Isomonodromic deformation of Lamé connections, Painlevé VI equation and Okamoto symmetry

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Abstract. A Lamé connection is a logarithmic sl(2, Ĉ)-connection \((E, \nabla)\) over an elliptic curve \(X: \{ y^2 = x(x-1)(x-t) \}, \ t \neq 0, 1, \) having a single pole at infinity. When this connection is irreducible, we show that it is invariant under the standard involution and can be pushed down to a logarithmic sl(2, Ĉ)-connection on \(\mathbb{P}^1\) with poles at 0, 1, \(t\), and \(\infty\). Therefore the isomonodromic deformation \((E_t, \nabla_t)\) of an irreducible Lamé connection, when the elliptic curve \(X_t\) varies in the Legendre family, is parametrized by a solution \(q(t)\) of the Painlevé VI differential equation \(P_{VI}\). The variation of the underlying vector bundle \(E_t\) along the deformation is computed in terms of the Tu moduli map: it is given by another solution \(\tilde{q}(t)\) of \(P_{VI}\), which is related to \(q(t)\) by the Okamoto symmetry \(s_2 s_1 s_2\) (Noumi–Yamada notation). Motivated by the Riemann–Hilbert problem for the classical Lamé equation, we raise the question whether the Painlevé transcendents do have poles. Some of these results were announced in [6].

Keywords: complex ordinary differential equations, isomonodromic deformations, Lamé differential equation, Painlevé equation.

In memory of Andrei Bolibrukh

Motivations

The classical Lamé equation (here in the Legendre form)

\[
\frac{d^2 u}{dx^2} + \frac{1}{2} \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t} \right) \frac{du}{dx} + \frac{c-n(n+1)x}{4x(x-1)(x-t)} u = 0
\]

with \(t, n, c \in \mathbb{C}, \ t \neq 0, 1, \) determines a projective structure on the elliptic curve

\(X: \{ y^2 = x(x-1)(x-t) \}, \quad t \in \mathbb{C} \setminus \{0, 1\},\)

having a Fuchsian singularity at the point \(\omega_{\infty}\) at infinity. The projective charts on the affine part of the curve are local determinations of the maps

\[
\phi: X \setminus \{\omega_{\infty}\} \to \mathbb{P}^1; \quad (x, y) \mapsto \frac{u_1(x)}{u_2(x)},
\]

where \(u_1\) and \(u_2\) range over independent solutions of (1). Put \(\vartheta := 2n\). Then there is a local coordinate \(z\) at \(\omega_{\infty}\) such that one of the projective charts around this
point is given by
\[ \phi = z^\vartheta \text{ when } \vartheta \notin \mathbb{Z}, \]
\[ \phi = z^m \text{ or } \phi = 1/z^m + \log(z) \text{ when } \vartheta = \pm m, m \in \mathbb{Z}_{>0}, \]
\[ \phi = z \text{ (regular) when } \vartheta = 0. \]
The monodromy of any projective chart \( \phi = u_1/u_2 \) after analytic continuation along any loop \( \gamma \) is given by \( \rho(\gamma) \circ \phi \), where
\[ \rho: \pi_1(X \setminus \{\omega_\infty\}) \rightarrow \text{PGL}(2, \mathbb{C}) \]
is the projective monodromy representation of (1) computed in the basis \( (u_1, u_2) \).
The following natural question goes back to Poincaré (see [1]): which topological representations
\[ \rho: \pi_1(\text{once-punctured torus}) \rightarrow \text{PGL}(2, \mathbb{C}) \]
can occur as the monodromy of a Fuchsian projective structure?
Every Fuchsian projective structure on the once-punctured torus (that is, having moderate growth at the puncture) is of the above form: the parameter \( t \) stands for the underlying complex structure, \( n \) (or \( \vartheta = 2n + 1 \)) for the Fuchsian type of the puncture, and \( c \) is an accessory parameter. The number of parameters fits with the dimension of the space of such topological representations up to conjugacy (see [2]). One thus expects that a generic representation should be the monodromy of some Lamé equation. The corresponding question for regular projective structures on complete curves has been answered only recently in [3] by pants decomposition and gluing methods. Our initial aim, from which the present work evolved, was to use the isomonodromy method to answer the Lamé case as a test. As we shall see, we actually reduce the question to the existence of poles for Painlevé VI transcendentns, which looks difficult, though of a different nature.

The Lamé equation may be viewed as a logarithmic \( \text{sl}(2, \mathbb{C}) \)-connection on the trivial vector bundle \( \mathcal{O} \oplus \mathcal{O} \) over the elliptic curve \( X \), having a single pole at \( \omega_\infty \): every eigenvector of the residual matrix connection at \( \omega_\infty \) provides a cyclic vector by going back to the scalar elliptic form. On the other hand, the Riemann–Hilbert correspondence asserts that every representation
\[ \rho: \pi_1(X \setminus \{\omega_\infty\}) \rightarrow \text{SL}(2, \mathbb{C}) \]
is the monodromy of a logarithmic \( \text{sl}(2, \mathbb{C}) \)-connection \( \nabla \) on some vector bundle \( E \) of rank 2 over \( X \), having a single pole at \( \omega_\infty \). Our initial question takes the following form: given a topological representation (3), can we choose the complex structure of \( X \) in such a way that the realizing connection \( (E, \nabla) \) is defined on the trivial bundle? Now the question fits perfectly into the setting of isomonodromic deformations.
Starting with a ‘Lamé connection’ \( (E_0, \nabla_0) \) on \( X_{t_0} \), we consider its isomonodromic deformation \( (E_t, \nabla_t) \) arising when the complex structure of the curve \( X_t \) varies (here the deformation parameter \( t \) must be regarded as an element of the universal covering \( T \simeq \mathbb{H} \rightarrow \mathbb{P}^1 \setminus \{0, 1, \infty\} \), which is the Teichmüller space of the once-punctured torus). If the pole of the Lamé connection is not apparent, then this deformation is unique and is characterized by the fact that its monodromy representation is locally constant. Equivalently, the deformation is induced by the unique
integrable logarithmic connection on the universal curve $X \to T$ with the following properties: this connection has a single pole along the section $t \mapsto \omega_\infty \in X_t$ and its restriction to the initial fibre $X_{t_0}$ coincides with $(E_0, \nabla_0)$. Our initial question takes the following form: *for which connections (or representations) can we ensure that the underlying vector bundle $E_t$ becomes trivial for some convenient parameter $t$ along the isomonodromic deformation of a Lamé connection?*

In this paper we shall compute the variation of the vector bundle $E_t$ along the isomonodromic deformation of the initial Lamé connection. This variation is given by a solution of the Painlevé VI equation with appropriate parameters, and the bundle $E_t$ becomes trivial only when this Painlevé transcendent has a pole (other than 0, 1 and $\infty$). We are finally led to the following question, which seems to be open: *do Painlevé transcendents have poles?*

Actually, much more interesting is the corresponding question for regular connections over curves of genus $g > 1$. In accordance with [4], regular projective structures correspond there to regular connections on the maximally unstable indecomposable $\text{SL}(2, \mathbb{C})$-bundle (an $\text{sl}(2, \mathbb{C})$-oper in the sense of [5], §2.7). The main result of [3] can be rephrased as follows: *this special bundle occurs along the isomonodromic deformation if and only if the monodromy is irreducible and does not lie in $\text{SU}(2, \mathbb{C})$ (up to $\text{SL}(2, \mathbb{C})$-conjugacy). Can we prove this directly by computing the variation of the bundle?*

Another interesting question for $g > 1$ is whether a given topological representation $\pi_1(X^{\text{top}}) \to \text{SL}(2, \mathbb{C})$ can be realized as the monodromy of a connection on the trivial bundle for an appropriate choice of the complex structure of $X$? In the case when the image $\Gamma$ of this representation is a discrete subgroup, this provides an embedding $X \to \text{SL}(2, \mathbb{C})/\Gamma$: the fundamental matrix of the associated linear system determines an equivariant map $\tilde{X} \to \text{SL}(2, \mathbb{C})$ (where $\tilde{X} \to X$ is the universal covering). The existence of compact curves in quotients of $\text{SL}(2, \mathbb{C})$ is still an open problem. Thus the isomonodromic approach yields a common geometrical framework for questions arising in various contexts.

Some of our results were announced in [6].

§ 1. The main result

Isomonodromic deformations of meromorphic connections on the Riemann sphere have been the subject of extensive studies (see [7], [8]). In this situation the underlying vector bundle is constant on a Zariski-open subset of the parameter space (see [9]), which enables one to compute the isomonodromy condition explicitly in the form of Schlesinger equations. Painlevé equations arise after further reduction in the simplest case, that of rank 2 with 4 poles. To observe continuous deformations of the underlying bundle, one must switch to connections over curves of genus $g > 0$. The simplest non-trivial case (regular connections of rank 1 over an elliptic curve) was considered in [10], [11]. It was observed that the variation of the underlying line bundle along an isomonodromic deformation is a Painlevé transcendent. In the present paper we study the next most difficult case, that of logarithmic connections of rank 2 with a single pole over an elliptic curve.

Throughout the paper, a *Lamé connection* is a pair $(E, \nabla)$ consisting of a locally trivial holomorphic vector bundle $E$ of rank 2 over an elliptic curve belonging to
the Legendre family
\[ X : \{ y^2 = x(x-1)(x-t) \}, \quad t \in \mathbb{C} \setminus \{0, 1\}, \] (4)
and of a traceless logarithmic connection \( \nabla \) having (at most) a single pole at the point \( \omega_{\infty} \) at infinity:
\[ \nabla : E \to E \otimes \Omega_X^1([\omega_{\infty}]), \quad \det(E) = \mathcal{O}_X, \quad \text{tr}(\nabla) = d : \mathcal{O}_X \to \Omega_X^1 \]
(here we identify the vector bundles \( E, \Omega \) with the corresponding sheaves of holomorphic sections). From now on, such connections \((E, \nabla)\) will be considered up to holomorphic bundle isomorphisms. The exponent \( \vartheta \in \mathbb{C} \) of \((E, \nabla)\) is defined (up to a sign) as the difference between the eigenvalues \( \pm \vartheta/2 \) of the residual matrix of the connection at \( \omega_{\infty} \).

The underlying vector bundle of a Lamé connection has trivial determinant bundle because the connection is traceless. By the results of Atiyah [12], almost all vector bundles of rank 2 with trivial determinant over \( X \) are decomposable, that is, they take the form
\[ E = L \oplus L^{-1}, \quad \text{where} \quad L \in \text{Pic}(X). \] (5)
The complete list is obtained by adding four extra bundles \( E_i, i = 0, 1, t, \infty \). The set of semistable bundles consists of all decomposable bundles with \( L \in \text{Pic}^0(X) \), that is, all bundles of the form
\[ E = L \oplus L^{-1}, \quad \text{where} \quad L = \mathcal{O}_X([\omega] - [\omega_{\infty}]), \quad \omega = (x, y) \in X, \] (6)
with the four indecomposable bundles. The corresponding moduli space is \( \mathbb{P}^1 \) (see [13]) and the quotient map is given by
\[ \lambda : \begin{cases} E \mapsto x & \text{in the notation of (6),} \\ E_i \mapsto i, \quad i = 0, 1, t, \infty. \end{cases} \] (7)
If we denote the 2-torsion points of \( X \) by \( \omega_i = (i, 0) \in X, i = 0, 1, t, \infty \), then \( E_i \) is the unique non-trivial extension
\[ 0 \to L_i \to E_i \to L_i \to 0, \quad \text{where} \quad L_i = \mathcal{O}([\omega_i] - [\omega_{\infty}]), \] (8)
and the moduli map \( \lambda \) identifies \( E_i \) with the trivial extension \( L_i \oplus L_i \). In particular, the point \( \lambda = \infty \) corresponds to both the trivial vector bundle and \( E_{\infty} \).

The isomonodromic deformation of a Lamé connection is defined as follows. Consider the universal covering \( T \to \mathbb{P}^1 \setminus \{0, 1, \infty\} \) (the Teichmüller space of the punctured torus) and the universal Legendre curve \( \mathcal{X} \to T \) over this parameter space: its fibre \( X_t \) at any point \( t \) is precisely the curve \( \{ y^2 = x(x-1)(x-t) \} \) (by an abuse of notation, we write \( t \) for a point of \( T \) and for its projection on \( \mathbb{P}^1 \)). The point \( \omega_{\infty} \) at infinity determines a section \( T \to \mathcal{X} \) of this fibration. For every Lamé connection \((E_0, \nabla_0)\) on \( X_{t_0} \) there is a unique flat logarithmic connection \((E, \nabla)\) on the total space \( \mathcal{X} \) having the section \( \omega_{\infty} \) as the polar set and inducing the initial connection \((E_0, \nabla_0)\) on \( X_{t_0} \) (see [7], [14], [15]). The deformation \( t \mapsto (E_t, \nabla_t) \) induced by the
family \( t \mapsto X_t \) is the isomonodromic deformation of \((E_0, \nabla_0)\). If the pole of \( \nabla_0 \) is not an apparent singular point (that is, a point with local monodromy \( \pm I \)), then \( t \mapsto (E_t, \nabla_t) \) is precisely the unique deformation having a constant monodromy representation. The exponent \( \vartheta \) of the Lamé connection is constant along such a deformation. Finally, one can speak of the variation of the underlying vector bundle \( E_t \) along the deformation: just consider the moduli map \( t \mapsto \lambda(E_t) \) defined above. Here is our main result.

**Theorem 1.** Let \((E_t, \nabla_t)\) be the isomonodromic deformation of an irreducible Lamé connection. Then the underlying vector bundle \( E_t \) is semistable for a Zariski-open subset of the parameter space \( T \) (see [15]) and its Tu invariant \( t \mapsto \lambda(E_t) \in \mathbb{P}^1 \), which is defined by (7), is a solution of the Painlevé VI differential equation

\[
\frac{d^2 \lambda}{dt^2} = \frac{1}{2} \left( \frac{1}{\lambda} + \frac{1}{\lambda - 1} + \frac{1}{\lambda - t} \right) \left( \frac{d\lambda}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{\lambda - t} \right) \left( \frac{d\lambda}{dt} \right)
+ \frac{\lambda(\lambda - 1)(\lambda - t)}{t^2(t - 1)^2} \left( \frac{\kappa_0^2}{2} - \frac{\kappa_0^2}{2} \frac{t}{\lambda^2} + \frac{\kappa_1^2}{2} \frac{t - 1}{(\lambda - 1)^2} + \frac{1 - \kappa_1^2}{2} \frac{t(t - 1)}{(\lambda - t)^2} \right)
\]

with parameters \((\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) = (\vartheta/4, \vartheta/4, \vartheta/4, \vartheta/4)\), where \( \vartheta \) is the exponent of the Lamé connection.

By ‘Zariski-open’ we mean that the exceptional values of \( t \) form a discrete subset of the parameter space \( T \). This property follows directly from [15].

It is already known that isomonodromic deformations of (generic) Lamé connections are parametrized by the Painlevé VI equation with parameters specified above. Namely, the isomonodromic deformation equations on the elliptic curve were computed directly in [16], [17], and the elliptic form of the Painlevé VI equation (see [18]) was recognized.

Our approach to these results is quite different. We first prove, using the Riemann–Hilbert correspondence, that every irreducible Lamé connection can be pushed down via the 2-fold covering \( X \to \mathbb{P}^1 \) to a logarithmic connection of rank 2 with four poles on \( \mathbb{P}^1 \) and the poles are the ramification values 0, 1, \( t \) and \( \infty \) of the covering. Thus we are back to the classical case of Fuchs: the isomonodromic deformation is parametrized by a solution (to be denoted by \( q(t) \)) of the Painlevé VI equation with parameters \((\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) = (1/2, 1/2, 1/2, (\vartheta - 1)/2)\). This already explains why the Painlevé VI equation arises in the Lamé case. So far, no computation is needed. Elementary birational geometry is used to go back to the initial deformation \((E_t, \nabla_t)\), and the moduli \( \lambda \) of the vector bundles \( E_t \) can be expressed in terms of \( q(t) \): we recognize in \( \lambda(E_t) \) the image of \( q(t) \) under the Okamoto symmetry \( s_1 s_2 s_1 \) (see [19]). This automatically implies that \( \lambda(E_t) \) is also a solution of the Painlevé VI equation, but with new parameters \((\vartheta/4, \vartheta/4, \vartheta/4, \vartheta/4)\).

A similar assertion holds for the classical Painlevé VI interpretation in terms of the logarithmic \( \mathfrak{sl}(2, \mathbb{C}) \)-connections with four poles on \( \mathbb{P}^1 \), when we consider the parabolic bundle determined by the eigendirections of the residual matrix of the connection (see [20], [21]).

More generally, we can start with the isomonodromic deformation of a logarithmic connection of rank 2 with four poles on \( \mathbb{P}^1 \), parametrized by any Painlevé VI
transcendent $q(t)$. Then one can lift this deformation to the Legendre elliptic
curve $X_t$ as a logarithmic connection of rank 2 with poles at the ramification
points $\omega_j$ in such a way that the moduli $\lambda(E_t)$ of the corresponding vector bundles
are obtained from $q(t)$ by the Okamoto symmetry $s_1 s_2 s_1$. This provides a new geo-
metric interpretation of this strange symmetry. We thus obtain in a natural way an
isomonodromic deformation problem (a Lax pair) for the general elliptic form of the
Painlevé VI equation, just by considering those traceless logarithmic connections
of rank 2 on $X_t$ (with poles at the second-order points $\omega_i$) that moreover commute
with the elliptic involution $(x, y) \mapsto (x, -y)$. This was also considered in [22].

When we set $\vartheta = 0$, all Lamé connections with vanishing exponent are reducible,
but regular ones can still be pushed down to $\mathbb{P}^1$. Our result remains valid in this
case and we arrive at the following corollary.

**Corollary 2 ([10], [11]).** Let $t \mapsto (L_t, \nabla_t)$ be the isomonodromic deformation of
a regular connection of rank 1 on the Legendre deformation $X_t$, and let $E_t = \mathcal{O}([\omega(t)] - [\omega_\infty])$ be the underlying line bundle, where $\omega(t) = (x(t), y(t)) \in X_t$. Then
the function $t \mapsto x(t)$ is a solution of the Painlevé VI equation with parameters
$(\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) = (0, 0, 0, 0)$.

In this case one can compute explicitly the variation of the line bundle $L_t$ by
means of elliptic functions and obtain the following result.

**Corollary 3 (Picard [23], see [24]).** The general solution of the Painlevé VI equation
with parameters $(\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) = (0, 0, 0, 0)$ is given by

$$
t \mapsto x(t), \quad \text{where } (x(t), y(t)) := \pi(c_0 \cdot \omega_0 + c_1 \cdot \omega_1), \quad c_0, c_1 \in \mathbb{C},
$$

where $\pi : \mathbb{C} \to X_t$ is the universal covering and the $\omega_i(t)$ are the half-periods of $X_t$.

The Painlevé transcendents described in Corollary 3 have poles if and only if
either $c_0$ or $c_1$ is non-real.

**Structure of the paper.** The paper is self-contained in the sense that it contains
several background/survey sections, namely:

- § 8 (appendix) contains basic facts about bundles, connections, moduli spaces,
parabolic structures, elementary transformations;
- § 4 describes how the Painlevé VI equation arises as an isomonodromy condition
for linear differential equations;
- § 5 gives a geometric and modular description of the phase portrait of the
Painlevé VI equation, including Okamoto’s space of initial conditions.

The original part of the paper is concentrated in the following sections.

- § 2: construction of Lamé connections by elliptic pullback of some systems
on $\mathbb{P}^1$;
- § 3: how to compute the Tu invariant of the bundle in terms of the parabolic
structure of the downward system;
- § 6: why most of the Lamé connections are elliptic pullbacks (a Riemann–
Hilbert approach);
- § 7: conclusion of the proof of the main theorem.
§ 2. Our main construction: elliptic pullback

Here we construct Lamé connections by lifting to the elliptic two-fold covering \( \pi: X \rightarrow \mathbb{P}^1 \) certain sl(2, \( \mathbb{C} \))-connections having logarithmic poles at the critical values of \( \pi \). Later we will prove that all irreducible Lamé connections can be obtained in this way. This will be used to parametrize their isomonodromic deformations by means of Painlevé VI solutions in an explicit way.

Let us fix exponents \( \theta = (\theta_0, \theta_1, \theta_t, \theta_\infty) \) and consider a logarithmic sl(2, \( \mathbb{C} \))-connection \((E, \nabla)\) on \( \mathbb{P}^1 \) with poles at 0, 1, \( t \), \( \infty \) and with prescribed exponents (that is, the eigenvalues of the residual matrix at \( i \) are equal to \( \pm \theta_i/2 \) for \( i = 0, 1, t, \infty \)). Such a connection will be called a Heun connection.

An important piece of data to be used later is the parabolic structure \( l = (l_0, l_1, l_t, l_\infty) \). It is defined by the eigenlines \( l_i \in \mathbb{P}(E|_i) \) of the residue of \( \nabla \) at \( i \) with respect to the eigenvalue \( -\theta_i/2 \), where \( i = 0, 1, t, \infty \). If \( \nabla \) does indeed have poles at each of the \( \omega_i \), the parabolic structure is perfectly well defined by the connection and the choice of exponents \( \theta_i \) (they are defined up to a sign). However, it is important to allow non-singular points in our construction if we want to fit with the usual Painlevé VI phase space (see [25]). If \( \theta_i = 0 \) and the corresponding point is non-singular, then every line \( l_i \in \mathbb{P}(E|_i) \) is an eigenline and we have to choose one for our construction. The set of data \((E, \nabla, l)\) with the properties above is called a parabolic Heun connection with parameter \( \theta \).

**Example 4.** If \( E \) is a trivial bundle, then \( \nabla \) is defined by a Fuchsian system

\[
\frac{dY}{dx} = \left( \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t} \right) Y, \quad A_i \in \text{sl}(2, \mathbb{C}).
\]  

(10)

The residual matrix at \( x = i \) is equal to \( A_i \) for \( i = 0, 1, t, \infty \), where \( A_\infty \) is defined by

\[
A_0 + A_1 + A_t + A_\infty = 0.
\]  

(11)

Restrictions on the exponents are given by the equalities \( \det(A_i) = -\theta_i^2/4 \), and the parabolic structure is given by \( l_i = \ker(A_i + (\theta_i/2)I) \).

To motivate the following construction, we point out that for the special exponents

\[
\theta = \begin{pmatrix}
1 & 1 & 1 & 1 + \vartheta \\
2 & 2 & 2 & 2
\end{pmatrix}
\]

it will provide a Lamé connection with exponent \( \vartheta \) at infinity.

**Step 1.** We pull back the connection \((E, \nabla)\) to the elliptic covering

\[ \pi: X \rightarrow \mathbb{P}^1; \quad (x, y) \mapsto x. \]

This yields a logarithmic sl(2, \( \mathbb{C} \))-connection on \( X \) of the form

\[ (\tilde{E}, \tilde{\nabla}) := \pi^*(E, \nabla) \]

with poles at the points \( \omega_0, \omega_1, \omega_t, \omega_\infty \) of ramification and twice the initial exponents, 2\( \theta_i \). The parabolic structure \( \tilde{l} := \pi^*l \) corresponds to eigenlines with respect to the eigenvalues \( -\theta_i \). The point \( i \) has unipotent monodromy (resp. is non-singular)
for $\nabla$ if and only if the point $\omega_i$ is for $\tilde{\nabla}$. We have already noticed that for the exponents $\theta_i = 1/2, i = 0, 1, t$, the corresponding singular points $\omega_i$ of $(\tilde{E}, \tilde{\nabla})$ are projectively apparent, that is, they have local monodromy $-I$. They will disappear in the next two steps.

Remark that we could choose an initial connection on $\mathbb{P}^1$ with a single pole at $\infty$ so that its lifting will be of Lamé type. But the monodromy would then be trivial and the Lamé connection ‘very reducible’.

**Step 2.** We make a convenient birational bundle modification

$$\phi: \tilde{E} \rightarrow E'$$

such that the new connection

$$(\tilde{E}', \tilde{\nabla}') := \phi_*(\tilde{E}, \tilde{\nabla})$$

is still logarithmic and has poles at $\omega_i$ with eigenvalues $\theta_i$ and $1-\theta_i$ for $i = 0, 1, t, \infty$. This is done by successively applying elementary transformations $\text{elm}_{l_i}^+$ with respect to the parabolic structure $\tilde{l}$ over each singular point (see §8.10 for the definition and properties of $\text{elm}_{l_i}^+$):

$$\phi = \text{elm}_{l_0}^+ \circ \text{elm}_{l_1}^+ \circ \text{elm}_{l_t}^+ \circ \text{elm}_{l_{\infty}}^+.$$  

The new connection is not traceless: we have

$$\det(\tilde{E}') = \mathcal{O}_X([\omega_0] + [\omega_1] + [\omega_t] + [\omega_{\infty}]) \simeq \mathcal{O}_X(4[\omega_{\infty}]),$$

and the trace $\text{tr}(\tilde{\nabla}')$ is the unique logarithmic connection on $\mathcal{O}_X(4[\omega_{\infty}])$ having at each $\omega_i$ a pole with residue $+1$ and trivial monodromy. Over the affine chart $X^*$, $\text{tr}(\tilde{\nabla}')$ is defined in a convenient trivialization of the line bundle by

$$d - \left(\frac{dx}{x} + \frac{dx}{x-1} + \frac{dx}{x-t}\right).$$

**Step 3.** We now twist $(\tilde{E}, \tilde{\nabla})$ by an appropriate connection of rank 1 in order to restore the property of being traceless. To do this, we choose the unique square root $(L, \zeta)$ of $(\det(\tilde{E}'), \text{tr}(\tilde{\nabla}'))$ defined on the line bundle $L = \mathcal{O}_X(2[\omega_{\infty}])$: $\zeta$ is given over the affine chart by

$$d - \left(\frac{dx}{2x} + \frac{dx}{2(x-1)} + \frac{dx}{2(x-t)}\right).$$

The resulting sl(2, $\mathbb{C}$)-connection

$$(\tilde{E}'', \tilde{\nabla}'') := (\tilde{E}', \tilde{\nabla}') \otimes (L, \zeta)^{\otimes(-1)}$$

has exponent $2\theta_i - 1$ at $\omega_i, i = 0, 1, t, \infty$. For the special parameters

$$\theta = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} + \frac{\vartheta}{2}\right),$$

$(\tilde{E}'', \tilde{\nabla}'')$ is a Lamé connection (that is, having a single pole at $\infty$) with exponent $\vartheta$ (at $\omega_{\infty}$). We shall see that all irreducible Lamé connections can be obtained in this way. When $\vartheta$ is an odd integer, we note that $\omega_{\infty}$ is a unipotent singular point for $\tilde{\nabla}''$ if and only if $\infty$ is unipotent for $\nabla$. When $\vartheta$ is even, $\omega_{\infty}$ is always apparent for the Lamé connection $\tilde{\nabla}''$ (never unipotent non-trivial).
§ 3. Computing the vector bundle of an elliptic pullback

In the notation of § 2 we would like to determine the vector bundle $\tilde{E}''$ over the elliptic curve $X$ in terms of the initial connection $(E, \nabla)$. In fact, the construction of $\tilde{E}''$ depends only on $E$ and the parabolic structure $l = (l_0, l_1, l_t, l_\infty)$. For simplicity we restrict our attention to irreducible connections, although the general case could be handled by similar arguments. The goal of this section is to prove the following theorem.

**Theorem 5.** Let $(E, \nabla, l)$ be an irreducible parabolic connection and let $(\tilde{E}'', \tilde{\nabla}'')$ be its elliptic pullback. Then $\tilde{E}''$ is semistable and one of the following cases holds.

1) The bundle $E$ is trivial and no three lines $l_i$ coincide. In particular, the cross-ratio

$$c = \frac{l_t - l_0}{l_1 - l_0} \frac{l_1 - l_\infty}{l_t - l_\infty} \in \mathbb{P}^1$$

is well defined and we have

$$\lambda(\tilde{E}'') = t \frac{c - 1}{c - t}.$$ 

Precisely, $\tilde{E}''$ is indecomposable if and only if

- either $c = t$ (the diagonal case),
- or $c = 0, 1, \infty$ and only two of the lines $l_i$ coincide (the other two being mutually distinct).

2) $E = \mathcal{O}(-1) \oplus \mathcal{O}(1)$ and none of the lines $l_i$ coincides with the line subbundle $\mathcal{O}(1)$. Then $\tilde{E}''$ either is trivial or coincides with the indecomposable bundle $E_0$ depending on whether or not all the lines $l_i$ lie on the line subbundle $\mathcal{O}(-1) \hookrightarrow E$.

For the applications we have in mind, it should be emphasized that along the isomonodromic deformation $(E_t, \nabla_t)$ of such a connection $(E, \nabla)$, the set of parameters $t$ for which $E_t$ is non-trivial is always a strict (possibly empty) analytic subset of the parameter space $T$ (see [15]).

We start the proof of Theorem 5 by justifying the restrictions on $E$ and $l$. They result from the following lemma.

**Lemma 6.** Let $X$ be a curve of genus $g$, $E$ a vector bundle, $\nabla : \nabla : E \to E \otimes \Omega(D)$ a logarithmic connection with a reduced effective divisor $D$, and $L \subset E$ a line bundle which is not $\nabla$-invariant. Then the integer

$$\nu := \deg(E) + \deg(D) + 2g - 2 - 2 \deg(L) \geq 0$$

is a bound for the number of those poles of $\nabla$ where $L$ coincides with an eigenline.

Here we mean the eigenline of the residue of $\nabla$.

**Proof.** The composite

$$L \xrightarrow{\text{inclusion}} E \xrightarrow{\nabla} E \otimes \Omega(D) \xrightarrow{\text{quotient}} E/L \otimes \Omega(D) =: L'$$

determines a homomorphism $L \to L'$ of line bundles and, therefore, a section

$$\phi \in H^0(X, L' \otimes L^{-1}) = H^0(X, \det(E) \otimes \Omega(D) \otimes L^{-2}).$$
Since $L$ is not $\nabla$-invariant, $\phi$ is a non-trivial section vanishing at precisely the following points:

- the non-singular points of $\nabla$ where $L$ is stabilized: $\nabla(L) \subset L \otimes \Omega(D)$;
- the poles of $\nabla$ where $L$ coincides with an eigenline of the residue of $\nabla$.

This gives the result. □

In our situation, $g = 0$ and $\deg(D) = 4$. We deduce that $2 - 2\deg(L) \geq 0$ for every line bundle $L \subset E$ (since $\nabla$ is irreducible, no line bundle $L$ can be $\nabla$-invariant). Therefore $E$ is either trivial or equal to $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. In the former case, every embedding $\mathcal{O} \hookrightarrow E$ passes through at most two eigenlines. In the latter, $L = \mathcal{O}(1)$ passes through no eigenlines.

3.1. Projective bundles. To compute the modular invariant $\lambda(\tilde{E}''')$ for the elliptic pullback, it is convenient to work with the associated projective bundle. Given any vector bundle $E$ over a curve $X$, we denote the associated $\mathbb{P}^1$-bundle by $\mathbb{P}(E)$. Another vector bundle $E'$ gives rise to the same $\mathbb{P}^1$-bundle if and only if $E' = L \otimes E$ for some line bundle $L$. When $E$ and $E'$ are both determinant-free, $L$ is a 2-torsion point of the Jacobian. Hence there are at most $2^{2g}$ determinant-free vector bundles $E'$ giving rise to the same $\mathbb{P}^1$-bundle, where $g$ is the genus of $X$. Line bundles $L \subset E$ are in a one-to-one correspondence with sections $\sigma : X \to \mathbb{P}(E)$. The total space $S$ of the bundle $\mathbb{P}(E)$ is a ruled surface, and every section $\sigma$ determines a curve on $S$, to be denoted by $\sigma$ for simplicity. The normal bundle of $\sigma$ in $S$ is identified with the line bundle $\det(E) \otimes L^{-2}$, where $L \subset E$ is the corresponding line bundle, so that the self-intersection is given by

$$\sigma \cdot \sigma = \deg(E) - 2\deg(L).$$

Recall that $E$ is semistable if and only if $\deg(E) - 2\deg(L) \geq 0$ for every line bundle $L \subset E$. If $L'$ is any line bundle distinct from $L$, then the composite

$$L' \to E \to E/L \simeq \det(E) \otimes L^{-1}$$

is a non-trivial homomorphism of line bundles. It determines a non-trivial holomorphic section of $\det(E) \otimes L^{-1} \otimes L'^{-1}$ vanishing at those points where $L$ and $L'$ coincide. This yields a formula for the intersection number of the corresponding sections $\sigma$ and $\sigma'$:

$$\sigma \cdot \sigma' = \deg(E) - \deg(L) - \deg(L') = \frac{1}{2}(\sigma \cdot \sigma + \sigma' \cdot \sigma') \geq 0.$$

A vector bundle $E$ is decomposable (that is, of the form $E = L \oplus L'$) if and only if $\mathbb{P}(E)$ admits two non-intersecting sections $\sigma$ and $\sigma'$ (they correspond to $L$ and $L'$) or, equivalently, two sections $\sigma$ and $\sigma'$ having opposite self-intersection numbers. In this case the $\mathbb{P}^1$-bundle $\mathbb{P}(E)$ may be viewed as the fibrewise compactification $L' \otimes L^{-1}$ of the line bundle $L' \otimes L^{-1}$ obtained by adding a section at infinity. To be precise, $\sigma$ is the zero section and $\sigma'$ is the section at infinity in $\mathbb{P}(E) \simeq L' \otimes L^{-1}$.

If $X$ is an elliptic curve (for example, $X : \{y^2 = x(x-1)(x-t)\}$) and $\det(E) = \mathcal{O}_X$, then $E$ is semistable if and only if $\mathbb{P}(E)$ has a section with zero self-intersection. The four indecomposable bundles defined in (8) correspond to the same $\mathbb{P}^1$-bundle $\mathbb{P}(E_0)$, which is given by the unique non-trivial extension

$$0 \to \mathcal{O}_X \to E_0 \to \mathcal{O}_X \to 0.$$
The corresponding ruled surface $S_0$ is characterized by the existence of one and only one section with zero self-intersection. By the results of Atiyah [12] (see also [13]), all other semistable determinant-free vector bundle are decomposable: the corresponding $\mathbb{P}^1$-bundles are of the form

$$\mathbb{P}(E) = \mathcal{O}_X([\omega] - [-\omega]),$$

where $\pm \omega \in X$ are the two points of the fibre $\pi^{-1}(\lambda(E))$ (see the definition (7)). We note that the modular invariant $\lambda(E)$ is determined by $\mathbb{P}(E)$ up to the action of the 2-torsion points of the elliptic curve $X$: the determinant-free vector bundles with $\mathbb{P}^1$-bundle $\mathbb{P}(E)$ are the four semistable bundles with modular invariants

$$\lambda, \quad \frac{t}{\lambda}, \quad \frac{\lambda - t}{\lambda - 1}, \quad \text{and} \quad \frac{\lambda - 1}{\lambda - t}.$$

### 3.2. Ruled surfaces and elliptic pullback.

Let us now describe the construction of §2 in terms of ruled surfaces. We consider a rational ruled surface $p: S = \mathbb{P}(E) \to \mathbb{P}^1$ equipped with the parabolic structure $l$ determined by a point $l_i$ on the fibre $S|_i = p^{-1}(i)$ for $i = 0, 1, t, \infty$.

**Step 1.** The elliptic ruled surface $\tilde{S} = \mathbb{P}(\tilde{E}) \to \mathbb{P}^1$ is obtained as the two-fold ramified covering $\Pi: \tilde{S} \to S$ branching along the four fibres $S|_i$. It makes the following diagram commutative:

$$
\begin{array}{ccc}
\tilde{S} & \xrightarrow{\Pi} & S \\
\downarrow{\tilde{\phi}} & & \downarrow{p} \\
X & \xrightarrow{\pi} & \mathbb{P}^1 \\
\end{array}
$$

We equip $\tilde{S}$ with a parabolic structure $\tilde{l}$ by putting $\tilde{l}_i = \Pi^{-1}(l_i) \in \tilde{S}|_{\omega_i} = \tilde{p}^{-1}(\omega_i)$ for $i = 0, 1, t, \infty$.

**Step 2 (and 3).** The birational transformation

$$
\begin{array}{ccc}
\tilde{S} & \xrightarrow{\phi} & \tilde{S}' \\
\downarrow{\tilde{\phi}} & & \downarrow{\tilde{\phi}} \\
X & \xrightarrow{\tilde{\phi}} & \tilde{S}' \\
\end{array}
$$

is obtained by blowing up the four points $\tilde{l}_i$ and then blowing down the strict transforms of the four fibres. Step 3 is irrelevant from the projective point of view since it suffices to multiply $\tilde{E}'$ by a line bundle in order to obtain $\tilde{E}''$. Therefore $\tilde{S}'' = \tilde{S}'$.

### 3.3. Elliptic ruled surfaces and elementary transformations.

By Lemma 6, the parabolic ruled surface $(S, l)$ we start with is of the following form:

- either $S = \mathbb{P}^1 \times \mathbb{P}^1$ and no three points $l_i \in S$ lie on the same horizontal line;
- or $S = F_2$ is the second Hirzebruch surface, that is, the total space of $\mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(1))$, and none of the points $l_i$ lies on the ‘negative’ section $\sigma$ corresponding to $\mathcal{O}(1)$.

In either case we write $\tilde{S}'$ for the corresponding elliptic pullback constructed in §3.2.
Proposition 7. When $S = \mathbb{P}^1 \times \mathbb{P}^1$, choose a coordinate $w$ and consider the cross-ratio
\[ c = \frac{w_t - w_0}{w_1 - w_0} \frac{w_1 - w_\infty}{w_t - w_\infty} \in \mathbb{P}^1, \]
where the points $w_i$ are defined by the formula $l_i = (i, w_i)$. Then the following three assertions hold.

- If $c \neq 0, 1, t, \infty$, then $\tilde{S}'$ is a decomposable ruled surface
\[ \tilde{S}' = \mathcal{O}_X([w] - [-\omega]), \]
where $\pm \omega = (x, \pm y) \in X$ are the two points over $x = t(c - 1)/(c - t)$.

- If $c = t$ (all four points $l_i$ lie on the same irreducible curve of bidegree $(1, 1)$), then $\tilde{S}'$ is the indecomposable ruled surface $S_0$.

- If $c = 0, 1, \infty$, then at least two points $l_i$ lie on the same horizontal line. Moreover, if the two other points lie on another horizontal line, then $\tilde{S}'$ is a trivial bundle. Otherwise $\tilde{S}'$ is the indecomposable ruled surface $S_0$.

When $S = \mathbb{P}_2$, the following two assertions hold.

- If all four points $l_i$ lie on a section with self-intersection +2 (that is, on a section induced by an arbitrary embedding $\mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$), then $\tilde{S}'$ is a trivial bundle.

- Otherwise $\tilde{S}'$ is the indecomposable ruled surface $S_0$.

Proof. We first consider the generic case when $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $c \neq 0, 1, t, \infty$. Then one can choose the vertical coordinate $w$ in such a way that
\[ l_0 = (0, 0), \quad l_1 = (1, 1), \quad l_t = (t, c) \quad \text{and} \quad l_\infty = (\infty, \infty). \]

One easily checks by computation that there is a unique curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(2, 2)$ intersecting each fibre $x = i$ at the point $l_i$ with multiplicity 2. The equation of $C$ is of the form $F = 0$, where
\[ F(x, w) = ((c - t)x - t(c - 1))w^2 + 2(t - 1)cxw - cx((c - 1)x - (c - t)). \quad (12) \]
The discriminant with respect to the variable $w$ is given by
\[ \text{disc}_w(F) = 4c(c - 1)(c - t)x(x - 1)(x - t). \]

Since it does not vanish identically and its roots are simple, the curve $C$ is reduced, irreducible and smooth. Its lifting $\tilde{C}$ to the trivial bundle $X \times \mathbb{P}^1$ splits into the union of two distinct sections, $\tilde{\sigma}_0$ and $\tilde{\sigma}_\infty$, which intersect each other at exactly the four points $\tilde{l}_i$ without multiplicity. After elementary transformations with centres $\tilde{l}_i$ we obtain disjoint sections $\sigma_0$ and $\sigma_\infty$ of $\tilde{S}'$. This already means that $\tilde{S}'$ is the compactification of a line bundle. To determine this line bundle, we consider the horizontal section $w = \infty$ of $\mathbb{P}^1 \times \mathbb{P}^1$ passing through $l_\infty$. It intersects the $(2, 2)$-curve $C$ at two points $(l_\infty$ and the point $s = (x, \infty)$ with coordinate $x = t(c - 1)/(c - t))$ and lifts to a section $\tilde{\sigma}$ of $X \times \mathbb{P}^1$ intersecting $\tilde{\sigma}_0$ over 0
and, say, \( \omega \), and then intersecting \( \tilde{\sigma}_\infty \) over 0 and \(-\omega\), where \( \pm \omega \in X \) are the two points of \( X \) over \( x \). After elementary transformations with centres \( \tilde{l}_i \) we obtain a section \( \sigma \) of \( \tilde{S}' \) intersecting \( \sigma_0 \) at \( \omega \) and \( \sigma_\infty \) at \(-\omega\). Thus \( \tilde{S}' = \mathcal{O}(\omega - [-\omega]) \).

This argument is summarized in Fig. 1.

![Diagram](image)

**Figure 1.** Lifting \( \mathbb{P}(\nabla) \) to the elliptic curve \( E_t \)

When \( c = t \), the curve \( F = 0 \) degenerates to twice the (1,1)-curve passing through all the points \( l_i \), namely, the diagonal section \( \sigma(x) = x \). Its lifting to \( \tilde{S} \) is the graph \( \tilde{\sigma} \) of the two-fold covering \( X \to \mathbb{P}^1 \) with self-intersection +4. After
elementary transformations we obtain a section $\tilde{\sigma}'$ of $\tilde{S}'$ having zero self-intersection. More precisely, the normal bundle of $\tilde{\sigma}'$ is trivial. Indeed, the section $\tilde{\sigma}$ is induced in $\tilde{S} = \mathbb{P}(\tilde{E})$, $\tilde{E} = O_X \oplus O_X$, by the line bundle $\tilde{L} \subset \tilde{E}$ generated by its meromorphic section $(x, y) \mapsto \begin{pmatrix} x \\ 1 \end{pmatrix}$ whose divisor is $-2[\omega_\infty]$. The normal bundle of $\tilde{\sigma}$ in $\tilde{S}$ is therefore given by

$$N_{\tilde{\sigma}} = \det(\tilde{E}) \otimes \tilde{L}^{-2} = O_X \oplus O_X (-2[\omega_\infty])^{-2} = O_X (4[\omega_\infty]).$$

After elementary transformations we obtain that

$$N_{\tilde{\sigma}'} = \det(\tilde{E}') \otimes L'^{-2}$$

$$= O_X ([\omega_0] + [\omega_1] + [\omega_t] + [\omega_\infty]) \otimes O_X ([\omega_0] + [\omega_1] + [\omega_t] - [\omega_\infty])^{-2} = O_X.$$

If $\tilde{S}'$ were decomposable, it would be a trivial $\mathbb{P}^1$-bundle, and we must now exclude this possibility. Consider the centres $\tilde{l}'_i \in \tilde{S}'$ of the inverse elementary transformations. If $\tilde{S}'$ were the trivial $\mathbb{P}^1$-bundle, most of horizontal sections would avoid the four points $\tilde{l}'_i$ and would define, back on $\tilde{S} = X \times \mathbb{P}^1$, a section with self-intersection $-4$. This is impossible. Thus $\tilde{S}'$ is the indecomposable bundle $S_0$.

We now assume that $c = \infty$ (the other cases $c = 0$ or 1 are similar). We are in one of the following three cases:

1) $w_0 = 0$, $w_1 = 1$ and $w_t = w_\infty = \infty$;
2) $w_0 = w_1 = 0$, $w_t = 1$ and $w_\infty = \infty$;
3) $w_0 = w_1 = 0$ and $w_t = w_\infty = \infty$.

The last case is easy since the three horizontal sections $\sigma_0$, $\sigma_1$ and $\sigma_\infty$ (which are defined by $w = 0$, $w = 1$ and $w = \infty$ respectively) are transformed on the elliptic pullback $\tilde{S}'$ into disjoint sections $\tilde{\sigma}'_0$ and $\tilde{\sigma}'_\infty$ and a third section $\tilde{\sigma}'_1$ that intersects $\tilde{\sigma}'_0$ at $\omega_t$ and $\omega_\infty$, and $\tilde{\sigma}'_\infty$ at $\omega_0$ and $\omega_1$. We promptly deduce that

$$\tilde{S}' = \mathcal{O}_X ([\omega_t] + [\omega_\infty] - [\omega_0] - [\omega_1]) = X \times \mathbb{P}^1.$$

We now study the first case, where only $w_t$ and $w_\infty$ coincide (this is similar to the diagonal case). The section $w = \infty$ of $S$ induces a section $\tilde{\sigma}'_\infty$ of $\tilde{S}'$ having a trivial normal bundle. If $\tilde{S}'$ were decomposable, it would be the trivial bundle, and a generic constant section $\tilde{\sigma}'$ would provide a section $\tilde{\sigma}$ of the trivial bundle $\tilde{S}$ having self-intersection $-4$, a contradiction. Thus $\tilde{S}'$ is the indecomposable bundle $S_0$.

We finally consider the case when $S = \mathbb{F}_2$. As above, we easily see that the exceptional section $\sigma_\infty$, which is induced by $\mathcal{O}_{\mathbb{F}_2}(1)$, yields a section $\tilde{\sigma}'_\infty$ of $\tilde{S}'$ having a trivial normal bundle. Again, as in the diagonal case, consider the centres $\tilde{l}'_i \in \tilde{S}'$ of the elementary transformations inverse to $\phi$; they are contained in $\tilde{\sigma}'_\infty$. If $\tilde{S}'$ is the trivial bundle, then its constant sections $\tilde{\sigma}'$ give rise to a pencil of sections $\tilde{\sigma}$ of $\tilde{S}$ having the four points $\tilde{l}'_i$ as base points. A special member of this pencil is given by the union of $\tilde{\sigma}_\infty$ and the four fibres over the points $\omega_i$, and the pencil itself consists of all sections of $\tilde{S}$ passing through the four points $\tilde{l}'_i$ and disjoint from $\tilde{\sigma}_\infty$ (plus the special one). The elliptic involution $\tau: (x, y) \mapsto (x, -y)$
permutes these sections, and so acts on the parameter space $\mathbb{P}^1$. This action has at least two fixed points, namely, $\tilde{\sigma}_\infty$ and another section $\tilde{\sigma}_0$ that can be pushed down as a section $\sigma_0$ of $S$. By construction, $\sigma_0$ passes through all the points $l_i$ and does not intersect $\sigma_\infty$; it is a $+2$-curve, as required. Conversely, when all $l_i$ lie on a $+2$-curve $\sigma_0$, we obtain a second section $\tilde{\sigma}_0'$ of $\tilde{S}'$, which yields a trivialization. We note that the pencil considered above comes from a pencil not of sections of $S$, but of curves that intersect each fibre twice. □

§ 4. Isomonodromic deformations and the Painlevé VI equation

In this section we recall how isomonodromic deformations of logarithmic sl(2, $\mathbb{C}$)-connections $(E_t, \nabla_t)$ over the 4-punctured sphere are parametrized by Painlevé VI solutions, and how we can use this parametrization to compute the variation of the bundle $E''_t$ of the corresponding elliptic pullback. We first recall what an isomonodromic deformation is.

4.1. Isomonodromic deformations and flat connections. Let $X_0$ be a complex projective curve and let $(E_0, \nabla_0)$ be a vector bundle of rank 2 over $X_0$ equipped with a (flat) logarithmic sl(2, $\mathbb{C}$)-connection with a reduced effective polar divisor $D_0$. For every topologically trivial analytic deformation $(X_t, D_t)$ of the punctured curve there is a unique deformation $(E_t, \nabla_t)$ of the vector bundle and the connection such that the monodromy data remain constant. This follows from the Riemann–Hilbert correspondence (Proposition 24 with parameters). The monodromy data consist of the monodromy representation complemented by the ‘parabolic structure’ at apparent singular points. Equivalently, if we denote the total space of the deformation by $X$ and the corresponding (smooth) divisor by $D \subset X$, then the deformation $(E_t, \nabla_t)$ is induced by the unique flat logarithmic sl(2, $\mathbb{C}$)-connection $(E, \nabla)$ with polar divisor $D$ inducing $(E_0, \nabla_0)$ on the slice $X_0$.

In this paper we consider the case of the 4-punctured sphere and the once-punctured torus. Their deformations are parametrized by the corresponding Teichmüller spaces that are both isomorphic to the Poincaré half-plane $\mathbb{H}$. To be precise, we start with the isomonodromic deformation of a logarithmic connection $(E_t, \nabla_t)$ over the Riemann sphere with poles at 0, 1, $t$ and $\infty$, where $t$ ranges over the universal covering $T \simeq \mathbb{H} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Consider the deformation $(\tilde{E}_t, \tilde{\nabla}_t)$ of the corresponding elliptic pullback constructed in § 2. We easily see that $(\tilde{E}_t, \tilde{\nabla}_t)$ is still an isomonodromic deformation of some logarithmic connection over the Legendre family of elliptic curves $X_t$ with poles contained in the ramification locus $\{\omega_0, \omega_1, \omega_t, \omega_\infty\}$ of the elliptic curve. The parameter space $T$ is now understood as the Teichmüller space of the torus. For the special parameters $\theta = (1/2, 1/2, 1/2, 1/2 + \theta/2)$ we obtain the isomonodromic deformation of a Lamé connection.

4.2. The Painlevé VI equation and Fuchsian equations. Although we do not really need it, it is interesting to recall how the Painlevé VI equation was originally derived as an isomonodromic equation for Fuchsian projective structures on the 4-punctured sphere with one extra branch point. After normalizing the singular points as 0, 1, $t$ and $\infty$ by a Möbius transformation, the corresponding
Fuchsian second-order differential equation $u_{xx} + f(x)u_x + g(x)u = 0$ takes the form

$$
\begin{cases}
   f(x) = \frac{1 - \kappa_0}{x} + \frac{1 - \kappa_1}{x - 1} + \frac{1 - \kappa_t}{x - t} - \frac{1}{x - q}, \\
   g(x) = -\frac{t(t-1)H}{x-t} + \frac{q(q-1)p}{x-q} + \rho(\kappa_\infty + \rho) \\
   x(x-1)
\end{cases}
$$

(13)

Here $q \not\in \{0, 1, t, \infty\}$ is the branch point, $\kappa_i$ is the local exponent at $i = 0, 1, t, \infty$, and $\rho$ is fixed by the relation

$$
\kappa_0 + \kappa_1 + \kappa_t + \kappa_\infty + 2\rho = 1.
$$

(14)

Note that the parameters $p$ and $H$ are residues of $g$:

$$
H = -\text{Res}_{x=t} g(x) \quad \text{and} \quad p = \text{Res}_{x=q} g(x).
$$

(15)

The singular point $q$ has exponent 2. This point is apparent (that is, a branch point of the projective chart) if and only if the parameter $H$ is given by

$$
H = \frac{q(q-1)(q-t)}{t(t-1)} \left( p^2 - \left( \frac{\kappa_0}{q} + \frac{\kappa_1}{q-1} + \frac{\kappa_t - 1}{q-t} \right) p + \frac{\rho(\kappa_\infty + \rho)}{q(q-1)} \right).
$$

(16)

Under these assumptions, the local charts $\phi = u_1/u_2$, where $u_1$ and $u_2$ range over independent solutions of (13), form a projective atlas on the complement of 0, 1, $t$, $q$ and $\infty$ in the Riemann sphere. At each of the singular points $i = 0, 1, t, \infty$ one of the projective charts takes the form $\phi = z^{\kappa_i}$ (or, possibly, $\phi = 1/z^m + \log(z)$ in the case when $\kappa_i = \pm m \in \mathbb{Z}_{>0}$) for a convenient local coordinate $z$ at $i$. At the point $q$ one of the projective charts takes the form $\phi = z^2$ (a simple branch point). Conversely, every projective structure on the Riemann sphere having five singularities with moderate growth, one of which is a simple branch point, is conjugate by a Möbius transformation to an element of the family above. Such projective structures are characterized by the following data:

- the positions $t$ and $q$ of the singular points,
- the exponents $\kappa = (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty)$,
- the monodromy representation of a projective chart $\phi$ up to conjugacy.

A small deformation of an equation (13) with moving singular points $t$ and $q$ is said to be isomonodromic if the projective charts have constant monodromy representation (up to conjugacy). Such deformations are characterized by the following classical theorem.

**Theorem 8 (Fuchs–Malmquist).** A deformation of (13) parametrized by the position $t$ of the singular point is isomonodromic if and only if the exponents $\kappa_i$ remain fixed and the parameters $(p(t), q(t))$ satisfy the non-autonomous Hamiltonian system

$$
\frac{\partial q}{\partial t} = \frac{\partial H}{\partial p} \quad \text{and} \quad \frac{\partial p}{\partial t} = -\frac{\partial H}{\partial q}.
$$

(17)

The first Hamiltonian equation (17) is of the form

$$
p = \frac{1}{2} \left( \frac{t(t-1)}{q(q-1)(q-t)} \frac{\partial q}{\partial t} + \frac{\theta_0}{q} + \frac{\theta_1}{q-1} + \frac{\theta_t - 1}{q-t} \right).
$$

(18)
Substituting (18) in the second equation (17), we obtain the Painlevé VI equation (9) with parameters $\kappa$. From a chronological point of view, the Painlevé VI equation was first derived by Fuchs, and the Hamiltonian form was discovered later by Malmquist.

4.3. The Painlevé VI equation and Fuchsian systems. Let us now recall how Painlevé VI solutions correspond to isomonodromic deformations of logarithmic sl(2, $\mathbb{C}$)-connections ($E_t, \nabla_t$) with singular points 0, 1, $t$ and $\infty$ over the Riemann sphere. Let ($E_t, \nabla_t$) be such a deformation. Assume that it is irreducible (this depends only on the monodromy and not on the value of $t$). By Lemma 6, the underlying bundle $E_t$ is either trivial or equal to $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. It turns out that $E_t$ must be trivial for all values of $t$ outside a discrete subset of $T$: there are no non-trivial irreducible isomonodromic deformations of such connections on the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ (see [15]). It is thus enough to consider isomonodromic deformations of sl(2, $\mathbb{C}$)-Fuchsian systems

$$\frac{dY}{dx} = \left( \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t} \right)Y, \quad A_i \in \text{sl}(2, \mathbb{C}).$$

(19)

The residual matrix at the singular point $x = \infty$ is given by

$$A_0 + A_1 + A_t + A_\infty = 0.$$  

(20)

We denote the eigenvalues of $A_i$ by $\pm \theta_i/2$. Then

$$A_i = \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix}, \quad \text{where} \quad a_i^2 + b_ic_i = \frac{\theta_i^2}{4}, \quad i = 0, 1, t, \infty.$$  

(21)

After a change of variables $Y := MY$, where $M \in \text{SL}(2, \mathbb{C})$, we normalize

$$A_\infty = \begin{pmatrix} \theta_\infty/2 & 0 \\ * & -\theta_\infty/2 \end{pmatrix}.$$  

(22)

We exclude the case when $A_\infty = 0$: the connection $\nabla$ must be singular at $\infty$. Then the following theorem holds.

**Theorem 9.** A small deformation $A_i = A_i(t)$ of the system (10), normalized by (22), is isomonodromic if and only if the eigenvalues $\pm \theta_i/2$ are constant and the function $q := tb_0/(tb_0 + (t-1)b_1)$ satisfies the differential equation

$$\frac{dq}{dt} = -2a_0 \frac{q-1}{t-1} - 2a_1 \frac{q}{t} + (1 - \theta_\infty) \frac{q(q-1)}{t(t-1)}.$$  

(23)

and the Painlevé VI equation (9) with parameters

$$\kappa = (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) = (\theta_0, \theta_1, \theta_t, \theta_\infty - 1).$$

We first deduce Theorem 9 from the Fuchs–Malmquist theorem (Theorem 8).
Proof. The vector \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) is an eigenvector with eigenvalue \(-\theta_\infty/2\) at \(\infty\). Since the system (10) is irreducible, this vector is not invariant and can be chosen for a cyclic vector to derive a scalar Fuchsian equation. Namely, if \( Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \) is a solution of (10), then the function
\[
u := \sqrt{x^\theta_0(x-1)^\theta_1(x-t)^\theta_t}y_1
\]
satisfies the scalar equation (13) with exponents
\[
(\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) = (\theta_0, \theta_1, \theta_t, \theta_\infty - 1)
\]
and parameters
\[
q = \frac{tb_0}{tb_0 + (t-1)b_1},
\]
\[
p = \frac{a_0 + \theta_0/2}{q} + \frac{a_1 + \theta_1/2}{q-1} + \frac{a_t + \theta_t/2}{q-t}.
\]
In fact, the Darboux coordinates have the following interpretation. The point \( x = q \) is the unique other point at which \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) is again an eigenvector of the system (10). The corresponding eigenvalue is
\[
-p + \frac{\theta_0}{2q} + \frac{\theta_1}{2(q-1)} + \frac{\theta_t}{2(q-t)}.
\]
It follows from Theorem 8 that a deformation of the system (10) is isomonodromic if and only if the auxiliary variables \( p \) and \( q \) defined by (25) and (26) satisfy the Hamiltonian system (17) with the non-autonomous Hamiltonian \( H(t,p,q) \) defined in (16).

Let us now explain how to reconstruct the system (10) uniquely (up to a gauge transformation) from a solution \( q(t) \) of the Painlevé VI equation. We first introduce an auxiliary variable \( p \) by the formula (18). This gives us a unique scalar equation (13) from which one can reconstruct a Fuchsian system by the standard method. The resulting system is defined (up to a gauge transformation) by the formula (10) with equations (21) and
\[
\begin{align*}
a_0 &= \frac{p}{t} - \frac{\kappa_0}{2}, \\
a_1 &= -\frac{p}{t-1} + \frac{q-1}{t-1}(\rho + \kappa_\infty) - \frac{\kappa_1}{2}, \\
a_t &= \frac{p}{t(t-1)} \left( \frac{q-t}{t-1} \right) (\rho + \kappa_\infty) - \frac{\kappa_t}{2},
\end{align*}
\]
\[
\begin{align*}
b_0 &= -\frac{q}{t}, \\
b_1 &= \frac{q-1}{t-1},
\end{align*}
\]
where \( p = q(q-1)(q-t)p \) and \( \rho \) is defined by (14). The coefficients \( c_i \) of this system can immediately be deduced from the equations (21). The standard formulae given by Jimbo and Miwa (see [26], pp. 199, 200) assume that \( \theta_\infty \neq 0 \) so that the matrix \( A_\infty \) can be further normalized as a diagonal matrix by an additional gauge transformation. The resulting formulae are much more complicated than those above. The way we obtain the formulae (27) is described in §§5.3–5.5.
4.4. The vector bundle of an elliptic pullback. Coming back to our initial problem and wishing to apply Theorem 5, we would like to parametrize the parabolic structure \( l \) induced by \( \nabla_t \) (or, equivalently, by the system (10)) in terms of the Painlevé VI transcendent \( q(t) \) that parametrizes the deformation. We deduce from (21) and (27) that the eigenline \( l_i \) associated with the eigenvalue \(-\theta_i/2\) over the pole \( i = 0, 1, t \) is given by

\[
l_i = \left(-b_i : a_i + \frac{\theta_i}{2}\right),
\]

whence

\[
\begin{align*}
    l_0 &= \left(1 : \frac{p}{q}\right), \\
    l_1 &= \left(1 : \frac{p}{q-1} + \rho + \kappa_\infty\right), \\
    l_t &= \left(1 : \frac{p}{q-t} + (\rho + \kappa_\infty)t\right), \\
    l_\infty &= (0 : 1)
\end{align*}
\]

(recall that \( l_\infty \) was normalized by (22)). These expressions for \( l_i \) no longer hold for a gauge-equivalent system (10) (for example, through the Jimbo–Miwa normalization), but the cross-ratio

\[
c = \frac{l_t - l_0}{l_1 - l_0} \frac{l_1 - l_\infty}{l_t - l_\infty} = t \frac{(q-1)p + \rho + \kappa_\infty}{(q-t)p + \rho + \kappa_\infty}
\]

depends only on the system (22) up to gauge transformations. We note that the formula (29) gives an elegant definition of the auxiliary variable \( p \) in terms of the parabolic structure of the connection, \( q \) and \( t \).

**Corollary 10.** Let \((E_t, \nabla_t)\) be the isomonodromic deformation defined by the Painlevé VI solution \( q(t) \) as above, and let \((\tilde{E}_t'', \nabla_t'')\) be the elliptic pullback of this deformation. Then the bundle \( \tilde{E}_t'' \) is semistable and has invariant

\[
\lambda(\tilde{E}_t'') = q(t) + \frac{\rho + \kappa_\infty}{p(t)}.
\]

**Proof.** By construction, the bundle \( E_t \) is trivial. By Theorem 5, \( \tilde{E}_t'' \) is semistable and has invariant

\[
\lambda(\tilde{E}_t'') = t \frac{c - 1}{c - t} = q + \frac{\rho + \kappa_\infty}{p}.
\]

All computations above hold only under the generic assumptions that \( q \neq 0, 1, t, \infty \). On the other hand, it is well known that constant solutions \( q(t) \equiv 0, 1, t \) or \( \infty \) correspond to isomonodromic deformations of reducible connections. Thus Corollary 10 is enough for proving Theorem 1. However, we can be more precise and check for special values of \( q \) and \( p \) whether \( \tilde{E}'' \) is indecomposable or not.
§ 5. The geometry of the Painlevé VI equation

Here we introduce some moduli space $\mathcal{M}_\theta^{t_0}$ of $\text{sl}(2, \mathbb{C})$-connections with poles at $0, 1, t_0$ and $\infty$ and eigenvalues $\theta$. It contains all irreducible connections. This space was originally used in Okamoto’s work [27] to construct a good space of initial conditions for the Painlevé VI equation from which the Painlevé property can be read off geometrically. In [25] this space was identified with a moduli space of connections, extending the dictionary established in § 4.3. We recall this construction and then use it to determine the vector bundle $\tilde{E}_t$ of the elliptic pullback for special values of $p$ and $q$.

5.1. Okamoto’s space of initial conditions. The Painlevé property, which characterizes the Painlevé equations among all differential equations of the form $\lambda'' = F(t, \lambda, \lambda')$, says that all Painlevé VI solutions can be analytically continued as meromorphic solutions along any path avoiding $0, 1$ and $\infty$. Painlevé VI solutions become meromorphic and global on the universal covering of the 3-punctured sphere.

The (naive) space of initial conditions $\mathbb{C}^2 \ni (q(t_0), q'(t_0))$ for the Painlevé VI equation at some point $t_0 \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ fails to describe all solutions in a neighbourhood of $t_0$. Painlevé VI solutions are meromorphic and some of them have a pole at $t_0$; we have to add them. The good space of initial conditions is

$$\mathcal{M}_\theta^{t_0} := \{\text{germs of meromorphic } P_{VI}^\theta \text{-solutions at } t\}. \quad (30)$$

To construct it, Okamoto considers the phase portrait of the Painlevé VI equation in the variables

$$(t, q, q') \in (\mathbb{P}^1 \setminus \{0, 1, \infty\}) \times \mathbb{C}^2$$

(introducing an auxiliary equation $dq/dt = q'$): it is defined by a rational vector field that determines a singular holomorphic foliation on any rational compactification. For example, we can start with $\mathbb{P}^1 \times \mathbb{P}^2$ and observe that the singularities of the foliation are located on the special fibres $t = 0, 1, \infty$ (these do not concern us) and at the infinity of the $\mathbb{P}^2$-factor. The latter are degenerate and lie along a one-dimensional section of the $t$-projection; we must blow up this section in order to reduce the degeneracy of the singular points. After nine successive blow-ups, Okamoto obtains a fibre bundle

$$t : \mathcal{M}_\theta^t \to \mathbb{P}^1 \setminus \{0, 1, \infty\} \quad (31)$$

(we just ignore what happens over $t = 0, 1, \infty$) which is not locally trivial as an analytic bundle (but is as a topological bundle). There is a non-vertical divisor $Z \subset \mathcal{M}_\theta^t$ consisting of vertical leaves (with respect to the $t$-projection) and singular points. On the complement $\mathcal{M}_\theta^t := \mathcal{M}_\theta^t \setminus Z$ of this divisor, the Painlevé foliation is transversal to $t$ and induces a local analytic trivialization of the bundle

$$t : \mathcal{M}_\theta^t \to \mathbb{P}^1 \setminus \{0, 1, \infty\}. \quad (32)$$

By construction, the fibre $\mathcal{M}_\theta^t$ over any point $t \neq 0, 1, \infty$ may be interpreted as a set of germs of meromorphic $P_{VI}^\theta$-solutions. Actually, for special parameters $\theta$, 

there are leaves staying at the infinity of the affine chart \((q, q')\). These cannot be regarded as meromorphic solutions. They should rather be viewed as ‘constant solutions \(q \equiv \infty\)’. The divisor \(Z\) actually coincides with the reduced polar divisor of the closed 2-form defined in the affine chart as

\[
dt \wedge dH + dp \wedge dq,
\]

where \(H\) is defined by (16). The kernel of this 2-form determines the Painlevé fibration.

We now describe the parameter space \(\mathcal{M}_t^\theta\) starting with the Hirzebruch ruled surface \(\mathbb{F}_2\). Define a reduced divisor \(Z_t \subset \mathbb{F}_2\) as the union of the section \(\sigma: \mathbb{P}^1 \to \mathbb{F}_2\) having self-intersection \(-2\) and the four fibres over 0, 1, \(t_0\) and \(\infty\). Next, we fix two points on each vertical component of \(Z_t\), none of which lies on the horizontal component. Blowing up these eight points, we obtain the compact space \(\mathcal{M}_t^\theta\). Let us preserve the notation \(Z_t\) for the strict transform of this divisor. The complement \(\mathcal{M}_t^\theta : = \mathcal{M}_t^\theta \setminus Z_t\) is the space of initial conditions. It remains to define the position of these eight points as a function of \(\theta\) and \(t\) (see § 5.6).

5.2. Projective structures and Riccati foliations. We go back to the approach of Fuchs, where the Painlevé transcendents parametrize isomonodromic deformations of Fuchsian projective structures with 4 + 1 singular points (see § 4.2). Such a structure can be defined by a Fuchsian second-order differential equation (13). One can also determine it by the data consisting of a logarithmic \(\text{sl}(2, \mathbb{C})\)-connection \((E, \nabla)\) together with a line subbundle \(L \subset E\) which is not \(\nabla\)-invariant and plays the role of a cyclic vector (see § 4.3). In [5] these data are referred to as ‘\(\text{sl}(2, \mathbb{C})\)-opers’. A more geometrical picture inspired by the works of Ehresman uses the notion of a ‘projective oper’. This is a triple \((\mathbb{P}(E), \mathbb{P}(\nabla), \sigma)\), where \(\mathbb{P}(E)\) is the associated \(\mathbb{P}^1\)-bundle, \(\mathbb{P}(\nabla)\) is the induced projective connection and \(\sigma: \mathbb{P}^1 \to \mathbb{P}(E)\) is the section corresponding to \(L\). For example, the system (10) determines the Riccati equation

\[
\frac{dy}{dx} = -b(x)y^2 - 2a(x)y + c(x)
\]

on a trivial \(\mathbb{P}^1\)-bundle by the formula \((1 : y) = (y_1 : y_2)\), where

\[
Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.
\]

We prefer to consider the associated phase portrait, that is, the singular holomorphic foliation \(\mathcal{F}_0\) induced by this equation on the ruled surface \(\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1\) and also called a Riccati foliation (see [28] or § 8.8). Its singular points are located at the poles of the Riccati equation. Precisely, in the notation (21), the singular points are

\[
\begin{align*}
\ell_i & = \left(-b_i : a_i + \frac{\theta_i}{2}, \frac{a_i - \theta_i}{2} : c_i\right), \\
\ell'_i & = \left(-b_i : a_i - \frac{\theta_i}{2}, \frac{a_i + \theta_i}{2} : c_i\right).
\end{align*}
\]
From the point of view of foliations\(^1\) we say that \(l_i\) and \(l'_i\) have exponents \(\theta_i\) and \(-\theta_i\) respectively. They correspond to the eigenlines of the system with eigenvalues \(-\theta_i/2\) and \(\theta_i/2\) (pay attention to the sign). If \(\theta_i = 0\), then either the singular point of the system is logarithmic and the two singular points of \(F_1\) coincide, or the singular point of the system is apparent and the Riccati foliation is non-singular. In the latter case we shall introduce an additional parabolic structure (see below). The section \(\sigma\) defined by \(y = \infty\) plays the role of a cyclic vector. It has two tangencies with the Riccati foliation, namely, at \(x = q\) and \(x = \infty\) (where \(\sigma\) passes through a singular point of the foliation), see the bottom diagram in Fig. 2.

\[^1\text{For example, } 1/\theta_i\text{ is the Camacho–Sad index of } F_1\text{ along the fibre } x = i\text{ at } l_i\text{ (see [28]).}\]
In this picture the foliation $\mathcal{F}_0$ is regular, transversal to the $\mathbb{P}^1$-fibre over a generic point $x$ and therefore transversely projective (see [29]). Thus it induces a projective structure on the section $\sigma$ that projects to the base $\mathbb{P}^1$. Clearly, the resulting projective structure on (a Zariski-open subset of) $\mathbb{P}^1$ is preserved by birational bundle transformations, and it is natural to look for the simplest birational model.

In the triple $(\mathcal{F}_0, \mathcal{F}_0, \sigma)$ above, the point $x = \infty$ artificially plays a special role since we require in the normalization (22) that the section $\sigma$ passes through the singular point $(x, y) = (\infty, \infty)$ of the foliation. It is more natural to apply an elementary transformation at this point and obtain the following more symmetric picture (see [30]): a Riccati foliation $\mathcal{F}_1$ on the Hirzebruch ruled surface $\mathbb{F}_1$ having singular points over $0, 1, t$ and $\infty$, and the section $\sigma$ now has a simple tangency with the foliation at the point $x = q$. In fact, $\sigma : \mathbb{P}^1 \to \mathbb{F}_1$ is the unique negative section (that is, with self-intersection $-1$). The exponents (eigenvalues) of the foliation are now given by

$$\kappa = (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) := (\theta_0, \theta_1, \theta_t, \theta_\infty - 1).$$

Over each pole $x = i$ of the Riccati equation, the foliation $\mathcal{F}_1$ has one or two singular points depending on the exponent $\kappa_i$; when $q = i$, one of the singular points accidentally lies on $\sigma$ (see the middle diagram in Fig. 2).

We now add a parabolic structure which will be convenient in our further application of elementary transformations. It is a necessary tool in order to get a smooth moduli space when one of the exponents $\kappa_i$ vanishes.

The Okamoto space $\mathcal{M}_t^\Theta$ of initial conditions can be viewed as the moduli space of such Riccati foliations $\mathcal{F}_1$. To be precise, we fix the exponents $\kappa$ and the parameter $t$ and consider the data $(\mathbb{F}_1, \mathcal{F}_1, \sigma, l)$, where

- $\mathbb{F}_1$ is the Hirzebruch ruled surface equipped with the ruling structure $\mathbb{F}_1 \to \mathbb{P}^1$,
- $\mathcal{F}_1$ is a regular Riccati foliation on $\mathbb{F}_1$ which is transversal to the ruling structure outside $x = 0, 1, t, \infty$,
- over each $i = 0, 1, t, \infty$, either the fibre $x = i$ is invariant and there is one singular point for each exponent $\pm \kappa_i$, or $\kappa_i = 0$ and the foliation is regular and transversal to the ruling,
- $l = (l_0, l_1, l_t, l_\infty)$, where either $l_i$ is a singular point with exponent $\kappa_i$ over $x = i$, or $\kappa_i = 0$, $\mathcal{F}_i$ is regular over $x = i$ and $l_i$ is any point of the fibre.

Such data correspond exactly to projectivizations of semistable parabolic connections, which were used in [25] to construct the moduli space $\mathcal{M}_t^\Theta$.

5.3. Riccati foliations on $\mathbb{F}_2$. Putting $y = x(x-1)(x-t)u'/u$ in the scalar equation (13), we obtain the Riccati equation

$$\frac{dy}{dx} = -\frac{y^2}{x(x-1)(x-t)} + \left(\frac{\kappa_0}{x} + \frac{\kappa_1}{x-1} + \frac{\kappa_t}{x-t} + \frac{1}{x-q}\right)y$$

$$- \frac{q(q-1)(q-t)p}{x-q} - q(q-1)p + t(t-1)H - \rho(\kappa_\infty + \rho)(x-t). \quad (34)$$

Recall that $\kappa_0 + \kappa_1 + \kappa_t + \kappa_\infty + 2\rho = 1$ and $H$ is defined by the formula (16). The equation (34) defines a Riccati foliation $\mathcal{F}_2$, which naturally compactifies on the
Hirzebruch surface $\mathbb{F}_2$ given by two charts

$$(x, y) \in (\mathbb{P}^1 \setminus \{\infty\}) \times \mathbb{P}^1 \quad \text{and} \quad (x, \tilde{y}) \in (\mathbb{P}^1 \setminus \{0\}) \times \mathbb{P}^1$$

with transition map $\tilde{y} = y/x^2$. The singular points of the foliation lie on the five fibres:

$$x = 0, 1, t, q \quad \text{and} \quad \infty,$$

with corresponding exponents (up to a sign)

$$\kappa_0, \kappa_1, \kappa_t, \kappa_q = 1 \quad \text{and} \quad \kappa_\infty.$$

To be precise, the singular points in the first chart $(x, y)$ are given by

$$s_0 = (0, 0), \quad s_1 = (1, 0), \quad s_t = (t, 0),$$

$$s'_0 = (0, t\kappa_0), \quad s'_1 = (1, (1-t)\kappa_1), \quad s'_t = (t, t(t-1)\kappa_t),$$

$$s_q = (q, p), \quad s'_q = (q, \infty).$$

Here $s_i$ (resp. $s'_i$) is the singular point with exponent $\kappa_i$ (resp. $-\kappa_i$) and $p = q(q-1)(q-t)p$. At $x = \infty$, the singular points are given in the chart $(x, \tilde{y})$ by the equations

$$\begin{aligned}
  s_\infty &= (\infty, -\rho), \\
  s'_\infty &= (\infty, -\rho - \kappa_\infty).
\end{aligned}$$

The ‘cyclic vector’ is given by the section $\sigma$ defined in these charts by $y = \infty$ and $\tilde{y} = \infty$ respectively.

5.4. Riccati foliations on $\mathbb{F}_1$. After an elementary transformation centred at the nodal singular point $s_q = (q, p)$ of the section $\mathcal{F}_0$ we obtain a Riccati foliation $\mathcal{F}_1$ on the Hirzebruch surface $\mathbb{F}_1$ with poles $x = 0, 1, t, \infty$ and exponents $\kappa_0, \kappa_1, \kappa_t, \kappa_\infty$ respectively, as in §5.2. The apparent singular point has disappeared.

If we define $\mathbb{F}_1$ by the usual charts

$$(x, y) \in (\mathbb{P}^1 \setminus \{\infty\}) \times \mathbb{P}^1 \quad \text{and} \quad (x, \tilde{y}) \in (\mathbb{P}^1 \setminus \{0\}) \times \mathbb{P}^1$$

with transition map $\tilde{y} = y/x$, then the negative section $\sigma$ is given by the formulae $y = \infty$ and $\tilde{y} = \infty$ respectively and the Riccati foliation $\mathcal{F}_1$ is induced by the equation

$$\begin{aligned}
  \frac{dy}{dx} &= \frac{(qy - p)(qy - p + t\kappa_0)}{tqx} \\
  &\quad - \frac{((q-1)y - p)((q-1)y - p + (1-t)\kappa_1)}{(t-1)(q-1)(x-1)} \\
  &\quad + \frac{((q-t)y - p)((q-t)y - p + t(t-1)\kappa_t)}{t(t-1)(q-t)(x-t)} - \rho(\kappa_\infty + \rho).\end{aligned}$$
This equation is obtained from (34) by setting \( y := (x - q)y + p \). The singular points in the first chart \((x, y)\) are now given by

\[
\begin{align*}
  s_0 &= \left(0, \frac{p}{q}\right), \\
  s'_0 &= \left(0, \frac{p - t\kappa_0}{q}\right), \\
  s_1 &= \left(1, \frac{p}{q - 1}\right), \\
  s'_1 &= \left(1, \frac{p - (1 - t)\kappa_1}{q - 1}\right),
\end{align*}
\]

(38)

and the singular points at \( x = \infty \) are given in the chart \((x, \tilde{y})\) by

\[
\begin{align*}
  s_\infty &= (\infty, -\rho), \\
  s'_\infty &= (\infty, -\rho - \kappa_\infty)
\end{align*}
\]

(39)

(here again \( s_i \) has exponent \( \kappa_i \)).

5.5. Riccati foliations on \( \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1 \). Finally, by making the last elementary transformation centred at the singular point \( s'_\infty \) of \( \mathcal{F}_1 \), we obtain a Riccati foliation \( \mathcal{F}_0 \) with eigenvalues

\[
\theta = (\theta_0, \theta_1, \theta_t, \theta_\infty) = (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty + 1).
\]

The underlying ruled surface depends on the relative position of the singular point \( s'_\infty \) of \( \mathcal{F}_1 \) with respect to the negative section \( \sigma_{-1} \): the Riccati foliation \( \mathcal{F}_0 \) is defined on

- \( \mathbb{F}_0 \) when \( s'_\infty \notin \sigma_{-1} \) (generic case),
- \( \mathbb{F}_2 \) when \( s'_\infty \in \sigma_{-1} \) (codimension 1 case).

In particular, \( q = \infty \) in the latter case.

If \( q \neq \infty \), then the Riccati equation defining the foliation \( \mathcal{F}_0 \) on \( \mathbb{F}_0 \) can be obtained from (37) by setting \( y := y - (\kappa_\infty + \rho) \). We obtain the equation

\[
\frac{dy}{dx} = \frac{(qy - p)(qy - p + t\kappa_0)}{tqx} - \frac{((q - 1)(y - \rho - \kappa_\infty) - p)((q - 1)(y - \rho - \kappa_\infty) - p + (1 - t)\kappa_1)}{(t - 1)(q - 1)(x - 1)} + \frac{((q - t)(y - t(\kappa_\infty + \rho) - p)((q - t)(y - t(\kappa_\infty + \rho)) - p + t(t - 1)\kappa_t)}{t(t - 1)(q - t)(x - t)}.
\]

(40)

This is precisely the projectivization of the Fuchsian system in (27). The singular points \( s_i \) with eigenvalues \( \kappa_i \) correspond to the parabolic structure \( l \) defined by the
formulae (28):
\[
\begin{align*}
  s_0 &= \left(0, \frac{p}{q}\right), \\
  s_1 &= \left(1, \frac{p}{q-1} + \rho + \kappa_{\infty}\right), \\
  s_t &= \left(t, \frac{p}{q-t} + (\rho + \kappa_{\infty})t\right), \\
  s_{\infty} &= (\infty, \infty),
\end{align*}
\]
and
\[
\begin{align*}
  s'_0 &= \left(0, \frac{p - t\kappa_0}{q}\right), \\
  s'_1 &= \left(1, \frac{p - (1 - t)\kappa_1}{q-1} + \rho + \kappa_{\infty}\right), \\
  s'_t &= \left(t, \frac{p - t(t - 1)\kappa_t}{q-t} + (\rho + \kappa_{\infty})t\right).
\end{align*}
\]

(41)

5.6. The moduli space $\mathcal{M}_i^\kappa$. With each pair $(p, q) \in \mathbb{C} \times (\mathbb{P}^1 \setminus \{0, 1, t, \infty\})$ we have associated a Riccati foliation $\mathcal{F}_1$ on the Hirzebruch surface $\mathbb{F}_1$ with poles 0, 1, $t$, $\infty$ and exponents $\kappa$.

Conversely, let $\mathcal{F}_1$ be such a Riccati foliation on $\mathbb{F}_1$ with poles 0, 1, $t$ and $\infty$. Then the unique tangency $x = q$ between the negative section $\sigma_{-1}: \mathbb{P}^1 \to \mathbb{F}_1$ and the foliation $\mathcal{F}_1$ uniquely determines $q \in \mathbb{P}^1$. Now, if $q \neq 0, 1, t, \infty$, then we perform an elementary transformation at this tangency point and thus define a Riccati foliation $\mathcal{F}_2$ on $\mathbb{F}_2$. There is a unique section $\sigma_2: \mathbb{P}^1 \to \mathbb{F}_2$ with self-intersection +2 passing through the singular points $s_0$, $s_1$ and $s_t$ of $\mathcal{F}_2$. Choose a chart $(x, y) \in \mathbb{C} \times \mathbb{P}^1$ such that the section $\sigma_2$ and the negative section $\sigma_{-2}$ are given by the equations $y = 0$ and $y = \infty$ respectively. In fact, $y$ is uniquely determined if we normalize the $y^2$-coefficient of the Riccati equation as in (34). We then observe that all singular points $s_i$ and $s'_i$ for $i = 0, 1, t, \infty$ depend only on the parameters $\kappa$ and $t$; the singular point $s_q$ given by $(x, y) = (q, p)$ is the only one depending on the particular foliation $\mathcal{F}_1$. Define $p$ by the formula $p = q(q-1)(q-t)p$. We note that the Riccati foliation $\mathcal{F}_1$ is characterized by the position of the nodal singular point of $\mathcal{F}_2$ on the Hirzebruch surface $\mathbb{F}_2$ (after the normalization above).

Theorem 11 [25]. Fix the parameters
\[ \kappa = (\kappa_0, \kappa_1, \kappa_t, \kappa_{\infty}) \]
and consider Riccati foliations on the Hirzebruch surface $\mathbb{F}_1$ having at most simple poles at 0, 1, $t$ and $\infty$, with exponents $\pm \kappa_i$ over $x = i$ such that the negative section $\sigma_{-1}$ of $\mathbb{F}_1$ is not $\mathcal{F}$-invariant (the semistability condition). Add moreover a parabolic structure $l = (l_0, l_1, l_t, l_{\infty})$ which consists of the following data over each point $i = 0, 1, t, \infty$:

- either of the singular point $l_i \in \mathbb{F}_1$ with exponent $\kappa_i$;
- or, when $\kappa_i = 0$ and there is actually no singular point, of any point of the fibre $x = i$ (the only case when the parabolic structure is relevant, that is, not determined by the foliation itself).
The moduli space $\mathcal{M}_t^{\kappa}$ of such pairs $(\mathcal{F}, \mathcal{I})$ up to bundle automorphisms is a quasi-projective rational surface that can be described as follows. Start with the Hirzebruch surface $\mathbb{F}_2$ and write $Z_t$ for the reduced divisor consisting of the four fibres $x = i$ together with the negative section $\sigma_{-2}$. Define eight points $s_i$ and $s'_i$ by the formulae

\[
\begin{aligned}
{s_0} &= (0, 0), & {s_1} &= (1, 0), & {s_t} &= (t, 0), \\
{s'_0} &= (0, t\kappa_0), & {s'_1} &= (1, (1-t)\kappa_1), & {s'_t} &= (t, t(t-1)\kappa_t)
\end{aligned}
\]

in the first chart $(x, y)$ and

\[
\begin{aligned}
{s_\infty} &= (\infty, -\rho), \\
{s'_\infty} &= (\infty, -\rho - \kappa_\infty)
\end{aligned}
\]

in the chart $(x, \bar{y} = y/x^2)$. Carry out the following operations for each $i = 0, 1, t, \infty$.

- If $\kappa_i \neq 0$, then blow up the two points $s_i$ and $s'_i$.
- If $\kappa_i = 0$, then successively blow up the point $s_i = s'_i$ and the point where the exceptional divisor meets the strict transform of the fibre $x = i$.

Denote the resulting eight-point blow-up of $\mathbb{F}_2$ by $\overline{\mathcal{M}}_t^{\kappa}$. Then the moduli space $\mathcal{M}_t^{\kappa}$ is the complement in $\overline{\mathcal{M}}_t^{\kappa}$ of the strict transform of the divisor $Z_t$.

The eight points $s_i$ and $s'_i$ in this theorem are nothing but the singular points of the foliation on $\mathbb{F}_2$ described in §5.3. When the nodal singular point $(x, y) = (q, p)$ tends to $s_i$ (resp. $s'_i$), the singular point $s'_i$ (resp. $s_i$) of the corresponding foliation on $\mathbb{F}_1$ (see §5.4) tends to the negative section $\sigma_{-1}$. The exact limit depends on the way in which the nodal point tends to $s_i$ or $s'_i$. If $\kappa_i \neq 0$, then the fibre of $\mathcal{M}_t^{\kappa}$ over $x = i$ consists of two disjoint copies $S_i$ and $S'_i$ of the affine line $\mathbb{C}$ which contain $s_i$ and $s'_i$ respectively: they stand for the moduli spaces of those foliations $\mathcal{F}$ whose singular points $s'_i$ (resp. $s_i$) lie on the negative section $\sigma_{-1}$. If $\kappa_i = 0$, then the fibre of $\mathcal{M}_t^{\kappa}$ over $x = i$ is the union of an affine line $S_i \simeq \mathbb{F}$ and a projective line $S'_i \simeq \mathbb{P}^1$ that intersect each other transversally at one point and are projected to the point $s_i = s'_i$. The component $S_i$ consists of those foliations whose point $s_i$ lies on the negative section $\sigma_{-1}$. The compact component $S'_i$ consists of those parabolic foliations $(\mathcal{F}, \mathcal{I})$ which actually have no pole over $x = i$. If we neglect the parabolic structure, then the rational curve $S'_i$ blows down to a quadratic singular point.

Sketch of the proof of Theorem 11. Let $\mathcal{F}$ be a Riccati foliation on $\mathbb{F}_1$ having at most simple poles over $x = 0, 1, t, \infty$ such that the negative section $\sigma_{-1}$ is not invariant (semistability). In the standard chart $(x, y)$ the foliation is given by

\[
\frac{dy}{dx} = \frac{(a_1x + a_0)y^2 + (b_2x^2 + b_1x + b_0)y + (c_3x^3 + c_2x^2 + c_1x + c_0)}{x(x-1)(x-t)},
\]

where $a_k, b_k, c_k \in \mathbb{C}$ and the coefficients $a_0$ and $a_1$ do not vanish simultaneously. Bundle automorphisms are given by changes of coordinates of the form $y := ay + bx + c$, $a, b, c \in \mathbb{C}$, $a \neq 0$. The single tangency between $\mathcal{F}$ and the negative
section $\sigma_{-1}$ is given by $x = q := -a_0/a_1$. By making the change of coordinates $y := ay$, one can achieve the following normalization:

$$
either a_1x + a_0 = x - q, \quad \text{or} \quad a_1x + a_0 = \tilde{q}x - 1,$$

where $\tilde{q} = 1/q$. We first assume that $q \neq \infty$, whence we can assume that $a_1x + a_0 = x - q$. Using a change of coordinates of the form $y := ay$, one can achieve the following normalization:

$$
either a_1x + a_0 = x - q, \quad \text{or} \quad a_1x + a_0 = 1/q x - 1,$$

where $1/q = \frac{1}{q}$. We first assume that $q \neq \infty$, whence we can assume that $a_1x + a_0 = x - q$. Using a change of coordinates of the form $y := y + bx + c$, we can further assume that $b_1 = b_2 = 0$. Incidentally, assuming that $q \neq \infty$, we obtain a unique normal form

$$\frac{dy}{dx} = (x - q)y^2 + b_0y + c_0 \frac{dx}{x} + \frac{c_1}{1 - t}(x - 1) + \frac{c_t}{t(t - 1)(x - t)} + c_\infty. \quad (45)$$

It follows from (37) that

$$b_0 = -2p + (q - 1)(q - t)\kappa_0 + q(q - t)\kappa_1 + q(q - 1)\kappa_t.$$

The residue at $x = 0$ is found from the formula

$$dy = -qy^2 + b_0y + c_0 \frac{dx}{x} + (\text{holomorphic at } x = 0), \quad (46)$$

and the exponents $\pm \kappa_0$ are determined by the discriminant

$$\Delta_0 := \frac{b_0^2 + 4qc_0}{t^2} = \kappa_0^2.$$

We similarly get

$$\Delta_1 := \frac{b_0^2 + 4(q - 1)c_1}{(t - 1)^2} = \kappa_1^2, \quad \Delta_t := \frac{b_0^2 + 4(q - t)c_t}{t^2(t - 1)^2} = \kappa_t^2,$$

$$\Delta_\infty := 1 - 4c_\infty = \kappa_\infty^2.$$ 

Once the parameter $\kappa$ is fixed, one can uniquely determine $c_0$, $c_1$, $c_t$ and $c_\infty$ as functions of $\kappa$, $q$ and $b_0$ (that is, $p$) provided that $q \neq 0, 1, t, \infty$. In a neighbourhood of $q = 0$ one can still express $c_1$, $c_t$ and $c_\infty$ as functions of $\kappa$, $q$ and $b_0$, and the moduli space of such foliations is locally isomorphic to the surface

$$\{(q, b_0, c_0) \in \mathbb{C}^3, b_0^2 + 2qc_0 = (t\kappa_0)^2\}.$$

We promptly see that the moduli space consists, over $q = 0$, of an affine line parametrized by $c_0$ over each of the points $b_0 = \pm t\kappa_0$.

If $\kappa_0 \neq 0$, then the graph

$$c_0 = -\frac{(b_0 - t\kappa_0)(b_0 + t\kappa_0)}{4q}.$$
is clearly obtained by blowing up the two points and then deleting the level \( c_0 = \infty \), that is, the strict transform of \( q = 0 \). The formula (46) for the residue shows that the following conditions are equivalent over \( q = 0 \):

- \( b_0 = t\kappa_0 \) (resp. \( b_0 = -t\kappa_0 \)),
- \( p = 0 \) (resp. \( p = t\kappa_0 \)),
- the singular point \( s'_0 \) (resp. \( s_0 \)) of the corresponding foliation \( F_1 \) lies on the negative section \( y = \infty \).

If \( \kappa_0 = 0 \), then the graph

\[
c_0 = -\frac{b^2_0}{4q}
\]

is obtained after two blow-ups, and the affine line parametrized by \( c_0 \) is the set of all foliations \( F_1 \) whose singular point \( s_0 = s'_0 \) lies on the negative section \( y = \infty \). The surface with equation \( b^2_0 + 4qc_0 = 0 \) has a singular point at \((q, b_0, c_0) = (0, 0, 0)\). This corresponds to the case when the residue (46) vanishes, that is, \( F_1 \) actually has no singular point at \( x = 0 \). The parabolic data at \( x = 0 \) provide a desingularization of the surface. Indeed, the moduli space of pairs \((F_1, I)\) is locally parametrized in a neighbourhood of \( q = 0 \) by the set

\[
\{(q, b_0, c_0, s_0) \in \mathbb{C}^3 \times \mathbb{P}^1, \ b^2_0 + 2qc_0 = 0, \ 2qs_0 = b_0 \}
\]

(or it would be better to write \( 2qu_0 = b_0v_0 \), where \((u_0 : v_0) = s_0\)). The parabolic data \( s_0 = b_0/(2q) \) parametrize the exceptional divisor \( S'_0 \) (thus realizing the blow-up). The intersection \( S_0 \cap S'_0 \) is given by \((q, b_0, c_0, s_0) = (0, 0, 0, \infty)\) and describes the foliation without singular points at \( x = 0 \) whose parabolic structure lies on the negative section \( y = \infty \).

The study of \( q = 1, t, \infty \) is similar except that one must choose another normalization for \( q = \infty \):

\[
\frac{dy}{dx} = \frac{(\tilde{q}x - 1)y^2 + \tilde{b}_2x^2y}{x(x - 1)(x - t)} + \frac{\tilde{c}_0}{x} + \frac{\tilde{c}_1}{x - 1} + \frac{\tilde{c}_t}{x - t} + \tilde{c}_\infty,
\]

where \((\tilde{q}, \tilde{b}_2) = (1/q, b_0/q^2)\) is the other chart of \( \mathbb{F}_2 \).

The deformation \( t \mapsto \mathcal{M}_t^\kappa \) is analytically (but not algebraically) trivial and the trivialization is given by the Painlevé flow (see [31], [25]). The good phase space for the Painlevé VI equation is the (locally analytically trivial) fibration

\[
t: \mathcal{M}_t^\kappa \to \mathbb{P}^1 \setminus \{0, 1, \infty\}.
\]

The map \( q: \mathcal{M}_t^\kappa \to \mathbb{P}^1 \) is regular (when none of the \( \kappa_i \) vanish) and endows \( \mathcal{M}_t^\kappa \) (for \( t \) fixed) with the structure of an affine \( \mathbb{A}^1 \)-bundle with double fibres over \( 0, 1, t, \infty \). Finally, we note that the formula (37) defines an explicit section (universal Riccati foliation)

\[
(t, q, p) \mapsto (\mathbb{F}_1, F_1)
\]

over \( \mathcal{M}_t^\kappa \setminus \{q = 0, 1, t, \infty\} \).
5.7. The moduli space $\mathcal{M}_t^\theta$ of sl(2, $\mathbb{C}$)-connections. By §5.5 we have an isomorphism

$$\text{elm}_{s'_\infty} : \mathcal{M}_t^\kappa \overset{\sim}{\longrightarrow} \mathcal{M}_t^\theta$$

from the previous moduli space to the moduli space $\mathcal{M}_t^\theta$ of sl(2, $\mathbb{C}$)-connections $(E, \nabla)$ with exponents

$$\theta = (\theta_0, \theta_1, \theta_t, \theta_{\infty}) := (\kappa_0, \kappa_1, \kappa_t, \kappa_{\infty} + 1).$$

We recall that for every point of this moduli space the underlying vector bundle $E$ is either trivial or equal to $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. It is actually trivial on a Zariski-open subset of $\mathcal{M}^\theta$. The locus of the non-trivial bundles is Malgrange’s theta divisor $\Theta$ defined for fixed $t$ by the exceptional divisor $S_\infty$ arising from the blow-up of $s_\infty$. Indeed, the points of the last divisor correspond to those foliations $F_1$ for which $s'_\infty$ lies on the negative section $\sigma_{-1}$; applying $\text{elm}_{s'_\infty}$ gives a foliation $F_0$ on the Hirzebruch surface $F_2 = \mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(1))$. In fact we have $F_0 = F_2$ in this case: we are back to the foliation of §5.3, where $s_q \rightarrow s_\infty$. To be precise, putting

$$p = \frac{\lambda - \rho q}{(q - 1)(q - t)}$$

in (13) and letting $q$ tend to $\infty$, we obtain the Heun equation

$$g(x) = \frac{\rho(\rho + \kappa_{\infty} + 1)x - \lambda \kappa_{\infty} - \rho((t + 1)\rho + \kappa_1 + t\kappa_t)}{x(x - 1)(x - t)}.$$

For our special choice of the parameters $\theta = (1/2, 1/2, 1/2, 1/2 + \vartheta/2)$ we obtain the Lamé equation (1) with $n = \vartheta/2$ and $c = 2\lambda(\vartheta - 1) + (t + 1)\vartheta/2(\vartheta/2 - 1)$.

Finally, the open set $\mathcal{M}^\theta \setminus \Theta$ (resp. the closed subset $\Theta$) may be viewed as the moduli space of Riccati foliations $F_0$ on $F_0$ (resp. $F_2$) with simple poles $(0, 1, t, \infty)$ and exponents $(\theta_0, \theta_1, \theta_t, \theta_{\infty})$, excluding those foliations for which the section passing through $s_\infty$ (resp. the exceptional section) is totally $F_0$-invariant.

5.8. Okamoto symmetries. There are many birational transformations

$$(\kappa, t, p, q) \mapsto (\bar{\kappa}, \bar{t}, \bar{p}, \bar{q})$$

that induce biregular diffeomorphisms between moduli spaces

$$\mathcal{M}^\kappa \rightarrow \mathcal{M}^{\bar{\kappa}},$$

equivariant with respect to the projection $t$. They were studied in [32]. Some of them are the classical Schlesinger transformations arising from geometrical transformations of connections (resp. Riccati foliations); together with a strange extra symmetry, they generate the full group of Okamoto symmetries.

5.8.1. Change of signs. First, we can change the ‘spin structure’, that is, the signs of the parameters $\pm \kappa_i$. This does not change the Riccati foliation or the coefficients
$t$, $q$, $b_0$ of the normal form (45), but the variable $p$ is modified as follows:

$$
\begin{align*}
\begin{cases}
(q, p) &\mapsto (q, p - \frac{\kappa_0}{q}), \\
(q, p) &\mapsto (q, p - \frac{\kappa_1}{q-1}), \\
(q, p) &\mapsto (q, p - \frac{\kappa_t}{q-t}), \\
(q, p) &\mapsto (q, p) 
\end{cases}
\end{align*}
$$

5.8.2. Permutation of poles. One can permute the four poles of the Riccati foliation. The resulting group of order 24 is generated by the elements

$$(01): \begin{cases}
(k_0, k_1, k_t, k_\infty) &\mapsto (k_1, k_0, k_t, k_\infty), \\
t &\mapsto 1-t, \\
(q, p) &\mapsto (1-q, -p),
\end{cases}$$

$$(1t): \begin{cases}
(k_0, k_1, k_t, k_\infty) &\mapsto (k_0, k_t, k_1, k_\infty), \\
t &\mapsto \frac{1}{t}, \\
(q, p) &\mapsto \left(\frac{q}{t}, tp\right),
\end{cases}$$

$$(0\infty)(1t): \begin{cases}
(k_0, k_1, k_t, k_\infty) &\mapsto (k_\infty, k_t, k_1, k_0), \\
t &\mapsto t, \\
(q, p) &\mapsto \left(\frac{t}{q}, \frac{t(qp + \rho)}{q}\right)
\end{cases}$$

(the change of variable is given by $\tilde{x} = 1 - x$, $x/t$ and $t/x$ respectively; we get $\tilde{b}_0 = b_0$, $b_0/t^2$ and $(t(q-1)(q-t) - tb_0)/q^2$ respectively).

Together with sign changes, we already get a linear group of order 384 acting on our moduli space.

5.8.3. Elementary transformations. Let $\mathcal{F}$ be a Riccati foliation on $\mathbb{P}_1$ representing a point of $\mathcal{M}_t^\kappa$. An elementary transformation centred at one of the singular points (say, $s_0$) of $\mathcal{F}$ gives a new Riccati foliation with simple poles over 0, 1, $t$, $\infty$ and with the shifted parameters $\tilde{\kappa} = (\kappa_0 - 1, \kappa_1, k_t, k_\infty)$. The resulting bundle is either
the trivial bundle $\mathbb{F}_0$, or the Hirzebruch surface $\mathbb{F}_2$. However, after two (or, more generally, an even number of) such elementary transformations we are back to $\mathbb{F}_1$.

Indeed, since the type $n$ of the Hirzebruch surface $\mathbb{F}_n$ shifts by $\pm 1$ under each elementary transformation, we must only exclude the possibility that, for example, $(\mathbb{F}_1, \mathcal{F}) \rightarrow (\mathbb{F}_2, \mathcal{F}') \rightarrow (\mathbb{F}_3, \mathcal{F}'')$.

This would mean that each of the two successive elementary transformations has centre on the negative section. In this case, the negative section $\sigma_{-3}$ of $\mathbb{F}_3$ is the strict transform of $\sigma_{-1}$. Since $\sigma_{-1}$ is not $\mathcal{F}$-invariant, we obtain that $\sigma_{-3}$ is not $\mathcal{F}''$-invariant. But then Proposition 26 (see §8.8) gives a negative tangency, a contradiction. Thus we have defined a biregular transformation

\[ \text{elm}_{s_{\infty}} \circ \text{elm}_{s_0} : \begin{cases} (\kappa_0, \kappa_1, \kappa_t, \kappa_{\infty}) \mapsto (\kappa_0 - 1, \kappa_1, \kappa_t, \kappa_{\infty} - 1), \\ t \mapsto t, \\ (q, p) \mapsto (\tilde{q}, \tilde{p}). \end{cases} \]

Omitting the huge formula, we note that $\tilde{q}$ is given by the unique tangency point between $\mathcal{F}$ and the only section $\sigma$ of $\mathbb{F}_1$ that has self-intersection $+1$ and passes through $s_0$ and $s_{\infty}$.

More generally, given any quadruple

\[ n = (n_0, n_1, n_t, n_{\infty}) \in \mathbb{Z}^4, \quad n = n_0 + n_1 + n_t + n_{\infty} \in 2\mathbb{Z}, \]

we construct a biregular transformation

\[ \text{elm}^n_{t} : \begin{cases} (\kappa_0, \kappa_1, \kappa_t, \kappa_{\infty}) \mapsto (\kappa_0 - n_0, \kappa_1 - n_1, \kappa_t - n_t, \kappa_{\infty} - n_{\infty}), \\ t \mapsto t, \\ (q, p) \mapsto (\tilde{q}, \tilde{p}), \end{cases} \]

where

\[ \text{elm}^n_{t} = \text{elm}^{n_0}_{s_{0}} \circ \text{elm}^{n_1}_{s_{1}} \circ \text{elm}^{n_t}_{s_{t}} \circ \text{elm}^{n_{\infty}}_{s_{\infty}} \]

with the convention that $\text{elm}^{-n_i}_{s_i} := \text{elm}^{-n_i}_{s_i}$ when $n_i < 0$. As above, there is a unique section $\sigma$ with self-intersection $\sigma \cdot \sigma = n - 1$ and tangency of multiplicity $n_i$ with the foliation at each point $s_i$; the extra tangency between $\mathcal{F}$ and $\sigma$ is at $x = \tilde{q}$.

We now get an infinite affine group of transformations. We denote it by $H$.

5.8.4. *The Okamoto symmetry.* To generate the full group $G$ of biregular transformations described in [32], we need an extra symmetry,

\[ \begin{cases} (\kappa_0, \kappa_1, \kappa_t, \kappa_{\infty}) \mapsto (\kappa_0 + \rho, \kappa_1 + \rho, \kappa_t + \rho, \kappa_{\infty} + \rho), \\ t \mapsto t, \\ (q, p) \mapsto (q + \frac{\rho}{p}, p), \end{cases} \]

(called $s_2$ in [19]) or any of its conjugates. So far there is no geometric interpretation of this symmetry as long as we interpret $\mathcal{M}^\kappa$ as the moduli space of rank-2
connections (or Riccati foliations). To derive the full Okamoto group from natural transformations of connections, one has to deal with isomonodromic deformations of connections of rank 3 (see [26]) or more (see [19]).

The conjugate of the Okamoto symmetry above by the change of signs $(+, +, +, -)$ is equal to $s_2 s_1 s_2$ in the notation of [19] and is given by

$$
\begin{cases}
(k_0, k_1, k_t, k_\infty) \mapsto (k_0 + \rho + k_\infty, k_1 + \rho + k_\infty, k_t + \rho + k_\infty, -\rho), \\
t \mapsto t, \\
(q, p) \mapsto \left(q + \frac{\rho + k_\infty}{p}, p\right).
\end{cases}
$$

We recognize in $\tilde{q} = q + (\rho + k_\infty)/p$ the Tu invariant of the underlying vector bundle of the elliptic pullback (see Corollary 10).

5.9. Special configurations. For special values of the parameter $\kappa$ the moduli space $M^\kappa_t$ contains complete rational curves (independently of $t$). They arise from the curves on $\mathbb{F}_2$ avoiding the negative section and passing through $s_i$ or $s_i'$ for each value of $i = 0, 1, t, \infty$. They correspond either to the locus of reducible connections or to the locus of connections with an apparent singular point. It turns out that there are no other complete curves in $M^\kappa_t$ (see [25]). In particular, there are no such curves for generic values of $\kappa$. Here are some examples.

Suppose that $\kappa_t = 1$ and the foliation $\mathcal{F}$ has an apparent singular point at $x = t$. Then the pole disappears after one elementary transformation centred at $s_t$. We thus obtain a Riccati foliation $\mathcal{F}_0$ of hypergeometric type on $\mathbb{F}_0$ (it cannot be $\mathbb{F}_1$ by Proposition 26), that is, a foliation with poles at $0, 1, \infty$ and exponents $(\kappa_0, \kappa_1, \kappa_\infty)$. Conversely, a foliation $\mathcal{F}$ as above can be recovered from $\mathcal{F}_0$ by an elementary transformation at any point of the fibre $x = t$. This gives us a rational family of foliations $\mathcal{F}$ (parametrized by the fibre $x = t$). The corresponding rational curve $C$ in the moduli space $M^\kappa_t$ is given by the equation

$$
q(q - 1)p^2 - ((q - 1)\kappa_0 + q\kappa_1)p + \frac{(\kappa_0 + \kappa_1)^2 - \kappa_\infty^2}{4}.
$$

This is the only curve $C$ on $\mathbb{F}_2$ with the following properties:

- $q: C \to \mathbb{P}^1$ has degree 2;
- $C$ does not intersect the negative section $\sigma_{-2}$;
- $C$ intersects the fibre $q = i$ at both $s_i$ and $s_i'$ for $i = 0, 1, \infty$;
- $C$ intersects the fibre $q = t$ twice at $s_t$;
- $C$ is singular at $s_t$: it has two smooth branches.

**Proof.** To compute this family in the moduli space, we start with the Riccati foliation $\mathcal{F}_0$ that can be normalized to

$$
\frac{dy}{dx} = -y^2 - (\kappa_0 + x\kappa_\infty)y + cx, \quad c = \frac{\kappa_1^2 - (\kappa_0 + \kappa_\infty)^2}{4}
$$

(we exclude some reducible cases here) and choose a parabolic structure $s_t = (t, \lambda)$ over $x = t$. The horizontal section $y = \lambda$ is sent by the elementary transformation $\text{elm}_{s_t}: \mathbb{F}_0 \to \mathbb{F}_1$ to the negative section: the corresponding value
$q = (\lambda^2 + \kappa_0 \lambda)/(c - \kappa_\infty \lambda)$ corresponds to the unique tangency point between $F_0$ and $y = \infty$. We have already noted that the map $\lambda \mapsto q(\lambda)$ has degree 2. We now claim that the foliation $F = \text{elm}_{s'_t} F_0$ determines a point over $s_0$ (resp. $s'_0$) in the moduli space $M^\Theta_t$ if and only if the parabolic structure $s_t = (t, \lambda)$ and the singular point $s'_0$ (resp. $s_0$) of $F_0$ lie on the same horizontal section, namely, on $y = \lambda$. Indeed, this section is transformed by $\text{elm}_{s_t}$ into the negative section of $\mathbb{F}_1$. Thus the curve $C$ passes once through each of the points $s_0$, $s'_0$. The same holds over $q = 1$ and $\infty$. Finally, when $s_t$ is an apparent singular point for $F_1$, it cannot lie on the negative section $\sigma_{-1}$ (otherwise an application of $\text{elm}_{s_t}$ would give a hypergeometric foliation on $\mathbb{F}_2$, having by Proposition 26 a tangency of multiplicity $-1$ with the negative section, a contradiction). We can now compute the equation of the curve $C$. Since $C$ has degree 2 and does not intersect the negative section of $\mathbb{F}_2$, the equation of $C$ takes the form $P^2 + A(q)P + B(q)$, where $A$ and $B$ are polynomials of degrees 2 and 4 respectively. The condition that $C$ passes through all the points $s_i$ and $s'_i$, except $s'_t$, determines not only $C$, but a pencil of curves; they are all smooth and vertical at $s_t$ (and thus escape from $M^\Theta_t$ over this point), except one of them which has a singularity with normal crossing at $s_t$. $\square$

In the case when $\kappa_t = 2$, the foliations $F$ with apparent singular points at $x = t$ are obtained as follows. Take the hypergeometric foliation $F_0$ on $\mathbb{F}_1$ with exponents $(\kappa_0, \kappa_1, \kappa_\infty)$, choose a parabolic structure $s_t$ at $x = 0$ and apply $\text{elm}_{s_t}$ twice. We again get a rational curve in the moduli space which projects down to a curve $C \subset \mathbb{F}_2$ with the following properties:

- $q: C \to \mathbb{P}^1$ has degree 4;
- $C$ does not intersect the negative section $\sigma_{-2}$;
- $C$ intersects the fibre $q = i$ at both $s_i$ and $s'_i$ for $i = 0, 1, \infty$;
- $C$ intersects the fibre $q = t$ three times at $s_t$ and once at $s'_t$.

5.10. Bolibrukh–Heu transversality. A remarkable result of Bolibrukh ([9], §5.2, Proposition 5.6) asserts in our context that the isomonodromic deformation $t \mapsto (E_t, \nabla_t) \in \mathcal{M}_t^\Theta$ of an irreducible $\text{sl}(2, \mathbb{C})$-connection is ‘mostly’ defined on the trivial bundle.

**Theorem 12** (Bolibrukh). Let $t \mapsto (E_t, \nabla_t) \in \mathcal{M}_t^\Theta$ be a local isomonodromic deformation. Then one of the following cases holds.

- The underlying bundle $E_t$ is trivial outside a discrete subset of the parameter space $T$.
- $E_t \equiv \mathcal{O}(-1) \oplus \mathcal{O}(1)$ and the destabilizing subsheaf $\mathcal{O}(1)$ is $\nabla_t$-invariant.

In particular, the former case holds when $(E_t, \nabla_t)$ is irreducible. Bolibrukh proved a more general result for certain logarithmic connections of arbitrary rank on the Riemann sphere. The rank 2 case was considered by Heu [15] in full generality (for regular or irregular $\text{sl}(2, \mathbb{C})$-connections on an arbitrary Riemann surface). We will use the following corollary.

**Proposition 13.** Let $t \mapsto (E_t, \nabla_t) \in \mathcal{M}_t^\Theta$ be a local isomonodromic deformation of an $\text{sl}(2, \mathbb{C})$-connection. Assume that $E_t$ is trivial and two eigenlines $l_i$ and $l_j$, $i, j \in \{0, 1, t, \infty\}$, coincide along the whole deformation. Then the connection
\((E_t, \nabla_t)\) is reducible: the constant line bundle \(L_t \subset E_t\) determined by the lines \(l_i = l_j\) is \(\nabla_t\)-invariant.

Proof. Applying \(\text{elm}_s \circ \text{elm}_s\) to the deformation, we obtain an isomonodromic deformation on the Hirzebruch surface \(\mathbb{F}_2\). By Theorem 12, this is possible only when the connection is reducible. \(\square\)

Remark 14. The following more general result holds. Consider an irreducible Riccati foliation \(\mathcal{F}_0\) in \(\mathcal{M}_t^0\) defined on the trivial bundle \(\mathbb{P}^1 \times \mathbb{P}^1\). The sections \(\sigma_d\) with self-intersection \(d \geq 0\) \((d\ \text{even})\) form a \((d+1)\)-dimensional family. The smooth curve \(\sigma_d\) has exactly \(d + 2\) tangencies with \(\mathcal{F}_0\) counting multiplicities. Then the tangency locus can be completely contained in the fibres over \(\{0, 1, t, \infty\}\) only for isolated points of the parameter space \(T\). When \(d = 0\), this yields the previous proposition. When \(d = 2\), we obtain, for example, that all four parabolics cannot lie on a curve of bidegree \((1, 1)\) along the deformation.

\section*{§ 6. Lamé connections}

The aim of this section is to give a rough description of the moduli space of Lamé connections up to biregular bundle transformations, using the Riemann–Hilbert correspondence. Here we fix an elliptic curve

\[ X : \{y^2 = x(x-1)(x-t)\}. \]

We will say which Lamé connections are invariant under the elliptic involution

\[ \sigma : X \to X; \quad (x, y) \mapsto (x, -y). \]

In § 7 we will see that \(\sigma\)-invariant Lamé connections can be pushed down by the double covering

\[ \pi : X \to \mathbb{P}^1; \quad (x, y) \mapsto x \]

to logarithmic connections with poles at the four ramification points \(i = 0, 1, t, \infty\).

Let \((E, \nabla)\) be a Lamé connection on the elliptic curve \(X\), thus having a simple pole at \(\omega_\infty\). When the exponent \(\vartheta\) is not an integer, the connection can be reduced to the following matrix form:

\[ \nabla : W \mapsto dW - \Omega W, \quad \Omega = \begin{pmatrix} \frac{\vartheta}{2} \frac{dz}{z} & 0 \\ 0 & -\frac{\vartheta}{2} \frac{dz}{z} \end{pmatrix}, \]

where \(z \in (\mathbb{C}, 0)\) is any local coordinate of \(X\) at \(\omega_\infty\), and \(W \in \mathbb{C}^2\) is a convenient local holomorphic trivialization of \(E\). On the other hand, if \(\vartheta \in \mathbb{Z}\) \((\text{say, } \vartheta = n \in \mathbb{Z}_{\leq 0})\), then the pole is said to be resonant and the matrix form can be reduced (by a local gauge transformation as above) to

\[ \text{either } \Omega = \begin{pmatrix} n/2 & 0 \\ 0 & -n/2 \end{pmatrix} \frac{dz}{z} \quad \text{or } \Omega = \begin{pmatrix} n/2 & z^n/2 \\ 0 & -n/2 \end{pmatrix} \frac{dz}{z}. \quad (47) \]
The point $\omega_{\infty}$ is said to be an apparent singular point for $\nabla$ (actually regular when $n = 0$) in the former case, and a logarithmic singular point in the latter.

The connection $\nabla$ is regular on the affine part $X^* = X \setminus \{\omega_{\infty}\}$ of $X$ and inherits a monodromy representation

$$\rho: \pi_1(X^*) \to \text{SL}(2, \mathbb{C}),$$

which is well defined by $(E, \nabla)$ up to $\text{SL}(2, \mathbb{C})$-conjugacy. We fix a loop $\delta \in \pi_1(X^*)$ which goes to $\omega_{\infty}$, turns around once, and comes back to the initial point. Then the matrix $\rho(\delta)$ is called the local monodromy of $(E, \nabla)$ around $\omega_{\infty}$. It is conjugate to

$$\begin{pmatrix} e^{i\vartheta} & 0 \\ 0 & e^{-i\vartheta} \end{pmatrix} \quad \text{(or $\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in the logarithmic case).}$$

All of this is obviously independent of the choice of the base point for the fundamental group. We note that the singular point $\omega_{\infty}$ is apparent if and only if the local monodromy $\rho(\delta)$ is equal to $\pm I$, that is, it belongs to the centre of $\text{SL}(2, \mathbb{C})$; this can occur only when $\vartheta \in \mathbb{Z}^*$. 

6.1. The Riemann–Hilbert correspondence. For every exponent $\vartheta \in \mathbb{C}$ we have an analytic map

$$\text{RH}: \begin{cases} \text{Lamé connections over } X \\ \text{with exponent } \vartheta \end{cases} / \sim \to \begin{cases} \rho: \pi_1(X^*) \to \text{SL}(2, \mathbb{C}), \\ \text{tr}(\rho(\delta)) = 2 \cos(\pi \vartheta) \end{cases} / \sim,$$

$$(E, \nabla) \mapsto \rho,$$

which sends every Lamé connection (considered up to holomorphic bundle isomorphisms) to its monodromy representation (considered up to $\text{SL}(2, \mathbb{C})$-conjugacy). This map is almost one-to-one: it is surjective, and its restriction to the set of connections without apparent singular points (that is, connections with $\rho(\delta) \neq \pm I$) is injective.

If $(E, \nabla)$ has an apparent singular point at $\omega_{\infty}$, where $\vartheta = n \in \mathbb{Z}_{>0}$, then all horizontal sections have meromorphic extension to $\omega_{\infty}$. Holomorphic sections of this kind are contained in a line subbundle (say, $L$) of $E$ near $\omega_{\infty}$. In terms of the normal form (47), $L$ is the constant line bundle generated by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Except in very special cases, $L$ cannot be extended to a line bundle $L \subset E$ on the whole of $X$: it is only defined in a neighbourhood of $\omega_{\infty}$. The fibre of $L$ at $\omega_{\infty}$ coincides with the eigenline of the residual matrix associated with the positive eigenvalue $n/2$. By the Riemann–Hilbert correspondence over the punctured curve $X^*$, any two Lamé connections with the same monodromy representation are conjugate to each other by a gauge transformation over $X^*$. This conjugacy extends to a global gauge transformation if and only if it conjugates the corresponding local line bundles as defined above. One can restore the injectivity of the Riemann–Hilbert map in the following way. Regard the monodromy representation $\rho$ as an action of the fundamental group $\pi_1(X^*, p)$ on the space $E_p \simeq \mathbb{C}^2$ of germs of solutions at the base point $p$. Then analytic continuation of local holomorphic solutions at $\omega_{\infty}$ to the base point $p$. 


along (half-)δ yields a one-dimensional subspace \( L_p \subset E_p \). In other words, \( L_p \) is obtained by analytic continuation of \( L \) (as a \( \nabla \)-invariant line bundle) along \( \delta \). The Lamé connection (with an apparent singular point) is characterized by the pair

\[
(\rho, L_p) \in \text{Hom}(\pi_1(X^*, p), SL(E_p)) \times \mathbb{P}(E_p)
\]

up to conjugacy:

\[
(\rho, L_p) \sim (M^{-1} \rho M, M^{-1} L_p), \quad M \in SL(2, \mathbb{C}).
\]

This is a kind of parabolic structure for the space of representations.

### 6.2. The Fricke moduli space.

We recall how to describe the moduli space of representations following Fricke (see [2]). Fix standard generators \( \alpha, \beta \in \pi_1(X^*) \) of the fundamental group in such a way that their commutator \( \delta := [\alpha, \beta] = \alpha \beta \alpha^{-1} \beta^{-1} \) represents a small loop turning once around the puncture, as before. We neglect the base point since it plays no role in our discussion. A representation \( \rho \) is determined by the images of generators,

\[
A := \rho(\alpha) \quad \text{and} \quad B := \rho(\beta).
\]

The ring of polynomial functions on \( SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \) that are invariant under \( SL(2, \mathbb{C}) \)-conjugacy, is generated by the functions

\[
a := \text{tr}(A), \quad b := \text{tr}(B) \quad \text{and} \quad c := \text{tr}(AB).
\]

For example, the trace of the commutator \( \delta = [\alpha, \beta] \) is given by

\[
d := \text{tr}([A, B]) = a^2 + b^2 + c^2 - abc - 2,
\]

whence we have

\[
\rho \text{ is reducible} \quad \iff \quad a^2 + b^2 + c^2 - abc - 2 = 2.
\]

The geometric quotient of \( \text{Hom}(\pi_1(X^*_t), SL(2, \mathbb{C})) \) by the action of \( SL(2, \mathbb{C}) \)-conjugacy is identified with \( \mathbb{C}^3 \) by means of the composite

\[
\text{Hom}(\pi_1(X^*_t), SL(2, \mathbb{C})) \rightarrow SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \rightarrow \mathbb{C}^3,
\]

\[
\rho \mapsto (A, B) \mapsto (a, b, c).
\]

To be precise (see [33], [2]), if \( a^2 + b^2 + c^2 - abc - 2 \neq 2 \), then the fibre over \((a, b, c)\) consists of the single \( SL(2, \mathbb{C}) \)-conjugacy class of the irreducible representation defined by the matrices

\[
A = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & \gamma^{-1} \\ -\gamma & b \end{pmatrix}, \quad \text{where} \quad \gamma + \gamma^{-1} = c.
\]

This normal form is obtained in any basis of the form \((v, -\gamma B \cdot v)\), where \( v \) is an eigenvector for the product \( A \cdot B \) with eigenvalue \( \gamma \). It depends only on the choice of the root \( \gamma \). The commutator is therefore given by

\[
[A, B] = \begin{pmatrix}
-2\gamma^2 + 1 & a - b\gamma \\
\gamma^2 & \frac{\gamma^2}{\gamma} \\
\frac{a\gamma - b}{\gamma} & \gamma
\end{pmatrix}.
\]
One can check by a direct computation that the matrices $A$ and $B$ above share a common eigenvector if and only if $d = 2$.

**Corollary 15.** A Lamé connection is reducible if and only if $\theta \in \mathbb{Z}$.

The elliptic involution $\sigma : X \to X$, $(x, y) \mapsto (x, -y)$, acts on our moduli space by sending the SL(2, $\mathbb{C}$)-class defined by $(A, B)$ to that defined by $(A^{-1}, B^{-1})$. This may be seen by choosing a fixed point of $\sigma$ for the base point of the fundamental group. The following lemma is one of the key points in our construction.

**Lemma 16.** An SL(2, $\mathbb{C}$)-conjugacy class is stabilized by the elliptic involution $\sigma$ if and only if it consists of either irreducible or Abelian representations. In other words, for every pair $(A, B)$ generating an irreducible or Abelian subgroup of SL(2, $\mathbb{C}$) there is a matrix $M \in \text{SL}(2, \mathbb{C})$ such that

$$M^{-1}AM = A^{-1} \quad \text{and} \quad M^{-1}BM = B^{-1}. $$

In the irreducible case $M$ is unique up to a sign and $\text{tr}(M) = 0$.

**Proof.** In SL(2, $\mathbb{C}$) we have $\text{tr}(A^{-1}) = \text{tr}(A)$ and

$$\text{tr}(A^{-1}B^{-1}) = \text{tr}(B^{-1}A^{-1}) = \text{tr}((AB)^{-1}) = \text{tr}(AB),$$

so that the involution acts trivially on the quotient, that is, on the triples $(a, b, c)$. Since irreducible SL(2, $\mathbb{C}$)-classes are characterized by the corresponding triples $(a, b, c)$, they are $\sigma$-invariant. Another way to prove this is to note that the matrix $M$ in the statement of the lemma must transpose the two eigenvectors of each of the matrices $A, B$. We look for an element $\overline{M}$ of PSL(2, $\mathbb{C}$) transposing the corresponding points of $\mathbb{P}^1$, that is, sending a quadruple $(a_1, a_2, b_1, b_2)$ to the quadruple $(a_2, a_1, b_2, b_1)$. Since the cross-ratios of these quadruples coincide, such an element $\overline{M}$ exists. Moreover, $\overline{M}$ is an involution since its square fixes the four points and, therefore, $\text{tr}(M) = 0$. One must study the degenerate cases when $a_1 = a_2$ and/or $b_1 = b_2$ separately; we omit this discussion. In the reducible case we have, for example, $a_1 = b_1$ and $\overline{M}$ exists if and only if $a_2 = b_2$, which characterizes the Abelian case. Finally, $\sigma$ interchanges the upper-triangular and lower-triangular SL(2, $\mathbb{C}$)-representations but stabilizes the diagonal ones. □

When the matrices $A$ and $B$ are in the normal form above, the matrix $M$ in the lemma is given, up to a sign, by the formula

$$M = \begin{pmatrix} \gamma^2 - 1 & a - b\gamma \\ \frac{2\gamma}{a\gamma - b} & \frac{2\gamma}{\gamma^2 - 1} \end{pmatrix}. \quad (48)$$

We resume our discussion of the non-resonant case.

**Corollary 17.** Given an elliptic curve $X$ and $\theta \notin \mathbb{Z}$, the Lamé connections on $X$ with exponent $\theta$ are in one-to-one correspondence with points of the smooth affine hypersurface

$$S_d := \{(a, b, c) \in \mathbb{C}^3; \ a^2 + b^2 + c^2 - abc - 2 = d\}, \quad d = 2 \cos(\pi \theta).$$

They are irreducible and $\sigma$-invariant.
Proof. The connections are irreducible ($\vartheta \notin \mathbb{Z}$) and have no apparent singular points. Therefore the Riemann–Hilbert correspondence is injective and the desired assertions follow from Lemma 16. □

6.3. Resonant cases. We now complete the picture by studying the case of resonant parameters $\vartheta \in \mathbb{Z}$.

6.3.1. $\vartheta \in \mathbb{Z}\backslash 2\mathbb{Z}$. The monodromy representation is irreducible and is characterized by the corresponding triple $(a, b, c)$. The local monodromy around $\omega_{\infty}$ is unipotent, with repeated eigenvalue $-1$, and is given by the commutator

$$[A, B] = \left( \begin{array}{cc} a^2 + b^2 + \gamma^2 - abc & \gamma^{-2}(a - b\gamma) \\ a - b\gamma^{-1} & \gamma^{-2} \end{array} \right), \quad \gamma + \gamma^{-1} = c.$$

We have $\text{tr}([A, B]) = a^2 + b^2 + c^2 - abc - 2 = -2$ and $[A, B] = -I$ precisely when $(a, b, c) = (0, 0, 0)$, the unique singular point of the surface.

**Proposition 18.** The Lamé connections with exponent $\vartheta \in \mathbb{Z}\backslash 2\mathbb{Z}$ having a logarithmic singular point are in one-to-one correspondence with the smooth points of the Markov affine hypersurface

$$S_{-2} = \{(a, b, c) \in \mathbb{C}^3; \ a^2 + b^2 + c^2 - abc = 0\}.$$

They are irreducible and $\sigma$-invariant.

**Proof.** The same as for Corollary 17. □

When $(a, b, c) = (0, 0, 0)$, the image of the monodromy representation is the dihedral group of order 8 (that is, quaternionic):

$$A = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \quad \text{and} \quad B = \left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right),$$

and the singular point $\omega_{\infty}$ of the connection is apparent: $[A, B] = -I$. In this case the Lamé connection is not characterized by its monodromy representation. Indeed, we have the following proposition.

**Proposition 19.** The Lamé connections over the singular point $(a, b, c) = (0, 0, 0)$ are in one-to-one correspondence with $\mathbb{P}^1$. They have the same monodromy representation into the dihedral subgroup of order 8 in $\text{SL}(2, \mathbb{C})$, and thus are irreducible. All of them are $\sigma$-invariant.

**Proof.** The Lamé connection is determined by its monodromy representation $\rho$ (acting on the space of solutions $E_p \simeq \mathbb{C}^2$) and the line $L_p \subset E_p$ corresponding to the solutions holomorphic at $\omega_{\infty}$ after analytic continuation along $\delta$. In other words, the connection is determined by a triple

$$(A, B, L) \in \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \times \mathbb{P}^1.$$

Another triple $(\tilde{A}, \tilde{B}, \tilde{L})$ represents a connection gauge equivalent to the initial one if and only if

$$(\tilde{A}, \tilde{B}, \tilde{L}) = (M^{-1} A M, M^{-1} B M, M^{-1} L), \quad M \in \text{SL}(2, \mathbb{C}).$$
The monodromy representation, being irreducible here, has centralizer $\pm I$ acting trivially on $\mathbb{P}^1$: once the monodromy representation $(A, B)$ is fixed, the gauge equivalence classes of connections are in one-to-one correspondence with $\mathbb{P}^1 \supset L$.

One can easily check that the action of $\sigma$ on Lamé connections induces the following action on the corresponding triples:

$$\sigma: (A, B, L) \mapsto (A^{-1}, B^{-1}, (AB)^{-1}L).$$

It turns out that the matrix $M$ in Lemma 16 (see (48)) takes the following form when $(a, b, c) = (0, 0, 0)$ (and, for example, $\gamma = i$):

$$M = A \cdot B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

It conjugates $\sigma(A, B, L)$ to $(A, B, L)$, thus proving the $\sigma$-invariance of the corresponding connection, a kind of miracle. □

Remark 20. In fact, the moduli space of Lamé connections with a fixed exponent $\vartheta \in \mathbb{Z} \setminus 2\mathbb{Z}$ may be viewed, over the surface $S_{-2}: \{a^2 + b^2 + c^2 - abc = 0\}$, as the minimal resolution obtained by a single blow-up of the singular point $(0, 0, 0)$: the exceptional divisor stands for the set of connections with apparent singular point considered in Proposition 19. This will follow immediately from a similar result obtained in [25] for connections over $\mathbb{P}^1$ after our descent construction. Let us also give some direct arguments. For every point $p$ in the smooth part of the affine surface $S_{-2}$, we have a one-dimensional subspace $L_p \subset E_p$ of all solutions holomorphic at $\omega_\infty$ after analytic continuation along $\delta$. Using the local model (47) in the logarithmic case, one can check that $L_p$ coincides with an eigenspace of the local monodromy $[A, B]$, namely,

$$L = \mathbb{C} \cdot \left(\begin{pmatrix} \gamma^2 + 1 \\ -a\gamma^2 + b\gamma \end{pmatrix}\right).$$

The exceptional divisor of $S_{-2}$ is given by the equation $a^2 + b^2 + c^2 = 0$ in the homogeneous coordinates $(a : b : c)$ and can be parametrized by

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^2; \quad s \mapsto (i(s^2 + 1) : s^2 - 1 : 2s).$$

One can easily verify that the line $L_p$ tends to $\mathbb{C} \cdot \left(\begin{pmatrix} 1 \\ s \end{pmatrix}\right)$ as the representation $\rho$ tends to the point $s$ in the parametrization above.

6.3.2. $\vartheta \in 2\mathbb{Z}$. In this case, several non-Hausdorff phenomena occur in the moduli spaces of representations and connections. The Hausdorff quotient is given by the Cayley affine hypersurface

$$S_2 = \{(a, b, c) \in \mathbb{C}^3; \; a^2 + b^2 + c^2 - abc = 4\}.$$

The singular points of $S_2$ are

$$(a, b, c) = (2, 2, 2), \; (2, -2, -2), \; (-2, 2, -2) \; \text{and} \; (-2, -2, 2).$$
They play the same role in the sense that they are permuted when one changes signs of the generators:

\[(A, B), (A, -B), (-A, B) \text{ and } (-A, -B).\]

Over a smooth point \((a, b, c) \in S_2\) there are exactly three distinct \(\text{SL}(2, \mathbb{C})\)-conjugacy classes, namely:

\[(A, B) = \left(\begin{pmatrix} \alpha & \lambda \\ 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} \beta & \mu \\ 0 & \beta^{-1} \end{pmatrix}\right), \left(\begin{pmatrix} \alpha & 0 \\ \lambda & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} \beta & 0 \\ \mu & \beta^{-1} \end{pmatrix}\right)\]

(genuine upper- and lower-triangular) and

\[(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}), \quad \text{where } \begin{cases} \alpha + \alpha^{-1} = a, \\ \beta + \beta^{-1} = b, \\ \alpha\beta + (\alpha\beta)^{-1} = c. \end{cases}\]

In the triangular cases, \(\lambda\) and \(\mu\) can be chosen arbitrarily provided that \([A, B] \neq I\), that is, \(b\lambda + a\mu \neq 0\).

Each of the two triangular representations corresponds to a unique Lamé connection (with a logarithmic singular point). Being permuted by \(\sigma\), they are not \(\sigma\)-invariant. The moduli space of these triangular connections is a two-fold covering of the smooth part \(S_2^*\) of \(S_2\), the two sheets of which are permuted around each of the four singular points.

When \(\vartheta = 0\), the diagonal representation corresponds to a unique (regular) Lamé connection which is \(\sigma\)-invariant.

When \(\vartheta \neq 0\), Lamé connections over the diagonal representation have an apparent singular point. There are exactly three equivalence classes of such connections corresponding to the following choices of the line bundle \(L\) (see the proof of Proposition 19):

\[\mathbb{P}^1 \ni L = (1 : 0), (0 : 1) \text{ or } (1 : 1)\]

(any choice \(L = (1 : s), s \in \mathbb{C}^*,\) is equivalent to \(L = (1 : 1)\)). The involution \(\sigma\) permutes the first two connections but fixes the ‘generic’ third one: in both situations it suffices to choose the matrix

\[M = \begin{pmatrix} 0 & i\gamma^{-1} \\ i\gamma & 0 \end{pmatrix}\]

(again see the proof of Proposition 19). Incidentally, the set of Lamé connections with diagonal monodromy splits into the union of another two-fold covering of \(S_2^*\), with Galois involution \(\sigma\), and a copy of \(S_2^*\) on which \(\sigma\) acts trivially.

Finally, over each smooth point \((a, b, c) \in S_2^*\) there are exactly five Lamé connections (or three if \(\vartheta = 0\), only one of which is \(\sigma\)-invariant.

Now consider a singular point, say, \((a, b, c) = (2, 2, 2)\) (we recall that the four singular points play the same role). There are infinitely many distinct \(\text{SL}(2, \mathbb{C})\)-conjugacy classes over this point in the fibre defined by the unipotent pairs

\[(A, B) = \left(\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\right), \quad (r : s) \in \mathbb{P}^1,\]
and the central pair \((A, B) = (I, I)\) (when \((r, s) = (0, 0)\)). If \(\vartheta = 0\), then these representations correspond bijectively to Lamé connections that are \(\sigma\)-invariant. If \(\vartheta \neq 0\), then for every unipotent representation there are exactly two Lamé connections given by \(L = (1 : 0)\) and \(L = (s : 1)\) (all choices of \(s \in \mathbb{C}\) are equivalent) and one Lamé connection with trivial monodromy; all of them are \(\sigma\)-invariant.

**Remark 21.** When there is an apparent singular point and the direction \(L\) is fixed by the monodromy, we get a \(\nabla\)-invariant line bundle (again denoted by \(L\)) of positive degree, whence it follows that the bundle \(E\) is unstable. Indeed, in this case we have \(\vartheta = \pm 2m, m \in \mathbb{Z}_{>0}\), and the \(\nabla\)-horizontal sections of \(L\) are holomorphic and vanish to order \(m\) at \(\omega_\infty\). Then \(\text{deg}(L) = m\) by the Fuchs relation. We do not want to consider this kind of deformations in this paper.

We resume a part of our discussion.

**Proposition 22.** If \(\vartheta = 2m \in \mathbb{Z}\), then all semistable and \(\sigma\)-invariant Lamé connections have an apparent singular point at \(\omega_\infty\) (or are regular at \(\omega_\infty\) when \(m = 0\)) and their monodromy data belong to the following list:

- \((A, B) = \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}\right)\), where \((\alpha, \beta) \neq (\pm 1, \pm 1)\), and \(L = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)\) when \(m \neq 0\);

- \((A, B) = \left(\pm \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right)\), where \((s, t) \in \mathbb{P}^1\), and \(L = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)\) when \(m \neq 0\).

When \(m = 0\), we must also add the four connections with monodromy \((A, B) = (\pm I, \pm I)\).

### 6.4. Irreducible Lamé connections are elliptic pullbacks.

We now check that \(\sigma\)-invariant representations actually come from representations of the 4-punctured sphere by means of the elliptic covering. To be precise, let us consider the elliptic pullback construction of §2 from the monodromy representation point of view. Given a connection \((E, \nabla) \in \mathcal{M}_t^\theta\) with exponents

\[
\theta = \left(\begin{smallmatrix} 1 & 1 & 1 & 1 + \vartheta \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{smallmatrix}\right),
\]

we consider its monodromy representation

\[
\pi_1(\mathbb{P}^1 \setminus \{0, 1, t, \infty\}) \to \text{SL}(2, \mathbb{C}).
\]

It is determined by matrices \((M_0, M_1, M_t, M_\infty)\) satisfying the equations

\[
(M_0)^2 = (M_1)^2 = (M_t)^2 = -I \quad \text{and} \quad \text{tr}(M_\infty) = -2 \sin \frac{\pi \vartheta}{2}.
\]

The monodromy of its elliptic pullback is therefore given by the matrices

\[
A = M_0M_1 \quad \text{and} \quad B = M_1M_t
\]

(see [34], §2, for details), whose commutator

\[
[A, B] = -M_0(M_\infty)^2M_0^{-1}
\]
has the following trace:

\[ \text{tr}([A, B]) = - \text{tr}((M_{\infty})^2) = 2 - (\text{tr}(M_{\infty}))^2 = 2 \cos(\pi \vartheta). \]

Clearly, this representation is \( \sigma \)-invariant since when \( M := \pm M_1 \) we obtain from (49) that

\[ M^{-1}AM = A^{-1} \quad \text{and} \quad M^{-1}BM = B^{-1}. \]

Conversely, suppose that \((A, B)\) determines the monodromy of a \( \sigma \)-invariant Lamé connection. Then there is a matrix \( M \) conjugating \((A, B)\) to \((A^{-1}, B^{-1})\). It is clear from the previous sections that we can assume that \( M \) has zero trace: \( M^2 = -I \).

Then it is straightforward to check that \((A, B)\) is the elliptic pullback of the following representation:

\[ M_0 = -AM, \quad M_1 = M, \quad M_t = -MB \quad \text{and} \quad M_{\infty} = B^{-1}MA^{-1}. \]

If the monodromy \((A, B)\) is irreducible, then \((M_0, M_1, M_t, M_{\infty})\) is the unique (up to a sign) quadruple whose elliptic pullback gives the representation \((A, B)\).

**Corollary 23.** Let \((E, \nabla)\) be an irreducible Lamé connection with exponent \( \vartheta \notin 2\mathbb{Z} \). Then there is a unique (up to isomorphism) connection \((E_t, \nabla_t) \in M^\theta_t\) with exponents

\[ \theta = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} + \vartheta \right) \]

such that \((E, \nabla)\) is the elliptic pullback of \((E_t, \nabla_t)\).

**Proof.** Consider the monodromy representation \((A, B)\) of \((E, \nabla)\). There is a unique quadruple \((M_0, M_1, M_t, M_{\infty})\) lifting to the representation \((A, B)\) such that \( \text{tr}(M_{\infty}) = -2\sin(\pi \vartheta/2) \) (one must choose the sign of \( M \) and, therefore, of the quadruple properly). Assume that the non-resonance condition holds: \( \vartheta \notin \mathbb{Z} \). By the Riemann–Hilbert correspondence there is a unique (up to isomorphism) connection \((E, \nabla) \in M^\theta_t\) with prescribed monodromy and exponents. By construction, the elliptic pullback of \((E, \nabla)\) must have exponent \( \vartheta \) and monodromy representation \((A, B)\), the same as those of \((E, \nabla)\). Again by the (uniqueness part of the) Riemann–Hilbert correspondence, the elliptic pullback of \((E, \nabla)\) must be isomorphic to \((E, \nabla)\). This proves the corollary in the non-resonant case. When \( \vartheta \in \mathbb{Z} \setminus 2\mathbb{Z} \), the same proof holds provided that the singular point is logarithmic (that is, has infinite monodromy). However, when the pole of \((E, \nabla)\) becomes apparent, we must use the parabolic structure to restore the injectivity of the Riemann–Hilbert correspondence. We do not give details, but the key point in the proof is given by Proposition 19. □

**§ 7. Proof of Theorem 1**

We now give the proof of Theorem 1 in detail. Let \( t \mapsto (E_t, \nabla_t) \) be an isomonodromic deformation of an irreducible Lamé connection with exponent \( \vartheta \). By Corollary 23, it is the elliptic pullback of an isomonodromic deformation \( t \mapsto (E_t, \nabla_t) \in M^\theta_t \), where \( \theta = (1/2, 1/2, 1/2, 1/2 + \vartheta/2) \). By the Bolibrukh transversality theorem (see § 5.10) there is an open set of parameters for which the bundle \( E_t \) is trivial and
the parabolic directions \((l_0, l_1, l_t, l_\infty)\) are pairwise distinct; moreover, they do not lie on a curve of degree \((1, 1)\). Therefore the cross-ratio
\[
c = \frac{l_t - l_0}{l_1 - l_0} \frac{l_1 - l_\infty}{l_t - l_\infty} \in \mathbb{P}^1 \setminus \{0, 1, t, \infty\}
\]
is not special and we can apply Proposition 7 and Corollary 10, which yield an explicit formula for the Tu invariant:
\[
\lambda(E_t) = q + \frac{\rho + \kappa_\infty}{p},
\]
where \(t \mapsto (p(t), q(t))\) are the invariants of \((E_t, \nabla_t)\). In particular, \(\lambda(E_t)\) coincides with the Okamoto symmetry \(s_2 s_1 s_2\) of \(q(t)\) (see \(\S\) 5.8) and, therefore, is also a solution of the Painlevé VI equation with parameters
\[
\tilde{\kappa} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}, \frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right).
\]

\(\S\) 8. Appendix: flat logarithmic sl(2, \(\mathbb{C}\))-connections

Here we recall the basic facts about flat logarithmic connections. More details can be found in [14], [35], [30], [15].

A meromorphic connection of rank \(r\) over a smooth complex manifold \(X\) is a pair \((E, \nabla)\), where \(E\) is a locally trivial holomorphic vector bundle of rank \(r\) over \(X\) and \(\nabla\) is a \(\mathbb{C}\)-linear morphism of sheaves,
\[
\nabla: \mathcal{E} \to \mathcal{M}(K) \otimes \mathcal{O} \mathcal{E}
\]
(where \(\mathcal{E}\) is the sheaf of meromorphic sections of \(E\), and \(\mathcal{M}(K)\) is the sheaf of meromorphic sections of the canonical bundle \(K\)) satisfying the Leibnitz rule
\[
\nabla(fv) = df \otimes v + f \nabla(v)
\]
for all sections \(f\) of the structure sheaf \(\mathcal{O}\) and all sections \(v\) of the vector bundle \(E\). From an analytic point of view, if \(E\) is given by the charts
\[
U_i \times \mathbb{C}^r \ni (x, Y_i), \quad X = \bigcup_i U_i,
\]
glued by the transition maps
\[
Y_i = M_{i,j} Y_j, \quad M_{i,j} \in \text{GL}(r, \mathcal{O}(U_i \cup U_j)),
\]
then \(\nabla\) is given in these trivializing charts by a tuple of differential operators of the form
\[
Y_i \mapsto dY_i - \Omega_i Y_i, \quad \Omega_i \in \text{gl}(r, \mathcal{M}(K)(U_i)),
\]
satisfying the compatibility conditions
\[
\Omega_j = M_{i,j}^{-1} \Omega_i M_{i,j} + M_{i,j}^{-1} dM_{i,j}.
\]
We adopt this analytic point of view throughout the paper. Meromorphic connections \((E, \nabla)\) are considered up to holomorphic isomorphisms of vector bundles.
8.1. The polar divisor. We say that $\nabla$ has a pole at a point $x \in U_i \subset X$ if at least one of the entries of the corresponding matrix $\Omega_i$ has a pole at $x$. The order of the pole is the maximal order over all entries. It is easily seen to be independent of the choices of the chart $U_i$ and the local trivialization $Y_i$. The polar divisor $D = (\nabla)_\infty$ of $\nabla$ is a well-defined positive divisor on $X$.

8.2. Flatness and the monodromy representation. A horizontal section (or a solution) of $(E, \nabla)$ is any section $v$ of $E$ satisfying $\nabla(v) = 0$. In a chart, horizontal sections $Y_i(x)$ are solutions of the Pfaffian system $dY_i = \Omega_i Y_i$. The connection $(E, \nabla)$ is said to be flat (or integrable) if the following equality holds in every chart:

$$d\Omega_i + \Omega_i \wedge \Omega_i = 0$$

(in one chart is actually enough). This is equivalent to the existence of a basis $B = (v_1, \ldots, v_r)$ of horizontal holomorphic sections for $\nabla$ at every regular point. In other words, a connection is flat if and only if it is locally trivial at every regular point, that is, it is given by $Y_i \mapsto dY_i$ ($\Omega_i \equiv 0$) in a convenient local trivialization of $E$. This basis $B$ admits analytic continuation along any path in $X \setminus D$ (just by gluing the local trivializations of $\nabla$ using transition maps of the bundle). Therefore, fixing a point $x_0 \in X \setminus D$ and a basis $B$ as above in a neighbourhood of $x_0$, we get the monodromy representation of $(E, \nabla)$ with respect to $B$, that is, a homomorphism

$$\rho_{\nabla, B}: \pi_1(X \setminus D, x_0) \to \text{GL}(r, \mathbb{C}), \quad \gamma \mapsto M_\gamma,$$

defined in the following way. If $B_\gamma$ is the new basis of horizontal sections around $x_0$ obtained after analytic continuation along $\gamma$, then $M_\gamma$ is given by

$$B_\gamma = BM_\gamma.$$

If we change the basis of horizontal sections $B$ to another basis $B' = MB$, $M \in \text{GL}(r, \mathbb{C})$, then the new monodromy representation is given by

$$\rho_{\nabla, B'}(\gamma) = M \cdot \rho_{\nabla, B}(\gamma) \cdot M^{-1} \quad \forall \gamma \in \pi_1(X \setminus D, x_0).$$

Therefore the monodromy representation is well defined by $\nabla$ up to $\text{GL}(r, \mathbb{C})$-conjugacy and we shall simply write $\rho_{\nabla}$ for any representative of the conjugacy class.

8.3. Flat logarithmic connections. A flat connection is said to be logarithmic if it has only simple poles (that is, $D$ is reduced) and the connection matrix $\Omega$ in every chart is such that its differential $d\Omega$ also has simple poles. The latter condition is equivalent to the fact that the connection has a product structure in the neighbourhood of any smooth point of the polar divisor $D$: there are local coordinates $(x_1, \ldots, x_n)$ on $X$ and a local trivialization $Y_i$ such that $D$ is given by $x_2 = \cdots = x_n = 0$ and the connection matrix depends on only one variable $x := x_1$. Therefore the $\text{GL}(r, \mathbb{C})$-conjugacy class of the residual matrix is constant along each irreducible component of $D$, and one can speak of the eigenvalues $\{\theta_1, \ldots, \theta_r\}$ of the connection at each pole, that is, at every component of $D$. A pole is said to be resonant if at least two of the eigenvalues differ by an integer:
\[ \theta_i - \theta_j \in \mathbb{Z}, \ i \neq j. \]  
At every smooth point of a non-resonant pole, the connection matrix can be further reduced to its principal part

\[
\Omega = \begin{pmatrix}
\theta_1 & & \\
& \ddots & \\
0 & & \theta_r
\end{pmatrix} \frac{dx}{x}.
\]

In the resonant case, for each pair \( \theta_i - \theta_j \in \mathbb{Z}_{\geq 0} \) one can reduce the \((i, j)\)-entry of \( \Omega \) to a resonant monomial \( c \cdot x^{\theta_i - \theta_j} \cdot \frac{dx}{x} \). For each irreducible component \( D_j \) of the divisor \( D \) we fix a path \( \delta_j \) in \( X \setminus D \) joining the base point \( x_0 \) to a smooth point of \( D_j \). Consider a loop \( \gamma_j \) in \( X \setminus D \) based at \( x_0 \), going first very close to \( D_j \) along \( \delta_j \), turning once around \( D_j \), and going back to \( x_0 \) along \( \delta_j^{-1} \). The conjugacy class of \( \rho(\gamma_j) \) is independent of the choices and is called the local monodromy of \((E, \nabla)\) around \( D_j \); the eigenvalues are given by \( \{e^{2i\pi \theta_1}, \ldots, e^{2i\pi \theta_r}\} \). If the local monodromy is diagonalizable, then so is the residual matrix; the converse is not true.

### 8.4. Trace and twist.

The **trace** of a connection \((E, \nabla)\) is the meromorphic connection \((\text{det}(E), \text{tr}(\nabla))\) of rank 1, where \(\text{det}(E)\) is the determinant of \(E\), defined in the previous notation by the transition maps \(\text{det}(M_{i,j})\), and \(\text{tr}(\nabla)\) is given by \(y_i \mapsto dy_i - \text{tr}(\Omega_i)y_i\). The connection \((E, \nabla)\) is said to be **traceless** if its trace is the trivial connection \(y \mapsto dy\) (on the trivial bundle). The polar divisor of the trace is bounded by that of the initial connection. The **twist** of \((E, \nabla)\) by a connection \((L, \zeta)\) of rank 1 is a connection of rank \(r\) given by their tensor product \((L \otimes E, \zeta \otimes \nabla)\).

If \((L, \zeta)\) is defined in the same open covering \(U_i\) by \(y_i \mapsto dy_i - \omega_iy_i\) with transition maps \(y_i = m_{i,j}y_j\), then the twist is given by the matrices \(\Omega_i + \omega_i \cdot I\) with transition maps \(Y_i = m_{i} \cdot M_{i}Y_{j}\). We have

\[
\text{det}(L \otimes E) = \text{det}(L)^{\otimes r} \otimes \text{det}(E) \quad \text{and} \quad \text{tr}(\zeta \otimes \nabla) = \text{tr}(\zeta)^{\otimes r} \otimes \text{tr}(\nabla).
\]

The trace of a flat (resp. logarithmic) connection is flat (resp. logarithmic).

### 8.5. \(\text{sl}(2, \mathbb{C})\)-connections.

For convenience of notation we now restrict ourselves to flat logarithmic \(\text{sl}(2, \mathbb{C})\)-connections (that is, those that are traceless of rank 2). Their monodromy representation takes values in \(\text{SL}(2, \mathbb{C})\). For each irreducible component \(D_j\) of the polar divisor \(D\), the **exponent** \(\theta_j \in \mathbb{C}\) is defined (up to a sign) as the difference between the two eigenvalues \(\pm \theta_j/2\) of the residual matrix. The corresponding local monodromy matrix has trace \(2 \cos(\pi \theta)\). The component \(D_j\) is resonant if and only if \(\theta_j \in \mathbb{Z}\). In this case (say, when \(\theta_j = n \in \mathbb{Z}_{\leq 0}\)) the connection matrix can be reduced to one of the following:

\[
\text{either} \quad \Omega = \begin{pmatrix}
\frac{n}{2} & 0 \\
0 & -\frac{n}{2}
\end{pmatrix} \frac{dx}{x}, \quad \text{or} \quad \Omega = \begin{pmatrix}
\frac{n}{2} & x^n \\
0 & -\frac{n}{2}
\end{pmatrix} \frac{dx}{x}
\]

in the neighbourhood of any smooth point of \(D_j\). The corresponding local monodromy matrix is of the form

\[
\pm \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \quad \text{or} \quad \pm \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix},
\]
respectively, where $\pm := (-1)^n$. The pole is said to be *apparent* in the former case (there is no pole when $n = 0$) and *logarithmic* in the latter. In both cases we note that the bounded solutions
\[ v = \left( \frac{cx^{n/2}}{0} \right), \quad c \in \mathbb{C}, \]
form a one-dimensional subspace in the space of solutions.

8.6. The Riemann–Hilbert correspondence. We define the *Riemann–Hilbert correspondence* as a map
\[
RH: \begin{cases} 
\text{flat logarithmic} & \to \ \pi_1(X \setminus D, x_0) \to \text{SL}(2, \mathbb{C}), \\
\text{sl}(2, \mathbb{C})\text{-connections over } X & \text{with polar divisor } D & \text{having trace } 2 \cos(\pi \theta_j) \\
& \text{and exponent } \theta_j \text{ over } D_j & \text{around } D_j
\end{cases} / \sim \to \begin{cases} 
\rho, \\
(E, \nabla) \mapsto \rho,
\end{cases}
\]
which sends every connection (considered up to holomorphic bundle isomorphisms) to its monodromy representation (considered up to conjugacy). This map is surjective provided that $D$ has normal crossings [14] or $X$ has dimension $\leq 2$ [30]. Moreover, it is injective provided that none of the exponents is a non-zero integer. In fact, the lack of injectivity comes from apparent singular points. One can restore the injectivity in the resonant case by enriching the monodromy data as follows. For every $\theta_j \in \mathbb{Z} \setminus \{0\}$ fix a basis $B$ of solutions near $x_0$ and consider the one-dimensional subspace $L_j \subset \mathbb{C}^2$ of those solutions that are bounded around $D_j$ after analytic continuation along the path $\delta_j$. The full monodromy data, which characterize the connection up to isomorphism, consist of the monodromy transformation $\rho$ and the set of all $L_j \in \mathbb{P}^1 (= \mathbb{P}(\mathbb{C}^2))$, where $j$ ranges over the set $J^{\text{res}}$ of all indices such that $\theta_j \in \mathbb{Z} \setminus \{0\}$. Any base change $B' = MB$, $M \in \text{SL}(2, \mathbb{C})$, yields new monodromy data
\[
\rho' = M \cdot \rho \cdot M^{-1} \quad \text{and} \quad L'_j = M \cdot L_j \quad \forall j \in J^{\text{res}}
\]
(proposition 24). Assume that $D$ is a reduced divisor with normal crossings and let $D_j, \gamma_j, \delta_j$ be as above. Then the set of flat logarithmic sl(2, $\mathbb{C}$)-connections $(E, \nabla)$ with polar divisor $D$ and exponents $\theta_j$ around $D_j$ modulo isomorphisms is in one-to-one correspondence with the set of pairs $(\rho, (L_j)_{j \in J^{\text{res}}})$, where
- the homomorphism $\rho \in \text{Hom}(\pi_1(X \setminus D, x_0), \text{SL}(2, \mathbb{C}))$ is such that $\text{tr}(\rho(\gamma_j)) = 2 \cos(\pi \theta_j)$ for all $j \in J$,
- the line $L_j \in \mathbb{P}^1$ is $\rho(\gamma_j)$-invariant for all $j \in J^{\text{res}}$, modulo the SL(2, $\mathbb{C}$)-action defined by (50).

8.7. Reducible $\text{gl}(2, \mathbb{C})$-connections. A line subbundle $L \subset E$ is said to be $\nabla$-invariant if it is generated by $\nabla$-horizontal sections. In this case the connection $\nabla$ induces a meromorphic connection $\nabla|_L$ on $L$. The connection $(E, \nabla)$ is said to
be reducible if it admits such an invariant line bundle, and irreducible otherwise. When the connection is reducible, its monodromy representation is also reducible: the monodromy group has a common eigenvector. In the logarithmic case with normal crossings of the polar divisor, the converse is true: \((E, \nabla)\) is reducible if and only if \(\rho\) is.

8.8. Projective \(\text{sl}(2, \mathbb{C})\)-connections and Riccati foliations. A meromorphic connection \((E, \nabla)\) of rank 2 induces a projective \(\text{sl}(2, \mathbb{C})\)-connection \((\mathbb{P}(E), \mathbb{P}(\nabla))\) on \(X\). If the linear connection is given in the trivializing chart \(Y_i\) by

\[ Y_i \mapsto dY_i - \Omega_i Y_i, \quad \Omega_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}, \]

where \(\alpha_i, \beta_i, \gamma_i\) and \(\delta_i\) are meromorphic 1-forms on \(U_i\), then the projective connection \(\mathbb{P}(\nabla)\) is defined in the projective trivializing chart \(\mathbb{P}(Y_i) = (1 : z_i) \in \mathbb{P}^1\) by

\[ z_i \mapsto dz_i + \beta_i z_i^2 + (\alpha_i - \delta_i) z_i - \gamma_i. \]

Another linear connection \((E', \nabla')\) defines the same projective connection if and only if it is the twist of \((E, \nabla)\) by some meromorphic connection \((L, \zeta)\) of rank 1.

Conversely, if \(H^2(X, \mathcal{O}^*) = 0\) (which always holds when \(X\) is a curve), then every \(\mathbb{P}^1\)-bundle \(P\) is the projectivization \(P = \mathbb{P}(E)\) of some vector bundle of rank 2. Moreover, the formulae above show that for every meromorphic projective \(\text{sl}(2, \mathbb{C})\)-connection on \(P\) and every meromorphic (linear) connection \(\zeta\) on the line bundle \(L = \det(E)\), there is a unique meromorphic linear connection \(\nabla\) on \(E\) lifting the projective connection from \(\mathbb{P}(E)\) with prescribed trace \(\text{tr}(\nabla) = \zeta\) on \(\det(E)\).

When \(X\) is a curve, there are two topological types of \(\mathbb{P}^1\)-bundles: the topological type of \(\mathbb{P}(E)\) is determined by the parity of \(\text{deg}(E) \in \mathbb{Z}/2\mathbb{Z}\). We note that topological triviality is a condition for the existence of a square root \(L\) of \(\det(E) \in \text{Pic}(X)\). In other words, a \(\mathbb{P}^1\)-bundle \(P\) is topologically trivial if and only if \(P\) can be lifted as an \(\text{SL}(2, \mathbb{C})\)-vector bundle: setting \(E := E \otimes L^{\otimes(-2)}\), we have \(P = \mathbb{P}(E)\) with \(\det(E) = \mathcal{O}\). This \(\text{SL}(2, \mathbb{C})\)-lifting depends on the choice of a square root: it is well defined up to points of order 2 in \(\text{Pic}(X)\), and there are \(2^{2g}\) possible liftings over a curve \(X\) of genus \(g\). Finally, every meromorphic projective connection on \(P\) lifts uniquely to a linear \(\text{sl}(2, \mathbb{C})\)-connection \(\nabla\) on \(E\) (with the same pole divisor).

When \(X\) is a curve, the total space of \(\mathbb{P}(E)\) is a ruled surface \(S \to X\), and the Riccati equation \(\mathbb{P}(\nabla) = 0\) determines a singular foliation \(\mathcal{F}\) on \(S\) whose leaves are the graphs of horizontal sections of the projective connection. The pair \((S, \mathcal{F})\) is called a Riccati foliation (see [28]). This foliation is regular and transversal to the ruling outside the polar locus of \(\nabla\). Over the poles of the projective connection, the \(\mathbb{P}^1\)-fibre is the disjoint union of a vertical fibre of \(\mathcal{F}\) and one or two singular points. To be precise, when \(\nabla\) is logarithmic (with simple poles), the singular points correspond to eigenlines of the linear connection \(\nabla\). Let \(\theta\) and \(\theta'\) be the eigenvalues of \(\nabla\) at some pole \(x \in X\). We denote the corresponding eigenlines by \(l\) and \(l'\). The exponent (or the Camacho–Sad index of the vertical leaf) at the singular point \(l\) of the Riccati foliation is \(\kappa = \theta' - \theta\). Let \(\sigma : X \to S\) be a section.
Proposition 25. Let \( X \) be a curve, \((S, \mathcal{F})\) a Riccati foliation with simple poles over \( X \), and \( \sigma \) an \( \mathcal{F} \)-invariant section. Then
\[
\sigma \cdot \sigma = \sum_i \kappa_i,
\]
where \( \kappa_i \) are the exponents of the singular points passed through by \( \sigma \), and \( i \) ranges over the set of all invariant fibres of \( \mathcal{F} \).

This is a particular case of the Camacho–Sad formula (see [28], p. 37).

Proof. Viewed as a projective connection, there is a unique lifting \((E, \nabla)\) of the projective connection such that the \( \nabla \)-invariant line bundle \( L \) corresponding to \( \sigma \) is the trivial bundle, and the connection induced by \( \nabla \) on \( L \) is the trivial connection: the eigenvalues of \( \nabla \) over the pole \( i \) are given by \( 0 \) and \( \kappa_i \). Then Fuchs relations give
\[
\deg(E) = \sum_i \kappa_i,
\]
and we have
\[
\sigma \cdot \sigma = \deg(E) - 2\deg(L) = \deg(E).
\]
\( \square \)

Proposition 26. Let \( X \) be a curve of genus \( g \), \((S, \mathcal{F})\) a Riccati foliation over \( X \) with \( n \) poles (counted with multiplicities), and \( \sigma: X \to S \) a section which is not \( \mathcal{F} \)-invariant. Then the number of tangencies between \( \sigma \) and the fibration (including the singular points lying on \( \sigma \)) is given by
\[
tang(\mathcal{F}, \sigma) = 2g - 2 + n + \sigma \cdot \sigma
\]
(counted with multiplicities).

This is a particular case of Proposition 2 in [28], p. 37.

Proof. Choose any lifting \((E, \nabla)\) of the projective connection and apply Lemma 6 to the line bundle \( L \subset E \) corresponding to \( \sigma \). \( \square \)

8.9. Stability of bundles and connections. A vector bundle of rank 2 over a curve \( X \) is said to be stable (resp. semistable) if
\[
\deg(E) - 2\deg(L) > 0 \quad (\text{resp. } \geq 0)
\]
for all line subbundles \( L \subset E \). This notion is invariant under projective equivalence: the \( \mathbb{P}^1 \)-bundle \( \mathbb{P}(E) \) is stable (resp. semistable) if
\[
\sigma \cdot \sigma > 0 \quad (\text{resp. } \geq 0)
\]
for all sections \( \sigma: X \to \mathbb{P}(E) \).

Similarly, we say that a connection \((E, \nabla)\) is stable (resp. semistable) if
\[
\deg(E) - 2\deg(L) > 0 \quad (\text{resp. } \geq 0)
\]
for all $\nabla$-invariant line subbundles $L \subset E$. Again, this notion is invariant under projective equivalence: the projective connection $(\mathbb{P}(E), \mathbb{P}(\nabla))$ is stable (resp. semistable) if

$$\sigma \cdot \sigma > 0 \quad \text{(resp. } \geq 0)$$

for all $\mathbb{P}(\nabla)$-invariant sections $\sigma$. In particular, all irreducible connections $(E, \nabla)$ are stable even if the bundle $E$ is unstable. However, for a semistable connection, the bundle $E$ cannot be arbitrarily unstable: by Lemma 6, the stability index is bounded by

$$\deg(E) - 2 \deg(L) \geq 2 - 2g - \deg(D),$$

where $D$ is the polar divisor of $\nabla$.

### 8.10. Meromorphic and elementary gauge transformations.

We recall the definition of an elementary transformation of a vector bundle $E$ of rank 2 over a curve $X$. For every point $p \in X$ and every line $l \in \mathbb{P}(E_p)$ in the fibre over $p$, one usually defines two birational bundle transformations $\text{elm}_{p,l}^+: E \to E^+$ and $\text{elm}_{p,l}^-: E \to E^-$, which are unique up to post-composition with a bundle isomorphism. When restricted to the punctured curve $X^* = X \setminus \{p\}$, both transformations $\text{elm}_{p,l}^\pm$ induce isomorphisms. In a neighbourhood of $p$ they can be described as follows. Choose a local coordinate $x: U \to \mathbb{C}$ at $p$ and a trivialization $Y: E|_U \to \mathbb{C}^2$ such that the line $l$ is spanned by $Y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This, in particular, induces a trivialization of $E|_{X^*}$ on $U^* = U \setminus \{p\}$. The elementary transformations $\text{elm}_{p,l}^\pm$ can be defined by the commutative diagram

$$\begin{array}{ccc}
E & \xrightarrow{\text{elm}_{p,l}^\pm} & E^+ \\
| & \downarrow \text{id} & \downarrow \text{id} \\
E^\pm & \xleftarrow{\text{id}} & E^+|_{U^*} \\
\end{array}$$

where

$$\phi^+(Y) = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} Y \quad \text{and} \quad \phi^-(Y) = \begin{pmatrix} 1/x & 0 \\ 0 & 1 \end{pmatrix} Y.$$ 

All three bundles $E$ and $E^\pm$ are constructed by gluing the local trivial bundle $U \times \mathbb{C}^2$ to the same restricted bundle $E|_{X^*}$ through different bundle isomorphisms (the identity or $\phi^\pm$) over the punctured neighbourhood $U^*$. The isomorphisms $E|_{X^*} \to E^\pm|_{X^*}$ given by this construction extend as birational bundle transformations. We have

$$\det(E^\pm) = \det(E) \otimes \mathcal{O}(\pm [p]).$$

On the other hand, $\text{elm}_{p,l}^\pm$ induce the same birational $\mathbb{P}^1$-bundle transformation

$$\text{elm}_{p,l}: P = \mathbb{P}(E) \to P'$$

since $\phi^+$ and $\phi^-$ coincide both in $\text{PGL}(2, \mathcal{O}(U^*))$ and $\text{PGL}(2, \mathcal{M}(U))$. 

We claim that our construction depends only on the ‘parabolic structure’ \((p, l)\), not on the choice of the local trivialization \(Y\). Indeed, for another choice \(\tilde{Y} = M \cdot Y\), \(M \in \text{GL}(2, \mathcal{O}(U))\), we must verify that \(\phi^+ (\tilde{Y}) = \phi^+(M \cdot Y) = \tilde{M} \cdot \phi^+(Y)\), where \(\tilde{M} \in \text{GL}(2, \mathcal{O}(U))\).

If \(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\), then \(\tilde{M} = \begin{pmatrix} a & b/x \\ xc & d \end{pmatrix}\); since \(l\) must be spanned by \(\tilde{Y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\), we have \(b(0) = 0\) and \(\tilde{M}\) is holomorphic with \(\text{det}(\tilde{M}) = \text{det}(M) \neq 0\).

A similar calculation shows that the lines \(l^\pm \subset E_p^\pm\) defined by \(\phi^\pm = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\) in the construction above are independent of our choices. In other words, if \(E\) is equipped with a parabolic structure \(l \subset E_p\) over \(p\), then the elementary transformations induce a birational transformation

\[
\text{elm}^\pm_p: (E, l) \rightarrow (E^\pm, l^\pm)
\]

of parabolic bundles which is well defined up to left and right composition with parabolic bundle isomorphisms. It also follows from the computations above that the composites

\[
\text{elm}^\pm_p \circ \text{elm}^\mp_p: (E, l) \rightarrow (E', l')
\]

are parabolic bundle isomorphisms. In this sense, \(\text{elm}^+_p\) and \(\text{elm}^-_p\) are inverse to each other. We can also consider a general parabolic bundle \((E, l)\) of rank 2 over \((X, S)\), where \(S \subset X\) is a finite subset and \(l: S \rightarrow \mathbb{P}(E|_S)\) is a section of the projective bundle induced over \(S\). The elementary transformations \(\text{elm}^\pm_p: (E, l) \rightarrow (E^\pm, l^\pm)\) of parabolic bundles over \((X, S)\) are defined for \(p \in S\) in the same way as above (note that \(\text{elm}^\pm_{p, l(p)}\) induces an isomorphism of parabolic bundles over \((X^*, S^*)\)), and as the identity for \(p \notin S\). Finally, if \(p_1, p_2 \in S\) are two distinct points, then the elementary transformations \(\text{elm}^\pm_{p_1}\) and \(\text{elm}^\pm_{p_2}\) commute (up to parabolic bundle isomorphisms), so that one can define \(\text{elm}^\pm_{S'}\) for any subset \(S' \subset S\).

We now describe the action of elementary transformations on parabolic connections \((E, \nabla, l)\), where \((E, l)\) is a parabolic bundle over \((X, S)\) in the notation above and \(\nabla\) is a meromorphic connection on \(E\). Let \(p \in S\) and denote by \(\nabla^\pm\) the pushforward of \(\nabla\) under the elementary transformations \(\text{elm}^\pm_p: (E, l) \rightarrow (E^\pm, l^\pm)\). Then \(\nabla^\pm\) are meromorphic connections on \(E^\pm\). In the notation above, if \(\nabla\) is given in the coordinates \((x, Y)\) by the formula

\[
Y \mapsto dY - \Omega Y, \quad \Omega = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},
\]

then \(\nabla^\pm\) is defined in the local trivializations \(Y^\pm = \phi^\pm(Y)\) of \(E^\pm\) by \(Y^\pm \mapsto dY^\pm - \Omega^\pm Y^\pm\), where

\[
\Omega^+ = \begin{pmatrix} \alpha & \beta \\ x & \gamma \end{pmatrix} \quad \text{and} \quad \Omega^- = \begin{pmatrix} \alpha - \frac{dx}{x} & \beta \\ x & \gamma \end{pmatrix}.
\]
If \( p \) is not a pole of \( \nabla \), then \( \nabla^\pm \) has a logarithmic pole at \( p \). When \( p \) is a pole of order \( k \) for \( \nabla \), there are two cases:

- if \( l(p) \) is an eigenvector of \( \nabla \) at \( p \) (that is, of \( x^k\Omega \) at \( x = 0 \)), then \( \nabla^\pm \) has a pole of order \( k \) or \( k - 1 \) at \( p \);
- otherwise \( \nabla^\pm \) has a pole of order \( k + 1 \) at \( p \).

In either case, \( l^\pm(p) \) is an eigenvector of \( \nabla^\pm \) at \( p \).

When \( \nabla \) is a logarithmic connection, the connections \( \nabla^\pm \) are logarithmic if and only if either \( p \) is regular, or \( p \) is a pole and \( l(p) \) is an eigenline for \( \nabla \). One can then choose the coordinate \( Y \) in such a way that \( \nabla \) is given by \( Y \mapsto dY - AY \frac{dx}{x} \) with

\[
\Omega = \begin{pmatrix} \frac{dx}{x} & 0 \\ 0 & \frac{dx}{x} \end{pmatrix}, \quad \begin{pmatrix} (\theta + n)\frac{dx}{x} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \theta \frac{dx}{x} \\ x^n \frac{dx}{x} \end{pmatrix}, \quad \begin{pmatrix} \theta \frac{dx}{x} \\ x^n \frac{dx}{x} \end{pmatrix},
\]

or \( \begin{pmatrix} \theta \frac{dx}{x} \\ x^n \frac{dx}{x} \end{pmatrix} \) or \( \begin{pmatrix} \theta \frac{dx}{x} \\ x^n \frac{dx}{x} \end{pmatrix} \),

including the regular case \( A = 0 \) with the restriction \( n > 0 \) in the middle case. Then \( \nabla^\pm \) are given in the coordinates \( Y^\pm = \phi^\pm(Y) \) by \( Y^\pm \mapsto dY^\pm - A^\pm Y^\pm \frac{dx}{x} \), where

\[
A^+ = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 + 1 \end{pmatrix}, \quad \begin{pmatrix} (\theta + n) & x^n \\ 0 & \theta + 1 \end{pmatrix}, \quad \begin{pmatrix} \theta & 0 \\ x^{n+1} & (\theta + n + 1) \end{pmatrix},
\]

\[
A^- = \begin{pmatrix} \theta_1 - 1 & 0 \\ 0 & \theta_2 \end{pmatrix}, \quad \begin{pmatrix} (\theta + n - 1) & x^{n-1} \\ 0 & \theta \end{pmatrix}, \quad \begin{pmatrix} \theta - 1 & 0 \\ x^{n+1} & (\theta + n) \end{pmatrix},
\]

respectively. We resume.

- If \( p \) is a regular point of \( \nabla \), that is, \( A = 0 \), then the connections \( \nabla^\pm \) are logarithmic with eigenvalues 0 and \( \pm 1 \).
- If \( p \) is a pole and \( \theta \) is an eigenvalue of \( \nabla \) at \( p \), then there is one and only one eigenline \( l(p) \) associated with \( \theta \) except in the diagonal case when \( \theta_1 = \theta_2 = \theta \). When \( \nabla \) is traceless, the last case does not occur and (the eigenspace of) each eigenvalue determines a parabolic structure over \( p \).
- If \( \{\theta_1, \theta_2\} \) are the eigenvalues of \( \nabla \) at \( p \) and \( l(p) \) is the eigenline associated with \( \theta_2 \), then \( \nabla^+ \) (resp. \( \nabla^- \)) has eigenvalues \( \{\theta_1, \theta_2 + 1\} \) (resp. \( \{\theta_1 - 1, \theta_2\} \)). The connections \( \nabla^\pm \) are of diagonal type if and only if \( \nabla \) is. The parabolic structure \((E^\pm, \nabla^\pm, l^\pm)\) over \( p \) corresponds to the eigenvalue \( \theta_1 \).

The trace of the connection is changed by the formula

\[
\text{tr}(\nabla^\pm) = \text{tr}(\nabla) \otimes \zeta^\pm,
\]

where \( \zeta \) is the unique logarithmic connection on \( O_X(\pm[p]) \) having a single pole at \( p \) with residue \( \pm 1 \) and trivial monodromy. Indeed, the monodromy is not changed by a birational bundle transformation.
Bibliography

[1] H. Poincaré, “Sur les groupes des équations linéaires”, *Acta Math.* 4 (1884), 201–312.

[2] W. M. Goldman, “The modular group action on real $SL(2)$-characters of a one-holed torus”, *Geom. Topol.* 7 (2003), 443–486.

[3] D. Gallo, M. Kapovich, and A. Marden, “The monodromy groups of Schwarzian equations on closed Riemann surfaces”, *Ann. of Math.* (2) 151:2 (2000), 625–704.

[4] R. C. Gunning, “Special coordinate coverings of Riemann surfaces”, *Math. Ann.* 170 (1967), 67–86.

[5] A. Beilinson and V. Drinfeld, *Opers*, arXiv: math/0501398.

[6] F. Loray, “Okamoto symmetry of Painlevé VI equation and isomonodromic deformation of Lamé connections”, *Algebraic, analytic and geometric aspects of complex differential equations and their deformations. Painlevé hierarchies*, RIMS Kôkyûroku Bessatsu, vol. B2, Res. Inst. Math. Sci. (RIMS), Kyoto 2007, pp. 129–136.

[7] B. Malgrange, “Sur les déformations isomonodromiques. I. Singularités régulières”, *Mathematics and physics* (Paris, 1979/1982), Progr. Math., vol. 37, Birkhäuser Boston, Boston, MA 1983, pp. 401–426.

[8] J. Palmer, “Zeros of the Jimbo, Miwa, Ueno tau function”, *J. Math. Phys.* 40:12 (1999), 6638–6681.

[9] A. A. Bolibrukh, “The Riemann–Hilbert problem”, *Uspekhi Mat. Nauk* 45:2(272) (1990), 3–47; English transl., *Russian Math. Surveys* 45:2 (1990), 1–58.

[10] N. J. Hitchin, “Twistor spaces, Einstein metrics and isomonodromic deformations”, *J. Differential Geom.* 42:1 (1995), 30–112.

[11] F. Loray, M. van der Put, and F. Ulmer, “The Lamé family of connections on the projective line”, *Ann. Fac. Sci. Toulouse Math.* (6) 17:2 (2008), 371–409.

[12] M. F. Atiyah, “Vector bundles over an elliptic curve”, *Proc. London Math. Soc.* (3) 7 (1957), 414–452.

[13] L. W. Tu, “Semistable bundles over an elliptic curve”, *Adv. Math.* 98 (1993), 1–26.

[14] P. Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Math., vol. 163, Springer-Verlag, Berlin–New York 1970.

[15] V. Heu, “Stability of rank 2 vector bundles along isomonodromic deformations”, *Math. Ann.* 344:2 (2009), 463–490.

[16] S. Kawai, “Isomonodromic deformation of Fuchsian projective connections on elliptic curves”, *Nagoya Math. J.* 171 (2003), 127–161.

[17] A. M. Levin and M. A. Olshanetsky, “Painlevé–Calogero correspondence”, *Calogero–Moser–Sutherland models* (Montréal, QC, 1997), CRM Ser. Math. Phys., Springer, New York 2000, pp. 313–332.

[18] Yu. I. Manin, “Sixth Painlevé equation, universal elliptic curve, and mirror of $\mathbb{P}^2$”, *Geometry of differential equations*, Amer. Math. Soc. Transl. Ser. 2, vol. 186, Amer. Math. Soc., Providence, RI 1998, pp. 131–151.

[19] M. Noumi and Y. Yamada, “A new Lax pair for the sixth Painlevé equation associated with $\mathfrak{so}(8)^*$”, *Microlocal analysis and complex Fourier analysis*, World Sci. Publ., River Edge, NJ 2002, pp. 238–252.

[20] D. Arinkin and S. Lysenko, “Isomorphisms between moduli spaces of $SL(2)$-bundles with connections on $\mathbb{P}^1 \setminus \{x_1, \ldots, x_4\}$”, *Math. Res. Lett.* 4:2 (1997), 181–190.

[21] F. Loray, M.-H. Saito, and C. Simpson, “Foliations on the moduli space of rank two connections on the projective line minus four points”, *Geometric and differential Galois theories*, Sémin. Congr., vol. 27, Soc. Math. France, Paris 2013, pp. 117–170.
[22] A. Zotov, “Elliptic linear problem for the Calogero–Inozemtsev model and Painlevé VI equation”, *Lett. Math. Phys.* **67**:2 (2004), 153–165.

[23] É. Picard, “Mémoire sur la théorie des fonctions algébriques de deux variables”, *J. Math. Pures et Appl.* (4) **5** (1889), 135–319.

[24] M. Mazzocco, “Picard and Chazy solutions to the Painlevé VI equation”, *Math. Ann.* **321**:1 (2001), 157–195.

[25] M. Inaba, K. Iwasaki, and M.-H. Saito, “Dynamics of the sixth Painlevé equation”, *Théories asymptotiques et équations de Painlevé, Sémin. Congr.* , vol. 14, Soc. Math. France, Paris 2006, pp. 103–167.

[26] P. Boalch, “From Klein to Painlevé via Fourier, Laplace and Jimbo”, *Proc. London Math. Soc.* (3) **90**:1 (2005), 167–208.

[27] K. Okamoto, “Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé”, *Japan. J. Math. (N. S.)* **5**:1 (1979), 1–79.

[28] M. Brunella, *Birational geometry of foliations*, Monografías de Matemática, Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro 2000.

[29] D. Cerveau, A. Lins-Neto, F. Loray, J. V. Pereira, and F. Touzet, “Complex codimension one singular foliations and Godbillon–Vey sequences”, *Mosc. Math. J.* **7**:1 (2007), 21–54.

[30] F. Loray and J. V. Pereira, “Transversely projective foliations on surfaces: existence of minimal form and prescription of monodromy”, *Internat. J. Math.* **18**:6 (2007), 723–747.

[31] M.-H. Saito, T. Takebe, and H. Terajima, “Deformation of Okamoto–Painlevé pairs and Painlevé equations”, *J. Algebraic Geom.* **11**:2 (2002), 311–362.

[32] K. Okamoto, “Studies on the Painlevé equations. I. Sixth Painlevé equation $P_{VI}$”, *Ann. Mat. Pura Appl.* (4) **146** (1987), 337–381.

[33] R. C. Churchill, “Two generator subgroups of $\text{SL}(2, \mathbb{C})$ and the hypergeometric, Riemann, and Lamé equations. Differential algebra and differential equations”, *J. Symbolic Comput.* **28**:4-5 (1999), 521–545.

[34] S. Cantat and F. Loray, “Dynamics on character varieties and Malgrange irreducibility of Painlevé VI equation”, *Ann. Inst. Fourier (Grenoble)* **59**:7 (2009), 2927–2978.

[35] D. Novikov and S. Yakovenko, “Lectures on meromorphic flat connections”, *Normal forms, bifurcations and finiteness problems in differential equations*, NATO Sci. Ser. II Math. Phys. Chem., vol. 137, Kluwer, Dordrecht 2004, pp. 387–430.

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