The optimal lower bound estimation of the number of closed geodesics on Finsler compact space form $S^{2n+1}/\Gamma$

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Abstract

Let $M = S^{2n+1}/\Gamma$, $\Gamma$ is a finite group which acts freely and isometrically on the $(2n+1)$-sphere and therefore $M$ is diffeomorphic to a compact space form. In this paper, we first investigate Katok’s famous example about irreversible Finsler metrics on the spheres to study the topological structure of the contractible component of the free loop space on the compact space form $M$, then we apply the result to establish the resonance identity for homologically visible contractible minimal closed geodesics on every Finsler compact space form $(M,F)$ when there exist only finitely many distinct contractible minimal closed geodesics on $(M,F)$. As its applications, using this identity and the enhanced common index jump theorem for symplectic paths proved by Duan, Long and Wang in [13], we show that there exist at least $2n+2$ distinct closed geodesics on every compact space form $S^{2n+1}/\Gamma$ with a bumpy irreversible Finsler metric $F$ under some natural curvature condition, which is the optimal lower bound due to Katok’s example.

Key words: Contractible closed geodesics; Resonance identity; Compact space forms; Morse theory; the enhanced common index jump theorem

AMS Subject Classification: 53C22, 58E05, 58E10.

1 Introduction

Let $M = S^{2n+1}/\Gamma$, $\Gamma$ is a finite group which acts freely and isometrically on the $(2n+1)$-sphere and therefore $M$ is diffeomorphic to a compact space form which is typically a non-simply connected

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manifold. In papers [44] and [27], the existence of at least two distinct non-contractible closed geodesics on every bumpy $S^n/\Gamma$ with $n \geq 2$ was proved, however, this paper is concerned with the total number of closed geodesics on Finsler $S^{2n+1}/\Gamma$, the main ingredients are the investigations of the topological structure of the contractible component of the free loop space on $S^{2n+1}/\Gamma$ which yields a new resonance identity for homologically visible contractible minimal closed geodesics on Finsler $S^{2n+1}/\Gamma$, and the enhanced common index jump theorem for symplectic paths discovered by Duan, Long and Wang in [13].

A closed curve on a Finsler manifold is a closed geodesic if it is locally the shortest path connecting any two nearby points on this curve. As usual, on any Finsler manifold $(M, F)$, a closed geodesic $c : S^1 = \mathbb{R}/\mathbb{Z} \to M$ is prime if it is not a multiple covering (i.e., iteration) of any other closed geodesics. Here the $m$-th iteration $c^m$ of $c$ is defined by $c^m(t) = c(mt)$. The inverse curve $c^{-1}$ of $c$ is defined by $c^{-1}(t) = c(1-t)$ for $t \in \mathbb{R}$. Note that unlike Riemannian manifold, the inverse curve $c^{-1}$ of a closed geodesic $c$ on an irreversible Finsler manifold need not be a geodesic. We call two prime closed geodesics $c$ and $d$ distinct if there is no $\theta \in (0, 1)$ such that $c(t) = d(t + \theta)$ for all $t \in \mathbb{R}$. For a closed geodesic $c$ on $(M, F)$, denote by $P_c$ the linearized Poincaré map of $c$. Recall that a Finsler metric $F$ is bumpy if all the closed geodesics on $(M, F)$ are non-degenerate, i.e., $1 \notin \sigma(P_c)$ for any closed geodesic $c$.

Let $\Lambda M$ be the free loop space on $M$ defined by

$$\Lambda M = \left\{ \gamma : S^1 \to M \mid \gamma \text{ is absolutely continuous and } \int_0^1 F(\gamma, \dot{\gamma})^2 dt < +\infty \right\},$$

endowed with a natural structure of Riemannian Hilbert manifold on which the group $S^1 = \mathbb{R}/\mathbb{Z}$ acts continuously by isometries (cf. Shen [42]).

It is well known (cf. Chapter 1 of Klingenberg [22]) that $c$ is a closed geodesic or a constant curve on $(M, F)$ if and only if $c$ is a critical point of the energy functional

$$E(\gamma) = \frac{1}{2} \int_0^1 F(\gamma, \dot{\gamma})^2 dt.$$

Based on it, many important results on this subject have been obtained (cf. [1], [18]-[19], [37]-[38]). In particular, in 1969 Gromoll and Meyer [17] used Morse theory and Bott’s index iteration formulae [7] to establish the existence of infinitely many distinct closed geodesics on $M$, when the Betti number sequence $\{\beta_k(\Lambda M; \mathbb{Q})\}_{k \in \mathbb{Z}}$ is unbounded. Then Vigué-Poirrier and Sullivan [50] further proved in 1976 that for a compact simply connected manifold $M$, the Gromoll-Meyer condition holds if and only if $H^*(M; \mathbb{Q})$ is generated by more than one element.
However, when \( \{\beta_k(\Lambda M; Q)\}_{k \in \mathbb{Z}} \) is bounded, the problem is quite complicated. In 1973, Katok [21] endowed some irreversible Finsler metrics to the compact rank one symmetric spaces

\[
S^n, \ \mathbb{R}P^n, \ \mathbb{C}P^n, \ \mathbb{H}P^n \text{ and } \text{CaP}^2,
\]
each of which possesses only finitely many distinct prime closed geodesics (cf. also Ziller [51],[52]). On the other hand, Franks [15] and Bangert [3] together proved that there are always infinitely many distinct closed geodesics on every Riemannian sphere \( S^2 \) (cf. also Hingston [19], Klingenberg [23]). These results imply that the metrics play an important role on the multiplicity of closed geodesics on those manifolds.

In 2004, Bangert and Long [6] (published in 2010) proved the existence of at least two distinct closed geodesics on every Finsler \( S^2 \). Subsequently, such a multiplicity result for \( S^n \) with a bumpy Finsler metric was proved by Duan and Long [10] and Rademacher [41] independently. Furthermore in a recent paper [12], Duan, Long and Wang proved the same conclusion for any compact simply-connected bumpy Finsler manifold, and in [13] they obtained the optimal lower bound estimation of the number of closed geodesics on any compact simply-connected bumpy Finsler manifold under some curvature conditions. We refer the readers to [11], [20], [33], [40]-[46] and the references therein for more interesting results and the survey papers of Long [32], Taimanov [43], Burns and Matveev [8] and Oancea [36] for more recent progresses on this subject.

As for the multiplicity of closed geodesics on non-simply connected manifolds whose free loop space possesses bounded Betti number sequence, Ballman et al. [2] proved in 1981 that every Riemannian manifold with the fundamental group being a nontrivial finitely cyclic group and possessing a generic metric has infinitely many distinct closed geodesics. In 1984, Bangert and Hingston [4] proved that any Riemannian manifold with the fundamental group being an infinite cyclic group has infinitely many distinct closed geodesics. Since then, there seem to be very few works on the multiplicity of closed geodesics on non-simply connected manifolds. The main reason is that the topological structures of the free loop spaces on these manifolds are not well known, so that the classical Morse theory is difficult to be applicable.

Motivated by the studies on simply connected manifolds, in particular, the resonance identity proved by Rademacher [37], and based on Westerland’s works [47], [48] on loop homology of \( \mathbb{R}P^n \), Xiao and Long [49] in 2015 investigated the topological structure of the non-contractible loop space and established the resonance identity for the non-contractible closed geodesics on \( \mathbb{R}P^{2n+1} \) by use of \( \mathbb{Z}_2 \) coefficient homology. As an application, Duan, Long and Xiao [14] proved the existence of at least two distinct non-contractible closed geodesics on \( \mathbb{R}P^3 \) endowed with a bumpy and irreversible
Finsler metric. Subsequently in [44], Taimanov used a quite different method from [49] to compute the rational equivariant cohomology of the non-contractible loop spaces in compact space forms $S^n/\Gamma$ and proved the existence of at least two distinct non-contractible closed geodesics on $\mathbb{R}P^2$ endowed with a bumpy irreversible Finsler metric. Then in [26], Liu combined Fadell-Rabinowitz index theory with Taimanov’s topological results to get many multiplicity results of non-contractible closed geodesics on positively curved Finsler $\mathbb{R}P^n$. In [28], Liu and Xiao established the resonance identity for the non-contractible closed geodesics on $\mathbb{R}P^n$, and together with [14] and [44] proved the existence of at least two distinct non-contractible closed geodesics on every bumpy $\mathbb{R}P^n$ with $n \geq 2$. Furthermore, Liu, Long and Xiao [27] proved that every bumpy Finsler compact space form $S^n/\Gamma$ possesses two distinct closed geodesics in each of its nontrivial classes.

Based on the works of [13] and [27], it is natural to ask whether we can improve the lower bound of the total number of closed geodesics on bumpy Finsler compact space forms under some natural conditions. Note that the only non-trivial group which acts freely on $S^{2n}$ is $\mathbb{Z}_2$ and $S^{2n}/\mathbb{Z}_2 = \mathbb{R}P^{2n}$ (cf. P.5 of [44]). In [16], Ginzburg, Gurel and Macarini have obtained the optimal lower bound of the total number of closed geodesics on bumpy Finsler $\mathbb{R}P^{2n}$, so the left interesting part about this problem is to estimate the lower bound of the total number of closed geodesics on bumpy Finsler compact space form $S^{2n+1}/\Gamma$. This paper is devoted to answer this question. To this end, we first investigate Katok’s example about irreversible Finsler metrics on the spheres and compute the $S^1$-equivariant Betti number sequence of the contractible component of the free loop space on the compact space form $S^{2n+1}/\Gamma$ in Section 4, and then in Section 5 we use this topological result to establish the following resonance identity.

**Theorem 1.1** Let $M = S^{2n+1}/\Gamma$ and $e$ be the identity in $\pi_1(M)$. Suppose the Finsler manifold $(M, F)$ possesses only finitely many distinct contractible minimal closed geodesics, among which we denote the distinct homologically visible contractible minimal closed geodesics by $c_1, \ldots, c_r$ for some integer $r > 0$, where a contractible closed geodesic $c$ is called minimal if it is not an iteration of any other contractible closed geodesics. Then we have

$$
\sum_{j=1}^{r} \frac{\hat{\chi}(c_j)}{i(c_j)} = B(\Lambda_e M, \Lambda^0 M; \mathbb{Q}) = \frac{n+1}{2n},
$$

where $\Lambda_e M$ is the contractible loop space of $M$, $\Lambda^0 M = \{\text{constant point curves in } M\} \cong M$ and the mean Euler number $\hat{\chi}(c_j)$ of $c_j$ is defined by

$$
\hat{\chi}(c_j) = \frac{1}{n_j} \sum_{m=1}^{n_j} \sum_{l=0}^{4n} (-1)^{l+i(c_j^m)} k_l c_j^m (c_j^m) \in \mathbb{Q},
$$
and $n_j = n_{c_j}$ is the analytical period of $c_j$, $k^c_j(c_j^m)$ is the local homological type number of $c_j^m$, $i(c_j)$ and $\hat{i}(c_j)$ are the Morse index and mean index of $c_j$ respectively.

In particular, if the Finsler metric $F$ on $M = S^{2n+1}/\Gamma$ is bumpy, then (1.1) has the following simple form

$$
\sum_{j=1}^{r} \left( \frac{(-1)^{i(c_j)}k^c_j(c_j) + (-1)^{i(c_j^2)}k^c_j(c_j^2)}{2} \right) \frac{1}{\hat{i}(c_j)} = \frac{n + 1}{2n}.
$$

(1.2)

Based on Theorem 1.1, we use Morse theory and the enhanced common index jump theorem for symplectic paths proved by Duan, Long and Wang in [13] to prove our main multiplicity results of contractible closed geodesics on $(S^{2n+1}/\Gamma, F)$.

**Theorem 1.2** Let $M = (S^{2n+1}/\Gamma, F)$ be a bumpy Finsler compact space form. If the number of contractible closed geodesics is finite and the flag curvature $K$ satisfies

$$
\left( \frac{\lambda_1 + \lambda_2}{1 + \lambda_1} \right)^2 < K \leq 1,
$$

then there exist at least $(2n + 2)$ distinct contractible closed geodesics with even Morse indices, and $2n$ ones of them are non-hyperbolic.

**Remark 1.1** Due to Katok’s famous example, the lower bound of the number of closed geodesics obtained in the above theorem is optimal, cf. [52] and Section 4 below.

This paper is organized as follows. In Section 2, we review the critical point theory of closed geodesics and apply the splitting theorem on critical modules to the contractible component of the free loop space of $(S^{2n+1}/\Gamma, F)$. Then in Section 3, we recall the precise iteration formulae and the enhanced common index jump theorem of symplectic paths, which also work for Morse indices of orientable closed geodesics. In Section 4, we investigate Katok’s famous example about irreversible Finsler metrics on the spheres to study the topological structure of the contractible component of the free loop space on the compact space form $S^{2n+1}/\Gamma$, then in Section 5 we apply the result to establish the resonance identity in Theorem 1.1. Finally in Section 6, we give the proof of our main Theorem 1.2.

In this paper, let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{Q}^c$ denote the sets of natural integers, non-negative integers, integers, rational numbers and irrational numbers respectively. We also use notations $E(a) = \min\{k \in \mathbb{Z} | k \geq a\}$, $\lfloor a \rfloor = \max\{k \in \mathbb{Z} | k \leq a\}$, $\varphi(a) = E(a) - \lfloor a \rfloor$ and $\{a\} = a - \lfloor a \rfloor$ for any $a \in \mathbb{R}$. Throughout this paper, we use $\mathbb{Q}$ coefficients for all homological and cohomological modules.
2 Critical point theory for closed geodesics

Let $M = (M, F)$ be a compact Finsler manifold, the space $\Lambda = \Lambda M$ of $H^1$-maps $\gamma : S^1 \to M$ has a natural structure of Riemannian Hilbert manifolds on which the group $\gamma \in \Lambda$ acts continuously by isometries. This action is defined by $(s \cdot \gamma)(t) = \gamma(t + s)$ for all $\gamma \in \Lambda$ and $s, t \in S^1$. For any $\gamma \in \Lambda$, the energy functional is defined by

$$E(\gamma) = \frac{1}{2} \int_{S^1} F(\gamma(t), \gamma'(t))^2 dt.$$  \hspace{1cm} (2.1)

It is $C^{1,1}$ and invariant under the $S^1$-action. The critical points of $E$ of positive energies are precisely the closed geodesics $\gamma : S^1 \to M$. The index form of the functional $E$ is well defined along any closed geodesic $c$ on $M$, which we denote by $E''(c)$. As usual, we denote by $i(c)$ and $\nu(c)$ the Morse index and nullity of $E$ at $c$. For a closed geodesic $c$ we set $\Lambda(c) = \{ \gamma \in \Lambda \mid E(\gamma) < E(c) \}$.

Recall that respectively the mean index $\bar{i}(c)$ and the $S^1$-critical modules of $c^m$ are defined by

$$\bar{i}(c) = \lim_{m \to \infty} \frac{i(c^m)}{m}, \quad \mathcal{C}_*(E, c^m) = H_*((\Lambda(c^m) \cup S^1 \cdot c^m)/S^1, \Lambda(c^m)/S^1; \mathbb{Q}).$$  \hspace{1cm} (2.2)

We call a closed geodesic $c$ satisfying the isolation condition, if the following holds:

**Lemma 2.1** (cf. Satz 6.11 of [38] and [6]) Suppose $c$ is a prime closed geodesic on a Finsler manifold $M$ satisfying (Iso). Then there exist $U_{c^m}$ and $N_{c^m}$, the so-called local negative disk and the local characteristic manifold at $c^m$ respectively, such that $\nu(c^m) = \dim N_{c^m}$ and

$$\mathcal{C}_d(E, c^m) = H_q((\Lambda(c^m) \cup S^1 \cdot c^m)/S^1, \Lambda(c^m)/S^1)$$

$$= (H_{i(c^m)}(U_{c^m} \cup \{c^m\}, U^-_{c^m}) \otimes H_{q-i(c^m)}(N_{c^m} \cup \{c^m\}, N_{c^m}))/\mathbb{Z}_m.$$

(i) When $\nu(c^m) = 0$, there holds

$$\mathcal{C}_q(E, c^m) = \begin{cases} \mathbb{Q}, & \text{if } i(c^m) - i(c) \in 2\mathbb{Z} \text{ and } q = i(c^m), \\ 0, & \text{otherwise}, \end{cases}$$

(ii) When $\nu(c^m) > 0$, there holds

$$\mathcal{C}_q(E, c^m) = H_{q-i(c^m)}(N_{c^m} \cup \{c^m\}, N_{c^m})\epsilon(c^m)\mathbb{Z}_m,$$

where $\epsilon(c^m) = (-1)^{i(c^m) - i(c)}$. 

\hspace{1cm} 6
In the following, we let $M = S^{2n+1}/\Gamma$ and $e$ be the identity in $\pi_1(M)$, $\Gamma$ acts freely and isometrically on the $(2n + 1)$-sphere and therefore $M$ is diffeomorphic to a compact space form. Then the free loop space $\Lambda M$ possesses a natural decomposition

$$\Lambda M = \bigsqcup_{g \in \pi_1(M)} \Lambda_g M,$$

where $\Lambda_g M$ is the connected components of $\Lambda M$ whose elements are homotopic to $g$. We set $\Lambda_e(c) = \{\gamma \in \Lambda_e M \mid E(\gamma) < E(c)\}$. Note that for a contractible minimal closed geodesic $c$, $c^m \in \Lambda_e M$ for every $m \in \mathbb{N}$. Here a contractible closed geodesic $c$ is called minimal if it is not an iteration of any other contractible closed geodesics.

Now we restrict the energy functional $E$ on the contractible component $\Lambda_e M$ and study the Morse theory on $\Lambda_e M$. Thus we define the $S^1$-critical modules of $c^m$ for $E|_{\Lambda_e M}$ as

$$\overline{C}_q(E, c^m; [e]) = H_q\left(\left(\Lambda_e(c^m) \cup S^1 \cdot c^m\right)/S^1, \Lambda_e(c^m)/S^1; \mathbb{Q}\right).$$

Then by the same proof of Lemma 2.1 we have:

**Proposition 2.1** Suppose $c$ is a contractible minimal closed geodesic on Finsler $M = S^{2n+1}/\Gamma$ satisfying (Iso). Then there exist $U^-_{c^m}$ and $N_{c^m}$, the so-called local negative disk and the local characteristic manifold at $c^m$ respectively, such that $\nu(c^m) = \dim N_{c^m}$ and

$$\overline{C}_q(E, c^m; [e]) = H_q\left(\left(\Lambda_e(c^m) \cup S^1 \cdot c^m\right)/S^1, \Lambda_e(c^m)/S^1\right) = (H_{i(c^m)}(U^-_{c^m} \cup \{c^m\}, U^-_{c^m}) \otimes H_{q-i(c^m)}(N_{c^m} \cup \{c^m\}, N_{c^m}))^{\mathbb{Z} m},$$

(i) When $\nu(c^m) = 0$, there holds

$$\overline{C}_q(E, c^m; [e]) = \begin{cases} \mathbb{Q}, & \text{if } i(c^m) - i(c) \in 2\mathbb{Z} \text{ and } q = i(c^m), \\ 0, & \text{otherwise}, \end{cases}$$

(ii) When $\nu(c^m) > 0$, there holds

$$\overline{C}_q(E, c^m; [e]) = H_{q-i(c^m)}(N_{c^m} \cup \{c^m\}, N_{c^m})\epsilon(c^m)^{\mathbb{Z} m},$$

where $\epsilon(c^m) = (-1)^{i(c^m) - i(c)}$.

As usual, for $m \in \mathbb{N}$ and $l \in \mathbb{Z}$ we define the local homological type numbers of $c^m$ by

$$k_l^{\epsilon(c^m)}(c^m) = \dim H_l(N_{c^m} \cup \{c^m\}, N_{c^m})\epsilon(c^m)^{\mathbb{Z} m}. \quad (2.3)$$
Based on works of Rademacher in [37], Long and Duan in [33] and [11], we define the *analytical period* $n_c$ of the closed geodesic $c$ by

$$n_c = \min \{ j \in \mathbb{N} \mid \nu(c^j) = \max_{m \geq 1} \nu(c^m), \ i(c^{m+j}) - i(c^m) \in 2\mathbb{Z}, \forall m \in \mathbb{N} \}. \quad (2.4)$$

Then by [33] and [11], we have

$$k_l^{(c^{m+hnc})} = k_l^{(c^m)}, \quad \forall m, h \in \mathbb{N}, l \in \mathbb{Z}. \quad (2.5)$$

For more detailed properties of the analytical period $n_c$ of a closed geodesic $c$, we refer readers to the two Section 3s in [33] and [11].

### 3 The enhanced common index jump theorem of symplectic paths

In [29] of 1999, Y. Long established the basic normal form decomposition of symplectic matrices. Based on this result he further established the precise iteration formulae of indices of symplectic paths in [30] of 2000. Note that this index iteration formulae works for Morse indices of iterated closed geodesics (cf. [25], [24] and Chap. 12 of [31]). Since every closed geodesic on odd dimensional Finsler manifolds is orientable, then by Theorem 1.1 of [24] the initial Morse index of a closed geodesic $c$ on a Finsler compact space form $S^{2n+1}/\Gamma$ coincides with the index of a corresponding symplectic path.

As in [30], denote by

$$N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \quad \text{for } \lambda = \pm 1, \ b \in \mathbb{R}, \quad (3.1)$$

$$D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \text{for } \lambda \in \mathbb{R} \setminus \{0, \pm 1\}, \quad (3.2)$$

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \text{for } \theta \in (0, \pi) \cup (\pi, 2\pi), \quad (3.3)$$

$$N_2(e^{i\sqrt{-1}}, B) = \begin{pmatrix} R(\theta) & B \\ 0 & R(\theta) \end{pmatrix}, \quad \text{for } \theta \in (0, \pi) \cup (\pi, 2\pi) \text{ and}$$

$$B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \text{ with } b_j \in \mathbb{R}, \text{ and } b_2 \neq b_3. \quad (3.4)$$

Here $N_2(e^{i\sqrt{-1}}, B)$ is non-trivial if $(b_2 - b_3) \sin \theta < 0$, and trivial if $(b_2 - b_3) \sin \theta > 0$. 

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As in [30], the $\odot$-sum (direct sum) of any two real matrices is defined by

\[
\begin{pmatrix}
A_1 & B_1 \\
C_1 & D_1
\end{pmatrix}_{2i \times 2i} \odot \begin{pmatrix}
A_2 & B_2 \\
C_2 & D_2
\end{pmatrix}_{2j \times 2j} = \begin{pmatrix}
A_1 & 0 & B_1 & 0 \\
0 & A_2 & 0 & B_2 \\
C_1 & 0 & D_1 & 0 \\
0 & C_2 & 0 & D_2
\end{pmatrix}.
\] (3.5)

For every $M \in \text{Sp}(2n)$, the homotopy set $\Omega(M)$ of $M$ in $\text{Sp}(2n)$ is defined by

\[
\Omega(M) = \{N \in \text{Sp}(2n) \mid \sigma(N) \cap U = \sigma(M) \cap U \equiv \Gamma \text{ and } \nu_\omega(N) = \nu_\omega(M) \forall \omega \in \Gamma\},
\]

where $\sigma(M)$ denotes the spectrum of $M$, $\nu_\omega(M) \equiv \text{dim}_C \ker_C(M - \omega I)$ for $\omega \in U$. We denote by $\Omega^0(M)$ the path connected component of $\Omega(M)$ containing $M$.

**Lemma 3.1** (cf. [30], Lemma 9.1.5 and List 9.1.12 of [31]) For $M \in \text{Sp}(2n)$ and $\omega \in U$, the splitting number $S^\pm_M(\omega)$ (cf. Definition 9.1.4 of [31]) satisfies

\[
S^\pm_M(\omega) = 0, \quad \text{if } \omega \notin \sigma(M).
\] (3.6)

\[
S^\pm_{N_1(1,a)}(1) = \begin{cases} 
1, & \text{if } a \geq 0, \\
0, & \text{if } a < 0.
\end{cases}
\] (3.7)

For any $M_i \in \text{Sp}(2n_i)$ with $i = 0$ and 1, there holds

\[
S^\pm_{M_0 \odot M_1}(\omega) = S^\pm_{M_0}(\omega) + S^\pm_{M_1}(\omega), \quad \forall \omega \in U.
\] (3.8)

For every $\gamma \in \mathcal{P}_\tau(2n) \equiv \{\gamma \in C([0, \tau], \text{Sp}(2n)) \mid \gamma(0) = I_{2n}\}$, we extend $\gamma(t)$ to $t \in [0, m\tau]$ for every $m \in \mathbb{N}$ by

\[
\gamma^m(t) = \gamma(t - j\tau)\gamma(\tau)^j \quad \forall \ j \tau \leq t \leq (j + 1)\tau \quad \text{and} \quad j = 0, 1, \ldots, m - 1,
\] (3.9)

as in P.114 of [29]. As in [35] and [31], we denote the Maslov-type indices of $\gamma^m$ by $(i(\gamma, m), \nu(\gamma, m))$.

Then the following decomposition theorem and precise iteration formula for symplectic paths are proved in [29] and [30].

**Theorem 3.1** (cf. Theorem 1.8.10, Lemma 2.3.5 and Theorem 8.3.1 of [31]) For every $M \in \text{Sp}(2n)$, there exists a continuous path $f \in \Omega^0(M)$ such that $f(0) = M$ and

\[
f(1) = N_1(1, 1)^{op-} \odot I_{2p_0} \odot N_1(1, -1)^{op+} \odot N_1(-1, 1)^{op-} \odot (-I_{2q_0}) \odot N_1(-1, -1)^{op+} \odot N_2(e^{\alpha_1\sqrt{-1}}, A_1) \odot \cdots \odot N_2(e^{\alpha_r\sqrt{-1}}, A_r) \odot N_2(e^{\beta_1\sqrt{-1}}, B_1) \odot \cdots \odot N_2(e^{\beta_{r_0}\sqrt{-1}}, B_{r_0}) \odot R(\theta_1) \odot \cdots \odot R(\theta_r) \odot D(\pm 2)\gamma^m,
\] (3.10)
More precisely, by (4.10), (4.40) and (4.41) in [35], we have 

where \( \theta_1/2\pi \in (0, 1) \) for \( 1 \leq j \leq r \); \( N_2(e^{\alpha_j \sqrt{-1}}, A_j) \)'s are nontrivial and \( N_2(e^{\beta_j \sqrt{-1}}, B_j) \)'s are trivial, and non-negative integers \( p_-, p_0, p_+, q_-, q_0, q_+, r, r_*, r_0, h \) satisfy 

\[
p_- + p_0 + p_+ + q_- + q_0 + q_+ + r + 2r_* + 2r_0 + h = n.
\]

Let \( \gamma \in \mathcal{P}_\tau(2n) = \{ \gamma \in C([0, \tau], \text{Sp}(2n)) \mid \gamma(0) = I \} \). Denote the basic normal form decomposition of \( M \equiv \gamma(\tau) \) by (3.10). Then we have 

\[
i(\gamma^m) = m(i(\gamma) + p_- + p_0 - r) + 2 \sum_{j=1}^r E \left( \frac{m\theta_j}{2\pi} \right) - r
\]

\[
- p_- - p_0 - \frac{1}{2}(-1)^m(q_0 + q_+) + 2 \sum_{j=1}^{r_*} \varphi \left( \frac{m\alpha_j}{2\pi} \right) - 2r_*.
\]

The common index jump theorem (cf. Theorem 4.3 of [35]) for symplectic paths established by Long and Zhu in 2002 has become one of the main tools to study the multiplicity and stability problems of closed solution orbits in Hamiltonian and symplectic dynamics. Recently, the following enhanced version of it has been obtained by Duan, Long and Wang in [13], which will play an important role in the proofs in Section 6.

**Theorem 3.2 (cf. Theorem 3.5 of [13])** (The enhanced common index jump theorem for symplectic paths) Let \( \gamma_k \in \mathcal{P}_{\tau_k}(2n) \) for \( k = 1, \cdots, q \) be a finite collection of symplectic paths. Let \( M_k = \gamma(\tau_k) \). We extend \( \gamma_k \) to \([0, +\infty) \) by (3.9) inductively. Suppose 

\[
i(\gamma_k, 1) > 0, \quad \forall \, k = 1, \cdots, q.
\]

Then for every integer \( \bar{m} \in \mathbb{N} \), there exist infinitely many \( (q + 1) \)-tuples \( (N, m_1, \cdots, m_q) \in \mathbb{N}^{q+1} \) such that for all \( 1 \leq k \leq q \) and \( 1 \leq m \leq \bar{m} \), there holds 

\[
\nu(\gamma_k, 2m_k - m) = \nu(\gamma_k, 2m_k + m) = \nu(\gamma_k, m), \quad (3.12)
\]

\[
i(\gamma_k, 2m_k + m) = 2N + i(\gamma_k, m), \quad (3.13)
\]

\[
i(\gamma_k, 2m_k - m) = 2N - i(\gamma_k, m) - 2(S_{M_k}^+(1) + Q_k(m)), \quad (3.14)
\]

\[
i(\gamma_k, 2m_k) = 2N - (S_{M_k}^+(1) + C(M_k) - 2\Delta_k), \quad (3.15)
\]

where 

\[
\Delta_k = \sum_{0 < \{m_k \theta_k/\pi\} \leq \delta} S_{M_k}^- (e^{\sqrt{-1} \theta_k}), \quad Q_k(m) = \sum_{e^{\sqrt{-1} \theta_k} \in \sigma(M_k), \{m_k \theta_k/\pi\} = \{0\}} S_{M_k}^- (e^{\sqrt{-1} \theta_k}). \quad (3.16)
\]

More precisely, by (4.10), (4.40) and (4.41) in [35], we have 

\[
m_k = \left( \left\lfloor \frac{N}{Mi(\gamma_k, 1)} \right\rfloor + \chi_k \right) \bar{M}, \quad 1 \leq k \leq q,
\]

\[10\]
where $\chi_k = 0$ or $1$ for $1 \leq k \leq q$ and $\frac{M_\theta}{\bar{M}} \in \mathbb{Z}$ whenever $e^\frac{-\theta}{\bar{M}} \in \sigma(M_k)$ and $\frac{\theta}{\bar{M}} \in \mathbb{Q}$ for some $1 \leq k \leq q$. Furthermore, given $M_0 \in \mathbb{N}$, by (iv) of Remark 3.6 of [13], we may further require $M_0|N$, and by (4.20) in Theorem 4.1 of [35], for any $\epsilon > 0$, we can choose $N$ and $\{\chi_k\}_{1 \leq k \leq q}$ such that

$$\left| \left\{ \frac{N}{M i(\gamma_k, 1)} \right\} - \chi_k \right| < \epsilon, \quad 1 \leq k \leq q. \quad (3.18)$$

### 4 Katok’s metrics on spheres and compact space forms

In this section, we compute the $S^1$-equivariant Betti number sequence of $(\Lambda_e M, \Lambda^0 M)$ via Katok’s famous metrics on $S^n$, where $M = S^{2n+1}/\Gamma$, $\Lambda^0 M = \{\text{constant point curves in } M\} \cong M$ and $\Lambda_e M$ is the contractible component of the free loop space on $M$.

In 1973, Katok [21] constructed his famous irreversible Finsler metrics on $S^n$ which possess only finitely many distinct prime closed geodesics. His examples were further studied closely by Ziller [52] in 1982, from which we borrow most of the notations.

For $S^{2n-1}$ with the canonical Riemannian metric $g$, any closed one-parameter group of isometries is conjugate to a diagonal matrix

$$\phi_t = \text{diag}(R(pt/p_1), \ldots , R(pt/p_n)),$$

where $p_i \in \mathbb{Z}, p = p_1 \cdots p_n$, the $p_i$ are relatively prime and $R(\omega)$ is a rotation in $\mathbb{R}^2$ with angle $\omega$. For $S^{2n}$ with the canonical Riemannian metric $g$, the same is true if the matrix is enlarged by one row and one column with a 1 in the diagonal. Let $T S^n$ and $T^* S^n$ be its tangent bundle and cotangent bundle respectively. Define $H_0, H_1 : T^* S^n \to \mathbb{R}$ by

$$H_0(x) = \|x\|_* \quad \text{and} \quad H_1(x) = x(V), \forall x \in T^* S^n,$$

where $\| \cdot \|_*$ denotes the dual norm of $g$ and $V$ is the vector field generated by $\phi_t$. Let

$$H_\alpha = H_0 + \alpha H_1 \quad \text{for} \quad \alpha \in (0, 1).$$

Then $\frac{1}{2} H_\alpha^2$ is homogeneous of degree two and the Legendre transform

$$L \frac{1}{2} H_\alpha^2 = DF \left( \frac{1}{2} H_\alpha^2 \right) : T^* S^n \to TS^n$$

is a global diffeomorphism. Hence,

$$N_\alpha = H_\alpha \circ L^{-1} \frac{1}{2} H_\alpha^2.$$
defines a Finsler metric on \( S^n \). Since \( H_\alpha(-x) \neq H_\alpha(x) \), \( N_\alpha \) is not reversible. It was proved that \((S^n, N_\alpha)\) with \( \alpha \in (0,1) \setminus \mathbb{Q} \) possesses precisely \( 2\left[\frac{n+1}{2}\right] \) distinct prime closed geodesics and all of them are irrationally elliptic(cf. Katok [21] and pp.137-139 of Ziller [52] for more details).

Consider a finite group \( \Gamma \) which acts freely and isometrically on \((S^n, g)\). For \((S^{2n-1}, g)\), any element \( h \) of \( \Gamma \) has the form

\[
h = \text{diag}(R(\alpha_1), \cdots, R(\alpha_n))
\]

for some \( \alpha_i \in [0, 2\pi) \), where \( 1 \leq i \leq n \). Note that the only non-trivial group which acts freely on \( S^{2n} \) is \( \mathbb{Z}_2 \)(cf. P.5 of [44]), then \( h \) is the identity or the antipodal map for \((S^n, g)\). Thus \( h \circ \phi_t = \phi_t \circ h \) for any \((S^n, g)\).

**Lemma 4.1** For any element \( h \in \Gamma \), \( h : (S^n, N_\alpha) \to (S^n, N_\alpha) \) is an isometry.

**Proof:** For any \( x \in T_p^* S^n \), we have \( H_\alpha \circ (h^{-1})^*(x) = H_\alpha(x) \). In fact, we have \((h^{-1})^*(x) \in T_{h(p)}^* S^n \) and

\[
H_\alpha \circ (h^{-1})^*(x) = \| (h^{-1})^*(x) \|_* + \alpha ((h^{-1})^*(x))(V_{h(p)}) \\
= \| x \|_* + \alpha x ((h^{-1})^*(V_{h(p)})) \\
= \| x \|_* + \alpha x (V_p) \\
= H_\alpha(x) \quad (4.1)
\]

where the second identity is due to the fact that \( h \) is isomorphic for the canonical metric \( \| \cdot \|_* \) and the third identity is due to the fact that \( h_*(V_p) = V_{h(p)} \) which follows by the definition of \( V \) and \( h \circ \phi_t = \phi_t \circ h \).

To prove that \( h \) is an isometry of \((S^n, N_\alpha)\), i.e.,

\[
H_\alpha \circ L_{\frac{1}{2}H_\alpha}^{-1} \circ h_*(X) = H_\alpha \circ L_{\frac{1}{2}H_\alpha}^{-1} (X), \forall X \in T_p S^n,
\]

it is sufficient by (4.1) to prove

\[
L_{\frac{1}{2}H_\alpha}^{-1} \circ h_*(X) = (h^{-1})^* \circ L_{\frac{1}{2}H_\alpha}^{-1} (X), \forall X \in T_p S^n. \quad (4.2)
\]

In fact, (4.2) is equivalent to

\[
h_* \circ L_{\frac{1}{2}H_\alpha}^{-1} (x) = L_{\frac{1}{2}H_\alpha}^{-1} \circ (h^{-1})^*(x), \forall x \in T_p S^n. \quad (4.3)
\]
Note that for any \( x \in T_p^n S^n \),

\[
L_{\frac{1}{2}\alpha}^2(x) = DF\left( \frac{1}{2} H_\alpha^2 \right)(x)
\]

\[
= H_\alpha(x) \cdot DF(H_\alpha)(x)
\]

\[
= H_\alpha(x)(\star x/\| \star x \| + \alpha V_p)
\]

where \( \star x \) is the canonical identification of \( x \)(cf. p. 143 in [52]). Thus we get

\[
L_{\frac{1}{2}\alpha}^2 \circ (h^{-1})^*(x) = H_\alpha \circ (h^{-1})^*(x) \cdot ((h^{-1})^* x) \| \star ((h^{-1})^* x) \| + \alpha V_h(p))
\]

\[
= H_\alpha(x)((h^{-1})^* x) \| \star ((h^{-1})^* x) \| + \alpha V_h(p))
\]

\[
= H_\alpha(x)(h_\star(\star x)/\| h_\star(\star x) \| + \alpha V_h(p))
\]

\[
= H_\alpha(x)(h_\star(\star x)/\| \star x \| + \alpha V_h(p))
\]

\[
= h_\star \circ L_{\frac{1}{2}\alpha}^2(x),
\]

where the second identity is due to (4.1), the third identity is due to the fact that \( \star ((h^{-1})^* x) = h_\star(\star x) \), the fourth identity is due to the fact that \( h \) is isomorphic for the canonical metric \( \| \cdot \| \) and the fifth identity is due to the fact that \( h_\star(V_p) = V_h(p) \). Then (4.3), and then (4.2) as well as Lemma 4.1 are proved.

By Lemma 4.1, we can endow \( S^n / \Gamma \) a Finsler metric induced by \( N_\alpha \), which is still denoted by \( N_\alpha \) for simplicity. Therefore the natural projection

\[
\pi : (S^n, N_\alpha) \to (S^n / \Gamma, N_\alpha)
\]

is locally isometric.

In order to compute the \( S^1 \)-equivariant Betti number sequence of \( (\Lambda \epsilon M, \Lambda^0 M) \) for \( M = S^{2n+1} / \Gamma \), we review the \( S^1 \)-equivariant Betti numbers of the free loop space pair \( (\Lambda S^{2n+1}, \Lambda^0 S^{2n+1}) \).

**Lemma 4.2** (cf. Theorem 2.4 and Remark 2.5 of [37] and [18], cf. also Lemma 2.5 of [11]) Let \( (S^{2n+1}, F) \) be a \( (2n + 1) \)-dimensional Finsler sphere. Then the \( S^1 \)-equivariant Betti numbers of \( (\Lambda S^{2n+1}, \Lambda^0 S^{2n+1}) \) are given by

\[
b_j \equiv \text{rank} H_j(\Lambda S^{2n+1}/S^1, \Lambda^0 S^{2n+1}/S^1; \mathbb{Q}) = \begin{cases} 2, & \text{if } j \in \mathcal{K} \equiv \{ 2nk \mid 2 \leq k \in \mathbb{N} \}, \\ 1, & \text{if } j \in \{ 2n + 2k \mid k \in \mathbb{N}_0 \} \setminus \mathcal{K}, \\ 0 & \text{otherwise}. \end{cases} \quad (4.4)
\]

Combining Lemma 4.2 with the fact that \( \pi : (S^n, N_\alpha) \to (S^n / \Gamma, N_\alpha) \) is locally isometric, we can give the \( S^1 \)-equivariant Betti numbers of \( (\Lambda \epsilon M, \Lambda^0 M) \) as follows.
Proposition 4.1 For \( M = S^{2n+1}/\Gamma \), the \( S^1 \)-equivariant Betti numbers of \((\Lambda_e M, \Lambda^0 M)\) are given by

\[
\beta_j \equiv \text{rank} \, H_j(\Lambda_e M/S^1, \Lambda^0 M/S^1; \mathbb{Q}) = \begin{cases} 
2, & \text{if } j \in K \equiv \{2nk \mid 2 \leq k \in \mathbb{N} \}, \\
1, & \text{if } j \in \{2n+2k \mid k \in \mathbb{N}_0 \} \setminus K, \\
0, & \text{otherwise}.
\end{cases} \tag{4.5}
\]

and the average \( S^1 \)-equivariant Betti number of \((\Lambda_e M, \Lambda^0 M)\) satisfies

\[
\bar{B}(\Lambda_e M, \Lambda^0 M; \mathbb{Q}) \equiv \lim_{q \to +\infty} \frac{1}{q} \sum_{k=0}^{q} (-1)^k \beta_k = \frac{n+1}{2n}. \tag{4.6}
\]

Proof: Since the Betti numbers \( \beta_j \) are topological invariants of \( M \), they are independent of the choice of the Finsler metric \( F \) on it. To estimate them, it suffices to choose a special Finsler metric \( F = N\alpha \) for \( \alpha \in (0, 1) \setminus \mathbb{Q} \), i.e., the Katok metrics.

Since all prime closed geodesics on \((S^{2n+1}, N\alpha)\) are irrationally elliptic, denoted by \( \{c_j\}_{j=1}^{2n+2} \).

By the decomposition in Theorem 3.1, the linearized Poincaré map \( P_{c_j} \) can be connected to \( f_{c_j}(1) \) in \( \Omega^0(P_{c_j}) \) satisfying

\[
f_{c_j}(1) = R(\theta_{j1}) \circ \cdots \circ R(\theta_{j(2n)}), \quad 1 \leq j \leq 2n+2, \tag{4.7}
\]

where \( \frac{\theta_{jk}}{2\pi} \in \mathbb{Q}^c \) for \( 1 \leq j \leq 2n+2 \) and \( 1 \leq k \leq 2n \). Then by Theorem 3.1, we obtain

\[
i(c_j) \in 2\mathbb{Z}, \quad 1 \leq j \leq 2n+2, \tag{4.8}
\]

\[
i(c_j^m) = m(i(c_j) - 2n) + 2 \sum_{k=1}^{2n} \left\lfloor \frac{m\theta_{jk}}{2\pi} \right\rfloor + 2n, \quad \nu(c_j^m) = 0, \quad \forall \ m \geq 1, \tag{4.9}
\]

where (4.8) follows from Theorem 8.1.7 of [31] and the symplectic additivity of iterated indices.

By (4.8) and (4.9), there holds

\[
i(c_j^m) \in 2\mathbb{Z}, \quad \forall \ 1 \leq j \leq 2n+2, \quad m \geq 1. \tag{4.10}
\]

Define

\[
M_p = \sum_{j=1}^{2n+2} M_p(j) \equiv \sum_{j=1}^{2n+2} \# \{m \geq 1 \mid i(c_j^m) = p, \ \overline{C}_p(E, c_j^m) \neq 0 \}, \quad p \in \mathbb{Z}.
\]

Then the following Morse inequality (cf. Theorem I.4.3 of [9]) holds

\[
M_p \geq b_p, \tag{4.11}
\]

\[
M_p - M_{p-1} + \cdots + (-1)^p M_0 \geq b_p - b_{p-1} + \cdots + (-1)^p b_0, \quad \forall \ p \in \mathbb{N}_0. \tag{4.12}
\]
where the $b'_j$s are given in Lemma 4.2.

By Lemma 2.1 and (4.10) we obtain

$$M_p = \sum_{j=1}^{2n+2} M_p(j) = \sum_{j=1}^{2n+2} \#\{m \geq 1 \mid i(c^m_j) = p\}, \quad \forall \ p \in \mathbb{Z}, \quad \text{(4.13)}$$

which together with (4.10), yields

$$M_p = b_p = 0, \quad \forall \ p = 1 \text{ (mod2)}, \ p \in \mathbb{N}_0. \quad \text{(4.14)}$$

Then by the Morse inequality (4.12), we obtain

$$M_p = b_p, \quad \forall \ p = 0 \text{ (mod2)}, \ p \in \mathbb{N}_0. \quad \text{(4.15)}$$

On the other hand, the set of contractible closed geodesics on $(M, N_\alpha) = (S^{2n+1}/\Gamma, N_\alpha)$ is

$$\{c^m_j \mid 1 \leq j \leq 2n+2, m \in \mathbb{N}\}$$

since the closed geodesics $c'_j$s found in [52] are great circles of $S^{2n+1}$.

Then $c_j$ is a contractible minimal closed geodesic on $(S^{2n+1}/\Gamma, N_\alpha)$. Note that (4.10) yields

$$i(c^m_j) - i(c_j) \in 2\mathbb{Z}, \quad \forall \ 1 \leq j \leq 2n+2, \ m \geq 1. \quad \text{(4.16)}$$

Combining (4.16) with Proposition 2.1, we know the Morse type numbers of the contractible component of the free loop space on $(S^{2n+1}/\Gamma, N_\alpha)$ are just the Morse type numbers $M_p'$. Then we can apply Morse inequality to the contractible component of the free loop space on $(S^{2n+1}/\Gamma, N_\alpha)$ and obtain

$$M_p \geq \beta_p, \quad \text{(4.17)}$$

$$M_p - M_{p-1} + \cdots + (-1)^p M_0 \geq \beta_p - \beta_{p-1} + \cdots + (-1)^p \beta_0, \quad \forall \ p \in \mathbb{N}_0. \quad \text{(4.18)}$$

Combining (4.14) with (4.17), we have

$$\beta_p = 0, \quad \forall \ p = 1 \text{ (mod2)}, \ p \in \mathbb{N}_0. \quad \text{(4.19)}$$

Then by the Morse inequality (4.18), we obtain

$$M_p = \beta_p, \quad \forall \ p = 0 \text{ (mod2)}, \ p \in \mathbb{N}_0, \quad \text{(4.20)}$$

which together with (4.15) yields

$$\beta_p = b_p, \quad \forall \ p = 0 \text{ (mod2)}, \ p \in \mathbb{N}_0. \quad \text{(4.21)}$$

It together with (4.19) and Lemma 4.2 completes our proofs. □
5 Resonance identity for contractible closed geodesics on \( (S^{2n+1}/\Gamma, F) \)

In this section, we apply Proposition 4.1 to obtain the resonance identity for homologically visible contractible minimal closed geodesics on a Finsler \( M = (S^{2n+1}/\Gamma, F) \) claimed in Theorem 1.1, provided the number of all distinct contractible minimal closed geodesics on \( M \) is finite. Note that if the number of all distinct contractible minimal closed geodesics on \( M \) is finite, then the total number of all distinct closed geodesics on \( M \) is finite.

Firstly we have

**Definition 5.1** Let \((M, F)\) be a compact Finsler manifold. A contractible closed geodesic \( c \) on \( M \) is homologically visible, if there exists an integer \( k \in \mathbb{Z} \) such that \( \bar{C}_k(E, c; [e]) \neq 0 \). We denote by \( \text{CG}_{hv}(M, F) \) the set of all distinct homologically visible contractible minimal closed geodesics on \((M, F)\).

**Proof of Theorem 1.1.** Recall that we denote the homologically visible contractible minimal closed geodesics by \( \text{CG}_{hv}(M) = \{c_1, \ldots, c_r\} \) for some integer \( r > 0 \) when the number of distinct contractible minimal closed geodesics on \( M = (S^{2n+1}/\Gamma, F) \) is finite.

Note also that we have \( \hat{i}(c_j) > 0 \) for all \( 1 \leq j \leq r \), otherwise there is a homologically visible contractible closed geodesic \( c \) on \( M \) satisfying \( \hat{i}(c) = 0 \). Then \( i(c^m) = 0 \) for all \( m \in \mathbb{N} \) by Bott iteration formula and \( c \) must be an absolute minimum of the energy functional \( E \) in its free homotopy class, since otherwise there would exist infinitely many distinct closed geodesics on \( M \) by Theorem 3 on p.385 of [5]. It also follows that this homotopy class must be non-trivial, which contradicts to that \( c \) is contractible.

Denote the contractible component of the free loop space on \( M \) by \( \Lambda_eM \). Then the set of contractible closed geodesics on \( M = (S^{2n+1}/\Gamma, F) \) is \( \{c^m_j \mid 1 \leq j \leq r, m \in \mathbb{N}\} \).

Let

\[
M_q \equiv M_q(\Lambda_eM, \Lambda^0M) = \sum_{1 \leq j \leq r, m \geq 1} \dim \overline{C}_q(E, c^m_j; [e]), \quad q \in \mathbb{Z}.
\]  

The Morse series of \((\Lambda_eM, \Lambda^0M)\) is defined by

\[
M(t) = \sum_{h=0}^{+\infty} M_h t^h.
\]

**Claim 1.** \( \{m_h\} \) is a bounded sequence.

In fact, by (2.5), we have

\[
M_h = \sum_{j=1}^{r} \sum_{m=1}^{n_j} \sum_{l=0}^{4n} k_{l}^e(c^m_j) (c^m_j) ^{#} \left\{ s \in \mathbb{N} \cup \{0\} \mid h - i(c^m_j+sn_j) = l \right\},
\]
and by Theorems 10.1.2 of [31], we have \(|i(c_j^{m+sn_j}) - (m + sn_j)\hat{i}(c_j)| \leq 2n\), then

\[
\# \left\{ s \in \mathbb{N} \cup \{0\} \mid h - i(c_j^{m+sn_j}) = l \right\} \\
= \# \left\{ s \in \mathbb{N} \cup \{0\} \mid l + i(c_j^{m+sn_j}) = h, \ |i(c_j^{m+sn_j}) - (m + sn_j)\hat{i}(c_j)| \leq 2n \right\} \\
\leq \# \left\{ s \in \mathbb{N} \cup \{0\} \mid 2n \geq |h - l - (m + sn_j)\hat{i}(c_j)| \right\} \\
= \# \left\{ s \in \mathbb{N} \cup \{0\} \mid \frac{h - l - 2n - m\hat{i}(c_j)}{n_j\hat{i}(c_j)} \leq s \leq \frac{h - l + 2n - m\hat{i}(c_j)}{n_j\hat{i}(c_j)} \right\} \\
\leq \frac{4n}{n_j\hat{i}(c_j)} + 1. \tag{5.4}
\]

Hence Claim 1 follows by (5.3) and (5.4).

We now use the method in the proof of Theorem 5.4 of [34] to estimate

\[M^q(-1) = \sum_{h=0}^{q} M_h(-1)^h.\]

By (5.2) and (2.5) we obtain

\[
M^q(-1) = \sum_{h=0}^{q} M_h(-1)^h \\
= \sum_{j=1}^{r} \sum_{m=1}^{n_j} \sum_{l=0}^{4n} \sum_{h=0}^{q} (-1)^h k_l^L(c_j)(c_j^m) \# \left\{ s \in \mathbb{N} \cup \{0\} \mid h - i(c_j^{m+sn_j}) = l \right\} \\
= \sum_{j=1}^{r} \sum_{m=1}^{n_j} \sum_{l=0}^{4n} (-1)^{l+i(c_j^m)} k_l^L(c_j^m) \# \left\{ s \in \mathbb{N} \cup \{0\} \mid l + i(c_j^{m+sn_j}) \leq q \right\}.
\]

On the one hand, we have

\[
\# \left\{ s \in \mathbb{N} \cup \{0\} \mid l + i(c_j^{m+sn_j}) \leq q \right\} \\
= \# \left\{ s \in \mathbb{N} \cup \{0\} \mid l + i(c_j^{m+sn_j}) \leq q, \ |i(c_j^{m+sn_j}) - (m + sn_j)\hat{i}(c_j)| \leq 2n \right\} \\
\leq \# \left\{ s \in \mathbb{N} \cup \{0\} \mid 0 \leq (m + sn_j)\hat{i}(c_j) \leq q - l + 2n \right\} \\
= \# \left\{ s \in \mathbb{N} \cup \{0\} \mid 0 \leq s \leq \frac{q - l + 2n - m\hat{i}(c_j)}{n_j\hat{i}(c_j)} \right\} \\
\leq \frac{q - l + 2n}{n_j\hat{i}(c_j)} + 1.
\]
On the other hand, we have

\[
\begin{align*}
\# \left\{ s \in \mathbb{N} \cup \{0\} \mid l + i(c_j^{m+sn_j}) \leq q \right\} &= \# \left\{ s \in \mathbb{N} \cup \{0\} \mid l + i(c_j^{m+sn_j}) \leq q, \ |i(c_j^{m+sn_j}) - (m + sn_j)i(c_j)| \leq 2n \right\} \\
&\geq \# \left\{ s \in \mathbb{N} \cup \{0\} \mid i(c_j^{m+sn_j}) \leq (m + sn_j)i(c_j) + 2n \leq q - l \right\} \\
&\geq \# \left\{ s \in \mathbb{N} \cup \{0\} \mid 0 \leq s \leq \frac{q - l - 2n - mi(c_j)}{nj(i(c_j))} \right\} \\
&\geq \frac{q - l - 2n}{nj(i(c_j))} - 1.
\end{align*}
\]

Thus we obtain

\[
\lim_{q \to +\infty} \frac{1}{q} M^q(-1) = \sum_{j=1}^{r} \sum_{m=1}^{4n} \sum_{l=0}^{4n} (-1)^{l+i(c_j^m)} k_i^{\epsilon(c_j^m)}(c_j^m) \frac{1}{nj(i(c_j))} = \sum_{j=1}^{r} \frac{\hat{\chi}(c_j)}{i(c_j)}.
\]

Since \( m_h \) is bounded, by Morse inequality (cf. Theorem I.4.3 of [9]) we then obtain

\[
\lim_{q \to +\infty} \frac{1}{q} M^q(-1) = \lim_{q \to +\infty} \frac{1}{q} \sum_{k=0}^{q} (-1)^k \beta_k = \bar{B}(\Lambda e M, \Lambda^0 M; \mathbb{Q}),
\]

Thus by (4.6) we get

\[
\sum_{j=1}^{r} \frac{\hat{\chi}(c_j)}{i(c_j)} = \frac{n + 1}{2n},
\]

which proves (1.1) of Theorem 1.1. For the special case when each \( c_j^m \) is non-degenerate with \( 1 \leq j \leq r \) and \( m \in \mathbb{N} \), we have \( n_j = 2 \) and \( k_i^{\epsilon(c_j^m)}(c_j^m) = 1 \) when \( l = 0 \) and \( i(c_j^m) - i(c_j) \in 2\mathbb{Z} \), and \( k_i^{\epsilon(c_j^m)}(c_j^m) = 0 \) for all other \( l \in \mathbb{Z} \). Then (1.1) has the following simple form

\[
\sum_{j=1}^{r} \left( \frac{(-1)^{i(c_j)} k_i^{\epsilon(c_j)}(c_j) + (-1)^{i(c_j^2)} k_i^{\epsilon(c_j^2)}(c_j^2)}{2} \right) \frac{1}{i(c_j)} = \frac{n + 1}{2n},
\]

which proves (1.2) of Theorem 1.1. \( \square \)

### 6 Proof of Theorem 1.2

In order to prove Theorem 1.2, let \( M = (S^{2n+1}/\Gamma, F) \) with a bumpy, irreversible Finsler metric \( F \).

We make the following assumption

**\textbf{(FCCG)}** Suppose that there exist only finitely many contractible minimal closed geodesics \( \{c_k\}_{k=1}^{q} \) with \( i(c_k) \geq 2 \) for \( k = 1, 2, \cdots, q \) on \( M \).
Since the flag curvature $K$ of $(S^{2n+1}/\Gamma, F)$ satisfies $\left(\frac{\lambda}{\lambda+1}\right)^2 < K \leq 1$ by assumption, then every contractible closed geodesic $c_k$ must satisfy

$$i(c_k) \geq 2n \geq 2,$$

by Theorem 3 and Lemma 3 of [39]. Thus (FCCG) holds.

**Step 1** There exist at least $2n$ distinct non-hyperbolic contractible closed geodesics, all of which possess even Morse indices.

Since by the assumption (FCCG), there exist only finitely many contractible minimal closed geodesics on the bumpy manifold $M$, any closed geodesic $c_k$ among $\{c_k\}_{k=1}^q$ must have positive mean index (cf. Section 5), i.e.,

$$\hat{i}(c_k) > 0, \quad 1 \leq k \leq q,$$

which implies that $i(c^m_k) \to +\infty$ as $m \to +\infty$. So the positive integer $\bar{m}$ defined by

$$\bar{m} = \max_{1 \leq k \leq q} \left\{ \min\{m_0 \in \mathbb{N} \mid i(c^m_{k+l}) \geq i(c^l_k), \quad \forall \ l \geq 1, m \geq m_0\} \right\}$$

is well-defined and finite.

For the integer $\bar{m}$ defined in (6.2), it follows from Theorem 3.2 that there exist infinitely many $q+1$-tuples $(N, m_1, \cdots, m_q) \in \mathbb{N}^{q+1}$ such that for any $1 \leq k \leq q$, there holds

$$\bar{m} + 2 \leq \min\{2m_k, 1 \leq k \leq q\},$$

$$i(c^{2m_k-m}_k) = 2N - i(c^m_k), \quad 1 \leq m \leq \bar{m},$$

$$i(c^{2m_k}_k) = 2N - C(M_k) + 2\Delta_k,$$

$$i(c^{2m_k+m}_k) = 2N + i(c^m_k), \quad 1 \leq m \leq \bar{m},$$

where $M_k = P_{c_k} \in Sp(4m)$ is the linearized Poincaré map of the contractible minimal closed geodesic $c_k$. Here note that in the bumpy case, by (3.16) we have $S^+_{M_k}(1) = 0$ and $Q_k(m) = 0, \forall \ m \geq 1$.

On one hand, there holds $i(c^m_k) \geq i(c_k)$ for any $m \geq 1$ by the Bott-type formulae (cf. [7] and Theorem 9.2.1 of [31]). Thus by (6.4), (6.6) and the assumption $i(c_k) \geq 2$ for $1 \leq k \leq q$ in the theorem, it yields

$$i(c^{2m_k-m}_k) = 2N - i(c^m_k) \leq 2N - i(c_k) \leq 2N - 2, \quad 1 \leq m \leq \bar{m},$$

$$i(c^{2m_k+m}_k) = 2N + i(c^m_k) \geq 2N + i(c_k) \geq 2N + 2, \quad 1 \leq m \leq \bar{m},$$
On the other hand, by the definition $(6.2)$ of $\bar{m}$ and $(6.4)$, $(6.6)$, for $2m_k > m \geq \bar{m} + 1$, we obtain
\[ i(c_k^{2m_k - m}) \leq i(c_k^{2m_k - 1}) = 2N - i(c_k) \leq 2N - 2, \quad (6.9) \]
\[ i(c_k^{2m_k + m}) \geq i(c_k^{2m_k + 1}) = 2N + i(c_k) \geq 2N + 2. \quad (6.10) \]

In summary, by (6.4)-(6.10), for $1 \leq k \leq q$, we have proved
\[ i(c_k^{2m_k - m}) \leq 2N - 2, \quad 1 \leq m < 2m_k, \quad (6.11) \]
\[ i(c_k^{2m_k}) = 2N - C(M_k) + 2\Delta_k, \quad (6.12) \]
\[ i(c_k^{2m_k + m}) \geq 2N + 2, \quad \forall m \geq 1. \quad (6.13) \]

**Claim 1:** For $N \in \mathbb{N}$ in Theorem 3.2 satisfying $(6.11)$-$(6.13)$ and $2N \tilde{B}(\Lambda_e M, \Lambda^0 M; \mathbb{Q}) \in \mathbb{N}$, we have
\[ \sum_{1 \leq k \leq q} 2m_k \tilde{\chi}(c_k) = 2N \tilde{B}(\Lambda_e M, \Lambda^0 M; \mathbb{Q}). \quad (6.14) \]

In fact, let $\epsilon < \frac{1}{1 + 2M \sum_{1 \leq k \leq q} |\tilde{\chi}(c_k)|}$, by Theorem 1.1 and (3.17)-(3.18), it yields
\[ \left| 2N \tilde{B}(\Lambda_e M, \Lambda^0 M; \mathbb{Q}) - \sum_{k=1}^{q} 2m_k \tilde{\chi}(c_k) \right| = \left| \sum_{k=1}^{q} \frac{2N \tilde{\chi}(c_k)}{i(c_k)} - \sum_{k=1}^{q} 2\tilde{\chi}(c_k) \left( \frac{N}{M i(c_k)} \right) + \chi_k \right| M \]
\[ \leq 2M \sum_{k=1}^{q} |\tilde{\chi}(c_k)| \left( \frac{N}{M i(c_k)} \right) - \chi_k \right| \quad < \quad 2M \epsilon \sum_{k=1}^{q} |\tilde{\chi}(c_k)| \quad < \quad 1, \]
which proves Claim 1 since each $2m_k \tilde{\chi}(c_k)$ is an integer.

Now by Proposition 2.1, it yields
\[ \sum_{m=1}^{2m_k} (-1)^{i(c_k^m)} \dim \overline{C}_{i(c_k^m)}(E, c_k^m ; [e]) \]
\[ = \sum_{i=0}^{m_k - 1} \sum_{m=2i+1}^{2i+2} (-1)^{i(c_k^m)} \dim \overline{C}_{i(c_k^m)}(E, c_k^m ; [e]) \]
\[ = \sum_{i=0}^{m_k - 1} \sum_{m=1}^{2} (-1)^{i(c_k^m)} \dim \overline{C}_{i(c_k^m)}(E, c_k^m ; [e]) \]
\[ = m_k \sum_{m=1}^{2} (-1)^{i(c_k^m)} \dim \overline{C}_{i(c_k^m)}(E, c_k^m ; [e]) \]
\[ = 2m_k \tilde{\chi}(c_k), \quad \forall 1 \leq k \leq q, \quad (6.15) \]
where the second equality follows from Proposition 2.1 and the fact \(i(c_k^{m+2}) - i(c_k^m) \in 2\mathbb{Z}\) for all \(m \geq 1\) from Theorem 3.1, and the last equality follows from Proposition 2.1, (1.2) and (2.3).

So, for \(1 \leq k \leq q\), by (6.11), (6.13) and Proposition 2.1, we know that all \(c_k^{2m_k-m}\)'s with \(2m_k > m \geq 1\) only have contributions to the alternating sum \(\sum_{p=0}^{2N-2} (-1)^p M_p\), and all \(c_k^{2m_k+m}\)'s with \(m \geq 1\) have no contributions to \(\sum_{p=0}^{2N+1} (-1)^p M_p\).

Thus for the Morse-type numbers \(M_p\)'s in (5.1), by (6.15) we have

\[
\sum_{p=0}^{2N+1} (-1)^p M_p = \sum_{k=1}^{q} \sum_{1 \leq m \leq 2m_k} (-1)^i(c_k^m) \dim \overline{C}_{i(c_k^m)}(E, c_k^m; [\varepsilon])
\]

\[
= \sum_{k=1}^{q} \sum_{m=1}^{2m_k} (-1)^i(c_k^m) \dim \overline{C}_{i(c_k^m)}(E, c_k^m; [\varepsilon])
- \sum_{1 \leq k \leq q} (-1)^i(c_k^{2m_k}) \dim \overline{C}_{i(c_k^{2m_k})}(E, c_k^{2m_k}; [\varepsilon])
= \sum_{k=1}^{q} 2m_k \hat{\chi}(c_k)
- \sum_{1 \leq k \leq q} (-1)^i(c_k^{2m_k}) \dim \overline{C}_{i(c_k^{2m_k})}(E, c_k^{2m_k}; [\varepsilon]). \tag{6.16}
\]

In order to exactly know whether the iterate \(c_k^{2m_k}\) of \(c_k\) has contribution to the alternative sum \(\sum_{p=0}^{2N+1} (-1)^p M_p\), \(1 \leq k \leq q\), we set

\[
N_+^e = \# \{ 1 \leq k \leq q \mid i(c_k^{2m_k}) \geq 2N + 2, \ i(c_k^{2m_k}) - i(c_k) \in 2\mathbb{Z}, \ i(c_k) \in 2\mathbb{Z} \}, \tag{6.17}
\]

\[
N_+^o = \# \{ 1 \leq k \leq q \mid i(c_k^{2m_k}) \geq 2N + 2, \ i(c_k^{2m_k}) - i(c_k) \in 2\mathbb{Z}, \ i(c_k) \in 2\mathbb{Z} - 1 \}, \tag{6.18}
\]

\[
N_-^e = \# \{ 1 \leq k \leq q \mid i(c_k^{2m_k}) \leq 2N - 2, \ i(c_k^{2m_k}) - i(c_k) \in 2\mathbb{Z}, \ i(c_k) \in 2\mathbb{Z} \}, \tag{6.19}
\]

\[
N_-^o = \# \{ 1 \leq k \leq q \mid i(c_k^{2m_k}) \leq 2N - 2, \ i(c_k^{2m_k}) - i(c_k) \in 2\mathbb{Z}, \ i(c_k) \in 2\mathbb{Z} - 1 \}. \tag{6.20}
\]

Here note that \(\overline{B}(\Lambda_\varepsilon M, \Lambda^0 M; \mathbb{Q}) = \frac{n+1}{2n}\), and by Theorem 3.2, we can suppose that \(N\) is a multiple of \(n\). Thus by Claim 1, (6.16), Proposition 2.1, the definitions of \(N_+^e\) and \(N_+^o\) and Morse inequality, we have

\[
\frac{N(n+1)}{n} + N_+^o - N_+^e = \sum_{k=1}^{q} 2m_k \hat{\chi}(c_k) - (-N_+^o + N_+^e) = \sum_{p=0}^{2N+1} (-1)^p M_p \leq \sum_{p=0}^{2N+1} (-1)^p \beta_p = \frac{N(n+1)}{n} - n, \tag{6.21}
\]

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where the Betti numbers $\beta_p$’s are given by (4.5). Then

$$N_+^e \geq n.$$  \hfill (6.22)

Similar to (6.11)-(6.13), it follows from Theorem 3.2 that there exist also infinitely many $(q+1)$-tuples $(N', m_1', \cdots, m_q') \in \mathbb{N}^{q+1}$ such that for any $1 \leq k \leq q$, there holds

$$i(c_k^{2m'_k-m}) \leq 2N' - 2, \quad 1 \leq m < 2m'_k, \quad (6.23)$$

$$i(c_k^{2m'_k}) = 2N' - C(M_k) + 2\Delta'_k, \quad (6.24)$$

$$i(c_k^{2m'_k+m}) \geq 2N' + 2, \quad \forall m \geq 1. \quad (6.25)$$

where, furthermore, $\Delta_k$ and $\Delta'_k$ satisfy the following relationship

$$\Delta'_k + \Delta_k = C(M_k), \quad \forall 1 \leq k \leq q, \quad (6.26)$$

which exactly follows from (especially the term (c) of) Theorem 4.2 of [35] (cf. [13] for more details).

Similarly, we define

$$N_+^{e^*} = \# \{1 \leq k \leq q \mid i(c_k^{2m'_k}) \geq 2N' + 2, \quad i(c_k^{2m'_k}) - i(c_k) \in 2\mathbb{Z}, \quad i(c_k) \in 2\mathbb{Z} \}, \quad (6.27)$$

$$N_+^{o^*} = \# \{1 \leq k \leq q \mid i(c_k^{2m'_k}) \geq 2N' + 2, \quad i(c_k^{2m'_k}) - i(c_k) \in 2\mathbb{Z}, \quad i(c_k) \in 2\mathbb{Z} - 1 \}, \quad (6.28)$$

$$N_-^{e^*} = \# \{1 \leq k \leq q \mid i(c_k^{2m'_k}) \leq 2N' - 2, \quad i(c_k^{2m'_k}) - i(c_k) \in 2\mathbb{Z}, \quad i(c_k) \in 2\mathbb{Z} \}, \quad (6.29)$$

$$N_-^{o^*} = \# \{1 \leq k \leq q \mid i(c_k^{2m'_k}) \leq 2N' - 2, \quad i(c_k^{2m'_k}) - i(c_k) \in 2\mathbb{Z}, \quad i(c_k) \in 2\mathbb{Z} - 1 \}. \quad (6.30)$$

So by (6.24) and (6.26) it yields

$$i(c_k^{2m'_k}) = 2N' - C(M_k) + 2(C(M_k) - \Delta_k) = 2N' + C(M_k) - 2\Delta_k. \quad (6.31)$$

So by definitions (6.17)-(6.20) and (6.27)-(6.30) we have

$$N_+^e = N_+^{e^*}, \quad N_+^o = N_+^{o^*}. \quad (6.32)$$

Thus, through carrying out the similar arguments to (6.21)-(6.22), by Claim 1, Proposition 2.1, the definitions of $N_+^{e^*}$ and $N_+^{o^*}$ and Morse inequality, we have

$$\frac{N'(n + 1)}{n} + N_+^{o^*} - N_+^{e^*} = \sum_{k=1}^{q} 2m'_k \hat{\chi}(c_k) - (-N_+^{o^*} + N_+^{e^*})$$

$$= \sum_{p=0}^{2N'+1} (-1)^p M_p$$

$$\leq \sum_{p=0}^{2N'+1} (-1)^p \beta_p = \frac{N'(n + 1)}{n} - n, \quad (6.33)$$
which, together with (6.32), implies
\[ N_+^e = N_+^{e'} \geq n. \quad (6.34) \]

So by (6.22) and (6.34) it yields
\[ q \geq N_+^e + N_-^e \geq 2n. \quad (6.35) \]

In addition, any hyperbolic contractible closed geodesic \( c_k \) must have \( i(c_k^{2m_k}) = 2N \) since there holds \( C(M_k) = 0 \) and \( \Delta_k = 0 \) in the hyperbolic case. However, by (6.17) and (6.19), there exist at least \((N_+^e + N_-^e)\) contractible closed geodesic with even indices \( i(c_k^{2m_k}) \geq 2N + 2 \) or \( i(c_k^{2m_k}) \leq 2N - 2 \). So all these \((N_+^e + N_-^e)\) contractible closed geodesics are non-hyperbolic. Then (6.35) shows that there exist at least \(2n\) distinct non-hyperbolic contractible closed geodesics. And (6.17), (6.19) and (6.35) show that all these non-hyperbolic contractible closed geodesics and their iterations have even Morse indices. This completes the proof of Step 1.

We denote these \(2n\) non-hyperbolic contractible closed geodesics by \( \{c_k\}_{k=1}^{2n} \).

**Step 2.** There exist at least two distinct contractible closed geodesics different from those found in Step 1 with even Morse indices.

In fact, for those \(2n\) distinct contractible closed geodesics \( \{c_k\}_{k=1}^{2n} \) found in Step 1, there holds \( i(c_k^{2m_k}) \neq 2N \) by the definitions of \( N_+^e \) and \( N_-^e \), which, together with (6.11) and (6.13) yields
\[ i(c_k^{m}) \neq 2N, \quad m \geq 1, \quad k = 1, \ldots, 2n. \quad (6.36) \]

Then by Proposition 2.1 it yields
\[ \sum_{1 \leq k \leq 2n \atop m \geq 1} \dim C_{2N}(E, c_k^m; [e]) = 0. \quad (6.37) \]

Therefore, noting that \(N\) is a multiple of \(n\), by (6.37), (4.5) of Proposition 4.1 and Morse inequality, we obtain
\[
\sum_{2n+1 \leq k \leq q} \dim C_{2N}(E, c_k^{2m_k}; [e]) = \sum_{2n+1 \leq k \leq q \atop m \geq 1} \dim C_{2N}(E, c_k^m; [e]) = \sum_{1 \leq k \leq q \atop m \geq 1} \dim C_{2N}(E, c_k^m; [e]) = M_{2N} \geq \beta_{2N} = 2. \quad (6.38)
\]

where the first equality follows from (6.11), (6.13) and Proposition 2.1.

Now by (6.38) and Proposition 2.1, it yields that there exist at least two contractible closed geodesic \(c_{2n+1}\) and \(c_{2n+2}\) with \( i(c_k^{2m_k}) = 2N \) and \( i(c_k^{2m_k}) - i(c_k) \in 2\mathbb{Z} \) for \( k = 2n + 1 \) and \( 2n + 2 \).
Thus both $c_{2n+1}$ and $c_{2n+2}$ are different from $\{c_k\}_{k=1}^{2n}$ by (6.36), they and all of their iterates have even Morse indices. This completes the proof of Step 2.

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