On the gauge dependence of gravitational waves generated at second order from scalar perturbations

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We revisit and clarify the gauge dependence of gravitational waves generated at second order from scalar perturbations. In a universe dominated by a perfect fluid with a constant equation-of-state parameter $w$, we compute the energy density of such induced gravitational waves in the Newtonian, comoving, and uniform curvature gauges. Huge differences are found between the Newtonian and comoving gauge results for any $w \geq 0$. This is always caused by the perturbation of the shift vector. Interestingly, the Newtonian and uniform curvature gauge calculations give the same energy density for $w > 0$. In the case of $w = 0$, the uniform curvature gauge result differs only by a factor from that of the comoving gauge, but deviates significantly from that of the Newtonian gauge. Our calculation is done analytically for $w = 0$ and $w = 1/3$, and our result is consistent with the previous numerical one.

I. INTRODUCTION

After the detection of gravitational waves (GWs) by the LIGO and Virgo collaborations \cite{ligo, virgo}, it is becoming more and more important to study GWs generated from various sources. Among a variety of sources, scalar (density) perturbations at quadratic order \cite{ballesteros, antusch} are of particular interest. Though the detectability of the GWs induced at second order depends on the cosmological scenarios under consideration, at least we know that their sources, i.e., scalar perturbations, do exist in the Universe. Since the energy-density spectrum of induced GWs is determined from the cosmic expansion history and the primordial spectrum of scalar perturbations, it can be a powerful probe for different scenarios having particular features in these respects \cite{mukhanov, starobinsky, puetzfeld, chin, min}. Almost all previous studies on scalar-induced GWs have employed the Newtonian gauge for the scalar perturbations at linear order. At second order in cosmological perturbation theory, however, tensor perturbations are dependent on the gauge choice of the scalar perturbations, as emphasized earlier in Ref. \cite{wu}. This is in contrast to the tensor perturbations at linear order. One would thus notice that there is no a priori reason for using only the Newtonian gauge. The source term for induced GWs during the matter-dominated era was obtained in the comoving gauge in Ref. \cite{chiueh}, but explicit solutions for the induced GWs have not been discussed in depth there. More recently, the explicit calculation of induced GWs in different gauges has been presented in Ref. \cite{tomikawa}, where, interestingly enough, a significant gauge dependence has been reported. The main result of Ref. \cite{tomikawa} was obtained numerically, using the standard cosmological model with the best-fitting cosmological parameters. Here the following questions would arise:

- Can we understand this gauge dependence more analytically?
- How does this gauge dependence depend on the background cosmological evolution (more specifically, the equation-of-state parameter)?
- How does the gauge dependence depend on the input form of the primordial power spectrum of the scalar perturbations?

To clarify these points, we consider a universe filled with a single perfect fluid with a constant equation-of-state parameter, and evaluate the energy density of induced GWs in three representative gauges: the Newtonian gauge, the comoving gauge, and the uniform curvature gauge. Our calculation is done analytically for matter-dominated and radiation-dominated universes following Ref. \cite{bassett} and numerically in the other cases.

This paper is organized as follows. In the next section, we derive the basic formula for the energy density of induced GWs. In Sec. III, we introduce the action approach to GWs induced from scalar perturbations to derive the gauge-ready form of the source term. Then, in Sec. IV we discuss the gauge dependence of induced GWs by evaluating their evolution analytically and numerically for different values of the equation-of-state parameter. Our conclusion is drawn in Sec. V.

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II. INDUCED GRAVITATIONAL WAVES

We start with deriving the expression for the energy density of induced GWs. The evolution equation for the induced GWs \( h_{ij}(\eta, \vec{x}) \) is of the form

\[
h''_{ij} + 2\mathcal{H}h'_{ij} - \partial^2 h_{ij} = \Lambda_{ijkl}S_{kl},
\]

where a dash denotes differentiation with respect to the conformal time \( \eta \) and \( \Lambda_{ijkl} \) is the projection tensor which extracts the transverse-traceless part of the source term \( S_{kl}(\eta, \vec{x}) \). A more explicit expression for \( \Lambda_{ijkl}S_{kl} \) will be given shortly below. We consider a universe filled with a perfect fluid whose equation of state parameter \( w \) is the Fourier transform of \( S \).

The Fourier components of \( h_{ij} \) are defined by

\[
h_{ij}(\eta, \vec{x}) = \sum_A \int \frac{d^3k}{(2\pi)^{3/2}} h_A(\eta, \vec{k}) e^{(A)}_{ij}(\vec{k}) e^{i\vec{k} \cdot \vec{x}} \quad (A = +, \times). \tag{2}
\]

The two polarization tensors \( e^{(A)}_{ij}(\vec{k}) \) are defined as

\[
e^{(+)}_{ij}(\vec{k}) = \frac{1}{\sqrt{2}} \left[ e_i(\vec{k})e_j(\vec{k}) - \tau_i(\vec{k})\tau_j(\vec{k}) \right], \tag{3}
\]

\[
e^{(\times)}_{ij}(\vec{k}) = \frac{1}{\sqrt{2}} \left[ e_i(\vec{k})\tau_j(\vec{k}) + \tau_i(\vec{k})e_j(\vec{k}) \right], \tag{4}
\]

where \( e_i(\vec{k}) \) and \( \tau_j(\vec{k}) \) are unit vectors orthogonal to each other and to \( \vec{k} \). It follows from the definition of the polarization tensors that \( \vec{k}^i e^{(A)}_{ij}(\vec{k}) = 0 \) and \( e^{(A)}_{ij}(\vec{k}) e^{(A')}_{ij}(\vec{k}) = \delta_{AA'} \).

Using the polarization tensors one can write the right-hand side of Eq. (1) as

\[
\Lambda_{ijkl}S_{kl}(\eta, \vec{x}) = \sum_A \int \frac{d^3k}{(2\pi)^{3/2}} e^{(A)}_{ij}(\vec{k}) e^{(A)}_{lm}(\vec{k}) S_{lm}(\eta, \vec{k}), \tag{5}
\]

where

\[
S_{ij}(\eta, \vec{k}) := \int \frac{d^3x}{(2\pi)^{3/2}} S_{ij}(\eta, \vec{x}) e^{-i\vec{k} \cdot \vec{x}} \tag{6}
\]

is the Fourier transform of \( S_{ij}(\eta, \vec{x}) \). We write \( S_A(\eta, \vec{k}) := e^{(A)}_{ij}(\vec{k}) S_{ij}(\eta, \vec{k}) \).

The formal solution to Eq. (1) in the Fourier domain is given by

\[
h_A(\eta, \vec{k}) = \frac{1}{a(\eta) \eta} \left[ \int_0^\eta G_k(\eta, \tilde{\eta}) a(\tilde{\eta}) S_A(\tilde{\eta}, \vec{k}) d\tilde{\eta} \right], \tag{7}
\]

where the Green’s function is expressed in terms of the Bessel functions as

\[
G_k(\eta, \tilde{\eta}) = \frac{\pi}{2} \eta^{1/2} \tilde{\eta}^{1/2} \left[ Y_\nu(k\eta) J_\nu(k\tilde{\eta}) - J_\nu(k\eta) Y_\nu(k\tilde{\eta}) \right], \tag{8}
\]

with

\[
\nu := \frac{3(1 - w)}{2(1 + 3w)}. \tag{9}
\]

The energy density of GWs is given by

\[
\rho_{GW}(\eta) = \frac{M_{Pl}^2}{2} \sum_A \frac{1}{a^2} \langle h'_A(\eta, \vec{x}) h'_A(\eta, \vec{x}) \rangle, \tag{10}
\]

where \( \langle \cdots \rangle \) denotes a spatial average and

\[
h_A(\eta, \vec{x}) := \int \frac{d^3k}{(2\pi)^{3/2}} h_A(\eta, \vec{k}) e^{i\vec{k} \cdot \vec{x}}. \tag{11}
\]
In the subhorizon regime, $k \eta \gg 1$, the time derivative of $h_A$ is approximately given by

$$h'_A(\eta, \vec{k}) \simeq \frac{1}{a(\eta)} \int_0^\eta \partial_\eta G_k(\eta, \tilde{\eta}) a(\tilde{\eta}) S_A(\eta, \vec{k}) d\tilde{\eta}.$$  \hspace{1cm} (15)

We find the following approximate expression for the time derivative of the Green’s function,

$$\partial_\eta G_k(\eta, \tilde{\eta}) \simeq G_k(\eta, \tilde{\eta}) := \frac{\pi}{2} (k \eta)^{1/2} (k \tilde{\eta})^{1/2} [J_{\nu-1}(k \eta) J_{\nu}(k \tilde{\eta}) - J_{\nu-1}(k \eta) Y_\nu(k \tilde{\eta})].$$  \hspace{1cm} (16)

More explicitly, for $\nu = 1/2 (w = 1/3)$ we have

$$G_k(\eta, \tilde{\eta}) = \cos[k(\eta - \tilde{\eta})],$$  \hspace{1cm} (17)

and for $\nu = 3/2 (w = 0)$ we have

$$G_k(\eta, \tilde{\eta}) = \cos[k(\eta - \tilde{\eta})] + \frac{\sin[k(\eta - \tilde{\eta})]}{k \tilde{\eta}}.$$  \hspace{1cm} (18)

It then follows that

$$\rho_{GW}(\eta) = \frac{M_{Pl}^2}{2} \frac{1}{V a^2(\eta)} \sum_A \int d^3 k \int_0^\eta d\eta' \int_0^\eta d\eta'' a(\eta') a(\eta'') G_k(\eta, \eta') G_k(\eta, \eta'') S_A(\eta', \vec{k}) S_A(\eta'', \vec{k}).$$  \hspace{1cm} (19)

We are interested in the case where $S_A$ is of the form

$$S_A(\eta, \vec{k}) = \epsilon_{ij}^{(A)}(\vec{k}) \int \frac{d^3 q}{(2\pi)^3} q^i q^j A(\vec{q}) A(\vec{k} - \vec{q}) F(\vec{k}, \vec{q}, \eta),$$  \hspace{1cm} (20)

where $A(\vec{q})$ is a Gaussian random field and $F(\vec{k}, \vec{q}, \eta)$ is some function. In the actual calculation, $A(\vec{q})$ will be the primordial amplitude of scalar perturbations and $F$ contains the information of their time evolution. The power spectrum $P(k)$ is defined by

$$\langle A(\vec{k}) A^*(\vec{q}) \rangle = \frac{2\pi^2}{k^3} \delta^{(3)}(\vec{k} - \vec{q}) P(k).$$  \hspace{1cm} (21)

(Now $\langle \cdots \rangle$ denotes an average over the whole distribution.) Using Wick’s theorem, the two-point correlator of the source term $S_A$ can be written as

$$\langle S_A(\eta', \vec{k}) S_A^*(\eta'', \vec{k'}) \rangle = \int \frac{d^3 q}{(2\pi)^3} (\epsilon_{ij}^{(A)}(\vec{k}) q^i q^j)^2 \frac{8\pi^4}{q^3} \delta^{(3)}(\vec{k} - \vec{q}) \delta^{(3)}(\vec{k'} - \vec{q}) P(q) P(|\vec{k} - \vec{q}|) F(\vec{k}, \vec{q}, \eta') F(\vec{k}, \vec{q}, \eta'').$$  \hspace{1cm} (22)

Using this and $[\delta^{(3)}(\vec{k} - \vec{q})]^2 = [V/(2\pi)^3] \delta^{(3)}(\vec{k} - \vec{q})$, one can write the energy density of GWs \cite{19} as

$$\rho_{GW}(\eta) = \frac{2M_{Pl}^2}{a^4(\eta)} \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} \int_0^\eta d\eta' \int_0^\eta d\eta'' a(\eta') a(\eta'') G_k(\eta, \eta') G_k(\eta, \eta'') \times q^4 \sin^4 \theta \frac{\pi^4}{q^3} P(q) P(|\vec{k} - \vec{q}|) F(\vec{k}, \vec{q}, \eta') F(\vec{k}, \vec{q}, \eta''),$$  \hspace{1cm} (23)
where $\theta$ is the angle between $\vec{q}$ and $\vec{k}$.

The final step is to extract from the above expression the energy density parameter of GWs, $\Omega_{GW}(\eta, k)$, defined by

$$\rho_{GW}(\eta) = 3M_{Pl}^2H^2\int\Omega_{GW}(\eta, k)d\ln k.$$  \hfill (24)

In practice, $F$ depends on $\vec{k}$, $\vec{q}$, and $\eta$ through $q\eta$ and $|\vec{k} - \vec{q}|\eta$. Therefore, it is convenient to introduce dimensionless variables $u := |\vec{k} - \vec{q}|/k$ and $v := q/k$. Using these variables, the energy density parameter is expressed as

$$\Omega_{GW}(\eta, k) = \frac{1}{3H^2a^4(\eta)}\int_0^\infty dv \int_{|1-v|}^{1+v} du \int_0^\eta d\eta' \int_0^{\eta} d\eta'' a(\eta')dG_k(\eta, \eta')G_k(\eta, \eta'')$$

$$\times \frac{k^4v^2}{4u^2} \left[ 1 - \left( \frac{1 + v^2 - u^2}{2v} \right)^2 \right] ^2 P(ku)P(kv)F(u, v, k\eta')F(u, v, k\eta'')$$

$$= \frac{k^2}{12H^2} \int_0^\infty dv \int_{|1-v|}^{1+v} du \frac{v^2}{u^2} \left[ 1 - \left( \frac{1 + v^2 - u^2}{2v} \right)^2 \right] ^2 P(ku)P(kv)\mathcal{I}(u, v, k\eta),$$  \hfill (25)

where

$$\mathcal{I}(u, v, k\eta) = \frac{k}{a(\eta)}\int_0^\eta d\eta' a(\eta')G_k(\eta, \eta')F(u, v, k\eta').$$  \hfill (26)

The information of the initial conditions for scalar perturbations is encoded in the power spectrum $P$, while the time evolution of the perturbations determines the form of the integral $\mathcal{I}$. The two distinct effects are thus separated. The gauge difference is essentially imprinted in $\mathcal{I}$. This can be evaluated analytically for $w = 0$ and $w = 1/3$ \cite{25} and numerically for the other values of $w$.

### III. ACTION APPROACH TO INDUCED GRAVITATIONAL WAVES

To derive the gauge-ready form of the source term for induced GWs in a universe filled with an irrotational barotropic perfect fluid, it is convenient to employ the action-based approach, describing the fluid in terms of a shift-symmetric k-essence field.

Our action is given by

$$S = \int d^4x\sqrt{-g}\left[ \frac{M_{Pl}^2}{2} R + P(X) \right],$$  \hfill (27)

where $X := -\rho\mu^\nu\partial_\mu\phi\partial_\nu\phi/2$. The k-essence field is equivalent to a cosmological perfect fluid whose energy density and pressure are given respectively by $\rho = 2XP_X - P$ and $p = P$. (Here and hereafter we write $\partial P/\partial X = P_X$.) Therefore, a $w = p/\rho = \text{const}$ fluid can be mimicked by \cite{26, 27}

$$P \propto X^{(1 + w)/2w}.$$  \hfill (28)

Hereafter we will assume that $w \geq 0$. Note that the $w = 0$ case appears to be singular, but a careful inspection shows that the limit in fact makes sense \cite{28}.

The metric in the 3 + 1 Arnowitt-Deser-Misner (ADM) form is given by

$$ds^2 = -N^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt).$$  \hfill (29)

We only fix the spatial coordinate, while leaving the temporal gauge degree of freedom unfixed, as we are interested only in the latter gauge difference. We therefore write the ADM variables in terms of the scalar and tensor perturbations as

$$N = 1 + \delta N, \quad N_i = \partial_i\chi, \quad g_{ij} = a^2e^{-2\phi}(e^h)_{ij},$$  \hfill (30)

where $(e^h)_{ij} = \delta_{ij} + h_{ij} + h_{ik}h_{kj}/2 + \cdots$. The perturbed scalar field is written as

$$\phi = \phi(t) + \delta\phi.$$  \hfill (31)
We will omit the bar from the background value if unnecessary. Using the temporal gauge degree of freedom one can eliminate one of \( \chi, \psi, \) and \( \delta \phi, \)

The background equations are given by

\[
3M_{Pl}^2 H^2 = 2XP - P, \tag{32}
\]
\[
M_{Pl}^2 \left( 3H^2 + 2\dot{H} \right) = -P, \tag{33}
\]
\[
\frac{d}{dt} \left( a^3 \dot{\phi} P_X \right) = 0. \tag{34}
\]

In order to derive the equations of motion for the perturbations, we substitute the metric \([30]\) and the scalar field \([31]\) to the action \([27]\) and expand it to third order. At third order we only need the terms containing one tensor and two scalars, because the variation of such terms with respect to \( h_{ij} \) leads to the source terms for GWs induced by scalar perturbations. Thus, the action that suffices for our purpose is

\[
S = \int dtd^3x \left[ \mathcal{L}^{(2)}_h + \mathcal{L}^{(2)}_s + \mathcal{L}^{(3)}_{ssh} \right], \tag{35}
\]

where

\[
\mathcal{L}^{(2)}_s = a^3 M_{Pl}^2 \left[ -3\dot{\psi}^2 - \frac{\psi \partial^2 \psi}{a^2} - 2H \delta \chi \frac{\partial^2 \chi}{a^2} - 2\psi \frac{\partial^2 \chi}{a^2} - 6H \delta N \psi + 2\delta N \frac{\partial^2 \psi}{a^2} \right]
\]
\[
+ a^3 \left( XP + 2X^2 P_{XX} - 3M_{Pl}^2 H^2 \right) \delta N^2 + \frac{a^3}{2} \left( P_X + 2XP_{XX} \right) \left( \delta \phi^2 - 2\dot{\phi} \delta N \delta \phi \right)
\]
\[
- 3a \dot{\phi} P_X \delta \phi + \frac{a P_X}{2} \left( \delta \phi \partial^2 \delta \phi + 2\dot{\phi} \delta \phi \partial^2 \chi \right), \tag{36}
\]
\[
\mathcal{L}^{(3)}_{ssh} = a M_{Pl}^2 \left[ 2 \left( H \delta N + \psi \right) \chi_{ij} h_{ij} + \frac{1}{2} \left( \delta N + 3 \psi \right) \chi_{ij} h_{ij} - \psi_i \psi_{j} h_{ij} + 2\delta N \psi \psi_{ij} h_{ij} + \frac{1}{4a^2} \partial^2 \left( \chi_i \chi_j \right) h_{ij} \right]
\]

\[
+ \frac{a P_X}{2} h_{ij} \left( \partial_i \delta \phi \partial_j \delta \phi + 2\dot{\phi} \partial_i \chi \partial_j \delta \phi \right). \tag{38}
\]

The quadratic Lagrangian \( \mathcal{L}^{(2)}_s \) yields the linearized equations of motion for the scalar perturbations. The variation of the above action with respect to \( h_{ij} \) gives the equation of motion for \( h_{ij} \) sourced by the scalar perturbations, which takes the form of Eq. \([31]\). Now it is straightforward to obtain

\[
S_{ij}(\eta, \vec{x}) = \frac{1}{a^2} \partial^2 (\chi_i \chi_j) + 8 \delta N_i (\psi_j) - 4 \psi_i \psi_j + \frac{8}{a} \left( H \delta N + \psi \right) \chi_{ij} - \frac{2}{a^2} \frac{d}{d\eta} \left[ a \left( \delta N + 3 \psi \right) \chi_{ij} \right]
\]

\[
+ \frac{4XP_X}{M_{Pl}^2 H^2} \left[ Q_{,i} Q_{,j} + \frac{2H}{a} \chi_{,i} Q_{,j} \right], \tag{39}
\]

where we defined \( Q(t, \vec{x}) := H \delta \phi / \dot{\phi} \). Note that one can simplify the expression further by using the background equation and write \( 4XP_X/M_{Pl}^2 H^2 = 6(1 + w) \). This is the gauge-ready form of the source term for induced GWs. Moving to the Fourier domain, we have

\[
S_A(\eta, \vec{k}) = e_{ij}^{(A)}(\vec{k}) \int \frac{d^3q}{(2\pi)^{3/2}} q^i \left[ \frac{k^2}{a^2} \chi(\eta, \vec{q}) \chi(\eta, \vec{k} - \vec{q}) + 4 \delta N(\eta, \vec{q}) \psi(\eta, \vec{k} - \vec{q}) + 4 \psi(\eta, \vec{q}) \delta N(\eta, \vec{k} - \vec{q}) + \cdots \right]. \tag{40}
\]

This can be recast in the form of Eq. \([20]\) by separating the scalar perturbations into the transfer functions and the primordial amplitudes.
IV. THE GAUGE DEPENDENCE

Now let us investigate the gauge dependence of induced GWs. Specifically, we consider the Newtonian gauge ($\chi = 0$), the comoving gauge ($\delta \phi = 0$), and the uniform curvature gauge ($\zeta = 0$). The gauge dependence can be seen clearly by evaluating the integral $I(u, v, k\eta)$. In radiation-dominated (RD) and matter-dominated (MD) universes this can be done analytically. For the other values of $w$, one needs to perform numerical integration to evaluate $I$ precisely.

A. Newtonian Gauge

We start with reproducing the standard Newtonian gauge result. The Newtonian gauge is defined by

$$\chi = 0.$$  

(41)

Following the conventional notation we write $\delta N = \Phi$ and $\psi = \Psi$.

From the Lagrangian (36) we obtain

$$\Psi'' + 2\mathcal{H}\Psi' + \frac{k^2}{3} (\Psi - \Phi) + \mathcal{H}\Phi' + \frac{3}{2} (1 - w) \mathcal{H}^2 \Phi - \frac{3}{2} (1 + w) \mathcal{H}^2 \left[ \frac{Q'}{\mathcal{H}} + \frac{3}{2} (1 - w) Q \right] = 0,$$

(42)

$$- 3\mathcal{H}\Psi' - k^2 \Psi + \frac{3(1 - w)}{2w} \mathcal{H}^2 \Phi - \frac{3(1 + w)}{2w} \mathcal{H}^2 \left[ \frac{Q'}{\mathcal{H}} + \frac{3}{2} (1 - w) Q \right] = 0,$$

(43)

$$Q'' + \mathcal{H} (2Q' - \Phi' - 3w\Psi') + wk^2 Q = 0,$$

(44)

where we moved to the Fourier domain and used the conformal time. The solution to the above set of equations is given by

$$\Phi = \Psi = A\Phi(\vec{k}) f_\Phi(u, v, k\eta),$$

(45)

$$Q = A\frac{3(1 + w)}{A\Phi(\vec{k})} \left[ 2f_\Phi + (1 + 3w) \eta f_\Phi' \right],$$

(46)

where

$$f_\Phi(u, v, k\eta) := \Gamma(\nu + 2) \left( \frac{\sqrt{w} k\eta}{2} \right)^{\nu - 1} J_{\nu + 1}(\sqrt{w} k\eta),$$

(47)

$A\Phi(\vec{k})$ is the amplitude of $\Phi$ at $\eta = 0$, and we discarded another independent solution that diverges at $\eta = 0$. We thus have $\Phi = \Psi = A\Phi$ and $Q = 2A\Phi/[3(1 + w)]$ at $\eta = 0$. Substituting the above result to the source term (40), one can compute $F(u, v, k\eta)$.

In a RD universe, we find

$$\frac{1}{54} F_{\text{RD}, \chi}(u, v, k\eta) = \frac{18}{u^2 v^2 k^4 \eta^4} \cos \left( \frac{uk\eta}{\sqrt{3}} \right) \cos \left( \frac{vk\eta}{\sqrt{3}} \right)$$

$$+ \frac{2\sqrt{3}}{u^2 v^2 k^2 \eta^2} (u^2 k^2 \eta^2 - 9) \sin \left( \frac{uk\eta}{\sqrt{3}} \right) \cos \left( \frac{vk\eta}{\sqrt{3}} \right)$$

$$+ \frac{2\sqrt{3}}{u^2 v^2 k^2 \eta^2} (v^2 k^2 \eta^2 - 9) \sin \left( \frac{uk\eta}{\sqrt{3}} \right) \sin \left( \frac{vk\eta}{\sqrt{3}} \right)$$

$$+ \frac{1}{u^2 v^2 k^2 \eta^2} [54 - 6(u^2 + v^2) k^2 \eta^2 + u^2 v^2 k^4 \eta^4] \sin \left( \frac{uk\eta}{\sqrt{3}} \right) \sin \left( \frac{vk\eta}{\sqrt{3}} \right),$$

(48)

Following Ref. [25] one can evaluate analytically the integral $I(u, v, k\eta)$ for $k\eta \gg 1$:

$$I_{\text{RD}, \chi}(u, v, k\eta) = \frac{1}{4k\eta} [I_1 \cos(k\eta) + I_2 \sin(k\eta)],$$

(49)
where
\[ I_1 := \frac{27}{2} \frac{u^2 + v^2 - 3}{u^3 v^3} \left[ -4uv + (u^2 + v^2 - 3) \ln \frac{3 - (u + v)^2}{3 - (u - v)^2} \right], \]  
\[ I_2 := \frac{27}{2} \pi \frac{(u^2 + v^2 - 3)^2}{u^3 v^3} \Theta(u + v - \sqrt{3}), \]  
with \( \Theta \) being the step function. Its oscillation average is therefore
\[ \overline{I_{RD,\chi}} = \frac{I_2^2 + I_3^2}{32(k\eta)^2}. \]  

In a MD universe, we take the \( w \to 0 \) limit in Eqs. (45) and (46) and obtain \( \Phi = \Psi = A \), \( Q = (2/3) A \). Thus, it is easy to see
\[ F_{MD,\chi} = \frac{20}{3}, \]  
and thus
\[ I_{RD,\chi} = \frac{20[k\eta - \sin(k\eta)]}{(k\eta)^2}. \]  

B. Comoving Gauge v.s. Newtonian Gauge

The comoving gauge is defined by
\[ \delta \phi = 0. \]  

We write the comoving curvature perturbation as \( \zeta = -\psi \).

The linear equations of motion in the comoving gauge are
\[ 6\mathcal{H} \zeta' + 2k^2 \zeta + \frac{2}{a} \mathcal{H} k^2 \chi + \frac{3(1 - w)}{w} \mathcal{H}^2 \delta N = 0, \]  
\[ \delta N = \frac{\zeta'}{\mathcal{H}}, \]  
\[ \zeta'' + 2\mathcal{H} \zeta' + \frac{k^2}{3} \zeta + \frac{k^2}{3a} (\chi' + \mathcal{H} \chi) - \mathcal{H} \delta N' + \frac{k^2}{3} \delta N - \frac{3}{2} (1 - w) \mathcal{H}^2 \delta N = 0. \]  

The solution regular at \( \eta = 0 \) is given by
\[ \zeta = A_1(k) f_\zeta(k, \eta), \quad \delta N = \frac{1 + 3w}{2} A_1(k) \eta f'_\zeta, \quad \frac{k}{a} \chi = -\frac{A_1(k)}{2} \left[ \frac{3(1 + w) f'_\zeta}{w} + (1 + 3w) k \eta f_\zeta \right], \]  
where
\[ f_\zeta(k, \eta) := \Gamma(\nu + 1) \left( \frac{\sqrt{w} k \eta}{2} \right)^{-\nu} J_\nu(\sqrt{w} k \eta). \]  

It is well-known that the primordial amplitude in the comoving gauge is related to that in the Newtonian gauge by
\[ A_1(k) = \frac{5 + 3w}{3(1 + w)} A_\Phi(k). \]  

To understand the gauge dependence in a RD universe, it will be helpful to see the behavior of the scalar perturbations for \( k\eta \gg 1 \). For \( w = 1/3 \), we have
\[ \zeta = A_\zeta \frac{\sin(k\eta/\sqrt{3})}{k\eta/\sqrt{3}}, \quad \delta N \approx A_\zeta \cos(k\eta/\sqrt{3}), \quad \frac{k}{a} \chi \approx -\sqrt{3} A_\zeta \sin(k\eta/\sqrt{3}). \]
This should be contrasted with the behavior of the Newtonian gauge perturbations in a RD universe for $k\eta \gg 1$:

$$
\Phi = \Psi \approx -9A_\Phi \frac{\cos(k\eta/\sqrt{3})}{k^2\eta^2}, \quad Q \approx \frac{3}{2}A_\Phi \frac{\sin(k\eta/\sqrt{3})}{k\eta/\sqrt{3}}.
$$

We see that in the comoving gauge the source term contains the terms that do not decay at late times. Accordingly, we have

$$
F_{RD,\delta\phi}(u,v,k\eta) = \left\{ -2 - \frac{12}{u^2v^2k^2\eta^2} \left[ 3 - 2(u^2 + v^2) \right] \right\} \cos\left( \frac{uk\eta}{\sqrt{3}} \right) \cos\left( \frac{vk\eta}{\sqrt{3}} \right)
$$

$$
+ \frac{2\sqrt{3}}{u^3v^2k^3\eta^4} \left\{ 6 \left[ 3 - 2(u^2 + v^2) \right] + u^2 \left( -3 + u^2 + 2v^2 \right) k^2\eta^2 \right\} \sin\left( \frac{uk\eta}{\sqrt{3}} \right) \cos\left( \frac{vk\eta}{\sqrt{3}} \right)
$$

$$
+ \frac{2\sqrt{3}}{u^2v^3k^3\eta^4} \left\{ 6 \left[ 3 - 2(u^2 + v^2) \right] + v^2 \left( -3 + 2u^2 + v^2 \right) k^2\eta^2 \right\} \cos\left( \frac{uk\eta}{\sqrt{3}} \right) \sin\left( \frac{vk\eta}{\sqrt{3}} \right)
$$

$$
+ \frac{1}{u^3v^3k^3\eta^4} \left\{ -36 \left[ 3 - 2(u^2 + v^2) \right] + k^2\eta^2 \left[ 6(u^2 + v^2) - k^2\eta^2u^2v^2 \right] \right\} \sin\left( \frac{uk\eta}{\sqrt{3}} \right) \sin\left( \frac{vk\eta}{\sqrt{3}} \right),
$$

which does not decay at late times, in contrast to the Newtonian gauge result (48). It then follows that

$$
I_{RD,\delta\phi} = \left( \frac{2}{3} \right)^2 I_{RD,\chi} - \frac{3}{2u^2v^2k\eta} \left\{ 3(2u^2 - 3uv + 2v^2) \cos\left( \frac{u-v}{\sqrt{3}} k\eta \right) + 3(2u^2 + 3uv + 2v^2) \cos\left( \frac{u+v}{\sqrt{3}} k\eta \right) \right\}
$$

$$
+ \frac{\sqrt{3}}{2uv} \left\{ (u-v) \sin\left( \frac{u-v}{\sqrt{3}} k\eta \right) - (u+v) \sin\left( \frac{u+v}{\sqrt{3}} k\eta \right) \right\},
$$

where we took the limit $k\eta \gg 1$. If we had only the first term in Eq. (65), the energy density of induced GWs would always be gauge-invariant, as the factor $(2/3)^2$ is canceled in the final result due to the relation (61). The first line decays as $\sim \eta^{-1}$, while the second line just oscillates, and hence the latter in fact dominates at late times, resulting in a large gauge dependence. This is essentially due to the first term in the source (39).

The large gauge dependence we have observed is in fact generic to the other values of $w$ ($>0$). In the Newtonian gauge, we have, for $k\eta \gg 1$,

$$
\Phi = \Psi \sim \eta^{-\nu-3/2}, \quad Q \sim \eta^{-\nu-1/2}.
$$

However, in the comoving gauge we have

$$
\zeta \sim \eta^{-\nu-1/2}, \quad \delta N \sim \frac{k}{a} \chi \sim \eta^{-\nu+1/2},
$$

which shows that the first term in the source (39) always overwhelms the other contributions and causes a large gauge dependence.

Let us then consider a MD universe. Since one has $w$ in the denominator in Eq. (59), the $w \to 0$ limit in the k-essence description of a fluid seems particularly subtle in the comoving gauge. However, for $w \ll 1$, $f_\zeta$ is approximated by

$$
f_\zeta = 1 - \frac{w}{10} k^2\eta^2,
$$

and using this one finds

$$
\zeta = A_\zeta, \quad \delta N = 0, \quad \frac{k}{a} \chi = -\frac{1}{5} A_\zeta k\eta
$$

in the $w \to 0$ limit [23]. Thus, one can safely take the $w \to 0$ limit. Note that $(k/a)\chi$ grows in time. This is again different from the behavior of the Newtonian gauge variables in a MD universe: $\Phi$, $\Psi$, and $Q$ remain constant. This difference gives rise to a growing contribution in $F$ and $I$:

$$
F_{MD,\delta\phi} = 2 - \frac{k^2\eta^2}{25},
$$

$$
I_{MD,\delta\phi} = \left( \frac{3}{5} \right)^2 I_{MD,\chi} \frac{k\eta}{5}.
$$
If we had only the first term in Eq. (71), there would be no gauge dependence in induced GWs, given that Eq. (61) accounts for the factor \((3/5)^2\). However, this term decays as \(\sim \eta^{-1}\), and so the second term dominates at late times. Therefore, there is a large gauge dependence also in this case. Again, this is caused by the first term in the source \(39\).

C. Uniform Curvature Gauge v.s. Newtonian Gauge

Finally, let us consider the uniform curvature gauge defined by

\[
\psi = 0, \tag{72}
\]

The evolution of the scalar perturbations in the uniform curvature gauge is governed by

\[
\mathcal{H}Q' + \frac{3}{2}(1-w)\mathcal{H}^2Q - \frac{2w}{3(1+w)} \frac{\mathcal{H}}{a} k^2 \chi - \frac{1-w}{1+w} \mathcal{H}^2 \delta N = 0, \tag{73}
\]

\[
\delta N = \frac{3(1+w)}{2} Q, \tag{74}
\]

\[
Q'' + \mathcal{H}(2Q' - \delta N') + wk^2 \left( Q + \frac{\mathcal{H}}{a} \chi \right) = 0. \tag{75}
\]

The non-decaying solution is given by

\[
Q = A_Q(\vec{k}) f_Q(\eta, k), \quad \delta N = \frac{3(1+w)}{2} A_Q(\vec{k}) f_Q, \quad \frac{k}{a} \chi = \frac{3(1+w)}{2w} A_Q(\vec{k}) \frac{f_Q'}{k}, \tag{76}
\]

where

\[
f_Q(\eta, k) = \Gamma(\nu+1) \left( \frac{\sqrt{w} k \eta}{2} \right)^{-\nu} J_\nu(\sqrt{w} k \eta). \tag{77}\]

This function is the same as Eq. (60). The primordial amplitude \(A_Q\) is related to \(A_\zeta\) (and \(A_\Phi\)) by

\[
A_Q(\vec{k}) = -A_\zeta(\vec{k}) = \frac{5 + 3w}{3(1+w)} A_\Phi(\vec{k}). \tag{78}\]

In a RD universe, we have

\[
Q = \frac{\delta N}{2} = A_Q \frac{\sin(\kappa \eta/\sqrt{3})}{\kappa \eta/\sqrt{3}}, \quad \frac{k}{a} \chi \approx 6 A_Q \frac{\cos(\kappa \eta/\sqrt{3})}{\kappa \eta} \tag{79}\]

for \(k \eta \gg 1\). Therefore, unlike in the comoving gauge, the perturbations in the uniform curvature gauge decay as \(\sim \eta^{-1}\). It is then straightforward to compute

\[
F_{\text{RD},\zeta}(\vec{k}, \eta) = \frac{12(a^2 + \nu^2 - 3)}{u^2 v^2 k^2 \eta^2} \cos \left( \frac{u k \eta}{\sqrt{3}} \right) \cos \left( \frac{v k \eta}{\sqrt{3}} \right) - \frac{12 \sqrt{3} (u^2 + \nu^2 - 3)}{u^3 v^2 k^3 \eta^3} \sin \left( \frac{u k \eta}{\sqrt{3}} \right) \cos \left( \frac{v k \eta}{\sqrt{3}} \right)
- \frac{12 \sqrt{3} (a^2 + \nu^2 - 3)}{u^2 v^2 k^3 \eta^3} \cos \left( \frac{u k \eta}{\sqrt{3}} \right) \sin \left( \frac{v k \eta}{\sqrt{3}} \right) + \frac{36 (a^2 + \nu^2 - 3)}{u^3 v^3 k^3 \eta^4} \sin \left( \frac{u k \eta}{\sqrt{3}} \right) \sin \left( \frac{v k \eta}{\sqrt{3}} \right). \tag{80}\]

This expression is clearly different from the Newtonian gauge result \(48\). However, integrating this to get \(I\) we find

\[
I_{\text{RD},\psi} = \left( \frac{2}{3} \right)^2 I_{\text{RD},\chi}. \tag{81}\]

Taking into account the relation \(78\), we see from Eq. \(81\) that the comoving gauge and the uniform curvature gauge give the identical result on the energy density of induced GWs.

To see whether this is accidental or not, we evaluate \(I\) numerically for the other values of \(w(\lesssim 1)\). Examples of our numerical investigation are presented in Figs. 1, 3. In Figs. 1 and 2 we present the comparison of \(I\) in the
Newtonian and uniform curvature gauges for different values of \( w \) with \( u, \nu, k\eta \) being fixed. We also show in Fig. 3 the comparison of \( I \) as a function of \( k\eta \) for \( w = 2/3 \). These results imply that the following relation holds:

\[
I,\psi = \left( \frac{3(1+w)}{5+3w} \right)^2 I,\chi.
\]  

(82)

We thus conclude that the Newtonian gauge and the uniform curvature gauge give the identical result on \( \Omega_{GW} \) for \( w > 0 \).

This is, however, not true in the case of \( w = 0 \). Again, there is a subtlety regarding \( w \) in the denominator, but this can be circumvented in the same way as in the comoving gauge. Since \( f_Q = 1 - \frac{w k^2 \eta^2}{10} \) for \( w \ll 1 \), we have

\[
\frac{k}{a} \chi = -\frac{3}{10} A_Q k \eta.
\]  

(83)
FIG. 3. Comparison of $I(2,2,k\eta)$ computed in the Newtonian and uniform curvature gauges for $w = 2/3$.

Therefore, the evolution of the scalar perturbations in a MD universe is very similar to that in the comoving gauge. This yields the following result on $F$ and $I$:

$$F_{MD,\psi} = \frac{3}{2} - \frac{9}{100} k^2 \eta^2, \quad I_{MD,\psi} = \left( \frac{3}{5} \right)^2 I_{MD,\chi} - \frac{9}{20} k \eta.$$  \hspace{1cm} (84)

Therefore, in a MD universe, the uniform curvature gauge and the Newtonian gauge give very different results. Rather, the uniform curvature gauge is similar to the comoving gauge and their results are different only by a factor at late times: $I_{MD,\psi} \approx (9/4) I_{MD,\delta\phi}$. Here again, this is due to the first term in the source (39). The factor $9/4$ comes from the coefficients of the solution of $\chi$: $(3/10)^2 = (9/4) \times (1/5)^2$. It is exactly this factor that explains the difference between the comoving and uniform curvature gauges in the numerical calculation in [24].

V. CONCLUDING REMARKS

In this paper, we have revisited the issue of the gauge dependence of gravitational waves (GWs) induced at second order from scalar perturbations. We have evaluated the energy density of induced GWs in different gauges in a universe dominated by a perfect fluid whose equation-of-state parameter $w$ is constant, and arrived at the following conclusions: (i) the amplitude of induced GWs in the comoving gauge is significantly larger than that in the Newtonian gauge for any $w \geq 0$, and this huge gauge dependence is a consequence of the presence of the shift vector; (ii) for $w > 0$ the Newtonian gauge result agrees with that of the uniform curvature gauge; (iii) for $w = 0$ the uniform curvature gauge result differs only by a factor from that of the comoving gauge, but deviates significantly from that of the Newtonian gauge. Our calculation has been done analytically for $w = 0$ and $w = 1/3$ using the method of Ref. [25]. The above conclusions are consistent with the previous numerical result [24]. The gauge dependence has been clarified based only on the evolution of the perturbations, and hence our result is robust against the input form of the primordial power spectrum of the scalar perturbations.

For simplicity and clarity, we have focused on the ideal case with $w = \text{const}$ rather than the realistic and conventional cosmological setup. Nevertheless, we believe that the present paper would be of help to gaining a deeper understanding of the gauge dependence of scalar-induced GWs.

As was noted in Ref. [24], the appropriate gauge one should choose depends on what quantity one measures in each observation. Given that there is a large gauge dependence of second-order GWs, it would be important to address this issue and identify the true observables.

Note added: While we were in the final stage of this work, the paper by J. O. Gong [28] appeared in the arXiv, where the gauge dependence of induced gravitational waves was studied analytically by comparing the Newtonian
and comoving gauge results in a matter-dominated universe. Our conclusion agrees with his where we overlap.

**ACKNOWLEDGMENTS**

We are grateful to K. Inomata and R. Saito for fruitful discussions. The work of KT was supported by the Rikkyo University Special Fund for Research. The work of TK was supported by MEXT KAKENHI Grant Nos. JP15H05888, JP17H06359, JP16K17707, and JP18H04355.
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