Description of the Energy Eigenstates of the 1D Hubbard Model in Terms of Rotated-Electron Site Distribution Configurations

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In this paper we describe the pseudoparticles, holons, and spinons whose occupancy configurations describe the energy eigenstates of the one-dimensional (1D) Hubbard model in terms of rotated electrons. Rotated electrons are related to electrons by a mere unitary transformation such that rotated electron double occupation is a good quantum number and the effective electronic lattice occupied by rotated electrons is identical to the real-space lattice occupied by electrons. Moreover, we find that the band-momentum pseudoparticle description associated with the Bethe-ansatz Takahashi’s thermodynamic equations is related by Fourier transform to a local pseudoparticle representation in terms of occupancy configurations of spatial coordinates. Such spatial coordinates correspond to an effective pseudoparticle lattice. There is an effective pseudoparticle lattice for each pseudoparticle branch with finite occupancy in a given state. This description introduces the local pseudoparticles whose spatial coordinate is the conjugate of the band momentum. We describe the energy eigenstates in terms of local pseudoparticle site distribution configurations in such lattices. Moreover, we relate both the local pseudoparticle internal structure and the latter configurations to the rotated-electron site distribution configurations which describe the energy eigenstates. The electron - rotated-electron unitary transformation is such that the latter configurations are independent of the value of the on-site Coulombian repulsion. Our findings provide useful information about the relation of the exotic pseudoparticles, holons, and spinons that diagonalize the non-perturbative many-electron problem to the original electrons. We provide an example showing how the derived local pseudoparticle representation can be used in the evaluation of finite-energy few-electron spectral function expressions.

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I. INTRODUCTION

Low-dimensional, correlated systems have attracted much attention in recent years due to a large variety of unconventional electronic properties directly connected with electronic correlations. In this context one-dimensional (1D) systems are of special interest, essentially stimulated by theoretical predictions like charge-spin separation for the 1D Hubbard model \[9\]. While the 1D Hubbard model \[9\] is the prototype of an exactly solvable model for correlated electrons, such a charge-spin separation corresponds to the description of the energy eigenstates in terms of occupancy configurations of holons and spinons. Recently the charge-spin separation of that model was shown to occur for all values of excitation energy, the corresponding holon and spinon description being extended to the complete set of energy eigenstates which span the Hilbert space \[9\]. On the other hand, quasi-1D materials are ideal model systems which allow the study of basic physical concepts in one dimension. In the last years there has been a renewed interest on the unconventional spectral properties of these materials \[10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34\]. Moreover, recent angle-resolved ultraviolet photoemission spectroscopy revealed very similar spectral fingerprints from both high-$T_c$ superconductors and quasi-1D compounds \[15\]. The similarity of the ultraviolet data for these two different systems could be evidence of the occurrence of the above mentioned charge-spin separation associated with holons and spinons. The anomalous temperature dependence of the spectral function could also indicate a dimensional crossover \[15, 35, 36, 37\]. The results of Refs. \[38, 39\] also suggest that the unconventional spectral properties observed in two-dimensional (2D) materials could have a 1D origin. Thus the holons and spinons could play an important role in spectral properties of both 1D and 2D low-dimensional materials.

Recently it was found that for the 1D Hubbard model there are exact selection rules which limit the number of holons and spinons created by application of rotated electron operators onto any eigenstate of the spin $\sigma$ electron number \[13\]. The concept of rotated electron is associated with a unitary transformation introduced in Ref. \[40\]. For such rotated electrons double occupation is a good quantum number for all values of the on-site Coulombian repulsion $U$. In order to distinguish the rotated-electron description from the electron representation, the concept of effective electronic lattice was introduced \[13\]. It is such that rotated electrons occupy the effective electronic lattice, whereas electrons occupy the real-space lattice. However, these lattices are identical, having the same lattice constant $a$ and length $L = a N_a$, where $N_a$ designates the number of sites. The occurrence of rotated electrons leads to clear finger prints in the few-electron spectral properties. Indeed, the few-electron spectral functions include several upper
Hubbard bands corresponding to different and separated energy scales. The \(D^{th}\) band is spanned by excited states of rotated-electron double occupation \(D_r\). However, in general only the first few upper bands have a significant amount of spectral weight. Although the above-mentioned selection rules refer to rotated-electron operators, their existence induces restrictions in the numbers of holons, spinons, and pseudoparticles contained in few-electron excitations. For instance, according to the results found in Ref. \[13\], about 99\% of the upper Hubbard band spectral weight of the one-electron addition spectral function corresponds to the first upper Hubbard band. Interestingly, rotated electron double occupation \(D_r\) equals the number of holons of \(\eta\) spin projection \(−1/2\) and charge \(−2e\). Here \(-e\) denotes the electronic charge and the \(\eta\) spin is a \(SU(2)\) algebra associated with the charge degrees of freedom \[51\] \[42\]. The charge \(−2e\) and \(+2e\) holons refer to all values of the on-site Coulomb repulsion \(U\) and are related to the large \(U\) dublons and holons respectively, studied in Ref. \[13\].

The related to the electron - rotated electron unitary transformation is part of the non-perturbative diagonalization of the quantum problem. Throughout this paper the designation rotated-electron site distribution configuration (and electron site distribution configuration) refers to the effective electronic lattice (and real-space lattice). A property of deep physical meaning is that the energy eigenstates are described by the the same rotated-electron site distribution configurations for all values of \(U/t\), where \(t\) is the first-neighbor transfer integral. This is in contrast to the complex electron site distribution configurations of these states, which change upon variations of the value of the on-site repulsion. Fortunately, the electron - rotated electron unitary transformation becomes the unit transformation as \(U/t \rightarrow \infty\). Thus one can reach the \(U/t\) independent rotated-electron site distribution configurations which describe the energy eigenstates by studying the corresponding electron site distribution configurations for the 1D Hubbard model in the limit \(U/t \rightarrow \infty\). In the present paper we study such \(U/t\) independent rotated electron configurations. In part, this is achieved by considering the corresponding problem of the electron real-space lattice charge (and spin) sequences which describe the energy eigenstates of the 1D Hubbard model in the limit \(U/t \rightarrow \infty\). The problem of the 1D Hubbard model in the limit of \(U/t \rightarrow \infty\) has been previously studied by many authors \[2\] \[3\] \[4\] \[5\] \[40\] \[44\] \[45\] \[46\] \[47\] \[48\] \[49\] \[50\] \[51\] \[52\]. In such a limit there is a huge degeneracy of \(\eta\)-spin and spin occupancy configurations. Thus there are several choices for complete sets of energy eigenstates with the same energy and momentum spectra. However, only one of these choices is associated with the rotated-electron site distribution configurations reached by the electron - rotated electron canonical unitary transformation. It refers to the complete set of energy eigenstates generated from the corresponding energy eigenstates of the 1D Hubbard model for finite values of \(U/t\) by turning off adiabatically the parameter \(t/U\). We call the obtained set of states of the model \textit{band-momentum energy eigenstates}. The electron occupancy configurations which describe these states have not been studied so far. In addition to study this problem, in this paper we discuss the relation of the obtained states to an alternative choice of energy eigenstates whose electron site distribution configurations were studied in Ref. \[51\].

Concerning the open questions whose clarification our results are useful for, one should mention the evaluation of few-electron spectral functions for finite values of excitation energy. Except in the limit \(U/t \rightarrow \infty\) \[3\] \[4\] \[5\], this is an important open problem of interest for the further understanding of the unconventional spectral properties observed in low-dimensional materials. The relationship of the pseudoparticles, holons, and spinons to the electrons is a complex problem of crucial importance for the evaluation of these spectral functions. As discussed in later sections, the concepts of local pseudoparticle and effective pseudoparticle lattices introduced in this paper, as well as the relationship of the energy-eigenstate pseudoparticle occupancy configurations in these effective lattices to the rotated-electron occupancy configurations, provide useful information about such a relationship. As open questions are concerned, we emphasize that the studies of this paper are a necessary step and a valuable contribution for the evaluation of the overlap between few-electron excitations and energy eigenstates, as further discussed in Sec. VI. Elsewhere the tools and concepts introduced in this paper are used in the study of few-electron spectral distributions for finite values of momentum and excitation energy \[53\] \[54\]. Importantly, predictions obtained recently by application of a preliminary version of the theory introduced in this paper and in Refs. \[53\] \[54\] seem to describe both qualitatively and quantitatively the one-electron removal spectral lines observed in real quasi-1D materials by photoemission experiments for finite values of the excitation energy \[55\].

The paper is organized as follows: In Sec. II we introduce the 1D Hubbard model and summarize the concept of rotated electron as well as the description of the model in terms of holons, spinons, and pseudoparticles. In Sec. III we introduce the complete set of energy eigenstates of the 1D Hubbard model in the limit \(U/t \rightarrow \infty\) which refers to the electron - rotated electron canonical unitary transformation. This includes the introduction of a set of basic properties which turn out to be useful for finding the internal structure of the local \(\alpha, \nu\) pseudoparticle, which is one of the subjects of Sec. IV. In Sec. IV we introduce a complete basis of local states which we express in terms of rotated-electron local charge, spin, and \(c\) pseudoparticle sequences. In that section we also find the rotated-electron site distribution configurations which describe the internal structure of the local \(\alpha, \nu\) pseudoparticle. The local \(c\) pseudoparticle and \(\alpha, \nu\) pseudoparticle effective lattices are introduced in Sec. V. In that section we also express the energy eigenstates in terms of the Fourier-transform superpositions of local charge, spin, and \(c\) pseudoparticle sequences introduced in the previous section. A practical application of the derived local pseudoparticle representation...
is given in Sec. VI. Finally, in Sec. VII we present the concluding remarks.

II. THE 1D HUBBARD MODEL, ROTATED ELECTRONS, AND SUMMARY OF THE PSEUDOPARTICLE, HOLON, AND SPINON DESCRIPTION

In this section we introduce the model used in our studies. In addition, we discuss the concept of rotated electron and summarize some basic information about the pseudoparticle, holon, and spinon description which is useful for the studies of this paper.

A. THE MODEL

In a chemical potential \( \mu \) and magnetic field \( H \) the 1D Hubbard Hamiltonian can be written as,

\[
\hat{H} = \hat{H}_{SO(4)} + \sum_{\alpha} \mu_{\alpha} \hat{S}_z^\alpha
\]

(1)

where the Hamiltonian,

\[
\hat{H}_{SO(4)} = \hat{H}_H - \left( \frac{U}{2} \right) \hat{N} + \left( \frac{U}{4} \right) \hat{N}_a,
\]

(2)

has \( SO(4) \) symmetry \([41, 42, 56]\) and

\[
\hat{H}_H = \hat{T} + U \hat{D},
\]

(3)

is the simple Hubbard model. The operators

\[
\hat{T} = -t \sum_{j=1}^{N_a} \sum_{\sigma=\uparrow, \downarrow} \sum_{\delta=-1, +1} \hat{c}_{j, \sigma}^\dagger \hat{c}_{j+\delta, \sigma},
\]

(4)

and

\[
\hat{D} = \sum_{j=1}^{N_a} \hat{n}_{j, \uparrow} \hat{n}_{j, \downarrow},
\]

(5)

on the right-hand side of Eq. (3) are the kinetic energy operator and the electron double occupation operator respectively. The operator \( \hat{n}_{j, \sigma} = \hat{c}_{j, \sigma}^\dagger \hat{c}_{j, \sigma} \) on the right-hand side of Eq. (5) counts the number on spin \( \sigma \) electrons at real-space lattice site \( j = 1, 2, 3, ..., N_a \), where the number of lattice sites \( N_a \) is even and large, \( N_a/2 \) is odd, and we consider periodic boundary conditions. The associated spin \( \sigma \) electron number operator reads,

\[
\hat{N}_\sigma = \sum_j \hat{n}_{j, \sigma}.
\]

(6)

The operators \( \hat{c}_{j, \sigma}^\dagger \) and \( \hat{c}_{j, \sigma} \) which appear in the above equations are the spin \( \sigma \) electron creation and annihilation operators at site \( j \) respectively. Moreover, on the right-hand side of Eq. (1), \( \mu_e = 2\mu \), \( \mu_s = 2\mu_0 H \), \( \mu_0 \) is the Bohr magneton, and \( \hat{S}_z^\sigma = -\frac{1}{2}[\hat{N}_\sigma - \hat{N}] \) and \( \hat{S}_z^\tau = -\frac{1}{2} [\hat{N}_\uparrow - \hat{N}_\downarrow] \) are the diagonal generators of the \( SU(2) \) \( \eta \)-spin \( \hat{S}^c \) and spin \( \hat{S}^\eta \) algebras \([41, 42]\) respectively. In the latter expressions \( \hat{N} = \sum_{\sigma} \hat{N}_\sigma \) is the electron number operator and the operator \( \hat{N}_\sigma \) counts the number of spin \( \sigma \) electrons and is given in Eq. (5). We denote by \( N_\uparrow \) and \( N_\downarrow \) the number of spin-up electrons and spin-down electrons respectively, and by \( N = N_\uparrow + N_\downarrow \) the number of electrons. The lattice constant is denoted by \( a \) and thus the length of the system is \( L = N_a a \). We consider electronic densities and the spin densities given by \( n = n_\uparrow + n_\downarrow \) and \( m = n_\uparrow - n_\downarrow \) respectively, where \( n_\sigma = N_\sigma/L \) and \( n = N/L \). These densities
and spin densities belong to the domains defined by the following inequalities \(0 \leq n \leq 1/a; \ 1/a \leq n \leq 2/a\) and \(-n \leq m \leq n;\ - (2/a - n) \leq m \leq (2/a - n)\) respectively.

The Hamiltonian \(\hat{H}_{SO(4)}\) defined in Eq. (2) commutes with the six generators of the \(\eta\)-spin \(S_c\) and spin \(S_s\) algebras \[41, 42, 56\]. While the expressions of the diagonal generators were provided above, the off-diagonal generators of these two algebras read,

\[
\hat{S}^{c+} = \sum_j (-1)^j c^\dagger_{j,\uparrow} c_{j,\uparrow}; \quad \hat{S}^{c-} = \sum_j (-1)^j c_{j,\uparrow} c_{j,\downarrow},
\]

and

\[
\hat{S}^{s+} = \sum_j c^\dagger_{j,\uparrow} c_{j,\uparrow}; \quad \hat{S}^{s-} = \sum_j c_{j,\uparrow} c_{j,\downarrow},
\]

respectively. We note that the Bethe-ansatz solution of the 1D Hubbard model refers to the Hilbert subspace spanned by the lowest-weight states (LWSs) of the \(\eta\)-spin and spin algebras, i.e. such that \(S^\alpha = -S^\alpha\) \[56\].

Finally, the momentum operator is given by,

\[
\hat{P} = -\frac{i}{2} \sum_{j} \sum_{\sigma} \left[ c_{j,\sigma} c_{j+1,\sigma} - c_{j+1,\sigma} c_{j,\sigma} \right],
\]

and commutes with the Hamiltonians introduced in Eqs. (1) and (2).

**B. ROTATED ELECTRON OPERATORS**

Rotated electrons are associated with a unitary transformation first introduced by Harris and Lange \[40, 47\]. Such a transformation plays a key role in the expression of the holon and spinon number operators in terms of electronic operators for the whole parameter space of the model \[13\]. The electron operators that occur in the 1D Hubbard model \[41, 42\] are defined by \(c_{j,\sigma}^\dagger\), while the rotated electron operator \(\tilde{c}_{j,\sigma}^\dagger\) is given by,

\[
\tilde{c}_{j,\sigma}^\dagger = \hat{V}^\dagger(U/t) c_{j,\sigma}^\dagger \hat{V}(U/t),
\]

where \(\hat{V}(U/t)\) is the Harris and Lange unitary operator. As mentioned in Sec. I, such rotated electrons conserve double occupation for all finite values of \(U/t\). The operators \(\hat{V}^\dagger(U/t)\) and \(\hat{V}(U/t)\) can be written as,

\[
\hat{V}^\dagger(U/t) = e^{-\tilde{S}}; \quad \hat{V}(U/t) = e^{\tilde{S}}.
\]

These operators are uniquely defined by the following two equations,

\[
\hat{H}_H = \hat{V}^\dagger(U/t) \hat{H}_H \hat{V}(U/t) = \hat{H}_H + [\hat{H}_H, \tilde{S}] + \frac{1}{2} [[\hat{H}_H, \tilde{S}], \tilde{S}] + ..., \quad (12)
\]

and

\[
[\hat{H}_H, \hat{V}^\dagger(U/t) \hat{D} \hat{V}(U/t)] = [\hat{H}_H, \hat{D}] = 0. \quad (13)
\]

In these equations the Hamiltonian \(\hat{H}_H\) is given in Eq. (3) and and the rotated-electron double occupation operator \(\hat{D}\) reads,

\[
\hat{D} \equiv \hat{V}^\dagger(U/t) \hat{D} \hat{V}(U/t) = \sum_j c_{j,\uparrow}^\dagger \tilde{c}_{j,\uparrow} c_{j,\downarrow}^\dagger \tilde{c}_{j,\downarrow}, \quad (14)
\]
where \( \hat{D} \) is the electron double occupation operator given in Eq. \( 5 \). Note that \( c_{j, \sigma}^\dagger \) and \( c_{j, \sigma}^\dagger \) are only identical in the \( U/t \to \infty \) limit where electron double occupation becomes a good quantum number. The expression of any rotated operator \( \hat{O} \) such that,

\[
\hat{O} = \hat{V}^\dagger(U/t) \hat{O} \hat{V}(U/t),
\]

in terms of the rotated electron operators \( c_{j, \sigma}^\dagger \) and \( c_{j, \sigma}^\dagger \) is the same as the expression of the corresponding general operator \( \hat{O} \) in terms of the electron operators \( c_{j, \sigma}^\dagger \) and \( c_{j, \sigma}^\dagger \) respectively. Equations \( 12 \) and \( 13 \) can be used to derive an expression for the unitary operator \( \hat{V}(U/t) \) in terms of electronic elementary operators order by order in \( t/U \). The authors of Ref. \[47\] carried out this expansion up to eighth order (see foot note \[12\] of that reference).

Operators which commute with the Harris and Langue unitary operator have the same expressions in terms of both electronic and rotated-electronic operators. Examples of such operators are the six generators of the \( SU(2) \) \( \eta \)-spin and spin algebras. Thus it follows that the commutators \([\hat{V}(U/t), \tilde{S}_\alpha \tilde{S}_\alpha] \) and \([\hat{V}(U/t), \hat{N}_\sigma] \) vanish, where \( \tilde{S}_\alpha \tilde{S}_\alpha \equiv \tilde{\eta} \tilde{\eta} \) and \( \tilde{\eta} \tilde{\eta} = \tilde{\ell} \tilde{\ell} \tilde{\eta} \tilde{\eta} \) are the square \( \eta \)-spin and spin operators respectively, and \( \hat{N}_\sigma \) is the spin \( \sigma \) electron number operator \[6\]. The Harris and Langue unitary operator also commutes with the momentum operator \[4\].

C. SUMMARY OF THE PSEUDOPARTICLE, HOLON, AND SPINON DESCRIPTION

Let us summarized the aspects of the holon, spinon, and band-momentum pseudoparticle description introduced in Ref. \[11\] and further investigated in Refs. \[12, 13\] which are useful for the studies of this paper. Such a description is extracted directly from combination of the Bethe-ansatz Takahashi’s thermodynamic equations with the \( SO(4) \) symmetry of the model and the symmetries associated with the concept of rotated diagonalization. Indeed, holons, spinons, and band-momentum pseudoparticles emerge naturally from the non-perturbative diagonalization of the model. The first step of such a diagonalization is precisely the Hilbert-space unitary rotation which maps electrons onto rotated electrons and is such that rotated-electron double occupation is a good quantum number. Let us denote by \( N_c \) the number of rotated-electron singly occupied sites. Thus the numbers \( N_{c, \eta} \) \( N_{c, \eta} = [N - N_c]/2 \), and \( [N_{c, \eta} - N_c]/2 \) equal the number of rotated-electron singly occupied sites, doubly occupied sites, and empty sites respectively. (Here \( N_{c, \eta} = (2 N_{a, \eta} - N_c) \).) Importantly, once rotated electron double occupation is a good quantum number also all these numbers are good quantum numbers for all values of \( U/t \). There are precisely \( N_c \) \( \eta \)-pseudoparticles and \( N_c \) spinons. These \( \eta \)-pseudoparticles (and spinons) describe the charge part (and spin part) of the \( N_c \) rotated electrons which singly occupy lattice sites. Thus the charge and spin degrees of freedom of these \( N_c \) rotated electrons separate. For each \( \eta \)-pseudoparticle there is a chargeon and a rotated-electronic hole. The chargeon corresponds to the charge part of the rotated electron which singly occupies the site. The non-perturbative organization of the electronic degrees of freedom associates the \( [N - N_c] \) rotated electrons which doubly occupy \( [N - N_c]/2 \) sites with \( [N - N_c]/2 \) holons of \( \eta \)-spin projection \(-1/2\). Indeed each rotated-electron doubly occupied site corresponds to a spin singlet rotated electron pair which is nothing but a \(-1/2\) holon. Finally, the \( [N_{c, \eta} - N_c]/2 \) rotated-electron empty sites are nothing but the \( [N_{c, \eta} - N_c]/2 \) holons of \( \eta \)-spin projection \(+1/2\). Such an analysis confirms the importance played by the unitary operator \( \hat{V}(U/t) \) in the description of the quantum objects whose occupancy configurations describe all energy eigenstates of the model.

Let us summarize some of the properties of the quantum objects which we have briefly related to rotated-electrons above. One should distinguish the total \( \eta \) spin (and spin) of the system, which we denote by \( S_c \) (and \( S_s \)) and the corresponding \( \eta \)-spin (and spin) projection, which we denote by \( S^\pm_\eta \) (and \( S^\pm_\eta \)) from the \( \eta \) spin (and spin) carried by the elementary quantum objects. We call \( s_c \) (and \( s_s \)) the \( \eta \) spin (and spin) carried by the holons, spinons, and \( \eta \)-pseudoparticles and \( \sigma_c \) (and \( \sigma_s \)) their \( \eta \)-spin (and spin) projection. The holons are such that \( s_c = 1/2, s_s = 0 \), and \( \sigma_c = \pm 1/2 \) whereas the spinons have \( s_s = 1/2 \) and \( \sigma_s = \pm 1/2 \). Through out this paper we denote the holons and spinons according to their \( \pm 1/2 \) \( \eta \)-spin and spin projections respectively. The \( 1/2 \) and \( 1/2 \) holons have charge \(-2e\) and \( +2e \) respectively. It is shown in later sections that the band-momentum \( c \)-pseudoparticle is associated with the local \( c \)-pseudoparticle. The latter local quantum object describes the chargeon and the corresponding rotated electronic hole of each rotated-electron singly occupied site. There are also \( 2\nu \)-holon (and \( 2\nu \)-spinon) composite \( c, \nu \) pseudoparticles (and \( s, \nu \) pseudoparticles). The \( c, \nu \) pseudoparticle has \( s_c = 0 \) and \( s_s = 0 \) (and the \( s, \nu \) pseudoparticle has \( s_s = 0 \) and no charge degrees of freedom) and contains an equal number \( \nu \) of \(-1/2 \) and \(+1/2 \) \( \eta \)-spin projection holons (and spin projection spinons). The \( c \)-pseudoparticle has \( s_c = 0 \) and no spin degrees of freedom. The \( \pm 1/2 \) holons (and \( \pm 1/2 \) spinons) which are not part of \( 2\nu \)-holon composite \( c, \nu \) pseudoparticles (and \( 2\nu \)-spinon composite \( s, \nu \) pseudoparticles) are called \( \pm 1/2 \) Yang holons (and \( \pm 1/2 \) HL spinons). In the designations \( HL \) spinon and Yang
holon, HL stands for Heilmann and Lieb and Yang refers to C. N. Yang respectively, who are the authors of Refs. [11, 12].

All energy eigenstates can be specified by the $c$ pseudoparticle band-momentum distribution function $N_c(q)$, by the set of $\alpha, \nu$ pseudoparticle band-momentum distribution functions $\{N_{\alpha,\nu}(q)\}$ where $\alpha = c, s$ and $\nu = 1, 2, \ldots$, and by the set of numbers $L_{c, -1/2}$ of $s_c = 1/2$, $s_s = 0$, and $-1/2$ Yang holons ($\alpha = c$) and $s_s = 1/2$ and $-1/2$ HL spinons ($\alpha = s$). An important concept is that of CPHS ensemble subspace where CPHS stands for $c$ pseudoparticle, holon, and spinon. This is a Hilbert subspace spanned by all states with fixed values for the $-1/2$ Yang holon number $L_{c, -1/2}$, $-1/2$ HL spinon number $L_{s, -1/2}$, $c$ pseudoparticle number $N_c$, and for the sets of $\alpha, \nu$ pseudoparticle numbers $\{N_{c,\nu}\}$ and $\{N_{s,\nu}\}$ corresponding to the $\nu = 1, 2, 3, \ldots$ branches.

The variable $q$ appearing in the above distribution functions $N_c(q)$ and $\{N_{\alpha,\nu}(q)\}$ where $\alpha = c, s$ and $\nu = 1, 2, \ldots$ is the continuum band momentum associated with the discrete band-momentum values $q_j$ which are of the following form,

\[
q_j = \frac{2\pi}{L} I_j^c; \quad j = 1, \ldots, N_c, \\
q_j = \frac{2\pi}{L} I_j^{\alpha,\nu}; \quad j = 1, \ldots, N_{\alpha,\nu}; \quad \alpha = c, s, \nu = 1, 2, \ldots. \tag{16}
\]

The $I_j^c$ numbers are integers (half-odd integers), if $\frac{N_c}{2} - \sum_{\alpha=c,s} \sum_{\nu=1}^{\infty} N_{\alpha,\nu}$ is odd (even) and the $I_j^{\alpha,\nu}$ numbers are integers (half-odd integers), if $N_{\alpha,\nu}^{*}$ is odd (even). The discrete index $j$ can have the following values $j = 1, 2, 3, \ldots, N_c$ and $j = 1, 2, 3, \ldots, N_{\alpha,\nu}$ for the $c$ and $\alpha, \nu$ pseudoparticles respectively, where the number of lattice sites $N_c$ and the number

\[
N_{\alpha,\nu}^{*} = N_{\alpha,\nu} + N_{\alpha,\nu}^{h}, \tag{17}
\]

where

\[
N_{\alpha,\nu}^{h} = L_{\alpha} + 2 \sum_{\nu' = \nu + 1}^{\infty} (\nu' - \nu)N_{\alpha,\nu'}, \tag{18}
\]

give the total number of corresponding discrete band-momentum values $q_j$. On the right-hand side of Eq. (17) $N_{\alpha,\nu}$ and $N_{\alpha,\nu}^{h}$ are the number of $\alpha, \nu$ pseudoparticles and $\alpha, \nu$ pseudoparticle holes respectively. On the right-hand side of Eq. (18) $L_{\alpha}$ is the number of Yang holons ($\alpha = c$) and HL spinons ($\alpha = s$) such that $L_{\alpha} = 2S_{\alpha}$ where $S_{\alpha}$ is the $\eta$-spin ($\alpha = c$) and spin ($\alpha = s$) respectively [11]. On the other hand, the number of $c$ pseudoparticle holes is $N_{\alpha}^{h} = [N_{\alpha} - N_c]$, as given below in Eq. (23). The pseudoparticles obey a Pauli principle respecting the band-momentum occupancies, i.e. a discrete band-momentum value $q_j$ can either be unoccupied or singly occupied by a pseudoparticle.

It follows from Eq. (16) that the discrete band-momentum values $q_j$ are such that,

\[
q_{j+1} - q_j = \frac{2\pi}{L}. \tag{19}
\]

The band-momentum distribution functions $N_c(q)$ and $\{N_{\alpha,\nu}(q)\}$ where $\alpha = c, s$ and $\nu = 1, 2, \ldots$ read $N_c(q_j) = 1$ and $N_{\alpha,\nu}(q_j) = 1$ for occupied values of the discrete band momentum $q_j$ and $N_c(q_j) = 0$ and $N_{\alpha,\nu}(q_j) = 0$ for unoccupied values of that band momentum. The limiting band-momentum value which defines the continuum band-momentum $q$ range $-q_{\alpha,\nu}$ $\leq$ $q$ $\leq$ $q_{\alpha,\nu}$ of the $\alpha, \nu$ pseudoparticle branch is given by,

\[
q_{\alpha,\nu} = \frac{\pi}{L} [N_{\alpha,\nu}^{*} - 1] \approx \frac{\pi N_{\alpha,\nu}^{*}}{L}. \tag{20}
\]

The $c$-pseudoparticle continuum band-momentum limiting values read

\[
q_c^{\pm} = \pm q_c, \tag{21}
\]

for $\frac{N_c}{2} - \sum_{\alpha=c,s} \sum_{\nu=1}^{\infty} N_{\alpha,\nu}$ even and

\[
q_c^+ = q_c + \frac{\pi}{L}; \quad q_c^- = -q_c + \frac{\pi}{L}, \tag{22}
\]
for $\frac{N_c}{2} = \sum_{\nu=1}^{N_c} N_{\alpha,\nu} \text{odd. Here}$

$$q_c = \frac{\pi}{a} \left[1 - \frac{1}{N_c}\right] \approx \frac{\pi}{a}.$$  (23)

The numbers $L_{\alpha,\sigma}$ of $\sigma_c$ Yang holons ($\alpha = c$) and of $\sigma_s$ HL spinons ($\alpha = s$) such that $L_\alpha = \sum_{\sigma=\pm 1/2} L_{\alpha,\sigma}$ are given by,

$$L_{\alpha,\sigma} = S_\alpha - 2\sigma \alpha S_\sigma = \frac{L_\alpha}{2} - 2\sigma \alpha S_\sigma^z; \quad \alpha = c, s, \quad \sigma = \pm 1$$  (24)

where the values $S_\alpha$ of \eta spin ($\alpha = c$) or spin ($\alpha = s$) of the system can be written as,

$$S_\alpha = \frac{1}{2} [L_{\alpha,+1/2} + L_{\alpha,-1/2}] = \frac{1}{2} M_\alpha - \sum_{\nu=1}^{\infty} \nu N_{\alpha,\nu}; \quad \alpha = c, s.$$  (25)

Here $M_\alpha$ denotes the total number of holons $M_c$ and of spinons $M_s$ and $N_{\alpha,\nu}$ gives the number of $\alpha, \nu$ pseudoparticles. The \eta-spin ($\alpha = c$) and spin ($\alpha = s$) projections read,

$$S_c^z = -\frac{1}{2} [L_{c,+1/2} - L_{c,-1/2}] = -\frac{1}{2} [M_{c,+1/2} - M_{c,-1/2}];$$

$$S_s^z = -\frac{1}{2} [L_{s,+1/2} - L_{s,-1/2}] = -\frac{1}{2} [M_{s,+1/2} - M_{s,-1/2}],$$  (26)

respectively, where $M_{c,\sigma}$ and $M_{s,\sigma}$ give the total number of $\sigma_c$ holons and of $\sigma_s$ spinons respectively. These numbers can be written as follows,

$$M_{\alpha,\sigma} = L_{\alpha,\sigma} + \sum_{\nu=1}^{\infty} \nu N_{\alpha,\nu}; \quad \alpha = c, s.$$  (27)

The holon and spinon numbers $M_c$ and $M_s$ respectively, can be expressed as,

$$M_c = \sum_{\sigma=\pm 1/2} M_{c,\sigma} = N^h_c = [N_a - N_c]; \quad M_s = \sum_{\sigma=\pm 1/2} M_{s,\sigma} = N_c,$$  (28)

where $N_c$ and $N^h_c$ are the numbers of $c$ pseudoparticles and $c$ pseudoparticle holes respectively.

The above equations are valid for all energy eigenstates. Let us now consider the particular case of a ground state. In the case of electronic densities $n$ and spin densities $m$ such that $0 \leq n \leq 1/a$ and $0 \leq m \leq n$ respectively, a ground state belongs to a CPHS ensemble subspace with the values for the pseudoparticle, $\pm 1/2$ holon, and $\pm 1/2$ spinon numbers given by $L_{c,-1/2} = M_{c,-1/2} = 0$, $M_{c,+1/2} = L_{c,+1/2} = (N_a - N^0_c)$, $N_c = N^0_c$, and $N_{c,\nu} = 0$ in the charge sector. In the spin sector the numbers are $M_{s,-1/2} = N^0_{s,1} = N^0_s$, $M_{s,+1/2} = N^0_{s,1} - N^0_s$, $N_{s,\nu} = 0$, and $L_{s,-1/2} = N_{s,\nu} = 0$. In these expressions the electron numbers $N^0_c$, $N^0_s$, and $N^0 = N^0_c + N^0_s$ refer to the ground state. In the case of the ground-state CPHS ensemble subspace the expression of the number $N^*_{\alpha,\nu}$ given in Eqs. 17 and 18 also simplifies 12 and reads,

$$N^*_{c,\nu} = (N_a - N^0_c); \quad N^*_{s,1} = N^0_s; \quad N^*_{s,\nu} = (N^0_{s,1} - N^0_s), \quad \nu > 1,$$  (29)

whereas $N^*_c$ is given by $N^*_c = N_c$ for all energy eigenstates. It follows that also the expressions of the limiting band-momentum values (20)–(23) simplify in the case of the ground-state CPHS ensemble subspace. Disregarding corrections of the order of $1/L$, in the case of the ground state these limiting band-momentum values can be written as $\pm q_c^0$ and $\pm q_{s,\nu}^0$, where

$$q_c^0 = \pi/a; \quad q_{s,1}^0 = k_F^1; \quad q_{s,\nu}^0 = [\pi/a - 2k_F^\nu]; \quad q_{s,\nu}^0 = [k_F^1 - k_F^\nu], \quad \nu > 1.$$  (30)

On the right-hand side of Eq. 30 the Fermi momenta are such that $2k_F^1 = k_F^+ - k_F^-$, $k_F^1 = \pi n^1$, and $k_F^\nu = \pi n^\nu$. In most situations one can disregard the $1/L$ corrections and use the band-momentum limiting values given in Eq. 30.
III. ENERGY EIGENSTATES AND BASIC PROPERTIES ASSOCIATED WITH THE PSEUDOPARTICLE INTERNAL STRUCTURE AND EMPTY SITES

As mentioned in Secs. I and II, the energy eigenstates of the 1D Hubbard model are described by the same rotated-electron site distribution configurations for all values of $U/t$. Fortunately, since the electron-rotated electron unitary transformation becomes the unit transformation as $U/t \to \infty$, one can reach the $U/t$ independent rotated-electron site distribution configurations which describe the energy eigenstates by studying the corresponding electron site distribution configurations for the 1D Hubbard model in the limit $U/t \to \infty$. Thus the main goal of this section is to extract from the physics associated with the $U/t \to \infty$ limit of the model useful information for the construction of the rotated-electron occupancy configurations that describe the energy eigenstates for finite values of $U/t$.

In the limit $U/t \to \infty$ there is a huge degeneracy of $\eta$-spin and spin occupancy configurations. Such a degeneracy results from the simple form that the kinetic energy $T$ and potential energy $V$ spectra associated with the operators $\hat{C}_\alpha, \nu$ and $\hat{C}_{\alpha, \nu}$ respectively, have in the limit $U/t \to \infty$,

$$E_H = T + V; \quad T = 2t \sum_{j=1}^{N_e} N_e(q_j) [-2t \cos q_j]; \quad V/U = D,$$

where $E_H$ is the energy spectrum of the Hamiltonian $\hat{H}$ and the electron double occupation $D$ is a good quantum number which equals rotated-electron double occupation in that limit. The kinetic energy equals that of a system of free spin-less fermions $\hat{C}_\alpha, \nu$. The corresponding momentum spectrum is given in Eq. (A3) of Appendix A. A short discussion of the relation of our results to some of the well known concepts of the $U/t \to \infty$ physics is presented in that Appendix. Since the energy spectrum (31) is independent of the $\eta$-spin and spin occupancy configurations, there are several choices for complete sets of energy eigenstates with the same energy and momentum spectra. However, only one of these choices is associated with the rotated-electron site distribution configurations which describe the energy eigenstates for all values of $U/t$. The discussion presented in Appendix A and in this section about the relation of the energy-eigenstate rotated-electron description introduced in this paper to other choices of energy eigenstates and to the $t/U << 1$ physics associated with the leading-order $t/U$ corrections to the energy spectrum $\hat{C}_\alpha, \nu$, contextualizes the quantum problem studied in this paper and contributes to its deeper understanding.

The pseudoparticle band momentum obeys well defined boundary conditions which are a necessary condition for the fulfillment of the periodic boundary conditions for the original electrons. However, such a pseudoparticle band-momentum boundary conditions are not a sufficient condition to ensure the electronic periodic boundary conditions. A second condition imposes that the internal structure of the local $\alpha, \nu$ pseudoparticles introduced in later sections must be of a specific form. In this section we introduce a set of properties which are rather useful both for the construction of the specific rotated-electron site distribution configurations which describe the internal structure of such local $\alpha, \nu$ pseudoparticles and the definition of the corresponding pseudoparticle empty sites in terms of occupancy configurations of rotated-electron sites. As further discussed in Sec. VI, the construction of the rotated-electron site distribution configurations which describe the energy eigenstates and the introduction of the associated concepts of local pseudoparticle and effective pseudoparticle lattice are necessary steps for the evaluation of few-electron spectral functions at finite values of excitation energy.

Our choice of energy eigenstates corresponds to the band-momentum energy eigenstates associated with the thermodynamic Bethe-ansatz equations introduced by Takahashi $\hat{5}$. As discussed in Appendix A, for $U/t \to \infty$ there are other choices for complete sets of energy eigenstates. One of the alternative complete set of energy and momentum eigenstates for the model in the limit $U/t \to \infty$ can be explicitly constructed for the Harris-Lange model mentioned in Appendix A by the method used in Ref. $\hat{51}$. Both the band-momentum energy eigenstates and the symmetrized energy eigenstates used in that reference are superpositions of charge (and spin) sequences formed by local electron distribution configurations of doubly occupied and empty sites (and spin-down and spin-up singly occupied sites). The expression of both these two sets of energy eigenstates ensures the periodic boundary conditions for the original electronic problem.

Throughout this paper we denote the rotated-electron doubly occupied and empty sites by $\bullet$ and $\circ$ (and the spin-down and spin-up rotated-electron singly occupied sites by $\downarrow$ and $\uparrow$) respectively. In the present $U/t \to \infty$ limit such a concept also refers to electrons. Often we add an index to these symbols which defines the position of the doubly occupied site or empty site (and spin-down singly occupied site or spin-up singly occupied site). In the case of the symmetrized energy eigenstates the charge and spin sequences are properly symmetrized owing to the periodic boundary conditions of the original problem. (This justifies the designation of symmetrized energy eigenstates.) The procedure used in such a symmetrization involves powers of suitable charge and spin operators $\hat{T}_C$ and $\hat{T}_S$ respectively, for cyclic permutation of the electron site distribution configurations of the local charge and spin sequences $\hat{51}$. For
instance, let

\[ (\bullet, \bullet, \circ, \circ, \ldots, \circ, \bullet, \bullet) , \]  

(32)

and

\[ (\downarrow, \uparrow, \downarrow, \uparrow, \ldots, \downarrow, \downarrow, \uparrow) , \]  

(33)

be a charge and a spin sequence respectively. Then the operators \( \hat{T}_C \) and \( \hat{T}_S \) are such that,

\[ \hat{T}_C (\bullet, \bullet, \circ, \circ, \ldots, \circ, \bullet, \bullet) = (\circ, \bullet, \bullet, \circ, \circ, \ldots, \circ, \bullet, \bullet) , \]  

(34)

and

\[ \hat{T}_S (\downarrow, \uparrow, \downarrow, \uparrow, \ldots, \downarrow, \downarrow, \uparrow) = (\uparrow, \downarrow, \downarrow, \uparrow, \ldots, \downarrow, \downarrow, \uparrow) , \]  

(35)

respectively. The powers \( [\hat{T}_C]^{K_C} \) and \( [\hat{T}_S]^{K_S} \) were called in Ref. [51] \( K_C \) and \( K_S \) respectively. This introduces the charge momentum \( k_C \) and spin momentum \( k_S \) respectively, such that,

\[
\begin{align*}
    k_C &= \frac{2\pi}{K_C a} m_C ; \\
    m_C &= 0, 1, \ldots, K_C - 1 , \\
    k_S &= \frac{2\pi}{K_S a} m_S ; \\
    m_S &= 0, 1, \ldots, K_S - 1 .
\end{align*}
\]  

(36)

The symmetrized energy eigenstates are classified according to their charge and spin sequence, their charge momentum \( k_C \) and spin momentum \( k_S \), and a number \( N_C \) of momenta which in the notation of Ref. [51] equals the number of charges. The latter discrete momenta are closely related to the discrete band-momentum values occupied by \( c \) pseudoparticle holes in the pseudoparticle representation of the band-momentum energy eigenstates studied in later sections. Moreover, the number \( N_C \) of charges equals both the number \( N_c^h \) of \( c \) pseudoparticle holes and the number \( M_c \) of holons. Thus this is a good quantum number for both the band-momentum and symmetrized energy eigenstates. However, the charge momentum \( k_C \) and spin momentum \( k_S \) are eigenvalues of the charge momentum operator \( \hat{k}_C \) and spin momentum operator \( \hat{k}_S \) respectively, which in general do not commute with the set of operators \( \{ N_{\alpha, \nu}(q_j) \} \) of the band-momentum basis. Thus in general the band-momentum energy eigenstates are not eigenstates of the charge momentum operator \( \hat{k}_C \) and spin momentum operator \( \hat{k}_S \). Exceptions are the band-momentum energy eigenstates with occupancy of a single \( c, \nu \) pseudoparticle (and a single \( s, \nu \) pseudoparticle) and with no finite occupancy of Yang holons (and HL spinons) and of \( c, \nu' \) pseudoparticles (and \( s, \nu' \) pseudoparticles) belonging to other branches such that \( \nu' \neq \nu \). We find below that in this case the corresponding band-momentum energy eigenstate is also an eigenstate of the charge (and spin) momentum operator of eigenvalue \( k_C = \pi/a \) (and \( k_S = \pi/a \)).

According to Eq. [19] the discrete values of the band momentum \( q_j \) are such that \( q_{j+1} - q_j = 2\pi/L \) and \( q_j = [2\pi/L] I_j^c \) or \( q_j = [2\pi/L] I_j^{s, \nu} \) where the numbers \( I_j^c \) and \( I_j^{s, \nu} \) with \( j = 1, 2, \ldots, N_c \) and \( j = 1, 2, \ldots, N_s^{\alpha, \nu} \) respectively, are integers or half-odd integers as a result of the following boundary conditions,

\[ e^{iq_j L} = (e^{i\pi})^{\sum_{\alpha = c, s} \sum_{\nu = 1}^{N_{\alpha, \nu}}} N_{\alpha, \nu} , \]  

(37)

in the case of the \( c \) pseudoparticle branch and,

\[ e^{iq_j L} = (e^{i\pi})^{\sum_{\alpha = c, s} \sum_{\nu = 1}^{N_{\alpha, \nu}}} [1 + N_{\alpha, \nu}^*] \]  

(38)

in the case of the \( \alpha, \nu \) pseudoparticle branches. Here \( N_{\alpha, \nu} \) and \( L_{\alpha, \nu} \) denote the number of \( \alpha, \nu \) pseudoparticles and of Yang holons (\( \alpha = c \)) or HL spinons (\( \alpha = s \)) respectively, and the number \( N_{\alpha, \nu}^* \) is given in Eq. [17]. According to Eq. [20], in the case of the \( \alpha, \nu \) pseudoparticle branches the \( j = 1 \) minimum and \( j = N_{\alpha, \nu}^* \) maximum index of the band-momentum values \( q_j \) are such that,

\[ -q_1 = q_{N_{\alpha, \nu}^*} = q_{\alpha, \nu} = \frac{\pi}{L} [N_{\alpha, \nu}^* - 1] . \]  

(39)
In the case of the $c$ pseudoparticles the limiting values are $q_1 = q_c^-$ and $q_{N_a} = q_c^+$ where the band momenta $q_c^\pm$ are defined in Eqs. (21), (23).

There is a holon, spinon, $c$ pseudoparticle separation for the whole parameter space of the 1D Hubbard model \[11\]. In the case of the $t/U \to 0$ limit the description of the $c$ pseudoparticle excitation sector is very similar for both the representations in terms of band-momentum energy eigenstates and of symmetrized energy eigenstates. The separation of the charge and spin excitation sectors occurs for these two alternative representations. The construction of the band-momentum energy eigenstates also involves superpositions of charge (and spin) sequences associated with rotated-electron distribution configurations of doubly occupied and empty sites (and spin-down and spin-up singly occupied sites). Such superpositions are also due to the periodic boundary conditions but are not in general generated by the above symmetrization procedure. In spite of these general similarities, the symmetrized and band-momentum energy eigenstates of the 1D Hubbard model in the limit of $t/U \to 0$ are different states. The difference between the symmetrized and band-momentum energy eigenstates refers in general both to the form of the local electron distribution configurations of doubly occupied and empty sites (and spin-down and spin-up singly occupied sites) which describe the charge (and spin) sequences and to the form of the superposition of these local sequences which describes the energy eigenstates.

Let us introduce eight properties which correspond to a first step of the introduction of the concepts of local pseudoparticle and effective pseudoparticle lattices. However, the precise definition of these concepts involves clarification of several issues and is only fulfilled in Sec. V. These properties are used in the ensuing section in finding the electron site distribution configurations of the local charge and spin sequences whose Fourier-transform superpositions describe the band-momentum energy eigenstates of the model in the limit $t/U \to 0$. These properties follow in part from symmetries and features of the pseudoparticle, holon, and spin description and related rotated electron representation studied in Refs. \[11, 12, 13\] and from well known properties associated with the $t/U \to 0$ physics. We emphasize that the properties given below also apply to finite values of $t/U$ provided that electrons are replaced by rotated electrons. These useful properties read:

1-III The numbers of electron doubly occupied sites, empty sites, spin-down singly occupied sites, and spin-up singly occupied sites are good quantum numbers whose values are equal to the total numbers of $-1/2$ holons, $+1/2$ holons, $-1/2$ spinons, and $+1/2$ spinons respectively, of the band-momentum energy eigenstates.

2-III The kinetic energy $T$ given in Eq. (31) arises from the movements of the singly occupied sites relative to the doubly occupied and empty sides which do not change double occupation. These movements are fully described by the $c$ pseudoparticles which are associated with the charge degrees of freedom of these sites. In the limit $t/U \to 0$ these quantum objects acquire a spin-less fermion spectrum. On the other hand, the electron distribution configurations of doubly occupied and empty sites and of spin-down and spin-up singly occupied sites do not contribute to the kinetic energy $T$. Moreover these electron site distribution configurations must remain unchanged in spite of the movements of the $c$ pseudoparticles. Alternatively, we can consider that the electron doubly occupied and empty sites move relative to the singly occupied sites. In this case one describes the movements of the $c$ pseudoparticles in terms of the movements of the corresponding $c$ pseudoparticle holes. This is the choice of Ref. \[5\] for the case of the symmetrized energy eigenstates. We recall that the numbers $N_c$ of $c$ pseudoparticles, $N_c^h$ of $c$ pseudoparticle holes, $M_s$ of spinons, and $M_c$ of holons are such that $N_c + N_c^h = N_a$, $N_c = M_s$, and $N_c^h = M_c$ and thus these two alternative descriptions are fully equivalent.

3-III We call local charge sequences and local spin sequences the occupancy configurations of $\pm 1/2$ holons (and corresponding electron distribution configurations of doubly occupied and empty sites) and the occupancy configurations of the $\pm 1/2$ spinons (and corresponding electron distribution configurations of singly occupied sites of spin projection $\pm 1/2$) respectively. These local charge (and spin) sequences can also be expressed in terms of occupancy configurations of $\pm 1/2$ Yang holons and local $c, \nu$ pseudoparticles (and $\pm 1/2$ HL spinons and local $s, \nu$ pseudoparticles) belonging to $\nu = 1, 2, 3...$ branches. From the general properties introduced in Ref. \[11\] one finds that the electron description of these quantum objects in terms of distribution configurations of electron doubly occupied and empty sites (and spin-down and spin-up singly occupied sites) is as follows: A $-1/2$ Yang holon (and $-1/2$ HL spinon) is described by a doubly occupied site (and a spin-down singly occupied site); A $+1/2$ Yang holon (and $+1/2$ HL spinon) is described by an empty site (and a spin-up singly occupied site); A local $c, \nu$ pseudoparticle (and a local $s, \nu$ pseudoparticle) is described by a number $\nu$ of doubly occupied sites and a number $\nu$ of empty sites (and a number $\nu$ of spin-down singly occupied sites and a number $\nu$ of spin-up singly occupied sites). The electron distribution configurations of doubly occupied and empty sites (and of spin-down and spin-up singly occupied sites) of any local charge (and spin) sequence can be expressed in terms of a corresponding occupancy configuration of $\pm 1/2$ Yang holons and local $c, \nu$ pseudoparticles (and $\pm 1/2$ HL spinons and local $s, \nu$ pseudoparticles) belonging
to the $\nu = 1, 2, 3...$ branches. The form of Eqs. 4-III and 5-III of Appendix A reveals that a $-1/2$ holon carries momentum $\pi/a$ and that in the limit $t/U \rightarrow 0$ each doubly occupied site must also be associated with a momentum $\pi/a$ respectively. Therefore, the expressions of the local charge sequences must include a phase factor operator $\exp(i\pi \sum_j j \hat{D}_j)$ such that the $j$ summation runs over the sites doubly occupied and empty and the local double occupation operator has eigenvalues 1 and 0 if the site $j$ is doubly occupied and empty respectively.

4-III The number of different possible discrete locations of the electron distribution configurations of doubly occupied and empty sites ($\alpha = c$) or of spin-down and spin-up singly occupied sites ($\alpha = s$) describing a local $\alpha, \nu$ pseudoparticle equals the number $N_{\alpha, \nu}$ given in Eq. 7-III. Each of these possible positions corresponds to a different local charge ($\alpha = c$) or spin ($\alpha = s$) sequence. The number $N_{\alpha, \nu}$ is directly provided by the Bethe-ansatz solution II since it also equals the number of different discrete band momentum values $q_j$, where $j = 1, 2, ..., N_{\alpha, \nu}$, of the band-momentum $\alpha, \nu$ pseudoparticle band.

5-III In case of energy eigenstates with finite occupancy of $-1/2$ and $+1/2$ Yang holons (and $-1/2$ and $+1/2$ HL spinons) the electron distribution configurations of doubly occupied and empty sites (and spin-down and spin-up singly occupied sites) describing the local $c, \nu$ pseudoparticle (and local $s, \nu$ pseudoparticle) must remain unchanged under the application of the off-diagonal generators of the $SU(2)$ $\eta$-spin algebra given in Eq. 7-III (and off-diagonal generators of the $SU(2)$ spin algebra given in Eq. 8-III). Moreover, in case of states with no $+1/2$ or $-1/2$ Yang holons (and no $+1/2$ or $-1/2$ HL spinons) application of the operators $\hat{S}_c^+=\sum_j(1)^j\hat{c}_{j,\uparrow}^\dagger\hat{c}_{j,\downarrow}$ or $\hat{S}_c=\sum_j(1)^j\hat{c}_{j,\uparrow}\hat{c}_{j,\downarrow}$ (and $\hat{S}_s^+=\sum_j\hat{c}_{j,\uparrow}^\dagger\hat{c}_{j,\downarrow}$ or $\hat{S}_s=\sum_j\hat{c}_{j,\uparrow}\hat{c}_{j,\downarrow}$) onto these states must give zero. These requirements result from the value of the $\eta$ spin of the $c, \nu$ pseudoparticles (and spin of the $s, \nu$ pseudoparticles) which is given by $s_{c} = 0$ (and $s_{s} = 0$) 11. On the other hand, the transformations generated by application of these off-diagonal generators onto the electron distribution configurations of doubly occupied and empty sites (and spin-down and spin-up singly occupied sites) describing the $\pm1/2$ Yang holons (and $\pm1/2$ HL spinons) must be the ones defined by the $\eta$-spin (and spin) algebra 11.

6-III Let us consider a local charge (and spin) sequence with no Yang holons (and no HL spinons) and consisting of $\nu$ electron doubly occupied sites and $\nu$ electron empty sites (and $\nu$ electron spin-down singly occupied sites and $\nu$ electron spin-up singly occupied sites). If such a sequence describes a single local $c, \nu$ pseudoparticle (and local $s, \nu$ pseudoparticle) it is properly symmetrized in such way that the distribution configurations associated with the internal structure of that quantum object remain unchanged under cyclic permutations. Furthermore, property 5-III imposes that a band-momentum energy eigenstate with the above Yang holon and $c, \nu$ pseudoparticle (and HL spinon and $s, \nu$ pseudoparticle) numbers is an eigenstate of the charge (and spin) momentum operator $\hat{k}_C$ (and $\hat{k}_S$) of eigenvalue $k_C = \pi/a$ (and $k_S = \pi/a$).

7-III The electron distribution configurations of the $\nu$ doubly occupied sites and $\nu$ empty sites (and $\nu$ spin-down singly occupied sites and $\nu$ spin-up singly occupied sites) which describe the internal structure a local $c, \nu$ pseudoparticle (and $s, \nu$ pseudoparticle) should be the same for all local charge ($\alpha = c$) or spin ($\alpha = s$) sequences involved in the description of the $4^{N_{\nu}}$ band-momentum energy eigenstates. This is a necessary condition for the indiscernible character of local $\alpha, \nu$ pseudoparticles with the same lattice position but involved in occupancy configurations describing different local charge ($\alpha = c$) or spin ($\alpha = s$) sequences. This indiscernible character of the local $\alpha, \nu$ pseudoparticles follows from the corresponding indiscernible character of the $\alpha, \nu$ pseudoparticles of band-momentum $q_j$, which are indistinguishable quantum objects.

8-III Since in the limit $t/U \rightarrow 0$ there is nearest-neighbor hopping only and it does not change double occupation, the charge, spin, $C$ pseudoparticle separation studied in Ref. 11 implies that both the local charge and spin sequences of the energy eigenstates must be separately conserved. Moreover, the periodic boundary conditions of the original electronic problem are ensured if both the pseudoparticle band-momentum discrete values obey Eqs. 4-III and 5-III and the requirements of the basic properties 6-III and 7-III are fulfilled. The band-momentum energy eigenstates are Fourier transform superpositions of local charge sequences, spin sequences, and $C$ pseudoparticle sequences corresponding to Slater determinants involving the pseudoparticle band momentum $q_j$ and the spatial coordinate $x_j$ of these quantum objects. Both in the case of the $c$ pseudoparticle branch and of the $\alpha, \nu$ pseudoparticle branches with finite occupancy in a given state, such spatial coordinate is the conjugate of the band momentum $q_j$ of the above Fourier transforms. The spatial coordinate $x_j$ of the $c$ pseudoparticles (and $\alpha, \nu$ pseudoparticles) is associated with an effective pseudoparticle lattice of length $L$, number of lattice sites $N_{\alpha}$ (and $N_{\alpha, \nu}$ given in Eq. 11) and lattice constant $a$ (and $a_{\alpha, \nu} = L/N_{\alpha, \nu}$). The possible values of these spatial coordinates are $x_j = a j$ where $j = 1, 2, 3, ..., N_{\alpha}$ (and $x_j = a_{\alpha, \nu} j$
where \( j = 1, 2, 3, \ldots, N^*_\alpha, \nu \). The effective pseudoparticle lattices arise because the occupancy configurations of the different pseudoparticle branches are separately conserved.

These eight basic properties play an important role in the mechanisms which in the \( t/U \to 0 \) limit determine the choice of the electronic site distribution configurations of the local charge, spin, and \( c \) pseudoparticle sequences whose Fourier transform superpositions describe the band-momentum energy eigenstates. Property 1-III follows from well known properties of the 1D Hubbard model in the limit \( t/U \to 0 \). Property 2-III is consistent with the finding of Ref. \[12\] that in the limit \( t/U \to 0 \) only the \( c \) pseudoparticles move and carry kinetic energy, whereas the \( \pm 1/2 \) holons and \( \pm 1/2 \) spinons correspond to unchanged occupancy configurations in that limit. Such a property results from well-established features of the Bethe-ansatz solution \[2, 5, 12\]. Property 3-III is a consequence of the combination of property 1-III with the relation of Yang holons, HL spinons, and \( \alpha, \nu \) pseudoparticles to \( \pm 1/2 \) holons and \( \pm 1/2 \) spinons. In property 4-III the number \( N^*_\alpha, \nu \) given in Eq. \[17\] plays a central role. The expression of that number is valid for all values of \( U/t \) and is provided by the Bethe ansatz solution. This is consistent with the electron site distribution configurations which describe the local \( \alpha, \nu \) pseudoparticles in the limit \( t/U \to 0 \) being the same as the rotated-electron configurations which describe these local \( \alpha, \nu \) pseudoparticles for finite values of \( t/U \). Property 5-III follows from general symmetries of the 1D Hubbard model which also are valid for all values of \( U/t \). This basic property is also consistent with the above equivalence of the electron site distribution configurations in the limit of \( t/U \to 0 \) and of the rotated-electron configurations for finite values of \( t/U \). Property 6-III is a consequence of the periodic boundary conditions in the particular case when a local charge or spin sequence corresponds to a single \( c, \nu \) or \( c, \nu \) pseudoparticle respectively. Property 7-III results from the indiscernible character of the \( \alpha, \nu \) pseudoparticles. In the case of band-momentum pseudoparticles such an indiscernible character is implicit in the description of the band-momentum energy eigenstates in terms of pseudoparticle occupancy configurations \[11, 12\]. Finally, property 8-III is related to the periodic boundary conditions of the original electronic problem and to the existence in this limit of nearest-neighbor hopping only which does not change double occupation. Such a property is also related to the charge, spin, \( c \) pseudoparticle separation studied in Ref. \[11\] for all energy scales and for the whole parameter space of the 1D Hubbard model. In addition, in the case of the \( \alpha, \nu \) pseudoparticle branches, we find below that the occurrence of the effective pseudoparticle lattices mentioned in this property is related to the separation of the internal and translational degrees of freedom of the local \( \alpha, \nu \) pseudoparticles.

Since in the \( t/U \to 0 \) limit the rotated electrons become the electrons, often below we refer to the rotated-electron site distribution configurations only. However, whenever referring to rotated electrons we mean implicitly that in the limit \( t/U \to 0 \) the electron site distribution configurations which describe the band-momentum energy eigenstates are the same.

**IV. THE PSEUDOPARTICLE INTERNAL STRUCTURE AND COMPLETE SET OF LOCAL STATES IN TERMS OF CHARGE, SPIN, AND \( c \) PSEUDOPARTICLE SEQUENCES**

In this section we find the rotated-electron site distribution configurations which describe the local charge, spin, and \( c \) pseudoparticle sequences. This requires the study of the internal structure of the local \( \alpha, \nu \) pseudoparticles. In addition, such an analysis requires the definition of the positions in the effective electronic lattice of the rotated-electron site distribution configurations which describe both the local \( c \) and \( \alpha, \nu \) pseudoparticles and their empty sites. We find that each specific site distribution configurations of a local charge, spin, and \( c \) pseudoparticle sequence defines a local state. The set of all different possible such states constitutes a complete set of local states. In the ensuing section we express the energy eigenstates of the model as a Fourier-transform superposition of these local states. Such a study reveals the connection of the energy eigenstates to the rotated electron distribution configurations of doubly occupied sites, empty sites, spin-down singly occupied sites, and spin-up singly occupied sites which describe the local states.

**A. THE LOCAL \( \alpha, \nu \) PSEUDOPARTICLE INTERNAL STRUCTURE**

In the following we use in part the basic properties 1-III to 8-III to study the rotated-electron site distribution configurations which describe the local pseudoparticles introduced in this paper. We start by restricting our study to rotated-electron site distribution configurations which describe the lowest-weight states of both the \( SU(2) \) SU-spin and spin algebras. These states have no \(-1/2\) Yang holons and no \(-1/2\) HL spinons. Thus in this case the number of Yang holons (and of HL spinons) is such that \( L_c = L_{c, +1/2} \) (and \( L_s = L_{s, +1/2} \)).

In the limit \( U/t \to \infty \) the \( N_c \) \( c \) pseudoparticles behave as spin-less fermions in a lattice of \( N_c = N^c_c + N^b_c = M_s + M_c \) sites. Such a lattice is nothing but the effective \( c \) pseudoparticle lattice mentioned in property 8-III. As in the case of
the band-momentum $q_j$, such an effective $c$ pseudoparticle lattice and its site coordinates $x_j$ remain the same for the whole parameter space of the model. Also the $c$ pseudoparticle occupancy configurations of such a lattice describing a given energy eigenstate are the same for all values of $U/t$, density $n$, and spin density $m$. At a fixed value $N_c = M_s$ of the numbers $N_c$ of $c$ pseudoparticles and $M_s$ of spinons the number of occupancy configurations of the local $c$ pseudoparticles in such an effective lattice is given by,

$$\frac{(N_a)}{N_c} = \frac{N_a!}{N_c!M_s^M}, \quad (40)$$

Each of these $c$ pseudoparticle occupancy configurations of the effective $c$ pseudoparticle lattice defines a local $c$ pseudoparticle sequence.

Let $x_{ij} = a_j$ where $l = 1, 2, ..., N_c$ be the actual set of occupied coordinates of the effective $c$ pseudoparticle lattice out of the available $x_j = a_j$ coordinate sites where $j = 1, 2, ..., N_a$. We note that these $N_a$ sites have the same coordinates and lattice constant $a$ as the $N_a$ sites of the effective electronic lattice. Moreover, the set of coordinates $x_{ij} = a_j$ where $l = 1, 2, ..., N_c$ corresponding to the occupied sites of the effective $c$ pseudoparticle lattice are precisely the same as the coordinates of the rotated electron singly occupied sites. Thus we can define each of the occupancy configurations whose total number is given in Eq. (40) by the coordinates in units of the lattice constant $a$ of the rotated-electron singly occupied sites,

$$(j_1, j_2, ..., j_{N_c}). \quad (41)$$

The positions occupied by the rotated-electron doubly occupied and empty sites are then these left over by the rotated-electron singly occupied sites. Since the sequence (11) also gives the positions of the local $c$ pseudoparticles in their effective lattice, the sites left over by these pseudoparticles define the positions of the local $c$ pseudoparticle holes. Thus the relation of the numbers $N_c$ and $N_c^b$ of local $c$ pseudoparticles and local $c$ pseudoparticle holes respectively, to the rotated-electron site distribution configurations of a given energy eigenstate confirms that there are $N_c$ rotated-electron singly occupied sites and $N_c^b = [N_c - N_c]$ rotated-electron doubly-occupied and empty sites, as mentioned in Sec. II. From the use of Eqs. (27) and (28) we find that these numbers can be expressed as follows,

$$N_c^b = N_a - N_c = L_c + 2 \sum_{\nu=1}^{\infty} \nu N_{c,\nu};$$

$$N_c = L_c + 2 \sum_{\nu=1}^{\infty} \nu N_{s,\nu}. \quad (42)$$

The local charge sequences (and spin sequences) introduced in property 3-III involve only the rotated-electron doubly-occupied and empty site distribution configurations (and spin-down and spin-up rotated-electron singly occupied site distribution configurations) of $N_c^b = [N_a - N_c]$ sites (and $N_c$ sites) out of a total number $N_a$ of sites. As a result of the independent conservation of the charge and spin sequences, the rotated-electron site distribution configurations of a charge (and spin) sequence is for any energy eigenstate obtained simply by omitting the $N_c$ singly occupied sites (and $N_c^b = [N_a - N_c]$ doubly-occupied and empty sites). Thus a local charge (and spin) sequence corresponds to the rotated-electron occupancy configurations of a site domain of a number $N_c^b = [N_a - N_c]$ of sites (and $N_c$ of sites). It follows that the number of sites of such a domain depends on the specific state under consideration. In the case of the ground state numbers provided in Sec. II, the number of sites of the effective electronic lattice which belong to the charge and spin sequences is $[N_a - N_c]$ and $N_c$ respectively.

Also the effective $\alpha, \nu$ pseudoparticle lattices mentioned in property 8-III and introduced below, result from independent conservation laws associated with each $\alpha, \nu$ pseudoparticle branch such that $\alpha = c, s$ and $\nu = 1, 2, ...$ branch. These effective $c, \nu$ pseudoparticle (and $s, \nu$ pseudoparticle) lattices are generated below by omission of a number

$$\sum_{\nu'=1}^{\infty} \left(\nu + \nu' - |\nu - \nu'|\right) N_{c,\nu'} - N_{c,\nu}$$

and

$$\sum_{\nu'=1}^{\infty} \left(\nu + \nu' - |\nu - \nu'|\right) N_{s,\nu'} - N_{s,\nu}$$

of sites out of the total number $N_c^b = [N_a - N_c]$ (and $N_c$) of sites of the local charge sequence (and spin sequence). Again the number of sites of the local charge (and spin) sequence which contribute to an effective $c, \nu$ pseudoparticle (and $s, \nu$ pseudoparticle) lattice depends on the specific state under consideration. For instance, in the case of a ground state the number of sites of the local charge sequence (and spin sequence) which contribute to an effective $c, \nu$ pseudoparticle (and $s, \nu$ pseudoparticle) lattice is $[N_a - N_c]$ (and $N_c$) for the $s, 1$ pseudoparticle branch and $[N_c^0 - N_c^0]$ for the $s, \nu$ pseudoparticle branches such that $\nu > 1$ out of the $[N_a - N_c]$ sites (and $N_c^0 = [N_c^0 + N_c^0]$ sites) of such a sequence. On the other hand, the
number of sites of the effective \(c\) pseudoparticle lattice equals the number \(N_\alpha\) of sites of the effective electronic lattice and is the same for all energy eigenstates.

The expressions of Eq. 12 show that the value of the number \(L_c\) of Yang holons (and \(L_s\) of HL spinons) of the local charge (and spin) sequence is uniquely determined by the values of the numbers of \(c\) pseudoparticles and \(c, \nu\) pseudoparticles (\(c\) pseudoparticles and \(s, \nu\) pseudoparticles) of the same sequence. Thus for given number \(N^c_\alpha = [N_\alpha + N_c]\) of rotated-electron doubly-occupied and empty sites (and \(N_c\) of rotated-electron singly occupied sites) we can uniquely define the rotated-electron site distribution configurations of the local charge (and spin) sequence by providing the sites occupied by local \(c, \nu\) pseudoparticles (and \(s, \nu\) pseudoparticles) belonging to the branches \(\nu = 1, 2, ...\) with finite occupancy. The charge-sequence (and spin-sequence) sites of the effective electronic lattice left over by these local pseudoparticles define the positions of the Yang holons (and HL spinons). It is useful to introduce the charge-sequence site index \(h\) and the spin-sequence site index \(l\) such that,

\[
h = 1, 2, ..., [N_\alpha - N_c]; \quad l = 1, 2, ..., N_c.
\]

The ordering of the charge-sequence (and spin-sequence) index corresponds to the location order from the left to the right-hand side of the corresponding site in the effective electronic lattice. The position in the effective electronic lattice of a spin-sequence site of index \(l\) and of a charge-sequence site of index \(h\) are given by,

\[
x_{ji} = j_l a, \quad l = 1, 2, ..., N_c; \quad x_{jh} = j_h a, \quad h = 1, 2, ..., [N_\alpha - N_c],
\]

where \(j_l\) are the indices of Eq. 11 which define the location of the rotated-electron singly occupied sites and \(j_h\) are the indices which define the location of the rotated-electron doubly occupied and empty sites. The location of the latter sites corresponds to the sites left over by the rotated-electron singly occupied sites.

From both the above analysis and property 3-III we conclude that one can uniquely specify a given rotated-electron site distribution configuration by providing the position of the \(N_c\) sites occupied by local \(c\) pseudoparticles, \(2 \sum_{\nu=1}^\infty \nu N_{c, \nu}\) sites occupied by local \(c, \nu\) pseudoparticles, and \(2 \sum_{\nu=1}^\infty \nu N_{s, \nu}\) sites occupied by local \(s, \nu\) pseudoparticles. The \(N_c\) sites occupied by \(c\) pseudoparticles define the location of the rotated-electron singly occupied sites. The \([N_\alpha - N_c]\) rotated-electron doubly-occupied and empty sites are the sites left over in the charge and spin sequences respectively, by the sites occupied by local \(c, \nu\) pseudoparticles and \(s, \nu\) pseudoparticles respectively. As mentioned above, to start with we consider that all Yang holons (and HL spinons) have \(\eta\)-spin projection (and spin projection) \(+1/2\) and thus correspond to rotated-electron empty sites (and rotated-electron spin-up singly occupied sites). The generalization to rotated-electron site distribution occupancies associated with states containing both \(\pm1/2\) Yang holons (and \(\pm1/2\) HL spinons) is straightforward and is introduced later in this section.

The Yang holons (and HL spinons) have no internal structure and are the simplest of the quantum objects which occupy the charge (and spin) sequence of the effective electronic lattice. According to property 1-III, the \(+1/2\) Yang holons (and \(+1/2\) HL spinons) correspond to rotated-electron empty sites (and spin-up rotated-electron singly occupied sites) of these sequences. On the other hand, a local \(\alpha, \nu\) pseudoparticle has internal structure and thus is a more involved quantum object than a Yang holon or a HL spinon. This internal structure corresponds to the rotated-electron distribution configurations of the \(\nu\) doubly occupied sites and \(\nu\) empty sites \((\alpha = c)\) or \(\nu\) spin-down singly occupied sites and \(\nu\) spin-up singly occupied sites \((\alpha = s)\) which according to property 3-III describe such a local \(\alpha, \nu\) pseudoparticle. If in the rotated-electron site distribution configurations which describe the local \(c, \nu\) pseudoparticles we replace doubly occupied sites \(\bullet\) and empty sites \(\circ\) by spin-down singly occupied sites \(\downarrow\) and spin-up singly occupied sites \(\uparrow\) respectively, and omit the phase factors generate by the operator \(\exp(i \pi \sum_j J_{\nu j})\) mentioned in property 3-III, we obtain the corresponding distribution configurations of the \(s, \nu\) pseudoparticles. Therefore, often we consider the rotated-electron site distribution configurations which describe the local \(c, \nu\) pseudoparticles only.

It is useful to classify the \(2\nu\) lattice sites of the local charge sequence involved in the description of a local \(c, \nu\) pseudoparticle into two sets of \(\nu\) sites each. Let us denote the index of these two sets of \(\nu\) sites by \(h_{j, g}\) and \(h_{j, \nu + g}\) respectively. Here \(g = 1, 2, ..., \nu\) and the index \(j = 1, 2, ..., N^{c*, \nu}_{\nu}\) refers to the position \([h_j a]\) of the local \(c, \nu\) pseudoparticle in the effective electronic lattice where,

\[
h_j = \frac{j_{h, \nu} + j_{h, \nu + 1}}{2},
\]

and the numbers \(j_{h, \nu}\) are the indices \(j_b\) of Eq. 11 which define the location of the rotated-electron doubly occupied/empty sites in such a lattice. Combination of the values of the indices \(j = 1, 2, ..., N^{c*, \nu}_{\nu}\) and \(g = 1, 2, ..., \nu\)
fully defines the position of the above $2\nu$ sites. The index $g$ is such that $h_{j,1} < h_{j,2} < ... < h_{j,\nu}$ and $h_{j,\nu+1} < h_{j,\nu+2} < ... < h_{j,2\nu}$ respectively, where $h_{j,\nu} < h_{j,\nu+1}$. Equivalently, often we denote these $2\nu$ indices simply by $h_{j,x}$, where $x = 1, 2, ..., 2\nu$ and $h_{j,1} < h_{j,2} < ... < h_{j,2\nu}$. The set of $2\nu$ indices $\{h_{j,1}, h_{j,2}, ..., h_{j,2\nu}\}$ is in general a sub-set of the $[N_a - N_c]$ charge-sequence indices $h$ given in Eq. (48). The latter indices define the location of the charge-sequence rotated-electron doubly occupied and empty sites. We can also define the charge-sequence position of the local $c, \nu$ pseudoparticle which is defined as,

$$\tilde{h}_j = \frac{h_{j,\nu} + h_{j,\nu+1} - 1}{2}. \quad (46)$$

The same definitions hold for the local $s, \nu$ pseudoparticles with the indices $h_{j,g}$ and $h_{j,\nu+g}$ replaced by the indices $l_{j,g}$ and $l_{j,\nu+g}$ respectively, and thus the equivalent indices $h_{j,x}$ replaced by $l_{j,x}$. The position of the local $s, \nu$ pseudoparticle in the effective electronic lattice is defined as $[l_j[a]$ where,

$$l_j = \frac{j_l_{i,\nu} + j_l_{i,\nu+1}}{2}, \quad (47)$$

$j = 1, 2, ..., N^{*}_{s,\nu}$, and $j_l$ are the indices of Eq. (11) which define the location of the rotated-electron singly occupied sites. On the other hand, the spin-sequence position of the local $s, \nu$ pseudoparticle is defined as,

$$\bar{l}_j = \frac{l_{j,\nu} + l_{j,\nu+1} - 1}{2}, \quad (48)$$

where again $j = 1, 2, ..., N^{*}_{s,\nu}$. We note that the indices $\tilde{h}_j$ (and $\tilde{l}_j$) given in Eq. (46) (and Eq. (49)) which define the charge-sequence (and spin-sequence) position of the local $c, \nu$ pseudoparticle (and local $s, \nu$ pseudoparticle) are always positive integer numbers $1, 2, 3, ...$. According to Eq. (49) (and Eq. (14)) the position $[h_j[a]$ (and $[l_j[a]$) of the local $c, \nu$ pseudoparticle (and local $s, \nu$ pseudoparticle) in the $N_a$-site effective electronic lattice refers to a single point inside the $2\nu$-site domain associated with such a quantum object. On the other hand, the charge-sequence position (and spin-sequence position) of the local $c, \nu$ pseudoparticles (and local $s, \nu$ pseudoparticles) defined by the index of Eq. (46) (and Eq. (48)) refers to the location of that quantum object relative to the $[N_a - N_c]$ sites (and $N_c$ sites) of the charge (and spin) sequence only.

Below we clarify the following two issues: First, we find the rotated-electron distribution occupancies of the $2\nu$ sites which describe a local $\alpha, \nu$ pseudoparticle. Second, we find the rotated-electron site distribution occupancies which define the $N_{\alpha,\nu}^h$ charge-sequence ($\alpha = c$) or spin-sequence ($\alpha = s$) empty sites corresponding to the local $\alpha, \nu$ pseudoparticle branch. This study reveals that the position in the effective electronic lattice of the sites associated with the $N_{\alpha,\nu}^h$ charge-sequence or spin-sequence empty sites of each local $\alpha, \nu$ pseudoparticle branch is uniquely determined by the position of the local $c$ pseudoparticles, local $c, \nu$ pseudoparticles, and local $s, \nu$ pseudoparticles. The same holds for the locations of the Yang holons and HL spinons, as discussed above. Below we also confirm that for fixed values of the numbers $N_{\alpha,\nu}$ and $N_{\alpha,\nu}^*$ the number of occupancy configurations of the local $\alpha, \nu$ pseudoparticles is given by,

$$\binom{N_{\alpha,\nu}^*}{N_{\alpha,\nu}} = \frac{N_{\alpha,\nu}^*!}{N_{\alpha,\nu}^*! \cdot N_{\alpha,\nu}^h!}. \quad (49)$$

We classify each of the local $\alpha, \nu$ pseudoparticle occupancy configurations by providing the indices $h_j$ or $l_j$ given in Eqs. (45) or (47) respectively, corresponding to the $N_{c,\nu}$ and $N_{s,\nu}$ pseudoparticle locations in the effective electronic lattice,

$$(h_1,h_2,...,h_{N_{c,\nu}}), \quad (50)$$

and

$$(l_1,l_2,...,l_{N_{s,\nu}}), \quad (51)$$

respectively. From the combination of Eqs. (49) and (50) it follows that the number of occupancy configurations of the local $c$ and $\alpha, \nu$ pseudoparticles of a CPHS ensemble subspace is given by,

$$\mathcal{N}_{CPHS-es} = \binom{N_a}{N_{c}} \prod_{\alpha = c,s} \prod_{\nu = 1}^{\infty} \binom{N_{\alpha,\nu}^*}{N_{\alpha,\nu}}. \quad (52)$$
Let us denote each local state representing a specific local charge, spin, and $c$ pseudoparticle sequence whose number for a given CPHS ensemble subspace is given in Eq. 52 by,

$$\{|j_1, j_2, ..., j_N_c, \nu\}; \{(h_1, h_2, ..., h_{N_c, \nu})\}; \{(l_1, l_2, ..., l_{N_c, \nu})\}. \quad (53)$$

Here

$$\{(h_1, h_2, ..., h_{N_c, \nu})\} = (h_1, h_2, ..., h_{N_c, 1}); (h_1, h_2, ..., h_{N_c, 2}); (h_1, h_2, ..., h_{N_c, 3}); ...,$$

(54)

gives the positions in the effective electronic lattice of the local $c, \nu$ pseudoparticles belonging to branches with finite occupancy in the state under consideration and

$$\{(l_1, l_2, ..., l_{N_c, \nu})\} = (l_1, l_2, ..., l_{N_c, 1}); (l_1, l_2, ..., l_{N_c, 2}); (l_1, l_2, ..., l_{N_c, 3}); ...,$$

(55)

provides the positions in the same lattice of the local $s, \nu$ pseudoparticles belonging to branches with finite occupancy in the same state.

We emphasize that the number $\nu$ of such local states which equals the number of different local pseudoparticle occupancy configurations of a CPHS ensemble subspace indeed equals the dimension of such a subspace. It is straightforward to confirm from the results of Ref. [11] that the number of band-momentum energy eigenstates that span such a CPHS ensemble subspace also has the same value. This suggests that the set of local states of form 55 is complete in that subspace, as is confirmed below. We find later on in this section that the generalization of the present analysis to rotated-electron site distribution configurations describing energy eigenstates with finite occupancies of $\pm 1/2$ Yang holons and $\pm 1/2$ HL spinons requires the introduction of the numbers $L_{c, -1/2}$ of $-1/2$ Yang holons and $L_{s, -1/2}$ of $-1/2$ HL spinons in the labelling of the local states 55.

In order to find the rotated-electron site distribution configurations which describe the internal structure of a local $c, \nu$ pseudoparticle we use mainly the basic properties 3-III, 5-III, 6-III, and 7-III. Often we consider the specific case of a local $c, \nu$ pseudoparticle. The internal structure of such a quantum object is studied by considering first a charge sequence constituted by $\nu$ rotated-electron doubly occupied sites and $\nu$ rotated-electron empty sites. The use of properties 5-III and 6-III leads to the finding of the specific rotated-electron site distribution configurations which describe the local $c, \nu$ pseudoparticle. Property 7-III then states that the obtained rotated-electron site distribution configurations describe a local $c, \nu$ pseudoparticle in any charge sequence.

According to property 3-III, the internal structure of a local $c, \nu$ pseudoparticle involves a number $\nu$ of rotated-electron doubly occupied sites and an equal number of rotated-electron empty sites. There is a number $\binom{2\nu}{\nu}$ of different distribution configurations of these $\nu$ rotated-electron doubly occupied sites and $\nu$ rotated-electron empty sites. For simplicity let us restrict the present preliminary analysis to charge sequences with $\nu$ rotated-electron doubly occupied sites and an equal number of rotated-electron empty sites and without Yang holons. Below we find that a local $c, \nu$ pseudoparticle is a superposition of a number $2^\nu$ of the above rotated-electron site distribution configurations. For $\nu = 1$ the local $c, 1$ pseudoparticle is a properly symmetrized superposition of the two available rotated-electron site distribution configurations. For $\nu = 2$ the local $c, 2$ pseudoparticle is a properly symmetrized superposition of four out of the six available rotated-electron site distribution configurations. Thus states including all possible six different distribution configurations are a superposition of two states including two local $c, 1$ pseudoparticles and a local $c, 2$ pseudoparticle respectively. For $\nu = 3$ the local $c, 3$ pseudoparticle is a properly symmetrized superposition of eight out of the twenty available rotated-electron site distribution configurations. States including all possible twenty different distribution configurations are a superposition of three states including (i) three local $c, 1$ pseudoparticles, (ii) a local $c, 1$ pseudoparticle and a local $c, 2$ pseudoparticle, and (iii) a local $c, 3$ pseudoparticle. In the general case one has that the local $c, \nu$ pseudoparticle is a properly symmetrized superposition of $2^\nu$ out of the $\binom{2\nu}{\nu}$ available rotated-electron site distribution configurations. States including all possible $\binom{2\nu}{\nu}$ different distribution configurations are a superposition of several states including different numbers $N_{c, \nu'}$ of $c, \nu'$ pseudoparticles such that $\nu' = 1, 2, ..., \nu$ and $\sum_{\nu'=1}^\nu N_{c, \nu'} = \nu$.

Among the $2^\nu$ rotated-electron site distribution configurations of a $c, \nu$ pseudoparticle there is always a distribution configuration where the first $\nu$ sites of index $h_j, g$ and $g = 1, 2, ..., \nu$ are doubly occupied by rotated electrons and the last $\nu$ sites of index $h_j, \nu+g$ and $g = 1, 2, ..., \nu$ are free of rotated electrons. We represent such a local $c, \nu$ pseudoparticle rotated-electron site distribution configuration of a charge sequence by,

$$\{\bullet_{h_j, 1}, \bullet_{h_j, 2}, \bullet_{h_j, \nu}, \circ_{h_j, 1+\nu}, ..., \circ_{h_j, 2\nu}\}. \quad (56)$$
The same applies to a local $s, \nu$ pseudoparticle, the rotated-electron site distribution configuration of a spin sequence corresponding to the one given in Eq. (56) being,

\[
\left( \downarrow j_1, \ldots, \downarrow j_\nu, \uparrow j_{1+\nu}, \ldots, \uparrow j_\nu \right).
\]

We confirm below that the $c, \nu$ pseudoparticle located in the same lattice at position $[l_j, a]$ and the $s, \nu$ pseudoparticle located in the same lattice at position $[l_j, a]$ are described by the following properly symmetrized superposition of $2\nu$ rotated-electron site distribution configurations,

\[
\prod_{x=1}^{2\nu} e^{i\pi h_{j, x} \hat{D}_{j, x}} \prod_{g=1}^\nu (1 - \hat{T}_{c, \nu, j, g}) \left( \bullet_{h_{j, 1}}, \ldots, \bullet_{h_{j, \nu}}, \circ_{h_{j, 1+\nu}}, \ldots, \circ_{h_{j, \nu}} \right); \quad j = 1, 2, \ldots, N_{c, \nu},
\]

and

\[
\prod_{g=1}^\nu (1 - \hat{T}_{s, \nu, j, g}) \left( \downarrow j_1, \ldots, \downarrow j_\nu, \uparrow j_{1+\nu}, \ldots, \uparrow j_\nu \right); \quad j = 1, 2, \ldots, N_{s, \nu},
\]

respectively. Here and according to property 3-III, the site-$h_{j, x}$ rotated-electron double occupation operator $\hat{D}_{j, x}$ has eigenvalue 1 and 0 when that site is doubly occupied by rotated electrons and free of rotated electrons respectively, and the operator $\hat{T}_{c, \nu, j, g}$ (and $\hat{T}_{s, \nu, j, g}$) acts on the pair of sites of indices $h_{j, g}$ and $h_{j, g+\nu}$ (and $l_{j, g}$ and $l_{j, g+\nu}$) only. This operator always acts on rotated-electron site distribution configurations of the particular form illustrated in Eq. (56) (and in Eq. (57)). From the application of this operator onto such a rotated-electron site distribution configuration a new distribution configuration is generated where the site of index $h_{j, g}$ is free of rotated electrons, the site of index $h_{j, g+\nu}$ is doubly occupied by rotated electrons, and the occupancy of the other $2(\nu - 1)$ sites remains unchanged, i.e.

\[
\hat{T}_{c, \nu, j, g} \left( \bullet_{h_{j, 1}}, \ldots, \bullet_{h_{j, \nu}}, \circ_{h_{j, 1+\nu}}, \ldots, \circ_{h_{j, \nu}} \right) = \left( \bullet_{h_{j, 1}}, \ldots, \bullet_{h_{j, \nu}}, \circ_{h_{j, 1+\nu}}, \ldots, \circ_{h_{j, \nu}} \right).
\]

The same transformation law,

\[
\hat{T}_{s, \nu, j, g} \left( \downarrow j_1, \ldots, \downarrow j_\nu, \uparrow j_{1+\nu}, \ldots, \uparrow j_\nu \right) = \left( \downarrow j_1, \ldots, \downarrow j_\nu, \uparrow j_{1+\nu}, \ldots, \uparrow j_\nu \right),
\]

is associated with the application of the operator $\hat{T}_{s, \nu, j, g}$ onto a rotated-electron site distribution configuration of the type illustrated in Eq. (57).

The $2\nu$ internal rotated-electron site distribution configurations on the right-hand side of Eq. (55) (and Eq. (59)) are generated by considering that in each of the $g = 1, 2, \ldots, \nu$ pairs of sites of indices $h_{j, g}$ and $h_{j, g+\nu}$, the site of index $h_{j, g}$ is doubly occupied by rotated electrons and the site of index $h_{j, g+\nu}$ is free of rotated electrons and vice versa. In general we call $h_{j, x} \leftrightarrow h_{j, x'}$ site pair a superposition of two rotated-electron distribution configurations of a rotated-electron doubly occupied site and a rotated-electron empty site where $h_{j, x}$ and $h_{j, x'}$, such that $x, x' = 1, \ldots, 2\nu$ and $x' > x$, are the indices of the two lattice sites involved. In the first rotated-electron site distribution configuration the sites of indices $h_{j, x}$ and $h_{j, x'}$ are doubly occupied by rotated electrons and free of rotated electrons respectively. In the second rotated-electron site distribution configuration the sites of indices $h_{j, x}$ and $h_{j, x'}$ are free of rotated electrons and doubly occupied by rotated electrons respectively. The rotated-electron distribution configurations of all the remaining sites of the charge sequence except these two are identical. The

\[
h_{j, x} \leftrightarrow h_{j, x'}; \quad x, x' = 1, \ldots, 2\nu; \quad x' > x,
\]

site pair is defined as a superposition of these two rotated-electron site distributions configurations where the first and the second distributions are multiplied by a phase factor of 1 and $-1$ respectively.

In figure 1 we represent a $h_{j, x} \leftrightarrow h_{j, x'}$ site pair by two vertical lines connected by a horizontal line. The two vertical lines have the same height and connect the corresponding site of the charge sequence to the horizontal line. Thus the two sites of a charge sequence connected by three such lines are assumed to be involved in a superposition of two rotated-electron site distribution configurations multiplied by the phase factors of 1 and $-1$ respectively, where
according to the above recipe one of these sites is doubly occupied by rotated electrons and the other one is free of rotated electrons and vice versa. Similarly, we call

\[ l_j, x \leftrightarrow l_j, x'; \quad x, x' = 1, ..., 2\nu; \quad x' > x, \]  

(63)
site pair an equivalent superposition of two rotated-electron site distribution configurations of a spin sequence. The definitions are the same provided that we replace rotated-electron doubly occupied sites and rotated-electron empty sites by spin-down rotated-electron singly occupied sites and spin-up rotated-electron singly occupied sites respectively. Such a rotated-electron singly occupied site pair is also graphically represented by the two vertical lines connected by a horizontal line plotted in Fig. 1. The concept of a site pair plays an important role in the description of the rotated-electron site distribution configurations of a local \( \alpha, \nu \) pseudoparticle.

It is confirmed below that the rotated-electron site distribution configurations which describe a local \( \alpha, \nu \) pseudoparticle always involve a number \( \nu \) of site pairs such that \( x = g \) and \( x' = g + \nu \) where \( g = 1, ..., \nu \). Since there are two possible rotated-electron site occupancies for each pair and the number of pairs of each local \( \alpha, \nu \) pseudoparticle is \( \nu \), the total number of different internal rotated-electron site distribution configurations is indeed \( 2^\nu \). For example, in the case of a \( c, \nu \) pseudoparticle located in the effective electronic lattice at the position \([h, j]\), the \( 2^\nu \) rotated-electron site distribution configurations superposed in expression (65) are the following,

\[
\left[ \prod_{x=1}^{2} e^{i\pi h_{j,x} \hat{D}_{j,x}} \right] (1 - \tilde{\mathcal{T}}_{c,1,j,g} (\bullet_{h_{j,1}}, \circ_{h_{j,2}})) = -e^{i\pi h_{j,1} (\bullet_{h_{j,1}}, \circ_{h_{j,2}})} + e^{i\pi h_{j,2} (\circ_{h_{j,1}}, \bullet_{h_{j,2}})}
\]  

(64)
in the case of the \( c, 1 \) pseudoparticle,

\[
\left[ \prod_{x=1}^{3} e^{i\pi h_{j,x} \hat{D}_{j,x}} \right] \left[ \prod_{y=1}^{2} (1 - \tilde{\mathcal{T}}_{c,2,j,g}) \right] (\bullet_{h_{j,1}}, \bullet_{h_{j,2}}, \circ_{h_{j,3}}, \circ_{h_{j,4}}) =
\]

\[
+ e^{i\pi h_{j,1} + h_{j,2}} (\bullet_{h_{j,1}}, \bullet_{h_{j,2}}, \circ_{h_{j,3}}, \circ_{h_{j,4}}) - e^{i\pi h_{j,2} + h_{j,3}} (\circ_{h_{j,1}}, \bullet_{h_{j,2}}, \bullet_{h_{j,3}}, \circ_{h_{j,4}}) 
\]

\[
+ e^{i\pi h_{j,3} + h_{j,4}} (\circ_{h_{j,1}}, \bullet_{h_{j,2}}, \bullet_{h_{j,3}}, \bullet_{h_{j,4}}) - e^{i\pi h_{j,1} + h_{j,4}} (\bullet_{h_{j,1}}, \circ_{h_{j,2}}, \bullet_{h_{j,3}}, \bullet_{h_{j,4}}),
\]

(65)
for the \( c, 2 \) pseudoparticle, and

\[
\left[ \prod_{x=1}^{4} e^{i\pi h_{j,x} \hat{D}_{j,x}} \right] \left[ \prod_{y=1}^{2} (1 - \tilde{\mathcal{T}}_{c,3,j,g}) \right] (\bullet_{h_{j,1}}, \bullet_{h_{j,2}}, \bullet_{h_{j,3}}, \circ_{h_{j,4}}, \circ_{h_{j,5}}, \circ_{h_{j,6}}) =
\]

\[
- e^{i\pi h_{j,1} + h_{j,2} + h_{j,3}} (\bullet_{h_{j,1}}, \bullet_{h_{j,2}}, \bullet_{h_{j,3}}, \circ_{h_{j,4}}, \circ_{h_{j,5}}, \circ_{h_{j,6}}) 
\]

\[
+ e^{i\pi h_{j,2} + h_{j,3} + h_{j,4}} (\circ_{h_{j,1}}, \bullet_{h_{j,2}}, \bullet_{h_{j,3}}, \circ_{h_{j,4}}, \circ_{h_{j,5}}, \circ_{h_{j,6}}) 
\]

\[
- e^{i\pi h_{j,3} + h_{j,4} + h_{j,5}} (\circ_{h_{j,1}}, \circ_{h_{j,2}}, \bullet_{h_{j,3}}, \circ_{h_{j,4}}, \circ_{h_{j,5}}, \circ_{h_{j,6}}) 
\]

\[
+ e^{i\pi h_{j,1} + h_{j,3} + h_{j,5}} (\bullet_{h_{j,1}}, \circ_{h_{j,2}}, \bullet_{h_{j,3}}, \bullet_{h_{j,4}}, \circ_{h_{j,5}}, \circ_{h_{j,6}}) 
\]

\[
- e^{i\pi h_{j,1} + h_{j,2} + h_{j,3} + h_{j,4}} (\bullet_{h_{j,1}}, \bullet_{h_{j,2}}, \bullet_{h_{j,3}}, \bullet_{h_{j,4}}, \circ_{h_{j,5}}, \circ_{h_{j,6}}) 
\]

\[
+ e^{i\pi h_{j,2} + h_{j,3} + h_{j,4} + h_{j,5}} (\circ_{h_{j,1}}, \bullet_{h_{j,2}}, \bullet_{h_{j,3}}, \bullet_{h_{j,4}}, \circ_{h_{j,5}}, \circ_{h_{j,6}}) 
\]

\[
- e^{i\pi h_{j,1} + h_{j,2} + h_{j,3} + h_{j,4} + h_{j,5}} (\circ_{h_{j,1}}, \bullet_{h_{j,2}}, \bullet_{h_{j,3}}, \bullet_{h_{j,4}}, \circ_{h_{j,5}}, \circ_{h_{j,6}}) 
\]

\[
+ e^{i\pi h_{j,2} + h_{j,3} + h_{j,4} + h_{j,5} + h_{j,6}} (\circ_{h_{j,1}}, \bullet_{h_{j,2}}, \bullet_{h_{j,3}}, \bullet_{h_{j,4}}, \circ_{h_{j,5}}, \circ_{h_{j,6}}),
\]

(66)
in the case of a \( c, 3 \) pseudoparticle. The \( 2^{4} = 16 \) rotated-electron site distribution configurations of a \( c, 4 \) pseudoparticle located in the effective electronic lattice at position \([h, j]\) are shown and discussed in Appendix B. Similar expressions for local \( s, 1, s, 2, s, 3 \), or \( s, 4 \) pseudoparticles are obtained by replacing in Eqs. (64) and in Eq. (65) of Appendix B respectively, the \( \bullet \) and \( \circ \) sites by \( \downarrow \) and \( \uparrow \) sites respectively, the \( h_{j,g} \) indices by \( l_{j,g} \) indices, and the local rotated-electron double occupation operator phase factors by one.

Before using properties 3-\text{III}, 6-\text{III}, and 7-\text{III} to confirm the validity of the local pseudoparticle expressions (65)–(66), let us check whether these expressions conform to the requirement of property 5-\text{III}. According to that requirement, application of the off-diagonal generators of the \( SU(2) \) \( \eta \)-spin algebra given in Eq. (7) onto the superposition of the
2Ω internal rotated-electron site distribution configurations of a c, ν pseudoparticle located in the effective electronic lattice at position \([l_j, a]\) must give zero. It is straightforward to confirm from the analysis of the changes in the rotated-electron site distribution configurations generated by these operators that this requirement is fulfilled and then,

\[
\hat{S}_c^{\pm} \left[ \prod_{x=1}^{2\nu} e^{i\pi h_{j, x} D_{j, x}} \right] \left[ \prod_{g=1}^{\nu} (1 - \hat{T}_{c, \nu, j, g}) \right] \times (\bullet_{h_{j, 1}}, ..., \bullet_{h_{j, \nu}}, \circ_{h_{j, 1+\nu}}, ..., \circ_{h_{j, 2\nu}}) = 0; \quad j = 1, 2, ..., N_{c, \nu}.
\]  

(67)

The same is valid for application of the off-diagonal generators of the SU(2) spin algebra given in Eq. \([8]\) onto the superposition of the 2Ω internal rotated-electron site distribution configurations of a s, ν pseudoparticle located in the effective electronic lattice at position \([l_j, a]\),

\[
\hat{S}_c^{s} \left[ \prod_{g=1}^{\nu} (1 - \hat{T}_{s, \nu, j, g}) \right] \left( \downarrow_{l_{j, 1}}, ..., \downarrow_{l_{j, \nu}}, \uparrow_{l_{j, 1+\nu}}, ..., \uparrow_{l_{j, 2\nu}} \right) = 0; \quad j = 1, 2, ..., N_{s, \nu}.
\]  

(68)

In order to confirm the validity of the pseudoparticle expressions \([55]-[58]\), let us follow property 5-III and consider again a local charge sequence with no Yang holons and constituted by a number ν of rotated-electron doubly occupied sites and a number ν of rotated-electron empty sites. If we request that the rotated-electron distribution configurations of these 2ν sites describe a single c, ν pseudoparticle, according to the restrictions imposed by the Bethe-ansatz solution \([1]\), the associated local charge sequence must be characterized by the following numbers,

\[
M_c = 2\nu; \quad N_{c, \nu} = 1; \quad N_{c, \nu}^h = 0; \quad N_{c, \nu} = 0, \quad \nu' \neq \nu.
\]  

(69)

Such a local charge sequence has a single c, ν pseudoparticle and no c, ν pseudoparticle empty sites. This corresponds to an effective c, ν pseudoparticle lattice constituted by a single site located in the effective electronic lattice at \([l_j, a]\). That site is occupied by a c, ν pseudoparticle. In this limiting case in order to ensure periodic boundary conditions for the original electronic problem it is required that the charge sequence should be properly symmetrized by the method used in Ref. [51]. Such a symmetrization involves the operator \(\hat{T}_C\) of Eq. \([34]\).

A result of key importance is that the fulfillment of the requirement of property 5-III imposes that the superposition of rotated-electron distribution configurations of the ν doubly occupied sites and ν empty sites (and ν spin-down singly occupied sites and ν spin-up singly occupied sites) of such a c, ν pseudoparticle (and s, ν pseudoparticle) must be expressed in the form of a number ν of \(h_{1, x} \leftrightarrow h_{1, x'}\) site pairs (and \(l_{1, x} \leftrightarrow l_{1, x'}\) site pairs) where \(x, x' > x\) and have the values \(x, x' = 1, 2, ..., 2\nu\). There is a number \((2\nu - 1)!!\) of possible different choices for these \(h_{1, x} \leftrightarrow h_{1, x'}\) and \(l_{1, x} \leftrightarrow l_{1, x'}\) site pairs. Fortunately, the use of requirement 6-III reveals that only one of these \((2\nu - 1)!!\) choices corresponds to the c, ν pseudoparticle (and s, ν pseudoparticle). The remaining \((2\nu - 1)!! - 1\) choices can be expressed as a superposition of several states described by different numbers \(N_{c, \nu'}\) of c, ν' pseudoparticles such that \(\sum_{\nu' = 1}^{\nu'} \nu' N_{c, \nu'} = \nu\) (and \(N_{s, \nu} = 0, \nu'\) pseudoparticles such that \(\sum_{\nu' = 1}^{\nu'} \nu' N_{s, \nu'} = \nu\)). Moreover, the fact that the symmetrized rotated-electron site distribution configuration which describes the c, ν pseudoparticle (and s, ν pseudoparticle) involves a number ν of site pairs implies that it corresponds to a state with charge (and spin) momentum \(k_C = \pi/a\) (and \(k_S = \pi/a\)). The possible values of such a charge momentum are given in Eq. \([36]\). For other values of charge momentum that state cannot be described in terms of a number ν of site pairs and thus the requirement introduced in property 5-III is not fulfilled.

For instance, when \(\nu = 1\) application of the basic properties 3-III, 5-III, and 6-III leads uniquely to the following description of the c, 1 pseudoparticle,

\[
\left[ \prod_{x=1}^{2} e^{i\pi h_{1, x} D_{1, x}} \right] \sum_{\nu' = 0}^{\nu} e^{i\pi \nu C} \left( \hat{T}_C \right)^{\nu C} (\bullet_{h_{1, 1}}, \circ_{h_{1, 2}}).
\]  

(70)

This is the product of a rotated-electron double-occupation operator phase factor with a symmetrized charge sequence. The charge sequence defined by Eq. \([70]\) can also be written as,

\[
\left[ \prod_{x=1}^{2} e^{i\pi h_{1, x} D_{1, x}} \right] (1 - \hat{T}_{c, 1, 1, 1}) (\bullet_{h_{1, 1}}, \circ_{h_{1, 2}}),
\]  

(71)
in agreement with the general expression (58) and the local $c,1$ pseudoparticle expression (60). For $j = 1$, $x = 1$, and $x' = 2$ Fig. 1 describes the $h_{1,1} \leftrightarrow h_{1,2}$ site pair associated with this $c,1$ pseudoparticle.

Next we consider the case $\nu = 2$. The fulfillment of the requirement introduced in property 5-III alone imposes in this case that the superposition of rotated-electron distribution configurations of the two doubly occupied sites and two empty sites representative of a $c,2$ pseudoparticle must be expressed in the form of two $h_{1,x} \leftrightarrow h_{1,x'}$ site pairs where $x$ and $x'$ are such that $x' > x$ and have the values $x, x' = 1, 2, 3, 4$. There are three possible different choices for the superposition of rotated-electron distribution configurations formed by two site pairs. These choices correspond to the following sets: (a) $\{ h_{1,1} \leftrightarrow h_{1,4}; h_{1,2} \leftrightarrow h_{1,3} \}$, (b) $\{ h_{1,1} \leftrightarrow h_{1,3}; h_{1,2} \leftrightarrow h_{1,4} \}$, and (c) $\{ h_{1,1} \leftrightarrow h_{1,2}; h_{1,3} \leftrightarrow h_{1,4} \}$. According to property 6-III, the charge sequence (c) is excluded because it describes two $c,1$ pseudoparticles and thus there remain two possible choices for the $c,2$ pseudoparticle. Figure 2 represents the charge sequences corresponding to the choices (a) and (b). Each of the $h_{1,x} \leftrightarrow h_{1,x'}$ site pairs are represented as in Fig. 1 with $j = 1$. The figure shows how the rotated-electron site distribution configurations of these two charge sequences change as a result of a cyclic permutation which transforms the first site of the charge sequence to the last site and multiplies the final state by a phase factor of $-1$. This phase factor arises from the required $k_C = \pi/a$ charge momentum. We note that the single $h_{1,1} \leftrightarrow h_{1,2}$ site pair descriptive of the $c,1$ pseudoparticle given in expressions (60) and (61) transforms into itself under such a transformation. On the other hand, in the present case the rotated-electron site distribution configuration (a) transforms onto two $c,1$ pseudoparticles whereas the rotated-electron site distribution configuration (b) transforms onto itself. Thus according to property 6-III only the rotated-electron site distribution configuration (b) is properly symmetrized and describes the $c,2$ pseudoparticle. Moreover, only this sequence can be written as a symmetrized charge sequence of charge momentum $k_C = \pi/a$. Again, such a symmetrized charge sequence can also be written in the form given in the general Eq. (58) and in the local $c,2$ pseudoparticle expression (60). The two equivalent expressions read,

$$
\left[ \prod_{x=1}^{4} e^{i\pi h_{1,x} \hat{D}_{1,x}} \right] \sum_{\nu_C=0}^{3} e^{i\pi \nu_C} \left( \hat{T}_C \right)^{\nu_C} (\hat{g})^{h_{1,1} \leftrightarrow \bullet h_{1,2} \leftrightarrow \circ h_{1,3} \leftrightarrow \circ h_{1,4})
$$

$$
= \left[ \prod_{x=1}^{4} e^{i\pi h_{1,x} \hat{D}_{1,x}} \right] \left( \prod_{g=1}^{2} (1 - \hat{T}_{C,2,1,g}) \right) (\hat{g})^{h_{1,1} \leftrightarrow \bullet h_{1,2} \leftrightarrow \circ h_{1,3} \leftrightarrow \circ h_{1,4}).
$$

(72)

Let us now consider the case $\nu = 3$. Again the fulfillment of the requirement introduced in property 5-III imposes that the superposition of rotated-electron distribution configurations of the three doubly occupied sites and three empty sites representative of a $c,3$ pseudoparticle should be expressed in the form of three $h_{1,x} \leftrightarrow h_{1,x'}$ site pairs where $x$ and $x'$ are such that $x' > x$ and have the values $x, x' = 1, 2, 3, 4, 5, 6$. There are fifteen different possible choices for superposition of rotated-electron site distribution configurations formed by three site pairs. However, only one of these charge sequences can be written in the form of symmetrized charge sequence of charge momentum $k_C = \pi/a$. For example, in Fig. 3 we represent the charge sequences (a) $\{ h_{1,1} \leftrightarrow h_{1,3}; h_{1,2} \leftrightarrow h_{1,5}; h_{1,4} \leftrightarrow h_{1,6} \}$ and (b) $\{ h_{1,1} \leftrightarrow h_{1,4}; h_{1,2} \leftrightarrow h_{1,5}; h_{1,3} \leftrightarrow h_{1,6} \}$. The figure shows how the rotated-electron site distribution configurations of these two charge sequences change as a result of a cyclic permutation which transforms the first site of the charge sequence to the last site and multiplies the final state by a phase factor of $-1$. Both the rotated-electron site distribution configuration (a) and all remaining possible rotated-electron site distribution configurations constituted by three $h_{1,x} \leftrightarrow h_{1,x'}$ site pairs except the rotated-electron site distribution configuration (b) do not transform onto themselves. The latter rotated-electron site distribution configuration transforms onto itself, as confirmed by analysis of the figure. Thus according to property 6-III only the rotated-electron site distribution configuration (b) is properly symmetrized and describes the local $c,3$ pseudoparticle. Only this sequence can be written in terms of a symmetrized charge sequence of charge momentum $k_C = \pi/a$. Again, such symmetrized charge sequence can also be written in the form given in the general Eq. (58) and in the local $c,3$ expression (60). The two equivalent expressions are the following,
Superposition of several charge sequences is an eigenstate of the charge momentum operator with eigenvalue $k_{\nu}$. Also, the $c, \nu$ pseudoparticles belonging to branches such that $\nu > 3$ are a superposition of several such symmetric charge sequences. However, always the state obtained from the superposition of several charge sequences is an eigenstate of the charge momentum operator with eigenvalue $k_{\nu} = \pi/a$. That state can also be expressed in the general form given in Eq. (58).

As a last example we provide the expression in terms of symmetric charge sequences of the rotated-electron site distribution configurations representative of a $c, 4$ pseudoparticle. This can again be written both (i) in the form given in Eqs. (58) and in Eq. (B1) of Appendix B and (ii) as a superposition of symmetric charge sequences. The two equivalent expressions are in this case given by:

\[
\prod_{x=1}^{6} e^{i\pi h_{c,x} \cdot \vec{D}_{j,x}} \left\{ \sum_{\nu_C=0}^{5} e^{i\pi\nu_C \left( \hat{T}_{C} \right)^{\nu_C} (\bullet_{h_{1,1}}, \bullet_{h_{1,2}}, \bullet_{h_{1,3}}, \odot_{h_{1,4}}, \odot_{h_{1,5}}, \odot_{h_{1,6}})} \right\} + \sum_{\nu_C=0}^{1} e^{i\pi\nu_C \left( \hat{T}_{C} \right)^{\nu_C} (\odot_{h_{1,1}}, \bullet_{h_{1,2}}, \odot_{h_{1,3}}, \bullet_{h_{1,4}}, \odot_{h_{1,5}}, \bullet_{h_{1,6}})} = \prod_{x=1}^{6} e^{i\pi h_{c,x} \cdot \vec{D}_{j,x}} \left[ \prod_{g=1}^{3} (1 - \hat{T}_{C,3,1,g}) \right] (\bullet_{h_{1,1}}, \bullet_{h_{1,2}}, \bullet_{h_{1,3}}, \bullet_{h_{1,4}}, \odot_{h_{1,5}}, \odot_{h_{1,6}}, \odot_{h_{1,7}}, \odot_{h_{1,8}}).
\]

Also in this case one needs more than one symmetric charge sequence to reach the $c, 3$ expression upon application of the operator $\hat{T}_{C}$ onto these charge sequences two and six times respectively. However, in agreement with properties 5-III and 6-III, the state obtained from the superposition of these two charge sequences is an eigenstate of the charge momentum operator with eigenvalue $k_{\nu} = \pi/a$. Also the $c, \nu$ pseudoparticles belonging to branches such that $\nu > 3$ are a superposition of several such symmetric charge sequences. However, always the state obtained from the superposition of several charge sequences is an eigenstate of the charge momentum operator with eigenvalue $k_{\nu} = \pi/a$. That state can also be expressed in the general form given in Eq. (58).

Also in this case one needs more than one symmetric charge sequence to reach the $c, 4$ expression given in Appendix A. Again, the state obtained from the superposition of the involved charge sequences is an eigenstate of the charge momentum operator with eigenvalue $k_{\nu} = \pi/a$.

For the case of charge (and spin) sequences constituted by several local $c, \nu$ pseudoparticles (and local $s, \nu$ pseudoparticles) belonging to one or several $\nu = 1, 2, ...$ branches and by Yang holons (and HL spinons) the internal structure of the local $c, \nu$ pseudoparticles (and local $s, \nu$ pseudoparticles) is given by Eq. (58) (and Eq. (59)). This result follows from the basic property 7-III. According to that property the rotated-electron distribution configurations of the $\nu$ doubly occupied sites and $\nu$ empty sites (and $\nu$ spin-down singly occupied sites and $\nu$ spin-up singly occupied sites) which describe the internal structure, a $c, \nu$ pseudoparticle (and $s, \nu$ pseudoparticle) located in the effective electronic lattice at position $[h_{j}, a]$ (and $[l_{j}, a]$) are the same as for the above limiting charge (and spin) sequence constituted by a single pseudoparticle. We will in the ensuing section confirm that in agreement with property 8-III, this ensures the periodic boundary conditions of the original electronic problem provided that the band-momentum energy eigenstates are suitable Fourier-transform superpositions of the local charge sequences with band-momentum values obeying Eqs. (37) and (38).

However, whether inside the domain of $2\nu$ sites of a local $c, \nu$ pseudoparticle there can be sites with index $j_{h}$ such that $j_{h,1} < j_{h} < j_{h,2}$, and which sites are occupied by Yang holons or other local $c, \nu$ pseudoparticles remains an open question. This problem is absent in the limiting case we considered here where the charge sequence is constituted by a number $2\nu$ of sites of the effective electronic lattice. This issue is closely related to the definition of the local $c, \nu$ pseudoparticle charge-sequence empty sites which is discussed in the following. A similar problem occurs for the spin sequences.
B. THE LOCAL $\alpha, \nu$ PSEUDOPARTICLE EMPTY SITES AND THE MOVING PSEUDOPARTICLE ELEMENTARY STEPS

In the definition of the $N_{c,\nu}^h$ charge-sequence empty sites of a local $c, \nu$ pseudoparticle it is useful to consider different charge sequences where all remaining local $c, \nu'$ pseudoparticles except that specific local $c, \nu$ pseudoparticle keep their charge-sequence positions $h_j$ defined in Eq. (42). Below we call steady local $c, \nu'$ pseudoparticles the former local pseudoparticles whose charge-sequence positions $h_j$ remain unchanged. Obviously, any local pseudoparticle can be chosen to be the moving pseudoparticle. This just refers to a set of different charge sequences where the charge-sequence position of that quantum object changes whereas the charge-sequence positions of the remaining local pseudoparticles remain unchanged. We note that according to Eq. (46) the charge-sequence position of a local $c, \nu$ pseudoparticle can remain unchanged in spite of the movements of some of its $2\nu$ sites, as we confirm below. Concerning the definition of the $N_{c,\nu}^h$ charge-sequence empty sites of a local $c, \nu$ pseudoparticle, from the use of properties 1-III to 8-III and of other properties of the 1D Hubbard model we find the following distribution configuration properties:

1-IV Let us consider a set of rotated-electron doubly occupied and/or rotated-electron empty sites located at positions $[j_h, a]$ where according to Eqs. (13) and (14) the index $h$ has the values $h = 1, 2, ..., [N_a - N_c]$. Moreover, we consider that all of these sites are located inside the domain of sites of a local $c, \nu$ pseudoparticle and thus their indices $j_h$ are such that $j_{h+1} < j_h < j_{h+2}$ and $j_h \neq j_{h+s}$ where $x = 1, 2, ..., 2\nu$. This property states that the set of rotated-electron doubly occupied/empty sites of index $j_h$ obeying these inequalities cannot correspond to sites occupied by Yang holons or to any of the $2\nu'$ sites of the charge sequence associated with local $c, \nu'$ pseudoparticles belonging to branches such that $\nu' > \nu$. Thus such a set of rotated-electron doubly occupied or empty sites can only correspond to sub-sets of $2\nu'$ sites of the charge sequence associated with local $c, \nu'$ pseudoparticles belonging to branches such that $\nu' \leq \nu$.

2-IV Let us consider a local $c, \nu$ pseudoparticle and a local $c, \nu'$ pseudoparticle belonging to the same charge sequence and to branches such that $\nu \leq \nu'$. Moreover, we consider that there are no charge-sequence local pseudoparticles inside the $2\nu'$-site domain of the $c, \nu'$ pseudoparticle other than the $c, \nu$ pseudoparticle. The description of cases involving more than two local pseudoparticles can be easily reached by generalization of the two-pseudoparticle situation considered here. Then the $2\nu$ sites of the charge sequence associated with the local $c, \nu$ pseudoparticle are located either outside the domain of $2\nu'$ sites of the local $c, \nu'$ pseudoparticle limited in the left and right ends by the sites of index $h_{j,1}$ and $h_{j,2\nu}$ respectively, or are all located inside that domain. If the $2\nu$ sites are located inside the domain of these $2\nu'$ sites there is a number $2\nu' - 1$ of permitted positions for the $c, \nu$ pseudoparticle. In each of these permitted positions all of the $2\nu$ sites of the local $c, \nu$ pseudoparticle are located between two first neighboring sites of indices $h_{j, x}$ and $h_{j', x+1}$ of the local $c, \nu'$ pseudoparticle. These permitted positions correspond to the neighboring sites of indices $h_{j, x}$ and $h_{j', x+1}$ such that $x = 1, ..., 2\nu' - 1$.

3-IV Let us consider again the local $c, \nu$ pseudoparticle and the local $c, \nu'$ pseudoparticle referred to in property 2-IV. The charge-sequence position $h_j$ given in Eq. (42) of such a steady local $c, \nu'$ pseudoparticle remains unchanged in all possible $2\nu' - 1$ positions of the local $c, \nu$ pseudoparticle while passing its $2\nu'$-site domain. The inverse statement is even stronger. It is required that when a local $c, \nu'$ pseudoparticle belonging to a branch such that $\nu' \geq \nu$ passes a steady local $c, \nu$ pseudoparticle, the position in the effective electronic lattice of all the $2\nu$ sites of the latter pseudoparticle must remain unchanged. Moreover, the $2\nu' - 1$ possible different positions of the $2\nu$ sites of the local $c, \nu$ pseudoparticle inside the site domain of the local $c, \nu'$ pseudoparticle must be the same when the local $c, \nu$ pseudoparticle passes the steady local $c, \nu'$ pseudoparticle from its left to its right-hand side and vice versa. Also the relative positions of the two quantum objects when the local $c, \nu$ pseudoparticle passes the steady local $c, \nu'$ pseudoparticle and when the local $c, \nu'$ pseudoparticle passes the steady local $c, \nu$ pseudoparticle must be the same. However, note that although these relative positions are the same, when a local $c, \nu'$ pseudoparticle belonging to a branch such that $\nu' \geq \nu$ passes a steady local $c, \nu$ pseudoparticle it does not occupy any of the $2\nu$ sites of the effective electronic lattice occupied by the latter local pseudoparticle. Thus the positions in the lattice of these $2\nu$ sites remain indeed unchanged. A last requirement is that the positions of all $2\nu'$ sites of a steady local $c, \nu'$ pseudoparticle belonging to a branch such that $\nu' \geq \nu$ must be the same in the rotated-electron site distribution configurations before and after it is passed by the moving local $c, \nu$ pseudoparticle. If the local $c, \nu$ pseudoparticle is moving from the left to the right-hand side of the steady local $c, \nu'$ pseudoparticle, before and after is meant here as the rotated-electron site distribution configurations where the $2\nu$ sites of the former quantum object are located outside the domain of $2\nu'$ sites of the steady local pseudoparticle and on its right and left hand sides respectively.

4-IV Corresponding rules are also valid for the case of the spin sequences and local $s, \nu$ pseudoparticles.
A simple example of a domain constituted by fourteen sites of a permitted charge sequence is represented in Fig. 4. In such a figure the $h_{j, g} \leftrightarrow h_{j, g+\nu}$ site pairs of the local $c, \nu$ pseudoparticles are represented as in Figs. 1 to 3. The six sites with no vertical lines correspond to empty sites associated with $+1/2$ Yang holons. There are two local $c, 1$ pseudoparticles and a local $c, 2$ pseudoparticle in the charge-sequence domain represented in the figure.

In order to confirm the validity of the counting of rotated-electron site distribution occupancies leading to expression (19), let us use the properties 1-IV to 4-IV in the classification of the different $N_{c, \nu}$ possible positions of a local $c, \nu$ pseudoparticle in the effective electronic lattice. Our analysis can be easily generalized to the case of a $s, \nu$ pseudoparticle. Our study includes the introduction of the concept of 2$\nu$-site caterpillar step, where the 2$\nu$-leg caterpillar is nothing but the 2$\nu$-site pseudoparticle block associated with the moving $c, \nu$ pseudoparticle. For simplicity we consider a charge sequence such that $N_{c, \nu} = 1$ for the $\nu$ branch. However, the generalization of our results to branches with finite occupancy of several local pseudoparticles is straightforward. As mentioned above, the possible positions of such a pseudoparticle correspond to a number $N_{c, \nu}^*$ of different charge sequences where the charge-sequence positions of all remaining quantum objects remain unchanged. Thus these $N_{c, \nu}^*$ charge sequences differ in the position of the $c, \nu$ pseudoparticle relative to the other quantum objects. The above property 3-IV states that the charge-sequence position $h_j$ given in Eq. (19) of all quantum objects except that of the moving $c, \nu$ pseudoparticle must remain unchanged in all these $N_{c, \nu}^*$ charge sequences. Let us confirm that the charge-sequence position of a steady local $c, \nu$ pseudoparticle belonging to a branch such that $\nu' > \nu$ indeed remains unchanged when it is passed by a local $c, \nu$ pseudoparticle, in spite of changes of the positions in the effective electronic lattice of some of its $2\nu$ sites.

We start by considering the case of a local $c, \nu$ pseudoparticle where $\nu = \nu'$ denotes the largest $\nu$ value of the $c, \nu$ pseudoparticle branches with finite occupancy in the charge sequence. According to Eq. (17), in this case the number of local $c, \nu$ pseudoparticle empty sites is given by $N_{c, \nu}^h = L_{c}$ and equals the number of Yang holons. The different possible positions of a $c, \nu$ pseudoparticle can be achieved by elementary steps in the charge sequence where each of its $2\nu$ sites moves forward by a single site of the charge sequence. If the movement is from the left to the right-hand side, the $2\nu th$ pseudoparticle site of index $h_{j, 2\nu}$ moves into the site previously occupied by a Yang holon. The remaining $2\nu - 1$ pseudoparticle sites of index $h_{j, x}$ move into the site previously associated with the pseudoparticle site of index $h_{j, x+1}$ where $x = 1, \ldots, 2\nu - 1$. Finally, the Yang holon removed from the site newly associated with the pseudoparticle site of index $h_{j, 2\nu}$ moves into the site previously associated with the first pseudoparticle site of index $h_{j, 1}$. We call such an elementary collective step of the $2\nu$ pseudoparticle sites $2\nu$-leg caterpillar step. According to such an analogy the $2\nu$ legs refer to these sites and the caterpillar refers to the local pseudoparticle. While each of these legs moves forward by a charge-sequence site, the whole caterpillar itself also moves forward by a single charge-sequence site. The net result of such a $2\nu$-leg caterpillar step is that the compact domain of $2\nu$ pseudoparticle sites interchanges position with a Yang holon. When such a moving $c, \nu$ pseudoparticle passes a steady local $c, \nu$ pseudoparticle, each of its $2\nu$ sites jump the $2\nu$ sites of the latter local pseudoparticle. Thus the $2\nu$-leg caterpillar moves forward like the $2\nu$ sites of such a steady local $c, \nu$ pseudoparticle do not exist in the charge sequence. As we discuss below, the construction of the corresponding effective $c, \nu$ pseudoparticle lattice mentioned in property 8-III involves omission of the sites of the effective electronic lattice belonging to the $2\nu-site$ domains of such a local $c, \nu$ pseudoparticles. Moreover, for such an effective $c, \nu$ pseudoparticle lattice the $2\nu$-site domain of the $c, \nu$ pseudoparticles is seen as a point-like occupied site. Thus in the particular case of the $\nu$ branch such an effective lattice has $N_{c, \nu} = L_{c} + N_{c, \nu}$ sites and a number $N_{c, \nu}^h = L_{c}$ free of $c, \nu$ pseudoparticles, as we confirm below.

Let us consider now the case of a general local $c, \nu$ pseudoparticle belonging to a branch such that $1 \leq \nu \leq \nu'$. For simplicity we keep the assumption that $N_{c, \nu} = 1$ for the $\nu$ branch. We consider a given charge sequence where the $c, \nu$ pseudoparticle is located in the effective electronic lattice at position $[h_{j, \nu}]$. It follows from property 4-III that the $N_{c, \nu}^h$ remaining possible positions of the local $c, \nu$ pseudoparticle in the charge sequence define the positions of the local $c, \nu$ pseudoparticle charge-sequence empty sites. Out of these $N_{c, \nu}^h = L_{c} + 2 \sum_{\nu' = \nu+1}^{\nu'} (\nu' - \nu)N_{c, \nu'}$ possible positions, $L_{c}$ is the number of positions associated with Yang holons. Again, the different positions of the moving local $c, \nu$ pseudoparticle can be achieved by $2\nu$-leg caterpillar steps. The net result of such a $2\nu$-leg caterpillar step is that the compact domain constituted by the $2\nu$ $c, \nu$ pseudoparticle sites interchanges position with a Yang holon or with a site of a $h_{j', \nu'} \leftrightarrow h_{j', \nu' + \nu}$ pair of a steady local $c, \nu'$ pseudoparticle belonging to a branch such that $\nu' > \nu$. A $2\nu$-leg caterpillar step is defined here as in the case of the above moving $c, \nu$ pseudoparticle. If the movement is from the left to the right-hand side, the local $c, \nu'$ pseudoparticle site of index $h_{j, 2\nu}$ moves into the site previously occupied by a Yang holon or into one of the two sites previously associated with a local $c, \nu'$ pseudoparticle site $h_{j', \nu'} \leftrightarrow h_{j', \nu' + \nu}$ site pair. The remaining $2\nu - 1$ sites of the local $c, \nu$ pseudoparticle of index $h_{j, x}$ such that $x = 1, \ldots, 2\nu - 1$, move into the site previously associated with the local $c, \nu$ pseudoparticle site of index $h_{j, x+1}$ where again $x = 1, \ldots, 2\nu - 1$. Finally, the Yang holon or $c, \nu'$ pseudoparticle site associated with the local $h_{j', \nu'} \leftrightarrow h_{j', \nu' + \nu}$ site pair which was removed from the site newly associated with the local $c, \nu$ pseudoparticle site of index $h_{1, 2\nu}$ moves into the site previously associated with the local $c, \nu$ pseudoparticle site of index $h_{j, 1}$. Again, the net result of such a $2\nu$-leg caterpillar step is that the local $c, \nu$ pseudoparticle moves forward by a single charge-sequence site.

Let us consider that in the initial rotated-electron site distribution configuration the moving $c, \nu$ pseudoparticle
is located on the left-hand side of the steady local $c, \nu'$ pseudoparticle. Thus in the final state just after passing the latter pseudoparticle, the $c, \nu$ pseudoparticle will occupy in the charge sequence a domain of $2\nu$ sites located on the right-hand side of the $c, \nu'$ pseudoparticle site domain. According to property 3-IV, in both the initial and final rotated-electron site distribution configurations all $2\nu'$ sites of the latter quantum object remain the same. For simplicity we consider that in the initial rotated-electron site distribution configuration, the $2\nu'$ charge-sequence sites located just on the right-hand side of the steady local $c, \nu'$ pseudoparticle are occupied by Yang holons. However, our results can be generalized to rotated-electron site distribution configurations where these sites are occupied by a third pseudoparticle. Another case which we could consider is when the moving local pseudoparticle passes two steady local $c, \nu'$ and $c, \nu''$ pseudoparticles such that one of them is located within the site domain of the other one. From the systematic use of the properties 1-IV to 4-IV one can describe the movements of a local $c, \nu$ pseudoparticle when it passes any rotated-electron site distribution configuration involving several steady local pseudoparticles.

According to property 2-IV the $2\nu$ sites of a local $c, \nu$ pseudoparticle can have $2\nu' - 1$ different positions inside the $2\nu'$-site domain of a local $c, \nu'$ pseudoparticle belonging to a branch such that $\nu' > \nu$. We consider that in the initial rotated-electron site distribution configuration, the local $c, \nu$ and $c, \nu'$ pseudoparticles are located side by side in the charge sequence. Thus these two quantum objects occupy together a compact $2(\nu + \nu')$-site domain. By simple counting arguments it is straightforward to realize that in order to reach each of these permitted positions each of the $2\nu$ sites of the local $c, \nu$ pseudoparticle must perform a jump of $2\nu$ sites in the charge sequence somewhere inside the $2\nu'$-site domain of the steady local $c, \nu'$ pseudoparticle. This result is consistent with the expression of the number $2\left(\sum_{\nu'=\nu+1}^{\nu+\infty}(\nu' - \nu)N_{\alpha, \nu'}\right)$ of the available positions for the moving local $c, \nu$ pseudoparticle involving sites of steady local $c, \nu'$ pseudoparticles belonging to branches such that $\nu' > \nu$. This expression reveals that when located inside the site domain of such a pseudoparticle each of the $2\nu$ sites of the local $c, \nu$ pseudoparticle can move into a possible $2(\nu' - \nu)$ sites out of the $2\nu'$ total number of sites of the steady local $c, \nu'$ pseudoparticle. This agrees with the above result obtained from counting arguments alone. We note that the number of charge-sequence sites jumped by each local $c, \nu$ caterpillar leg equals precisely $2\nu$, the number of sites of such a $c, \nu$ pseudoparticle. Thus these jumps ensure that the position in the effective electronic lattice of all the $2\nu'$ sites of the steady local $c, \nu'$ pseudoparticle remains unchanged after the $c, \nu$ passes it, as requested by the property 3-IV. This result also applies to the general situation when a moving local $c, \nu$ pseudoparticle passes a domain containing several local steady $c, \nu'$ pseudoparticles belonging to branches such that $\nu' > \nu$. Also in the general situation that each of the legs of the moving caterpillar uses $2(\nu' - \nu)$ possible sites of each steady local $c, \nu'$ pseudoparticle, out of a total of $2\nu'$ sites.

The question which arises from the occurrence of the above jump mechanism is which are the $2\nu$ sites of the steady local $c, \nu'$ pseudoparticle that are jumped by each site of the moving local $c, \nu$ pseudoparticle. Fortunately, the requirements mentioned in the property 3-IV provide the needed information for the precise definition of the $2\nu' - 1$ permitted positions of the local $c, \nu$ pseudoparticle inside the steady local $c, \nu'$ pseudoparticle. We find that in order to reach these permitted positions each site of the local $c, \nu$ pseudoparticle must perform two independent jumps of $\nu$ sites in the charge sequence. Let us suppose that the local $c, \nu$ pseudoparticle is moving from the left to the right-hand side. These jumps bring the local $c, \nu$ pseudoparticle to and from respectively, a position where its charge-sequence location $h_j$, as given by Eq. (40), coincides with that of the steady local $c, \nu'$ pseudoparticle. In such an event each of the $2\nu$ legs of the caterpillar jumps a number $\nu$ of sites and reaches a position where its $2\nu$ legs occupy $2\nu$ sites of the charge sequence located around the center of the steady local $c, \nu'$ pseudoparticle. In order to move forward to the right-hand side from that position, each of the $2\nu$ legs of the caterpillar jump again a number of $\nu$ sites of the charge sequence. Thus the possible positions of a local $c, \nu$ pseudoparticle when it passes from the left to the right-hand side of a steady local $c, \nu'$ pseudoparticle are the following: First the local $c, \nu$ pseudoparticle performs a $2\nu$-leg caterpillar step. The net result of such an elementary collective step is that the compact domain of $2\nu$ sites of the local $c, \nu$ pseudoparticle interchanges position with the site of index $h_{j',1}$ of the local $c, \nu'$ pseudoparticle. Thus this first elementary collective step brings the local $c, \nu$ pseudoparticle inside the domain of $2\nu'$ sites of the steady local $c, \nu'$ pseudoparticle. In the reached rotated-electron site distribution configuration all the $2\nu$ sites of the local $c, \nu$ pseudoparticle are located between the sites of index $h_{j',1}$ and $h_{j',2}$ of the steady local $c, \nu'$ pseudoparticle. This procedure is repeated $\nu' - 1$ times until the $2\nu$ sites of the $c, \nu$ pseudoparticle are located between the sites of index $h_{j',\nu'-1}$ and $h_{j',\nu'}$ of the steady local $c, \nu'$ pseudoparticle. In each of the corresponding $2\nu$-leg caterpillar steps the local $c, \nu$ pseudoparticle moves forward by a single lattice site of the charge sequence. However, in the $\nu'$th step each site of the $c, \nu$ pseudoparticle jumps $\nu$ sites of the charge sequence. This jump brings it to a rotated-electron site distribution configuration where the charge-sequence position $h_j$ of the local $c, \nu$ pseudoparticle given in Eq. (40) coincides with that of the steady local $c, \nu'$ pseudoparticle. Let $h_{j,x}$ where $x = 1,...,2\nu$ denote the indices of the $2\nu$ sites of the $c, \nu$ pseudoparticle while it was located between the sites of index $h_{j',\nu'-1}$ and $h_{j',\nu'}$ of the local $c, \nu'$ pseudoparticle. In the next collective movement the local $c, \nu$ pseudoparticle site of index $h_{j',2\nu}$ jumps into the site which was associated with the local $c, \nu'$ pseudoparticle site of index $h_{j',\nu+\nu'}$. Out of the remaining $2\nu - 1$ sites of the local $c, \nu$ pseudoparticle, $\nu$ of these sites of indices $h_{j,x}$ such that $x = \nu, ... , 2\nu - 1$ jump into the local $c, \nu'$ pseudoparticle sites of indices $h_{j',\nu'-1}$ where again $x = \nu, ..., 2\nu - 1$. The remaining $\nu - 1$ sites of the local $c, \nu$ pseudoparticle of
indices \( h_{j,x} \) such that \( x = 1, \ldots, \nu - 1 \) jump into the sites which were occupied by the local \( c, \nu \) pseudoparticle sites of indices \( h_{j,x} \) where \( x = \nu + 2, \ldots, 2\nu \). As a result of these events the local \( c, \nu' \) pseudoparticle site of index \( h_{j',\nu'} \) jumps \( \nu - 1 \) charge-sequence sites from the right to the left-hand side and thus moves by a block of \( \nu \) sites in such a direction. In contrast, all the remaining \( c, \nu' \) pseudoparticle sites of indices \( h_{j',x} \) such that \( x = 1, \ldots, \nu' - 1, \nu' + 1, \ldots, 2\nu' \) jump \( \nu - 1 \) charge-sequence sites from the left to the right-hand side and move by a block of \( \nu \) sites in such a direction. These collective jumps reach a configuration where the \( 2\nu \) sites of the \( c, \nu \) pseudoparticle are located between the sites of \( h_{j',\nu'} \) and \( h_{j',1+\nu'} \) of the \( c, \nu' \) pseudoparticle. In order to reach the next permitted position each of the \( 2\nu \) local \( c, \nu \) pseudoparticle sites jump again a number \( \nu \) of lattice sites of the charge sequence. This brings the local \( c, \nu \) pseudoparticle to a distribution configuration where its \( 2\nu \) sites are located between the sites of index \( h_{j',\nu'} \) and \( h_{j',2+\nu'} \) of the steady local \( c, \nu' \) pseudoparticle. These two jumps are followed by \( \nu' - 1 \) \( 2\nu'-\text{leg} \) caterpillar steps where the \( c, \nu \) pseudoparticle moves again by a single lattice site of the charge sequence. The result of each of these elementary collective steps is that the compact domain of \( 2\nu \) \( c, \nu \) pseudoparticle sites interchanges position with one of the sites of the \( c, \nu' \) pseudoparticle. Step number \( \nu' - 2 \) of these \( \nu' - 1 \) \( 2\nu'-\text{leg} \) caterpillar steps leads to a distribution configuration where the \( 2\nu \) sites of the \( c, \nu \) pseudoparticle are located between the sites of index \( h_{j',2\nu' - 1} \) and \( h_{j',2\nu'} \) of the steady local \( c, \nu' \) pseudoparticle. Finally, the last step brings the local \( c, \nu \) pseudoparticle to a distribution configuration where it is located outside and on the right-hand side of the domain of \( 2\nu' \) sites of the steady local \( c, \nu' \) pseudoparticle. The same intermediate positions are reached when the \( c, \nu \) pseudoparticle passes the domain of \( 2\nu' \) sites of the \( c, \nu' \) pseudoparticle from the right to the left-hand side.

We have just studied a situation where a local \( c, \nu \) pseudoparticle passes a steady local \( c, \nu' \) pseudoparticle belonging to a branch such that \( \nu > \nu' \). For simplicity we assumed that in the initial rotated-electron site distribution configuration the domain of \( 2\nu \) charge-sequence sites located just on the right-hand side of the steady local \( c, \nu' \) pseudoparticle are occupied by Yang holons. However, our results can be generalized to more complicated situations. An example is a situation where a local \( c, \nu \) pseudoparticle passes a steady local \( c, \nu' \) pseudoparticle which has as first neighbor in the charge sequence a second steady local \( c, \nu'' \) pseudoparticle. Another example is when a local \( c, \nu \) pseudoparticle passes a steady local \( c, \nu' \) pseudoparticle containing inside its \( 2\nu'-\text{site} \) domain a third steady local \( c, \nu'' \) pseudoparticle belonging to a branch such that \( \nu'' \leq \nu' \). These more general situations can be described by the use of the properties 1-IV to 4-IV. One always finds that in order to move forward, the \( 2\nu' \) sites of the local \( c, \nu \) pseudoparticle use \( 2(\nu' - \nu) \) sites out of the total number of \( 2\nu' \) sites of each steady local \( c, \nu' \) pseudoparticle belonging to branches such that \( \nu' > \nu \). Moreover, when passing the central region of the \( 2\nu' \)-site domain of such a local \( c, \nu' \) pseudoparticle, the \( 2\nu \) legs of the \( c, \nu \) caterpillar always perform the two \( \nu'-\text{site} \) jumps described above.

In summary, the number of charge-sequence empty sites of a \( c, \nu \) pseudoparticle equals the number of sites occupied by Yang holons plus a number of sites given by \( 2(\nu' - \nu) \) for each \( c, \nu' \) pseudoparticle of the charge sequence belonging to branches such that \( \nu' > \nu \). The number of permitted positions of the local \( c, \nu \) pseudoparticle inside each of the site domains of such a \( c, \nu' \) pseudoparticle is \( 2\nu' - 1 \). In these permitted positions the \( c, \nu \) pseudoparticle is located between the sites \( h_{j',x} \) and \( h_{j',x+1} \) such that \( x = 1, \ldots, 2\nu' - 1 \) of the local \( c, \nu' \) pseudoparticle. In order to reach or to leave a position between the sites of index \( h_{j',\nu'} \) and \( h_{j',1+\nu'} \) of steady local \( c, \nu' \) pseudoparticle belonging to a branch such that \( \nu' > \nu \), each of the \( 2\nu \) legs of the moving caterpillar always performs a jump of \( \nu \) sites in the charge sequence. Otherwise it moves throughout the charge sequence by \( 2\nu'-\text{leg} \) caterpillar steps. In such steps the \( 2\nu'-\text{site} \) block of the local \( c, \nu \) pseudoparticle moves forward by a charge-sequence site. The net result of such an event is that the \( 2\nu'-\text{site} \) domain of the moving quantum object interchanges position with either a Yang holon or with a site of a \( c, \nu' \) pseudoparticle belonging to a branch such that \( \nu' > \nu \). The present definition of the permitted positions of a local \( c, \nu \) pseudoparticle when passing the remaining quantum objects which occupy a charge sequence provides a consistent, complete, and unique description for the rotated-electron distribution configurations of doubly occupied sites and empty sites which describe the charge sequence. The results obtained for the charge sequences and corresponding \( c, \nu \) pseudoparticles are also valid for the spin sequences and their \( s, \nu \) pseudoparticles.

The events studied above are illustrated in Figs. 5 and 6 for the case when a local \( c, 1 \) pseudoparticle passes a steady local \( c, 1 \) pseudoparticle and a steady local \( c, 2 \) pseudoparticle respectively, from the left to the right-hand side. If instead we consider that the local \( c, 1 \) pseudoparticle is steady, Fig. 6 illustrates the possible positions of a local \( c, 2 \) pseudoparticle when it passes from the right to the left-hand side a steady local \( c, 1 \) pseudoparticle. In this case the position in the effective electronic lattice of the two sites of the \( c, 1 \) pseudoparticle remains unchanged and the different relative positions of the two quantum objects are reached by movements of the \( c, 2 \) pseudoparticle. Importantly, we emphasize that in this case the \( c, 2 \) pseudoparticle does not occupy the sites occupied by the \( c, 1 \) pseudoparticle. Again this is consistent with property 3-IV.

The charge sequence (and the spin sequence) is obtained by omission of the rotated-electron singly occupied sites (and rotated-electron doubly occupied/empty sites). Such a procedure follows from the independent conservation of the charge and spin sequence. We note that each of the \( c, \nu \) pseudoparticle (and \( s, \nu \) pseudoparticle) branch occupancy configurations are also independently conserved. Thus it follows our above analysis that the construction of the effective \( c, \nu \) pseudoparticle lattice mentioned in property 8-III also involves omission of the sites of the effective
electronic lattice which are jumped by the movements of a $c, \nu$ pseudoparticle, since these sites are not seen by such a moving quantum object. The omitted sites correspond to the $2\nu'$-site domains of local $c, \nu'$ pseudoparticles of branches such that $\nu' < \nu$ and to $2\nu$-site domains belonging to the $2\nu'$ sites of each $c, \nu'$ pseudoparticles such that $\nu' > \nu$. Moreover, for such an effective $c, \nu$ pseudoparticle lattice the $2\nu$-site domain of the $c, \nu$ pseudoparticles is seen as a point-like occupied site. Thus such an effective lattice has $N_{c, \nu}^h = L_c + \sum_{\nu' > \nu} 2(\nu' - \nu) N_{c, \nu'} + N_{c, \nu}$ sites, a number $N_{c, \nu}^h = L_c + 2 \sum_{\nu' > \nu} (\nu' - \nu) N_{c, \nu'}$ of which are empty of $c, \nu$ pseudoparticles, as we confirm below. A similar analysis holds for the $s, \nu$ pseudoparticles.

C. COMPLETE SET OF CHARGE, SPIN, AND $c$ PSEUDOPARTICLE SEQUENCE LOCAL STATES

We have just found the rotated-electron distribution configurations of the $2\nu$ sites which describe a local $\alpha, \nu$ pseudoparticle. Moreover, we also defined the $N_{c, \nu}^h$ charge-sequence empty sites of such a local $\alpha, \nu$ pseudoparticle. Although our results can be generalized to the case of the local spin sequences and corresponding local $s, \nu$ pseudoparticles, often our analysis focused explicitly on the case of the local charge sequences and associated local $c, \nu$ pseudoparticles. An important result which follows from our findings is that indeed the positions of the $N_{c, \nu}^h$ charge-sequence empty sites of each local $c, \nu$ pseudoparticle branch are fully determined by the location of the $c$ pseudoparticles and $c, \nu$ pseudoparticles alone. Our results thus confirm that for fixed values of the numbers $N_{c, \nu}$ and $N_{c, \nu}^h$ the number of occupancy configurations of the local $c, \nu$ pseudoparticles is given by Eq. (10) for $\alpha = c$. It follows that we can classify the local $c, \nu$ pseudoparticle occupancy configurations for $\nu = 1, 2, ...$ branches with finite occupancy in a given state by providing the indices $h_j$ given in Eqs. (15). These indices correspond to the $N_{c, \nu}^h$ pseudoparticle locations provided in Eq. (50). Our results also confirm that the number of local pseudoparticle occupancy configurations of a CPHS ensemble subspace is indeed given by Eq. (22). Finally, since our results can be generalized to spin sequences as well, they also validate the representation associated with the local state occupancy configurations for a CPHS ensemble subspace is indeed given in Eq. (22). Our results also confirm that the set of these local states whose number is given in Eq. (22) constitutes a complete set of states in the corresponding CPHS ensemble subspace.

Thus in the case of CPHS ensemble subspaces with no $-1/2$ Yang holons and no $-1/2$ HL spinons the local states defined in Eqs. (26) - (28) constitute a complete set of states. Our main goal here is the generalization of these local states to CPHS ensemble subspaces with finite occupancies of both $\pm 1/2$ Yang holons and $\pm 1/2$ HL spinons. We start by expressing the complete set of local states defined in Eqs. (26) - (28) in terms of the following more basic states,

$$|\{j_1, j_2, ..., j_{N_c}\}; \{\bullet_{j_{1,1}}, ..., \bullet_{j_{1,\nu}}, \bigtriangledown_{j_{1,\nu}}, ..., \bigtriangledown_{j_{1,2\nu}}\}; \{\downarrow_{j_{1,1}}, ..., \downarrow_{j_{1,\nu}}, \uparrow_{l_{1,1}}, ..., \uparrow_{l_{1,2\nu}}\}\rangle,$$

where the charge sequence refers to the following sets of rotated-electron distribution configurations of doubly-occupied and empty sites,

$$\{\bullet_{j_{1,1}}, ..., \bullet_{j_{1,\nu}}, \bigtriangledown_{j_{1,\nu}}, ..., \bigtriangledown_{j_{1,2\nu}}\} = (\bullet_{h_{1,1}}, \bigtriangledown_{h_{1,1}}), (\bullet_{h_{2,1}}, \bigtriangledown_{h_{2,1}}), ..., (\bullet_{h_{N_{c,\nu}}, \nu}, \bigtriangledown_{h_{N_{c,\nu}}, \nu});$$

$$\{\bigtriangledown_{h_{1,1}}, ..., \bigtriangledown_{h_{1,\nu}}, \bullet_{h_{1,\nu}}, ..., \bullet_{h_{1,2\nu}}\} = (\bullet_{h_{1,1}}, \bigtriangledown_{h_{1,1}}), (\bullet_{h_{1,2}}, \bigtriangledown_{h_{1,2}}), ..., (\bullet_{h_{N_{c,\nu}}, \nu}, \bigtriangledown_{h_{N_{c,\nu}}, \nu});$$

$$\{\downarrow_{j_{1,1}}, ..., \downarrow_{j_{1,\nu}}, \uparrow_{l_{1,1}}, ..., \uparrow_{l_{1,2\nu}}\} = (\downarrow_{h_{1,1}}, ..., \downarrow_{h_{1,\nu}}, \uparrow_{l_{1,1}}, ..., \uparrow_{l_{1,2\nu}});$$

and the same follows for the spin sequence with the index $h$ replaced by the index $l$, rotated-electron doubly occupied sites $\bullet$ replaced by spin-down rotated-electron singly occupied sites $\bigtriangledown$, and sites free of rotated electrons $\circ$ replaced by spin-up rotated-electron singly occupied sites $\uparrow$. The states defined in Eqs. (26) - (28) provide the location of the $2\nu$ sites associated with each of the $\alpha, \nu$ pseudoparticles whose branches have finite occupancy in the corresponding charge and spin sequence. However, we note that these states include only rotated-electron site distribution configurations of the simple type given in Eqs. (20) and (21). The general charge and spin sequences can be obtained from these states by application of suitable operators, as we discussed above. These are thus the most basic and simple states. Our first task is expressing the states defined in Eqs. (26) - (28) in terms of these simple states.

Each local $c, \nu$ and $s, \nu$ pseudoparticle is described by the superposition of $2^n$ rotated-electron site distribution configurations given in Eqs. (18) and (19) respectively. The type of configuration superposition given in these equations is common to the description of all local $\alpha, \nu$ pseudoparticles of a charge or spin sequence. Thus the proper description of the the charge and spin sequence of each of the states defined in Eqs. (26) - (28) involves the
consider first the case of the charge sequences. According to property 1-III the ± SU(N) electron empty sites (and rotated-electron doubly occupied sites). The application of the off-diagonal generators of local charge sequence is described by a superposition order to achieve the Yang holon occupancy configurations required by the case of states associated with CPHS ensemble subspaces spanned by states with both finite occupancies of η sites of such a charge sequence respectively. In the case of lowest weight state (and highest weight state) of the holons correspond to single rotated-electron doubly occupied lattice sites and single rotated-electron empty lattice precisely the same occupancy configurations of local c spins algebra all Yang holons have the same numbers were given above and are illustrated in Eqs. (60) and (61). The local states given in both Eqs. (53)-(55) and in Eq. (77) provide a complete basis of states for CPHS ensemble subspaces spanned by lowest-weight states of both the SU(2) η-spin and spin algebras. Let us now consider the general case of states associated with CPHS ensemble subspaces spanned by states with both finite occupancies of ±1/2 Yang holons and ±1/2 HL spinons. The description of the local charge and spin sequences with finite occupancies of ±1/2 Yang holons and ±1/2 HL spinons respectively, is quite similar to the ones we studied above. For simplicity let us consider first the case of the charge sequences. According to property 1-III the −1/2 Yang holons and +1/2 Yang holons correspond to single rotated-electron doubly occupied lattice sites and single rotated-electron empty lattice sites of such a charge sequence respectively. In the case of a lowest weight state (and highest weight state) of the η-spin algebra all Yang holons have the same η-spin projection σ_c = +1/2 (and σ_c = −1/2) and correspond to rotated-electron empty sites (and rotated-electron doubly occupied sites). The application of the off-diagonal generators of the SU(2) η-spin algebra given in Eq. (4) leads to flips of the Yang holons η-spins [11]. Following property 5-III, in order to achieve the Yang holon occupancy configurations required by the SU(2) η-spin algebra, we find that each local charge sequence is described by a superposition

\[ \begin{aligned}
\sum_{\alpha=c,s} L_c \sum_{\nu=1}^{\infty} N_{\alpha,\nu} & \left( (j_1, j_2, \ldots, j_{N_c}); \{ (h_1, h_2, \ldots, h_{N_c, \nu}) \}; \{ (l_1, l_2, \ldots, l_{N_\nu, \nu}) \} \right) \\
= & \left[ \prod_{\nu=1}^{N_c} \prod_{j'=1}^{N_{\nu, \nu}} \prod_{x=1}^{h_j} e^{i\pi h_{j', x}} \hat{D}_{j', x} \right] \left[ \prod_{\alpha=c,s} \prod_{\nu'=1}^{\infty} \prod_{j''=1}^{N_{\nu', \nu''}} (1 - \hat{T}_{\alpha, \nu', j'', g''}) \right] \\
& \times \left( (j_1, j_2, \ldots, j_{N_c}); \{ \bullet_{h_{j_1, 1}}, \bullet_{h_{j_1, 2}}, \sigma_{h_{j_1, 1+\nu}}, \sigma_{h_{j_1, 2}} \}; \{ \downarrow_{l_{j_1, 1}}, \downarrow_{l_{j_1, 2}}, \uparrow_{l_{j_1, 1+\nu}}, \uparrow_{l_{j_1, 2}} \} \right),
\end{aligned} \]

(77)

where the operator \( \hat{D}_{j', x} \) measures the number of rotated-electron doubly occupied sites in the charge-sequence site of index \( h_{j', x} \) (whose value is given by 1 or 0) and the transformation laws for application of the operator \( \hat{T}_{\alpha, \nu', j'', g''} \) were given above and are illustrated in Eqs. (60) and (61).

The local states given in both Eqs. (53)-(55) and in Eq. (77) provide a complete basis of states for CPHS ensemble subspaces spanned by lowest-weight states of both the SU(2) η-spin and spin algebras. Let us now consider the general case of states associated with CPHS ensemble subspaces spanned by states with both finite occupancies of ±1/2 Yang holons and ±1/2 HL spinons. The description of the local charge and spin sequences with finite occupancies of ±1/2 Yang holons and ±1/2 HL spinons respectively, is quite similar to the ones we studied above. For simplicity let us consider first the case of the charge sequences. According to property 1-III the −1/2 Yang holons and +1/2 Yang holons correspond to single rotated-electron doubly occupied lattice sites and single rotated-electron empty lattice sites of such a charge sequence respectively. In the case of a lowest weight state (and highest weight state) of the η-spin algebra all Yang holons have the same η-spin projection \( \sigma_c = +1/2 \) (and \( \sigma_c = −1/2 \)) and correspond to rotated-electron empty sites (and rotated-electron doubly occupied sites). The application of the off-diagonal generators of the SU(2) η-spin algebra given in Eq. (4) leads to flips of the Yang holons η-spins [11]. Following property 5-III, in order to achieve the Yang holon occupancy configurations required by the SU(2) η-spin algebra, we find that each local charge sequence is described by a superposition

\[ \begin{aligned}
\left( \frac{L_c}{L_c, -1/2} \right) = \frac{L_c!}{L_c, -1/2! L_c, +1/2!},
\end{aligned} \]

(78)
of Yang holon occupancy configurations. Here \( L_c = 2S_c \) stands for the total number of Yang holons in the charge sequence. All the \( \left( \frac{L_c}{L_c, -1/2} \right) \) states associated with these configurations have the same set of \( L_c \) lattice sites occupied by Yang holons and the same numbers of −1/2 Yang holons and +1/2 Yang holons. However, these states differ in the distributions of the −1/2 Yang holons and +1/2 Yang holons over the \( L_c \) lattice sites. The same analysis holds for spin sequences with finite occupancies of ±1/2 HL spinons. Thus there are \( \prod_{\alpha=c,s} (L_\alpha + 1) \) states with precisely the same occupancy configurations of local c pseudoparticles and local α, ν pseudoparticles. Each of these occupancy configurations is described by a state of form given in Eq. (77). On the other hand, we denote each of the \( \prod_{\alpha=c,s} (L_\alpha + 1) \) general states by,

\[ \left( \frac{L_c}{L_c, -1/2}; L_s, -1/2 \right) (j_1, j_2, \ldots, j_{N_c}); \{ (h_1, h_2, \ldots, h_{N_c, \nu}) \}; \{ (l_1, l_2, \ldots, l_{N_\nu, \nu}) \} ,
\]

(79)

where \( L_c, -1/2 \) and \( L_s, -1/2 \) are the corresponding numbers of −1/2 Yang holons and −1/2 HL spinons respectively. We note that Eq. (22) remains valid for the general local states (29). Thus the values of the numbers \( L_c \) and \( L_s \) are determined by the values of the numbers of c pseudoparticles and of α, ν pseudoparticles. The values of the numbers \( L_c, -1/2 \) and \( L_s, -1/2 \) were added to the local states (24) because they are needed for the specification of the corresponding CPHS ensemble subspace. In contrast, the values of the numbers \( L_c, +1/2 \) and \( L_s, +1/2 \) are dependent and given by \( L_c, +1/2 = L_c - L_c, -1/2 \) and \( L_s, +1/2 = L_s - L_c, -1/2 \) and are not explicitly provided in the expression of the local states (24). Each of these normalized states is given by,
superpositions of local states of general form given in Eq. (80), as discussed in the ensuing section. The concept of effective pseudoparticle lattice, whose spatial coordinates are associated with the pseudoparticle translational degrees of freedom only. According to the properties 1-IV to 4-IV, in addition to the $2\nu$-site internal structure of a local pseudoparticle, the $2\nu$-site domains of the remaining local pseudoparticles of the same branch. Thus in the description of the motion of a local $c, \nu$ pseudoparticle, the $2\nu$-site domains of the remaining local pseudoparticles of the same branch. Thus in the description of the motion of a local $c, \nu$ pseudoparticle, the $2\nu$-site domains of the remaining local pseudoparticles of the same branch. Thus in the description of the motion of a local $c, \nu$ pseudoparticle, the $2\nu$-site domains of the remaining local pseudoparticles of the same branch. Thus in the description of the motion of a local $c, \nu$ pseudoparticle, the $2\nu$-site domains of the remaining local pseudoparticles of the same branch. Thus in the description of the motion of a local $c, \nu$ pseudoparticle, the $2\nu$-site domains of the remaining local pseudoparticles of the same branch. Thus in the description of the motion of a local $c, \nu$ pseudoparticle, the $2\nu$-site domains of the remaining local pseudoparticles of the same branch. Thus in the description of the motion of a local $c, \nu$ pseudoparticle, the $2\nu$-site domains of the remaining local pseudoparticles of the same branch. Thus in the description of the motion of a local $c, \nu$ pseudoparticle, the $2\nu$-site domains of the remaining local pseudoparticles of the same branch. Thus in the description of the motion of a local $c, \nu$ pseudoparticle, the $2\nu$-site domains of the remaining local pseudoparticles of the same branch. Thus in the description of the motion of a local $c, \nu$ pseudoparticle, the $2\nu$-site domains of the remaining local pseudoparticles of the same branch. Thus in the description of the motion of a local $c, \nu$ pseudoparticle, the $2\nu$-site domains of the remaining local pseudoparticles of the same branch. Thus in the description of the motion of a local $c, \nu$ pseudoparticle, the $2\nu$-site
domains of these objects play the role of the occupied sites of an effective $c, \nu$ pseudoparticle lattice. From the point of view of the motion of these objects in such an effective $c, \nu$ pseudoparticle lattice these $2\nu$-site domains are point-like sites without internal structure. Roughly speaking, these point-like pseudoparticle occupied sites correspond to the center of mass of the $2\nu$-site block associated with each local $c, \nu$ pseudoparticle. This description corresponds to a separation of the pseudoparticle translational and internal degrees of freedom. The latter degrees of freedom are described by the $2\nu$-site rotated-electron distribution configurations studied in the previous section. Such internal structure plays a key role both in the fulfillment of the $c, \nu$ pseudoparticle transformations under application of the off-diagonal generators of the $SU(2)$ $\eta$-spin algebra. In addition, such an internal structure is a necessary condition for the fulfillment of the periodic boundary conditions of the original electronic problem. On the other hand, the $c, \nu$ pseudoparticle translational degrees of freedom are closely related to the spatial coordinates of the effective pseudoparticle lattice. These spatial coordinates play the role of conjugate variable relative to the band momentum $q_j$. Such a band momentum obeys Eq. (13) to ensure periodic boundary conditions for the original electronic problem. An important point is that the pseudoparticle spatial coordinate occupancy configuration of each $c, \nu$ pseudoparticle branch with finite occupancy in a given energy eigenstate is independently conserved. As the charge (and spin) sequence corresponds to the rotated-electron doubly occupied and empty sites (and rotated-electron singly occupied sites) only, one can also introduce an independent effective $c, \nu$ pseudoparticle lattice for each of these occupied pseudoparticle branches, whose coordinates correspond to some of the sites of the local charge sequence only.

2-V Within the above separation of the pseudoparticle translational and internal degrees of freedom, the moving $c, \nu$ pseudoparticle sees the $2\nu$-site domains of each of the local pseudoparticles belonging to the same branch as point-like occupied pseudoparticle sites. On the other hand, according to the property 3-IV a moving $c, \nu$ pseudoparticle jumps the $2\nu'$-site domains representative of each local $c, \nu'$ pseudoparticles belonging to branches such that $\nu' \leq \nu$. This property can be understood as follows: Since in its movements the $c, \nu$ pseudoparticle sees the $2\nu$-site domains of the remaining $c, \nu$ pseudoparticles of the same branch as point-like unreachable occupied pseudoparticle sites, such a $2\nu$-site domain width is the smallest width seen by the $c, \nu$ pseudoparticle. Thus such a width plays the role of a site domain width uncertainty. As a result, pseudoparticle site domains of width smaller than the $2\nu$-site domain are not seen by the $c, \nu$ pseudoparticle. This is consistent with the exact property that the moving $c, \nu$ pseudoparticle does not see (i.e., jumps) smaller $2\nu'$-site $c, \nu'$ pseudoparticle domains corresponding to branches such that $\nu' < \nu$. On the other hand, again according to property 3-IV, the $2\nu$-leg caterpillar step movements of the $c, \nu$ pseudoparticle uses $2(\nu' - \nu)$ sites only, out of the $2\nu'$-site domain of each local $c, \nu'$ pseudoparticle belonging to branches such that $\nu' > \nu$. Thus, again consistently with the above site domain width uncertainty, in their motion throughout the effective electronic lattice, a local $c, \nu$ pseudoparticle does not see a number $2\nu$ of sites out of the $2\nu'$ sites of such a $\nu' > \nu$ local $c, \nu'$ pseudoparticle.

3-V Let us introduce the concepts of empty sites and lattice constant of the effective $c, \nu$ pseudoparticle lattice which follow from the above analysis. For the moving $c, \nu$ pseudoparticle all the $N_{c, \nu}^*$ charge-sequence empty sites corresponding to a number $L_c$ of $\pm 1/2$ Yang holons and a number $2\sum_{\nu'=1}^{\infty} (\nu' - \nu) N_{c, \nu'}$ of sites belonging to the $2\nu'$-site domains of local $c, \nu'$ pseudoparticles of the charge sequence such that $\nu' > \nu$ are seen as equivalent and indiscernible point-like empty sites of an effective $c, \nu$ pseudoparticle lattice. Moreover, according to the property 1-V such a quantum object also sees the $N_{c, \nu}$ $2\nu$-site domains of pseudoparticles belonging to the $c, \nu$ pseudoparticle branch as equivalent and indiscernible point-like occupied sites. Thus while running through all its possible positions in the effective electronic lattice the moving $c, \nu$ pseudoparticle sees all the $N_{c, \nu}^* = N_{c, \nu} + N_{c, \nu}^h$ sites of the effective $c, \nu$ pseudoparticle lattice as equivalent and indiscernible point-like sites. Therefore, these sites are equally spaced for the effective $c, \nu$ pseudoparticle lattice. The value of the corresponding lattice constant is uniquely defined as follows: In its movements a $c, \nu$ pseudoparticle does not see (i) the $2\sum_{\nu'=1}^{\infty} \nu' N_{c, \nu'}$ sites occupied by $c, \nu'$ pseudoparticles belonging to branches such that $\nu' \leq \nu$; (ii) a number $2\nu \sum_{\nu'=\nu+1}^{\infty} N_{c, \nu'}$ of sites belonging to the $2\nu'$-site domains of $c, \nu'$ pseudoparticles belonging to branches such that $\nu' > \nu$; and (iii) the $N_{c, \nu}$ sites singly occupied by rotated electrons. Thus the $c, \nu$ pseudoparticle jumps all the above sites of the effective electronic lattice and sees the $2\nu$-site pseudoparticle domains of the $c, \nu$ pseudoparticle branch as point-like occupied sites. On the other hand, in order to pass all the $N_{c, \nu}^* = N_{c, \nu} + N_{c, \nu}^h$ sites of the effective $c, \nu$ pseudoparticle lattice and return to its original position, the $c, \nu$ pseudoparticle should run through a distance which equals the length $L$ of the real-space and effective electronic lattice. Therefore, a necessary condition to ensure the periodic boundary conditions of the original electronic problem is that the length of the effective $c, \nu$ pseudoparticle lattice must equal the length $L$ of the effective electronic lattice. This determines uniquely the value of the lattice constant of the effective $c, \nu$ pseudoparticle lattice which reads $a_{c, \nu} = L/N_{c, \nu}^*$. That the length of the effective $c, \nu$ pseudoparticle lattice is $L$ is consistent with the spacing $q_{j+1} - q_j = 2\pi/L$ given in Eq. (15) for the corresponding pseudoparticle discrete band-momentum values provided by the exact Bethe-ansatz solution [11]. Such a discrete band momentum $q_j$ is the conjugate of the coordinate $x_j = a_{c, \nu} j$
of the effective $c, \nu$ pseudoparticle lattice where $j = 1, 2, ..., N^*_{c, \nu}$.

4-V A similar analysis is valid for the local $s, \nu$ pseudoparticles, provided that the above mentioned $N_c$ sites singly occupied by rotated electrons are replaced by the $[N_a - N_c]$ sites doubly occupied by rotated electrons and free of rotated electrons. The lattice constant of the effective $s, \nu$ pseudoparticle lattice is given by $a_{s, \nu} = L/N^*_{s, \nu}$. On the other hand, the effective $c$ pseudoparticle lattice has $N_a$ lattice sites and its lattice constant $a$ is the same as for the effective electronic lattice. The positions of the $N_c$ $c$ pseudoparticles and $N_h$ $c$ pseudoparticle holes in this effective lattice equal the corresponding positions of the rotated-electron singly occupied sites and rotated-electron doubly occupied/empty sites, as already discussed in previous sections.

5-V The set of pseudoparticle occupancy configurations of the effective $c$ pseudoparticle, $c, \nu$ pseudoparticle, and $s, \nu$ pseudoparticle lattices belonging to branches $\nu = 1, 2, ...$ with finite occupancy in a given local state of form (80) together with the numbers $L_{c, -1/2}$ and $L_{s, -1/2}$ of such a state contain the same information as the corresponding description of the same local state in terms of the rotated-electron site distribution configurations.

Since the value of the number $N^*_{\alpha, \nu}$ given in Eqs. (17) and (18) is distinct for different CPHS ensemble subspaces, it follows from the expression of the effective $\alpha, \nu$ pseudoparticle lattice constants,

$$a_{\alpha, \nu} = a \frac{N_a}{N^*_{\alpha, \nu}} = \frac{L}{N^*_{\alpha, \nu}}; \quad \alpha = c, s; \quad \nu = 1, 2, 3, ...,$$

(81)

that the value of such constants changes accordingly.

Let us introduce the following notation for the spatial coordinates of the effective $c$ pseudoparticle lattice,

$$x_j = a_j, \quad j = 1, 2, 3, ..., N_a,$$

(82)

where the index $j$ was called $j_l$ and $j_h$ in Eq. (44) for the $c$ pseudoparticle occupied sites ($l = 1, 2, 3, ..., N_c$) and empty sites ($h = 1, 2, 3, ..., [N_a - N_c]$) respectively. We recall that the number of these occupied and empty sites of the effective $c$ pseudoparticle lattice equals the number of rotated-electron singly occupied sites and rotated-electron doubly occupied/empty sites respectively. Moreover, let us introduce the following notation for the spatial coordinates of the effective $c, \nu$ pseudoparticle and $s, \nu$ pseudoparticle lattices,

$$x_j = a_{c, \nu} j, \quad j = 1, 2, 3, ..., N^*_{c, \nu},$$

(83)

and

$$x_j = a_{s, \nu} j, \quad j = 1, 2, 3, ..., N^*_{s, \nu},$$

(84)

respectively.

The band-momentum limiting values given in Eq. (20) for the $\alpha, \nu$ pseudoparticle bands can be expressed in terms of the corresponding lattice constants $a_{\alpha, \nu}$ as follows,

$$q_{\alpha, \nu} = \frac{\pi}{a_{\alpha, \nu}} [1 - 1/N_a] \approx \frac{\pi}{a_{\alpha, \nu}}.$$

(85)

Also the $c$ pseudoparticle limiting band-momentum value (23) can be written as,

$$q_c = \frac{\pi}{a} [1 - 1/N_a] \approx \frac{\pi}{a}.$$

(86)

Thus the domain of available pseudoparticle band-momentum values corresponds to an effective first-Brillouin zone associated with an underlying effective pseudoparticle lattice. The fact that the spatial coordinate introduced in Eqs. (82), (83), and (84) is the conjugate of the pseudoparticle band momentum given in Eq. (16) is used in Refs. [53, 54] in Fourier analysis involving pseudoparticle creation and annihilation operators.
The ground-state plays an important role in the study of finite-energy few-electron spectral functions [52, 53]. Thus let us use the ground-state Eqs. (29) and (30) in order to find the corresponding values for the effective $\alpha, \nu$ pseudoparticle lattice constants in the case of electronic densities $n$ and spin densities $m$ such that $0 < n < 1/a$ and $0 < m < n$ respectively. We find that in the case of a ground state corresponding to an electronic density and a spin density whose values are within these domains these constants read,

$$\alpha^0_{c, \nu} = \frac{a}{\delta}; \quad \alpha^0_{s, 1} = \frac{a}{n\uparrow}; \quad \alpha^0_{s, \nu} = \frac{a}{m},$$

where $\delta = [1/a - n]$ is the doping concentration away from half filling.

Moreover, the ground-state number $N^*_{\alpha, \nu}$ given in Eq. (30) and the ground-state limiting band-momentum values can be written in terms of the effective pseudoparticle lattice constants as follows,

$$N^*_{\alpha, \nu} = \frac{L}{\alpha^0_{\alpha, \nu}},$$

and

$$q^0_{\alpha, \nu} = \frac{\pi}{\alpha^0_{\alpha, \nu}}.$$

respectively.

We note that when the effective pseudoparticle lattice constants given in Eq. (87) diverge, as is the case for $\alpha^0_{c, \nu}$ as $\delta = [1/a - n] \to 0$ and for $\alpha^0_{s, \nu}$ as $m = [n\uparrow - n\downarrow] \to 0$ when $\nu > 1$, the corresponding number $N^*_{\alpha, \nu}$ is zero. This just means that in these limits these bands and corresponding effective lattices do not exist for states belonging to the ground-state CPHS ensemble subspace.

Finally, let us relate the new introduced effective pseudoparticle lattices to previous results on the 1D Hubbard model in the limit $U/t \to \infty$. Indeed the effective $c$ and $s, 1$ pseudoparticle lattices introduced here are related to known properties of the model in that limit. For instance, it is well known that in such a limit and at zero spin density the charge and spin degrees of freedom of the 1D Hubbard model can be described by two independent systems of $N$ spin-less fermions and $N/2$ spin-down spins [2, 3, 4, 5, 11, 12]. The spin-less fermions can be associated with an effective lattice with $j = 1, 2, 3, ..., N/2$ sites, whereas the $N/2$ spin-down spins correspond to a squeezed effective lattice with $j = 1, 2, 3, ..., N/2$ sites. The spin-less fermion and spin occupancy configurations of these effective lattices describe electron occupancy configurations of the associated real-space lattice. In the limit $U/t \to \infty$ such spin-less fermion and spin effective lattices are directly related to the effective $c$ pseudoparticle and $s, 1$ pseudoparticle lattices respectively, introduced above. On the other hand, within our pseudoparticle description of the quantum problem these effective pseudoparticle lattices are well defined for all finite values of $U/t$. In that case the pseudoparticle occupancy configurations of these effective lattices describe rotated-electron site distribution configurations. In the limit $U/t \to \infty$ the energy eigenstates with finite occupancies in the effective $c, \nu$ pseudoparticle lattices have finite values of electron double occupation. Thus according to the energy spectrum given in Eq. (31) these states have in such a limit an infinite excitation energy relative to the ground state and do not contribute to the finite-energy physics. This explains why these states are unimportant for the $U/t \to \infty$ physics and are in general not considered in that limit [2, 3, 4, 5]. At zero spin density the effective $s, \nu$ pseudoparticle lattices such that $\nu > 1$ do not exist for the ground state because $N^*_{s, \nu} = 0$ for such a state. However, at zero spin density excitations with finite occupancy of $s, \nu$ pseudoparticles belonging to branches such that $\nu > 1$ have a gapless energy spectrum. In the limit $U/t \to \infty$ the spin excitations involving both occupancy configurations of $s, 1$ pseudoparticles and $s, \nu$ pseudoparticles belonging to branches such that $\nu > 1$ are often described by the isotropic Heisenberg chain which describes these excitations [2, 3, 4, 5].

B. CONSTRUCTION OF THE ENERGY EIGENSTATES

Our last goal is the construction of the energy eigenstates in terms of the local charge, spin, and $c$ pseudoparticle sequences studied in previous sections. The energy eigenstates are described in terms of band-momentum pseudoparticle, Yang holon, and HL spinon occupancy configurations in Ref. [11] by direct use of the Bethe-ansatz solution and $SO(4)$ symmetry of the model. In the previous section we expressed the local states given in both Eqs. (77) and Eq. (78) in terms of rotated-electron site distribution configurations and showed that they provide a complete basis of...
states. In order to relate these local states to the energy eigenstates we should first express the former states in terms of local $c$ and $\alpha, \nu$ pseudoparticle occupancy configurations in the corresponding effective pseudoparticle lattices. We then Fourier transform the obtained states into band-momentum space $g_j$ with respect to the spatial coordinates of the effective $c$ and $\alpha, \nu$ pseudoparticle lattices.

As in the case of Eq. (14) for the case of the effective $c$ pseudoparticle lattice, we denote by $j_l$ and $j_h$ where $l = 1, 2, ..., N_{\alpha, \nu}$ and $h = 1, 2, ..., N^h_{\alpha, \nu}$ respectively, the occupied and empty sites respectively, of the effective $\alpha, \nu$ pseudoparticle lattices. The spatial coordinate of these occupied and empty pseudoparticle sites is given by,

\[
x_{j_l} = a_{\alpha, \nu} j_l, \quad l = 1, 2, ..., N_{\alpha, \nu}; \quad \alpha = c, s;
\]

\[
x_{j_h} = a_{\alpha, \nu} j_h, \quad h = 1, 2, ..., N^h_{\alpha, \nu}; \quad \alpha = c, s,
\]

where $j_l$ is the index which defines the position of the occupied pseudoparticle sites and $j_h$ the location of the empty pseudoparticle sites. We note that the empty sites of the effective pseudoparticle lattices correspond to the sites left over by the occupied sites and vice versa. Thus we can uniquely specify a given effective pseudoparticle lattice site distribution configuration by providing the location of the occupied sites (or empty sites) only. Here we choose the representation in terms of the locations in the effective $c$ and $\alpha, \nu$ pseudoparticle lattices of the occupied sites $x_{j_l}$ of Eqs. (44) and (90). In such a representation the states given in Eq. (80) are denoted as,

\[
| \{ L_{c,-1/2}, L_{s,-1/2}; \{ x_{j_1}, x_{j_2}, ..., x_{j_{N_{c,s}}} \} \} \rangle \equiv | \{ x_{j_1}, x_{j_2}, ..., x_{j_{N_{c,s}}} \} \rangle \equiv | \{ x_{j_1}, x_{j_2}, ..., x_{j_{N_{c,s}}} \} \rangle \equiv | \{ x_{j_1}, x_{j_2}, ..., x_{j_{N_{c,s}}} \} \rangle .
\]

Here $(x_{j_1}, x_{j_2}, ..., x_{j_{N_{c,s}}})$ is the set of spatial coordinates associated with the set of indices $(j_1, j_2, ..., j_{N_{c,s}})$ given in Eq. (11). These spatial coordinates correspond to the set of sites occupied by $c$ pseudoparticles in the corresponding effective lattice. Moreover,

\[
\{ (x_{j_1}, x_{j_2}, ..., x_{j_{N_{c,s}}} ) \} = (x_{j_1}, x_{j_2}, ..., x_{j_{N_{c,s}}}) ; (x_{j_1}, x_{j_2}, ..., x_{j_{N_{c,s}}}) ; (x_{j_1}, x_{j_2}, ..., x_{j_{N_{c,s}}}) ; \cdots,
\]

\[
\alpha = c, s,
\]

gives the local pseudoparticle coordinates of each of the effective $\alpha, \nu$ pseudoparticle lattices with finite occupancy in the associated effective electronic lattice.

We emphasize that from analysis of the specific rotated-electron site distribution configurations relative to a given state of form (77) one can construct by use of the above properties 1-V to 3-V the $c$ pseudoparticle, $c, s$ pseudoparticle, and $s, \nu$ pseudoparticle occupancy configurations in the effective $c, c, \nu$, and $s, \nu$ pseudoparticle lattices respectively, of the corresponding state of form (11). This relation was already mentioned in the above property 5-V and the inverse statement is obviously also true. Thus Eqs. (77) and (11) refer to two different representations of the same local states. These local states constitute a complete set of states. However, the form of these local states does not ensure the periodic boundary conditions of the original electronic problem.

The representation of the local states in terms of the effective pseudoparticle lattice occupancy configurations given in Eq. (11) is the most suitable starting point for construction of the energy and momentum eigenstates. Such a construction is fulfilled by Fourier transforming the local states given in Eq. (11) into band-momentum space with respect to the spatial coordinates of the effective pseudoparticle lattices given in Eqs. (82), (83), and (84). Provided that the discrete band-momentum values obey the boundary conditions associated with Eqs. (82) and (83), the form of the obtained states ensures the periodic boundary conditions of the original electronic problem. Such procedures lead to the following expression of the energy eigenstates in terms of the local states of Eq. (11),
The permutations $\mathcal{P}$ on the right-hand side of Eq. (93) generate a Slater determinant of the band momenta and the spatial coordinates of the $N_c$ pseudoparticles and $N_{\alpha,\nu}$ pseudoparticles belonging to branches $\alpha = c, s$ and $\nu = 1, 2, \ldots$ with finite occupancy in the corresponding local charge and spin sequences. On the left-hand side of Eq. (93) the set of pseudoparticle occupied band-momentum values specify the energy eigenstate. There $(q_{j_1}, q_{j_2}, \ldots, q_{j_{N_c}})$ are the set of occupied band momentum values out of the $N_n$ available discrete $q_j$ values of the $c$ pseudoparticle band such that $j = 1, \ldots, N_a$. Moreover,

$$\{q_{j_1}, q_{j_2}, \ldots, q_{j_{N_c}}\} = (q_{j_1}, q_{j_2}, \ldots, q_{j_{N_{\alpha,s}}}); (q_{j_1}, q_{j_2}, \ldots, q_{j_{N_{\alpha,2}}}); (q_{j_1}, q_{j_2}, \ldots, q_{j_{N_{\alpha,3}}}); \ldots,$$

$$\alpha = c, s,$$

are the set of $N_{\alpha,\nu}$ occupied band momentum values out of the $N_{\alpha,\nu}^*$ available discrete $q_j$ values of the $\alpha, \nu$ pseudoparticles and $\nu$ pseudoparticles bands of branches $\alpha = c, s$ and $\nu = 1, 2, \ldots$, such that $j = 1, \ldots, N_{\alpha,\nu}^*$. Such band-momentum pseudoparticle occupancy configurations are discussed in Ref. [11].

Since equations (77) and (91) refer to two different representations of the same states, the energy eigenstates (93) are Fourier-transform superpositions involving permutations of the local charge-sequence and spin-sequence states (91). The latter local states correspond to rotated-electron site distribution configurations. Thus combination of the general expressions (80) and (93) provides important information about the relationship between the band-momentum pseudoparticles and the rotated-electron occupancy configurations. We emphasize that the permutations on the right-hand side of Eq. (93) are associated with different locations for the local $\alpha, \nu$ pseudoparticles but that the internal structure of these quantum objects remains invariant under these permutations. Both the Fourier transforms and permutations of the general expression (93) do not touch the internal structure of the local $\alpha, \nu$ pseudoparticles and refer only to their translational degrees of freedom. This results from the point-like character of the occupied sites of the effective pseudoparticle lattices. Thus the local and band-momentum $\alpha, \nu$ pseudoparticles have the same internal structure.

While the complete set of symmetrized states used in Ref. [51] are energy eigenstates of the 1D Hubbard model in the limit $U/t \rightarrow \infty$ only, the states given in Eqs. (93) and (94) are energy eigenstates of that model for all values of $U/t$. The local charge, spin, and $c$ pseudoparticle sequences defined in expressions (77) and (80) and used in the construction of the latter states, describe rotated-electron site distribution configurations which in the limit $U/t \rightarrow \infty$ refer to similar electron configurations. Thus in contrast to the complete set of symmetrized energy eigenstates used in Ref. [51], the complete set of energy eigenstates of form (93) are in the limit $U/t \rightarrow \infty$ the band-momentum states associated with the unitary operator $V(U/t)$.

In the case of the energy eigenstates given in Eq. (93), a first condition for the fulfillment of the periodic boundary conditions of the original electronic problem is that in all the local charge and spin sequences described by the states (80), the local $c, \nu$ pseudoparticles and $s, \nu$ pseudoparticles are described by the same rotated-electron $2\nu$-site distribution configurations given in Eqs. (83) and (84) respectively. These are obtained from the proper symmetrization of the corresponding single pseudoparticle $2\nu$-site sequences. A second condition to ensure these periodic boundary conditions is that energy eigenstates given in Eq. (93) are Fourier transform superpositions of these charge and spin sequences of the form given on the right-hand side of that equation. Provided that the discrete values of the band momentum $q_j$ of these Slater determinant superpositions obey Eqs. (97) and (98), the periodic boundary conditions of the original electronic problem are ensured. Note that the general expression provided in Eq. (93) includes suitable
permutations of the local sequences. In the same equation the rotated-electron site distribution configurations of the local charge and spin sequences associated with the states \( |\mathbf{71}\rangle \) or \( |\mathbf{84}\rangle \) are reexpressed in terms of pseudoparticle occupancy configurations of the corresponding effective pseudoparticle lattices.

Let us consider the specific case of a ground state with electronic densities and spin densities belonging to the domains \( 0 \leq n \leq 1/a \) and \( 0 \leq m \leq n \) respectively. Such a state is a superposition of local normalized states \( |\mathbf{80}\rangle \) which have no finite occupancies of \(-1/2\) Yang holons, \(-1/2\) HL spinons, \(c, \nu\) pseudoparticles, and \(s, \nu\) pseudoparticles belonging to branches such that \( \nu > 1 \) \([\mathbf{12}, \mathbf{15}]\). Thus in the case of such a ground state these local states are of the following simplified form,

\[
2^{N_{c},1/2} |(0, 0); (x_{j1}, x_{j2}, ..., x_{jN_{c}}); (x_{j1}, x_{j2}, ..., x_{jN_{s},1})
\]

\[
= \prod_{j=1}^{N_{s},1} (1 - \hat{\mathbf{T}}(s,1,j',1)) |(0, 0); (j_1, j_2, ..., j_{N_{s}}); \{(\downarrow_{j',1}, \uparrow_{l_1,2})\}
\]

where

\[
\{(\downarrow_{j_1,1}, \uparrow_{l_1,2})\} = (\downarrow_{l_1,1}, \uparrow_{l_1,2}), (\downarrow_{l_1,1}, \uparrow_{l_2,2}), ..., (\downarrow_{l_{N_{s},1},1}, \uparrow_{l_{N_{s},1},2}),
\]

and the operators \( \hat{\mathbf{T}}(s,1,j'',1) \) are the same as in the general Eq. \( |\mathbf{80}\rangle \). The states defined in Eqs. \( |\mathbf{84}\rangle \) and \( |\mathbf{86}\rangle \) provide the position of the two rotated-electron singly occupied sites associated with the internal structure of each local \( s, 1 \) pseudoparticle.

These local states have an alternative representation in terms of the \( c \) and \( s, 1 \) pseudoparticle occupancy configurations of the effective \( c \) and \( s, 1 \) pseudoparticle lattices respectively. Within such an effective pseudoparticle lattice representation the local states \( |\mathbf{80}\rangle \) are denoted by \(|(0, 0); (x_{j1}, x_{j2}, ..., x_{jN_{c}}); (x_{j1}, x_{j2}, ..., x_{jN_{s},1})\rangle\). Here \((x_{j1}, x_{j2}, ..., x_{jN_{s},1})\) are the spatial coordinates which define the positions of the local \( s, 1 \) pseudoparticles in the effective \( s, 1 \) pseudoparticle lattice. These spatial coordinates correspond to the indices \((l_1, l_2, ..., l_{N_{s},1})\) on the left-hand side of Eq. \( |\mathbf{85}\rangle \). The sites of such an effective \( s, 1 \) pseudoparticle lattice are equally spaced, the corresponding ground-state lattice constant \( a_{s,1} \) being given in Eq. \( |\mathbf{87}\rangle \). The ground state expression is a particular case of the Fourier-transform superposition of local states given in Eq. \( |\mathbf{88}\rangle \). In the case of the ground-state such an expression simplifies to,

\[
|\mathbf{GS}\rangle = N_{c}^{N_{0}^{0}/2} (N_{s,1}^{0})^{N_{0}^{0}/2} |(0, 0); (q_{j1}, q_{j2}, ..., q_{jN_{0}}); (q_{j1}, q_{j2}, ..., q_{jN_{0}})\rangle
\]

\[
= \sum_{c_1 < c_2 < ... < c_{N_{0}}} \sum_{P} (-1)^{P} e^{i\sigma_{1} \sum_{i=1}^{N_{0}} c_{P}^{(i)} q_{j_{i}}} \sum_{g_{1} < g_{2} < ... < g_{N_{0}}} \sum_{P} (-1)^{P} e^{i\sigma_{1} \sum_{j'=1}^{N_{0}} g_{P}^{(i)} q_{j_{i}}}
\]

\[
\times |(0, 0); (x_{j1}, x_{j2}, ..., x_{jN_{0}}); (x_{j1}, x_{j2}, ..., x_{jN_{0}})\rangle,
\]

where \( N_{c} = N_{0}^{0}, N_{s,1} = N_{4}^{0} \), and \( N_{s,1}^{*} = N_{4}^{0} \).

VI. PRACTICAL APPLICATION OF THE DERIVED LOCAL PSEUDOPARTICLE REPRESENTATION

The concepts of local pseudoparticle and of effective pseudoparticle lattice introduced in this paper are applied in the studies of Refs. \([\mathbf{53}, \mathbf{54}]\) to the evaluation of finite-energy few-electron spectral function expressions. In this paper we identified the conjugate variable of the pseudoparticle band momentum obtained naturally from the Bethe-ansatz solution in Ref. \([\mathbf{11}]\). Such a conjugate variable is nothing but the spatial coordinate of the effective pseudoparticle lattice. We found that there is one of such lattices for each pseudoparticle branch. Moreover, we studied the pseudoparticle site distribution configurations which describe the energy eigenstates. We have also related these configurations to the rotated-electron site distribution configurations which describe the same states. Since electrons and rotated electrons are related by a mere unitary transformation, our results provide useful information about the relation between electronic excitations and energy eigenstates.

The results obtained in this paper are used in Refs. \([\mathbf{57}, \mathbf{54}]\) in the evaluation of the overlap between few-electron excitations and energy eigenstates. As discussed in Ref. \([\mathbf{58}]\), the pseudoparticles studied here have residual interactions.
In that reference it is found that a canonical unitary transformation maps the pseudoparticles onto non-interacting pseudofermions. (This transformation is other than the electron - rotated-electron canonical unitary transformation associated with the operator.) Interestingly, the latter quantum objects are more suitable for the evaluation of matrix elements between the ground state and excited states. The pseudoparticle - pseudofermion unitary transformation maps the $c$ pseudoparticle (and $\alpha, \nu$ pseudo-particle) discrete band-momentum value $q_j$ onto the $c$ pseudoparticle (and $\alpha, \nu$ pseudofermion) discrete shifted-momentum value $\bar{q}_j = q_j + Q_{\alpha,\nu}(q_j)/L$ (and $\bar{q} = q_j + Q_{\alpha,\nu}(q_j)/L$). Here $Q_{\alpha,\nu}(q)/L$ and $Q_{\alpha,\nu}(q)/L$ are momentum functionals which depend on the excited state occupancy configurations and involve the two-pseudofermion phase shifts. Although the momenta $Q_{\alpha,\nu}(q)/L$ and $Q_{\alpha,\nu}(q)/L$ are of order $1/L$ and play a crucial role in the spectral properties of the quantum problem, the above transformation is such that the differences $|Q_{\alpha,\nu}(q_{j+1}) - Q_{\alpha,\nu}(q_j)|/L$ and $|Q_{\alpha,\nu}(q_{j+1}) - Q_{\alpha,\nu}(q_j)|/L$ are of order $|1/L|^2$ and thus vanish within our large-$L$ description. Therefore, to first order in $1/L$ the shifted-momentum values are such that $q_{j+1} - q_j = 2\pi/L$, like the corresponding band-momentum values. It follows that there is no level crossing between the sequence of $\{q_j\}$ discrete values and the corresponding sequence of $\{\bar{q}_j\}$ discrete values such that $j = 1, 2, ..., N_\alpha$ and $j = 1, 2, ..., N_\alpha^*$. Moreover, the presence of the extra momentum term $Q_{\alpha,\nu}(q)/L$ or $Q_{\alpha,\nu}(q)/L$ in the shifted-momentum expression does not affect the underlying effective $\alpha, \nu$ pseudofermion lattice. Indeed, that momentum just imposes a twisted boundary condition such that each $c$ pseudofermion (and $\alpha, \nu$ pseudofermion) hopping from site $N_\alpha$ to site 0 (and site $N_\alpha^*, \nu$ to site 0) of such an effective lattice will acquire a phase $e^{iQ_{\alpha,\nu}(q)}$ (and $e^{iQ_{\alpha,\nu}(q)}$). Thus the effective pseudofermion lattice introduced here is invariant under the pseudofermion - pseudofermion transformation studied in Ref. 53.

In the case of the Hilbert subspace where the few-electron excitations are contained there is an one-to-one correspondence between the pseudofermion shifted-momentum occupancy configurations (and pseudofermion site distribution configurations) and the corresponding pseudoparticle band-momentum occupancy configurations (and pseudoparticle site distribution configurations) that describe the energy eigenstates. Thus the results found in the present paper are useful for the description of these states in terms of pseudofermion occupancy configurations as well. Since the pseudofermions are non-interacting, it is found in Ref. 53 that the wave function of the excited states of the ground-state normal-ordered 1D Hubbard model factorizes for all values of $U/t$. Importantly, such a factorization is used in Ref. 53, where the few-electron spectral functions are expressed as a convolution of single pseudofermion spectral functions. This factorization involves the pseudofermion branches with finite occupancy in the excited states. Thus due to such a factorization the problem of the evaluation of finite-energy few-electron spectral functions is reduced to the problem of the evaluation of pseudofermion spectral functions. The latter problem involves the derivation of matrix elements between the ground state and excited states for pseudofermion operators.

The effective pseudoparticle lattices introduced in this paper play an important role in the evaluation of the pseudofermion matrix elements. It turns out that the derivation of the above matrix elements involves the anticommutator $\{f_{\bar{q},\alpha,\nu}^\dagger, f_{\bar{q}',\alpha',\nu'}\}$ where the operators $f_{\bar{q},\alpha,\nu}^\dagger$ and $f_{\bar{q},\alpha,\nu}$ create and annihilate respectively, a $\alpha, \nu$ pseudofermion of shifted momentum $\bar{q}$, and similar operators can be introduced for the $c$ pseudofermions. We note that because of the functional character of the momenta $Q_{\alpha,\nu}(q)/L$ and $Q_{\alpha,\nu}(q)/L$ these anticommutation relations have not the usual fermionic form. This justifies the designation pseudofermion. Furthermore, without the concept of local pseudoparticle and effective pseudoparticle lattice introduced in this paper the evaluation of these anticommutators would be a complex open problem. However, our results imply the existence of local pseudofermions which correspond to the local pseudoparticles introduced in this paper. Fortunately both the invariance of the pseudoparticle effective lattice under the pseudoparticle - pseudofermion transformation and the associated concept of local pseudofermion allows the expression of the above anticommutators in terms of the local pseudofermion operator anticommutators $\{f_{\bar{x}_{j\alpha,\nu}, \alpha,\nu}^\dagger, f_{\bar{x}_{j',\alpha',\nu'}}\}$ associated with spatial coordinates $x_j$ and $x_{j'}$ of the effective $\alpha, \nu$ and $\alpha', \nu'$ lattices respectively. Such an expression is very useful for the evaluation of the pseudofermion matrix elements between the ground state and excited states. As in the case of the pseudoparticles, also the shifted-momentum and local pseudofermion descriptions are related by a simple Fourier transform. Thus one can express the above shifted-momentum pseudofermion operator anticommutators in terms of the local pseudofermion operator anticommutators as follows.

$$\{f_{\bar{q},\alpha,\nu}^\dagger, f_{\bar{q}',\alpha',\nu'}\} = \frac{1}{L} \sum_{j=1}^{N_\alpha^*} \sum_{j'=1}^{N_{\alpha'}^*} e^{i\bar{q}x_j - i\bar{q}'x_{j'}} \{f_{\bar{x}_{j\alpha,\nu}, \alpha,\nu}^\dagger, f_{\bar{x}_{j',\alpha',\nu'}}\}. \tag{98}$$

As a result of the introduction of the concept of effective pseudofermion (and pseudofermion) lattice the derivation of the local operator anticommutators on the right-hand side of this equation is trivial and expressions for the band-momentum anticommutators operators can be easily obtained by performing the $j$ and $j'$ summations.
Corresponding expressions are found for the c pseudofermions such that,

\[ \{ f_{j,c}, f_{j',c} \} = \frac{1}{L} \sum_{j=1}^{N_a} \sum_{j'=1}^{N_a} e^{i q x_j - i q' x_{j'}} \{ f_{j,c}, f_{j',c} \}. \]  

(99)

and the anticommutators involving \( \alpha, \nu \) pseudofermions and \( c \) pseudofermions vanish. (Also the anticommutators given in Eq. (98) vanish for \( \alpha, \nu \neq \alpha', \nu' \).)

We note that the pseudofermion description introduced in Ref. [53] is a generalization for all values of \( U/t \) of the well known \( U/t \to \infty \) quantum-object description used in Refs. \[2, 3, 4, 5\]. For instance, in the limit \( U/t \to \infty \) the \( c \) pseudofermions become the spin-less fermions of Refs. \[2, 3, 4, 5\] and the spinons become the spins of these references. The \( s, 1 \) pseudofermions describe a spin pair of opposite projections. Also the remaining \( s, \nu \) pseudofermions describe 2\( \nu \) spins, \( \nu \) of each direction. In the limit \( U/t \to \infty \) the general \( c \) pseudofermion anticommutation relations become nothing but the spin-less fermion anticommutation relations provided in the first (unnumbered) equation of Sec. IV of Ref. \[2\]. As discussed in Ref. \[53\] and confirmed in Ref. \[54\], except for the \( U/t \) dependence of the two-pseudofermion phase shifts involved in the general expressions of the momentum functionals \( Q_c(q)/L \) and \( Q_{\alpha, \nu}(q)/L \), the evaluation of the pseudofermion matrix elements for finite values of \( U/t \) is formally similar to the derivation of the spin-less fermion matrix elements presented in Ref. \[2\] for \( U/t \to \infty \). This similarity results from the facts that both the energy-eigenstate rotated-electron occupancy configurations and corresponding pseudoparticle, holon, and spinon occupancy configurations introduced in this paper are independent of the value of \( U/t \) and that in the limit \( U/t \to \infty \) considered in Ref. \[2\] electrons and rotated electrons are the same quantum objects. This example reveals how the local pseudoparticle (and pseudofermion) representation introduced in this paper is used for practical calculations at finite \( U/t \). The only formal difference between the finite \( U/t \) studies of Refs. \[53, 54\] and those of Ref. \[2\] for \( U/t \to \infty \) is that in the former references the evaluation of the spectral function for each pseudofermion branch proceeds as in latter reference for the spin-less fermions. In contrast, the studies of Ref. \[2\] use the well known \( U/t \to \infty \) connection between the spin degrees of freedom of the 1D Hubbard model and the spin 1/2 isotropic Heisenberg chain \[57\] to evaluate the spin part of the one-electron spectral function. Such direct connection does not exist for finite values of \( U/t \). However, the method used in Refs. \[53, 54\] for finite values of \( U/t \) is also valid for \( U/t \to \infty \) and leads to the same final results as the analysis of Ref. \[2\].

VII. CONCLUDING REMARKS

In this paper we studied the relation between the description of the energy eigenstates of the 1D Hubbard model in terms of rotated-electron occupancy configurations and of Yang holon, HL spinons, and pseudoparticle band momentum occupancy configurations. Such a relation involves the concepts of local pseudofermion and effective pseudoparticle lattices. We introduced the pseudoparticle spatial coordinates \( x_j \) associated with and conjugate of the pseudoparticle band-momentum \( q_j \). The band-momentum pseudoparticle description is provided by the Bethe-ansatz Takahashi’s thermodynamic equations \[5\]. The pseudoparticle spatial coordinates introduced in this paper correspond to effective pseudoparticle lattices whose length and lattice constant are independent of the value of \( U/t \). The introduction of the concept of the pseudoparticle effective lattice involved the study rotated-electron distribution configurations of doubly occupied and empty sites (and of spin-down and spin-up singly occupied sites) which describe the internal structure of the local \( c, \nu \) pseudoparticles (and local s, \( \nu \) pseudoparticles). In what the translational degrees of freedom are concerned, the local \( s, \nu \) pseudoparticle is a point-like quantum object, its internal structure being the same as that of the corresponding band-momentum pseudoparticle. We also found that the spatial coordinates of the occupied and empty sites of the effective \( c \) pseudoparticle lattice are the same as the coordinates of the sites singly occupied by rotated electrons and rotated-electron doubly occupied/empty sites respectively. The energy eigenstates can be described in terms of pseudoparticle site distribution configurations in the corresponding effective pseudoparticle lattice. Our results reveal that there is an one-to-one correspondence between the local pseudoparticle site distribution configurations in the effective pseudoparticle lattices which describe a given energy eigenstate and the rotated-electron site distribution configurations which describe the same state.

As mentioned in Sec. II, the electron site distribution configurations which describe the energy eigenstates are very complex and dependent on the value of \( U/t \). This is confirmed by the \( U/t \) dependence of the double occupation expectation value studied in Ref. \[12\]. However, the electron - rotated-electron canonical unitary transformation is such that it maps these complex and \( U/t \) dependent electron site distribution configurations onto the corresponding \( U/t \) independent rotated-electron site distribution configurations studied in this paper. This is an important property of the non-perturbative diagonalization of the 1D Hubbard model. According to the results obtained in this paper, this property implies that the local pseudoparticle site distribution configurations of the effective pseudoparticle
lattices which describe the energy eigenstates are also independent of the value of $U/t$. Moreover, since the spatial coordinate of these effective lattices is the conjugate of the pseudoparticle band-momentum obtained from analysis of the Bethe-ansatz solution \[11\], it is again such a property which justifies why the pseudoparticle band-momentum occupancies which describe the energy eigenstates are independent of the values of $U/t$. The invariance associated with the commutation of the momentum operator with the electron - rotated-electron unitary operator is related to the $U/t$ independence of the band-momentum pseudoparticle, rotated-electron, and local pseudoparticle occupancy configurations which describe the energy eigenstates. In this paper we profitted from such a $U/t$ independence of the pseudoparticle and rotated-electron occupancy configurations and extracted some of our results from analysis of the $U/t \rightarrow \infty$ problem. In that limit the quantum problem simplifies because the $\eta$-spin and spin occupancy configurations are degenerated and electrons and rotated electrons are the same quantum object and thus electron double occupation is a good quantum number.

We also provided preliminary information about the application of the concepts of local pseudoparticle and effective pseudoparticle lattice introduced in this paper to the evaluation of the spectral properties of few-electron operators. The study of the finite-energy few-electron spectral functions is required for the further understanding of the unusual finite-energy/frequency spectral properties observed in quasi-1D materials \[15,17,18,21,30\], which are far from being well understood. Indeed there are clear indications that electronic correlations effects might play an important role in the finite-energy physics of these low-dimensional materials \[15,17,18,21,30\]. For low values of the excitation energy the microscopic electronic properties of these materials are usually described by systems of coupled chains. On the other hand, for finite values of the excitation energy larger than the transfer integrals for electron hopping between the chains, 1D lattice models like the 1D Hubbard model are expected to provide a good description of the physics of these materials. The studies of the present paper are a first necessary step for the evaluation of few-electron spectral function expressions for finite values of excitation energy, as confirmed in Refs. \[53,54\]. As mentioned in Sec. I, application of a preliminary version of the theory introduced here and in these references to the study of the one-electron removal spectral properties of quasi-1D materials leads to a quantitative and successful description of the exotic independent charge and spin spectral lines observed in photoemission experiments for all values of excitation energy \[55\].

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APPENDIX A: RELATION TO THE LARGE $U/t$ PHYSICS AND OTHER CHOICES OF ENERGY EIGENSTATES FOR $U/t \rightarrow \infty$

In this Appendix we review some aspects of the large $U/t$ physics that are useful for the contextualization of the problems studied in this paper. We start by considering the simple 1D Hubbard Hamiltonian given in Eq. (3) which differs from the Hamiltonians given in Eqs. (1) and (2) in the definition of the zero-energy level. The energy spectrum of the stationary states of the model \[3\] can be written as a functional of the pseudoparticle band-momentum distribution functions and $-1/2$ Yang holon numbers such that \[11,12\].

$$E_H = -2t \sum_{j=1}^{N_a} N_c(q_j) \cos k(q_j) + U L_{c,-1/2}$$

This energy functional depends on these distribution functions through the momentum-rapidity functional $k(q_j)$ and rapidity functional $\Lambda_{c,\nu}(q_j) \[11,12\]$. The momentum spectrum $P$ of the energy eigenstates has a simpler expression, being the linear superposition of the pseudoparticle band-momentum distribution functions and $-1/2$ Yang holon number. It reads,
\[ P = \sum_{j=1}^{N_c} N_c(q_j) q_j + \sum_{j=1}^{N_s,\nu} N_s,\nu(q_j) q_j + \sum_{j=1}^{N^*_c,\nu} N^*_c,\nu(q_j) \left( \frac{\pi}{a} - q_j \right) + \frac{\pi}{a} M_{c,-1/2}. \]  

(A2)

As in the case of the momentum values of the quasi-particles of a Fermi liquid, the momentum \( P \) given in Eq. (A2) is additive relative to the band-momentum values of the pseudoparticles.

The momentum rapidity functional \( k(q_j) \) and the rapidity functional \( \Lambda_{c,\nu}(q_j) \) on the right-hand side of Eq. (A2) can be defined in terms of the Takahasi’s thermodynamic equations [7] rewritten in functional form [11, 12]. These quantities are functionals of the set of band-momentum distribution functions \( N_c(q_j) \) and \( \{ N_{\alpha,\nu}(q_j) \} \), where \( \alpha = c, s \) and \( \nu = 1, 2, \ldots \), whose possible different occupancy configurations classify the energy eigenstates. In the limit \( U/t \to \infty \) these functionals have the same value for all energy eigenstates and read \( k(q_j) = q_j \) and \( \Lambda_{c,\nu}(q_j) = 0 \). Thus in such a limit the energy and momentum expressions (A1) and (A2) respectively, simplify to expression (31) and

\[ P = \sum_{j=1}^{N_c} N_c(q_j) q_j + \sum_{j=1}^{N_s,\nu} N_s,\nu(q_j) q_j + \sum_{j=1}^{N^*_c,\nu} N^*_c,\nu(q_j) \left( \frac{\pi}{a} - q_j \right) + \frac{\pi}{a} D, \]  

(A3)

respectively, where the electron double occupation \( D \) is a good quantum number such that \( D = M_{c,-1/2} \) in that limit. For finite values of \( U/t \) the latter relation is replaced by \( D_r = M_{c,-1/2} \), where \( D_r \) is the rotated-electron double occupation. According to Eq. (27) the number \( M_{c,-1/2} \) of -1/2 holons can be written as

\[ M_{c,-1/2} = D_r = L_{c,-1/2} + \sum_{\nu=1}^{\infty} \nu N_{c,\nu}. \]  

(A4)

In the case of zero spin density, electronic densities \( 0 \leq n \leq 1/a \), and on-site repulsion \( U >> t \) the ground state energy \( E_0 \) of the 1D Hubbard model is given by [40],

\[ E_0 = -\frac{2Nc t}{\pi} \sin(\pi n) - \frac{t^2}{U} 4N n \ln 2 \left( 1 - \frac{\sin(2\pi n)}{2\pi n} \right), \]  

(A5)

and the ground state electron double occupation \( D_0 \) can be written as [12, 40],

\[ D_0 = \frac{\partial E_0}{\partial U} = \left( \frac{t}{U} \right)^2 4N n \ln 2 \left( 1 - \frac{\sin(2\pi n)}{2\pi n} \right). \]  

(A6)

Here the term \(-\frac{2Nc t}{\pi} \sin(\pi n)\) is the kinetic energy associated with the hopping processes which do not change electron double occupation. On the other hand, the energy term \(-\frac{t^2}{U} 4N n \ln 2 \left( 1 - \frac{\sin(2\pi n)}{2\pi n} \right)\) includes both kinetic and potential energy contributions. It arises from hopping processes which change electron double occupation and lead to the ground-state electron double occupation value given in Eq. (A6). Thus the physics associated with this second energy term corresponds to excitation processes of higher order in \( t/U \) relatively to the \( t/U \to 0 \) limit where electron double occupation is a good quantum number and the ground-state double electron occupation of the 1D Hubbard model is exactly zero. If we include energy contributions of the order \( t(t/U)^1 \) the ground state contains a small but finite electron double occupation expectation value given in Eq. (A6) and the corresponding quantum problem is not equivalent to the physical situation of interest for the rotated-electron studies of this paper. Indeed, electrons equal rotated electrons when the limit \( t/U \to 0 \) is reached and double occupation is a good quantum number. Only in that limit do the electron occupancy configurations which describe the band-momentum energy eigenstates equal the corresponding rotated-electron configurations which are valid for all values of \( U/t \).
Note that at half filling the electronic density reads \( n = 1/a \), the energy term of order \( t(t/U)^0 \) on the right-hand side of Eq. \((A5)\) vanishes, and the ground-state energy and electron double occupation expressions given in Eqs. \((A5)\) and \((A6)\) simplify to \( E_0 = -\frac{t^2}{4} 4N_a \ln 2 \) and \( D_0 = \left( \frac{t}{U} \right)^2 4N_a \ln 2 \) respectively. It is well known that this energy can be associated with an isotropic Heisenberg model \([57]\). The corresponding ground state leads to energy contributions of the order \( t(t/U)^4 \) and thus contains a small but finite electron double occupation expectation value, \( D_0 = \left( \frac{t}{U} \right)^2 4N_a \ln 2 \). It follows that the usual description of the large-\(U/t\) half-filling Hubbard model in terms of an isotropic Heisenberg model is not equivalent to our limit. In that limit only hopping processes which do not change the value of electron double occupation should be considered.

To leading order in the parameter \( t/U \), the energy spectrum of the 1D Hubbard model in the limit \( t/U \to 0 \) is of the form given in Eq. \((31)\). The permitted hopping processes lead to contributions in the eigenstate energies of order \( t(t/U)^{-1} \) and \( t(t/U)^0 \). These contributions are associated with the energy terms \( \frac{N_2}{2N} \int_{q_{\rm c}}^{q_{\rm f}} dq N_{e}(q)[-2t \cos q] \) and \( Ud \) respectively, on the right-hand side of Eq. \((31)\). In the particular case of the ground state there are no contributions of order \( t(t/U)^{-1} \) because the electron double occupation eigenvalue \( D_0 \) is zero. In the limit \( t/U \to 0 \) only hopping processes such that the electron singly occupied sites can move relatively to the electron doubly-occupied and empty site distribution configurations without changing these configurations are permitted. In such a limit there is a huge degeneracy of \( n \)-spin and spin occupancy configurations. Thus there are several choices for complete sets of energy eigenstates with the same energy and momentum spectra given in Eqs. \((31)\) and \((A3)\) respectively. This is because in this limit there are also many choices for complete sets of compatible observables. The 1D Hubbard model in the limit of \( U/t \to \infty \) has been studied in the literature by many authors \([2, 3, 4, 5, 40, 44, 45, 46, 48, 49, 50, 51]\). In the case of evaluation of quantities describing the physics of the model in the limit of \( U/t \to \infty \), the alternative use of different complete sets of states leads to the same final expressions for correlation functions and other quantities of physical interest.

In this paper we are interested in one of these choices of energy eigenstates only. It corresponds to the complete set of \( 4^{N_{c}} \) energy eigenstates generated from the corresponding energy eigenstates of the finite-\(U/t\) 1D Hubbard model by turning off adiabatically the parameter \( t/U \). Only that set of states correspond to the states obtained from the finite \( t/U \) states by the electron - rotated-electron unitary transformation. These states are common eigenstates of both the 1D Hubbard model as \( U/t \to \infty \) and of the set of number operators \( \{L_{a, -1/2} \}, \{\tilde{N}_{a}(q_{j}) \} \), and \( \{\tilde{N}_{a, \nu}(q_{j}) \} \) with \( \alpha = c, s \) and \( \nu = 1, 2, 3, \ldots \) in the same limit. Together with the Hamiltonian these operators constitute a complete set of compatible and commuting hermitian operators \([12]\). These operators also commute with the momentum operator \( \hat{P} \). For these energy eigenstates the band momentum \( q_{j} \) of the \( c \) pseudoparticles and \( c, \nu \) and \( s, \nu \) pseudoparticles such that \( \nu = 1, 2, \ldots \) is a good quantum number. This justifies the designation of band-momentum energy eigenstates.

However, in the limit \( t/U \to 0 \) there are other choices for complete sets of energy and momentum eigenstates which are due to the periodic boundary conditions of the original electronic problem. In such a limit the 1D Hubbard model can be mapped onto a problem for which the number of doubly occupied sites is conserved \([40, 50]\). Since in the limit \( t/U \to 0 \) the ground state has zero double occupation, one usually introduces the concepts of lower and upper Hubbard bands which are associated with the Hilbert subspaces spanned by states with zero and finite values of the double occupation respectively \([40, 50]\). In the particular case of excitations involving creation of an electron the upper Hubbard band corresponds to the Hilbert subspace spanned by states with a single doubly occupied site. Within the \( t/U \to 0 \) scheme used in Refs. \([40, 50]\) one spectrally decomposes the elementary electronic operators into those which solely act in the upper or lower Hubbard bands, and eliminate perturbatively those parts which couple the two bands. This is achieved by application of a suitable transformation that eliminates these parts to a given order in \( t/U \). Such a transformation is nothing but the electron - rotated-electron unitary transformation in the limit of small values of \( t/U \). This procedure leads to an effective Hamiltonian which is equivalent to the 1D Hubbard model to lowest order in \( t/U \). In reference \([51]\) that effective Hamiltonian was called Harris-Lange model. In reference \([12]\) the concepts of lower and upper Hubbard bands were generalized to all values of \( t/U \) and associated with the rotated-electron double occupation.

### APPENDIX B: DESCRIPTION OF A LOCAL \( c, 4 \) PSEUDOPARTICLE IN TERMS OF ROTATED-ELECTRON SITE DISTRIBUTION CONFIGURATIONS

In this Appendix we illustrate for the specific case of a local \( c, 4 \) pseudoparticle how the rotated-electron site distribution configurations which describe a local \( \alpha, \nu \) pseudoparticle always involve a number \( \nu \) of site pairs such that \( x = g \) and \( x' = g + \nu \) where \( g = 1, \ldots, \nu \). In the present case we have that \( \alpha = c \) and thus the rotated-electron site distribution configurations correspond to doubly occupied and empty sites. Moreover, since \( \nu = 4 \) a local \( c, 4 \) pseudoparticle involves four site pairs such that \( x = g \) and \( x' = g + 4 \) where \( g = 1, 2, 3, 4 \). Once there are two possible
rotated-electron site occupancies for each pair and the number of pairs of each local $c$, 4 pseudoparticle is four, the total number of different internal rotated-electron site distribution configurations is $2^4 = 16$. According to the general expression (4) these 16 different internal rotated-electron site distribution configurations of doubly occupied and empty sites are such that,

$$\prod_{x=1}^{8} e^{i\pi h_{j,x} \hat{D}_{j,x}} \prod_{g=1}^{4} (1 - \hat{T}_{c,4,j,g})$$

\begin{align*}
&\times \left( \bullet_{h_{j,1}} \bullet_{h_{j,2}} \bullet_{h_{j,3}} \bullet_{h_{j,4}} \bullet_{h_{j,5}} \bullet_{h_{j,6}} \bullet_{h_{j,7}} \bullet_{h_{j,8}} \right) = \\
&+ e^{i\pi h_{j,1} + h_{j,2} + h_{j,3} + h_{j,4}} \left( \bullet_{h_{j,1}} \bullet_{h_{j,2}} \bullet_{h_{j,3}} \bullet_{h_{j,4}} \bullet_{h_{j,5}} \bullet_{h_{j,6}} \bullet_{h_{j,7}} \bullet_{h_{j,8}} \right) \\
&- e^{i\pi h_{j,1} + h_{j,2} + h_{j,3} + h_{j,6}} \left( \bullet_{h_{j,1}} \bullet_{h_{j,2}} \bullet_{h_{j,3}} \bullet_{h_{j,4}} \bullet_{h_{j,6}} \bullet_{h_{j,7}} \bullet_{h_{j,8}} \right) \\
&+ e^{i\pi h_{j,1} + h_{j,2} + h_{j,5}} \left( \bullet_{h_{j,1}} \bullet_{h_{j,2}} \bullet_{h_{j,3}} \bullet_{h_{j,4}} \bullet_{h_{j,5}} \bullet_{h_{j,7}} \bullet_{h_{j,8}} \right) \\
&- e^{i\pi h_{j,1} + h_{j,2} + h_{j,4}} \left( \bullet_{h_{j,1}} \bullet_{h_{j,2}} \bullet_{h_{j,3}} \bullet_{h_{j,4}} \bullet_{h_{j,5}} \bullet_{h_{j,6}} \bullet_{h_{j,8}} \right) \\
&+ e^{i\pi h_{j,1} + h_{j,3} + h_{j,4}} \left( \bullet_{h_{j,1}} \bullet_{h_{j,2}} \bullet_{h_{j,3}} \bullet_{h_{j,4}} \bullet_{h_{j,5}} \bullet_{h_{j,6}} \bullet_{h_{j,7}} \bullet_{h_{j,8}} \right) \\
&- e^{i\pi h_{j,1} + h_{j,3} + h_{j,5}} \left( \bullet_{h_{j,1}} \bullet_{h_{j,2}} \bullet_{h_{j,3}} \bullet_{h_{j,4}} \bullet_{h_{j,5}} \bullet_{h_{j,6}} \bullet_{h_{j,7}} \bullet_{h_{j,8}} \right) \\
&+ e^{i\pi h_{j,1} + h_{j,4} + h_{j,5}} \left( \bullet_{h_{j,1}} \bullet_{h_{j,2}} \bullet_{h_{j,3}} \bullet_{h_{j,4}} \bullet_{h_{j,5}} \bullet_{h_{j,6}} \bullet_{h_{j,7}} \bullet_{h_{j,8}} \right) \\
&- e^{i\pi h_{j,1} + h_{j,4} + h_{j,6}} \left( \bullet_{h_{j,1}} \bullet_{h_{j,2}} \bullet_{h_{j,3}} \bullet_{h_{j,4}} \bullet_{h_{j,5}} \bullet_{h_{j,6}} \bullet_{h_{j,7}} \bullet_{h_{j,8}} \right) \\
&+ e^{i\pi h_{j,2} + h_{j,3} + h_{j,4}} \left( \bullet_{h_{j,1}} \bullet_{h_{j,2}} \bullet_{h_{j,3}} \bullet_{h_{j,4}} \bullet_{h_{j,5}} \bullet_{h_{j,6}} \bullet_{h_{j,7}} \bullet_{h_{j,8}} \right) \\
&- e^{i\pi h_{j,2} + h_{j,4} + h_{j,5}} \left( \bullet_{h_{j,1}} \bullet_{h_{j,2}} \bullet_{h_{j,3}} \bullet_{h_{j,4}} \bullet_{h_{j,5}} \bullet_{h_{j,6}} \bullet_{h_{j,7}} \bullet_{h_{j,8}} \right) \\
&+ e^{i\pi h_{j,2} + h_{j,5} + h_{j,6}} \left( \bullet_{h_{j,1}} \bullet_{h_{j,2}} \bullet_{h_{j,3}} \bullet_{h_{j,4}} \bullet_{h_{j,5}} \bullet_{h_{j,6}} \bullet_{h_{j,7}} \bullet_{h_{j,8}} \right) \\
&- e^{i\pi h_{j,2} + h_{j,6} + h_{j,7}} \left( \bullet_{h_{j,1}} \bullet_{h_{j,2}} \bullet_{h_{j,3}} \bullet_{h_{j,4}} \bullet_{h_{j,5}} \bullet_{h_{j,6}} \bullet_{h_{j,7}} \bullet_{h_{j,8}} \right) \\
&+ e^{i\pi h_{j,3} + h_{j,4} + h_{j,5}} \left( \bullet_{h_{j,1}} \bullet_{h_{j,2}} \bullet_{h_{j,3}} \bullet_{h_{j,4}} \bullet_{h_{j,5}} \bullet_{h_{j,6}} \bullet_{h_{j,7}} \bullet_{h_{j,8}} \right) \\
&- e^{i\pi h_{j,3} + h_{j,5} + h_{j,6}} \left( \bullet_{h_{j,1}} \bullet_{h_{j,2}} \bullet_{h_{j,3}} \bullet_{h_{j,4}} \bullet_{h_{j,5}} \bullet_{h_{j,6}} \bullet_{h_{j,7}} \bullet_{h_{j,8}} \right) \\
&+ e^{i\pi h_{j,4} + h_{j,5} + h_{j,6}} \left( \bullet_{h_{j,1}} \bullet_{h_{j,2}} \bullet_{h_{j,3}} \bullet_{h_{j,4}} \bullet_{h_{j,5}} \bullet_{h_{j,6}} \bullet_{h_{j,7}} \bullet_{h_{j,8}} \right) \\
&- e^{i\pi h_{j,4} + h_{j,6} + h_{j,7}} \left( \bullet_{h_{j,1}} \bullet_{h_{j,2}} \bullet_{h_{j,3}} \bullet_{h_{j,4}} \bullet_{h_{j,5}} \bullet_{h_{j,6}} \bullet_{h_{j,7}} \bullet_{h_{j,8}} \right) \\
\end{align*}

On the left-hand side of this equation the site-$h_{j,x}$ rotated-electron double occupation operator $\hat{D}_{j,x}$ has eigenvalue 1 and 0 when that site is doubly occupied by rotated electrons and free of rotated electrons respectively, and the operator $\hat{T}_{c,4,j,g}$ acts on the pair of sites of indices $h_{j,g}$ and $h_{j,g+4}$ only. This operator is designed to act onto the rotated-electron eight-site distribution configuration of the particular form illustrated on the left-hand side of Eq. [B4]. This specific configuration is such that the first four sites are doubly occupied by rotated electrons and the following four sites are free of rotated electrons. From the application of this operator onto such a rotated-electron eight-site distribution configuration a new distribution configuration is generated where the site of index $h_{j,g}$ is free of rotated electrons, the site of index $h_{j,g+4}$ is doubly occupied by rotated electrons, and the occupancy of the other six sites remains unchanged. This operation is repeated according to the products on the left-hand side of Eq. [B4] and leads to the 16 internal rotated-electron site distribution configurations on the right-hand side of the same equation. These configurations are generated by considering that in each of the $g = 1, 2, 3, 4$ pairs of sites of indices $h_{j,g}$ and $h_{j,g+4}$, the site of index $h_{j,g}$ is doubly occupied by rotated electrons and the site of index $h_{j,g+4}$ is free of rotated electrons and vice versa. The phase factors appearing on both sides of the above equation result from the momentum $\pi/a$ associated with each rotated-electron doubly occupied site. This is the momentum of the $-1/2$ holon corresponding to such a site. Finally note that a similar description is obtained for the $s$, 4 pseudoparticle provided that one replaces these phase factors by one and the rotated-electron doubly occupied and empty sites by spin-down and spin-up rotated-electron singly occupied sites respectively.

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FIG. 1: Graphical representation of a charge \( h_{j,x} \leftrightarrow h_{j,x'} \) site pair or of a spin \( l_{j,x} \leftrightarrow l_{j,x'} \) site pair. In the particular case of \( x = 1 \) and \( x' = 2 \) the figure represents the \( h_{j,1} \leftrightarrow h_{j,2} \) site pair of a \( c,1 \) pseudoparticle or of the \( l_{j,1} \leftrightarrow l_{j,2} \) site pair of a \( s,1 \) pseudoparticle.

FIG. 2: Graphical representation of charge sequences of a zero \( \eta \)-spin energy eigenstate with no Yang holons and two rotated-electron doubly occupied sites and two rotated-electron empty sites only. The \( h_{j,g} \leftrightarrow h_{j,g+\nu} \) site pairs are represented as in Fig. 1. We note that the relative height of different site pairs has no physical meaning. The goal of using site pairs of different height is just to clearly define the two sites of each pair. In figures (a) and (b) two possible rotated-electron doubly occupied site/empty site pair distribution configurations are represented. The figures show how these distribution configurations change as a result of a cyclic permutation which transforms the first site of the charge sequence onto its last site. The distribution configuration (a) transforms onto itself whereas the distribution configuration (b) transforms onto two \( c,1 \) pseudoparticles. Thus according to property 6-III the distribution configuration (a) describes a \( c,2 \) pseudoparticle. Note that the distribution configuration (b) describes two \( c,1 \) pseudoparticles. The distribution occupancies of the figure describe alternatively \( s,1 \) and \( s,2 \) pseudoparticles. In this case the spin sequences correspond to a zero spin energy eigenstate with no HL spinons and with two spin-down singly occupied sites and two spin-up singly occupied sites only.
FIG. 3: Graphical representation of charge sequences of a zero $\eta$-spin energy eigenstate with no Yang holons and three rotated-electron doubly occupied sites and three rotated-electron empty sites only. The $h_{j,g} \leftrightarrow h_{j,g+\nu}$ site pairs are represented as in Fig. 1. In figures (a) and (b) it is shown how two possible rotated-electron doubly occupied site-empty site pair distribution configurations change as a result of a cyclic permutation which transforms the first site of the charge sequence onto its last site. The distribution configuration (a) transforms onto itself whereas the distribution configuration (b) transforms onto a non-equivalent distribution configuration. Thus according to property 6-III the distribution configuration (a) describes a $c, 3$ pseudoparticle. The figures represent alternatively spin sequences of a zero spin energy eigenstate with no HL spinons and with three spin-down singly occupied sites and three spin-up singly occupied sites only.

FIG. 4: Graphical representation of a domain of a charge sequence of the effective electronic lattice including fourteen sites. The $h_{j,g} \leftrightarrow h_{j,g+\nu}$ site pairs are represented as in Fig. 1. The six sites with no vertical lines correspond to Yang holons. There are two local $c, 1$ pseudoparticles described by $h_{j,1} \leftrightarrow h_{j,2}$ site pairs and a local $c, 2$ pseudoparticle described by a $h_{j,1} \leftrightarrow h_{j,3}$ and a $h_{j,2} \leftrightarrow h_{j,4}$ site pairs. Alternatively, if we replace the Yang holons by HL spinons and the $c, \nu$ pseudoparticles by $s, \nu$ pseudoparticles the figure represents the domain of a spin sequence with six HL spinons, two $s, 1$ pseudoparticles, and a $s, 2$ pseudoparticle.
FIG. 5: An illustration of the possible positions of a local $c, 1$ pseudoparticle when it passes from the left to the right-hand side of a steady local $c, 1$ pseudoparticle. The $h_{j, 1} \leftrightarrow h_{j, g+\nu}$ site pairs are represented as in Fig. 1. In the three rotated-electron site distribution configurations (a)-(c) the charge-sequence position of the steady pseudoparticle is labelled by a $X$ and remains unchanged. These three rotated-electron site distribution configurations correspond to three different charge sequences. Alternatively, the figure represents the corresponding electron site distribution configurations of two $s, 1$ pseudoparticles in three different spin sequences.
FIG. 6: An illustration of the possible positions of a local \( c,1 \) pseudoparticle when it passes from the left to the right-hand side of a steady local \( c,2 \) pseudoparticle. The \( h_{j,g} \leftrightarrow h_{j,g+\nu} \) site pairs are represented as in Fig. 1. In the five rotated-electron site distribution configurations (a)-(e) the charge-sequence position of the steady \( c,2 \) pseudoparticle is labelled by a \( X \) and remains unchanged. If instead we consider a steady local \( c,1 \) pseudoparticle, figures (a)-(e) illustrate the possible positions of a local \( c,2 \) pseudoparticle when it passes from the right to the left-hand side of a steady local \( c,1 \) pseudoparticle. However, in this second case the charge-sequence position of the local \( c,1 \) pseudoparticle should remain unchanged and the \( X \) point of figures (a)-(e) should be moved accordingly. In both cases these five distribution configurations correspond to five different charge sequences. Alternatively, the figure represents the corresponding electron site distribution configurations of a \( s,1 \) pseudoparticle and a \( s,2 \) pseudoparticle.