Error Estimate for Two-Dimensional Coupled Burgers’ Equations By Weak Galerkin Finite Element Method

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Abstract: A weak Galerkin finite element method (WG-FEM) can be considered a general finite element methods for solving partial differential equations (PDEs) by approximating the differential operators as distributions in weak forms. A weak Galerkin finite element method is used in this work for solving two Dimensional Burgers’ equations in lowest order Raviart-Thomas element with polynomails of constant basis. Both the continuous and discrete time WG-FEM are analysed. The optimal order estimates in $H^1$-error and $L^2$ -error are obtained. Numerical results are applied to clarify the theoretical analysis.

1. Introduction

In this work, the nonlinear time dependent for the two dimensional coupled Burgers’ problem considered as [2].

$$\frac{\partial u}{\partial t} - \epsilon(u_{xx} + u_{yy}) + u u_x + v u_y = f(x,y,t), \quad (x,y,t) \in \Omega \times (0,T], \quad (1)$$

$$\frac{\partial v}{\partial t} - \epsilon(v_{xx} + v_{yy}) + u v_x + v v_y = g(x,y,t), \quad (x,y,t) \in \Omega \times (0,T], \quad (2)$$

with Dirchlet boundary conditions

$$u(x,y,t) = \zeta(x,y,t), \quad (x,y,t) \in \partial \Omega \times (0,T], \quad (3)$$

and initial conditions

$$v(x,y,t) = \eta(x,y,t), \quad (x,y,t) \in \partial \Omega \times (0,T], \quad (4)$$

$$u(x,y,0) = u^0(x,y), \quad (x,y) \in \Omega. \quad (5)$$

$$v(x,y,0) = v^0(x,y), \quad (x,y) \in \Omega, \quad (6)$$

where $\Omega = \{(x,y), a \leq x \leq b, c \leq y \leq d\}$ is the computational domain and $\partial \Omega$ is it’s boundary, $u(x,y,t)$ and $v(x,y,t)$ are the velocity components, $u^0, v^0, \zeta$ and $\eta$ are known functions, $\frac{\partial u}{\partial t}$ is unsteady term, $u u_x$ is the nonlinear convection term, $\epsilon (u_{xx} + u_{yy})$ is the diffusion term, and $\epsilon = \frac{1}{Re}$ is diffusion constant and $Re$ is the Reynolds number. $f, g \in L^2(\Omega, t)$.

Various methods have been introduced for numerical solution of Burgers’ equation. These methods mainly include finite difference method (FDM) , finite volume, finite element method (FEM), boundary element, decomposition method, homotopy method, differential quadrature methods, etc., see ([3]-[12],[15],[16],[26],[27],[28],[29]) and the references therein. Recently, the WG-FEM has
attracted much attention in the field of numerical partial differential equations, see ([1],[13],[14],[17],[18],[19],[20],[21],[22],[23],[24],[25],[30]). In this work, the WG-FEM is applied for solving two dimensional coupled Burgers’ equations, a weak form presented for problem (1-6) with conservation form for non linear terms and based on this weak form, the continuous and discrete time WG-FEM are establish, also we derive the order error estimates in the discrete $L^2$-norm, respectively.

Rest of the paper is organized as follows. In Section 2, we introduce the definition of discrete weak derivative, discrete weak gradient and weak finite element spaces. In section 3, we present a variational form and weak variational form for problem (1-6). Section 4 is used to drive an order error estimate for the continuous and discrete time WG-FEM. In Section 5, numerical results are presented to proof the effectiveness of the WG-FEM and emphasis our theoretical analysis.

2. A Weak Galerkin Finite Element Spaces

In this section, some of weak differential operators, such as weak derivative and weak gradient are presented. Then we introduce some important weak function spaces which are useful in the error analysis of WGFEM. Let $\mathcal{P}_h(\Omega)$ be a partition of the domain $\Omega$ with mesh size $h = \max_{K \in \mathcal{T}_h} h_K$, where $h_K$ is longest side of $K$.

Let $(u, v) = \sum_{K \in \mathcal{T}_h} (u, v)_K = \sum_{K \in \mathcal{T}_h} \int_K uv \, dK$, $\|u\|^2 = \|u\|_H^2 = (u, u)_h$. For each triangle $K \in \mathcal{T}_h$ and $\partial K$ denote the interior and boundary of $K$ respectively, the function $w = \{w_0, w_b\}$ is called a weak function defined on $K$, where $w_0 \in L^2(K)$ and $w_b \in L^2(\partial K)$. The space of weak functions and corresponding vector space defined on $K$ are given by\n
$$W(K) = \{ w = \{w_0, w_b\} | w_0 \in L^2(K), w_b \in L^2(\partial K) \}. \tag{8}$$

We set\n
$$H(div, K) = \{ w, w \in (L^2(K))^2, \nabla \cdot w \in L^2(K) \}. \tag{9}$$

**Definition 2.1** Let $w \in W(K)$, the weak derivative operator of $w$ in the direction $\chi_j$ is defined as a linear functional $\frac{\partial w}{\partial \chi_j}$ on $H^1(K)$ such that\n
$$\int_K \frac{\partial w}{\partial \chi_j} q dx = -\int_K w_b \frac{\partial q}{\partial \chi_j} dx + \int_{\partial K} w_b q n_j ds, \quad \forall q \in H^1(K). \tag{10}$$

**Definition 2.2** Let $w \in W(K)$, the weak gradient operator of $w$ is defined as a linear functional $\nabla w \in H(div, K)$ on each element $K$, by the following equation:\n
$$\int_K \nabla w \cdot q dK = -\int_K w_b (\nabla \cdot q) dK + \int_{\partial K} w_b (q \cdot n) ds, \quad \forall q \in H(div, K). \tag{11}$$

Denote by $P_j(K)$ and $P_l(\partial K)$ the set of polynomials in $K$ and $e \in \partial K$ with degree less than or equal $j, l$ respectively. A discrete weak function $w = \{w_0, w_b\}$ refers to a polynomial with two components, the first component $w_0$ represent the interior $K$ and $w_b$ is defined on each edge $e, e \in \partial K$. Note that $w_b$ may or may not equal to the trace of $w_0$ on $\partial K$. There are two trial finite element spaces defined as:

$$U_h = \{ u = \{u_0, u_b\} : \{u_0, u_b\}|_K \in P_j(K) \times P_l(\partial K), \forall e \in \partial K, \forall K \in \mathcal{T}_h \}, \tag{12}$$

$$V_h = \{ v = \{v_0, v_b\} : \{v_0, v_b\}|_K \in P_j(K) \times P_l(\partial K), \forall e \in \partial K, \forall K \in \mathcal{T}_h \}. \tag{13}$$

Define two test spaces by,

$$U_h^0 = \{ u = \{u_0, u_b\} \in U_h : u_b|_{\partial K \cap \partial \Omega} = 0 \}, \tag{14}$$

and\n
$$V_h^0 = \{ v = \{v_0, v_b\} \in V_h : v_b|_{\partial K \cap \partial \Omega} = 0 \}. \tag{15}$$
Let $V_r(K) \subseteq [P_r(K)]^2$. In this paper we take the indices $j = l = k \geq 0$, $r = k + 1$, and $V_{r-k+1}(K) \subseteq [P_{k+1}(K)]^2$, the space $V_r(K)$ is usual Raviart-Thomas element $RT_k(K)$ of order $k$. These elements are referred as $\{P_k(K)^2, P_k(\partial K)^2, V_r(K)\}$ element in our numerical experiments. The Raviart-Thomas element $[17]$ $RT_k(K)$ of order $k$ is of the following form

$$RT_k(K) = P_k(K)^2 + \bar{P}_k(K)x.$$ 

**Definition 2.3** Let $w \in W(K)$, the discrete weak derivative operator of $w$ in the direction $x_j$ is defined as unique polynomial $\frac{\partial w_j}{\partial x_j}$ on $P_r(K)$ such that

$$\int_K \frac{\partial w_j}{\partial x_j} q \, dx = -\int_K w \frac{\partial q}{\partial x_j} \, dx + \int_{\partial K} w \eta \, n \, ds, \quad \forall q \in P_r(K). \quad (16)$$

**Definition 2.4** Let $w \in W(K)$, the discrete weak gradient operator of $w$ is defined as unique polynomial $\nabla w = \{\nabla w_0, \nabla w_1\}$ on each element $K$, by the following equation:

$$\int_K \nabla w \cdot q \, dK = -\int_K w_0(\nabla q) \, dK + \int_{\partial K} w_0(q \cdot n) \, ds, \quad \forall q \in V_r(K). \quad (17)$$

In this paper, for any $j \geq 0$, the seminorm is given by

$$|w|_{r,\Omega} = \left( \sum_{|i| = s} \int_{\Omega} |\partial^i w|^2 \, d\Omega \right)^{1/2},$$

with the norm

$$\|w\|_{r,\Omega} = \left( \sum_{i=0}^r |\partial^i w|^2 \, d\Omega \right)^{1/2},$$

if $r = 0$, we have

$$\|w\|_{0,\Omega} = \left( \int_{\Omega} |u|^2 \, d\Omega \right)^{1/2}.$$ 

For simplicity we use $\partial_d$, $\nabla_d$ refer to a discrete weak derivative or gradient and $\|\|$ for the $L^2 -$ norm.

In this paper we use two projection, the first $\Pi_h \in \mathcal{V}_r(K)\times P_h$ is projection of $P_r(K)\times P_0(\partial K)$ and the second projection $\Pi_h$ which satisfy the following properties, for each $K \in T_h$ and $\Pi_h u \in V_r(K)$.

$$\Pi_h u(x) = u(x), \quad x \in (x_i, y_j), \quad i = 1,2, \ldots, n \quad \text{and} \quad j = 1,2, \ldots, m, \quad (18)$$

$$\Pi_h u \leq Ch^{s+1} \quad \|u\|_{s+1}, \quad (19)$$

$$\Pi_h u \leq Ch^s \quad \|u\|_{s}, \quad (20)$$

**Definition 2.5** If $u \in H^1_0(\Omega) \cap H^{s+1}(\Omega)$, $Q_h u \in U_0^h$ or $V_0^h$ we have

$$\|Q_h u - u\|_{\Omega} \leq Ch^s \quad \|u\|_{s}, \quad (22)$$

$$\|\nabla Q_h u - \nabla u\|_{\Omega} \leq Ch^s \quad \|u\|_{s+1}, \quad (23)$$

**Lemma 2.1** For any $u \in H(\text{div}, \Omega)$ we have

$$\sum_{K \in T_h} (-\nabla \cdot u, w_k)_K = \sum_{K \in T_h} (\Pi_h u, \nabla_d w)_K, \quad \forall w = \{w_0, w_1\} \in U_0^h(j, l). \quad (24)$$

3. **Variational form and Weak Variational form**

Multiplying equations (1),(2) by $w, \rho \in H^1_0(\Omega)$ respectively and integrating by part we get,

$$(u_t, w) + \epsilon (\nabla u, \nabla w) + (u u_t, w) + (\nabla u, \nabla w) = (f, w), \quad (25)$$

$$(v_t, \rho) + \epsilon (\nabla v, \nabla \rho) + (u v, \rho) + (\nabla v, \nabla \rho) = (g, \rho), \quad \forall w, \rho \in H^1_0(\Omega),$$

and

$$(u(x, y, 0), w) = (u^0, w), \quad (v(x, y, 0), \rho) = (v^0, \rho).$$

We can write the nonlinear terms $u u_t$ and $v v_t$ in conservation form and integration by part as
\[
(u u_x, w) = \frac{1}{2} ((u^2)_x, w) = -\frac{1}{2} (u u, w_x), \quad (v v_y, \rho) = \frac{1}{2} ((v^2)_y, \rho). \]
Substituting in to equation (3.1) the Variational form is find \( u, v \in H^1(0, T; H^1_0(\Omega)) \) such that
\[
\begin{cases}
(u_t, w) + a(u, w) = (f, w), \\
\end{cases}
\]
and
\[
\begin{cases}
(v_t, \rho) + a(v, \rho) = (g, \rho), \\
\end{cases}
\]
where \( a(u, w) = (\nabla u, \nabla w), \quad a(v, \rho) = (\nabla v, \nabla \rho) \), and for \( \alpha, \beta > 0 \) the properties (bounded and coercive) holds. i.e.
\[
|\alpha(u, w)| \leq \beta \| \nabla u \| \| \nabla w \|.
\]

4. Error Analysis

4.1. Continuous Time Weak Galerkin Finite Element Method

In this subsection we analysis continuous time WG-FEM for coupled Burgers’ equations and derive error estimation in \( H^1 \) – norm and \( L^2 \) – norm respectively. Based on variational formulation (26), (27), the continuous time weak Galerkin finite element method is find \( u_h(t) = (u_0(., t), u_h(., t)) \in U_h^0 \) and \( v_h(t) = (v_0(., t), v_h(., t)) \in V_h^0 \) satisfying \( u_h = Q_h \xi, \quad v_h = Q_h \eta \) and \( u_h(0) = Q_h u^0, \quad v_h(0) = Q_h v^0 \) such that
\[
\begin{align*}
(u_{h,t}, w_0)_h + \epsilon (\nabla u_h, \nabla w_0)_h - \frac{1}{2} (u_h u_h, \frac{\partial w_0}{\partial x})_h + (v_h \frac{\partial u_h}{\partial y}, w_0)_h = (f, w_0)_h, \\
(v_{h,t}, \rho_0)_h + \epsilon (\nabla v_h, \nabla \rho_0)_h + (u_h \frac{\partial v_h}{\partial x}, \rho_0)_h - \frac{1}{2} (v_h v_h, \frac{\partial \rho_0}{\partial y})_h = (g, \rho_0)_h
\end{align*}
\]
\[
\forall \rho \in V_h^0.
\]
\[
u_h(x, y, 0) = \nu_h^0(x, y) \quad \text{and} \quad v_h(x, y, 0) = \nu_h^0(x, y),
\]
where \( \nu_h^0, \nu_h^0 \) are proper approximation of functions \( u^0, v^0 \) respectively.

Lemma 4.1 Let \( u(t), v(t) \in H^1(0, T; H^2(\Omega)) \) be the solution of problem (1.1)-(1.2) then, we have
\[
(u_{h,t}, w_0)_h + \epsilon (\Pi_h \nabla u, \nabla w_0)_h - \frac{1}{2} (\Pi_h u^2, \frac{\partial w_0}{\partial x})_h + (v h \frac{\partial u_h}{\partial y}, w_0)_h = (f, w_0)_h, \\
\forall w \in U_h^0.
\]
\[
(v_{h,t}, \rho_0)_h + \epsilon (\Pi_h \nabla v, \nabla \rho_0)_h - \frac{1}{2} (\Pi_h v^2, \frac{\partial \rho_0}{\partial y})_h + (u_h \frac{\partial v_h}{\partial x}, \rho_0)_h = (g, \rho_0)_h, \\
\forall \rho \in V_h^0.
\]

Proof. Multiply equation (1) by \( w_0 \) and integration we get
\[
(u_{h,t}, w_0)_h - \epsilon (\nabla u, w_0)_h + (u u_x, w_0)_h + (v v_y, w_0)_h = (f, w_0)_h.
\]
For third term we can written as
\[
(u u_x, w_0)_h = \frac{1}{2} ((u^2)_x, w_0)_h,
\]
it follows from (18) that
\[
((u^2)_x, w_0)_K = ((\Pi_h u^2)_x, w_0)_K,
\]
from definition of weak derivative operator in direction \( x \) we have
\[
((\Pi_h u^2)_x, w_0)_K = -((\Pi_h u^2, \frac{\partial w_0}{\partial x})_K + (w_0, q_n u)_K.
\]
By summing both side for (34) and using Lemma(2.6), it leads to
\[
((u^2)_x, w_0)_h = -((\Pi_h u^2, \frac{\partial w_0}{\partial x})_h, \quad \forall w = (w_0, w_b) \in U_h^0.
\]
Similarly for second term we have
\[
-\epsilon (\nabla u, w_0)_h = -\epsilon (\nabla (\Pi_h u), w_0)_h = \epsilon (\Pi_h \nabla u, \nabla w_0)_h.
\]
Substituting (35),(36) in equation (33) we complete the proof

Lemma 4.2 Let \( u(x, y, t), v(x, y, t), u_h(x, y, t) \) and \( v_h(x, y, t) \) are solutions of problem (1)-(2) and (29)-(30) respectively and let \( u(t), v(t) \in H^1(0, T, H^1(\Omega)) \) and \( \zeta_h = Q_h \xi, \eta_h = Q_h \eta \) then there exist constant \( c, \) such that
\[ e^u_t \| ^2 + e \int_0^t \| \nabla d e^u (s) \| ^2 \, ds + \frac{1}{2} \int_0^t \| \frac{\partial d e^u}{\partial x} \| ^2 \, ds \leq \| e^u_0 (\cdot , 0) \| ^2 + \| \nabla \| ^2 (k+1) \int_0^t \| \nabla (s) \| ^2 + \| e^u_0 (\cdot , 0) \| ^2 + \| \nabla \| ^2 (k+1) \int_0^t \| \nabla (s) \| ^2 + \| e^u_0 (\cdot , 0) \| ^2 + \| \nabla \| ^2 (k+1) \int_0^t \| \nabla (s) \| ^2.
\]

Proof. Subtracting (29) from (31) and using the fact \((Q_h u_t, w_0)_h = (u_t, w_0)_h\), we get
\[
(\nabla u u_t - u_t u, w_0)_h + e (\Pi_h v u, \nabla d w)_h - e (\nabla u u, \nabla d w)_h = \frac{1}{2} (\Pi_h u^2, \frac{\partial d e^u}{\partial x})_h - \frac{1}{2} (u_h u_h, \frac{\partial d e^u}{\partial x})_h + (v_h (\frac{\partial d u}{\partial y}), w_0)_h - (v u y, w_0)_h.
\] (37)

Add and subtract the term \(e (\nabla d Q_h u, \nabla d w)_h\), we get
\[
((Q_h u - u_h)_h, w_0)_h + e (\nabla d (Q_h u - u_h), \nabla d w)_h = \frac{1}{2} (\Pi_h u^2, \frac{\partial d e^u}{\partial x})_h - \frac{1}{2} (u_h u_h, \frac{\partial d e^u}{\partial x})_h + (v_h (\frac{\partial d u}{\partial y}), w_0)_h - (v u y, w_0)_h + e (\nabla d Q_h u, \nabla d e^u)_h - e (\Pi_h v u, \nabla d w e^u)_h.
\] (38)

To represent the error, we introduce the following notations:
\[
e^u = \Phi_h u - u_h = (Q_h u - u_h)_h, Q_h u - u_h = (e^u_0, e^u),
\]
\[
e^v = \Phi_h v - v_h = (Q_h v - v_h)_h, Q_h v - v_h = (e^v_0, e^v),
\]
taking \(w = e^u\) in equation (38) and using property (28), we have
\[
\frac{1}{2} \frac{d}{dt} \| e^u_0 \| ^2 + e \| \nabla d e^u \| ^2 \leq \frac{1}{2} (\Pi_h u^2, \frac{\partial d e^u}{\partial x})_h - \frac{1}{2} (u_h u_h, \frac{\partial d e^u}{\partial x})_h
\]
\[
+ (v_h (\frac{\partial d u}{\partial y}), e^u)_h - (v u y, e^u)_h + e (\nabla d Q_h u, \nabla d e^u)_h - e (\Pi_h v u, \nabla d w e^u)_h
\] (39)

To estimate \(l_1\) and \(l_2\), we add and subtract \(\frac{1}{2} (Q_h u Q_h u, \frac{\partial d e^u}{\partial x})_h\) \(Q_h v u, e^u_0\) and using fact \(Q_h u\) is good approximation of \(u\), we get
\[
l_1 = \frac{1}{2} (\Pi_h u^2 - u_h^2, \frac{\partial d e^u}{\partial x})_h - \frac{1}{2} (u_h^2 - (Q_h u)^2, \frac{\partial d e^u}{\partial x})_h = l_{11} - l_{12}.
\]
\[
l_2 = (v_h (Q_h u, v_h, e^u_0)_h - (Q_h u u_h, v_h, e^u)_h = l_{21} - l_{22},
\]
\[
\frac{1}{2} \frac{d}{dt} \| e^u_0 \| ^2 + e \| \nabla d e^u \| ^2 \leq l_{11} + l_{12} + l_{21} + l_{22} \leq \frac{1}{2} (\Pi_h u^2 - u_h^2, \frac{\partial d e^u}{\partial x})_h + (v (Q_h u u_h, v_h, e^u_0)_h + e (\nabla d Q_h u, \nabla d e^u)_h - e (\Pi_h v u, \nabla d w e^u)_h.
\] (40)

To estimate \(l_3\) we add and subtract \(e (\nabla u, \nabla d e^u)_h\), equation (40) become
\[
\frac{1}{2} \frac{d}{dt} \| e^u_0 \| ^2 + e \| \nabla d e^u \| ^2 \leq l_{11} + l_{12} + l_{21} + l_{22} + l_{31} + l_{32} + l_{33} + l_{34} + l_{35} + l_{36}.
\] (41)

By using Young’s inequality for \(l_{12}, l_{22}, l_{121}, l_{121}, l_{121}, and l_{122}\), we get
\[
|l_{12}| = \frac{1}{2} (u_h^2 - (Q_h u)^2, \frac{\partial d e^u}{\partial x})_h \leq \frac{1}{2} \| u_h^2 - (Q_h u)^2 \| ^2 + \| \frac{\partial d e^u}{\partial x} \| ^2,
\]
\[
|l_{22}| = (Q_h v Q_h u u_h - v_h \frac{\partial d e^u}{\partial y}, e^u_0)_h \leq \frac{1}{2} \| Q_h v Q_h u u_h - v_h \frac{\partial d u}{\partial y} \| ^2 + \| e^u_0 \| ^2,
\]
\[
|l_{111}| = \frac{1}{2} (\Pi_h u^2 - u_h^2, \frac{\partial d e^u}{\partial x})_h \leq \frac{1}{2} \| \Pi_h u^2 - u_h^2 \| ^2 + \| \frac{\partial d e^u}{\partial x} \| ^2,
\]
\[
|l_{21}| = (v (Q_h u u_h - v_h), e^u_0)_h \leq \frac{1}{4} \| v \| ^2 \| Q_h u u_h - v_h \| ^2 + \| e^u_0 \| ^2,
\]
\[
|l_{31}| = e (\nabla d Q_h u - \nabla u, \nabla d e^u)_h \leq \| \nabla d Q_h u - \nabla u \| ^2 + \| \nabla d e^u \| ^2,
\]
\[
|l_{32}| = e (\nabla u - \nabla u, \nabla d e^u)_h \leq \| \nabla u - \nabla u \| ^2 + \| \nabla d e^u \| ^2,
\]
substituting $l_{12}, l_{22}, l_{11}, l_{21}, l_{31},$ and $l_{32}$ in equation (41) with noting that $\| u^2_K - (Q_n u)^2 \|_2^2$ and $\| Q_n v_H u_y - v_H \frac{\partial u_H}{\partial y} \|^2$ are nonnegative terms, we get

$$\frac{1}{2} \frac{d}{dt} \| e^u_0 \|^2 + e \| \nabla_d e^u \|^2 + 2 \| \frac{\partial e^u}{\partial x} \|^2 \leq \frac{1}{4} \| \Pi u^2 \|_2^2 - u^2 \|^2 + \frac{1}{4} \| v \|^2_2 (Q_n u_y - u_y) \|^2$$

$$+ \frac{1}{2} \| e^u_0 \|^2 + \frac{1}{\epsilon} \| \nabla_d Q_n u - v \|^2_2 + \epsilon \| \nabla u - \Pi_n \nabla u \|^2_2.$$  

(42)

With (20), (21) and (23), it yields

$$\frac{1}{2} \frac{d}{dt} \| e^u_0 \|^2 + \frac{1}{\epsilon} \| \nabla_d e^u \|^2 + \frac{1}{\epsilon} \| \frac{\partial e^u}{\partial x} \|^2 \|^2 \leq C(\| e^u_0 \|^2 + h^{2(k+1)} \| u \|^2_{k+2}).$$

Integrating w.r.t. $t$ and by Grönwall lemma, we get

$$\| e^u_0 \|^2 + \epsilon \int_0^t \| \nabla_d e^u(s) \|^2 ds + \int_0^t \| \frac{\partial e^u}{\partial x}(s) \|^2 ds \leq \| e^u_0 \|^2 + Ch^{2(k+1)} \int_0^t \| u(s) \|^2_{k+2} ds.$$

In the same way for the second equation.

**Theorem 4.1** Under the assumption of Lemma (4.2), there exist constant $C$, such that

$$\| u - u^0 \|^2 + \epsilon \int_0^t \| \nabla u(s) - \nabla_d u_n(s) \|^2 ds + \int_0^t \| u(s) - \frac{\partial u_n(s)}{\partial x} \|^2 ds \leq C(\| u - Q_n u \|^2 + \| Q_n u - u^0 \|^2 + \epsilon \int_0^t \| \nabla u(s) - \nabla_d Q_n u(s) \|^2 ds + \epsilon \int_0^t \| \nabla_d Q_n u(s) \|^2 ds + \epsilon \int_0^t \| \nabla u(s) - \frac{\partial u_n(s)}{\partial x} \|^2 ds).$$

(46)

With (20),(21), lemma (4.2) and definition (2.5), we obtain

$$\| u - u^0 \|^2 + \epsilon \int_0^t \| \nabla u(s) - \nabla_d u_n(s) \|^2 ds$$

$$+ \int_0^t \| u(s) - \frac{\partial u_n(s)}{\partial x} \|^2 ds \leq \| u^0 \|^2 + C h^{2(k+1)} \int_0^t \| u(s) \|^2_{k+2} ds.$$

(47)

In the same way for Equation (45).

**Theorem 4.2** Let $u(x, y, t), v(x, y, t), u_n(x, y, t)$ and $v_n(x, y, t)$ are solutions of problem (1)-(2) and (29)-(30) respectively and let $u(t), v(t) \in H^3(0, T, H^{r+1}(\Omega))$ and $\zeta_n = Q_n \zeta, \eta_n = Q_n \eta$ then there exist constant $C$ such that

$$\int_0^T \| (e^u_0)_{\epsilon} \|^2 ds + \epsilon \| \nabla_d e^u \|^2 + \| e^u \|^2 \leq \| (e^u_{\epsilon}) \|^2 + \epsilon \| \nabla_d e^u_{\epsilon}(., 0) \|^2 + Ch^{2(k+1)} \int_0^T \left( \| u \|^2_{k+2} + \int_0^T \| u \|^2_{k+2} + \int_0^T \| u \|^2_{k+2} \right) ds.$$

(48)

$$\int_0^T \| (e^v_0)_{\epsilon} \|^2 ds + \epsilon \| \nabla_d e^v \|^2 + \| e^v \|^2 \leq \| (e^v_{\epsilon}) \|^2 + \epsilon \| \nabla_d e^v_{\epsilon}(., 0) \|^2 + Ch^{2(k+1)} \int_0^T \left( \| v \|^2_{k+2} + \int_0^T \| v \|^2_{k+2} + \int_0^T \| v \|^2_{k+2} \right) ds.$$

(49)

**Proof.** Taking $e^u = (e^u_0)_{\epsilon}$ in equation (40), we obtain

$$\int_0^T \| (e^u_0)_{\epsilon} \|^2 + \frac{1}{2} \frac{d}{dt} \| \nabla_d e^u \|^2 + R_1 + R_2 \leq \frac{1}{2} (\| \Pi u^2 \|_2^2 - u^2 \|^2 + \frac{1}{\epsilon} \| \nabla u - \Pi \nabla u \|^2_2) \| (e^u_{\epsilon}) \|^2 + (v(Q_n u_y - u_y), (e^u_{\epsilon}))_n$$
To estimate $R_1 \sim R_5$, Cauchy, Young’s inequalities, Definition (2.5) and Eq.(21), we have

$$|R_1| = \frac{1}{2} (u_h^2 - (Q_h u)^2) \frac{\partial_d (e^{(u)})}{\partial x} h$$
$$= \frac{1}{2} \frac{d}{dt} (u_h^2 - (Q_h u)^2) + u_h \frac{\partial_d (e^{(u)})}{\partial x} h \\
\leq \frac{1}{2} \frac{d}{dt} (u_h^2 - (Q_h u)^2) + \frac{1}{2} \frac{d}{dt} (\frac{\partial_d (e^{(u)})}{\partial x})^2. \tag{51}$$

$$|R_2| = (Q_h v Q_h u - v_h \frac{\partial_d (e^{(u)})}{\partial x} (e^{(u)})_t) \leq \frac{1}{2} (Q_h v Q_h u - v_h \frac{\partial_d (e^{(u)})}{\partial x})^2. \tag{52}$$

$$|R_3| = \frac{1}{2} (\Pi h u^2 - u^2) \frac{\partial_d (e^{(u)})}{\partial x} h$$
$$= \frac{1}{2} \frac{d}{dt} (\Pi h u^2 - u^2) \frac{\partial_d (e^{(u)})}{\partial x} h \\
\leq \frac{1}{2} \frac{d}{dt} (\Pi h u^2 - u^2) \frac{\partial_d (e^{(u)})}{\partial x} h + \frac{1}{2} \frac{d}{dt} (\Pi h u^2 - u^2)^2 + \frac{1}{2} \frac{d}{dt} (\frac{\partial_d (e^{(u)})}{\partial x})^2. \tag{53}$$

$$|R_4| = (v'(Q_h u - u) + (e^{(u)})_t) \leq \frac{1}{4} v'' \| Q_h u - u \|^2 + \| (e^{(u)})_t \|^2 \leq C h^{2(k+1)} \| u \|^2_{k+2} + \| (e^{(u)})_t \|^2. \tag{54}$$

$$|R_5| = \epsilon \| \nabla_d Q_h u, \nabla_d (e^{(u)}) \|_h - \epsilon \Pi h \nabla u, \nabla_d (e^{(u)}) \|_h$$
$$= \epsilon \| \nabla_d Q_h u - \Pi h \nabla u, \nabla_d (e^{(u)}) \|_h$$
$$\leq \epsilon \frac{d}{dt} (\nabla_d Q_h u - \Pi h \nabla u, \nabla_d (e^{(u)}) \|_h - \epsilon \frac{d}{dt} (\nabla_d Q_h u - \Pi h \nabla u, \nabla_d (e^{(u)}) \|_h)$$
$$\leq \epsilon \frac{d}{dt} (\nabla_d Q_h u - \Pi h \nabla u, \nabla_d (e^{(u)}) \|_h + C h^{2(k+1)} \| u \|^2_{k+2} + \| (e^{(u)})_t \|^2 + \epsilon \| \nabla_d (e^{(u)}) \|^2 \leq C h^{2(k+1)} \| u \|^2_{k+2} + \| (e^{(u)})_t \|^2 + \epsilon \| \nabla_d (e^{(u)}) \|^2 + \epsilon \frac{d}{dt} (\nabla_d Q_h u - \Pi h \nabla u, \nabla_d (e^{(u)}) \|_h) + \epsilon \frac{d}{dt} (\nabla_d Q_h u - \Pi h \nabla u, \nabla_d (e^{(u)}) \|_h) + \frac{1}{2} \frac{d}{dt} (\Pi h u^2 - u^2) \frac{\partial_d (e^{(u)})}{\partial x} h. \tag{55}$$

Substitution $R_1 \sim R_5$ in Eq.(50) with noting that $\| \frac{d}{dt} (u_h^2 - (Q_h u)^2) \|^2$ and $\| Q_h v Q_h u - v_h \frac{\partial_d (e^{(u)})}{\partial x} \|^2$ are nonnegative, we get

$$\| (e^{(u)})_t \|^2 + \frac{d}{dt} (\| \nabla d (e^{(u)}) \|^2 + \frac{1}{2} \| \frac{\partial_d (e^{(u)})}{\partial x} \|^2 + \frac{1}{2} \frac{d}{dt} (u_h^2 - (Q_h u)^2) \| (e^{(u)})_t \|^2 + \frac{1}{2} \frac{d}{dt} (u_h^2 - (Q_h u)^2) \frac{\partial_d (e^{(u)})}{\partial x} h \leq C h^{2(k+1)} \| u \|^2_{k+2} + C h^{2(k+1)} \| u \|^2_{k+2} + \frac{3}{2} \| \frac{\partial_d (e^{(u)})}{\partial x} \|^2 + \epsilon \| \nabla_d (e^{(u)}) \|^2 + \| (e^{(u)})_t \|^2 + \epsilon \frac{d}{dt} (\nabla_d Q_h u - \Pi h \nabla u, \nabla_d (e^{(u)}) \|_h + \epsilon \frac{d}{dt} (\nabla_d Q_h u - \Pi h \nabla u, \nabla_d (e^{(u)}) \|_h) + \frac{1}{2} \frac{d}{dt} (\Pi h u^2 - u^2) \frac{\partial_d (e^{(u)})}{\partial x} h. \tag{56}$$

Integrating with respect to t, we get

$$\frac{1}{2} \int_0^t \| (e^{(u)})_t \|^2 ds + \frac{d}{dt} (\| \nabla_d (e^{(u)}) \|^2 + \frac{1}{2} (u_h^2 - (Q_h u)^2) \| (e^{(u)})_t \|^2 + \frac{1}{2} \frac{d}{dt} (u_h^2 - (Q_h u)^2) \| (e^{(u)})_t \|^2 + \frac{1}{2} \frac{d}{dt} (u_h^2 - (Q_h u)^2) \frac{\partial_d (e^{(u)})}{\partial x} h \leq \frac{1}{2} \| \nabla_d (e^{(u)}) \|^2 \| (e^{(u)})_t \|^2 + \frac{1}{2} \| \nabla_d (e^{(u)}) \|^2 \| (e^{(u)})_t \|^2 + \frac{1}{2} \| \nabla_d (e^{(u)}) \|^2 \| (e^{(u)})_t \|^2 \| (e^{(u)})_t \|^2 + \frac{1}{2} \| \nabla_d (e^{(u)}) \|^2 \| (e^{(u)})_t \|^2 \leq \frac{1}{2} \| \nabla_d (e^{(u)}) \|^2 \| (e^{(u)})_t \|^2 \tag{57}.$$
\[ + C h^{2(k+1)} \left( \int_0^t \| u \|_{K_2}^2 \, ds + \int_0^t \| u_t \|_{K_2}^2 \, ds \right) + \int_0^t \| e_u \|_2^2 + \int_0^t \| \nabla_d(e_u) \|_2^2 \, ds + \int_0^t \frac{\partial_d(e_u)}{\partial x} \right\|_2 \, ds + 2e \| \nabla_d Q \nabla u - \nabla u \|_2^2 + \frac{\epsilon}{8} \| \nabla_d(e_u) \|_2^2 + 2e \| \nabla u - \nabla_h \nabla u \|_2^2 + \frac{\epsilon}{8} \| \nabla_d(e_u) \|_2^2 + \frac{1}{4} \| \nabla_h^2 u - u \|_2^2 \].

With (20), (21) and estimate (43), it yields

\[ \frac{1}{2} \int_0^t \| (e_u^0)_{t} \|_2^2 \, ds + \frac{\epsilon}{2} \| \nabla_d e_u \|_2^2 + \frac{\epsilon}{2} \| \nabla_d e_u(., 0) \|_2^2 + C h^{2(k+1)} \left( \int_0^t \| u \|_{K_2}^2 + \int_0^t \| u_t \|_{K_2}^2 \, ds \right) + \frac{1}{2} \int_0^t \| e_u \|_2^2 + \frac{\epsilon}{2} \| \nabla_d e_u \|_2^2 \right\]^2. \]

By rearranging

\[ \int_0^t \| (e_u^0)_{t} \|_2^2 \, ds + \epsilon \| \nabla_d e_u \|_2^2 + \frac{\epsilon}{2} \| e_u \|_2 \leq \frac{\epsilon}{2} \| e_u \|_2 \| \nabla_d e_u(., 0) \|_2^2 + \frac{\epsilon}{2} \| \nabla_d e_u(., 0) \|_2^2 \]

\[ + C h^{2(k+1)} \left( \int_0^t \| u \|_{K_2}^2 + \int_0^t \| u_t \|_{K_2}^2 \, ds \right) + \frac{1}{2} \int_0^t \| e_u \|_2^2 + \frac{\epsilon}{2} \| \nabla_d e_u \|_2^2 \right\]^2. \]

4.2. Discrete Time Weak Galerkin Finite Element Method

In this subsection we derive error estimation in the H^1 -norm and L^2 -norm respectively for Burgers' equation. Let 0 = t_0 < t_1 < \ldots < t_N = T be a partition of time interval [0, T] and the time level t = t_n = n\tau where n \in \mathbb{Z}^+ and U_n \in U_n(t, \lambda) is the approximate solution of u(t_n). The backward Euler weak Galerkin method is defined by replacing the time derivative in equation (29), (30) by a backward difference quotient \( \partial_t U_n = (U_n - U_{n-1})/\tau \)

\[ (\partial_t U_n, w_n) + \epsilon(\nabla D U_n, \nabla w_n) + \frac{1}{2}(U_n, \partial \partial_{x} w_n, w_n) + (W_n, \partial \partial_{x} w_n, w_n) = (f, w_n), \quad \forall w \in U_n^0, \]

\[ (\partial_t V_n, \rho_n) + \epsilon(\nabla D V_n, \nabla \rho_n) + (U_n, \partial \partial_{x} \rho_n, \rho_n) = \frac{1}{2}(W_n, \partial \partial_{x} \rho_n, \rho_n) = (g, \rho_n), \quad \forall \rho \in U_n^0, \]

\[ U_0(x, y, 0) = u_0(x, y) \quad \text{and} \quad V_0(x, y, 0) = v_0(x, y), \]

where u_0, v_0 are proper approximation of functions u and v respectively.

**Theorem 4.3** Let u(x, y, t), v(x, y, t), U_n(x, y, t) and V_n(x, y, t) are solutions of problem (1), (2), (61) and (62) respectively and let u(t), v(t) \in H^2(0, T; H^2+1(\Omega)) then there exists constant C such that

\[ \| (e_u^0) \|_2^2 + \sum_{j=1}^{n} \epsilon \| \nabla_d e_u^j \|_2^2 \leq \frac{\epsilon}{2} \| e_u^0 \|_2^2 + C \left( h^{2(k+1)} \max_{j=1, n} \| u(t_j) \|_{K_2}^2 + \tau^2 \int_0^{T} \| u_t(t) \|_{K_2}^2 \right). \]

\[ \| (e_u^0) \|_2^2 + \sum_{j=1}^{n} \epsilon \| \nabla_d e_u^j \|_2^2 \leq \frac{\epsilon}{2} \| e_u^0 \|_2^2 + C \left( h^{2(k+1)} \max_{j=1, n} \| u(t_j) \|_{K_2}^2 + \tau^2 \int_0^{T} \| v_t(t) \|_{K_2}^2 \right). \]

**Proof.** Set t = t_n in equation (31), we obtain

\[ (u_t(t_n), w_0) + \epsilon(\Pi_h \nabla u(t_n), \nabla w_n) + \frac{1}{2}(\Pi_h u^2(t_n), \partial \partial_{x} w_n, w_n) + (v(t_n) u_y(t_n), w_0) = (f, w_0). \]

Subtract (61) from (65) with the fact (\partial_t (Q_0 u(t_n)), w_n) = (\partial_t u(t_n), w_n), we have

\[ (\partial_t (Q_0 u(t_n) - U_n^0), w_n) + \epsilon(\nabla D (Q_0 u(t_n) - U_n^0), \nabla w_n) = \frac{1}{2}(\Pi_h u^2(t_n), \partial \partial_{x} w_n, w_n) - \frac{1}{2}(U_n u_n, \partial \partial_{x} w_n, w_n) - \epsilon(\Pi_h \nabla u(t_n), \nabla w_n) + (\partial_t u(t_n) - u_t(t_n), w_n). \]

Let e_u^0 = Q_0 u(t_n) - U_n and taking w = e_u^0 in (66), we get

\[ \frac{1}{\tau} \| (e_u^0)_{t} \|_2^2 + \frac{1}{\tau} \| (e_u^{n-1})_{0} - (e_u^{n-1})_{0} \|_2^2 + \epsilon \| \nabla_d e_u^0 \|_2^2 \leq \frac{1}{2}(\Pi_h u^2(t_n), \partial \partial_{x} w_n, w_n) - \frac{1}{2}(U_n u_n, \partial \partial_{x} w_n, w_n) + (\nabla \partial \partial_{x} w_n, (e_u^0)_{0}) - (v(t_n) u_y(t_n), (e_u^0)_{0}) + \epsilon(\Pi_h \nabla u(t_n), \nabla e_u^0_{n}) - (\Pi_h \nabla u(t_n), \nabla e_u^0_{n}) \]

\[ + \epsilon(\Pi_h \nabla u(t_n), \nabla e_u^0_{n}) - \epsilon(\Pi_h \nabla u(t_n), \nabla e_u^0_{n}). \]
Estimate $J1, J2$ and $J3$ similar to estimate $I1, I2$ and $I3$ therefor, we have

$$J1 = \frac{1}{2} (\Pi_n u^2(t_n) - u^2(t_n), (e^{u^2}_n)^h) - \frac{1}{2} ((U^2_n - (Q_n u^2), \frac{\partial e^{u^2}_n}{\partial x} )h),$$

$$J2 = (v(t_n)(Q_n u_y(t_n) - u_y(t_n)), (e^{v}_n)^0 h - (Q_n v(t_n)Q_n u_y(t_n) - V_n \frac{\partial u}{\partial y}, (e^{v}_n)^0 h),$$

$$J3 = \epsilon (\nabla Q_n u(t_n) - \nabla u(t_n), \nabla_d e^{u^2}_n h - \epsilon (\nabla u(t_n) - \Pi_n \nabla u(t_n), \nabla_d e^{u^2}_n h),$$

substituting $J1$ and $J2$ in equation (67), we have

$$1 \| (e^{u^2}_n)^0 \|^2 - \frac{1}{\tau} ((e^{u^2}_{n-1})_0 (e^{u^2}_n)^0 h + \epsilon \| \nabla_d e^{u^2}_n \|^2 + J12 + J22 \leq \frac{1}{2} (\Pi_n u^2(t_n) - u^2(t_n), \frac{\partial e^{u^2}_n}{\partial x} )h + (v(t_n)(Q_n u_y(t_n) - u_y(t_n)), (e^{v}_n)^0 h) +$$

$$\epsilon (\nabla Q_n u(t_n) - \nabla u(t_n), \nabla_d e^{u^2}_n h) + \epsilon (\nabla u(t_n) - \Pi_n \nabla u(t_n), \nabla_d e^{u^2}_n h) + (\partial_t u(t_n) - u_t(t_n), (e^{u^2}_n)^0 h)$$

$$= J11 + J21 + J31 + J32 + J4.$$ 

By using Young's inequality

$$J12 = (U^2_n - (Q_n u)^2, \frac{\partial e^{u^2}_n}{\partial x}) \leq \frac{1}{4} \| U^2_n - (Q_n u)^2 \|^2 + \frac{1}{4} \| \frac{\partial e^{u^2}_n}{\partial y} \|^2,$$

$$J22 = (Q_n v(t_n)Q_n u_y(t_n) - V_n \frac{\partial u}{\partial y}, (e^{v}_n)^0 h) \leq \frac{1}{2} \| Q_n v(t_n)Q_n u_y(t_n) - V_n \frac{\partial U}{\partial y} \|^2$$

$$J11 = \frac{1}{2} (\Pi_n u^2(t_n) - u^2(t_n), \frac{\partial e^{u^2}_n}{\partial x} )h \leq \frac{1}{4} \| \Pi_n u^2(t_n) - u^2(t_n) \|^2 + \frac{1}{4} \| \frac{\partial e^{u^2}_n}{\partial x} \|^2,$$

$$J21 = (v(t_n)(Q_n u_y(t_n) - u_y(t_n)), (e^{v}_n)^0 h) \leq \| v(t_n) \|_w (Q_n u_y(t_n) - u_y(t_n) \|_w ^2$$

Substituting $J11, J12, J21, J22, J31, J32, J4$ in (68), with that $\| U^2_n - (Q_n u)^2 \|^2$, $\| Q_n v(t_n)Q_n u_y(t_n) - V_n \frac{\partial u}{\partial y} \|^2$, $\| \Pi_n u^2(t_n) - u^2(t_n) \|^2$ are nonnegative terms, we get

$$\| (e^{u^2}_n)^0 \|^2 + \frac{\tau}{2} \| \nabla_d e^{u^2}_n \|^2 \leq \| (e^{u^2}_{n-1})_0 \|^2 + \frac{1}{2} \| (e^{u^2}_n)^0 \|^2 + \frac{1}{4} \| \Pi_n u^2(t_n) - u^2(t_n) \|^2$$

$$+ \frac{\tau}{2} \| v(t_n) \|_w, \| Q_n u_y(t_n) - u_y(t_n) \|^2 + \tau \| \nabla Q_n u(t_n) - \nabla u(t_n) \|^2$$

$$+ \epsilon \| \nabla u(t_n) - \Pi_n \nabla u(t_n) \|^2 + \| \partial_t u(t_n) - u_t(t_n) \|^2.$$ 

(69)

$$\| (e^{u^2}_n)^0 \|^2 + \epsilon \| \nabla_d e^{u^2}_n \|^2 \leq \| (e^{u^2}_{n-1})_0 \|^2 + S_1(t_n) + 2 \| \partial_t u(t_n) - u_t(t_n) \|^2.$$ 

(70)

Where

$$S_1(t_n) = \frac{\tau}{2} \| \Pi_n u^2(t_n) - u^2(t_n) \|^2 + 2 \tau \| v(t_n) \|_w, \| Q_n u_y(t_n) - u_y(t_n) \|^2$$

$$+ 2 \epsilon \| \nabla Q_n u(t_n) - \nabla u(t_n) \|^2 + 2 \epsilon \| \nabla u(t_n) - \Pi_n \nabla u(t_n) \|^2,$$

by repeated application and with equation (20),(21) and (23)
This implies
\[ \parallel \sum_{j=1}^{n} S_j(t_j) \geq C \rho h^{2(k+1)} \parallel u(t) \parallel_{k+2}. \] (72)

So that
\[ \sum_{j=1}^{n} \parallel u(t_j) \parallel_{k+2}^{2} \leq \rho r \max_{j=1,\ldots,n} \parallel u(t_j) \parallel_{k+2} \leq \rho r \max_{j=1,\ldots,n} \parallel u(t_j) \parallel_{k+2}. \] (73)

We can write
\[ q^{j} = u_{\varepsilon}(t_{j}) - \tilde{u}(t_{j}) = (u_{\varepsilon} - \frac{1}{\varepsilon}(u^{j} - u^{j-1})). \]

By Young's inequality
\[ \parallel q^{j} \parallel_{k+2}^{2} \leq \tau \int_{t_{j-1}}^{t_{j}} \parallel u_{tt} \parallel dt \leq \tau \int_{t_{j-1}}^{t_{j}} (\int_{t_{j-1}}^{t} \parallel u_{tt} \parallel dt) dt \leq \tau \int_{t_{j-1}}^{t_{j}} (\int_{t_{i}}^{t_{j-1}} \parallel u_{tt} \parallel dt) dt = \tau \int_{t_{j-1}}^{t_{j}} \parallel u_{tt} \parallel dt. \] (74)

Substitute (73),(74) in (71) and discrete Grönwall lemma the proof is complete.

**Theorem 4.4** Let \( u(x,y,t), v(x,y,t), U_{n}(x,y,t) \) and \( V_{n}(x,y,t) \) are solutions of problem (1),(2), (61) and (62) respectively and let \( u(t), v(t) \in H^{2}(0,T,H^{2+1}(\Omega)) \) then there exist constant \( C \) such that
\[ \parallel \nabla_{x} u_{n} \parallel \leq C \parallel \eta_{n} \parallel \parallel u_{0} \parallel + \parallel \nabla_{x} u_{n} \parallel \]
\[ + h^{2(k+1)} \parallel u(t) \parallel_{k+2} + \parallel u_{i}(t) \parallel_{k+2} + \tau^{2} \int_{0}^{t} \parallel u_{tt} \parallel dt \parallel_{k+2} ds. \] (75)

Where
\[ \parallel u \parallel_{k+2} = \max_{j=1,\ldots,n} \parallel u(t_{j}) \parallel_{k+2}, \quad \text{and} \quad \parallel u_{i} \parallel_{k+2} = \max_{j=1,\ldots,n} \parallel u_{i}(t_{j}) \parallel_{k+2}. \]

**Proof.** Taking \( w = \tilde{u} \) in error equation (66)
\[ (\tilde{\partial}_{t}(e_{u}^{n}), \tilde{\partial}_{t}(e_{u}^{n})) + \epsilon(\nabla_{x} u_{n}, \nabla_{x} \tilde{u}_{n}(e_{u}^{n})) = \frac{1}{2} (\Pi u_{n}, \frac{\partial_{t} \tilde{u}_{n}(e_{u}^{n})}{\partial_{t}})_{h} - \frac{1}{2} (u_{n}, \frac{\partial_{t} \tilde{u}_{n}(e_{u}^{n})}{\partial_{t}})_{h} \]
\[ + (\gamma_{h} \frac{\partial_{t} \tilde{u}_{n}(e_{u}^{n})}{\partial_{t}}, \tilde{u}_{n}(e_{u}^{n}))_{h} - (u_{n}, u_{n})_{h} \]
\[ + \epsilon(\nabla_{x} u_{n}(t_{n}), \nabla_{x} \tilde{u}_{n}(e_{u}^{n}))_{h} - \epsilon(\Pi u_{n}, \nabla_{x} \tilde{u}_{n}(e_{u}^{n}))_{h} \]
\[ + (\tilde{u}_{n}(t_{n}) - u_{n}(t_{n}), \tilde{u}_{n}(e_{u}^{n}))_{h}. \]

This implies
\[ \parallel \tilde{u} \parallel_{k+2}^{2} + \epsilon(\nabla_{x} u_{n}(t_{n}), \frac{\partial_{t} \tilde{u}_{n}(e_{u}^{n})}{\partial_{t}}_{h} - \frac{\partial_{t} \tilde{u}_{n}(e_{u}^{n})}{\partial_{t}}_{h})_{h} \leq K1 + K2 + K3 + K4. \] (79)

Similarity for Theorem (4.3), Equation (79) become
\[ \parallel \tilde{u} \parallel_{k+2}^{2} + \epsilon(\nabla_{x} u_{n}(t_{n}), \frac{\partial_{t} \tilde{u}_{n}(e_{u}^{n})}{\partial_{t}}_{h} - \frac{\partial_{t} \tilde{u}_{n}(e_{u}^{n})}{\partial_{t}}_{h})_{h} + K12 + K22 \leq K11 + K21 + K31 + K32. \] (80)

By Young's inequality
\[ |K_{12}| = \frac{1}{2} \left( U_n^2 - (Q_n u(t_n)) \right), \quad \frac{\partial_u \tilde{e}_{0}^u}{\partial y} \leq \frac{1}{4} \left[ U_n^2 - (Q_n u(t_n))^2 \right] \| + \frac{1}{4} \left[ \frac{\partial_u \tilde{e}_{0}^u}{\partial x} \right] \| ,
\]
\[ |K_{22}| = (Q_n v(t_n) Q_n u(t_n) - V_n u(t_n) \tilde{e}_{0}^u) \leq \frac{1}{2} \left[ Q_n v(t_n) Q_n u(t_n) - V_n u(t_n) \tilde{e}_{0}^u \right] \| + \frac{1}{2} \left[ \frac{\partial_y \tilde{e}_{0}^u}{\partial y} \right] \| ,
\]
\[ |K_{11}| = \frac{1}{2} \left( \Pi_n u^2(t_n) - u^2(t_n) \right), \quad \frac{\partial_u \tilde{e}_{0}^u}{\partial x} \leq \frac{1}{4} \left[ \Pi_n u^2(t_n) - u^2(t_n) \right] \| + \frac{1}{4} \left[ \frac{\partial_y \tilde{e}_{0}^u}{\partial y} \right] \| ,
\]
\[ |K_{21}| = (v(t_n) (Q_n u(t_n) - u(t_n))) \tilde{e}_{0}^u \leq \frac{1}{2} \left[ v(t_n) \right] \| + \frac{1}{2} \left[ \frac{\partial_y \tilde{e}_{0}^u}{\partial y} \right] \| ,
\]
\[ |K_{4}| = (\tilde{e}_{0}^u - u(t_n)) \frac{\partial_u \tilde{e}_{0}^u}{\partial y} \leq \frac{1}{4} \left[ \tilde{e}_{0}^u - u(t_n) \right] \| + \frac{1}{4} \left[ \frac{\partial_y \tilde{e}_{0}^u}{\partial y} \right] \| .
\]

By Young’s inequality,
\[ \epsilon (\nabla \Phi, \nabla \Phi) \leq \frac{\epsilon}{2} \left( \nabla \Phi, \nabla \Phi \right) + \frac{\epsilon}{2} \left( \nabla \Phi, \nabla \Phi \right) + \epsilon \| v(t_n) \| \| u(t_n) \| + \epsilon \| \Phi \| \| \Phi \| + \epsilon \| \tilde{e}_{0}^u \| \| \tilde{e}_{0}^u \| + \epsilon \| \tilde{e}_{0}^u \| \| \tilde{e}_{0}^u \| + \epsilon \| \tilde{e}_{0}^u \| \| \tilde{e}_{0}^u \| .
\]

After cancelation, we can write Equation (81) as
\[ \frac{\epsilon}{2} \left( \nabla \Phi, \nabla \Phi \right) + \epsilon \| v(t_n) \| \| u(t_n) \| + \epsilon \| \Phi \| \| \Phi \| + \epsilon \| \tilde{e}_{0}^u \| \| \tilde{e}_{0}^u \| + \epsilon \| \tilde{e}_{0}^u \| \| \tilde{e}_{0}^u \| .
\]

Multiply by \( \frac{1}{\alpha} \)
\[ \frac{\alpha}{2} \| \nabla \Phi \| \leq \frac{\beta}{2} \| \nabla \Phi \| + \epsilon \| \tilde{e}_{0}^u \| \| \tilde{e}_{0}^u \| + \epsilon \| \tilde{e}_{0}^u \| \| \tilde{e}_{0}^u \| + \epsilon \| \tilde{e}_{0}^u \| \| \tilde{e}_{0}^u \| .
\]

This implies
\[ \frac{\alpha}{2} \| \nabla \Phi \| \leq \frac{\beta}{2} \| \nabla \Phi \| + \epsilon \| \tilde{e}_{0}^u \| \| \tilde{e}_{0}^u \| + \epsilon \| \tilde{e}_{0}^u \| \| \tilde{e}_{0}^u \| + \epsilon \| \tilde{e}_{0}^u \| \| \tilde{e}_{0}^u \| .
\]
\[ + \tau \sum_{j=1}^{n} \| \nabla_d (e_j^{(n)}) \|_2^2 + \tau \sum_{j=1}^{n} \| (w_j^{(n)})_t \|_2^2 + C \tau^2 \int_0^{\tau_n} \| u_{tt} \|_{k+2}^2 \, ds. \]

By rearranging
\[ \| \nabla_d (e_n^{(n)}) \|_2^2 \leq C (\| \nabla_d (e_0^{(0)}) \|_2^2 + \tau h^{2(k+1)}) \| u \|_{k+2}^2 + \tau^2 \int_0^{\tau_n} \| u_{tt} \|_{k+2}^2 \, ds \]
\[ + \| w_0^{(n)} \|_2^2 + \| w_1^{(n)} \|_2^2 + \tau \sum_{j=1}^{n} \| (w_j^{(n)})_t \|_2^2 \]
\[ + \tau \sum_{j=1}^{n} \| \nabla_d (e_j^{(n)}) \|_2^2 + \tau \sum_{j=1}^{n} \| (w_j^{(n)})_t \|_2^2. \]

We have
\[ \| (w_j^{(n)})_t \|_2^2 \leq C \tau f^{(j)}_{t-1} \| (w_j)^\theta \|_{k+2}^2 \, ds. \]
(83)
\[ \| (w_j^{(n)})_t \|_2^2 \leq C h^{2(k+1)} \| u_t \|_{k+2}^2. \]
(84)
And
\[ \sum_{j=1}^{n} \| \nabla_d e_j^{(n)} \|_2^2 \leq C \left( \| e_0^{(0)} \|_2^2 + \| \nabla_d e_0^{(0)} \|_2^2 \right) + h^{2(k+1)} \left( \| u(t) \|_{k+2}^2 + \| u_t(t) \|_{k+2}^2 + \tau^2 \int_0^{\tau_n} \| u_{tt} \|_{k+2}^2 \, ds \right) \]
\[ + \tau^2 \int_0^{\tau_n} \| u_{tt} \|_{k+2}^2 \, ds. \]
(85)

Then by substitution (84),(85) and (86) into (83) we have
\[ \| (w_j^{(n)})_t \|_2^2 \leq C h^{2(k+1)} \| u_t \|_{k+2}^2. \]
(87)

Simplified equation (87) we get the error estimation (76) which completes the proof.

5. Numerical Experiments

In this section, we use the following norms:
\[ \| \nabla_d e_h \| = \left( \sum_{K \in \mathcal{T}_h} \int_K |\nabla_d e_h|^2 \, dx \right)^{1/2} \quad H^1 \text{ semi-norm}, \]
\[ \| e_0 \| = \left( \sum_{K \in \mathcal{T}_h} \int_K |e_0|^2 \, dx \right)^{1/2} \quad \text{Element-based } L^2 \text{ norm}, \]
\[ \| e_b \| = \left( \sum_{e \in \mathcal{E}_h} h_e \int_{e_h} |e_b|^2 \, ds \right)^{1/2} \quad \text{Edge-based } L^2 \text{ norm}, \]
\[ \| e_0 \|_{\infty} = \sup_{x \in K} |e_0(x)|, \quad \| e_b \|_{\infty} = \sup_{x \in \Omega} |e_b(x)|. \]

To represent the error \( e^{(n)} = Q_h u - u_h \) between the \( L^2 \) projection \( Q_h u \) of the exact solution \( u \) and the numerical solution \( u_h \), we use the the discrete weak space \( u_h(0,0) \) and Raviart-Thomas element \( V_h(K) = RT_0 \) as space of discrete weak gradient [13].

5.1. Test problem 1

In this subsection, we present the test problem to illustrate the backward Euler WG finite elements method for the time dependent coupled Burgers’ equations (1)- (2) over a square domain \( \Omega : [0,1] \times [0,1] \). The exact solutions of coupled Burgers equation can be generated by using the Hopf Cole transformation see [26] which are:
\[ u(x,y,t) = \frac{3}{4} - \frac{1}{4 \left(1 + e^{-\frac{-4x+4y-t}{3\epsilon}}\right)}, \]
\[ v(x,y,t) = \frac{3}{4} + \frac{1}{4 \left(1 + e^{-\frac{-4x+4y-t}{3\epsilon}}\right)}. \]
We chose the approximation space $U_h$ or $V_h$

$$U_h = \begin{cases} 
    u_0 \in P_0(K), & \text{for all } K \in T_h \\
    \text{for all } K \in T_h & u_b \in P_0(e) \\
    u_b = 0 & \text{for all } K \in T_h \text{ and } e \subset K \notin \partial \Omega \\
    u_b = \zeta_h & \text{for all } K \in T_h \text{ and } \partial K \notin \partial \Omega.
\end{cases}$$

Table 1 and 2 show the $L^2$ and $H^1$ error with convergence rate for the WG solution on triangle meshes, the triangles mesh is obtained by divided each diagonal line with a negative slope. In the test we use $\tau = 0.01$ and $\epsilon = 1.14$ to check the order of convergence corresponding to time step size $\tau$ and mesh size $h = \frac{1}{n}$ (see Figure 4), the numerical results show that the WG solutions with constant space ($k = 0$) equivalent to slandered finite element solutions with linear space ($k = 1$) with convergence rate $O(h)$ in $H^1$ and $L^2$ norm respectively, this results are show in Figure 1, 2 and 3.

$$\begin{array}{|c|c|c|c|}
\hline
h & |\nabla \epsilon| & |\epsilon|_{L^2,K} & |\epsilon|_{L^2,\partial K} \\
\hline
1/2 & 1.9083e-03 & 8.5436e-05 & 5.0696e-04 \\
1/4 & 1.0157e-03 & 3.0473e-05 & 2.2705e-04 \\
1/8 & 5.1980e-04 & 1.1338e-05 & 8.9754e-05 \\
1/16 & 2.6310e-04 & 6.9307e-06 & 5.4778e-05 \\
1/32 & 1.2190e-04 & 6.2042e-06 & 3.9415e-05 \\
\hline
\end{array}$$

Table 1: $L^2$ and $H^1$ error with convergence rate for $u$.

$$\begin{array}{|c|c|c|c|}
\hline
h & |\nabla \epsilon| & |\epsilon|_{L^2,K} & |\epsilon|_{L^2,\partial K} \\
\hline
1/2 & 4.7662e-03 & 1.6356e-05 & 1.0062e-03 \\
1/4 & 2.4140e-03 & 4.4744e-05 & 2.2705e-04 \\
1/8 & 1.2190e-03 & 1.1338e-05 & 8.9754e-05 \\
1/16 & 6.1228e-04 & 6.9307e-06 & 5.4778e-05 \\
1/32 & 3.0755e-04 & 6.2042e-06 & 3.9415e-05 \\
\hline
\end{array}$$

Table 2: $L^2$ and $H^1$ error with convergence rate for $v$. 

![Figure 1: Numerical and Exact solutions for u and v in case ($T = 1, \tau = 0.01, \epsilon = 1.14$)](image-url)
Figure 2: The Error $\| \nabla u \|$ for $u(x, y, t)$.

Figure 3: The Error $\| \nabla v \|$ for $v(x, y, t)$.
5.2. Test problem 2

In this subsection, we test the of two dimension coupled Burgers’ equations (1)- (2) over a square domain \( \Omega: [0,1] \times [0,1] \). The exact solutions of two dimension coupled Burgers’ equation [6] are:

\[
\begin{align*}
    u(x, y, t) &= -2\epsilon \frac{2\pi e^{-5\pi^2t\epsilon} \cos(2\pi x) \sin(\pi y)}{2 + e^{-5\pi^2t\epsilon} \sin(2\pi x) \sin(\pi y)} \\
    v(x, y, t) &= -2\epsilon \frac{\pi e^{-5\pi^2t\epsilon} \sin(2\pi x) \cos(\pi y)}{2 + e^{-5\pi^2t\epsilon} \sin(2\pi x) \sin(\pi y)}.
\end{align*}
\]

Different computational meshes are used and the time step for computation satisfies

\[
\Delta t = cfl \times \text{min} h^2
\]

where \( cfl \) is a parameter dependent on the problem and \( \text{min} h \) is the shortest length of all the triangles. The boundary and initial conditions are taken from the exact solution, To check the convergence we set \( \epsilon = 0.0001 \).

Table 3 and 4 show that the \( L^2 \) and \( H^1 \) error with respect to the velocity \( u \) and \( v \), while table 5 and 6 show the comparison between WG-FEM and LDG(Local Discontinuous Galerkin) [6] in \( L^{\infty} \) error with \( \epsilon = 0.1 \) and \( cfl = 0.05 \), both methods use a linear element and mesh size \( h = \frac{1}{n}, \ n = 5,10,15,20, \) figure 5 and 6 show that the match the numerical solution with exact with respect to \( u \) and \( v \).

| \( h \) | \( \| \nabla u \| \) | \( \| u \|_{L^2(K)} \) | \( \| u \|_{L^2, e(K)} \) |
|-------|----------------|-----------------|----------------|
| 1/2   | 7.3211e-06     | 4.3719e-07      | 1.9827e-04     |
| 1/4   | 6.2856e-06     | 7.4686e-07      | 1.3193e-04     |
| 1/8   | 3.7400e-06     | 8.5142e-07      | 6.4875e-05     |
| 1/16  | 2.0586e-06     | 7.2417e-07      | 2.7605e-05     |
| 1/32  | 1.3431e-06     | 6.2042e-07      | 1.9415e-05     |

Table 3: \( L^2 \) and \( H^1 \) error for \( u \) in case \( T = 1, \epsilon = 0.0001 \) and \( cfl = 0.05 \).

| \( h \) | \( \| \nabla v \| \) | \( \| v \|_{L^2(K)} \) | \( \| v \|_{L^2, e(K)} \) |
|-------|----------------|-----------------|----------------|
| 1/2   | 5.4542e-06     | 4.1089e-07      | 1.5056e-04     |
| 1/4   | 2.3448e-06     | 4.0939e-07      | 5.2950e-05     |
| 1/8   | 1.4174e-06     | 3.6926e-07      | 2.3645e-05     |
| 1/16  | 7.8946e-07     | 3.2682e-07      | 1.0094e-05     |
| 1/32  | 1.3431e-07     | 2.2042e-07      | 1.0005e-05     |

Table 4: \( L^2 \) and \( H^1 \) error for \( v \) in case \( T = 1, \epsilon = 0.0001 \) and \( cfl = 0.05 \).

| \( h \) | \( \| u \|_{\infty, \text{LDG}} \) | \( \| u \|_{L^\infty, K} \) | \( \| u \|_{L^\infty, e(K)} \) |
|-------|----------------|-----------------|----------------|
| 1/5   | 3.2995e-03     | 1.3028e-04      | 0              |
| 1/10  | 8.5481e-04     | 2.8206e-05      | 0              |
| 1/15  | 1.5239e-04     | 1.1931e-05      | 1.2958e-07     |
| 1/20  | 6.8944e-05     | 6.5827e-06      | 7.8937e-07     |

Table 5: \( L^{\infty} \) error for \( u \) in case \( T = 1, \epsilon = 0.1 \) and \( cfl = 0.05 \).

| \( h \) | \( \| v \|_{\infty, \text{LDG}} \) | \( \| v \|_{L^\infty, K} \) | \( \| v \|_{L^\infty, e(K)} \) |
|-------|----------------|-----------------|----------------|
| 1/5   | 3.1717e-03     | 4.2835e-05      | 5.5983e-05     |
| 1/10  | 7.8166e-04     | 6.5973e-06      | 2.1992e-05     |
| 1/15  | 1.3796e-04     | 2.6331e-06      | 1.6046e-05     |
| 1/20  | 5.5051e-05     | 1.4301e-06      | 1.2764e-05     |

Table 6: \( L^{\infty} \) error for \( v \) in case \( T = 1, \epsilon = 0.1 \) and \( cfl = 0.05 \).
Figure 5: match the Numerical solution with Exact solution for a test problem 2.

Figure 6: Numerical and Exact solution for a test problem 2.

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