OPS-QFTs: A new type of quaternion Fourier transforms based on the orthogonal planes split with one or two general pure quaternions

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Abstract. We explain the orthogonal planes split (OPS) of quaternions based on the arbitrary choice of one or two linearly independent pure unit quaternions \( f, g \). Next we systematically generalize the quaternionic Fourier transform (QFT) applied to quaternion fields to conform with the OPS determined by \( f, g \), or by only one pure unit quaternion \( f \), comment on their geometric meaning, and establish inverse transformations.

Keywords: Clifford geometric algebra, quaternion geometry, quaternion Fourier transform, inverse Fourier transform, orthogonal planes split

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References [2, 3, 4, 5] give background on quaternion Fourier transformations. For details and proofs of the orthogonal planes split we refer the reader to [6].

Gauss, Rodrigues and Hamilton’s four-dimensional (4D) quaternion algebra \( \mathbb{H} \) is defined over \( \mathbb{R} \) with three imaginary units:

\[
ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \quad i^2 = j^2 = k^2 = ijk = -1.
\] (1)

Every quaternion can be written explicitly as

\[
q = qr + qi + qj + qk \in \mathbb{H}, \quad q_r, q_i, q_j, q_k \in \mathbb{R},
\] (2)

and has a quaternion conjugate (equivalent to reversion in \( \text{Cl}^{+}_{3,0} \))

\[
\tilde{q} = q_r - qi - qj - qk, \quad \tilde{pq} = \tilde{q}\tilde{p},
\] (3)

which leaves the scalar part \( q_r \) unchanged. This leads to the norm of \( q \in \mathbb{H} \)

\[
|q| = \sqrt{q\tilde{q}} = \sqrt{q_r^2 + q_i^2 + q_j^2 + q_k^2}, \quad |pq| = |p||q|.
\] (4)

The part \( q = q - q_r = (q - \tilde{q})/2 = qi + qj + qk \) is called a pure quaternion, and it squares to the negative number \(-(q_r^2 + q_i^2 + q_k^2)\). Every unit quaternion (i.e. \(|q| = 1\)) can be written as

\[
q = q_r + qi + qj + qk = q_r + \sqrt{q_i^2 + q_j^2 + q_k^2} \hat{q} = \cos \alpha + \sin \alpha \hat{q} = e^{\alpha \hat{q}},
\]

\[
\cos \alpha = q_r, \quad \sin \alpha = \sqrt{q_i^2 + q_j^2 + q_k^2}, \quad \hat{q} = \frac{qi + qj + qk}{\sqrt{q_i^2 + q_j^2 + q_k^2}}, \quad \hat{q}^2 = -1.
\] (5)

The inverse of a non-zero quaternion is

\[
q^{-1} = \frac{\tilde{q}}{|q|^2} = \frac{\tilde{q}}{q\tilde{q}}.
\] (6)

The symmetric scalar part of a quaternion is defined as

\[
\text{Sc}(q) = q_r = \frac{1}{2}(q + \tilde{q}), \quad \text{Sc}(pq) = \text{Sc}(qp) = p_r q_r - p_i q_i - p_j q_j - p_k q_k \quad \text{for} \quad p \in \mathbb{H}.
\] (7)
with linearity
\[ \text{Sc}(\alpha p + \beta q) = \alpha \text{Sc}(p) + \beta \text{Sc}(q), \quad \forall p, q \in \mathbb{H}, \quad \alpha, \beta \in \mathbb{R}. \]  
(8)
The scalar part and the quaternion conjugate allow the definition of the \( \mathbb{R}^4 \) inner product of two quaternions \( p, q \) as
\[ \text{Sc}(pq) = \text{Sc}(p)x + \text{Sc}(q)x + \text{Sc}(p)y + \text{Sc}(q)y + \text{Sc}(p)z + \text{Sc}(q)z + \text{Sc}(p)w + \text{Sc}(q)w \in \mathbb{R}. \]  
(9)

We consider an arbitrary pair of linearly independent nonorthogonal pure quaternions \( f, g, f^2 = g^2 = -1, f \neq \pm g \). The orthogonal 2D planes split (OPS) is then defined with respect to the linearly independent pure unit quaternions \( f, g \) as
\[ q_\pm = \frac{1}{2}(q \pm fg). \]  
(10)

We observe, that \( ffg = q_+ - q_- \), i.e. under the map \( f() \) the \( q_+ \) part is invariant, but the \( q_- \) part changes sign.

Both parts are two-dimensional, and span two completely orthogonal planes. The \( q_+ \) plane is spanned by the orthogonal quaternions \( \{ -g, 1 + fg \} \) whereas the \( q_- \) plane is e.g. spanned by \( \{ f + g, 1 - fg \} \).

**Lemma 1** (Orthogonality of two OPS planes). Given two quaternions \( q, p \) and applying the OPS with respect to two linearly independent pure unit quaternions \( f, g \) we get zero for the scalar part of the mixed products
\[ \text{Sc}(p_+q_-) = 0, \quad \text{Sc}(p_-q_+) = 0. \]  
(11)

The set \( \{ f - g, 1 + fg, f + g, 1 - fg \} \) forms an orthogonal basis of \( \mathbb{H} \) interpreted as \( \mathbb{R}^4 \). We can therefore use the following representation for every \( q \in \mathbb{H} \) by means of four real coefficients \( q_1, q_2, q_3, q_4 \in \mathbb{R} \)
\[ q = q_1(1 + fg) + q_2(f - g) + q_3(1 - fg) + q_4(f + g), \]  
(12)
\[ q_1 = \text{Sc}(q(1 + fg)^{-1}), \quad q_2 = \text{Sc}(q(f - g)^{-1}), \quad q_3 = \text{Sc}(q(1 - fg)^{-1}), \quad q_4 = \text{Sc}(q(f + g)^{-1}). \]

As an example we have for \( f = i, g = j \) we obtain
\[ q_1 = \frac{1}{2}(q_i + q_k), \quad q_2 = \frac{1}{2}(q_i - q_j), \quad q_3 = \frac{1}{2}(q_r - q_k), \quad q_4 = \frac{1}{2}(q_i + q_j). \]  
(13)

**Theorem 1** (Determination of \( f, g \) from given analysis planes). Assume that two desired analysis planes are given by a set of four orthogonal quaternions \( \{ a, b \} \) and \( \{ c, d \} \). Without restriction of generality \( a^2 = c^2 = -1 \).

For the \( \{ a, b \} \) plane to become the \( q_- \) plane and the \( \{ c, d \} \) plane the \( q_+ \) plane we need to set
\[ f = ba, \quad g = \text{Sc}(f\bar{a})a - \text{Sc}(f\bar{c})c. \]  
(14)

For the opposite assignment of the \( \{ a, b \} \) plane to become the \( q_+ \) plane and the \( \{ c, d \} \) plane the \( q_- \) plane, we only need to change the sign of \( g \), i.e. we can set
\[ f = ba, \quad g = \text{Sc}(f\bar{c})c - \text{Sc}(f\bar{a})a. \]  
(15)

The map \( f() \) rotates the \( q_- \) plane by \( 180^\circ \) around the \( q_+ \) axis plane. This interpretation of the map \( f() \) is in perfect agreement with Coxeter’s notion of half-turn in \([1]\).

The following identities hold
\[ e^{\alpha f}q_\pm e^{\beta g} = q_\pm e^{(\beta + \alpha)g} = e^{(\alpha + \beta)f}q_\pm. \]  
(16)

The general double sided orthogonal planes (i.e. 2D subspaces) split quaternion Fourier transform (OPS-QFT) is defined as
\[ \mathcal{F}_{\mathcal{S}} f, g \{ h \}(\omega) = \int_{\mathbb{R}^2} e^{-j\omega_1 x_1} h(x) e^{-j\omega_2 x_2} d^2x, \]  
(17)
where \( h \in L^1(\mathbb{R}^2, \mathbb{H}) \), \( d^2x = dx_1 dx_2 \) and \( x, \omega \in \mathbb{R}^2 \). The OPS-QFT \([17]\) is invertible
\[ h(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{j\omega_1 x_1} \mathcal{F}_{\mathcal{S}} f, g \{ h \} \{ \omega \} e^{j\omega x} d^2\omega, \]  
(18)
Linearity of the integral allows us to use the OPS split \( h = h_+ + h_\pm = \frac{1}{2}(h \pm fhg) \)

\[
\mathcal{F}^{f,g}\{h\}(\omega) = \mathcal{F}^{f,g}\{h_+\}(\omega) + \mathcal{F}^{f,g}\{h_\pm\}(\omega) = \mathcal{F}^{f,g}\{h\}(\omega) + \mathcal{F}^{\dagger,f}\{h\}(\omega),
\]

(19)

since by its construction the operators of the Fourier transformation \( \mathcal{F}^{f,g} \), and of the OPS with respect to \( f, g \) commute. From (16) follows

**Theorem 2 (OPS-QFT of \( h_\pm \)).** The QFT of the \( h_\pm \) OPS split parts, with respect to two linearly independent unit quaternions \( f, g \), of a quaternion module function \( h \in L^2(\mathbb{R}^2, \mathbb{H}) \) have the complex forms

\[
\mathcal{F}^{f,g}\{h_\pm\}(\omega) = \int_{\mathbb{R}^2} h_\pm e^{-\frac{1}{2}(x_1 \omega_1 + x_2 \omega_2)} d^2 x = \int_{\mathbb{R}^2} e^{-f(x_1 \omega_1 + x_2 \omega_2)} h_\pm d^2 x.
\]

(20)

The geometric interpretation of the integrand \( e^{-f(x_1 \omega_1 + x_2 \omega_2)} \) of the QFT\(^{f,g}\) in (17) is: The integrand means to locally rotate by the phase angle \(- (x_1 \omega_1 + x_2 \omega_2) \) in the \( q_- \) plane, and by phase angle \(- (x_1 \omega_1 - x_2 \omega_2) = x_2 \omega_2 - x_1 \omega_1 \) in the \( q_+ \) plane.

The *phase angle* OPS-QFT with a straight forward two phase angle interpretation is

\[
\mathcal{F}^{f,g}\{h\}(\omega) = \int_{\mathbb{R}^2} e^{-\frac{1}{2}(x_1 \omega_1 + x_2 \omega_2)} h(x) e^{-g\frac{1}{2}(x_1 \omega_1 - x_2 \omega_2)} d^2 x.
\]

(21)

where again \( h \in L^1(\mathbb{R}^2, \mathbb{H}) \), \( d^2 x = dx_1 dx_2 \) and \( x, \omega \in \mathbb{R}^2 \). Its inverse is given by

\[
h(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\frac{1}{2}(x_1 \omega_1 + x_2 \omega_2)} \mathcal{F}^{f,g}\{h\}(\omega) e^{g\frac{1}{2}(x_1 \omega_1 - x_2 \omega_2)} d^2 \omega.
\]

(22)

The geometric interpretation of the integrand of (21) is a local phase rotation by angle \(- (x_1 \omega_1 + x_2 \omega_2)/2 - (x_1 \omega_1 - x_2 \omega_2)/2 = -x_1 \omega_1 \) in the \( q_- \) plane, and a second local phase rotation by angle \(- (x_1 \omega_1 + x_2 \omega_2)/2 + (x_1 \omega_1 - x_2 \omega_2)/2 = -x_2 \omega_2 \) in the \( q_+ \) plane.

If we apply the OPS\(^{f,g}\) split to (21) we obtain the following two parts

\[
\mathcal{F}^{f,g}_{D_+}\{h\} = \int_{\mathbb{R}^2} h_+ e^{f x_2 \omega_2} d^2 x = \int_{\mathbb{R}^2} e^{-f x_2 \omega_2} h_+ d^2 x, \quad \mathcal{F}^{f,g}_{D^-}\{h\} = \int_{\mathbb{R}^2} h_- e^{-g x_1 \omega_1} d^2 x = \int_{\mathbb{R}^2} e^{-f x_1 \omega_1} h_- d^2 x.
\]

(23)

The OPS with respect to, e.g., \( f = g = i \) gives

\[
q_\pm = \frac{1}{2}(q \pm iq), \quad q_+ = qj + qk = (q_j + q_k)i, \quad q_- = q_r + q_i,
\]

(24)

where the \( q_+ \) plane is two-dimensional and manifestly orthogonal to the 2D \( q_- \) plane. The above corresponds to the simplex/perplex split of \( \mathbb{H} \). \( e^{qf}q e^{-\beta f} \) means a rotation by angle \( \alpha + \beta \) in the \( q_- \) plane followed by a rotation by angle \( \alpha - \beta \) in the orthogonal \( q_+ \) plane.

A variant of the OPS-QFT with \( g = \bar{f} \) is therefore

\[
\mathcal{F}^{f,\bar{f}}\{h\}(\omega) = \int_{\mathbb{R}^2} e^{-f x_1 \omega_1} h(x) e^{-\bar{f} x_2 \omega_2} d^2 x,
\]

(25)

where \( h \in L^1(\mathbb{R}^2, \mathbb{H}) \), \( d^2 x = dx_1 dx_2 \) and \( x, \omega \in \mathbb{R}^2 \). The immediate geometric interpretation is that the integrand \( e^{-f x_1 \omega_1} h(x) e^{-\bar{f} x_2 \omega_2} \) leads to a local phase rotation by angle \(- (x_1 \omega_1 + x_2 \omega_2) \) in the \( q_- \) plane combined with a second local phase rotation by angle \( x_2 \omega_2 - x_1 \omega_1 \) in the \( q_+ \) plane. The inverse transform is given by

\[
h(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{f x_1 \omega_1} \mathcal{F}^{f,\bar{f}}\{h\}(\omega) e^{\bar{f} x_2 \omega_2} d^2 \omega.
\]

(26)

The *phase angle* OPS-QFT, using \( g = f \), with a straight forward two phase angle interpretation is

\[
\mathcal{F}^{f,f}_{D_+}\{h\}(\omega) = \int_{\mathbb{R}^2} e^{-\frac{1}{2}(x_1 \omega_1 + x_2 \omega_2)} h(x) e^{-\frac{1}{2}(x_1 \omega_1 - x_2 \omega_2)} d^2 x.
\]

(27)
where again $h \in L^1(\mathbb{R}^2, \mathbb{H})$, $d^2x = dx_1dx_2$ and $x, \omega \in \mathbb{R}^2$. Its inverse is given by

$$h(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{if(x_1\omega_1 + x_2\omega_2)} \mathcal{F}_D^f \{h\}(\omega) e^{-if(x_1\omega_1 + x_2\omega_2)}d^2\omega. \quad (28)$$

The geometric interpretation of the integrand of (27) is a local phase rotation by angle $-(x_1\omega_1 + x_2\omega_2)/2$ in the $q_-$ plane, and a second local phase rotation by angle $-(x_1\omega_1 + x_2\omega_2)/2 + (x_1\omega_1 - x_2\omega_2)/2 = -x_2\omega_2$ in the $q_+$ plane.

If we apply the OPS\(^{\pm}\) split to (27) we obtain the following two parts

$$\mathcal{F}_D^f \{h\} = \int_{\mathbb{R}^2} h_+e^{if_2\omega_2}d^2x = \int_{\mathbb{R}^2} e^{-if_2\omega_2}h_+d^2x, \quad \mathcal{F}_D^{-f} \{h\} = \int_{\mathbb{R}^2} h_-e^{-if_2\omega_2}d^2x = \int_{\mathbb{R}^2} e^{-if_2\omega_2}h_-d^2x. \quad (29)$$

The geometric interpretation of the integrand of (27) is a local phase rotation by angle $-(x_1\omega_1 + x_2\omega_2)/2$ in the $q_-$ plane, and a second local phase rotation by angle $-(x_1\omega_1 + x_2\omega_2)/2 + (x_1\omega_1 - x_2\omega_2)/2 = -x_2\omega_2$ in the $q_+$ plane.

The general OPS-QFT involving quaternion conjugation is defined as

$$\mathcal{F}_c^g \{h\}(\omega) = \int_{\mathbb{R}^2} e^{-if_1\omega_1}h(x)e^{-if_2\omega_2}d^2x, \quad (30)$$

where $h \in L^1(\mathbb{R}^2, \mathbb{H})$, $d^2x = dx_1dx_2$ and $x, \omega \in \mathbb{R}^2$. The inverse is given by

$$h(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-if_2\omega_2} \mathcal{F}_c^g \{h\}(\omega)e^{-if_1\omega_1}d^2\omega. \quad (31)$$

This approach results in the following OPS theorem.

**Theorem 3 (OPS-QFT $\mathcal{F}_c^g$ of $h_\pm$).** The OPS-QFT $\mathcal{F}_c^g$ (30) of the $h_\pm = \frac{1}{2}(h \pm fhg)$ OPS split parts, with respect to two linearly independent unit quaternions $f, g$, of a quaternion module function $h \in L^2(\mathbb{R}^2, \mathbb{H})$ have the complex forms

$$\mathcal{F}_c^g \{h\} = \int_{\mathbb{R}^2} \sim e^{-if_2\omega_2}d^2x = \int_{\mathbb{R}^2} e^{-if_1\omega_1}h(x)e^{-if_2\omega_2}d^2x. \quad (32)$$

The variant of the general OPS-QFT involving quaternion conjugation, using $g = f$, is defined as

$$\mathcal{F}_c^f \{h\}(\omega) = \int_{\mathbb{R}^2} e^{-if_1\omega_1}h(x)e^{-if_2\omega_2}d^2x, \quad (33)$$

where $h \in L^1(\mathbb{R}^2, \mathbb{H})$, $d^2x = dx_1dx_2$ and $x, \omega \in \mathbb{R}^2$. The inverse is given by

$$h(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-if_2\omega_2} \mathcal{F}_c^f \{h\}(\omega)e^{-if_1\omega_1}d^2\omega. \quad (34)$$

This results in the following OPS theorem.

**Theorem 4 (OPS-QFT $\mathcal{F}_c^f$ of $h_\pm$).** The OPS-QFT $\mathcal{F}_c^f$ (33) of the $h_\pm = \frac{1}{2}(h \pm fhf)$ OPS split parts, with respect to the unit quaternion $f$, of a quaternion module function $h \in L^2(\mathbb{R}^2, \mathbb{H})$ have complex forms

$$\mathcal{F}_c^f \{h\} = \int_{\mathbb{R}^2} \sim e^{-if_2\omega_2}d^2x = \int_{\mathbb{R}^2} e^{-if_1\omega_1}h(x)e^{-if_2\omega_2}d^2x. \quad (35)$$

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