$ stands for the second fundamental form of $M$ and $H$ for the mean curvature vector field. We refer the interested readers to [Sato] and the references therein for further discussion about totally umbilical submanifolds. The Gaussian equation implies
\[
\text{Rm}(X, Y, Z, W) = \text{Rm}(X, Y, Z, W) - \langle h(X, W), h(Y, Z) \rangle + \langle h(X, Z), h(Y, W) \rangle.
\]
Let \( \{e_1, ..., e_n, \bar{e}_{n+1}, ..., \bar{e}_m\} \) is an orthonormal basic of $T_p\overline{M}$ such that \( \{e_1, ..., e_n\} \) is an orthonormal basic of $T_pM$. We have
\[
g(\mathfrak{R}(e_i \wedge e_j), e_k \wedge e_\ell) = g(\overline{\mathfrak{R}}(e_i \wedge e_j), e_k \wedge e_\ell) + |H|^2 (g_{ik}g_{j\ell} - g_{i\ell}g_{kj}).
\]
Suppose that $\overline{\mu}_1 \leq \overline{\mu}_2 \leq \ldots \leq \overline{\mu}_{\binom{m}{2}}$ are eigenvalues of the curvature operator $\overline{\mathfrak{R}}$ of $\overline{M}$. Due to (5.1), we see that all eigenvalues of $\mathfrak{R}$ are of form $|H|^2 + \overline{\mu}_i$, for some $1 \leq i \leq \binom{m}{2}$. Theorem A in [PW21a] implies

**Theorem 5.3.** Given $n \geq 3$ and $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$. Let $M^n$ be a closed immersed submanifold in a Riemannian manifold $\overline{M}$. If $M$ is totally umbilical and
\[ \overline{\mu}_1 + \ldots + \overline{\mu}_{n-p} > -(n-p)|H|^2 \]
then $b_1(M) = \ldots = b_p(M) = 0$ and $b_{n-p}(M) = \ldots = b_{n-1}(M) = 0$. Here $\overline{\mu}_1 \leq \overline{\mu}_2 \leq \ldots \leq \overline{\mu}_{\binom{m}{2}}$ are eigenvalues of the curvature operator $\overline{\mathfrak{R}}$ of $\overline{M}$.

This yields immediately the following result.

**Corollary 5.4.** Given $n \geq 3$. Let $M^n$ be a closed immersed submanifold in a Riemannian manifold $\overline{M}$. If $M$ is totally umbilical and
\[ \overline{\mu}_1 + \ldots + \overline{\mu}_{n-\lceil \frac{n}{2} \rceil} > - \left( n - \lceil \frac{n}{2} \rceil \right) |H|^2 \]
then $b_1(M) = \ldots = b_{n-1}(M) = 0$. Here $\overline{\mu}_1 \leq \overline{\mu}_2 \leq \ldots \leq \overline{\mu}_{\binom{m}{2}}$ are eigenvalues of the curvature operator $\overline{\mathfrak{R}}$.

Using similar arguments, we also can derive several vanishing results for harmonic forms on submanifolds. We leave them as easy exercises for the interested readers.

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Gunhee Cho
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, SANTA BARBARA
552 UNIVERSITY RD, ISLA VISTA, CA 93106.
E-mail address: gunhee.cho@math.ucsb.edu

Nguyen Thac Dung
FACULTY OF MATHEMATICS - MECHANICS - INFORMATICS
VIETNAM NATIONAL UNIVERSITY, UNIVERSITY OF SCIENCE, HANOI
HANOI, VIET NAM AND
THANG LONG INSTITUTE OF MATHEMATICS AND APPLIED SCIENCES (TIMAS)
THANG LONG UNIVERSITY
NGHIEM XUAN YEM, HOANG MAI
HANOI, VIETNAM
E-mail address: dungmath@gmail.com or dungmath@vnu.edu.vn

Tran Quang Huy
FACULTY OF MATHEMATICS - MECHANICS - INFORMATICS
VIETNAM NATIONAL UNIVERSITY, UNIVERSITY OF SCIENCE, HANOI
HANOI, VIET NAM
E-mail address: tranquanghuy11061998@gmail.com