Linear Instability Analysis of an Electrified Viscoelastic Liquid Sheet in Compressible Ambient Gas

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Abstract. The linear instability analysis of an electrified viscoelastic liquid sheet exposed to a compressible ambient gas in the presence of a transverse electric field is theoretically investigated. The dispersion relations for the sinuous and varicose mode are obtained. The combined effects of Mach number and Euler number on the instability of the liquid sheet are checked. Our results demonstrate that there is a critical Mach number, below which the growth rate increases and above which the growth rate decreases with the increase of the Mach number. A larger Euler number leads to a smaller critical Mach number. The maximum growth rate nearly stays constant for Mach number that is smaller than the critical Mach number at large Euler numbers. Besides, the electric field enhances the instability of charged electrified liquid sheets, and the maximum growth rate is a linear increasing function of the Euler number. Furthermore, the compressibility of the gas flow will weaken the effect of the electric field on the instability of liquid sheets.

1. Introduction
The instability of a liquid sheet exposed to a gas flow in the presence of a transverse electric field is of vitally practical importance in the electro-spraying and electro-spinning application.

The linear instability analysis of an inviscid liquid sheet was firstly studied by Squire[1] and Hagerty & Shea[2]. They proved that the surface tension displays a stabilizing effect on the disturbances of the liquid sheet. Li & Tankin[3] performed a study on the instability of viscous liquid sheets. It was shown that there exists an additional viscosity-enhanced instability that is induced by liquid viscosity at low Weber numbers. Liu et al.[4] studied the linear instability of viscoelastic sheets, and found that viscoelastic sheets are much more unstable than Newtonian liquid sheets at the same Weber number and Reynolds number. Sayed & Syam[5] investigated the electro-hydrodynamic instability of a dielectric, compressible liquid sheet streaming into a compressible gas. Yang et al.[6][7] evaluated the instability of electrified viscoelastic liquid sheets in quiescent gas medium. They found that the disturbance growth rate of the electrified viscoelastic liquid sheet is much higher than that of the electrified Newtonian liquid sheet. No analytical approach, however, has yet been made for the linear instability of a viscoelastic sheet exposed to a compressible gas flow in the presence of a transverse electric field. The comprehensive effects of Mach number and Euler number on the instability of viscoelastic sheet were considered in the present paper.
2. Mathematic solutions

Consider a viscoelastic incompressible liquid sheet of thickness \(2a\), density \(\rho_l\), surface tension \(\sigma\), surrounded by a compressible gas medium of density \(\rho_g\). In the undisturbed state, the sheet moves at a constant velocity through a inviscid gas flow with a velocity of \(U_g\), the liquid sheet has a relative velocity \(\vec{U} = (U,0) = (U_i - U_g,0)\), where \(U_i\) is the unperturbed axial velocity of liquid sheet. The interface displacement is \(\eta_j\), where \(j = 1\) represents the upper interface and \(j = 2\) represents the lower interface.

The governing equations for the liquid sheet are the conservation laws of mass and momentum, as given below:

\[
\nabla \cdot \vec{U} = 0 \quad -a + \eta_1 < y < a + \eta_2
\]

\[
\rho_l (\frac{\partial}{\partial t} + \vec{U} \cdot \nabla) \vec{U} = -\nabla p - \nabla \cdot \tau, \quad -a + \eta_1 < y < a + \eta_2
\]

Where \(t\) is the time, \(\rho\) is the liquid pressure, \(\tau\) is the extra stress tensor of liquid. The Oldroyd B-constitutive equation is adopted, which is written in objective reference frames as:

\[
\tau + \lambda_1 \frac{D\tau}{Dt} + \frac{1}{2} \mu (\dot{\tau} \cdot \dot{\tau}) - \frac{1}{2} \mu_t (\tau \cdot \dot{\tau} + \dot{\tau} \cdot \tau) + \frac{1}{2} \nu_1 (\tau : \dot{\tau}) \delta = -\mu [\ddot{\gamma} + \lambda_2 \frac{D\gamma}{Dt} - \mu_2 (\dot{\gamma} \cdot \dot{\gamma}) + \frac{1}{2} \nu_2 (\dot{\gamma} : \dot{\gamma}) \delta]
\]

\[
\left[ \hat{\gamma}, \hat{\omega}, \frac{D\mathbf{F}}{Dt} \right] = \nabla \mathbf{U} + (\nabla \mathbf{U})^T, \nabla \mathbf{U} - (\nabla \mathbf{U})^T, \frac{\partial \mathbf{F}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{F} + \frac{1}{2} (\hat{\omega} \cdot \mathbf{F} - \mathbf{F} \cdot \hat{\omega})
\]

Where \(\mu\) is the zero shear viscosity, \(\lambda_1\) is the stress relaxation time, \(\lambda_2\) is the deformation retardation time, and \(\delta\) is the unit tensor. The quantities \(\mu, \mu_t, \mu_2, \nu_1,\) and \(\nu_2\) are time constants.

The kinematic boundary conditions at the interface can be expressed as followings:

\[
F_{j+} + \mathbf{U} \cdot \nabla F_{j} = 0, \quad y = (-1)^{\nu_{j+}} a + \eta_j
\]

Where \(F_{j} = y - \eta_j - (-1)^{\nu_{j+}} a\) is the interface function of the liquid and gas.

As the gas phase is inviscid, the liquid shear stress at the interface should vanish:

\[
\left[ \eta_j \cdot (-p \delta + \tau) \right] \times \mathbf{n}_j = 0
\]

Where \(\mathbf{n}_j = \nabla F_{j} / |\nabla F_{j}|\) is the surface unit normal vector.

Correspondingly, the governing equations of inviscid gas phases are

\[
\nabla \cdot (\rho_g \mathbf{U}_g) = 0 \quad y < -a + \eta_1 \text{ or } y > a + \eta_2
\]

\[
\rho_g (\frac{\partial}{\partial t} + \mathbf{U}_g \cdot \nabla) \mathbf{U}_g = -\nabla p_g, \quad y < -a + \eta_1 \text{ or } y > a + \eta_2
\]

Where \(\rho_g\) is the gas density, \(\mathbf{U}_g\) is the gas velocity vector, \(p_g\) is the gas pressure.

The boundary conditions for the gas phase near the interface of the sheet are regarded as:
The stress at the interface is continuous, thus the normal dynamic boundary condition becomes:

\[ n_j \cdot (-p\sigma + \tau) + p_g + E_j + (-1)^{j+1} \sigma (\nabla \cdot n_j) = 0, \quad y = (-1)^{j+1} a + \eta_j \]  

(10)

For the liquid is a perfect conductor, the electric potential must satisfy the Laplace equation:

\[ \nabla^2 V_g = 0 \]  

(11)

As to the electric field, the electrical potential at the electrode equals zero:

\[ V_g \bigg|_{y=a+d} = 0, \quad V_g \bigg|_{y=a} = V_0 \]  

(12)

The dependent variables in Eqs. (1)-(12) can be presented as the sum of the basic value and the unsteady perturbation. Furthermore, the solutions of the unsteady perturbation are sought in the decomposition of normal mode:

\[ \begin{aligned} & (U, U_g, p, p_g, \rho, \rho_g, \eta, \tau, V_g) = (\bar{U}, \bar{U}_g, \bar{p}, \bar{p}_g, \bar{\rho}, \bar{\rho}_g, 0, 0, \bar{V}_g) + \\ & \left[ \eta_0 (\hat{U}, \hat{U}_g, \hat{p}, \hat{p}_g, \hat{\rho}, \hat{\rho}_g, \hat{\eta}, \hat{\tau}, \hat{V}_g) \cdot \exp(ikx + \alpha t) + c.c. \right] \end{aligned} \]  

(13)

Where \( \eta_0 \) is the initial amplitude of disturbance at the interface. \( k \) is the wave number. The frequency \( \omega = \omega_r + i \omega_i \) is a complex number, whose real part, \( \omega_r \), represents the temporal growth rate. The symbol ‘c.c.’ represents the complex conjugate aiming to eliminate the imaginary parts and make the solutions suitable for the practice.

Substituting Eq. (13) into the governing equations (1)-(12), and collecting the coefficient of the term \( \eta_0 \), The dispersion relation of the non-dimensional wave number \( K \) and complex frequency \( \Omega \) \((\Omega = \Omega_r + i \Omega_i)\) is obtained.

Sinuous mode:

\[ \frac{KQ\Omega^2}{\sqrt{Ma^2 \Omega^2 + K^2}} + K^3 \left( \frac{1}{We} - \frac{Eu}{K^2 D^3} \right) + \left( \frac{L^2 + K^2}{Re_{eff}^2} \right) \tanh(K) = 0 \]  

(14)

Varicose mode:

\[ \frac{KQ\Omega^2}{\sqrt{Ma^2 \Omega^2 + K^2}} + K^3 \left( \frac{1}{We} - \frac{Eu}{K^2 D^3} \right) + \left( \frac{L^2 + K^2}{Re_{eff}^2} \right) \coth(K) = 0 \]  

(15)

Where the effective Reynolds number \( Re_{eff} \) and the total wavenumber \( L \) are defined as:

\[ Re_{eff} = \frac{1 + (iK + \Omega) \cdot El \cdot Re}{1 + (iK + \Omega) \cdot \lambda \cdot El \cdot Re} \]  

(16)

\[ L = \sqrt{K^2 + Re(iK + \Omega)} \]  

(17)

To ensure proper simplification of the analysis and consistency of the results, all flow parameters are non-dimensionalized. The density ratio, the ratio of distance between horizontal electrode and the
liquid sheet to sheet thickness, the Reynolds number, the Weber number, the electrical Euler number, the Elasticity number, the time ratio, and the Mach number are respectively defined as $Q = \frac{\rho_g}{\rho_l}$, $D = d / a$, $Re = \frac{\rho_l Ua}{\mu}$, $We = \frac{\rho_1 U^2 a}{\sigma}$, $Eu = \frac{\varepsilon_0 V^2}{\rho U^2 a^2}$, $El = \frac{\lambda_u \mu}{\rho a^2}$, $\lambda = \frac{\lambda_0}{\lambda_1}$ and $Ma = \frac{U_g}{C}$. The non-dimensional dispersion relation is solved using the software Maple®.

3. Results and discussion
The linear instability of an electrified viscoelastic sheet in compressible gas streams is studied in this section. In the present study, we found that the effects of Weber number, Reynolds number, gas-to-liquid density ratio, Elasticity number and the time ratio on the linear instability of the sheets in compressible gas streams are essentially similar to those in incompressible gas streams. Thus, we will focus on the effects of the Mach number and electrical Euler number.

Figure 1 shows the dispersion curves of sinuous and varicose mode. As shown in the Figure, the temporal growth rate increases with the increased wavenumber until the maximum is reached, after which the growth rate decreases until the wavenumber reaches the cut-off value. Besides, the growth rate for the sinuous mode is greater than that for the varicose mode. Thus, only the sinuous mode was considered below.

![Figure 1. Dispersion relations of sinuous and varicose mode at $Q = 0.001$, $D = 10$, $Re = 1000$, $We = 500$, $Eu = 0.1$, $El = 0.1$, $\lambda = 0.1$, $Ma = 0.5$.](image)

The effect of the Mach number on the growth rate of the liquid sheet is depicted in Figure 2. It can be found that, as $Ma$ is increased, the maximum growth rate notably increases at $Ma < 1$ but decreases at $Ma > 1$, and the dominant wavenumber displaces toward larger value, meanwhile the unstable region broadens and the cut-off wavenumber increases. In addition, there exists a critical Mach number for a fixed Euler number, below which the maximum growth rate increases and above which it decreases with the Mach number. As shown in Figure 3, the critical Mach number $Ma_{cr}$ is around the value of 1 and decreases as the Euler number increases. Notably, the maximum growth rate nearly stays constant at $Ma < Ma_{cr}$ for large Euler number ($Eu = 1$).
The effect of the Mach number on the growth rate at \( Q = 0.001, \ D = 10, \ Re = 1000, \ We = 500, \ Eu = 0.1, \ El = 0.1, \ \lambda = 0.1 \)

![Figure 2](image)

Figure 2 The effect of the Mach number on the growth rate at \( Q = 0.001, \ D = 10, \ Re = 1000, \ We = 500, \ Eu = 0.1, \ El = 0.1, \ \lambda = 0.1 \)

The effect of Euler number on the growth rate is shown in Figure 4. Obviously, a larger Euler number leads to a larger maximum growth rate, indicating that the electric field can promote the instability of charged electrified liquid sheets. Nevertheless, the dominant and cut-off wavenumber nearly keep constant for different Euler numbers. It is worth noting that the Euler number does not affect the growth rate when the wavenumber is larger than 1 under the situation of \( Ma = 1.2 \).

![Figure 4](image)

Figure 4 The effect of Euler number on the growth rate for (a) \( Ma = 0.5 \) and (b) \( Ma = 1.2 \) at \( Q = 0.001, \ D = 10, \ Re = 1000, \ We = 500, \ El = 0.1, \ \lambda = 0.1 \).

As shown in Figure 5, the maximum growth rate grows with the increase of the Euler number linearly. Moreover, the slope of the linear curve is decreased with the increased Mach number, thus we can draw a conclusion that the compressibility of the gas flow will weaken the effect of the electric field on the instability of liquid sheets.

![Figure 5](image)
4. Conclusions

A linear instability of an electrified viscoelastic liquid sheet moving in a co-flowing compressible gas and a transverse electric field has been performed. The dispersion relations for the sinuous and varicose mode have been derived. The result indicates that the sinuous mode is more dominant than the varicose mode.

The combined effects of Mach number and Euler number on the instability are discussed. There exists a critical Mach number $Ma_{cr}$, below which the maximum growth rate increases and above which the growth rate decreases with the increased Mach number. In addition, the critical Mach number is closed to the value of 1 and decreased with the increase of Euler number. The maximum growth rate nearly keeps constant at $Ma < Ma_{cr}$ under the situation of $Eu = 1$. Moreover, a larger Euler number causes a larger growth rate but has virtually no effect on the dominant and cut-off wavenumber. Under the situation of $Ma = 1.2$, the Euler number does not affect the growth rate when the wavenumber is larger than 1. Furthermore, the maximum growth rate is a linear increasing function of the Euler number, the slope of which decreases with the increasing Mach number. It is concluded that the compressibility of the gas flow will weaken the effect of the electric field on the instability of the liquid sheets.

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