ENDOMORPHISM ALGEBRAS OF JACOBIANS OF CERTAIN SUPERELLIPTIC CURVES

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1. Introduction

Let $K$ be a field of characteristic zero, $\overline{K}$ its algebraic closure. We use $X, Y, Z$ to denote smooth algebraic varieties over $\overline{K}$. If $X$ is an abelian variety over $\overline{K}$, we write $\text{End}(X)$ for its ring of absolute endomorphisms and $\text{End}^0(X)$ for its endomorphism algebra $\text{End}(X) \otimes \mathbb{Q}$. Given an abelian variety $Y$ over $\overline{K}$, we write $\text{Hom}(X, Y)$ for the group of all $\overline{K}$-homomorphisms from $X$ to $Y$.

Let $p \in \mathbb{N}$ be a prime, $q = p^r$, $n = mp^s \geq 5$ where $m, r, s \in \mathbb{N}$ and $p \nmid m$.

Let $f(x) \in K[x]$ be a polynomial of degree $n$ without multiple roots. We write $\mathcal{R}_f = \{ \alpha_i \}_{1 \leq i \leq n}$ for the set of roots of $f(x)$ in $\overline{K}$, and $\text{Gal}(f)$ for the Galois group $\text{Gal}(K(\mathcal{R}_f)/K)$. The Galois group may be viewed as a certain permutation group of $\mathcal{R}_f$, i.e., as a subgroup of the group of permutations $\text{Perm}(\mathcal{R}_f) \cong S_n$.

Let $\zeta_q \in \overline{K}$ be a primitive $q$-th root of unity, and

$$P_q(x) = x^q - 1 + \ldots + 1 \in \mathbb{Z}[x].$$

Then $P_q(x) = \prod_{i=1}^r \Phi_{p^i}(x)$, where $\Phi_{p^i}$ denote the $p^i$-th cyclotomic polynomial.

Hence $\mathbb{Q}[x]/P_q(x)\mathbb{Q}[x]$ is isomorphic to $\prod_{i=1}^r \mathbb{Q}(\zeta_{p^i})$, a direct product of cyclotomic fields.

Let $C_{f,q}$ be a smooth projective model of the affine curve $y^q = f(x)$. We denote by $\delta_q$ the nontrivial periodic automorphism of $C_{f,q}$:

$$\delta_q : C_{f,q} \to C_{f,q}, \quad (x, y) \mapsto (x, \zeta_q y).$$

It follows from Albanese functoriality that $\delta_q$ induces an automorphism of the Jacobian $J(C_{f,q})$ of $C_{f,q}$, and by an abuse of notation we also denote the induced automorphism of $J(C_{f,q})$ by $\delta_q$. It will be shown (Lemma 2.7) that $P_q(x)$ is the minimal polynomial of $\delta_q$ over $\mathbb{Z}$ in $\text{End}(J(C_{f,q}))$. This gives rise to an embedding

$$\prod_{i=1}^r \mathbb{Q}(\zeta_{p^i}) \cong \mathbb{Q}[t]/P_q(t)\mathbb{Q}[t] \cong \mathbb{Q}[\delta_q] \subseteq \text{End}^0(J(C_{f,q})), \quad t \mapsto \delta_q.$$ 

We prove that if $\text{Gal}(f)$ is large enough, then the embedding above is actually an isomorphism.

**Theorem 1.1.** Let $K$ be a field of characteristic zero, $f(x) \in K[x]$ an irreducible polynomial of degree $n \geq 5$. Suppose $\text{Gal}(f)$ is either the full symmetric group $S_n$ or the alternating group $A_n$. Then $\text{End}^0(J(C_{f,q})) = \mathbb{Q}[\delta_q] = \prod_{i=1}^r \mathbb{Q}(\zeta_{p^i})$. 

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Remark 1.2. Notice that the cases $p \nmid n$ or $q \mid n$, i.e., $s = 0$ or $s \geq r$, respectively, have been covered in [10]. So it remains to show that Theorem [11] holds in the case $0 < s < r$. Henceforth we will assume that $0 < s < r$ throughout the rest of the paper.

Remark 1.3. Replacing $K$ by a suitable quadratic extension if necessary, we may and will assume that $\text{Gal}(f) = \mathbb{A}_n$, which is simple nonabelian since $n \geq 5$. After that, we also note that $K(\zeta_q)$ is linearly disjoint from $K(\mathcal{O}_f)$, because $K(\zeta_q)/K$ is abelian while the Galois group of $K(\mathcal{O}_f)/K$ is simple nonabelian. It follows that the Galois group $\text{Gal}(f)$ remains $\mathbb{A}_n$ if the field $K$ is replaced with $K(\zeta_q)$. So we further assume that $\zeta_q \in K$ for the rest of this paper.

This paper is organized as follows. In Section 2 we establish some preliminary results about the curve $C_{f,q}$ and its Jacobian $J(C_{f,q})$. In Section 3 we introduce the abelian subvariety $J^{(f,p')}$ and show inductively that $J(C_{f,q})$ is isogenous to a product of abelian subvarieties $J^{(f,p')}$.

In Section 4 we show that $J^{(f,p')}$ is absolutely simple with endomorphism algebra $\mathbb{Q}(\zeta_{p'})$, which will enable us to prove the main Theorem.

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2. Preliminaries

We keep all the notation and assumptions of the previous section. In particular, $\deg(f) = n = mp^s$, where $p \nmid m$ and $0 < s < r$. It is known (cf. [8] page 3358) that on the curve $C_{f,q}$ there are exactly $p^s$ points at infinity, which we denote by $\infty_j$ with $1 \leq j \leq p^s$. We also write $S_\infty$ for the set of points at infinity on $C_{f,q}$.

Lemma 2.1. (i) The curve $C_{f,q}$ has genus $g(C_{f,q}) = ((q-1)(n-1)+1-p^s)/2$.

(ii) The automorphism $\delta_q$ acts transitively on $S_\infty$, and the stabilizer of each $\infty_j$ is the cyclic group $\langle \delta_q^p \rangle$ generated by $\delta_q^p$.

Proof. The formula for $g(C_{f,q})$ is a special case of [1] equation (4)) (See also [8] Proposition 1]). The assertion about the action of $\delta_q$ on $S_\infty$ also follows from an explicit description of $S_\infty$ given in [3], which we summarize here. Recall that $\mathcal{O}_f = \{\alpha_i\}_{1 \leq i \leq n}$ is the set of roots of $f(x)$. So over the algebraic closure $\overline{K}$, $C_{f,q}$ is given by the equation $y^q = c_n \prod_{i=1}^{n} (x - \alpha_i)$, where $c_n \in K$ is the leading coefficient of $f(x)$. Let $a, b$ be a pair of integers such that $am + bp^r - s = 1$ and $0 < a < p^{r-s}$. Let us consider the following birational transformation:

$$x = u^{-a}v^{-p^{r-s}}, \quad y = u^b v^{-m}.$$  

Then the defining equation for the curve $C_{f,q}$ over $\overline{K}$ changes into

$$u^{p^s} = c_n \prod_{i=1}^{n} \left(1 - \alpha_i u^{-a}v^{p^{r-s}}\right).$$

It is easy to show that in $(u, v)$-coordinates

$$S_\infty = \{ (\zeta_{p^s}^j, 0) \mid 1 \leq j \leq p^s \}$$

where $\zeta_{p^s} = \zeta_{q}^{p^{r-s}}$ is a primitive $p^s$-th root of unity. We set $\zeta_{\infty_j} = (\zeta_{p^s}^j, 0)$. To study the action of $\delta_q$ on $S_\infty$, we first find that the inverse to the birational transformation
\[ u = x^{-m}y^{p^r-s}, \quad v = x^{-b}y^{-a}. \]

So in \((u, v)\)-coordinates, \(\delta_q\) is given by
\[ \delta_q : C_{f,q} \rightarrow C_{f,q}, \quad (u, v) \mapsto (\zeta_q^{p^r-s} u, \zeta_q^{-a} v) = (\zeta_q^r u, \zeta_q^{-a} v). \]

Now assertion (ii) follows easily. \(\square\)

Given a smooth algebraic variety \(X\) over \(\bar{K}\), we write \(\Omega^1(X)\) for the \(\bar{K}\)-vector space of the differentials of the first kind on \(X\). By functoriality, \(\delta_q\) induces a periodic \(\bar{K}\)-linear automorphism \(\delta_q^* : \Omega^1(C_{f,q}) \rightarrow \Omega^1(C_{f,q})\). Clearly, \(\delta_q^*\) is semi-simple therefore \(\Omega^1(C_{f,q})\) splits into a direct sum of eigenspaces \(\Omega^1(C_{f,q})_i\), where
\[ \Omega^1(C_{f,q})_i := \{ \omega \in \Omega^1(C_{f,q}) \mid \delta_q^*(\omega) = \zeta_q^{-i} \omega \} \]
is the eigenspace corresponding to eigenvalue \(\zeta_q^{-i}\). We use \(n_i\) to denote the dimension of the eigenspace \(\Omega^1(C_{f,q})_i\).

If \(z \in \mathbb{R}\) is a real number, then we write \([z]_S\) for the greatest integer that is strictly less than \(z\). More explicitly, if \(z \notin \mathbb{Z}\), then \([z]_S\) coincides with the floor function \([z]\); if \(z \in \mathbb{Z}\), then \([z]_S = z - 1\).

**Proposition 2.2.** (i) The set
\[ \{ \omega_{i,j} = x^{j-i}dx/y^{q-i} \mid 0 < i, 0 < j, ni + jq < nq \} \]
is a basis of \(\Omega^1(C_{f,q})\).
(ii) \(n_i = [mi/q]_{S} = [mi/p^{r-s}]_{S}\).

*Proof.* Part (i) of the proposition is a restatement of \(8\) Proposition 2] where we note that on page 3360 of \(8\), the conditions on \((i, j)\) for the differential form \(x^i dx/y^j\) to be holomorphic hold without the assumption \(n = \deg f > q\). For part (ii), each \(\omega_{i,j} = x^{j-i}dx/y^{q-i}\) is an eigenvector of \(\delta_q^*\) with eigenvalue \(\zeta_q^{-i}\). For a fixed \(0 < i < q\), the set of differential forms \(\omega_{q-i,j} \in \Omega^1(C_{f,q})\) forms a basis of \(\Omega^1(C_{f,q})\).

The number of \(j\)'s that satisfy \(j > 0\) and \(n(q - i) + jq < nq\) is exactly \([ni/q]_S\). \(\square\)

**Remark 2.3.** Let \(d(n, q)\) be the greatest common divisor of the integers in the set
\[ \left\{ n_i = \left[ \frac{mi}{p^{r-s}} \right]_{S}, \quad 0 < i < q, p \nmid i \right\}. \]

If we write \(m = kp^{r-s} + c\), with \(0 < c < p^{r-s}\), then
\[ n_1 = \left[ \frac{kp^{r-s} + c}{p^{r-s}} \right] = k, \]
\[ n_{p^{r-s}+1} = \left[ \frac{(kp^{r-s} + c)(p^{r-s} + 1)}{p^{r-s}} \right] = kp^{r-s} + k + c, \]
\[ n_{p^{r-s}-1} = \left[ \frac{(kp^{r-s} + c)(p^{r-s} - 1)}{p^{r-s}} \right] = kp^{r-s} - k + c - 1. \]

It follows that \(d(n, q) = 1\) when \(0 < s < r\).

**Remark 2.4.** Similar to the situation with \(C_{f,q}\), the automorphism \(\delta_q\) of \(J(C_{f,q})\) induces a \(\bar{K}\)-linear automorphism of \(\Omega^1(J(C_{f,q}))\) which we again denote by \(\delta_q^*\). If \(D\) is a divisor of degree 0 on \(C_{f,q}\), we write \(\text{cl}(D)\) for the linear equivalence class.
of \(D\). Let \(P_i\) be the \(\delta_q\)-invariant point on \(C_{f,q}(\overline{K})\) with \((x,y)\)-coordinates \((\alpha_i,0)\) where \(\alpha_i \in \mathfrak{R}_f\). For a fixed \(i\), the map

\[
\tau : C_{f,q} \to J(C_{f,q}), \quad P \mapsto \text{cl}((P) - (P_i))
\]

is an embedding of algebraic varieties over \(K\), and the induced map

\[
\tau^* : \Omega^1(J(C_{f,q})) \to \Omega^1(C_{f,q})
\]

is an isomorphism which commutes with \(\delta_q^*\). This allows us to identify \(\Omega^1(J(C_{f,q}))\) with \(\Omega^1(C_{f,q})\) via \(\tau^*\). In particular, the dimension of the eigenspace corresponding to the eigenvalue \(\zeta_{q^{-1}}\) for the automorphism \(\delta_q^* : \Omega^1(J(C_{f,q})) \to \Omega^1(J(C_{f,q}))\) is exactly \(n_i = [m_i/q]_S\) as given by part (ii) of Proposition 2.2.

Remark 2.5. Clearly, \(\zeta_{q^{-1}}\) is an eigenvalue of \(\delta_q^*\) if and only if \(n_i > 0\). By part (ii) of Proposition 2.2, \(\zeta_{q^{-1}}\) is an eigenvalue of \(\delta_q^*\) for all \(p^r - p^{r-1} \leq i \leq p^r - 1\) (recall that \(n = mp^s \geq 5\)). Also note that 1 is not an eigenvalue. Taking into account that \(\delta_q^2 = 1\), one sees that

\[
P_q(x) = \frac{x^q - 1}{x - 1} = x^{q-1} + \cdots + x + 1
\]

is the minimal polynomial over \(\mathbb{Q}\) of \(\delta_q^*\) on \(\Omega^1(J(C_{f,q}))\).

**Lemma 2.6.** The minimal polynomial over \(\mathbb{Q}\) of \(\delta_q\) in \(\text{End}^0(J(C_{f,q}))\) is \(P_q(t)\). We have natural isomorphisms

\[
\prod_{i=1}^r Q(\zeta_{p^i}) \cong Q[t]/P_q(t)Q[t] \cong Q[\delta_q] \subseteq \text{End}^0(J(C_{f,q})), \quad t \mapsto \delta_q,
\]

\[
\mathbb{Z}[t]/P_q(t)\mathbb{Z}[t] \cong \mathbb{Z}[\delta_q] \subseteq \text{End}(J(C_{f,q})), \quad t \mapsto \delta_q.
\]

**Proof.** When \(p \nmid \deg f\), the result was obtained in [10, Lemma 4.8]. In our case here, instead of having one point at infinity on curve \(C_{f,q}\) that is fixed by \(\delta_q\) as [10, Lemma 4.8] dealt with, we have \(p^s\) points at infinity on which \(\delta_q\) acts transitively (Lemma 2.2). But it turns out that this cause no extra hindrance, and the proof of [10, Lemma 4.8] can be carried out here in exactly the same way. \(\square\)

We denote by \(B\) the set of all \(\delta_q\)-invariant points on the affine curve \(y^q = f(x)\). More explicitly, we set \(B := \{P_i = (\alpha_i,0) \mid \alpha_i \in \mathfrak{R}_f\}\).

**Lemma 2.7.** Let \(D = \sum_{P \in B} a_P(P)\) be a divisor on \(C_{f,q}\) of degree 0 that is supported on \(B\). Then \(D\) is principal if and only if \(a_P \equiv a_Q \mod q\) for any two points \(P,Q \in B\).

**Proof.** Again, this is a generalized version of [10, Lemma 4.7], and a similar proof applies. \(\square\)

### 3. Cyclic covers and Jacobians

Recall that we assumed that \(K\) contains the \(q\)-th roots of unity. Given a curve \(C_{f,q}\) and its Jacobian \(J(C_{f,q})\) as in previous sections, we consider the abelian subvariety

\[
J^{(f,q)} := \mathcal{P}_q/p(\delta_q)(J(C_{f,q})) \subseteq J(C_{f,q}).
\]
Clearly, $J^{(f,q)}$ is a $\delta_q$-invariant abelian subvariety which is defined over $K$. In addition, $\Phi_q(\delta_q)(J^{(f,q)}) = 0$, where $\Phi_q(x)$ denotes the $q$-th cyclotomic polynomial. Hence we have the following embeddings

$$i : \mathcal{O} = \mathbb{Z}[\zeta_q] \hookrightarrow \text{End}(J^{(f,q)}) \quad E = \mathbb{Q}(\zeta_q) \hookrightarrow \text{End}^0(J^{(f,q)}) \quad \zeta_q \mapsto \delta_q|_{J^{(f,q)}}.$$  

**Remark 3.1.** If $q = p$, then $\mathcal{P}_{q/p}(x) = \mathcal{P}_1(x) = 1$ and therefore $J^{(f,p)} = J(C_{f,q}).$

Let $\text{End}_K(J^{(f,q)})$ be the ring of all $K$-endomorphisms of $J^{(f,q)}$. By Remark 1.3, $i(\mathcal{O}) \subseteq \text{End}_K(J^{(f,q)})$. Let $\lambda = (1 - \zeta_q)\mathcal{O}$, and note that $\lambda$ is the only prime ideal in $\mathcal{O}$ that divides $p\mathcal{O}$. We write $J^{(f,q)}_{\lambda}$ for the group of $\lambda$-torsions of the abelian variety $J^{(f,q)}$:

$$J^{(f,q)}_{\lambda} := \{x \in J^{(f,q)} \mid i(c)x = 0, \forall c \in \lambda\}.$$  

Given $z \in J^{(f,q)}$, $i(1 - \zeta_q)(z) = 0$ if and only if $z$ is fixed by $\delta_q|_{J^{(f,q)}}$. It follows that $J^{(f,q)}_{\lambda} = (J^{(f,q)})^\lambda$, the subgroup of $J^{(f,q)}$ fixed by $\delta_q$. Clearly $J^{(f,q)}_{\lambda}$ is a vector space over the finite field $k(\lambda) := \mathcal{O}/\lambda \cong \mathbb{F}_p$. It is also a Gal($\overline{K}/K$) sub-module of $J^{(f,q)}[p]$, where we denote by $A[p]$ the subgroup of the abelian group $A$ generated by elements of order $p$. We would like to understand the structure of $J^{(f,q)}_{\lambda}$ as a Galois module.

Recall that $\mathfrak{A}_f = \{\alpha_i\}_{i=1}^n$ is the set of roots of $f(x)$. Let us define

$$\mathbb{F}_p^{\mathfrak{A}_f} := \{\phi : \mathfrak{A}_f \rightarrow \mathbb{F}_p\},$$

$$(\mathbb{F}_p^{\mathfrak{A}_f})^0 := \{\phi \in \mathbb{F}_p^{\mathfrak{A}_f} \mid \sum_{i=1}^n \phi(\alpha_i) = 0\}.$$  

The space $\mathbb{F}_p^{\mathfrak{A}_f}$ is naturally equipped with a structure of Gal($f$)-module in which both $(\mathbb{F}_p^{\mathfrak{A}_f})^0$ and the space of constant functions $\mathbb{F}_p \cdot 1$ are submodules. Since $p \mid n$, the space $\mathbb{F}_p \cdot 1$ is a Gal($f$)-submodule of $(\mathbb{F}_p^{\mathfrak{A}_f})^0$. We write $\mathcal{V}_{\mathfrak{A}_f}$ for the quotient Gal($f$)-module $\mathbb{F}_p^{\mathfrak{A}_f} / (\mathbb{F}_p \cdot 1)$, and $(\mathbb{F}_p^{\mathfrak{A}_f})^0$ for the image of $(\mathbb{F}_p^{\mathfrak{A}_f})^0$ under the projection map $\mathbb{F}_p^{\mathfrak{A}_f} \rightarrow \mathcal{V}_{\mathfrak{A}_f}$. Clearly $(\mathbb{F}_p^{\mathfrak{A}_f})^0$ is a codimension one submodule of $\mathcal{V}_{\mathfrak{A}_f}$.

Let $\text{Div}^0_B(C_{f,q})$ be the group of divisors of degree 0 of $C_{f,q}$ supported on the set of points $B = \{(\alpha_i, 0)\}_{i=1}^n$. Clearly $\text{Div}^0_B(C_{f,q})$ is a Gal($\overline{K}/K$)-submodule of $\text{Div}^0(C_{f,q})$ for which the Galois action factors through Gal($f$). As an abelian group, $\text{Div}^0_B(C_{f,q})$ is free of rank $n - 1$. Let $\text{Pic}^0_B(C_{f,q})$ be the image of $\text{Div}^0_B(C_{f,q})$ under the canonical map of Gal($\overline{K}/K$)-modules $\text{Div}^0(C_{f,q}) \rightarrow \text{Pic}^0(C_{f,q})$. Then $\text{Pic}^0_B(C_{f,q})$ inherits the structure of a Gal($f$)-module from $\text{Div}^0_B(C_{f,q})$. As before, we write $\text{Pic}^0_B(C_{f,q})[p]$ for the subgroup of $\text{Pic}^0_B(C_{f,q})$ generated by elements of order $p$. It is readily seen to be a Gal($f$)-submodule of $\text{Pic}^0_B(C_{f,q})$.

**Lemma 3.2.** The Gal($f$)-module $\text{Pic}^0_B(C_{f,q})[p]$ is canonically isomorphic to $\mathcal{V}_{\mathfrak{A}_f}$. So $\dim_{\mathbb{F}_p} \text{Pic}^0_B(C_{f,q})[p] = n - 1$. Moreover, $\text{Pic}^0_B(C_{f,q})[p]$ is a Gal($\overline{K}/K$)-submodule of $J^{(f,q)}_{\lambda}$.

**Proof.** By Lemma 2.1, the automorphism $\delta_q$ acts transitively on the set $S_{\infty}$ of points at infinity of curve $C_{f,q}$, and $\delta_q^p$ fixes $\infty_j$ for all $\infty_j \in S_{\infty}$. Further, the set $B$ consists all $\delta_q$ invariant points on $C_{f,q}$. Pick a point $\infty \in S_{\infty}$ and consider the
following divisor classes for $1 \leq i \leq n$,
\[
D_i = P_{q/p}((P_i) - (\infty)) = p^{r-1}(P_i) - p^{r-s-1}\sum_{j=1}^{p^s} (\infty_j).
\]

Note that $pD_i = p^r(P_i) - p^{r-s}\sum_{j=1}^{p^s} (\infty_j) = \text{Div}(x - \alpha_i) = 0$.

Recall that in the proof of Lemma 2.1 we find a pair of integers $(a, b)$ such that $a m + b p^{r-s} = 1$. Then
\[
\text{Div}(y^n(x - \alpha)^b) = a \sum_{k=1}^{n} (P_k) + b p^r(P_i) - (a m + b p^{r-s})\sum_{j=1}^{p^s} (\infty_j)
\]
\[
= a \sum_{k=1}^{n} (P_k) + b p^r(P_i) - \sum_{j=1}^{p^s} (\infty_j).
\]

It follows that the divisor $\sum_{j=1}^{p^s} (\infty_j)$ is linearly equivalent to $a \sum_{k=1}^{n} (P_k) + b p^r(P_i)$. Therefore, the divisor class
\[
D_i = p^{r-1}(P_i) - p^{r-s-1}\sum_{j=1}^{p^s} (\infty_j)
\]
\[
= p^{r-1}(P_i) - p^{r-s-1}\left(a(\sum_{k=1}^{n} (P_k)) + b p^r(P_i)\right)
\]
\[
= -a p^{r-s-1}\sum_{k=1}^{n} (P_k) + p^{r-1}(1 - b p^{r-s})(P_i)
\]
\[
= a m p^{r-1}(P_i) - a p^{r-s-1}\sum_{k=1}^{n} (P_k).
\]

In particular, $D_i$ is supported on $B$, so $D_i \in \text{Pic}^0_B(C_{f,q})[p]$. Let $\phi_i$ be the function on $\mathcal{R}_f$ given by $\phi_i(\alpha_j) = \delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$, and $0$ otherwise. The $\phi_i$’s form a basis of $\mathbb{F}_p^{\mathcal{R}_f}$. This allows us to define an $\mathbb{F}_p$-linear map $\pi: \mathbb{F}_p^{\mathcal{R}_f} \to \text{Pic}^0_B(C_{f,q})[p]$ by specifying the image of the basis elements,
\[
\pi(\phi_i) = D_i = P_{q/p}((P_i) - (\infty)) = a m p^{r-1}(P_i) - a p^{r-s-1}\sum_{k=1}^{n} (P_k).
\]

It is clear from the right hand side of (3.1) that $\pi$ is also a Gal($f$)-equivariant map.

By Lemma 2.7, given an element $\sum_{i=1}^{n} x_i \phi_i \in \mathbb{F}_p^{\mathcal{R}_f}$,
\[
\pi(\sum_{i=1}^{n} x_i \phi_i) = 0 \iff \sum_{i=1}^{n} x_i \left(a m p^{r-1}(P_i) - a p^{r-s-1}\sum_{k=1}^{n} (P_k)\right) \sim 0
\]
\[
\iff \forall (i, j), x_i a m p^{r-1} \equiv x_j a m p^{r-1} \mod q
\]
\[
\iff \forall (i, j), x_i = x_j \quad \text{since } p \nmid am
\]
\[
\iff \sum_{i=1}^{n} x_i \phi_i \in \mathbb{F}_p \cdot 1
\]
Therefore, \( \pi \) induces a \( \mathbb{F}_p \)-linear embedding \( \bar{\pi} : V_{\mathfrak{N} f} = \mathbb{F}_p^{\mathfrak{N} f}/(\mathbb{F}_p, 1) \to \text{Pic}^0_B(C_{f,q})[p] \) of \( \text{Gal}(f) \)-modules. On the other hand, recall that \( \text{Pic}^0_B(C_{f,q}) \) is defined to be the homomorphic image of \( \text{Div}^0_B(C_{f,q}) \), which is free abelian of rank \( n-1 \). It follows that \( \text{Pic}^0_B(C_{f,q}) \) can be generated by \( n-1 \) elements. Hence \( \dim_{\mathbb{F}_p} \text{Pic}^0_B(C_{f,q})[p] \leq n-1 \). Note that \( \dim_{\mathbb{F}_p} V_{\mathfrak{N} f} = n-1 \). We conclude that \( \bar{\pi} \) is an isomorphism. It is clear from our construction that \( \text{Pic}^0_B(C_{f,q})[p] \subseteq J^{(f,q)}_\lambda \).

The case when \( q = p \) of the previous lemma was treated by B. Poonen and E. Schaefer (see \([4]\) and \([6]\)).

**Remark 3.3.** Let \( T_p(J^{(f,q)}) \) be the \( p \)-adic Tate module (cf. \([2]\) p.170, \([7]\) 1.2)) of \( J^{(f,q)} \) defined as the projective limit of Galois modules \( J^{(f,q)}[p^i] \). Let \( \mathcal{O}_\lambda = \mathcal{O} \otimes \mathbb{Z}_p \) be the completion of \( \mathcal{O} \) with respect to the \( \lambda \)-adic topology. By \([3]\) prop 2.2.1, \( T_p(J^{(f,q)}) \) is a free \( \mathcal{O}_\lambda \) module of rank \( 2 \dim J^{(f,q)}/[E : \mathbb{Q}] = 2 \dim J^{(f,q)}/(p^r - p^{r-1}) \).

**Proposition 3.4.** The abelian variety \( J^{(f,q)} \) has dimension \( \dim \left( J^{(f,q)} \right) = (p^r - p^{r-1})(n-1)/2 \). There is a \( K \)-isogeny \( J^{(f,q)} \to J^{(f,q)}/p \times J^{(f,q)} \), and the \( \text{Gal}(\bar{K}/K) \)-module \( J^{(f,q)}_\lambda \) coincides with \( \text{Pic}^0_B(C_{f,q})[p] \). In particular, the \( \text{Gal}(\bar{K}/K) \)-action on \( J^{(f,q)}_\lambda \) factors through \( \text{Gal}(f) \).

**Proof.** Once again the \( p \nmid \deg f \) version of the Proposition has already been done in \([10]\) Lemma 4.11. The proof goes exactly the same way in our case. The idea is to do an argument on the dimensions based on Lemma \(2.6\) and Lemma \(3.2\). See the proof of \([10]\) Lemma 4.11 for more details. \( \Box \)

**Corollary 3.5.** There is a \( K \)-isogeny

\[
\text{J}(C_{f,q}) \to \text{J}(C_{f,p}) \times \prod_{i=2}^{r} \text{J}(f,p^i) = \prod_{i=1}^{r} \text{J}(f,p^i).
\]

**Proof.** This follows from induction on \( i \) and the fact that \( \text{J}(f,p) = J(C_{f,p}) \), as pointed out in Remark \(3.1\). \( \Box \)

**Remark 3.6.** Notice that Theorem \([11]\) will follow if we show that \( \text{End}^0(J^{(f,p^i)}) \cong \mathbb{Q}(\zeta_{p^i}) \) for all \( 1 \leq i \leq r \). The case that \( i \leq s \) has already been treated in \([10]\) Theorem 4.17. \( \Box \)

Let \( V \) be a vector space over a field \( E \), let \( G \) be a group and \( \rho : G \to \text{Aut}_E(V) \) a linear representation of \( G \) in \( V \). We write \( \text{End}_G(V) \) for the \( E \)-algebra of \( G \)-equivariant endomorphisms of \( V \). The equality \( J^{(f,q)}_\lambda = \text{Pic}^0_B(C_{f,q})[p] \), together with Lemma \(3.2\) enables us to determine \( \text{End}_{\text{Gal}(f)}(J^{(f,q)}_\lambda) \).

**Lemma 3.7.** Let \( n = mp^s \geq 5 \). Assume that \( 0 < s < r \) and \( \text{Gal}(f) \) is either \( S_n \) or \( A_n \). Then \( \text{End}_{\text{Gal}(f)}(J^{(f,q)}_\lambda) = \mathbb{F}_p \cdot \text{Id} \).

**Proof.** By Proposition \(3.3\) and Lemma \(3.2\), we have \( J^{(f,q)}_\lambda = \text{Pic}^0_B(C_{f,q})[p] \cong V_{\mathfrak{N} f} \) as \( \text{Gal}(f) \)-modules. It is enough to prove \( \text{End}_{\text{Gal}(f)}(V_{\mathfrak{N} f}) = \mathbb{F}_p \cdot \text{Id} \). Recall that when \( p \nmid n \), \( (\mathbb{F}_p^{(p^i)})^{00} \) is a codimension one submodule of \( V_{\mathfrak{N} f} \). For simplicity, we write \( V \) for \( V_{\mathfrak{N} f} \), and \( W \) for \( (\mathbb{F}_p^{(p^i)})^{00} \). It is known (Mortimer \([3]\)) that \( W \) is a simple \( \text{Gal}(f) \)-module with \( \text{End}_{\text{Gal}(f)}(W) = \mathbb{F}_p \cdot \text{Id}_W \). Given \( \theta \in \text{End}_{\text{Gal}(f)}(V_{\mathfrak{N} f}) \), the map \( \theta|_W : W \to \theta(W) \) is either zero or an isomorphism of \( \text{Gal}(f) \)-modules. In the latter
case, unless \( \theta(W) = W \), the intersection \( \theta(W) \cap W \) would be a proper nonzero submodule of \( W \), contradicting the simplicity of \( W \). Hence \( \theta|_W \in \text{End}_{\text{Gal}(f)}(W) = \text{F}_p \cdot \text{Id}_{W} \). By subtracting an element of \( \text{F}_p \cdot \text{Id}_{V} \), we may and will assume that \( \theta|_W = 0 \). In order to show that \( \text{End}_{\text{Gal}(f)}(V) = \text{F}_p \cdot \text{Id}_{V} \), it is enough to show that \( \theta(V) = 0 \). Recall that the \( \text{F}_p \)-vector space \( V \) is generated the set \( \{ \bar{\phi}_i \}_{i=1}^n \), where \( \bar{\phi}_i \) denotes the equivalent class of \( \phi_i \) modulo \( \text{F}_p : \sum_{i=1}^n \phi_i \). We identify \( \text{Gal}(f) \) with the permutation group on \( n \) letters \( \{1, \ldots, n\} \) and adopt the convention that \( \forall g \in \text{Gal}(f), \bar{\phi}_i \in V, g(\bar{\phi}_i) = \bar{\phi}_{g(i)} \). By definition of \( W = (\text{F}_p^{\mathbb{Q}/f})^0 \), given \( g \in \text{Gal}(f) \) and \( \bar{\phi} \in V \), the element \( g\bar{\phi} - \phi \in W \). Thus \( \theta(g\bar{\phi}) = \theta(\bar{\phi}) \). Assume that

\[
\theta(g\bar{\phi}) = \theta(\bar{\phi}) = \sum_{i=1}^n a_i \bar{\phi}_i, \quad \text{and write } \quad h = g^{-1}.
\]

Then

\[
g\theta(\bar{\phi}) = g \sum_{i=1}^n a_i \bar{\phi}_i = \sum_{i=1}^n a_i g(\bar{\phi}_i) = \sum_{i=1}^n a_{h(i)} \bar{\phi}_i.
\]

It follows that

\[
\theta(g\bar{\phi}) = g \theta(\bar{\phi}) \iff \sum_{i=1}^n a_i \bar{\phi}_i = \sum_{i=1}^n a_{h(i)} \bar{\phi}_i \\
\iff \sum_{i=1}^n a_i \bar{\phi}_i = \sum_{i=1}^n a_{h(i)} \bar{\phi}_i \quad \text{mod } \text{F}_p : \sum_{i=1}^n \phi_i \\
\iff a_i - a_{j} = a_{h(i)} - a_{h(j)} \quad \forall 1 \leq i, j \leq n.
\]

Since \( \text{Gal}(f) \) is either \( S_n \) or \( A_n \), it is doubly transitive. Fix an index \( i \). For any pair \( (j, k) \) such that neither \( j \) nor \( k \) equals to \( i \), there exist \( g_{jk} \in \text{Gal}(f) \) such that \( h_{jk}(i) = g_{jk}^{-1}(i) = i \) and \( h_{jk}(j) = g_{jk}^{-1}(j) = k \). It follows from \( \theta g_{jk} = g_{jk} \theta \) that \( a_i - a_j = a_i - a_k \). Thus \( a_j = a_k \) for all pairs \( (j, k) \) where neither entry is equal to \( i \). Varying the index \( i \) shows that there exist an \( a \in \text{F}_p \) such that \( a_j = a \) for all \( 1 \leq j \leq n \). Hence \( \theta(\bar{\phi}) = \sum_{i=1}^n a_i \bar{\phi}_i = a \sum_{i=1}^n \bar{\phi}_i = 0. \)

4. The endomorphism algebra of \( J^{(f, q)} \)

Let \( E \) be a number field that is normal over \( \mathbb{Q} \) with ring of integers \( \mathcal{O} \). Let \( K \) be a field of characteristic zero that contains a subfield isomorphic to \( E. \) Let \( (X, i) \) be a pair such that \( X \) is an abelian variety over \( K \) with an embedding \( i(\mathcal{O}) \subseteq \text{End}_K(X) \). The tangent space \( \text{Lie}_K(X) \) to \( X \) at the identity carries a natural structure of an \( E \otimes_{\mathbb{Q}} K \)-module. For each field embedding \( \tau : E \hookrightarrow K, \) we put

\[
\text{Lie}_K(X)_\tau = \{ z \in \text{Lie}_K(X) \mid i(e)z = \tau(e)z, \forall e \in E \}
\]

\[
n_\tau(X, E) = \dim_K \text{Lie}_K(X)_\tau
\]

Put \( X = J^{(f, q)} \) and \( E = \mathbb{Q}(\zeta_q) \). Let us consider the induced operator \( (\delta_q|_{J^{(f, q)}})^* : \Omega^1(J^{(f, q)}) \to \Omega^1(J^{(f, q)}) \). Applying [10] Theorem 3.10 to \( Y = J(C_{f, q}), Z = J^{(f, q)}, \) and \( P(t) = P_{q/p}(t) \), we see that the spectrum of \( (\delta_q|_{J^{(f, q)}})^* \) consists of primitive \( q \)-th roots of unity \( \zeta_q \) with \( n_i > 0 \), and the multiplicity of \( \zeta_q \) equals \( n_i \) (see part (ii) of Proposition [22]). Let \( \tau : E = \mathbb{Q}(\zeta_q) \hookrightarrow K \) be the embedding that sends \( \zeta_q \) to \( \zeta_q^{-i} \); then \( n_{\tau}(J^{(f, q)}, \mathbb{Q}(\zeta_q)) = n_i \).

Let \( \mathfrak{C}_X \) denote the center of the endomorphism algebra \( \text{End}^0(X) \). We quote the following theorem (cf [9] Theorem 2.3).
Theorem 4.1. If $E/\mathbb{Q}$ is Galois, $i(E)$ contains $\mathfrak{C}_X$ and $\mathfrak{C}_X \neq i(E)$, then there exists a nontrivial automorphism $\kappa : E \to E$ such that $n_\tau(X,E) = n_{\tau\kappa}(X,E)$ for all embeddings $\tau : E \to K$.

For simplicity, let $[a,b]_\mathbb{Z}$ denote the set of integers in the closed interval $[a,b] \subseteq \mathbb{R}$ that are not divisible by $p$, where $p$ should be apparent from the context. We note that $[1,p]_\mathbb{Z}$ is the usual set of representatives of the group $(\mathbb{Z}/p\mathbb{Z})^\times$, which is isomorphic to $\text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$.

The following two lemmas, together with Theorem 4.1, enable us to prove that $i(\mathbb{Q}(\zeta_q))$ coincides with $\mathfrak{C}_{\mathfrak{H}(f,a)}$ whenever $i(\mathbb{Q}(\zeta_q))$ contains $\mathfrak{C}_{\mathfrak{H}(f,a)}$.

Lemma 4.2. Let $p$ be an odd prime and $q = p^r$ be a power of $p$. Given $k \in (\mathbb{Z}/q\mathbb{Z})^\times$, let $\theta_k$ denote the action of $k$ on $(\mathbb{Z}/q\mathbb{Z})^\times$ by multiplication, i.e., $\theta_k(u) = ku$ for any $u \in (\mathbb{Z}/q\mathbb{Z})^\times$. Let $f : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{R}$ be a monotonic function on $[1,q]_\mathbb{Z}$. The following statements are equivalent:

(i) $f \circ \theta_k = f$.
(ii) $k = 1$ or $f$ is a constant function.

Proof. Assume that $k \neq 1$. It suffices to show that if $f \circ \theta_k = f$, then $f$ is a constant function. Let $\text{ord}(k)$ be the order of the element $k$ in $(\mathbb{Z}/q\mathbb{Z})^\times$. Note that $f \circ \theta_k = f$ is equivalent to $f \circ \theta_j = f$ for all $0 \leq j < \text{ord}(k)$. Since $p$ is an odd prime, $(\mathbb{Z}/q\mathbb{Z})^\times$ is a cyclic group. Given $k$ and $k'$ in $(\mathbb{Z}/q\mathbb{Z})^\times$ with $\text{ord}(k) \mid \text{ord}(k')$, the cyclic group $\langle k \rangle$ is a subgroup of $\langle k' \rangle$. Suppose the lemma holds for $k$; a fortiori it holds true if one replace $k$ by $k'$. Thus one may assume that $\text{ord}(k)$ is a prime. Note if $\text{ord}(k) = 2$, then $k = q - 1$. Hence $f(q - 1) = f(1)$, and $f$ is constant by monotonicity. So we further assume that $\text{ord}(k)$ is an odd prime.

Any element in $\langle k \rangle$ other than the identity is a generator of $\langle k \rangle$. Without loss of generality, we assume that $k$ is the smallest in the set

$$S = \{ i \mid 1 < i < q, \exists j \text{ such that } i \equiv k^j \pmod{q} \}.$$ 

In other words, $S$ is the set of representatives strictly between 1 and $q$ for non-identity elements of the cyclic group $\langle k \rangle$. Also let $m_S$ denote the largest element of the set $S$. Notice that $[q/k] < m_S$ by choice of $m_S$, and $k < m_S$, since we have more than two elements in $S$.

From the equality $f(kx) = f(x)$, we conclude $f(m_S) = f(1)$, and $f(x)$ is constant on $[1,m_S]_\mathbb{Z}$ by monotonicity. Notice that $f(q - 1) = f(q - m_S) - (q - 1)m_S \equiv 0 \pmod{q}$. If $q - m_S \leq m_S$, i.e., $m_S > (q - 1)/2$, then $f(q - m_S) = f(1)$, thus $f(q - 1) = f(q - m_S) = f(1)$. By monotonicity again, we conclude that $f(x)$ is constant.

So furthermore, assume that $2m_S \leq q - 1$. In particular, $[q/k] \geq [q/m_S] \geq 2$. Let $c = [q/k]$ if $p \mid [q/k]$, and $c = [q/k] - 1$ otherwise. Clearly, $q - 2k < ck < q$. By construction, $c \in [1,m_S]_\mathbb{Z}$, which implies that $f(c) = f(1)$. Therefore, $f(ck) = f(c) = f(1)$ and $f(x)$ is constant on $[1,ck]_\mathbb{Z}$. On the other hand,

$$ck > q - 2k > q - 2m_S.$$ 

Hence $f(q - 2k) = f(q - 2m_S) = f(1)$. This shows that $f$ is constant on $[1,q - 2]_\mathbb{Z}$.

Last, $f(q - 1) = f(q - k)$ but clearly, $q - k \in [1,q - 2]_\mathbb{Z}$. Therefore, if $f \circ \theta_k = f$, then $f$ is constant on $[1,2^{r-1}]_\mathbb{Z}$.

Lemma 4.3. Let $p = 2$ and $q = 2^r$. Given $k \neq 1$ in $(\mathbb{Z}/q\mathbb{Z})^\times$, let $\theta_k$ be as defined in the previous lemma. Let $f : (\mathbb{Z}/2^r\mathbb{Z})^\times \to \mathbb{R}$ be a monotonic function on $[1,2^r]_\mathbb{Z}$. If $f \circ \theta_k = f$, then $f$ is constant on $[1,2^{r-1}]_\mathbb{Z}$. 

ENDOMORPHISM ALGEBRA OF CERTAIN JACOBIANS 9
Proof. For \( r = 1, 2 \), the case is trivial since \([1, 2^{r-1}]_2\) consists of only one element. For \( r \geq 3 \), \((\mathbb{Z}/2^r\mathbb{Z})^\times \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{r-2}\mathbb{Z}\). The order of any nontrivial element in \((\mathbb{Z}/2^r\mathbb{Z})^\times\) is a power of 2. So the cyclic group \((k)\) contains one of the 3 elements of order two: \(2^{r-1} - 1, 2^{r-1} + 1, 2^r - 1\). Call it \(x_k\). It follows from the identity \(f \circ \theta_k = f\) that \(f(1) = f(x_k)\). The lemma now follows from monotonicity of \(f\) since \(x_k \geq 2^{r-1} - 1\). \(\square\)

Corollary 4.4. If \(E = \mathbb{Q}(\zeta_q)\) contains \(\mathcal{C}_{J(f,q)}\), then \(\mathcal{C}_{J(f,q)} = \mathbb{Q}(\zeta_q)\).

Proof. If \(q = 2\), then \(E = \mathbb{Q}(\zeta_2) = \mathbb{Q}\). Since \(\mathcal{C}_{J(f,q)}\) is a subfield of \(E\), we see that \(\mathcal{C}_{J(f,q)} = \mathbb{Q} = E\). Further assume that \(q > 2\). Suppose that \(\mathcal{C}_{J(f,q)} \neq E\). It follows from Theorem 4.1 that there exists an nontrivial field automorphism \(\kappa : \mathbb{Q}(\zeta_q) \rightarrow \mathbb{Q}(\zeta_q)\) such that \(n_\tau(X,E) = n_\tau(\kappa(X,E))\) for all embeddings \(\tau : E \rightarrow K\). Clearly \(\kappa(\zeta_q) = \zeta_q^s\) for some \(s\) in \([1, q-2]_2\). It follows that if we define \(f : (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{R}\) where \(i \mapsto n_i\), then \(f(\kappa(i)) = f(i)\) for all \(i \in (\mathbb{Z}/q\mathbb{Z})^\times\). Notice that \(f\) is a nondecreasing function on the set of representatives. Recall that \(n \geq 5\). If \(p\) is odd, then part (ii) of Lemma 2.2 \(\square\)

Proposition 4.5. Let \(K\) be a field of characteristic zero. Let \(p\) be prime, \(q = p^r\) be a power of \(p\), and \(n = mp^s \geq 5\) with \(0 < s < r\) and \(p \nmid m\). Suppose that \(\text{Gal}(f)\) is either \(S_n\) or \(A_n\). Then

\[
\text{End}^0(J(f,q)) \cong \mathbb{Q}(\zeta_q), \quad \text{End}(J(f,q)) \cong \mathbb{Z}[\zeta_q]
\]

Proof. By remark 1.3 we may assume that \(\zeta_q \in K\) and \(\text{Gal}(f) = A_n\). Since \(n \geq 5\), the group \(\text{Gal}(f)\) contains no non-trivial normal subgroups. We write \(\text{End}^0(J(f,q),i)\) for the centralizer of \(i(\mathbb{Q}(\zeta_q))\) inside \(\text{End}^0(J(f,q))\). Applying [11, Theorem 3.12(ii)(2)], and combining with remark 2.3 and Lemma 3.7 we get

\[
\text{End}^0(J(f,q),i) = i(\mathbb{Q}(\zeta_q)) \cong \mathbb{Q}(\zeta_q), \quad \text{End}(J(f,q),i) = i(0) \cong 0.
\]

In particular, \(\mathbb{Q}(\zeta_q)\) contains the center of \(\text{End}^0(J(f,q))\). If follows from Corollary 4.4 that \(i(\mathbb{Q}(\zeta_q))\) coincides with the center of \(\text{End}^0(J(f,q))\). Thus

\[
\text{End}^0(J(f,q)) = \text{End}^0(J(f,q),i) = i(\mathbb{Q}(\zeta_q))
\]

If \(\delta_q|_{J(f,q)}\) is the restriction of \(\delta_q\) to \(J(f,q)\), viewed as an automorphism of \(J(f,q)\), then \(i(0) \cong \mathbb{Z}[\delta_q|_{J(f,q)}]\) is the maximal order in \(\mathbb{Q}(\zeta_q)\) and \(\mathbb{Z}[\delta_q|_{J(f,q)}] \subseteq \text{End}(J(f,q))\), and we conclude that \(\mathbb{Z}[\delta_q|_{J(f,q)}] = \text{End}(J(f,q))\). \(\square\)

Proof of Theorem 1.1. We need to consider only the case \(s > 0\) (see Remark 1.2). By Corollary 3.5 it suffices to show \(\text{End}^0(J(f,q^i)) = \mathbb{Q}(\zeta_{q^i})\) for all \(1 \leq i \leq r\), which follows from Proposition 4.5 and Remark 3.6 \(\square\)

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