(de)Tails of Toda CFT

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Abstract

The relation between the partition function of $\mathcal{N} = 2$ gauge theories in 4d and conformal Toda field theory in 2d is explained for the case where the 4d theory is a linear quiver with “quiver tails”. That is when the 4d theory has gauge groups of different rank. We propose an identification of a subset of the states of Toda CFT which represent the Coulomb-branch parameters of the different rank gauge multiplets and study their three-point functions and descendants.
1 Introduction

The Alday-Gaiotto-Tachikawa (AGT) relation \[1\] expresses the partition function of certain 4d gauge theories with \( \mathcal{N} = 2 \) supersymmetry on \( S^4 \) in terms of correlation functions in 2d Liouville theory. More precisely, such 4d theories are conjectured to have S-duality symmetries, and the identification is between the partition function calculated in one S-duality frame and the Liouville correlation function calculated via the conformal bootstrap with a particular pants decomposition of the Riemann surface.\(^1\)

The gauge theory partition function on \( S^4 \) is given by an integral over Coulomb branch parameters for each of the gauge groups with certain classical, 1-loop, and instanton contributions [2]. The identification is between:

| Gauge theory                  | Liouville                  |
|-------------------------------|----------------------------|
| Coulomb branch parameters     | Primary states in intermediate channels |
| 1-loop corrections            | Three point functions      |
| Instanton contributions       | Contributions of descendants, captured by the conformal blocks. |

The gauge theories related to Liouville theory in this way have gauge group \( SU(2)^k \) for some \( k \). The AGT relation was generalized by Wyllard [3] to the case of other gauge groups, namely \( SU(N)^k \). The identification goes as before, with Liouville theory replaced by \( A_{N-1} \) Toda CFT [4, 5].

We would like to address a hybrid case, when the gauge group is a product \( \prod_i SU(N_i) \) with different values of \( N_i \). We will consider only the case when the theory has a Lagrangean in at least one duality frame, so we exclude the theories with \( T_N \) factors [6], and when the theories are conformal (apart for explicit mass terms). The classification of these theories was done by by Gaiotto and Witten [6, 7] and involves a linear quiver of gauge groups \( SU(N) \) with extra quiver tails of lower rank gauge groups, as we review below.

We would like to reproduce the partition function of such a theory from a correlation function of \( A_{N-1} \) Toda CFT. Since the AGT relation identifies the coulomb branch parameters with primary states in intermediate channels along the surface, for groups of rank smaller than \( N \) in the tail, the space of states in the corresponding intermediate channel should be smaller than the full \( N - 1 \) dimensional space of Toda primaries. The purpose of this manuscript is to propose what this reduced space of states is, so that we can generalize the AGT correspondence to this case too.

In the next section we review the classification of quiver tails and its relation to semi-degenerate representations of Toda CFT [8]. A careful analysis of the four-dimensional

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\(^1\)Some refer to the AGT relation as the identification between instanton partition functions and Liouville conformal blocks, which is also true.
theory leads to a guess on the allowed intermediate states in the relevant channels, which is backed by an analysis of the Seiberg-Witten curve. In Section 3 we study the simplest theory of this type, with gauge group $SU(2) \times SU(3)$. In that case one can write down a Ward identity for the 3-point function \cite{9} whose solution is exactly the space of states we proposed. The three point function and its relation to the 1-loop determinant is discussed in much more generality in the following sections. So here we concentrate on the contribution of level 1 descendants and show that it agrees with the instanton correction to the gauge theory expression. We note, in particular that the sum over W-algebra descendants of the restricted state representing $SU(2)$ reproduces the same instanton contribution as do the Virasoro descendants in Liouville CFT. In this case we also write down the precise Seiberg-Witten curve and match the mass parameters and intermediate states to a semiclassical limit of the Toda states.

In Section 4 we evaluate the relevant 3-point functions. We start with the expression for the 3-point function of a “simple puncture” state and two generic (“full puncture”) states \cite{10, 11} and consider the limit when these two states become semi-degenerates. We show that in that limit the 3-point function will generically vanish, except for the special states that we guessed in Section 2 where we find a pole of the expected order.

In Section 5 we combine the expressions we found for the 3-point functions for the entire tail and show how they reproduce the correct 1-loop part in the gauge theory calculation.

We discuss some further issues in Section 6.

### 2 Quiver tails and semi-degenerate representations

#### 2.1 Classification of quiver tails

Let us recall here the classification of conformal $SU(N)$ linear quiver tails \cite{6, 7}.

An $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theory is conformal when the total number of matter fields in the fundamental and anti-fundamental representations is $N_F = 2N$. Any subgroup of the flavor symmetry may be gauged, but then we should ask whether the resulting gauge theory is still conformal. The simplest theories constructed in this way have some number $k$ of $SU(N)$ gauge multiplets arranged along a line with nearest neighbors having bi-fundamental matter charged under the two groups. At the end of this linear quiver we can add bi-fundamental matter between the first and last group, getting a quiver ring. It is also possible to add $N$ fundamental fields to the first and $N$ anti-fundamental to the last, giving a linear quiver.

In the latter configuration, at each end of the linear quiver there is an $SU(N)$ flavor symmetry, and we can try to gauge a subgroup of it. For example, if $N$ is even we can gauge an $SU(N/2)$ subgroup. This gauge theory will be conformal, since there are $N$ fields in the fundamental of $SU(N/2)$. Such constructions can be done at either end of the quiver and
involve a finite series of groups of decreasing rank. The most general linear quiver tail has gauge group

\[ SU(N_1) \times SU(N_2) \times \cdots \times SU(N_{k-1}) \times SU(N_k) , \quad N_k = N . \]  

(2.1)

The requirement of conformality is that the sequence of integers \( N_1, \cdots, N_k \) is convex, i.e.

\[ N_1 \geq N_2 - N_1 \geq N_3 - N_2 \geq \cdots \geq N_k - N_{k-1} . \]  

(2.2)

The \( SU(N_i) \) group couples to \( N_{i-1} + N_{i+1} \) fields\(^2\) from the two bi-fundamentals so that by adding \( 2N_i - (N_{i-1} + N_{i+1}) \geq 0 \) extra fundamental fields, we get a conformal theory.

The most general linear quiver tail is illustrated in Figure 1. The circles are gauge multiplets, the boxes are fundamental fields and the horizontal lines are bi-fundamentals. On the right this quiver is terminated on \( N \) anti-fundamentals, but it can also be continued with more \( SU(N) \) gauge fields and another quiver tail. The flavor symmetry is given by the product of \( U(2N_i - (N_{i-1} + N_{i+1})) \), an extra \( U(N) \) from the fundamental field on the right and \( k - 1 \) factors of \( U(1) \). We will also add masses to the fields, where each of the fundamental fields can have a mass and likewise the bi-fundamentals. There are in total \( \sum_i (2N_i - N_{i-1} + N_{i+1}) \) = \( N_i \) mass parameters for the fundamental fields and \( k - 1 \) mass parameters for the bi-fundamentals. In our conventions all masses as well as Coulomb branch parameters are purely imaginary.

We can encode the structure of the quiver tail in a Young-diagram with \( N \) boxes by taking the \( r \)th row to be of length \( N_l - N_{l-1} \). It will be useful also to look at the columns in the diagram and we label the Young-diagram by the sequence of heights \([n_1, n_2, \cdots, n_{N_l}]\).

The longest quiver tail has \( N_1 = 2 \) and \( N_l - N_{l-1} = 1 \) for all \( l > 1 \) and has \( N - 1 \) rows. It has two columns with \([N - 1, 1]\) and is also referred to as a simple puncture. The shortest quiver has \( k = 1 \), so it’s a single row, or \([1, 1, \cdots, 1]\), also known as a full puncture.

### 2.2 Semi-degenerate representations of Toda CFT

The classification of quiver tails in terms of Young-diagrams matches the classification of physical semi-degenerate representations of conformal Toda theory \[^3\]. Semi-degenerate fields have descendants which are null vectors, the restriction to physical semi-degnerates is that all the null states are at level 1 (or are descendants of other null states). If the state satisfies \( N - 1 \) independent null conditions the state is completely degenerate, and if they are all at level 1, it's the identity state.

A generic primary state of \( A_{N-1} \) Toda CFT is given by \( N - 1 \) continuous parameters (modulo \( S_N \) Weyl reflections). It can be easily written in the orthonormal basis in terms of an \( N \)-vector whose components sum to zero. Due to the constraints satisfied by the semi-degenerate states, they will have fewer continuous parameters.

\[^2\]Here \( N_0 = 0 \).
To be specific, the vertex operator for a primary state of Toda CFT is given (ignoring normal ordering issues) by
\[ e^{\langle \alpha, \phi \rangle} , \tag{2.3} \]
where \( \phi \) is the Toda field defined on the root space of the \( A_{N-1} \) algebra and the vector \( \alpha \) has the form
\[ \alpha = \vec{Q} + \gamma , \tag{2.4} \]
where \( \vec{Q} \) is the background charge\(^3\)
\[ \vec{Q} = Q \rho , \quad Q = b + \frac{1}{b} , \tag{2.5} \]
and \( \rho \) is the Weyl vector of \( A_{N-1} \), that is half the sum of all the positive roots. \( b \) is the coupling constant of the action of Toda. The Virasoro central charge is
\[ c = N - 1 + Q^2 N(N^2 - 1) . \tag{2.6} \]

For a generic state of Toda, \( \gamma \) is purely imaginary and breaks the entire \( S_N \) Weyl group. For the physical (level 1) degenerate fields, the weight \( \alpha \) is invariant under a subgroup of the Weyl group.\(^4\) Since \( \vec{Q} \) is real and breaks the Weyl group, \( \gamma \) is not purely imaginary anymore.

\(^3\)While many other quantities are vectors, we write the arrow only over \( Q \) to distinguish the vector \( Q \rho \) from the constant \( Q \).

\(^4\)For more details on semi-degenerate states see \cite{8, 12}.
To see the relation to Young-diagrams, for a state where $\alpha$ breaks $S_N$ to $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_N}$ we associate the Young-diagram $[n_1, n_2, \ldots, n_N]$. Such a state will depend on $N_1 - 1$ continuous imaginary parameters, one for each column in the diagram, with the weighted sum vanishing. We write explicit expressions for such states below in \( (2.7) \).

### 2.3 Subspaces of primaries of Toda CFT

According to the AGT correspondence we should associate to each gauge theory a Riemann surface with punctures, decorated by mass parameters. In the case of linear quivers, the surface is a punctured sphere. The number of punctures is the number of gauge group components plus three.\(^5\)

\(^5\) $S_I$ is of course trivial. We included them to get the sum $\sum n_i = N$.\(^6\)

\(^6\) A simple way to do the counting it to recall that the complex moduli parameters of the surface are mapped to the gauge couplings.
Since the classification of quiver tails matches the physical semi-degenerate fields of Toda, it is natural to expect that for each of the tails there will be one puncture with the semi-degenerate state. Indeed the analysis of the Seiberg-Witten curve for theories with a quiver tail gives a special singular point, and many other regular singularities ("simple punctures"). In the case of the theory in Figure 1 there are \( k + 1 \) simple punctures, one full puncture, corresponding to the right-most matter in the quiver and a special puncture where we will insert the semi-degenerate state.

We expect to be able to reproduce the partition function of this gauge theory by studying Toda CFT on this Riemann surface. We consider the trivalent graph in Figure 2 representing the successive fusion of the semi-degenerate state \( \alpha^{(0)} \) with simple-puncture states \( \mu^{(l)} \). The intermediate states \( \alpha^{(l)} \) with \( 1 \leq l \leq k \) have to be integrated over. They should be related to the Coulomb branch parameters \( a^{(l)} \), which take values in \( i\mathbb{R}^{N_l - 1}/S_{N_l} \), and therefore the states \( \alpha^{(l)} \) should be restricted to such a subspace of primaries of Toda.

We can now state in a precise way the question we would like to answer:

1. How are the mass parameters \( m_l \) and \( \hat{m}_l \) in the gauge theory encoded in the states \( \mu^{(l)}, \alpha^{(0)} \) and \( \alpha^{(k+1)} \).

2. What are the allowed states \( \alpha^{(l)} \) in the intermediate channels and how are they related to the Coulomb branch parameters \( a^{(l)} \).

Our proposal is the following:

- The mass parameters determine \( \alpha^{(0)}, \mu^{(l)} \) and \( \alpha^{(k+1)} \) according to (2.11) and (2.12).
- The intermediate states \( \alpha^{(l)} \) are semi-degenerate states. Their Young-diagrams is gotten from that of \( \alpha^{(l-1)} \) by combining the first two rows into a single row.
- The Coulomb branch parameters \( a^{(l)} \) are encoded in a subset of the directions of the vector \( \alpha^{(l)} \) which are not degenerate. The other directions are related to the mass parameters to the right of that state, see (2.11) and (2.12).

To start we should specify how we encode the mass parameters of the fundamental fields in \( \hat{m} \). There are in total \( N_1 \) fundamental fields for the different groups (excluding the \( N \) fundamentals on the right), one for each column in the Young-diagram. We label \( \hat{m}_1 \) the mass of the right most fundamental field in the quiver and \( \hat{m}_{N_1} \) the left most.

The state \( \alpha^{(0)} \) has \( N_1 - 1 \) continuous parameters and a symmetry \( S_{n_1} \times \cdots \times S_{N_{N_1}} \). We may parametrize \( \alpha^{(0)} \) in the orthonormal basis as

\[
\alpha^{(0)} = \vec{Q} + (\beta_{N_1}^{(0)} + \delta_{n_{N_1},1}, \ldots, \beta_{N_1}^{(0)} + \delta_{n_{N_1},n_{N_1}}, \ldots, \beta_{1}^{(0)} + \delta_{n_{1},1}, \ldots, \beta_{1}^{(0)} + \delta_{n_{1},n_{1}}),
\]

where

\[
\sum_{j=1}^{N_1} n_j \beta_j^{(0)} = 0.
\]
The parameters $\delta_{n,j} = (2j - n - 1)Q/2$ are necessary to make the state $\alpha^{(0)}$ invariant under the desired subgroup of $S_N$.

The Young-diagram has $2N_1 - N_2$ columns of unit height (for which $n_i = 1$ and $\delta_{1,1} = 0$). We write the $\beta^{(0)}_i$ parameters associated to them at the left of the vector, with the parameters for the other columns of the Young-diagram further on the right.

Our claim is that the state $\alpha^{(l)}$ will have a Young-diagram with rows of length $N_{l+1}, N_{l+2} - N_{l+1}, \ldots, N - N_{k-1}$. There are $2N_{l+1} - N_{l+2}$ columns of unit height and the others are of height $n_j - l$, where $n_j$ are the heights of the columns in the original Young-diagram (including only the columns for which $n_j - l > 1$).

We parameterize the intermediate states of this type in the orthonormal basis as

$$
\alpha^{(l)} = \tilde{Q} + (\tilde{\beta}^{(l)}_1 + \gamma^{(l)}_1, \ldots, \tilde{\beta}^{(l)} + \gamma^{(l)}_{N_l} \cdot \beta^{(l)}_{N_{l+1} - N_l} + \delta_{N_{l+1} - N_l - 1}, \ldots, \beta^{(l)}_{N_{l+2} - N_{l+1} - 1} + \delta_{N_{l+2} - N_{l+1} - 1}\cdot)
$$

$$
\beta^{(l)}_1 + \delta_{N_1 - 1}, \ldots, \beta^{(l)}_1 + \delta_{N_1 - 1} - l)
$$

where

$$
N_l \tilde{\beta}^{(l)} = - \sum_{i=1}^{N_{l+1} - N_l} (n_i - l) \beta^{(l)}_i, \quad \text{and} \quad \sum_{i=1}^{N_l} \gamma^{(l)}_i = 0.
$$

The simple puncture states $\mu^{(l)}$ are also semi-degenerate with Young-diagrams with columns $[N - 1, 1]$ and one continuous parameter $\mu_l$. We take them to be proportional to the first fundamental weight $\mu^{(l)} = (Q/2 - \mu_l)N\omega_1$.

The identification we propose is that $\gamma^{(l)}$ are the $N_l - 1$ Coulomb branch parameters $a^{(l)}_i$ of $SU(N_l)$. The parameters $\beta^{(l)}_i$ as well as the parameters of the simple punctures are given by the mass parameters as the solution to the following set of linear equations

$$
\begin{align*}
\mu_l - \beta^{(l-1)}_i + \beta^{(l)}_i &= 0, & l = 1, \ldots, n_i, \\
\mu_l - \bar{\beta}^{(l-1)} + \bar{\beta}^{(l)} &= m_{l-1}, & l = 2, \ldots, n_1.
\end{align*}
$$

with the initial values

$$
\beta^{(n_1)}_i = \bar{\beta}^{(n_1)}_i = \hat{m}_1, \quad \beta^{(n_1)} = 0.
$$

These equations can be solved recursively from large to small $l$.

The examples we concentrate on in the following sections are of Toda theory with $N + 1$ simple punctures and one full puncture, so that $\alpha^{(0)}$ also has a Young-diagram with columns

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Footnotes:

7In [12] these terms were grouped together into $Q\rho_I$.

8$\beta^{(n_1)}_i$ with these indices do not appear in the parametrization of the states above, but are useful for writing the equations.
Then the solution to the conditions (2.11) is

\[ \beta^{(1)} = \frac{l + 1}{N} \left( \hat{m}_1 + \sum_{i=l}^{N-2} m_i \right), \quad N \mu_l = l m_{l-1} - \hat{m}_1 - \sum_{i=l}^{N-2} m_i, \quad m_0 = -\hat{m}_2. \] (2.13)

In the rest of this manuscript we test this identification by studying the fusion rules of two semi-degenerate states and a simple puncture. We also compare the product of three point functions appearing in the tail with the 1-loop partition function arising from integrating out the fundamental and bi-fundamental fields in the gauge theory.

### 2.4 The Seiberg-Witten curve

As a first test, though, we would like to check the consistency of the construction under decoupling of one of the gauge groups. We consider sending to zero the coupling of the first gauge multiplet, that of $SU(N_1)$. In terms of the Riemann surface this is a degeneration limit where the neck connecting the first two punctures (with insertions $\alpha^{(0)}$ and $\mu^{(1)}$) is stretched to infinite length. In the limit this neck will look like a single puncture on the sphere with the state $\alpha^{(1)}$ and the same $k + 1$ other operators as before.

On the gauge theory side we have the gauge group $SU(N_2) \times \cdots \times SU(N_k)$. The Coulomb branch parameters of $SU(N_1)$ and $m_1$ are now frozen as mass parameters for $SU(N_2)$. Looking at how we defined $\alpha^{(0)}$ and $\mu^{(1)}$ before exactly matches the definition of $\alpha^{(1)}$ and $\mu^{(2)}$ in terms of the new masses. The identification of all the other parameters $\alpha^{(l)}$ and $\mu^{(i+1)}$ for $i > 1$ is not affected by the decoupling of the first $SU(N_1)$, since their definitions depended only on parameters that appear in the quiver to the right of $SU(N_2)$.

This can be done in more detail in the semiclassical approximation (ignoring $b$ and $b^{-1}$) by examining the Seiberg-Witten curve for these theories, which was written down in [13, 8]

\[ 0 = \prod_{i=1}^{N} (v - \check{M}_i) z^{k+1} + c_k \left( v^{N} - M_k v^{N-1} - u^{(k)}_2 v^{N-2} - \cdots - u^{(k)}_{N-1} v - u^{(k)}_N \right) z^k + \cdots \]
\[ + c_j \left( \prod_{i=1}^{N_j+1-N_j} (v - \check{M}_i)^{n_{i-j}} \right) \left( v^{N_j} - M_j v^{N_j-1} - \cdots - u^{(j)}_{N_{j-1}} v - u^{(j)}_{N_j} \right) z^j + \cdots \]
\[ + c_0 \prod_{i=1}^{N_1} (v - \check{M}_i)^{n_i} \] (2.14)

where $c_j$ and $u_s^{(j)}$ with $s = 2, \ldots, N_j$ are the gauge coupling and a set of Coulomb branch parameters for the $j$-th gauge group. $\check{M}_i$ are the masses of the $N$ fundamental hypermultiplets coupling to the rightmost gauge group. $\check{M}_i$ and $M_i$ are related to the masses of the fundamental hypermultiplets of the tail and the bifundamental matter.

The physical masses of BPS states can be read from the poles in the Seiberg-Witten differential. The
This curve was rewritten by Gaiotto in a different form [6]. First we collect all the terms with definite power of \(v\)

\[
\Delta(z) v^N + M(z) v^{N-1} + R_{N-2}(z) v^{N-2} + \cdots + R_{N-1}(z) = 0
\]

(2.15)

and then shift \(v\) as \(v \rightarrow v - \frac{M(z)}{N\Delta(z)}\). Considering the change of coordinate \(v = xz\) the curve can be written as

\[
x^N + \Phi^{(N-1)}(z)x^{N-2} + \cdots + \Phi^{(N)}(z) = 0.
\]

(2.16)

In this way the Seiberg-Witten curve is written as an \(N\)'th cover of a Reimann surface which in this case is the \(k + 3\) punctured sphere.

The functions \(\Phi^{(I)}(z)\) have poles of order \(I\) at the location of each of the punctures \(z = z_i\)

\[
\Phi^{(I)}(z) = \frac{\phi^{(I)}_{i,0}}{(z - z_i)^I} + \frac{\phi^{(I)}_{i,-1}}{(z - z_i)^{I-1}} + \cdots
\]

(2.17)

and it was shown in [11, 8] that in the semiclassical approximation the coefficient of the leading poles are the charges under the W-symmetry of the Toda state inserted at the puncture

\[
\bar{\phi}^{(I)}_{i,0} \sim w^I_i.
\]

(2.18)

It is straight-forward to read the parameters of the Toda states from the Seiberg-Witten curve, which is written explicitly in (2.14). One does has to take into account the shift in \(v\) needed to write (2.16), which act by linear transformations on the mass parameters. Furthermore, as mentioned in footnote 9, these parameters are not exactly the physical masses in the theory. In Section 3.4 we do it in detail for the \(SU(2) \times SU(3)\) theory. Here we study the allowed intermediate states in the general curve, but due to this difficulty, we discuss only the type of states, not the exact values of the mass parameters.

Looking at (2.14) and expanding around \(z \sim 0\) we can identify in the semiclassical approximation [8]

\[
\alpha^{(0)} \sim \left( \frac{M^{(0)}_{N_1}}{n_{N_1}}, \ldots, \frac{M^{(0)}_{N_1}}{n_{N_1}}, \frac{M^{(0)}_{1}}{n_1}, \ldots, \frac{M^{(0)}_{1}}{n_1} \right)
\]

(2.19)

where \(\hat{M}^{(0)}_i = \hat{M}_i - \bar{M}^{(0)}\) and \(\bar{M}^{(0)} = \frac{1}{N} \sum_{j=1}^{N_1} n_j \hat{M}_j\). Considering the parameterization (2.7), we have \(\beta^{(0)}_i = \hat{M}^{(0)}_i\).
In order to analyze the internal state $\alpha^{(l)}$, we decouple the first $l$ gauge groups by setting $c_j = 0$ for $j < l$. The Seiberg-Witten curve becomes

$$0 = \prod_{i=1}^{N} (v - \hat{M}_i) z^{k+1} + c_k \left( v^N - M_k v^{N-1} - u^{(k)}_2 v^{N-2} - \cdots - u^{(k)}_{N-1} v - u^{(k)}_{N} \right) z^k + \cdots$$

$$+ c_l \left( \prod_{i=1}^{N_{i+1}-N_i} (v - \bar{M}_i)^{n_i - l} \right) \left( \prod_{i=1}^{N_i} (v - r_i) \right) z^l$$

(2.20)

where $r_i$, with $i = 1, \ldots, N_i$ are the roots of the polynomial

$$v^{N_i} - M_i v^{N_i-1} - u^{(l)}_2 v^{N_i-2} - \cdots - u^{(l)}_{N_i-1} v - u^{(l)}_{N_i}$$

(2.21)

and thus satisfy $\sum_{i=1}^{N_i} r_i = M_i$. $r_i$ are the $N_i$ parameters of the Coulomb branch of the $SU(N_i)$ gauge group. Defining $\hat{M}_i^{(l)} = \hat{M}_i - \bar{M}^{(l)}$ and $r_i^{(l)} = r_i - \bar{M}^{(l)}$ with

$$\bar{M}^{(l)} = \frac{1}{N} \left( \sum_{i=1}^{N_{i+1}-N_i} (n_i - l) \hat{M}_i + M_i \right)$$

(2.22)

we can identify semiclassically

$$\alpha^{(l)} \sim (r_1^{(l)}, \ldots, r_{N_i}^{(l)}, \hat{M}_1^{(l)}, \hat{M}_2^{(l)}, \hat{M}_3^{(l)}, \hat{M}_4^{(l)}, \hat{M}_5^{(l)}, \hat{M}_6^{(l)}; n_{N_{i+1}-N_i-1}^{(l)}, \cdots, n_{1-l}^{(l)})$$

(2.23)

The analysis of the Seiberg-Witten curve thus confirm that the internal states $\alpha^{(l)}$ are of the form described in (2.9).

As mentioned above, the mass parameters as they appear in these states are not the physical masses, but are related to them by linear transformations. In fact, we can view our identification of states in Section 2.3 as filling in the information needed to complete the writing of the curve in Gaiotto form. The functions $\Phi^{(l)}(z)$ are completely determined by knowing their behavior near all their singularities. The leading poles give the W-charges of the external states. The lower order poles (like $\phi_{i, -1}^{(l)}$) satisfy certain constraints among them which mirror the degeneracy conditions of the Toda states and after solving for these constraints, the remaining information parametrizes the Coulomb branch.

3 The case of $SU(2) \times SU(3)$

We study now in most detail the case of $SU(2) \times SU(3)$ gauge theory, whose quiver diagram is shown in Figure 4. The $SU(2)$ gauge group with Coulomb branch parameter $a^{(l)}$ couples to a single fundamental field with mass $\hat{m}_2$ and a bi-fundamental field of mass $m_1$. The $SU(3)$
Figure 4: The simplest linear quiver tail with gauge group $SU(2) \times SU(3)$.

On the Toda side we have semi-degenerate external states $\alpha^{(0)}$ and $\mu^{(1)}$ which fuse to the primary $\alpha^{(1)}$. That is fused with $\mu^{(2)}$ to the intermediate state $\alpha^{(2)}$ which is finally fused with $\mu^{(3)}$ to the generic external state $\alpha^{(3)}$. We parameterize these states as

$$
\alpha^{(l)} = \bar{Q} + (2\beta^{(0)}, \beta^{(0)} - Q/2, \beta^{(0)} + Q/2) \\
\beta^{(1)} = -2\bar{\beta}^{(1)}, \gamma^{(1)} = -\gamma^{(1)} \tag{3.1}
$$

$$
\mu^{(l)} = \bar{Q} + (-2\mu_l, \mu_l - Q/2, \mu_l + Q/2) \quad l = 1, 2, 3.
$$

### 3.1 Fusion rules from Ward identity

In the preceding section we proposed a map of the parameters between the quiver tails in $\mathcal{N} = 2$ conformal gauge theories and correlation functions in Toda CFT with certain semi-degenerate states in the intermediate channels. The space of primary states in the intermediate channels is restricted by the conditions $\alpha^{(l)}$ and $\mu^{(l)}$. Normally we should sum over all primary states in the intermediate channel. Our claim is that only the states we wrote
down in (2.9) appear in the fusion rules of the other states (to the left) in the correlation function.

In the case of the SU(2) × SU(3) quiver, this claim amounts to saying that the fusion of two semi-degenerate states α(0) and µ(1) gives a state α(1), which while not degenerate, does satisfy a condition restricting one of its two continuous parameters, β(1), to be related to β(0) and µ1 and in turn, to the masses in the gauge theory picture.

Being semi-degenerate means that the Verma module of the state α(0), has a null states at level 1. This degeneracy condition allows to write a Ward identity on the three point function of α(0), µ(1) and α(1). Due to the fact that µ(1) is also semi-degenerate, this Ward identity can be written as a condition on the state α(1). This Ward identity was written in [9] and we present here a slightly different derivation of it.

The degeneracy condition of α(0) is given by
\[ \langle \alpha(0) | \left( W_1 - \frac{3w_0}{2\Delta_0} L_1 \right) \sim 0 \] (3.2)
where
\[ \langle \alpha(0) | L_0 = \Delta_0 \langle \alpha(0) | , \quad \langle \alpha(0) | W_0 = w_0 \langle \alpha(0) | . \] (3.3)

We would like to use the equation satisfied by α(0) to derive a constraint on α(1), so we need to commute the operators through V_µ(1)(z). The action of L_n and W_n on the primaries is given by
\[
\begin{align*}
[L_n, V_\mu(1)(z)] &= z^{n+1} \partial V_\mu(1) + \Delta_\mu (n + 1) z^n V_\mu(1)(z) , \\
[W_n, V_\mu(1)(z)] &= z^n \left( \frac{w_\mu}{2} (n + 1)(n + 2) + (n + 2) z \hat{W}_- + z^2 \hat{W}_- \right) V_\mu(1)(z) .
\end{align*}
\] (3.4)

By virtue of the degeneracy condition of the state µ, which can be written as \[ [W_-1, V_\mu(1)] = \frac{3w_\mu}{2\Delta_\mu} [L_-1, V_\mu(1)] , \] the combinations
\[ e_n = L_n - z L_{n-1} , \quad f_n = W_n - z W_{n-1} - z \frac{3w_\mu}{2\Delta_\mu} L_{n-1} , \] (3.5)
have simple commutation relations with V_µ(1)
\[
\begin{align*}
[e_n, V_\mu(1)(z)] &= z^n \Delta_\mu V_\mu(1)(z) , \\
[f_n, V_\mu(1)(z)] &= - z^n \frac{n-2}{2} w_\mu V_\mu(1)(z) .
\end{align*}
\] (3.6)

The degeneracy condition (3.2) can be written as
\[ \langle \alpha(0) | \left( W_1 - \frac{3w_0}{2\Delta_0} L_1 \right) = \langle \alpha(0) | \left( f_1 - \frac{3w_0}{2\Delta_0} e_1 + z \left( - \frac{1}{2} w_0 + \frac{3w_\mu}{2\Delta_\mu} \Delta_0 \right) \right) \sim 0 . \] (3.7)
Permuting through $V_{\mu(1)}(z)$ we get

$$\langle \alpha^{(0)} | \left( W_1 - \frac{3w_0}{2\Delta_0} L_1 \right) V_{\mu(1)}(z) \rangle = \langle \alpha^{(0)} | V_{\mu(1)} \left( f_1 - \frac{3w_0}{2\Delta_0} e_1 + z \left( -\frac{1}{2} w_0 + \frac{3w_\mu}{2\Delta_\mu} \Delta_0 + \frac{1}{2} w_\mu - \frac{3w_0}{2\Delta_0} \Delta_\mu \right) \right) \rangle = \langle \alpha^{(0)} | V_{\mu(1)} \left( W_1 - \frac{3w_0}{2\Delta_0} L_1 - z \left( W_0 + \frac{3}{2} \left( \frac{w_\mu}{\Delta_\mu} + \frac{w_0}{\Delta_0} \right) L_0 + \frac{w_0}{2} - \frac{3w_\mu}{2\Delta_\mu} \Delta_0 - \frac{w_\mu}{2} + \frac{3w_0}{2\Delta_0} \Delta_\mu \right) \right) \rangle . \quad (3.8)$$

Finally we can write the Ward identity by multiplying by $|\alpha^{(1)}\rangle$ on the right. The degeneracy condition implies that this correlation function should vanish. Given that $\alpha^{(1)}$ is a primary, it is annihilated by $L_1$ and $W_1$ and satisfies $L_0|\alpha^{(1)}\rangle = \Delta_1|\alpha^{(1)}\rangle$ and $W_0|\alpha^{(1)}\rangle = w_1|\alpha^{(1)}\rangle$. We get an equation relating $w_1$ and $\Delta_1$ by

$$\left( w_1 + \frac{3}{2} \left( \frac{w_\mu}{\Delta_\mu} - \frac{w_0}{\Delta_0} \right) \right) (\Delta_1 - \Delta_0 - \Delta_\mu) - w_0 + w_\mu \right) \langle \alpha^{(0)} | V_{\mu(1)}(z) |\alpha^{(1)}\rangle = 0 . \quad (3.9)$$

Therefore, for the three point function not to vanish, the prefactor has to. This is indeed the equation found by [9] which restricts the allowed primaries $\alpha^{(1)}$ in the intermediate channel of the $A_2$ description of $SU(2) \times SU(3)$ gauge theory.

To understand this equation it is useful to write the quantum numbers of the states as

$$\Delta = \frac{1}{2} \langle \alpha, 2\tilde{Q} - \alpha \rangle = Q^2 + \sum_{i<j} \langle \alpha - Q, h_i \rangle \langle \alpha - Q, h_j \rangle ,$$

$$w = \kappa \prod_{i=1}^3 \langle \alpha - Q, h_i \rangle , \quad \kappa = i \sqrt{\frac{48}{22 + 5c}} . \quad (3.10)$$

Here $h_i$ are the weights of the fundamental representation, which for $SU(3)$ are

$$h_1 = \frac{1}{3} \left( 2, -1, -1 \right) , \quad h_2 = \frac{1}{3} \left( -1, 2, -1 \right) , \quad h_3 = \frac{1}{3} \left( -1, -1, 2 \right) . \quad (3.11)$$

For the degenerate states parameterized in the orthonormal basis as in (3.1) we have

$$\Delta_0 = 3 \left( \frac{Q^2}{4} - (\beta^{(0)})^2 \right) , \quad w_0 = \frac{2}{3} \kappa \beta^{(0)} \Delta_0 .$$

$$\Delta_\mu = 3 \left( \frac{Q^2}{4} - \mu_1^2 \right) , \quad w_\mu = \frac{2}{3} \kappa \mu_1 \Delta_\mu . \quad (3.12)$$

Plugging these relations into (3.9) we find the equation for the components of $\alpha^{(1)}$

$$\prod_{i=1}^3 \left( \langle \alpha^{(1)} - Q, h_i \rangle + \mu_1 - \beta^{(0)} \right) = 0 . \quad (3.13)$$
The three different solutions to this equation are related, of course, by Weyl reflections. Using the parametrization of $\alpha^{(1)}$ in (3.1) we can choose the solution

$$\beta^{(1)} = \beta^{(0)} - \mu_1.$$  \hfill (3.14)

As we see, though $\alpha^{(1)}$ is not a degenerate state, one of its components is fixed by the external parameters $\beta^{(0)}$ and $\mu_1$. The other parameter $\gamma^{(1)}_1 = -\gamma^{(2)}_2$ is free, and should match the Coulomb branch parameter of the $SU(2)$ gauge group.

### 3.2 One loop contribution

Within the AGT correspondence the product of the 1-loop determinants of the vector and hypermultiplets matches the three point functions associated to the fusion of successive primary states in the CFT.

In the following two sections we study the three point functions of semi-degenerate states in Toda CFT which are needed for the calculation of the quiver tail. In particular we show in Section 4 that imposing the condition (3.14) gives a pole in the three point function, whose residue is the desired three point function. In Section 5 we combine all the terms together and show in a more general setup that the 1-loop partition function agrees with the product of three point functions in Toda CFT. It is easy to plug in the states parameterized above in (3.1) into the resulting expressions, and we will not copy it here.

The upshot of that calculation is that it allows us to identify the gauge theory and Toda parameters as

$$\gamma_i^{(1)} = a_i^{(1)} , \quad \gamma_i^{(2)} = a_i^{(2)} , \quad \beta^{(3)}_i + \mu_3 = \tilde{m}_i ,$$

$$3\beta^{(0)} = m_1 + \hat{m}_1 - \hat{m}_2 , \quad \frac{3}{2} \beta^{(1)} = m_1 + \hat{m}_1 ,$$

$$3\mu_1 = -m_1 - \hat{m}_1 - \hat{m}_2 , \quad 3\mu_2 = 2m_1 - \hat{m}_1 , \quad 3\mu_3 = \sum_{i=1}^3 \tilde{m}_i.$$  \hfill (3.15)

### 3.3 Instantons and descendants

Within the AGT correspondence, the instanton corrections to the gauge theory partition function [16, 17] match the contributions of descendants to the conformal blocks of the CFT. We turn now to analyzing this question for our case of the $SU(2) \times SU(3)$ gauge theory and its description within $A_2$ Toda CFT.

The relevant Toda correlator we are calculating is on the five-punctured sphere represented by the graphs in Figure 4. There are two intermediate states, $\alpha^{(1)}$ and $\alpha^{(2)}$ and the
conformal blocks capture the contributions of all W-algebra descendants of these primaries

\[
\mathcal{F}(q^{(1)}, q^{(2)}) = \sum_{x, x', y, y'} (q^{(1)})|x|(q^{(2)})|y| \frac{\langle \alpha^{(0)}|V_{\mu^{(1)}}|x; \alpha^{(1)} \rangle}{\langle \alpha^{(0)}|V_{\mu^{(1)}}|\alpha^{(1)} \rangle} X^{-1}_{x|x'}(\alpha^{(1)}) X^{-1}_{y|y'}(\alpha^{(2)}) \frac{\langle x'; \alpha^{(1)}|V_{\mu^{(2)}}|y; \alpha^{(2)} \rangle}{\langle \alpha^{(1)}|V_{\mu^{(2)}}|\alpha^{(2)} \rangle}.
\]

The expansion parameters \(q^{(i)}\) are the complex structure moduli of the Riemann surface, related to the gauge groups couplings \(\tau^{(i)}\) by \(q^{(i)} = e^{2\pi i \tau^{(i)}}\). The sum is taken over all the descendants \(|x; \alpha\rangle = L_{-x_1} \ldots L_{-x_r} W_{-\tilde{x}_1} \ldots W_{-\tilde{x}_r}|\alpha\rangle\), represented by the vectors \(x = (x_1, \ldots, x_r, \tilde{x}_1, \ldots, \tilde{x}_r)\) with \(x_1 \leq \cdots \leq x_r\) and \(\tilde{x}_1 \leq \cdots \leq \tilde{x}_r\). The level of the descendant is given by \(|x| = x_1 + \cdots + x_r + \tilde{x}_1 + \cdots + \tilde{x}_r\). \(X^{-1}_{x|x'}(\alpha)\) is the inverse of the Shapovalov matrix defined as \(X_{x|x'}(\alpha) = \langle x; \alpha|x'; \alpha\rangle\). The Shapovalov matrix is block diagonal, where any block correspond to a set of descendants with a given level. Restricting to a fixed level we define

\[
\mathcal{F}(q^{(1)}, q^{(2)}) = \sum_{n_1, n_2} \mathcal{F}_{n_1, n_2}(q^{(1)})^{n_1}(q^{(2)})^{n_2}.
\]

The contribution of a single instanton to the \(SU(3)\) gauge theory was matched with the level one conformal block in \(A_2\) Toda CFT in [3] and two instantons were matched with level two in [18]. The generalization to the linear quiver with \(SU(3) \times SU(3)\) gauge group was done in [9]. Here we study the case of \(SU(2) \times SU(3)\) and the novel question, as with the primaries discussed above, is how does \(A_2\) Toda reproduce the \(SU(2)\) part of the theory, which is also captured by Liouville theory. To address this question we can focus on the \(q^{(1)}\) dependance and ignore all descendants of \(\alpha^{(2)}\)

\[
\mathcal{F}(q^{(1)}) = \sum_{n_1} \mathcal{F}_{n_1, 0}(q^{(1)})^{n_1} = \sum_{x, x'} (q^{(1)})|x| \frac{\langle \alpha^{(0)}|V_{\mu^{(1)}}|x; \alpha^{(1)} \rangle}{\langle \alpha^{(0)}|V_{\mu^{(1)}}|\alpha^{(1)} \rangle} X^{-1}_{x|x'}(\alpha^{(1)}) \frac{\langle x'; \alpha^{(1)}|V_{\mu^{(2)}}|\alpha^{(2)} \rangle}{\langle \alpha^{(1)}|V_{\mu^{(2)}}|\alpha^{(2)} \rangle}.
\]

We will restrict our analysis to level one, where there are two descendants \(L_{-1}|\alpha^{(1)}\) and \(W_{-1}|\alpha^{(1)}\). The Shapovalov matrix for the level one states is a \(2 \times 2\) matrix which was evaluated in [3] [18], as were the ratio of 3-point functions of the descendants and primaries. We plug in the expression for the state \(\alpha^{(1)}\) satisfying the condition (3.14) and use the degeneracy condition for \(\alpha^{(0)}\) and \(\mu^{(1)}\) and \(\mu^{(2)}\) to find (in the parametrization (3.11))

\[
\mathcal{F}_{1, 0} = -6 \frac{(\beta^{(0)} - \beta^{(1)})^2}{Q^2 - 4(\gamma^{(1)})^2} \prod_{i=1}^{3} (\gamma_i^{(2)}) + \mu_2 + \frac{\beta^{(1)}}{2} + D,
\]

where

\[
D = \frac{1}{4} \sum_{i=1}^{3} (\gamma_i^{(2)})^2 - \frac{1}{2} (\gamma_1^{(1)})^2 + \frac{3}{4} (\beta^{(0)})^2 - 2\mu_1^2 - 2\mu_2^2 + \beta^{(0)} \mu_1 + \beta^{(0)} \mu_2 + \mu_1 \mu_2 + \frac{3}{8} Q^2
\]
In the gauge theory the SU(2) vector multiplet with Coulomb branch parameter $a^{(1)}$ couples to one fundamental and three bi-fundamental fields. This is equivalent to four fundamental hypermultiples with masses equal to $\hat{m}_2, m_1 + a^{(2)}_1, m_1 + a^{(2)}_2, m_1 + a^{(2)}_3$. An explicit instanton counting for SU(2) gauge theory coupled to 4 fundamental hypermultiplet with these masses gives \cite{19, 20, 3}

$$Z_{\text{inst}}(q^{(1)}) = 1 + q^{(1)}{\frac{2\hat{m}_2 \prod_{i=1}^{3} (a^{(2)}_i + m^{(2)}_i)}{Q^2 - 4a^{(1)}_1}} + \mathcal{O}((q^{(1)})^2) \quad (3.21)$$

Using the dictionary relating the Toda parameters and the gauge theory parameters (3.15), derived from comparing the one-loop partition function and Toda three point functions, we find that the ratio in (3.19) matches exactly the partition function for a single instanton in SU(2) (3.21). It is tempting to conjecture that a similar relation persists to higher order in the instanton expansion such that the SU(2) instanton partition functions and $A_2$ Toda conformal blocks would be related by

$$\mathcal{F}(q^{(1)}) = (1 - q^{(1)})^{-D} Z_{\text{inst}}(q^{(1)}) \quad (3.22)$$

This is essentially the same as we would get by considering Liouville theory, where at level one there is a single state in the Virasoro Verma module. The relation we derived involved summing over two W-algebra descendants and, unlike in Liouville, the denominator in (3.19) is not the conformal dimension of the Toda state $\alpha^{(1)}$, yet it agrees with the Liouville dimension and gauge theory expression.

### 3.4 The Seiberg-Witten curve for SU(2) × SU(3)

The Seiberg-Witten curve (2.14) for the SU(2) × SU(3) theory is

$$0 = (v - \hat{M}_1)(v - \hat{M}_2)(v - \hat{M}_3)z^3 + c_2(v^3 - M_2v^2 - u^{(2)}_2v - u^{(2)}_3)z^2 + c_1(v - \hat{M}_1)(v^2 - M_1v - u^{(1)}_2)z + c_0(v - \hat{M}_1)^2(v - \hat{M}_2). \quad (3.23)$$

We can easily follow the procedure outlined in Section 2.4 and write this curve in Gaiotto form as

$$x^3 + \Phi^{(2)}(z)x + \Phi^{(3)}(z) = 0 \quad (3.24)$$

$\Phi^{(i)}(z)$ has poles at $z = 0, 1, A, B, \infty$ where we set

$$c_0 = -AB, \quad c_1 = A + B + AB, \quad c_2 = -1 - A - B. \quad (3.25)$$

\footnote{With a different $D$.}
The parameters of the Toda states are related to the Laurent coefficients of $\Phi^{(I)}$ giving

\[
\begin{align*}
\mu_1 &= \frac{(B - A - AB)\hat{M}_1 + B\hat{M}_2 - (A + B + AB)M_1 + A(1 + A + B)M_2 - A^2 \sum_{i=1}^{3} \hat{M}_i}{3(1 - A)(A - B)}, \\
\mu_2 &= \frac{(A + B - AB)\hat{M}_1 - AB\hat{M}_2 + (A + B + AB)M_1 - (1 + A + B)M_2 + \sum_{i=1}^{3} \hat{M}_i}{3(1 - A)(1 - B)}, \\
\mu_3 &= \frac{(A - B - AB)\hat{M}_1 + A\hat{M}_2 - (A + B + AB)M_1 + B(1 + A + B)M_2 - B^2 \sum_{i=1}^{3} \hat{M}_i}{3(A - B)(1 - B)},
\end{align*}
\]

(3.26)

As mentioned in Section 2.4 (see footnote 9), these complicated relations are a consequence of a complicated relation between the physical masses and the parameters in (3.23) (as well as the shift in $v$ to (3.24)). These expressions agree with our identifications in (3.15) once we relate the parameters in (3.23) to the physical masses as

\[
\begin{align*}
M_1 &= \frac{A\hat{m}_1 + A(1 + B)\hat{m}_2 - (A + 2B - AB)m_1 + \frac{4}{3}(A + 4B + 4AB)\sum_{i=1}^{3} \hat{m}_i}{A + B + AB}, \\
M_2 &= \frac{(1 + A)\hat{m}_1 A\hat{m}_2 + (A - 2)m_1 + (1 + A + 2B)\sum_{i=1}^{3} \hat{m}_i}{1 + A + B}, \\
\hat{M}_1 &= \hat{m}_1 + \frac{2}{3} \sum_{i=1}^{3} \hat{m}_i, \\
\hat{M}_2 &= \hat{m}_2 - m_1 + \frac{2}{3} \sum_{i=1}^{3} \hat{m}_i, \\
\hat{M}_i &= \hat{m}_i.
\end{align*}
\]

(3.27)

With this, the parameters $\mu_i$ as well as $\beta^{(0)}$ satisfy the relations in (3.15).

According to our prescription, of the two internal states in the Toda description of this theory $\alpha^{(1)}$ has one parameter $\beta^{(1)}$ fixed by the external data and one free parameter and $\alpha^{(2)}$ is completely free. In Section 2.4 we explained how to see the constraint on the internal states from the Seiberg-Witten curve. We consider the limit of $c_0 = 0$ in (3.23) which gives

\[
0 = \prod_{i=1}^{3} (v - \hat{M}_i)z^3 + c_2 \left(v^3 - u_2^{(2)}v - u_3^{(2)}\right)z^2 + c_1(v - \hat{M}_1)(v - r_1)(v - r_2)z
\]

(3.28)

where $r_1, r_2$ are the roots of the polynomial

\[
v^2 - M_1 v - u_2^{(1)}
\]

(3.29)

and thus satisfy $r_1 + r_2 = M_1$. Defining $\hat{M}_1^{(1)} = \hat{M}_1 - \hat{M}(1)$ and $r_i^{(1)} = r_i - \hat{M}(1)$ with $\hat{M}(1) = \frac{1}{3}(\hat{M}_1 + M_1)$ we can identify semiclassically

\[
\alpha^{(1)} \sim (r_1^{(1)}, r_2^{(1)}, \hat{M}_1^{(1)})
\]

(3.30)

In terms of our parametrization of the states in (3.1) we have

\[
\beta^{(1)} = \hat{M}_1^{(1)} = \frac{1}{3} \left(2\hat{M}_1 - M_1\right)
\]

(3.31)
After performing the redefinition (3.27) we get
\[ \beta^{(1)} = \frac{2}{3}(\tilde{m}_1 + m_1), \] (3.32)
as in equation (3.15).

### 4 Three point functions of semi-degenerate states

In the previous section we showed how a Ward identity constrains the fusion of a semi-degenerate state and a simple puncture state to a third state satisfying certain restrictions on its parameters. The derivation involved details of the \( W_3 \) algebra. In this section we show a simpler route to the same conclusion. We look at the three point functions of two generic states and one simple puncture conjectured by Fateev and Litvinov \[ \text{[10, 11]} \] and consider the limit when the two generic states become semi-degenerate.

While one would expect to find a pole in the three point function, as we shall see, generically the result is zero. To cancel this zero we impose the condition (2.11) on the momenta of the states and find extra zeroes in the denominator, giving the expected pole, whose coefficient is then interpreted as the three point function of the semi-degenerate states.

The three point function of two generic and one simple puncture state proportional to the last fundamental weight is \[ \text{[10, 11]} \]

\[
C_{FL}(\alpha, (\frac{Q}{2} - \mu)N\omega_{N-1}, \alpha') = \left[ \pi \bar{\mu} \gamma(b^2)b^{2b^2} \right]^{(2\bar{Q}^- - \alpha' - \alpha, \rho)/b} \times \frac{(\Upsilon(b))^{N-1} \Upsilon((\frac{Q}{2} - \mu)) \prod_{e>0} \Upsilon((\bar{Q}^- - \alpha, e)) \Upsilon((\bar{Q}^- - \alpha', e))}{\prod_{ij} \Upsilon((\frac{Q}{2} - \mu - (\alpha - \bar{Q}, h_i) + (\alpha' - \bar{Q}, h_j))}. \]

(4.1)
The product in the numerator is over all the positive roots and in the denominator over all the weights of the fundamental representation. The two special functions in this expression are \( \gamma(x) = \Gamma(x)/\Gamma(1-x) \) and \( \Upsilon(x) \), which is defined in the appendix. \( \bar{\mu} \) is the cosmological constant of Toda theory, not to be confused with the parameter of the state \( \frac{Q}{2} - \mu \).

For the simple puncture proportional to the first fundamental weight the expression is

\[
C_{FL}(\alpha, (\frac{Q}{2} - \mu)N\omega_1, \alpha') = \left[ \pi \bar{\mu} \gamma(b^2)b^{2b^2} \right]^{(2\bar{Q}^- - \alpha' - \alpha, \rho)/b} \times \frac{(\Upsilon(b))^{N-1} \Upsilon((\frac{Q}{2} - \mu)) \prod_{e>0} \Upsilon((\bar{Q}^- - \alpha, e)) \Upsilon((\bar{Q}^- - \alpha', e))}{\prod_{ij} \Upsilon((\frac{Q}{2} - \mu - (\alpha - \bar{Q}, h_i) + (\alpha' - \bar{Q}, h_j))}. \]

(4.2)

Using the reflection relation \( \Upsilon(x) = \Upsilon(Q - x) \), we see that the denominators of the above expressions are equal upon replacement of \( \mu \to -\mu \).

Now we take \( \alpha \) and \( \alpha' \) to be semi-degenerate states. Since keeping track of indices is really tedious in this calculation we will concentrate on the case when the two states are “hooks”，
\[\alpha = \tilde{\mathcal{Q}} + \left( -\frac{n}{N-n} \beta + \gamma_1, \ldots, -\frac{n}{N-n} \beta + \gamma_{N-n}, \beta + \delta_{n,1}, \ldots, \beta + \delta_{n,n} \right) \] (4.3)

and likewise \(\alpha'\).

Specializing to the “hook” states \(\alpha\) and \(\alpha'\) we find

\[C_{FL}(2\tilde{\mathcal{Q}} - \alpha, (\frac{Q}{2} - \mu) N\omega_1, \alpha') = \left[ \pi \mu \gamma(b^2) \beta^{2b^2} \right]^{(\alpha - \alpha', \mathcal{R}/b)} \left( \Upsilon(b) \right)^{N-1} \Upsilon(N(\frac{Q}{2} - \mu)) \]

\[\times \frac{\prod_{i<n} \Upsilon(\gamma_i - \gamma_j) \prod_{i<n} \Upsilon(\gamma_i' - \gamma_j')}{\prod_{i<n} \Upsilon(\gamma_i - \gamma_j) \prod_{i<n} \Upsilon(\gamma_i' - \gamma_j')} \times \frac{\prod_{j<n} \Upsilon(\gamma_j - \beta + \gamma_i)}{\prod_{j<n} \Upsilon(\gamma_j' - \beta' + \gamma_i')} \times \frac{\prod_{i<j} \Upsilon(\delta_{n,i} - \delta_{n,j}) \prod_{i<j} \Upsilon(\delta_{n,i} - \delta_{n,j})}{\prod_{i<j} \Upsilon(\delta_{n,i} - \delta_{n,j}) \prod_{i<j} \Upsilon(\delta_{n,i} - \delta_{n,j})} \times \frac{\prod_{j<n} \Upsilon(\gamma_j - \delta_{n,j})}{\prod_{j<n} \Upsilon(\gamma_j' - \delta_{n,j})} \times \frac{\prod_{j<n} \Upsilon(\gamma_j - \delta_{n,j})}{\prod_{j<n} \Upsilon(\gamma_j' - \delta_{n,j})} \times \frac{\prod_{j<n} \Upsilon(\gamma_j - \delta_{n,j})}{\prod_{j<n} \Upsilon(\gamma_j' - \delta_{n,j})} \times \frac{\prod_{j<n} \Upsilon(\gamma_j - \delta_{n,j})}{\prod_{j<n} \Upsilon(\gamma_j' - \delta_{n,j})} \times \frac{\prod_{j<n} \Upsilon(\gamma_j - \delta_{n,j})}{\prod_{j<n} \Upsilon(\gamma_j' - \delta_{n,j})} \times \frac{\prod_{j<n} \Upsilon(\gamma_j - \delta_{n,j})}{\prod_{j<n} \Upsilon(\gamma_j' - \delta_{n,j})} \] (4.4)

Recall that \(\delta_{n,j} = (j - (n + 1)/2)Q\), such that the difference \(\delta_{n,j} - \delta_{n,i}\) is an integer multiple of \(Q\), where \(\Upsilon\) has a zero. Therefore the numerator on the last line of (4.4) has a zero of order \(\frac{1}{2}(n(n - 1) + n'(n' - 1))\).

We found that for generic values of the momenta \(\beta\) and \(\gamma\) the 3-point function vanishes. For specific values of the momentum we can get extra zeroes in the denominator which may cancel the numerator. If we impose

\[\beta - \beta' = \mu, \] (4.5)

and if \(n - n'\) is odd, the argument of the Upsilon functions in the denominator of the last line of (4.4) will also be integer multiples of \(Q\) giving \(nn'\) zeros in the denominator and the overall order of the zero is

\[\frac{(n - n')^2 - n - n'}{2}. \] (4.6)

Let us assume \(n > n'\), then equation (4.5) represents \(n'\) conditions on the state \(\alpha'\). This should correspond to a pole of order \(n'\) in the three point function. Solving that (4.6) is \(-n'\) yields

\[n' = n - 1. \] (4.7)

If we consider two general semi-degenerate states a similar argument can be applied separately to each column. This confirms our claim in Section 2 that the consecutive states

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\textsuperscript{12}In comparison with our notation in [2.9], we omit the subscript from \(\beta_1\) and take \(\bar{\beta} = -n\beta/(N - n)\).

\textsuperscript{13}Note that in our notations, \(\beta, \beta'\) and \(\mu\) are all purely imaginary.
in the intermediate channels $\alpha^{(l)}$ correspond to Young-diagrams with all the columns of length greater than one, shortened by one, and the first row extended accordingly.

Going back to the “hook” states with $n' = n - 1$, using (4.5), the three point function (4.4) can now be simplified as

$$C_{FL}(2\bar{Q} - \alpha, (\frac{Q}{2} - \mu) N \omega_1, \alpha') = \left[\pi \mu \gamma(b^2) b^{2-2\beta} \right]^{(\alpha - \alpha', \beta)/b} (\Upsilon(b))^{N-1} \Upsilon(N(\frac{Q}{2} - \mu))
\times \frac{\Upsilon(nQ)}{\Upsilon(0)} \prod_{i<j \leq N-n} \Upsilon(\gamma_i - \gamma_j) \prod_{i<j \leq N-n'} \Upsilon(\gamma_j' - \gamma_i')
\times \prod_{i \leq N-n} \prod_{j \leq n} \Upsilon(\frac{Q}{2} - \frac{N-n}{n} \beta + \gamma_i - \gamma_i') \prod_{i \leq N-n'} \prod_{j \leq n'} \Upsilon(\frac{N-n}{n'} \beta' - \gamma_i' + \delta_n, j)
\times \prod_{i \leq N-n} \prod_{j \leq n} \Upsilon(\frac{Q}{2} - \frac{N-n}{n} \beta + \gamma_i - \gamma_i') \prod_{i \leq N-n'} \prod_{j \leq n'} \Upsilon(\frac{N-n}{n'} \beta' - \gamma_i' + \delta_n, j)
$$

(4.8)

Using that $\Upsilon(Q-x) = \Upsilon(x)$ this can be further simplified to

$$C_{FL}(2\bar{Q} - \alpha, (\frac{Q}{2} - \mu) N \omega_1, \alpha') = \left[\pi \mu \gamma(b^2) b^{2-2\beta} \right]^{(\alpha - \alpha', \beta)/b} (\Upsilon(b))^{N-1} \Upsilon(N(\frac{Q}{2} - \mu))
\times \frac{\Upsilon(nQ)}{\Upsilon(0)} \prod_{i<j \leq N-n} \Upsilon(\gamma_i - \gamma_j) \prod_{i<j \leq N-n'} \Upsilon(\gamma_j' - \gamma_i')
\times \prod_{i \leq N-n} \prod_{j \leq n} \Upsilon(\frac{Q}{2} - \frac{N-n}{n} \beta - \gamma_i + \gamma_i') \prod_{i \leq N-n'} \prod_{j \leq n'} \Upsilon(\frac{N-n}{n'} \beta' - \gamma_i' + \gamma_i')
$$

(4.9)

As explained above, this expression has a pole of order $n - 1$, and the desired three point function is the residue at the pole.

## 5 Quiver tail in Toda CFT

We are now ready to check the one-loop part of the AGT correspondence, and our identification of the space of allowed intermediate primary states. We employ the formula derived in the previous section and therefore consider a “simple tail”. That is our name for the gauge theory with $SU(2) \times SU(3) \times \cdots \times SU(N)$ gauge symmetry, such that the Riemann surface has one full puncture and $N + 1$ simple punctures.

### 5.1 1-loop partition function

We start with the gauge theory calculation. There are $N - 1$ vector multiplets with Coulomb branch parameters $a^{(l)}$, $l = 1, \cdots, N-1$. Together with the Vandermonde determinant, each vector multiplet contributes to the 1-loop partition function [16] [17]

$$|Z^{1\text{loop}}^{\text{vector}}| = \frac{\prod_{i<j} |a^{(l)}_i - a^{(l)}_j|^2}{\prod_{i<j} |\Gamma_b(a^{(l)}_i - a^{(l)}_j + 1/b) \Gamma_b(a^{(l)}_i - a^{(l)}_j + b)|^2} = \prod_{i<j} \Upsilon(a^{(l)} - a^{(l)}_i) \Upsilon(a^{(l)} - a^{(l)}_j).
$$

(5.1)
The bi-fundamental fields charged under $SU(l) \times SU(l+1)$ contribute

$$|Z_{\text{bi-fund}}^{1\text{-loop}}|^2 = \prod_{i=1}^{l+1} \prod_{j=1}^{l+1} |\Gamma_b(Q_i + a_i^{(t-1)} - a_j^{(t)} - m_{t-1})|^2 = \frac{1}{\prod_{i=1}^{l} \prod_{j=1}^{l+1} \gamma(Q_i + a_i^{(t-1)} - a_j^{(t)} - m_{t-1})} \prod_{i=1}^{l} \prod_{j=1}^{l+1} \gamma(Q_i + a_i^{(N-1)} - a_j^{(N)} - m_{N-1})$$

(5.2)

Lastly we have the contributions of the fundamental and anti-fundamental fields. For the “simple quiver tail” there is one fundamental field charged under $SU(2)$, one under $SU(N)$ and $N$ anti-fundamental fields also charged under $SU(N)$. Their contribution is

$$|Z_{\text{fund}}^{1\text{-loop}}|^2 = \prod_{i=1}^{2} \prod_{j=1}^{N} |\Gamma_b(Q + \hat{a}_i - \hat{m}_j)|^2 \prod_{i=1}^{N} |\Gamma_b(Q + a_i^{(N-1)} - \hat{m}_1)|^2 \prod_{i=1}^{N} \prod_{j=1}^{N} |\Gamma_b(Q + a_i^{(N-1)} - \hat{m}_j)|^2$$

$$= \prod_{i=1}^{2} \gamma(Q + a_i^{(1)}) \prod_{i=1}^{N} \gamma(Q + a_i^{(N-1)} - \hat{m}_1) \prod_{i=1}^{N} \gamma(Q + a_i^{(N-1)} - \hat{m}_j)$$

(5.3)

The full 1-loop contribution is the product of the $N-1$ vector multiplets (5.1), $N-2$ bi-fundamental fields (5.2) and the fundamentals (5.3). We now show how these terms arise from the 3-point functions of semi-degenerate states in Toda CFT.

5.2 Product of Toda 3-point functions

According to our prescription the external states are $\alpha^{(0)}$, $\alpha^{(N)}$ and $\mu^{(l)}$ and the states in the intermediate channels are $\alpha^{(l)}$. The states are parameterized as in (2.9), which is the same as (4.3) with $n_l = N - l - 1$. The states $\mu^{(l)} = (Q/2 - \mu_i)N\omega_1$ and the intermediate states satisfy the constraints (4.5). We use (4.9) to evaluate the consecutive three point functions. Up to a momentum independent constant the first three point function is

$$C_{FL}(2Q - \alpha^{(0)}, \mu^{(1)}, \alpha^{(1)}) \propto \left[ \pi \bar{\mu} \gamma(b^2) b^{2b^2} \right]^{(\alpha^{(0)} - \alpha^{(1)}, \rho)/b} \gamma(N(\frac{Q}{2} - \mu_1)) \gamma(N(\frac{Q}{2} + \beta^{(0)})) \prod_{i<j \leq 2} \gamma(\gamma_j^{(1)} - \gamma_i^{(1)}) \prod_{i \leq 2} \gamma(N\beta^{(0)} - \frac{N}{2} \beta^{(1)} + \gamma_j^{(1)}) \prod_{i \leq 2} \gamma(N\frac{3}{2} \beta^{(1)} - \gamma_i^{(1)} + \frac{N-1}{2} Q)$$

(5.4)
The next \( N - 3 \) three point functions, with \( l = 2, \cdots N - 2 \), are of the form

\[
C_{FL}(2\vec{Q} - \alpha^{(l-1)}, \mu^{(l)}, \alpha^{(l)}) \propto \left[ \pi \bar{\mu} \gamma(b^2) b^{2-2b^2} \right]^{(\alpha^{(l-1)} - \alpha^{(l)}, \rho)/b} \Gamma \left( N \left( \frac{Q}{2} - \mu_i \right) \right) \\
\times \frac{\prod_{i \leq j \leq l+1} \Gamma \left( \gamma^{(l)}_{i} - \gamma^{(l)}_{j} \right)}{\prod_{i \leq j} \Gamma \left( \frac{Q}{2} + \frac{N}{l+1} \beta^{(l-1)} - \gamma^{(l)}_{i} + \gamma^{(l)}_{j} \right)} \\
\times \frac{\prod_{i \leq j+1} \Gamma \left( \frac{N}{l+1} \beta^{(l-1)} - \gamma^{(l)}_{i} + \gamma^{(l)}_{j} \right)}{\prod_{i \leq j} \Gamma \left( \frac{N}{l+1} \beta^{(l-1)} - \gamma^{(l)}_{i} + \gamma^{(l)}_{j} \right)}
\]  

(5.5)

Note that the ratio on the last line will cancel between consecutive terms in the product over \( l \) and likewise the prefactor in the square bracket. The last two intermediate states, \( \alpha^{(N-2)} \) and \( \alpha^{(N-1)} \) are not degenerate. \( \alpha^{(N-2)} \) still has one component \( \beta^{(N-2)} \) and \( N - 1 \) \( \gamma^{(N-2)} \) components. For \( \alpha^{(N-1)} \) there is no \( \beta^{(N-1)} \), but only \( \gamma^{(N-1)} \). The three point function is still the same as above, with \( l = N - 1 \) if we define \( \beta^{(N-1)} = \beta^{(N-2)} - \mu_{N-1} \).

The last three point function is between two non-degenerate states and a simple puncture

\[
C_{FL}(2\vec{Q} - \alpha^{(N-1)}, \mu^{(N)}, \alpha^{(N)}) \propto \left[ \pi \bar{\mu} \gamma(b^2) b^{2-2b^2} \right]^{(\alpha^{(N-1)} - \alpha^{(N)}, \rho)/b} \\
\times \frac{\Gamma \left( N \left( \frac{Q}{2} - \mu_N \right) \right) \prod_{i \leq j \leq N} \Gamma \left( \gamma^{(N-1)}_{i} - \gamma^{(N-1)}_{j} \right) \prod_{i \leq j \leq N} \Gamma \left( \beta^{(N)}_{j} - \beta^{(N)}_{i} \right)}{\prod_{i \leq j \leq N} \Gamma \left( \frac{Q}{2} - \mu_N + \gamma^{(N-1)}_{i} - \beta^{(N)}_{j} \right)}
\]  

(5.6)

Combining all the terms together we find (with \( \gamma^{(0)} = 0 \))

\[
\prod_{i=1}^{N} C_{FL}(2\vec{Q} - \alpha^{(l-1)}, \mu^{(l)}, \alpha^{(l)}) \propto \left[ \pi \bar{\mu} \gamma(b^2) b^{2-2b^2} \right]^{(\alpha^{(0)} - \alpha^{(N)}, \rho)/b} \\
\times \prod_{i=1}^{N-1} \prod_{j \leq l+1} \left| \Gamma \left( \gamma^{(l)}_{i} - \gamma^{(l)}_{j} \right) \right|^2 \\
\times \frac{\prod_{i \leq j \leq l+1} \Gamma \left( \frac{Q}{2} + \frac{N}{l+1} \beta^{(l-1)} - \gamma^{(l)}_{i} + \gamma^{(l)}_{j} \right)}{\prod_{i \leq j \leq N} \Gamma \left( \frac{Q}{2} + \beta^{(N)}_{j} - \gamma^{(N-1)}_{i} \right) \prod_{i \leq j \leq N} \Gamma \left( \beta^{(N)}_{j} - \beta^{(N)}_{i} \right)}
\]  

(5.7)

The numerator on the second line matches the product of 1-loop determinants of the vector multiplets \( (5.1) \) with the identification \( \gamma^{(l)}_{i} = a^{(l)}_{i} \). The denominator for \( l > 1 \) is the same as the 1-loop determinant of the bi-fundamental fields \( (5.2) \), with the identification

\[
\frac{N}{l} \beta^{(l-1)} - \frac{N}{l+1} \beta^{(l)} = m_{l-1} \quad l = 2, \cdots N - 1.
\]  

(5.8)

Note that using the parametrization \( (2.9) \) this can also be written as

\[
\mu_{l} - \bar{\beta}^{(l-1)} + \bar{\beta}^{(l)} = m_{l-1}.
\]  

(5.9)
The terms in the numerator of the last line depend only on the external states and can be removed by field redefinitions. The denominator in the second line for \( l = 1 \) and that in the last line are the same as the fundamental and anti-fundamental fields in (5.3) with

\[
N \beta^{(0)} - \frac{N}{2} \beta^{(1)} = -\hat{m}_2, \quad \beta^{(N-1)} = \hat{m}_1, \quad \mu_N + \beta^{(N)}_i = \tilde{m}_i. \tag{5.10}
\]

Lastly we have the relation (4.5)

\[
\beta^{(l-1)} - \beta^{(l)} = \mu_l. \tag{5.11}
\]

This gives a matching between the product of three point functions in Toda CFT and the one-loop partition function of the gauge theory, up to terms which can be absorbed in overall normalizations. The expressions (2.11) and (2.12) are the natural generalization of these conditions to a general quiver tail.

6 Discussion

We have generalized the AGT correspondence to the case of 4d \( \mathcal{N} = 2 \) linear quiver theories with quiver tails. They can be described within 2d conformal \( A_{N-1} \) Toda field theory, where \( N - 1 \) is the rank of the largest gauge group. As proposed in [9], one should consider the correlation function of simple punctures on the sphere with two special punctures with specific semi-degenerate states. The smaller gauge groups in the tail are represented by subspaces of states of Toda CFT which are, or are not degenerate.

These subspaces of states arose as the result of the successive fusion of the degenerate state at the special puncture with the states at the simple punctures. As we have shown in Section 3 in the case of the \( SU(2) \times SU(3) \) quiver, the restriction to this subspace arises as a consequence of a Ward identity for the degeneracy condition of the special state. The same should be true more generally, with a special state satisfying \( N - N_1 \) degeneracy conditions and therefore there will be this number of conditions on the state \( \alpha^{(1)} \), reducing the space of states to be \( N_1 - 1 \) dimensional. For \( A_3 \) Toda one should be able to use the explicit algebra written down in [21, 22] to derive these conditions.

The other way to see the fusion rules is from studying the three point function of generic states and its degenerations. The result of Section 4 is that indeed the desired subspace arises in the three point function. We did it in full detail for the “simple tail”, but it seems to work more generally. Still, it may be that in certain special cases there would be extra states allowed in the OPE. Comparing these three point functions to the 1-loop partition function in the gauge theory allowed us to identify the full map of parameters between the two picture.

In the case of \( SU(2) \times SU(3) \) we studied also the contribution to the conformal blocks from level 1 states. By the AGT duality, the sum over descendants of the Virasoro algebra
is equal to the contribution of $SU(2)$ instantons and $W_3$ descendants agree with $SU(3)$ instantons. In our case we found that when restricting to the one-dimensional subspace of $A_2$ Toda primaries, for which the three point function does not vanish, the sum over both descendants of the $W_3$ algebra at level one adds up to the same answer as the $SU(2)$ instantons (with an extra polynomial remnant). Though we summed over two descendants, we reproduces the same answer that one gets from the single descendant at level one of Liouville.

It is compelling to postulate alternative 2d descriptions of quiver tails in addition to the one presented in this paper. One possibility is to couple Toda theories of different rank, so the $SU(2) \times SU(3)$ would be described by Liouville coupled to $A_2$ Toda. Clearly the spaces of primaries of these two theories agrees with the Coulomb branch parameters of the two groups without the need to restrict to a subspace. Likewise, the descendants are known to reproduce the instanton partition functions. What is needed is to find a way to couple the two 2d CFTs in a consistent way, which will give the desired answer.

It would be interesting to generalize our construction to other linear theories which are not conformal, along the lines of \cite{23}.

With the map we proposed in this paper it is possible now to study observables in these conformal theories with quiver tails. One can introduce surface operators $\cite{24, 25, 26, 27, 28, 29, 30, 31, 32}$, Wilson loops and ‘t Hooft loops $\cite{24, 33, 12, 34, 35}$ and domain walls $\cite{12, 36}$ and see how they behave when coupled to the lower rank gauge groups in the tail.

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**A Special Functions**

The function $\Gamma_b(x)$ is a close relative of the double Gamma function studied in $\cite{37, 38}$. It can be defined by means of the integral representation

$$\log \Gamma_b(x) = \int_0^\infty \frac{dt}{t} \left( \frac{e^{-xt} - e^{-Qt/2}}{(1 - e^{-bt})(1 - e^{-t/b})} - \frac{(Q - 2x)^2}{8e^t} - \frac{Q - 2x}{t} \right). \tag{A.1}$$
We use the following relation satisfied by this function

\[ \Gamma_b(x + b)\Gamma_b(x + 1/b) = x \Gamma_b(x)\Gamma_b(x + b + 1/b). \]  \hspace{1cm} (A.2)

The Υ function may be defined in terms of \( \Gamma_b \) (with \( Q = b + 1/b \))

\[ \Upsilon(x) \equiv \frac{1}{\Gamma_b(x)\Gamma_b(Q - x)}. \]  \hspace{1cm} (A.3)

An integral representation convergent in the strip \( 0 < \text{Re}(x) < Q \) is

\[ \log \Upsilon(x) = \int_0^\infty \frac{dt}{t} \left[ \left( \frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2(Q - x)t}{\sinh^2 t} \right]. \]  \hspace{1cm} (A.4)

Important properties we need are the obvious reflection \( \Upsilon(Q - x) = \Upsilon(x) \) and that at integer multiples of \( Q \) it has zeros.

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