The Geometric Origin of the CP Phase

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Abstract

The complex phase present in CP-violating systems such as neutral kaons is shown to be of geometrical origin. It is also concluded that the complex phase of the Cabibbo–Kobayashi–Maskawa (CKM) matrix is a Berry-like phase.
Following the original analogy presented in Ref. [1] between the physics of the weak decay of neutral K-mesons (kaons) and a classical system of oscillators it was shown [2] that this analogy can be promoted to an equivalence between the Schroedinger dynamics of a quantum system with a finite number of basis states and classical dynamics. The equivalence is an isomorphism that connects in univocal way both dynamical systems. There have been several other proposals and realizations [3],[4],[5],[6],[7] of the equivalence between the physics of the neutral K-meson quantum system and classical systems of oscillators either electrical or mechanical. In particular, in Ref. [3], Rosner and Slezak presented the emulation of CP violation in the kaon system provided by the two-dimensional motion of a Foucault pendulum. In their conclusions one reads: “the classical emulation of CP violation via the Foucault pendulum leaves us with one big puzzle.” This statement is because for this emulation they need to consider an asymmetry in the coupling between the two modes of the pendulum, that classically is imposed by the rotation of Earth. In the case of kaon mixing, that effect is “thought to arise from a complex phase in the CKM matrix.” The authors finally asked themselves if that phase is an indication of a new fundamental asymmetry from beyond the standard model (SM).

Motivated by this comment we analyze in this note the connection between the complex phase present in CP-violating systems such as the quantum one of neutral kaons, and a geometrical phase such as the latitude effect acquired by the Foucault pendulum. Moreover, we prove that the complex phase of the CKM matrix, origin of the CP-violation in the framework of the SM, is a Berry-like phase [8].

As is well known, the Berry phase appears when a quantum system evolves and returns to its initial physical state in an adiabatic way. After this evolution, it acquires a memory of this evolution in the form of a time-independent geometrical phase in the quantum wave function.

As we have stated, our analysis originates in the Foucault pendulum as an emulation of the CP-violating system [3]. The motion of the Foucault pendulum acquires a latitude effect in the form of a phase shift [9]. This phase was shown [10] to be no more than an example of the classical version of the Berry phase, namely the Hannay phase [11].

Let us remember the dynamics of the Foucault pendulum, both from the standard way based upon a non-inertial reference frame and also in the way that the Hannay phase is explicitly shown to appear.

The equation of motion of the Foucault pendulum related to the kaon system needs slightly different natural frequencies in both directions of motion and vastly different damping in these directions in order to emulate the characteristics of neutral kaons [3]. Consequently the mass matrix corresponding to the Foucault pendulum satisfying CPT invariance (N_{21} =
−\mathcal{N}_{12}) reads [3]

\[ \mathcal{N} = \begin{pmatrix} \omega_1 - i\gamma_1 & i\Omega' \\ -i\Omega' & \omega_2 - i\gamma_2 \end{pmatrix}. \] (1)

Here \( \Omega' \) is in general complex.

The matrix for the kaon system, written in terms of the effective Hamiltonian (Eq. (53) of Ref. [2]) is

\[ \mathcal{H} = \begin{pmatrix} \Sigma + \Delta \cos \alpha & i\Delta \sin \alpha \\ -i\Delta \sin \alpha & \Sigma - \Delta \cos \alpha \end{pmatrix} \] (2)

where

\[ \Sigma = \frac{\mu_S + \mu_L}{2} = \frac{m_S - m_L - i(\gamma_S - \gamma_L)}{2}; \quad \Delta = \frac{\mu_S - \mu_L}{2} = \frac{m_S - m_L + i(\gamma_S - \gamma_L)}{2} \] (3)

with \( m_S \) and \( m_L \) being the short and long \( K^0 \)-masses respectively and \( \gamma_S, \gamma_L \) the corresponding decay rates. Consequently,

\[ \Omega' = \Delta \sin \alpha; \] (4)

the phase \( \alpha \) is given by

\[ \exp(i\alpha) = \frac{1 - \epsilon}{1 + \epsilon} \Rightarrow \alpha \approx 2i\epsilon \] (5)

where \( \epsilon \) is the traditional parameter that measures CP violation in mixing.

From Eq. (1) one finds, after simple algebra and using the fact that \( \epsilon \) is very small (\( \sim 10^{-3} \)), that

\[ \Re(\Omega') = \frac{m_L - m_S}{2} \sin(2\Im(\epsilon)), \] (6)

\[ \Im(\Omega') = \frac{\gamma_L - \gamma_S}{2} \sin(2\Im(\epsilon)). \] (7)

This \( \Omega' \) plays the role of \( \Omega = \Omega_E \sin \theta \) in the simple Foucault pendulum. In this last expression, \( \Omega_E \) is the Earth daily-rotation frequency. It includes the latitude effect on the pendulum motion measured by the latitude of the observation point, \( \theta \). This effect is the manifestation of the Hannay geometrical phase. Now, the inclusion of dissipation, always present for kaons, leads to a complex \( \Omega' \). The real part, being proportional to the mass difference, is related to the Hermitian part of the Hamiltonian, while the imaginary part, proportional to the decay rates, comes from its non-Hermitian part [12]. Due to the approximation introduced by the smallness of \( \epsilon \), the phase is the same for both parts of \( \Omega' \).

At this point it is worth remembering that the latitude effect we have already mentioned refers to the fact that after the Earth has completed one day’s revolution, the plane of oscillation of the Foucault pendulum will not complete a \( 2\pi \) rotation. There is a “defect” in the angle of rotation of the pendulum plane measured by \( \sin \theta \), with \( \theta \) the latitude where
the pendulum is maintained. The analysis of the Foucault pendulum’s motion can be done without using the so-called fictitious forces appearing in non-inertial frames. In fact \[10\] one can develop a geometrical model of the pendulum in which it keeps a constant direction of oscillation manifested as a parallel displacement along a curved surface of the Earth with a rotation frequency much lower that the oscillation one. As we said before, this is an example of the Hannay phase \[11\]. Then, one can conclude that in our case of the kaon system, \((23\epsilon)\) plays exactly the role of the latitude, the geometrical phase.

This result can be rephrased by saying that the Earth’s motion is to the Foucault pendulum as the weak interaction Hamiltonian (the cause of the splitting of masses between \(K_S\) and \(K_L\) and the differences between mean lives) is to the kaon system. Moreover, the geometrical phase related to the latitude phase is directly related to the \(\epsilon\) parameter that measures the CP-violation in kaons.

Let us now go to the quantum analysis of the Berry phase in our case of interest — neutral kaons. This analysis starts at the quark level where the CKM matrix is active.

In the Berry analysis one starts with a quantum Hamiltonian that depends on a real parameter, \(R\), that is adiabatically varied (it changes with a frequency that is much smaller than any other frequency present in the problem).

In our case of the kaon system, certainly this small frequency characterizing the adiabatic change is the weak interaction part of the Hamiltonian, \(H_W\), that is a small perturbation to the strong interaction Hamiltonian whose scale is the \(K^0\) mass. On the other hand, the \(K^0 - \bar{K}^0 - K^0\)-oscillations provide the cycle that enters in the Berry analysis. In order to perform the analysis we refer first to quarks and their charged current interactions measured by the CKM matrix. In entering the explicit calculation to show our assertions, let us remember the general structure of the CKM matrix:

\[
CKM = \begin{pmatrix}
V_{ud} & V_{us} & V_{ub} \\
V_{cd} & V_{cs} & V_{cb} \\
V_{td} & V_{ts} & V_{tb}
\end{pmatrix},
\tag{8}
\]

where each element of the matrix measures the vertex of both quarks involved with the \(W\) vector boson. As was proposed in Ref. \[13\] we consider a sequence of quantum weak transitions connected with a SM contribution to the \(K^0 - \bar{K}^0\) oscillation: namely, the sequence of Fig. 1, that is the cycle \(d \rightarrow c \rightarrow s \rightarrow u \rightarrow d\). The chosen sequence means that one has

\[
|d > < d| H_W |u > < u| H_W |s > < s| H_W |c > < c| H_W |d > = V_{ud} V_{us}^* V_{cs} V_{cd}^* |d > .
\tag{9}
\]

Note that we have maintained only the CKM contribution to each vertex. In the same way,
Figure 1: One of the sequences of weak vertices considered

now for the sequence $s \rightarrow u \rightarrow d \rightarrow c \rightarrow s$, one obtains

$$|s><s|H_W|u><u|H_W|d><d|H_W|c><c|H_W|s> = V_{us}V^{*}_{ud}V_{dc}V^{*}_{cs}|s>$$. \hfill (10)

Both previous equations are the ones related to the way quarks behave in the $K^0 \rightarrow \bar{K}^0 \rightarrow K^0$ oscillation and in the $\bar{K}^0 \rightarrow K^0 \rightarrow \bar{K}^0$ one, respectively. In order to go from quarks to kaons, one has to add the contributions of the antiquark cycles, that are

$$|\bar{s}><\bar{s}|H_W|\bar{c}><\bar{c}|H_W|\bar{d}><\bar{d}|H_W|\bar{u}><\bar{u}|H_W|\bar{s}> = V_{cs}V^{*}_{cd}V_{ud}V^{*}_{us}|\bar{s}>$$, \hfill (11)

and

$$|\bar{d}><\bar{d}|H_W|\bar{c}><\bar{c}|H_W|\bar{s}><\bar{s}|H_W|\bar{u}><\bar{u}|H_W|\bar{d}> = V_{cd}V^{*}_{cs}V_{us}V^{*}_{ud}|\bar{d}>$$, \hfill (12)

respectively.

Consequently, the amplitude for the cyclic oscillation $K^0 - \bar{K}^0 - K^0$ is proportional to

$$(V_{ud}V^{*}_{us}V_{cs}V^{*}_{cd})^2 = (A + iB)^2$$ \hfill (13)

while for the cyclic oscillation $\bar{K}^0 - K^0 - \bar{K}^0$ the amplitude is proportional to

$$(V_{us}V^{*}_{ud}V_{cd}V^{*}_{cs})^2 = (A - iB)^2$$, \hfill (14)

which is exactly the complex conjugate of the previous one. Clearly, the difference between the two previous expressions is proportional to the amount of CP-violating effects of the weak interaction. It results in

$$2AB = \Im (V_{ud}V^{*}_{us}V_{cs}V^{*}_{cd}) = J_{CP}$$, \hfill (15)

the Jarlskog invariant measuring CP violation $[14]$. 
Notice that $J_{CP}$, being invariant, does not depend (except for a sign) on the pair of quantum states chosen. Moreover, $\epsilon$ is related to $J_{CP}$ by

$$|\epsilon| \propto A_{ut} \Im(V_{ud}V_{us}^*V_{ts}^*) + A_{ct} \Im(V_{cd}V_{cs}^*V_{ts}^*) + A_{tt} \Im(V_{td}V_{ts}^*V_{td}^*V_{ts}^*) ,$$  \hspace{1cm} (16)$$

where the imaginary parts are related to the Jarlskog invariant and the $A$’s are numerical coefficients arising from integrations over virtual momenta.

Recalling Eqs. (6) and (7) we unveil the geometric character of the CKM matrix elements through the Jarlskog invariant.

In summary, we have shown that the complex phase of the CKM matrix, responsible for the CP violation present in the SM, is of geometrical origin, entirely similar to the quantum Berry phase. This result opens some new roads in the analysis and possible measurements of CP-violation effects.

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