New Deformation of quantum oscillator algebra: Representation and some applications

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Abstract

This work addresses the study of the oscillator algebra, defined by four parameters $p$, $q$, $α$, and $ν$. The time-independent Schrödinger equation for the induced deformed harmonic oscillator is solved; explicit analytic expressions of the energy spectrum are given. Deformed states are built and discussed with respect to the criteria of coherent state construction. Various commutators involving annihilation and creation operators and their combinatorics are computed and analyzed. Finally, the correlation functions of matrix elements of main normal and antinormal forms, pertinent for quantum optics analysis, are computed.

1 Introduction

The deformation of the harmonic oscillator algebra whose applications in physics are presently rather technical but nonetheless very promising, possesses an important and useful representation theory in connection to that of their classical limit algebra. From the other side, there are some hopes that, in physical studies of nonlinear phenomena, the deformed oscillator can play the same role as the usual boson oscillator in usual nonrelativistic quantum mechanics. This could explain why various quantum deformations of boson oscillator commutation relations have attracted a great attention during the last few years (see [1]- [17] and references therein). This might be also due to the fact that there exist correspondences between quantum groups, quantum algebras, statistical mechanics, quantum field theory, conformal field theory, quantum and nonlinear optics and noncommutative geometry, etc. Furthermore, such a connection is extended to coherent states (cs) deducible from the study of quantum groups and, therefore, from the deformation of Heisenberg algebra. As a pertinent application in quantum optics, cs can be used to compute matrix elements, $\nu\langle z|a^m a^n|z\rangle_\nu$, $\nu\langle z|a^n a^m|z\rangle_\nu$, corresponding to normal and antinormal forms, respectively [17], (also called symmetric form), by using the normal product technique.
Note that the most spread, in the literature, multiparameter deformation of the harmonic oscillator is the so-called \((p, q)\)-deformed oscillator. For more details, see [5]. To cite an example, let us mention the \((p, q)\)-deformed oscillator algebra defined by the following Chakrabarti and Jagannathan’s commutation relations [7]:

\[
\begin{align*}
aa^\dagger - qa^\dagger a &= p^{-N}, \quad aa^\dagger - p^{-1}a^\dagger a = q^{N}, \\
[N, a] &= -a, \quad [N, a^\dagger] = a^\dagger,
\end{align*}
\]

where \(p, q \in \mathbb{R}; a, a^\dagger\) and \(N = a^\dagger a\) are the annihilation, creation and number operators, respectively. This deformation has been generalized in various ways in the literature [1] [6] and [8].

This paper extend the result on [8] by introducing two new positive parameters functions \(\phi_1\) and \(\phi_2\). The spectrum and states of this new class of quantum oscillator algebra are given. We also analyzed the coherent states and correlation functions useful for the study of quantum optics properties.

The work is organized as follows. In section 2, we give the preliminary and definitions permitted to define properly the mean result of our work. Section 3 is devoted to compute related deformed states. Relevant correlation functions as well as new identities are also built in section 4. We give in Section 5 a discussion of a particular case of algebra [3]. We end with the concluding and remarks in section 6.

### 2 New class of deformation quantum oscillator algebra

We start from the following mains definitions of the \((p, q)\)-deformed oscillator algebra and its generalization.

- The \((p, q)\)-oscillator algebra is generated by three elements \(a, a^\dagger\) and \(N\) obeying the relation [7]

\[
\begin{align*}
aa^\dagger - p^{-1}a^\dagger a &= q^{N}, \quad aa^\dagger - qa^\dagger a = p^{-N}, \\
[N, a] &= -a, \quad [N, a^\dagger] = a^\dagger.
\end{align*}
\]

- The \((q, p; \alpha, \beta, l)\)-deformed canonical commutation relations defined in [6] is written as follows

\[
\begin{align*}
aa^\dagger - q^{-l}a^\dagger a &= p^{-\alpha N - \beta}, \quad aa^\dagger - p^{-l}a^\dagger a = q^{\alpha N + \beta} \\
[N, a] &= -\frac{l}{\alpha} a, \quad [N, a^\dagger] = \frac{l}{\alpha} a^\dagger.
\end{align*}
\]

It is worth noticing that further generalization involving a new parameters \(\phi_1\) and \(\phi_2\) generate a richer algebra, with novel interesting properties, as we will see in the sequel. In this case, we arrive at the following definition.

**Definition1: (The deformed algebra)** The deformed algebra generated by the operators \(\{1, a, a^\dagger, N\}\) is defined in this paper by the relations

\[
\begin{align*}
aa^\dagger - q^\nu a^\dagger a &= \phi_1(p, q)p^{-\alpha N}, \quad aa^\dagger - p^{-\nu}a^\dagger a = \phi_2(p, q)q^{\alpha N}, \\
[N, a^\dagger] &= \frac{\nu}{\alpha} a^\dagger, \quad [N, a] = -\frac{\nu}{\alpha} a.
\end{align*}
\]

where \(\phi_1(p, q)\) and \(\phi_2(p, q)\) are two non singular and real valued positive functions of deformation parameters \(p\) and \(q\).

**Remark1:** It’s important to notify immediately that in the limit when \(\nu, \alpha \to 1\) and \(\phi_1(p, q) = \phi_2(p, q) = 1\), one recovers the algebra studied by Chakrabarti et al [7]. In the same manner if \(\phi_1(p, q) = \phi_2(p, q) = 1\), one recovers the algebra studied by Chakrabarti et al [7].
Proposition 1

The states $|n\rangle$ are built as follows: if $pq < 1$ and $\phi_2(p,q) < \phi_1(p,q)$

$$|n\rangle = \frac{p^{nu(n-1)/4}}{\sqrt{\tau^p_{n-1,q}(\phi_1)(\phi_1^{-1}(p,q)\phi_2(p,q)(pq)^{\alpha\chi_0+n\nu};(pq)^{\nu})}} a^\dagger |0\rangle, \quad n \geq 0,$$

and if $pq > 1$ and $\phi_1(p,q) < \phi_2(p,q)$

$$|n\rangle = \frac{q^{-nu(n-1)/4}}{\sqrt{\tau^q_{n-1,q}(\phi_1)(\phi_2^{-1}(p,q)\phi_1(p,q)(pq)^{-\alpha\chi_0-n\nu};(pq)^{-\nu})}} a^\dagger |0\rangle, \quad n \geq 0.$$

The states (6) and (7) satisfy the orthogonality and completeness conditions

$$\langle m|n\rangle = \delta_{mn}, \quad \sum_{n=0}^{\infty} |n\rangle \langle n| = 1.$$

In order to understand and study properly this new oscillator algebra we calculate the actions of the deformed operators $a$, $a^\dagger$ and $N$ on $\mathcal{F}$ and get

- if $pq < 1$ and $\phi_2(p,q) < \phi_1(p,q)$

$$a|n\rangle = \tau^p_{n-1,q}(\phi_1(p,q)) p^{-(n-1)\nu/2} \sqrt{1 - \frac{\phi_2(p,q)}{\phi_1(p,q)} (pq)^{\alpha\chi_0+n\nu}} |n-1\rangle.$$

- if $pq > 1$ and $\phi_1(p,q) < \phi_2(p,q)$

$$a|n\rangle = \tau^q_{n-1,q}(\phi_2(p,q)) q^{(n-1)\nu/2} \sqrt{1 - \frac{\phi_1(p,q)}{\phi_2(p,q)} (pq)^{-\alpha\chi_0-n\nu}} |n-1\rangle.$$

- if $pq < 1$ and $\phi_2(p,q) < \phi_1(p,q)$

$$a^\dagger |n\rangle = \tau^p_{n,q-1}(\phi_1(p,q)) p^{-n\nu/2} \sqrt{1 - \frac{\phi_2(p,q)}{\phi_1(p,q)} (pq)^{\alpha\chi_0+(n+1)\nu}} |n+1\rangle.$$

- if $pq > 1$ and $\phi_1(p,q) < \phi_2(p,q)$

$$a^\dagger |n\rangle = \tau^q_{n,q-1}(\phi_2(p,q)) q^{n\nu/2} \sqrt{1 - \frac{\phi_1(p,q)}{\phi_2(p,q)} (pq)^{-\alpha\chi_0-(n+1)\nu}} |n+1\rangle.$$
Let $\tau_{p,q}(t) = \frac{tp^\alpha q^\nu + \nu}{p^\nu - q^\nu}$, it gets even

$$N|n\rangle = (\chi_0 + n)|n\rangle. \quad (13)$$

From (9)-(12), one can deduce that

$$aa^\dagger|n\rangle = \frac{\phi_1(p,q)p^{-\alpha \chi_0 -(n+1)\nu} - \phi_2(p,q)q^\alpha \chi_0 + (n+1)\nu}{p^{-\nu} - q^\nu}|n\rangle, \quad (14)$$

$$a^\dagger a|n\rangle = \frac{\phi_1(p,q)p^{-\alpha \chi_0 - n\nu} - \phi_2(p,q)q^\alpha \chi_0 + n\nu}{p^{-\nu} - q^\nu}|n\rangle. \quad (15)$$

Finally we may conclude that, the states $|n\rangle$ solve the time-independent Schrödinger equation of the deformed oscillator Hamiltonian $H = a^\dagger a + aa^\dagger$, i.e. $H|n\rangle = E_{n,\alpha}^\nu(p,q)|n\rangle$, with the corresponding eigenvalue $E_{n,\alpha}^\nu(p,q)$ given by

$$E_{n,\alpha}^\nu(p,q) = \tau_{p^{-1},q}(\phi_1(p,q))p^{-(n+1)\nu}\left\{1 + p^{-\nu} - \frac{\phi_2(p,q)}{\phi_1(p,q)}(pq)^{\alpha \chi_0 + n\nu}(1 + q^{\nu})\right\}, \quad \text{with } pq < 1 \text{ and } \phi_2(p,q) < \phi_1(p,q) \quad (16)$$

$$E_{n,\alpha}^\nu(p,q) = \tau_{q,p^{-1}}(\phi_2(p,q))q^{(n+1)\nu}\left\{1 + q^{\nu} - \frac{\phi_1(p,q)}{\phi_2(p,q)}(pq)^{\alpha \chi_0 + n\nu}(1 + p^{-\nu})\right\}, \quad \text{with } pq > 1 \text{ and } \phi_1(p,q) < \phi_2(p,q). \quad (17)$$

It follows from (16) and (17) that the spectrum of the deformed Hamiltonian $H$ is symmetric under the change $q \rightarrow p^{-1}, \phi_1(p,q) \rightarrow \phi_2(p,q)$.

**Proposition 2** For $A$ and $B \in L(L)$, where $L(L)$ is the set of linear operators acting on Fock space $L$ we consider the $p^\nu$ and $q^{-\nu}$ commutators defined by: $[A, B]_{q^\nu} = AB - q^\nu BA$ and $[A, B]_{p^{-\nu}} = AB - p^{-\nu}BA$, the following brackets hold

$$[a, a^\dagger m_{+1}]_{q^\nu} = a^\dagger m\left(\frac{\phi_1(p,q)p^{-\alpha N(p^{-m+1}\nu - q^{\nu}) - \phi_2(p,q)q^\alpha N(q^{m+1}\nu - q^{\nu})}{p^{-m} - q^{\nu}}\right),$$

$$[a, a^\dagger m_{+1}]_{p^{-\nu}} = a^\dagger m\left(\frac{\phi_1(p,q)p^{-\alpha N(p^{-m+1}\nu - p^{-\nu}) - \phi_2(p,q)q^\alpha N(q^{m+1}\nu - p^{-\nu})}{p^{-m} - q^{\nu}}\right). \quad (19)$$

**Proof** The proof can be performed by using equation (16). ■

For $n, m \in \mathbb{N} \setminus \{0\}$, the expressions of operators $a^n a^\dagger m$ and $a^\dagger m a^n$ are given in different cases by the following relations

- For $n < m$

  $$a^n a^\dagger m = p^{-\nu(\frac{n}{2})} T_{q^{-1},p}(\phi_2(p,q))\left\{\frac{\phi_1(p,q)}{\phi_2(p,q)}(pq)^{\alpha N + \nu}; (pq)^\nu\right\}_n a^\dagger m - n \quad (20)$$

  if $pq < 1$ and $\phi_2(p,q) < \phi_1(p,q)$.

  $$a^n a^\dagger m = q^{\nu(\frac{n}{2})} T_{q^{-1},p}(\phi_2(p,q))\left\{\frac{\phi_1(p,q)}{\phi_2(p,q)}(pq)^{-\alpha N - \nu}; (pq)^{-\nu}\right\}_n a^\dagger m - n, \quad (21)$$
if $pq > 1$ and $\phi_1(p, q) < \phi_2(p, q)$,

$$a^\dagger m a^n = p^{-\nu(2)} \left[ (p) S_{a^1}^{-m} T_{p-1,q}(\phi_1(p, q)) \right] \frac{n}{\phi_1(p, q)(pq)^{\alpha N+\nu}; (pq)^\nu} a^\dagger m - n,$$  \hspace{1cm} (22)

if $pq < 1$ and $\phi_2(p, q) < \phi_1(p, q)$,

$$a^\dagger m a^n = q^n \frac{n}{\phi_2(p, q)(pq)^{-\alpha N-\nu}; (pq)^\nu} a^\dagger m - n,$$  \hspace{1cm} (23)

if $pq > 1$ and $\phi_1(p, q) < \phi_2(p, q)$.

- If $n > m$,

$$a^\dagger a^m = p^{\nu(2)} (p) S_{a^1}^{n} \left[ \frac{n}{\phi_1(p, q)(pq)^{\alpha N+\nu}; (pq)^\nu} a^\dagger a^{n-m},$$  \hspace{1cm} (24)

if $pq < 1$ and $\phi_2(p, q) < \phi_1(p, q)$,

$$a^\dagger a^m = q^{-\nu(2)} \left[ q S_{a^1}^{-1} T_{q,p-1}(\phi_2(p, q)) \right] \frac{n}{\phi_2(p, q)(pq)^{-\alpha N-\nu}; (pq)^\nu} a^\dagger a^{n-m},$$  \hspace{1cm} (25)

if $pq > 1$ and $\phi_1(p, q) < \phi_2(p, q)$.

- If $n = m$,

$$a^\dagger a^m = p^{-\nu(2)} \left[ T_{p-1,q}(\phi_1(p, q)) \right] \frac{n}{\phi_1(p, q)(pq)^{\alpha N+\nu}; (pq)^\nu} a^\dagger a^{n-m},$$  \hspace{1cm} (28)

if $pq < 1$ and $\phi_2(p, q) < \phi_1(p, q)$,

$$a^\dagger a^m = q^{-\nu(2)} \left[ T_{q,p-1}(\phi_2(p, q)) \right] \frac{n}{\phi_2(p, q)(pq)^{-\alpha N-\nu}; (pq)^\nu} a^\dagger a^{n-m},$$  \hspace{1cm} (29)

if $pq > 1$ and $\phi_1(p, q) < \phi_2(p, q)$.

- If $n = m$,

$$a^\dagger a^m = p^{\nu(2)} \left[ T_{p-1,q}(\phi_1(p, q)) \right] \frac{n}{\phi_1(p, q)(pq)^{\alpha N-\nu}; (pq)^\nu} a^\dagger a^{n-m},$$  \hspace{1cm} (30)

if $pq < 1$ and $\phi_2(p, q) < \phi_1(p, q)$,

$$a^\dagger a^m = q^{-\nu(2)} \left[ T_{q,p-1}(\phi_2(p, q)) \right] \frac{n}{\phi_2(p, q)(pq)^{\alpha N-\nu}; (pq)^\nu} a^\dagger a^{n-m},$$  \hspace{1cm} (31)

if $pq > 1$ and $\phi_1(p, q) < \phi_2(p, q)$. In the above expressions, $(p, q) S_{a^1}$ is the translation operator defined as follows

$$\left( p, q \right) S_{a^1} = p S_{a^1} q S_{a^1}, \quad p S_{a^1} p^{\alpha N} = p^{\alpha N+\nu}, \quad q S_{a^1} q^{\alpha N} = q^{\alpha N+\nu}, \quad \forall n \in \mathbb{N}.$$  \hspace{1cm} (32)
The operator $\mathcal{T}_{p,q}(t)$ acts on the vacuum state $|0\rangle$ as follows

$$\mathcal{T}_{p,q}(t)|0\rangle = \frac{tp^{\alpha \chi_0 + \nu}}{p^{\nu} - q^{\nu}} |0\rangle,$$

where the product $(a; q)_l = (1 - a)(1 - aq) \ldots (1 - aq^{l-1}), l = 1, 2, 3, \ldots$ This results reveal to be useful for deducing the commutators between $a^n$ and $a^{tm}$

- For $n < m$,

$$[a^n, a^{tm}]_{q^\nu} = p^{-\nu\left(\frac{n}{2}\right)}(1 - q^{\nu} S_{a^1}^m) \mathcal{T}_{p^{-1}, q}^n (\phi_1(p, q)) \times \left(\frac{\phi_2(p, q)}{\phi_1(p, q)}(pq)^{\alpha N + \nu}; (pq)^\nu\right)_n a^{tm-n},$$

if $pq < 1$ and $\phi_2(p, q) < \phi_1(p, q)$ and

$$[a^n, a^{tm}]_{q^\nu} = q^{\nu\left(\frac{n}{2}\right)}(1 - q^{\nu} S_{a^1}^{-m}) \mathcal{T}_{q^{-1}, p}^n (\phi_2(p, q)) \times \left(\frac{\phi_1(p, q)}{\phi_2(p, q)}(pq)^{-\alpha N - \nu}; (pq)^{-\nu}\right)_n a^{tm-n},$$

if $pq > 1$ and $\phi_1(p, q) < \phi_2(p, q)$.

- For $n > m$,

$$[a^n, a^{tm}]_{q^\nu} = p^{\nu\left(\frac{m}{2}\right)}(p; q) S_{a^1}^n - q^{\nu} \mathcal{T}_{p^{-1}, q}^m (\phi_1(p, q)) \times \left(\frac{\phi_2(p, q)}{\phi_1(p, q)}(pq)^{\alpha N}; (pq)^\nu\right)_m a^{n-m},$$

if $pq < 1$ and $\phi_2(p, q) < \phi_1(p, q)$ and

$$[a^n, a^{tm}]_{q^\nu} = q^{-\nu\left(\frac{n}{2}\right)}(p; q) S_{a^1}^n - q^{\nu} \left[q^{\nu}\mathcal{T}_{q^{-1}, p}^m (\phi_2(p, q))\right] \times \left(\frac{\phi_1(p, q)}{\phi_2(p, q)}(pq)^{-\alpha N}; (pq)^{-\nu}\right)_m a^{n-m},$$

if $pq > 1$ and $\phi_1(p, q) < \phi_2(p, q)$. Noting that by similar computation we can derive the $p^{-\nu}$ commutators between $a^n$ and $a^{tm}$.

### 3 Coherent states

In this part we investigate some class of deformed states. These states are the eigenstates of the annihilation operator so called coherent states.

**Proposition 3** The CS associated with the algebra [3] and [4] with $(\chi_0 = 0$ and $\nu = \alpha )$ are given in different case by

1. if $pq < 1$, $\phi_2(p, q) < \phi_1(p, q)$

$$|z\rangle_{\nu} = \mathcal{N}_{\nu}^{-1/2}(|z|^2) \sum_{n=0}^{\infty} \frac{p^{\nu(n-1)/4} z^n}{\sqrt{\tau^n(\phi_1^{-1}(p, q)\phi_2(p, q)(pq)^\nu; (pq)^\nu)_n}} |n\rangle, z \in \mathbb{D}_{\nu},$$

where

$$\mathcal{N}_{\nu}(x) = \sum_{n=0}^{\infty} \frac{p^{\nu\left(\frac{n}{2}\right)}(\phi_1^{-1}(p, q)\phi_2(p, q)(pq)^\nu; (pq)^\nu)_n}{(\phi_1(p, q))^n} \left(1 - \frac{(pq)^\nu}{x}\right)^n,$$

(37)
2. if \( pq > 1 \), \( \phi_1(p, q) < \phi_2(p, q) \)

\[
|z|_\nu = N_\nu^{-1/2}(|z|^2) \sum_{n=0}^{\infty} \frac{q^{-n\nu(n-1)}/4z^n}{\sqrt{\tau^n(\phi_2^{-1}(p, q)\phi_1(p, q)(pq)^{-\nu}; (pq)^{-\nu})_n}}|n|, \quad z \in D_\nu,
\]

where

\[
N_\nu(x) = \sum_{n=0}^{\infty} \frac{q^{-n\nu(n-1)/4}}{(\phi_2^{-1}(p, q)\phi_1(p, q)(pq)^{-\nu}; (pq)^{-\nu})_n} \left( \frac{1 - (pq)^{-\nu}}{\phi_2(p, q)} \right)^n x^n
\]

In the above expressions the convergence domain \( D_\nu \) of the serie \( N_\nu(x) \) is given by

\[
D_\nu = \{ z \in \mathbb{C} : |z|^2 < R_\nu \}, \quad \text{with } R_\nu = \infty, \quad \nu > 0.
\]

\( R_\nu \) is the radius associated with the same series.

In the limit, when \( \phi_1(p, q) = 1 = \phi_1(p, q), p, q \to 1 \), the series \( N_\nu(x) \) is reduced to the usual exponential function \( e^x \). Also, when \( \phi_1(p, q) = 1 = \phi_1(p, q), p \to 1, \nu \to 1 \) the series \( N_\nu(x) \) is reduced to the \( q \)-exponential function \( e_q(x) \). The most important of property of \( |z|_\nu \) is that if \( |z'|_\nu \) is another CS, then

\[
\nu(z'|z)_\nu = \frac{N_\nu(z^2)}{\sqrt{N_\nu(|z|^2)}},
\]

which means that such states are not orthogonal.

**Proposition** The CS defined in (36) and (38) are normalized, are continuous in \( z \) and solve the unity i.e., \( \int_{D_\nu} \frac{d^2W_\nu(|z|^2)}{\pi} z\nu \nu(z) = 1 \). The resolution of unity assumes the existence of a positive weight a function \( W_\nu(|z|^2) \) such that \( W_\nu(x) = N_\nu(x)\hat{W}_\nu(x), \ x = |z|^2 \).

**Proof:** From (31), when \( z' \to z \), (36) and (38) are normalized. For the continuity we can see that \( |||z'|_\nu - |z|_\nu||^2 = 2(1 - \Re \nu(z'|z)_\nu) \), so

\[
|||z'|_\nu - |z|_\nu||^2 \to 0 \quad \text{as } |z| \to 0,
\]

since \( \nu(z'|z)_\nu \to 0 \) as \( |z'| \to 0 \). The resolution of the unity can be seen by using the relation

\[
\int_{D_\nu} \frac{d^2W_\nu(|z|^2)}{\pi} z\nu \nu(z) = \sum_{n,m=0}^{\infty} \frac{p^{n\nu(n-1)}/4 + m\nu(m-1)/4 - n+m}{\sqrt{(\phi_2(p, q)/\phi_1(p, q)^\nu ; (pq)^\nu)_n}\sqrt{(\phi_2(p, q)/\phi_1(p, q)^\nu ; (pq)^\nu)_m}} \times \int_{D_\nu} z^n z^m d^2W_\nu(|z|^2) \frac{d^2W_\nu(|z|^2)}{\pi N_\nu(|z|^2)},
\]

where \( \hat{W}_\nu(x) \) has to be determined from the equations

\[
\int_{D_\nu} x^n \hat{W}_\nu(x) dx = p^{-n\nu(n-1)/2} \left( \frac{\phi_2(p, q)}{\phi_1(p, q)} \right) \left( \frac{1 - (pq)^{-\nu}}{1 - (pq)^{-\nu}} \right)^n,
\]

if \( pq < 1 \), \( \phi_2(p, q) < \phi_1(p, q) \) and

\[
\int_{D_\nu} x^n \hat{W}_\nu(x) dx = q^{-n\nu(n-1)/2} \left( \frac{\phi_1(p, q)}{\phi_2(p, q)} \right) \left( \frac{1 - (pq)^{-\nu}}{1 - (pq)^{-\nu}} \right)^n,
\]

if \( pq > 1 \), \( \phi_1(p, q) < \phi_2(p, q) \). If \( n \) is extended to \( s - 1 \), \( s \in \mathbb{C} \), then the problem can be formulated to the classical Stieltjes power moment problem when \( 0 < pq < 1 \) or Hausdorff power moment problem when \( pq > 1 \).
4 Matrix elements

Let us now compute correlation functions with matrix elements of normal and antinormal forms pertaining to quantum optics.

- For \( n < m \), if \( pq < 1 \), \( \phi_2(p, q) < \phi_1(p, q) \), the normal form is defined by

\[
\nu(z|a^\dagger m a^n|z)_{\nu} = N^{-1}_{\nu}(|z|^2) \sum_{r,s=0}^{\infty} \frac{p^{\nu(r-1)/4+sv(s-1)/4} F^{rs}_{mn}}{\left(\frac{s}{\phi_1(p,q)}(pq)^{\nu};(pq)^{\nu}\right)_r \left(\frac{s}{\phi_2(p,q)}(pq)^{\nu};(pq)^{\nu}\right)_s} F^{rs}_{mn}, \tag{44}
\]

where the matrix elements \( F^{rs}_{mn} \) are given by

\[
F^{rs}_{mn} = |r|a^m a^n|sangle = C_s C^{-1}_{m-n+s} p^{\nu(z)}(\frac{s}{\phi_1(p,q)}(pq)^{\nu};(pq)^{\nu})_n \delta_{r,m-n+s}, \tag{45}
\]

and

\[
C^2_n = p^{\nu(z)}(\frac{1 - (pq)^{\nu}}{\phi_2(p,q)})^n \left(\frac{\phi_1(p,q)}{\phi_2(p,q)}(pq)^{\nu};(pq)^{\nu}\right)_n \tag{46}
\]

if \( pq > 1 \), \( \phi_1(p,q) < \phi_2(p,q) \)

\[
\nu(z|a^\dagger m a^n|z)_{\nu} = N^{-1}_{\nu}(|z|^2) \sum_{r,s=0}^{\infty} \frac{q^{-\nu(r-1)/4-sv(s-1)/4} F^{rs}_{mn}}{\left(\frac{s}{\phi_1(p,q)}(pq)^{-\nu};(pq)^{-\nu}\right)_r \left(\frac{s}{\phi_2(p,q)}(pq)^{-\nu};(pq)^{-\nu}\right)_s} F^{rs}_{mn}, \tag{49}
\]

where the matrix elements \( F^{rs}_{mn} \) are given by

\[
F^{rs}_{mn} = |r|a^m a^n|sangle = \tilde{C}_s \tilde{C}^{-1}_{m-n+s} q^{-\nu(z)}(\frac{s}{\phi_2(p,q)}(pq)^{-\nu};(pq)^{-\nu})_n \delta_{r,m-n+s}, \tag{47}
\]

\[
\tilde{C}^2_n = q^{-\nu(z)}(\frac{1 - (pq)^{-\nu}}{\phi_1(p,q)})^n \left(\frac{\phi_2(p,q)}{\phi_1(p,q)}(pq)^{-\nu};(pq)^{-\nu}\right)_n \tag{48}
\]

The antinormal form if \( pq < 1 \), \( \phi_2(p,q) < \phi_1(p,q) \), is given by

\[
\nu(z|a^n a^\dagger m|z)_{\nu} = N^{-1}_{\nu}(|z|^2) \sum_{r,s=0}^{\infty} \frac{p^{\nu(r-1)/4+sv(s-1)/4} G^{rs}_{mn}}{\left(\frac{s}{\phi_1(p,q)}(pq)^{\nu};(pq)^{\nu}\right)_r \left(\frac{s}{\phi_2(p,q)}(pq)^{\nu};(pq)^{\nu}\right)_s} G^{rs}_{mn}, \tag{49}
\]

where the matrix elements \( G^{rs}_{mn} \) are given by

\[
G^{rs}_{mn} = |r|a^n a^\dagger m|sangle = C_s C^{-1}_{m-n+s} p^{\nu(z)}(\frac{s}{\phi_1(p,q)}(pq)^{s+m}\nu);(pq)^{s+m}\nu)_n \delta_{r,m-n+s}. \tag{50}
\]

if \( pq > 1 \), \( \phi_1(p,q) < \phi_2(p,q) \)

\[
\nu(z|a^n a^\dagger m|z)_{\nu} = N^{-1}_{\nu}(|z|^2) \sum_{r,s=0}^{\infty} \frac{q^{-\nu(r-1)/4-sv(s-1)/4} G^{rs}_{mn}}{\left(\frac{s}{\phi_2(p,q)}(pq)^{-\nu};(pq)^{-\nu}\right)_r \left(\frac{s}{\phi_1(p,q)}(pq)^{-\nu};(pq)^{-\nu}\right)_s} G^{rs}_{mn}, \tag{49}
\]
where the matrix elements $G_{mn}^{rs}$ are given by

$$G_{mn}^{rs} = \langle r | a^m a^n | s \rangle$$

$$= \frac{C_s C_{m-n-s} q^{-\nu(z)} \left( \frac{\phi_2(p,q) p^{(s+1)} v}{1-(pq)^v} \right)_n \left( \frac{\phi_1(p,q)}{\phi_2(p,q)} (pq)^{-m} ; (pq)^v \right)_{n,r,m-n+s}}.$$  

(51)

- For $n > m$, if $pq > 1$, $\phi_2(p,q) < \phi_1(p,q)$, the normal form is defined by

$$\nu(z | a^m a^n | z)_\nu = N_{\nu}^{-1} (|z|)^2 \sum_{r,s=0}^{\infty} p r^{\nu(r-1)/4 + s v(s-1)/4} \tilde{G}_{mn}^{rs},$$  

(52)

where the matrix elements $\tilde{G}_{mn}^{rs}$ are given by

$$\tilde{G}_{mn}^{rs} = \langle r | a^m a^n | s \rangle$$

$$= C_r C_{r-n-m} p^{\nu(z)} \left( \frac{\phi_1(p,q) p^{(n-s)} v}{1-(pq)^v} \right)_n \left( \frac{\phi_2(p,q)}{\phi_1(p,q)} (pq)^{-r} ; (pq)^v \right)_{n,m,s,r-n-m},$$  

(53)

if $pq > 1$, $\phi_1(p,q) < \phi_2(p,q)$,

$$\nu(z | a^m a^n | z)_\nu = N_{\nu}^{-1} (|z|)^2 \sum_{r,s=0}^{\infty} q r^{\nu(r-1)/4 - s v(s-1)/4} \tilde{G}_{mn}^{rs},$$  

(55)

where the matrix elements $\tilde{G}_{mn}^{rs}$ are given by

$$\tilde{G}_{mn}^{rs} = \langle r | a^m a^n | s \rangle$$

$$= C_r C_{r-n-m} q^{\nu(z)} \left( \frac{\phi_2(p,q) p^{(n-s)} v}{1-(pq)^v} \right)_n \left( \frac{\phi_1(p,q)}{\phi_2(p,q)} (pq)^{-r} ; (pq)^v \right)_{n,m,s,r-n-m},$$  

(54)

The antinormal form if $pq < 1$, $\phi_2(p,q) < \phi_1(p,q)$, is expressed by the formula:

$$\nu(z | a^m a^n | z)_\nu = N_{\nu}^{-1} (|z|)^2 \sum_{r,s=0}^{\infty} p r^{\nu(r-1)/4 + s v(s-1)/4} \tilde{G}_{mn}^{rs},$$  

(55)

where the matrix elements $\tilde{F}_{mn}^{rs}$ are given by

$$\tilde{F}_{mn}^{rs} = \langle r | a^m a^n | s \rangle$$

$$= C_s C_{r+n-m} p^{\nu(z)} \left( \frac{\phi_1(p,q) p^{(s+1)} v}{1-(pq)^v} \right)_n \left( \frac{\phi_2(p,q)}{\phi_1(p,q)} (pq)^{-r} ; (pq)^v \right)_{n,m,s,r+n-m},$$  

(56)

if $pq < 1$, $\phi_1(p,q) < \phi_2(p,q)$,

$$\nu(z | a^m a^n | z)_\nu = N_{\nu}^{-1} (|z|)^2 \sum_{r,s=0}^{\infty} q r^{\nu(r-1)/4 - s v(s-1)/4} \tilde{G}_{mn}^{rs},$$  

(55)

where the matrix elements $\tilde{G}_{mn}^{rs}$ are given by

$$\tilde{G}_{mn}^{rs} = \langle r | a^m a^n | s \rangle$$

$$= C_s C_{r+n-m} q^{\nu(z)} \left( \frac{\phi_2(p,q) p^{(s+1)} v}{1-(pq)^v} \right)_n \left( \frac{\phi_1(p,q)}{\phi_2(p,q)} (pq)^{-r} ; (pq)^v \right)_{n,m,s,r+n-m},$$  

(55)
5 Discussions

This part addressed the study of the previous deformation in the particular case that we present. Let us reexpressed the relations (39) as

\[
\begin{align*}
aa^\dagger n - q^\nu a^\dagger n a &= [n\nu]^{1,1}_{p,q,\alpha} a^\dagger n - p^{\alpha N}\nu \\
\end{align*}
\]

(58)

where the quantity $[n]^{\phi_1,\phi_2}_{p,q,\nu}$ is the deformed number given by

\[
[n\nu]^{\phi_1,\phi_2}_{p,q,\nu} = \frac{\phi_1(p,q)p^{-\nu} - \phi_2(p,q)q^\nu}{p^{-\nu} - q^\nu}.
\]

(59)

Then it appears important to emphasize that: By setting $C_1$ as the Casimir operator and defined the regular operators functions

\[
\hat{\phi}_1(p,q,C_1) = (1 + 2\gamma C_1)p^{-\beta}, \quad \hat{\phi}_2(p,q,C_1) = (1 + 2\gamma C_1)q^\beta
\]

(60)
such that $\hat{\phi}_1|n\rangle = (1 + 2\gamma \nu^2)p^{-\beta}|n\rangle$ and $\hat{\phi}_2|n\rangle = (1 + 2\gamma \omega^2)q^\beta|n\rangle$, where $\nu^2$ is the eigenvalue of the operator $C_1$. The commutation relations (39) take the form

\[
[a\dagger - q^\nu a\dagger] = \hat{\phi}_1(p,q,C_1)p^{-\alpha N}, \quad [a\dagger - p^{-\nu} a\dagger] = \hat{\phi}_2(p,q,C_1)q^{\alpha N}.
\]

(61)

The eigen-equation associated with the Hamiltonian operator $H^{\nu,\gamma}_{\alpha,\beta}(p,q) = a\dagger a + aa\dagger$, i.e. $H^{\nu,\gamma}_{\alpha,\beta}(p,q)|n\rangle = E^{\nu,\gamma}_{\alpha,\beta}(p,q)|n\rangle$, is such that the corresponding eigenvalue $E^{\nu,\gamma}_{\alpha,\beta}(p,q)$ is written as

\[
E^{\nu,\gamma}_{\alpha,\beta}(p,q) = (1 + 2\gamma \nu^2)\{q^{\alpha \chi_0 + \nu + \beta} + (1 + \nu)p^{-\nu} + \alpha \chi_0 + \beta\}_{(p,q,\nu)}
\]

(62)
or, equivalently,

\[
E^{\nu,\gamma}_{\alpha,\beta}(p,q) = (1 + 2\gamma \nu^2)\{p^{-\alpha \chi_0 - \nu - \beta} + (1 + \nu)p^{-\nu} + \alpha \chi_0 + \beta\}_{(p,q,\nu)}.
\]

(63)

where

\[
[n;\gamma C_1]_{(p,q,\nu)}! = \frac{(p^{-\nu}q^\nu)_{n}}{n!(p^{-\nu} - q^\nu)^n}.
\]

(64)

Indeed, for $n,m \in \mathbb{N}\setminus\{0\}$, we derive the next commutators:

- For $n < m$, $p < 1$ and $q > 1$,

\[
[a^n,(a\dagger)^m]_{q^\nu} = \frac{(1 + 2\gamma C_1)^n}{(p^{-\nu} - q^\nu)^n} \left[ ((p^{-\alpha N - \beta - \nu},q^{\alpha N + \beta + \nu});(p^{-\nu},q^\nu))_{n} \right. \\
- \left. q^\nu ((p^{-\alpha N - \beta + (m-1)\nu},q^{\alpha N + \beta - (m-1)\nu});(p^{-\nu},q^\nu))_{n} \right] (a\dagger)^{m-n}.
\]

There follows:

\[
\sum_{n=0}^{m-1} [a^n,a\dagger^m]_{q^\nu} = (p,q)S_{a\dagger} \left( 1 - q^\nu_{(p,q)}S_{a\dagger}^{-m} \right) \sum_{n=0}^{m-1} \frac{(p^{-\alpha N - \beta},q^{\alpha N + \beta});(p^{-\nu},q^\nu))_{n}}{(p^{-\nu},q^\nu);(p^{-\nu},q^\nu))_{n}} a\dagger^{m-n} \\
- (p,q)S_{a\dagger} \left( 1 - q^\nu_{(p,q)}S_{a\dagger}^{-m} \right) \mathcal{L}_{m-1}[(p^{-\alpha N - \beta},q^{\alpha N + \beta});(p^{-\nu},q^\nu);a\dagger^{-1}]a\dagger^{m},
\]

(65)
where \( \mathcal{L}_m \) is a deformed hypergeometric function given by

\[
\mathcal{L}_m[(\lambda, \sigma); (p, q); z] := \sum_{n=0}^{m} \frac{((\lambda, \sigma); (p, q))_n z^n}{((p, q); (p, q))_n},
\]  

(66)

By using the \((p, q)\)-binomial theorem given by

\[
\sum_{n=0}^{\infty} \frac{((a, b); (p, q))_n z^n}{((p, q); (p, q))_n} = \frac{((p, b z); (p, q))_{\infty}}{((p, a z); (p, q))_{\infty}},
\]

(67)

and for \( m \to \infty \), the relation (65) is reduced to the

\[
\mathcal{L}_\infty = \frac{((p-\nu, q^{\alpha N+\beta} a^{\dagger -1}); (p-\nu, q^{\nu}))_{\infty}}{((p-\nu, p^{-\alpha N-\beta} a^{\dagger -1}); (p-\nu, q^{\nu}))_{\infty}}, \quad ||a^{\dagger -1}|| < 1,
\]

(68)

- For \( n > m \), by using the identity

\[
[m; \gamma C_1]_{(p, q; \nu)}! = (-1)^m (1 + 2\gamma C_1)^m (p^{-1} q)^{m \nu + m(m-1)/2} \frac{((p^{-\nu}, q^{-\nu}); (p^{-\nu}, q^{-\nu}))_{m}}{(p-\nu - q^{-\nu})^m},
\]

(69)

we infer

\[
\sum_{m=0}^{n-1} \frac{[a^n, a^{\dagger m}]_{q^{-\nu}}((p^{\nu}, 0); (p^{\nu}, q^{-\nu}))_{m}}{[m; \gamma C_1]_{(p, q; \nu)}!} = ((p, q) T_{a}^{-\nu} - q^{-\nu}) \mathcal{L}_{n-1}[((p^{-\alpha N-\beta}, q^{\alpha N+\beta}); (p^{\nu}, q^{\nu}); a^{-1}] a^n,
\]

(70)

where

\[
\mathcal{L}_{n-1}[(p^{-\alpha N-\beta}, q^{\alpha N+\beta}); (p^{\nu}, q^{\nu}); a^{-1}] = \sum_{m=0}^{n-1} \frac{((p^{-\alpha N-\beta}, q^{\alpha N+\beta}); (p^{\nu}, q^{\nu}))_{m} a^{-m}}{(0, q^{-\nu}), (p^{\nu}, q^{\nu})_{m} a^{-m}.
\]

(71)

When \( n \to \infty \), (71) becomes

\[
\mathcal{L}_{\infty} = \binom{p^{-\alpha N-\beta}, q^{\alpha N+\beta}}{0, q^{-\nu}} (p^{\nu}, q^{\nu}); a^{-1}] ||a^{-1}|| < 1.
\]

(72)

- For \( n = m \), we obtain

\[
\sum_{n=0}^{\infty} \frac{((1, 0), (p^{-\nu}, q^{\nu}))_{n} a^n (a^{\dagger})^{2n} (p^{-\nu} - q^{\nu})^{n}}{[n; \gamma C_1]!} = \binom{p^{-\alpha N-\beta}, q^{\alpha N+\beta} + \nu}{0, 1} (p^{\alpha N+\beta}, q^{\alpha N-\beta} - (p^{-\nu}, q^{\nu}); (1 + 2\gamma C_1)(q^{p^{-\nu}})^{\alpha N+\beta},
\]

(73)

6 Conclusion

In this paper, associated relevant properties have been investigated for the induced deformed harmonic oscillator; the energy spectrum has been explicitly computed. Deformed coherent states have been built and discussed with respect to the criteria of coherent state construction. Various commutators involving annihilation and creation operators and their combinatorics have been computed and analyzed. Finally, the correlation functions of matrix elements of main normal and antinormal forms, pertinent for quantum optics analysis, have been computed.
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