INCOMPRESSIBLE HYDRODYNAMIC APPROXIMATION WITH VISCOUS HEATING TO THE BOLTZMANN EQUATION

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ABSTRACT. The incompressible Navier-Stokes-Fourier system with viscous heating was first derived from the Boltzmann equation in the form of the diffusive scaling by Bardos-Levermore-Ukai-Yang (2008). The purpose of this paper is to justify such an incompressible hydrodynamic approximation to the Boltzmann equation in \( L^2 \cap L^\infty \) setting in a periodic box. Based on an odd-even expansion of the solution with respect to the microscopic velocity, the diffusive coefficients are determined by the incompressible Navier-Stokes-Fourier system with viscous heating and the super Burnett functions. More importantly, the remainder of the expansion is proven to decay exponentially in time via an \( L^2 - L^\infty \) approach on the condition that the initial data satisfies the mass, momentum and energy conversation laws.

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1. Introduction

1.1. The problem. This paper is concerned with the connection between the incompressible fluid dynamical equations with viscous heating and the Boltzmann equation in a periodic box. In the diffusive regime, the time evolution of the dilute gas is governed by the following rescaled Boltzmann equation:

\[
\epsilon \partial_t F + v \cdot \nabla_x F = \frac{1}{\epsilon} Q(F, F), \quad x \in \mathbb{T}^3, \quad v \in \mathbb{R}^3,
\]

with initial data

\[
F(0, x, v) = F_0(x, v), \quad x \in \mathbb{T}^3, \quad v \in \mathbb{R}^3.
\]

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Here, \( F(t, x, v) \geq 0 \) is the distribution function of particles at time \( t \in \mathbb{R}_+ \), position \( x \in [-\pi, \pi]^3 = \mathbb{T}^3 \) and velocity \( v \in \mathbb{R}^3 \), and \( \epsilon > 0 \) is the Knudsen number which is proportional to the mean free path.

\( Q(\cdot, \cdot) \) in (1.1) is the Boltzmann collision operator, which for the hard sphere model takes the following non-symmetric form:

\[
Q(F, H) = \int_{\mathbb{R}^3 \times S^2_+} \left( F(v') H(v') - F(v) H(v) \right) |(v - v_*) \cdot \omega| dv_* d\omega,
\]

where \( S^2_+ = \{ \omega \in S^2 : (v - v_*) \cdot \omega \geq 0 \} \) and \( (v, v_*), (v', v'_*) \), denote velocities of two particles before and after an elastic collision respectively, satisfying

\[
v' = v - [(v - v_*) \cdot \omega] \omega, \quad v'_* = v_* + [(v - v_*) \cdot \omega] \omega.
\]

Recently, there have been great interest \cite{2, 13, 16, 17, 20, 35, 42} in studying the rescaled heating to the Boltzmann equation in a periodic box. Let

\[
\mu(v) = M_{[1,0,1]} = \frac{1}{(2\pi)^3} e^{-\frac{|v|^2}{2}}.
\]

The odd-even expansion \cite{4} suggests that the solution of (1.1) can be written as

\[
F = \mu + \epsilon \sqrt{\mu} \left\{ f_1 + \epsilon f_2 + \epsilon^2 f_3 + \epsilon^3 f_4 + \epsilon^4 f_5 + \epsilon^5 f_6 + \epsilon^{4-\beta} R \right\}, \quad 0 < \beta < 1/2,
\]

where

\[
f_1, f_3 \text{ and } f_5 \text{ are odd in } v, \quad \text{while } f_2, f_4 \text{ and } f_6 \text{ are even in } v.
\]

Plugging (1.3) into (1.1) and comparing the coefficients on both side of the resulting equation, we obtain for \( \epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3, \epsilon^4 \) and \( \epsilon^5 \)

\[
L f_1 = 0,
\]

\[
\partial_t f_1 + v \cdot \nabla_x f_1 + L f_2 = \Gamma(f_1, f_1),
\]

\[
\partial_t f_2 + v \cdot \nabla_x f_2 + L f_3 = \Gamma(f_1, f_2) + \Gamma(f_2, f_1),
\]

\[
\partial_t f_3 + v \cdot \nabla_x f_3 + L f_4 = \Gamma(f_2, f_3) + \Gamma(f_3, f_2) + \Gamma(f_4, f_1) + \Gamma(f_1, f_4),
\]

\[
\partial_t f_4 + v \cdot \nabla_x f_4 + L f_5 = \Gamma(f_3, f_4) + \Gamma(f_4, f_3) + \Gamma(f_5, f_2) + \Gamma(f_2, f_5) + \Gamma(f_5, f_1),
\]

and the equation for the remainder \( R \)

\[
\epsilon \partial_t R + v \cdot \nabla_x R + \frac{1}{\epsilon} LR = \{ \Gamma(f_1, R) + \Gamma(R, f_1) \} + \epsilon \{ \Gamma(f_2, R) + \Gamma(R, f_2) \} + \epsilon^2 \{ \Gamma(f_3, R) + \Gamma(R, f_3) \} + \epsilon^3 \{ \Gamma(f_4, R) + \Gamma(R, f_4) \} + \epsilon^4 \{ \Gamma(f_5, R) + \Gamma(R, f_5) \} + \epsilon^5 \{ \Gamma(f_6, R) + \Gamma(R, f_6) \} + \epsilon^{4-\beta} \Gamma(R, R) - \epsilon^{1+\beta} \{ \partial_t f_5 + v \cdot \nabla_x f_6 \} - \epsilon^{2+\beta} \partial_t f_6,
\]

with

\[
R(0, x, v) = R_0(x, v).
\]
Here, the linear collision operator $L$ and nonlinear collision operator $\Gamma$ are defined as

$$Lg = -\frac{1}{\sqrt{\mu}}\{Q(\mu, \sqrt{\mu}g) + Q(\sqrt{\mu}g, \mu)\},$$

and

$$\Gamma(g, h) = \frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}g, \sqrt{\mu}h),$$

respectively. The null space of $L$ denoted by $\mathcal{N}(L)$ is generated by $[\sqrt{\mu}, v, \sqrt{\mu}v^2]$, thus for any function $g(t, x, v)$, we can decompose it as follows

$$g = Pg + \{I - P\}g,$$

where $Pg$ is the $L^2_v$–projection of $g$ on the null space for $L$ for given $(t, x)$ and we can further denote $Pg$ by

$$Pg = \left\{ \rho_g(t, x) + v \cdot u_g(t, x) + \frac{|v|^2 - 3}{2} \theta_g(t, x) \right\} \sqrt{\mu}.$$ 

Here $\rho_g(t, x), u_g(t, x)$, and $\theta_g(t, x)$ also represent the density, velocity, and temperature fluctuation physically respectively. It is traditional to call $Pg$ the macroscopic part and $\{I - P\}g$ the microscopic part. In addition, the linearized Boltzmann collision operator $L$ satisfies

$$Lg = \nu(v)g - Kg,$$

where $\nu(v)$ is called the collision frequency which is given by

$$\nu(v) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} |(v - v_s) \cdot \omega| \mu(u)B(\theta)dv_sd\omega \sim \langle v \rangle = \sqrt{1 + |v|^2},$$

and operator $K = K_2 - K_1$ is defined as in the following

$$\begin{cases} [K_1g](v) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} |(v - v_s) \cdot \omega| \mu^{\frac{1}{2}}(v_s)\mu^{\frac{1}{2}}(v)g(v_s)dv_sd\omega, \\
[K_2g](v) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} |(v - v_s) \cdot \omega| \mu^{\frac{1}{2}}(v_s) \left\{ \mu^{\frac{1}{2}}(v'_s)g(v') + \mu^{\frac{1}{2}}(v'_s)g(v'_s) \right\} dv_sd\omega. \end{cases} \tag{1.13}$$

It is well known that $L \geq 0$ and there exists $\delta_0 > 0$ such that

$$\langle Lg, g \rangle \geq \delta_0 \|\{I - P\}g\|^2_\nu.$$

For later use, we also define the following Burnett functions $A(v)$ and $B(v)$ as

$$A(v) = (A(v)_{ij})_{3 \times 3} = \left\{ v \otimes v - \frac{1}{3}|v|^2I \right\} \sqrt{\mu}, \quad B(v) = (B_j(v))_{3 \times 1} = \frac{|v|^2 - 5}{2} v \sqrt{\mu},$$

where $I$ is the identity matrix.
From (1.4), (1.5), (1.6), (1.7), (1.8), (1.9) and (1.10), one can further define $f_1, f_2, f_3, f_4, f_5$, and $f_6$ as follows:

$$
\begin{align*}
  f_1 &= u_1 \cdot v \sqrt{\mu}, \\
  f_2 &= \left\{ \rho_1 + \frac{1}{6}(|u_1|^2 + 3\theta_1)(|v|^2 - 3) \right\} \sqrt{\mu} - L^{-1}[A(v) \cdot \nabla_x u_1 + \frac{1}{2}A(v) : u_1 \otimes u_1], \\
  f_3 &= u_2 \cdot v \sqrt{\mu} + \rho_1 u_1 \cdot v \sqrt{\mu} + L^{-1}\{ -v \cdot \nabla_x f_2 + \Gamma(f_1, f_2) + \Gamma(f_2, f_1) \}, \\
  f_4 &= \left\{ \rho_2 + \frac{1}{6}(2u_1 \cdot u_2 + 3\theta_2)(|v|^2 - 3) + \frac{1}{6}\rho_1 u_1^2(|v|^2 - 3) \right\} \sqrt{\mu} + \rho_1 \theta_1 \frac{|v|^2 - 3}{2} \sqrt{\mu} + \frac{\theta_1^2}{2} \sqrt{\mu} + L^{-1}\{ -\partial_t f_2 - v \cdot \nabla_x f_3 + \Gamma(f_2, f_2) + \Gamma(f_1, f_3) + \Gamma(f_3, f_1) \}, \\
  f_5 &= u_3 \cdot v \sqrt{\mu} + \rho_1 u_2 \cdot v \sqrt{\mu} + \rho_2 u_1 \cdot v \sqrt{\mu} + L^{-1}\{ -\partial_t f_3 - v \cdot \nabla_x f_4 + \Gamma(f_2, f_3) + \Gamma(f_3, f_2) + \Gamma(f_1, f_4) + \Gamma(f_4, f_1) \}, \\
  f_6 &= L^{-1}\{ -\partial_t f_4 - v \cdot \nabla_x f_5 + \sum_{i+j=6} \{ \Gamma(f_i, f_j) + \Gamma(f_j, f_i) \} \}.
\end{align*}
$$

Here, we can take $P f_6 = 0$, since the expansion is truncated. It is worth stressing that the macroscopic parts of $f_1, f_2, f_3, f_4$ and $f_5$ stem from the following Taylor expansion of the local Maxwellian:

$$
M_{[1 + \epsilon_1^2 \rho_1 + \epsilon_2 \rho_2, \epsilon u_1 + \epsilon_3^{u_2} + \epsilon_7^{u_3} \cdot \epsilon^{\theta_1} + \epsilon_8^{\theta_2}]} \nonumber
= \frac{1 + \epsilon_1^2 \rho_1 + \epsilon_2^2 \rho_2}{(2\pi(1 + \epsilon_1^2 \theta_1 + \epsilon_2^2 \theta_2))^{3/2}} e^{-\frac{|v - u_1 - 3u_2 - 5u_3|^2}{2(1 + \epsilon_1^2 \theta_1 + \epsilon_2^2 \theta_2)}} e^{\left\{ \mu v \cdot \epsilon \frac{u_1}{2} + \frac{1}{6} \left( \rho_1 ((|u_1|^2 + 3\theta_1)(|v|^2 - 3) + \frac{1}{2}A(v) : u_1 \otimes u_1) \right) \right\} + \epsilon^3 \left\{ u_2 \cdot v + \rho_1 u_1 \cdot v + \theta_1 u_1 \cdot B(v) + \frac{1}{6} P_1(v) : u_1 \otimes u_1 \otimes u_1 \right\} + \epsilon^4 \left\{ \rho_2 + \frac{1}{6} (2u_1 \cdot u_2 + 3\rho_1 \theta_1 + 3\theta_2)(|v|^2 - 3) + \theta_1^2 \frac{(|v|^2 - 3)(|v|^2 - 5)}{4} + A(v) : u_1 \otimes u_2 + \frac{1}{2} \rho_1 (v \otimes v - I) : u_1 \otimes u_1 + \frac{1}{2} \theta_1 \frac{(|v|^2 - 7v \otimes v - I)(|v|^2 - 5)}{2} : u_1 \otimes u_1 + \frac{1}{24} P_2(v) : u_1 \otimes u_1 \otimes u_1 \right\} + \epsilon^5 \left\{ u_3 \cdot v + \rho_1 u_2 \cdot v + \rho_2 u_1 \cdot v + \theta_1 u_2 \cdot B(v) + \theta_2 u_1 \cdot B(v) + \rho_1 \theta_1 u_1 \cdot B(v) + \frac{1}{2} P_1(v) : u_1 \otimes u_1 \otimes u_2 + \frac{1}{6} \rho_1 P_1(v) : u_1 \otimes u_1 \otimes u_1 + \frac{1}{6} \theta_1 (v \otimes v \cdot \frac{|v|^2 - 9}{2} - 3I \otimes \frac{|v|^2 - 3}{2} + 6v \otimes I) : u_1 \otimes u_1 \otimes u_1 + \frac{1}{120} P_3(v) : u_1 \otimes u_1 \otimes u_1 \otimes u_1 \right\} + O(\epsilon^6) \right\},
$$

or $M_{[1 + \epsilon_1^2 \rho_1 + \epsilon_2 \rho_2, \epsilon u_1 + \epsilon_3^{u_2} + \epsilon_7^{u_3} \cdot \epsilon^{\theta_1} + \epsilon_8^{\theta_2}]}$.
where $P_1\sqrt{\mu}, P_2\sqrt{\mu}$ and $P_3\sqrt{\mu}$ are the super Burnett functions given by
\[
\begin{aligned}
P_1 &= v \otimes v \otimes v - 3I \otimes v, \\
P_2 &= v \otimes v \otimes v \otimes v - 6I \otimes v \otimes v + 3I, \\
P_3 &= v \otimes v \otimes v \otimes v \otimes v - 10I \otimes v \otimes v \otimes v + 15I \otimes v.
\end{aligned}
\]

Moreover, here and in the sequel, we define $M = (a_{ij})$ and $N = (b_{ij})$. It is also straightforward to check that $P_1(v) : u_1 \otimes u_1 \otimes u_1$, $P_2(v) : u_1 \otimes u_1 \otimes u_1 \otimes u_1$, $P_3(v) : u_1 \otimes u_1 \otimes u_1 \otimes u_1 \otimes u_1, \left(\frac{|u|^2-7}{2}v \otimes v - I\frac{|u|^2-5}{2}\right)$ and $(v \otimes v \otimes v|v|^2-3 - 3I \otimes v|v|^2-3 + 6v \otimes I)$ : $u_1 \otimes u_1 \otimes u_1$ all belong to the orthogonal complement of $\mathcal{N}(L)$.

In addition, (1.6), (1.7) and (1.8) give rise to the following so-called incompressible Navier-Stokes-Fourier equations with viscous heating
\[
\begin{aligned}
&\nabla \cdot u_1 = 0, \\
&\partial_t u_1 + u_1 \cdot \nabla u_1 + \nabla p_1 = \mu_\star \Delta u_1, \quad p_1 = \rho_1 + \theta_1, \\
&\partial_t \left(\frac{3}{2} \theta_1 - \rho_1\right) + u_1 \cdot \nabla \left(\frac{3}{2} \theta_1 - \rho_1\right) = \kappa_\star \Delta \theta_1 + \frac{1}{2} \mu_\star \left|\nabla u_1 + (\nabla u_1)^T\right|^2, \\
&\rho_1(x, 0) = \rho_{1,0}(x), \quad u_1(0, x) = u_{1,0}(x), \quad \theta_1(0, x) = \theta_{1,0}(x),
\end{aligned}
\] (1.15)

and (1.8), (1.9) and (1.10) lead us to
\[
\begin{aligned}
&\partial_t \rho_1 + \nabla \cdot u_2 + \nabla \cdot (\rho_1 u_1) = 0, \\
&\partial_t u_2 + u_1 \nabla \cdot u_2 + \nabla u_1 \cdot u_2 + u_1 \cdot \nabla u_2 + \nabla \left(\rho_2 + \theta_2 - \frac{1}{3} u_1 \cdot u_2\right) \\
&= \mu_\star \Delta u_2 + \frac{\mu_\star}{3} \nabla \cdot \nabla \cdot \{I - P_0\} u_2 + \nabla \cdot \langle \partial_t f_2, L^{-1} A(v) \rangle \\
&- \nabla \cdot \left(\Gamma(f_2, f_2), L^{-1} A(v)\right) - \nabla \cdot \left(\Gamma(f_1, \{I - P\} f_3) + \Gamma(\{I - P\} f_3, f_1), L^{-1} A(v)\right) \\
&- \partial_t (\rho_1 u_1) - 2 \nabla \cdot (\rho u_1 \otimes u_1) - \nabla (\rho_1 \theta_1 + \frac{5}{2} \theta_1^2 - \frac{1}{3} \rho_1 |u_1|^2) \\
&+ \mu_\star \Delta (\rho_1 u_1) + \frac{\mu_\star}{3} \nabla \cdot \nabla \cdot (\rho_1 u_1), \\
&\partial_t \left(\frac{3}{2} \theta_2 - \rho_2\right) + \frac{5}{2} \nabla \cdot (u_1 \theta_2) + \frac{5}{6} \nabla \cdot (u_1 u_1 \cdot u_2) \\
&= \kappa_\star \Delta \theta_2 + \kappa_\star (\rho_1 \theta_1 + \theta_1^2) + \frac{2 \kappa_\star}{3} (\Delta (u_1 \cdot u_2) + \kappa_\star \Delta (\rho_1 u_1^2) - \frac{1}{2} \partial_t (2 u_1 \cdot u_2 + \rho_1 u_1^2 + 3 \rho_1 \theta_1) \\
&- \frac{5}{2} \nabla \cdot (u_1 (\rho_1 \theta_1 + \theta_1^2)) - \frac{5}{6} \nabla \cdot (u_1 \rho u_1^2) \\
&- \nabla \cdot \langle L^{-1} \{ - \partial_t f_3 - v \cdot \nabla \{I - P\} f_4 + \Gamma(f_2, f_3) + \Gamma(f_3, f_2) \}, B(v) \rangle \\
&- \nabla \cdot \langle L^{-1} \{ \Gamma(f_1, \{I - P\} f_3) + \Gamma(\{I - P\} f_3, f_1) \}, B(v) \rangle, \\
&\partial_t \rho_2 + \nabla \cdot u_3 + \nabla \cdot (\rho_1 u_2) + \nabla \cdot (\rho_2 u_1) = 0, \\
&\rho_2(0, x) = \rho_{2,0}(x), \quad u_2(0, x) = u_{2,0}(x), \quad \theta_2(0, x) = \theta_{2,0}(x).
\end{aligned}
\] (1.16)
Moreover, \( \mu_s = \frac{1}{10} (L^{-1}[A(v)], A(v)) \) and \( \kappa_s = \frac{1}{3} (L^{-1}[B(v)], B(v)) \) represent the viscosity and heat conductivity, respectively. It should be pointed out that

\[
\frac{1}{2} \mu_s \left| \nabla u_1 + (\nabla u_1)^T \right|^2 \overset{\text{def}}{=} \frac{1}{2} \mu_s \text{trace}\left((\nabla u_1 + (\nabla u_1)^T)^2\right)
\]

\[
\overset{\text{def}}{=} \frac{1}{2} \mu_s (\nabla u_1 + (\nabla u_1)^T) : (\nabla u_1 + (\nabla u_1)^T)
\]

\[
= \mu_s \left( \sum_{i,j} \partial_j u_i^1 \partial_i u_i^1 + \sum_i (\partial_i u_i^1)^2 \right)
\]

is the viscous heating term, which does not appear in the classical INSF equations, cf. [2] [20].

1.3. **Main results.** For \( l \geq 0 \), denote \( u_l = \langle v \rangle^l = (1 + |v|^2)^{l/2} \). We now state our main results as follows:

**Theorem 1.1.** Let \( F_0(x, v) = \mu + \epsilon \sqrt{\mu} \left\{ \sum_r \epsilon^{r-1} f_r(0, x, v) + \epsilon^{4-\beta} R_0(x, v) \right\} \geq 0 \) with \( 0 < \beta < 1/2 \). Assume

\((A_1) : f_r(0, x, v) (r = 1, 2, \cdots, 6) \) possess the zero-mean hydrodynamic fields:

\[
(f_r(0, x, v), [1, v, (v^2 - 3)] \sqrt{\mu}) = 0,
\]

namely,

\[
\left\{ \begin{array}{l}
\int_{T^3} \rho_{1,0} dx = \int_{T^3} \rho_{2,0} dx = 0, \\
\int_{T^3} (3\theta_{1,0} + |u_{1,0}|^2) dx = \int_{T^3} (3\theta_{2,0} + 2u_{1,0} \cdot u_{2,0} + \rho_{1,0}|u_{1,0}|^2 + 3\rho_{1,0}\theta_{1,0}) dx = 0, \\
\int_{T^3} u_{1,0} dx = \int_{T^3} (u_{2,0} + \rho_{1,0}u_{1,0}) dx = \int_{T^3} (u_{3,0} + \rho_{1,0}u_{2,0} + \rho_{2,0}u_{1,0}) dx = 0,
\end{array} \right.
\]

in particular, the velocity fields also satisfy

\[
P_0 u_{r,0} = u_{r,0} \text{ for } r = 1, 2, 3,
\]

and there exists a sufficiently small \( \epsilon_0 > 0 \) such that

\[
\|u_{1,0}\|_{H^4} + \|\theta_{1,0}\|_{H^4} \leq \epsilon_0;
\]

\((A_2) : \) for \( l > 3/2, \epsilon^{3/2} \|w_l R_0\|_\infty + \|R_0\|_2 \) is sufficiently small and

\[
(R_0(x, v), [1, v, v^2] \sqrt{\mu}) = 0.
\]

Then the Cauchy problem \((1.1)\) and \((1.2)\) admits a unique global solution

\[
F(t, x, v) = \mu + \epsilon \sqrt{\mu} \left\{ \sum_{r=1}^{6} \epsilon^{r-1} f_r + \epsilon^{4-\beta} R \right\} \geq 0,
\]
with \( f_r \) \((r = 1, 2, \cdots, 6)\) satisfying (1.14), (1.15) and (1.16) and \( R \) satisfying (1.11) and (1.12), respectively. Moreover, there exists a constant \( \lambda > 0 \) and a polynomial \( P \) with \( P(0) = 0 \) such that for any \( t \geq 0 \) and \( l > 3/2 \)

\[
e^{3/2} \|w_1 R(t)\|_\infty + \|R(t)\|_2 \\
\leq C e^{-\lambda t} \left\{ e^{3/2} \|w_1 R_0\|_\infty + \|R_0\|_2 + e^\beta P (\|u_{1,0,\theta_{1,0}}\|_{H^{16}} + \|u_{2,0,\theta_{2,0}}\|_{H^{14}}) \right\}.
\]

A great amount of effort has been paid on the study of the hydrodynamic approximation (limits) to the Boltzmann equation, since the pioneering work by Hilbert, who introduce his famous expansion in terms of Knudsen number \( \epsilon \) in [25] to explore the connection between the fluid dynamics and the Boltzmann equation. Grad [19] and Nishida [44] investigated the asymptotic equivalence of the Boltzmann equation and the compressible Euler equations for gas dynamics, while Caflisch [6] and Lachowiz [33] also studied the same issue by different methods. Mathematical descriptions on the closeness of the Chapman-Enskog expansion [7] of the Boltzmann equation to the solutions of the compressible Navier-Stokes equations were obtained by Lachowiz [34], Kawashima-Matsumura-Nishida [32] and Liu-Yang-Zhao [38].

In the context of diffusive scaling, the problem can be faced only in the low mach number regime, in this situation, the Boltzmann solution shall be close to the INSF system, see in particular, a formal derivation by Bardos-Golse-Levermore [1] and [2] for a general momentum argument of deriving global Levay solution of INSF from global renormalized solution [9] of the Boltzmann equation with additional assumption which remained unverified. Later on, there are a huge number of papers concerning this topic, see [16, 18, 30, 31, 35, 36, 42, 45, 3]. We point out that some of assumptions in [2] have been removed in those works. A full proof for the INSF limits of the Boltzmann equation has been given by Golse-Saint-Raymond [17]. There also have been extensive investigation on the convergence of the smooth solutions of the INSF system to the Boltzmann equation, see [3, 8, 14, 20, 39, 47].

We also mention that when the solutions of the Boltzmann equation are a small perturbation of some nontrivial profiles, for instance, some basic wave patterns, stationary solutions, time-periodic solutions, etc., the time-asymptotic equivalence of the Boltzmann equation and the compressible Navier-Stokes equation are also studied, cf. [10, 11, 26, 27, 28, 29, 40, 41, 48, 49, 50, 51] and the references cited therein.

Recently, a new model called the INSF system with viscous heating was derived by Bardos-Levermore-Ukai-Yang [4]. The aim of the present paper is to justify such an incompressible hydrodynamic approximation to the Boltzmann equation in a periodic box via an \( L^2 - L^\infty \) method developed in [12, 13, 21, 22, 23, 24]. We now outline a few key points of the paper which are distinct to some extent with the previous work by Bardos-Levermore-Ukai-Yang [4]:

- The odd-even decomposition of the rescaled Boltzmann equation is more complex and accurate, namely, we expand the solution of the Boltzmann equation up to sixth order with a remainder.
- To determine the diffusive coefficients, we introduce the super Burnett functions which play a vital role in defining the macroscopic parts of the diffusive coefficients.
- A good structure of the INSF system with viscous heating is observed so that the smooth solutions of the macroscopic equations are obtained via an elementary energy method.
- We design an elaborate space \( X_\delta \) to capture the properties of the solution of the remainder equation in \( L^2 \cap L^\infty \) setting.

The organization of the paper is as follows. Section 2 contains some elementary identities and estimates regarding the Boltzmann collision operators. We provide a direct approach to derive the INSF equations with viscous heating and present the construction of the diffusive coefficients in Section 3. Sections 4 and 5 are devoted to the \( L^2 \) and \( L^\infty \) estimates of the linear
equation of the remainder, respectively. The proof of our main result Theorem 1.1 is concluded in Section 6.

1.4. Notations and Norms. Throughout this paper, \( C > 0 \) stands for the generic positive constant and \( \lambda, \lambda_1, \lambda_2 \) as well as \( \lambda_0 \) denote some generic positive (generally small) constants, where \( C \) may take different values in different places. \( D \leq E \) means that there is a generic constant \( C > 0 \) such that \( D \leq C E \). \( D \sim E \) means \( D \lesssim E \) and \( E \lesssim D \). Let \( 1 \leq p \leq \infty \), we denote \( \| \cdot \|_p \) either the \( L^p(\mathbb{T}^3 \times \mathbb{R}^3) \)-norm or the \( L^p(\mathbb{T}^3) \)-norm, and denote \( \| \cdot \|_\nu \equiv \| \cdot \|_2^{1/2} \). Moreover, \( (\cdot , \cdot) \) denotes the \( L^2 \) inner product in \( \mathbb{T}^3 \times \mathbb{R}^3 \) or \( \mathbb{T}^3 \) with the \( L^2 \) norm \( \| \cdot \|_2 \), and \( (\cdot , \cdot) \) stands for the \( L^2 \) inner product in \( \mathbb{R}^3 \).

2. Preliminary

In this section, we give some basic identities and significant estimates which will be used in the later proofs. The first one is concerned with the relations between the nonlinear operator \( \Gamma \) and linear operator \( L \).

**Lemma 2.1.** It holds that

\[
\Gamma(Pg, Pg) = \frac{1}{2} L \left\{ \frac{(Pg)^2}{\mu} \right\}, \quad \Gamma(Pg, (Pg)^2 \mu^{-1/2}) + \Gamma((Pg)^2 \mu^{-1/2}, Pg) = \frac{1}{3} L \left\{ \frac{(Pg)^3}{\mu} \right\}. \tag{2.1}
\]

**Proof.** The first identity in (2.1) has been proved in [20, pp.648-649], the second one can be verified similarly, we omit the details for simplicity. This completes the proof of Lemma 2.1 \( \Box \)

The following significant relations are quoted from [2, Lemma 4.4, pp.711] and [4, Proposition 2.5, pp.17] as well as [3] (2.36)-(2.36), pp.16-17.

**Lemma 2.2.** It holds that

\[
( A_{ij}(v), L^{-1} A_{kl}(v) ) = \frac{1}{10} \langle A(v) : L^{-1} A(v) \rangle \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{3} \delta_{ij} \delta_{kl} \right)
= \mu_* \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{3} \delta_{ij} \delta_{kl} \right), \tag{2.2}
\]

\[
( B_i(v), L^{-1} B_j(v) ) = \frac{1}{3} \langle B(v) \cdot L^{-1} B(v) \rangle \delta_{ij} = \kappa_* \delta_{ij}, \tag{2.3}
\]

\[
( A_{ij}(v), v_k L^{-1} B_l(v) ) = ( A_{ik}(v), v_j L^{-1} B_l(v) ) = \frac{2}{3} \kappa_* ( \delta_{ik} \delta_{jl} - \delta_{ij} \delta_{kl} ), \tag{2.4}
\]

and

\[
( L^{-1} A_{ij}(v), v_k L^{-1} B_l(v) ) = \frac{1}{10} \langle L^{-1} A(v) : v \otimes L^{-1} B(v) \rangle \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{3} \delta_{ij} \delta_{kl} \right). \tag{2.5}
\]

In addition, it follows

\[
( \Gamma(v_i \sqrt{\mu}, L^{-1} A_{kl}(v)) + \Gamma(L^{-1} A_{kl}(v), v_i \sqrt{\mu}) , L^{-1} B_j(v) ) + \langle v_i A_{kl}(v), L^{-1} B_j(v) \rangle = ( A_{ij}(v) : L^{-1} A_{kl}(v) ). \tag{2.6}
\]

Let us now report the result which can be directly proved by the definition of \( \Gamma \).

**Lemma 2.3.** Let \( p_1(v) \) and \( p_2(v) \) be any polynomials in \( v \), then for any functions \( a(t, x) \) and \( b(t, x) \), there exist constants \( c_1, c_2 \in (0, 1/4) \) such that

\[
|ab| \mu^{c_2} \lesssim |\Gamma(a p_1(v) \sqrt{\mu}, b p_2(v) \sqrt{\mu})| \lesssim |ab| \mu^{c_1}.
\]

The following lemma is devoted to the \( L^p \) estimates of the nonlinear operator \( \Gamma \).
Lemma 2.4. It holds that for \( l \geq 0 \),
\[
\|\nu^{-1} w_l \Gamma(f_1, f_2)\|_{\infty} \leq C \|w_l f_1\|_{\infty} \|w_l f_2\|_{\infty},
\]
and for \( l > \frac{3}{2} \),
\[
\|\nu^{-1/2}(\Gamma(f_1, f_2) + \Gamma(f_2, f_1))\|_{2}^2 \leq C \|w_l \nu f_1\|_{\infty}^2 \|f_2\|_{\nu}^2.
\]
In particular, it holds that
\[
\|\nu^{-1/2}(\Gamma(f, f))\|_{2}^2 \leq C \|w_l f\|_{\infty}^2 \|f\|_{\nu}^2,
\]
for \( l > \frac{3}{2} \).

Proof. The proof of (2.7) has been given in [21, Lemma 5, pp.730], and the proofs for (2.8) and (2.9) are similar as that of [37, Lemma 2.3, pp.12].

Recall the definition for \( K \) in (1.13), one can rewrite
\[
[Kf](v) = \int_{\mathbb{R}^3} k(v, v') f(v') dv' = \int_{\mathbb{R}^3} [k_2(v, v') - k_1(v, v')] f(v') dv',
\]
with
\[
k_1(v, v') = \int_{S^2} |(v - v') \cdot \omega| \sqrt{\mu(v)} \sqrt{\mu(v')} d\omega,
\]
and
\[
k_2(v, v') = C |v - v'|^{-1} \exp \left(-\frac{1}{8} |v - v'|^2 - \frac{1}{8} \left(\frac{|v|^2 - |v'|^2}{|v - v'|^2}\right)^2\right).
\]
The following lemma which states the estimates of \( k(v, v') \) is borrowed from Lemma 3 of [21, pp.727] and [15, Lemma 3.3.1, pp.49].

Lemma 2.5. It holds that
\[
\int_{\mathbb{R}^3} k(v, v') dv' \leq \frac{C}{1 + |v|},
\]
and moreover, for any \( l \geq 0 \),
\[
w_l(v) \int_{\mathbb{R}^3} k(v, v') e^{\varepsilon |v - v'|^2} \frac{dv'}{w_l(v')} \leq \frac{C}{1 + |v|},
\]
where \( \varepsilon > 0 \) and sufficiently small.

Finally, we cite the \( L^p - L^q \)-estimate on the Riesz potential [5] on torus.

Lemma 2.6. Assume \( f \in H^s(\mathbb{T}^3) \) with \( \int_{\mathbb{T}^3} f dx = 0 \), define
\[
N_\alpha(f) = \Delta^\alpha \partial_\gamma^\alpha f,
\]
if \( |\gamma| \leq \alpha < 3 \), \( p > 1 \), then we have
\[
\|N_\alpha(f)\|_{L^q} \leq C \|f\|_{L^p}.
\]
Here \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha - |\gamma|}{3} \).

3. INSF equations with viscous heating and diffusive coefficients

One purpose of this section is to show the derivation of the INSF equations (1.15) and (1.16), although a formal one has been given in [4, Section 3, pp.19], here we will propose a more direct approach to derive the INSF equations (1.15). In addition, the \( H^s(\mathbb{T}^3) \) estimates of the solutions of the system (1.15) and (1.16) as well as the estimates for the coefficients \( f_1, f_2, \ldots, f_6 \) will also be deduced.
3.1. Derivation of INSF with viscous heating. Let us now derive the equations \([1.15]\).

Taking the inner product of \([1.6], (1.7)\) and \([1.8]\) with \([v, \sqrt{\mu}, v, \sqrt{\mu}, \frac{|v|^2 - 5}{2}, \sqrt{\mu}]\) with respect to \(v\) over \(\mathbb{R}^3\), respectively, one has

\[
\langle v \cdot \nabla_x f_1, v \sqrt{\mu} \rangle = 0,
\]
\[
\langle \partial_t f_1 + v \cdot \nabla_x f_2, v \sqrt{\mu} \rangle = 0,
\]
\[
\left\langle \partial_t f_2 + v \cdot \nabla_x f_3, \frac{|v|^2 - 5}{2} \sqrt{\mu} \right\rangle = 0.
\]

Substituting \([1.14]_1, [1.14]_2\) and \([1.14]_3\) into the above equations, we further obtain

\[
\nabla_x \cdot u_1 = 0,
\]
\[
\partial_t u_1 + \left\langle v \cdot \nabla_x \left\{ p_1 + \frac{1}{6}(|u_1|^2 + 3\theta_1)(|v|^2 - 3) \right\} \sqrt{\mu}, v \sqrt{\mu} \right\rangle
- \nabla_x \cdot \left\langle A(v) : \nabla_x u_1, L^{-1}A(v) \right\rangle + \frac{1}{2} \nabla_x \cdot \left\langle A(v) : u_1 \otimes u_1, A(v) \right\rangle = 0,
\]

and

\[
\partial_t \left\langle p_1 + \frac{1}{6}(|u_1|^2 + 3\theta_1)(|v|^2 - 3) \right\rangle \sqrt{\mu}, \frac{|v|^2 - 5}{2} \sqrt{\mu} \right\rangle
+ \nabla_x \cdot \left\langle -v \cdot \nabla_x f_2 + \Gamma(f_1, f_2) + \Gamma(f_2, f_1), L^{-1}B(v) \right\rangle = 0,
\]

respectively.

Next, by applying \([2.2]\) in Lemma \([2.2]\) we see that \([3.1]\) is equivalent to

\[
\partial_t u_1 + u_1 \cdot \nabla_x u_1 + \nabla_x p_1 = \mu_4 \Delta u_1,
\]

with \(p_1 = p_1 + \theta_1\).

As to \([3.2]\), the first term on the left hand side gives rise to

\[
\partial_t \left( \frac{1}{2} |u_1|^2 + \frac{3}{2} \theta_1 - \rho_1 \right).
\]

We now calculate \(\nabla_x \cdot \left\langle -v \cdot \nabla_x f_2, L^{-1}B(v) \right\rangle\) and \(\nabla_x \cdot \left\langle \Gamma(f_1, f_2) + \Gamma(f_2, f_1), L^{-1}B(v) \right\rangle\) as follows. One can see that

\[
\nabla_x \cdot \left\langle v \cdot \nabla_x \left\{ L^{-1}A(v) : \nabla_x u_1 \right\}, L^{-1}B(v) \right\rangle = 0.
\]

Indeed, from \([2.5]\), it follows

\[
\sum_{i,j,k,l=1}^{3} \partial_{kl} \left\langle v_k \left\{ L^{-1}A_{ij}(v) \partial_i u_1^j \right\}, L^{-1}B_l(v) \right\rangle
= \sum_{i,j,k,l=1}^{3} \frac{1}{10} \langle L^{-1}A(v) : v \otimes L^{-1}B(v) \rangle \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{3} \delta_{ij} \delta_{kl} \right) \partial_{kl} u_1^j = 0.
\]

Consequently, we have by applying \([2.3]\) that

\[
\nabla_x \cdot \left\langle -v \cdot \nabla_x f_2, L^{-1}B(v) \right\rangle
= -\nabla_x \cdot \left\langle v \cdot \nabla_x \left\{ p_1 + \frac{1}{6}(|u_1|^2 + 3\theta_1)(|v|^2 - 3) \right\} \sqrt{\mu}, L^{-1}B(v) \right\rangle
- \nabla_x \cdot \left\langle v \cdot \nabla_x \left\{ -L^{-1}A(v) : \nabla_x u_1 + \frac{1}{2} A(v) : u_1 \otimes u_1 \right\}, L^{-1}B(v) \right\rangle
= -\kappa_4 \Delta \theta_1 - \kappa_5 \frac{\kappa_4}{3} \Delta (|u_1|^2) - \frac{1}{2} \nabla_x \cdot \left\langle v \cdot \nabla_x \left\{ A(v) : u_1 \otimes u_1 \right\}, L^{-1}B(v) \right\rangle.
\]
By virtue of \[2.1\], we now calculate

\[
\nabla_x \cdot \langle \Gamma(f_1, f_2) + \Gamma(f_2, f_1), L^{-1}B(v) \rangle \\
= \nabla_x \cdot \langle \Gamma(f_1, \mathbf{P}f_2) + \mathbf{P} \Gamma(f_2, f_1), L^{-1}B(v) \rangle \\
+ \nabla_x \cdot \langle \Gamma(f_1, \{I - \mathbf{P}\}f_2) + \{I - \mathbf{P}\} \Gamma(f_2, f_1), L^{-1}B(v) \rangle \\
= \nabla_x \cdot \left\{ u_1 \cdot v \left\{ \rho_1 + \frac{1}{6} \left| u_1 \right|^2 + 3 \theta_1 \left| v \right|^2 - 3 \right\} \right\} \sqrt{\mu}, B(v) \\
+ \nabla_x \cdot \left\langle \left\{ -L^{-1}A(v) : \nabla_x u_1, \frac{1}{2} A(v) : u_1 \otimes u_1 \right\} \right\} \sqrt{\mu}, L^{-1}B(v) \\
+ \nabla_x \cdot \left\{ \left\langle -L^{-1}A(v) : \nabla_x u_1, \frac{1}{2} A(v) : u_1 \otimes u_1 \right\}, u_1 \cdot v \sqrt{\mu} \right\}, L^{-1}B(v) \\
\right. \\
= \nabla_x \cdot \left[ u_1 \left( \frac{1}{2} \theta_1 + \frac{5}{6} \left| u_1 \right|^2 \right) \right] + \frac{1}{2} \nabla_x \cdot \left\{ \left\langle (u_1 \cdot v \sqrt{\mu}, A(v) : u_1 \otimes u_1), L^{-1}B(v) \right\rangle \\
+ \frac{1}{2} \nabla_x \cdot \left\langle (A(v) : u_1 \otimes u_1, u_1 \cdot v \sqrt{\mu}), L^{-1}B(v) \right\rangle \\
- \nabla_x \cdot \left\langle (u_1 \cdot v \sqrt{\mu}, -L^{-1}A(v) : \nabla_x u_1), L^{-1}B(v) \right\rangle \\
- \nabla_x \cdot \left\langle (L^{-1}A(v) : \nabla_x u_1, u_1 \cdot v \sqrt{\mu}), L^{-1}B(v) \right\rangle \right) \\
\right.
\]

(3.6)

Using the second identity in \[2.1\], one sees that

\[
\frac{1}{2} \nabla_x \cdot \left\langle (u_1 \cdot v \sqrt{\mu}, A(v) : u_1 \otimes u_1), L^{-1}B(v) \right\rangle + \frac{1}{2} \nabla_x \cdot \left\langle (A(v) : u_1 \otimes u_1, u_1 \cdot v \sqrt{\mu}), L^{-1}B(v) \right\rangle \\
= \frac{1}{6} \nabla_x \cdot \langle L(v \otimes v \otimes v : u_1 \otimes u_1 \otimes u_1 \sqrt{\mu}), L^{-1}B(v) \rangle - \frac{1}{6} \nabla_x \cdot \langle L(u_1 \cdot v \left| v \right|^2 \left| u_1 \right|^2 \sqrt{\mu}), L^{-1}B(v) \rangle \\
= -\frac{1}{3} \nabla_x \cdot (u_1 \left| u_1 \right|^2). \\
\]

(3.7)

For the remaining terms in \[3.5\] and \[3.6\], we will show that

\[
\frac{\kappa_s}{3} \Delta \langle \left| u_1 \right|^2 \rangle + \frac{1}{2} \nabla_x \cdot \langle v : \nabla_x \{A(v) : u_1 \otimes u_1\}, L^{-1}B(v) \rangle \\
+ \nabla_x \cdot \langle (u_1 \cdot v \sqrt{\mu}, L^{-1}A(v) : \nabla_x u_1), L^{-1}B(v) \rangle \\
+ \nabla_x \cdot \langle (L^{-1}A(v) : \nabla_x u_1, u_1 \cdot v \sqrt{\mu}), L^{-1}B(v) \rangle \\
= \mu_s \nabla_x \cdot \left( u_1 (\nabla_x u_1 + (\nabla_x u_1)^T) \right). \\
\]

(3.8)

To confirm this, we first get from \[2.4\] that

\[
\sum_{i,k,l} \{ \langle A_{ki}(v), v_i L^{-1}B_j(v) \rangle - \langle A_{ki}(v), v_i L^{-1}B_j(v) \rangle \} u_i \partial_k u^k_i \\
= \frac{2 \kappa_s}{3} \sum_{i,k,l} (\delta_{ij} \delta_{kl} - \delta_{ik} \delta_{lj}) u_i^j \partial_k u^k_i = \frac{2 \kappa_s}{3} \sum_k u_j \partial_k u^k_i - \frac{2 \kappa_s}{3} \sum_i u_i^j \partial_j u^i_i = -\frac{2 \kappa_s}{3} \sum_i u_i^j \partial_j u^i_i, \\
\]
Then the left hand side of (3.8) can be further rewritten as
\[-\nabla_x \cdot \left\{ \sum_{i,k,l} \left\{ \langle A_{ki}(v), v_i L^{-1} B(v) \rangle - \langle A_{kl}(v), v_i L^{-1} B(v) \rangle \right\} u_i^k \partial u_1^l \right\}
\]
\[+ \nabla_x \cdot \left\{ \sum_{i,k,l} \langle A_{ki}(v), v_i L^{-1} B(v) \rangle u_i^k \partial u_1^l \right\}
\]
\[+ \nabla_x \cdot \langle \Gamma \left( u_1 \cdot v \sqrt{\mu}, L^{-1} A(v) : \nabla_x u_1 \right), L^{-1} B(v) \rangle
\]
\[+ \nabla_x \cdot \langle \Gamma \left( L^{-1} A(v) : \nabla_x u_1, u_1 \cdot v \sqrt{\mu} \right), L^{-1} B(v) \rangle
\]
\[= \nabla_x \cdot \left\{ \sum_{i,k,l} \langle A_{ki}(v), v_i L^{-1} B(v) \rangle u_i^k \partial u_1^l \right\}
\]
\[+ \nabla_x \cdot \langle \Gamma \left( u_1 \cdot v \sqrt{\mu}, L^{-1} A(v) : \nabla_x u_1 \right), L^{-1} B(v) \rangle
\]
\[+ \nabla_x \cdot \langle \Gamma \left( u_1 \cdot v \sqrt{\mu}, L^{-1} A(v) : \nabla_x u_1 \right), L^{-1} B(v) \rangle
\]
\[= \sum_j \partial_j \left\{ u_i^j \langle A_{ij}(v), L^{-1} A_{kl}(v) \rangle \partial u_1^k \right\},\]
according to (2.6). Hence (3.8) is valid. We now conclude from (3.4), (3.5), (3.6), (3.7) and (3.8) that
\[\partial_t \left( \frac{1}{2} |u_1|^2 + \frac{3}{2} \theta_1 - \rho_1 \right) + \nabla_x \left[ u_1 \left( \frac{5}{2} \theta_1 + \frac{1}{2} |u_1|^2 \right) \right] = \kappa_s \Delta \theta_1 + \mu_s \nabla_x \cdot \left( u_1 (\nabla_x u_1 + (\nabla_x u_1)^T) \right).\]
(3.9)

Lastly, the subtraction of (3.9) and $u_1 \cdot (3.3)$ yields the energy equation (1.15).
Likewise, one can see that (1.10) follows from the inner products (1.8), $\sqrt{\mu}$, (1.9), $v \sqrt{\mu}$, (1.10), $\frac{s^2}{2} - \sqrt{\mu}$ and (1.10), $\sqrt{\mu}$, we refer to [20] section 4, pp.643 for more details.

3.2. Diffusive coefficients. The diffusive coefficients $f_1, f_2, \cdots, f_6$ will be determined by solving the Cauchy problem (1.15) and (1.16) on torus $T^3$.

**Proposition 3.1.** Assume the condition $(A_1)$ in Theorem (7) is valid, then there exist unique functions $f_1, f_2, \cdots, f_6$ with zero mean hydrodynamic fields, which read
\[(f_r(t, x, v), [1, v, (v^2 - 3)] \sqrt{\mu}) = 0, \quad r = 1, 2, \cdots, 6,\]
such that $f_1, f_2, \cdots, f_6$ satisfy (1.15), (1.16) and (1.14). Moreover, for $1 \leq r \leq 6$, and for any $s \geq 2$ and $l \geq 0$, there exists $\lambda_0 > 0$ and a polynomial $P$ with $P(0) = 0$ such that
\[\sum_{a_0 + a_0 \leq s} \| \partial_t^{a_0} \nabla^a_x f_r(t) \|_2 \leq e^{-\lambda_0 t} P(||u_1, 0, \theta_1, 0||_{H^{2s+2(r-1)}} + ||u_2, 0, \theta_2, 0||_{H^{2s}}), \quad 1 \leq r \leq 3,\]
\[\sum_{a_0 + a_0 \leq s} \| \partial_t^{a_0} \nabla^a_x f_r(t) \|_2 \leq e^{-\lambda_0 t} P(||u_1, 0, \theta_1, 0||_{H^{2s+2r}} + ||u_2, 0, \theta_2, 0||_{H^{2s+2(r-1)}}), \quad 4 \leq r \leq 6,\]
\[\sum_{a_0 + a_0 \leq s} \| w_i \partial_t^{a_0} \nabla^a_x f_r(t) \|_\infty \leq e^{-\lambda_0 t} P(||u_1, 0, \theta_1, 0||_{H^{2s+2r}} + ||u_2, 0, \theta_2, 0||_{H^{2s+2}}), \quad 1 \leq r \leq 3,\]
and
\[\sum_{a_0 + a_0 \leq s} \| w_i \partial_t^{a_0} \nabla^a_x f_r(t) \|_\infty \leq e^{-\lambda_0 t} P(||u_1, 0, \theta_1, 0||_{H^{2s+2r+2}} + ||u_2, 0, \theta_2, 0||_{H^{2s+2r}}), \quad 4 \leq r \leq 6.\]
(3.10)
Proof. To determine $f_1$ and $f_2$, we start by solving the system (1.15). The basic tool in the proof of the existence of (1.15) is the Galerkin approximation and the classical energy method [10, 13], we skip the details for brevity. In what follows, we first show that

$$
\sum_{\alpha_0 + \alpha \leq 2} \| \partial_t^{\alpha_0} \nabla_x^\alpha u_1 \|_2 + \sum_{\alpha_0 + \alpha \leq 2, \alpha_0 \leq 1} \| \partial_t^{\alpha_0} \nabla_x^\alpha [\rho_1, \theta_1] \|_2 \leq C e^{-\lambda_0 t} P (\|u_{1,0}, \theta_{1,0}\|_{H^1})
$$

(3.11)

under the a priori assumption

$$
\sum_{\alpha_0 + \alpha \leq 2} \| \partial_t^{\alpha_0} \nabla_x^\alpha u_1 \|_2 + \sum_{\alpha_0 + \alpha \leq 2, \alpha_0 \leq 1} \| \partial_t^{\alpha_0} \nabla_x^\alpha [\rho_1, \theta_1] \|_2 \leq \varepsilon_0.
$$

(3.12)

Taking the inner product of $\partial_t^{\alpha_0} \nabla_x^\alpha (1.15)_2$ with $\partial_t^{\alpha_0} \nabla_x^\alpha u_1$ ($\alpha_0 + \alpha \leq 2$) over $\mathbb{T}^3$ and employing $\nabla_x \cdot u_1 = 0$ and Sobolev’s inequality, one has

$$
\frac{1}{2} \frac{d}{dt} \| \partial_t^{\alpha_0} \nabla_x^\alpha u_1 \|_2^2 + \mu_\ast \| \nabla_x \partial_t^{\alpha_0} \nabla_x^\alpha u_1 \|_2^2 \leq \sum_{\alpha_0 \leq 2} \| \partial_t^{\alpha_0} \nabla_x^\alpha u_1 \|_2^2 \leq C \sum_{\alpha_0 \leq 2} \| \partial_t^{\alpha_0} \nabla_x^\alpha u_1 \|_2^2.
$$

(3.13)

Taking the summation of (3.13) over $\alpha_0 + \alpha \leq 2$ and using (3.12), we obtain for some $\lambda_1 > 0$

$$
\sum_{\alpha_0 + \alpha \leq 2} \frac{d}{dt} \| \partial_t^{\alpha_0} \nabla_x^\alpha u_1 \|_2^2 + 2 \lambda_1 \sum_{\alpha_0 + \alpha \leq 2} \| \partial_t^{\alpha_0} \nabla_x^\alpha u_1 \|_2^2 \leq 0,
$$

(3.14)

where we have also used the following Sobolev inequality on torus

$$
\| u \|_p \leq C \| \nabla_x u \|_2 \text{ with } p \in [2, 6] \text{ for } u \in H^1(\mathbb{T}^3) \text{ and } \int_{\mathbb{T}^3} u dx = 0.
$$

(3.15)

(3.14) further implies

$$
\sum_{\alpha_0 + \alpha \leq 2} \| \partial_t^{\alpha_0} \nabla_x^\alpha u_1 \|_2 \leq C e^{-\lambda_1 t} \sum_{\alpha_0 + \alpha \leq 2} \| \partial_t^{\alpha_0} \nabla_x^\alpha u_{1,0} \|_2.
$$

(3.16)

We note immediately that the the temporal derivatives of the above initial data are understood by the equations (1.15)_2, for instance, $\partial_t \partial_{xx}^\alpha u_{1,0}$ is defined as

$$
\lim_{t \to 0^+} \partial_t \nabla_x^\alpha u_1 (t, x) = \lim_{t \to 0^+} \nabla_x \{ -P_0 (u_1 \cdot \nabla_x u_1) + \mu_\ast \Delta u_1 \} = \nabla_x \{ -P_0 (u_{1,0} \cdot \nabla_x u_{1,0}) + \mu_\ast \Delta u_{1,0} \}.
$$

(3.17)

With this, one can formally view the two spatial derivatives as “equivalent” to one temporal derivative and therefore there exists a polynomial $P$ with $P(0) = 0$ such that

$$
\sum_{\alpha_0 + \alpha \leq 2} \| \partial_t^{\alpha_0} \nabla_x^\alpha u_{1,0} \|_2 \leq P (\|u_{1,0}\|_{H^1}).
$$

As to the estimates for $\theta_1$ and $\rho_1$, we first get from equation (1.15)_2 that

$$
\rho_1 (t, x) = -\Delta^{-1} \nabla_x \cdot (u_1 \cdot \nabla_x u_1) - \theta_1.
$$

(3.18)

Inserting (3.18) into (1.15)_3, one has

$$
\frac{5}{2} \partial_t \theta_1 + \partial_t \Delta^{-1} \nabla_x \cdot (u_1 \cdot \nabla_x u_1) + u_1 \cdot \nabla_x \left( \frac{5}{2} \theta_1 + \Delta^{-1} \nabla_x \cdot (u_1 \cdot \nabla_x u_1) \right)
= \kappa_\ast \Delta \theta_1 + \frac{1}{2} \mu_\ast \| \nabla_x u_1 + (\nabla_x u_1)^T \|^2.
$$

(3.19)
We now get the inner product of $\partial_t^\alpha \nabla^\alpha_2 (3.19)$ $(\alpha_0 + \alpha \leq 2, \alpha_0 \leq 1)$ with $\partial_t^\alpha \nabla^\alpha_1 \theta_1$ over $T^3$ that
\[
\frac{5}{2} (\partial_t \partial_t^\alpha \nabla^\alpha_1 \theta_1, \partial_t^\alpha \nabla^\alpha_2 \theta_1) + (\partial_t \partial_t^\alpha \nabla^\alpha_1 \Delta^{-1} \nabla_\alpha \cdot (u_1 \cdot \nabla u_1), \partial_t^\alpha \nabla^\alpha \theta_1) \\
+ (\partial_t^\alpha \nabla^\alpha_1 [u_1 \cdot \nabla_\alpha (5/2 \theta_1 + \Delta^{-1} \nabla_\alpha \cdot (u_1 \cdot \nabla u_1)]), \partial_t^\alpha \nabla^\alpha \theta_1) \\
= \kappa_s (\partial_t^\alpha \nabla^\alpha_1 \Delta \theta_1, \partial_t^\alpha \nabla^\alpha \theta_1) + \frac{\mu_s}{2} \left( \partial_t^\alpha \nabla^\alpha_1 \nabla_\alpha u_1 + (\nabla_\alpha u_1)^T \right)^2, \partial_t^\alpha \nabla^\alpha \theta_1). \\
\]

By integration by parts and using (3.15) and Lemma (2.6), one has
\[
\frac{d}{dt} \sum_{\alpha_0 + \alpha \leq 2, \alpha_0 \leq 1} \| \partial_t^\alpha \nabla^\alpha_1 \theta_1 \|^2 + 2 \lambda_2 \sum_{\alpha_0 + \alpha \leq 2, \alpha_0 \leq 1} \| \nabla_\alpha \partial_t^\alpha \nabla^\alpha_1 \theta_1 \|^2 \\
\leq C \sum_{\alpha_0 + \alpha \leq 2} \| \partial_t^\alpha \nabla^\alpha_1 u_1 \|^2 \sum_{\alpha_0 + \alpha \leq 2} \| \partial_t^\alpha \nabla^\alpha_1 u_1 \|^2_{H^1}, \\
\]
for some $\lambda_2 > 0$. On the other hand, (3.9) and (1.18) imply
\[
\int_{T^3} (3 \theta_1 + |u_1|^2) dx = 0, \\
\]
using this and Poincaré’s inequality, one further gets from (3.20) that
\[
\frac{d}{dt} \sum_{\alpha_0 + \alpha \leq 2, \alpha_0 \leq 1} \| \partial_t^\alpha \nabla^\alpha_1 \theta_1 \|^2 + 2 \lambda_2 \sum_{\alpha_0 + \alpha \leq 2, \alpha_0 \leq 1} \| \partial_t^\alpha \nabla^\alpha_1 \theta_1 \|^2 \\
\leq C \sum_{\alpha_0 + \alpha \leq 2} \| \partial_t^\alpha \nabla^\alpha_1 u_1 \|^2 \sum_{\alpha_0 + \alpha \leq 2} \| \partial_t^\alpha \nabla^\alpha_1 u_1 \|^2_{H^1}. \\
\]
As a sequence, according to (3.13), it follows that
\[
\sum_{\alpha_0 + \alpha \leq 2, \alpha_0 \leq 1} \| \partial_t^\alpha \nabla^\alpha_1 \theta_1 \|^2 \leq e^{-2 \lambda_2 t} \sum_{\alpha_0 + \alpha \leq 2, \alpha_0 \leq 1} \| \partial_t^\alpha \nabla^\alpha_1 \theta_1 \|^2 \\
+ P^2(\|u_1, 0\|_{H^1})e^{-2 \lambda_2 t} \int_0^t e^{(2 \lambda_2 - 2 \lambda_1)s} \| \partial_s^\alpha \nabla^\alpha_1 u_1 \|^2_{H^1} ds \\
\leq e^{-2 \lambda_2 t} \sum_{\alpha_0 + \alpha \leq 2, \alpha_0 \leq 1} \| \partial_t^\alpha \nabla^\alpha_1 \theta_1 \|^2 + e^{-2 \lambda_2 t} P^2(\|u_1, 0\|_{H^1}), \\
\]
provided $0 < \lambda_2 < \lambda_1$. Moreover, (3.18) gives
\[
\sum_{\alpha_0 + \alpha \leq 2, \alpha_0 \leq 1} \| \partial_t^\alpha \nabla^\alpha_1 \rho_1 \|^2 \leq \varepsilon_0 \sum_{\alpha_0 + \alpha \leq 2} \| \partial_t^\alpha \nabla^\alpha_1 u_1 \|^2 + C \sum_{\alpha_0 + \alpha \leq 2, \alpha_0 \leq 1} \| \partial_t^\alpha \nabla^\alpha_1 \theta_1 \|^2. \\
\]
As (3.17), we define
\[
\partial_t \nabla^\alpha_1 \theta_1, 0 = -\frac{2}{5} \lim_{t \to 0^+} \partial_t \nabla^\alpha_1 \Delta^{-1} \nabla_\alpha \cdot (u_1 \cdot \nabla u_1) + \frac{2}{5} \nabla^\alpha_1 \lim_{t \to 0^+} \left\{ \kappa_s \Delta \theta_1 + \frac{1}{2} \mu_s |\nabla_\alpha u_1 + (\nabla_\alpha u_1)^T|^2 \right\} \\
- \frac{2}{5} \lim_{t \to 0^+} \nabla^\alpha_x \left\{ u_1 \cdot \nabla_x \left( \frac{5}{2} \theta_1 + \Delta^{-1} \nabla_\alpha \cdot (u_1 \cdot \nabla u_1) \right) \right\}. \\
\]
Consequently,
\[
\sum_{\alpha_0 + \alpha \leq 2, \alpha_0 \leq 1} \| \partial_t^\alpha \nabla^\alpha_1 \theta_1 \|^2 \leq P(\|u_1, 0, \theta_1, 0\|_{H^1}). \\
\]
\[
(3.23) 
\]
Letting $\lambda_2 < \lambda_1$ and taking $\lambda_0 = \min\{\lambda_2, \lambda_1\}$, we thereby obtain from (3.21), (3.22) and (3.23) that
\[
\sum_{\alpha_0 + \alpha \leq 2, \alpha_0 \leq 1} \| \partial_t^\alpha \nabla_x^\alpha [\rho_1, \theta_1] \|_2 \leq e^{-\lambda_0 t} P(\| [u_{1,0}, \theta_{1,0}] \|_{H^1}).
\] (3.24)

Therefore, (3.11) follows from (3.10) and (3.23). It should be pointed out that we can obtain the exponential decay of $\| \partial_t^2 u_1 \|_2$ but $\| \partial_t^3 [\rho_1, \theta_1] \|_2$, this phenomenon is essentially determined by the different structure of the equations (1.15) and (3.10). In what follows, we shall not include the highest time-derivatives of $u_1$ or $u_2$ in order to make our presentation more easy to read.

With (3.11) in hand, we now turn to deduce the following higher order $L^2$ estimates:
\[
\sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s-1} \| \partial_t^\alpha \nabla_x^\alpha [\rho_1, u_1, \theta_1] \|_2 \leq C e^{-\lambda_0 t} P(\| [u_{1,0}, \theta_{1,0}] \|_{H^2}), \quad s \geq 3.
\] (3.25)

Since $\sum_{\alpha_0 + \alpha = s, \alpha_0 \leq s-1} \| \partial_t^\alpha \nabla_x^\alpha [\rho_1, u_1, \theta_1] \|_2$ may not be small for $s \geq 3$, we need to proceed differently. From (1.15), it follows for $\alpha_0 + \alpha = s \geq 3, \alpha_0 \leq 1$ and $\eta > 0$
\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \sum_{\alpha_0 + \alpha = s} \| \partial_t^\alpha \nabla_x^\alpha u_1 \|_2^2 + \mu_s \sum_{\alpha_0 + \alpha = s} \| \nabla_x \partial_t^\alpha \nabla_x^\alpha u_1 \|_2^2 \\
\leq \sum_{\alpha_0' + \alpha' \leq \alpha} \sum_{\alpha_0 + \alpha = s} \| (\partial_t^\alpha \nabla_x^\alpha u_1 \otimes \partial_t^\alpha \nabla_x^\alpha u_1) \|_3 \\
\leq \sum_{\alpha_0' + \alpha' \leq 1} \sum_{\alpha_0 + \alpha = s} \| (\partial_t^\alpha \nabla_x^\alpha u_1 \otimes \partial_t^\alpha \nabla_x^\alpha u_1) \|_3 
+ \sum_{\alpha_0' + \alpha' > s-1} \sum_{\alpha_0 + \alpha = s} \| (\partial_t^\alpha \nabla_x^\alpha u_1 \otimes \partial_t^\alpha \nabla_x^\alpha u_1) \|_3 \\
\leq \left( \sum_{\alpha_0' + \alpha' \leq 1} \| \partial_t^\alpha \nabla_x^\alpha u_1 \|_{H^1} + \eta \right) \sum_{\alpha_0 + \alpha = s} \| \nabla_x \partial_t^\alpha \nabla_x^\alpha u_1 \|_2^2 + C \eta \sum_{\alpha_0' + \alpha' \leq s-1} \| \partial_t^\alpha \nabla_x^\alpha u_1 \|_2^2 ,
\end{aligned}
\]

where the last inequality holds due to (3.13) and Cauchy-Schwarz’s inequality with $\eta > 0$. Then one sees that $u_1$ enjoys the estimates (3.25), using a method of induction on $s \geq 3$. The corresponding estimates for $\rho_1$ and $\theta_1$ can be obtained in the similar way as deriving (3.24).

We get at once from (3.25) and the definitions for $f_1$ and $f_2$ in (1.14) that
\[
\begin{cases}
\sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s-1} \| \partial_t^\alpha \nabla_x^\alpha f_1 \|_2 \leq C e^{-\lambda_0 t} P(\| u_{1,0} \|_{H^{2s}}), \\
\sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s-1} \| \partial_t^\alpha \nabla_x^\alpha f_2 \|_2 \leq C e^{-\lambda_0 t} P(\| [u_{1,0}, \theta_{1,0}] \|_{H^{2s+2}}).
\end{cases}
\] (3.26)

In addition, from Sobolev’s inequality, it follows that
\[
\sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s-1} \| u_1 \partial_t^\alpha \nabla_x^\alpha f_1 \|_\infty \leq C e^{-\lambda_0 t} P(\| u_{1,0} \|_{H^{2s+2}}),
\]
and
\[
\sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s-1} \| u_1 \partial_t^\alpha \nabla_x^\alpha f_2 \|_\infty \leq C e^{-\lambda_0 t} P(\| [u_{1,0}, \theta_{1,0}] \|_{H^{2s+4}}),
\]
for any \( l \geq 0 \). Since \( L^{-1} \) preserves decay in \( v \), c.f. [6], one also has by Lemma 2.3 and (1.14):

\[
\sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s-1} \| w_l \partial_t^{\alpha_0} \nabla_x^\alpha (I - P) f_3 \|_2 \leq C e^{-\lambda_0 t} P(||u_{1,0}, \theta_{1,0}||_{H^{2s+4}}), \tag{3.27}
\]

and

\[
\sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s-1} \| w_l \partial_t^{\alpha_0} \nabla_x^\alpha (I - P) f_3 \|_\infty \leq C e^{-\lambda_0 t} P(||u_{1,0}, \theta_{1,0}||_{H^{2s+6}}).
\]

Let us now turn to estimate \( u_2, f_4, f_5 \) and \( f_6 \). To do this, we first rewrite (1.16) and (1.18) as

\[
\left\{ \begin{array}{l}
\partial_t \rho_1 + \nabla_x \cdot (I - P_0) u_2 + \nabla_x \rho_1 \cdot u_1 = 0, \\
\partial_t P_0 u_2 + \nabla_x u_1 \cdot P_0 u_2 + u_1 \cdot \nabla_x P_0 u_2 + \nabla_x \left( \rho_1 + \rho_2 - \frac{1}{3} u_1 \cdot P_0 u_2 \right) \\
= \mu_\alpha \Delta P_0 u_2 + \mu_\alpha \nabla_x \cdot (I - P_0) u_2 - \partial_t (I - P_0) u_2 - u_1 \cdot \nabla_x (I - P_0) u_2 \\
- \nabla_x \cdot \Gamma(f_2, f_2, L^{-1} A(v)) - \nabla_x \cdot \Gamma(f_1, (I - P) f_3, f_3, f_3, (I - P) f_3, f_3, L^{-1} A(v)) \\
- \partial_t (\rho_1 u_1) - 2 \nabla_x \cdot (\rho u_1 \otimes u_1) - \nabla_x (\rho_1 u_1 + \frac{5}{2} \rho_1^2 - \frac{1}{3} \rho_1 |u_1|^2) \\
+ \mu_\alpha (\rho_1 u_1) + \mu_\alpha \nabla_x \cdot (\rho_1 u_1), \\
u_2(0, x) = u_{2,0}(x) = P_0 u_{2,0}(x).
\end{array} \right.
\tag{3.28}
\]

The proof for the existence of the above \textit{linear} system is quite standard. In what follows we will show that

\[
\sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s-1} \| \partial_t^{\alpha_0} \nabla_x^\alpha u_2 \|_2 \leq C e^{-\lambda_0 t} P(||u_{1,0}, \theta_{1,0}||_{H^{2s+2}}), \quad s \geq 3, \tag{3.29}
\]

under the condition (3.25).

To begin with, observe that (1.16) and (1.18) imply

\[
\int_{\mathbb{T}^3} (u_2 + \rho_1 u_1) dx = \int_{\mathbb{T}^3} (u_{2,0} + \rho_{1,0} u_{1,0}) dx = 0.
\]

Thus we know thanks to (1.17)

\[
\int_{\mathbb{T}^3} P_0 (u_2 + \rho_1 u_1) dx = \int_{\mathbb{T}^3} (I - P_0) (u_2 + \rho_1 u_1) dx = 0. \tag{3.30}
\]

In light of (3.28) and (3.30), we have by standard elliptic estimates and Poincaré’s inequality

\[
\sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s-1} \| \partial_t^{\alpha_0} \nabla_x^\alpha (I - P_0) (u_2 + \rho_1 u_1) \|_2 \\
\leq C \sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s-1} \| \partial_t^{\alpha_0} \nabla_x^\alpha \partial_t \rho_1 \|_2 \\
\leq C e^{-\lambda_0 t} P(||u_{1,0}, \theta_{1,0}||_{H^{2s+2}}),
\]

for any \( l \geq 0 \).
which further yields

\[
\sum_{\alpha_0 + s \leq \alpha, \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^\alpha (I - P_0) u_2 \|_2^2 \\
\leq C \sum_{\alpha_0 + s \leq \alpha, \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^\alpha [\partial_t \rho_1, \rho_1, u_1] \|_2 \leq C \sum_{\alpha_0 + s + 1, \alpha_0 \leq s} \| \partial_t^{\alpha_0} \nabla_x^\alpha [\rho_1, u_1] \|_2 \\
\leq C e^{-\lambda_1 t} P \left( \| [u_{1,0}, \theta_{1,0}] \|_{H^{2+s}} \right),
\]

(3.31)

according to (3.29).

To prove (3.29), it remains now to estimate $P_0 u_2$. Taking the inner product of $\partial_t^{\alpha_0} \nabla_x^\alpha (3.28)_2$ with $\nabla_x \partial_t^{\alpha_0} \nabla_x^\alpha P_0 u_2$ and applying Lemma 2.4 as well as Sobolev’s inequality, one has for $\alpha_0 + \alpha = s, \alpha_0 \leq s - 1$ and $s \geq 3$

\[
\frac{1}{2} \frac{d}{dt} \sum_{\alpha_0 + \alpha = s} \| \partial_t^{\alpha_0} \nabla_x^\alpha P_0 u_2 \|_2^2 + \mu_s \sum_{\alpha_0 + \alpha = s} \| \nabla_x \partial_t^{\alpha_0} \nabla_x^\alpha P_0 u_2 \|_2^2 \\
\leq 2 \sum_{\alpha_0 \leq \alpha_0, \alpha_0 \leq \alpha, \alpha_0 \leq s} \sum_{\alpha_0 + \alpha = s} |(\partial_t^{\alpha_0} \nabla_x^{\alpha^0} u_1 \otimes \partial_t^{\alpha_0} \nabla_x^{\alpha_0 - \alpha^0} P_0 u_2, \nabla_x \partial_t^{\alpha_0} \nabla_x^\alpha P_0 u_2)| \\
+ 2 \sum_{\alpha_0 + \alpha = s} \sum_{\alpha_0 \leq \alpha, \alpha_0 \leq \alpha, \alpha_0 \leq s} |(\partial_t^{\alpha_0} \nabla_x^{\alpha^0} (u_1 \otimes \{I - P_0\} u_2), \nabla_x \partial_t^{\alpha_0} \nabla_x^\alpha P_0 u_2)| \\
+ \sum_{\alpha_0 + \alpha = s} \sum_{\alpha_0 \leq \alpha, \alpha_0 \leq \alpha, \alpha_0 \leq s} |(\partial_t^{\alpha_0} \nabla_x^{\alpha^0} (\Gamma(f_1, \{I - P\} f_3) + \Gamma((I - P) f_3, f_1), L^{-1} A(v)), \nabla_x \partial_t^{\alpha_0} \nabla_x^\alpha P_0 u_2)| \\
+ \sum_{\alpha_0 + \alpha = s} \sum_{\alpha_0 \leq \alpha, \alpha_0 \leq \alpha, \alpha_0 \leq s} |(\partial_t^{\alpha_0} \nabla_x^{\alpha^0} \partial_t (\rho_1 u_1) + 2 \nabla_x : (\rho u_1 \otimes u_1) + \mu_s \Delta (\rho_1 u_1), \partial_t^{\alpha_0} \nabla_x^\alpha P_0 u_2)| \\
\leq \left( \sum_{\alpha_0 \leq \alpha_0 \leq 1} \| \partial_t^{\alpha_0} \nabla_x^{\alpha^0} u_1 \|_{H^1} + \eta \right) \sum_{\alpha_0 + \alpha = s} \| \nabla_x \partial_t^{\alpha_0} \nabla_x^\alpha P_0 u_2 \|_2^2 + C_\eta \sum_{\alpha_0 + \alpha \leq \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^\alpha u_1 \|_2^2 \\
+ C_\eta \sum_{\alpha_0 + \alpha \leq \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^\alpha (I - P_0) u_2 \|_2^2 + C_\eta \sum_{\alpha_0 + \alpha \leq \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^\alpha \rho_1 \|_2^2 \\
+ C_\eta \sum_{\alpha_0 + \alpha \leq \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^\alpha f_2 \|_2^2 + C_\eta \sum_{\alpha_0 + \alpha \leq \alpha_0 \leq s} \| \partial_t^{\alpha_0} \nabla_x^\alpha f_1 \|_2^2 \\
+ C_\eta \sum_{\alpha_0 + \alpha \leq \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^\alpha (I - P) f_3 \|_2^2.
\]

(3.32)

Recalling (3.30), one has

\[
\sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^\alpha P_0 u_2 \|_2 \\
\leq \sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^\alpha P_0 (u_2 + \rho_1 u_1) \|_2 + \sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^\alpha P_0 (\rho_1 u_1) \|_2 \\
\leq C \sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^\alpha P_0 \nabla_x (u_2 + \rho_1 u_1) \|_2 + \sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^\alpha P_0 (\rho_1 u_1) \|_2.
\]

(3.33)
Plugging (3.25), (3.26), (3.27), (3.31) and (3.33) into (3.32) leads us
\[
\sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^{\alpha} P_{0} u_2 \|_2 \leq Ce^{-\lambda_0 t} P \left( \| [u_{1,0}, \theta_{1,0}] \|_{H^{2s+4}} + \| u_{2,0} \|_{H^{2s}} \right), \quad s \geq 3.
\]
Therefore (3.29) is valid. Once the estimates for \( u_{2} \) are obtained, one can immediately show that
\[
\begin{align*}
\sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^{\alpha} f_3 \|_2 & \leq Ce^{-\lambda_0 t} P \left( \| [u_{1,0}, \theta_{1,0}] \|_{H^{2s+4}} + \| u_{2,0} \|_{H^{2s}} \right), \\
\sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^{\alpha} (I - P) f_4 \|_2 & \leq Ce^{-\lambda_0 t} P \left( \| [u_{1,0}, \theta_{1,0}] \|_{H^{2s+6}} + \| u_{2,0} \|_{H^{2s+2}} \right),
\end{align*}
\]
and
\[
\begin{align*}
\sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^{\alpha} f_3 \|_{\infty} & \leq Ce^{-\lambda_0 t} P \left( \| [u_{1,0}, \theta_{1,0}] \|_{H^{2s+6}} + \| u_{2,0} \|_{H^{2s+2}} \right), \\
\sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^{\alpha} (I - P) f_4 \|_{\infty} & \leq Ce^{-\lambda_0 t} P \left( \| [u_{1,0}, \theta_{1,0}] \|_{H^{2s+8}} + \| u_{2,0} \|_{H^{2s+4}} \right),
\end{align*}
\]
according to the definition (1.14). We now turn to estimate \( \rho_2 \) and \( \theta_2 \). From (1.16), it follows
\[
\nabla_x \rho_2 = \frac{1}{3} \nabla_x (u_1 \cdot u_2) - \nabla_x \theta_2 - (u_1 \nabla_x \cdot u_2 + \nabla_x u_1 \cdot u_2 + u_1 \cdot \nabla_x u_2) + \mu_s \Delta u_2 \\
- \partial_t u_2 + \frac{\mu_s}{3} \nabla_x \nabla_x \cdot u_2 + \nabla_x \cdot (\partial_t f_2, L^{-1} A(v)) - \nabla_x \cdot (\Gamma(f_2, f_2, L^{-1} A(v)) \\
- \nabla_x \cdot (\Gamma(f_1, (I - P) f_3) + \Gamma((I - P) f_3, f_1), L^{-1} A(v)) \\
- \left\{ \partial_t (\rho_1 u_1) + 2 \nabla_x \cdot (\rho u_1 \otimes u_1) + \nabla_x (\rho_1 \theta_1 + \frac{5}{2} \theta_1^2 - \frac{1}{3} \rho_1 |u_1|^2) \right\} \\
+ \mu_s \Delta (\rho_1 u_1) + \frac{\mu_s}{3} \nabla_x \nabla_x \cdot (\rho_1 u_1)
\]
definition \(- \nabla_x \theta_2 + R_{\rho_2}.
\]
here \( R_{\rho_2} \) denotes the summation of all the other terms except \( \nabla_x \theta_2 \) on the right hand side of the above identity. Inserting \( \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \theta_2 \) (3.35) (\( \alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1 \)) into \( \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \theta_2 \) (1.16) and taking the inner product of the resulting equation with \( \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \theta_2 \), one has
\[
\frac{5}{2} \langle \partial_t \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \theta_2, \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \theta_2 \rangle - \langle \partial_t^{\alpha_0} \nabla_x^{\alpha} R_{\rho_2}, \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \theta_2 \rangle \\
+ \frac{5}{2} \langle \partial_t \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \cdot (u_1 \theta_2), \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \theta_2 \rangle + \frac{5}{6} \langle \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \cdot (u_1 u_2), \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \theta_2 \rangle \\
- \kappa_s \langle \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \Delta \theta_2, \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \theta_2 \rangle + \kappa_s \langle \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \Delta (\rho_1 u_1^2 \theta_1^2), \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \theta_2 \rangle \\
+ \frac{2 \kappa_s}{3} \langle \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \Delta (u_1 u_2), \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \theta_2 \rangle \\
+ \frac{\kappa_s}{3} \langle \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \Delta (u_1 u_1^2), \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \theta_2 \rangle \\
- \frac{1}{2} \langle \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \partial_t (2 u_1 u_2 + \rho u_1^2 u_2 + 3 \rho_1 \theta_1), \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \theta_2 \rangle \\
+ \frac{5}{2} \langle \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \cdot (u_1 (\rho_1 \theta_1^2)), \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \theta_2 \rangle - \frac{5}{6} \langle \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \cdot (u_1 u_1^2), \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \theta_2 \rangle \\
- \langle \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \cdot (L^{-1} \{ - \partial_t f_3 - v \cdot \nabla \{ I - P \} f_4 + \Gamma(f_2, f_3) + \Gamma(f_3, f_2) \}, B(v)), \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \theta_2 \rangle \\
- \langle \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \cdot (L^{-1} \{ \Gamma(f_1, \{ I - P \} f_4) + \Gamma(\{ I - P \} f_4, f_1) \}, B(v)), \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \theta_2 \rangle.
\]
By integration by parts and using Cauchy-Schwartz’s inequality with \( \eta > 0 \), we deduce

\[
\frac{d}{dt} \| \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \theta_2 \|_2^2 + \lambda_3 \| \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x^2 \theta_2 \|_2^2 \\
\leq C \eta \sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^{\alpha} [\rho_1, u_1, \theta_1, u_2] \|_2^2 \\
+ C \eta \sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^{\alpha} [f_1, f_2, f_3, \{I - P\} f_4] \|_2^2 + \eta \| \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \theta_2 \|_{H^1}^2.
\]

Poincaré’s inequality further yields

\[
\| \partial_t^{\alpha_0} \nabla_x^{\alpha} \nabla_x \theta_2 \|_2^2 \leq C \sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1} e^{-\lambda_3 t} \int_0^t e^{\lambda_3 s} \| \partial_t^{\alpha_0} \nabla_x^{\alpha} [\rho_1, u_1, \theta_1, u_2] \|_2^2 ds
\]

\[
+ C \sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1} e^{-\lambda_3 t} \int_0^t e^{\lambda_3 s} \| \partial_t^{\alpha_0} \nabla_x^{\alpha} [f_1, f_2, f_3, \{I - P\} f_4] \|_2^2 ds.
\]  

(3.36)

Invoking (1.18), (1.16) and (1.16), one has

\[
\int_{T_3} (3 \theta_2 + 2 u_1 \cdot u_2 + \rho_1 |u_1|^2 + 3 |\theta_1|) \rho_2 dx = 0, \int_{T_3} \rho_2 dx = 0.
\]  

(3.37)

Combing now (3.26), (3.29), (3.31), (3.36) and (3.37) and taking \( \lambda_3 < \lambda_0 \), we deduce

\[
\sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^{\alpha} [\rho_1, \theta_2] \|_2 \leq C e^{-\lambda_0 t} P \left( \| [u_{1,0}, \theta_{1,0}] \|_{H^{2+s}} + \| [u_{2,0}, \theta_{2,0}] \|_{H^{2+s}} \right)
\]

\[
+ C \sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^{\alpha} (2 u_1 \cdot u_2 + \rho_1 u_2^2 + 3 |\theta_1|) \|_2
\]

\[
+ C \sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^{\alpha} (2 u_1 \cdot u_2 + \rho_1 u_2^2 + 3 |\theta_1|) \|_2
\]

\[
\leq C e^{-\lambda_0 t} P \left( \| [u_{1,0}, \theta_{1,0}] \|_{H^{2+s}} + \| [u_{2,0}, \theta_{2,0}] \|_{H^{2+s}} \right).
\]  

(3.38)

We now conclude from (1.14), (3.26), (3.27), (3.29) and (3.38) as well as Lemma 2.4 that

\[
\sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^{\alpha} P f_4 \|_2 \leq C e^{-\lambda_0 t} P \left( \| [u_{1,0}, \theta_{1,0}] \|_{H^{2+s}} + \| [u_{2,0}, \theta_{2,0}] \|_{H^{2+s}} \right),
\]  

(3.39)

and

\[
\sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^{\alpha} (I - P) f_5 \|_2 \leq C e^{-\lambda_0 t} P \left( \| [u_{1,0}, \theta_{1,0}] \|_{H^{2+s}} + \| [u_{2,0}, \theta_{2,0}] \|_{H^{2+s}} \right).
\]  

(3.40)

As to \( Pf_5 \), it suffices to determine \( u_3 \). Since the expansion (1.3) is truncated at \( f_6 \), we can assume \( P_0 u_3 = 0 \). Moreover, (1.11) and (1.18) imply \( \int_{T_3} [u_3 + \rho_2 u_1 + \rho_1 u_2] dx = 0 \). Those ensure us to obtain from (1.16) that

\[
\sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^{\alpha} u_3 \|_2 \leq C \sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^{\alpha} \partial_t \rho_1, \rho_1, \rho_2, u_1, u_2 \|_2
\]

\[
\leq C e^{-\lambda_0 t} P \left( \| [u_{1,0}, \theta_{1,0}] \|_{H^{2+s}} + \| [u_{2,0}, \theta_{2,0}] \|_{H^{2+s}} \right).
\]  

(3.41)

Therefore one deduces from (3.26), (3.27), (3.31), (3.33), (3.41) and (3.11)

\[
\sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s - 1} \| \partial_t^{\alpha_0} \nabla_x^{\alpha} f_5 \|_2 \leq C e^{-\lambda_0 t} P \left( \| [u_{1,0}, \theta_{1,0}] \|_{H^{2+s}} + \| [u_{2,0}, \theta_{2,0}] \|_{H^{2+s}} \right),
\]
and
\[ \sum_{\alpha_0 + \alpha \leq s, \alpha_0 \leq s-1} \| \partial_t^{\alpha_0} \nabla_x^{\alpha} f_0 \|_2 \leq C e^{-\lambda_0 t} P \left( \| [u_{1,0}, \theta_{1,0}] \|_{H^{2+s+12}} + \| u_{2,0} \|_{H^{2s+10}} \right). \]

Finally, using Lemma 2.8 and the fact that $L^{-1}$ preserves the decay in $v$, we see that (3.10) is also valid. This ends the proof of Proposition 3.1.

\[ \square \]

4. $L^2$–Theory

In this section, we will study the solutions of the linear equation of the remainder which satisfies (1.11) in $L^2$ setting. The main purpose of this section is to prove the following:

**Proposition 4.1.** Assume $g_1, g_2 \in L^2(\mathbb{R}_+ \times \mathbb{T}^3 \times \mathbb{R}^3)$ and for all $t > 0$,
\[ \int_{\mathbb{T}^3 \times \mathbb{R}^3} g_1(t, x, v)[1, v, v^2] \sqrt{\mu} dv dx = 0. \]  
(4.1)

Then, for $g = \epsilon g_1 + g_2$ and for any sufficiently small $\epsilon$, there exists a unique solution to the problem
\[
\begin{cases}
\epsilon \partial_t f + v \cdot \nabla_x f + \epsilon^{-1} Lf = g, & x \in \mathbb{T}^3, \ v \in \mathbb{R}^3, \\
f(0, x, v) = f_0(x, v), & x \in \mathbb{T}^3, \ v \in \mathbb{R}^3,
\end{cases}
\]
(4.2)
such that
\[ \int_{\mathbb{T}^3 \times \mathbb{R}^3} f(t, x, v)[1, v, v^2] \sqrt{\mu} dv dx = 0, \quad \text{for all } t \geq 0. \]  
(4.3)

Moreover, there is $0 < \lambda \ll 1$ such that for $t \geq 0$,
\[
\begin{align*}
\| e^{\lambda t} f(t) \|_2^2 &+ \epsilon^2 \int_0^t \| e^{\lambda \tau} (I - P) f(\tau) \|_2^2 d\tau + \int_0^t \| e^{\lambda \tau} Pf(\tau) \|_2^2 d\tau \\
&\leq \| f_0 \|_2^2 + \int_0^t \| \nu^{-\frac{1}{2}} e^{\lambda \tau} (I - P) g \|_2^2 + \epsilon^2 \int_0^t \| e^{\lambda \tau} Pf(\tau) \|_2^2 d\tau.
\end{align*}
\]  
(4.4)

To prove Proposition 4.1, let us first show that the macroscopic part of the solution of (4.2) can be dominated by its microscopic part, for results in this direction, we have

**Lemma 4.1.** Assume $g = \epsilon g_1 + g_2$ with $g_1$ satisfying (1.1) and $f$ satisfies (4.1) and (4.3). Then there exists a function $G(t)$ such that, for all $t \geq 0$, $G(t) \lesssim \epsilon \| f(t) \|_2$ and
\[ \int_0^t \| Pf(\tau) \|_0^2 d\tau \lesssim G(t) - G(0) + \int_0^t \| \nu^{-1/2} g(\tau) \|_2^2 d\tau + \epsilon^2 \int_0^t \| (I - P) f(\tau) \|_0^2 d\tau. \]

**Proof.** The proof is the same as Lemma 3.9 in [13, pp.45] or Lemma 6.1 in [20, pp.656] with some trivial modification. $\square$

We are now in a position to complete

**The proof of Proposition 4.1.** Notice that (4.2) is a linear problem, whose global existence is easy to be seen, in what follows, we only prove (4.4). Let $y(t) = e^{\lambda t} f(t)$ with $\lambda > 0$. We multiply (4.2) by $e^{\lambda t}$, so that $y$ satisfies
\[
\partial_t y + \epsilon^{-1} v \cdot \nabla_x y + \epsilon^{-2} Ly = \lambda y + \epsilon^\lambda e^{-1} g, \quad y(0, x, v) = f_0(x, v), \quad x \in \mathbb{T}^3, \ v \in \mathbb{R}^3.
\]  
(4.5)
Taking the inner product of (4.5) with \( y \) over \( T^3 \times \mathbb{R}^3 \) and integrating the resulting equation with respect to time, one has

\[
\frac{1}{2} \| y(t) \|_2^2 + \epsilon^{-2} \int_0^t \| \{ I - P \} y(s) \|_2^2 \, ds \\
\leq (\lambda + \eta) \int_0^t \| y(s) \|_2^2 + \| y(0) \|_2^2 + \int_0^t \epsilon^{\lambda s} \| \nu^{-\frac{1}{2}} \{ I - P \} g \|_2^2 \, ds + \epsilon^{-2} C_\eta \int_0^t \epsilon^{\lambda s} \| g \|_2^2 \, ds. \tag{4.6}
\]

Applying Lemma 4.1 to (4.5), we deduce

\[
\int_0^t \| Py(s) \|_2^2 \, ds \lesssim G(t) - G(0) + \epsilon^{-2} \int_0^t \| \{ I - P \} y(s) \|_2^2 \, ds + \int_0^t \epsilon^{\lambda s} \| g \|_2^2 \, ds + \lambda \int_0^t \| y \|_2^2 \, ds, \tag{4.7}
\]

where \( G(t) \lesssim \epsilon \| y(t) \|_2^2 \). (4.4) thereby follows from a linear combination of (4.6) and (4.7). This finishes the proof of Proposition 4.1.

\[\Box\]

5. \( L^\infty \)-Theory

This section is dedicated to obtaining the \( L^\infty \)-estimates of the solution to the linear equation (4.12). More precisely, we are going to prove the following:

**Proposition 5.1.** Assume \( f \) satisfies

\[
\begin{align*}
[\partial_t + \epsilon^{-1} v \cdot \nabla_x + \epsilon^{-2} \nu(v)] f &= \epsilon^{-2} K f + \epsilon^{-1} g, \\
f(0, x, v) &= f_0(x, v), \quad x \in T^3, \quad v \in \mathbb{R}^3.
\end{align*}
\tag{5.1}
\]

Then, for \( l \geq 0 \), there exists \( \lambda > 0 \) such that

\[
\| \epsilon^{\lambda t} w_l f(t) \|_\infty \lesssim \epsilon^{-\lambda} \| \epsilon^{\frac{\lambda}{2} t} w_l f_0 \|_\infty + \epsilon^{\frac{\lambda}{2}} \epsilon^{-\lambda} \sup_{0 \leq s \leq t} \| \epsilon^{\lambda s} \nu^{-1} w_l g(s) \|_\infty + \epsilon^{-\lambda} \sup_{0 \leq s \leq t} \| \epsilon^{\lambda s} f(s) \|_2. \tag{5.2}
\]

**Proof.** Notice that the equations of the characteristics for (5.1) are

\[
\frac{dX_s(t) \, ds}{ds} = \frac{V(s)}{\epsilon}, \quad \frac{dV(s)}{ds} = 0,
\]

with initial data \( [X(t; t, x, v), V(t; t, x, v)] = [x, v] \). By this, we write \( X_s(t) = X_s(s; t, x, v) = x + \frac{s - t}{\epsilon} v \) and \( V(s) = V(s; t, x, v) = v \). Let \( h = w_l f \), we then get from Duhamel’s principle that

\[
\epsilon^{3/2} h(t, x, v) = \epsilon^{3/2} \epsilon^{-\nu(\gamma; t, s)} h_0(\frac{x - \frac{v}{\epsilon} t}{\epsilon}, v) + \int_0^t e^{-\nu(\gamma; t, s)} \left[ \epsilon^{-1/2} K \nu h + \epsilon^{1/2} w_l g \right] (s, x + \frac{(s - t) v}{\epsilon}, v) \, ds,
\]

where \( K \nu(\cdot) = w_l K(\frac{\nu}{w_l}) \) and \( h_0(x, v) = f_0(x, v) w_l \). Direct calculation yields

\[
| \epsilon^{3/2} h(t, x, v) | \leq \epsilon^{3/2} \epsilon^{-\nu(\gamma; t, s)} | h_0(\frac{x - \frac{v}{\epsilon} t}{\epsilon}, v) | + C \epsilon^{5/2} \epsilon^{-\lambda} \sup_{0 \leq s \leq t} \{ \epsilon^{\lambda s} \| \nu^{-1} w_l g(s) \|_\infty \}
\]

\[
+ \int_0^t e^{-\nu(\gamma; t, s)} e^{-1/2} \left[ K \nu h(s, x + \frac{(s - t) v}{\epsilon}, v) \right] \, ds.
\tag{5.3}
\]
Here, $\nu_0$ is a constant and satisfies $0 < \nu_0 \leq \nu(v)$ and $\lambda \leq \frac{\nu_0}{2x}$. We further iterate this formula to evaluate $J_1$ as

$$J_1 \leq \int_0^t e^{-\frac{\nu(v)(t-s)}{\epsilon^2}} \frac{1}{\epsilon^2} \int_{\mathbb{R}^3} k_w(v, v') e^{3/2} h(s, x + \frac{(s-t)v}{\epsilon}, v') dv' ds$$

$$\leq \int_0^t e^{-\frac{\nu(v)(t-s)}{\epsilon^2}} \frac{1}{\epsilon^2} \left\{ e^{3/2} e^{-\frac{\nu_0}{2x}} \|h_0\|_\infty + C e^{5/2} e^{-\lambda \tau} \right\} \sup_{0 \leq \tau \leq s} \left\{ e^{\lambda \tau} \|\nu^{-1} w_l g(\tau)\|_\infty \right\} ds \int_{\mathbb{R}^3} k_w(v, v') dv'$$

$$+ \int_0^t e^{-\frac{\nu(v)(t-s)}{\epsilon^2}} \frac{1}{\epsilon^2} \int_0^s e^{-\frac{\nu(v')(s-t)}{\epsilon^2}} e^{-5/2} \int_{\mathbb{R}^3} k_w(v, v') k_w(v', v'') dv' dv'' ds$$

$$\times |h(\tau, x + \frac{(s-t)v}{\epsilon} + \frac{(s-t)v'}{\epsilon}, v')| dv' dv'' d\tau ds$$

$$\leq C e^{3/2} e^{-\frac{\nu_0}{2x}} \|h_0\|_\infty + C e^{5/2} e^{-\lambda \tau} \sup_{0 \leq \tau \leq t} \left\{ e^{\lambda \tau} \|\nu^{-1} w_l g(\tau)\|_\infty \right\}$$

$$+ \int_0^t e^{-\frac{\nu(v)(t-s)}{\epsilon^2}} \frac{1}{\epsilon^2} \int_0^s e^{-\frac{\nu(v)(s-t)}{\epsilon^2}} e^{-5/2} \int_{\mathbb{R}^3} k_w(v, v') k_w(v', v'') h(\tau, X_\epsilon(\tau; \tau, X_\epsilon(s), v'), v'') dv' dv'' d\tau ds,$$

where $X_\epsilon(\tau; \tau, X_\epsilon(s), v') = x + \frac{(s-t)v}{\epsilon} + \frac{(s-t)v'}{\epsilon}$, and $k_w(v, v') = \frac{w_l(v)k(v, v')}{w_l(v)}$ with $k(v, v')$ given by (2.10). To compute $J_2$, we first split it into

$$J_2 = \int_0^t e^{-\frac{\nu(v)(t-s)}{\epsilon^2}} \int_{s-\epsilon^2}^s e^{-\frac{\nu(v')(s-t)}{\epsilon^2}} \cdots d\tau ds + \int_0^t e^{-\frac{\nu(v)(t-s)}{\epsilon^2}} \int_0^s e^{-\frac{\nu(v)(t-s)}{\epsilon^2}} \cdots d\tau ds \equiv J_{2,1} + J_{2,2},$$

where $\kappa$ is positive and sufficiently small.

Let us now turn to compute $J_{2,1}$ and $J_{2,2}$. For $J_{2,1}$, it is straightforward to see that

$$J_{2,1} \leq \int_0^t \int_{s-\epsilon^2}^s C_K e^{-\frac{\nu_0(t-s)}{2x}} e^{-5/2} \|h(\tau)\|_\infty d\tau ds$$

$$\leq C_K e^{-\frac{\nu_0}{2x}} \int_0^t \int_{s-\epsilon^2}^s \left\{ e^{\frac{\nu_0}{2x}} e^{-5/2} \|h(\tau)\|_\infty \right\} d\tau ds$$

$$\leq C_K e^{-\frac{\nu_0}{2x}} \int_0^t \int_{s-\epsilon^2}^s \left\{ e^{\frac{\nu_0}{2x}} \|h(\tau)\|_\infty \right\} d\tau ds \quad (5.5)$$

As to $J_{2,2}$, the estimates are divided into following three cases:

Case 1: $|v| \geq N$ with $N$ being positive and large. In this case, Lemma 2.5 implies

$$\int_{\mathbb{R}^6} k_w(v, v') k_w(v', v'') dv' dv'' \leq C (1 + |v|)^{-1} \leq \frac{C}{1 + N}.$$  

Therefore

$$J_{2,2} \leq \frac{C}{1 + N} e^{-\frac{\nu_0}{2x}} e^{3/2} \sup_{0 \leq \tau \leq t} \left\{ e^{\frac{\nu_0}{2x}} \|h(\tau)\|_\infty \right\} \int_0^t e^{-\frac{\nu_0(t-s)}{2x}} e^{-2} \int_0^s e^{-\frac{\nu_0(s-t)}{2x}} e^{-2} d\tau ds$$

$$\leq \frac{C}{1 + N} e^{-\frac{\nu_0}{2x}} e^{3/2} \sup_{0 \leq \tau \leq t} \left\{ e^{\frac{\nu_0}{2x}} \|h(\tau)\|_\infty \right\}.$$  

(5.6)
Case 2: \( |v| < N, |v'| \geq 2N \), or \( |v'| \leq 2N, |v''| \geq 3N \). Observe that we have either \( |v' - v| \geq N \) or \( |v'' - v'| \geq N \) and either one of the following holds accordingly for \( \varepsilon > 0 \)

\[
|k_w(v, v')| \leq Ce^{-\frac{\varepsilon}{8}N^2} |k_w(v, v')e^{\frac{\varepsilon}{8}|v-v'|^2}|, \quad |k_w(v', v'')| \leq Ce^{-\frac{\varepsilon}{8}N^2} |k_w(v', v'')e^{\frac{\varepsilon}{8}|v'-v''|^2}|
\]

from which and Lemma 2.5 it follows that

\[
J_{2,2} \leq \int_0^t e^{-\frac{\mu(t-s)}{c^2}} e^{-\frac{\mu\tau}{2c^2}} \epsilon^{-5/2} \left\{ \int_{|v| \leq N, |v'| \geq 2N} + \int_{|v'| \leq 2N, |v''| \geq 3N} \right\} \leq Ce^{-\frac{\varepsilon}{8}N^2} e^{-\frac{\mu\tau}{2c^2}} \epsilon^{-2} \int_0^s e^{-\frac{\mu\tau(s-t)}{2c^2}} e^{-\frac{\mu\tau}{2c^2}} d\tau ds
\]

\[
\leq Ce^{-\frac{\varepsilon}{8}N^2} \epsilon^{-\frac{\mu(s-t)}{2c^2}} \epsilon^{-3/2} \sup_{0 \leq \tau \leq t} \left\{ e^{-\frac{\mu\tau}{2c^2}} \|h(\tau)\|_\infty \right\} \int_0^s e^{-\frac{\mu\tau(s-t)}{2c^2}} e^{-\frac{\mu\tau}{2c^2}} d\tau ds \quad (5.7)
\]

Case 3: \( |v| \leq N, |v'| \leq 2N, |v''| \leq 3N \). In this situation, the velocity domain is bounded and most importantly there is a lower bound \( s - \tau > \kappa e^2 \), which ensures us to convert the \( L^\infty \) norm into \( L^2 \) norm. To do so, for any large \( N > 0 \), we first choose a number \( m(N) \) to define

\[
k_{w,m}(p, v') \equiv 1_{|p-v'| \geq \frac{1}{m}} 1_{|v'| \leq m} k_w(p, v'),
\]

such that \( \sup_p \int_{\mathbb{R}^3} |k_{w,m}(p, v') - k_w(p, v')| dv' \leq \frac{1}{N} \). We then split

\[
k_w(v, v')k_w(v', v'') = \{k_w(v, v') - k_{w,m}(v, v')\}k_w(v', v'') + \{k_w(v', v'') - k_{w,m}(v', v'')\}k_{w,m}(v, v') + k_{w,m}(v, v')k_{w,m}(v', v''),
\]

one can use such an approximation to bound the above \( J_{2,2} \) by

\[
\frac{Ce^{-\frac{\mu\tau}{2c^2}}}{N} \epsilon^{3/2} \sup_{0 \leq \tau \leq t} \left\{ e^{-\frac{\mu\tau}{2c^2}} \|h(\tau)\|_\infty \right\} \left\{ \sup_{|v'| \leq 2N} \int |k_w(v', v'')| dv'' + \sup_{|v'| \leq 2N} \int |k_{w,m}(v, v')| dv' \right\}
\]

\[
+ C \int_0^t \int_0^{s-\kappa e^2} \int_{|v'| \leq 2N, |v''| \leq 3N} e^{-\frac{\mu\tau(s-t)}{c^2}} e^{-\frac{\mu\tau}{c^2}} x^{-5/2} k_{w,m}(v, v')k_{w,m}(v', v'') |h(\tau, X_\epsilon(\tau; X_\epsilon(s), v'), v'')| dv' dv'' \quad (5.8)
\]
Next, by a change of variable $y = X_\epsilon(\tau; X_\epsilon(s), v') = x + \frac{(s-\tau)v + (s-\tau)v'}{\epsilon}$, and for $s - \tau \geq \kappa \epsilon^2$, we can further control the last term in (5.8) by:

$$C_N \int_0^t \int_0^{s-\kappa \epsilon^2} e^{-\frac{\nu_0(t-s)}{\epsilon^2}} e^{-\frac{\nu_0(s-\tau)}{\epsilon^2}} e^{-5/2} \int_{|v''| \leq 3N} \left\{ \int_{|y - X_\epsilon(s)| \leq 2(s-\tau)N} |h(\tau, y, v'')|^2 dy \right\}^{1/2} dv'' d\tau ds$$

$$\leq C_N \int_0^t \int_0^{s-\kappa \epsilon^2} e^{-\frac{\nu_0(t-s)}{\epsilon^2}} e^{-\frac{\nu_0(s-\tau)}{\epsilon^2}} e^{-5/2} \left[ \frac{s - \tau}{\epsilon} \right]^{3/2} \\ \times \left\{ \int_{|v''| \leq 3N} \int_{T^3} |h(\tau, y, v'')|^2 dy dv'' \right\}^{1/2} d\tau ds$$

$$\leq C_N e^{-\lambda t} \sup_{s \geq 0} \left\{ e^{\lambda s} \| f(s) \|_2 \right\} \int_0^t \int_0^{s-\kappa \epsilon^2} e^{-\frac{\nu_0(t-s)}{2\epsilon^2}} e^{-\frac{\nu_0(s-\tau)}{2\epsilon^2}} e^{-\lambda t} e^{-\frac{s - \tau}{\epsilon} \left[ \frac{(s - \tau)}{\epsilon} \right]^{3/2}} d\tau ds$$

(5.9)

where we have used the fact that $\lambda \leq \frac{\nu_0}{2\epsilon^2}$.

Inserting (5.1), (5.3), (5.6), (5.7), (5.8), and (5.9) into (5.3), one can see that (5.2) is true, which concludes the proof of Proposition 5.1.

\[ \square \]

### 6. Global existence and time decay

In this final section, we will prove the global existence and exponential time decay of the solutions to the equation (1.11) in $L^2 \cap L^\infty$-framework. That is we intend to complete

The proof of Theorem 1.1. Recall the Cauchy problem for the linearized equation (4.2) or (5.1), to prove the global existence of (1.11) with $R(0, x, v) = R_0(x, v)$, let us first design the following iteration sequence

$$\begin{cases}
\epsilon \partial_t R^{\ell+1} + v \cdot \nabla_x R^{\ell+1} + \frac{1}{\epsilon} LR^{\ell+1} = g(R^{\ell}), \\
R^{\ell+1}(0, x, v) = R_0(x, v), \quad R^0 = R_0(x, v), \quad x \in T^3, \quad v \in \mathbb{R}^3,
\end{cases}$$

(6.1)

where $g(R^{\ell})$ is defined by

$$g(R^{\ell}) = \left\{ \Gamma(f_1, R^{\ell}) + \Gamma(R^{\ell}, f_1) \right\} + \epsilon \left\{ \Gamma(f_2, R^{\ell}) + \Gamma(R^{\ell}, f_2) \right\}$$

$$\quad + \epsilon^2 \left\{ \Gamma(f_3, R^{\ell}) + \Gamma(R^{\ell}, f_3) \right\} + \epsilon^3 \left\{ \Gamma(f_4, R^{\ell}) + \Gamma(R^{\ell}, f_4) \right\}$$

$$\quad + \epsilon^4 \left\{ \Gamma(f_5, R^{\ell}) + \Gamma(R^{\ell}, f_5) \right\} + \epsilon^5 \left\{ \Gamma(f_6, R^{\ell}) + \Gamma(R^{\ell}, f_6) \right\}$$

$$\quad + \epsilon^{4+\beta} \Gamma(R^{\ell}, R^{\ell}) - \epsilon^{1+\beta} \{ \partial_t f_5 + v \cdot \nabla f_6 \} - \epsilon^{2+\beta} \partial_t f_6.$$

(6.2)

Clearly, (6.2) satisfies the conditions listed in Proposition 4.1 with $g = g(R^{\ell})$.

It is important to note that the iteration scheme (6.2) does not provide us the positivity of the solution of the original equation (1.11), however it coincides with the linearized equation (4.2)
so that Propositions 4.1 and 5.1 can be directly used. Let us now define the following energy functional

$$\mathcal{E}(f)(t) = e^{2\lambda} \|w t f(t)\|_\infty^2 + e^{2\lambda} \|f(t)\|_2^2,$$

and dissipation rate

$$\mathcal{D}(f)(t) = e^{-2} e^{2\lambda} \|\{I - P\} f(t)\|_\nu^2 + e^{2\lambda} \|P f(t)\|_2^2.$$

For later use, we also define a Banach space

$$X_\delta(t) = \left\{ f \mid \sup_{0 \leq s \leq t} \mathcal{E}(f)(s) + \int_0^t \mathcal{D}(f)(s)ds < \delta, \ \delta > 0 \right\},$$

equipped with the norm

$$\|f\|_{X_\delta} = \sup_{0 \leq s \leq t} \mathcal{E}(f)(s) + \int_0^t \mathcal{D}(f)(s)ds.$$

We now show that $R^{t+1} \in X_\delta$ if $R^t \in X_\delta$. For this, on the one hand, we know from (4.4) and (5.2) with $f = R^{t+1}$ and $g = g(R^t)$ that (6.2) admits a unique solution $R^{t+1}$ satisfying

$$\sup_{0 \leq s \leq t} \mathcal{E}(R^{t+1})(s) + \int_0^t \mathcal{D}(R^{t+1})(s)ds \leq C \mathcal{E}(f)(0) + C e^\lambda \sup_{0 \leq s \leq t} e^{2\lambda} \|\nu^{-1} w g(R^t)(s)\|_\infty^2$$

$$+ C \int_0^t e^{2\lambda} \|\nu^{-1/2} (I - P) g(R^t)(s)\|_2^2 ds + C e^{-2} \int_0^t e^{2\lambda} \|\nu^{-1/2} P g(R^t)(s)\|_2^2 ds. \quad (6.3)$$

On the other hand, thanks to Lemmas 2.4 and 2.3 as well as Proposition 3.1, it follows for $\lambda_0 > \lambda > 0$ and $l > 3/2$

$$\int_0^t e^{2\lambda} \|\nu^{-1/2} (I - P) g(R^t)(s)\|_2^2 ds \leq C \sup_{0 \leq s \leq t} \|w \nu f_1(s)\|_\infty^2 \int_0^t e^{2\lambda} \|R^t(s)\|_\nu^2 ds + C \sum_{i=2}^6 e^{2(i-1)} \|w \nu f_i(s)\|_\infty^2 \int_0^t e^{2\lambda} \|R^t(s)\|_\nu^2 ds$$

$$+ C e^{8-2\beta} \sup_{0 \leq s \leq t} \|w R^t(s)\|_\nu^2 \int_0^t e^{2\lambda} \|R^t(s)\|_\nu^2 ds + e^{2+2\beta} \int_0^t e^{2\lambda} \|\{I - P\} (\partial_t f_5 + v \cdot \nabla_x f_6)\|_2^2 ds$$

$$+ e^{4+2\beta} \int_0^t e^{2\lambda} \|\partial_t f_6\|_2^2 ds \leq C \left( \sup_{0 \leq s \leq t} \mathcal{E}(R^t)(s) + \varepsilon_0^2 + e^2 P^2 (\|[u_{1,0}, \theta_{1,0}]\|_{H^{14}} + \|[u_{2,0}, \theta_{2,0}]\|_{H^{12}}) \right) \int_0^t \mathcal{D}(R^t)(s)ds$$

$$+ e^{2+2\beta} \int_0^t e^{2\lambda} e^{-2\lambda_0} P^2 (\|[u_{1,0}, \theta_{1,0}]\|_{H^{14}} + \|[u_{2,0}, \theta_{2,0}]\|_{H^{12}}) ds, \quad (6.4)$$

$$e^{-2} \int_0^t e^{2\lambda} \|\nu^{-1/2} P g(R^t)(s)\|_2^2 ds \leq C e^{2\beta} \int_0^t e^{2\lambda} \|P (\partial_t f_5 + v \cdot \nabla_x f_6)\|_2^2 ds \quad (6.5)$$

$$\leq C e^{2\beta} \int_0^t e^{2\lambda} e^{-2\lambda_0} P^2 (\|[u_{1,0}, \theta_{1,0}]\|_{H^{14}} + \|[u_{2,0}, \theta_{2,0}]\|_{H^{12}}) ds \leq C e^{2\beta} P^2 (\|[u_{1,0}, \theta_{1,0}]\|_{H^{14}} + \|[u_{2,0}, \theta_{2,0}]\|_{H^{12}}),$$
and
\[
\sup_{0 \leq s \leq t} e^{2\lambda s} \left\| \nu^{-1} w_l g(R^\ell(s)) \right\|_\infty^2
\leq C \sup_{0 \leq s \leq t} \left\| w_l f_1(s) \right\|_\infty^2 \sup_{0 \leq s \leq t} e^{2\lambda s} \left\| w_l R^\ell(s) \right\|_\infty^2
\]
\[
+ C \sum_{i=2}^6 e^{2(i-1)} \sup_{0 \leq s \leq t} \left\| w_l f_i(s) \right\|_\infty^2 \sup_{0 \leq s \leq t} e^{2\lambda s} \left\| w_l R^\ell(s) \right\|_\infty^2
\]
\[
+ C e^{8-2\beta} \sup_{0 \leq s \leq t} \left\| w_l R^\ell(s) \right\|_\infty^2 \sup_{0 \leq s \leq t} e^{2\lambda s} \left\| w_l R^\ell(s) \right\|_\infty^2
\]
\[
+ e^{2+2\beta} \sup_{0 \leq s \leq t} e^{2\lambda s} \left\| w_l \{ \partial_t f_5 + v \cdot \nabla_x f_6 \} \right\|_\infty^2 + e^{4+2\beta} \sup_{0 \leq s \leq t} e^{2\lambda s} \left\| w_l \partial_t f_6 \right\|_\infty^2 ds
\]
\[
\leq C \left\{ \sup_{0 \leq s \leq t} \mathcal{E}(R^\ell(s)) + \varepsilon_0^2 + e^2 P^2 (\|[u_{1,0},\theta_{1,0}]\|_{H^{16}} + \|[u_{2,0},\theta_{2,0}]\|_{H^{14}}) \right\} \sup_{0 \leq s \leq t} \mathcal{E}(R^\ell(s))
\]
\[
+ e^{2+2\beta} \sup_{0 \leq s \leq t} \left\{ e^{-2\lambda s} e^{2\lambda s} P^2 (\|[u_{1,0},\theta_{1,0}]\|_{H^{16}} + \|[u_{2,0},\theta_{2,0}]\|_{H^{14}}) \right\}.
\]

To this end, one has from \((6.3)\), \((6.4)\), \((6.5)\) and \((6.6)\) that
\[
X_\delta(R^{\ell+1})(t) \leq C \mathcal{E}(R_0(0)) + e^{2\beta} \left\{ \varepsilon_0^2 + e^2 P^2 (\|[u_{1,0},\theta_{1,0}]\|_{H^{16}} + \|[u_{2,0},\theta_{2,0}]\|_{H^{14}}) \right\} + C X_\delta(R^\ell)(t)
\]
\[
+ C X_\delta^2(R^\ell)(t),
\]
which further implies \(X_\delta(R^{\ell+1})(t) < \delta\) if \(R^\ell \in X_\delta\) with \(\delta, \varepsilon_0, \epsilon\) and \(\mathcal{E}(R_0)\) being small enough.

In what follows we prove the strong convergence of the iteration sequence \((R^\ell)\) \(_{\ell=0}^\infty\) constructed above. To do this, by taking difference of the equations that \(R^{\ell+1}\) and \(R^\ell\) satisfy, we deduce that
\[
e^\ell \nu_l[R^{\ell+1} - R^\ell] + v \cdot \nabla_x [R^{\ell+1} - R^\ell] + \frac{1}{\epsilon} L[R^{\ell+1} - R^\ell] = g(R^\ell) - g(R^{\ell-1}),
\]
with \(R^{\ell+1} - R^\ell = 0\) initially. By the same fashion as for obtaining \((6.7)\), one obtains
\[
X_\delta(R^{\ell+1} - R^\ell)(t) \leq C \left\{ \varepsilon_0^2 + e^2 P^2 (\|[u_{1,0},\theta_{1,0}]\|_{H^{16}} + \|[u_{2,0},\theta_{2,0}]\|_{H^{14}}) \right\} X_\delta(R^\ell - R^{\ell-1})(t)
\]
\[
+ C \left\{ X_\delta(R^\ell) + X_\delta(R^{\ell-1}) \right\} X_\delta(R^\ell - R^{\ell-1})(t).
\]
Thus \((R^\ell)\) \(_{\ell=0}^\infty\) is a Cauchy sequence in \(X_\delta\) for \(\delta\) suitably small. Moreover, take \(R\) as the limit of the sequence \((R^\ell)\) \(_{\ell=0}^\infty\) in \(X_\delta\), then \(R\) satisfies
\[
\sup_{0 \leq s \leq t} \mathcal{E}(R)(s) + \int_0^t D(R)(s) ds \leq CE(R)(0) + C e^{2\beta} \left\{ P^2 (\|[u_{1,0},\theta_{1,0}]\|_{H^{16}} + \|[u_{2,0},\theta_{2,0}]\|_{H^{14}}) \right\}.
\]
The proof for the uniqueness of the solution obtained above is standard, and the proof of the positivity of \(\mu + \epsilon \sqrt{\nu} \left\{ \sum_{i=1}^6 e^{i-1} f_i + e^{4-i} R \right\} \) is the same as that of Section 3.8 in [13] pp.66 and thus will be omitted. This ends the proof of Theorem I.4.

\[\square\]

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