On the reducibility of Schlesinger isomonodromic families

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Abstract

We obtain some sufficient conditions for reducibility of a Schlesinger isomonodromic family with the (block) upper-triangular monodromy to the same (block) upper-triangular form via a constant gauge transformation. We also obtain integral representations of hypergeometric type for entries of upper-triangular solutions of the Schlesinger equation.

1 Introduction

Let us consider on the Riemann sphere \( \overline{\mathbb{C}} = \mathbb{C} \cup \{ z = \infty \} \) a Fuchsian system of \( p \) linear differential equations If the infinity is not contained in a set of singularities of the coefficient matrix, then such a system can be written in the form

\[
\frac{dy}{dz} = \left( \sum_{i=1}^{n} B_i^0 \right) y, \quad \sum_{i=1}^{n} B_i^0 = 0, \tag{1}
\]

where \( y(z) \in \mathbb{C}^p, B_1^0, \ldots, B_n^0 \) are constant matrices of size \( p \times p \) (they are residue matrices of the system), \( a_0^1, \ldots, a_0^n \in \mathbb{C} \) are singular points of the system.

An important characteristic of a linear system is the monodromy, or monodromy representation. In a neighborhood of a non-singular point \( z_0 \) consider a fundamental matrix \( Y(z) \) of the system (1). An analytic continuation of the matrix \( Y(z) \) along an arbitrary loop \( \gamma \) starting at \( z_0 \) and contained in \( \overline{\mathbb{C}} \setminus \{ a_0^1, \ldots, a_0^n \} \) transforms this matrix in, generally speaking, another fundamental matrix \( \tilde{Y}(z) \). Two fundamental matrices are connected by a non-degenerate transition matrix \( G_\gamma \) corresponding to the loop \( \gamma \):

\[
\tilde{Y}(z) = Y(z)G_\gamma.
\]

The map \([\gamma] \mapsto G_\gamma^{-1}\) depends on the homotopic class \([\gamma]\) of the loop \( \gamma \) only and thus defines a representation

\[
\chi : \pi_1(\overline{\mathbb{C}} \setminus \{ a_0^1, \ldots, a_0^n \}, z_0) \longrightarrow \text{GL}(p, \mathbb{C})
\]

of the fundamental group of the space \( \overline{\mathbb{C}} \setminus \{ a_0^1, \ldots, a_0^n \} \) in the space of non-degenerate complex matrices of size \( p \times p \). The representation \( \chi \) is called the monodromy of the system (1).

By the monodromy matrix of the Fuchsian system (1) at a singular point \( \alpha_i^0 \) (with respect to the fundamental matrix \( Y(z) \)) one understands the matrix \( G_i \) corresponding to
a simple loop $\gamma_i$ encircling the point $a_i^0$, so that $G_i^{-1} = \chi([\gamma_i])$. The matrices $G_1, \ldots, G_n$ are generators of the monodromy group of the system (I).

If instead of the fundamental matrix $Y(z)$ we consider another fundamental matrix $Y'(z) = Y(z)C$, $C \in \text{GL}(p, \mathbb{C})$, then the corresponding monodromy matrices are of the form $G_i' = C^{-1}G_iC$. Thus monodromy matrices are defined up to a simultaneous conjugation.

The main object of the present work is a Schlesinger isomonodromic family

$$
\frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{B_i(a)}{z - a_i}\right) y, \quad B_i(a^0) = B_i^0,
$$

of Fuchsian systems holomorphically depending on a parameter $a = (a_1, \ldots, a_n) \in D(a^0)$, where $D(a^0)$ is a small polydisk centered at the point $a^0 = (a_1^0, \ldots, a_n^0)$ of the space $\mathbb{C}^n \setminus \bigcup_{i \neq j}\{a_i = a_j\}$. The monodromy of systems of this family is the same for all $a \in D(a^0)$ and the residue matrices $B_i(a)$ satisfy the Schlesinger equation [16]

$$
dB_i(a) = - \sum_{j=1, j \neq i}^{n} \frac{[B_i(a), B_j(a)]}{a_i - a_j} d(a_i - a_j), \quad i = 1, \ldots, n.
$$

The Schlesinger equation is integrable in the Frobenius sense in the polydisk $D(a^0)$, i.e., for any initial data $B_i^1, \ldots, B_i^n$ it has the unique solution $B_i(a), \ldots, B_n(a)$ such that $B_i(a^0) = B_i^0$ (see [2] theorem 14.2]). According to the Malgrange theorem [14] the matrix valued functions $B_i(a)$ can be continued meromorphically onto the universal cover $Z$ of the space $\mathbb{C}^n \setminus \bigcup_{i \neq j}\{a_i = a_j\}$. The polar set $\Theta \subset Z$ of the continued matrix functions $B_i(a)$ is called the Malgrange $\Theta$-divisor ($\Theta$ depends on the initial data $B_i(a^0) = B_i^0$). Moreover there exists a function $\tau$, holomorphic on the whole space $Z$, whose zero set coincides with $\Theta$. This function is called the $\tau$-function of the Schlesinger equation and Miwa’s formula [8] (see also [2] theorem 17.1) holds for it:

$$
d\ln \tau(a) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{\text{tr}(B_i(a)B_j(a))}{a_i - a_j} d(a_i - a_j).
$$

If the system (I) is meromorphically equivalent to a Fuchsian system

$$
\frac{d\tilde{y}}{dz} = \left(\sum_{i=1}^{n} \frac{\tilde{B}_i}{z - a_i^0}\right) \tilde{y}, \quad \sum_{i=1}^{n} \tilde{B}_i = 0
$$

(under a meromorphically invertible gauge transformation $\tilde{y} = \Gamma(z)y$) whose residue matrices $\tilde{B}_i$ have a block upper-triangular form

$$
\tilde{B}_i = \begin{pmatrix}
\tilde{B}_i^1 & * & * \\
0 & \tilde{B}_i^2 & * \\
& \ddots & \ddots \\
0 & \cdots & 0 & \tilde{B}_i^k
\end{pmatrix}, \quad i = 1, \ldots, n,
$$

then the monodromy matrices of the (transformed and, consequently, initial) system have the same block upper-triangular form. The inverse, generally speaking, is fulfilled in a
weaker sense. Namely, as shown by S. Malek \[12\], if the monodromy matrices of the Fuchsian system (1) have a block upper-triangular form

\[
G_i = \begin{pmatrix} G^1_i & * & * \\ 0 & G^2_i & * \\ \vdots & \vdots & \ddots \\ 0 & \ldots & 0 & G^k_i \end{pmatrix}, \quad i = 1, \ldots, n, \tag{3}
\]

then the system is meromorphically equivalent to a Fuchsian system whose residue matrices $\tilde{B}_i$ are of the block upper-triangular form

\[
\tilde{B}_i = \begin{pmatrix} B'_i & * & * \\ 0 & B''_i \end{pmatrix}.
\]

But a subsequent simultaneous reducibility of the blocks $B'_i$ (or $B''_i$) in a correspondence with the reducibility (3) of the matrices $G_i$ can not to take place (see [1, proposition 5.1.1], [6]).

Similar connections between the meromorphic reducibility of systems and the block structure of monodromy matrices take place for Schlesinger isomonodromic families as well (more details on the parametric reducibility will be presented elsewhere). Here, for subsequent applications to integral representations of solutions of the Schlesinger equation, we are interested in sufficient conditions for the reducibility of the residue matrices $B_i(a)$ of the Schlesinger isomonodromic family (2) to the same block upper-triangular form (3) as its monodromy matrices do have, with respect to a constant gauge transformation. As a technique tool we employ the approach that uses holomorphic vector bundles and meromorphic connections.

\section{Fuchsian systems and logarithmic connections in holomorphic vector bundles}

According to the Levelt theorem \[11\], in a neighborhood of each singular point $a^0_i$ of the system (1) there exists a fundamental matrix of the form

\[
Y(z) = U_i(z)(z - a^0_i)^{\Lambda_i}(z - a^0_i)^{E_i}, \tag{4}
\]

where $U_i(z)$ is a holomorphically invertible matrix at the point $a^0_i$, $\Lambda_i = \operatorname{diag}(\lambda^1_i, \ldots, \lambda^p_i)$ is a diagonal integer matrix whose elements $\lambda^1_i$ organize a non-increasing sequence, $E_i = (1/2\pi i) \ln G_i$ is an upper-triangular matrix (the normalized logarithm of the corresponding monodromy matrix) whose eigenvalues $\rho^j_i$ satisfy the condition

\[
0 \leq \operatorname{Re} \rho^j_i < 1.
\]

Such the fundamental matrix is called the \textit{Levelt matrix}, and one also says that its columns form the \textit{Levelt basis} in the solution space of the Fuchsian system (in a neighborhood of the singular point $a^0_i$). The diagonal elements $\lambda^1_i$ of the matrix $\Lambda_i$ are called the (Levelt) \textit{valuations} and the complex numbers $\beta^j_i = \lambda^j_i + \rho^j_i$ are called the (Levelt) \textit{exponents} of the
Fuchsian system at the singular point \( a_i^0 \). It is not difficult to check that the exponents of the Fuchsian system at the point \( a_i^0 \) coincide with the eigenvalues of the residue matrix \( B_i^0 \).

Let us recall some notions concerning holomorphic vector bundles and meromorphic connections. In an analytic interpretation, a holomorphic bundle \( E \) of rank \( p \) (over the Riemann sphere) is defined by a cocycle \( \{ g_{\alpha \beta}(z) \} \), that is, a collection of holomorphic matrix functions, corresponding to a covering \( \{ U_{\alpha} \} \) of the Riemann sphere:

\[
g_{\alpha \beta} : U_{\alpha} \cap U_{\beta} \rightarrow \text{GL}(p, \mathbb{C}), \quad U_{\alpha} \cap U_{\beta} \neq \emptyset.
\]

These functions satisfy conditions

\[
g_{\alpha \beta} = g_{\beta \alpha}^{-1}, \quad g_{\alpha \beta} g_{\beta \gamma} g_{\gamma \alpha} = I \quad (\text{for } U_{\alpha} \cap U_{\beta} \cap U_{\gamma} \neq \emptyset).
\]

Two holomorphically equivalent cocycles \( \{ g_{\alpha \beta}(z) \}, \{ g'_{\alpha \beta}(z) \} \) define the same bundle. The equivalence of cocycles means that there exists a set \( \{ h_{\alpha}(z) \} \) of holomorphic functions \( h_{\alpha} : U_{\alpha} \rightarrow \text{GL}(p, \mathbb{C}) \) such that

\[
h_{\alpha}(z) g_{\alpha \beta}(z) = g'_{\alpha \beta}(z) h_{\beta}(z).
\]

A section \( s \) of the bundle \( E \) is determined by a set \( \{ s_{\alpha}(z) \} \) of vector functions \( s_{\alpha} : U_{\alpha} \rightarrow \mathbb{C}^p \) that satisfy the conditions \( s_{\alpha}(z) = g_{\alpha \beta}(z)s_{\beta}(z) \) on intersections \( U_{\alpha} \cap U_{\beta} \neq \emptyset \).

A meromorphic connection \( \nabla \) in a holomorphic vector bundle \( E \) is determined by a set \( \{ \omega_{\alpha} \} \) of matrix meromorphic differential 1-forms that are defined in corresponding neighborhoods \( U_{\alpha} \) and satisfy gluing conditions

\[
\omega_{\alpha} = (dg_{\alpha \beta})g_{\alpha \beta}^{-1} + g_{\alpha \beta} \omega_{\beta} g_{\alpha \beta}^{-1} \quad (\text{for } U_{\alpha} \cap U_{\beta} \neq \emptyset).
\]

Under a transition to an equivalent cocycle \( \{ g'_{\alpha \beta} \} \) connected with the initial one by the relations \( 5 \), the 1-forms \( \omega_{\alpha} \) of the connection \( \nabla \) are transformed into the corresponding 1-forms

\[
\omega'_{\alpha} = (dh_{\alpha})h_{\alpha}^{-1} + h_{\alpha} \omega_{\alpha} h_{\alpha}^{-1}.
\]

Inversely, the existence of holomorphic matrix functions \( h_{\alpha} : U_{\alpha} \rightarrow \text{GL}(p, \mathbb{C}) \) such that the matrix 1-forms \( \omega_{\alpha} \) and \( \omega'_{\alpha} \) (satisfying conditions \( 6 \) for \( g_{\alpha \beta} \) and \( g'_{\alpha \beta} \) respectively) are connected by the relation \( 7 \), indicates the equivalence of cocycles \( \{ g_{\alpha \beta} \} \) and \( \{ g'_{\alpha \beta} \} \).

A set \( \{ s_{\alpha}(z) \} \) of vector functions satisfying linear differential equations \( ds_{\alpha} = \omega_{\alpha} s_{\alpha} \) in the corresponding \( U_{\alpha} \) by virtue of conditions \( 6 \) determines a section of the bundle \( E \), which is called horizontal with respect to the connection \( \nabla \). Thus horizontal sections of a holomorphic vector bundle with a meromorphic connection are determined by solutions of local systems of linear differential equations. The monodromy of a connection characterizes ramification of horizontal sections under their analytic continuation along loops in \( \mathbb{C} \) not containing singular points of the connection 1-form. It is defined similarly to the monodromy of a linear differential system. A connection is called logarithmic or Fuchsian, if all singular points of its 1-form are poles of the first order.

The degree \( \deg E \) (which is an integer) of the holomorphic vector bundle \( E \) with the meromorphic connection \( \nabla \) one can define as the sum

\[
\deg E = \sum_{i=1}^{n} \text{res}_{\alpha_i^0} \text{tr} \, \omega_i.
\]
of residues of local 1-forms $\text{tr} \omega_i$ by the all singular points of the connection, where $\omega_i$ is the local 1-form of the connection $\nabla$ in a neighborhood of its singular point $a_i^0$.

Further we say about holomorphic vector bundles with logarithmic connections in view of their close relation with Fuchsian systems. If a bundle is holomorphically trivial (all matrices of the cocycle can be taken as identity matrices), then by virtue of the conditions (6) the matrix 1-forms of a logarithmic connection coincide on non-empty intersections $U_\alpha \cap U_\beta$. Hence horizontal sections of such a bundle are solutions of a global Fuchsian system of linear differential equations defined on the whole Riemann sphere. Inversely, the Fuchsian system (1) determines the logarithmic connection in the holomorphically trivial vector bundle of rank $p$ over $\mathbb{C}$. Clear, such bundle has the standard definition by the cocycle that consists of the identity matrices while the connection is defined by the matrix 1-form

$$\omega^0 = \sum_{i=1}^{n} \frac{B_i^0}{z-a_i^0} \, dz$$

of coefficients of the system. But for us it will be more convenient to use the following coordinate description.

At first we consider a covering $\{U_\alpha\}$ of the punctured Riemann sphere $\mathbb{C} \setminus \{a_0^0, \ldots, a_n^0\}$ and a corresponding set of constant matrix functions $g_{\alpha\beta}(z) \equiv \text{const}$, which are expressed in terms of the monodromy matrices $G_1, \ldots, G_n$ of the system (1) via operations of multiplication and taking the inverse matrix (see [2, Lect. 8]). In this case matrix differential 1-forms $\omega'_\alpha$ defining a connection will be equal to zero. Further the covering $\{U_\alpha\}$ is complemented by small neighborhoods $O_i$ of the singular points $a_i^0$ of the system, thus we obtain a covering of the Riemann sphere $\mathbb{C}$. To non-empty intersections $O_i \cap U_\alpha$ there correspond matrix functions $g_{i\alpha}(z) = Y_i(z)$ of the cocycle, where $Y_i(z)$ is a germ of a fundamental matrix of the system whose monodromy matrix at the point $a_i^0$ is equal to $G_i$. (So, for the analytic continuations of a chosen germ to non-empty intersections $O_i \cap U_\alpha \cap U_\beta$ the cocycle relations $g_{i\alpha}g_{\alpha\beta} = g_{i\beta}$ hold.) Matrix differential 1-forms $\omega'_i$ determining the connection in neighborhoods $O_i$ coincide with the 1-form $\omega^0$ of coefficients of the system. In order to prove holomorphic equivalence of the cocycle $\{g_{i\alpha}, g_{\alpha\beta}\}$ to the identity cocycle, it remains to check existence of holomorphic matrix functions

$$h_\alpha : U_\alpha \rightarrow \text{GL}(p, \mathbb{C}), \quad h_i : O_i \rightarrow \text{GL}(p, \mathbb{C}),$$

such that

$$\omega'_\alpha = (dh_\alpha)h_\alpha^{-1} + h_\alpha\omega_\alpha h_\alpha^{-1}, \quad \omega'_i = (dh_i)h_i^{-1} + h_i\omega_i h_i^{-1}.$$ \quad (8)

Since we have $\omega_\alpha = \omega^0$ and $\omega'_\alpha = 0$ for all $\alpha$, the first equation of the (8) is rewritten as a linear system

$$d(h_\alpha^{-1}) = \omega^0 h_\alpha^{-1},$$

which has a holomorphic solution $h_\alpha^{-1} : U_\alpha \rightarrow \text{GL}(p, \mathbb{C})$ because the 1-form $\omega^0$ is holomorphic in $U_\alpha$. The second equation of the (8) has a holomorphic solution $h_i(z) \equiv I$, as $\omega_i = \omega'_i = \omega^0$.

One says that the bundle $E$ has a subbundle $E' \subset E$ of rank $k < p$ that is stabilized by the connection $\nabla$, if the pair $(E, \nabla)$ admits a coordinate description $\{g_{\alpha\beta}\}, \{\omega_\alpha\}$ of
the following block-uppertriangular form:

\[ g_{\alpha \beta} = \begin{pmatrix} g^1_{\alpha \beta} & \ast \\ 0 & g^2_{\alpha \beta} \end{pmatrix}, \quad \omega_\alpha = \begin{pmatrix} \omega^1_\alpha & \ast \\ 0 & \omega^2_\alpha \end{pmatrix}, \]

where \( g^1_{\alpha \beta} \) and \( \omega^1_\alpha \) are blocks of size \( k \times k \) (then the cocycle \( \{ g^1_{\alpha \beta} \} \) defines the subbundle \( E' \) and the 1-forms \( \omega^1_\alpha \) define the restriction \( \nabla' \) of the connection \( \nabla \) to the subbundle \( E' \)).

**Example 1.** Consider the Fuchsian system (1) with a reducible monodromy representation and corresponding holomorphically trivial vector bundle with the logarithmic connection. Let us demonstrate that to the monodromy subrepresentation there corresponds a holomorphic vector subbundle that is stabilized by the connection.

We use the above coordinate description of the bundle and connection with the cocycle \( \{ g'_{\alpha \beta}, \omega'_{\alpha} \} \) and set \( \{ \omega'_i, \omega'_i \} \) of matrix 1-forms. We can pass to the equivalent cocycle (preserving the previous notations) changing the matrices \( g'_{\alpha \beta} \) to the matrices \( S^{-1}g'_{\alpha \beta}S \) and the matrices \( g'_{i \alpha} \) to the matrices \( g'_{i \alpha}S \), where \( S \) is a constant non-degenerate matrix reducing the monodromy matrices \( G_1, \ldots, G_n \) of the system to the same block upper-triangular form \( G'_1, \ldots, G'_n \). Then the matrices \( g'_{\alpha \beta} \) are block upper-triangular (and \( \omega'_\alpha = 0 \)), and the matrices \( g'_{i \alpha} \) have the form

\[ g'_{i \alpha}(z) = M_i(z)(z - a^0_i)^{E_i}, \quad E_i = (1/2\pi i) \ln G'_i, \]

where \( M_i(z) \) are meromorphic matrices in neighborhoods of the corresponding points \( a^0_i \). For the latter the following factorizations hold: \( M_i(z) = V_i(z)P_i(z) \), where the matrix \( V_i(z) \) is holomorphically invertible at the point \( a^0_i \), \( P_i(z) \) is a polynomial upper-triangular matrix in \( (z - a^0_i)^{\pm 1} \) (see, for example, [7, Lemma 1]). Thus changing the matrices \( g'_{i \alpha}(z) \) to

\[ V_i^{-1}(z)g'_{i \alpha}(z) = P_i(z)(z - a^0_i)^{E_i} \]

and the matrix 1-forms \( \omega'_i \) to

\[ V_i^{-1}\omega'_i V_i - V_i^{-1}(dV_i) = (dP_i)P_i^{-1} + P_i \frac{E_idz}{z - a^0_i}P_i^{-1}, \]

we pass to the holomorphically equivalent coordinate description whose cocycle matrices and matrix 1-forms of the connection have the same block upper-triangular form.

**Remark 1.** From the definition of the subbundle \( E' \) that is stabilized by the logarithmic connection \( \nabla' \), one can see that the set of exponents of the restriction \( \nabla' \) at each singular point \( a^0_i \) (the set of eigenvalues of the residue matrix of the corresponding 1-form) is a subset of exponents of the connection \( \nabla \). Therefore the degree \( \deg E' \) of the subbundle \( E' \) is equal to the sum of exponents from these subsets over all singular points of the connection.

The following auxiliary lemma points to a certain block structure of the residue matrices of a Fuchsian system in the case when the corresponding holomorphically trivial vector bundle with a logarithmic connection has a holomorphically trivial subbundle that is stabilized by the connection.
Lemma 1. If $E$ is the holomorphically trivial vector bundle of rank $p$ over $\overline{\mathbb{C}}$ having a holomorphically trivial subbundle $E' \subset E$ of rank $k$ that is stabilized by the connection $\nabla$, then the corresponding Fuchsian system (1) is reduced to a block upper-triangular form via a constant gauge transformation $\tilde{y}(z) = Cy(z)$, $C \in \text{GL}(p, \mathbb{C})$. That is,

$$CB^0_iC^{-1} = \begin{pmatrix} B^i_1 & * \\ 0 & * \end{pmatrix}, \quad i = 1, \ldots, n,$$

where $B^i_1$ is a block of size $k \times k$.

Proof. Let $\{s_1, \ldots, s_p\}$ be a basis of global sections of the bundle $E$ (which are linear independent at each point $z \in \overline{\mathbb{C}}$) such that the 1-form of the connection $\nabla$ in this basis is the 1-form $\omega^0$ of coefficients of the Fuchsian system. Consider also a basis $\{s'_1, \ldots, s'_p\}$ of global holomorphic sections of the bundle $E'$ such that $s'_1, \ldots, s'_k$ are sections of the subbundle $E'$, $(s'_1, \ldots, s'_p) = (s_1, \ldots, s_p)C^{-1}$, $C \in \text{GL}(p, \mathbb{C})$.

Now choose a basis $\{h_1, \ldots, h_p\}$ of sections of the bundle $E$ such that they are horizontal with respect to the connection $\nabla$ and $h_1, \ldots, h_k$ are sections of the subbundle $E'$ (it is possible since $E'$ is stabilized by the connection). Let $Y(z)$ be a fundamental matrix of the Fuchsian system whose columns are the coordinates of the sections $h_1, \ldots, h_p$ in the basis $\{s_1, \ldots, s_p\}$. Then

$$\tilde{Y}(z) = CY(z) = \begin{pmatrix} k \times k & * \\ 0 & * \end{pmatrix}$$

is a block upper-triangular matrix, since its columns are the coordinates of the sections $h_1, \ldots, h_p$ in the basis $\{s'_1, \ldots, s'_p\}$. Consequently, the transformation $\tilde{y}(z) = Cy(z)$ reduces the initial system to a block upper-triangular form. \hfill \Box

3 On the reducibility of Fuchsian systems and their isomonodromic families

Let monodromy matrices of the Fuchsian system (1) have the block upper-triangular form (3). Denote by $m^s \times m^s$ size of the blocks $G^s_i, \ldots, G^s_n$ ($s = 1, \ldots, k$). There holds the following sufficient condition of reducibility of the residue matrices of the system to the same block upper-triangular form.

Theorem 1. If the exponents $\beta^j_i$ of the Fuchsian system (1) satisfy the condition

$$\text{Re} \beta^j_i > -1/n(p - m^k), \quad i = 1, \ldots, n, \quad j = 1, \ldots, p, \quad (9)$$

then there exists a constant matrix $C \in \text{GL}(p, \mathbb{C})$ such that the matrices $CB^0_iC^{-1}$ have the same block upper-triangular form as the monodromy matrices of the system.

Proof. We use a geometric interpretation (exposed in the previous section) according to which to the Fuchsian system (1) there corresponds a holomorphically trivial vector bundle $E$ of rank $p$ over the Riemann sphere endowed a logarithmic connection $\nabla$. Since the monodromy matrices of the system are block upper-triangular, there exists a flag $E^1 \subset E^2 \subset \ldots \subset E^k = E$ of subbundles of ranks $m^1, m^1 + m^2, \ldots, m^1 + \ldots + m^k = p$ correspondingly that are stabilized by the connection $\nabla$ (see Example 1).
Let us estimate the degree of each subbundle $E^s$, $s \leq k$, using Remark 1. The degree of the holomorphically trivial vector bundle $E_k$ is equal to zero and for $s < k$ we have:

$$\deg E^s = \sum_{i=1}^{n} \sum_{j \in J_i, |J_i|=m^1+...+m^s} \text{Re} \beta^j_i > -(m^1 + \ldots + m^s)/(p - m^k) \geq -1.$$ 

Therefore, $\deg E^s = 0$ (the degree of a subbundle of a holomorphically trivial vector bundle is non-positive, see [2, Prop. 11.1]), and all the subbundles $E^1 \subset \ldots \subset E^k$ are holomorphically trivial (a subbundle of a holomorphically trivial vector bundle is holomorphically trivial, if its degree is equal to zero, see [2, Corollary 11.1]). Now the assertion of the theorem follows from Lemma 1.

Remark 2. Although the inequalities (9) point out to the boundedness of real parts of the exponents from below, from these inequalities and the classical Fuchs relation [11]

$$\sum_{i=1}^{n} \sum_{j=1}^{p} \beta^j_i = 0$$

(which says that the degree of the holomorphically trivial vector bundle $E$ is equal to zero) it follows also the boundedness from above.

Corollary 1. If the monodromy representation of the Fuchsian system (1) is upper-triangular and its exponents $\beta^j_i$ satisfy the condition

$$\text{Re} \beta^j_i > -1/n(p - 1), \quad i = 1, \ldots, n, \quad j = 1, \ldots, p,$$

then there exists a constant matrix $C \in \text{GL}(p, \mathbb{C})$ such that all the matrices $CB^0_iC^{-1}$ are upper-triangular.

The monodromy representation of a Fuchsian system is called a $B$-representation, if it is reducible and the Jordan form of each monodromy matrix $G_i$ consists of one Jordan box only. In this case we have the following assertion.

Proposition 1. If the monodromy representation of the Fuchsian system (1) is an upper-triangular $B$-representation, then there exists a constant matrix $C \in \text{GL}(p, \mathbb{C})$ such that all the matrices $CB^0_iC^{-1}$ are upper-triangular.

Proof. Since the monodromy representation of the Fuchsian system is a $B$-representation, then at each singular point $a_i$ the system has only one exponent (of multiplicity $p$): $\beta^1_i = \ldots = \beta^p_i = \beta_i$ (see the proof of Theorem 11.2 from [2]). Then we have $p \sum_{i=1}^{n} \beta_i = 0$ because of triviality of the vector bundle $E$ of rank $p$ corresponding to the given Fuchsian system. Now, as in the proof of Theorem 1, from the upper-triangularity of the monodromy matrices it follows the existence of a flag $E^1 \subset E^2 \subset \ldots \subset E^p = E$ of subbundles of ranks $1, 2, \ldots, p$ correspondingly that are stabilized by the connection $\nabla$. But each subbundle $E^s$ is holomorphically trivial, since its degree $\deg E^s = s \sum_{i=1}^{n} \beta_i$ is equal to zero. To complete the proof it remains to apply again Lemma 1.

Let us say a few words about the reducibility of Schlesinger isomonodromic families [2]. First of all, it follows from the result of S. Malek (mentioned in Introduction) that if the monodromy of such a family is reducible, then via a meromorphically invertible
(in $z$) gauge transformation $\tilde{y} = \Gamma(z, a)y$ this family can be transformed to a Schlesinger isomonodromic family whose residue matrices $\tilde{B}_i(a)$ have the block-uppertriangular form

$$\tilde{B}_i(a) = \begin{pmatrix} B'_i(a) & B''_i(a) \\ 0 & B''_i(a) \end{pmatrix}.$$ 

Indeed, for $a = a^0$ the initial system is reduced to the required form via a meromorphically invertible transformation. The transformed block upper-triangular system can be included in the Schlesinger isomonodromic family with the residue matrices $\tilde{B}_i(a)$ of the same block upper-triangular form (because they are solutions of the Schlesinger equation and they are block upper-triangular for $a = a^0$). Since the initial and obtained families have the same monodromy, they are connected by a meromorphically invertible transformation $\tilde{y} = \Gamma(z, a)y$. S. Malek [13] has shown also that the matrix $\Gamma(z, a)$ is holomorphic with respect to the variable $a$ in $D(a^0)$, with the exception of some analytic subset of codimension one, and the entries of $\Gamma(z, a)$ are rational functions in $z, a_1, \ldots, a_n$ and entries of residue matrices $B_i(a)$ (see also [5]).

We can formulate the following analogs of Corollary 1 and Proposition 1.

**Proposition 2.** If the monodromy representation of the Fuchsian system (1) is upper-triangular and its exponents $\beta^j_i$ satisfy the condition

$$\text{Re} \beta^j_i > -1/n(p-1), \quad i = 1, \ldots, n, \quad j = 1, \ldots, p,$$

then for the Schlesinger isomonodromic deformation (2) of this system there exists a constant matrix $C \in \text{GL}(p, \mathbb{C})$ such that all the matrices $CB_i(a)C^{-1}$ are upper-triangular.

**Proposition 3.** If the monodromy representation of the Fuchsian system (1) is an upper-triangular $B$-representation, then for the Schlesinger isomonodromic deformation (2) of this system there exists a constant matrix $C \in \text{GL}(p, \mathbb{C})$ such that all the matrices $CB_i(a)C^{-1}$ are upper-triangular.

**Proof.** Both propositions are proved with the same method. The residue matrices $B^0_i$ of the Fuchsian system satisfying the condition of the proposition is transformed to an upper-triangular form $CB^0_iC^{-1}$. It is not difficult to see that matrices $CB_i(a)C^{-1}$ also satisfy the Schlesinger equation. As they are upper-triangular for $a = a^0$: $CB_i(a^0)C^{-1} = CB^0_iC^{-1}$, then by virtue of the Schlesinger equation they remain upper-triangular for all $a \in D(a^0)$. Indeed, any partial derivative of solutions of the Schlesinger equation is expressed via partial derivatives of lower orders in terms of matrix operations of addition and multiplication. Therefore from the upper-triangularity of solutions for $a = a^0$ it follows that their partial derivatives of any order are upper-triangular for $a = a^0$. Hence such solutions are upper-triangular for all $a \in D(a^0)$. 

Since the eigenvalues $\beta^j_i$ of the residue matrices $B_i(a)$ of the Schlesinger isomonodromic family (2) do not depend on $a$ (see, for example, [3]), for the $\tau$-function of an upper-triangular family (as a consequence of Miwa’s formula) there holds the relation

$$d \ln \tau(a) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{\alpha_{ij}}{a_i - a_j} d(a_i - a_j),$$

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where \( \alpha_{ij} = \beta_i^1 \beta_j^1 + \ldots + \beta_i^p \beta_j^p \). Thus \( \tau(a) = \prod_{i<j}(a_i - a_j)^{\alpha_{ij}} \) is a non-zero holomorphic function on the universal cover \( Z \) of the space \( \mathbb{C}^n \setminus \bigcup_{i \neq j} \{a_i = a_j\} \). This implies that the Malgrange \( \Theta \)-divisor of an upper-triangular Schlesinger isomonodromic family is empty and one can holomorphically continue all matrices \( B_i(a) \) to the whole space \( Z \).

### 4 Triangular Schlesinger families in small dimensions

In the previous section there were proposed some sufficient conditions of the reducibility of the Schlesinger isomonodromic family \([2]\) to an upper-triangular form via a constant gauge transformation. Now we describe integral representations for entries of the residue matrices of such a family in the dimensions \( p = 2 \) and \( p = 3 \).

In the case \( p = 2 \) upper-triangular residue matrices \( B_i(a) \) have the form

\[
B_i(a) = \begin{pmatrix}
\beta_i^1 & b_i(a) \\
0 & \beta_i^2
\end{pmatrix}, \quad i = 1, \ldots, n,
\]

where the exponents \( \beta_i^1, \beta_i^2 \) are constant. Then the functions \( b_i(a) \) by virtue of the Schlesinger equation satisfy the following system of homogeneous linear differential equations:

\[
\frac{db_i(a)}{a_i - a_j} = - \sum_{j=1, j \neq i}^n (\beta_i b_j(a) - \beta_j b_i(a)) \frac{d(a_i - a_j)}{a_i - a_j}, \quad i = 1, \ldots, n,
\]

(10)

where \( \beta_i = \beta_i^1 - \beta_i^2 \).

The system (10) is the Jordan-Pochhammer system, i.e., a linear Pfaffian system for the vector function \( b(a) = (b_1(a), \ldots, b_n(a))^\top \):

\[
\frac{db}{\Omega b}, \quad b(a) \in \mathbb{C}^n,
\]

(11)

with the meromorphic (holomorphic in \( D(a^0) \)) coefficient matrix 1-form

\[
\Omega = \sum_{1 \leq j < k \leq n} J_{jk}(\beta) \frac{d(a_j - a_k)}{a_j - a_k},
\]

(12)

where \( J_{jk}(\beta) \) are constant \((n \times n)\)-matrices. Each matrix \( J_{jk}(\beta) \) has only four non-zero entries: in the \( j \)-th row the entry with the number \( j \) is equal to \( \beta_k \) while the entry with the number \( k \) is equal to \( -\beta_j \), and in the \( k \)-th row the entry with the number \( j \) is equal to \( -\beta_k \) while the entry with the number \( k \) is equal to \( \beta_j \).

Any solution of the system (11), (12) can be represented in the following form (10), (9):

\[
b_i(a) = \beta_i \int_{\gamma} (t - a_1)^{\beta_1} \cdots (t - a_n)^{\beta_n} \frac{dt}{t - a_i}, \quad i = 1, \ldots, n,
\]

(13)

where \( \gamma \in H_1(\mathbb{C} \setminus \{a_1, \ldots, a_n\}, \mathbb{L}^\ast) \) is a twisted cycle whose coefficient takes value in the local system \( \mathbb{L}^\ast \) which is dual to the local system \( \mathbb{L} \) associated to the multi-valued function \( F(t) = (t - a_1)^{\beta_1} \cdots (t - a_n)^{\beta_n} \). (To define correctly integration of multi-valued functions
along cycles, one should use 1-homologies with coefficients in a local system; see details, for example, in [14] or [15].

In the case $p = 3$ upper-triangular residue matrices $B_i(a)$ have the form

$$B_i(a) = \begin{pmatrix} \beta^1_i & u_i(a) & b_i(a) \\ 0 & \beta^2_i & v_i(a) \\ 0 & 0 & \beta^3_i \end{pmatrix}, \quad i = 1, \ldots, n,$$

where the exponents $\beta^1_i, \beta^2_i, \beta^3_i$ are constant. By virtue of the Schlesinger equation, the vector functions $u(a) = (u_1(a), \ldots, u_n(a))^\top$ and $v(a) = (v_1(a), \ldots, v_n(a))^\top$ satisfy the Jordan-Pochhammer systems

$$du = \Omega^u u, \quad dv = \Omega^v v, \quad (14)$$

where

$$\Omega^u = \sum_{1 \leq j < k \leq n} J_{jk}(\beta^u) \frac{d(a_j - a_k)}{a_j - a_k}, \quad \beta^u_i = \beta^1_i - \beta^2_i,$$

$$\Omega^v = \sum_{1 \leq j < k \leq n} J_{jk}(\beta^v) \frac{d(a_j - a_k)}{a_j - a_k}, \quad \beta^v_i = \beta^2_i - \beta^3_i.$$

Therefore for the functions $u_i(a), v_i(a)$ there hold integral representations similar to (13).

At the same time, the functions $b_i(a)$ satisfy the following system of non-homogeneous linear differential equations:

$$db_i(a) = - \sum_{j=1, j \neq i}^n (\beta_i b_j(a) - \beta_j b_i(a)) \frac{d(a_i - a_j)}{a_i - a_j} + \theta_i, \quad i = 1, \ldots, n, \quad (15)$$

where

$$\theta_i = - \sum_{j=1, j \neq i}^n (u_i(a)v_j(a) - u_j(a)v_i(a)) \frac{d(a_i - a_j)}{a_i - a_j}, \quad \beta_i = \beta^1_i - \beta^3_i.$$

Denote by $b(a)$ the vector function $(b_1(a), \ldots, b_n(a))^\top$ and by $\theta$ the vector valued differential 1-form $(\theta_1, \ldots, \theta_n)^\top$, and rewrite the system (15) in the matrix form

$$db = \Omega b + \theta, \quad (16)$$

where the matrix 1-form $\Omega$ has the form (12). For any solutions $u(a), v(a)$ of the systems (14), the system (16) is integrable in the polydisk $D(a^0)$, which follows from the integrability of the Schlesinger equation. Hence its solution can be represented in the form $b(a) = Y(a)c(a)$, where $Y(a)$ is a fundamental matrix of the corresponding homogeneous system and the vector function $c(a)$ satisfies the relation

$$dc = Y^{-1}\theta,$$

whose right part is necessarily a closed differential 1-form. Therefore,

$$c(a) = \text{const} + \int_{a^0}^a Y^{-1}\theta.$$
and the integral does not depend on a path connecting the points $a^0$ and $a \in D(a^0)$.

In the (generic) case, when all $\beta_i \neq 0$, for the entries of the matrix $Y^{-1}$ one can propose integral representations analogous to (13). Namely, the following lemma holds.

**Lemma 2** (A. Varchenko). Let all $\beta_i \neq 0$ and $Y^*(a)$ be a fundamental matrix of a linear Pfaffian system

$$db^* = -\Omega b^*, \quad -\Omega = \sum_{1 \leq j < k \leq n} J_{jk}(-\beta) \frac{d(a_j - a_k)}{a_j - a_k}. \quad (17)$$

Then $Y^{-1} = (Y^*)^T \Lambda$, where $\Lambda = \text{diag} \left( \frac{1}{\beta_1}, \ldots, \frac{1}{\beta_n} \right)$.

**Proof.** Since for a pair of vectors $u, w \in \mathbb{C}^n$ the scalar products $(u, J_{jk}(\beta)w)$ and $(J_{jk}(\beta)u, w)$ are equal to

\[
(u, J_{jk}(\beta)w) = u_j(\beta_k w_j - \beta_j w_k) + u_k(\beta_j w_k - \beta_k w_j),
\]

\[
(J_{jk}(\beta)u, w) = w_j(\beta_k u_j - \beta_j u_k) + w_k(\beta_j u_k - \beta_k u_j),
\]

then for the symmetric bilinear form

\[
(u, w)_\beta = \sum_{i=1}^n \frac{1}{\beta_i} u_i w_i,
\]

one has $(u, J_{jk}(\beta)w)_\beta = (J_{jk}(\beta)u, w)_\beta$ for all $j < k$, or

\[
(J_{jk}(\beta)u, w)_\beta + (u, J_{jk}(-\beta)w)_\beta = 0.
\]

Consequently, if $b(a)$ is a solution of the system (11), (12), and $b^*(a)$ is a solution of the system (17), then the function $(b(a), b^*(a))_\beta$ is constant. Therefore,

\[
(Y^*)^T \Lambda Y = C
\]

is a constant non-degenerate matrix (choosing a suitable fundamental matrix $Y$, one can suppose that the matrix $C$ is the identity matrix). □

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