The Secant-Newton Map is Optimal
Among Contracting \(n^{th}\) Degree Maps
for \(n^{th}\) Root Computation

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Abstract
Consider the problem: given a real number \(x\) and an error bound \(\varepsilon\), find an interval such that it contains \(\sqrt[n]{x}\) and its width is less than \(\varepsilon\). One way to solve the problem is to start with an initial interval and to repeatedly update it by applying an interval refinement map on it until it becomes narrow enough. In this paper, we prove that the well known Secant-Newton map is optimal among a certain family of natural generalizations.

Keywords: \(n^{th}\) root, interval mathematics, secant, Newton, contracting

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1 Introduction
Computing the \(n^{th}\) root of a given real number is a fundamental operation. Naturally, various numerical methods have been developed \[16, 12, 13, 1, 8, 4, 1, 15, 14, 11, 2, 7, 3, 5, 9, 10\]. In this paper, we consider an interval version of the problem \[12, 1, 13\]: given a real number \(x\) and an error bound \(\varepsilon\), find an interval such that it contains \(\sqrt[n]{x}\) and its width is less than \(\varepsilon\). One way to solve the problem starts with an initial interval

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and repeatedly updates it by applying a refinement map on it until it becomes narrow enough (see below).

\[
\text{in: } x > 0, \ \varepsilon > 0 \\
\text{out: } I, \text{ interval such that } \sqrt[n]{x} \in I \text{ and } \text{width}(I) \leq \varepsilon \\
I \leftarrow \left[\min(1, x), \max(1, x)\right] \\
\text{while } \text{width}(I) > \varepsilon \\
I \leftarrow R(I, x) \\
\text{return } I
\]

A well known refinement map \(R^\ast\), tailored for \(n^{th}\) root computation, is obtained by combining the secant map and the Newton map where the secant/Newton map is used for determining the lower/upper bound of the refined interval, that is,

\[
R^\ast : [L, U], x \mapsto \left[L + \frac{x - L^n}{L^{n-1} + L^{n-2}U + \ldots + LU^{n-2} + U^{n-1}}, U + \frac{x - U^n}{nU^{n-1}}\right]
\]

which can be easily derived from Figure 1.

![Figure 1: Derivation of Secant-Newton map](image-url)

A question naturally arises. **Is there any refinement map which is better than Secant-Newton?** In order to answer the question rigorously, one first needs to fix a search space, that is, a family of maps in which we search for a better map. In this paper, we will consider the family of all the “natural generalizations” of the Secant-Newton map. The above picture shows that the Secant-Newton map is contracting, that is, \(L \leq L' \leq \sqrt[n]{x} \leq U' \leq U\). Furthermore, it "scales properly," that is, if we multiply \(\sqrt[n]{x}\), \(L\) and \(U\) by a number, say \(s\), then \(L'\) and \(U'\) are also multiplied by \(s\). This is due to the fact that the numerators are \(n^{th}\) degree forms in \(\sqrt[n]{x}\), \(L\) and \(U\) and the denominators are \(n - 1^{th}\) degree forms. These observations suggest the following
The main contribution of this paper is the finding that the Secant-Newton map is optimal among all the contracting $n^{th}$ degree maps. By optimal, we mean that the output interval of the Secant-Newton map is always a subset of the output interval of any other contracting $n^{th}$ degree map, and in fact is almost always a proper subset. This result generalizes the previous result on the square root computation by Erascu-Hong [6], where the Secant-Newton map was shown to be optimal among all the contracting quadratic maps. The new contributions, beyond the straightforward adaptation of [6], are as follows.

• We found that the precise notion of the “optimality” for the square-root case in [6] could not be extended straightforwardly to the $n^{th}$ root case. It had to be modified in a subtle but crucial way. See Theorem 1 (b).
• Furthermore, we found that the proof techniques used in [6] could not be straightforwardly extended. In fact, it turns out that only a small part of the proof technique could be straightforwardly generalized. However, the rest of the proof could not be generalized. Thus, we developed several new proof techniques. See Lemmas 4, 5, 6, 7 and 8.

The paper is structured as follows. In Section 2, we precisely state the main claim of the paper. In Section 3, we prove the main claim.

2 Main Result

In this section, we will make a precise statement of the main result informally described in the previous section. For this, we recall a few notions and notations.

Definition 1 ($n^{th}$ degree map). We say that a map

$$R : [L, U], x \mapsto [L', U']$$

such that

$$L \leq L' \leq \sqrt[n]{x} \leq U' \leq U,$$

which we will call contracting $n^{th}$ degree maps. By choosing values for the parameters $p = (p_0, \ldots, p_{2n})$ and $q = (q_0, \ldots, q_{2n})$, we get each member of the family. For instance, the Secant-Newton map can be obtained by setting

$$p = (-1, 0, 0, \ldots, 1, 1, \ldots, 1)$$

$$q = (-1, 0, 0, \ldots, n, 0, 0, \ldots, 0).$$

The main contribution of this paper is the finding that the Secant-Newton map is optimal among all the contracting $n^{th}$ degree maps. By optimal, we mean that the output interval of the Secant-Newton map is always a subset of the output interval of any other contracting $n^{th}$ degree map, and in fact is almost always a proper subset.

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- We found that the precise notion of the “optimality” for the square-root case in [6] could not be extended straightforwardly to the $n^{th}$ root case. It had to be modified in a subtle but crucial way. See Theorem 1 (b).
- Furthermore, we found that the proof techniques used in [6] could not be straightforwardly extended. In fact, it turns out that only a small part of the proof technique could be straightforwardly generalized. However, the rest of the proof could not be generalized. Thus, we developed several new proof techniques. See Lemmas 4, 5, 6, 7 and 8.

The paper is structured as follows. In Section 2 we precisely state the main claim of the paper. In Section 3 we prove the main claim.
is an $n^{th}$ degree map if it has the following form
\[
L' = L + \frac{x + p_0 L^n + p_1 L^{n-1} U + \cdots + p_n U^n}{p_{n+1} L^{n+1} + p_{n+2} L^{n+2} U + \cdots + p_{2n} U^{n+1}}
\]
\[
U' = U + \frac{x + q_0 U^n + q_1 U^{n-1} L + \cdots + q_n L^n}{q_{n+1} U^{n+1} + q_{n+2} U^{n+2} L + \cdots + q_{2n} L^{n+1}}.
\]
We will denote such a map by $R_{p,q}$.

**Definition 2** (Secant-Newton map). The Secant-Newton map is the $n^{th}$ degree map $R_{p^*,q^*}$ where $p^* = (-1,0,0,...,0,1,1,...,1)$ and $q^* = (-1,0,0,...,0,n,0,0,...,0)$, namely
\[
R_{p^*,q^*} : [L,U], x \mapsto [L^*,U^*]
\]
where
\[
L^* = L + \frac{x - L^n}{n L^{n+1} + L^{n+2} U + \cdots + U^{n+1}}
\]
\[
U^* = U + \frac{x - U^n}{n U^{n+1} L + \cdots + U^{n+1}}
\]

**Definition 3** (Contracting $n^{th}$ degree map). We say that a map
\[
R : [L,U], x \mapsto [L',U']
\]
is a contracting $n^{th}$ degree map if it is an $n^{th}$ degree map and
\[
\forall_{L,U,x} \quad 0 < L \leq \sqrt[n]{x} \leq U \quad \Rightarrow \quad L \leq L' \leq \sqrt[n]{x} \leq U' \leq U.
\]

Now we are ready to state the main result of the paper.

**Theorem 1** (Main Result). Let $R_{p,q}$ be a contracting $n^{th}$ degree map which is not $R_{p^*,q^*}$ (Secant-Newton). Then we have
\[
(a) \quad \forall_{L,U,x} \quad 0 < L \leq \sqrt[n]{x} \leq U \quad \Rightarrow \quad R_{p^*,q^*}([L,U],x) \subseteq R_{p,q}([L,U],x)
\]
\[
(b) \quad \forall_{L,U,x} \quad 0 < L \leq \sqrt[n]{x} \leq U \quad \Rightarrow \quad R_{p^*,q^*}([L,U],x) \subset R_{p,q}([L,U],x)
\]

**Remark 1.** The small circle above the universal quantifier in the second claim indicates that the statement holds for almost all values of $L,U,x$. Equivalently, this means the set of exceptions has measure zero.

**Remark 2.** The first claim states that the Secant-Newton map is never worse than any other contracting $n^{th}$ degree map. The second claim states that the Secant-Newton map is almost always better than all the other contracting $n^{th}$ degree maps.

**Remark 3.** This paper is a sequel to the square root case study of Erascu-Hong [8]. Erascu-Hong proved the following two results for the square-root case ($n = 2$):
\[
\forall_{L,U,x} \quad 0 < L \leq \sqrt[n]{x} \leq U \quad \Rightarrow \quad R_{p^*,q^*}([L,U],x) \subseteq R_{p,q}([L,U],x)
\]
\[
\forall_{L,U,x} \quad 0 < L < \sqrt[n]{x} < U \quad \Rightarrow \quad R_{p^*,q^*}([L,U],x) \subset R_{p,q}([L,U],x)
\]
These look very similar to the two claims in Theorem \[\|\] above for the \(n\)th root case (arbitrary \(n\)). In fact, the first claims look identical to each other, which means that the claim for the \(n = 2\) generalizes to arbitrary \(n\) without any change. On the other hand, the second claims have subtle but important differences.

\[
\begin{align*}
n = 2 & : \quad \forall L, U, x \quad \cdots \quad L < \sqrt[n]{x} < U \quad \cdots \\
n = \text{arbitrary} & : \quad \forall L, U, x \quad \cdots \quad L \leq \sqrt[n]{x} \leq U \quad \cdots 
\end{align*}
\]

Note that \(\forall\) is replaced with \(\forall\) and \(<\) with \(\leq\). These subtle changes are necessary because, to our surprise, the claim for \(n = 2\) does not hold in general. For example, consider the following \(n = 3\) case:

\[
p = (-1, 0, 0, 0, 2, \frac{1}{2}, 1), \quad q = (1, 0, 0, 0, 3, 0, 0), \quad L = 1, \quad U = 2, \quad x = \left(\frac{3}{2}\right)^3
\]

This implies \(0 < L < \sqrt[3]{x} < U\), and

\[
R_{p^*, q^*}([L, U], x) = R_{p, q}([L, U], x)
\]

\[
\iff L' = L^* \wedge U^* = U^* \\
\iff L + \frac{x - L^3}{p_4L^2 + p_5LU + p_6U^2} = L + \frac{x - L^3}{L^2 + LU + U^2} \\
\wedge \\
U + \frac{x - L^3}{q_4U^2 + q_5UL + q_6L^2} = U + \frac{x - L^3}{q_4U^2 + q_5UL + q_6L^2} \\
\iff p_4L^2 + p_5LU + p_6U^2 = L^2 + LU + U^2 \\
\wedge \\
q_4U^2 + q_5UL + q_6L^2 = 3U^2 \\
\iff (p_4 - 1)L^2 + (p_5 - 1)LU + (p_6 - 1)U^2 = 0 \\
\wedge \\
(q_4 - 3)U^2 + q_5UL + q_6L^2 = 0 \\
\iff (2 - 1)(1)^2 + (\frac{1}{2} - 1)(1)(2) + (1 - 1)(2)^2 = 0 \\
\wedge \\
(3 - 3)(2)^2 + (3)(1)(2) + (0)(1)^2 = 0 \\
\iff 0 = 0 \quad \wedge \quad 0 = 0 \\
\iff \text{true.}
\]

3 Proof

In this section, we will prove the main result (Theorem \[\|\]). For the sake of easy readability, the proof will be divided into several lemmas, which are interesting on their own. The main theorem follows immediately from the last two lemmas (Lemmas \[\|\] and \[\|\]).

This paper is a sequel to \[\|\] where square-root (\(n = 2\)) was considered. Hence we initially hoped that the proof techniques developed in \[\|\] would be generalizable straightforwardly to the \(n\)th root case. It turns out that a part of the proof could indeed be straightforwardly generalized (Lemmas \[\|\] and \[\|\]). However, the rest of the proof could not be generalized at all. Thus, we had to develop several new proof techniques (Lemmas \[\|\] and \[\|\]).
Lemma 2. Let $R_{p,q}$ be a contracting $n^{th}$ degree map. Then we have

$$p_0 = -1 \land p_1 = \cdots = p_n = 0,$$

$$q_0 = -1 \land q_1 = \cdots = q_n = 0.$$ 

Proof. Let $R_{p,q}$ be a contracting $n^{th}$ degree map. Then $p,q$ satisfy the condition (1). The proof essentially consists of instantiating the condition (1) on $x = L^n$ and $x = U^n$. By instantiating the condition (1) with $x = L^n$ and recalling the definition of $L'$, we have

$$\forall_{L,U} \ 0 < L \leq U \implies L + \frac{L^n + p_1 L^{n-1} U + \cdots + p_n U^n}{p_{n+1} L^{n-1} + p_{n+2} L^{n-2} U + \cdots + p_{2n} U^{n-1}} = L.$$ 

By simplifying, removing the denominator and collecting, we have

$$\forall_{L,U} \ (L, U) \in D \implies g(L, U) = 0,$$ 

where

$$D = \{(L, U) : 0 < L \leq U\},$$

$$g(L, U) = (1 + p_0) L^n + p_1 L^{n-1} U + \cdots + p_n U^n.$$ 

Since the bivariate polynomial $g$ is zero over the 2-dim real domain $D$, it must be identically zero. Thus its coefficients $1 + p_0, p_1, \ldots, p_n$ must be all zero.

By instantiating the condition (1) with $x = U^n$ and recalling the definition of $U'$, we have

$$\forall_{L,U} \ 0 < L \leq U \implies U + \frac{U^n + q_0 U^n + q_1 U^{n-1} L + \cdots + q_n L^n}{q_{n+1} U^{n-1} + q_{n+2} U^{n-2} L + \cdots + q_{2n} L^{n-1}} = U.$$ 

By simplifying, removing the denominator and collecting, we have

$$\forall_{L,U} \ (L, U) \in D \implies h(L, U) = 0,$$ 

where

$$D = \{(L, U) : 0 < L \leq U\},$$

$$h(L, U) = (1 + q_0) U^n + q_1 U^{n-1} L + \cdots + q_n L^n.$$ 

Since the bivariate polynomial $h$ is zero over the 2-dim real domain $D$, it must be identically zero. Thus its coefficients $1 + q_0, q_1, \ldots, q_n$ must be all zero. □

Lemma 3. Let $R_{p,q}$ be a contracting $n^{th}$ degree map. Then we have

$$L' = L + \frac{x - L^n}{p_{n+1} L^{n-1} + p_{n+2} L^{n-2} U + \cdots + p_{2n} U^{n-1}},$$

$$U' = U + \frac{x - U^n}{q_{n+1} U^{n-1} + q_{n+2} U^{n-2} L + \cdots + q_{2n} L^{n-1}}.$$ 

Proof. Let $R_{p,q}$ be a contracting $n^{th}$ degree map. From Lemma 2 we have

$$p_0 = -1 \land p_1 = \cdots = p_n = 0,$$

$$q_0 = -1 \land q_1 = \cdots = q_n = 0.$$
Recalling the definition of $L'$ and $U'$, we have

$$L' = L + \frac{x - L^n}{p_{n+1}L^{n-1} + p_{n+2}L^{n-2}U + \ldots + p_{2n}U^{n-1}}$$

$$U' = U + \frac{x - U^n}{q_{n+1}U^{n-1} + q_{n+2}U^{n-2}L + \ldots + q_{2n}L^{n-1}}$$

Lemma 4. Let $R_{p,q}$ be a contracting $n^{th}$ degree map. Then we have

$$\forall_{L,U,x} \ 0 < L \leq \sqrt[n]{x} \leq U \implies L' \leq L'$$

Proof. Let $R_{p,q}$ be a contracting $n^{th}$ degree map. Then we have

$$\forall_{L,U,x} \ 0 < L \leq \sqrt[n]{x} \leq U \implies L' \leq \sqrt[n]{x}$$

From Lemma 3 we have

$$\forall_{L,U,x} \ 0 < L \leq \sqrt[n]{x} \leq U \implies L + \frac{x - L^n}{p_{n+1}L^{n-1} + p_{n+2}L^{n-2}U + \ldots + p_{2n}U^{n-1}} \leq \sqrt[n]{x}$$

By considering only the case $L < \sqrt[n]{x}$, we have

$$\forall_{L,U,x} \ 0 < L \leq \sqrt[n]{x} \leq U \implies \sqrt[n]{x} - L \leq \sqrt[n]{x} - L$$

Since $\sqrt[n]{x^{n-1}} + \sqrt[n]{x^{n-2}}L + \ldots + \sqrt[n]{x} > 0$ for $0 < L < \sqrt[n]{x}$, we have

$$\forall_{L,U,x} \ 0 < L \leq \sqrt[n]{x} \leq U \implies \frac{1}{\sqrt[n]{x^{n-1}} + \sqrt[n]{x^{n-2}}L + \ldots + \sqrt[n]{x}} \leq \frac{1}{\sqrt[n]{x^{n-1}} + \sqrt[n]{x^{n-2}}L + \ldots + \sqrt[n]{x}}$$

By considering only the case $\sqrt[n]{x} = U$, we have

$$\forall_{L,U} \ 0 < L \leq U \implies \frac{1}{p_{n+1}L^{n-1} + p_{n+2}L^{n-2}U + \ldots + p_{2n}U^{n-1}} \leq \frac{1}{U^{n-1} + U^{n-2}L + \ldots + L^{n-1}}$$

Since $x - L^n \geq 0$ for $L \leq \sqrt[n]{x}$, we have

$$\forall_{L,U} \ 0 < L \leq \sqrt[n]{x} \land L < U \implies \frac{x - L^n}{p_{n+1}L^{n-1} + p_{n+2}L^{n-2}U + \ldots + p_{2n}U^{n-1}} \leq \frac{x - L^n}{U^{n-1} + U^{n-2}L + \ldots + L^{n-1}}$$

Since $x - L^n = 0$ when $L = U$, we have

$$\forall_{L,U} \ 0 < L \leq \sqrt[n]{x} \implies \frac{x - L^n}{p_{n+1}L^{n-1} + p_{n+2}L^{n-2}U + \ldots + p_{2n}U^{n-1}} \leq \frac{x - L^n}{U^{n-1} + U^{n-2}L + \ldots + L^{n-1}}$$

By adding $L$ on both sides, we have

$$\forall_{L,U} \ 0 < L \leq \sqrt[n]{x} \implies L + \frac{x - L^n}{p_{n+1}L^{n-1} + p_{n+2}L^{n-2}U + \ldots + p_{2n}U^{n-1}} \leq L + \frac{x - L^n}{L^{n-1} + U^{n-2}L + \ldots + L^{n-1}}$$

Thus

$$\forall_{L,U} \ 0 < L \leq \sqrt[n]{x} \implies L' \leq L'$$

\qed
Lemma 5. If
\[ \forall_{L,U,x} \ 0 < L \leq \sqrt[n]{x} < U \implies A \geq B \]
then
\[ \forall_{L,U} \ 0 < L < U \implies C \geq B \]
where
\[
A = \frac{1}{U^{n-1} + U^{n-2} \sqrt[n]{x} + \ldots + \sqrt[n]{x}^{n-1}},
\]
\[
B = \frac{n+1}{q_{n+1} U^{n-1} + q_{n+2} U^{n-2} L + \ldots + q_{2n} L^{n-1}},
\]
\[
C = \frac{1}{n U^{n-1}}.
\]

Proof. Assume
\[ \forall_{L,U,x} \ 0 < L \leq \sqrt[n]{x} < U \implies A \geq B. \] (2)
We need to show
\[ \forall_{L,U} \ 0 < L < U \implies C \geq B. \]
Let \( L, U \) be arbitrary but fixed such that \( 0 < L < U \). We need to prove that \( C \geq B \).
We will prove by contradiction, and thus assume \( C < B \). In order to derive a contradiction, we will try to find \( \sqrt[n]{x} \) such that \( 0 < L \leq \sqrt[n]{x} < U \) is true but \( A \geq B \) is false, which contradicts the assumption (2).
Let \( B - C = d \). If \( A - C < d \) then \( A \geq B \) is false. Thus it suffices to find \( \sqrt[n]{x} \) such that \( 0 < L \leq \sqrt[n]{x} < U \) and \( A - C < d \), that is, \( f(\sqrt[n]{x}) < 0 \) where
\[
f(z) = \frac{1}{U^{n-1} + U^{n-2} z + \ldots + z^{n-1}} - \frac{1}{n U^{n-1}} - d.
\]
We consider two cases:
Case 1: \( f(L) < 0 \). Let \( \sqrt[n]{x} = L \). Trivially \( f(\sqrt[n]{x}) < 0 \).
Case 2: \( f(L) \geq 0 \). It is obvious that \( f \) is continuous and monotonically decreasing over \([L, U]\). It is also obvious that \( f(U) = -d < 0 \). Hence there exists a unique real root \( \alpha \) of \( f \) in \([L, U]\). Let \( \sqrt[n]{x} = \frac{L + U}{2} \). Then clearly \( f(\sqrt[n]{x}) < 0 \).
Thus we have derived the desired contradiction in both cases. Hence \( C \geq B \) and the lemma is proved. \( \blacksquare \)
**Lemma 6.** Let $R_{p,q}$ be a contracting $n^{th}$ degree map. Then we have
\[
\forall_{L,U,x} \quad 0 < L \leq \sqrt[n]{x} \leq U \implies U^* \leq U'.
\]

**Proof.** Let $R_{p,q}$ be a contracting $n^{th}$ degree map. Then we have
\[
\forall_{L,U,x} \quad 0 < L \leq \sqrt[n]{x} \leq U \implies \sqrt[n]{x} \leq U'.
\]

From Lemma 5 we have
\[
\forall_{L,U,x} \quad 0 < L \leq \sqrt[n]{x} \leq U \implies \sqrt[n]{x} \leq U + \frac{x - U^n}{q_{n+1}U^{n-1} + q_{n+2}U^{n-2}L + \ldots + q_{2n}L^{n-1}}
\]
\[
\forall_{L,U,x} \quad 0 < L \leq \sqrt[n]{x} \leq U \implies U - \sqrt[n]{x} \geq \frac{q_{n+1}U^{n-1} + q_{n+2}U^{n-2}L + \ldots + q_{2n}L^{n-1}}{U^n - x}
\]
\[
\forall_{L,U,x} \quad 0 < L \leq \sqrt[n]{x} \leq U \implies U - \sqrt[n]{x} \geq \frac{(U - \sqrt[n]{x}) (U^{n-1} + U^{n-2} \sqrt[n]{x} + \ldots + \sqrt[n]{x}^{n-1})}{q_{n+1}U^{n-1} + q_{n+2}U^{n-2}L + \ldots + q_{2n}L^{n-1}}
\]

By considering only the case $\sqrt[n]{x} < U$, we have
\[
\forall_{L,U,x} \quad 0 < L \leq \sqrt[n]{x} < U \implies 1 \geq \frac{U^{n-1} + U^{n-2} \sqrt[n]{x} + \ldots + \sqrt[n]{x}^{n-1}}{q_{n+1}U^{n-1} + q_{n+2}U^{n-2}L + \ldots + q_{2n}L^{n-1}}
\]

Since $U^{n-1} + U^{n-2} \sqrt[n]{x} + \ldots + \sqrt[n]{x}^{n-1} > 0$ for $0 < L \leq \sqrt[n]{x} < U$, we have
\[
\forall_{L,U,x} \quad 0 < L \leq \sqrt[n]{x} < U \implies \frac{1}{U^{n-1} + U^{n-2} \sqrt[n]{x} + \ldots + \sqrt[n]{x}^{n-1}} \geq \frac{1}{q_{n+1}U^{n-1} + q_{n+2}U^{n-2}L + \ldots + q_{2n}L^{n-1}}
\]

By Lemma 5 we have
\[
\forall_{L,U} \quad 0 < L < U \implies \frac{1}{nU^{n-1}} \geq \frac{1}{q_{n+1}U^{n-1} + q_{n+2}U^{n-2}L + \ldots + q_{2n}L^{n-1}}
\]

Since $x - U^n \leq 0$ for $\sqrt[n]{x} \leq U$, we have
\[
\forall_{L,U,x} \quad 0 < L \leq \sqrt[n]{x} < U \wedge L < U \implies \frac{x - U^n}{nU^{n-1}} \leq \frac{x - U^n}{q_{n+1}U^{n-1} + q_{n+2}U^{n-2}L + \ldots + q_{2n}L^{n-1}}
\]

Since $x - U^n = 0$ when $L = U$, we have
\[
\forall_{L,U,x} \quad 0 < L \leq \sqrt[n]{x} \leq U \implies \frac{x - U^n}{nU^{n-1}} \leq \frac{x - U^n}{q_{n+1}U^{n-1} + q_{n+2}U^{n-2}L + \ldots + q_{2n}L^{n-1}}
\]

By adding $U$ on both sides, we have
\[
\forall_{L,U,x} \quad 0 < L \leq \sqrt[n]{x} \leq U \implies U + \frac{x - U^n}{nU^{n-1}} \leq U + \frac{x - U^n}{q_{n+1}U^{n-1} + q_{n+2}U^{n-2}L + \ldots + q_{2n}L^{n-1}}
\]

Thus
\[
\forall_{L,U,x} \quad 0 < L \leq \sqrt[n]{x} \leq U \implies U^* \leq U'.
\]

Now we are ready to prove the two claims in Main Theorem. The following lemma (Lemma 7) will prove the claim (a) and the subsequent lemma (Lemma 8) will prove the claim (b).
Lemma 7 (Main Theorem (a)). Let $R_{p,q}$ be a contracting $n^{th}$ degree map which is not $R_{p^*,q^*}$ (Secant-Newton). Then we have
\[ \forall (L,U,x) \quad 0 < L \leq \sqrt[n]{x} \leq U \implies R_{p^*,q^*}([L,U],x) \subseteq R_{p,q}([L,U],x). \]

Proof. Let $R_{p,q}$ be a contracting $n^{th}$ degree map which is not $R_{p^*,q^*}$ (Secant-Newton), that is, $p \neq p^*$ or $q \neq q^*$. Let $L, U, x$ be arbitrary but fixed such that $0 < L \leq \sqrt[n]{x} \leq U$. From Lemmas 8 and 9 we have
\[ L' \leq L^* \land U^* \leq U'. \]
Hence $R_{p^*,q^*}([L,U],x) \subseteq R_{p,q}([L,U],x)$. Main Theorem (a) has been proved.

Lemma 8 (Main Theorem (b)). Let $R_{p,q}$ be a contracting $n^{th}$ degree map which is not $R_{p^*,q^*}$ (Secant-Newton). Then we have
\[ \forall (L,U,x) \quad 0 < L \leq \sqrt[n]{x} \leq U \implies R_{p^*,q^*}([L,U],x) \subseteq R_{p,q}([L,U],x). \]

Proof. Let $R_{p,q}$ be a contracting $n^{th}$ degree map which is not $R_{p^*,q^*}$ (Secant-Newton), that is, $p \neq p^*$ or $q \neq q^*$. We need to show
\[ \forall (L,U,x) \quad 0 < L \leq \sqrt[n]{x} \leq U \implies R_{p^*,q^*}([L,U],x) \subseteq R_{p,q}([L,U],x). \]
It suffices to find a non-zero polynomial $f$ in the variables $L, U, x$ such that
\[ \forall (L,U,x) \neq 0 \quad 0 < L \leq \sqrt[n]{x} \leq U \implies R_{p^*,q^*}([L,U],x) \subseteq R_{p,q}([L,U],x) \]
since the solution set of $f(L,U,x) = 0$ has measure zero. We consider two cases:

Case 1: $p \neq p^*$. Let
\[ f = (x - L^n) \left( (p_{n+1} + 1)L^{n-1} + (p_{n+2} - 1)L^{n-2}U + \cdots + (p_{2n} - 1)U^{n-1} \right). \]
Note that $f$ is a non-zero polynomial. Let $L, U, x$ be such that $f(L,U,x) \neq 0$ and $0 < L \leq \sqrt[n]{x} \leq U$. We need to show that $R_{p^*,q^*}([L,U],x) \subseteq R_{p,q}([L,U],x)$. From Lemma 8 we already have $R_{p^*,q^*}([L,U],x) \subseteq R_{p,q}([L,U],x)$. Thus it suffices to show that
\[ R_{p^*,q^*}([L,U],x) \neq R_{p,q}([L,U],x). \]
Note
\[ f(L,U,x) \neq 0 \]

\[ \implies (x - L^n) \left( (p_{n+1} + 1)L^{n-1} + (p_{n+2} - 1)L^{n-2}U + \cdots + (p_{2n} - 1)U^{n-1} \right) \neq 0 \]

\[ \implies (x - L^n) \left( (p_{n+1}L^{n-1} + p_{n+2}L^{n-2}U + \cdots + p_{2n}U^{n-1}) - (L^{n-1} + L^{n-2}U + \cdots + U^{n-1}) \right) \neq 0 \]

\[ \implies \frac{x - L^n}{L_{n+1}L^{n-1} + p_{n+2}L^{n-2}U + \cdots + p_{2n}U^{n-1}} \neq \frac{x - L^n}{L^{n-1} + L^{n-2}U + \cdots + U^{n-1}} \]

\[ \implies L' \neq L^* \]

\[ \implies R_{p^*,q^*}([L,U],x) \neq R_{p,q}([L,U],x). \]
Case 2: $q \neq q^*$. Let

$$f = (x - U^n) \left( (q_{n+1} - n) U^{n-1} + q_{n+2} U^{n-2} L + \cdots + q_{2n} L^{n-1} \right).$$

Note that $f$ is a non-zero polynomial. Let $L, U, x$ be such that $f(L, U, x) \neq 0$ and $0 < L \leq \sqrt[n]{x} \leq U$. We need to show that $R_{p^*, q^*}([L, U], x) \subsetneq R_{p, q}([L, U], x)$. From Lemma 7, we already have $R_{p^*, q^*}([L, U], x) \subseteq R_{p, q}([L, U], x)$. Thus it suffices to show that

$$R_{p^*, q^*}([L, U], x) \neq R_{p, q}([L, U], x).$$

Note

$$f(L, U, x) \neq 0 \implies (x - U^n) \left( (q_{n+1} - n) U^{n-1} + q_{n+2} U^{n-2} L + \cdots + q_{2n} L^{n-1} \right) \neq 0 \implies (x - U^n) \left( (q_{n+1} U^{n-1} + q_{n+2} U^{n-2} L + \cdots + q_{2n} L^{n-1}) - n U^{n-1} \right) \neq 0 \implies \frac{x - U^n}{n U^{n-1}} \neq \frac{x - U^n}{U^n} \implies U' \neq U^* \implies R_{p^*, q^*}([L, U], x) \neq R_{p, q}([L, U], x).$$

Main Theorem (b) has been proved. \(\square\)

4 Conclusion

In this paper we extended a previous work on the optimal square root computation by Erascu-Hong [6] to arbitrary $n$th root computation. The contributions are as follows.

- We proved that the well known Secant-Newton refinement map is “optimal” among its natural generalizations, that is, among the maps that are contracting and are certain rational functions.

- We found that the precise notion of the “optimality” for the square-root case in [6] could not be extended straightforwardly to the $n$th root case. It had to be modified in a subtle but crucial way.

- Furthermore, we found that the proof techniques used in [6] could not be straightforwardly extended. In fact, it turns out that only a small part of the proof technique could be straightforwardly generalized. However, the rest of the proof could not be generalized. Thus, we developed several new proof techniques.

This work motivates several interesting further questions.

- What about dropping the condition “contracting”? The Secant-Newton map is a particular instance of interval Newton map with slope where $m$ is chosen to be $U$. If one chooses a different $m$ value (from $U$), then the interval Newton map with slope is not contracting. In practice, one remedies this by intersecting the result of the map with $[L, U]$ before the next iteration. This trivially ensures that the resulting map is contracting. This motivates a larger family of maps where a map is defined as a quadratic map composed with intersection with $[L, U]$. One asks what the optimal map is among the larger family of maps.
What about broadening the scope of this problem? The problem tackled in this paper could be recast as follows: given a positive number $x$, find the positive real root of the polynomial equation $f(y) = y^n - x$, using an interval refinement map. This motivates the following natural generalization: find a real root of an arbitrary polynomial equation in a given interval, using an interval refinement map. Again, one could ask what the optimal refinement map is among a naturally chosen family of maps.

We leave them as open problems/challenges for future research.

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