SPIN POLYNOMIAL INVARIANTS
FOR DOLGACHEV SURFACES

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I. Introduction

The definition of Spin–polynomial invariants [PT], [Ty] uses the same machinery as Donaldson’s invariants. One considers anti self dual connections \( a \) for which a coupled Dirac operator \( D_a \) has nontrivial kernel. For generic metrics the gauge equivalence classes of such connections form a subspace of the moduli space of ASD–connections which then can be used much the same way as the moduli space itself to define invariants of the differentiable structure of the underlying four dimensional manifold \( X \). To do all of this, one of course has to fix a Spin\(^C\)–structure on \( X \).

From the technical point of view these invariants appear to be better accessible to algebraic geometric computations than the polynomial invariants themselves. This is because the corresponding bundles come with a section. In principle this reduces the computation of the invariants to the study of the zeroes of such a section and the associated extension groups.

In this paper we apply this seemingly simple recipe to a family of well known algebraic surfaces, the Dolgachev surfaces, which have been studied by a number of authors [Dol], [FM1], [OV], [Ba1]. To add some spice, there are a number of problems to tackle with. A first one is the chamber structure: Since \( b_2^+ = 1 \), the invariants depend a priori on the chosen metric and it seems hard to give a description of this dependence in general. However, for small Chern classes this dependence sometimes can be controlled. In our case we get a diffeomorphism invariant, if we choose the Chern classes \( c_2 = 2 \) and \( c_1 = K_S + 2nk \). This invariant then is a polynomial

\[
q_S(n) = a(n)Q^2 + b(n)Qk^2 + c(n)k^4
\]

in the intersection form \( Q \) and a primitive class \( k \), which is a rational multiple of the canonical class \( K_S \) of our surface \( S \).

To compute this invariant we resort to algebraic geometric techniques. The appropriate moduli parametrize stable bundles together with sections. Actually we also have to consider a second moduli parametrizing stable sheaves together with maps to the canonical line bundle. These objects lead to projective moduli spaces \( M^{PH}(c_1, c_2) \). The \( \mu \)–classes on \( M^{PH}(c_1, c_2) \) are pulled back from the corresponding \( \mu \)–classes on the moduli \( \mathcal{M}^H(c_1, c_2) \) of stable sheaves along the natural map.

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forgetting the section or cosection. An intersection product of these algebraically
defined classes gives the same result as the differential geometric computation. This
is a consequence of the corresponding statement for ordinary Donaldson polynomi-
als ([Mo], [Li]). The final result of our computation then is:

**Theorem.** For a simply connected Dolgachev surface $S$ with multiple fibres of mul-
tiplicities $p$ and $q$ the coefficients $a(n)$ and $b(n)$ are given by

\[
a(n) = 3n
\]
\[
b(n) = (2p^2q^2 - 2p^2 - 2q^2 - 1)n.
\]

The coefficient $b(n)$ thus carries exactly the same information as does Donald-
son’s $\Gamma$–invariant for Dolgachev surfaces (cf. [Ba1]). It suggests a strong relation-
ship between the Spin–invariants and Donaldson’s original ones. Our methods are
not good enough to get the third coefficient, which we expect to finish the diffeo-
morphism classification of Dolgachev surfaces. As we understand, this classification
was recently obtained by R. Friedman [Fr] by different methods.

Nevertheless the methods developed in this paper are of interest in their own.
The main technical problems in dealing with the moduli spaces are: They have the
wrong dimension and furthermore consist of various components most of which are
singular at every point. To compensate for the wrong dimension we use bundle the-
ory. This replaces counting of intersection points with determining an appropriate
top Chern class. The nonreduced structure is determined by explicitly constructing
infinitesimal extensions and showing that any further extension is obstructed. This
explicitness is necessary in order to understand whether and if so which sections or
cosections extend.

The paper is organized as follows: The next chapter briefly reviews the definition
of the Spin–polynomials and provides machinery for dealing with the singularities in
the moduli spaces. The third chapter then gives the definition of the diffeomorphism
invariant for Dolgachev surfaces. Chapter four does the necessary bundle theory,
which resembles the one developped in [Ba2]. The final chapter then handles the
combinatorics.

II. Spin$^C$ POLYNOMIAL INVARIANTS AND WALL STRUCTURES

IIa Spin$^C$ polynomial invariants.

Let $X$ be a four dimensional Riemannian manifold. An integral lift $C$ of the
second Stiefel-Whitney class $w_2(X)$ defines a Spin$^C$–structure on $X$, which in turn
defines a pair $W^+, W^-$ of rank–2 complex Hermitian vector bundles satisfying:
$c_1(\Lambda^2W^\pm) = C$. A choice of a connection $\nabla$ on the line bundle $\Lambda^2W^+$ gives rise to a
Dirac operator

\[
D^{C,\nabla} : \Gamma(W^+) \to \Gamma(W^-),
\]

which, when coupled with a connection $a$ on a Hermitian rank–2 bundle $E$, gives a
Fredholm operator

\[
D^{C,\nabla}_a : \Gamma(W^+ \otimes E) \to \Gamma(W^- \otimes E).
\]

We will assume its index to be nonpositive,

\[
\text{ind}(D^{C,\nabla}) = \dim \ker(D^{C,\nabla}) - \dim \coker(D^{C,\nabla}) < 0.
\]
The family $\mathcal{D}_a^{C,\nabla}$ of Dirac operators over the space $\mathcal{A}$ of hermitian connections on $E$ can be used to define a subspace of the moduli space of anti self dual (ASD) connections (see also [PT]):

### 2.1 Definition.

Fixing $C$, $\nabla$ and the metric $g$ on $X$, the moduli of 1–instantons is the subspace

$$\mathcal{M}_1^{g,C,\nabla} = \{ [a] \in \mathcal{M}_{ASD} | \dim(\ker \mathcal{D}_a^{C,\nabla}) > 0 \}$$

of the moduli of ASD connections on the adjoint $SO(3)$-bundle, which as usual is identified with the moduli of Hermite-Einstein bundles on $E$.

This moduli space provides invariants of the smooth structure of the manifold $X$ (compare [PT] for an integer and [Ty] for a polynomial invariant). It is convenient to also consider another moduli space, which should be thought of as a resolution of those singularities of $\mathcal{M}_1^{g,C,\nabla}$ which arise from a big kernel of of the Dirac operator. This space has to be used in order to apply general position and bordism arguments to the moduli of 1–instantons (cf. ch.1 of [PT]). The map which assigns to a connection $a$ and a section $\sigma$ of the bundle $(W^+ \otimes E)$ the self dual part of the curvature of $a$ and the evaluation of the Dirac operator on $\sigma$

$$(a, \sigma) \mapsto (F_a^+, \mathcal{D}_a^{C,\nabla}(\sigma))$$

is equivariant with respect to the standard action of the gauge group $G_E$ of the bundle $E$. It therefore gives rise to a section $s$ (see 1.1.26 of [PT]) of a corresponding (Hilbert) bundle over the space

$$\mathcal{P} = \mathcal{A}_\lambda \times_{G_E} S(\Gamma(W^+ \otimes E)).$$

### 2.2 Definition.

The moduli space of pairs $\mathcal{MP}^{g,C,\nabla}$ is the zero set of this section. A point in this moduli of pairs thus consists of a pair of objects: an ASD connection and an element of the projectivisation of the kernel of the coupled Dirac operator. (Note that the central $U(1)$ in the gauge group $G_E$ acts trivially on the first factor in $\mathcal{P}$ and in the standard way on the second.)

### 2.3 Transversality Theorem [PT, ch 2.3].

For a generic metric $g$ on $X$ and a generic connection $\nabla$ on $\Lambda^2 W^+$ the moduli $\mathcal{MP}^{g,C,\nabla}$ of pairs is a smooth manifold of dimension

$$\dim(\mathcal{MP}^{g,C,\nabla}) = 2(4c_2(E) - c_1(E)^2) + 2(\text{ind} \mathcal{D}_a^{C,\nabla} - 1).$$

There is an obvious map $\pi : \mathcal{MP}^{g,C,\nabla} \to \mathcal{M}_1^{g,C,\nabla}$ with fibre $\mathbb{P}(\ker \mathcal{D}_a^{C,\nabla})$ over the point $[a] \in \mathcal{M}_1^{g,C,\nabla}$. The tangent space at a point in the moduli of pairs is conveniently described in the following exact sequence:

$$0 \to T_{a,\sigma} \mathcal{MP}^{g,C,\nabla} \to (\ker \mathcal{D}_a^{C,\nabla})/ <\sigma > \oplus H^1(AdE) \overset{\iota}{\to} \text{coker} \mathcal{D}_a^{C,\nabla}$$

with $i(\sigma',\alpha) = \varpi(\alpha \ast \sigma)$. Here $\ast$ is multiplication of (vector valued) spinors with (matrix valued) 1–forms, followed by spinor multiplication and $\varpi$ is an orthogonal projection to the space of negative harmonic spinors $\text{coker} \mathcal{D}_a$. Finally we denote by $<\sigma>$ the line in a vector space corresponding to a point $\sigma$ of its projectivisation.
This exact sequence globalizes to a fibre of $\pi$ as an exact sequence of bundles over $\mathbb{P}(\ker D_a^{C,\nabla})$:

$$0 \to T\mathcal{M}^{g,C,\nabla}_{|\pi^{-1}_a} \to T\mathbb{P}(\ker D_a^{C,\nabla}) \oplus H^1(AdE) \to \text{coker } D_a^{C,\nabla} \otimes \mathcal{O}_{\mathbb{P}(\ker D_a^{C,\nabla})}(1).$$

A neighborhood of the point $[a]$ in the moduli space of ASD–instantons can be described as a neighborhood of the origin in the zero set of the obstruction map

$$\psi : H^1_a(AdE) \to H^2_a(AdE)$$

in the Kuranishi description of the moduli space (cf. [FU], cor. 4.8). Here $H^i_a(AdE)$ denotes the homology in the Atiyah complex

$$\Omega^0(AdE) \xrightarrow{d_a} \Omega^1(AdE) \xrightarrow{d_a} \Omega^2(AdE)$$

associated to the connection operator $d_a$. There is parallel description for the moduli space of pairs:

**2.4 Lemma.** A neighborhood of the point $([a],\sigma)$ of the space $\mathcal{M}^{g,C,\nabla}$ can be described as a neighborhood of the zero set of the obstruction map:

$$(\ker D_a)/<\sigma> \oplus H^1(AdE) \xrightarrow{\Psi} \text{coker } D_a^{C,\nabla} \oplus H^2(AdE),$$

$$\Psi(\rho, \alpha') = (\varpi(\alpha * \sigma) + O(|\alpha|^2), \psi).$$

This globalizes to a neighborhood of the $\mathbb{P}(\ker D_a) \times o \subset \mathbb{P}(\ker D_a) \times H^1_a(AdE)$ which can be described through the zero set of an obstruction section of the bundle

$$\text{coker } D_a^{C,\nabla} \otimes \mathcal{O}_{\mathbb{P}(\ker D_a^{C,\nabla})}(1) \oplus H^2_a(AdE).$$

**Proof.** By [FrU], lemma 4.7, there is a diffeomorphism $\chi$ of $\Omega^1(AdE)$ such that

$$F^{a+\chi(\alpha)} = d^+_a(\alpha) + \psi(\alpha).$$

The linearisation of this diffeomorphism at zero is the identity map. Furthermore,

$$D_{a+\chi(\alpha)}(\sigma + \rho) = D^+_a(\sigma) + \chi(\alpha) * \sigma.$$

Extend the diffeomorphism $\chi$ to a diffeomorphism of the space $\Omega^1(AdE) \times \Gamma(W^+ \otimes E)$:

$$(\alpha, \sigma') \mapsto (\chi(\alpha), \chi_\alpha(\sigma'))$$

to get the standard Kuranishi description:

$$D_{a+\chi(\alpha)}(\sigma + \chi_\alpha(\sigma')) = \varpi(\chi(\alpha) * \sigma).$$

Now the leading term in the first component of $\Psi$ comes from the leading term of the nonlinear term:

$$\varpi(\chi(\alpha) * \sigma) = \varpi(\alpha * \sigma) + O(|\alpha|^2) * \sigma. \quad \Box$$
The map $\Psi$ is not an honest obstruction map since it has the linear term. However, it is convenient since it is explicit.

This lemma in particular makes it possible to define an orientation of the space $\mathcal{MP}$ at a smooth point having fixed an orientation of the moduli space of instantons. The latter is given by an orientation of $H^1_{\text{ad}}(AdE)$. Using the standard orientation of complex projective space one gets an orientation of $\mathbb{P}(\text{ker} D_a) \times H^1_{\text{ad}}(AdE)$. This induces a natural orientation on our moduli space of pairs, as it is the zero set of a (transversal) section of a complex vector bundle on an oriented manifold. In the special case of a Hodge metric the orientation of the moduli of ASD-connections will be chosen in such a way that it coincides with the one arising from the complex structure on the moduli space of stable bundles.

In what follows we fix the Chern classes $c_i$ of the bundle and sometime omit them from subscripts. Denote by $\mathcal{V}^g_{\Sigma_i}^{C,\nabla}$ the codimension-2 submanifold of the manifold $\mathcal{M}^g_{\Sigma_i}^{C,\nabla}$ dual to $\mu(\Sigma)$. It is a lift of a codim-2 submanifold in the moduli of bundles over the Riemann surface $\Sigma \subset X$ (cf. [Do3]). Let $\mathcal{MP}^g_{\Sigma_i}^{C,\nabla}$ be its preimage in $\mathcal{MP}^{g,C,\nabla}$. The submanifold $\mathcal{V}^g_{\Sigma_i}^{C,\nabla}$ is obtained as the zero set of a certain section $s_{\Sigma_i}$ of the line bundle $\mathcal{L}_{\Sigma_i}$ representing $\mu(\Sigma)$. Using such submanifolds in the moduli of pairs one can apply the yoga of the Donaldson invariants to define the Spin$^C$ polynomial invariants (cf. [Ty]). As usual, genericity assumptions on the metric have to be made. In particular there should be no reducible connection in the moduli space (i.e. $b^+_2(X) \geq 1$). Suppose the moduli of pairs $\mathcal{MP}^{g,C,\nabla}$ is a manifold of dimension $2d$. The Spin polynomial invariant $q_{c_1,c_2}^{C} \in \text{Sym}^d(H^2(x))$ then is defined (cf. [Ty]) by counting oriented intersection points

\begin{equation}
(\ast) \quad q_{c_1,c_2}^{C}(\Sigma_1, \ldots, \Sigma_d) = \#\{\cap \mathcal{V}^g_{\Sigma_i}^{C,\nabla}\}.
\end{equation}

**2.5 Theorem** [Ty]. Suppose $b^+_2(X) > 1$. Then the Spin polynomial invariant $q_{c_1,c_2}^{C}$ is independent of the metric $g$ or the connection $\nabla$. For a diffeomorphism $f : X \to Y$ of 4-manifolds one has

$$f^* q_{Y,c_1,c_2}^{C} = q_{X,f^*c_1,f^*c_2}^{f^*C}.$$ 

The dependence on the Spin$^C$ structure is described in the formula

$$q_{c_1+2\delta,c_2}^{C+2\delta} = q_{c_1+2\delta,c_2+c_1\delta+\delta^2}^{C}.$$ 

In case $b^+_2(X) = 1$ the right hand side of $(\ast)$ actually depends on the metric. For small values of $p_1 = c_1^2 - 4c_2$ this dependence can be controlled. In particular this leads to chamber structures as in [Do1]. A special case will be considered below.

**II.b The Case of a Hodge metric.**

Consider now the case where our favorite 4-manifold $X$ has the structure of an algebraic surface $S \subset \mathbb{P}^N$. In this case the moduli of irreducible ASD connections can be identified with the moduli of stable holomorphic bundles (cf. [Do2, UY, DK]). The natural choice of a Spin$^C$ structure then is the anticanonical class $C = c_1(S) = -K_S$. With this choice the kernel of the coupled Dirac operator is isomorphic to the sum of the zeroth and second cohomology group of the corresponding stable bundle. The cokernel is isomorphic to its first cohomology group.
So the existence of a nontrivial kernel for the Dirac operator coupled with an ASD-connection is equivalent to the nonvanishing of either the zeroth or the second cohomology group of the corresponding stable bundle. This leads to a description of the moduli of pairs as the moduli space $MP^H(c_1,c_2)$ of stable bundles together with a section or a cosection. By a cosection of an algebraic bundle $E$ we mean a section of $E^\vee \otimes K_S$. We will denote by $MP^H_{a,b}(c_1,c_2)$ the stratum of $MP^H(c_1,c_2)$ where $h^0(S;E) = a$ and $h^2(S;E) = b$. It maps to the stratum $M_{a+b}^H(c_1,c_2)$ of the moduli space $M^H(c_1,c_2)$ of stable bundles. For simplicity, we will always assume the first Chern class $c_1$ to be odd and the very ample divisor $H$ (or the Hodge metric $H$) to be suitably generic such that there are no properly slope semistable sheaves (or reducible connections) in the compactified moduli. Also for simplicity assume that $MP_{1,1}$ is empty. Then one can compare the orientations of these components defined in the previous section with their natural orientations as complex spaces. It turns out that the orientation of $MP_{1,0}$ will coincide with its natural complex orientation and the orientation of $MP_{0,1}$ will differ from the natural complex one by $(-1)^{\chi(E)+1}$.

For a description of $MP^H_{1,0}$ in algebraic geometric terms we will make simplifying assumptions: Suppose $c_1$ is odd, so there are no properly slope semistable sheaves and the moduli space $M^H(c_1,c_2)$ of stable sheaves is projective. Assume furthermore the existence of a universal sheaf $E_M$ on $S \times M^H(c_1,c_2)$. Then $MP^H_{1,0}(c_1,c_2)$ and $MP^H_{0,1}(c_1,c_2)$ are defined as projective cones

$$\text{Proj}(\text{Ext}^2_{PM}(E_M, p_S^*\omega_S)) \quad \text{and} \quad \text{Proj}(\text{Ext}^2_{PM}(E_M, p_S^*\omega_S)).$$

(We denote by $P(V)$ the space of lines and by $\text{Proj}(V)$ the space of hyperplanes, which agrees with the Grothendieck definition, cf. [Ha, II.7].) Here we take $\text{Sym}^i(\text{Ext}^2_{PM}(E_M, p_S^*\omega_S))$ as the graded sheaf with $\text{Sym}^0(.) = \mathcal{O}_{\text{supp}(.)}$. The maps $p_S$ and $PM$ denote the projection of the product $S \times M^H(c_1,c_2)$ to the respective factors.

This definition leads to a nice behaviour with respect to base change: For $g : Y \to M^H(c_1,c_2)$ we have

$$g^*\text{Ext}^2_{PM}(E_M, p_S^*\omega_S) \cong \text{Ext}^2_{Py}(E_Y, p_S^*\omega_S)$$

because the fiber $S$ of $PM$ is two dimensional and smooth (cf. [BPS,2.2]).

A Hodge metric and a harmonic connection on $\Lambda^2 W_\pm$ in general are not generic and thus the transversality theorem usually fails to hold. A method to deal with such a situation was developed in [PT, ch 3] in the special case where no $\mu$-classes appeared. But the $\mu$-classes can be included without any difficulties into this method by replacing the section $s$ of the bundle $F$ in [PT, 3.1.17] by the section $s + \sum_i s_i$ of $F \oplus (\oplus_i L_{S_i})$. Applying the same arguments one gets the following analog of [PT, prop. 3.2.4]:

2.6 Proposition. Denote by $\mu^H$ the intersection of $VP_{\Sigma_i}$, $i = 1, \ldots, d$, for the Hodge metric in consideration, i.e. $\mu^H \subset MP_{1,0} \cup MP_{0,1}$. Set $\mu^H_{1,0} = \mu^H \cap MP_{1,0}$. For a generic deformation of the parameters $g$ and $\nabla$ the following formula gives the (0-dimensional) fundamental class of the manifold $\mu^H_{1,0}$ which is the deformation of $\mu^H_{1,0}$:

$$[\mu^H_{g,\nabla}] = [\mu \cap (\text{can}(\mu^H))].$$
Here, \( c_* \) denotes the total Chern class and \( \text{can}(Y) \) the canonical class ([Fu, Ex. 4.2.6]) of a variety \( Y \). An analog statement is true for a deformation of \( \mu_{g,C,S} = \mu_{g,C,S}^0 \cap MP_{0,1} \). Instead of using the stable bundle \( E \) one has to use \( E^* \otimes K_S \). When combining the two contributions, one as of course to take the orientations into account.

In the example of Dolgachev surfaces below the stratum \( \mu_{1,1}^H \) in general will not be empty, but \( \mu_3^H \) will be. (This happens for small \( |n| < pq \) only). That \( \mu_{1,1}^H \setminus \mu_3^H \) actually deforms into the empty set if the orientations do not coincide, follows from the following consideration: The local model of a point of degree \( \mu \) be empty, but can determine \( \mu \) implies that the total Chern class of this virtual bundle is 1. So we need only determine \( \text{can}(\mu^H) \). If \( Z \) and thus the point in \( \mathcal{M}_{1,0}^H \) is reduced, this canonical class is just the Euler characteristic of the projective space \( \mathbb{P}(H^0(\mathcal{E})) \). Otherwise we can analyse the scheme \( \mu^H \) locally over a nonreduced point \( \xi \) in \( \mathcal{M}_{1,0} \), where in general there are imbedded subschemes: Inductively, over subschemes \( \xi_n \) of \( \xi \), we see that the obstruction to extending \( \pi^{-1}(\xi_n) \) to a scheme over \( \xi_{n+1} \) vanishes along a projective subspace of \( \mathbb{P}(H^0(\mathcal{E}_{\xi_n})^\vee) \). The canonical classes add. \( \square \)
II.c Wall structure.

In the case $b_2^+(X) = 1$ reducible ASD-connections will appear in one-parameter families of metrics on $X$ although they are absent in the moduli space for generic metrics. So there will be a singularity in the bordism moduli space defined by a path of metrics if the path meets a metric which allows for reducible ASD-connections (cf. [D1]). Any decomposition $E = M \oplus \left(c_1 - M\right)$ with $c_1^2 - 4M \cdot (c_1 - M) \geq p_1(adE)$ defines a one-codimensional subspace in the space $C$ of all metrics. Sending a metric to a selfdual harmonic form gives a period map $C \to \mathbb{P}(H^2_+)$ to the projectivisation of the cone $H^2_+ \subset H^2(X; \mathbb{R})$ of positive vectors. A decomposition of $E$ as above defines a wall $(c_1 - 2M)^\perp$ in $\mathbb{P}(H^2_+)$. A connected component in the complement of the set of all walls is called a chamber. In the case of moduli space of pairs not every wall is effective in the sense that intersections of $\mu$–classes give different results on both sides of the wall.

Our aim here is to prove that for $p_1 = c_1^2 - 4c_2 > -7$ or $p_1 = -8, w_2 \neq 0$ the Spin–polynomial invariants depend only on the chamber to which the metric maps.

Consider a reducible ASD-connection of type $M \oplus (c_1 - M)$ with $(c_1 - 2M)^2 = p_1(adE), c_1 \cong w_2(adE) \mod 2$. It lies in the moduli space itself, not in its Taubes compactification. If a path of metrics crosses the wall $(c_1 - 2M)^\perp$ then the corresponding instanton bordism $\cup_t M^g_{\text{ASD}}$ will have a singularity at the point given by $M \oplus (c_1 - M)$. A neighborhood of this singularity in the moduli of ASD-connections can be described as a (real) cone over the complex projective space $\mathbb{P}(H^1(Ad(M \oplus (c_1 - M)))$ (without loss of generality we can assume the second cohomology to vanish). In other terms we have a smooth bordism between the moduli of ASD-connections

$$M^g_{\text{ASD}} \quad \text{and} \quad M^g_{\text{ASD}} \cup \mathbb{P}(H^1(Ad(M \oplus (c_1 - M))).$$

The same picture is true for the moduli spaces of $1$–instantons with only a minor modification: Now the $1$–instantons describe a homology class of $\mathbb{P}(H^1(Ad(M \oplus (c_1 - M)))$. If this homology class happens to vanish, then the wall has no effect on the Spin$^C$–invariants. In order that this homology class does not vanish one has to have either $\text{ind} D_{2M} > 0$ or $\text{ind} D_{2(c_1 - M)} > 0$ (cf. [PT,1.4.2]). Recall that the index of the Dirac operator on a manifold $X$ with Spin$^C$–structure $C$ coupled with a connection on a line bundle $L$ is given by the formula

$$\text{ind} D_L = (\{(C + c_1(L))^2 - \text{Sign}(X)\})/8.$$

2.8 Remark. This condition is effective and will be applied lateron to a Dolgachev surface $S$. We will take $c_2 = 2$ and $c_1$ a rational multiple of the canonical class $K_S$. Suppose $M$ is perpendicular to $K_S$. Then one gets

$$\text{ind} D_{2M} = \left(\{(C + 2M)^2 - \text{Sign}(S)\}/8 \right) = 4M^2 - 8 = 0$$

and similarly $\text{ind} D_{2(c_1 - M)} = 0$. So these particular walls have no effect on the Spin$^C$–invariants.

To analyse the link of the singularity in the bordism moduli space one can apply Porteus formula [Fu, ch. 14]. The link comes as the fundamental class of the degeneracy locus

$$D = M^g \cap R(H^1(Ad(M \oplus (c_1 - M))).$$
of a morphism between bundles $E \to F$: Consider the index bundle of the family of coupled Dirac operators over the link of the singularity in the bordism moduli space. This index bundle can be described as an equivariant complex of $S^1$–bundles

$$\ker \mathcal{D}_{(M \oplus (c_1 - M))} \to \text{coker } \mathcal{D}_{(M \oplus (c_1 - M))}$$

(with linear actions of weights $+1$ and $-1$ on the former and the latter) over the $S^1$–space $H^1(\text{Ad}(M \oplus (c_1 - M)))$ (with linear action of weight two). Because of the transversality theorem [PT, 1.3] we may assume:

$$\ker \mathcal{D}_{(c_1 - M)} = \text{coker } \mathcal{D}_M = 0.$$ 

We twist the above complex by a one-dimensional $S^1$–representation of weight 1 to get representations which descend to bundles over $\mathbb{P} H^1(\text{Ad}(M \oplus (c_1 - M)))$. These bundles can be written as a formal difference

$$\left( \text{ind}(\mathcal{D}_{(c_1 - M)}) \otimes \mathcal{O}(-1) \right) - \text{ind}(\mathcal{D}_M).$$

and Porteus formula gives for $k = \text{codim } \mathcal{M}_1^C$

$$[D] = c_k(\text{ind}(\mathcal{D}_{(c_1 - M)}) \otimes \mathcal{O}(-1) - (\text{ind}(\mathcal{D}_M))).$$

This fundamental class is given in terms of indexes and in particular is independent of any metric. To get formulae describing the change in the polynomial invariant, one of course has to evaluate products of $\mu$–classes on this homology class. However, the result again is independent of any metric. It only depends on the wall defined by the reducible ASD-connection.

This method fortunately works in one more case to the effect that the invariants for $p_1(adE) = -8$ are functions depending only on chambers in cohomology: Suppose now the reducible connection $A_r$ on the bundle $M \oplus (c_1 - M)$ to satisfy $M_*(c_1 - M) = p_1(E) + 4$. This means it lies in the top stratum of the compactification of the moduli space. First we describe the link of this singularity in the compactification.

The Spin$^C(4)$–structure on our manifold determines a principal $U(2)$-bundle $P^+$ corresponding to $W^+$. Denote by $P_{M \oplus (c_1 - M)}$ the principal bundle corresponding to our reducible bundle. Reducibility provides an isomorphism

$$P_{M \oplus (c_1 - M)} = U(2) \times_{T^2} (P_M \times P_{(c_1 - M)}),$$

where $P_M, P_{(c_1 - M)}$ are the principal bundles corresponding to $M$ and $c_1 - M$, respectively. Let $Gl$ be the subspace of all hermitian isomorphisms in the bundle of homomorphisms $\text{Hom}(M \oplus (c_1 - M), W^+)$ (recall both arguments are equipped with hermitian metrics). The space $Gl$ is a principal $T^2$–bundle

$$((P^+ \times P_{M \oplus (c_1 - M)})/U(2) = (P^+ \times (P_M \times P_{(c_1 - M)}))/(T^2)$$

over the space $P^+/S^1 \times S^1 = \mathbb{P}(W^+)$. Here $T^2 \subset U(2)$ be the center, considered as holonomy.
centralizer of a 1-instanton over $S^4$. It also acts on the space $Gl$. The "diagonal" subgroup $S^1_\Delta \subset T^2 \times S^1$, defined by

$$e^{i\phi} \mapsto \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}, \quad \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$$

acts trivially on $Gl$ and in fact we have an action of

$$T^2 = T^2 \times S^1/S^1.$$ 

We will use the space $Gl \times \mathbb{R}_{\geq 0}$ (where $\mathbb{R}_{\geq 0}$ parametrises the concentration of a glued instanton on the 4-sphere) as the space of parameters needed to describe the neighborhood of ideal connections in the Taubes compactification (cf. [Do3, ch. 5]). Since in the Taubes compactification all glueing parameters disappear for ideal connections, one has to consider the space $Gl \times \mathbb{R}_{\geq 0}/Gl \times 0$. The neighborhood of an ideal connection of type $(A_r, x)$ for $x \in X$ in the compactified moduli space is described by

$$(Gl \times \mathbb{R}_{\geq 0}/Gl \times 0) \times H^1(ad(A_r))/T^2$$

(cf. Th 4.53 in [loc. cit.]).

2.9 Lemma. The space

$$(Gl \times \mathbb{R}_{\geq 0}/Gl \times 0) \times H^1(ad(A_r))/T^2$$

is a fibration $\nabla$ over $X$ with fiber a real cone over an algebraic subvariety $V$ of $\mathbb{P}(S^2(E^*)) \otimes H^1(ad(A_r)))$. The variety $V$ is the image under the Segre embedding

$$\mathbb{P}(S^2(E^*)) \times \mathbb{P}(H^1(ad(A_r)) \rightarrow \mathbb{P}(S^2(E^*) \otimes H^1(ad(A_r))$$

of the subspace $R \times \mathbb{P}(H^1(ad(A_r))$ where $R$ is a rational norm curve in $\mathbb{P}(S^2(E^*))$.

Proof. First note that the diagonal $S^1 \subset T^2$ acts trivially on $H^1(ad(A_r))$, so we can factor it out to get

$$((Gl/S^1) \times \mathbb{R}_{\geq 0}/(Gl/S^1) \times 0) \times H^1(ad(A_r))/S^1.$$ 

The space $Gl/S^1$ is isomorphic to the principal $S^1$-bundle associated to the line bundle $T_p\mathbb{P}(W^+) \otimes p^*\mathcal{L}^{-1}$. Indeed, the principal $S^1$-bundle $P^+/S^1$ is associated with the line bundle $T_p\mathbb{P}(W^+)$ on $\mathbb{P}(W^+)$ and the principal $S^1$-bundle $(P_M \times P_{(c_1-M)})/(S^1)$ is associated with a line bundle $\mathcal{L}$ with Chern class $2M - c_1$ on $X$. Therefore

$$((P^+ \times (P_M \times P_{(c_1-M)}))/(T^2))/S^1 = ((P^+/S^1) \times (P_M \times P_{(c_1-M)})/S^1)/(S^1)$$

and the statement follows from the definition of tensor product of line bundles. Now one can identify the space

$$((Gl/S^1) \times \mathbb{R}_{\geq 0}/(Gl/S^1) \times 0)$$

in the following way. Take the obvious $S^1$-equivariant map

$$f : ((Gl/S^1) \times \mathbb{R}_{\geq 0} \rightarrow T_p\mathbb{P}(W^+) \otimes p^*\mathcal{L}^{-1}.$$
It projects \((Gl/S^1) \times 0\) to \(Gl/T^2 = \mathbb{P}(W^+)\). We now construct a \(S^1\)-equivariant map of total spaces
\[ v : T_p \mathbb{P}(W^+) \otimes p^* \mathcal{L}^{-1} \to S^2(E^*) \otimes \det E \otimes p^* \mathcal{L}^{-1} \]
as the twist of the obvious map of total spaces \(\mathcal{O}_{\mathbb{P}(E)}(2) \to S^2(E^*)\) with the line bundle \(\det E \otimes p^* \mathcal{L}^{-1}\). It is easy to see that the composite \(S^1\)-equivariant map \(vf\) projects \((Gl/S^1) \times 0\) to the zero-section of \(S^2(E^*) \otimes \det E \otimes p^* M^{-2}\).

To include the vector space \(H^1(ad(A_r))\) into the picture consider the composite \(vf \times id_{H^1(ad(A_r))}:\)
\[(Gl/S^1) \times \mathbb{R}_{\geq 0} \times H^1(ad(A_r)) \to S^2(E^*) \otimes \det E \otimes p^* \mathcal{L}^{-1} \times H^1(ad(A_r))\]
which leads to the Segre imbedding
\[S^2(E^*) \otimes \det E \otimes p^* \mathcal{L}^{-1} \times H^1(ad(A_r)) \to S^2(E^*) \otimes \det E \otimes p^* \mathcal{L}^{-1} \otimes H^1(ad(A_r)).\]

\[\square\]

2.10 Corollary. The singularity in the bordism of instanton spaces \(\cup_t \mathcal{M}_{1^t}^q\) given by a reducible connection of the considered type is the space \(\nabla\).

2.11 Lemma. Suppose the characteristic classes of the underlying bundle \(E\) satisfy \(p_1(Ad E) = c_1(E)^2 - 4c_2(E) > -7\) or \(p_1(Ad E) = -8, w_2(Ad E) \neq 0\). Then the \(\mathrm{Spin}^C\)-invariants depend only on chambers, not on metrics.

Proof. We have determined the links of the singularities in the bordism moduli spaces. Now consider a path of metrics such that the periods of the metrics cross a wall. The change in the invariants then is computed by evaluating certain cohomology classes on these links which do not depend on the metric in which we cross the wall. This gives a unique value for the \(\mathrm{Spin}\)-polynomial for any metric with a given period. \[\square\]

A diffeomorphism invariant for Dolgachev surfaces

We want to evaluate some \(\mathrm{Spin}^C\)-invariant in the case of simply connected Dolgachev surfaces, i.e. of relatively minimal elliptic surfaces with geometric genus \(p_g = 0\). As was shown in the original paper of Dolgachev [Dol], there are exactly two multiple fibers of coprime multiplicity \(p\) and \(q\). Here we allow \(p\) or \(q\) to be 1, noting that in this case the surface is rational: A canonical divisor is linearly equivalent to \(F - F_p - F_q\), where \(F\) and \(F_p, F_q\) denote a generic and the respective reduced multiple fibers. The ray \(\mathbb{Q}[F] \subset H^2(S; \mathbb{Q})\) of rational multiples of the Poincaré dual \([F]\) of \(F\) is generated by a primitive element \(k = 1/pq[F] = a[F_p] + b[F_q]\) with integers satisfying \(aq + bp = 1\).

Let \(q_{\mathbb{Q}}(n)\) denote the \(\mathrm{Spin}^C\)-invariant \(q_{\mathbb{K}; s + 2nk; 2}^H\), evaluated on a chamber in \(H \in H^2_+(S; \mathbb{R})\) containing \(k\) in its closure. Recall that the invariant is a function on the set of chambers, i.e. the components of
\[H^2_+(S; \mathbb{R}) \setminus \cup_I e^1, \quad I = \{e \in H^2(S; \mathbb{Z}) \mid -8 \leq (c_1 - 2e)^2 \leq -1\}.

In the present case we can, however, say more:

In the present case we can, however, say more:
3.1 Theorem. The polynomial \( q_S(n) \in \text{Sym}^4(H^2(S)) \) is a diffeomorphism invariant of Dolgachev surfaces, i.e. if \( F : S \to S' \) is a diffeomorphism of Dolgachev surfaces, then
\[
f^*q_{S'}(n)(x_1, \ldots, x_4) = \pm q_S(n)(f_*(x_1), \ldots, f_*(x_4)).
\]
Furthermore, \( q_S(n) \) is a polynomial of the form
\[
q_S(n) = a(n)Q^2 + b(n)Qk^2 + c(n)k^4
\]
in the intersection form \( Q \in \text{Sym}^2(H^2(S)) \) and in \( k \).

Proof. The ray \( Q[F] \) is invariant under diffeomorphisms for nonrational Dolgachev surfaces, i.e. \( f^*Q[F] = Q[F'] \) for a diffeomorphism between rational Dolgachev surfaces ([FM1, thm. 6A]). So the set of chambers containing \( k \) in its closure is mapped to itself. It remains to show that the invariant takes the same value on neighboring such chambers. To see this note that \( k^\perp \subset H^2(S; \mathbb{Z}) \) carries an even form. So the respective chambers are separated by walls \( e^\perp \) with \( e^2 = -2 \). The result now follows from the fact (cf. 2.8) that chambers separated by walls perpendicular to \((-2)\)-classes \( e \in k^\perp \) have the same invariant. Thus \( q_S(n) \) is an element in \( \text{Sym}^4(H^2(S)) \) which is invariant under the diffeomorphism group of \( S \). It is clear that \( \pm k \) is an invariant linear form on \( H_2(S) \). So it suffices to determine the invariant polynomials on \( \ker k \). Now \( \ker k \) is isomorphic to the lattice \( -\tilde{E}_8 \) with radical generated by \( 1/pqF \). By [FM1, III.2] the transvection \( T_y(x) = x + (x.y)F \) can be realized by a diffeomorphism of \( S \) for \( x, y \in -\tilde{E}_8 \). From this it follows that invariant polynomial functions on \( -\tilde{E}_8 \) are pulled back from invariant polynomial functions on \( -\tilde{E}_8/\text{rad} = -E_8 \). These are well known to be generated by invariant polynomials in degree 2 (corresponding to \( Q \)), 8, 12, 14, 18, 20, 24, 30. The theorem thus is established for nonrational surface. In the case of a rational minimally elliptic surface one has to use the fact that the diffeomorphism group acts transitively on the set
\[
\{ \kappa \in H^2(S; \mathbb{Z}) | \kappa^2 = 0, \kappa \text{ indivisible}, \kappa^\perp \text{ carries an even form} \}.
\]
(cf. [FM1, II.2.4] and [Wa].) \( \Box \)

Bundle theory, multiplicities

IV.a Description of the bundles.

In this section the bundles relevant for the computation of the spin polynomial invariant will be described. The discussion is completely analogous to the one in [Ba2, ch. II], so we will be brief in our presentation.

Let \( S \) denote a relatively minimal simply connected Dolgachev surface. The surface \( S \) is projective and we will assume it to be suitably generic. It will suffice that \( S \) be nodal in the sense of [FM2, ch.2]: The multiple fibres have smooth reduction and all singular nonmultiple fibres are irreducible rational curves with a single ordinary double point. The attribute \( \text{vertical} \) will be used for any divisor linearly equivalent to a rational multiple of a generic fiber. Such a vertical divisor is linearly equivalent to \( lF + mF_p + nF_q \) for uniquely determined integers \( l, m \) and \( n \) with \( 0 \leq m \leq n \) and \( 0 \leq p \leq q \).
4.1 Lemma. The vertical line bundle $\mathcal{O}_S(lF + mF_p + nF_q) = \mathcal{L}$ has cohomology groups of the following dimensions:

$$h^0(S, \mathcal{L}) = \max\{l + 1, 0\}, \quad h^2(S, \mathcal{L}) = \max\{-l, 0\},$$

$$h^1(S, \mathcal{L}) = \max\{l, -1 - l, 0\}.$$ 

Proof. By definition, $h^0(S, \mathcal{L}) = h^0(\mathbb{P}^1, \pi_*\mathcal{L}) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(l))$. For $h^2$ use Serre duality, for $h^1$ Riemann Roch. □

Let $H$ be an ample divisor "near $K_S$" (i.e. assume $H - 2(H \cdot F)F$ is ample, too). We will be interested in those $H$–stable bundles $\mathcal{E}$ with Chern classes $c_1 = D$ a vertical divisor and $c_2 = 2$ which admit a section or a cosection. These bundles represent the closed points in the subscheme $\mathcal{M}_1^H(D, 2)$ of the moduli space $\mathcal{M}^H(D, 2)$. The choice of ample divisor implies that $\mathcal{E} \in \mathcal{M}_1^H(D, 2)$ arises as an extension

$$(*) \quad 0 \to \mathcal{O}_S(C) \to \mathcal{E} \to J_Z(D - C) \to 0$$

with $Z \subset S$ a 2–cluster, i.e. a zero dimensional subscheme of length 2, and $C$ an effective vertical divisor. In particular the direct image sheaf $\pi_*(\mathcal{E})$ is locally free on $\mathbb{P}^1$ of rank either 1 or 2. The bundle will be called of type 1 or 2, correspondingly. These bundles have been described in [Ba2, prop 3.4 and 3.6]:

4.2 Proposition (Properties of type 1 bundles). Let $\mathcal{E} \in \mathcal{M}_1^H(D, 2)$ be a type 1 stable bundle. Then the divisor $C$ and the subscheme $Z$ are uniquely determined. On the other hand, for a given vertical divisor $C$ there exists a semistable bundle as above if and only if $2C - D$ is linearly equivalent to $-F + \alpha F_p + \beta F_q$ with nonnegative integers $\alpha$ and $\beta$ such that $\alpha/p + \beta/q < 1$. In particular these bundles are parameterized by a Zariski–open subset of Hilb$^2(S)$, which has the expected dimension 4. The Zariski tangent space of the moduli space $\mathcal{M}^H(D, 2)$ at $\mathcal{E}$ has dimension $5 + h^2(S; \mathcal{O}_S(D - 2C))$. □

4.3 Proposition (Properties of type 2 bundles). Let $\mathcal{E} \in \mathcal{M}^H(D, 2)$ be a type 2 stable bundle. Then it arises as an extension

$$(**) \quad 0 \to \mathcal{O}_S(C_1) \oplus \mathcal{O}_S(C_2) \to \mathcal{E} \to J_{Z \subset I}(D - C_1) \to 0.$$ 

Here $I$ denotes the minimal vertical divisor containing $Z$ and $J_{Z \subset I}$ denotes the corresponding ideal sheaf. The divisor $I$ and the line bundles $\mathcal{O}_S(C_i)$ are uniquely determined by $\mathcal{E}$. Furthermore, if the divisor $I$ contains a nonmultiple fiber $F$, then $h^0(\mathcal{O}_{F \cap Z}) \geq 2$. The bundles $(**)$ admit at most 4–dimensional moduli. □

Remarks. 1. The subspace $\mathcal{M}_1^H(D, 2)$ of the moduli space $\mathcal{M}^H(D, 2)$ consists of several components in general. The type–1 components are indexed by the pairs $(\alpha, \beta)$ in the proposition. A listing of the different type–2 components can be obtained by considering which possible $Z \subset I$ may occur. For a generic bundle in these components $I$ is a generic fibre and the possible $\mathcal{O}_S(C_i)$ are characterized by the conditions $H.C_1 < H.(C_2 + F) < H.(C_2 + 2F)$ and $h^0(\mathcal{O}_S(C_1 - C_2)) = h^0(\mathcal{O}_S(C_2 - C_1)) = 0$. These conditions are equivalent with: $C_1 = C_2 - F + \alpha F_p + \beta F_q$ with $0 < \alpha, \beta$ and $\alpha + \beta p < \alpha q$. It turns out that the other strata of
type–2 bundles are indeed contained as subspaces in at least one of the components mentioned above.

It should also be pointed out that the singular locus of $\mathcal{M}^H(D,2)$ contains only type–1 and type–2 bundles. Moreover, the components of $\mathcal{M}^H(D,2)$ as Weil divisors in $\mathcal{M}^H(D,2)$ are not reduced in general. These singular structures will be explored lateron.

**IV.b Some families of stable sheaves.**

We will evaluate the invariant $q_S(n)$ at four classes $x_1, \ldots, x_4 \in H_2(S;\mathbb{Z})$ with $x_4 = F$. So it suffices to construct the divisor $\mu(F) \subset \mathcal{M}^H(D,2)$ and the restriction of a universal sheaf to $S \times \mu(F)$. For this we use the following observation in [FM2, ch 3.9]: the divisor $\mu(F)$ geometrically is represented by the locus

$$\mu(F) = \{[\mathcal{E}] \in \mathcal{M}^H_1(D,c) \mid h^0(\mathcal{E}_F \otimes \theta_F) \neq 0\}$$

with $\theta_F \in \text{Pic}^0(F) \setminus \{0\}$. In the present case, where the bundles are all given as extensions

$$0 \rightarrow \mathcal{O}_S(C) \rightarrow \mathcal{E} \rightarrow \mathcal{J}_{ZCD} \rightarrow 0,$$

this means $\mu(F) = \{[\mathcal{E}] \mid Z \cap F \neq \emptyset\}$. A component $C$ of $\mu(F)$ then has to be counted with multiplicity $\min_C h^1(\mathcal{E}_F \otimes \theta_F)$, so if $C$ generically parametrizes type–$n$ bundles, it has to be counted with multiplicity $n$.

Here is the construction of a family parametrizing stable bundles which generically are of the form

$$0 \rightarrow \mathcal{O}_S \oplus \mathcal{O}_S(-F + \alpha F_p + \beta F_q) \rightarrow \mathcal{E} \rightarrow \mathcal{J}_{ZCF} \rightarrow 0$$

for $0 < \alpha, \beta$ with $pq > \alpha q + \beta p$. Let $T$ denote the tautological divisor in $F \times \text{Hilb}^2(F)$. Then (compare [Ba2, 3.7]) there is a unique nontrivial extension, which indeed is locally free:

$$0 \rightarrow \mathcal{O}_{S \times \text{Hilb}^2(F)} \rightarrow \tilde{\mathcal{E}} \rightarrow \mathcal{J}_T(\beta F_q) \rightarrow 0.$$

Let $\tau : \mathcal{M}_F \rightarrow \text{Hilb}^2(F)$ be the $\mathbb{P}^1$–bundle $\text{Proj}(pr_2^*(\tilde{\mathcal{E}}|_{\alpha F_p \times \text{Hilb}^2(F)}))$. The family of type–2 bundles then is described as follows (cf. [Ba2, 3.8]):

**4.4 Proposition.** Assume by symmetry $\alpha q < \beta p$. Then the kernel $\mathcal{U}$ of the natural surjection

$$(\text{id} \times \tau)^* \tilde{\mathcal{E}}(\alpha F_p,0) \rightarrow \mathcal{O}_{\alpha F_p}(\alpha F_p) \boxtimes \mathcal{O}_{\mathcal{M}_F}(1)$$

is a bundle parametrizing stable type 2 bundles. The generic member of this family has a presentation

$$0 \rightarrow \mathcal{O}_S \oplus \mathcal{O}_S(-F + \alpha F_p + \beta F_q) \rightarrow \mathcal{E} \rightarrow \mathcal{J}_{ZCF} \rightarrow 0.$$

The induced map of $\mathcal{M}_F$ to the moduli space $\mathcal{M}^H(\alpha F_p + \beta F_q,2)$ is injective on geometric points. The Chern classes of $\mathcal{U}$ are

$$c_1(\mathcal{U}) = (\alpha F_p + \beta F_q) \times \mathcal{M}_F - S \times \tau^* K_{\text{Hilb}^2(F)}$$

$$c_2(\mathcal{U}) = \tau^* T + \alpha F_p \times c_1(\mathcal{O}_{\mathcal{M}}(1)).$$

Note that this construction was not symmetric in $p$ and $q$. This is reflected in the computation (cf. [Ba2, 3.10]) of products of the classes

$$\mu(A) = A\setminus (c_2(\mathcal{U}) - \frac{1}{4} c_1^2(\mathcal{U})).$$
4.5 Corollary. Let $\mathcal{U}$ be the bundle in (4.4). Then for an effective divisor $A$ in $S$ we get

$$\mu(A)^3 = 3\phi_2(\alpha, \beta)(A.F)(A.k)^2,$$

where $k \in H^2(S; \mathbb{Z})$ denotes the primitive class $\frac{1}{pq}F$ and $\phi_2(\alpha, \beta) = \alpha(q - \beta)pq$. □

The family $\mathcal{E}_X$ of type–1 stable sheaves to be constructed will be over the blow up $X$ of $S \times F$ along the diagonally imbedded curve $F \subset F \times F \subset S \times F$ with exceptional divisor $E$. The sheaf will at a generic point $x$ parametrize stable bundles of type 1 of the form

$$0 \to \mathcal{O}_S(D) \to \mathcal{E}_x \to \mathcal{J}_Z \to 0$$

with $D = -F + \alpha F_p + \beta F_q$ and $Z \in \text{Hilb}^2(S)$, the image of $x$ under the canonical map $X \to \text{Hilb}^2(S)$.

Set $d = pq - \alpha q - \beta p$ and denote by $\delta_p$ the integer

$$\delta_p = \left\lfloor \frac{d}{2q} \right\rfloor - 1.$$

In [Ba2, 3.16] a subscheme $\mathcal{R}(\mathcal{L}_p)$ of $pF_p \times pF_p$ was constructed containing the diagonal subscheme $\Delta_{2\delta_p} \subset 2\delta_p F_p \times 2\delta_p F_p$. Let $\mathcal{J}_\Delta|\mathcal{R}(\mathcal{L}_p)$ denote the corresponding ideal sheaf. The main properties of this subscheme are:

4.6 Lemma. The space $\mathcal{R}(\mathcal{L}_p)$ is flat and finite of degree $\delta_p(\delta_p + 1)$ over $F_p \times F_p$ and

$$\text{Ext}^*_\mathcal{O}_S(\mathcal{J}_\Delta|\mathcal{R}(\mathcal{L}_p) \otimes \mathcal{O}_F, \mathcal{O}_S \times X) =
\begin{cases}
\mathcal{O}_{\mathcal{R}(\mathcal{L}_p) \times F}(-D, pr_1^*(D + (2\delta_p + 1)F_p)) & \text{for } * = 2 \\
0 & \text{else.} \quad \square
\end{cases}$$

This subscheme can be used to construct a geometric representativ of the second Chern class of $\mathcal{E}_X$ (cf. [Ba2, 3.18]):

4.7 Proposition. There is a unique nontrivial extension

$$0 \to \mathcal{O}_{S \times X}(D, L) \to \mathcal{E}_X \to \mathcal{J}_\Gamma \to 0$$

giving a flat $X$–family of stable sheaves. The subscheme $\Gamma \subset S \times X$ is the union of $(\mathcal{R}(\mathcal{L}_p) \cup \mathcal{R}(\mathcal{L}_q)) \times F$ with the graphs $\gamma(X)$ and $\rho(X)$ of the maps $\gamma, \rho : X \to S$ factoring through the projection maps $pr_i : S \times F \to S$ ($i = 1, 2$). The divisor $L$ is given by $-L = E + \gamma^*(D + 2\delta_p F_p + 2\delta_q F_q - K_S)$. □

4.8 Corollary. Let $\mathcal{E}_X$ be the sheaf in (4.7). Then for an effective divisor $A$ in $S$ we get $\mu(A)^3 = 3(A.F)(A.A) + 6\phi_1(A.F)(A.k)^2$ with

$$\phi_1 = \delta_p q(\delta_p q + q - d) + \delta_q p(\delta_q p + p - d) + \frac{d^2}{4} + \frac{d}{2}(pq - p - q).$$

Proof. The divisor $\mu(A)$ on $X$ is the pullback along the map $X \to \text{Hilb}^2(S)$ of the divisor

$$\mathfrak{G} = \mathfrak{A} + \delta_p(\delta_p + 1)(A.F_p)\mathfrak{F}_p + \delta_q(\delta_q + 1)(A.F_q)\mathfrak{F}_q + \frac{1}{4}(A.D)(\mathfrak{D} + 2\delta_p\mathfrak{F}_p + 2\delta_q\mathfrak{F}_q - \mathfrak{K}_S + \mathfrak{T})$$

$$= \mathfrak{A} + \mathfrak{F} + \mathfrak{T}.$$
The fraktur-letters $\fraktur F$ and $\fraktur A$ denote the divisors in $\text{Pic} \text{Hilb}^2(S) \cong \text{Pic}(S) \oplus \frac{1}{2}\mathbb{Z}\fraktur F$ corresponding to divisors $F$ and $A$ in $S$. The divisor $\fraktur X$ is the exceptional locus of the Chow map $\text{Hilb}^2(S) \to S^2(S)$. So we get $\mu(A)^3 = \fraktur F^3, \fraktur X = 3(A.A)(A.F) + 6x(A.F)^2 - 24y^2(A.F)$. The latter formula is obtained by applying the following multiplication rules for divisors in $\text{Hilb}^2(S)$: \hfill $\square$

4.9 Lemma. Let $S$ be an algebraic surface and $A, B, C, D$ divisors on $S$. Then on $\text{Hilb}^2(S)$ one has the following intersection products:

1. $\fraktur A \cdot \fraktur B \cdot \fraktur C \cdot \fraktur D = (A.B)(C.D) + (A.C)(B.D) + (A.D)(B.C)$
2. $\fraktur A \cdot \fraktur B \cdot \fraktur X = 0$
3. $\fraktur A \cdot \fraktur B \cdot \fraktur X^2 = -8(A.B)$
4. $\fraktur A \cdot \fraktur X^3 = 8(A.K_S)$
5. $\fraktur X^4 = -8(K_S^2 + c_2(S))$.

Proof. Without loss of generality, we may assume the divisors in $S$ very ample and in general position. The first two formulas then are obtained via the Schubfach principle. The last three follow from the explicit description of the twofold cover of $\text{Hilb}^2(S)$ as the blow up of $S \times S$ along the diagonal. \hfill $\square$

IV.c Infinitesimal extensions of bundles and sections.

The subspace $\mathcal{M}_1^H(D, 2) \subset \mathcal{M}^H(D, 2)$ is not reduced in general. Let $\mathcal{M}(\alpha, \beta)$ be an irreducible component parametrizing type-1 bundles of the form

$0 \to \mathcal{O}_S(C) \to \mathcal{E} \to \mathcal{J}_Z(D - C) \to 0$

with $2C - D = -F + \alpha F_p + \beta F_q$.

4.10 Proposition. The generic point of $\mathcal{M}(\alpha, \beta)$ has length 1 if $\alpha = \beta = 0$, length 4 if $\alpha \beta \neq 0$ and length 2 else.

Proof. Again the proof goes along the lines of [Ba2, ch. 5], hence we will be brief. The bundle is obtained from the unique nonsplit extension

$0 \to \mathcal{O}_S(C + 2F) \to \overline{\mathcal{E}} \to \mathcal{O}_S(D - C) \to 0$

by elementary transformation along a sheaf $\mathcal{L}_I$ supported on two disjoint generic fibers $F$ and $F'$ which restrict to line bundles of degree one on either component. Note that the restriction of $\overline{\mathcal{E}}$ to a generic fiber is nonsplit! (For simplicity we henceforth assume $D - C = 0$. This can be achieved by tensoring with a line bundle.) The extension group $\text{Ext}^1(\overline{\mathcal{E}}, \mathcal{E})$ is zero if $\alpha$ and $\beta$ both are, is 2-dimensional, if neither of $\alpha$ nor $\beta$ vanishes and one dimensional else. This follows from Riemann–Roch and the computation of $\dim \text{Hom}(\overline{\mathcal{E}}, \mathcal{E}) = 3$ and $\dim \text{Hom}(\overline{\mathcal{E}}, \mathcal{E}(K_S)) \in \{1, 2, 3\}$. The inclusion $j : \mathcal{E} \to \overline{\mathcal{E}}$ induces maps $j^* : \text{Ext}^1(\overline{\mathcal{E}}, \mathcal{E}) \to \text{Ext}^1(\mathcal{E}, \overline{\mathcal{E}})$ and $j_* : \text{Ext}^1(\mathcal{E}, \mathcal{E}) \to \text{Ext}^1(\mathcal{E}, \overline{\mathcal{E}})$. Consider the commuting diagram:

\[
\begin{array}{ccc}
\cdots & \longrightarrow & \text{Ext}^1(\mathcal{E}, \mathcal{E}) \\
\downarrow & & \downarrow j^* \\
\text{Hom}(\mathcal{E}, \mathcal{E}) & \longrightarrow & \text{Ext}^1(\mathcal{E}, \overline{\mathcal{E}})
\end{array}
\]
Now \( j^* \) is injective and the image of \( T \) is the 4-dimensional tangent space to the reduction \( \mathcal{M}(\alpha, \beta)_{\text{red}} \). From (4.2) we know the dimension of \( \text{Ext}^1(\mathcal{E}, \mathcal{E}) \) to be 5 (iff \( \alpha \beta = 0 \)) or 6. In particular, \( \text{im} j^* = \text{im} j_\ast \) if and only if \( \alpha + \beta = 0 \). It remains to consider two cases:

**Suppose** \( \alpha + \beta \neq 0 \). Then every nonsplit extension in \( \text{Ext}^1(\mathcal{E}, \mathcal{E}) \) lifts to a nontrivial deformation of \( \mathcal{E} \) over the double point \( Z_2 = \text{Spec} \mathbb{C}[t]/(t^2) \) which are not tangent to \( \mathcal{M}(\alpha, \beta)_{\text{red}} \). We can write down explicit representatives for \( \text{Ext}^1(\mathcal{E}, \mathcal{E}) \) the following way: For \( \alpha \neq 0 \) let \( \mathcal{F} \) denote the nonsplit extension

\[
0 \to \mathcal{O}(F + \beta F_q) \to \mathcal{F} \to \mathcal{O} \to 0.
\]

On the bundle \( \mathcal{F} \oplus \mathcal{F}(\alpha F_p) \) define an endomorphism \( \varphi \) with \( \varphi^2 = 0 \) by

\[
\varphi = \begin{pmatrix} \tau & 0 \\ \text{id} \otimes \sigma & -\tau' \end{pmatrix}.
\]

Here \( \sigma \) is a nontrivial section of \( \mathcal{O}(\alpha F_p) \) and \( \tau : \mathcal{F} \to \mathcal{O}(2F + \beta F_q) \subset \mathcal{F} \) is a nontrivial map chosen in such a way that the support of its cokernel is disjoint from \( F_p \). Finally, \( \tau' \in \text{End} \mathcal{F}(\alpha F_p) \) satisfies \( \tau' \circ (\text{id} \otimes \sigma) = (\text{id} \otimes \sigma) \circ \tau \). It is easy to check the isomorphisms \( \text{coker} \varphi \cong \mathcal{F} \cong \mathcal{E} \). In particular, the sheaf \( \mathcal{F} \oplus \mathcal{F}(\alpha F_p) \), together with the endomorphism \( \varphi \) defines an infinitesimal deformation \( \mathcal{E}_{Z_2} \) of \( \mathcal{E} \) over the double point. The infinitesimal deformation \( \mathcal{E}_{Z_2} \) then is the kernel of a map of \( \mathcal{E}_{Z_2} \) to an infinitesimal deformation of \( \mathcal{L}_I \). Now suppose \( \mathcal{E}_{Z_3} \) extends \( \mathcal{E}_{Z_2} \) over the triple point \( \text{Spec} \mathbb{C}[t]/(t^3) \). Then as in the proof of [Ba2, 5.1] it follows \( h^0(\mathcal{E}_{Z_3}|_{\mathcal{p}F_p}) = 2 \). The induced map \( Z_3 \to \mathcal{M}^H(D, 2) \) thus cannot factor through \( \mathcal{M}_1^H(D, 2) \).

**Suppose** \( \alpha + \beta = 0 \). Then consider the bundle \( \mathcal{A}_n \) over \( S \) which inductively is obtained as the unique nonsplit extension

\[
0 \to \mathcal{O}_S(nF) \to \mathcal{A}_{n+1} \to \mathcal{A}_n \to 0
\]

with \( \mathcal{A}_1 = \mathcal{O}_S \). The restriction of \( \mathcal{A}_n \) to each fibre is indecomposable. Let \( t : \mathcal{A}_4 \to \mathcal{A}_1 \) be a map of rank 2 factoring through multiplication \( \mathcal{A}_2 \to \mathcal{A}_2(2F) \) with a section of \( \mathcal{O}_S(2F) \) vanishing at two generic fibres \( F \) and \( F' \). Let \( x \) and \( y \) be reduced points on \( F \) and \( F' \). Then there is a commuting diagram of surjections

\[
\begin{array}{ccc}
\mathcal{A}_4 & \xrightarrow{\varphi} & \mathcal{G} = \left( \mathcal{H}_F^2 \cap \mathcal{O}_F(x) \right) \oplus \left( \mathcal{H}_F^2 \cap \mathcal{O}_F'(y) \right) \\
\downarrow & & \downarrow \text{id} \oplus \text{id} \\
\mathcal{A}_2 & \xrightarrow{\varphi} & \mathcal{O}_F(x) \oplus \mathcal{O}_F'(y)
\end{array}
\]

of \( \mathcal{O}_S[t]/(t^2) \)-modules (with \( t : \mathcal{G} \to \mathcal{G} \) the zero map). Here \( \mathcal{H}_F^2 \) denotes the unique rank 2 indecomposable bundle on \( F \) with trivial determinant and \( (A \dashv B) \) denotes a (unique) indecomposable bundle, together with a short exact sequence

\[
0 \to A \to (A \dashv B) \to B \to 0.
\]

The kernel restricts to \( \ker \varphi|_{\mathcal{F} \cup \mathcal{F}'} = \mathcal{O}_F(-x) \oplus \mathcal{O}_F'(-y) \oplus \mathcal{G} \). The point now is that the map \( \varphi : \mathcal{H}_F^2 \cap \mathcal{F} \to \mathcal{O}_F(x) \oplus \mathcal{O}_F'(y) \) extends to a map \( \mathcal{G} \to \mathcal{O}_F(x) \oplus \mathcal{O}_F'(y) \). Then let \( \mathcal{E}_{Z_2} \) be the kernel of the induced map \( \ker \varphi \to \mathcal{O}_F(x) \oplus \mathcal{O}_F'(y) \). It is easy to verify that this sheaf defines a map \( Z_2 \to \mathcal{M}^H(D, 2) \) which does not factor through the subscheme of 1-instantons. \( \square \)

Using this explicit description of infinitesimal deformations of bundles it is easy to find which sections extend to these deformations.
4.12 Corollary. Let $\xi \in \mathcal{M}(\alpha, \beta)$ denote the generic point and $E_\xi$ the pull back of a universal bundle to $S \times \xi$. Then $\text{length} \text{Ext}^2_{\mathcal{O}_S}(E_\xi, p^*_S \omega_S)$ is given by the sum $\sum_{\gamma \in \Delta} h^0(S; \mathcal{O}_S(X))$ over the set $\Delta = \{C, (C - \alpha F_p), (C - \beta F_q), (C - \alpha F_p - \beta F_q)\}$. □

Now let $\mathcal{N}(\alpha, \beta)$ denote an irreducible component in $\mathcal{M}^1(D, 2)$ parametrizing type–2 bundles generically of the form

$$0 \to \mathcal{O}_S(C) \oplus \mathcal{O}_S(C - F + \alpha F_p + \beta F_p) \to E \to J_{Z_{CF}} \to 0$$

with $0 < \alpha, \beta$ and $\alpha q + \beta p < pq$. One easily computes $\text{Ext}^2(E, E) = 0$ and therefore $\mathcal{M}^H(D, 2)$ is generically smooth along $\mathcal{N}(\alpha, \beta)$.

4.13 Proposition. The generic point of $\mathcal{N}(\alpha, \beta)$ has length 1.

Proof. It suffices to construct an infinitesimal deformation $\mathcal{E}_{Z_2}$ such that no section of $E$ extends (compare the proof of 4.11). Let $t : \mathcal{A}_2 \to \mathcal{A}_2$ be a rank–1 map factoring through multiplication with a section of $\mathcal{O}_S(F)$ vanishing at $F$ and set $J^\gamma_{Z_{CF}} = \text{Hom}(J_{Z_{CF}}, \mathcal{O}_F)$. Then there is a commuting diagram

$$\begin{array}{ccc}
\mathcal{A}_2 \oplus \mathcal{A}_2 & \xrightarrow{\varphi} & (\mathcal{H}^2_F \oplus \mathcal{H}^2_F) \oplus J^\gamma_{Z_{CF}} \\
\downarrow & & \downarrow \\
\mathcal{A}_1 \oplus \mathcal{A}_1 & \xrightarrow{\varphi} & J^\gamma_{Z_{CF}}
\end{array}$$

of $\mathcal{O}_S[t]/t^2$–modules. The kernel restricts to $\ker \varphi|_F = J_{Z_{CF}} \oplus J^\gamma_{Z_{CF}}$ and the map $\varphi : \mathcal{H}^2_F \oplus \mathcal{H}^2_F \to J^\gamma_{Z_{CF}}$ extends to a map $(\mathcal{H}^2_F \oplus \mathcal{H}^2_F) \oplus J^\gamma_{Z_{CF}} \to J^\gamma_{Z_{CF}}$. The sheaf $\mathcal{E}_{Z_2}$ then is the kernel of the induced map $\ker \varphi \to J^\gamma_{Z_{CF}}$. □

4.14 Corollary. If $\xi \in \mathcal{N}(\alpha, \beta)$ denotes the generic point and $E_\xi$ the pull back of a universal bundle to $S \times \xi$. Then $\text{length} \text{Ext}^2_{\mathcal{O}_S}(E_\xi, p^*_S \omega_S)$ is $h^0(S; \mathcal{O}_S(C) \oplus \mathcal{O}_S(C + \alpha F_p + \beta F_p))$. □

V. Combinatorics

We finally combine the information of the various strata to compute the first two coefficients in the Spin$^C$–invariant for Dolgachev surfaces. The result is:

5.1 Theorem. The first two coefficients in the polynomial $q_S(n)$ are $a(n) = 3n$ and $b(n) = (2p^2 q^2 - 2p^2 - 2q^2 - 1)n$.

Remarks. - Here we use the symmetrization convention which gives

$$(Q^2 + Qk^2 + k^4)(A, A, A, A) = (A.A)^2 + (A.A)(A.k)^2 + (A.k)^4,$$

which actually differs from the one used in [Ba2].
- In the case $p = q = 1$ one can actually determine the third coefficient to be $c(n) = 21n$. Our methods, however, do not allow to compute the third coefficient in general. Such a computation probably would give the diffeomorphism classification of Dolgachev surfaces, which Friedman ([Fr]) recently obtained by other methods.
means. Regarding the problem of diffeomorphism classification, the above coefficients contain exactly the same information as the $\Gamma$–invariant of Donaldson (cf. [Ba1]).

The above convention gives $Q^2(A, A, A, F) = (A.A)(A.F)$ and $Qk^2(A, A, A, F) = \frac{1}{2}(A.F)(A.k)^2$. To prove the theorem, we first gather the contributions of the different strata into a closed formula. We will assume $p$ to be odd. The map

$$(\alpha, \beta) \mapsto \begin{cases} \left( \frac{\alpha-1}{2}, \frac{\beta-1}{2} \right) & \text{if } \alpha \text{ is odd} \\ \left( \frac{p-\alpha-1}{2}, \frac{q-\beta-1}{2} \right) & \text{if } \alpha \text{ is even} \end{cases}$$

defines a bijection of

$$\square = \{ (\sigma, \tau) \in \mathbb{Z}^2 \mid 0 \leq \sigma \leq \frac{p-1}{2}, 0 \leq \tau \leq \frac{q-1}{2} \}$$

with the set

$$\{ (\alpha, \beta) \mid (\alpha - 1)q + (\beta - 1)p \text{ even}, \frac{\alpha}{p} + \frac{\beta}{q} < 1 \}$$

indexing the type–1 components $M(\alpha, \beta)$. Similarly, the map

$$(\alpha, \beta) \mapsto \begin{cases} \left( \frac{\alpha-1}{2}, \frac{q-\beta-1}{2} \right) & \text{if } \alpha \text{ is odd} \\ \left( \frac{p-\alpha-1}{2}, \frac{\beta-1}{2} \right) & \text{if } \alpha \text{ is even} \end{cases}$$

defines a bijection of $\square$ with the indexing set

$$\{ (\alpha, \beta) \mid \alpha q + \beta p \text{ odd}, \frac{\alpha}{p} + \frac{\beta}{q} < 1 \}$$

of type–2 components $N(\alpha, \beta)$. (Note that for $\alpha = 0$ or $\beta = 0$ there is no component $N(\alpha, \beta)$. But then the formula (4.5) also vanishes.)

Next come multiplicities. These multiplicities merely count the number of sections or cosections of a generic bundle parametrized in $M(\alpha, \beta)$ or $N(\alpha, \beta)$. We fix a number $n = lpq + Aq + Bp$ with our usual convention $0 \leq \frac{A}{p}, \frac{B}{q} < 1$.

5.2 Lemma. The multiplicity of the component $M(\alpha, \beta)$ or $N(\alpha, \beta)$ is given by

$$m(\sigma, \tau, n) = (l + 1 + H_p(\sigma, A) + H_q(\tau, B)) c(\sigma, \frac{p-1}{2}) c(\tau, \frac{q-1}{2})$$

with $H_p(\sigma, A) = \begin{cases} \frac{1}{2} & \text{if } \sigma \geq p - A \\ -\frac{1}{2} & \text{if } \sigma \geq A \\ 0 & \text{else} \end{cases}$

and $c(\sigma, \frac{p-1}{2}) = \begin{cases} 1 & \text{if } \sigma = \frac{p-1}{2} \\ 2 & \text{else.} \end{cases}$

Proof. Let $E$ be a generic bundle in $M(\alpha, \beta)$. Then because of (3.14) one has to compute

$$\sum h^0(E(X)) - h^2(E(X))$$
with \( I = \{0, -\alpha F_p, -\beta F_q, -\alpha F_p - \beta F_q\} \). One verifies the claim via checking the different cases: whether or not \( 2A \leq p \), whether or not \( 2B \leq q \) and whether or not \( \alpha \) is odd. For example, if \( 2A \leq p \), \( 2B \leq q \) and \( \alpha \) odd, one has a presentation

\[
0 \to \mathcal{O}(lF+(A+\sigma)F_p+(B+\tau)F_q) \to \mathcal{E} \to \mathcal{J}_Z(lF+(A-\sigma)F_p+(B-\tau)F_q+K_S) \to 0
\]

and the claim follows by applying (3.1).

For the components \( \mathcal{N}(\alpha, \beta) \) the divisor \( \mu(F) \) already has multiplicity 2 (compare IV.b) which has to be multiplied with \( h^{0}(\mathcal{E}) - h^{2}(\mathcal{E}) \). Again one applies (3.1) to \( \mathcal{E} \), which in the example above has a presentation

\[
0 \to \mathcal{O}_S(lF+(A+\sigma)F_p+(B-\tau-1)F_q) \oplus \mathcal{O}_S(lF+(A-\sigma-1)F_p+(B+\tau)F_q) \to \mathcal{E} \to \mathcal{J}_{\mathbb{Z}F} \to 0. \quad \Box
\]

The correction term \( s_q = s_q(\tau) = d(\sigma, \tau) - \left[ \frac{d(\sigma, \tau)}{2q} \right] 2q \) takes odd values in the interval \( 0 \leq s_q(\tau) < 2q \). Using this correction term we get the following expression for the function \( \phi_1 \):

\[
\phi_1 = \frac{1}{4}(2dpq - d^2) - \frac{1}{4}(2qs_q - s_q^2) - \frac{1}{4}(2ps_p - s_p^2)
\]

where \( d \) as a function of \( \sigma \) and \( \tau \) is given by

\[
d = \begin{cases} 
    pq - (2\sigma + 1)q - (2\tau + 1)p & \text{if } 2\sigma q + 2\tau p < pq - p - q \\
    -pq + (2\sigma + 1)q - (2\tau + 1)p & \text{else}
\end{cases}
\]

The contribution of type–2 bundles is given by

\[
\phi_2 = \begin{cases} 
    (2\sigma + 1)(2\tau + 1)pq & \text{if } 2\sigma q + 2\tau p < pq - p - q \\
    (p - 2\sigma - 1)(q - 2\tau - 1)pq & \text{else}
\end{cases}
\]

To get decent formulae, define the functions

\[
T_p = T_p(\sigma) = (2\sigma + 1)(2p - 2\sigma - 1) \\
S_p = S_p(\sigma) = 2ps_p - s_p^2 \\
R = R_p(\sigma, \tau) = \max(pq - p - q - 2\sigma q - 2\tau p, 0)
\]

and their analogs \( T_q \) and \( S_q \) by interchanging the rôles of \( (p, \sigma) \) and \( (q, \tau) \). Then we can sum up everything to get the

**5.3 Lemma.** The coefficients \( a(n) \) and \( b(n) \) are given by the formulae

\[
a(n) = \sum \Box m(\sigma, \tau, n) \\
b(n) = \sum \Box m(\sigma, \tau, n) \Phi(\sigma, \tau)
\]

with \( \Phi = \Phi(\sigma, \tau) = 12\phi_1 + 6\phi_2 = 3(q^2T_p - S_p) - 3(p^2T_q + S_q) + 6pqR - 3p^2q^2 \).

The theorem now follows from lemmata (5.4)–(5.7) below.
5.4 Lemma. \[ \sum m(\sigma, \tau, n) = n \]

The proof is left to the reader. \(\square\)

5.5 Lemma. \[ 3 \sum \Box mS_q = n(2q^2 + 1) + \begin{cases} -Bp(2q^2 + 1) + 3p \sum_{0 \leq \tau < B} S_q & \text{if } 2B \leq q \\ (q - B)p(2q^2 + 1) + 3p \sum_{0 \leq \tau < q - B} S_q & \text{else} \end{cases} \]

5.6 Lemma. \[ 3 \sum \Box m(p^2T_q - S_q) = n(2q^2 + 1)(p^2 - 1) + X_q \]

with \(X_q = \begin{cases} -Bp(2q^2 + 1)(p^2 - 1) + 3p \sum_{0 \leq \tau < B} (p^2T_q - S_q) & \text{if } 2B \leq q \\ (q - B)p(2q^2 + 1)(p^2 - 1) + 3p \sum_{0 \leq \tau < q - B} (p^2T_q - S_q) & \text{else} \end{cases} \)

Proof of 5.5. For \(0 \leq \tau < \frac{q}{2}\) each term \(S_q(\tau)\) is a product \(x(2q - x)\) with \(0 < x \leq q\) an odd integer and each such value is obtained exactly once. So we get

\[6 \sum_{0 \leq \tau < \left[\frac{q}{2}\right]} S_q = \begin{cases} q(q - 1)(2q - 1) & \text{if } q \text{ is odd} \\ q(2q^2 + 1) & \text{else} \end{cases}\]

In particular the "\((l + 1)cc"-summand adds up to

\[3 \sum \Box (l + 1)c(\sigma, \frac{p - 1}{2})c(\tau, \frac{q - 1}{2})S_q = (l + 1)pq(2q^2 + 1).\]

Similarly one gets the two other components

\[6 \sum \Box H_p(\sigma, A)c(\sigma, \frac{p - 1}{2})c(\tau, \frac{q - 1}{2})S_q = (2A - p)q(2q^2 + 1)\]

\[6 \sum \Box H_q(\tau, B)c(\sigma, \frac{p - 1}{2})c(\tau, \frac{q - 1}{2})S_q = \begin{cases} -pq(2q^2 + 1) + 6p \sum_{0 \leq \tau < B} S_q & \text{if } 2B \leq q \\ pq(2q^2 + 1) - 6p \sum_{0 \leq \tau < q - B} S_q & \text{else} \end{cases} \]

Proof of 5.6. The values \(S_q(\tau)\) are merely permutations of the values of \(T_q(\tau)\) for \(0 \leq \tau < \left[\frac{q}{2}\right]\) and \(S_q(\frac{q - 1}{2}) = T_q(\frac{q - 1}{2})\). So the claim is immediate from (3.5). \(\square\)
5.7 Lemma. \[ 6pq \sum R = n(p^2 - 1)(q^2 - 1) - X - X_q. \]

Proof. For \(2\sigma q + 2\tau p < pq - p - q\) the product \(c(\sigma, \frac{p-1}{2})c(\tau, \frac{q-1}{2}) = 4\) is constant on the index set. So we get

\[ \sum R = \sum 4(l + 1)R + \Xi_A + \Xi_B \]

with \(\Xi_B = \begin{cases} 
\sum (-2R) + \sum_{0 \leq B \leq \sigma < \frac{q}{2}} 2R & \text{if } 2B \leq q \\
\sum 2R - \sum_{0 \leq \sigma < q-B \leq \sigma < \frac{q}{2}} (-2R) & \text{else} 
\end{cases} \)

and an analog expression for \(\Xi_A\). Setting \(z(\tau) = \left[ \frac{pq - p - q - 2\tau p}{2q} \right]\) we get

\[ \sum_{0 \leq \tau < y} \sum_{0 \leq \tau < \frac{q}{2}} R = \sum_{0 \leq \tau < y} \sum_{0 \leq \sigma < z(\tau)} (pq - p - q - 2\sigma q - 2\tau p) \]
\[ = \sum_{0 \leq \tau < B} (z(\tau) + 1)(pq - p - q - 2\tau p - qz(\tau)) \]
\[ = \sum_{0 \leq \tau < B} \frac{1}{4p} ((q^2(p^2 - 1) - (p^2T_q - S_q)) \].

Using again (5.5), we may replace \(y\) by the different choices we have and get

\[ 6pq \sum 4(l + 1)R = (l + 1)(p^2 - 1)(q^2 - 1), \]

\[ \Xi_B = \begin{cases} 
(-\frac{1}{12}(q^2 - 1) + \frac{q}{4}B)(p^2 - 1) - \sum_{0 \leq \tau < B} \frac{1}{4q}(p^2T_q - S_q) & \text{if } 2B \leq q \\
(\frac{1}{12}(q^2 - 1) - \frac{q}{4}(q - B))(p^2 - 1) + \sum_{0 \leq \tau < q - B} \frac{1}{4q}(p^2T_q - S_q) & \text{else} 
\end{cases} \]

The result follows. \(\square\)

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