Skyrmion Excitation in Two-Dimensional Spinor Bose-Einstein Condensate

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We study the properties of coreless vortices (skyrmion) in spinor Bose-Einstein condensate. We find that this excitation is always energetically unstable, it always decays to an uniform spin texture. We obtain the skyrmion energy as a function of its size and position, a key quantity in understanding the decay process. We also point out that the decay rate of a skyrmion with high winding number will be slower. The interaction between skyrmions and other excitation modes are also discussed.

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I. INTRODUCTION

Topological objects have been attracting interest from various fields of physics for several decades. Roughly speaking, there are two kinds of topological excitations in two-dimensional space. The configuration of the first kind, such as vortex or monopole, has a natural singularity and depends on azimuthal angles even at infinity. These excitations have infinite kinetic energy unless they are coupled to other fields vanishing at infinity. In order to obtain a topological structure with finite energy, the configuration of the second kind must be uniform at infinity, which means that the topological structure should be defined in a compactified space. The skyrmion is an example of the second kind topological structure, and the skyrmion excitation in n-dimensional space exists when the nth homotopy group of the internal space is nontrivial.

Since its introduction in the 1960s in nuclear physics, and its application in QCD, skyrmions have been found in condensed matter systems such as quantum Hall effect and high temperature superconductivity. The achievement of Bose-Einstein condensate (BEC) in dilute Bose gases provide an opportunity to investigate many-body theory in new system. So far some topological excitations such as vortices and vortex rings in scalar BEC have been observed in a number of labs. Recently the realization of a spinor BEC, whose spin degrees of freedom are unfrozen has generated interest in studying richer topological excitations in such systems.

The mean field description of spin-1 BEC was proposed in pioneer works of Ho and Ohmi et al. The symmetry group of the ground state order parameter of an antiferromagnetic spin-1 BEC is found to be $U(1) \times S^2/Z^2$, where $U(1)$ denotes the global phase angle and $S^2$ is a unit sphere denoting all orientation of the spin quantization axis. The additional $Z^2$ is because of the reflection of spin quantization axis is equivalent to the change the global phase by $\pi$. The general form of such a spinor field is

$$\zeta = \frac{1}{\sqrt{2}} \begin{pmatrix} -m_x + im_y \\ \sqrt{2}m_z \\ m_x + im_y \end{pmatrix}$$

where $\vec{m}$ is the Bloch vector. It should be pointed out that the second homotopy group of $S^2$ is homomorphic to integer group, this not only tells us of the existence of monopole excitation in three-dimensional antiferromagnetic BEC, which had been investigated by H.T.C. Stoof et al., but also implies the existence of the skyrmion excitation in two-dimensional case.

It is naturally to ask if this excitation mode is energetically stable or not. In this paper, we answer the question within the Thomas-Fermi approximation and a variational method, and we find that the skyrmion is always energetically unstable. In other words, in presence of any energy dissipation mechanism, the spin texture will always decay. However, this result does not imply that the skyrmion can not be created, a skyrmion with half topological charge has been successfully generated in a recent experiment by adiabatic deformation of the magnetic trap. Although the half skyrmion created in this experiment is different from that discussed in this paper, it is believed that a skyrmion with integer winding number can also be created in the near future.

Furthermore, it is useful to discover whether the skyrmion decays by expanding to infinity size or shrinking to tiny size after it is created, and to understand how its center of mass moves. To answer these two questions, we need to obtain the energy of skyrmion as a function of its size and position, this energy function is the central result of our paper. We also find that the presence of a vortex influences skyrmion’s motion. Additionally based on the results of Ref. we will point out that the decay rate of high winding number skyrmions will be much slower than those with winding number 1.

II. THE ENERGETIC STABILITY OF $Q = 1$ SKYRMION

When we only focus on the energy property of skyrmions, we can first neglect the interaction in spin channel which disrupt the $S^2$ order parameter, and the energy functional for such BEC can be simplified as following:
\[ K(\varphi, \zeta) = \int \int d^2 r \left( \frac{k_r^2}{2m} |\nabla \varphi|^2 + \frac{k_\zeta^2}{2m} |\nabla \zeta|^2 - (\mu - V_{\text{trap}}(\vec{r}))|\varphi|^2 + \frac{4\pi h^2 a_{sc} N}{m} |\varphi|^4 \right) \]  

(2)

\( \mu \) is the chemical potential and \( V_{\text{trap}} \) is the confining trap potential. In this section, we will use conformal mapping to construct a general skyrmion excitation with winding number \( Q = 1 \). These variational wave functions are used to show that such skyrmions are always energetically unstable.

Equation (2) is a nonlinear sigma model coupled to a \( \varphi^4 \) model in an external potential. An \( \sigma \)-skyrmion is an instanton solution to the \( 1+1 \) dimensional nonlinear sigma model, this skyrmion minimizes the energy functional in the sector of each homotopy class, and can be constructed from the help of fractional linear mapping which maps the compactified complex space into itself, and the most general form of this mapping is

\[ \Omega = f(z) = \prod_{i=1}^{N} \frac{a_i z + b_i}{c_i z + d_i} \]  

(3)

with the constraint \( a_i d_i - b_i c_i = 1 \), where \( N \) is the winding number. Complex number \( \omega \) are mapped to vectors on the unit sphere via

\[ m_x = \frac{\Omega + \overline{\Omega}}{1 + |\Omega|^2}, m_y = -i \frac{\Omega - \overline{\Omega}}{1 + |\Omega|^2}, m_z = \frac{|\Omega|^2 - 1}{|\Omega|^2 + 1} \]  

(4)

Owing to the conformal invariance of the nonlinear sigma model, its classical solutions are infinitely degenerate and the total energy is independent of the parameters in the analytical function \( f(z) \). The coupling between the spinor field \( \zeta \) and the superfluid field \( \varphi \) breaks the conformal invariance, and skyrmions are not classical solutions to our model. The spin texture defined by equation (3) can however be used in a variational calculation. The texture of spinor field contributes an effective potential, which changes the density profile. Given an effective potential, the minimum of the energy functional, is function of the parameters in the conformal mapping \( f(z) \). This function will tell us the information about the energetic stability of skyrmion excitation.

We first consider the \( Q = 1 \) case, \( \Omega = f(z) = \frac{az + b}{cz + d} \). It is not difficult to show that

\[ |\nabla \zeta|^2 = 8 \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2} = \frac{8}{|cz + d|^2 + |az + b|^2} \]  

(5)

On the condition \( ad - bc = 1 \), the effective potential is non-singular implying the density remains finite at the skyrmion core. This result is consistent with the coreless character of skyrmion excitation. The effective potential has two barrier localized at \( z = -\frac{a}{c} \) and \( z = -\frac{b}{a} \), with the height \( |c|^4 \) and \( |a|^4 \) respectively, where the density of the condensate should be smaller than the surrounding. Recall that in the repulsive case the interaction constant \( g \) is positive, both the interaction energy and the kinetic energy favor homogenous density profile, too much undulation will no doubt increase the energy, so the choice \( c = 0 \) helps to decrease the energy. This value of \( c \) also produces the most symmetric texture. The function \( f(z) \) is reduced to \( a^2(z - z_1) \), where the parameters \( a \) and \( z_1 \) characterizes the size and location of the skyrmion.

For simplicity, we first force \( z_1 = 0 \) to be at the center of the trap. Non-zero \( z_1 \) will be discussed later. The effective potential from the spin texture now is \( \frac{1}{|\frac{a}{c}|^2 + |a|^2 |z|^2} \), and the full potential is showed in Figure 1 for different \( a \). Notice that the barrier height of the effective potential is proportional to \( |a|^4 \), and that integrating the potential energy over the whole two-dimension space results in a constant. In the limit \( a \to 0 \), the effective potential spreads uniformly throughout the whole space. In the contrary limit of infinite \( a \), the effective potential becomes a \( \delta \) function.

We now investigate the energetic stability of skyrmion in the framework of TF approximation where the term \( |\nabla \varphi|^2 \) is neglected. In this approximation, the density profile is

\[ n(\vec{r}) = \left( \mu - a_{\text{HO}}^2 r^2 - \frac{8}{(\frac{1}{\alpha})^2 + a^2 r^2} \right) \frac{1}{2g} \Theta \left( \mu - a_{\text{HO}}^2 r^2 - \frac{8}{(\frac{1}{\alpha})^2 + a^2 r^2} \right) \]  

(6)

in which \( a_{\text{HO}} = \left( \frac{\omega}{\hbar} \right)^2 \) and \( g = 8\pi a_{sc} N \), \( \omega \) is the frequency of the harmonic trap, and \( \Theta(x) \) is a step function.

The normalization condition and the constraint that the
condensate density is non-negative give the following two equations:

\[ \frac{1}{2g} \int_0^{2\pi} \int_0^r r dr d\theta \left[ \mu - a^2 H \right] - \frac{8}{\left( \frac{1}{8} + a^2 r^2 \right)} = 1 \]  
\[ \mu - a^2 H \right) - \frac{8}{\left( \frac{1}{8} + a^2 x^2 \right)} = 0 \]

where \( x \) is the value of \( r \) at which the Thomas-Fermi density vanishes. Solving the two equations we obtain the relation between the chemical potential \( \mu \), Thomas-Fermi radius \( x \) and the skyrmion’s size \( a \). One can substitute these relationships back to the energy functional and then obtain the minimal energy \( E \) as a function of size \( a \) which is plotted in Figure 2.

Figure 2 shows that the minimal energy \( E \) has two minima occurring at zero and infinity, and there exists a critical size \( a_c \) which corresponds to the maximal value of \( E \). When \( a \) is quite large, the TF approximation fails, we use variational density profile to obtain a more accurate result \[14\], which is shown in Figure 3.

Physically, we can understand the two figures in the following way. When the size parameter \( a \) is sufficiently small, the spin configuration within the TF radius becomes uniform, the effective potential becomes flat inside the TF radius and just becomes a uniform shift of the chemical potential. So the energy approaches the ground state energy as \( a \rightarrow 0 \). In the opposite limit, the barrier is quite high and thus the density almost vanishes at the center of the skyrmion, but at the same time the width of the barrier becomes narrow. The energy cost of the low density region is proportional to the volume and decreases as \( a \) becoming larger. In this case the skyrmion can be observable by imaging the density profile, and observing the low density core. These figures tell us that the \( Q = 1 \) skyrmion is energetically unstable, in presence of any weak dissipation, it will either expand to become unobservable, or shrink to an infinitesimal size. As a semi-classical object the dissipative dynamics of a skyrmion depends on whether or not its initial size is larger than the critical size \( a_c \).

For the same reason, the position \( z_1 \) of the skyrmion tends to move toward the edge of the condensate where the density is lower and the energy cost \( \int V_{eff} |\varphi|^2 \) is smaller.

### III. DISSIPATION DYNAMICS

Although we believe that the energetically unstable skyrmion will decay in presence of any weak dissipation, the exact description of the dynamics depends on the details of the dissipation mechanism. A possible source of dissipation is interaction with the non-condensed component. Since skyrmion is a global topological object and the system is in a quite low temperature, this may not be important. Another source of dissipation is the spontaneous emission of other excitation modes, such as phonons mode or vortices, similar to the spontaneous emission of an excited atom. This mechanism is observed in numerical studies of superfluid vortex reconnection.

![FIG. 1: The trap potential added with the effective potential induced by \( Q = 1 \) skyrmion, in the unit of \( \hbar \omega \), is plotted as a function of \( x/l \) for \( a/\sqrt{a_{HO}} = 1, 1.5, 2 \) respectively. Here \( l = 1/\sqrt{a_{HO}} \).](image1)

![FIG. 2: The minimal energy \( E \) in the unit of \( \hbar \omega \) vs. the size of skyrmion \( a \) in the unit of \( L = \sqrt{a_{HO}} \). The parameter \( g/\sqrt{a_{HO}} \) is set to 2.](image2)

![FIG. 3: The minimal energy \( E \) in the unit of \( \hbar \omega \) vs. size \( a \) in the unit of \( L \), the result is obtained using variational method.](image3)
Let us discuss the process in more detail. When $a$ increases, the peak of effective potential grows and atoms are pushed away from the center of the skyrmion. The kinetic energy of these atoms will lead to oscillatory motion of the condensate density. If the trap harmonic potential is anisotropic, the extracted atoms can also rotate to form a vortex. These modes take energy away from the skyrmion, leading to a change of the skyrmion size.

IV. INTERACTION BETWEEN A VORTEX AND A SKYRMION

We rewrite $\varphi$ as $\Psi e^{i\theta}$. In equation (2), the phase field $\theta$ is not directly coupled to the spinor field $\zeta$, but they both interact with the density field. Since both vortices and skyrmion have low density cores, it is favorable that the skyrmion core fits entirely within the vortex, thus one expects that the presence of vortex change the dominant trend of the evolvement of the skyrmion from expanding to shrinking, and one should find the skyrmion attracted to the vortex center instead of moving toward the cloud’s edge.

In principle, we can integrate out the $\Psi$ field in the action and obtain the effective Hamiltonian to describe the interaction between a vortex and a skyrmion. Unfortunately the $\Psi^4$ term makes this procedure difficult. However in the TF approximation, where the derivative terms of $\Psi$ are neglected, the action is a quadratic form of the density field $n$. We can integrate it out and obtain an effective Hamiltonian describing the interaction between $\theta$ and $\zeta$:

$$H_{\text{int}} = -\frac{1}{2g} \left( \frac{\hbar^2}{2m} \right)^2 \int d^2 \vec{r} \left( |\nabla \theta|^2 |\nabla \zeta|^2 \right)$$

The interaction energy will be smallest when the maximum of $|\nabla \theta|$ and $|\nabla \zeta|$ coincide. This result is consistent with the above picture.

V. $Q > 1$ SKYRMION

In this section we discuss skyrmions with $Q > 1$, demonstrating their differences from the $Q = 1$ skyrmion. Following the logic of the second section, we choose the most symmetric case, $f(z) = a^2 z^n$, finding the effective potential

$$V_n = 8\frac{n^2 \mu^{2n-2}}{[\frac{1}{a} + a^2 r^{2n}]^2}$$

The barrier of $V_n$ ($n > 2$) lies on a circle

$$r = \left( \frac{2n - 2}{2n + 2} \left( \frac{1}{a} \right)^4 \right)^{\frac{1}{2n}}$$

with the height

$$V_{\text{max}} = 2 \left( n^2 - 1 \right) \left( \frac{2n + 1}{2n - 1} a^4 \right)^{\frac{1}{2n}}$$

this structure is markedly different from $V_1$ whose peak is always localized at its center. In the center, the effective potentials is always zero for any $n \geq 2$. Figure 4 shows $V_2$ for different value of $a$. As $a$ is decreasing, the location of the off-center peak of the potential approaches infinity as $(\frac{1}{a})^\frac{4}{2n}$ and its height decreases as $a^\frac{4}{n}$. Because the size of condensate is always finite, this implies that it is energetically favorable for a skyrmion to increase to a larger size so that the peak of the effective potential is outside the TF radius and the effective potential becomes small and uniform inside the condensate. What is different from the $Q = 1$ case is the atoms must tunnel through the barrier of the effective potential as the size increases. Thus, the rate of the process will be characterized by the WKB tunnelling rate. The situation has been studied carefully in Ref. 7, and it showed that the tunnelling rate of such process may be long enough to form a dynamic metastable skyrmion.

![FIG. 4: The trap potential added with the effective potential induced by $Q = 2$ skyrmion, in the unit of $\hbar \omega$, is plotted as a function of $x/l$ for $a/\sqrt{\hbar \omega} = 1, 3$ and 5 respectively](image)

VI. CONCLUSION

In summary, we constructed and studied the skyrmion excitation mode of the spinor field and showed that any finite size skyrmion is always unstable. Through a variational study of the skyrmion excitation energy for different size and position, we find that it is energetically favorable for skyrmion to expand to infinite size or shrinking to infinitesimal size, resolution in a uniform spin texture in most part of the condensate. We discussed the interplay between skyrmion modes and other excitation modes: (1) the emission phonons can dissipate the skyrmion energy and change its size, and (2) the vortex has attractive interactions with a skyrmion. We also showed that the behavior of $Q > 1$ skyrmions are quite different from $Q = 1$ skyrmion since the effective potentials induced by their interactions...
configuration have some markedly difference. However, all these discussions are concentrated on properties of the skyrmion mode itself and a qualitative prediction on the skyrmion dynamics is made from the energetic consideration. To study the detail of skyrmion dynamic process, it is important to consider the interaction in spin channel, which cause spin fluctuations exceeding the $S^2$ internal space.\[8\]

In the end, noticing that $\zeta$ represents local spin-gauge degrees of freedom, we remark that the model we explored in this paper shares some features of Yang-Mills theory, that is, the instanton solutions having the same winding number are saddle points of the Lagrangian and are energetically degenerate although they have different shape and size, because of the local scaling invariance of the pure gauge theory. However, when the coupling to matter field breaks the conformal symmetry, the instanton solutions are no longer classical solutions and have different actions. In addition, the dynamic term of gauge field is absent in our model. The conclusion of energetic instability of any finite size skyrmion is determined by the above two features.

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