THE CUP PRODUCT ON HOCHSCHILD COHOMOLOGY FOR
LOCALIZATIONS OF FILTERED KOSZUL ALGEBRAS

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Abstract. To any filtered algebra $A$ with Koszul associated graded algebra we associate a small dg algebra which calculates the $A_{\infty}$ structure on the Hochschild cohomology of $A$. In particular, it calculates the cup product on Hochschild cohomology. This dg algebra is, as an algebra, simply the tensor product of $A$ and the Koszul dual of its associated graded algebra. We then show that the Hochschild cohomology algebra of any Ore localization of $A$ can be calculated by a localization of the dg algebra associated to $A$. As an application we directly calculate the Hochschild cohomology algebra of the universal enveloping algebra of the Heisenberg Lie algebra.

1. Introduction

Fix a base field $k$ of arbitrary characteristic. By an algebra, complex, etc. we will mean a $k$-algebra, $k$-complex, etc., and an unadorned tensor product $\otimes$ is a tensor product over $k$. This paper provides a means of calculating the Hochschild cohomology ring of an algebra which is close to Koszul. In particular, we are interested in algebras which come equipped with a filtration such that the associated graded algebra is Koszul, i.e. arbitrary PBW deformations of Koszul algebras. Such algebras will be called filtered Koszul.

Recall that for any algebra $A$ the categories of $A$-bimodules and (left) $A^e = A \otimes A^{\text{op}}$-modules are canonically isomorphic. The Hochschild cohomology of an algebra $A$ is the Ext algebra $\text{Ext}_{A^e}(A, A)$, where $A$ is the regular bimodule $A \otimes A$. The multiplication on $\text{Ext}_{A^e}(A, A)$ is usually referred to as the cup product and Hochschild cohomology is more commonly denoted $\text{HH}^*(A) := \text{Ext}_{A^e}(A, A)$. We let $\text{HH}^*(A, M)$ denote $\text{Ext}_{A^e}(A, M)$ for an arbitrary bimodule $M$. (It is shown that the cup product and Yoneda product agree at [2, Proposition 1.1].) This cohomology has particular significance to the deformation theory of noncommutative, and commutative, algebras. There is an additional graded Lie algebra structure on $\text{HH}^*(A)$ that interacts with the cup product to measure to what degree, and in what directions, the algebra $A$ admits deformations. See, for example, [5] [10] [1].

A Koszul algebra is a connected graded, finitely generated, algebra $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ such that $\text{Ext}_A(A, A)$ is generated by $\text{Ext}_A^1(k, k)$. Such algebras can be described more tangibly as graded algebras which are generated in degree 1, have defining relations in degree 2, and satisfy a certain homological purity condition. In the case that $A$ is Koszul we fully understand the algebra structure on $\text{Ext}_A(A, A)$, and call it the Koszul dual of $A$. We fix the notation $\Lambda := \text{Ext}_A(k, k)$ for the Koszul dual.外

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dual of \( A \), as opposed to the more conventional \( A^! \). We view \( \Lambda \) as a (homologically) graded algebra \( \Lambda = \bigoplus_i \Lambda_i \), with \( \Lambda_i = \text{Ext}_A^i(k,k) \).

In addition to having an easily understood Ext algebra, Koszul algebras admit a canonical \( A \)-bimodule resolution

\[
K = \ldots \xrightarrow{d} A \otimes (\Lambda^2)^* \otimes A \xrightarrow{d} A \otimes (\Lambda^1)^* \otimes A \xrightarrow{d} A \otimes A \to 0 \tag{1}
\]
of \( A \). For any finitely generated left (resp. right) module \( M \), applying the functor \( - \otimes_A M \) (resp. \( M \otimes_A - \)) to the above resolution results in a resolution of \( M \) over \( A \), and in the case that \( M = k \) this resolution is minimal. Examples of Koszul algebras include skew polynomial rings, Jordan planes, and Sklyanin algebras.

Analogous constructions to the ones described above exist for filtered Koszul algebras. Let \( A \) be a filtered Koszul algebra and let \( \text{gr}A \) denote its associated graded algebra. In this case the Koszul dual algebra must be replaced with a Koszul dual (curved) dg algebra, which we also denote by \( \Lambda = (\Lambda, d_\Lambda, c_\Lambda) \). In short, a dg algebra is a chain complex with a compatible algebra structure, and for a curved dg algebra we require that \( d_\Lambda^2 \) is simply inner, \( d_\Lambda^2 = [c_\Lambda, -] \), as opposed to \( d_\Lambda^2 = 0 \). Our use of the curvature is rather mild, and one will not be harmed in thinking simply of dg algebras.

As a graded ring, \( \Lambda \) is the Koszul dual \( \text{Ext}_{\text{gr}A}^\bullet(k,k) \) of the associated graded algebra. The differential \( d_\Lambda \) and curvature \( c_\Lambda \) take account of all the information lost in the grading process \( A \mapsto \text{gr}A \). We call \( \Lambda \) the Koszul dual (curved) dg algebra to \( A \). The most standard examples of filtered Koszul algebras are universal enveloping algebras of Lie algebras, Weyl algebras, and Clifford algebras. More information on curved dg algebras and filtered Koszul algebras can be found in Section 3.

Although we have a concrete understanding of many homological constructions for filtered Koszul algebras, very little has been written on their Hochschild cohomology. This is especially true if we consider the many algebra structures Hochschild cohomology carries. In this paper we give an approach to the Hochschild cohomology algebras of filtered Koszul algebras. This approach is then shown to extended to localizations of such algebras. Our main result is the following.

**Theorem A.** Let \( A \) be filtered Koszul and let \( \Lambda \) be the Koszul dual of \( A \), i.e. the Koszul dual algebra of the associated graded algebra of \( A \) along with the (curved) dg structure described above. Let \( B = AS^{-1} \) be any Ore localization of \( A \) with respect to a denominator set \( S \). Then there is a canonically defined degree 1 element \( e \in \Lambda \otimes A \) such that

\[
(\Lambda \otimes B, d_\Lambda \otimes \text{id}_B - [e, -])
\]

is a dg algebra and

\[
H^\bullet(\Lambda \otimes B, d_\Lambda \otimes \text{id}_B - [e, -]) = HH^\bullet(B) \tag{2}
\]
as an algebra.

The element \( e \) is defined in Definition 6.1 and acts on \( \Lambda \otimes B \) by way of the localization map \( \Lambda \otimes A \to \Lambda \otimes B \). A more refined and useful result than Theorem A is actually true. In Theorem 7.4 it is shown that, for the algebra \( A \) itself, \( (\Lambda \otimes A, d_\Lambda \otimes \text{id}_A - [e, -]) \) is quasi-isomorphic to the endomorphism dg algebra \( \text{Hom}_A^\bullet(K,K) \). Here \( K \) is the Koszul resolution, described at (1). This implies, for example, that the equality (2) is one of \( A_\infty \) algebras. Definitions and basic
properties for $A_\infty$ algebras are given in Section 9.3. The extension to localizations is given in Section 8.

Before proving Theorem A we also give a slight generalization of [32, Theorem 9.1], which appears in Propositions 6.5 and 8.2.

**Theorem B.** Let $A$ be filtered Koszul and $\Lambda$ be its Koszul dual (curved) dg algebra. Let $B = AS^{-1}$ be any Ore localization with respect to a denominator set $S$. The homology of the complex

$$\left( \Lambda \otimes B^e, d_\Lambda \otimes \text{id}_B - [e, -] \right)$$

is equal to $\text{Ext}^B_{B^e}(B, B^e)$ as a $B$-bimodule.

It is worth mentioning that what we really do is define a functor

$$\Lambda \tilde{\otimes} - : M \mapsto (\Lambda \otimes M, d_\Lambda \otimes \text{id}_M - [e, -])$$

which replaces the Hochschild cochain complex as a model for $\text{RHom}_A(A, -)$, and is dg algebra valued whenever $M$ is an algebra extension of $A$, i.e. an algebra with an $A$-bimodule structure induced by an algebra map $A \to M$ (see Section 6 and the beginning of Section 7). The hope is that this paper will provide the reader with an approach to the Hochschild cohomology algebra of Koszul algebras that is both easily understood and natural, in the sense that it is basis free.

Finally, we should call the reader’s attention to the multitude of papers dedicated to the Hochschild cohomology of several classes of rings, such as [2, 6, 9, 27]. In the paper [2], in particular, the authors relate the multiplication on the Hochschild cohomology of a Koszul (path) algebra $A$ to a comultiplicative structure on a special collection of right ideals in $A$. Also, Witherspoon and coauthors have given an extensive analysis of the Hochschild cohomology of group algebras and smash products. A non exhaustive list of their publications would include [28, 29, 30]. It may also be appropriate to mention that many of the methods used in this paper owe an intellectual debt to the work of Keller, Lefèvre-Hasegawa, and Van den Bergh.

The present paper is organized as follows: Sections 2 through 5 are dedicated to background material and a presentation of Koszul resolutions via twisting cochains. In Section 6 the functor $\Lambda \tilde{\otimes} -$ is defined and shown to calculate $HH^\bullet(A, M)$. In Section 7 it is shown that $\Lambda \otimes A$ is a dg algebra calculating the algebra structure on $HH^\bullet(A)$. Section 8 gives an analysis of Hochschild cohomology for localizations and Section 9 gives a short presentation of $A$-infinity algebras. Finally, Section 10 is dedicated to the example of the universal enveloping algebra $U(h)$ of the Heisenberg Lie algebra.

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2. Notations

Let $B$ be an arbitrary ring. By a “$B$-module” we mean a left $B$-module unless stated otherwise. We will always use the cohomological indexing convention

$$X = \cdots \to X^{n-1} \to X^n \to X^{n+1} \to \cdots$$
for chain complexes. Given $B$-complexes $X$ and $Y$ we write $\text{Hom}_B(X, Y)$ for the standard Hom complex

$$\text{Hom}_B(X, Y) = \bigoplus_{n \in \mathbb{Z}} \left( \prod_i \text{Hom}_B(X^i, Y^{i+n}) \right)$$

For any homogenous function $\theta \in \text{Hom}_B(X, Y)$ of degree $n$, the differential $d$ is given by the formula $d(\theta) = d_Y \theta - (-1)^n \theta d_X$.

The grading on a graded ring $A$ will be seen as internal and will be denoted by a lower index $A = \bigoplus_i A_i$. A graded $B$-module will be referred to as homologically graded if it is to be viewed as a chain complex with vanishing differential. When doing computations with graded modules we assume that each element $x$ is homogenous of a particular degree, and we let $|x|$ denote the degree of such an element.

Sweedler’s notation will be used to denote the comultiplication on a coalgebra $C$. So the element $\Delta(c)$ will be written $\Delta(c) = c_1 \otimes c_2$, with the sum implicit. To say this more clearly, “$c_1 \otimes c_2$” is simply shorthand for some expression of the element $\Delta(c) = \sum_i c_{i1} \otimes c_{i2}$ in the tensor product $C \otimes C$. Higher iterations of the comultiplication will be denoted using similar notation. For example, the element $$(\Delta \otimes \text{id}) \Delta(c) = (\text{id} \otimes \Delta) \Delta(c)$$

will be denoted $c_1 \otimes c_2 \otimes c_3$. Again, there is an implicit sum. If $C$ is graded, and $c \in C$ is homogeneous, then the $c_1, c_2, \text{etc.}$ will always be taken to be homogeneous.

3. Reminders on DG Algebras, DG Coalgebras, and Filtered Koszul Algebras

3.1. **Dg algebras and coalgebras.** Recall that a dg algebra is a chain complex $(A, d)$ equipped with a unit $k \to A$ and associative multiplication $\mu : A \otimes A \to A$ which are both chain maps. On elements, this means the unit 1 is a cycle and that $d$ satisfies

$$d(fg) = d(f)g + (-1)^{|f|} fd(g),$$

i.e. that $d$ is a graded derivation. A dg coalgebra is defined dually to be a complex $(C, d)$ with a coalgebra structure such that each structure map $C \to k$, $\Delta : C \to C \otimes C$, is a chain map. We will call a dg (co)algebra locally finite if it is finite dimensional in each homological degree. A dg algebra $A$ (resp. dg coalgebra $C$) is said to be augmented (resp. coaugmented) if it comes equipped with a dg map $A \xrightarrow{\text{aug}} k$ (resp. $k \xrightarrow{\text{coaug}} C$).

Given an arbitrary dg algebra $A$ and dg coalgebra $C$ the hom complex $\text{Hom}_k(C, A)$ becomes a dg algebra under the convolution product

$$f \ast g := \mu_A(f \otimes g) \Delta_C : c \mapsto (-1)^{|c_1||g|} f(c_1)g(c_2).$$

In particular the dual $C^* = \text{Hom}_k(C, k)$ is an algebra. One can check that the dual $A^* = \text{Hom}_k(A, k)$ of any locally finite dg algebra is a dg coalgebra under the coproduct $\Delta(\gamma) = \gamma \mu$. The double dual of a locally finite dg (co)algebra $A$ is naturally isomorphic to $A$ via the standard map

$$\text{ev} : A \to (A^*)^*$$

$$a \mapsto (\phi \mapsto (-1)^{|a||\phi|} \phi(a)).$$

(4)
The tensor product of dg (co)algebras is again a dg (co)algebra under the differential \( d_{A \otimes A'} = d_A \otimes id_{A'} + id_A \otimes d_{A'} \).

3.2. Graded Koszul duality with signs. As stated in the introduction, a Koszul algebra is a finitely generated connected graded algebra \( A \), i.e. a graded algebra of the form
\[
A = k \oplus A_1 \oplus A_2 \oplus \cdots,
\]
such that \( \text{Ext}_k(A, k) \) is generated by \( \text{Ext}_k^1(A, k) \) as an algebra. Here \( k = \mathbb{A}k \) denotes the graded simple module \( A/(A_{\geq 1}) \). The Koszul dual of a Koszul algebra \( A \) is the algebra \( \text{Ext}_k(A, k) \). To avoid confusion with the filtered case, we denote the Koszul dual by \( E \) for the moment.

Any Koszul algebra will have a quadratic presentation \( A = k(V)/R \). Let us fix a Koszul algebra with such a presentation. Here \( R \subset V \otimes V \) is the subspace of quadratic relations for \( A \). It is well known that we have a presentation \( E \cong k(V^*)/(R^+) \) given by the identification \( E^1 = V^* \). We give here a description of the Koszul dual which takes into account the homological grading on the implicit Koszul resolution of \( k \), which gives rise to the Koszul dual. In particular, we identify \( E \) with the dual algebra of some particular homologically graded coalgebra, i.e. a dg coalgebra with vanishing differential.

We let \( T(V) = \oplus_{n \geq 0} V^\otimes n \) denote the tensor coalgebra on \( V \). Recall that the comultiplication on \( T(V) \) is defined by "separation of tensors"
\[
v = (v_1 \otimes \ldots \otimes v_n) \mapsto (1) \otimes (v) + (v) \otimes (1) + \sum_{1 \leq j \leq n-1} (v_1 \otimes \ldots \otimes v_j) \otimes (v_{j+1} \otimes \ldots \otimes v_n).
\]

We consider \( T(V) \) to be homologically graded by taking \( V \) to be in degree \(-1\). (So the more cumbersome notation \( T(\Sigma V) \) may be more appropriate here.) The following lemma is well known. See for example [13 Section 4.7], [16 Sections 3.1.3-3.2.2]. In any case, we sketch the proof for the reader’s convenience.

**Lemma 3.1.** The graded subspace \( C \) of \( T(V) \) defined by \( C^0 = k, C^{-1} = V, \) and
\[
C^{-i} = \bigcap_{i_1 + i_2 = i - 2} V^\otimes i_1 \otimes R \otimes V^\otimes i_2
\]
for all \( i \geq 2 \), is a graded subcoalgebra.

**Proof.** We will show that \( C \) is closed under the coproduct on \( T(V) \). We grade \( T(V) \) by negated tensor degree so that the inclusion \( C \to T(V) \) is a graded map. Let \( c \) be a homogenous element in \( C \). Since \( C^{-1} \) and \( C^0 \) are equal to \( T(V)^{-1} = V \) and \( T(V)^0 = k \) respectively, it is trivial to show that \( \Delta(c) \subset C \otimes C \) whenever \( |c| \) is 0, 1, or 2.

Let us assume \( |c| = -n \leq -3 \). For \( i, j \geq 0 \), let \( \Delta_{ij}(c) \) denote the component of \( \Delta(c) \) in
\[
T(V)^{-i} \otimes T(V)^{-j} = (V^\otimes i) \otimes (V^\otimes j).
\]
So we have \( \Delta(c) = \sum_{ij} \Delta_{ij}(c) \), and \( \Delta(c) \in C \otimes C \) if and only if each \( \Delta(c)_{ij} \) is in \( C^{-i} \otimes C^{-j} \).

Since \( C^{-n} \) is the intersection \([13]\), we can write \( c \) as a sum
\[
c = \sum_i v_{l_1} \otimes \ldots \otimes v_{l_{k-1}} \otimes r_{l_k} \otimes v_{l_{k+2}} \ldots \otimes v_{l_n}
\]
for any $k$ between 1 and $n - 1$, where the $r_{i_k} \in R$. Now by letting $k$ vary we see that

$$\Delta_{ij}(c) \in (C^{-i} \otimes T(V)^{-j}) \cap (T(V)^{-i} \otimes C^{-j}).$$

One can verify that this final intersection is equal to $C^{-i} \otimes C^{-j}$. 

Let us outline our identification $E = C^*$. Consider $k(V^*)$, the free algebra on the degree 1 space $V^*$. We have the canonical algebra isomorphism

$$k(V^*) \rightarrow (T(V))^* \quad \quad f_1 \otimes \ldots \otimes f_n \mapsto (v_1 \otimes \ldots \otimes v_n) \mapsto (-1)^{n(n-1)/2} f_1(v_1) \ldots f_n(v_n).$$

(6)

Here the $f_i$ are in $V^*$, the $v_i$ are in $V$, the function $f_1 \otimes \ldots \otimes f_n$ will vanish off $V^* \otimes n$, and the exponent $n(n-1)/2 = \sum_{i=0}^{n-1} i$ comes from commuting the degree $-1$ variables $v_i$ past the degree 1 maps $f_i$.

If we then compose the isomorphism $[\Box]$ with the dual of the restriction $C \rightarrow T(V)$ we get an algebra map $k(V^*) \rightarrow C^*$. One can verify that the kernel of this map is the ideal $(R^\perp)$ and so we get an isomorphism $E \rightarrow C^*$. This isomorphism simply sends a monomial $f_1 \ldots f_n$ in $E$ to the function $f_1 \ldots f_n : C \rightarrow k$, $v_1 \otimes \ldots \otimes v_n \mapsto (-1)^{n(n-1)/2} f_1(v_1) \ldots f_n(v_n)$.

Algebraically, one can see the isomorphism $E \rightarrow C^*$ as the unique algebra map defined as the identity on the generators $E^1 = V^* = (C^{-1})^*$. It is via this isomorphism that we identify $E$ with $C^*$ as a graded algebra.

**Remark 3.2.** The sign conventions we employ here make no difference in the presentation of the Koszul dual, since we will simply replace each relation $r$ produced via the unsigned identification $V^* \otimes V^* \cong (V \otimes V)^*$ with the negated relation under the signed identification $V^* \otimes V^* \cong (V \otimes V)^*$. The identity map on the generators $V^*$ will then provide an isomorphism

$$k(V^*)/(R^\perp) \rightarrow k(V^*)/(-R^\perp).$$

The conventions do make a difference once we start considering differentials and curvature.

### 3.3. (Augmented) filtered Koszul algebras

The class of algebras we will be interested are the following.

**Definition 3.3** (Filtered Koszul algebras). A $\mathbb{Z}_{\geq 0}$-filtered algebra $A = \cup_{i \geq 0} F_i A$ such that $\text{gr}A$ is Koszul is called a filtered Koszul algebra.

To distinguish between filtered Koszul and standard Koszul algebras we may refer to standard Koszul algebras as graded Koszul. The class of filtered Koszul algebras includes the class of graded Koszul algebras, since we can give any Koszul algebra the filtration $F_i A = \sum_{i=0}^n A_i$. In this case $\text{gr}A = A$.

Let $A$ be a filtered Koszul algebra. Let $E$ denote the Koszul dual algebra $\text{Ext}_{\text{gr}A}(k,k)$ of $\text{gr}A$. So we have $A = k(V)/ (R)$ and $E \cong k(V^*)/(R^\perp)$. Recall our identification of $E$ with the dual $C^*$ of the intersection coalgebra of Lemma 3.1. In particular, we have identified $E^2$ with $C^{*2} = R^*$ by sending a monomial $f_1 f_2$ in $E^2$ (where the $f_i \in V^*$) to the function $\sum r_i \otimes r'_i \mapsto - \sum f_1(r_i) f_2(r'_i)$.

Suppose for the moment that $A$ is augmented, i.e. has some fixed algebra map $\epsilon : A \rightarrow k$. For example, we could consider $A$ to be a universal enveloping algebra of a Lie algebra $g$ with the standard augmentation $U(g) \rightarrow k$ sending $g$ to 0. This will provide splittings $A = k \oplus \ker(\epsilon)$ and $F_1 A = V \oplus k$, and hence also
provide an embedding $V \to A$. This embedding then produces a second embedding $V \otimes V \to A \otimes A$, and finally a third embedding

$$R \subset V \otimes V \to A \otimes A$$

of the relations for the associated graded ring $grA$ into $A \otimes A$. We omit the proof of the following lemma.

**Lemma 3.4.** Suppose $A$ is filtered graded ring $grA$ augmented, as above. The restriction of the multiplication $\mu : A \otimes A \to A$ on $A$ to the subspace $R \subset A \otimes A$ has image in the set of generators $V$. Whence we get a canonically defined map

$$\mu|_R : R \to V.$$  

If we take $\alpha_1 := -\mu|_R$, then we get a presentation

$$A = k(V)/(r + \alpha_1(r))_{r \in R}.$$  

It is well known that the function $V^* = E^1 \to R^* = E^2$ given by precomposing with $\alpha_1$, $f \mapsto f\alpha_1$, extends to a dg algebra structure $d^A$ on $E$, and that $(E, d^A)$ calculates the Ext algebra $Ext_A(k, k)$. This result appears in Priddy’s original work on Koszul resolutions [25, Theorem 4.3]. (See also [25, Proposition 2.2], [23, Section 5.4], and Proposition 3.7 below.) In the case that $A$ is the universal enveloping algebra of a Lie algebra $\mathfrak{g}$, for example, the restriction (7) is given by the Lie bracket and the dg algebra $(E, d^A)$ is the Chevalley-Eilenberg dg algebra of $\mathfrak{g}$, $(E, d^A) = (\wedge^* \mathfrak{g}^*, d|$).

3.4. *Nonaugmented filtered Koszul algebras and curved dg structures.* The following is just a reiteration of the work of [25, Proposition 2.2]. We take $A$ filtered Koszul with $grA = k(V)/(R)$.

In the case of a nonaugmented filtered Koszul algebra $A$, we will have to choose some section $V \to F_1A$ of the sequence $0 \to k \to F_1A \to V \to 0$ in order to identify $V$ with a subspace in $A$. This will also provide a splitting $F_1A = V \oplus k$. We have the following weak analog of Lemma 3.4.

**Lemma 3.5.** Let $A$ be filtered Koszul, but not necessarily augmented, and fix some section $V \to F_1A$. Then restricting the multiplication on $A$ to the subspace $R \subset V \otimes V \subset A \otimes A$ produces a map

$$\mu|_R : R \to F_1A = V \oplus k.$$  

Take $\alpha_0 : V \to k$ to be the unique functions so that $\mu|_R = [-\alpha_1 - \alpha_0]^T$. Then we get a presentation

$$A = k(V)/(r + \alpha_1(r) + \alpha_0(r))_{r \in R}.$$  

One can see from the Lemmas 3.4 and 3.5 that $A$ is augmented if and only if the section $V \to F_1A$ can be chosen so that $\alpha_0 = 0$.

As in the previous subsection, we take $E$ to be the algebra of extensions $Ext_{grA}(k, k)$ for the associated graded algebra $grA$, which we have assumed to be Koszul. The function $V^* = E^1 \to R^* = E^2$ given by precomposition with $\alpha_1$, $f \mapsto f\alpha_1$, will again extend to a well defined derivation $d^A$ on $E$ [25, Proposition 2.2]. However, we will not be fortunate enough to have that $d^A$ is square $0$. Instead, $d^A d^A$ will simply be inner. Whence we introduce the notion of a curved dg (co)algebra.
Definition 3.6 (Curved dg (co)algebras). A curved dg algebra is a graded algebra $B = \bigoplus_{i \in \mathbb{Z}} B^i$ along with a degree 1 graded derivation $d_B$, and a degree 2 element $c_B \in B^2$, so that
$$d_B^2 = [c_B, -] \quad \text{and} \quad d_B(c_B) = 0.$$ (Here $d_B^2$ is the square $d_B^2 = d_B d_B$.) Dually, a curved dg coalgebra is a graded coalgebra $D = \bigoplus_{i \in \mathbb{Z}} D^i$ along with a degree 1 coderivation $d_D$, and degree 2 function $f_D : D \to k$ satisfying
$$d_D^2 = (f_D \otimes \text{id} - \text{id} \otimes f_D)\Delta \quad \text{and} \quad f_D d_D = 0.$$ We may denote a curved dg algebra (resp. coalgebra) as a triple $(B, d_B, c_B)$ (resp. $(D, d_D, f_D)$).

Note that the curvature element $c_B$ in a curved dg algebra $B$ is not uniquely determined by $d_B$ in general. Indeed, when $B$ is graded commutative, any degree 2 element $c_B \in B^2$ will have $[c_B, -] = 0$ and hence produce a curved dg algebra $(B, 0, c_B)$.

As with dg algebras and coalgebras, we have some standard constructions. Given a curved dg algebra $B$ and a curved dg coalgebra $D$ the set of graded maps $\text{Hom}_k(D, B) = \bigoplus_n \prod_i \text{Hom}_k(D^i, B^{i+n})$ becomes a curved dg algebra under the convolution product, standard derivation $d(\xi) = d_B\xi - (-1)^{|\xi|}\xi d_D$, and curvature $c_{\text{Hom}} = c_B c_D - 1_B f_D$.

[24] Section 6.2]. In particular, the graded dual $D^*$ of any curved dg coalgebra becomes a curved dg algebra with curvature element $c_{D^*} = -f_D$. The graded dual of any locally finite curved dg algebra $B$ becomes a curved dg coalgebra with the obvious coproduct, derivation $d(\eta) = -(-1)^{|\eta|}\eta d_B$, and curvature function $f_{B^*} = -ev_{c_B}$. When $D$ is locally finite, the evaluation map $ev : D \to D^{**}$ provides an isomorphism of curved dg coalgebras between $D$ and its double dual. Finally, the tensor product $B \otimes B'$ of curved dg algebras will again be a curved dg algebra with $d_{B \otimes B'} = d_B \otimes \text{id}_{B'} + \text{id}_B \otimes d_{B'}$ and $c_{B \otimes B'} = c_B \otimes 1 + 1 \otimes c_{B'}$.

Theoretically, curved dg structures arise as deformations of dg algebras. For example, a cocycle in the second Hochschild cohomology of a dg algebra $B$ will correspond to a curved dg $k[t]/(t^2)$-algebra, or more generally curved $A_\infty$ $k[t]/(t^2)$-algebra, which reduces to $B$ at $t = 0$.

Returning to our filtered Koszul algebra $A = k(V)/(r + \alpha_1(r) + \alpha_0(r))$ of [9], we note that $\alpha_0$ defines a function $R \to k$, and hence an element in $E^2 = R^*$. Take $c^A = -\alpha_0$. Recall that the derivation $d^A$ was defined on the generators of $E$ as the function $E^1 = V^* \to E^2 = R^*$, $f \mapsto f\alpha_1$. In [25], Positselski proves the following

Proposition 3.7 ([25] Proposition 2.2]). Suppose $A$ is filtered Koszul. The triple $(\text{Ext}_{grA}(k, k), d^A, c^A)$ defines a curved dg algebra structure on the algebra of extensions $\text{Ext}_{grA}(k, k)$ of the Koszul algebra $grA$.

Proof. The proof is the same as in [25], which essentially shows that the intersection coalgebra $C$ of Lemma 3.3 is a curved dg coalgebra with $d_C|C^{-1} = \alpha_1$ and $f_C = \alpha_0$. We only note here that the sign on the curvature has changed due to the our signed identification with $C^*$ (see Remark 3.2).

It is this structure which we view as the Koszul dual of $A$. Here we could take $A$ to be a Weyl algebra or Clifford algebra. It is well known that Weyl algebras are simple, and hence admit no augmentation.
An augmented PBW deformation

\[ A_n(k) = k\langle x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \rangle / (\langle \frac{\partial}{\partial x_j}, x_i \rangle - \delta_{ij}) \]

we have \( \text{gr}A = k[x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}] \) and \( \text{Ext}_{\text{gr}A}(k, k) = k[\lambda_1, \ldots, \lambda_n, \theta_1, \ldots, \theta_n] \).

In the second algebra, the variables \( \lambda_i \) and \( \theta_j \) are the duals of the \( x_i \) and \( \frac{\partial}{\partial x_i} \) respectively. We consider these functions to have homological degree 1, and the algebra \( k[\lambda_1, \ldots, \lambda_n, \theta_1, \ldots, \theta_n] \) is the free graded commutative algebra with these generators. (So the variables anti-commute.)

Recall our identification of \( \text{Ext}^2_{\text{gr}A}(k, k) = (V^* \otimes V^*) / R^\perp \) with \( R^* \) is given by sending a monomial \( f_1 f_2 \) to the function \( \sum_i r_i \otimes r'_i \mapsto -\sum f_1(r_i) f_2(r'_i) \).

So \( \lambda \theta \) gets identified with the function \( R \rightarrow k \) defined on basis elements by

\[
\begin{align*}
    x_k \otimes x_l - x_l \otimes x_k &\mapsto 0 \\
    \lambda c \theta : \frac{\partial}{\partial x_k} \otimes x_l - x_l \otimes \frac{\partial}{\partial x_k} &\mapsto \delta_{il} \delta_{jk} \\
    \frac{\partial}{\partial x_k} \otimes \frac{\partial}{\partial x_l} - \frac{\partial}{\partial x_l} \otimes \frac{\partial}{\partial x_k} &\mapsto 0
\end{align*}
\]

Whence, in this case, the curvature element \( c^{A_n(k)} = -\alpha_0 \) in \( k[\lambda_1, \ldots, \lambda_n, \theta_1, \ldots, \theta_n]^2 \) will be the sum

\[
c^{A_n(k)} = \sum_{i=1}^n \lambda_i \theta_i.
\]

The corresponding Koszul dual of \( A_n(k) \) will be the curved dg algebra

\[
(k[\lambda_1, \ldots, \lambda_n, \theta_1, \ldots, \theta_n], 0, c^{A_n(k)}).
\]

3.6. Example: PBW deformations of skew polynomial rings. Take \( V = \langle x_1, \ldots, x_n \rangle \) and let \( k_Q[V] \) denote the skew polynomial ring

\[
k_Q[V] = k[x_1, \ldots, x_n] / (x_j x_k - q_{jk} x_k x_j)
\]

for \( Q = [q_{jk}] \) a multiplicatively skew symmetric matrix \( q_{jk} = q_{kj}^{-1} \) with \( q_{jj} = 1 \).

The Koszul dual of the skew polynomial ring in the skew exterior algebra

\[
\text{Ext}_{k_Q[V]}(k, k) = \bigwedge_{\lambda} V^* := k[\lambda_1, \ldots, \lambda_n] / (\lambda_k \lambda_j + q_{jk} \lambda_j \lambda_k, \lambda_j^2).
\]

An augmented PBW deformation \( A \) of \( k_Q[V] \) will be given by some constants \( c_i^{jk} \) so that the relations on our PBW deformation \( A \) will be given by

\[
A = k[x_1, \ldots, x_n] / (x_j x_k - q_{jk} x_k x_j - \sum_i c_i^{jk} x_i).
\]

Thus see that \( \alpha : R \rightarrow V \) will be the function \( x_j \otimes x_k - q_{jk} x_k \otimes x_j \mapsto -\sum_i c_i^{jk} x_i \).

Now, on the Koszul dual, the product \( \lambda_i \lambda_j \) in \( (\bigwedge_{\lambda} V^*)^2 \) is identified with the function

\[
x_m \otimes x_m - q_{im} x_m \otimes x_l \mapsto -(\lambda_i(x_l)) \lambda_j(x_m) - q_{im} \lambda_i(x_m) \lambda_j(x_l)) = -\delta_{il} \delta_{jm} + q_{im} \delta_{im} \delta_{jl},
\]
i.e. the negated dual of the relations $x_i x_j - q_{ij} x_j x_i$. So

$$d_A(\lambda_i) = \lambda_i \alpha_1$$

$$= (x_j \otimes x_k - q_{jk} x_k \otimes x_j \mapsto - \sum_i c_{ik}^j \lambda_i(x_i))$$

$$= (x_j \otimes x_k - q_{jk} x_k \otimes x_j \mapsto - c_{ik}^j)$$

$$= \sum_{j<k} c_{jk}^i \lambda_j \lambda_k.$$

So, in the final analysis, the differential $d_A$ is given by the same constants as those defining the map $\alpha_1$, and hence the relations on $A$. We will come back to this example in Section 6.

4. Twisting Cochains

We first give the definition in the non-curved setting, then address the curved situation independently.

**Definition 4.1 (Twisting cochain).** Let $A$ be an augmented dg algebra, with augmentation $\epsilon$, and $C$ be a coaugmented coalgebra, with coaugmentation $u$. A degree 1 linear map $\pi : C \to A$ is called a twisting cochain if

i) There are containments $u(k) \subset \ker \pi$ and $\text{im}(\pi) \subset \ker \epsilon$.

ii) The map $\pi$ satisfies the equation $- (d_A \pi + \pi d_C) + \mu(\pi \otimes \pi) \Delta = 0$.

Condition i) is equivalent to the requirement that $\pi$ factors $C \to C/u(k) \to \ker(\epsilon) \to A$. In other sources, the formula in ii) may appear as $d_A \pi + \pi d_C + \mu(\pi \otimes \pi) \Delta = 0$.

One can mediate between the two perspectives by replacing $\pi$ with $-\pi$. Assuming $k$ is of characteristic $\neq 2$, this alternate form of condition ii) is exactly the statement that $\pi$ is a solution to the Maurer-Cartan equation

$$d(\pi) + \frac{1}{2} [\pi, \pi] = 0,$$

where $[,]$ denotes the graded commutator on the dg algebra $\text{Hom}_k(C, A)$. We are using this alternate formulation simply because it is the formula that arises naturally in our setting.

**Remark 4.2.** Despite the fact that the use of twisting cochains in noncommutative algebra is something of a novelty, the idea is not at all new. Twisting cochains appear in works of topologists dating back at least to the 1950’s.

Given a twisting cochain $\pi : C \to A$ we can form the twisted tensor products $A \otimes_{\pi} C$, $C \otimes_{\pi} A$, and $A \otimes_{\pi} C \otimes_{\pi} A$. These are the chain complexes with underlying graded spaces $A \otimes C$, $C \otimes A$, and $A \otimes C \otimes A$ and differentials

$$d_{A \otimes C} = (\mu(id \otimes \pi) \otimes idC)(id_A \otimes \Delta),$$

$$d_{C \otimes A} = (id_C \otimes \mu(\pi \otimes id))(\Delta \otimes id_A),$$

and

$$d_{A \otimes C \otimes A} + (\mu(id_A \otimes \pi) \otimes idC \otimes idA - id_A \otimes idC \otimes \mu(\pi \otimes id)) (id_A \otimes \Delta \otimes id_A)$$

respectively. Any of the above differentials will be denoted $d_{\pi}$ by abuse of notation.

Note that a sign appears when applying these differentials to elements, since $\pi$ is a degree 1 map. For example, if $a \in A$ and $c \in C$ then

$$d_{\pi}(a \otimes c) = d_A(a) \otimes c + (-1)^{|a|}a \otimes d_C(c) + (-1)^{|a|}a \pi(c_1) \otimes c_2$$
whereas
\[ d_\pi (c \otimes a) = d_C(c) \otimes a + (-1)^{|c|} c \otimes d_A(a) - (-1)^{|c|} c_1 \otimes \pi(c_2) a. \]

Supposing \( A \) is concentrated in degree 0, we can consider the bar dg coalgebra
\[ \mathcal{B}A = \cdots \to A \otimes A \otimes A \to A \otimes A \to \to k \to 0 \]
with differential given by \( d_{\mathcal{B}}|A = 0 \) and
\[ d_{\mathcal{B}}|A^\otimes n : a_1 \otimes \ldots \otimes a_n \mapsto \sum_i (-1)^i a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n. \]

We have the canonical twisting cochain \( \pi : \mathcal{B}A \to A \) which is the identity on \( A = \mathcal{B}^{-1} A \) and 0 elsewhere. The twisted tensor products
\[ A \otimes_\pi \mathcal{B}A, \ A \otimes_\pi A, \ A \otimes_\pi \mathcal{B}A \otimes_\pi A \]
will then recover the standard bar resolutions of [7, 4, 20], for example. More information on twisting cochains and twisted tensor products can be found in [13, Section 4.3] and [16, Chapter 2].

4.1. Twisting cochains in the curved/nonaugmented case. As we saw in Section 3.4, if we would like to understand the Koszulity of a nonaugmented algebra we will have to deal with curved dg (co)algebras. Here we follow [24, Section 6.2]. The reader may also refer to [21].

**Definition 4.3** (Twisting cochains with curvature). A degree 1 linear map \( \pi : C \to A \) from a curved dg coalgebra to a curved dg algebra is called a twisting cochain if
\[ (c_A \epsilon_C - 1_A f_C)(c_A \pi + \pi d_C) + \mu(\pi \otimes \pi) \Delta = 0 \]
holds.

Here, again, we differ from some other references by a sign. This discrepancy can be alleviated if we replace \( \pi \) with \(-\pi\). In this setting we still get a twisted tensor products \( A \otimes_\pi C \otimes_\pi A \). The differential on this complex is, oddly enough, given by the same formula as in the non-curved setting:
\[ d_{A \otimes C \otimes A} + (\mu(id_A \otimes \pi) \otimes id_C \otimes id_A - id_A \otimes id_C \otimes \mu(\pi \otimes id)) (id_A \otimes \Delta \otimes id_A). \] (10)

Since the formula (10) does not appear in the given references explicitly, we verify below that it does in fact produce a differential when \( A \) is concentrated in degree 0, i.e. when \( A \) is just an algebra. This is the only setting in which we will be using twisting cochains.

As one can see from the above formula, the curvature disappears at this level, and the awkward distinction between the augmented and non-augmented algebras becomes irrelevant. Indeed, for the remainder of the paper we will be able to provide a uniform analysis of the augmented and non-augmented situations.

4.2. Verifying (10) produces a differential when \( A \) is concentrated in degree zero. In this case the differential and curvature on \( A \) vanish and the twisting cochain identity will appears as
\[ -f_C - \pi d_C + \mu(\pi \otimes \pi) \Delta = 0. \] (11)
Whence the formula \([\text{(10)}]\) reduces to
\[ d_\pi = id_A \otimes d_C \otimes id_A + (\mu(id_A \otimes \pi) \otimes id_C \otimes id_A) - id_A \otimes id_C \otimes \mu(\pi \otimes id)) (id_A \otimes \Delta \otimes id_A). \]

Take \(d = d_C\). On elements the differential then appears as
\[ d_\pi(x \otimes c \otimes y) = x \otimes d_C(c) \otimes y + x\pi(c_1) \otimes c_2 \otimes y - (-1)^{|\pi|} x \otimes c_1 \otimes \pi(c_2)y. \]

Squaring this operation then gives
\[
d_\pi^2(x \otimes c \otimes y) = x \otimes d^2(c) \otimes y + x\pi(d(c)_1) \otimes d(c)_2 \otimes y + (-1)^{|d(c)|} x \otimes d(c)_1 \otimes \pi(d(c)_2)y + x\pi(c_1) \otimes d(c)_2 \otimes y + x\pi(c_1) \pi(c_2) \otimes c_2 \otimes y - (-1)^{|c_2|} x\pi(c_1) \otimes c_2 \otimes \pi(c_3)y - (-1)^{|c_1|} x \otimes d(c)_1 \otimes \pi(c_2)y + (-1)^{|c_1| + |c_2|} x \pi(c_1) \otimes c_2 \otimes \pi(c_3)y + (-1)^{|c_1| + |c_2| + |c_1|} x \otimes c_1 \otimes \pi(c_1)\pi(c_2)y.
\]

Using the coderivation identities \(d_C(c_1) \otimes d_C(c_2) = d_C(c_1) \otimes c_2 + (-1)^{|c_1|} c_1 \otimes d_C(c_2)\) and \(d_\pi^2(c) = f_{C}(c_1)c_2 - c_1 f_{C}(c_2)\), and the fact \(|d_C(c)| = |c| + 1\) we can reorganize the above expression to get
\[
d_\pi^2(x \otimes c \otimes y) = x \otimes f_{C}(c_1) \otimes c_2 \otimes y - x \otimes c_1 \otimes f_{C}(c_1)y + x\pi(d(c)_1) \otimes c_2 \otimes y + (-1)^{|c_1|} x\pi(c_1) \otimes d(c)_2 \otimes y + x\pi(c_1) d(c)_2 \otimes y + x\pi(c_1) \pi(c_2) \otimes c_2 \otimes y - (-1)^{|c_2|} x\pi(c_1) \otimes c_2 \otimes \pi(c_3)y - (-1)^{|c_1|} x \otimes d(c)_1 \otimes \pi(c_2)y - (-1)^{|c_1| + |c_2|} x \pi(c_1) \otimes c_2 \otimes \pi(c_3)y + (-1)^{|c_1| + |c_2| + |c_1|} x \otimes c_1 \otimes \pi(c_1)\pi(c_2)y.
\]

Finally, noting that \(\pi(c) = 0\) whenever \(|c| \neq -1\), since \(\pi\) is degree 1, we get
\[
d_\pi^2(x \otimes c \otimes y) = x \otimes f_{C}(c_1) \otimes c_2 \otimes y - x \otimes c_1 \otimes f_{C}(c_1)y + x\pi(d(c)_1) \otimes c_2 \otimes y + (-1)^{|c_1|} x\pi(c_1) \otimes d(c)_2 \otimes y + x\pi(c_1) d(c)_2 \otimes y + x\pi(c_1) \pi(c_2) \otimes c_2 \otimes y - (-1)^{|c_2|} x\pi(c_1) \otimes c_2 \otimes \pi(c_3)y - (-1)^{|c_1| + |c_2|} x \pi(c_1) \otimes c_2 \otimes \pi(c_3)y + (-1)^{|c_1| + |c_2| + |c_1|} x \otimes c_1 \otimes \pi(c_1)\pi(c_2)y.
\]

by \([\text{11}]\). So \(d_\pi^2\) is in fact 0. In the augmented case, we can arrive at the other two twisted tensor products \(C \otimes_A A\) and \(A \otimes_A C\) by applying the functors \(k \otimes_A -\) and \(- \otimes_A k\) to the bimodule complex \(A \otimes_A A \otimes_A A\). So we see that the differentials on these complexes are, in fact, also square zero.
5. Koszul Resolutions Via Twisting Cohains

In this section we give a presentation of Koszul resolutions based on the work of Keller and Lefèvre-Hasegawa. The original presentation, in the case that $A$ is graded Koszul, appears in [13, Section 4.7] and [12].

Let $A$ be a filtered Koszul algebra and $\Lambda = (\Lambda, d, c, \alpha)$ be its Koszul dual (curved) dg algebra of Proposition 3.7. Let $\text{gr}A = k\langle V \rangle/(R)$ and $\Lambda = k\langle V^* \rangle/(R^*)$ be dual minimal presentations, and let $\{x_i\}$ and $\{\lambda_i\}$ be dual bases for $V$ and $V^*$ respectively. Recall the intersection coalgebra $C = \cdots \oplus (R \otimes V \otimes R) \oplus R \oplus V \oplus k$ of Section 3.2. The following Lemma was essentially proved in [25].

**Lemma 5.1.** There is a natural curved dg structure on $C$ given by $f_C = \alpha_0$ and $d_C|C^{-2} = \alpha_1$ so that the identification $\Lambda = C^*$ is one of curved dg algebras.

In the statement of the above lemma we are using the fact that $C^{-2} = R$. The definitions of the $\alpha_i$ are recalled in the first lines of the proof.

**Proof.** Recall from (9) that we have presentations

$$\text{gr}A = k\langle V \rangle/(R) \text{ and } A = k\langle V \rangle/(r + \alpha_1(r) + \alpha_0(r))_{r \in R},$$

after choosing some embedding $V \to F_1A$.

Let us suppose that such a curved dg coalgebra structure on $C$ exists, i.e. one satisfying $f_C = \alpha_0$ and $d_C|C^{-2} = \alpha_1$. Then the induced curved dg structure on the chain dual $C^*$ will be given by

$$d_{C^*}|C^1 : f \mapsto (f \mapsto (-1)^{|f|} f d_C = f \alpha_1$$

and $c_{C^*} = f_C = \alpha_0$. Since $d_{C^*}$ and $d_\Lambda$ agree on the generators, they agree on all of $\Lambda = C^*$. So $\Lambda = C^*$ as a curved dg algebra. To see that a curved dg structure on $C$ satisfying the proposed conditions actually exists, one can verify that the graded coalgebra isomorphism $ev : C \to C^{**} = \Lambda^*$ induces the proposed curved dg structure on $C$.

**Lemma/Definition 5.2** (The twisting cochain $\pi$). Let $\pi : \Lambda^* \to A$ be the composition of the projection $\Lambda^* \to \Lambda^{*-1} = V^{**}$ with the inclusion $V^{**} \to A$, $ev_{v} \mapsto v$. The map $\pi : \Lambda^* \to A$ is a twisting cochain.

**Proof.** Let us view $\pi$ as a map $\pi : C \to A$ via the evaluation isomorphism $ev : C \cong C^{**} = \Lambda^*$. Under this identification, $\pi$ is defined on degree 1 as the embedding $C^1 = V \to A$ given by the presentation (9). We need to verify the formula

$$-f_C - \pi d_C + \mu(\pi \otimes \pi) \Delta = 0.$$

It suffices to check that the above equation holds when evaluated at a homogeneous degree $-2$ element in $C$, since the left hand side vanishes on elements of all other degrees. Recalling that $C^{-2} = R$, we evaluate on a relation $r = \sum_i r_i \otimes r_i'$ to get

$$(-f_C - \pi d_C + \mu(\pi \otimes \pi) \Delta)(r) = -f_C(r) - \pi d_C(r) + \mu(\pi(r) \otimes \pi(1) + \pi(1) \otimes \pi(r) - \sum_i \pi(r_i) \otimes \pi(r_i')) = -a_0(r) - \pi(\alpha_1(r)) - \sum_i r_i r_j = -a_0(r) - \alpha_1(r) + \alpha_1(r) + \alpha_0(r) = 0.$$
One can use \cite[Proposition 2.2.4.1]{10} to show that the twisted tensor product
\[ A \otimes \pi \Lambda^* \otimes \pi A \]
provides a resolution for \( A \Lambda A \). Alternatively, in the case where \( A \) is graded Koszul the above resolution is easily seen to recover the standard Koszul resolution \cite[proof of Proposition 3.3]{11}, \cite[Section 4.7]{12}. In general, one can employ the filtration
\[ F_i(A \otimes \pi \Lambda^* \otimes \pi A) = \sum_{i_1 + i_2 + i_3 = i} F_{i_1}A \otimes (\Lambda^*)^{-i_2} \otimes F_{i_3}A \]
and an easy spectral sequence argument to see that \( H^{<0}(A \otimes \pi \Lambda^* \otimes \pi A) = 0 \). The fact that \( H^0(A \otimes \pi \Lambda^* \otimes \pi A) = A \) is apparent.

**Notation 5.3** (The Koszul resolution \( K \)). If \( A \) is filtered Koszul we write \( K = K(A) \) for the bimodule resolution \( A \otimes \pi \Lambda^* \otimes \pi A \) of \( A \). Elements in \( K \) will often be denoted \( x \otimes \phi \otimes y \), where \( x, y \in A \) and \( \phi \in \Lambda^* \).

6. A Complex Calculating \( HH^*(A, M) \)

Let \( A \) be filtered Koszul with Koszul dual (curved) dg algebra \( \Lambda \). Recall our dual presentations \( \text{gr} A = k(V)/(R) \) and \( \Lambda = k(V^*)/(R^\perp) \) and dual bases \( \{x_i\} \) and \( \{\lambda_i\} \) for \( V \) and \( V^* \) respectively.

**Definition 6.1** (The identity element). For \( A \) and \( \Lambda \) as above, we take \( e := \sum_i \lambda_i \otimes x_i \). We call it the identity element in \( \Lambda \otimes A \).

To see that \( e \) is invariant under change of basis we can note that \( e \) is the preimage of the identity map under the canonical isomorphism \( V^* \otimes V \to \text{Hom}_k(V, V) \), \( f \otimes x \mapsto (y \mapsto f(y)x) \). (Whence the name “identity element”.) Alternatively, we can find \( e \) as the image of \( 1 \in k \) under the standard coevaluation map \( k \to V^* \otimes V \) that appears in studies of tensor categories.

Let \( M = (M, d_M) \) be a dg \( \Lambda \)-bimodule. That is, a chain complex and \( \Lambda \)-bimodule for which the structure map \( \Lambda \otimes M \to M \) is a chain map. Given \( \phi \in M^* \) and \( f \in \Lambda \) we define the left and right \( \Lambda \)-actions
\[ f \cdot \phi := (m \mapsto (-1)^{|\phi||f|} |f||m| \phi(mf)) \]
and
\[ \phi \cdot f := (m \mapsto \phi(fm)). \]

These actions are compatible with the standard differential on the chain dual \( M^* = \text{Hom}_k(M, k) \) and give it the structure of a dg \( \Lambda \)-bimodule. If \( M \) is locally finite, then the natural graded isomorphism \( M \to M^{**} \), \( m \mapsto ev_m \), is an isomorphism of dg bimodules. We let \([,]\) denote the graded commutator \([f, m] := fm - (-1)^{|f||m|}mf\).

The graded space \( A \otimes \Lambda \otimes A \) is given the (non dg) \( \Lambda \otimes A \)-bimodule structure induced by the inner \( A \)-bimodule structure on \( A \otimes A \) and the regular bimodule structure on \( \Lambda \).

**Lemma 6.2.** Let \( A \) be filtered Koszul with Koszul dual (curved) dg algebra \( \Lambda \). The differential \( d_\pi \) on \( K = A \otimes \pi \Lambda^* \otimes \pi A \) is the operation \( d_{A \otimes \Lambda^* \otimes A} - [e, -] \). In terms of the dual bases \( \{x_i\} \) and \( \{\lambda_i\} \), \( d_\pi \) is the map
\[ x \otimes \phi \otimes y \mapsto x \otimes d_{\Lambda^*}(\phi) \otimes y - \sum_i x \otimes \lambda_i \phi \otimes x_i y - (-1)^{|\phi|} x x_i \otimes \phi \lambda_i \otimes y. \]
Proof. Since we have a basis \( \{ e_i \} \) for \((\Lambda^*)^{-1}\), for any homogeneous \( \phi \in \Lambda^* \) we can write the coproduct \( \Delta(\phi) \) as
\[
\Delta(\phi) = \phi_1 \otimes \phi_2 \\
= \sum_i e_i \otimes \eta_i \quad \text{mod} \quad \left( (\Lambda^*)^{|\phi|-1} \otimes \Lambda^* \right) \\
= \sum_i \xi_i \otimes e_i \quad \text{mod} \quad \left( \Lambda^* \otimes (\Lambda^*)^{|\phi|-1} \right).
\]
In particular,
\[
(\pi \otimes id)\Delta(\phi) = \sum_i x_i \otimes \eta_i
\]
and
\[
(id \otimes \pi)\Delta(\phi) = \sum_i \left(-1\right)^{|\phi|+1} \xi_i \otimes x_i,
\]
where the \( x_i \) are now degree 0 elements in \( A \). The sign \( (-1)^{|\phi|+1} \) appearing in the second expression comes from the fact that \((id \otimes \pi)(\xi_i \otimes x_i) = (-1)^{|\xi_i|}\xi_i \otimes x_i \) and that \( |\xi_i| = |\phi| + 1 \). So the differential \( d_\pi \) on the twisted tensor product becomes
\[
d_\pi : x \otimes \phi \otimes y \mapsto x \otimes d_\Lambda^*(\phi) \otimes y + \sum_i x x_i \otimes \eta_i \otimes y + \left(-1\right)^{|\phi|} x \otimes \xi_i \otimes x_i y. \tag{12}
\]
We have
\[
\phi \cdot \lambda_i(f) = \phi(\lambda_i f) \\
= \phi \mu(\lambda_i \otimes f) \\
= \Delta(\phi)(\lambda_i \otimes f) \\
= \left( \sum_j e_{x_j} \otimes \eta_j \right)(\lambda_i \otimes f) \\
= (-1)^{|\phi|+1} e_{x_i}(\lambda_i) \eta_i(f) \\
= -(-1)^{|\phi|+1} \lambda_i(x_i) \eta_i(f) \\
= -(-1)^{|\phi|+1} \eta_i(f),
\]
and
\[
\lambda_i \cdot \phi(f) = (-1)^{|\phi|+|f|} \phi(f \lambda_i) \\
= (-1)^{|\phi|+|f|} \Delta(\phi)(f \otimes \lambda_i) \\
= -(-1)^{|\phi|} \phi(f).
\]
Whence the expression \( \text{(12)} \) can be rewritten in the desired form
\[
d_\pi : x \otimes \phi \otimes y \mapsto x \otimes d_\Lambda^*(\phi) \otimes y - \sum_i x \otimes \lambda_i \phi \otimes x_i y + (-1)^{|\phi|} x x_i \otimes \phi \lambda_i \otimes y.
\]
This gives the equality \( d_\pi = d_{\Lambda \otimes \Lambda^*} - [e] \).

For any \( A \)-bimodule \( M \) we let \( \Lambda \otimes A \) act on the left and right of \( \Lambda \otimes M \) in the obvious way. That is, we let \( \Lambda \) act on itself and let \( A \) act on \( M \) independently.

Lemma 6.3. For an arbitrary \( A \)-bimodule \( M \) there is a natural graded isomorphism
\[
\Lambda \otimes M \cong \text{Hom}_{\Lambda^*}(K, M) \tag{13}
\]
\[
\text{taking any } f \otimes m \in \Lambda \otimes M \text{ to the function }
K \rightarrow M \\
x \otimes \phi \otimes y \mapsto ev_f(\phi)_{xmy}.
\]
The differential on \( \Lambda \otimes M \) induced by this isomorphism is \( d_{\Lambda \otimes M} = [e, -] \).
Proof. It is apparent that (13) is an isomorphism. The differential on the Hom complex \( \text{Hom}_{A^e}(K, M) \) applied to the image of an element \( f \otimes m \in \Lambda \otimes M \) produces the function

\[
x \otimes \phi \otimes y \mapsto (1)^{|f|} ev_f(d(\phi)) xmy - (1)^{|f|} \left( \sum_i (ev_f(\lambda_i \phi)xmx_iy - (1)^{|f|} ev_f(\phi \lambda_i) xx_iy) \right).
\]

This expression reduces to

\[
x \otimes \phi \otimes y \mapsto d(ev_f(\phi)) xmy - \left( \sum_i \lambda_i ev_f(\phi)xmx_iy - (1)^{|f|} ev_f(\phi \lambda_i) xx_iy \right).
\]

Since the natural isomorphism \( \Lambda \rightarrow \Lambda^{**} \), \( f \mapsto ev_f \), is one of dg \( \Lambda \)-bimodules, the above calculation shows that the differential on \( \Lambda \otimes M \) induced by the isomorphism (13) is

\[
f \otimes m \mapsto d(f) \otimes m - \left( \sum_i \lambda_i f \otimes x_i m - (1)^{|f|} f \lambda_i \otimes mx_i \right).
\]

\[\Box\]

**Definition 6.4** (The functor \( \Lambda \tilde{\otimes} - \)). Let \( A \) be filtered Koszul. We define the functor

\[
\Lambda \tilde{\otimes} : A\text{-bimod} \rightarrow k\text{-complexes}
\]

to be the one sending a bimodule \( M \) to \( (\Lambda \otimes M, d_{\Lambda \otimes M} - [e, -]) \), and a bimodule map \( \eta : M \rightarrow N \) to \( id_{\Lambda} \otimes \eta : \Lambda \otimes M \rightarrow \Lambda \otimes N \).

Lemma 6.3 tells us that there is a natural isomorphism of functors \( \Lambda \tilde{\otimes} \rightarrow \text{Hom}_{A^e}(K, -) \).

Recall that there is an \( A^e \)-complex structure on \( \text{Hom}_{A^e}(K, A^e) \) induced by the inner bimodule structure on \( A^e = A \otimes A^{op} \). This bimodule structure induces a bimodule structure on \( HH^\bullet(A, A^e) \). The following result provides a slight generalization of [31, Theorem 9.1] to allow for filtered, not just graded, Koszul algebras.

**Proposition 6.5.** Let \( A \) be filtered Koszul and \( \Lambda \) be its Koszul dual (curved) dg algebra. The complex

\[
\Lambda \tilde{\otimes} M = (\Lambda \otimes M, d_{\Lambda \otimes M} - [e, -])
\]

calculates \( HH^\bullet(A, M) \). If we take \( M = A^e \), the above complex calculates the \( A \)-bimodule structure on \( HH^\bullet(A, A^e) \).

Proof. The previous lemma tells us that the complex \( \Lambda \tilde{\otimes} M \) calculates \( HH^\bullet(A, M) \). When \( M = A^e \), the natural isomorphism (13) is one of \( A^e \)-complexes. So the second claim is clear.

\[\Box\]

**Remark 6.6.** The definition of the differential on \( K = A \otimes \Lambda^* \otimes A \) given in Lemma 6.2 looks very similar to the standard Koszul resolution defined in [31, Section 3], modulo some signs. The two resolutions are, in fact, exactly the same. The superficial differences are entirely accounted for by our choice to employ signs when commuting graded variables.
6.1. Example continued: PBW deformations of skew polynomials. Take $k = \mathbb{C}$. Consider $kQ[Q_{12}, Q_{23}, Q_{31}]$ (from Subsection 3.6) with skewing constants $q_{12} = q_{23} = q_{31} = q$ for some $q \neq 0$. This algebra has a particular deformation

$$
\tilde{A}_q = k\langle x_1, x_2, x_3 \rangle / (x_i x_j - qx_j x_i - x_k),
$$

where the indices $\{i, j, k\}$ are taken to be cyclically ordered in the relations. In the case that $q = 1$ this algebra is isomorphic to the universal enveloping algebra $A_1 = U(sl_2)$ of the Lie algebra $sl_2$, and when $q = -1$ the algebra is isomorphic to the universal enveloping algebra $U(sl_2^q)$ of the color Lie algebra $sl_2^q$ [3 Section 5]. It was shown in [3] that the category of finite dimensional representation of $U(sl_2^q)$ is semisimple, as is the case with $U(sl_2)$. And so we know a bit about its cohomology of $A_q$ at these points.

Let us suppose now that $q \neq 0, 1$ (so $\tilde{A}_q \neq U(sl_2)$) and employ the isomorphic presentation

$$
A_q = k\langle x_1, x_2, x_3 \rangle / (x_i x_j - qx_j x_i - (1 - q)x_k),
$$

where the $\{i, j, k\}$ are cyclically ordered. The algebra $A_q$ has two specific 1-dimensional representation $p_0$ and $p_1$, which are both $k$ as a vector space with left actions $x_i \cdot 1 = 0$ and $x_i \cdot 1 = 1$ respectively. We also have the one dimensional bimodule $p_{10} = \text{Hom}_k(p_0, p_1)$ which is $k$ as a vector space with left and right actions $x_i \cdot 1 = 1$ and $1 \cdot x_i = 0$ respectively.

Recall that we have an isomorphism $\text{Ext}_{A_q}(p_0, p_1) = HH^*(A_q, p_{10})$ [34 Lemma 9.1.9]. Recall also that $\Lambda$ is the dg algebra $\Lambda^{Q_q}(\lambda_1, \lambda_2, \lambda_3)$ with differential

$$
d_\Lambda : \lambda_i \mapsto (1 - q)\lambda_j \lambda_k, \text{ where } \{i, j, k\} \text{ is cyclically ordered}
$$

(see Subsection 3.6). This algebra is concentrated in degrees 0 to 3, and $\dim \Lambda^0 = \dim \Lambda^3 = 1$ while $\dim \Lambda^1 = \dim \Lambda^2 = 3$.

One can check $d_A^3 = d_A^2 = 0$. So, according to Proposition 6.3, we can calculate $\text{Ext}_{A_q}(p_0, p_1)$ as the homology of the complex $\Lambda \otimes p_{10}$ with differential

$$
d^0(1 \otimes 1) = -(\sum_i \lambda_i \otimes x_i \cdot 1 - \lambda_i \otimes 1 \cdot x_i)
$$

$$
= -(\lambda_1 \otimes 1 + \lambda_2 \otimes 1 + \lambda_3 \otimes 1)
$$

$$
d^1(\lambda_i \otimes 1) = d_\Lambda(\lambda_i) \otimes 1 - \lambda_j \lambda_k \lambda_i \otimes 1
$$

$$
= (1 - q)\lambda_j \lambda_k \lambda_i \otimes 1 + q\lambda_i \lambda_j \lambda_k \otimes 1 - \lambda_k \lambda_i \lambda_j \otimes 1
$$

$$
d^2(\lambda_i \lambda_j \otimes 1) = \lambda_k \lambda_i \lambda_j \otimes 1,
$$

where again $\{i, j, k\}$ is taken to be cyclically ordered. The map $d^0$ is injective while the map $d^2$ is surjective. So, $\text{Ext}_{A_q}^1(k, k) = \text{Ext}_{A_q}^2(k, k) = 0$ if and only if the map $d^1$ is rank 2.

In the ordered bases $\{\lambda_1, \lambda_2, \lambda_3\}$ and $\{\lambda_2 \lambda_3, \lambda_3 \lambda_1, \lambda_1 \lambda_2\}$ the differential $d^1$ is given by the matrix

$$
d^1 = \begin{bmatrix}
(1 - q) & 0 & -1 \\
-1 & (1 - q) & q \\
q & -1 & (1 - q)
\end{bmatrix}.
$$

This matrix is of rank 1 or 2, and is rank 2 if and only if $q$ solves the equation $q^2 - q + 1 = 0$, i.e. if and only if $q = e^{\pm \pi/3}$. So

$$
\text{Ext}_{A_q}^1(p_0, p_1) = k \text{ when } q = e^{\pm \pi/3} \text{ and } \text{Ext}_{A_q}^1(p_0, p_1) = 0 \text{ otherwise.}
$$

Whence we conclude that there exists some non-split extension $0 \to p_1 \to M \to p_0 \to 0$ exactly when $q = e^{\pm \pi/3}$. 

proof. The natural isomorphism identifies the product on $\Lambda\otimes A$. The fact that $\Lambda\otimes B$ is an algebra extension of $A$, i.e. an algebra $B$ with $A$-bimodule structure induced by an algebra map $A \to B$. It is simply the sum of an inner derivation and the tensor derivation $d_{\Lambda\otimes B}$, and sums of derivations are derivations. Therefore the above complex is a dg algebra. In particular, $\Lambda\otimes A$ is a dg algebra.

**Notation 7.1.** Let $\mathcal{C}(A)$ denote the dg algebra $\Lambda\otimes A$. Elements in $\mathcal{C}(A)$ will generally be denoted $f \otimes a$, where $f \in \Lambda$ and $a \in A$.

Freeness of $K$ over $A^*$ allows us to also identify $\text{Hom}_{A^*}(K, A)$ with the set of graded homs $\text{Hom}_k(\Lambda^*, A)$, as a graded vector space. Note that, since $\Lambda^*$ is a graded coalgebra, $\text{Hom}_k(\Lambda^*, A)$ is an algebra under the convolution product.

**Proposition 7.2.** The product on $\text{Hom}_{A^*}(K, A)$ induced by the natural isomorphism $\text{Hom}_{A^*}(K, A) \cong \text{Hom}_k(\Lambda^*, A)$ takes an element $\theta \otimes \eta \in \text{Hom}_{A^*}(K, A) \otimes \text{Hom}_{A^*}(K, A)$ to the function

$$\theta \ast \eta = \left( x \otimes \phi \otimes y \mapsto (-1)^{||\phi||||\eta||} x\theta(\phi_1)\eta(\phi_2)y \right).$$

This product gives $\text{Hom}_{A^*}(K, A)$ the structure of a dg algebra.

**Proof of Product Formula.** The natural isomorphism $\text{Hom}_{A^*}(K, A) \cong \text{Hom}_k(\Lambda^*, A)$ sends a function $\theta$ to its restriction $\theta|\Lambda^*$. The inverse map sends an element $\tau \in \text{Hom}_k(\Lambda^*, A)$ to the function $x \otimes \phi \otimes y \mapsto x\tau(\phi)y$. So the product $\theta \ast \eta$ is the function

$$x \otimes \phi \otimes y \mapsto (-1)^{||\phi||||\eta||} x\theta(\phi_1)\eta(\phi^*_2)y = (-1)^{||\eta||||\phi||} x\theta(\phi_1)\eta(\phi_2)y.$$

The product on $\text{Hom}_{A^*}(K, A)$ described above will (still) be called the convolution product and will be denoted $\ast$. The fact that $\text{Hom}_{A^*}(K, A)$, equipped with the convolution product, is a dg algebra will follow from the first portion of the next lemma and the fact that $\mathcal{C}(A)$ is a dg algebra.

**Lemma 7.3.** The natural isomorphism $\mathcal{C}(A) \cong \text{Hom}_{A^*}(K, A)$ of Lemma 6.2 identifies the product on $\mathcal{C}(A)$ with the convolution product on $\text{Hom}_{A^*}(K, A)$. Therefore, it is an isomorphism of dg algebras.

**Proof.** Let $f \otimes a$ and $g \otimes b$ be elements in $\mathcal{C}(A)$. By abuse of notation we let $f \otimes a$ and $g \otimes b$ denote their images in $\text{Hom}_{A^*}(K, A)$ as well. For any $\phi \in \Lambda^* \subset K$ we have

$$(f \otimes a) \ast (g \otimes b)(\phi) = (-1)^{||\phi||||g||} ev_f(\phi_1) \cdot a \cdot ev_g(\phi_2) \cdot b$$

$$= (-1)^{||\phi||||g||} ev_f(\phi_1)ev_g(\phi_2)ab$$

$$= ev_{fg}(\phi)ab$$

$$= (fg \otimes ab)(\phi).$$

Since the functions $(f \otimes a) \ast (g \otimes b)$ and $fg \otimes ab$ agree on the generators $\Lambda^*$ they agree on $K$. ■
Theorem 7.4. The map \( \sigma : \mathcal{C}(A) \cong \text{Hom}_{A^e}(K, A) \rightarrow \text{Hom}_{A^e}(K, K) \) defined by

\[
\theta \mapsto (x \otimes \phi \otimes y \mapsto (-1)^{|\phi| |\theta|} x \otimes \phi_1 \otimes \theta(\phi_2)y)
\]

is a quasi-isomorphism of dg algebras.

Proof. First, let us verify that \( \sigma \) is a map of chain complexes. Since all maps will be \( A^e \)-linear, it suffices to show equality of functions on the generating subspace \( \Lambda^* \subset K \). For simplicity of notation let us first assume that \( A \) is graded Koszul. Equivalently, we are assuming \( d_A = 0 \).

Let \( \theta \) be in \( \text{Hom}_{A^e}(K, A) \) and let \( \phi \) be an arbitrary element on \( \Lambda^* \). We have

\[
\sigma(d(\theta))(\phi) = (-1)^{\omega_1} 1 \otimes \phi_1 \otimes \pi(\phi_2) \theta(\phi_3) + (-1)^{\omega_2} 1 \otimes \phi_1 \otimes \theta(\phi_3) \pi(\phi_2) \quad (14)
\]

and

\[
d(\sigma(\theta))(\phi) = d_K(\sigma(\theta))(\phi) + (-1)^{|\theta|+1} \sigma(\theta)(d_K(\phi))
\]

\[
= (-1)^{\chi_1} \phi_1 \otimes \phi_2 \otimes \theta(\phi_3) + (-1)^{\chi_2} \phi_1 \otimes \pi(\phi_2) \theta(\phi_3) + (-1)^{\chi_3} \phi_1 \otimes \pi(\phi_2) \theta(\phi_3) + (-1)^{\chi_4} \phi_1 \otimes \theta(\phi_2) \pi(\phi_3) \quad (15)
\]

The exponents are

\[
\begin{align*}
\omega_1 &= |\theta| + 1 + (|\theta| + 1)|\phi_1| \\
\omega_2 &= |\theta| + 1 + (|\theta| + 1)|\phi_1| + |\phi_2| + 1 \\
\chi_1 &= |\theta|(|\phi_1| + |\phi_2|) \\
\chi_2 &= |\phi_1| + 1 + |\theta|(|\phi_1| + |\phi_2|) \\
\chi_3 &= |\theta| + 1 + |\theta||\phi_2| \\
\chi_4 &= |\theta| + 1 + |\theta||\phi_1| + (|\phi_1| + |\phi_2|) + 1.
\end{align*}
\]

Since \( \pi(\phi) = 0 \) whenever \( |\phi_{\ell}| \neq 1 \) we can replace \( \chi_1 \) and \( \chi_2 \) with

\[
\begin{align*}
\chi_1 &= |\theta|(1 + |\phi_2|) \\
\chi_2 &= |\phi_1| + 1 + |\theta|(|\phi_1| + 1).
\end{align*}
\]

Now \( \chi_1 + 1 = \chi_3, \chi_2 = \omega_1, \chi_4 = \omega_2 \), and the equality \( \sigma(d(\theta))(\phi) = d(\sigma(\theta))(\phi) \) becomes clear. This verifies that \( \sigma \) is a chain map.

In the case that \( A \) is filtered Koszul, and \( d_A \) is not necessarily vanishing, we must add the term

\[
(-1)^{|\theta|+1+(|\theta|+1)|\phi_1|} 1 \otimes \phi_1 \otimes \theta(d_A(\phi_2))
\]

to equation (14) and the term

\[
(-1)^{|\phi_1|+1+|\theta|(|\phi_1|+|\phi_2|)} 1 \otimes d_A(\phi_1) \otimes \theta(\phi_2)
\]

\[
+ (-1)^{|\theta|+1+(|\theta|+1)|\phi_1|} 1 \otimes d_A(\phi_1) \otimes \theta(\phi_2)
\]

\[
+ (-1)^{|\theta|+1+(|\theta|+1)|\phi_1|} 1 \otimes \phi_1 \otimes \theta(d_A(\phi_2))
\]

to equation (15). Here we have used the fact that \( d_A \) is a coderivation so that

\[
\Delta(d_A(\phi)) = d_A(\phi_1) \otimes \phi_2 + (-1)^{|\phi_1|} \phi_1 \otimes d_A(\phi_2).
\]

The second correction term \( 17 \) reduces to give \( 16 = 17 \). So \( \sigma \) is still a chain map even when we allow \( A \) to be filtered.

For \( \theta \) and \( \eta \) in \( \text{Hom}_{A^e}(K, A) \) the composition \( \sigma(\theta) \sigma(\eta) \) is the map

\[
\phi \mapsto (-1)^{|\theta|+|\phi_1|+|\theta||\phi_1|+|\eta||\phi_1|+|\phi_2|} 1 \otimes \phi_1 \otimes \theta(\phi_2) \eta(\phi_3),
\]

which is exactly equal to \( \sigma(\theta \ast \eta) \). So \( \sigma \) is an algebra map. To see that \( \sigma \) is a quasi-isomorphism simply note that it provides a section for the quasi-isomorphism \( \text{Hom}_{A^e}(K, K) \cong \text{Hom}_{A^e}(K, A) \) induced by the quasi-isomorphism of \( A^e \)-complexes \( K \cong A \).
The following corollary is immediate.

**Corollary 7.5.** The homology algebra of the dg algebra
\[ C(A) = (\Lambda \otimes A, d_{\Lambda \otimes A} - [e, -]) \]
is the Hochschild cohomology $HH^\cdot(A)$ equipped with the cup product.

**Remark 7.6.** It is a perfectly reasonable exercise to prove directly that $\text{Hom}_{\Lambda \otimes A}(K, A)$, equipped with the convolution product, is a dg algebra. Therefore Theorem 7.4 can be proved without any reference to $C(A)$ or the identity element $e$.

**Remark 7.7.** Let $C\cdot(\Lambda)$ denote the Hochschild cochain complex for $A$ as defined in [7, Section 5.2]. Theorem 7.4 can alternately be proved by showing that the quasi-isomorphism $C\cdot(\Lambda) \to \text{Hom}_{\Lambda \otimes A}(K, A)$ dual to the canonical embedding of $K$ into the bar resolution for $A$ (see [26, Proposition 3.9]) maps the cup product of elements in $C\cdot(\Lambda)$ to the convolution product of their images.

### 8. Hochschild Cohomology of Localizations

Let $A$ and $\Lambda$ be as in Section 5. We will say an algebra $B$ is a flat extension of $A$ if $B$ comes equipped with an algebra map $A \to B$ and it is flat over $A$ on the left and right independently. Note that we always have a surjective $B$-bimodule map $B \otimes A B \to B$ given by the multiplication on $B$. Recall that $K = A \otimes \Lambda^* \otimes \Lambda$ denotes the Koszul resolution of $A$ and $e$ denotes the identity element in $A \otimes \Lambda$.

**Lemma 8.1.** Suppose $B$ is a flat extension of $A$ and that the map $B \otimes A B \to B$ is an isomorphism. Then the complex $B \otimes_A K \otimes A B$ is a projective $B$-bimodule resolution of $B$.

**Proof.** The complex $B \otimes_A K \otimes A B$ will be free over $B^e$, since $K$ is free over $A^e$. Since the functors $B \otimes A -$ and $- \otimes_A B$ are exact they commute with homology. Hence $B \otimes_A K \otimes A B$ will have homology
\[ B \otimes_A H(K) \otimes_A B = B \otimes_A A \otimes_A B = B \otimes_A B = B. \]

It is clear that $B \otimes_A K \otimes A B$ is the complex
\[ (B \otimes \Lambda^* \otimes B, d_{B \otimes \Lambda^* \otimes B} - [e, -]). \]

For simplicity we will denote this complex by $K^B$. The proofs of the next results are the same as those of Lemma 6.3, Proposition 6.5, and Theorem 7.4, and will be omitted.

**Proposition 8.2.** For $B$ as in the previous lemma and any $B$-bimodule $M$, there is a natural isomorphism
\[ (\Lambda \otimes M, d_{\Lambda \otimes M} - [e, -]) \xrightarrow{\cong} \text{Hom}_{B^e}(K^B, M). \]

Consequently, the above complex calculates $HH^\cdot(B, M)$. In the case that $M = B^e$ this complex calculates the bimodule structure on $HH^\cdot(B, B^e)$.

The isomorphism [18] is defined in the same way as [13]. We will let $C(B)$ denote the dg algebra $(\Lambda \otimes B, d_{\Lambda \otimes B} - [e, -])$. As was the case previously, $\text{Hom}_{B^e}(K^B, B) \cong \text{Hom}_k(\Lambda^*, B)$ comes equipped with a convolution product and the isomorphism $C(B) \cong \text{Hom}_{B^e}(K^B, B)$ is one of dg algebras.
Theorem 8.3. The map $\sigma : \mathcal{C}(B) \cong \text{Hom}_{B^e}(K^B, B) \to \text{Hom}_{B^e}(K^B, K^B)$ defined by
\[ \theta \mapsto \left( x \otimes \phi \otimes y \mapsto (-1)^{1+|\theta|} x \otimes \phi_1 \otimes \theta(\phi_2)y \right) \]
is a quasi-isomorphism of dg algebras, and $H(\mathcal{C}(B)) = HH^\bullet(B)$ as an algebra.

Of particular interest is the case in which $B$ is an Ore localization of $A$ with respect to some denominator set $S \subset A$, as defined in [S Ch. 10]. In this case we will write $B = AS^{-1}$. If $A$ is commutative, Ore localization is simply the standard localization. However, in the case that $A$ is noncommutative we need to place some normality conditions on the elements of our multiplicative set $S$ in order to form $AS^{-1}$. The localization $AS^{-1}$ is flat over $A$ [S Corollary 10.13] and it is easy to check that $AS^{-1} \otimes_A AS^{-1} \to AS^{-1}$ is an isomorphism.

Before giving the next result let us note that, in the case that $A$ is commutative, $\mathcal{C}(A)$ is a left $A$-module. In fact, the dg algebra map $A = \mathcal{C}(A)^0 \to \mathcal{C}(A)$ makes it into a dg $A$-algebra. Hence the homology $HH^\bullet(A)$ is an $A$-algebra as well.

Corollary 8.4. Let $S \subset A$ be a denominator set. Then
\[ \mathcal{C}(AS^{-1}) = (A \otimes AS^{-1}, d_{A \otimes AS^{-1}} - [e, -]) \]
is a dg algebra calculating the Hochschild cohomology algebra $HH^\bullet(AS^{-1})$. In the case that $A$ is commutative $HH^\bullet(AS^{-1}) = AS^{-1} \otimes_A HH^\bullet(A)$ as an algebra.

Proof. The first statement is clear. In the commutative case we have $AS^{-1} \otimes_A \mathcal{C}(A) = \mathcal{C}(AS^{-1})$ and flatness of the localization gives
\[ HH^\bullet(AS^{-1}) = H(\mathcal{C}(AS^{-1})) = H(AS^{-1} \otimes_A \mathcal{C}(A)) = AS^{-1} \otimes_A HH^\bullet(A). \]

This final statement about commutative algebras can be deduced more directly from the fact that hom functors commute with flat base change for commutative algebras. Although we do not seek to make the following notion explicit, the dg algebra $\mathcal{C}(AS^{-1})$, along with the obvious dg algebra map $\mathcal{C}(A) \to \mathcal{C}(AS^{-1})$, can be thought of as a localization of $\mathcal{C}(A)$ with respect to the set $S \subset A \subset \mathcal{C}(A)$.

9. A REMARK ON THE $A_\infty$ STRUCTURE ON HOCHSCHILD COHOMOLOGY

Definition 9.1 ($A_\infty$ algebra). An $A_\infty$ algebra is a graded space $\Sigma = \oplus_i \Sigma^i$ equipped with operations
\[ m_n : \Sigma^{\otimes n} \to \Sigma, \]
for all $n \geq 1$, of respective degrees $2 - n$, satisfying the equations
\[ 0 = \sum_{r+s+t=n} m_{n-s+1}(id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) \]
for all $n > 0$.

Some introductory notes on $A_\infty$ algebras can be found in [11, 13, 17]. We refer the reader to these articles for basic results and detailed references. Some applications and calculations of $A_\infty$ structures appear in [14, 18, 19].

A standard fact, due to Kadeishvili, states that the homology of any dg algebra has a unique $A_\infty$ structure (up to non-unique isomorphism). Therefore any Ext
algebra $\text{Ext}_B(M, M)$, for any algebra $B$ and $B$-module $M$, has an $A_\infty$ structure. This follows from the fact that, if we let $P$ be a projective resolution of $M$, we have $\text{Ext}_B(M, M) = H(\text{Hom}_B(P, P))$. The $A_\infty$ structure on $\text{Ext}_B(M, M)$ is independent of our choice of resolution $P$. Furthermore, it lifts the algebra structure on $\text{Ext}_B(M, M)$ in the sense that $m_2$ is simply the Yoneda product. Another standard fact is that a quasi-isomorphism of dg algebras induces an isomorphism of $A_\infty$ algebras on their homologies. So it is clear that Theorem 8.3 can be used to prove a slightly stronger result than was given.

**Theorem 9.2.** Let $A$ be a filtered Koszul algebra and $S$ be a denominator set in $A$. Then the dg algebra

$$\mathcal{E}(AS^{-1}) = (\Lambda \otimes AS^{-1}, d_{\Lambda \otimes AS^{-1}} - [e, -])$$

calculates the $A_\infty$ structure on $HH^*(AS^{-1})$.

**Remark 9.3.** The fact that the $A_\infty$ structure on an Ext algebra is independent of the choice of resolution does not seem to appear in the standard references. However, any two resolutions $P$ and $P'$ are summands of a third $P''$, and hence produce dg quasi-isomorphisms

$$\text{End}(P) \xrightarrow{\sim} \text{End}(P'') \text{ and } \text{End}(P'') \xrightarrow{\sim} \text{End}(P').$$

The fact that quasi-isomorphisms of dg algebras induce $A_\infty$ algebra isomorphisms on homology then implies $H(\text{End}(P)) \cong H(\text{End}(P''))$.

10. **Example 1: The Heisenberg Lie Algebra**

Suppose $k$ is characteristic 0. Let $U = U(h)$ be the universal enveloping algebra of the Heisenberg Lie algebra $h$. Explicitly, $h$ is the 3 dimensional Lie algebra $h = \langle x_1, x_2, x_3 \rangle$ with $x_3 = [x_1, x_2]$ and $[x_1, x_3] = [x_2, x_3] = 0$. The dg algebra dual to $U$ is the exterior algebra $\Lambda = \bigwedge \langle \lambda_1, \lambda_2, \lambda_3 \rangle$, where the element $\lambda_i$ is dual to $x_i$. The differential on $\Lambda$ sends $\lambda_3$ to $\lambda_1 \lambda_2$ and all other monomials to 0.

It is appropriate to mention, before beginning, that the Hochschild homology of $U(h)$ is calculated in Nuss’ thesis [22, Theorem 3.2, pg. 48], as a vector space. One can then use Van den Bergh’s duality [33] to deduce the vector space structure on Hochschild cohomology. The presentation by Nuss does not look especially similar to the one given here, particularly in degrees 1 and 2. However, the two presentations of $HH^*(U)$ are abstractly isomorphic, simply because they are both of countable dimension in each degree.

In the five subsections below we demonstrate, in detail, the six points of the following theorem.

**Theorem 10.1.** Let $h = \langle x_1, x_2, x_3 \rangle$ denote the Heisenberg Lie algebra and $U = U(h)$ denote its universal enveloping algebra. Take $Z = k[x_3]$. We have the following description of $HH^*(U)$:

1. $HH^0(U) = Z$.
2. $HH^1(U)$ is the direct sum $FH^1 \oplus TH^1$ of a free, infinitely generated, $Z$-module and free, infinitely generated, $Z/(x_3)$-module.
3. $HH^2(U)$ is a free, infinitely generated $Z/(x_3^2)$-module.
4. $HH^3(U) = \lambda_1 \lambda_2 \lambda_3 / U/(x_3)$, i.e. is an infinitely generated $Z/(x_3)$-module.
5. The cup product $HH^1(U) \otimes HH^1(U) \to HH^2(U)$ has image $x_3 \cdot HH^2(U)$.
6. The cup product $HH^1(U) \otimes HH^2(U) \to HH^3(U)$ is surjective.
Note that points (2), (3) and (5) together imply that \( HH^\bullet(U) \) is infinitely generated as a \( \mathcal{Z} \)-algebra with an infinite number of generators in degrees 1 and 2.

Of course, our statement that a \( \mathcal{Z} \)-module is a free \( \mathbb{Z}/(x_3^n) \)-module it to say that it is annihilated by \( x_3^n \) and free after we mod out by the annihilator. The details are given below.

10.1. **The DG Algebra \( \mathcal{E}(U) \) and \( HH^0(U) \).** Since it will simplify our presentation, we simply give the center \( HH^0(U) \) here, before we begin our computations. The proof is omitted, although it can easily be verified, especially from the presentation of \( d^0 \) given below.

**Lemma/Definition 10.2.** The center of \( U \), which we denote by \( \mathcal{Z} \), is the polynomial ring \( k[x_3] \). We have \( HH^0(U) = \mathcal{Z} \).

Note that \([x_1,-]\) is a derivation on \( U \) with \([x_1,x_1] = \delta_2 x_3\). Similarly, \([x_2,x_1] = -\delta_2 x_3\) and \([x_3,-]=0\), since \( x_3 \) is central. Indeed, for a monomial \( a = x_1^{n_1} x_2^{n_2} x_3^{n_3} \) we have

\[
[x_1,a] = n_2 x_1^{n_1} x_2^{n_2-1} x_3^{n_3+1} \text{ whenever } n_2 > 0
\]

and

\[
[x_2,a] = -n_1 x_1^{n_1-1} x_2^{n_2} x_3^{n_3+1} \text{ whenever } n_1 > 0.
\]

So, under the vector space isomorphism \( U = k[x_1,x_2,x_3] \) the operations \([x_1,-]\) and \([x_2,-]\) can be written as the differential operators \( x_3 \partial_2 \) and \(-x_3 \partial_1 \) respectively. As far as computing the homology as a \( \mathcal{Z} \)-module, we identify \( U \) with the simultaneous \( h \)-representation and \( \mathcal{Z} \)-module (or simply \( \mathcal{Z} \otimes h \)-representation) \( k[x_1,x_2,x_3] \) where \( h \) acts by the prescribed operators.

Observe that \( U = k[x_1,x_2,x_3] \) is free over \( \mathcal{Z} \) with basis given by monomials in \( x_1 \) and \( x_2 \). Whence \( \mathcal{E}(U) \) is seen to have a \( \mathcal{Z} \)-basis of monomials \( \{ \lambda^M \otimes x^N \} \), for \( M = (m_1,m_2,m_3) \) and \( N = (n_1,n_2) \) and \( \lambda^M \otimes x^N = \lambda^{m_1} \lambda^{m_2} \lambda^{m_3} \otimes x_1^{n_1} x_2^{n_2} \).

The dg algebra \( \mathcal{E}(U) \) calculating \( HH^\bullet(U) \) is the tensor algebra

\[
\bigwedge (\lambda_1,\lambda_2,\lambda_3) \otimes U = U \bigoplus (\lambda_1,\lambda_2,\lambda_3) \otimes U \bigoplus (\lambda_1 \lambda_3) \otimes U \bigoplus (\lambda_1 \lambda_2 \lambda_3) \otimes U.
\]

The differential is specified on the \( \mathcal{Z} \)-basis \( \lambda^M \otimes x^N \) by

\[
d^0(x^N) = -(\sum_i \lambda_i \otimes x_i x^N - \lambda_i \otimes x^N x_i) \\
= -\lambda_1 \otimes [x_1,x^N] - \lambda_2 \otimes [x_2,x^N] - \lambda_3 \otimes [x_3,x^N] \\
= -\lambda_1 \otimes [x_1,x^N] - \lambda_2 \otimes [x_2,x^N] \\
= -\lambda_1 \otimes x_3 \partial_2(x^N) + \lambda_2 \otimes x_3 \partial_1(x^N).
\]

(19)

in degree 0,

\[
d^1(\lambda_1 \otimes x^N) = -\lambda_2 \lambda_1 \otimes [x_2,x^N] - \lambda_3 \lambda_1 \otimes [x_3,x^N] \\
= -\lambda_1 \lambda_3 \otimes x_3 \partial_1(x^N) \\
= -\lambda_1 \lambda_3 \otimes x_3 \partial_2(x^N)
\]

(20)

\[
d^1(\lambda_2 \otimes x^N) = -\lambda_1 \lambda_2 \otimes [x_1,x^N] \\
= -\lambda_1 \lambda_3 \otimes x_3 \partial_2(x^N) \\
= -\lambda_1 \lambda_3 \otimes x_3 \partial_2(x^N)
\]

\[
d^1(\lambda_3 \otimes x^N) = \lambda_1 \lambda_2 \otimes x^N - \lambda_1 \lambda_3 \otimes [x_1,x^N] - \lambda_2 \lambda_3 \otimes [x_2,x^N] \\
= \lambda_1 \lambda_2 \otimes x^N + \lambda_2 \lambda_3 \otimes x_3 \partial_1(x^N) + \lambda_3 \lambda_1 \otimes x_3 \partial_2(x^N)
\]

in degree 1, and

\[
d^2(\lambda_1 \lambda_2 \otimes x^N) = 0 \\
d^2(\lambda_2 \lambda_3 \otimes x^N) = \lambda_1 \lambda_2 \lambda_3 \otimes x_3 \partial_2(x^N) \\
d^2(\lambda_3 \lambda_1 \otimes x^N) = -\lambda_1 \lambda_2 \lambda_3 \otimes x_3 \partial_1(x^N)
\]

(21)

in degree 2.
10.2. Calculating $HH^1(U)$. In calculating the degree 1 boundaries it will be helpful to have some notation.

**Notation 10.3.** For a polynomial $a$ in $U$, which we are identifying with $k[x_1, x_2, x_3]$, we let $\int_a$ denote the antiderivative of $a$ with respect to $x_1$ with 0 constant term. For example $\int_2 x_1^3 x_2^5 = \frac{1}{6} x_1^3 x_2^6$.

**Lemma 10.4.** An element $\xi = \lambda_1 \otimes a_1 + \lambda_2 \otimes a_2 + \lambda_3 \otimes a_3$ is in $B^1$ if and only if

1. $a_3 = 0$
2. $x_3$ divides $a_2$ and $a_3$
3. $\int_a a_1 - \int a_2 = 0$.

**Proof.** From (12) it is clear that these conditions are necessary in order for $\xi$ to be a boundary, and when they are satisfied we will have $\xi = d^0(\int_1 a_2/x_3)$. \hfill $\blacksquare$

**Lemma 10.5.** Any degree 1 cycle is as sum of the following types of elements:

Type 1) $\lambda_1 \otimes a$, where $a \in \mathbb{Z}k[x_2]$. 
Type 2) $\lambda_2 \otimes a$, where $a \in \mathbb{Z}k[x_1]$. 
Type 3) $\lambda_1 \otimes \int_a a - \lambda_2 \otimes \int_2 a$, where $a \in k[x_1, x_2]$. 
Type 3.5) $\lambda_1 \otimes x_3 \int_a a - \lambda_2 \otimes x_3 \int_2 a$, where $a \in \mathbb{Z}k[x_1, x_2]$. 
Type 4) $\lambda_1 \otimes f x_1 + \lambda_3 \otimes f x_3$, where $f \in \mathbb{Z}$.

**Proof.** First, note that each of these elements is a cycle. Now, let $\xi = \lambda_1 \otimes b_1 + \lambda_2 \otimes b_2 + \lambda_3 \otimes b_3$ be a cycle. Since applying the differential then composing with the projection onto $\lambda_3 \lambda_1 \otimes U + \lambda_2 \lambda_3 \otimes U$ yields the element $\lambda_3 \lambda_1 \otimes x_3 \partial_2(b_3) + \lambda_2 \lambda_3 \otimes x_3 \partial_1(b_3)$

we conclude that $\partial_2(b_3) = \partial_1(b_3) = 0$. That is to say, $b_3$ is in $\mathbb{Z}$. So, by subtracting an element of type 4 we may assume $\xi = \lambda_1 \otimes b_1 + \lambda_2 \otimes b_2 + \lambda_3 \otimes \delta$, where $\delta$ is a unit or 0. Since the image of $\lambda_1 \otimes b_1 + \lambda_2 \otimes b_2$ will be divisible by $x_3$, we conclude that $\delta$ must also be divisible by $x_3$ in order for this element to be a cycle, and hence $\delta = 0$. So we may take $\xi = \lambda_1 \otimes b_1 + \lambda_2 \otimes b_2$.

By further subtracting elements of types 1 and 2 we may assume $b_1$ is divisible by $x_1$ and $b_2$ is divisible by $x_2$. So $\xi = \lambda_1 \otimes (\sum_{i,j \geq 0} \frac{a_{ij}}{i+1}) x_1 - \lambda_2 \otimes (\sum_{i,j \geq 0} \frac{a'_{ij}}{j+1}) x_2$,

where the $a_{ij}$ and $a'_{ij}$ are of homogenous degree $i$ with respect to $x_1$ and $j$ with respect to $x_2$, and $d(\xi) = -\lambda_1 \lambda_2 \otimes x_3 (\sum a_{ij} - \sum a'_{ij})$.

Since $U$ is torsion free over $\mathbb{Z}$, we must have $\sum a_{ij} = \sum a'_{ij} = a$ for some polynomial $a$. Hence

$\xi = \lambda_1 \otimes (\sum_{ij} \frac{a_{ij}}{(i+1)}) x_1 - \lambda_2 \otimes (\sum_{ij} \frac{a'_{ij}}{(j+1)}) x_2 = \lambda_1 \otimes \int_1 a - \lambda_2 \otimes \int_2 a$.

So $\xi$ is the sum of elements of types 3 and 3.5. \hfill $\blacksquare$

**Proposition 10.6.** We have $HH^1(U) = FH^1 \oplus TH^1$ where $FH^1$ is the free $\mathbb{Z}$-module with generators

\begin{equation}
\{ \lambda_1 \otimes x_2^{n_2}, \lambda_2 \otimes x_1^{n_1}, \lambda_1 \otimes x_1 - \lambda_3 \otimes x_3 : n_i \geq 0 \},
\end{equation}
and $TH^1$ is the torsion $\mathbb{Z}$-module with annihilator $(x_3)$ and generators
\[ \{ \lambda_1 \otimes \frac{x^{N+1}}{n_1+1} - \lambda_2 \otimes \frac{x^{N+1}}{n_2+1} : N = (n_1, n_2), n_i \geq 0 \}. \] (23)

Let us remark here that we can substitute the generator $\lambda_2 \otimes x_2 - \lambda_3 \otimes x_3$ for our generator $\lambda_1 \otimes x_1 - \lambda_3 \otimes x_3$ to get the same result.

**Proof.** First note that, from the description of $Z^1$ given in Lemma 10.5 and the fact that $C^1(U)$ is a free $\mathbb{Z}$-module on our given basis, the submodule of cycles $Z^1$ is free on the basis
\[ \{ \lambda_1 \otimes x_2^{n_2}, \lambda_2 \otimes x_1^{n_1}, \lambda_1 \otimes x_1 - \lambda_3 \otimes x_3 : n_i \geq 0 \} \]
\[ \{ \lambda_1 \otimes \frac{x^{N+1}}{n_1+1} - \lambda_2 \otimes \frac{x^{N+1}}{n_2+1} : N = (n_1, n_2), n_i \geq 0 \}. \]
From the description of $B^3$ above it then becomes clear that we have a surjective map $HH^1 = Z^1/B^1 \to FH^1 \oplus TH^1$, and one similarly constructs a map backwards. \qed

**10.3. Calculating $HH^2(U)$ and $HH^3(U)$.**

**Lemma 10.7.** Any degree 2 boundary is a $\mathbb{Z}$-linear combination of the following types of elements:

- **Type 1)** $\lambda_1 \lambda_2 \otimes x_3 a$, where $a \in k[x_1, x_2]$.
- **Type 2)** $\lambda_1 \lambda_2 \otimes 1$.
- **Type 2.5)** $\lambda_1 \lambda_2 \otimes x^N + \lambda_2 \lambda_3 \otimes n_1 x_3 x^{N-(1,0)} + \lambda_3 \lambda_1 \otimes n_2 x_3 x^{N-(0,1)}$, where $x^M$ is taken to be zero when an entry in $M = (n_1, n_2)$ is negative, and we only consider $N = (n_1, n_2)$ with $n_1 + n_2 > 0$.

**Proof.** For $i = 1, 2$ and $a \in k[x_1, x_2]$, $d(\lambda_i \otimes a)$ is an element of type 1, and $d(\lambda_3 \otimes a)$ is a sum of elements of types 2 and 2.5. Furthermore, each of these elements are boundaries as we have
\[
d(-\lambda_1 \otimes \int a) = \lambda_1 \lambda_2 \otimes x_3 a \\
d(\lambda_1 \otimes 1) = \lambda_1 \lambda_2 \otimes 1 \\
d(\lambda_3 \otimes x^N) = \lambda_1 \lambda_2 \otimes x^N + \lambda_2 \lambda_3 \otimes n_1 x_3 x^{N-(1,0)} + \lambda_3 \lambda_1 \otimes n_2 x_3 x^{N-(0,1)}.
\]

**Lemma 10.8.** A degree 2 cycle is a $\mathbb{Z}$-linear combination of the following types of elements:

- **Type 1)** $\lambda_1 \lambda_2 \otimes a$, where $a \in k[x_1, x_2]$.
- **Type 2)** $\lambda_2 \lambda_3 \otimes x_1^n$, $n \geq 0$.
- **Type 3)** $\lambda_3 \lambda_1 \otimes x_2^n$, $n \geq 0$.
- **Type 4)** $\lambda_2 \lambda_3 \otimes \int a + \lambda_3 \lambda_1 \otimes \int a$, where $a \in k[x_1, x_2]$.

**Proof.** For any cycle $\xi$, by subtracting a $\mathbb{Z}$-linear combination of elements of types 1, 2, and 3 we may assume
\[
\xi = \lambda_2 \lambda_3 \otimes a_2 + \lambda_3 \lambda_1 \otimes a_1
\]
with $a_2$ divisible by $x_2$ and $a_1$ divisible by $x_1$. Whence
\[
d(\xi) = d(\lambda_2 \lambda_3 \otimes a_2) + d(\lambda_3 \lambda_1 \otimes a_1) \\
= \lambda_1 \lambda_2 \lambda_3 \otimes x_3 \partial_2(a_2) - \lambda_1 \lambda_2 \lambda_3 \otimes x_3 \partial_1(a_1) \\
= x_3(\lambda_1 \lambda_2 \lambda_3 \otimes \partial_2(a_2) - \partial_1(a_1)).
\]
Since $U$ is torsion free over $\mathbb{Z}$ this implies $\partial_2(a_2) = \partial_1(a_1) = a$ and $a_1 = \int_{a_2} a$ and $a_2 = \int_{a_1} a$, since we already know $a_1$ and $a_2$ have vanishing constant terms.

It will be helpful to have the following Lemma in deducing the degree 2 cohomology.

**Lemma 10.9.** The second homology $HH^2(U)$ is annihilated by $x_3^2$.

**Proof.** We simply check that each of the generators, given in Lemma 10.8, is annihilated by $x_3^2$. A cycle of type 1 becomes a boundary of type 1 after multiplying by $x_3$, and hence is annihilated on cohomology. A cycle of type 2 multiplied by $x_3^2$ is of the form

$$\lambda_2 \lambda_3 \otimes x_3^2 x_1^n$$

$$= \lambda_2 \lambda_3 \otimes x_3 \partial_1 \left( \frac{1}{(n+1)} x_3 x_1^{n+1} \right)$$

$$= \lambda_2 \lambda_3 \otimes x_3 \partial_1 \left( \frac{1}{(n+1)} x_3 x_1^{n+1} \right) + \lambda_3 \lambda_1 \otimes n_2 x_3 \partial_2 (x_3 x_1^{n+1})$$

$$= \lambda_2 \lambda_3 \otimes x_3 \partial_1 \left( \frac{1}{(n+1)} x_3 x_1^{n+1} \right) + \lambda_3 \lambda_1 \otimes n_2 x_3 \partial_2 (x_3 x_1^{n+1})$$

$$= (\lambda_1 \lambda_2 \otimes x_3 x_1^{n+1}) + \lambda_2 \lambda_3 \otimes x_3 \partial_1 \left( \frac{1}{(n+1)} x_3 x_1^{n+1} \right) + \lambda_3 \lambda_1 \otimes n_2 x_3 \partial_2 (x_3 x_1^{n+1})$$

$$= -\lambda_1 \lambda_2 \otimes x_3 x_1^{n+1},$$

which is a sum of boundaries, and hence is zero in cohomology. Similarly, we see that elements of type 3 and 4 are also annihilated by $x_3^2$.

Note that from our $\mathbb{Z}$-basis for $\mathcal{C}^2(U)$, the elements of types 1-4 above are all $\mathbb{Z}$-linearly independent, and so we can produce an easy basis for $Z^2$ from Lemma 10.8.

**Proposition 10.10.** The second Hochschild cohomology $HH^2(U)$ is the free $\mathbb{Z}/(x_3^2)$-module $TH^2$ with basis

$$\{ \lambda_2 \lambda_3 \otimes x_1^n, \lambda_3 \lambda_1 \otimes x_2^m, \lambda_2 \lambda_3 \otimes \frac{x_1^{n+0.1}}{(n+1)} + \lambda_3 \lambda_1 \otimes \frac{x_1^{n+1.0}}{(n+1)} \}_{l,m,N}.$$ 

**Proof.** By the previous lemma, we have the obvious $\mathbb{Z}$-module map $f : TH^2 \to HH^2(U)$, sending the basis elements to their corresponding generators of types 2, 3, and 4. The generators of type 1 are in the image of this map since we have the boundaries

$$\lambda_1 \lambda_2 \otimes a + \lambda_2 \lambda_3 \otimes x_3 \partial_1 (a) + \lambda_3 \lambda_1 \otimes x_3 \partial_2 (a),$$

and $\lambda_1 \lambda_2 \otimes 1$. So the map $f$ is surjective. Note that $Z^2$ is the free $\mathbb{Z}$-module on the basis elements given by Lemma 10.8. So we have the $\mathbb{Z}$-module map $g : Z^2 \to TH^2$ defined on generators of type 1 by

$$g(\lambda_1 \lambda_2 \otimes a) = -(\lambda_2 \lambda_3 \otimes x_3 \partial_1 (a) + \lambda_3 \lambda_1 \otimes x_3 \partial_2 (a))$$

and on the generators of types 2, 3, and 4 in the obvious way. This map is clearly surjective and is seen to annihilate all boundaries (since $TH^2$ is annihilated by $x_3^2$). Whence we get an induced $\mathbb{Z}$-linear map $\tilde{g} : HH^2(U) \to TH^2$. These maps are mutually inverse since they are mutually inverse on the $\mathbb{Z}$-generators. And so we have $HH^2(U) = TH^2$.

The calculation of $HH^3(U)$ is trivial, and so we state it here explicitly.

**Proposition 10.11.** The third Hochschild cohomology $HH^3(U)$ is the free $\mathbb{Z}/(x_3)$-module with basis $\{ \lambda_1 \lambda_2 \otimes x^N : N \in \mathbb{Z}_{\geq 0}^2 \}$. 

Proof. This is clear from the definition of $d^2$ given at (21), and the fact that the partial differentials $\partial_i : k[x_1, x_2] \to k[x_1, x_2]$ are surjective.

10.4. A multiplication table for $HH^\bullet(U)$.

Lemma 10.12. For any homogeneous $\xi$ in $HH^\bullet(U)$ of degree $\geq 1$ we have $\xi^2 = 0$.

Proof. One can calculate directly that $\xi^2 = 0$ for degree 1 cycles, or simply note that this follows by graded commutativity. For cocycles of degree $> 1$ this follows simply by the fact that the cohomology vanishes in degree $> 3$. ■

Below we give a list of products of degree 1 basis elements for $HH^\bullet(U)$. Let us note that the following operations are happening in cohomology, not in $\mathcal{C}(U)$. The elements below represent classes in $HH^\bullet(U)$.

- $\lambda_1 \otimes x_1^n \cdot \lambda_2 \otimes x_1^m = -(\lambda_2 \lambda_3 \otimes n_1 x_3 x^{N-1} + \lambda_3 \lambda_2 \otimes n_2 x_3 x^{N-1})$
- $\lambda_1 \otimes x_2^n \cdot (\lambda_1 \otimes x_1 - \lambda_3 \otimes x_3) = \lambda_3 \lambda_1 \otimes x_3 x_2^n$
- $\lambda_1 \otimes x_2^n \cdot (\lambda_1 \otimes x_1 - \lambda_3 \otimes x_3) = \lambda_3 \lambda_1 \otimes x_3 x_2^n$

This list, along with graded commutativity of Hochschild cohomology and the previous lemma, gives a complete multiplication table. We leave it to the reader to verify these facts, but give a computation of the final product here, as it is the most difficult.

Proof of the final product. In cohomology we have

$$(\lambda_1 \otimes x_1 - \lambda_3 \otimes x_3) \cdot (\lambda_1 \otimes x_1 - \lambda_3 \otimes x_3) = -(\lambda_2 \lambda_3 \otimes n_1 x_3 x^{N-1} + \lambda_3 \lambda_2 \otimes n_2 x_3 x^{N-1})$$

This list, along with graded commutativity of Hochschild cohomology and the previous lemma, gives a complete multiplication table. We leave it to the reader to verify these facts, but give a computation of the final product here, as it is the most difficult.

From the above list of products we see that for any basic element $\xi$ in $HH^2(U)$ we have $x_3 \xi = \eta_1 \cdot \eta_2$ for some basic $\eta_i$ in degree 1. So we have the following proposition.

Proposition 10.13. The multiplication map $HH^1(U) \otimes HH^1(U) \to HH^2(U)$ has image exactly $x_3 \cdot HH^2(U)$. ■
What is left is to present the multiplication table for $HH^1(U) \otimes HH^2(U)$. We have

- $\lambda_1 \otimes x_2^{n_2} \cdot \lambda_2 \lambda_3 \otimes x_1^{n_1} = \lambda_1 \lambda_2 \lambda_3 \otimes x^N$
- $\lambda_1 \otimes x_2^{n_2} \cdot \lambda_2 \lambda_3 \otimes x_2^{n_2} = 0$
- $\lambda_1 \otimes x_2^{n_2} \cdot (\lambda_2 \lambda_3 \otimes \frac{z_{M+(1,0)}}{(m_2+1)}) + \lambda_3 \lambda_1 \otimes \frac{z_{M+(1,0)}}{(m_1+1)}) = \lambda_1 \lambda_2 \lambda_3 \otimes \frac{z_{M+(0,0)+1}}{(m_2+1)}$
- $\lambda_2 \otimes x_1^{n_1} \cdot \lambda_2 \lambda_3 \otimes x_1^{n_1} = 0$
- $\lambda_2 \otimes x_1^{n_1} \cdot \lambda_3 \lambda_1 \otimes x_2^{n_2} = \lambda_1 \lambda_2 \lambda_3 \otimes x^N$
- $\lambda_2 \otimes x_1^{n_1} \cdot (\lambda_2 \lambda_3 \otimes \frac{z_{M+(1,0)}}{(m_2+1)}) + \lambda_3 \lambda_1 \otimes \frac{z_{M+(1,0)}}{(m_1+1)}) = \lambda_1 \lambda_2 \lambda_3 \otimes \frac{z_{M+(n_1+1,0)}}{(m_1+1)}$
- $(\lambda_1 \otimes x_1 - \lambda_3 \otimes x_3) \cdot (\lambda_2 \lambda_3 \otimes \frac{z_{M+(1,0)}}{(m_2+1)}) + \lambda_3 \lambda_1 \otimes \frac{z_{M+(1,0)}}{(m_1+1)}) = \lambda_1 \lambda_2 \lambda_3 \otimes \frac{z_{M+(1,1)}}{(m_2+1)}$
- $(\lambda_1 \otimes x_1 - \lambda_3 \otimes x_3) \cdot (\lambda_2 \lambda_3 \otimes \frac{z_{M+(1,0)}}{(m_2+1)}) + \lambda_3 \lambda_1 \otimes \frac{z_{M+(1,0)}}{(m_1+1)}) = \lambda_1 \lambda_2 \lambda_3 \otimes \frac{z_{M+(1,1)}}{(m_2+1)}$
- $(\lambda_1 \otimes x_1 - \lambda_3 \otimes x_3) \cdot (\lambda_2 \lambda_3 \otimes \frac{z_{M+(1,0)}}{(m_2+1)}) + \lambda_3 \lambda_1 \otimes \frac{z_{M+(1,0)}}{(m_1+1)}) = \lambda_1 \lambda_2 \lambda_3 \otimes \frac{z_{M+(1,1)}}{(m_2+1)}$
- $(\lambda_1 \otimes x_1 - \lambda_3 \otimes x_3) \cdot (\lambda_2 \lambda_3 \otimes \frac{z_{M+(1,0)}}{(m_2+1)}) + \lambda_3 \lambda_1 \otimes \frac{z_{M+(1,0)}}{(m_1+1)}) = \lambda_1 \lambda_2 \lambda_3 \otimes x^{L+M+(1,1)}$

The computations are more straightforward than those for degree 1 elements, and are left to the interested reader. Our final proposition is clear from the above table.

**Proposition 10.14.** The product $HH^1(U) \otimes HH^2(U) \rightarrow HH^3(U)$ is surjective.

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