Degenerations of nilpotent associative commutative algebras

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ABSTRACT
We give a complete description of degenerations of complex 5-dimensional nilpotent associative commutative algebras. As corollary, we have the description of all rigid algebras in the variety of 5-dimensional commutative Leibniz algebras.

1. Introduction
There are many results related to the algebraic and geometric classification of low-dimensional algebras in the varieties of Jordan, Lie, Leibniz and Zinbiel algebras; for algebraic classifications see, for example, [1, 10, 14, 19–22, 25, 29]; for geometric classifications and descriptions of degenerations see, for example, [1–30]. Degeneration of algebras is an interesting subject, which has been studied in various papers. In particular, there are many results concerning degenerations of algebras of small dimensions in a variety defined by a set of identities. One of the important problems in this direction is a description of the so-called rigid algebras. These algebras are of special interest, since the closures of their orbits under the action of the generalized linear group form irreducible components of the variety under consideration (with respect to the Zariski topology). For example, rigid algebras in the varieties of all 4-dimensional Leibniz algebras [18], all 4-dimensional nilpotent Novikov algebras [20], all 4-dimensional nilpotent terminal algebras [21], all 4-dimensional nilpotent bicommutative algebras [22], all 6-dimensional nilpotent binary Lie algebras [1], and in some other varieties, were classified. There are fewer works in which the full information about degenerations was given for some variety of algebras. This problem was solved for 2-dimensional pre-Lie algebras [6], for 2-dimensional terminal algebras [9], for 3-dimensional Novikov algebras [7], for 3-dimensional Jordan algebras [15], for 3-dimensional Jordan superalgebras [5], for 3-dimensional Leibniz and 3-dimensional anticommutative algebras [19], for 4-dimensional Lie algebras [8], for 4-dimensional Lie superalgebras [4], for 4-dimensional Zinbiel and 4-dimensional nilpotent Leibniz algebras [23], for 5-dimensional nilpotent Tortkara algebras [14], for 6-dimensional nilpotent Lie algebras [16, 30], for 6-dimensional nilpotent Malcev algebras [24], for 7-dimensional 2-step nilpotent Lie algebras [3], and for all 2-
dimensional algebras [25]. Here we construct the graphs of primary degenerations for the variety of complex 5-dimensional nilpotent associative commutative algebras.

2. Degenerations of algebras

Given an \( n \)-dimensional vector space \( V \), the set \( \text{Hom}(V \otimes V, V) \cong V^* \otimes V^* \otimes V \) is a vector space of dimension \( n^3 \). This space inherits the structure of the affine variety \( \mathbb{C}^n \). Indeed, let us fix a basis \( e_1, \ldots, e_n \) of \( V \). Then any \( \mu \in \text{Hom}(V \otimes V, V) \) is determined by \( n^3 \) structure constants \( c_{ij}^k \in \mathbb{C} \) such that \( \mu(e_i \otimes e_j) = \sum_{k=1}^n c_{ij}^k e_k \). A subset of \( \text{Hom}(V \otimes V, V) \) is Zariski-closed if it can be defined by a set of polynomial equations in the variables \( c_{ij}^k \) (\( 1 \leq i,j,k \leq n \)).

The general linear group \( \text{GL}(V) \) acts by conjugation on the variety \( \text{Hom}(V \otimes V, V) \) of all algebra structures on \( V \):

\[
(g \ast \mu)(x \otimes y) = g\mu(g^{-1}x \otimes g^{-1}y),
\]

for \( x, y \in V, \mu \in \text{Hom}(V \otimes V, V) \) and \( g \in \text{GL}(V) \). Clearly, the \( \text{GL}(V) \)-orbits correspond to the isomorphism classes of algebra structures on \( V \). Let \( T \) be a set of polynomial identities which is invariant under isomorphism. Then the subset \( \mathbb{L}(T) \subset \text{Hom}(V \otimes V, V) \) of the algebra structures on \( V \) which satisfy the identities in \( T \) is \( \text{GL}(V) \)-invariant and Zariski-closed. It follows that \( \mathbb{L}(T) \) decomposes into \( \text{GL}(V) \)-orbits. The \( \text{GL}(V) \)-orbit of \( \mu \in \mathbb{L}(T) \) is denoted by \( O(\mu) \) and its Zariski closure by \( \overline{O(\mu)} \).

Let \( A \) and \( B \) be two \( n \)-dimensional algebras satisfying the identities from \( T \) and \( \mu, \lambda \in \mathbb{L}(T) \) represent \( A \) and \( B \), respectively. We say that \( A \) degenerates to \( B \) and write \( A \rightarrow B \) if \( \lambda \in \overline{O(\mu)} \). Note that in this case we have \( \overline{O(\lambda)} \subset \overline{O(\mu)} \). Hence, the definition of degeneration does not depend on the choice of \( \mu \) and \( \lambda \). If \( A \not\approx B \), then the assertion \( A \rightarrow B \) is called a proper degeneration. We write \( A \not\rightarrow B \) if \( \lambda \not\in \overline{O(\mu)} \).

Let \( A \) be represented by \( \mu \in \mathbb{L}(T) \). Then \( A \) is rigid in \( \mathbb{L}(T) \) if \( O(\mu) \) is an open subset of \( \mathbb{L}(T) \). Recall that a subset of a variety is called irreducible if it cannot be represented as a union of two nontrivial closed subsets. A maximal irreducible closed subset of a variety is called an irreducible component. It is well known that any affine variety can be represented as a finite union of its irreducible components in a unique way. The algebra \( A \) is rigid in \( \mathbb{L}(T) \) if and only if \( \overline{O(\mu)} \) is an irreducible component of \( \mathbb{L}(T) \).

In the present work, we use the methods applied to Lie algebras in [8, 16, 17, 30]. First of all, if \( A \rightarrow B \) and \( A \not\approx B \), then \( \dim \text{Der}(A) < \dim \text{Der}(B) \), where \( \text{Der}(A) \) is the Lie algebra of derivations of \( A \). We will compute the dimensions of the algebras of derivations and will check the assertion \( A \rightarrow B \) only for those \( A \) and \( B \) such that \( \dim \text{Der}(A) < \dim \text{Der}(B) \). Secondly, if \( A \rightarrow C \) and \( C \rightarrow B \), then \( A \rightarrow B \). If there is no \( C \) such that \( A \rightarrow C \) and \( C \rightarrow B \) are proper degenerations, then the assertion \( A \rightarrow B \) is called a primary degeneration. If \( \dim \text{Der}(A) < \dim \text{Der}(B) \) and there are no \( C \) and \( D \) such that \( C \rightarrow A, B \rightarrow D, C \not\rightarrow D \) and one of the assertions \( C \rightarrow A \) and \( B \rightarrow D \) is a proper degeneration, then the assertion \( A \not\rightarrow B \) is called a primary non-degeneration. It suffices to prove only primary degenerations and nondegenerations to describe degenerations in the variety under consideration. It is easy to see that any algebra degenerates to the algebra with zero multiplication. From now on we use this fact without mentioning it.

To prove primary degenerations, we will construct families of matrices parametrized by \( t \in \mathbb{C}^* \). Namely, let \( A \) and \( B \) be two algebras represented by the structures \( \mu \) and \( \lambda \) from \( \mathbb{L}(T) \), respectively. Let \( e_1, \ldots, e_n \) be a basis of \( V \) and \( c_{ij}^k(t) \) (\( 1 \leq i,j,k \leq n \)) be the structure constants of \( \lambda \) in this basis. If there exist \( a_i(t) \in \mathbb{C} \) (\( 1 \leq i,j \leq n, t \in \mathbb{C}^* \)) such that \( E_i(t) = \sum_{j=1}^n a_i(t)e_j \) (\( 1 \leq i \leq n \)) form a basis of \( V \).
for any \( t \in \mathbb{C}^* \), and the structure constants \( c^k_{i,j}(t) \) of \( \mu \) in the basis \( E^i_1, ..., E^i_n \) satisfy \( \lim_{t \to 0} c^k_{i,j}(t) = c^k_{i,j} \), then \( A \to B \). In this case, \( E^i_1, ..., E^i_n \) is called a parametric basis for \( A \to B \).

To prove primary nondegenerations we will use the following lemma (see [16]).

**Lemma 1.** Let \( B \) be a Borel subgroup of \( GL(V) \) and \( R \subset \mathbb{L}(T) \) be a \( B \)-stable closed subset. If \( A \to B \) and \( A \) can be represented by \( \mu \in R \) then there is \( \lambda \in R \) that represents \( B \).

In what follows, each time we need to prove some primary nondegeneration \( \mu \not\sim \lambda \), we will define \( R \) by a set of polynomial equations in structure constants \( c^k_{i,j} \) in such a way that the structure constants of \( \mu \) in the basis \( e_1, ..., e_n \) satisfy these equations. We will omit everywhere the verification of the fact that \( R \) is stable under the action of the subgroup of lower triangular matrices and of the fact that \( \lambda \not\in R \) for any choice of basis of \( V \). To simplify our equations, we will use the notation \( A_i = \langle e_1, ..., e_n \rangle \), \( i = 1, ..., n \) and write simply \( A_pA_q \subset A_r \) instead of \( c^k_{i,j} = 0 \) \( (i \geq p, j \geq q, k < r) \).

If the number of orbits under the action of \( GL(V) \) on \( \mathbb{L}(T) \) is finite, then the graph of primary degenerations gives the whole picture. In particular, the description of rigid algebras and irreducible components can be easily obtained.

### 3. Nilpotent associative commutative algebras

The algebraic classification of 5-dimensional nilpotent associative commutative algebras was given in [29] (see, Table 1). Also, in the same paper, it was proved that the variety of all 5-dimensional nilpotent associative commutative algebras has only one irreducible component. The main result of the present section is the following theorem.

**Theorem 2.** The graph of all degenerations in the variety of 5-dimensional nilpotent associative commutative algebras is given in Figure 1 below.

**Proof.** Tables 2 and 3 give the proofs for all primary degenerations and nondegenerations.

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**Table 1.** 5-dimensional nilpotent associative commutative algebras.

| \( A \) | \( T \cap A \) | Multiplication table |
|-------|--------|-------------------|
| \( A_{01} \) | 5 | \( e^1_1 = e_2, \quad e^1_2 = e_4, \quad e_1 e_2 = e_4, \quad e_1 e_4 = e_5, \quad e_2 e_3 = e_5 \) |
| \( A_{02} \) | 6 | \( e^1_1 = e_3, \quad e^1_2 = e_4, \quad e^1_3 = e_5, \quad e_1 e_3 = e_5, \quad e_1 e_4 = e_5 \) |
| \( A_{03} \) | 6 | \( e^1_1 = e_3, \quad e^1_2 = e_4, \quad e^1_3 = e_5, \quad e_1 e_3 = e_5, \quad e_2 e_4 = e_5 \) |
| \( A_{04} \) | 7 | \( e^1_1 = e_3, \quad e_1 e_2 = e_4, \quad e_1 e_4 = e_5, \quad e_2 e_3 = e_5 \) |
| \( A_{05} \) | 7 | \( e^1_1 = e_3, \quad e^1_2 = e_4, \quad e^1_3 = e_5, \quad e_1 e_2 = e_3, \quad e_1 e_3 = e_4 \) |
| \( A_{06} \) | 7 | \( e^1_1 = e_2, \quad e^1_2 = e_3, \quad e^1_3 = e_5, \quad e^1_4 = e_5 \) |
| \( A_{07} \) | 7 | \( e_1 e_2 = e_4, \quad e_2 e_3 = e_5, \quad e_1 e_4 + e_5 \) |
| \( A_{08} \) | 8 | \( e^1_1 = e_3, \quad e^1_2 = e_4, \quad e^1_3 = e_5, \quad e^1_4 = e_4, \quad e_1 e_2 = e_5 \) |
| \( A_{09} \) | 8 | \( e^1_1 = e_2, \quad e^1_2 = e_3, \quad e^1_3 = e_4, \quad e^1_4 = e_5 \) |
| \( A_{10} \) | 9 | \( e^1_1 = e_3, \quad e^1_2 = e_4, \quad e^1_3 = e_5, \quad e^1_4 = e_5 \) |
| \( A_{11} \) | 9 | \( e^1_1 = e_3, \quad e^1_2 = e_4, \quad e^1_3 = e_4, \quad e^1_5 = e_5 \) |
| \( A_{12} \) | 11 | \( e_1 e_2 = e_4, \quad e_1 e_3 = e_5, \quad e_2 e_3 = e_5 \) |
| \( A_{13} \) | 8 | \( e^1_1 = e_4, \quad e^1_2 = e_5, \quad e^1_3 = e_5 \) |
| \( A_{14} \) | 9 | \( e^1_1 = e_5, \quad e^1_2 = e_4, \quad e_1 e_3 = e_4 \) |
| \( A_{15} \) | 9 | \( e_1 e_2 = e_3, \quad e^1_3 = e_5 \) |
| \( A_{16} \) | 10 | \( e^1_1 = e_3, \quad e^1_2 = e_4, \quad e^1_3 = e_5 \) |
| \( A_{17} \) | 10 | \( e^1_1 = e_4, \quad e^1_2 = e_5, \quad e^1_3 = e_5 \) |
| \( A_{18} \) | 11 | \( e^1_1 = e_2, \quad e^1_2 = e_3 \) |
| \( A_{19} \) | 11 | \( e^1_1 = e_3, \quad e^1_2 = e_4 \) |
| \( A_{20} \) | 12 | \( e^1_1 = e_3, \quad e^1_2 = e_4 \) |
| \( A_{21} \) | 11 | \( e^1_1 = e_3, \quad e^1_2 = e_5 \) |
| \( A_{22} \) | 12 | \( e^1_1 = e_4, \quad e^1_2 = e_4 \) |
| \( A_{23} \) | 14 | \( e_1 e_2 = e_3 \) |
| \( A_{24} \) | 17 | \( e^1_1 = e_2 \) |
Figure 1. The graph of degenerations of 5-dimensional nilpotent associative commutative algebras.
Table 2. Degenerations of 5-dimensional nilpotent associative commutative algebras.

| $\mathcal{A}_0$ → $\mathcal{A}_1$ | $E_1$ = te_1 | $E_1^* =$ | $E_2^* =$ | $E_3^* =$ |
|----------------|-------------|----------|------------|------------|
| $\mathcal{A}_0$ = A1 | $e_1 = t^2 e_2$ | $e_4 - t^{-3} e_3$ | $e_1 = t e_1 + 2 t - 1$ | $1 - 5 t + 5 t^2 e_3^* + \frac{3 + 9 t - 7 t^2}{2(2 - 3 t)^2} e_4^*$ | $-1 + t + t^2 e_1 + \frac{3 + 9 t - 7 t^2}{4t(-2 + 3t)^3} e_4^*$ |
| $\mathcal{A}_0$ = A2 | $e_4$ | $e_1$ | $e_1$ | $e_1$ |
| $\mathcal{A}_0$ = A3 | $-t e_2$ | $e_1$ | $e_1$ | $-\frac{1}{3} e_1$ |
| $\mathcal{A}_0$ = A4 | $t e_3$ | $e_1$ | $e_1$ | $e_1$ |
| $\mathcal{A}_0$ = A5 | $-t e_3 + t e_4$ | $e_1$ | $e_1$ | $e_1$ |
| $\mathcal{A}_0$ = A6 | $e_5$ | $e_1$ | $e_1$ | $e_1$ |
| $\mathcal{A}_0$ = A7 | $e_5$ | $e_1$ | $e_1$ | $e_1$ |
| $\mathcal{A}_0$ = A8 | $-t e_4$ | $e_1$ | $e_1$ | $e_1$ |
| $\mathcal{A}_0$ = A9 | $t e_5 + t e_4$ | $e_1$ | $e_1$ | $e_1$ |
| $\mathcal{A}_0$ = A10 | $e_1$ + $\frac{1}{2} e_2$ | $e_1$ | $e_1$ | $e_1$ |
| $\mathcal{A}_0$ = A11 | $e_1$ + $\frac{1}{2} e_4$ | $e_1$ | $e_1$ | $e_1$ |
| $\mathcal{A}_0$ = A12 | $e_1$ + $\frac{1}{2} e_4$ | $e_1$ | $e_1$ | $e_1$ |
| $\mathcal{A}_0$ = A13 | $e_1$ + $\frac{1}{2} e_4$ | $e_1$ | $e_1$ | $e_1$ |
| $\mathcal{A}_0$ = A14 | $e_1$ + $\frac{1}{2} e_4$ | $e_1$ | $e_1$ | $e_1$ |
| $\mathcal{A}_0$ = A15 | $e_1$ + $\frac{1}{2} e_4$ | $e_1$ | $e_1$ | $e_1$ |
| $\mathcal{A}_0$ = A16 | $e_1$ + $\frac{1}{2} e_4$ | $e_1$ | $e_1$ | $e_1$ |
| $\mathcal{A}_0$ = A17 | $e_1$ + $\frac{1}{2} e_4$ | $e_1$ | $e_1$ | $e_1$ |
| $\mathcal{A}_0$ = A18 | $e_1$ + $\frac{1}{2} e_4$ | $e_1$ | $e_1$ | $e_1$ |
| $\mathcal{A}_0$ = A19 | $e_1$ + $\frac{1}{2} e_4$ | $e_1$ | $e_1$ | $e_1$ |
| $\mathcal{A}_0$ = A20 | $e_1$ + $\frac{1}{2} e_4$ | $e_1$ | $e_1$ | $e_1$ |
| $\mathcal{A}_0$ = A21 | $e_1$ + $\frac{1}{2} e_4$ | $e_1$ | $e_1$ | $e_1$ |

(continued)
Table 3. Non-degenerations of 5-dimensional nilpotent associative commutative algebras.

| Nondegeneration | Arguments |
|-----------------|-----------|
| A_{23} \Leftrightarrow A_{20} | \mathcal{R} = \{ A_1^i = 0, A_1^j = 0, A_1 \cdot A_2 \subseteq A_3 \} |
| A_{22} \Leftrightarrow A_{16} | \mathcal{R} = \{ A_1 A_2 = 0, c_{12}^1 \cdot c_{12}^2 = c_{12}^2 \cdot c_{12}^1 \} |
| A_{21} \Leftrightarrow A_{20} | \mathcal{R} = \{ \dim \text{Ann}(A_{21}) = 1 \} |
| A_{20} \Leftrightarrow A_{16} | \mathcal{R} = \{ A_1^i \subseteq A_1, A_1^j = 0 \} |
| A_{19} \Leftrightarrow A_{18} | \mathcal{R} = \{ A_1^k = 0 \} |
| A_{18} \Leftrightarrow A_{17} | \mathcal{R} = \{ A_1^i \subseteq A_1, A_1 \cdot A_2 \subseteq A_3, f_1 = e_1, f_2 = e_2, f_3 = e_3, f_4 = e_4, f_5 = e_5 \} |
| A_{17} \Leftrightarrow A_{12} | \mathcal{R} = \{ A_1 A_3 = 0, A_1^i = 0 \} |

4. Commutative Leibniz algebras

Recall that an algebra \( L \) is called a Leibniz algebra if the identity \((xy)z = (xz)y + x(yz)\) holds in \( L \). Let us consider the class of commutative Leibniz algebras. Let \( \mathcal{C}^{\text{leib}}_n \) be the variety of \( n \)-dimensional commutative Leibniz algebras. It is easy to see that every commutative Leibniz algebra is a 2-step nilpotent associative and commutative algebra. Therefore, it is a central extension of an algebra with zero multiplication. Using the algebraic classification of complex 5-dimensional nilpotent associative commutative algebras and Theorem 2, we have the following

**Theorem 4.** Let \( L \) be a non-trivial complex 5-dimensional commutative Leibniz algebra. Then \( L \) is isomorphic to an algebra from the following list:

\[ A_{07}, A_{09}, A_{11}, A_{12}, A_{15}, A_{16}, A_{17}, A_{19}, A_{20}, A_{21}, A_{22}, A_{23}, A_{24}. \]

The variety \( \mathcal{C}^{\text{leib}}_5 \) has 3 irreducible components defined by the rigid algebras \( A_{07}, A_{16}, A_{21} \).

Analyzing rigid 5-dimensional commutative Leibniz algebras, we can obtain the following useful lemmas.

**Lemma 5.** Let \( A \) be a commutative algebra with the basis \( \{ e_i : 1 \leq i \leq n \} \) and the multiplication given by \( e_i e_j = e_{i+j} \), then \( A \) is rigid in the variety \( \mathcal{C}^{\text{leib}}_n \).

**Proof.** Let \( \mathbb{C}^k \) be the \( k \)-dimensional algebra with zero multiplication. Then its maximal nonsplit central extension is of dimension \( k + k^2 \), and as the 1-dimensional central extensions of \( \mathbb{C}^k \) are characterized by \( \text{Aut}(\mathbb{C}^k) = \text{GL}_k \)-orbits on the Grassmannian of \( k \)-dimensional subspaces on \( H^2(\mathbb{C}^k, \mathbb{C}) = \mathbb{C}^{k^2} \), there is a unique extension of \( \mathbb{C}^k \) of this dimension up to isomorphism (about central extensions of algebras see, for example, [1, 10, 14, 20, 21]). Now, as commutative Leibniz
algebras are extensions of algebras with zero multiplication, one can see that \( A \) has maximal dimension of \( A^2 \) among the algebras in the variety \( \mathcal{Lieb}_{\frac{n+1}{2}} \) and any algebra with the same dimension of \( A^2 \) in the variety \( \mathcal{Lieb}_{\frac{n+1}{2}} \) is isomorphic to \( A \).

\[ \square \]

**Lemma 7.** Let \( A \) be a commutative algebra with the basis \( \{e_0, e_i\}_{1 \leq i \leq n} \) and the multiplication given by \( e_i^2 = e_0 \), then \( A \) is rigid in the variety \( \mathcal{Lieb}_{n+1} \).

**Proof.** The algebra \( A \) has minimal dimension of \( \text{Ann} A \) among the algebras in the variety \( \mathcal{Lieb}_{n+1} \). Any algebra with the 1-dimensional annihilator in \( \mathcal{Lieb}_{n+1} \) must be an extension of the \( n \)-dimensional algebra with zero multiplication and the cocycle defining this extension is a nondegenerate symmetric bilinear form. This form can be diagonalized, giving us the multiplication in \( A \).

\[ \square \]

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**References**

[1] Abdelwahab, H., Calderón, A. J., Kaygorodov, I. (2019). The algebraic and geometric classification of nilpotent binary lie algebras. *Int. J. Algebra Comput.* 29(06):1113–1129. DOI: 10.1142/S0218196719500437.

[2] Alvarez, M. A. (2018). On rigid 2-step nilpotent lie algebras. *Algebra Colloq.* 25(2):349–360.

[3] Alvarez, M. A. (2018). The variety of 7-dimensional 2-step nilpotent lie algebras. *Symmetry* 10(1):26. DOI: 10.3390/sym10010026.

[4] Alvarez, M. A., Hernández, I. (2018). On degenerations of lie superalgebras. *Linear Multilinear Algebra.* DOI: 10.1080/03081087.2018.1498060.

[5] Alvarez, M. A., Hernández, I., Kaygorodov, I. (2019). Degenerations of Jordan superalgebras. *Bull. Malays. Math. Sci. Soc.* 42(6):3289–3301. DOI: 10.1007/s40840-018-0664-3.

[6] Beneš, T., Burde, D. (2009). Degenerations of pre-Lie algebras. *J. Math. Phys.* 50(11):112102. DOI: 10.1063/1.3246608.

[7] Beneš, T., Burde, D. (2014). Classification of orbit closures in the variety of three-dimensional novikov algebras. *J. Algebra Appl.* 13(02):1350081–1350033. DOI: 10.1142/S0219498813500813.

[8] Burde, D., Steinhoff, C. (1999). Classification of orbit closures of 4–dimensional complex lie algebras. *J. Algebra* 214(2):729–739. DOI: 10.1006/jabr.1998.7714.

[9] Calderón, A. J., Fernández Ouaridi, A., Kaygorodov, I. (2018). The classification of 2-dimensional rigid algebras. *Linear Multilinear Algebra.* DOI: 10.1080/03081087.2018.1519009.

[10] Fernández Ouaridi, Kaygorodov, I., Khrypchenko, M., Volkov, Y. Degenerations of nilpotent algebras. arXiv:1905.05361.

[11] Gorbatevich, V. (1991). On contractions and degeneracy of finite-dimensional algebras. *Soviet Math. (Iz. VUZ).* 35(10):17–24.
[12] Gorbatsevich, V. (1994). Anticommutative finite-dimensional algebras of the first three levels of complexity. St. Petersburg Math. J. 5:505–521.

[13] Gorshkov, I., Kaygorodov, I., Khrypchenko, M. (2019). The geometric classification of nilpotent tortkara algebras. Commun. Algebra. DOI: 10.1080/00927872.2019.1635612.

[14] Gorshkov, I., Kaygorodov, I., Kytmanov, A., Salim, M. (2019). The variety of nilpotent tortkara algebras. J. Sib. Fed. Univ. Math. Phys. 12(2):173–184.

[15] Gorshkov, I., Kaygorodov, I., Yu, P. (2019). Degenerations of Jordan algebras and marginal algebras. Algebra Colloq.

[16] Grunewald, F., O’Halloran, J. (1988). Varieties of nilpotent lie algebras of dimension less than six. J. Algebra 112(2):315–325. DOI: 10.1016/0021-8693(88)90093-2.

[17] Grunewald, F., O’Halloran, J. (1988). A characterization of orbit closure and applications. J. Algebra 116(1):163–175. DOI: 10.1016/0021-8693(88)90199-8.

[18] Ismailov, N., Kaygorodov, I., Volkov, Y. (2018). The geometric classification of liebniz algebras. Int. J. Math. 29(05):1850035. DOI: 10.1142/S0129167X18500350.

[19] Ismailov, N., Kaygorodov, I., Volkov, Y. (2019). Degenerations of liebniiz and anticommutative algebras. Can. Math. Bull. 62(3):539–549. DOI: 10.4153/S0008439519000018.

[20] Karimjanov, I., Kaygorodov, I., Khudoyberdiyev, A. (2019). The algebraic and geometric classification of nilpotent Novikov algebras. J. Geom. Phys. 143:11–21. DOI: 10.1016/j.geomphys.2019.04.016.

[21] Kaygorodov, I., Khrypchenko, M., Popov, Y. The algebraic and geometric classification of nilpotent terminal algebras. arXiv:1909.00358

[22] Kaygorodov, I., Páez-Guillán, P., Voronin, V. The algebraic and geometric classification of nilpotent bicommutative algebras. arXiv:1903.08997.

[23] Kaygorodov, I., Popov, Y., Pozhidaev, A., Volkov, Y. (2018). Degenerations of zinbiel and nilpotent leibniz algebras. Linear Multilinear Algebra 66(4):704–716. DOI: 10.1080/03081087.2017.1319457.

[24] Kaygorodov, I., Popov, Y., Volkov, Y. (2018). Degenerations of binary-Lie and nilpotent malcev algebras. Commun. Algebra 46(11):4928–4941. DOI: 10.1080/00927872.2018.1459647.

[25] Kaygorodov, I., Volkov, Y. (2019). The variety of 2-dimensional algebras over an algebraically closed field. Can. J. Math-J. 71(4):819–842. DOI: 10.4153/S0008414X18000056.

[26] Kaygorodov, I., Volkov, Y. (2019). Complete classification of algebras of level two. Moscow Math. J. 19(3):485–521. DOI: 10.17323/1609-4514-2019-19-3-485-521.

[27] Khudoyberdiyev, A. (2015). The classification of algebras of level two. J. Geom. Phys. 98:13–20. DOI: 10.1016/j.geomphys.2015.07.020.

[28] Khudoyberdiyev, A., Omirov, B. (2013). The classification of algebras of level one. Linear Algebra Appl. 439(11):3460–3463. DOI: 10.1016/j.laa.2013.09.020.

[29] Mazzola, G. (1980). Generic finite schemes and hochschild cocycles. Comment. Math. Helv. 55(1):2, 267–293. DOI: 10.1007/BF02566686.

[30] Seeley, C. (1990). Degenerations of 6-dimensional nilpotent lie algebras over C. Commun. Algebra 18:3493–3505.