AXIOMATIC FOUNDATIONS FOR THE PRINCIPLE OF
ENTROPY INCREASE

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February 18, 2009

Abstract
We show that the principle of entropy increase may be exactly founded on a few axioms
valid not only for quantum and classical statistics, but also for a wide range of statistical
processes.

Keywords: Information conservation, Divisibility of the system, Correlation information.
PACS: 02.50.-r, 05.30.Ch, 05.70.-a

1 Introduction
The second law of thermodynamics, or the principle of entropy increase, is an exact law of
nature. To explore its foundation is one of the most important topics in physics for more than
a century. From the daily life we know, matter always approaches its equilibrium state. Text
books[1]-[4] tell us that the equilibrium state is a state with maximum entropy, and the state of
a macroscopic system with larger entropy is more probable. It almost explains the above daily
life experience. However, whether the system always goes from a less probable state to a more
probable state, or why the principle of entropy increase works, is still an open question. The
H-theorem of Boltzmann is a classical proof for definite approaching to equilibrium. It is based
on a model of colliding classical particle system for the macroscopic matter, therefore is not
general enough, even from the view point of the classical statistical physics. Recently we gave
an exact and general proof for the principle of entropy increase by use of general principles of
quantum theory[5]. In this paper, we would put the principle on an axiomatic basis, consistent
with classical and quantum mechanics, but not special for them. We collect definitions and
axioms in section 2, lemmas and theorems in section 3. Among them, theorem 4 is exactly the
principle of entropy increase. Section 4 is a discussion.
2 Definitions and axioms

Definition 1: The evolution is a process not interrupted by measurement.

Definition 2: A system is a collection of objects, its evolution in time is determined by itself, and is not correlated with any other object.

Definition 3: The state of a system at a given time is a property of the system at that time, which determines which observables are certain, what are their value, as well as the change of this property itself at that time.

It means the existence of a state is conditional. If there is a state for the system under consideration at a time \( t_0 \), it must have a state at any other time \( t \) in the course of evolution which is determined by the original state at the time \( t_0 \). The differential equation governing the state evolution is of the first order in time, which in turn means that two different states keep different during the evolution. These points are true both for classical and quantum mechanics.

On a set of independent events one may define probability distribution.

Definition 4: Independence between two states means: if the system stays in one of them the observer can definitely not see the property which defines another state.

Definition 5: A set of states for a system is complete if and only if any other state of the system depends on at least one state in it.

In classical mechanics, two different states are always independent from each other; while in quantum mechanics, independence of states means they are orthogonal to each other. Two different but nonorthogonal states depend on each other in the sense, that there is a nonzero probability \( W_{ab} \) to see the property of one state \( a \) while the system stays in another state \( b \). In classical mechanics \( W_{ab} = \delta_{ab} \), while in quantum mechanics the relation is relaxed to be \( W_{ab} = W_{ba} \). Both in classical and in quantum mechanics, two independent states keep independent from each other through their evolution. Since an orbital state of a particle occupies a finite volume \( h^3 \) in its phase space, quantum state is numerable. Classical state is innumerable in its original form. However, it is an effective method in classical statistical mechanics, to let the phase volume of a state finite, so that make the state numerable, and let the phase volume approaches zero at the final stage of derivation. If classical theory is applicable to the problem, this method always gives right answer.

Definition 6: The state with a certain value of a given observable is called an eigenstate of this observable, the corresponding value of the observable is called its eigenvalue.

In classical mechanics, every state is an eigenstate of all observables. But in quantum mechanics, eigenstate of an observable has to be solved from the eigenequation of the observable. However, in both cases a set of all independent eigenstates for a given observable is complete.

Definition 7: If a set of observables may be measured simultaneously with certain outcome, and the result is complete enough to determine the state of the measured system, the set is called a complete set of observables for the system.

Definition 8: Measurement of a complete set of observables on a system is called a complete
measurement.

While a complete measurement determines the state of the system, an incomplete measurement cannot determine the state but may determine a probability distribution of the system over a set of different states. In classical mechanics, it means a probability distribution over a complete set of independent states. In quantum mechanics one handles this situation by a Hermitian operator, called the density operator. Its eigenstates are independent of each other, and the eigenvalues may be regarded as probabilities of finding that the system stays in the corresponding states respectively. In both cases, an incomplete measurement determines a probability distribution of the system over a complete set of independent states. Denoting the $n$th state of the set by $|n\rangle$ and the probability of the system staying in the state $|n\rangle$ by a non-negative number $W_n$, we have the normalization relation $\sum_n W_n = 1$, and

**Definition 9:** The information of a system is given by

$$I \equiv \sum_n W_n \ln W_n.$$  

The summation is over the complete set of independent states, and the information means the amount of information for short.

In classical and quantum physics, it is always possible to divide a macroscopic system into subsystems, so that every subsystem is still large enough to be macroscopic but is already macroscopically uniform, and the microscopic non-uniformity is still negligible.

**Definition 10:** The uniform system is a system, in which a kind of observable (intensive observable) takes the same value everywhere, and the values of other kinds of observables (extensive observables) are proportional to each other.

**Definition 11:** The divisible system is either a uniform system or a system which may be divided into uniform subsystems.

For a divisible system one may define the entropy.

**Definition 12:** In unit of Boltzmann constant $k$ the entropy is given by

$$S = -\sum_i I_i,$$

in which subscript $i$ specifies its subsystems all being uniform. It is the negative sum of the information of these subsystems.

We assume our system satisfies the following axioms:

**Axiom 1:** The system is in one of its states at a given time.

**Axiom 2:** The probability $W_{ab}$ of finding the property of state $a$ in the state $b$ equals the probability $W_{ba}$ of finding the property of state $b$ in the state $a$.

**Axiom 3:** Two eigenstates of the same observable with different eigenvalues are independent from each other.

**Axiom 4:** The set of all independent eigenstates for a given observable is complete.

**Axiom 5:** The set of independent states for a system is numerable.
Axiom 6: The state at a time determines the state at any other time for the same system throughout the whole process of evolution.

Axiom 7: Independent states keep independent from each other during the evolution in time.

Axiom 8: The state of the system after a complete measurement is the one determined by the outcome of the measurement.

Axiom 9: The system may be described by a probability distribution over a complete set of independent states.

Axiom 10: The system is divisible.

In classical and quantum statistical physics, these axioms are exactly satisfied. We expect they may still be satisfied in future new physics, and also be satisfied in some processes other than those in physics. It would make the results derived from them not only exactly true in physics but also applicable to a wide range of other problems.

3 Lemmas and theorems

Now let us remind you some mathematical inequalities. One can find them and their proofs elsewhere[5, 6]. Mathematically, we define $0 \ln 0 \equiv \lim_{\xi \to 0} (\xi \ln \xi) = 0$.

Lemma 1. For any non-negative number $x$ we have

$$x \ln x \geq x - 1,$$

the equality holds when and only when $x = 1$.

Lemma 2. For sets $[w_i]$ and $[x_i]$ of non-negative numbers with $\sum_i x_i = 1$, we have

$$\sum_i x_i w_i \ln \sum_{i'} x_i' w_i' \leq \sum_i x_i w_i \ln w_i.$$

Lemma 3. For sets $[W_i]$ and $[T_{ij}]$ of non-negative numbers with

$$\sum_i W_i = 1 \quad \text{and} \quad \sum_j T_{ij} = \sum_j T_{ij} = 1,$$

we have

$$W_j' \equiv \sum_i W_i T_{ij} \geq 0 \quad \text{for every } j,$$

$$\sum_j W_j' = 1,$$

and

$$\sum_j W_j' \ln W_j' \leq \sum_i W_i \ln W_i.$$

Lemma 4. For positive numbers $[W_{ij}]$, $W_i = \sum_j W_{ij}$ and $W_j' = \sum_i W_{ij}$, with $\sum_{ij} W_{ij} = 1$, we have

$$\sum_i W_i = 1, \quad \sum_j W_j' = 1,$$
and
\[ \sum_{ij} W_{ij} \ln W_{ij} \geq \sum_i W_i \ln W_i + \sum_j W'_j \ln W'_j. \]  
(10)
The equality holds when and only when \( W_{ij} = W_i W'_j \) for all \( ij \), it is that the \( W_{ij} \) may be factorized.

Consider a system. At time \( t_0 \), we do not know its state, but know the probability distribution over a complete set of its independent states. Its information is therefore given by (1). We have

**Theorem 1 (information conservation):** The information of a system does not change in the course of evolution.

**Proof:** From the definition 9 we see, the information of a system relates only to the probability distribution over a complete set of its independent states, irrelevant to the contents of these states. Since, according to the axiom 7, the independence of states does not change in the evolution, the probability distribution and therefore the information of the system does not change either. It is
\[ \mathcal{I}(t) = \mathcal{I}(t_0), \]  
(11)
in which \( \mathcal{I}(t) \) and \( \mathcal{I}(t_0) \) are information of the system at two different time \( t \) and \( t_0 \) respectively, in its course of evolution. The theorem is proven.

According the axioms 3 and 4, a complete set \([L] \) of observables for the system has a complete set of independent eigenstates. Denoting this set by \([|m\rangle]\), and the probability of finding the property of state \(|m\rangle\) by \( W'_m \), the information of the complete set \([L] \) of observables for the system is defined by
\[ \mathcal{I}_{[L]} \equiv \sum_m W'_m \ln W'_m. \]  
(12)
Denoting the probability of finding the property of state \(|m\rangle\) in the state \(|n\rangle\) by \( W_{m,n} \), we have
\[ W'_m = \sum_n W_{m,n} W_n, \]  
(13)
\[ \sum_m W'_m = 1, \]  
(14)
and by axiom 2 also
\[ \sum_n W_{m,n} = \sum_m W_{m,n} = 1. \]  
(15)
Since probabilities \( W_{m,n} \) are non-negative, according to lemma 3 and equation (1) we obtain
\[ \mathcal{I}_{[L]} \leq \mathcal{I}, \]  
(16)
and therefore have proven

**Theorem 2:** The information of a given complete set of observables for the system is not more than the information of the system itself.

Now, let us divide the system into two parts \( a \) and \( b \). Suppose \([L_i] \), with \( i = a \) or \( b \), is a complete set of observables of part \( i \), \(|n_i\rangle\) is their \( n_i \)th eigenstate, and \([|n_i\rangle]\) is a complete
set of independent states for part \( i \). Therefore \([L_a, L_b]\) is a complete set of observables, and \([|n_a n_b]\) is a complete set of independent states, both for the system. Denoting the probability of finding the property of state \(|n_a n_b\rangle\) in the state \(|n\rangle\) by \(W_{n_a n_b, n}\), the probability of finding part \( a \) in the state \(|n_a\rangle\) and part \( b \) in the state \(|n_b\rangle\) is

\[
W_{n_a n_b} = \sum_n W_{n_a n_b, n} W_n ,
\]

with normalization

\[
\sum_{n_a n_b} W_{n_a n_b} = 1 .
\]

According to the theorem 2, the information of observables \([L_a, L_b]\) for the system is

\[
I_{L_a, L_b} = \sum_{n_a, n_b} W_{n_a n_b} \ln W_{n_a n_b} \leq I .
\]

The probability of finding part \( a \) in the state \(|n_a\rangle\) and the probability of finding the part \( b \) in the state \(|n_b\rangle\) are

\[
W_a = \sum_{n_b} W_{n_a n_b} \quad \text{and} \quad W_b = \sum_{n_a} W_{n_a n_b} .
\]

respectively. In (18-20), it is understood that the summation is over those \( n_a \) and \( n_b \) only, for which \( W_{n_a n_b} > 0 \). According to axiom 8, after the measurement of \([L_i]\), the state of part \( i \) would be one in the set \([|n_i]\)\. The probability for the presence of state \(|n_i\rangle\) is \(W_{n_i}\). The information for part \( i \) is

\[
I_i = \sum_{n_i} W_{n_i} \ln W_{n_i} .
\]

From lemma 4 and equations (19,21) we see

\[
I_a + I_b \leq I .
\]

The equality holds when and only when \( W_{n_a n_b} = W_{n_a} W_{n_b} \) for all \( n_a \) and \( n_b \), it is that the probability distribution is factorized. The later means two parts of the system do not correlate with each other. We may further subdivide the parts and apply (22) to them again and again, the result is the statement

\[
\sum_i I_i \leq I ,
\]

in which the summation is over all parts of the system. Therefore we have proven

**Theorem 3**: The sum of information of all parts of the system is no more than the information of the system itself.

According to the axiom 10, we may make every part of the system be uniform. In this case, by the definition 12 for the entropy and the equation (23), we see

\[
S \geq -I
\]
for a system. The equality holds when and only when parts of the system are not correlated to each other. We then arrive at

**Theorem 4 (principle of entropy increase):** The entropy of a system if changes can only increase.

**Proof:** We would prove the theorem operationally. Suppose we measured the entropy $S(t_0)$ of the system at the beginning time $t_0$. According to the definition 12, it means that we measured the entropy of every uniform part of the system and summed them up. This operation had to destroy the correlation between these parts, and made the probability distribution of the system factorized to a product of probability distributions of its parts. By eq.(24) and the discussion after it, we see

$$S(t_0) = -I(t_0), \quad (25)$$

in which $I(t_0)$ is the information of the system at time $t_0$ after the measurement. After this first measurement the system evolves according to its own dynamics with information conservation (theorem 1). In this course various parts of the system become correlated because the interaction between them. The probability distribution of the system will not keep being factorized to the product of probability distributions of its parts. Let us measure the entropy $S(t)$ at the time $t > t_0$ in the course, by eqs.(24), (11) and (25), we see

$$S(t) \geq -I(t) = -I(t_0) = S(t_0), \quad (26)$$

in which $I(t)$ is the information of the system just before the measurement at time $t$. If the evolution of the system is interrupted by measurements at times $t_0 < t_1 < t_2 < ... < t_{n-1} < t$, by the arguments resulting in (26) we see $S(t) \geq S(t_{n-1}) \geq ... \geq S(t_2) \geq S(t_1) \geq S(t_0)$. Anyway, we have

$$S(t) \geq S(t_0). \quad (27)$$

The interaction between different parts of the system makes these parts be correlated, which in turn makes the entropy of the system strictly increase. The theorem is therefore proven.

### 4 Discussion

The proof here is quite general. Beside the axioms stated in section 2, nothing is assumed. Both classical and quantum statistics satisfy these axioms. Therefore, the principle of entropy increase is exactly true in them. The axioms are not too special, and may also be satisfied by a wide class of statistical processes. It means, perhaps we can find some statistical science other than physics, in which the principle of entropy increase is applicable as well.

From the proof we learn that the entropy of a system increases only because that, when one considers it he always neglects the correlation information between different parts of the system. It emphasizes the importance of the correlation information in a complete statistical science.

This work is supported by the National Nature Science Foundation of China with Grant number 10305001.
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