ON VANISHING THEOREMS FOR VECTOR BUNDLE VALUED
p-FORMS AND THEIR APPLICATIONS

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Abstract. Let $F : [0, \infty) \to [0, \infty)$ be a strictly increasing $C^2$ function with $F(0) = 0$. We unify the concepts of $F$-harmonic maps, minimal hypersurfaces, maximal spacelike hypersurfaces, and Yang-Mills Fields, and introduce $F$-Yang-Mills fields, $F$-degree, $F$-lower degree, and generalized Yang-Mills-Born-Infeld fields (with the plus sign or with the minus sign) on manifolds. When $F(t) = t, \frac{1}{2}t^2, \sqrt{1 + 2t} - 1, \text{ and } 1 - \sqrt{1 + 2t}$, the $F$-Yang-Mills field becomes an ordinary Yang-Mills field, $p$-Yang-Mills field, a generalized Yang-Mills-Born-Infeld field with the plus sign, and a generalized Yang-Mills-Born-Infeld field with the minus sign on a manifold respectively. We also introduce the $E_{F,g}$-energy functional (resp. $F$-Yang-Mills functional) and derive the first variational formula of the $E_{F,g}$-energy functional (resp. $F$-Yang-Mills functional) with applications. In a more general frame, we use a unified method to study the stress-energy tensors that arise from calculating the rate of change of various functionals when the metric of the domain or base manifold is changed. These stress-energy tensors are naturally linked to $F$-conservation laws and yield monotonicity formulae, via the coarea formula and comparison theorems in Riemannian geometry. Whereas a “microscopic” approach to some of these monotonicity formulae leads to celebrated blow-up techniques and regularity theory in geometric measure theory, a “macroscopic” version of these monotonicity inequalities enables us to derive some Liouville type results and vanishing theorems for $p$-forms with values in vector bundles, and to investigate constant Dirichlet boundary value problems for 1-forms. In particular, we obtain Liouville theorems for $F$–harmonic maps (which include harmonic maps, $p$-harmonic maps, exponentially harmonic maps, minimal graphs and maximal space-like hypersurfaces, etc), $F$–Yang-Mills fields, extended Born-Infeld fields, and generalized Yang-Mills-Born-Infeld fields (with the plus sign and with the minus sign) on manifolds etc. As another consequence, we obtain the unique constant solution of the constant Dirichlet boundary value problems on starlike domains for vector bundle-valued 1-forms satisfying an $F$-conservation law, generalizing and refining the work of Karcher and Wood on harmonic maps. We also obtain generalized Chern type results for constant mean curvature type equations for $p$–forms on $\mathbb{R}^m$ and on manifolds $M$ with the global doubling property by a different approach. The case $p = 0$ and $M = \mathbb{R}^m$ is due to Chern.

1. Introduction

A theorem due to Garber, Ruijsenaars, Seiler and Burns [GRSB] states that every harmonic map $u : \mathbb{R}^m \to S^m$ with finite energy must be constant($m > 2$).

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This result has been generalized by Hildebrandt [Hi] and Sealey [Se1] to harmonic maps into arbitrary Riemannian manifolds from more general domains, for example from a hyperbolic $m$-space form, or from $\mathbb{R}^m$ with certain globally conformal flat metrics, where $m > 2$. In the context of harmonic maps, the stress-energy tensor was introduced and studied in detail by Baird and Eells [BE]. Following Baird-Eells [BE], Sealey [Se2] introduced the stress-energy tensor for vector bundle valued $p$-forms and established some vanishing theorems for $L^2$ harmonic $p$-forms. Liouville type theorems for vector bundle valued harmonic forms or for forms satisfying certain conservation laws have been treated by [KW] and [Xi1]. These follow immediately from monotonicity formulae. A similar technique was also used by [EF1] and [EF2] to show nonexistence of $L^2$-eigenforms of the Laplacian (on functions and differential forms) on certain complete noncompact manifolds of nonnegative sectional curvature.

On the other hand, in [Ar], M. Ara introduced the $F$-harmonic map and its associated stress-energy tensor. Let $F: [0, \infty) \rightarrow [0, \infty)$ be a $C^2$ function such that $F' > 0$ on $[0, \infty)$, and $F(0) = 0$. A smooth map $u: M \rightarrow N$ between two Riemannian manifolds is said to be an $F$-harmonic map if it is a critical point of the following $F$-energy functional $E_F$ given by

\[ E_F(u) = \int_M F \left( \frac{|du|^2}{2} \right) dv \]

with respect to any compactly supported variation, where $|du|$ is the Hilbert-Schmidt norm of the differential $du$ of $u$, and $dv$ is the volume element of $M$. When $F(t) = \frac{1}{p}(2t)^\frac{2}{p}$, $(1 + 2t)^\alpha$ ($\alpha > 1$, $\dim M = 2$), and $e^t$, the $F$-harmonic map becomes a harmonic map, a $p$-harmonic map, an $\alpha$-harmonic map, and an exponentially harmonic map respectively. One of these striking features is that we can use, for example $p$-harmonic maps to study topics or problems that do not seem to be approachable by ordinary harmonic maps (in which $p = 2$) (see e.g. [We2,3, LWe]).

In addition to the above examples, $F$-energy functionals and their critical points arise widely in geometry and physics. Recall that a minimal hypersurface in $\mathbb{R}^{m+1}$, given as the graph of the function $u$ on a Euclidean domain satisfies the following differential equation and is a solution of Plateau’s problem (for any closed $m - 1$-dimensional submanifold in the minimal graph as a given boundary):

\[ \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \]  

If a maximal spacelike hypersurface in Minkowski space $\mathbb{R}^{n,1}$ (with the coordinate $(t, x^1, \ldots, x^n)$ and the metric $ds^2 = dt^2 - \sum_{i=1}^n (dx^i)^2$) is given as the graph of the function $v$ on a Euclidean domain, then the function $v$ satisfies

\[ \text{div} \left( \frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} \right) = 0. \]

Obviously the solutions $u$ and $v$ are $F$-harmonic maps from a domain in $\mathbb{R}^m$ to $\mathbb{R}$ with $F = \sqrt{1 + 2t} - 1$ and $F = 1 - \sqrt{1 - 2t}$ respectively, with respect to any compactly supported variation. In [Ca], Calabi showed that equations (1.2) and (1.3) are equivalent over any simply connected domain in $\mathbb{R}^2$. Along the lines of
Calabi, Yang [Ya] showed that, for \( m = 3 \), equations (1.2) and (1.3) over a simply connected domain are, respectively, equivalent instead to the vector equations

\[
\nabla \times \left( \frac{\nabla \times A}{\sqrt{1 + |\nabla \times A|^2}} \right) = 0
\]

(1.4)

(\( A \) is a vector field in \( \mathbb{R}^3 \) and \( \nabla \times (\cdot) \) is the curl of \((\cdot)\) which arise in the nonlinear electromagnetic theory of Born and Infeld [BI]. This observation leads Yang [Ya] to give a generalized treatment of equations of (1.2) and (1.3) expressed in terms of differential forms as follows:

\[
\delta \left( \frac{d\omega}{\sqrt{1 + |d\omega|^2}} \right) = 0, \quad \omega \in A^p(\mathbb{R}^m)
\]

(1.5)

and

\[
\delta \left( \frac{d\sigma}{\sqrt{1 - |d\sigma|^2}} \right) = 0, \quad \sigma \in A^q(\mathbb{R}^m)
\]

(1.6)

(where \( d \) is the exterior differential operator and \( \delta \) is the codifferential operator), and a reformulation of Calabi’s equivalence theorem in arbitrary \( n \) dimensions. Born-Infeld theory is of contemporary interest due to its relevance in string theory ([BN], [DG], [Ke], [LY], [Ya], [SiSiYa]). It is easy to verify that the solutions of (1.5) and (1.6) are critical points of the following Born-Infeld type energy functionals

\[
E_{BI}^+(\omega) = \int_{\mathbb{R}^m} \sqrt{1 + |d\omega|^2} - 1 \ dv
\]

(1.7)

and

\[
E_{BI}^-(\sigma) = \int_{\mathbb{R}^m} 1 - \sqrt{1 - |d\sigma|^2} \ dv
\]

(1.8)

respectively. By choosing a sequence of cutoff functions and integrating by parts, Sibner-Sibner-Yang [SiSiYa] established a Liouville theorem for the \( L^2 \) exterior derivative \( d\omega \) of a solution \( \omega \) of (1.5). They also introduced Yang-Mills-Born-Infeld fields and obtained a Liouville type result for finite-energy solutions of a generalized self-dual equation reduced from the Yang-Mills-Born-Infeld equation on \( \mathbb{R}^4 \).

In this paper, we unify the concepts of \( F \)-harmonic maps, minimal hypersurfaces in Euclidean space, maximal spacelike hypersurfaces in Minkowski space, and Yang-Mills Fields, and introduce \( F \)-Yang-Mills fields, \( F \)-degree, \( F \)-lower degree, and generalized Yang-Mills-Born-Infeld fields (with the plus sign or with the minus sign) on manifolds (cf. Definitions 3.2, 4.1, 6.1 and 8.1). When \( F(t) = t, \frac{1}{p}(2t)^\frac{p}{2}, \sqrt{1 + 2t} - 1, \) and \( 1 - \sqrt{1 - 2t} \), the \( F \)-Yang-Mills field becomes an ordinary Yang-Mills field, a \( p \)-Yang-Mills field, a generalized Yang-Mills-Born-Infeld field with the plus sign, and a generalized Yang-Mills-Born-Infeld field with the minus sign on a manifold respectively. We also introduce the \( E_{F,g} \)-energy functional (resp. \( F \)-Yang-Mills functional) and derive the first variational formula of the \( E_{F,g} \)-energy functional (resp. \( F \)-Yang-Mills functional) (Lemmas 2.5 and 3.1) with applications.

In a more general frame, we use a unified method to study the stress-energy tensors that arise from calculating the rate of change of various functionals when the metric of the domain or base manifold is changed. These stress-energy tensors lead to a fundamental integral formula (2.10), are naturally linked to \( F \)-conservation laws. For example, we prove that every \( F \)-Yang-Mills field satisfies an \( F \)-conservation
law. In particular, every $p-$Yang-Mills field satisfies a $p$-conservation law (cf. Theorem 3.1 and Corollary 3.1). As an immediate consequence, the simplified integral formula (2.11), from (2.10) holds for vector bundle valued forms satisfying an $F-$conservation law in general, and holds for $F-$Yang-Mills field in particular. This yields monotonicity inequalities, via the coarea formula and comparison theorems in Riemannian geometry (cf. Theorem 4.1 and Proposition 4.1). Whereas a “microscopic” approach to monotonicity formulae leads to celebrated blow-up techniques due to E. de-Giorgi [Gi] and W.L. Fleming [Fl], and regularity theory in geometric measure theory(cf. [FF,A,SU,PS,HL,Lu]). For example, the regularity results of Allard [A] depend on the monotonicity formulae for varifolds. The regularity results of Schoen and Uhlenbeck [SU] depend on the monotonicity formulae for harmonic maps which they derived for energy minimizing maps; monotonicity properties are also dealt with by Price and Simon [PS] for Yang-Mills fields, and by Hardt-Lin [HL] and Luckhaus [Lu] for $p$-harmonic maps. A “macroscopic” version of these monotonicity formulae enable us to derive some Liouville type results and vanishing theorems under suitable growth conditions on Cartan-Hadamard manifolds or manifolds which possess a pole with appropriate curvature assumptions (e.g. Theorems 5.1 and 5.2). In particular, our results are applicable to $F-$harmonic maps, $F-$Yang-Mills fields, extended Born-Infeld fields, and generalized Yang-Mills-Born-Infeld fields (with the plus sign or with the minus sign) on manifolds, and obtain the first vanishing theorem for $p-$Yang-Mills fields (cf. Theorems 5.3-5.8). In fact, we introduce the following $E_{F,g}-$energy functional

$$E_{F,g}(\sigma) = \int_M F\left(\frac{\lvert d\nabla \sigma \rvert^2}{2}\right)d\nu_g$$

for forms $\sigma \in A^{p-1}(\xi)$ with values in a Riemannian vector bundle $\xi$, or study an even more general functional $\mathcal{E}_{F,g}(\omega)$ for forms $\omega \in A^p(\xi)$ (see (2.5)), introduced by Lu-Shen-Cai [LSC]. Naturally, the stress-energy tensor associated with $E_{F,g}(\sigma)$ or $\mathcal{E}_{F,g}(\omega)$ plays an important role in establishing Liouville type results for extremals of $E_{F,g}$ or forms satisfying an $F-$conservation law.

Our growth assumptions in Liouville type theorems in the general settings (cf. (5.1), (5.4), Theorems 5.1 and 5.2) are weaker than the assumption of finite energy for harmonic maps due to Garber, Ruijsenaars, Seiler and Burns [GRSB], Sealey [Se1], and others, or finite $F$-energy for $F$-harmonic maps due to M. Kassi [Ka], or $L^p$ growth for vector bundle valued forms due to J.C. Liu [Li1], or the slowly divergent $F$-energy condition(e.g. (5.3)) for harmonic maps and Yang-Mills fields that was first introduced by H.S. Hu in [Hu1,2], for $F$-harmonic maps due to Liao and Liu [LL2], and for an extremal of $\mathcal{E}_{F,g}$-energy functional treated by M. Lu, W.W. Shen and K.R. Cai [LSC](see Theorem 10.1, Examples 10.1 and 10.2 in Appendix).

Furthermore, our estimates in the monotonicity formulae are sharp in the sense that in special cases, they recapture the monotonicity formulae of harmonic maps [SU] and Yang-Mills field [PS] (cf. Corollary 4.1. and Remark 4.2).

In addition to establishing vanishing theorems and Liouville type results, the monotonicity formulae may be used to investigate the constant Dirichlet boundary-value problem as well. We obtain the unique constant solution of the constant Dirichlet boundary value problem on starlike domains for vector bundle-valued 1-forms satisfying an $F$-conservation law (cf. Theorem 6.1), generalizing and refining the work of Karcher and Wood on harmonic maps [KW]. Notice that our
constant boundary-value result holds for any starlike domain, while the original result in [KW] was stated for a disc domain. For an extended Born-Infeld field \( \omega \in A^p(\mathbb{R}^m) \) with the plus sign, we give an upper bound of the Born-Infeld type energy \( E_{BI}^+(\omega; G(\rho)) \) of the p-form \( \omega \) over its "graph" \( G(\rho) \) in \( \mathbb{R}^{m+k} \) (cf. Proposition 7.1). This recaptures the volume estimate for the minimal graph of \( f \) due to P. Li and J.P. Wang, when \( \omega = f \in A^1(\mathbb{R}^m) = C^\infty(\mathbb{R}^m) \) (cf. [LW]).

As further applications, we obtain vanishing theorems for extended Born-Infeld fields (with the plus sign or with the minus sign) on manifolds under an appropriate growth condition on \( E_{BI}^+ \)-energy, and for generalized Yang-Mills-Born-Infeld fields (with the plus sign or with the minus sign) on manifolds under an appropriate growth condition on \( \mathcal{Y} \mathcal{M}^2_{BI} \)-energy. (cf. Theorems 7.1, 8.1, and 8.2, Propositions 7.2 and 8.1). The case \( M = \mathbb{R}^m \) and \( d\omega \in L^2 \), where \( \omega \) is a Born-Infeld field (hence \( \omega \) has finite \( E_{BI}^+ \)-energy, by the inequality \( \sqrt{1 + t^2} - 1 \leq \frac{t^2}{2} \) for any \( t \in \mathbb{R} \)) is due to L. Sibner, R. Sibner and Y.S. Yang(cf. [SiSiYa]).

Being motivated by the work in [Wei1,2] and [LWW], we consider constant mean curvature type equations for \( p \)-forms on \( \mathbb{R}^m \) and thereby obtain generalized Chern type results for constant mean curvature type equations for \( p \)-forms on \( \mathbb{R}^m \) and on manifolds with the global doubling property by a different approach(cf. Theorems 9.1-9.4). The case \( p = 0 \) and \( M = \mathbb{R}^m \) is due to Chern (cf. Corollary 9.1).

This paper is organized as follows. Generalized \( F \)-energy functionals and \( F \)-conservation laws are given in section 2. In section 3, we introduce \( F \)-Yang-Mills fields. In section 4, we derive monotonicity formulae. Liouville type results and vanishing theorems are established in three subsections 5.1-5.3 of section 5. In section 6, we treat constant Dirichlet Boundary-Value Problems for vector valued 1-forms. Extended Born-Infeld fields and exact forms are presented in section 7. In section 8, we introduce generalized Yang-Mills-Born-Infeld fields (with the plus sign and with the minus sign) on manifolds. Generalized Chern type results on manifolds are investigated in sections 9. In the last section, we provide an appendix of a theorem on \( \mathcal{E}_{F,g} \)-energy growth.

Throughout this paper let \( F : [0, \infty) \to [0, \infty) \) be a strictly increasing \( C^2 \) function with \( F(0) = 0 \), and let \( M \) denote a smooth \( m \)-dimensional Riemannian manifold (mostly \( m > 2 \)); all data will be assumed smooth for simplicity unless otherwise indicated.

### 2. Generalized \( F \)-Energy Functionals and \( F \)-Conservation Laws

Let \( (M, g) \) be a smooth Riemannian manifold. Let \( \xi : E \to M \) be a smooth Riemannian vector bundle over \( (M, g) \), i.e. a vector bundle such that at each fiber is equipped with a positive inner product \( \langle \ , \ \rangle_E \). Set \( A^p(\xi) = \Gamma(A^p T^* M \otimes E) \) the space of smooth \( p \)-forms on \( M \) with values in the vector bundle \( \xi : E \to M \). The exterior differential operator \( d^E : A^p(\xi) \to A^{p+1}(\xi) \) relative to the connection \( \nabla^E \) is given by

\[
d^E \sigma (X_1, \ldots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \nabla^E_{X_i} (\sigma(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})) + \sum_{i<j} (-1)^{i+j} \sigma([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1})
\] (2.1)
where the symbols covered by ~ are omitted. Since the Levi-Civita connection on TM is torsion-free, we also have

\[(d^\omega \sigma)(X_{1}, ..., X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} (\nabla_{X_{i}} \sigma)(X_{1}, ..., \hat{X}_{i}, ..., X_{p+1})\]

For two forms \(\omega, \omega' \in A^{p}(\xi)\), the induced inner product is defined as follows:

\[
\langle \omega, \omega' \rangle = \sum_{i_{1} < \cdots < i_{p}} \langle \omega(e_{i_{1}}, ..., e_{i_{p}}), \omega'(e_{i_{1}}, ..., e_{i_{p}}) \rangle_{E}
\]

\[
= \frac{1}{p!} \sum_{i_{1}, ..., i_{p}} \langle \omega(e_{i_{1}}, ..., e_{i_{p}}), \omega'(e_{i_{1}}, ..., e_{i_{p}}) \rangle_{E}
\]

where \(\{e_{1}, \cdots e_{m}\}\) is a local orthonormal frame field on \((M, g)\). Relative to the Riemannian structures of \(E\) and \(TM\), the codifferential operator \(\delta^{\nabla} : A^{p}(\xi) \rightarrow A^{p-1}(\xi)\) is characterized as the adjoint of \(d\) via the formula

\[
\int_{E} (d^\omega \sigma, \rho) dv_{g} = \int_{E} (\sigma, \delta^{\nabla} \rho) dv_{g}
\]

where \(\sigma \in A^{p-1}(\xi), \rho \in A^{p}(\xi)\), one of which has compact support, and \(dv_{g}\) is the volume element associated with the metric \(g\) on \(TM\). Then

\[
(\delta^{\nabla} \omega)(X_{1}, ..., X_{p-1}) = - \sum_{i} \langle \nabla_{e_{i}} \omega(e_{i}, X_{1}, ..., X_{p-1})
\]

For \(\omega \in A^{p}(\xi)\), set \(|\omega|^{2} = \langle \omega, \omega \rangle\) defined as in (2.3). The authors of [LSC] defined the following \(E_{F,g}\)-energy functional given by

\[
E_{F,g}(\omega) = \int_{M} F\left(\frac{|\omega|^{2}}{2}\right) dv_{g}
\]

where \(F : [0, +\infty) \rightarrow [0, +\infty)\) is as before. For our purpose, we also allow the domain of \(F\) to be \([0, c)\), where \(c\) is a positive number. In fact, we will study the case \(F : [0, \frac{1}{2}) \rightarrow [0, 1)\) in Section 7.

The stress-energy associated with the \(E_{F,g}\)-energy functional is defined as follows (cf. [BE], [Ba], [Ar], [LSC]):

\[
S_{F,g}(X, Y) = F\left(\frac{|\omega|^{2}}{2}\right) g(X, Y) - F'\left(\frac{|\omega|^{2}}{2}\right) (\omega \odot \omega)(X, Y)
\]

where \(\omega \odot \omega\) denotes a 2–tensor defined by:

\[
\langle \omega \odot \omega \rangle(X, Y) = \langle i_{X} \omega, i_{Y} \omega \rangle
\]

Here \(\langle \cdot, \cdot \rangle\) is the induced inner product on \(A^{p-1}(\xi)\), and \(i_{X} \omega\) is the interior multiplication by the vector field \(X\) given by

\[
(i_{X} \omega)(Y_{1}, ..., Y_{p-1}) = \omega(X, Y_{1}, ..., Y_{p-1})
\]

for \(\omega \in A^{p}(\xi)\) and any vector fields \(Y_{i}\) on \(M\), \(1 \leq l \leq p - 1\). When \(F(t) = t\) and \(\omega = du\) for a map \(u : M \rightarrow N\), \(S_{F,g}\) is just the stress-energy tensor introduced in [BE].

For two 2–tensors \(T_{1}, T_{2} \in \Gamma(\otimes^{2} T^{*} M)\), their inner product is defined as follows:

\[
\langle T_{1}, T_{2} \rangle = \sum_{i,j} T_{1}(e_{i}, e_{j}) T_{2}(e_{i}, e_{j})
\]

where \(\{e_{i}\}\) is an orthonormal basis with respect to \(g\).
Suppose \( M \) is a complete Riemannian manifold. We calculate the rate of change of the \( \mathcal{E}_{F,g} \)-energy integral \( \mathcal{E}_{F,g}(\omega) \) when the metric \( g \) on the domain or base manifold is changed. To this end, we consider a compactly supported smooth one-parameter variation of the metric \( g \), i.e. a smooth family of metrics \( g_s \) such that \( g_0 = g \). Set \( \delta g = \partial g_s / \partial s |_{s=0} \). Then \( \delta g \) is a smooth 2-covariant symmetric tensor field on \( M \) with compact support.

**Lemma 2.1.** For \( \omega \in A^p(\xi) \) (\( p \geq 1 \)), then
\[
\frac{d\mathcal{E}_{F,g_s}(\omega)}{ds}
\bigg|_{s=0} = \frac{1}{2} \int_M \langle S_{F,\omega}, \delta g \rangle dv_g
\]
where \( S_{F,\omega} \) is as in (2.6).

**Proof.** From [Ba], we know that
\[
\frac{d| \omega |^2_{g_s}}{ds}
\bigg|_{s=0} = -\langle \omega \otimes \omega, \delta g \rangle
\]
and
\[
\frac{d}{ds} \langle \nabla g_s, \omega \rangle |_{s=0} = \frac{1}{2} \langle g, \delta g \rangle dv_g
\]
Then
\[
\frac{d\mathcal{E}_{F,g_s}(\omega)}{ds}
\bigg|_{s=0} = \int_M F' \left( \frac{| \omega |^2}{2} \right) \frac{d| \omega |^2_{g_s}}{ds}
\bigg|_{s=0} dv_g + \int_M F \left( \frac{| \omega |^2}{2} \right) \frac{d}{ds} \langle \nabla g_s, \omega \rangle |_{s=0}
\]
\[
= \frac{1}{2} \int_M \langle F \left( \frac{| \omega |^2}{2} \right) g - F' \left( \frac{| \omega |^2}{2} \right) \omega \otimes \omega, \delta g \rangle dv_g
\]
\[
= \frac{1}{2} \int_M \langle S_{F,\omega}, \delta g \rangle dv_g
\]
\[\square\]

**Remark 2.1.** When \( F(t) = t \), the above result was derived by Sanini in [San] and by Baird in [Ba].

For a vector field \( X \), we denote by \( \theta_X \) its dual one form, i.e., \( \theta_X(\cdot) = g(X, \cdot) \). By definition, the 2-tensor \( \nabla \theta_X \) is given by
\[
(\nabla \theta_X)(Y, Z) = (\nabla_Y \theta_X)(Z)
\]
(2.9)
\[
= Y(\theta_X(Z)) - \theta_X(\nabla_Y Z)
\]
\[
= g(\nabla_Y X, Z)
\]

**Lemma 2.2.** (cf. [Xi1])
\[
\nabla_X \left( \frac{| \omega |^2}{2} \right) = \langle i_X d^\nabla \omega + d^\nabla i_X \omega, \omega \rangle - \langle \omega \otimes \omega, \nabla \theta_X \rangle
\]
\[
\langle d^\nabla i_X \omega, \omega \rangle = \sum_{j_1 < \cdots < j_{p-1}} \langle \omega(e_{j_1}, \ldots, e_{j_{p-1}}), (\nabla e_i, \omega)(X, e_{j_1}, \ldots, e_{j_{p-1}}) \rangle + \langle \omega \otimes \omega, \nabla \theta_X \rangle
\]

Next, we have the following result in which \( F(t) = t \) is known in [Se2] and [Xi1]:
Lemma 2.3. Let $\omega \in A^p(\xi)$ ($p \geq 1$) and let $S_{F,\omega}$ be the stress-energy tensor defined by (2.6), then for any vector field $X$ on $M$, we have

$$
(\text{div } S_{F,\omega})(X) = F'(\frac{|\omega|^2}{2})(\delta \nabla \omega, i_X \omega) + F'(\frac{|\omega|^2}{2})(i_X d\nabla \omega, \omega) - \langle i_{\text{grad}(F'(\frac{|\omega|^2}{2}))} \omega, i_X \omega \rangle
$$

where $\text{grad}(\bullet)$ is the gradient vector field of $\bullet$.

Proof. By using Lemma 2.2 and (2.9), we derive the following

$$
(\text{div } S_{F,\omega})(X) = \sum_{i=1}^{m} \nabla e_i S_{F,\omega}(e_i, X) - S_{F,\omega}(e_i, \nabla e_i, X)
$$

$$
= \sum_{i=1}^{m} \nabla e_i (F(\frac{|\omega|^2}{2})(e_i, X) - F'(\frac{|\omega|^2}{2})(i_{e_i}, i_X \omega))
$$

$$
- F'(\frac{|\omega|^2}{2})(e_i, \nabla e_i, X) + F'(\frac{|\omega|^2}{2})(i_{e_i}, \omega, i_X \omega)
$$

$$
= \sum_{i=1}^{m} e_i F(\frac{|\omega|^2}{2})(e_i, X) - e_i F'(\frac{|\omega|^2}{2})(i_{e_i}, i_X \omega)
$$

$$
- F'(\frac{|\omega|^2}{2})e_i (i_{e_i}, \omega, i_X X) + F'(\frac{|\omega|^2}{2})(i_{e_i}, \omega, i_{\nabla e_i} \omega)
$$

$$
= \nabla X F(\frac{|\omega|^2}{2}) - \sum_{i=1}^{m} e_i (F'(\frac{|\omega|^2}{2}))(i_{e_i}, \omega, i_X \omega)
$$

$$
- F'(\frac{|\omega|^2}{2})e_i (i_{e_i}, \omega, i_X \omega) + F'(\frac{|\omega|^2}{2})(i_{e_i}, \omega, i_{\nabla e_i} \omega)
$$

$$
= F'(\frac{|\omega|^2}{2})(i_X d\nabla \omega + d\nabla i_X \omega, \omega) - F'(\frac{|\omega|^2}{2})(\omega \odot \omega, \nabla \theta X)
$$

$$
- \langle i_{\text{grad}(F'(\frac{|\omega|^2}{2}))} \omega, i_X \omega \rangle + F'(\frac{|\omega|^2}{2})(\delta \nabla \omega, i_X \omega)
$$

$$
- F'(\frac{|\omega|^2}{2}) \sum_{j_1, \ldots, j_{p-1}, \ldots} \langle \omega(e_{j_1}, e_{j_2}, \ldots, e_{j_{p-1}}), (\nabla e_i)(X, e_{j_1}, \ldots, e_{j_{p-1}}) \rangle
$$

$$
= F'(\frac{|\omega|^2}{2})(i_X d\nabla \omega + d\nabla i_X \omega, \omega) - \langle i_{\text{grad}(F'(\frac{|\omega|^2}{2}))} \omega, i_X \omega \rangle
$$

$$
+ F'(\frac{|\omega|^2}{2})(\delta \nabla \omega, i_X \omega) - F'(\frac{|\omega|^2}{2})(d\nabla i_X \omega, \omega)
$$

Definition 2.1. $\omega \in A^p(\xi)$ ($p \geq 1$) is said to satisfy an $F$–conservation law if $S_{F,\omega}$ is divergence free, i.e. the $(0,1)$–type tensor field $\text{div } S_{F,\omega}$ vanishes identically ($\text{div } S_{F,\omega} \equiv 0$).

Lemma 2.4. ([Ba]) Let $T$ be a symmetric $(0,2)$–type tensor field. Let $X$ be a vector field, and $\theta_X$ be its dual 1-form, then

$$
\text{div}(i_X T) = (\text{div } T)(X) + \langle T, \nabla \theta_X \rangle
$$
Proof. Let \( \{ e_i \} \) be a local orthonormal frame field. Then

\[
\text{div}(i_X T) = \sum_{i=1}^{m} \left( \nabla_{e_i} (i_X T) \right)(e_i) \\
= \sum_{i=1}^{m} \left( \nabla_{e_i} (T(X, e_i)) - T(X, \nabla_{e_i} e_i) \right) \\
= \sum_{i=1}^{m} (\nabla_{e_i} T)(X, e_i) + \sum_{i=1}^{m} T(\nabla_{e_i} X, e_i) \\
= (\text{div} T)(X) + \sum_{i,j=1}^{m} T(e_i, e_j) g(\nabla_{e_i} X, e_j)
\]

This via (2.9) proves the Lemma. \( \square \)

Let \( D \) be any bounded domain of \( M \) with \( C^1 \)–boundary. By applying \( T = S_{F,\omega} \) to Lemma 2.4 and using Stokes’ Theorem, we immediately have the following

\[
(2.10) \quad \int_{\partial D} S_{F,\omega}(X, \nu) ds_g = \int_D \langle S_{F,\omega}, \nabla \theta_X \rangle + (\text{div} S_{F,\omega})(X) \ dv_g
\]

where \( \nu \) is unit outward normal vector field along \( \partial D \) with \( (m-1) \)-dimensional volume element \( ds_g \). In particular, if \( \omega \) satisfies an \( F \)–conservation law, we have

\[
(2.11) \quad \int_{\partial D} S_{F,\omega}(X, \nu) ds_g = \int_D \langle S_{F,\omega}, \nabla \theta_X \rangle dv_g
\]

It should be pointed out that the formulae (2.10) and (2.11) were also derived in [LSC]. We will give some important applications of (2.11) later.

Now we introduce a new \( E_{F,g} \)-energy functional as follows: For \( \sigma \in A^{p-1}(\xi) \)

\[
(2.12) \quad E_{F,g}(\sigma) = \int_M F\left( \frac{|d\nabla \sigma|^2}{2} \right) dv_g
\]

This functional includes the functionals for \( F \)–harmonic maps (in which \( \sigma \) is a map between two Riemannian manifolds), and Born-Infeld fields (in which \( \sigma \) is the potential of an electric field or a magnetic field and \( M = \mathbb{R}^3 \); cf. [Ya]) as its special cases, etc.

**Lemma 2.5** (The First Variation Formula for \( E_{F,g} \)-energy functional).

\[
\frac{d E_{F,g}(\sigma_t)}{dt} \bigg|_{t=0} = - \int_M \langle \tau_F(\sigma), \eta \rangle dv_g
\]

for any \( \eta \in A^{p-1}(\xi) \) with compact support, where \( \sigma_t = \sigma + t \eta \) and \( \tau_F(\sigma) = -\delta^\nabla (F'\left( \frac{|d\nabla \sigma|^2}{2} \right) d\nabla \sigma) \). Furthermore, the Euler-Lagrange equation of \( E_{F,g} \) is

\[
(2.13) \quad F'\left( \frac{|d\nabla \sigma|^2}{2} \right) \tau(\sigma) + i \text{grad}(F'\left( \frac{|d\nabla \sigma|^2}{2} \right)) d\nabla \sigma = 0
\]

where \( \tau(\sigma) = -\delta^\nabla d\nabla \sigma \).
Proof: We compute
\[
\frac{dE_{F,\sigma}(\sigma + t\eta)}{dt}|_{t=0} = \int_M \left( F'(\frac{|d\nabla \sigma|^2}{2})\langle d\nabla \sigma, d\nabla \eta \rangle dv_g \right)
\]
\[
= \int_M \langle \delta^\nabla (F'(\frac{|d\nabla \sigma|^2}{2})d\nabla \sigma), \eta \rangle dv_g
\]
\[
= -\int_M \langle \tau_F(\sigma), \eta \rangle dv_g
\]
where
\[
\tau_F(\sigma) = -\delta^\nabla (F'(\frac{|d\nabla \sigma|^2}{2})d\nabla \sigma)
\]
\[
= \sum_{i=1}^m \nabla_{e_i}(F'(\frac{|d\nabla \sigma|^2}{2})d\nabla \sigma)(e_i, \cdots, \cdot)
\]
\[
= \sum_{i=1}^m e_i(F'(\frac{|d\nabla \sigma|^2}{2}))d\nabla \sigma(e_i, \cdots, \cdot) + F'(\frac{|d\nabla \sigma|^2}{2})(\nabla_{e_i}d\nabla \sigma)(e_i, \cdots, \cdot)
\]
\[
= F'(\frac{|d\nabla \sigma|^2}{2})\tau(\sigma) + i_{\text{grad}(F'(\frac{|d\nabla \sigma|^2}{2}))}d\nabla \sigma
\]
\]

From Lemma 2.3 and the above expression (2.13) for \(\tau_F(\sigma)\), we immediately have the following

**Corollary 2.1.** For \(\sigma \in A^{p-1}(\xi)\), we have
\[
\langle \text{div} S_{F,\sigma^2 \sigma}, (X) \rangle = -\langle \tau_F(\sigma), i_X d\nabla \sigma \rangle + F'(\frac{|d\nabla \sigma|^2}{2})\langle i_X (d\nabla)^2 \sigma, d\nabla \sigma \rangle
\]
In particular, if \(\tau_F(\sigma) = 0\) and \((d\nabla)^2 \sigma = 0\), then \(\text{div} S_{F,\sigma^2 \sigma} = 0\).

**Remark 2.2.** In some cases, the condition \((d\nabla)^2 \sigma = 0\) is satisfied automatically. For example, if \(\sigma \in A^{p-1}(M) := \Gamma(\Lambda^{p-1}T^* M)\), or \(\sigma = d\varphi \in A^1(\varphi^{-1}TN)\), where \(\varphi : M \to N\) is a smooth map, then we have \((d\nabla)^2 \sigma = 0\).

**Corollary 2.2.** ([BE], [Ka]) Let \(\varphi : M \to N\) be an \(F\)-harmonic map. Then \(\text{div} S_{F,\varphi} = 0\). In particular, if \(F(t) = t\) and \(\varphi : M \to N\) is a harmonic map, we have \(\text{div} S_{\text{Id},\varphi} = 0\).

### 3. F-Yang-Mills Fields

In this section we introduce \(F\)-Yang-Mills functionals and \(F\)-Yang-Mills fields. Just as \(F\)-harmonic maps play a role in the space of maps between Riemannian manifolds, so do \(F\)-Yang-Mills fields in the space of curvature tensors (associated with connections on the adjoint bundles of principal \(G\)-bundles) over Riemannian manifolds. Let \(P\) be a principal bundle with compact structure group \(G\) over a Riemannian manifold \(M\). Let \(Ad(P)\) be the adjoint bundle
\[
Ad(P) = P \times_{\text{Ad}} \mathcal{G}
\]
where \(\mathcal{G}\) is the Lie algebra of \(G\). Every connection \(\rho\) on \(P\) induces a connection \(\nabla\) on \(Ad(P)\). We also have the Riemannian connection \(\nabla^M\) on the tangent bundle \(TM\), and the induced connection on \(\Lambda^p T^* M \otimes Ad(P)\). An \(Ad_G\) invariant inner product on \(\mathcal{G}\) induces a fiber metric on \(Ad(P)\) and making \(Ad(P)\) and \(\Lambda^p T^* M \otimes Ad(P)\)
into Riemannian vector bundles. Although $\rho$ is not a section of $\Lambda^1 T^*M \otimes \text{Ad}(P)$, via its induced connection $\nabla$, the associated curvature $R^\nabla$, given by $R^\nabla_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, is in $A^2(\text{Ad}(P))$. Let $C$ be the space of connections $\nabla$ on $\text{Ad}(P)$. We now introduce

**Definition 3.1.** The $F$–Yang–Mills functional is the mapping $\mathcal{YM}_F : C \rightarrow \mathbb{R}^+$ given by

$$
\mathcal{YM}_F(\nabla) = \int_M F(\frac{1}{2}||R^\nabla||^2)dv
$$

where the norm is defined in terms of the Riemannian metric on $M$ and a fixed $\text{Ad}_G$-invariant inner product on the Lie algebra $G$ of $G$. That is, at each point $x \in M$, its norm

$$
||R^\nabla||_x^2 = \sum_{i<j}||R^\nabla_{e_i,e_j}||_x^2
$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $T_x(M)$ and the norm of $R^\nabla_{e_i,e_j}$ is the standard one on $\text{Hom}(\text{Ad}(P), \text{Ad}(P))$—namely, $(S,U) \equiv \text{trace}(S^T U)$. 

**Definition 3.2.** A connection $\nabla$ on the adjoint bundle $\text{Ad}(P)$ is said to be an $F$-Yang–Mills connection and its associated curvature tensor $R^\nabla$ is said to be an $F$-Yang–Mills field, if $\nabla$ is a critical point of $\mathcal{YM}_F$ with respect to any compactly supported variation in the space of connections on $\text{Ad}(P)$. A connection $\nabla$ is said to be a $p$-Yang–Mills field and its associated curvature tensor $R^\nabla$ is said to be a $p$-Yang–Mills field, if $\nabla$ is a critical point of the $p$-Yang–Mills functional $\mathcal{YM}_p$ with respect to any compactly supported variation, where $\mathcal{YM}_p(\nabla) = \frac{1}{p} \int_M ||R^\nabla||^p dv$, and $p \geq 2$.

**Lemma 3.1** (The First Variation Formula for $F$-Yang–Mills functional $\mathcal{YM}_F$). Let $A \in A^1(\text{Ad}(P))$ and $\nabla^t = \nabla + tA$ be a family of connections on $\text{Ad}(P)$. Then

$$
\frac{d}{dt} \mathcal{YM}_F(\nabla^t)|_{t=0} = \int_M (\delta \mathcal{L}(F(\frac{1}{2}||R^\nabla||^2)R^\nabla), A) dv
$$

Furthermore, The Euler–Lagrangian equation for $\mathcal{YM}_F$ is

$$
F'(\frac{1}{2}||R^\nabla||^2)\delta \mathcal{L} R^\nabla - i_{\text{grad} F'(\frac{1}{2}||R^\nabla||^2)} R^\nabla = 0
$$
or

$$
\delta \mathcal{L} F'(\frac{1}{2}||R^\nabla||^2)R^\nabla = 0
$$

**Proof.** By assumption, the curvature of $\nabla^t$ is given by

$$
R^{\nabla t} = R^\nabla + t(d^\nabla A) + t^2[A,A]
$$

where $[A,A] \in A^2(\text{Ad}(P))$ is given by $[A,A]_{X,Y} = [A_X, A_Y]$. Indeed, for any local vector fields $X, Y$ on $M$. with $[X,Y] = 0$, we have

$$
R^{\nabla t}_{XY} = (\nabla_X + tA_X)(\nabla_Y + tA_Y) - (\nabla_Y + tA_Y)(\nabla_X + tA_X)
$$

$$
= R^\nabla_{XY} + t[\nabla_X, A_Y] - t[\nabla_Y, A_X] + t^2[A_X, A_Y]
$$

$$
= R^\nabla_{XY} + t(\nabla_X (A_Y) - t\nabla_Y (A_X) + t^2[A, A]_{X,Y}
$$

$$
= R^\nabla_{XY} + t(d^\nabla A)_{X,Y} + t^2[A, A]_{X,Y}
$$
Thus
\[ F\left(\frac{1}{2}\|R^\nabla\|^2\right) = F\left(\frac{1}{2}\|R^\nabla\|^2 + t(R^\nabla, d^\nabla A) + \varepsilon(t^2)\right) \]
where \(\varepsilon(t^2) = o(t^2)\) as \(t \to 0\). Therefore
\[ \mathcal{Y}^\nabla \mathcal{M}_F(\nabla^t) = \int_M F\left(\frac{1}{2}\|R^\nabla\|^2 + t(R^\nabla, d^\nabla A) + \varepsilon(t^2)\right) dv \]
and
\[ \frac{d}{dt} \mathcal{Y}^\nabla \mathcal{M}_F(\nabla^t)|_{t=0} = \int_M F'\left(\frac{1}{2}\|R^\nabla\|^2\right)(R^\nabla, d^\nabla A) dv \]
This derives the Euler-Lagrange equation for \(\mathcal{Y}^\nabla \mathcal{M}_F\) as follows
\[ 0 = \delta^\nabla \left(F'\left(\frac{1}{2}\|R^\nabla\|^2\right)R^\nabla\right) \]
\[ = -\sum_{i=1}^m (\nabla_{e_i} F'\left(\frac{1}{2}\|R^\nabla\|^2\right)R^\nabla)(e_i, \cdot) \]
\[ = F'\left(\frac{1}{2}\|R^\nabla\|^2\right)\delta^\nabla R^\nabla - i_{\text{grad}(F'\left(\frac{1}{2}\|R^\nabla\|^2\right))}R^\nabla \]

**Example 3.1.** The Euler-Lagrangian equation for \(p\)-Yang-Mills functional \(\mathcal{Y}^\nabla \mathcal{M}_p\) is
\begin{equation} \delta^\nabla (\|R^\nabla\|^{p-2}R^\nabla) = 0 \tag{3.4} \end{equation}

or \(\|R^\nabla\|^{p-2}\delta^\nabla R^\nabla - i_{\text{grad}(\|R^\nabla\|^{p-2})}R^\nabla = 0\)

**Theorem 3.1.** Every \(F\)-Yang-Mills field \(R^\nabla\) satisfies an \(F\)-conservation law.

**Proof.** It is known that \(R^\nabla\) satisfies the Bianchi identity
\begin{equation} d^\nabla R^\nabla = 0 \tag{3.5} \end{equation}
Therefore, by Lemma 2.3, Lemma 3.1 and (3.5), we immediately derive the desired
\[ \text{div} S_{F,R^\nabla} = 0 \]

**Definition 3.3.** \(\omega \in A^k(\xi) (k \geq 1)\) is said to satisfy a \(p\)-conservation law \((p \geq 2)\) if \(S_{F,\omega}\) is divergence free for \(F(t) = \frac{1}{p}(2t)^\frac{p}{2}\), i.e. for any vector field \(X\) on \(M\), we have
\begin{equation} |\omega|^{p-2}\langle \delta^\nabla \omega, i_X \omega \rangle + |\omega|^{p-2}\langle i_X d^\nabla \omega, \omega \rangle - \langle i_{\text{grad}(|\omega|^{p-2})} \omega, i_X \omega \rangle = 0 \tag{3.6} \end{equation}
As an immediate consequence, one has

**Corollary 3.1.** Every \(p\)-Yang-Mills field \(R^\nabla\) satisfies a \(p\)-conservation law.

The \(F\)-conservation law is crucial to our subsequent development. \(F\)-Yang-Mills fields in the cases \(F(t) = \sqrt{1+2t} - 1\) and \(F(t) = 1 - \sqrt{1-2t}\) will be explored in section 8.
4. Monotonicity Formulae

In this section, we will establish monotonicity formulae on Cartan-Hadamard manifolds or more generally on complete manifolds with a pole. We recall a Cartan-Hadamard manifold is a complete simply-connected Riemannian manifold of non-positive sectional curvature. A pole is a point \( x_0 \in M \) such that the exponential map from the tangent space to \( M \) at \( x_0 \) into \( M \) is a diffeomorphism. By the radial curvature \( K \) of a manifold with a pole, we mean the restriction of the sectional curvature function to all the planes which contain the unit vector \( \partial \) tangent to the unique geodesic joining \( x_0 \) to \( x \) and pointing away from \( x_0 \). Let the tensor \( g - dr \otimes dr = 0 \) on the radial direction \( \partial \), and is just the metric tensor \( g \) on the orthogonal complement \( \partial \perp \). We'll use the following comparison theorems in Riemannian geometry:

**Lemma 4.1.** (cf. [GW]) Let \((M, g)\) be a complete Riemannian manifold with a pole \( x_0 \). Denote by \( K_r \) the radial curvature \( K_r \) of \( M \).

(i) If \(-\alpha^2 \leq K_r \leq -\beta^2 \) with \( \alpha > 0 \), \( \beta > 0 \), then

\[
\beta \coth(\beta r)[g - dr \otimes dr] \leq \text{Hess}(r) \leq \alpha \coth(\alpha r)[g - dr \otimes dr]
\]

(ii) If \( K_r = 0 \), then

\[
\frac{1}{r}[g - dr \otimes dr] = \text{Hess}(r)
\]

(iii) If \(-\frac{A}{(1+r^2)\sqrt{r}} \leq K_r \leq \frac{B}{(1+r^2)\sqrt{r}} \) with \( \epsilon > 0 \), \( A \geq 0 \), and \( 0 \leq B < 2\epsilon \), then

\[
\frac{1 - \frac{B}{r}}{r}[g - dr \otimes dr] \leq \text{Hess}(r) \leq \frac{e^{\frac{r}{\epsilon}}}{r}[g - dr \otimes dr]
\]

(iv) If \(-Ar^q \leq K_r \leq -Br^q \) with \( A \geq B > 0 \) and \( q > 0 \), then

\[
B_0 r^q[g - dr \otimes dr] \leq \text{Hess}(r) \leq (\sqrt{A} \coth \sqrt{A})r^q[g - dr \otimes dr]
\]

for \( r \geq 1 \), where \( B_0 = \min \{ 1, -\frac{A}{B} + (B + \frac{B}{2})^2 \} \).

**Proof.** (i), (ii), and (iv) are treated in section 2 of [GW].

(iii) Since for every \( \epsilon > 0 \),

\[
\frac{d}{ds} \left( -\frac{1}{2\epsilon}(1+s^2)^{-\epsilon} \right) = \frac{s}{(1+s^2)^{1+\epsilon}},
\]

we have

\[
\int_0^\infty \frac{A}{(1+s^2)^{1+\epsilon}} ds = \frac{A}{2\epsilon} < \infty \quad \text{and} \quad \int_0^\infty \frac{B}{(1+s^2)^{1+\epsilon}} ds = \frac{B}{2\epsilon} < 1.
\]

Now the assertion is an immediate consequence of Quasi-isometry Theorem due to Greene-Wu [GW, p.57] in which \( 1 \leq q \leq e^{\frac{n}{m}} \) and \( 1 - \frac{A}{B} \leq \mu \leq 1 \).

In analogous to [Ka], (in which (iv) is employed) for a given function \( F \), we introduce the following

**Definition 4.1.** The \( F \)-degree \( d_F \) is defined to be

\[
d_F = \sup_{t \geq 0} \frac{tF'(t)}{F(t)}
\]

For the most part of this paper, \( d_F \) is assumed to be finite, unless otherwise stated.
Lemma 4.2. Let $M$ be a complete manifold with a pole $x_0$. Assume that there exist two positive functions $h_1(r)$ and $h_2(r)$ such that

$$h_1(r)[g - dr \otimes dr] \leq Hess(r) \leq h_2(r)[g - dr \otimes dr]$$

on $M \setminus \{x_0\}$. If $h_2(r)$ satisfies

$$rh_2(r) \geq 1$$

Then

$$\langle S_{F, \omega}, \nabla \theta X \rangle \geq (1 + (m - 1)rh_1(r) - 2pd_F rh_2(r))F(\frac{||\omega||^2}{2})$$

where $X = r \nabla r$.

Proof. Choosing an orthonormal frame $\{e_i, \frac{\partial}{\partial r}\}_{i=1, \ldots, m-1}$ around $x \in M \setminus \{x_0\}$. Take $X = r \nabla r$. Then

$$\nabla_{\frac{\partial}{\partial r}} X = \frac{\partial}{\partial r}$$

$$\nabla e_i X = r \nabla e_i \frac{\partial}{\partial r} = rHess(r)(e_i, e_j)e_j$$

Using (2.6), (2.9), (4.4) and (4.5), we have

$$\langle S_{F, \omega}, \nabla \theta X \rangle = F(\frac{||\omega||^2}{2})(1 + \sum_{i=1}^{m-1} rhess(r)(e_i, e_i))$$

$$- \sum_{i,j=1}^{m-1} F'(\frac{||\omega||^2}{2})(\omega \otimes \omega)(e_i, e_j)rHess(r)(e_i, e_j)$$

$$- F'(\frac{||\omega||^2}{2})(\omega \otimes \omega)(\frac{\partial}{\partial r}, \frac{\partial}{\partial r})$$

By (4.1), we get

$$\langle S_{F, \omega}, \nabla \theta X \rangle \geq F(\frac{||\omega||^2}{2})(1 + (m - 1)rh_1(r))$$

$$- F(\frac{||\omega||^2}{2}) \sum_{i=1}^{m-1} (\omega \otimes \omega)(e_i, e_i)rh_2(r)$$

$$- F(\frac{||\omega||^2}{2})(\omega \otimes \omega)(\frac{\partial}{\partial r}, \frac{\partial}{\partial r})$$

$$\geq F(\frac{||\omega||^2}{2})(1 + (m - 1)rh_1(r) - 2pd_F rh_2(r))$$

$$+ F(\frac{||\omega||^2}{2})(rh_2(r) - 1)(i_{\omega\frac{\partial}{\partial r}}\omega, i_{\omega\frac{\partial}{\partial r}}\omega)$$

The last step follows from the fact that

$$\sum_{i=1}^{m-1} (\omega \otimes \omega)(e_i, e_i) + (\omega \otimes \omega)(\frac{\partial}{\partial r}, \frac{\partial}{\partial r})$$

$$= \sum_{1 \leq j_1 < \ldots < j_{p-1} \leq m} \sum_{i=1}^{p} \langle \omega(e_i, e_{j_1}, \ldots, e_{j_{p-1}}), \omega(e_i, e_{j_1}, \ldots, e_{j_{p-1}}) \rangle$$

$$\leq p||\omega||^2,$$
where \( e_m = \frac{\partial}{\partial r} \). Now the Lemma follows immediately from (4.2) and (4.7). \( \square \)

**Theorem 4.1.** Let \((M, g)\) be an \( m \)-dimensional complete Riemannian manifold with a pole \( x_0 \). Let \( \xi : E \to M \) be a Riemannian vector bundle on \( M \) and \( \omega \in A^p(\xi) \). Assume that the radial curvature \( K_r \) of \( M \) satisfies one of the following three conditions:

(i) \( -\alpha^2 \leq K_r \leq -\beta^2 \) with \( \alpha > 0 \), \( \beta > 0 \) and \((m - 1)\beta - 2pd_F \geq 0\);

(ii) \( K_r = 0 \) with \( m - 2pd_F > 0 \);

(iii) \( \frac{A}{(1 + r^2)^{\alpha}} \leq K_r \leq \frac{B}{(1 + r^2)^{\beta}} \) with \( \epsilon > 0 \), \( A \geq 0 \), \( 0 < B < 2\epsilon \), and \( m - (m - 1)\beta - 2pe^F d_F > 0 \).

If \( \omega \) satisfies an \( F \)-conservation law, then

\[
\frac{1}{\rho_1} \int_{B_{\rho_1}(x_0)} F\left(\frac{\omega^2}{2}\right) dv \leq \frac{1}{\rho_2} \int_{B_{\rho_2}(x_0)} F\left(\frac{\omega^2}{2}\right) dv
\]

for any \( 0 < \rho_1 \leq \rho_2 \), where

\[
\lambda = \begin{cases} 
0 - 2pd_F & \text{if } K_r \text{ satisfies (i)} \\
0 - 2pd_F & \text{if } K_r \text{ satisfies (ii)} \\
m - (m - 1)\beta - 2pe^F d_F & \text{if } K_r \text{ satisfies (iii)}.
\end{cases}
\]

**Proof.** Take a smooth vector field \( X = r\nabla r \) on \( M \). If \( K_r \) satisfies (i), then by Lemma 4.1 and the increasing function \( \beta r \coth(\beta r) \to 1 \) as \( r \to 0 \), (4.2) holds. Now Lemma 4.2 is applicable and by (4.3), we have on \( B_{\rho}(x_0) \setminus \{x_0\} \), for every \( \rho > 0 \),

\[
\langle S_{F, \omega}, \nabla \theta_X \rangle \geq (1 + (m - 1)\beta r \coth(\beta r) - 2pd_F \cdot \alpha \coth(\alpha r)) F\left(\frac{\omega^2}{2}\right)
\]

\[
= (1 + \beta r \coth(\beta r)(m - 1 - 2pd_F \cdot \frac{\alpha \coth(\alpha r)}{\beta r \coth(\beta r)})) F\left(\frac{\omega^2}{2}\right)
\]

\[
> (1 + 1 \cdot (m - 1 - 2pd_F \cdot \frac{\alpha}{\beta} \cdot 1)) F\left(\frac{\omega^2}{2}\right) = \lambda F\left(\frac{\omega^2}{2}\right),
\]

provided that \( m - 1 - 2pd_F \cdot \frac{\alpha}{\beta} \geq 0 \), since \( \beta r \coth(\beta r) > 1 \) for \( r > 0 \), and \( \frac{\coth(\alpha r)}{\coth(\beta r)} < 1 \), for \( 0 < \beta < \alpha \), and coth is a decreasing function. Similarly, from Lemma 4.1 and Lemma 4.2, the above inequality holds for the cases (ii) and (iii) on \( B_{\rho}(x_0) \setminus \{x_0\} \).

Thus, by the continuity of \( \langle S_{F, \omega}, \nabla \theta_X \rangle \) and \( F\left(\frac{\omega^2}{2}\right) \), and (2.6), we have for every \( \rho > 0 \),

\[
\langle S_{F, \omega}, \nabla \theta_X \rangle \geq \lambda F\left(\frac{\omega^2}{2}\right) \quad \text{in} \quad B_{\rho}(x_0)
\]

\[
\rho F\left(\frac{\omega^2}{2}\right) \geq S_{F, \omega}(X, \frac{\partial}{\partial r}) \quad \text{on} \quad \partial B_{\rho}(x_0)
\]

It follows from (2.11) and (4.10) that

\[
\rho \int_{\partial B_{\rho}(x_0)} F\left(\frac{\omega^2}{2}\right) ds \geq \lambda \int_{B_{\rho}(x_0)} F\left(\frac{\omega^2}{2}\right) dv
\]

Hence we get from (4.11) the following

\[
\frac{\int_{\partial B_{\rho}(x_0)} F\left(\frac{\omega^2}{2}\right) ds}{\int_{B_{\rho}(x_0)} F\left(\frac{\omega^2}{2}\right) dv} \geq \frac{\lambda}{\rho}
\]
The coarea formula implies that
\[
\frac{d}{d\rho} \int_{B_\rho(x_0)} F\left(\frac{\omega}{2}\right)dv = \int_{\partial B_\rho(x_0)} F\left(\frac{\omega}{2}\right)ds
\]
Thus we have
\[
\frac{d}{d\rho} \int_{B_\rho(x_0)} F\left(\frac{\omega}{2}\right)dv = \int_{\partial B_\rho(x_0)} F\left(\frac{\omega}{2}\right)ds
\]
for a.e. \( \rho > 0 \). By integration (4.13) over \([\rho_1, \rho_2]\), we have
\[
\ln \int_{B_{\rho_2}(x_0)} F\left(\frac{\omega}{2}\right)dv - \ln \int_{B_{\rho_1}(x_0)} F\left(\frac{\omega}{2}\right)dv \geq \ln \rho_2^\lambda - \ln \rho_1^\lambda
\]
This proves (4.8). \( \square \)

**Remark 4.1.** (a) The Theorem is obviously trivial when \( \lambda \leq 0 \). (b) A study of Laplacian comparison on Cartan-Hadamard manifolds with \( \text{Ric}_M \leq -\beta^2 \) has been made in [Di]. By employing our techniques, as in the proofs of Lemma 4.2 and Theorem 4.1, some monotonicity formulas under appropriate curvature conditions, can be derived. (c) Whereas curvature assumptions (i) to (iii) cannot be exhaustive, our method is unified in the following sense: Regardless how radial curvature varies, as long as we have Hessian comparison estimates (4.1) with bounds satisfying (4.2), and the factor \( 1 + (m-1)c_{hf_1}(r) - 2pd_F r h_2(r) \geq c > 0 \) in (4.3) for some constant \( c \), and \( \omega \) satisfies an \( F^- \) conservation law, then we obtain a monotonicity formula (4.8) for \( E_{F,\theta}(\omega) \)-energy, for an appropriate \( \lambda > 0 \).

**Corollary 4.1.** Suppose \( M \) has constant sectional curvature \( -\alpha^2 \) (\( \alpha^2 \geq 0 \)). Let \( m - 1 - 2pd_F \geq 0 \), if \( \alpha \neq 0 \), and \( m - 2pd_F > 0 \) if \( \alpha = 0 \). Let \( \omega \in A^p(\xi) \) be a \( \xi \)-valued \( p \)-form on \( M^n \) satisfying an \( F^- \) conservation law. Then
\[
\frac{1}{\rho_1^{m-2pd_F}} \int_{B_{\rho_1}(x_0)} F\left(\frac{\omega}{2}\right)dv \leq \frac{1}{\rho_2^{m-2pd_F}} \int_{B_{\rho_2}(x_0)} F\left(\frac{\omega}{2}\right)dv
\]
for any \( x_0 \in M \) and \( 0 < \rho_1 \leq \rho_2 \).

**Proof.** In Theorem 4.1, if we take \( \alpha = \beta \neq 0 \) for the case (i) or \( a = 0 \) for the case (ii), this corollary follows from (4.8) immediately. \( \square \)

**Remark 4.2.** When \( F(t) = t \) and \( \omega \) is the differential of a harmonic map or the curvature form of a Yang-Mills connection, then we recover the well-known monotonicity formulae for the harmonic map or Yang-Mills field (cf. [PS]).

**Proposition 4.1.** Let \( (M, g) \) be an \( m \)-dimensional complete Riemannian manifold whose radial curvature satisfies
\[
(iv) - Ar^{2q} \leq K_r \leq -B r^{2q} \text{ with } A \geq B > 0 \text{ and } q > 0.
\]
Let \( \omega \in A^p(\xi) \) satisfy an \( F^- \) conservation law, and \( \delta := (m-1)B_0 - 2pd_F \sqrt{A} \text{ coth } \sqrt{A} \geq 0 \), where \( B_0 \) is given in Lemma 4.1. Suppose (4.15) holds. Then
\[
\frac{1}{\rho_1^{1+q}} \int_{B_{\rho_1}(x_0)-B_1(x_0)} F\left(\frac{\omega}{2}\right)dv \leq \frac{1}{\rho_2^{1+q}} \int_{B_{\rho_2}(x_0)-B_1(x_0)} F\left(\frac{\omega}{2}\right)dv
\]
for any \( 1 \leq \rho_1 \leq \rho_2 \).
Proof. Take $X = r\nabla r$. Applying Lemma 4.1, (4.2), and (4.3), we have
\[ \langle S_{F, \omega}, \nabla \theta_X \rangle \geq F\left(\frac{|\omega|^2}{2}\right)(1 + \delta r^{q+1}) \]
and
\[
S_{F, \omega}(X, \frac{\partial}{\partial r}) = F\left(\frac{|\omega|^2}{2}\right) - F'(\frac{|\omega|^2}{2})\langle i \frac{\partial}{\partial r}, \omega \rangle \quad \text{on} \quad \partial B_1(x_0) \\
S_{F, \omega}(X, \frac{\partial}{\partial r}) = \rho F\left(\frac{|\omega|^2}{2}\right) - \rho F'(\frac{|\omega|^2}{2})\langle i \frac{\partial}{\partial r}, \omega \rangle \quad \text{on} \quad \partial B_\rho(x_0)
\]
It follows from (2.11) that
\[
\rho \int_{\partial B_{\rho}(x_0)} F\left(\frac{|\omega|^2}{2}\right) - F'(\frac{|\omega|^2}{2})\langle i \frac{\partial}{\partial r}, \omega \rangle \, ds - \int_{\partial B_1(x_0)} F\left(\frac{|\omega|^2}{2}\right) - F'(\frac{|\omega|^2}{2})\langle i \frac{\partial}{\partial r}, \omega \rangle \, ds \\
\geq \int_{B_\rho(x_0) - B_1(x_0)} (1 + \delta r^{q+1})F\left(\frac{|\omega|^2}{2}\right).
\]
Whence, if
\[ \int_{\partial B_1(x_0)} F\left(\frac{|\omega|^2}{2}\right) - F'(\frac{|\omega|^2}{2})\langle i \frac{\partial}{\partial r}, \omega \rangle \, ds \geq 0, \]
then
\[ \rho \int_{\partial B_{\rho}(x_0)} F\left(\frac{|\omega|^2}{2}\right) \, ds \geq (1 + \delta) \int_{B_\rho(x_0) - B_1(x_0)} F\left(\frac{|\omega|^2}{2}\right) \, dv \]
for any $\rho > 1$. Coarea formula then implies
\[ \frac{d}{d\rho} \int_{B_\rho(x_0) - B_1(x_0)} F\left(\frac{|\omega|^2}{2}\right) \, dv \geq \frac{1 + \delta}{\rho} \, d\rho \]
for a.e. $\rho \geq 1$. Integrating (4.16) over $[\rho_1, \rho_2]$, we get
\[
\ln \left( \int_{B_{\rho_2}(x_0) - B_1(x_0)} F\left(\frac{|\omega|^2}{2}\right) \, dv \right) - \ln \left( \int_{B_{\rho_1}(x_0) - B_1(x_0)} F\left(\frac{|\omega|^2}{2}\right) \, dv \right) \\
\geq (1 + \delta) \ln \rho_2 - (1 + \delta) \ln \rho_1
\]
Hence we prove the proposition. \qed

Corollary 4.2. Let $K_r$ and $\delta$ be as in Proposition 4.1, and $\omega$ satisfy an $F$-conservation law. Suppose
\[ d_F \left| i \frac{\partial}{\partial r} \omega \right|^2 \leq \frac{|\omega|^2}{2} \]
on $\partial B_1$, or $F\left(\frac{|\omega|^2}{2}\right) - F'(\frac{|\omega|^2}{2})|i \frac{\partial}{\partial r} \omega|^2 \geq 0$ on $\partial B_1$. Then (4.14) holds.

Proof. The assumption (4.17) implies that (4.15) holds, and the assertion follows from Proposition 4.1. \qed

5. VANISHING THEOREMS AND LIOUVILLE TYPE RESULTS

In this section we list some results in the following three subsections, that are immediate applications of the monotonicity formulae in the last section.
5.1. Vanishing theorems for vector bundle valued $p$-forms.

**Theorem 5.1.** Suppose the radial curvature $K_r$ of $M$ satisfies the condition in Theorem 4.1. If $\omega \in A^p(\xi)$ satisfies an $F$–conservation law and

\[
\int_{B_\rho(x_0)} F\left(\frac{|\omega|^2}{2}\right) \, dv = o(\rho^\lambda) \quad \text{as } \rho \to \infty
\]

where $\lambda$ is given by (4.9), then $F\left(\frac{|\omega|^2}{2}\right) \equiv 0$, and hence $\omega \equiv 0$. In particular, if $\omega$ has finite $\mathcal{E}_{F,g} – $energy, then $\omega \equiv 0$.

**Definition 5.1.** $\omega \in A^p(\xi)$ is said to have slowly divergent $\mathcal{E}_{F,g} – $energy, if there exists a positive continuous function $\psi(r)$ such that

\[
\int_0^\infty \frac{dr}{r^\psi(r)} = +\infty
\]

for some $\rho_1 > 0$, and

\[
\lim_{\rho \to \infty} \int_{B_\rho(x_0)} \frac{F\left(\frac{|\omega|^2}{2}\right)}{\psi(r(x)))} \, dv < \infty
\]

**Remark 5.1.** (1) Hesheng Hu introduced the notion of slowly divergent energy (in which $F(t) = t$, $\omega = du$, or $\omega = R^V$), and made a pioneering study in [Hu1,2]. (2) In [LL2] and [LSC], the authors established some Liouville results for $F$–harmonic maps or forms with values in a vector bundle satisfying an $F$–conservation law under the condition of slowly divergent energy. Obviously Theorem 5.1 improves all these growth conditions, as its special cases of $F$, and expresses the growth condition more explicitly (cf. Theorem 10.1, Examples 10.1 and 10.2 in Appendix).

**Theorem 5.2.** Suppose $M$ and $\delta$ satisfy the condition in Proposition 4.1. If $\omega \in A^p(\xi)$ satisfies an $F$–conservation law, (4.15) holds, and

\[
\int_{B_\rho(x_0)} F\left(\frac{|\omega|^2}{2}\right) \, dv = o(\rho^{1+\delta}) \quad \text{as } \rho \to \infty
\]

then $\omega \equiv 0$ on $M – B_1(x_0)$. In particular, if $\omega$ has finite $\mathcal{E}_{F,g} – $energy, then $\omega \equiv 0$ on $M – B_1(x_0)$.

Notice that Theorem 5.2 only asserts that $\omega$ vanishes in an open set of $M$. If $\omega$ possesses the unique continuation property, then $\omega$ vanishes on $M$ everywhere (cf. Corollaries 5.2 and 5.4).

5.2. Liouville theorems for $F$-harmonic maps. Let $u : M \to N$ be an $F$–harmonic map. Then its differential $du$ can be viewed as a 1-form with values in the induced bundle $u^{-1}TN$. Since $\omega = du$ satisfies an $F$–conservation law, we obtain the following Liouville-type

**Theorem 5.3.** Let $N$ be a Riemannian manifold. Suppose the radial curvature $K_r$ of $M$ and $\lambda$ satisfy the condition in Theorem 4.1 in which $p = 1$. Then every $F$–harmonic map $u : M \to N$ with the following growth condition is a constant.

\[
\int_{B_\rho(x_0)} F\left(\frac{|du|^2}{2}\right) \, dv = o(\rho^\lambda) \quad \text{as } \rho \to \infty
\]

In particular, every $F$–harmonic map $u : M \to N$ with finite $F$–energy is a constant.
**Proof.** This follows at once from Theorem 5.1 in which \( p = 1 \) and \( \omega = du \). \( \square \)

**Remark 5.2.** This is in contrast to a Liouville Theorem for \( F \)-harmonic maps into a domain of strictly convex function by a different approach (cf. Theorem 12.1 in [We2]).

**Theorem 5.4** (Liouville Theorem for \( p \)-harmonic maps). Let \( N \) be a Riemannian manifold. Suppose the radial curvature \( K_r \) of \( M \) satisfies one of the following three conditions:

(i) \(-\alpha^2 \leq K_r \leq -\beta^2 \) with \( \alpha > 0 \), \( \beta > 0 \) and \((m-1)\beta - p\alpha \geq 0\);

(ii) \( K_r = 0 \) with \( m - p > 0 \);

(iii) \( \frac{A}{(1+r^2)^{p-2}} \leq K_r \leq \frac{B}{(1+r^2)^{p-2}} \) with \( \epsilon > 0 \), \( A \geq 0 \), \( 0 < B < 2\epsilon \), and \( m - (m-1)\frac{B}{2} - pe^\frac{\beta}{2} > 0 \).

Then every \( p \)-harmonic map \( u : M \to N \) with the following \( p \)-energy growth condition (5.6) is a constant.

\[
\frac{1}{p} \int_{B_\rho(x_0)} |du|^p \, dv = o(\rho^{\lambda}) \quad \text{as } \rho \to \infty
\]

where

\[
\lambda = \begin{cases} 
    m - p\frac{\beta}{2}, & \text{if } K_r \text{ satisfies (i)} \\
    m - p, & \text{if } K_r \text{ satisfies (ii)} \\
    m - (m-1)\frac{B}{2} - pe^\frac{\beta}{2}, & \text{if } K_r \text{ satisfies (iii)}. 
\end{cases}
\]

In particular, every \( p \)-harmonic map \( u : M \to N \) with finite \( p \)-energy is a constant.

**Proof.** This follows immediately from Theorem 5.3 in which \( F(t) = \frac{1}{p}(2t)^{\frac{p}{p-1}} \) and \( d_F = \frac{p}{p-1} \).

**Remark 5.3.** The case \( \frac{1}{p} \int_{B_\rho(x_0)} |du|^p \, dv = o((\ln \rho)^q) \) as \( \rho \to \infty \) for some positive number \( q \) is due to Liu-Liao [LL1].

**Corollary 5.1.** Let \( M, N, K_r, \lambda \) and the growth condition (5.6) be as in Theorem 5.4, in which \( p = 2 \). Then every harmonic map \( u : M \to N \) is a constant.

**Theorem 5.5.** Let \( M, N, K_r, \) and \( \delta \) satisfy the condition of Proposition 4.1 in which \( p = 1 \). Suppose (4.15) holds for \( \omega = du \). Then every \( F \)-harmonic map \( u : M \to N \) with the following growth condition is a constant on \( M - B_1(x_0) \):

\[
\int_{B_\rho(x_0)} F\left(\frac{|du|^2}{2}\right) dv = o(\rho^{\lambda+\delta}) \quad \text{as } \rho \to \infty
\]

on \( M - B_1(x_0) \). In particular, if \( u \) has finite \( F \)-energy, then \( u \equiv \text{const} \) on \( M - B_1(x_0) \).

**Proof.** This follows at once from Proposition 4.1.

**Proposition 5.1.** Let \( (M, g) \) be an \( m \)-dimensional complete Riemannian manifold whose radial curvature satisfies \(-A r^{2q} \leq K_r \leq -B r^{2q} \) with \( A \geq B > 0 \) and \( q > 0 \). If \( \delta := (m-1)B_0 - p/\sqrt{A} \coth \sqrt{A} \geq 0 \), where \( B_0 \) is given in Lemma 4.1. Suppose (4.15) holds for \( \omega = du \). Then every \( p \)-harmonic map \( u : M \to N \) with the growth condition \( \frac{1}{p} \int_{B_\rho(x_0)} |du|^p \, dv = o(\rho^{\lambda+\delta}) \) as \( \rho \to \infty \) is a constant on \( M - B_1(x_0) \). In particular, if \( u \) has finite \( p \)-energy, then \( u \equiv \text{const} \) on \( M - B_1(x_0) \).
Corollary 5.2. Let \( M, N, K_r, \delta, (4.15) \), and the growth condition be as in Proposition 5.1, in which \( p = 2 \). Then every harmonic map \( u : M \rightarrow N \) is a constant.

Proof. This follows immediately from Proposition 5.1 and the unique continuation property of a harmonic map. \( \square \)

5.3. Applications in \( F \)-Yang-Mills fields. Let \( R^\nabla \) be an \( F \)-Yang-Mills field, associated with an \( F \)-Yang-Mills connection \( \nabla \) on the adjoint bundle \( \text{Ad}(P) \) of a principle \( G \)-bundle over a manifold \( M \). Then \( R^\nabla \) can be viewed as a 2-form with values in the adjoint bundle over \( M \), and by Theorem 3.1, \( \omega = R^\nabla \) satisfies an \( F \)-conservation law.

Theorem 5.6 (Vanishing Theorem for \( F \)-Yang-Mills fields). Let \( M, K_r, \) and \( \lambda \) satisfy the condition in Theorem 4.1 in which \( p = 2 \). Suppose \( F \)-Yang-Mills field \( R^\nabla \) satisfies the following growth condition

\[
(5.9) \quad \int_{B_\rho(x_0)} F\left( \frac{|R^\nabla|^2}{2} \right) dv = o(\rho^\lambda) \quad \text{as } \rho \rightarrow \infty.
\]

Then \( R^\nabla \equiv 0 \) on \( M \). In particular, every \( F \)-Yang-Mills field \( R^\nabla \) with finite \( F \)-Yang-Mills energy vanishes on \( M \).

Proof. This follows at once from Theorem 5.1 in which \( p = 2 \) and \( \omega = R^\nabla \). \( \square \)

Theorem 5.7 (Vanishing Theorem for \( p \)-Yang-Mills fields). Suppose the radial curvature \( K_r \) of \( M \) satisfies the one of the following conditions:

(i) \(-\alpha^2 \leq K_r \leq -\beta^2 \) with \( \alpha > 0 \), \( \beta > 0 \) and \((m - 1)\beta - 2\rho_0 \geq 0\);

(ii) \( K_r = 0 \) with \( m - 2p > 0 \);

(iii) \(-\frac{\lambda}{(1 + \tau)^{m-1}} \leq K_r \leq \frac{\lambda B}{(1 + \tau)^{m-1}} \) with \( \epsilon > 0 \), \( A \geq 0 \), and \( 0 < B < 2\epsilon \), and \( m - (m - 1)\frac{p^2}{2} - 2p\epsilon \lambda > 0 \).

Then every \( p \)-Yang-Mills field \( R^\nabla \) with the following growth condition vanishes:

\[
(5.10) \quad \frac{1}{p} \int_{B_\rho(x_0)} |R^\nabla|^p dv = o(\rho^\lambda) \quad \text{as } \rho \rightarrow \infty
\]

where

\[
(5.11) \quad \lambda = \begin{cases} 
  m - 2p, & \text{if } K_r \text{ satisfies (i)} \\
  m - 2p, & \text{if } K_r \text{ satisfies (ii)} \\
  m - (m - 1)\frac{p^2}{2} - 2p\epsilon \lambda, & \text{if } K_r \text{ satisfies (iii)}.
\end{cases}
\]

In particular, every \( p \)-Yang-Mills field \( R^\nabla \) with finite \( \mathcal{YM}_p \)-energy vanishes on \( M \).

Corollary 5.3. Let \( M, N, K_r, \lambda, \) and the growth condition (5.12) be as in Theorem 5.10, in which \( p = 2 \). Then every Yang-Mills field \( R^\nabla \equiv 0 \) on \( M \).

Theorem 5.8. Suppose \( M, K_r, \) and \( \delta \), satisfy the same conditions of Proposition 4.1 in which \( p = 2 \), and (4.15) holds for \( \omega = R^\nabla \). Then every \( F \)-Yang-Mills field \( R^\nabla \) with the following growth condition vanishes on \( M - B_1(x_0) \):

\[
(5.12) \quad \int_{B_\rho(x_0)} F\left( \frac{|R^\nabla|^2}{2} \right) dv = o(\rho^{1+\delta}) \quad \text{as } \rho \rightarrow \infty
\]

In particular, if \( R^\nabla \) has finite \( F \)-Yang-Mills energy, then \( R^\nabla \equiv 0 \) on \( M - B_1(x_0) \).
Proof. This follows immediately from Proposition 4.1. □

Proposition 5.2. Let \((M,g)\) be an \(m\)-dimensional complete Riemannian manifold whose radial curvature satisfies 
\[-Ar^q \leq K_r \leq -Br^q\] 
with \(A \geq B > 0\) and \(q > 0\). Let \(\delta := (m-1)B_0 - 2p\sqrt{A} \coth \sqrt{A} \geq 0\), where \(B_0\) is given in Lemma 4.1, and let (4.15) hold for \(\omega = R^\nabb\). Then every \(p\)-Yang-Mills field \(R^\nabb\) with the growth condition 
\[\frac{1}{p} \int_{B_\rho(x_0)} |R^\nabb|^p dv = o(\rho^{1+\delta})\] 
as \(\rho \to \infty\) vanishes on \(M - B_1(x_0)\). In particular, if \(R^\nabb\) has finite \(p\)-Yang-Mills energy, then \(R^\nabb \equiv 0\) on \(M - B_1(x_0)\).

Corollary 5.4. Let \(M, K_r, \delta, (4.15)\), and the growth condition be as in Proposition 5.2, in which \(p = 2\). Then every Yang-Mills field \(R^\nabb\) is \(0\) on \(M\).

Proof. This follows at once from Proposition 5.2, and the unique continuation property of Yang-Mills field. □

Further applications will be treated in Section 8.

6. CONSTANT DIRICHLET BOUNDARY-VALUE PROBLEMS

To investigate the constant Dirichlet boundary-value problems for 1-forms, we begin with

Definition 6.1. The \(F\)-lower degree \(l_F\) is given by

\[l_F = \inf_{t \geq 0} \frac{tF'(t)}{F(t)}\]

Definition 6.2. A bounded domain \(D \subset M\) with \(C^1\) boundary \(\partial D\) is called starlike if there exists an interior point \(x_0 \in D\) such that

\[(\partial_{r_{x_0}}, \nu)|_{\partial D} \geq 0\]

where \(\nu\) is the unit outer normal to \(\partial D\), and the vector field \(\partial_{r_{x_0}}\) is the unit vector field such that for any \(x \in D \setminus \{x_0\} \cup \partial D\), \(\partial_{r_{x_0}}(x)\) is the unit vector tangent to the unique geodesic joining \(x_0\) to \(x\) and pointing away from \(x_0\).

It is obvious that any convex domain is starlike.

Theorem 6.1. Suppose \(M\) satisfies the same condition of Theorem 4.1 and \(D \subset M\) is a bounded starlike domain with \(C^1\) boundary. Assume that the \(F\)-lower degree \(l_F \geq 1/2\). If \(\omega \in A^1(\xi)\) satisfies an \(F\)-conservation law and annihilates any tangent vector \(\eta\) of \(\partial D\), then \(\omega\) vanishes on \(D\).

Proof. By assumption, there exists a point \(x_0 \in D\) such that the distance function \(r_{x_0}\) satisfies (6.1). Take \(X = r\nabb\), where \(r = r_{x_0}\). From the proofs of Theorem 4.1, we know that

\[(S_F,\omega, \nabb X) \geq cF\left(\frac{\omega^2}{2}\right)\]
where $c$ is a positive constant. Since $\omega \in A^1(\xi)$ annihilates any tangent vector $\eta$ of $\partial D$, we easily derive the following on $\partial D$

\begin{equation}
S_{F,\omega}(X,\nu) = rS_{F,\omega}(\frac{\partial}{\partial r},\nu)
\end{equation}

(6.3)

\begin{align*}
&= r(F(\frac{|\omega|^2}{2})\frac{\partial}{\partial r},\nu) - F'(\frac{|\omega|^2}{2})(\omega(\frac{\partial}{\partial r}),\omega(\nu)) \\
&= r\left(\frac{\partial}{\partial r},\nu\right)(F(\frac{|\omega|^2}{2}) - F'(\frac{|\omega|^2}{2}|\omega|^2)) \\
&\leq r\left(\frac{\partial}{\partial r},\nu\right)(1-2l_F)F(\frac{|\omega|^2}{2}) \leq 0
\end{align*}

From (2.11), (6.2) and (6.3), we have

\begin{equation*}
0 \leq \int_D cF(\frac{|\omega|^2}{2})dv \leq 0
\end{equation*}

which implies that $\omega \equiv 0$. □

**Corollary 6.1.** Suppose $M$ and $D$ satisfy the same assumptions of Theorem 6.1. Let $u : \overline{D} \to N$ be a $p$–harmonic map ($p \geq 1$) into an arbitrary Riemannian manifold $N$. If $u|_{\partial D}$ is constant, then $u|_{\partial D}$ is constant.

**Proof.** For a $p$–harmonic map $u$, we have $F(t) = \frac{1}{p}(2t)^{\frac{p}{2}}$. Obviously $d_F = l_F = \frac{p}{2}$. Take $\omega = du$. This corollary follows immediately from Theorem 6.1. □

**Remark 6.1.** When $M = \mathbb{R}^m$ and $D = B_\rho(x_0)$, this result, Corollary 6.1, recaptures the work of Karcher and Wood on the constant Dirichlet boundary-value problem for harmonic maps [KW]. The result of Karcher and Wood was also generalized to harmonic maps with potential by Chen [Ch] and $p$–harmonic maps with potential by Liu [Li2] for disc domains.

### 7. Extended Born-Infeld fields and Exact Forms

In this section, we will establish Liouville type theorems for solutions of the extended Born-Infeld equations (1.5) and (1.6) proposed by [Ya]. Using Hodge star operator $*$, we can rewrite the equations (1.5) and (1.6) as

\begin{equation}
\frac{d\omega}{\sqrt{1 + |d\omega|^2}} = 0, \quad \omega \in A^p(\mathbb{R}^m)
\end{equation}

(7.1)

and

\begin{equation}
\frac{d\sigma}{\sqrt{1 - |d\sigma|^2}} = 0, \quad \sigma \in A^q(\mathbb{R}^m)
\end{equation}

(7.2)

respectively. As pointed out in the introduction, the solutions of (7.1) and (7.2) are critical points of the $E_{BI}^+$-energy functional and the $E_{BI}^-$-energy functional respectively. Notice that the $E_{BI}^+$-energy functional and the $E_{BI}^-$-energy functional are $E_{F,\sigma}$-energy functionals with $F(t) = \sqrt{1 + 2t} - 1$ ($t \in [0, +\infty)$), and $F(t) = 1 - \sqrt{1 - 2t}$ ($t \in [0, 1/2]$) respectively.
Definition 7.1. The extended Born-Infeld energy functional with the plus sign on a manifold $M$ is the mapping $E^+_{BI} : A^p(M) \to \mathbb{R}^+$ given by

\begin{equation}
E^+_{BI}(\omega) = \int_M \sqrt{1 + |d\omega|^2} - 1 \; dv
\end{equation}

and the extended Born-Infeld energy functional with the minus sign on a manifold $M$ is the mapping $E^-_{BI} : A^q(M) \to \mathbb{R}^+$ given by

\begin{equation}
E^-_{BI}(\sigma) = \int_M \sqrt{1 - |d\sigma|^2} - 1 \; dv
\end{equation}

A critical point $\omega$ of $E^+_{BI}$ (resp. $\sigma$ of $E^-_{BI}$) with respect to any compactly supported variation is called an extended Born-Infeld field with the plus sign (resp. with the minus sign) on a manifold.

Obviously Corollary 2.1 implies that the solutions of (7.1) and (7.2) satisfy $F$-conservation laws.

Now we recall the equivalence between (7.1) and (7.2) found by [Ya] as follows:

Let $\omega \in A^p(\mathbb{R}^m)$ be a solution of (7.1) with $0 \leq p \leq m - 2$. Then

\begin{equation}
\tau = \pm \ast \left( \frac{d\omega}{1 + |d\omega|^2} \right)
\end{equation}

is a closed $(m - p - 1)$-form. Since the de Rham cohomology group $H^{m-p-1}(\mathbb{R}^m) = 0$, there exists an $(m - p - 2)$-form $\sigma$ such that $\tau = d\sigma$. It is easy to derive from (7.5) the following

\begin{equation}
|d\omega|^2 = \frac{|d\sigma|^2}{1 - |d\sigma|^2}
\end{equation}

and

\begin{equation}
d\omega = \pm (-1)^p(m-p) \ast \frac{d\sigma}{\sqrt{1 - |d\sigma|^2}}
\end{equation}

The Poincaré Lemma implies that $\sigma$ satisfies (7.2) with $q = m - p - 2$. Using (7.6), we get

\begin{equation}
1 - \sqrt{1 - |d\sigma|^2} = \frac{\sqrt{1 + |d\omega|^2} - 1}{\sqrt{1 + |d\omega|^2}}
\end{equation}

\begin{equation}
\leq \sqrt{1 + |d\omega|^2} - 1
\end{equation}

Conversely, a solution $\sigma$ of (7.2) with $0 \leq q \leq m - 2$ gives us a solution $\omega \in A^p(\mathbb{R}^m)$ of (7.1) with $p = m - q - 2$, and

\begin{equation}
\sqrt{1 + |d\omega|^2} - 1 = \frac{1 - \sqrt{1 - |d\sigma|^2}}{\sqrt{1 - |d\sigma|^2}}
\end{equation}

Let's first consider the equation (7.1) and let $\omega$ be a solution of (7.1). Choose an orthonormal basis $\omega_1, ..., \omega_k$ of $A^p(\mathbb{R}^m)$ consisting of constant differential forms, where $k = \binom{m}{p}$, and for each $1 \leq \alpha \leq k$,

$$
\omega_\alpha = dx_{j_1} \wedge \cdots \wedge dx_{j_p}
$$
for some \(1 \leq j_1 < \cdots < j_p \leq m\). Then we may write \(\omega = \sum_{\alpha=1}^{k} f^\alpha \omega_\alpha\). So \(\omega\) may be regarded as a map \(\omega : \mathbb{R}^m \to A^p(\mathbb{R}^m) \simeq \mathbb{R}^k\) where \(k = \binom{m}{p}\). Let \(M = (x, \omega(x))\) be the graph of \(\omega\) in \(\mathbb{R}^{m+k}\) and let \(G(\rho)\) be the extrinsic ball of radius \(\rho\) of the graph centered at the origin of \(\mathbb{R}^{m+k}\) given by

\[
G(\rho) = M \cap B^{m+k}(\rho)
\]

Set

\[
\omega_\rho = \sum_{\alpha=1}^{k} f^\alpha_\rho \omega_\alpha
\]

where

(7.10)

\[
f^\alpha_\rho(x) = \begin{cases} 
\rho & \text{if } f^\alpha > \rho \\
f^\alpha(x) & \text{if } |f^\alpha(x)| \leq \rho \\
-\rho & \text{if } f^\alpha < \rho 
\end{cases}
\]

For any \(\delta > 0\), let \(\phi\) be a nonnegative cut-off function defined on \(\mathbb{R}^m\) given by

(7.11)

\[
\phi = \begin{cases} 
1 & \text{on } B^m(\rho) \\
\frac{(1+\delta)r(x)}{\delta} & \text{on } B^m((1+\delta)\rho) \setminus B^m(\rho) \\
0 & \text{on } \mathbb{R}^m \setminus B^m((1+\delta)\rho)
\end{cases}
\]

**Proposition 7.1.** Let \(\omega \in A^p(\mathbb{R}^m)\) be an extended Born-Infeld field with the plus sign on \(\mathbb{R}^m\). Then the Born-Infeld type energy of \(\omega\) over \(G(\rho)\) satisfies the upper bound

\[
E_{BI}^+(\omega; G(\rho)) \leq m \sqrt{k \omega_m \rho^m}
\]

where \(k = \binom{m}{p}\) and \(\omega_m\) is the volume of the unit ball in \(\mathbb{R}^m\).

**Proof.** Taking inner product with \(\phi \omega_\rho\), we may get from (1.5) or (7.1) that

\[
0 = \int_{\mathbb{R}^m} \left\langle d^*(\frac{d\omega}{\sqrt{1 + |d\omega|^2}}), \phi \omega_\rho \right\rangle \, dx
\]

\[
= \int_{\mathbb{R}^m} \left\langle \frac{d\omega}{\sqrt{1 + |d\omega|^2}}, d(\phi \omega_\rho) \right\rangle \, dx
\]

\[
= \int_{\mathbb{R}^m} \left\langle \frac{d\omega}{\sqrt{1 + |d\omega|^2}}, d\phi \wedge \omega_\rho \right\rangle \, dx + \int_{\mathbb{R}^m} \phi \left( \frac{d\omega}{\sqrt{1 + |d\omega|^2}} \right) \, d\omega_\rho \, dx
\]

Using the fact that \(|d\phi| = |\nabla \phi| \leq \frac{1}{\delta \rho}\) and \(|\omega_\rho| \leq \sqrt{k} \rho\), we have

\[
\int_{B^m((1+\delta)\rho)} \phi \left( \frac{d\omega}{\sqrt{1 + |d\omega|^2}} \right) \, d\omega_\rho \, dx \leq \int_{B^m((1+\delta)\rho)} \frac{|d\phi||\omega_\rho||d\omega|}{\sqrt{1 + |d\omega|^2}} \, dx
\]

\[
\leq \frac{\sqrt{k}}{\delta} \text{Vol}(B^m((1+\delta)\rho) - B^m(\rho))
\]

So

\[
\int_{B^m(\rho) \cap \{ |f^\alpha| \leq \rho \}} \frac{|d\omega|^2}{\sqrt{1 + |d\omega|^2}} \, dx \leq \frac{\sqrt{k}}{\delta} \text{Vol}(B^m((1+\delta)\rho) - B^m(\rho))
\]
Because $G(\rho) \subset M \cap (B^m(\rho) \times [-\rho, \rho])$, we have

$$E_{B^i}^+(\omega; G(\rho)) \leq \int_{B^m(\rho) \cap \{|f^\alpha| \leq \rho\}} \sqrt{1 + |d\omega|^2} - 1 \, dx$$

$$\leq \int_{B^m(\rho) \cap \{|f^\alpha| \leq \rho\}} \sqrt{1 + |d\omega|^2} \, dx - Vol(B^m(\rho) \cap \{|f^\alpha| \leq \rho\})$$

$$\leq \int_{B^m(\rho) \cap \{|f^\alpha| \leq \rho\}} \frac{|d\omega|^2 + 1}{\sqrt{1 + |d\omega|^2}} \, dx - Vol(B^m(\rho) \cap \{|f^\alpha| \leq \rho\})$$

$$\leq \frac{\sqrt{k}}{\delta} Vol(B^m((1 + \delta)\rho) - B^m(\rho))$$

$$= \frac{\sqrt{k}}{\delta} \omega_m((1 + \delta)^m \rho^m - \rho^m)$$

Let $\delta \to 0$, we have

$$E_{B^i}^+(\omega; G(\rho)) \leq m\sqrt{k} \omega_m \rho^m$$

\[\square\]

**Remark 7.1.** When $\omega = f \in A^0(\mathbb{R}^m) = C^\infty(\mathbb{R}^m)$, the above result is the volume estimate for the minimal graph of $f$ (cf. [LW]).

**Lemma 7.1.** (i) If $F(t) = \sqrt{1 + 2t} - 1$ with $t \in [0, +\infty)$, then $d_F = 1$ and $l_F = 1/2$.

(ii) If $F(t) = 1 - \sqrt{1 - 2t}$ with $t \in [0, 1/2)$, then $d_F = +\infty$ and $l_F = 1$.

**Proof.** (i) For $F(t) = \sqrt{1 + 2t} - 1$, we have

$$\frac{tF'(t)}{F(t)} = \frac{t}{\sqrt{1 + 2t} - 1}$$

$$= \frac{\sqrt{1 + 2t} + 1}{2\sqrt{1 + 2t}} \quad \text{for} \quad t \in (0, +\infty)$$

Hence,

$$\frac{tF'(t)}{F(t)} = \frac{1}{2} + \frac{1}{2\sqrt{1 + 2t}} \quad \text{for} \quad t \in [0, +\infty).$$

By definition, we get $d_F = 1$ and $l_F = 1/2$.

(ii) For $F(t) = 1 - \sqrt{1 - 2t}$, we have

$$\frac{tF'(t)}{F(t)} = \frac{t}{1 - 2t(1 - \sqrt{1 - 2t})}$$

$$= \frac{1 + \sqrt{1 - 2t}}{2\sqrt{1 - 2t}} \quad \text{for} \quad t \in (0, \frac{1}{2}).$$

Hence,

$$\frac{tF'(t)}{F(t)} = \frac{1}{2} + \frac{1}{2\sqrt{1 - 2t}} \quad \text{for} \quad t \in [0, \frac{1}{2}).$$

By definition, we obtain $d_F = +\infty$ and $l_F = 1$.

By applying Corollary 4.1 to $M = \mathbb{R}^m$ and $F = \sqrt{1 + 2t} - 1$, we immediately get the following:
Theorem 7.1. Let $\omega \in A^p(\mathbb{R}^m)$ be an extended Born-Infeld field with the plus sign on $\mathbb{R}^m$. If $m > 2p$ and $\omega$ satisfies the following growth condition
\[
\int_{B_\rho(x_0)} \sqrt{1 + |d\omega|^2} - 1 \ dx = o(\rho^{m-2p}) \quad \text{as } \rho \to \infty
\]
for some point $x_0 \in \mathbb{R}^m$, then $d\omega = 0$, and $\omega$ is exact. In particular, if $\omega$ has finite $E_{BI}^+$-energy, then $\omega$ is exact.

Remark 7.2. In [SiSiYa], the authors proved the following: Let $\omega$ be a solution of (7.1). If $d\omega \in L^2(\mathbb{R}^m)$ ($m \geq 3$) or $d\omega \in L^2(\mathbb{R}^2) \cap H^1$ on $\mathbb{R}^2$, where $H^1$ is the Hardy space, then $d\omega \equiv 0$. In view of the inequality $\sqrt{1 + t^2} - 1 \leq \frac{t^2}{2}$ for any $t \geq 0$, it is clear that being in $L^2$ ensures finite $E_{BI}^+$-energy.

Using the duality between solutions of (7.1) and (7.2), we have

Proposition 7.2. Let $\sigma \in A^q(\mathbb{R}^m)$ be a $q$–form with $\frac{m-4}{2} < q < m - 2$. If $\sigma$ is an extended Born-Infeld field with the minus sign on $\mathbb{R}^m$, and $\sigma$ satisfies the following growth
\[
(7.14) \quad \int_{B_\rho(x_0)} \frac{1 - \sqrt{1 - |d\sigma|^2}}{\sqrt{1 - |d\sigma|^2}} \ dx = o(\rho^{2q-m+4}) \quad \text{as } \rho \to \infty
\]
then $d\sigma = 0$, and $\sigma$ is exact. In particular, if $\sigma$ has finite $E_{BI}^-$-energy, then $\sigma$ is exact.

Proof. By the duality between (7.1) and (7.2), we get a solution $\omega$ from the solution $\sigma$ of (7.2), where $\omega$ satisfies (7.1) and (7.9). Since $p = m - q - 2$, the condition $q > \frac{m-4}{2}$ is equivalent to $m > 2p$. Obviously (7.9) and (7.14) imply
\[
\int_{B_\rho(x_0)} \sqrt{1 + |d\omega|^2} - 1 \ dx = o(\rho^{m-2p}) \quad \text{as } \rho \to \infty
\]
Therefore Theorem 7.1 implies that $d\omega = 0$ which is equivalent to $d\sigma = 0$. \hfill $\square$

Proposition 7.3. Let $\sigma \in A^q(\mathbb{R}^m)$ be a $q$–form with $q < \frac{m-2}{2}$. Suppose that $\sigma$ is an extended Born-Infeld field with the minus sign on $\mathbb{R}^m$, satisfying
\[
(7.15) \quad |d\sigma|^2 \leq 1 - \frac{(q + 1)^2}{(m - q - 1)^2}
\]
Then
\[
(7.16) \quad \frac{1}{\rho_1^2} \int_{B_{\rho_1}(x_0)} 1 - \sqrt{1 - |d\sigma|^2} \ dx \leq \frac{1}{\rho_2^2} \int_{B_{\rho_2}(x_0)} 1 - \sqrt{1 - |d\sigma|^2} \ dx
\]
for any $0 < \rho_1 \leq \rho_2$.

Proof. Let $F(t) = 1 - \sqrt{1 - 2t}$. For the distance function $r$ on $\mathbb{R}^m$, we have
\[
(7.17) \quad Hess(r) = \frac{1}{r} [g - dr \otimes dr]
\]
where $g$ is the standard Euclidean metric. Taking $X = r \nabla r$, using (4.6) and (7.17), we have at those points $x \in \mathbb{R}^m$, where $d\sigma(x) \neq 0$,
\[
\langle S_{F,d\sigma}, \nabla \theta_X \rangle = mF\left(\frac{|d\sigma|^2}{2}\right) - qF\left(\frac{|d\sigma|^2}{2}\right)|d\sigma|^2
\]
\[
(7.18) \quad = (m - q \frac{F\left(\frac{|d\sigma|^2}{2}\right)|d\sigma|^2}{F\left(\frac{|d\sigma|^2}{2}\right)})F\left(\frac{|d\sigma|^2}{2}\right)
\]
From (7.13), it is easy to see that (7.15) is equivalent to, for every \( x \in \mathbb{R}^m \),

\[
m - q \frac{F'(\frac{1}{2})}{F'(\frac{1}{2})} |d\sigma|^2 = m - q \left( 1 + \frac{1}{\sqrt{1 - |d\sigma|^2}} \right) \geq \frac{m}{q + 1}
\]

which implies

\[
\langle S_{F,d\sigma}, \nabla \theta_x \rangle \geq \frac{m}{q + 1} F\left( \frac{|d\sigma|^2}{2} \right) \quad \text{on} \quad B_\rho(x_0)
\]

Therefore we can prove this Proposition by using (7.20) in the same way as we prove Theorem 4.1, via (4.10). \( \square \)

**Corollary 7.1.** In addition to the same hypotheses of Proposition 7.3, if \( \sigma \) satisfies

\[
\int_{B_\rho(x_0)} 1 - \sqrt{1 - |d\sigma|^2} \, dx = o\left( \rho^{q+1} \right) \quad \text{as} \quad \rho \to \infty
\]

then \( d\sigma \equiv 0 \), and \( \sigma \) is exact. In particular, if \( \sigma \) has finite \( E_{BI}^- \) energy, then \( \sigma \) is exact.

**8. Generalized Yang-Mills-Born-Infeld Fields (with the plus sign and with the minus sign) on Manifolds**

In [SiSiYa], L. Sibner, R. Sibner and Y.S. Yang consider a variational problem which is a generalization of the (scalar valued) Born-Infeld model and at the same time a quasilinear generalization of the Yang-Mills theory. This motivates the study of Yang-Mills-Born-Infeld fields on \( \mathbb{R}^4 \), and they prove that a generalized self-dual equation whose solutions are Yang-Mills-Born-Infeld fields has no finite-energy solution except the trivial solution on \( \mathbb{R}^4 \). In this section, we introduce the following

**Definition 8.1.** The generalized Yang-Mills-Born-Infeld energy functional with the plus sign on a manifold \( M \) is the mapping \( \mathcal{YM}^+_{BI} : \mathcal{C} \to \mathbb{R}^+ \) given by

\[
\mathcal{YM}^+_{BI}(\nabla) = \int_M \sqrt{1 + ||R^\nabla||^2} - 1 \, dv
\]

and the generalized Yang-Mills-Born-Infeld energy functional with the minus sign on a manifold \( M \) is the mapping \( \mathcal{YM}^-_{BI} : \mathcal{C} \to \mathbb{R}^+ \) given by

\[
\mathcal{YM}^-_{BI}(\nabla) = \int_M 1 - \sqrt{1 - ||R^\nabla||^2} \, dv
\]

The associate curvature form \( R^\nabla \) of a critical connection \( \nabla \) of \( \mathcal{YM}^+_{BI} \) (resp. \( \mathcal{YM}^-_{BI} \)) is called a generalized Yang-Mills-Born-Infeld field with the plus sign (resp. with the minus sign) on a manifold.

By applying \( F(t) = \sqrt{1 + 2t} - 1 \) and \( F(t) = 1 - \sqrt{1 - 2t} \) to Theorem 3.1, we obtain

**Corollary 8.1.** Every generalized Yang-Mills-Born-Infeld field (with the plus sign or with the minus sign) on a manifold satisfies an \( F \)-conservation law.
**Theorem 8.1.** Let the radial curvature $K_r$ of $M$ satisfy one of the three conditions (i), (ii), and (iii) in Theorem 4.1 in which $p = 2$ and $d_F = 1$. Let $R^\nabla$ be a generalized Yang-Mills-Born-Infeld field with the plus sign on $M$. If $R^\nabla$ satisfies the following growth condition

$$\int_{B_\rho(x_0)} \sqrt{1 + \|R^\nabla\|^2} - 1 \ dv = o(\rho^\lambda) \quad \text{as} \ \rho \to \infty$$

where

$$\lambda = \begin{cases} 
  m - \frac{4\beta}{\beta} & \text{if } K_r \text{ satisfies (i)}; \\
  m - 4 & \text{if } K_r \text{ satisfies (ii)}; \\
  m - (m - 1)\frac{2}{m^2} - 4e^{\frac{\rho}{m}} & \text{if } K_r \text{ satisfies (iii)}
\end{cases}$$

then its curvature $R^\nabla \equiv 0$. In particular, if $R^\nabla$ has finite $\mathcal{YM}^{+}_{BI}$-energy, then $R^\nabla \equiv 0$.

**Proof.** By applying Corollary 8.1 and $F(t) = \sqrt{1 + 2t} - 1$ to Theorem 4.1 in which $d_F = 1$, by Lemma 7.1(i), and $p = 2$, for $R^\nabla \in A^2(AdP)$, the result follows immediately.

**Theorem 8.2.** Suppose $M$ has constant sectional curvature $-\alpha^2$ ($\alpha^2 \geq 0$). Let $R^\nabla$ be a generalized Yang-Mills-Born-Infeld field with the plus sign on $M$. If $m > 4$ and $R^\nabla$ satisfies the following growth condition

$$\int_{B_\rho(x_0)} \sqrt{1 + \|R^\nabla\|^2} - 1 \ dv = o(\rho^{m-4}) \quad \text{as} \ \rho \to \infty$$

then its curvature $R^\nabla \equiv 0$. In particular, if $R^\nabla$ has finite $\mathcal{YM}^{+}_{BI}$-energy, then $R^\nabla \equiv 0$.

**Proof.** This follows at once by applying $\alpha = \beta$ in conditions (i) and (ii) of Theorem 8.1.

**Corollary 8.2.** Let $R^\nabla$ be a Yang-Mills-Born-Infeld field with the plus sign on $\mathbb{R}^m$. If $m > 4$ and $R^\nabla$ satisfies the following growth condition

$$\int_{B_\rho(x_0)} \sqrt{1 + \|R^\nabla\|^2} - 1 \ dx = o(\rho^{m-4}) \quad \text{as} \ \rho \to \infty$$

then its curvature $R^\nabla \equiv 0$. In particular, if $R^\nabla$ has finite $\mathcal{YM}^{+}_{BI}$-energy, then $R^\nabla \equiv 0$.

If we replace $d\sigma$ with $R^\nabla$ and set $q = 2$ in Proposition 7.3, by a similar argument, we obtain the following

**Proposition 8.1.** Let $R^\nabla$ be a Yang-Mills-Born-Infeld field with the minus sign on $\mathbb{R}^m$. Suppose $m > 6$,

$$\|R^\nabla\|^2 \leq \frac{m^2 - 6m}{m^2 - 6m + 9}$$

and

$$\int_{B_\rho(x_0)} 1 - \sqrt{1 - \|R^\nabla\|^2} \ dx = o(\rho^{\frac{m}{2}}) \quad \text{as} \ \rho \to \infty$$

Then $R^\nabla = 0$. 
It is well-known that there are no nontrivial Yang-Mills fields in \( \mathbb{R}^m \) with finite Yang-Mills-energy for \( m \geq 5 \) (in contrast with \( \mathbb{R}^4 \), where the problem is conformally invariant and one obtains Yang-Mills fields with finite Yang-Mills-energy by pullback from \( S^4 \) (cf. [JT])). In Corollary 8.2, for the case \( m \geq 5 \), we obtain a similar result for Yang-Mills-Born-Infeld field (with the plus sign) on \( \mathbb{R}^m \). It’s natural to ask if there exists a nontrivial Yang-Mills-Born-Infeld field (with the plus sign) on \( \mathbb{R}^4 \) with finite \( \mathcal{YM}^+_{BI} \)-energy.

9. Generalized Chern Type Results on Manifolds

A Theorem of Chern states that every entire graph \( x_{m+1} = f(x_1, \ldots, x_m) \) on \( \mathbb{R}^m \) of constant mean curvature is minimal in \( \mathbb{R}^{m+1} \). In this section, we view functions as 0-forms and consider the following constant mean curvature type equation for \( p \)-forms \( \omega \) on \( \mathbb{R}^m \) (\( p < m \)) and on manifolds with the global doubling property by a different approach (being motivated by the work in [We1,2] and [LWW]):

\[
\delta \left( \frac{d\omega}{\sqrt{1 + |d\omega|^2}} \right) = \omega_0
\]

where \( \omega_0 \) is a constant \( p \)-form (Thus when \( p = 0 \), (9.1) is just the equation describing graphic hypersurface with constant mean curvature). Equivalently, (9.1) may be written as

\[
d \ast \left( \frac{d\omega}{\sqrt{1 + |d\omega|^2}} \right) = \xi_0
\]

where \( \xi_0 \) is a constant \((m - p)\)-form.

**Theorem 9.1.** Suppose \( \omega \) is a solution of (9.2) on \( \mathbb{R}^m \). Then \( \xi_0 = 0 \).

**Proof.** Obviously, for every \((m - p)\)-plane \( \Sigma \) in \( \mathbb{R}^m \), there exists a volume element \( d\Sigma \) of \( \Sigma \), such that \( \xi_0|\Sigma = c d\Sigma \), for some constant \( c \). Let \( i : \Sigma \hookrightarrow \mathbb{R}^m \) be the inclusion mapping. If follows from (9.2) and Stokes’ Theorem that for every ball \( B(x_0, r) \) of radius \( r \) centered at \( x_0 \) in \( \Sigma \subset \mathbb{R}^m \), and its boundary \( \partial B(x_0, r) \) with the surface element \( dS \), we have

\[
\begin{align*}
0 \leq |c|\omega_{m-p} & \omega_{m-p} \\
= \left| \int_{B(x_0,r)} \frac{c \ d\Sigma}{} \right| \\
= \left| \int_{B(x_0,r)} i^* \xi_0 \right| \\
= \left| \int_{B(x_0,r)} d i^* \left( \ast \frac{d\omega}{\sqrt{1 + |d\omega|^2}} \right) \right| \\
= \left| \int_{\partial B(x_0,r)} i^* \left( \frac{d\omega}{\sqrt{1 + |d\omega|^2}} \right) \right| \\
\leq \int_{\partial B(x_0,r)} \frac{d\omega}{\sqrt{1 + |d\omega|^2}} \ dS \\
\leq (m - p) \omega_{m-p} \omega_{m-p-1}
\end{align*}
\]
where \( \omega_{m-p} \) is the volume of the unit ball in \( \Sigma \). Hence we get

\[
0 \leq |c| \leq \frac{m-p}{r}
\]

which implies that \( c = 0 \) by letting \( r \to \infty \).

This generalizes the work of Chern:

**Corollary 9.1.** ([Che]) Let \( p = 0 \) in Theorem 9.1. Then the graph of \( \omega \) over \( \mathbb{R}^m \) is a minimal hypersurface in \( \mathbb{R}^{m+1} \).

**Proof.** As \( p = 0 \), we may assume that \( \omega = f \) for some function \( f \) on \( \mathbb{R}^m \). Then (9.1) is equivalent to

\[
\text{div} \left( \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) = c
\]

where \( c \) is a constant. Now the assertion follows from Theorem 9.1.

**Corollary 9.2.** Let \( p = 0 \) and \( m \leq 7 \) in Theorem 9.1. Then the graph of \( \omega \) over \( \mathbb{R}^m \) is a hyperplane in \( \mathbb{R}^{m+1} \).

**Proof.** This follows at once from Corollary 9.1 and Bernstein Theorems for minimal graphs (cf. [Be], [Al], [Gi] and [Si]).

**Corollary 9.3.** Let \( p = 0 \) and \( |\nabla \omega|(x) \leq \beta \) (for all \( x \in \mathbb{R}^m \), where \( \beta > 0 \) is a constant) in Theorem 9.1. Then the graph of \( \omega \) over \( \mathbb{R}^m \) is a hyperplane in \( \mathbb{R}^{m+1} \), for all \( m \geq 2 \).

**Proof.** This follows at once from Corollary 9.1 and Harnack’s Theorem due to Moser (cf. [Mo], p.591).

In fact, we can give a further generalization.

**Theorem 9.2.** Let \( \omega \) be a differential form of degree \( p \) on \( \mathbb{R}^m \), satisfying

\[
d \ast \left( \frac{d\omega}{\sqrt{1 + |d\omega|^2}} \right) = \xi
\]

where \( \xi \) is a differential form of degree \( m-p \) on \( \mathbb{R}^m \). Suppose there exists an \((m-p)\)-plane \( \Sigma \) in \( \mathbb{R}^m \), with the volume element \( d\Sigma \), such that \( \xi|_{\Sigma} = g(x)d\Sigma \), off a bounded set \( K \) in \( \Sigma \), where \( g \) is a continuous function on \( \Sigma \setminus K \) with \( c = \inf_{x \in \Sigma \setminus K} |g(x)| \). Then \( c = 0 \).

**Proof.** We consider two cases:

Case 1. \( g \) assumes both positive and negative values: By the intermediate value theorem, \( g \) assumes value 0 at some point, and thus \( c = \inf_{x \in \Sigma \setminus K} |g(x)| = 0 \).

Case 2. \( g \) is a nonpositive or nonnegative function: Since \( K \subset \Sigma \) is bounded, choose a sufficiently large \( r_0 < r \) so that \( K \subset B(x_0, r_0) \), where \( B(x_0, r_0) \) is the ball of radius \( r_0 \) centered at \( x_0 \) in \( \Sigma \subset \mathbb{R}^m \). Let \( 0 \leq \psi \leq 1 \) be the cut off function such that \( \psi \equiv 1 \) on \( B(x_0, r_0) \) and \( \psi \equiv 0 \) off \( B(x_0, 2r) \subset \Sigma \), and \( |\nabla \psi| \leq \frac{C}{r} \) (cf. also Lemma 1 in [We1]). Let \( i : \Sigma \hookrightarrow \mathbb{R}^m \) be the inclusion mapping. Multiplying (9.5) by \( \psi \), and applying the divergence theorem, we have
\[ c\omega_{m-p}(r^{m-p} - r_0^{m-p}) \leq \int_{B(x_0, r) \setminus B(x_0, r_0)} \psi(x)g(x)d\Sigma \]
\[ = \int_{B(x_0, r) \setminus B(x_0, r_0)} \psi \psi^* \xi \]
\[ = \int_{B(x_0, r) \setminus B(x_0, r_0)} \psi d\psi \left( \ast \left( \frac{d\omega}{\sqrt{1 + |d\omega|^2}} \right) \right) \]
\[ \leq \int_{B(x_0, 2r) \setminus B(x_0, r_0)} |\nabla \psi| \left( \frac{d\omega}{\sqrt{1 + |d\omega|^2}} \right) d\Sigma \]
\[ + \int_{\partial B(x_0, r_0)} \left( \frac{d\omega}{\sqrt{1 + |d\omega|^2}} \right) dS \]
\[ \leq \omega_{m-p} C_1 2^{m-p} r^{m-p-1} + (m-p)\omega_{m-p} r_0^{m-p-1} \]

where \( \omega_{m-p} \) is the volume of the unit ball in \( \Sigma \). Hence
\[ 0 \leq c\omega_{m-p}(1 - \frac{r_0^{m-p}}{r^{m-p}}) \leq \frac{\omega_{m-p} C_1 2^{m-p}}{r} + \frac{(m-p)\omega_{m-p} r_0^{m-p-1}}{r^{m-p}} \]
implies that \( c = 0 \) by letting \( r \to \infty \).

**Corollary 9.4.** There does not exist a solution of (9.5) such that \( \xi|\Sigma = g(x)d\Sigma \), off a bounded set \( K \) in some \( (m-p) \)-plane \( \Sigma \) in \( \mathbb{R}^m \) with \( c > 0 \), where \( g \) is a continuous real-valued (not necessary nonnegative or nonpositive) function, and \( c = \inf_{x \in \Sigma \setminus K} |g(x)| \).

**Corollary 9.5.** Let \( f \) be a function satisfying
\[ \text{div} \left( \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) = c \]
on a bounded subset \( K \subset \mathbb{R}^m \), where \( c \equiv \text{const.} \). Then \( c = 0 \). In particular, every graph of \( f \) of constant mean curvature off a cylinder \( \mathbb{R}^m \setminus (B(x_0, r_0) \times \mathbb{R}) \) is minimal.

**Proof.** This follows at once from Theorem 9.2 in which \( \omega = df \) and \( p = 0 \). In particular, we choose \( K = B(x_0, r_0) \). \( \square \)

**Remark 9.1.** This result, Corollary 9.4, recaptures Corollary 9.1, a theorem of Chern, in which \( K \) is an empty set. Notice that Chern’s result was also generalized to graphs with higher codimension and parallel mean curvature in Euclidean space by Salavessa [Sa1].

Next we consider the following equation
\[ (9.6) \quad \delta \left( \frac{d\sigma}{\sqrt{1 - |d\sigma|^2}} \right) = \rho_0 \]
which generalizes the constant mean curvature equation for spacelike hypersurfaces.

**Theorem 9.3.** Let \( \sigma \) be a differential form of degree \( q \) \((\leq m-1)\) on \( \mathbb{R}^m \), satisfying
\[ (9.7) \quad d* \left( \frac{d\sigma}{\sqrt{1 - |d\sigma|^2}} \right) = \tau_0 \]
where \( \tau_0 \) is a constant \((m-q)\)-form. If

\[
\frac{1}{\sqrt{1 - |d\sigma|^2}} = o(r)
\]

where \( r \) is the distance from the origin, then \( \tau_0 = 0 \).

**Proof.** Obviously, for every \((m-q)\)-plane \( \Sigma \) in \( \mathbb{R}^m \), there exists a volume element \( d\Sigma \) of \( \Sigma \), such that \( \tau_0|_{\Sigma} = c d\Sigma \). Let \( i : \Sigma \hookrightarrow \mathbb{R}^m \) be the inclusion mapping. For every ball \( B(x_0, r) \) of radius \( r \) centered at \( x_0 \) in \( \Sigma \subset \mathbb{R}^m \), and its boundary \( \partial B(x_0, r) \), by using (9.7) and Stokes’ Theorem, we have

\[
|c| \omega_{m-q} r^{m-q} = \left| \int_{B(x_0, r)} c d\Sigma \right| \leq \int_{\partial B(x_0, r)} \frac{d\sigma}{\sqrt{1 - |d\sigma|^2}} dS \leq (m-q) \sup_{\partial B(x_0, r)} \left\{ \frac{1}{\sqrt{1 - |d\sigma|^2}} \right\} \omega_{m-q} r^{m-q-1}
\]

where \( \omega_{m-q} \) is the volume of the unit ball in \( \Sigma \). Hence

\[
|c| \leq \frac{m-q}{r} \sup_{\partial B(x_0, r)} \left\{ \frac{1}{\sqrt{1 - |d\sigma|^2}} \right\}
\]

implies that \( c = 0 \) by letting \( r \to \infty \). \( \square \)

**Remark 9.2.** (1) When \( q = 0 \), (9.6) describes spacelike graphic hypersurface with constant mean curvature. It is known that \( \frac{1}{\sqrt{1 - |d\sigma|^2}} \) is bounded iff the Gauss image of the hypersurface is bounded (cf. [Xi2,3]). Such kind of Chern type results under growth conditions were obtained in [Do], [Sa2] for spacelike graphs as well. (2) A similar generalized Chern type result can be established for the following more general equation

\[
\delta^\nabla (F' \left( \frac{|d\nabla \sigma|^2}{2} \right) d\nabla \sigma) = \rho_0
\]

Using a different technique or idea (cf. [We1,2], [LWW]), one can extend the above results to complete noncompact manifold \( M \) that has the global doubling property, i.e., \( \exists D(M) > 0 \) such that \( \forall r > 0, \forall x \in M \)

\[
Vol(B(x, 2r)) \leq D(M) Vol(B(x, r))
\]

Examples of complete manifolds with the global doubling property include complete noncompact manifolds of nonnegative Ricci curvature, in particular Euclidean space \( \mathbb{R}^m \).

**Theorem 9.4.** Let \( \omega \) be a differential form of degree \( p \) on \( M \) that has the global doubling property, and satisfies

\[
d \ast \left( \frac{d\omega}{\sqrt{1 + |d\omega|^2}} \right) = \xi
\]

where \( \xi \) is a differential form of degree \( m-p \) on \( M \). Suppose there exists an \((m-p)\)-dimensional submanifold \( \Sigma \) in \( M \), with the volume element \( d\Sigma \), such that
\( \xi_{|\Sigma} = g(x)d\Sigma \), off a bounded set \( K \) in \( \Sigma \), where \( g \) is a continuous function on \( \Sigma \setminus K \) with \( c = \inf_{x \in \Sigma \setminus K} |g(x)| \). Then \( c = 0 \).

**Proof.** Proceed as in the proof of Theorem 9.2, it suffices to show the result holds for \( g \geq 0 \) or \( g \leq 0 \). Let \( K \subset B(x_0, r_0) \), where \( B(x_0, r_0) \) is the geodesic ball of radius \( r_0 \) in \( M \), centered at \( x_0 \). Let \( 0 \leq \psi \leq 1 \) be the cut off function such that \( \psi \equiv 1 \) on \( B(x_0, r_0) \) and \( \psi \equiv 0 \) off \( B(x_0, 2r) \), and \( |\nabla \psi| \leq \frac{\rho}{r} \) (cf. also Lemma 1 in [We1]). Let \( i : \Sigma \hookrightarrow M \) be the inclusion mapping. Then multiplying both sides of the equation (9.10) by \( \psi \), integrating over the annulus \( B(x_0, 2r) \setminus B(x_0, r_0) \subset M \setminus K \), and applying Stokes’ Theorem, we have

\[
\begin{align*}
\int_{B(x_0, r) \setminus B(x_0, r_0)} \psi(x)g(x) d\Sigma & \leq \int_{B(x_0, 2r) \setminus B(x_0, r_0)} \psi g(x) d\Sigma \\
& = \int_{B(x_0, 2r) \setminus B(x_0, r_0)} \psi i^* \xi \\
& = \int_{B(x_0, 2r) \setminus B(x_0, r_0)} \psi i^* \left( \star \left( \frac{d\omega}{\sqrt{1 + |d\omega|^2}} \right) \right) \\
& \leq \int_{B(x_0, 2r) \setminus B(x_0, r_0)} |\nabla \psi| \frac{d\omega}{\sqrt{1 + |d\omega|^2}} d\Sigma \\
& + \int_{\partial B(x_0, r_0)} \frac{d\omega}{\sqrt{1 + |d\omega|^2}} |dS|
\end{align*}
\]

where \( dS \) is the area element of \( \partial B(x_0, r) \). Hence, dividing (9.11) by \( \text{Vol}(B(x_0, r)) \), one has

\[
c(1 - \frac{\text{Vol}(B(x_0, r_0))}{\text{Vol}(B(x_0, r))}) \leq \frac{\text{Vol}(\partial B(x_0, r_0))}{\text{Vol}(B(x_0, r))} + \frac{C_1 D(M)}{r} \to 0
\]

as \( r \to \infty \), since \( M \) has infinite volume (by Lemma 5.1 in [LWW]).

\( \square \)

## 10. Appendix: A Theorem on \( E_{F,g} \)-Energy Growth

In this Appendix, we provide a theorem on \( E_{F,g} \)-energy growth, with examples (cf. Examples 10.1 and 10.2). These in particular, imply that our growth assumptions (5.1) and (5.4) in Liouville type results are weaker than the existing growth conditions such as finite \( E_{F,g} \)-energy, slowly divergent \( E_{F,g} \)-energy (cf. (5.3)), (10.6), and (10.7).

We say that \( f(r) \sim g(r) \) as \( r \to \infty \), if \( \limsup_{r \to \infty} \frac{f(r)}{g(r)} = 1 \), and \( f(r) \sim g(r) \) as \( r \to \infty \), otherwise. We say that \( f(r) \asymp g(r) \) for large \( r \), if there exist positive constants \( k_1 \) and \( k_2 \) such that \( k_1 g(r) \leq f(r) \leq k_2 g(r) \) for all large \( r \), and \( f(r) \not\asymp g(r) \) for large \( r \) otherwise.

**Lemma 10.1.** Let \( \psi(r) > 0 \) be a continuous function such that

\[
\int_{\rho_0}^{\infty} \frac{dr}{r \psi(r)} = +\infty
\]
for some $\rho_0 > 0$. Then

(i) $\psi(r)$ can not go to infinity faster than $r^\lambda$, i.e., $\lim_{r \to \infty} \frac{\psi(r)}{r^\lambda} \neq \infty$; for any $\lambda > 0$.

(ii) If $\lim_{r \to \infty} \frac{\psi(r)}{r^\lambda}$ exists for some $\lambda > 0$, then

\begin{equation}
\lim_{r \to \infty} \frac{\psi(r)}{r^\lambda} = 0,
\end{equation}

$f(r) \sim g(r)$, and $\psi(r) \not\sim r^\lambda$.

Proof. Suppose on the contrary, i.e. $\lim_{r \to \infty} \frac{\psi(r)}{r^\lambda} = c < \infty$, where $c \neq 0$ (resp. $\lim_{r \to \infty} \frac{\psi(r)}{r^\lambda} = \infty$ ). Then there would exist $\rho_1 > 0$ such that if $r \geq \rho_1$, $\psi(r) > c^2 r^\lambda$ (resp. $\psi(r) > kr^\lambda$, where $k > 0$ is a constant.) This would lead to

\[
\int_{\rho_1}^{\infty} \frac{dr}{r^\lambda} \leq \frac{2}{c} \int_{\rho_1}^{\infty} \frac{dr}{r^{1+\lambda}} \left(\text{resp. } k \int_{\rho_1}^{\infty} \frac{dr}{r^{1+\lambda}}\right) < \infty,
\]

contradicting (5.2), by the continuity of $\psi(r)$ if $\rho_0 < \rho_1$.

\[\square\]

**Theorem 10.1.** Let $\omega \in A^p(\xi)$ have slowly divergent $\mathcal{E}_{F,\varphi}$ energy. That is,

\begin{equation}
\lim_{\rho \to \infty} \int_{B_\rho(x_0)} \frac{F(\frac{\langle \omega \rangle^2}{2})}{\psi(r(x))} dv < \infty
\end{equation}

for some continuous function $\psi(r) > 0$ satisfying (5.2).

(i) For any $\lambda > 0$, $\lim_{r \to \infty} \frac{\psi(r)}{r^\lambda} \neq \infty$.

(ii) If $\lim_{r \to \infty} \frac{\psi(r)}{r^\lambda}$ exists for some $\lambda > 0$, then

\begin{equation}
\int_{B_\rho(x_0)} F(\frac{\langle \omega \rangle^2}{2}) dv = o(\rho^\lambda) \text{ as } \rho \to \infty.
\end{equation}

Proof. In view of Lemma 10.1 and (10.1), we have for every $\epsilon > 0$, there exists $\rho_2 > 0$, such that if $r > \rho_2$, then

\begin{equation}
\psi(r) < \frac{\epsilon}{2L + 1} r^\lambda,
\end{equation}

where $L := \lim_{\rho \to \infty} \int_{B_\rho(x_0)} \frac{F(\frac{\langle \omega \rangle^2}{2})}{\psi(r(x))} dv$ (by assumption $0 \leq L < \infty$). Hence by the definition of $L$, there exists $\rho_3 > 0$ such that if $\rho > \rho_3$, then

\begin{equation}
\int_{B_\rho(x_0)} F(\frac{\langle \omega \rangle^2}{2}) dv < L + 1
\end{equation}

Since $\lim_{\rho \to \infty} \frac{1}{\rho^\lambda} \int_{B_{\rho_2}(x_0)} F(\frac{\langle \omega \rangle^2}{2}) dv = 0$, we have for every $\epsilon > 0$, there exists $\rho_4 > 0$, such that if $\rho > \rho_4$, then

\begin{equation}
\frac{1}{\rho^\lambda} \int_{B_{\rho_2}(x_0)} F(\frac{\langle \omega \rangle^2}{2}) dv < \frac{\epsilon}{2}.
\end{equation}
It follows that for every $\epsilon > 0$, one can choose $\rho_5 = \max\{\rho_2, \rho_3, \rho_4\}$, such that if $\rho > \rho_5$, then via (10.2) (10.3) and (10.4), we have

\[
\frac{1}{\rho^\lambda} \int_{B_\rho(x_0)} F\left(\frac{|\omega|^2}{2}\right) dv = \frac{1}{\rho^\lambda} \int_{B_{\rho_2}(x_0)} F\left(\frac{|\omega|^2}{2}\right) dv + \int_{B_\rho(x_0) \setminus B_{\rho_2}(x_0)} \frac{F\left(\frac{|\omega|^2}{2}\right)}{\psi(r(x))} \psi(r) \rho^\lambda dv
\]

\[
< \frac{\epsilon}{2} + \frac{\epsilon}{2(L+1)} \int_{B_{\rho(x_0) \setminus B_{\rho_2}(x_2)}} F\left(\frac{|\omega|^2}{2}\right) \frac{r^\lambda}{\psi(r(x))} \rho^\lambda dv
\]

\[
\leq \frac{\epsilon}{2} + \frac{\epsilon}{2(L+1)} \int_{B_{\rho_2}(x_0)} F\left(\frac{|\omega|^2}{2}\right) dv
\]

\[
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]

That is, (5.1) holds. \hfill \Box

**Example 10.1.** Let $\omega \in A^p(\xi)$ have the growth rate

\[
\lim_{\rho \to \infty} \int_{B_\rho(x_0)} \frac{F\left(\frac{|\omega|^2}{2}\right)}{(\ln r(x))^q} dv < \infty
\]

for some number $q \leq 1$. Then $\omega$ has slowly divergent $\mathcal{E}_{F,g}$ energy (5.3), as $\psi(r) = (\ln r)^q$ satisfies (5.2) for any number $q \leq 1$. Furthermore, as an immediate consequence of Theorem 10.1, $\omega$ has the growth rate

\[
\int_{B_\rho(x_0)} F\left(\frac{|\omega|^2}{2}\right) dv = o(\rho^\lambda) \quad \text{as} \quad \rho \to \infty
\]

for any $\lambda > 0$. The following is an example of $\psi(r)$ that does not satisfy (5.2), yet $\omega$ has the growth rate (5.1):

**Example 10.2.** Let $\omega \in A^p(\xi)$ have the growth rate

\[
\lim_{\rho \to \infty} \int_{B_\rho(x_0)} \frac{F\left(\frac{|\omega|^2}{2}\right)}{(\ln r(x))^q} dv < \infty
\]

for some number $q' > 1$. Then $\psi(r) = (\ln r)^{q'}$ does not satisfy (5.2) for any number $q' > 1$. Since $(\ln \rho)^{q'}$ goes to infinity slower than $\rho^\lambda$ for any $q', \lambda > 0$, it is evident that $\omega$ has the growth rate (5.1), via (10.3) for any $\lambda > 0$.

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