We prove that the distribution of eigenvectors of generalized Wigner matrices is universal both in the bulk and at the edge. This includes a probabilistic version of local quantum unique ergodicity and asymptotic normality of the eigenvector entries. The proof relies on analyzing the eigenvector flow under the Dyson Brownian motion. The key new ideas are: (1) the introduction of the eigenvector moment flow, a multi-particle random walk in a random environment, (2) an effective estimate on the regularity of this flow based on the maximum principle.

**Keywords**: Universality, Quantum unique ergodicity, Eigenvector moment flow.

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1 Introduction

Wigner has envisioned that the laws of the eigenvalues of large random matrices are new paradigms for universal statistics of large correlated quantum systems. Although this vision has not been proved for any truly interacting quantum system, it is generally considered to be valid for a wide range of models. For example, the quantum chaos conjecture by Bohigas-Giannoni-Schmit [6] asserts that eigenvalue statistics of the Laplace operator on a domain or manifold are given by the random matrix statistics provided that the corresponding classical dynamics are chaotic. Similarly, one expects that the eigenvalue statistics of random Schrödinger operators (Anderson tight binding models) are given by the random matrix statistics in the delocalization regime. Unfortunately, both conjectures are far beyond the reach of the current mathematical technology.

In Wigner’s original theory, the eigenvector behaviour plays no role. As suggested by the Anderson model, random matrix statistics coincide with delocalization of eigenvectors. A strong notion of delocalization, at least in terms of “flatness of the eigenfunctions”, is the quantum ergodicity. For the Laplacian on a negative curved compact Riemannian manifold, Shnirel’man [30], Colin de Verdière [10] and Zelditch [35] proved that quantum ergodicity holds. More precisely, let \( \{\psi_k\}_{k \geq 1} \) denote an orthonormal basis of eigenfunctions of the Laplace-Beltrami operator, associated with increasing eigenvalues, on a negative curved manifold \( M \) (or more generally, assume only that the geodesic flow of \( M \) is ergodic) with volume measure \( \mu \). Then for any open set \( A \subset M \) one has

\[
\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{j : \lambda_j \leq \lambda} \left| \int_A |\psi_j(x)|^2 \mu(dx) - \int_A \mu(dx) \right|^2 = 0
\]

where \( N(\lambda) = |\{j : \lambda_j \leq \lambda\}| \). Quantum ergodicity was also proved for \( d \)-regular graphs under certain assumptions on the injectivity radius and spectral gap of the adjacency matrices [3]. Random graphs are considered a good paradigm for many ideas related to quantum chaos [23].

An even stronger notion of delocalization is the quantum unique ergodicity conjecture (QUE) proposed by Rudnick-Sarnak [29], i.e., for any negatively curved compact Riemannian manifold \( M \), the eigenstates become equidistributed with respect to the volume measure \( \mu \): for any open \( A \subset M \) we have

\[
\int_A |\psi_k(x)|^2 \mu(dx) \xrightarrow{k \to \infty} \int_A \mu(dx).
\]

Some numerical evidence exists for both eigenvalue statistics and the QUE, but a proper understanding of the semiclassical limit of chaotic systems is still missing. One case for which QUE was rigorously proved concerns arithmetic surfaces, thanks to tools from number theory and ergodic theory on homogeneous spaces [19, 20, 25]. For results in the case of general compact Riemannian manifolds whose geodesic flow is Anosov, see [2].

A major class of matrices for which one expects that Wigner’s vision holds is the Wigner matrices, i.e., random matrices with matrix elements distributed by identical mean-zero random variables. For this class of matrices, the Wigner-Dyson-Mehta conjecture states that the local statistics are independent of the laws of the matrix elements and depend only on the symmetry class. This conjecture was recently solved for an even more general class: the generalized Wigner matrices for which the distributions of matrix entries can vary and have different variances. (See [16, 17] and [15] for a review. For earlier results on this conjecture for Wigner matrices, see [13, 33] for the bulk of the spectrum and [18, 31, 32] for the edge). One key ingredient of the method initiated in [13] proceeds by interpolation between Wigner and Gaussian ensembles through Dyson’s Brownian motion, a matrix process which induces an autonomous evolution of eigenvalues. The
fundamental conjecture for the Dyson Brownian motion, the Dyson conjecture, states that the time to local equilibrium in of order \( t \gtrsim 1/N \), where \( N \) is the size of the matrix. This conjecture was resolved in [18] (see [13] for the earlier results) and is the underlying reason for the universality.

Concerning the eigenvectors distribution, complete delocalization was proved in [18] for generalized Wigner matrices in the following sense: with very high probability

\[
\max |u_i(\alpha)| \leq \frac{(\log N)^C \log \log N}{\sqrt{N}},
\]

where \( C \) is a fixed constant and the maximum ranges over all coordinates \( \alpha \) of the \( L^2 \)-normalized eigenvectors, \( u_1, \ldots, u_N \) (a stronger estimate was obtained for Wigner matrices in [12], see also [7] for a delocalization bound for the Laplacian on deterministic regular graphs). Although this bound prevents concentration of eigenstates onto a set of size less than \( N(\log N)^{-C \log \log N} \), it does not imply the “complete flatness” of type (1.1). In fact, if the eigenvectors are distributed by the Haar measure on the orthogonal group, the weak convergence

\[
\sqrt{N} u_i(\alpha) \to \mathcal{N},
\]

holds, where \( \mathcal{N} \) is a standard Gaussian random variable and the eigenvector components are asymptotically independent. Since the eigenvectors of GOE are distributed by the Haar measure on the orthogonal group, this asymptotic normality [12] holds for GOE (and a similar statement holds for GUE). For Wigner ensembles, by comparing with GOE, this property was proved for eigenvectors in the bulk by Knowles-Yin and Tao-Vu [21, 34] under the condition that the first four moments of the matrix elements of the Wigner ensembles match those of the standard normal distribution. For eigenvectors near the edges, the matching condition can be reduced to only the first two moments [21].

In this paper, we develop a completely new method to show that this asymptotic normality (1.2) and independence of eigenvector components hold for generalized Wigner matrices without any moment matching condition. In particular, even the second moments of the matrix elements of the Wigner ensembles match those of the standard normal distribution. For eigenvectors near the edges, the matching condition can be reduced to only the first two moments [21].

The key idea in this new approach is to analyze the “Dyson eigenvector flow”. More precisely, the Dyson Brownian motion is induced by the dynamics in which matrix elements undergo independent Brownian motions. The same dynamics on matrix elements yield a flow on the eigenvectors. This eigenvector flow, which we will call the *Dyson eigenvector flow*, was computed in the context of Brownian motion on ellipsoids [27], real Wishart processes [8], and for GOE/GUE in [4] (see also [1]). This flow is a diffusion process on a compact Lie group \( O(N) \) or \( U(N) \) endowed with a Riemannian metric. This diffusion process roughly speaking can be described as follows. We first randomly choose two eigenvectors, \( u_i \) and \( u_j \). Then we randomly rotate these two vectors on the circle spanned by them with a rate \((\lambda_i - \lambda_j)^{-2}\) depending on the eigenvalues. Thus the eigenvector flow depends on the eigenvalue dynamics. If we freeze the eigenvalue flow, the eigenvector flow is a diffusion with time dependent singular coefficients depending on the eigenvalues.

Due to its complicated structure, the Dyson eigenvector flow has never been analyzed. Our key observation is that the dynamics of the moments of the eigenvector entries can be viewed as a multi-particle random walk in a random environment. The number of particles of this flow is one half of the degree of polynomials in the eigenvector entries, and the (dynamic) random environment is given by jump rates depending on the eigenvalues. We shall call this flow the *eigenvector moment flow*. If there is only one particle, this flow is the random walk with the random jump rate \((\lambda_i - \lambda_j)^{-2}\) between two integer locations \( i \) and \( j \). This one
dimensional random walk process was analyzed in [14] locally for the purpose of the single gap universality between eigenvalues. An important result of [14] is the Hölder regularity of the solutions. In higher dimensions, the jump rates depend on the locations of nearby particles and the flow is not a simple tensor product of the one dimensional process. Fortunately, we find that this flow is reversible with respect to an explicit equilibrium measure. The Hölder regularity argument in [14] can be extended to any dimension to prove that the solutions of the moment flow are locally Hölder continuous. From this result and the local semicircle law (more precisely, the isotropic local semicircle law proved in [22] and [5]), one can obtain that the bulk eigenvectors generated by a Dyson eigenvector flow satisfy local quantum unique ergodicity, and the law of the entries of the eigenvectors are Gaussian.

Instead of showing the Hölder regularity, we will directly prove that the solution to the eigenvector moment flow converges to a constant. This proof is based on a maximum principle for parabolic differential equations and the local isotropic law [5] mentioned previously. It yields the convergence of the eigenvector moment flow to a constant for \( t \gtrsim N^{-1/4} \) with explicit error bound. This immediately implies that all eigenvectors (in the bulk and at the edge) generated by a Dyson eigenvector flow satisfy local quantum unique ergodicity, and the law of the entries of the eigenvectors are Gaussian. For bulk eigenvectors, we will prove that the eigenvector moment flow reaches equilibrium for \( t \gtrsim N^{-1} \), which is optimum.

In order to prove that the eigenvectors of the original matrix ensemble satisfy quantum ergodicity, it remains to approximate the Wigner matrices by Gaussian convoluted ones, i.e., matrices which are a small time solution to the Dyson Brownian motion. We invoke the Green function comparison theorem in a version similar to the one stated in [21]. For bulk eigenvectors, we can remove this small Gaussian component by a continuity principle instead of the Green function comparison theorem: we will show that the Dyson Brownian motion preserves the detailed behavior of eigenvalues and eigenvectors up to time \( N^{-1/2} \) directly by using the Itô formula. This approach is much more direct and there is no need to construct moment matching matrices.

The eigenvector moment flow developed in this paper can be applied to other random matrix models. For example, the local quantum unique ergodicity holds for covariance matrices and a certain class of Erdős-Rényi graphs. To avoid other technical issues, in this paper we only consider generalized Wigner matrices. Before stating the results and giving more details about the proof, we recall the definition of the considered ensemble.

**Definition 1.1.** A generalized Wigner matrix \( H_N \) is an Hermitian or symmetric \( N \times N \) matrix whose upper-triangular matrix elements \( h_{ij} = \overline{h_{ji}}, \, i \leq j \), are independent random variables with mean zero and variance \( \sigma_{ij}^2 = \mathbb{E}(|h_{ij}|^2) \) satisfying the following additional two conditions:

(i) Normalization: for any \( j \in [1,N] \), \( \sum_{i=1}^{N} \sigma_{ij}^2 = 1 \).

(ii) Non-degeneracy: there exists a constant \( C \) independent of \( N \), such that \( C^{-1}N^{-1} \leq \sigma_{ij}^2 \leq CN^{-1} \) for all \( i, j \in [1,N] \). In the Hermitian case, we furthermore assume that, for any \( i < j \), \( \mathbb{E}((h_{ij})^*h_{ij}) \geq cN^{-1} \) in the sense of inequality between \( 2 \times 2 \) positive matrices, where \( h_{ij} = (\Re(h_{ij}), \Im(h_{ij})) \).

Moreover, we assume that all moments of the entries are finite: for any \( p \in \mathbb{N} \) there exists a constant \( C_p \) such that for any \( i,j,N \) we have

\[
\mathbb{E}(|\sqrt{N}h_{ij}|^p) < C_p.
\]

In the following, \( (u_i)_{i=1}^{N} \) denotes an orthonormal eigenbasis for \( H_N \), a matrix from the (real or complex) generalized Wigner ensemble. The eigenvector \( u_i \) is associated with the eigenvalue \( \lambda_i \), where \( \lambda_1 \leq \ldots \leq \lambda_N \).
Theorem 1.2. Let \((H_N)_{N \geq 1}\) be a sequence of generalized Wigner matrices, \(m \in \mathbb{N}\) and \(I \subset [1, N]\), \(|I| = m\). Then for any unit vector \(\mathbf{q}\) in \(\mathbb{R}^N\), we have
\[
\sqrt{N}(|\langle q, u_k \rangle|)_{k \in I} \rightarrow (|A_j|)_{j=1}^m
\]
in the symmetric case, and
\[
2\sqrt{N}(|\langle q, u_k \rangle|)_{k \in I} \rightarrow (|A_j^{(1)} + iA_j^{(2)}|)_{j=1}^m
\]
in the Hermitian case,

in the sense of convergence of moments, where all \(A_j, A_j^{(1)}, A_j^{(2)}\), are independent standard Gaussian random variables. This convergence holds uniformly in \(I\) and \(|q| = 1\). More precisely, for any polynomial \(P\) in \(m\) variables, there exists \(\varepsilon = \varepsilon(P) > 0\) such that for large enough \(N\) we have
\[
\sup_{I \subset [1, N], |I|=m, |q|=1} \left| \mathbb{E} \left( P \left( (N|\langle q, u_k \rangle|^2)_{k \in I} \right) \right) - \mathbb{E} \left( P \left( (|A_j|^2)_{j=1}^m \right) \right) \right| \leq N^{-\varepsilon}, \tag{1.4}
\]
respectively for the real and complex generalized Wigner ensembles.

This convergence of moments implies in particular joint weak convergence. Choosing \(\mathbf{q}\) an element of the canonical basis, Theorem 1.2 implies in particular that any entry of an eigenvector is asymptotically normally distributed, modulo the (arbitrary) phase choice. Because the above convergence holds for any \(|q| = 1\), asymptotic joint normality of the eigenvector entries also holds. Since eigenvectors are defined only up to a phase, we define the equivalence relation \(u \sim v\) if \(u = \pm v\) in the symmetric case and \(u = \lambda v\) for some \(|\lambda| = 1\) in the Hermitian case.

Corollary 1.3 (Asymptotic normality of eigenvectors for generalized Wigner matrices). Let \((H_N)_{N \geq 1}\) be a sequence of generalized Wigner matrices, \(\ell \in \mathbb{N}\). Then for any \(k \in [1, N]\) and \(J \subset [1, N]\), \(|J| = \ell\), we have
\[
\sqrt{N} \langle u_k(\alpha) \rangle_{\alpha \in J} \rightarrow (A_{j,\ell})_{j=1}^N \quad \text{for the real generalized Wigner ensemble,}
\]
\[
2\sqrt{N} \langle u_k(\alpha) \rangle_{\alpha \in J} \rightarrow (A_{j,\ell}^{(1)} + iA_{j,\ell}^{(2)})_{j=1}^N \quad \text{for the complex generalized Wigner ensemble,}
\]
in the sense of convergence of moments modulo \(\sim\), where all \(A_j, A_j^{(1)}, A_j^{(2)}\), are independent standard Gaussian variables. More precisely, for any polynomial \(P\) in \(\ell\) variables (resp. \(Q\) in \(2\ell\) variables) there exists \(\varepsilon\) depending on \(P\) (resp. \(Q\)) such that, for large enough \(N\),
\[
\sup_{J \subset [1, N], |J| = \ell, k \in [1, N]} \left| \mathbb{E} \left( P \left( \sqrt{N} \langle e^{i\omega u_k(\alpha)} \rangle_{\alpha \in J} \right) \right) - \mathbb{E} \left( (A_{j,\ell})_{j=1}^N \right) \right| \leq N^{-\varepsilon}, \tag{1.5}
\]
for the symmetric (resp. Hermitian) generalized Wigner ensembles. Here \(\omega\) is independent of \(H_N\) and uniform on the binary set \(\{0, \pi\}\) (resp. \(\{0, 2\pi\}\)).
Corollary 1.4 (Local quantum unique ergodicity for generalized Wigner matrices). Let \((H_N)_{N \geq 1}\) be a sequence of generalized (real or complex) Wigner matrices. Then there exists \(\varepsilon > 0\) such that for any \(\delta > 0\), there exists \(C > 0\) such that the following holds: for any \((a_N)_{N \geq 1}\), \(a_N : [1, N] \to [-1, 1]\) with \(\sum_{\alpha=1}^{N} a_N(\alpha) = 0\) and \(k \in [1, N]\), we have

\[
P\left(\left|\frac{N}{a_N}\langle u_k, a_N u_k \rangle\right| > \delta\right) \leq C \left(N^{-\varepsilon} + |a_N|^{-1}\right).
\]  

(1.6)

Under the condition that the first four moments of the matrix elements of the Wigner ensembles match those of the standard normal distribution, (1.6) can also be proved from the results in [21, 34]; the four moment matching conditions was reduced to two moments for eigenvectors near the edges [21].

The quantum ergodicity for a class of sparse regular graphs was proved by Anantharaman-Le Masson [3], partly based on pseudo-differential calculus on graphs from [24]. The main result in [3] is for deterministic graphs, but for the purpose of this paper we only state its application to random graphs (see [3] for details and more general statements). If \(u_1, \ldots, u_N\) are the \((L^2\)-normalized) eigenvectors of the discrete Laplacian of a uniformly chosen \((q+1)\)-regular graph with \(N\) vertices, then for any fixed \(\delta > 0\) we have, for any \(q \geq 1\) fixed,

\[
P\left(\exists k : |\langle u_k, a_N u_k \rangle| > \delta\right) \underset{N \to \infty}{\longrightarrow} 0,
\]

where \(a_N\) may be random (for instance, it may depend on the graph). The results in [3] were focused on very sparse deterministic regular graphs and are very different from our setting for generalized Wigner matrices.

Notice that our result (1.6) allows the test function to have a very small support and it is valid for any \(k\). This means that eigenvectors are flat even in “microscopic scales”. However, the equation (1.6) does not imply that all eigenvectors are completely flat simultaneously with high probability, i.e., we have not proved the following statement:

\[
P\left(\sup_{1 \leq k \leq N} |\langle u_k, a_N u_k \rangle| > \delta\right) \to 0
\]

for \(a_N\) with support of order \(N\). This strong form of QUE, however, holds for the Gaussian ensembles.

Our next task is to show that the eigenvector moment flow reaches equilibrium in the bulk at the optimal time scale \(t \gtrsim N^{-1}\). In the following statement, \(M_N\) is any \(N \times N\) symmetric or Hermitian matrix satisfying the isotropic local semicircle law. Let \(G_N^{(s)}\) (resp. \(G_N^{(h)}\)) be a sequence of \(N \times N\) random matrices from the Gaussian orthogonal (resp. unitary) ensemble (normalized with limiting spectral measure supported on \((-2, 2)\), for example).

**Theorem 1.5.** Assume that the distributions of \((M_N)_{N \geq 1}\) are in the “good set” \(\mathcal{A}\) (see (6.12)). Let \(u\) be an eigenbasis of \(M_N + \sqrt{t} G_N^{(s)}\). Let \(\varepsilon\) be any arbitrarily small positive constants. For any \(N^{-1+\varepsilon} \leq t \leq 1\), asymptotic normality and local quantum unique ergodicity hold for the bulk eigenvectors (see (6.16) for a precise statement). Similar results holds for \(M_N + \sqrt{t} G_N^{(h)}\).

This theorem means that, the initial structure of bulk eigenvectors completely disappears with the addition of a small noise, provided that the initial matrix satisfies a strong form of semicircle law. The precise conditions are encoded in the good set \(\mathcal{A}\) (see (6.12)) to be specified in Section 6. Note that by standard perturbation theory Theorem 1.5 in general does not hold for \(t \ll N^{-1}\). Recall that Dyson’s conjecture states that the relaxation time to local equilibrium for bulk eigenvalues under the DBM is \(t \sim N^{-1}\). Thus Theorem 1.5 is the analogue of this conjecture in the context of bulk eigenvectors. We will prove Theorem...
by using a localized version of the previously mentioned maximum principle. This relies on a finite speed of propagation estimate for the eigenvector moment flow which will be explained in Section 6.

In the following section, we will define the Dyson vector flow and, for the sake of completeness, prove the well-posedness of the eigenvector stochastic evolution. In Section 3 we will introduce the eigenvector moment flow and prove the existence of an explicit reversible measure. In Section 4, we will prove Theorem 1.2 under the additional assumption that $H_N$ is the sum of a generalized Wigner matrix and a Gaussian matrix with small variance. The proof in this section relies on a maximum principle for the eigenvector moment flow. We will prove Theorem 1.2 by using the Green function comparison theorem in Section 5. Relaxation to equilibrium in the bulk for $t \gtrsim N^{-1}$ will be proved in Section 6, where the localized maximum principle will be developed. The appendix contains a continuity estimate for the Dyson Brownian motion up to time $N^{-1/2}$.

### 2 The Dyson Vector Flow

This section first mentions the stochastic differential equation for the eigenvectors under the Dyson Brownian motion. This evolution is given by (2.3) and (2.5). We then give a concise form of the generator for this Dyson vector flow. We will follow the usual slight ambiguity of terminology by naming both the matrix flow and the eigenvalue flow a Dyson Brownian motion. In case we wish to distinguish them, we will use matrix Dyson Brownian motion for the matrix flow.

**Definition 2.1.** Hereafter is our choice of normalization for the Dyson Brownian motion.

(i) Let $B^{(s)}$ be a $N \times N$ matrix such that $B^{(s)}_{ij}$ ($i < j$) and $B^{(s)}_{ij}/\sqrt{2}$ are independent standard Brownian motions, and $B^{(s)}_{ij} = B^{(s)}_{ji}$. The $N \times N$ symmetric Dyson Brownian motion $H^{(s)}$ with initial value $H^{(s)}_0$ is defined as

$$H^{(s)}_t = H^{(s)}_0 + \frac{1}{\sqrt{N}} B^{(s)}_t,$$

(2.1)

(ii) Let $B^{(h)}$ be a $N \times N$ matrix such that $\Re(B^{(h)}_{ij}), \Im(B^{(h)}_{ij})(i < j)$ and $B^{(h)}_{ii}/\sqrt{2}$ are independent standard Brownian motions, and $B^{(h)}_{ji} = (B^{(h)}_{ij})^*$. The $N \times N$ Hermitian Dyson Brownian motion $H^{(h)}$ with initial value $H^{(h)}_0$ is

$$H^{(h)}_t = H^{(h)}_0 + \frac{1}{2N} B^{(h)}_t.$$

**Definition 2.2.** We refer to the following stochastic differential equations as the Dyson Brownian motion for (2.2) and (2.4) and the Dyson vector flow for (2.3) and (2.5).

(i) Let $\lambda_0 \in \Sigma_N = \{\lambda_1 < \cdots < \lambda_N\}, u_0 \in O(N)$, and $B^{(s)}$ be as in Definition 2.1. The symmetric Dyson Brownian motion/vector flow with initial condition $(\lambda_1, \ldots, \lambda_N) = \lambda_0, (u_1, \ldots, u_N) = u_0$, is

$$d\lambda_k = \frac{dB^{(s)}_{kk}}{\sqrt{N}} + \left( \frac{1}{N} \sum_{\ell \neq k} \frac{1}{\lambda_k - \lambda_\ell} \right) dt,$$

$$du_k = \frac{1}{\sqrt{N}} \sum_{\ell \neq k} dB^{(s)}_{kl} u_\ell - \frac{1}{2N} \sum_{\ell \neq k} (\lambda_k - \lambda_\ell)^2 u_k.$$

(2.2)
(ii) Let \( \lambda_0 \in \Sigma_N, \ u_0 \in U(N) \), and \( B^{(h)} \) be as in Definition 2.7. The Hermitian Dyson Brownian motion/vector flow with initial condition \( (\lambda_1, \ldots, \lambda_N) = \lambda_0, (u_1, \ldots, u_N) = u_0, \) is

\[
d\lambda_k = \frac{dB^{(h)}_{kk}}{\sqrt{2N}} + \left( \frac{1}{N} \sum_{\ell \neq k} \frac{1}{\lambda_k - \lambda_\ell} \right) dt, \tag{2.4}
\]

\[
du_k = \frac{1}{\sqrt{2N}} \sum_{\ell \neq k} \frac{dB^{(h)}_{k\ell}}{\lambda_k - \lambda_\ell} u_\ell - \frac{1}{2N} \sum_{\ell \neq k} \frac{dt}{(\lambda_k - \lambda_\ell)^2} u_k. \tag{2.5}
\]

The theorem below contains the following results. (a) The above stochastic differential equations admit a unique strong solution, this relies on classical techniques and an argument originally by McKean [26]. (b) The matrix Dyson Brownian motion induces the eigenvalues Dyson Brownian motion and the eigenvector Dyson flow. This statement was already proved in [4]. (c) For calculation purpose, one can condition on the trajectory of the eigenvalues to study the eigenvectors evolution. For the sake of completeness, this theorem is proved in the appendix.

With a slight abuse of notation, we will write \( \lambda_t \) either for \( (\lambda_1(t), \ldots, \lambda_N(t)) \) or for the \( N \times N \) diagonal matrix with entries \( \lambda_1(t), \ldots, \lambda_N(t) \).

Theorem 2.3. The following statements about the Dyson Brownian motion and eigenvalue/vector flow hold.

(a) Existence and strong uniqueness hold for the system of stochastic differential equations (2.4), (2.5). Let \( (\lambda_t, u_t)_{t \geq 0} \) be the solution. Almost surely, for any \( t \geq 0 \) we have \( \lambda_t \in \Sigma_N \) and \( u_t \in O(N) \).

(b) Let \( (H_t)_{t \geq 0} \) be a symmetric Dyson Brownian motion with initial condition \( H_0 = u_0 \lambda_0 u_0^* \), \( \lambda_0 \in \Sigma_N \). Then the processes \( (H_t)_{t \geq 0} \) and \( (u_t \lambda_t u_t^*)_{t \geq 0} \) have the same distribution.

(c) Existence and strong uniqueness hold for (2.2). For any \( T > 0 \), let \( \nu_T^{H_0} \) be the distribution of \( (\lambda_t)_{0 \leq t \leq T} \) with initial value the spectrum of a matrix \( H_0 \). For \( 0 \leq T \leq T_0 \) and any given continuous trajectory \( \lambda = (\lambda_t)_{0 \leq t \leq T_0} \subset \Sigma_N \), existence and strong uniqueness holds for (2.3) on \( [0, T] \). Let \( \mu_T^{H_0, \lambda} \) be the distribution of \( (u_t)_{0 \leq t \leq T} \) with initial value the eigenvectors of a matrix \( H_0 \) and \( \lambda \) given.

Let \( F \) be continuous bounded, from the set of continuous paths on \( [0, T] \) on \( N \times N \) symmetric matrices to \( \mathbb{R} \). Then for any initial matrix \( H_0 \) we have

\[
\mathbb{E}^{H_0}(F((H_t)_{0 \leq t \leq T})) = \int d\nu_T^{H_0}(\lambda) \int d\mu_T^{H_0, \lambda}(u) F((u_t \lambda_t u_t^*)_{0 \leq t \leq T}). \tag{2.6}
\]

The exact analogous statements hold in the Hermitian setting.

We will omit the subscript \( T \) when it is obvious. The previous theorem reduces the study of the eigenvector dynamics to the stochastic differential equations (2.3) and (2.5). The following lemma gives a concise form of the generators of these diffusions. It is very similar to the well-known forms of the generator for the Brownian motion on the unitary/orthogonal groups up to the following difference: weights vary depending on eigenvalue pairs.
We will need the following notations (the dependence in $t$ will often be omitted for $c_{k\ell}$, $1 \leq k < \ell \leq N$):

\[
c_{k\ell}(t) = \frac{1}{N(\lambda_k(t) - \lambda_\ell(t))^2},
\]

\[
u_k \partial u_\ell = \sum_{\alpha=1}^{N} u_k(\alpha) \partial u_{\ell(\alpha)}, \quad u_k \partial \pi_\ell = \sum_{\alpha=1}^{N} u_k(\alpha) \partial \pi_{\ell(\alpha)},
\]

\[
X_{k\ell}^{(s)} = u_k \partial_{u_\ell} - \nu_\ell \partial_{u_k}, \quad X_{k\ell}^{(h)} = \nu_k \partial_{u_\ell} - \nu_\ell \partial_{u_k}.
\]

**Lemma 2.4.** For the diffusion (2.3) (resp. (2.5)), the generators acting on smooth functions $f((u_i(\alpha))_{1 \leq i, \alpha \leq N})$:

\[
\mathbb{R}^{N^2} \to \mathbb{R} \quad \text{resp.} \quad \mathbb{C}^{N^2} \to \mathbb{R}
\]

are respectively

\[
L_i^{(s)} = \sum_{1 \leq k < \ell \leq N} c_{k\ell}(t) X_{k\ell}^{(s)},
\]

\[
L_i^{(h)} = \frac{1}{2} \sum_{1 \leq k < \ell \leq N} c_{k\ell}(t) \left( X_{k\ell}^{(h)} + X_{k\ell}^{(h)} X_{k\ell}^{(h)} \right).
\]

The above lemma means $d\mathbb{E}(g(u_t))/dt = \mathbb{E}(L_i^{(s)} g(u_t))$ (resp. $d\mathbb{E}(g(u_t))/dt = \mathbb{E}(L_i^{(h)} g(u_t))$) for the stochastic differential equations (2.3) (resp. (2.5)). It relies on a direct calculation via Itô’s formula. The details are given in the appendix.

## 3 The Eigenvector Moment Flow

### 3.1 The moment flow

Our observables will be moments of projections of the eigenvectors onto a given direction. More precisely, for any fixed $q \in \mathbb{R}^N$ and for any $1 \leq k \leq N$, define

\[
z_k(t) = \sqrt{N} \langle q, u_k(t) \rangle = \sum_{\alpha=1}^{N} q(\alpha) u_k(t, \alpha).
\]

With this $\sqrt{N}$ normalization, the typical size of $z_k$ is of order 1. We assume that the eigenvalues trajectory $(\lambda_k(t), 0 \leq t \leq T_0)_{k=1}^{N}$ is given and remains in the simplex $\Sigma(N)$. Furthermore, $u$ is the unique strong solution of the stochastic differential equation (2.3) (resp. (2.5)) with the given eigenvalues trajectory. Let $P^{(s)}(t) = P^{(s)}(z_1, \ldots, z_N)(t)$ and $P^{(h)}(t) = P^{(h)}(z_1, \ldots, z_N)(t)$ be smooth functions. Then a simple calculation yields

\[
X_{k\ell}^{(s)} P^{(s)} = (z_k \partial_{z_\ell} - z_\ell \partial_{z_k}) P^{(s)},
\]

\[
X_{k\ell}^{(h)} P^{(h)} = (z_k \partial_{z_\ell} - \pi_\ell \partial_{\pi_k}) f, \quad X_{k\ell}^{(h)} P^{(h)} = (\pi_k \partial_{\pi_\ell} - z_\ell \partial_{z_k}) P^{(h)}.
\]
For $m \in [1, N]$, denote by $j_1, \ldots, j_m$ positive integers and let $i_1, \ldots, i_m$ in $[1, N]$ be $m$ distinct indices. The test functions we will consider are:

$$P(s)_{i_1, \ldots, i_m}^{j_1, \ldots, j_m}(z_1, \ldots, z_N) = \prod_{\ell=1}^m z_{i_\ell}^{2j_\ell},$$

$$P(h)_{i_1, \ldots, i_m}^{j_1, \ldots, j_m}(z_1, \ldots, z_N) = \prod_{\ell=1}^m z_{i_\ell}^{j_\ell}.$$

For any $m$ fixed, linear combinations of such polynomial functions are stable under the action of the generator. More precisely, the following formulas hold.

(i) In the symmetric setting, one can use \((3.1)\) to evaluate the action of the generator. If neither $k$ nor $\ell$ are in $\{i_1, \ldots, i_m\}$, then $(X_{k\ell}^{(s)})^2 P(s)_{i_1, \ldots, i_m}^{j_1, \ldots, j_m} = 0$; the other cases are covered by:

$$X_{k\ell}^{(s)} P(s)_{i_1, \ldots, i_m}^{j_1, \ldots, j_m} = 2j_1 (2j_1 - 1) P(s)_{i_1, \ldots, i_m}^{j_1, \ldots, j_m} - 2j_1 P(s)_{i_1, \ldots, i_m}^{j_1, \ldots, j_m} \text{ when } \ell \notin \{i_1, \ldots, i_m\},$$

$$X_{i_1 i_2}^{(s)} P(s)_{i_1, \ldots, i_m}^{j_1, \ldots, j_m} = 2j_1 (2j_1 - 1) P(s)_{i_1, \ldots, i_m}^{j_1, \ldots, j_m} + 2j_2 (2j_2 - 1) P(s)_{i_1, \ldots, i_m}^{j_1, j_2 - 1, \ldots, j_m} - (2j_1 (2j_2 + 1) + 2j_2 (2j_1 + 1)) P(s)_{i_1, i_2, \ldots, i_m}^{j_1, j_2, \ldots, j_m}.$$

(ii) In the Hermitian setting, we note that the polynomials $P^{(h)}$ are invariant under the permutation $z_i \to z_i$. Thus the action of the generator $L_{i_\ell}^{(h)}$ \((2.8)\) on such functions $P^{(h)}$ simplifies to

$$L_{i_\ell}^{(h)} P^{(h)} = \sum_{k < \ell} c_{k\ell} X_{k\ell}^{(h)} X_{i_\ell}^{(h)} P^{(h)}.$$

Then \((3.2)\) yields

$$X_{k\ell}^{(h)} X_{i_\ell}^{(h)} P^{(h)}_{i_1, \ldots, i_m}^{j_1, \ldots, j_m} = j_1^2 P^{(h)}_{i_1, \ldots, i_m}^{j_1 - 1, \ldots, j_m} - j_1 P^{(h)}_{i_1, \ldots, i_m}^{j_1, \ldots, j_m} \text{ when } \ell \notin \{i_1, \ldots, i_m\},$$

$$X_{i_1 i_2}^{(h)} X_{i_\ell}^{(h)} P^{(h)}_{i_1, \ldots, i_m}^{j_1, \ldots, j_m} = j_1^2 P^{(h)}_{i_1, \ldots, i_m}^{j_1 - 1, j_2 + 1, \ldots, j_m} + j_2^2 P^{(h)}_{i_1, \ldots, i_m}^{j_1, j_2 - 1, \ldots, j_m} - (j_1 (j_2 + 1) + j_2 (j_1 + 1)) P^{(h)}_{i_1, i_2, \ldots, i_m}^{j_1, j_2, \ldots, j_m}.$$

We now take expectations and properly normalize the polynomials by defining

$$Q_{\lambda, t}^{H_0(s)}_{i_1, \ldots, i_m}^{j_1, \ldots, j_m} = \mathbb{E} H_0 \left( P^{(s)}_{i_1, \ldots, i_m}^{j_1, \ldots, j_m}(t) \right) | \lambda \prod_{\ell=1}^m a(2j_\ell) \text{ where } a(n) = \prod_{k \leq n, k \text{ odd}} k,$$

$$Q_{\lambda, t}^{H_0(h)}_{i_1, \ldots, i_m}^{j_1, \ldots, j_m} = \mathbb{E} H_0 \left( P^{(h)}_{i_1, \ldots, i_m}^{j_1, \ldots, j_m}(t) \right) | \lambda \prod_{\ell=1}^m (2j_\ell j_\ell)!^{-1}. \tag{3.3}$$

Note that $a(2n) = \mathbb{E} \mathcal{N}^{2n} = \mathbb{E} \mathcal{N}_1^2 + \mathbb{E} \mathcal{N}_2^{2n}$, with $\mathcal{N}, \mathcal{N}_1, \mathcal{N}_2$ independent standard Gaussian random variables. The above discussion implies the following evolution of the (time dependent) expectation $Q^{(s)}$ (resp. $Q^{(h)}$) along the Dyson eigenvector flow \((2.3)\) (resp. \((2.5)\)).
(i) Symmetric case: \( L^{(s)}_t Q^{H_0,(s)}_{\lambda,t} = \sum_{k,\ell} C_{k,\ell} (X_{k,\ell}^{(s)})^2 Q^{H_0,(s)}_{\lambda,t} \) where
\[
(X_{k,\ell}^{(s)})^2 Q^{H_0,(s)}_{\lambda,t} = 2j_1 Q^{H_0,(s)}_{\lambda,t} (j_1 - 1, ..., j_m) - 2j_1 Q^{H_0,(s)}_{\lambda,t} (j_1 - 1, ..., j_m) \text{ when } \ell \notin \{i_1, ..., i_m\},
\]
\[
(X_{i_1 i_2}^{(s)})^2 Q^{H_0,(s)}_{\lambda,t} = 2j_1 (2j_2 + 1) Q^{H_0,(s)}_{\lambda,t} (j_1 - 1, j_2 + 1, ..., j_m) + 2j_2 (2j_1 + 1) Q^{H_0,(s)}_{\lambda,t} (j_1 + 1, j_2 - 1, ..., j_m)
- (2j_1 (2j_2 + 1) + 2j_2 (2j_1 + 1)) Q^{H_0,(s)}_{\lambda,t} (j_1, j_2, ..., j_m).
\]

(ii) Hermitian case: \( L^{(h)}_t Q^{H_0,(h)}_{\lambda,t} = \sum_{k,\ell} C_{k,\ell} (X_{k,\ell}^{(h)})^2 Q^{H_0,(h)}_{\lambda,t} \) where
\[
(X_{k,\ell}^{(h)})^2 Q^{H_0,(h)}_{\lambda,t} = j_1 Q^{H_0,(h)}_{\lambda,t} (j_1 - 1, j_2 + 1, ..., j_m) - j_1 Q^{H_0,(h)}_{\lambda,t} (j_1 - 1, ..., j_m) \text{ when } \ell \notin \{i_1, ..., i_m\},
\]
\[
(X_{i_1 i_2}^{(h)})^2 Q^{H_0,(h)}_{\lambda,t} = j_1 (2j_2 + 1) Q^{H_0,(h)}_{\lambda,t} (j_1 - 1, j_2 + 1, ..., j_m) + j_2 (2j_1 + 1) Q^{H_0,(h)}_{\lambda,t} (j_1 + 1, j_2 - 1, ..., j_m)
- (j_1 (2j_2 + 1) + j_2 (2j_1 + 1)) Q^{H_0,(h)}_{\lambda,t} (j_1, j_2, ..., j_m).
\]

Thanks to the scalings \((3.3)\) and \((3.4)\), in the right hand sides of the above four equations, the sum of the coefficients vanishes. This allows us to interpret them as a multi-particle random walk (in a random environment) in the next subsection.

3.2 Multi-particle random walk. Consider the following notation, \( \eta : [1, N] \to \mathbb{N} \) where \( \eta_j := \eta(j) \) is interpreted as the number of particles at the site \( j \). Thus \( \eta \) denotes the configuration space of particles. We denote \( N(\eta) = \sum_j \eta_j \).

Define \( \eta^{i,j} \) to be the configuration by moving one particle from \( i \) to \( j \). If there is no particle at \( i \) then \( \eta^{i,j} = \eta \). Notice that there is a direction and the particle is moved from \( i \) to \( j \). Given \( n > 0 \), there is a one to one correspondence between \((1)\) \{\((i_1, j_1), ..., (i_m, j_m)\)\} with distinct \( i_k \)'s and positive \( j_k \)'s summing to \( n \), and \((2)\) \( N(\eta) = n \): we map \{\((i_1, j_1), ..., (i_m, j_m)\)\} to \( \eta \) with \( \eta_{i_k} = j_k \) and \( \eta_{i^c} = 0 \) if \( \ell \notin \{i_1, ..., i_m\} \).

We define
\[
f^{H_0,(s)}_{\lambda,t}(\eta) = Q^{H_0,(s)}_{\lambda,t} (j_1, ..., j_m)^t, \quad f^{H_0,(h)}_{\lambda,t}(\eta) = Q^{H_0,(h)}_{\lambda,t} (j_1, ..., j_m)^t,
\]
\[(3.5)\]
if the configuration of \( \eta \) is the same as the one given by the \( i,j \)'s. The dependence in the initial matrix \( H_0 \) will often be omitted: \( f^{(s)}_{\lambda,t} = f^{H_0,(s)}_{\lambda,t} \). \( f^{(h)}_{\lambda,t} = f^{H_0,(h)}_{\lambda,t} \). The following theorem summarizes the results from the previous subsection. It also defines the eigenvector moment flow, through the generators \((3.7)\) and \((3.8)\). It is a multi-particles random walk (with \( n = N(\eta) \) particles) in a random environment, the jump rates depending on the eigenvalues.

Theorem 3.1 (Eigenvector moment flow). Let \( q \in \mathbb{R}^N \), \( z_k = \sqrt{N(q, u_k(t))} \) and \( c_{ij}(t) = \frac{1}{N(\lambda_i, \lambda_j)^2(t)} \).

(i) Suppose that \( u \) is the solution to the symmetric Dyson vector flow \((2.3)\) and \( f^{(s)}_{\lambda,t}(\eta) \) is given by \((3.5)\) where \( \eta \) denote the configuration \{\((i_1, j_1), ..., (i_m, j_m)\)\}. Then \( f^{(s)}_{\lambda,t} \) satisfies the equation
\[
\partial_t f^{(s)}_{\lambda,t} = B^{(s)}(t) f^{(s)}_{\lambda,t},
\]
\[
B^{(s)}(t) f^{(s)}(\eta) = \sum_{i \neq j} c_{ij}(t) 2\eta_i (1 + 2\eta_j) \left( f(\eta^{i,j}) - f(\eta) \right).
\]
\[(3.6)\]
(ii) Suppose that $u$ is the solution to the Hermitian Dyson vector flow \( \sum_{i,j} = \sum_{i<j} \), and $f^{(h)}_{k,t}$ is given by \( \{k\} \). Then it satisfies the equation

$$\partial_t f^{(h)}_{k,t} = \mathcal{B}^{(h)}(t)f^{(h)}_{k,t},$$

$$\mathcal{B}^{(h)}(t)f(\eta) = \sum_{i \neq j} c_{ij}(t)\eta_i(1 + \eta_j)\left(f(\eta^i) - f(\eta)\right).$$\hspace{1cm}(3.8)$$

An important property of the eigenvector moment flow is reversibility with respect to a simple, explicit, equilibrium measure, as proved below. In the Hermitian case, this is simply the uniform measure on the configuration space.

Recall that a measure $\pi$ on the configuration space is said to be reversible with respect to a generator $L$ if $\sum_\eta \pi(\eta)g(\eta)Lf(\eta) = \sum_\eta \pi(\eta)f(\eta)Lg(\eta)$ for any functions $f$ and $g$. We then define the Dirichlet form by

$$D^\pi(f) = -\sum_\eta \pi(\eta)f(\eta)Lf(\eta).$$

**Proposition 3.2.** For the eigenvector moment flow, the following properties hold.

(i) Define a measure on the configuration space by assigning the weight

$$\pi^{(s)}(\eta) = \prod_{x=1}^{N} \phi(\eta_x), \quad \phi(k) = k \left(1 - \frac{1}{2k}\right).$$\hspace{1cm}(3.9)$$

Then $\pi^{(s)}$ is a reversible measure for $\mathcal{B}^{(s)}$ and the Dirichlet form is given by

$$D^{\pi^{(s)}}(f) = \sum_\eta \pi^{(s)}(\eta) \sum_{i \neq j} c_{ij}\eta_i(1 + 2\eta_j)\left(f(\eta^i) - f(\eta)^2\right).$$

(ii) The uniform measure $(\pi^{(h)}(\eta) = 1$ for all $\eta)$ is reversible with respect to $\mathcal{B}^{(h)}$. The associated Dirichlet form is

$$D^{\pi^{(h)}}(f) = \frac{1}{2} \sum_\eta \sum_{i \neq j} c_{ij}\eta_i(1 + 2\eta_j)\left(f(\eta^i) - f(\eta)^2\right).$$

**Proof.** We first consider (i), concerning the symmetric eigenvector moment flow. The measure $\pi^{(s)}$ is reversible for $\mathcal{B}^{(s)}$ for any choice of the coefficients satisfying $c_{ij} = c_{ji}$ if and only if, for any $i < j$,

$$\sum_\eta \pi^{(s)}(\eta)g(\eta)\left(2\eta_i(1 + 2\eta_j)f(\eta^i) + 2\eta_j(1 + 2\eta_i)f(\eta^i)\right) = \sum_\eta \pi^{(s)}(\eta)f(\eta)\left(2\eta_i(1 + 2\eta_j)g(\eta^i) + 2\eta_j(1 + 2\eta_i)g(\eta^i)\right).$$

A sufficient condition is clearly that both of the following equations hold:

$$\sum_\eta \pi^{(s)}(\eta)g(\eta)2\eta_i(1 + 2\eta_j)f(\eta^i) = \sum_\eta \pi^{(s)}(\eta)f(\eta)2\eta_j(1 + 2\eta_i)g(\eta^i),$$

$$\sum_\eta \pi^{(s)}(\eta)g(\eta)2\eta_j(1 + 2\eta_i)f(\eta^i) = \sum_\eta \pi^{(s)}(\eta)f(\eta)2\eta_i(1 + 2\eta_j)g(\eta^i).$$
Consider the left hand side of the first above equation. Let $\xi = \eta^{ij}$. If $\xi_i > 0$ then $\eta = \xi^{ij}$, $\eta_i = \xi_i + 1$ and $\eta_j = \xi_j - 1$. For the right hand side of the second equation, we make the change of variables $\xi = \eta^{ij}$. This proves that the above equations are equivalent to

$$\sum_{\xi} \pi^{(s)}(\xi^{ij}) g(\xi^{ij}) 2(\xi_j + 1)(2\xi_j - 1)f(\xi) = \sum_{\xi} \pi^{(s)}(\xi) f(\xi) 2\xi_j (1 + 2\xi_i) g(\xi^{ij}),$$

$$\sum_{\xi} \pi^{(s)}(\xi^{ij}) g(\xi^{ij}) 2(\xi_j + 1)(2\xi_j - 1)f(\xi) = \sum_{\xi} \pi^{(s)}(\xi) f(\xi) 2\xi_i (1 + 2\xi_j) g(\xi^{ij}).$$

If the measure if of type $\pi^{(s)}(\eta) = \prod_x \phi(\eta_x)$ and we note $\xi_i = a, \xi_j = b$, this first equation is equivalent to

$$\phi(a + 1)\phi(b - 1)2(a + 1)(2b - 1) = \phi(a)\phi(b)2b(2a + 1),$$

and second equation yields the same condition, transposing $a$ and $b$. This holds for all $a$ and $b$ if $\phi(k + 1) = ((2k + 1)/(2k + 2))\phi(k)$, which gives \((4.9\), normalizing with the initial data $\phi(0) = 1$. In the case (ii), the same reasoning yields that $\phi$ is constant.

Finally, the Dirichlet form calculation is standard: for example, for (i), $\sum_{ij} \eta^{(s)}(\eta)\mathcal{B}^{(s)}(f^2)(\eta) = 0$ by reversibility. Noting $\mathcal{B}^{(s)}(f^2)(\eta) = 2f(\eta)\mathcal{B}^{(s)}f(\eta) - \sum_{ij} \pi^{(s)}(\eta)2\eta_i (1 + 2\eta_j)(f(\eta) - f(\eta^{ij}))^2$ allows to conclude. 

\[ \Box \]

4 MAXIMUM PRINCIPLE

From now on we only consider the symmetric ensemble. The Hermitian case can be treated with the same arguments and only notational changes. Given a typical path $A$, in this section we will prove that the solution to the eigenvector moment flow \((3.7)\) converges uniformly to 1 for $t = N^{-1/4+\varepsilon}$. It is clear that the maximum (resp. minimum) of $f$ over $\eta$ decreases (resp. increases). We can quantify this decrease (resp. increase) in terms of the maximum and minimum themselves (see \((4.11)\)). This yields an explicit convergence speed to 1 by a Gronwall argument.

4.1 Isotropic local semicircle law. Fix a (small) $\omega > 0$ and define

$$S = S(\omega, N) = \{ z = E + i\eta \in \mathbb{C} : |E| \leq \omega^{-1}, N^{-1+\omega} \leq \eta \leq \omega^{-1} \}. \quad (4.1)$$

In the statement below, we will also need $m(z)$, the Stieltjes transform of the semicircular distribution, i.e.

$$m(z) = \int \frac{\varphi(s)}{s - z} ds = \frac{-z + \sqrt{z^2 - 4}}{2}, \quad \varphi(s) = \frac{1}{2\pi} \sqrt{(4 - s^2)^+},$$

where the square root is chosen so that $m$ is holomorphic in the upper half plane and $m(z) \to 0$ as $z \to \infty$. The following isotropic local semicircle law (Theorem 4.2 in [5]) gives very useful bounds on $(q, u_k)$ for any eigenvector $u_k$ via estimates on the associated Green function.

Theorem 4.1 (Isotropic local semicircle law [5]). Let $H$ be an element from the generalized Wigner ensemble and $G(z) = (H - z)^{-1}$. Suppose that \((1.3)\) holds. Then for any (small) $\xi > 0$ and (large) $D > 0$ we have, for large enough $N$,

$$\sup_{|q| = 1, z \in S} \mathbb{P} \left( |\langle q, G(z)q \rangle - m(z) | > N^\xi \left( \sqrt{\frac{\text{Im} m(z)}{N\eta}} + \frac{1}{N\eta} \right) \right) \leq N^{-D}. \quad (4.2)$$
An important consequence of this theorem, to be used in multiple occasions, is the following isotropic delocalization of eigenvectors: under the same assumptions as Theorem 4.1, for any \(\xi > 0\) and \(D > 0\), we have

\[
\sup_{|q|=1} \mathbb{P} \left( \left| \langle (q, u_k) \right| > N^{-1+\xi} \right) \leq N^{-D}.
\]

Note that in the case of the Stieltjes transform the error term was shown to be of lower order in the previous work [18]: under the same assumptions as in Theorem 4.1 we have

\[
\sup_{z \in \mathbb{S}} \mathbb{P} \left( \left| \frac{1}{N} \text{Tr} G(z) - m(z) \right| < \frac{N^\xi}{N^\eta} \right) \leq N^{-D}.
\]

### 4.2 Maximum Principle and regularity.

Let \(H_0\) be a symmetric generalized Wigner matrix with eigenvalues \(\lambda_0\) and an eigenbasis \(u_0\). Assume that \(\lambda, u\) satisfy (2.2) with initial condition \(\lambda_0, u_0\). Let \(G(z, t) = (u^* \lambda u - z)^{-1}(t)\) be the Green function. For \(\omega > \xi > 0\) and \(q \in \mathbb{R}^N\), consider the following three conditions (remember the notation (4.1) for \(\mathbb{S}(\omega, N)\)):

\[
A_1(q, \omega, \xi, N) = \left\{ \left| \langle (q, G(z, t)q) \right| - m(z) \right| < N^\xi \left( \sqrt{\frac{3m(z)}{N^\eta}} + \frac{1}{N^\eta} \right), \right.
\]

\[
\frac{1}{N} \text{Tr} G(z, t) - m(z) \left| < \frac{N^\xi}{N^\eta} \right. \text{ for all } t \in [0, 1], z \in \mathbb{S}(\omega, N) \}, \quad (4.4)
\]

\[
A_2(\omega, N) = \left\{ |\lambda_k(t) - \gamma_k| < N^{-\frac{1}{2} + \omega}(k)^{-\frac{1}{2}} \text{ for all } t \in [0, 1], k \in [1, N] \right\}, \quad (4.5)
\]

\[
A_3(\omega, N) = \left\{ |(q, u_k(t))|^2 < N^{-1+\omega} \text{ for all } t \in [0, 1], k \in [1, N] \right\}.
\]

Then these conditions hold with high probability.

**Lemma 4.2.** For any \(\omega > \xi > 0, \nu > 0\) and \(N\) large enough, we have

\[
\inf_{q \in \mathbb{R}^N} \mathbb{P} (A_1(q, \omega, \xi, N) \cap A_2(\omega, N) \cap A_3(\omega, N)) \geq 1 - N^{-2\nu},
\]

where the probability denotes the law of the random variable \(H_0\) and the paths of \(\lambda, u\).

**Proof.** For any fixed time, by (2.2), (4.2) and (4.3), the condition (4.4) holds with probability \(1 - N^{-C}\) for any \(C\). As \(C\) can be arbitrary, the same condition hold for any time and \(z\) in a discrete set of size \(N^{C/2}\), say. By continuity argument, we can extend it to all \(z \in \mathbb{S}\) and time between 0 and 1 and this proves (4.4). From (4.4), the other two conditions (4.5), (4.6) hold by the standard argument to prove rigidity, see [18] for details.

We define the set

\[
A(q, \omega, \xi, \nu, N) = \left\{ (H_0, \lambda) : \mathbb{P} \left( A_1(q, \omega, \xi, N) \cap A_2(\omega, N) \cap A_3(\omega, N) \mid (H_0, \lambda) \right) \geq 1 - N^{-\nu} \right\}.
\]

From the previous lemma, one easily sees that for any \(\omega > \xi\) and \(\nu\) we have, for large enough \(N\),

\[
\inf_{q \in \mathbb{R}^N} \mathbb{P} (A(q, \omega, \xi, \nu, N)) \geq 1 - N^{-\nu}.
\]
Theorem 4.3. Let $n \in \mathbb{N}$ and $f$ be a solution of the eigenvector moment flow (3.6) with initial matrix $H_0$ and path $\lambda$ in $A(q, \omega, \xi, \nu, N)$ for some $\nu > 2$. Let $t = N^{-1/4+\delta}$, where $\delta \in (\frac{1}{2}, 1/4]$ and we assume that $\omega > \xi$ and $n\omega < 1/2$. Then for any $\varepsilon > 0$ and large enough $N$ we have

$$\sup_{\eta, N^2(\eta) = n} |f_t(\eta) - 1| \leq CN^{n\omega + \varepsilon - 2\delta}. \tag{4.7}$$

The constant $C$ depends on $\varepsilon, \omega, \delta$ and $n$ but not on $q$.

We have the following asymptotic normality for eigenvectors of a Gaussian divisible Wigner ensemble with a small Gaussian component.

**Corollary 4.4.** Let $\delta$ be an arbitrarily small constant and $t = N^{-1/4+\delta}$. Let $H_t$ be the solution to (3.6) and $(u_1(t), \ldots, u_N(t))$ be an eigenbasis of $H_t$. The initial condition $H_0$ is assumed to be a symmetric generalized Wigner matrix. Then for any polynomial $P$ in $m$ variables and any $\varepsilon > 0$ such that for large enough $N$,

$$\sup_{I \subset [1, N], |I| = m, |q| = 1} \left| \mathbb{E} \left( P \left( (N(q, t)) k \right) \right) - \mathbb{E} \left( (N_j^m) k \right) \right| \leq CN^{\varepsilon - 2\delta}. \tag{4.8}$$

**Proof.** Since $H_0$ is a generalized Wigner matrices, the isotropic local semicircle law, Theorem 4.1, holds for all time with $\xi$ arbitrarily small. With $\omega = 2\xi$, and noticing that Lemma 4.2 holds for arbitrary large $\nu > 0$, (4.7) implies that (4.8) holds.

**Proof of Theorem 4.3.** We begin with the case $n = 1$. Let $f_s(k) = f_s(\eta)$, where $\eta$ is the configuration with one particle at the lattice point $k$. Assume that

$$\max_{k \in [1, N]} f_s(k) = f_s(k_0)$$

for some $k_0$ ($k_0$ is not unique in general). Using (3.6) and the above maximum property, for any $\eta > 0$ we have

$$\partial_s f_s(k_0) = \frac{1}{N} \sum_{j \neq k_0} \frac{f_s(j) - f_s(k_0)}{(\lambda_j - \lambda_{k_0})^2} \leq \frac{1}{N\eta} \sum_{j \neq k_0} \frac{\eta f_s(j)}{(\lambda_j - \lambda_{k_0})^2 + \eta^2} - f_s(k_0) \frac{1}{N\eta} \sum_{j \neq k_0} \frac{\eta}{(\lambda_j - \lambda_{k_0})^2 + \eta^2}. \tag{4.9}$$

Notice that

$$\frac{1}{N} \sum_{1 \leq j \leq N} \frac{\eta f_s(j)}{(\lambda_j - \lambda_{k_0})^2 + \eta^2} = \mathbb{E} \left( \sum_{j=1}^{N} \frac{\eta(q, u_j)^2}{(\lambda_j - \lambda_{k_0})^2 + \eta^2} \left| (H_0, \lambda) \right| \right).$$

From the definition of $A(q, \omega, \xi, \nu, N)$, for $N^{-1+\omega} < \eta < 1$ we therefore have

$$\frac{1}{N} \sum_{1 \leq j \leq N, j \neq k_0} \frac{\eta f_s(j)}{(\lambda_j - \lambda_{k_0})^2 + \eta^2} = \mathbb{E} \left( \frac{N\xi(\mathcal{M}(\lambda_{k_0} + i\eta))^{1/2}}{(N\eta)^{1/2}} + \frac{N^\omega}{N\eta} \right),$$

where the error $N^\omega/(N\eta)$ comes from the missing term $j = k_0$ and we have used that for $(H_0, \lambda) \in A(q, \omega, \xi, \nu, N)$, $N(q, u_j)^2$ is bounded by $N^\omega$ with very high probability. For the same reason, we have

$$\frac{1}{N} \sum_{1 \leq j \leq N, j \neq k_0} \frac{\eta}{(\lambda_j - \lambda_{k_0})^2 + \eta^2} = \mathbb{E} \left( \frac{N^\omega}{N\eta} \right).$$
Using these estimates, (4.9) yields
\[ \partial_s f_s(k_0) - 1 \leq -c \frac{\Im m(\lambda_{k_0} + i\eta)}{\eta} (f_s(k_0) - 1) + O \left( \frac{N^{\frac{5}{2}} \Im m(\lambda_{k_0} + i\eta)^{1/2}}{N^{1/2} \eta^{3/2}} \right) + O \left( \frac{N^\omega}{N_{\eta}} \right). \]

Moreover, from the definition of \( A(q,\omega,\xi,\nu,N) \), we know that \(-2 - N^{-\frac{2}{1+\omega}} \leq \lambda_{k_0} \leq 2 + N^{-\frac{2}{1+\omega}} \). As our final choice of \( \eta \) will satisfy \( N^{-\frac{2}{1+\xi}} \leq \eta \leq 1 \), this implies that
\[ \Im m(\lambda_{k_0} + i\eta) \geq c \sqrt{\eta}. \]

Let \( S_s = \sup_k (f_s(k) - 1) \). Note that there may be some \( s \) for which \( S_s \) is not differentiable (at times when the maximum is obtained for at least two distinct indices). But if we denote
\[ S'_s = \limsup_{u \to t} \frac{S_t - S_u}{t - u}, \]
the above reasoning shows
\[ S'_s \leq -c \frac{\sqrt{\eta}}{\sqrt{\eta}} S_s + C \frac{N^{\xi}}{N^{1/2} \eta^{3/2}} + C \frac{N^\omega}{N_{\eta}} \leq -c \frac{\sqrt{\eta}}{\sqrt{\eta}} S_s + C \frac{N^\omega}{N^{1/2} \eta^{3/2}}. \]
We chose \( \eta = N^{-\frac{1}{2} + 2\delta - \varepsilon} \) for some small \( \varepsilon \in (0, 2\delta - \omega) \) and \( t = N^{-\frac{1}{2} + \delta} \). The Gronwall inequality gives
\[ S_t \leq C \left( e^{-N^{r/2}} + N^{\omega + \varepsilon - 2\delta} \right). \]
We can do the same reasoning for the minimum of \( f \). This concludes the proof for \( n = 1 \).

For \( n \geq 2 \) the same argument works and we will proceed by induction. Let \( \xi \) satisfy
\[ \max_{N(\eta) = n} f_s(\eta) = f_s(\xi). \]
Assume \( \xi \) is associated to \( j_r \) particles at site \( k_r, 1 \leq r \leq m \) for some \( m \leq n \), where the \( k_r \)'s are distinct and \( j_r \geq 1 \). Then
\[ \partial_s f_s(\xi) \leq C \sum_{r=1}^{m} \left( \frac{1}{N} \eta \sum_{j \neq k_r} \frac{\eta f_s(\xi^{k_r,j})}{(\lambda_{k_r} - \lambda_j)^2 + \eta^2} - f_s(\xi) \frac{1}{N} \eta \sum_{j \neq k_r} \frac{\eta}{(\lambda_{k_r} - \lambda_j)^2 + \eta^2} \right), \]
where \( \xi^{k_r,j} \) is defined in Section 3.2. We now estimate the first term on the right hand side (the second term was estimated in the previous \( n = 1 \) step). By (4.6), for \( (H_0, \lambda) \in A(q,\omega,\xi,\nu,N), \) \( N(q, u_j)^2 \) is bounded by \( N^\omega \) with very high probability. Thus we have
\[ \frac{1}{N} \sum_{j \neq k_r} \frac{\eta f_s(\xi^{k_r,j})}{(\lambda_{k_r} - \lambda_j)^2 + \eta^2} = \frac{1}{N} \sum_{j \neq \{k_1,\ldots,k_m\}} \frac{\eta f_s(\xi^{k_r,j})}{(\lambda_{k_r} - \lambda_j)^2 + \eta^2} + O \left( \frac{N^\omega}{N_{\eta}} \right). \]

Moreover, by definition the above sum can be estimated by
\[ \mathbb{E} \left( \left( \frac{(N(q, u_j)^2)^{j_r-1}}{a(2j_r - 2)} \prod_{1 \leq r \leq m, r \neq r} (N(q, u_j)^2)^{j_r} a(2j_r) \right) \left( \frac{1}{N} \sum_{j \neq \{k_1,\ldots,k_m\}} \frac{\eta(N(q, u_j)^2)}{(\lambda_j - \lambda_{k_r})^2 + \eta^2} \right) \right) \mathbb{E}^{(H_0, \lambda)}, \]
where \( \mathbb{E} \) denotes the ensemble average over \( (H_0, \lambda) \).
where we first used that extending the indices to $1 \leq j \leq N$ induces an error $O(N^\omega (N\eta)^{-1})$ and the bound $N(q, u_j)^2 \leq N^\omega$ holds with very high probability. We have also used that for $(H_0, \lambda) \in A(q, \omega, \xi, \nu, N)$, we can replace $\Im(q, G(\lambda_k + i\eta), q)$ by $3m(\lambda_k + i\eta) + O(N^\Xi(N\eta)^{-1/2})$. This yields
\[
\frac{1}{N} \sum_{j \neq k_r} \frac{\eta f_s(\xi^{k_r})}{(\lambda_{k_r} - \lambda_j)^2 + \eta^2} = f_s(\xi \setminus k_r) \Im(m(\lambda_{k_r} + i\eta) + O\left(\frac{N^{n\omega}}{(N\eta)^{1/2}}\right), \tag{4.13}
\]
where $\xi \setminus k_r$ stands for the configuration $\xi$ with one particle removed from site $k_r$. By induction assumption, we can use \[14\] to estimate $f_s(\xi \setminus i_r)$ for $s \in (t/2, t)$. We have thus proved that
\[
\partial_s (f_s(\xi) - 1) \leq -\frac{c}{\sqrt{\eta}} (f_s(\xi) - 1) + O\left(\frac{N^{n\omega}}{N^{1/2}\eta^{3/2}}\right) + O\left(\frac{N^{(n-1)\omega - 2\delta}}{\eta}\right).
\]
on $(t/2, t)$. Notice that by our assumptions on the parameters $\omega, \delta, \eta$, and $\xi$, the first error term always dominates the second. One can now bound $|f_s(\xi) - 1|$ in the same way as in the $n = 1$ case.

If $\omega$ can be chosen arbitrarily small (this is true for generalized Wigner matrices), Theorem \[13\] gives $\sup_{\eta, N(\eta) = n} |f_t(\eta) - 1| \to 0$ for any $t = N^{-1/4+\epsilon}$. This could be improved to $t = N^{-1/3+\epsilon}$ by allowing $\eta$ to depend on $k_0$ in the previous reasoning (chose $\eta = N^{-2/3+\epsilon} k_0^{1/3}$).

More generally, our proof shows that the following equation \[14\] (with the convention \[10\]) holds. Let
\[
\Delta_1(k, \eta) = \mathbb{E}((q, G(\lambda_k + i\eta)q) - 3m(\lambda_k + i\eta) | (H_0, \lambda)),
\]
\[
\Delta_2(k, \eta) = \mathbb{E}(N^{-1} \text{Tr}(G(\lambda_k + i\eta) - 3m(\lambda_k + i\eta) | (H_0, \lambda)),
\]
where all variables depend on $t$. Then the following maximum inequality holds:
\[
S'_t \leq \max_{k: S_t = f_t(k)} \inf_{\eta > 0} \left\{ -\frac{3m(\lambda_k + i\eta)}{\eta} S_t + \frac{|\Delta_1(k, \eta)|}{\eta} + \frac{|\Delta_2(k, \eta)| (S_t + 1)}{\eta} + \frac{N^\omega (S_t + 1)}{N\eta^2} \right\}. \tag{4.14}
\]
Similar inequalities for a general number of particles can be obtained.

5 Proof of the main results

5.1 A comparison theorem for eigenvectors. Corollary \[12\] asserts the asymptotic normality of eigenvector components for Gaussian divisible ensembles for $t$ not too small. In order to prove Theorem \[12\] we need to remove the small Gaussian components of the matrix elements in this Gaussian divisible ensemble. Similar questions occurred in the proof of universality conjecture for Wigner matrices and several methods were developed for this purpose (see, e.g., [11] and [33]). In this paper, we will use the Green function comparison theorem introduced in [17, Theorem 2.3]. Although this method was invented mainly for identifying the probability distributions of eigenvalues, with additional argument it applies to eigenvectors as well [21]. Roughly speaking, [21, Theorem 1.10] states that the distributions of eigenvectors for two generalized Wigner ensembles are identical provided the first four moments of the matrix elements are identical and a level repulsion bound holds for one of the two ensembles (By extending the argument of [33], a similar comparison result for eigenvectors of two Wigner ensembles was obtained independently in [34]). We need the following extension of [21, Theorem 1.10] concerning the probability distributions of projections of eigenvectors. We first recall the following definition.
Definition 5.1 (Level repulsion estimate). Fix an energy $E$ such that $\gamma_k \leq E \leq \gamma_{k+1}$ for some $j \in [1, N]$. A generalized Wigner ensemble is said to satisfies the level repulsion at the energy $E$ if there exist $\alpha_0 > 0$ and $\delta > 0$ such that for any $0 < \alpha < \alpha_0$, we have

$$\mathbb{P} \left( \left| \{ j : \lambda_j \in [E - N^{-2/3} - \alpha \hat{k}^{-1/3}, E + N^{-2/3} - \alpha \hat{k}^{-1/3}] \} \right| \geq 2 \right) \leq N^{-\delta \alpha^{2/3}} - \alpha^{-1/3},$$

A matrix ensemble is said to satisfy the level repulsion estimate uniformly if this property holds for any energy $E \in (-2, 2)$.

The following theorem is slight extension of [21, Theorem 1.10] with the following modifications: (1) We slightly weaken the fourth moment matching condition. (2) The original theorem was only for components of eigenvectors; we allow the eigenvector to project to a fixed direction. (3) We state it for all energies in the entire spectrum. (4) We include an error bound for the comparison. (4) We state it only for eigenvectors with no involvement of eigenvalues. Theorem 5.2 can be proved using the argument in [21]; the only modification is to replace the local semicircle law used in [21] by the isotropic local semicircle law, Theorem 4.1. Since this type of argument based on the Green function comparison theorem has been done several times, we will not repeat it here. Notice that near the edge, the four moment matching condition can be replaced by just two moments. But for applications in this paper, this improvement will not be used and so we refer the interested reader to [21].

Theorem 5.2 (Eigenvector Comparison Theorem). Let $H^\gamma$ and $H^w$ be generalized Wigner ensembles where $H^\gamma$ satisfies the level repulsion estimate uniformly. Suppose that the first three off-diagonal moments of $H^\gamma$ and $H^w$ are the same, i.e.

$$\mathbb{E}^\gamma(h_{ij}^3) = \mathbb{E}^w(h_{ij}^3) \quad \text{for } i \neq j$$

and that the first two diagonal moments of $H^\gamma$ and $H^w$ are the same, i.e.

$$\mathbb{E}^\gamma(h_{ii}^2) = \mathbb{E}^w(h_{ii}^2).$$

Assume also that the fourth off-diagonal moments of $H^\gamma$ and $H^w$ are almost the same, i.e., there is an $\alpha > 0$ such that

$$\left| \mathbb{E}^\gamma(h_{ij}^4) - \mathbb{E}^w(h_{ij}^4) \right| \leq N^{-2-\alpha} \quad \text{for } i \neq j.$$

Then there is $\varepsilon > 0$ depending on $\alpha$ such that for any integer $k$, any $q_1, \ldots, q_k$ and any choice of indices $1 \leq j_1, \ldots, j_k \leq N$ we have

$$(\mathbb{E}^\gamma - \mathbb{E}^w) \Theta \left( N\langle q, u_{j_1} \rangle^2, \ldots, N\langle q, u_{j_k} \rangle^2 \right) = O(N^{-\varepsilon}),$$

where $\Theta$ is a smooth function that satisfies

$$|\partial^m \Theta(x)| \leq C(1 + |x|)^C$$

for some arbitrary $C$ and all $m \in \mathbb{N}^k$ satisfying $|m| \leq 5$. 

5.2 Proof of Theorem 5.2. We now summarize our situation: Given a generalized Wigner ensemble $\hat{H}$, we wish to prove that (1.4) holds for the eigenvectors of $\hat{H}$. We have proved in (1.8) that this estimate holds for any Gaussian divisible ensemble of type $H_0 + \sqrt{t} U$, and therefore by simple rescaling for any ensemble of type

$$H_t = e^{-t/2}H_0 + (1 - e^{-t})^{1/2} U,$$
where $H_0$ is any initial generalized Wigner matrix and $U$ is an independent standard GOE matrix, as long as $t \geq N^{-1/4+\delta}$. We fix $\delta$ a small number, say, $\delta = 1/8$. Now we construct a generalized Wigner matrix $H_0$ such that the first three moments of $H_t$ match exactly those of the target matrix $\hat{H}$ and the differences between the fourth moments of the two ensembles are less than $N^{-c}$ for some $c$ positive. This existence of such an initial random variable is guaranteed by, say, Lemma 3.4 of [16]. By the eigenvector comparison theorem, Theorem 5.2 we have proved (1.4) and this concludes our proof of Theorem 1.2.

5.3 Proof of Corollary 1.3. Let $\mathcal{N} = (\mathcal{X}_1, \ldots, \mathcal{X}_N)$ be a Gaussian vector with covariance $\text{Id}$. Let $m, \ell \in \mathbb{N}$, $k \in [\alpha N, (1 - \alpha) N]$ and $\{i_1, \ldots, i_{\ell}\} := J \subset [1, N]$. For $q$ such that $q_i = 0$ if $i \notin J$, consider the polynomial in $\ell$ variables:

$$Q(q_{i_1}, \ldots, q_{i_{\ell}}) = \mathbb{E} \left( |\langle \mathcal{X}, q \rangle|^m \right) - \mathbb{E} \left( |\langle q, \mathcal{N} \rangle|^{2m} \right).$$

From (1.4), there exists $\varepsilon > 0$ such that

$$\sup_{|q_{i_1}|^2 \leq \frac{1}{\ell}, \ldots, |q_{i_{\ell}}|^2 \leq \frac{1}{\ell}} |Q(q_{i_1}, \ldots, q_{i_{\ell}})| \leq \sup_{|q| = 1} |Q(q_{i_1}, \ldots, q_{i_{\ell}})| \leq N^{-\varepsilon},$$

where, for the first inequality, we note that the maximum of $Q$ in the unit ball is achieved on the unit sphere. Noting $R(q_{i_1}) = Q(q_{i_1}, \ldots, q_{i_{\ell}})$ with the coefficients of the polynomial $R$ depending on $q_{i_2}, \ldots, q_{i_{\ell}}$, the above bound implies that all the coefficients of $R$ are bounded by $C_1 N^{-\varepsilon}$ for some universal constant $C_1$ (indeed, one recovers the coefficients of $R$ from its evaluation at $\ell + 1$ different points, by inverting a Vandermonde matrix).

By iterating the above bound on the coefficients finitely many times ($\ell$ iterations), we conclude that there is a universal constant $C_\ell$ such that all coefficients of $Q$ are bounded by $C_\ell N^{-\varepsilon}$. This means that for any $k \in [\alpha N, (1 - \alpha) N]$ and $J \subset [1, N]$ with $|J| = \ell$,

$$\left| \mathbb{E} \left( \prod_{\alpha \in J} \left( \sqrt{N} u_k(\alpha) \right)^{m_\alpha} \right) - \mathbb{E} \left( \prod_{\alpha \in J} \left( \mathcal{X}_\alpha \right)^{m_\alpha} \right) \right| \leq C N^{-\varepsilon}$$

whenever the integer exponents $m_\alpha$ satisfy $\sum m_\alpha = m$. Here $C$ depends only on $m$, not on the choice of $k$ or $J$. This concludes the proof of (1.3), in the case of a monomial $P$ with even degree. If $P$ is a monomial of odd degree, (1.3) is trivial: the left hand side vanishes thanks to the uniform phase choice $e^{i\omega}$. This concludes the proof of Corollary 1.3.

5.4 Proof of Corollary 1.4. A second moment calculation yields

$$\mathbb{E} \left( \frac{N}{|a_N|} (u_k, a_N u_k)^2 \right)^2 = \frac{1}{|a_N|^2} \mathbb{E} \left( \left( \sum_{\alpha} a_N(\alpha) (\langle u_k(\alpha) \rangle - 1) \right)^2 \right) \leq \max_{\alpha \neq \beta} \mathbb{E} \left( \frac{1}{|a_N|} \max_{\alpha} \mathbb{E} \left( \left( \langle u_k(\alpha) \rangle^2 - 1 \right)^2 \right) \right).$$

From (1.5), the first term of the right hand side is bounded by $N^{-\varepsilon}$ and the second term is bounded by $1/|a_N|$. The Markov inequality then allows us to conclude the proof of Corollary 1.4.
6 Relaxation to equilibrium for \( t \gtrsim N^{-1} \)

The maximum inequality (4.14) allowed to prove convergence of the eigenvector moment flow along the whole spectrum, in Section 4, for \( t \gtrsim N^{-1/4} \). Assume that, for some reason, the maximum of this flow is always obtained for configurations supported in the bulk. Then we can make the approximation \( \Im \lambda_k + i \eta \sim 1 \) in (4.14), and we obtain

\[
S_t' \leq -\frac{1}{\eta^3} S_t + \frac{N^\xi}{N^{1/2} \eta^{3/2}}
\]

assuming the optimal isotropic local semicircle law with a tiny error \( N^{\xi}/\sqrt{N\eta} \). Choosing \( \eta = N^{-1+\varepsilon} \) for some small \( \varepsilon > 0 \) then gives, by Gronwall, a relaxation time of order \( \gtrsim N^{-1} \).

The purpose of this section is to make the previous argument rigorous: by finite speed of propagation of the eigenvector moment flow, this process can be modified into another process which achieves its maximum in the bulk.

6.1 Finite speed of propagation. We assume that the following two conditions hold, for some (small) fixed parameter \( \xi > 0 \). The definition of \( c_{ij}(s) \), was given in (2.7).

(i) For some \( \rho > 0 \), we have

\[
\sup_{0 \leq s \leq 1} \frac{1}{s + N^{-1}} \int_0^s \frac{1}{N^2} \sum_{i < j} c_{ij}(u) du \leq N^\rho. \tag{6.1}
\]

(ii) There is a constant \( C \) such that for any \( |i - j| \geq N^\xi \) and \( 0 \leq s \leq 1 \),

\[
c_{ij}(s) \leq C \frac{N}{(i-j)^2}. \tag{6.2}
\]

The above two conditions are not restrictive, in the sense that they hold with large probability, for small enough \( \rho \). This is the content of the following lemma essentially proved (in greater generality) in [14].

**Lemma 6.1.** There exists a small \( \rho_0 > 0 \) such that, for any \( 0 < \rho < \rho_0 \), there exists \( \varepsilon > 0 \) such that (6.1) holds with probability \( 1 - N^{-\varepsilon} \). Furthermore, for any \( \xi > 0 \), there exists \( c > 0 \) such that (6.2) holds with probability \( 1 - e^{-c(\log N)^2} \).

The following cutoff of the dynamics will be useful. Let \( 1 \ll \ell \ll N \) be a parameter to be specified later. We split the time dependent operator \( \mathcal{B} \) defined in (3.7) into a short-range and a long-range part: \( \mathcal{B} = \mathcal{S} + \mathcal{L} \), with

\[
(\mathcal{S} f)(\eta) = \sum_{|j-k| \leq \ell} c_{jk}(s) 2 \eta_j (1 + 2 \eta_k) \left( f(\eta^{i,k}) - f(\eta) \right),
\]

\[
(\mathcal{L} f)(\eta) = \sum_{|j-k| > \ell} c_{jk}(s) 2 \eta_j (1 + 2 \eta_k) \left( f(\eta^{i,k}) - f(\eta) \right).
\]

Notice that \( \mathcal{S} \) and \( \mathcal{L} \) are time dependent. Moreover, \( \mathcal{S} \) is also reversible with respect to \( \pi \) (the proof of Proposition 3.2 applies to any symmetric \( c_{ij} \)'s). Denote by \( U_{\mathcal{S}}(s, t) \) the semigroup associated with \( \mathcal{S} \) from time \( s \) to time \( t \), i.e.

\[
\partial_t U_{\mathcal{S}}(s, t) = \mathcal{S}(t) U_{\mathcal{S}}(s, t)
\]
for any $s \leq t$, and $U_\mathcal{B}(s, s) = \text{Id}$. The notation $U_\mathcal{B}(s, t)$ is analogous. In the following lemma, we prove that the short-range dynamics provide a good approximation of the global dynamics. Lemmas 6.2 and 6.3 follow the same proof as in [14], where they were shown for $n = 1$.

**Lemma 6.2.** Suppose that the coefficients of $\mathcal{B}$ satisfy (6.2) for some $\xi > 0$ and let $\ell \gg N^\xi$. Suppose that the initial data is the delta function at an arbitrary configuration $\eta$. Then for any $s \geq 0$ we have

$$\| (U_\mathcal{B}(0, s) - U_\mathcal{S}(0, s)) \delta_\eta \|_1 \leq C \frac{Ns}{\ell},$$

where $C$ only depends on $\xi$ (in particular not on $\eta$).

**Proof.** By the Duhamel formula we have

$$U_\mathcal{B}(0, s) \delta_\eta = U_\mathcal{S}(0, s) \delta_\eta + \int_0^s U_\mathcal{B}(s', s) L(s') U_\mathcal{S}(0, s') \delta_\eta ds'.$$

Notice that for $\ell \gg N^\xi$ we can use (6.2) to get

$$\| L^s \|_1 \leq \sum_{\eta} \sum_{|j-k| \geq \ell} c_{jk} \eta_j(1 + 2\eta_k)(|f(\eta^{-1}^j)| + |f(\eta)|) \leq C N\ell^{-1} \| f \|_1.$$

Since $U_\mathcal{B}$ and $U_\mathcal{S}$ are contractions in $L^1$, this yields

$$\int_0^s \| U_\mathcal{B}(s', s) L(s') U_\mathcal{S}(0, s') \delta_\eta \|_1 ds' \leq C N\ell^{-1} \int_0^s \| \delta_\eta \|_1 ds' \leq C \frac{Ns}{\ell},$$

which concludes the proof.

The following lemma gives a decay estimate for the short-range dynamics. It is not optimal, but sufficient for our purpose. Before stating it, on the set of configurations with $n$ particles we define the following distance:

$$d(\eta, \xi) = \min_{x, y: C(x) = \eta, C(y) = \xi} \sum_i |x_i - y_i|,$$

where $C$ is the map from coordinates to the configuration defined by $C(x)_i = \sum_k 1_{x_k = i}$.

**Lemma 6.3.** For any configurations $\eta, \xi$, define $r_s(\eta, \xi) = (U_\mathcal{S}(0, s) \delta_\eta)(\xi)$. Assume that (6.1) holds. Then uniformly in $\eta, \xi$ we have

$$r_s(\eta, \xi) \leq C \exp \left( -\frac{d(\xi, \eta)}{\ell N^{1+\frac{2N}{N-1}}(s + N-1)^{1/2}} \right).$$

**Proof.** Define

$$f(s) = \sum_\xi \phi(\xi) \pi(\xi) r_s^2(\eta, \xi), \quad \phi(\xi) = e^{d(\xi, \eta)/\theta}.$$
with some parameter $\theta \geq \ell$ to be specified later. Clearly, $f(0) = \pi(\eta)$ is of order one. Differentiating $f$ yields
\[
f'(s) = 2 \sum_{\xi} \phi(\xi) \pi(\xi) \sum_{|j-k| \leq \ell} c_{jk}(s)2\xi_j(1+2\xi_k)r(s, \xi) (r(s, \xi^{j,k}) - r(s, \xi))
\]
\[
= - \sum_{\xi} \pi(\xi) \sum_{|j-k| \leq \ell} c_{jk}(s)2\xi_j(1+2\xi_k) (r(s, \xi^{j,k}) - r(s, \xi)) (\phi(\xi^{j,k}) - \phi(\xi)) r(s, \xi)
\]
\[
= - \sum_{\xi} \pi(\xi) \sum_{|j-k| \leq \ell} c_{jk}(s)2\xi_j(1+2\xi_k) (\phi(\xi^{j,k}) - \phi(\xi))^2 (r(s, \xi)^2 - r(s, \xi) \phi(\xi^{j,k}) + r(s, \xi) - r(s, \xi)^2)
\]
\[
+ \sum_{\xi} \pi(\xi) \sum_{|j-k| \leq \ell} c_{jk}(s)2\xi_j(1+2\xi_k) (r(s, \xi^{j,k}) - r(s, \xi)) (\phi(\xi) - \phi(\xi^{j,k})) r(s, \xi),
\]
where we used the reversibility property (Proposition 3.2) in the second equality. We now use the Cauchy-Schwarz inequality to bound the last term and absorb the quadratic term in $r(s, \xi^{j,k}) - r(s, \xi)$ into the first term that is negative. This yields
\[
f'(s) \leq \sum_{\xi} \pi(\xi) \sum_{|j-k| \leq \ell} c_{jk}(s)2\xi_j(1+2\xi_k) (\phi(\xi^{j,k}) - \phi(\xi))^2 r(s, \xi)^2.
\]
Assuming $\ell \leq \theta$, we have $|\phi(\xi^{j,k}) - \phi(\xi)| \leq C\frac{\ell}{\theta} \phi(\xi)$ for $|j-k| \leq \ell$. Thus the previous equation gives
\[
f'(s) \leq C \left( \frac{\ell}{\theta} \right)^2 \sum_{|j-k| \leq \ell} c_{jk}(s) f(s).
\]
From a Gronwall argument this yields
\[
f(s) \leq \exp \left( C \left( \frac{\ell}{\theta} \right)^2 \int_0^s \sum_{|j-k| \leq \ell} c_{jk}(u) du \right) f(0) \leq C \exp \left( C \left( \frac{\ell}{\theta} \right)^2 N^{2+\rho} (s + N^{-1}) \right),
\]
where we used the assumption \[6.1\]. We therefore proved $f(s) \leq C$, provided that we choose $\theta = \ell N^{1+\frac{\rho}{2}} (s + N^{-1})^{1/2}$. This concludes the proof of the lemma. \[\square\]

6.2 Flat initial condition at the edge. Let $\alpha > 0$ be a fixed small number. We define the following cut and average operators on the space of functions of configurations with $n$ points: any $a \in [1, N/2]$,
\[
(Cut_a(f))(\eta) = f(\eta) \text{ if } \eta \subset [a, N + 1 - a], \quad \text{1 otherwise},
\]
\[
Av(f) = \frac{1}{\|\alpha N, 2\alpha N\|} \sum_{a \in [\alpha N, 2\alpha N]} Cut_a(f).
\]
We can write
\[
Av(f)(\eta) = a_\eta f(\eta) + (1 - a_\eta) \tag{6.3}
\]
for some coefficient $a_\eta \in [0, 1]$ ($a_\eta = 0$ if $\eta \not\subset [\alpha N, (1 - \alpha)N]$, $1$ if $\eta \subset [2\alpha N, (1 - 2\alpha)N]$). We will only use the elementary property
\[
|a_\eta - a_\xi| \leq C \frac{d(\eta, \xi)}{N}. \tag{6.4}
\]
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For a general number of particles \( n \), consider now the following modification of the eigenvector moment flow \( (3.6) \). We only keep the short-range dynamics (depending on a parameter \( \ell \)) and modify the initial condition to be flat when there is a particle close to the edge:

\[
\begin{align*}
\partial_t g_{\lambda,t} &= \mathcal{J}(t)g_{\lambda,t}, \\
g_{\lambda,0}(\eta) &= (Avf_{\lambda,0})(\eta),
\end{align*}
\]

We will abbreviate \( g_{\lambda,t}(\eta) \) by \( g_t(\eta) \), and \( f_{\lambda,t}(\eta) \) by \( f_t(\eta) \) (for \( n = 1 \), \( f_1(k) \) and \( g_t(k) \) where \( \eta \) is the configuration with 1 particle at \( k \)).

For small time \( t \), by finite speed of propagation we will prove that \( g = 1 \) (up to exponentially small corrections) close to the edge, so that the maximum principle for the dynamics \( (3.5) \) can be localized in the bulk.

We first prove that for these modified dynamics, the isotropic law holds, in the following sense. Notice that the following result is deterministic, the only probabilistic content being the choice \( (H_0, \lambda) \) that is in

\[
A(q, \omega, \xi, \rho, N) = A(q, \omega, \xi, \nu, N) \cap \left\{ \sup_{0 \leq s \leq 1} \frac{1}{s + N^{-1}} \int_0^s \frac{1}{N^2} \sum_{i<j} c_{ij}(u) du \leq N^\rho \right\}.
\]

From lemmas \[4.2\] and \[6.1\] for any \( \nu > 0 \), \( \omega > \xi > 0 \), and \( \rho > 0 \), the set \( A \) has measure at least \( 1 - N^{-z} \) for some \( z > 0 \).

**Lemma 6.4.** There exists (small) positive constants \( \omega_0, \xi_0, \rho_0 \) such that the following holds. Assume \((H_0, \lambda) \in A(q, \omega, \xi, \nu, N)\) where \( 0 < \omega < \omega_0 \), \( 0 < \xi < \xi_0 \), \( 0 < \rho < \rho_0 \), \( \nu > 2 \). Let \( \delta \in (0, 1/2) \). Let \( z \) satisfy \(-3 < \Re(z) < 3, N^{-1+2\omega} < 3z < N^{-1+\delta}, 0 \leq t \leq N^{-2} - 2\delta \) and \( N^{1+\frac{\delta}{2}} t \leq \ell \leq N^{-\frac{3}{2}}/\sqrt{\ell} \) (where \( \ell \) is the short-range dynamics cutoff parameter). Then we have

\[
\left| \sum_{k=1}^{N} \frac{1}{N} g_t(k) - \Im(z) \right| \leq C N^{\xi+\omega+\rho} \sqrt{\frac{\Im(z)}{N\eta}},
\]

where \( C \) depends only on \( \xi, \omega, \nu, \rho \). Moreover, consider the case of \( n \) particles. Let \( k_0 \in [1, N] \) and \( z = \lambda_{k_0} + i\eta \). Then for any configuration \( \eta \) containing at least one particle at \( k_0 \) we have

\[
\sum_{k=1}^{N} \frac{1}{N} g_t(\eta_{k_0}^k) = \Im(z) (a_n f_t(\eta_{k_0})) + (1 - a_n) + O \left( N^{\xi+\omega+\rho} \sqrt{\frac{\Im(z)}{N\eta}} \right)
\]

where \( \eta_{k_0} \) stands for the configuration \( \eta \) with one particle removed from site \( k_0 \).

**Proof.** For the sake of clarity, we first prove \[6.7\]. We can bound the left hand side of \[6.7\] by \(|(i)| + |(ii)| + |(iii)|\) where

\[
\begin{align*}
(i) &= \sum_{k=1}^{N} \frac{1}{N} \frac{(U_\mathcal{J}(0, t) Av f_0)(k)}{z - \lambda_k}, \\
(ii) &= \sum_{k=1}^{N} \frac{1}{N} \frac{(Av U_\mathcal{J}(0, t) f_0)(k)}{z - \lambda_k}, \\
(iii) &= \sum_{k=1}^{N} \frac{1}{N} \frac{(Av U_\mathcal{J}(0, t) f_0)(k)}{z - \lambda_k} - \Im(z).
\end{align*}
\]
The term (i) will be controlled by finite speed of propagation, (ii) thanks to Lemma 6.2 and (iii) thanks to the isotropic local semicircle law. To bound (i), we write

\[(U_{\mathcal{O}}(0,t)\text{Av}_{f_0}(k) - (\text{Av}U_{\mathcal{O}}(0,t)f_0)(k) = \frac{1}{\alpha N} \sum_{a \in \mathbb{N}, 2\alpha N} (U_{\mathcal{O}}(0,t)\text{Cut}_a f_0 - \text{Cut}_a U_{\mathcal{O}}(0,t)f_0)(k). \quad (6.9)\]

Let \(\varepsilon > 0\) be a small fixed constant. By Lemma 6.3 if \(|i - k| > \ell N^{1+\frac{2}{\omega}+\varepsilon t^{1/2}}\) then \((U_{\mathcal{O}}(0,t)\delta_k)(i) \leq C \exp(-N^{\varepsilon})\). For \(k > a + \ell N^{1+\frac{2}{\omega}+\varepsilon t^{1/2}}\), by reversibility this implies that

\[(U_{\mathcal{O}}(0,t)\text{Cut}_a f_0)(k) = \frac{1}{\pi(k)}(U_{\mathcal{O}}(0,t)\text{Cut}_a f_0, \delta_k)_{\pi} = \frac{1}{\pi(k)}(\text{Cut}_a f_0, U_{\mathcal{O}}(0,t)\delta_k)_{\pi} = \frac{1}{\pi(k)}(f_0, U_{\mathcal{O}}(0,t)\delta_k)_{\pi} + O(e^{-N^{\varepsilon/2}}) = (\text{Cut}_a U_{\mathcal{O}}(0,t)f_0)(k) + O(e^{-N^{\varepsilon/2}}). \quad (6.10)\]

For \(k < a - \ell N^{1+\frac{2}{\omega}+\varepsilon t^{1/2}}\), in the same way we obtain

\[(U_{\mathcal{O}}(0,t)\text{Cut}_a f_0)(k) = (\text{Cut}_a U_{\mathcal{O}}(0,t)f_0)(k) + O(e^{-N^{\varepsilon/2}}). \quad (6.11)\]

For \(a - \ell N^{1+\frac{2}{\omega}+\varepsilon t^{1/2}} \leq k \leq a + \ell N^{1+\frac{2}{\omega}+\varepsilon t^{1/2}}\), we use the obvious bound

\[|(U_{\mathcal{O}}(0,t)\text{Cut}_a f_0)(k) - (\text{Cut}_a U_{\mathcal{O}}(0,t)f_0)(k)| \leq 2 \sup_k f_0(k) \leq CN^\omega. \quad (6.12)\]

Equations (6.10), (6.11), (6.12) together imply that (6.9) is bounded by \(CN^{\omega+\frac{2}{\omega}+\varepsilon t^{1/2}}\), thus

\[|\langle i \rangle | \leq C N^{\omega+\frac{2}{\omega}+\varepsilon t^{1/2}} 3m(z) \leq C N^{\omega+\varepsilon} 3m(z) \leq \sqrt{\frac{3m(z)}{N \eta}}. \]

provided that \(\ell \sqrt{t} \leq 1/(N\eta)^{1/2}\).

To bound the term (ii), we can directly use reversibility and Lemma 6.2 to have

\[|\langle (\text{Av}U_{\mathcal{O}}(0,t)f_0)(k) - (\text{Av}U_{\mathcal{O}}(0,t)f_0)(k)| \leq |\langle (U_{\mathcal{O}}(0,t)f_0)(k) - (U_{\mathcal{O}}(0,t)f_0)(k)\rangle_{\pi} = \frac{1}{\pi(k)}|\langle f_0, (U_{\mathcal{O}}(0,t) - U_{\mathcal{O}}(0,t)\delta_k)_{\pi} \rangle | \leq C N^{\omega} \frac{N t}{\ell}. \]

This proves that \(|\langle ii \rangle | \leq C N^{\omega+\varepsilon} \sqrt{\frac{3m(z)}{N \eta}}\), if \(N t / \ell \leq 1/(N\eta)^{1/2}\).

Concerning the error term (iii), we proceed as follows. Let \(m_0\) be the index such that \(|\Re(z) - \gamma_{m_0}| = \inf_{1 \leq i \leq N}\{|\Re(z) - \gamma_i|\}\). Then

\[\sum_{k=1}^N \frac{1}{N} (\text{Av}U_{\mathcal{O}}(0,t)f_0)(k) = \sum_{|k-m_0| \leq \sqrt{N \eta} / \ell} \frac{1}{N} (\text{Av}U_{\mathcal{O}}(0,t)f_0)(k) + O\left(\frac{N^\omega}{N} \sum_{i > \sqrt{N \eta} / \ell} \frac{\eta}{i^2 + (i/N)^2}\right)\]

where we use that \(\|f_0\|_\infty \leq N^\omega\). For any function \(f\) we write \((\text{Av})f(k) = a_k f(k) + (1-a_k)\) with the notation
from (6.3). We obtain

\[ \sum_{k=1}^{N} \frac{1}{N} \left( \text{Av} U_{\mathcal{D}}(0,t) f_0(k) \right)_{z-\lambda_k} = \sum_{|k-m_0| \leq N \sqrt{\eta}} \frac{1}{N} \left( a_k f_k(k) + (1-a_k) \right)_{z-\lambda_k} + O(\sqrt{\eta}). \]

Moreover, the first sum above is equal to

\[ a_{k_0} \sum_{k=1}^{N} \frac{1}{N} \frac{f_k(k)}{z-\lambda_k} + (1-a_{k_0}) \sum_{k=1}^{N} \frac{1}{N} \frac{1}{z-\lambda_k} + O(\sqrt{\eta}) = \sum_{k=1}^{N} \frac{1}{N} \frac{f_k(k)}{z-\lambda_k} + O(\sqrt{\eta}). \]

where we used \((H_0, \lambda) \in A(\eta, \alpha, \omega, \nu, N)\). From (6.4), the second sum in (6.13) can be bounded by \(O(\sqrt{\eta})\). Gathering all estimates, we obtain that (6.7) holds if we have

\[ \max \left( \ell \sqrt{\eta}, \frac{N t}{\ell}, \sqrt{\eta} \right) \leq \frac{1}{\sqrt{\eta}}. \]

Our initial assumptions on \(\eta, t, \ell\) were chosen such that the above three restrictions hold.

In the case of general \(n\), to prove (6.8), we proceed in the same way. As the term of type (i) is also bounded by finite speed of propagation, we just need to prove that

\[ \sum_{k=1}^{N} \frac{1}{N} \left( \text{Av} U_{\mathcal{D}}(0,t) f_0(k) \right)_{z-\lambda_k} = \sum_{|k-m_0| \leq N \sqrt{\eta}} \frac{1}{N} \left( a_k f_k(k) + (1-a_k) \right)_{z-\lambda_k} + O(\sqrt{\eta}). \]

Thanks to Lemma 6.2, it is sufficient to prove the above estimate replacing \(U_{\mathcal{D}}\) by \(U_{\mathcal{R}}\). We also can restrict the summation to \(|k-k_0| \leq N \sqrt{\eta}\). Then, similarly to the \(n = 1\) case, we write

\[ \text{Av} U_{\mathcal{R}}(0,t) f_0(k) = \text{Av} f_k(k) = a_{\eta_0} f_k(k) + (1-a_{\eta_0}) = \left( a_{\eta_0} f_k(k) (1-a_{\eta_0}) \right) + ((a_{\eta_0} - a_{\eta_0}) f_k(k) + (a_{\eta_0} - a_{\eta_0})). \]

Using (6.4) and \(|k-k_0| \leq N \sqrt{\eta}\) to bound the above second term, we are left with proving that

\[ a_{\eta_0} \sum_{k=1}^{N} \frac{1}{N} \frac{f_k(k)}{z-\lambda_k} + (1-a_{\eta_0}) \sum_{k=1}^{N} \frac{1}{N} \frac{1}{z-\lambda_k} = \sum_{k=1}^{N} \frac{1}{N} \frac{f_k(k)}{z-\lambda_k} + (1-a_{\eta_0}) + O \left( N^{\xi+n+\rho} \sqrt{\eta} \right). \]

The second sum above is properly estimated by \(m(z)\) because we are in a good set. Concerning the first sum, its contribution is not trivial if \(a_{\eta_0} \neq 0\), in particular \(k_0 \in [\alpha N, (1-\alpha) N]\). Then \(\Im m(z) \sim 1\) and this first sum can be estimated exactly as in (6.12), (6.13). This concludes the proof.

6.3 Localized maximum principle. The following deterministic result states that, for a typical initial conditions and a generic eigenvalue path, the relaxation time of the bulk eigenvectors is of order at most \(N^{-1+\delta}\) for any \(\delta > 0\).
Theorem 6.5. Let \( n \in \mathbb{N} \), and \( \delta, \alpha > 0 \) be arbitrarily small constants, and \( t = N^{-1+\delta} \). Then there exists constants \( \omega_0, \rho_0 \) such that the following holds.

Let \( (H_0, \lambda) \in \mathcal{A}(q, \omega, \xi, \rho, \nu, N) \) (not necessary a generalized Wigner matrix) with \( 0 < \xi < \omega < \omega_0 \), \( \rho < \rho_0 \), \( \nu > 2 \). Let \( f \) be a solution of the eigenvector moment flow \((6.6)\) with initial matrix \( H_0 \) and path \( \lambda \). Then there exists \( \epsilon > 0 \) such that for large enough \( N \) we have

\[
\sup_{\eta : N(\eta) = n, \eta \in [\alpha N, (1-\alpha) N]} |f_t(\eta) - 1| \leq CN^{-\epsilon}.
\]

(6.14)

If the initial condition is a generalized Wigner matrix, the matrix Dyson Brownian motion is again a generalized Wigner ensemble after rescaling. In this case, the asymptotic normality of the eigenvectors was already proved in Theorem 1.2 and therefore the conclusion of Theorem 6.5 was proved as well. The key point of Theorem 6.5 lies in that it holds for deterministic initial matrices, provided that the local isotropic semicircle law and level repulsion estimate hold. More precisely, assume \( H_0 \) have distribution \( \mu_N \) (on the set of \( N \times N \) symmetric matrices) such that \( \langle \mu_N \rangle_{N \geq 1} \) is in the set

\[
\mathcal{A} = \{ \langle \mu_N \rangle_{N \geq 1} \forall \omega, \xi, \rho > 0, \exists \epsilon > 0, N_0 : \forall N \geq N_0, \inf_{q \in \mathbb{R}^N} \int d\nu_M(M) d\nu_t^M(\lambda) 1 \left( (M, \lambda) \in \mathcal{A}(q, \omega, \xi, \rho, \nu, N) \right) \geq 1 - N^{-\epsilon} \} \tag{6.15}
\]

where \( \nu_t^M \) is eigenvalues path distribution (see Theorem 2.3), and the set \( \mathcal{A} \) consists in initial matrices and eigenvalues paths satisfying a local isotropic law and level repulsion estimate (see \( 6.6 \)). Then the asymptotic normality \((1.4)\) in the bulk holds, where \( u \) is the eigenbasis of \( H_0(t) \) and \( t = N^{-1+\delta} \) for any polynomial \( P \), we have

\[
\sup_{t \in [\alpha N, (1-\alpha) N]} |E \left( P \left( \langle N \rangle_q, u_k \right)^2 \right)_{k \in \eta} \rangle - E \left( P \left( \langle A \rangle_{\eta}^{2m} \right)_{j=1} \right) | \leq N^{-\epsilon}. \tag{6.16}
\]

This proves Theorem 1.5 stating that the Dyson eigenvector flow reaches local equilibrium (i.e., Gaussian statistics) after time \( t \gtrsim 1/N \).

Proof of Theorem 6.5. As \( \alpha \) is arbitrary we just need to prove the result for \( \alpha \) replaced by \( 3\alpha \). Moreover, we only need to prove \((6.14)\) with \( f_t(\eta) \) replaced by \( g_t(\eta) \) solving the cutoff dynamics \((6.5)\). Indeed, using reversibility of both \( U_\tau \) and \( U_{\lambda} \) with respect to \( \pi \), we have

\[
f_t(\eta) - g_t(\eta) = \frac{1}{\pi(\eta)} \langle f_0, (U_{\lambda}(0, t) - U_{\tau}(0, t)) \delta_\eta \rangle_{\pi} + \frac{1}{\pi(\eta)} \langle f_0 - g_0, U_{\tau}(0, t) \delta_\eta \rangle_{\pi}.
\]

From Lemma 6.2 the first term is bounded by \( N^{1+\omega} t/\ell \). From Lemma 6.3 the second term is exponentially small (remember that \( f_0(\xi) = g_0(\xi) \) if \( \xi \subset [2\alpha N, (1-2\alpha) N] \)). We therefore just need to show that

\[
\sup_{\eta : N(\eta) = n, \eta \in [3\alpha N, (1-3\alpha) N]} |g_t(\eta) - 1| \leq CN^{-\epsilon}.
\]

We will prove that such an estimate holds for any \( \alpha > 0 \) by induction on \( n \). Assume there is just one particle. Following the idea from the proof of Theorem 1.3 for a given \( 0 \leq s \leq t \) let \( k_0 \) be an index such that \( g_s(k_0) = \sup_k g_s(k) \). We consider two possible cases: if \( g_s(k_0) - 1 \leq N^{-10} \) then there is nothing to prove.
We then have
\[
\partial_s g_s(k_0) = (\mathcal{F}(s)g_s)(k_0) = \frac{1}{N} \sum_{j \neq k_0, |j - k_0| \leq \ell} \frac{g_s(j) - g_s(k_0)}{(\lambda_k - \lambda_{k_0})^2} \leq \frac{1}{\eta} \sum_{j \neq k_0, |j - k_0| \leq \ell} \frac{\eta g_s(j)}{N (\lambda_k - \lambda_{k_0})^2 + \eta^2} \frac{g_s(k_0)}{\eta} \sum_{j \neq k_0, |j - k_0| \leq \ell} \frac{1}{N (\lambda_k - \lambda_{k_0})^2 + \eta^2}.
\]

If \( \ell \gg N\eta \) (which we obviously can assume, as we will choose \( \eta = N^{1+c} \) for some small \( c > 0 \), extending the above sums to all indices \( j \) induces an error \( \eta N^{1+\omega}/\ell \). This fact combined to Lemma 6.1 and the hypothesis \( (H_0, \lambda) \in \mathcal{A}(q, \omega, \xi, \rho, \nu, N) \) we proved (here \( z = \lambda_0 + \eta \))
\[
\partial_s g_s(k_0) - 1) \leq -\frac{3m(z)}{\eta} (g_s(k_0) - 1) + O\left( \frac{(3m(z))^{1/2}}{\eta^{3/2}N^{1/2}} N^{\omega+\rho+x} \right) + O\left( \frac{N^{1+\omega}}{\ell} \right).
\]
As \( k_0 \in \left[ 0, (1 - \frac{\alpha}{2})N \right] \), we have \( 3m(z) \sim 1 \). Moreover, the second error term is dominated by the first one. As a consequence, noting as previously \( S_\alpha = \sup_k (g_s(k) - 1) \), we proved that if \( S_\alpha \geq N^{-10} \) then
\[
\partial_s s_\alpha \leq -\frac{c}{\eta} s_\alpha + C \frac{N^{\omega+\rho+x}}{\eta^{3/2}N^{1/2}}.
\]
We chose \( \eta = N^{-1+\frac{\delta}{2}} \) and \( \max(\omega, \xi, \rho_0, \rho_0) \leq \delta/10 \). By Gronwall’s lemma, this concludes the proof for \( n = 1 \).

For general \( n \), as in the 1-particle case we can assume that \( \sup_\eta g_\ell(\eta) \) is achieved for some \( \xi \subset \left[ \frac{\alpha}{2}N, (1 - \frac{\alpha}{2})N \right] \). Then the analogue of (4.11) holds with \( f \) replaced by \( g \). The first sum in (4.11) then can be evaluated using (6.3):
\[
\frac{1}{N\eta} \sum_{j \neq k_0} \frac{\eta g_s(\xi, j)}{(\lambda_k - \lambda_j)^2 + \eta^2} = 3m(\lambda_k + \eta) (a_\xi f_s(\xi) + (1 - a_\xi)).
\]
From the result at rank \( n - 1 \) with \( \alpha \) replaced by \( \alpha/10 \), we know that for \( s \in [t/2, t] \) we have
\[
f_s(\xi) \leq g_s(\xi) + O(N^{-\epsilon}) = 1 + O(N^{-\epsilon}).
\]
This proves that
\[
\partial_s g_s(\xi) - 1) \leq -\frac{3m(z)}{\eta} (g_s(\xi) - 1) + O\left( \frac{(3m(z))^{1/2}}{\eta^{3/2}N^{1/2}} N^{\omega+\rho+x} + \frac{3m(z)}{\eta} N^{-\epsilon} \right).
\]
One now can conclude the proof as in the \( n = 1 \) case.

**APPENDIX A  CONTINUITY ESTIMATE FOR \( t \lesssim N^{-1/2} \)**

The main result in Section 6, Theorem 5.5, asserts the asymptotic normality of eigenvector components for Gaussian divisible ensembles for \( t \geq N^{-1} \). To prove Theorem 1.2 for bulk eigenvectors, in this appendix we remove the small Gaussian components of the matrix elements. As we saw in Section 5, one way to proceed consists in a Green function comparison theorem. Here, we proceed in a different way: the Dyson
Brownian motion preserves the local structure of generalized Wigner matrices up to time $N^{-1/2}$ (see the lemma hereafter). This approach is much more direct and there is no need to construct moment matching matrices. It provides a completely dynamical proof of Theorem 1.2 for bulk eigenvectors.

We remark that although this proof is very simple, the fact that the Dyson Brownian motion preserves the detailed behaviour of eigenvalues and eigenvectors is surprising and even contradictory. Consider for example the eigenvalue flow. It was proved that this spectral dynamics take very general initial data to local equilibrium for any time $t \geq N^{-1}$. So how can we prove that the changes of the eigenvalues up to time $N^{-1/2}$ is less than the accuracy $N^{-1}$? The answer is that we only prove the preservation of the Dyson Brownian motion for matrix models. In other words, the matrix structure gives this preservation of the local structure.

We start with the following matrix stochastic differential equation which is an Ornstein-Uhlenbeck version of the Dyson Brownian motion. Let $H_t = (h_{ij}(t))$ be a symmetric $N \times N$ matrix. The dynamics of the matrix entries are given by the stochastic differential equations

$$\frac{dh_{ij}(t)}{dt} = \frac{dB_{ij}(t)}{\sqrt{N}} - \frac{1}{2N}h_{ij}(t)dt,$$

(A.1)

where $B$ is symmetric with $(B_{ij})_{i \leq j}$ a family of independent Brownian motions. The parameter $s_{ij} > 0$ can take any positive values, but in this paper, we choose $s_{ij}$ to be the variance of $h_{ij}(0)$. Clearly, for any $t \geq 0$ we have $E(h_{ij}(t)^2) = s_{ij}$ and thus the variance of the matrix element is preserved in this flow. We will call this system of stochastic differential equations (A.1) a generalized Dyson Brownian motion. For this flow, the following continuity estimate holds.

**Lemma A.1.** Suppose that we have $c/N \leq s_{ij} \leq C/N$ for some fixed constants $c$ and $C$, uniformly in $i$ and $j$. Denote $\partial_{ij} = \partial_{h_{ij}}$. Suppose that $F$ is a smooth function of the matrix elements $(h_{ij})_{i \leq j}$ satisfying

$$\sup_{0 \leq s \leq t, i \leq j, \theta} \mathbb{E}\left((N^{3/2}|h_{ij}(s)^3| + \sqrt{N}|h_{ij}(s)||\partial_{ij}^3 F(\theta H_s)|\right) \leq M,$$

(A.2)

where $(\theta H)_{ij} = \theta_{ij} h_{ij}$, $\theta_{k\ell} = 1$ unless $\{k, \ell\} = \{i, j\}$ and $0 \leq \theta_{ij} \leq 1$. Then

$$\mathbb{E}F(H_t) - \mathbb{E}F(H_0) = O(tN^{1/2})M.$$  

**Proof.** By Itô’s formula, we have

$$\partial_t \mathbb{E}F(H_t) = -\frac{1}{2N} \sum_{i \leq j} \left(\frac{1}{s_{ij}} \mathbb{E}(h_{ij}(t) \partial_{ij} F(H_t)) - \mathbb{E}(\partial_{ij}^2 F(H_t))\right).$$

A Taylor expansion yields

$$\mathbb{E}(h_{ij}(t) \partial_{ij} F(H_t)) = \mathbb{E}h_{ij}(t) \partial_{ij} F(h_{ij}(t) = 0) + \mathbb{E}(h_{ij}(t)^2 \partial_{ij}^2 F(h_{ij}(t) = 0) + O\left(\sup_{\theta} \mathbb{E}(|h_{ij}(t)^3 \partial_{ij}^3 F(\theta H_t)|)\right)$$

$$= s_{ij} \mathbb{E}(\partial_{ij}^2 F(h_{ij}(t) = 0) + O\left(\sup_{\theta} \mathbb{E}(|h_{ij}(t)^3 \partial_{ij}^3 F(\theta H_t)|)\right),$$

$$\mathbb{E}(\partial_{ij}^2 F(H_t)) = \mathbb{E}(\partial_{ij}^2 F(h_{ij}(t) = 0) + O\left(\sup_{\theta} \mathbb{E}(|h_{ij}(t) \partial_{ij}^2 F(\theta H)|)\right).$$
Together with the condition $c/N \leq s_{ij} \leq C/N$, we have

$$\partial_t \mathbb{E}(H_t) = N^{1/2} O \left( \sup_{i \in \mathcal{I}, \theta} \mathbb{E}(N^{3/2} |h_{ij}(t)^3| + N^{1/2} |h_{ij}(t)|) |\partial^2_{ij} F(\Theta H_t)| \right).$$

Integration over time finishes the proof. \hfill \Box

The previous lemma implies the following eigenvalues and eigenvectors continuity estimate for the dynamics (A.1).

**Corollary A.2.** Let $\alpha > 0$ be arbitrarily small, $\delta \in (0, 1/2)$ and $t = N^{-1+\delta}$. Denote by $H_t$ the solution of (A.7) with a symmetric generalized Wigner matrix $H_0$ as the initial condition. Let $\mu_t$ be the law of $H_t$. Let $m$ be any positive integer and $\Theta : \mathbb{R}^{2m} \to \mathbb{R}$ be a smooth function satisfying

$$\sup_{k \in [0,5], x \in \mathbb{R}} |(\Theta^{(k)}(x))(1 + |x|)^{-C} < \infty$$

for some $C > 0$. Denote by $(u_1(t), \ldots, u_N(t))$ the eigenvectors of $H_t$ associated with the eigenvalues $\lambda_1(t) \leq \ldots \leq \lambda_N(t)$. Then there exists $\varepsilon > 0$ (depending only on $\Theta, \delta$ and $\alpha$) such that, for large enough $N$,

$$\sup_{t \in [\alpha N, (1-\alpha) N], |l| = m, |q| = 1} |(\mathbb{E}^t - \mathbb{E}^{t_0}) \Theta \left( (N\lambda_k - \gamma_k), N\langle q, u_k \rangle^2 \right)_k | \leq N^{-\varepsilon}.$$

**Proof.** One may try to apply Lemma A.1 directly for $F(H) = (\lambda, u)$, but the third derivative of this function seems hard to bound. Instead, we can prove the continuity estimate when $F$ is a product of Green functions of $H$, which in turn implies the continuity estimate for eigenvalues and eigenvectors. In the following, the fact that (i) and (ii) imply (A.2) relies on classical techniques [21]. The crucial condition is (i), i.e., comparison of Green functions up to some scale smaller than microscopic, $\eta = N^{-1-\varepsilon}$. In Section 5 such a comparison was shown by moment matching. Hereafter, Lemma A.1 allows to prove this Green function comparison by a dynamic approach.

Let $v$ and $w$ refer to two generalized Wigner ensembles. Consider the following statements.

(i) Green functions comparison up to a very small scale. For any $\kappa > 0$ there exists $\xi, \varepsilon > 0$ such that for any $N^{-1-\xi} < \eta < 1$ and any smooth function $F$ with polynomial growth, we have

$$\sup_{|q| = 1, E_1, \ldots, E_m \in (-2 + \kappa, 2 - \kappa)^m} \left| \mathbb{E}^{v,w} F \left( \langle \langle q, G(z_k)q \rangle \rangle^m \right) \right| \leq C N^{-\varepsilon} \left( \frac{1}{N\eta} + \frac{1}{\sqrt{N\eta}} \right),$$

for some $C = C(\kappa, F) > 0$. Here $z_k = E_k + i\eta$.

(ii) Level repulsion estimate. For both ensembles $v$ and $w$ and for any $\kappa > 0$ the following holds. There exists $\xi_0 > 0$ such that for any $0 < \xi < \xi_0$ there exists $\delta > 0$ satisfying

$$\mathbb{P} \left( |\lambda_i - E| \geq N^{-1-\xi}, |\lambda_i - E| \geq N^{-1-\xi} \right) \leq N^{-\xi - \delta},$$

for any $E \in (-2 + \kappa, 2 - \kappa)$. Here the probability measure can be either the ensemble $v$ or $w$.

From Section 5 in [21], if (i) and (ii) hold then for any $\alpha > 0$ and $\Theta$ satisfying (A.3) there exists $\varepsilon > 0$ such that for large enough $N$ we have

$$\sup_{t \in [\alpha N, (1-\alpha) N], |l| = m, |q| = 1} \left| \mathbb{E}^{v,w} \Theta \left( (N\lambda_k - \gamma_k), N\langle q, u_k \rangle^2 \right)_k \right| \leq N^{-\varepsilon}. \quad \text{(A.4)}$$
The level repulsion condition condition (ii) was proved in the generalized Wigner context [14, equation (5.32)]. We therefore only need to check the main assumption (i), which is a consequence of Lemma A.1 and the isotropic local semicircle law, Theorem A.1. Indeed, we need to find a good bound $M$ in (A.2) for a function $F$ of type given in (i). For simplicity we only consider the case

$$F(H) = \langle q, G(z)q \rangle,$$

where $z = E + i\eta$ with $N^{-1-\xi} < \eta < 1$ and $-2 + \kappa < E < 2 - \kappa$. The general case

$$F(H) = \langle q_1, G(z_1)q_1 \rangle \ldots \langle q_k, G(z_k)q_k \rangle$$

is analogous. We have

$$\partial^2_{ij} \langle q, G(z)q \rangle = -\sum_{\alpha, \beta} \sum_{a, b} q_a G(z)_{\alpha a} G(z)_{\beta b} q_b,$$

where $\{\alpha_k, \beta_k\} = \{i, j\}$ or $\{j, i\}$. From the isotropic local semicircle law (4.2) the following four expressions

$$\sum_{a} q_a G(z)_{\alpha a}, \quad G(z)_{\beta b}, \quad G(z)_{\beta' b}, \quad \sum_{b} G(z)_{\beta' b} q_b$$

are bounded by $N^{2\xi}((N\eta)^{-1} + (N\eta)^{-1/2})$ with very high probability provided that $N^{-1+\xi} \leq \eta \leq 1$. Moreover, by a dyadic argument explained in [17] Section 8, we have for any $y \leq \eta$,

$$|\langle q, G(E + i\eta)q \rangle| \leq C\log N \frac{N^2}{y} \Im \langle q, G(E + i\eta)q \rangle.$$

Consequently, we proved that uniformly in $E \in (-2 + \kappa, 2 - \kappa)$, $N^{-1-\xi} \leq \eta \leq 1$, we have

$$\partial^2_{ij} \langle q, G(E + i\eta)q \rangle = O(N^{2\xi}(N\eta)^{-1} + (N\eta)^{-1/2})$$

with very high probability. The hypothesis (A.2) therefore holds with $M = C(\xi)N^{2\xi}((N\eta)^{-1} + (N\eta)^{-1/2})$. As $\xi$ is arbitrarily small, Lemma A.1 proves that for any $\delta \in (0, 1/2)$ and $t = N^{-1+\delta}$ there exists some $\varepsilon > 0$ with

$$|\mathbb{E} F(H_t) - \mathbb{E} F(H_0)| \leq N^{-\varepsilon}((N\eta)^{-1} + (N\eta)^{-1/2}).$$

Thus assumption (i) holds and the Corollary is proved. \qed

To complete the proof of Theorem A.2 for bulk eigenvectors by a dynamical approach, we proceed as follows. Let $H_0$ be a generalized Wigner matrix. For $\delta \in (0, 1/2)$ and $t = N^{-1+\delta}$, let $H_t$ be the solution of (A.1) at time $t$. On the one hand, from Corollary A.2 we have

$$\sup_{t \in [\alpha N, (1-\alpha)N], |I|=m, |q|=1} \left| \mathbb{E} \left( P \left( \left( N \langle q, u_k(t) \rangle \right)^2 \right)_{k \in I} \right) - \mathbb{E} \left( P \left( \left( N \langle q, u_k \rangle \right)^2 \right)_{k \in I} \right) \right| \leq N^{-\varepsilon}.$$

On the other hand, the entry $h_{ij}(t)$ of $H_t$ is distributed as

$$e^{-2\pi i \eta_{ij}} h_{ij}(0) + \left( s_{ij} \left( 1 - e^{-\frac{1}{2\pi i \eta_{ij}}} \right) \right)^{1/2}.$$

(A.5)
where \((\mathcal{N}^{(ij)})_{i \leq j}\) are independent standard Gaussian random variables. For any \(\nu < \frac{1}{2} \inf_{i,j} s_{ij} \left(1 - e^{-\frac{i}{s_{ij}}} \right),\) let \(W_0\) be a random matrix with entry \((W_0)_{ij}\) distributed as

\[
e^{-\frac{2\nu}{s_{ij}}} h_{ij}(0) + \left(s_{ij} \left(1 - e^{-\frac{i}{s_{ij}}} \right) - \nu \right)^{1/2} \mathcal{N}^{(ij)}_1 \quad \text{if } i \neq j,
\]

\[
e^{-\frac{2\nu}{s_{ij}}} h_{ij}(0) + \left(s_{ij} \left(1 - e^{-\frac{i}{s_{ij}}} \right) - 2\nu \right)^{1/2} \mathcal{N}^{(ij)}_1 \quad \text{if } i = j,
\]

where \((\mathcal{N}^{(ij)}_1)_{i \leq j}\) are independent standard Gaussian random variables, independent from \(H_0\). Then \(W_0\) is a generalized Wigner matrix modulo scaling: for any \(i\) we have \(\sum_j \text{Var}(W_0)_{ij} = 1 - (N + 1)\nu\). Moreover from \((A.3)\) \(h_{ij}(t)\) is distributed as

\[
(W_0)_{ij} + \nu^{1/2} \mathcal{N}^{(ij)}_2 \quad \text{if } i \neq j,
\]

\[
(W_0)_{ij} + (2\nu)^{1/2} \mathcal{N}^{(ij)}_2 \quad \text{if } i = j,
\]

where \((\mathcal{N}^{(ij)}_2)_{i \leq j}\) are independent standard Gaussian random variables, independent of \(W_0\). This proves that \(H_t\) is distributed as \(W_t\), where \((W_t)_{s \geq 0}\) satisfies \((2.1)\) and \(t' = N\nu\). We choose \(\nu = N^{-2+\xi}\) for some \(\xi \in (0,1)\) and apply Theorem \(6.5\) to \(W_t\); this yields

\[
\sup_{I \subseteq [\alpha N, (1-\alpha)N] |I| = m, |q| = 1} \left| \mathbb{E} \left( P \left( \left( (N(q, u(t))^2 \right)_{k \in I} \right) \right) - \mathbb{E} \left( \left( \mathcal{N}^{(2)}_{j=1} \right) \right) \right| \leq N^{-\varepsilon}.
\]

We have thus proved Theorem \(1.2\) by a dynamic approach, in the bulk case.

### Appendix B Generator of the Dyson vector flow

**B.1 Proof of Theorem 2.3.** We first consider the symmetric case.

(a) For any \(\varepsilon > 0\), let \(\tau_\varepsilon = \inf \{ t \geq 0 \mid |\lambda_i - \lambda_j| = \varepsilon \text{ for some } i \neq j \text{ or } |\lambda_i| = \varepsilon^{-1} \text{ for some } i \} \) and \(\phi_\varepsilon\) be a smooth function on \(\mathbb{R}\) such that \(\phi_\varepsilon(x) = x^{-1}\) if \(x \geq \varepsilon\). Then, as all of the following coefficients are Lipschitz, pathwise existence and uniqueness holds for the system of stochastic differential equations

\[
d\lambda_k = \frac{dB^{(s)}_{kk}}{\sqrt{N}} + \left( \frac{1}{N} \sum_{\ell \neq k} \phi_\varepsilon(\lambda_k - \lambda_\ell) - \frac{1}{2} \lambda_k \right) dt,
\]

\[
du_k = \frac{1}{\sqrt{N}} \sum_{\ell \neq k} (dB^{(s)}_{kk}) \phi_\varepsilon(\lambda_k - \lambda_\ell) u_\ell - \frac{1}{2N} \sum_{\ell \neq k} \phi_\varepsilon(\lambda_k - \lambda_\ell)^2 u_k dt.
\]

Consequently, if one can prove that \(\tau_\varepsilon \to \infty\) almost surely as \(\varepsilon \to 0\), then existence and strong uniqueness for the system \((2.2), (2.3)\) easily follow. This non-explosion nor collision result follows from Proposition 1 in \([28]\). It immediately yields \(\lambda \in \Sigma_N\) for any \(t \geq 0\).

To prove that \(u_t \in \mathcal{O}(N)\) for any \(t \geq 0\), we consider the stochastic differential equations satisfied by \(u_t \cdot u_j\),
1 ≤ i ≤ j ≤ N. Itô’s formula yields

\[ d(u_i \cdot u_j) = \frac{1}{\sqrt{N}} \sum_{k \in \{i,j\}} \left( \frac{dB_{jk}^{(s)}}{\lambda_j - \lambda_k} u_i \cdot u_k + \frac{dB_{ik}^{(s)}}{\lambda_i - \lambda_k} u_j \cdot u_k \right) + \frac{1}{\sqrt{N}} \frac{dB_{jk}^{(s)}}{\lambda_j - \lambda_i} (|u_i|^2 - |u_j|^2) \]

\[ - \frac{1}{2N} \left( \sum_{k \neq i} \frac{1}{(\lambda_j - \lambda_k)^2} + \sum_{k \neq j} \frac{1}{(\lambda_i - \lambda_k)^2} + \frac{1}{(\lambda_j - \lambda_i)^2} \right) u_i \cdot u_j \, dt, \quad i \neq j, \]

\[ d(|u_i|^2) = \frac{2}{\sqrt{N}} \sum_{k \neq i} \frac{dB_{ik}^{(s)}}{\lambda_i - \lambda_k} u_i \cdot u_k. \]

For the same reason as previously, existence and strong uniqueness hold for the above system, and \( u_i \cdot u_j = 0 \) (\( i \neq j \)), \( |u_i|^2 = 1 \) is an obvious solution (remember that \( u_0 \in O(N) \)), which completes the proof.

(b) Let \( \tilde{H}_t^{(s)} = u_i \lambda_i u_i^* \). On the one hand, Itô’s formula gives

\[ d\tilde{H}_t^{(s)} = (u \lambda (du)^* + u (d\lambda) u^* + (du) \lambda u^*)_{km} + \frac{1}{N} \left( \frac{\lambda}{\lambda_i - \lambda_j} \right) u_s(k) u_s(m) \, dt. \tag{B.1} \]

On the other hand, the evolution equations for \( \lambda \) and \( u \) is

\[ d\lambda = dM_{\lambda} + dD_{\lambda} \]

\[ (dM_{\lambda})_{ij} = \frac{dB^{(s)}_{ij}}{\sqrt{N}} 1_{i=j}, \quad (dD_{\lambda})_{ij} = \frac{1}{2} \left( \sum_{\ell \neq i} \frac{1}{\lambda_i - \lambda_{\ell}} \right) dt 1_{i=j}, \]

\[ du = u(dM_{u} + dD_{u}) \]

\[ (dM_{u})_{ij} = \frac{1}{\sqrt{N}} \frac{dB^{(s)}_{ij}}{\lambda_i - \lambda_j} 1_{i \neq j}, \quad (dD_{u})_{ij} = -\frac{1}{2N} \sum_{\ell \neq i} \frac{dt}{(\lambda_i - \lambda_{\ell})^2} 1_{i \neq j}. \]

Consequently, after defining the diagonal matrix process \( D \) by

\[ (dD)_{ij} = \frac{1}{N} \sum_{\ell \neq i} \frac{\lambda_{\ell}}{(\lambda_i - \lambda_{\ell})^2} dt 1_{i=j}, \]

the equation (B.1) can be written

\[ d\tilde{S} = u(\lambda (dM_u)^* + (dM_u) \lambda + dM_\lambda) u^* + u(\lambda (dD_u)^* + (dD_u) \lambda + dD_\lambda + dD) u^*. \]

We have \( \lambda (dM_u)^* + (dM_u) \lambda + dM_\lambda = \frac{1}{\sqrt{N}} dB^{(s)} \) and \( \lambda (dD_u)^* + (dD_u) \lambda + dD_\lambda + dD = 0 \), so

\[ d\tilde{S} = \frac{1}{\sqrt{N}} u(dB^{(s)}) u^*. \]

As \( u_t \in O(N) \) almost surely for any \( t \geq 0 \), by Lévy’s criterion, the process \( M \) defined by \( M_0 = 0 \) and \( dM_t = u(dB^{(s)}) u^* \) is a symmetric Dyson Brownian motion. This concludes the proof: \( (\tilde{H}_t^{(s)})_{t \geq 0} \) and \( (H_t^{(s)})_{t \geq 0} \) have the same law, as they are both solution of the same stochastic differential equation, for which weak uniqueness holds.

(c) Existence and strong uniqueness for (2.2) has a proof strictly identical to (a). For a given continuous trajectory \( (\lambda_t)_{t \geq 0} \subset \Sigma_N \), existence and strong uniqueness for (2.3) is elementary, because \( \sup_{t \in [0,T], i \neq j} |\lambda_i - \lambda_j|^{-1} < \infty \) and the coefficients are Lipschitz for any given \( t \in [0,T] \).
Let $X'$ be the solution of (2.2), and $(u'_t(X'))_{t \geq 0}$ be the solution of (2.3) for given $X$. If the initial conditions match, we have

$$E((X'_t, u'_t(X')) = (X_t, u_t) \text{ for all } t \geq 0) = 1,$$

(B.2)

because $(X'_t, u'_t(X'))$ is a solution of the system of stochastic differential equations (2.2, 2.3) for which strong uniqueness holds. Equations (B.2) together with (B.3) yields

$$E(F((H_{1t}^{(s)})_{0 \leq t \leq T})) = E(F((u'_t(X'_t(u'_t(X'_t)) \ast)_{0 \leq t \leq T})).$$

(B.3)

As strong uniqueness holds, $(X'_t)_{0 \leq t \leq T}$ is a measurable function (called $f$) of $((B_{it}^{(s)})_{0 \leq t \leq T})_{i=1}^N$, and $(u'_t(X'))_{0 \leq t \leq T}$ is a measurable function of $((B_{ij}^{(s)})_{0 \leq i,j \leq T})_{i<j}$ and $(X'_t)_{0 \leq t \leq T}$ (called $g$). We therefore have (for some Wiener measures $\omega_1, \omega_2$) for any bounded continuous function $G$

$$E(G((X'_t)_{0 \leq t \leq T}, (u'_t(X'))_{0 \leq t \leq T})) = \int \int d\omega_1(B_1)d\omega_2(B_2)G(f(B_1), g(B_1), B_2)$$

$$= \int \int d\nu_T(\lambda)d\omega_2(B_2)G(\lambda, g(B_2)) = \int \int d\nu_T(\lambda)d\mu_T(u(X))G(\lambda, u(X)).$$

Together with (B.3), this concludes the proof. We used the independence of the diagonal of $B(t)$ with the other entries in the first equality above.

**B.2 Proof of Lemma 2.4.** We consider the Hermitian setting, the symmetric one being slightly easier. Let $f$ be a smooth function of the matrix entries, $u_k(\alpha) = x_{k\alpha} + iy_{k\alpha}, 1 \leq k, \alpha \leq N$. We denote $\langle \cdot, \cdot \rangle' = (d/dt)\langle \cdot, \cdot \rangle$. Itô’s formula yields

$$\frac{d}{dt} E(f) = E((I) + (II) + (III)),$$

(I) = \sum_{k, \alpha} (-\frac{1}{2} \sum_{\beta \neq k} c_{\beta k} (x_{k\alpha} \partial_{x_{k\alpha}} + y_{k\alpha} \partial_{y_{k\alpha}})) f,$$

(II) = \frac{1}{2} \sum_{k, \alpha, \beta} \langle (x_{k\alpha}, x_{k\beta})', \partial_{x_{k\alpha}, x_{k\beta}} \rangle + \langle (y_{k\alpha}, y_{k\beta})', \partial_{y_{k\alpha}, y_{k\beta}} \rangle + \langle (x_{k\alpha}, y_{k\beta})', \partial_{x_{k\alpha}, y_{k\beta}} \rangle + \langle (y_{k\alpha}, x_{k\beta})', \partial_{y_{k\alpha}, x_{k\beta}} \rangle f,$$

(III) = \sum_{k, \ell, \alpha, \beta} \langle (x_{k\alpha}, \partial_{x_{k\alpha}}), \partial_{x_{k\alpha}} \rangle + \langle (y_{k\alpha}, y_{k\beta})', \partial_{y_{k\alpha}, y_{k\beta}} \rangle + \langle (x_{k\alpha}, \partial_{y_{k\alpha}}), \partial_{y_{k\alpha}} \rangle + \langle (y_{k\alpha}, \partial_{y_{k\alpha}}), \partial_{y_{k\alpha}} \rangle f.$$

Substituting $\partial_x = \partial_u + \partial_{\bar{u}}$ and $\partial_y = i(\partial_u - \partial_{\bar{u}})$ gives

(I) = -\frac{1}{2} \sum_{k, \ell, \alpha, \beta} c_{\ell k} (u_k(\alpha) \partial_{u_k(\alpha)} + \bar{u}_k(\alpha) \partial_{\bar{u}_k(\alpha)} + u_{\ell}(\alpha) \partial_{u_{\ell}(\alpha)} + \bar{u}_{\ell}(\alpha) \partial_{\bar{u}_{\ell}(\alpha)})) f$$

$$= -\frac{1}{2} \sum_{k, \ell, \alpha, \beta} c_{\ell k} (u_k(\alpha) \partial_{u_k(\alpha)} + \bar{u}_k(\alpha) \partial_{\bar{u}_k(\alpha)} + u_{\ell}(\alpha) \partial_{u_{\ell}(\alpha)} + \bar{u}_{\ell}(\alpha) \partial_{\bar{u}_{\ell}(\alpha)})) f.$$

Moreover, from the stochastic differential equation (2.3), we obtain

$$\langle x_{k\alpha}, x_{k\beta} \rangle' = \langle y_{k\alpha}, y_{k\beta} \rangle' = \frac{1}{2} \sum_{\ell \neq k} c_{\ell k} \Re(u_{\ell}(\alpha) \bar{u}_{\ell}(\beta)), \langle x_{k\alpha}, y_{k\beta} \rangle' = \langle y_{k\alpha}, x_{k\beta} \rangle' = \frac{1}{2} \sum_{\ell \neq k} c_{\ell k} \Im(u_{\ell}(\alpha) u_{\ell}(\beta)).$$
It implies that

\[(II) = \frac{1}{2} \sum_{k < \ell, \alpha, \beta} c_{k\ell} \left( u_\ell(\alpha) u_\ell(\beta) \partial_{u_k(\alpha)} \pi_k(\beta) + \pi_k(\alpha) u_k(\beta) \partial_{u_\ell(\alpha)} \pi_\ell(\beta) + u_k(\alpha) \pi_k(\beta) \partial_{u_\ell(\alpha)} \pi_\ell(\beta) \right) f \]

\[= \frac{1}{2} \sum_{k < \ell} c_{k\ell} \left( u_\ell \partial_{u_k} \pi_\ell + \pi_k \partial_{u_\ell} u_\ell + u_k \partial_{u_\ell} \pi_k \right) f.\]

Finally, concerning the term (III), a calculation yields, for \(k \neq \ell\),

\[\langle x_{k\alpha}, x_{\ell\beta} \rangle' = -\langle y_{k\alpha}, y_{\ell\beta} \rangle' = -\frac{1}{2} c_{k\ell} \Im((u_\ell(\alpha) u_k(\beta)). \langle x_{k\alpha}, y_{\ell\beta} \rangle' = \langle x_{\ell\beta}, y_{k\alpha} \rangle' = -\frac{1}{2} c_{k\ell} \Im((u_\ell(\alpha) u_k(\beta)).\]

We therefore get

\[(III) = -\frac{1}{2} \sum_{k < \ell, \alpha, \beta} c_{k\ell} \left( u_\ell(\alpha) u_k(\beta) \partial_{u_k(\alpha)} u_\ell(\beta) + \pi_k(\alpha) \pi_k(\beta) \partial_{u_\ell(\alpha)} \pi_\ell(\beta) + u_k(\alpha) u_\ell(\beta) \partial_{u_k(\alpha)} \pi_k(\beta) + \pi_k(\alpha) \pi_k(\beta) \partial_{u_\ell(\alpha)} \pi_\ell(\beta) \right) f \]

\[= -\frac{1}{2} \sum_{k < \ell} c_{k\ell} \left( u_\ell \partial_{u_k} u_\ell + \pi_k \partial_{u_\ell} u_k - \pi_k \partial_{u_\ell} u_k - u_k \partial_{u_\ell} u_k - u_\ell \partial_{u_k} u_k + u_k \partial_{u_\ell} \pi_k - u_k \partial_{u_\ell} \pi_k \right) f.\]

Gathering our estimates for (I), (II) and (III) yields

\[\frac{d}{dt} E(f) = \frac{1}{2} \sum_{k < \ell} c_{k\ell} E \left( \left( \partial_{u_k} u_\ell - \pi_k \partial_\ell \pi_k \right) (\pi_k \partial_\ell \pi_k - u_\ell \partial_{u_k} u_k) + \left( \pi_k \partial_\ell \pi_k - u_\ell \partial_{u_k} \right) \left( u_k \partial_{u_\ell} \pi_k - u_k \partial_{u_\ell} \pi_k \right) \right) f.\]

This completes the proof.

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