Simulation of Heat Transfer in the Liquid Core of the Earth

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Abstract. The results of numerical simulation of unsteady heat transfer of an electrically conductive liquid in a spherical layer modeling the Earth’s liquid core are presented. The evolution of the structure of the flow of an electrically conductive fluid, the temperature field, magnetic induction, and the distribution of local Nusselt numbers is studied. The gravitational acceleration vector is directed along the radius to the center of the spherical layer.

1. Introduction

In [1], stationary results of numerical modeling of convective heat transfer of an electrically conductive fluid in a spherical layer are presented taking into account the heat of joule dissipation and internal heat sources when heat is applied to the surface of the inner sphere. According to P. Roberts, the ratio of the inner diameter of the spherical layer to the outer was taken $d/D = 1/2.8$.

This paper presents the non-stationary results of numerical modeling of convective heat transfer of an electrically conductive fluid in a spherical layer without taking into account the heat of joule dissipation and internal heat sources. On both surfaces of the spherical layer, constant temperature values were set (boundary conditions of the first kind. The inner surface of the layer is more heated). The ratio of the inner diameter of the spherical layer to the outer was taken $d/D = 1/2.5$ (according to the classical Jeffreys-Gutenberg model [2]).

2. Mathematical model

Convective heat transfer of an electrically conductive fluid is described by a system of differential equations of magnetic hydrodynamics and heat transfer, namely: motion, taking into account electromagnetic, inertial, viscous and lifting forces; energy, magnetic induction and continuity equations. The Boussinesq approximation is used.

The mathematical formulation of the problem in dimensionless form in the spherical coordinate system, taking into account the symmetry with respect to longitude, in the variables vortex $- \omega$, the stream function $- \psi$, the temperature $- \theta$, has the form [3]:

$$
\frac{1}{Ho} \frac{\partial \omega}{\partial t} + \frac{1}{r^2 \sin \theta} \left[ \frac{\partial \psi}{\partial \theta} \frac{\partial \omega}{\partial r} - \frac{\partial \psi}{\partial r} \frac{\partial \omega}{\partial \theta} - \frac{\omega}{r} \frac{\partial \psi}{\partial \theta} + \omega \frac{\partial \theta}{\partial \theta} \frac{\partial \psi}{\partial \theta} \right] =
$$

$$
= \frac{1}{Re} \left[ \frac{\partial^2 \omega}{\partial r^2} + \frac{2 \omega}{r} \frac{\partial \theta}{\partial r} + \frac{1}{r^2} \frac{\partial \omega}{\partial \theta} + \frac{1}{r^2} \frac{\partial \theta}{\partial \theta} \frac{\partial \omega}{\partial \theta} - \frac{\omega}{r^2 \sin^2 \theta} \right].
$$
\[- \frac{Gr}{Re' r} \frac{\partial \theta}{\partial r} + \frac{S}{Re_m} \left[ \frac{B_r}{r} \frac{\partial^2 \theta}{\partial r^2} + 2 \frac{B_r}{r} \frac{\partial B_\theta}{\partial r} + \frac{\partial B_\theta}{\partial \theta} + \frac{B_\theta}{r} \frac{\partial B_r}{\partial \theta} \right] \]

\[- \frac{B_\theta}{r} \frac{\partial^2 B_r}{\partial \theta^2} - \frac{1}{r} \frac{\partial B_r}{\partial r} \frac{\partial B_\theta}{\partial \theta} + \frac{B_\theta}{r} \frac{\partial B_\theta}{\partial \theta} + \frac{1}{r^2} \frac{\partial B_\theta}{\partial \theta} \frac{\partial B_\theta}{\partial \theta} \]

\[ \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2} \frac{\partial \psi}{\partial \theta} - \cot \theta \frac{\partial \psi}{\partial \theta} = - \omega \sin \theta \]

\[ \frac{1}{Ho} \frac{\partial B_r}{\partial r} + \frac{1}{r} \sin \theta \left( \frac{\partial B_\theta}{\partial \theta} - \frac{\partial B_r}{\partial r} \right) + \frac{1}{Pe} \left( \frac{\partial^2 \theta}{\partial r^2} + \frac{2}{r} \frac{\partial \theta}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \theta}{\partial \theta^2} \right) \cot \theta \frac{\partial \theta}{\partial r} \]

When conducting a computational experiment, the boundary conditions of the first kind were set for temperature: the temperature on the inner surface of the spherical layer \( \Gamma_1 \) \( (r = 1) \) and the temperature at the outer \( \Gamma_2 \) \( (r = r_2) \):

\[ \theta |_{\Gamma_1} = 1; \quad \theta |_{\Gamma_2} = 0 . \]

On the axis of symmetry: \[ \frac{\partial \theta}{\partial \theta} |_{\theta = 0, \pi} = 0 . \]

The boundary conditions for the stream function, the vortex and the magnetic induction were as follows [4]:

\[ \Psi |_{\Gamma_2} = \Psi |_{\theta = 0, \pi} = \omega |_{\theta = 0, \pi} = 0 ; \quad \frac{\partial B_r}{\partial \theta} |_{\theta = 0, \pi} = 0 ; \quad \frac{\partial B_\theta}{\partial \theta} |_{\theta = 0, \pi} = 0 ; \quad \frac{\partial B_\theta}{\partial \theta} |_{\theta = 0, \pi} = 0 . \]

The boundary conditions for the vortex at the boundaries of the layer assume a linear change in it along the normal. The local and averaged Nusselt numbers at the boundaries of the layer were calculated by the formulas:

\[ Nu_1 = - \left. \frac{\partial \theta}{\partial r} \right|_{\Gamma_1}, \quad Nu_2 = - \left. \frac{\partial \theta}{\partial r} \right|_{\Gamma_2} ; \quad Nu_1 = \frac{1}{2} \int_0^\pi \left. \frac{\partial \theta}{\partial r} \right|_{\Gamma_1} \sin \theta d\theta, \quad Nu_2 = \frac{1}{2} \int_0^\pi \left. \frac{\partial \theta}{\partial r} \right|_{\Gamma_2} \sin \theta d\theta. \]

The numerical solution of the problem was carried out by the finite element method; the solution algorithm is presented in [5]. When calculating in space, a grid of 90x90 was used. An implicit difference scheme was applied over time. Time step \( \Delta \tau = 0.1 ; \) the difference analogue of the system of nonlinear partial differential equations is a system of nonlinear algebraic equations, which was solved by the Seidel method using lower relaxation. The calculations were performed with the following values of dimensionless similarity numbers: \( Re = Pe = 10^2 ; \) \( Re_m = 10^2 ; \) \( S = 10^2 ; \) \( Ho = 10. \)

The accepted homogeneous number \( Ho = 10 \) means that the contribution of the convective component of the acceleration of fluid motion in the spherical layer is ten times larger than the contribution of the local component. For accepted values dimensionless similarity numbers, the Hartmann number \( G = (S \cdot Re)^{0.5} = 10^3 . \) This
mean that a strong magnetic field \((G = 10^3)\) acts on the liquid and at the same time it has a large electrical conductivity \((Re_m=10^2)\).

In figure 1, 2 shows the results of non-stationary calculations.

3. Results

In figure 1 shows the results for time instants: \(1 - \tau = 1; 2 - \tau = 2; 3 - \tau = 3\).

![Figure 1. Calculated fields.](image)

(a) – temperature; (b) – stream function; (c) – tension of vortex; (d, e) – radial and meridional components of magnetic induction; (f) – distribution of Nusselt numbers.

At the initial stage of heating \((\tau = 1)\), heat transfer in the layer is carried out by heat conduction. Isotherms are concentric circles (figure 1, a; 1). Over time \((\tau = 2; 3)\), the temperature field is rearranged. This is especially evident for time \(\tau = 3\), where the main temperature change occurs in the region of the poles and in the equatorial plane (figure 1, a; 3). In this case, the mechanism of energy transfer in the layer changes from thermal conductivity to convection (figure 1, a; 2, 3). The maximum value of the temperature in the layer \(\theta_{max} = 1\). The distribution of local Nusselt numbers (figure 1, f; 1) is characteristic of the heat conduction regime (the red line on the inner surface and the green line on
the outer one). Over time, the distribution of local Nusselt numbers changes and becomes characteristic of the convection mode (figure 1, f; 2; 3). For \(\tau = 2\) two minimums and three maximums take place on the outer surface of the layer (figure 1, f; 2). For \(\tau = 3\) one minimum and two maximums take place on the inner surface of the layer (figure 1, f; 3), and two minimums and one maximum occur on the outer surface. Values of averaged and intervals of change of local Nusselt numbers:

\[
1 - \overline{Nu}_1 = 2.719; \quad \overline{Nu}_1 = 0.029; 2.719 \leq Nu_1 \leq 2.722; 0.029 \leq Nu_2 \leq 0.029.
\]

\[
2 - \overline{Nu}_2 = 2.317; \quad \overline{Nu}_2 = 1.525; 1.232 \leq Nu_1 \leq 2.824; 0.141 \leq Nu_2 \leq 0.248.
\]

\[
3 - \overline{Nu}_3 = 6.338; \quad \overline{Nu}_3 = 2.525; 1.544 \leq Nu_1 \leq 8.285; 0.143 \leq Nu_3 \leq 9.083.
\]

Heat transfer on the inner surface of the layer is more intense than on the outer.

The structure of the fluid flow in the layer (starting from \(\tau = 3\)) is represented by four convective cells (figure 1, b; 3) – two small-scale in the field of poles and two large-scale in the field of equator (red color – the fluid moves counterclockwise, the values of the stream function are positive; blue – fluid moves clockwise, values are negative). The maximum value of the stream function is \(|\psi| = 3.48 \cdot 10^4\). The vortex field is represented by two small-scale vortices in the field of poles (figure 1, c; 2) and four large-scale vortices (figure 1, c; 3). The maximum value of the vortex is \(|\phi| = 2.69\).

The field of the radial component of magnetic induction in the computational domain (figure 1, d; 3) is represented by eight “magnetic cells” \(|B_r| = 5.01 \cdot 10^{-5}\). The field of the meridional component of magnetic induction is shown in figure 1, f. For \(\tau = 1\); 2 these fields are almost identical; on the outer surface of the layer, the values of the meridional component of magnetic induction are positive, and on the inner – negative. For \(\tau = 3\) significant changes occur (figure 1, e; 3) in contrast to the results presented in figure 1, e; 1, 2. The contours of the meridional component of magnetic induction are bent in the entire region, forming a “magnetic cell” in the equator region, resembling the shape of a triangle \(|B_r| = 1.11 \cdot 10^{-5}\).

In figure 2 shows the calculation results for time: \(1 - \tau = 5; 2 - \tau = 10; 3 - \tau = 15\).

The results obtained (figure 2) differ significantly from the results shown in figure 1. For the considered time instants, heat transfer in the layer is carried out by convection (figure 2, a, f), the hydrodynamic structure (fields of the stream function and vortex) is stabilized (figure 2, b, c). Significant changes in temperature gradients occur in the region of the poles and the equatorial plane. The distribution of local Nusselt numbers (figure 2, f) on the outer surface of the layer has two minima and one maximum at \(\theta \sim \pi/2\), and on the inside – two maxima and one minimum at \(\theta \sim \pi/2\). The maximum value of the temperature in the layer \(\theta = 1\).

The averaged values and intervals of the local Nusselt numbers are as follows:

\[
1 - \overline{Nu}_1 = 6.886; \quad \overline{Nu}_1 = 2.891; 1.794 \leq Nu_1 \leq 9.243; 0.101 \leq Nu_2 \leq 7.528.
\]

\[
2 - \overline{Nu}_2 = 6.911; \quad \overline{Nu}_2 = 2.835; 1.801 \leq Nu_1 \leq 9.285; 0.087 \leq Nu_2 \leq 7.464.
\]

\[
3 - \overline{Nu}_3 = 6.908; \quad \overline{Nu}_3 = 2.830; 1.801 \leq Nu_1 \leq 9.280; 0.087 \leq Nu_3 \leq 7.454.
\]

Heat transfer on the inner surface of the layer is more intense than on the outer.

Two convective cells (figure 2, b) and four vortices (figure 2, c) are formed in the layer. In a convective cell and a large-scale vortex of the northern hemisphere the fluid moves clockwise, and the south – against. Maximum values of the stream function \(|\psi| = 4.85 \cdot 10^{-1}\); vortex \(|\phi| = 2.87\).

The field of the radial component of magnetic induction (figure 2, d; 2; 3) is represented by two “magnetic cells”, the values of which are positive in the northern hemisphere and negative in the southern hemisphere \(|B_r| = 5.08 \cdot 10^{-4}\).
Figure 2. Calculated fields.
(a) – temperature; (b) – stream function; (c) – tension of vortex; (d, e) – radial and meridional components of magnetic induction; (f) – distribution of Nusselt numbers.

The field of the meridional component of magnetic induction is shown in figure 2, e ($|B_\theta| = 4.21 \cdot 10^{-2}$).

In figure 3 shows the results of calculations of the magnetic induction field for time: 1 – $\tau = 20$; 2 – $\tau = 25$; 3 – $\tau = 30$.

Figure 3. Calculated fields.

An increase in time does not lead to a change in the temperature fields, the stream function, the vortex, and the distribution of Nusselt numbers in comparison with the results presented in figure 2. The maximum value of the temperature in the layer $\theta_{max} = 1$. For $\tau = 20; 25; 30$ values of averaged and intervals of change of local Nusselt numbers reach stationary values:

$\overline{Nu}_1 = 6.908; \overline{Nu}_2 = 2.830; 1.801 \leq Nu_1 \leq 9.279; 0.087 \leq Nu_2 \leq 7.453$; heat transfer on the inner surface of the layer is more intense than on the outer. $|F| = 4.84 \cdot 10^{-1}; |\phi| = 2.87; |B_r| = 8.14 \cdot 10^{-1}; |B_\theta| = 6.64 \cdot 10^{-2}.$
From the obtained data (\(\tau = 20; 25; 30\)) it follows that thermal and hydrodynamic processes (temperature fields, stream functions, vortex, distribution of Nusselt numbers) reach their stationary state even at \(\tau = 20\), and the magnetic induction field continues to change in time.

For \(\tau = 40; 50; 60\) maximum values \(\theta_{\text{max}}\); \(\theta\); \(\theta\); as well as the averaged values and intervals of variation of local Nusselt numbers are the same as for \(\tau = 20; 25; 30\). The maximum values of the components of magnetic induction increase with time: \(B_r = 1.03\); \(B_\theta = 8.38 \cdot 10^{-2}\).

For \(\tau = 70; 90; 110\) maximum values \(\theta_{\text{max}}\); \(\theta\); \(\theta\) as well as the averaged values and intervals of variation of local Nusselt numbers are the same as for \(\tau = 40; 50\) and 60. The values of the components of magnetic induction increase: \(B_r = 1.58\); \(B_\theta = 1.28 \cdot 10^{-1}\).

For \(\tau = 130; 150; 200\) maximum values \(\theta_{\text{max}}\); \(\theta\); \(\theta\) as well as the averaged values and intervals of variation of local Nusselt numbers are the same as for \(\tau = 70; 90\) and 110. The values of the components of magnetic induction increase: \(B_r = 2.77\); \(B_\theta = 2.24 \cdot 10^{-1}\).

4. Conclusion

Analysis of the results allows us to draw the following conclusions:

1. The evolution of heat transfer and the magnetohydrodynamic structure of the flow of an electrically conductive fluid in a spherical layer is investigated;
2. For the considered values of dimensionless similarity numbers, it turned out that the temperature field and the hydrodynamic structure of the fluid flow in the layer reach their stationary state in a short period of time (already at \(\tau = 20\)). Whereas the magnetic induction field continues to change in time;
3. In the distribution of local Nusselt numbers, there are points of intersection of the curves on the outer and inner boundaries of the spherical layer (starting from time moment \(\tau = 3\) of four). These points are characterized by equal heat fluxes. It turned out that over time, the location of two external points shifts to the poles;
4. The mathematical model and the results obtained can be useful in modeling thermal and magnetohydrodynamic processes in the liquid core of the Earth and other planets.

References
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