ON HAMILTONIAN FLOWS ON EULER-TYPE EQUATIONS

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Abstract. Properties of Hamiltonian symmetry flows on hyperbolic Euler-type Liouvillean equations $\mathcal{E}_{\text{EL}}'$ are analyzed. Description of their Noether symmetries assigned to the integrals for these equations is obtained. The integrals provide Miura transformations from $\mathcal{E}_{\text{EL}}'$ to the multi-component wave equations $\mathcal{E}$. By using these substitutions, we generate an infinite-Hamiltonian commutative subalgebra $\mathfrak{A}$ of local Noether symmetry flows on $\mathcal{E}$ proliferated by weakly nonlocal recursion operators. We demonstrate that the correlation between the Magri schemes for $\mathfrak{A}$ and for the induced “modified” Hamiltonian flows $\mathfrak{B} \subset \text{sym} \mathcal{E}'_{\text{EL}}$ is such that these properties are transferred to $\mathfrak{B}$ and the recursions for $\mathcal{E}'_{\text{EL}}$ are factorized. Two examples associated with the 2D Toda lattice are considered.

Introduction. In this paper, we consider the problem of constructing pairs of commutative hierarchies of Hamiltonian evolution equations related by Miura-type transformations and identified with Lie subalgebras of the Noether symmetry algebras for Euler–Lagrange-type systems. By using two standard schemes ([3, 15, 16]), which are the Miura substitutions defined by the integrals of Liouvillean hyperbolic equations and construction of the second Hamiltonian structure by a Miura transformation, we restrict the exposition to the class of the Euler–Lagrange Liouvillean hyperbolic systems $\mathcal{E}'_{\text{EL}}$. We obtain an explicit description of their Noether symmetries assigned to the integrals for $\mathcal{E}'_{\text{EL}}$; these flows are Hamiltonian w.r.t. the (first) structures derived from the Lagrangians. Then we analyze the properties of the Magri schemes for two hierarchies: $\mathfrak{A}$, which is composed by symmetries of the wave equation, and $\mathfrak{B} \subset \text{sym} \mathcal{E}'_{\text{EL}}$, correlated by the Miura maps.
Two examples are discussed. First, we relate the Korteweg–de Vries equation

\begin{align}
    s_t &= -\beta s_{xxx} + \frac{3}{2} s_x^2, \\
    w_t &= -\beta w_{xxx} + 3ww_x, \quad w = s_x, \quad \beta = \text{const},
\end{align}

and multi-component modified KdV equations (see [14]) with the wave equation and the two-dimensional Toda lattice (2DTL, in particular, associated with a semisimple Lie algebra, see [4]), respectively, and factorize the Lenard recursion operator for Eq. (1) to a product of two vector-valued operators. Second, we demonstrate that the Boussinesq equation

\begin{align}
    \begin{cases}
        u_t &= \frac{1}{3} v_{xxx} + \frac{4}{3} v_x^2, \\
        v_t &= u_x
    \end{cases}
    \quad \begin{cases}
        U_t &= V_x, \\
        V_t &= \frac{1}{3} U_{xxx} + \frac{8}{3} UU_x
    \end{cases}
    \quad \begin{cases}
        U &= v_x, \\
        V &= u_x
    \end{cases}
\end{align}

and the modified Boussinesq equations ([6]), as well as the Hamiltonian structures for their local commutative hierarchies, are obtained from the geometry of the ambient wave and 2DTL equations, respectively.

The relationship between the Hamiltonian and Lagrangian approaches towards integrable evolution equations was discussed in [17], where Lagrangian representations were derived by using the Legendre transform from the sequence of Hamiltonian functionals for an evolution equation at hand. Therefore, that concept was closed w.r.t. evolutionary systems and the Miura-type transformations between them, whichever representation it might be. Our approach is opposite to [17]. Indeed, we interpret (bi-)Hamiltonian hierarchies of evolution equations as flows defined by subalgebras of the Noether symmetry algebras for ambient hyperbolic Euler-type systems (see also [9]). By using the canonical variables for the Euler-type systems and treating the differential constraint between coordinates \( u \) and momenta \( m \) as the rule that defines the Clebsch potentials, we establish the relation between the potential and nonpotential components of the hierarchies that describe the evolution of coordinates and momenta, respectively. The adjoint linearization \((U^*_m)^*\) of this constraint defines the first Hamiltonian structures for these hierarchies such that their Magri schemes are correlated; the second Hamiltonian structures obtained from the Miura transformations supply the recursion operators both for the evolutionary hierarchies and the ambient Euler-type systems.

The paper is organized as follows. In Sec. 1, we relate the differential constraint between coordinates and momenta for hyperbolic Euler equations with the Hamiltonian operators for their Noether symmetry algebras. Then we consider substitutions from Euler-type Liouvillean
equations $\mathcal{E}_\text{EL}$ generated by their integrals; in this case, the Miura transformation to a subalgebra $\mathfrak{A}$ of the Noether symmetries of the Euler wave equation $\mathcal{E}$ generates the second Hamiltonian structure on $\mathfrak{A}$. Thus we obtain the pair $\mathfrak{A}$, $\mathfrak{B}$ of sequences of Hamiltonian flows on $\mathcal{E}$ and $\mathcal{E}_\text{EL}$, respectively, which are correlated by the Miura map. Then we discuss properties of the flows within $\mathfrak{A}$ and $\mathfrak{B}$ and the corresponding recursions. Finally, we demonstrate which of these properties for $\mathfrak{A}$ such as the locality, commutativity, (bi-)Hamiltonianity, etc., are transferred to $\mathfrak{B}$.

In Sec. 2, we recall necessary facts about geometry of the 2DTL $u_{xy} = \exp(Ku)$; all notions and notation follow [1]. We consider the hierarchy of KdV equation (1) and the sequence of multi-component analogues ([14]) of the modified KdV equations associated with the 2DTL. We obtain a factorization of the Lenard recursion operator for KdV to a product of vector-valued operators; also, we prove that the higher flows for the multi-component mKdV are local and pairwise commute.

In Sec. 3, we demonstrate that the hierarchy $\mathfrak{B}$ of higher flows for modified Boussinesq equation (8) is a local commutative subalgebra of Noether symmetries of 2DTL (9) associated with the algebra $\mathfrak{sl}_3(\mathbb{C})$ such that integrals (7) for Eq. (9) induce the second Hamiltonian structure (10) for Eq. (2); a factorization of the recursion operator for 2DTL is obtained.

1. Hierarchies and Miura transformations

Consider a first-order Lagrangian $\mathcal{L} = [L(u, u_x, u_y; x, y) \, dx \wedge dy]$ with the density $L = -\frac{1}{2} \sum_{i,j} \bar{\kappa}_{ij} u_i^x u_j^y + H(u; x, y)$, where $\bar{\kappa}$ is a real, nondegenerate, symmetric, constant $(r \times r)$-matrix. Choose the variable $y$ for the “time” coordinate, leave $x$ for the spatial coordinate, and denote by $m_j = \partial L/\partial u_j^y$ the $j$th conjugate coordinate (momentum) for the $j$th dependent variable $u_j$ for any $1 \leq j \leq r$:

$$m_j = -\frac{1}{2} (\bar{\kappa} u_x)^*.$$  \hfill (3)

We claim that the adjoint inverse linearization $D_x^{-1} \circ \bar{\kappa}^{-1}$ of differential constraint (2) between the coordinates and momenta is a Hamiltonian structure for the hyperbolic Euler equation $\mathcal{E}_\text{EL} = \{ E_u(\mathcal{L}) = 0 \}$ itself and also for the algebra $\text{sym} \, \mathcal{L}$ of Noether symmetries for $\mathcal{E}_\text{EL}$. Consider the Legendre transform $H \, dx \wedge dy = \langle m, \partial L/\partial u_y \rangle - \mathcal{L}$ and hence assign the Hamiltonian $\mathcal{H}(u, m) = [H \, dx]$ to the Lagrangian $\mathcal{L}$. The hyperbolic Euler equation $\mathcal{E}_\text{EL}$ is equivalent to the system $u_y = 1 \cdot E_m(\mathcal{H})$, $m_y = -1 \cdot E_u(\mathcal{H})$ that involves the canonical Hamiltonian structure $\left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$. Owing to the relations $\frac{1}{2} E_m = \bar{\kappa}^{-1} \cdot D_x^{-1} \cdot E_u$,.
\[ E_u = \frac{1}{2} D_x \circ \bar{\kappa} \cdot E_m, \] the dynamical equations are separated. Indeed, they are evolutionary: \( u_y = A_1 \circ E_u([H[u] \, dx]) \), \( m_y = -\frac{1}{2} \bar{\kappa} \circ E_m([H[m] \, dx]) \), that is, they are obtained by using the pair of mutually inverse Hamiltonian operators \( A_1 = \bar{\kappa}^{-1} \cdot D_x^{-1} \) and \( \bar{A}_1 = D_x \cdot \bar{\kappa} \).

Consider a hyperbolic Euler equation \( \mathcal{E}_{EL} \) and suppose that it admits a symmetry flow \( u_t = \phi(u_x, \ldots) \equiv \phi(u_x) \); here \( t \) is a parameter along the integral trajectories. Then the evolution \( m_t \) of the momenta is described by the nonpotential equation \( m_t = -\frac{1}{2} \bar{\kappa} \cdot D_x \{ \phi(m) \} \); see Eq. (11) for an example of symmetry flows on the wave equation \( s_{xy} = 0 \).

**Lemma 1** \((11)\). Let \( \mathcal{E} = \{ u_{xy} = f(u; x, y) \} \) be a quasilinear hyperbolic equation and \( \mathcal{H} = H \, dx + \ldots \) be its conservation law: \( d_u(\mathcal{H}) = \nabla(u_{xy} - f) \). Then the generating section \( \psi \equiv \nabla^*(1) \) of the conservation law \( \mathcal{H} \) is \( \psi = -D_x(E_u(H \, dx)) \).

By the Noether theorem \((13)\), the Noether symmetries for self-adjoint Euler systems \( \mathcal{E}_{EL} = \{ E_u(\mathcal{L}) = 0 \} \) coincide with the generating sections of their conservation laws; by \( [13] \), the correlation between the Noether symmetries \( \varphi_\mathcal{L} \) and the generating sections \( \psi \) for nonselfadjoint Euler equations \( \mathcal{E}_{EL} \approx \{ \bar{\kappa}^{-1} \cdot E_u(\mathcal{L}) = 0 \} \) with a unit symbol is \( \psi = \bar{\kappa} \varphi_\mathcal{L} \).

Let \( \phi \) be a Noether flow on an Euler-type hyperbolic system: \( u_t = \frac{1}{2} E_m(\mathcal{H}), m_t = -\frac{1}{2} E_u(\mathcal{H}), \) where \( \mathcal{H} = [H \, dx] \) and \( H \) is a conserved density. Hence we have \( u_t = \bar{\kappa}^{-1} \cdot D_x^{-1} E_u(\mathcal{H}), m_t = -\frac{1}{2} D_x \cdot \bar{\kappa} E_m(\mathcal{H}), \) i.e., both equations \( u_t = \phi \) and \( m_t = -\frac{1}{2} \bar{\kappa} D_x \{ \phi \} \) are Hamiltonian simultaneously and their Hamiltonian operators \( A_1 \) and \( \bar{A}_1 \) are mutually inverse (e.g., \([10]\)). The induced evolution of momenta \( m \) can be calculated in two distinct ways: by variation of the Hamiltonian, \( m_t = -E_u(\mathcal{H}) = -\ell^u_m(\mathcal{H}) \), or by using relation \( [3] \) explicitly: \( m_t = \ell^u_m(\mathcal{E}_m(\mathcal{H})) \). Two evolutions are correlated for hyperbolic Euler equations since the condition \( \ell^u_m = -\ell^u_m \) holds. Further, let \( R \) be a recursion for the hyperbolic Euler equation \( \mathcal{E}_{EL} = \{ E_u(\mathcal{L}) = 0 \} \) such that \( R \) generates the sequence \( u_{ti} = \phi_i[u_x] \) composed by its commuting Hamiltonian symmetries originating from some \( \phi_{-1} \). The recursion \( R \) is shared by all equations \( u_{ti} = \phi_i \); the operator \( R^* \) maps the velocities \( m_{ti} \) of evolution of the momenta (see \([12]\)).

From now on, we investigate a particular class of Liouvillean hyperbolic Euler equations \([3] [4] [7]\) which admit nontrivial integrals, that is, functionals \( w \in \ker D_y \) belonging to the kernel of the total derivative \( D_y \) restricted onto the equation \( \mathcal{E}'_{EL} \) at hand. Our reasonings hold up to the involution \( x \leftrightarrow y \) if it is a symmetry of \( \mathcal{E}'_{EL} \), otherwise the exposition admits the mirror copy with \( x \) and \( y \) replaced.
Suppose that the integrals \( w[\bar{m}] \) depend on the momenta \( \bar{m} \) for a Liouvillean Euler-type equation. Then the equation itself and its symmetry algebra are mapped by the Miura transformation \( m = w[\bar{m}] \) to the multi-component wave equation \( E \) and its symmetry algebra, respectively. The number of components within the wave equation equals the number of the integrals \( w \) involved in the Miura map. Also, two other mappings are well defined: the covariant mapping \( w_* : \text{sym} E'_{EL} \rightarrow \text{sym} E \) of the symmetry flows (see Proposition 4 in [3]) and the contravariant mapping \( w^* \) of the Hamiltonians \( H \, dx \) (and hence, of the Hamiltonian flows). Indeed, by the transformation \( m = w[\bar{m}] \), the variational derivatives w.r.t. the momenta \( m \) and the modified momenta \( \bar{m} \) are correlated by the rule \( E_{\bar{m}} = \Box \circ E_m \), where \( \Box = (\ell_{\bar{m}} m)^* \). Hence from Lemma 1 we deduce

**Proposition 1.** The adjoint linearization \( \Box = (\ell_{\bar{m}} m)^* \) of the integrals \( w \) for a Liouvillean Euler-type equation \( E'_{EL} = \{ E_u(L') = 0 \} \) w.r.t. its momenta \( \bar{m} \) factors its Noether symmetries: \( \phi_{L'} = \Box(E_m(H[m])) \), where \( H \) is arbitrary.

Examples are found in [13, 14]. Recently, the correlation between the structure of generators of the symmetry algebra for Liouville and hyperbolic systems and their integrals was analyzed in [2].

Consider the substitution \( m = w[\bar{m}] \) between an Euler-type Liouvillean system \( E'_{EL} \) and the multi-component wave equation \( E \). The pair \( (w^*, w_*) \) generates the second Hamiltonian structure \( \hat{A}_2 \) for Noether symmetries of the target equation \( E \), see [16]. Namely, let \( \mathfrak{A} \) be a sequence of Noether flows on \( E \) such that \( H_i[m] \) are the (Hamiltonian) conserved densities for the first structures \( ((\ell_{\bar{m}} m)^*)^{\pm1} \). Hence for any Hamiltonian flow \( \phi_{i-1} = E_m(H_i) \) on \( E \) we obtain the flow \( u_i = \phi_{i-1} \) on \( E'_{EL} \). Now use the condition \( (\ell_{\bar{m}} m)^* = -\ell_{\bar{m}} m \), hence the evolution of the modified momenta \( \bar{m} \) is well defined. The evolution \( \Theta \Box(\phi_{i-1})(w) \) of the substitution \( w \) along the flow \( \phi_i \in \mathfrak{A} \) is correlated with the flow \( \phi_i \) that succeeds \( \phi_{i-1} \) in \( \mathfrak{A} \) if the second Hamiltonian structure \( \hat{A}_2 \) in \( \mathfrak{A} \) satisfies the operator equation

\[
\ell_{\bar{m}} m \circ (\ell_{\bar{m}} m)^* \circ (\ell_{\bar{m}} m)^* = \hat{A}_2;
\]

and conversely, the operator \( \hat{A}_2 \) defined in [4] is always a Hamiltonian structure for \( \mathfrak{A} \). In other words, Eq. (4) is the condition of reducibility of the structure \( \hat{A}_2 \) to the canonical form “d/dx”. Condition (4) specifies the set of admissible Miura transformations \( w = w[\bar{m}] \) and modified Euler equations \( E'_{EL} \) for a given \( \hat{A}_2 \).

**Remark 1.** If \( \hat{A}_2 \) is compatible with \( \hat{A}_1 \), then \( \mathfrak{A} \) is commutative and therefore the recursion \( R = (\hat{A}_2 \circ \hat{A}_1^{-1})^* \) for \( E \) is a recursion for \( \mathfrak{A} \).
Thence, the bi-Hamiltonian hierarchy $\mathfrak{A}$ is infinite-Hamiltonian. Namely, let $\varepsilon = \left\{ u_t = \hat{A}_1(\psi[u]) \right\}$ be a Hamiltonian equation and let $R$ be a recursion for $\varepsilon$; in our case, the operator $R$ proliferates the commuting symmetry flows on the wave equation $E$. Then $\varepsilon$ is also Hamiltonian w.r.t. the operators $\hat{A}_i = R^{-1}\hat{A}_1$ provided that they are skew-adjoint; it is indeed so for Eq. (4). The proof is based on the following argument. By [12, 18], the relation $\ell_\varepsilon A + A\ell_{\varepsilon}^* = 0$ holds for Hamiltonian operators $A$ for $\varepsilon$; the converse is also true under regularity assumptions. Since $R$ is a recursion for $\varepsilon$, we have $\ell_\varepsilon R = R\ell_\varepsilon$. Indeed, from the operator equality $\ell_\varepsilon R = \bar{R}\ell_\varepsilon$ that holds for some $\bar{R}$, we obtain $\bar{R} = R$ by commuting this equality with $t$. Therefore, the identity $\ell_\varepsilon(RA) + (RA)\ell_{\varepsilon}^* = 0$ holds and thus the operator $RA$ is also Hamiltonian. Now proceed by induction. These reasonings are rigorous for those operators $\hat{A}_i$ which are differential (i.e., local w.r.t. $D_x$), otherwise the above relations must be extended to a nonlocal setting. The operators $\hat{A}_i$ are not always compatible; a pair of Hamiltonian differential operators may generate a nonlocal sequence $\mathfrak{A}$. Further, we discuss the locality aspects in more details.

Consider the sequence $\mathfrak{B} \subset \text{sym} \mathcal{E}_{EL}'$ of Noether symmetries $\varphi_i$ proliferated by the recursion $R' = \Box \circ A_1 \circ \Box^* \circ \ell_m^*$ such that the integrals $w$ define the Miura map $\mathfrak{B} \rightarrow \mathfrak{A}$ to a bi-Hamiltonian sequence of Noether symmetries of the multi-component wave equation $E$. The sequences $\mathfrak{A}$ and $\mathfrak{B}$ share the Hamiltonians, and the second Hamiltonian structure $\hat{A}_2$ for $\mathfrak{A}$ is induced by the first Hamiltonian structure $\hat{B}_1 = (\ell_m^*)^*$ for $\mathfrak{B}$ according to the following diagram (5):

![Diagram](image-url)

The Magri schemes (15) for $\mathfrak{A}$ and $\mathfrak{B}$ and factorization (4), see [16], of $\hat{A}_2$ are indeed standard. Now we see that the first Hamiltonian structures $\hat{A}_1$ and $\hat{B}_1$ originate from the Lagrangians for the ambient Euler-type equations and the operator $\Box$ is constructed by using the integrals of the Liouvillean systems $\mathcal{E}_{EL}'$. 

(5)
If the Hamiltonian operators $\hat{A}_1 = (\ell^*_m)$ and $\hat{A}_2$, see (4), are compatible, then the functionals $\mathcal{H}_i[m]$ are in involution w.r.t. both structures, the flows $\phi_i$ commute ([13]), and $\mathfrak{A}$ is infinite-Hamiltonian by Remark 1.

We also have

**Proposition 2.** Let a recursion $R$ for an evolution equation $\varepsilon = \{u_i = \phi[u_i]\}$ be constructed by using a single layer of Abelian nonlocal variables (that is, expressions which are nonlocal w.r.t. the variables in $\varepsilon$ and which are produced by a set of conservation laws $\eta_\alpha$ for $\varepsilon$, see [1]).

Then $R$ is weakly nonlocal: $R = \text{differential part} + \sum_{\alpha} \varphi_\alpha D_x^{-1} \circ \psi_\alpha$, where $\varphi_\alpha \in \text{sym} \varepsilon$ and $\psi_\alpha$ is the generating section of the conservation law $\eta_\alpha$ for any $\alpha$.

The proof of Proposition 2 follows from the definitions; it is generalized easily to the case of multiple layers of the Abelian nonlocal variables defined by using (nonlocal) conservation laws. Hence we conclude that the recursion $R^*: \Phi_i \mapsto \Phi_{i+1}$ for the hierarchy $\mathfrak{A}$ is weakly nonlocal owing to the factorization $R^* = \hat{A}_2 \circ \hat{A}_1^{-1}$ and Eq. (4).

**Proposition 3.** If the operators $\hat{A}_1$ and $\hat{A}_2$ are compatible, then the flows $\phi_i \in \text{sym} \varepsilon$ are Noether.

Indeed, the homological formulation (e.g., [12]) of the Magri scheme and the triviality ([11]) of the Poisson cohomologies w.r.t. the structure $\hat{A}_1$ guarantee the existence of the conservation laws $\mathcal{H}_i$ for $\varepsilon$; now refer Lemma 1 and the Noether theorem ([8, 13]). We also conclude that $\mathfrak{A}$ is local in $w$.

Further, we demonstrate which of the above properties can be transferred on $\mathfrak{B}$. Obviously, $\mathfrak{A}$ and $\mathfrak{B}$ are local simultaneously. The restrictions of the mappings $B_2: \psi_i \mapsto \varphi_{i+1}$ and $\hat{B}_2: \varphi_i \mapsto \psi_{i+1}$ onto the functionals $H[w] dx$ that depend on $\bar{u}$ through the substitution $w = w[m]$ are Hamiltonian operators, and the functionals $\mathcal{H}_i[w[m]]$ are in involution w.r.t. $\hat{B}_2$ since it is so for $\hat{A}_3$; from Remark 1 we see why the Jacobi identity holds for the operators $B_2$ and $\hat{B}_2$ if the Hamiltonians $\mathcal{H}$ depend on $\bar{u}$ arbitrarily.

**Proposition 4.** Let the above notation and assumptions hold. If the flows $\phi_i \in \mathfrak{A}$ commute and $\ker \hat{A}_2 = 0$, then the sequence $\mathfrak{B}$ is also commutative.

**Proof.** By Proposition 1 every Noether symmetry of a Liouvillean Euler hyperbolic equation $\mathcal{E}'_{EL}$ factors by the operator $\Box$ (up to the involution $x \leftrightarrow y$ if it is a symmetry of $\mathcal{E}'_{EL}$). The commutator of two Noether symmetries $\Box(\phi'), \Box(\phi'')$ is a Noether symmetry again:
\{\Box(\phi'), \Box(\phi'')\} = \Box(\phi_{\{1,2\}}), \text{ although the structure of } \phi_{\{1,2\}} \text{ may contain addends which are unusual w.r.t. the Jacobi bracket of } \phi' \text{ and } \phi'', \text{ see Eq. (3) below. From Eq. (3) it follows that } \mathcal{E}\Box(\phi_{\{1,2\}})(w) = \hat{A}_2(\phi_{\{1,2\}}). \text{ Hence if the operator } A_2 \text{ is an arbitrary real, constant, nondegenerate } (r \times r)\text{-matrix and the symmetry condition } \kappa_{ij} \equiv a_i k_{ij} = \kappa_{ji} \text{ holds for elements of the matrix } \kappa = ||\kappa_{ij}||. \text{ Then the matrix } K \text{ is symmetrizable; denote its inverse by } K^{-1} = ||k_{ij}||. \text{ The } r\text{-component two-dimensional Toda lattice associated with the nondegenerate symmetric matrix } K \text{ is } \mathcal{E}_{\text{Toda}} = \{u_{xy} = \exp(Ku)\}. \text{ The density } L_{\text{Toda}} \text{ of its Lagrangian } L_{\text{Toda}} = -\frac{1}{2}\langle u_y, \kappa u_x \rangle - \langle \tilde{a}, \exp(Ku) \rangle; \text{ denote by } \vartheta = \kappa u_x \text{ the momenta obtained from } \mathcal{E}_{\text{Toda}}. \text{ The component } w = \frac{1}{2}\langle u_x, \kappa u_x \rangle - \langle \tilde{a}, u_{xx} \rangle \text{ of the energy–momentum tensor for } \mathcal{E}_{\text{Toda}} \text{ vanishes w.r.t. the restriction } \hat{D}_y \text{ of the total derivative } D_y \text{ on } \mathcal{E}_{\text{Toda}}: \hat{D}_y(w) = 0. \text{ The conformal weights } \tilde{\Delta} = \{\Delta^1, \ldots, \Delta^r\} \text{ of the fields } \exp(u) \text{ are } \Delta^i = \sum_{j=1}^r k^{ij}. \text{ Consider the case when } w \text{ and its differential consequences } W \text{ are unique solutions which depend on the derivatives of } u \text{ to the equation } \hat{D}_y(\Omega) = 0; \text{ then we say that } K \text{ is generic.} \text{ The Noether symmetries of } \mathcal{E}_{\text{Toda}} \text{ associated with a generic symmetrizable } (r \times r)\text{-matrix } K \text{ are } \varphi = \Box(\mathbf{E}_w(H(x,W)\,dx)) \text{ up to the transformation } x \leftrightarrow y \text{ (13), see also [5]). Here the vector–valued,}
first-order differential operator $\Box$ is $\Box = (\ell^i w_i)^* = u_x + \Delta D_x$ and $H$ is a smooth function. The operator $R_{\text{Toda}} = \Box \circ \ell_s$, where $s_x = w$ and $s_{xy} = 0$, is a nonlocal recursion operator for the algebra sym $E_{\text{Toda}}$ (3).

Set $\phi_{-1} = 1$ and generate three symmetry sequences for $i \geq 0$: $\varphi_i = \Box (\phi_{i-1}) \in \text{sym } E_{\text{Toda}}$, $\phi_i = R_{\text{pKdV}} (\phi_{i-1}) \in \text{sym } E_{\text{pKdV}}$, and $\Phi_i = \mathcal{E}_{\phi_i}(w) \in \text{sym } E_{\text{KdV}}$, where the KdV equations are (1) and $\beta \equiv \langle \vec{u}, \Delta \rangle$. The evolutions $\Phi_i$ and $\phi_i$ are elements of the local commutative bi–Hamiltonian hierarchies for the (p)KdV equations $w_{t_i} = \Phi_1$ and $s_{t_i} = \phi_1$, respectively. The equation $u_{t_1} = \varphi_1 = \Box (\phi_0)$ is an $r$-component analogue of the potential modified KdV equation, being a Noether symmetry of the Toda lattice $E_{\text{Toda}}$ as well. Equation $u_{t_1} = \varphi_1$ is Hamiltonian w.r.t. the operator $B_1 = D_x^{-1} \kappa^{-1}$. The $r$-component analogue $E_{\text{mKdV}} = \{ \varphi_{t_1} = \kappa D_x (\varphi_1) \}$ of the modified KdV equation is Hamiltonian w.r.t. $B_1 = \kappa D_x$. The integral $w[\varphi]$ defines a Miura transformation between the higher modified KdV equations and the hierarchy $\mathfrak{A}$ of Eq. (1). Indeed, relation (1) holds for the transformation $w = w[\varphi]$ and the Hamiltonian operator $\hat{A}_2 = -\beta D_x^3 + D_x \circ w + w \cdot D_x$ for the KdV equation $E_{\text{KdV}}$. Hence, the times $t_i$ of the evolutions $u_{t_i} = \varphi_i$ are correlated with the times in the hierarchy $\mathfrak{A}$ for $E_{\text{pKdV}}$ by this map, and the four equations $E_{(p)(m)\text{KdV}}$ and their higher analogues as well share the same set of the Hamiltonians $\mathcal{H}_i = \{ H_{t_i} \} dx$.

**Proposition 5** (14). The factorizations $R_{\text{Toda}} = \Box \circ \ell_s$ and $R_{\text{pKdV}} = \ell_s \circ \Box$ hold for the recursions $R_{\text{Toda}} : \varphi_i \mapsto \varphi_{i+1}$ and $R_{\text{pKdV}} : \phi_i \mapsto \phi_{i+1}$. Every flow $\varphi_k = \Box \circ E_w(H_{k-1}) \in \text{sym } E_{\text{Toda}}$ is a Noether symmetry of the Toda equation, and $\varphi_k$ is Hamiltonian w.r.t. the Hamiltonian structure $B_1 = \kappa^{-1} \cdot D_x^{-1}$ and the Hamiltonian $H_{k-1} = \{ H_{k-1} \} dx$ for the $(k - 1)$th higher KdV equation.

**Proposition 6.** The symmetries $\varphi_k$ commute.

**Proof.** The KdV hierarchy $\mathfrak{A}$ is commutative and local in $s_x = w$. Let $\varphi' = \Box (\varphi'(x, W))$ and $\varphi" = \Box (\varphi''(x, W))$, then the Jacobi bracket of $\varphi'$ and $\varphi"$ is $\{ \varphi', \varphi" \} = \Box (\varphi_{(1,2)}(x, W))$, where (see 3 for the scalar case)

$$\varphi_{(1,2)} = \mathcal{E}_{\varphi'}(\varphi') - \mathcal{E}_{\varphi"}(\varphi') + D_x(\varphi') \varphi" - \varphi' D_x(\varphi"").$$

We have $[\mathcal{E}_{\varphi_{k_1}}, \mathcal{E}_{\varphi_{k_2}}](w) = \{ \Phi_{k_1}, \Phi_{k_2} \} = 0$. We also have $\varphi_{k_i} = \Box (\varphi_{k_i-1})$ and $\mathcal{E}_{(\varphi_{k_1}, \varphi_{k_2})}(w) = \hat{A}_2(\varphi_{(k_1, k_2)})$. Obviously, the Hamiltonian operator $\hat{A}_2$ for KdV is injective. Therefore, $\varphi_{(k_1, k_2)} = 0$ and $\{ \varphi_{k_1}, \varphi_{k_2} \} = 0$. 


\[ \varphi_{k_i} = \Box(0) = 0. \] We also conclude that the operator \( R_{\text{Toda}} \) is a recursion for the \( i \)th equation \( u_t^i = \varphi_i \) within the commutative symmetry subalgebra \( \mathfrak{B} \), here \( i \geq 0 \) is arbitrary. \hfill \Box

3. The Boussinesq hierarchy

The first Hamiltonian structure for the Boussinesq equation (2) is \( \hat{A}_1 = \left( \begin{array}{cc} 0 & D_x \vspace{1mm} \\ -D_x & 0 \end{array} \right) \). Hence \( u, v \) such that \( U = u_x, V = u_x \) are the potentials satisfying the potential Boussinesq equation, see Eq. (2). Further on, we use the notation \( u = U(u, v) \) and \( U = U(U, V) \). The Lagrangian functional with the density \( L_{\text{Bouss}} = -\frac{1}{2} u_x v_y - \frac{1}{2} v_x u_y \) for the ambient two-component wave equation \( \mathcal{E} = \{v_{xy} = 0, u_{xy} = 0\} \) is obtained straightforwardly. A Miura transformation for Eq. (2) is given by the formulas (6)

\[
U = a^2 + ab + b^2 + 2a_x + bx, \quad V = -2a(b(a + b) + bx) - D_x(b_x + ab + b^2)
\]

(7)

such that the nonpotential modified Boussinesq equation is

\[
a_t = \frac{1}{3} D_x(a^2 - 2ab - 2b^2 - a_x - 2b_x), \quad b_t = \frac{1}{3} D_x(-2a^2 - 2ab + b^2 + 2a_x + bx).
\]

Equation (8) is Hamiltonian w.r.t. the Hamiltonian operator \( \hat{B}_1 = -\frac{1}{3} \left( \begin{array}{cc} 0 & D_x \vspace{1mm} \\ -D_x & 0 \end{array} \right) \cdot D_x \) and the Casimir \( \mathcal{H}_0 = [U dx] \). (We note that \( \hat{B}_1 \) contains the Cartan matrix \( K_{\mathfrak{sl}_3} \) of the algebra \( \mathfrak{sl}_3(\mathbb{C}) \), see [9].) Also, recall that the second sequence of flows for Eq. (2) starts with the Casimir \( \mathcal{H}_0 = [U dx] \). Now we see that the integral constraints \([U dx] = \int_{-\infty}^{+\infty} u_x \, dx = \text{const} \), \([V dx] = \int_{-\infty}^{+\infty} u_x \, dx = \text{const} \) used in the inverse scattering problem method are natural. Moreover, note that the Hamiltonians \( \mathcal{H}_0 \) and \( \mathcal{H}_0 \) define nonzero symmetry flows on Eq. (2), which are the shifts of the potential variables \( u \) and \( v \), respectively.) Therefore, we introduce the potentials \( \alpha, \beta \) such that \( a = \frac{1}{3}(2\alpha_x - \beta_x) \), \( b = \frac{1}{3}(-\alpha_x + 2\beta_x) \) and denote \( \alpha = \text{t}(\alpha, \beta) \) and \( a = \text{t}(a, b) \). The Hamiltonian flows \( u_{t_{i-1}} = \phi_{i-1} \) are mapped to the flows \( \alpha_{t_i} = \varphi_i = \Box(\phi_{i-1}) \) by the operator

\[
\Box(\ell a)^* = \left( \begin{array}{cc} 2a + b & -4ab - 2b^2 - 2b_x + b D_x \\
-4a^2 - 4ab + 2b_x + b D_x & \end{array} \right).
\]

The ambient Euler equation \( \mathcal{E}_{\text{EL}}' \) is described by solving the equations \( \{ \varphi_i \in \text{sym} \mathcal{E}_{\text{EL}}' \}, i \geq 0 \), w.r.t. the Hamiltonian \( \mathcal{H}_{\text{EL}}' \); solving the first two, we obtain \( \mathcal{H}_{\text{EL}}' = c_1 \exp(\beta - \alpha) + c_2 \exp(\alpha) + c_3 \exp(-\beta), \) where \( c_1, c_2, c_3 \in \mathbb{R} \), and hence we get the 2DTL. The functionals \( \mathcal{U}(a) \) are integrals for \( \mathcal{E}_{\text{EL}}' \) (hence it is Liouvillian) if \( c_2 = 0 \). Thus we obtain the
system
\[
\alpha_{xy} = -c_1 \exp(\beta - \alpha) - c_3 \exp(-\beta), \quad \beta_{xy} = c_1 \exp(\beta - \alpha) - 2c_3 \exp(-\beta),
\]
which is transformed to the 2DTL (9) \(\alpha_{xy} = \exp(K_3 \alpha)\) by a change of variables. Recall that \(\phi_i \in \text{im} \mathcal{E}_U\) and \(\Box = (\ell^*_U)^*\). Therefore the flows \(\phi_i = \Box(\phi_{i-1})\) are Noether symmetries of \(\mathcal{E}_{EL}\). Then the Miura transformation \(U = U[\alpha]\) and Eq. (4) provide the natural factorization of the second Hamiltonian structure (18)
\[
\hat{A}_2 = \begin{pmatrix}
-2D_x^3 + 2U D_x + U_x & 3V D_x + 2V_x \\
3V D_x + V_x & 2 \frac{3}{2} D_x^2 - \frac{5}{3} (U D_x^3 + D_x \circ U) + U_{xx} D_x + D_x \circ U_{xx} + \frac{2}{3} U D_x \circ U
\end{pmatrix}
\]
for Boussinesq equation (2). The initial terms of the Boussinesq hierarchy and the correlated sequence of modified Boussinesq flows that share the Hamiltonians \(\mathcal{H}_i\) are shown in diagram (5). Here the Hamiltonian densities are \(H_0 = V, \quad H_1 = \frac{1}{3} V^2 + \frac{1}{6} U_x^2 + \frac{2}{9} U^3, \quad \text{and} \quad H_2 = \frac{1}{12} U_{xx} V_{xx} - \frac{5}{12} U U_{xx} V - \frac{5}{16} U_x^3 V + \frac{5}{36} U^3 V + \frac{5}{36} V^3, \quad \text{etc.} \) From Proposition (4) we conclude that the elements \(\phi_i\) of the sequence \(\mathcal{B}\) of modified Boussinesq Noether flows on 2DTL (9) are local in \(\alpha\) and pairwise commute. The proof is analogous to Proposition (6); we note that the explicit formula, Eq. (5), for the bracket \(\phi_i \{ \phi_j \}\) is not used and hence may be not known. Finally, we obtain the factorizations \(R_{\text{pmBous}} = \Box \circ \hat{A}_1^{-1} \circ \Box^* \circ \ell^*_\alpha\) and \(R_{\text{mBous}} = \ell^*_\alpha \circ \Box \circ \hat{A}_1^{-1} \circ \Box^*\) for the recursion operators \(R_{\text{pmBous}} = R_{\text{mBous}}^*\) that proliferate higher modified Boussinesq equations (8). By Proposition (2) the recursions for \(\mathcal{A}\) and \(\mathcal{B}\) are weakly nonlocal.

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REFERENCES

[1] Bocharov A. V., Chetverikov V. N., Duzhin S. V., et al. Symmetries and conservation laws for differential equations of mathematical physics. AMS, Providence, RI, 1999. I. Krasil’shchik and A. Vinogradov, eds.
[2] Demskoi D. K., Startsev S. Ya. // Fundam. Appl. Math. 10 (2004), n.1, 29–37.
[3] Zhiber A. V., Sokolov V. V. // Russ. Math. Surveys 56 (2001), n.1, 61–101.
[4] Leznov A. N., Smirnov V. G., Shabat A. B. // Teor. matem. fizika 51 (1982), n.1, 10–21.
[5] Meshkov A. G. // Teor. matem. fizika 63 (1985), n.3, 323–332.
[6] Pavlov M. V. // Fundam. Appl. Math. 10 (2004), n.1, 175–182.
[7] Shabat A. B., Yamilov R. I. Exponential systems of type I and the Cartan matrices. Preprint. Ufa, 1981.
[8] Barnich G., Brandt F., Henneaux M. // Commun. Math. Phys. 174 (1995), 57–92.
[9] Fordy A. P, Gibbons J. // J. Math. Phys. 21 (1980), n.10, 2508–2510; J. Math. Phys. 22 (1981), n.6, 1170–1175.
[10] Gardner C. S. // J. Math. Phys. 12 (1971), n.8, 1548–1551.
[11] Getzler E. // Duke Math. J. 111 (2002), 535–560.
[12] Kersten P., Krasil’shchik I., Verbovetsky A. // J. Geom. Phys. 50 (2004), n.1-4, 273–302.
[13] Kiselev A. V. // Acta Appl. Math. 83 (2004), n.1-2, 175–182.
[14] Kiselev A. V., Ovchinnikov A. V. // J. Dynamical and Control Systems 10 (2004), n.3, 431–451.
[15] Magri F. // J. Math. Phys. 19 (1978), n.5, 1156–1162.
[16] Miura R. M. // J. Math. Phys. 9 (1968), 1202–1204.
[17] Nutku Y., Pavlov M. V. // J. Math. Phys. 43 (2002), n.3, 1441–1459.
[18] Olver P. J. Applications of Lie groups to differential equations, 2nd ed., Springer–Verlag, NY, 1993.

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