Critical Localization and Strange Nonchaotic Dynamics: The Fibonacci Chain

Surendra Singh Negi and Ramakrishna Ramaswamy

School of Physical Sciences
Jawaharlal Nehru University, New Delhi 110 067, INDIA

(March 30, 2022)

Abstract

The discrete Schrödinger equation with a quasiperiodic dichotomous potential specified by the Fibonacci sequence is known to have a singular continuous eigenvalue spectrum with all states being critically localized. This equation can be transformed into a quasiperiodic skew product dynamical system. In this iterative mapping which is entirely equivalent to the Schrödinger problem, critically localized states correspond to fractal attractors which have all Lyapunov exponents equal to zero. This provides an alternate means of studying the spectrum, as has been done earlier for the Harper equation. We study the spectrum of the Fibonacci system and describe the scaling of gap widths with potential strength.

I. INTRODUCTION

The phenomenon of localization, first discussed in the context of disordered systems by Anderson [1], has been of great interest in condensed matter physics for several decades. The Anderson model [1] is the tight–binding Hamiltonian
\[ H = \sum_i W_i c_i^\dagger c_i + \sum_{i,j} I_{ij} c_i^\dagger c_j, \quad (1) \]

where \( c_i^\dagger \) is the creation operator at the lattice site indexed by \( i \), and \( W_i \) and \( I_{ij} \) are on-site and hopping matrix elements. In the original model, \( I_{ij} = I \) for \( i, j \) nearest neighbours and zero otherwise, and \( W_i \) is chosen randomly in the interval \([-W/2, W/2]\). \( I \) and \( W \) can be combined into a dimensionless ratio which is a measure of the disorder. For \( \frac{W}{T} > \left[ \frac{W}{T} \right]_c \), a critical value, all the states are exponentially localized. Below the critical value, there is a mobility edge and states can be extended. For ordered systems, such as the analogous system with a periodic potential, states are extended and the energy spectrum is a band \[3\].

There are systems that lie between these extremes, for instance crystals with additional periodic modulation which is incommensurate with the period of the underlying lattice \[6,8\]. The lack of strict translational order has an effect similar to disorder, and thus it is of interest to know the nature of the states in such systems. In this paper we study the spectrum of a 1–d chain, the potential at each site being determined by a quasiperiodic sequence deriving from the Fibonacci numbers \[9–12\].

The paradigm for studies of quasiperiodic Schrödinger problems has been the Harper equation \[13\]

\[ \psi_{n+1} + \psi_{n-1} + V_n \psi_n = E \psi_n, \quad (2) \]

which is the discrete Schrödinger equation for a particle in the quasiperiodic potential \( V_n = 2\varepsilon \cos 2\pi (n\omega + \phi_0) \), with \( \omega \) being irrational. This is a tight–binding model describing the motion of an electron in two dimensional lattice in the presence of a magnetic field, but is also known to arise in numerous other contexts. As a result, it has been extensively studied both in the physics \[14–21\] as well as in the mathematics \[22–26\] literature (as the “almost–Mathieu” equation). Since the results that are known for this system have considerable bearing on the present work, we first briefly review the salient features of the Harper system.

Transfer matrices provide a powerful methodology for the calculation of the band spectrum \[27\] in matrix form, the discrete Schrödinger equation, Eq.(2) can be written as
\[
\begin{pmatrix}
\psi_{n+1} \\
\psi_n
\end{pmatrix}
= T_n
\begin{pmatrix}
\psi_n \\
\psi_{n+1}
\end{pmatrix}
\] (3)

where the transfer matrix \( T_n \) is given by
\[
T_n = \begin{pmatrix}
E - V_n & -1 \\
1 & 0
\end{pmatrix}
\] (4)

For 1–d chains, the state vector at the final site is obtained from the starting value by matrix multiplication,
\[
\begin{pmatrix}
\psi_{N+1} \\
\psi_N
\end{pmatrix}
= T
\begin{pmatrix}
\psi_1 \\
\psi_0
\end{pmatrix}
\] (5)

with \( T = T_N \cdot T_{N-1} \cdot T_{N-2} \cdots T_1 \). The eigenvalues of \( T_n \) can be calculated by the Cayley-Hamilton theorem as
\[
\lambda = \frac{E - V_n \pm \sqrt{(E - V_n)^2 - 4}}{2}.
\] (6)

From this equation it is evident that \( \lambda \) can be complex if \( |E - V_n| < 2 \): this corresponds to extended states. If \( |E - V_n| > 2 \), then \( \lambda \) is real, corresponding to localized states. There is a metal-insulator transition at the so-called mobility edge \( |E - V_n| = 2 \).

In order to determine the actual nature of the spectrum for irrational \( \omega \), a barrage of mathematical techniques have been applied to the study of the Harper equation \([13,28,32]\). One of the main results has been the discovery of duality by André and Aubry \([8]\). Since the Harper potential contains no harmonics, there can be a solution to Eq. (2) such that
\[
\psi_n = \exp\{ikn\} \sum_{m=-\infty}^{\infty} f_m \exp\{im(2\pi \omega n + \phi_0)\}
\] (7)

where the Fourier coefficients \( f_m \) are given by
\[
f_m = \exp\{-ikm\} \sum_{n=-\infty}^{\infty} \psi_n \exp\{-in(2\pi \omega m + \phi_0)\}.
\] (8)

Substituting the value of \( \psi_n \) in the above equation yields
\[ f_{m+1} + f_{m-1} + \frac{2}{\epsilon} \cos(2\pi \omega m + k) f_m = \frac{1}{\epsilon} E f_m, \quad (9) \]

which, written as

\[ f_{m+1} + f_{m-1} + 2\bar{\epsilon} \cos(2\pi \omega m + k) f_m = \bar{E} f_m \quad (10) \]

is similar in form to Eq. (2), with \( \bar{\epsilon} \equiv 1/\epsilon \) and \( \bar{E} \equiv E/\epsilon \). The equations are identical when

\[ \epsilon = 1, \psi_0 = k \]

with \( \psi \equiv f \). Now if \( f_m \) is localized then the sum \( \sum_{m=-\infty}^{\infty} |f_m|^2 \) is finite, and therefore the function

\[ f(x) = \exp\{ikx\} \sum_{m=-\infty}^{\infty} f_m \exp\{im2\pi \omega x\} \quad (11) \]

converges. Thus \( f(x) \) behaves like a Bloch function and represents an extended state. For \( x = n + \frac{\phi_0}{2\pi \omega}, \) \( f(x) = \exp\{ik \frac{\phi_0}{2\pi \omega}\} \psi_n \). Hence if \( \psi_n \) is extended then \( \sum_n |\psi_n|^2 \) diverges. Therefore, if \( \psi_n \) is extended then \( f_n \) is localized.

The duality of the model is this correspondence which maps the region \( \epsilon > 1 \) to the region \( \epsilon < 1 \), with the point \( \epsilon = 1 \) being “self dual”. Thus states for this value of \( \epsilon \) are neither extended nor localized: they are critical.

This criticality has two manifestations. The spectrum of the Harper equation at \( \epsilon = 1 \) is singular continuous: the eigenvalues form a Cantor set. The wave–functions for these critical states are power–law localized. As a function of \( \omega \), the spectrum of the Harper equation, which was first studied in some detail by Hofstadter [33] has a remarkable shape. For every rational value, it consists of a finite number of bands, while for irrational values of \( \omega \) it consists of an apparently fractal set of points, giving what is termed the Hofstadter butterfly.

A number of other quasiperiodic potentials [16–19] have since been shown to support critical states. One widely studied system is the Kohmoto model, defined at lattice site \( n \) through the rule [16,17]
\[ V_n = \alpha \quad 0 \leq \{n\omega\} \leq \omega \]
\[ = -\alpha \quad \omega < \{n\omega\} \leq 1, \quad (12) \]

(the notation is \(\{y\} \equiv y \mod 1\),) which is quasiperiodic if \(\omega\) is an irrational number. For the case of \(\omega = \gamma = (\sqrt{5} - 1)/2\), the golden mean ratio, one obtains the so-called Fibonacci chain.

This latter potential is the subject of the present paper. Our approach is based on the equivalence between the discrete Schrödinger equation and a derived iterative mapping for the amplitude ratio of the wave function at neighbouring sites. This equivalence was first noted by Bondeson et al. \[34\] who showed that the Schrödinger equation with a quasiperiodic potential could be transformed into a dynamical system with quasiperiodic forcing. Ketoja and Satija \[29\] extended this analysis to the Harper equation, Eq. (2), obtaining the entirely equivalent Harper map,

\[ x_{k+1} = \frac{-1}{x_k - E + 2\epsilon \cos 2\pi \phi_k} \quad (13) \]
\[ \phi_{k+1} = \{\phi_k + \omega\}, \quad (14) \]

using the transformation \(\psi_{i-1}/\psi_i \rightarrow x_i\). This is a skew-product driven iterative mapping of the infinite strip \((\infty, \infty) \otimes [0, 1]\) to itself. If the frequency \(\omega\) is an irrational number, the driving is quasiperiodic in time. For the Kohmoto model, the analogous corresponding map is

\[ x_{k+1} = \frac{-1}{x_k - E + V_k} \quad (15) \]

with \(V_k\) given by Eq. (12). In either case, the quantum problem is meaningful only when \(E\) is an eigenvalue, but in the above map, \(E\) appears only as a parameter. Such driven maps, which were first introduced by Grebogi et al. \[35\], have been studied extensively in the context of strange nonchaotic dynamics \[30\]. In quasiperiodically driven dissipative systems, the dynamics can (for appropriate parameter values) be on fractal sets which concurrently have non-positive Lyapunov exponents: such motion is on strange nonchaotic attractors (SNAs).
As has now been demonstrated in several studies, localized states correspond to SNAs \[29–31\]. The transition from extended to localized states as a function of potential strength can be viewed as a transition to SNAs. Critical states have a special significance: these are SNAs on which all Lyapunov exponents are zero. In this paper we use this fact to study the states of the Fibonacci chain for which it is known explicitly that all states are critical, by determining the eigenvalue spectrum of the system through the condition that the Lyapunov exponent for the specified value of $E$ be zero. This provides a simpler alternative than the transfer matrix techniques, and in addition, gives a characterization of the states of the system which is not accessible to purely quantum mechanical methods \[31\].

In the following section we briefly review the relevant features of SNAs and the methods used to study them. Section III contains the main results of this work for the Fibonacci chain. Section IV contains a discussion of related potentials deriving from abstract aperiodic sequences, followed by a brief summary.

II. STRANGE NONCHAOTIC ATTRACTIONS

Strange nonchaotic attractors are frequently found in nonlinear systems where the forcing is quasiperiodic \[36\]. There are similarities between SNAs and both periodic as well as chaotic attractors. Like the former, they are characterized by zero or nonpositive Lyapunov exponents, and like the latter, they have a fractal structure \[35\]. Owing to fractal structure of SNAs, the dynamics is strictly aperiodic. Numerous examples are known now of systems with SNAs. These include, in addition to discrete quasiperiodically forced maps, a number of continuous dynamical systems such as driven pendulums and oscillators \[36,37\].

In the mapping first discussed by Grebogi, Ott, Pelikan and Yorke \[35\]

\[
x_{i+1} = 2\alpha \cos 2\pi \theta_i \tanh x_i \\
\theta_{i+1} = \{\omega + \theta_i\}.
\]

the reasoning that establishes the existence of strange nonchaotic motion is as follows. For $\omega$
an irrational number, there are no periodic orbits in this system. The mapping $x \to 2\alpha \tanh x$ is 1–1 and contracting, taking the real line into the interval $[-2\alpha, 2\alpha]$. The dynamics in $\theta$ is ergodic in the unit interval and therefore the attractor of the dynamical system must be contained in the strip $[-2\alpha, 2\alpha] \otimes [0,1]$. This region contains an invariant subspace, namely the line $x = 0$. A point $x_n$ with the corresponding $\phi_n = 1/4$ will map to $(x_{n+1} = 0, \theta_{n+1} = \omega + 1/4)$, and hence the subsequent iterates will all remain in this invariant subspace. At the same time, the line $x = 0$ can be made unstable by increasing $\alpha$: the transverse Lyapunov exponent is $\lambda = \ln |\alpha|$. Therefore, for $|\alpha| > 1$ the transverse Lyapunov exponent is positive and the line $x = 0$ is no longer attracting. The total Lyapunov exponent can, however, become negative for sufficiently large $\alpha$ so there is an attractor which lies in the region $x = \pm 2\alpha$, with a dense set of points on the line $x = 0$, as well as points off this line. The above arguments can be put on firm mathematical footing to show that the attractor is both strange and nonchaotic [38] for $\alpha > 1$. For the Harper map, similar arguments [32,39] can be advanced to suggest that for large coupling the dynamics is on SNAs.

In systems such as the quasiperiodically driven circle map [40] or the logistic map [41],

\begin{align}
    x_{n+1} &= \alpha [1 + \epsilon \cos 2\pi \theta_n] x_n (1 - x_n) \\
    \theta_{n+1} &= \{ \theta_n + \omega \},
\end{align}

SNAs are frequently observed in the neighborhood of the transition to chaos [40,41], so it is important to determine the scenarios through which they can be formed, as well as the different methods used for their characterization. To date there are several routes known for the creation of SNAs; these have been reviewed recently [36]. SNAs can be characterized by calculating the Lyapunov exponents, fractal dimensions, as well as correlation functions and related quantities [42].

For map of Harper type, namely of the form Eq. (13-14) or Eq. (15), the nontrivial Lyapunov exponent can be easily computed as

\begin{equation}
    \lambda = \lim_{N \to \infty} \sum_{j=0}^{N-1} \ln x_{i+1}^2
\end{equation}
while the other Lyapunov exponent is trivially zero. Note that since the dynamics is invertible, there can be no chaos in this system.

III. THE FIBONACCI CHAIN

The Fibonacci chain is the simplest example of a quasicrystal in one-dimension, and can be thought of as a two-component lattice. Using the two symbols $a$ and $b$, with initial sequences $S_0 = b, S_1 = a$, the recursive substitution $a \rightarrow ab, b \rightarrow a$ gives

$$S_{l+1} = \{S_l \cdot S_{l-1}\} \quad (21)$$

with $\{\cdot\}$ representing concatenation. It is clear that the sequence $S_k$ has $F_k$ elements, where $F_k$ is a Fibonacci number specified by $F_{k+1} = F_k + F_{k-1}$, $F_0 = 1, F_1 = 1$. Labeling sites of a lattice of length $F_k$ by the element of a sequence $S_k$, and specifying the site potential $V_l = \alpha$ if the symbol is $a$, $V_l = -\alpha$ if the symbol is $b$ gives the Fibonacci chain of length $F_k$. We are interested in the eigenvalue spectrum of this chain in the limit $k \rightarrow \infty$.

The Fibonacci chain is the particular case of $\omega = \gamma \equiv (\sqrt{5} - 1)/2$, in the more general Kohmoto model, Eq. (12). For the Fibonacci chain, and more generally for any irrational $\omega$ in the Kohmoto model, it is known that all states are critical for any $\alpha$ [16,17]. We can therefore use the alternate ‘quantization condition’, $\{E : \lambda(E, \omega, \alpha) = 0\}$ to obtain the eigenvalue spectrum of the Kohmoto model, as shown in Fig. 1 for the value $\alpha = 1$. The Cantor set structure which is also evident in the spectrum for all irrational $\omega$ is more clearly seen in the dependence of $\lambda$ on $E$ at fixed $\omega, \alpha$, as in Fig. 2. The curve $\lambda(E)$ meets the line $\lambda = 0$ on a Cantor set of points, namely the spectrum, and in the spectral gaps, the curve is parabolic in shape.

The gaps of the spectrum can be studied for any irrational $\omega$; we consider here the case of the golden mean since this case has been considered in detail earlier. The behavior of gap widths as a function of $\alpha$ has been of interest, not just in the Kohmoto model, but in the Harper equation [13] and other related models potentials derived from aperiodic sequence
as well \[13,14\].

Note that since the Lyapunov exponent is strictly nonpositive in this system, the dynamics is entirely on attractors (strange as well as non–strange). Following Johnson and Moser \[25\] it is possible to define a winding number as a function of \(E\) as \[13\]

\[\Omega(E) = \lim_{N \to \infty} \frac{1}{2N} W(x_1, \ldots, x_N) \tag{22}\]

where \(W(x_1, \ldots, x_N)\) is an indicator function that counts the number of changes of sign in the orbit \(x_1, \ldots, x_N\). Since the variable \(x_k\) is the ratio of the wavefunction at neighbouring sites, \(\Omega(E)\) essentially counts the number of nodes in the wavefunction per unit length, and is therefore the normalized integrated density of states. The winding number is shown as a function of \(E\) for \(\alpha = 1\) in the Fibonacci chain in Fig. 3; within a spectral gap, the winding number which remains constant, can be expressed as \(p + q\gamma\), with \(p, q\) integers \[26\]. This provides a labeling of the gaps which are indicated for a few of the larger gaps in Fig. 3.

For fixed \(\omega\), the ordering of the individual eigenstates do not change with the potential strength \(\alpha\). However gap widths \(w\) vary with \(\alpha\) in a complicated manner. The width of the largest gap, namely the one marked \(A\) in Fig. 2, scales linearly: \(w_A \sim \alpha\). Those marked \(B\) and \(C\) asymptote to constant width, \(w_B, w_C \sim \alpha^0\), while all other gaps decrease in width, but as power, \(w \sim \alpha^{-\mu}\), the exponent \(\mu\) depending on the particular gap. The dependence of width on \(\alpha\) is shown for gaps \(A - D\) in Fig. 4. In the case of Harper equation it has been shown \[31\] that each gap can be indexed by distinct topological integer index, which determines the scaling of the widths. Scaling results are also available for the Thue–Morse and period–doubling potentials \[19\]. In the present case it has not been possible to deduce any simple number–theoretic dependence of the gap exponents as in the Harper case, though clearly, here too the gaps never close since the spectrum is always singular continuous.

The depths characterising the gaps, namely the minimum value that the LE takes inside a gap also vary with \(\alpha\) in a similar fashion, increasing for \(A\), asymptoting to constant for \(B\) and \(C\), and decreasing as a power for all other gaps.
IV. SUMMARY AND DISCUSSION

The equivalence between critical SNAs and critically localized states provides a new means of determining the spectrum of discrete quasiperiodic systems. We have, in previous studies [31,43], extensively applied this technique to the study of the Harper equation, and here we make application to the case of the Kohmoto model, specifically the case of the Fibonacci chain, where the potential takes values ±α in a quasiperiodic manner as described in Section III.

Similar dichotomous potentials which have been studied in the context of critical localization are the Thue-Morse, period-doubling and Rudin-Shapiro sequences [19]. The methods employed here, namely the derivation of an iterative mapping equivalent to the discrete Schrödinger equation and the identification of critically localized states with critical SNAs can be applied to this entire class of problems.

The present method makes it possible to empirically study the eigenvalue spectrum of such systems in considerable detail. As in related quasiperiodic problems [19,31], the scaling of the gap widths with potential strength α depends on the particular gap being considered. For the Fibonacci chain the scaling is \( w \sim \alpha^\mu \), with the exponent \( \mu = 1 \) for the largest gap, \( \mu = 0 \) for the next two, and \( \mu \leq -1 \) for the remaining gaps. The organization of the gaps, which can be uniquely labeled by two integers, remains incompletely understood, but it is likely that, as in the Harper problem [31], the exponent governing the power–law decay of gap widths will be related to the gap labels. These aspects, and the application of the present technique to more general aperiodic sequences is the subject of ongoing work [46].

ACKNOWLEDGMENT

This work is supported by a grant from the Department of Science and Technology.
REFERENCES

[1] P.W. Anderson, Phys. Rev. 109, 1492 (1958); Rev. Mod. Phys. 50, 11978.

[2] B. L. Altshuler and B. D. Simon, Universalities: From Anderson localization to quantum chaos, Elsevier Science Publisher, B. V. 1994.

[3] M. Hike, J. Phys. A, Math Jr. 30, L367 (1997).

[4] N. F. Mott, Metal-Insulator Transitions, (Taylor and Francis, London 1990)

[5] M. Ya. Azbel, Sov. Phys. JETP, 19, 634 (1964); Phys. Rev. Lett. 43, 1954 (1979).

[6] J. B. Sokoloff, Phys. Rep. 126, 189 (1985).

[7] D. J. Thouless, Phys. Rev. B 28, 4722 (1983).

[8] G. André and S. Aubry, Ann. Isr. Phys. Soc., 3, 133 (1980).

[9] M. Kohmoto and J. Banavar, Phys. Rev. B 34, 563 (1986).

[10] Y. Liu, W. Sitrakool, Phys. Rev. B 43, 1110 (1991).

[11] G. Gumbs and M. K. Ali, Phys. Rev. Lett. 60, 1081 (1998).

[12] M. Kohmoto, B. Sutherland and C. Tang, Phys. Rev. B 35, 1020 (1987).

[13] P. G. Harper, Proc. Phys. Soc. London A 68, 874 (1955).

[14] D. J. Thouless, J. Phys. C. Solid State Physics 5, 77 (1972).

[15] A. Barelli, J. Bellissard, P. Jacquod and D. L. Shepelyansky Phys. Rev. Lett. 23, 4752 (1996).

[16] M. Kohmoto, Phys. Rev. Lett. 51, 1198 (1983).

[17] S. Ostlund, R. Pandit, D. Rand, H. J. Schellnhuber, and E. D. Siggia, Phys. Rev. Lett. 50, 1873 (1983).
[18] M. Kohmoto, L. P. Kadanoff, and C. Tang, Phys. Rev. Lett. 50, 1870 (1983).

[19] J. M. Luck, Phys. Rev. B 39, 5834 (1989).

[20] A. G. Abanov, J. C. Talstra, P. B. Wiegmann, Phys. Rev. Lett. 81, 2112 (1998).

[21] T. Geisel, R. Ketzmerick, G. Petschel, Phys. Rev. Lett. 66, 1651 (1991).

[22] Ya. G. Sinai, Journal of Statistical Physics 46, 861 (1987).

[23] R. del Rio, S. Jitomirskaya, Y. Last, and B. Simon, Phys. Rev. Lett. 75, 117 (1995).

[24] J. Bellisard, R. Lima, and D. Testard, Comm. Math. Phys. 88, 107 (1983).

[25] R. Johnson and J. Moser, Comm. Math. Phys. 84, 403 (1982).

[26] M. Waldschmidt, P. Moussa, J. -M. Luck and C. Itzykson From Number Theory to Physics, Springer-Verlag, 1989.

[27] H.J. Stockmann, Quantum Chaos : An Introduction, (Cambridge University Press, Cambridge, 1999).

[28] J. Ketoja and I. Satija, Phys. Lett. A 194, 64 (1994).

[29] J. Ketoja and I. Satija, Phys. Rev. Lett. 75, 2762 (1995).

[30] A. Prasad, R. Ramaswamy, I. Satija, and N. Shah, Phys. Rev. Lett. 83, 4530 (1999).

[31] S. S. Negi and R. Ramaswamy, Phys. Rev. E, submitted.

[32] J. Ketoja and I. Satija, Physica D 109, 70 (1997). Phys. Rev. Lett. 83, 4530 (1999).

[33] D. R. Hofstadter, Phys. Rev. B 14, 2239 (1976).

[34] A. Bondeson, E. Ott, and T. M. Antonsen, Phys. Rev. Lett. 55, 2103 (1985).

[35] C. Grebogi, E. Ott, S. Pelikan, and J. Yorke, Physica D 13, 261 (1984).

[36] A. Prasad, S. S. Negi, and R. Ramaswamy, Int. J. Bifurcation and Chaos, to be pub-
lished, 2001.

[37] A Venkatesan, M Lakshmanan, A Prasad and R Ramaswamy, Phys. Rev. E 61, 3641 (2000).

[38] G. Keller, Fund. Math. 151, 139 (1996).

[39] S. S. Negi and R. Ramaswamy, Pramana J. Phys. 56, 47 (2001).

[40] O. Sosnovtseva, U. Feudel, J. Kurths, and A. Pikovsky, Phys. Lett. A 218, 255 (1996).

[41] A. Prasad, V. Mehra and R. Ramaswamy, Phys. Rev. E 57, 1576 (1998).

[42] A. Pikovsky and U. Feudel, Chaos 5, 253 (1995).

[43] S. S. Negi and R. Ramaswamy, Phys. Rev. E, submitted.

[44] F. Piechon, M. Benakli and A. Jagannathan, 1995 LANL archives, cond-mat/9502068.

[45] P. C. Ferreira, F. P. Mancini and M. H. R. Tragtenberg, 2000 LANL archives, cond-mat/0002329.

[46] S S Negi, Ph D thesis, Jawaharlal Nehru University, New Delhi, 2001.
Figure Captions
FIG. 1. Phase diagram for the Kohmoto model for $\alpha = 1$.

FIG. 2. Lyapunov exponent versus energy at $\alpha = 1$ for the Fibonacci chain, namely $\omega = (\sqrt{5} - 1)/2$. The largest visible gaps are labelled $A, B, C$ and $D$ respectively. The dynamics for $\lambda = 0$ corresponds to SNAs.
FIG. 3. Integrated density of states as a function of energy $E$ for $\omega = \gamma = (\sqrt{5} - 1)/2$ at $\alpha = 1$.

On the plateaux corresponding to the gaps $A, B, C$ and $D$ the rotation number can be expressed as $p + q\gamma$, with $p, q$ integers. These are indicated for the largest gaps.

FIG. 4. Scaling of the gap widths, $w$ in the Fibonacci chain for the largest gaps as a function of $\alpha$. 