Heat conduction in anisotropic media.
Nonlinear self-adjointness and conservation laws

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Abstract
Nonlinear self-adjointness of the anisotropic nonlinear heat equation is investigated. Mathematical models of heat conduction in anisotropic media with a source are considered and a class of self-adjoint models is identified. Conservation laws corresponding to the symmetries of the equations in question are computed.

Keywords: Heat conduction, Anisotropic, Nonlinear self-adjointness, Symmetries, Conservation laws.

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Introduction

1.1 Formulation of the problem

The nonlinear second-order evolution equation

\[ u_t = (f(u)u_x)_x + (g(u)u_y)_y + (h(u)u_z)_z \]  

(1.1)

describes the heat conduction in anisotropic materials whose physical characteristics such as the thermal conductivity are affected by the temperature. Physical applications and interesting mathematical properties of Eq. (1.1) and of its extension to the case of existence of an external source \( q(u) \),

\[ u_t = (f(u)u_x)_x + (g(u)u_y)_y + (h(u)u_z)_z + q(u) \]  

(1.2)

are discussed in [1] (see also [2], Section 10.9).

The functions \( f(u) \), \( g(u) \), \( h(u) \) are positive according to their physical meaning. So, we consider Eqs. (1.1) and (1.2) with arbitrary positive coefficients \( f(u) \), \( g(u) \), \( h(u) \). Furthermore, we will assume that these coefficients are linearly independent and that none of them is constant, i.e.

\[ f'(u) \neq 0, \quad g'(u) \neq 0, \quad h'(u) \neq 0. \]  

(1.3)

Note, that Eq. (1.1) has a conservation form whereas Eq. (1.2) with \( q(u) \neq 0 \) does not have such form. A conservation form is useful in many respects, e.g. in qualitative and numerical analysis. Moreover, possibility of different conservation forms can be helpful. Therefore we will construct various conservation laws for Eq. (1.1) using the method of nonlinear self-adjointness [3] and investigate the question on existence of conservation laws for Eq. (1.2) with specific values of the source term \( q(u) \neq 0 \).

1.2 Nonlinear self-adjointness

Recall the definition of nonlinear self-adjointness. Let us consider a second-order partial differential equation

\[ F(x, u, u_{(1)}, u_{(2)}) = 0, \]  

(1.4)

where \( u \) is the dependent variable, \( u_{(1)} \) and \( u_{(2)} \) are the sets of the first-order partial derivatives \( u_i \) and the second-order derivatives \( u_{ij} \) of \( u \) with respect to the independent variables \( x = (x^1, \ldots, x^n) \). The adjoint equation to Eq. (1.4) is

\[ F^*(x, u, v, u_{(1)}, v_{(1)}, u_{(2)}, v_{(2)}) = 0, \]  

(1.5)
where $F^*$ is defined by

$$F^*(x, u, v, u(1), u(2), v(2)) = \frac{\delta(vF)}{\delta u}.$$  \hspace{1cm} (1.6)

Here $v$ is a new dependent variable and $v(1), v(2)$ are the sets of its partial derivatives. Furthermore, $\delta(vF)/\delta u$ denotes the variational derivative of $vF$:

$$\frac{\delta(vF)}{\delta u} = \frac{\partial(vF)}{\partial u} - D_i \left( \frac{\partial(vF)}{\partial u_i} \right) + D_i D_k \left( \frac{\partial(vF)}{\partial u_{ik}} \right) - \cdots,$$

where the total differentiations are extended to the new dependent variable $v$:

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + v_i \frac{\partial}{\partial v} + u_{ij} \frac{\partial}{\partial u_j} + v_{ij} \frac{\partial}{\partial v_j} + \cdots.$$  \hspace{1cm} (1.7)

Eq. (1.4) is said to be \textit{nonlinearly self-adjoint} \cite{3} if the adjoint equation (1.5) is satisfied for all solutions $u$ of the original equation (1.4) upon a substitution

$$v = \varphi(x, u), \quad \varphi \neq 0.$$  \hspace{1cm} (1.8)

The condition that the function $\varphi$ that does not vanish is significant. The condition for the nonlinear self-adjointness can be written in the form

$$F^*(x, u, \varphi, u(1), \varphi(1), u(2), \varphi(2)) = \lambda F(x, u(1), u(2)),$$

where $\lambda = \lambda(x, u, u(1), \ldots)$ is an undetermined variable coefficient and $\varphi(1), \varphi(2)$ denote the derivatives of the function $\varphi$. E.g. $\varphi(1)$ is the set of the first-order total derivatives

$$D_i(\varphi) = \frac{\partial \varphi(x, u)}{\partial x^i} + u_i \frac{\partial \varphi(x, u)}{\partial u}, \quad i = 1, \ldots, n.$$  

Eq. (1.9) should be satisfied identically in all variables $x, u, u(1), u(2)$.

1.3 Conserved vector associated with symmetries

The general result on construction of conserved vectors associated with symmetries of nonlinearly self-adjoint equations demonstrated in \cite{3} leads to the following statement for the second-order equation (1.4).

Let (1.4) be nonlinearly self-adjoint and admit a one-parameter point transformation group with the generator

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta(x, u) \frac{\partial}{\partial u}.$$  \hspace{1cm} (1.10)

\begin{footnote}{In general, the substitution (1.8) can be of the form $v = \varphi(x, u, u(1))$.}

\end{footnote}
Then the vector
\[ C^i = W \left[ \frac{\partial L}{\partial u_i} - D_j \left( \frac{\partial L}{\partial u_{ij}} \right) \right] + D_j(W) \frac{\partial L}{\partial u_{ij}}, \quad i = 1, \ldots, n, \] (1.11)
is a conserved vector for Eq. (1.4), i.e. satisfies the conservation equation
\[ \left[ D_i(C^i) \right]_{(1.4)} = 0. \] (1.12)
Here
\[ W = \eta - \xi^j u_j \] (1.13)
and \( L \) is the formal Lagrangian for Eq. (1.4) given by
\[ L = vF. \] (1.14)
It is assumed that the variable \( v \) and its derivatives are eliminated from the right-hand side of Eq. (1.11) by using the substitution (1.8), where the function \( \varphi(x, u) \) is found by solving the nonlinear self-adjointness condition (1.9).

2 Investigation of nonlinear self-adjointness

2.1 Substitution (1.8) for Equation (1.1)

Since Eq. (1.1) has the conservation form (1.12), it is nonlinearly self-adjoint by Theorem 8.1 from [3]. Let us find the corresponding substitution (1.8).

We write Eq. (1.1) in the form (1.4):
\[ F \equiv -u_t + f(u)u_{xx} + g(u)u_{yy} + h(u)u_{zz} + f'(u)u_x^2 + g'(u)u_y^2 + h'(u)u_z^2 = 0, \] (2.1)
insert the expression for \( F \) in (1.6) and after simple calculations obtain the following adjoint equation (1.5) to Eq. (1.1):
\[ F^* \equiv v_t + f(u)v_{xx} + g(u)v_{yy} + h(u)v_{zz} = 0. \] (2.2)

In our case the substitution (1.8) has the form
\[ v = \varphi(t, x, y, z, u). \] (2.3)

Its derivatives are written
\[
\begin{align*}
v_t & \equiv D_t(\varphi) = \varphi_u u_t + \varphi_t, \quad v_x & \equiv D_x(\varphi) = \varphi_u u_x + \varphi_x, \\
v_y & \equiv D_y(\varphi) = \varphi_u u_y + \varphi_y, \quad v_z & \equiv D_z(\varphi) = \varphi_u u_z + \varphi_z, \\
v_{xx} & \equiv D_x^2(\varphi) = \varphi_u u_{xx} + \varphi_{uu} u_x^2 + 2\varphi_{ux} u_x + \varphi_{xx}, \\
v_{yy} & \equiv D_y^2(\varphi) = \varphi_u u_{yy} + \varphi_{uu} u_y^2 + 2\varphi_{uy} u_y + \varphi_{yy}, \\
v_{zz} & \equiv D_z^2(\varphi) = \varphi_u u_{zz} + \varphi_{uu} u_z^2 + 2\varphi_{uz} u_z + \varphi_{zz}. 
\end{align*}
\] (2.4)
Now we take the nonlinear self-adjointness condition (1.9), where $F$ and $F^*$ are given by (2.1) and (2.2), respectively:

$$v_t + f(u)v_{xx} + g(u)v_{yy} + h(u)v_{zz} = \lambda \left[-u_t + f(u)u_{xx} + g(u)u_{yy} + h(u)u_{zz} + f'(u)u_x^2 + g'(u)u_y^2 + h'(u)u_z^2\right]$$  \hspace{1cm} (2.5)

where the corresponding derivatives of $v$ in the left-hand side should be replaced with their expressions (2.4). First we compare the coefficients for $u_t$ in both sides of Eq. (2.5) and obtain

$$\lambda = -\varphi_u.$$

Then the coefficients for $u_{xx}, u_{yy}, u_{zz}$ yield:

$$f(u)\varphi_u = -f(u)\varphi_u, \quad g(u)\varphi_u = -g(u)\varphi_u, \quad h(u)\varphi_u = -h(u)\varphi_u.$$  

By our assumption, the functions $f(u), g(u), h(u)$ do not vanish. Therefore the above equations yield that $\varphi_u = 0$. Hence, $\varphi = \varphi(t, x, y, z)$ and therefore

$$\lambda = 0, \quad v_t = \varphi_t, \quad v_{xx} = \varphi_{xx}, \quad v_{yy} = \varphi_{yy}, \quad v_{zz} = \varphi_{zz}.$$  

Then Eq. (2.5) becomes

$$\varphi_t + f(u)\varphi_{xx} + g(u)\varphi_{yy} + h(u)\varphi_{zz} = 0.$$  \hspace{1cm} (2.6)

Since $f(u), g(u), h(u)$ are linearly independent and obey the conditions (1.3), whereas $\varphi$ does not depend on $u$, Eq. (2.6) yields

$$\varphi_t = 0, \quad \varphi_{xx} = 0, \quad \varphi_{yy} = 0, \quad \varphi_{zz} = 0.$$  \hspace{1cm} (2.7)

The general solution of Eqs. (2.7) is given by

$$\varphi = a_1 xyz + a_2 xy + a_3 xz + a_4 yz + a_5 x + a_6 y + a_7 z + a_8$$

with arbitrary constant coefficients $a_1, \ldots, a_8$. This proves the following.

**Proposition 2.1.** Eq. (1.1) satisfies the nonlinear self-adjointness condition (1.9) with the substitution (2.3) of the form

$$v = a_1 xyz + a_2 xy + a_3 xz + a_4 yz + a_5 x + a_6 y + a_7 z + a_8.$$  \hspace{1cm} (2.8)
2.2 Two-dimensional equation with a source

Let us consider Eq. (1.2), for the sake of simplicity, in the case of two spatial variables $x, y$:

$$u_t = (f(u)u_x)_x + (g(u)u_y)_y + q(u).$$  \hspace{1cm} (2.9)

The adjoint equation has the form

$$F^* \equiv v_t + f(u)v_{xx} + g(u)v_{yy} + q'(u)v = 0.$$  \hspace{1cm} (2.10)

Repeating the calculations of Section 2.1 we obtain the following equation for the nonlinear self-adjointness of Eq. (2.9) (compare with Eq. (2.6)):

$$\varphi_t + f(u)\varphi_{xx} + g(u)\varphi_{yy} + q'(u)\varphi = 0.$$  \hspace{1cm} (2.11)

If $f(u), g(u)$ and $q(u)$ are arbitrary functions, Eq. (2.11) yields (compare with Eqs. (2.7)):

$$\varphi_t = 0, \quad \varphi_{xx} = 0, \quad \varphi_{yy} = 0, \quad \varphi = 0.$$  \hspace{1cm} (2.12)

These equations show that a substitution of the form (1.8) does not exist. Indeed, the last equation in (2.12) contradicts the condition $\varphi \neq 0$. Hence, Eq. (2.9) with the arbitrary source $q(u)$ is not nonlinearly self-adjoint with the substitution of the form (1.8).

However, Eq. (2.9) with sources of particular forms can be nonlinearly self-adjoint. For example, let

$$q'(u) = rf(u), \quad r = \text{const}.$$  \hspace{1cm} (2.13)

Then Eq. (2.11) becomes

$$\varphi_t + f(u)[\varphi_{xx} + r\varphi] + g(u)\varphi_{yy} = 0$$

and yields (compare with Eqs. (2.12)):

$$\varphi_t = 0, \quad \varphi_{yy} = 0, \quad \varphi_{xx} + r\varphi = 0.$$  \hspace{1cm} (2.14)

The solution to Eqs. (2.14) has the form

$$\varphi = a(x)y + b(x),$$  \hspace{1cm} (2.15)

where $a(x)$ and $b(x)$ arbitrary solutions of the linear second-order ODE

$$w'' + rw = 0.$$  \hspace{1cm} (2.16)

Eq. (2.13) shows that the source strength increases together with the temperature, i.e. $q'(u) > 0$, if $r = \omega^2 > 0$, and decreases, $q'(u) < 0$, if $r = -\delta^2 < 0$. Having this in mind and denoting

$$\mathcal{F}(u) = \int f(u)du$$  \hspace{1cm} (2.17)
we consider two particular forms of Eq. (2.9):

\[ u_t = (f(u)u_x)_x + (g(u)u_y)_y + \omega^2 F(u), \quad \omega = \text{const.} \quad (2.18) \]

and

\[ u_t = (f(u)u_x)_x + (g(u)u_y)_y - \delta^2 F(u), \quad \delta = \text{const.} \quad (2.19) \]

In the case (2.18) Eq. (2.16) is written

\[ w'' + \omega^2 w = 0 \]

and yields

\[ w = C_1 \cos(\omega x) + C_2 \sin(\omega x). \]

Hence

\[ a(x) = A_1 \cos(\omega x) + A_2 \sin(\omega x), \quad b(x) = B_1 \cos(\omega x) + B_2 \sin(\omega x) \]

with arbitrary constants \( A_1, A_2, B_1, B_2 \). We substitute these expressions in Eq. (2.15) and arrive at the following statement.

**Proposition 2.2.** Eq. (2.18) satisfies the nonlinear self-adjointness condition (1.9) with the substitution (2.3) of the form

\[ v = (A_1 y + B_1) \cos(\omega x) + (A_2 y + B_2) \sin(\omega x). \quad (2.20) \]

In the case (2.19) Eq. (2.16) is written

\[ w'' - \delta^2 w = 0 \]

and yields

\[ w = C_1 e^{\delta x} + C_2 e^{-\delta x}. \]

Proceeding as above we arrive at the following statement.

**Proposition 2.3.** Eq. (2.19) satisfies the nonlinear self-adjointness condition (1.9) with the substitution (2.3) of the form

\[ v = (A_1 y + B_1) e^{\delta x} + (A_2 y + B_2) e^{-\delta x}. \quad (2.21) \]

### 2.3 Remark on materials with specific anisotropy

The situation is different if the conditions (1.3) are not satisfied. Let, e.g. \( g(u) \) be a positive constant, \( g = k \). Then Eq. (1.1) has the form

\[ u_t = (f(u)u_x)_x + ku_{yy} + (h(u)u_z)_z. \quad (2.22) \]
In this case Eqs. (2.7) are replaced by the following equations:

\[ \varphi_t + k \varphi_{yy} = 0, \quad \varphi_{xx} = 0, \quad \varphi_{zz} = 0. \]  
(2.23)

The second and third equations of the system (2.23) yield

\[ \varphi = \alpha(t, y)xz + \beta(t, y)x + \gamma(t, y)z + \sigma(t, y). \]

The first equation (2.23) shows that \( \alpha(t, y), \beta(t, y), \gamma(t, y) \) and \( \sigma(t, y) \) solve the adjoint equation

\[ v_t + kv_{yy} = 0 \]  
(2.24)

to the linear heat equation

\[ u_t - ku_{yy} = 0. \]  
(2.25)

Thus, we have demonstrated the following statement.

**Proposition 2.4.** Eq. (2.22) satisfies the nonlinear self-adjointness condition (1.9) with the substitution (2.3) of the form

\[ v = \alpha(t, y)xz + \beta(t, y)x + \gamma(t, y)z + \sigma(t, y), \]  
(2.26)

where \( \alpha(t, y), \beta(t, y), \gamma(t, y) \) and \( \sigma(t, y) \) are any solutions of the adjoint equation (2.24) to the linear heat equation (2.25).

Combining Propositions 2.2 and 2.3 with Proposition 2.4, we obtain the following statements.

**Proposition 2.5.** The equation

\[ u_t = (f(u)u_x)_x + ku_{yy} + \omega^2 F(u), \quad f(u) = F'(u), \]  
(2.27)

satisfies the nonlinear self-adjointness condition (1.9) with the substitution (2.3) of the form

\[ v = \alpha(t, y)\cos(\omega x) + \beta(t, y)\sin(\omega x), \]  
(2.28)

where \( \alpha(t, y) \) and \( \beta(t, y) \) are any solutions of the adjoint equation (2.24) to the linear heat equation (2.25).

**Proposition 2.6.** The equation

\[ u_t = (f(u)u_x)_x + ku_{yy} - \delta^2 F(u), \quad f(u) = F'(u), \]  
(2.29)

satisfies the nonlinear self-adjointness condition (1.9) with the substitution (2.3) of the form

\[ v = \alpha(t, y)e^{\delta x} + \beta(t, y)e^{-\delta x}, \]  
(2.30)

where \( \alpha(t, y) \) and \( \beta(t, y) \) are any solutions of the adjoint equation (2.24) to the linear heat equation (2.25).
3 Conservation laws

3.1 Computation of conserved vectors for Equation (1.1)

Here we construct the conserved vector (1.11) for Eq. (1.1):

\[ u_t = f(u) u_{xx} + g(u) u_{yy} + h(u) u_{zz} + f'(u) u_x^2 + g'(u) u_y^2 + h'(u) u_z^2, \]  

(3.1)

associated with its translational symmetries. We specify the notation by writing the symmetry generator (1.10) in the form

\[ X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial u}. \]  

(3.2)

Then the expression (1.13) becomes

\[ W = \eta - \xi^1 u_t - \xi^2 u_x - \xi^3 u_y - \xi^4 u_z \]  

(3.3)

and the conservation equation (1.12) means that the following equation holds on the solutions of Eq. (3.1):

\[ D_t (C^1) + D_x (C^2) + D_y (C^3) + D_z (C^4) = 0. \]  

(3.4)

The formal Lagrangian (1.14) for Eq. (3.1) is

\[ \mathcal{L} = v [f(u) u_{xx} + g(u) u_{yy} + h(u) u_{zz} + f'(u) u_x^2 + g'(u) u_y^2 + h'(u) u_z^2 - u_t]. \]  

(3.5)

Due to the specific dependence of the formal Lagrangian (3.5) on the derivatives \( u_i, u_{ij} \), the components of the vector (1.11) are written

\[ C^1 = W \frac{\partial \mathcal{L}}{\partial u_t}, \]
\[ C^2 = W \left[ \frac{\partial \mathcal{L}}{\partial u_x} - D_x \left( \frac{\partial \mathcal{L}}{\partial u_{xx}} \right) \right] + D_x(W) \frac{\partial \mathcal{L}}{\partial u_{xx}}, \]
\[ C^3 = W \left[ \frac{\partial \mathcal{L}}{\partial u_y} - D_y \left( \frac{\partial \mathcal{L}}{\partial u_{yy}} \right) \right] + D_y(W) \frac{\partial \mathcal{L}}{\partial u_{yy}}, \]
\[ C^4 = W \left[ \frac{\partial \mathcal{L}}{\partial u_z} - D_z \left( \frac{\partial \mathcal{L}}{\partial u_{zz}} \right) \right] + D_z(W) \frac{\partial \mathcal{L}}{\partial u_{zz}}. \]

Substituting here the explicit expression (3.5) of \( \mathcal{L} \) we obtain:

\[ C^1 = -Wv, \]
\[ C^2 = W \left[ f'(u) u_x v - f(u) v_x \right] + f(u) v D_x(W), \]
\[ C^3 = W \left[ g'(u) u_y v - g(u) v_y \right] + g(u) v D_y(W), \]
\[ C^4 = W \left[ h'(u) u_z v - h(u) v_z \right] + h(u) v D_z(W). \]  

(3.6)
Eqs. (1.1) and (1.2) with arbitrary coefficients \( f(u), g(u), h(u) \) are invariant under the groups of translations of \( t, x, y, z \) with the generators

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = \frac{\partial}{\partial z}.
\] (3.7)

Eq. (1.1) has also a dilation symmetry. Moreover, both equations (1.1) and (1.2) may have more symmetries in certain particular cases [2], but we don’t consider them here.

Let us apply the formula (3.6) to the symmetry \( X_2 \). The corresponding quantity (3.3) equals \( W = -u_x \). We substitute it in (3.6) and obtain

\[
C_1 = vu_x,
\]

\[
C_2 = -f'(u)vu_x^2 + f(u)u_xv_x - f(u)vu_{xx},
\]

\[
C_3 = -g'(u)vu_xu_y + g(u)u_xv_y - g(u)vu_{xy},
\]

\[
C_4 = -h'(u)vu_xu_z + h(u)u_xv_z - h(u)vu_{xz}.
\] (3.8)

We have to substitute here the expression (2.8) for \( v \),

\[
v = a_1 xyz + a_2 xy + a_3 xz + a_4 yz + a_5 x + a_6 y + a_7 z + a_8.
\] (2.8)

Since \( v \) is a given function whereas \( u \) is any solution of Eq. (3.1), we want to simplify the conserved vector (3.8) by transforming it in an equivalent conserved vector which conserved density \( C_1 \) contains \( u \) instead of \( u_x \). To this end, we use the identity \( vu_x = D_x(uv) - uv_x \) and write \( C_1 \) in (3.8) in the form

\[
\tilde{C}_1 = C_1 + C_2(uv),
\]

where

\[
\tilde{C}_1 = -uv_x.
\] (3.9)

Then we transfer the term \( D_x(uv) \) from \( C_1 \) to \( C_2 \) using the usual procedure (see, e.g. [3], Section 8.1). Namely, since the total differentiations commute with each other, we have

\[
D_t(\tilde{C}_1 + D_x(uv)) + D_x(C_2) = D_t(\tilde{C}_1) + D_x(C_2 + D_t(uv)).
\]

Therefore the conservation equation (3.4) for the vector (3.8) can be equivalently rewritten in the form

\[
D_t(\tilde{C}_1) + D_x(\tilde{C}^2) + D_y(C^3) + D_z(C^4) = 0,
\] (3.10)

where \( \tilde{C}_1 \) has the form (3.9) and \( \tilde{C}^2 \) is given by

\[
\tilde{C}^2 = C^2 + D_t(uv).
\]
Let us work out the above expression for \( \tilde{C}^2 \). Invoking that \( D_t(v) = 0 \) due to Eq. (2.8), and using Eqs. (1.1), (3.1) we have

\[
\tilde{C}^2 = C^2 + vu_t = C^2 + v[f(u)u_{xx} + f'(u)u_x^2 + Dy(g(u)u_y) + D_z(h(u)u_z)].
\]

We substitute here the expression of \( C^2 \) from Eqs. (3.8) and obtain:

\[
\tilde{C}^2 = f(u)u_xv_x + vD_y(g(u)u_y) + vD_z(h(u)u_z).
\]

We simplify the latter expression for \( \tilde{C}^2 \) by noting that

\[
vD_y(g(u)u_y) = D_y(g(u)vv_y) - g(u)uyv_y,
\]

\[
vD_z(h(u)u_z) = D_z(h(u)vuz) - h(u)uzv_z.
\]

Therefore we transfer the terms \( D_y(g(u)vv_y) \) and \( D_z(h(u)vuz) \) to \( C^3 \) and \( C^4 \), respectively, and obtain:

\[
\tilde{C}^2 = f(u)u_xv_x - g(u)uyv_y - h(u)uzv_z.
\] (3.11)

Now the components \( C^3 \) and \( C^4 \) of the vector (3.8) become:

\[
\tilde{C}^3 = C^3 + D_x(g(u)vv_y), \quad \tilde{C}^4 = C^4 + D_x(h(u)vuz).
\]

After substituting here the expressions of \( C^3 \) and \( C^4 \) from (3.8) we have

\[
\tilde{C}^3 = g(u)(u_xv_y + uyv_x), \quad \tilde{C}^4 = h(u)(u_xv_z + uzv_x).
\]

Combining these expressions with (3.9), (3.11) and ignoring the tilde, we arrive at the following conserved vector which is equivalent to (3.8):

\[
C^1 = -uv_x,
C^2 = f(u)u_xv_x - g(u)uyv_y - h(u)uzv_z, \quad (3.12)
C^3 = g(u)(u_xv_y + uyv_x),
C^4 = h(u)(u_xv_z + uzv_x).
\]

The vector (3.12) involves the first-order derivatives of the variable \( v \) given by Eq. (2.8). Therefore the vector (3.12) contains seven parameters \( a_1, \ldots, a_7 \). In fact, it is a linear combination of seven linearly independent conserved vectors obtained from (3.12) setting by turns one of the parameters \( a_i \) equal to 1 and the others equal to 0. But some of these seven vectors are trivial in the sense that their divergence is identically zero, i.e. the conservation equation (3.4) is
satisfied identically. For example, setting in (3.12) \( a_6 = 1, a_1 = \cdots = a_7 = 0 \), i.e. \( v = y \), we obtain the vector

\[
C^1 = 0, \quad C^2 = -g(u)u_y, \quad C^3 = g(u)u_x, \quad C^4 = 0.
\]

For this vector Eq. (3.4) is satisfied identically,

\[
D_t(C^1) + D_x(C^2) + D_y(C^3) + D_z(C^4) = -D_x(g(u)u_y) + D_y(g(u)u_x) \equiv 0.
\]

Let us single out the nontrivial conserved vectors. Since \( v \) given by Eq. (2.8), the conservation equation (3.4) for the vector (3.12) is written as

\[
D_t(C^1) + D_x(C^2) + D_y(C^3) + D_z(C^4) = v_x F,
\]

where \( F \) is given by Eq. (2.1),

\[
F = -u_t + (f(u)u_x)_x + (g(u)u_y)_y + (h(u)u_z)_z.
\]

Then, specifying the expression of \( v_x \) from (2.8), we write (3.13) in the form

\[
D_t(C^1) + D_x(C^2) + D_y(C^3) + D_z(C^4) = (a_1 y z + a_2 y + a_3 z + a_5) F. \tag{3.14}
\]

Eq. (3.14) shows that we have only four nontrivial conserved vectors. They correspond to \( a_1, a_2, a_3 \) and \( a_5 \), i.e. they are obtained from (3.12) setting by turns one of these four parameters to be equal to 1, the others equal to 0. For example, the nontrivial conserved vector (3.12) corresponding to \( a_5 \) is

\[
C^1 = -u, \quad C^2 = f(u)u_x, \quad C^3 = g(u)u_y, \quad C^4 = h(u)u_z. \tag{3.15}
\]

The conservation equation (3.4) for the vector (3.15) coincides with Eq. (1.1).

Thus, the nontrivial conserved vectors are obtained by substituting in (3.12) the expression (2.8) for \( v \) with \( a_4 = a_6 = a_7 = 0 \). The resulting vector

\[
C^1 = -(a_1 y z + a_2 y + a_3 z + a_5) u,
\]

\[
C^2 = (a_1 y z + a_2 y + a_3 z + a_5) f(u)u_x - (a_1 z + a_2) x g(u)u_y - (a_1 y + a_3) x h(u)u_z,
\]

\[
C^3 = (a_1 z + a_2) g(u)(xu_x + yu_y) + (a_3 z + a_5) g(u)u_y,
\]

\[
C^4 = (a_1 y + a_3) h(u)(xu_x + zu_z) + (a_2 y + a_5) h(u)u_z
\]

is the linear combination with the coefficients \( a_5, a_1, a_2, a_3 \) of four linearly independent vectors, namely the vector (3.15) and the following three vectors:

\[
C^1 = -yz u, \quad C^2 = yz f(u)u_x - xz g(u)u_y - xy h(u)u_z,
\]

\[
C^3 = zg(u)(xu_x + yu_y), \quad C^4 = yh(u)(xu_x + zu_z); \tag{3.17}
\]
\[ C^1 = -yu, \quad C^2 = yf(u)u_x - xg(u)u_y, \]
\[ C^3 = g(u)(xu_x + yu_y), \quad C^4 = yh(u)u_z; \quad (3.18) \]
\[ C^1 = -zu, \quad C^2 = zf(u)u_x - xh(u)u_z, \]
\[ C^3 = zg(u)u_y, \quad C^4 = h(u)(xu_x + zu). \quad (3.19) \]

The conserved vectors associated with the symmetry \( X_3 \) from (3.7) can be obtained from the above results merely by the permutation \( x \leftrightarrow y \), followed by the permutations \( f \leftrightarrow g \) and \( C_2 \leftrightarrow C_3 \). This procedure maps the vector (3.12) to the following conserved vector:

\[ C^1 = -uv_y, \]
\[ C^2 = f(u)(yu_x + xu_y), \quad (3.20) \]
\[ C^3 = g(u)yv_x - f(u)xv_x - h(u)zu, \]
\[ C^4 = h(u)(yu_z + zu). \]

Accordingly, Eq. (3.14) becomes the following conservation equation for the vector (3.20):

\[ D_t(C^1) + D_x(C^2) + D_y(C^3) + D_z(C^4) = (a_1xz + a_2x + a_4z + a_6)F. \quad (3.21) \]

It shows that the nontrivial conserved vectors are obtained by substituting in (3.20) the expression (2.8) for \( v \) with \( a_3 = a_5 = a_7 = 0 \). The resulting vector

\[ C^1 = -(a_1xz + a_2x + a_4z + a_6)u, \]
\[ C^2 = (a_1z + a_2)f(u)(xu_x + yu_y) + (a_4z + a_6)f(u)u_x, \quad (3.22) \]
\[ C^3 = (a_1xz + a_2x + a_4z + a_6)g(u)y \]
\[ - (a_1z + a_2)yf(u)u_x - (a_1x + a_4)yh(u)u_z, \]
\[ C^4 = (a_1x + a_4)h(u)(yu_y + zu) + (a_2x + a_6)h(u)u_z \]

is the linear combination with the coefficients \( a_6, a_1, a_2, a_4 \) of four linearly independent vectors, namely the vector (3.15) and the following three vectors:

\[ C^1 = -xz u, \quad C^2 = zf(u)(xu_x + yu_y), \]
\[ C^3 = xzg(u)y - yzf(u)u_x - xyh(u)u_z, \quad (3.23) \]
\[ C^4 = xh(u)(yu_y + zu). \]
\[ C^1 = -xu, \quad C^2 = f(u)(xu_x + yu_y), \]
\[ C^3 = xg(u)u_y - yf(u)u_x, \quad C^4 = xh(u)u_z; \quad (3.24) \]
\[ C^1 = -zu, \quad C^2 = zf(u)u_x, \quad (3.25) \]
\[ C^3 = zg(u)u_y - yh(u)u_z, \quad C^4 = h(u)(yu_y + zu_z). \]

Proceeding as above with the symmetry \( X_4 \) from (3.7) we obtain, in addition to (3.15), (3.17)-(3.19) and (3.23)-(3.25), the following conserved vectors:
\[ C^1 = -xyu, \quad C^2 = yf(u)(xu_x + zu_z), \]
\[ C^3 = xg(u)(yu_y + zu_z), \quad (3.26) \]
\[ C^4 = xyh(u)u_z - yzf(u)u_x - xzg(u)u_y; \]
\[ C^1 = -xu, \quad C^2 = f(u)(xu_x + zu_z), \]
\[ C^3 = xg(u)u_y, \quad C^4 = xh(u)u_z - zf(u)u_x; \quad (3.27) \]
\[ C^1 = -yu, \quad C^2 = yf(u)u_x, \quad (3.28) \]
\[ C^3 = g(u)(yu_y + zu_z), \quad C^4 = yh(u)u_z - zg(u)u_y. \]

Finally, we turn to the time-translational symmetry \( X_1 \) from (3.7). In this case \( W = -u_t \). Replacing \( u_t \) by the right-hand side of Eq. (1.1) we obtain from the first equation (3.6):
\[ C^1 = v [D_x(f(u)u_x) + D_y(g(u)u_y) + D_z(h(u)u_z)]. \quad (3.29) \]

Now we observe that
\[ vD_x(f(u)u_x) = D_x[vf(u)u_x - v_xF(u)], \]
where we denote \( F(u) = \int f(u)du \) and use the equation \( v_{xx} = 0 \) resulting from the representation (2.8) of \( v \). Transforming likewise two other terms in (3.29) we write \( C^1 \) in the divergent form:
\[ C^1 = D_x[vf(u)u_x - v_xF(u)] + D_y[vg(u)u_y - v_yG(u)] + D_z[vh(u)u_z - v_zH(u)], \]
where \( G(u) = \int g(u)du, \quad H(u) = \int h(u)du \). Now we can transfer all terms of \( C^1 \) to the components \( C^2, C^3, C^4 \) and obtain \( C^1 = 0 \). The calculation shows that after this transfer we will have \( C^1 = C^2 = C^3 = C^4 = 0 \). Hence, \( X_1 \) does not lead to a non-trivial conservation law.
Thus, we have proved the following statement.

**Theorem 3.1.** The translational symmetries (3.7) of Eq. (3.1) with arbitrary coefficients $f(u), g(u), h(u)$ provide ten linearly independent conserved vectors (3.15), (3.17)-(3.19) and (3.23)-(3.28).

**Remark 3.1.** Eq. (2.22) is nonlinearly self-adjoint with the substitution (2.30) containing arbitrary solutions $\alpha(t, y), \beta(t, y), \gamma(t, y), \sigma(t, y)$ of the adjoint equation (2.24) to the one-dimensional linear heat equation. Therefore the conserved vector constructed by the above procedure for Eq. (2.22) will contain arbitrary solutions of Eq. (2.24).

### 3.2 Conserved vectors for Equation (2.18)

As mentioned in Section 1.1, the anisotropic heat equation (1.2) with an external source $q(u) \neq 0$ does not have a conservation form. However, Eq. (1.2) can be rewritten in a conservation form

$$D_t(C^1) + D_x(C^2) + D_y(C^3) = 0$$

if it is nonlinearly self-adjoint, for example, in the special cases (2.18) and (2.19). We will find here the conservation form for Eq. (2.18). The calculations are similar for Eq. (2.19).

We write Eq. (2.18) in the form

$$u_t = f(u)u_{xx} + g(u)u_{yy} + f'(u)u_x^2 + g'(u)u_y^2 + \omega^2 F(u), \quad \omega = \text{const.}, \quad (3.30)$$

and have the formal Lagrangian

$$\mathcal{L} = v[f(u)u_{xx} + g(u)u_{yy} + f'(u)u_x^2 + g'(u)u_y^2 + \omega^2 F(u) - u_t]. \quad (3.31)$$

For this formal Lagrangian Eqs. (1.11) yield (cf. Eqs. (3.6))

$$C^1 = -Wv,$$

$$C^2 = W[f'(u)u_x v - f(u)v_x] + f(u)vD_x(W), \quad (3.32)$$

$$C^3 = W[g'(u)u_y v - g(u)v_y] + g(u)vD_y(W).$$

Eq. (3.30) admits the three-dimensional Lie algebra spanned by the operators $X_1, X_2, X_3$ from (3.7). Let us apply the formula (3.32) to the symmetry $X_2$. In this case $W = -u_x$ and (3.32) is written (see Eqs. (3.8))

$$C^1 = vu_x,$$

$$C^2 = -f'(u)vu_x^2 + f(u)u_x u_x v - f(u)v u_{xx}, \quad (3.33)$$

$$C^3 = -g'(u)vu_x u_y + g(u)u_x u_y v - g(u)v u_{xy},$$
where \( v \) should be replaced by its expression (2.20),
\[
v = (A_1 y + B_1) \cos(\omega x) + (A_2 y + B_2) \sin(\omega x) .
\] (2.20)

Let us simplify the vector (3.33) in the same way as in Section 3.2. We write
\[
C^1 = \tilde{C}^1 + D_x(uv),
\]
where
\[
\tilde{C}^1 = -uv_x ,
\] (3.34)
and replace \( C^2 \) by
\[
\tilde{C}^2 = C^2 + D_t(uv) .
\]
Hence
\[
\tilde{C}^2 = C^2 + vu_t
\]
\[
= C^2 + v\left[ f(u)u_{xx} + f'(u)u_x^2 + D_y(g(u)u_y) + \omega^2 F(u) \right] .
\]
The substitution of the expression (3.32) for \( C^2 y \) yields:
\[
\tilde{C}^2 = f(u)u_x v_x + \omega^2 F(u) + vD_y(g(u)u_y) .
\]
One can verify that the following equation holds:
\[
vD_y\left(g(u)u_y\right) = D_y \left[ vg(u)u_y - \mathcal{G}(u)v_y \right] .
\]
It is obtained by introducing the function \( \mathcal{G}(u) = \int g(u)du \) and noting that
the equation \( v_{yy} = 0 \) is valid for the representation (2.20) of \( v \). Hence \( \tilde{C}^2 \) can be reduced to
\[
\tilde{C}^2 = f(u)u_x v_x + \omega^2 F(u) ,
\] (3.35)
whereas \( C^3 \) becomes
\[
\tilde{C}^3 = C^3 + D_x \left[ vg(u)u_y - \mathcal{G}(u)v_y \right] .
\]
The latter equation upon inserting the expression (3.32) for \( C^3 y \) yields:
\[
\tilde{C}^3 = g(u)u_y v_x - \mathcal{G}(u)v_{xy} .
\] (3.36)

Collecting Eqs. (3.34)-(3.36) and ignoring the tilde we arrive at the vector
\[
C^1 = -uv_x ,
\]
\[
C^2 = f(u)u_x v_x + \omega^2 F(u)v ,
\] (3.37)
\[
C^3 = g(u)u_y v_x - \mathcal{G}(u)v_{xy} .
\]
where \( v \) should be replaced by its expression (2.20). The vector (3.37) satisfies the conservation equation in the following form:

\[
D_t(C^1) + D_x(C^2) + D_y(C^3) = v_x \left[ (f(u)u_x)_x + (g(u)u_y)_y + \omega^2 F(u) - u_t \right].
\]

The expression (2.20) for \( v \) contains four arbitrary constants \( A_1, A_2, B_1, B_2 \). Accordingly, the vector (3.37) is a linear combination of four linearly independent vectors. Hence, we have demonstrated the following statement.

**Theorem 3.2.** The invariance of Eq. (3.18) with respect to the one-parameter group of translations of \( x \) with the generator \( X_2 \) provides the following four linearly independent conserved vectors:

\[
C^1 = \sin(\omega x) u, \quad C^2 = -\sin(\omega x) f(u)u_x + \omega \cos(\omega x) F(u), \\
C^3 = -\sin(\omega x) g(u)u_y;
\]

(3.38)

\[
C^1 = \cos(\omega x) u, \quad C^2 = -\cos(\omega x) f(u)u_x - \omega \sin(\omega x) F(u), \\
C^3 = -\cos(\omega x) g(u)u_y;
\]

(3.39)

\[
C^1 = y \sin(\omega x) u, \quad C^2 = -y \sin(\omega x) f(u)u_x + \omega y \cos(\omega x) F(u), \\
C^3 = -y \sin(\omega x) g(u)u_y + \sin(\omega x) G(u);
\]

(3.40)

\[
C^1 = y \cos(\omega x) u, \quad C^2 = -y \cos(\omega x) f(u)u_x - \omega y \sin(\omega x) F(u), \\
C^3 = -y \cos(\omega x) g(u)u_y + \cos(\omega x) G(u).
\]

(3.41)

**Remark 3.2.** The conserved vectors provided by the generator \( X_3 \) of the group of translations of \( y \) can be computed likewise. The generator \( X_1 \) of the time-translations provides only the trivial conserved vector.

**Remark 3.3.** Using the conserved vectors (3.38)-(3.41) one can write Eq. (2.18) in four different conservation forms. For example, the vector (3.38) satisfies the conservation equation

\[
D_t(C^1) + D_x(C^2) + D_y(C^3) = \sin(\omega x) \left[ u_t - (f(u)u_x)_x - (g(u)u_y)_y - \omega^2 F(u) \right].
\]

Accordingly, Eq. (2.18) can be replaced by the following conservation equation:

\[
D_t [\sin(\omega x) u] - D_x [\sin(\omega x) f(u)u_x + \omega \cos(\omega x) F(u)] \\
- D_y [\sin(\omega x) g(u)u_y] = 0.
\]

(3.42)
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