Geometry behind Robertson type uncertainty principles

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Abstract

A nontrivial lower bound for the determinant of the covariance matrix of quantum mechanical observables has previously been presented by Gibilisco and Andai. In that inequality the Riemannian metric occurred just in the lower bound. We endow the state space with special Riemannian metrics, such that both sides of the uncertainty relation can be derived only from the metric and we prove an uncertainty relation for these metrics. From our result the relation between the quantum covariance, the metric dependent symmetric and antisymmetric covariances follows easily furthermore we give a better lower bound for the determinant of the covariance matrix.

Introduction

The states of an n-level system are identified with the set of n × n self adjoint positive semidefinite matrices with trace 1, and the physical observables are identified with n × n self adjoint matrices. The concept of uncertainty was introduced by Heisenberg in 1927 [7], who demonstrated the impossibility of simultaneous measurement of position (q) and momentum (p).

He considered Gaussian distributions (f(q)), and defined uncertainty of f as its width Df. If the width of the Fourier transform of f is denoted by DF(f), then the first formalization of the uncertainty principle can be written as

\[ D_f D_{F(f)} = \text{constant}. \]

In 1927 Kennard generalized Heisenberg’s result [10], he proved the inequality

\[ \text{Var}_D(A) \text{Var}_D(B) \geq \frac{1}{4} \]

for observables A, B which satisfy the relation [A, B] = −i, for every state D, where Var_D(A) = Tr(DA^2) − (Tr(DA))^2. In 1929 Robertson [18] extended Kennard’s result for arbitrary two observables A, B

\[ \text{Var}_D(A) \text{Var}_D(B) \geq \frac{1}{4} |\text{Tr}(D[A, B])|^2. \]
In 1930 Scrödinger [20] improved this relation including the correlation between observables $A, B$
\[
\text{Var}_D(A) \text{Var}_D(B) - \text{Cov}_D(A, B)^2 \geq \frac{1}{4} \left| \text{Tr}(D [A, B]) \right|^2,
\]
where for a given state $D$ the (symmetrized) covariance of the observables $A$ and $B$ is defined as
\[
\text{Cov}_D(A, B) = \frac{1}{2} \left( \text{Tr}(DAB) + \text{Tr}(DBA) \right) - \text{Tr}(D) \text{Tr}(DB).
\]
The Schrödinger uncertainty principle can be formulated as
\[
\det \begin{pmatrix}
\text{Cov}_D(A, A) & \text{Cov}_D(A, B) \\
\text{Cov}_D(B, A) & \text{Cov}_D(B, B)
\end{pmatrix} \geq \det \left[ \frac{i}{2} \begin{pmatrix}
\text{Tr}(D [A, A]) & \text{Tr}(D [A, B]) \\
\text{Tr}(D [B, A]) & \text{Tr}(D [B, B])
\end{pmatrix} \right].
\]
For the set of observables $(A_i)_{1, \ldots, N}$ this inequality was generalized by Robertson in 1934 [19] as
\[
\det \left( [\text{Cov}_D(A_h, A_j)]_{h,j=1,\ldots,N} \right) \geq \det \left( -\frac{1}{2} \text{Tr}(D [A_h, A_j]) \right)_{h,j=1,\ldots,N}.
\]
Gibilisco and Isola in 2006 conjectured that
\[
\det \left( [\text{Cov}_D(A_h, A_j)]_{h,j=1,\ldots,N} \right) \geq \det \left( \frac{f(\beta)}{2} \langle i [D, A_h], i [D, A_j] \rangle_D \right)_{h,j=1,\ldots,N},
\]
holds [5], where the scalar product $\langle \cdot, \cdot \rangle_{D,f}$ is induced by an operator monotone function $f$, according to Petz classification theorem [17]. We note that if the density matrix is not strictly positive, then the scalar product $\langle \cdot, \cdot \rangle_{D,f}$ is not defined. The inequality (1) was studied first only in the case $N = 1$ for special functions $f$. The cases $f(x) = f_{\text{SLD}}(x) = \frac{1+x}{1+x^2}$ and $f(x) = f_{\text{WY}}(x) = \frac{1}{2} (\sqrt{x} + 1)^2$ were proved by Luo in [12] and [13]. The general case of the Conjecture was been proven by Hansen in [6] and shortly after by Gibilisco et al. with a different technique in [3].

In the case $N = 2$ the inequality was proved for $f = f_{\text{WY}}$ by Luo, Q. Zhang and Z. Zhang [16] [14] [15]. The case of Wigner–Yanase–Dyson metric, where $f_\beta(x) = \frac{\beta (1-\beta)(x-1)^2}{(x-1)(x-1+\beta)}$ ($\beta \in [-1, 2] \setminus \{0, 1\}$) was proved independently by Kosaki [11] and by Yanagi, Furuichi and Kuriyama [21]. The general case is due to Gibilisco, Imparato and Isola [5] [3].

For arbitrary $N$ the conjecture was proved by Andai [11] and Gibilisco, Imparato and Isola [3].

In inequality (1) the Riemannian metric occurs on right hand side of the equation only, while the left hand side is just the quantum covariance. In this paper we endow the state space with special Riemannian metrics, such that both sides of the inequality can be derived alone from the metric and we prove an uncertainty relation for these metrics. From our result the relation between the quantum covariance, the metric dependent symmetric and antisymmetric covariances follows easily.

\section{Riemannian metrics on the state space}

Let us denote by $\mathcal{M}_n$ the set of $n \times n$ self adjoint positive definite matrices and by $\mathcal{M}_n^1$ the interior of the $n$-level quantum mechanical state space i.e. the set of $n \times n$ self adjoint positive definite trace one matrices. Let $\mathcal{M}_{n,sa}$ be the set of observables of the $n$-level quantum system, namely the set of $n \times n$ self adjoint matrices, and let $\mathcal{M}_{n,sa}^{(0)}$ be the set observables with zero trace. Spaces $\mathcal{M}_n$ and $\mathcal{M}_n^1$ are form convex sets in the space of self adjoint matrices, and they are obviously differentiable manifolds [8]. The tangent space of $\mathcal{M}_n$ at a given state $D$ can be identified with $\mathcal{M}_{n,sa}$ and the tangent space of $\mathcal{M}_n^1$ with $\mathcal{M}_{n,sa}^{(0)}$. 
Let \((K_m)_{m \in \mathbb{N}}\) be a family of Riemannian metrics on \(M^1_n\). This family of metrics is said to be monotone if
\[
K^{(m)}_{T(D)}(T(X), T(X)) \leq K^{(n)}_D(X, X)
\]
holds for every completely positive, trace preserving linear map \(T : M_n(\mathbb{C}) \to M_m(\mathbb{C})\) (such a mapping is called a stochastic mapping), for every \(D \in M^1_n\) and for all \(X \in M^1_{n,sa}\) and \(m, n \in \mathbb{N}\). Let us denote by \(\mathcal{F}_{op}\) the set of operator monotone functions \(f : \mathbb{R}^+ \to \mathbb{R}\) with the property \(f(x) = x f(x^{-1})\) for every positive argument \(x\) and with the normalization condition \(f(1) = 1\).

**Theorem 1.** Petz classification theorem \([17]\). There exists a bijective correspondence between the monotone family of metrics \((K^{(n)})_{n \in \mathbb{N}}\) and functions \(f \in \mathcal{F}_{op}\). The metric is given by
\[
K^{(n)}_D(X, Y) = \text{Tr} \left( X \left( R^\frac{1}{2}_{n,D} f(L_{n,D} R^{-1}_{n,D}) R^\frac{1}{2}_{n,D} \right)^{-1}(Y) \right)
\]
for all \(n \in \mathbb{N}\) where \(L_{n,D}(X) = DX, R_{n,D}(X) = XD\) for all \(D, X \in M_n(\mathbb{C})\).

Equation (2) may be considered as a scalar product \(\langle \cdot, \cdot \rangle_{D,f}\). For operator monotone functions \(f \in \mathcal{F}_{op}\) we also introduce the notation
\[
m_f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \quad (x, y) \mapsto y f \left( \frac{x}{y} \right).
\]

The metric defined in Theorem 1 can be extended to the space \(M_n\). For every \(D \in M_n\) and matrices \(A, B \in M_{n,sa}\) let us define
\[
\langle A, B \rangle_{D,f} = \text{Tr} \left( X \left( R^\frac{1}{2}_{n,D} f(L_{n,D} R^{-1}_{n,D}) R^\frac{1}{2}_{n,D} \right)^{-1}(Y) \right),
\]
with this notion the pair \((M_n, \langle \cdot, \cdot \rangle_{D,f})\) will be a Riemannian manifold for every operator monotone function \(f \in \mathcal{F}_{op}\).

Using these symbols we have the following Theorem, the proof of which will be given after Theorem 3.

**Theorem 2.** For every \(D \in M_n\), \(A, B \in M_{n,sa}\) and \(f \in \mathcal{F}_{op}\) we have
\[
\langle A, B \rangle_{D,f} = \sum_{k,l=1}^n A_{lk} B_{kl} \frac{1}{m_f(\lambda_k, \lambda_l)}.
\]

After this Theorem, one can see, that \(\langle \cdot, \cdot \rangle_{f}\) is a Riemannian metric, since the function \(m_f\) is symmetric and smooth. This observation led us to the following definition.

**Definition 1.** Introduce the notation
\[
\mathcal{C}_M = \left\{ g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \quad | \quad g \text{ is asymmetric smooth function, with analytical extension defined on a neighborhood of } \mathbb{R}^+ \times \mathbb{R}^+ \right\}
\]
implicating the correspondence with Cencov-Morozova functions. Fix a function \(g \in \mathcal{C}_M\). Define for every \(D \in M_n\) and for every \(A, B \in M_{n,sa}\)
\[
\langle A, B \rangle_{D,g} = \text{Tr} \left( Ag(L_{n,D}, R_{n,D})(B) \right).
\]
Theorem 3. For every $D \in \mathcal{M}_n$, $A, B \in \mathcal{M}_{n,sa}$ and $g \in \mathcal{C}_M$ we have

$$(A, B)_{D,g} = \sum_{k,l=1}^{n} A_{lk} B_{kl} g(\lambda_k, \lambda_l).$$

For every $g \in \mathcal{C}_M$ the function $(\cdot, \cdot)_{g}$ is a Riemannian metric on $\mathcal{M}_n$.

Proof. Note, that the operators $L_{n,D}$ and $R_{n,D}$ are self adjoint and positive definite with respect to the Hermitian form $(X, Y) \mapsto \text{Tr} X^* Y$. For an analytical function $f$ the operators $f(L_{n,D})$ and $f(R_{n,D})$ are well defined, and by the Riesz-Dunford operator calculus we have

$$f(L_{n,D}) = \frac{1}{2\pi i} \oint \int f(\xi) (\xi \text{id}_{\mathcal{M}_{n,sa}} - L_{n,D})^{-1} d\xi,$$

where we integrate once around the spectrum $D$. The operators $L_{n,D}$ and $R_{n,D}$ commute, therefore a similar reasoning gives the following formula

$$g(L_{n,D}, R_{n,D}) = \frac{1}{(2\pi i)^2} \oint \int f(\xi, \eta) (\xi \text{id} - L_{n,D})^{-1} \circ (\eta \text{id} - R_{n,D})^{-1} d\xi d\eta,$$

or equivalently

$$g(L_{n,D}, R_{n,D})(B) = \frac{1}{(2\pi i)^2} \oint \int f(\xi, \eta) (\xi - D)^{-1} B(\eta - D)^{-1} d\xi d\eta.$$

Assuming the matrix $D$ to be the form of $D = \text{Diag}(\lambda_1, \ldots, \lambda_n)$ for matrix units $E_{ij}$ and $E_{kl}$ we have

$$(E_{ij}, E_{kl})_{D,g} = \text{Tr} \left( \frac{1}{(2\pi i)^2} \oint \int g(\xi, \eta) E_{ij}(\xi - D)^{-1} E_{kl}(\eta - D)^{-1} d\xi d\eta \right).$$

Since for arbitrary matrices $A, B \in \mathcal{M}_{n,sa}$ we have $A = \sum_{i,j=1}^{n} A_{ij} E_{ij}$ and $B = \sum_{k,l=1}^{n} B_{kl} E_{kl}$ therefore

$$(A, B)_{D,g} = \sum_{i,j,k,l=1}^{n} \delta_{kj} \delta_{il} A_{ij} B_{kl} g(\lambda_j, \lambda_l) = \sum_{k,l=1}^{n} A_{lk} B_{kl} g(\lambda_k, \lambda_l).$$

For every $g \in \mathcal{C}_M$ and $D \in \mathcal{M}_n$ the function $(A, B) \mapsto (A, B)_{D,g}$ is a positive bilinear map, and for every smooth vector field $\gamma: \mathcal{M}_n \to \mathcal{M}_{n,sa}$ the function

$$\mathcal{M}_n \to \mathbb{R} \quad D \mapsto (\gamma(D), \gamma(D))_{D,g}$$

is smooth, therefore $(\cdot, \cdot)_{g}$ defines a Riemannian metric on $\mathcal{M}_n$.

Proof. To prove Theorem 2 assume that $f \in \mathcal{F}_\text{op}$, $D \in \mathcal{M}_n$ and $A, B \in \mathcal{M}_{n,sa}$. If we define

$$g: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \quad (x, y) \mapsto (\sqrt{gy}(xy^{-1}) \sqrt{g})^{-1},$$

then we have $g(x, y) = \frac{1}{m_f(x, y)}$ and

$$(A, B)_{D, f} = (A, B)_{D,g}.$$

Applying Theorem 3 we have

$$A_{lk} B_{kl} g(\lambda_k, \lambda_l) = \sum_{k,l=1}^{n} A_{lk} B_{kl} \frac{1}{m_f(\lambda_k, \lambda_l)},$$

which proves Theorem 2. 

\[\square\]
2 Covariances derived from the metric

For observable $A \in M_{n,sa}$ and a state $D \in \mathcal{M}_n$, we define $A_0 = A - \text{Tr}(DA)I$, where $I$ is the $n \times n$ identity matrix. Using this transformation we have $\text{Tr}DA_0 = 0$.

**Definition 2.** For observables $A, B \in M_{n,sa}$, state $D \in \mathcal{M}_n$ and function $f \in F^n_{op}$ we define the quantum covariance of $A$ and $B$

$$q\text{Cov}_D(A, B) = \frac{1}{2}(\text{Tr}(DAB) + \text{Tr}(DBA)) - \text{Tr}(DA)\text{Tr}(DB)$$

the asymmetric quantum covariance as

$$q\text{Cov}_{D,f}^{as}(A, B) = \frac{f(0)}{2} \langle i [D, A] , i [D, B] \rangle_{D,f}$$

and the symmetric quantum covariance as

$$q\text{Cov}_{D,f}^{s}(A, B) = \frac{f(0)}{2} \langle \{D, A \} , \{D, B \} \rangle_{D,f}.$$  

**Theorem 4.** Assume that the state $D \in \mathcal{M}_n$ is of the form $D = \text{Diag}(\lambda_1, \ldots, \lambda_n)$. For every $A, B \in M_{n,sa}$ and $f \in F_{op}$ we have

$$\text{Cov}_D(A, B) = \frac{1}{2} \sum_{k,l=1}^{n} \frac{\lambda_k + \lambda_l}{m_f(\lambda_k, \lambda_l)} A_{kl} B_{kl} - \text{Tr}(DA) \text{Tr}(DB) = \sum_{k,l=1}^{n} \frac{\lambda_k + \lambda_l}{2} [A_0]_{lk} [B_0]_{kl}$$

$$q\text{Cov}_{D,f}^{as}(A, B) = \frac{f(0)}{2} \sum_{k,l=1}^{n} \frac{(\lambda_k - \lambda_l)^2}{m_f(\lambda_k, \lambda_l)} A_{kl} B_{kl} = \frac{f(0)}{2} \sum_{k,l=1}^{n} \frac{(\lambda_k - \lambda_l)^2}{m_f(\lambda_k, \lambda_l)} [A_0]_{lk} [B_0]_{kl}$$

$$q\text{Cov}_{D,f}^{s}(A, B) = \frac{f(0)}{2} \sum_{k,l=1}^{n} \frac{(\lambda_k + \lambda_l)^2}{m_f(\lambda_k, \lambda_l)} A_{kl} B_{kl}$$

$$q\text{Cov}_{D,f}^{s}(A, B) = \frac{f(0)}{2} \sum_{k,l=1}^{n} \frac{(\lambda_k + \lambda_l)^2}{m_f(\lambda_k, \lambda_l)} [A_0]_{lk} [B_0]_{kl} + 2f(0) \text{Tr}(DA) \text{Tr}(DB).$$

**Proof.** Simple matrix computation, we show the case of asymmetric quantum covariance.

$$q\text{Cov}_{D,f}^{as}(A, B) = \frac{f(0)}{2} \langle DA - AD, DB - BD \rangle_{D,f}$$

$$= \frac{f(0)}{2} \sum_{k,l=1}^{n} \frac{1}{m_f(\lambda_k, \lambda_l)} (DA - AD)_{lk} (DB - BD)_{kl}$$

Since $(DA - AD)_{lk} = (\lambda_k - \lambda_l)A_{lk}$ we immediately have the formula in the Theorem.

All of the covariances in the Theorem have the form of

$$\sum_{k,l=1}^{n} \alpha_{kl} A_{kl} B_{kl},$$

where $\alpha$ is a symmetric function. Since $[A_0]_{kl} = A_{kl} - \delta_{kl} \text{Tr}(DA)$ we have the equality

$$\sum_{k,l=1}^{n} \alpha_{kl} [A_0]_{kl} [B_0]_{kl} = \sum_{k,l=1}^{n} \alpha_{kl} A_{kl} B_{kl} + \left( \sum_{k=1}^{n} \alpha_{kk} \right) \text{Tr}(DA) \text{Tr}(DB)$$

$$- \left( \sum_{k=1}^{n} \alpha_{kk} A_{kk} \right) \text{Tr}(DB) - \left( \sum_{k,k=1}^{n} \alpha_{kk} B_{kk} \right) \text{Tr}(DA).$$

Using this equation one can express $\sum_{k,l=1}^{n} \alpha_{kl} A_{kl} B_{kl}$ in terms of normalised observables.
It is worth noting that for operator monotone function \( f_0(x) = \frac{1 + x}{2} \) we have
\[
\text{Cov}_D(A_0, B_0) = \text{qCov}^s_{D,f_0}(A_0, B_0).
\]

**Definition 3.** For an operator monotone function \( f \in \mathcal{F}_\text{op} \) let us define the following \( \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) functions
\[
g_{cl}(x, y) = \frac{x + y}{2}, \quad g^{as}_f(x, y) = \frac{f(0)(x - y)^2}{2xf\left(\frac{x}{2}\right)}, \quad g_f^s(x, y) = \frac{f(0)(x + y)^2}{2xf\left(\frac{x}{2}\right)}.
\]

A simple combination of Theorems 4 and 3 gives the following Corollary.

**Corollary 1.** For every \( f \in \mathcal{F}_\text{op} \), \( D, g \in \mathcal{M}_n \) and for every matrices \( A, B \in \mathcal{M}_{n,sa} \) the following equalities hold.
\[
(A_0, B_0)_{D,g_{cl}} = \text{Cov}_D(A_0, B_0)
\]
\[
(A_0, B_0)_{D,g^{as}_f} = \text{qCov}^{as}_{D,f}(A_0, B_0) = \text{qCov}^{as}_{D,f}(A, B)
\]
\[
(A_0, B_0)_{D,g_f^s} = \text{Cov}^s_{D,f}(A_0, B_0)
\]

### 3 Uncertainty relations

**Theorem 5.** Consider a density matrix \( D \in \mathcal{M}_n \), functions \( g_1, g_2 \in \mathcal{C}_\mathcal{M} \) such that
\[
g_1(x, y) \geq g_2(x, y) \quad \forall x, y \in \mathbb{R}^+
\]
holds and an \( N \)-tuple of nonzero matrices \( (A^{(k)})_{k=1,\ldots,N} \in \mathcal{M}_{n,sa} \). Define the \( N \times N \) matrices \( \text{Cov}_{D,g_1} \) and \( \text{Cov}_{D,g_2} \) with entries
\[
[\text{Cov}_{D,g_k}]_{ij} = (A^{(i)}_0, A^{(j)}_0)_{D,g_k} \quad k = 1, 2.
\]
We have
\[
\det(\text{Cov}_{D,g_1}) \geq \det(\text{Cov}_{D,g_2}) + \det(\text{Cov}_{D,g_1} - \text{Cov}_{D,g_2}) + R(D, g_1, g_2, N),
\]
where
\[
R(D, g_1, g_2, N) = \sum_{k=1}^{N-1} \binom{N}{k} \sqrt[\frac{k}{N}]{\det(\text{Cov}_{D,g_1})} \sqrt[\frac{N-k}{N}]{\det(\text{Cov}_{D,g_1} - \text{Cov}_{D,g_2})}.
\]

**Proof.** We can assume that the density matrix \( D \) is diagonal, \( D = \text{Diag}(\lambda_1, \ldots, \lambda_n) \). The matrix \( \text{Cov}_{D,g_k} \) \( (k = 1, 2) \) is obviously real and symmetric. First we prove that for every \( g \in \mathcal{C}_\mathcal{M} \) the matrix \( \text{Cov}_{D,g} \) is positive definite. Consider a vector \( x \in \mathbb{C}^N \) and define an \( n \times n \) matrix as \( C = \sum_{a=1}^{N} x_a A^{(a)}_0 \). Then we have
\[
(x, \text{Cov}_{D,g} x) = \sum_{a,b=1}^{N} x_a x_b \text{Cov}_{D,f}(A^{(a)}, A^{(b)})
\]
\[
= \sum_{a,b=1}^{N} \sum_{h,j=1}^{n} g(\lambda_a, \lambda_j) x_a x_b \left[A^{(a)}_0\right]_{hj} \left[A^{(b)}_0\right]_{jh}
\]
\[
= \sum_{h,j=1}^{n} g(\lambda_h, \lambda_j) \left[\sum_{a=1}^{N} x_a \left[A^{(a)}_0\right]_{jh}\right] \left[\sum_{b=1}^{N} x_b \left[A^{(b)}_0\right]_{jh}\right]
\]
\[
= \sum_{h,j=1}^{n} g(\lambda_h, \lambda_j) |C_{hj}|^2 \geq 0.
\]
We can repeat our arguments providing Equation (4) for the matrix $\text{Cov}_{D,g_1} - \text{Cov}_{D,g_2}$ using $g_1 - g_2$ instead of $g$. This leads us to the conclusion that $\text{Cov}_{D,g_1} - \text{Cov}_{D,g_2}$ is a real, symmetric, positive definite matrix.

Using the Minkowski determinant inequality (see for example [2] p. 70 or [9]) for real symmetric positive matrices $\text{Cov}_{D,g_1}$ and $(\text{Cov}_{D,g_1} - \text{Cov}_{D,g_2})$ we have

$$\left\{ \det(\text{Cov}_{D,g_2} + (\text{Cov}_{D,g_1} - \text{Cov}_{D,g_2})) \right\}^{\frac{1}{N}} \geq \left\{ \det(\text{Cov}_{D,g_1}) \right\}^{\frac{1}{N}} \left\{ \det(\text{Cov}_{D,g_1} - \text{Cov}_{D,g_2}) \right\}^{\frac{1}{N}},$$

which gives the inequality stated in the Theorem.

Using the inequality (3) in the previous Theorem one can compare the different covariances.

**Theorem 6.** Consider a density matrix $D \in M_1^n$, functions $f_1, f_2 \in F_{op}$ such that

$$\frac{f_1(t)}{f_2(t)} \geq \frac{f_1(t)}{f_2(t)} \quad \forall t \in \mathbb{R}^+$$

holds and an $N$-tuple of nonzero matrices $(A^{(k)})_{k=1,\ldots,N} \in M_{n,sa}$. Define the $N \times N$ matrices $\text{Cov}_{D}$, $q\text{Cov}^{as}_{D,f_k}$ and $q\text{Cov}^{s}_{D,f_k}$ ($k = 1, 2$) with entries

$$[\text{Cov}_{D}]_{ij} = \text{Cov}_{D}(A_0^{(i)}, A_0^{(j)})$$

$$[q\text{Cov}^{as}_{D,f_k}]_{ij} = q\text{Cov}^{as}_{D,f_k}(A_0^{(i)}, A_0^{(j)}) \quad k = 1, 2$$

$$[q\text{Cov}^{s}_{D,f_k}]_{ij} = q\text{Cov}^{s}_{D,f_k}(A_0^{(i)}, A_0^{(j)}) \quad k = 1, 2.$$

We have

$$\det(\text{Cov}_{D}) \geq \det(q\text{Cov}^{as}_{D,f_k}) \quad k = 1, 2$$

$$\det(\text{Cov}_{D}) \geq \det(q\text{Cov}^{s}_{D,f_k}) \quad k = 1, 2$$

$$\det(q\text{Cov}^{as}_{D,f_k}) \geq \det(q\text{Cov}^{as}_{D,f_k}) \quad k = 1, 2$$

$$\det(q\text{Cov}^{s}_{D,f_k}) \geq \det(q\text{Cov}^{s}_{D,f_k})$$

**Proof.** If for functions $f_1, f_2 \in F_{op}$

$$\frac{f_1(t)}{f_2(t)} \geq \frac{f_1(t)}{f_2(t)} \quad \forall t \in \mathbb{R}^+$$

holds then we have the following inequalities for all $x, y \in \mathbb{R}^+$.

$$g_{cl}(x, y) \geq g^{as}_{f_k}(x, y) \quad k = 1, 2$$

$$g_{cl}(x, y) \geq g^{s}_{f_k}(x, y) \quad k = 1, 2$$

$$g^{as}_{f_k}(x, y) \geq g^{as}_{f_k}(x, y) \quad k = 1, 2$$

$$g^{s}_{f_k}(x, y) \geq g^{s}_{f_k}(x, y)$$

Applying Theorem 5 for these inequalities we have the inequalities for different covariance matrices stated in the Theorem.
Corollary 2. Using the same notation as in the previous theorem for any operator monotone function $f \in \mathcal{F}_\text{op}$ we have
\[
\det(\text{Cov}_D) \geq \det(q\text{Cov}^*_D,f) \geq \det(q\text{Cov}^{as}_D,f).
\]
The generalised form of the uncertainty relation mentioned in the Introduction (see Equation (1) in these terms is
\[
\det(\text{Cov}_D) \geq \det(q\text{Cov}^{as}_D,f).
\]

Theorem 7. With same notation as above let us consider operator monotone functions $f_1, f_2 \in \mathcal{F}_\text{op}$. If
\[
\frac{f_1(t)}{f_1(0)} \geq \frac{f_2(t)}{f_2(0)} \times \frac{(t+1)^2}{(t-1)^2} \forall t \in \mathbb{R}^+
\]
holds, then
\[
\det(q\text{Cov}^{as}_{D,f_1}) \geq \det(q\text{Cov}^{as}_{D,f_2}).
\]
If
\[
\frac{f_1(t)}{f_1(0)} \leq \frac{f_2(t)}{f_2(0)} \times \frac{(t+1)^2}{(t-1)^2} \forall t \in \mathbb{R}^+
\]
holds, then
\[
\det(q\text{Cov}^{as}_{D,f_1}) \geq \det(q\text{Cov}^{as}_{D,f_2}).
\]

Proof. If for functions $f_1, f_2 \in \mathcal{F}_\text{op}$
\[
\frac{f_1(t)}{f_1(0)} \geq \frac{f_2(t)}{f_2(0)} \times \frac{(t+1)^2}{(t-1)^2} \forall t \in \mathbb{R}^+
\]
holds then we have the following inequalities for all $x, y \in \mathbb{R}^+$.
\[
g^{as}_{f_1}(x,y) \geq g^{as}_{f_2}(x,y)
\]
Theorem 5 provides the desired inequality for asymmetric and symmetric covariance matrices in this case. The second inequality can be proved similarly.

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