ON FREE BOUNDARY MINIMAL ANNULI IN A
HALF-BALL

DONG-HWI SEO

Abstract. We consider a uniqueness problem of embedded free boundary minimal annuli in the three-dimensional unit half-ball and show that the half of the critical catenoid is the only such surfaces. Using this result and a symmetry principle, we obtain several characterizations of the critical catenoid among embedded free boundary minimal annuli in the three-dimensional unit ball. For example, we show that if a boundary component of such surfaces is invariant under the reflections through two coordinate planes, then the critical catenoid is the only such surfaces.

1. Introduction

A free boundary minimal surface $M$ in the unit ball $B^3 \subset \mathbb{R}^3$ is a minimal surface in $\mathbb{B}^3$ whose boundary $\partial M$ meets $\mathbb{B}^3$ orthogonally along $\partial M$. This topic was initially considered by Nitsche in [19]. Recently, increasing attention has been paid to the topic after a new construction of free boundary minimal surfaces in a ball via extremal metrics of the normalized first Steklov eigenvalues of compact surfaces with boundary by the work of Fraser and Schoen [7, 8], Petrides [20], and Matthiesen and Petrides [17]. For recent survey on this topic, see [15].

In 1985, Nitsche showed that every immersed free boundary minimal disk in a ball is congruent to the equatorial disk [19]. In the same paper, Nitsche claimed the uniqueness conjecture, which is now well-known by Fraser and Li [6], as follows.

Conjecture. The only embedded free boundary minimal annulus in $\mathbb{B}^3$ is the critical catenoid.

Fraser and Schoen showed that if an immersed free boundary minimal annulus in $\mathbb{B}^n, n \geq 3$, has the first Steklov eigenvalue 1, then it is congruent to the critical catenoid [8]. Using this result, McGrath showed that if an embedded free boundary minimal annulus $\Sigma$ in $\mathbb{B}^3$ is invariant under the reflections through the three coordinate planes, $\Sigma$ is congruent to the critical catenoid [18]. Later, Kusner and McGrath extended this result by replacing the three coordinate planes with the antipodal map [13]. On the other hand, using Alexandrov reflection method, Barbosa and Silva showed if $\partial \Sigma$
is invariant under the reflections through the three coordinate planes, then
$\Sigma$ is congruent to the critical catenoid [11].

In this paper, we consider the uniqueness problem of embedded free boundary minimal annuli in a half-ball as follows. Note that the condition on the boundary is necessary (see Remark 3 in Section 5).

**Theorem 1.** Let $\Sigma'$ be an embedded free boundary minimal annulus in the half-ball in $\mathbb{R}^3$ of radius 1, centered at the origin. If one boundary component is contained in the open hemisphere and the other boundary component is contained in the equatorial disk, then $\Sigma'$ is congruent to the half of the critical catenoid, $\{x_1^2 + x_2^2 = \cosh^2 x_3\} \cap \{x_3 \tanh x_3 < 1\} \cap \{x_3 > 0\}$.

The main idea of the proof comes from the existence of a coordinate plane with the following properties: This plane does not meet $\partial \Sigma$ (Proposition 1) and every plane perpendicular to this plane meets each boundary component at most two points (Corollary 1). This observation is derived from convexity of the boundary components of $\Sigma$ and the concept of flux for minimal surfaces (see Section 4).

Using this theorem and the symmetry principle (Theorem 5), we can obtain the following uniqueness theorem with a symmetry assumption on the boundary (see also Corollary 2).

**Theorem 2.** If $\partial \Sigma$ is invariant under the reflections through two coordinate planes, then $\Sigma$ is congruent to the critical catenoid.

In [12], Kapouleas and Li showed that if an embedded free boundary minimal surface $M$ in $B^3$ has a rotationally invariant boundary component, then $M$ is congruent to either equatorial disk or the critical catenoid. If we restrict the topology of $M$ by annulus, Theorem 2 and the symmetry principle give the following.

**Theorem 3.** If a component of $\partial \Sigma$ is invariant under the reflections through two coordinate planes, then $\Sigma$ is congruent to the critical catenoid.

If we add a congruence assumption on the boundary components, we can reduce a symmetry assumption in Theorem 2.

**Theorem 4.** If $\partial \Sigma$ is consists of two congruent components and it is invariant under the reflection through a coordinate plane, then $\Sigma$ is congruent to the critical catenoid.

In Section 2, we introduce some basic facts about Steklov eigenvalue problem. In Section 3, we prove the symmetry principle for a compact orientable embedded free boundary minimal surface in the three-dimensional unit ball. In Section 4, we prove several properties of $\partial \Sigma$. In Section 5 and Section 6, we prove uniqueness results for the critical catenoid.

2. **Background: Steklov eigenvalue problem**

Let $M^n, n \geq 2$ be a compact Riemannian manifold with boundary $\partial M$. Then, the Steklov eigenvalue problem, which is introduced by Steklov in
is to find $\sigma \in \mathbb{R}$ for which there exists a function $u \in C^\infty(M)$ satisfying
\[
\begin{cases}
\Delta u = 0 & \text{in } M \\
\frac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial M,
\end{cases}
\]
where $\nu$ is the outward unit conormal vector of $M$ along $\partial M$. It is known that the eigenvalues $\sigma$ of this problem are discrete and it form a sequence, $0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \to \infty$ (see, for example, [10]). Note that constant functions are Steklov eigenfunctions with eigenvalue $\sigma_0 = 0$. In addition, for $i = 0, 1, \ldots$, we have
\[
\sigma_{i+1}(M) = \inf_{f \in C^\infty(\partial M) \setminus \{0\}} \left\{ \frac{\int_M |\nabla \hat{f}|^2}{\int_{\partial M} f^2} \left| \int_{\partial M} fu_k = 0 \text{ for } k = 0, \ldots, i \right. \right\},
\]
where $\hat{f} \in C^\infty(M)$ is the harmonic extension of $f$ and $u_k$ is a Steklov eigenfunction corresponding to $\sigma_k$.

If $M$ is an immersed free boundary minimal surface in $\mathbb{B}^n, n \geq 3$, we have the following interesting properties.

**Lemma 1.** Any coordinate functions, $x_i, i = 1, 2, 3$, are Steklov eigenfunctions of $M$ with eigenvalue $1$. It implies $\sigma_1(M) \leq 1$.

*Proof.* See [7, Lemma 2.2] or [15, Theorem 2.2].

**Lemma 2.** If $u$ is a first Steklov eigenfunction of $M$ with $\sigma_1(M) < 1$, then we have
\[
\int_{\partial M} u = \int_{\partial M} u x_i = 0 \text{ for all } i = 1, 2, 3.
\]

*Proof.* The conclusion immediately follows from (2) with the previous lemma.

\[
\square
\]

3. **Symmetry principle**

In this section, we show that the following symmetry principle. This theorem roughly says that a symmetry of the boundary of an embedded free boundary minimal surface in a ball engenders a global symmetry. It is reminiscent of the symmetry principle for Bryant surfaces (see [9, Theorem 12]).

**Theorem 5** (Symmetry principle). Let $M$ be a compact orientable embedded free boundary minimal surface in $\mathbb{B}^3$. If $A \in O(3)$ maps a boundary component of $M$ into another boundary component (the two components can be identical), then $M$ is invariant under $A$.

We will prove this theorem by Björling uniqueness theorem (see, for example, [5, Theorem 1 in p.125]) with the following lemma.
Lemma 3 (Boundary symmetry). Let \((\partial M)_1, \cdots, (\partial M)_n\) be the boundary components of \(M\). If \(A((\partial M)_i) = (\partial M)_j\) (\(i\) can be identical to \(j\)), then the Gauss map \(g\) of \(M\) satisfies
\[
(4) \quad g \circ A(p) = A \circ g(p)
\]
for any \(p \in (\partial \Sigma)_i\).

Proof. We begin by recalling an orientation of \(g\). By free boundary condition, we can define the outward unit conormal vector \(\nu\) of \(M\) along \(\partial M\) by the position vector. Since \(M\) is orientable, we can define an oriented unit tangent vector \(t\) of \(\partial M\). Then, \(t \wedge \nu\) becomes an oriented Gauss map of \(M\) along \(\partial M\).

Now we show that \(t \circ A(p) = (\det(A)) \cdot A \circ t(p)\) for all \(p \in (\partial M)_i\). Let \(U_k, k = 1, \cdots, n\) be the convex sets in \(S^2\) whose boundaries are \((\partial M)_k\), respectively. Then, \(M \cup \bigcup_{k=1}^n U_k\) is orientable and we can define the oriented Gauss map of \(U_k\) by an orientation of \(M\). We may assume the oriented Gauss map of \(U_i\) is the position vector. Since \(A((\partial M)_i) = (\partial M)_j\), we have \(A(U_i) = U_j\) and \(A\) maps the outward unit conormal vector and the oriented Gauss map of \(U_i\) along \(\partial U_i = (\partial M)_i\) into those of \(U_j\) along \(\partial U_j\). Thus, if \(A\) is orientation-preserving, \(t \circ A(p) = A \circ t(p)\). Otherwise, \(t \circ A(p) = -A \circ t(p)\).

Sum up the arguments, if \(A\) is orientation-preserving, \(A \circ g(p) = A \circ (t \wedge \nu)(p) = (A \circ t)(A \circ \nu)(p) = (t \circ A)(\nu \circ A)(p) = g \circ A(p)\). For the second to the last equality, we used \(\nu(q) = q\) for all \(q \in \partial M\). By a similar argument, we can also obtain the conclusion when \(A\) is orientation-reversing. \(\square\)

Proof of Theorem 5

By Lemma 3, \(A\) maps the Björling data of \((\partial M)_i\) into those of \((\partial M)_j\). Applying Björling uniqueness theorem, we can show that \(A(M) = M\). \(\square\)

Remark 1. By Theorem 5 if a boundary component of \(M\) is invariant under the reflection through a coordinate plane, then \(M\) is invariant under the reflection. Moreover, if a component of \(\partial M\) is rotationally symmetric, \(M\) is rotationally symmetric. Then one can show that \(M\) is the equatorial disk or the critical catenoid (see [12, Corollary 3.9]).

4. PROPERTIES OF BOUNDARY

In this section, we prove three propositions about \(\partial \Sigma\), where \(\Sigma\) is an embedded free boundary minimal annulus in \(\mathbb{B}^3\) as in the introduction section. Before introducing the propositions, we recall two-piece property obtained by Lima and Menezes [16] and convexity of the boundary components of \(\Sigma\) by Kusner and McGrath [13].

Theorem 6 (Two-piece property, [16]). Every plane passing through the origin dissects \(\Sigma\) into two components.

Proof. See [16, Theorem A]. \(\square\)
Using two-piece property and the argument of Fraser and Schoen [8, Proposition 8.1], Kusner and McGrath obtained the following lemma [13].

**Lemma 4** ([13]). Each boundary component of $\Sigma$ is convex in $S^2$. In other words, there are at most two intersection points of a boundary component and a great circle.

**Proof.** See [13, Corollary 7]. \(\square\)

Let $\partial \Sigma = (\partial \Sigma)_1 \cup (\partial \Sigma)_2$, where $(\partial \Sigma)_i, i = 1, 2,$ are distinct boundary components of $\Sigma$. By Lemma 4, each $(\partial \Sigma)_i$ is a convex curve in $S^2$. Then, we define the following concepts, which are motivated from flux of a minimal surface.

**Definition 1.** We define a function $F$ from a closed curve in $S^2$ into $R^3$ by
\[
F(C) = \int_C x,
\]
where $C$ is a closed curve in $S^2$ and $x$ is the position vector of $R^3$. In addition, we define $f_i \in S^2, i = 1, 2,$ by
\[
f_i := \frac{F((\partial \Sigma)_i)}{|F((\partial \Sigma)_i)|}.
\]

*Flux* is used to define a map from one-dimensional homology classes of a minimal surface in $R^3$ into $R^3$. It is defined by the integral of an oriented unit conormal vectors along a representative closed curve (see, for example, [4, Corollary 1.8]). Note that $(\partial \Sigma)_1$ and $(\partial \Sigma)_2$ are homologous and the position vectors of them become outward unit conormal vectors of $\Sigma$ along them. Thus, we have
\[
F((\partial \Sigma)_1) + F((\partial \Sigma)_2) = 0.
\]
Then, we have
\[
f_1 + f_2 = 0.
\]
Using (7) or (8), we obtain the following three propositions with a corollary.

**Proposition 1.** There exists a great circle of $S^2$ that does not meet $\partial \Sigma$.

**Proof.** We begin with the following claim:

**Claim 1.** A convex closed curve $C$ in $S^2$ does not meet the equatorial disk perpendicular to $F(C)$.

**Proof of Claim 7.** Suppose not. Then, there is a point $p \in C$, say $p = (1, 0, 0)$, such that the closed hemisphere centered at $-p$, say $H_1 := \{x_1 \leq 0\} \cap S^2$, contains the direction of $F(C)$. Denote the open hemisphere $\{x_1 > 0\} \cap S^2$ by $H_2$. We show that
\[
L(C \cap H_1) < L(C \cap H_2),
\]
where $L(C')$ is the length of a curve $C'$ in $\mathbb{S}^2$. Since $C$ is a convex curve in $\mathbb{S}^2$, $C \cap \{x_1 = 0\}$ is at most two points. If $C \cap \{x_1 = 0\}$ is the empty set or a point, $L(C \cap \tilde{H}_1) = 0 < L(C) = L(C \cap H_2)$. Otherwise, $L(C \cap H_2) \geq \pi$, which is the length of a great semicircle. Since $L(C) < 2\pi$ (see [11, Lemma 4]), we obtain (9).

Using (9), $C$ is divided by $C_1 := C \cap \tilde{H}_1$, $C_2 := C \cap \{x_1 > c_1\}$ for some $c_1 > 0$ such that $L(C_1) = L(C_2)$, and $C_3 := C - (C_1 \cup C_2)$. Then we construct subsets $C_1'$ and $C_2'$ of the great circle $G = \{x_3 = 0\} \cap \mathbb{S}^2$ by

(10) $C_1' := G \cap \{c_2 < x_1 < 0\}$ for some $c_2 < 0$ such that $L(C_1') = L(C_1)$

and

(11) $C_2' := G \cap \{x_1 > c_3\}$ for some $c_3 > 0$ such that $L(C_2') = L(C_2)$.

Then we have $L(\{x_1 > c\} \cap C_2) \geq L(\{x_1 > c\} \cap C_2')$ for all $c \geq 0$. Thus, by Fubini theorem, we have

(12) $\int_{C_2} x_1 = \int_0^\infty L(\{x_1 > c\} \cap C_2) dc \geq \int_0^\infty L(\{x_1 > c\} \cap C_2') dc = \int_{C_2'} x_1$.

Likewise, we have

(13) $\int_{C_1} x_1 \geq \int_{C_1'} x_1$.

Thus,

(14) $\int_C x_1 \geq \int_{C_1'} x_1 + \int_{C_2'} x_1 > 0$,

where the last inequality comes from the previous argument with Fubini theorem. It is contrary to the assumption that $\tilde{H}_1$ contains the direction of $F(C)$.

Using (7), Claim 1 implies that the equatorial disk perpendicular to $F((\partial \Sigma)_1)$ and $F((\partial \Sigma)_2)$ does not meet $\partial \Sigma$. It completes the proof.

Remark 2. Claim 1 provides a constructive proof of the existence of a great circle that does not meet a given convex curve in $\mathbb{S}^2$ (see [11, Lemma 4 and Theorem 2]).

Proposition 2. For each $i = 1, 2$, $f_i$ is contained in the interior of the spherical convex hull of $(\partial \Sigma)_i$, respectively.

Proof. Suppose $f_i \in (\partial \Sigma)_i$. By convexity of $(\partial \Sigma)_i$, we can find a great circle $G_1$ passing through $f_i$ such that $(\partial \Sigma)_i$ is contained in the one of closed hemispheres with boundary $G_1$. Let $G_1 := \{ax_1 + bx_2 + cx_3 = 0\} \cap \mathbb{S}^2$, and we may assume $(\partial \Sigma)_i$ is contained in $\{ax_1 + bx_2 + cx_3 \geq 0\}$. Note that $(\partial \Sigma)_i \neq G_1$. Then, $(a, b, c) \cdot \int_{(\partial \Sigma)_i} xds > 0$, so $f_i$ is contained in $\{ax_1 + bx_2 + cx_3 > 0\}$. This contradicts $f_i \in G_1$. Thus, $f_i \notin (\partial \Sigma)_i$.

Now we assert that if there is a geodesic segment of $\mathbb{S}^2$ connecting $f_i$ and two points $p, q \in (\partial \Sigma)_i$, then it has endpoints $p$ and $q$. Suppose not. We
may assume the endpoints are \( f_1 \) and \( q \). By convexity of \((\partial \Sigma)_i\), the great circle \( G_2 \) passing through \( f_1, p, \) and \( q \), dissects \((\partial \Sigma)_i\) into two components. We may assume \( f_1 = (0, 0, 1) \), \( G_2 = \{ x_1 = 0 \} \cap S^2 \), and \( p \in G_2 \cap \{ x_2 < 0 \} \).

By Claim 1 \((\partial \Sigma)_i \subset \{ x_3 > 0 \}\). Then, \( p, q \in \{ x_2 < 0 \} \cap \{ x_3 > 0 \} \cap G_2 \).

In addition, by Lemma 4 \((\partial \Sigma)_i \) does not meet \( G_2 \cap \{ x_2 > 0 \} \). Thus, for a fixed component of \((\partial \Sigma)_i \setminus G_2 \), there exists a great circle passing through \( f_1 \) such that they only meet at one point. Furthermore, such circumstance occurs if and only if they meet at the point tangentially. If there exists a great circle \( G_3 \) passing through \( f_1 \) such that \((\partial \Sigma)_i\) is contained in the one of closed hemispheres with boundary \( G_3 \), we can obtain a contradiction by a similar argument as in the previous paragraph. Otherwise, we can find a great circle that meets \((\partial \Sigma)_i\) at least four points. This also contradicts Lemma 4. Thus, we proved the assertion, which completes the proof. \( \square \)

**Proposition 3.** Let \( H_i \) be the open hemisphere centered at \( f_1 \). Let \( \Pi_i \) be a plane that bounds a convex spherical cap of \( S^2 \) whose center lies in \( S^2 \setminus H_i \). Then, \( \Pi_i \) meets \((\partial \Sigma)_i\) at most two points.

**Proof.** Suppose not. Let \( C_i = \Pi_i \cap S^2 \) and \( p_1, p_2, p_3 \in \Pi_i \cap (\partial \Sigma)_i \). We may assume that \( f_1 = (0, 0, 1) \) and the center of the convex spherical cap is \( a_i = (0, a_2, a_3) \) with \( a_2 > 0 \) and \( a_3 \leq 0 \). Additionally, we may assume that \( p_2 \) is contained in the shortest arc in \( C_i \) connecting \( p_1 \) and \( p_3 \). It is clear that \( \Pi_i \) does not pass through \( f_1 \) by assumption. Then the great circle passing through \( f_1 \) and \( p_2 \) meets \( \{ x_3 = 0 \} \cap \{ x_2 > 0 \} \) at a point, say \( q_1 \in S^2 \). Note that the geodesic segment between \( p_2 \) and \( q_1 \) is contained in the convex spherical cap. Thus, the segment meets the geodesic segment between \( p_1 \) and \( p_3 \) at a point, say \( q_2 \). Then, \( q_2 \) is contained in the interior of the spherical convex hull of \((\partial \Sigma)_i\) because it is contained in the geodesic segment between \( p_1, p_3 \in (\partial \Sigma)_i \). Thus, \( p_2 \in (\partial \Sigma)_i \) is contained in the geodesic segment between the two interior points, \( q_2 \) and \( f_1 \), of the spherical convex hull (see Proposition 2), which is impossible. \( \square \)

**Corollary 1.** If a plane \( \Pi \) has a normal vector perpendicular to the line passing through \( f_1 \) and \( f_2 \), then \( \Pi \) meets each boundary component at most two points.

**Proof.** If \( \Pi \) does not pass through \( f_1 \) and \( f_2 \), by Proposition 3 we obtained the desired conclusion. Otherwise, \( \Pi \) passes through the origin, so by two-piece property (Theorem 6) or Lemma 4 the corollary follows. \( \square \)

5. **Symmetry conditions by coordinate planes**

In this section, we prove Theorem 1, 2, and 3 by using propositions in the previous section. To this end, we will use the method of proof by contradiction by assuming \( \sigma_1(\Sigma) < 1 \). Then, by the work of Fraser and Schoen, \( \Sigma \) is congruent to the critical catenoid [8].

We first recall some basic properties of a first Steklov eigenfunction, which are used in McGrath’s work [18].
Definition 2 (Nodal domain). Let \( u \) be a Steklov eigenfunction of \( \Sigma \). Then the nodal set of \( u \) is \( \mathcal{N} = \{ p \in \Sigma \mid u(p) = 0 \} \). A nodal domain of \( u \) is a component of \( \Sigma \setminus \mathcal{N} \).

Lemma 5 (Courant nodal domain theorem, [14, 18]). If \( u \) is a first Steklov eigenfunction, then \( u \) has exactly two nodal domains.

Proof. See [14] or [18, Lemma 2.2]. \( \square \)

Lemma 6 (Symmetry of a first eigenfunction, [18]). If \( \Sigma \) is invariant under the reflection through a plane and \( \sigma_1(\Sigma) < 1 \), then a first Steklov eigenfunction is invariant under the reflection.

Proof. See [18, Lemma 3.2]. \( \square \)

The following lemma is obtained by Proposition 1.

Lemma 7. If \( \sigma_1(\Sigma) < 1 \), then a first Steklov eigenfunction \( u \) is sign-changing in one of boundary components of \( \Sigma \). Furthermore, the nodal set of \( u \) in this component of \( \partial \Sigma \) is exactly two points.

Proof. Suppose \( u \) does not change its sign in each boundary component of \( \Sigma \). Since \( \int_{\partial \Sigma} u = 0 \) (Lemma 2), \( u \) has different signs in \( (\partial \Sigma)_1 \) and \( (\partial \Sigma)_2 \). By Proposition 1 we can find a plane \( \{ ax_1 + bx_2 + cx_3 = 0 \} \) that does not meet \( \partial \Sigma \). Since \( ax_1 + bx_2 + cx_3 \) has different signs in \( (\partial \Sigma)_1 \) and \( (\partial \Sigma)_2 \), we have

\[
\int_{\partial \Sigma} u(ax_1 + bx_2 + cx_3) \neq 0. \tag{15}
\]

On the other hand, \( \sigma_1(\Sigma) < 1 \) implies that \( \int_{\partial \Sigma} uax_1 = \int_{\partial \Sigma} ubx_2 = \int_{\partial \Sigma} ucx_3 = 0 \) (see Lemma 2), which leads a contradiction with (15). Therefore, \( u \) is sign-changing in at least one component of \( \partial \Sigma \), say \( (\partial \Sigma)_1 \). Note that \( \mathcal{N} \) consists of finite number of loops and arcs between boundary components (see [3] or [8, Theorem 2.3]). In addition, if \( u \) does not change sign at \( p \in \mathcal{N} \cap (\partial \Sigma)_1 \) along \( (\partial \Sigma)_1 \), these loops and arcs meet \( (\partial \Sigma)_1 \) transversely at \( p \). Then, by Courant nodal domain theorem, \( (\partial \Sigma)_1 \) has exactly two points in \( \mathcal{N} \) and the proof is complete. \( \square \)

Using the previous lemma and Proposition 3 (or Corollary 1), we can obtain the following theorem.

Theorem 7. If \( \Sigma \) is invariant under the reflection through two coordinate planes, then \( \Sigma \) is congruent to the critical catenoid.

Proof. Suppose \( \sigma_1(\Sigma) < 1 \). Let \( u \) be a first Steklov eigenfunction and \( \mathcal{N} \) be the nodal set of \( u \). Using Lemma 7 we may assume that \( u \) is sign-changing in \( (\partial \Sigma)_1 \) and let \( p_1, p_2 \in \mathcal{N} \cap (\partial \Sigma)_1 \).

By assumption, we say \( \Sigma \) is invariant under the reflection \( R_{\Pi} \) through a plane \( \Pi \). Then, \( R_{\Pi}(f_1) = f_1 \) or \( R_{\Pi}(f_1) = f_2 \). For the former case, \( \Pi \) is a plane passing through \( f_1 \) and the origin. For the latter case, \( \Pi \) is the plane.
perpendicular to $f_1$. For simplicity, we may assume $f_1 = (0, 0, 1)$. Then, it is enough to consider the following two cases.

**Case 1** The Reflection planes are $\{x_1 = 0\}$ and $\{x_2 = 0\}$. Let $\Pi_1 = \{x_1 = 0\}$ and $\Pi_2 = \{x_2 = 0\}$. Then, $N \cap (\partial \Sigma) \subset \Pi_1 \cup \Pi_2$, because if it is not true, $(\partial \Sigma)$ contains at least four nodal points by symmetry of $u$ (see Lemma 6), contrary to Proposition 3. Now, assume $p_1 \in \Pi_1$. Then, $p_2 \in \Pi_1$ by symmetry of $u$. Then, also by symmetry of $u$, $u$ does not change its sign in $(\partial \Sigma)$, contrary to our assumption.

**Case 2** One of the reflection planes is $\{x_3 = 0\}$. Let $\Pi_1 = \{x_3 = 0\}$. By symmetry of $u$ (see Lemma 6), we have $p_3, p_4 \in N \cap (\partial \Sigma)$ and $R_{\Pi_1}(\{p_1, p_2\}) = \{p_3, p_4\}$. Then, we have a plane $\Pi_2$ passing through $p_1, p_2, p_3, p_4$ which is perpendicular to $\Pi_1$. Let $\Pi_2 = \{x_2 = c\}$. By Corollary 1, $|\Pi_2 \cap \partial \Sigma| \leq 4$, thus $\Pi_2 \cap \partial \Sigma = \{p_1, p_2, p_3, p_4\}$. By symmetry of $u$, the signs of $u$ do not change in $(\partial \Sigma) \cap \{x_2 > c\}$ and $(\partial \Sigma) \cap \{x_2 < c\}$. In addition, the two signs are different because of $\int_{\partial \Sigma} u(x_2 - c) \neq 0$. Thus, we have

\begin{equation}
\int_{\partial \Sigma} u(x_2 - c) \neq 0.
\end{equation}

On the other hand, $\sigma_1(\Sigma) < 1$ implies that $\int_{\partial \Sigma} u = \int_{\partial \Sigma} u x_2 = 0$ (see Lemma 2), which leads a contradiction with (16).

By both of Case 1 and 2, we have $\sigma_1(\Sigma) = 1$. Therefore, by the theorem of Fraser and Schoen (see [8, Theorem 1.2]), $\Sigma$ is congruent to the critical catenoid.

Now, we can prove Theorem 1, 2, and 3.

**Proof of Theorem 1**

$\Sigma'$ can be extended to the embedded free boundary minimal annulus in the unit ball in $\mathbb{R}^3$ with the reflection symmetry through the plane. By Case 2 in Theorem 7, the surface is the critical catenoid, which is the desired conclusion.

**Remark 3.** The condition on the boundary in Theorem 1 is necessary. In [2], Carlotto, Franz, and Schulz constructed an embedded free boundary minimal surface $\Sigma_1$ in $\mathbb{B}^3$ with only one boundary component, genus 1, and the dihedral symmetry $D_2$. By the symmetry, this surface is invariant under the reflection through a coordinate plane $\Pi$. Then, a component of $\Sigma \setminus \Pi$ is an embedded free boundary minimal annulus in the half-ball that has a boundary component that meets both of $\Pi$ and an open hemisphere bounded by $\Pi$ orthogonally.

**Proofs of Theorem 2 and Theorem 3**

The proofs follow from Theorem 7 with the symmetry principle (Theorem 5).
Using Case 2 in the proof of Theorem 7 and Kusner and McGraths’ theorem (see [13] or Theorem 8) with the symmetry principle, we obtain the following corollary.

**Corollary 2.** If either the antipodal map or the reflection through a plane maps a boundary component of \( \Sigma \) into the other boundary component, then \( \Sigma \) is congruent to the critical catenoid.

6. Conditions with congruent boundary components

Recall the uniqueness theorem by Kusner and McGrath.

**Theorem 8 ([13]).** If \( \Sigma \) is invariant under the antipodal map, then \( \Sigma \) is congruent to the critical catenoid.

**Proof.** See [13, Theorem 2].

We give a proof of Theorem 4 by combining several results in this paper and Theorem 8. The proof based on the observation about congruence of the piece of \( \Sigma \) in some hemisphere and the piece of \( \Sigma \) in the other hemisphere.

**Proof of Theorem 4**

We may assume that \( f_1 = (0,0,1) \) and \( f_2 = (0,0,-1) \) by [8]. Since \((\partial \Sigma)_1 \) and \((\partial \Sigma)_2 \) are congruent, there is \( A \in O(3) \) such that \( A((\partial \Sigma)_1) = (\partial \Sigma)_2 \). Let \( \Sigma_1 = \Sigma \cap \{ x_3 \geq 0 \} \) and \( \Sigma_2 = \Sigma \cap \{ x_3 \leq 0 \} \). By the symmetry principle (Theorem 5), \( A(\Sigma_1) = \Sigma_2 \). In particular, for any \( p \in \Sigma \cap \{ x_3 = 0 \} \), we have

\[
A(p) \in \Sigma \cap \{ x_3 = 0 \} \quad \text{and} \quad g \circ A(p) = A \circ g(p).
\]

Since \( A(f_1) = f_2 \), \( A \) can be represented by \( B \circ R_{\{ x_3 = 0 \}} \), where \( B \) is a map in \( O(3) \) that preserves \( x_3 \)-axis and \( R_{\{ x_3 = 0 \}} \) is the reflection through \( \{ x_3 = 0 \} \). Note that \( B \) is either a rotation about \( x_3 \)-axis or the reflection through a plane that passes through \( f_2 \) and the origin.

**Case 1** \( B \) is a rotation about \( x_3 \)-axis.

Let \( B \) be the rotation about \( x_3 \)-axis by an angle \( \theta \). Let \( p \in \Sigma \cap \{ x_3 = 0 \} \) and \( g(p) = (x,y,z) \).

If \( B \) is an irrational rotation, the orbit of \( p \) under \( A \) is dense in the circle in \( \{ x_3 = 0 \} \) of radius \( |p| \), centered at the origin. By (17), the orbit of \( p \) is contained in \( \Sigma \cap \{ x_3 = 0 \} \) and \( x_3 \)-coordinates of its Gauss map image are either \( z \) or \(-z\). Note that \( \{ q \in \{ A^n(p), n \in \mathbb{N} \}| g(q) \cdot (0,0,1) = z \} \) and \( \{ r \in \{ A^n(p), n \in \mathbb{N} \}| g(r) \cdot (0,0,1) = -z \} \) are dense in the circle. By continuity of \( g, z = 0 \) and \( \Sigma \) meets \( \{ x_3 = 0 \} \) perpendicularly. By Theorem 4 \( \Sigma \) is congruent to the critical catenoid.

Now, we may assume \( \theta = \frac{b}{a} \cdot 2\pi \), where \( a \) and \( b \) are relatively prime.

If \( a \) is odd, applying (17), we obtain \( g(p) = g \circ A^a(p) = A^a \circ g(p) = (x,y,-z) \). Thus \( z = 0 \) and, by Theorem 1, the \( \Sigma \) is congruent to the critical catenoid.

If \( a = 2(2k-1), k \in \mathbb{N} \), applying (17), we obtain \(-p = A^{a/2}(p) \in \Sigma \cap \{ x_3 = 0 \} \) and \( g(-p) = g \circ A^{a/2}(p) = A^{a/2} \circ g(p) = -g(p) \). By Björling uniqueness
theorem (see [5, Theorem 1 in p.125]), \( \Sigma \) is invariant under the antipodal map. Then, by Theorem 3, \( \Sigma \) is congruent to the critical catenoid.

If \( a = 2^m(2k-1), m, k \in \mathbb{N} \) with \( m \geq 2 \), we consider the symmetry assumption on \( \partial \Sigma \). By Corollary 2, we only need to consider the case that \( (\partial \Sigma)_1 \) is invariant under the reflection through a plane, say \( \Pi_1 \). Since \( R_{\Pi_1}((\partial \Sigma)_1) = (\partial \Sigma)_1, R_{\Pi_1}(f_1) = f_1 \). Thus, we may assume \( \Pi_1 = \{ x_1 = 0 \} \). Then, by the symmetry principle, \( \Sigma \) is invariant under the reflection through \( \{ x_1 = 0 \} \). Now, by (17), we obtain \( g(-p) = g \circ A^{a/2}(p) = A^{a/2} \circ g(p) = (-x, -y, z) \). Then we have \( R_{\{x_2=0\}}(p) = R_{\{x_1=0\}}(-p) \in \Sigma \cap \{ x_3 = 0 \} \)

and \( g \circ R_{\{x_2=0\}}(p) = g \circ R_{\{x_1=0\}}(-p) = R_{\{x_1=0\}} \circ g(-p) = (x, -y, z) = R_{\{x_2=0\}} \circ g(p) \). Then, by Björling uniqueness theorem, \( \Sigma \) is also invariant under the reflection through \( \{ x_2 = 0 \} \). Then, by Theorem 7, \( \Sigma \) is congruent to the critical catenoid.

**Case 2** \( B \) is a reflection through a plane passing through \( f_2 \) and the origin.

Let \( \Pi_1 \) be the reflection plane of \( B \) and \( \Pi_2 \) be the reflection plane in the assumption on \( \partial \Sigma \). Using Corollary 2 as in Case 1, we may assume \( \Pi_2 := \{ x_1 = 0 \} \). Then, \( R_{\Pi_1} \circ R_{\Pi_2} \) is a rotation about \( x_3 \)-axis. Then, we define \( B' := R_{\Pi_1} \circ R_{\Pi_2} = B \circ R_{\Pi_2} \) and \( A' := B' \circ R_{\{x_3=0\}}. \) Then (17) holds for \( A' \) instead of \( A \), because the symmetry principle and the boundary symmetry (see Lemma 3) give \( A'(p) = B' \circ R_{\{x_3=0\}}(p) = B \circ R_{\Pi_2}(p) = A \circ R_{\Pi_2}(p) \in \Sigma \cap \{ x_3 = 0 \} \)

and \( g \circ A'(p) = g \circ A \circ R_{\Pi_2}(p) = (A \circ g) \circ R_{\Pi_2}(p) = A \circ (R_{\Pi_2} \circ g)(p) = B \circ (R_{\Pi_2} \circ R_{\{x_3=0\}}) \circ g(p) = A' \circ g(p) \). Thus, Case 2 reduces to Case 1.

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Research Institute for Natural Sciences, Hanyang University, 222 Wangsimni-ro, Seongdong-gu, Seoul, 04763, Republic of Korea

Email address: donghwi.seo26@gmail.com