Abstract We give a necessary and sufficient mean condition for the quotient of two Jensen functionals and define a new class \( \Lambda_{f,g}(a,b) \) of mean values where \( f, g \) are continuously differentiable convex functions satisfying the relation \( f''(t) = tg''(t), t \in \mathbb{R}^+ \). Then we asked for a characterization of \( f, g \) such that the inequalities \( H(a,b) \leq \Lambda_{f,g}(a,b) \leq A(a,b) \) or \( L(a,b) \leq \Lambda_{f,g}(a,b) \leq I(a,b) \) hold for each positive \( a, b \), where \( H, A, L, I \) are the harmonic, arithmetic, logarithmic and identric means, respectively. For a subclass of \( \Lambda \) with \( g''(t) = t^s, s \in \mathbb{R}, \) this problem is thoroughly solved.

1. Introduction

1.1 It is said that the mean \( P \) is intermediate relating to the means \( M \) and \( N \), \( M \leq N \) if the relation

\[
M(a,b) \leq P(a,b) \leq N(a,b),
\]

holds for each two positive numbers \( a, b \).

It is also well known that

\[
\min\{a,b\} \leq H(a,b) \leq G(a,b) \leq L(a,b) \leq I(a,b) \leq A(a,b) \leq S(a,b) \leq \max\{a,b\},
\]

(1)

where

\[
H = H(a,b) := 2(1/a + 1/b)^{-1}; \quad G = G(a,b) := \sqrt{ab}; \quad L = L(a,b) := \frac{b-a}{\log b - \log a};
\]

\[
I = I(a,b) := (b^b/a^a)^{1/(b-a)}/e; \quad A = A(a,b) := \frac{a+b}{2}; \quad S = S(a,b) := a^{a/b}b^{b/a},
\]

are the harmonic, geometric, logarithmic, identric, arithmetic and Gini mean, respectively.

An easy task is to construct intermediate means related to two given means \( M \) and \( N \) with \( M \leq N \). For instance, for an arbitrary mean \( P \), we have that

\[
M(a,b) \leq P(M(a,b), N(a,b)) \leq N(a,b).
\]

The problem is more difficult if we have to decide whether the given mean is intermediate or not. For example, the relation

\[
L(a,b) \leq S_s(a,b) \leq I(a,b),
\]

1
holds for each positive \(a\) and \(b\) if and only if \(0 \leq s \leq 1\), where the Stolarsky mean \(S_s\) is defined by (cf [4])

\[
S_s(a, b) := \left( \frac{b^s - a^s}{s(b - a)} \right)^{1/(s-1)}.
\]

Also,

\[
G(a, b) \leq A_s(a, b) \leq A(a, b),
\]

holds if and only if \(0 \leq s \leq 1\), where the Hölder mean of order \(s\) is defined by

\[
A_s(a, b) := \left( \frac{a^s + b^s}{2} \right)^{1/s}.
\]

An inverse problem is to find best possible approximation of a given mean \(P\) by elements of an ordered class of means \(S\). A good example for this topic is comparison between the logarithmic mean and the class \(A_s\) of Hölder means of order \(s\). Namely, since \(A_0 = \lim_{s \to 0} A_s = G\) and \(A_1 = A\), it follows from (1) that

\[
A_0 \leq L \leq A_1.
\]

Since \(A_s\) is monotone increasing in \(s\), an improving of the above is given by Carlson [2]:

\[
A_0 \leq L \leq A_{1/2}.
\]

Finally, Lin shoved in [3] that

\[
A_0 \leq L \leq A_{1/3},
\]

is the best possible approximation of the logarithmic mean by the means from the class \(A_s\).

Numerous similar results have been obtained recently. For example, an approximation of Seiffert’s mean by the class \(A_s\) is given in [6], [8].

In this article we shall give best possible approximations for a whole variety of elementary means (1) by the class \(\lambda_s\) defined below (see Thm 3.).

1. 2. Let \(f, g\) be twice continuously differentiable (strictly) convex functions on \(\mathbb{R}^+\). By definition (cf [1], p. 5),

\[
\bar{f}(a, b) := f(a) + f(b) - 2f\left( \frac{a + b}{2} \right) > 0, \quad a \neq b,
\]

and

\[
\bar{f}(a, b) = 0,
\]

if and only if \(a = b\).

It turns out that the expression

\[
\Lambda_{f,g}(a, b) := \frac{\bar{f}(a, b)}{g(a, b)} = \frac{f(a) + f(b) - 2f\left( \frac{a + b}{2} \right)}{g(a) + g(b) - 2g\left( \frac{a + b}{2} \right)},
\]

represents a mean of two positive numbers \(a, b\); that is, the relation

\[
\min\{a, b\} \leq \Lambda_{f,g}(a, b) \leq \max\{a, b\},
\]

(2)
holds for each $a, b \in \mathbb{R}^+$, if and only if the relation
\[ f''(t) = tg''(t), \quad \text{(3)} \]
holds for each $t \in \mathbb{R}^+$.

Let $f, g \in C^\infty(0, \infty)$ and denote by $\Lambda$ the set \{(f, g)\} of convex functions satisfying the relation (3). There is a natural question how to improve the bounds in (2); in this sense we come upon the following intermediate mean problem:

**Open question** Under what additional conditions on $f, g \in \Lambda$, the inequalities 
\[ H(a, b) \leq \Lambda_{f,g}(a, b) \leq A(a, b), \]
or, more tightly, 
\[ L(a, b) \leq \Lambda_{f,g}(a, b) \leq I(a, b), \]
hold for each $a, b \in \mathbb{R}^+$?

As an illustration, consider the function $f_s(t)$ defined to be 
\[ f_s(t) = \begin{cases} 
(t^s - st + s - 1)/s(s - 1), & s(s - 1) \neq 0; \\
t - \log t - 1, & s = 0; \\
t \log t - t + 1, & s = 1.
\end{cases} \]

Since 
\[ f'_s(t) = \begin{cases} 
\frac{t^{s-1} - 1}{s-1}, & s(s - 1) \neq 0; \\
1 - \frac{1}{t}, & s = 0; \\
\log t, & s = 1,
\end{cases} \]
and 
\[ f''_s(t) = t^{s-2}, \quad s \in \mathbb{R}, \quad t > 0, \]
it follows that $f_s(t)$ is a twice continuously differentiable convex function for $s \in \mathbb{R}, \quad t \in \mathbb{R}^+$.

Moreover, it is evident that $(f_{s+1}, f_s) \in \Lambda$.

We shall give in the sequel a complete answer to the above question concerning the means 
\[ \tilde{f}_{s+1}(a, b)/\tilde{f}_s(a, b) := \lambda_s(a, b) \]
defined by 
\[ \lambda_s(a, b) = \begin{cases} 
\frac{a^{s+1} + b^{s+1} - 2(\frac{a+b}{a+b})^{s+1}}{s+1} & s \in \mathbb{R}/\{-1, 0, 1\}; \\
2 \log \frac{a+b}{a-b} - \log a - \log b & s = -1; \\
a \log a + b \log b - (a+b) \log \frac{a+b}{a-b} & s = 0; \\
2 \log \frac{a+b}{a-b} - \log a - \log b & s = 0; \\
(b-a)^2 & s = 1.
\end{cases} \]

Those means are obviously symmetric and homogeneous of order one.
As a consequence we obtain some new intermediate mean values; for instance, we show that the inequalities
\[ H(a, b) \leq \lambda_{-1}(a, b) \leq G(a, b) \leq \lambda_0(a, b) \leq L(a, b) \leq \lambda_1(a, b) \leq I(a, b), \]
hold for arbitrary \( a, b \in \mathbb{R}^+ \).

Note that
\[ \lambda_{-1} = \frac{2G^2 \log(A/G)}{A - H}; \quad \lambda_0 = A \frac{\log(S/A)}{\log(A/G)}; \quad \lambda_1 = \frac{1}{2} A - H \frac{\log(S/A)}{2 \log(S/A)}. \]

2. Results

We prove firstly the following

**Theorem 1** Let \( f, g \in C^2(I) \) with \( g'' > 0 \). The expression \( \Lambda_{f,g}(a, b) \) represents a mean of arbitrary numbers \( a, b \in I \) if and only if the relation
\[ f''(t) = tg''(t) \] (3)
holds for \( t \in I \).

**Remark 1** In the same way, for arbitrary \( p, q > 0, p + q = 1 \), it can be deduced that the quotient
\[ \Lambda_{f,g}(p, q; a, b) := \frac{pf(a) + qf(b) - f(pa + qb)}{pg(a) + qg(b) - g(pa + qb)} \]
represents a mean value of numbers \( a, b \) if and only if (3) holds.

A generalization of the above assertion is the next

**Theorem 2** Let \( f, g : I \to \mathbb{R} \) be twice continuously differentiable functions with \( g'' > 0 \) on \( I \) and let \( p = \{p_i\}, i = 1, 2, \ldots, \sum p_i = 1 \) be an arbitrary positive weight sequence. Then the quotient of two Jensen functionals
\[ \Lambda_{f,g}(p, x) := \frac{\sum_{i=1}^n p_i f(x_i) - f(\sum_{i=1}^n p_i x_i)}{\sum_{i=1}^n p_i g(x_i) - g(\sum_{i=1}^n p_i x_i)}, \quad n \geq 2, \]
represents a mean of an arbitrary set of real numbers \( x_1, x_2, \ldots, x_n \in I \) if and only if the relation
\[ f''(t) = tg''(t) \]
holds for each \( t \in I \).

**Remark 2** It should be noted that the relation \( f''(t) = tg''(t) \) determines \( f \) in terms of \( g \) in an easy way. Precisely,
\[ f(t) = tg(t) - 2G(t) + ct + d, \]
where \( G(t) := \int_1^t g(u)du \) and \( c \) and \( d \) are constants.

Our results concerning the means \( \lambda_s(a, b), s \in \mathbb{R} \) are included in the following

**Theorem 3** For the class of means \( \lambda_s(a, b) \) defined above, the following assertions hold for each \( a, b \in \mathbb{R}^+ \).
1. The means $\lambda_s(a, b)$ are monotone increasing in $s$;
2. $\lambda_s(a, b) \leq H(a, b)$ for each $s \leq -4$;
3. $H(a, b) \leq \lambda_s(a, b) \leq G(a, b)$ for $-3 \leq s \leq -1$;
4. $G(a, b) \leq \lambda_s(a, b) \leq L(a, b)$ for $-1/2 \leq s \leq 0$;
5. there is a number $s_0 \in (1/12, 1/11)$ such that $L(a, b) \leq \lambda_s(a, b) \leq I(a, b)$ for $s_0 \leq s \leq 1$;
6. there is a number $s_1 \in (1.03, 1.04)$ such that $I(a, b) \leq \lambda_s(a, b) \leq A(a, b)$ for $s_1 \leq s \leq 2$;
7. $A(a, b) \leq \lambda_s(a, b) \leq S(a, b)$ for each $2 \leq s \leq 5$;
8. there is no finite $s$ such that the inequality $S(a, b) \leq \lambda_s(a, b)$ holds for each $a, b \in \mathbb{R}^+$.

The above estimations are best possible.

3. Proofs

**Proof of Theorem 1** We prove firstly the necessity of the condition (3).

Since $\Lambda_{f,g}(a, b)$ is a mean value for arbitrary $a, b \in I; \ a \neq b$, we have

$$\min\{a, b\} \leq \Lambda_{f,g}(a, b) \leq \max\{a, b\}.$$ 

Hence

$$\lim_{{b \to a}} \Lambda_{f,g}(a, b) = a. \quad (4)$$

From the other hand, due to l’Hospital’s rule we obtain

$$\lim_{{b \to a}} \Lambda_{f,g}(a, b) = \lim_{{b \to a}} \left( \frac{f'(b) - f'(\frac{a+b}{2})}{g'(b) - g'(\frac{a+b}{2})} \right) = \lim_{{b \to a}} \left( \frac{2f''(b) - f''(\frac{a+b}{2})}{2g''(b) - g''(\frac{a+b}{2})} \right)$$

$$= \frac{f''(a)}{g''(a)}. \quad (5)$$

Comparing (4) and (5) the desired result follows.

Suppose now that (3) holds and let $a < b$. Since $g''(t) > 0$ for $t \in [a, b]$ by the Cauchy mean value theorem there exists $\xi \in (\frac{a+t}{2}, t)$ such that

$$\frac{f'(t) - f'(\frac{a+t}{2})}{g'(t) - g'(\frac{a+t}{2})} = \frac{f''(\xi)}{g''(\xi)} = \xi. \quad (6)$$

But,

$$a \leq \frac{a+t}{2} < \xi < t \leq b,$$

and, since $g'$ is strictly increasing, $g'(t) - g'(\frac{a+t}{2}) > 0$, $t \in [a, b]$. 

Therefore, by (6) we get
\[ a(g'(t) - g'(\frac{a + t}{2})) \leq f'(t) - f'(\frac{a + t}{2}) \leq b(g'(t) - g'(\frac{a + t}{2})). \]  \hspace{1cm} (7)

Finally, integrating (7) over \( t \in [a, b] \) we obtain the assertion from Theorem 1.

**Proof of Theorem 2** We shall give a proof of this assertion by induction on \( n \).

By Remark 1, it holds for \( n = 2 \).

Next, it is not difficult to check the identity
\[
\sum_{i=1}^{n} p_i f(x_i) - f(\sum_{i=1}^{n} p_i x_i) = (1 - p_n)(\sum_{i=1}^{n-1} p'_i f(x_i) - f(\sum_{i=1}^{n-1} p'_i x_i)) \\
+ [(1 - p_n)f(T) + p_n f(x_n) - f((1 - p_n)T + p_n x_n)],
\]
where
\[
T := \sum_{i=1}^{n-1} p'_i x_i; \quad p'_i := p_i/(1 - p_n), \quad i = 1, 2, \ldots, n - 1; \quad \sum_{i=1}^{n-1} p'_i = 1.
\]

Therefore, by induction hypothesis and Remark 1, we get
\[
\sum_{i=1}^{n} p_i f(x_i) - f(\sum_{i=1}^{n} p_i x_i) \leq \max\{x_1, x_2, \ldots, x_{n-1}\}(1 - p_n)(\sum_{i=1}^{n-1} p'_i f(x_i) - f(\sum_{i=1}^{n-1} p'_i x_i)) \\
+ \max\{T, x_n\}[(1 - p_n)g(T) + p_n g(x_n) - g((1 - p_n)T + p_n x_n)] \\
\leq \max\{x_1, x_2, \ldots, x_n\}((1 - p_n)(\sum_{i=1}^{n-1} p'_i g(x_i) - g(\sum_{i=1}^{n-1} p'_i x_i)) \\
+ [(1 - p_n)g(T) + p_n g(x_n) - g((1 - p_n)T + p_n x_n))] \\
= \max\{x_1, x_2, \ldots, x_n\}(\sum_{i=1}^{n} p_i g(x_i) - g(\sum_{i=1}^{n} p_i x_i)).
\]

The inequality
\[
\min\{x_1, x_2, \ldots, x_n\} \leq A_{f,g}(p, x),
\]
can be proved analogously.

For the proof of necessity, put \( x_2 = x_3 = \cdots = x_n \) and proceed as in Theorem 1.

**Remark** It is evident from (3) that if \( I \subseteq \mathbb{R}^+ \) then \( f \) has to be also convex on \( I \). Otherwise, it shouldn't be the case. For example, the conditions of Theorem 2 are satisfied with \( f(t) = t^3/3, g(t) = t^2, t \in \mathbb{R} \).

Hence, for an arbitrary sequence \( \{x_i\}_1^n \) of real numbers, we obtain
\[
\min\{x_1, x_2, \ldots, x_n\} \leq \frac{\sum_{i=1}^{n} p_i x_i^3 - (\sum_{i=1}^{n} p_i x_i)^3}{3(\sum_{i=1}^{n} p_i x_i^2 - (\sum_{i=1}^{n} p_i x_i)^2)} \leq \max\{x_1, x_2, \ldots, x_n\}.
\]
Because the above inequality does not depend on \( n \), a probabilistic interpretation of the above result is contained in the following.

**Theorem 4.** For an arbitrary probability law \( F \) of random variable \( X \) with support on \((-\infty, +\infty)\), we have

\[
(EX)^3 + 3(\min X) \sigma_X^2 \leq EX^3 \leq (EX)^3 + 3(\max X) \sigma_X^2.
\]

**Proof of Theorem 3, part 1** We shall prove a general assertion of this type. Namely, for an arbitrary positive sequence \( x = \{x_i\} \) and an associated weight sequence \( p = \{p_i\}, i = 1, 2, \ldots \), denote

\[
\chi_s(p, x) := \begin{cases} 
\frac{\sum p_i x_i - (\sum p_i x_i)^s}{s^{s-1}}, & s \in \mathbb{R}/\{0, 1\}; \\
\log(\sum p_i x_i) - \sum p_i \log x_i, & s = 0; \\
\sum p_i x_i \log x_i - (\sum p_i x_i) \log(\sum p_i x_i), & s = 1.
\end{cases}
\]

For \( s \in \mathbb{R}, r > 0 \) we have

\[
\chi_s(p, x) \chi_{s+r+1}(p, x) \geq \chi_{s+1}(p, x) \chi_{s+r}(p, x),
\]

which is equivalent to

**Theorem 3a** The sequence \( \{\chi_{s+1}(p, x)/\chi_s(p, x)\} \) is monotone increasing in \( s, s \in \mathbb{R} \).

This assertion follows applying the result from ([5], Theorem 2) which states that

**Lemma 1** For \(-\infty < a < b < c < +\infty\), the inequality

\[
(\chi_b(p, x))^{c-a} \leq (\chi_a(p, x))^{c-b} (\chi_c(p, x))^{b-a},
\]

holds for arbitrary sequences \( p, x \).

Putting there \( a = s, b = s + 1, c = s + r + 1 \) and \( a = s, b = s + r, c = s + r + 1 \), we successively obtain

\[
(\chi_{s+1}(p, x))^{r+1} \leq (\chi_s(p, x))^r \chi_{s+r+1}(p, x),
\]

and

\[
(\chi_{s+r}(p, x))^{r+1} \leq \chi_s(p, x) (\chi_{s+r+1}(p, x))^r.
\]

Since \( r > 0 \), multiplying those inequalities we get the relation (4) i.e. the proof of Theorem 3a.

The part 1. of Theorem 3 follows for \( p_1 = p_2 = 1/2 \).

A general way to prove the rest of Theorem 3 is to use an easy-checkable identity

\[
\frac{\lambda_s(a, b)}{A(a, b)} = \lambda_s(1 + t, 1 - t),
\]

with \( t := \frac{b-a}{b+r-a} \).
Since $0 < a < b$, we get $0 < t < 1$. Also,

$$
\frac{H(a, b)}{A(a, b)} = 1 - t^2; \quad \frac{G(a, b)}{A(a, b)} = \sqrt{1 - t^2}; \quad \frac{L(a, b)}{A(a, b)} = \frac{2t}{\log(1 + t) - \log(1 - t)}; \quad (5)
$$

$$
\frac{I(a, b)}{A(a, b)} = \exp\left(\frac{(1 + t) \log(1 + t) - (1 - t) \log(1 - t)}{2t}\right); \quad \frac{S(a, b)}{A(a, b)} = \exp\left(\frac{1}{2}((1 + t) \log(1 + t) + (1 - t) \log(1 - t))\right).
$$

Therefore, we have to compare some one-variable inequalities and to check their validness for each $t \in (0, 1)$.

For example, we shall prove that the inequality

$$
\lambda_s(a, b) \leq L(a, b)
$$

holds for each positive $a, b$ if and only if $s \leq 0$.

Since $\lambda_s(a, b)$ is monotone increasing in $s$, it is enough to prove that

$$
\frac{\lambda_0(a, b)}{L(a, b)} \leq 1.
$$

By the above formulae, this is equivalent to the assertion that the inequality

$$
\phi(t) \leq 0 \quad (6)
$$

holds for each $t \in (0, 1)$, with

$$
\phi(t) := \frac{\log(1 + t) - \log(1 - t)}{2t}((1 + t) \log(1 + t) + (1 - t) \log(1 - t)) + \log(1 + t) + \log(1 - t).
$$

We shall prove that the power series expansion of $\phi(t)$ have non-positive coefficients. Thus the relation (6) will be proved.

Since

$$
\frac{\log(1 + t) - \log(1 - t)}{2t} = \sum_{k=0}^{\infty} \frac{t^{2k}}{2k + 1}; \quad \log(1 + t) + \log(1 - t) = -t^2 \sum_{k=0}^{\infty} \frac{t^{2k}}{k + 1};
$$

$$
(1 + t) \log(1 + t) + (1 - t) \log(1 - t) = t^2 \sum_{k=0}^{\infty} \frac{t^{2k}}{(k + 1)(2k + 1)},
$$

we get

$$
\phi(t)/t^2 = \sum_{n=0}^{\infty} \left(- \frac{1}{n + 1} + \sum_{k=0}^{n} \frac{1}{(2n - 2k + 1)(k + 1)(2k + 1)}\right) t^{2n} = \sum_{0}^{\infty} c_n t^{2n}.
$$

Hence,

$$
c_0 = c_1 = 0; \quad c_2 = -1/90,
$$
and, after some calculation, we get
\[ c_n = \frac{2}{(n+1)(2n+3)} \left( (n+2) \sum_{k=1}^{n} \frac{1}{2k+1} - (n+1) \sum_{k=1}^{n} \frac{1}{2k} \right), \quad n > 1. \]

Now, one can easily prove (by induction, for example) that
\[ d_n := (n+2) \sum_{k=1}^{n} \frac{1}{2k+1} - (n+1) \sum_{k=1}^{n} \frac{1}{2k} \]
is a negative real number for \( n \geq 2 \). Therefore \( c_n \leq 0 \), and the proof of the first part is done.

For \( 0 < s < 1 \) we have
\[ \frac{\lambda_s(a,b)}{L(a,b)} - 1 = \frac{(1-s)((1+t)^{s+1} + (1-t)^{s+1} - 2) \log \frac{1+t}{1-t}}{2t(1+s)(2 - (1+t)^s - (1-t)^s)} - 1 = \frac{1}{6} st^2 + O(t^4) \quad (t \to 0). \]

Therefore, \( \lambda_s(a,b) > L(a,b) \) for \( s > 0 \) and sufficiently small \( t := (b-a)/(b+a) \).

Similarly, we shall prove that the inequality
\[ \lambda_s(a,b) \leq I(a,b), \]
holds for each \( a, b; 0 < a < b \) if and only if \( s \leq 1 \).

As before, it is enough to consider the expression
\[ \frac{I(a,b)}{\lambda_1(a,b)} = e^{\mu(t)} \nu(t) := \psi(t), \]
with
\[ \mu(t) = \frac{(1+t) \log(1+t) - (1-t) \log(1-t)}{2t} - 1; \quad \nu(t) = \frac{(1+t) \log(1+t) + (1-t) \log(1-t)}{t^2}. \]

It is not difficult to check the identity
\[ \psi'(t) = -e^{\mu(t)} \phi(t)/t^3. \]

Hence by (6), we get \( \psi'(t) > 0 \) i. e. \( \psi(t) \) is monotone increasing for \( t \in (0, 1) \).

Therefore
\[ \frac{I(a,b)}{\lambda_1(a,b)} \geq \lim_{t \to 0^+} \psi(t) = 1. \]

By monotonicity it follows that \( \lambda_s(a,b) \leq I(a,b) \) for \( s \leq 1 \).

For \( s > 1 \), \( \frac{b-a}{b+a} = t \), we have
\[ \lambda_s(a,b) - I(a,b) = \left( \frac{1}{6} (s-1)t^2 + O(t^4) \right) A(a,b) \quad (t \to 0^+). \]
Hence, $\lambda_s(a, b) > I(a, b)$ for $s > 1$ and $t$ sufficiently small.

From the other hand,

$$\lim_{t \to 1^-} \left[ \frac{\lambda_s(a, b)}{I(a, b)} - 1 \right] = \frac{e(s - 1)(2^{s+1} - 2)}{2(s + 1)(2^s - 2)} - 1 := \tau(s).$$

Examining the function $\tau(s)$, we find out that it has the only real zero at $s_0 \approx 1.0376$ and is negative for $s \in (1, s_0)$.

**Remark 2** Since $\psi(t)$ is monotone increasing, we also get

$$\frac{I(a, b)}{\lambda_1(a, b)} \leq \lim_{t \to 1^-} \psi(t) = \frac{4\log 2}{e}.$$ 

Hence

$$1 \leq \frac{I(a, b)}{\lambda_1(a, b)} \leq \frac{4\log 2}{e}.$$ 

A calculation gives $\frac{4\log 2}{e} \approx 1.0200$.

Note also that

$\lambda_2(a, b) \equiv A(a, b)$.

Therefore, applying the assertion from the part 1., we get

$\lambda_s(a, b) \leq A(a, b)$, $s \leq 2$; $\lambda_s(a, b) \geq A(a, b)$, $s \geq 2$.

Finally, we give a detailed proof of the part 7.

We have to prove that $\lambda_s(a, b) \leq S(a, b)$ for $s \leq 5$. Since $\lambda_s(a, b)$ is monotone increasing in $s$, it is sufficient to prove that the inequality

$\lambda_5(a, b) \leq S(a, b)$

holds for each $a, b \in \mathbb{R}^+$.

Therefore, by the transformation given above, we get

$$\log \frac{\lambda_2}{A} = \log \left[ \frac{2(1 + t)^6 + (1 - t)^6 - 2}{3(1 + t)^5 + (1 - t)^5 - 2} \right] = \log \left[ \frac{2 + 15t^2 + t^4}{15} \right] \leq \log \left[ \frac{1 + t^2 + t^4/4}{1 + t^2/2} \right] = \log(1 + t^2/2) = t^2/2 - t^4/8 + t^6/24 - \cdots \leq t^2/2 + t^4/12 + t^6/30 + \cdots = \frac{1}{2}((1 + t) \log(1 + t) + (1 - t) \log(1 - t)) = \log S/A,$$

and the proof is done.

Further, we have to show that $\lambda_s(a, b) > S(a, b)$ for some positive $a, b$ whenever $s > 5$. 
Indeed, since
\[(1 + t)^s + (1 - t)^s - 2 = \left(\frac{s}{2}\right)t^2 + \left(\frac{s}{4}\right)t^4 + O(t^6),\]
for \(s > 5\) and sufficiently small \(t\), we get
\[
\frac{\lambda_s}{A} = \frac{s - 1}{s + 1} \left(\frac{s + 1}{2}\right)t^2 + \left(\frac{s - 1}{4}\right)t^4 + O(t^6)
\]
\[
\left.\quad = 1 + (s - 1)(s - 2)t^2/12 + O(t^4)\right|_{\text{for } s > 5} = 1 + (\frac{s - 1}{6})t^2 + O(t^4).
\]

Similarly,
\[
\frac{S}{A} = \exp\left(\frac{1}{2}((1 + t) \log(1 + t) + (1 - t) \log(1 - t))\right) = \exp(t^2/2 + O(t^4)) = 1 + t^2/2 + O(t^4).
\]

Hence,
\[
\frac{1}{A}(\lambda_s - S) = \frac{1}{6}(s - 5)t^2 + O(t^4),
\]
and this expression is positive for \(s > 5\) and \(t\) sufficiently small, i.e. \(a\) sufficiently close to \(b\).

As for the part 8., applying the above transformation we obtain
\[
\frac{\lambda_s(a,b)}{S(a,b)} = \frac{s - 1}{s + 1} \left(\frac{s + 1}{2}\right)t^2 + \left(\frac{s - 1}{4}\right)t^4 - 2 \exp\left(-\frac{1}{2}((1 + t) \log(1 + t) + (1 - t) \log(1 - t))\right),
\]
where \(0 < a < b\), \(t = \frac{b - a}{b + a}\).

Since for \(s > 5\),
\[
\lim_{t \to 1^-} \frac{\lambda_s}{S} = \frac{s - 1}{s + 1} \frac{2^s - 1}{2^s - 2^s}
\]
and the last expression is less than one, it follows that the inequality \(S(a,b) < \lambda_s(a,b)\) cannot hold whenever \(\frac{b}{a}\) is sufficiently large.

The rest of the proof is straightforward.

Acknowledgment. The author is indebted to the referees for valuable suggestions.

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