On the order of regular graphs with fixed second largest eigenvalue

Jae Young Yang\(^1\)*, Jack H. Koolen\(^{2,3}†\)

\(^1\) School of Mathematical Sciences, Anhui University, 111 Jiulong Road, Hefei, 230039, Anhui, PR China
\(^2\) School of Mathematical Sciences, University of Science and Technology of China, 96 Jinzhai Road, Hefei, 230026, Anhui, PR China
\(^3\) Wen-Tsun Wu Key Laboratory of CAS, 96 Jinzhai Road, Hefei, 230026, Anhui, PR China

e-mail: piez@naver.com, koolen@ustc.edu.cn

Abstract

Let \(v(k, \lambda)\) be the maximum number of vertices of a connected \(k\)-regular graph with second largest eigenvalue at most \(\lambda\). The Alon-Boppana Theorem implies that \(v(k, \lambda)\) is finite when \(k > \frac{\lambda^2 + 4}{4}\). In this paper, we show that for fixed \(\lambda \geq 1\), there exists a constant \(C(\lambda)\) such that \(2k + 2 \leq v(k, \lambda) \leq 2k + C(\lambda)\) when \(k > \frac{\lambda^2 + 4}{4}\).

\(*\) J.Y. Yang is partially supported by the National Natural Science Foundation of China (No. 11371028).

†J.H. Koolen is partially supported by the National Natural Science Foundation of China (No. 11471009 and No. 11671376).
Keywords: smallest eigenvalue, Hoffman graph, Alon-Boppana Theorem, co-edge-regular graph

AMS classification: 05C50, 05C75, 05C62

1 Introduction

Let \( v(k, \lambda) \) be the maximum order of a connected \( k \)-regular graph with second largest eigenvalue at most \( \lambda \). For \( \lambda \geq 2\sqrt{k-1} \), it is known that \( v(k, \lambda) \) is infinite from the existence of infinite families of bipartite regular Ramanujan graphs [11]. The Alon-Boppana Theorem [1, 5, 6, 8, 12, 14, 15] states:

**Theorem 1.1.** For any integer \( k \geq 3 \) and real number \( \lambda < 2\sqrt{k-1} \), the number \( v(k, \lambda) \) is finite.

In this paper, we will look at the behavior of \( v(k, \lambda) \) when \( \lambda \) is fixed and \( k \) goes to infinity. Our main theorem is:

**Theorem 1.2.** Let \( \lambda \) be an integer at least 1. Then there exists a constant \( C_1(\lambda) \) such that

\[
2k + 2 \leq v(k, \lambda) \leq 2k + C_1(\lambda)
\]

holds for all \( k > \frac{\lambda^2 + 4}{4} \).

For fixed real number \( \lambda \geq 1 \), define \( T(\lambda) \) as

\[
T(\lambda) := \limsup_{k \to \infty} v(k, \lambda) - 2k.
\]

Because of Theorem 1.2, \( T(\lambda) \) is well-defined. We will show that \( T(\lambda) \geq 2\lambda \) holds for fixed positive integer \( \lambda \).

The proof of Theorem 1.2 is based on the following proposition. In order to state this proposition, we need to introduce the next notion. For a vertex \( x \) of a graph \( G \), let \( \Gamma_i(x) \) be the set of vertices which are at distance \( i \) from \( x \).

**Proposition 1.3.** Let \( \lambda \) be a real number at least 1. Then there exists a constant \( M(\lambda) \geq \lambda^3 \) such that, if \( G \) is a graph satisfying

(i) every pair of vertices at distance 2 has at least \( M(\lambda) \) common neighbors,

(ii) the smallest eigenvalue of \( G \), \( \lambda_{\text{min}}(G) \), satisfies \( \lambda_{\text{min}}(G) \geq -\lambda \),

\( \lambda_{\text{min}}(G) \geq -\lambda \),
then $G$ has diameter 2 and $|\Gamma_2(x)| \leq \lfloor |\lambda| |\lambda^2| \rfloor$ for all $x \in V(G)$.

Remark 1.4. Neumaier [13] mentioned that Hoffmann gave a very large bound on the intersection number $c_2$ of strongly regular graphs. This may imply that Proposition 1.3 was already known by Hoffmann. However, we could not find it in the literature.

To prove Proposition 1.3, we use a combinatorial object named Hoffman graphs. The definition and basic properties of Hoffman graphs are given in Section 2. In Section 3, we prove Proposition 1.3. In Section 4, we present some known facts on the number $v(k, \lambda)$, and, in Section 5, we prove Theorem 1.2 by using Proposition 1.3. In Section 6, we discuss the behavior of the number $T(\lambda)$ for a fixed positive integer $\lambda$. In the last section, we give two more applications of Proposition 1.3 for the classes of co-edge regular graphs and amply regular graphs.

2 Hoffman graphs

In this section, we introduce the definition and basic properties of Hoffman graphs. Hoffman graphs were defined by Woo and Neumaier [16] following an idea of Hoffmann [7]. For more details or proofs, see [9, 10, 16].

2.1 Definition and properties of Hoffman graphs

Definition 2.1. A Hoffman graph $\mathfrak{h}$ is a pair $(H, \ell)$ of a graph $H$ and a labeling map $\ell : V(H) \to \{\text{fat}, \text{slim}\}$ satisfying two conditions:

(i) the vertices with label fat are pairwise non-adjacent,

(ii) every vertex with label fat has at least one neighbor with label slim.

The vertices with label fat are called fat vertices, and the set of fat vertices of $\mathfrak{h}$ are denoted by $V_{\text{fat}}(\mathfrak{h})$. The vertices with label slim are called slim vertices, and the set of slim vertices are denoted by $V_{\text{slim}}(\mathfrak{h})$. Now, we give some definitions.
Definition 2.2. For a Hoffman graph $\mathbf{h}$, a Hoffman graph $\mathbf{h}_1 = (H_1, \ell_1)$ is called an induced Hoffman subgraph of $\mathbf{h}$ if $H_1$ is an induced subgraph of $H$ and $\ell(x) = \ell_1(x)$ for all vertices $x$ of $H_1$.

Definition 2.3. Two Hoffman graphs $\mathbf{h} = (H, \ell)$ and $\mathbf{h}' = (H', \ell')$ are called isomorphic if there exists a graph isomorphism $\psi$ from $H$ to $H'$ such that $\ell(x) = \ell'(\psi(x))$ for all vertices $x$ of $H$.

Definition 2.4. For a Hoffman graph $\mathbf{h} = (H, \ell)$, let $A(H)$ be the adjacency matrix of $H$ with a labeling in which the fat vertices come last. Then

$$A(H) = \begin{pmatrix} A_{\text{slim}} & C \\ C^T & O \end{pmatrix},$$

where $A_{\text{slim}}$ is the adjacency matrix of the subgraph of $H$ induced by slim vertices and $O$ is the zero matrix.

The real symmetric matrix $S(\mathbf{h}) = A_{\text{slim}} - CC^T$ is called the special matrix of $\mathbf{h}$, and the eigenvalues of $\mathbf{h}$ are the eigenvalues of $S(\mathbf{h})$.

For a Hoffman graph $\mathbf{h}$, we focus on its smallest eigenvalue in this paper. Let $\lambda_{\text{min}}(\mathbf{h})$ denote the smallest eigenvalue of $\mathbf{h}$. Now, we discuss some spectral properties of $\lambda_{\text{min}}(\mathbf{h})$ without proofs.

Lemma 2.5. [16, Corollary 3.3] If $\mathbf{h}'$ is an induced Hoffman subgraph of $\mathbf{h}$, then $\lambda_{\text{min}}(\mathbf{h}') \geq \lambda_{\text{min}}(\mathbf{h})$ holds.

Theorem 2.6. [10, Theorem 2.2] Let $\mathbf{h}$ be a Hoffman graph. For a positive integer $p$, let $G(\mathbf{h}, p)$ be the graph obtained from $\mathbf{h}$ by replacing every fat vertex of $\mathbf{h}$ by a complete graph $K_p$ of $p$ slim vertices, and connecting all vertices of the $K_p$ to all neighbors of the original fat vertex by edges. Then

$$\lambda_{\text{min}}(G(\mathbf{h}, p)) \geq \lambda_{\text{min}}(\mathbf{h}),$$

and

$$\lim_{p \to \infty} \lambda_{\text{min}}(G(\mathbf{h}, p)) = \lambda_{\text{min}}(\mathbf{h}).$$
2.2 Quasi-clique and associated Hoffman graph

In this subsection, we introduce two terminologies, quasi-clique and associated Hoffman graph. Most of this section is explicitly formulated in [10]. Note that the term quasi-clique in this paper is different from the term quasi-clique in [16].

For the rest of this section, let \( \tilde{K}_{2m} \) be the graph consisting of a complete graph \( K_{2m} \) and a vertex which is adjacent to exactly \( m \) vertices of the \( K_{2m} \).

For a positive integer \( m \) at least 2, let \( G \) be a graph which does not contain \( \tilde{K}_{2m} \) as an induced subgraph. For a positive integer \( n \) at least \( (m+1)^2 \), let \( \mathcal{C}(n) \) be the set of maximal cliques of \( G \) with at least \( n \) vertices. Define the relation \( \equiv_n^m \) on \( \mathcal{C}(n) \) by \( C_1 \equiv_n^m C_2 \) if every vertex \( x \in C_1 \) has at most \( m-1 \) non-neighbors in \( C_2 \) and every vertex \( y \in C_2 \) has at most \( m-1 \) non-neighbors in \( C_1 \) for \( C_1, C_2 \in \mathcal{C}(n) \).

**Lemma 2.7.** [10] Lemma 3.1] Let \( m, n \) be two integers at least 2 such that \( n \geq (m+1)^2 \). Then the relation \( \equiv_n^m \) on \( \mathcal{C}(n) \) is an equivalence relation.

For a maximal clique \( C \in \mathcal{C}(n) \), let \( [C]^m_n \) denote the equivalence class containing \( C \). Now, we are ready to define the term quasi-clique.

**Definition 2.8.** Let \( m, n \) be two integers at least 2 such that \( n \geq (m+1)^2 \). For a maximal clique \( C \in \mathcal{C}(n) \), we define the quasi-clique \( Q[C]^m_n \) with respect to the pair \( (m, n) \) of \( G \), as the subgraph of \( G \) induced on the vertices which have at most \( m-1 \) non-neighbors in \( C \).

By [10] Lemma 3.2 and [10] Lemma 3.3, the quasi-clique \( Q[C]^m_n \) is well-defined for \( C \in \mathcal{C}(n) \).

Now we introduce the associated Hoffman graphs. In the next proposition, we present a result which is needed to show Proposition 1.3.

**Definition 2.9.** Let \( m, n \) be two integers at least 2 such that \( n \geq (m+1)^2 \). Let \( [C_1]^m_n, [C_2]^m_n, \ldots, [C_t]^m_n \) be all the equivalence classes of \( G \) under \( \equiv_n^m \). The associated Hoffman graph \( g = g(G, m, n) \) is the Hoffman graph with the following properties.

1. \( V_{\text{sim}}(g) = V(G) \), and \( V_{\text{fat}}(g) = \{F_1, \ldots, F_t\} \), where \( t \) is the number of equivalence classes of \( G \) under \( \equiv_n^m \).
(ii) the induced Hoffman subgraph of $g$ on $V_{\text{slim}}(g)$ is isomorphic to $G$,

(iii) the fat vertex $F_i$ is adjacent to all vertices of the quasi-clique $Q[C_i]_m$
for $i = 1, 2, \ldots, t$.

**Proposition 2.10.** [10, Proposition 4.1] There exists a positive integer $n = n(m, \phi, \sigma, p) \geq (m + 1)^2$ such that for any integer $q \geq n$, and any Hoffman graph $h$ with at most $\phi$ fat vertices and at most $\sigma$ slim vertices, the graph $G(h, p)$ is an induced subgraph of $G$, provided that the graph $G$ satisfies the following conditions:

(i) the graph $G$ does not contain $\tilde{K}_{2m}$ as an induced subgraph,

(ii) the associated Hoffman graph $g = g(G, m, q)$ contains $h$ as an induced Hoffman subgraph.

3 Main tool

In this section, we prove Proposition 1.3, which is the main tool of this paper. Before we prove Proposition 1.3, we first show two lemmas.

Let $H$ be a graph. Define $q(H)$ the Hoffman graph obtained by attaching one fat vertex to all vertices of $H$. Then $\lambda_{\min}(q(H)) = -\lambda_{\max}(\overline{H})$, where $\lambda_{\max}(\overline{H})$ is the maximal eigenvalue of the complement $\overline{H}$ of $H$. The Perron-Frobenius Theorem implies the following lemma.

**Lemma 3.1.** Let $H$ be a graph with an isolated vertex $x$. If $\lambda_{\min}(q(H)) \geq -\lambda$ for some real number $\lambda \geq 1$, then $H$ has at most $\lceil \lambda^2 \rceil + 1$ vertices.

**Proof.** Let $n$ be the number of vertices of $H$. Since $x$ is an isolated vertex of $H$, $x$ is adjacent to all other vertices of $H$ in the complement $\overline{H}$ of $H$. By the Perron-Frobenius Theorem, we have

$$\lambda_{\max}(\overline{H}) \geq \lambda_{\max}(K_{1, n-1}) = \sqrt{n - 1}$$

This shows the lemma. \qed

6
Lemma 3.2. Let $\lambda$ be a real number at least 1. Then there exist minimum positive integers $t'(\lambda)$ and $m'(\lambda)$ such that both $\lambda_{\min}(K_{2,t'(\lambda)}) < -\lambda$ and $\lambda_{\min}(\tilde{K}_{2m'(\lambda)}) < -\lambda$ hold.

Proof. Since $\lambda_{\min}(K_{2,t}) = -\sqrt{2t}$ and $\lambda_{\min}(\tilde{K}_{2m})$ is the smallest eigenvalue of the matrix

$$
\begin{pmatrix}
    m-1 & m & 0 \\
    m & m-1 & 1 \\
    0 & m & 0
\end{pmatrix},
$$

it is easily checked that

$$
\lim_{t \to \infty} \lambda_{\min}(K_{2,t}) = \lim_{m \to \infty} \lambda_{\min}(\tilde{K}_{2m}) = -\infty.
$$

This shows the existence of $t'(\lambda)$ and $m'(\lambda)$.

Proof of Proposition 1.3. First, we consider the Hoffman graph $h(\lfloor \lambda + 1 \rfloor)$ with $\lfloor \lambda + 1 \rfloor$ fat vertices adjacent to one slim vertex. Then $\lambda_{\min}(h(\lfloor \lambda + 1 \rfloor)) = -\lfloor \lambda + 1 \rfloor < -\lambda$, so there exists an positive integer $p_0$ such that $\lambda_{\min}(G(h(\lfloor \lambda + 1 \rfloor), p_0)) < -\lambda$ by Theorem 2.6.

Next, let $\{H_1, \ldots, H_r\}$ be the set of pairwise non-isomorphic graphs on $\lfloor \lambda^2 \rfloor + 2$ vertices with an isolated vertex. By Lemma 3.1, $\lambda_{\min}(q(H_i)) < -\lambda$ holds for all $i = 1, \ldots, r$. For each $i = 1, \ldots, r$, there exists positive integers $p_i$ such that $\lambda_{\min}(G(q(H_i), p_i)) < -\lambda$ by Theorem 2.6. Set $p' = \max p_i$.

For the two integers $t' = t'(\lambda)$ and $m' = m'(\lambda)$ of Lemma 3.2, let $n' = n(m', \lfloor \lambda + 1 \rfloor, \lfloor \lambda^2 + 2 \rfloor, p')$ where $n(m', \lfloor \lambda + 1 \rfloor, \lfloor \lambda^2 + 2 \rfloor, p')$ is the integer in Proposition 2.10. This means that the associated Hoffman graph $g(G, m', n')$ does not contain any of the Hoffman graphs in the set $\{h(\lfloor \lambda + 1 \rfloor) \cup \{q(H_i) \mid i = 1, \ldots, r\}$ as induced subgraphs. This implies that the following two conditions hold:

(i) for each vertex $x$ of $g(G, m', n')$ and one of its fat neighbor $f_x$, the number of vertices which is adjacent to $f_x$ and non-adjacent to $x$ is at most $\lfloor \lambda^2 \rfloor$,

(ii) every vertex $x$ of $g(G, m', n')$ has at most $\lfloor \lambda \rfloor$ fat neighbors.
Now we want to assume that for any two distinct non-adjacent vertices $x$ and $y$ of $G$, they have a common fat neighbor in $g(G, m', n')$. To do so, let $M(\lambda)$ be the number $\max\{R(n', t'), \lfloor\lambda^3 + 1\rfloor\}$, where $R(n', t')$ denotes the Ramsey number. Recall that the Ramsey number $R(s, t)$ is the minimal positive integer $n$ such that any graph with order $n$ contains a clique of order $s$ or a coclique of order $t$. The property of Ramsey number implies that for two vertices $x, y$ at distance 2, their common neighborhood contains a clique of size $n'$ or a coclique of size $t'$. Hence, there exists a fat vertex which is adjacent to both $x$ and $y$ in $g(G, m', n')$. From (i) and (ii), we conclude that $|\Gamma_2(x)| \leq \lfloor\lambda\rfloor\lfloor\lambda^2\rfloor$ for all $x \in V(G)$.

Assume that there exists a vertex $y \in \Gamma_3(x)$ for some $x$. Then there exists a vertex $z$ such that $z \in \Gamma_1(x)$ and $z \in \Gamma_2(y)$. The common neighborhood of $y$ and $z$ have at least size $M(\lambda)$ and is contained in $\Gamma_2(x)$. This is impossible. Hence, $G$ has diameter 2.

4 Some known facts on the number $v(k, \lambda)$

In this section we give some known facts on the number $v(k, \lambda)$. We start from the case $\lambda < 0$. If a connected graph $G$ is not complete, $G$ contains $K_{1,2}$ as an induced subgraph. Then by interlacing, $G$ has second largest eigenvalue at least 0. It implies that if a graph $G$ has negative second largest eigenvalue, $G$ is a complete graph. Thus, $v(k, \lambda) = k + 1$ for $\lambda < 0$ and the unique graph with the equality case is the complete graph $K_{k+1}$.

For $\lambda = 0$, a regular graph with non-positive second largest eigenvalue is a complete multipartite graph [2, Corollary 3.5.4]. Among $k$-regular complete multipartite graphs, we can check that the complete bipartite graph $K_{k,k}$ maximizes the number of vertices. Hence $v(k, 0) = 2k$ and the unique graph with the equality case is the complete multipartite graph $K_{k,k}$.

For $\lambda = 1$, let $G$ be a regular graph with second largest eigenvalue at most 1. Then the complement of $G$ is a regular graph with smallest eigenvalue at least $-2$. Since such regular graphs are classified in [3], we can find the all values of $v(k, 1)$ [4, Theorem 3.2]. Especially, $v(k, 1) = 2k + 2$ when $k \geq 11$. The equality case is obtained by the complement of the line graph of $K_{2,k+1}$. Note that $2k + 2 \leq v(k, 1) \leq 2k + 6$ for all $k$. 

8
For other values of $\lambda > 1$, Cioab˘ a et al. [4] found several values of $v(k, \lambda)$ by using a linear programming method. Using the method of Cioab˘ a et al., it can be shown that $v(k, \lambda) \leq (\lambda + 2)k + \lambda^3 + \lambda^2 - \lambda$ if $k$ is large enough. Theorem 1.2 improves this result significantly.

5 Proof of the main theorem

Now, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $G$ be a $k$-regular graph with second largest eigenvalue $\lambda$ and $v(k, \lambda)$ vertices. Since $v(k, 1) \geq 2k + 2$ and $\lambda \geq 1$, $v(k, \lambda) \geq 2k + 2$. We only need to show Theorem 1.2 for large enough $k$, so we may assume that $k > \lambda(\lambda + 1)(\lambda + 2)$. Now, we consider the complement $\overline{G}$ of $G$. Then $\overline{G}$ is a $l$-regular graph with smallest eigenvalue $-1 - \lambda$ and $v(k, \lambda)$ vertices, where $l = v(k, \lambda) - k - 1 \geq k + 1$.

Suppose that $l \geq k + C_1(\lambda)$, where $C_1(\lambda) = M(\lambda + 1) - 1$, where $M(\lambda)$ is the constant of Proposition 1.3. Let $x$ be a vertex of $\overline{G}$. Then the set of non-neighbors of $x$ has size $k$ and has at least $M(\lambda + 1)$ neighbors in the neighborhood of $x$ since $\overline{G}$ is $l$-regular and $l \geq k + C_1(\lambda)$. It implies that the set of non-neighbors of $x$ is exactly $\Gamma_2(x)$. By Proposition 1.3, $G$ has diameter 2 and $|\Gamma_2(x)| \leq (\lambda + 1)(\lambda^2 + 2\lambda)$. This contradicts to the assumption $k > (\lambda + 1)(\lambda^2 + 2\lambda)$. Hence, $l \leq k + C_1(\lambda) - 1$ and $v(k, \lambda) = 1 + k + l \leq 2k + C_1(\lambda)$. □

6 The behavior of $T(\lambda)$

Recall that, for fixed real number $\lambda \geq 1$, $T(\lambda)$ is defined as

$$T(\lambda) := \limsup_{k \to \infty} v(k, \lambda) - 2k.$$

Now we give a result on $T(\lambda)$.

The complement of the line graph of $K_{2,a+1}$, denoted by $\overline{L(K_{2,a+1})}$, is an $a$-regular graph which has $2a + 2$ vertices and spectrum $\{[a]^1, [1]^a, [-1]^a, [-a]^1\}$. We consider the coclique extension of this graph.
Definition 6.1. For an integer $q > 1$, the $q$-coclique extension $\tilde{G}_q$ of a graph $G$ is the graph obtained from $G$ by replacing each vertex $x \in G$ by a coclique $\tilde{X}$ with $q$ vertices, such that $\tilde{x} \in \tilde{X}$ and $\tilde{y} \in \tilde{Y}$ are adjacent if and only if $x$ and $y$ are adjacent in $G$.

If $\tilde{G}_q$ is the $q$-coclique extension of $G$, then $\tilde{G}_q$ has adjacency matrix $A \otimes J_q$, where $J_q$ is the all one matrix of size $q$ and $\otimes$ denotes the Kronecker product. This shows that, if a graph $G$ has spectrum 

$$\{[\lambda_0]^{m_0}, [\lambda_1]^{m_1}, \ldots, [\lambda_n]^{m_n}\},$$

then the $q$-coclique extension $\tilde{G}_q$ of $G$ has spectrum 

$$\{[q\lambda_0]^{m_0}, [q\lambda_1]^{m_1}, \ldots, [q\lambda_n]^{m_n}, [0]^{(q-1)(m_0+m_1+\cdots+m_n)}\}.$$ 

Hence the $q$-coclique extension of $L(K_{2,a+1})$ is a $qa$-regular graph with order $2qa + 2q$ and spectrum

$$\{[qa]^1, [qa]^a, [qa]^a, [qa]^1, [0]^{(q-1)(2a+2)}\}.$$ 

This implies that the $\lambda$-coclique extension of $L(K_{2,a+1})$ has second largest eigenvalue $\lambda$, and that $v(k, \lambda) \geq 2k + 2\lambda$ when $k$ is a multiple of $\lambda$. Hence we have:

**Lemma 6.2.** Let $\lambda$ be a positive integer. Then $T(\lambda) \geq 2\lambda$.

Moreover, we have a conjecture on $T(\lambda)$ as follows:

**Conjecture 6.3.** Let $\lambda$ be a positive integer. Then $T(\lambda) = 2\lambda$.

For $\lambda = 1$, this conjecture is true as $T(1) = 2$.

### 7 Applications

In this section, we introduce two applications of Proposition 1.3. We first consider co-edge regular graphs with parameters $(v, k, c_2)$, which are $k$-regular graphs with $v$ vertices and the property such that every pair of non-adjacent vertices has exactly $c_2$ common neighbors. By applying Proposition 1.3 and Theorem 1.1, we obtain the following theorem.
Theorem 7.1. Let $\lambda \geq 1$ be a real number. Let $G$ be a connected co-edge regular graph with parameters $(v, k, c_2)$. Then there exists a real number $C_2(\lambda)$ (only depending on $\lambda$) such that, if $G$ has smallest eigenvalue at least $-\lambda$, then $c_2 > C_2(\lambda)$ implies that $v - k - 1 \leq \frac{(\lambda - 1)^2}{4} + 1$ holds.

Proof. Let $\ell = v - k - 1$, then $|\Gamma_2(x)| = \ell$ for all $x \in V(G)$ since $G$ has diameter 2. We can apply Proposition 1.3 for $C(\lambda) = M(\lambda) - 1$ to obtain that either $c_2 \leq C(\lambda)$ or $\ell = v - k - 1 \leq \lfloor \lambda \rfloor \lfloor \lambda^2 \rfloor$ holds. Suppose $\ell > \frac{(\lambda - 1)^2}{4} + 1$.

The complement of $G$ is $\ell$-regular and has second largest eigenvalue at most $\lambda - 1$, and therefore has at most $v(\ell, \lambda - 1)$ vertices. As $v(\ell, \lambda - 1)$ is a finite number by Theorem 1.1, we see that the theorem follows, if we take $C_2(\lambda) = \max\{\max\{v(\ell, \lambda - 1) - \ell - 1 | \frac{(\lambda - 1)^2}{4} + 1 < \ell \leq \lfloor \lambda \rfloor \lfloor \lambda^2 \rfloor\}, C(\lambda)\}$. \hfill $\Box$

An edge-regular graph with parameters $(v, k, a_1)$ is a $k$-regular graph with $v$ vertices such that any two adjacent vertices have exactly $a_1$ common neighbors. Note that the complement of a co-edge regular graph is edge-regular.

Remark 7.2. (i) Let $\ell$ be an integer at least 3 and let $\lambda := 2\sqrt{\ell - 1}$. Take the infinite family of the bipartite $\ell$-regular Ramanujan graphs, as constructed in [11]. The graphs in this family are clearly edge-regular with $a_1 = 0$ and have second largest eigenvalue at most $2\sqrt{\ell - 1}$. Let $\Gamma$ be a graph in this family with $v$ vertices. Then the complement of $\Gamma$ is co-edge-regular with parameters $(v, v - \ell - 1, v - 2\ell)$ and has smallest eigenvalue at least $-1 - 2\sqrt{\ell - 1}$.

This example shows that the upper bound for $v - k - 1$ in Theorem 7.1 cannot be improved.

(ii) For $\lambda = 2$, we find $C_2(2) = 8$, by [2, Theorem 3.12.4(iv)].

An amply regular graph with parameters $(v, k, a_1, c_2)$ is a $k$-regular graph with $v$ vertices such that any two adjacent vertices have exactly $a_1$ common neighbors and any two vertices at distance 2 have $c_2$ common neighbors. We call an amply regular graph with diameter 2 strongly regular. Neumaier [13] proved the following theorem which is called the $\mu$-bound for strongly regular graphs.
Theorem 7.3. ([13, Theorem 3.1]) Let \( G \) be a coconnected strongly regular graph with parameters \((v, k, a_1, c_2)\) and integral smallest eigenvalue \(-\lambda \leq -2\). Then
\[
c_2 \leq \lambda^3(2\lambda - 3).
\]
The condition \(-\lambda \leq -2\) implies that \( G \) is not a union of cliques of the same size. Since the only strongly regular graphs which are not coconnected, are the complete multipartite graphs, we obtain the following theorem.

Theorem 7.4. Let \( G \) be an amply regular graph with parameters \((v, k, a_1, c_2)\). Let \( \lambda \geq 2 \) be an integer. Then there exists a real number \( C_3(\lambda) \) such that if \( G \) has smallest eigenvalue at least \(-\lambda\), then \( c_2 \leq C_3(\lambda) \) or \( G \) is a complete multipartite graph.

Proof. Let \( C_3(\lambda) = \max\{M(\lambda) - 1, \lambda^3(2\lambda - 3)\} \). If \( c_2 > C_3(\lambda) \), then \( G \) has diameter 2 by Proposition 1.3. By Theorem 7.3, \( G \) is not coconnected. Hence, \( G \) is a complete multipartite graph.

\[ \square \]

References

[1] N. Alon, Eigenvalues and expanders, Combinatorica, 6:83–96, 1986.

[2] A.E. Brouwer, A.M. Cohen, and A. Neumaier, Distance-Regular Graphs, Springer-Verlag, Berlin, 1989.

[3] F.C. Bussemaker, D. Cvetković, and J.J. Seidel, Graphs related to exceptional root systems, Report TH Eindhoven 76-WSK-05, 1976.

[4] S.M. Cioabă, J.H. Koolen, H. Nozaki, and J.R. Vermette, Maximizing the order of a regular graph of given valency and second eigenvalue, SIAM J. Discrete Math., 30(3):1509–1525, 2016.

[5] S.M. Cioabă, On the extreme eigenvalues of regular graphs, J. Combin. Theory Ser. B, 96:367–373, 2016.

[6] J. Friedman, Some geometric aspects of graphs with their eigenfunctions, Duke Math. J., 69:487–525, 1993.
[7] A.J. Hoffman, On graphs whose least eigenvalue exceeds $-1 - \sqrt{2}$, *Linear Algebra Appl.*, 16:153–165, 1977.

[8] S. Hoory, N. Linial, and A. Wigderson, Expander graphs and their applications, *Bull. Amer. Math. Soc.*, 46:439–561, 2006.

[9] H.J. Jang, J.H. Koolen, A. Munemasa and T. Taniguchi, On fat Hoffman graphs with smallest eigenvalue at least $-3$, *Ars Math. Contemp.*, 7:105–121, 2014.

[10] H.K. Kim, J.H. Koolen, and J.Y. Yang, A structure theory for graphs with fixed smallest eigenvalue, *Linear Algebra Appl.*, 540:1–13, 2016.

[11] A. Marcus, D.A. Spielman, and N. Srivastava, Interlacing families I: Bipartite Ramanujan graphs of all degrees, *Ann. of Math.*, 182:307–325, 2015.

[12] B. Mohar, A strengthening and a multipartite generalization of the Alon-Boppana-Serre theorem, *Proc. Amer. Math. Soc.*, 138:3899–3909, 2010.

[13] A. Neumaier, Strongly regular graphs with smallest eigenvalue $-m$, *Arch. Math.(Basel)*, 33(4):392–400, 1979–80.

[14] A. Nilli, On the second eigenvalue of a graph, *Discrete Math.*, 91:207–210, 1991.

[15] A. Nilli, Tight estimates for eigenvalues of regular graphs, *Electron. J. Combin.*, 11:#N9, 2004.

[16] R. Woo, and A. Neumaier, On graphs whose smallest eigenvalue is at least $-1 - \sqrt{2}$, *Linear Algebra Appl.*, 226–228:577–591, 1995.