TWISTED EXPONENTIAL SUMS OF POLYNOMIALS IN ONE VARIABLE

CHUNLEI LIU AND WENXIN LIU

Abstract. The twisted $T$-adic exponential sum associated to a polynomial in one variable is studied. An explicit arithmetic polygon in terms of the highest two exponents of the polynomial is proved to be a lower bound of the Newton polygon of the $C$-function of the twisted $T$-adic exponential sum. This bound gives lower bounds for the Newton polygon of the $L$-function of twisted $p$-power order exponential sums.

1. Introduction

Let $p$ be a prime number, $q$ a power of $p$, and $\mathbb{F}_q$ the finite field with $q$ elements. Let $W$ be the Witt ring scheme, $\mathbb{Z}_q = W(\mathbb{F}_q)$, and $\mathbb{Q}_q = \mathbb{Z}_q[\frac{1}{p}]$. Let $\mu_{q-1}$ be the group of $(q - 1)$-th roots of unity in $\mathbb{Z}_q$, $\omega : x \mapsto \hat{x}$ the Teichmüller lifting from $\mathbb{F}_q$ to $\mu_{q-1}$, and $\chi = \omega^{-u}$ with $u \in \mathbb{Z}^n/(q - 1)$ a character of $(\mathbb{F}_q^\times)^n$ into $\mu_{q-1}$.

Let $\triangle \supseteq \{0\}$ be an integral convex polytope in $\mathbb{R}^n$, and $I$ the set of vertices of $\triangle$ different from the origin. Let

$$f(x) = \sum_{u \in \triangle} (a_u x^u, 0, 0, \ldots) \in W(\mathbb{F}_q[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]) \text{ with } \prod_{u \in I} a_u \neq 0,$$

where $x^u = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$ if $u = (u_1, u_2, \ldots, u_n) \in \mathbb{Z}^n$.

**Definition 1.1.** For any positive integer $l$, the sum

$$S_{f, \chi}(l, T) = \sum_{x \in (\mathbb{F}_q^\times)^n} \chi(\text{Norm}_{\mathbb{F}_q/d}(x))(1 + T)^{\text{Tr}_{\mathbb{F}_q/d}(f(x))} \in \mathbb{Z}_q[[T]]$$

is called a twisted $T$-adic exponential sum of $f(x)$. And the function

$$L_{f, \chi}(s, T) = \exp \left( \sum_{l=1}^{\infty} S_{f, \chi}(l, T)^{\frac{s}{l}} \right)$$

is called a $L$-function of twisted $T$-adic exponential sums.

We have

$$L_{f, \chi}(s, T) = \prod_{x \in (\mathbb{Z}_q^\times)^n \otimes \mathbb{F}_q} \frac{1}{1 - \chi(\text{Norm}_{\mathbb{F}_q/(\mathbb{F}_q)}}(1 + T)^{\text{Fr}_x s^m}.$$
where $\mathbb{G}_m$ is the multiplicative group $xy = 1$, $m = \deg(x)$ and $Fr_x = Tr_{\mathbb{Q}_m/\mathbb{Q}_p}(f(x))$.

That Euler product formula gives

$$L_{f,\chi}(s, T) \in 1 + s\mathbb{Z}_q[[T]][[s]].$$

The theory of $T$-adic exponential sums without twists was developed by Liu-Wan [LWn], and the theory of twisted $T$-adic exponential sums was developed by Liu [Liu2].

Let $m \geq 1$, $\zeta_p^m$ a primitive $p^m$-th root of unity, and $\pi_m = \zeta_p^m - 1$. Then the specialization $L_{f,\chi}(s,\pi_m)$ is the $L$-function of twisted $p$-power order exponential sums $S_{f,\chi}(l,\pi_m)$. These sums were studied by Liu [Liu], with the $m = 1$ case studied by Adolphson-Sperber [AS,AS2], and if $\chi$ is trivial, they were studied by Liu-Wei [LW].

Define

$$C_{f,\chi}(s, T) = \exp\left(\sum_{l=1}^{\infty} - (q^l - 1)^{-n} S_{f,\chi}(l, T) \frac{s^l}{T}\right).$$

Call it a $C$-function of twisted $T$-adic exponential sums.

We have

$$L_{f,\chi}(s, T) = \prod_{i=0}^{n} C_{f,\chi}(q^i s, T)(-1)^{n-i+1}^{\binom{n}{i}},$$

and

$$C_{f,\chi}(s, T)(-1)^{n-1} = \prod_{j=0}^{+\infty} L_{f,\chi}(q^j s, T)^{\binom{n+j-1}{j}}.$$

So we have

$$C_{f,\chi}(s, T) \in 1 + s\mathbb{Z}_q[[T]][[s]].$$

We view $C_{f,\chi}(s, T)$ as a power series in the single variable $s$ with coefficients in the $T$-adic complete field $\mathbb{Q}_q((T))$. The $C$-function $C_{f,\chi}(s, T)$ was shown $T$-adic entire in $s$ by Liu [Liu2].

Let $C(\Delta)$ be the cone generated by $\Delta$, $M(\Delta) = C(\Delta) \cap \mathbb{Z}^n$, and $\deg_\Delta$ the degree function on $C(\Delta)$, which is $\mathbb{R}_{\geq 0}$ linear and takes the values 1 on each co-dimension 1 face not containing 0. Let $u \in \mathbb{Z}^n/(q-1)$, and

$$M_u(\Delta) := \frac{1}{q-1} (M(\Delta) \cap u).$$

**Definition 1.2.** Let $b$ be the least positive integer such that $p^b u = u$. Order elements of $\bigcup_{i=0}^{b-1} M_{p^i u}(\Delta)$ such that

$$\deg_\Delta(x_1) \leq \deg_\Delta(x_2) \leq \cdots.$$

A convex function on $\mathbb{R}_{\geq 0}$ which is linear between consecutive integers with initial value 0 is called the infinite $u$-twisted Hodge polygon of $\Delta$ if its slopes between consecutive integers are the numbers

$$\deg_\Delta(x_{b(i+1)}) + \deg_\Delta(x_{b(i+2)}) + \cdots + \deg_\Delta(x_{b(i+1)}) \quad b(i+1), \quad i = 0, 1, \cdots.$$
We denote it by $H_{\triangle,u}^\infty$.

The twisted Hodge polygon for Laurent polynomials can be found in the literature, see Adolphson-Sperber [AS,AS2]. Liu [Liu2] proved the following.

**Theorem 1.3.** We have

$$
T - \text{adic NP of } C_{f,\chi}(s, T) \geq \text{ord}_p(q)(p - 1)H_{\triangle,u}^\infty,
$$

where NP is the short for Newton polygon.

If $\triangle$ is dimension one, the Newton polygon of the $L$-function $L_{f,\chi}(s, \pi_m)$ for $m = 1$ was studied by Blache-Féard-Zhu [BFZ], and if $\chi$ is trivial, it was studied by Liu-Liu-Niu [LLN] for $m \geq 1$.

From now on, we assume that $\triangle = [0, d], \chi = \omega^u$ with $1 \leq u \leq q - 1$, and $q = p^b$. Write

$$
u = u_0 + u_1p + \cdots + u_{b-1}p^{b-1}, 0 \leq u_i \leq p - 1.
$$

**Definition 1.4.** For $a \in \mathbb{N}, 1 \leq i \leq b$,

$$
\delta(i)(n) = \begin{cases} 1, & pl \equiv n - u_{b-i}(d) \text{ for some } l < d\{\frac{n}{d}\} \\ 0, & \text{otherwise.} \end{cases}
$$

where $\{\cdot\}$ is the fractional part of a real number.

**Definition 1.5.** A convex function on $\mathbb{R}_{\geq 0}$ which is linear between consecutive integers with initial value 0 is called the twisted arithmetic polygon of $\triangle = [0, d]$ if its slopes between consecutive integers are the numbers

$$
\omega_{\triangle,u}(n) = \frac{1}{b} \sum_{i=1}^{b} \left( \left\lfloor \frac{(p - 1)n + u_{b-i}(d)}{d} \right\rfloor - \delta(i)(n) \right), a \in \mathbb{N},
$$

where $\left\lfloor \cdot \right\rfloor$ is the least integer equal or greater than a real number. We denote it by $p_{\triangle,u}$.

Liu-Niu [LN] proved the following.

**Theorem 1.6.** If $p > 4d$,

$$
T - \text{adic NP of } C_{f,\chi}(s, T) \geq bp_{\triangle,u}.
$$

By a result of Li [Li], $L_{f,\chi}(s, \pi_m)$ is a polynomial with degree $p^{m-1}d$ if $p \mid d$. Combined this result with the above theorem, one can infer the following.

**Theorem 1.7.** If $p > 4d$, then

$$
\pi_m - \text{adic NP of } L_{f,\chi}(s, \pi_m) \geq bp_{\triangle,u} \text{ on } [0, p^{m-1}d],
$$

with equality holding for a generic $f$ of degree $d$.

We assume that the second highest exponent of $f$ is $k$. So $k \leq d - 1$, and

$$
f(x) = (a_dx^d, 0, 0, \cdots) + \sum_{i=1}^{k} (a_ix^i, 0, 0, \cdots) \in W(F_q[x]) \text{ with } a_da_k \neq 0.
$$
For $a \in \mathbb{N}$, define

$$\varpi_{d,[0,k],u}(a) = \frac{1}{b} \sum_{i=1}^{b} \left( \frac{pa + u_{b-i}}{d} - \left\lfloor \frac{a}{d} \right\rfloor + \left\lfloor \frac{ra_i}{k} \right\rfloor - \left\lfloor \frac{ra}{k} \right\rfloor \right) + \frac{1}{b} \sum_{i=1}^{b} \sum_{j=0}^{r_a} \left( 1 \left( \frac{r_{j,i}}{d} \right) - 1 \left( \frac{r_j}{d} \right) \right) - \frac{1}{b} \sum_{i=1}^{b} \sum_{j=0}^{r_a-1} \left( 1 \left( \frac{r_{j,i}}{d} \right) - 1 \left( \frac{r_j}{d} \right) \right),$$

where $r_a = d\left\{ \frac{a}{d} \right\}$, $r_{a,i} = d\left\{ \frac{pa + u_{b-i}}{d} \right\}$ for $a \in \mathbb{N}, 1 \leq i \leq b$.

**Definition 1.8.** A convex function on $\mathbb{R}_{\geq 0}$ which is linear between consecutive integers with initial value 0 is called the twisted arithmetic polygon of $\{d\} \cup \{0,k\}$ if its slopes between consecutive integers are the numbers $\varpi_{d,[0,k],u}(a), a \in \mathbb{N}$. We denote it by $p_{d,[0,k],u}$.

We can prove the following.

**Theorem 1.9.** We have $p_{d,[0,k],u} \geq p_{\triangle,u}$.

The main result of this paper is the following.

**Theorem 1.10.** If $p > d(2d + 1)$, then

$$T - \text{adic NP of } C_f(s,T) \geq \text{ord}_p(q) p_{d,[0,k],u}.$$  

**Corollary 1.11.** If $p > d(2d + 1)$, then

$$\pi_m - \text{adic NP of } L_{f,\chi}(s,\pi_m) \geq \text{ord}_p(q) p_{d,[0,k],u} \text{ on } [0, p^{m-1}d].$$

2. **The T-Adic Dwork Theory**

In this section we review the $T$-adic analogue of Dwork theory on exponential sums.

We can write

$$\frac{u}{q - 1} = -(u_0 + \frac{u_1p + \cdots}{q - 1}), \quad u_i = u_{b+i} \text{ for } i \geq 0.$$  

and $p^iu = q_i(q - 1) + s_i$ for $i \in \mathbb{N}$ with $0 \leq s_i < q - 1$, then $s_{b-l} = u_l + u_{l+1}p + \cdots + u_{b+l-1}p^{b-l}$ for $0 \leq l \leq b - 1$ and $s_i = s_{b+i}$.

Write $C_u = \{ v \in \mathbb{N} | v \equiv u \text{ (mod } q-1) \}$. Let

$$B_u = \{ \sum_{v \in C_u} b_v \pi^{v - \frac{1}{q-1}} a^\frac{-1}{q-1} : b_v \in \mathbb{Z}_q[[\pi^{\frac{1}{q-1}}]] \text{ and ord}_v b_v \to \infty \text{ as } v \to \infty \}.$$  

Define $B = \bigoplus_{i=1}^{b} B_{p^i u}$, then $B = \bigoplus_{i=1}^{b} B_{p^i u}$ has a basis represented by

$$\prod_{1 \leq i \leq b} \{ x^j \frac{a}{q^{n-1} + j} \}_{j \in \mathbb{N}}.$$
Note that the Galois group of \( \mathbb{Q}_q \) over \( \mathbb{Q}_p \) can act on \( B \) but keeping \( \pi^{1/d} \) as well as the variable \( x \) fixed. Let \( \sigma \in \text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p) \) be the Frobenius element such that \( \sigma(\zeta) = \zeta^p \) if \( \zeta \) is a \((q-1)\)-th root of unity and \( \Psi_p \) the operator on \( B \) defined by the formula

\[
\Psi_p \left( \sum_{i \in \mathbb{N}} c_i x^i \right) = \sum_{i \in \mathbb{N}} c_{pi} x^i.
\]

The Frobenius operator \( \Psi \) on \( B \) is defined by \( \Psi := \sigma^{-1} \circ \Psi_p \circ E_f \), where \( E_f(x) := E(\pi^{\hat{a}_d} x^d) \prod_{i=1}^k E(\pi^{\hat{a}_i} x^i) \).

Note that \( \Psi : B_u \to B_{p^{-1} u} = B_{p^{b-1} u} \), hence \( \Psi \) is well defined. It follows that \( \Psi^b \) operates on \( B_u \) and is linear over \( \mathbb{Z}_q[[\pi^{1/(q-1)}]] \). Moreover it is completely continuous in the sense of Serre [Se].

**Theorem 2.1** \((T\text{-adic Dwork trace formula})\).

\[
C_{f,\chi}(s, T) = \det(1 - \Psi^b s|B_u/\mathbb{Z}_q[[\pi^{1/(q-1)}]])
\]

3. Key Estimate

In order to study

\[
C_{f,\chi}(s, T) = \det(1 - \Psi^b s|B_u/\mathbb{Z}_q[[\pi^{1/(q-1)}]])
\]

we first study

\[
\det(1 - \Psi s|B/\mathbb{Z}_p[[\pi^{1/(q-1)}]]) = \sum_{i=0}^{\infty} (-1)^i c_i s^i.
\]

We are going to show that

**Theorem 3.1.** If \( p > d(2d+1) \), then we have

\[
\text{ord}_\pi(c_{b^2 m}) \geq b^2 p_d, [0, k], u(m).
\]

Consider the operator \( \Psi_p \circ E_f(x) \) on \( B \), we have

\[
\Psi_p \circ E_f(x)(x^{\frac{s_i}{\pi^{1/(q-1)}}} + j) = \Psi_p \left( \sum_{l=0}^{\infty} \gamma l x^{\frac{s_i}{\pi^{1/(q-1)}}} + j + l \right)
= \sum_{p^l+s_i-j \geq 0} \gamma_{p^l+s_i-j} x^{\frac{s_i}{\pi^{1/(q-1)}} + j + l}
= \sum_{l=0}^{\infty} \gamma_{p^l+s_i-j} x^{\frac{s_i}{\pi^{1/(q-1)}} + j + l}.
\]

Then the matrix of \( \Psi_p \circ E_f(x) \) on \( B \) with respect to the basis \( \prod_{1 \leq i \leq b} \{x^{\frac{s_i}{\pi^{1/(q-1)}} + j} \}_{j \in \mathbb{N}} \) is

\[
(G(k,l)(i,j))_{1 \leq k, i \leq b, l, j \in \mathbb{N}}.
\]
where
\[ G_{(k,l)(i,j)} = \begin{cases} \gamma_{\tau+l+u_{i,j}} - 1, & k = i - 1; \\ 0, & \text{otherwise.} \end{cases} \]

Fix a normal basis \( \tilde{\xi}_1, \cdots, \tilde{\xi}_b \) of \( \mathbb{F}_q \) over \( \mathbb{F}_p \). Let \( \xi_1, \cdots, \xi_b \) be their Teichmüller lifts. Then \( \tilde{\xi}_1, \cdots, \tilde{\xi}_b \) is a normal basis of \( \mathbb{Q}_q \) over \( \mathbb{Q}_p \), and \( \sigma \) acts on \( \tilde{\xi}_1, \cdots, \tilde{\xi}_b \) as a permutation. Write
\[ \xi_v^{\sigma - 1}G_{(k,l)(i,j)} = \sum_{w=1}^{b} G_{((k,l),w)((i,j),v)}\xi_w. \]

It is easy to see that \( G_{((k,l),w)((i,j),v)} = 0 \) if \( k \neq i - 1 \). For \( k = i - 1 \), write
\[ G_{((i-1,l),w)((i,j),v)} = G^{(i)}_{(l,w)(j,v)}. \]

Write \( G^{(i)} = (G^{(i)}_{(l,w)(j,v)})_{i,j \in \mathbb{N}, 1 \leq w, v \leq b} \), then the matrix of the operator \( \Psi \) on \( B \) over \( \mathbb{Z}_p[[\frac{1}{\tau + \gamma_j}]] \) with respect to the basis \( \{\xi_v x^{\frac{1}{\tau + \gamma_j}}\}^{1 \leq i, v \leq b, j \in \mathbb{N}} \) is
\[
G = \begin{pmatrix}
0 & G^{(1)} & 0 & \cdots & 0 \\
0 & 0 & G^{(2)} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & G^{(b-1)} \\
G^{(b)} & 0 & 0 & \cdots & 0
\end{pmatrix}.
\]

Hence we have
\[
det(1 - \Psi|B/\mathbb{Z}_p[[\frac{1}{\tau + \gamma_j}]]) = det(1 - Gs) = \sum_{m=0}^{\infty} (-1)^m c_ms^m,
\]
with \( c_m = \sum_{F \in \mathbb{F}_m} \det(F) \), where \( F \) runs over all principle \( m \times m \) submatrix of \( G \).

For every principle submatrix \( F \) of \( G \), write \( F^{(i)} = F \cap G^{(i)} \) as the submatrix of \( G^{(i)} \). For principle \( bm \times bm \) submatrix \( F \) of \( G \), by linear algebra, if one of \( F^{(i)} \) is not \( m \times m \) submatrix of \( G^{(i)} \), then at least one row or column of \( F \) are 0 since \( F \) is principle.

Let \( \mathcal{F}_m \) be the set of all \( bm \times bm \) principle submatrices \( F \) of \( G \) with \( F^{(i)} \) all \( m \times m \) submatrices of \( G^{(i)} \) for each \( 1 \leq i \leq b \).

**Lemma 3.2.** We have
\[
c_{bm} = \sum_{F \in \mathcal{F}_m} \det(F) = \sum_{F \in \mathcal{F}_m} (-1)^{\tau(b-1)} \prod_{i=1}^{b} \det(F^{(i)}).
\]

**Corollary 3.3.**
\[
c_{b^2m} = \sum_{F \in \mathcal{F}_{bm}} \prod_{i=1}^{b} \det(F^{(i)}) = \sum_{F \in \mathcal{F}_{bm}} \prod_{i=1}^{b} \prod_{\tau \in \mathbb{R}_i} sgn(\tau) \prod_{(l,w) \in R_i} G^{(i)}_{(l,w)\tau(l,w)},
\]
where $R_i$ runs over all subsets of $\mathbb{N} \times \{1, 2, \cdots, b\}$ with cardinality $bm$, $\tau$ runs over all permutations of $R_i$, $1 \leq i \leq b$.

So Theorem $3.1$ is reduced to the following.

**Theorem 3.4.** Let $p > d(2d + 1)$. Then we have

$$\sum_{i=1}^{b} \text{ord}_\pi \left( \sum_{\tau} \text{sgn}(\tau) \prod_{(l,\omega) \in R_i} G_{(l,\omega)\tau{l,\omega}}^{(i)} \right) \geq b^2 p_{d,[0,k],u}(m).$$

Let $O(\pi^\alpha)$ denote any element of $\pi$-adic order $\geq \alpha$.

**Lemma 3.5.** For any $1 \leq i \leq b$ and $1 \leq w, v \leq b$, we have

$$G_{(l,w)(j,v)}^{(i)} = O(\pi^{\lfloor \frac{pl+ub_{v}-j}{d}\rfloor + \lfloor \frac{d\{(pl+ub_{v}-j)/k\}}{d}\rfloor}).$$

**Proof.** Write

$$E_f(x) = \sum_{i \in \mathbb{N}} \gamma_i x^i.$$

Then

$$\gamma_i = \sum_{j_1 \leq j_2 \leq j_3 \leq \cdots \leq j_k} \prod_{j=1}^{k} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_k} = O(\pi^{\lfloor \frac{1}{d}\rfloor + \lfloor \frac{\pi}{d}\rfloor}).$$

Since

$$\xi_{v}^{-1} G_{(i-1,l)(i,j)}^{(i-1)} = \xi_{v}^{-1} \gamma_{pl+ul_{d}-j} = \sum_{w=1}^{b} G_{(l,w)(j,v)}^{(i)} \xi_{w},$$

we have

$$\text{ord}_\pi(G_{(l,w)(j,v)}^{(i)}) = \text{ord}_\pi(\gamma_{pl+ul_{d}-j}).$$

The lemma now follows. \qed

By the above lemma, Theorem $3.4$ is reduced to the following.

**Theorem 3.6.** Let $p > d(2d + 1)$. For $1 \leq i \leq b$, let $R_i \subset \mathbb{N} \times \{1, 2, \cdots, b\}$ be a subset of cardinality $bm$, and $\tau$ a permutation of $R_i$. Then

$$\sum_{i=1}^{b} \sum_{(l,\omega) \in R_i} \left( \left\lfloor \frac{pl+ub_{v}-\tau(l)}{d}\right\rfloor + \left\lfloor \frac{d\{(pl+ub_{v}-\tau(l))/k\}}{d}\right\rfloor \right) \geq b^2 p_{d,[0,k],u}(m),$$

where $\tau(l)$ is defined by $\tau(l, \omega) = (\tau(l), \tau(\omega)).$

**Proof.** By definition, we have

$$p_{d,[0,k],u}(m) = \sum_{a=0}^{m-1} \omega_{d,[0,k],u}(a) = \sum_{i=1}^{b} p_{d,[0,k],u}^{(i)}(m),$$
where
\[
p_{d,[0,k],u}^{(i)}(m) = \frac{1}{b} \sum_{a=0}^{m-1} \left( \frac{p_d + u_{b-i} - \tau(l)}{d} - \left\lfloor \frac{\tau(l)}{k} \right\rfloor + \frac{r_{1,i} - \tau(l)}{k} \right) + \sum_{1 \leq l \leq m} \left( \frac{p_d + u_{b-i} - \tau(l)}{d} - \left\lfloor \frac{\tau(l)}{k} \right\rfloor \right).
\]

Then it suffices to show that for 1 \leq i \leq b, and for any permutation \( \tau \) of \( R_i \), we have
\[
\sum_{(l,\omega) \in R_i} \left( \left\lfloor \frac{p_d + u_{b-i} - \tau(l)}{d} \right\rfloor + \left\lfloor \frac{\tau(l)}{k} \right\rfloor \right) \geq b^2 p_{d,[0,k],u}^{(i)}(m).
\]

Note that
\[
\sum_{(l,\omega) \in R_i} \left( \left\lfloor \frac{p_d + u_{b-i} - \tau(l)}{d} \right\rfloor + \left\lfloor \frac{\tau(l)}{k} \right\rfloor \right) = \sum_{(l,\omega) \in R_i} \left( \left\lfloor \frac{p_d + u_{b-i} - \tau(l)}{d} \right\rfloor + \frac{\tau(l)}{d} \right) + \sum_{(l,\omega) \in R_i} \left( \left\lfloor \frac{\tau(l)}{k} \right\rfloor \right)
\]

And
\[
\sum_{(l,\omega) \in R_i} \left( \left\lfloor \frac{d-k}{k} \right\rfloor + \left\lfloor \frac{\tau(l)}{k} \right\rfloor \right) \geq \sum_{(l,\omega) \in R_i} \left\lfloor \frac{r_{1,i} - \tau(l)}{k} \right\rfloor \geq \sum_{(l,\omega) \in R_i} \left\lfloor \frac{r_{1,i}}{k} \right\rfloor - \left\lfloor \frac{\tau(l)}{k} \right\rfloor \geq \sum_{(l,\omega) \in R_i} \left\lfloor \frac{(m-1)}{k} \right\rfloor - \sum_{(l,\omega) \in R_i} \left\lfloor \frac{\tau(l)}{k} \right\rfloor.
\]

We have
\[
\sum_{(l,\omega) \in R_i} \left( \left\lfloor \frac{p_d + u_{b-i} - \tau(l)}{d} \right\rfloor + \left\lfloor \frac{r_{1,i} - \tau(l)}{k} \right\rfloor \right) = b \sum_{l=0}^{m-1} \left( \left\lfloor \frac{p_d + u_{b-i} - \tau(l)}{d} \right\rfloor + \frac{r_{1,i} - \tau(l)}{k} \right) + \sum_{l \geq m} \left( \left\lfloor \frac{p_d + u_{b-i} - \tau(l)}{d} \right\rfloor + \frac{r_{1,i} - \tau(l)}{k} \right) - \sum_{0 \leq l \leq m} \left( \left\lfloor \frac{p_d + u_{b-i} - \tau(l)}{d} \right\rfloor + \frac{r_{1,i} - \tau(l)}{k} \right)
\]
\[
\geq b \sum_{l=0}^{m-1} \left( \left\lfloor \frac{p_d + u_{b-i} - \tau(l)}{d} \right\rfloor + \frac{r_{1,i} - \tau(l)}{k} \right) + N \left( \left\lfloor \frac{p_d + u_{b-i} - \tau(l)}{d} \right\rfloor - \left\lfloor \frac{m-1}{d} \right\rfloor + \frac{d-1}{k} \right) - N \left( \left\lfloor \frac{p_d + u_{b-i} - \tau(l)}{d} \right\rfloor - \left\lfloor \frac{m-1}{d} \right\rfloor + \frac{d-1}{k} \right)
\]
where $N = \# \{ (l, \omega) \in R_l ; l \geq m \} = \# \{ (l, \omega) \notin R_l ; 0 \leq l < m \}$.

Similarly, we have

$$\sum_{(l, \omega) \in R_l} \frac{1}{\lVert \rho \rVert_{(l+1)}} \geq b \sum_{l=0}^{m-1} \frac{1}{\lVert \rho \rVert_{(l+1)}} - N,$$

and

$$\sum_{(l, \omega) \in R} \frac{1}{\lVert \rho \rVert_{(l+1)}} \leq b \sum_{l=0}^{m-1} \frac{1}{\lVert \rho \rVert_{(l+1)}} + N.$$

Therefore,

$$\sum_{(l, \omega) \in R_l} \left( \frac{pl + u_{b-l} - \tau(l)}{d} \right) \geq b \sum_{l=0}^{m-1} \left( \frac{pl + u_{b-l} - \tau(l)}{d} \right) + N \left( \frac{p}{d} - 1 - 2(d-1) \right) = b^2 p_{d,[0,k],a}(m) + N \left( \frac{p}{d} - 2d + 1 \right) \geq b^2 p_{d,[0,k],a}(m).$$

\[ \square \]

4. PROOF OF THE MAIN RESULT

In this section we prove Theorem 1.10, which says that, if $p > d(2d+1)$, then

$$T - \text{adic NP of } C_{f,\chi}(s, T) \geq b p_{d,[0,k],a}.$$  

Lemma 4.1. We have

$$T - \text{adic NP of } \det(1 - \Psi^b s^b | B/\mathbb{Z}_p[[\pi^{1/(q-1)}]]) = T - \text{adic NP of } \det(1 - \Psi^b s^b | B/\mathbb{Z}_p[[\pi^{1/(q-1)}]]).$$

Proof. The lemma follows from the following:

$$\prod_{\zeta^b=1} \det(1 - \Psi^b s^b | B/\mathbb{Z}_p[[\pi^{1/(q-1)}]]) = \det(1 - \Psi^b s^b | B/\mathbb{Z}_p[[\pi^{1/(q-1)}]]) = \text{Norm}(\det(1 - \Psi^b s^b | B/\mathbb{Z}_p[[\pi^{1/(q-1)}]]),$$

where the Norm is the norm map from $\mathbb{Q}_q[[\pi^{1/(q-1)}]]$ to $\mathbb{Q}_p[[\pi^{1/(q-1)}]]$.  

\[ \square \]

Lemma 4.2. We have

$$T - \text{adic NP of } C_{f,\chi}(s, T)^b = T - \text{adic NP of } \det(1 - \Psi^b s^b | B/\mathbb{Z}_q[[\pi^{1/(q-1)}]]).$$
Proof. Let $\sigma$ act on $\mathbb{Q}_q[[T]]$ coordinate-wise. Hence
\begin{align*}
S_{f, \chi^p}(l, T) &= \sum_{x \in \mathbb{F}_{q^l}^\times} \chi(\text{Norm}_{\mathbb{F}_{q^l}/\mathbb{F}_q}(x)) (1 + T)^{\text{Tr}_{\mathbb{Q}_q^l/\mathbb{Q}_q}(f(x))} \\
&= S_{f, \chi}(l, T)^{\sigma},
\end{align*}
therefore $C_{f, \chi^p}(s, T) = C_{f, \chi}(s, T)^{\sigma}$, which yields that the T-adic Newton polygons of $C_{f, \chi^p}(s, T)$ and $C_{f, \chi}(s, T)$ coincide with each other. Hence the lemma follows from the following
\begin{align*}
\prod_{i=1}^{b} C_{f, \chi}(s, T)^{\sigma_i} &= \prod_{i=1}^{b} C_{f, \chi^p}(s, T) \\
&= \prod_{i=1}^{b} \det(1 - \Psi^b s | B_{u}/\mathbb{Z}_q[[\pi^{\frac{1}{q^l-1}}]]) \\
&= \det(1 - \Psi^b s | B/\mathbb{Z}_q[[\pi^{\frac{1}{q^l-1}}]]).
\end{align*}
\[\square\]

Corollary 4.3. The T-adic Newton polygon of $C_{f, \chi}(s, T)$ is the lower convex closure of the points
\[\left(i, \frac{1}{b} \text{ord}_{\mathbb{Q}_q^l} c_{b^i}\right), \ i = 0, 1, \ldots\]

Proof. By Lemma 4.11 the T-adic Newton polygon of
\[\det(1 - \Psi^b s | B_{u}/\mathbb{Z}_q[[\pi^{\frac{1}{q^l-1}}]])\]
is the lower convex closure of the points
\[\left(bi, \text{ord}_{\mathbb{Q}_q^l} c_{bi}\right), \ i = 0, 1, \ldots .\]
Hence the T-adic Newton polygon of
\[\det(1 - \Psi^b s | B/\mathbb{Z}_q[[\pi^{\frac{1}{q^l-1}}]])\]
is the convex closure of the points
\[\left(i, \text{ord}_{\mathbb{Q}_q^l} c_{bi}\right), \ i = 0, 1, \ldots .\]
By Lemma 4.2 the T-adic Newton polygon of $C_{f, \chi}(s, T)^b$ is the lower convex closure of the points
\[\left(bi, \text{ord}_{\mathbb{Q}_q^l} c_{b^i}\right), \ i = 0, 1, \ldots .\]
The lemma is proved. \[\square\]

We now prove Theorem 1.10.
Proof of Theorem 1.10. By Theorem 2.1, we have
\[C_{f, \chi}(s, T) = \det(1 - \Psi^b s | B_{u}/\mathbb{Z}_q[[\pi^{\frac{1}{q^l-1}}]]) ,\]
Then by Corollary 4.3, the $T$-adic Newton polygon of $C_{f,\chi}(s, T)$ is the lower convex closure of the points
\[(i, \frac{1}{b} \text{ord}_\pi c_{b^i}), \ i = 0, 1, \cdots .\]
Therefore the result follows from Theorem 3.1, which says that, if $p > d(2d + 1)$, then we have
\[\text{ord}_\pi(c_{b^m}) \geq b^2 p_{d,[0,k],u}(m).\]
\[\square\]

We conclude this section by proving Corollary 1.11.

\textbf{Proof of Corollary 1.11}  Assume that $L_{f,\chi}(s, \pi_m) = \prod_{i=1}^{p^{m-1}d} (1 - \beta_i s)$. Then
\[C_{f,\chi}(s, \pi_m) = \prod_{j=0}^{\infty} L_{f,\chi}(q^j s, \pi_m) = \prod_{j=0}^{\infty} \prod_{i=1}^{p^{m-1}d} (1 - \beta_i q^j s).\]
Therefore the slopes of the $q$-adic Newton polygon of $C_{f,\chi}(s, \pi_m)$ are the numbers
\[j + \text{ord}_q(\beta_i), \ 1 \leq i \leq p^{m-1}d, j = 0, 1, \cdots .\]
It is well-known that $\text{ord}_q(\beta_i) \leq 1$ for all $i$. Therefore,
\[q - \text{adic NP of } L_{f,\chi}(s, \pi_m) = q - \text{adic NP of } C_{f,\chi}(s, \pi_m) \text{ on } [0,p^{m-1}d].\]
It follows that
\[\pi_m - \text{adic NP of } L_{f,\chi}(s, \pi_m) = \pi_m - \text{adic NP of } C_{f,\chi}(s, \pi_m) \text{ on } [0,p^{m-1}d].\]
By the integrality of $C_{f,\chi}(s, T)$ and Theorem 1.10, we have
\[\pi_m - \text{adic NP of } C_{f,\chi}(s, \pi_m) \geq T - \text{adic NP of } C_{f,\chi}(s, T) \geq \text{ord}_p(q)p_{d,[0,k],u}.\]
Therefore,
\[\pi_m - \text{adic NP of } L_{f,\chi}(s, \pi_m) \geq \text{ord}_p(q)p_{d,[0,k],u} \text{ on } [0,p^{m-1}d].\]
\[\square\]

\section{Comparison between arithmetic polygons}

In this section we prove Theorem 1.9, which says that
\[p_{d,[0,k],u} \geq p_{\Delta,u}.\]

\textbf{Proof of Theorem 1.9}  It is clear that $p_{d,[0,k]}(0) = p_{\Delta}(0)$. It suffices to show that, for $m \in \mathbb{N}$, we have
\[p_{d,[0,k],u}(m + 1) \geq p_{\Delta,u}(m + 1).\]
By a result in Liu-Niu [LN], we have
\[p_{\Delta,u}(m + 1) = \sum_{i=1}^{b} p^{(i)}_{\Delta,u}(m + 1),\]
where
\[
p^{(i)}(m+1) = \frac{1}{b} \left( \sum_{a=0}^{m} \left( \left\lfloor \frac{pa + ub_{b-1}}{d} \right\rfloor - \left\lfloor \frac{a}{d} \right\rfloor \right) + \sum_{a=0}^{r_m} \left( \left\lfloor \frac{a}{d} \right\rfloor \right) \right).
\]

By definition, we have
\[
p_d([0,k], u)(m + 1) = \sum_{a=0}^{m} \varpi_d([0,k], u)(a) = \sum_{i=1}^{b} p^{(i)}_{d,[0,k], u}(m + 1),
\]
where
\[
p^{(i)}_{d,[0,k], u}(m+1) = \frac{1}{b} \sum_{a=0}^{m} \left( \left\lfloor \frac{pa + ub_{b-1}}{d} \right\rfloor - \left\lfloor \frac{a}{d} \right\rfloor + \left\lfloor \frac{r_{a,i}}{k} \right\rfloor - \left\lfloor \frac{r_a}{k} \right\rfloor + 1 \left( \frac{r_{a,i}}{k} \right) - 1 \left( \frac{r_a}{k} \right) \right).
\]

Then it suffices to show that
\[
p^{(i)}_{d,[0,k], u}(m + 1) \geq p^{(i)}_{\triangle,u}(m + 1).
\]

For \(m \geq 0, 1 \leq i \leq b\), let \(A_{i1} = \{0 \leq a \leq r_m | a \neq r_{L,i} \text{ for some } 0 \leq l \leq r_m\}\).

Note that
\[
\{a : 0 \leq a \leq r_m\} = A_{i1} \cup A_{i2},
\]
where \(A_{i2} = \{r_{a,i} | 0 \leq a \leq r_m\}\). And
\[
\{r_{a,i} | 0 \leq a \leq r_m\} = A_{i2} \cup A_{i3},
\]
where \(A_{i3} = \{r_{a,i} > r_m | 0 \leq a \leq r_m\}\), so we have \(|A_{i1}| = |A_{i3}|\). Then
\[
\sum_{a=0}^{m} \left( \left\lfloor \frac{r_{a,i}}{k} \right\rfloor - \left\lfloor \frac{r_a}{k} \right\rfloor + 1 \left( \frac{r_{a,i}}{k} \right) - 1 \left( \frac{r_a}{k} \right) \right)
\]
\[
= \sum_{a=0}^{r_m} \left( \left\lfloor \frac{r_{a,i}}{k} \right\rfloor - \left\lfloor \frac{r_a}{k} \right\rfloor + 1 \left( \frac{r_{a,i}}{k} \right) - 1 \left( \frac{r_a}{k} \right) \right)
\]
\[
= \sum_{r_{a,i} \in A_{i3}} \left\lfloor \frac{r_{a,i} - r_m}{k} \right\rfloor + \sum_{a \in A_{i1}} \left( \left\lfloor \frac{r_m - a}{k} \right\rfloor - 1 \left( \frac{r_m}{k} \right) - 1 \left( \frac{a}{k} \right) \right)
\]
\[
\geq \sum_{r_{a,i} \in A_{i3}} \left\lfloor \frac{r_{a,i} - r_m}{k} \right\rfloor.
\]

Therefore, we have
\[
b(p^{(i)}_{d,[0,k], u}(m + 1) - p^{(i)}_{\triangle,u}(m + 1))
\]
\[
= \sum_{a=0}^{m} \left( \left\lfloor \frac{pa + ub_{b-1}}{d} \right\rfloor - \left\lfloor \frac{a}{d} \right\rfloor + \left\lfloor \frac{r_{a,i}}{k} \right\rfloor - \left\lfloor \frac{r_a}{k} \right\rfloor + 1 \left( \frac{r_{a,i}}{k} \right) - 1 \left( \frac{r_a}{k} \right) \right)
\]
\[
- \left( \sum_{a=0}^{m} \left( \left\lfloor \frac{pa + ub_{b-1}}{d} \right\rfloor - \left\lfloor \frac{a}{d} \right\rfloor + \sum_{a=0}^{r_m} \left( \left\lfloor \frac{a}{d} \right\rfloor \right) \right) \right).
\]
\[
\sum_{a=0}^{m} \left( \left\lceil \frac{a}{d} \right\rceil - \left\{ \frac{R_{a,i}}{d} \right\} \right) + \sum_{\frac{r_{a,i}}{k} \in A_{i3}} \left( \left\lceil \frac{R_{a,i} - R_m}{k} \right\rceil - \left\lceil \frac{R_{0,i} - R_m}{k} \right\rceil \right) \left\{ \frac{u_{b_i}}{d} \right\} + \sum_{\frac{r_{a,i}}{k} \in A_{i3}} \left( \left\lceil \frac{R_{a,i} - R_m}{k} \right\rceil - \left\lceil \frac{R_{0,i} - R_m}{k} \right\rceil \right) \left\{ \frac{u_{b_i}}{d} \right\}
\]

\[
= \sum_{a=0}^{m} \left( \left\lceil \frac{a}{d} \right\rceil - \left\{ \frac{R_{a,i}}{d} \right\} \right) + \sum_{a=1}^{r_m} \left( \left\lceil \frac{R_{a,i} - R_m}{k} \right\rceil - \left\lceil \frac{R_{0,i} - R_m}{k} \right\rceil \right) \left\{ \frac{u_{b_i}}{d} \right\}
\]

where for \( 1 \leq i \leq b \),

\[
\delta_{m,1}^{(i)} = \begin{cases} 0, & \text{if there exists } 0 \leq a \leq r_m \text{ such that } pa + ub_{-i} \equiv 0 \pmod{d}; \\ -1, & \text{otherwise}. \end{cases}
\]

\[
\delta_{m,2}^{(i)} = \begin{cases} -1, & \text{if there exists } 1 \leq a \leq r_m \text{ such that } pa + ub_{-i} \equiv 0 \pmod{d}; \\ 0, & \text{otherwise}. \end{cases}
\]

The theorem now follows.

\[ \square \]

References

[AS] A. Adolphson and S. Sperber, On twisted exponential sums, Math. Ann., 290 (1991), 713-726.

[AS2] A. Adolphson and S. Sperber, Twisted exponential sums and Newton polyhedra, J. reine angew. Math. 443 (1993), 151-177.

[Li] W.-C. W. Li, Character sums over \( p \)-adic fields, J. Number Theory 74 (1999), no.2, 181-229.

[Liu] C. Liu, The \( L \)-functions of twisted Witt coverings, J. Number Theory, 125 (2007), 267-284.

[Liu2] C. Liu, T-adic exponential sums under diagonal base change, arXiv:0909.1111

[LLN] C. Liu, Wenxin Liu and Chuanze Niu, T-adic exponential sums in one variable, arXiv:0901.0354

[LN] C. Liu and C. Niu, Twisted T-adic exponential sums of polynomial in one variable.

[LW] C. Liu and D. Wei, The \( L \)-functions of Witt coverings, Math. Z., 255 (2007), 95-115.

[LWn] C. Liu and D. Wan, T-adic exponential sums, Algebra & Number Theory, to appear.

[BFZ] R. Blache, E. Férard and J.H. Zhu, Hodge-Stickelberger polygons for \( L \)-functions of exponential sums of \( P(x^s) \), Math. Res. Letter, 15 (2008), 1053-1071.

[Se] J-P. Serre, Endomorphismes complètement continus des espaces de Banach \( p \)-adiques, Publ. Math., IHES., 12(1962), 69-85.