NEWTON-PUISEUX ROOTS OF JACOBIAN DETERMINANTS

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Abstract. Let \( f(x, y), g(x, y) \) denote either a pair of holomorphic function germs, or a pair of monic polynomials in \( x \) whose coefficients are Laurent series in \( y \). A polar root is a Newton-Puiseux root, \( x = \gamma(y) \), of the Jacobian \( J = f_yg_x - f_xg_y \), but not a root of \( f \cdot g \).

We define the tree-model, \( T(f, g) \), for the pair, using the set of contact orders of the Newton-Puiseux roots of \( f \) and \( g \). Our main results (§2) describe how the \( \gamma \)'s climb, and leave, the tree (like vines). We also show by two examples (§5) that when the tree has what we call collinear points or bars, the way the \( \gamma \)'s leave the tree is not an invariant of the tree; this phenomenon is in sharp contrast to that in the one function case where the tree \( T(f) \) completely determines how the polar roots split away (§10, §11).

Our results yield a factorisation of the Jacobian determinant in \( \mathbb{C}\{x, y\} \) (§6). As in the one function case, the factors need not be invariants, nor irreducible. However, some factors do yield invariant truncations and intersection multiplicities (§7).

Take two holomorphic germs \( f, g : (\mathbb{C}^2, O) \to (\mathbb{C}, O) \), and a coordinate system \((x, y)\). The Newton-Puiseux factorisations are of the form

\[
\begin{align*}
 f(x, y) &= u(x, y) \cdot y^{E_1} \cdot \prod_{i=1}^{p} [x - \alpha_i(y)], \quad E_1 \geq 0, \\
 g(x, y) &= u'(x, y) \cdot y^{E_2} \cdot \prod_{j=1}^{q} [x - \beta_j(y)], \quad E_2 \geq 0,
\end{align*}
\]

(0.1)

where \( u, u' \) are units, \( \alpha_i, \beta_j \) are fractional power series with \( O_y(\alpha_i) > 0, O_y(\beta_j) > 0 \).

We shall also write \( \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q \) as \( \lambda_1, \ldots, \lambda_N \), \( N := p + q \).

Definition 0.1. A polar root of the pair \((f, g)\), relative to the coordinate system \((x, y)\), is a Newton-Puiseux root, \( x = \gamma(y) \), \( O_y(\gamma) > 0 \), of the Jacobian determinant \( J(x, y) := J(f, g)(x, y) := \begin{vmatrix} f_y & f_x \\ g_y & g_x \end{vmatrix} \),

which is not one of the \( \lambda_k \)'s, that is:

\( J(\gamma(y), y) = 0, \quad f(\gamma(y), y)g(\gamma(y), y) \neq 0 \).

Polar curves play an important rôle in Singularity Theory; they have been intensively studied by many authors from different perspectives. See, e.g., [3], [6], [11], [13].

It is easy to see that the Newton-Puiseux roots of \( J(x, y) \) are the polar roots plus the multiple roots of the product function \( f(x, y)g(x, y) \).

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In this paper we shall assume that \( f(x, y)g(x, y) \) has only simple roots: \( \lambda_i \neq \lambda_j \) if \( i \neq j \).

We shall use the contact orders \( O(\lambda_s, \lambda_t) := O_y(\lambda_s(y) - \lambda_t(y)) \), \( 1 \leq s, t \leq N \), to associate to the pair \((f, g)\) a combinatorial object: the tree model \( T(f, g) \). We use the tree model to analyse the contact orders between the polar roots and the roots \( \lambda_i \), so that we can visualize how the polar roots climb (like vines) along the tree and how they leave it. Each bar of the tree gives rise to a rational function \( M(z) \) of one complex variable. This function depends not only on the orders of contact between the \( \lambda_i \)’s but also on the coefficients. In particular, its zeros play an important rôle since we can identify them with some of the places where the polar roots leave the tree. But it may happen that this function \( M(z) \) is identically zero. Then the bar is called collinear and, as we show in \( \S 5 \), it is impossible, in general, to know precisely how the polar roots climb this bar and leave the tree.

We have divided our results into two parts. Part 1, in \( \S 2 \), contains three theorems (Theorems T, N, and C) and their corollaries. They describe the positions of the polar roots relative to \( T(f, g) \). The proofs depend heavily on the classical Theorem of Rouché. We show, in particular, that if a polar root leaves the tree on a non-collinear bar, it must do so at a “pure” zero of the associated rational function (see the next section for this terminology). This will be used in Part 2 to obtain a factorisation of the Jacobian determinant.

Part 2, in \( \S 6 \) and \( \S 7 \), contains results which describe how \( J(x, y) \) can be factored in \( \mathbb{C}\{x, y\} \), and how to compute the intersection multiplicities of each factor with the zero sets of \( f, g, \) and \( f \cdot g \). The factors can be reducible. Amongst the objects we come across, we shall carefully distinguish those which are invariants of the tree from those which are not (\( \S 7 \)).

*Our results generalise that in the one function case.* More specifically, by taking \( g(x, y) = y \), \( J(x, y) \) reduces to \( f_x \), and \( T(f, y) = T(f) \), the tree-model defined in \([10]\). The curve \( f_x = 0 \) is called a polar curve, it has been studied since the time of M. Noether. The components, defined by the irreducible factors of \( f_x \) in \( \mathbb{C}\{x, y\} \), are called the polar branches. In \([15]\), Pham showed that the Zariski equisingularity type \( \text{[17]} \) of the polar curve, and that of the polar branches, need not be determined by that of \( f = 0 \). (If \( f \) is generic in its equisingularity class then the equisingularity class of the generic polar curve is described in \([10]\).)

However, for the contact orders of the polar roots with the roots of \( f = 0 \), the story is different. The set of contact orders, \( C(f, f_x) := \{O(\alpha_i, \gamma_j)\} \), between the roots of \( f \) and that of \( f_x \) can be calculated using the tree-model \( T(f) \) alone. This is proved in \([10]\), and, for irreducible \( f \), also in \([14]\).

Therefore, the contact order set \( C(f, f_x) \) is an invariant of the equisingularity type of \( f \).

(Attention should be paid to the rather subtle distinction between polar roots, which are studied in this paper, and the well-established notion of polar branches; the former are fractional power series, the latter are primes in \( \mathbb{C}\{x, y\} \) generated by the former. This difference between polar roots and polar branches has eluded some experts.)

A more detailed account of the one function case will be given in Section \( \S 8.2 \).

The main results of this paper have been announced in \([12]\).

*In the general case*, \( T(f, g) \) may have collinear points and bars (in the one function case, all bars of \( T(f) \) are purely non-collinear), and then we encounter a completely new phenomenon. Namely, it may not be possible anymore to know precisely where some of the polar roots leave
the tree. That is, the set of contact orders \( \{O(\lambda_i, \gamma_j)\} \) between the roots \( \lambda_i \) of \( f \cdot g \) and the polar roots \( \gamma_j \) need not be determined by \( T(f, g) \). We give examples in §3.

There are many other tree-models in the one function case (Cassas-Alvero, Eggars, Wall, etc). The models \( T(f), T(f, g) \), we use here can better express how the polar roots split away from the tree. Our definitions are simple but do not use the language of Graph Theory.

**Conventions:** A fractional power series \( \lambda(y) \) will also be called an "arc". If \( O(\lambda, \mu) > q \), we write \( \lambda \equiv \mu \mod q^+ \). We use \( O(y^+) \) to represent a quantity which, as \( y \to 0 \), has the same order as \( y^e \), for some \( e > 0 \). Finally, "\( + \cdots \)" will mean "plus higher order terms".

1. The tree model \( T(f, g) \).

The tree model is a geometric object that allows us to visualize the numerical data given by the contact orders \( O(\lambda_i, \lambda_j) \) between the roots of \( f \cdot g \) and then between a given arc \( x = \xi(y) \) and the \( \lambda_i \)'s. The construction of \( T(f, g) \) is as follows (compare [10]). First, draw a horizontal bar, denoted by \( B^*_n \), and call it the ground bar (the soil). Then draw a vertical line segment on \( B^*_n \) as the main trunk of the tree. Mark \([p, q]\) alongside the trunk to indicate that \( p \alpha_i \)'s and \( q \beta_j \)'s are bundled together.

Let \( h_0 := \min \{O(\lambda_i, \lambda_j)|1 \leq i, j \leq N\} \). Then draw a bar, \( B_0 \), on top of the main trunk. Call \( h(B_0) := h_0 \) the height of \( B_0 \). We define \( h(B_n) := 0 \).

The roots \( \lambda_k, 1 \leq k \leq N \), are divided into equivalence classes modulo \( h_0^+ \). We then represent each equivalence class by a vertical line segment drawn on top of \( B_0 \). Each is called a trunk.

If a trunk consists of \( s \alpha_i \)'s and \( t \beta_j \)'s \( (s \geq 0, t \geq 0, s + t \geq 1) \), we say it has bimultiplicity \([s, t]\), and mark \([s, t]\) alongside. We call \( s + t \) the total multiplicity.

Now, the same construction is repeated recursively on each trunk, getting more bars, then more trunks, etc. The height of each bar, the bimultiplicity and the total multiplicity of each trunk, are defined likewise.

The construction terminates at the stage where the bars have infinite height. We shall omit drawing bars of infinite height.

**Example 1.1.** Take constants \( A \neq 0 \neq B \), integers \( 0 < e < E \). Then consider

\[
\begin{align*}
f(x, y) &= (x + y)(x - ye^{e+1} + Ay^{E+1})(x + ye^{e+1} + By^{E+1}), \\
g(x, y) &= (x - y)(x - ye^{e+1} - Ay^{E+1})(x + ye^{e+1} - By^{E+1}).
\end{align*}
\]

The tree model \( T(f, g) \) is shown in Fig.1 with \( h(B_0) = 1, h(B_1) = e + 1, h(B_2) = h(B_3) = E + 1 \). There are six roots \( \lambda_k \), hence six bars of infinite height. The notations "\( \circ \)" and "\( \times \)" will be defined in Convention 2.2.
Tracing upward from the main trunk to a bar of infinite height amounts to identifying a root $\lambda_k$. The heights of the bars coming across on the way up are the contact orders of $\lambda_k$ with the other roots.

Take a bar $B$, with finite height $h := h(B)$. Take a root $\lambda_k$ whose modulo $h^+$ class is a trunk on $B$. Let $\lambda_B(y)$ denote $\lambda_k(y)$ with all terms $y^e$, $e \geq h$, omitted. (In particular, $\lambda_B(0) = 0$.) Clearly, $\lambda_B$ depends only on $B$, not on the choice of $\lambda_k$. We can then write

$$\lambda_k(y) = \lambda_B(y) + cy^{h(B)} + \cdots, \quad c \in \mathbb{C},$$

where $c$ is uniquely determined by $\lambda_k$.

We say the trunk $T$ which contains $\lambda_k$ grows on $B$ at $c$. If $T$ has bimultiplicity $[s, t]$, we also say $c$ has bimultiplicity $[s, t]$ on $B$. The main trunk grows on $B^*$ at 0.

Let $B^*$ be the bar on top of $T$. As in [9], we say $B^*$ is a postbar of $B$, supported at $c$, and write: $B \perp c B^*$. In case $c$ need not be specified, we simply write: $B \perp B^*$.

We say $B'$ lies above $B$ if there is a postbar sequence: $B \perp B_1 \perp \cdots \perp B_q \perp B'$; in this case, if $B \perp c B_1$, we also say $B'$ lies over $c$, or $c$ lies below $B'$.

**Definition 1.2.** Take any arc $\xi$. If $\xi$ has the form

$$\xi(y) = \lambda_B(y) + ay^{h(B)} + \cdots, \quad a \in \mathbb{C},$$

then we say $\xi$ climbs over $B$ at $a$ (like a vine). In this case, if no trunk grows at $a$ we say $\xi$ leaves the tree on $B$ at $a$.

If $O(\xi, \lambda_B) < h(B)$, we say $\xi$ is bounded by $B$.

**Convention 1.3.** Take a bar $B$. We shall identify $z \in \mathbb{C}$ with the arc $\lambda_B(y) + zy^{h(B)}$; and shall also use $B$ to denote the set of such arcs. (Intuitively, $B$ is a copy of $\mathbb{C}$.)

## 2. The main results.

In this section we describe the possible positions of a polar root of the pair $(f, g)$ with respect to the tree $T(f, g)$. Take a bar $B$, $h(B) < \infty$. Take a germ $F(x, y)$ and a generic $z \in \mathbb{C}$. Let

$$\nu_F(B) := O_y(F(\lambda_B(y) + zy^{h(B)}, y)).$$
and, for any fractional power series $\eta(y)$, 
$$
\nu_F(\eta) := O_y(F(\eta(y), y)).
$$
In particular, $\nu_f(B_z) = E_1$, $\nu_g(B_z) = E_2$, by (1.1).

Let $T_k$, $1 \leq k \leq l$, be the set of trunks on $B$. Suppose $T_k$ grows at $z_k$, having bimultiplicity $[p_k, q_k]$. We write 
$$
\Delta_B(z_k) := \begin{vmatrix} \nu_f(B) & p_k \\ \nu_g(B) & q_k \end{vmatrix}, \quad 1 \leq k \leq l,
$$
and call 
$$
\mathcal{M}_B(z) := \sum_{k=1}^{l} \frac{\Delta_B(z_k)}{z - z_k}, \quad z \in \mathbb{C},
$$
the rational function associated to $B$. (Those terms with $\Delta_B(z_k) = 0$ can be omitted.)

**Definition 2.1.** We say $z_k$, $1 \leq k \leq l$, is a **collinear point** on $B$ if $\Delta_B(z_k) = 0$, otherwise, $z_k$ is called **non-collinear**.

Let $C(B)$ and $N(B)$ denote respectively the set of collinear and non-collinear points:

(2.1) 
$$
C(B) \cup N(B) = \{z_1, \ldots, z_l\}.
$$
Their (finite) cardinal numbers are denoted by $c(B)$ and $n(B)$ respectively.

**Convention 2.2.** A collinear point will be indicated by $\circ$; a non-collinear point by $\times$.

**Definition 2.3.** We call $B$ a **collinear bar** if $\Delta_B(z_k) = 0$ for all $k$, $1 \leq k \leq l$. Otherwise we call $B$ **non-collinear**. We say $B$ is **purely non-collinear** if $C(B) = \emptyset \neq N(B)$.

In Example 1.1, $B_1$ and $B_2$ are collinear, $B_3$ are purely non-collinear.

If $\mathcal{M}_B(z) = 0$, we say $z$ is a **mero-zero** on $B$. Let $m_B(z)$ denote its multiplicity. Let $M(B)$ denote the set of mero-zeros on $B$. We write:

$$
m_B(z) := \sum_{z \in M(B)} m_B(z).
$$

Suppose $N(B) \neq \emptyset$. A non-collinear $z_k$ is a simple pole, hence not a mero-zero:

(2.2) 
$$
N(B) \cap M(B) = \emptyset, \quad n(B) \geq m(B) + 1.
$$
On the other hand it may happen that $C(B) \cap M(B) \neq \emptyset$. If $z \in M(B) \setminus C(B)$, we say $z$ is a **pure mero-zero**.

It can happen that $\Delta(z_k) = 0$ for all $k$. In this case $\mathcal{M}_B \equiv 0$, $N(B) = \emptyset$, and $M(B) = \mathbb{C}$.

It can also happen that $M(B) = \emptyset$. For example, for $f(x, y) = x$ and $g(x, y) = x^2 - y^2$, there is only one bar $B$ of height 1 and we have:

$$
\mathcal{M}_B(z) = \frac{\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}}{z} + \frac{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}{z - 1} + \frac{\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}}{z + 1} = \frac{2}{z(z^2 - 1)}.
$$

Take a non-collinear bar $B$. We define formally the total multiplicity function by

$$
\tau_B(z) = \begin{cases} p_k + q_k, & \text{if } z = z_k; \\
0, & \text{otherwise},
\end{cases}
$$
and the *mero-multiplicity function* by

\[
\mu_B(z) = \begin{cases} 
  m_B(z), & \text{if } z \in M(B); \\
  -1, & \text{if } z \in N(B); \\
  0, & \text{otherwise.}
\end{cases}
\]

We also write

\[
\tau(B) := \sum_{z \in \mathbb{C}} \tau_B(z), \quad \mu(B) := \sum_{z \in \mathbb{C}} \mu_B(z).
\]

Note that, obviously,

\[
\mu(B) = m(B) - n(B), \quad \text{a negative integer.}
\]

Let \(T_B(z)\) denote the total number of polar roots (counting multiplicities) which climb over \(B\) at \(z\), and let \(T(B)\) denote that of those which climb over \(B\).

**Theorem T.** Let \(B\) be a non-collinear bar. Then

\[
T_B(z) = \tau_B(z) + \mu_B(z), \quad z \in \mathbb{C},
\]

and, consequently,

\[
T(B) = \tau(B) + \mu(B).
\]

**Corollary 2.4.** If a polar root climbs over \(B\) at \(z\), then \(z \in N(B) \cup C(B) \cup M(B)\).

**Corollary 2.5.** Suppose \(z\) is a pure mero-zero on \(B\). Then there are exactly \(m_B(z)\) polar roots (counting multiplicities) climbing over \(B\) at \(z\). (Thus, they all leave the tree at \(z\).)

**Corollary 2.6.** Suppose \(\sum_{z \in N(B)} \Delta_B(z_k) \neq 0\). Then \(m(B) + 1 = n(B)\) and

\[
T(B) = \sum_{k=1}^{l} (p_k + q_k) - 1.
\]

In particular, if \(pE_2 - qE_1 \neq 0\) then the total number of polar roots is \(p + q - 1\).

**Corollary 2.7.** Suppose that \(B\) is on the top of trunk \(T\) of bimultiplicity \([s,0]\) (resp. \([0,t]\)) and \(\nu_f(B) \neq 0\) (resp. \(\nu_f(B) \neq 0\)). Then \(B\) is purely non-collinear, \(m(B) + 1 = n(B)\), and \(T(B) = s - 1\) (resp. \(t - 1\)).

Now we shall study how the polar roots climb the tree, and where they leave. The simplest cases have already been dealt with in the above corollaries. (See also §\(\mathbb{N}\).)

**Theorem N.** Take \(z \in N(B)\). Let \(B^*\) be the postbar of \(B\) supported at \(z\). Then

\[
m(B^*) + 1 = n(B^*).
\]

In particular, \(B^*\) is non-collinear. Moreover, every polar root which climbs over \(B\) at \(z\) must also climb over \(B^*\). That is, there is no polar root, \(\gamma\), such that

\[
h(B) < O(\gamma, \lambda_{B^*}) < h(B^*).
\]
Definition 2.8. Take $c \in C(B)$. A set of non-collinear bars $\{\bar{B}_1, \ldots, \bar{B}_l\}$ is called a (non-collinear) cover of $c$ if the following holds:

(i) Each $\bar{B}_s$ lies over $c$ and is minimal in the sense that there is a sequence

$$B \perp B_1^* \perp \cdots \perp B_{r(s)}^* \perp \bar{B}_s, \quad B \perp c \perp B_1^*,$$

where either $r(s) = 0$ (i.e. $B \perp c \bar{B}_s$), or else all $B_i^*$, $1 \leq i \leq r(s)$, are collinear.

(ii) Each root $\lambda_k$ climbing over $B$ at $c$ also climbs over a (unique) $\bar{B}_s$.

In Fig. 1, $\{B_2, B_3\}$ is a cover of $0 \in C(B_0)$. In Fig. 2, $\{B_2, B_3\}$ is a cover of $c \in C(B_0)$.

Take a bar $\hat{B}$ of maximal height. Since all $\lambda_k$ are simple roots, every trunk growing on $\hat{B}$ has bimultiplicity either $[1, 0]$ or $[0, 1]$. Since $\nu_f(\hat{B}) \neq 0 \neq \nu_g(\hat{B})$, $\hat{B}$ is purely non-collinear. It follows that every $c$ has a (unique) cover.

Theorem C. Let $B$ be a non-collinear bar. Take $c \in C(B)$ with cover $\{\bar{B}_1, \ldots, \bar{B}_l\}$. Then there are exactly

$$m_B(c) + \sum_{s=1}^l [n(\bar{B}_s) - m(\bar{B}_s)]$$

polar roots which climb over $B$ at $c$, bounded by every $\bar{B}_s$, $1 \leq s \leq l$.

Finally, let us imagine bars as tiles, collinear points as holes, and introduce two definitions. By a partial repair of $B$, $C(B) \neq \emptyset$, we mean a sequence, beginning with $\hat{B}_0 := B$,

$$(2.11)\quad \hat{B}_0 \perp \hat{B}_1 \perp \cdots \perp \hat{B}_t,$$

where $\hat{B}_t$ is purely non-collinear, $\hat{B}_{i+1}$ is supported at a collinear point of $\hat{B}_i$, $0 \leq i \leq t - 1$.

The repair of $B$, denoted be $\mathcal{R}(B)$, is the set of all bars, other than $B$, which appear in any partial repair. Let us write

$$(2.12)\quad \mathcal{R}(B) = \{\hat{B}_1, \ldots, \hat{B}_r\}.$$

Note that if a bar appears in more than one sequence (2.11), it is merely taken as one element of $\mathcal{R}(B)$. In Fig.2, $\mathcal{R}(B_0) = \{B_1, B_2, B_3, B_4\}$.

A polar root is called a weed on $B$ if it climbs over $B$ at a point in $C(B) \cup M(B)$, and, whenever it climbs over a bar, say $\hat{B}_s$, in $\mathcal{R}(B)$, it does so at a point in $C(\hat{B}_s) \cup M(\hat{B}_s)$.

Let $w(B)$ denote the number of weeds on $B$ (counting multiplicities).
Corollary 2.9. Let $B$ be a non-collinear bar with repair $\mathcal{R}(B) = \{\hat{B}_1, \ldots, \hat{B}_r\}$. Then

$$(2.13) \quad w(B) = m(B) + \sum_{s=1}^r n(\hat{B}_s).$$

(In this formula, a collinear $\hat{B}_s$ yields $n(\hat{B}_s) = 0$.)

Let $\mathcal{A}(B)$ denote the set consisting of $B$ and all $B'$ of finite height lying above $B$. We say $B' \in \mathcal{A}(B)$ is basic if either $B' = B$ or else $B'$ is a non-collinear bar, supported at a non-collinear point. In Fig.2, $B_0, B_5$, and $B_6$ are the basics in $\mathcal{A}(B_0)$.

If $B$ is purely non-collinear we define $\mathcal{R}(B) := \emptyset$, $w(B) := m(B)$.

Corollary 2.10. Let $B$ be a non-collinear bar. Let $\{B'_1, \ldots, B'_L\}$ be the set of all basics in $\mathcal{A}(B)$. Then $\mathcal{T}(B) = \sum_{i=1}^L w(B'_i)$.

Note that, by Theorem N, a collinear bar is never supported at a non-collinear point, hence we have a disjoint union:

$$(2.14) \quad \mathcal{A}(B) = \bigcup_{i=1}^L [\{B'_i\} \cup \mathcal{R}(B'_i)].$$

Remark 2.11. Suppose the ground bar $B_0$ is collinear. Take a polar root $\gamma$. Either $\gamma$ climbs over some non-collinear bar, or else it is bounded by all non-collinear bars of minimal height. This is because every bar of maximal height is non-collinear. Let $\{\hat{B}_1, \cdots, \hat{B}_s\}$ be the cover of $0 \in B_0$. Let us write $J(x, y) = y^{E*} J_0(x, y), J_0$ regular in $x$, say of order $K$. Therefore the total number of polar roots bounded by all non-collinear bars of minimal height is $K - \sum_{i=1}^s \mathcal{T}(\hat{B}_i)$.

3. Lemmas.

Take $z_k \in N(B) \cup C(B)$, with bimultiplicity $[p_k, q_k]$. Suppose $B \perp B^*$ at $z_k$.

Lemma 3.1. All arcs $\xi$ climbing over $B$ at $z_k$, bounded by $B^*$, yield a constant determinant:

$$\det \begin{pmatrix} \nu_f(B) & p_k \\ \nu_g(B) & q_k \end{pmatrix} = \det \begin{pmatrix} \nu_f(\xi) & p_k \\ \nu_g(\xi) & q_k \end{pmatrix} = \det \begin{pmatrix} \nu_f(B^*) & p_k \\ \nu_g(B^*) & q_k \end{pmatrix}.$$ 

In particular, the determinants vanish if and only if $z_k \in C(B)$; in this case, there is a common ratio:

$$[\nu_f(B) : \nu_g(B)] = [\nu_f(\xi) : \nu_g(\xi)] = [\nu_f(B^*) : \nu_g(B^*)].$$

The proof is short. By assumption,

$$\lambda_B(y) = \lambda_B(y) + z_k y^{h(B)} + \cdots \quad \xi(y) - \lambda_B^*(y) = a y^{h(B) + e} + \cdots,$$

where $a \neq 0$, $h(B) < h(B) + e < h(B^*)$.

Let $\zeta(y) := \lambda_0(y) + z y^{h(B)}, z$ a generic number.

The number of roots $\alpha_i$ with $O(\alpha, \xi) = h(B) + e$ is precisely $p_k$, and $O(\alpha, \zeta) = h(B)$. For any other root $\alpha_k$, $O(\alpha_k, \xi) = O(\alpha_k, \zeta)$. Hence,

$$\nu_f(\xi) = \sum_{j=1}^p O(\alpha_j, \zeta) + p_k \cdot e = \nu_f(B) + p_k \cdot e.$$
Similarly, \( \nu_{g}(\xi) = \nu_{g}(B) + q_{k} \cdot e \). This completes the proof.

**Corollary 3.2.** Take \( c \in C(B) \), with cover \( \{ \bar{B}_{1}, \ldots, \bar{B}_{l} \} \). For all arcs \( \xi \) which climb over \( B \) at \( c \), bounded by every \( \bar{B}_{s} \), \( 1 \leq s \leq l \), and for all \( B_{j}^{*} \) in the sequences (2.8) for each \( \bar{B}_{s} \), there is a constant ratio:

\[
(3.1) \quad [\nu_{f}(\xi) : \nu_{g}(\xi)] = [\nu_{f}(B) : \nu_{g}(B)] = [\nu_{f}(B_{j}^{*}) : \nu_{g}(B_{j}^{*})] = [\nu_{f}(\bar{B}_{s}) : \nu_{g}(\bar{B}_{s})].
\]

All \( B_{j}^{*} \) are collinear. A recursive application of Lemma 3.1 completes the proof.

**Lemma 3.3.** Take any \( a \notin N(B) \cup C(B) \). For all arcs \( \xi \) climbing over \( B \) at \( a \),

\[
(3.2) \quad \frac{y \frac{d}{dx} f(x, y)}{f(x, y)} = \nu_{f}(B) + O(y^{+}), \quad \frac{y \frac{d}{dx} g(x, y)}{g(x, y)} = \nu_{g}(B) + O(y^{+}).
\]

Indeed, we can write

\[
f(\xi(y), y) = by^{e} + \cdots, \quad b \neq 0, \quad e = \nu_{f}(B),
\]

and then the lemma follows.

4. Proofs.

We show Theorems T, N, and C. The proofs depend heavily on the classical Theorem of Rouché:

\[
\frac{1}{2\pi i} \int_{C} \frac{d}{dz} \log \mathcal{M}(z) \, dz = N - P, \quad \text{the argument index in } C,
\]

where \( N, P \) denote respectively the number of zeros and poles in a contour \( C \).

Take a non-collinear bar \( B \). Define the following meromorphic function

\[
\mathcal{M}_{B}(z, y) := \begin{vmatrix} \nu_{f}(B) & \sum_{i=1}^{p} \frac{y^{h_{i}(B)}}{x - \alpha_{i}(y)} \\ \nu_{g}(B) & \sum_{j=1}^{q} \frac{y^{h_{j}(B)}}{x - \beta_{j}(y)} \end{vmatrix},
\]

where we have made the substitution \( x = \lambda_{B}(y) + z y^{h(B)} \). It is easy to see that

\[
\mathcal{M}_{B}(z) = \mathcal{M}_{B}(z, 0) \quad (\neq 0),
\]

whence, by Rouché’s Theorem,

\[
\frac{1}{2\pi i} \int_{C} \frac{d}{dz} \log \mathcal{M}_{B}(z, y) \, dz = \frac{1}{2\pi i} \int_{C} \frac{d}{dz} \log \mathcal{M}_{B}(z) \, dz, \quad |y| \text{ small}.
\]

Using the following identities (derived from (1.1)):

\[
\frac{f_{x}}{f} = \sum \frac{1}{x - \alpha_{i}} + \frac{u_{x}}{u}, \quad \frac{g_{x}}{g} = \sum \frac{1}{x - \beta_{j}} + \frac{u'_{x}}{u'},
\]

we can write \( J(x, y) \) as

\[
(4.1) \quad J(x, y) = y^{-1} f g \left| \begin{array}{c} \frac{y f_{y}}{y g_{y}} \\ \frac{f_{x}}{g_{x}} \end{array} \right| = y^{-h(B) - 1} f g [\mathcal{M}_{B}(z, y) + \mathcal{P}_{B}(z, y)],
\]
where
\[
P_B(z, y) := \begin{vmatrix}
\frac{yf_y - \nu_f(B)}{y_{f_y} - \nu_g(B)} & y^{h(B)}\frac{f_y}{g_y} \\
y^{h(B)}\frac{f_y}{g_y} & \frac{\nu_g(B)}{\nu_f(B)}
\end{vmatrix} + y^{h(B)} \begin{vmatrix}
\frac{\nu_g(B)}{\nu_f(B)} & \nu_f(B) \\
\nu_g(B) & \frac{\nu_f(B)}{\nu_g(B)}
\end{vmatrix} \frac{\mu}{y^2}.
\]

Here \(P_B(z, y)\) is a meromorphic function of \(z, y\) which is, by Lemma 3.3, well-defined at a generic point \((z, 0)\).

**Lemma 4.1.** \(P_B(z, 0) = 0\) for \(z \notin N(B) \cup C(B)\).

This is not obvious. Although the second summand clearly vanishes when \(y = 0\), the second column of the first determinant may not.

Take any \(\xi\) climbing over \(B\) at a point \(a \notin N(B) \cup C(B)\). Let us evaluate \(P_B\) along \(\xi\). The first determinant vanishes by Lemma 3.3. Since \(\xi\) is arbitrary, Lemma 4.1 follows.

**Corollary 4.2.** Take \(a \in \mathbb{C}\), and a small \(\varepsilon > 0\). Take \(y \in \mathbb{C}\), \(|y| \ll \varepsilon\). Then
\[
\frac{1}{2\pi i} \int_{|z-a|=\varepsilon} \frac{d}{dz} \log [\mathcal{M}_B(z, y) + P_B(z, y)] \, dz = \frac{1}{2\pi i} \int_{|z-a|=\varepsilon} \frac{d}{dz} \log \mathcal{M}_B(z) \, dz = \mu_B(a).
\]

Now let us take \(a = z_k\) on \(B\). There are \(\tau_B(z_k)\) roots of \(f(x, y)g(x, y)\) in the contour \(|z - z_k| = \varepsilon\). It follows from the above corollary that \(J(x, y)\) has \(\tau_B(z_k) + \mu_B(z_k)\) roots in the contour. Since \(\varepsilon\) is arbitrarily small, these roots must all climb over \(B\) at \(z_k\). This completes the proof of Theorem T.

To prove Theorem N, we can permute the indices in (2.1), if necessary, so that \(z = z_1\) with bimultiplicity \([p_1, q_1]\). Let \(B^*\) be the postbar of \(B\) supported at \(z_1\), and let

\[N(B^*) = \{z_1^*, \ldots, z_s^*\}, \quad C(B^*) = \{z_{s+1}^*, \ldots, z_{s+t}^*\},\]

where \(s = n(B^*), \quad t = c(B^*)\). Then, clearly,
\[
\sum_{k=1}^{s+t} p_k^* = p_1, \quad \sum_{k=1}^{s+t} q_k^* = q_1,
\]
and, since \(z_1\) is non-collinear,
\[
D^* := \sum_{k=1}^{s+t} \frac{\nu_f(B)}{\nu_g(B)} \begin{vmatrix}
p_k^* & p_1 \\
q_k^* & q_1
\end{vmatrix} \neq 0.
\]

Moreover,
\[
D^* = \begin{vmatrix}
\nu_f(B^*) & p_1 \\
\nu_g(B^*) & q_1
\end{vmatrix} = \sum_{k=1}^{s} \nu_f(B^*) \begin{vmatrix}
p_k^* & p_1 \\
q_k^* & q_1
\end{vmatrix},
\]
the first equality follows from Lemma 4.1, the second holds because \(z_{s+j}^*\) are collinear.

Now consider \(\mathcal{M}_{B^*}(z)\), the rational function associated to \(B^*\). Its numerator is a polynomial of degree \(s - 1\) with leading coefficient \(D^* \neq 0\). We have proved (2.7).

The second part of Theorem N follows from (2.7) and Theorem T. Indeed, let us calculate the number of polar roots climbing over \(B\) at \(z\) and the same number over \(B^*\). By Theorem T the former equals \(p_1 + q_1 - 1\) and the latter equals \(\sum p_k^* + \sum q_k^* + m(B^*) - n(B^*)\).

For Theorem C the ideas of the proof are the same as before. Calculating the argument index of \(\mathcal{M}_B = y^{h(B)+1}f^{-1}g^{-1}J(x, y)\) on \(B\) at \(c \in C(B)\) yields \(m_B(c)\). Calculating the index
on each $\tilde{B}_s$, $1 \leq s \leq l$, in a large contour yields $m(\tilde{B}_s) - n(\tilde{B}_s)$. The total deficit is the number in (2.10), proving Theorem C.

For Corollary 2.9 we take a cover of each $c \in C(B)$. If $\tilde{B}_s$ in (2.4) is not purely non-collinear we take the covers of its collinear points. This process is repeated recursively. Now, adding all the collinear bars $B'_s$ appearing in (2.9) to the above covers yields the repair $R(B)$. Hence Corollary 2.3 follows from Theorems C and T.

Take a polar root $\gamma$ which climbs over $B$. By Theorem N, there is a unique basic $B'_s$ in $A(B)$ for which $\gamma$ is accounted for in $w(B'_s)$, $1 \leq s \leq L$. This proves Corollary 2.10.

### 5. What Theorem C Does Not Say

Theorem C does not say precisely where the polar roots leave the tree. The number of polar roots given in Theorem C is determined by the tree, but their orders of contact with the tree need not be. It follows that the contact structure of the two curve germs, i.e. $T(f, g)$, does not give full information on how to factor the Jacobian into irreducible factors in $\mathbb{C}[x, y]$. (See Theorem F below.) We shall use two examples to show that in our case, contrary to the one function case see Section 5.2, the way these polar roots leave the tree need not be an invariant of the tree; the coefficients of the $\lambda_i$’s may also play a rôle.

First, consider Example 1.1. The tree model is shown in Fig.1, with $\nu_f(B_1) = \nu_g(B_1) = E + e + 3$, $B_1$ being collinear. By Theorem T, there are four polar roots climbing over $B_0$, all at 0.

In the following, we take $e < E < 2e$.

To decide how many polar roots climb over $B_1$, we put $x = zy^{e+1}$. An easy calculation yields

$$\frac{yg_y}{g} - \frac{yf_y}{f} = 2y^{E-e}\left[\frac{ez^2y^{2e-E}}{z^2y^{2e}-1} - \frac{(E-e)A(z-1)}{(z-1)^2 - A^2y^{2(E-e)}} - \frac{(E-e)B(z+1)}{(z+1)^2 - B^2y^{2(E-e)}}\right];$$

and

$$\frac{g_z}{g} - \frac{f_z}{f} = 2y^{E-e}\left[\frac{y^{2e-E}}{z^2y^{2e}-1} + \frac{A}{(z-1)^2 - A^2y^{2(E-e)}} + \frac{B}{(z+1)^2 - B^2y^{2(E-e)}}\right].$$

Hence

$$J(x, y) = y^{-e-2} \cdot f \cdot g \cdot \begin{vmatrix} yf_y & f_y \\ yg_y & g_y \end{vmatrix} = 2y^{E-2e-2} \cdot f \cdot g \cdot \Delta(z, y),$$

where

$$\Delta(z, 0) = \begin{vmatrix} 2e + 3 & 1 \\ (z-1)^{-1} & (z+1)^{-1} \end{vmatrix} = (z^2 - 1)^{-2}[(A + B)(2E + 3)z^2 + 2(A - B)(E + e + 3)z + (2e + 3)(A + B)].$$

Observe that if $A + B \neq 0$, there are two zeros. This means that two polar roots climb over $B_1$, the remaining two are bounded by $B_1$. If, however, $A + B = 0$, then there is only one zero. This means that one polar root climbs over $B_1$, three are bounded by $B_1$.

Thus, in general, one cannot tell the positions of polar roots relative to collinear bars.
Example 5.1. Take integers e, E, N, 2e > E > e > 0, N ≥ 0. Let
\[ f(x, y) := [x^2 - y^{2(e+1)}][(x - y)^2 - y^{2(e+1)+N}], \quad g(x, y) := [x + y^{E+1}][x + y]. \]
The tree has four bars, \( B_4, B_1, B_2, B_3; \) \( h(B_1) = 1, h(B_2) = e + 1, h(B_3) = e + 1 + N; \) and
\[ \mathcal{M}_{B_1}(z) = \frac{8}{z^2 - 1}, \quad \mathcal{M}_{B_2}(z) = \frac{-2(e + 2)}{z(z^2 - 1)}, \]
where \( B_2 \) is supported at a collinear point \( 0 \in B_1. \)

By Theorem C, three polar roots, say \( \gamma_i, 1 \leq i \leq 3, \) climb over \( B_1 \) at 0, bounded by \( B_2. \)

Let us write \( x = Xy. \) The arcs \( \eta_i(y) := y^{-1}\gamma_i(y), i = 1, 2, 3, \) are Newton-Puiseux roots of the equation
\[ X^3(8 + \cdots) - Xy^E(2E + 2 + \cdots) - y^{2e}(2e + 2 + \cdots) = 0, \]
where the dotted terms are in the maximal ideal. If \( 3E < 4e \) then the Newton Polygon of this equation has vertices \((3, 0), (1, E), \) and \((0, 2e). \) Then two \( \eta_i's \) have order \( \frac{E}{2}, \) one has order \( 2e - E. \) Thus, two polar roots have order \( \frac{E}{2} + 1, \) one has order \( 2e - E + 1. \)

We can take \( e = 7. \) Then \( E_1 := 8, E_2 := 9 \) both satisfy the above inequality. Let
\[ g_k(x, y) = (x + y^{E_k+1})(x + y), \quad k = 1, 2. \]
Then \( T(f, g_1) = T(f, g_2), \) but, as \( E_1 \neq E_2, \) the polar roots split away from the trees at different heights between \( B_1 \) and \( B_2. \)

The above examples show that the factorization can depend on the coefficients of the Newton-Puiseux roots, not merely on the contact structure.

6. Factors of \( J(x, y) \) in \( \mathbb{C}\{x, y\}. \)

We can introduce an additional structure on \( T(f, g) \) and use it to define a factorisation of \( J(x, y) \) in \( \mathbb{C}\{x, y\}. \) The factors are not irreducible, in general.

Definition 6.1. Take \( B, \bar{B}. \) We say \( B \) is conjugate to \( \bar{B}, \) writing as \( B \sim \bar{B}, \) if, and only if \( h(B) = h(\bar{B}) \) and there exists an irreducible \( p(x, y) \in \mathbb{C}\{x, y\}, \) of which one (Newton-Puiseux) root climbs over \( B \) and one climbs over \( \bar{B}. \)

Lemma 6.2. Suppose \( B \sim \bar{B}. \) Take any irreducible \( q(x, y) \in \mathbb{C}\{x, y\}. \) If \( q(x, y) \) has a root climbing over \( B \) then it also has a root climbing over \( \bar{B}. \)

Proof. Take an integer \( D \) such that the roots of \( p(x, y) \) and \( q(x, y) \) can all be written in the following form:
\[ (6.1) \quad \lambda(y) := c_1 y^{n_1/D} + c_2 y^{n_2/D} + \cdots, \quad 0 < \frac{n_1}{D} < \frac{n_2}{D} < \cdots. \]

Of course, here we allow \( D, n_1, n_2, \ldots, \) to have common factors.

Let \( \theta \) be any \( Dth \) root of unity: \( \theta^D = 1. \) Each \( \theta \) yields a transformation (conjugation) on arcs of form \((6.1): \)
\[ \theta(\lambda)(y) := c_1 \theta^{n_1} y^{n_1/D} + c_2 \theta^{n_2} y^{n_2/D} + \cdots. \]
As in [16] (p.107), it is easy to see that the $\theta$’s permute transitively the roots of $p(x,y)$, and also that of $q(x,y)$. The contact order is preserved; in particular,

\[ O(\alpha, \beta) = O(\theta(\alpha), \theta(\beta)), \quad \text{if} \quad p(\alpha(y), y) = q(\beta(y), y) = 0. \]

Lemma 6.2 follows immediately. That $\sim$ is an equivalence relation also follows.

Thus, one can simply use any irreducible component of $f(x,y)g(x,y)$ as $q(x,y)$ to identify an equivalence class of bars at any given height.

Let $\mathfrak{B} := \mathfrak{B}(f, g)$ denote the set of all equivalence classes of bars.

Take $\mathfrak{B} \in \mathfrak{B}$. If some $B \in \mathfrak{B}$ is collinear (resp. non-collinear) then every $\bar{B} \in \mathfrak{B}$ is collinear (resp. non-collinear); in this case we say $\mathfrak{B}$ is collinear (resp. non-collinear).

Let $C(\mathfrak{B})$, $N(\mathfrak{B})$ denote respectively the collinear and non-collinear classes of bars.

Let us take an integer $D$ so that $\lambda_k$, $1 \leq k \leq N$, can all be written in the form $[\bar{B}^\alpha]$.

Take $\theta$, $\theta^D = 1$. Take $z \in B$ (Convention [L3]). If $h(B) = \frac{\theta B}{B}$, then $\theta(z) = \theta^\alpha z$, i.e.,

\[ \theta(\lambda_B(y) + zy^{h(B)}) = \lambda_B(y) + \theta^\alpha zy^{h(B)}; \quad \bar{B} \sim B. \]

If $z_k \in C(B)$ (resp. $N(B)$), having bimultiplicity $[p_k, q_k]$, then $\bar{z}_k := \theta(z_k) \in C(\bar{B})$ (resp. $N(\bar{B})$), having bimultiplicity $[\bar{p}_k, \bar{q}_k] = [p_k, q_k]$. Hence

\[ \nu_f(B) = \nu_f(\bar{B}), \quad \nu_q(B) = \nu_q(\bar{B}). \]

Observe also that

\[ \mathcal{M}_B(\theta^n z) = \theta^{-n} \mathcal{M}_B(z), \]

whence $\theta$ induces a bijection between the pure mero-zeros of $B$ and $\bar{B}$, preserving the mero-multiplicity.

Now take $\mathfrak{B}$, non-collinear, and consider the product

\[ P_{\mathfrak{B}}(x,y) := \prod_j [x - \gamma_j(y)], \]

taking over all $j$ such that $\gamma_j$ leaves the tree on some $B \in \mathfrak{B}$.

**Lemma 6.3.** Take any non-collinear $\mathfrak{B} \in \mathfrak{B}$. Then $P_{\mathfrak{B}}(x,y) \in \mathbb{C}\{x,y\}$, and hence

\[ P(x,y) := \prod_{\mathfrak{B} \in N(\mathfrak{B})} P_{\mathfrak{B}}(x,y) \]

is a factor of $J_{(f,g)}(x,y)$ in $\mathbb{C}\{x,y\}$.

Take a polar root $\gamma$, leaving the tree on $B \in \mathfrak{B}$. Take an irreducible $p(x,y) \in \mathbb{C}\{x,y\}$ having $\gamma$ as a root. Every root of $p(x,y)$ leaves the tree on some $\bar{B} \in \mathfrak{B}$. Hence $P_{\mathfrak{B}}(x,y)$ is a product of factors like $p(x,y)$. This completes the proof.

Take $\mathfrak{B}$, non-collinear, $h(\mathfrak{B}) := h(B) > 0$, $B \in \mathfrak{B}$. Consider the product

\[ Q_{\mathfrak{B}}(x,y) := \prod_j [x - \gamma_j(y)], \]

taking over all $j$ such that $\gamma_j$ climbs over some $B \in \mathfrak{B}$ at some $c \in C(B)$, and is bounded by every bar of the cover of $c$. If $\mathfrak{B}$ is purely non-collinear, we define $Q_{\mathfrak{B}}(x,y) := 1$. 
As for the ground bar $B_1$, it may be collinear or not. Let
\[ Q_{B_1}(x, y) := \prod [x - \gamma_j(y)], \]
taking over all $j$ such that $\gamma_j$ is bounded by every non-collinear bar of minimal height.

**Theorem F.** The above defined $Q_B(x, y)$ and $Q_{B_1}(x, y)$ are in $\mathbb{C}\{x, y\}$, and hence
\[ J_{(f, g)}(x, y) = \text{unit} \cdot y^E \cdot Q_{B_1}(x, y) \cdot \prod_{B \in N(\mathbb{B})} P_B(x, y) \cdot Q_B(x, y), \quad E \geq 0, \]
h($\mathbb{B}$) > 0, is a decomposition in $\mathbb{C}\{x, y\}$.

7. Tree-invariants and Mero-invariants

In this section we analyse how the factorisation (6.2) is determined by the tree. For this we first define what is meant by a tree-invariant of the pair $(f, g)$. We say $(f, g)$ and $(f', g')$ are equivalent if the following two conditions are satisfied.

**Condition 1.** (Compare [17].) The roots of $f$ and $f'$, and of $g$ and $g'$, are in one-one correspondence: $\alpha_i \rightarrow \alpha'_i$, $1 \leq i \leq p$; $\beta_j \rightarrow \beta'_j$, $1 \leq j \leq q$. The contact order is preserved:
\[ O(\alpha_i, \alpha_k) = O(\alpha'_i, \alpha'_k), \quad O(\alpha_i, \beta_j) = O(\alpha'_i, \beta'_j), \quad O(\beta_j, \beta_s) = O(\beta'_j, \beta'_s). \]

This is equivalent to saying that there is a bijection between the bars and trunks of $T(f, g)$ and that of $T(f', g')$ which preserves the heights and bimultiplicities. If a bar $B$ corresponds to a bar $B'$, then $\nu_f(B) = \nu_{f'}(B')$ and $\nu_g(B) = \nu_{g'}(B')$ and hence $B$ is collinear iff so is $B'$.

In particular, if $B$ corresponds to $B'$, both non-collinear, then $C(B) \cup N(B)$ and $C(B') \cup N(B')$ are in one-one correspondence, say $z_k \rightarrow z'_k$, as in (2.1), such that
\[ \Delta_B(z_k) = \Delta_{B'}(z'_k), \quad 1 \leq k \leq l. \]

**Condition 2.** If $B$ corresponds to $B'$, non-collinear, then $m(B) = m(B')$; and if $c \in C(B)$ corresponds to $c' \in C(B')$ then $m_B(c) = m_{B'}(c')$.

We say that an object associated to $(f, g)$ is a tree-invariant of $(f, g)$ if it depends only on the equivalence class of $(f, g)$.

**Remark 7.1.** If $T$ is a trunk of bimultiplicity $[s, 0]$ or $[0, t]$. Let $B$ be the bar on top of $T$. Then Condition 2 for $B$ is satisfied automatically by Corollary [27].

Take $B \in \mathbb{B}$, non-collinear. Let $m^*(B)$ denote the number of pure mero-zeros on $B$ (counting multiplicities). Let
\[ m^*(\mathbb{B}) := \sum_{B \in \mathbb{B}} m^*(B); \quad \nu_f(\mathbb{B}) := \nu_f(B); \quad \nu_g(\mathbb{B}) := \nu_g(B). \]

Let $P_B$ denote the curve germ defined by $P_B(x, y) = 0$.

As a consequence of Condition 1, an equivalence between $T(f, g)$ and $T(f', g')$ induces a bijection between $\mathfrak{B}(f, g)$ and $\mathfrak{B}(f', g')$. Suppose $\mathbb{B}$ corresponds to $\mathbb{B}'$. Then $\mathbb{B}$ is non-collinear if, and only if $\mathbb{B}'$ is; in this case $m^*(\mathbb{B}) = m^*(\mathbb{B}')$, by Condition 2.
**Theorem I.** If $\mathbb{B}$ is non-collinear, then

$$I(C_f, P_B) = \nu_f(\mathbb{B})m^*(\mathbb{B}); \quad I(C_g, P_B) = \nu_g(\mathbb{B})m^*(\mathbb{B});$$

and

$$I(C, P_B) = [\nu_f(\mathbb{B}) + \nu_g(\mathbb{B})]m^*(\mathbb{B}),$$

where $C_f = f^{-1}(0)$, $C_g = g^{-1}(0)$, and $C = C_f \cup C_g$. These intersection multiplicities are tree-invariants of $(f, g)$. Similarly the orders (in $x$) of the factors in (6.2) are tree-invariants.

If a polar root leaves the tree on a non-collinear bar, it must do so at a pure mero-zero (Corollary 2.4). Hence the intersection multiplicities $I(C_f, P_B)$, $I(C_g, P_B)$ are tree-invariants. The above formulae follow from Corollary 2.5.

By Theorem C, the number of the $\gamma_j$'s in the definition of $Q_B(x, y)$ is a tree-invariant. This number is the order. The last statement of Theorem I is proved.

**Attention.** Example 1.1, as analysed in §5, shows that similar intersection multiplicities defined by $Q_B(x, y)$ need not be tree-invariants. We have no formula for them either.

**Addendum.** Take a non-collinear $\mathbb{B}$. Let

$$P_B(x, y) = p_1(x, y)^{e_1} \cdots p_s(x, y)^{e_s}, \quad e_i \geq 1,$$

be the irreducible decomposition of $P_B(x, y)$ in $\mathbb{C}\{x, y\}$. Then, clearly,

$$s \leq e_1 + \cdots + e_s \leq m^*(B), \quad B \in \mathbb{B}.$$

Note that $m^*(B)$ is a tree-invariant, but $s, e_i$ are not.

The following example shows that the Zariski equisingularity types ([17]) of $P_B(x, y)$ need not be tree-invariants.

**Example 7.2.** (Compare Example(4.4) in [10].) Take $g = g' = g'' = y$, and

$$f = x^3 - y^4, \quad f' = x^3 - y^4 - 3xy^5, \quad f'' = x^3 - y^4 - 3xy^6.$$

Note that $(f, g)$, $(f', g')$, and $(f'', g'')$ are equivalent. There is only one bar of height $\frac{4}{3}$ in each tree, for which

$$P_B(x, y) = x^2; \quad P_B'(x, y) = x^2 - y^5; \quad P_B''(x, y) = x^2 - y^6.$$

We now define the truncation relative to $T(f, g)$, $P_B^\top$, as follows. Take any arc $\xi$, not a root of $f \cdot g$, leaving the tree on $B$ at $a$. We call

$$\xi^\top(y) := \lambda_B(y) + ay^{h(B)}$$

the truncation of $\xi$ relative to $T(f, g)$. If $\xi$ is one of the roots $\lambda_k$, we define $\xi^\top := \xi$.

Take $h(x, y) \in \mathbb{C}\{x, y\}$, regular in $x$, say of order $r$, with factorisation

$$h(x, y) = \text{unit} \cdot \prod_{i=1}^r (x - \xi_i), \quad O(\xi_i) > 0.$$
We define the truncation of \( h(x, y) \) relative to \( T(f, g) \) by
\[
h^\top(x, y) := \prod_{i=1}^{r} (x - \xi_i^\top(y)).
\]
A simple conjugation argument shows that \( h^\top(x, y) \in \mathbb{C}\{x, y\} \). Note that in Example 7.2, \( P^\top_B = P^\top_B' = P^\top_B'' = x^2 \).

We say \((f, g)\) and \((f', g')\) are mero-equivalent if the following condition is also satisfied.

**Condition 3.** Suppose \( B \) corresponds to \( B' \), non-collinear. Then there exists a bijection between the pure mero-zeros of \( B \) and \( B' \), preserving the mero-multiplicity.

It is easy to see that in this case, \( P^\top_B \) and \( P^\top_{B'} \) are Zariski equisingular.

We say a function germ associated to \( T(f, g) \) is a mero-invariant if its Zariski equisingularity type depends only on the mero-equivalence class of \((f, g)\).

**Addendum.** In Theorem I the intersection multiplicities do not change when \( P_B \) is replaced by the curve germ defined by \( P^\top_B(x, y) = 0 \). The truncations \( P^\top_B(x, y) \) are mero-invariants.

8. Discussions

8.1. Let \( f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \) be given. We can take a generic constant \( c \) and substitute \( y \) by \( y + cx \). Then \( f, g \) are mini-regular in \( x \), that is, \( E_1 = E_2 = 0 \) and \( O(f) = p, O(g) = q \). In the factorisations \( \binom{11}{10} \), \( O(\alpha_i) \geq 1 \) and \( O(\beta_j) \geq 1 \). Since \( c \) is generic, \( J(x, y) \) is also mini-regular in \( x \), say of order \( m \). Hence there are exactly \( m \) polar roots, all of order \( \geq 1 \). These are called the "generic" polar roots.

The following example shows in particular that \( m \) need not be a tree-invariant.

**Example 8.1.** Take \( f(x, y) = x^2 - G(x, y)^2, g(x, y) = x - 2G(x, y) \), where
\[
G(x, y) := \int_{0}^{y} (x - t^2)^2 dt, \quad G(x, 0) = 0, \quad G(0, y) = \frac{1}{5}y^5.
\]
Then \( J(f, g) = -2(2x - G)G_y = -2(2x - G)(x - y^2)^2 \), having \( m = 3 \).

The ground bar is collinear. The other bar of \( T(f, g) \) is non-collinear, having height 5. Two of the three polar roots are bounded by this bar; they leave the tree at height 2.

Next, take \( f' = x^2 - G(x^2, y)^2, g' = x - 2G(x^2, y) \). Then \( T(f', g') = T(f, g) \), and
\[
J(f', g') = -2(2x - G(x^2, y))(x^2 - y^2)^2, \quad m = 5.
\]

Four polar roots are bounded by the bar of height 5; they leave the tree at height 1.
8.2. One function case. Our results generalise that of the one function case. Let \( f(x, y) \) be regular in \( x, g(x, y) = y \). Then \( J(x, y) = f_x, T(f, g) = T(f) \), the tree-model defined in (10). All bars are purely non-collinear (Corollary 2.7). Theorem T reduces to Lemma(3.3) of (10): if there are \( l \) trunks growing on \( B \), then there are \( l - 1 \) polar roots leaving the tree on \( B \).

Let us first suppose \( f(x, y) \) is irreducible (Merle’s case). Let \( \{ \frac{a_1}{p_1}, \frac{a_2}{p_2}, \ldots, \frac{a_g}{p_g} \} \) denote the Puiseux characteristic sequence of \( f \), as in (10) (p.308). The tree \( T(f) \) has a simple form (10). Since \( f \) is irreducible, all bars of the same height are obviously conjugate. Let \( \mathbb{B}_s \) denote the class of bars with height \( \frac{aq}{p_1 \cdots p_r}, 1 \leq s \leq g \).

Take \( \mathbb{B}_s \). Take any polar root, \( \gamma \), which leaves the tree on some \( B \in \mathbb{B}_s \). Note that \( \nu_f(B) = O(f(\gamma(y), y) \). There is a sequence of postbars, \( B_1 \perp B_2 \perp \cdots \perp B_s \), where \( B_s := B \), with heights \( h(B_k) = \frac{aq}{p_1 \cdots p_k}, 1 \leq k \leq s \).

Given \( k \), there are \( p_k \) trunks growing on \( B_k \), and \( p_k - 1 \) polar roots leaving the tree on \( B_k \).

Hence the total number of such \( \gamma \) is \( (p_s - 1)p_{s-1} \cdots p_1 \). It follows that

\[
f_x = \text{unit} \cdot \prod_{i=1}^{g} P_{\mathbb{B}_i}(x, y); \quad I(C_f, P_{\mathbb{B}_s}) = \nu_f(\mathbb{B}_s)[p_s - 1]p_{s-1} \cdots p_1.
\]

This is Merle’s theorem (14) (true also for non-generic polars).

Next consider the general case where \( f \) may be reducible (Garcia-Barroso’s case). Let \( H \) denote the set of contact orders, \( O(\alpha_i, \alpha_j) \), between the Newton-Puiseux roots \( \alpha_i \) of \( f \). Note that \( H \) is just the set of heights \( h(B) \), for all \( B \).

Take \( h \in H \). Unlike Merle’s case, the bars with the same height \( h \) may have more than one conjugate classes. Let them be denoted by \( \mathbb{B}^{(h)}_1, \ldots, \mathbb{B}^{(h)}_r \). Theorem F implies:

\[
f_x = \text{unit} \cdot \prod_{h \in H} \prod_{i=1}^{r(h)} P_{\mathbb{B}^{(h)}_i}(x, y).
\]

This is Garcia-Barroso’s theorem (14) (true also for non-generic polars). An equivalence class of bars is a ”black point” of Eggers (8), the height is the ”valuation”.

Example 7.2 also shows the Pham phenomenon in the one function case. Note that \( f, f', \) and \( f'' \) all have Puiseux exponent \( \frac{1}{3} \), hence \( T(f) = T(f') = T(f'') \). There are two polar roots in each tree. For \( f \), they generate the same factor \( x: f_x = 3x^2 \); for \( f' \), an irreducible factor: \( f'_x = 3(x^2 - y^3) \); and for \( f'' \), two distinct factors: \( f''_x = 3(x - y^3)(x + y^3) \).

This explains why in the factorisations of Merle, Garcia-Barroso and our Theorem F, the factors need not be invariants of the tree.

A brief summary: In the one function case, the factors of \( f_x \) in \( \mathbb{C}\{x, y\} \) are not invariants of \( T(f) \) (14), but the set of contact orders, \( C(f, f_x) \), is (14). For a general pair \((f, g)\), however, if there are collinear points or bars, then the contact order set need not be an invariant (14).

8.3. Meromorphic Case. One can easily extend our results to the meromorphic case to generalise the results of Assi, see (3) (and also (1) or (4)).

Let \( F(X, Y), G(X, Y) \) be a given pair of monic polynomials in \( X \):

\[
F(X, Y) = X^p + a_1(Y)X^{p-1} + \cdots + a_p(Y), \quad G(X, Y) = X^q + \cdots + b_q(Y),
\]

where \( a_i(Y), b_j(Y) \in \mathbb{C}(Y)^* \), the field of fractional power series of \( Y \). The Newton-Puiseux Theorem asserts that \( \mathbb{C}(Y)^* \) is algebraically closed (14), p.98).
For simplicity, let us assume \( a_i, b_j \) are Laurent series.

Take a large integer \( s \). We then use the substitution \( X = xy^{-s}, Y = y \), and define
\[
f(x, y) := F(xy^{-s}, y), \quad g(x, y) := G(xy^{-s}, y).
\]
Then, clearly, \( y^{\nu s}f(x, y) \) and \( y^{\nu s}g(x, y) \) are holomorphic and hence \( f \) and \( g \) can be factored in the form (0.1) with \( u = y^{-\nu s}, u' = y^{-\nu s}, \) (not units). Moreover,
\[
XYJ_{(F, G)}(X, Y) = xyJ_{(f, g)}(x, y);
\]
and \( X = \xi(Y) \) is a root of \( J(X, Y) \) if and only if \( x = y^\nu \xi(y) \) is one of \( J(x, y) \).

We have thus reduced the meromorphic case to the holomorphic case. As before, we can define the tree-model, \( T(f, g) \), the associated rational functions, etc. An important observation is that in this case, \( \nu_f(B), \nu_g(B) \) can be negative, or zero. As an example, let us take
\[
F(X, Y) = X^4 - Y^{-2}X^2 + 1, \quad G(X, Y) = X^2 - Y^{-1}X,
\]
with \( X = xy^{-2}, Y = y \). The tree \( T(f, g) \) has a bar, \( B \), with \( h(B) = 2, \nu_f(B) = \nu_g(B) = 0 \). A bar \( B \) with \( \nu_f(B) = \nu_g(B) = 0 \) is obviously collinear; thus, if \( B \) is also of maximal height, then any collinear point which lies below \( B \) has no cover.

Theorems T, N, and C remain true. Corollary(2.7), Corollary(2.8) and formula (2.14) remain true if we assume that for every bar, \( B' \), lying above \( B, \nu_f(B') \neq 0 \neq \nu_g(B') \).

Theorem(JF1), in \([1]\), (and the Theorem in \([2]\), Section 9, p9,) yields factors under the hypothesis \( \Omega_B(G) = 1 \) (the notation of \([1]\)). The contact set (defined in \([1]\), p.129, lines 5 and 21) is essentially the tree-model defined in \([1]\), a bud (\([1]\)) is essentially a bar in \([1]\); Theorem(DF1) of \([1]\) is Lemma(3.3) of \([1]\). In our terminology, the condition \( \Omega_B(G) = 1 \) means that there is no Newton-Puiseux root of \( G \) climbing over \( B \), i.e. \( B \) has bimultiplicity \([s, 0] \), so this case is not different to the one function case. In particular, (see Corollary \([27]\)) \( B \) is purely non-collinear and \( \tau(B) > 0 \), except (in the meromorphic case) when \( \nu_B(G) = 0 \), that is \( S(G, B) = 0 \) in the terminology of \([1]\). In general, however, the tree \( T(F \cdot G) \) of the product function can obviously have many “buds” \( B \) which do not have this property.

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