One-loop renormalization of higher-derivative 2D dilaton gravity

E. Elizalde
Department E.C.M. and I.F.A.E., Faculty of Physics, University of Barcelona
Diagonal 647, 08028 Barcelona
and Blanes Center for Advanced Studies, CSIC, 17300 Blanes, Spain

S. Naftulin
Institute for Single Crystals, 60 Lenin Ave., 310141 Kharkov, Ukraine
and Torselin, Sci. and Manufacture Centre, 310105 Kharkov, Ukraine

S.D. Odintsov
Department E.C.M., Faculty of Physics, University of Barcelona
Diagonal 647, 08028 Barcelona, Spain

Abstract

A theory of higher-derivative 2D dilaton gravity which has its roots in the massive higher-spin mode dynamics of string theory is suggested. The divergences of the effective action to one-loop are calculated, both in the covariant and in the conformal gauge. Some technical problems which appear in the calculations are discussed. An interpretation of the theory as a particular D=2 higher-derivative σ-model is given. For a specific case of higher-derivative 2D dilaton gravity, which is one loop multiplicatively renormalizable, static configurations corresponding to black holes are shown to exist.
1. **Introduction.** It is well known by now that the one-dimensional string (or the one-dimensional $\sigma$-model) describes the so-called 2D gravity. Such a theory has been very popular recently as a toy model for the study of formal questions of quantum gravity, for the investigation of the black hole structure and Hawking radiation, for its interesting connections with conformal field theory, and so on. A huge volume of literature on this field exists already and it is generally expected that the study of 2D gravity will help us in the construction of a consistent theory of 4D quantum gravity.

Different models of 2D gravity have been considered. A very popular one, which follows from a one-dimensional $\sigma$-model, is described by the action

$$S = -\int d^2 x \sqrt{g} \left[ c_1(\Phi) g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + c_2(\Phi) R + V(\Phi) \right],$$

(1)

where $\Phi$ is the dilaton field. However, this action is actually the one-dimensional analog of the action which describes the massless modes of the string. In order to take into account the first level massive higher-spin modes, one has to modify the standard bosonic $\sigma$-model, including also all possible terms with quartic derivatives [1]. Different approaches have been developed so far [1, 2] for the description of the massive string excitations, but only in the linear field approach.

It seems quite reasonable to formulate the theory of 2D gravity which stems from the massive mode string dynamics and to study this theory along the same lines as the usual 2D dilaton gravity. Of course such a theory will be one with higher derivatives. Its investigation may be useful in order to understand general properties of quantum gravity, and also perhaps for obtaining the dynamics of massive string modes, since it seems that this 2D gravity with higher derivatives should be easier to understand than real strings.

Motivated by these considerations, we start from the following action which includes all possible quartic derivative terms:

$$S = -\int d^2 x \sqrt{g} \left[ a_1(\Phi) g^{\mu\nu} g^{\alpha\beta} \partial_\mu \Phi \partial_\nu \Phi \partial_\alpha \Phi \partial_\beta \Phi + a_2(\Phi) g^{\mu\nu} g^{\alpha\beta} \partial_\mu \Phi \partial_\alpha \Phi \nabla_\nu \partial_\beta \Phi + a_3(\Phi) g^{\mu\nu} \partial_\mu \Phi \partial_\alpha \Phi \nabla_\nu \partial_\beta \Phi + a_4(\Phi) g^{\mu\nu} \partial_\mu \Phi \partial_\alpha \Phi \partial_\beta \Phi + a_5(\Phi) g^{\mu\nu} \Box \partial_\mu \Phi \partial_\nu \Phi + a_6(\Phi) \Box \Phi + a_7(\Phi) \Box^2 \Phi + a_8(\Phi) g^{\mu\nu} \epsilon^{\alpha\beta} \nabla_\mu \partial_\alpha \Phi \nabla_\nu \partial_\beta \Phi + a_9(\Phi) \epsilon^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + a_{10}(\Phi) \epsilon^{\mu\nu} g^{\alpha\beta} \partial_\mu \Phi \partial_\alpha \Phi \nabla_\nu \partial_\beta \Phi + a_{11}(\Phi) R g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + a_{12}(\Phi) g^{\mu\nu} \partial_\mu R \partial_\nu \Phi + a_{13}(\Phi) R \Box \Phi + a_{14}(\Phi) \Box R + a_{15}(\Phi) R^2 + a_{16}(\Phi) \epsilon^{\mu\nu} \partial_\mu R \partial_\nu \Phi + C_1(\Phi) g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + C_2(\Phi) R + C_3(\Phi) \Box \Phi + V(\Phi) \right].$$

(2)

Here we suppose that all the functions of the dilaton $\Phi$ (coefficients) are analytic, $\epsilon^{\mu\nu}$ is an antisymmetric tensor, the dimensions are: $[a] = L^2$, $[C] = L^0$, $[V] = L^{-2}$, and the minus sign in front of the action is chosen for convenience.
Integrating the action (2) by parts and dropping total derivatives (in full analogy with string theory [1]), we obtain

\[
S = -\int d^2x \sqrt{g} \left[ Z_1(\Phi) g^{\mu\nu} g^{\alpha\beta} \partial_\mu \Phi \partial_\nu \Phi \partial_\alpha \Phi \partial_\beta \Phi + Z_2(\Phi) g^{\mu\nu} g^{\alpha\beta} \partial_\mu \Phi \partial_\alpha \Phi \nabla_\nu \partial_\beta \Phi \\
+ Z_3(\Phi) g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + Z_4(\Phi) R g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + Z_5(\Phi) R^2 \Phi + Z_6(\Phi) \right].
\]

(3)

In the next sections we will discuss the quantum structure of the action (3), which is going to be our starting point. In particular, we will calculate the one-loop divergences of the theory given by (3) in covariant and conformal gauges.

2. One-loop renormalization. Let us start the calculation of the one-loop divergences of the theory under discussion. In what follows we shall assume that \(Z_6(\Phi)\) has no zeros, and that

\[
\Delta \equiv \det \begin{pmatrix} 4Z_6 & -Z_5 \\ -Z_5 & Z_3 \end{pmatrix} \neq 0.
\]

(4)

Such restriction appears in the course of the calculation, but its precise physical meaning is not clear. We are going to work in the background field method, according to which we set

\[
\Phi \rightarrow \Phi + \varphi, \quad g_{\mu\nu} \rightarrow g_{\mu\nu} + h_{\mu\nu},
\]

(5)

where \(\varphi\) and \(h_{\mu\nu}\) are quantum fields, and the following notations will be used: \(h = g^{\mu\nu} h_{\mu\nu}\) and \(h_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h\). Working in the covariant effective action formalism —developed for 2D dilaton gravity [1] in Refs. [3]— the problem reduces to the calculation of the tr log from the fourth order differential operator \(\hat{H}\), which is essentially the second functional derivative of the action (3).

This operator contains \(3 \times 3\)-matrices acting on the space of quantum fields \(\Phi^i \equiv \{\varphi, h, h_{\mu\nu}\}\). If minimal gauge conditions are used, it takes the form

\[
\hat{H}_{ij} = \hat{K}_{ij} \square^2 + \hat{L}^{\mu\nu\lambda} \nabla_\mu \nabla_\nu \nabla_\lambda + \hat{M}^{\mu\nu} \nabla_\mu \nabla_\nu + \hat{U}^{\lambda} \nabla_\lambda + \hat{Y}_{ij}.
\]

(6)

Without loss of generality, we take \(\hat{L}^{\mu\nu\lambda}\) and \(\hat{M}^{\mu\nu}\) to be fully symmetric in their greek indices.

As the matrices \(\hat{K}, \hat{L}\) and \(\hat{M}\) do not commute with the covariant derivative, a naive choice of their components can get changed when integrating by parts. The remedy to this situation is found in ’t Hooft and Veltman’s procedure [4], which yields unique Hermitian matrices, according to the rule

\[
\hat{K} \rightarrow \hat{K}' = \frac{1}{2} (\hat{K} + \hat{K}^T),
\]

\[
\hat{L}^{\mu\nu\lambda} \rightarrow \hat{L}'^{\mu\nu\lambda} = \frac{1}{2} (\hat{L}^{\mu\nu\lambda} - \hat{L}^{\nu\mu\lambda}) + \frac{2}{3} (g^{\nu\lambda} \nabla^\mu \hat{K}^T + g^{\mu\lambda} \nabla^\nu \hat{K}^T + g^{\mu\nu} \nabla^\lambda \hat{K}^T),
\]

\[
\hat{M}^{\mu\nu} \rightarrow \hat{M}'^{\mu\nu} = \frac{1}{2} (\hat{M}^{\mu\nu} - \hat{M}^{\nu\mu}).
\]

(3)
\[ \hat{M}^{\mu\nu} \rightarrow \hat{M}^{\prime\mu\nu} = \frac{1}{2}(\hat{M}^{\mu\nu} + \hat{M}^{T\mu\nu}) - \frac{3}{2} \nabla_\lambda \hat{L}^{T\mu\nu\lambda} \nabla^{\mu\nu} + \nabla^{\mu} \nabla^{\nu} \hat{K}^{T} + \nabla^{\nu} \nabla^{\mu} \hat{K}^{T} + g^{\mu\nu} \Box \hat{K}^{T}, \quad (7) \]

and so on. Notice that the matrices \( \hat{U}^{\lambda} \) and \( \hat{Y}^{\mu\nu} \) have no effect on the one-loop divergences.

The divergent part of the Tr log \( \hat{H} \) (modulo surface terms) may be expressed as follows

\[ \frac{i}{2} \text{Tr} \log \hat{H} \left| \text{div} \right| = \frac{i}{2} \text{Tr} \log \left[ \hat{1} \Box^{2} + \hat{S}^{\mu\nu\lambda} \nabla_\mu \nabla_\nu \nabla_\lambda + \hat{N}^{\mu\nu} \nabla_\mu \nabla_\nu + \hat{U}^{\lambda} \nabla_\lambda + \hat{Y} \right] \left| \text{div} \right| = \frac{1}{32} \epsilon \int d^{2}x \sqrt{g} \left[ 2 \text{Tr} \left( \hat{S}^{\mu\nu\lambda} \hat{S}^{\mu\nu\lambda} \right) + 3 \text{Tr} \left( \hat{S}^{\lambda} \hat{S}_{\lambda} \right) - 16 \text{Tr} \hat{N}_{\nu} \right], \quad (8) \]

where \( \hat{S}^{\mu\nu\lambda} \) and \( \hat{N}^{\mu\nu} \) are assumed to be completely symmetric in their greek indices, \( \hat{S}^{\lambda} \equiv \hat{S}_{\nu}^{\nu} \), and \( \epsilon = 2\pi(n - 2) \). So far we have assumed that a minimal gauge of the standard type exists, what is not evident in the case under discussion. Expanding the action (3) in powers of the quantum fields, one verifies that there appear no terms of the form \( \bar{\psi} \Box^{2} \bar{\psi} \), so that the matrix \( \hat{K} \) in Eq. (6) is degenerate. (In higher dimensions, the corresponding term in standard \( R^{2} \)-gravity comes from the Weyl tensor squared, which vanishes identically for \( d = 2 \)). Possible solutions of this problem are the following. (i) One may gauge the field \( \bar{\psi}_{\mu\nu} \) away by adopting the conformal gauge and by working in this conformal gauge. (ii) One may instead invent some procedure in order to make the matrix \( \hat{K} \) in (4) non-degenerate “by hand”, for instance, by adding some term which should not influence the divergences. (iii) And one may also consider a (non-standard) non-linear gauge, for example, including curvature terms in the gauge condition. In what follows we will apply the first two procedures and will show that both give equivalent off-shell expressions for the one-loop divergences (up to surface terms).

In order to modify at the quantum level the second variation of the action, let us consider the following term

\[ \delta S = -\frac{\xi}{2} \int d^{2}x \sqrt{g} Z_{6}(\Phi) \bar{h}^{\mu\nu} \Box[2 \bar{R}_{\mu\nu} - \bar{R} \bar{g}_{\mu\nu}]. \quad (9) \]

Here the field \( \Phi \) is classical, while the quantities with tildes contain both background and quantum components, and hence must be Taylor expanded (to first order in fluctuations). The weight factor \( \xi \) is arbitrary. As the expression in the square brackets on the r.h.s. of Eq. (9) is zero precisely when \( d = 2 \), the divergent part of the effective action should not depend on \( \xi \). (Sometimes a non-essential renormalization of the background field is compulsory in order to eliminate the \( \xi \)-dependence (3).)

Adding the term (9) to the initial action (as has been proposed for the 2D dilaton gravity (1) in Refs. (5, 6)) may actually change the structure of the operator \( \tilde{H} \) (8). Finally, at the end of the calculations one can put \( \xi \) equal to zero. However, since the intermediate expressions acquire a pole at \( \xi = -1 \) or \( \xi = 0 \), we will consider the region \( \xi > 0 \) only. Note also that
it is not evident that the ghost operator will become both minimal and non-degenerate for
$\xi \neq 0$, and this should be checked directly for the gauge under discussion.

Let us choose the gauge-fixing action in the form

$$S_{GF} = -\int d^2x \sqrt{g} \chi^\mu C_{\mu \nu} \chi^\nu, \quad (10)$$

where

$$\chi^\mu = -\nabla_\nu \tilde{h}^{\mu \nu} + \frac{1}{2(1 + \xi)} \nabla^\mu h - \frac{1}{2(1 + \xi)} \frac{Z_5}{Z_6} \nabla^\mu \varphi,$$

$$\tilde{C}_{\mu \nu} = (\xi g_{\mu \nu} \Box + \nabla_\mu \nabla_\nu - \xi R_{\mu \nu}) Z_6.$$

The one-loop effective action is given by the standard expression

$$\Gamma_{\text{div}} = i \frac{2}{2} \text{Tr} \log \tilde{H} - i \text{Tr} \log \tilde{M} + i \frac{2}{2} \text{Tr} \log \tilde{C}, \quad (11)$$

where

$$\tilde{H}_{ij} = (S^{(2)} + \delta S^{(2)} + S_{GF})_{ij}, \quad \tilde{M}_{\mu \nu} = \tilde{C}_{\mu \lambda} \frac{\delta \chi^\lambda}{\delta u^\nu},$$

where $u^\nu$ are the gauge transformation parameters.

It is interesting to note that the divergent part of the last term in (11) is known to be a
$\lambda$-dependent surface term $\lambda = -1/(1 + \xi)$, which is well defined only for $\lambda > -1$, i.e. $\xi > 0$. Since we are not interested in surface divergences, this term will not be important for our purposes. The calculation of the divergences being extremely tedious, we restrict ourselves to the particular case $\Phi = \text{const.}$, while arguing that $\Gamma_{\text{div}}$ is off-shell identical to that of the less technical case of the conformal gauge. For $\Phi = \text{const.}$ the minimal quartic operator (6) reads (only its non-zero components are written)

$$\tilde{K}_{\varphi \varphi} = 2Z_3 - \frac{Z_2^2}{2(1 + \xi) Z_6}, \quad \tilde{K}_{\varphi h} = \tilde{K}_{h \varphi} = -\frac{\xi Z_5}{2(1 + \xi)}, \quad \tilde{K}_{hh} = \frac{\xi Z_6}{2(1 + \xi)},$$

$$\tilde{M}_{\varphi \varphi} = -\xi Z_6 P^{\rho \sigma, \alpha \beta}, \quad P^{\rho \sigma, \alpha \beta} \equiv g^{\rho \sigma} g^{\alpha \beta} - \frac{1}{2} g^{\rho \sigma} g^{\alpha \beta};$$

$$\tilde{M}_{\varphi h} = -2C_1 g^{\mu \nu} + (Z_3 - 2Z_4 + 2Z'_6) R g^{\mu \nu},$$

$$\tilde{M}_{h h} = 2C_1 g^{\mu \nu} - \frac{1}{2} (Z_5 + 2Z'_6) R g^{\mu \nu}, \quad \tilde{M}_{\mu \nu} = Z_6 R g^{\mu \nu},$$

$$\tilde{M}_{\alpha \beta} = \left[ \frac{C_2}{2} + (1 + \xi) Z_6 R \right] P^{\rho \sigma, \alpha \beta} g^{\mu \nu} + [(4\xi - 2) Z_6 R - C_2] P^{\rho \sigma, \mu \lambda} P^{\alpha \beta, \nu \kappa} g_{\mu \lambda}. \quad (12)$$

Now, there is a simple way to see how the $\bar{h}h$ sector decouples, namely to set $\xi \to \infty$, for instance, and indeed the explicit calculation shows that the $\bar{h}h$ terms only give $\xi$-independent total derivatives.

The ghost operator has the form

$$\tilde{M}^{\mu \nu} = \xi \Box^2 g_{\mu \nu} + R \nabla_\mu \nabla_\nu. \quad (13)$$
Discarding $\xi$-independent surface terms, one easily gets the following contribution to \((11)\)

$$
- \frac{1}{2\epsilon} \int d^2x \sqrt{g} \left[ \frac{2}{\xi} R + \cdots \right].
$$

\((14)\)

This is clearly a surface term as well, but we have kept it to demonstrate that it cancels out the corresponding term stemming from Tr log $\tilde{H}$. Performing some algebra (and using Eq. \((8)\)), we obtain

$$
\Gamma_{\text{div}}[\Phi = \text{const}] = - \frac{1}{2\epsilon} \int d^2x \sqrt{g} \left[ \frac{1}{\Delta} (4C_1Z_6 + 2C_2Z_5) \right.
\quad + \left. \frac{2}{\Delta} (Z_5'^2 - 5Z_3Z_6 + 2Z_4Z_6 + 2Z_5Z_6' - 2Z_6Z_5') \right].
$$

\((15)\)

All the $\xi$-dependent terms including the surface ones have cancelled exactly. This is a pleasant surprise since, generally speaking, we would have expected that a renormalization of the metric would be needed in order to eliminate $\xi$ from the effective action (as in the theory \((\ref{eq:11})\), see \([5]\)). We will see below that the result of the calculation in the conformal gauge at $\Phi = \text{const.}$ exactly coincides with Eq. \((13)\).

Now, working in the conformal gauge, we split the fields according to

$$
\Phi \rightarrow \Phi + \varphi, \quad g_{\mu\nu} \rightarrow e^{2\sigma} g_{\mu\nu}.
$$

\((16)\)

Under the conformal transformation, our basic action \((\ref{eq:5})\) becomes

\[ S = - \int d^2x \sqrt{g} e^{-2\sigma} \left\{ Z_1(\Phi)g^{\mu\nu}g^{\alpha\beta} \partial_\mu \Phi \partial_\nu \Phi \partial_\alpha \Phi \partial_\beta \Phi + Z_2(\Phi)g^{\mu\nu}g^{\alpha\beta} \partial_\mu \Phi \partial_\nu \Phi \partial_\lambda \Phi \partial_\beta \Phi \right. \]

\[ - Z_2(\Phi)g^{\mu\nu}g^{\alpha\beta} \partial_\mu \Phi \partial_\nu \Phi \partial_\alpha \Phi \partial_\beta \Phi \partial_\sigma + Z_3(\Phi)g^{\mu\nu}g^{\alpha\beta}g^{\lambda\sigma} \partial_\mu \Phi \partial_\nu \Phi \partial_\alpha \Phi \partial_\beta \Phi \partial_\lambda \Phi \partial_\sigma + 2Z_3(\Phi)g^{\mu\nu}g^{\alpha\beta} \partial_\mu \Phi \partial_\nu \Phi \partial_\alpha \Phi \partial_\beta \Phi \partial_\sigma
\]

\[ - 4Z_3(\Phi)g^{\mu\nu}g^{\alpha\beta} \partial_\mu \Phi \partial_\nu \Phi \partial_\alpha \Phi \partial_\beta \Phi \partial_\lambda \Phi \partial_\sigma + 2Z_3(\Phi)g^{\mu\nu}g^{\alpha\beta} \partial_\mu \Phi \partial_\nu \Phi \partial_\alpha \Phi \partial_\beta \Phi \partial_\beta \Phi
\]

\[ + Z_4(\Phi)Rg^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - 2Z_4(\Phi)g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi \partial_\sigma + Z_5(\Phi)R \partial_\Phi
\]

\[ - 2Z_5(\Phi) \Box \sigma \partial_\Phi + Z_6(\Phi)R^2 - 4Z_6(\Phi)R \partial_\sigma + 4Z_6(\Phi) \Box \sigma
\]

\[ + e^{2\sigma} \left[ C_1(\Phi)g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + C_2(\Phi)R - 2C_2(\Phi) \Box \sigma + V(\Phi)e^{4\sigma} \right].
\]

\((17)\)

The corresponding ghost contribution to the divergences are just the surface terms and hence they will be dropped. Thus, we have an effective theory of two scalars on a curved background.

Expanding in powers of the quantum fields $\Phi^i \equiv \{\varphi, \sigma\}$, we get

$$
\hat{K}_{ij} = 2 \begin{pmatrix} Z_3 & -Z_5 \\ -Z_5 & 4Z_6 \end{pmatrix}, \quad \hat{L}^{\mu\nu} = \hat{A}_{ij} g^{\lambda(\mu} g^{\nu)}
$$

$$
\hat{A}^\lambda_\varphi = 4(Z_3' - Z_2) \partial^\lambda \Phi, \quad \hat{A}^\lambda_\sigma = 4(2Z_4 - Z_5') \partial^\lambda \Phi, \quad \hat{A}^\lambda_\varphi = 4(Z_3' - Z_5') \partial^\lambda \Phi,
$$

$$
\hat{M}_{ij} = g_{\mu\nu} \hat{M}_{ij},
$$
\[\hat{M}_{\phi} = -4C_1 + 2(Z_3 - 2Z_4 + 2Z'_5)R + 2(Z''_3 - 8Z_1 - Z'_2)\partial^3 \Phi \partial_\lambda \Phi + 6(Z'_3 - Z_2)\Box \Phi,\]
\[\hat{M}_{\sigma} = \hat{M}_{\sigma \phi} = -4C'_2 - 4(Z_5 + 2Z'_6)R + 4(2Z_4 - Z_3 - 2Z'_5)\Box \Phi\]
\[\hat{M}_{\sigma \sigma} = 32Z_6R + 16(Z_5 + Z'_6)\Box \Phi + 8(2Z_4 - Z_3 + 2Z''_6)\partial^3 \Phi \partial_\lambda \Phi.\] (18)

Following the procedure (7)–(8), after some algebra the one-loop effective action can be found to be (having dropped the surface terms)

\[\Gamma_{\text{div}} = -\frac{1}{2\epsilon} \int d^2x \sqrt{g} \left[ \frac{1}{\Delta}(4C_1Z_6 + 2C'_2Z_5) + \frac{2}{\Delta}(Z^2_3 - 5Z_3Z_6 + 2Z_4Z_6 + 2Z'_5Z''_5 - 2Z'_6Z'_5)R \right.\]
\[+ \frac{1}{\Delta} \left( \frac{3}{2}Z^2_3 + 16Z_1Z_6 - 2Z_2Z_5 - 2Z_3Z_4 - 2Z'_4 + 2Z_5Z'_3 + 2Z_5Z'_4 - 6Z_6Z'_2 \right.\]
\[+ 2Z_6Z''_3) \partial^3 \Phi \partial_\lambda \Phi + \left( \frac{2(Z_2Z_6 - Z_5Z'_5 + Z_6Z'_3)}{\Delta} \right) \partial^3 \Phi \partial_\lambda \Phi \right] \equiv -\frac{1}{2\epsilon} \int d^2x \sqrt{g} \left[ A_1(\Phi) + A_2(\Phi)R + A_3(\Phi)\partial^3 \Phi \partial_\lambda \Phi \right].\] (19)

As can be easily seen, for \(\Phi = \text{const.}\) Eq. (19) coincides with the covariant gauge effective action (13). Notice also that higher-derivative divergences do not appear.

The theory under discussion is one-loop multiplicatively renormalizable in the usual sense if the following conditions are fulfilled

\[A_1(\Phi) = a_1V(\Phi), \quad A_2(\Phi) = a_2C_2(\Phi), \quad A_3(\Phi) = a_3C_1(\Phi),\] (20)

where \(a_1, a_2\) and \(a_3\) are arbitrary constants. Many different sets of dilatonic functions in (8) satisfy the conditions (20). A simple example of a multiplicatively renormalizable theory is given by

\[Z_i = e^\Phi, \quad i = 1, \ldots, 6, \quad C_1 = \text{const}, \quad C_2 = \text{const}, \quad V = \text{const},\] (21)

or the even more immediate one \(Z_i = 1, i = 1, \ldots, 6, C_1(\Phi), C_2(\Phi)\) and \(V(\Phi)\) being arbitrary. Some other families of solutions can also be given explicitly.

It is easy to write the generalized renormalization group equations for our theory. In particular, all generalized \(\beta\)-functions corresponding to the higher-derivative terms are zero, and we have a large freedom because all the functions \(Z_i\) are free parameters of the theory, in the generalized renormalization group.

To be noted also is the fact that one cannot obtain the one-loop renormalization of low-derivative 2D dilaton gravity (8) as a particular case of the theory (8). For that purpose it is necessary to put the \(Z_i\) in (8) equal to zero, but such a restriction contradicts condition (1). The reason is that higher-derivative terms give essential contributions to the renormalization of the low-derivative terms (but not to their own renormalization!). This fact completely
changes the structure of renormalization, as compared with that of low-derivative dilaton gravity \[3\].

3. **σ-model interpretation.** The use of the conformal gauge indicates the possibility to interpret the theory under discussion as a certain \(D = 2\) σ-model with higher derivatives. Indeed, choosing the conformal gauge \(g_{\mu\nu} \rightarrow e^{2\sigma} g_{\mu\nu}\), one can represent the theory (17) as

\[
S = \int d^2x \sqrt{g} \left[ T(X) + R\psi(X) + G_{ab}(X)g^{\mu\nu}\partial_\mu X^a \partial_\nu X^b + C(X)R^2 + R\partial^a U_a(X) \\
+ Rg^{\mu\nu}\partial_\mu X^a \partial_\nu X^b W_{ab}^{(1)}(X) + g^{\mu\nu}g^{\alpha\beta}\partial_\mu X^a \partial_\nu X^b \partial_\alpha X^c \partial_\beta X^d T_{abcd}^{(1)}(X) \\
+ g^{\mu\nu}\Box X^a \partial_\nu X^b X^c T_{abc}^{(1)}(X) + g^{\mu\nu}g^{\alpha\beta}\nabla_\mu X^a \partial_\nu X^b \partial_\beta X^c T_{abc}^{(2)}(X) \\
+ \Box X^a \Box X^b M_{ab}^{(1)}(X) + g^{\mu\nu}g^{\alpha\beta}\nabla_\mu X^a \nabla_\nu \partial_\beta X^b M_{ab}^{(2)}(X) \right],
\]

(22)

where

\[
X^a = (\sigma, \Phi), \quad T(X) = -V(\Phi)e^{2\sigma}, \quad \psi(X) = -C_2(\Phi), \quad C(X) = -e^{2\sigma} Z_6(\Phi),
\]

\[
G_{ab} = -\begin{pmatrix} 0 & C_2(\Phi) & C_1(\Phi) \\ C_2(\Phi) & 0 & 0 \\ C_1(\Phi) & 0 & 0 \end{pmatrix}, \quad (U_\Phi, U_\sigma) = -e^{2\sigma} (Z_5(\Phi), -4Z_6(\Phi)),
\]

\[
W_{ab}^{(1)} = -e^{2\sigma} \begin{pmatrix} 0 & 0 \\ 0 & Z_4(\Phi) \end{pmatrix},
\]

and the non-zero components of the remaining tensors are

\[
F_{\Phi\Phi\Phi\Phi}^{(1)} = -e^{-2\sigma} Z_1(\Phi), \quad F_{\Phi\Phi\Phi\sigma}^{(1)} = -e^{-2\sigma} Z_2(\Phi), \quad F_{\Phi\Phi\sigma\sigma}^{(1)} = -2e^{-2\sigma} Z_3(\Phi),
\]

\[
T_{\Phi\Phi\sigma}^{(1)} = -2e^{-2\sigma} Z_3(\Phi), \quad T_{\Phi\sigma\Phi}^{(1)} = 2e^{-2\sigma} Z_4(\Phi), \quad T_{\Phi\Phi\Phi}^{(2)} = -e^{-2\sigma} Z_2(\Phi), \quad T_{\Phi\Phi\Phi}^{(2)} = 4e^{-2\sigma} Z_3(\Phi),
\]

\[
M_{\sigma\sigma}^{(1)} = -4e^{-2\sigma} Z_6(\Phi), \quad M_{\sigma\Phi}^{(1)} = 2e^{-2\sigma} Z_5(\Phi), \quad M_{\Phi\Phi}^{(2)} = -e^{-2\sigma} Z_3(\Phi).
\]

(23)

Thus, we have arrived to the particular case of \(D = 2\) higher-derivative σ-model considered in Ref. [1]. The field \(M^{(2)}_{ab}\) is of Stueckelberg type, and can be gauged away by integrating by parts (this changes some other higher-derivative terms in (22)). The background field equations in the linear approximation (which describe the first massive level of the corresponding string) can be easily taken from [1] for our specific values of the functions under discussion.

4. **Dilatonic solutions.** To simplify the analysis a bit, we start from the following version of action \[3\]

\[
S = -\int d^2x \sqrt{g} \left[ Z_1 g^{\mu\nu}g^{\alpha\beta}\partial_\mu \Phi \partial_\nu \Phi \partial_\alpha \Phi \partial_\beta \Phi + Z_5 R \Box \Phi + Z_6 R^2 \\
+ C_1(\Phi) g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + V(\Phi) \right],
\]

(24)
where $Z_1$, $Z_5$ and $Z_6$ are constants, and always $Z_5 \neq 0$ and $Z_6 \neq 0$. The corresponding equations of motion are

$$ \frac{\delta S}{\delta \Phi} = -4Z_1 \nabla_\alpha \left( g^{\mu\nu} g^{\alpha\beta} \partial_\mu \Phi \partial_\nu \Phi \partial_\beta \Phi \right) + Z_5 \Box R + V'(\Phi) + C_1(\Phi)g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - 2\nabla_\nu [C_1(\Phi)g^{\mu\nu} \partial_\mu \Phi] = 0, $$

$$ \frac{1}{2} g^{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}} = -Z_1 g^{\mu\alpha} g^{\nu\beta} \partial_\mu \Phi \partial_\nu \Phi \partial_\alpha \Phi \partial_\beta \Phi + Z_5 \Box^2 \Phi - Z_5 R \Box \Phi $$

$$ -Z_6 R^2 + 2Z_6 \Box R + V(\Phi) = 0. $$

We will deal with black hole type metrics of the ordinary kind (using the gauge [7])

$$ ds^2 = -g(r)dt^2 + g^{-1}(r)dr^2 \rightarrow -dt^2 + dr^2, \ r \rightarrow \infty. $$

Taking into account the $t$-independence of $\Phi$ (and meaning now $(\ )' \equiv \partial_\tau$), we obtain

$$ -4Z_1 \left[ g^2(\Phi')^3 \right]' + Z_5 (gg''')' + V'(\Phi)/\Phi' - [C_1(\Phi)g\Phi']' - C_1(\Phi) (g\Phi')' = 0, $$

$$ -Z_1 g^2(\Phi')^4 + Z_5 [g(g\Phi'')'' - Z_5 g''(g\Phi')' - Z_6 (g'')^2 + 2Z_6 (gg'')'] + V(\Phi) = 0. $$

From previous analysis of standard dilatonic gravity, we can expect to find solutions of these quite involved differential equations (27) for string potentials of the form

$$ V \sim \Lambda + e^{\lambda\Phi(r)}. $$

The solutions are of the kind

$$ \Phi(r) = a + br, $$

being $\Lambda$, $\lambda$, $a$ and $b$ some constants. This is indeed the case. By expanding $g(r)$ in series

$$ g = c + \frac{\alpha}{r} - \frac{\alpha_1}{r^2} + \cdots, \ c = g(\infty), $$

it is not difficult to see that indeed solutions of (27) are found in the following two cases.

(a) Case $Z_1 \neq 0$, $C_1 = \text{const}$. The equations of motion reduce to (with $\Phi= a + br$)

$$ -4Z_1 b^4 g^2 + Z_5 bg'''' + V(\Phi) - 2C_1 b^2 g = k, $$

$$ -Z_1 b^4 g^2 + Z_5 bg'''' - Z_6 (g'')^2 + 2Z_6 (gg'')' + V(\Phi) = 0, $$

where $k$ is an arbitrary constant (of integration). A solution is found for arbitrary $\Lambda$, with

$$ b^2 = -\frac{C_1}{3Z_1 c}, $$

(in other words, satisfying $\Phi'(r)^2 \simeq -C_1/(3Z_1 g(1/r)))$, with the only restriction that $\alpha = 0$, i.e., that

$$ \frac{dg}{dr^{-1}} \bigg|_{r^{-1}=0} = 0. $$
This solution is obtained by substituting the expansion (30) into the equations of motion (27), and it is exact up to terms of order $O(r^{-4})$. That all terms up to this order can be matched for both equations with so few requirements is not trivial at all, as can be seen immediately by comparing with different ansätze.

(b) Case $Z_1 = 0$, $C_1(\Phi) \sim V(\Phi)$. This is also an interesting situation and the result is very similar to the previous one. The equations of motion can now be written as (again with $\Phi = a + br$)

\begin{align*}
Z_5b (gg'')' + V'(\Phi) - 2b^2 [C_1(\Phi)g]' + C_1'(\Phi)b^2g &= 0, \\
Z_5bgg'' - Z_6(g''^2) + 2Z_6(gg'')' + V(\Phi) &= 0.
\end{align*}

As before, a solution which is exact up to order $O(r^{-4})$ is obtained for $b^2 = \frac{V'(\infty)}{g(\infty) C_1'(\infty)}$ (35)

(this is a non-zero constant), for a potential of the form (28). (Actually, to match the terms up to order four in $r^{-1}$, also potentials e.g. of the form $V = V_0 \left(1 - e^{-v/r^4}\right)$ would do, but these particular forms are too connected with the approximation one is working at).

In contrast, it is also easy to check that an ansatz of the type $\Phi = a + \log(r - r_0)$ does not lead to any solution in the first case, unless $g = \text{const.}$, a trivial situation. In the second case, one could say that a solution is obtained (in principle) to order $O(r^{-4})$, since all terms do vanish to this order. However, it is not really meaningful as an approximation to a series solution, since already the first non-vanishing terms cannot be compensated in any way.

5. **Concluding remarks.** Of course many questions are still left for future investigation of such higher-derivative $D = 2$ $\sigma$-model. Some of them have been listed in Ref. [1]. One of particular importance for us concerns the relations between the $\beta$-functions corresponding to the couplings in (22) or, more precisely, which of these $\beta$-functions are independent? The answer for the case of the standard $\sigma$-model (first three terms in (22)) is well known (see for example [8]), but not for the full higher-derivative $\sigma$-model (22). As we could see from the calculations in the previous section, only $T(X)$, $\psi(X)$ and $G_{ab}(X)$ are getting renormalized in the one-loop approximation, and only through higher-derivative terms (or in the case of $T(X)$ through a mixture of higher-derivative and lower-derivative terms).

The other interesting question which remains open concerns the interpretation of the condition of the vanishing of the $\beta$-functions in the model under discussion. However, all these questions should be first understood for the case of the general higher-derivative $\sigma$-model [1].
Note, finally, that the model discussed in this work provides a big arena for the study of 2D black holes and their properties, like Hawking radiation \([9]\), etc., and surely deserves further study.

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