AN ORIENTED COMPETITION MODEL ON $\mathbb{Z}^2_+$. 

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Abstract. We consider a two-type oriented competition model on the first quadrant of the two-dimensional integer lattice. Each vertex of the space may contain only one particle of either Red type or Blue type. A vertex flips to the color of a randomly chosen southwest nearest neighbor at exponential rate 2. At time zero there is one Red particle located at (1, 0) and one Blue particle located at (0, 1). The main result is a partial shape theorem: Denote by $R(t)$ and $B(t)$ the red and blue regions at time $t$. Then (i) eventually the upper half of the unit square contains no points of $B(t)/t$, and the lower half no points of $R(t)/t$; and (ii) with positive probability there are angular sectors rooted at (1, 1) that are eventually either red or blue. The second result is contingent on the uniform curvature of the boundary of the corresponding Richardson shape.

Key words: competition, shape theorem, first passage percolation.

1. Introduction.

In this paper we study a model where two species Red and Blue compete for space on the first quadrant of $\mathbb{Z}^2$. At time $t > 0$ every vertex of $\mathbb{Z}^2$ is in one of the three possible states: vacant, occupied by a Red particle, or occupied by a Blue particle. An unoccupied vertex $z = (x, y)$ may be colonized from either $(x, y - 1)$ or $(x - 1, y)$ at rate equal to the number of occupied south-west neighbors; at the instant of first colonization, the vertex flips to the color of a randomly chosen occupied south-west neighbor. Once occupied, a vertex remains occupied forever, but its color may flip: the flip rate is equal to the number of south-west neighbors occupied by particles of the opposite color. The state of the system at any time $t$ is given by the pair $R(t), B(t)$ where $R(t)$ and $B(t)$ denote the set of sites occupied by Red and Blue particles respectively. The set $R(t) \cup B(t)$ evolves precisely as the occupied set in the oriented Richardson model, and thus, for any initial configuration with only finitely many occupied sites, the growth of this set is governed by the Shape Theorem, which states that the set of occupied vertices scaled by time converges to a deterministic set $S$ (see for example

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A rigorous construction and more detailed description of the oriented competition model is given in Section 2.2. The simplest interesting initial configuration has a single Red particle at the vertex \((1, 0)\), a single Blue particle at \((0, 1)\), and all other sites unoccupied. We shall refer to this as the default initial configuration. When the oriented competition process is started in the default initial configuration, the red and blue particles at \((1, 0)\) and \((0, 1)\) are protected: their colors can never be flipped. Thus, both colors survive forever w.p.1. Computer simulations for the oriented competition model started in the default and other finite initial configurations suggest that the shapes of the regions occupied by the Red and Blue types stabilize as times goes to infinity – see Figure 1 for snapshots of two different realizations of the model, each started from the default initial configuration. A peculiar feature of the stabilization is that the limit shapes of the red and blue regions are partly deterministic and partly random: The southeast corner of the occupied region is always equally divided between the red and blue populations, with boundary lying along the line \(y = x\). However, the outer section seems to stabilize in a random union of angular wedges rooted at a point near the center of the Richardson shape. Although the location of the root appears to be deterministic, both the number and angles of the outer red and blue regions vary quite dramatically from one simulation to the next.

The purpose of this paper is to prove that stabilization of Red and Blue zones occurs with positive probability. (We conjecture that in fact it occurs with probability 1, but we have been unable to prove this.) To state our result precisely, we shall need several facts about the limit shape \(S\) of the oriented Richardson model \(Z(t) := R(t) \cup B(t)\). The proof of the Shape
Theorem 2 shows that $S$ is a compact, convex subset of the first quadrant of $\mathbb{R}^2$. It is generally believed – but has not been proved – that the outer boundary $\partial o S$ of $S$ (the portion of $\partial S$ that lies in the interior of the first quadrant) is uniformly curved, that is, for every point $x$ in this part of the boundary there is a circle of finite radius passing through $x$ that contains $S$ in its interior. In section 3 we shall prove the following.

**Lemma 1.** The Richardson shape $S$ has the points $(1,0)$ and $(0,1)$ on its boundary, and the point $(1,1)$ in its interior.

It will follow by convexity that the unit square $Q = [0,1]^2$ lies entirely in $S$. Define $Q_1$ and $Q_2$ to be the subsets of $Q$ that lie (strictly) above and below the main diagonal $x = y$.

For any subset $Z \subset \mathbb{R}^2$, define

$$\hat{Z} = \{x \in \mathbb{R}^2 : \text{dist}(x, Z) \leq 1/2\},$$

where dist denotes distance in the $L^\infty$-norm on $\mathbb{R}^2$. For any set $Z \subset \mathbb{R}^2$ and any scalar $s > 0$, let $Z/s = \{y/s : y \in Z\}$.

**Theorem 1.** With probability one, for all large $t$

$$Q_1 \subset \hat{R}(t)/t \quad \text{and} \quad Q_2 \subset \hat{B}(t)/t.$$

Furthermore, if the outer boundary $\partial o S$ of the Richardson limit shape $S$ is uniformly curved, then for every $\epsilon > 0$ the following holds with positive probability: There exist random angular sectors $A_1, \ldots, A_n$ rooted at $(1,1)$ that do not intersect the open unit square $Q^o$ such that

(a) eventually $A_i$ is either Red or Blue, and
(b) the complement of $\bigcup A_i$ in $S \setminus Q$ has angular measure less than $\epsilon$.

Another competition model on $\mathbb{Z}^d$ (non-oriented version) was studied in [5]. It was shown that if the process starts with finitely many particles of both types (Red and Blue), then the two types coexist with positive probability under the condition that the shape set of the corresponding non-oriented Richardson model is uniformly curved. The behavior of the oriented model differs from that of the model considered in [5] in that the limit shape contains the deterministic component $(1)$.  

2. **Preliminaries**

2.1. **Graphical Constructions.** The competition model, the Richardson model, and the competition model in a hostile environment may be built using the same percolation structure $\Pi$. For details on percolation structures see [3]. Here we briefly describe the construction of $\Pi$. To each directed edge $xy$ such that $x \in \mathbb{Z}^2_+ \setminus \{(0,0)\}$, and $y = x + (0,1)$ or $y = x + (1,0)$
is assigned rate-1 Poisson process. The Poisson processes are mutually independent. Above each vertex $x$ is drawn a timeline, on which are placed marks at the occurrence times $T_i^{xy}$ of the Poisson processes attached to directed edges emanating from $x$; at each such mark, an arrow is drawn from $x$ to $y$. A directed path through the percolation structure $\Pi$ may travel upward, at speed 1, along any timeline, and may (but does not have to) jump across any outward-pointing arrow that it encounters. A reverse path is a directed path run backward in time: thus, it moves downward along timelines and jumps across inward-pointing arrows. A voter-admissible path is a directed path that does not pass any inward-pointing arrows. Observe that for each vertex $z$ and each time $t > 0$ there is a unique voter-admissible path beginning at time $0$ and terminating at $(z, t)$: its reverse path is gotten by traveling downward along timelines, starting at $(z, t)$, jumping across all inward-pointing arrows encountered along the way.

For each $(z, t)$ denote by $\Gamma(z, t)$ the collection of reverse paths on percolation structure $\Pi$ originating at $(z, t)$ and terminating in $(\mathbb{Z}^2_+, 0)$. We also use $\Gamma(z, t)$ to denote the set of ends of all paths in the collection. There exists a unique reverse voter-admissible path $\tilde{\gamma}(z, t) = \tilde{\gamma}$ in $\Gamma(z, t)$. We say that a path $\gamma$ has attached end, or $\gamma$ is attached, if it terminates in $(R(0) \cup B(0), 0)$. For $s \in [0, t]$ denote by $\gamma(s)$ the location of the path in $\mathbb{Z}^2_+$ at time $t - s$, i.e. $\gamma(s) = z'$ if $(z', t - s) \in \gamma$. We can now put an order relation on $\Gamma(z, t)$ as follows. For two reverse paths $\gamma_1, \gamma_2 \in \Gamma(z, t)$, let $\tau_i = \inf\{s > 0 : \gamma_i(s) = \tilde{\gamma}(s)\}$, $i = 1, 2$, and set $\gamma_1 \prec \gamma_2$ if $\tau_1 \leq \tau_2$. The order relation sets a priority on assigning an ancestor. A vertex $z$ is occupied by a particle at time $t$ if and only if there is at least one attached reverse path originating at $(z, t)$. The set of terminating points of all attached paths in $\Gamma(z, t)$ is referred as the set of potential ancestors of the particle at $(z, t)$. The maximal element $\tilde{\gamma}$ in the set of attached paths uniquely determines the ancestor. Let $(z', 0)$ be the terminating point of $\tilde{\gamma}$. Then the particle at $(z', 0)$ is said to be the ancestor of the particle at $(z, t)$. Note that $(z, t)$ is vacant if and only if the set of attached paths is empty.

2.2. The simplest oriented growth model. Denote by $Z(t)$ the set of vertices occupied by time $t$, and fix an initial configuration $Z(0) = \{(0, 1), (1, 0)\}$. The Richardson model can be built using percolation structure as follows. Set $Z(t)$ to be the set of vertices $z$ in $\mathbb{Z}_+^2$ such that there is a directed path in $\Pi$ that starts at $(Z(0), 0)$ and terminates at $(z, t)$. For $z \in \mathbb{R}_+^2$ let $T(z) = \inf\{t : z \in \hat{Z}(t)\}$, and let $\mu(z) = \lim_{n \to \infty} n^{-1} T(nz)$. The limit exists almost surely by subadditivity. The growth of $\hat{Z}(t)$ is governed by a Shape Theorem. A weakened version of the standard Shape theorem may be obtained by using subadditivity arguments. The problem with a standard version of the Shape theorem was that $\mu(z)$ was not known to be continuous on the boundaries of $\mathbb{R}_+^2$. In [6] J. Martin showed that $\mu(z)$
is continuous on all of $\mathbb{R}^2_+$, and established the Shape theorem. Furthermore, large deviations results for the Richardson model follow from papers by Kesten [4] and Alexander [1].

**Theorem 2.** There exists a non-random compact convex subset $S$ of $\mathbb{R}^2_+$ such that for $\alpha \in (1/2, 1)$, constants $c_1, c_2 > 0$ (depending on $\alpha$) and all $t > 0$

$$P[S(t - t^\alpha) \subset \hat{Z}(t) \subset S(t + t^\alpha)] > 1 - c_1 t^2 \exp\{-c_2 t^{(\alpha-1/2)}\}.$$ 

Let $\hat{S}$ be the limit set of the South-West oriented Richardson model. This process starts with two particles at the vertices $(-1, 0)$ and $(0, -1)$, and lives in the third quadrant of $\mathbb{Z}^2$. It is easy to see that $\hat{S} = -S$. For $\epsilon > 0$ define the cone $K_\epsilon$ rooted at $(1, 1)$ by

$$K_\epsilon = \{z \in \mathbb{R}^2 : \arg\{z - (1, 1)\} \in (-\pi/2 + \epsilon, \pi - \epsilon)\}.$$ 

The following lemma follows from an elementary geometric argument. The proof is identical to the proof of Lemma 4 in [7].

**Lemma 2.** Suppose that $\partial^* S$ is uniformly curved. For every $\epsilon > 0$ and $\alpha \in (1/2, 1)$ there exists $c > 0$ so that if $z \in \partial S \cap K_\epsilon$, then for all $t_1, t_2 > 0$ we have

$$S(t_1 + t_1^\alpha) \cap (z(t_1 + t_2) + \hat{S}(t_2 + t_2^\alpha)) \subset D(z t_1, c(t_1 + t_2)^{(\alpha+1)/2}).$$

3. Growth and competition in hostile environment.

Suppose that at time zero every vertex of $\mathbb{Z}^2_+$ except the origin contains a particle. There are two distinguished particles located at $(1, 0)$ and $(0, 1)$, say Black particles. All other vertices are occupied by White particles. Every vertex flips to the color of a randomly chosen south-west nearest neighbor with exponential rate $2$. Thus, at time $t$ the color of a vertex $z$ is uniquely determined by its voter-admissible path. The set of Black particles $Q(t)$ is defined to be the set of all vertices $z$ such that the unique reverse voter-admissible path beginning at $(z, t)$ terminates at $\{(1, 0), (0, 1)\}$. Note that every vertex $(z_1, 0)$ on the horizontal coordinate axes and every vertex $(0, z_2)$ on the vertical coordinate axes eventually flips to Black color and stays Black forever. Thus, almost surely for all large $t$ vertex $(z_1, z_2)$ is Black. By subadditivity, a shape theorem should hold for the growth model. Computer simulations of the growth model suggest that the shape set is a square (see the first picture on Figure 3). Below it is shown that the limit shape is exactly $Q$.

**Proposition 1.** For every $\alpha \in (1/2, 1)$ there exist $c_1, c_2$ such that for all $t > 0$

$$P[Q(t - t^\alpha) \subset \hat{Q}(t) \subset Q(t + t^\alpha)] > 1 - c_1 t^2 \exp\{-c_2 t^{(\alpha-1/2)}\}.$$
Proof. Recall that for every \( t > 0 \) and \( z \in \mathbb{Z}^2 \), there exists a unique reverse voter-admissible path \( \tilde{\gamma}(z,t) \) starting at \((z,t)\). The path travel downward, at rate 1, and jumps across all inward-pointing arrows. Until the path hits the horizontal (vertical) axis the number of horizontal (vertical) jumps is distributed as Poisson process with parameter 1. Thus, there exist constants \( c_1 \) and \( c_2 \) such that for every \( z \in Q(t-t^\alpha) \)

\[
P(\tilde{\gamma}(z,t) \text{ terminates in } \{(1,0),(0,1)\}) \geq 1 - c_1 \exp\{-c_2 t^{(\alpha-1/2)}\}.
\]

For the same reason, there exist constants \( c_1 \) and \( c_2 \) such that for every \( z \in Q^\alpha(t+t^\alpha) \),

\[
P(\tilde{\gamma}(z,t) \text{ terminates in } \{(1,0),(0,1)\}) \leq c_1 \exp\{-c_2 t^{(\alpha-1/2)}\}.
\]

The proposition follows from the fact that the number of vertices in \( Q(t-t^\alpha) \) is of order at most \( O(t^2) \) and the number of vertices on the boundary of \( Q(t+t^\alpha) \) is of order at most \( O(t) \). \( \square \)

If the growth models \( Q(t) \) and \( S(t) \) are coupled on the same percolation structure \( \Pi \), then clearly \( Q(t) \subseteq S(t) \), and thus \( Q \subseteq S \). Lemma 4 asserts that \( S \) is strictly larger than \( Q \).

Proof of Lemma 4. The following argument was communicated to the authors by Yuval Peres. We consider a representation of the Richardson model as a first passage percolation model. To each edge of the lattice associate a mean one exponential random variable, also called a passage time of the edge. The variables are mutually independent. For every pair of vertices \( z_1 = (x_1,y_1), z_2 = (x_2,y_2) \) such that \( x_1 \leq x_2 \) and \( y_1 \leq y_2 \) define the passage time \( T(z_1,z_2) \) from \( z_1 \) to \( z_2 \) as the infimum over traversal times of all North-East oriented paths from \( z_1 \) to \( z_2 \). The traversal time of an oriented path is the sum of the passage times of its edges. In the first passage percolation description of the Richardson model, let

\[
Z(t) = \{ z \in \mathbb{Z}^2 : T((1,0),z) \leq t \text{ or } T((0,1),z) \leq t \}.
\]

It is enough to show that for some \( \epsilon > 0 \), the vertex \((1,1)\) is in \((1-\epsilon)S\). Consider a sequence of vertices \( z_n = (n,n) \) on the main diagonal of the first quadrant of \( \mathbb{Z}^2 \). By the shape theorem, it suffices to prove that almost surely for infinitely many \( n \)'s the occupation times of \( z_n \) satisfy \( T(z_n) \leq n(1-\epsilon) \).

Consider vertices \((0,2),(2,0)\), and \((1,1)\). There are exactly four oriented distinct paths from the origin to these vertices. Each such path has two edges and expected passage time equal to 2. Let \( \gamma_{(1)} \) be the path with the smallest passage time among these four paths. Denote by \( X_1 \) the terminal point of \( \gamma_{(1)} \), and denote its passage time by \( T_1 \). By symmetry \( P(X_1 = (0,2)) = P(X_1 = (2,0)) = 1/4 \) and \( P(X_1 = (1,2)) = 1/2 \). It easy to see that \( ET_1 < 1 \). Indeed, let \( \gamma_0 \) be the path obtained by the following procedure. Start at the origin and make two oriented steps each time moving
in the direction of the edge with minimal passage time (either north or east). Clearly

\[ ET_1 < E\tau(\gamma_0) = 1 \]

where \( \tau(\gamma_0) \) is the total passage time of \( \gamma_0 \). Restart at \( X_1 \) and repeat the procedure. Denote by \( X_2 \) the displacement on the second step and by \( T_2 \) the passage time of the time minimizing path from \( X_1 \) to \( X_1 + X_2 \). Note that \( W_k = \sum_{k=1}^{\infty} X_k \) is a random walk on \( \mathbb{Z}_+^2 \). The random walk visits the main diagonal infinitely often in such a way that \( W_k = (k, k) \). Furthermore, if \( S_k = \sum_{k=1}^{\infty} T_k \), then by SLLN for some \( \epsilon > 0 \) almost surely for all large \( k \) we have \( S_k \leq (1 - \epsilon)k \). This finishes the proof.

Suppose now that at time zero there is one Red particle at \((1, 0)\) and one Blue particle at \((0, 1)\). As in the growth model, all other vertices are occupied by White particles. The flip rules are the same as in the growth model: every vertex flips to the color of a randomly chosen south-west nearest neighbor with exponential rate 2. At time \( t > 0 \) the color of a vertex \( z \) is uniquely determined by its voter-admissible path. The Red cluster \( R(1)(t) \) and the Blue cluster \( B(1)(t) \) are defined to be the sets of all vertices \( z \) such that the unique reverse voter-admissible path beginning at \((z, t)\) terminates respectively at \((1, 0)\) and \((0, 1)\). For \( t > 0 \) define

\[
K_1(t^\alpha) = \{ z = (z_1, z_2) \in \mathbb{Z}_+^2 : z_1 - z_2 > t^\alpha \},
\]

\[
K_2(t^\alpha) = \{ z = (z_1, z_2) \in \mathbb{Z}_+^2 : z_2 - z_1 > t^\alpha \}.
\]

**Proposition 2.** For every \( \alpha \in (1/2, 1) \) there exist \( c_1, c_2 > 0 \) such that for all \( t > 0 \)

\[
P[Q(t - t^\alpha) \cap K_1(t^\alpha) \subset \hat{R}^{(1)}(t)] > 1 - c_1 t^2 \exp\{-c_2 t^{(\alpha-1)/2}\},
\]
The result (2) immediately follows from the above observations.

\[ P[Q(t - t^\alpha) \cap K_2(t^\alpha) \subset \hat{B}(t)] > 1 - c_1 t^2 \exp\{-c_2 t^{(\alpha-1/2)}\}. \]

**Proof.** We only show (2), since the proof of (3) is identical. First, observe that by Proposition 1 there exist \( c_1, c_2 > 0 \)

\[ P[Q(t - t^\alpha) \subset \hat{R}(t) \cup \hat{B}(t)] > 1 - c_1 t^2 \exp\{-c_2 t^{(\alpha-1/2)}\}. \]

Second, note that with probability exponentially close to one voter admissible paths of all vertices \( z \in Q(t - t^\alpha) \cap K_1(t^\alpha) \cap \mathbb{Z}_+^2 \) terminate below main diagonal. That is, there exist \( c_1, c_2 > 0 \) such that

\[ P(\hat{\gamma}_{z,t}(t) \in K_1(0)) \geq 1 - c_1 \exp\{-c_2 t^{(\alpha-1/2)}\}. \]

The result (2) immediately follows from the above observations. \( \square \)

4. Oriented Competition model. Proof of Theorem 1

If the competition model and the competition model in hostile environment are constructed on the same percolation structure \( \Pi \), then almost surely for all \( t > 0 \)

\[ R^{(1)}(t) \subseteq R(t), \ B^{(1)}(t) \subseteq B(t). \]

Hence it follows from Proposition 2 that almost surely for all large \( t \)

\[ Q(t - t^\alpha) \cap K_1(t^\alpha) \subset \hat{R}(t), \]

\[ Q(t - t^\alpha) \cap K_2(t^\alpha) \subset \hat{B}(t). \]

Thus, asymptotically (as \( t \) goes to infinity) the square \( Q \subset \mathcal{S} \) is colored deterministically. In particular, the region below the main diagonal is red, and the region above the diagonal is blue. This proves the first part of Theorem 1.

The next question is what happens in the region \( \mathcal{S} \setminus Q \). For each \( z \in (\partial\mathcal{S}) \setminus Q \) and any \( g > 0 \), define the angular sector \( \mathcal{A}(z; g) \subset \mathcal{S} \setminus Q \) of angular measure \( g \) rooted at \((1,1)\) and centered at \( z \) by

\[ \mathcal{A}(z; g) := \{y \in \mathcal{S} : |\arg\{y - (1,1)\} - \arg\{z - (1,1)\}| < g/2\}. \]

Fix \( \epsilon > 0, \alpha \in (1/2,1) \) and \( \beta \in (1/2,1) \) such that \((\alpha + 1)/2 < \beta\). For \( g > 0 \) and \( t \geq 1 \), let \( A_1 \subset A_2 \) be angular sectors with common center \( z \) and angular measures \( r < r + t^{\beta - 1} \), respectively, and such that \( A_2 \subset K_t \). Define by \( A_1^s \) and \( A_2^s \) the complements of the sectors in \( \mathcal{S} \setminus Q \). Fix \( \delta \in (0,1) \), and set

\[ \mathcal{R}_0 = \mathcal{R}_0^t = A_2(t - t^\alpha), \]
\[ B_0 = B_0^t = A_2^s(t + t^\alpha), \]
\[ B_1 = B_1^t = A_1(t(1 + \delta) + (t(1 + \delta))^{\alpha}), \]
\[ \mathcal{R}_1 = \mathcal{R}_1^t = A_1(t(1 + \delta) - (t(1 + \delta))^{\alpha}). \]
Lemma 3. There exist constants $c_1, c_2 > 0$ such that the following is true, for any $t \geq 1$. If the initial configuration $\xi, \zeta$ is such that $\hat{\xi} \supset R_\alpha^t$ and $\hat{\zeta} \subset B_0^t$, then

\begin{equation}
1 - P_{\xi, \zeta} [\hat{\mathcal{B}}(\delta t) \subset B_1^t] \leq c_1 t^2 \exp\{-c_2 (\delta t)^{\alpha - 1/2}\}.
\end{equation}

Lemma 3 implies that once an angular segment is occupied by one of the two types, it must remain so (except near its boundary) for a substantial amount of time. Thus, Theorem 1 immediately follows from Lemma 3 and Theorem 2. For more details, see analogous construction in [5] (Section 4.3, pg. 14-15).

Let $a = (1,1) \in S$, be the right upper corner vertex of $Q$, and let $b$ be a point on the boundary of $S$ such that $b \in K_\alpha$. For $r, q \in \mathbb{R}$ denote by $I(r, q)$ an interval with ends at $r$ and $q$ and by $L(r, q)$ a line segment starting at $r$ and passing through $q$. Let $b(\delta)$ be a point in the interval $I(a, b)$ such that $|b' - a| = |b - a|/(1 + \delta)$ where $| \cdot |$ is an Euclidean norm. For a point $r \in \mathbb{R}_+$ let $\hat{\tau}$ be the nearest vertex with integer coordinates. That is, $\text{dist}(r, \hat{\tau}) \leq 1/2$ (if there is more than one such vertex, choose the vertex with the smallest coordinates).

Suppose that at time zero the initial configuration $R(0), B(0)$ is such that $\hat{\mathcal{R}}(0) \cup \hat{\mathcal{B}}(0) \approx \mathcal{S}t$ for large $t > 0$. Then, by the shape theorem, $\hat{\mathcal{R}}(\delta t) \cup \hat{\mathcal{B}}(\delta t) \approx \mathcal{S}t(1 + \delta)$. Consider the line segment $L(at(1 + \delta), bt(1 + \delta))$ starting at $at(1 + \delta)$ and passing through $bt(1 + \delta)$. Fix a point $r \in L(a, b)$ such that $rt(1 + \delta) \in R(\delta t) \cup B(\delta t)$. Note that $rt(1 + \delta) \in L(at(1 + \delta), bt(1 + \delta))$.

In Claims 1, 2 and 3 below, it is shown that if $\partial^\alpha S$ is uniformly curved, then with probability exponentially close to one the ancestor of $rt(1 + \delta)$ (if exists) is in the $(\delta t)^\beta$ neighborhood of $I(at, bt)$ for some $\beta \in (3/4, 1)$. In particular if $r \in I(b', b)$, then the ancestor of $rt(1 + \delta)$ (if exists) is in the $(\delta t)^\beta$ neighborhood of $bt$. Observe that this implies the statement of the Lemma 3.

Consider three cases:

1. $r \in I(a, b')$;
2. $r \in I(b', b)$ and $rt(1 + \delta) \in \mathcal{S}(t(1 + \delta) - (t(1 + \delta))^\alpha)$;
3. $rt(1 + \delta) \in \mathcal{S}(t(1 + \delta) + (t(1 + \delta))^\alpha) \setminus \mathcal{S}(t(1 + \delta) - (t(1 + \delta))^\alpha)$.

The Claims 1, 2 and 3 deal with the three cases respectively.

Claim 1. There exist constants $c_1, c_2 > 0$ such that for every $b \in \partial S \cap K_\alpha$ and for every $r \in I(a, b')$, if the initial configuration $R(0), B(0)$ is such that $\mathcal{S}(t - t^\alpha) \subset \hat{\mathcal{R}}(0) \cup \hat{\mathcal{B}}(0) \subset \mathcal{S}(t + t^\alpha)$ then with probability at least $1 - c_1 \exp\{-c_2 (\delta t)^{(\alpha - \frac{1}{2})}\}$ the ancestor of $r(1 + \delta)t$ exists and is located in the $(\delta t)^\alpha$ neighborhood of $r(1 + \delta)t - a\delta t \in I(at, bt)$. 
Proof. Recall that the voter admissible path is a continuous time random walk with exponential waiting times between jumps and drift $-a$. By standard large deviations results, with probability exponentially close to one the voter admissible reverse path $\tilde{\gamma} \in \Gamma(r(1+\delta)t, \delta t)$ is attached to a vertex in the disk of radius $(\delta t)^\alpha$ centered at $(rt(1+\delta) - a\delta t) \in I(at, bt)$. That is, for some constants $c_1, c_2 > 0$,

$$P[|\tilde{\gamma}(t) - (rt(1+\delta) - a\delta t)| > (\delta t)^\alpha] \leq c_1 \exp\{-c_2(\delta t)^{(\alpha - \frac{1}{2})}\}.$$ 

\[\square\]

**Claim 2.** There exist constants $c_1, c_2 > 0$ such that for every $b \in \partial S \cap K_t$ and for every $r \in I(\beta', b)$ with $rt(1+\delta) \in S(t(1+\delta) - (t(1+\delta))^\alpha)$ if the initial configuration $R(0), B(0)$ is such that $S(t - t^\alpha) \subset \hat{R}(0) \cup \hat{B}(0) \subset S(t + t^\alpha)$, then with probability at least $1 - c_1 \exp\{-c_2(\delta t)^{(\alpha - \frac{1}{2})}\}$ the ancestor of $r(1+\delta)t$ exists and is located in the $(\delta t)^\beta$ neighborhood of $bt$.

**Proof.** The heuristics of the proof are as follows. For $t_1 \in (0, \delta t)$ consider a subset $\Gamma_1(rt(1+\delta), r\delta t)$ of the set of reverse paths $\Gamma(rt(1+\delta), r\delta t)$ that contains only those paths that coincide with the reverse voter admissible path $\tilde{\gamma}$ on $\mathbb{Z}^2 \times (\delta t - t_1, \delta t)$. That is, for every $\gamma \in \Gamma_1(rt(1+\delta), r\delta t)$, for all $0 < s < t_1$, $\gamma(s) = \tilde{\gamma}(s)$ The set of ends of $\Gamma_1(rt(1+\delta), r\delta t)$ is obtained by constructing reverse oriented Richardson process on the subset $\mathbb{Z}^2_+ \times (0, t_2)$ of the percolation structure. The process starts with one occupied vertex at $\tilde{\gamma}(t_1)$, and runs backward in time for $t_2 = \delta t - t_1$ units of time. By making an appropriate choice of $t_1$ and $t_2$, we show that with probability exponentially close to one the ancestor vertex of $(rt(1+\delta), r\delta t)$ exists and is located in the $(\delta t)^\beta$ neighborhood of $bt$. Denote by

$$\kappa = \frac{|r - b|}{|\beta' - b|}.$$ 

Consider $L(0, b(1+\delta)t)$, a line in $\mathbb{R}^2$ connecting the origin $0$ and the point $b(1+\delta)t$. There exists a unique point $r' \in L(0, b(1+\delta)t)$ between $bt$ and $b(1+\delta)t$ such that

$$rt(1+\delta) - r' = (\kappa \delta t) a$$

$$\mu(r' - bt) = (1 - \kappa)\delta t.$$ 

Set $t_1 = \kappa \delta t - (\delta t)^\alpha$. Note that on the percolation structure, if we start at $(rt(1+\delta), \delta t)$, and follow the reverse voter admissible path for $t_1$ units of time, then with probability exponentially close to one the end of the path is located in an Euclidean disk with center at $r' + (\delta t)^\alpha a$ and radius $(\delta t)^\alpha \epsilon_1/4$ where $\epsilon_1$ is chosen so that $a(1 + \epsilon_1) \in S$. That is, for some constants $c_1, c_2 > 0$,

$$P[|\tilde{\gamma}(t_1) - (r' + (\delta t)^\alpha a)| > (\delta t)^\alpha \epsilon_1/4] \leq c_1 \exp\{-c_2(\delta t)^{(\alpha - \frac{1}{2})}\}.$$ 

Observe also that if we start a reverse oriented Richardson process (i.e. South-West oriented Richardson process) from any vertex $z$ in the $(\delta t)^\alpha \epsilon_1/4$
AN ORIENTED COMPETITION MODEL ON $\mathbb{Z}_+^2$.

neighborhood of $r' + (\delta t)^\alpha a$, and run it backward in time for $t_2 = (1 - \kappa)\delta t + (\delta t)^\alpha$ units of time, then

\begin{equation}
(5) \quad P[\Gamma(z, t_2) \cap (R(0) \cup B(0)) = \emptyset] < c_1 \exp\{-c_2(\delta t)^{(\alpha - \frac{1}{2})}\}.
\end{equation}

Indeed, since

\begin{equation}
S(t - t^\alpha) \cap (z + \hat{S}(t_2 - (\delta t)^\alpha \epsilon_1/4)) \neq \emptyset,
\end{equation}

(5) follows by by Theorem 2.

Also, by Theorem 2 and by Lemma 2,

\begin{equation}
P[\Gamma(z, t_2) \cap (R(0) \cup B(0)) \not\subset D(bt, (\delta t)^\beta)] < c_1 \exp\{-c_2(\delta t)^{(\alpha - \frac{1}{2})}\}.
\end{equation}

Thus, with probability exponentially close to one the intersection of the set $\Gamma_1(rt(1 + \delta t), \delta t)$ with $R(0) \cup B(0)$ is non-empty and belongs to a disk of radius $(\delta t)^\beta$ and center at $bt$. \hfill \Box

**Claim 3.** There exist constants $c_1, c_2 > 0$ such that for every $b \in \partial S \cap K_\epsilon$ and for every $r \in L(b', b)$ with $rt(1 + \delta) \in S(t(1 + \delta) + (t(1 + \delta))^\alpha) \setminus S(t(1 + \delta) - (t(1 + \delta))^\alpha)$, if the initial configuration $R(0), B(0)$ is such that $S(t - t^\alpha) \subset \hat{R}(0) \cup \hat{B}(0) \subset S(t + t^\alpha)$, then with probability at least $1 - c_1 \exp\{-c_2(\delta t)^{(\alpha - \frac{1}{2})}\}$ the set of potential ancestors of $\hat{r}(1 + \delta)t$ is either empty or it is contained in the $(\delta t)^\beta$ neighborhood of $bt$.

**Proof.** If the ancestor of $rt(1 + \delta)$ exists, it is located in the set of ends of $\Gamma(rt(1 + \delta), \delta t)$. The set of ends of $\Gamma(rt(1 + \delta), \delta t)$ is obtained by constructing reverse oriented Richardson process starting with one occupied vertex at $rt(1 + \delta)$, and running the process on the subset $\mathbb{Z}_+^2 \times (0, \delta t)$ of the percolation structure backward in time for $\delta t$ units of time. Then by Theorem 2 and Lemma 2

\begin{equation}
P[\Gamma(rt(1 + \delta), \delta t) \cap (R(0) \cup B(0)) \not\subset D(bt, (\delta t)^\beta)] \leq c_1 \exp\{-c_2(\delta t)^{(\alpha - \frac{1}{2})}\}.
\end{equation}

\hfill \Box

The Claims 1, 2 and 3 imply the statement of Lemma 3. This finishes the proof of Theorem 1.

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