SHARP INEQUALITY FOR BOUNDED SUBMARTINGALES AND THEIR DIFFERENTIAL SUBORDINATES

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Submitted July 13, 2007, accepted in final form December 8, 2008

AMS 2000 Subject classification: Primary: 60G42. Secondary: 60G46.
Keywords: Martingale, submartingale, distribution function, tail inequality, differential subordination.

Abstract
Let $\alpha$ be a fixed number from the interval $[0, 1]$. We obtain the sharp probability bounds for the maximal function of the process which is $\alpha$-differentially subordinate to a bounded submartingale. This generalizes the previous results of Burkholder and Hammack.

1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, equipped with a discrete filtration $(\mathcal{F}_n)$. Let $f = (f_n)_{n=0}^{\infty}$, $g = (g_n)_{n=0}^{\infty}$ be adapted integrable processes taking values in a certain separable Hilbert space $\mathcal{H}$. The difference sequences $d f = (d f_n)$, $d g = (d g_n)$ of these processes are given by

$$d f_0 = f_0, \quad d f_n = f_n - f_{n-1}, \quad d g_0 = g_0, \quad d g_n = g_n - g_{n-1}, \quad n = 1, 2, \ldots.$$ 

Let $g^*$ stand for the maximal function of $g$, that is, $g^* = \max_n |g_n|$. The following notion of differential subordination is due to Burkholder. The process $g$ is differentially subordinate to $f$ (or, in short, subordinate to $f$) if for any nonnegative integer $n$ we have, almost surely,

$$|d g_n| \leq |d f_n|.$$

We will slightly change this definition and say that $g$ is differentially subordinate to $f$, if the above inequality for the differences holds for any positive integer $n$.

Let $\alpha$ be a fixed nonnegative number. Then $g$ is $\alpha$-differentially subordinate to $f$ (or, in short, $\alpha$-subordinate to $f$), if it is subordinate to $f$ and for any positive integer $n$ we have

$$\mathbb{E}(d g_n | \mathcal{F}_{n-1}) \leq \alpha \mathbb{E}(d f_n | \mathcal{F}_{n-1}).$$

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This concept was introduced by Burkholder in [2] in the special case \( \alpha = 1 \). In general form, it first appeared in the paper by Choi [3].

In the sequel it will sometimes be convenient to work with simple processes. A process \( f \) is called simple, if for any \( n \) the variable \( f_n \) is simple and there exists \( N \) such that \( f_N = f_{N+1} = f_{N+2} = \ldots \). Given such a process, we will identify it with the finite sequence \( (f_n)_{n=0}^N \).

Assume that the processes \( f \) and \( g \) are real-valued and fix \( \alpha \in [0, 1] \). The objective of this paper is to establish a sharp exponential inequality for the distribution function of \( g^\alpha \) under the assumption that \( f \) is a submartingale satisfying \( ||f||_\infty \leq 1 \) and \( g \) is \( \alpha \)-subordinate to \( f \). To be more precise, for any \( \lambda > 0 \) define the function \( V_{\alpha,\lambda} : [-1, 1] \times \mathbb{R} \to \mathbb{R} \) by the formula

\[
V_{\alpha,\lambda}(x_0, y_0) = \sup \mathbb{P}(g^\alpha \geq \lambda).
\]  

Here the supremum is taken over all pairs \((f, g)\) of integrable adapted processes, such that \((f_0, g_0) \equiv (x_0, y_0)\) almost surely, \( f \) is a submartingale satisfying \( ||f||_\infty \leq 1 \) and \( g \) is \( \alpha \)-subordinate to \( f \). The filtration must also vary, as well as the probability space, unless it is nonatomic. Our main result is an explicit formula for the functions \( V_{\alpha,\lambda} \), \( \lambda > 0 \). Usually we will omit the index \( \alpha \) and write \( V_\lambda \) instead of \( V_{\alpha,\lambda} \).

Let us discuss some related results which appeared in the literature. In [1] Burkholder studied the analogous question in the case of \( f, g \) being Hilbert space-valued martingales. The paper [1] contains also a related one-sided sharp exponential inequality for real martingales. This work was later extended by Hammack [4], who established a similar (two-sided) inequality under the assumption that \( f \) is a submartingale bounded by 1 and \( g \) is \( \nu \)-valued, \( \nu \geq 1 \), and strongly 1-subordinate to \( f \). Both papers present applications to stochastic integrals.

The paper is organized as follows. In the next section we introduce a family of special functions \( U_\lambda, \lambda > 0 \) and study their properties. This enables us to establish the inequality \( V_\lambda \leq U_\lambda \) in Section 3. Then we prove the reverse inequality in the last section.

Throughout the paper, \( \alpha \) is a fixed number from the interval \([0, 1]\). All the considered processes are assumed to be real valued.

## 2 The explicit formulas

Let \( S \) be the strip \([-1, 1] \times \mathbb{R} \). Consider the following subsets of \( S \): for \( 0 < \lambda \leq 2 \),

- \( A_\lambda = \{(x, y) \in S : |y| \geq x + \lambda - 1\} \)
- \( B_\lambda = \{(x, y) \in S : 1 - x \leq |y| < x + \lambda - 1\} \)
- \( C_\lambda = \{(x, y) \in S : |y| < 1 - x \text{ and } |y| < x + \lambda - 1\} \)

For \( \lambda \in (2, 4) \), define

- \( A_\lambda = \{(x, y) \in S : |y| \geq ax + \lambda - a\} \)
- \( B_\lambda = \{(x, y) \in S : ax + \lambda - a > |y| \geq x - 1 + \lambda\} \)
- \( C_\lambda = \{(x, y) \in S : x - 1 + \lambda > |y| \geq 1 - x\} \)
- \( D_\lambda = \{(x, y) \in S : 1 - x > |y| \geq -x + 3 + \lambda \text{ and } |y| < x - 1 + \lambda\} \)
- \( E_\lambda = \{(x, y) \in S : -x - 3 + \lambda > |y|\} \).
Finally, for $\lambda \geq 4$, let
\[
A_\lambda = \{(x, y) \in S : |y| \geq \alpha x + \lambda - \alpha\},
\]
\[
B_\lambda = \{(x, y) \in S : \alpha x + \lambda - \alpha > |y| \geq x - 1 + \lambda\},
\]
\[
C_\lambda = \{(x, y) \in S : x - 1 + \lambda > |y| \geq -x + 3 + \lambda\},
\]
\[
D_\lambda = \{(x, y) \in S : -x + 3 + \lambda > |y| \geq 1 - x\},
\]
\[
E_\lambda = \{(x, y) \in S : 1 - x > |y|\}.
\]

Let $H : S \times (-1, \infty) \to \mathbb{R}$ be a function given by
\[
H(x, y, z) = \frac{1}{\alpha + 2} \left[ \left(1 + \frac{x + 1 + |y|}{1 + z}(\alpha + 1)(x + 1 - |y|)\right) \right].
\]  
(2)

Now we will define the special functions $U_\lambda : S \to \mathbb{R}$. For $0 < \lambda \leq 2$, let
\[
U_\lambda(x, y) = \begin{cases} 
\frac{1}{2} & \text{if } (x, y) \in A_\lambda, \\
\frac{2 - 2x}{1 + \lambda - \lambda^2} & \text{if } (x, y) \in B_\lambda, \\
1 - \frac{(\alpha - 1) - |y| + \lambda \cdot 2\lambda^2 - 4}{2(1-x)(1-a)(\lambda-2)} & \text{if } (x, y) \in C_\lambda, \\
\frac{2(1-x)}{\lambda} \left[ 1 - \left(1-a\right)(\lambda-2) \right] - \frac{(1-x)^2 - |y|^2}{\lambda^2} & \text{if } (x, y) \in D_\lambda, \\
a_\lambda H(x, y, \lambda - 3) + b_\lambda & \text{if } (x, y) \in E_\lambda,
\end{cases}
\]  
(3)

where
\[
a_\lambda = -\frac{(1 + \alpha)(\lambda - 2)^2}{\lambda^2}, \quad b_\lambda = 1 - \frac{4(\lambda - 2)(1 - \alpha)}{\lambda^2}.
\]  
(4)

For $\lambda \geq 4$, set
\[
U_\lambda(x, y) = \begin{cases} 
1 & \text{if } (x, y) \in A_\lambda, \\
\frac{2 - 2x}{1 + \lambda - \lambda^2} - \frac{1}{4} (1-x)(1-a) & \text{if } (x, y) \in B_\lambda, \\
\frac{4}{1+x+y} & \text{if } (x, y) \in C_\lambda, \\
\frac{4}{1+x+y} & \text{if } (x, y) \in D_\lambda, \\
a_\lambda H(x, y, 1) + b_\lambda & \text{if } (x, y) \in E_\lambda,
\end{cases}
\]  
(6)

where
\[
a_\lambda = -b_\lambda = -\frac{(1 + \alpha)}{2} \exp\left(\frac{4 - \lambda}{2\alpha + 2}\right).
\]  
(7)

For $\alpha = 1$, the formulas (3), (4), (6) give the special functions constructed by Hammack [4]. The key properties of $U_\lambda$ are described in the two lemmas below.

**Lemma 1.** For $\lambda > 2$, let $\phi_\lambda, \psi_\lambda$ denote the partial derivatives of $U_\lambda$ with respect to $x, y$ on the interiors of $A_\lambda, B_\lambda, C_\lambda, D_\lambda, E_\lambda$, extended continuously to the whole of these sets. The following statements hold.
(i) The functions $U_\lambda$, $\lambda > 2$, are continuous on $S \setminus \{(1, \pm \lambda)\}$.

(ii) Let

$$S_\lambda = \{(x, y) \in [-1, 1] \times \mathbb{R} : |y| \neq \alpha x + \lambda - \alpha \text{ and } |y| \neq x + \lambda - 1\}.$$ 

Then

$$\phi_\lambda, \psi_\lambda, \lambda > 2, \text{ are continuous on } S_\lambda. \quad (8)$$

(iii) For any $(x, y) \in S$, the function $\lambda \mapsto U_\lambda(x, y)$, $\lambda > 0$, is left-continuous.

(iv) For any $\lambda > 2$ we have the inequality

$$\phi_\lambda \leq -\alpha|\psi_\lambda|. \quad (9)$$

(v) For $\lambda > 2$ and any $(x, y) \in S$ we have $\chi_{\{|y| \geq \lambda\}} \leq U_\lambda(x, y) \leq 1$.

Proof. We start with computing the derivatives. Let $y' = y/|y|$ stand for the sign of $y$, with $y' = 0$ if $y = 0$. For $\lambda \in (2, 4)$ we have

$$\phi_\lambda(x, y) = \begin{cases} 0 & \text{if } (x, y) \in A_\lambda, \\
-\frac{(2\lambda-4)\alpha}{2\lambda - 2y} & \text{if } (x, y) \in B_\lambda, \\
-\frac{2\lambda - 2y}{(1+\lambda - x - |y|)^2} + \frac{2\lambda - 4(1-\alpha)}{\lambda^2} & \text{if } (x, y) \in C_\lambda, \\
-\frac{2\lambda}{\lambda} \left[1 - \frac{(1-\alpha)(\lambda - 2)}{\lambda^2}\right] + \frac{2(1-x)}{\lambda^2} & \text{if } (x, y) \in D_\lambda, \\
-c_\lambda(x + |y| + 1)^{-\alpha/(\alpha+1)}(x + 1 + \frac{\alpha}{\alpha+1}|y|) & \text{if } (x, y) \in E_\lambda, \\
\end{cases}$$

$$\psi_\lambda(x, y) = \begin{cases} 0 & \text{if } (x, y) \in A_\lambda, \\
\frac{2\lambda - 4}{2\lambda - 2y} y' & \text{if } (x, y) \in B_\lambda, \\
\frac{2\lambda y'}{(1+\lambda - x - |y|)^2} & \text{if } (x, y) \in C_\lambda, \\
\frac{2\lambda y'}{\lambda} & \text{if } (x, y) \in D_\lambda, \\
c_\lambda(x + |y| + 1)^{-\alpha/(\alpha+1)} \frac{y'}{1+\alpha} & \text{if } (x, y) \in E_\lambda, \\
\end{cases}$$

where

$$c_\lambda = 2(1+\alpha)(\lambda - 2)^{\alpha/(\alpha+1)}\lambda^{-2}.$$ 

Finally, for $\lambda \geq 4$, set

$$\phi_\lambda(x, y) = \begin{cases} 0 & \text{if } (x, y) \in A_\lambda, \\
-\frac{\alpha}{4} & \text{if } (x, y) \in B_\lambda, \\
-\frac{1}{4} & \text{if } (x, y) \in C_\lambda, \\
x + 1 + 2\alpha & \text{if } (x, y) \in D_\lambda, \\
c_\lambda(x + |y| + 1)^{-\alpha/(\alpha+1)}(x + 1 + \frac{\alpha}{\alpha+1}|y|) & \text{if } (x, y) \in E_\lambda, \\
\end{cases}$$

$$\psi_\lambda(x, y) = \begin{cases} 0 & \text{if } (x, y) \in A_\lambda, \\
\frac{1}{4} y' & \text{if } (x, y) \in B_\lambda, \\
\frac{2\lambda - 4}{(1+\lambda - x - |y|)^2} y' & \text{if } (x, y) \in C_\lambda, \\
\frac{1}{8} \exp \left(\frac{x + |y| + 3 - \lambda}{2(\alpha + 1)}\right) y' & \text{if } (x, y) \in D_\lambda, \\
c_\lambda(x + |y| + 1)^{-\alpha/(\alpha+1)} \frac{y'}{1+\alpha} & \text{if } (x, y) \in E_\lambda, \\
\end{cases}$$
where
\[ c_\lambda = (1 + \alpha)2^{-(2\alpha+3)/(\alpha+1)} \exp \left( \frac{4 - \lambda}{2(\alpha + 1)} \right). \]

Now the properties (i), (ii), (iii) follow by straightforward computation. To prove (iv), note first that for any \( \lambda > 2 \) the condition (9) is clearly satisfied on the sets \( A_\lambda \) and \( B_\lambda \). Suppose \((x, y) \in C_\lambda \). Then \( \lambda - |y| \in [0, 4] \), \( 1 - x \leq \min\{|\lambda - |y||, 4 - \lambda + |y||\} \) and (9) takes form
\[ -2(\lambda - |y|) + \frac{2\lambda - 4}{\lambda^2}(1 - \alpha)(1 - x + \lambda - |y|)^2 + 2\alpha(1 - x) \leq 0, \]

or
\[ -2(\lambda - |y|) + \frac{1 - \alpha}{4} (1 - x + \lambda - |y|)^2 + 2\alpha(1 - x) \leq 0, \quad (10) \]
depending on whether \( \lambda < 4 \) or \( \lambda \geq 4 \). As \((2\lambda - 4)/\lambda^2 \leq \frac{1}{4} \), it suffices to show (10). If \( \lambda - |y| \leq 2 \), then, as \( 1 - x \leq \lambda - |y| \), the left-hand side does not exceed
\[ -2(\lambda - |y|) + (1 - \alpha)(\lambda - |y|)^2 + 2\alpha(\lambda - |y|) = (\lambda - |y|)(-2 + (1 - \alpha)(\lambda - |y|) + 2\alpha) \]
\[ \leq (\lambda - |y|)(-2 + 2(1 - \alpha) + 2\alpha) = 0. \]
Similarly, if \( \lambda - |y| \in (2, 4] \), then we use the bound \( 1 - x \leq 4 - \lambda + |y| \) and conclude that the left-hand side of (10) is not greater than
\[ -2(\lambda - |y|) + 4(1 - \alpha) + 2\alpha(4 - \lambda + |y|) = -2(\lambda - |y| - 2)(1 + \alpha) \leq 0 \]
and we are done with the case \((x, y) \in C_\lambda \).

Assume that \((x, y) \in D_\lambda \). For \( \lambda \in (2, 4) \), the inequality (9) is equivalent to
\[ -\frac{2}{\lambda} \left[ 1 - \frac{(1 - \alpha)(\lambda - 2)}{\lambda} \right] + \frac{2 - 2x}{\lambda^2} \leq -\frac{2\alpha |y|}{\lambda^2}, \]
or, after some simplifications, \( \alpha |y| + 1 - x \leq 2 + a\lambda - 2\alpha. \) It is easy to check that \( \alpha |y| + 1 - x \) attains its maximum for \( x = -1 \) and \( |y| = \lambda - 2 \) and then we have the equality. If \((x, y) \in D_\lambda \) and \( \lambda \geq 4 \), then (9) takes form \(-2(\alpha + 1 + x) \leq -\alpha(1 - x) \), or \((x + 1)(\alpha + 1) \geq 0 \). Finally, on the set \( E_\lambda \), the inequality (9) is obvious.

(v) By (9), we have \( \phi_\lambda \leq 0 \), so \( U_\lambda(x, y) \geq U_\lambda(1, y) = x_{\lambda(\lambda \geq y)} \). Furthermore, as \( U_\lambda(x, y) = 1 \) for \( |y| \geq \lambda \) and \( \psi_\lambda(x, y) \) is clearly non-negative on \( S_\lambda \), the second estimate follows.

\[ \square \]

**Lemma 2.** Let \( x, h, y, k \) be fixed real numbers, satisfying \( x, x + h \in [-1, 1] \) and \( |k| \leq |h| \). Then for any \( \lambda > 2 \) and \( \alpha \in (0, 1) \),
\[ U_\lambda(x + h, y + k) \leq U_\lambda(x, y) + \phi_\lambda(x, y)h + \psi_\lambda(x, y)k. \quad (11) \]

We will need the following fact, proved by Burkholder; see page 17 of [1].

**Lemma 3.** Let \( x, h, y, k, z \) be real numbers satisfying \( |k| \leq |h| \) and \( z > -1 \). Then the function
\[ F(t) = H(x + th, y + tk, z), \]
defined on \( \{ t : |x + th| \leq 1 \} \), is convex.
Proof of the Lemma. Consider the function
\[ G(t) = G_{x,y,h,k}(t) = U_\lambda(x + th, y + tk), \]
defined on the set \( \{t : |x + th| \leq 1\} \). It is easy to check that \( G \) is continuous. As explained in (II), the inequality (11) follows once the concavity of \( G \) is established. This will be done by proving the inequality \( G'' \leq 0 \) at the points, where \( G \) is twice differentiable and checking the inequality \( G'_+(t) \leq G'_-(t) \) for those \( t \), for which \( G \) is not differentiable (even once). Note that we may assume \( t = 0 \), by a translation argument \( G''_{x,y,h,k}(t) = G''_{x+th,y+tk,h,k}(0) \), with analogous equalities for one-sided derivatives. Clearly, we may assume that \( h \geq 0 \), changing the signs of both \( h, k \), if necessary.

Due to the symmetry of \( U_\lambda \), we are allowed to consider \( y \geq 0 \) only.

We start from the observation that \( G''(0) = 0 \) on the interior of \( A_\lambda \) and \( G'_+(0) \leq G'_-(0) \) for \( (x, y) \in A_\lambda \cap \overline{B}_\lambda \). The latter inequality holds since \( U_\lambda \equiv 1 \) on \( A_\lambda \) and \( U_\lambda \leq 1 \) on \( B_\lambda \). For the remaining inequalities, we consider the cases \( \lambda \in (2,4) \), \( \lambda \geq 4 \) separately.

The case \( \lambda \in (2,4) \). The inequality \( G''(0) \leq 0 \) is clear for \( (x, y) \) lying in the interior of \( B_\lambda \). On \( C_\lambda \), we have
\[ G''(0) = -\frac{4(h + k)(h\lambda - y - k(1 - x))}{(1 - x - y + \lambda^2)^3} \leq 0, \]
which follows from \(|k| \leq h\) and the fact that \( \lambda - y \geq 1 - x \). For \((x, y)\) in the interior of \( D_\lambda \),
\[ G''(0) = \frac{-h^2 + k^2}{\lambda^2} \leq 0, \]
as \(|k| \leq h\). Finally, on \( E_\lambda \), the concavity follows by Lemma [3].

It remains to check the inequalities for one-sided derivatives. By Lemma (II)(ii), the points \((x, y)\), for which \( G \) is not differentiable at 0, do not belong to \( S_\lambda \). Since we excluded the set \( A_\lambda \cap \overline{B}_\lambda \), they lie on the line \( y = x - 1 + \lambda \). For such points \((x, y)\), the left derivative equals
\[ G'_-(0) = -\frac{2\lambda - 4}{\lambda^2}(ah - k), \]
while the right one is given by
\[ G'_+(0) = \frac{-h + k}{2(\lambda - y)} + \frac{(2\lambda - 4)(1 - a)h}{\lambda^2}, \]
or
\[ G'_+(0) = -\frac{2h}{\lambda} \left[ 1 - \frac{(1 - a)(\lambda - 2)}{\lambda} \right] + \frac{2(1 - x)h + 2yk}{\lambda^2}, \]
depending on whether \( y \geq 1 - x \) or \( y < 1 - x \). In the first case, the inequality \( G'_+(0) \leq G'_-(0) \) reduces to
\[ (h - k) \left( \frac{1}{2(\lambda - y)} - \frac{2(\lambda - 2)}{\lambda^2} \right) \geq 0, \]
while in the remaining one,
\[ \frac{2}{\lambda^2}(h - k)(y - (\lambda - 2)) \geq 0. \]
Both inequalities follow from the estimate \( \lambda - y \leq 2 \) and the condition \(|k| \leq h\).
The case $\lambda \geq 4$. On the set $B_3$ the concavity is clear. For $C_3$, we have that the formula (12) holds. If $(x, y)$ lies in the interior of $D_3$, then

$$G''(0) = \frac{1}{8} \exp \left( \frac{3 + x + y - \lambda}{2(\alpha + 1)} \right) \left[ \frac{1 - x}{2(\alpha + 1)} \cdot (-h^2 + k^2) - \left( 2 - \frac{1}{\alpha + 1} \right) (h^2 + hk) \right] \leq 0,$$

since $|k| \leq h$ and $(1-x)/(\alpha+1) \leq 2$. The concavity on $E_3$ is a consequence of Lemma 3. It remains to check the inequality for one-sided derivatives. By Lemma 1(ii), we may assume $y = x + \lambda - 1$, and the inequality $G_+'(0) \leq G'_-(0)$ reads

$$\frac{1}{2} (h - k) \left( \frac{1}{\lambda - y} - \frac{1}{2} \right) \geq 0,$$

an obvious one, as $\lambda - y \leq 2$. □

3 The main theorem

Now we may state and prove the main result of the paper.

**Theorem 1.** Suppose $f$ is a submartingale satisfying $||f||_{\infty} \leq 1$ and $g$ is an adapted process which is $\alpha$-subordinate to $f$. Then for all $\lambda > 0$ we have

$$P(g^* \geq \lambda) \leq EU_\lambda(f_0, g_0). \quad (13)$$

**Proof.** If $\lambda \leq 2$, then this follows immediately from the result of Hammack [4]; indeed, note that $U_\lambda$ coincides with Hammack’s special function and, furthermore, since $g$ is $\alpha$-subordinate to $f$, it is also $1$-subordinate to $f$.

Fix $\lambda > 2$. We may assume $\alpha < 1$. It suffices to show that for any nonnegative integer $n$,

$$P(|g_n| \geq \lambda) \leq EU_\lambda(f_0, g_0). \quad (14)$$

To see that this implies (13), fix $\varepsilon > 0$ and consider a stopping time $\tau = \inf \{ k : |g_k| \geq \lambda - \varepsilon \}$. The process $f^\tau = (f_{t \wedge \tau})$, by Doob’s optional sampling theorem, is a submartingale. Furthermore, we obviously have that $||f^\tau||_{\infty} \leq 1$ and the process $g^\tau = (g_{t \wedge \tau})$ is $\alpha$-subordinate to $f^\tau$. Therefore, by (14),

$$P(|g_n^\tau| \geq \lambda - \varepsilon) \leq EU_{\lambda - \varepsilon}(f_{0}^\tau, g_{0}^\tau) = EU_{\lambda - \varepsilon}(f_{0}, g_{0}).$$

Now if we let $n \to \infty$, we obtain $P(g^* \geq \lambda) \leq EU_{\lambda}(f_{0}, g_{0})$ and by left-continuity of $U_\lambda$ as a function of $\lambda$, (13) follows.

Thus it remains to establish (14). By Lemma 1(v), $P(|g_n| \geq \lambda) \leq EU_{\lambda}(f_n, g_n)$ and it suffices to show that for all $1 \leq j \leq n$ we have

$$EU_{\lambda}(f_j, g_j) \leq EU_{\lambda}(f_{j-1}, g_{j-1}). \quad (15)$$

To do this, note that, since $|d g_j| \leq |d f_j|$ almost surely, the inequality (11) yields

$$U_{\lambda}(f_j, g_j) \leq U_{\lambda}(f_{j-1}, g_{j-1}) + \phi_{\lambda}(f_{j-1}, g_{j-1}) d f_j + \psi_{\lambda}(f_{j-1}, g_{j-1}) d g_j \quad (16)$$

with probability 1. Assume for now that $\phi_{\lambda}(f_{j-1}, g_{j-1}) d f_j$, $\psi_{\lambda}(f_{j-1}, g_{j-1}) d g_j$ are integrable. By $\alpha$-subordination, the condition (9) and the submartingale property $\mathbb{E}(d_j | \mathcal{F}_{j-1}) \geq 0$, we have

$$\mathbb{E}[\phi_{\lambda}(f_{j-1}, g_{j-1}) d f_j + \psi_{\lambda}(f_{j-1}, g_{j-1}) d g_j | \mathcal{F}_{j-1}]$$
\[
\begin{align*}
&\leq \phi_\lambda(f_{j-1}, g_{j-1})\mathbb{E}(df_j | \mathcal{F}_{j-1}) + |\psi_\lambda(f_{j-1}, g_{j-1})| \cdot |\mathbb{E}(dg_j | \mathcal{F}_{j-1})| \\
&\leq [\phi_\lambda(f_{j-1}, g_{j-1}) + \alpha|\psi_\lambda(f_{j-1}, g_{j-1})|] \mathbb{E}(df_j | \mathcal{F}_{j-1}) \leq 0.
\end{align*}
\]

Therefore, it suffices to take the expectation of both sides of (16) to obtain (15). Thus we will be done if we show the integrability of \(\phi_\lambda(f_{j-1}, g_{j-1})df_j\) and \(\psi_\lambda(f_{j-1}, g_{j-1})dg_j\). In both the cases \(\lambda \in (2, 4), \lambda \geq 4\), all we need is that the variables

\[
\frac{2\lambda - 2|g_{j-1}|}{(1 - f_{j-1} - |g_{j-1}| + \lambda)^2} df_j \quad \text{and} \quad \frac{2 - 2f_{j-1}}{(1 - f_{j-1} - |g_{j-1}| + \lambda)^2} dg_j
\]

(17)

are integrable on the set \(K = \{ |g_{j-1}| < f_{j-1} + \lambda - 1, \ |g_{j-1}| \geq \lambda - 1 \}\), since outside it the derivatives \(\phi_\lambda, \psi_\lambda\) are bounded by a constant depending only on \(\alpha, \lambda\) and \(|df_j|, |dg_j|\) do not exceed 2. The integrability is proved exactly in the same manner as in [4]. We omit the details. \(\square\)

We will now establish the following sharp exponential inequality.

**Theorem 2.** Suppose \(f\) is a submartingale satisfying \(\|f\|_\infty \leq 1\) and \(g\) is an adapted process which is \(\alpha\)-subordinate to \(f\). In addition, assume that \(\|g\|_\infty \leq \|f\|_\infty\) with probability 1. Then for \(\lambda \geq 4\) we have

\[
P(g^* \geq \lambda) \leq \gamma e^{-\lambda/(2\alpha + 2)},
\]

(18)

where

\[
\gamma = \frac{1 + \alpha}{2\alpha + 4} \left( \alpha + 1 + 2^{-\alpha - 1} \right) \exp\left( \frac{2}{\alpha + 1} \right).
\]

The inequality is sharp.

This should be compared to Burkholder’s estimate (Theorem 8.1 in [11])

\[
P(g^* \geq \lambda) \leq \frac{\epsilon^2}{4} e^{-\lambda}, \quad \lambda \geq 2,
\]

in the case when \(f, g\) are Hilbert space-valued martingales and \(g\) is subordinate to \(f\). For \(\alpha = 1\), we obtain the inequality of Hammack [4],

\[
P(g^* \geq \lambda) \leq \frac{(8 + \sqrt{2})\epsilon}{12} e^{-\lambda/4}, \quad \lambda \geq 4.
\]

**Proof of the inequality (18).** We will prove that the maximum of \(U_\lambda\) on the set \(K = \{(x, y) \in S : |y| \leq |x|\}\) given by the right hand side of (18). This, together with the inequality (13) and the assumption \(P((f_0, g_0) \in K) = 1\), will imply the desired estimate. Clearly, by symmetry, we may restrict ourselves to the set \(K^+ = K \cap \{y \geq 0\}\). If \((x, y) \in K^+\) and \(x \geq 0\), then it is easy to check that

\[
U_\lambda(x, y) \leq U_\lambda((x + y)/2, (x + y)/2).
\]

Furthermore, a straightforward computation shows that the function \(F : [0, 1] \rightarrow \mathbb{R}\) given by \(F(s) = U_\lambda(s, s)\) is nonincreasing. Thus we have \(U_\lambda(x, y) \leq U_\lambda(0, 0)\). On the other hand, if \((x, y) \in K^+\) and \(x \leq 0\), then it is easy to prove that \(U_\lambda(x, y) \leq U_\lambda(-1, x + y + 1)\) and the function \(G : [0, 1] \rightarrow \mathbb{R}\) given by \(G(s) = U_\lambda(-1, s)\) is nondecreasing. Combining all these facts we have that for any \((x, y) \in K^+\),

\[
U_\lambda(x, y) \leq U_\lambda(-1, 1) = \gamma e^{-\lambda/(2\alpha + 2)}.
\]

(19)

Thus (18) holds. The sharpness will be shown in the next section. \(\square\)
4 Sharpness

Recall the function $V_{\lambda} = V_{a,\lambda}$ defined by (1) in the introduction. The main result in this section is Theorem 3 below, which, combined with Theorem 1, implies that the functions $U_{\lambda}$ and $V_{\lambda}$ coincide. If we apply this at the point $(-1, 1)$ and use the equality appearing in (19), we obtain that the inequality (18) is sharp.

**Theorem 3.** For any $\lambda > 0$ we have

$$U_{\lambda} \leq V_{\lambda}. \quad (20)$$

The main tool in the proof is the following "splicing" argument. Assume that the underlying probability space is the interval $[0, 1]$ with the Lebesgue measure.

**Lemma 4.** Fix $(x_0, y_0) \in [-1, 1] \times \mathbb{R}$. Suppose there exists a filtration and a pair $(f, g)$ of simple adapted processes, starting from $(x_0, y_0)$, such that $f$ is a submartingale satisfying $||f||_{\infty} \leq 1$ and $g$ is $\alpha$-subordinate to $f$. Then $V_{\lambda}(x_0, y_0) \geq \mathbb{E}V_{\lambda}(f_{\infty}, g_{\infty})$ for $\lambda > 0$.

**Proof.** Let $N$ be such that $(f_N, g_N) = (f_{\infty}, g_{\infty})$ and fix $\epsilon > 0$. With no loss of generality, we may assume that $\sigma$-field generated by $f$, $g$ is generated by the family of intervals $\{[a_i, a_{i+1}): i = 1, 2, \ldots, M - 1\}$, $0 = a_1 < a_2 < \ldots < a_M = 1$. For any $i \in \{1, 2, \ldots, M - 1\}$, denote $x^i_0 = f_N(a_i)$, $y^i_0 = g_N(a_i)$. There exists a filtration and a pair $(f^i, g^i)$ of adapted processes, with $f^i$ being a submartingale bounded in absolute value by 1 and $g$ being $\alpha$-subordinate to $f$, which satisfy $f^i_0 = x^i_0\chi_{[0, 1]}$, $g^i_0 = y^i_0\chi_{[0, 1]}$ and $\mathbb{P}((g^i)^* \geq \lambda) > \mathbb{E}V_{\lambda}(f^i_0, g^i_0) - \epsilon$. Define the processes $F, G$ by $F_k = f_k$, $G_k = g_k$ if $k \leq N$ and

$$F_k(\omega) = \sum_{i=1}^{M-1} f^i_{k-N}(\omega - a_i)/(a_{i+1} - a_i)\chi_{[a_i, a_{i+1})}(\omega),$$

$$G_k(\omega) = \sum_{i=1}^{M-1} g^i_{k-N}(\omega - a_i)/(a_{i+1} - a_i)\chi_{[a_i, a_{i+1})}(\omega)$$

for $k > N$. It is easy to check that there exists a filtration, relative to which the process $F$ is a submartingale satisfying $||F||_{\infty} \leq 1$ and $G$ is an adapted process which is $\alpha$-subordinate to $F$. Furthermore, we have

$$\mathbb{P}(G^* \geq \lambda) \geq \sum_{i=1}^{M-1} (a_{i+1} - a_i)\mathbb{P}((g^i)^* \geq \lambda)$$

$$> \sum_{i=1}^{M-1} (a_{i+1} - a_i)(\mathbb{E}V_{\lambda}(f^i_0, g^i_0) - \epsilon) = \mathbb{E}V_{\lambda}(f_{\infty}, g_{\infty}) - \epsilon.$$

Since $\epsilon$ was arbitrary, the result follows. \qed

**Proof of Theorem 3** First note the following obvious properties of the functions $V_{\lambda}$, $\lambda > 0$: we have $V_{\lambda} \in [0, 1]$ and $V_{\lambda}(x, y) = V_{\lambda}(x, -y)$. The second equality is an immediate consequence of the fact that if $g$ is $\alpha$-subordinate to $f$, then so is $-g$.

In the proof of Theorem 3 we repeat several times the following procedure. Having fixed a point $(x_0, y_0)$ from the strip $S$, we construct certain simple finite processes $f, g$ starting from $(x_0, y_0)$, take their natural filtration $(\mathcal{F}_n)$, apply Lemma 4 and thus obtain a bound for $V_{\lambda}(x_0, y_0)$. All the constructed processes appearing in the proof below are easily checked to satisfy the conditions.
of this lemma: the condition $\|f\|_\infty \leq 1$ is straightforward, while the $\alpha$-subordination and the fact that $f$ is a submartingale are implied by the following. For any $n \geq 1$, either $df_n$ satisfies $\mathbb{E}(df_n \mid \mathcal{F}_{n-1}) = 0$ and $dg_n = \pm df_n$, or $df_n \geq 0$ and $dg_n = \pm adf_n$.

We will consider the cases $\lambda \leq 2$, $2 < \lambda < 4$, $\lambda \geq 4$ separately. Note that by symmetry, it suffices to establish (20) on $S \cap \{y \geq 0\}$.

The case $\lambda \leq 2$. Assume $(x_0, y_0) \in A_\lambda$. If $y_0 \geq \lambda$, then $g^* \geq \lambda$ almost surely, so $V_\lambda(x_0, y_0) \geq 1 = U_\lambda(x_0, y_0)$. If $\lambda > y_0 \geq \alpha x_0 - \alpha + \lambda$, then let $(f_0, g_0) \equiv (x_0, y_0)$,

$$df_1 = (1 - x_0)\chi_{[0,1]} \quad \text{and} \quad dg_1 = adf_1,$$

Then we have $g_1 = y_0 + \alpha - \alpha x_0 \geq \lambda$, which implies $g^* \geq \lambda$ almost surely and (20) follows. Now suppose $(x_0, y_0) \in A_\lambda$ and $y_0 < \alpha x_0 - \alpha + \lambda$. Let $(f, g) \equiv (x_0, y_0)$,

$$df_1 = \frac{y_0 - x_0 + 1 - \lambda}{1 - \alpha} \chi_{[0,1]}, \quad dg_1 = adf_1$$

and

$$df_2 = dg_2 = \beta \chi_{[0,1 - \beta/2]} + (\beta - 2)\chi_{[1 - \beta/2,1]},$$

where

$$\beta = \frac{\alpha x_0 - y_0 - \alpha + \lambda}{1 - \alpha} \in [0, 2].$$

Then $(f_2, g_2)$ takes values $(-1, \lambda - 2)$, $(1, \lambda)$ with probabilities $\beta/2$, $1 - \beta/2$, respectively, so, by Lemma 4

$$V_\lambda(x_0, y_0) \geq \frac{\beta}{2} V_\lambda(-1, \lambda - 2) + (1 - \frac{\beta}{2}) V_\lambda(1, \lambda) = \frac{\beta}{2} V_\lambda(-1, 2 - \lambda) + 1 - \frac{\beta}{2}.$$  

(25)

Note that $(-1, 2 - \lambda) \in A_\lambda$. If $2 - \lambda \geq \alpha \cdot (-1) - \alpha + \lambda$, then, as already proved, $V_\lambda(-1, 2 - \lambda) = 1$ and $V_\lambda(x_0, y_0) \geq 1 = U_\lambda(x_0, y_0)$. If the converse inequality holds, i.e., $2 - \lambda < -2\alpha + \lambda$, then we may apply (25) to $x_0 = -1$, $y_0 = 2 - \lambda$ to get

$$V_\lambda(-1, 2 - \lambda) \geq \frac{\beta}{2} V_\lambda(-1, 2 - \lambda) + 1 - \frac{\beta}{2},$$

or $V_\lambda(-1, 2 - \lambda) \geq 1$. Thus we established $V_\lambda(x_0, y_0) = 1$ for any $(x_0, y_0) \in A_\lambda$.

Suppose then, that $(x_0, y_0) \in B_\lambda$. Let

$$\beta = \frac{2(1 - x_0)}{1 - x_0 - y_0 + \lambda} \in [0, 1]$$

and consider a pair $(f, g)$ starting from $(x_0, y_0)$ and satisfying

$$df_1 = -dg_1 = \frac{x_0 - y_0 - 1 + \lambda}{2} \chi_{[0,\beta]} + (1 - x_0)\chi_{[\beta,1]}.$$  

(27)

On $[0, \beta)$, the pair $(f_1, g_1)$ lies in $A_\lambda$; Lemma 4 implies $V_\lambda(x_0, y_0) \geq \beta = U_\lambda(x_0, y_0)$.

Finally, for $(x_0, y_0) \in C_\lambda$, let $(f, g)$ start from $(x_0, y_0)$ and

$$df_1 = -dg_1 = \frac{x_0 - \lambda + 1 + y_0}{2} \chi_{[0,\gamma]} + \frac{y_0 - x_0 + 1}{2} \chi_{[\gamma,1]},$$

where

$$\gamma = \frac{2(1 - x_0)}{1 - x_0 - y_0 + \lambda} \in [0, 1].$$
where
\[ \gamma = \frac{y_0 - x_0 + 1}{\lambda} \in [0, 1] . \]

On \([0, \gamma)\), the pair \((f_1, g_1)\) lies in \(A_\lambda\), while on \([\gamma, 1]\) we have \((f_1, g_1) = ((x_0 + y_0 + 1)/2, (x_0 + y_0 - 1)/2) \in B_\lambda\). Hence
\[ V_\lambda(x_0, y_0) \geq \gamma \cdot 1 + (1 - \gamma) \cdot \frac{1 - x_0 - y_0}{\lambda} = U_\lambda(x_0, y_0) . \]

The case \(2 < \lambda < 4\). For \((x_0, y_0) \in A_\lambda\) we prove (20) using the same processes as in the previous case, i.e., the constant ones if \(y_0 \geq \lambda\) and the ones given by (21) otherwise. The next step is to establish the inequality
\[ V_\lambda(-1, \lambda - 2) \geq U_\lambda(-1, \lambda - 2) = \frac{1 + \alpha}{2} + \frac{1 - \alpha}{2} \cdot \left( \frac{4 - \lambda \gamma^2}{\lambda} \right) . \tag{28} \]

To do this, fix \(\delta \in (0, 1]\) and set
\[ \beta = \frac{\delta(1 - \alpha)}{\lambda}, \quad \kappa = \frac{4 - \lambda - \delta(1 + \alpha)}{\lambda} \cdot \beta, \quad \gamma = \beta + (1 - \beta) \cdot \frac{\delta(1 + \alpha)}{4}, \quad \nu = \kappa \cdot \frac{\lambda}{4} . \]

We have \(0 \leq \nu \leq \kappa \leq \beta \leq \gamma \leq 1\). Consider processes \(f, g\) given by \((f_0, g_0) \equiv (-1, \lambda - 2), (df_1, dg_1) \equiv (\delta, \alpha \delta)\),
\[ df_2 = -dg_2 = \frac{\lambda - \delta(1 - \alpha)}{2} \chi_{(0, \beta)} - \frac{\delta(1 - \alpha)}{2} \chi_{(\beta, 1)}, \]
\[ df_3 = dg_3 = -\left( \lambda - 2 + \frac{\delta(1 + \alpha)}{2} \chi_{(0, \kappa)} + \left( 2 - \frac{\lambda + \delta(1 + \alpha)}{2} \right) \chi_{(\kappa, \beta)} \right), \]
\[ + \left( 2 - \frac{\delta(1 + \alpha)}{2} \right) \chi_{(\beta, \gamma)} - \frac{\delta(1 + \alpha)}{2} \chi_{(\gamma, 1)}, \]
\[ df_4 = -dg_4 = \left( -2 + \frac{\lambda}{2} \right) \chi_{(0, \nu)} + \frac{\lambda}{2} \chi_{(\nu, \kappa)}. \]

As \((f_4, g_4)\) takes values \((1, \lambda), (1, 0)\) and \((-1, \lambda - 2)\) with probabilities \((\gamma - \beta) + (\kappa - \nu), \beta - \kappa\) and \(1 - \gamma + \nu\), respectively, we have
\[ V_\lambda(-1, \lambda - 2) \geq \gamma - \beta + \kappa - \nu + (1 - \gamma + \nu)V_\lambda(-1, \lambda - 2), \]
or
\[ V_\lambda(-1, \lambda - 2) \geq \frac{\gamma - \beta + \kappa - \nu}{\gamma - \nu} = \frac{1 + \alpha}{2} + \frac{1 - \alpha}{2} \cdot \left( \frac{4 - \lambda \gamma^2}{\lambda} \right) - \frac{\delta(1 - \alpha^2)}{\lambda^2} . \]

As \(\delta\) is arbitrary, we obtain (28). Now suppose \((x_0, y_0) \in B_\lambda\) and recall the pair \((f, g)\) starting from \((x_0, y_0)\) given by (22) and (23) (with \(\beta\) defined in (24)). As previously, it leads to (25), which takes form
\[ V_\lambda(x_0, y_0) \geq \frac{\beta}{2} \left[ \frac{1 + \alpha}{2} + \frac{1 - \alpha}{2} \cdot \left( \frac{4 - \lambda \gamma^2}{\lambda} \right) \right] + 1 - \frac{\beta}{2} \]
\[ = \frac{\beta(1 - \alpha)}{4} \left[ \left( \frac{4 - \lambda \gamma^2}{\lambda} \right) - 1 \right] + 1 = \frac{(ax_0 - \alpha - y_0 + \lambda(4 - 2\lambda))}{\lambda^2} + 1 = U_\lambda(x_0, y_0) . \]
For \((x_0, y_0) \in C_\lambda\), consider a pair \((f, g)\), starting from \((x_0, y_0)\) defined by \((27)\) (with \(\beta\) given by \((26)\)). On \([0, \beta]\) we have \((f_1, g_1) = ((x_0 + y_0 + 1 - \lambda)/2, (x_0 + y_0 - 1 + \lambda)/2) \in B_\lambda\), so Lemma\(^4\) yields

\[
V_\lambda(x_0, y_0) \geq \beta V_\lambda \left( \frac{x_0 + y_0 + 1 - \lambda}{2}, \frac{x_0 + y_0 - 1 + \lambda}{2} \right)
\]

\[
= \frac{2(1 - x_0)}{1 + \lambda - x_0 - y_0} \cdot \left\{ \frac{1}{2} \left[ a \left( \frac{x_0 + y_0 - 1 - \lambda}{2} \right) - \frac{x_0 + y_0 - 1 - \lambda}{2} \right] \cdot \frac{2\lambda - 4}{\lambda^2} \right\}
\]

\[
= U_\lambda(x_0, y_0).
\]

For \((x_0, y_0) \in D_\lambda\), set \(\beta = (y_0 - x_0 + 1)/\lambda \in [0, 1]\) and let a pair \((f, g)\) be given by \((f_0, g_0) \equiv (x_0, y_0)\) and

\[
df_1 = dg_1 = \frac{-x_0 + y_0 + 1 - \lambda}{2} \chi_{\{0, \beta\}} + \frac{-x_0 + y_0 + 1}{2} \chi_{\{\beta, 1\}}.
\]

As \((f_1, g_1)\) takes values

\[
\left( \frac{x_0 + y_0 + 1 - \lambda}{2}, \frac{x_0 + y_0 - 1 + \lambda}{2} \right) \in B_\lambda \text{ and } \left( \frac{x_0 + y_0 + 1}{2}, \frac{x_0 + y_0 - 1}{2} \right) \in C_\lambda
\]

with probabilities \(\beta\) and \(1 - \beta\), respectively, we obtain \(V_\lambda(x_0, y_0)\) is not smaller than

\[
\beta V_\lambda \left( \frac{x_0 + y_0 + 1 - \lambda}{2}, \frac{x_0 + y_0 - 1 + \lambda}{2} \right) + (1 - \beta)V_\lambda \left( \frac{x_0 + y_0 + 1}{2}, \frac{x_0 + y_0 - 1}{2} \right)
\]

\[
= \frac{y_0 - x_0 + 1}{\lambda} \cdot \left\{ 1 - \left[ a \left( \frac{x_0 + y_0 - 1 - \lambda}{2} \right) - \frac{x_0 + y_0 - 1 - \lambda}{2} \right] \cdot \frac{2\lambda - 4}{\lambda^2} \right\}
\]

\[
+ \frac{\lambda - y_0 + x_0 - 1}{\lambda} \left[ \frac{1 - x_0 - y_0}{\lambda} - \frac{(1 - x_0 - y_0)(1 - a)(\lambda - 2)}{\lambda^2} \right]
\]

\[
= I + II + III + IV,
\]

where

\[
I + III = \frac{y_0 - x_0 + 1}{\lambda} + \frac{(\lambda - y_0 + x_0 - 1)(1 - x_0 - y_0)}{\lambda^2} = \frac{2(1 - x_0)}{\lambda} - \frac{(1 - x_0)^2 - y_0^2}{\lambda^2}
\]

and

\[
II + IV = \frac{(1 - a)(\lambda - 2)}{\lambda^3} \left[ (y_0 - x_0 + 1)(y_0 + x_0 - 1 - \lambda) - (1 - x_0 - y_0)(\lambda - y_0 + x_0 - 1) \right]
\]

\[
= - \frac{(1 - a)(\lambda - 2)}{\lambda^3} \cdot \lambda(2 - 2x_0).
\]

Combining these facts, we obtain \(V_\lambda(x_0, y_0) \geq U_\lambda(x_0, y_0)\).

For \((x_0, y_0) \in E_\lambda\) with \((x_0, y_0) \neq (-1, 0)\), the following construction will turn to be useful. Denote \(w = \lambda - 3\), so, as \((x_0, y_0) \in E_\lambda\), we have \(x_0 + y_0 < w\). Fix positive integer \(N\) and set \(\delta = \delta_N = (w - x_0 - y_0)/[N(\alpha + 1)]\). Consider sequences \((x_j^N)_{j=1}^{N+1}, (p_j)_{j=1}^{N+1}\), defined by

\[
x_j^N = x_0 + y_0 + (j - 1)\delta(\alpha + 1), j = 1, 2, \ldots, N + 1,
\]
and $p_1^N = (1 + x_0)/(1 + x_0 + y_0)$,

$$p_{j+1}^N = \frac{(1 + x_j^N)(1 + x_j^N + \delta (\alpha + 1))p_j^N}{(1 + x_j^N + \delta (\alpha + 1))/2} + \frac{\delta}{1 + x_{j+1}^N}, \ j = 1, 2, \ldots, N. \tag{29}$$

We construct a process $(f, g)$ starting from $(x_0^N, y_0^N)$ such that for $j = 1, 2, \ldots, N + 1$,

the variable $(f_{3j}, g_{3j})$ takes values $(x_j^N, 0)$ and $(-1, 1 + x_j^N)$

with probabilities $p_j^N$ and $1 - p_j^N$, respectively. \tag{30}

We do this by induction. Let

$$df_1 = -dg_1 = y_0 \chi_{\{0, p_1^N\}} + (1 - x_0) \chi_{\{p_1^N, 1\}}, \ df_2 = dg_2 = df_3 = dg_3 = 0.$$  

Note that (30) is satisfied for $j = 1$. Now suppose we have a pair $(f, g)$, which satisfies (30) for $j = 1, 2, \ldots, n, n \leq N$. Let us describe $f_n$ and $g_n$ for $k = 3n + 1, 3n + 2, 3n + 3$. The difference $df_{3n+1}$ is determined by the following three conditions: it is a martingale difference, i.e., satisfies $E(dg_{3n+1} \mid \mathcal{F}_{3n}) = 0$; conditionally on $\{f_{3n} = x_n^N\}$, it takes values in $[-1 - x_n^N, \delta (\alpha + 1)/2]$; and vanishes on $\{f_{3n} \neq x_n^N\}$. Furthermore, set $dg_{3n+1} = df_{3n+1}$. Moreover,

$$df_{3n+2} = \delta \chi_{\{f_{3n+1} = -1\}}, \ dg_{3n+2} = \frac{g_{3n+1}}{g_{3n+1}} \cdot df_{3n+2}.$$  

Finally, the variable $df_{3n+3}$ satisfies $E(dg_{3n+3} \mid \mathcal{F}_{3n+2}) = 0$, and, in addition, the variable $f_{3n+3}$ takes values in $\{-1, x_n^N + \delta (\alpha + 1)\} = \{-1, x_{n+1}^N\}$. The description is completed by

$$dg_{3n+3} = -\frac{g_{3n+2}}{g_{3n+2}} \cdot df_{3n+3}.$$  

One easily checks that $(f_{3n+3}, g_{3n+3})$ takes values in $\{(x_{n+1}^N, 0), (-1, 1 + x_{n+1}^N)\}$; moreover, since

$$E_{f_{3n+3}} = E_{f_{3n}} + Ed_{f_{3n+2}} = x_n^N p_n^N - (1 - p_n^N) + \delta E(f_{3n+1} = -1)$$

$$= x_n^N p_n^N - (1 - p_n^N) + \delta \left(1 - p_n^N + p_n^N \cdot \frac{\delta (\alpha + 1)}{2(1 + x_n^N) + \delta (\alpha + 1)}\right)$$

$$= p_n^N \cdot \frac{(x_n^N + 1)(1 + x_n^N + \delta (\alpha + 1)/2)}{1 + x_n^N + \delta (\alpha + 1)/2} + \delta - 1,$$

we see that $P(f_{3n+3} = x_{n+1}^N) = p_{n+1}^N$ and the pair $(f, g)$ satisfies (29) for $j = n + 1$. Thus there exists $(f, g)$ satisfying (29) for $j = 1, 2, \ldots, N + 1$. In particular, $(f_{3(N+1)}, g_{3(N+1)})$ takes values $(w, 0), (-1, w + 1) \in D_3$, with probabilities $p_{N+1}^N, 1 - p_{N+1}^N$. By Lemma 4,

$$V_3(x_0, y_0) \geq p_{N+1}^N V_3(w, 0) + (1 - p_{N+1}^N) V_3(-1, w + 1). \tag{31}$$

Recall the function $H$ defined by (2). The function $h : [x_0 + y_0, w] \to \mathbb{R}$ given by $h(t) = H(x_0 + y_0, t)$, satisfies the differential equation

$$h'(t) + \frac{\alpha + 2}{\alpha + 1} \cdot \frac{h(t)}{1 + t} = \frac{1}{(\alpha + 1)(1 + t)}.$$
As we assumed $x_0 + y_0 > -1$, the expression $(h(x + \delta) - h(x))/\delta$ converges uniformly to $h'(x)$ on $[x_0 + y_0, \lambda - 3]$. Therefore there exist constants $\epsilon_N$, which depend only on $N$ and $x_0 + y_0$ satisfying $\lim_{N \to \infty} \epsilon_N = 0$ and for $1 \leq j \leq N$,

$$
\left| \frac{h(x^N_j) - h(x^N_{j+1})}{(\alpha + 1)\delta_N} + \frac{[\frac{\alpha + 2}{\alpha + 1}(1 + x^N_j) - \frac{\delta(\alpha + 1)}{2}]h(x^N_j)}{(1 + x^N_{j+1})(1 + x^N_j + \frac{\delta(\alpha + 1)}{2})} \right| \leq \epsilon_N,
$$

or, equivalently,

$$
\left| h(x^N_{j+1}) - \frac{(1 + x^N_j)(1 + x^N_j + \frac{\delta(\alpha + 1)}{2})h(x^N_j)}{(1 + x^N_j)(1 + x^N_j + \frac{\delta(\alpha + 1)}{2})} \right| \leq (\alpha + 1)\delta_N \epsilon_N.
$$

Together with (29), this leads to

$$
|h(x^N_{j+1}) - p^N_{j+1}| \leq \frac{(1 + x^N_j)(1 + x^N_j + \frac{\delta(\alpha + 1)}{2})}{(1 + x^N_{j+1})(1 + x^N_j + \frac{\delta(\alpha + 1)}{2})}|h(x^N_j) - p^N_j| + (\alpha + 1)\delta_N \epsilon_N.
$$

Since $p^N_1 = h(x^N_1)$, we have

$$
|h(w) - p^N_{N+1}| \leq (\alpha + 1)N\delta_N \epsilon_N = (\lambda - 3 - x_0 - y_0)\epsilon_N
$$

and hence $\lim_{N \to \infty} p^N_{N+1} = h(w)$. Combining this with (31), we obtain

$$
V_{\lambda}(x_0, y_0) \geq h(w)(V_{\lambda}(w, 0) - V_{\lambda}(-1, w + 1)) + V_{\lambda}(-1, w + 1).
$$

As $w = \lambda - 3$, it suffices to check that we have

$$
a_{\lambda} = V_{\lambda}(\lambda - 3, 0) - V_{\lambda}(-1, \lambda - 2) \quad \text{and} \quad b_{\lambda} = V_{\lambda}(-1, \lambda - 2),
$$

where $a_{\lambda}$, $b_{\lambda}$ were defined in (5). Finally, if $(x_0, y_0) = (-1, 0)$, then considering a pair $(f, g)$ starting from $(x_0, y_0)$ and satisfying $df_1 \equiv \delta$, $dg_1 \equiv \alpha \delta$, we get

$$
V(-1, 0) \geq V(-1 + \delta, \alpha \delta).
$$

(32)

Now let $\delta \to 0$ to obtain $V(-1, 0) \geq U(-1, 0)$.

The case $\lambda \geq 4$. We proceed as in previous case. We deal with $(x_0, y_0) \in A_{\lambda}$ exactly in the same manner. Then we establish the analogue of (28), which is

$$
V(-1, \lambda - 2) \geq U_{\lambda}(-1, \lambda - 2) = \frac{1 + \alpha}{2}.
$$

(33)

To do this, fix $\delta \in (0, 1)$ and set

$$
\beta = \frac{4 - 2\delta}{4 - \delta(1 + \alpha)}, \quad \gamma = \beta \cdot \left(1 - \frac{\delta(\alpha + 1)}{4}\right).
$$

Now let a pair $(f, g)$ be defined by $(f_0, g_0) \equiv (-1, \lambda - 2)$, $(df_1, dg_1) \equiv (\delta, \alpha \delta)$,

$$
df_2 = -dg_2 = -\frac{\delta(1 - \alpha)}{2} \chi_{[0, \beta)} + (2 - \delta)\chi_{[\beta, 1]},
$$

and...
apply Lemma 4 and check that it gives

\[ V(-1, \lambda - 2) \geq \gamma V(-1, \lambda - 2) + (\beta - \gamma) V(1, \lambda), \]

or

\[ V(-1, \lambda - 2) \geq \frac{\beta - \gamma}{1 - \gamma} \left( \frac{\alpha + 1}{2} - \delta \right). \]

It suffices to let \( \delta \to 0 \) to obtain (33). The cases \((x_0, y_0) \in B_\lambda, C_\lambda\) are dealt with using the same processes as in the case \( \lambda \in (2, 4) \). If \((x_0, y_0) \in D_\lambda\), then Lemma 4 applied to the pair \((f, g)\) given by \((f_0, g_0) \equiv (x_0, y_0)\),

\[ df_1 = -dg_1 = -(1 + x_0) \chi_{[0(1-x_0)/2]} + (1 - x_0) \chi_{[(1-x_0)/2, 1]}, \]

yields

\[ V(x_0, y_0) \geq \frac{1 - x_0}{2} V(-1, x_0 + y_0 + 1). \quad (34) \]

Furthermore, for any number \( y \) and any \( \delta \in (0, 1) \), we have

\[ V(-1, y) \geq V(-1, y + \alpha \delta), \]

which is proved in the same manner as (32). Hence, for large \( N \), if we set \( \delta = (\lambda - 3 - x_0 - y_0)/(N(\alpha + 1)) \), the inequalities (34) and (35) give

\[ V(x_0, y_0) \geq \frac{1 - x_0}{2} V(-1, x_0 + y_0 + 1) \geq \frac{1 - x_0}{2} V(-1 + \delta, x_0 + y_0 + 1 + \alpha \delta) \geq \frac{1 - x_0}{2} \left( 1 - \frac{\delta}{2} \right) V(-1, x_0 + y_0 + 1 + (\alpha + 1) \delta) \geq \frac{1 - x_0}{2} \left( 1 - \frac{\delta}{2} \right)^N V(-1, x_0 + y_0 + 1 + N(\alpha + 1) \delta) = \frac{1 - x_0}{2} \left( 1 - \frac{\lambda - 3 - x_0 - y_0}{2N(\alpha + 1)} \right)^N V(-1, \lambda - 2) = \frac{(1 - x_0)(1 + \alpha)}{4} \left( 1 - \frac{\lambda - 3 - x_0 - y_0}{2N(\alpha + 1)} \right)^N. \]

Now take \( N \to \infty \) to obtain \( V_\lambda(x_0, y_0) \geq U_\lambda(x_0, y_0) \).

Finally, if \((x_0, y_0) \in E_\lambda\) we use the pair \((f, g)\) used in the proof of the case \((x_0, y_0) \in E_\lambda, \lambda \in (2, 4)\), with \( \omega = 1 \). Then the process \((f, [g])\) ends at the points \((1, 0)\) and \((-1, 2)\) with probabilities, which can be made arbitrarily close to \(H(x_0, y_0, 1)\) and \(1 - H(x_0, y_0, 1)\), respectively. It suffices to apply Lemma 4 and check that it gives \( V_\lambda(x_0, y_0) \geq U_\lambda(x_0, y_0) \).

Acknowledgement: The results were obtained while the author was visiting Université de Franche-Comté in Besançon, France.

References

[1] D. L. Burkholder, *Explorations in martingale theory and its applications*, Ecole d’Ete de Probabilités de Saint-Flour XIX—1989, 1–66, Lecture Notes in Math., 1464, Springer, Berlin, 1991. [MR1108183]
[2] D. L. Burkholder, *Strong differential subordination and stochastic integration*, Ann. Probab. 22 (1994), 995-1025. MR1288140

[3] C. Choi, *A submartingale inequality*, Proc. Amer. Math. Soc. 124 (1996), 2549-2553. MR1353381

[4] W. Hammack, *Sharp inequalities for the distribution of a stochastic integral in which the integrator is a bounded submartingale*, Ann. Probab. 23 (1995), 223-235. MR1330768