Entropy and multifractal analysis of multiplicity distributions from \textit{pp} simulated events up to LHC energies

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Using three different Monte Carlo generators of high energy proton-proton collisions (HIJING, NEXUS, and PSM) we study the energy dependence of multiplicity distributions of charged particles including the LHC energy range. Results are used for calculation of the information entropy, Renyi’s dimensions and other multifractal characteristics of particle production.

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I. INTRODUCTION

Energy dependence of multiplicity distribution (MD) of particles produced in high energy collisions of hadrons and nuclei is important issue of multiparticle dynamics \cite{1}. Though MD contains information about particle correlations in an integrated form it provides general and sensitive tool to probe the dynamics of the interaction. Analysis of the multiplicity data from \textit{pp} and \textit{AA} collisions will be thus important part of the physics programme at the Large Hadron Collider (LHC) at CERN. In this new energy domain some basic questions concerning MD remain still open. Current theoretical understanding of the interplay between soft and semihard mechanisms of particle production is insufficient to provide reliable estimates even of the elementary quantity characterizing probability distribution $P_n$ of the $n$-particle event - the average multiplicity of particles $<n>$ produced in \textit{pp} collisions. Different scenarios vary over a wide range of values \cite{2}. Notwithstanding such uncertainty, large multiplicity of particles makes it feasible to study specific characteristics of MD which are relevant to fractal properties of the multiparticle dynamics. Fractality is usually connected with investigations of multiplicity in limited intervals of phase-space and demonstrated as a power-law in resolution dependence of the multiplicity moments \cite{3,4}. Such behaviour is characteristic for effects of nonstatistical fluctuations \cite{5} and/or intermittency \cite{6}. Besides fractality, there exists evidence in favour of multifractality \cite{7,8} in

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the structure of inelastic events. Some authors brought suggestions \[9\] that the property could be expressed in simple
thermodynamic terms. Namely, it was shown \[10\] that constant specific heat, widely used in standard thermodynamics,
reflects multifractal character of various stochastic systems in a reasonable approximation. In hadronic interactions,
its constancy relative to the q-order number of the multifractal characteristics (the generalized dimensions) was
indicated in Ref. \[11\]. Universality of the multifractal specific heat with respect to various hadron-nucleus reactions
was demonstrated \[12\] by means of the method proposed by Takagi \[13\]. Though exploiting similar variables and
methods supports multifractal interpretation of data, conclusion about the methodology is far from being unique. In
such situation, alternative approaches to the above aspects of multiparticle production are also needed. They can be
helpful in better understanding of multifractality in high energy collisions and useful in extracting dynamical origin
of its basic phenomenological observation - the power law dependence of the corresponding measures with respect to
the fractal resolution.

In this paper we employ and further extend our earlier study of MD \[14, 15\] based on robust characteristics of MD
such as Shannon’s information entropy

\[ S = - \sum P_n \ln P_n \]  \\

(1)

and its generalization, Rényi’s order-q information entropy \[16, 17\]

\[ I_q = \frac{1}{1 - q} \ln \sum (P_n)^q \]  \\

(2)

for which \(I_1 = S\). They contain information on the multiplicity moments \(<n^k> = \sum n^k P_n\) in a non-trivial way. The
outline of the paper is as follows. In Section 2 we show how regularities in energy dependence of the Shannon entropy
provide independent constraint on the energy dependence of \(<n>\). Section 3 is devoted to multifractal interpretation
of observed regularities in terms of the Rényi’s order-q entropy. We use three different Monte Carlo (MC) generators
of high energy proton-proton collisions, HIJING \[18\], NEXUS \[19, 20\], and PSM \[21\] to study energy dependence of
MD up to the LHC energies. Results of MC simulations are reported in Section 4. Summary and discussion of the
results is presented in Section 5.

II. ENTROPY

Entropy is important characteristic of systems with many degrees of freedom. It seems quite natural to use it in
description of high energy multiparticle production processes. In particular, entropy of MD is an effective variable
characterizing inelastic collisions with many particles produced. The simple relation

\[ S - \ln <n> \simeq -\int_0^\infty \psi(z) \ln \psi(z) dz \]  \hspace{1cm} (3)

between the entropy \( S \) and the average multiplicity \(<n>\) is valid with good accuracy for large enough \(<n>\). The function \( \psi(z) \) is related to the probability \( P_n \) by the formula

\[ \psi \left( z = \frac{n}{<n>} \right) \equiv <n> P_n. \]  \hspace{1cm} (4)

The KNO scaling \[22\] of MD postulates energy independence of \( \psi(z) \) what can be expressed according to Eq. \[3\] as follows

\[ S - \ln <n> \simeq \text{const.}(\sqrt{s}). \]  \hspace{1cm} (5)

Last relation implies that all information about the energy dependence of MD is contained in its first moment \(<n>\). Such behaviour was indeed observed in hadron-hadron collisions up to the ISR energy range \[23\]. However, beyond this range, the KNO scaling of MD of charged hadrons produced in \( pp/p\bar{p} \) collisions was shown to be significantly violated \[24, 25\]. The observed scaling violation, i.e. break down of Eq. \[5\] results via Eq. \[3\] in non-trivial correlation between \( S \) and \(<n>\). Experimental situation concerning the difference \( S - \ln <n> \) calculated for MD of charged particles in the full phase space is illustrated in Fig. 1. Here, due to the charge conservation, \(<n> = <n_{ch}/2>\) is half of the average multiplicity of charged particles. Recently published Tevatron data \[26\] from \( p\bar{p} \) collisions are also included in the figure. The values of the difference \[3\] show clear increase with energy and thus confirm violation of the KNO scaling observed at lower ISR energies.

Let us note that increase of \( S - \ln <n> \) can not continue ad infinitum. It was shown \[14\] that this difference is asymptotically bounded by unity from above. The extremal value of the r.h.s. of the relation \[3\] is obtained from the principle of maximal entropy applied to all continuous functions \( \psi(z) \). If the functions \[4\] satisfy the conditions

\[ \int_0^\infty \psi(z) dz = \int_0^\infty z \psi(z) dz = 1 \]  \hspace{1cm} (6)

valid for all normalized MD with given \(<n>\), the extremum is reached for the KNO function \( \psi(z) = \exp(-z) \) which corresponds to the geometrical distribution. The difference

\[ S - \ln <n> \leq (1+<n>) \ln(1+\frac{1}{<n>}) = 1 + O(<n>^{-1}) \]  \hspace{1cm} (7)

is thus bounded from above by the expression which is calculated for the geometrical MD. The bounding values are represented by the curve in Fig.1. Last relation shows that energy dependence of the information entropy can be used as an independent constraint on energy dependence of the average multiplicity.
Energy dependence of the entropy $S$ in high energy collisions was first studied in Ref. [14]. Using MD of charged secondaries produced in $pp$ and $p\bar{p}$ collisions in the energy range $\sqrt{s} \leq 900$ GeV, monotonous increase of $S$ with center-of-mass energy $\sqrt{s}$ was found. For $\sqrt{s} > 20$ GeV, the linearity

$$S = D_1 Y_m$$

(8)

with the maximum rapidity $Y_m = \ln(\sqrt{s}/m_\pi)$ of the hadrons produced is valid. Here $m_\pi$ is the pion mass and $D_1$ is an energy independent constant. Recently published Tevatron data [26] extend the validity of these findings up to $\sqrt{s} = 1.8$ TeV. The experimental situation is shown in Fig. 2 where the published errors of MD have been taken into account. Value of $D_1$ is within ±2% error band consistent with the predicted one $D_1 \equiv S/Y_m = 0.417 \pm 0.009$ [14] shown by the full/dashed lines in Fig. 2. On closer inspection, however, one can see that $D_1$ calculated from E735 data is systematically above the UA5 values. It is connected with higher tails of the E735 MD as compared to the UA5 data. The discrepancy might be due to the sizable systematic uncertainty in both E735 and UA5 since in both experiments the full phase space MD were obtained by computer simulation from data measured in a restricted range of rapidity.

As the bounding value in $\ln$ tends to unity, the observed monotonous increase of the entropy $S = D_1 \ln(\sqrt{s}/m_\pi)$, if valid at higher energies, will also govern the energy dependence of $<n>$:

$$<n> \approx \exp(S) = (\sqrt{s}/m_\pi)^{D_1}.$$  

(9)

Such asymptotic power law behaviour of the average multiplicity should be contrasted with other approaches to multiparticle production. In particular in Ref. [2], $<n>$ was predicted to increase as a second order polynomial in $\ln s$. Difference between these two predictions will be substantial at the top energy of LHC, because according to Eq. $<n> \approx 110$ at $\sqrt{s} = 14$ TeV, while according to the parametrization used in Ref. [2] $<n> \approx 70$ at this energy.

### III. MULTIFRACTALITY

Fractal geometry is nowadays widely used in many branches of physics [17]. In astronomy analysis of galaxy and cluster distributions it has led to surprising result that galaxy correlations up to $150 h^{-1}$Mpc are scale invariant and not homogenous [27]. In multiparticle dynamics methods introduced originally for description of the fractal properties of stochastic systems [28] are used extensively [1]. In particular, study of MD in small rapidity bins using the scaled factorial moments $F_q$ revealed typical linear behaviour

$$\ln F_q = -\tau_q \ln \delta$$

(10)
in terms of the bin resolution $\delta$. Such behaviour was interpreted as (multi)fractal property of particle production. Another approach to the multifractality is directly connected to the Rényi’s order-$q$ information entropy $I_q^2$. Energy dependence of $I_q$ in high energy collisions was first studied in Ref. 15. Using MD of charged secondaries produced in $pp$ and $p\bar{p}$ collisions in the energy range $\sqrt{s} \leq 900$ GeV it was found that increase of $I_q$ with center-of-mass energy $\sqrt{s}$ is similar for various values of $q$. For $\sqrt{s} > 20$ GeV, the observed asymptotic linearity

$$I_q = D_q Y_m$$  \hspace{1cm} (11)$$

with maximum rapidity of the hadrons produced, $Y_m = \ln(\sqrt{s}/m_\pi)$, generalize Eq. 8. Let us briefly summarize multifractal interpretation of Eq. 11. In contrast to standard intermittency and multifractal analysis we consider fractal resolution $\delta$ related to the total energy $\sqrt{s}$ available and not to the phase-space binning. During hadron-hadron interaction with many particles produced, the energy dissipates into $N = \sqrt{s}/m_\pi$ discrete sites each of the size $m_\pi$. The site labeled by $n$ is occupied with the probability $P_n$. Since most of the sites are unoccupied the overlay of many inelastic events can be visualized as a fractal with overall extent $\sqrt{s}$ characterized by a local mass distribution function. The sufficient condition to produce such self-similar (hierarchical) structure is that the probability $P_n$ exhibits some type of scale invariant behaviour. In particular, if at sufficiently high energies, i.e. at high enough resolution $\delta = 1/N$, the $P_n$ acquires a power law dependence on the resolution $\delta$, the quantity $\sum(P_n)^q$ will scale with $\delta$ like

$$\sum(P_n)^q \sim \delta^{-(1-q)D}.$$  \hspace{1cm} (12)$$

If $D > 0$ is independent of $\delta$, one usually speaks about fractality 17 of the distribution $P_n$. The multifractals 17,28 generalize the notion of fractals for $D = D_q$ depending on $q$. Spectrum of generalized dimensions $D_q$ which for multifractals is a decreasing function of $q$ 28 has the following meaning. $D_0$ corresponds to the capacity (box dimension) of the support of the measure $P_n$, information dimension $D_1$ 16 characterizes scaling of the information entropy 11 and $D_q$'s for integer $q \geq 2$ can be related to the scaling behaviour of $q$-point correlation integrals 15,28. The observed approximate independence of the generalized dimensions

$$D_q = -I_q/\ln \delta$$  \hspace{1cm} (13)$$

on the energy (resolution $\delta$) as well as their decrease with increasing $q$ show 15 that for $\sqrt{s} \geq 20$ GeV the full phase space MDs of charged particles from non-single-diffractive hadron-hadron collisions are indeed multifractal. Let us note that multifractality besides predicting $D_q$ to be decreasing functions of $q$ does not, in general, provide any further information about the $q$-dependence of the spectrum of the generalized dimensions $D_q$. In particular knowledge of
say $D_1$ and $D_2$ is insufficient to predict scaling behaviour of the higher $q$ correlation integrals. It is thus gratifying to find out that this could be, at least in principle, achieved within interpretation of multifractality in thermodynamical terms \[10\]. Latter is based on analogy between l.h.s. of Eq. \[12\] and partition function

$$Z(q) \equiv \sum (P_n)^q \quad \text{(14)}$$

with $q$ playing the rôle of inverse temperature $q \equiv T^{-1}$ and $V \equiv -\ln \delta$ representing volume. The thermodynamic limit of infinite volume $V \rightarrow \infty$ is then equivalent to the limit of increasing resolution $\delta \rightarrow 0$. In the constant specific heat approximation the $q$-dependence of the generalized dimensions $D_q$ acquires particularly simple form \[10\]

$$D_q \simeq (a - c) + c \frac{\ln q}{(q - 1)}.$$

(15)

The coefficient $c$ represents multifractal specific heat and $a = D_1$. Regular behaviour of this type is expected to occur for multifractals for which, in classical analogy with specific heat of gases and solids, the multifractal specific heat $c$ is independent of temperature \[20\] in a wide range of $q$.

We have examined validity of the approximation given by Eq. \[15\] for $D_q$ defined by Eqs. \[11\] and \[2\]. In Fig. 3a we present $q$-dependence of generalized dimensions calculated from the Tevatron data \[26\] at $\sqrt{s} = 300$, 546, 1000, and 1800 GeV. One can see from the figure that the values of $D_q$ reveal indeed linear increase as a function of $\ln(q)/(q - 1)$. This behaviour makes it possible to define the slope parameter $c$ in the region $q \geq 1$. Similar $D_q$ dependencies for data from CERN ISR and $Sp\bar{p}S$ Collider experiments \[23, 24, 25\] are shown in Fig. 3b. The dashed line coincides with the full line in Fig. 3a indicating position of $D_q$ values calculated from E735 data. Both data sets obtained by Tevatron and CERN experiments reveal approximately the same slope while their intercepts are mutually shifted. The shift is due to larger values of $D_1$ for E735 data, as already shown in Fig. 2. As pointed in the previous section, the discrepancy in intercepts is connected with systematically larger high multiplicity tails of the data from Tevatron when compared to the data from the CERN $Sp\bar{p}S$ Collider. This might be connected with the mutual systematic uncertainties of the experimental procedures when extending measured data into the full phase space region.

Fitting the slope parameters for $q \geq 1$ at each separate energy, we have determined the values of multifractal specific heat $c$. The results are presented in the lower part of Fig. 2. The errors in determination of $c$ were calculated from error bars of MD quoted in literature. They represent $\sim 10 - 15\%$ of the established values. For $\sqrt{s} \geq 20$ GeV, the multifractal specific heat is within the estimated errors approximately energy independent quantity and achieves the value $c \approx 0.08$. Note that this number practically coincides with the slope obtained from the electron-positron multiplicity data \[30\] what is indicated by the full line in Fig. 3b. Since our study concerns the full phase space and
it is performed in a different sense than the usual intermittency analysis, we obtain smaller value of $c$ in comparison with the specific heat ($c \sim 0.26$) determined from multifractal properties of the factorial moments [31]. While the multifractal specific heat reported in Ref. [12] reveals some kind of universality with respect to various interactions, energy dependence of $D_q$ obtained with the same method [13] seems to be significant. Contrary to this, our method gives smaller values of the generalized dimensions $D_q$ which are approximately energy independent for $\sqrt{s} \geq 20$ GeV.

Energy independence of the multifractal specific heat confirms that, in addition to the information dimension $D_1$, there appears to be yet another parameter $c$ which could be used as universal characteristic of particle production in hadron-hadron interactions at high energies. Knowing $D_1$ and $c$, all other $D_q$ can be thus deduced from Eq. 15.

IV. MONTE CARLO SIMULATIONS

We have exploited three different Monte Carlo models in our investigation of the proton-proton interactions beyond up to date accessible energies.

1. The HIJING [18] Monte Carlo code is based on QCD-inspired models for multiple jets production. It allows to study jets and the associated particle multiplicities. The model includes minijet production, soft excitation, nuclear shadowing of parton distribution functions and jet interaction in the dense nuclear matter.

2. The NEXUS [19, 20] Monte Carlo code is based upon the hypothesis that the behaviour of the high energy interactions is universal. Basic building blocks of hadron-hadron or nucleus-nucleus scattering are parton ladders coupled softly to the nucleons. It relays on a consistent multiple scattering approach in the sense that most of the dynamics follows from a formula for the total cross section expressed in terms of cut diagrams.

3. The Parton String Model (PSM) [21] includes in its initial stage both soft and semihard components which lead to the formation of color strings. Collectivity is taken into account considering the possibility of strings in color representations higher than triplet or anti-triplet by means of string fusion. String breaking leads to the production of secondaries.

We have analyzed $\sqrt{s}$-dependence of the average multiplicity $<n_{ch}>$, the entropy $S$, the difference $S - \ln(<n_{ch}>/2)$, and the multifractal specific heat $c$ for the charged particles in the energy interval $25$ GeV $< \sqrt{s} < 14$ TeV. The quantities have been obtained for 14 points from this interval using 10000 simulated events for each one. The results of the energy dependence of $<n_{ch}>$ are shown in Fig. 4. Though all three models give similar values in the range $\sqrt{s} < 1$ TeV, their predictions significantly differ for $\sqrt{s} > 1$ TeV. The average multiplicity simulated by HIJING and
PSM codes is well approximated by the power dependence

\[ <n_{ch}> = a s^b \]  \hspace{1cm} (16)

with \( b \approx 0.018 \) for HIJING and PSM. The NEXUS predictions fall well bellow and can be fitted much better by the second order polynomial in logarithm \( \sqrt{s} \):

\[ <n_{ch}> = a_0 + a_1 \ln \sqrt{s} + a_2 \ln^2 \sqrt{s}. \]  \hspace{1cm} (17)

The above parametrization corresponds to the prediction of the average multiplicity given in Ref. [2].

Let us now compare both parametrizations from the point of view of the observed regularity in the information dimension \( D_1 \). The energy dependence of \( D_1 = S/Y_m \) is displayed in Fig. 5. For HIJING and PSM the behaviour is consistent with the experimentally observed ratio \( S/Y_m = 0.417 \pm 0.009 \) extrapolated towards super-high energies. This supports the conjecture that information entropy per unit rapidity should stay constant in the full phase space up to the LHC energy region and even beyond. In such case, energy dependence of the average multiplicity should achieve the asymptotic power law behaviour (9). The HIJING and PSM simulations of the average multiplicity confirm this tendency. On the other hand, results of simulations with the NEXUS code show slow but continuous decrease of \( D_1 = S/Y_m \) with the energy. The decrease of \( D_1 \) is connected with non-power like, namely the logarithmic energy dependence of the average multiplicity. Using the general limit on MD one can say: either the entropy \( S \) must slow down and violate the entropy scaling as indicated by the NEXUS model, or, providing the entropy scaling stays valid, the average multiplicity must grow faster, similar as in the HIJING and PSM models, and violate thus the parametrization of Ref. [2].

The above statement relies in details on the energy dependence of the difference \( S - \ln(<n_{ch}>/2) \) which is depicted in Fig. 6. Though for all energies considered the difference does not reach the bounding value (17), its saturation in the LHC energy range is almost complete for the PSM model. The NEXUS prediction falls well bellow at the LHC energy and is closer to the KNO-scaling behaviour given by Eq. (5). In the case of the HIJING we have checked the sensitivity of \( S - \ln(<n_{ch}>/2) \) to the interactions dynamics, changing in particular number of the produced jets \( N_{jets} \). There are points of regime changes clearly visible in Fig. 6. They correspond to various maximal number of produced jets, which had been set to \( N_{jets} = 0, 1, \) and \( 5 \) for \( \sqrt{s} < 0.1 \text{ TeV}, 0.1 \text{ TeV} < \sqrt{s} < 1 \text{ TeV}, \) and \( \sqrt{s} > 1 \text{ TeV} \), respectively. Character of these changes demonstrates that the quantity \( S - \ln(<n_{ch}>/2) \) is sensitive to the dynamic of the interaction, in particular, to the number of produced jets. Increase of \( N_{jets} \) results in larger number of parton-parton collisions which play role in the KNO scaling violation [26]. The violation seems however not to destroy the
self-similarity of parton dynamics as manifested by the regularity in behaviour of the information entropy of MD including the data from Tevatron (Fig. 2).

Multifractal character of particle production consistent with approximate energy independence of the multifractal specific heat $c$ is predicted by the PSM, HIJING, and NEXUS models in the LHC energy region and beyond. We demonstrate this in Fig. 7. The tendency towards a constant value of $c$ confirmed by the simulation results is similar with the experimental situation shown in Fig. 2. It will be interesting to see whether this will really hold true for the forthcoming data from LHC.

V. SUMMARY AND DISCUSSION

Having enumerated experimentally observed features of MD in multi-hadron production such as regularities in the information entropy and generalized dimensions, we have studied the multifractality of inelastic hadron-hadron collisions at high energies. We have applied the multifractal specific heat approach to characterize this property in a quantitative way. The generalized dimensions were calculated using the resolution $\delta = m_\pi / \sqrt{s}$ defined as minimal fraction of the center-of-mass energy which can be carried away by the outgoing hadrons. The experimental data on MD are consistent with the constant specific heat approximation $c \sim 0.08$ with respect to both, the multifractal temperature $T \equiv q^{-1}$ and the energy $\sqrt{s}$.

We have extended our study of MD in $pp$ collisions up to LHC energies exploiting three different Monte Carlo generators of high energy hadron-hadron and nucleus-nucleus collisions (HIJING, NEXUS, and PSM). The HIJING and PSM predictions confirm approximate energy independence of the entropy dimension $D_1 \equiv S/Y_m = 0.417 \pm 0.009$ in the full phase-space up to the LHC energy and even beyond. These models give $S_{LHC} \sim 4.65$ at $\sqrt{s}=14$ TeV and a power-like energy behaviour of the average multiplicity. Simulations with NEXUS show monotonous decrease of $S/Y_m$ which corresponds to a slower (logarithmic) increase of the average multiplicity with energy. In this view, HIJING and PSM, unlike to NEXUS, prefer self-similar character of multi-hadron production. All three models predict approximately constant value of the multifractal specific heat and thus suggest multifractal character of proton-proton interactions in the considered energy region. The information about self-similarity and multifractal structure is contained in MDs and can be quantified by the behaviour of both the information entropy and the multifractal specific heat in a model independent way.
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FIG. 1: The energy dependence of the difference $S - \ln(<n_{ch}>/2)$ for charged particles (experimental data). The full circles correspond to data from E735 experiment.

FIG. 2: The energy dependence of the entropy dimension $D_1$ and the multifractal specific heat $c$ for charged particles. The triangles correspond to data from E735 experiment.
FIG. 3: The generalized dimensions $D_q$ as function of $\ln(q)/(q-1)$ for charged particles. Data are taken (a) from E735 experiment [26] and (b) from Refs. [23, 24, 25, 30].
FIG. 4: The energy dependence of the average multiplicity simulated by MC for charged particles.

FIG. 5: The energy dependence of $D_1 = S/Y_m$ of charged particles. The points correspond to the MC simulations.
FIG. 6: The energy dependence of the difference $S - \ln(\langle n_{ch} \rangle / 2)$ of charged particles. The points correspond to the MC simulations.

FIG. 7: The multifractal specific heat $c$ calculated from MC simulations.