Quantum asymmetric cryptography with symmetric keys

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Based on quantum encryption, we present a new idea for quantum public-key cryptography (QPKC) and construct a whole theoretical framework of a QPKC system. We show that the quantum-mechanical nature renders it feasible and reasonable to use symmetric keys in such a scheme, which is quite different from that in conventional public-key cryptography. The security of our scheme is analyzed and some features are discussed. Furthermore, the state-estimation attack to a prior QPKC scheme is demonstrated.

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I. INTRODUCTION

In the 1970s, the concept of public-key cryptography (PKC), also called asymmetric cryptography, was proposed\textsuperscript{[1, 2]}. It represented the radical revision of cryptographic thinking and transformed the world of information security. Before the appearance of PKC, the tool for keeping the secrecy of communications was symmetric cryptography, where two parties involved in the communication must previously share a sequence of secret bits (i.e., the key) to encrypt and decrypt the message. In this condition how to securely distribute such a key between the users becomes an intractable problem. On the contrary, in PKC there are two different keys $e$ and $d$ (this is the reason why it is also known as asymmetric cryptography), called the public key and the private key respectively. Just as their names imply, $e$ would be published and anyone can access it freely, whereas $d$ is only known to its owner. As described by Rivest, Shamir, and Adleman when they presented the famous RSA scheme\textsuperscript{[2]}, a PKC system generally satisfies the following four conditions: (C1) A message encrypted with $e$ can be correctly decrypted with $d$; (C2) Both the encryption and the decryption are easy to compute; (C3) It is difficult to compute $d$ from the public $e$; (C4) A message encrypted with $d$ can also be correctly decrypted with $e$. Armed with these properties, PKC can be conveniently utilized by users, who do not need previously share a secret key anymore. Therefore, PKC can resolve the difficulty of key distribution in symmetric cryptography, and then the latter can be used to encrypt the messages. This kind of hybrid cryptosystem is generally used in our practical implementations. Furthermore, PKC is also the most suitable choice for another important application of cryptography, that is, digital signature\textsuperscript{[3]}.

The security of PKC lies on computational complexity assumptions, which is reflected by the condition (C3). Equivalently, the reliability of a PKC scheme is based on certain mathematically difficult problems such as integer factorization, discrete logarithm, etc. However, most of such problems are not difficult in the context of quantum computation anymore\textsuperscript{[4, 5]}. As a result, most of PKC schemes will be broken by future quantum computer. It is natural to ask, at that time, what is the substitution for PKC to distribute a key? One possible way is to exploit quantum mechanics, which is called quantum key distribution (QKD) or quantum cryptography\textsuperscript{[6]}. QKD has a unique property, that is, the potential eavesdropping would be exposed by the users, and consequently it can achieve unconditional security in theory. This security is assured by fundamental principles in quantum mechanics instead of hardness of computational problems.

In fact, QKD can only realize one application of PKC, i.e., key distribution. But about digital signature, the other important application, what can we do? Obviously we do not want to give up the significant flexibility of PKC even in the era of quantum computer. To this end the research is progressing along two directions. One is to look for difficult problems under quantum computation (especially the existing quantum algorithms\textsuperscript{[4, 5]}) and construct PKC based on them\textsuperscript{[7, 8, 9, 10]}. In these schemes the key is still composed of classical bits, and it follows that the flexibility of PKC is retained. But the fact that their security lies on unproved computational assumptions is unchanged. For simplicity, we call this kind of cryptosystems the first class of quantum PKC (QPKC class I). The other direction pursues PKC with perfect security by adding more quantum elements in the schemes, which is just like that of QKD\textsuperscript{[11, 12]}. In these schemes the security is assured by physical laws instead of unproved assumptions. However, the keys generally contain qubits, which are, at least within current techniques, more difficult to deal with, and then the flexibility of PKC would be reduced to some extent. We call these cryptosystems the second class of quantum PKC (QPKC class II). In our opinion, both classes of QPKC are of significance for the future applications. Class I is more practical, whereas class II is more ideal and still needs more related researches. In this paper we study the latter.
Recently, G. M. Nikolopoulos presented a novel QPKC scheme (GMN scheme) based on single-qubit rotations \[11\]. In this scheme the public key consists of polarization qubits. Each qubit is generated by rotating a standard state \(|0\rangle\) by a random angle. All these angles (represented by bits) form the corresponding private key. According to Holevo’s theorem \[13\], little information can be elicited by measuring these qubits even when many copies of public key are served, which is far from obtaining the exact value of the corresponding private key. This basic idea is similar to the one proposed by Gottesman \[12\].

In this paper we will point out a potential security problem in GMN scheme, and propose a new theoretical framework for QPKC based on quantum encryption \[14, 15, 16\]. In our scheme two qubits from a Bell state are served, which is far from obtaining the exact value of the corresponding private key. Consequently, a straightforward strategy for Eve arises, that is, trying to estimate the private key to certain accuracy by measuring the public key and using the result to obtain plaintext from the ciphertext.

Now we show what Eve can obtain by above strategy. To see the particular accuracy to which Eve can estimate the private key, we can use some results in the research of state estimation \[17, 18, 19\]. In GMN scheme, all the single-qubit states lie on the \(x-z\) plane of Bloch sphere. In this condition by optimal collective measurements the obtainable fidelity between the estimation result and the object state is \[18, 19\]

\[
F = \frac{1}{2} + \frac{1}{2^{M+1}} \sum_{i=0}^{M-1} \sqrt{\binom{M}{i}\binom{M}{i+1}} \approx 1 - \frac{1}{4M} \tag{1}
\]

where \(M\) denotes the number of the object state. That is to say, if Eve has \(M\) identical unknown states \(|\psi\rangle\) on the \(x-z\) plane, she can obtain a known state \(|\psi'\rangle\) so that

\[
|\langle \psi | \psi' \rangle|^2 = F \tag{2}
\]

It can be see that the guessed state \(|\psi'\rangle\) will be very close to the object state \(|\psi\rangle\) when \(M\) is large.

Suppose Eve can get \(K\) public keys in GMN scheme. Without loss of generality, we take one state \(|\psi_{s_j}\rangle\) as our example. In this condition Eve has \(K\) identical qubits in this state. Thus she can obtain a guessed state \(|\psi'_{s_j}\rangle\) by optimal collective measurements so that

\[
|\langle \psi_{s_j} | \psi'_{s_j} \rangle|^2 \approx 1 - 1/(4K). \tag{4}
\]

Note that here state \(|\psi'_{s_j}\rangle\) is known to Eve and it means an approximate value of the integer \(s_j\) in the private key. As a result, Eve can construct a measurement basis \(B_{s_j} = \{|\psi'_{s_j}\rangle, |\psi'_{s_j}^-\rangle\}\) and measure any single qubit in it (\(|\psi'_{s_j}^-\rangle\) is the state orthogonal with \(|\psi'_{s_j}\rangle\)). In the following we will show that this basis brings Eve the chance to extract information of the plaintext.

In the process of encryption the sender (say Alice) will get a copy of Bob’s public key, and use the qubit in state \(|\psi_{s_j}\rangle\) to encrypt the \(j\)th bit of her plaintext \(m_j\) \((m_j = 0\text{ or } 1)\). The corresponding ciphertext is the quantum state \(\hat{R}^j(m_j\pi)|\psi_{s_j}\rangle\), which implies that the plaintext 0 and 1 will be encrypted into the ciphertext \(|\psi'_{s_j}\rangle\) and \(|\psi'_{s_j}^-\rangle\) respectively. Thus Eve can intercept the ciphertext sent by Alice and measure it in the basis \(B_{s_j}\), concluding the results \(|\psi_{s_j}\rangle\) and \(|\psi_{s_j}^-\rangle\) represent the plaintext 0 and 1 respectively. Since \(|\psi_{s_j}\rangle\) and \(|\psi_{s_j}^-\rangle\) might be very close on the Bloch sphere, Eve will obtain the correct plaintext \(m_j\) with a high probability, i.e. \(P_e = F\). Equivalently,
where $A$ and $E$ represent Alice and Eve respectively, and $I(A, E)$ denotes the mutual information between them.

Though the detection of eavesdropping is not involved in Ref. [11], we must consider the disturbance brought by Eve's intervention in view of its importance in quantum cryptography. In above attack, to avoid being discovered by Alice and Bob, Eve can resend her measurement result $|\psi_{a'}\rangle$ or $|\psi_{a''}\rangle$ to Bob after the measurement. In this condition an error occurs with the probability

$$P_e = 2F(1 - F)$$  \hspace{1cm} (4)

Here “error” means the case where the bit sent by Alice is different from the one received by Bob.

From Eqs.(3) and (4) it can be seen that Eve can obtain the amount of the information she can obtained about $m_j$ equals

$$I(A, E) = H(A) - H(A|E) = 1 - 2[F \log F + (1 - F) \log(1 - F)]$$  \hspace{1cm} (3)

where $A$ and $E$ represent Alice and Eve respectively, and $I(A, E)$ denotes the mutual information between them.

Before the description of our QPKC scheme, it is necessary to introduce several basic assumptions about this system. That is, (A1) there is a believable center (Trent) in the QPKC system; (A2) Trent can authenticate every user’s identity in the communications between them, which can be realized by quantum authentication protocols; (A3) the information transmitted in the classical channels can be eavesdropped, but cannot be modified. These assumptions are reasonable and generally accepted in PKC (e.g. A1 and A2) and quantum cryptography (e.g. A3).

This scheme consists of the following four stages.

**Stage 1: Key generation.** Trent generates a pair of keys, i.e., the public key $c$ and the private key $d$, for each user. Without loss of generality, consider Bob as our example. The particular process is as follows.

1. Trent prepares a sequence of qubit pairs $S_1 = \{(p_1, q_1), (p_2, q_2), ..., (p_n, q_n)\}$. Each pair is in the Bell state

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$$  \hspace{1cm} (5)

Two qubit sequences $S_p = \{p_1, p_2, ..., p_n\}$ and $S_q = \{q_1, q_2, ..., q_n\}$ will be used as Bob’s public key and private key, respectively.

2. To securely transmit $S_q$ to Bob, Trent also generates a certain quantity of decoy states $S_d = \{d_1, d_2, ..., d_k\}$, where every qubit is randomly in one of the states $\{|0\rangle, |1\rangle, +\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), -\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}$. Please note that here the meaning of decoy state is somewhat different from that, as widely studied now, used in the way to resolve the problem of Photon-Number-Splitting (PNS) attack in a practical QKD implementation. However, the tasks of both kinds of decoy states are the same, that is, helping users discover potential at-
tacks. We will discuss the role of above decoy states in detail in Section IV.

3. Trent inserts each qubit in $S_q$ into a random position of the sequence $S_q$, obtaining a new qubit sequence $S_{qd}$. Then Trent sends $S_{qd}$ to Bob via a quantum channel.

4. After Bob received all these qubits, Trent tells Bob the position and the basis (i.e. $B_z = \{ |0\rangle, |1\rangle \}$ or $B_x = \{ |+\rangle, |-\rangle \}$) of each decoy state.

5. Bob measures all decoy states in their corresponding bases, and then announces the measurement results to Trent. By comparing these results with the initial states of these qubits, Trent can judge whether the transmitted sequence is disturbed.

6. If no eavesdropping occurs, Bob obtains his private key $d$, i.e. the sequence $S_q$. At the same time, Trent stores Bob’s public key $e$, i.e. $S_p$, for future usage. Otherwise the communication may be insecure and will abort.

In the following stages we can see that the keys might be not enough for encrypting a long message, or be consumed gradually. But whenever it is not enough to be used, Trent can generate new Bell-state pairs to refuel the keys.

**Stage 2: Encryption.** Suppose a user, say Alice, wants to send an $r$-bit message $m = \{m_1, m_2, ..., m_r\}$ to Bob, where $m_i = 0$ or 1, and $r \leq n$. Then Alice can encrypt it according to the following steps.

1. Alice requests Trent to send her $r$ qubits of Bob’s public key.

2. Trent sends the first $r$ qubits of the sequence $S_p$ to Alice. Here we use $S_p$ to denote this part of sequence, i.e. $S_p = \{p_1, p_2, ..., p_r\}$. Similar to that in Stage 1, Trent also utilizes decoy states so that these qubits are securely transmitted to Alice.

3. Alice generates an $r$-qubit sequence $L = \{l_1, l_2, ..., l_r\}$ with states $\{|m_1\rangle, |m_2\rangle, ..., |m_r\rangle \}$ respectively, which corresponds to her message to be encrypted.

4. Alice encrypts her message $L$ with the public key $S_p$. More concretely, Alice uses one qubit in $S_p$ to encrypt her corresponding message qubit via a CNOT operation. For example, to encrypt $|l_i\rangle$, Alice performs a CNOT gate $C_{p_i l_i}$, (the first subscript $p_i$ denotes the controller and the second $l_i$ represents the target) on qubits $p_i$ and $l_i$, that is

$$C_{p_i l_i} |\Phi^+\rangle_{p,q} |m_i\rangle_{l_i} = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)_{p,q,i} \quad (6)$$

where $\overline{m}_i = 1 - m_i$.

5. After the encryption of all her message qubits, Alice sends the sequence $L$ (the ciphertext) to Bob through a quantum channel.

**Stage 3: Decryption.** After Bob received all these qubits, he can execute the following steps to recover the message $m$.

1. For each qubit in the ciphertext $L$, Bob performs a CNOT operation $C_{q_i l_i}$ to decrypt it. Then the state changes into

$$C_{q_i l_i} \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)_{p,q,i} |m_i\rangle_{l_i} \quad (7)$$

2. Bob measures each qubit in $L$ in basis $B_x$. From Eq.(7) we can see that the measurement results exactly compose the message $m$. Thus the message sent by Alice is recovered and the decryption is finished.

**Stage 4: Key recycling.** There is a good property in the above communication, that is, the states of Bob’s keys are still unchanged after the processes of encryption and decryption. Therefore, the keys can be recycled according to the following steps.

1. Alice sends Bob’s public key, i.e. the qubit sequence $S_p$ to Trent.

2. To ensure the security of these recycled key qubits, Trent randomly selects a certain number of them from $S_p$ as the test qubits, and measures each of them in $B_z$ or $B_x$ at random.

3. Trent tells Bob the position and the measurement basis of each test qubit.

4. Bob measures his corresponding qubits in the same bases and announces his results. Because every two corresponding qubits in two keys should be in Bell state $|\Phi^+\rangle$, the measurement results would exhibit deterministic correlations. For example, they are equal in the measurement in both bases $B_z$ and $B_x$.

5. By comparing their measurement results Trent can judge whether these qubits are attacked. If they are not, Trent and Bob store the remaining qubits to refuel the public key and the private key. Otherwise the recycled key qubits would be discarded.

Now we have described the QPKC scheme based on quantum encryption. It can be seen that both the qubits in public key and the ones in private key come from Bell state $|\Phi^+\rangle$, and are in the same state (i.e. the maximally mixed state $\rho = 1/2(|0\rangle\langle 0| + |1\rangle\langle 1|)$). Therefore, an interesting event happens. That is, this QPKC scheme essentially use a pair of symmetric keys. In fact the basic idea of this scheme is similar to that of quantum Vernam cipher [24]. In conventional cryptography, as we know, the Vernam cipher (i.e. one-time pad) [24] can never be used in PKC because its decryption key and encryption key are equal, and they can be copied at will. But in the quantum context things become totally different. That is, one cannot obtain the decryption key (i.e. private key) by replicating a copy of the encryption key (i.e. public key) even though they are in the same state, which is guaranteed by quantum no-cloning theorem [25].

Finally, about this QPKC scheme, there are some issues to be clarified.

1. In fact the public key obtained by Alice is a subsequence of $S_p$. After Alice received these qubits, it is necessary for Trent to tell Bob which sub-sequence of $S_p$ was sent to Alice so that Bob can use his corresponding qubits to decrypt Alice’s ciphertext. By this way Bob can correctly decrypt every ciphertext even though there are
multiple ciphertexts received simultaneously from different senders.

2. In above description Alice and Bob do not detect the potential eavesdropping to the ciphertext, which may happen when it was transmitted in the channel. As we will show in Section IV, Eve cannot obtain the message from the ciphertext. But she can still do a denial-of-service (DoS) attack to disturb the communication \cite{24,22}. To enable Alice and Bob to discover this kind of attack, the method of message authentication can be introduce to this scheme. For example, when Alice wants to send message $m$ to Bob, she computes the message digest $H(m)$ via a public Hash function (e.g. MD5, SHA-1, et al.) \cite{23} first, and then sends both $m$ and $H(m)$ to Bob by above QPKC system. Thus after Bob received the corresponding two parts $m'$ and $H'(m)$ he can detect eavesdropping by verifying whether $H'(m)$ is the message digest of $m'$. By this means Alice and Bob can discover the eavesdropping to the ciphertext.

3. Till now we have not consider the noise in quantum channels. As we know, quantum state will change because of the unavoidable decoherence in a noisy channel. In this condition, the technologies of entanglement purification \cite{25,22} and quantum privacy amplification \cite{30} can be introduced in this scheme to improve the quality of the Bell state of these EPR pairs (i.e. the keys) after the transmission of one of the keys. Therefore, our QPKC scheme can be used even in a noisy circumstances.

### IV. Security Analysis

In a QPKC system the aim of Eve is to obtain Bob’s private key, which can be used to decrypt the ciphertext, or alternatively, obtain the plaintext without the private key. Therefore, it must be ensured that the above two events cannot occur in a secure QPKC system. The following discussions will be based on this fact.

In above QPKC scheme some familiar and reliable manners are utilized to guarantee its security. For example, BB84-type qubits \cite{31} are used as the decoy states to protect the transmitted sequence, and conjugate-bases measurements to identify the state of recycled key qubits. Note that in our scheme every public-key qubit is only used to encrypt one message bit (or qubit), so there is no correlation between different ciphertexts. As a result, we have no need to consider the conventional attack strategies such as chosen-plaintext attack and chosen-ciphertext one. In the following we will briefly discuss the security with respect to different stages of this scheme.

**Key generation.** In this stage Trent prepares EPR pairs in $|\Phi^+\rangle$ and sends one qubit in each pair (i.e. the sequence $S_p$) to Bob as his private key. Because Trent is believable we only need to consider the attack from an outside eavesdropper (Eve). In this process Eve has the chance to obtain Bob’s private key, with which she can decrypt any ciphertext sent to Bob. However, Eve’s goal will not be achieved because of the usage of decoy states.

The reasons are as follows.

First, quantum no-cloning theorem \cite{25} ensures that Eve cannot replicate the qubits in the private key. For simplicity, consider one EPR pair $(p_i, q_i)$, where $p_i$ is a qubit in sequence $S_p$ (public key) and $q_i$ is the one in $S_p$ (private key). Obviously one cannot generate a new qubit $q'_i$, a copy of $q_i$, when $q_i$ is transmitted in the channel so that both $(p_i, q'_i)$ and $(p_i, q_i)$ are in Bell state $|\Phi^+\rangle$. This is guaranteed by fundamental laws in quantum mechanics. This point is very different from that in conventional PKC systems, in which the private key can never be transmitted in the public channel because it is in the form of bits and can be easily copied.

Second, since both the decoy qubits and the private-key ones are in the same state, i.e. the maximally mixed state $\rho = 1/2(|0\rangle\langle 0| + |1\rangle\langle 1|)$, these two kinds of qubits cannot be distinguished. That is to say, any attack operation which is expected to be performed on the private-key qubits will be also inevitably executed on the decoy ones. As a result, the attack would leave a trace on the decoy states and then be discovered by legal users. For example, Eve may want to entangle her ancilla into the Bell state by a collective operation on it and qubit $q_i$, and subsequently use the ancilla to decrypt the ciphertext which was encrypted by $q_i$. More concretely, Eve prepares an ancilla $|0\rangle_a$, and performs a CNOT operation $C_{q,a}$ when $q_i$ is transmitted in the channel. That is,

$$C_{q,a} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{p,q,a}|0\rangle_a = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)_{p,q,a}. \quad (8)$$

And then resend $q_i$ to Bob. When Alice uses $p_i$, the corresponding public-key qubit, to encrypt a message bit $m_i$, the state of the whole system changes into

$$C_{p,i} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{p,q,a} |m_i\rangle_{l_i} = \frac{1}{\sqrt{2}}(|000m_i\rangle + |111\rangle)_{p,q,a,l_i}. \quad (9)$$

In this condition Eve can correctly obtain $|m_i\rangle$ if she intercepts $l_i$, the ciphertext qubit, when it is transmitted to Bob and performs the following operation

$$C_{a,l_i} \frac{1}{\sqrt{2}}(|000m_i\rangle + |111\rangle)_{p,q,a} |m_i\rangle_{l_i} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)_{p,q,a}|m_i\rangle_{l_i}. \quad (10)$$

which means Eve gets the plaintext $m_i$.

However, the above attack will bring disturbance to the decoy states. For example, consider decoy state $|+\rangle$. When Alice intercepts it and performs her first CNOT operation on it and her ancilla $|0\rangle$, they will come into Bell state $|\Phi^+\rangle$, which results in a totally random result when Bob measures the decoy state to detect eavesdropping.

Therefore, the above attack will be inevitably discovered by Bob and Trent. In fact, BB84-type particles can
reliably guarantee the security of a quantum sequence, which has been reflected by the proved security of BB84 protocol \cite{31,32,33}. Equivalently, any effective attack will be disclosed by the detection via those particles.

**Encryption.** As introduced in Section III, we use symmetric keys in our QPKC scheme. That is, the public key and the private one are in the same state. Therefore, anyone who has the public key can also decrypt the ciphertext encrypted by this key. In this stage Eve has the chance to touch the public key when it is transmitted from Trent to Alice. However, similar to that in Stage I, Eve can never replicate those qubits and the decoy states ensure the security of the public key. Consequently, any effective attack on the public key will be discovered by legal users.

Now let us observe what Eve can obtain from the ciphertext when it is transmitted from Alice to Bob. From above analysis, it can be seen that Eve cannot elicit any helpful information from the transmitted key qubits, including both the public key and the private key, if she does not want to bring disturbance to the decoy states. In this condition Eve can obtain nothing about the plaintext from the ciphertext because all ciphertext qubits are in the same state $\rho = 1/2(|0\rangle\langle 0| + |1\rangle\langle 1|)$ in spite of the value (0 or 1) of corresponding message bit.

The classical Hash function is used in this stage. We should emphasize that, though Hash functions are not perfect (e.g., collisions might be found by some advanced algorithms \cite{34,35}), it does not decrease the security of the whole QPKC system. In this stage, as shown above, Eve cannot obtain the plaintext at all. The usage of Hash function is just to protect the scheme against DoS attack. In fact it plays the role like message authentication code (MAC). As a result, general Hash functions such as MD5, SHA-1, et al. can disclose a potential DoS attack.

**Decryption.** In this stage Eve has no chance to attack because no qubits are transmitted in the channel. After Bob obtained the plaintext, he can judge whether DoS attack occurred with the help of Hash function.

**Key recycling.** In this stage Alice sends the public key back to Trent. This situation, as far as Eve is concerned, is similar to that in the beginning of Stage II. But here we should also consider the attack from Alice. Because the recycled public-key qubits will be reused in later applications where another one (say Charlie) sends his message to Bob, Alice can do something for future illegal decryption when these qubits are still in her hand. For example, Alice can entangle her ancilla into each Bell states and use it to decrypt the ciphertext sent by Charlie later (similar to Eve’s strategy in Stage I and the ones in Refs. \cite{36,37}).

Taking above threat into account, we have to ensure that the states of the public-key qubits Alice sent back are unchanged (that is, each qubit is still in Bell state $|\Phi^+\rangle$ with its corresponding particle in Bob’s hand). In our scheme we use the manner of conjugate-bases measurements to detect eavesdropping, which can resist attacks from both Eve and Alice. This manner has been widely used in quantum cryptography and its reliability has been proved \cite{38,39}. Here we will not repeat the analysis any more.

Finally, it is well known that, in a practical QKD system, Eve may attack only a little part of the transmitted particles so that the introduced disturbance will be covered up by channel noises. In this case Eve can elicit a small amount of information about the key. And at the same time, legal users cannot ascertain whether there is an eavesdropper in the channel because the error rate introduced by Eve is small enough. At that time, the users can perform privacy amplification \cite{40,41} on the raw key and then obtain a final key with unconditional security. In our QPKC scheme, similar problem also exists. Eve may attack only a little part of the key qubits and then obtain some information about the plaintext. In this condition we introduce entanglement purification \cite{28,29} and quantum privacy amplification \cite{30} in our scheme, which makes it possible to achieve unconditional security in theory.

**V. DISCUSSIONS AND CONCLUSIONS**

Compared with the previous QPKC system (GMN scheme) \cite{11}, our scheme has the following features.

1. The roles of public key and private key are equal. When Rivest, Shamir, and Adleman presented the famous RSA scheme \cite{2}, as described in Section I, they pointed out four basic conditions which a PKC system generally satisfies. Among them the last condition (C4) requires that the users can also use private key to encrypt a message and use public key to decrypt it correctly. This requirement opens the door for an important application of PKC, i.e. digital signature. But this aim is not achieved in GMN scheme. The problem is resolved in our scheme because both public key and private key are quantum one and in the same state. Therefore, *this feature makes it possible to construct a quantum signature protocol based on our scheme*. Of course to design such a protocol is a complex work \cite{42,43} and it is beyond the scope of this paper.

2. The manner to verify the identity of public key is presented in our scheme. In both schemes public key is quantum one and its identity should be authenticated when the message sender received it from Trent (or a key-distribution center, i.e. KDC, called in Ref. \cite{11}). This is a crucial point for the security of whole QPKC system. However, authentication is still an open question in GMN scheme because of the complexity of the public-key states. In our scheme this problem is resolved from two aspects. On the one hand, the decoy-states detection is utilized to protect the public-key qubits from being attacked by Eve. On the other hand, because the key qubits are from the same Bell state $|\Phi^+\rangle$, entanglement purification and quantum privacy amplification can be easily performed on them in the sense that they are existing
technologies for Bell states \cite{28,29,30}. Through these manners high-fidelity Bell state can be finally obtained even under a noisy channel, or equivalently, the state of public key can be authenticated. On the contrary, the public-key states are different from each other and even unknown for the message sender in GMN scheme, which makes it very hard to perform quantum privacy amplification on them.

3. The state-estimation attack is invalid for our scheme. In GMN scheme, as discussed in Section II, Eve can estimate the state of public key by measuring multiple copies of them, and then obtain much information about the plaintext. However, in our scheme any two qubits from different public key belong to different EPR pairs and there are no correlations between them. Even though the same qubit is reused in subsequential encryption, it is independent with itself in previous usage because its state is identified in the process of recycling. As a result, Eve cannot get more useful information from multiple public key than that from one. In fact, as pointed out in Section IV, no one can obtain a copy of private key (or qubits with which Eve can correctly decrypt a certain ciphertext) from the public key without introducing disturbance. This is guaranteed by fundamental laws in quantum mechanics.

4. The keys can be reused and refuelled whenever it is needed.

We have to confess that, apart from above features, there is also a disadvantage of our scheme. That is, private key consists of qubits in stead of bits as in GMN scheme, which presents a burden to the user to store them. However this is not a fatal problem because quantum storage seems necessary in a QPKC system. For example, many copies of public key must be stored by Trent or KDC for a long time.

One may argue that our scheme does not look like a practical PKC system (e.g. any familiar conventional PKC such as the famous RSA scheme) for the following two reasons: 1. Some QKD-like strategies for eavesdropping detection are used to guarantee the security; 2. It uses symmetric keys. We emphasize that all these facts have their roots in the quantum nature of QPKC. Now let us give further interpretations about above two questions.

1. As we know, the quantum-mechanical nature of qubits renders eavesdropping detectable, which is the root of the unconditional security of quantum cryptography. To obtain this advantage in a quantum protocol, an eavesdropping-detection process is absolutely necessary. It is also the fact in QPKC. For example, in a QPKC system the public key, generally composed by qubits \cite{11}, must be authenticated after the transmission in a public channel. Otherwise Eve may correctly decrypt the corresponding ciphertext by a prior attack on this key (e.g. replacing it with her own qubits or entangling ancillas into it). Therefore, we have to introduce some QKD-like strategies to protect the security of the public key, which exactly reflects the essential characteristic of quantum cryptograph. On the contrary, the classical public key in RSA scheme can be easily authenticated by conventional technologies such as digital signature \cite{3}. Note that there is no such strategies in GMN scheme because the content of public-key authentication is not contained in Ref. \cite{11}.

2. By choosing Bell-state qubits as the keys we initially intended to avoid the state-estimation attack as in GMN scheme. In fact Bell states have a special feature which is suitable for QPKC. That is, these states can be authenticated by existing technologies (especially entanglement purification and quantum privacy amplification), which is an important issue in QPKC but still not resolved in the previous scheme. We know that people can never use equal keys in a conventional PKC system because in this condition anyone can get the private key just by replicating a copy of the public key, and then decrypt all corresponding ciphertexts. Thus we really need to design two different keys so that Eve cannot obtain the private key from the public one. However, in the quantum circumstance, things go very differently. On the one hand, quantum no-cloning theorem does not allow the replication of qubits any more. On the other hand, the authentication of public key is necessary in QPKC and, at the same time, whenever the authentication is successful it generally ensures that Eve cannot read any information from public key. In this condition, therefore, we have no need to design two different keys any more. That is, equal keys are competent for QPKC. In fact we have shown that it is feasible to use symmetric keys in QPKC system, which touches on the very nature of the quantum state.

In conclusion, as a subsequent study of Ref. \cite{11}, we gave a new elementary idea for QPKC and constructed a whole theoretical framework of a QPKC system. It was shown that symmetric keys could be used in QPKC, which is quite different from that in conventional PKC. The security and features of this scheme were discussed. In addition, a possible attack to GMN scheme \cite{11} was demonstrated. Combining the unconditional security of QKD and the significant flexibility of PKC, QPKC has been an expected goal of the scholars in the field of quantum cryptography for a long time. But to design a practical QPKC scheme, or alternatively, to demonstrate its feasibility, is still a difficult work. This study can be seen as a step towards this direction.

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[44] In section I we divided QPKC into two classes. We believe that the public key in QPKC class II must be quantum one. Otherwise a classical public key has not the ability to detect eavesdropping, and then cannot obtain unconditional security in theory, which is the characteristic of QPKC class I.