TRANSCENDENCE AND CONTINUED FRACTION EXPANSION OF VALUES OF HECKE–MAHLER SERIES

YANN BUGEAUD AND MICHEL LAURENT

Abstract. Let \( \theta \) and \( \rho \) be real numbers with \( 0 \leq \theta, \rho < 1 \) and \( \theta \) irrational. We show that the Hecke–Mahler series

\[
F_{\theta, \rho}(z_1, z_2) = \sum_{k_1 \geq 1} \sum_{k_2=1}^{[k_1 \theta + \rho]} z_1^{k_1} z_2^{k_2},
\]

where \( [\cdot] \) denotes the integer part function, takes transcendental values at any algebraic point \((\beta, \alpha)\) with \(0 < |\beta|, |\beta \alpha \theta| < 1\). This extends earlier results of Mahler (1929) and Loxton and van der Poorten (1977), who settled the case \( \rho = 0 \). Furthermore, for positive integers \( b \) and \( a \), with \( b \geq 2 \) and \( a \) congruent to \( 1 \) modulo \( b - 1 \), we give the continued fraction expansion of the number

\[
\frac{(b-1)^2}{b} F_{\theta, \rho} \left( \frac{1}{b}, \frac{1}{a} \right) + \frac{[\theta + \rho](b-1)}{b^2 a},
\]

from which we derive a formula giving the irrationality exponent of \( F_{\theta, \rho}(1/b, 1/a) \).

À la mémoire du Professeur Andrzej Schinzel

Contents

1. Introduction and main results ............................ 1
2. Sturmian words ........................................... 7
3. Continued fraction expansion ............................. 9
4. Functional equations and expansions of Hecke-Mahler series .... 15
5. When the slope has unbounded partial quotients ............. 21
6. Functional transcendence .................................. 23
7. Transcendence of Hecke–Mahler series at algebraic points ... 25
References .................................................. 28

1. Introduction and main results

Throughout, \([\cdot]\) and \(\lceil \cdot \rceil\) are, respectively, the integer part and the upper integer part functions. For a real number \( \theta \) in \((0,1)\), set

\[
h_{\theta}(z) = \sum_{k \geq 1} [k \theta] z^k,
\]

2010 Mathematics Subject Classification. 11J04, 11J70, 11J81.

Key words and phrases. rational approximation, continued fraction, Mahler’s method, transcendence, Sturmian sequence.
where $z$ is a complex number with $|z| < 1$, and
\[ F_\theta(z_1, z_2) = \sum_{k_1 \geq 1} \sum_{k_2 = 1}^{[k_1 \theta]} z_1^{k_1} z_2^{k_2}, \]
where $z_1, z_2$ are complex numbers with $|z_1| < 1, |z_1 z_2^\theta| < 1$. The series $h_\theta(z)$ has been introduced by Hecke [9] in 1922. Böhmer [4] proved in 1927 that, if $\theta$ has unbounded partial quotients, then $h_\theta(1/b)$ is transcendental, for every integer $b \geq 2$. Two years later, in his fundational paper [12], Mahler introduced the two-variables series $F_\theta(z_1, z_2)$ (note that Mahler and most of his followers used $\omega$ in place of $\theta$, while we keep the notation from [7]) and, among other results, he established that $h_\theta(\beta)$ is transcendental for every quadratic irrational number $\theta$ and every complex non-zero algebraic number $\beta$ in the open unit disc. This has been extended to every irrational number $\theta$ in $(0, 1)$ by Loxton and van der Poorten [11] (see also [13, Section 2.9]) nearly fifty years later.

We adopt a slightly different point of view to generalize the functions $h_\theta$ and $F_\theta$. Let $\theta$ and $\rho$ be real numbers with $0 \leq \theta, \rho < 1$ and $\theta$ irrational. For $n \geq 1$, set
\[ s_n := s_n(\theta, \rho) = \lfloor n \theta + \rho \rfloor - \lfloor (n-1) \theta + \rho \rfloor, \quad s'_n := s'_n(\theta, \rho) = \lceil n \theta + \rho \rceil - \lceil (n-1) \theta + \rho \rceil. \]
Then, the infinite words
\[ s_{\theta, \rho} := s_1 s_2 s_3 \ldots, \quad s'_{\theta, \rho} := s'_1 s'_2 s'_3 \ldots \]
are, respectively, the lower and upper Sturmian words of slope $\theta$ and intercept $\rho$, written over the alphabet $\{0, 1\}$. For complex numbers $\alpha, \beta$ with $|\beta \alpha^\theta| < 1$, write
\[ \xi_{s_{\theta, \rho}}(\beta, \alpha) = \sum_{n \geq 1} s_n \alpha^n \beta^{n-1} s_n = \sum_{n \geq 1} s_n \alpha^n [n \theta + \rho]. \]
Observe that, setting
\[ F_{\theta, \rho}(z_1, z_2) = \sum_{k_1 \geq 1} \sum_{k_2 = 1}^{[k_1 \theta + \rho]} z_1^{k_1} z_2^{k_2}, \]
we have
\[ \xi_{s_{\theta, \rho}}(\beta, \alpha) = (1 - \beta) F_{\theta, \rho}(\beta, \alpha) + \beta^2 \alpha [\theta + \rho], \]
for any $\beta, \alpha$ satisfying $|\beta| < 1, |\beta \alpha^\theta| < 1$. The notation $\xi_{s_{\theta, \rho}}$ was introduced in [7] and we keep it in the present work. The transcendence of $F_{\theta, \rho}(\beta, \alpha)$ for nonzero algebraic numbers $\alpha, \beta$ has been widely studied, after the pioneering works of Mahler [12] and Loxton and van der Poorten [11] in the case $\rho = 0$. Borwein and Borwein [5, Theorem 0.4] established that, if the slope $\theta$ has infinitely many partial quotients greater than or equal to 3, then $\xi_{s_{\theta, \rho}}(1/b, 1/a)$ is transcendental for every positive integers $a, b$ with $b \geq 2$. Komatsu [10] (see also [14]) proved that, if the slope $\theta$ has unbounded partial quotients, then $\xi_{s_{\theta, \rho}}(\beta, \alpha)$ is transcendental for every nonzero complex algebraic numbers $\alpha, \beta$ with $|\beta \alpha^\theta| < 1$, under some technical condition. Lastly, Ferenčzi and Mauduit [8] used combinatorial properties of Sturmian sequences and Ridout’s $p$-adic extension of Roth’s theorem to show that $\xi_{s_{\theta, \rho}}(1/b, 1)$ is transcendental for every integer $b \geq 2$. 
Our first main theorem is a considerable extension of all these results.

**Theorem 1.1.** Let \( \theta \) and \( \rho \) be real numbers with \( 0 \leq \theta, \rho < 1 \) and \( \theta \) irrational. Let \( \alpha, \beta \) be nonzero complex algebraic numbers such that \( |\beta\alpha^\theta| < 1 \) and \( \beta \neq 1 \). Then, the complex number \( \xi_{s\theta,\rho}(\beta, \alpha) \) is transcendental. In particular, if \( |\beta| < 1 \), then the complex numbers

\[
h_{\theta,\rho}(\beta), \quad F_{\theta,\rho}(\beta, \alpha).
\]

are transcendental.

Since, for every \( \alpha \) in the open unit disc, we have

\[
\xi_{s\theta,\rho}(1, \alpha) = \frac{\alpha}{1 - \alpha},
\]

the assumption \( \beta \neq 1 \) in Theorem 1.1 is necessary.

When the slope \( \theta \) has unbounded partial quotients in its continued fraction expansion, Theorem 1.1 was proved by Komatsu [10], under some mild additional assumption on \( \alpha \) and \( \beta \). For the sake of completeness, we display a complete proof in Section 5.

Adamczewski and Bugeaud [2, Proposition 11.1] proved that the Diophantine exponent (which measures the repetitions occurring at the beginning or near the beginning of an infinite word, see [2, p. 70]) of a Sturmian sequence is infinite if and only if its slope \( \theta \) has unbounded partial quotients, independently of the value of its intercept \( \rho \). Under this assumption, the \( p \)-adic Schmidt Subspace Theorem applies to show that, for every \( \rho \) in \([0, 1)\) and every nonzero algebraic number \( \beta \) in the open unit disc, the complex number \( \xi_{s\theta,\rho}(\beta, 1) \) is either transcendental, or lies in \( \mathbb{Q}(\beta) \); see [1, Theorem 1]. A different, more involved, application of the \( p \)-adic Schmidt Subspace Theorem allows us to get the same conclusion if \( \theta \) has bounded partial quotients; details will be given in a subsequent paper.

The proof of Theorem 1.1 follows Mahler’s method and its extension by Loxton and van der Poorten [11]. A key point is the following construction, leading to a chain of functional equations. For an irrational real number \( \theta \) in \((0, 1)\), write

\[
\theta = [0; a_1, a_2, \ldots], \quad \theta_k = [0; a_{k+1}, a_{k+2}, \ldots], \quad k \geq 0,
\]

in such a way that

\[
\theta_0 = \theta, \quad \theta_{k+1} = \frac{1}{\theta_k} - a_{k+1} = \left\{ \frac{1}{\theta_k} \right\}, \quad k \geq 0,
\]

where \( \{\cdot\} \) denotes the fractional part function. Let \( (p_k/q_k)_{k \geq 0} \) denote the sequence of convergents to \( \theta \). An elementary calculation yields the equation

\[
F_{\theta_k,0}(z_1, z_2) = -F_{\theta_{k+1},0}(z_1^{a_{k+1}+1}z_2, z_1) + \frac{z_1^{a_{k+1}+1}z_2}{(1 - z_1^{a_{k+1}+1}z_2)(1 - z_1)}.
\]

When \( \theta \) is a quadratic irrational, the sequence \( (\theta_k)_{k \geq 1} \) is ultimately periodic, and this chain of functional equation yields a single functional equation. Namely, assuming that \( \theta_{k+s} = \theta_k \) for \( k \geq 0 \) and an even positive integer \( s \), we end up with a functional equation of the form

\[
F_{\theta,0}(z_1, z_2) = F_{\theta,0}(z_1^{q_s}z_2^{p_s}, z_1^{q_s-1}z_2^{p_s-1}) + R(z_1, z_2),
\]
where \( R(z_1, z_2) \) is in \( \mathbb{Q}(z_1, z_2) \) and which has been treated by Mahler [12]. In general, we have a system of functional equations

\[
F_{\theta, 0}(z_1, z_2) = (-1)^k F_{\theta, 0}(z_1^{q_k} z_2^{p_k}, z_1^{q_k-1} z_2^{p_k-1}) + R_k(z_1, z_2), \quad k \geq 1.
\]

Loxton and van der Poorten [11] developed a general theory which applies to such chains of equations under some technical constraints. These assumptions may be satisfied (for a suitable subsequence of the indices \( k \)) if we assume that the sequence \( (a_k)_{k \geq 1} \) is bounded. By means of our new result on the structure of Sturmian sequences [7] (see Proposition 2.2 below), we are able to show that this approach also works for the more general series \( F_{\theta, \rho}(z_1, z_2) \). The unbounded case, treated in Section 5, is related to the second part of our paper devoted to continued fractions expansions.

We stress an immediate consequence of Theorem 1.1. For more on \( \beta \)-expansions of real numbers, the reader is directed to [1] and the references given therein.

**Corollary 1.2.** Let \( \alpha \) and \( \beta \) be real algebraic numbers with \( \beta > 1 \). Then, the \( \beta \)-expansion of \( \alpha \) is not given by a Sturmian sequence.

Theorem 1.1 asserts that any power series whose sequence of coefficients is a Sturmian sequence of integers sends non-zero algebraic points in the unit disc to transcendental points. This is not the case for every automatic series, as shown by Adamczewski and Faverjon [3, Section 8.1], who gave the example of an automatic series taking an algebraic value at any point of the form \( \phi^{1/3^\ell} \), where \( \phi = (1 - \sqrt{5})/2 \) and \( \ell \geq 1 \).

Let \( a \) and \( b \) be positive integers with \( b \geq 2 \). By Theorem 1.1, the real numbers \( \xi_{\theta, \rho}(1/b, 1/a) \) and \( \xi_{\theta', \rho}(1/b, 1/a) \) are transcendental. We now deal with the continued fraction expansion of the real numbers \( \xi \) of the form

\[
(b - 1)\xi_{\theta, \rho}(1/b, 1/a) \quad \text{or} \quad (b - 1)\xi_{\theta', \rho}(1/b, 1/a).
\]

When \( a = 1 \), we will recover the expansion of Sturmian numbers obtained in [7].

We denote by \( (b_k)_{k \geq 1} \) the sequence of digits of the number

\[
(1.1) \quad \rho - \theta = \sum_{k \geq 0} b_{k+1}(q_k \theta - p_k),
\]

written in the Ostrowski numeration system with base \( \theta \) (normalized as in Theorem 2.1 of [7] or in Theorem 4.2 when \( \rho \) is of the form \(-m \theta + p\), with \( m, p \) nonnegative integers). We set (by convention, an empty sum is equal to zero)

\[
(1.2) \quad t_k = \sum_{j=1}^{k} b_j q_{j-1} - 1, \quad \tilde{t}_k = \sum_{j=1}^{k} b_j p_{j-1}, \quad r_k = q_k - t_k, \quad \tilde{r}_k = p_k - \tilde{t}_k, \quad k \geq 0.
\]
For $k \geq 0$, set

$$c_k = \begin{cases} \frac{b^{a_1-b_1-1}}{b^{k+q_k-1}a_{k+1}a_{k+1}^{p_{k+1}}}, & \text{when } k = 0, \\ \frac{b^{r_k+a_{k+1}}(b^{a_{k+1}}a_{k+1})^{-1}}{b^{r_k+a_{k+1}a_{k+1}^{p_{k+1}}} - 1}, & \text{when } k \geq 1, \end{cases}$$

$$d_k = b^{r_k}a^{r_k} - 1,$$

$$e_k = b^{r_k}a^{r_k} - 1,$$

$$f_k = b^{r_k}a^{r_k}(b^{a_{k+1}}a_{k+1})^{-1}.$$

When $a = 1$, the four sequences $(c_k)_{k \geq 0}, (d_k)_{k \geq 0}, (e_k)_{k \geq 0}, (f_k)_{k \geq 0}$ coincide with the corresponding ones introduced in [7]. We point out that some elements of these sequences may be non-positive, exactly in the same situations as in [7]. For example, $f_k$ is equal to 0 when $b_{k+1} = 0$ and $c_{k+1}$ is equal to 0 when $a_{k+2} = b_{k+2} + 1$. In the case where $a_{k+2} = b_{k+2}$, we have $b_{k+1} = 0$, thus $r_k + q_k = r_{k+1} + q_{k+1} = r_{k+1} + q_k + p_{k+1} + p_k$, so that

$$c_{k+1} = b^{r_k+a_{k+1}a_{k+1}}(b^{a_{k+1}}a_{k+1})^{-1} - 1$$

$$= \frac{b^{r_k}a^{r_k} - b^{r_k+a_{k+1}a_{k+1}}a^{r_k}a_{k+1} + p_{k+1}}{b^{r_k+a_{k+1}a_{k+1}}a_{k+1} - 1} = -b^{r_k}a^{r_k} = -e_k - 1$$

is negative. Notice as well that $e_k$ is always positive, because $b \geq 2, a \geq 1, r_k \geq 1$, and that $d_k$ is non-negative and vanishes if and only if $t_k = t_k = 0$, that is to say when $b_1 = \ldots = b_k = 0$.

Keeping this in mind, and with some abuse of language, the next theorem asserts that

$$[0; c_0, d_0, 1, e_0, f_0, c_1, d_1, 1, e_1, f_1, c_2, \ldots]$$

is an (improper) continued fraction expansion of $\xi$. In order to rule out non-positive elements in the sequence

$$c_0, d_0, 1, e_0, f_0, c_1, d_1, 1, e_1, f_1, c_2, \ldots$$

we apply to it some contraction rules. The precise statement is as follows.

**Theorem 1.3.** Let $a$ and $b$ be positive integers. Assume that $b \geq 2$ and that $a$ is congruent to 1 modulo $b - 1$. Let $A_1, A_2, A_3, \ldots$ be the sequence of positive integers obtained from the sequence $c_0, d_0, 1, e_0, f_0, c_1, d_1, 1, e_1, f_1, c_2, \ldots$ after the application of the following rules:

(i) For any $k \geq 0$ such that $a_{k+2} = b_{k+2}$, replace the string of the 9 consecutive terms

$$c_k, d_k, 1, e_k, f_k = 0, c_{k+1} = -e_k - 1, d_{k+1} = d_k, 1, e_{k+1}$$

by the single element $c_k + e_{k+1} + 1$.

(ii) Replace any three consecutive elements of this new sequence of the form $x, 0, y$ by the integer $x + y$ ($x$ and $y$ may vanish) and continue the reduction until one obtains positive integers.

Then, the continued fraction expansion of $\xi$ is given by

$$\xi = [0; A_1, A_2, A_3, \ldots].$$
Observe that the sequence \((A_j)_{j \geq 1}\) is well-defined. Indeed, \(c_k\) and \(c_{k+1}\) cannot be both negative, since we cannot have simultaneously \(a_{k+1} = b_{k+1}\) and \(a_{k+2} = b_{k+2}\) by Ostrowski numeration rules. The process (ii) enables us to get rid of the 0 after having ruled out the negative terms using (i). The occurrences of 0, after performing the rule (i), are fully described thanks to the six cases displayed in Section 7 of [7], which remain unchanged in our setting. For convenience, we reproduce the list below.

(ii) \(b_k = 0\) and \(a_{k+2} = b_{k+2}\) with \(k \geq 1\), corresponding to the string 1,\(e_{k-1}, f_{k-1} = 0, c_k + e_{k+1} + 1, f_{k+1}\), where \(e_{k-1} > 0, c_k + e_{k+1} + 1 > 0, f_{k+1} > 0\).

(ii) \(b_{k+1} = 0, t_{k+1} \geq 1\) and \(a_{k+2} = b_{k+2} + 2\) with \(k \geq 0\), corresponding to the string 1,\(e_k, f_k = 0, c_{k+1}, d_{k+1}\), where \(e_k > 0, c_{k+1} > 0, d_{k+1} > 0\).

(ii) \(b_{k+1} \geq 1\) and \(a_{k+2} = b_{k+2} + 1\) with \(k \geq 0\), corresponding to the string \(e_k, f_k, c_{k+1} = 0, d_{k+1}, 1\), where \(e_k > 0, f_k > 0, d_{k+1} > 0\).

(ii) \(b_{k+1} = 0, t_{k+1} \geq 1\) and \(a_{k+2} = b_{k+2} + 1\) with \(k \geq 0\), corresponding to the string 1,\(e_k, f_k = 0, c_{k+1} = 0, d_{k+1}, 1\), where \(e_k > 0, d_{k+1} > 0\).

(ii) \(t_{k+1} = 0\) and \(a_{k+2} \geq b_{k+2} + 2\) with \(k \geq 0\), corresponding to the string 1,\(e_k, f_k = 0, c_{k+1}, d_{k+1} = 0, 1, e_{k+1}\), where \(e_k > 0, c_{k+1} > 0, e_{k+1} > 0\).

(ii) \(t_{k+1} = 0\) and \(a_{k+2} = b_{k+2} + 1\) with \(k \geq 0\), corresponding to the string 1,\(e_k, f_k = 0, c_{k+1} = 0, d_{k+1} = 0, 1, e_{k+1}\), where \(e_k > 0, e_{k+1} > 0\).

As a simple example, we obtain the

\[\xi = [0; c_0 + 1, e_0, f_0, c_1, d_1, 1, e_1, f_1, c_2, \ldots].\]

**Proof.** Observe that \(d_0 = 0\), while all the other elements of the sequence \(c_0, d_0, 1, e_0, f_0, c_1, d_1, 1, e_1, f_1, c_2, \ldots\) are positive. \qed

When \(a\) and \(b\) are positive integers with \(b \geq 2\) and \(a\) not congruent to 1 modulo \(b - 1\), we get the regular continued fraction expansion of \(1/\xi - (c_0 + 1) = 1/\xi - (b^{e_1} - b^a - 1)/(b - 1)\).

As a consequence of Theorem 1.3, we obtain an expression for the irrationality exponent of any real number \(\xi\) as above in terms of its slope and its intercept.

Keep our notation and define

\[\nu_k(1) = 2 + \frac{t_k}{r_{k+1}}, \quad \nu_k(2) = 2 + \frac{r_k}{r_{k+1} + t_k},\]

\[\nu_k(3) = 1 + \frac{q_{k+1}}{r_{k+1} + q_k}, \quad \nu_k(4) = 1 + \frac{r_{k+2}}{q_{k+1}}.\]
Put
\[ \nu(1) = \limsup_{k \to +\infty} \{\nu_k(1) : a_{k+1} - b_{k+1} \geq 1 \} \quad \text{and} \quad a_{k+2} - b_{k+2} \geq 1 \}, \]
\[ \nu(2) = \limsup_{k \to +\infty} \{\nu_k(2) : a_{k+2} - b_{k+2} \geq 1 \}, \]
and, for \( j = 3, 4, \)
\[ \nu(j) = \limsup_{k \to +\infty} \nu_k(j). \]

**Theorem 1.5.** Let \( b \geq 2 \) and \( a \geq 1 \) be integers. The irrationality exponent of \( \xi_{s_{\theta, \rho}}(1/b, 1/a) \) (resp., of \( \xi_{s_{\theta, \rho}}(1/b, 1/a) \)) is equal to
\[ \max\{\nu(1), \nu(2), \nu(3), \nu(4)\}. \]

Theorem 1.5 extends [7, Theorem 2.4] which covers the case \( a = 1 \).

2. STURMIAN WORDS

We collect in this Section some important properties of the Sturmian words \( s_{\theta, \rho} \) and \( s'_{\theta, \rho} \), obtained in [7]. Recall that \( (p_k/q_k)_{k \geq 0} \) is the sequence of convergents to \( \theta = [0; a_1, a_2, \ldots] \) and that the sequences \( (b_k)_{k \geq 1}, (r_k)_{k \geq 0}, (t_k)_{k \geq 0}, (\tilde{r}_k)_{k \geq 0}, (\tilde{t}_k)_{k \geq 0} \) are defined in (1.1) and (1.2).

**Lemma 2.1.** We have
\[ r_0 = 1, \quad \tilde{r}_0 = 0, \quad r_1 = a_1 - b_1, \quad \tilde{r}_1 = 1, \]
and the following recursion formulae hold for any \( k \geq 0 \):
\[ r_{k+1} = r_k + (a_{k+1} - b_{k+1} - 1)q_k + q_k - 1, \]
\[ \tilde{r}_{k+1} = \tilde{r}_k + (a_{k+1} - b_{k+1} - 1)p_k + p_k - 1. \]
It follows that
\[ r_{k+1} = 1 - q_k + \sum_{j=0}^{k} (a_{j+1} - b_{j+1})q_j = q_k + t_{k+1}, \quad k \geq 0, \]
\[ \tilde{r}_{k+1} = 1 - p_k + \sum_{j=0}^{k} (a_{j+1} - b_{j+1})p_j = p_k + \tilde{t}_{k+1}, \quad k \geq 0. \]

Moreover, we have \( 0 \leq t_k < q_k \), \( 0 \leq \tilde{t}_k \leq p_k \), \( 1 \leq r_k \leq q_k \), and \( 0 \leq \tilde{r}_k \leq p_k \), for every \( k \geq 0 \).

**Proof.** Notice that the classical recurrence relations \( q_{j+1} = a_{j+1}q_j + q_{j-1} \) and \( p_{j+1} = a_{j+1}p_j + p_{j-1} \) for any \( j \geq 0 \), arising from the theory of continued fractions, yield the formulae
\[ \sum_{j=0}^{k} a_{j+1}q_j = a_1 + \sum_{j=1}^{k} q_{j+1} - q_j - 1 = q_k + k - 1, \quad k \geq 0, \]
and
\[ \sum_{j=0}^{k} a_{j+1}p_j = \sum_{j=1}^{k} p_{j+1} - p_j - 1 = p_k + k - 1, \quad k \geq 0. \]
Then, we deduce the formulae of Lemma 2.1 from (1.2), (2.1) and (2.2). Notice finally that the Ostrowski numeration rules \((0 \leq b_1 \leq a_1 - 1, 0 \leq b_k \leq a_k, \text{ for } k \geq 1, \text{ and } b_{k+1} = a_{k+1} \text{ implies } b_k = 0, \text{ for every } k \geq 1)\) yield by induction on \(k\) the required inequalities. \(\square\)

Let

\[
c_{\theta} := s_{\theta, \theta} = s'_{\theta, \theta},
\]

be the characteristic word of slope \(\theta\). For \(k \geq 1\), we denote by \(M_k\) the prefix of length \(q_k\) of \(c_{\theta}\). Set \(M_0 = 0\) and \(M_{-1} = 1\).

**Proposition 2.2.** Define inductively two sequences of finite words \((T_k)_{k \geq 0}\) and \((R_k)_{k \geq 0}\) on \(\{0, 1\}\) by letting \(T_0\) be the empty word, \(R_0 = 0\), and by the recursion formulae

\[
T_{k+1} = M_k^{b_{k+1}} T_k
\]

and

\[
R_{k+1} = \begin{cases} R_k M_k^{a_{k+1}-b_{k+1}-1} M_{k-1} & \text{if } b_{k+1} < a_{k+1}, \\ R_{k-1} & \text{if } b_{k+1} = a_{k+1}, \end{cases}
\]

for any \(k \geq 0\). Then, \(T_k\) (resp. \(R_k\)) has length \(t_k\) (resp. \(r_k\)) and contains \(\tilde{t}_k\) (resp. \(\tilde{r}_k\)) letters 1. Set

\[
V_k = R_k T_k, \quad k \geq 0.
\]

The word \(V_k\) has length \(q_k\), contains \(p_k\) letters 1, and its first \(q_k - 1\) letters coincide with those of \(s_{\theta, \rho}\) (or \(s'_{\theta, \rho}\)). Moreover \(M_k = T_k R_k\) and the sequence \((V_k)_{k \geq 0}\) satisfies the recurrence relations

\[
V_{-1} = 1, \quad V_0 = 0, \quad V_1 = V_0^{a_1-b_1-1} V_{-1} V_0^{b_1}, \quad V_{k+1} = V_k^{a_{k+1}-b_{k+1}} V_{k-1} V_k^ {b_{k+1}}, \quad k \geq 1.
\]

**Proof.** Proposition 2.2 is a reformulation of the results of [7, Section 3], with the exception of the assertions concerning the number of letters 1. These follow from Lemma 2.1, combined with the recursion formulae (2.3) and (2.4) established in [7, Lemma 3.3], by observing that the word \(M_k\) contains \(p_k\) letters 1. Notice that when \(a_{k+1} = b_{k+1}\), we have \(b_k = 0\), so that

\[
\tilde{r}_k = \tilde{r}_{k-1} + (a_k - 1)p_{k-1} + p_{k-2} = \tilde{r}_{k-1} + p_k - p_{k-1},
\]

\[
\tilde{r}_{k+1} = \tilde{r}_k + (a_{k+1} - b_{k+1} - 1)p_{k} + p_{k-1} = \tilde{r}_k - p_k + p_{k-1} = \tilde{r}_{k-1}.
\]

It follows that \(R_{k+1} = R_{k-1}\) contains \(\tilde{r}_{k+1} = \tilde{r}_{k-1}\) letters 1, as claimed. \(\square\)

**Definition 2.3.** The sequence \((b_k)_{k \geq 1}\) is called the formal intercept of the Sturmian word \(s_{\theta, \rho}\) of slope \(\theta\) and intercept \(\rho\).

The next lemma will be used in Sections 3 and 5.

**Lemma 2.4.** As \(k\) tends to infinity, we have

\[
r_k \theta - \tilde{r}_k = O(1), \quad t_k \theta - \tilde{t}_k = O(1).
\]
Proof. Lemma 2.1 yields the formula
\[ r_k \theta - \tilde{r}_k = q_k \theta - p_k - \sum_{j=0}^{k-1} b_{j+1} (q_j \theta - p_j) \]
\[ = \theta - \rho + (q_k \theta - p_k) + \sum_{j \geq k} b_{j+1} (q_j \theta - p_j), \]
recalling the Ostrowski expansion
\[ \rho - \theta = \sum_{j \geq 0} b_{j+1} (q_j \theta - p_j). \]
This shows that \( r_k \theta - \tilde{r}_k = O(1) \). Since \( |q_k \theta - p_k| \leq 1 \), we get the second estimate. □

3. Continued fraction expansion

The main goal of this Section is to prove Theorem 1.3, and to give further results on the convergents of \( \xi \).

For a finite word \( W = w_1 \ldots w_\ell \) over the alphabet \( \{0, 1\} \) and variables \( a, b \), set
\[ W(b, a) = \sum_{n=1}^{\ell} w_n b^{\ell-n} a^{\sum_{h=n+1}^{\ell} w_h} = b^\ell a^{\sum_{h=1}^{\ell} w_h} \sum_{n=1}^{\ell} w_n (1/b)^n (1/a)^{\sum_{h=1}^{n} w_h}. \]
Note that the exponent \( \sum_{h=1}^{n} w_h \) counts the number of letters 1 in the prefix of length \( n \) of the word \( W \).

Now, if \( x = x_1 x_2 \ldots \) is an infinite word over the alphabet \( \{0, 1\} \), recall that we have set
\[ \xi_x(\beta, \alpha) = \sum_{n \geq 1} x_n \beta^n \alpha^{\sum_{h=1}^{n} x_h}. \]
When \( x \) is an ultimately periodic word, \( \xi_x(\beta, \alpha) \) is a rational function in the two variables \( \beta \) and \( \alpha \). Set \( a = 1/\alpha \) and \( b = 1/\beta \). More precisely, we have the

Lemma 3.1. Let \( Y = y_1 \ldots y_\ell \) and \( Z = z_1 \ldots z_\ell \) be two finite words over \( \{0, 1\} \). Put \( \tilde{r} = y_1 + \cdots + y_\ell \) and \( \tilde{s} = z_1 + \cdots + z_\ell \). Then
\[ YZ(b, a) = b^{\tilde{r}} a^{\tilde{s}} Y(b, a) + Z(b, a), \]
where \( YZ = y_1 \ldots y_\ell z_1 \ldots z_\ell \) stands for the concatenation of the two words \( Y \) and \( Z \). Moreover, if \( |b^{\tilde{r}} a^{\tilde{s}}| > 1 \), then
\[ \xi_{Z\infty}(\beta, \alpha) = \frac{Z(b, a)}{b^{\tilde{r}} a^{\tilde{s}} - 1} \quad \text{and} \quad \xi_{YZ\infty}(\beta, \alpha) = \frac{YZ(b, a) - Y(b, a)}{b^{\tilde{r}} a^{\tilde{s}} (b^{\tilde{r}} a^{\tilde{s}} - 1)}, \]
where \( Z\infty \) stands for the concatenation of infinitely many copies of \( Z \).

Proof. The first formula immediately follows from the definition.
By setting \( x = YZ^\infty \) and writing \( n = r + js + m \) for \( n \geq r + 1 \), we obtain by periodicity

\[
\xi_{x}(\beta, \alpha) = \sum_{n \geq 1} x_n \beta^n \alpha \sum_{h=1}^{n} x_h
\]

\[
= \sum_{n=1}^{r} y_n \beta^n \alpha \sum_{h=1}^{n} y_h + \sum_{j \geq 0} \sum_{m=1}^{s} z_m \beta^{r+j s + m} \alpha \sum_{h=1}^{m} z_h
\]

\[
= \sum_{n=1}^{r} y_n \beta^n \alpha \sum_{h=1}^{n} y_h + \beta^r \alpha^{r} \sum_{m=1}^{s} z_m \beta^{m} \alpha \sum_{h=1}^{m} z_h
\]

\[
= Y(b, a) \frac{b^s a^s - 1}{b^s a^s - 1} + Z(b, a)
\]

\[
= YZ(b, a) - Y(b, a)
\]

When \( Y \) is the empty word, we obtain the formula \( \xi_{Z^\infty}(\beta, \alpha) = \frac{Z(b, a)}{b^s a^s - 1} \).

We use Lemma 3.1 in order to construct rational fractions in \( a \) and \( b \) associated to four sequences of periodic words which approach the Sturmian word \( s = s_{\theta, \rho} \). For any \( k \geq 0 \), define

\[
(1)_k = \frac{(b - 1)(R_{k+1}(b, a) - R_k(b, a))}{b^k a^k ((b^k a^{k+1}-r_k b^k a^{k+1} - 1)},
\]

which is associated to the word \( R_k(M_k^{a_{k+1}-b_{k+1}-1} M_{k-1})^\infty \) whenever \( a_{k+1} - b_{k+1} \geq 1 \). Next, set

\[
(2)_k = \frac{(b - 1)(R_{k+1}T_k)(b, a)}{b^k a^{k+1} + t_k a^k a^{k+1} - 1},
\]

associated to the purely periodic word \( (R_{k+1}T_k)^\infty \). The third approximation is

\[
(3)_k = \frac{(b - 1)((R_{k+1}M_k)(b, a) - R_{k+1}(b, a))}{b^k a^{k+1} (b^k a^{k+1} - 1)},
\]

associated to the word \( R_{k+1}M_k^\infty \). Put finally

\[
(4)_k = \frac{(b - 1)V_{k+1}(b, a)}{b^k a^{k+1} (b^k a^{k+1} - 1)},
\]

associated to the purely periodic word \( V_{k+1}^\infty = (R_{k+1}T_{k+1})^\infty \).

We now give an analogue of [7, Lemma 7.1] in our framework. We use the notation \( \frac{P}{Q} = c \cdot \frac{P'}{Q'} + \frac{P''}{Q''} \) between fractions to mean that both relations \( P = cP' + P'' \) and \( Q = cQ' + Q'' \) hold true. Similarly, \( (2)_k + (1)_k \) stands below for the fraction whose numerator (resp. denominator) is the difference between the numerators (resp. denominators) of \( (2)_k \) and \( (1)_k \). It is convenient to define formally \( (3)_{-1} = \frac{b-1}{a} \) and \( (4)_{-1} = \frac{0}{a-1} \). Then, we have the

**Lemma 3.2.** For any \( k \geq 0 \), we have the following relations:

\[
(1)_k = c_k \cdot (4)_{k-1} + (3)_{k-1},
\]
\[(2)_k \cdot (1)_k = d_k \cdot (1)_k + (4)_k - 1, \]
\[(2)_k = 1 \cdot ((2)_k \cdot (1)_k) + (1)_k, \]
\[(3)_k = e_k \cdot (2)_k + (2)_k \cdot (1)_k, \]
\[(4)_k = f_k \cdot (3)_k + (2)_k. \]

**Proof.** We compute
\[(1)_0 = \frac{b - 1}{b^2 - b_1 a}, \quad (2)_0 \cdot (1)_0 = \frac{0}{b - 1}, \quad (2)_0 = \frac{b - 1}{b^2 - b_1 a - 1}, \]
\[(3)_0 = \frac{(b - 1)^2}{b^2 - b_1 a(2 - 1)}, \quad (4)_0 = \frac{b^2 (b - 1)}{b^2 a - 1}. \]

We have
\[c_0 = \frac{b^2 - b_1 a - b}{b - 1}, \quad d_0 = 0, \quad e_0 = b - 1, \quad f_0 = \frac{b^2 - 1}{b - 1}, \]
so that the five above relations are verified for \(k = 0\).

Assume now that \(k \geq 1\). The third relation is obvious. Let us check the four remaining relations. The denominators of \((1)_k, (2)_k \cdot (1)_k, (2)_k, (3)_k, (4)_k\) are respectively
\[Q_{(1)}(k) = b^{r_k+1}a^{\tilde{r}_k+1} - b^{r_k}a^{\tilde{r}_k}, \quad Q_{(2)}(1)_k = b^{r_k+1+t_k}a^{\tilde{r}_k+1+\tilde{t}_k} - 1 - (b^{r_k+1}a^{\tilde{r}_k+1} - b^{r_k}a^{\tilde{r}_k}), \]
\[Q_{(2)}(2)_k = b^{r_k+1+t_k}a^{\tilde{r}_k+1+\tilde{t}_k} - 1, \quad Q_{(3)}(3)_k = b^{r_k+1}a^{\tilde{r}_k+1}(b^{q_k}a^{p_k} - 1), \quad Q_{(4)}(4)_k = b^{q_k+1}a^{p_k+1} - 1. \]

Using Lemma 2.1, we check that
\[Q_{(1)}(k) - Q_{(3)}_{(k+1)} = \frac{b^{r_k+1}a^{\tilde{r}_k+1} - b^{r_k}a^{\tilde{r}_k} - b^{r_k+q_k-1}a^{\tilde{r}_k+p_k-1} + b^{r_k}a^{\tilde{r}_k}}{b^{q_k}a^{p_k} - 1} = c_k, \]
where
\[c_k = \frac{Q_{(2)}(k) - Q_{(1)}(1)_k - Q_{(4)}_{(k+1)} - Q_{(1)}(k)}{Q_{(1)}(k)}. \]

As required. Similarly, we have to check that \(d_k = \frac{Q_{(2)}(k) - Q_{(1)}(k) - Q_{(4)}_{(k+1)}}{Q_{(1)}(k)}\), or equivalently \(d_k + 1 = \frac{Q_{(2)}(k) - Q_{(4)}_{(k+1)}}{Q_{(1)}(k)}\). Now,
\[Q_{(2)}(2)_k - Q_{(4)}_{(k+1)} = b^{r_k+1+t_k}a^{\tilde{r}_k+1+\tilde{t}_k} - 1 - (b^{q_k}a^{p_k} - 1) = b^{r_k}a^{\tilde{r}_k} = d_k + 1, \]
\[Q_{(3)}(3)_k + Q_{(1)}(1)_k = b^{r_k+1+q_k}a^{\tilde{r}_k+1+p_k} - b^{r_k}a^{\tilde{r}_k} = b^{r_k}a^{\tilde{r}_k} = e_k + 1, \]
writing again \(q_k = r_k + t_k\) and \(p_k = \tilde{r}_k + \tilde{t}_k\). For the fifth relation, we have to show that \(e_k = \frac{Q_{(2)}(k) + Q_{(1)}(k)}{Q_{(2)}(k)}\), or equivalently that \(e_k + 1 = \frac{Q_{(2)}(k) + Q_{(1)}(k)}{Q_{(2)}(k)}\). But
\[Q_{(2)}(4)_k - Q_{(2)}(3)_k = \frac{b^{q_k+1}a^{p_k+1} - b^{r_k+1+t_k}a^{\tilde{r}_k+1+\tilde{t}_k}}{b^{q_k+1}a^{p_k+1}(b^{q_k}a^{p_k} - 1)} = b^{r_k}a^{\tilde{r}_k} = f_k, \]
where
since
\[ q_{k+1} = r_{k+1} + t_{k+1} = r_k + t_k + b_{k+1}q_k \]
and
\[ p_{k+1} = \tilde{r}_{k+1} + \tilde{t}_{k+1} = \tilde{r}_k + \tilde{t}_k + b_{k+1}p_k. \]
It remains to deal with the numerators. For the first relation, we have to show that
\[ c_k V_k(b, a) + R_k M_{k-1}(b, a) - R_k(b, a) = R_{k+1}(b, a) - R_k(b, a). \]
Assume first that \( a_{k+1} - b_{k+1} \geq 1 \). Using Lemma 3.1, Proposition 2.2 and noting that \((R_k T_k)^{a_{k+1} - b_{k+1} - 1} R_k M_{k-1} = R_{k+1}\) by (2.4), we compute
\[
c_k V_k(b, a) = V_k(b, a) \times b^r k^{q_k - a^r k + p_k} (b^{q_k a^p k})^{a_{k+1} - b_{k+1} - 1} - 1
\]
and
\[
\begin{align*}
\text{as required. Assume finally that } a_{k+1} &= b_{k+1}. \hspace{1cm} &
\text{Then, } c_k &= -b^r - a^r \tilde{k} - 1 \text{ and we have}

V_k(b, a) \times (-b^r - a^r \tilde{k} - 1) &= -V_k R_{k-1}(b, a) + R_{k-1}(b, a)
\]
\[
= -R_k T_k R_{k-1}(b, a) + R_{k-1}(b, a)
\]
\[
= -R_k M_{k-1}(b, a) + R_{k+1}(b, a),
\]
since \( T_k = T_{k-1} \) in that case, thanks to (2.3).

The second relation for the numerators reads
\[
(d_k + 1)(R_k b_{k+1}(b, a) - R_k(b, a)) = R_{k+1} T_k(b, a) - V_k(b, a).
\]
To that purpose, write
\[
b^{q_k} a^{q_k} (R_{k+1}(b, a) - R_k(b, a))
\]
\[
= b^{q_k} a^{q_k} R_{k+1}(b, a) + T_k(b, a) - (b^{q_k} a^{q_k} R_k(b, a) + T_k(b, a))
\]
\[
= R_{k+1} T_k(b, a) - R_k T_k(b, a) = R_{k+1} T_k(b, a) - V_k(b, a),
\]
The third relation for the numerators is obvious, while the fourth writes
\[
(e_k + 1) R_{k+1} T_k(b, a) = R_{k+1} M_k(b, a) - R_{k+1}(b, a) + (R_{k+1}(b, a) - R_k(b, a))
\]
\[
= R_{k+1} M_k(b, a) - R_k(b, a),
\]
which follows from the equalities
\[
b^{r_k} a^{q_k} R_{k+1} T_k(b, a) + R_k(b, a) = R_{k+1} T_k R_k(b, a) = R_{k+1} M_k(b, a),
\]
by Lemma 3.1 (with \( Z = R_k \) and \( Y = R_{k+1} T_k \)) and Proposition 2.2. The fifth relation writes
\[
f_k ((R_{k+1} M_k)(b, a) - R_{k+1}(b, a)) = V_{k+1}(b, a) - (R_{k+1} T_k)(b, a).
\]
Notice that
\[
V_{k+1} = R_{k+1} T_{k+1} = R_{k+1} M_k^{b_{k+1}} T_k,
\]
so that
\[
V_{k+1}(b,a)-(R_{k+1}T_k)(b,a)
= (V_{k+1}(b,a) - T_k(b,a)) - ((R_{k+1}T_k)(b,a) - T_k(b,a))
= b^k a^k ((R_{k+1}M_k^{b_{k+1}})(b,a) - R_{k+1}(b,a)),
\]
thanks to Lemma 3.1. Now, we can write
\[
(R_{k+1}M_k^{b_{k+1}})(b,a) - R_{k+1}(b,a)
= \sum_{j=0}^{b_{k+1}+1-1} ((R_{k+1}M_k^{j+1})(b,a) - (R_{k+1}M_k^j)(b,a))
= \left( \sum_{j=0}^{b_{k+1}+1-1} b^{jq_k} a^{jp_k} \right) ((R_{k+1}M_k)(b,a) - R_{k+1}(b,a)),
\]
by factoring $M_k^j$ on the right and applying again Lemma 3.1. The fifth relation immediately follows and Lemma 3.2 has been fully checked.

We have now all the tools to prove Theorems 1.3 and 1.5.

**Proof of Theorem 1.3.** At this stage, the proof of Theorem 1.3 follows the argumentation of Section 7 in [7]. We briefly take it again.

Let us number $\alpha_1, \alpha_2, \ldots$ the cyclic sequence
\[
e_0, d_0, 1, e_0, f_0, c_1, d_1, 1, e_1, f_1, c_2, d_2, 1, e_2, f_2, \ldots
\]
and define two sequences $(P_j)_{j \geq -1}$ and $(Q_j)_{j \geq -1}$ by the recurrence relations
\[
P_{-1} = b - 1, \quad P_0 = 0, \quad P_j = \alpha_j P_{j-1} + P_{j-2}, \quad j \geq 1,
\]
\[
Q_{-1} = 0, \quad Q_0 = b - 1, \quad Q_j = \alpha_j Q_{j-1} + Q_{j-2}, \quad j \geq 1.
\]
Equivalently, we have the matrices equalities
\[
\begin{pmatrix} Q_j \\ P_j \end{pmatrix} = \begin{pmatrix} b-1 & 0 \\ 0 & b-1 \end{pmatrix} \begin{pmatrix} \alpha_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} \alpha_j & 1 \\ 1 & 0 \end{pmatrix}, \quad j \geq 1.
\]
Lemma 3.2 tells us that the sequence $(P_j)_{j \geq 1}$ (resp. $(Q_j)_{j \geq 1}$) coincide with the sequence of numerators (resp. denominators) of
\[
(1)_0, (2)_0, \ldots \ldots (1)_0, (2)_0, (3)_0, (4)_0, (1)_1, (2)_1, (3)_1, (4)_1, \ldots .
\]
Now, we reduce by associativity the (formal) infinite product
\[
\begin{pmatrix} \alpha_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_3 & 1 \\ 1 & 0 \end{pmatrix} \cdots ,
\]
thanks to the processes (i) and (ii) of Theorem 1.3. For shortness, write
\[
E(\alpha) = \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix}.
\]

(i) If $a_{k+2} = b_{k+2}$, we reduce the product of the nine consecutive factors :
\[
E(c_k)E(d_k)E(1)E(e_k)E(f_k)E(c_{k+1})E(d_{k+1})E(1)E(e_{k+1}) = E(c_k)E(d_k)E(1)E(e_k)E(0)E(-e_k - 1)E(d_k)E(1)E(e_{k+1}) = E(c_k + e_{k+1} + 1).
\]
If $c_k = \alpha_i$, we thus have
\[
\begin{pmatrix}
Q_{l+8} & Q_{l+7} \\
P_{l+8} & P_{l+7}
\end{pmatrix}
= \begin{pmatrix}
Q_{l-1} & Q_{l-2} \\
P_{l-1} & P_{l-2}
\end{pmatrix}
\begin{pmatrix}
(c_k + e_{k+1} + 1) & 1 \\
1 & 0
\end{pmatrix},
\]
and we jump from $\frac{P_{l-1}}{Q_{l-1}} = (4)_{k-1}$ to $\frac{P_{l+8}}{Q_{l+8}} = (3)_{k+1}$, thanks to the elementary matrix $E(c_k + e_{k+1} + 1)$.

(ii) Reduction (i) enables us to transform the infinite product $E(\alpha_1)E(\alpha_2) \cdots$ into the product
\[
E(\alpha_1)E(\alpha_2) \cdots = E(\alpha'_1)E(\alpha'_2) \cdots,
\]
where $\alpha'_1, \alpha'_2, \ldots$ are non-negative. We may encounter some zeroes (these precisely occur in the six cases displayed after Theorem 1.3). If $\alpha'_i = 0$ say, we replace $\alpha'_{i-1}, 0, \alpha'_i$ by $\alpha'_{i-1} + \alpha'_i$, thanks to the matrices equality
\[
E(\alpha'_{i-1})E(0)E(\alpha'_i) = E(\alpha'_{i-1} + \alpha'_i).
\]

We finally end up with a product
\[
E(\alpha_1)E(\alpha_2) \cdots = E(A_1)E(A_2) \cdots
\]
where $A_1, A_2, \ldots$ are positive. Moreover, every convergent $[0; A_1, \ldots, A_n]$ equals one of the five fractions $(1)_k, (2)_k, (3)_k, (4)_k$, and infinitely many of these convergents are of the form $(j)_k$ for some $1 \leq j \leq 4$. Now, by Lemma 3.1, $(j)_k = (b-1)\xi R_{k+1} \cdots (1/b, 1/a)$ for some ultimately periodic word $R_{k+1} \cdots$ sharing a large prefix with $s_{\theta, \rho}$ (or $s'_{\theta, \rho}$). It follows that
\[
\xi = [0, A_1, A_2, \ldots].
\]

Since $a$ and $b$ are integers, observe that the numbers
\[
c_0, d_0, 1, e_0, f_0, c_1, d_1, 1, e_2, f_2, \ldots
\]
are integers, except possibly $c_0 = \frac{\alpha_1 - b + a - b}{b-1}$. But $c_0$ is a non-negative integer if we assume that $a$ is congruent to 1 modulo $b - 1$. Then, the $A_j$ are positive integers and $A_1, A_2, \ldots$ is the sequence of partial quotients of $\xi$. Theorem 1.3 is proved. \hfill $\Box$

**Proof of Theorem 1.5.** Assume first that $a$ is congruent to 1 modulo $b - 1$. Let $\xi$ denote one of the numbers $(b-1)\xi_{s_{\theta, \rho}}(1/b, 1/a)$ or $(b-1)\xi'_{s'_{\theta, \rho}}(1/b, 1/a)$ and let $\xi'$ be the corresponding number with $a = 1$. We denote by $(P_j/Q_j)_{j \geq 1}$ (resp. $(P'_j/Q'_j)_{j \geq 1}$) the sequence of convergents to $\xi$ (resp. $\xi'$). We claim that
\[
Q'_j \gg Q_j^\varphi, \quad \text{where } \varphi = \frac{\log b}{\log ba^8},
\]
as $j$ tends to infinity. Indeed, it follows from the proof of Theorem 1.3 that each convergent $P_j/Q_j$ coincides with one of the fractions
\[
(1)_k, (2)_k, (3)_k, (4)_k,
\]
for some $k$. Moreover, if $P_j/Q_j = (3)_k$ say, then $P'_j/Q'_j = (3)'_k$, where $(3)'_k$ stands for the corresponding fraction with $a = 1$, observing that the reductions (i) and
(ii) occurring in Theorem 1.3 are independent of $a$ and $b$ (they depend only on the two sequences $(a_k)_{k \geq 1}$ and $(b_k)_{k \geq 1}$). It follows that
\[(ba^\theta)^{r_{k+1}+q_k} \ll Q_j = \frac{b^{r_{k+1}a^{r_{k+1}}(b^\theta a^{p_k} - 1)}}{b - 1} \ll (ba^\theta)^{r_{k+1}+q_k},\]
by using Lemma 2.4, while
\[b^{r_{k+1}+q_k} \ll Q'_j = \frac{b^{r_{k+1}}(b^\theta - 1)}{b - 1} \leq b^{r_{k+1}+q_k}.\]

Then, (3.1) holds true in this case. The other cases are similar.

Now, the theory of continued fractions and (3.1) yield that the irrationality exponents $\mu(\xi)$ and $\mu(\xi')$ of $\xi$ and $\xi'$ are equal, since they are given by the formulae
\[
\mu(\xi) = 1 + \limsup_{j \to +\infty} \frac{\log Q_{j+1}}{\log Q_j} = 1 + \limsup_{j \to +\infty} \frac{\log Q'_{j+1}}{\log Q'_j} = \mu(\xi').
\]
As already mentioned, Theorem 1.5 holds true for $\xi'$, and thus for $\xi$, by [7, Theorem 2.4].

If $a$ is not assumed to be congruent to 1 modulo $b - 1$, then $A_1$ is a positive rational number whose denominator divides $b - 1$, while $A_2,A_3,\ldots$ are positive integers. Define
\[P_j/Q_j = [0; A_1,\ldots, A_j], \quad j \geq 1,
\]
or equivalently
\[
\begin{pmatrix} Q_j & Q_{j-1} \\ P_j & P_{j-1} \end{pmatrix} = \begin{pmatrix} A_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} A_j & 1 \\ 1 & 0 \end{pmatrix}, \quad j \geq 1.
\]
Then, the $P_j$ are integers and the $Q_j$ are rational numbers with denominators dividing $b - 1$. The sequence $(P_j/Q_j)_{j \geq 1}$ does not necessarily coincide with the sequence of convergents to $\xi$. However, the inequalities
\[
|\xi - P_j/Q_j| \leq \frac{1}{Q_jQ_{j+1}}, \quad j \geq 1,
\]
remain true, and the above argumentation remains valid. \qed

4. Functional equations and expansions of Hecke-Mahler series

We give in this Section analytical formulae involving Hecke-Mahler series which will reveal to be useful for the proof of Theorem 1.1.

Let us begin with a relation between the fractions $(3)_k$ and $(4)_k$. Here and unless otherwise stated, we consider $\alpha$ and $\beta$ as variables and work in the ring of power series $\mathbb{Q}[[\alpha, \beta]]$. We have
\[
(4.1) \quad (3)_k = (4)_{k-1} + (-1)^k \left( \frac{1}{\beta} - 1 \right) \frac{\beta^{r_{k+1}+q_k} \alpha^{r_{k+1}+p_k}}{1 - \beta^{q_k} \alpha^{p_k}}.
\]

We stress that here and in this section the $+$ sign between fractions denotes the usual addition and not the Farey addition, denoted by $\dot{+}$ in the previous section.
The equality (4.1) is proved in [6, Lemma 2.2], but only with the case \( \alpha = 1 \). The general case is similar.

For \( k \geq 0 \), set \( v_k = V_k^{\infty} \),

\[
\sigma_k = \sum_{n=1}^{q_k} s_n \beta^n \alpha^{\sum_{h=1}^{n} s_h}, \quad \text{and} \quad \gamma_k = \beta^{q_k} \alpha^{p_k},
\]

so that

\[
\xi_{v_k}(\beta, \alpha) = \frac{\sigma_k}{1 - \gamma_k} = \frac{(4)_k - 1}{\beta - 1},
\]

for \( k \geq 1 \). We have

\[
\sigma_0 = 0, \quad \sigma_1 = \beta^{a_1 - b_1} \alpha, \quad \gamma_0 = \beta, \quad \gamma_1 = \beta^{a_1} \alpha.
\]

The recursion relations between the words \( V_k \) yield the

**Lemma 4.1.** For any \( k \geq 1 \), the numerators \( \sigma_k \) satisfy the linear recurrence relation

\[
\sigma_{k+1} = \frac{1 - \gamma_{k+1} - \gamma_k^{a_{k+1} - b_{k+1}}(1 - \gamma_{k-1})}{1 - \gamma_k} \sigma_k + \gamma_k^{a_{k+1} - b_{k+1}} \sigma_{k-1}.
\]

It follows that

\[
(4)_k - (4)_{k-1} = (-1)^k \frac{\alpha \beta \left( \frac{1}{\beta} - 1 \right)^2 \prod_{h=0}^{k} \gamma_h^{a_h+b_h+1}}{(1 - \gamma_{k+1})(1 - \gamma_k)}
\]

\[
= (-1)^k \frac{\alpha \beta \left( \frac{1}{\beta} - 1 \right)^2 \beta^{\sum_{h=0}^{k} (a_h+b_h+1)q_h \alpha \sum_{h=0}^{k} (a_h+b_h+1)\gamma_h}}{(1 - \beta^{q_k+1} \alpha^{p_k+1})(1 - \beta^{q_k} \alpha^{p_k})}
\]

\[
= (-1)^k \left( \frac{1}{\beta} - 1 \right)^2 \frac{\beta^{r_{k+1}+q_k \alpha \gamma_{k+1}+p_k} - \beta^{r_{k+2}+q_k+1 \alpha \gamma_{k+2}+p_k} + (1 - \beta^{q_k} \alpha^{p_k})}{(1 - \beta^{q_k+1} \alpha^{p_k+1})(1 - \beta^{q_k} \alpha^{p_k})}.
\]

and that

\[
(3)_{k+1} - (3)_k = (-1)^k \left( \frac{1}{\beta} - 1 \right)^2 \frac{\beta^{r_{k+1}+q_k \alpha \gamma_{k+1}+p_k} - \beta^{r_{k+2}+q_k+1 \alpha \gamma_{k+2}+p_k} + (1 - \beta^{q_k} \alpha^{p_k})}{(1 - \beta^{q_k+1} \alpha^{p_k+1})(1 - \beta^{q_k} \alpha^{p_k})}.
\]

**Proof.** Note that \( V_k \) has length \( q_k \) and contains \( p_k \) letters \( 1 \). Then, we deduce from the word equation

\[
V_{k+1} = V_k^{a_{k+1} - b_{k+1}} V_k^{b_{k+1}}
\]

the equality

\[
\sigma_{k+1} = (1 + \gamma_k + \cdots + \gamma_k^{a_{k+1} - b_{k+1} - 1}) \sigma_k + \gamma_k^{a_{k+1} - b_{k+1} + 1} \sigma_{k-1}
\]

\[
+ \gamma_k^{a_{k+1} - b_{k+1}} \gamma_k (1 + \cdots + \gamma_k^{b_{k+1} - 1}) \sigma_k
\]

\[
= 1 - \frac{\gamma_k^{a_{k+1} - b_{k+1} + 1} \gamma_k (1 + \cdots + \gamma_k^{b_{k+1} - 1}) \sigma_k + \gamma_k^{a_{k+1} - b_{k+1} + 1} \sigma_{k-1}}{1 - \gamma_k}
\]

\[
= 1 - \frac{\gamma_k^{a_{k+1} - b_{k+1} + 1} (1 - \gamma_{k-1}) \sigma_k + \gamma_k^{a_{k+1} - b_{k+1} + 1} \sigma_{k-1}}{1 - \gamma_k}.
\]
since $\gamma_{k+1}^a \gamma_{k-1}^b = \gamma_{k+1}^a$. Observe now that the denominators $1 - \gamma_k$ satisfy obviously the same linear relation

$$1 - \gamma_{k+1} = \frac{1 - \gamma_{k+1} - \gamma_k^{a_k+1} - b_{k+1}(1 - \gamma_{k-1})}{1 - \gamma_k} (1 - \gamma_k) + \gamma_k^{a_k+1} - b_{k+1}(1 - \gamma_{k-1}).$$

It follows that

$$(1 - \gamma_k) \sigma_{k+1} - (1 - \gamma_{k+1}) \sigma_k = -\gamma_k^{a_k+1} - b_{k+1}((1 - \gamma_{k-1}) \sigma_k - (1 - \gamma_k) \sigma_{k-1}).$$

Going down inductively to $k = 1$, we obtain

$$(1 - \gamma_k) \sigma_{k+1} - (1 - \gamma_{k+1}) \sigma_k = (-1)^k ((1 - \gamma_0) \sigma_1 - (1 - \gamma_1) \sigma_0) \prod_{h=1}^k \gamma_h^{a_h+1} - b_{h+1} = (-1)^k \left( \frac{1}{\beta} - 1 \right) \beta^{r_{k+1}q_k} \alpha \tilde{r}_{k+1} + p_k,$$

by using Lemma 2.1 and noting that $\sigma_0 = 0$ and $\sigma_1 = \beta^{a_1-b_1} \alpha = \gamma_0^{a_1-b_1} \alpha$. The formulae for $(4)_k - (4)_{k-1}$ immediately follow. For the difference $(3)_{k+1} - (3)_k$, we use moreover the equality (4.1) to obtain

$$(3)_{k+1} - (3)_k =$$

$$(4)_k - (4)_{k-1} + (-1)^k \left( \frac{1}{\beta} - 1 \right)^2 \left( \frac{\beta^{r_{k+2}q_k+1} \alpha \tilde{r}_{k+2} + p_k}{1 - \beta^{q_k+1} \alpha p_k} - \frac{\beta^{r_{k+1}q_k} \alpha \tilde{r}_{k+1} + p_k}{1 - \beta^{q_k} \alpha p_k} \right)$$

$$= (-1)^k \left( \frac{1}{\beta} - 1 \right)^2 \frac{\beta^{r_{k+2}q_k+1} \alpha \tilde{r}_{k+2} + p_k}{(1 - \beta^{q_k} \alpha p_k)(1 - \beta^{q_k+1} \alpha p_k + 1)} \left( 1 - \beta^{q_k+1} \alpha p_k \right) + \beta^{r_{k+1}q_k} \alpha \tilde{r}_{k+1} + p_k \left( 1 - \beta^{q_k} \alpha p_k \right)$$

$$= (-1)^k \left( \frac{1}{\beta} - 1 \right)^2 \frac{\beta^{r_{k+1}q_k+1} \alpha \tilde{r}_{k+1} + p_k}{(1 - \beta^{q_k} \alpha p_k)(1 - \beta^{q_k+1} \alpha p_k + 1)} \left( 1 - \beta^{q_k} \alpha p_k \right) \left( 1 - \beta^{q_k+1} \alpha p_k + 1 \right).$$

The proof is complete.

\(\square\)

**Corollary 4.2.** We have the following formulae for the Hecke-Mahler series

$$\xi_{\theta, \rho}(\beta, \alpha) = (1 - \beta)\alpha \sum_{k \geq 0} (-1)^k \prod_{h=0}^k \frac{\gamma_h^{a_h+1} - b_{h+1}}{1 - \gamma_{k+1}} (1 - \gamma_k)$$

$$= (1 - \beta)\alpha \sum_{k \geq 0} (-1)^k \frac{\beta^{\sum_{h=0}^k (a_h+1-b_{h+1})} \alpha \sum_{h=0}^k (a_h+1-b_{h+1}) p_h}{(1 - \beta^{q_k+1} \alpha p_k + 1)(1 - \beta^{q_k} \alpha p_k)}$$

$$= \frac{1 - \beta}{\beta} \sum_{k \geq 0} (-1)^k \frac{\beta^{r_{k+1}q_k} \alpha \tilde{r}_{k+1} + p_k}{(1 - \beta^{q_k+1} \alpha p_k + 1)(1 - \beta^{q_k} \alpha p_k)}.$$
and
\[ \xi_{s_{\theta_p}}(\beta, \alpha) = \frac{1 - \beta}{\beta} \left( \frac{\beta^{r_1+1} \alpha - \beta^{r_2+q_1} \alpha^{\tilde{r}_2+p_1} (1 - \beta^{q_0} \alpha^p_0)}{(1 - \beta^{q_1} \alpha^p_1) (1 - \beta^{q_0} \alpha^p_0)} \right) + \sum_{k \geq 1} (-1)^k \frac{\beta^{r_{k+1}+q_{k+1}+p_{k+1} + p_k} - \beta^{r_{k+1}+q_{k+1}+p_{k+1}+p_{k+1}} (1 - \beta^{q_k} \alpha^p_k)}{(1 - \beta^{q_k} \alpha^p_k) (1 - \beta^{q_k} \alpha^p_k)}. \]

Proof. We use the telescopic sums
\[ \xi_{s_{\theta_p}}(\beta, \alpha) = \frac{(4)_0}{\beta - 1} + \sum_{k \geq 1} \frac{(4)_k - (4)_{k-1}}{\beta - 1} = \frac{(3)_1}{\beta - 1} + \sum_{k \geq 1} \frac{(3)_{k+1} - (3)_k}{\beta - 1}, \]
which, combined with Lemma 4.1, give rise to the terms in the sums with index \( k \geq 1 \). It remains to compute \((4)/\left(\frac{1}{\beta} - 1\right)\) and \((3)/\left(\frac{1}{\beta} - 1\right)\). We have the equalities
\[ \frac{(4)_0}{\beta - 1} = \frac{\sigma_1}{1 - \gamma_1} = \frac{\alpha \beta a_1 - b_1}{1 - \alpha \beta a_1} = \alpha (1 - \beta) \frac{\gamma_0^{a_1 - b_1}}{(1 - \gamma_0)(1 - \gamma_1)} = \frac{1 - \beta}{\beta} \frac{\beta^{r_1+q_0} \alpha^{r_1+p_0}}{(1 - \beta^{q_0} \alpha^p_0)}, \]
which establish the three first expressions for \( \xi_{s_{\theta_p}}(\beta, \alpha) \).

We now deal with
\[ \frac{(3)_1}{\beta - 1} = \xi_{R_2 M_1^\infty}(\beta, \alpha). \]
Assume first that \( a_2 - b_2 \geq 1 \). Then,
\[ R_2 M_1^\infty = R_1 M_1^{a_2 - b_2 - 1} M_0 M_1^\infty = 0^{a_1 - b_1 - 1} (0^{a_1 - 1})^{a_2 - b_2 - 1} 1 (0^{a_2 - 1})^\infty. \]
It follows that
\[ \xi_{R_2 M_1^\infty}(\beta, \alpha) = \beta^{a_1 - b_1} \alpha \left( 1 + \beta^{a_1} \alpha + \cdots + (\beta^{a_1} \alpha)^{a_2 - b_2 - 1} \right) \]
\[ + \beta^{a_1 - b_1 + a_1 (a_2 - b_2 - 1)} \alpha^{a_2 - b_2} \frac{\beta^{a_1} \alpha}{1 - \beta^{a_1} \alpha} = \frac{\beta u_1 \alpha v_1 - \beta u_2 \alpha v_2 + \beta u_3 \alpha v_3}{1 - \beta^{a_1} \alpha}, \]
with
\[ u_1 = a_1 - b_1 = r_1, \quad v_1 = 1, \]
\[ u_2 = a_1 - b_1 + a_1 (a_2 - b_2) = r_2 + q_1 - 1, \quad v_2 = a_2 - b_2 + 1 = \tilde{r}_2 + p_1, \]
\[ u_3 = a_1 a_2 + 1 + a_1 - b_1 - b_2 a_1 = r_2 + q_1, \quad v_2 = a_2 - b_2 + 1 = \tilde{r}_2 + p_1. \]
Thus,
\[ \frac{(3)_1}{\beta - 1} = \frac{1 - \beta}{\beta} \frac{\beta^{r_1+1} \alpha - \beta^{r_2+q_1} \alpha^{\tilde{r}_2+p_1} (1 - \beta)}{(1 - \beta^{a_1} \alpha)(1 - \beta)}. \]
as asserted. The fourth expression for \( \xi_{\theta, \rho}(\beta, \alpha) \) is established when \( a_2 - b_2 \geq 1 \). In the case \( a_2 = b_2 \), we have \( R_2 = 0, r_1 = a_1 \). The computations are similar and simpler.

\( \Box \)

As an example, for the characteristic Sturmian word \( c_\theta \) we have \( b_k = 0 \) for every \( k \geq 1 \). Then, it follows from (2.1) and (2.2) that

\[
\sum_{h=0}^{k} (a_{h+1} - b_{h+1}) q_h = q_{k+1} + q_k - 1 \quad \text{and} \quad \sum_{h=0}^{k} (a_{h+1} - b_{h+1}) p_h = p_{k+1} + p_k - 1.
\]

Thus, we recover the known formula

\[
\xi_{c_\theta}(\beta, \alpha) = \left( \frac{1}{\beta} - 1 \right) \sum_{k \geq 0} (-1)^k \frac{\beta^{q_{k+1}+q_k} \alpha^{p_{k+1}+p_k}}{(1 - \beta^{q_{k+1}+q_k}) (1 - \beta^{q_k} \alpha^{p_k})},
\]

which is usually obtained as a consequence of the functional equation for the Hecke-Mahler series.

Conversely, a functional chain of equations of Mahler’s type can be deduced from our formula for an arbitrary Sturmian word \( s \). For \( m \geq 0 \), put \( \theta_m = [0, a_{m+1}, a_{m+2}, \ldots] \) and denote by \( s_m \) the Sturmian word with slope \( \theta_m \) and formal intercept \( b_{m+1}, b_{m+2}, \ldots \) (see Definition 2.3). Observe that \( s = s_0 \). With our notation, we have \( \xi_{s}(\beta, \alpha) = \xi_{s_0}(\gamma_0, \gamma_{-1}) \), where \( \gamma_{-1} = \beta^{-1} \alpha^{p_{-1}} = \alpha \).

**Proposition 4.3.** For any \( m \geq 1 \), we have the relation of Mahler’s type

\[
\xi_{s}(\beta, \alpha) = (1 - \beta) \alpha \sum_{k=0}^{m-1} (-1)^k \beta^{\sum_{h=0}^{k} (a_{h+1} - b_{h+1}) q_h} \alpha^{\sum_{h=0}^{k} (a_{h+1} - b_{h+1}) p_h} \\
\times \xi_{s_m}(\beta^{q_{m-1}+q_{m}} \alpha^{p_{m-1}+p_{m-1}}, \frac{1 - \beta^{q_{m-1}+q_{m}}} {1 - \beta^{q_{m}} \alpha^{p_{m-1}}}, \frac{1 - \beta^{q_{m}} \alpha^{p_{m-1}}}{1 - \beta^{q_{m}} \alpha^{p_{m-1}}}, \xi_{s_m}(\gamma_{m}, \gamma_{m-1})}
\]

\[= (1 - \beta) \alpha \left( \sum_{k=0}^{m-1} (-1)^k \sum_{h=0}^{k} \gamma_h \right) \alpha^{\sum_{h=0}^{m-1} (a_{h+1} - b_{h+1})} \times \xi_{s_m}(\gamma_{m}, \gamma_{m-1})
\]

Proof. We truncate the sum giving \( \xi_{s}(\beta, \alpha) \) at the order \( m \) and consider the remaining terms

\[
(1 - \beta) \alpha \sum_{k \geq m} (-1)^k \beta^{\sum_{h=0}^{k} (a_{h+1} - b_{h+1}) q_h} \alpha^{\sum_{h=0}^{k} (a_{h+1} - b_{h+1}) p_h} \\
\times \sum_{h=0}^{m} (-1)^k \beta^{\sum_{h=0}^{k} (a_{m+h+1} - b_{m+h+1}) q_{m+h}} \alpha^{\sum_{h=0}^{k} (a_{m+h+1} - b_{m+h+1}) p_{m+h} + 1}
\]

\[\times \xi_{s_m}(\gamma_{m}, \gamma_{m-1})
\]
Now, we claim that the last factor
\[ \sum_{k \geq 0} (-1)^k \frac{\beta \sum_{h=0}^{k} (a_{m+h+1} - b_{m+h+1}) q_{m+h} \alpha \sum_{h=0}^{k} (a_{m+h+1} - b_{m+h+1}) p_{m+h}}{(1 - \beta^{q_{m+k+1}} \alpha^{p_{m+k+1}})(1 - \beta^{q_{m+k}} \alpha^{p_{m+k}})} \]
is equal to
\[ (1 - \beta^{q_m} \alpha^{p_m})^{-1} (\beta^{q_{m-1}} \alpha^{p_{m-1}})^{-1} \xi_{s_m} (\beta^{q_m} \alpha^{p_m}, \beta^{q_{m-1}} \alpha^{p_{m-1}}). \]
Indeed, let \((u_n/v_n)_{n \geq 0}\) be the convergents of \(\theta_m\). We have
\[
\frac{u_0}{v_0} = 0, \quad \frac{u_1}{v_1} = \frac{1}{a_{m+1}}, \quad \frac{u_2}{v_2} = \frac{a_{m+2}}{a_{m+1}a_{m+2} + 1}, \ldots,
\]
and we easily check that, for any \(h \geq 0\), we have
\[
q_{m+h} = v_h q_m + u_h q_{m-1} \quad \text{and} \quad p_{m+h} = v_h p_m + u_h p_{m-1}.
\]
It follows that we can write the exponents in a form involving the convergents of \(\theta_m\):
\[
\sum_{h=0}^{k} (a_{m+h+1} - b_{m+h+1}) q_{m+h} = \left( \sum_{h=0}^{k} (a_{m+h+1} - b_{m+h+1}) v_h \right) q_m + \left( \sum_{h=0}^{k} (a_{m+h+1} - b_{m+h+1}) u_h \right) q_{m-1}
\]
and
\[
\sum_{h=0}^{k} (a_{m+h+1} - b_{m+h+1}) p_{m+h} = \left( \sum_{h=0}^{k} (a_{m+h+1} - b_{m+h+1}) v_h \right) p_m + \left( \sum_{h=0}^{k} (a_{m+h+1} - b_{m+h+1}) u_h \right) p_{m-1}.
\]
Thus
\[
\beta \sum_{h=0}^{k} (a_{m+h+1} - b_{m+h+1}) q_{m+h} \alpha \sum_{h=0}^{k} (a_{m+h+1} - b_{m+h+1}) p_{m+h} = \left( \beta^{q_m} \alpha^{p_m} \right)^{k+1} \left( \beta^{q_{m-1}} \alpha^{p_{m-1}} \right)^{k+1} \xi_{s_m} (\beta^{q_m} \alpha^{p_m}, \beta^{q_{m-1}} \alpha^{p_{m-1}})^{k+1}
\]
and
\[
1 - \beta^{q_{m+k+1}} \alpha^{p_{m+k+1}} = 1 - (\beta^{q_m} \alpha^{p_m})^{k+1} (\beta^{q_{m-1}} \alpha^{p_{m-1}})^{k+1} \quad \text{and} \quad 1 - \beta^{q_{m+k}} \alpha^{p_{m+k}} = 1 - (\beta^{q_m} \alpha^{p_m})^{k} (\beta^{q_{m-1}} \alpha^{p_{m-1}})^{k}.
\]
The last claim follows from the equalities
\[
(1 - \beta)\alpha \left( \sum_{k=0}^{m-1} (-1)^k \frac{\prod_{h=0}^{k} \gamma_h^{a_{h+k+1} - b_{h+1}}}{(1 - \gamma_{k+1})(1 - \gamma_k)} \right) = \frac{\sigma_m}{1 - \gamma_m} - \frac{\sigma_0}{1 - \gamma_0} = \frac{\sigma_m}{1 - \gamma_m}.
\]
The proof is complete. \(\square\)
5. When the slope has unbounded partial quotients

The purpose of this Section is to establish Theorem 1.1 when the slope $\theta$ has unbounded partial quotients. In this case, an application of Liouville’s inequality is sufficient to conclude. We use the logarithmic Weil height $h$ and Liouville’s inequality under the form

$$
\log |\zeta| \geq -[\mathbb{Q}(\zeta):\mathbb{Q}] h(\zeta),
$$

for any nonzero algebraic number $\zeta$. There is some similarity with the proof of [10, Theorem 6].

We make use of the approximations $\frac{\beta}{1-\beta} (4)_{k-1}$ and $\frac{\beta}{1-\beta} (3)_{k}$ to $\xi = \xi_{\alpha,\beta}(\beta, \alpha)$, considered in Section 4. If $U$ and $V$ are positive quantities depending upon $k$, let us write $U \asymp V$ to indicate that there exist positive constants $c, c'$ such that the inequalities $cU \leq V \leq c'U$ hold for large $k$.

**Lemma 5.1.** We have the two estimates

$$
|\xi - \frac{\beta}{1-\beta} (4)_{k-1}| \asymp (|\beta\alpha^k|)_{u_k+q_k}
$$

with $u_k = \begin{cases} r_{k+1} & \text{if } a_{k+2} - b_{k+2} \geq 1, \\ r_k + q_{k+1} & \text{if } a_{k+2} = b_{k+2}, \end{cases}$

and

$$
|\xi - \frac{\beta}{1-\beta} (3)_{k}| \asymp (|\beta\alpha^k|)_{v_k+q_{k+1}}
$$

with $v_k = \begin{cases} r_{k+1} + q_k & \text{if } a_{k+2} - b_{k+2} \geq 2, \\ r_k + 2q_k & \text{if } a_{k+2} - b_{k+2} = 1, a_{k+3} - b_{k+3} \geq 1, \\ r_{k+1} + q_k & \text{if } a_{k+2} = 1, b_{k+2} = 0, a_{k+3} = b_{k+3}, \\ r_k & \text{if } a_{k+2} = b_{k+2}. \end{cases}$

**Proof.** Let us set, for $k \geq 1$,

$$
\Gamma_k = \frac{1 - \beta}{\beta} (-1)^k \frac{\beta r_{k+1} + q_k \alpha^{r_{k+1}+p_k}}{(1 - \beta q_{k+1} \alpha^{p_k+1})(1 - \beta q_k \alpha^{p_k})},
$$

and

$$
\Delta_k = \frac{1 - \beta}{\beta} (-1)^k \frac{\beta r_{k+2} + q_{k+1} + q_k \alpha^{r_{k+2}+q_{k+1}+p_{k+1}} - \beta r_{k+2} + q_{k+1} \alpha^{r_{k+2}+p_{k+1}} - \beta r_{k+1} + q_{k+1} \alpha^{r_{k+1}+p_{k+1}}(1 - \beta q_k \alpha^{p_k})}{(1 - \beta q_{k+1} \alpha^{p_k+1})(1 - \beta q_k \alpha^{p_k})}.
$$

Recalling Lemma 4.1 and the representations of $\xi = \xi_{\alpha,\beta}(\beta, \alpha)$ given in Corollary 4.2, we have

$$
\xi - \frac{\beta}{1-\beta} (4)_{k-1} = \sum_{h \geq k} \Gamma_h, \quad \text{and} \quad \xi - \frac{\beta}{1-\beta} (3)_{k} = \sum_{h \geq k} \Delta_h.
$$

We now estimate the two above sums. For the sum $\sum_{h \geq k} \Gamma_h$, observe that the two sequences of exponents $r_{h+1} + q_h = 1 + \sum_{j=0}^{h} (a_{j+1} - b_{j+1})q_j$, $h = k, k+1, \ldots$
and
\[ \tilde{r}_{h+1} + p_n = 1 + \sum_{j=0}^{h} (a_{j+1} - b_{j+1}) p_j, \quad h = k, k + 1, \ldots \]

occurring in the quantities \( \Gamma_h \), are non-decreasing. Moreover, \( r_{h+1} + q_h = r_{h+2} + q_{h+1} \) if and only if \( a_{h+2} = b_{h+2} \) and \( r_{h+2} + q_{h+1} \geq r_{h+1} + q_h + q_{h+1} \) if \( a_{h+2} > b_{h+2} \). Notice also that we cannot have \( r_{h+1} + q_h = r_{h+2} + q_{h+1} = r_{h+3} + q_{h+2} \), since the simultaneous equalities \( a_{h+2} = b_{h+2} \) and \( a_{h+3} = b_{h+3} \) are forbidden according to Ostrowski’s rules.

In view of Lemma 2.4, we have
\[ |\Gamma_h| \asymp (|\beta \alpha^\theta|)^{r_{h+1}+q_h}. \]

In order to estimate \( \sum_{h \geq k} \Gamma_h \), we distinguish two cases. Assume first that \( a_{k+2} - b_{k+2} \geq 1 \). Then
\[ |\Gamma_k| \asymp (|\beta \alpha^\theta|)^{r_{k+1}+q_k} \quad \text{and} \quad |\Gamma_h| \ll (|\beta \alpha^\theta|)^{r_{k+1}+q_k+q_{k+1}}, \quad h \geq k + 1. \]

Taking into account the preceding observations, it follows that
\[ |\sum_{h \geq k} \Gamma_h| \asymp (|\beta \alpha^\theta|)^{r_{k+1}+q_k}. \]

Assume secondly that \( a_{k+2} = b_{k+2} \). Then,
\[
|\Gamma_k + \Gamma_{k+1}| = \frac{1 - \beta}{\beta} \left| \frac{\beta^{r_{k+1}+q_k} \alpha^{p_k} \beta^{r_{k+1}+q_{k+1}+p_k}}{1 - \beta \alpha^{p_k} \alpha^{p_{k+1}}} \left( 1 - \frac{1}{1 - \beta q_k \alpha^{p_k}} \right) \right| = \frac{1 - \beta}{\beta} \cdot \frac{\beta^{r_{k+1}+q_k} \alpha^{p_k} \beta^{r_{k+1}+q_{k+1}+p_k} (\beta^{q_{k+2}} - \beta \alpha^{p_k})}{(1 - \beta q_k \alpha^{p_k})(1 - \beta q_{k+1} \alpha^{p_{k+1}})},
\]

so that
\[ |\Gamma_k + \Gamma_{k+1}| \asymp (|\beta \alpha^\theta|)^{r_{k+1}+2q_k} = (|\beta \alpha^\theta|)^{r_k + q_k + q_{k+1}}. \]

Now, since \( a_{k+3} > b_{k+3} \), we get
\[ r_{k+3} + q_{k+2} - (r_k + q_{k+1}) = r_{k+3} + q_{k+2} - (r_{k+2} + q_{k+1}) \geq q_{k+2}, \]

so that
\[ |\sum_{h \geq k+2} \Gamma_h| \ll (|\beta \alpha^\theta|)^{r_k + q_k + q_{k+1}}. \]

It follows that
\[ |\sum_{h \geq k} \Gamma_h| \asymp (|\beta \alpha^\theta|)^{r_k + q_k + q_{k+1}}. \]

We now briefly deal with the sum \( \sum_{h \geq k} \Delta_h \). Observe that
\[
|\Delta_h| \asymp \begin{cases} |\beta \alpha^\theta|^{r_{h+1}+q_{h+1}+q_h} & \text{if} \quad a_{h+2} - b_{h+2} \geq 2, \\ |\beta \alpha^\theta|^{r_{h+1}+q_{h+1}+2q_h} & \text{if} \quad a_{h+2} - b_{h+2} = 1, \\ |\beta \alpha^\theta|^{r_{h+1}+q_{h+1}} & \text{if} \quad a_{h+2} = b_{h+2}. \end{cases}
\]

Looking at the absolute value of \( \Delta_k \) and \( \Delta_{k+1} \) according to the above cases, we check that
\[ |\sum_{h \geq k} \Delta_h| \asymp |\Delta_k|, \]
unless \( a_{k+2} = 1, b_{k+2} = 0 \) and \( a_{k+3} = b_{k+3} \), in which case

\[
| \sum_{h \geq k} \Delta_k | \asymp | \Delta_{k+1} |.
\]

It follows that

\[
| \sum_{h \geq k} \Delta_k | \asymp | \beta a^\theta |^{v_k+q_{k+1}},
\]

as asserted. Lemma 5.1 is proved.

We are now able to prove Theorem 1.1 when \( \theta \) has unbounded partial quotients. Assume on the contrary that \( \xi \) is algebraic. We distinguish two cases.

Assume first that \( r_{k+1}/q_k \) takes arbitrarily large values and set

\[
\zeta = \xi - \frac{\beta}{1 - \beta} (4)_{k-1}.
\]

Lemma 5.1 yields, for large \( k \), that \( \zeta \) is non-zero and that

\[
\log | \zeta | \ll -(u_k + q_k) \ll -r_{k+1},
\]

since we have always \( u_k \geq r_{k+1} \). But the algebraic number \( \zeta \) has height \( h(\zeta) \ll q_k \). This contradicts Liouville’s inequality (5.1), provided that we have chosen \( k \) such that \( r_{k+1}/q_k \) is large enough.

Assume now that the sequence \( (r_{k+1}/q_k)_{k \geq 1} \) is bounded. Set now

\[
\zeta = \xi - \frac{\beta}{1 - \beta} (3)_{k}.
\]

Again Lemma 5.1 implies that \( \zeta \) is non-zero and that

\[
\log | \zeta | \ll -(u_k + q_{k+1}) \ll -q_{k+1},
\]

when \( k \) is large enough. But the algebraic number \( \zeta \) has now height

\[
q_k \ll r_{k+1} + q_k \ll q_k,
\]

by assumption. We get a final contradiction with Liouville’s inequality 5.1, provided that we have chosen \( k \) such that \( q_{k+1}/q_k \) is large enough.

6. Functional transcendence

A general idea underlying Mahler’s method is that the transcendence of a function \( f(z) \) over \( \mathbb{Q}(z) \) is transferred to the transcendence of the value of \( f \) at every nonzero algebraic point in the open unit disc. Therefore, we need a functional transcendence statement.

**Proposition 6.1.** Let \( \theta, \rho \) be real numbers such that \( 0 \leq \theta, \rho < 1 \) and \( \theta \) irrational. Then, the function \( z \mapsto \xi_{sp,\rho}(z, 1) \) is transcendental over \( \mathbb{C}(z) \). Consequently, the function \( (z_1, z_2) \mapsto \xi_{sp,\rho}(z_1, z_2) \) is transcendental over \( \mathbb{C}(z_1, z_2) \).

**Proof.** Observe that an algebraic function, say \( f(z) \), holomorphic in the open unit disc, can be analytically prolonged in a neighborhood of a point \( z_0 \) on the unit circle, if we assume that \( z_0 \) is not a root of the discriminant of the minimal polynomial of \( f(z) \) over \( \mathbb{C}(z) \).

\[
\text{TRANSCENDENCE OF VALUES OF HECKE–MAHLER SERIES}
\]
Therefore, it is sufficient to show that $z \mapsto \xi_{\theta, \rho}(z, 1)$ cannot be prolongated beyond the unit circle. The case $\rho = 0$ has been treated by Hecke [9]. His argument extends easily to an arbitrary value of $\rho$. For the sake of completeness, we give the details below. Set

$$F(z) = \sum_{n \geq 1} (n\theta + \rho)z^n.$$  

Recall that if, for a power series $\sum_{n \geq 1} c_n z^n$, we have

$$\lim_{t \to +\infty} \frac{1}{t} \sum_{n=1}^{t} c_n = c,$$

then

$$\lim_{r \to 1^-} (1 - r)^{+\infty} \sum_{n=1}^{+\infty} c_n r^n = c,$$

where $r \to 1^-$ means that the real number $r$ tends to 1 and is less than 1. Let $t$ be a positive integer. Write

$$S(t) = \sum_{n=1}^{t} (n\theta + \rho)e^{2i\pi n\alpha}$$

and take $\alpha = q\theta + p$, for integers $p, q$ with $q$ nonzero. We have

$$S(t) = \sum_{n=1}^{t} (n\theta + \rho)e^{2i\pi n(q\theta + p)} = e^{-2i\pi q\rho} \sum_{n=1}^{t} (n\theta + \rho)e^{2i\pi q(n\theta + \rho)}.$$  

As $\theta$ is irrational, the sequence $(\{n\theta + \rho\})_{n \geq 1}$ is equidistributed in $[0, 1]$, thus

$$\lim_{t \to +\infty} \frac{1}{t} \sum_{n=1}^{t} f(\{n\theta + \rho\}) = \int_{0}^{1} f(x)dx,$$

for every continuous function $f$. Consequently,

$$\lim_{t \to +\infty} \frac{1}{t} S(t) = \frac{e^{-2i\pi q\rho}}{2i\pi q}.$$  

It then follows that

$$\lim_{r \to 1^-} (1 - r)^{+\infty} \sum_{n=1}^{+\infty} (n\theta + \rho)(re^{2i\pi (q\theta + p)})^n = \lim_{r \to 1^-} (1 - r)F(re^{2i\pi (q\theta + p)}) = \frac{e^{-2i\pi q\rho}}{2i\pi q}.$$  

Since the set of points of the form $q\theta + p$ is dense modulo one, the function $F$ cannot be prolongated beyond the unit circle. The same conclusion holds for the function $z \mapsto \sum_{n \geq 1} (n\theta + \rho)z^n$. $\square$
7. Transcendence of Hecke–Mahler series at algebraic points

Loxton and van der Poorten [11] (see also [13, Section 2.9]) obtained a general transcendence theorem for chains of functional equations of Mahler’s type, from which they deduced [11, Theorem 8] the transcendence of $F_{\theta,0}(\beta, \alpha)$, for every irrational number $\theta$ in $(0,1)$ and every nonzero complex algebraic numbers $\alpha, \beta$ with $|\beta \alpha^\theta| < 1$ and $\beta^{p_k} \alpha^{q_k} \neq 1$ for $k \geq 1$, where $p_k/q_k$ is the $k$-th convergent to $\theta$.

We follow the presentation of Nishioka [13], with some simplification and modernization. In her book, the size $\| \alpha \|$ of an algebraic number $\alpha$ is the maximum of the absolute values of the conjugates of $\alpha$ and of its denominator. The function $\log \| \cdot \|$ is thus comparable to the logarithmic Weil height $h$, which we are using.

For a $2 \times 2$ matrix $\Omega = (\omega_{i,j})$ with nonnegative integer coefficients and a point $(z_1, z_2)$ in $\mathbb{C}^2$, we define an application $\Omega : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ by

$$\Omega(z_1, z_2) = (z_1^{\omega_{1,1}} z_2^{\omega_{1,2}}, z_1^{\omega_{2,1}} z_2^{\omega_{2,2}}).$$

Let $(\Omega_k)_{k \geq 1}$ be a sequence of matrices with nonnegative integer coefficients. Let $K$ be a number field and $\alpha_1, \alpha_2$ nonzero elements in $K$. Write

$$(\alpha_1^{(k)}, \alpha_2^{(k)}) = \Omega_k(\alpha_1, \alpha_2), \quad k \geq 1.$$ 

Let $f_k(z_1, z_2)$, $k \geq 0$, be in $\mathbb{Z}[[z_1, z_2]]$ with bounded coefficients. Write

$$f_k(z_1, z_2) = \sum_{\lambda_1, \lambda_2 \geq 0} s^{(k)}_{\lambda_1, \lambda_2} z_1^{\lambda_1} z_2^{\lambda_2}, \quad k \geq 0,$$

and $s^{(k)} = (s^{(k)}_{\lambda_1, \lambda_2})_{\lambda_1, \lambda_2 \geq 0}$. For a collection $\mathbf{s} = (s_{\lambda_1, \lambda_2})_{\lambda_1, \lambda_2 \geq 0}$ of variables, set

$$F(z_1, z_2; \mathbf{s}) = \sum_{\lambda_1, \lambda_2 \geq 0} s_{\lambda_1, \lambda_2} z_1^{\lambda_1} z_2^{\lambda_2}.$$ 

Then, we have

$$F(z_1, z_2; \mathbf{s}^{(k)}) = f_k(z_1, z_2), \quad k \geq 0.$$ 

Assume that there exist positive real numbers $r_1, r_2, \ldots$ such that $(r_k)_{k \geq 1}$ tends to infinity and

(i) Every coefficient of $\Omega_k$ is $\ll r_k$, for $k \geq 1$.

(ii) There exist positive real numbers $\eta_1, \eta_2$ which are linearly independent over the rationals and such that

$$\log |\alpha_i^{(k)}| \sim -\eta_i r_k, \quad i = 1, 2, \quad \text{as } k \text{ tends to infinity}.$$ 

(iii) For $k \geq 1$, there exist $a_k, b_k$ in $K$ such that

$$f_k(\Omega_k(\alpha_1, \alpha_2)) = a_k f_0(\alpha_1, \alpha_2) + b_k$$ 

and

$$h(a_k), h(b_k) \ll r_k.$$
(iv) If \( p \) is a positive integer, \( P_0(z_1, z_2; \mathfrak{s}), \ldots, P_p(z_1, z_2; \mathfrak{s}) \) are polynomials in \( z_1, z_2 \) and in the variables \( s_{\lambda_1, \lambda_2} \), with coefficients in \( K \), and
\[
E(z_1, z_2; \mathfrak{s}) = \sum_{j=0}^{p} P_j(z_1, z_2; \mathfrak{s}) F(z_1, z_2; \mathfrak{s})^j = \sum_{\lambda_1, \lambda_2 \geq 0} P_{\lambda_1, \lambda_2}(\mathfrak{s}) z_1^{\lambda_1} z_2^{\lambda_2},
\]
then there exist nonnegative \( \lambda_1, \lambda_2 \) with the following property: If \( k \) is sufficiently large and \( P_0(z_1, z_2; \mathfrak{s}^{(k)}), \ldots, P_p(z_1, z_2; \mathfrak{s}^{(k)}) \) are not all zero, then \( P_{\lambda_1, \lambda_2}(\mathfrak{s}^{(k)}) \) is non-zero.

Assumptions (i), (ii), and (iii) correspond exactly to Assumptions (I), (II), and (III) in [13]. Our assumption (iv) is a simplified version of Assumption (V) in [13]. Assumption (IV) in [13] is clearly satisfied since the coefficients of the series \( f_k \) are integers and are bounded.

**Theorem 7.1** (Loxton–van der Poorten). Under the above assumption, the complex number \( f_0(\alpha_1, \alpha_2) \) is transcendental.

Our presentation slightly differs from that of [11], where the authors have to cope with admissibility conditions on \( \alpha_1 \) and \( \alpha_2 \). Here, we have expressed Assumption (iii) with \( a_k, b_k \) in \( K \), and not with functions \( a_k(\alpha_1, \alpha_2), b_k(\alpha_1, \alpha_2) \) in \( K(\alpha_1, \alpha_2) \), in which case we should have excluded the pairs \((\alpha_1, \alpha_2)\) at which these functions are not defined. To overcome this difficulty, Nishioka [13, p. 77] assumes that \( \alpha_1 \) and \( \alpha_2 \) are in the open unit disc, but this is quite restrictive.

We show how Theorem 7.1 applies to establish Theorem 1.1 when the slope \( \theta \) has bounded partial quotients.

For \( m \geq 0 \), recall that \( \theta_m = [0, a_{m+1}, a_{m+2}, \ldots] \) and that \( s_m \) denotes the Sturmian word with slope \( \theta_m \) and formal intercept \( b_{m+1}, b_{m+2}, \ldots \) (see Definition 2.3), as in Section 4. Let us start by the following estimates.

**Proposition 7.2.** Let \( \alpha \) and \( \beta \) be complex numbers such that \( 0 < |\beta\alpha^\theta| < 1 \) and \( \beta \neq 1 \). If there exists \( \ell \) such that \( \beta^\ell \alpha^\ell = 1 \), then put \( m_0 = \ell + 1 \), otherwise put \( \ell = -1 \) and \( m_0 = 1 \). For any \( m \geq m_0 \), there exist \( A_m \) and \( B_m \) such that
\[
\xi_{s_m}(\beta^{\ell m} \alpha^{m_p}, \beta^{\ell m-1} \alpha^{p_{m-1}}) = A_m \xi_{s}(\beta, \alpha) + B_m
\]
and
\[
h(A_m), h(B_m) \ll_{\alpha, \beta} q_m.
\]

**Proof of Proposition 7.2.** Here, \( \alpha \) and \( \beta \) denote complex numbers satisfying \( 0 < |\beta\alpha^\theta| < 1 \) and \( \beta \neq 1 \). Recall that \( \gamma_k = \beta^{q_k} \alpha^{p_k} \), for \( k \geq 0 \). Let \( m \geq m_0 \) be an integer. Then, \( \gamma_m \neq 1 \) and
\[
A'_m = \frac{\sigma_m}{1 - \gamma_m}, \quad B'_m = (-1)^m \frac{\prod_{h=0}^{m-1} \gamma_h^{a_{h+1} - b_{h+1}}}{(1 - \gamma_m) \gamma_{m-1}}
\]
are well defined. Proposition 4.3 asserts that
\[
\xi_{s}(\beta, \alpha) = (1 - \beta)\alpha \left( A'_m + B'_m \xi_{s_m}(\gamma_m, \gamma_{m-1}) \right).
\]
It is sufficient to prove that
\[
h(A'_m), h(B'_m) \ll_{\alpha, \beta} q_m
\]
to establish the proposition. For \( k = 0, \ldots, m - 1 \), we have

\[
h\left( \prod_{h=0}^{k} \gamma_h \right) \leq \sum_{h=0}^{k} a_{h+1} h(\gamma_h) \leq \sum_{h=0}^{k} a_{h+1} (g_k h(\beta) + p_k h(\alpha)).
\]

\[
\leq (q_{k+1} + q_k) h(\beta) + (p_{k+1} + p_k) h(\alpha),
\]

by (2.1) and (2.2). This implies that \( h(B'_m) \ll_{\alpha, \beta} q_m \).

To estimate the height of

\[
\sigma_m = \sum_{n=1}^{q_m} s_n \beta^n \alpha^{\sum_{h=1}^{n} s_h},
\]

first note that its denominator is bounded from above by \( q_m \) times the product of the denominators of \( \alpha \) and \( \beta \). Let \( M \geq 2 \) be an upper bound for the moduli of the conjugates of \( \alpha \) and \( \beta \). Then, the modulus of any conjugate of \( \sigma_m \) is at most equal to \( M^2 + \ldots + M^{2q_m} \), thus less than \( M^{2q_m+1} \). We conclude that \( h(A'_m) \ll_{\alpha, \beta} q_m \), as asserted.

We are now equipped to complete the proof of Theorem 1.1.

**Proof of Theorem 1.1 when \( \theta \) has bounded partial quotients.** Let \( \theta \) and \( \rho \) be as in the statement of the theorem. Assume that \( \theta \) has bounded partial quotients. Let \( \alpha_1, \alpha_2 \) be complex numbers such that \( 0 < |\alpha_1 \alpha_2^{\rho}| < 1 \) and \( \alpha_1 \neq 1 \).

Let \( M \) be a positive integer such that \( a_k, b_k \leq M \) for \( k \geq 1 \). Recall that \( p_k/q_k \) denotes the \( k \)-th convergent to \( \theta \). Let \( m_0 \) be as in Proposition 7.2. By compactness, there exist an increasing sequence \( (\nu_k)_{k \geq 1} \) of positive integers, with \( \nu_1 \geq m_0 \), integers \( g_1, g_2, \ldots, a'_1, a'_2, \ldots \) in \( \{1, \ldots, M\} \), and integers \( b'_1, b'_2, \ldots \) in \( \{0, \ldots, M\} \) such that

\[
(a_{\nu_1}, a_{\nu_1-1}, a_{\nu_1-2}, \ldots) \to (g_1, g_2, g_3, \ldots), \quad k \to \infty,
\]

\[
(a_{\nu_1+1}, a_{\nu_1+2}, a_{\nu_1+3}, \ldots) \to (a'_1, a'_2, a'_3, \ldots), \quad k \to \infty,
\]

and

\[
(b_{\nu_1+1}, b_{\nu_1+2}, b_{\nu_1+3}, \ldots) \to (b'_1, b'_2, b'_3, \ldots), \quad k \to \infty.
\]

As \( k \) tends to infinity, the Sturmian word \( s_{\nu_k} \) with slope \( \theta_{\nu_k} \) and intercept \( b_{\nu_k+1}, b_{\nu_k+2}, \ldots \) tends to the Sturmian word \( s' \) with slope \([0; a'_1, a'_2, \ldots]\) and formal intercept \( b'_1, b'_2, \ldots \).

Set

\[
\phi := [0; g_1, g_2, \ldots].
\]

Observe that \( \phi \) is irrational and, by the theory of continued fractions,

\[
\lim_{k \to +\infty} \frac{p_{\nu_k - 1}}{p_{\nu_k}} = \lim_{k \to +\infty} \frac{q_{\nu_k - 1}}{q_{\nu_k}} = \phi.
\]

Define

\[
\Omega_k = \begin{pmatrix} q_{\nu_k} & p_{\nu_k} \\ p_{\nu_k - 1} & q_{\nu_k - 1} \end{pmatrix}, \quad r_k = q_{\nu_k}, \quad k \geq 1.
\]

Assumption (i) is satisfied.
Since $0 < |\alpha_1 \alpha_2^\theta| < 1$, 
\[
\lim_{k \to +\infty} \frac{\log |\alpha_1^{(k)}|}{r_k} = \frac{q_{\nu_k} \log |\alpha_1| + p_{\nu_k} \log |\alpha_2|}{r_k} = \log |\alpha_1| + \theta \log |\alpha_2|,
\]
\[
\lim_{k \to +\infty} \frac{\log |\alpha_2^{(k)}|}{r_k} = \frac{q_{\nu_k-1} \log |\alpha_1| + p_{\nu_k-1} \log |\alpha_2|}{r_k} = \phi(\log |\alpha_1| + \theta \log |\alpha_2|).
\]
and \(\phi\) is irrational, Assumption (ii) is satisfied.

Put
\[
f_0(z_1, z_2) = \xi_{\theta}(z_1, z_2), \quad f_k(z_1, z_2) = \xi_{\gamma \nu_k}(z_1, z_2), \quad k \geq 1.
\]
The coefficients of \(f_k\) are in \{0, 1\} for \(k \geq 0\). It follows from Proposition 7.2 that Assumption (iii) is satisfied.

As noted above, we have
\[
\lim_{k \to +\infty} f_k(z_1, z_2) = \xi_{\theta}(z_1, z_2).
\]
Furthermore, it follows from Proposition 6.1 that \(\xi_{\theta}(z_1, z_2)\) is a transcendental function over \(\mathbb{C}(z_1, z_2)\).

Let \(p\) be a positive integer and \(P_0(z_1, z_2; \bar{s}), \ldots, P_p(z_1, z_2; \bar{s})\) be polynomials as in (iv). Let \(E(z_1, z_2; \bar{s})\) be as above. Setting
\[
\xi_{\theta}(z_1, z_2) = \sum_{\lambda_1, \lambda_2 \geq 0} \sigma_{\lambda_1, \lambda_2} \frac{1}{z_1^{\lambda_1} z_2^{\lambda_2}},
\]
we have
\[
\lim_{k \to +\infty} P_j(z_1, z_2; \bar{s}^{(k)}) = P_j(z_1, z_2; \bar{s}), \quad \lim_{k \to +\infty} P_{\lambda_1, \lambda_2}(\sigma^{(k)}) = P_{\lambda_1, \lambda_2}(\sigma).
\]
If \(P_j(z_1, z_2; \sigma), 0 \leq j \leq p,\) are all zero, then the polynomials \(P_j(z_1, z_2; \sigma^{(k)})\) vanish identically for \(k\) sufficiently large, thus Assumption (iv) is clearly satisfied. Otherwise, \(E(z_1, z_2; \sigma)\) is not zero, since \(\xi_{\theta}\) is a transcendental function. Consequently, there exist nonnegative \(\lambda_1, \lambda_2\) such that \(P_{\lambda_1, \lambda_2}(\sigma)\) is nonzero. Hence, \(P_{\lambda_1, \lambda_2}(\sigma^{(k)})\) is not zero for all \(k\) sufficiently large and Assumption (iv) is satisfied. All this shows that Theorem 7.1 applies and yields Theorem 1.1 when the slope \(\theta\) has bounded partial quotients. \(\square\)

REFERENCES

[1] B. Adamczewski and Y. Bugeaud, *Dynamics for \(\beta\)-shifts and Diophantine approximation*, Ergodic Theory Dynam. Systems 27 (2007), 1695–1711.
[2] B. Adamczewski et Y. Bugeaud, *Nombres réels de complexité sous-linéaire : mesures d’irrationalité et de transcendance*, J. reine angew. Math. 658 (2011), 65–98.
[3] B. Adamczewski and C. Faverjon, *Méthode de Mahler : relations linéaires, transcendance et applications aux nombres automatiques*, Proc. Lond. Math. Soc. 115 (2017), 55–90.
[4] P. E. Böchner, *Über die Transzendenz gewisser dyadischer Brüche*, Math. Ann. 96 (1927), 367–377.
[5] J. M. Borwein and P. B. Borwein, *On the generating function of the integer part: \([n\alpha + \gamma]\)*, J. Number Theory 43 (1993), 293–318.
[6] Y. Bugeaud, D. H. Kim, M. Laurent, and A. Nogueira, *On the Diophantine nature of the elements of Cantor sets arising in the dynamics of contracted rotations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 22 (2021), 1691–1704.

[7] Y. Bugeaud and M. Laurent, *Combinatorial structure of Sturmian words and continued fraction expansions of Sturmian numbers*. Preprint.

[8] S. Ferenczi and Ch. Mauduit, *Transcendence of numbers with a low complexity expansion*, J. Number Theory 67 (1997), 146–161.

[9] E. Hecke, "Uber analytische Funktionen und die Verteilung von Zahlen mod. eins, Abh. Math. Sem. Hamburg 1 (1922), 54–76.

[10] T. Komatsu, *A certain power series and the inhomogeneous continued fraction expansions*, J. Number Theory 59 (1996), 291–312.

[11] J. H. Loxton and A. J. van der Poorten, *Arithmetic properties of certain functions in several variables III*, Bull. Austral. Math. Soc. 16 (1977), 15–47.

[12] K. Mahler, *Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen*, Math. Ann. 101 (1929), 342–366.

[13] Ku. Nishioka, Mahler Functions and Transcendence. Lecture Notes in Math. 1631, Springer, 1996.

[14] Ku. Nishioka, I. Shiokawa and J. Tamura, *Arithmetical propertise of a certain power series*, J. Number Theory 42 (1992), 61–87.