On the coverings of closed non-orientable Euclidean manifolds $B_3$ and $B_4$.

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Abstract

There are only 10 Euclidean forms, that is flat closed three dimensional manifolds: six are orientable $G_1, \ldots, G_6$ and four are non-orientable $B_1, \ldots, B_4$. The aim of this paper is to describe all types of $n$-fold coverings over the non-orientable Euclidean manifolds $B_3$ and $B_4$, and calculate the numbers of non-equivalent coverings of each type. The manifolds $B_3$ and $B_4$ are uniquely determined among non-orientable forms by their homology groups $H_1(B_3) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$ and $H_1(B_4) = \mathbb{Z}_4 \times \mathbb{Z}$.

We classify subgroups in the fundamental groups $\pi_1(B_3)$ and $\pi_1(B_4)$ up to isomorphism. Given index $n$, we calculate the numbers of subgroups and the numbers of conjugacy classes of subgroups for each isomorphism type and provide the Dirichlet generating functions for the above sequences.

Key words: Euclidean form, platycosm, flat 3-manifold, non-equivalent coverings, crystallographic group.

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Introduction

Let $\mathcal{M}$ be a connected manifold with fundamental group $G = \pi_1(\mathcal{M})$. Two coverings

$$p_1 : \mathcal{M}_1 \to \mathcal{M} \text{ and } p_2 : \mathcal{M}_2 \to \mathcal{M}$$

are said to be equivalent if there exists a homeomorphism $h : \mathcal{M}_1 \to \mathcal{M}_2$ such that $p_1 = p_2 \circ h$. According to the general theory of covering spaces, any $n$-fold covering is uniquely determined by a subgroup of index $n$ in the group $G$. The equivalence classes of $n$-fold coverings of $\mathcal{M}$ are in one-to-one correspondence with the conjugacy classes of subgroups of index $n$ in the fundamental group $\pi_1(\mathcal{M})$. See, for example, ([7], p. 67). In such a way the following natural problems arise: to describe the isomorphism classes of subgroups of finite index in the fundamental group of a given manifold and to enumerate the finite index subgroups and their conjugacy classes with respect to isomorphism type.

We use the following notations: let $s_G(n)$ denote the number of subgroups of index $n$ in the group $G$, and let $c_G(n)$ be the number of conjugacy classes of such subgroups. Similarly, by $s_{H,G}(n)$ denote the number of subgroups of index $n$ in the group $G$, which are isomorphic to $H$, and by $c_{H,G}(n)$ the number of conjugacy classes of such subgroups. So, $c_G(n)$ coincides with the number of nonequivalent $n$-fold coverings over a manifold $\mathcal{M}$ with fundamental group $\pi_1(\mathcal{M}) \cong G$, and $c_{H,G}(n)$ coincides with the number of nonequivalent $n$-fold coverings $p : \mathcal{N} \to \mathcal{M}$, where $\pi_1(\mathcal{N}) \cong H$ and $\pi_1(\mathcal{M}) \cong G$. The numbers $s_G(n)$ and $c_G(n)$, where $G$ is the fundamental group of closed orientable or non-orientable surface, were found in ([14], [15], [16]). In the paper [18], a general method for calculating the number $c_G(n)$ of conjugacy classes of subgroups in an arbitrary finitely generated group $G$ was given. Asymptotic formulas for $s_G(n)$ in many important cases were obtained in [12].

The values of $s_G(n)$ for the wide class of 3-dimensional Seifert manifolds were calculated in [9] and [10]. The present paper is a part of the series of our papers devoted to enumeration of finite-sheeted coverings of coverings over closed Euclidean 3-manifolds. These manifolds are also known as flat 3-dimensional manifolds or Euclidean 3-forms.

The class of such manifolds is closely related to the notion of Bieberbach group. Recall that a subgroup of isometries of $\mathbb{R}^3$ is called Bieberbach group if it is discrete, cocompact and torsion free. Each 3-form can be represented as a quotient $\mathbb{R}^3/G$ where $G$ is a Bieberbach group. In this case, $G$ is isomorphic to the fundamental group of the manifold, that is $G \cong \pi_1(\mathbb{R}^3/G)$. Classification of three dimensional Euclidean forms up to homeomorphism is presented in [22]. The class of such manifolds consists of six orientable $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$, $\mathcal{G}_5$, $\mathcal{G}_6$, and four non-orientable ones $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$. One can find the correspondence between Wolf and Conway-Rossetti notations of these Euclidean 3-manifolds and their homology groups in Table 1 in [3].

In our previous paper [3] we describe isomorphism types of finite index subgroups $H$ in the fundamental group $G$ of manifolds $\mathcal{B}_1$ and $\mathcal{B}_2$. Further, we calculate the respective numbers $s_{H,G}(n)$ and $c_{H,G}(n)$ for each isomorphism type $H$. In subsequent articles [4] and [5] similar questions were solved for manifolds $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$ and $\mathcal{G}_5$.

The aim of the present paper is to solve the same questions for manifolds $\mathcal{B}_3$ and $\mathcal{B}_4$. This manifolds are uniquely defined among non-orientable Euclidean forms by their
homology groups $H_1(B_3) = \mathbb{Z}_2^2 \oplus \mathbb{Z}$ (coincides with $H_1(G_2)$, but the manifold $G_2$ is orientable) and $H_1(B_4) = \mathbb{Z}_4 \oplus \mathbb{Z}$. To describe these manifolds through Bieberbach group, consider the following isometries of $\mathbb{R}^3$:

$$
S_1: (x, y, z) \mapsto (x + 1, y, z),
S_2: (x, y, z) \mapsto (-x, y + 1, z),
S_3: (x, y, z) \mapsto (-x, -y, z + 1),
\tilde{S}_3: (x, y, z) \mapsto (-x + 1/2, -y, z + 1).
$$

The Bieberbach groups $\pi_1(B_3)$ and $\pi_1(B_4)$ are generated by triples $S_1, S_2, S_3$ and $S_1, S_2, \tilde{S}_3$ respectively.

To describe $B_3$ and $B_4$ in more geometric terms we do the following. Take the cube $[0, 1]^3$ in $\mathbb{R}^3$ (it serves as the fundamental domain for both manifolds). Glue its faces $x = 0$ and $x = 1$ by parallel shift $(x, y, z) \mapsto (x + 1, y, z)$. Glue the faces $y = 0$ and $y = 1$ by mirror symmetry $(x, y, z) \mapsto (-x + 1, y, z)$ followed by parallel shift $(x, y, z) \mapsto (x, y + 1, z)$. Finally, in case of $B_3$ glue the face $z = 0$ to face $z = 1$ by the central symmetry $(x, y, z) \mapsto (-x + 1, -y + 1, z)$ followed with parallel shift $(x, y, z) \mapsto (x, y, z + 1)$. In case of $B_4$ split the face $z = 0$ into two equal rectangles (longer side parallel to $OY$), self-align each of them by the central symmetry $(x, y, z) \mapsto (-x + 1, -y + 1, z)$ and $(x, y, z) \mapsto (-x + 3/2, -y + 1, z)$ respectively. Then shift the face $z = 0$ to $z = 1$ with $(x, y, z) \mapsto (x, y, z + 1)$.

In the present paper, we classify subgroups in the fundamental groups $\pi_1(B_3)$ and $\pi_1(B_4)$ up to isomorphism. Given index $n$, we calculate the numbers of subgroups and the numbers of conjugacy classes of subgroups for each isomorphism type. Also, we provide the Dirichlet generating functions for all the above sequences.

Numerical methods to solve these and similar problems for the three-dimensional crystallographic groups were developed by the Bilbao group [2]. The convenience of language of Dirichlet generating series for this kind of problems was demonstrated in [21]. The first homologies of all the three-dimensional crystallographic groups are determined in [19].

**Notations**

Suppose $G$ is a group, $u, v$ are its elements and $H, F$ are its subgroups. We use $u^v$ instead of $vu^{-1}v$ and $[u, v]$ in place of $uuv^{-1}$ for the sake of brevity. By $H^v$ denote the subgroup $\{u^v | u \in H\}$. By $H^F$ denote the family of subgroups $H^v, v \in F$. By $Ad_v : G \to G$ denote the automorphism, given by $u \to u^v$.

By $s_{H,G}(n)$ we denote the number of subgroups of index $n$ in the group $G$, isomorphic to the group $H$; by $c_{H,G}(n)$ the number of conjugacy classes of subgroups of index $n$ in the group $G$, isomorphic to the group $H$. Through this paper usually $G$ and $H$ are fundamental groups of manifolds $G_i$ or $B_i$, in this case we omit $\pi_1$ in indexes.

Also we will need the following number-theoretic functions. Given a fixed $n$ we widely use summation over all representations of $n$ as a product of two or three positive integer
factors $\sum_{ab=n}$ and $\sum_{abc=n}$. The order of factors is important. Also we consider this sum vanishes if $n$ is not integer.

To start with, this is the natural language to express the function $\sigma_0(n)$ – the number of representations of number $n$ as a product of two factors

$$\sigma_0(n) = \sum_{ab=n} 1.$$  

We will also need the following generalizations of $\sigma_0$:

$$\sigma_1(n) = \sum_{ab=n} a; \quad \sigma_2(n) = \sum_{ab=n} \sigma_1(a) = \sum_{abc=n} a; \quad d_3(n) = \sum_{ab=n} \sigma_0(a) = \sum_{abc=n} 1;$$

$$\chi(n) = \sum_{ab=n} a\sigma_1(b) = \sum_{ab=n} a\sigma_0(a) = \sum_{abc=n} ab; \quad \omega(n) = \sum_{ab=n} a\sigma_1(a) = \sum_{abc=n} a^2b.$$

1Formulation of main results

The main goal of this paper is to prove the following four theorems.

**Theorem 1.** Every subgroup $\Delta$ of finite index $n$ in $\pi_1(B_3)$ have one of the following isomorphism types: $\pi_1(G_1) \cong \mathbb{Z}^3$, $\pi_1(G_2)$, $\pi_1(B_1)$, $\pi_1(B_2)$, $\pi_1(B_3)$ or $\pi_1(B_4)$. The respective numbers of subgroups are given by the formulas

(i) $s_{G_1,B_3}(n) = \omega(n/4)$,  
(ii) $s_{G_2,B_3}(n) = \omega(n/2) - \omega(n/4)$,  
(iii) $s_{B_1,B_3}(n) = 2\chi(n/2) - 2\chi(n/4)$,  
(iv) $s_{B_2,B_3}(n) = 4\chi(n/2) - 4\chi(n/8)$,  
(v) $s_{B_3,B_3}(n) = \chi(n) - 3\chi(n/2) + 2\chi(n/4)$,  
(vi) $s_{B_4,B_3}(n) = 2\chi(n/2) - 6\chi(n/4) + 4\chi(n/8)$.

**Theorem 2.** The numbers of non-equivalent $n$-fold covering over $B_3$ with respect to homeomorphism type (i.e. the numbers of conjugacy classes of subgroups of index $n$ in $\pi_1(B_3)$ with respect to isomorphism type of a subgroup) are given by

(i) $c_{G_1,B_3}(n) = \frac{1}{4}(\omega(n/4) + 3\sigma_2(n/4) + 9\sigma_2(n/8)),$  
(ii) $c_{G_2,B_3}(n) = \frac{1}{2}(\sigma_2(n/2) + 2\sigma_2(n/4) - 3\sigma_2(n/8) + d_3(n/2) - d_3(n/4) + d_3(n/8) - 3d_3(n/16) + 2d_3(n/32)),$  
(iii) $c_{B_1,B_3}(n) = \sigma_2(n/2) - \sigma_2(n/8) + d_3(n/2) - d_3(n/8),$  
(iv) $c_{B_2,B_3}(n) = 2\sigma_2(n/4) - 2\sigma_2(n/8) + d_3(n/4) - d_3(n/8),$  
(v) $c_{B_3,B_3}(n) = d_3(n) - d_3(n/2) - d_3(n/4) + d_3(n/8),$  
(vi) $c_{B_4,B_3}(n) = 2d_3(n/2) - 4d_3(n/4) + 2d_3(n/8).$
Theorem 3. Every subgroup $\Delta$ of finite index $n$ in $\pi_1(B_4)$ have one of the following isomorphism types: $\pi_1(G_1) \cong \mathbb{Z}^3$, $\pi_1(G_2)$, $\pi_1(B_1)$, $\pi_1(B_2)$ or $\pi_1(B_3)$. The respective numbers of subgroups are

\begin{align*}
(i) \quad & s_{G_1,B_4}(n) = \omega\left(\frac{n}{4}\right), \\
(ii) \quad & s_{G_2,B_4}(n) = \omega\left(\frac{n}{2}\right) - \omega\left(\frac{n}{4}\right), \\
(iii) \quad & s_{B_1,B_4}(n) = 2\chi\left(\frac{n}{2}\right) - 2\chi\left(\frac{n}{4}\right), \\
(iv) \quad & s_{B_2,B_4}(n) = 4\chi\left(\frac{n}{4}\right) - 4\chi\left(\frac{n}{8}\right), \\
(v) \quad & s_{B_4,B_4}(n) = \chi(n) - 5\chi\left(\frac{n}{2}\right) + 8\chi\left(\frac{n}{4}\right) - 4\chi\left(\frac{n}{8}\right).
\end{align*}

Theorem 4. The numbers of non-equivalent $n$-fold covering over $B_4$ with respect to homeomorphism type (i.e. the numbers of conjugacy classes of subgroups of index $n$ in $\pi_1(B_4)$ with respect to isomorphism type of a subgroup) are

\begin{align*}
(i) \quad & c_{G_1,B_4}(n) = \frac{1}{4}\left(\omega\left(\frac{n}{4}\right) + 3\sigma_2\left(\frac{n}{4}\right) + 9\sigma_2\left(\frac{n}{8}\right)\right), \\
(ii) \quad & c_{G_2,B_4}(n) = \frac{1}{2}\left(\sigma_2\left(\frac{n}{2}\right) + 2\sigma_2\left(\frac{n}{4}\right) - 3\sigma_2\left(\frac{n}{8}\right) + d_3\left(\frac{n}{2}\right) - d_3\left(\frac{n}{4}\right) - 3d_3\left(\frac{n}{8}\right) + 5d_3\left(\frac{n}{16}\right) - 2d_3\left(\frac{n}{32}\right)\right), \\
(iii) \quad & c_{B_1,B_4}(n) = \sigma_2\left(\frac{n}{2}\right) - \sigma_2\left(\frac{n}{8}\right) + d_3\left(\frac{n}{2}\right) - 2d_3\left(\frac{n}{4}\right) + d_3\left(\frac{n}{8}\right), \\
(iv) \quad & c_{B_2,B_4}(n) = 2\sigma_2\left(\frac{n}{4}\right) - 2\sigma_2\left(\frac{n}{8}\right) + d_3\left(\frac{n}{4}\right) - 2d_3\left(\frac{n}{8}\right) + d_3\left(\frac{n}{16}\right), \\
(v) \quad & c_{B_4,B_4}(n) = d_3(n) - 3d_3\left(\frac{n}{2}\right) + 3d_3\left(\frac{n}{4}\right) - d_3\left(\frac{n}{8}\right).
\end{align*}

Also we present an alternative proof for the previously known results (see [17]) about the enumeration of subgroups and conjugacy classes of subgroups of the fundamental group of Klein bottle.

Theorem 5 (Klein bottle). Let $\pi_1(K) = \langle x, y : yxy^{-1} = x^{-1} \rangle$ be the fundamental group of Klein bottle. Then each subgroup of finite index in $\pi_1(K)$ is isomorphic to either $\pi_1(K)$ or $\mathbb{Z}^2$. The respective numbers of subgroups and conjugacy classes of subgroups are

\begin{align*}
& s_{\mathbb{Z}^2,\pi_1(K)}(n) = \sigma_1\left(\frac{n}{2}\right), \\
& c_{\mathbb{Z}^2,\pi_1(K)}(n) = \frac{1}{2}\left(\sigma_1\left(\frac{n}{2}\right) + \sigma_0\left(\frac{n}{2}\right) + \sigma_0\left(\frac{n}{4}\right)\right), \\
& s_{\pi_1(K),\pi_1(K)}(n) = \sigma_1(n) - \sigma_1\left(\frac{n}{2}\right), \\
& c_{\pi_1(K),\pi_1(K)}(n) = \sigma_0(n) - \sigma_0\left(\frac{n}{4}\right).
\end{align*}

For Dirichlet generating series for the sequences, provided by Theorems 1–5 see Appendix.
2 Preliminaries

Further we use the following representations for the fundamental groups $\pi(B_3)$ and $\pi(B_4)$, see [22] or [20].

$$\pi_1(B_3) = \langle x, y, z : yxy^{-1} = xz^{-1} = x^{-1}, zyz^{-1} = y^{-1} \rangle.$$ \hspace{1cm} (2.1)

$$\pi_1(B_4) = \langle x, y, z : yxy^{-1} = xz^{-1} = x^{-1}, zyz^{-1} = xy^{-1} \rangle.$$ \hspace{1cm} (2.2)

Below we represent the free abelian groups of rank two and three by the set of pairs and triples of integer numbers respectively. Given a subgroup $H$ of an abelian group $G$, we say that two elements $u, v \in G$ are congruent modulo $H$ if $u - v \in H$. In this case we write $u \equiv v \mod H$.

We need the following version of Proposition 1 in [4].

**Proposition 1.**

a) The subgroups of index $n$ in $\mathbb{Z}^2$ are in one-to-one correspondence with the matrices $\begin{pmatrix} b & d \\ 0 & a \end{pmatrix}$, where $ab = n$, $0 \leq d < a$. A subgroup $\Delta$ of index $n$ is generated by the rows $(0, a)$ and $(b, d)$ of corresponding matrix $\Delta = \langle (0, a), (b, d) \rangle$. Consequently, the number of such subgroups is $\sigma_1(n)$.

b) The subgroups of index $n$ in $\mathbb{Z}^3$ are in one-to-one correspondence with the matrices $\begin{pmatrix} c & e & f \\ 0 & b & d \\ 0 & 0 & a \end{pmatrix}$, where $abc = n$, $0 \leq d < a$ and $0 \leq e < b$. A subgroup $\Delta$ of index $n$ is generated by the rows $(0, 0, a)$, $(0, b, d)$ and $(c, e, f)$ of corresponding matrix $\Delta = \langle (0, 0, a), (0, b, d), (c, e, f) \rangle$. Consequently, the number of such subgroups is $\sigma_2(n)$.

**Corollary 1.** Let $\Delta$ be a subgroup of finite index $n$ in $\mathbb{Z}^2$ and $\begin{pmatrix} b & d \\ 0 & a \end{pmatrix}$ be its corresponding matrix as described in Proposition 1. Then the set of elements $\{(i, j) | 0 \leq i < a, 0 \leq j < b\}$ is a complete set of coset representatives in $\mathbb{Z}^2/\Delta$.

**Corollary 2.** Given an integer $n$, by $S(n)$ denote the number of pairs $(\Delta, \nu)$, where $\Delta$ varies over all subgroup of index $n$ in $\mathbb{Z}^2$ and $\nu$ is a coset of $\mathbb{Z}^2/\Delta$ with $2\nu = 0$. Then

$$S(n) = \sigma_1(n) + 3\sigma_1\left(\frac{n}{2}\right).$$

For the proof see Corollary 1 in [4].

We also need the following fact.

**Lemma 1.** Let $G$ be an abelian group and $H$ its subgroup of finite index. Let $\phi : G \rightarrow G$ be an endomorphism of $G$, such that $\phi(H) \subseteq H$ and the index $|G : \phi(G)|$ is also finite. Then the cardinality of kernel of $\phi : G/H \rightarrow G/H$ is equal to the index $|G : (H + \phi(G))|$. 
Proof. Indeed, 

\[ |\ker_\phi(G/H)| = \frac{|G/H|}{|\phi(G)/(\phi(G) \cap H)|} \]

By the Second Isomorphism Theorem \( \phi(G)/(\phi(G) \cap H) \cong (\phi(G) + H)/H \). So

\[ |\ker_\phi(G/H)| = \frac{|G/H|}{(\phi(G) + H)/H} = |G/(\phi(G) + H)|. \]

\[ \square \]

**Remark 1.** Combining Lemma \( \square \) and Corollary \( \square \) we get the following observation. Consider a finite index subgroup \( H \leq \mathbb{Z}^2 \). The number of \( \nu \in \mathbb{Z}^2/H \) such that \( 2\nu = 0 \) is equal to \( |\mathbb{Z}^2/\langle(2,0), (0,2), H\rangle| \). This can be proved by application of Lemma \( \square \) to the subgroup \( H \) and the endomorphism \( \phi: g \mapsto 2g, g \in \mathbb{Z}^2 \). Since for each \( H \) the numbers \(|\{\nu|\nu \in \mathbb{Z}^2/H, 2\nu = 0\}\}| and \( |\mathbb{Z}^2/\langle(2,0), (0,2), H\rangle| \) coincide, their sums taken over all subgroups \( H \) also coincide, that is

\[ S(n) = \sum_{H \leq \mathbb{Z}^2, |\mathbb{Z}^2/H|=n} |\{\nu|\nu \in \mathbb{Z}^2/H, 2\nu = 0\}| = \sum_{H \leq \mathbb{Z}^2, |\mathbb{Z}^2/H|=n} |\mathbb{Z}^2/\langle(2,0), (0,2), H\rangle|. \]

**Corollary 3.** Let \( \ell: \mathbb{Z}^3 \mapsto \mathbb{Z}^3 \) be an automorphism of \( \mathbb{Z}^3 \), given by \( \ell(x,y,z) = (-x,y,z) \). Then the number of subgroups \( \Delta < \mathbb{Z}^3, |\mathbb{Z}^3: \Delta| = n \) with \( \ell(\Delta) = \Delta \) is \( \sigma_2(n) + 3\sigma_2(\frac{n}{2}) \).

**Proof.** By Proposition \( \square \) any subgroup \( \Delta \) of finite index in \( \mathbb{Z}^3 \) is generated by the rows of the corresponding matrix \( \begin{pmatrix} c & e & f \\ 0 & b & d \\ 0 & 0 & a \end{pmatrix} \). So the condition \( \ell(\Delta) = \Delta \) is equivalent to \( \langle(0,0,a),(0,b,d),(c,e,f)\rangle = \langle(0,0,a),(0,b,d),(-c,e,f)\rangle \). Replace \( (c,e,f) \) with \( (-c,-e,-f) \). Obviously, this does not change the group. It follows that \( (-c,-e,-f) \equiv (-c,e,f) \mod \langle(0,0,a),(0,b,d)\rangle \). The latter implies \( (0,2e,2f) \equiv 0 \mod \langle(0,0,a),(0,b,d)\rangle \). Denote \( H = \langle(0,0,a),(0,b,d)\rangle \). Fix the index \( k = |\mathbb{Z}^2:H| \). By Corollary \( \square \) the number of such pairs \( (H,(0,e,f)) \) equals \( \sigma_1(k) + 3\sigma_1(\frac{k}{2}) \). Summing over all the possible values \( k \mid n \) get the result. \( \square \)

**Definition 1.** Let \( G \) and \( H \) be some groups, \( \phi, \psi \) be automorphism of \( G \) and \( H \) respectively. We call the pairs \((G,\phi)\) and \((H,\psi)\) isomorphic if there exists an isomorphism \( \xi: G \mapsto H \) such that \( \xi \circ \phi = \psi \circ \xi \).

**Definition 2.** Let \( \phi \) and \( \psi \) be automorphisms of \( \mathbb{Z}^2 \). By \( f_{\mathbb{Z}^2,\phi}(n) \) we denote the number of subgroups \( H < \mathbb{Z}^2 \) such that \( |\mathbb{Z}^2:H| = n \) and \( \phi(H) \leq H \). Similarly by \( f_{\mathbb{Z}^2,\phi,\psi}(n) \) we denote the number of subgroups \( H < \mathbb{Z}^2 \) such that \( |\mathbb{Z}^2:H| = n \), \( \phi(H) \leq H \) and the pair of \( H \) and the restriction of \( \phi \) to \( H \) is isomorphic to \((H,\psi)\).

The following automorphisms will be of the most importance throughout the article.

**Notation.** By \( \ell \) and \( j \) denote the automorphisms of \( \mathbb{Z}^2 \), given by \( \ell: (u,v) \mapsto (u,-v) \) and \( j: (u,v) \mapsto (u,u-v) \) respectively.
Proposition 2. The following identities hold

\[ f_{\mathbb{Z}^2, \ell, \ell}(n) = \sigma_0(n), \quad f_{\mathbb{Z}^2, \ell, j}(n) = \sigma_0\left(\frac{n}{2}\right), \quad f_{\mathbb{Z}^2, \ell}(n) = \sigma_0(n) + \sigma_0\left(\frac{n}{2}\right); \]

\[ f_{\mathbb{Z}^2, j, \ell}(n) = \sigma_0\left(\frac{n}{2}\right), \quad f_{\mathbb{Z}^2, j, j}(n) = \sigma_0(n) - 2\sigma_0\left(\frac{n}{2}\right) + 2\sigma_0\left(\frac{n}{4}\right), \quad f_{\mathbb{Z}^2, j}(n) = \sigma_0(n) - \sigma_0\left(\frac{n}{2}\right) + 2\sigma_0\left(\frac{n}{4}\right). \]

Proof. Given a subgroup \( \Delta \) of index \( n \) in \( \mathbb{Z}^2 \) consider its corresponding matrix \( \begin{pmatrix} b & d \\ 0 & a \end{pmatrix} \) as described in Proposition \( \square \). Denote \( X_\Delta = (0, a) \) and \( Y_\Delta = (b, d) \).

Assume a subgroup \( \Delta \) is preserved by \( \ell \). Then \( ((0, a), (b, d)) = ((0, -a), (b, -d)) \), or equivalently \( 2d \equiv 0 \ mod \ a \). Recalling that \( 0 \leq d < a \), we have \( 2d = 0 \) or \( 2d = a \), the latter is possible only in the case of even \( a \). In the first case \( \ell(X_\Delta) = -X_\Delta \) and \( \ell(Y_\Delta) = Y_\Delta \), so the automorphism \( \phi : \mathbb{Z} \to \Delta \) given by \( \phi((1, 0)) = Y_\Delta \) and \( \phi((0, 1)) = X_\Delta \) provides an isomorphism \( (\Delta, \ell) \cong (\mathbb{Z}^2, \ell) \).

In case \( 2d = a \) we have \( \ell(X_\Delta) = -X_\Delta \) and \( \ell(Y_\Delta) = (b, -\frac{d}{2}) = Y_\Delta - X_\Delta \). So \( \ell(Y_\Delta - X_\Delta) = Y_\Delta = (Y_\Delta - X_\Delta) + X_\Delta \). Note that elements \( X_\Delta \) and \( Y_\Delta - X_\Delta \) generate the same group \( (X_\Delta, Y_\Delta - X_\Delta) = \langle X_\Delta, Y_\Delta \rangle = \Delta \). So an automorphism \( \phi : \mathbb{Z} \to \Delta \) given by \( \phi((1, 0)) = Y_\Delta - X_\Delta \) and \( \phi((0, 1)) = X_\Delta \) provides an isomorphism \( (\Delta, \ell) \cong (\mathbb{Z}^2, \ell) \).

So each factorization \( ab = n \) in the case of an odd \( a \) provides one \( \ell \)-invariant subgroup \( \Delta \), moreover \( (\Delta, \ell) \cong (\mathbb{Z}^2, \ell) \). In the case of an even \( a \) we get two subgroups \( (\Delta_1, \ell) \cong (\mathbb{Z}^2, \ell) \) and \( (\Delta_2, \ell) \cong (\mathbb{Z}^2, j) \). That is

\[ f_{\mathbb{Z}^2, \ell, \ell}(n) = \sum_{a|n} 1 = \sigma_0(n), \quad f_{\mathbb{Z}^2, \ell, j}(n) = \sum_{a|n, 2|a} 1 = \sigma_0\left(\frac{n}{2}\right), \quad f_{\mathbb{Z}^2, \ell}(n) = f_{\mathbb{Z}^2, \ell, \ell}(n) + f_{\mathbb{Z}^2, \ell, j}(n). \]

The enumeration of \( j \)-invariant subgroups \( \Delta \) follows the similar way. If \( j(\Delta) = \Delta \) then \( ((0, a), (b, d)) = ((0, -a), (b, -d)) \), or equivalently \( b - 2d \equiv 0 \ mod \ a \). Define integers \( m \) and \( k \) by \( b - 2d = ma \) and \( 2k = m \) or \( 2k + 1 = m \) depending on the parity of \( m \). Then \( j(X_\Delta) = -X_\Delta \) and \( j(Y_\Delta) = Y_\Delta + mX_\Delta \). So \( j(Y_\Delta + kX_\Delta) = (Y_\Delta + kX_\Delta) + (m - 2k)X_\Delta \). Also note that elements \( X_\Delta \) and \( Y_\Delta + kX_\Delta \) generate the same group \( (X_\Delta, Y_\Delta + kX_\Delta) = \langle X_\Delta, Y_\Delta \rangle = \Delta \). So the automorphism \( \phi : \mathbb{Z} \to \Delta \) given by \( \phi((1, 0)) = Y_\Delta + kX_\Delta \) and \( \phi((0, 1)) = X_\Delta \) provides an isomorphism \( (\Delta, j) \cong (\mathbb{Z}^2, \ell) \) or \((\Delta, j) \cong (\mathbb{Z}^2, j) \) in cases \( m \) is even or odd respectively.

Now for each factorization \( ab = n \) we find out, what solutions the equation \( b - 2d = ma \) have under the restriction \( 0 \leq d < a \). In case \( a \) is odd there are a unique solution for each factorization, moreover the parities of \( b \) and \( m \) coincide. In case \( a \) is even and \( b \) is odd there are no solutions. In case \( a \) and \( b \) are even there are two solutions for each factorization, one with even and one with odd \( m \). That is

\[ f_{\mathbb{Z}^2, j, j}(n) = \sum_{a|n, 2|a} 1 = \sigma_0(n) - 2\sigma_0\left(\frac{n}{2}\right) + 2\sigma_0\left(\frac{n}{4}\right); \]

\[ f_{\mathbb{Z}^2, j}(n) = \sum_{a|n, 2|a} 1 = \sigma_0\left(\frac{n}{2}\right); \quad f_{\mathbb{Z}^2, j, j}(n) = f_{\mathbb{Z}^2, j, \ell}(n) + f_{\mathbb{Z}^2, j, j}(n). \]
Observation. Indeed, it was proven above that for an arbitrary involutory isomorphism \( f : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2 \) the pair \((\mathbb{Z}^2, f)\) is isomorphic to one of the following four: \((\mathbb{Z}^2, \text{id})\), \((\mathbb{Z}^2, -\text{id})\), \((\mathbb{Z}^2, \ell)\) or \((\mathbb{Z}^2, j)\). But we do not use this fact further.

2.1 The structure of subgroups in the fundamental group of Klein bottle.

The purpose of this chapter is to provide the enumeration of subgroups and the conjugacy classes of subgroups in the fundamental group of Klein bottle.

Notation. By \( \Gamma \) denote the group, generated by \( x, y \) with the relation \( yxy^{-1} = x^{-1} \). Note that \( \Gamma \) is isomorphic to the fundamental group of Klein bottle (see, for example, [7], p.72).

Lemma 2. (i) Each element \( g \in \Gamma \) can be represented in the canonical form \( g = x^a y^b \) for some integer \( a, b \).

(ii) The product of two canonical forms is given by the formula

\[
x^a y^b \cdot x^c y^d = \begin{cases} x^{a+c} y^{b+d} & \text{if } b \equiv 0 \mod 2 \\ x^{a-c} y^{b+d} & \text{if } b \equiv 1 \mod 2 \\ \end{cases}
\]

or shortly \( x^a y^b \cdot x^c y^d = x^{a+(-1)^b c} y^{b+d} \).

(iii) The representation in the canonical form \( g = x^a y^b \) for each element \( g \in \Gamma \) is unique.

Proof. Parts (i) and (ii) follows routinely from the definition of the group, thus we concentrate on (iii). First, note that the subgroup, generated by \( x \) is normal in \( \Gamma \). Indeed, \( x^y = x^{-1} \) – this is exactly the relation of the group. So \( x^g \in \{ x, x^{-1} \} \) for any \( g \in \langle x, y \rangle = \Gamma \). This means that the factorization \( \phi : \Gamma \mapsto \Gamma / \langle x \rangle \) is well-defined. Obviously, \( \phi(y) \) generates \( \phi(\Gamma) \cong \mathbb{Z} \).

Secondly, note that \( x \) have the infinite order in \( \Gamma \). This follows from the Magnus Theorem (see [13]). Indeed, the relation \( yxy^{-1}x \) is cyclically reduced and contains \( x \), thus \( x \) generates a free subgroup. For more direct proof see ([6], Th. 10).

Assume the canonical representation of some element is not unique, that is \( x^a y^b = x^c y^d \) holds for some \( (a, b) \neq (c, d) \). Applying \( \phi \) to both parts one gets \( b = d \), consequently \( a \neq c \). But \( x^a y^b = x^c y^d \) implies \( x^a = x^c \). This contradicts the infinite order of \( x \). \( \square \)

Proposition 3. The subgroups of index \( n \) in \( \Gamma \) are in one-to-one correspondence with the matrices \( \begin{pmatrix} b & d \\ 0 & a \end{pmatrix} \), where \( ab = n \), \( 0 \leq d < a \). A subgroup \( \Delta \) of index \( n \) is generated by elements \( x^a \) and \( x^d y^b \) where \( a, b, d \) are elements of the corresponding matrix. Finally, \( \Delta \cong \mathbb{Z}^2 \) if \( b \) is even and \( \Delta \cong \Gamma \) if \( b \) is odd.
Proof. We consider all elements of $\Gamma$ to be represented in the canonical form, provided by Lemma 2. To build the map from subgroups $\Delta$ of index $n$ to matrices of prescribed form do the following. Consider the minimal positive integer $a$, such that $X_{\Delta} = x^a \in \Delta$. Such $a$ exists since index of $\Delta$ in $\Gamma$ is finite.

Now consider the minimal positive integer $b$, for which there exists integer $s$ such that the element $x^s y^b \in \Delta$ belongs to $\Delta$. Put $Y_{\Delta} = X_{\Delta}^{-[n/a]}, x^s y^b = x^d y^b$, where $[r]$ denotes the maximal integer $t$ such that $t \leq r$. Note that $Y_{\Delta} \in \Delta$ and $d$ satisfy $0 \leq d < a$. Correspondence part is done.

To show that different matrices provide different subgroups $\Delta$ consider the following sets:

\[
\{x^{ia+jd}y^{jb} | i, j \in \mathbb{Z}\},
\]

in case of even $b$ and

\[
\{x^{ia}y^{jb} | i, j \in \mathbb{Z}, 2 \mid j\} \cup \{x^{ia+jd}y^{jb} | i, j \in \mathbb{Z}, 2 \nmid j\}.
\]

in case of odd $b$. Direct verification through equation (2.3) shows that the above sets are subgroups in $\Gamma$. They are obviously generated by $X_{\Delta}, Y_{\Delta}$.

Now we prove the isomorphism part, we do it separately for different parities of $b$. Fix a subgroup $\Delta$, consider corresponding matrix, suppose $b$ is odd. Denote $X = x^a$ and $Y = x^d y^b$. Note that $X$ and $Y$ satisfy the relation $YXY^{-1} = X^{-1}$. Thus the map $x \to X, y \to Y$ can be extended to an epimorphism $\Gamma \to \Delta$, so we only have to prove this epimorphism is an isomorphism. As it is shown in the proof of Lemma 2 by means of relation $YXY^{-1} = X^{-1}$ one can reduce any element of $\langle X, Y \rangle$ to $X^s Y^t$ for some integers $s, t$. We have to prove that this representation is unique. Suppose $X^s Y^t = x^u y^v$, using Equation (2.3) and definition of $X, Y$ one can show

\[
X^s Y^t = \begin{cases} x^{sa} y^{tb} & \text{if } t \equiv 0 \mod 2 \\ x^{sa+d} y^{tb} & \text{if } t \equiv 1 \mod 2 \end{cases}
\]

or

\[
(u, v) = \begin{cases} (sa, tb) & \text{if } t \equiv 0 \mod 2 \\ (sa + d, tb) & \text{if } t \equiv 1 \mod 2 \end{cases}
\]

All we need from the previous formula is to show that at most one pair $(s, t)$ corresponds to a pair $(u, v)$, and the pair $(u, v)$ is unique in virtue of Lemma 2. Thus each element $g \in \langle X, Y \rangle$ can be uniquely represented in the form $g = X^s Y^t$, so $x \to X, y \to Y$ spawns an isomorphism $\Gamma \mapsto \langle X, Y \rangle$.

The case of even $b$ is similar. □

Remark 2. Let $\Gamma_+ = \langle x, y^2 \rangle$ be the subgroup of index 2 in $\Gamma$. It follows from the above consideration that each abelian subgroup of finite index in $\Gamma$ lies in $\Gamma_+$.

Proof of Theorem 5. By Proposition 8 a subgroup $\Delta \cong \mathbb{Z}^2$ of index $n$ is given by a matrix $\begin{pmatrix} b & d \\ 0 & a \end{pmatrix}$, where $ab = n, b$ is even and $0 \leq d < a$. Then

\[
s_{\mathbb{Z}^2, \pi_1(K)}(n) = \sum_{ab=n, 2 \mid b} a = \sigma_1\left(\frac{n}{2}\right).
\]
To enumerate conjugacy classes we identify subgroups with their corresponding matrix, and consider how the conjugation changes the corresponding matrix. Obviously, $a$ and $b$ are invariant. Also, $Ad_x$ preserves $d$ and $Ad_y$ maps $d \mapsto a - d$. That is for a fixed factorization $ab = n$ there are $\frac{a+1}{2}$ or $\frac{a+2}{2}$ conjugacy classes in case of an odd or even $a$ respectively. Thus

$$c_{2^2,\pi_1(K)}(n) = \sum_{ab=n, 2|b} \begin{cases} \frac{a+1}{2} & \text{if } 2 \nmid a \\ \frac{a+2}{2} & \text{if } 2 \mid a \end{cases} = \frac{1}{2} \left( \sigma_1\left(\frac{n}{2}\right) + \sigma_0\left(\frac{n}{2}\right) + \sigma_0\left(\frac{n}{4}\right) \right).$$

In case $\Delta \cong \pi_1(K)$ arguing similarly we get

$$s_{\pi_1(K),\pi_1(K)}(n) = \sum_{ab=n, 2|b} a = \sigma_1(n) - \sigma_1\left(\frac{n}{2}\right).$$

Also, $Ad_x$ maps $d \mapsto d + 2$ (keep in mind that $d$ is a residue modulo $a$) and $Ad_y$ maps $d \mapsto a - d$. That is for a fixed factorization $ab = n$ there are 1 or 2 conjugacy classes in case of an odd or even $a$ respectively. Thus

$$c_{\pi_1(K),\pi_1(K)}(n) = \sum_{ab=n, 2|b} \begin{cases} 1 & \text{if } 2 \nmid a \\ 2 & \text{if } 2 \mid a \end{cases} = \sigma_0(n) - \sigma_0\left(\frac{n}{4}\right).$$

The structure of groups $\pi_1(B_3)$ and $\pi_1(B_4)$

The following two propositions provides the canonical form of an element in $\pi_1(B_3) = \langle x, y, z : yxy^{-1} = zxz^{-1} = x^{-1}, yz^{-1} = y^{-1} \rangle$ and $\pi_1(B_4) = \langle x, y, z : yxy^{-1} = zxz^{-1} = x^{-1}, yz^{-1} = xy^{-1} \rangle$, they are counterparts of Lemma 2. As before, $\Gamma = \langle x, y : yxy^{-1} = x^{-1} \rangle$.

**Proposition 4.** (i) Each element $g \in \pi_1(B_3)$ can be represented in the canonical form $g = x^a y^b z^c$ for some integer $a, b, c$.

(ii) The product of two canonical forms is given by

$$x^a y^b z^c \cdot x^d y^e z^f = \begin{cases} x^{a+d} y^{b+e} z^{c+f} & \text{if } b \equiv 0 \mod 2, \ c \equiv 0 \mod 2 \\ x^{a-d} y^{b+e} z^{c+f} & \text{if } b \equiv 1 \mod 2, \ c \equiv 0 \mod 2 \\ x^{a-d} y^{b-e} z^{c+f} & \text{if } b \equiv 0 \mod 2, \ c \equiv 1 \mod 2 \\ x^{a+d} y^{b-e} z^{c+f} & \text{if } b \equiv 1 \mod 2, \ c \equiv 1 \mod 2 \end{cases},$$

or shortly

$$x^a y^b z^c \cdot x^d y^e z^f = x^{a+(-1)^{b+e}d} y^{b+(-1)^{c}e} z^{c+f}.$$
(iii) The canonical epimorphism \( \phi : \pi_1(\mathcal{B}_3) \to \Gamma \) given by \( \phi : x^ay^bz^c \to y^bz^c \) is well-defined.

(iv) The representation in the canonical form \( g = x^ay^bz^c \) for each element \( g \in \pi_1(\mathcal{B}_3) \) is unique.

To prove part (iv) we need the following lemma.

**Lemma 3.** The groups \( \pi_1(\mathcal{B}_3) \) and \( \pi_1(\mathcal{B}_4) \) are torsion-free (that is, contain no elements of finite order).

**Proof.** This is a particular case of the well-known statement that a finite order isometry of Euclidean space has a fixed point, see, for example, ([6] Th. 10). Since \( \pi_1(\mathcal{B}_3) \) and \( \pi_1(\mathcal{B}_4) \) are the fundamental groups of closed Euclidean manifolds, thus they act as isometry groups on the common universal covering \( \mathbb{E}^3 \); that is have no fixed points. \( \square \)

**Proof of Proposition 4.** Items (i–iii) follows routinely from the representation 2.1 of the group \( \pi_1(\mathcal{B}_3) \). To prove (iv) assume the opposite, that is \( x^a y^b z^c = x'^a y'^b z'^c \) for some triples \((a,b,c) \neq (a',b',c')\). Applying \( \phi \) to both parts and using the uniqueness of the canonical form for \( \Gamma \) (see Lemma 2) one gets \((b,c) = (b',c')\). In turn, \( a \neq a' \) \& \( x^a = x'^a \) is a contradiction with the infinite order of \( x \), provided by Lemma 3. \( \square \)

Note that the group \( \Gamma \) appearing both in Propositions 4 and 5 is the same group – the fundamental group of Klein bottle. This reflects the fact that both manifolds are circle bundles over the Klein bottle.

**Proposition 5.** (i) Each element \( g \in \pi_1(\mathcal{B}_4) \) can be represented in the canonical form \( g = x^a y^b z^c \) for some integer \( a,b,c \).

(ii) The product of two canonical forms is given by

\[
\begin{align*}
    x^a y^b z^c \cdot x^d y^e z^f &= \begin{cases} 
    x^{a+d} y^{b+e} z^{c+f} & \text{if } b \equiv 0 \mod 2, \ c \equiv 0 \mod 2 \\
    x^{a-d} y^{b+e} z^{c+f} & \text{if } b \equiv 1 \mod 2, \ c \equiv 0 \mod 2 \\
    x^{a-d} y^{b-e} z^{c+f} & \text{if } b, e \equiv 0 \mod 2, \ c \equiv 1 \mod 2 \\
    x^{a-d+1} y^{b-e} z^{c+f} & \text{if } b \equiv 0 \mod 2, \ c, e \equiv 1 \mod 2 \\
    x^{a+d} y^{b-e} z^{c+f} & \text{if } e \equiv 0 \mod 2, \ b, c \equiv 1 \mod 2 \\
    x^{a+d+1} y^{b-e} z^{c+f} & \text{if } b \equiv 1 \mod 2, \ c, e \equiv 1 \mod 2
    \end{cases}
\end{align*}
\]

Or shortly

\[
x^a y^b z^c \cdot x^d y^e z^f = x^{a+(-1)^{b+c} + \frac{1-(-1)^{b+c}}{2} \cdot (-1)^c} y^{b+(1)^c} z^{c+f}.
\]

(iii) The canonical epimorphism \( \phi : \pi_1(\mathcal{B}_4) \to \Gamma \), given by \( \phi : x^a y^b z^c \to y^b z^c \) is well-defined.

(iv) The representation in the canonical form \( g = x^a y^b z^c \) for each element \( g \in \pi_1(\mathcal{B}_4) \) is unique.
Next proposition provides effective means to enumerate the finite index subgroups of the group $\pi_1(\mathcal{B}_3)$. It plays the same role for the group $\pi_1(\mathcal{B}_3)$ as Proposition 1 for $\mathbb{Z}^2$ and $\mathbb{Z}^3$ or Proposition 3 for $\Gamma$.

**Proposition 6.** The subgroups $\Delta$ of index $n$ in $\pi_1(\mathcal{B}_3)$ are in one-to-one correspondence with the integer matrices
\[
\begin{pmatrix}
c & e & f \\
0 & b & d \\
0 & 0 & a
\end{pmatrix}
\]
such that

(i) $a, b, c > 0$ and $abc = n$;

(ii) $0 \leq d, f < a$;

(iii) if case $b$ is odd, $e$ is even and $0 \leq e < 2b$; if $b$ is even then $0 \leq e < b$;

(iv) if $b$ is even and $e$ is odd, then $2d \equiv 0 \mod a$;

(v) if $b$ is odd and $c$ is even, then $2f \equiv 0 \mod a$;

(vi) if both $b, c$ are odd, then $2f \equiv 2d \mod a$.

Herewith the subgroup $\Delta$ is generated by elements $x^a$, $x^d y^b$ and $x^f y^e z^c$, where $a, b, c, d, e, f$ are elements of the corresponding matrix.

**Proof.** Below we consider all elements of the group $\pi_1(\mathcal{B}_3)$ to be represented in the canonical form provided by Proposition 5. To build the map from subgroups $\Delta$ of index $n$ to matrices of prescribed form do the following. Consider the minimal positive integer $a$, such that the element $X_\Delta = x^a \in \Delta$. Such an exponent $a$ exists since index of $\Delta$ in $\pi_1(\mathcal{B}_3)$ is finite.

Consider the minimal positive integer $b$ such that there exists integer $s$ such that the element $x^s y^b \in \Delta$ belongs to $\Delta$. Put $Y_\Delta = X_\Delta^{-[s/a]} x^s y^b = x^d y^b$. Note that $Y_\Delta \in \Delta$ and $d$ satisfy $0 \leq d < a$.

Unfortunately our final step depends upon the parity of $b$. In case $b$ is even consider the triple $(c, e, f)$ such that $c, e, f$ are non-negative integers, $x^f y^e z^c \in \Delta$ and the triple $(c, e, f)$ is lexicographically minimal among such triples. Note that $e < b$, otherwise the element $Y_\Delta^{-1} x^f y^e z^c$ have same $c$ and smaller nonnegative $e$; multiplying by a suitable degree of $X_\Delta$ one can make the exponent at $x$ also nonnegative. The matrix is built. Put $Z_\Delta = x^f y^e z^c$.

The case $b$ is odd is done in a similar way, with the sole difference that we consider triples $(c, e, f)$ with even $e$, and use the element $Y_\Delta^{-2} x^f y^e z^c$ to prove $e < 2b$.

To prove the remaining properties (iv–vi) we do the following. Note that if an element $x^u$ belongs to $\Delta$, then $a \mid u$. Otherwise the existence of element $X_\Delta^{-[u/a]} x^u \in \Delta$ contradicts the definition of $a$. In case $b, c$ are even and $e$ is odd consider an element $Y_\Delta Z_\Delta Y_\Delta^{-1} Z_\Delta^{-1} = x^{2d} \in \Delta$; thus $a \mid 2d$. In case $b$ is even and $c, e$ are odd consider an element $Y_\Delta Z_\Delta Y_\Delta^{-1} Z_\Delta^{-1} = x^{2d} \in \Delta$; then $a \mid 2d$. Together this two cases cover the case $b$ is even $e$ is odd. In case $b$ is odd and $e$ is even consider the element $Y_\Delta Z_\Delta Y_\Delta^{-1} Z_\Delta^{-1}$ =
$x^{-2f} \in \Delta$; we have $a | 2f$. In case both $b, c$ are odd $Y_{\Delta} Z_{\Delta} Y_{\Delta} Z_{\Delta}^{-1} = x^{2d-2f} \in \Delta$; this implies $a | 2d - 2f$.

To build a correspondence from matrices to subgroups in all cases do the following: denote $X = x^a, Y = x^d y^b, Z = x^f y^e z^c$ and consider the set

$$\{X^u Y^v Z^w | u, v, w \in \mathbb{Z}\}$$

Direct calculus using (3.4) shows that this set is closed by multiplication, then it forms a subgroup of index $n$. $\square$

Next proposition is an exact analog of the previous one for manifold $B_4$.

**Proposition 7.** The subgroups $\Delta$ of index $n$ in $\pi_1(B_4)$ are in one-to-one correspondence with the integer matrices

$$\begin{pmatrix}
  c & e & f \\
  0 & b & d \\
  0 & 0 & a
\end{pmatrix}$$

such that

(i) $a, b, c > 0$ and $abc = n$;

(ii) $0 \leq d, f < a$;

(iii) if case $b$ is odd, $e$ is even and $0 \leq e < 2b$; if $b$ is even then $0 \leq e < b$;

(iv) if $b$ is even and $e$ is odd, then $2d \equiv 0 \mod a$;

(v) if $b$ is odd and $c$ is even, then $2f \equiv 0 \mod a$;

(vi) if both $b, c$ are odd, then $2(d - f) \equiv 1 \mod a$.

Herewith the subgroup $\Delta$ is generated by elements $x^a, x^d y^b$ and $x^f y^e z^c$, where $a, b, c, d, e, f$ are elements of the corresponding matrix.

**Observation.** Note that the sole difference between Proposition 6 and Proposition 7 is in the item (vi). In particular, $2(d - f) \equiv 1 \mod a$ implies $a$ is odd. Thus $n$ is necessarily odd in this case, because $n = abc$.

The proof literally repeats the proof of Proposition 6, so we omit it. Next proposition shows that the invariants introduced in Proposition 6 are sufficient to determine the the isomorphism type of a subgroup.

**Proposition 8.** Let $\Delta$ be a subgroup of finite index in $\pi_1(B_3)$ described by matrix

$$\begin{pmatrix}
  c & e & f \\
  0 & b & d \\
  0 & 0 & a
\end{pmatrix}$$

determined in Proposition 7. Then

(1) If $b, c, e$ are even then $\Delta \cong \mathbb{Z}^3$.

(2) If $c$ is odd and $b, e$ are even then $\Delta \cong \pi_1(G_2)$.

(3) If $e$ is odd and $d \equiv 0 \mod a$ ($b$ is necessarily even in this case) then $\Delta \cong \pi_1(B_1)$.
(4) If \( e \) is odd and \( d \equiv \frac{a}{2} \) mod \( a \) (and \( b \) are necessarily even in this case) then \( \Delta \cong \pi_1(B_2) \).

(5) If \( c, e \) are even, \( b \) is odd and \( f \equiv 0 \) mod \( a \) then \( \Delta \cong \pi_1(B_1) \).

(6) If \( c, e \) are even, \( b \) is odd and \( f \equiv \frac{a}{2} \) mod \( a \) (\( a \) is necessarily even in this case) then \( \Delta \cong \pi_1(B_2) \).

(7) If \( c, b \) are odd and \( f - d \equiv 0 \) mod \( a \) (\( e \) is necessarily even in this case) then \( \Delta \cong \pi_1(B_3) \).

(8) If \( c, b \) are odd and \( f - d \equiv \frac{a}{2} \) mod \( a \) (\( a \) and \( e \) are necessarily even in this case) then \( \Delta \cong \pi_1(B_3) \).

**Remark.** Note that the case \( b \) and \( e \) are both odd is prohibited by Proposition 6. First we need the following lemma.

**Lemma 4.** For a triple of integers \((p, q, r)\) there is at most one integer triple \((s, t, u)\) such that \(x^py^qz^r = X_\Delta Y_\Delta Z_\Delta^u\).

**Proof.** Since among \(X_\Delta, Y_\Delta, Z_\Delta\) only the latter one have a non-zero exponent at \(z\) in the canonical representation, \(r\) at most uniquely determines \(u\). The element \(x^py^qz^rZ_\Delta^{-u}\) have zero exponent at \(z\), thus its exponent at \(y\) at most uniquely determines \(t\), similar way for \(p\). \(\square\)

**Proof of Proposition 6** Consider a subgroup \(\Delta\) and define \(X_\Delta, Y_\Delta, Z_\Delta\) as in the proof of Proposition 6. We need the following lemma.

By Proposition 6 the subgroup \(\Delta\) is generated by \(X_\Delta, Y_\Delta, Z_\Delta\). We use the following standard representations of crystallographical groups (see [22] or [20])

\[
\begin{align*}
\mathbb{Z}^3 &= \langle x, y, z : x^1y^{-1}z = x^1z^{-1}y = y^1z^{-1} = 1 \rangle, \\
\pi_1(G_2) &= \langle x, y, z : x^1y^{-1}z = 1, x^1z^{-1}y = 1 \rangle, \\
\pi_1(B_1) &= \langle x, y, z : x^1y^{-1}z = y^1z^{-1} = 1, x^1z^{-1} = 1 \rangle, \\
\pi_1(B_2) &= \langle x, y, z : x^1y^{-1}z = 1, x^1z^{-1} = 1 \rangle, \\
\pi_1(B_3) &= \langle x, y, z : x^1y^{-1}z = 1, x^1z^{-1}y = 1 \rangle, \\
\pi_1(B_4) &= \langle x, y, z : x^1y^{-1}z = 1, y^1z^{-1} = 1 \rangle.
\end{align*}
\]

Below \(G\) is one of the groups from the list 3.6. Our immediate goal is to build the bijection \(\xi\) from the set \(\{x, y, z\}\) of generators of \(G\) to the set \(\{X_\Delta, Y_\Delta, Z_\Delta\}\) of generators of \(\Delta\), such that the relations of \(G\) hold for their images \(\xi(x), \xi(y), \xi(z)\). This means that \(\xi\) can be extended to an epimorphism \(\xi : G \rightarrow \Delta\).

In case \(G\) is one of the groups \(\mathbb{Z}^3, \pi_1(G_2), \pi_1(B_3), \pi_1(B_4)\), put \(\xi(x) = X_\Delta, \xi(y) = Y_\Delta, \xi(z) = Z_\Delta\). For the cases \(\pi_1(B_1)\) and \(\pi_1(B_2)\) the following makes the job:

- if \(c\) is even and \(e\) is odd then \(\xi(x) = X_\Delta, \xi(y) = Y_\Delta, \xi(z) = Z_\Delta\);
- if both \(c, e\) are odd then \(\xi(x) = Y_\Delta, \xi(y) = X_\Delta, \xi(z) = Z_\Delta\);
• finally, if $b$ is odd then $\xi(x) = X_\Delta$, $\xi(y) = Z_\Delta$, $\xi(z) = Y_\Delta$.

Direct verification through (3.4) shows that relations (3.6) hold.

Now we need to prove that the epimorphism $\xi$ is really an isomorphism. We claim that in each of the groups $\mathbb{Z}^3$, $\pi_1(G_2)$, $\pi_1(B_1)$, $\pi_1(B_2)$, $\pi_1(B_3)$, $\pi_1(B_4)$ any element $g$ can be uniquely represented in the canonical form $g = x^s y^t z^u$. This is obvious for $\mathbb{Z}^3$; for $\pi_1(G_2)$ see (3, Proposition 2); for $\pi_1(B_1)$ and $\pi_1(B_2)$ see (3, Propositions 1 and 6 respectively); for $\pi_1(B_3)$ and $\pi_1(B_4)$ this is Proposition 4 and Proposition 5 above. Thus each element $g \in \Delta$ can be represented in the form $g = X_\Delta^s Y_\Delta^t Z_\Delta^u$. If $\xi$ have non-trivial core then some $g \in \Delta$ have two different representations. Recall that $\Delta$ is a subgroup of $\pi_1(B_3)$, then by Proposition 4 any element $g$ can be represented in the form $g = x^s y^t z^u$. Thus the existence of two different representations $g = X_\Delta^s Y_\Delta^t Z_\Delta^u$ contradicts Lemma 4.

Next proposition is a twin of Proposition 8 for the group $\pi_1(B_4)$.

**Proposition 9.** Let $\Delta$ be a subgroup of finite index in $\pi_1(B_4)$ described by matrix
\[
\begin{pmatrix}
c & e & f \\
ob & d & \\
0 & 0 & a
\end{pmatrix}
\]
determined in Proposition 8. Then

1. If $b, c, e$ are even then $\Delta \cong \mathbb{Z}^3$.
2. If $c$ is odd and $b, e$ are even then $\Delta \cong \pi_1(G_2)$.
3. If $e$ is odd and $d \equiv 0 \mod a$ ($b$ is necessarily even in this case) then $\Delta \cong \pi_1(B_1)$.
4. If $e$ is odd and $d \equiv \frac{a}{2} \mod a$ ($a$ and $b$ are necessarily even in this case) then $\Delta \cong \pi_1(B_2)$.
5. If $c, e$ are even, $b$ is odd and $f \equiv 0 \mod a$ then $\Delta \cong \pi_1(B_1)$.
6. If $c, e$ are even, $b$ is odd and $f \equiv \frac{a}{2} \mod a$ ($a$ is necessarily even in this case) then $\Delta \cong \pi_1(B_2)$.
7. If $c, b$ are odd (recall that in this case necessarily $a, e$ are odd and $2(d - f) \equiv 1 \mod a$) then $\Delta \cong \pi_1(B_4)$.

The proof repeats exactly the proof of Proposition 8 so we omit it.

### 4 Proof of Theorem 1 and Theorem 2

**Overall scheme of the proof**

In this chapter we prove the Theorems 1 and 2 separately for each isomorphism type.

**Notations.** By $\Lambda$ denote the subgroup $\Lambda = \langle x, y^2, z^2 \rangle$ in the group $\pi_1(B_3) = \langle x, y, z \rangle$.

Note that $\Lambda \cong \mathbb{Z}^3$, $\Lambda \lhd \pi_1(B_3)$ and $\pi_1(B_3)/\Lambda \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$; the cosets are represented by 1, $y$, $z$ and $yz$. 

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Definition 3. Recall that $\Delta^\Lambda$ denotes the set of subgroups $\{\Delta^\lambda \mid \lambda \in \Lambda\}$, further we call it an intermediate conjugacy class of $\Delta$. Denote the number of intermediate conjugacy classes of subgroups $\Delta$ of a given isomorphism type $H$ by $c_{H, \pi_1(B_3)}^\Lambda(n)$.

Recall that $Ad_g$ provides an automorphism $u \mapsto u^g$ of the group $\pi_1(B_3)$.

Definition 4. Given an isomorphism type $H$ of a subgroup $\Delta$, consider intermediate conjugacy classes $\Delta^\Lambda$. The number of intermediate conjugacy classes $\Delta^\Lambda$ preserved by $Ad_1$, $Ad_y$, $Ad_z$ and $Ad_{xyz}$ denote $N_{H, \pi_1(B_3),1}(n)$, $N_{H, \pi_1(B_3),y}(n)$, $N_{H, \pi_1(B_3),z}(n)$ and $N_{H, \pi_1(B_3),yz}(n)$ respectively. In particular, $N_{H, \pi_1(B_3),1}(n) = c_{H, \pi_1(B_3)}^\Lambda(n)$, but we introduce this notation for uniformity. Further we omit $H$ and $\pi_1(B_3)$ as indexes if the context is clear.

Our calculation of $s_{H, \pi_1(B_3)}(n)$ will be straightforward, while the calculation of $c_{H, \pi_1(B_3)}^\Lambda(n)$ will take the several steps. First we find $c_{H, \pi_1(B_3)}^\Lambda(n)$. Essentially, Proposition 3 claims that a subgroup $\Delta$ is uniquely determined by the choice of three items. The first is a factorization $abc = n$, the second is a coset of the element $x^d$ in $\langle x \rangle/\langle x^a \rangle$ (which we simplified by the restriction $0 \leq d < a$) and the third is a coset of $x^f y^e$ in $\langle x, y \rangle/\langle x^a, x^d y^b \rangle$ (similarly, this was simplified by restricting $0 \leq e < b$ and $0 \leq f < a$). The conjugation by generators of $\Lambda$ has the following properties:

- the replacement $\Delta \mapsto \Delta^x$ maps $x^d \mapsto x^d[x, y^b]$ and $x^f y^e \mapsto x^f[x, y^e z^e]y^e$; where we denote $[u, v] = uvv^{-1}v^{-1}$;
- the replacement $\Delta \mapsto \Delta^y^2$ maps $x^f y^e \mapsto x^f y^e[y^2, z^e]$;
- the replacement $\Delta \mapsto \Delta^z^2$ is always identity transformation.

This allows to calculate $c_{H, \pi_1(B_3)}^\Lambda(n)$ uniformly for different isomorphism types $H$. Calculations of $N_1(n)$, $N_y(n)$, $N_z(n)$ and $N_{yz}(n)$ use Corollary 3 and similar considerations. An application of the Burnside Lemma finishes the job.

4.1 Case $\Delta \cong \pi_1(G_1) \cong \mathbb{Z}^3$

By Proposition 6 and Proposition 8 every subgroup $\Delta \cong \mathbb{Z}^3$ of index $n$ in $\pi_1(B_3)$ corresponds to an integer matrix

$$
\begin{pmatrix}
c & e & f \\
0 & b & d \\
0 & 0 & a
\end{pmatrix}
$$

where $a, b, c > 0, abc = n; b, c, e$ are even and $0 \leq d, f < a, 0 \leq e < b$. Thus

$$
\begin{align*}
s_{G_1, B_3}(n) &= |\{(a, b, c, d, e, f) \in \mathbb{Z}^6 \mid a, b, c > 0, abc = n, 2 \mid b, c, e, 0 \leq d, f < a, 0 \leq e < b\}| \\
&= \sum_{abc=n^2} a^2b = \omega(\frac{n}{4}).
\end{align*}
$$

Note that $\Delta \cong \mathbb{Z}^3$ implies $\Delta \leq \Lambda$; recall that $\Lambda \cong \mathbb{Z}^3$, therefore $\Delta^\Lambda = \{\Delta\}$. Indeed, this is another way to get the equality $s_{G_1, B_3}(n) = \omega(\frac{n}{4})$. So, $c_{G_1, B_3}^\Lambda(n) = N_1(n) = \omega(\frac{n}{4})$. 

Since $\Lambda = \langle x, y^2, z^2 \rangle \cong \mathbb{Z}^3$, we use the additive notation for $\Lambda$, representing $x$, $y^2$ and $z^2$ by $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ respectively. Then $Ad_y$, $Ad_z$ and $Ad_{yz}$ are given by $Ad_y : (u, v, w) \mapsto (-u, v, w)$, $Ad_z : (u, v, w) \mapsto (-u, -v, w)$ and $Ad_{yz} : (u, v, w) \mapsto (u, -v, w)$. Thus by Corollary $\pi_1(N_2) = N_2(n) = N_{yz}(n) = \sigma_2(\frac{n}{4}) + 3\sigma_2(\frac{n}{8})$. Applying the Burnside Lemma one gets

$$c_{G_1, B_3}(n) = \frac{1}{4}(\omega(\frac{n}{4}) + 3\sigma_2(\frac{n}{4}) + 9\sigma_2(\frac{n}{8})).$$

### 4.2 Case $\Delta \cong \pi_1(G_2)$

By Proposition $\mathbb{Q}$ and Proposition $\mathbb{S}$ every subgroup $\Delta \cong \pi_1(G_2)$ of index $n$ in $\pi_1(B_3)$ corresponds to an integer matrix

\[
\begin{pmatrix}
c & e & f \\
0 & b & d \\
0 & 0 & a
\end{pmatrix}
\]

where $a, b, c > 0$, $abc = n$; $b, e$ are even, $c$ is odd and $0 \leq d, f < a$, $0 \leq e < b$. Thus

\[
s_{G_1, B_3}(n) = |\{(a, b, c, d, e, f) \in \mathbb{Z}^6 | a, b, c > 0, abc = n, 2 \mid b, e, 2 \nmid c, 0 \leq d, f < a, 0 \leq e < b\}|
\]

\[
= \sum_{\substack{abc=n, \\
2 \nmid b, 2 \nmid c}} a^2b - \sum_{\substack{abc=n, \\
2 \nmid b, 2 \nmid c}} a^2b = \omega(\frac{n}{2}) - \omega(\frac{n}{4}).
\]

So for the case $\Delta \cong \pi_1(G_2)$ we are done with Theorem $\mathbb{I}$ in this case, and proceed to Theorem $\mathbb{I}$.

**Notations.** Recall that $X_\Delta = x^a$, $Y_\Delta = x^dy^h$ and $Z_\Delta = x^fz^e$. Put $H_\Delta = \langle X_\Delta, Y_\Delta \rangle$ and $J_H = \langle x, y^2 \rangle / \langle x^2, y^4, H_\Delta \rangle$.

Observe that $|J_H| \in \{1, 2, 4\}$ for every $H$.

To enumerate intermediate conjugacy classes note that $X_\Delta = X_\Delta^\lambda$ and $Y_\Delta = Y_\Delta^\lambda$ for all $\lambda \in \Lambda$. Also $[Z_\Delta, x] = x^2$, $[Z_\Delta, y^2] = y^4$ and $[Z_\Delta, z^2] = 1$. Since $\Lambda = \langle x, y^2, z^2 \rangle$, the family of intermediate conjugacy classes $\{\Delta^\lambda | H = \pi_1(G_2), |\pi_1(B_3) : H| = n \}$ is enumerated by a choice of the following invariants:

- a factorization $abc = n$, where $b$ is even and $c$ is odd,
- a subgroup $H < \langle x, y^2 \rangle$ such that $|\langle x, y \rangle : H| = ab$,
- a coset of the element $Z_\Delta z^{-c}$ in $J_H$.

So we have to enumerate the sequences of choices. Note that the number of sequences for a fixed $c$ is provided by Remark $\mathbb{I}$ namely the number of pairs $(H, h)$ such that $|Z^2 : H| = \frac{n}{2c}$, $h \in I_H$ is equal to $\sigma_1(\frac{n}{2c}) + 3\sigma_1(\frac{n}{4c})$. Summing over all odd values of $c$ one gets

\[
c_{G_2, B_3}(n) = \sigma_2(\frac{n}{2}) + 2\sigma_2(\frac{n}{4}) - 3\sigma_2(\frac{n}{8}).
\]

Now we find $N_2(n)$, $N_3(n)$ and $N_{yz}(n)$. Note that $Ad_z$ preserves any $\Delta^\lambda$, in other words $N_2(n) = N_1(n)$ and $N_3(n) = N_{yz}(n)$.
From this point until the end of the subsection \( H \) always denotes a subgroup of \( \langle x, y^2 \rangle \cong \mathbb{Z}^2 \). For the latter we use the additive notation, representing \( x \) and \( y^2 \) by \((1, 0)\) and \((0, 1)\) respectively.

**Notation.** There are 4 cosets in the coset decomposition \( \mathbb{Z}^2/(2\mathbb{Z})^2 \), represented by elements \((0, 0)\), \((1, 0)\), \((0, 1)\) and \((1, 1)\) respectively. We denote this cosets by \( M_{0, 0}, M_{1, 0}, M_{0, 1}, M_{1, 1} \). We define the type of a subgroup \( H < \mathbb{Z}^2 \) as follows:

- \( H \) have the type \((0, 0)\) if \( H \subseteq M_{0, 0} \)
- \( H \) have the type \((1, 0)\) if \( H \subseteq M_{0, 0} \cup M_{1, 0} \) but \( H \not\subseteq M_{0, 0} \)
- \( H \) have the type \((0, 1)\) if \( H \subseteq M_{0, 0} \cup M_{0, 1} \) but \( H \not\subseteq M_{0, 0} \)
- \( H \) have the type \((1, 1)\) if \( H \subseteq M_{0, 0} \cup M_{1, 1} \) but \( H \not\subseteq M_{0, 0} \)
- finally, \( H \) is of total type if none of the previous cases hold.

Note that \( |J_H| \) depends only on the type of a subgroup \( H \), and equals 4,2,2,2,1 respectively.

Denote by \( G_{0,0}(n), G_{1,0}(n), G_{0,1}(n), G_{1,1}(n) \) and \( G_{\text{total}}(n) \) the number of subgroups \( n \) of respective type having index \( n \) in \( \mathbb{Z}^2 \) and preserved by automorphism \( (u, v) \mapsto (u, -v) \).

To calculate \( N_y(n) \) we need to enumerate pairs \((H, h)\) of described above form, such that the composition of correspondences \( H, h \rightarrow \Delta \rightarrow \Delta^y \rightarrow (H', h') \) acts identically on them. Recalling all definitions one gets that \( H' = H^y \) and \( h' = h[Z_\Delta, y] \). Note that \([Z_\Delta, y]\) does not depends on \( \Delta \), moreover \([Z_\Delta, y] = [z, y] = y^2 \). So the condition \( h' = h \) is equivalent to \( y^2 \in J_H \), that is holds true for all \( h \in J_H \) if \( H \) have type \((0, 1)\) or total, and for none \( h \in J_H \) if \( H \) have any other type. Recall that \( |J_H| \) is equal to 2 and 1 is case \( H \) have type \((0, 1)\) and total respectively. So we have to find the sum of \( 2G_{0,1}(k) + G_{\text{total}}(k) \) over all possible \( k = |\langle x, y^2 \rangle : H| \).

Fix an index \( k = |\mathbb{Z}^2 : H| \). Note that \( Ad_y \) have the form \( (u, v) \mapsto (u, -v) \) on \( \langle x, y^2 \rangle = \langle (1, 0), (0, 1) \rangle \). So we claim the following isomorphisms:

\[
((1, 0), (0, 1)), Ad_y \cong ((2, 0), (0, 1)), Ad_y \cong ((1, 0), (0, 2)), Ad_y \cong ((2, 0), (0, 2)), Ad_y \cong (\mathbb{Z}^2, \ell), (\ell, (1, 0), (0, 2)), Ad_y \cong (\mathbb{Z}^2, j).
\]

That is

\[
G_{0,0}(k) + G_{1,0}(k) + G_{0,1}(k) + G_{1,1}(k) + G_{\text{total}}(k) = f_{\mathbb{Z}^2, \ell}(k),
\]

\[
G_{0,0}(k) + G_{1,0}(k) = G_{0,0}(k) + G_{0,1}(k) = f_{\mathbb{Z}^2, \ell}(k) = f_{\mathbb{Z}^2, j}(k),
\]

\[
G_{0,0}(k) + G_{1,1}(k) = f_{\mathbb{Z}^2, j}(k),
\]

\[
G_{0,0}(k) = f_{\mathbb{Z}^2, \ell}(k).
\]

Substituting values from Proposition\[2\] we have \( 2G_{0,1}(k) + G_{\text{total}}(k) = f_{\mathbb{Z}^2, \ell}(k) - f_{\mathbb{Z}^2, j}(k) = \sigma_0(k) + d_2(\frac{k}{4}) - 2\sigma_0(\frac{k}{8}) \). Summing over all values \( k \) such that \( 2k \mid n \) and \( \frac{n}{2k} \) is odd, one gets \( N_y(n) = d_3(\frac{n}{2}) - d_3(\frac{n}{4}) + d_3(\frac{n}{8}) - 3d_3(\frac{n}{16}) + 2d_3(\frac{n}{32}) \).
Finally,
\[
c_{G_2, B_3} = \frac{1}{2}\left(\sigma_2\left(\frac{n}{2}\right) + 2\sigma_2\left(\frac{n}{4}\right) - 3\sigma_2\left(\frac{n}{8}\right) + d_3\left(\frac{n}{2}\right) - d_3\left(\frac{n}{4}\right) + d_3\left(\frac{n}{8}\right) - 3d_3\left(\frac{n}{16}\right) + 2d_3\left(\frac{n}{32}\right)\right).
\]

4.3 Case $\Delta \cong \pi_1(B_1)$

The difficulty of this case is that two types of matrices lead to the same isomorphism type $\pi_1(B_1)$, see cases 3 and 5 of Proposition 8. We consider these cases separately.

4.3.1 Case 3 of Proposition 8

In this case the corresponding matrix of a subgroup $\Delta$ has the form \[
\begin{pmatrix}
c & e & f \\
0 & b & 0 \\
0 & 0 & a
\end{pmatrix},
\]
where $b$ is even and $e$ is odd. So for a fixed triple $(a, b, c)$ there are $\frac{b}{2}$ choices for $e$ and $a$ choices for $f$. That is
\[
\sum_{\substack{a, b, c = n \mod 2 \mid b, \, a, \
| a | b}} \frac{ab}{2} = \chi\left(\frac{n}{2}\right).
\]

Now we enumerate conjugacy classes of subgroups $\Delta$ matching case 3. When replacing a subgroup $\Delta$ with a conjugated one $\Delta^g, g \in \pi_1(B_3)$ the numbers $a, b, c$ are not changed. The pair $(e, f)$ is changed in the following way.

First assume that $c$ is odd. Taking into account that $\pi_1(B_3) = \langle x, y, z \rangle$ it is sufficient to describe the action of $Ad_x, Ad_y, Ad_z$:

\[
Ad_x : (e, f) \mapsto (e, f), \quad Ad_y : (e, f) \mapsto (e + 2, -f), \quad Ad_z : (e, f) \mapsto (-e, -f).
\]

Keep in mind that $e$ and $f$ are residues modulo $b$ and $a$ respectively, for example, $-f$ refers to the number $a - f$. Also recall that $e$ is always odd; the parity of $e$ is well-defined since $b$ is even.

So, in case $b \equiv 0 \mod 4$ there are $a$ orbits under the action of the group generated by $Ad_x, Ad_y$ and $Ad_z$; each residue $s$ modulo $a$ corresponds to the orbit \{(4i + 1, s)\mid 0 \leq i < \frac{b}{4}\} \cup \{(4i + 3, -s)\mid 0 \leq i < \frac{b}{4}\}.

In case $b \equiv 2 \mod 4$ the orbits are of the form \{(2i + 1, s)\mid 0 \leq i < \frac{b}{2}\} \cup \{(2i + 1, -s)\mid 0 \leq i < \frac{b}{2}\}. Thus there are $\frac{a+1}{2}$ orbits if $a$ is odd, and $\frac{a+2}{2}$ orbits if $a$ is even.

So, the number of conjugacy classes for a fixed odd $c$ is

\[
\sum_{\substack{a, b = \frac{c}{2} \mod \frac{b}{2} \mid c}} \begin{cases}
\frac{a + 1}{2} & \text{if } 2 \nmid a, \, 4 \nmid b \\
\frac{a + 2}{2} & \text{if } 2 \mid a, \, 4 \nmid b \\
a & \text{if } 4 \mid b
\end{cases}
\]
\[
= \frac{1}{2}\left(\sum_{a, b = \frac{c}{2} \mod \frac{b}{2} \mid c} a \text{ if } 4 \nmid b + \sum_{a, b = \frac{c}{2} \mod \frac{b}{2} \mid c} 1 \text{ if } 2 \nmid a\right)
\]
\[
= \frac{1}{2}\left(\sigma_1\left(\frac{n}{2c}\right) + \sigma_1\left(\frac{n}{4c}\right) + \sigma_0\left(\frac{n}{2c}\right) - \sigma_0\left(\frac{n}{8c}\right)\right).
\]
Now assume that \( c \) is even. Then

\[
Ad_x : (e, f) \mapsto (e, f + 2), \quad Ad_y : (e, f) \mapsto (e, -f), \quad Ad_z : (e, f) \mapsto (-e, -f).
\]

Thereby for fixed \((a, b, c)\) conjugacy classes are enumerated by pairs of the following form. The first element of the pair is a class of odd residues \( \{e, -e\} \) modulo \( b \), the second element is a parity of \( f \). There are \( \frac{b+2}{4} \) and \( \frac{b+2}{4} \) choices of the first element in the cases \( b \equiv 0 \mod 4 \) and \( b \equiv 2 \mod 4 \) respectively. There are 1 and 2 choices of parity in case \( a \) is odd and even respectively.

Thus the number of conjugacy classes for a fixed even \( c \) is

\[
\sum_{ab=\frac{b+2}{4}} \left\{ \begin{array}{ll}
\frac{b+2}{4} & \text{if } 4 \nmid b \\
\frac{b}{4} & \text{if } 4 \mid b
\end{array} \right\} \times \left\{ \begin{array}{ll}
1 \text{ if } 2 \nmid a \\
2 \text{ if } 2 \mid a
\end{array} \right\} = \frac{1}{2} \left( \sum_{ab=\frac{b+2}{4}} \left\{ \begin{array}{ll}
\frac{b}{4} & \text{if } 2 \nmid a \\
2 & \text{if } 2 \mid a
\end{array} \right\} \right) = \frac{1}{2} \left( \sigma_1 \left( \frac{n}{2c} \right) + \sigma_1 \left( \frac{n}{4c} \right) + \sigma_0 \left( \frac{n}{2c} \right) - \sigma_0 \left( \frac{n}{8c} \right) \right).
\]

It is noteworthy that we got the same function for both parities of \( c \). One can suggests that there is a single argument to cover both cases which the authors failed to find.

\[
c_{B_1, B_3}^{\text{case 3}}(n) = \sum_{c \mid n} \frac{1}{2} \left( \sigma_1 \left( \frac{n}{2c} \right) + \sigma_1 \left( \frac{n}{4c} \right) + \sigma_0 \left( \frac{n}{2c} \right) - \sigma_0 \left( \frac{n}{8c} \right) \right) = \frac{1}{2} \left( \sum_{ab=\frac{b+2}{4}} \left\{ \begin{array}{ll}
\frac{b+2}{4} & \text{if } 2 \nmid a \\
2 & \text{if } 2 \mid a
\end{array} \right\} \right) \times \left\{ \begin{array}{ll}
1 \text{ if } 2 \nmid a \\
2 \text{ if } 2 \mid a
\end{array} \right\} = \frac{1}{2} \left( \sigma_2 \left( \frac{n}{2} \right) + \sigma_2 \left( \frac{n}{4} \right) + d_3 \left( \frac{n}{2} \right) - d_3 \left( \frac{n}{8} \right) \right).
\]

### 4.3.2 Case 5 of Proposition 8

This section follows the same scheme as the previous one. Here the corresponding matrix of a subgroup \( \Delta \) has the form

\[
\begin{pmatrix}
c & e & 0 \\
0 & b & d \\
0 & 0 & a
\end{pmatrix}
\]

where \( b \) is odd, \( c \) is even and \( e \) is an even residue modulo \( 2b \). So for a fixed triple \((a, b, c)\) there are \( b \) choices for \( e \) and \( a \) choices for \( d \). That is

\[
s_{B_1, B_3}^{\text{case 5}}(n) = \sum_{ab=\frac{b+2}{4}} \left\{ \begin{array}{ll}
\frac{b+2}{4} & \text{if } 2 \nmid a \\
2 & \text{if } 2 \mid a
\end{array} \right\} \times \left\{ \begin{array}{ll}
1 \text{ if } 2 \nmid a \\
2 \text{ if } 2 \mid a
\end{array} \right\} = \chi \left( \frac{n}{2} \right) - 2 \chi \left( \frac{n}{4} \right).
\]

As in previous section, for a fixed triple \((a, b, c)\) the subgroups \( \Delta \) are enumerated by pairs \((d, e)\). Conjugation acts on pairs in the following way:

\[
Ad_x : (d, e) \mapsto (d + 2, e), \quad Ad_y : (d, e) \mapsto (-d, e), \quad Ad_z : (d, e) \mapsto (-d, -e).
\]

Thus conjugacy classes are enumerated by pairs, consisting of the parity of \( d \) and the class \( \{e, -e\} \) of residue \( e \) modulo \( 2b \). There are 1 and 2 choices for the first element of such a pair in case \( a \) is odd and even respectively. Also we have \( \frac{b+1}{2} \) choices for the second element.
Then the number of conjugacy classes for a fixed even $c$ is

$$
\sum_{\substack{ab=\frac{n}{2} \\ 2|b}} \frac{b+1}{2} \times \left\{ \begin{array}{ll}
1 & \text{if } 2 \nmid a \\
2 & \text{if } 2 \mid a
\end{array} \right.
= \frac{1}{2} (\sigma_1\left(\frac{n}{c}\right) - \sigma_1\left(\frac{n}{2c}\right) - 2\sigma_1\left(\frac{n}{4c}\right) + \sigma_0\left(\frac{n}{c}\right) - \sigma_0\left(\frac{n}{4c}\right)).
$$

Summing over all even values of $c$ one gets

$$
c_{B_1B_3}^{case\ 5}(n) = \sum_{c|n, 2|c} \frac{1}{2} (\sigma_1\left(\frac{n}{c}\right) - \sigma_1\left(\frac{n}{2c}\right) - 2\sigma_1\left(\frac{n}{4c}\right) + \sigma_0\left(\frac{n}{c}\right) - \sigma_0\left(\frac{n}{4c}\right))
= \frac{1}{2} (\sigma_2\left(\frac{n}{2}\right) - \sigma_2\left(\frac{n}{4}\right) - 2\sigma_2\left(\frac{n}{8}\right) + d_3\left(\frac{n}{2}\right) - d_3\left(\frac{n}{8}\right)).
$$

Finally

$$
s_{B_1B_3}(n) = s_{B_1B_3}^{case\ 3}(n) + s_{B_1B_3}^{case\ 5}(n) = 2\chi\left(\frac{n}{2}\right) - 2\chi\left(\frac{n}{4}\right),
c_{B_1B_3}(n) = c_{B_1B_3}^{case\ 3}(n) + c_{B_1B_3}^{case\ 5}(n) = \sigma_2\left(\frac{n}{2}\right) - \sigma_2\left(\frac{n}{4}\right) + d_3\left(\frac{n}{2}\right) - d_3\left(\frac{n}{8}\right).
$$

### 4.4 Case $\Delta \cong \pi_1(B_2)$

This section follows the same scheme as Section 4.3. Two types of matrices lead to the same isomorphism type $\pi_1(B_2)$, see cases 4 and 6 of Proposition 8. We consider these cases separately.

#### 4.4.1 Case 4 of Proposition 8

In this case the corresponding matrix of a subgroup $\Delta$ has the form $\begin{pmatrix} c & e & f \\ 0 & b & \frac{a}{2} \\ 0 & 0 & a \end{pmatrix}$, where $a, b$ are even and $e$ is odd. So for a fixed triple $(a, b, c)$ there are $\frac{b}{2}$ choices for $e$ and $a$ choices for $f$. That is

$$
s_{B_2B_3}^{case\ 4}(n) = \sum_{\substack{abc=n, \\ 2|a,b}} \frac{ab}{2} = 2\chi\left(\frac{n}{4}\right).
$$

Now we enumerate conjugacy classes of subgroups $\Delta$ matching case 4 of Proposition 8. When replacing a subgroup $\Delta$ with a conjugated one $\Delta^g, g \in \pi_1(B_3)$ the numbers $a, b, c$ are not changed. Wherein the pair $(e, f)$ is changed in the following way.

First assume that $c$ is odd. Then

$$
A_{dx} : (e, f) \mapsto (e, f), \quad A_{dy} : (e, f) \mapsto (e + 2, -f), \quad A_{dz} : (e, f) \mapsto (-e, -f).
$$

Recall that $e, f$ first appeared in Proposition 8 and their geometrical meaning is the following. Consider the set $\mathbb{Z}^2$ and its subset $Odd = \{(2u + 1, v)\} \subset \mathbb{Z}^2$. Also consider the group $G$ of bijections $Odd \mapsto Odd$, generated by $$(2u + 1, v) \mapsto (2u + 1 + b, v + \frac{a}{2})$$
and \((2u + 1, v) \mapsto (2u + 1, v + a)\). The set of pairs \((e, f)\) is the set of representatives of orbits of \(Odd\) under the action of \(G\).

So we have to calculate the number of orbits of pairs \(\{(2u + 1, v) \mid u, v \in \mathbb{Z}\}\) under the action of the group, generated by the mappings

\[
(2u + 1, v) \mapsto (2u + 3, -v), \quad (2u + 1, v) \mapsto (-2u - 1, -v),
(2u + 1, v) \mapsto (2u + 1 + b, v + \frac{a}{2}), \quad (2u + 1, v) \mapsto (2u + 1, v + a).
\]

Omitting routine calculations we claim that the number of orbits is \(\frac{b}{2}\) unless simultaneously \(b \equiv 2 \mod 4\) and \(4 \mid a\); in the latter case the number of orbits is \(\frac{a + 2}{2}\).

So, the number of conjugacy classes for a fixed odd \(c\) is

\[
\sum_{\frac{ab}{2} = \frac{c}{2}} \frac{a}{2} + \sum_{\frac{ab}{2} = \frac{c}{2}} \frac{b}{2} = \sigma_1\left(\frac{n}{4c}\right) + \sigma_0\left(\frac{n}{8c}\right) - \sigma_0\left(\frac{n}{16c}\right).
\]

Now assume that \(c\) is even. Then

\[
Ad_x : (e, f) \mapsto (e, f + 2), \quad Ad_y : (e, f) \mapsto (e, -f), \quad Ad_z : (e, f) \mapsto (-e, -f).
\]

Following the above argument we have to calculate the number of orbits of pairs \(\{(2u + 1, v) \mid u, v \in \mathbb{Z}\}\) under the action of the group generated by the mappings

\[
(u, v) \mapsto (u, v + 2), \quad (u, v) \mapsto (u + b, v + \frac{a}{2}),
(u, v) \mapsto (u, v + a).
\]

Again omitting calculations, the number of orbits is \(\frac{b}{2}\) unless simultaneously \(b \equiv 2 \mod 4\) and \(4 \mid a\); in the latter case the number of orbits is \(\frac{b + 2}{2}\).

So, the number of conjugacy classes for a fixed even \(c\) is

\[
\sum_{\frac{ab}{2} = \frac{c}{2}} \frac{b}{2} + \sum_{\frac{ab}{2} = \frac{c}{2}} \frac{a}{2} = \sigma_1\left(\frac{n}{4c}\right) + \sigma_0\left(\frac{n}{8c}\right) - \sigma_0\left(\frac{n}{16c}\right).
\]

Summing over possible values of \(c\) one gets

\[
c_{B_2, B_3}^{case 4}(n) = \sum_{c \mid n} \left( \sigma_1\left(\frac{n}{4c}\right) + \sigma_0\left(\frac{n}{8c}\right) - \sigma_0\left(\frac{n}{16c}\right) \right) = \sigma_2\left(\frac{n}{4}\right) + d_3\left(\frac{n}{8}\right) - d_3\left(\frac{n}{16}\right).
\]

### 4.4.2 Case 6 of Proposition 8

In this case the corresponding matrix of a subgroup \(\Delta\) has the form

\[
\begin{pmatrix}
c & e & a \\
0 & b & d \\
0 & 0 & a
\end{pmatrix},
\]

where \(b\) is odd, \(a, c\) are even and \(e\) is an even residue modulo \(2b\). So for a fixed triple \((a, b, c)\) there are \(b\) choices for \(e\) and \(a\) choices for \(d\). That is

\[
s_{B_2, B_3}^{case 6}(n) = \sum_{ab = n, \atop 2|b, 2|a, c} ab - \sum_{ab = n, \atop 2|a, c} ab = 2\chi\left(\frac{n}{4}\right) - 4\chi\left(\frac{n}{8}\right).
\]
As in the previous section, for a fixed triple \((a, b, c)\) the subgroups \(\Delta\) are enumerated by pairs \((d, e)\). Conjugation acts on pairs in the following way:

\[
Ad_x : (d, e) \mapsto (d + 2, e), \quad Ad_y : (d, e) \mapsto (-d, e + 2), \quad Ad_z : (d, e) \mapsto (-d, -e).
\]

Thus conjugacy classes are enumerated by invariants, each consisting of the parity of \(d\) and the class \(\{e, -e\}\) of residue \(e\) modulo \(2b\). Since \(a\) is always even, there are 2 choices for the first element of such pair. Also we have \(\frac{n+1}{2}\) choices for the second element.

Then the number of conjugacy classes for a fixed even \(c\) is

\[
\sum_{ab=\frac{c}{2}, \frac{b}{2}|a} (b + 1) = \sigma_1(\frac{n}{2c}) - 2\sigma_1(\frac{n}{4c}) + \sigma_0(\frac{n}{2c}) - \sigma_0(\frac{n}{4c}).
\]

Summing over all even values of \(c\) we get

\[
c_{case \ 5}^{c_{case \ 5}}(n) = \sum_{c|n, 2|c} \left( \sigma_1(\frac{n}{2c}) - 2\sigma_1(\frac{n}{4c}) + \sigma_0(\frac{n}{2c}) - \sigma_0(\frac{n}{4c}) \right) = \sigma_2(\frac{n}{4}) - 2\sigma_2(\frac{n}{8}) + d_3(\frac{n}{4}) - d_3(\frac{n}{8}).
\]

Finally,

\[
s_{B_2, B_3}(n) = s_{case \ 4}^{case \ 4}(n) + s_{case \ 6}^{case \ 6}(n) = 4\chi(\frac{n}{4}) - 4\chi(\frac{n}{8}),
\]

\[
c_{B_2, B_3}(n) = c_{case \ 4}^{case \ 4}(n) + c_{case \ 6}^{case \ 6}(n) = 2\sigma_2(\frac{n}{4}) - 2\sigma_2(\frac{n}{8}) + d_3(\frac{n}{4}) - d_3(\frac{n}{16}).
\]

### 4.5 Case \(\Delta \cong \pi_1(B_3)\)

In this case the corresponding matrix of a subgroup \(\Delta\) has the form \(\begin{pmatrix} c & e & d \\ 0 & b & d \\ 0 & 0 & a \end{pmatrix}\), where \(b, c\) are odd, \(e\) is even and \(0 \leq e < 2b, 0 \leq d < a\). Thus

\[
s_{B_2, B_3}(n) = \left| \{(a, b, c, d, e) \in \mathbb{Z}^6 \mid abc = n, 2 \nmid b, c, 2 \mid e, 0 \leq d < a, 0 \leq e < 2b \} \right|
= \sum_{ab=\frac{n}{2b}, c} ab = \chi(n) - 3\chi(\frac{n}{2}) + 2\chi(\frac{n}{4}).
\]

To enumerate the conjugacy classes consider subgroups \(\Delta\) corresponding to some fixed factorization \(abc = n\). We identify such subgroups with the pairs \((d, e)\) whence \(f = d\). Conjugations act as follows:

\[
Ad_x : (d, e) \mapsto (d + 2, e), \quad Ad_y : (d, e) \mapsto (-d, e + 2), \quad Ad_z : (d, e) \mapsto (-d, -e).
\]

Thus for a fixed factorization \(abc = n\) the conjugacy classes are enumerated by the parity of \(d\); namely there is only one conjugacy class if \(a\) is odd and two if \(a\) is even. Then

\[
c_{B_2, B_3}(n) = \sum_{abc=\frac{n}{2b}, c} \left\{ \begin{array}{ll}
1 & \text{if } 2 \nmid a \\
2 & \text{if } 2 \mid a
\end{array} \right\} = d_3(n) - d_3(\frac{n}{2}) - d_3(\frac{n}{4}) - d_3(\frac{n}{8}).
\]
4.6 Case \( \Delta \cong \pi_1(B_4) \)

This section is similar to Section 4.5 with the sole difference that in all summations the terms corresponding to odd values of \( a \) vanish.

Indeed, the corresponding matrix of a subgroup \( \Delta \) has the form

\[
\begin{pmatrix}
c & e & f + \frac{a}{2} \\
0 & b & d \\
0 & 0 & a
\end{pmatrix},
\]

where \( b, c \) are odd, \( a, e \) are even and \( 0 \leq e < 2b, 0 \leq d, f < a \) and \( d \equiv f \mod a \). Then

\[
s_{B_4,B_3}(n) = \left| \{(a, b, c, d, e, f) \in \mathbb{Z}^6 | abc = n, 2 \nmid b, c, 2 \nmid a, e, 0 \leq d = f < a, 0 \leq e < 2b \} \right|
\]

\[
= \sum_{\substack{a,b,c,d,e,f \in \mathbb{Z}^6 \ni abc = n, 2 \nmid b, c, 2 \nmid a, e, 0 \leq d = f < a, 0 \leq e < 2b \}} ab = 2\chi(\frac{n}{2}) - 6\chi(\frac{n}{4}) + 4\chi(\frac{n}{8}),
\]

in a similar way

\[
c_{B_4,B_3}(n) = \sum_{\substack{a,b,c,d,e,f \in \mathbb{Z}^6 \ni abc = n, 2 \nmid b, c, 2 \nmid a, e, 0 \leq d = f < a, 0 \leq e < 2b \}} 2 = 2d_3(\frac{n}{2}) - 4d_3(\frac{n}{4}) + 2d_3(\frac{n}{8}).
\]

5 The proof of Theorem 3 and Theorem 4

Most sections of this chapter follow the same logic as their counterparts in Section 4.

5.1 Case \( \Delta \cong \pi_1(G_1) \cong \mathbb{Z}^3 \)

As in 4.4 we have

\[
s_{G_1,B_4}(n) = \omega(\frac{n}{4}), \quad \text{and} \quad c_{G_1,B_4}(n) = \frac{1}{4}\left(\omega(\frac{n}{4}) + 3\sigma_2(\frac{n}{4}) + 9\sigma_2(\frac{n}{8})\right).
\]

5.2 Case \( \Delta \cong \pi_1(G_2) \)

The proof in this section follows exactly Section 4.2 until we explore the condition \( h' = h \), which is equivalent to \([z, y] \in H \) (for the definition of \( H \) and \( J_H \) see Notation in Section 4.2). \( N_y(n) \) is defined in Section 4. Note that in the group \( \pi_1(B_4) \) the identity \([z, y] = xy^2 \) holds (compare with \([z, y] = y^2 \) in case of \( \pi_1(B_3) \)). Thus \( N_y(n) \) appears to be the sum of \( 2G_{1,1}(k) + G_{\text{total}}(k) \) taken over all possible indexes \( k = |\langle x, y^2 \rangle : H | \) (compare to \( 2G_{0,1}(k) + G_{\text{total}}(k) \) in case of \( \pi_1(B_3) \)). Direct calculations leads to the answer

\[
s_{G_2,B_4}(n) = \omega(\frac{n}{2}) - \omega(\frac{n}{4}),
\]

\[
c_{G_2,B_4}(n) = \frac{1}{2}\left(\sigma_2(\frac{n}{2}) + 2\sigma_2(\frac{n}{4}) - 3\sigma_2(\frac{n}{8}) + d(\frac{n}{2}) - d(\frac{n}{4}) - 3d(\frac{n}{8}) + 5d(\frac{n}{16}) - 2d(\frac{n}{32})\right).
\]
5.3 Case $\triangle \cong \pi_1(B_1)$

5.3.1 Case 3 of Proposition [9]

Discrepancy with Section 4.3.1 is in the action of conjugation. Namely

$$Ad_x : (e, f) \mapsto (e, f), \quad Ad_y : (e, f) \mapsto (e + 2, -f + 1), \quad Ad_z : (e, f) \mapsto (-e, -f + 1).$$

in case $c$ is odd;

$$Ad_x : (e, f) \mapsto (e, f + 2), \quad Ad_y : (e, f) \mapsto (e, -f), \quad Ad_z : (e, f) \mapsto (-e, -f + 1).$$

in case $c$ is even.

Enumerating the orbits of $(e, f)$ for a fixed value of $c$ we get

$$\sum_{a, b = \frac{n}{2c} \text{ if } 2 \nmid a, 4 \nmid b} \frac{a + 1}{2} \left( \sum_{a, b = \frac{n}{2c} \text{ if } 2 \mid a, 4 \mid b} \frac{a}{2} \right) = \frac{1}{2} \left( \sum_{a, b = \frac{n}{2c} \text{ if } 4 \mid a} \frac{b}{2} \right) + \frac{1}{2} \left( \sum_{a, b = \frac{n}{2c} \text{ if } 2 \mid a} \frac{a}{2} \right)$$

$$= \frac{1}{2} \left( \sigma_1 \left( \frac{n}{2c} \right) + \sigma_1 \left( \frac{n}{4c} \right) + \sigma_0 \left( \frac{n}{2c} \right) - 2\sigma_0 \left( \frac{n}{4c} \right) + \sigma_0 \left( \frac{n}{8c} \right) \right)$$

in case of an odd $c$, and

$$\sum_{a, b = \frac{n}{2c} \text{ if } 2 \nmid a, 4 \mid b} \frac{b + 2}{4} \left( \sum_{a, b = \frac{n}{2c} \text{ if } 2 \mid a} \frac{b}{2} \right) = \frac{1}{2} \left( \sum_{a, b = \frac{n}{2c} \text{ if } 4 \mid b} \frac{b}{2} \right) + \frac{1}{2} \left( \sum_{a, b = \frac{n}{2c} \text{ if } 2 \mid b} \frac{b}{2} \right)$$

$$= \frac{1}{2} \left( \sigma_1 \left( \frac{n}{2c} \right) + \sigma_1 \left( \frac{n}{4c} \right) + \sigma_0 \left( \frac{n}{2c} \right) - 2\sigma_0 \left( \frac{n}{4c} \right) + \sigma_0 \left( \frac{n}{8c} \right) \right).$$

As before we see a coincidence of the functions, derived from different combinatorics.

Summing this over all values of $c$ we get

$$s_{B_3B_4}^\text{case 3}(n) = \sum_{a, b = \frac{n}{2c}, 2 \mid b} \chi(\frac{n}{2}), \quad \text{and} \quad c_{B_3B_4}^\text{case 3}(n) = \frac{1}{2} \left( \sigma_2 \left( \frac{n}{2} \right) + \sigma_2 \left( \frac{n}{4} \right) + d_3 \left( \frac{n}{2} \right) - 2d_3 \left( \frac{n}{4} \right) + d_3 \left( \frac{n}{8} \right) \right).$$

5.3.2 Case 5 of Proposition [9]

The sole difference from Section 4.3.2 is in the action of the conjugation

$$Ad_x : (d, e) \mapsto (d + 2, e), \quad Ad_y : (d, e) \mapsto (-d, e), \quad Ad_z : (d, e) \mapsto (-d + 1, -e).$$
This leads to the number of conjugacy classes for a fixed $c$

$$\sum_{\substack{ab=n, \\ 2 \nmid a}} 2^{b+1} = 2 \bigg\{ \frac{n}{2} \bigg( \sigma_1\left( \frac{n}{c} \right) - \sigma_1\left( \frac{n}{2c} \right) - 2\sigma_1\left( \frac{n}{4c} \right) + \sigma_0\left( \frac{n}{c} \right) - 2\sigma_0\left( \frac{n}{2c} \right) + \sigma_0\left( \frac{n}{4c} \right) \bigg) \bigg\}. $$

Summing over all even values of $c$ finishes the job. So

$$s_{\text{case 5}}(n) = \sum_{\substack{abc=n, \\ 2 \nmid b}} \frac{ab}{2} = \chi\left( \frac{n}{2} \right) - 2\chi\left( \frac{n}{4} \right),$$

$$c_{\text{case 5}}(n) = \frac{1}{2} \left( \sigma_2\left( \frac{n}{2} \right) - \sigma_2\left( \frac{n}{4} \right) - 2\sigma_2\left( \frac{n}{8} \right) + d_3\left( \frac{n}{2} \right) - 2d_3\left( \frac{n}{4} \right) + d_3\left( \frac{n}{8} \right) \right).$$

Finally

$$s_{B_3, B_4}(n) = 2\chi\left( \frac{n}{2} \right) - 2\chi\left( \frac{n}{4} \right), \quad c_{B_3, B_4}(n) = \sigma_2\left( \frac{n}{2} \right) - \sigma_2\left( \frac{n}{4} \right) + d_3\left( \frac{n}{2} \right) - 2d_3\left( \frac{n}{4} \right) + d_3\left( \frac{n}{8} \right).$$

### 5.4 Case $\Delta \cong \pi_1(B_2)$

#### 5.4.1 Case 4 of Proposition 9

This section follows Section 4.4.1 exactly up to the explicit formulas for the action of conjugation which in the present case is the following.

First assume that $c$ is odd. Then

$$\text{Ad}_x : (e, f) \mapsto (e, f), \quad \text{Ad}_y : (e, f) \mapsto (e + 2, -f + 1), \quad \text{Ad}_z : (e, f) \mapsto (-e, -f + 1).$$

So we have to calculate the number of orbits of pairs $\{(2u + 1, v) | u, v \in \mathbb{Z}\}$ under the action of the group generated by the mappings

$$(2u + 1, v) \mapsto (2u + 3, -v + 1), \quad (2u + 1, v) \mapsto (-2u - 1, -v + 1),$$

$$(2u + 1, v) \mapsto (2u + 1 + b, v + \frac{a}{2}), \quad (2u + 1, v) \mapsto (2u + 1, v + a).$$

Then the number of orbits is $\frac{a}{2}$ unless simultaneously $a, b \equiv 2 \mod 4$; in the latter case the number of orbits is $\frac{a+2}{2}$. Compare with the case of the group $\pi_1(B_3)$, where the number of orbits is $\frac{a}{2}$ unless simultaneously $b \equiv 2 \mod 4$ and $4 \nmid a$; in the latter case the number of orbits is $\frac{a+2}{2}$.

So, the number of conjugacy classes for a fixed odd $c$ is

$$\sum_{\substack{a=\frac{4}{2}, \\ 2 \nmid a, b}} \frac{a}{2} + \sum_{\substack{a=\frac{4}{2}, \\ a, b \equiv 2 \mod 4}} 1 = \sigma_1\left( \frac{n}{4c} \right) + \sigma_0\left( \frac{n}{4c} \right) - 2\sigma_0\left( \frac{n}{8c} \right) + \sigma_0\left( \frac{n}{16c} \right).$$

Now assume that $c$ is even. Then

$$\text{Ad}_x : (e, f) \mapsto (e, f + 2), \quad \text{Ad}_y : (e, f) \mapsto (e, -f), \quad \text{Ad}_z : (e, f) \mapsto (-e, -f + 1).$$
As in Section 4.4.1 we have to calculate the number of orbits of pairs \{(2u + 1, v) | u, v \in \mathbb{Z}\} under the action of the group generated by the mappings

\[(2u + 1, v) \mapsto (2u + 1, v + 2), \quad (2u + 1, v) \mapsto (2u + 1, -v), \quad (2u + 1, v) \mapsto (-2u - 1, -v + 1), \quad (2u + 1, v) \mapsto (2u + 1 + b, v + \frac{a}{2}), \quad (2u + 1, v) \mapsto (2u + 1, v + a).\]

The number of orbits is \(\frac{b}{2}\) unless \(a, b \equiv 2 \mod 4\), in the latter case the number is \(\frac{b}{2} + 1\). So, the number of conjugacy classes for a fixed even \(c\) is

\[
\sum_{\substack{ab=\frac{b}{2} \pm 1 \mod 2 \mid a,b}} 1 = \sigma_1\left(\frac{n}{4c}\right) + \sigma_0\left(\frac{n}{4c}\right) - 2\sigma_0\left(\frac{n}{8c}\right) + \sigma_0\left(\frac{n}{16c}\right).
\]

Finalizing our calculations we get

\[
s^4_{B_2, B_4}(n) = 2\chi\left(\frac{n}{4}\right),
\]

\[
c^4_{B_2, B_4}(n) = \sum_{c | n} \sigma_1\left(\frac{n}{4c}\right) + \sigma_0\left(\frac{n}{4c}\right) - 2\sigma_0\left(\frac{n}{8c}\right) + \sigma_0\left(\frac{n}{16c}\right) - \sigma_2\left(\frac{n}{4}\right) + d_3\left(\frac{n}{4}\right) - 2d_3\left(\frac{n}{8}\right) + d_3\left(\frac{n}{16}\right).
\]

5.4.2 Case 6 of Proposition 9

The distinction with Section 4.4.2 is in the action of conjugation

\[Ad_x : (d,e) \mapsto (d+2,e), \quad Ad_y : (d,e) \mapsto (-d,e), \quad Ad_z : (d,e) \mapsto (-d,-e).\]

This leads to \(b\) orbits (compare with \(b + 1\) in Section 4.4.2). Thus

\[
s^6_{B_2, B_4}(n) = 2\chi\left(\frac{n}{4}\right) - 4\chi\left(\frac{n}{8}\right), \quad c^6_{B_2, B_4}(n) = \sum_{\substack{abc=n, \ 2 | a,c, 2 | b}} b = \sigma_2\left(\frac{n}{4}\right) - 2\sigma_2\left(\frac{n}{8}\right).
\]

Finally we have

\[
s_{B_2, B_4}(n) = s^4_{B_2, B_4}(n) + s^6_{B_2, B_4}(n) = 4\chi\left(\frac{n}{4}\right) - 4\chi\left(\frac{n}{8}\right),
\]

\[
c_{B_2, B_4}(n) = c^4_{B_2, B_4}(n) + c^6_{B_2, B_4}(n) = 2\sigma_2\left(\frac{n}{4}\right) - 2\sigma_2\left(\frac{n}{8}\right) + d_3\left(\frac{n}{4}\right) - 2d_3\left(\frac{n}{8}\right) + d_3\left(\frac{n}{16}\right).
\]

5.5 Case \(\Delta \cong \pi_1(B_4)\)

This section is similar to Section 4.5 with two differences. First, the summation is taken over all triples \((a, b, c) : abc = n\), where \(a, b, c\) are odd. In particular, all sums vanishes if \(n\) is even.

Second, the conjugation acts as follows:

\[Ad_x : (d,e) \mapsto (d + 2, e), \quad Ad_y : (d,e) \mapsto (-d,e + 2), \quad Ad_z : (d,e) \mapsto (-d + 1, -e).\]
But this leads to the same result: for a fixed triple of off positives \((a, b, c)\) there is only one orbit. So we have

\[
s_{B_4,B_4}(n) = \begin{cases} 
\chi(n) & \text{if } 2 \nmid n \\
0 & \text{if } 2 \mid n \end{cases} = \chi(n) - 5\chi\left(\frac{n}{2}\right) + 8\chi\left(\frac{n}{4}\right) - 4\chi\left(\frac{n}{8}\right),
\]

\[
c_{B_4,B_4}(n) = \begin{cases} 
d_3(n) & \text{if } 2 \nmid n \\
0 & \text{if } 2 \mid n \end{cases} = d_3(n) - 3d_3\left(\frac{n}{2}\right) + 3d_3\left(\frac{n}{4}\right) - d_3\left(\frac{n}{8}\right).
\]

### Appendix

Given a sequence \(\{f(n)\}_{n=1}^{\infty}\), the formal power series

\[
\hat{f}(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}
\]

is called a Dirichlet generating function for \(\{f(n)\}_{n=1}^{\infty}\). To reconstruct the sequence \(f(n)\) from \(\hat{f}(s)\) one can use Perron’s formula ([1], Th. 11.17). Given sequences \(f(n)\) and \(g(n)\) we call their convolution \((f * g)(n) = \sum_{k|n} f(k)g\left(\frac{n}{k}\right)\). In terms of Dirichlet generating series the convolution of sequences corresponds to the multiplication of generating series \(\hat{f} \cdot \hat{g}(s) = \hat{f}(s)\hat{g}(s)\). For the above facts see, for example, ([1], Ch. 11–12).

Here we present the Dirichlet generating functions for the sequences \(s_{H,G}(n)\) and \(c_{H,G}(n)\). Since theorems 1–4 provide the explicit formulas, the remainder is done by direct calculations.

Consider the Riemann zeta function \(\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\). Following [1] note that

\[
\begin{align*}
\tilde{\sigma}_0(s) &= \zeta^2(s), \\
\tilde{\sigma}_1(s) &= \zeta(s)\zeta(s-1), \\
\tilde{\sigma}_2(s) &= \zeta^2(s)\zeta(s-1), \\
\tilde{d}_3(s) &= \zeta^3(s), \\
\tilde{\chi}(s) &= \zeta(s)\zeta(s-1)^2, \\
\tilde{\omega}(s) &= \zeta(s)\zeta(s-1)^2\zeta(s-2).
\end{align*}
\]

Dirichlet generating functions for the sequences provided by Theorem 5 are

| Table 1. Dirichlet generating functions for the sequences, related to Klein bottle. |
|-----------------|-----------------|
| \(\hat{s}_{\mathbb{Z}_2,\pi_1(k)}\) | \(\hat{c}_{\mathbb{Z}_2,\pi_1(k)}\) |
| \(s_{\pi_1(k),\pi_1(k)}\) | \(c_{\pi_1(k),\pi_1(k)}\) |
| \(2^{-s}\zeta(s)\zeta(s-1)\) | \(2^{-s-1}\zeta(s)\left(\zeta(s-1) + (1 + 2^{-s})\zeta(s)\right)\) |
| \((1 - 2^{-s})\zeta(s)(s - 1)\) | \((1 - 2^{-s})(1 + 2^{-s})\zeta^2(s)\) |

In Tables 2 and 3 we provide the Dirichlet generating functions for the sequences given by Theorems 1–4.

Table 2. Dirichlet generating functions for the sequences \(s_{H,G}(n)\).
Table 3. Dirichlet generating functions for the sequences $c_{H,G}(n)$.

| $H$ | $G$ | $\pi_1(B_3)$ | $\pi_1(B_4)$ |
|-----|-----|--------------|--------------|
| $\pi_1(G_1)$ | $4^{-s}\zeta(s)\zeta(s-1)\zeta(s-2)$ | $4^{-s}\zeta(s)\zeta(s-1)\zeta(s-2)$ |
| $\pi_1(G_2)$ | $2^{-s}(1-2^{-s})\zeta(s)\zeta(s-1)\zeta(s-2)$ | $2^{-s}(1-2^{-s})\zeta(s)\zeta(s-1)\zeta(s-2)$ |
| $\pi_1(B_1)$ | $2^{-s+1}(1-2^{-s})\zeta(s)\zeta(s-1)\zeta(s-2)$ | $2^{-s+1}(1-2^{-s})\zeta(s)\zeta(s-1)^2$ |
| $\pi_1(B_2)$ | $4^{-s+1}(1-2^{-s})\zeta(s)\zeta(s-1)^2$ | $4^{-s+1}(1-2^{-s})\zeta(s)\zeta(s-1)^2$ |
| $\pi_1(B_3)$ | $(1-2^{-s})(1-2^{-s+1})\zeta(s)\zeta(s-1)^2$ | does not exist |
| $\pi_1(B_4)$ | $2^{-s+1}(1-2^{-s})(1-2^{-s+1})\zeta(s)\zeta(s-1)^2$ | $(1-2^{-s})(1-2^{-s+1})^2\zeta(s)\zeta(s-1)^2$ |

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