Salce’s problem on cotorsion pairs is undecidable

Sean Cox

Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, Richmond, Virginia, USA

Correspondence
Sean Cox, Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, 1015 Floyd Ave, Richmond, VA 23284, United States. Email: scox9@vcu.edu

Abstract
In the 1970’s, Salce introduced the notion of a cotorsion pair of classes of abelian groups, and asked whether every such pair is complete (that is, has enough injectives and projectives); we refer to this as Salce’s Problem (for Ab). We prove that it is consistent, relative to the consistency of Vopěnka’s Principle (VP), that the answer is affirmative. Combined with a previous result of Eklof and Shelah, this shows that Salce’s Problem for Ab, and in fact for R-Mod when R is hereditary, is independent of the ZFC axioms (modulo the consistency of VP).

MSC (2020)
03E75, 16B70, 16E30, 18G25 (primary), 16D40, 16D90 (secondary)

1 | INTRODUCTION

Cotorsion pairs, also called cotorsion theories, were introduced by Salce [15] in the setting of abelian groups. For a class \( C \) of \( R \)-modules,

\[
\downarrow C = \{ X : \text{Ext}_R^1(X, C) = 0 \text{ for all } C \in C \}
\]

and

\[
C_{\downarrow} = \{ Y : \text{Ext}_R^1(C, Y) = 0 \text{ for all } C \in C \}.
\]
A cotorsion pair is a pair \((\mathcal{A}, \mathcal{B})\) of classes such that \(\mathcal{A} = \perp \mathcal{B}\) and \(\mathcal{A}^{\perp} = \mathcal{B}\). The cotorsion pair \((\mathcal{A}, \mathcal{B})\) is called complete if, for every module \(M\), there exists a short exact sequence

\[
0 \rightarrow B \rightarrow A \rightarrow M \rightarrow 0
\]

for some \(A \in \mathcal{A}\) and \(B \in \mathcal{B}\). Cotorsion pairs that are complete provide for a nice approximation theory. The pair

\[\text{(Projective Modules, All Modules)}\]

is the simplest example of a complete cotorsion pair.

Salce’s original paper dealt exclusively with the category \(\text{Ab}\) of abelian groups (that is, \(\mathbb{Z}\)-modules), and he asked whether every cotorsion pair in this category is complete [15, Problem 2]. We shall refer to this as Salce’s Problem, or sometimes as Salce’s Problem for \(\text{Ab}\), to distinguish it from the still-open generalized version (see Question 5.2). Our main result is:

**Theorem 1.1.** It is consistent, relative to the consistency of Vopěnka’s Principle (VP), that the answer to Salce’s Problem is affirmative.

In fact, it is consistent relative to VP that for any ring \(R\) and any cotorsion pair \((\mathcal{A}, \mathcal{B})\) of \(R\)-modules, if \(\perp \mathcal{B}\) is downward closed under elementary submodules for all \(B \in \mathcal{B}\), then \((\mathcal{A}, \mathcal{B})\) is both cogenerated and generated by a set, and hence\(^8\) complete.

VP is a well-studied principle that is equivalent to the assertion (scheme) that ‘every subfunctor of an accessible functor is accessible’ (see [1] for many other characterizations). The history of VP is amusing; the historical remarks in Chapter 6 of Adámek–Rosický [1] refer to VP as ‘a practical joke which misfired’. VP fits in the large cardinal hierarchy just below the huge cardinals.

Eklof–Shelah [6] proved that it is consistent, relative to the consistency of the Zermelo–Fraenkel axioms of mathematics (ZFC), that the answer to Salce’s Problem is negative. Together with Theorem 1.1 this yields:

**Corollary 1.2.** Salce’s Problem (for \(\text{Ab}\), and more generally for \(R\)-Mod with \(R\) hereditary) is independent of the ZFC axioms (assuming that ZFC + VP is consistent).

There are three main ingredients to the proof of Theorem 1.1. First, if VP is consistent, then VP is also consistent with the statement

\[
\diamondsuit_\lambda(S) \text{ holds for all regular uncountable } \lambda \text{ and all stationary } S \subseteq \lambda, \tag{*}
\]

where \(\diamondsuit_\lambda(S)\) is the well-known combinatorial principle isolated by Ronald Jensen (see the Appendix). Because of a nice ‘VP-preservation’ theorem of Brooke–Taylor [3], this part of the argument is a standard forcing construction, and is relegated to the Appendix.

---

\(^1\) We also say the cotorsion pair has enough projectives if this property holds. Salce proved this is equivalent to the pair having enough injectives; that is, that for every module \(M\), there is a short exact sequence \(0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0\) with \(A \in \mathcal{A}\) and \(B \in \mathcal{B}\).

\(^2\) This always holds if the ring is hereditary; indeed, if \(R\) is hereditary and \(X\) is any \(R\)-module, then \(\perp X\) is downward closed under all submodules.

\(^8\) By Eklof–Trlifaj [8], see below.
The second key ingredient of Theorem 1.1 is the following theorem, which employs VP and (set-theoretic) ‘elementary submodel’ arguments:

**Theorem 1.3.** Vopěnka’s Principle implies that for every ring \( R \) and every class \( \mathcal{B} \) of \( R \)-modules: if \( \downarrow B \) is downward closed under elementary submodules for all \( B \in \mathcal{B} \), then there is a set \( B_0 \subseteq B \) such that

\[
\downarrow B = \downarrow B_0.
\]

Rephrasing Theorem 1.3 in terms of cotorsion pairs yields:

**Corollary 1.4.** Vopěnka’s Principle implies that if \((A, B)\) is a cotorsion pair, and \( \downarrow B \) is downward closed under elementary submodules for all \( B \in \mathcal{B} \), then \((A, B)\) is cogenerated by a set.†

Theorem 1.3 and Corollary 1.4 can be generalized to other categories; see Section 5.

The third ingredient of Theorem 1.1 is:

**Theorem 1.5** (Šaroch–Trlifaj [16]; Eklof–Trlifaj [7]). If (*) holds, then every cotorsion pair that is cogenerated by a set, and whose left coordinate is downward closed under elementary submodules, is also generated by a set, and hence (by Eklof–Trlifaj [8]) complete.

We also prove a result using ‘only’ supercompact cardinals, which are weaker in consistency strength than Vopěnka’s Principle. A class \( \mathcal{C} \) of modules is a Kaplansky class ([10]) if there exists a cardinal \( \kappa \) such that, for every \( C \in \mathcal{C} \) and every \( X \subseteq C \) with \(|X| < \kappa\), there is a \(< \kappa\)-presented submodule \( C_0 \) of \( C \) such that \( X \subseteq C_0 \), and both \( C_0 \) and \( C/C_0 \) are in \( \mathcal{C} \).

**Theorem 1.6.** If there is a proper class of supercompact cardinals, then for every set \( S \) of modules over any ring, \( \downarrow S \) is a Kaplansky class.

The key to Theorem 1.6 is Lemma 3.1 (p. 4), which may be of interest elsewhere; roughly speaking, supercompactness of \( \kappa \) yields many elementary submodels \( \mathcal{U} \) of the universe of size \(< \kappa \) such that ‘intersection with \( \mathcal{U} \)’ commutes with the Ext functor (for objects that \( \mathcal{U} \) has access to).

Kaplansky classes that are also closed under direct limits are in fact deconstructible (see [17]). And cotorsion pairs whose left coordinate is deconstructible are complete (see [13]). Hence:

**Corollary 1.7.** If there is a proper class of supercompact cardinals, then all cotorsion pairs that are cogenerated by a set, and whose left coordinate is closed under direct limits, are complete.

Corollary 1.7 can be viewed as a variant of a theorem of El Bashir [9], who proved that under Vopěnka’s Principle, every class of modules that is closed under direct sums and direct limits is a covering class. In particular, his theorem yielded that under Vopěnka’s Principle, cotorsion pairs whose left coordinate is closed under direct limits are complete. Corollary 1.7 uses a weaker assumption than Vopěnka’s Principle, but at the expense of only being able to prove it for cotorsion pairs cogenerated by a set.

† Using the terminology of Göbel–Trlifaj [13]; ‘generated by a set’ in the terminology of Eklof–Trlifaj [8] and Salce [15]. See Section 2 for details.
Section 2 includes preliminaries, Section 3 proves Theorem 1.6, and Section 4 proves Theorem 1.1. Sections 3 and 4 are entirely independent of each other, but Section 3 appears first because it is slightly more concrete than the proof from Vopěnka’s Principle. Section 5 discusses generalizations and open problems, and the Appendix proves that CON(VP) implies CON(VP + (*)).

2 | PRELIMINARIES

All notation and terminology agrees with Jech [14] and Göbel–Trlifaj [13]. By ‘R-module’ we will officially mean left R-module.

If C is a class, then both

\[ (⊥C, (⊥C)^⊥) \] (1)

and

\[ (⊥C^⊥, C^⊥) \] (2)

are cotorsion pairs; (1) is called the cotorsion pair cogenerated by C, and (2) is called the cotorsion pair generated by C, using the terminology of Göbel–Trlifaj [13].

† The terminology is inconsistent across the literature; for example, the meanings of ‘cogenerated by’ and ‘generated by’ given above are switched, for example, in Eklof–Trlifaj [8].

We say that a cotorsion pair is cogenerated by a set if it is of the form (1) where C is a set (as opposed to a proper class), and generated by a set if it is of the form (2) where C is a set. Around the turn of the millennium, Eklof and Trlifaj proved the following landmark theorem, which was key to one of the solutions of the Flat Cover Conjecture ([2]):

Theorem 2.1 (Eklof–Trlifaj [8]). All cotorsion pairs generated by sets are complete.

For structures \( \mathfrak{A} \) and \( \mathfrak{B} \) in a fixed first order signature, \( \mathfrak{A} < \mathfrak{B} \) means that \( \mathfrak{A} \) is an elementary substructure of \( \mathfrak{B} \). For a ring \( R \), the language for \( R \)-modules consists of the usual language for abelian groups, together with, for each \( r \in R \), a function symbol for scalar multiplication by \( r \) (so for infinite \( R \), the language for \( R \)-modules has cardinality \( |R| \)). For an infinite cardinal \( \lambda \), \( H_\lambda \) refers to the set of all sets of hereditary cardinality \( < \lambda \), and \( \mathcal{S}_\lambda \) refers to the structure \( (H_\lambda, \in) \).

The reader more familiar with the \( V_\alpha \) hierarchy can just as well use those instead. The equality \( V_\lambda = H_\lambda \) holds for the closed unbounded class of \( \lambda \) that are fixed points of the \( \beth \) function. The structure \( \mathcal{S}_\lambda \) is a \( \Sigma_1 \)-elementary substructure of the universe for every uncountable cardinal \( \lambda \).

For a regular uncountable cardinal \( \kappa \) and a cardinal \( \lambda \geq \kappa \), let \( \mathcal{P}_\kappa(H_\lambda) \) denote the set of \( N \subset H_\lambda \) such that \( |N| < \kappa \), \( N \cap \kappa \in \kappa \), and \( \mathfrak{N} = (N, \in) \) is an elementary substructure of \( \mathcal{S}_\lambda \). Any such \( \mathfrak{N} \) is extensional, so Mostowski’s collapsing theorem applies. For such \( \mathfrak{N} \), let \( \mathcal{H}(\mathfrak{N}) \) denote the Mostowski collapse of \( \mathfrak{N} \), and let

\[ \sigma_{\mathfrak{N}} : \mathcal{H}(\mathfrak{N}) \rightarrow_{\text{iso}} \mathfrak{N} < \mathcal{S}_\lambda \]

denote the inverse of the collapsing map, which can be viewed as an elementary embedding from \( \mathcal{H}(\mathfrak{N}) \rightarrow \mathcal{S}_\lambda \). For \( b \in \mathfrak{N} \), \( b_{\mathfrak{N}} \) will denote \( \sigma_{\mathfrak{N}}^{-1}(b) \); that is, \( b_{\mathfrak{N}} \in \mathcal{H}(\mathfrak{N}) \) is the image of \( b \) under the transitive collapsing map for \( \mathfrak{N} \). A set \( S \subset \mathcal{P}_\kappa(H_\lambda) \) is called stationary (in \( \mathcal{P}_\kappa(H_\lambda) \)) if, for every
$p_1, \ldots, p_k \in H_\lambda$, there exists an $\mathfrak{R} \in S$ such that $\{p_1, \ldots, p_k\} \subset \mathfrak{R} < H_\lambda$.\footnote{There are other equivalent ways of defining this kind of stationarity. See Lemma 0 part (a) of Foreman–Magidor–Shelah [12].} Note that if $|\mathfrak{R}| < \kappa$, then $H(\mathfrak{R})$ is both an element and subset of $H_\kappa$; in particular, $b_\mathfrak{R}$ is of hereditary cardinality $< \kappa$ for all $b \in \mathfrak{R}$.

**Fact 2.2.** Suppose $R$ is a ring of size $< \kappa$, $M$ is an $R$-module, $\mathfrak{R} \in \mathcal{P}_\kappa^*(H_\lambda)$, and

$$\{R, M\} \subset \mathfrak{R} < H_\lambda.$$

Then:

1. $R$ is also a subset of $\mathfrak{R}$; in fact any $X \in \mathfrak{R}$ such that $|X| < \kappa$ is also a subset of $\mathfrak{R}$;
2. $\mathfrak{R} \cap M$ is an elementary submodule of $M$;
3. $\sigma_{\mathfrak{R}} \upharpoonright M_{\mathfrak{R}}$ is an elementary embedding (in the language of $R$-modules) from $M_{\mathfrak{R}} \to M$ with image $\mathfrak{R} \cap M$.

**Proof.** Part 1 follows from the fact that $\mathfrak{R} \cap \kappa$ is transitive, which was part of the definition of $\mathcal{P}_\kappa^*(H_\lambda)$; see [4] for details. It follows that the signature of $M$ is both an element and a subset of $\mathfrak{R}$, which ensures that $\mathfrak{R}$ is closed under Skolem functions for $M$. The last part is just because for any $b \in \mathfrak{R}$, $\sigma_{\mathfrak{R}}(b_{\mathfrak{R}}) = b \cap \mathfrak{R}$. \hfill $\square$

The following lemma will be used only in the proof of Theorem 1.6.

**Lemma 2.3.** Let $R$ be a ring.

(i) Suppose $H$ is a transitive set and $\pi : A \to B$ is a function such that $\{R, A, B, \pi\} \subset H$. Then the statement $\pi$ is an $R$-module homomorphism from $A \to B$ is absolute between $(H, \in)$ and the universe of sets.

(ii) If $\mu$ is an uncountable cardinal such that $R \in H_\mu$, then $S_\mu$ correctly computes $\text{Ext}^n_R$ for all $n \in \mathbb{N}$; that is, for all $R$-modules $X, Y \in S_\mu$, $\text{Ext}^n_R(X, Y)$ as computed in $S_\mu$ is the real $\text{Ext}^n_R(X, Y)$.

**Proof.** The statement $'\pi$ is an $R$-module homomorphism from $A \to B'$ is a $\Sigma_0$ statement in the language of set theory, so is absolute between all transitive sets (see Jech [14]). Now suppose $\mu$ is an uncountable cardinal, and $A \in S_\mu$ is an $R$-module. Since $A$ and $R$ both have cardinality $< \mu$, there is a projective resolution $\tilde{P}$ of $A$ of hereditary cardinality $< \mu$, and hence $\tilde{P} \in S_\mu$. Moreover, for all $R$-modules $X, Y$ in $S_\mu$, the $< \mu$-closure of $S_\mu$ ensures that $\text{Hom}_R(X, Y) \subset S_\mu$ (and part (i) ensures that $S_\mu$ and the universe agree about which elements of $S_\mu$ count as $R$-module homomorphisms). This ensures that $S_\mu$ correctly computes the relevant homology groups that define $\text{Ext}^n_R(X, Y)$. \hfill $\square$

## 3 PROOF OF THEOREM 1.6

Following Viale [18], we say that $\mathfrak{R} \in \mathcal{P}_\kappa^*(H_\lambda)$ is 0-guessing if $\mu := (\text{ordtype of } \mathfrak{R} \cap \lambda)$ is a cardinal, and the transitive collapse $S(\mathfrak{R})$ of $\mathfrak{R}$ is equal to $S_\mu$. A classic result of Magidor shows
that if $\kappa$ is supercompact, then the 0-guessing sets are stationary in $\varphi^*_\kappa(H_\lambda)$ for all $\lambda \geq \kappa$ (Magidor's lemma was phrased somewhat differently; see [4] for a quick proof).

**Lemma 3.1.** If $R$ is a ring of size $< \kappa$, $A$ and $B$ are $R$-modules, and

$$\{R, A, B\} \subset \mathcal{N} < \mathcal{H}_\lambda$$

is such that $\mathcal{N}$ is a 0-guessing model with $\mathcal{N} \cap \kappa \in \kappa$, then:

(i) for all $n \in \mathbb{N}$:

$$\mathcal{N} \cap \text{Ext}^n_R(A, B) \cong \text{Ext}^n(\mathcal{N} \cap A, \mathcal{N} \cap B).$$

In particular,

$$\text{Ext}^n_R(A, B) = 0 \text{ if and only if } \text{Ext}^n(\mathcal{N} \cap A, \mathcal{N} \cap B) = 0;$$

(ii) every homomorphism from $\mathcal{N} \cap A \to \mathcal{N} \cap B$ lifts to a homomorphism from $A \to B$. In particular, if $B$ is also a subset of $\mathcal{N}$, every homomorphism from $\mathcal{N} \cap A \to \mathcal{N} \cap B$ lifts to a homomorphism from $A \to B$.

**Proof.** Since $|R| < \kappa$, we will without loss of generality assume $R \in H_\kappa$. Since $R \in \mathcal{N} \cap H_\kappa$ and $\mathcal{N} \cap \kappa$ is transitive, it follows that $R \subset \mathcal{N}$ and $R$ is not moved by the transitive collapsing map of $\mathcal{N}$. Also, since $R \subset \mathcal{N}$, $\mathcal{N} \cap M$ is an $R$-submodule of $M$ for all $R$-modules $M \in \mathcal{N}$.

Let $G^n := \text{Ext}^n_R(A, B)$. As discussed above, $\sigma : \mathcal{N} \ni \mathcal{N} \to \mathcal{H}_\lambda$ denotes the inverse of the Mostowsk collapse. Since $A$ and $B$ are in $\mathcal{N}$, it follows that $G^n \in \mathcal{N}$ for all $n \in \mathbb{N}$, and these objects are all in the range of $\sigma$. Recall that for $b \in \mathcal{N}$ — that is, for $b$ in the range of $\sigma$ — we use $b_{\mathcal{N}}$ to denote $\sigma^{-1}(b)$. Then by Fact 2.2,

$$R_{\mathcal{N}} = R, A_{\mathcal{N}} \simeq \mathcal{N} \cap A, B_{\mathcal{N}} \simeq \mathcal{N} \cap B, \text{ and } G^n_{\mathcal{N}} \simeq \mathcal{N} \cap G^n.$$  

(3)

By elementarity of $\sigma$, $\mathcal{H}(\mathcal{N}) \models 'G^n_{\mathcal{N}} = \text{Ext}^n_{R_{\mathcal{N}}}(A_{\mathcal{N}}, B_{\mathcal{N}})'$. Since $\mathcal{N}$ is 0-guessing, $\mathcal{H}(\mathcal{N})$ is of the form $\mathcal{H}_\mu$, so by Lemma 2.3, $\mathcal{H}(\mathcal{N}) = \mathcal{H}_\mu$ is correct about $\text{Ext}^n_{R_{\mathcal{N}}}$ for all $n \in \mathbb{N}$. In particular, $G^n_{\mathcal{N}}$ really is $\text{Ext}^n_{R_{\mathcal{N}}}(A_{\mathcal{N}}, B_{\mathcal{N}})$. This, together with (3), proves part (i).

For part (ii), suppose $\phi : \mathcal{N} \cap A \to \mathcal{N} \cap B$ is a homomorphism, and let $\bar{\phi} := \sigma^{-1}[\phi]$ be the pointwise preimage of $\phi$ via $\sigma$. Since the domain and range of $\phi$ are subsets of $\mathcal{N}$, it follows that $\bar{\phi}$ is a total function from $A_{\mathcal{N}} \to B_{\mathcal{N}}$, and is in fact an $R = R_{\mathcal{N}}$-module homomorphism. Since $\bar{\phi}$ is a subset of $A_{\mathcal{N}} \times B_{\mathcal{N}} = A_{\mathcal{N}} \times B_{\mathcal{N}} \in \mathcal{H}(\mathcal{N})$, and $\mathcal{H}(\mathcal{N}) = \mathcal{H}_\mu$, it follows (by hereditary closure of $\mathcal{H}_\mu$) that $\bar{\phi}$ is an element of $\mathcal{H}(\mathcal{N})$. And, by Lemma 2.3, $\bar{\phi}$ is an $R$-module homomorphism from $A_{\mathcal{N}} \to B_{\mathcal{N}}$ is downward absolute from the universe to the transitive set $\mathcal{H}(\mathcal{N})$. Let $\hat{\phi} := \sigma(\bar{\phi})$; by elementarity of $\sigma$, $\mathcal{H}(\mathcal{N}) \models \hat{\phi} \text{ is an R-module homomorphism from } A \to B'$; and this is upward absolute to the universe (again by Lemma 2.3). And $\hat{\phi}$ extends $\phi$ because

$$\bar{\phi} = \sigma(\bar{\phi}) = \sigma(\sigma^{-1}[\phi]).$$

\[\text{□}\]

\[\text{†That is, } \bar{\phi} := \{\sigma^{-1}(a, b) : (a, b) \in \phi\}.\]
Now to prove Theorem 1.6: suppose \( R \) is a ring, \( S \) is a set of \( R \)-modules, and \( \kappa \) is a supercompact cardinal such that \( R \in H_\kappa \) and \( \bigcup S \subseteq H_\kappa \).

We claim that \( \perp S \) is a \( \kappa \)-Kaplansky class. Assume \( M \in \perp S \), and \( X \subseteq M \) is such that \( |X| < \kappa \). Fix a cardinal \( \lambda > \kappa \) such that \( M \in H_\lambda \). By Magidor’s lemma mentioned above, there is a 0-guessing \( G \in \mathcal{P}^*(H_\lambda) \) such that \( G \subseteq \mathcal{S}_\lambda \), \( X \in G \), and \( S \in G \). By Fact 2.2, \( X \) is also a subset of \( G \). So it will suffice to prove that \( G \cap M \) and \( \frac{M}{G \cap M} \) are both in \( \perp S \).

Since \( S \) is an element of \( G \) and has size \( < \kappa \), Fact 1 ensures that \( S \subseteq G \); and then another application of Fact 1 (this time using that \( \bigcup S \) is in \( H_\kappa \)) ensures that every \( S \subseteq S \) is both an element and a subset of \( G \). Consider any such \( S \). By part (i) of Lemma 3.1,

\[
\text{Ext}(G \cap M, G \cap S = S) = 0, \quad \text{so } G \cap M \subseteq \perp S.
\]

(4)

To show that \( \frac{M}{G \cap M} \subseteq \perp S \), consider the short exact sequence

\[
0 \longrightarrow G \cap M \longrightarrow M \longrightarrow \frac{M}{G \cap M} \longrightarrow 0
\]

and the associated exact sequence

\[
\begin{array}{ccc}
\text{Hom}(G \cap M, S) & \longrightarrow & \text{Hom}(M, S) \longrightarrow \text{Hom}(\frac{M}{G \cap M}, S) \\
\text{Ext}(G \cap M, S) & \longrightarrow & \text{Ext}(M, S) = 0 \longrightarrow \text{Ext}(\frac{M}{G \cap M}, S).
\end{array}
\]

(5)

Observe that the diagonal map is surjective, since \( \text{Ext}(M, S) = 0 \). Furthermore, since \( S \) is both an element and subset of \( G \), part (ii) of Lemma 3.1 ensures that the restriction map \( \text{Hom}(M, S) \rightarrow \text{Hom}(G \cap M, S) \) is also surjective. Then by exactness, the lower right term in the diagram (5) must be 0.

\[\text{PROOF OF THEOREM 1.1}\]

4.1 Proof of Theorem 1.3

\( V \) denotes the universe of sets. We will say \( P \) is a class relation if \( P \) is a definable subclass of \( V^n \) for some (meta-mathematical) natural number \( n \), possibly defined with some suppressed parameters.

† Recall from Section 2 that for a (partially) elementary submodel \( G \) of the universe of sets, if \( b \in G \), then \( b_{\mathcal{E}} \) denotes the image of \( b \) under the transitive collapse of \( G \). The following lemma is an immediate consequence of Corollary A.2 of Cox [4].

\[\text{That is, } P \text{ is a class relation if } P \subseteq V^n \text{ for some natural number } n, \text{ and there is a formula } \phi(u_1, \ldots, u_n, w_1, \ldots, w_k) \text{ in the language of set theory, and parameters } p_1, \ldots, p_k, \text{ such that } \]

\[P = \left\{ (x_1, \ldots, x_n) : \phi(x_1, \ldots, x_n, p_1, \ldots, p_k) \right\}. \]
**Lemma 4.1.** Assume Vopěnka’s Principle. Let $P \subseteq V^n$ be an $n$-ary class relation. Then there is a proper class of cardinals $\kappa$ (depending on $P$) with the following property: for every $$(a_1, \ldots, a_n) \in V^n \times H_\kappa,$$

there exists an $\mathcal{R}$ such that $|\mathcal{R}| < \kappa$, $\mathcal{R} \cap \kappa \in \kappa$, $\{a_1, \ldots, a_k, r\} \subset \mathcal{R} \prec_{\Sigma_1} (V, \in)$, and $$(a_1, \ldots, a_n) \in P \iff ((a_1)_R, \ldots, (a_k)_R) \in P.$$ Note that, if $\mathcal{R}$ is as in the conclusion of the lemma, then the transitive collapsing map of $\mathcal{R}$ fixes $r$, since $r \in \mathcal{R} \cap H_\kappa$ and $\mathcal{R} \cap \kappa$ is transitive (Fact 2.2). In the application below, the role of the $r$ will be played by the ring. Also note that if $|\mathcal{R}| < \kappa$, then the transitive collapse of $\mathcal{R}$ is both an element and subset of $H_\kappa$.

**Remark 4.2.** In concrete categories like $R$-Mod, if $M$ is an $R$-module and $R \cup \{R, M\} \subset \mathcal{R} \prec_{\Sigma_1} (V, \in)$, then we can talk about the $R$-module $\mathcal{R} \cap M$, which is isomorphic to $M_\mathcal{R}$. Similarly, if $f : A \to B$ is an $R$-module homomorphism and $f \in \mathcal{R}$, then we can make sense of

$$f \upharpoonright \mathcal{R} : \mathcal{R} \cap A \to \mathcal{R} \cap B,$$

which is isomorphic (in the arrow category) to $f_\mathcal{R}$.

But the transitive collapsed versions make sense even for non-concrete categories, and under VP can be arranged to be legitimate objects and morphisms from that category. For example, suppose $\{a, f\} \subset \mathcal{R} \prec_{\Sigma_1} (V, \in)$ where $a$ is an object and $f$ is a morphism in the possibly non-concrete category $C$. Then $a_\mathcal{R}$ and $f_\mathcal{R}$ are always defined (simply as the image of $a$, $f$ under the transitive collapse map of $\mathcal{R}$, respectively). And, if $\mathcal{R}$ correctly reflects the definition of the category — as is the case, for example, if the $P$ from Lemma 4.1 specifies the objects and morphisms of $C$, and $\mathcal{R}$ is as in the conclusion of that lemma — then $a_\mathcal{R}$ and $f_\mathcal{R}$ will be in the category $C$ as well (and have any other properties that $a$ and $f$ had that were specified by $P$).

We now prove Theorem 1.3. Assume Vopěnka’s Principle, and suppose $R$ is a ring and $B$ is a class of $R$-modules such that $\perp B$ is downward closed under elementary submodules for every $B \in B$. Let $P$ be the 2-ary class relation

$$P := \{(M, B) : M, B \in R\text{-Mod}, B \in B, \text{ and } M \notin \perp B\}.$$ By Lemma 4.1, there is a $\kappa$ such that $R \in H_\kappa$ and $\kappa$ has the required properties listed in the lemma with respect to $P$. Let $B^{< \kappa} := \{B \in B : |B| < \kappa\}$. We claim that $\perp B = \perp (B^{< \kappa})$, (6) which will complete the proof since $B^{< \kappa}$ has a set of representatives. The $\subseteq$ direction of (6) is trivial. To see the $\supseteq$ direction, suppose $M$ is an $R$-module and $M \notin \perp B$; then $\text{Ext}(M, B) \neq 0$ for some $B \in B$. So

$$(M, B) \in P.$$
By the property of $\kappa$, there is an $\mathcal{R}$ of size $< \kappa$ such that $\{M, B, R\} \subset \mathcal{R} <_{\Sigma_1} (V, \in)$, $\mathcal{R} \cap \kappa \in \kappa$, and $(M_{\mathcal{R}}, B_{\mathcal{R}}) \in P$; so

$$B_{\mathcal{R}} \in B$$

(7)

and

$$M_{\mathcal{R}} \notin \perp (B_{\mathcal{R}}).$$

(8)

(Viewed another way — assuming without loss of generality that $B$ is closed under isomorphisms — this just says that $\mathcal{R} \cap B \in B$ and $\mathcal{R} \cap M \notin \perp (\mathcal{R} \cap B)$).

We claim that $M \notin \perp (B_{\mathcal{R}})$; this will finish the proof, since $B_{\mathcal{R}}$ is a $< \kappa$-sized element of $B$. Suppose toward a contradiction that $M \in \perp (B_{\mathcal{R}})$. Since $R = R_{\mathcal{R}}$ (so in particular $R \subset \mathcal{R}$), Fact 2.2 ensures that $M_{\mathcal{R}}$ is an $R = R_{\mathcal{R}}$-module and

$$\sigma_{\mathcal{R}} | M_{\mathcal{R}} : M_{\mathcal{R}} \to M$$

is an elementary embedding in the language of $R$-modules, so $M_{\mathcal{R}}$ is isomorphic to an elementary submodule of $M$. Since $M \in \perp (B_{\mathcal{R}})$ and $\perp (B_{\mathcal{R}})$ is downward closed under elementary submodules, it follows that $M_{\mathcal{R}} \in \perp (B_{\mathcal{R}})$. But this contradicts (8). This completes the proof of Theorem 1.3.

4.2 Proof of Theorem 1.1

Assume $VP$ is consistent. By the result of the Appendix, $VP$ is also consistent with (*), so we may assume that both $VP$ and (*) hold. Suppose $(A, B)$ is a cotorsion pair such that $\perp B$ is downward closed under elementary submodules for all $B \in B$. By Theorem 1.3, there is a set $B_0 \subseteq B$ such that

$$\perp B = \perp B_0.$$ 

So the cotorsion pair

$$(A, B) = (\perp B, B) = (\perp B_0, B)$$

is cogenerated by the set $B_0$, and $\perp B$ is downward closed under elementary submodules for all $B \in B_0$. It follows that $\perp B_0$ is downward closed under elementary submodules. By (*) and the Šaroch–Trlifaj Theorem 1.5, the cotorsion pair $(\perp B_0, B)$ — which is cogenerated by the set $B_0$ — is also generated by a set, and hence complete.

5 Generalizations and Open Problems

Keeping Lemma 4.1 and Remark 4.2 in mind, Theorem 1.3 can be easily generalized to any category where the Ext functor make sense, so long as for every $b \in B$, the class $\perp b$ is closed under the $(-)_{\mathcal{R}}$ operation for sufficiently many $\mathcal{R} <_{\Sigma_1} (V, \in)$.
We end with some questions:

**Question 5.1** (Trlifaj, personal correspondence). Is there (a ZFC-provable) example of a ring and a cotorsion pair of modules over that ring that is *not* generated by a set?†

**Question 5.2** (General Salce Problem). Is there (a ZFC-provable) example of a ring and a cotorsion pair of modules over that ring that is *not* complete?‡

By Theorem 1.1, an affirmative answer to either question would require the ring to *not* be (ZFC-provably) hereditary. And by Theorem 1.6, an affirmative answer to either question could *not* be a (ZFC-provably) cotorsion pair of the form

\[
\left( \perp S, (\perp S)^\perp \right)
\]

where \( S \) is a set and \( \perp S \) is closed under direct limits.

We showed that consistency of the large cardinal principle VP implies the consistency of an affirmative solution to Salce’s question. This raises:

**Question 5.3.** Does the (scheme) *all* cotorsion pairs in the category of abelian groups are complete carry large cardinal consistency strength? Does the conclusion of Theorem 1.3 carry large cardinal consistency strength?

**APPENDIX: CONSISTENCY OF VOPĚNKA’S PRINCIPLE WITH DIAMOND ON ALL STATIONARY SETS**

For a regular uncountable \( \lambda \) and a stationary \( S \subseteq \lambda \), Jensen’s \( \Diamond_\lambda(S) \) principle asserts that there is a sequence

\[
\langle X_\alpha : \alpha \in S \rangle
\]

such that \( X_\alpha \subseteq \alpha \) for all \( \alpha \in S \), and for every \( X \subseteq \lambda \), the set

\[
\{ \alpha \in S : X \cap \alpha = X_\alpha \}
\]

is stationary. Starting with a model of Vopěnka’s Principle (VP), we briefly sketch how to produce a model of VP that also satisfies

\[
\forall \lambda \in \text{REG} \cap [\omega_1, \infty) \forall S \subseteq \lambda \text{ stationary } \Rightarrow \Diamond_\lambda(S).
\]  

(A1)

It is a folklore fact that if \( \lambda \) is regular and uncountable, then there is a \( < \lambda \)-directed closed poset \( D_\lambda \) forcing \( \Diamond_\lambda(S) \) holds for all stationary \( S \subseteq \lambda \); and if \( \lambda^{<\lambda} = \lambda \), then \( D_\lambda \) also has the \( \lambda^+ \)-chain condition (\( D_{<\lambda} \) is just a \( < \lambda \)-support product of adding Cohen subsets of \( \lambda \); in fact, adding a single Cohen subset of \( \lambda \) suffices, see [11]).

† Or, equivalently, whose left coordinate is not deconstructible.

‡ Equivalently, whose left coordinate is not a special precovering class?
Assume VP holds. By Corollary 26 of Brooke–Taylor [3], we may without loss of generality assume GCH holds as well. Define an Easton support iteration

$$\langle P_\alpha, \dot{Q}_\alpha : \alpha < \text{ORD} \rangle$$

where $P_\alpha$ forces ‘if $\alpha$ is regular and uncountable, $\dot{Q}_\alpha = \dot{D}_\alpha$; otherwise $\dot{Q}_\alpha$ is the trivial forcing’. Let $P$ be the (class-sized) direct limit of this iteration. By Theorem 25 of Brooke–Taylor [3], $P$ preserves VP. So we just need to show that $P$ forces (A1). Suppose $\lambda$ is regular in the extension; then Diamond holds at all stationary subsets of $\lambda$ in $V^{P_{\lambda+1}}$ by design, since the poset used at stage $\lambda$ was $D_{\lambda}^{V_{P_{\lambda}}}$. Note also that $P_{\lambda+1} = P_{\lambda+}$ because $\dot{Q}_\alpha$ is trivial for $\alpha \in (\lambda, \lambda^+)$. Since $\dot{Q}_\lambda(S)$ holds for all stationary $S \subseteq \lambda$ is clearly preserved by forcings that add no new subsets of $\lambda$, it suffices to show that $P_{\lambda+1} = P_{\lambda^+}$ forces the tail of the iteration to be $< \lambda^+$-directed closed. Now the GCH assumption, regularity of $\lambda$, and the fact that it is an Easton support iteration ensure that $|P_\lambda| \leq \lambda$, so $P_\lambda$ is (at worst) $\lambda^+$-cc. Hence, since $D_\lambda$ is also forced to be $\lambda^+$-cc, $P_{\lambda^+} = P_{\lambda+1} = P_\lambda \ast D_\lambda$ is also $\lambda^+$-cc. Then the assumptions of Proposition 7.12 of Cummings [5] are satisfied (with his $\beta$ and $\kappa$ both interpreted as our $\lambda^+$), which guarantees that $P_{\lambda^+}$ forces the tail of the iteration to be $< \lambda^+$-directed closed.

**Acknowledgements**

Many thanks to Jan Trlifaj for helpful correspondence regarding cotorsion pairs and deconstructibility.

**Journal Information**

The *Bulletin of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

**References**

1. J. Adámek and J. Rosický, *Locally presentable and accessible categories*, Lond. Math. Soc. Lect. Note Ser., vol. 189, Cambridge Univ. Press, Cambridge, 1994.
2. L. Bican, R. El Bashir, and E. Enochs, *All modules have at covers*, Bull. Lond. Math. Soc. **33** (2001), no. 4, 385–390.
3. A. D. Brooke-Taylor, *Indestructibility of Vopěnka’s Principle*, Arch. Math. Logic **50** (2011), no. 5-6, 515–529.
4. S. Cox, *Maximum deconstructibility in module categories*, J. Pure Appl. Algebra **226** (2022) 106934.
5. J. Cummings, *Iterated forcing and elementary embeddings*, Handbook of set theory, vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 775–883.
6. P. C. Eklof and S. Shelah, *On the existence of precovers*, Illinois J. Math. **47** (2003), no. 1-2, 173–188. Special issue in honor of Reinhold Baer (1902–1979).
7. P. C. Eklof and J. Trlifaj, *Covers induced by Ext*, J. Algebra **231** (2000), no. 2, 640–651.
8. P. C. Eklof and J. Trlifaj, *How to make Ext vanish*, Bull. Lond. Math. Soc. **33** (2001), no. 1, 41–51.
9. R. El Bashir, *Covers and directed colimits*, Algebr. Represent. Theory **9** (2006), no. 5, 423–430.
10. E. E. Enochs and J. A. López-Ramos, *Kaplansky classes*, Rend. Sem. Mat. Univ. Padova **107** (2002), 67–79.
11. M. Eskew, (https://mathoverow.net/users/11145/monroe-eskew), *Forcing Diamond*, MathOverflow. URL: https://mathoverow.net/q/125308 (version: 2013-03-22).
12. M. Foreman, M. Magidor, and S. Shelah, *Martin’s maximum, saturated ideals, and nonregular ultrfilters*, I, Ann. of Math. (2) **127** (1988), no. 1, 1–47.
13. R. Göbel and J. Trlifaj, *Approximations and endomorphism algebras of modules. Volume I*, Second revised and extended edition, De Gruyter Exp. Math., vol. 41, Walter de Gruyter GmbH & Co. KG, Berlin, 2012.

14. T. Jech, *Set theory*, Springer Monographs in Mathematics, Springer, Berlin, 2003. The third millennium edition, revised and expanded.

15. L. Salce, *Cotorsion theories for abelian groups*, Symposia Mathematica, Vol. XXIII (Conf. Abelian Groups and their Relationship to the Theory of Modules, INDAM, Rome, 1977), Academic Press, London–New York, 1979, pp. 11–32.

16. J. Šaroch and J. Trlifaj, *Completeness of cotorsion pairs*, Forum Math. **19** (2007), no. 4, 749–760.

17. J. Št’ovíček, *Deconstructibility and the Hill lemma in Grothendieck categories*, Forum Math. **25** (2013), no. 1, 193–219.

18. M. Viale, *Guessing models and generalized Laver diamond*, Ann. Pure Appl. Logic **163** (2012), no. 11, 1660–1678.