Consensus Problems in Complex-Weighted Networks*

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Abstract

Consensus problems for multi-agent systems in the literature have been studied under real (mostly nonnegative) weighted networks. Complex-valued systems can be used to model many phenomena in applications including complex-valued signals and the motion in a plane. A natural question that arises is whether we can establish consensus under complex-weighted networks in some sense. In this paper, we provide a positive answer to this question. More precisely, we show that in a complex-weighted network all agents can achieve modulus consensus in which the states of all agents reach the same modulus. Necessary and sufficient conditions for modulus consensus are given in both continuous-time and discrete-time cases, which explicitly reveal how the connectedness of networks and structural properties of complex weights jointly affect modulus consensus. As a special case, the bipartite consensus problems on signed networks are revisited. Moreover, our modulus consensus results are used to study circular formation problems in a plane. We first study the control problem of circular formation with relative positions that requires all the agents converge to a common circle centered at a given point and are distributed along the circle in a desired pattern, expressed by the prespecified angle separations and ordering among agents. It is shown that the circular formation with relative positions can be achieved if and only if the communication digraph has a spanning tree. It has the unspecified radius and absolute phases. To completely determine the circular formation, we discuss the control problem of circular formation with absolute positions. By using the pinning control strategy, we find that the circular formation with absolute positions can be achieved via a single local controller if and only if the communication digraph has a spanning tree.

Keywords: Modulus consensus, complex-weighted digraphs, circular formation, pinning control.

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1 Introduction

In the past decade there has been increasing interest in studying consensus problems of multi-agent systems. This is partly due to the broad applications of consensus problems in many areas including flocking, distributed computing, sensor networks and formation control [7, 13, 24, 25, 26, 27, 30, 35]. Research on consensus problems heavily depend on the set of values that the weights between agents may take.

In most current research on consensus problems, the agreement among agents is achieved by the cooperative relationship between agents. Such a system is often called cooperative system. It is convenient to use a weighted graph to model the interaction among the agents. The cooperative relationship between agents is characterized by edges with positive weights. It has been shown that algebraic graph theory, in particular graph Laplacians, provides a natural framework to deal with consensus problems. There has been a substantial body of research on the standard consensus problems. We review some basic results related closely to our current work. For undirected graphs, it is well-known [25, 26] that Laplacian-based protocols asymptotically solve the average-consensus problem if and only if the interaction graph is connected. Whereas for digraphs, a necessary and sufficient condition for solving the general consensus (not necessarily average consensus) problem is that the interaction digraph has a spanning tree [2, 32]. Note that these results are valid for both continuous-time and discrete-time cases. Further extensions have been made to include stochastic interaction topology, delay effects, dynamic agents and quantization effects, see, e.g., [11, 15, 16, 18, 19, 20, 23, 34, 36], to name a few.

In the standard consensus problems discussed above, it is always assumed that only cooperative relationship exists between the agents. However in many real world scenarios some agents cooperate, while others compete. For example, in social networks, friendly and hostile relationships among the agents are ubiquitous [10, 22]. We encounter this case in truth estimation for autonomous networks in engineering applications [9, 14]. This kind of relationships can be represented by signed graphs, i.e., graphs in which the weights of edges can be negative. A positive weight indicates the corresponding two agents are friends while a negative one indicates they are enemies. Having observed this, a recent work [1] studies the bipartite consensus problems on signed graphs, meaning that all agents converge to a consensus value which is the same for all except for the sign. It turns out that the structural balancedness of signed graphs plays a critical role in bipartite consensus problems. In social networks, structural balance is a well-known property [10, 22]. A signed graph is said to be (structurally) balanced if there exists a bipartition of the signed graph into two subcommunities such that any edge between the two subcommunities has negative weights and any edge within each subcommunity has positive weights. In particular, it has been shown in [1] that for a connected signed graph (or a strongly connected signed digraph), Laplacian-based protocol solves the bipartite consensus problem if and only if the signed (directed) graph is balanced. These results are obtained only for continuous-time case.
Motivated by the above works, especially [1, 25, 26, 32], the aim of this paper is to study consensus problems on complex-weighted networks, i.e., the weight associated to each edge is a complex number. Unlike for the real-weighted networks, the adjacency matrix of complex-weighted networks is complex-valued. A general notion of consensus introduced in this paper is called *modulus* consensus. By modulus consensus, we mean that all limiting values of the agents have the same modulus. It is shown that the notion of structural balance for complex-weighted graphs also plays a key role for obtaining modulus consensus results. The complex-weighted graph is said to be balanced if the weights of all cycles are positive. This definition of balance is consistent with the balance of the signed graphs when the weights of graphs are restricted to the real field. With both continuous-time and discrete-time Laplacian-based models, some necessary and sufficient conditions for modulus consensus are obtained in terms of the properties of complex-weighted graphs. We further consider the pinning modulus consensus problems. The modulus consensus results under switching topology are also obtained.

For the special case of signed digraphs, the bipartite consensus results are derived, which supplement the results in [1]. As an important application, our modulus consensus results are used to design distributed protocols, achieving circular formation in a planar space. We first study the circular formation with relative positions, which requires that all the agents converge to a common circle with given center, distributed in the circle with predefined angle separations and ordering. It is shown that the circular formation with relative positions can be achieved if and only if the communication digraph has a spanning tree. The case of switching topology is also discussed. Note that the obtained circular formation has the unspecified radius and absolute phases. To completely determine the achieved circular formation, we discuss the circular formation with absolute positions. By using the pinning control strategy, we find that all the agents can converge to the circular formation with absolute positions via a single local controller if and only if the communication digraph has a spanning tree. We note that complex-weighted networks have appeared in the study of leader-follower planar formation [17].

The remainder of this paper is organized as follows. Section 2 provides some preliminary results on complex-weighted graphs which is useful in establishing our main results. In Section 3 we present the modulus consensus results and illustrate them by several examples. We further discuss the case of switching topology. For the special case of signed digraphs, the bipartite consensus results are also derived. The circular formation problems are studied in Section 4. This paper is concluded in Section 5.

The notation used in the paper is quite standard. Let $\mathbb{R}$ be the field of real numbers and $\mathbb{C}$ the field of complex numbers. For a complex matrix $A \in \mathbb{C}^{n \times n}$, $A^*$ denotes the conjugate transpose of $A$. We use $\bar{z}$ to denote the complex conjugate of a complex number $z$. The modulus of $z$ is denoted by $|z|$. Let $\mathbf{1} \in \mathbb{R}^n$ be the $n$-dimensional column vector of ones. For $x = [x_1, \ldots, x_n]^T \in \mathbb{C}^n$, let $\|x\|_1$ be its 1-norm.
Denote by $\mathbb{T}$ the unit circle, i.e., $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. It is easy to see that $\mathbb{T}$ is an abelian group under multiplication. For $\zeta = [\zeta_1, \ldots, \zeta_n]^T \in \mathbb{T}^n$, let $D_\zeta := \text{diag}(\zeta)$ denote the diagonal matrix with $i$th diagonal entry $\zeta_i$. Finally, we have $j = \sqrt{-1}$.

2 Preliminaries

In this section we present some useful results on complex-weighted graphs, which are instrumental in the investigation of modulus consensus problems. We believe that these results themselves are also interesting from the graph theory point of view. Before proceeding, we introduce some basic concepts of complex-weighted graphs.

The digraph associated with a complex matrix $A = [a_{ij}]_{n \times n}$ is denoted by $G(A) = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, n\}$ is the vertex set and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set. An edge $(j, i) \in \mathcal{E}$, i.e., there exists an edge from $j$ to $i$ if and only if $a_{ij} \neq 0$. The matrix $A$ is usually called the adjacency matrix of the digraph $G(A)$. Moreover, we assume that $a_{ii} = 0$, for $i = 1, \ldots, n$, i.e., $G(A)$ has no self-loop. For easy reference, we say $G(A)$ is complex, real and nonnegative if $A$ is complex, real and nonnegative, respectively. Let $\mathcal{N}_i$ be the neighbor set of agent $i$, defined as $\mathcal{N}_i = \{j : a_{ij} \neq 0\}$. A directed path in $G(A)$ from $i_1$ to $i_k$ is a sequence of distinct vertices $i_1, \ldots, i_k$ such that $(i_l, i_{l+1}) \in \mathcal{E}$ for $l = 1, \ldots, k-1$. A cycle is a path such that the origin and terminus are the same. The weight of a cycle is defined as the product of weights on all its edges. A cycle is said to be positive if it has a positive weight. The following definitions are used throughout this paper.

- A digraph is said to be balanced if all cycles are positive.
- A digraph has a directed spanning tree if there exists at least one vertex (called a root) which has a directed path to all other vertices.
- A digraph is strongly connected if for any two distinct vertices $i$ and $j$, there exists a directed path from $i$ to $j$.

It is clear that being strongly connected is stronger than having a directed spanning tree. All vertices of a strongly connected graph can serve as roots. They are equivalent when $A$ is Hermitian.

For a complex digraph $G(A)$, the Laplacian matrix $L = [l_{ij}]_{n \times n}$ of $G(A)$ is defined by $L = D - A$ where $D = \text{diag}(d_1, \ldots, d_n)$ is the modulus degree matrix of $G(A)$ with $d_i = \sum_{j \in \mathcal{N}_i} |a_{ij}|$. This definition appears in the literature on gain graphs (see, e.g., [28]), which can be thought as a generalization of standard Laplacian matrix of nonnegative graphs. We need the following definition on switching equivalence [28, 37].


Definition 2.1. Two graphs $\mathcal{G}(A_1)$ and $\mathcal{G}(A_2)$ are said to be switching equivalent, written as $\mathcal{G}(A_1) \sim \mathcal{G}(A_2)$, if there exists a vector $\zeta = [\zeta_1, \ldots, \zeta_n]^T \in \mathbb{T}^n$ such that $A_2 = D_\zeta^{-1}A_1D_\zeta$.

It is not difficult to see that the switching equivalence is an equivalence relation. We can see that switching equivalence preserves connectivity and balancedness. To establish our results, we need to investigate the properties of eigenvalues of the Laplacian matrix $L$.

2.1 Properties of the Laplacian

For brevity, we say $A$ is essentially nonnegative if $\mathcal{G}(A)$ is switching equivalent to a graph with a nonnegative adjacency matrix. By definition, it is clear to see that $A$ is essentially nonnegative if and only if there exists a diagonal matrix $D_\zeta$ such that $D_\zeta^{-1}A_1D_\zeta$ is nonnegative. By the Geršgorin disk theorem [12, Theorem 6.1.1], we see that all the eigenvalues of the Laplacian matrix $L$ of $A$ have nonnegative real parts and zero is the only possible eigenvalue with zero real part. We next further discuss the properties of eigenvalues of $L$ in terms of $\mathcal{G}(A)$.

Lemma 2.2. Zero is an eigenvalue of $L$ with an eigenvector $\zeta \in \mathbb{T}^n$ if and only if $A$ is essentially nonnegative.

Proof. (Sufficiency) Assume that $A$ is essentially nonnegative. That is, there exists a diagonal matrix $D_\zeta$ such that $A_1 = D_\zeta^{-1}AD_\zeta$ is nonnegative. Let $L_1$ be the Laplacian matrix of the nonnegative matrix $A_1$ and thus $L_1\mathbf{1} = 0$. A simple observation shows that these two Laplacian matrices are similar, i.e., $L_1 = D_\zeta^{-1}LD_\zeta$. Therefore, $L\zeta = 0$.

(Necessity) Let $L\zeta = 0$ with $\zeta \in \mathbb{T}^n$. Then we have $LD_\zeta\mathbf{1} = 0$ and so $D_\zeta^{-1}LD_\zeta\mathbf{1} = 0$. We can verify that $D_\zeta^{-1}LD_\zeta \in \mathbb{R}^{n \times n}$ has nonpositive off-diagonal entries. This implies that $A_1 = D_\zeta^{-1}AD_\zeta$ is nonnegative and thus $A$ is essentially nonnegative.

If we take the connectedness into account, then we can derive a stronger result.

Proposition 2.3. Zero is a simple eigenvalue of $L$ with an eigenvector $\xi \in \mathbb{T}^n$ if and only if $A$ is essentially nonnegative and $\mathcal{G}(A)$ has a spanning tree.

Proof. The proof follows from a sequence of equivalences:

$$(1) \iff (2) \iff (3) \iff (4).$$

Conditions (1)-(4) are given in the following.

(1) $A$ is essentially nonnegative and $\mathcal{G}(A)$ has a spanning tree.
(2) There exists a diagonal matrix $D_\zeta$ such that $A_1 = D_\zeta^{-1}AD_\zeta$ is nonnegative and $\mathcal{G}(A_1)$ has a spanning tree.

(3) There exists a diagonal matrix $D_\zeta$ such that $L_1 = D_\zeta^{-1}LD_\zeta$ has a simple zero eigenvalue with an eigenvector being $1$.

(4) $L$ has a simple zero eigenvalue with an eigenvector $\zeta \in \mathbb{T}^n$.

Here, the second one is from [29, Lemma 3.1] and the last one follows from the similarity.

Here a key issue is how to verify the essential nonnegativity of $A$. Thanks to the concept of balancedness of digraphs, we can derive a necessary and sufficient condition for $A$ to be essentially nonnegative. To this end, for a complex matrix $A$, we denote by $A_H = (A + A^*)/2$ the Hermitian part of $A$. Clearly, we have $A = A_H$ when $A$ is Hermitian.

**Proposition 2.4.** The complex matrix $A = [a_{ij}]_{n \times n}$ is essentially nonnegative if and only if $\mathcal{G}(A_H)$ is balanced and $a_{ij}a_{ji} \geq 0$ for all $1 \leq i, j \leq n$.

**Proof.** Since $A_H$ is Hermitian, it follows from [37] that $\mathcal{G}(A_H)$ is balanced if and only if $A_H$ is essentially nonnegative. Therefore, to complete the proof, we next show that $A$ is essentially nonnegative if and only if $A_H$ is essentially nonnegative and $a_{ij}a_{ji} \geq 0$ for all $1 \leq i, j \leq n$.

**Sufficiency:** By the condition that $a_{ij}a_{ji} \geq 0$, we have that $|a_{ij}a_{ji}| = \bar{a}_{ij}\bar{a}_{ji}$. Multiplying both sides by $a_{ij}$, we obtain that $|a_{ji}|a_{ij} = |a_{ij}|a_{ji}$. Consequently, for a diagonal matrix $D_\zeta$ with $\zeta = [\zeta_1, \ldots, \zeta_n]^T \in \mathbb{T}^n$, we have for $a_{ij} \neq 0$

$$\zeta_i^{-1}a_{ij} + \bar{a}_{ji}\zeta_j = \frac{1}{2} \frac{|a_{ji}|}{|a_{ij}|} \zeta_i^{-1}a_{ij}\zeta_j.$$ (1)

It thus follows that $D_\zeta^{-1}A_HD_\zeta$ being nonnegative implies $D_\zeta^{-1}AD_\zeta$ being nonnegative, which proves the sufficiency.

**Necessity:** Now assume that $A$ is essentially nonnegative. That is, there exists a diagonal matrix $D_\zeta$ such that $D_\zeta^{-1}AD_\zeta$ is nonnegative. Then we have

$$a_{ij}a_{ji} = (\zeta_i^{-1}a_{ij}\zeta_j)(\zeta_j^{-1}a_{ji}\zeta_i) \geq 0$$

from which we know that relation (1) follows. This implies that $D_\zeta^{-1}A_HD_\zeta$ is nonnegative. This concludes the proof.

The above proposition deals with the balancedness of $\mathcal{G}(A_H)$, instead of $\mathcal{G}(A)$ itself. The reason is that $\mathcal{G}(A)$ being balanced is not a sufficient condition for $A$ being essentially nonnegative, as shown in the following example.
Example 2.5. Consider the complex matrix \( A \) given by
\[
A = \begin{bmatrix}
0 & 2 & 0 \\
1 & 0 & 0 \\
-j & j & 0
\end{bmatrix}.
\]
It is straightforward that \( G(A) \) only has a positive cycle of length two and thus is balanced. However, we can check that \( A \) is not essentially nonnegative.

We next turn our attention to the case that \( A \) is not essentially nonnegative. When \( G(A) \) has a spanning tree and \( A \) is not essentially nonnegative, what we can only obtain from Proposition 2.3 is that either zero is not an eigenvalue of \( L \), or zero is an eigenvalue of \( L \) with no an associated eigenvector in \( \mathbb{T}^n \). To provide further understanding, we here consider the special case that \( A \) is Hermitian. In this case, \( L \) is also Hermitian. Then all eigenvalues of \( L \) are real. Let \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \) be the eigenvalues of \( L \). The positive semidefiniteness of \( L \), i.e., the fact that \( \lambda_1 \geq 0 \), can be obtained by the following observation. For \( z = [z_1, \ldots, z_n]^T \in \mathbb{C}^n \), we have
\[
z^* L z = \sum_{i=1}^{n} \bar{z}_i \left( \sum_{j \in \mathcal{N}_i} |a_{ij}| z_i - \sum_{j \in \mathcal{N}_i} a_{ij} z_j \right)
= \frac{1}{2} \sum_{(j,i) \in \mathcal{E}} (|a_{ij}| |z_i|^2 + |a_{ij}| |z_j|^2 - 2a_{ij} \bar{z}_i z_j)
= \frac{1}{2} \sum_{(j,i) \in \mathcal{E}} |a_{ij}| |z_i - \varphi(a_{ij}) z_j|^2
\]
where \( \varphi : \mathbb{C} \setminus \{0\} \to \mathbb{T} \) is defined by \( \varphi(a_{ij}) = \frac{a_{ij}}{|a_{ij}|} \). Based on (2), we have the following lemma.

Lemma 2.6. Let \( A \) be Hermitian. Assume that \( G(A) \) has a spanning tree. Then \( L \) is positive definite, i.e., \( \lambda_1 > 0 \), if and only if \( A \) is not essentially nonnegative.

Proof. We only show the sufficiency since the necessity follows directly from Proposition 2.3. Assume the contrary. Then there exists a nonzero vector \( y = [y_1, \ldots, y_n]^T \in \mathbb{C}^n \) such that \( Ly = 0 \). By (2),
\[
y^* L y = \frac{1}{2} \sum_{(j,i) \in \mathcal{E}} |a_{ij}| \left| y_i - \frac{a_{ij}}{|a_{ij}|} y_j \right|^2 = 0.
\]
This implies that \( y_i = \frac{a_{ij}}{|a_{ij}|} y_j \) for \((j, i) \in \mathcal{E}\) and so \( |y_i| = |y_j| \) for \((j, i) \in \mathcal{E}\). Note that for \( G(A) \) with \( A \) being Hermitian, having a spanning tree is equivalent to the strong connectivity. Then we conclude that \( |y_i| = |y_j| \) for all \( i, j = 1, \ldots, n \). Without loss of generality, we assume that \( y \in \mathbb{T}^n \). It follows from Lemma 2.2 that \( A \) is essentially nonnegative, a contradiction. \( \square \)
On the other hand, for the general case that $A$ is not Hermitian, we cannot conclude that $L$ has no zero eigenvalue when $\mathcal{G}(A)$ has a spanning tree and $A$ is not essentially nonnegative. Example 3.9 provides such an example.

2.2 Perturbations to the Laplacian

We start with the definition of the diagonal matrix $E_{ii}$, which has the property that only $i$th diagonal element, say $\varepsilon$, is positive and the remaining elements are all zero. In this subsection, we study the property of a perturbed Laplacian matrix $L + E_{ii}$ for some $1 \leq i \leq n$. For an essentially nonnegative matrix $A$, Proposition 2.3 shows that $L$ has a simple zero eigenvalue if and only if $\mathcal{G}(A)$ has a spanning tree. It is natural to ask whether in this case there exists an index $i$ such that $L + E_{ii}$ has no zero eigenvalue. The following lemma gives a positive answer.

**Lemma 2.7.** For an essentially nonnegative matrix $A$, all eigenvalues of $L + E_{ii}$ have positive real parts if and only if $i$ is the root of a spanning tree of $\mathcal{G}(A)$.

**Remark 2.8.** The real analogue of Lemma 2.7 has been extensively used in the literature on pinning synchronization, see, e.g., [4, 33]. We want to mention that our result is also very closely related to the classical theory on irreducibly diagonally dominant matrices [12, Corollary 6.2.27]. We here give an alternative proof by using the well-known Kirchhoff’s Matrix-Tree Theorem.

**Proof.** Since $A$ is essentially nonnegative, we can prove this lemma for nonnegative $A$ without loss of generality.

(Sufficiency) Suppose that $\mathcal{G}(A)$ has a spanning tree. For simplicity, let vertex 1 be the root of a spanning tree. We need to show that all eigenvalues of $L + E_{11}$ have positive real parts. In light of the Geršgorin disk theorem [12, Theorem 6.1.1], we can see that it is equivalent to show that zero is not the eigenvalue of $L + E_{11}$. Suppose, by way of contradiction, that there exists a nonzero $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$ such that $(L + E_{11})x = 0$. On the other hand, by the well-known Kirchhoff’s Matrix-Tree Theorem in graph theory [5], we know that there exists a nonnegative vector $c = [c_1, \ldots, c_n]^T \in \mathbb{R}^n$ such that $c^TL = 0$ and $c_1 > 0$. In fact, $c_1$ is the sum of the weights of all possible spanning tree of $\mathcal{G}(A)$ rooted at vertex 1. As a result, we have $0 = c^T(L + E_{11})x = c^TE_{11}x = c_1\varepsilon x_1$, which implies that $x_1 = 0$. Again from $(L + E_{11})x = 0$, we see that $Lx = 0$, which, together with the existence of spanning tree for $\mathcal{G}(A)$, concludes that $x \in \text{span}\{1\}$. Thus, we have $x = 0$ by $x_1 = 0$. We get a contradiction.

(Necessity) Suppose that all eigenvalues of $L + E_{ii}$ have positive real parts. The proof of the sufficiency implies that we only need to show that $\mathcal{G}(A)$ has a spanning tree. By contradiction, we assume that $\mathcal{G}(A)$ has no spanning tree. Let $T_1$ be a maximal tree of $\mathcal{G}(A)$ in the sense of the number of contained vertices. Let $V_1$ be the vertex set of $T$. 

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We see that $V_1$ can be partitioned into two vertex subset $V_{11}$ and $V_{12}$ satisfying that $V_{11}$ contains the root of $T_1$ and $G[V_{11}]$, the subgraph of $G(A)$ induced by $V_{11}$, is strongly connected. Moreover, all directed edges between $V_{11}$ and $V_{12}$ are from $V_{11}$ to $V_{12}$. Note that $V_{12}$ can be empty. Let $V_2$ be the set of remaining vertices, i.e., $V_2 = \{1, \ldots, n\} \setminus V_1$. Since $T_1(A)$ contains no spanning tree, we have that $V_2$ is nonempty. In view of the fact that $T_1$ is the maximal tree of $G(A)$, we see that there exists no directed edge between $V_{11}$ to $V_2$ and the only possible directed edges between $V_{12}$ and $V_2$ are from $V_2$ to $V_{12}$. As a result, by relabeling the vertices if necessary, we can conclude that the standard Laplacian matrix $L$ of $G(A)$ takes the partitioned form:

$$L = \begin{bmatrix} L_{11} & 0 & 0 \\ * & L_{22} & * \\ 0 & 0 & L_{33} \end{bmatrix}$$

where “*” denotes possible nonzero entries, $L_{11}$ and $L_{33}$ correspond to the standard Laplacian matrices of $G[V_{11}]$ and $G[V_2]$, respectively. Since both $L_{11}$ and $L_{33}$ have zero eigenvalue, we have that $y_1^T L_{11} = 0$ and $y_3^T L_{33} = 0$ for two nonzero vectors $y_1$ and $y_3$ with compatible dimensions. Consequently, both the linear independent vectors $[y_1^T, 0, 0]^T$, $[0, 0, y_3^T]^T \in \mathbb{R}^n$ are left eigenvectors of $L$ corresponding to the zero eigenvalue. This is a contradiction to the existence of an index $i$ such that all eigenvalues of $L + E_{ii}$ have positive real parts. We complete the proof.

**Remark 2.9.** It is worth mentioning that the above results for the Laplacian matrix $L$ cannot be deduced from the standard approaches such as the Geršgorin disk theorem and diagonal dominance analysis [12]. This is because these standard arguments only use the information about the absolute values of the off-diagonal entries.

## 3 Modulus consensus

In this section, we first formulate the basic consensus problems to be studied. Then the necessary and sufficient conditions for modulus consensus are derived with the help of the preliminary results established in Section 2.

### 3.1 Consensus problems

For a group of $n$ agents, we consider the continuous-time (CT) consensus protocol over complex field

$$\dot{z}_i(t) = u_i(t), \ t \geq 0$$

(3)

where $z_i(t) \in \mathbb{C}$ and $u_i(t) \in \mathbb{C}$ are the state and input of agent $i$, respectively. We also consider the corresponding discrete-time (DT) protocol over complex field

$$z_i(k+1) = z_i(k) + u_i(k), \ k = 0, 1, \ldots$$

(4)
The communications between agents are modeled as a complex graph $G(A)$. The control input $u_i$ is designed, in a distributed way, as

$$u_i = -\kappa \sum_{j \in N_i} (|a_{ij}| z_i - a_{ij} z_j),$$

where $\kappa > 0$ is a fixed control gain. Then we have the following two systems described as

$$\dot{z}_i(t) = -\kappa \sum_{j \in N_i} (|a_{ij}| z_i - a_{ij} z_j)$$

and

$$z_i(k+1) = z_i(k) - \kappa \sum_{j \in N_i} (|a_{ij}| z_i - a_{ij} z_j).$$

Denote by $z = (z_1, \ldots, z_n)^T \in \mathbb{C}^n$ the aggregate position vector of $n$ agents. With the Laplacian matrix $L$ of $G(A)$, these two systems can be rewritten in more compact forms:

$$\dot{z}(t) = -\kappa Lz(t) \quad (5)$$

in the CT case and

$$z(k+1) = z(k) - \kappa Lz(k) \quad (6)$$

in the DT case. Inspired by the consensus in real-weighted networks [1, 25, 26], we introduce the following definition.

**Definition 3.1.** We say that the CT system (5) (or the DT system (6)) reaches the **modulus consensus** if $\lim_{t \to \infty} |z_i(t)| = a > 0$ (or $\lim_{k \to \infty} |z_i(k)| = a > 0$) for $i = 1, \ldots, n$.

### 3.2 Consensus conditions

In this subsection we establish the necessary and sufficiency conditions for modulus consensus. Let $A$ be an essentially nonnegative complex matrix. If $G(A)$ has a spanning tree, then it follows from Proposition 2.3 that $L$ has a simple eigenvalue at zero with an associated eigenvector $\zeta \in \mathbb{T}^n$. Thus, we have $A_1 = D_{\zeta}^{-1} A D_{\zeta}$ is nonnegative and $D_{\zeta}^{-1} L D_{\zeta}$ has a simple eigenvalue at zero with an associated eigenvector $1$. In the standard consensus theory [31], it is well-known that $D_{\zeta}^{-1} L D_{\zeta}$ has a nonnegative left eigenvector $\nu = [\nu_1, \ldots, \nu_n]^T$ corresponding to eigenvalue zero, i.e., $\nu^T (D_{\zeta}^{-1} L D_{\zeta}) = 0$ and $\nu_i \geq 0$ for $i = 1, \ldots, n$. We assume that $\|\nu\|_1 = 1$. Letting $\eta = D_{\zeta}^{-1} \nu = [\eta_1, \ldots, \eta_n]^T$, we have $\|\eta\|_1 = 1$ and $\eta^T L = 0$. We first state a necessary and sufficient condition for modulus consensus of the CT system (5).

**Theorem 3.2.** The CT system (5) reaches modulus consensus if and only if $A$ is essentially nonnegative and $G(A)$ has a spanning tree. In this case, we have

$$\lim_{t \to \infty} z(t) = (\eta^T z(0)) \zeta.$$
Proof. Assume that $A$ is essentially nonnegative and $\mathcal{G}(A)$ has a spanning tree. By Proposition 2.3, we have $L$ has a simple eigenvalue at zero with an associated eigenvector $\zeta \in \mathbb{T}^n$. Thus, we conclude that $A_1 = D_\zeta^{-1}AD_\zeta$ is nonnegative and $D_\zeta^{-1}LD_\zeta$ has a simple eigenvalue at zero with an eigenvector $\mathbf{1}$. Let $z = D_\zeta x$. By system (5), we can see that $x$ satisfies the system
\[
\dot{x} = -\kappa D_\zeta^{-1}LD_\zeta x.
\]
Note that this is the standard consensus problem. From [31], it follows that
\[
\lim_{t \to \infty} x(t) = \nu^T x(0) \mathbf{1} = \nu^T D_\zeta^{-1}D_\zeta x(0) \mathbf{1}.
\]
This is equivalent to
\[
\lim_{t \to \infty} z(t) = (\nu^T D_\zeta^{-1}D_\zeta x(0)) \mathbf{1} = (\eta^T D_\zeta^{-1}D_\zeta x(0)) \zeta.
\]

To show the other direction, we now assume that the system (5) reaches modulus consensus but $\mathcal{G}(A)$ does not have a spanning tree. Let $T_1$ be a maximal subtree of $\mathcal{G}$. Note that $T_1$ is a spanning tree of subgraph $\mathcal{G}_1$ of $\mathcal{G}(A)$. Denote by $\mathcal{G}_2$ the subgraph induced by vertices not belonging to $\mathcal{G}_1$. It is easy to see that there does not exist edge from $\mathcal{G}_1$ to $\mathcal{G}_2$ since otherwise $T_1$ is not a maximal subtree. All possible edges between $\mathcal{G}_1$ and $\mathcal{G}_2$ are from $\mathcal{G}_2$ to $\mathcal{G}_1$, and moreover we can see that there is no directed path from a vertex in $\mathcal{G}_2$ to the root of $T_1$ by $T_1$ being a maximal subtree again. Therefore it is impossible to reach the modulus consensus between the root of $T_1$ and vertices of $\mathcal{G}_2$. This implies that the system (5) cannot reach modulus consensus. We obtain a contradiction. Hence $\mathcal{G}(A)$ have a spanning tree. On the other hand, since the system (5) reaches modulus consensus we can see that the solutions $y = [y_1, \ldots, y_n]^T$ of the equation $Ly = 0$ always have the property $|y_i| = |y_j|$ for all $i, j = 1, \ldots, n$. Namely, zero is an eigenvalue of $L$ with an eigenvector $\zeta \in \mathbb{T}^n$. It thus follows from Lemma 2.2 that $A$ is essentially nonnegative. We complete the proof of Theorem 3.2.

For $\mathcal{G}(A)$, define the maximum modulus degree $\Delta$ by $\Delta = \max_{1 \leq i \leq n} d_i$. We are now in a position to state the modulus consensus result for the DT system (6).

**Theorem 3.3.** Assume that the input gain $\kappa$ is such that $0 < \kappa < 1/\Delta$. Then the DT system (6) reaches modulus consensus if and only if $A$ is essentially nonnegative and $\mathcal{G}(A)$ has a spanning tree. In this case, we have
\[
\lim_{k \to \infty} z(k) = (\eta^T D_\zeta z(0)) \zeta.
\]

**Proof.** Assume that $A$ is essentially nonnegative and $\mathcal{G}(A)$ has a spanning tree. By Propositions 2.3, we have $L$ has a simple eigenvalue at zero with an associated eigenvector $\zeta \in \mathbb{T}^n$. Thus, we conclude that $A_1 = D_\zeta^{-1}AD_\zeta$ is nonnegative and $D_\zeta^{-1}LD_\zeta$ has
a simple eigenvalue at zero with an associated eigenvector $1$. Let $z = D_ζ x$. By system (6), we can see that $x$ satisfies the system
\[ x(k + 1) = (I - κD_ζ^{-1}LD_ζ)x(k). \]
Note that this is the standard consensus problem. From [31], it follows that
\[ \lim_{k \to \infty} x(k) = ν^T x(0)1 = ν^T D_ζ^{-1}z(0)1. \]
This is equivalent to
\[ \lim_{k \to \infty} z(k) = (ν^T D_ζ^{-1}z(0))D_ζ1 = (η^T z(0))ζ. \]

To show the other direction, we now assume that the system (6) reaches modulus consensus. Using the same arguments as for the CT system (5) above, we can see that $G(A)$ have a spanning tree. On the other hand, based on the Geršgorin disk theorem [12, Theorem 6.1.1], all the eigenvalues of $-κL$ are located in the union of the following $n$ disks:
\[ \left\{ z \in \mathbb{C} : \left| z + κ \sum_{j \in \mathcal{N}_i} |a_{ij}| \right| \leq κ \sum_{j \in \mathcal{N}_i} |a_{ij}| \right\}, \quad i = 1, \ldots, n. \]
Clearly, all these $n$ disks are contained in the largest disk defined by
\[ \{ z \in \mathbb{C} : |z + κΔ| \leq κΔ \}. \]
Noting that $0 < κ < 1/Δ$, we can see that the largest disk is contained in the region \( \{ z \in \mathbb{C} : |z + 1| < 1 \} \cup \{0\} \). By translation, we have all the eigenvalues of $I - κL$ are located in the following region:
\[ \{ z \in \mathbb{C} : |z| < 1 \} \cup \{1\}. \]
Since the system (6) reaches modulus consensus we can see that 1 must be the eigenvalue of $I - κL$. All other eigenvalue of $I - κL$ have the modulus strictly smaller than 1. Moreover, if $y = [y_1, \ldots, y_n]^T$ is an eigenvector of $I - κL$ corresponding to eigenvalue 1, then $|y_i| = |y_j| > 0$ for $i, j = 1, \ldots, n$. That is, zero is an eigenvalue of $L$ with an eigenvalue $ζ \in \mathbb{T}^n$. It thus follows from Lemma 2.2 that $A$ is essentially nonnegative. We complete the proof of Theorem 3.3.

**Remark 3.4.** With the help of Proposition 2.4, we have that the condition that $A$ is essentially nonnegative can be examined by the condition that $G(A_H)$ is balanced. For the special case when $A$ is Hermitian, Theorems 3.2 and 3.3 take a simpler form. As an example, we consider the CT system (5) with $A$ being Hermitian. In this case, it follows from Proposition 2.4 that $A$ is essentially nonnegative if and only if $G(A)$ is balanced. Then we have that the CT system (5) reaches modulus consensus if and only if $G(A)$ has a spanning tree and is balanced. In this case,
\[ \lim_{t \to \infty} z(t) = \frac{1}{n}(ζ^*z(0))ζ. \]
In addition, in view of Lemma 2.6 it yields that $\lim_{t \to \infty} z(t) = 0$ when $G(A)$ has a spanning tree and is unbalanced.
3.3 Pinning modulus consensus

In this subsection, our aim is to make all agents achieve the modulus consensus with a specified modulus via pinning a single agent. Consider an essentially nonnegative matrix $A$. Lemma 2.2 shows that there exists a vector $\zeta \in \mathbb{T}^n$ satisfying $L\zeta = 0$. Based on systems (5) and (6) we respectively discuss the following CT system and DT system:

\[
\dot{z}(t) = -\kappa Lz(t) - \kappa E_{ii}(z(t) - \tilde{z}\zeta)
\] (7)

and

\[
z(k + 1) = z(k) - \kappa Lz(k) - \kappa E_{ii}(z(k) - \tilde{z}\zeta),
\] (8)

where the diagonal matrix $E_{ii}$ is defined in Section 2.2 and the specified modulus is given by $|\tilde{z}|$ with $\tilde{z} \in \mathbb{C}$. We can see that the pinning controller is only added to a single agent $i$. We are now in a position to state the main result for the CT system (7).

**Theorem 3.5.** Let $A$ be essentially nonnegative. Then the CT system (7) achieves the modulus consensus with a specified modulus $|\tilde{z}|$ if and only if $i$ is the root of a spanning tree of $G(A)$. In this case,

\[
\lim_{t \to \infty} z(t) = \tilde{z}\zeta.
\]

**Proof.** Letting $e(t) = z(t) - \tilde{z}\zeta$, the CT system (7) leads to that

\[
\dot{e}(t) = -\kappa(L + E_{ii})e(t).
\] (9)

Note that the condition that the CT system (7) achieves the modulus consensus with a specified modulus $|\tilde{z}|$ is equivalent to the condition that the system (9) is asymptotically stable. In view of Lemma 2.7 this is equivalent to the condition that $i$ is the root of a spanning tree of $G(A)$.

Similarly, using Lemma 2.7 we obtain the corresponding result for the DT system (8).

**Theorem 3.6.** Assume that

\[
0 < \kappa < \frac{1}{\Delta + \varepsilon}.
\]

Let $A$ be essentially nonnegative. Then the CT system (8) achieves the modulus consensus with a specified modulus $|\tilde{z}|$ if and only if $i$ is the root of a spanning tree of $G(A)$. In this case,

\[
\lim_{k \to \infty} z(k) = \tilde{z}\zeta.
\]
3.4 Illustrative examples

In this subsection we present some examples to illustrate our main results.

**Example 3.7.** Consider the complex graph $\mathcal{G}(A)$ illustrated in Figure 1 with adjacency matrix

$$
A = \begin{bmatrix}
0 & 0 & -j & 0 \\
1 & 0 & 0 & 0 \\
0 & j & 0 & 0 \\
0 & 1 + j & 0 & 0
\end{bmatrix}.
$$

It is trivial that $\mathcal{G}(A)$ has a spanning tree. Since $\mathcal{G}(A_H)$ is balanced, Proposition 2.4 implies that $A$ is essentially nonnegative. Define $\zeta = [1, 1, j, e^{j\frac{\pi}{4}}]^T \in \mathbb{T}^4$. Then we have

$$
A_1 = D_{\zeta}^{-1} A D_{\zeta} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & \sqrt{2} & 0 & 0
\end{bmatrix}.
$$

The Laplacian matrix $L$ of $A$ is given by

$$
L = \begin{bmatrix}
1 & 0 & j & 0 \\
-1 & 1 & 0 & 0 \\
0 & -j & 1 & 0 \\
0 & -1 - j & 0 & \sqrt{2}
\end{bmatrix}.
$$

The set of eigenvalues of $L$ is $\{0, \sqrt{2}, 3/2 + \sqrt{3}j/2, 3/2 - \sqrt{3}j/2\}$. The vector $\zeta$ is an eigenvector associated with eigenvalue zero. A simulation under system (5) is given in Figure 2, which shows that the modulus consensus is reached asymptotically. This confirms the analytical results of Theorems 3.2 and 3.3.

**Example 3.8.** Consider the complex graph $\mathcal{G}(A)$ illustrated in Figure 3 with Hermitian adjacency matrix

$$
A = \begin{bmatrix}
0 & 1 - j & -1 - j \\
1 + j & 0 & -j \\
-1 + j & j & 0
\end{bmatrix}.
$$
It is easy to verify that $\mathcal{G}(A)$ is strongly connected and balanced. It follows that $A$ is essentially nonnegative. Define

$$\zeta = [e^{-j\pi/2}, e^{-j\pi/4}, e^{j\pi/4}]^T \in \mathbb{T}^3.$$  

Then we have

$$A_1 = D_\zeta^{-1}AD_\zeta = \begin{bmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 0 & 1 \\ \sqrt{2} & 1 & 0 \end{bmatrix}.$$  

The Laplacian matrix $L$ of $A$ is given by

$$L = \begin{bmatrix} 2\sqrt{2} & -1 + j & 1 + j \\ -1 - j & 1 + \sqrt{2} & j \\ 1 - j & -j & 1 + \sqrt{2} \end{bmatrix}.$$  

The set of eigenvalues of $L$ is $\{0, 3.41, 4.24\}$. The vector $\zeta$ is an eigenvector associated with eigenvalue zero. The simulation in Figure 4 shows that the modulus consensus is reached asymptotically. This agrees with the analytical results in Remark 3.4.

**Example 3.9.** Consider the complex graph $\mathcal{G}(A)$ illustrated in Figure 5 with adjacency
Figure 4: Modulus consensus process of the agents.

Figure 5: Unbalanced graph.

matrix

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 - j & 0 & 0 \\
0 & j & 0 & 0 & 0 & 0 \\
j & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & - j & 0
\end{bmatrix}
\]

We can see that \( \mathcal{G}(A) \) has a spanning tree and \( A \) is not essentially nonnegative since \( \mathcal{G}(A_H) \) is unbalanced. We can verify that zero is an eigenvalue of \( L \). The simulation in Figure 6 shows that the modulus consensus cannot be reached.

### 3.5 Modulus consensus under switching topology

We now extend the modulus consensus results obtained in Section 3.2 to the case of switching topology. Consider \( m \) possible topologies defined by complex adjacency matrices \( \{A_1, \ldots, A_m\} \), for which we assume that \( A_p \) is essentially nonnegative for \( p = 1, \ldots, m \). We further assume that there exists a common diagonal matrix \( D_\xi \) with \( \xi \in \mathbb{T}^n \) such that all the \( D_\xi^{-1}A_pD_\xi \)'s are nonnegative. This can be guaranteed by the restriction that for any \( i \neq j \in \{1, \ldots, n\} \) the entries \( a_{p,ij} \) have the same arguments.
Figure 6: Trajectories of the agents which mean that modulus consensus cannot be reached.

for all $p \in \{1, \ldots, m\}$. We first consider the CT system under switching topology

$$\dot{z}(t) = -\kappa L_{\sigma(t)} z(t)$$  

(10)

where $\sigma(t) : [0, \infty) \to \{1, \ldots, m\}$ is a right continuous piecewise constant switching signal with switching times $\{t_i, i = 0, 1, \ldots\}$.

**Theorem 3.10.** The system (10) achieves modulus consensus if there exists a subsequence $\{t_{i_k}\}$ of $\{t_i, i = 0, 1, \ldots\}$ such that $t_{i_{k+1}} - t_{i_k}$ is uniformly bounded for $k \geq 0$ and the union of digraphs $\mathcal{G}(A_{\sigma(t)})$ over each time interval $[t_{i_k}, t_{i_{k+1}})$ has a spanning tree.

**Proof.** By assumption on $\{A_1, \ldots, A_m\}$, we know that they can be simultaneously transformed into nonnegative matrices by a common diagonal matrix $D_\zeta$. Taking the transformation $z(t) = D_\zeta x(t)$, system (5) now becomes

$$\dot{x}(t) = -\kappa D_\zeta^{-1} L_{\sigma(t)} D_\zeta x(t)$$

which is the standard consensus problem since $D_\zeta^{-1} L_{\sigma(t)} D_\zeta$ is the standard Laplacian matrix. From [32, Theorem 3.12], the result follows. \qed

We now consider the DT system under switching topology

$$z(k + 1) = z(k) - \kappa L_{\sigma(k)} z(k).$$  

(11)

We assume that $0 < \kappa < 1/(\max_{1 \leq p \leq m} \Delta_p)$ where $\Delta_p$ is maximum modulus degree of $\mathcal{G}(A_p)$ for $p = 1, \ldots, m$.

**Theorem 3.11.** The system (11) achieves modulus consensus if there exists a subsequence $\{k_j\}$ of $\{k, k = 0, 1, \ldots\}$ such that $k_{j+1} - k_j$ is uniformly bounded for $j \geq 0$ and the union of digraphs $\mathcal{G}(A_{\sigma(k)})$ over each time interval $[k_j, k_{j+1})$ has a spanning tree.

**Proof.** The result follows from [32, Theorem 3.10] together with the common switching equivalence transformation used in the proof of Theorem 3.10. \qed
3.6 Bipartite consensus revisited

Recently, bipartite consensus problems have been studied in [1]. In this subsection, we derive some bipartite consensus results from our modulus consensus results obtained in Section 3.2. We will see that these bipartite consensus results improve the existing results in the literature.

Let $G(A)$ be a signed graph, i.e., $A = [a_{ij}]_{n \times n} \in \mathbb{R}^{n \times n}$ and $a_{ij}$ can be negative. By bipartite consensus, we mean on a signed graph, all agents converge to a consensus value whose absolute value is the same for the all agents except for the sign. The state $z$ is now restricted to the field of real numbers $\mathbb{R}$, denoted by $x$. Then the two systems (5) and (6) reduces to the standard consensus systems:

\[
\dot{x}(t) = -\kappa Lx(t) \quad (12)
\]

and

\[
x(k+1) = x(k) - \kappa Lx(k). \quad (13)
\]

With the above two systems and based on Theorems 3.2 and 3.3 in Section 3.2, we can derive the bipartite consensus results on signed graphs.

**Corollary 3.12.** Let $G(A)$ be a signed digraph. Then the CT system (12) achieves bipartite consensus asymptotically if and only if $A$ is essentially nonnegative and $G(A)$ has a spanning tree. In this case, for any initial state $x(0) \in \mathbb{R}^n$, we have

\[
\lim_{t \to \infty} x(t) = (\eta^T x(0))\sigma
\]

where $\sigma = [\sigma_1, \ldots, \sigma_n]^T \in \{\pm 1\}^n$ such that $D_\sigma AD_\sigma$ is nonnegative matrix and $\eta^T L = 0$ with $\eta = [\eta_1, \ldots, \eta_n]^T \in \mathbb{R}^n$ and $\|\eta\|_1 = 1$.

**Corollary 3.13.** Let $G(A)$ be a signed digraph. Then the DT system (13) with $0 < \kappa < 1/\Delta$ achieves bipartite consensus asymptotically if and only if $A$ is essentially nonnegative and $G(A)$ has a spanning tree. In this case, for any initial state $x(0) \in \mathbb{R}^n$, we have

\[
\lim_{k \to \infty} x(k) = (\eta^T x(0))\sigma
\]

where $\eta$ and $\sigma$ are defined as in Corollary 3.12.

**Remark 3.14.** Corollary 3.12 indicates that bipartite consensus can be achieved under a condition weaker than that given in Theorems 1 and 2 in [1]. In addition, we also obtain a similar necessary and sufficient condition for bipartite consensus of the DT system (13). The bipartite consensus problems under switching topology can be discussed in the same manner as in the previous subsection and will be omitted here.

4 Circular formation

As an important application, our modulus consensus results are used to study the circular formation problems in this section.
4.1 Circular formation with relative positions

We consider a group of $n$ agents moving in a plane labeled as $1, \ldots, n$. The positions of agents are denoted by complex numbers $z_1, \ldots, z_n \in \mathbb{C}$. Let $z = [z_1, \ldots, z_n]^T$ be the aggregate position vector of $n$ agents. The circular formation with relative positions we are concerned with is defined as follows.

**Definition 4.1.** Given $q \in \mathbb{C}$ and $\theta = [0, \theta_2, \ldots, \theta_n] \in [0, 2\pi)^n$, we say the group of $n$ agents forms a circular formation $[q, \theta]$ if their positions are such that $z_i - q = (z_1 - q)e^{j\theta_i}$ for $i = 2, \ldots, n$.

Clearly, in a circular formation $[q, \theta]$, all agents are distributed in a circle centered at $q$ with a prespecified angle $\theta$ separations between agents. Note that the circular formation $[q, \theta]$ has the unspecified radius and absolute phases. This is the reason why it is called the circular formation with relative positions. Our goal is to design distributed algorithms to realize the circular formation. Let $G(A)$ be a graph with nonnegative adjacency matrix $A = (a_{ij})_{n \times n}$. The graph $G(A)$ can be undirected or directed. We define the standard Laplacian matrix $L = (l_{ij})_{n \times n}$ of $A$ by

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j \\ d_i = \sum_{k \in N_i} a_{ik}, & j = i. \end{cases}$$

Then we have $\Delta = \max_{1 \leq i \leq n} l_{ii}$. For given circular formation $[q, \theta]$, we define

$$D_\zeta = \text{diag}(\zeta) \text{ with } \zeta = [1, e^{j\theta_2}, \ldots, e^{j\theta_n}].$$

It is easy to see that $\zeta \in \mathbb{T}^n$. We are now in a position to present our algorithms: for the CT system

$$\dot{z}(t) = -\kappa D_\zeta LD_\zeta^{-1}(z(t) - q1) \quad (14)$$

and for the DT system

$$z(k+1) = z(k) - \kappa D_\zeta LD_\zeta^{-1}(z(k) - q1). \quad (15)$$

We now state the circular formation result for the CT system (14).

**Theorem 4.2.** A group of $n$ agents under the CT system (14) with initial state $z(0)$ asymptotically reaches the circular formation $[q, \theta]$ if and only if the communication topology $G(A)$ has a spanning tree. In this case, for $i = 2, \ldots, n$, we have

$$\lim_{t \to \infty} (z_i(t) - q) = \lim_{t \to \infty} (z_1(t) - q)e^{j\theta_i} = (\nu^T D_\zeta^{-1}(z(0) - q1)) e^{j\theta_i}$$

where $\nu \in \mathbb{R}^n$ is a nonnegative vector satisfying $\nu^T L = 0$ and $\|\nu\|_1 = 1$. 

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Proof. Assume that $\mathcal{G}(A)$ has a spanning tree. Define $\delta_i(t) = z_i(t) - q$ and let $\delta(t) = [\delta_1(t), \ldots, \delta_n(t)]^T$. Then $\delta(t)$ satisfies the following system
\[
\dot{\delta}(t) = -\kappa D \zeta L D^{-1}_\zeta \delta(t).
\]
(16)

Let $A = D \zeta A D^{-1}_\zeta$. It is trivial that $A$ is essentially nonnegative. As a result, we can apply Theorem 3.2 to get
\[
\lim_{t \to \infty} \delta(t) = (\nu^T D^{-1}_\zeta \delta(0)) \zeta.
\]

This is equivalent to
\[
\lim_{t \to \infty} (z_i(t) - q) = \lim_{t \to \infty} (z_1(t) - q) e^{i\theta_i} = (\nu^T D^{-1}_\zeta (z(0) - q1)) e^{i\theta_i}.
\]

Conversely, if the circular formation $[q, \theta]$ can be achieved, then system (16) can reach modulus consensus. It thus follows from Theorem 3.2 that $\mathcal{G}(A)$ has a spanning tree.

Similarly, we can obtain the circular formation result for the DT system (15). We omit the proof.

**Theorem 4.3.** Let $0 < \kappa < 1/\Delta$. A group of $n$ agents under the system (15) with initial state $z(0)$ asymptotically reaches the circular formation $[q, \theta]$ if and only if the communication topology $\mathcal{G}(A)$ has a spanning tree. In this case, for $i = 2, \ldots, n$, we have
\[
\lim_{k \to \infty} (z_i(k) - q) = \lim_{k \to \infty} (z_1(k) - q) e^{i\theta_i} = (\nu^T D^{-1}_\zeta (z(0) - q1)) e^{i\theta_i}
\]
where the vector $\nu$ is defined as in Theorem 4.2.

**Remark 4.4.**

(i) In the case that the communication topology $\mathcal{G}(A)$ is undirected, the sufficient and necessary condition for the circular formation in Theorems 4.2 and 4.3 can be restated that $\mathcal{G}(A)$ is connected since having a spanning tree is equivalent to the connectedness for an undirected graph. In this case, we have $\nu = \frac{1}{n} \mathbf{1}$.

Also, it follows from Theorems 4.2 and 4.3 that the radius of the reached circle is depending on the initial state $z(0)$ and moreover, we have
\[
|\nu^T D^{-1}_\zeta (z(0) - q1)| \leq \max_{1 \leq i \leq n} |z_i(0) - q|.
\]

(ii) By introducing a moving center, we can also study the moving circular formation, i.e., the agents reach a predefined circular formation while moving. The two systems we consider here are
\[
\dot{z}(t) = -\kappa D \zeta L D^{-1}_\zeta (z(t) - w_0(t) \mathbf{1}) + \dot{w}_0(t) \mathbf{1}
\]
Figure 7: A tree

\begin{align*}
\mathbf{A} = &\begin{bmatrix}
0 & 1 & 2 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 2 & 0 & 0
\end{bmatrix}
\end{align*}

Figure 8: Reaching the circular formation \([0, \theta_5]\).

\[z(k+1) = z(k) - \kappa D_\zeta L D_\zeta^{-1} (z(k) - w_0(k)1) + (w_0(k+1) - w_0(k))1\]

where \(w_0 \in \mathbb{C}\) is the moving center. Similar to Theorems 4.2 and 4.3, the moving circular formation for these two systems can be obtained. To avoid repetitions, we omit them.

**Example 4.5.** Consider a group of five agents. The communication graph \(G(A)\) is illustrated in Figure 7 with adjacency matrix

\begin{align*}
\mathbf{A} = &\begin{bmatrix}
0 & 1 & 2 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 2 & 0 & 0
\end{bmatrix}
\end{align*}

We consider the circular formation \([0, \theta_5]\) defined by \(\theta_5 = [0, 0.4\pi, 0.8\pi, 1.2\pi, 1.6\pi]\). Note that \(G(A)\) is undirected and connected. We now can apply Theorems 4.2 and 4.3 to achieve the defined circular formation \([0, \theta_5]\). A simulation under the CT system (14) is shown in Figure 8. A simulation for moving circular formation is also shown in Figure 9.
4.2 Circular formation with relative positions under switching topology

We consider \( m \) possible communication topologies defined by nonnegative adjacency matrices \( \{A_1, \ldots, A_m\} \). We first consider the continuous-time circular formation algorithm under switching topology

\[
\dot{z}(t) = -\kappa D_\zeta L_{\sigma(t)} D_\zeta^{-1}(z(t) - q1) \tag{17}
\]

where \( \sigma(t) : [0, \infty) \to \{1, \ldots, m\} \) is a right continuous piecewise constant switching signal with switching times \( \{t_i, i = 0, 1, \ldots\} \). In light of Theorem 3.10, we obtain the following result.

**Theorem 4.6.** The system (17) achieves the circular formation \([q, \theta]\) if there exists a subsequence \( \{t_{ik}\} \) of \( \{t_i, i = 0, 1, \ldots\} \) such that \( t_{ik+1} - t_{ik} \) is uniformly bounded for \( k \geq 0 \) and the union of digraphs \( \mathcal{G}(A_{\sigma(t)}) \) over each time interval \([t_{ik}, t_{ik+1})\) has a spanning tree.

We now consider the discrete-time circular formation algorithm under switching topology

\[
z(k + 1) = z(k) - \kappa D_\zeta L_{\sigma(k)} D_\zeta^{-1}(z(k) - q1). \tag{18}
\]

Invoking Theorem 3.11, we have the following result.

**Theorem 4.7.** The system (18) achieves the circular formation \([q, \theta]\) if there exists a subsequence \( \{k_j\} \) of \( \{k, k = 0, 1, \ldots\} \) such that \( k_{j+1} - k_j \) is uniformly bounded for \( j \geq 0 \) and the union of digraphs \( \mathcal{G}(A_{\sigma(k)}) \) over each time interval \([k_j, k_{j+1})\) has a spanning tree.
4.3 Circular formation with absolute positions

In the above two subsections, we investigate the circular formation problem with relative positions, where all agents are required to converge to a common circle of predefined center, with preassigned angular separations and ordering along the circle. The radius and the angular positions of the circle are not prespecified and are determined by the initial condition and the graph topology. In some applications, however, we wish to set the specific positions of the circle completely. To completely determine the achieved circular formation, we now study the circular formation with absolute positions. We apply the pinning control strategy to one of the agents in system (14). Hence the pinning controlled CT system can be expressed as

\[
\dot{z}(t) = -\kappa D\zeta D\zeta^{-1}(z(t) - q1) - \kappa E_{ii}((z(t) - q1) - p1)
\]  

where the diagonal matrix \( E_{ii} \) is defined in Section 2.2 and \( p \in \mathbb{C} \). Recall that \( E_{ii} \) has the property that only the \( i \)th diagonal entry \( \varepsilon \) is positive and the remaining diagonal entries are all zero. Hence only the agent \( i \) is pinned. We say the agents reach the absolute-position circular formation \([q, \theta, p]\) if the circular formation \([q, \theta]\) is achieved with \( q + p \) being the absolute position of the pinned agent. The modulus of \( p \), namely \( |p| \), is the radius of the achieved circle. Analogously, the pinning controlled DT system is described by

\[
z(k + 1) = z(k) - \kappa D\zeta D\zeta^{-1}(z(k) - q1)
- \kappa E_{ii}((z(k) - q1) - p1).
\]  

We now study under what condition the pinning controlled systems (19) and (20) can achieve the absolute-position circular formation \([q, \theta, p]\).

We can now state the main results of this subsection. First, we consider the CT case.

**Theorem 4.8.** The CT system (19) achieves the absolute-position circular formation \([q, \theta, p]\) if and only if \( i \) is the root of a spanning tree of \( \mathcal{G}(A) \).

**Proof.** (Sufficiency) Assume that \( \mathcal{G}(A) \) has a spanning tree. Without loss of generality, let agent 1 be the root of a spanning tree. We now impose the pinning controller on agent 1. Let \( x(t) = D\zeta^{-1}(z(t) - q1) \). Noting that \( E_{11} = \text{diag}(\varepsilon, 0, \ldots, 0) \), it follows from system (19) that \( x(t) \) satisfies the system:

\[
\dot{x}(t) = -\kappa Lx(t) - \kappa E_{11}(x(t) - p1).
\]  

Now applying Theorem 3.5 to conclude that

\[
\lim_{t \to \infty} x(t) = p1
\]

which implies that

\[
\lim_{t \to \infty} (z(t) - q1) = p\zeta.
\]
That is,
\[
\lim_{t \to \infty} (z_1(t) - q) = p
\]
and
\[
\lim_{t \to \infty} (z_i(t) - q) = pe^{j\theta_i} \quad \text{for } i = 2, \ldots, n.
\]

(Necessity) We assume that the system (19) achieves the absolute-position circular formation \([q, \theta, p]\). This, together with the above proof, implies that system (21) achieves the modulus consensus with a specified modulus \(|p|\). Then it follows from Theorem 3.5 that \(i\) is the root of a spanning tree of \(G(A)\).

Then we consider the DT case. The proof is similar to the above arguments together with Theorem 3.6 and is omitted here.

**Theorem 4.9.** Let
\[
0 < \kappa < \frac{1}{\Delta + \varepsilon}.
\]
Then the DT system (20) achieves the absolute-position circular formation \([q, \theta, p]\) if and only if \(i\) is the root of a spanning tree of \(G(A)\).

**Remark 4.10.** We see that all agents can be pinned to the preassigned positions in the achieved circular formation only by pinning the root of a spanning tree. These results are based on the intuitive knowledge of the network topologies.

**Example 4.11.** Consider a group of five agents. The communication graph \(G(A)\) is illustrated in Figure 10 with adjacency matrix given by
\[
A = \begin{bmatrix}
0 & 0 & 2 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}.
\]
We consider the absolute-position circular formation \([0, \theta_5, p]\) defined by
\[
\theta_5 = [0, 0.5\pi, 0.75\pi, 1.25\pi, 1.5\pi]
\]
and \(p = 20j\). We see that \(G(A)\) has a spanning tree with a root vertex 1. In light of Theorems 4.8 and 4.9 the absolute-position circular formation \([0, \theta_5, p]\) can be achieved by the pinning control on agent 1. A simulation result is given in Figure 11.

**Remark 4.12.** Circular formation problems have been extensively studied in the literature, see, e.g., [3, 6, 8, 21]. Compared with the existing works, our results on the circular formation exhibit the following advantages: our algorithms are linear, very simple for analysis under both undirected and directed topology; we can control the angle separations and ordering between agents on the reached circle by simply adjusting the complex weights; the necessary and sufficient conditions for circular formation are obtained; all agents’ absolute positions in the reached circle can be controlled via a single controller; the discrete-time algorithm for circular formation is also investigated, which is, to the best of our knowledge, the first time to do so.
Figure 10: Communication graph.

Figure 11: Reaching the absolute-position circular formation $[0, \theta_5, p]$.

5 Conclusion

In this paper, we introduce a very general framework for investigating consensus problems in complex-weighted networks. With the careful choice of Laplacian matrix, necessary and sufficient conditions are obtained to reach modulus consensus which means that all agents reach an agreement in modulus but not in argument. Both continuous-time and discrete-time consensus protocols are considered. Our results reveal that modulus consensus relies not only on the connectedness of the networks, but also on the structural characterization of complex weights. The case of switching topology is also discussed. It has been shown that our results cover the bipartite consensus results on signed digraphs. As an interesting application, we use our consensus protocols to tackle the circular formation problems on both undirected and directed graphs. Necessary and sufficient conditions for the absolute-position circular formation are given as well. For the future works, important research directions include: generalize our results to the higher-order modulus consensus problems and introduce our approach to opinion dynamics in social networks.
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