THE STRONG NOVIKOV CONJECTURE FOR LOW DEGREE COHOMOLOGY

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ABSTRACT. We show that for each discrete group $\Gamma$, the rational assembly map

$$K_\ast(B\Gamma) \otimes \mathbb{Q} \rightarrow K_\ast(C^\ast_{\max}\Gamma) \otimes \mathbb{Q}$$

is injective on classes dual to $\Lambda^\ast \subset H^\ast(B\Gamma; \mathbb{Q})$, where $\Lambda^\ast$ is the subring generated by cohomology classes of degree at most 2. Our result implies homotopy invariance of higher signatures associated to classes in $\Lambda^\ast$. This consequence was first established by Connes-Gromov-Moscovici [4] and Mathai [9].

Our approach is based on the construction of flat twisting bundles out of sequences of almost flat bundles as first described in our work [5]. In contrast to the argument in [9], our approach is independent of (and indeed gives a new proof of) the result of Hilsum-Skandalis [6] on the homotopy invariance of the index of the signature operator twisted with bundles of small curvature.

1. INTRODUCTION

Throughout this paper, we use complex $K$-theory. Let $\Gamma$ be a discrete group and denote by $C^\ast_{\max}\Gamma$ the maximal group $C^\ast$-algebra of $\Gamma$. Recall the following form of the strong Novikov conjecture.

Conjecture 1.1. The Baum-Connes assembly map

$$A: K_\ast(B\Gamma) \rightarrow K_\ast(C^\ast_{\max}\Gamma)$$

is injective after tensoring with the rationals.

The Chern character

$$ch: K(-) \rightarrow H(-; \mathbb{Q})$$

is a natural transformation (of $\mathbb{Z}/2$-graded multiplicative (co-)homology theories) from $K$-homology to rational singular homology (both theories being defined in the homotopy theoretic sense). Let

$$\Lambda^\ast(\Gamma) \subset H^\ast(B\Gamma; \mathbb{Q})$$

be the subring generated by classes of degree at most 2.

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In this paper we verify the strong Novikov conjecture for classes dual to elements in $\Lambda^*(\Gamma)$.

**Theorem 1.2.** Let $h \in K_*(B\Gamma)$ be a $K$-homology class such that the map

$$\Lambda^*(\Gamma) \to \mathbb{Q}, \gamma \mapsto \langle \gamma, \text{ch}(h) \rangle,$$

given by the Kronecker pairing (i.e. only elements of equal degree are paired) is nonzero. Then

$$A(h) \neq 0.$$

As a corollary we obtain the following result on homotopy invariance of higher signatures ($\mathcal{L}(M)$ denotes the $L$-polynomial of $M$).

**Corollary 1.3 ([4], [9]).** Let $M$ be a closed connected oriented smooth manifold, let $\Gamma$ be a discrete group and let $f: M \to B\Gamma$ be a continuous map. Then for all $c \in \Lambda^*(\Gamma)$, the higher signature

$$\langle \mathcal{L}(M) \cup f^*(c), [M] \rangle$$

is an oriented homotopy invariant.

The discussion of this result in [9] is based on a theorem of Hilsum-Skandalis [6] saying that the index of the signature operator twisted with Hilbert $A$-module bundles ($A$ being a $C^*$-algebra) of small curvature is an oriented homotopy invariant. For flat twisting bundles this result was known before (see e.g. [7, 8, 10]). Our proof of Theorem 1.2 is independent from [6]. Indeed, we will illustrate in the last section of this paper how our methods allow the reduction of the Hilsum-Skandalis theorem to the case of flat twisting bundles.

### 2. Proof of the main theorem

By a standard suspension argument we may assume that $h \in K_0(B\Gamma)$. Because each discrete group is the direct limit of finitely presented groups and because the classifying space construction, the formation of $C^*_\text{max}$ and the $K$-theory functors commute with direct limits, it is enough to treat the case of finitely presented $\Gamma$.

Due to the geometric description of $K$-homology by Baum-Douglas [2], elaborated in [3], there is triple $(M, E, \phi)$, where $M$ is an even dimensional closed connected spin-manifold, $E \to M$ is a virtual (i.e. $\mathbb{Z}/2$-graded) hermitian complex vector bundle of finite dimension and $\phi: M \to B\Gamma$ is a continuous map, such that

$$\phi_* (E \cap [M]_K) = h .$$

(Strictly speaking, [3] only provides a spin$^c$-manifold; it is an exercise to obtain a spin manifold such that the associated spin$^c$-structure represents
the right $K$-homology class). In the formula, we consider $E$ as a class in $K^0(M)$, and $[M]_K$ is the $K$-theoretic orientation class of the spin manifold $M$.

As $\Gamma$ is finitely presented, we can and will assume that the map $\phi : M \to B\pi_1(M)$ induces an isomorphism of fundamental groups.

Now choose $h \in K_0(B\Gamma)$ as in the main theorem and let $(M, E, \phi)$ be a triple representing $h$. Let

$$\nu = E\Gamma \times_\Gamma C^*_{max} \Gamma$$

be the canonical flat $C^*_{max} \Gamma$-module bundle over $B\Gamma$.

Denoting by $S^\pm \to M$ the bundles of positive or negative complex spinors on $M$, respectively, the assembly map is described as follows.

**Proposition 2.1.** The element

$$A(h) \in K_0(C^*_{max} \Gamma)$$

is equal to the Mishchenko-Fomenko index of

$$D_{E \otimes \phi^*(\nu)} : \Gamma((S \otimes E)^+ \otimes \phi^*(\nu)) \to \Gamma((S \otimes E)^- \otimes \phi^*(\nu)),$$

the Dirac operator on $M$ twisted with the virtual bundle $E \otimes \phi^*(\nu)$. Here, $E$ is equipped with an arbitrary hermitian connection.

This description of the assembly map will be used in order to show that $A(h) \neq 0$.

For that purpose, let $c \in H^*(B\Gamma; \mathbb{Q})$ be such that

$$\langle c, \text{ch}(h) \rangle \neq 0 \in \mathbb{Q}.$$

We can assume that $c \in H^*(B\Gamma; \mathbb{Z})$. In order to keep the exposition transparent, let us first assume that $c \in H^2(B\Gamma; \mathbb{Z})$.

Let $L \to B\Gamma$ be a complex hermitean line bundle classified by $c$. We pick a unitary connection on the pull back bundle $L' = \tilde{\phi}^*(L) \to \tilde{M}$ and denote by $\omega \in \Omega^2(\tilde{M}; i\mathbb{R})$ its curvature form. Let $\pi : \tilde{M} \to M$ be the universal cover. Because the universal cover of $B\Gamma$ is contractible, the bundle $\pi^*(L') \to \tilde{M}$ is trivial. Fix a unitary trivialization $\pi^*(L') \cong \tilde{M} \times \mathbb{C}$. With respect to this trivialization, the induced connection on $\pi^*(L')$ is given by a 1-form $\eta \in \Omega^1(\tilde{M}; i\mathbb{R})$.

Because $U(1)$ is abelian, the curvature form of this connection is equal to $d\eta$ which in turn coincides with $\pi^*(\omega)$ by naturality. However, contrary to the form $\pi^*(\omega)$, the connection form $\eta$ is in general not invariant under the deck transformation group (this would imply that $\omega$ represents the zero class in $H^2(M)$).

We will use the bundle $\pi^*(L')$ in order to construct a flat $A$-module bundle $W \to M$ with an appropriate $C^*$-algebra $A$ along the lines of [5]. The
flat bundle $W$ will induce a holonomy representation

$$C^*_\max \pi_1(M) \to A$$

whose induced map in $K$-theory will be used to detect nontriviality of the element $A(h_i)$ appearing in the main theorem.

The details are as follows. For $t \in [0, 1]$ we consider the connection on $\widetilde{M} \times \mathbb{C}$ associated to the 1-form $t \cdot \eta$. The corresponding curvature form is equal to $t \cdot \pi^*(\omega)$ and this is invariant under deck transformations (note that this is in general not true for the forms $t \cdot \eta$ if $t \neq 0$).

We would like to use the bundle $L'$ to construct a family of almost flat bundles $(P_t)_{t \in [0, 1]}$ (cf. [5], Section 2) so that [5, Theorem 2.1] can be applied in order to obtain an “infinite product bundle”

$$V = \prod_{n \in \mathbb{N}} P_{1/n} \to M$$

whose quotient by the corresponding infinite sum bundle will be the desired flat bundle $W \to M$, cf. [5, Proposition 3.4]. However, by Chern-Weil theory it is in general impossible to produce a finite dimensional bundle on $M$ whose curvature form is equal to $t \cdot \omega$, $0 < t < 1$ (the associated Chern class would not be integral).

We bypass this difficulty by allowing infinite dimensional bundles. Consider the Hilbert space bundle

$$\mu = \widetilde{M} \times_\Gamma l^2(\Gamma) \to M.$$

Here, $\Gamma$ acts on the left of $l^2(\Gamma)$ by the formula

$$(\gamma \psi)(x) := \psi(x \cdot \gamma^{-1}), \ x, \gamma \in \Gamma.$$

The forms $t\eta \in \Omega^1(\widetilde{M})$ induce a family of connections $\nabla^t$ on $\mu$, the connection $\nabla^t$ being induced by the $\Gamma$-invariant connection on $\widetilde{M} \times l^2(\Gamma)$ which on the subbundle

$$\widetilde{M} \times \mathbb{C} \cdot 1 \subset \widetilde{M} \times l^2(\Gamma)$$

(identified canonically with $\widetilde{M} \times \mathbb{C}$) coincides with $(\gamma^{-1})^*(t\eta)$.

We wish to regard the bundles $(\mu, \nabla^t)$ as twisting bundles for the Dirac operator on $M$. The index of this Dirac operator will live in the $K$-theory of an appropriate $C^*$-algebra $A_t$ that we think of as the holonomy algebra of $\mu$ with respect to the connection $\nabla^t$.

To define these algebras, let us choose a base point $p \in M$ and a point $q \in \widetilde{M}$ above $p$. We identify the fibre over $p$ with the Hilbert space $l^2(\Gamma)$.

Now let

$$A_t \subset B(l^2(\Gamma))$$

be the norm-linear closure of all maps $l^2(\Gamma) \to l^2(\Gamma)$ that arise from parallel transport with respect to $\nabla^t$ along closed curves in $M$ based at $p$. Further,
we define a bundle $P_t \rightarrow M$ whose fibre over $x \in M$ is given by the norm-linear closure (in $\operatorname{Hom}(\mu_p, \mu_x)$) of all isomorphisms $\mu_p \rightarrow \mu_x$ arising from parallel transport with respect to $\nabla^t$ along smooth curves connecting $p$ with $x$. In this way we obtain, for each $t \in [0, 1]$, a locally trivial bundle $P_t$ consisting of free $A_t$-modules of rank 1 (the $A_t$-module structure given by precomposition) and equipped with $A_t$-linear connections. For the notions relevant in this context, we refer the reader e.g. to [11]. Parallel transport on $P_t$ along a curve connecting $x$ with $x'$ is induced by parallel transport $\mu_x \rightarrow \mu_x'$ with respect to $\nabla^t$. The bundle $\mu$ may be recovered from the "principal bundle" $P_t$ as an associated bundle, i.e. $\mu = P_t \times_{A_t} l^2(\Gamma)$.

The next lemma is crucial for the calculations which will follow.

Lemma 2.2. Each of the algebras $A_t$ carries a canonical trace

$$\tau_t : A_t \rightarrow \mathbb{C}$$

given by

$$\tau_t(\psi) := \langle \psi(1_e), 1_e \rangle$$

where $1_e \in l^2(\Gamma)$ is the characteristic function of the neutral element and $\langle -, - \rangle$ is the inner product on $l^2(\Gamma)$.

Proof. Let $\gamma$ and $\gamma'$ be two closed smooth curves based at $p \in M$ and let $\phi_\gamma$ and $\phi_{\gamma'}$ be parallel transport along $\gamma$ and $\gamma'$. We show that

$$\tau_t(\phi_\gamma \cdot \phi_{\gamma'}) = \tau_t(\phi_{\gamma'} \cdot \phi_\gamma).$$

We will assume from now on that $\gamma$ and $\gamma'$ represent elements in $\pi_1(M, p)$ that are inverse to each other (otherwise, both sides of the above equation are zero). We lift the composed curves $\gamma \cdot \gamma'$ and $\gamma' \cdot \gamma$ to curves $\chi$ and $\chi'$ starting at $q$ in the universal cover $\widetilde{M}$. By assumption, both $\chi$ and $\chi'$ are closed curves. We need to compare parallel transport of the element

$$(q, 1) \in \widetilde{M} \times \mathbb{C}$$

along $\chi$ and $\chi'$. Denoting the result of parallel transport along these curves by $(q, \xi_\chi)$ and $(q, \xi_{\chi'})$, respectively, we have

$$\xi_\chi = \exp \left( \int_{[0, 1]} -t \eta_\chi(\tau)(\dot{\chi}(\tau)) d\tau \right),$$

with the exponential map on the Lie group $S^1$. By use of Stoke's formula, the last expression is equal to

$$\exp \left( \int_D -d(\eta) dx \right) = \exp \left( \int_D -t \pi^*(\omega) dx \right),$$

where $D : D^2 \rightarrow \widetilde{M}$ is a disk with boundary $\chi$ (recall that $\widetilde{M}$ is simply connected). However, the curve $\chi'$ is obtained (up to reparametrization) from
by applying the deck transformation corresponding to $\gamma'$. This implies -

grace to the invariance of the form $\pi^*(\omega)$ under deck transformation - that

the last expression is equal to $\xi'\chi'$. Using continuity and linearity of $\tau_t$, this

shows our assertion. \hfill \Box

We now equip $E$ with a unitary connection and consider the twisted Dirac

operator

$$D_{E \otimes P_t} : \Gamma((S \otimes E)^+ \otimes P_t) \to \Gamma((S \otimes E)^- \otimes P_t).$$

The index $\text{ind}(D_{E \otimes P_t}) \in K_0(A_t)$ satisfies (up to sign) the equation

$$\tau_t(\text{ind}(D_{E \otimes P_t})) = \langle \text{ch}(A(M)) \cup \text{ch}(E) \cup \text{ch}_{\tau_t}(P_t), [M] \rangle$$

by the Mishchenko-Fomenko index theorem (see [11, Theorem 6.9]). Because the Poincaré dual of $\text{ch}(A(M))$ is equal to $\text{ch}([M]_K)$, the last expression equals $\langle \text{ch}_{\tau_t}(P_t), \text{ch}([E] \cap [M]_K) \rangle$. Recall that the choice of $(M, E, \phi)$ implies $\phi_* \text{ch}([E] \cap [M]_K) = h$. The construction of the connection $\nabla^t$ and

the definition of $\text{ch}_{\tau_t}$ (see [11, Definition 5.1.]) show

$$\text{ch}_{\tau_t}(P_t) = \exp(t\phi^*c) \in H^*(M; \mathbb{R})$$

so that finally

$$\tau_t(\text{ind}(D_{E \otimes P_t})) = \langle \exp(t\phi^*c), \text{ch}([E] \cap [M]_K) \rangle = \langle \exp(tc), \text{ch}(h) \rangle \in \mathbb{R}[t],$$

a polynomial which is different from zero by the assumption $\langle c, \text{ch}(h) \rangle \neq 0$.

We wish to use this calculation in order to detect nontriviality of $A(h) \in K_0(C^{\ast}_{\text{max}}\Gamma)$. This is done by constructing a flat bundle on $M$ out of the sequence of bundles $(P_{1/n})_{n \in \mathbb{N}}$ with connections. This sequence is almost flat in the sense of [5, Section 2]. Therefore, applying [5, Theorem 2.1.] we obtain a smooth “infinite product bundle”

$$V = \prod_{n \in \mathbb{N}} P_{1/n} \to M$$

equipped with a connection that we may think of as the product of the connections on $P_{1/n}$. After passing to the quotient by the infinite sum of the bundles $P_{1/n}$, we obtain a smooth bundle

$$W = (\prod P_{1/n}) / (\bigoplus P_{1/n}) \to M$$

of free modules of rank 1 over the $C^{\ast}$-algebra

$$A = (\prod A_{1/n}) / (\bigoplus A_{1/n})$$

which carries an induced flat connection. For more details of this discussion, we refer the reader to [5], especially to Section 2 and the statements before Proposition 3.4.
The flat bundle $W$ induces a unitary holonomy representation of $\pi_1(M)$. Because $\phi$ induces an isomorphism $\pi_1(M) \cong \Gamma$ by the choice of the triple $(M, \phi, E)$, we hence obtain a $C^*$-algebra map

$$\psi: C^*_{\max} \Gamma \to A$$

using the universal property of $C^*_{\max}$. The induced map in $K$-theory can be analyzed in terms of the $KK$-theoretic description of the index map (cf. [5, Lemma 3.1]). One concludes that $\psi^*(A(h)) \in K_0(A)$ is equal to the index of the twisted Dirac operator

$$D_{E \otimes W}: \Gamma((S \otimes E)^+ \otimes W) \to \Gamma((S \otimes E)^- \otimes W)$$

and this in turn equals the image of the index of the twisted Dirac operator

$$D_{E \otimes V}: \Gamma((S \otimes E)^+ \otimes V) \to \Gamma((S \otimes E)^- \otimes V)$$

under the canonical map $p_*: K_0(\prod A_{1/n}) \to K_0(A)$ induced by the canonical projection $p: \prod A_{1/n} \to A$.

We have a short exact sequence

$$K_0(\bigoplus A_{1/n}) \to K_0(\prod A_{1/n}) \xrightarrow{p_*} K_0(A)$$

where the left hand group is canonically isomorphic to the algebraic direct sum $\bigoplus_{n \in \mathbb{N}} K_0(A_{1/n})$. Furthermore, the traces $\tau_{1/n}: A_{1/n} \to \mathbb{C}$ all have norm 1 and hence induce a trace

$$\prod A_{1/n} \to \prod \mathbb{C}.$$ 

Using the canonical isomorphism $K_0(\bigoplus A_{1/n}) \cong \bigoplus K_0(A_{1/n})$, we finally get a trace map

$$\tau: \text{im} \ p_* \to (\prod \mathbb{C})/(\bigoplus \mathbb{C}).$$

Note that the direct sum on the right is understood in the algebraic sense: Any element has only finitely many components different from 0. Assuming $\text{ind}(D_{E \otimes W}) = 0$ we conclude that the element in $\prod \mathbb{C}/\bigoplus \mathbb{C}$ represented by

$$(\text{ind}(D_{E \otimes p_{1/n}}))_{n \in \mathbb{N}}$$

is equal to zero. Combining this with our previous calculation, this means that the polynomial

$$\langle \exp(tc), \text{ch}(h) \rangle \in \mathbb{R}[t]$$

is equal to zero for all but finitely many values $t = 1/n$, $n \in \mathbb{N}$. This implies that this polynomial is identically zero and hence in particular

$$\langle c, \text{ch}(h) \rangle = 0$$

in contradiction to our assumption. The proof of the main theorem is therefore complete, if $c \in H^2(B\Gamma; \mathbb{Z})$. 
We will now discuss the case of general $c \in \Lambda^*(\Gamma)$ (still assuming $h \in K_0(B\Gamma)$). If $$c = c_1 \cup \ldots \cup c_k$$ is a product of classes in $H^2(B\Gamma; \mathbb{Z})$, we replace the bundle $L$ in the above argument by the tensor product bundle $$L := L_1 \otimes L_2 \otimes \ldots \otimes L_k \to B\Gamma$$ where the line bundle $L_i \to B\Gamma$ is classified by $c_i$. In an analogous fashion as before, we get bundles $$P_{t_1, \ldots, t_k} \to M$$ of Hilbert-$A_{t_1, t_2, \ldots, t_k}$-modules working with the connection $$t_1 \eta_1(1) + t_2 \eta_2(2) + \ldots + t_k \eta_k(k)$$ on the bundle $\tilde{M} \times \mathbb{C}$ where $\eta(i)$ is a connection induced from $L_i$ and where each $t_i \in [0, 1]$.

Assuming that $A(h) = 0$ we can conclude in this case that the polynomial $$\langle \exp(t_1 c_1) \cdot \ldots \cdot \exp(t_k c_k), \text{ch}(h) \rangle \in \mathbb{R}[t_1, \ldots, t_k]$$ is equal to zero for all but finitely many $$(t_1, \ldots, t_k) = (1/n_1, \ldots, 1/n_k)$$ where $n_i \in \mathbb{N}$. Hence this polynomial is identically 0 and in particular $$\langle c_1 \cup \ldots \cup c_k, \text{ch}(h) \rangle = 0$$ again contradicting our assumption. In the most general case, $c$ being a sum $$c = c(1) + \ldots + c(k)$$ where each $c(i)$ is a product of two dimensional classes in $H^2(B\Gamma; \mathbb{Z})$, the assumption $$\langle c, \text{ch}(h) \rangle \neq 0$$ implies that already for one summand $c(i)$, we have $\langle c(i), \text{ch}(h) \rangle \neq 0$ so that we are reduced to the previous case.

Because we already used a suspension argument in order to restrict attention to classes in $K_0(B\Gamma)$, this finishes the proof of the main theorem.

3. HILSUM-SKANDALIS REVISITED

If $M$ is an oriented Riemannian manifold of even dimension, recall that $d + d^*$, the sum of the exterior differential and its formal adjoint, defines the signature operator $$D^{\text{sign}} : \Gamma(\Lambda^*(M)) \to \Gamma(\Lambda^*(M)),$$ where the $\pm$-signs indicate $\pm 1$-eigenspaces of the Hodge star operator.
If $E \to M$ is a Hilbert $A$-module bundle over $M$, where $A$ is some $C^*$-algebra, we obtain the twisted signature operator

$$D^\text{sign}_E : \Gamma(\Lambda^*_+(M) \otimes E) \to \Gamma(\Lambda^*_-(M) \otimes E)$$

which has an index $\text{ind} D^\text{sign}_E \in K_0(A)$. The following theorem says that this class is an oriented homotopy invariant, if $E$ has small curvature.

**Theorem 3.1** (Hilsum-Skandalis [6]). Let $M$ and $M'$ be closed oriented Riemannian manifolds of the same dimension and let $h : M' \to M$ be an orientation preserving homotopy equivalence. Then there exists a constant $c > 0$ with the following property: If $E \to M$ is a Hilbert $A$-module bundle with connection $\nabla$ so that the associated curvature form $\Omega \nabla \in \Omega^2(M; \text{End} E)$ satisfies the bound

$$\|\Omega\nabla\| < c,$$

(the norm being defined by the maximum norm on the unit sphere bundle in $\Lambda^2 M$ and the operator norm on each fibre $\text{End}(E_x, E_x)$), then we have

$$\text{ind} D^\text{sign}_{f^*(E)} = \text{ind} D^\text{sign}_E.$$

If $E$ is flat, i.e. $\Omega \nabla = 0$, this result was proved in [7, 8, 10]. In this section, we will explain briefly how this special case implies the general statement of Theorem 3.1. Our argument is again based on the construction of a flat bundle out of a sequence of almost flat bundles [5].

Assuming that no $c$ with the stated property exists, we find a sequence of $C^*$-algebras $A_n$ and a sequence of Hilbert $A_n$-module bundles $E_n \to M$ with connections $\nabla_n$ so that $\|\Omega \nabla_n\| < \frac{1}{n}$, but

$$\text{ind} D^\text{sign}_{f^*(E_n)} \neq \text{ind} D^\text{sign}_{E_n}$$

for all $n$. We obtain an almost flat sequence

$$f^*(E_n) \cup E_n \to M' \cup M$$

of Hilbert $A_n$-module bundles in the sense of [5, Section 2] over the disjoint union $M' \cup M$. Applying again the methods in [5], we obtain a flat Hilbert $A$-module bundle

$$W = (\prod f^*(E_n) \cup \prod E_n) / (\bigoplus f^*(E_n) \cup \bigoplus E_n) \to M' \cup M$$

where

$$A = (\prod A_n) / (\bigoplus A_n).$$

The index $\text{ind} D^\text{sign}_W$ of the signature operator on $(-M') \cup M$ (the minus-sign indicates reversal of orientation) twisted by $W$, vanishes by the results in [7, 8, 10].
This conclusion leads to the following contradiction. We have a canonical isomorphism

\[ K_0(\bigoplus A_n) \cong \bigoplus K_0(A_n) \]

and - assuming that each \( A_n \) is unital and stable, the last property easily being achieved by replacing each \( A_n \) by \( A_n \otimes \mathbb{K}(l^2(\mathbb{N})) \) - a canonical isomorphism

\[ K_0(\prod A_n) \cong \prod K_0(A_n), \]

compare the proof of [5, Proposition 3.6]. The signature operator on \(-M' \cup M\) twisted with the non-flat Hilbert \( \prod A_n \)-module bundle

\[ V := \prod f^*(E_n) \cup \prod E_n \]

has an index

\[ \text{ind}(D_{V^{\text{sign}}}) \in K_0(\prod A_n) \cong \prod K_0(A_n) \]

which is different from 0 for infinitely many factors by our assumption on the bundles \( E_n \). On the other hand, under the canonical map

\[ p_* : K_0(\prod A_n) \to K_0(A) \]

this index maps to the index of the signature operator twisted with the bundle \( W \) and this index was identified as 0. Now the contradiction arises from the fact that by the long exact \( K \)-theory sequence, the kernel of \( p_* \) is equal to \( K_0(\bigoplus A_n) = \bigoplus K_0(A_n) \) (algebraic direct sum) and therefore \( \text{ind}(D_{V^{\text{sign}}}) \) is not contained in it.

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