Vortex quantum dynamics of two dimensional lattice bosons

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Vortices, which are introduced into a boson superfluid by rotation or a magnetic field, tend to localize in a lattice configuration which coexists with superfluidity \textsuperscript{[1,2,3]}. In two dimensions a vortex lattice can melt by quantum fluctuations resulting in a non-superfluid Quantum Vortex Liquid (QVL). Present microscopic understanding of vortex dynamics of lattice bosons is insufficient to predict the actual melting density. A missing energy scale, which is difficult to obtain perturbatively or semiclassically, is the "bare" vortex hopping rate \( t_v \) on the dual lattice. Another puzzle is the temperature dependent Hall conductivity \( \sigma_H(T) \), which reflects the effective vortex Magnus dynamics in the QVL phase. In this paper we compute \( t_v \) and \( \sigma_H(T) \) by exact diagonalization of finite clusters near half filling. Mapping our effective Hamiltonian to the Boson Coloumb Liquid simulated by Ref. \[4\], we expect a QVL above a melting density of \( 6.5 \times 10^{-3} \) vortices per lattice site. The Hall conductivity near half filling reverses sign in a sharp transition accompanied by a vanishing temperature scale. At half filling, we show that vortices carry spin half degrees of freedom ('v-spins'), as a consequence of local non commuting SU(2) symmetries. Our findings could be realized in cold atoms, Josephson junction arrays and cuprate superconductors.

The model – We consider \( N_b \) hard core bosons (HCB) hopping on a square lattice of unit lattice constant and size \( N = L^x L^y \). An external vector potential \( \mathbf{A} \) modulates the hopping amplitude (Josephson energy) \( t \). The system is placed on a torus with periodic boundary conditions, as shown in Fig. 1. In the spin-\( \frac{1}{2} \) representation of HCB, the angular momentum raising and lowering operators \( S_r^\pm \) create and annihilate bosons; the occupation number is \( n_r = S_r^z + \frac{1}{2} \). The Hamiltonian we study is a gauged XXZ model,

\[
\mathcal{H} = -\frac{t}{4} \sum_{r,\eta} \left( e^{iA_\eta(r)} S_r^+ S_{r+\eta}^- + \text{H.c.} \right) + \frac{V}{2} \sum_{r,\eta} S_r^z S_{r+\eta}^z \tag{1}
\]

where \( \eta = \pm \hat{x}, \pm \hat{y} \), is the link direction on which the lattice gauge field \( A_\eta \) is defined. Here we only consider the superfluid regime of weak nearest neighbor repulsion \( 0 < V \ll t \).

In the absence of external magnetic field, the classical ground state of \( \mathcal{H} \), is a ferromagnet in the \( XY \) plane with a uniform \( z \)-magnetization density \( m_z = n_b - 1/2 \). The mean field superfluid stiffness is given by \( \rho_{sf} = t n_b \). Consequently, the superfluid transition temperature, which is proportional to \( \rho_s \), is maximal at half filling \[5\].

An important distinction between lattice hard core bosons and continuum models is the existence of a charge conjugation operator \( C \equiv \exp (i \pi \sum_r S_r^z) \) on the lattice. \( C \) transforms boson "particles" into "holes" \( n_i \to (1 - n_i) \), and the Hamiltonian into

\[
C \mathcal{H}[\mathbf{A},n_b] C = \mathcal{H}[\mathbf{-A}, 1 - n_b], \tag{2}
\]

where \( n_b = N_b / N \) is the filling fraction. A consequence of \( \mathcal{H}[\mathbf{A},n_b] \) is that the Hall conductivity is antisymmetric in \( n_b - 1/2 \):

\[
\sigma_H(n_b, T) = -\sigma_H(1 - n_b, T). \tag{3}
\]

In terms of vortex motion, this relation implies that below and above half filling vortices drift in opposite directions relative to the particle current. Sign reversal of Hall conductivity is familiar from tight binding electrons at half filling on bipartite lattices. Here, however, the mechanism of sign reversal is different: it arises from the hard-core interactions of bosons and occurs for any lattice structure.

FIG. 1: The Gauged Torus. Geometry of HCB Hamiltonian Eq. \[1\] which serves to extract its vortex mass and Hall conductivity. The torus surface is penetrated by a uniform magnetic field of one flux quantum, and threaded by two Aharonov Bohm fluxes \( \Theta = (\Theta^z, \Theta^y) \). For one flux quantum there is no translational symmetry on the torus. Red circles denote cycles of zero flux, and the vortex center \( R_c(\Theta) \) is localized on the antipodal point to their intersection, the null point.
Vortex effective Hamiltonian – Vortices in a two-dimensional superfluid act as charges in \((2 + 1)\)-dimensional electrodynamics, where the phonons of the superfluid become the photons of the electromagnetic theory. The duality between vortices and charges is discussed in refs. [3, 7, 8, 10, 11]. The role of speed of light is played by the phonon speed of sound, which in our model is \(c = \sqrt{2\alpha/a}/\hbar\), where \(a\) is the lattice constant. The vortex centers \(\mathbf{R}_v\) are modeled as point charges hopping on the dual lattice of plaquette centers, coupled minimally to the gauge field \(A^\mu = A^\mu_v + a^\mu\), where \(a^\mu\) is due to the average boson density, generating a dual magnetic flux per plaquette of \(2\pi n_b\), and where \(A^\mu\) is a dynamical field describing the fluctuations in the boson density and current. At low energies, vortex dynamics is described by a Harper Hamiltonian plus confining potential, 

\[
H^{R_v R_v} = -\sum_n e^{i A^\alpha_n} R_v - R_v + \mu N(R) \delta R_v R_v'. \tag{4}
\]

\(U_N(\mathbf{R})\) is an effective potential which binds the vortex to the ‘center of mass’ location \(R_v(\Theta^\alpha, \Theta^\beta)\), given by [12, 13],

\[
R_v^\alpha = \frac{L^\alpha}{N^\phi} e^{\phi_\beta} \left( n_\beta + \frac{\Theta^\beta}{2\pi} \right) \mod L^\alpha, \tag{5}
\]

\(N^\phi\) is the number of flux quanta, \(n_\alpha\) are integers, \(L^\alpha\) are the dimensions of the torus. The phase angles \(\Theta^\alpha\) are proportional to the solenoidal fluxes of the two elementary toroidal cycles, and a set of ‘null points’ at the intersections of cycles \(C_{x,y}\) is defined by requiring \(\oint C_{x,y} A^\alpha = 2\pi n_\alpha/N^\phi\), as shown for \(N^\phi = 1\) in Fig. 1. The vortex CM must lie antipodal to one of the \(N^\phi\) null points; we study the simple case \(N^\phi = 1\). The constant \(K\) is calculated variationally from Eq. (4) using spin coherent states. Minimizing the energy with respect to the position of the vortex centered at \(\mathbf{R}_v\), determines the effective potential \(U_N(\mathbf{R})\). For \(V = 0\) at we find \(K \simeq 39.2 t n_b (1 - n_b)\). We stress that inclusion of the potential \(U_N(\mathbf{R})\) is essential in order to extract the vortex hopping parameters from our small scale numerical calculations.

For a quantitative theory of vortices we need to evaluate the effective hopping \(t_v\). Since vortex tunneling between lattice sites depend on short range many-body correlations, we extract \(t_v\) from exact numerical diagonalizations of \(H\) on \(16 - 20\) sites clusters, in the presence of a single flux quantum. By tuning \(t_v\), we fit the lowest three eigenenergies \(E_n\) and eigenstates \(|\Psi_n\rangle\) of \(H\) to the lowest states of the effective Harper Hamiltonian [4]. This assume that \(|\Psi_n\rangle\) states correspond to zero point fluctuations of the vortex positions, since phonons are frozen out by the finite lattice gap of order \(2\pi t'/\sqrt{N}\).

Our results for \(t_v(n_b, V/t)\), for \(N = 20\) fit the analytical approximations,

\[
t_v(n_b, 0) = t - \frac{12.6}{V} \left( n_b - \frac{1}{2} \right)^2 + 164 \left( n_b - \frac{1}{2} \right)^4,
\]

\[
t_v(\frac{1}{2}, V) = t + 1.5 V + 2.7 V^2 - \frac{V^3}{t}. \tag{6}
\]

The system parameters were varied in the range \(|n_b - \frac{1}{2}| \leq 0.2\), and \(V/t < 0.5\). We find that at half filling, \(t_v\) varies very little between the \(N = 16\) and \(N = 20\) lattices. To further test our assignment of \(t_v\), we compare the vorticity density \(\langle \nabla \times \mathbf{j} \rangle\) of eigenstates of \(H\) to the probability density of eigenstates of \(H^v\). As shown in Fig. 2 using the fitted value of \(t_v\) we obtain similar distributions for both sets of wavefunctions.

Vortex tunneling – In [6] we find that near half filling, vortices are as light as bosons, \(t_v \approx t\). This implies that the vortex tunneling rate between two localized pinning potentials of strength \(V\), which are separated by distance \(d\), decays exponentially as \(\Gamma \sim V e^{-d/\lambda}\). The localization length \(\lambda \propto \sqrt{t/V}\) diverges at weak pinning. This result is to be contrasted with weakly interacting continuum Bose gas. There, the vortex tunneling rate between pinning sites is much smaller, and decays as a Gaussian \(\Gamma \sim e^{-2 d^2/\lambda}\) [15]. From this comparison, we conclude that at half filling \(n_b = 1/2\), the lattice and interactions enhance vortex mobility considerably.

Quantum Melting Transition – Having calculated \(t_v\), we can write down the effective multi-vortex Hamiltonian in the thermodynamic limit. We drop \(U_N\) at large \(N\). By [4], at half filling there is dual magnetic flux \(\pi\) per plaquette. In the magnetic Brillouin zone, there is a two-fold degenerate dispersion \(E_{k,s} = |s|, \downarrow\). We later return to

\[\text{FIG. 2: (a) The vorticity } \langle \nabla \times \mathbf{j} \rangle \text{ for the first three doublets of the HCB model, Eq. (1), with } N^\phi = 1 \text{ and } \Theta = 0 \text{ on a } 4 \times 4 \text{ lattice. The uniform background vorticity has been subtracted. (b) Single particle probability density of the lowest three excitations of } H^v\).\]
explain the origin of this ‘v-spin’ degeneracy. The vortex effective mass is \( M_v^{-1} = \frac{\partial^2 E}{\partial k^2} = t_v a^2/\hbar^2 \). Integrating out the phonon fluctuations \( A^\mu \), produces an instantaneous logarithmic (2D Coulomb) interaction between vortices, plus retarded (frequency-dependent) interactions \([11]\). These can be represented by a self energy \( \mathcal{H}^{\text{ret}}(\omega) \). Since we are interested in the short wavelength fluctuations which are responsible for the quantum melting of the vortex lattice, we ignore these retardation effects.

Thus, for half filled bosons and a vortex density \( n_v \) we arrive multivortex Hamiltonian

\[
\mathcal{H}^{\text{mv}} = \sum_{i,s=\uparrow,\downarrow} \frac{p_i^2}{2M_v} + \frac{\pi t}{4} \sum_{i \neq j} \log(|r_i - r_j|) - \frac{n_v \pi^2 t}{4} \sum_i |r_i|^2 + \mathcal{H}^{\text{ret}}(\omega). \tag{7}
\]

The single spin version of \( \mathcal{H}^{\text{mv}} \), after setting \( \mathcal{H}^{\text{ret}} \to 0 \), is the Boson Coulomb Liquid studied by Magro and Ceperly (MC) \([4]\) by diffusion Monte-Carlo simulation. Their dimensionless parameter which governs the phase diagram is \( r_s = \pi n_v a^2 \). We set their \( a_0 = (\frac{\hbar^2}{\pi m v^2} \pi)\) as the microscopic length which matches between their model and \( \mathcal{H}^{\text{mv}} \). MC found that below \( r_s \approx 12 \) the boson lattice undergoes quantum melting. Using our values of \( t_v \) in Eq. (6), the critical \( r_s \approx 12 \) translates into a vortex melting density of

\[
n_v^{\text{cr}} \leq \left(6.5 - 7.9 \frac{V}{t}\right) \times 10^{-3} \text{ vortices per site}. \tag{8}
\]

This is a suprisingly low vortex density, which implies that a QVL can be created at manageable rotation frequencies for cold atoms, and moderate magnetic fields for Josephson junction arrays and cuprate superconductors.

**Hall Conductance** - The temperature-dependent Hall conductance of the finite cluster is given by the thermally averaged Chern numbers \([19]\):

\[
\sigma_H(n_b, T) = \frac{1}{\pi} \sum_{n=0}^{\infty} \int_0^{2\pi} d^2 \Theta \frac{e^{-E_n/T}}{Z} \times \text{Im} \left( \frac{\partial \psi_n}{\partial \Theta_x} \frac{\partial \psi_n}{\partial \Theta_y} \right) \tag{9}
\]

\( E_n(\Theta), \) and \( |\psi_n(\Theta)\rangle \) are the exact spectrum and eigenstates of \([1]\). The results are matched at high temperatures with the ones obtained by the Kubo formula \([17]\). A typical Hall conductance as a function of filling for \( N_\phi = 1 \) is plotted in Fig. 3. At zero temperature, \( \sigma_H = N_b \) below half filling, reminiscent of the behavior in the continuum \( \sigma_H \propto N_b/N_v \), which holds irrespective of temperatures. However, for HCB \( \sigma_H(T, n_b) \) decreases with temperature. Moreover, \( \sigma_H \) reverses sign at half filling, as expected by \([3]\).

Our results show a striking general feature. We find that \( \sigma_H \) undergoes a sharp transition between \( \sigma_H > 0 \) and \( \sigma_H < 0 \) just below (above) half filling. As the temperature is lowered, the sign reversal of the Hall conductance happens across a narrower region around half filling. This suggests a singularity in the thermodynamic system with a vanishing energy scale. We define \( T_H(n_b) \) by \( \sigma_H(T_H) = \frac{1}{2} \sigma_H(0) \). In the inset of Fig. 3 we show that \( T_H \) seems to vanish with \( |n_b - \frac{1}{2}| \), although we cannot yet investigate this behavior further in larger systems.

**Spin-\(1/2 \) vortices** - Half filling is a special density for \( \mathcal{H} \). First, the Hall coefficient vanishes by (3), which implies that the vortices see no static Magnus field. Second, the external magnetic field creates a multitude of doublet degeneracies. To be precise, for any odd number \( N_\phi \) of flux quanta, there are \( N \) (the system size) distinct values of AB fluxes \( \Theta_i \) where all eigenstates are two-fold degenerate. We have found that these degeneracies are associated with non-commuting local symmetry operators

\[
\Pi^\alpha = \frac{1}{2} U^\alpha C P^\alpha [\mathbf{R}_\nu], \quad \alpha = x, y. \tag{10}
\]

\( C \) is the charge conjugation (see Eq. (5)), and \( U^\alpha \) is a pure gauge transformation. \( P^{\alpha(x)}(y) \) is a lattice reflection about the \( x(y) \) axis passing through the vorticity center \( \mathbf{R}_\nu \). For \( N \) discrete AB fluxes, \( \mathbf{R}_\nu \) can be placed on each one of the lattice positions, where \( \{\mathcal{H}, \Pi^\alpha\} = 0 \). These symmetries follow from the fact that \( C P^\alpha \) preserve the magnetic field and the AB fluxes. If \( \Theta_i \) are tuned by (5) to position \( \mathbf{R}_\nu \) on a lattice site, \( \Pi^\alpha \) sends \( \mathcal{H} \) to itself upto a pure gauge transformation \( (U^\alpha)^1 \).
yields the commutation rule
\[ \Pi^x \Pi^x = (-1)^{N_\phi} \Pi^y \Pi^y. \] (11)

We define the vector \( \Pi = (\Pi^x, \Pi^y, \Pi^z) \), where \( \Pi^2 = 2\Pi^x \Pi^y \). For any odd number \( N_\phi \) of vortices it is easy to show using (11) that each of the energy eigenstates is at least two-fold degenerate.

Since \( \Pi^2 = 3/4 \), they obey the algebra of spin half operators. Thus the doublets reflect the Kramers degeneracy expected for an odd number of interacting spin half degrees of freedom which we label \( v \)-spins. As shown by the form of \( \Pi^z \), the \( v \)-spins are attached to the vortex positions. The \( z \)-direction polarization corresponds to a boson charge density wave (CDW) modulation. Variational calculation shows the CDW to be exponentially localized in the vortex core [13]. Thus \( v \)-spin interactions between different vortices decay exponentially, and are very weak in the vortex lattice regime. We cannot determine their ordering configuration. If they interact ferromagnetically, a CDW order parameter can form. In any case, whatever the ordering tendencies, the low exchange energy ensures that \( v \)-spins excitations play an important role in the low temperature thermodynamics and transport coefficients of the multi vortex system in both lattice and QVL phases.

Nature of the QVL – Theoretical treatments of lattice bosons have found a myriad of vortex-antivortex condensate (VC) phases at all rational boson filling fractions, \( n_b = p/q \), due to \( q \)-fold degeneracies of the Harper hamiltonian on an infinite lattice [19, 20, 21, 22, 23]. VC’s are in effect, insulating phases where the dual Anderson-Higgs mechanism produces a Mott gap [7]. Some of these phases result in \( q \)-periodic CDWs. Therefore a possible candidate for the VC at half filling is the bipartite CDW i.e. the antiferromagnetic Ising state of [1] at \( V > t \). However, the QVL we study, which contains a net condensate (VC) phases at all rational boson filling fractions, differs from the proposed VC phases in two important respects.

(i) MC [4] have found that the liquid phase of \( \mathcal{H}_{\text{mv}} \) has vanishing condensate fraction. If their results (ignoring retardation effects) is relevant to the QVL, it should differ from a charge-gapped insulator. Whether it is a metal is an open possibility. Away from half filling, our results for \( \sigma_{yx} \) show that the vortices are subject to a strong magnetic field, which further suppresses their condensation. At low boson fillings and large vortex density, \( n_b/n_\phi < 1 \), there is evidence for fractional quantum hall phases [24, 25].

(ii) Away from the commensurate filling \( n_b = p/q \), the Hall conductivity is expected to cross zero, and be proportional to the excess density from \( p/q \). We found that the Hall conductance has a very different behavior: it has only one abrupt jump between a positive value below, and negative above half filling. We have computed the single flux Hall conductance on a finite lattice, but we expect it to represent the macroscopic Hall conductivity in the QVL phase where superfluid order parameter correlations are short ranged.

Summary – Vortices of hard core bosons near half filling are highly mobile logarithmically interacting charges. The vortex lattice is expected to melt into a QVL at around \( 7 \times 10^{-3} \) vortices per site. At boson density of half filling, the vortices are not subjected to an effective Magnus field, but carry local \( v \)-spin degrees of freedom, which effect the low energy correlations. Away from half filling, the Hall conductivity exhibits rapid variation accompanied by a vanishing energy scale. Although the issue is far from settled, we present arguments that the QVL is not a vortex condensate.

Acknowledgements. We thank Ehud Altman, Yosi Avron, David Ceperley, Herb Fertig, Gil Refael and Efrat Shimshoni for useful discussions. Support of the US Israel Binational Science Foundation is gratefully acknowledged. AA acknowledges the Aspen Center For Physics where some of the ideas were conceived.

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