DISCRETE HARDY SPACES AND HEAT SEMIGROUP ASSOCIATED WITH
THE DISCRETE LAPLACIAN

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Abstract. In this paper we study the behavior of some harmonic analysis operators associated with the discrete Laplacian $\Delta_d$ in discrete Hardy spaces $H^p(\mathbb{Z})$. We prove that the maximal operator and the Littlewood-Paley $g$ function defined by the semigroup generated by $\Delta_d$ are bounded from $H^p(\mathbb{Z})$ into $\ell^p(\mathbb{Z})$, $0 < p \leq 1$. Also, we establish that every $\Delta_d$-spectral multiplier of Laplace transform type is a bounded operator from $H^p(\mathbb{Z})$ into itself, for every $0 < p \leq 1$.

1. INTRODUCTION

Coifman and Weiss ([8]) defined Hardy spaces $H^p(X)$ when the underlying space $X$ is a space of homogeneous type. They extended the atomic decomposition theory for the classical Hardy spaces to this more general setting. The set $\mathbb{Z}$ of integer numbers endowed with the usual distance defined by the absolute value and the counting measure $\mu$ is a space of homogeneous type. Discrete Hardy spaces were mentioned in [8, p. 622] as an example of the general theory in [8].

Boza and Carro ([2], [3] and [4]) characterized Hardy spaces $H^p(\mathbb{Z})$, $0 < p \leq 1$, by using some maximal operators associated with a discretization of the classical Poisson integrals and approximations of the identity. Also, they described discrete Hardy spaces by using atoms and the discrete Hilbert transform. Boza and Carro established and took advantage of connections between discrete and continuous (classical) settings. Kanjin and Satake ([13]) developed molecular characterizations of discrete Hardy spaces. Komori ([14]) proved that every molecule can be decomposed in atoms without using the properties of the classical Hardy spaces $H^p(\mathbb{R})$. A discrete version for $H^p(\mathbb{Z})$ of the weak factorization results for Hardy spaces $H^p(\mathbb{R})$ due to Coifman, Rochberg and Weiss ([7]) and Miyachi ([17]), was proved by Boza ([1]). Eoff ([3, Theorem 1]) established that, for every $0 < p \leq 1$, the discrete Hardy space $H^p(\mathbb{Z})$ is isomorphic to the Paley-Wiener space $E^p$ that consists of all those entire functions of exponential type $\pi$ such that $\int_{\mathbb{Z}} |f(x)|^p dx < \infty$. Chen and Fang ([5]) extended Eoff’s result to higher dimensions when $p = 1$.

The discrete Laplacian $\Delta_d$ on $\mathbb{Z}$ is defined by

$$(\Delta_d f)(n) = -f(n + 1) + 2f(n) - f(n - 1), \quad n \in \mathbb{Z},$$

where $f$ is a complex function defined on $\mathbb{Z}$. For every $0 < p \leq \infty$, we denote by $\ell^p(\mathbb{Z})$ the usual Lebesgue space on $\mathbb{Z}$ with respect to the counting measure $\mu$. The operator $\Delta_d$ is bounded from $\ell^p(\mathbb{Z})$ into itself, for every $0 < p \leq \infty$, and it is a nonnegative operator in $\ell^2(\mathbb{Z})$.

We define the function

$$G(n, t) = e^{-2t} I_n(2t), \quad n \in \mathbb{Z} \text{ and } t > 0.$$ 

Here, for every $n \in \mathbb{Z}$, $I_n$ represents the modified Bessel function of the first kind and order $n$. The main properties of $I_n$ can be encountered in [15, Chapter 5].

For every $t > 0$ we consider the convolution operator $W_t$ defined by

$$W_t(f)(n) = \sum_{m \in \mathbb{Z}} G(n - m, t)f(m), \quad n \in \mathbb{Z},$$

for every $f \in \ell^p(\mathbb{Z})$, $1 \leq p \leq \infty$. In [5, Proposition 1] it was proved that the uniparametric family $\{W_t\}_{t>0}$ is a positive Markovian diffusion semigroup in the Stein’s sense ([20]) in $\ell^p(\mathbb{Z})$, $1 \leq p \leq \infty$. Moreover, $W_t = e^{-t\Delta_d}$, $t > 0$, that is, $-\Delta_d$ is the infinitesimal generator of $\{W_t\}_{t>0}$. 

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We now define the operators that we will study on Hardy spaces $H^p(Z), 0 < p \leq 1$. The maximal operator $W_*$ associated with \( \{W_t\}_{t>0} \) is defined by
\[
W_*(f) = \sup_{t>0} |W_t(f)|.
\]
The (vertical) Littlewood-Paley $g$-function for \( \{W_t\}_{t>0} \) is given by
\[
g(f)(n) = \left( \int_0^\infty |\omega_t W_t(f)(n)|^2 \frac{dt}{t} \right)^{1/2}, \quad n \in Z,
\]
for every $f \in \ell^p(Z), 1 \leq p < \infty$. If $f \in \ell^1(Z)$ the Fourier transform $F_Z(f)$ of $f$ is defined by
\[
F_Z(f)(\theta) = \sum_{n \in Z} f(n) e^{in\theta}, \quad \theta \in (-\pi, \pi).
\]
The Fourier transform $F_Z(f)$ can be extended from $\ell^1(Z)$ to $\ell^2(Z)$ as an isometry from $\ell^2(Z)$ into $L^2(-\pi, \pi)$. Moreover, the inverse operator $F_Z^{-1}$ of $F_Z$ is defined by
\[
F_Z^{-1}(\varphi)(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(\theta) e^{-in\theta} d\theta, \quad \varphi \in L^2(-\pi, \pi).
\]
It is no hard to see that, for every $\theta \in (-\pi, \pi), \Delta_d(e_\theta) = 2(1 - \cos \theta)e_\theta$, where $e_\theta(n) = e^{in\theta}, n \in Z$. Also, for every $f \in \ell^2(Z)$,
\[
F_Z(\Delta_d(f))(\theta) = 2(1 - \cos \theta)F_Z(f)(\theta), \quad \theta \in (-\pi, \pi).
\]
Thus, $F_m$ is bounded from $\ell^2(Z)$ into itself.

We now assume that $m \in L^\infty(0, \infty)$ and that there exists $\Psi \in L^\infty(0, \infty)$ such that
\[
m(\lambda) = \lambda \int_0^\infty e^{-\lambda \Psi(t)} dt, \quad \lambda \in (0, \infty).
\]
The multiplier $F_m$ is now called Laplace transform type multiplier (see [20]).

According to [20] the operators $W_*, g$ and $F_m$ are bounded from $\ell^p(Z)$ into itself for every $1 < p < \infty$. In [14] it was proved by using Calderón-Zygmund theory for Banach valued singular integrals that $W_*$ and $g$ are bounded operators from $\ell^p(Z, \omega)$ into itself, for every $1 < p < \infty$ and $\omega \in A_p(Z)$, and from $\ell^1(Z, \omega)$ into $\ell^1(\infty, Z, \omega)$, for every $\omega \in A_1(Z)$. Here if $\omega$ is a nonnegative sequence and $1 \leq p < \infty$, we say that a complex sequence $f \in \ell^1(\infty, Z, \omega)$ when
\[
\|f\|_{p, \omega} := \left( \sum_{n \in Z} |f(n)|^p \omega(n) \right)^{1/p} < \infty.
\]
On $\ell^p(Z, \omega)$ we consider the norm $\| \cdot \|_{p, \omega}$. Also, if $\omega$ is a nonnegative sequence, we say that a complex sequence $f \in \ell^1(\infty, Z, \omega)$ when
\[
\|f\|_{1, \omega} := \sup_{\lambda > 0} \lambda \omega(\{n \in Z : |f(n)| > \lambda \}) < \infty,
\]
where $\omega(E) = \sum_{n \in E} W(n)$, for every $E \subseteq Z$. On $\ell^1(\infty, Z, \omega)$ we consider the quasinorm $\| \cdot \|_{1, \omega}$.

For every $1 \leq p < \infty$, by $A_p(Z)$ we denote the Muckenhoupt class of weights. A nonnegative sequence $\omega \in A_p(Z)$ ([12 §8]) when there exists $C > 0$ such that, for every $M, N \in Z, M \leq N$,
\[
\left( \sum_{k=M}^{N} \omega(k) \right) \left( \sum_{k=M}^{N} \omega(k)^{-1/(p-1)} \right)^{p-1} \leq C(N - M + 1)^p, \quad 1 < p < \infty,
\]
and
\[
\left( \sum_{k=M}^{N} \omega(k) \right) \sup_{M \leq k \leq N} \omega(k)^{-1} \leq C(N - M + 1), \quad p = 1.
\]

The discrete Hilbert transform $H_\Delta$ is defined by
\[
H_\Delta(f)(n) = \sum_{m \in Z} \frac{f(m)}{n - m + 1/2}.
\]
when \( f \in C_0(\mathbb{Z}) \), where \( C_0(\mathbb{Z}) \) consists of all those complex sequences \( f \) such that \( f(n) = 0 \) when \( |n| \geq m \) for certain \( m \in \mathbb{N} \). It is well-known that if \( 1 < p < \infty \), \( H_d \) can be extended to \( \ell^p(\mathbb{Z}, \omega) \) as a bounded operator from \( \ell^p(\mathbb{Z}, \omega) \) into \( \ell^p(\mathbb{Z}, \omega) \) if and only if \( \omega \in A_p(\mathbb{Z}) \), and \( H_d \) can be extended to \( \ell^1(\mathbb{Z}, \omega) \) as a bounded operator from \( \ell^1(\mathbb{Z}, \omega) \) into \( \ell^1(\mathbb{Z}, \omega) \) if and only if \( \omega \in A_1(\mathbb{Z}) \) \([12]\). In \([8] \) \( \S 6 \) it is proved that \( H_d \) can be seen as a Riesz transform in the sense of Stein \([20]\) associated to the discrete Laplacian.

In this paper we prove that the maximal operator \( W_* \) and the Littlewood-Paley function \( g \) are bounded from the Hardy space \( H^p(\mathbb{Z}) \) into \( \ell^p(\mathbb{Z}) \), and the Laplace transform type multiplier \( T_m \) is bounded from \( H^p(\mathbb{Z}) \) into itself, for every \( 0 < p \leq 1 \).

As it was mentioned Hardy spaces \( H^p(\mathbb{Z}) \) were considered in \([8] \) p. 622 as an special case of the general theory developed in \([8]\). Boza and Carro \([2]\) and \([3]\) characterized Hardy spaces \( H^p(\mathbb{Z}) \) in different ways.

We recall some definitions and properties of the discrete Hardy spaces. Let \( \alpha > 0 \). The sequence space \( L_\alpha \) consists of all those complex sequences \( a \) such that

\[
\|a\|_{L_\alpha} := \sup_{n,m \in \mathbb{Z}, n \neq m} \left| a(n) - a(m) \right| / |n - m|^\alpha < \infty.
\]

\( L_\alpha \) is endowed with the quasinorm \( \| \cdot \|_{L_\alpha} \).

Let \( p, q \in (0, \infty) \) such that \( 0 < p < q \) and \( p \leq 1 \leq q \leq \infty \). A complex sequence \( b \) is said to be a \((p, q)\)-atom when there exist \( n_0 \in \mathbb{Z} \) and \( r_0 \geq 1 \) satisfying that

(i) The support of \( b \) is contained in the ball \( B_{B_\mathbb{Z}}(n_0, r_0) \);
(ii) \( \|b\|_q \leq \mu (B_{B_\mathbb{Z}}(n_0, r_0))^{1/q - 1/p} \), where \( 1/q \) is understood to be 0 when \( q = \infty \);
(iii) \( \sum_{n \in \mathbb{Z}} b(n) = 0 \).

Suppose that \( 0 < p < 1 \). If \( b \) is a \((p, q)\)-atom, then \( b \) defines an element \( L_b \) of the dual space \( (L_{1/p-1})' \) of \( L_{1/p-1} \) as follows

\[
\langle L_b, a \rangle = \sum_{n \in \mathbb{Z}} a(n)b(n), \quad a \in L_{1/p-1}.
\]

Let \( q \geq 1 \). The space \( H^{p,q}(\mathbb{Z}) \) is the subspace of the dual space \( (L_{1/p-1})' \) of \( L_{1/p-1} \) consisting of those linear functionals \( h \) defined in \( L_{1/p-1} \) such that there exist, for every \( j \in \mathbb{N} \), a \((p, q)\)-atom \( b_j \) and \( \lambda_j > 0 \) satisfying that \( \sum_{j \in \mathbb{N}} |\lambda_j|^q < \infty \) and \( h = \sum_{j \in \mathbb{N}} \lambda_j b_j \), where the series converges in \( (L_{1/p-1})' \). For every \( h \in H^{p,q}(\mathbb{Z}) \) we define \( \|h\|_{H^{p,q}(\mathbb{Z})} \) as follows

\[
\|h\|_{H^{p,q}(\mathbb{Z})} = \inf \left( \sum_{j \in \mathbb{N}} |\lambda_j|^p \right)^{1/p},
\]

where the infimum is taken over all those sequences \( \langle \lambda_j \rangle_{j \in \mathbb{N}} \) of nonnegative real numbers such that \( \sum_{j \in \mathbb{N}} |\lambda_j|^p < \infty \) and \( h = \sum_{j \in \mathbb{N}} \lambda_j b_j \) in \( (L_{1/p-1})' \), where \( b_j \) is a \((p, q)\)-atom, for every \( j \in \mathbb{N} \).

Suppose that \( h = \sum_{j \in \mathbb{N}} \lambda_j b_j \) in the sense of convergence in \( (L_{1/p-1})' \), where, for every \( j \in \mathbb{N} \), \( \lambda_j > 0 \) and \( b_j \) is a \((p, q)\)-atom, and \( \sum_{j \in \mathbb{N}} \lambda_j^q < \infty \). By using Hölder inequality we can see that \( \|b_j\|_p \leq 1, j \in \mathbb{N} \). Then, for every \( n, m \in \mathbb{N}, n < m \),

\[
\| \sum_{j=n}^{m} \lambda_j b_j \|_p \leq \sum_{j=n}^{m} |\lambda_j|^p \|b_j\|_p \leq \sum_{j=n}^{m} \lambda_j^p.
\]

Hence, the series \( \sum_{j \in \mathbb{N}} \lambda_j b_j \) converges in \( \ell^p(\mathbb{Z}) \). We write \( H = \sum_{j \in \mathbb{N}} \lambda_j b_j \) in \( \ell^p(\mathbb{Z}) \). We have that

\[
H(n) = \sum_{j \in \mathbb{N}} \lambda_j b_j(n), \quad n \in \mathbb{Z}.
\]

We consider, for every \( n \in \mathbb{Z} \), \( b^n = (b^n(m))_{m \in \mathbb{Z}} \), where \( b^n(n) = 1 \) and \( b^n(m) = 0, m \in \mathbb{Z}, m \neq n \). It is clear that \( b^n \in L_{1/p-1}, n \in \mathbb{Z} \). Then,

\[
|h(n)| = \lim_{m \to \infty} \sum_{j=0}^{m} \lambda_j \sum_{n \in \mathbb{Z}} b_j(m)b^n(m) = \sum_{j \in \mathbb{N}} \lambda_j b_j(n), \quad n \in \mathbb{Z}.
\]

It follows that \( H = h \). Hence, \( H^{p,q}(\mathbb{Z}) \) is contained in \( \ell^p(\mathbb{Z}) \). Moreover, this inclusion is continuous.
Let \( q > 1 \). The space \( H^{1,q}(Z) \) consists of all \( h \in \ell^1(Z) \) such that \( h = \sum_{j \in N} \lambda_j b_j \) in \( \ell^1(Z) \), where, for every \( j \in N \), \( \lambda_j > 0 \) and \( b_j \) is a \((1,q)\)-atom, and \( \sum_{j \in N} \lambda_j < \infty \). For every \( h \in H^{1,q}(Z) \),
\[
\|h\|_{H^{1,q}(Z)} = \inf \sum_{j \in N} \lambda_j
\]
where the infimum is taken over all those sequences \( (\lambda_j)_{j \in N} \) of nonnegative real numbers such that \( \sum_{j \in N} \lambda_j < \infty \) and \( h = \sum_{j \in N} \lambda_j b_j \) in \( \ell^1(Z) \), being \( b_j \) a \((1,q)\)-atom, for every \( j \in N \).

According to [8] Theorem A \( H^{p,q}(Z) = H^{p,\infty}(Z) \) algebraically and topologically, provided that \( 0 < p < q \) and \( p \leq 1 \leq q \leq \infty \).

By proceeding as in the continuous case, that is, in the \( H^p(\mathbb{R}) \) case (see [11] Theorem 7.7 and the following ones) and [8, p. 598]), from [6] Propositions 3 and 4 we can prove the following result.

**Theorem 1.1.** Let \( 1/2 < p \leq 1 \). Then, the operators \( W_\ast \) and \( g \) are bounded from \( H^{p,\infty}(Z) \) into \( \ell^p(Z) \).

In order to extend this property to values of \( p \) in \((0, 1/2]\) we consider the discrete Hardy spaces studied in [2] and [4].

Let \( 0 < p < q \) and \( p \leq 1 \leq q \leq \infty \). We say that a complex sequence \( b \) is a \((H,p,q)\)-atom when there exist \( n_0 \in Z \) and \( r_0 \geq 1 \) such that,

(i) The support of \( b \) is contained in the ball \( B_Z(n_0,r_0) \);

(ii) \( \|b\|_q \leq \mu(B_Z(n_0,r_0))^{1/q-1/p} \), where \( 1/q \) is understood to be 0 when \( q = \infty \);

(iii) \( \sum_{n \in Z} n^\alpha b(n) = 0 \), for every \( \alpha \in N \) such that \( \alpha \leq 1/p - 1 \).

We define the Hardy space \( H^{p,q}(Z) \) in the same way that \( H^{p,q}(Z) \) where \((H,p,q)\)-atoms replace \((p,q)\)-atoms. We have that \( H^{p,q}(Z) = H^{p,\infty}(Z) \) algebraically and topologically.

Hardy spaces \( H^{p,q}(Z) \) can be characterized by using discrete Hilbert transform. In [2] Theorems 3.10 and 3.14 it was established that a complex sequence \( h \) is in \( H^{p,\infty}(Z) \) if and only if \( h \in \ell^p(Z) \) and \( H_d(h) \in \ell^p(Z) \). Moreover, for every \( h \in H^{p,\infty}(Z) \), the quantities \( \|h\|_{H^{p,\infty}(Z)} \) and \( \|\lambda\|_p + \|H_d(h)\|_p \) are equivalent. In [2] and [4] the space \( H^{p,\infty}(Z) \) is also characterized by using maximal operators and square functions.

For every \( 0 < p \leq 1 \) we write \( \hat{H}^p(Z) \) for naming \( H^{p,\infty}(Z) \).

The main results of this paper are the following ones.

**Theorem 1.2.** Let \( 0 < p \leq 1 \). The operators \( W_\ast \) and \( g \) are bounded from \( H^p(Z) \) into \( \ell^p(Z) \).

**Theorem 1.3.** Assume that \( m \in L^\infty(0, \infty) \) and that \( m(\lambda) = \lambda \int_0^\infty e^{-\lambda \Psi(t)} dt, \lambda \in (0, \infty) \), where \( \Psi \in L^\infty(0, \infty) \). Then, the operator \( T_m \) is bounded from \( \ell^p(Z, \omega) \) into itself, for every \( 1 < p < \infty \) and \( \omega \in A_p(Z) \), from \( \ell^1(Z, \omega) \) into \( \ell^1,\infty(Z, \omega) \), when \( \omega \in A_1(Z) \), and from \( H^p(Z) \) into itself for every \( 0 < p \leq 1 \).

The multipliers of Laplace transform type \( T_m \) can be seen as special cases of the Fourier multipliers of Marcinkiewicz type considered in [13]. Then, we can deduce that \( T_m \) defines a bounded operator from \( H^p(Z) \) into itself from [13] Theorem 3. In the proof of [13] Theorem 3 Plancherel theorem for Fourier transform plays a key role. Our proof (see section 5) of \( H^p(Z) \)-boundedness of \( T_m \) is different from the one presented in [13] Theorem 3. We apply Proposition 2.8 and we do not use Plancherel theorem. Our procedure is more flexible because for instance it could be used in a weighted setting (see, for instance, [16] Theorem 4)). The study of discrete Hardy spaces with weights will be addressed in a future work.

The proofs of [6] Propositions 3 and 4) rely on the following integral representation of the modified Bessel function of first kind that is known as Schl"afli’s integral representation of Poisson type for \( I_\nu \) ([13] (5.10.22)],

\[
I_\nu(z) = \frac{z^\nu}{\sqrt{\pi 2^{\nu} \Gamma(\nu + 1/2)}} \int_{-1}^{1} e^{-zs} (1 - s^2)^{\nu - 1/2} ds, \quad \text{arg } z < \pi, \text{ and } \nu > -1/2.
\]

In order to prove Theorems 1.2 and 1.3 we will use, instead of (1.2), the following integral representation ([18] p. 456],

\[
I_\nu(t) = \frac{1}{\pi} \int_0^{\pi} e^{t \cos \theta} \cos(n \theta) d\theta, \quad t > 0, \text{ and } n \in \mathbb{Z}.
\]
After writing the operators \( W_\ast, g \) and \( T_m \) as Banach valued singular integrals of convolution type in the homogeneous group \((\mathbb{Z}, \cdot, \mu)\), we have to estimate derivatives of the respective kernels of those singular integrals. The integral representation \( (\ref{intrepr}) \) plays a key role to obtain the mentioned estimates.

This paper is organized as follows. After this introduction in Section 2 we establish some auxiliary results that will be useful in the sequel. Theorem 1.2 is proved in Section 3 for \( W_\ast \) and in Section 4 for \( g \). A proof for Theorem 1.3 is written in Section 5.

Throughout this paper by \( C \) and \( c \) we always represent positive constants that can change from one line to the other one, and by \( E[\alpha] \) we mean the highest integer number less or equal to \( \alpha \in \mathbb{R} \).

2. Auxiliary results

In this section we establish some properties that will be useful in the sequel.

The semigroup of operators \( \{W_t\}_{t > 0} \) generated by \(-\Delta_d \) in \( \ell^p(\mathbb{Z}) \) is defined by

\[
W_t(f)(n) = \sum_{m \in \mathbb{Z}} G(n - m, t) f(m), \quad t > 0,
\]

for every \( f \in \ell^p(\mathbb{Z}), 1 \leq p \leq \infty \), where \( G(m, t) = e^{-2tI_m}(2t), t > 0 \) and \( m \in \mathbb{Z} \).

According to \( (\ref{intrepr}) \) we have that

\[
G(m, t) = \frac{1}{\pi} \int_0^\pi \phi_t(\theta) \cos(m\theta) d\theta, \quad t > 0 \quad \text{and} \quad m \in \mathbb{Z},
\]

where \( \phi_t(\theta) = e^{-2t(1 - \cos \theta)}, t > 0 \) and \( \theta \in (0, \pi) \).

From now on, for every \( k \in \mathbb{N} \), we consider

\[
h_k(\theta) = \begin{cases} 
\sin \theta, & \text{if } k \text{ is odd}, \\
\cos \theta, & \text{if } k \text{ is even}, 
\end{cases} \quad \theta \in (0, \pi).
\]

Firstly we obtain a representation for the derivative of \( \phi_t, \ t > 0 \).

Lemma 2.1. Let \( k \in \mathbb{N}, \ h \geq 1 \). We have that, for every \( t > 0 \) and \( \theta \in (0, \pi) \),

\[
\partial_h^k \phi_t(\theta) = \phi_t(\theta) \sum_{(m_1, \ldots, m_k) \in \mathbb{N}^k} \alpha_{m_1, \ldots, m_k} \frac{m_1! \cdots m_k!}{m_1 + 2m_2 + \cdots + km_k = k} \frac{2^{m_1 + \cdots + m_k} k!}{m_1! \cdots m_k!} \sum_{j=1}^{E(k/2)} \sum_{j=1}^{E((k+1)/2)} \prod_{r=1}^k \psi(m_1 + \cdots + m_k) (\varphi(\theta)) \varphi^{(r)}(\theta)^{m_r}, \quad \theta \in (0, \pi).
\]

Proof. Let \( t \in (0, \infty) \). We define \( \Psi(z) = e^{2tz}, z \in \mathbb{R} \), and \( \varphi(\theta) = \cos \theta - 1, \ \theta \in (0, \pi) \). It is clear that \( \phi_t(\theta) = (\Psi \circ \varphi)(\theta), \ \theta \in (0, \pi) \).

By using Faà di Bruno formula \( (\ref{faadi}) \) we can write

\[
\partial_h^k \phi_t(\theta) = (\Psi \circ \varphi)^{(k)}(\theta)
\]

Since for each \( r \in \mathbb{N} \), \( \psi^{(r)}(z) = (2t)^r \Psi(z) \), and when \( r \geq 1 \),

\[
\varphi^{(r)}(\theta) = \cos(\theta + r \pi / 2) = (-1)^{E((r+1)/2)} h_r(\theta) = \begin{cases} 
(-1)^{(r+1)/2} \sin \theta, & \text{if } r \text{ is odd}, \\
(-1)^{r/2} \cos \theta, & \text{if } r \text{ is even},
\end{cases} \quad \theta \in (0, \pi),
\]

we can finish the proof of the lemma. \( \square \)

An immediate consequence of Lemma 2.1 is the following.
Lemma 2.2. Let $k \in \mathbb{N}$ be odd. Then $\partial^k_\theta \phi_l(0) = \partial^k_\theta \phi_l(\pi) = 0$, $t \in (0, \infty)$.

Next property follows by using partial integration and Lemma 2.2.

Lemma 2.3. Let $k \in \mathbb{N}$. We have that
\[
G(m, t) = \frac{(-1)^{[(k+1)/2]}}{\pi m^k} \int_0^\pi \partial^k_\theta \phi_l(\theta) h_k(m \theta) d\theta, \quad m \in \mathbb{Z} \setminus \{0\}, \ t > 0.
\]

In the next three propositions we collect some fundamental boundedness properties that will be useful for our proofs.

Proposition 2.4. Let $k \in \mathbb{N}$, $k \geq 1$. Then,
\[
\sup_{t \in (0, \infty)} \int_0^\pi |\partial^k_\theta \phi_l(\theta)| \theta^{k-1} d\theta < \infty.
\]

Proof. According to Lemma 2.1
\[
\partial^k_\theta \phi_l(\theta) = \sum_{(m_1, \ldots, m_k) \in \mathbb{N}^k} a_{m_1, \ldots, m_k} A_{m_1, \ldots, m_k}(t, \theta), \ t > 0, \ \theta \in (0, \pi),
\]
where, for each $(m_1, \ldots, m_k) \in \mathbb{N}^k$ such that $m_1 + 2m_2 + \ldots + km_k = k$, $a_{m_1, \ldots, m_k} \in \mathbb{R}$ and
\[
A_{m_1, \ldots, m_k}(t, \theta) = e^{-2(1 - \cos \theta) m_1 + \ldots + m_k (\cos \theta)^{\alpha_{m_1, \ldots, m_k}} (\sin \theta)^{\beta_{m_1, \ldots, m_k}}}, \ t > 0, \ \theta \in (0, \pi),
\]
being
\[
\alpha_{m_1, \ldots, m_k} = \sum_{j=1}^{E[k/2]} m_{2j} \quad \text{and} \quad \beta_{m_1, \ldots, m_k} = \sum_{j=1}^{E[(k+1)/2]} m_{2j-1}.
\]

In order to prove that
\[
\sup_{t \in (0, \infty)} \int_0^\pi |\partial^k_\theta \phi_l(\theta)| \theta^{k-1} d\theta < \infty,
\]
it is sufficient to see that, for every $(m_1, \ldots, m_k) \in \mathbb{N}^k$ such that $m_1 + 2m_2 + \ldots + km_k = k$, there exist $C_1, C > 0$ such that
\[
|A_{m_1, \ldots, m_k}(t, \theta)| \leq C t \theta^e e^{-ct \theta^2}, \ t > 0, \ \theta \in (0, \pi).
\]

Let $(m_1, \ldots, m_k) \in \mathbb{N}^k$ such that $m_1 + 2m_2 + \ldots + km_k = k$. Since $|\sin \theta| \leq |\theta|$, $\theta \in \mathbb{R}$, and $1 - \cos \theta \geq c \theta^2$, $\theta \in (0, \pi)$, we can write
\[
|A_{m_1, \ldots, m_k}(t, \theta)| \leq C e^{-ct \theta^2} \sum_{j=1}^{k} m_j \theta^{k-1+\beta_{m_1, \ldots, m_k}}
\]
\[
\leq C t \theta^e e^{-ct \theta^2} \sum_{j=1}^{k} m_j \leq C t \theta^e e^{-ct \theta^2}, \ t > 0, \ \theta \in (0, \pi).
\]

In the last inequality we have taken into account that
\[
(2.3)
\sigma_{m_1, \ldots, m_k} := k + \beta_{m_1, \ldots, m_k} - 2 \sum_{j=1}^{k} m_j \geq 0.
\]

Indeed, we can write
\[
k = \sum_{j=1}^{E[k/2]} 2jm_{2j} + \sum_{j=1}^{E[(k+1)/2]} (2j-1)m_{2j-1} \geq 2(\alpha_{m_1, \ldots, m_k} + \beta_{m_1, \ldots, m_k}) - \beta_{m_1, \ldots, m_k}
\]
\[
= 2 \sum_{j=1}^{k} m_j - \beta_{m_1, \ldots, m_k}.
\]

Moreover, $\sigma_{m_1, \ldots, m_k} = 0$ if, and only if $m_3 = m_4 = \ldots = m_k = 0$, that is, $k = m_1 + 2m_2$. \hfill \Box

From Proposition 2.4 it follows immediately that, for every $k \in \mathbb{N}$, $k \geq 1$, there exists $C > 0$ such that
\[
(2.4)
\sup_{t \in (0, \infty)} \left| \int_0^\pi \partial^k_\theta \phi_l(\theta) \theta^{k-1} \sin(z\theta) d\theta \right| \leq C, \quad z \in \mathbb{R}.
\]
Proposition 2.5. Let \( k \in \mathbb{N}, k \geq 1 \). Then,
\[
\sup_{\theta \in [0, \pi]} \left\| \frac{d^{k}}{d \theta^{k}} A_{m_{1}, \ldots, m_{k}}(t) \right\|_{L^{2}((0, \infty), dt/t)} < \infty.
\]

Proof. From \((2.1)\) we have, for every \( t > 0 \) and \( \theta \in (0, \pi) \),
\[
(2.5) \quad t \frac{d^{k}}{d \theta^{k}} A_{m_{1}, \ldots, m_{k}}(\theta) = \sum_{(m_{1}, \ldots, m_{k}) \in \mathbb{N}^{k}} \sum_{m_{1}+2m_{2}+\ldots+km_{k}=k} a_{m_{1}, \ldots, m_{k}} \left( -2t(1-\cos \theta) + \sum_{j=1}^{k} m_{j} \right) A_{m_{1}, \ldots, m_{k}}(t, \theta).
\]
Thus, in order to establish our result it is sufficient to see that, for every \((m_{1}, \ldots, m_{k}) \in \mathbb{N}^{k}\) such that \( m_{1}+2m_{2}+\ldots+km_{k}=k \), we can find \( C > 0 \), such that, for each \( z \geq 1 \),
\[
I_{m_{1}, \ldots, m_{k}}(z) := \left\| \int_{0}^{\pi} A_{m_{1}, \ldots, m_{k}}(t, \theta) \theta^{k-1} \sin(z \theta) d\theta \right\|_{L^{2}((0, \infty), dt/t)} \leq C.
\]
and
\[
J_{m_{1}, \ldots, m_{k}}(z) := \left\| \int_{0}^{\pi} t(1-\cos \theta) A_{m_{1}, \ldots, m_{k}}(t, \theta) \theta^{k-1} \sin(z \theta) d\theta \right\|_{L^{2}((0, \infty), dt/t)} \leq C.
\]
Let \((m_{1}, \ldots, m_{k}) \in \mathbb{N}^{k}\), with \( m_{1}+2m_{2}+\ldots+km_{k}=k \) and consider \( \sigma_{m_{1}, \ldots, m_{k}} \) as in \((2.3)\).
In the case that \( \sigma_{m_{1}, \ldots, m_{k}} > 0 \), by using \((2.2)\) and Minkowski integral inequality we obtain that
\[
I_{m_{1}, \ldots, m_{k}}(z) + J_{m_{1}, \ldots, m_{k}}(z) \leq C \int_{0}^{\pi} \int_{0}^{\pi} t \left| e^{-ct \theta^{2}} \right| \sigma_{m_{1}, \ldots, m_{k}}(1 + \theta^{2}) d\theta d\theta
\]
\[
\leq C \int_{0}^{\pi} \int_{0}^{\pi} \left( \int_{0}^{\infty} t e^{-ct \theta^{2}} d\theta \right)^{1/2} d\theta d\theta
\]
\[
\leq C \int_{0}^{\pi} \int_{0}^{\pi} \left( \sigma_{m_{1}, \ldots, m_{k}} \right)^{-1} d\theta \leq C, \quad z \in \mathbb{R}.
\]
Assume now that \( \sigma_{m_{1}, \ldots, m_{k}} = 0 \). Then \( m_{3} = m_{4} = \ldots = m_{k} = 0 \), \( k = m_{1}+2m_{2} \) and
\[
(2.6) \quad A_{m_{1}, \ldots, m_{k}}(t, \theta) = \frac{e^{-2t(1-\cos \theta)} m_{1}(\cos \theta)^{m_{2}}(\sin \theta)^{m_{1}}}{t^{m_{1}+m_{2}}}, \quad t > 0, \theta \in (0, \pi).
\]
Note that, since \( k \geq 1, m_{1}+m_{2} \geq 1 \).
For every \( z \in \mathbb{R} \) we can write
\[
(I_{m_{1}, \ldots, m_{k}}(z))^{2} = \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{\pi} e^{-2t(1-\cos \theta_{1}-\cos \theta_{2})} \frac{t^{2}(m_{1}+m_{2})-1}{2} \left( \cos \theta_{1} \cos \theta_{2} \right)^{m_{2}}(\sin \theta_{1} \sin \theta_{2})^{m_{1}}
\]
\[
\times (\theta_{1} \theta_{2})^{k-1} \sin(z \theta_{1}) \sin(z \theta_{2}) d\theta_{1} d\theta_{2} dt
\]
\[
= \frac{\Gamma(2(m_{1}+m_{2}))}{2^{2(m_{1}+m_{2})}} \int_{0}^{\infty} \int_{0}^{\pi} \frac{(\cos \theta_{1} \cos \theta_{2})^{m_{2}}(\sin \theta_{1} \sin \theta_{2})^{m_{1}}}{(\cos \theta_{1} \cos \theta_{2})^{2}(m_{1}+m_{2})} \frac{(\theta_{1} \theta_{2})^{k-1} \sin(z \theta_{1}) \sin(z \theta_{2})}{(\theta_{1} \theta_{2})^{2}} d\theta_{1} d\theta_{2}.
\]
Let \( F \) and \( G \) be the functions given by
\[
F(\theta_{1}, \theta_{2}) = \frac{(\cos \theta_{1} \cos \theta_{2})^{m_{2}}(\sin \theta_{1} \sin \theta_{2})^{m_{1}}}{(2(2-\cos \theta_{1} \cos \theta_{2}))^{2(m_{1}+m_{2})}}, \quad \theta_{1}, \theta_{2} \in (0, \pi),
\]
and
\[
H(\theta_{1}, \theta_{2}) = \frac{(\theta_{1} \theta_{2})^{m_{1}}}{(\theta_{1}^{2} + \theta_{2}^{2})^{2}(m_{1}+m_{2})}, \quad \theta_{1}, \theta_{2} \in (0, \pi).
\]
We have that
\[
|F(\theta_{1}, \theta_{2}) - H(\theta_{1}, \theta_{2})| \leq \frac{(\cos \theta_{1} \cos \theta_{2})^{m_{2}}(\sin \theta_{1})^{m_{1}}}{(2-\cos \theta_{1} \cos \theta_{2})^{2}(m_{1}+m_{2})} \left| (\sin \theta_{1})^{m_{1}} - \theta_{1}^{m_{1}} \right|
\]
\[
+ \frac{(\cos \theta_{1} \cos \theta_{2})^{m_{2}}(\sin \theta_{1})^{m_{1}}}{(2-\cos \theta_{1} \cos \theta_{2})^{2}(m_{1}+m_{2})} \left| (\sin \theta_{1})^{m_{1}} - \theta_{1}^{m_{1}} \right|
\]
\[
+ \frac{(\cos \theta_{1} \cos \theta_{2})^{m_{2}}(\sin \theta_{1})^{m_{1}}}{(2-\cos \theta_{1} \cos \theta_{2})^{2}(m_{1}+m_{2})} \left| (\sin \theta_{1})^{m_{1}} - \theta_{1}^{m_{1}} \right|
\]
\[
+ \frac{(\cos \theta_{1} \cos \theta_{2})^{m_{2}}(\sin \theta_{1})^{m_{1}}}{(2-\cos \theta_{1} \cos \theta_{2})^{2}(m_{1}+m_{2})} \left| (\sin \theta_{1})^{m_{1}} - \theta_{1}^{m_{1}} \right|
\]
\[
+ \frac{(\cos \theta_{1} \cos \theta_{2})^{m_{2}}(\sin \theta_{1})^{m_{1}}}{(2-\cos \theta_{1} \cos \theta_{2})^{2}(m_{1}+m_{2})} \left| (\sin \theta_{1})^{m_{1}} - \theta_{1}^{m_{1}} \right|
\]
\[
+ \frac{(\cos \theta_{1} \cos \theta_{2})^{m_{2}}(\sin \theta_{1})^{m_{1}}}{(2-\cos \theta_{1} \cos \theta_{2})^{2}(m_{1}+m_{2})} \left| (\sin \theta_{1})^{m_{1}} - \theta_{1}^{m_{1}} \right|
\]
\[
\begin{align*}
\text{Then, we get} & \quad \frac{(\theta_1 \theta_2)^m (\cos \theta_2)^{m_2}}{\left(\theta_1^2 + \theta_2^2\right)^{2(m_1+m_2)}} - \frac{(\theta_1 \theta_2)^{m_1}}{\left(\theta_1^2 + \theta_2^2\right)^{2(m_1+m_2)}} \\
& := \sum_{j=1}^{6} R_j(\theta_1, \theta_2), \quad \theta_1, \theta_2 \in (0, \pi).
\end{align*}
\]

We observe that if \(m_1 = 0\), then \(R_1(\theta_1, \theta_2) = R_2(\theta_1, \theta_2) = 0\) and in the case that \(m_2 = 0\), we have that \(R_3(\theta_1, \theta_2) = R_4(\theta_1, \theta_2) = 0\). By taking into account that \(|\sin \theta - \theta| \leq C\theta^3\) and \(|1 - \cos \theta - \theta^2/2| \leq c\theta^4\), \(\theta \in (0, \pi)\), and applying the mean value theorem we get, for each \(\theta \in (0, \pi)\),

\[
\begin{align*}
|\sin \theta|^{m_1} - \theta^{m_1} & \leq C\theta^{m_1+2}, \\
|1 - \cos \theta|^{m_2} & \leq C\theta^2,
\end{align*}
\]

and, if \(\alpha \in \mathbb{R}\),

\[
\left|\frac{1}{(2(1 - \cos \theta) + \alpha)^{2(m_1+m_2)}} - \frac{\theta^4}{(\theta^2 + \alpha)^{2(m_1+m_2)+1}} \right| \leq C \frac{\theta^2}{(\theta^2 + \alpha)^{2(m_1+m_2)}}.
\]

By considering these estimates and that \(0 \leq \sin \theta \leq \theta \) and \(|1 - \cos \theta| \geq C\theta^2\), \(\theta \in (0, \pi)\), we obtain that

\[
\begin{align*}
|F(\theta_1, \theta_2) - H(\theta_1, \theta_2)| & \leq C \left(\frac{(\theta_1 \theta_2)^m (\theta_1^2 + \theta_2^2)}{\left(\theta_1^2 + \theta_2^2\right)^{2(m_1+m_2)}} \right)^{k-1} \sin(z \theta_1) \sin(z \theta_2) d\theta_1 d\theta_2 \\
& \leq C \left(1 + \int_0^\pi \int_0^\pi \frac{(\theta_1 \theta_2)^{m_1+k-1}}{\left(\theta_1^2 + \theta_2^2\right)^{2(m_1+m_2)+1}} \sin(z \theta_1) \sin(z \theta_2) d\theta_1 d\theta_2 \right), \quad z \in \mathbb{R}.
\end{align*}
\]

Then, we get

\[
\begin{align*}
\left(\int_{m_1, \ldots, m_k} (z) \right)^2 & \leq C \left( \int_0^\pi \int_0^\pi |F(\theta_1, \theta_2) - H(\theta_1, \theta_2)| (\theta_1 \theta_2)^{k-1} d\theta_1 d\theta_2 \\
& + \quad \int_0^\pi \int_0^\pi H(\theta_1, \theta_2)(\theta_1 \theta_2)^{k-1} \sin(z \theta_1) \sin(z \theta_2) d\theta_1 d\theta_2 \right) \\
& \leq C \left( 1 + \int_0^\pi \int_0^\pi \frac{(\theta_1 \theta_2)^{m_1+k-1}}{\left(\theta_1^2 + \theta_2^2\right)^{2(m_1+m_2)+1}} \sin(z \theta_1) \sin(z \theta_2) d\theta_1 d\theta_2 \right), \quad z \in \mathbb{R}.
\end{align*}
\]

To analyze the last integral we proceed as follows:

\[
\begin{align*}
\left|\int_0^\pi \int_0^\pi \frac{(\theta_1 \theta_2)^{m_1+k-1}}{\left(\theta_1^2 + \theta_2^2\right)^{2(m_1+m_2)+1}} \sin(z \theta_1) \sin(z \theta_2) d\theta_1 d\theta_2 \right| \\
& = \left|\int_0^\pi \int_0^\pi \frac{(v_1 v_2)^{2(m_1+m_2)+1}}{(v_1^2 + v_2^2)^{2(m_1+m_2)+1}} \sin v_1 \sin v_2 d v_1 d v_2 \right| \\
& \leq C \left( \int_0^1 \int_0^1 d v_1 d v_2 + \int_0^\pi \int_0^\pi \frac{(v_1 v_2)^{2(m_1+m_2)+1}}{(v_1^2 + v_2^2)^{2(m_1+m_2)+1}} \sin v_1 \sin v_2 d v_1 d v_2 \right), \quad z \in \mathbb{R}.
\end{align*}
\]

By using partial integration we get

\[
I(z) := \int_0^\pi \int_1^\pi \frac{(v_1 v_2)^{2(m_1+m_2)+1}}{(v_1^2 + v_2^2)^{2(m_1+m_2)+1}} \sin v_1 \sin v_2 d v_1 d v_2
\]

\[
= \int_0^\pi \left[ \cos \frac{1}{(1 + v_2^2)^{2(m_1+m_2)+1}} - \cos \frac{1}{(v_2^2 + 1)^{2(m_1+m_2)+1}} \right] \frac{2(m_1+m_2)+1}{v_2} \sin v_2 d v_2 \\
+ \int_0^\pi \int_1^\pi \frac{v_1^{2(m_1+m_2)+1}}{(v_2^2 + v_1^2)^{2(m_1+m_2)+1}} d v_1 d v_2,
\]

Then, it follows that

\[
\begin{align*}
|I(z)| & \leq C \left( \int_0^1 \frac{1}{(1 + v_2^2)^{2(m_1+m_2)+1}} + \frac{1}{(z^2 + v_2^2)^{2(m_1+m_2)+1}} d v_2 + \int_0^\infty \int_1^\infty \frac{2(m_1+m_2)+1}{(v_2^2 + v_1^2)^{2(m_1+m_2)+1}} d v_1 d v_2 \right) \\
& \leq C \left( 1 + \frac{1}{z} \right) \leq C, \quad z \geq 1.
\end{align*}
\]

By combining the above estimates we conclude that \(I_{m_1, \ldots, m_k}(z) \leq C\), when \(z \geq 1\).
To deal with $J_{m_1,...,m_k}(z)$, $z \in \mathbb{R}$, we proceed as in the $I_{m_1,...,m_k}$-case, by taking into account that
\[
(J_{m_1,...,m_k}(z))^2 = \frac{\Gamma(2(m_1 + m_2) + 2)}{2^{2(m_1+m_2)+2}} \int_0^\pi \int_0^\pi \frac{(1 - \cos \theta_1)(1 - \cos \theta_2)}{(2 - \cos \theta_1 - \cos \theta_2)^2} \\
\times (\cos \theta_1 \cos \theta_2)^{m_2} (\sin \theta_1 \sin \theta_2)^{m_1} (\theta_1 \theta_2)^{k-1} \sin(z \theta_1) \sin(z \theta_2) \, d\theta_1 d\theta_2, \quad z \in \mathbb{R},
\]
and considering the functions $\tilde{F}$ and $H$ given by
\[
\tilde{F}(\theta_1, \theta_2) = \frac{(1 - \cos \theta_1)(1 - \cos \theta_2)}{(2 - \cos \theta_1 - \cos \theta_2)^2} F(\theta_1, \theta_2), \quad \theta_1, \theta_2 \in (0, \pi),
\]
and
\[
\tilde{H}(\theta_1, \theta_2) = \frac{(\theta_1 \theta_2)^2}{(\theta_1^2 + \theta_2^2)^2} H(\theta_1, \theta_2), \quad \theta_1, \theta_2 \in (0, \pi).
\]
in the role of $F$ and $H$, respectively.

**Proposition 2.6.** Let $k \in \mathbb{N}$, $k \geq 1$ and $\Psi \in L^\infty(0, \infty)$. Then,
\[
\sup_{z \geq 1} \left| \int_0^\infty \Psi(t) \int_0^\pi \partial_t \partial_\theta^5 \Phi(\theta) \theta^{k-1} \sin(z \theta) d\theta dt \right| < \infty.
\]

**Proof.** According to (2.3) it is sufficient to show that, for every $(m_1,...,m_k) \in \mathbb{N}^k$ such that $m_1 + 2m_2 + ... + km_k = k$, there exists $C > 0$, such that, for each $z \geq 1$,
\[
R_{m_1,...,m_k}(z) := \left| \int_0^\infty \Psi(t) \int_0^\pi A_{m_1,...,m_k}(t, \theta) \theta^{k-1} \sin(z \theta) d\theta dt \right| \leq C.
\]
and
\[
S_{m_1,...,m_k}(z) := \left| \int_0^\infty \Psi(t) \int_0^\pi (1 - \cos \theta) A_{m_1,...,m_k}(t, \theta) \theta^{k-1} \sin(z \theta) d\theta dt \right| \leq C.
\]
Let $(m_1,...,m_k) \in \mathbb{N}^k$, with $m_1 + 2m_2 + ... + km_k = k$ and let $\sigma_{m_1,...,m_k}$ be as in (2.3). When $\sigma_{m_1,...,m_k} > 0$, we can use the estimate (2.2) to get
\[
R_{m_1,...,m_k}(z) + S_{m_1,...,m_k}(z) \leq C \|\Psi\|_\infty \int_0^\pi \int_0^\infty \theta^{\sigma_{m_1,...,m_k} + 1} e^{-c \theta^2} (1 + t \theta^2) d\theta dt \\
\leq C \|\Psi\|_\infty \int_0^\pi \int_0^\infty \theta^{\sigma_{m_1,...,m_k} - 1} d\theta \leq C, \quad z \in \mathbb{R}.
\]

Suppose now that $\sigma_{m_1,...,m_k} = 0$ (recall that this means that $k = m_1 + 2m_2$) and consider the function
\[
H(t, \theta) := e^{-\theta^2} m_1 m_2 \theta^{m_1}, \quad t \in (0, \infty), \quad \theta \in (0, \pi).
\]
By taking into account (2.6), that $|\sin \theta - \theta| \leq C \theta^3$, $|2(1 - \cos \theta) - \theta^2| \leq C \theta^4$, and $c \theta^2 \leq 1 - \cos \theta \leq C \theta^2$, for $\theta \in (0, \pi)$, and by using the mean value theorem we obtain that
\[
|A_{m_1,...,m_k}(t, \theta) - H(t, \theta)| \leq t^{m_1 + m_2} \left( e^{-2(1 - \cos \theta)} (\cos \theta)^{m_2} [\sin(\theta)^{m_1} - \theta^{m_1}] \right) \\
+ |\theta^{m_1}(\cos \theta)^{m_2} \left( e^{-2(1 - \cos \theta)} - e^{-\theta^2} \right)\left( \theta^{m_1} - \theta^{m_1} \right)| \\
\leq C t^{m_1 + m_2} e^{-c \theta^2} \theta^{m_1}(\theta^2 + t \theta^4) \leq C t^{m_1 + m_2} e^{-c \theta^2} \theta^{m_1 + 2}, \quad t > 0, \quad \theta \in (0, \pi).
\]
We deduce that
\[
R_{m_1,...,m_k}(z) \\
\leq C \|\Psi\|_{L^\infty(0, \infty)} \left( \int_0^\infty \int_0^\pi t^{m_1 + m_2 - 1 - c \theta^2} \theta^{m_1 + k + 1} d\theta dt + \int_0^\infty \left| \int_0^\pi H(t, \theta) \theta^{k-1} \sin(z \theta) d\theta \right| \frac{dt}{t} \right) \\
\leq C \|\Psi\|_{L^\infty(0, \infty)} \left( \int_0^\pi \theta \theta d\theta + \int_0^\pi z^{2(m_1 + m_2)} \int_0^\pi t^{m_1 + m_2 - 1} \left| \int_0^\pi e^{-t \theta^2/z^2} e^{2(m_1 + m_2) - 1} \sin(v \theta) \frac{dv}{dt} \right| \theta d\theta \right) \\
\leq C \|\Psi\|_{L^\infty(0, \infty)} \left( 1 + z^{2(m_1 + m_2)} \int_0^\infty t^{m_1 + m_2 - 1} \int_0^1 e^{-t \theta^2/z^2} e^{2(m_1 + m_2) - 1} \sin(v \theta) \frac{dv}{dt} \right) \\
+ z^{2(m_1 + m_2)} \int_0^\infty t^{m_1 + m_2 - 1} \int_1^\pi e^{-t \theta^2/z^2} e^{2(m_1 + m_2) - 1} \sin(v \theta) \frac{dv}{dt} \right)
\[ \|\Psi\|_{L^\infty(0,\infty)} \leq C \int_0^{\infty} t^{m_1+m_2-1} \left| e^{-t^2/2} \sin \frac{1}{z} \right| dt \]

By making use of integration by parts we obtain that

\[ I(z) \leq z^{-2(m_1+m_2)} \left( \int_0^{\infty} t^{m_1+m_2-1} \left| e^{-t^2/2} \right| dt \right) \]

By combining the above estimates we conclude that \( R_{m_1,...,m_k}(z) \leq C, \ z \geq 1 \). The boundedness property for \( S_{m_1,...,m_k} \) can be established by proceeding in a similar way. \( \square \)

We will use the following generalizations of (2.4) and Propositions 2.3 and 2.6 in the proofs of our theorems.

**Proposition 2.7.** Let \( n \in \mathbb{N} \). For every \( k \in \mathbb{N}, \ k \geq \max\{n,1\} \), there exists \( C > 0 \) such that:

\[ \sup_{t \in (0,\infty)} \left| \int_0^{\pi} \partial_\theta^k \varphi_t(\theta) \theta^{k-n} h_n(\theta) d\theta \right| \leq C \ z^{n-1}, \quad z \in (0,\infty), \]

\[ \| \partial_\theta^k \varphi_t(\theta) \theta^{k-n} h_n(\theta) \|_{L^2((0,\infty),dt)} \leq C \ z^{n-1}, \quad z \geq 1, \]

and, for every \( \Psi \in L^\infty(0,\infty) \),

\[ \left| \int_0^{\infty} \Psi(t) \int_0^{\pi} \partial_\theta^k \varphi_t(\theta) \theta^{k-n} h_n(\theta) d\theta dt \right| \leq C \ z^{n-1}, \quad z \geq 1. \]

**Proof.** We proceed by induction. Note that the cases for \( n = 1 \) are the ones in (2.4) and Propositions 2.3 and 2.6. Let us show the property (2.7) for \( n = 0 \). Let \( k \in \mathbb{N}, \ k \geq 1 \). Partial integration leads to

\[ \int_0^{\pi} \partial_\theta^k \varphi_t(\theta) \theta^k \cos \pi z d\theta = \pi^k \partial_\theta^k \varphi_t(\pi z) - \frac{1}{z} \int_0^{\pi} \left[ \theta^k \partial_\theta^k \varphi_t(\theta) \right] \sin \pi z d\theta \]

Suppose now that \( n \geq 2 \) and, for every \( k \geq n - 1 \), (2.7) holds, when \( n \) is replaced by \( n - 1 \). Let \( k \geq n \). We observe that

\[ \int_0^{\pi} \partial_\theta^k \varphi_t(\theta) \theta^{k-n} d\theta = \sum_{i=0}^{k-n} a_{i,n} \theta^{k-n-i} \partial_\theta^i \varphi_t(\theta), \quad t > 0, \ \theta \in (0,\pi), \]

where \( a_{i,n} \in \mathbb{R}, \ i = 0, ..., k - n \), and by partial integration we get

\[ \int_0^{\pi} \partial_\theta^k \varphi_t(\theta) \theta^{k-n} h_n(\theta) d\theta = \sum_{i=0}^{k-n} a_{i,n} \left( \theta^{k-n-i} \partial_\theta^i \varphi_t(\theta) h_n(\theta) \right)_{\theta=0}^{\theta=\pi} \]

where \( a_{i,n} \in \mathbb{R}, \ i = 0, ..., k - n \), and by partial integration we get

\[ \int_0^{\pi} \partial_\theta^k \varphi_t(\theta) \theta^{k-n} h_n(\theta) d\theta = \sum_{i=0}^{k-n} a_{i,n} \left( \theta^{k-n-i} \partial_\theta^i \varphi_t(\theta) h_n(\theta) \right)_{\theta=0}^{\theta=\pi} \]

where \( a_{i,n} \in \mathbb{R}, \ i = 0, ..., k - n \).
For each $i = 0, \ldots, k - n - 1$, it is clear that $\theta^{k-n-i}\partial_{\theta}^{k-i-1}\phi_i(\theta)h_n(z\theta)_{\theta=0} = 0$, and, since $h_n(0) = 0$, when $n$ is odd and by Lemma \ref{lem:derivative_of_theta}, $\partial_{\theta}^{n-1}\phi_i(0) = 0$, when $n$ is even, $\partial_{\theta}^{n-1}\phi_i(0)h_n(0) = 0$. Hence

\[
\int_0^\pi \partial_{\theta}^k\phi_i(\theta)\theta^{k-n}h_n(z\theta)d\theta = \sum_{i=0}^{k-n} a_{i,n} (\theta^{k-n-i}\partial_{\theta}^{k-i-1}\phi_i(\theta)h_n(z\pi)
\]

(2.11)

$$+(-1)^n z\int_0^\pi \theta^{k-n-i}\partial_{\theta}^{k-i-1}\phi_i(\theta)h_{n-1}(z\theta)d\theta), \quad z, t \in (0, \infty).$$

As it was mentioned before, for every $\ell \in \mathbb{N}$ there exists a polynomial $p_{\ell}$ such that

$$\partial_{\theta}^\ell\phi_i(\pi) = p_{\ell}(t)e^{-4t}, \quad t > 0.$$

Then, for a certain polynomial $q_n$, we obtain

$$\int_0^\pi \partial_{\theta}^k\phi_i(\theta)\theta^{k-n}h_n(z\theta)d\theta = q_n(t)e^{-4t}h_n(z\pi)$$

and hence, by using the induction hypothesis we conclude that

$$\sup_{t \in (0, \infty)} \left| \int_0^\pi \partial_{\theta}^k\phi_i(\theta)\theta^{k-n}h_n(z\theta)d\theta \right| \leq C \left( \sup_{t \in (0, \infty)} |q_n(t)e^{-4t}| + z \sum_{i=0}^{k-n} \sup_{t \in (0, \infty)} \left| \int_0^\pi \theta^{k-n-i}\partial_{\theta}^{k-i-1}\phi_i(\theta)h_{n-1}(z\theta)d\theta \right| \right) \leq C(1 + z^{n-1}) \leq Cz^{n-1}, \quad z \geq 1.$$

From Propositions \ref{prop:inductive} and \ref{prop:inductive2} by taking into account (2.10) and (2.11) and proceeding by induction as before we obtain \ref{prop:inductive3} and \ref{prop:inductive4}. \qed

**Proposition 2.8.** Let $k \in \mathbb{N}$ and consider the function

$$H_k(z, t) := \frac{1}{2\pi} \int_0^\pi \partial_{\theta}^k\phi_i(\theta)h_k(z\theta)d\theta, \quad z, t \in (0, \infty).$$

We have that

(2.12)

$$\sup_{t > 0} |\partial_{\theta}^kH_k(z, t)| \leq Cz^{-k+1}, \quad z \in (0, \infty),$$

(2.13)

$$\|t\partial_{\theta}^kH_k(z, t)\|_{L^2((0, \infty), \phi)} \leq Cz^{-k+1}, \quad z \geq 1,$$

and, for every $\Psi \in L^\infty((0, \infty))$,

(2.14)

$$\left| \int_0^\infty \Psi(t)\partial_{\theta}^kH_k(z, t)dt \right| \leq Cz^{k+1}, \quad z \geq 1.$$

**Proof.** We can write

$$\partial_{\theta}^kH_k(z, t) = \sum_{j=0}^k \frac{c_{k,j}}{z^{k-j}} \int_0^\pi \partial_{\theta}^j\phi_i(\theta)\theta^{k-j}(\partial_{\theta}^{k-j}h_k)(z\theta)d\theta, \quad z, t \in (0, \infty),$$

for certain $c_{k,j} \in \mathbb{R}$, $j = 0, \ldots, k$.

Now we observe that, for every $r \in \mathbb{N}$,

$$\partial_{\theta}^r h_k(\theta) = (-1)^{\lfloor (r+1)/2 \rfloor} \left\{ \begin{array}{ll}
h_k(\theta), & \text{if } r \text{ is even}, \\
(-1)^k h_{k+1}(\theta), & \text{if } r \text{ is odd}, \end{array} \right. \quad \theta \in (0, \pi).$$

Then, for each $j = 0, \ldots, k$, if $k-j$ is even, that is, $k, j$ are both odd or even, $\partial_{\theta}^{k-j}h_k = (-1)^{(k-j)/2}h_k = (-1)^{(k-j)/2}h_j$. In the case that $k-j$ is odd we have that $\partial_{\theta}^{k-j}h_k = (-1)^{(k-j+1)/2+k}h_{k+1} = (-1)^{(k-j+1)/2+k}h_j$.

We deduce that

$$\partial_{\theta}^kH_k(z, t) = \sum_{j=0}^k \frac{c_{k,j}}{z^{k-j}} \int_0^\pi \partial_{\theta}^j\phi_i(\theta)\theta^{k-j}h_j(\theta)d\theta, \quad z, t \in (0, \infty),$$

for certain $c_{k,j} \in \mathbb{R}$, $j = 0, \ldots, k$. \qed
for certain $\tilde{c}_{k,j} \in \mathbb{R}$, $j = 0, ..., k$, and Proposition 2.2 allows us to conclude our result. \hfill $\square$

3. Proof of Theorem 1.2 for the maximal operator

We recall that the maximal operator $W_*$ associated to the semigroup $\{W_t\}_{t>0}$ is defined by

$$W_*(f) = \|W_t(f)\|_{L^\infty(0,\infty)},$$

for every $f \in \ell^p(\mathbb{Z})$, $1 \leq r \leq \infty$.

In [8] the behavior of $W_*$ in weighted $\ell^p$-spaces is studied by considering $W_*$ as a $L^\infty$-valued singular integral and by using vectorial Calderón-Zygmund theory ([19]). In [6, Proposition 3] it was proved that, for certain $C > 0$,

$$\|G(n, \cdot)\|_{L^\infty(0,\infty)} \leq \frac{C}{|n| + 1}, \quad n \in \mathbb{Z},$$

and

$$\|G(n + 1, \cdot) - G(n, \cdot)\|_{L^\infty(0,\infty)} \leq \frac{C}{|n|^2 + 1}, \quad n \in \mathbb{Z}.$$  

By using (3.1) and (3.2) a standard procedure (see, for instance, [8, p. 586]) allows to prove that, if $1/2 < p \leq 1$ there exists $C > 0$ such that

$$\|W_*(b)\|_p \leq C,$$

for every $(p, 2)$-atom $b$. Then, if $1/2 < p \leq 1$, since $\mathcal{H}^p(\mathbb{Z})$ is continuously contained in $\ell^2(\mathbb{Z})$ and $W_*$ is bounded from $\ell^2(\mathbb{Z})$ into itself, $W_*$ define a bounded operator from $\mathcal{H}^p(\mathbb{Z})$ into $\ell^p(\mathbb{Z})$ (Theorem 1.1). Our objective is to see that the this last property can be extended to those $p \in (0, 1/2]$. In order to do this we are going to improve the estimate (3.2).

For every $k \in \mathbb{N}$ we consider the function

$$F_k(z, t) = \frac{(-1)^{E[(k+1)/2]}}{\pi} H_k(z, t), \quad z, t \in (0, \infty),$$

where $H_k$ is the function defined in Proposition 2.8. According to Lemma 2.3 for each $k \in \mathbb{N}$, $F_k(n, t) = G(n, t)$, $n \in \mathbb{N}$, $n \geq 1$, and $t \in (0, \infty)$.

Let $0 < p \leq 1$ and $k = E[1/p]$. We take a $(H, p, 2)$-atom $b$ such that its support is contained in $B_{\mathbb{Z}}(n_0, r_0)$ and $\|b\|_2 \leq \mu(B_{\mathbb{Z}}(n_0, r_0))^{1/2 - 1/p}$, for certain $n_0 \in \mathbb{Z}$ and $r_0 \geq 1$. We write

$$\|W_*(b)\|_p^p = \sum_{n \in B_{\mathbb{Z}}(n_0, 2r_0)} |W_*(b)(n)|^p + \sum_{n \not\in B_{\mathbb{Z}}(n_0, 2r_0)} |W_*(b)(n)|^p.$$

Since $W_*$ is bounded from $\ell^2(\mathbb{Z})$ into itself we get

$$\sum_{n \in B_{\mathbb{Z}}(n_0, 2r_0)} |W_*(b)(n)|^p \leq \left( \sum_{n \in B_{\mathbb{Z}}(n_0, 2r_0)} |W_*(b)(n)|^2 \right)^{p/2} \mu(B_{\mathbb{Z}}(n_0, 2r_0))^{1 - p/2} \leq C \|b\|_2^p \mu(B_{\mathbb{Z}}(n_0, 2r_0))^{1 - p/2} \leq C.$$

On the other hand, if $n \in \mathbb{Z}$ and $n \geq n_0 + 2r_0$, we have that

$$W_t(b)(n) = \sum_{m \in B_{\mathbb{Z}}(n_0, n_0)} G(n - m, t)b(m) = \sum_{m \in B_{\mathbb{Z}}(n_0, n_0)} F_k(n - m, t)b(m)
= \sum_{m \in B_{\mathbb{Z}}(n_0, n_0)} \left( F_k(n - m, t) - \sum_{j=0}^{k-1} \frac{(\partial_j F_k)(n - n_0, t)(n_0 - m)^j}{j!} \right)b(m)
= \sum_{m \in B_{\mathbb{Z}}(n_0, n_0)} b(m) \int_{n-n_0}^{n-m} \int_{n-n_0}^{x_1} \cdots \int_{n-n_0}^{x_{k-1}} (\partial_z F_k)(z, t)dzdx_{k-1}dx_1, \quad t \in (0, \infty).$$

Then from (3.2) we deduce that

$$|W_*(b)(n)| \leq C \sum_{m \in B_{\mathbb{Z}}(n_0, n_0)} |b(m)| \int_{n-n_0}^{n-m} \int_{n-n_0}^{x_1} \cdots \int_{n-n_0}^{x_{k-1}} \frac{1}{|z|^{k+1}}dzdx_{k-1}dx_1.$$
In [6, Proposition 4] it was established that singular integral of convolution type whose kernel is given by

\[ f \]  

for every \( f \) (Theorem 1.1). In order to prove that \( g \) can be expressed as

\[ \sum_{m \in B_2(n_0, r_0)} |b(m)| \frac{|m - n_0|^k}{|n_0|^k+1} \leq C \frac{\|b\|^2}{|n - n_0|^{k+1}} \left( \sum_{m \in B_2(n_0, r_0)} |m - n_0|^{2k} \right)^{1/2} \]

we need to improve the property (4.2). Following the argument developed there, by taking into account the operator \( g \) is bounded from \( \mathcal{H}^p(\mathbb{Z}) \) into \( \ell^p(\mathbb{Z}) \).

If \( n \in \mathbb{Z} \) and \( n \leq n_0 - 2r_0 \), since \( G(\ell, t) = G(-\ell, t), \ell \in \mathbb{Z} \) and \( t \in (0, \infty) \), we get

\[ W_t(b)(n) = \sum_{m \in B_2(n_0, r_0)} G(m - n, t)b(m) = \sum_{m \in B_2(n_0, r_0)} b(m) \int_{n_0-n}^{n+1} \ldots \int_{n_0-n}^{n+1} (\partial_z^k F_k)(z, t)dzdx_{k-1}...dx_1, \ t \in (0, \infty), \]

and proceeding as before we obtain

\[ |W_*(b)(n)| \leq C \frac{r_0^{k+1-1/p}}{|n - n_0|^{k+1}}, \ n \in \mathbb{Z}, \ n \leq n_0 - 2r_0. \]

We conclude that

\[ \|W_*(b)\|^p \leq \sum_{n \notin B_2(n_0, 2r_0)} |W_*(b)(n)|^p \leq Cr_0^{(k+1)p-1} \sum_{n \notin B_2(n_0, 2r_0)} \frac{1}{|n - n_0|^{(k+1)p}} \leq C, \]

because \((k+1)p > 1\), where \( C > 0 \) is a constant independent of \( b \). Then, it follows that the operator \( W_* \) is bounded from \( \mathcal{H}^p(\mathbb{Z}) \) into \( \ell^p(\mathbb{Z}) \).

4. PROOF OF THEOREM [12] FOR THE LITTLEWOOD-PALY FUNCTION

The vertical Littlewood-Paley function associated to the semigroup \( \{W_t\}_{t>0} \) of operators defined by [11] can be expressed as

\[ g(f)(n) = \left\| t\partial_t W_t(f)(n) \right\|_{L^2((0, \infty), \mathfrak{V})}, \ n \in \mathbb{Z}, \]

for every \( f \in \ell^p(\mathbb{Z}) \) and \( 1 \leq p < \infty \).

In [6] it was proved that the operator \( g \) can be seen as a \( L^2((0, \infty), dt/t)\)-Calderón-Zygmund singular integral of convolution type whose kernel is given by

\[ \mathfrak{V}(n, t) = t\partial_t G(n, t), \ n \in \mathbb{Z} \text{ and } t \in (0, \infty). \]

In [6] Proposition 4 it was established that

\[ \|\mathfrak{V}(n, \cdot)\|_{L^2((0, \infty), \mathfrak{V})} \leq \frac{C}{|n| + 1}, \ n \in \mathbb{Z}, \]

and

\[ \|\mathfrak{V}(n + 1, \cdot) - \mathfrak{V}(n, \cdot)\|_{L^2((0, \infty), \mathfrak{V})} \leq \frac{C}{|n|^2 + 1}, \ n \in \mathbb{Z}. \]

A standard procedure allows us to see that \( g \) is bounded from \( \mathcal{H}^p(\mathbb{Z}) \) into \( \ell^p(\mathbb{Z}, \omega) \) when \( 1/2 < p \leq 1 \) (Theorem [13]). In order to prove that \( g \) is bounded from \( \mathcal{H}^p(\mathbb{Z}) \) into \( \ell^p(\mathbb{Z}, \omega) \) when \( 0 < p \leq 1/2 \) we need to improve the property [12].

As in Section 3 for every \((H, p, 2)\)-atom \( b \) with support in \( B_2(n_0, r_0) \) we write

\[ \|g(b)\|^p_p = \sum_{n \in B_2(n_0, 2r_0)} |g(b)(n)|^p + \sum_{n \notin B_2(n_0, 2r_0)} |g(b)(n)|^p. \]

Following the argument developed there, by taking into account the \( \ell^2 \)-boundedness of the operator \( g \) and estimation [2.13], we can establish that the operator \( g \) is bounded from \( \mathcal{H}^p(\mathbb{Z}) \) into \( \ell^p(\mathbb{Z}) \), for every \( 0 < p \leq 1 \).
5. Proof of Theorem 1.3

We observe from (5.3) that

\[ I_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\cos \theta} e^{-i\theta} d\theta = F^{-1}_Z(e^{i\cos \theta})(n), \quad n \in \mathbb{Z} \text{ and } t > 0. \]

Then,

\[ (5.1) \quad G(n, t) = F^{-1}_Z(e^{-2t(1-\cos \theta)})(n), \quad n \in \mathbb{Z} \text{ and } t > 0. \]

First we will establish that \( T_m \) is a Calderón-Zygmund singular integral operator. Hence, by using (5.1) we get

\[ \int_{-\pi}^{\pi} |m(2(1-\cos \theta))F_Z(f)(\theta)| d\theta \leq 2 \int_{-\pi}^{\pi} |F_Z(f)(\theta)|(1-\cos \theta) \int_{0}^{\infty} e^{-2t(1-\cos \theta)} |\Psi(t)| d\theta dt \]

\[ \leq C \|\Psi\|\mathcal{L}^\infty(0,\infty) \int_{-\pi}^{\pi} |F_Z(f)(\theta)| d\theta \leq C \|\Psi\|\mathcal{L}^\infty(0,\infty) \|F_Z(f)\|\mathcal{L}^2(-\pi,\pi) \]

\[ \leq C \|\Psi\|\mathcal{L}^\infty(0,\infty) \|f\|\ell^2(\mathbb{Z}). \]

By using (5.1) we get

\[ \int_{-\pi}^{\pi} \Psi(t) \int_{-\pi}^{\pi} F_Z(f)(\theta)(1-\cos \theta) e^{-2t(1-\cos \theta)} e^{-i\theta} d\theta dt \]

\[ = - \int_{0}^{\infty} \Psi(t) d\theta \int_{-\pi}^{\pi} F_Z(f)(\theta)e^{-2t(1-\cos \theta)} e^{-i\theta} d\theta dt \]

\[ = -2\pi \int_{0}^{\infty} \Psi(t) d\theta \int_{-\pi}^{\pi} F_Z(f)[F_Z(f)(G(\cdot, t))](n) dt \]

\[ = -2\pi \int_{0}^{\infty} \Psi(t) d\theta \sum_{m \in \mathbb{Z}} G(n-m, t)f(m)dt, \quad n \in \mathbb{Z}. \]

Hence, if \( f(m) = 0 \), for every \( m \in \mathbb{Z} \) with \( |m| > m_0 \), for certain \( m_0 \in \mathbb{N} \), we get

\[ T_m(f)(n) = \sum_{\ell \in \mathbb{Z}} K_m(n-\ell)f(\ell), \quad n \in \mathbb{Z}, \]

where \( K_m(n) = -\int_{0}^{\infty} \Psi(t) d\theta G(n, t)dt, \quad n \in \mathbb{Z}. \)

According to Lemma 2.3 we have that

\[ K_m(n) = -\frac{1}{\pi} \int_{0}^{\infty} \Psi(t) d\theta H_0(n, t)dt, \quad n \in \mathbb{Z}, \]

where \( H_0 \) is the function given in Proposition 2.8 for \( k = 0 \). Then, from (2.14) for \( k = 0 \) we see that

\[ |K_m(n)| \leq \frac{C}{|n|}, \quad n \in \mathbb{Z}, \quad n \neq 0. \]

Also we can write

\[ K_m(n + \ell) - K_m(n) = \int_{n}^{n+\ell} \partial_z J(z)dz, \]

where \( J(z) = \frac{1}{\pi} \int_{0}^{\infty} \Psi(t) d\theta H_1(z, t)dt, \quad z > 0. \) Here \( H_1 \) is the function in Proposition 2.8 for \( k = 1 \).

Again by (2.14) (for \( k = 1 \)) we deduce that

\[ |K_m(n + \ell) - K_m(n)| \leq C \int_{n}^{n+\ell} \frac{dz}{|z|^2} \leq C \frac{\ell}{n^2}, \quad n, \ell \in \mathbb{Z}, \quad n \cdot \ell > 0. \]

We have seen that \( T_m \) is a Calderón-Zygmund singular integral operator. Hence, \( T_m \) can be extended from \( \ell^p(\mathbb{Z}) \cap \ell^q(\mathbb{Z}, \omega) \) to \( \ell^p(\mathbb{Z}, \omega) \) as a bounded operator from \( \ell^q(\mathbb{Z}, \omega) \) into

(a) \( \ell^p(\mathbb{Z}, \omega) \), for every \( 1 < p < \infty \) and \( \omega \in A_p(\mathbb{Z}) \),
(b) \( \ell^{1,\infty}(\mathbb{Z}, \omega) \), when \( p = 1 \) and \( \omega \in A_1(\mathbb{Z}) \).
Also, it can be shown that $T_m$ defines a bounded operator from $H^p(Z)$ to $\ell^p(Z)$, for every $1/2 < p \leq 1$, by using (5.3) and for $0 < p \leq 1/2$ by means of (2.14) (see the argument in the proof of Theorem 1.2 for $W_*$ and $g$).

Next we are going to see that $T_m$ defines a bounded operator from $H^p(Z)$ into itself, $0 < p \leq 1$. In order to do this we will use molecules.

Discrete molecules were considered in [13] (see also [14]). Let $0 < p \leq 1 < q \leq \infty$, and $\alpha > 1/p - 1/q$. A complex sequence $M$ is said to be a $(H, p, q, \alpha)$-molecule centered in $n_0 \in Z$ when the following conditions are satisfied.

(i) $N_{p,q,\alpha}(M) = \|M\|^{1-\theta} \cdot \| - n_0 \| M \|^{\theta} < \infty$, where $\theta = (1/p - 1/q)/\alpha$,
(ii) $\sum_{n \in Z} n^j M(n) = 0, j \in \mathbb{N}, j \leq E[1/p] - 1$.

Note that the absolute convergence of the series in (ii) follows from the condition (i). In [13, Theorem 2] Hardy spaces $H^p(Z)$ were characterized by using molecules.

Let $0 < p \leq 1$ and $k = E[1/p]$. To establish that $T_m$ defines a bounded operator from $H^p(Z)$ into itself we see that there exists $C > 0$ such that if $b$ is a $(H, p, 2)$-atom associated to $n_0 \in Z$ and $r_0 \geq 1$, then $T_m(b)$ is a $(H, p, 2, k)$-molecule centered in $n_0$ and $N_{p,2,k}(T_m(b)) \leq C$.

Assume that $b$ is a $(H, p, 2)$-atom such that the support of $b$ is contained in $B_Z(n_0, r_0)$ and $\|b\|_2 \leq \mu(B_Z(n_0, r_0))^{1/2-1/p}$, where $n_0 \in Z$ and $r_0 \geq 1$. It is sufficient to consider $n_0 = 0$ because $T_m$ is a convolution operator.

Since $T_m$ is a bounded operator from $\ell^2(Z)$ into itself we have that

$$\|T_m(b)\|_2 \leq C \|b\|_2 \leq C \mu(B_Z(0, r_0))^{1/2-1/p}.$$

On the other hand we can write

$$\| \cdot ^k T_m(b) \|_2 = \left( \sum_{n \in B_Z(0, 2r_0)} + \sum_{n \not\in B_Z(0, 2r_0)} \right) |n|^{2k} |T_m(b)(n)|^2 = J_1 + J_2.$$

We have that

$$J_1 \leq C r_0^{2k} \|b\|_2^2 \leq C r_0^{2k} \mu(B_Z(0, r_0))^{1-2/p} \leq C \mu(B_Z(0, r_0))^{2k+1-2/p}.$$  

To estimate $J_2$ we proceed as in the proof of Theorem 1.2 for $W_*$ and $g$ by using (2.14). We obtain

$$\sum_{n \not\in B_Z(0, 2r_0)} |n|^{2k} |T_m(b)(n)|^2 \leq C \left( \sum_{n \not\in B_Z(0, 2r_0)} \frac{1}{|n|^{2k}} \|b\|_2 \sum_{m \in B_Z(0, r_0)} |m|^k \right)^2 \leq C \frac{\mu(B_Z(0, r_0))^{1-2/p} r_0^{2k+1}}{r_0^{2k+1-2/p}} \leq C \mu(B_Z(0, r_0))^{2k+1-2/p}.$$  

By combining the above estimates we get

$$\| \cdot ^k T_m(b) \|_2 \leq C \mu(B_Z(0, r_0))^{k+1-2/p}.$$  

We conclude that

$$N_{p,2,k}(T_m(b)) \leq C \mu(B_Z(0, r_0))^{(1-2-1/p)(1-\theta)+(k+1-2-1/p)\theta} = C,$$

where $\theta = (1/p - 1/2)/k$.

Next we prove that

$$\sum_{n \in Z} n^j T_m(b)(n) = 0, \quad j = 0, \ldots, k - 1.$$

According to [13, p. 303], \( \sum_{n \in Z} |n|^\theta |T_m(b)(n)| < \infty \), for every $j \in \mathbb{N}, j \leq k - 1$. Since $T_m(b) \in L^1(Z)$, $F_Z(T_m(b))$ is a continuous function in $(\pi, \pi)$. Also, $T_m(b) \in L^2(Z)$ and $F_Z(T_m(b))(\theta) = m(2(1 - \cos \theta))F_Z(b)(\theta)$, a.e. $\theta \in (-\pi, \pi)$. Since $m(2(1 - \cos \theta))F_Z(b)(\theta)$ is a continuous function in $(-\pi, \pi)$ \{0\}, we have that

$$F_Z(T_m(b))(\theta) = m(2(1 - \cos \theta))F_Z(b)(\theta), \quad \theta \in (-\pi, \pi) \setminus \{0\}.$$
Then, by taking into account that $m$ is a bounded function in $(0, \infty)$ and that $F_Z(b)(0) = \sum_{n \in \mathbb{Z}} b(n) = 0$, we get
\[
\lim_{\theta \to 0} F_Z(T_m(b))(\theta) = 0,
\]
and then we get (5.4) when $j = 0$, because $\lim_{\theta \to 0} F_Z(T_m(b))(\theta) = \sum_{n \in \mathbb{Z}} T_m(b)(n)$.

Let now $j \in \mathbb{N}$, $1 \leq j \leq k - 1$.

Since $\sum_{n \in \mathbb{Z}} |T_m(b)(n)||n|^j < \infty$, it follows that $F_Z(T_m(b)) \in C^j(-\pi, \pi)$ with
\[
\frac{d^j}{d\theta^j} F_Z(T_m(b))(\theta) = i^j \sum_{n \in \mathbb{Z}} n^j T_m(b)(n)e^{in\theta}, \quad \theta \in (-\pi, \pi),
\]
and
\[
\sum_{n \in \mathbb{Z}} n^j T_m(b)(n) = (-i)^j \lim_{\theta \to 0} \frac{d^j}{d\theta^j} F_Z(T_m(b))(\theta).
\]

On the other hand, we have that
\[
\frac{d^j}{d\theta^j} F_Z(T_m(b))(\theta) = \sum_{\ell=0}^{j} \binom{j}{\ell} \frac{d^{j-\ell}}{d\theta^{j-\ell}} m(2(1 - \cos \theta)) \frac{d^\ell}{d\theta^\ell} F_Z(b)(\theta), \quad \theta \in (-\pi, \pi) \setminus \{0\}.
\]

It is not hard to see that, for every $r \in \mathbb{N}$, there exists $C > 0$ such that
\[
\left| \frac{d^r}{d\theta^r} m(2(1 - \cos \theta)) \right| \leq \frac{C}{|\theta|^r}, \quad \theta \in (-\pi, \pi) \setminus \{0\}.
\]

By using Taylor theorem, since $\frac{d^r}{d\theta^r} F_Z(b)(0) = 0$, $\in \mathbb{N}$, $0 \leq \ell \leq k - 1$, we get, for a certain $C > 0$,
\[
\left| \frac{d^\ell}{d\theta^\ell} F_Z(b)(\theta) \right| \leq C|\theta|^{k-\ell}, \quad \theta \in (-\pi, \pi) \setminus \{0\} \text{ and } \ell = 0, 1, ..., k - 1.
\]

We conclude that $\lim_{\theta \to 0} \frac{d^\ell}{d\theta^\ell} F_Z(T_m(b))(\theta) = 0$ and hence, $\sum_{n \in \mathbb{Z}} n^j T_m(b)(n) = 0$. Thus, we have proved that the operator $T_m$ is bounded from $H^p(\mathbb{Z})$ into itself for every $p \in [0, 1]$.

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