Two-dimensional gauge theories of the symmetric group $S_n$ and branched n-coverings of Riemann surfaces in the large-n limit.

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Branched $n$-coverings of Riemann surfaces are described by a 2d lattice gauge theory of the symmetric group $S_n$ defined on a cell discretization of the surface. We study the theory in the large-$n$ limit, and we find a rich phase diagram with first and second order transition lines. The various phases are characterized by different connectivity properties of the covering surface. We point out some interesting connections with the theory of random walks on group manifolds and with random graph theory.

1. Introduction

The problem of counting in how many ways a Riemann surface can be covered $n$-times without allowing folds but allowing branch points is equivalent to computing the partition function of a lattice gauge theory defined on a discretization of the surface, with the symmetric group $S_n$ as the gauge group. The choice of the action for the gauge theory determines the branch-point structure of the coverings one wants to count[1–3].

Therefore 2d $S_n$ gauge theory is of interest not only because it is a simple but non-trivial non-abelian gauge theory, but also because it has a natural interpretation in terms of a 2d string theory, that is a theory of maps from a 2d world sheet to a 2d target space. Moreover the strings described by $S_n$ gauge theory, that can have branch points but not folds, are precisely the ones that enter the string-theoretic formulation of $SU(N)$ 2d gauge theory, discovered in [4,5]. Finally, branched coverings of Riemann surfaces appear naturally when one quantizes 2d Yang Mills theory in the unitary gauge. This gauge choice was shown in Refs.[6,7] to actually define a different theory, with a natural interpretation in terms of 2d strings.

In this work we will consider 2d $S_n$ gauge theory on a genus-zero Riemann surface, and in particular the theory that corresponds to coverings with quadratic branch points only, in the large-$n$ limit, and study its phase diagram. It turns out that one can obtain a non-trivial phase diagrams by letting the number of plaquettes $p$ of the discretized target Riemann surface scale with the order $n$ of the coverings as

$$p = An \log n$$

For example if one considers the theory in which exactly one quadratic branch point is placed on each plaquette, one finds that at $A = 1/2$ there is a phase transition separating a phase in which the covering has more than one connected component from one in which the covering surface is connected. By allowing each plaquette to have either no branch point or one quadratic branch point with a given probability, the phase diagram becomes more complex, with lines of first and second order phase transitions.

These results can be obtained by writing the partition function of the theory, as it is customary in 2d lattice gauge theories, as a sum over irreducible representations of $S_n$, and then looking for the representations that give the highest contribution in the large-$n$ limit. This procedure is outlined in Sec. 2.
Some insight into the nature of the various phase transitions can be gained from an exact correspondence, that we describe in Sec. 3, between 2d gauge theories on genus-zero Riemann surfaces and random walks on the gauge group. The correspondence is completely general with respect to the choice of the gauge group and the action. If one applies it to our gauge theory, one can map it into a particular random walk on $S_n$ that has been studied in the mathematical literature as an example of cutoff phenomenon. In Sec. 4 we will show that one of the phase transition lines we find corresponds to the cutoff phenomenon in this random walk.

Finally, another precise connection can be made with the theory of random graphs, and precisely with what are called phase transitions in random graphs. In Sec. 5 we will describe this connection, and use it to argue that the phases of our model are distinguished by the connectivity properties of the world-sheet.

Sec. 6 will be devoted to some final remarks. In this contribution we just present our results: for more details and derivations, we refer the reader to Ref. [9].

2. The model and the phase diagram on the sphere

The equivalence between 2d $S_n$ gauge theories on Riemann surfaces and the problem of counting branched coverings of such surfaces is described in detail in Ref. [3]. Such equivalence can be established for all possible branch point structure of the coverings. Here we will consider quadratic branch points only: On each plaquette of the target Riemann surface there is, with probability $x$, a quadratic branch point, or, with probability $1 - x$, no branch point.

The partition function on a disk depends on the branch point type on the boundary, that is, in the language of the gauge theory, on the holonomy $Q \in S_n$ at the boundary of the disk, and can be written as

$$Z_{n,\text{disk},p}(Q) = \frac{1}{n!} \sum_r d_r \text{ch}_r(Q) \left[ (1 - x) + x \frac{\text{ch}_r(2)}{d_r} \right]^p$$  (2)

where $p$ is the number of plaquettes, the sum runs over all the irreducible representations $r$ of $S_n$, $d_r$ is the dimension of the representation $r$ and $\text{ch}_r(2)$ is the character of a transposition in $r$. The problem of computing the partition function Eq.(2) in the large-$n$ limit consists in finding which representations give the largest contribution to the sum when $n \to \infty$.

The irreducible representations of $S_n$ are in one-to-one correspondence with the Young tableaux with $n$ boxes. The results of the analysis can be expressed in terms of the shape of the Young tableaux that dominate the sum in the large-$n$ limit. We will restrict ourselves to the case of the sphere, where $Q$ is the identical permutation.

If one lets $p$ scale with $n$ as in Eq.(1) one finds three phases in the $(x, A)$ plane, shown in Fig. 1:

- In phase I, the Young tableaux that dominate the sum have both rows and columns scaling as $n^{1/2}$.
- In phase II the sum is dominated by Young tableaux made of a single row of length $2Ax$ attached to a part of area $(1 - 2Ax)n$ whose rows and columns scale like $n^{1/2}$.
- In phase III the sum is dominated by a Young tableau made of a single row (or column) of length $n$.

The free energy in the large-$n$ limit can be written exactly, so that one can determine the order of the various transitions. It turns out that the (I,II) and (I,III) transitions are first order, while (II,III) is second order, characterized by a discontinuity of the second derivative of the free energy with respect to $A$.

What this type of analysis does not easily give is a characterization of the three phases in terms of some order parameter of clear physical meaning. This problem will be solved in part by mapping the theory first into a random walk on $S_n$, then into a classical problem of random graph theory. This will be done in the next two sections.
3. 2d gauge theories are random walks on the gauge group

A two-dimensional gauge theory on a disk is equivalent to a random walk on the gauge group manifold, the area of the disk being identified with the number of steps and the gauge theory action with the transition probability at each step.

This result is completely general with respect to the choice of the gauge group and the action, as is proven in Ref.[9]. Here we show how this result emerges in the $S_n$ gauge theory defined by Eq.(2) for $x = 1$ (the extension to general $x$ is trivial), that is where all plaquette variables are forced to be equal to a transposition.

As shown by Eq.(3) the partition function depends only on the holonomy $Q$ and the area of the disk. It follows that we can freely choose any cell decomposition of the disk made of $p$ plaquettes, e.g. the one shown in Fig. 2. To compute the partition function is to count the ways in which we can place permutations $P$ on all the links in such a way that

- the ordered product of the links around each plaquette is a transposition
- the ordered product of the links around the boundary of the disk is a permutation in the same conjugacy class as $Q$

Now we can use the gauge invariance of the theory to fix all the radial links to contain the identical permutation. At this point the links on the boundary are forced to contain transpositions: therefore the partition function with holonomy $Q$ is the number of ways in which one can write the permutation $Q$ as an ordered product of $p$ transpositions. This in turn can be seen as a random walk on $S_n$, in which, at each step, the permutation is multiplied by a transposition chosen at random: the gauge theory partition function for area $p$ and holonomy $Q$ is the probability that after $p$ steps the walker is in $Q$.

The cutoff phenomenon in random walks was discovered in Ref.[8] by studying this random walk on $S_n$ (more precisely the one corresponding to our model with $x = 1/n$). It was shown in Ref.[8] is that if the number of steps scales as $An \log n$, then in the large-$n$ limit for $A > 1/2$ the probability of finding the walker in any given element $Q \in S_n$ is just $1/n!$ for all $Q$: complete randomization has been achieved and all memory of the initial position of the walker has been erased.

In terms of the corresponding gauge theory, this statement translates into the following: for
A > 1/2, the partition function does not depend on the holonomy \( Q \), and at fixed holonomy \( Q \), does not depend on \( A \). This is true in particular for \( Q = 1 \), corresponding to the partition function on the sphere. Therefore the transition between phases II and III, found in Sec. 2, has a natural interpretation as a cutoff phenomenon in the corresponding random walk (see Ref.[9] for a discussion about generalizing the result of Ref.[8] to arbitrary values of \( x \)).

4. A connection with the theory of random graphs

The connection with the theory of random graphs is a consequence of the possibility, established in the previous section, of mapping one of the phase transitions of our theory of coverings into a cutoff phenomenon of a random walk. In fact it was shown in Ref. [10] that some phase transitions in random graphs can be interpreted as cutoff phenomena in certain random walks that naturally generate ensembles of random graphs.

Consider the random walk corresponding to our model, taken now with free boundary conditions, and take for simplicity \( x = 1 \): at each step the permutation is multiplied by a random transposition. From the point of view of coverings, a step in which the transposition \((ij)\) is used corresponds to adding a simple branch point that connects the two sheets \( i \) and \( j \) of the covering surface. One can think of the process as the construction of a graph of \( n \) sites, in which at each step one link, chosen at random, is added. After \( p = An \log n \) steps the expected number of links is equal to the number of steps (since the number of available links is \( O(n^2) \) the fact that the same link can be added more than once can be neglected in the large \( n \) limit; see Ref. [11]).

It is a classic result in the theory of random graphs [11,12] that if the number of links \( p \) is smaller than \( 1/2 \) \( n \log n \) then the graph is almost certainly disconnected while for \( p > 1/2 \) \( n \log n \) the graph is almost certainly connected, where “almost certainly” means that the probability is one in the limit \( n \to \infty \). Therefore we conclude that the transition at \( A = 1/2 \), for the model with free boundary conditions, is the point where the covering surface becomes connected.

Notice that the free boundary conditions are crucial for this argument to work: in the case, say, of the sphere, the corresponding random walk is forced to go back to the initial position in \( p \) steps, so that links in the graph are not added independently and the result of Ref.[11,12] do not apply. However there are strong arguments (see Ref.[9]) suggesting that connectedness of the covering surface can be used to distinguish the phases in the case of the sphere as well: phase III should correspond to a connected covering surface while phases I and II should not.

5. Conclusions

We have shown that the two dimensional gauge theory of the symmetric group, that describes the statistics of branched coverings of a Riemann surface, has a remarkably rich phase structure in the large-\( n \) limit.

The theory on the sphere can be studied by a variational approach that identifies which representations give the largest contribution in the large-\( n \) limit. If the number of branch points is taken to scale as \( n \log n \), three phases can be found, corresponding to different shapes of the dominant Young tableaux. The transitions between these lines can be located analytically and their order determined.

All two-dimensional gauge theories on a genus-0 surface can be mapped into random walks in the corresponding group manifold. In our case, this allows us to interpret one of the transition lines as a cutoff phenomenon in the corresponding random walk.

The theory on a disk, with free boundary conditions, can be studied with methods of the theory of random graphs: this allows one to show that there is a phase transition on a disk from a disconnected to a connected covering surface. From this one can argue, and with some limitations prove, that the connectness of the covering is what characterizes the different phases also on the sphere.

This last result is particularly interesting since it provides a description of the various phases in terms of geometric properties of the covering
surfaces, that is in terms of quantities that are natural when one considers the model as a two-dimensional string theory rather than as a Yang-Mills theory. Along this line, it was recently proposed in [13] that the phase transition we find is a close analogue of the Hagedorn phase transition in Matrix String Theory described in Refs. [14–18].

REFERENCES

1. I. K. Kostov and M. Staudacher, Phys. Lett. B394 (1997) 75 [hep-th/9611011].
2. I. K. Kostov, M. Staudacher and T. Wynter, Commun. Math. Phys. 191, 283 (1998) [hep-th/9703189].
3. M. Billo, A. D’Adda and P. Provero, arXiv:hep-th/0103242.
4. D. J. Gross and W. I. Taylor, Nucl. Phys. B400 (1993) 181 [hep-th/9301068].
5. D. J. Gross and W. I. Taylor, Nucl. Phys. B403 (1993) 395 [hep-th/9303046].
6. M. Billo, M. Caselle, A. D’Adda and P. Provero, Nucl. Phys. B 543 (1999) 141 [arXiv:hep-th/9809095].
7. M. Billo, A. D’Adda and P. Provero, Nucl. Phys. B 576 (2000) 241 [arXiv:hep-th/9911249].
8. P. Diaconis and M. Shahshahani, Z. Wahrsch. Verw. Gebiete 57 (1981) 159.
9. A. D’Adda and P. Provero, arXiv:hep-th/0110243.
10. T. Pak and V. H. Vu, Discrete Applied Math., 110 (2001) 251.
11. P. Erdős and A. Rényi, Publ. Math. Debrecen 6 (1959) 290.
12. P. Erdős and A. Rényi, “The Art of Counting” (Cambridge:MIT 1973).
13. G. W. Semenoff, arXiv:hep-th/0112043.
14. G. Grignani and G. W. Semenoff, Nucl. Phys. B 561, 243 (1999) [arXiv:hep-th/9903246].
15. G. Grignani, P. Orland, L. D. Paniak and G. W. Semenoff, Phys. Rev. Lett. 85, 3343 (2000) [arXiv:hep-th/0004194].
16. G. W. Semenoff, arXiv:hep-th/0009011.
17. G. Grignani, M. Orselli and G. W. Semenoff, JHEP 0107, 004 (2001) [arXiv:hep-th/0104112].
18. G. Grignani, M. Orselli and G. W. Semenoff, arXiv:hep-th/0110152.