A SIMPLE PROOF OF VOISIN’S THEOREM FOR CANONICAL CURVES

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ABSTRACT. We give a simple proof of Voisin’s Theorem for general canonical curves. This completely determines the terms of the minimal free resolution of the coordinate ring of such curves. In the case of curves of even genus, we further enhance Voisin’s Theorem by providing a structure theorem for the last syzygy space.

0. INTRODUCTION

In this paper, we give a simple proof of a theorem of Voisin [V2, V3] on the equations of general canonical curves. Using the techniques from our proof, we further give a substantial generalization of her result for curves of even genus. Recall that the classical Theorem of Noether–Babbage–Petri states that canonical curves are projectively normal, and that the ideal $I_{C/P^{g-1}}$ is generated by quadrics (with a few, well-understood, exceptions), see [AS] for a modern treatment. In the 1980s, M. Green realized that these classical results about the equations defining canonical curves should be the first case of a much more general statement about higher syzygies, and he made a very influential conjecture [G1] in this direction.

Whilst the general case of Green’s Conjecture remains open, in 2002 Voisin made a breakthrough by proving the conjecture for general curves of even genus [V2]. Voisin’s argument relies on an intricate study of the geometry of Hilbert schemes on a K3 surface. Recently, an algebraic approach to Voisin’s Theorem has been given, [AFPRW]. In this proof, the authors degenerate the K3 surface to the tangent developable, a singular surface whose hyperplane sections are cuspidal curves. The authors then apply the representation theory of an $SL_2$ action present in this special situation to establish Green’s conjecture for rational cuspidal curves. Explicit plethysm formulae play the key role, involving a change of basis between elementary symmetric polynomials and Schur polynomials. Maps which are simple to describe in one basis become rather complicated in the other, making the proof quite technical, see [AFPRW §5.5–5.7].

In this paper, we first give a simple proof of Voisin’s Theorem, using only basic homological algebra and without the need to degenerate. We further provide a structure theorem in the even genus case, describing in detail the extremal syzygy space. Our proof proceeds by direct computation on K3 surfaces. Let $X$ be a K3 surface over $\mathbb{C}$ with Picard group generated by an ample line bundle $L$ of even genus $g = 2k$, i.e. $(L)^2 = 2g - 2$. Define $K_{p,q}(X,L)$ as the middle cohomology of

$$\bigwedge^{p+1} H^0(X,L) \otimes H^0(X,L^\otimes q-1) \to \bigwedge^p H^0(X,L) \otimes H^0(X,L^\otimes q) \to \bigwedge^{p-1} H^0(X,L) \otimes H^0(X,L^\otimes q+1)$$

Voisin’s Theorem states that $K_{k,1}(X,L) = 0$, [V2]. This single vanishing suffices to prove Green’s Conjecture for general canonical curves in even genus.

Our proof is short and direct. Let $E$ be the rank two Lazarsfeld–Mukai bundle associated to a $g_{k+1}$ on a smooth curve $C \in |L|$, see [L]. The dual bundle $E^\vee$ fits into the exact sequence

$$0 \to E^\vee \to H^0(C,A) \otimes O_X \to i_* A \to 0,$$
for \( A \) a \( g^1_{k+1} \) on \( C \), where \( i : C \hookrightarrow X \) is the inclusion. The vector bundle \( E \) has invariants \( \det(E) = L, h^0(E) = k + 2, h^1(E) = h^2(E) = 0 \).

We deduce Voisin’s Theorem from the Künneth formula on \( X \times \mathbb{P}(H^0(E)) \). Our proof quickly reduces to showing that a certain square matrix is nonsingular. Since our matrix takes the form \( H^k(\text{Sym}^{k+1}G) \to H^k(\text{Sym}^2G \otimes G) \), for some bundle \( G \), the desired nonsingularity is automatic, see the proof of Proposition 1.5.

A few years after her breakthrough for even genus curves, Voisin deduced the odd genus case of generic Green’s conjecture out of the even genus case. In Section 3 we use our methods to give a short and streamlined version of Voisin’s argument:

**Theorem 0.1.** Let \( C \) be a general curve of genus \( g = 2k \) or \( g = 2k + 1 \). Then \( K_{k,1}(C, \omega_C) = 0 \).

In fact, our approach proves much more than Voisin’s theorem. In Section 2 we prove the following structure theorem for the last nonzero syzygy space for K3 surfaces of even genus.

**Theorem 0.2.** We have a natural isomorphism \( K_{k-1,1}(X, L) \cong \text{Sym}^{k-2}H^0(X, E) \).

Theorem 0.2 implies the vanishing \( K_{k,1}(X, L) = 0 \) by standard dimension computations, [F] §4.1, and thus is an enhancement of Voisin’s vanishing theorem. One should compare Theorem 0.2 to Schreyer’s Conjecture, as proven in [FK], describing the structure of the last syzygy space for curves of non-maximal gonality. Also note that it is possible to describe the map \( \text{Sym}^{k-2}H^0(X, E) \to K_{k-1,1}(X, L) \) explicitly, cf. [AN §3.4].

The structural result given by Theorem 0.2 implies a previously open conjecture known as the Geometric Syzygy Conjecture in even genus, see [vB3] where the statement is proven for \( g \leq 8 \). Recall the following important result:

**Theorem 0.3 ([AM], [G2]).** The ideal \( I_{C/P^{g-1}} \) of a canonical curve of Clifford index at least two is generated by quadrics of rank four.

This provides an enhancement of Petri’s theorem stating that \( I_{C/P^{g-1}} \) is generated by quadrics if \( \text{Cliff}(C) \geq 2 \). To extend Theorem 0.3 to higher syzygies, following [vB2] one defines the rank of a syzygy \( \alpha \in K_{p,1}(X, L) \) as the dimension of the minimal subspace \( V \subseteq H^0(X, L) \) such that \( \alpha \) is represented by an element of \( \bigwedge^p V \otimes H^0(X, L) \). Theorem 0.3 states that \( K_{1,1}(C, \omega_C) \) is spanned by syzygies of the minimal possible rank two.

For \( t \neq 0 \in H^0(X, E) \), syzygies corresponding to \( t^{k-2} \) have rank \( k + 1 = \dim H^0(L \otimes I_{Z(t)}) \), [vB2], [AN §3]. Theorem 0.2 thus implies \( K_{k-1,1}(X, L) \) is generated by syzygies of rank \( k + 1 \). Restricting to \( C \in |L| \), some of these syzygies drop rank by one, and the resulting syzygies continue to span \( K_{k-1,1}(C, \omega_C) \), see [V2] Prop. 7 and the unpublished [vBH] §11. Thus:

**Corollary 0.4 (Geometric Syzygy Conjecture in Even Genus).** Let \( C \) be a general curve of even genus \( g = 2k \). Then \( K_{k-1,1}(C, \omega_C) \) is generated by syzygies of the lowest possible rank \( k \).

Corollary 0.4 therefore provides an extension of Green’s theorem on quadrics [G2] to the space of linear syzygies of highest order.

It would appear to us to be very difficult to adapt degeneration methods to prove the structure Theorem 0.2 as opposed to merely establishing the vanishing from Voisin’s original result. For instance, the construction of Lazarsfeld–Mukai bundles fails on the (non-normal) tangent developable, so that it is not even clear how the bundle \( E \) degenerates to this surface.

There are no known conjectural candidates for an analogous result to Theorem 0.2 in odd genus \( g = 2k + 1 \). In this case, the dimension of \( K_{k-1,1}(X, L) \) is not given by a binomial coefficient, so this space cannot be of the form \( \text{Sym}^p(V) \) for any vector space \( V \). Furthermore, we no longer have uniqueness of the relevant Lazarsfeld-Mukai bundle in this situation.
The starting point for this paper is a universal version of the secant construction [EL2] §3. Our argument is largely formal, primarily using general results on vector bundles rather than a detailed study of the geometry of curves. As a result, we expect our approach to generalize well to the study of syzygies of higher dimensional varieties, for which previous approaches to Green’s conjecture do not seem applicable. See [EL1] for an application of vector bundle methods to syzygies of varieties of arbitrary dimension.

The main difficulty in extending our results to positive characteristic is that there no longer exist K3 surfaces of Picard rank one in this setting, [Hu] Ch. 17.

Acknowledgements I thank C. Voisin for helpful explanations and G. Farkas for numerous discussions. I thank R. Lazarsfeld for encouragement and for detailed comments. I thank D. Erman, D. Huybrechts and R. Yang for feedback on a draft. The author is supported by NSF grant DMS-1701245.

0.1. Preliminaries. We gather here a few facts. Let $0 \to F_1 \to F_2 \to F_3 \to 0$ be a short exact sequence of vector bundles over $\mathbb{C}$. From [V], for any $i$ we have exact sequences

$$
\ldots \to \bigwedge^i F_2 \otimes \text{Sym}^2(F_1) \to \bigwedge^{i-1} F_2 \otimes F_1 \to \bigwedge^i F_2 \to \bigwedge^i F_3 \to 0,
$$

$$
0 \to \text{Sym}^i(F_1) \to \text{Sym}^i(F_2) \to \text{Sym}^{i-1}(F_2) \otimes F_3 \to \text{Sym}^{i-2}(F_2) \otimes \bigwedge^2 F_3 \to \ldots
$$

We state two formulæ which we use without specific mention. Let $0 \to F_0 \to F \to G \to 0$ be a K3 surface of Picard rank one and even genus, $[Hu, \text{Ch. 17}]$. If $F$ is locally free. This follows from $[Ha, \text{III, Ex. 6.5}]$. We have an exact sequence $0 \to F \to G \to H \to 0$ of coherent sheaves on a quasi-projective variety, with $G$ locally free. Assume either $H$ is locally free or $H \simeq \mathcal{O}_D$ for a Cartier divisor $D$. Then $F$ is locally free. This follows from $[Ha, \text{III, Ex. 6.5}]$.

1. Voisin’s Theorem in Even Genus

Let $X$ be a K3 surface of Picard rank one and even genus $g = 2k$. Consider the unique rank two, Lazarsfeld–Mukai, bundle $E$ on $X$ as in the introduction. For general $s \in H^0(E)$, the zero-locus $Z(s)$ corresponds to a $g^k_{k+1}$ on a smooth $C \subset |L|$. For any $s \in H^0(E)$, $Z(s) \subseteq X$ is zero-dimensional and we have an exact sequence

$$
0 \to \mathcal{O}_X \xrightarrow{\delta} E \xrightarrow{\lambda_s} I_{Z(s)} \otimes L \to 0.
$$

Set $P := P(H^0(E)) \simeq P^{k+1}$. Consider $X \times P$ with projections $p : X \times P \to X$, $q : X \times P \to P$. Define $Z \subseteq X \times P$ as the locus $\{(x, s) \mid s(x) = 0\}$. Since $E$ is globally generated, $Z$ is a projective bundle over $X$ and hence smooth. Further $I_{Z(s)} \otimes L$ is globally generated, as it is a quotient of the globally generated bundle $E$. We have an exact sequence

$$
0 \to \mathcal{O}_X \boxtimes \mathcal{O}_P(-2) \xrightarrow{id} E \boxtimes \mathcal{O}_P(-1) \to \mathcal{O}^* L \otimes I_Z \to 0,
$$

where the first nonzero map is given by multiplication by

$$
id \in H^0(E \boxtimes \mathcal{O}_P(1)) \simeq H^0(E) \otimes H^0(E)^* \simeq \text{Hom}(H^0(E), H^0(E)).
$$

Note that $Z \to P$ is finite and flat, [Gro1] Prop. 6.1.5.
Applying R*

The sheaf

Lemma 1.1. Note H1 coincides with Pic(X) for X smooth. For this reason, it is essential that Pic(X) ∼= Z[L].

Let M := p*M_L, where M_L is the Kernel Bundle 0 → M_L → H0(L) ⊗ O_X → L → 0. By the well-known Kernel Bundle description of Koszul cohomology [EL2, §3], it suffices to show

H^1(X, \bigwedge^{k+1} M_L) = 0.

Note H^1(X × P, \bigwedge^{k+1} M) ∼= H^1(X, \bigwedge^{k+1} M_L) by the K"unneth formula, as H^1(O_X) = 0.

We now adapt [EL2, p. 615]. Let π : B → X × P be the blow-up along Z with exceptional divisor D. Then π_*O_B ∼= O_X×P, π_*D ∼= I_Z and R^iπ_*O_B = R^iπ_*D = 0 for i > 0, cf. [H3, V, Prop. 3.4 and Ex. 3.1]. Set p' := p ◦ π, q' := q ◦ π. We have canonical identifications

q'_*(p'^*L ⊗ I_D) ∼= q_*(p*L ⊗ I_Z), q'_*p'^*L ∼= q_*p*L.

Consider W := Coker (q'_*(p'^*L ⊗ I_D) → q'_*p'^*L) ∼= Coker (q_*(p*L ⊗ I_Z) → q_*p*L).

Lemma 1.1. The sheaf W is locally free of rank k.

Proof. Applying Rq_*, gives the exact sequence 0 → W → q_*(p*L|_Z) → R^1q_*(p*L ⊗ I_Z) → 0. For any s ∈ H^0(E), we have H^1(X, L ⊗ I_Z(s)) ∼= H^2(O_X). Thus q_*(p*L|_Z) and R^1q_*(p*L ⊗ I_Z) are locally free of ranks k+1 and 1, by Grauert’s Theorem [H3, III, §12]. The claim follows.

Since D is a divisor, we have a rank k vector bundle Γ := Ker (q'^*W → p'^*L|_D). As L ⊗ I_Z is globally generated for all s ∈ H^0(E), we have a natural surjection q'_*q_*(p'^*L ⊗ I_Z) → p'^*L ⊗ I_Z. Applying π* and noting that I_D is a quotient of π*I_Z, we have a surjection q'^*q'_*(p'^*L ⊗ I_D) → p'^*L ⊗ I_D. Let S be the vector bundle on B defined by the exact sequence

0 → S → q'^*q'_*(p'^*L ⊗ I_D) → p'^*L ⊗ I_D → 0.

We have an exact sequence 0 → S → p^*M → Γ → 0, giving the exact sequence

\ldots \rightarrow \bigwedge^{k-1} \pi^*M ⊗ Sym^2 S \rightarrow \bigwedge^k \pi^*M ⊗ S \rightarrow \bigwedge^{k+1} \pi^*M \rightarrow 0.

Remark. The Secant Sheaves defined in [EL2, §3] are only torsion-free in general. To apply [W], we need Γ to be locally free and hence we must pass to the blow-up B.

To prove Voisin’s Theorem it suffices to show

H^i(B, \bigwedge^{k+1-i} \pi^*M ⊗ Sym^i S) = 0 for 1 ≤ i ≤ k + 1.

One readily shows these vanishing for i < k + 1 (see Theorem [L6]). The crucial point is to show H^{k+1}(B, Sym^{k+1} S) = 0. To ease the notation, we set

G := q^*q_*(p*L ⊗ I_Z).

Lemma 1.2. We have H^{k+1}(X × P, Sym^{k+1} G) = 0 as well as a natural isomorphism

H^k(X × P, Sym^{k+1} G) ∼= Sym^k H^0(E).

Proof. The exact sequence 0 → q^*O(-2) → q^*E → q^*O(-1) → G → 0 gives the exact sequence

0 → Sym^k q^*E ⊗ q^*O(-k - 2) → Sym^{k+1} q^*E ⊗ q^*O(-k - 1) → Sym^{k+1} G → 0.
Since $q_* p^* E \simeq H^0(E) \otimes \mathcal{O}_P$ is trivial,
\[
H^k(\text{Sym}^{k+1} \mathcal{G}) \simeq H^{k+1}(\text{Sym}^k q_* p^* E \otimes q^* \mathcal{O}(-k - 2)) \simeq \text{Sym}^k H^0(E)
\]
The vanishing $H^{k+1}(\text{Sym}^{k+1} \mathcal{G}) = 0$ follows from
\[
H^{k+1}(\mathcal{O}_X \boxtimes \mathcal{O}_P(-k - 1)) = H^{k+2}(\mathcal{O}_X \boxtimes \mathcal{O}_P(-k - 2)) = 0,
\]
using $H^1(\mathcal{O}_X) = 0$.

The next lemma is a similar computation to the previous one.

**Lemma 1.3.** We have a natural isomorphism $H^k(\text{Sym}^k \mathcal{G} \otimes p^* L \otimes I_Z) \simeq \text{Sym}^k H^0(E)$.

**Proof.** We have the short exact sequence
\[
0 \to \text{Sym}^{k-1} q_* p^* E \otimes q^* \mathcal{O}(-k - 1) \otimes p^* L \otimes I_Z \to \text{Sym}^k q_* p^* E \otimes q^* \mathcal{O}(-k) \otimes p^* L \otimes I_Z
\]
\[
\to \text{Sym}^k \mathcal{G} \otimes p^* L \otimes I_Z \to 0,
\]
as well as the exact sequence $0 \to \mathcal{O}_X \boxtimes \mathcal{O}_P(-2) \to E \boxtimes \mathcal{O}_P(-1) \to p^* L \otimes I_Z \to 0$.

By the Künneth formula, $H^{k+2}(\mathcal{O}_X \boxtimes \mathcal{O}_P(-k - 3)) = 0$. We have
\[
H^{k+1}(\mathcal{O}_X \boxtimes \mathcal{O}_P(-k - 3)) = H^{k+1}(\mathcal{K}_P(-1)) \simeq H^0(\mathcal{O}_P(1)) \simeq H^0(E).
\]
Further, $H^{k+1}(E \boxtimes \mathcal{O}_P(-k - 2)) \simeq H^{k+1}(E \boxtimes \mathcal{K}_P) \simeq H^0(E)$. The map
\[
H^{k+1}(\mathcal{O}_X \boxtimes \mathcal{O}_P(-k - 3)) \to H^{k+1}(E \boxtimes \mathcal{O}_P(-k - 2))
\]
is identified with id : $H^0(E) \to H^0(E)$. Thus $H^{k+1}(L \boxtimes \mathcal{O}_P(-k - 1) \otimes I_Z) = 0$. We likewise have $H^k(L \boxtimes \mathcal{O}_P(-k - 1) \otimes I_Z) = 0$.

Using that $q_* p^* E \simeq H^0(E) \otimes \mathcal{O}_P$ is trivial, we have
\[
H^k(\text{Sym}^k \mathcal{G} \otimes p^* L \otimes I_Z) \simeq H^k(\text{Sym}^k q_* q^* p^* E \otimes q^* \mathcal{O}(-k) \otimes p^* L \otimes I_Z)
\]
\[
\simeq \text{Sym}^k H^0(E) \otimes H^k(L \boxtimes \mathcal{O}_P(-k) \otimes I_Z).
\]
To finish the proof, it suffices to show that the boundary map
\[
H^k(L \boxtimes \mathcal{O}_P(-k) \otimes I_Z) \to H^{k+1}(q^* \mathcal{K}_P)
\]
is an isomorphism, which follows from the fact that $H^i(E \boxtimes \mathcal{O}(-k - 1)) = 0$ for all $i$.

We now repeat the previous lemma, twisting instead by $\mathcal{G} := q^* q_* (p^* L \otimes I_Z)$.

**Lemma 1.4.** The evaluation morphism $\mathcal{G} \to p^* L \otimes I_Z$ induces an isomorphism
\[
H^k(\text{Sym}^k \mathcal{G} \otimes \mathcal{G}) \xrightarrow{\sim} H^k(\text{Sym}^k \mathcal{G} \otimes p^* L \otimes I_Z).
\]

**Proof.** We have the short exact sequences
\[
0 \to \text{Sym}^{k-1} q_* p^* E \otimes q^* \mathcal{O}(-k - 1) \otimes \mathcal{G} \to \text{Sym}^k q_* p^* E \otimes q^* \mathcal{O}(-k) \otimes \mathcal{G} \to \text{Sym}^k \mathcal{G} \otimes \mathcal{G} \to 0,
\]
and $0 \to q^* \mathcal{O}(-2) \to q^* q_* p^* E \otimes q^* \mathcal{O}(-1) \to \mathcal{G} \to 0$.

Using the second sequence, $H^{k+1}(\mathcal{G} \otimes q^* \mathcal{O}_P(-k - 1)) = H^k(\mathcal{G} \otimes q^* \mathcal{O}_P(-k - 1)) = 0$ and $H^k(\mathcal{G} \otimes q^* \mathcal{O}_P(-k)) \xrightarrow{\sim} H^{k+1}(q^* \mathcal{K}_P)$. Thus
\[
H^k(\text{Sym}^k \mathcal{G} \otimes \mathcal{G}) \simeq H^k(\text{Sym}^k q_* q^* p^* E \otimes q^* \mathcal{O}(-k) \otimes \mathcal{G}) \simeq \text{Sym}^k H^0(E)
\]
and the evaluation map gives an isomorphism $H^k(\text{Sym}^k \mathcal{G} \otimes \mathcal{G}) \xrightarrow{\sim} H^k(\text{Sym}^k \mathcal{G} \otimes p^* L \otimes I_Z)$.

As a corollary, we now deduce:

**Proposition 1.5.** The natural map gives an isomorphism
\[
H^k(\text{Sym}^{k+1} \mathcal{G}) \xrightarrow{\sim} H^k(\text{Sym}^k \mathcal{G} \otimes p^* L \otimes I_Z).
\]
Proof. By the previous lemmas, it suffices to show that the natural morphism
\[ H^k(\text{Sym}^{k+1} \mathcal{G}) \rightarrow H^k(\text{Sym}^k \mathcal{G} \otimes \mathcal{G}) \]
is injective. For a vector bundle \( \mathcal{F} \) the composition \( \text{Sym}^i \mathcal{F} \rightarrow \text{Sym}^{i-1} \mathcal{F} \otimes \mathcal{F} \rightarrow \text{Sym}^i \mathcal{F} \) of natural maps is just multiplication by \( i \). This completes the proof. \( \square \)

We now complete the proof that \( K_{k,1}(X, L) = 0 \).

**Theorem 1.6.** We have \( H^i(B, \wedge^{k+1-i} \pi^* \mathcal{M} \otimes \text{Sym}^i \mathcal{S}) = 0 \) for \( 1 \leq i \leq k+1 \).

**Proof.** Observe \( \pi^* \mathcal{G} \simeq q^* q'_* (p^* L \otimes I_D) \). From the defining sequence for \( \mathcal{S} \), we have an exact sequence
\[
0 \rightarrow \text{Sym}^i \mathcal{S} \rightarrow \text{Sym}^i \pi^* \mathcal{G} \rightarrow \text{Sym}^{i-1} \pi^* \mathcal{G} \otimes p^* L \otimes I_D \rightarrow 0.
\]
Using the projection formula, and recalling the identities \( \pi_* \mathcal{O}_B \simeq \mathcal{O}_{X \times \mathbf{P}} \), \( \pi_* I_D \simeq I_Z \) and \( R^j \pi_* \mathcal{O}_B = R^j \pi_* I_D = 0 \) for \( j > 0 \), we may identify
\[
H^f(B, \text{Sym}^i \pi^* \mathcal{G}) \rightarrow H^f(B, \text{Sym}^{i-1} \pi^* \mathcal{G} \otimes p^* L \otimes I_D)
\]
with the natural map \( H^f(X \times \mathbf{P}, \text{Sym}^i \mathcal{G}) \rightarrow H^f(X \times \mathbf{P}, \text{Sym}^{i-1} \mathcal{G} \otimes p^* L \otimes I_Z) \), for any \( \ell \). Taking the long exact sequence of cohomology for the sequence \( \square \) for \( i = k+1 \) and applying the previous lemmas, we immediately see \( H^{k+1}(B, \text{Sym}^{k+1} \mathcal{S}) = 0 \).

To complete the proof, it suffices to show
\[
H^i(\bigwedge^{k+1-i} \pi^* \mathcal{M} \otimes \text{Sym}^i \pi^* \mathcal{G}) = H^{i-1}(\bigwedge^{k+1-i} \pi^* \mathcal{M} \otimes \text{Sym}^{i-1} \pi^* \mathcal{G} \otimes p^* L \otimes I_D) = 0, \quad \text{for } 1 \leq i \leq k.
\]
The first vanishing follows from the exact sequence
\[
0 \rightarrow \text{Sym}^{i-1} q^* q_* p^* E \otimes q^* \mathcal{O}(-i-1) \rightarrow \text{Sym}^i q^* q_* p^* E \otimes q^* \mathcal{O}(-i) \rightarrow \text{Sym}^i \mathcal{G} \rightarrow 0,
\]
together with \( H^i(\bigwedge^{k+1-i} M_L \boxtimes \mathcal{O}_\mathbf{P}(-i)) = H^{i+1}(\bigwedge^{k+1-i} M_L \boxtimes \mathcal{O}_\mathbf{P}(-i-1)) = 0 \) for \( 1 \leq i \leq k \).

Next, from the exact sequence
\[
0 \rightarrow \text{Sym}^{i-2} q^* q_* p^* E \otimes q^* \mathcal{O}(-i) \otimes p^* L \otimes I_Z \rightarrow \text{Sym}^i q^* q_* p^* E \otimes q^* \mathcal{O}(-i+1) \otimes p^* L \otimes I_Z
\]
\[
\rightarrow \text{Sym}^{i-1} \mathcal{G} \otimes p^* L \otimes I_Z \rightarrow 0,
\]
it suffices to show \( H^{i-1}(\bigwedge^{k+1-i} M_L(L) \boxtimes \mathcal{O}(-i+1) \otimes I_Z) = H^i(\bigwedge^{k+1-i} M_L(L) \boxtimes \mathcal{O}(-i) \otimes I_Z) = 0 \) for \( 1 \leq i \leq k \). This follows from \( 0 \rightarrow \mathcal{O}_X \boxtimes \mathcal{O}_\mathbf{P}(-2) \rightarrow E \boxtimes \mathcal{O}_\mathbf{P}(-1) \rightarrow L \otimes I_Z \rightarrow 0 \), as \( H^0(X, M_L) = 0 \) if \( i = k \). \( \square \)

2. The Geometric Syzygy Conjecture

In this section, we use the techniques used in our proof of Voisin’s Theorem to resolve the Geometric Syzygy Conjecture for extremal syzygies of generic curves of even genus. We stick with the notation from Section \( \square \). As before, we consider the exact sequence \( 0 \rightarrow \mathcal{S} \rightarrow \pi^* \mathcal{M} \rightarrow \Gamma \rightarrow 0 \).

**Theorem 2.1.** The natural morphism \( H^1(B, \wedge^k \pi^* \mathcal{M} \otimes p^* L) \rightarrow H^1(B, \wedge^k \mathcal{G} \otimes p^* L) \) is surjective.

**Proof.** From the exact sequence
\[
\ldots \rightarrow \mathcal{S} \otimes \wedge^{k-1} \pi^* \mathcal{M} \otimes p^* L \rightarrow \wedge^k \mathcal{M} \otimes p^* L \rightarrow \wedge^k \Gamma \otimes p^* L \rightarrow 0,
\]
it suffices to show \( H^{1+i}(\text{Sym}^i \mathcal{S} \otimes \wedge^{k-1} \pi^* \mathcal{M} \otimes p^* L) = 0 \) for \( 1 \leq i \leq k \). From the exact sequence
\[
0 \rightarrow \text{Sym}^i \mathcal{S} \rightarrow \text{Sym}^i (q^* q'_* (p^* L \otimes I_D)) \rightarrow \text{Sym}^{i-1} (q^* q'_* (p^* L \otimes I_D)) \otimes p^* L \otimes I_D \rightarrow 0
\]
it suffices to have
\[ H^{1+i} \left( X \times P, \text{Sym}^i (q^* q_*(p^* L \otimes I_Z)) \otimes \wedge^{k-i} M \otimes p^* L \right) = 0, \]
\[ H^i \left( X \times P, \text{Sym}^{i-1} (q^* q_*(p^* L \otimes I_Z)) \otimes \wedge^{k-i} M \otimes p^* L \otimes 2 \otimes I_Z \right) = 0 \]
By taking \text{Sym}^i of the exact sequence
\[ 0 \to q^* O(-2) \to q^* q_* p^* E \otimes q^* O(-1) \to q^* q_*(p^* L \otimes I_Z) \to 0 \]
it suffices to show
\[ H^{2+i} \left( X \times P, \wedge^{k-i} M_L(L) \boxtimes \text{Sym}^{i-1} (q_* p^* E) (-i - 1) \right) = 0, \]
\[ H^{1+i} \left( X \times P, \wedge^{k-i} M_L(L) \boxtimes \text{Sym}^i (q_* p^* E) (-i) \right) = 0, \]
\[ H^{1+i} \left( X \times P, \left( \wedge^{k-i} M_L(2L) \boxtimes \text{Sym}^{i-2} (q_* p^* E) (-i) \right) \otimes I_Z \right) = 0, \]
\[ H^i \left( X \times P, \left( \wedge^{k-i} M_L(2L) \boxtimes \text{Sym}^{i-1} (q_* p^* E) (-i + 1) \right) \otimes I_Z \right) = 0. \]
Since \( q_* p^* E \simeq H^0(X, E) \otimes O_P \), the first two claims follow from the Küneth formula, for \( 1 \leq i \leq k \). For the last two claims, we use the short exact sequence
\[ 0 \to L^{-1} \boxtimes O(-2) \to E(L^{-1}) \boxtimes O(-1) \to I_Z \to 0, \]
so it suffices to have the four vanishings
\[ H^{2+i} \left( \wedge^{k-i} M_L(L) \boxtimes \text{Sym}^{i-2} (q_* p^* E) (-i - 2) \right) = 0, \]
\[ H^{1+i} \left( \wedge^{k-i} M_L \otimes E(L) \boxtimes \text{Sym}^{i-2} (q_* p^* E) (-i - 1) \right) = 0, \]
\[ H^{1+i} \left( \wedge^{k-i} M_L(L) \boxtimes \text{Sym}^{i-1} (q_* p^* E) (-i - 1) \right) = 0, \]
\[ H^i \left( \wedge^{k-i} M_L \otimes E(L) \boxtimes \text{Sym}^{i-1} (q_* p^* E) (-i) \right) = 0. \]
This follows from the Küneth formula, using \( H^1(X, L) = 0 \) if \( i = k \) in the first vanishing.

The Geometric Syzygy Conjecture in even genus now follows readily from Theorem 2.1.

**Lemma 2.2.** With notation as in Section 1, we have \( \wedge^k \Gamma \simeq I_D \otimes q^* O(k) \).

**Proof.** By taking determinants of the exact sequence \( 0 \to \Gamma \to q^* \mathcal{W} \to p^* L|_D \to 0 \), it suffices to show \( \det \mathcal{W} \simeq O_P(k) \). From \( 0 \to q_* (p^* L \otimes I_Z) \to q_* p^* L \to \mathcal{W} \to 0 \), we see \( \det \mathcal{W} = \det(q_* (p^* L \otimes I_Z))^{-1} \). We now deduce the claim from the exact sequence
\[ 0 \to O_P(-2) \to q_* p^* E \otimes O(-1) \to q_* (p^* L \otimes I_Z) \to 0. \]

**Corollary 2.3.** We have a natural isomorphism \( \text{Sym}^{k-2} H^0(X, E) \simeq K_{k-1,1}(X, L) \).

**Proof.** We have \( H^1(B, I_D \otimes q^* O_P(k)) \simeq H^1(X \times P, I_Z \otimes q^* O_P(k)) \). From
\[ 0 \to O \boxtimes O_P(-2) \to E \boxtimes O_P(-1) \to p^* L \otimes I_Z \to 0, \]
we obtain an isomorphism \( H^1(B, \wedge^k \Gamma \otimes p^* L) \simeq H^0(O_P(k - 2)) \simeq \text{Sym}^{k-2} H^0(X, E) \). Theorem 2.1 then gives a surjective map \( K_{k-1,2}(X, L) \to \text{Sym}^{k-2} H^0(X, E) \). Applying duality we have an injective map \( \text{Sym}^{k-2} H^0(X, E) \to K_{k-1,1}(X, L) \). But, Voisin’s Theorem in even genus, as proven in Section 1, implies \( \dim \text{Sym}^{k-2} H^0(X, E) = \dim K_{k-1,1}(X, L), \) [Em, §4.1], which completes the proof.
We end this section with a remark. From the sequence $0 \to \mathcal{S} \to \pi^* \mathcal{M} \to \Gamma \to 0$, the isomorphism $\psi : \mathcal{H}^1(\wedge^k \pi^* \mathcal{M} \otimes p^* L) \xrightarrow{\sim} \mathcal{H}^1(\wedge^k \Gamma \otimes p^* L)$ is Serre dual to

\[
\psi^* : \mathcal{H}^{k+2}(\wedge^k \mathcal{S} \otimes \omega_B) \to \mathcal{H}^{k+2}(\wedge^k \pi^* \mathcal{M} \otimes \omega_B).
\]

3. Voisin’s Theorem in Odd Genus

In this section we prove Green’s Conjecture for generic curves of odd genus $g = 2k + 1 \geq 9$, following a simplified version of the strategy of [V3]. Let $X$ be a K3 surface of Picard rank two, with $\text{Pic}(X)$ generated by a big and nef line bundle $L'$ with $(L')^2 = 2g$ together with a smooth rational curve $\Delta$ with $(L' \cdot \Delta) = 0$.

**Lemma 3.1.** Set $L_n := L' - n \Delta$ for $0 \leq n \leq 3$. Then $L_n$ is base-point free and is further ample for $0 < n \leq 2$. Furthermore, the linear system $|L_n|$ cannot be written as a sum of two pencils. In particular, smooth curves $C \in |L_n|$ are Brill–Noether general.

**Proof.** Since $(L_n)^2 \geq 0$ and $(L_n \cdot L') > 0$, we see $\dim |L_n| > 0$ by Riemann–Roch. We claim $L_n$ is nef, i.e. there is no rational, base component $R \sim aL' + b\Delta$ of $L_n$ with $(L_n \cdot R) < 0$. Otherwise, $R$ and $L_n - R$ would be effective and $a \neq 0$ so $a = 1$, but then $\dim |L_n| = \dim |L_n - R| = 0$. Since $(a \cdot \beta)$ is even for all $\alpha, \beta \in \text{Pic}(X)$, there are no divisors $E$ with $(L_n \cdot E) = 1$, hence the nef bundles $L_n$ are base-point free, [M] Prop. 8]. If $L_n$ is not ample for $n \neq 0$, there is a rational curve $R \sim aL' + b\Delta$ with $(L_n \cdot R) = 0$. Then $b = -\frac{aa}{n}$ with $a > 0$, so $(R)^2 = -2a^2g(\frac{2}{n^2} - 1) < -2$.

For the last claim it suffices to show that we cannot write $L_n = A_1 + A_2$ for divisors $A_i$ with $h^0(A_i) \geq 2$, $i = 1, 2$. Writing $A_1 = aL' + b_i\Delta$, we must have $a_j = 0$ for some $j \in \{1, 2\}$. But then $h^0(A_j) = 1$ which is a contradiction. \(\square\)

Setting $L := L_1 = L' - \Delta$, a general $C \in |L|$ is a curve of genus $g = 2k + 1$. To verify Green’s conjecture for $C$, it suffices to show $\mathcal{H}^1(\wedge^{k+1} M_L) = K_{k,1}(X, L) = 0$. We have an exact sequence $0 \to M_L \to M_{L'} \to \mathcal{O}(\Delta) \to 0$ of vector bundles on $X$. This gives an exact sequence

\[
0 \to \wedge^{k+1} M_L \to \wedge^{k+1} M_{L'} \to \wedge^k M_{L'}(-\Delta) \to 0.
\]

The induced map $pr_k : \mathcal{H}^1(\wedge^{k+1} M_{L'}) \to \mathcal{H}^1(\wedge^k M_{L'}(-\Delta))$ is called the projection map.

**Lemma 3.2.** Suppose the projection map $pr_k$ is injective. Then $\mathcal{H}^1(\wedge^{k+1} M_L) = 0$.

**Proof.** Indeed $\mathcal{H}^0(\wedge^k M_{L'}(-\Delta)) \subseteq \mathcal{H}^0(\wedge^k M_L) = 0$. \(\square\)

We have a rational resolution of singularities $\mu : X \to \hat{X}$, contracting $\Delta$ to a du Val singularity $p$ on a nodal K3 surface $\hat{X}$. Then $\hat{X}$ admits a line bundle $\hat{L}$ with $\mu^* \hat{L} = L'$.

We have $\text{Pic}(\hat{X}) = \text{Cl}(\hat{X}) \simeq \text{Cl}(X \setminus \Delta)$, so $\text{Cl}(\hat{X}) = \mathbb{Z}[\hat{L}]$ and $\hat{X}$ is factorial. Consider the rank two Lazarsfeld–Mukai bundle $\hat{E}$ on $\hat{X}$ induced by a $g^1_{k+2}$ on a general $C \in |\hat{L}|$. Set $E := \mu^* \hat{E}$, which is a rank two bundle on $X$ induced by a $g^1_{k+2}$ on a general $C \in |L'|$.

**Lemma 3.3.** We have $K_{k+1,1}(X, L') = 0$. Further, there is a natural isomorphism

\[
\text{Sym}^{k-1} \mathcal{H}^0(X, E) \simeq K_{k,1}(X, L').
\]

**Proof.** Set $P = P(\mathcal{H}^0(\hat{X}, \hat{E}))$ and let $\hat{Z} \subseteq \hat{X} \times P$ be the locus defined by $\{(x, s) \mid s(x) = 0\}$. Then $\hat{Z}$ is a projective bundle over $\hat{X}$, is a local complete intersection in $\hat{X} \times P$ and is finite and flat over $P$. As in Sections [1] and [2] we see $K_{k+1,1}(\hat{X}, \hat{L}) = 0$, and $\text{Sym}^{k-1} \mathcal{H}^0(\hat{E}) \simeq K_{k,1}(\hat{X}, \hat{L})$. Since $\hat{X}$ has rational singularities, $\mathcal{H}^0(X, nL') \simeq \mathcal{H}^0(X, n\hat{L})$ for all $n$, and the claim follows. \(\square\)

We will prove that we have a natural injection $\text{Sym}^{k-1} \mathcal{H}^0(X, E) \hookrightarrow \mathcal{H}^1(\wedge^k M_{L'}(-\Delta))$. By the above lemma, this implies injectivity of $pr_k$, so that Voisin’s Theorem follows.

**Lemma 3.4.** The natural map $d : \wedge^2 \mathcal{H}^0(E) \to \mathcal{H}^0(\wedge^2 E) = \mathcal{H}^0(L')$ does not vanish on decomposable elements.
Proof. Suppose \( v_1, v_2 \in H^0(E) \) with \( v_1 \wedge v_2 \neq 0, d(v_1 \wedge v_2) = 0 \). Then \( v_1, v_2 \) generate a rank one subbundle \( H_1 \subseteq E \) with at least two sections. We have already seen that this cannot occur. Let \( H_2 := E/H_1 \) and \( M_2 := H_2/T(H_2) \), where \( T(H_2) \) denotes the torsion subbundle. We have a short exact sequence \( 0 \to M_1 \to E \to M_2 \to 0 \), with \( M_1, M_2 \) rank-one, torsion-free sheaves and \( h^0(M_1) \geq 2 \). Then \( M_i = N_i \otimes I_i \), for \( N_i \) a line bundle and \( I_i \) an ideal sheaf of a 0-dimensional scheme for \( i = 1, 2 \). Since \( E \) is globally generated with \( h^2(E) = 0 \), we have \( h^0(N_2) \geq 2 \). But then \( \det(E) = L' = N_1 \otimes N_2 \) is a sum of line bundles with at least two sections. We have already seen that this cannot occur. □

To set things up, we need a lemma.

Lemma 3.5. The bundle \( E(\Delta) \) is isomorphic to the Lazarsfeld–Mukai bundle \( F \) corresponding to a \( g_k^1 \) on a general \( C' \in |L - \Delta| \). In particular, \( E(\Delta) \) is globally generated.

Proof. We claim that \( F \) is \( \mu \)-stable with respect to \( L - \Delta \). Otherwise, we have a filtration

\[
0 \to M \to F \to N \otimes I_\zeta \to 0
\]

where \( M, N \) are line bundles, \( I_\zeta \) is the ideal sheaf of a 0-dimensional scheme of length \( k - (M \cdot N) \), where \( h^0(N) \geq 2 \) and with \( \mu(M) \geq \mu(F) = g - 4 \geq \mu(N) \), cf. [LC]. We have \( h^2(M) = 0 \) since \( \mu(M) > 0 \) and further \( (M)^2 = \mu(M) - (M \cdot N) \geq (g - 4) - k \geq 0 \). Thus \( h^0(M) \geq 2 \) by Riemann–Roch, contradicting that \( L - \Delta = \det(F) \) cannot be written as a sum of two pencils.

The rank two bundle \( E(\Delta) \) is simple with \( \det E(\Delta) = L - \Delta \) and \( \chi(E(\Delta)) = k + 1 \). We claim that \( E(\Delta) \) is stable with respect to \( L - \Delta \). Since \( F \) is the unique stable bundle with these invariants, \( E(\Delta) \simeq F \). If \( E(\Delta) \) is not stable, choose a filtration \( 0 \to M' \to E(\Delta) \to N' \otimes I_\zeta' \to 0 \) as above. We again have \( h^0(M') \geq 2 \), and since \( N'(\Delta) \otimes I_\zeta' \) is a quotient of the globally generated bundle \( E \) and \( h^2(E) \neq 0 \), \( h^0(N'(\Delta)) \geq 2 \). This contradicts that \( L \) cannot be written as a sum of two pencils. □

Let \( N \) denote the rank \( k - 1 \) kernel bundle fitting into the sequence

\[
0 \to N \to H^0(E(\Delta)) \otimes \mathcal{O}_X \to E(\Delta) \to 0.
\]

For any \( t \in H^0(E) \), we have a map \( \wedge t : H^0(E(\Delta)) \to H^0(\wedge^2 E(\Delta)) = H^0(L) \), inducing a map \( N \to M_L \). This globalizes to a vector bundle morphism

\[
r : N \boxtimes \mathcal{O}_P(1) \to \mathcal{M},
\]

on \( X \times P \), where \( P := P(H^0(X, E)) \simeq P^{k+2} \), where \( p : X \times P \to X \), \( q : X \times P \to P \) are the projections and \( \mathcal{M} := p^*M_L \).

From Lemma 3.4, \( r \) fails to be injective on fibres precisely at points in the locus

\[
Z := \{ (x, s) \in X \times P \mid H^0(E(\Delta)) \}.
\]

Thus \( \text{Coker}(r) \) fails to be locally free along \( Z \subseteq X \times \Lambda \), where \( \Lambda := P(H^0(E(\Delta))) \subseteq P \).

We rectify the failure of \( \text{Coker}(r) \) to be locally free through a standard construction. Define \( \pi : B \to X \times P \) as the blow-up along the codimension four locus \( Z \) and let \( p' := p \circ \pi \), \( q' := q \circ \pi \). Let \( j : D \to B \) be the exceptional divisor. For any vector bundle \( A \) on \( X \times P \), \( H^j(B, \pi^*A(nD)) \simeq H^j(B, \pi^*A) \) for any \( j \) and for \( 1 \leq n \leq 3 \).

For \( p = (x, t) \in Z \), the kernel of \( r \otimes k(p) \) is isomorphic to \( \mathbb{C}(t) \subseteq H^0(X, E) \). Thus

\[
\text{Ker}(r|_Z) \simeq (q^*\mathcal{O}_P(-2))|_Z,
\]

where the inclusion \( (q^*\mathcal{O}_P(-2))|_Z \to (N \boxtimes \mathcal{O}_P(-1))|_Z \) is given by the section \( u \in H^0(Z, N \boxtimes \mathcal{O}_P(1)) \subseteq H^0(E) \otimes H^0(q^*\mathcal{O}_P(1)) \) obtained by restricting \( \text{id} \in H^0(E) \otimes H^0(q^*\mathcal{O}_P(1)) \) to \( Z \).
We now perform an elementary transformation on \(\mathcal{N} \boxtimes \mathcal{O}_P(-1)\). Define \(S\) as the dual bundle to \(\text{Im}(\pi^*r^\vee)\). Then \(S^\vee\) is a vector bundle of rank \(k-1\) fitting into the exact sequence

\[
0 \to S^\vee \to \pi^*(\mathcal{N} \boxtimes \mathcal{O}_P(1)) \to q^*\mathcal{O}_P(2)|_D \to 0.
\]

From the definition of \(S\), we have an exact sequence

\[
0 \to S \to \pi^*\mathcal{M} \to \Gamma \to 0,
\]

where \(\Gamma\) is locally free of rank \(k + 2\).

**Lemma 3.6.** We have a natural isomorphism \(H^2(B, \wedge^{k+2} \Gamma(p^*\Delta)(D)) \simeq \text{Sym}^{k-1}H^0(X, E)^\vee\).

**Proof.** We have \(\det\Gamma \simeq p^*(L^\vee) \otimes \det S^\vee\) so that \(\wedge^{k+2}(p^*\Delta) \simeq q^*\mathcal{O}_P(k-1)(-D)\). Thus \(H^2(B, \wedge^{k+2}(p^*(\Delta))(D)) \simeq H^2(q^*\mathcal{O}_P(k-1)) \simeq H^2(B, q^*\mathcal{O}_P(k-1)) \simeq \text{Sym}^{k-1}H^0(X, E)^\vee\). □

We have natural isomorphisms

\[
\text{K}_{k-1,1}(X, -\Delta, L) \simeq H^1(\wedge^k M_L(-\Delta)) \simeq H^0(\wedge^{k-1} M_L(L - \Delta)),
\]

\([\text{ELZ}]\; \S 3\]. We have \(H^0(\wedge^{k-1} M_L(L - \Delta))^\vee \simeq H^2(\wedge^{k+2} M_L(\Delta))\), since \(\wedge^{k-1} M_L^\vee \simeq \wedge^{k+2} M_L(L)\).

Taking exterior powers of 0 \(\to S \to \pi^*\mathcal{M} \to \Gamma \to 0\) and twisting by \(\mathcal{O}_B(D + p^*\Delta)\) induces

\[
\phi : H^2(\wedge^{k+2} M_L(D)) \to H^2(\wedge^{k+2}(p^*\Delta)(D)) \simeq \text{Sym}^{k-1}H^0(E)^\vee.
\]

To show \(\phi\) is surjective, it suffices to prove

\[
H^2+i(\text{sym}^{-i}(\mathcal{N})(p^*\Delta)(D)) = 0, \quad \text{for } 1 \leq i \leq k + 2.
\]

The blow-up of a projective space \(\mathcal{P}(V)\) along a subspace \(W \subseteq V\) is a projective bundle over \(\mathcal{P}(V/W)\), \([\text{ELZ}]\; \S 9.3.2\]. Thus \(B\) is a projective bundle \(\mathcal{P}(\mathcal{H})\) over \(\mathcal{P}(\mathcal{F})\), where \(\mathcal{F} := \text{Coker}(N \to H^0(E) \otimes \mathcal{O}_X)\). The projection morphism \(\chi : B \to \mathcal{P}(\mathcal{F})\) is defined over \(X\) with

\[
\chi^*\mathcal{O}_{\mathcal{P}(\mathcal{F})}(1) \simeq q^*\mathcal{O}_P(1)(-D).
\]

To describe \(\mathcal{H}\), let \(f : \mathcal{P}(\mathcal{F}) \to X\) be the projection and define \(\mathcal{P}\) by the exact sequence

\[
0 \to \mathcal{P}^\vee \to f^*\mathcal{O}_{\mathcal{P}(\mathcal{F})}(1) \to \mathcal{O}_{\mathcal{P}(\mathcal{F})}(1) \to 0.
\]

We have a surjection \(H^0(E) \otimes \mathcal{O}_{\mathcal{P}(\mathcal{F})} \to \mathcal{P}\) to 0.

The rank \(k\) bundle \(\mathcal{H}\) is defined by the exact sequence

\[
0 \to \mathcal{H} \to H^0(E) \otimes \mathcal{O}_{\mathcal{P}(\mathcal{F})} \to \mathcal{P} \to 0.
\]

We have a short exact sequence

\[
0 \to f^*N \to \mathcal{H} \to \mathcal{O}_{\mathcal{P}(\mathcal{F})}(-1) \to 0.
\]

The isomorphism \(B \simeq \mathcal{P}(\mathcal{H})\) gives an identification \(\mathcal{O}_{\mathcal{P}(\mathcal{H})}(1) \simeq q^*\mathcal{O}_P(1)\).

**Lemma 3.7.** We have \(S \simeq T_\chi \otimes q^*\mathcal{O}_P(-2)\), where \(T_\chi\) is the relative tangent bundle.

**Proof.** We have the relative Euler sequence

\[
0 \to \mathcal{O}_{\mathcal{P}(\mathcal{H})} \to \mathcal{O}_{\mathcal{P}(\mathcal{H})}(1) \to \chi^*\mathcal{H} \to \mathcal{T}_\chi \to 0.
\]

Twisting by \(\mathcal{O}_{\mathcal{P}(\mathcal{H})}(-2)\), we have the composite map \(\pi^*(\mathcal{N} \boxtimes \mathcal{O}_P(-1)) \to \chi^*\mathcal{H} \otimes q^*\mathcal{O}_P(-1) \to \mathcal{T}_\chi(-2)\). Dualizing, we obtain an exact sequence

\[
0 \to \Omega_\chi \otimes q^*\mathcal{O}_P(2) \to \pi^*(\mathcal{N} \boxtimes \mathcal{O}_P(1)) \to q^*\mathcal{O}_P(2)|_D \to 0,
\]

and comparison with the defining sequence for \(S^\vee\) gives \(S^\vee \simeq \Omega_\chi \otimes q^*\mathcal{O}_P(2)\). □

With these preparations, all but one of the required vanishings are immediate.
Proposition 3.8. We have $H^{2+i}(\bigwedge^{k+2-i} \pi^* M \otimes \text{Sym}^i(S)(p^* \Delta)(D)) = 0$ for $1 \leq i \leq k + 2$, $i \neq k + 1$.

Proof. Our proof is analogous to [V3 Lemma 6]. The Euler sequence $0 \to q^* \mathcal{O}_P(-2) \to \chi^* \mathcal{H} \otimes q^* \mathcal{O}_P(-1) \to S \to 0$ gives exact sequences
$$0 \to \chi^* \text{Sym}^{i-1} \mathcal{H}(-i - 1) \to \chi^* \text{Sym}^i \mathcal{H}(-i) \to \text{Sym}^i(S) \to 0,$$
where we have simplified notation by writing $(j)$ for twists by $q^* \mathcal{O}_P(j)$. We first claim
$$H^{2+i}(\bigwedge^{k+2-i} \pi^* M \otimes \chi^* \text{Sym}^i \mathcal{H}(-i)(p^* \Delta)(D)) = 0, \text{ for } 1 \leq i \leq k.$$

The fibres of $\chi$ are projective spaces of dimension $k - 1$ and $D$ has degree one on these fibres. Hence $R^j \chi_* q^* \mathcal{O}_P(-i)(D) = 0$ for all $j$ and $2 \leq i \leq k$.

For $i = 1$, the claim states $H^2(\bigwedge^{k+1} \pi^* M \otimes \chi^* \mathcal{H}(-1)(p^* \Delta)(D)) = 0$. We have $\omega_B \simeq q^* \omega_P(3D)$, so that this is equivalent to
$$H^{3+k}(\bigwedge^{2} \pi^* M \otimes \chi^* \text{Sym}^{k-1} \mathcal{H}(-k - 1)(p^* \Delta)(D)) = 0,$$
which is Serre dual to
$$H^1(\bigwedge^{2} \pi^* M \otimes \chi^* \text{Sym}^{k-1} \mathcal{H}(-2)(2D - p^* \Delta))$$
$$= H^1(\mathcal{P}(\mathcal{F}), f^*(\bigwedge^{2} M_L^\vee(-\Delta)) \otimes \text{Sym}^{k-1} \mathcal{H}^\vee \otimes \mathcal{O}(\mathcal{F})(-2)).$$

This space is isomorphic to $H^1(B, \bigwedge^{2} \pi^* M \otimes \chi^{k-3}(2D - p^* \Delta)) = 0$, as $\chi_* q^* \mathcal{O}_P(n) \simeq \text{Sym}^n \mathcal{H}^\vee$.

We are left with the case $i = k + 2$. The inclusion
$$\pi^*(\text{Sym}^{k+2} N(\Delta) \boxtimes \mathcal{O}_P(-(k + 2))) \rightarrow \text{Sym}^{k+2}(S)(p^* \Delta)$$
is an isomorphism off $D$. Since $\dim D = k + 3$, it suffices to show
$$H^{k+4}(X \times P^{k+2}, \text{Sym}^{k+2} N(\Delta) \boxtimes \mathcal{O}_P(-(k + 2))) = 0.$$

This follows from the K"unneth formula. \hfill \Box

It remains to deal with the case $i = k + 1$. The following lemma, stated in [V2, Prop. 6], was explained to us by C. Voisin.

Lemma 3.9. The multiplication map $H^0(X, E(-\Delta)) \otimes H^0(X, L - \Delta) \rightarrow H^0(X, E(L - 2\Delta))$ is surjective.

Proof. Let $C \in |L - \Delta|$ be a smooth curve. It suffices to prove surjectivity of the restricted multiplication map $H^0(E|_C(-\Delta)) \otimes H^0(K_C) \rightarrow H^0(E|_C(K_C - \Delta))$, [GP Observation 2.3]. Now $H^0(E|_C(-\Delta)) \simeq H^0(B) \oplus H^0(K_C - B)$ where $B$ is a $g^1_k$ on $C$, [V2]. The statement now follows from the following: for any base-point free line bundle $M$ on $C$ with $h^0(M) \geq 2$, the multiplication map $H^0(K_C) \otimes H^0(M) \rightarrow H^0(K_C + M)$ is surjective. Indeed, if $h^0(M) = 2$, this follows.
immediately from the base-point free pencil trick. Otherwise, let $Z$ be a general effective divisor of degree $h^0(M) - 2$. Thus $H^0(K_C) \otimes H^0(M - Z) \to H^0(K_C + M - Z)$ is surjective. Since this holds for any general such divisor, this proves the claim. \hfill \Box

We will make use of the following direct consequence of the previous lemma.

Lemma 3.10. We have $H^2(X, M_L(\Delta) \otimes N) = H^2(M_L(\Delta)) = 0$.

Proof. From the exact sequence

$$0 \to N \to H^0(E(-\Delta)) \otimes O_X \to E(-\Delta) \to 0,$$

it suffices to show $H^2(M_L(\Delta)) = 0$ and $H^1(M_L(E)) = 0$. The first vanishing follows immediately from the defining sequence for $M_L$.

The vanishing $H^1(M_L(E)) = 0$ is equivalent to surjectivity of the multiplication map $H^0(L) \otimes H^0(E) \to H^0(E(L))$. We have surjectivity of $H^0(L - \Delta) \otimes H^0(E(-\Delta)) \to H^0(E(L - 2\Delta))$ by Lemma 3.9. For a general $s \in H^0(E)$, we have the exact sequence $0 \to O_X \to E \to E \otimes I_{Z(s)} \to 0$. This implies $E|_{\Delta} \simeq \mathcal{O}^2$ is trivial. Thus $H^0(L - \Delta) \otimes H^0(E|_{\Delta}) \to H^0(E|_{\Delta}(L - \Delta))$. This shows $H^0(L - \Delta) \otimes H^0(E) \to H^0(E(L - \Delta))$. Since $H^0(L|_{\Delta} \otimes H^0(E) \to H^0(E|_{\Delta}(L))$, using the triviality of $E|_{\Delta}$, we obtain the vanishing $H^1(M_L(E)) = 0$. \hfill \Box

We now prove the remaining vanishing.

Proposition 3.11. We have $H^{k+3}(B, \pi^*M \otimes \text{Sym}^{k+1}(S)(p^*\Delta)(D)) = 0$. In particular we have a surjection $\phi : H^2(\wedge^{k+2}M_L(\Delta)) \to \text{Sym}^{k-1}H^0(E)^\vee$.

Proof. We write $(j)$ for twists by $q^*\mathcal{O}_P(j)$. We have the short exact sequence

$$0 \to \chi^*\text{Sym}^k\mathcal{H}(-2 - k) \to \chi^*\text{Sym}^{k+1}\mathcal{H}(-k - 1) \to \text{Sym}^{k+1}(S) \to 0.$$ 

We firstly claim that

$$H^{k+3}(\pi^*M \otimes \chi^*\text{Sym}^k\mathcal{H}(-2 - k)(p^*\Delta)(D)) \to H^{k+3}(\pi^*M \otimes \chi^*\text{Sym}^{k+1}\mathcal{H}(-k - 1)(p^*\Delta)(D))$$

is surjective. We have $O(D) \simeq O(1) \otimes \chi^*\mathcal{O}_P(F)(-1)$ and further $\omega_\chi \simeq \chi^*\det \mathcal{H}(-k)$. The map can thus be written as

$$H^4(\mathcal{P}, f^*M_L(\Delta) \otimes \text{Sym}^k\mathcal{H} \otimes R^{k-1}\chi^*O(-(k + 1))) \otimes \mathcal{O}_P(F)(-1)) \to H^4(\mathcal{P}, f^*M_L(\Delta) \otimes \text{Sym}^{k+1}\mathcal{H} \otimes R^{k-1}\chi^*O(-k)) \otimes \mathcal{O}_P(F)(-1)).$$

Using relative duality, this becomes

$$H^4(\mathcal{P}, f^*M_L(\Delta) \otimes \text{Sym}^k\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{O}_P(F)(-1)) \otimes \det \mathcal{H}) \to H^4(\mathcal{P}, f^*M_L(\Delta) \otimes \text{Sym}^{k+1}\mathcal{H} \otimes \mathcal{O}_P(F)(-1)) \otimes \det \mathcal{H}),$$

since $\chi^*O(n) \simeq \text{Sym}^n\mathcal{H}^\vee$. This map is surjective, since the composite $\text{Sym}^{k+1}\mathcal{H} \to \text{Sym}^k\mathcal{H} \otimes \mathcal{H} \to \text{Sym}^{k+1}\mathcal{H}$ of natural maps is multiplication by $k + 1$.

To conclude, it suffices that $H^{k+4}(\pi^*M \otimes \chi^*\text{Sym}^k\mathcal{H}(-2 - k - 2)(p^*\Delta)(D)) = 0$, or, equivalently $H^5(\mathcal{P}(F), f^*M_L(\Delta) \otimes \text{Sym}^k\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{O}_P(F)(-1)) \otimes \det \mathcal{H}) = 0$.

The same argument identifies this space with $H^{k+4}(\pi^*M \otimes \chi^*\mathcal{H}(-2k - 1 + p^*\Delta)(D))$, using $\chi^*O(n) \simeq \text{Sym}^n\mathcal{H}^\vee$ again. The required vanishing then follows from the exact sequence

$$0 \to p^*N \to \chi^*\mathcal{H} \to q^*\mathcal{O}_P(-1)(D) \to 0,$$

plus Lemma 3.10. \hfill \Box

Remark. Note that we do not need the geometry of Grassmannians, unlike [V23] §3, Fourth step.
Let $\psi^\vee : \text{Sym}^{k-1} \mathbb{H}^0(E) \xrightarrow{\sim} K_{k,1}(X, L + \Delta)$ and $\phi^\vee : \text{Sym}^{k-1} \mathbb{H}^0(E) \hookrightarrow K_{k-1,1}(X, -\Delta, L)$ be the duals to the maps from Lemma 3.3 and Proposition 3.11. By Lemma 3.2 to complete the proof of Voisin’s Theorem, it only remains to show that $\phi^\vee = pr_k \circ \psi^\vee$.

By duality and from the sequence $0 \to S \to \pi^* M \to \Gamma \to 0$, we identify $\phi^\vee$ with

$$
H^{k+2}(B, \wedge^{k-1} S \otimes \pi^* (L - \Delta \boxtimes \omega_B)(2D)) \to H^{k+2}(B, \pi^* \wedge^{k-1} M_L(L - \Delta) \boxtimes \omega_B)(2D)),
$$

which can, in turn, be identified with the natural map

$$
\phi^\vee : H^{k+2}(X, P, \wedge^{k-1} N(L - \Delta) \boxtimes \omega_B(1 - k)) \to H^{k+2}(X, P, \wedge^{k-1} M_L(L - \Delta) \boxtimes \omega_B).
$$

**Theorem 3.12.** With notation as above, we have $K_{k,1}(X, L) = 0$.

**Proof.** We need to show $\phi^\vee = pr_k \circ \psi^\vee$. To relate $\psi^\vee$ and $\phi^\vee$, let $\tilde{\pi} : \tilde{B} \to X \times P$ denote the blow-up in the codimension two locus $\tilde{\mathcal{Z}} := \{(x, s) \mid s(x) = 0\}$. Let $\tilde{\omega}$ denote the exceptional divisor. Define $\tilde{p} := p \circ \tilde{\pi}$ and $\tilde{q} := q \circ \tilde{\pi}$. We have the exact sequence

$$
0 \to \tilde{\pi}^*(\mathcal{O}_X(-\Delta) \boxtimes \mathcal{O}_P(-2))(\tilde{D}) \to \tilde{\pi}^*(\mathcal{E}(-\Delta) \boxtimes \mathcal{O}_P(-1)) \to \tilde{p}^* L \otimes I_{\tilde{D}} \to 0.
$$

We have $\tilde{q}_* (\tilde{p}^* L \otimes I_{\tilde{D}}) \simeq H^0(\mathbb{E}(-\Delta)) \boxtimes \mathcal{O}_P(-1)$. Define $\tilde{S}_L$ and $\tilde{S}_L$ by exact sequences

$$
0 \to \tilde{S}_L \to \tilde{q}^* \tilde{q}_* (\tilde{p}^* L' \otimes I_{\tilde{D}}) \to \tilde{p}^* L' \otimes I_{\tilde{D}} \to 0,
$$

$$
0 \to \tilde{S}_L \to H^0(\mathbb{E}(-\Delta)) \boxtimes \tilde{q}'_* \mathcal{O}_P(-1) \to \tilde{p}^* L \otimes I_{\tilde{D}} \to 0.
$$

Then $\psi^\vee$ is the natural map

$$
\psi^\vee : H^{k+3}(\wedge^{k+1} \tilde{S}_L' \otimes \omega_{\tilde{B}}) \to H^{k+3}(\tilde{\pi}^*(\wedge^{k+1} M_L \otimes \mathcal{O}_P) \otimes \omega_{\tilde{B}}).
$$

By taking exterior powers of the exact sequence

$$
0 \to \tilde{\pi}^*(N \boxtimes \mathcal{O}_P(-1)) \to \tilde{S}_L \to \tilde{\pi}^*(\mathcal{O}_X(-\Delta) \boxtimes \mathcal{O}_P(-2))(\tilde{D}) \to 0
$$

and using $H^0(X, \wedge^{k-1} N(L - 2\Delta)) = H^0(N^\vee(-\Delta)) = H^2(N(\Delta)) = 0$, identify $\phi^\vee$ with

$$
H^{k+2}(\wedge^{k+1} \tilde{S}_L(L - \Delta) \otimes \tilde{q}'_* \omega_P) \to H^{k+2}(\tilde{\pi}^*(\wedge^{k-1} M_L(L - \Delta) \boxtimes \omega_B)),
$$

induced by $\tilde{S}_L \hookrightarrow \tilde{p}^* M$. Using exterior powers of the defining sequence for $\tilde{S}_L$, this can be further identified with

$$
\phi^\vee : H^{k+3}(\wedge^{k+1} \tilde{S}_L(\wedge \Delta) \otimes \omega_{\tilde{B}}) \to H^{k+3}(\tilde{\pi}^*(\wedge^{k} M_L(-\Delta) \boxtimes \mathcal{O}_P) \otimes \omega_{\tilde{B}}),
$$

using $H^0(\mathcal{O}_X(-\Delta)) = H^1(\mathcal{O}_X(-\Delta)) = 0$.

Let $U \subseteq \tilde{B}$ be the complement of the codimension two locus $\tilde{\pi}^{-1}(X \times P)(H^0(\mathbb{E}(-\Delta)))$. We have an exact sequence $0 \to \tilde{S}_L|_U \to \tilde{S}_L'|_U \to \mathcal{O}_U(-\tilde{p}^* \Delta) \to 0$, giving a commutative diagram

$$
\begin{array}{ccc}
\wedge^{k+1} \tilde{S}_L'|_U & \xrightarrow{\phi^\vee} & \tilde{\pi}^*(\wedge^{k+1} M_L \otimes \mathcal{O}_P)|_U \\
\downarrow \cong & & \downarrow q_U \\
\wedge^k \tilde{S}_L|_U(-\tilde{p}^* \Delta) & \xrightarrow{\phi^\vee} & \tilde{\pi}^*(\wedge^{k} M_L(-\Delta) \boxtimes \mathcal{O}_P)|_U
\end{array}
$$

where $q : \tilde{\pi}^*(\wedge^{k+1} M_L \otimes \mathcal{O}_P) \to \tilde{\pi}^*(\wedge^{k} M_L(-\Delta) \boxtimes \mathcal{O}_P)$ is the projection. This diagram extends uniquely to $\tilde{B}$, giving the claim. \qed
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