Isotropic constants and Mahler volumes
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Abstract
This paper contains a few results related to volumes of projective perturbations and the Laplace transform on convex cones. First, it is shown that a sharp version of Bourgain’s slicing conjecture implies the Mahler conjecture for convex bodies that are not necessarily centrally-symmetric. Second, we find that by slightly translating the polar of a centered convex body, we may obtain another body with a bounded isotropic constant. Third, we provide a counter-example to a conjecture by Kuperberg on the distribution of volume in a body and in its polar.

1 Introduction
This paper describes interrelations between duality and distribution of volume in convex bodies. A convex body is a compact, convex subset $K \subseteq \mathbb{R}^n$ whose interior $\text{Int}(K)$ is non-empty. If $0 \in \text{Int}(K)$, then the polar body is defined by

$$K^\circ = \{ y \in \mathbb{R}^n ; \forall x \in K, \langle x, y \rangle \leq 1 \}.$$

The polar body $K^\circ$ is again a convex body with the origin in its interior, and moreover $(K^\circ)^\circ = K$. For $y \in \text{Int}(K^\circ)$ we may consider the convex body

$$\tau_y(K) := (K^\circ - y)^\circ,$$

a “dual translation” of $K$. A convex body $K \subseteq \mathbb{R}^n$ is investigated below in connection with the entire family of bodies $\{\tau_y(K) - x \}$ where $y$ ranges over $\text{Int}(K^\circ)$ and where $x \in \text{Int}(\tau_y(K))$. All members of this family are projective images of one another.

The Mahler volume of a convex body $K \subseteq \mathbb{R}^n$ with the origin in its interior is defined as

$$s(K) = \text{Vol}_n(K) \cdot \text{Vol}_n(K^\circ),$$

where $\text{Vol}_n$ is $n$-dimensional volume. In the class of convex bodies with barycenter at the origin, the Mahler volume is maximized for ellipsoids, as proven by Santalo [25], see also Meyer and Pajor [18]. For a general dimension $n$, the Mahler conjecture suggests that for any convex body $K \subseteq \mathbb{R}^n$ containing the origin in its interior,

$$s(K) \geq s(\Delta^n) = \frac{(n+1)^{n+1}}{(n!)^2},$$

(1)
where $\Delta^n \subseteq \mathbb{R}^n$ is any simplex whose vertices span $\mathbb{R}^n$ and add to zero. The conjecture was verified for convex bodies with certain symmetries in the works of Barthe and Fradelizi [3], Kuperberg [13] and Saint Raymond [26]. In two dimensions the conjecture was proven by Mahler [15], see also Meyer [17]. There is also a well-known version of the Mahler conjecture for centrally-symmetric convex bodies (i.e., when $K = -K$) that will not be much discussed here. It was proven by Bourgain and Milman [8] that for any convex body $K \subseteq \mathbb{R}^n$ with the origin in its interior,

$$s(K) \geq c^n \cdot s(\Delta^n)$$

for some universal constant $c > 0$. There are several beautiful, completely different proofs of the Bourgain-Milman inequality in addition to the original argument, including proofs by Kuperberg [13], by Nazarov [21] and by Giannopoulos, Paouris and Vritsiou [10]. The covariance matrix of a convex body $K \subseteq \mathbb{R}^n$ is the matrix $\text{Cov}(K) = (\text{Cov}_{ij})_{i,j=1,...,n}$ where

$$\text{Cov}_{ij} = \frac{\int_{K} x_i x_j \, dx}{\text{Vol}_n(K)} - \frac{\int_{K} x_i \, dx}{\text{Vol}_n(K)} \cdot \frac{\int_{K} x_j \, dx}{\text{Vol}_n(K)}.$$ 

The isotropic constant of a convex body $K \subseteq \mathbb{R}^n$ is the parameter $L_K > 0$ defined via

$$L_K^{2n} = \frac{\text{det} \, \text{Cov}(K)}{\text{Vol}_n(K)^2}. \quad (3)$$

We equip the space of convex bodies in $\mathbb{R}^n$ with the usual Hausdorff topology. The Mahler volume is a continuous function defined on the class of convex bodies in $\mathbb{R}^n$ containing the origin in their interior. A standard compactness argument shows that the minimum of the Mahler volume in this class is indeed attained. Below we present a variational argument in the class of translations and dual translations of $K$, that yields the following:

**Theorem 1.1.** Let $K \subseteq \mathbb{R}^n$ be a convex body which is a local minimizer of the Mahler volume. Then $\text{Cov}(K^\circ) \geq (n + 2)^{-2} \cdot \text{Cov}(K)^{-1}$ in the sense of symmetric matrices, and

$$L_K \cdot L_{K^\circ} \cdot s(K)^{1/n} \geq \frac{1}{n + 2}. \quad (4)$$

Consequently any global minimizer must satisfy $L_K \geq L_{\Delta^n}$ or $L_{K^\circ} \geq L_{\Delta^n}$.

It is well-known that $L_K > c$ for any convex body $K \subseteq \mathbb{R}^n$, where $c > 0$ is a universal constant. In fact the minimal isotropic constant is attained for ellipsoids. Bourgain’s slicing problem [4, 5] asks whether $L_K < C$ for a universal constant $C > 0$. The slicing conjecture has several equivalent formulations, and it seems to be related virtually to almost any asymptotic question about the distribution of volume in high-dimensional convex bodies. Currently the best known estimate is $L_K \leq C n^{1/4}$ for a convex body $K \subseteq \mathbb{R}^n$. This was shown in [11], slightly improving upon an earlier estimate by Bourgain [6, 7]. Two sources of information on the slicing problem are the recent book by Brazitikos, Giannopoulos,
Valettas and Vritsiou [9] and the survey paper by Milman and Pajor [19]. A strong version of the slicing conjecture is that for any convex body \( K \subseteq \mathbb{R}^n \),

\[
L_K \leq L_{\Delta^n} = \frac{(n!)^{\frac{1}{n}}}{(n + 1)^{\frac{n+1}{2n}} \cdot \sqrt{n + 2}}.
\]

This conjecture holds true in two dimensions. See also Rademacher [22] for supporting evidence. Theorem 1.1 admits the following:

**Corollary 1.2.** The strong version (5) of Bourgain’s slicing conjecture implies Mahler’s conjecture (1).

In order to see why inequality (4) implies Corollary 1.2, observe that by (4) and (5), for any local minimizer \( K \subseteq \mathbb{R}^n \) of the Mahler volume,

\[
\frac{1}{(n + 2)^n} \leq L^n_K \cdot L^n_{K^\circ} \cdot s(K) \leq L^n_{\Delta^n} \cdot s(K) = \frac{(n!)^2}{(n + 1)^{n+1} \cdot (n + 2)^n} \cdot s(K),
\]

which clearly yields (1). We are aware of two more conditional results in spirit of Corollary 1.2. Artstein-Avidan, Karasev and Ostrover [2] proved that the Mahler conjecture for centrally-symmetric bodies would follow from the Viterbo conjecture in symplectic geometry. It was recently shown that the Minkowski conjecture would follow from a strong version of the centrally-symmetric slicing conjecture, see Magazinov [16].

For the proof of Theorem 1.1 we use the Laplace transform in order to analyze the Mahler volume in the space of projective images of \( K \). The Laplace transform was also used in [11] in order to prove the isomorphic version of the slicing problem. We observe here the following variant of the result from [11]:

**Theorem 1.3.** Let \( K \subseteq \mathbb{R}^n \) be a convex body with barycenter at the origin and let \( 0 < \varepsilon < 1/2 \). Then there exists a convex body \( T \subseteq \mathbb{R}^n \) such that the following hold:

1. \((1 - \varepsilon)K \subseteq T \subseteq (1 + \varepsilon)K\).
2. The polar body \( T^\circ \) is a translation of \( K^\circ \).
3. \( L_T < C/\sqrt{\varepsilon} \), where \( C > 0 \) is a universal constant.

We say that two convex sets \( K_1 \subseteq \mathbb{R}^{n_1} \) and \( K_2 \subseteq \mathbb{R}^{n_2} \) are **affinely equivalent** if there exists an affine map \( T : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2} \), one-to-one on the set \( K_1 \), with \( K_2 = T(K_1) \). It is a curious fact that the quantity

\[
L_K^2 \cdot s(K)^{1/n}
\]

attains the same value \( 1/(n + 2) \) when \( K \subseteq \mathbb{R}^n \) is an ellipsoid and when \( K \subseteq \mathbb{R}^n \) is a simplex, see Alonso–Gutiérrez [1], where in these two examples and also in the next one we assume that the barycenter of \( K \) lies at the origin. Intriguingly,

\[
L_K^2 \cdot s(K)^{1/n} = 1/(n + 2)
\]
also when \( n = \ell(\ell + 1)/2 - 1 \) and \( K \subseteq \mathbb{R}^n \) is affinely equivalent to the collection of all symmetric, positive semi-definite \( \ell \times \ell \) matrices of trace one. This is not a mere coincidence. A common feature to these three examples is that they are hyperplane sections of \textit{convex homogenous cones}.

A convex cone in \( \mathbb{R}^{n+1} \) is a convex subset \( V \) such that \( \lambda x \in V \) for any \( x \in V \) and \( \lambda \geq 0 \). We say that a convex cone \( V \subseteq \mathbb{R}^{n+1} \) is proper if it is closed, has a non-empty interior, and is not the whole space \( \mathbb{R}^{n+1} \). A convex cone \( V \subseteq \mathbb{R}^{n+1} \) is \textit{homogenous} if for any \( x, y \in \text{Int}(V) \) there is an invertible, linear transformation \( T : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1} \) with \( T(x) = y \) and

\[
T(V) = V.
\]

Recall that the Santaló point of a convex body \( K \subseteq \mathbb{R}^n \) is the unique point \( x_0 \) in the interior of \( K \) such that the barycenter of \( (K - x_0)^\circ \) lies at the origin.

**Theorem 1.4.** Let \( K \subseteq \mathbb{R}^n \) be a convex body that is affinely equivalent to a hyperplane section of a convex, homogenous cone in \( \mathbb{R}^{n+1} \). Then,

(i) The barycenter of \( K \) is its Santaló point.

(ii) If the barycenter of \( K \) lies at the origin, then

\[
\text{Cov}(K^\circ) = (n + 2)^{-2} \cdot \text{Cov}(K)^{-1},
\]

and consequently \( L_K \cdot L_{K^\circ} \cdot s(K)^{1/n} = 1/(n + 2) \).

Thus, for instance, (6) also holds true for the convex set that consists of all positive-definite Hermitian or quaternionic-Hermitian matrices or of trace one. In all of the examples above (ball, simplex and convex collections of matrices) the convex body \( K^\circ \) has turned out to be a linear image of \( K \) as the cone is self-dual, thus \( L_K = L_{K^\circ} \). Apparently there are homogenous cones that are not self-dual (see, e.g., Vinberg [30]), yet we see from Theorem 1.4 that the barycenters and covariance matrices of hyperplane sections of homogenous cones automatically satisfy a certain duality property. Similar duality properties apply for higher moments as well.

The relation (7) between the covariance of a convex body and the covariance of the polar body reminds us of the quantity

\[
\phi(K) = \mathbb{E}(X, Y)^2 = \frac{1}{s(K)} \int_K \int_{K^\circ} \langle x, y \rangle^2 dx dy
\]

considered by Kuperberg [13]. Here, \( X \) and \( Y \) are independent random vectors, \( X \) is uniformly distributed in the convex body \( K \), while \( Y \) is uniformly distributed in \( K^\circ \). Assume that \( K \subseteq \mathbb{R}^n \) satisfies the assumptions of Theorem 1.4(ii). Then by the conclusion of Theorem 1.4,

\[
\phi(K) = \text{Tr}[\text{Cov}(K) \cdot \text{Cov}(K^\circ)] = n/(n + 2)^2.
\]
Conjecture 5.1 in [13] suggests that $\phi(K) \leq n/(n+2)$ for any centrally-symmetric convex body $K \subseteq \mathbb{R}^n$. This conjecture was verified in the case where $K$ is the unit ball of $l_p^n$ for $1 \leq p \leq \infty$, see Alonso-Gutiérrez [1]. Nevertheless, Kuperberg described Conjecture 5.1 in [13] as “perhaps less likely”. Our next proposition shows that this conjecture is indeed false in a sufficiently high dimension. A convex body $K \subseteq \mathbb{R}^n$ is unconditional if for any $x \in \mathbb{R}^n$, 
\[ x = (x_1, \ldots, x_n) \in K \iff (|x_1|, \ldots, |x_n|) \in K. \]

**Proposition 1.5.** For any $n \geq 1$ there exists an unconditional, convex body $K \subseteq \mathbb{R}^n$ with 
\[ \frac{1}{s(K)} \cdot \int_K x_1^2 dx \cdot \int_K x_2^2 dx \geq c, \]
where $c > 0$ is a universal constant. In particular, $\phi(K) \geq c$.

The example of Proposition 1.5 is optimal up to a universal constant as clearly $\phi(K) \leq 1$ for any centrally-symmetric convex body $K$ in any dimension. We say that $K \subseteq \mathbb{R}^n$ is 1-symmetric or that it has the symmetries of the cube if $K \subseteq \mathbb{R}^n$ is unconditional and furthermore for any permutation $\sigma \in S_n$ and a point $x \in \mathbb{R}^n$, 
\[ x = (x_1, \ldots, x_n) \in K \iff (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in K. \]

We find it plausible that Kuperberg’s conjecture or even the stronger versions from [13] hold true in the case of convex bodies that possess the symmetries of the cube. When the convex body $K \subseteq \mathbb{R}^n$ has these symmetries, not only we know that $\mathbb{E}\langle X, Y \rangle^2 \leq C/n$, but we may also prove that the random variable $\langle X, Y \rangle$ is approximately Gaussian when the dimension $n$ is large. This is a corollary of the results of [12].

Theorem 1.1 and Theorem 1.4 are proven in Sections 2 and 3. Section 4 discusses some examples, while Proposition 1.5 is proven in Section 5. In Sections 6 and 7 we prove Theorem 1.3. We continue with some notation and conventions that will be used below.

The relative interior of a convex set $K \subseteq X$ is its interior relative to the affine subspace spanned by $K$. We abbreviate $A + B = \{a + b; a \in A, b \in B\}$ and $x + A = \{x + a; x \in A\}$. We write $\langle x, y \rangle$ or $x \cdot y$ for the standard scalar product of $x, y \in \mathbb{R}^n$, and $|x| = \sqrt{x, x}$. A smooth function is $C^\infty$-smooth and we write log for the natural logarithm.

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## 2 Mahler volumes through the Laplace transform

In this section we discuss definitions and basic properties of Mahler volumes, the logarithmic Laplace transform and its Legendre transform. Let $K \subseteq \mathbb{R}^n$ be a compact, convex set,
and let \( p \) be a point belonging to the relative interior of \( K \). The Mahler volume of \( K \) with respect to the point \( p \), denoted by

\[
s_p(K) \in (0, \infty),
\]

is defined as follows: There exists a convex body \( K_1 \subseteq \mathbb{R}^n \) and an affine map \( T : \mathbb{R}^n \to \mathbb{R}^n \), that is invertible on \( K \) with \( T(K) = K_1 \), such that \( T(p) = 0 \). We may now set \( s_p(K) := s(K_1) \), and observe that this definition does not depend on the choice of \( K_1 \) and \( T \). Moreover, given any affine map that is injective on \( K \), we have

\[
s_T(K) = s_p(K).
\]

If the origin belongs to the interior of the convex body \( K \) then \( s(K) = s_0(K) \). For a compact, convex set \( K \subseteq \mathbb{R}^n \) we define

\[
\bar{s}(K) = s_{p_K}(K),
\]

where \( p_K \) is the Satanló point of \( K \). It is well-known (see, e.g., Schneider [28, Section 10.5]) that

\[
\bar{s}(K) = \inf_p s_p(K)
\]

where the infimum runs over all points \( p \) in the relative interior of \( K \). When \( V \subseteq \mathbb{R}^{n+1} \) is a proper, convex cone, its dual cone is defined via

\[
V^* = \{ y \in \mathbb{R}^{n+1} ; \forall x \in V, \langle x, y \rangle \leq 0 \}.
\]

The dual cone \( V^* \) is again a proper, convex cone, and additionally \((V^*)^* = V\).

**Lemma 2.1.** Let \( V \subseteq \mathbb{R}^{n+1} \) be a proper, convex cone. Let \( x_0 \in \text{Int}(V) \) and \( y_0 \in \text{Int}(V^*) \) satisfy \( \langle x_0, y_0 \rangle = -1 \). Then,

\[
s_{x_0}(K) = s_{y_0}(T) = \frac{1}{(n!)^2} \int_V e^{\langle x, y_0 \rangle} \, dx \cdot \int_{V^*} e^{\langle x_0, y \rangle} \, dy
\]

where \( K = \{ x \in V ; \langle x, y_0 \rangle = -1 \} \) and \( T = \{ y \in V^* ; \langle x_0, y \rangle = -1 \} \).

**Proof.** Assume first that there exists a unit vector \( e \in S^{n-1} \) with \( x_0 = e = -y_0 \). By Fubini’s theorem,

\[
\int_V e^{-\langle x, e \rangle} \, dx = \int_0^\infty \int_{\{ z \in V ; \langle z, e \rangle = t \}} e^{-\langle z, e \rangle} \, dz \, dt = \int_0^\infty \text{Vol}_n(tK) \cdot e^{-t} \, dt = n! \cdot \text{Vol}_n(K).
\]

By performing a similar computation for \( V^* \), we see that

\[
\int_V e^{-\langle x, e \rangle} \, dx \cdot \int_{V^*} e^{\langle e, y \rangle} \, dy = (n!)^2 \cdot \text{Vol}_n(K) \cdot \text{Vol}_n(T) = (n!)^2 \cdot \text{Vol}_n(K_1) \cdot \text{Vol}_n(T_1),
\]

where \( K_1 \) and \( T_1 \) are the images of \( K \) and \( T \) under the affine map that maps \( p \) to \( 0 \).
where we have set $K_1 = K - e$ and $T_1 = T + e$. Note that $K_1$ and $T_1$ are convex bodies in the $n$-dimensional linear space $E = e^\perp = \{x \in \mathbb{R}^n; \langle x, e \rangle = 0\}$. Both convex bodies contain the origin in their interior. Let us show that as subsets of the $n$-dimensional Euclidean space $E$, the two sets $K_1$ and $T_1$ satisfy

$$K_1 = T_1^\circ.$$  \hspace{1cm} (14)

Indeed, a given point $y \in E$ lies in $T_1$ if and only if $\langle y - e, x \rangle \leq 0$ for all points $x \in V$. By homogeneity, it suffices to look only at points $x \in K$, since any $x \in V$ takes the form $x = tz$ for some $z \in K, t \geq 0$. Thus, a given point $y \in E$ belongs to $T_1$ if and only if for all $x \in K_1$,

$$0 \geq \langle y - e, x + e \rangle = \langle x, y \rangle - \langle e, e \rangle = \langle x, y \rangle - 1.$$  \hspace{1cm} (15)

This proves (14). From (13) and (14) we obtain the conclusion of the proposition in the case where $x_0 = -y_0$ is a unit vector.

We move on to discuss the general case, which will be reduced to the case analyzed above using linear algebra. Since $\langle x_0, -y_0 \rangle > 0$, there exists a positive-definite, symmetric matrix $P \in \mathbb{R}^{n \times n}$ with $Px_0 = -y_0$. We may decompose $P = S^*S$ for some invertible matrix $S \in \mathbb{R}^{n \times n}$ and set

$$e = Sx_0.$$  \hspace{1cm} (16)

Note that $S^*e = S^*Sx_0 = Px_0 = -y_0$, and hence $(S^*)^{-1}y_0 = -e$. The vector $e \in \mathbb{R}^{n+1}$ is a unit vector as

$$-1 = \langle x_0, y_0 \rangle = \langle Sx_0, (S^*)^{-1}y_0 \rangle = -\langle e, e \rangle.$$  \hspace{1cm} (17)

Denoting $V_1 = S(V)$ we see that $V_1$ is a proper, convex cone with $V_1^* = (S^*)^{-1}V^*$. By changing variables \( \tilde{x} = Sx \) and $\tilde{y} = (S^*)^{-1}y$ we obtain

$$\int_V e^{\langle x, y_0 \rangle} dx \cdot \int_{V^*} e^{\langle x_0, y \rangle} dy = \int_{V_1} e^{-\langle \tilde{x}, e \rangle} d\tilde{x} \cdot \int_{V_1^*} e^{\langle e, \tilde{y} \rangle} d\tilde{y}.$$  \hspace{1cm} (18)

Denote $K_2 = \{x \in V_1; \langle x, -e \rangle = -1\}$ and $T_2 = \{y \in V_1^*; \langle e, y \rangle = -1\}$. We use (15) and the case of the proposition that was already proven and deduce that

$$\frac{1}{(n!)^2} \int_V e^{\langle x, y_0 \rangle} dx \cdot \int_{V^*} e^{\langle x_0, y \rangle} dy = s_e(K_2) = s_{-e}(T_2).$$  \hspace{1cm} (19)

However, $S(K) = K_2$ with $S(x_0) = e$ while $(S^*)^{-1}(T) = T_2$ with $(S^*)^{-1}(y_0) = -e$. Therefore $s_e(K_2) = s_{x_0}(K)$ and $s_{-e}(T_2) = s_{y_0}(T)$. Thus (11) follows from (15).

Remark 2.2. The sets $K$ and $T$ from Lemma 2.1 seem somewhat polar to each other, and this is indeed true when interpreted correctly: Set $\tilde{K} = K - x_0$ and $\tilde{T} = T - y_0$. Then $\tilde{K}$ is a convex body in the $n$-dimensional subspace $X = y_0^\perp$ while $\tilde{T}$ is a convex body in the $n$-dimensional subspace $Y = x_0^\perp$. Moreover,

$$\tilde{T} = \left\{ y \in Y; \forall x \in \tilde{K}, \langle x, y \rangle \leq 1 \right\}, \quad \tilde{K} = \left\{ x \in X; \forall y \in \tilde{T}, \langle x, y \rangle \leq 1 \right\}.$$  \hspace{1cm} (20)
Relation (17) is the precise duality relation that $\tilde{K}$ and $\tilde{T}$ satisfy. In particular, we conclude that the set $\tilde{K}$ is centrally-symmetric around the point $x_0$ if and only if the set $\tilde{T}$ is centrally-symmetric around the point $y_0$. Similarly, $x_0$ is the barycenter of $\tilde{K}$ if and only if $y_0$ is the Santaló point of $\tilde{T}$.

Let $V \subseteq \mathbb{R}^{n+1}$ be a proper, convex cone. For $x \in \text{Int}(V)$ and $y \in \text{Int}(V^*)$ we define

$$K_y = \{ z \in V ; \langle z, y_0 \rangle = -1 \} \quad \text{and} \quad T_x = \{ z \in V^* ; \langle x, z \rangle = -1 \}. \quad (18)$$

The notation (18) will accompany us throughout this paper. Observe that for any $t > 0$ and $y \in V^*$,

$$\int_V e^{(ty,x)} dx = \frac{1}{t^{n+1}} \int_V e^{(y,x)} dx. \quad (19)$$

By scaling, we obtain from Lemma 2.1 the following:

**Proposition 2.3.** Let $V \subseteq \mathbb{R}^{n+1}$ be a proper, convex cone. Let $x_0 \in \text{Int}(V)$ and $y_0 \in \text{Int}(V^*)$ and set $r = -1/\langle x_0, y_0 \rangle$. Then $K_{y_0}, T_{x_0} \subseteq \mathbb{R}^{n+1}$ are $n$-dimensional, compact, convex sets with

$$s_{rx_0}(K_{y_0}) = s_{ry_0}(T_{x_0}) = \frac{(-\langle x_0, y_0 \rangle)^{n+1}}{(n!)^2} \int_V e^{(x_0,y_0)} dx \cdot \int_{V^*} e^{(x_0,y)} dy. \quad (20)$$

The logarithmic Laplace transform of the proper, convex cone $V \subseteq \mathbb{R}^{n+1}$ is the function

$$\Phi_V(y) = \log \int_V e^{(y,x)} dx \quad (y \in \mathbb{R}^{n+1}). \quad (21)$$

The function $\Phi_V : \mathbb{R}^{n+1} \to \mathbb{R} \cup \{ +\infty \}$ attains a finite value at a point $x \in \mathbb{R}^{n+1}$ if and only if $x \in \text{Int}(V^*)$. This may be verified, for example, using formula (12). The function $\Phi_V$ is strictly-convex in $\text{Int}(V^*)$, as follows from the Cauchy-Schwartz inequality (see, e.g., [11]). It follows from (19) that $\Phi_V$ has the following homogeneity property: For any $t > 0$,

$$\Phi_V(ty) = \Phi_V(y) - (n + 1) \log t. \quad (22)$$

Differentiating (22) with respect to $t$ we obtain the useful relation

$$\langle \nabla \Phi_V(y), y \rangle = -(n + 1).$$

Proposition 2.3 tells us that for any $x \in \text{Int}(V)$ and $y \in \text{Int}(V^*)$,

$$\Phi_V(x) + \Phi_V(y) + (n + 1) \log(-\langle x, y \rangle) - 2 \log(n!) = \log s_{rx}(K_y) = \log s_{ry}(T_x), \quad (23)$$

for $r = -1/\langle x, y \rangle$. Thus, any local minimum of the Mahler volume is also a local minimum of the expression on the left-hand side of (23). We would like to compute the first and second variations at a local minimum. We could have proceeded by differentiating the
expression in (23) with respect to \( x \) and to \( y \), but we find it convenient to eliminate one variable by introducing the Legendre transform. The Legendre transform of a convex function \( \Phi : \mathbb{R}^{n+1} \to \mathbb{R} \cup \{+\infty\} \) is

\[
\Phi^*(x) = \sup_{y \in \mathbb{R}^{n+1}, \Phi(y) < \infty} \left[ \langle x, y \rangle - \Phi(y) \right] \quad (x \in \mathbb{R}^{n+1}).
\] (24)

A standard reference for the Legendre transform and convex analysis is Rockafellar [23]. The function \( \Phi^* : \mathbb{R}^{n+1} \to \mathbb{R} \cup \{+\infty\} \) is again convex. If \( \Phi \) is finite in a neighborhood of a point \( x \in \mathbb{R}^{n+1} \) and differentiable at \( x \), then

\[
\Phi^*(\nabla \Phi(x)) + \Phi(x) = \langle x, \nabla \Phi(x) \rangle.
\] (25)

In the case where \( \Phi = \Phi_V \), the supremum in (24) runs over \( y \in \text{Int}(V^*) \). Moreover, in this case we deduce from (22) that for any \( x \in \text{Int}(V) \),

\[
\Phi^*_V(x) = \sup_{s > 0, y \in \text{Int}(V^*)} \left[ (n + 1) \log s + s \langle x, y \rangle - \Phi_V(y) \right]
\] (26)

\[
= (n + 1) \log \left( \frac{n + 1}{e} \right) + \sup_{y \in \text{Int}(V^*)} \left[ -(n + 1) \cdot \log(-\langle x, y \rangle) - \Phi_V(y) \right].
\]

Note the difference between the function \( \Phi^*_V \), which is the Legendre transform of \( \Phi_V \), and the function \( \Phi^*_{V^*} \), which is the logarithmic Laplace transform of the dual cone \( V^* \).

**Corollary 2.4** ("Commuting the Laplace transform with convex duality"). Let \( V \subseteq \mathbb{R}^{n+1} \) be a proper, convex cone. Then the functions \( \Phi_{V^*} \) and \( \Phi^*_V \) attain finite values in \( \text{Int}(V) \), and their difference \( J := \Phi_{V^*} - \Phi^*_V \) satisfies

\[
J(x) = \kappa_n + \log \bar{s}(T_x) \quad \text{for } x \in \text{Int}(V),
\] (27)

where \( \kappa_n = 2 \log(n!) - (n + 1) \log \left( \frac{n + 1}{e} \right) \).

*Proof.* We already know that \( \Phi_{V^*} \) attains finite values in \( \text{Int}(V) \). Fix \( x \in \text{Int}(V) \). From (23) and (26),

\[
\Phi_{V^*}(x) - \Phi^*_V(x) - (n + 1) \log \left( \frac{n + 1}{e} \right) = \inf_{y \in \text{Int}(V^*)} \log s_{ry}(T_x)
\] (28)

where \( r = -1/\langle x, y \rangle \). When \( y \) ranges over the set \( \text{Int}(V^*) \), the point \( ry = -y/\langle x, y \rangle \) ranges over the entire relative interior of \( T_x \). From (10), the right-hand side of (28) equals \( \log \bar{s}(T_x) \), and (27) follows. The function \( \bar{s}(T_x) \) attains finite values in \( \text{Int}(V) \), as well as the function \( \Phi_{V^*} \), and by (27) also \( \Phi^*_V \) is finite in \( \text{Int}(V) \).

Observe that for any homogenous polynomial \( p_k(x) \) of degree \( k \) and any \( y \in \text{Int}(V^*) \),

\[
\int_V p_k(x) e^{\langle y, x \rangle} dx = \int_0^\infty \int_{\{x \in V : \langle x, y \rangle = -t|y|\}} p_k(x) e^{\langle y, x \rangle} dx dt
\]

\[
= \int_0^\infty (t|y|)^{k+n} e^{-t|y|} dt \cdot \int_{K_y} p_k(x) dx = \frac{(n + k)!}{|y|} \cdot \int_{K_y} p_k(x) dx.
\] (29)

We write \( b(K) \) for the barycenter of a convex body \( K \).
Lemma 2.5 ("Derivatives of \( \Phi_V \)). Let \( V \subseteq \mathbb{R}^{n+1} \) be a proper, convex cone. Then for any \( y \in \text{Int}(V^*) \),

\[
\nabla \Phi_V(y) = (n + 1) \cdot b(K_y)
\]

and the Hessian matrix is given by

\[
\nabla^2 \Phi_V(y) = (n + 2)(n + 1) \cdot \text{Cov}(K_y) + (n + 1) \cdot b(K_y) b^*(K_y),
\]

where we view \( z \in \mathbb{R}^{n+1} \) as a column vector while \( z^* \) is the corresponding row vector.

Proof. The function \( \Phi_V \) is clearly smooth in \( \text{Int}(V^*) \). By differentiating (21) we see that for any \( i = 1, \ldots, n+1 \),

\[
\frac{\partial \Phi_V(y)}{\partial y_i} = \frac{\int_V x_i e^{(y,x)} \, dx}{\int_V e^{(y,x)} \, dx} = \frac{(n + 1)!}{n!} \frac{\int_{K_y} x_i \, dx}{\text{Vol}_n(K_y)},
\]

where we used (29) twice in the last passage. This proves (30). By differentiating (21) one more time we see that for \( i, j = 1, \ldots, n+1 \),

\[
\frac{\partial^2 \Phi_V(y)}{\partial y_i \partial y_j} = \frac{\int_V x_i x_j e^{(y,x)} \, dx}{\int_V e^{(y,x)} \, dx} - \frac{\int_V x_i e^{(y,x)} \, dx}{\int_V e^{(y,x)} \, dx} \cdot \frac{\int_V x_j e^{(y,x)} \, dx}{\int_V e^{(y,x)} \, dx} - \frac{\int_V x_j e^{(y,x)} \, dx}{\int_V e^{(y,x)} \, dx} \cdot \frac{\int_V x_i e^{(y,x)} \, dx}{\int_V e^{(y,x)} \, dx} = (n + 2)(n + 1) \frac{\int_{K_y} x_i x_j \, dx}{\text{Vol}_n(K_y)} - (n + 1)^2 \frac{\int_{K_y} x_i \, dx}{\text{Vol}_n(K_y)} \cdot \frac{\int_{K_y} x_j \, dx}{\text{Vol}_n(K_y)},
\]

which is equivalent to (31).

The function \( \Phi_V \) is smooth and strictly-convex in \( \text{Int}(V^*) \), and it equals \(+\infty\) outside \( \text{Int}(V^*) \). Moreover, \( \nabla \Phi_V(y) \in \text{Int}(V) \) for any \( y \in \text{Int}(V^*) \), according to (30). The function \( \Phi^*_V \) is finite in \( \text{Int}(V) \), and from the standard theory of the Legendre transform (e.g., Rockafellar [23, Section 23]), for any \( x \in \text{Int}(V) \) there exists \( y \in \text{Int}(V^*) \) with \( \nabla \Phi_V(y) = x \).

Corollary 2.6. For any proper, convex cone \( V \subset \mathbb{R}^{n+1} \), the map \( \nabla \Phi_V : \text{Int}(V^*) \rightarrow \text{Int}(V) \) is a diffeomorphism.

Proof. We have just explained that the map \( \nabla \Phi_V : \text{Int}(V^*) \rightarrow \text{Int}(V) \) is onto. Since \( \Phi_V \) is strictly-convex, this map is one-to-one. The derivative of this smooth map is the matrix \( \nabla^2 \Phi_V(y) \), which is positive-definite by Lemma 2.5. Hence it is a diffeomorphism.

It follows from Corollary 2.6 and formula (25) that the function \( \Phi^*_V \) is differentiable in \( \text{Int}(V) \) and moreover, for any \( x \in \text{Int}(V) \) and \( y \in \text{Int}(V^*) \),

\[
\nabla \Phi^*_V(x) = y \iff \nabla \Phi_V(y) = x.
\]

In other words, the function \( \nabla \Phi^*_V \) is the inverse to the function \( \nabla \Phi_V \). Consequently the Hessian matrices are inverse to each other, that is, for any \( x \in \text{Int}(V) \),

\[
\nabla^2 \Phi^*_V(x) = [\nabla^2 \Phi_V(y)]^{-1},
\]

where \( y = \nabla \Phi^*_V(x) \). In this next section we will express the first and second derivatives of \( J = \Phi_V^* - \Phi_V^* \) in terms of first and second moments of hyperplane sections of the cones \( V \) and \( V^* \).
3 Projective perturbations and homogenous cones

In this section we prove Theorem 1.1 and Theorem 1.4. We say that two convex bodies $K, T \subseteq \mathbb{R}^n$ are projectively equivalent or that they are projective images of one another if $T$ is affinely-equivalent to a hyperplane section of the cone

$$V = \{(t, tx) \in \mathbb{R} \times \mathbb{R}^n ; t \geq 0, x \in K\}.$$  \hfill (34)

In other words, for a certain $y \in \text{Int}(V^*)$ the set $T$ is affinely equivalent to the convex set $K_y$ associated with the cone $V$. The family of affine images of a given convex body is typically of dimension $n^2 + n$, while the family of projective images is generally of dimension slightly larger, $n^2 + 2n$. Nevertheless, the family of projective images of a convex body $K \subseteq \mathbb{R}^n$ with a smooth boundary always contains bodies arbitrarily close to a Euclidean unit ball. In the case where $K \subseteq \mathbb{R}^n$ is a simplicial polytope, there are projective images of $K$ that are arbitrarily close to a simplex. A projective image of a polytope has the same number of vertices and faces as the original polytope, and in fact the boundary has the same combinatorial structure. A projective image of an ellipsoid is always an ellipsoid, and of a simplex is always a simplex. Our next lemma specializes the results of the previous section to the cone defined in (34). We use $x = (t, y) \in \mathbb{R} \times \mathbb{R}^n$ as coordinates in $\mathbb{R}^{n+1}$.

**Lemma 3.1.** Let $K \subseteq \mathbb{R}^n$ be a convex body with $b(K) = 0$ and let $V \subseteq \mathbb{R}^{n+1}$ be the proper, convex cone defined in (34). Denote $J = \Phi_{V^*} - \Phi_V^*$ and $e = (1, 0) \in \mathbb{R} \times \mathbb{R}^n \cong \mathbb{R}^{n+1}$. Then,

$$\nabla J(e) = (n + 1) \cdot (0, b(K^o)) \in \mathbb{R} \times \mathbb{R}^n \cong \mathbb{R}^{n+1}. \hfill (35)$$

In the case where $\nabla J(e) = 0$, the Hessian matrix satisfies

$$\frac{1}{n + 1} \cdot \nabla^2 J(e) = (n + 2) \cdot \text{Cov}(K^o) - \frac{1}{n + 2} \cdot \text{Cov}(K)^{-1}. \hfill (36)$$

A pedantic remark concerning formula (36): The left-hand side is a certain $(n + 1) \times (n + 1)$ matrix $A$, while the right-hand side is an $n \times n$ matrix $B$. What we actually mean, is that $A = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$. This is consistent with our choice of coordinates, which corresponds to the decomposition $\mathbb{R}^{n+1} \cong \mathbb{R} \times \mathbb{R}^n$.

**Proof of Lemma 3.1.** From (34) we deduce that

$$V^* = \{(t, tx) \in \mathbb{R} \times \mathbb{R}^n ; t \geq 0, x \in K^o\}.$$  

Recalling the notation of (18) from the previous section, we see that $T_e = \{-1\} \times K^o$. By applying Lemma 2.5 with the dual cone $V^*$, we obtain

$$\nabla \Phi_{V^*}(e) = (n + 1) \cdot b(T_e) = (n + 1) \cdot (-1, b(K^o)) \in \mathbb{R} \times \mathbb{R}^n.$$
Since $K_{-e} = \{1\} \times K$, when applying Lemma 2.5 with the cone $V$ we get
\[ \nabla \Phi_V(-e) = (n+1) \cdot b(K_{-e}) = (n+1) \cdot (1, b(K)) = (n+1) \cdot e. \] (37)
Recall that $\Phi_V(ty) = \Phi_V(y) - (n+1) \log t$ according to the homogeneity relation (22). By differentiation, we obtain $\nabla \Phi_V(ty) = \nabla \Phi_V(y)/t$, i.e., $\nabla \Phi_V$ is $(-1)$-homogenous. Therefore, from (37),
\[ \nabla \Phi_V(-(n+1) \cdot e) = e \quad \text{and hence} \quad \nabla \Phi^*(e) = -(n+1)e \] (38)
where we used (32) in the last passage. Consequently,
\[ \nabla J(e) = \nabla \Phi_V^*(e) - \nabla \Phi_V(e) = (n+1) \cdot (-1, b(K^\circ)) + (n+1)e = (n+1) \cdot (0, b(K^\circ)) \in \mathbb{R} \times \mathbb{R}^n, \]
proving (35). We move on to the proof of (36). Since $\nabla J(e) = 0$, then $b(K^\circ) = 0$ and $b(T_e) = -e$. According to Lemma 2.5,
\[ \nabla^2 \Phi_V(-e) = (n+2)(n+1)\text{Cov}(K_{-e}) + (n+1)ee^* = (n+2)(n+1)\text{Cov}(K) + (n+1)ee^* \] (39)
and
\[ \nabla^2 \Phi^*_V(e) = (n+2)(n+1)\text{Cov}(T_e) + (n+1)ee^* = (n+2)(n+1)\text{Cov}(K^\circ) + (n+1)ee^*. \] (40)
Since $\nabla \Phi_V$ is $(-1)$-homogenous, the Hessian $\nabla^2 \Phi_V$ is $(-2)$-homogenous, and $\nabla^2 \Phi_V(ty) = \nabla^2 \Phi_V(y)/t^2$. From (39) we obtain
\[ \nabla^2 \Phi_V(-(n+1) \cdot e) = \frac{n+2}{n+1} \cdot \text{Cov}(K) + \frac{1}{n+1} \cdot ee^*. \]
Thanks to (33) and (38) we know that
\[ \nabla^2 \Phi^*_V(e) = [\nabla^2 \Phi_V(-(n+1) \cdot e)]^{-1} = \frac{n+1}{n+2} \cdot \text{Cov}(K)^{-1} + (n+1) \cdot ee^*. \] (41)
Now (36) follows from (40), (41) and the fact that $J = \Phi_V^* - \Phi_V$. \hfill \Box

We proceed with a discussion and a proof of Theorem 1.4. Let $V \subseteq \mathbb{R}^{n+1}$ be a proper, convex cone. Denote by $\text{Aut}(V)$ the group of all invertible, linear transformations that preserve the cone $V$. Clearly $T \in \text{Aut}(V)$ implies that $T^* \in \text{Aut}(V^*)$ and vice versa, as for any $x \in \mathbb{R}^{n+1}$,
\[ \sup_{y \in V} \langle x, y \rangle = \sup_{y \in V} \langle x, Ty \rangle = \sup_{y \in \mathbb{R}^{n+1}} \langle T^*x, y \rangle. \]
The symmetries of the cone $V$ manifest themselves in the Laplace transform. That is, for any $T \in \text{Aut}(V)$ and $y \in \text{Int}(V^*)$,
\[ \Phi_V((T^*)y) = \log \int_V e^{\langle y, Tx \rangle} dx = \log \int_V e^{\langle y, x \rangle} dx - \log |\det T| = \Phi_V(y) - \log |\det T|. \] (42)
Consequently, for any $x \in \text{Int}(V)$ and $T \in \text{Aut}(V)$,
\[ \Phi^*_V(Tx) = \sup_{y \in \text{Int}(V^*)} [(x, T^*y) - \Phi_V(y)] = \Phi^*_V(x) - \log |\det T|. \] (43)
Proof of Theorem 1.4. We may assume that the barycenter of $K$ lies at the origin and define $V$ as in (34). Then $V$ is a convex, homogenous cone. From (42) and (43) we know that for any $x \in \text{Int}(V)$ and $T \in \text{Aut}(V)$,

$$J(Tx) = \Phi_V(Tx) - \Phi_V^*(Tx) = (\Phi_V(x) - \log |\det T|) - (\Phi_V^*(x) - \log |\det T|) = J(x).$$

However, for any $x, y \in \text{Int}(V)$ there is $T \in \text{Aut}(V)$ with $Tx = y$. Consequently $J : \text{Int}(V) \to \mathbb{R}$ is a constant function. In particular, the gradient and Hessian matrix of $J$ vanish. We may now apply the computations of Lemma 3.1. First, we deduce that $\bar{b}(K^o) = 0$, and hence the Santaló point of $K$ coincides with its barycenter. Second, we deduce that

$$(n + 2) \cdot \text{Cov}(K^o) = \frac{1}{n + 2} \cdot \text{Cov}(K)^{-1}. \quad (44)$$

In particular, $L_K^{2n} \cdot L_{K^o}^{2n} \cdot s(K)^2 = \det \text{Cov}(K) \cdot \det \text{Cov}(K^o) = (n + 2)^{-(2n)}$ by (3).

Theorem 1.1 will be deduced from the next proposition:

**Proposition 3.2.** Let $T \subseteq \mathbb{R}^n$ be a convex body which is a local minimizer of the Mahler volume $s(T)$ in the class of the projective images of $T$. Then,

$$(n + 2)^2 \cdot \text{Cov}(T^o) \geq \text{Cov}(T)^{-1}. \quad (44)$$

Moreover, if $T$ is a local maximizer of the Mahler volume in the class of projective images of $T$ with barycenter at the origin, then $(n + 2)^2 \cdot \text{Cov}(T^o) \leq \text{Cov}(T)^{-1}$.

**Proof.** A translation of $T$ is a particular case of projective image of $T$. Assume that $T$ is a local minimizer. From (10) we learn that the Santaló point of $T$ lies at the origin. Moreover, for any projective image $\bar{T}$ of $T$ that is sufficiently close to the body $T$,

$$\bar{s}(T) \geq \bar{s}(\bar{T}) = s(T).$$

Denote $K = T^o$, so $b(K) = 0$. Consider the proper, convex cone $V \subseteq \mathbb{R} \times \mathbb{R}^n$ defined in (34). For $e = (1, 0) \in \mathbb{R} \times \mathbb{R}^n$ we have

$$T_e = \{-1\} \times T.$$ 

Therefore $T_e$ is a projective image of $T$ for any $x \in \text{Int}(V)$. For any $x$ in some neighborhood of $e$ we know that

$$\bar{s}(T_e) \geq \bar{s}(T).$$

Recall from Corollary 2.4 that $J(x) = \Phi_V(x) - \Phi_V^*(x) = \kappa_n + \log \bar{s}(T_x)$. We thus conclude that $e$ is a local minimum of the function $J : \text{Int}(V) \to \mathbb{R}$. Thus $\nabla J(e) = 0$ and $\nabla^2 J(e) \geq 0$.

From Lemma 3.1 we obtain

$$(n + 2) \cdot \text{Cov}(T) - \frac{1}{n + 2} \cdot \text{Cov}(T^o)^{-1} = (n + 2) \cdot \text{Cov}(K^o) - \frac{1}{n + 2} \cdot \text{Cov}(K)^{-1} \geq 0.$$

This proves (44). Similarly, if $T$ is a local maximizer of $\bar{s}(T_x)$, then $\nabla J(e) = 0$ and $\nabla^2 J(e) \leq 0$, and from Lemma 3.1 we obtain $(n + 2)^2 \cdot \text{Cov}(T^o) \leq \text{Cov}(T)^{-1}$. \qed
Proof of Theorem 1.1. Assume that $K \subseteq \mathbb{R}^n$ is a local minimizer of the Mahler volume in the class of convex bodies containing the origin in their interior. In particular, $K$ is a local minimizer in the class of projective images of $K$, and by Proposition 3.2,

$$\text{Cov}(K^\circ) - (n + 2)^{-2} \cdot \text{Cov}(K)^{-1} \geq 0. \quad (45)$$

In order to prove (4), we use the fact that $\det(A) \geq \det(B)$ whenever $A \geq B \geq 0$. Thus, by (3) and (45),

$$L^2_K \cdot L^2_{K^\circ} \cdot s(K)^2 = \det \text{Cov}(K) \cdot \det \text{Cov}(K^\circ) \geq \frac{1}{(n + 2)^{2n}},$$

proving (4). Note that if $K$ is a global minimizer, then $s(K) \leq s(\Delta^n)$ and from (4),

$$L_K \cdot L_{K^\circ} \geq L^2_{\Delta^n}$$

where we used the fact that $L^2_{\Delta^n} \cdot s(\Delta^n)^{1/n} = 1/(n + 2)$. Hence $L_K \geq L_{\Delta^n}$ or $L_{K^\circ} \geq L_{\Delta^n}$ in the case of a global maximizer. \hfill \Box

Remark 3.3. Suppose that $K \subseteq \mathbb{R}^n$ is a convex body which is not an ellipsoid whose boundary is smooth with Gauss curvature that is always positive (i.e., the boundary is strongly-convex). Assume that the barycenter of $K$ lies at the origin, set $T = K^\circ$, and let $V$ be defined as in (34). Then by Corollary 2.4 with $x = e = (1, 0) \in \mathbb{R} \times \mathbb{R}^n$,

$$J(e) = \kappa_n + \log \bar{s}(T) < \kappa_n + \log s(B^n). \quad (46)$$

In the last passage we used the equality case in the Santaló inequality (see Meyer and Pajor[18]). We claim that for $J = \Phi_{V^*} - \Phi^*_V$,

$$\lim_{x \to \infty} J(x) = \kappa_n + \log s(B^n) > J(e), \quad (47)$$

where $\kappa_n$ is the constant from Corollary 2.4, i.e., $J(x) = \kappa_n + \log \bar{s}(T_x)$. Indeed, the behavior of the functional $J : \text{Int}(V) \to \mathbb{R}$ at infinity is simple to understand, as the corresponding hyperplane sections $T_x$ are very close to ellipsoids, and hence $\bar{s}(T_x)$ is very close to $s(B^n)$. From (46) and (47) we see that the infimum of $J$ is necessarily attained at some point $x \in \text{Int}(V)$. From Proposition 3.2 we see that when $T$ has a smooth and strongly-convex boundary, it has a projective image $\bar{T}$ whose barycenter and Santaló point are at the origin and

$$\text{Cov}(\bar{T}^\circ) \geq \text{Cov}(\bar{T})^{-1}/(n + 2)^2.$$
4 First examples

Let us begin this section by inspecting the simplest example, the case where

$$V = \mathbb{R}^{n+1}_+ = \{ x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}; \forall i, x_i \geq 0 \}. $$

The convex cone $V$ is homogeneous, with an automorphism group consisting of all diagonal transformations with positive entries on the diagonal. Moreover, in this case $V^* = -V$, and a homogeneous cone with this property is called a symmetric cone. For any $y \in \text{Int}(V^*)$, the set $K_y$ is an $n$-dimensional simplex. The logarithmic Laplace transform is given by

$$\Phi_V(y) = -\sum_{i=0}^{n} \log |y_i| \quad \text{for } y \in \text{Int}(V^*).$$

Since $\sup_{s>0}[-st + \log(s)] = -1 - \log(t)$ for any positive $t$, we have

$$\Phi^*_V(x) = -(n+1) - \sum_{i=0}^{n} \log |x_i| \quad \text{for } x \in \text{Int}(V).$$

It follows that $\Phi_{V^*}(x) - \Phi^*_V(x) = n + 1$ for any $x \in \text{Int}(V)$. From Corollary 2.4 and this example we conclude the following:

**Corollary 4.1.** The Mahler conjecture (1) is equivalent to the assertion that for any proper, convex cone $V \subseteq \mathbb{R}^{n+1}$ and a point $x \in \text{Int}(V)$ we have $\Phi_{V^*}(x) - \Phi^*_V(x) \geq n + 1$.

The second example we consider is where

$$V = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}; x_0 \geq \sqrt{\sum_{i=1}^{n} x_i^2} \right\}$$

is the Lorentz cone. Here again $V^* = -V$. For any $y \in \text{Int}(V^*)$, the set $K_y$ is an ellipsoid. Denoting $Q(x) = x_0^2 - \sum_{i=1}^{n} x_i^2$ for $x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$, we have

$$\Phi_V(y) = -\frac{n+1}{2} \log Q(y) + C_n, \quad \text{for } y \in \text{Int}(V^*),$$

as is dictated by the symmetries of the problem, where $C_n = \log(\pi^{n/2} \cdot \Gamma(n+1)/\Gamma(1+n/2))$. Moreover, here

$$\Phi_{V^*} - \Phi^*_V \equiv 2C_n - (n+1) \log \frac{n+1}{e}, \quad (48)$$

which by Santaló’s inequality is the maximal possible value of $\Phi_{V^*} - \Phi^*_V$ for any proper, convex cone $V \subseteq \mathbb{R}^{n+1}$. The right-hand side of (48) is asymptotically $(\log(2\pi) + o(1)) \cdot n$, according to Stirling’s approximation. The third example we consider is the cone of positive semi-definite matrices

$$V = \{ A \in \mathbb{R}^{n \times n}; A^* = A, A \geq 0 \} \quad (49)$$
This is a proper, convex cone in the linear space \( X_n \subseteq \mathbb{R}^{n \times n} \) of all real, symmetric matrices. We endow \( X_n \) with the scalar product

\[
\langle A, B \rangle = \text{Tr}[AB] = \sum_{i=1}^{n} A_{ii}B_{ii} + 2 \sum_{i<j} A_{ij}B_{ij}
\]

where \( A = (A_{ij})_{i,j=1,...,n} \) and \( B = (B_{ij})_{i,j=1,...,n} \). With this scalar product, we have \( V^* = -V \). The volume form in \( X_n \) which is induced by this scalar product is the form \( dA := 2^{(n-1)/4} \prod_{i<j} dA_{ij} \), up to a sign which corresponds to a choice of orientation in \( X_n \). The logarithmic Laplace transform is given by

\[
\Phi_V(-A) = \log \int_V e^{-\text{Tr}[AB]} dB = -\frac{n+1}{2} \cdot \log \det A + C_n
\]

for some constant \( C_n \). Indeed, for any map \( T \in \mathbb{R}^{n \times n} \) with \( \det(T) = 1 \) we know that \( \Phi_V(T^*AT) = \Phi_V(A) \). This shows that \( \Phi_V(A) \) depends only on the determinant of \( A \). The homogeneity property \( \Phi_V(tA) = -\log(t) \cdot n(n+1)/2 + \Phi_V(A) \) implies formula (50). The computation of \( C_n \) by induction on \( n \) is well-known and it is explained, e.g., in (Neretin, Lectures on Gaussian integrals, Section 5.7). For completeness, and since our notation is a bit different, let us include the standard computation.

**Lemma 4.2.** In the case where \( V = V_n \) is given by (49), we have

\[
C_n = \log \int_{V_n} e^{-\text{Tr}[A]} dA = \frac{n(n-1)}{4} \log(2\pi) + \sum_{k=1}^{n} \log \Gamma \left( \frac{k+1}{2} \right).
\]

**Proof.** For \( A \in X_n \) let us write

\[
A = \begin{pmatrix}
B & u \\
\frac{u^*}{s} & s
\end{pmatrix},
\]

where \( B \in X_{n-1} \) is a symmetric matrix, \( u \in \mathbb{R}^{n-1} \) and \( s \in \mathbb{R} \). Then \( dA = \pm 2^{(n-1)/2} dB \land du \land ds \), where we recall that \( dB := 2^{(n-1)(n-2)/4} \prod_{i<j} dB_{ij} \) while \( du = \prod_{i} du_i \). By setting \( v = u/s \), and \( D = B - svv^* \) we have that

\[
A = \begin{pmatrix}
\frac{B}{u^*} & u \\
\frac{D +svv^*}{s} & sv
\end{pmatrix} = \begin{pmatrix}
1 & v \\
0 & 1
\end{pmatrix} \cdot \begin{pmatrix}
D & 0 \\
0 & s
\end{pmatrix} \cdot \begin{pmatrix}
1 & 0 \\
0 & v^*
\end{pmatrix}.
\]

Note that \( du = s^{n-1} dv + \alpha \land ds \) for some \( \alpha \) and hence \( dB \land du \land ds = s^{n-1} dD \land dv \land ds \). Moreover, the map from \( A \in \text{Int}(V_n) \) to \( (D,v,s) \in \text{Int}(V_{n-1}) \times \mathbb{R}^{n-1} \times (0,\infty) \) is a diffeomorphism. Consequently,

\[
e^{C_n} = \int_{V_n} e^{-\text{Tr}[A]} dA = 2^{n-1/2} \int_{V_{n-1} \times \mathbb{R}^{n-1} \times (0,\infty)} e^{-\text{Tr}[D] - s|v|^2 - s^{n-1} dD \land dv \land ds}
\]

\[
= e^{C_{n-1}} \cdot 2^{n-1/2} \int_{0}^{\infty} s^{n-1} e^{-s} \left( \int_{\mathbb{R}^{n-1}} e^{-s|v|^2} dv \right) ds = e^{C_{n-1}} 2^{n-1/2} \int_{0}^{\infty} s^{n-1} e^{-s} \left( \frac{\pi}{s} \right)^{(n-1)/2} ds.
\]
It follows that
\[ C_n = C_{n-1} + \frac{n-1}{2} \log(2\pi) + \log \Gamma \left( \frac{n+1}{2} \right). \]

Since \( C_1 = 0 \), formula (51) follows by an easy induction. \[ \square \]

It follows that in the case where \( V \) is given by (49),
\[ \Phi_{V^*} - \Phi_V^* \equiv 2C_n - \frac{n(n+1)}{2} \log \frac{n+1}{2e} = \frac{n(n+1)}{2} \cdot \left[ \log(2\pi) - 1/2 + o(1) \right] \]
where the asymptotics as \( n \to \infty \) follows from Stirling’s formula. Since \( \log(2\pi) > 3/2 \), asymptotically this is not a counter-example to the Mahler conjecture as formulated in Corollary 4.1. Moreover, thanks to Theorem 1.4, we see that in high dimensions, the isotropic constant of the convex set of symmetric, positive-definite matrices of trace one is smaller than that of the simplex in the corresponding dimension. We proceed with the case of complex-valued matrices, where

\[ V = \{ A \in \mathbb{C}^{n \times n}; A^* = A, A \geq 0 \} \]

We write \( X_n \) for the space of Hermitian \( n \times n \) matrices, equipped with the scalar product \( \langle A, B \rangle = \text{Tr}[A^*B] \). This space has real dimension \( n^2 \), and we have \( V^* = -V \). The induced volume form is \( dA := 2^{n(n-1)/2} \prod_{i \leq j} dA_{ij} \), where \( A_{ii} \in \mathbb{R} \) and \( A_{ij} \in \mathbb{C} \) for \( i \neq j \). The logarithmic Laplace transform is given by
\[ \Phi_V(-A) = \log \int_V e^{-\text{Tr}[AB]} dB = -n \cdot \log \det A + C_n \]
where
\[ C_n = \frac{n(n-1)}{2} \log(2\pi) + \sum_{k=1}^{n-1} \log(k!) \]

Here,
\[ \Phi_{V^*} - \Phi_V^* \equiv 2C_n - n^2 \log \left( \frac{n}{e} \right) = n^2 \cdot \left[ \log(2\pi) - 1/2 + o(1) \right]. \]

We see again the numerical constant \( \log(2\pi) - 1/2 \) which appeared in the case of real, symmetric matrices. Does this numerical constant appear also in the quaternionic case?

A natural operation on convex cones is that of Cartesian products. If \( V_1 \subseteq \mathbb{R}^{n_1+1} \) and \( V_2 \subseteq \mathbb{R}^{n_2+1} \) are proper, convex cones, then so is the Cartesian product \( V_1 \times V_2 \subseteq \mathbb{R}^{n_1+1} \times \mathbb{R}^{n_2+1} \) whose dual is \( V_1^* \times V_2^* \). Moreover,
\[ \Phi_{V_1 \times V_2}(x, y) = \Phi_{V_1}(x) + \Phi_{V_2}(y) \quad (x \in V_1^*, y \in V_2^*) \]
and similarly \( \Phi_{V_1 \times V_2}^*(x, y) = \Phi_{V_1}^*(x) + \Phi_{V_2}^*(y) \). Write \( J_V(x) = \Phi_{V^*}(x) - \Phi_V^*(x) \) and let
\[ J_{n+1} := \inf_{V \subseteq \mathbb{R}^{n+1}} \inf_{x \in V} J_V(x) \]
where the first infimum runs over all proper, convex cones in \( \mathbb{R}^{n+1} \). Since \( J_{V_1 \times V_2}(x, y) = J_{V_1}(x) + J_{V_2}(y) \), we see that \( J_{n+m} \leq J_n + J_m \). The subadditivity property of \( J_n \) implies that

\[
\lim_{n \to \infty} \frac{J_n}{n} = \inf_{n \to \infty} \frac{J_n}{n}
\]

thanks to the Fekete lemma. Thus, in view of Corollary 4.1, the Mahler conjecture would follow from an asymptotic estimate of the form \( J_n \geq (1 + o(1)) \cdot n \). Let us assume now that for \( i = 1, 2 \), the cone \( V_i \subseteq \mathbb{R}^{n_i} \) takes the form

\[
V_i = \{ (t, tx); t \geq 0, x \in K_i \} \subseteq \mathbb{R} \times \mathbb{R}^{n_i}
\]

for some convex body \( K_i \subseteq \mathbb{R}^{n_i} \). Consider the hyperplane section of the cone \( V_1 \times V_2 \) that consists of all 4-tuples \((t, x, s, y)\) with \( t + s = 1 \). This hyperplane section is

\[
\{(t, tx, 1-t, (1-t)y); x \in K_1, y \in K_2, 0 \leq t \leq 1\},
\]

which is affinely equivalent to the geometric join of \( K_1 \) and \( K_2 \), defined via

\[
K_1 \hat{\diamond} K_2 := \sqrt{2} \cdot \{ (t - 1/2, tx, 1/2 - t, (1-t)y); x \in K_1, y \in K_2, 0 \leq t \leq 1\},
\]

The geometric join of \( K_1 \subseteq \mathbb{R}^{n_1} \) and \( K_2 \subseteq \mathbb{R}^{n_2} \) is an \((n_1 + n_2 + 1)\)-dimensional compact, convex set with a non-empty interior relative to the ambient linear subspace

\[
H_{n_1, n_2} = \{ (t, x, -t, y); t \in \mathbb{R}, x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2} \} \subseteq \mathbb{R}^{n_1+n_2+2}.
\]

Our definition of a geometric join is slightly different from the perhaps more standard notation in [14], yet the two definitions are affinely equivalent. The geometric join of \( \Delta^{n_1} \) and \( \Delta^{n_2} \) is an \((n_1 + n_2 + 1)\)-dimensional simplex. The geometric join of two polytopes is a polytope. The notion of geometric join is particularly suitable for duality:

**Proposition 4.3.** Let \( K_1 \subseteq \mathbb{R}^{n_1} \) and \( K_2 \subseteq \mathbb{R}^{n_2} \) be convex bodies containing the origin in their interiors. Then,

\[
(K_1 \hat{\diamond} K_2)^o = K_1^o \hat{\diamond} K_2^o,
\]

where we view \( K_1 \hat{\diamond} K_2 \) and \( K_1^o \hat{\diamond} K_2^o \) as convex bodies in the subspace \( H_{n_1, n_2} \subseteq \mathbb{R}^{n_1+n_2+2} \) which is equipped with the induced scalar product from \( \mathbb{R}^{n_1+n_2+2} \).

**Proof.** Define \( V_i \) as in (53) and write \( E = (e_{n_1}, e_{n_2}) \) where \( e_{n_i} = (0, 1) \in \mathbb{R} \times \mathbb{R}^{n_i} \) for \( i = 1, 2 \). The hyperplane sections of \( V_1^* \times V_2^* \) that are orthogonal to \( E \) are homothetic copies of \( K_1^o \hat{\diamond} K_2^o \). The corresponding hyperplane sections of \((V_1 \times V_2)^*\) are homothetic copies of \((K_1 \hat{\diamond} K_2)^o\). We now deduce (55) from the fact that \((V_1 \times V_2)^* = V_1^* \times V_2^* \) and \(|E| = \sqrt{2} \). \( \square \)
The geometric join may be viewed as a variant of Cartesian product. For example, it may be verified using Corollary 2.4 and the construction in the proof of Proposition 4.3 that
\[ \bar{s}(K_1 \diamond K_2) = C_{n_1, n_2} \cdot \bar{s}(K_1) \cdot \bar{s}(K_2) \] (56)
for any convex bodies \( K_1 \subseteq \mathbb{R}^{n_1} \) and \( K_2 \subseteq \mathbb{R}^{n_2} \), where
\[ C_{n_1, n_2} = \left( \frac{n_1! \cdot n_2!}{(n_1 + n_2 + 1)!} \right)^2 \cdot \frac{(n_1 + n_2 + 2)^{n_1 + n_2 + 2}}{(n_1 + 1)^{n_1 + 1} (n_2 + 1)^{n_2 + 1}}. \]
In comparison, for Cartesian products we have that for any convex bodies \( K_1 \subseteq \mathbb{R}^{n_1} \) and \( K_2 \subseteq \mathbb{R}^{n_2} \),
\[ \bar{s}(K_1 \times K_2) = \tilde{C}_{n_1, n_2} \cdot \bar{s}(K_1) \cdot \bar{s}(K_2) \]
where \( \tilde{C}_{n_1, n_2} = n_1! n_2! / (n_1 + n_2)! \). For more relations between the geometric join and various inequalities, see Rogers and Shephard [24].

5 Covariance of a body and its polar

In this section we prove Proposition 1.5. Let \( K \subseteq \mathbb{R}^n \) be a centrally-symmetric convex body. Then \( b(K) = b(K^\circ) = 0 \) and from Lemma 3.1 we see that the point \( e \) is a stationary point of the functional \( J \) defined with respect to the cone \( V \) from (34). Let us interpret this last statement in view of Corollary 2.4: The Mahler volume functional
\[ \tilde{K} \mapsto \bar{s}(\tilde{K}), \]
restricted to the class of projective images of \( K \), has a stationary point at \( K \). One may wonder whether this stationary point is in fact a local maximum. If it were the case, then by Proposition 3.2 we would have
\[ \phi(K) = \text{Tr}[\text{Cov}(K^\circ) \cdot \text{Cov}(K)] \leq n/(n + 2)^2. \] (57)
Inequality (57) amounts to Conjecture 5.1 from [13]. This local maximum property indeed holds in the case where \( K \) is the unit ball of \( \ell_p^n \) for \( 1 \leq p \leq \infty \), see Alonso-Gutíérrez [1]. However, Proposition 1.5 above implies that this local maximum property fails in general. The remainder of this section is concerned with the proof of Proposition 1.5. Set
\[ K_0 = B_1^{n-1} = \left\{ x \in \mathbb{R}^{n-1}; \sum_{i=1}^{n-1} |x_i| \leq 1 \right\}. \]
Define \( K_1 = K_0 \cap (\sqrt{3/n})B_2^{n-1} \) where \( B_2^{n-1} = \left\{ x \in \mathbb{R}^{n-1}; \sum_i |x_i|^2 \leq 1 \right\} \), and
\[ K = \left\{ (t, x) \in \mathbb{R} \times \mathbb{R}^{n-1}; |t| \leq 1, x \in (1 - |t|)K_0 + |t|K_1 \right\}. \] (58)
We claim that
\[ \text{Vol}_{n-1}(K_1) \geq \frac{1}{3} \cdot \text{Vol}_{n-1}(K_0). \] (59)
Indeed, a direct computation shows that $\int_{K_0} x_i^2 \, dx = 2\text{Vol}_{n-1}(K_0)/[n(n+1)]$ for all $i$. Therefore the average of the function $x \mapsto |x|^2$ on $K_0$ is at most $2/n$. Now (59) follows by the Markov-Chebychev inequality. From (58) and (59) we conclude that

$$\forall x_1 \in (-1, 1), \quad \frac{1}{3} \leq \frac{\text{Vol}_{n-1} (\{x \in \mathbb{R}^{n-1}; (x_1, x) \in K\})}{\text{Vol}_{n-1}(K_0)} \leq 1,$$

where again we use the coordinates $(x_1, x) \in \mathbb{R} \times \mathbb{R}^{n-1} \cong \mathbb{R}^n$. Consequently $\text{Vol}_n(K) \leq 2\text{Vol}_{n-1}(K_0)$ and

$$\int_K x_1^2 \frac{dx}{\text{Vol}_n(K)} = \int_{-1}^1 x_1^2 \cdot \frac{\text{Vol}_{n-1} (\{x \in \mathbb{R}^{n-1}; (x_1, x) \in K\})}{\text{Vol}_n(K)} \, dx_1 \geq \int_{-1}^1 \frac{x_1^2}{6} \, dx_1 = \frac{1}{9}. \quad (60)$$

We move on to discuss the integral of $x_1^2$ over $K^\circ$. We require the following:

**Lemma 5.1.** Let $X_1, \ldots, X_{n-1}$ be independent random variables, distributed uniformly in the interval $[-1, 1]$. Then with a probability of at least $1/6$, there exists a decomposition of $X = (X_1, \ldots, X_{n-1})$ as

$$X = (Y + Z)/2$$

with $Y \in (8/9)B_{\infty}^{n-1}$ and $Z \in \sqrt{3n/10} \cdot B_2^{n-1}$. Here, $B_{\infty}^{n-1} = [-1, 1]^{n-1}$.

**Proof.** We set

$$Y_i = \begin{cases} 
8/9 & \text{if } X_i > 4/9 \\
2X_i & \text{if } -4/9 \leq X_i \leq 4/9 \\
-8/9 & \text{if } X_i < -4/9
\end{cases}$$

and $Z_i = 2X_i - Y_i$. Then for any $i$,

$$\mathbb{E}Z_i^2 = \int_{4/9}^{5/9} (2t - 8/9)^2 \, dt = \int_0^{5/9} (2s)^2 \, ds = \frac{4}{3} \cdot \left(\frac{5}{9}\right)^3 < \frac{1}{4}.$$ 

Hence $\mathbb{E}|Z|^2 < n/4$, and consequently $\mathbb{P}(|Z| \geq \sqrt{3n/10}) \leq 5/6$ by the Markov-Chebyshev inequality. □

It follows from (58) that

$$K^\circ = \{(t, x) \in \mathbb{R} \times \mathbb{R}^{n-1}; x \in K_0^\circ \text{ and } x \in (1 - |t|) \cdot K_1^\circ\}.$$ 

Note that $K_0^\circ = B_{\infty}^{n-1}$ while $K_1^\circ$ is the convex hull of $B_{\infty}^{n-1}$ with $\sqrt{n/3} \cdot B_2^{n-1}$. Lemma 5.1 implies that for any $|t| \leq 1/20$,

$$\text{Vol}_{n-1} \left( \{x \in \mathbb{R}^{n-1}; (t, x) \in K^\circ\} \right) \geq \text{Vol}_{n-1} \left( \left\{ x \in B_{\infty}^{n-1}; x \in \frac{19}{20} K_1^\circ \right\} \right) \quad (61)$$

$$\geq \text{Vol}_{n-1} \left( \left\{ x \in B_{\infty}^{n-1}; x \in \frac{19}{20} \cdot B_{\infty}^{n-1} + \sqrt{n/3} \cdot B_2^{n-1} \right\} \right) \geq \frac{1}{6} \cdot \text{Vol}_{n-1} (B_{\infty}^{n-1}).$$
Write $\alpha(t) = (\{x \in \mathbb{R}^{n-1}; (t, x) \in K^\circ\})$. Then $\alpha$ is supported in $[-1, 1]$, and its maximum is attained at $t = 0$ by the Brunn-Minkowski inequality. From (61) we learn that $\alpha(t) \geq \alpha(0)/6$ for $|t| \leq 1/20$. Therefore,

$$\int_{K^\circ} \frac{x^2}{Vol_n(K^\circ)} \, dx = \int_{-1}^{1} \frac{s^2 \alpha(s) \, ds}{\int_{-1}^{1} \alpha(s) \, ds} \geq \frac{\int_{-1/20}^{1/20} s^2 \cdot (\alpha(0)/6) \, ds}{2\alpha(0)} \geq 10^{-6}. \quad (62)$$

Glancing at (58) we see that the compact set $K \subseteq \mathbb{R}^n$ is convex and unconditional. The conclusion (9) of Proposition 1.5 thus follows from (60) and (62). Since $K$ and $K^\circ$ are unconditional, their covariance matrices are positive-definite and diagonal. Hence, with $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$,

$$\phi(K) = \text{Tr}[\text{Cov}(K^\circ) \cdot \text{Cov}(K)] \geq \langle \text{Cov}(K^\circ)e_1, e_1 \rangle \cdot \text{Cov}(K)e_1 \geq c.$$  

This completes the proof of Proposition 1.5.

6 The floating body of a cone and self-convolution

In this section we describe certain relations between the floating body of a convex cone, its Laplace transform and its self-convolution. Given a convex set $A \subseteq \mathbb{R}^n$ and a parameter $\delta > 0$, Schütt and Werner [29] define the floating body $A_\delta$ as the intersection of all closed half-spaces $H \subseteq \mathbb{R}^n$ for which

$$Vol_n(A \cap H) \geq \delta.$$  

The floating body $A_\delta \subseteq A$ is closed and convex. When $V \subseteq \mathbb{R}^{n+1}$ is a proper, convex cone, by homogeneity we have

$$V_\delta = \delta^{-1/(n+1)} \cdot V_1 \quad \text{for all } \delta > 0. \quad (63)$$

Clearly $V_\delta \subseteq \text{Int}(V)$. Recall the logarithmic Laplace transform $\Phi_V$ and its Legendre transform $\Phi_V^*$.  

**Proposition 6.1.** For any proper, convex cone $V \subset \mathbb{R}^{n+1}$ and $\delta > 0$, we have

$$V_\delta = \{ x \in \text{Int}(V) ; \Phi_V^*(x) \leq \kappa_n - \log \delta \}$$

where $\kappa_n = \log \left[ (\frac{n+1}{e})^{n+1} / (n+1)! \right]$.

**Proof.** Recall from (12) above that for any $y \in \mathbb{R}^{n+1},$

$$e^{\Phi_V(y)} = \int_{V} e^{\langle y, x \rangle} \, dx = \frac{n!}{|y|} \cdot Vol_{n-1}(K_y) = (n+1)! \cdot Vol_n(C_y), \quad (64)$$

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where \( K_y = \{ x \in V ; \langle x, y \rangle = -1 \} \) while

\[
C_y = \{ x \in V ; \langle x, y \rangle \geq -1 \} .
\] (65)

A point \( x \in \mathbb{R}^{n+1} \) belongs to \( V_{\delta} \) if and only if the following holds: The point \( x \) belongs to \( \text{Int}(V) \) and for any \( y \in \text{Int}(V^*) \) with \( \langle x, y \rangle = -1 \),

\[
\text{Vol}_{n+1}(C_y) \geq \delta.
\]

By homogeneity, we see that for any \( x \in \text{Int}(V) \),

\[
x \in V_{\delta} \iff \forall y \in \text{Int}(V^*), \ \text{Vol}_{n+1}(C_y) \geq \frac{\delta}{(-\langle x, y \rangle)^{n+1}}.
\]

From (64), for any \( x \in \text{Int}(V) \), we have that \( x \in V_{\delta} \) if and only if

\[
(n + 1)! \cdot \delta \leq \inf_{y \in \text{Int}(V^*)} e^{\Phi_V(y)} \cdot (-\langle x, y \rangle)^{n+1} = \left( \frac{n + 1}{e} \right)^{n+1} \cdot e^{-\Phi_V(x)},
\]

where the last passage is the content of formula (26) above.

Given a point \( x \in \partial V_{\delta} \) we may look at the normal \( N(x) \) to the smooth hypersurface \( \partial V_{\delta} \) at the point \( x \), pointing outwards of \( V_{\delta} \), and satisfying

\[
|\langle N(x), x \rangle| = 1.
\]

Then \( N(x) = \nabla \Phi^*_V(x)/(n + 1) \), and by (25) and Proposition 6.1,

\[
\Phi_V(N(x)) = (n + 1) \log \frac{n + 1}{e} - \Phi^*_V(x) = \log((n + 1)!) + \log \delta.
\]

It follows that the \textit{polar hypersurface} to \( \partial V_{\delta} \), which is defined as the left-hand side of the following formula, satisfies

\[
\{ N(x) ; x \in \partial V_{\delta} \} = \{ y \in V^* ; \Phi_V(y) = \log((n + 1)!) + \log \delta \}.
\]

In other words, the level sets of the Laplace transform of the cone \( V \) are the polar hypersurfaces to the boundaries of the floating bodies \( V_{\delta} \).

In addition to the convex functions \( \Phi_V \) and \( \Phi^*_V \), we shall introduce yet another convex function that is canonically defined on a proper, convex cone \( V \). It is influenced by Schmuckenschläger’s work [27]. For a proper, convex cone \( V \subseteq \mathbb{R}^{n+1} \) and \( x \in \text{Int}(V) \) we define

\[
\Psi_V(x) = -\log(1_V * 1_V)(x) = -\log \text{Vol}_{n+1}(V \cap (x - V)),
\]

the \textit{self-convolution function} of the cone. Here \( 1_V \) is the characteristic function of the set \( V \), which attains the value 1 in \( V \) and vanishes elsewhere. Since \( V \) is convex, then the
convolution $1_V * 1_V$ is a log-concave function by the Brunn-Minkowski inequality and hence $\Psi_V$ is a convex function which is finite in $\text{Int}(V)$. Moreover,

$$\Psi_V(tx) = -(n + 1) \log t + \log Vol_{n+1}(V/t \cap (x - V/t)) = -(n + 1) \log t + \Psi_V(x). \quad (66)$$

Thus the convex function $\Psi_V$ has the same homogeneity as its sisters $\Phi_V^*$ and $\Phi^*_V$. The convex function $\Psi_V : \text{Int}(V) \to \mathbb{R}$ is not smooth in general. However, it is smooth when the boundary of $K_y$ is smooth and strongly convex for some (and hence for all) $y \in \text{Int}(V^*)$.

**Proposition 6.2.** Let $V \subseteq \mathbb{R}^{n+1}$ be a proper, convex cone. Then for any $x \in \text{Int}(V)$,

$$\Psi_V(x) \geq \Phi_V^*(x) + \kappa_n, \quad (67)$$

where $\kappa_n = \log(2^n (n + 1)!) - (n + 1) \log \left(\frac{n+1}{e}\right)$. There is equality in (67) if and only if $T_x$ is centrally-symmetric with respect to some point in $T_x$.

Moreover, consider the case where $T_x$ is centrally-symmetric with respect to some point in $T_x$, and where the boundary of $T_x$ is smooth and strongly convex. Then the equality in (67) is in fact to first order in $x$, and consequently in this case,

$$\nabla^2 \Psi_V(x) \geq \nabla^2 \Phi_V^*(x). \quad (68)$$

**Proof.** Set $y = \nabla \Phi_V^*(x)/(n + 1)$. Then $\langle x, y \rangle = -1$ by the usual homogeneity relation. For any point $z \in V \cap (x - V)$ the point $x - z$ also belongs to $V$. Therefore the convex body $V \cap (x - V)$ is centrally-symmetric around the point $x/2$. Consequently,

$$Vol_{n+1} \left( \left\{ z \in V \cap (x - V) ; \langle z - \frac{x}{2}, y \rangle \geq 0 \right\} \right) = \frac{1}{2} \cdot Vol_{n+1}(V \cap (x - V)). \quad (69)$$

Recall that $\langle x, y \rangle = -1$. From (69),

$$Vol_{n+1} \left( \left\{ z \in V ; \langle z, y \rangle \geq -\frac{1}{2} \right\} \right) \geq \frac{1}{2} \cdot Vol_{n+1}(V \cap (x - V)). \quad (70)$$

The left-hand side of (70) equals $Vol_{n+1}(C_{2y})$ while the right-hand side equals $e^{-\Psi_V(x)}/2$. Thanks to (64) we may rewrite (70) as

$$\Phi_V(2y) - \log(n + 1)! \geq -\Psi_V(x) - \log 2. \quad (71)$$

Recall that $y = \nabla \Phi_V^*(x)/(n + 1)$. Thanks to the elementary properties (22) and (25), we know that

$$\Phi_V^*(x) + \Phi_V(2y) = (n + 1) \log[(n + 1)/(2e)]. \quad (72)$$

Now (67) follows from (71) and (72).
Equality in (67) is equivalent to equality in (70). If equality holds in (70) then the closed convex set $V \cap (x - V)$ must contain the entire slice $K_{2y}$, or equivalently,

$$K_{2y} \cap (x - K_{2y}) \supseteq K_{2y}.$$ 

This means that $K_{2y}$ is centrally-symmetric around the point $x/2 \in K_{2y}$. This central symmetry condition is not only necessary but also sufficient for equality in (70), as it implies that $V \cap (x - V)$ is a double cone with base $K_{2y}$ and apices 0 and $x$, which leads to equality in (70). We have thus proven that equality holds in (67) if and only if $K_y$ is centrally-symmetric around $x$, which according to Remark 2.2 happens if and only if $T_x$ is centrally-symmetric around $y$.

Next, assume that $T_x$ has a smooth and strongly convex boundary, and moreover it is centrally-symmetric with respect to a certain point $y \in T_x$. Then $y \in \text{Int}(V^*)$ with $\langle x, y \rangle = -1$, and $K_y$ is symmetric around the point $x$. We thus see that the cone $V$ has a non-trivial symmetry, which is the linear map

$$S : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$

with $S(x) = x$ and $S(z) = -z$ for any $z \in x^\perp$. Since $S(V) = V$, also $\Phi^*_V \circ S = \Phi^*_V$ and $\Psi_V \circ S = \Psi_V$. It follows that at the point $x$, which is a fixed point of the symmetry $S$, both the gradient of $\Phi^*_V$ and the gradient of $\Psi_V$ are proportional to $x$. The homogeneity relations (22) and (66) yield

$$\langle \nabla \Psi_V(x), x \rangle = -(n + 1) = \langle \nabla \Phi^*_V(x), x \rangle.$$ 

Since the gradients are proportional, then necessarily $\nabla \Psi_V(x) = \nabla \Phi^*_V(x)$, and the equality in (67) is to first order. The Hessian inequality (68) follows. 

One may wonder whether there is an inequality of the form $\Psi_V \geq \Phi_{V^*}^* + c_n$, which becomes an equality at points $x$ for which $T_x$ has a point of central symmetry. Analogously to Corollary 6.2, this would imply an inequality of the form $\nabla^2 \Phi_{V^*}(x) \leq \nabla^2 \Psi_V(x)$ which can be used to bound from above the isotropic constant of the centrally-symmetric convex body $T_x$. The following is a crude reverse form of Proposition 6.2:

**Lemma 6.3.** Let $V \subseteq \mathbb{R}^{n+1}$ be a proper, convex cone. Then for any $x \in \text{Int}(V)$,

$$\Psi_V(x) \leq \Phi^*_V(x) + Cn,$$ 

(73)

where $C > 0$ is a universal constant.

**Proof.** Set $y = \nabla \Phi^*_V(x)/(n + 1)$. Then $b(K_y) = x$ by Lemma 2.5. By Fubini’s theorem,

$$\text{Vol}_{n+1}(V \cap (x - V)) = \frac{1}{|y|} \int_0^1 \text{Vol}_n[tK_y \cap (x - (1-t)K_y)]dt$$

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The barycenter of $K_y - x$ is at the origin, and by Milman and Pajor [20],

$$Vol_n[tK_y \cap (x - (1 - t)K_y)] = Vol_n(t(K_y - x) \cap (1 - t)(x - K_y)) \geq t^n(1 - t)^nVol_n(K_y).$$

Therefore,

$$e^{-\Psi_V(x)} = Vol_{n+1}(V \cap (x - V)) \geq \frac{Vol_n(K_y)}{|y|} \int_0^1 t^n(1 - t)^n dt = \frac{n! \cdot e^{\Phi_V(y)}}{(2n + 1) \cdot (2n)!},$$

where we used (29) in the last passage. As in (72) above, we know that

$$\Phi_V(x) + \Phi_V(y) = (n + 1) \log[(n + 1)/e].$$

Now (73) follows from (74), (75) and the fact that $(2n)! \cdot e^{n+1} \leq C \cdot n! \cdot \frac{(n+1)^{n+1}}{2n+1}$.

\section{The isomorphic slicing problem}

In this section we prove Theorem 1.3. We begin with a formula for the isotropic constant of a hyperplane section of $V$. Recall that for any $A \in \mathbb{R}^{(n+1)\times(n+1)}$ and $v \in \mathbb{R}^{n+1}$,

$$\det(A + vv^*) = \det(A) + v^*\text{Adj}(A)v$$

where Adj$(A)$ is the adjoint matrix.

\textbf{Lemma 7.1.} For any proper, convex cone $V \subset \mathbb{R}^{n+1}$ and $y \in \text{Int}(V^*)$,

$$\det \nabla^2 \Phi_V(y) = \kappa_n \cdot L^2_{K_y} \cdot e^{2\Phi_V(y)}$$

with $\kappa_n = (n + 1)^{n+1} \cdot (n + 2)^n/(n!)^2$.

\textbf{Proof.} By Lemma 2.5,

$$\frac{\nabla^2 \Phi_V(y)}{(n + 1)(n + 2)} = \text{Cov}(K_y) + \frac{b(K_y)b^*(K_y)}{n + 2}.$$}

The symmetric matrix Cov$(K_y)$ is of rank $n$, with the vector $y$ spanning its kernel. Therefore the adjoint matrix of Cov$(K_y)$ is

$$\det_n\text{Cov}(K_y) \cdot \frac{y y^*}{|y|^2},$$

where det$_n(A)$ stands for the sum of the determinants of all principal $n \times n$ minors of a matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$. Consequently,

$$\det \nabla^2 \Phi(y) = (n + 1)^{n+1} \cdot (n + 2)^n \cdot \frac{\det_n\text{Cov}(K_y)}{|y|^2} \cdot \frac{(y, b(K_y))^2}{n + 2}.$$
However, $\langle b(K_y), y \rangle = (\nabla \Phi_V(y), y)/(n + 1) = -1$ by Lemma 2.5 and the homogeneity relation (22). Hence, by (29),

$$L^{2n}_{K_y} = \frac{\det_n \text{Cov}(K_y)}{\text{Vol}_n(K^c_y)^2} = \frac{(n!)^2}{(n + 1)^{n+1} (n + 2)^n} \cdot \frac{\det \nabla^2 \Phi_V(y)}{e^{2\Phi_V(y)}},$$

and the formula follows. \qed

The role of the determinant $\nabla^2 \Phi_V$ is twofold: First, it appears in the expression for the isotropic constant in Lemma 7.1. Second, it is the Jacobian determinant of the map $\nabla \Phi_V : \text{Int}(V^*) \to \text{Int}(V)$. The next lemma describes a certain geometric property of this map.

**Lemma 7.2.** Let $V \subseteq \mathbb{R}^{n+1}$ be a proper, convex cone, let $y, z \in \text{Int}(V^*)$ and $\delta \in \{\pm 1\}$. Assume that $z \in y + \delta V^*$. Then,

$$\delta \cdot \langle \nabla \Phi_V(z) - \nabla \Phi_V(y), y \rangle \geq 0.$$

**Proof.** By approximation, we may assume that $z \in y + \delta \cdot \text{Int}(V^*)$. Then for any $x \in K_y$ we have $\langle x, z - y \rangle \neq 0$. This means that

$$K_y \cap K_z = \emptyset,$$

as there is no point $x \in K_y$ with $\langle x, z \rangle = \langle x, y \rangle = -1$. The convex set $K_y$ disconnects the cone $V$ into two connected components. In the case where $\delta = +1$, the set $K_z$ must be contained in the convex set $C_y$ defined in (65), and hence its barycenter satisfies,

$$b(K_z) \in C_y = \{x \in V; \langle x, y \rangle \geq -1\}.$$

By Lemma 2.5, we know that $\nabla \Phi_V(z) = (n + 1) \cdot b(K_z)$ and hence when $\delta = +1$,

$$\langle \nabla \Phi_V(z), y \rangle = (n + 1) \langle b(K_z), y \rangle \geq -(n + 1) = \langle \nabla \Phi_V(y), y \rangle,$$

(77)

where the last passage follows from the homogeneity of $\Phi_V$ as in (22). This completes the proof for the case $\delta = +1$. If $\delta = -1$ then $K_z$ must be contained in the convex set $V \setminus C_y = \{x \in V; \langle x, y \rangle < -1\}$. Therefore here

$$\langle \nabla \Phi_V(z), y \rangle = (n + 1) \langle b(K_z), y \rangle \leq -(n + 1) = \langle \nabla \Phi_V(y), y \rangle,$$

and the conclusion holds true in both cases. \qed

From the proof of Lemma 7.2 we obtain a simple geometric interpretation of the set $\nabla \Phi_V(y + V^*)$. Namely, this set consists of all barycenters of all hyperplane sections of the truncated cone $(n + 1) \cdot C_y$ that are disjoint from the base of this truncated cone. Our estimates in the argument below are based on the trivial inclusion of this set in $C_y$. 26
Lemma 7.3. Let $V \subseteq \mathbb{R}^{n+1}$ be a proper, convex cone, $y_0 \in \text{Int}(V^*)$ and $0 < \varepsilon < 1$. Then there exists a point \( y \in \text{Int}(V^*) \) such that \( y - y_0 \in V^* \cap (\varepsilon y_0 - V^*) \) and

\[
L_{K_y} \leq C/\sqrt{\varepsilon}
\]  

(78)

where \( C > 0 \) is a universal constant.

Proof. In this proof \( C, \tilde{C}, \tilde{C}, \tilde{C} > 0 \) denote various positive universal constants. We may assume that \( \varepsilon > e^{-n} \) as otherwise conclusion (78) follows from the trivial upper bound \( L_{K_{y_0}} \leq C\sqrt{n} \) (see, e.g., [9]). Define

\[
S = (y_0 + V^*) \cap (y_0 + \varepsilon y_0 - V^*). 
\]

According to Lemma 6.3,

\[
\text{Vol}_{n+1}(S) = \text{Vol}_{n+1}(V^* \cap (\varepsilon y_0 - V^*)) = \exp(-\Psi_{V^*}(\varepsilon y_0)) \geq \exp(-Cn - \Phi_{V^*}(\varepsilon y_0)).
\]

(79)

We would like to get rid of the expression \( \Phi_{V^*}(\varepsilon y_0) \), and replace it by \( \Phi_{V^*}(\varepsilon y_0) \) plus an error term. To this end, we may use the “commutation relation” of Corollary 2.4, according to which for any \( y \in \text{Int}(V^*) \),

\[
\Phi_{V^*}(y) - \Phi_{V^*}(0) = \log(n!^2 \cdot e^{n+1}) + \log(\tilde{s}_y) \geq n \log n - Cn + \log(\tilde{s}(K_y)).
\]

(80)

However, from the Bourgain-Milman inequality (2), we know that \( \tilde{s}(K_y) \geq (c/n)^n \) for some universal constant \( c > 0 \). Hence, from (79) and (80),

\[
\text{Vol}_{n+1}(S) \geq \exp(-\tilde{C}n - \Phi_{V^*}(\varepsilon y_0)).
\]

(81)

For any \( y \in S \) we know that \( y - (1+\varepsilon)y_0 \in -V^* \), and hence the scalar product of \( y - (1+\varepsilon)y_0 \) with \( \nabla \Phi_{V^*}(y_0 + \varepsilon y_0) \in V^* \) is non-negative. By the convexity of \( \Phi_{V^*} \), for any \( y \in S \),

\[
\Phi_{V^*}(y) \geq \Phi_{V^*}(y_0 + \varepsilon y_0) + \langle \nabla \Phi_{V^*}(y_0 + \varepsilon y_0), y - (1+\varepsilon)y_0 \rangle \geq \Phi_{V^*}(y_0 + \varepsilon y_0).
\]

(82)

From (81) and (82)

\[
\int_S e^{2\Phi^*_{V^*}(y)} dy \geq e^{-\tilde{C}n} \cdot e^{2\Phi_{V^*}(y_0 + \varepsilon y_0)} = e^{-\tilde{C}n} \cdot \left( \frac{\varepsilon}{1 + \varepsilon} \right)^{n+1} \cdot e^{\Phi_{V^*}(y_0)},
\]

(83)

where we used the homogeneity relation (22) in the last passage. According to Lemma 7.2, the image \( \nabla \Phi_{V^*}(S) \) is contained in the truncated cone \( (n+1) \cdot C_{y_0} \). Corollary 2.6 states that the map \( \nabla \Phi_{V^*} \) is a diffeomorphism. By changing variables,

\[
\text{Vol}_{n+1}((n+1) \cdot C_{y_0}) \geq \text{Vol}_{n+1}(\nabla \Phi_{V^*}(S)) = \int_S \det(\nabla^2 \Phi_{V^*}(y)) dy = \kappa_n \int_S e^{2\Phi_{V^*}(y)} L_{K_y}^{2n} dy,
\]

(84)
where $\kappa_n = (n+1)^{n+1} \cdot (n+2)^n / (n!)^2 \geq 1$ is the coefficient from Lemma 7.1. We know that $Vol_{n+1}(C_{y_0}) = e^{\Phi_V(y_0)}/(n+1)!$ according to (64). From (83) and (84),

$$
\frac{(n+1)^{n+1}}{(n+1)!} \cdot e^{\Phi_V(y_0)} \geq \min_{y \in S} L_{K_y}^{2n} \cdot e^{-\tilde{C}_n} \cdot \left( \frac{\varepsilon}{(1+\varepsilon)^2} \right)^{n+1} \cdot e^{\Phi_V(y_0)}.
$$

This implies that there exists $y \in S$ for which

$$
L_{K_y}^{2n} \leq \left( \frac{C_n}{\varepsilon} \right)^{n+1} \leq \left( \frac{C_n}{\varepsilon} \right)^n,
$$

where we used the assumption that $\varepsilon > e^{-n}$ in the last passage. This completes the proof of the lemma.

Given a convex body $K \subseteq \mathbb{R}^n$ with the origin in its interior, we consider the associated (non-symmetric) norm

$$
\|x\|_K = \inf \{ \lambda \geq 0 ; x \in \lambda K \} \quad (x \in \mathbb{R}^n).
$$

The supporting functional of $K$ is

$$
h_K(y) = \|y\|_{K^o} = \sup_{z \in K} \langle y, z \rangle = \sup_{0 \neq z \in \mathbb{R}^n} \frac{\langle y, z \rangle}{\|z\|_K} \quad (y \in \mathbb{R}^n).
$$

We write $\pi(t, x) = x$ for $(t, x) \in \mathbb{R} \times \mathbb{R}^n \cong \mathbb{R}^{n+1}$.

**Lemma 7.4.** Let $K \subseteq \mathbb{R}^n$ be a convex body with the origin in its interior, and set

$$
V = \{(t, tx) \in \mathbb{R} \times \mathbb{R}^n ; t \geq 0, x \in K \}.
$$

Then for any $y \in \text{Int}(V^*)$ and $x \in \mathbb{R}^n$, denoting $y = (y_1, \pi(y)) \in \mathbb{R} \times \mathbb{R}^n$ we have

$$
\|x\|_{\pi(K_y)} = -y_1 \|x\|_K - \langle x, \pi(y) \rangle,
$$

and setting $T = \pi(K_y) \subseteq \mathbb{R}^n$ we have

$$
T^o = -y_1 K^o - \pi(y).
$$

**Proof.** By definition,

$$
K_y = \{(t, tx) ; t \geq 0, x \in K, ty_1 + \langle tx, \pi(y) \rangle = -1 \}.
$$

It follows that

$$
\pi(K_y) = \left\{ x \in \mathbb{R}^n ; \|x\|_K \leq - \frac{1 + \langle x, \pi(y) \rangle}{y_1} \right\},
$$
from which (86) follows. Next, for any \( z \in \mathbb{R}^n \), we have that \( z \in T^\circ \) if and only if

\[
\langle z, x \rangle \leq \|x\|_{\Pi(K_y)} = -y_1\langle x, K \rangle - \langle x, \pi(y) \rangle \quad \text{for all } x \in \mathbb{R}^n.
\]  

(88)

Condition (88) is equivalent to \( \langle z + \pi(y), x \rangle \leq -y_1\langle x, K \rangle \) for all \( x \in \mathbb{R}^n \). Thus \( z \in T^\circ \) if and only if \( h_K(z + \pi(y)) \leq -y_1 \), or equivalently if and only if \( z + \pi(y) \in -y_1K^\circ \). The relation (87) follows.

**Proof of Theorem 1.3:** Define \( V \) as in (85). Since the barycenter of \( K \) lies at the origin, the point \( e = (1, 0) \in \mathbb{R} \times \mathbb{R}^n \) is the barycenter of \( K_{-e} = \{1\} \times K \). We will apply Lemma 7.3 with \( y_0 = -e \). By the conclusion of this lemma, there exists \( y \in \text{Int}(V^*) \) such that

\[
L_{K_y} \leq C/\sqrt{\varepsilon}.
\]

Moreover, \( y + e \in V^* \cap (-\varepsilon e - V^*) \). Since \( \langle z, e \rangle \leq 0 \) for any \( z \in V^* \), setting \( y_1 := \langle y, e \rangle \) we have

\[
-1 - \varepsilon \leq y_1 \leq -1.
\]  

(89)

Since \( y + e \in V^* \) we obtain from (89) that

\[
y + e \in \{z \in V^*; -\varepsilon \leq z_1 \leq 0\} = \{(t, tw) \in \mathbb{R} \times \mathbb{R}^n; -\varepsilon \leq t \leq 0, w \in -K^\circ \}.
\]

Thus \( \pi(y) \in \varepsilon K^\circ \). Since \( y + e \in -\varepsilon e - V^* \) we obtain from (89) that

\[
y + e \in \{z \in -\varepsilon e - V^*; -\varepsilon \leq z_1 \leq 0\} = \{(-\varepsilon - t, -tw) \in \mathbb{R} \times \mathbb{R}^n; -\varepsilon \leq t \leq 0, w \in -K^\circ \}.
\]

Thus \( \pi(y) \in -\varepsilon K^\circ \). To summarize,

\[
\pi(y) \in \varepsilon((-K^\circ) \cap K^\circ).
\]  

(90)

Denote

\[
T = -y_1 \cdot \pi(K_y).
\]

Then \( T \subseteq \mathbb{R}^n \) is a convex body that is affinely equivalent to \( K_y \), and hence \( L_T = L_{K_y} < C/\sqrt{\varepsilon} \). Moreover, by Lemma 7.4,

\[
T^\circ = K^\circ + \pi(y) / y_1
\]  

(91)

and \( T^\circ \) is a translate of \( K^\circ \). By (89) and (90) we know that \( \pi(y)/y_1 \) belongs to \( \varepsilon(-K^\circ) \cap \varepsilon K^\circ \). Hence, from (91),

\[
T^\circ \subseteq K^\circ + \varepsilon K^\circ \quad \text{and} \quad K^\circ \subseteq T^\circ + \varepsilon K^\circ.
\]

Equivalently,

\[
(1 - \varepsilon)K^\circ \subseteq T^\circ \subseteq (1 + \varepsilon)K^\circ.
\]

Since \( 0 < \varepsilon < 1/2 \), by dualizing this inclusion we obtain

\[
(1 - \varepsilon) \cdot K \subseteq \frac{1}{1 + \varepsilon} \cdot K \subseteq T \subseteq \frac{1}{1 - \varepsilon} \cdot K \subseteq (1 + 2\varepsilon) \cdot K,
\]

and the theorem follows by adjusting the universal constant \( C \). 

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