ANALYTICAL APPROXIMATION OF BLASIUS’ SIMILARITY SOLUTION WITH RIGOROUS ERROR BOUNDS

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Abstract. We use a recently developed method [2], [3] to find accurate analytic approximations with rigorous error bounds for the classical similarity solution of Blasius of the boundary layer equation in fluid mechanics, the two-point boundary value problem $f''' + f f'' = 0$ with $f(0) = f'(0) = 0$ and $\lim_{x \to \infty} f'(x) = 1$. The approximation is given in terms of a polynomial in $[0, \frac{5}{2}]$ and in terms of the error function in $[\frac{5}{2}, \infty)$. The two representations for the solution in different domains match at $x = \frac{5}{2}$ determining all free parameters in the problem, in particular $f''(0) = 0.469600 \pm 0.000022$ at the wall. The method can in principle provide approximations to any desired accuracy for this or wide classes of linear or nonlinear differential equations with initial or boundary value conditions. The analysis relies on controlling the errors in the approximation through contraction mapping arguments, using energy bounds for the Green’s function of the linearized problem.

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1. Introduction and main results

Finding exact expressions for solutions to nonlinear problems is an important area of research. Closed form solutions however exist only for a small sub-class of problems (essentially for integrable models). On the other-hand, if a problem involves some small parameter $\varepsilon$ (or a large parameter) and the limiting problem is exactly solvable, then there exist quite general asymptotic methods to obtain convenient expansions for the perturbed problem.

Indeed, consider for instance the question of finding the solution to $N[u, \varepsilon] = 0$, where $N$ is a (possibly nonlinear) differential operator in some space of functions satisfying boundary/initial conditions and that $u_0$ is the solution at $\varepsilon = 0$. Existence and uniqueness of a solution $u$ as well bounds on the error $E = u - u_0$ may be found as follows. We write

$$LE = -\delta - N_1(E)$$

where $L = \frac{dN}{du}|_{u=u_0}$, $\delta = N[u_0]$ and $N_1(E) = N(u_0 + E) - L E = O(\varepsilon^2)$, inverting $L$ in a suitable way, subject to the initial/boundary conditions, and using the contractive mapping theorem in an adapted norm to control the small nonlinearity $N_1(E)$.

A relatively general strategy has been recently been employed [2]-[3] in problems without explicit small parameters. The approach uses exponential asymptotic methods and classical orthogonal polynomial techniques to find a function $u_0$ which is a very accurate global approximation of the sought solution $u$, in the sense that $N(u_0)$ is very small in a suitable norm and the boundary conditions are satisfied up to small errors. Once this is accomplished, a perturbative approach similar with the one above applies with the role of $\varepsilon$ played by the norm of $\delta$ and one obtains an actual solution $u$ by controlling the equation satisfied by $E = u - u_0$.

In the present paper, we apply this strategy to the well-known Blasius similarity equation arising in boundary layer fluid-flow past a flat plate. We improve the methods in [2]-[3] by replacing the laborious detailed analysis of the Green’s function with softer energy methods.
The Blasius similarity solution satisfies

\[ f'''(x) + f(x)f''(x) = 0 \quad \text{for} \quad x \in (0, \infty) \]

with boundary conditions

\[ f(0) = 0, \quad f'(0) = 0, \quad \lim_{x \to +\infty} f'(x) = 1 \]

This equation and various modifications have garnered much attention since Blasius \[4\] derived it. Existence and uniqueness were first proved by Weyl in \[5\]. Issues of existence and uniqueness for this and related equations have been considered as well by many authors (see e.g. \[7, 8\], the latter being a review paper). Hodograph transformations \[6\] allow a convergent power series representation in the entire domain, but the convergence is slow at the edge of the domain and the representation is not quite convenient in finding an approximation to \( f \) directly. Empirically, there has been quite a bit of interest in obtaining simple expressions for Blasius and related similarity solutions. Liao \[9\] for instance introduced a formal method for an empirically accurate approximation; the theoretical basis for this procedure and its limitations remain however unclear. We are unaware of any rigorous error control for this or any other efficient approximation in terms of simple functions.

In \[5\], using a transformation introduced by Topfer \[11\], it is also proved that \( f \) in \([1, 2]\) can be expressed as

\[ f(x) = a^{-1/2}F\left(a^{-1/2}x\right), \]

where \( F \) satisfies the initial value problem

\[ F'''(x) + F(x)F''(x) = 0 \quad \text{for} \quad x \in (0, \infty) \]

with initial conditions

\[ F(0) = 0, \quad F'(0) = 0, \quad F''(0) = 1, \]

In \([3]\), \( \lim_{x \to +\infty} F'(x) = a \in \mathbb{R}^+ \) (cf. \([3, 11]\)). Conversely, if \( f(x) \) satisfies \([1, 2]\) with \( f''(0) = \beta > 0 \) (physically, this corresponds to positive wall stress), it may be checked that \( F(x) = \beta^{-1/3}f(\beta^{-1/3}x) \) satisfies \([3, 5]\). Because of this equivalence, it is more convenient to find an approximate analytical representation for the solution \( F \) of \([1, 5]\) and determine the value of \( a \). The solution \( f \) to the original problem is obtained through the transformation \([3]\); the stress at the wall is given by

\[ f''(0) = a^{-3/2} \]

1.1. **Approximate representation of the solution.** Let

\[ P(y) = \sum_{j=0}^{12} \frac{2}{5(j+2)(j+3)(j+4)}p_jy^j \]

\[ (1) \text{The equation in the original Blasius’ paper has a coefficient } \frac{1}{2} \text{ for } ff’; \text{ however the change of variable } x \to \sqrt{x}, f \to f/\sqrt{x} \text{ transforms (1) into Blasius’ original equation. Thus, } f''(0) = 0.469600 \pm 0.000022 \text{ transforms to } f''(0) = 0.3320574 \pm 0.000016 \text{ in the original variables.} \]
where \([p_0,\ldots,p_{12}]\) are given by

\[
\begin{align*}
-510 & \quad 10445149 & \quad -18523 & \quad 5934 & \quad -42998 & \quad 113448 & \quad 65173 & \quad 390101 & \quad -2326169 \\
-4134879 & \quad 7249 & \quad -1928001 & \quad 20880183 & \quad 1572554 & \quad 1546782 & \quad -32239 \\
& & & & & & & & 
\end{align*}
\]

Define

\[
t(x) = \frac{a}{2}(x+b/a)^2, \quad I_0(t) = 1 - \sqrt{\pi} e^{t^2} \text{erfc}(\sqrt{t}), \quad J_0(t) = 1 - \sqrt{2\pi} e^{2t} \text{erfc}(\sqrt{2t}),
\]

where \(\text{erfc}\) denotes the complementary error function and let

\[
q_0(t) = 2c\sqrt{\pi} e^{t^2} J_0(t) + c^2 e^{-2t} (2J_0 - I_0 - t_0^2),
\]

The theorem below provides an accurate representation of \(F\) of \([41],[5]\).

**Theorem 1.** Let \(F_0\) be defined by

\[
F_0(x) = \begin{cases} \frac{x^2}{2} + x^4 P(\frac{a}{2}x) & \text{for } x \in [0, \frac{5}{2}] \\ ax + b + \sqrt{\frac{a}{2}x}q_0(tx) & \text{for } x > \frac{5}{2} \end{cases}
\]

Then, there is a unique triple \((a,b,c)\) close to \((a_0,b_0,c_0) = (\frac{3221}{1946}, -\frac{2763}{1765}, \frac{377}{1613})\) in the sense that \((a,b,c) \in S\) where

\[
S = \left\{(a,b,c) \in \mathbb{R}^3 : \sqrt{(a-a_0)^2 + \frac{1}{4}(b-b_0)^2 + \frac{1}{4}(c-c_0)^2} \leq \rho_0 := 5 \times 10^{-5}\right\}
\]

with the property that \(F_0\) is a representation of the actual solution \(F\) to the initial value problem \([4],[5]\) within small errors. More precisely,

\[
F(x) = F_0(x) + E(x),
\]

where the error term \(E\) satisfies

\[
\|E''\|_\infty \leq 3.5 \times 10^{-6}, \quad \|E'\|_\infty \leq 4.5 \times 10^{-6}, \quad \|E\|_\infty \leq 4 \times 10^{-6} \quad \text{on } [0, \frac{5}{2}],
\]

and for \(x > \frac{5}{2}\)

\[
\begin{align*}
|E| & \leq 1.69 \times 10^{-5} t^{-2} e^{-3t} + \frac{d}{dx}E | \leq 9.20 \times 10^{-5} t^{-3/2} e^{-3t} \\
\left| \frac{d^2}{dx^2}E \right| & \leq 5.02 \times 10^{-4} t^{-1} e^{-3t}
\end{align*}
\]

**Remark 1.** Certainly, \(F\) is smooth since it is an actual solution of \([4],[5]\), which exists on \([0,\infty)\) and is unique, see \([5]\). However, the particular choice \((a,b,c) \in S\) in Theorem 1 needed in order for \(F = F_0 + E\) to solve \([4],[5]\) does not ensure continuity of the approximate solution \(F_0\) at \(x = \frac{5}{2}\). Nonetheless, if \(F_0,F'_0,F''_0\) are needed to be continuous, this can be achieved by a slightly different choice of \((a,b,c) \in S\) (see Remark 12), namely

\[
(a,b,c) = (1.6551904561499..., -1.565439826457..., 0.233728727537...).
\]

Note also that \((15)\) implies not only small absolute errors (that, in the far field hold even for the approximation of \(F''\) by zero) but also very small relative errors for \(x > 5/2\).
Definition 1. Let \( a_t = a_0 - \rho_0, a_r = a_0 + \rho_0, b_1 = b_0 - 2\rho_0, b_r = b_0 + 2\rho_0, \) \( c_1 = c_0 - 2\rho_0 \) and \( c_r = c_0 + 2\rho_0. \) We see that \( (a, b, c) \in S, \) implies that \( a \in [a_1, a_r], \) \( b \in [b_1, b_r], \) \( c \in (c_1, c_r). \) We define \( t_m = \frac{a}{2} \left( \frac{5}{2} + \frac{b}{a} \right)^2 \) and note that \( x \in \left[ \frac{5}{2}, \infty \right) \)
\[
\text{corresponds to } t \in [t_m, \infty) \text{ and } t_m \in \left( \frac{2}{m} \left[ \frac{5}{2} + \frac{b}{a_1} \right]^2, \frac{2}{m} \left[ \frac{5}{2} + \frac{b}{a_r} \right]^2 \right) =: (t_{m,l}, t_{m,r}) = (1.998859 \ldots, 1.999438 \ldots).
\]

Remark 2. The error bounds proved for \( E \) in Theorem 1 are likely a 10 fold overestimate. Comparison with the numerically calculated \( F \) suggests that \(|F - F_0|, |F' - F_0'| \) and \(|F'' - F_0''|\) in \([0, \frac{2}{5}]\) are at most \(2 \times 10^{-7}, 2 \times 10^{-7} \) and \(5 \times 10^{-7}\) respectively. Using the nonrigorous bounds on \( E \) and its derivatives reduces the \( \rho_0 \) in the definition of \( S \) from \(5 \times 10^{-5}\) to \(1.4 \times 10^{-5}\). It is thus likely that \((a, b, c) \approx (a_0, b_0, c_0)\) with five (rather than the proven four) digits accuracy.

The proof of Theorem 1 rests on the following three propositions, proved later in the paper.

**Proposition 2.** The error term \( E(x) = F(x) - F_0(x) \) verifies the equation
\[
\mathcal{E}[E] := E'' - F_0E'' - F''_0E = -F'''_0 - F_0'F''_0 - EE''
\]
and satisfies the bounds \((14)\) on \( I = [0, \frac{2}{5}]\)

**Proposition 3.** For given \((a, b, c)\) with \( a > 0, |c| < \frac{1}{2}, \) in the domain \( x \geq -\frac{b}{a} + \sqrt{\frac{2t}{a}}, \) for \( T \geq 1.99, \) which corresponds to \( t \geq T \geq 1.99, \) there exists unique solution to \((7)\) in the form
\[
F(x) = ax + b + \sqrt{\frac{a}{2t(x)}} q(t(x))
\]
that satisfies the condition \( \lim_{t \to \infty} \frac{q(t)}{\sqrt{t}} \to 0. \) Furthermore,
\[
q(t) = q_0(t) + \mathcal{E}(t)
\]
where \( \mathcal{E} \) is small and satisfies the following error bounds:
\[
|\mathcal{E}(t)| \leq 1.6667 \times 10^{-4} \frac{e^{-3t}}{9t^{3/2}}
\]
\[
|\mathcal{E}'(t) - \frac{1}{2t} \mathcal{E}(t)| \leq 1.6667 \times 10^{-4} \frac{e^{-3t}}{3t^{3/2}}
\]
\[
\left| \sqrt{t} \mathcal{E}''(t) - \frac{1}{2t} \mathcal{E}'(t) + \frac{1}{2t^{3/2}} \mathcal{E}(t) \right| \leq 1.6667 \times 10^{-4} t^{-1} e^{-3t}
\]

**Proposition 4.** There exists a unique triple \((a, b, c) \in S\) so that the functions in the previous two propositions: \( F_0(x) + E(x) \) for \( x \leq \frac{5}{2} \) and \( ax + b + \sqrt{\frac{a}{2t(x)}} q(t(x)) \) for \( x \geq \frac{5}{2} \) and their first two derivatives agree at \( x = \frac{5}{2}. \)

The proof of Theorem 1 follows from Propositions 2-3 in the following manner: Proposition 2 implies \( F(x) = F_0(x) + E(x) \) satisfies \((14)-(15)\) for \( x \in I; \) we note that \( F_0(0) = 0 = F_0'(0) \) and \( F_0''(0) = 1. \)
Proposition 4 implies $F(x) = ax + b + \sqrt{\frac{a}{2t(x)}}[g_0(t(x)) + \mathcal{E}(t(x))]$ satisfies (11) in a range of $x$ that includes $[\frac{5}{2}, \infty)$ when $(a, b, c) \in \mathcal{S}$. Further, Proposition 4 ensures that this is the same solution of the ODE (11) as the one in Proposition 2. Identifying $F_0(x)$ and $E(x)$ in Theorem 1 in this range of $x$ with $ax + b + \sqrt{\frac{a}{2t(x)}}g_0(t(x))$ and $\sqrt{\frac{a}{2t(x)}}\mathcal{E}(t(x))$, respectively, and relating $x$-derivatives to $t$ derivatives, the error bounds for $E, E'$ and $E''$ follow from the ones given for $\mathcal{E}$ in Proposition 4 for $(a, b, c) \in \mathcal{S}$. The proofs of Propositions 3, 4 are given in Sections 2-4 respectively.

2. Solution in the interval $I = [0, \frac{5}{2}]$ and proof of Proposition 2

The ansatz for $F_0$ in the compact set $I$ is obtained simply by projecting an empirically obtained high accuracy approximate solution on the subspace spanned by the first few Chebyshev polynomials. More precisely, to avoid estimating derivatives of an approximation, which are not well-controlled, we project instead the approximate third derivative $F''' = -F''F$ on the interval $I = [0, \frac{5}{2}]$. The rigorous control of the errors of the integrals of $F'''$ is a much simpler task, and this is the way we prove Proposition 2. For a given polynomial degree, a Chebyshev polynomial approximation of a function is known to be, typically, the most accurate in polynomial representation, in the sense of $L^\infty$.

We seek to control the error term $E$ in (13) by first estimating the remainder

$$R(x) = F'''_0(x) + F_0(x)F''_0(x),$$

which will be shown to be small ($\leq 0.673 \times 10^{-6}$). Then, we invert the principal part of the linear part of the equation for the error term $E$ by using initial conditions to obtain a nonlinear integral equation. The smallness of $R$ and careful bounds on the resolvent allow for using a contraction mapping argument to obtain the sharp estimates for $E$ and its derivatives stated in Proposition 2.

2.1. Estimating size of remainder $R(x)$ for $x \in I$. Since $P$ is a polynomial of degree twelve, $R(x)$ is a polynomial of degree 30. We estimate $R$ in $I$ in the following manner. We break up the interval into subintervals $\{[x_{j-1}, x_j]\}_{j=1}^{14}$ with $x_0 = 0$ and $x_{14} = \frac{5}{2}$, while $\{x_j\}_{j=1}^{13}$ is given by

$$\{0.0625, 0.125, 0.25, 0.375, 0.50, 0.75, 1.0, x_c, 1.5, 1.75, 2.0, 2.25, 2.40\},$$

where $x_c = 1.322040$ (3). The intervals were chosen based on how rapidly the polynomial $R(x)$ varies locally.

We re-expand $R(x)$ as polynomial in the scaled variable $\tau$, where $x = \frac{1}{2}(x_j + x_{j-1}) + \frac{1}{2}(x_j - x_{j-1})\tau$. and write

$$R(x) = P_3^{(j)}(\tau) + \sum_{k=4}^{30} a_k^{(j)} \tau^k$$

(2)Chebyshev polynomial approximation is on $[-1, 1]$ interval, but a linear scaling and shift of independent variables can accommodate any finite interval

(3)As will be found later, it is convenient to choose one of the subdivision points $x_c$ to be approximately, to the number of digits quoted, the value of $x$ where $F'''_0(x) = -2F''_0(x) + 1$ changes sign.
and determine the maximum $M_j$ and minimum $m_j$ of the third degree polynomial $F_3^{(j)}(t)$ for $\tau \in [-1,1]$ (using simple calculus). We estimate the $l^1$ error on the remaining coefficients:

$$E_R^{(j)} = \sum_{k=4}^{30} |a_k^{(j)}|$$

It follows that in the $j$-subinterval we have

$$m_j - E_R^{(j)} \leq R(x) \leq M_j + E_R^{(j)}$$

The maximum and minimum over any union of subintervals is found simply taking min and max of $m_j - E_R^{(j)}$ and $M_j + E_R^{(j)}$ over the the indices $j$ for subintervals involved. This elementary though tedious calculation\textsuperscript{[4]} yields

(25)  $-3.22 \times 10^{-7} \leq R(x) \leq 2.505 \times 10^{-7}$ for $x \in [0, x_c]$

$$4.6 \times 10^{-8} \leq R(x) \leq 4.06 \times 10^{-7} \text{ for } x \in [x_c, 2.0]$$

$$2.78 \times 10^{-7} \leq R(x) \leq 6.73 \times 10^{-7} \text{ for } x \in [2.0, 2.5]$$

We note that the remainder is at most $6.73 \times 10^{-7}$ in absolute value in the interval $I$. In a similar manner of finding maximum and minimum, bounds for the polynomials $F_0(x), F_0'$ and $F_0''(x)$ may also be found. For $x \in [0, \frac{1}{5}]$,

(26)  $-5 \times 10^{-10} \leq F_0(x) \leq 0.008 , -8 \times 10^{-12} \leq F_0'(x) \leq 0.13 ,$

$$0.99 \leq F_0''(x) \leq 1 + 2 \times 10^{-9}$$

while for $x \in \left[\frac{1}{5}, \frac{2}{5}\right],$

(27)  $0.03 \leq F_0(x) \leq 2.59 , 0.12 \leq F_0'(x) \leq 1.7 , 0.09 \leq F_0''(x) \leq 1$

### 2.2. Properties of some functions used in the sequel.

Based on the calculations above, one can also conclude that $F_0'' - 2F_0 + 1$ and $2F_0'' - 2F_0$ have only one zero in the interval $I$ in the following manner. Note that derivatives of these functions are $-F_0F_0'' - 2F_0' + R < 0$ and $-2F_0F_0'' - 2F_0' + 2R < 0$ respectively in $\left[\frac{1}{5}, \frac{2}{5}\right]$, where $R$ has been estimated in the previous subsection. From the bounds in the interval $[0, \frac{1}{5}]$, it is clear that $F_0'' - 2F_0 + 1$ and $2F_0'' - 2F_0$ are positive in $[0, \frac{1}{5}]$. Thus, we conclude that $F_0'' - 2F_0 + 1$ and $2F_0'' - 2F_0$ can have at most one zero in $I$. The values of $F_0'' - 2F_0 + 1$ at $x_c = 1.322040$ and 1.322041 have opposite signs, implying that there is a unique zero in $I$ in between two numbers (recalling that the derivative is negative). Similarly, we conclude there is a unique zero of $2F_0'' - 2F_0$ between 1.2314283 and 1.2314284. Similar arguments show that $F_0''(x) - 2F_0(x)$ only has one zero in $I$ at $x = 0.9399325 \cdots < x_c$.

\textsuperscript{[4]}The maximum and minimum found through analysis described here is found to be consistent with a numerical plot of the graph of $R(x)$, as must be the case. The calculations can be conveniently done with a computer algebra program, as they only involve operations with rational numbers.
2.3. Green’s function estimate for $x \in [x_l, x_r]$. Consider now the problem of solving the linear generally inhomogeneous equation

$$
L[\phi] := \phi''(x) + F_0(x)\phi'(x) + F'_0(x)\phi(x) = r(x)
$$

over a typical subinterval $[x_l, x_r] \subset I$, with initial conditions $\phi(x_l)$, $\phi'(x_l)$ and $\phi''(x_l)$ known. The solution of this inhomogeneous equation is given by the standard variation of parameter formula:

$$
\phi(x) = \sum_{j=1}^{3} \phi^{(j-1)}(x_l)\Phi_j(x) + \sum_{j=1}^{3} \Phi_j(x) \int_{x_l}^{x} \Psi_j(t)r(t)dt
$$

where $\{\Phi_j\}_{j=1}^{3}$ form a fundamental set of solutions to $L\phi = 0$ and $\{\Psi_j(x)\}_{j=1}^{3}$ are elements of the inverse of the fundamental matrix constructed from the $\Phi_j$ and their derivatives. The precise expressions are unimportant in the ensuing: we only need their smoothness in $x$. It also follows from the properties of $\Phi_j$ and $\Psi_j$ that

$$
\phi''(x) = \sum_{j=1}^{3} \phi^{(j-1)}(x_l)\Phi_j''(x) + \sum_{j=1}^{3} \Phi_j''(x) \int_{x_l}^{x} \Psi_j(t)r(t)dt,
$$

It is useful to write (30) in the following abstract form

$$
\phi''(x) = \sum_{j=1}^{3} \phi^{(j-1)}(x_l)\Phi_j''(x) + G[r](x)
$$

where from general properties of fundamental matrix and its inverse for the linear ODEs with smooth (in this case polynomial) coefficients $G$ is a bounded linear operator on $C([x_l, x_r])$; denote its norm by $M$.

$$
M = \|G\|
$$

Then, on the interval $[x_l, x_r]$ we have,

$$
\|\phi''\|_{\infty} \leq \sum_{j=1}^{3} \sup_{x \in [x_l, x_r]} \|\Phi_j''(x)\| + M\|r\|_{\infty}; \quad M_j = \sup_{x \in [x_l, x_r]} \|\Phi_j''(x)\|
$$

We will now estimate $M_j$ for $j = 1, 2, 3$ and $M$ indirectly, using “energy” bounds. Because of linearity of the problem, for the purposes of determining these bounds, it is enough to separately consider the cases (i)–(iii), when $r = 0$, $\phi^{(k-1)}(x_l) = 0$ for $1 \leq k \neq j \leq 3$, $\phi^{(j-1)}(x_l) = 1$ respectively, and, finally, (iv) when $\phi^{(k-1)}(x_l) = 0$ for $k = 1, 2, 3$ and $r(t) \neq 0$.

For all cases (i)–(iv), we return to the ODE

$$
\phi'' + F_0\phi' + F'_0\phi = r
$$

Multiplying by $2\phi''$, integrating from $x_l$ to $x$ and using initial conditions, it follows that

$$
(\phi''(x))^2 = (\phi''(x_l))^2 - \int_{x_l}^{x} \left\{2F_0(y)(\phi''(y))^2 + 2F'_0(y)\phi(y)\phi'(y) - 2\phi''(y)r(y)\right\}dy,
$$

(5) In particular, $\sum_{j=1}^{3} \Phi_j(x)\Psi_j(x) = 0$, $\sum_{j=1}^{3} \Phi'_j(x)\Psi_j(x) = 0$
We note further that, given $\phi(x_i)$ and $\phi'(x_i)$, $\phi(x)$ is determined from $\phi''(x)$ and the relation

$$\tilde{\phi}(x) := \phi(x) - \phi(x_i) - (x-x_i)\phi'(x_i) = \int_{x_i}^{x} (x-y)\phi''(y)dy$$

Using (36) in (35), it follows that

$$\phi''(x) \geq \frac{(\phi''(x_i))}{2} - \int_{x_i}^{x} 2F''_0(y)[\phi(x_i) + (y-x_i)\phi'(x_i)]\phi''(y)dy$$

$$- \int_{x_i}^{x} \left\{ 2F_0(y)(\phi''(y))^2 + 2F''_0(y)\tilde{\phi}(y)[\phi(y) - 2\phi''(y)\phi(y)] \right\} dy,$$

2.4. Case (i): Determination of $M_1$. In this case, we set $\phi(x_i) = 1$, $\phi'(x_i) = 0$, and Cauchy-Schwartz:

$$\phi''(x) \leq \int_{x_i}^{x} F''_0(y)dy + \int_{x_i}^{x} F''_0(y)\tilde{\phi}^2(y)dy$$

$$+ \int_{x_i}^{x} (\phi''(y))^2 \{2F''_0(y) - 2F_0(y)\} dy$$

It is convenient to define, for $x_i \leq \tau \leq x_r$,

$$D(\tau) = \sup_{x \in [x_i, \tau]} \left| \phi''(x) \right|^2$$

From (36) and the definition of $D$ we get

$$\|\phi''\|_\infty \leq \sqrt{D(x_r)}, \left| \tilde{\phi}(x) \right| \leq \frac{(x-x_i)^2}{2}\sqrt{D(x)}, (\phi''(x))^2 \leq D(x); \text{ in } [x_i, x_r],$$

Using (10) and (38) we see that

$$\phi''(x) \leq \left( \int_{x_i}^{x} F''_0(y)dy \right) + \int_{x_i}^{x} F''_0(y)\frac{(y-x_i)^4}{4}D(y)dy$$

$$+ \int_{x_i}^{x} (\phi''(y))^2 \{2F''_0(y) - 2F_0(y)\} dy$$

Define now

$$Q_1(x) = F''_0(x) \left( 2 + \frac{(x-x_i)^4}{4} \right) - 2F_0(x) \text{ if } 2F''_0(x) - 2F_0(x) > 0$$

$$\text{and } Q_1(x) = \frac{(x-x_i)^4}{4} \text{ if } 2F''_0(x) - 2F_0(x) \leq 0$$

Equations (10) and (11) imply the following inequality for for $0 \leq x_i \leq x \leq \tau \leq x_r$

$$\phi''(x) \leq \int_{x_i}^{x} F''_0(y)dy + \int_{x_i}^{x} D(y)Q_1(y)dy$$

Since the right side is independent of $x$, it follows that

$$D(\tau) \leq \int_{x_i}^{\tau} F''_0(y)dy + \int_{x_i}^{\tau} D(y)Q_1(y)dy$$

From Gronwall’s lemma, it follows that

$$D(\tau) \leq \left( \int_{x_i}^{\tau} F''_0(y)dy \right) \exp \left[ \int_{x_i}^{\tau} Q_1(y)dy \right]$$
Evaluating (45) at \( \tau = x_r \) immediately implies

\[ M_1 \leq (F_0'(x_r) - F_0'(x_l))^{1/2} \exp \left[ \frac{1}{2} \int_{x_l}^{x_r} Q_1(y)dy \right] \]  

2.5. **Case (ii): Determination of** \( M_2 \). In this case, we set \( \phi'(x_l) = 1, \phi(x_l) = 0 = \phi''(x_l) = 0 = r(x) \) in (57) to obtain by Cauchy-Schwartz

\[ (\phi''(x))^2 \leq \left( \int_{x_l}^{x} F_0''(y)(y - x_l)^2dy \right) + \int_{x_l}^{x} F_0''(y)\bar{\phi}^2(y)dy \]
\[ + \int_{x_l}^{x} (\phi''(y))^2 \{2F_0''(y) - 2F_0(y)\} dy \]

Again introducing \( D(x) \) and \( Q_1(x) \) as in case (i), we obtain the inequality

\[ D(\tau) \leq \int_{x_l}^{\tau} (y - x_l)^2F_0''(y)dy + \int_{x_l}^{\tau} D(y)Q_1(y)dy \]

and therefore, it follows from Gronwall’s Lemma that

\[ M_2 \leq \left( \int_{x_l}^{x_r} (y - x_l)^2F_0''(y)dy \right)^{1/2} \exp \left[ \frac{1}{2} \int_{x_l}^{x_r} Q_1(y)dy \right] \]  

2.6. **Case (iii): Determination of** \( M_3 \). In this case, we set \( \phi''(x_l) = 1, \phi(x_l) = 0 = \phi'(x_l) = 0 = r(x) \) in (57) to obtain

\[ (\phi''(x))^2 \leq 1 + \int_{x_l}^{x} F_0''(y)\bar{\phi}^2(y)dy + \int_{x_l}^{x} (\phi''(y))^2 \{2F_0''(y) - 2F_0(y)\} dy \]

Once again introducing \( D \) as in case (i) and defining

\[ Q_2(x) = \left( 1 + \frac{(x - x_l)^4}{4} \right) F_0''(x) - 2F_0(x), \text{ if } F_0''(x) - 2F_0(x) > 0 \]

and
\[ Q_2(x) = \frac{(x - x_l)^4}{4} F_0''(x), \text{ if } F_0''(x) - 2F_0(x) \leq 0 \]

imply

\[ D(\tau) \leq 1 + \int_{x_l}^{\tau} D(y)Q_2(y)dy \]

Gronwall’s Lemma and definition of \( D \) implies

\[ M_3 \leq \exp \left[ \frac{1}{2} \int_{x_l}^{x_r} Q_2(y)dy \right] \]  

2.7. **Case (iv): Determination of** \( M = \|G\| \). With \( \phi(x_l) = 0 = \phi'(x_l) = \phi''(x_l) = 0 \), (57) implies by Cauchy-Schwartz

\[ (\phi''(x))^2 \leq \int_{x_l}^{x} r^2(y)dy + \int_{x_l}^{x} F_0''(y)\bar{\phi}^2(y)dy \]
\[ + \int_{x_l}^{x} (\phi''(y))^2 \{F_0''(y) - 2F_0(y) + 1\} dy \]
We define $D$ as in (i) and also

\begin{equation}
Q(x) = F''_0(x) - 2F_0(x) + 1 + \frac{(x-x_1)^4}{4}F'_0(x) \text{ if } F''_0 - 2F_0 + 1 \geq 0
\end{equation}

\begin{equation}
Q(x) = \frac{(x-x_1)^4}{4}F'_0(x) \text{ if } F''_0 - 2F_0 + 1 < 0
\end{equation}

The inequality

\begin{equation}
D(\tau) \leq \int_{x_1}^{\tau} r^2(y)dy + \int_{x_1}^{\tau} D(y)Q(y)dy
\end{equation}

follows from (54), implying from Gronwall’s Lemma

\begin{equation}
\sqrt{D(x_r)} \leq \|r\|_\infty (x_r - x_i)^{1/2} \exp \left[ \frac{1}{2} \int_{x_i}^{x_r} Q(y)dy \right]
\end{equation}

implying

\begin{equation}
M \leq (x_r - x_i)^{1/2} \exp \left[ \frac{1}{2} \int_{x_i}^{x_r} Q(y)dy \right]
\end{equation}

2.8. Existence of $E$; error bounds for $x \in [x_1, x_r] \subset I$. Consider the decomposition

\begin{equation}
F(x) = F_0(x) + E(x)
\end{equation}

We seek to find error estimates for $E(x)$ and its first two derivatives for $x \in I$. For this purpose we break up $I$ into a number of subintervals. Note that for the first subinterval $x_i = 0$, where $E(x_i) = 0 = E'(x_i) = E''(x_i)$. Consider a typical subinterval $I = [x_i, x_r]$ where the bounds on $E(x_i)$, $E'(x_i)$ and $E''(x_i)$ on earlier subintervals have been already obtained. The equation for $E(x)$ on $[x_i, x_r]$ is

\begin{equation}
\mathcal{L}[E] := E''(x) + F_0(x)E''(x) + F'_0(x)E(x) = -E(x)E''(x) - R(x)
\end{equation}

Inverting $\mathcal{L}$ as described in previous subsection leads to the following integral equation:

\begin{equation}
E''(x) = \sum_{j=1}^{3} E^{(j-1)}(x_1)\Phi'_j(x) - \mathcal{G} [R] (x) + \mathcal{G} [EE''] (x) =: \mathcal{N} [E'''] (x)
\end{equation}

and where $E$ is given by

\begin{equation}
E(x) - E(x_i) - (x-x_1)E'(x_i) =: \tilde{E}(x) = \int_{x_i}^{x} (x-t)E''(t)dt
\end{equation}

Note that (62) implies

\begin{equation}
\|\tilde{E}\|_\infty \leq \frac{(x_r - x_i)^2}{2} \|E''\|_\infty
\end{equation}

We prove the following Lemma that, once some bounds are satisfied, ensures the existence, uniqueness and smoothness of the solution $E$ of (60) and provides estimates of $E$, $E'$ and $E''$.

**Lemma 5.** Assume that for some $\varepsilon > 0$ we have

\begin{equation}
M \left( \|E(x_i)\| + (x_r - x_i)\|E'(x_i)\| \right) (1 + \varepsilon) + \frac{1}{2}(x_r - x_i)^2 MB_0 (1 + \varepsilon)^2 < \varepsilon,
\end{equation}

\begin{equation}
M \left( \|E(x_1)\| + (x_r - x_1)\|E'(x_1)\| \right) + (x_r - x_1)^2 MB_0 (1 + \varepsilon) < 1,
\end{equation}

\begin{equation}
M \left( \|E(x_1)\| + (x_r - x_1)\|E'(x_1)\| \right) + (x_r - x_1)^2 MB_0 (1 + \varepsilon) < 1
\end{equation}
where

$$B_0 = M \| R \|_\infty + \sum_{j=1}^{3} M_j \| E^{(j)}(x_1) \|,$$

Then, there exists a unique solution $E''$ of (63) in a ball of radius $B_0(1 + \varepsilon)$ in the sup norm in $C([x_1, x_r])$.

Under these assumptions, $E$ is in $C^3([x_1, x_r])$ and satisfies (67) with initial conditions $E^{(j)}(x_1)$, $j = 0, 1, 2$ and

$$\| E'' \|_\infty \leq \| F_0 \|_\infty (1 + \varepsilon) B_0 + \| F'_0 \|_\infty \left( |E(x_1)| + (x_r - x_1)|E'(x_1)| \right)$$

$$+ \frac{1}{2} \| F''_0 \|_\infty (x_r - x_1)^2 B_0 (1 + \varepsilon) + \frac{1}{2} (x_r - x_1)^2 B_0^2 (1 + \varepsilon)^2 + \| R \|_\infty$$

Proof. Since $G$ is the Green’s function of a smooth linear ODE, $G$ maps $C([x_1, x_r])$ into itself; the same, clearly, holds for $N$. From the definitions of $M$ in (32) and of $M_j$, $j = 1, 2, 3$ in (63) (whose bounds will be obtained using (60), (69), (58) and (65)) it follows that

$$\| N [E''] \|_\infty \leq M \| R \|_\infty + \sum_{j=1}^{3} M_j \| E^{(j-1)}(x_1) \|$$

$$+ M (|E(x_1)| + |x_r - x_1||E'(x_1)|) \| E'' \|_\infty + \frac{(x_r - x_1)^2}{2} M \| E'' \|_\infty^2$$

and

$$\| N [E''] - \hat{N} [\hat{E}''] \|_\infty \leq M (|E(x_1)| + |x_r - x_1||E'(x_1)|) \| E'' - \hat{E}'' \|_\infty$$

$$+ \frac{(x_r - x_1)^2}{2} M \left( \| E'' \| + \| \hat{E}'' \|_\infty \right) \| E'' - \hat{E}'' \|_\infty$$

Using (64), (61) and (60) in (65) and (69) we see that $N$ is contractive in a ball of radius $(1 + \varepsilon) B_0$ in $C([x_1, x_r])$, implying existence and uniqueness of a solution to (62). Clearly, (61) is equivalent to (60); from (60) it follows that $E''$ is also continuous. Now, $E''$ is easily estimated from (60) in terms of lower order derivatives, and the result follows.

2.9. Determining $E$ using Lemma 5. In this section, we break the interval $[0.5/2]$ in a suitable way and show that Lemma 5 applies in all subintervals. The choice of subintervals is $I_1 = [0, x_c]$ $I_2 = [x_c, 2]$, $I_3 = [2, 5/2]$, where $x_c = 1.322040$ is, within number of digits quoted, the zero of $F''_0 - 2F_0 + 1$ obtained in (2.2).

2.9.1. Error estimates on $I_1$. On $I_1$, using (11), it is easily checked that

$$M \leq 3.03$$

while by (25) we have $\| R \|_{\infty, I_1} \leq 3.22 \times 10^{-7}$ Since the initial conditions on this interval are $E(x_1) = E'(x_1) = E''(x_1) = 0$, the $M_j$ do not contribute to (68), (69), and (60) implies

$$B_0 \leq 0.9757 \times 10^{-6}$$

The conditions (64) and (65) are satisfied for $\varepsilon = 3 \times 10^{-6}$ so that Lemma 5 implies

$$\| E'' \|_{\infty, I_1} \leq 0.976 \times 10^{-6}$$
On integration it follows that
\[ \|E'\|_{\infty,I_1} \leq 1.29 \times 10^{-6}, \|E\|_{\infty,I_1} \leq 0.853 \times 10^{-6} \]

2.9.2. Error estimates on \( I_2 \). On \( I_2 = [x_0, 2] \), using (11), it is easily checked that
\[ M_1 \leq 0.572, M_2 = 0.199, M_3 \leq 1.01, M \leq 0.825. \]

Since at \( x_0 \), we can apply (72) and (73) to bound \( E(x_0), E'(x_0), E''(x_0) \), using \( \|R\|_{\infty,I_2} \leq 4.06 \times 10^{-7} \), (66) implies
\[ B_0 \leq 2.0653 \times 10^{-6} \]

Lemma 5 applies if \( \varepsilon = 2 \times 10^{-6} \). Therefore, the solution \( E \) exists and is unique on \( I_2 \) and
\[ \|E''\|_{\infty,I_2} \leq 2.07 \times 10^{-6} \]

By integration and using the bounds on \( E(x_0), E'(x_0) \) obtained in the previous interval, we get
\[ \|E'\|_{\infty,I_2} \leq 2.7 \times 10^{-6}, \|E\|_{\infty,I_2} \leq 2.21 \times 10^{-6} \]

2.9.3. Error estimates on \( I_3 = [2, 2.5] \). On \( I_3 = [2, 2.5] \), using (11) we get
\[ M_1 \leq 0.3, M_2 \leq 0.0744, M_3 \leq 1.01, M \leq 0.708 \]

while \( \|R\|_{\infty,[2,2.5]} \leq 0.673 \times 10^{-6} \) by (20). Proceeding as in the previous interval we get
\[ B_0 \leq 3.431 \times 10^{-6} \]

Here Lemma 5 applies with \( \varepsilon = 3 \times 10^{-6} \) and thus
\[ \|E''\|_{\infty,I_3} \leq 3.44 \times 10^{-6} \]

Proceeding as in the previous intervals, we get
\[ \|E'\|_{\infty,I_3} \leq 4.42 \times 10^{-6}, \|E\|_{\infty,I_3} \leq 3.99 \times 10^{-6} \]

2.10. End of proof of Proposition 2. In the previous subsection we have shown that Lemma 5 applies on each of the intervals \( I_j \) entailing the existence and uniqueness of a solution \( E \) of the initial value problem (17)-(18) over the interval \( I \). The same calculations show the \( L^\infty \) norms of \( E^{(j)} \) satisfy the bounds in Theorem 1.

3. Solution in \( t \geq T \geq 1.99 \) for \( |c| < \frac{1}{4}, a > 0 \) and proof of Proposition 5

We decompose
\[ F(x) = ax + b + g(x) \]

Then, it is clear from (34) that \( g \) satisfies
\[ g''' + (ax + b + g)g'' = 0 \]

**Lemma 6.** Assume \( a > 0 \). Then any solution \( g \) to (53), for which \( g \to 0 \) as \( x \to +\infty \) has the following asymptotic behavior
\[ g''(x) \sim C \exp \left[ -\frac{a}{2} \left( x + \frac{b}{a} \right)^2 \right] \]

\[ g'(x) \sim \frac{C}{ax + b} \exp \left[ -\frac{a}{2} \left( x + \frac{b}{a} \right)^2 \right] \]
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(86) \( g(x) \sim \frac{C}{(ax+b)^2} \exp \left[ -\frac{a}{2} \left( x + \frac{b}{a} \right)^2 \right] \)

Proof. Eq. (83) implies

(87) \( g''(x) = \tilde{C} \exp \left[ -\frac{a}{2} \left( x + \frac{b}{a} \right)^2 + \int_{x_0}^{x} g(t) dt \right] \)

Since \( g(x) = o(1) \), \( \int_{x_0}^{x} g(t) dt = o(x) \) as \( x \to \infty \); \( a > 0 \) and (87) imply that for sufficiently large \( x \) we have

(88) \( |g''(x)| \leq |\tilde{C}| \exp \left[ -\frac{a}{2} \left( x + \frac{b}{a} \right)^2 + \varepsilon x \right] \)

Integrating \( g'' \) and using (88), we get for large \( x \),

(89) \( |g'(x) - C_1| \leq \frac{|\tilde{C}a|}{ax+b-\varepsilon} \exp \left[ \frac{a}{2} \left( x + \frac{b}{a} \right)^2 + \varepsilon x \right] \)

for some \( C_1 \). Similarly, for large \( x \) we get

(90) \( |g(x) - C_1x - C_2| \leq \frac{|\tilde{C}|}{(ax+b-\varepsilon)^2} \exp \left[ \frac{a}{2} \left( x + \frac{b}{a} \right)^2 + \varepsilon x \right] \)

Since \( g = o(1) \), we must have \( C_1 = 0, C_2 = 0 \), giving rise to an exponentially decaying a priori bounds on \( g \) in (90) (with \( C_1 = 0 = C_2 \)). We can then set \( x_0 = \infty \) in (87), and (84) follows, and by integration, (85) and (86) hold.

Lemma 7. For given \( a > 0 \), \( b \) and \( C \), for \( x_0 > 0 \) sufficiently large, there exists unique solution to (83) in \([x_0, \infty)\) with asymptotic behavior (84)-(86).

Proof. Take \( x_0 > 0 \) sufficiently large so that

(91) \( \frac{e^{-a/2(x_0+b/a)^2}}{(ax_0+b)^3} < \frac{1}{4|C|} \)

We define \( H(x) = \exp \left[ \frac{a}{2} (x+b/a)^2 \right] g''(x) \). Then, from Lemma 6, we see that the appropriate space for \( H \) is \( C([x_0, \infty)) \) with the sup norm. Lemma 6 also implies

(92) \( g(x) = \int_{\infty}^{x} \int_{\infty}^{y} \exp \left[ -\frac{a}{2} (s+b/a)^2 \right] H(s) ds dy \)

Eq. (92) immediately implies

(93) \( |g(x)| \leq \frac{e^{-\frac{a}{2}(x+b/a)^2}}{(ax+b)^2} \|H\|_\infty \)

From (83) we obtain the following integral equation:

(94) \( H(x) = C - \int_{\infty}^{x} \int_{\infty}^{y} H(s) g(s) ds dy =: N[H](x) \)

where \( g \) is determined from \( H \) using (92). We analyze (94) in \( C([x_0, \infty)) \). We consider the space of continuous functions in \([x_0, \infty)\), equipped with \( \|\cdot\|_\infty \) norm.
Clearly \( \mathcal{N} \) maps a ball of radius \( 2|C| \) in this space back to itself since (93), (94) and (95) implies
\[
\|\mathcal{N}[H]\|_\infty \leq |C| + \frac{e^{-a/2(x_0+b/a)^2}}{(ax_0 + b)^3} \|H\|_\infty \leq 2|C|
\]
Thus \( \mathcal{N} \) maps the ball \( B_C \) of radius \( 2|C| \) into itself and is contractive there since
\[
\|\mathcal{N}[H_1] - \mathcal{N}[H_2]\|_\infty \leq \frac{e^{-a/2(x_0+b/a)^2}}{(ax_0 + b)^3} (\|H_1\|_\infty + \|H_2\|_\infty) \|H_1 - H_2\|_\infty
\]
\[
\leq \frac{4|C|e^{-a/2(x_0+b/a)^2}}{(ax_0 + b)^3} \|H_1 - H_2\|_\infty =: \alpha \|H_1 - H_2\|_\infty ,
\]
where \( \alpha < 1 \). Thus, (94) has a unique solution in \( B_C \). Furthermore from (94), it is clear that \( H(x) - C \) as \( x \to +\infty \). Recalling the definition of \( H \) we see that the asymptotic behavior of \( g \) is as given by Lemma (6).

**Remark 3.** From Lemmas (3) and (7) it follows that for given \( a, b \) with \( a > 0 \), there exists a one parameter \( (C) \) family of solutions \( g \) to (83) for which \( g \to 0 \) as \( x \to \infty \). Any such solution has the asymptotic behavior given in Lemma (6).

We seek to prove Proposition (3). For that purpose, for \( a > 0 \), recalling the change of variable \( t = t(x) \) in (11), we make the transformation:
\[
F(x) = ax + b + \sqrt{\frac{a}{2t^3}} q(t)
\]
Note that the change of variable involves the parameters \( a \) and \( b, c \) only appears in the the solution \( q(t) \) as shall be seen shortly. The domain \( t \geq T \) corresponds to \( x \geq -\frac{b}{a} + \sqrt{\frac{2T}{a}} \). The change of variables (97) in (4) results in \( q(t) \) satisfying
\[
\frac{d^3}{dt^3} q + \left( 1 + \frac{q}{2t} \right) \frac{d^2}{dt^2} q + \left( \frac{1}{2t} + \frac{3}{4t^2} \right) \frac{dq}{dt} + \left( \frac{1}{2t^2} - \frac{3}{4t^3} \right) q + \frac{q^2}{4t^3} = 0
\]
Equation (98) admits two growing solutions \( \sqrt{t} \) and \( t \) corresponding to the freedom of changing \( a \) and \( b \). The only solution for which \( \frac{q}{\sqrt{t}} \to 0 \ t \to \infty \), as noted in Lemma (3) corresponds to \( q(t) \sim ct^{-1/2}e^{-t} \) for some \( c \).

From the general theory of representation of solutions by transseries (10) (12) it follows that any decaying solution to (88), has the following convergent function series representation for sufficiently large \( x \):
\[
q(t) = \sum_{n=1}^{\infty} \xi^n Q_n(t), \text{ where } \xi = \frac{ce^{-t}}{\sqrt{t}}
\]
where the functions \( Q_n \) are bounded (7). The equations for \( Q_n \) are obtained by formally plugging in (99) into (88), equating the different powers of \( \xi \) and requiring that \( Q_n \) be bounded as \( t \to \infty \). Only the equation for \( Q_1 \) is homogeneous while

\[(6) \text{A slight modification is needed to accommodate the present ODE which violates a non-degeneracy condition on the eigenvalues of the linearization; the changes are minor. Also, transseries are used to generate the appropriate ansatz and motivate our choice of \( q_0 \), but play no direct role in the proofs.}
\[(7) \text{More precisely, } Q_n \text{ are the Borel sums of their asymptotic power series.} \]
for \( n > 1 \), the equation for \( Q_n \) involves \( Q_j \) for \( 1 \leq j < n \) as a forcing term. The associated homogeneous equation does not admit any bounded solution, and thus the \( Q_n \)'s are uniquely determined from their equations and the boundedness condition. The multiplicative freedom of \( Q_1 \) is equivalent to choice of \( c \) and therefore without loss of generality, we may assume \( Q_1 \to 1 \) as \( t \to \infty \).

This motivates the choice of the approximation \( q_0 \) as a truncation of (99) (we choose to retain two terms in the expansion). To prove that this approximation is accurate, we define \( E = q - q_0 \) and show that \( E \) is small for \( t \geq T \) in an exponentially weighted \( L^\infty \) norm. We thus define

\[
E(t) = q(t) - q_0(t)
\]

where

\[
q_0(t) = \frac{ce^{-t}}{\sqrt{t}} Q_1(t) + \frac{c^2 e^{-2t}}{t} Q_2(t)
\]

where

\[
Q_1(t) = 2t I_0(t), \quad \text{where } I_0(t) := 1 - \sqrt{\pi t e^t} \text{erfc}(\sqrt{t}) = \frac{1}{2} \int_0^\infty \frac{e^{-st}}{(1 + s)^{3/2}} ds
\]

\[
Q_2(t) = -t I_0 - t I_0^2 + 2t J_0, \quad \text{where } J_0(t) := 1 - \sqrt{2\pi t e^t} \text{erfc}(\sqrt{2t}) = \frac{1}{4} \int_0^\infty \frac{e^{-st}}{(1 + s/2)^{3/2}} ds
\]

**Remark 4.** It is clear from (101)-(103) that \( q_0(t) \sim \frac{ce^{-t}}{\sqrt{t}} \) as \( t \to \infty \); furthermore since, by the change of variables we have \( g(x) = \sqrt{\frac{a}{3x(t)}} q(t(x)) \), by Lemma 8 with \( C = \sqrt{2a^{3/2}} c \), we have

\[
q(t) \sim q_0(t), \quad q'(t) \sim q'_0(t), \quad q''(t) \sim q''_0(t), \quad \text{as } t \to \infty
\]

To analyze the fully nonlinear equation (98), we write the differential equation for \( E \) which follows from (100) and (98)

\[
\frac{d^3}{dt^3} E + \left(1 + \frac{q_0}{2t}\right) \frac{d^2}{dt^2} E + \left(\frac{1}{2t} + \frac{3}{4t^2} - \frac{q_0}{4t^2}\right) \frac{dE}{dt} + \left(\frac{1}{2t^2} - \frac{3}{4t^3} + \frac{q''_0}{2t} - \frac{q'_0}{4t^2} + \frac{q_0}{2t^3}\right) E = -\frac{E}{2t} E' + \frac{E}{4t^2} E'' - \frac{E^2}{4t^3} - R
\]

where the remainder \( R = R(t) \) is given by

\[
R = \frac{d^3}{dt^3} q_0 + \left(1 + \frac{q_0}{2t}\right) \frac{d^2}{dt^2} q_0 + \left(\frac{1}{2t} + \frac{3}{4t^2} - \frac{q_0}{4t^2}\right) \frac{dq_0}{dt} + \left(\frac{1}{2t^2} - \frac{3}{4t^3}\right) q_0 + \frac{q'_0}{4t^3}
\]

\[
= \xi^3 R_3(t) + \xi^4 R_4(t), \quad \text{where } \xi = ct^{-1/2} e^{-t}
\]

where

\[
R_3(t) = \left(-\frac{3Q'_1}{4t^2} - \frac{Q'_1}{2t} + \frac{5Q_1}{2t} + \frac{13Q_1}{4t^2} + \frac{9Q_1}{4t^3}\right) Q_2 - \frac{2}{t} Q_1 Q'_2 + \frac{1}{2t} Q_1 Q'' + \frac{5}{4t^2} Q_1 Q'_2
\]

\[
R_4(t) = Q_2 \left(\frac{Q'_2}{2t} - \frac{2Q'_2}{t} - \frac{5Q_2}{4t^2}\right) + Q_2^2 \left(\frac{2}{t} + \frac{5}{2t^2} + \frac{3}{2t^3}\right)
\]
Using (106) and (103) in (108) we get, after some algebra,

\[ R_3(t) = J_0 - tI_0^2 - I_0^2 \]

\[ R_4(t) = \frac{t}{2}(I_0^2 + I_0^3 - 2I_0J_0) + \frac{1}{2}J_0 - \frac{1}{4}I_0 - \frac{1}{2}I_0J_0 + \frac{1}{4}I_0^3 \]

In the appendix (see equations (216) and (225)), it is shown that \( 0 \leq R_3(t) \leq R_{3,m} \leq 0.02057 \) and \( 0 \leq R_4(t) \leq R_{4,m} \leq 0.0009042 \) for \( t \geq 1.99 \). Instead of using a variation of parameter formula for the third order linear operator on the left of (105) and turn (105) into an integral equation, we find it convenient to define the auxiliary function

\[ h(t) = e^{t} \left( \sqrt{t}E'' - E' + \frac{E(t)}{2t^{3/2}} \right) \]

and analyze the equation for \( h \).

**Remark 5.** The choice \( h \) is motivated by the observation that

\[ \frac{d^2}{dx^2} \sqrt{\frac{a}{2t(x)}} \mathcal{E}(t(x)) = \sqrt{2a^{3/2}}e^{-t(x)}h(t(x)) \]

and thus by (112), (104) and Lemmas 6 and 7, \( g(x) = \sqrt{\frac{a}{2t(x)}}q(t(x)) \to 0 \) implies \( h(t) \to 0 \) as \( t \to \infty \).

Equation (99) can be rewritten as

\[ h' = -\frac{q_0 e^t}{2t}h + e^t B(t)\mathcal{E} - \frac{Eh}{2t} - t^{1/2} e^t R, \]

where

\[ B(t) = -\frac{q_0''(t)}{2t^{1/2}} + \frac{q_0'(t)}{4t^{3/2}} - \frac{q_0(t)}{4t^{5/2}} \]

The function \( \mathcal{E}(t) \) can be written in terms of \( h \) as follows:

\[ \mathcal{E}(t) = t^{1/2} \int_{\infty}^{t} \tau^{-1/2} \int_{\infty}^{\tau} s^{-1/2} e^{-s} h(s) ds d\tau \]

We write (113) in integral form

\[ h(t) = h_0(t) - \int_{\infty}^{t} \frac{q_0(\tau)e^\tau}{2\tau} h(\tau) d\tau + \int_{\infty}^{t} e^\tau B(\tau)\mathcal{E}(\tau) d\tau - \int_{\infty}^{t} \frac{h(\tau)\mathcal{E}(\tau)}{2\tau} d\tau =: \mathcal{N}[h](\tau), \]

where

\[ h_0(t) = -\int_{\infty}^{t} e^{\tau} \tau^{1/2} R(\tau) d\tau. \]

We will analyze (116) to find a unique exponentially decaying \( h \) (cf. (118)), and then determine \( \mathcal{E} \) from (115).

**Remark 6.** The functions, \( q_0, \mathcal{E}, h \) and \( R \) depend on \( c \) as well. For simplicity, our notation we will not show this dependence, except when needed.

We will prove that the operator \( \mathcal{N} \) defined in (116) is contractive in some small ball in the space \( \mathcal{H} \) defined as follows:
Using (106) and (117) we obtain

**Lemma 9.** For $t$ in $[T, \infty)$ we have

\begin{align}
\|h_0\| &\leq |c|^3 \left( \frac{1}{2} R_{3,m} + \frac{|c|e^{-T}}{3\sqrt{T}} R_{4,m} \right) \leq |c|^3 \left( \frac{1}{2} R_{3,m} + \frac{|c|e^{-T}}{3\sqrt{T}} R_{4,m} \right) \\
\|\partial_c h_0\| &\leq c^2 \left( \frac{3}{2} R_{3,m} + \frac{4|c|e^{-T}}{3\sqrt{T}} R_{4,m} \right)
\end{align}

where $R_{3,m}$ and $R_{4,m}$ are upper bounds for $|R_3(t)|$ and $|R_4(t)|$ in $[T, \infty)$.

**Proof.** Using (106) and (117) we obtain

\begin{align}
|h_0(t)| &\leq \int_t^\infty \left( |c|^3 e^{-2\tau} \tau^{-1} R_{3,m} + c^4 e^{-3\tau} |R_{3,m}^2| R_{4,m} \right) d\tau \\
&\leq \frac{|c|^3 e^{-2T}}{2T} R_{3,m} + \frac{c^4 e^{-3T}}{3T^{3/2}} R_{4,m}
\end{align}

implying (119). After differentiating (117) with respect to $c$, and using (106), (120) follows similarly.

**Lemma 10.**

\begin{align}
|\mathcal{E}(t)| &\leq \frac{1}{9t^{3/2}} e^{-3t} \|h\|, \quad |\partial_c \mathcal{E}(t)| \leq \frac{1}{9t^{3/2}} e^{-3t} \|\partial_c h\|
\end{align}

**Proof.** We note that

\begin{align}
\left| \int_\tau^\infty s^{-1/2} e^{-s} h(s) ds \right| &\leq \frac{1}{3} \tau^{-3/2} \|h\| e^{-\tau}
\end{align}

Using the inequality above it follows that

\begin{align}
|t^{1/2} \int_t^\infty \tau^{-1/2} \int_\tau^\infty s^{-1/2} e^{-s} h(s) ds| &\leq \frac{1}{9} t^{-3/2} e^{-3t} \|h\|
\end{align}

The bounds on $\partial_c \mathcal{E}$ are obtained in a similar way since (115) implies

\begin{align}
\partial_c \mathcal{E}(t; c) = t^{1/2} \int_\tau^t \tau^{-1/2} \int_\tau^\infty s^{-1/2} e^{-s} \partial_c h(s; c) ds d\tau
\end{align}

Let

\begin{align}
d_0 = \frac{e^{-T}}{\sqrt{T}} \left| Q_2(T) + 0.0944 e^{-6.159T} \right|
\end{align}

**Lemma 11.** For $t \geq T$, $q_0$ (cf. (101)) satisfies the following bounds

\begin{align}
|q_0| &\leq \frac{|c|e^{-t}}{\sqrt{T}} (1 + d_0 |c|) \leq \frac{|c|e^{-T}}{\sqrt{T}} (1 + d_0 |c|) =: q_{0,m} \\
|\partial_c q_0| &\leq \frac{e^{-t}}{\sqrt{T}} (1 + 2d_0 |c|) \leq \frac{e^{-T}}{\sqrt{T}} (1 + 2d_0 |c|) =: q_{0,c,m}
\end{align}
The result follows from Lemma 11, which implies

\[ \partial_t \left\{ q_0 - \frac{q_0}{2t} \right\} \leq \frac{e^{-t}}{\sqrt{t}} \left( 1 + \frac{3|c|e^{-T}}{4T^{3/2}} \right) \cdot \left( 1 + \frac{3|c|e^{-T}}{2T^{3/2}} \right) =: q_{0,d,c,m} \]

Proof. Using the integral representation for \( q_0 \), in the appendix it is shown (see (102) and (103)) that

\[ q_0(t) = \left\{ \frac{1}{2\sqrt{T}} \left( \left| q_{2}(T) \right| + 0.09044e^{-s_0T} \right) \right\} \]

where \( s_0 = 6.159 \cdots \).\implies (124) and (125). Straightforward calculations show that

\[ q_0'(t) - \frac{q_0(t)}{2t} = -\frac{ce^{-t}}{\sqrt{t}} \left( 1 - I_0 \right) - \frac{c^2 e^{-2t}}{2t} \left( 1 - I_1 - \frac{I_1^2}{t} + \frac{T^2}{4t^2} \right) \]

where \( I_0 \) is defined in (102) and \( I_1 = 2tI_0 \). The integral representation (102) implies that \( I_1 := 2tI_0 \in (0,1) \) and that \( 1 - I_1 = \frac{\tau}{	au + I_2} \), where

\[ I_2(t) = t \int_0^\infty \frac{e^{-st}ds}{(1 + s)^{5/2}} \]

and thus

\[ q_0(t) - \frac{q_0(t)}{2t} = -\frac{ce^{-t}}{\sqrt{t}} \left( 1 - I_0 \right) - \frac{c^2 e^{-2t}}{4t^2} \left( 3I_2 - 2I_1 + \frac{T^2}{2t} \right) \]

From the equation above and its \( c \)-derivative, and the fact that \( I_0, I_1, I_2 \) are in \( (0,1) \), \((126)\) and \((127)\) follow.

With \( d_0 \) given in \( (123) \) let

\[ d_q = \frac{|c|}{4} T^{-3/2} (1 + |c|d_0); \quad d_{q,c} = \frac{1}{4} T^{-3/2} (1 + 2|c|d_0) \]

Lemma 12. For \( t \geq T \geq 1 \), we have

\[ \left\| \int_\infty^t \frac{q_0(\tau)e^\tau}{2\tau} h(\tau)d\tau \right\| \leq d_q \|h\|, \quad \left\| \int_\infty^t \frac{\partial_\tau q_0(\tau)e^\tau}{2\tau} h(\tau)d\tau \right\| \leq d_{q,c} \|h\|, \]

Proof. The result follows from Lemma 11 which implies

\[ \left\| \frac{e^\tau q_0(\tau)}{2\tau} \right\| \leq \frac{c}{2T^{3/2}} (1 + |c|d_0) \]

\[ \left\| \frac{e^\tau \partial_\tau q_0(\tau)}{2\tau} \right\| \leq \frac{1}{2T^{3/2}} (1 + 2|c|d_0) \]

and noting that \( |h(t)| \leq t^{-1} e^{-2t} \|h\| \).

Let

\[ V(t;c) = \frac{2}{c} e^t B(t;c) \]
Lemma 13. B defined in \([113]\) satisfies the following inequalities for \(t \geq T \geq 1\)

\[
(133) \quad |B(t; c)| \leq \frac{|c|e^{-t}}{2t} \left( 1 + \frac{3|c|e^{-t}}{4T^{3/2}} \right) \leq \frac{|c|e^{-T}}{2T} \left( 1 + \frac{3|c|e^{-T}}{4T^{3/2}} \right) =: B_m
\]

\[
(134) \quad \partial_c B(t; c) \leq \frac{e^{-t}}{2t} \left( 1 + \frac{3|c|e^{-t}}{2T^{3/2}} \right) \leq \frac{e^{-T}}{2T} \left( 1 + \frac{3|c|e^{-T}}{2T^{3/2}} \right) =: B_{m,c}
\]

\[
(135) \quad \partial_c \left[ \frac{2t}{c} B(t; c) \right] \leq e^{-t} \left( 1 + \frac{ce^{-t}}{T^{3/2}} \right) \leq e^{-T} \left( 1 + \frac{ce^{-T}}{T^{3/2}} \right) =: B_{m,2,t}
\]

\[
(136) \quad \left| \partial_c \left[ \frac{2t}{c} B(t; c) \right] \right| \leq \frac{3e^{-2t}}{4T^{3/2}} \leq \frac{3e^{-2T}}{4T^{3/2}} =: B_{m,2,c}
\]

\[
V(t; c) \text{ defined in } \[132\] \text{satisfies }
\]

\[
(137) \quad V_m := 1 + \frac{3ce^{-T}}{4T^{3/2}} \geq |V(t; c)| \geq 1 - \frac{ce^{-T}}{4T^{3/2}} =: V_{\min}
\]

\[
(138) \quad \left| V'(t; c) \right| \leq \frac{ce^{-T}}{2T^{3/2}} =: V_{d,m}
\]

\[
(139) \quad \left| \partial_c V(t; c) \right| \leq \frac{e^{-T}}{4T^{3/2}} =: V_{c,m}
\]

Proof. Using \([101]\) in \([114]\) it follows that

\[
(140) \quad B(t; c) = -\frac{ce^{-t}}{2t} - \frac{e^2c^{-2t}}{8t^{5/2}} (3I_2 - I_1) , \text{ where } I_1 = 2tI_0 = t \int_0^\infty \frac{e^{-st}}{(1+s)^{3/2}}ds , I_2(t) = t \int_0^\infty \frac{e^{-st}ds}{(1+s)^{3/2}}
\]

\[
(141) \quad \partial_c B(t; c) = -\frac{e^{-t}}{2t} - \frac{c e^{-2t}}{4t^{5/2}} (3I_2 - I_1)
\]

\[
(142) \quad \partial_t \frac{2tB(t; c)}{c} = e^{-t} + \frac{c e^{-2t}}{4t^{3/2}} (3I_2 + I_1)
\]

\[
(143) \quad \partial_c \frac{2tB(t; c)}{c} = -\frac{e^{-2t}}{4t^{3/2}} (3I_2 - I_1)
\]

from which, again using the fact that \(I_2, I_1 \in (0,1)\), \([133]\)–\([136]\). To prove \([137]\)–\([139]\), note that

\[
(144) \quad V(t; c) := -\frac{2}{c}te^t B(t; c) = 1 + \frac{ce^{-t}}{4T^{3/2}} (3I_2 - I_1) ,
\]

\[
(145) \quad V'(t; c) = -\frac{ce^{-t}}{2T^{3/2}} I_1(t)
\]

\[
(146) \quad \partial_c V(t; c) = \frac{e^{-t}}{4T^{3/2}} (3I_2 - I_1)
\]
Lemma 14. For $T \geq 1$ we have

\begin{equation}
\left| \int_{t}^{T} e^{r} B(r) E(r) dr \right| \leq d_{B} \| h \|, \quad \left| \int_{1}^{T} e^{r} \partial_{c} B(r) E(r) dr \right| \leq d_{B, c} \| h \|
\end{equation}

where

\begin{equation}
d_{B} = \frac{|c| e^{-T}}{54T^{3/2}} \left(1 + |c| d_{1}\right), \quad d_{B, c} = \frac{e^{-T}}{54T^{3/2}} \left(1 + 2|c| d_{1}\right), \quad \text{where} \quad d_{1} = \frac{3e^{-T}}{4T^{3/2}}
\end{equation}

\begin{proof}
Using (122) and (133) in Lemma 13 the result follows immediately by integration.
\end{proof}

Lemma 15. For $T > 0$ we have,

\begin{equation}
\left| \int_{t}^{\infty} h(t) E(t) dt \right| \leq \frac{\| h \|^2}{90T^{3/2}} e^{-3T}
\end{equation}

\begin{proof}
Using Lemma 10 and the fact $\| h(t) \| \leq t^{-1} e^{-2t} \| h \|$ (which follows from the fact that $h \in \mathcal{H}$, the result follows by integration.
\end{proof}

Proposition 16. For $|c| \leq \frac{1}{4}$, $\varepsilon = 0.03$ and $T \geq 1.99$, there exists a unique solution to the integral equation (116) in a ball of size $(1 + \varepsilon) \| h_{0} \|$, implying that $\| h \| \leq (1 + \varepsilon) \| h_{0} \| \leq 1.6667 \times 10^{-4}$.

\begin{proof}
For $T \geq 1.99$, $|c| \leq \frac{1}{4}$ and $\varepsilon = 0.03$, by Lemmas 112, 14 and using $Q_{2}(1.99) = 0.0147 \cdots$ and inequalities $R_{3, m} \leq 0.0205666, R_{4, m} \leq 0.009042$ for $T \geq 1.99$, see (216) and (225) in the appendix, we get

\begin{equation}
\| N[h] \| \leq \| h_{0} \| + (d_{q} + d_{B})(1 + \varepsilon) \| h_{0} \| + \frac{e^{-3T}}{90T^{3/2}} (1 + \varepsilon)^{2} \| h_{0} \|^{2} \leq (1 + \varepsilon) \| h_{0} \|
\end{equation}

\begin{equation}
\| N[h_{1}] - N[h_{2}] \| \leq \left\{ d_{q} + d_{B} + \frac{e^{-3T}}{45T^{5/2}} (1 + \varepsilon) \| h_{0} \| \right\} \| h_{1} - h_{2} \|
\end{equation}

implying contractivity of the integral operator in the stated ball.
\end{proof}

Remark 7. For $|c| < \frac{1}{4}$ and $T \geq 1.99$, $h \in \mathcal{H}$ implies

\begin{equation}
|h(t)| \leq \| h \| t^{-e^{-2t}} \leq 1.6667 \times 10^{-4} \times T^{-1} e^{-2T} =: h_{m} \leq 1.5651 \times 10^{-6}, \forall t \geq T
\end{equation}

Remark 8. By uniqueness, this is the only solution with $h \to 0$ as $t \to \infty$; we have proved that such a solution $h \in \mathcal{H}$.

Remark 9. Proposition 16 extends to any $c$ if $T$ is large enough, as seen in the next proposition. This is likely to be useful in extending the present techniques to more general initial conditions than [5].

Proposition 17. For any $c$, there exists $T \geq 1$ large enough so that the integral equation (116) has a unique solution $h \in \mathcal{H}$.

\begin{proof}
It is clear from Lemmas 3, 12 and 14 that for any given $c$, the functions $d_{q}, d_{B}$ and $\| h_{0} \|$ are decreasing in $T$. Thus, the conditions (150) - (151) are met for any fixed $\varepsilon > 0$.
\end{proof}

\footnote{The values of the error function can be calculated using, for instance, [1], 7.1.28.}
Lemma 18. For $0 < a \leq a_r$, $|c| < \frac{1}{4}$ and $t \geq T \geq 1.99$, the function $E$ (see (115)) satisfies following bounds

$$
\left| \frac{a}{2t} E \right| \leq \sqrt{\frac{a}{2} \frac{1}{9t^2}} \|h\| \leq 1.69 \times 10^{-5} t^{-2} e^{-3t}
$$

$$
\left| \frac{d}{dx} \frac{a}{2t} E \right| \leq \frac{a}{3} e^{-3t} t^{-3/2} \|h\| \leq 9.20 \times 10^{-5} t^{-3/2} e^{-3t}
$$

$$
\left| \frac{d^2}{dx^2} \frac{a}{2t} E \right| \leq \sqrt{2} a^{3/2} \|h\| t^{-1} e^{-3t} \leq 5.02 \times 10^{-4} t^{-1} e^{-3t}
$$

Proof. From (115), the first statement follows immediately. The second statement follows from noting that the transformation (117) implies that

$$
\frac{d}{dx} \frac{a}{2t} E = a \left( E' - \frac{1}{2t} E \right) = a \int_\infty^t \tau^{-1/2} e^{-\tau} h(\tau) d\tau
$$

Furthermore, we can check

$$
\frac{d^2}{dx^2} \frac{a}{2t} E(t(x)) = \sqrt{2} a^{3/2} e^{-t} h,
$$

d and hence the third statement follows. □

Lemma 19. The function $h$ satisfies

$$
\|h'(t; \cdot)\| \leq 2d_q \|h\| + \frac{B_m}{9T^{1/2}} \|h\| + \frac{e^{-3T}}{18T^{5/2}} \|h\|^2 + |c|^3 \left( R_{3,m} + \frac{|c| e^{-T}}{T^{1/2}} R_{4,m} \right)
$$

Proof. We note from (113)

$$
\|h'(t; \cdot)\| \leq \sup_{t \geq T} \frac{|q| e^{-t}}{2t} \|h\| + \sup_{t \geq T} \frac{1}{9t} B(t) \|h\| + \frac{e^{-3T}}{18T^{5/2}} \|h\|^2 c^3 R_{3,m} + c^4 e^{-t} \frac{1}{T^{1/2}} R_{4,m}
$$

Using Lemma 12 and and 13 we get the statement follows. □

Lemma 20. The function $h$ satisfies

$$
\|\partial_c h(t; \cdot)\| \leq \left( 1 - d_q - d_B - \frac{e^{-3T}}{45T^{5/2}} \|h\| \right)^{-1} \left\{ \|\partial_c h_0\| + (d_{q,c} + d_{B,c}) \|h\| \right\}
$$

Proof. We note that (116) implies

$$
\partial_c h(t; \cdot) = \partial_c h_0(t; \cdot) - \int_\infty^t \frac{e^{-\tau}}{2\tau} \partial_c q_0(\tau; \cdot) h(\tau; \cdot) + \int_\infty^t e^\tau \partial_c B(\tau; \cdot) E(\tau; \cdot) - \int_\infty^t \frac{e^{-\tau}}{2\tau} q_0(\tau; \cdot) \partial_c h(\tau; \cdot) d\tau - \int_\infty^t e^\tau B(\tau; \cdot) \partial_c E(\tau; \cdot) d\tau
$$

Applying Lemmas 10, 9, 12 and 13 to (153) we get

$$
\|\partial_c h(t; \cdot)\| \leq \|\partial_c h_0\| + (d_{q,c} + d_{B,c}) \|h\| + \left( d_q + d_B + \frac{e^{-3T}}{45T^{5/2}} \|h\| \right) \|\partial_c h(\cdot; \cdot)\| \|h\| \leq \left( 1 - d_q - d_B - \frac{e^{-3T}}{45T^{5/2}} \|h\| \right)^{-1} \left\{ \|\partial_c h_0\| + (d_{q,c} + d_{B,c}) \|h\| \right\}.
$$

□
Remark 10. Since \( h \in \mathcal{H} \), Lemmas \([19,20]\) imply
\[
|h'(T; c)| \leq T^{-1} e^{-2T} \left( 2d_q \|h\| + \frac{B_m}{9} \|h\| + \frac{e^{-3T}}{18T^{3/2}} \|h\|^2 + |c|^3 \left( R_{3,m} + \frac{|c|e^{-T}}{T^{1/2}} R_{4,m} \right) \right) =: h_{d,m}
\]
\[
|\partial_c h(T; c)| \leq T^{-1} e^{-2T} \left( 1 - d_q - d_B - \frac{e^{-3T}}{45T^{3/2}} \|h\| \right)^{-1} \left\{ \|\partial_c h_0\| + (d_q + d_B) \|h\| \right\} =: h_{c,m}
\]

3.1. End of proof of Proposition \([3]\). For given \( |c| < \frac{1}{4} \) and \( T \geq 1.99 \), Proposition \([17]\) implies that \( E \) satisfies \([19]\) for \( t \geq T \geq 1.99 \). This implies the existence of a solution \( q = q_0 + E \) satisfying \([18]\) and having \( t^{1/2} e^{-t} \) decay as \( t \to \infty \). From Lemmas \([6,7]\) this is the only solution for which \( \frac{c}{\sqrt{t}} \to 0 \) as \( t \to \infty \), Proposition \([3]\) follows since the transformation \([27]\) for \( a > 0 \) in the regime \( t \geq T \geq 1.99 \) guarantees that \( F \) will satisfy \([4]\) for \( x \geq \frac{5}{2} \).

4. Matching and proof of Proposition \([3]\)

In order for the two representations \([59,9]\) and \([70,11]\) to coincide at \( x = \frac{5}{2} \) we match \( F \) and its two derivatives; from \([101,102,111]\) we get
\[
a = F'(\frac{5}{2}) - a \left( q'_0(t_m; c) - \frac{q_0(t_m; c)}{2t_m} \right) - a \int_{\tau=5}^{t_m} e^{-s} \frac{\sqrt{1}}{\sqrt{T}} h(\tau; c) d\tau =: N_1(a,b,c)
\]
\[
b = F'(\frac{5}{2}) \frac{5}{2} N_1 - \sqrt{\frac{a}{2t_m}} q_0(t_m; c) - \sqrt{\frac{a}{2}} \int_{\tau=5}^{t_m} s^{-1/2} \int_{\tau=5}^{t_m} s^{-1/2} e^{-s} h(s; c) ds := N_2(a,b,c)
\]
\[
c = \frac{1}{\sqrt{2e^{3/2}}} \left[ \left( V(t_m; c) + \frac{1}{c} h(t_m; c) \right) \right]^{-1} e^{t_m} F''(\frac{5}{2}) =: N_3(a,b,c)
\]

Definition 21. We define \( A = (a, b, c) \), \( A_0 = (a_0, b_0, c_0) \) and \( N(A) = (N_1, \frac{1}{2} N_2, \frac{5}{2} N_3) \). Define also
\[
S_A := \{ \|A - A_0\|_2 \leq \rho_0 := 5 \times 10^{-5} \}
\]
where \( \|\cdot\|_2 \) is the Euclidean norm and let
\[
J = \begin{pmatrix}
\partial_a N_1 & 2\partial_b N_1 & 2\partial_c N_1 \\
\frac{1}{2} \partial_a N_2 & \partial_b N_2 & \partial_c N_2 \\
\frac{1}{2} \partial_a N_3 & \partial_b N_3 & \partial_c N_3 
\end{pmatrix}
\]

Note 11. We see that \( A \in S_A \) implies \((a,b,c) \in S\). The system of equations \([157,158]\) is written as
\[
A = N[A]
\]
We define \( J = \frac{\partial N}{\partial A} \) to be the Jacobian and \( \|J\|_2 \) denotes the \( l^2 \) (Euclidean) norm of the matrix. We note that
\[
\|J\|_2^2 = (\partial_a N_1)^2 + 4 (\partial_b N_1)^2 + 4 (\partial_c N_1)^2 + \frac{1}{4} (\partial_a N_2)^2 + (\partial_b N_2)^2 + (\partial_c N_2)^2 + \frac{1}{4} (\partial_a N_3)^2 + (\partial_b N_3)^2 + (\partial_c N_3)^2
\]
The mean-value theorem implies
\[
\rho_0 = \| A_0 - N(A_0) \|_2 \leq \| A_0 - N(A_0) \|_2 + \| N(A) - N(A_0) \|_2 \leq \rho_0 (1 - \alpha) + \| J \|_2 \rho_0 \leq \rho_0
\]
and also, if \( A_1, A_2 \in S_A \):
\[
\| N(A_1) - N(A_2) \|_2 \leq \| J \|_2 \| A_1 - A_2 \|_2 \leq \alpha \| A_1 - A_2 \|_2
\]
Thus, (163) and (164) imply that \( N : S_A \to S_A \) and that it is contractive there; the result follows from the contractive mapping theorem.

4.1. Proof of Proposition 4. Proposition 4 follows from Lemma 22 once we show that (163) and (164) hold. In the following two subsections it will be shown that \( \alpha \leq 0.764 \) and that \( \| A_0 - N(A_0) \|_2 \leq 1.16 \times 10^{-5} \leq (1 - \alpha) \rho_0 \) thereby completing the proof of Proposition 4.

Remark 12. Note that the proof of Proposition 4 only requires smallness of the norms of \( h \) and \( E \) (we recall that \( F = F_0 + E \)) and on no further details about \( F \). If in some application \( F_0 \) needs to be made \( C^2 \), then this can be ensured by iterating \( N \) with \( h = E = 0 \); the first thirteen digits obtained in this way are given in (16).

4.2. Bounds on \( \| N(A_0) - A_0 \| \). We note that
\[
\| a_0 - N_1(a_0, b_0, c_0) \| \leq \| a_0 - F_0'(\frac{a_0}{2}) + a_0 \left( \frac{q_0(t_{m,0}; c_0) e^{-2t_{m,0} \| h \|}}{2t_{m,0}} \right) \| + \| E'(\frac{a_0}{2}) + \frac{a_0}{3} t_{m,0}^{-3/2} e^{-3t_{m,0} \| h \|} \| \leq 4.81 \times 10^{-6}
\]

(166)
\[
\| b_0 - N_2(a_0, b_0, c_0) \| \leq \| b_0 - F_0'(\frac{a_0}{2}) + \frac{a_0}{6} t_{m,0}^{-3/2} e^{-3t_{m,0} \| h \|} \| \leq 1.64 \times 10^{-5}
\]

(167)
\[
\| c_0 - N_3(a_0, b_0, c_0) \| \leq \| c_0 - \frac{1}{\sqrt{2a_0}} \left( V(t_{m,0}; c_0) \right) \| + \| E''(\frac{a_0}{2}) + \frac{a_0}{2} t_{m,0}^{-3/2} e^{-3t_{m,0} \| h \|} \| \leq 1.33 \times 10^{-5}
\]

(168)
\[
\| A_0 - N(A_0) \|_2 \leq 1.16 \times 10^{-5}
\]

implying
\[
\| A_0 - N(A_0) \|_2 \leq 1.16 \times 10^{-5}
\]
4.3. Bounds on the derivatives of $N_j$ and on $\|J\|_2$. We note that

$$
\partial_a N_1 = -\left( q'_0(t_m; c) - \frac{q_0(t_m; c)}{2t_m} \right) + a \left( \frac{25}{4} - \frac{b^2}{a^2} \right) t^{1/2} B(t_m; c) - \int_{-\infty}^{t_m} e^{-\tau} h(\tau; c) d\tau - \frac{a}{2t_m} \left( \frac{25}{4} - \frac{b^2}{a^2} \right) e^{-t_m h(t_m)}
$$

Remark 7 implies that the last two terms on the rhs of (170) are bounded by

$$
t_m^{-1/2} e^{-t_m h_m} \left[ 1 + \frac{a}{2} \left( \frac{25}{4} - \frac{b^2}{a^2} \right) \right].
$$

Applying now Lemmas 11, 13 to (170) we get

$$
\left| \partial_a N_1 \right| \leq \left| -q'_0(t_m,0, c_0) + \frac{q_0(t_m,0, c_0)}{2t_m} \right| + a_0 \left( \frac{25}{4} - \frac{b^2}{a^2} \right) t^{1/2} B(t_m,0; c_0)
$$

$$
+ \left\{ \frac{25}{4} (a_r - a_0) + \left( \frac{b^2}{a_l} - \frac{b^2}{a_0} \right) \right\} t^{1/2} B_m + e^{-t_m^1} h_m \left\{ 1 + \frac{a_r}{2} \left[ \frac{25}{4} - \frac{b^2}{a^2} \right] \right\}
$$

$$
+ \left\{ 2t^{1/2} B_m + a_0 \left( \frac{25}{4} - \frac{b^2}{a_0} \right) \left( B_m + \frac{c_r}{2} B_m,0, c_0 \right) \right\} (t_m,r - t_m,0)
$$

$$
+ \left( q_0, d,c,m + a_0 \left( \frac{25}{4} - \frac{b^2}{a_0} \right) \sqrt{t_m, r} B_m, c \right) (c_r - c_0) \leq 0.081
$$

The maximal value of the bounds is attained when $c = c_r$ and $T = t_m, l$, which we used to get the results above.

$$
\partial_c N_1 = \sqrt{2a} \left[ 2t_m B(t_m; c) - e^{-t_m h(t_m; c)} \right]
$$

hence using (152) and Lemma 13

$$
\left| \partial_b N_1 \right| \leq \sqrt{2a} \left( 2t_m, r B_m + e^{-t_m^1 h_m} \right) \leq 0.059
$$

$$
\partial_c N_1 = -a \partial_c \left\{ q'_0(t_m; c) - \frac{q_0(t_m)}{2t_m} \right\} - a \int_{-\infty}^{t_m} e^{-\tau} \sqrt{\tau} \partial_c h(\tau; c) d\tau
$$

implying from Lemmas 11 and equation (160),

$$
\left| \partial_c N_1 \right| \leq a_r q_0, d,c,m + a_r e^{-t_m^1 t_m^{1/2}} h_{c,m} \leq 0.163
$$

We now consider

$$
\partial_a N_2 = \frac{5}{2} \partial_a N_1 - \frac{1}{2} \sqrt{\frac{1}{2t_m}} q_0(t_m) - \frac{1}{2} \sqrt{\frac{a}{2t_m}} \left( \frac{25}{4} - \frac{b^2}{a^2} \right) \left( q'_0(t_m) - \frac{q_0(t_m)}{2t_m} \right)
$$

$$
- \frac{1}{2} \sqrt{\frac{1}{2t_m}} e(t_m) - \frac{1}{2} \sqrt{\frac{a}{2t_m}} \left( \frac{25}{4} - \frac{b^2}{a^2} \right) \int_{-\infty}^{t_m} e^{-\tau} h(\tau; c) d\tau
$$
It follows from Lemmas 9, 11, 13, Proposition 16 and equations 152 and 156 that

\[
\left| \partial_a N_2 \right| \leq \frac{5}{2} \left| \partial_a N_1 \right| + \frac{1}{2} \sqrt{\frac{1}{2a \tau m,l}} \left| q_0(t_m,0;c_0) + a_0 \left( \frac{25}{4} - \frac{b_2^2}{a_0^2} \right) \left( q_0'(t_m,0;c_0) - \frac{q_0(t_m,0;c_0)}{2t_m,0} \right) \right|
\]

\[
+ \frac{1}{2} \sqrt{\frac{1}{2a \tau m,l}} \left\{ \frac{25}{4} (a_r - a_0) + \left( \frac{b_2^2}{a_1} - \frac{b_5^2}{a_0} \right) \right\} q_{0,d,m} + \frac{e^{-t_m,r}}{2^{3/2} a_1^{1/2} t_m,l} \left( 1 + a_r \left[ \frac{25}{4} - \frac{b_2^2}{a_1^2} \right] \right) h_m
\]

\[
+ \frac{1}{2} \sqrt{\frac{1}{2a \tau m,l}} \left\{ q_{0,d,m} + \frac{q_0.m}{2t_m,l} + a_0 \left( \frac{25}{4} - \frac{b_2^2}{a_0^2} \right) 2t_m,r B_m \right\} (t_m,r - t_m,0)
\]

\[
+ \frac{1}{2} \sqrt{\frac{1}{2a \tau m,l}} \left\{ q_{0,c,m} + a_0 \left( \frac{25}{4} - \frac{b_2^2}{a_0^2} \right) q_{0,d,c,m} \right\} (c_r - c_0) \leq 0.232
\]

We also note that

\[
\partial_b N_2 = -\frac{5}{2} \partial_b N_1 - \left( q_0'(t_m;c) - \frac{q_0(t_m;c)}{2t_m} \right) - \int_{\infty}^{t_m} \frac{d\tau}{\sqrt{\tau}} e^{-\tau} h(\tau;c),
\]

and therefore Lemma 11 and 152 imply

\[
\left| \partial_b N_2 \right| \leq \frac{5}{2} \left| \partial_b N_1 \right| + q_{0,d,m} + e^{-t_m,r} t_m,l h_m \leq 0.168
\]

Now,

\[
\partial_c N_2 = -\frac{5}{2} \partial_c N_1 - \sqrt{\frac{a}{2t_m}} \partial_c q_0(t_m;c) - \sqrt{\frac{a}{2}} \int_{\infty}^{t_m} \tau^{-1/2} \int_{\infty}^{\tau} s^{-1/2} e^{-s} \partial_c h(s;c) ds d\tau
\]

Hence, 156 and Lemma 11 imply

\[
\left| \partial_c N_2 \right| \leq \frac{5}{2} \left| \partial_c N_1 \right| + \sqrt{\frac{a_r}{2t_m,l}} q_{0,c,m} + \sqrt{\frac{a_r^{-1}}{2}} t_m^{-1} h_{c,m} e^{-t_m,r} \leq 0.468
\]

Also,

\[
\partial_c N_3 = \frac{1}{2 \sqrt{2a} a^{3/2}} \left[ V(t_m;c) + \frac{1}{c} h(t_m;c) \right]^{-1} e^{t_m} F''(\frac{5}{2}) \left\{ -\frac{3}{a} + \left( \frac{25}{4} - \frac{b_2^2}{a^2} \right) \right\}
\]

\[
\times \left( 1 + \left[ V(t_m;c) + \frac{1}{c} h(t_m;c) \right]^{-1} \left[ V'(t_m;c) + \frac{1}{c} h'(t_m;c) \right] \right)
\]

Therefore, from Lemma 134, equations 152, 155 and the positivity of $V_{\min} - \frac{h_m}{c_l}$, $F_0''$ and $\frac{25}{4} - \frac{b_2^2}{a^2} - \frac{3}{a}$ (see Definition 11 26 and 274 and 367); in 367 we used $c = c_r$, $T = t_m,l$, which minimize $V_m$. It follows that

\[
\left| \partial_a N_3 \right| \leq \frac{1}{2 \sqrt{2a} a^{3/2}} \left[ V_{\min} - \frac{h_m}{c_l} \right]^{-1} e^{t_m,r} \left( F_0''(\frac{5}{2}) + |E''(\frac{5}{2})| \right) \left\{ \frac{25}{4} - \frac{b_2^2}{a_r^2} - \frac{3}{a} + \left( \frac{25}{4} - \frac{b_2^2}{a_r^2} \right) \right\}
\]

\[
\times \left[ V_{\min} - \frac{h_m}{c_l} \right]^{-1} \left[ V_{d,m} + \frac{1}{c_l} h_{d,m} \right] \leq 0.44
\]
Further,

\[
\partial_b N_3 = \frac{\sqrt{t_m}}{a^3} \exp(t_m) \left[ V(t_m; c) + \frac{1}{c} h(t_m; c) \right]^{-2} \left[ V(t_m; c) + \frac{1}{c} h(t_m; c) - V'(t_m; c) - \frac{1}{c} h'(t_m; c) \right]
\]

Using again Lemma 13 and equations (152), (155) we get

\[
\left| \partial_b N_3 \right| \leq \frac{\sqrt{t_m}}{a^2} \left[ \left| F'(\frac{y}{2}) \right| + \left| F''(\frac{y}{2}) \right| \right] \left[ V_{\text{min}} - \frac{h_m}{c} \right]^{-2} \times \left\{ V_m + \frac{1}{c} h_m + V_{d,m} + \frac{1}{c} h_{d,m} \right\} \leq 0.384
\]

Furthermore,

\[
\partial_c N_3 = -\frac{1}{\sqrt{2} a^{3/2}} \exp(t_m) \left[ F'(\frac{y}{2}) + |F''(\frac{y}{2})| \right] \left[ V_{\text{min}} - \frac{h_m}{c} \right]^{-2} \left[ \partial_c V(t_m; c) + \frac{1}{c} \partial_c h(t_m; c) - \frac{1}{c^2} h(t_m; c) \right]
\]

Lemma 13 and equations (152), (156) imply

\[
\left| \partial_c N_3 \right| \leq \frac{e^{t_m}}{\sqrt{2} a^{3/2}} \left[ F'(\frac{y}{2}) + |F''(\frac{y}{2})| \right] \left[ V_{\text{min}} - \frac{h_m}{c} \right]^{-2} \left[ V_{c,m} + \frac{1}{c} h_{c,m} + \frac{1}{c^2} h_m \right] \leq 0.0029
\]

By straightforward calculations we get \( \| J \|_2 \leq 0.764 \).

5. APPENDIX

5.1. Bounds on \( q_0 \) and \( \partial_c q_0 \). Using the integral representations of \( I_0 \) and \( J_0 \), (101), (102) and (103) imply

\[
q_0(t) = \frac{c}{\sqrt{t}} e^{-t} \int_0^\infty \frac{e^{-st}}{(1+s)^{3/2}} ds + c^2 e^{-2t} \int_0^\infty e^{-st} U(s) ds,
\]

where

\[
U(s) = \frac{1}{2(1+s/2)^{3/2}} - \frac{1}{2(1+s)^{3/2}} - \frac{s}{(2+s)^2 \sqrt{1+s}}
\]

Note that

\[
Q_2(t) = \int_0^\infty e^{-st} U(s) ds
\]

Making the change of variable

\[
s = -1 + \frac{(y-1)^2}{4y}
\]

which maps one-to-one \( (0, \infty) \) onto \( (3 + 2\sqrt{2}, \infty) \) we obtain from (189)

\[
U(s(y)) = \frac{4y^{3/2}(3-2\sqrt{2})(y-3-2\sqrt{2})}{(y-1)^3(1+y)^4} P_3(y)
\]

where

\[
P_3(y) = -y^3 + (17 + 10\sqrt{2})y^2 - (11 + 4\sqrt{2})y + 3 + 2\sqrt{2}
\]
This cubic has only one real root $y_0 = 30.604\ldots$ implying $s = s_0 = 6.159\ldots$
Since $P(y) > 0$ for $y < y_0$ and $P(y) < 0$ for $y > y_0$, $s > s_0$ corresponds to $y_0 > y > 3 + 2\sqrt{2}$ and $s > s_0$ corresponds to $y > y_0$, it follows from \[192\] that
\begin{equation}
U(s) > 0 \text{ in } (0, s_0) \text{ and } U(s) < 0 \text{ in } (s_0, \infty)
\end{equation}

Further, for $s > s_0$, (i.e. $y > y_0 \approx 30.604\ldots$), the functions
\begin{equation}
y^{-3}P_3(y), \quad \frac{y^{1/2}}{(y-1)} \quad \text{and} \quad \frac{4y^4}{(y-3-2\sqrt{2})(y-1)^2(1+y)^4}
\end{equation}
are decreasing. Therefore, it follows from \[192\] that for $s \geq s_0$, $(y > y_0)$
\begin{equation}
0 > U(s(y)) \geq -\frac{4y^{9/2}(3-2\sqrt{2})(y-3-2\sqrt{2})}{(y-1)^3(1+y)^4} \geq -\frac{4y_0^{9/2}(3-2\sqrt{2})(y_0-3-2\sqrt{2})}{(y_0-1)^3(1+y_0)^4} = -0.09437\ldots
\end{equation}
Therefore,
\begin{equation}
0 \geq \int_{s_0}^{\infty} U(s)e^{-st}ds > -\frac{0.0944}{t}e^{-s_0 t}
\end{equation}
Since $U(s)$ is being positive on $(0, s_0)$ (see \[194\])
\begin{equation}
\int_{0}^{s_0} U(s)e^{-st}ds
\end{equation}
is clearly a decreasing positive function of $t$ for $t \geq T$ at thus attains its maximum at $t = T$. Therefore for $t \geq T$ we have
\begin{equation}
\int_{0}^{\infty} U(s)e^{-st}dt \leq \int_{0}^{s_0} U(s)e^{-st}ds \leq \int_{0}^{\infty} U(s)e^{-sT}ds + \frac{0.0944}{t}e^{-s_0 T}
\end{equation}
We conclude that
\begin{equation}
\left|q_0(t)\right| \leq \frac{|c|e^{-t}}{\sqrt{t}} + c^2 e^{-2t} \frac{Q_2(T)}{T} + \frac{c^2}{T} e^{-2t} 0.0944 e^{-s_0 T}
\end{equation}
We note that that since $s_0 = 6.159\ldots$, for $T \geq 1.99$, $0.0944 e^{-s_0 T} \leq 4.5 \times 10^{-7}$. The numerical value $Q_2(1.99) = 0.0147\ldots$ can be easily obtained from rigorous formulas (e.g. \[7.1.28\]) By differentiating $q_0$ with respect to $c$ we get in a similar way
\begin{equation}
\left|\partial_c q_0(t)\right| \leq \frac{e^{-t}}{\sqrt{t}} + \frac{2|c|e^{-2t}}{T} Q_2(T) + \frac{2c}{T} e^{-2t} 0.0944 e^{-s_0 T}
\end{equation}
5.2. **Bounds on** $R_3$ **for** $t \geq T \geq 1.99$. In the formula \[109\] for $R_3$, $I_0$ and $J_0$ have integral representations, see \[102\] and \[103\]. Using these representations we obtain
\begin{equation}
I_0(t) = -\frac{1}{4} \int_{0}^{\infty} e^{-st} \int_{0}^{s} \frac{d\tau}{(1+\tau)^{3/2}(1+(s-\tau))^{3/2}} = -\int_{0}^{\infty} \frac{se^{-st}}{(s+2)^{3/2}\sqrt{1+s}} ds
\end{equation}
\begin{equation}
- tI_0(t) = -\int_{0}^{\infty} e^{-st}\partial_s \left(\frac{s}{(s+2)^{3/2}\sqrt{1+s}}\right) ds = \int_{0}^{\infty} e^{-st} \frac{3s^2 - 4}{2(s+2)^{3/2}(s+1)^{3/2}} ds
\end{equation}
Adding the expressions above, we obtain from \[109\]
\begin{equation}
R_3(t) = \int_{0}^{\infty} ds e^{-st} R_3(s) ds,\end{equation}
Clearly \( s \) is increasing in \( y \) and maps \( (3 + 2\sqrt{2}, \infty) \) to \((0, \infty)\). Thus

\[
R_3(s(y)) = \frac{4y^{3/2}(2 - \sqrt{2})(y - 3 - 2\sqrt{2})}{(y + 1)^6(y - 1)^3} P_3(y),
\]

where

\[
P_3(y) = 1 - 21 - 10\sqrt{2} + (42 - 2\sqrt{2})y^2 - (42 + 2\sqrt{2})y^3 + (10\sqrt{2} + 21)y^4 - y^5
\]

Elementary inequalities imply that \( P_3 \) has a real root \( y_0 \in (33.851, 33.852) \) corresponding to \( s = s_0 \in (6.9701, 6.9704) \). By factoring out \( y - y_0 \) and expanding the rest in powers of \( y - 5 \), we find

\[
P_3(y) = (y - y_0) \left( A_0 + A_1(y - 5) + A_2(y - 5)^2 + A_3(y - 5)^3 - (y - 5)^4 \right)
\]

where

\[
A_0 = -(y_0^2 + (10\sqrt{2} + 16)y_0^2 + (48\sqrt{2} + 38)y_0^3 + (238\sqrt{2} + 232)y_0 + 1139 + 1200\sqrt{2})
\]

\[
A_1 = -(y_0^2 + (11 + 10\sqrt{2})y_0^2 + (98\sqrt{2} + 93)y_0 + 728\sqrt{2} + 697)
\]

\[
A_2 = -(y_0^2 + (10\sqrt{2} + 6)y_0 + 148\sqrt{2} + 123)
\]

\[
A_3 = 1 + 10\sqrt{2} - y_0
\]

It is readily checked that for \( y_0 \) in the interval \((33.851, 33.852)\), the coefficients \( A_0, \ldots, A_3 \) are negative, implying that there has only one zero for \( y \geq 3 + 2\sqrt{2} > 5 \), namely \( y = y_0 \). This immediately implies that for \( s \in (0, s_0) \) where \( s_0 = 6.97 \cdots \) we have \( R_3(s) > 0 \), while for \( s > s_0 \), \( R_3 < 0 \).

We now minimize \( R_3(s(y)) \) for \( y > y_0 = 33.851 \cdots \). By simple estimates of the derivative, \( y^{-3}P_3(y) \) is seen to be decreasing, and \( 0 \geq y^{-5}P_3 > -1 \) for \( y \in [y_0, \infty) \). In this interval \( y - 3 - 2\sqrt{2} \leq y - 1 \) and thus for \( s > s_0 \), \( P_3(y) \geq 0 > 0 \) we obtain using (206), that

\[
R_3(s(y)) \geq -\frac{4y^{13/2}(2 - \sqrt{2})}{(y + 1)^6(y - 1)^2} \geq -\frac{4y_0^{13/2}(2 - \sqrt{2})}{(y_0 + 1)^6(y_0 - 1)^2} \geq -0.0107,
\]

since \( 4y^{13/2}(2 - \sqrt{2})(y + 1)^{-6}(y - 1)^{-2} \) is decreasing for \( y \in (y_0, \infty) \). Therefore since for \( t \geq T \),

\[
0 \leq \int_{s_0}^{s_0} e^{-st}R_3(s)ds \leq \int_{0}^{s_0} e^{-st}R_3(s)ds,
\]

\[
0 \leq -\int_{s_0}^{\infty} e^{-st}R_3(s)ds \leq -\int_{s_0}^{\infty} e^{-st}R_3(s)ds \leq \frac{0.0107}{T} e^{-s_0T},
\]

it follows that

\[
R_3(t) \leq R_3(T) + \frac{0.0107}{T} e^{-s_0T} \leq R_3(T) + 1.02 \times 10^{-8} =: R_{3,m} \leq 0.02057
\]
5.3. Estimating $R_4$. Using (102)–(103) in (110) we get

$$R_4(t) = \frac{1}{8t} (J_1 - I_1)(1 - I_1) - \frac{I_1}{16t^2} (J_1 - I_1^2) + \frac{I_3^3}{4}$$

where

$$I_1(t) = 2t I_0(t) = t \int_0^\infty \frac{e^{-st}}{(1 + s)^3/2} ds \in (0, 1)$$

$$J_1(t) = 4t J_0(t) = t \int_0^\infty \frac{e^{-st}}{(1 + s/2)^3/2} ds \in (0, 1)$$

Clearly $J_1(t) \geq I_1(t)$. This implies that

$$- \frac{1}{16t^2} (J_1 - I_1^2) \leq R_4(t) \leq \frac{1}{8t} (J_1 - I_1) + \frac{I_3^3}{4}$$

Now,

$$\frac{1}{t} (J_1 - I_1) = \int_0^\infty \frac{(1 + s)^{3/2} - (1 + s/2)^{3/2} e^{-st}}{(1 + s/2)^{3/2}(1 + s)^{3/2}} ds$$

which is clearly decreasing in $t$ as is $I_0$ (since they are Laplace transforms of positive functions), and therefore they attain their maximum at $t = T \geq 1.99$. Furthermore, it can be checked that

$$\frac{1}{16t^2} (J_1 - I_1^2) = \frac{1}{8t} \int_0^\infty e^{-st} \left\{ \frac{16s\sqrt{1 + s} + 10s^2\sqrt{1 + s} + 2s^3\sqrt{1 + s} + 3s^2\sqrt{4 + 2s + 8\sqrt{1 + s} + 8\sqrt{1 + s/2}}}{(2 + s)^3 \sqrt{4 + 2s} (1 + s)^{3/2}} \right\} ds$$

Arguing in the same way, we see that $\frac{1}{16t^2} (J_1 - I_1^2)$ is decreasing and attains its maximum at $t = T$, as does $\frac{I_3^3}{4}$. Evaluating $I_0$ and $J_0$ (see, once more, [1], 7.1.28) we get

$$0 \leq \frac{1}{8t} (J_1(t) - I_1(t)) + \frac{I_3^3(t)}{4} \leq \frac{1}{8T} (J_1(T) - I_1(T)) + \frac{I_3^3(T)}{4} \leq - \frac{1}{2} J_0(T) - \frac{1}{4} I_0(T) + \frac{I_3^3(T)}{4} \leq 0.009042$$

and

$$0 \leq \frac{1}{16t^2} (J_1 - I_1^2) \leq \frac{1}{16T^2} (J_1(T) - I_1^2(T)) = \frac{J_0(T)}{4T} - \frac{I_0^2(T)}{4} \leq 0.00572$$

and therefore, for $t \geq T \geq 1.99$, we have

$$|R_4(t)| \leq R_{4,m} \leq 0.009042$$

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