Abstract
Consider a non-negative function $f : \mathbb{R}^n \to \mathbb{R}^+$ such that $\int f \, d\gamma_n = 1$, where $\gamma_n$ is the $n$-dimensional Gaussian measure. If $f$ is semi-log-convex, i.e. if there exists a number $\beta \geq 1$ such that for all $x \in \mathbb{R}^n$, the eigenvalues of $\nabla^2 \log f(x)$ are at least $-\beta$, then $f$ satisfies an improved form of Markov’s inequality: For all $\alpha \geq e^3$,

$$\gamma_n \left( \{ x \in \mathbb{R}^n : f(x) > \alpha \} \right) \leq \frac{1}{\alpha} \cdot \frac{C\beta(\log \log \alpha)^4}{\sqrt{\log \alpha}},$$

where $C$ is a universal constant. The bound is optimal up to a factor of $C\sqrt{\beta}(\log \log \alpha)^4$, as it is met by translations and scalings of the standard Gaussian density.

In particular, this implies that the mass on level sets of a probability density decays uniformly under the Ornstein-Uhlenbeck semigroup. This confirms positively the Gaussian case of Talagrand’s convolution conjecture [Tal89].

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1 Introduction

Let \( n \geq 1 \) and equip \( \mathbb{R}^n \) with the standard Gaussian measure \( \gamma_n \). Consider a function \( f : \mathbb{R}^n \to \mathbb{R} \) in \( L^1(\gamma_n) \). The Ornstein-Uhlenbeck semi-group \( \{U_t : t \geq 0\} \) is defined by

\[
U_t f(x) = \mathbb{E} f \left( e^{-t} x + \sqrt{1 - e^{-2t}} Z \right),
\]

where \( Z \) has law \( \gamma_n \). One expects that the action of such a diffusion process serves to smoothen \( f \).

Indeed, Nelson’s hypercontractivity theorem \([\text{Nel73}]\) shows that \( U_t \) is a contraction from \( L^p(\gamma_n) \) to \( L^q(\gamma_n) \) for \( 1 < p \leq q \) and \( t \geq \frac{1}{2} \log \frac{q - 1}{p - 1} \).

The concept of hypercontractivity plays an important role in several mathematical fields. For example, in quantum field theory hypercontractivity can often be used to show that a Hamiltonian is essentially self-adjoint on its domain, laying the foundation for various constructions (see, e.g., \([\text{GRS75}]\)). We refer to the surveys \([\text{DGS92, Gro06}]\). In the study of partial differential equations, it is a key method in several approaches to establishing the existence and uniqueness of smooth solutions to evolution equations \([\text{Bre11}]\). Hypercontractivity is also a basic tool in establishing superconcentration \([\text{Cha14}]\).

In the present work, we assert a regularizing effect of \( U_t \) merely assuming that \( f \in L^1(\gamma_n) \). An important special case is when \( f \) is simply the indicator of a measurable subset of \( \mathbb{R}^n \). Assume now that \( f : \mathbb{R}^n \to \mathbb{R}^+ \) is non-negative. Certainly we have Markov’s inequality: For any \( \alpha \geq 1 \),

\[
\gamma_n \left( \{x : f(x) \geq \alpha \|f\|_1\} \right) \leq \frac{1}{\alpha},
\]

where we use \( \|f\|_1 = \int |f| \, d\gamma_n \). Of course, this bound is easily seen to be tight for any \( \alpha > 0 \) by taking \( f = 1_S \) for a measurable subset \( S \subseteq \mathbb{R}^n \) with \( \gamma_n(S) = 1/\alpha \). The “heat content” of \( f \) lies on a single level set, i.e. at a single “temperature.” A very natural question arises: Can a smoothed version of \( f \), i.e. \( U_t f \) for some \( t > 0 \), have its heat content concentrated near a single high temperature? Talagrand conjectured that this cannot be the case.\(^1\)

\textbf{Conjecture 1.1.} For every \( t > 0 \), there exists a function \( \psi_t : [1, \infty) \to [1, \infty) \) with \( \lim_{\alpha \to \infty} \psi_t(\alpha) = \infty \) such that for any measurable \( f : \mathbb{R}^n \to \mathbb{R}_+ \) and any \( \alpha > 1 \),

\[
\gamma_n \left( \{x : U_t f(x) > \alpha \|f\|_1\} \right) \leq \frac{1}{\alpha \psi_t(\alpha)}. \]

One should recall here that \( U_t \) preserves both positivity and the mean value; for non-negative \( f \), we have \( \|U_t f\|_1 = \|f\|_1 \). The conjecture posits a uniform bound on the tail of the smoothed function. Talagrand notes that the best rate of decay one can expect is \( \psi_t(\alpha) = c(t) \sqrt{\log \alpha} \) where \( c(t) \) is some function depending only on \( t \). We resolve the conjecture positively (Corollary 1.4 below) and achieve the bound

\[
\psi_t(\alpha) = \frac{c(t) \sqrt{\log \alpha}}{(\log \log \alpha)^4}.
\]

\(^1\)Talagrand actually made a stronger conjecture \([\text{Tal89}]\) that a similar statement should hold in the discrete cube. We refer the reader to Section 1.2. We attribute this weaker conjecture to him—with permission—to stress his role in predicting the phenomenon.
Ball, Barthe, Bednorz, Oleszkiewicz, and Wolff \([BBB+13]\) prove that Conjecture 1.1 holds in any fixed dimension; they achieve 
\[
\psi_t(\alpha) = C(t,n)\sqrt{\log \alpha/(\log \log \alpha)}
\]
where \(C(t,n)\) is a constant depending (exponentially) on the dimension \(n\).

The heat analogy. If one imagines the function \(f : \mathbb{R}^n \to \mathbb{R}_+\) as assigning an initial distribution of heat to space, Conjecture 1.1 asserts that if we allow the heat to diffuse for a short period of time, then the resulting distribution cannot have all the heat concentrated in a narrow range of high temperatures. More formally, we will see that for \(t > 0\),
\[
\int (U_t f) \mathbf{1}_{\{U_t f \in [\alpha, 2\alpha]\}} \, d\gamma_n \leq c(t) \frac{\log \log \alpha}{\sqrt{\log \alpha}} \|f\|_1.
\]

An isoperimetric perspective. A dual point of view is helpful in understanding the isoperimetric content of Conjecture 1.1. Fix \(t > 0\), let \(S \subseteq \mathbb{R}^n\) be an open subset, and consider the set of non-negative functions \(g : \mathbb{R}^n \to \mathbb{R}_+\) supported on \(S\), and such that \(\|U_t g\|_\infty \leq 1\). Our goal is to maximize \(\int g \, d\gamma_n\) subject to these constraints.

Clearly the choice \(g = 1_S\) has \(\int g \, d\gamma_n = \gamma_n(S)\). Conjecture 1.1 asserts that there should be a strategy that does much better. In fact, the largest function \(\psi_t\) achievable in Conjecture 1.1 is precisely the same as the largest function \(\psi_t\) such that the following holds for every open \(S \subseteq \mathbb{R}^n\):
\[
\sup_{\text{supp}(g) \subseteq S} \int g \, \frac{d\gamma_n}{\|U_t g\|_\infty} \geq \psi_t \left(\frac{1}{\gamma_n(S)}\right) \gamma_n(S).
\]

This dual characterization is a straightforward consequence of Hahn-Banach and self-adjointness of \(U_t\) as an operator on \(L^2(\gamma_n)\). We refer to this optimization problem as “isoperimetric” because the intuition is that to make \(g\) significantly larger subject to the constraint \(\|U_t g\|_\infty \leq 1\), one should concentrate \(g\) on the “boundary” of the set \(S\) where it may be allowed to take larger values due to the smoothing effect of \(U_t\).

Indeed, one can prove Conjecture 1.1 for \(n = 1\) via the dual (1) as follows: Given \(S \subseteq \mathbb{R}\), one should choose \(g\) to be a Dirac mass near the point of \(\mathbb{R} \setminus S\) which is closest to the origin. (Strictly speaking, one should take a sequence of points in \(S\) and a sequence of functions approximating Dirac masses at those points.) From the value \(\gamma_n(S)\), one can conclude that \(S\) contains a point sufficiently close to the origin. A simple calculation with the Gaussian density yields the desired bound.\(^2\)

1.1 Semi-log-convexity and anti-concentration of temperature

The resolution of Conjecture 1.1 arises from a more general phenomenon for semi-log-convex functions. Our main theorem follows.

Theorem 1.2. Let \(f : \mathbb{R}^n \to \mathbb{R}_+\) be a function with continuous second-order partial derivatives. Assume there is a \(\beta \geq 1\) such that for all \(x \in \mathbb{R}^n\),
\[
\nabla^2 \log f(x) \succeq -\beta \text{Id}.
\]

\(^2\)Completion of this sketch is Exercise 11.31 in O’Donnell’s book [O’D14]. It reflects an observation of O’Donnell and the second-named author from 2010.
Then for all $\alpha \geq e^3$,
\[
\gamma_n \left( \{ x \in \mathbb{R}^n : f(x) > \alpha \| f \|_1 \} \right) \leq \frac{1}{\alpha} \cdot \frac{C \beta (\log \log \alpha)^4}{\sqrt{\log \alpha}},
\]
where $C > 0$ is a universal constant.

We first explain how this resolves Conjecture 1.1 before moving on to a discussion of Theorem 1.2. Let $\{B_t\}$ be an $n$-dimensional Brownian motion with $B_0 = 0$, and let $P_t f(x) = \mathbb{E}[f(x + B_t)]$ denote the corresponding semigroup. A proof of the following standard fact is contained in the appendix.

**Lemma 1.3.** For any $f : \mathbb{R}^n \to \mathbb{R}_+$ in $L^1(\gamma_n)$ and $t > 0$, one has $\nabla^2 \log P_t f(x) \succeq -\frac{1}{4} \text{Id}$ for all $x \in \mathbb{R}^n$.

This rather immediately implies the following.

**Corollary 1.4.** There is a constant $C > 0$ such that for every $\rho \in (0, 1)$, the following holds. For every measurable $g : \mathbb{R}^n \to \mathbb{R}_+$ and every $\alpha \geq e^3$, one has
\[
\mathbb{P} \left( P_{1-\rho} g(B_\rho) > \alpha \| g \|_1 \right) \leq \frac{1}{\alpha} \cdot \frac{\rho}{1 - \rho} \frac{C(\log \log \alpha)^4}{\sqrt{\log \alpha}}.
\]

**Proof.** If we define $f(x) = g(\sqrt{\rho} x)$, then $P_{1-\rho} g(B_\rho)$ and $P_{(1-\rho)/\rho} f(Z)$ have the same law, where $Z$ is a standard $n$-dimensional Gaussian. Now combining Lemma 1.3 and Theorem 1.2 yields the desired result.

Corollary 1.4 yields a resolution to Conjecture 1.1 by noting that for any $t > 0$,
\[
\gamma_n \left( \{ x : U_t f(x) > \alpha \| f \|_1 \} \right) = \mathbb{P} \left( P_{1-e^{-2t}} f(B_{e^{-2t}}) > \alpha \| f \|_1 \right).
\]

**Translating the anti-concentration of Brownian motion.** Despite the fact that Theorem 1.2 is not a stochastic statement, the main theme of our paper is that the variance of Brownian motion can be translated into anti-concentration for the level sets of certain functionals on Gaussian space.

Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be as in the statement of Theorem 1.2 and additionally let us assume that $\int f \, d\gamma_n = 1$. Our goal is equivalent to bounding $\mathbb{P}(f(B_1) > \alpha)$, and it is easy to see that it would suffice to give an upper bound on $\mathbb{P}(f(B_1) \in [\alpha, 2\alpha])$.

Very roughly, this will be achieved as follows. We show that if $\mathbb{E}[f(B_1) 1_{\{f(B_1) \in [\alpha, 2\alpha]\}}]$ is large enough, then the values $\mathbb{E}[f(B_1) 1_{\{f(B_1) \in [2^k \alpha, 2^{k+1} \alpha]\}}]$ are proportionally large for approximately $\sqrt{\log \alpha}$ values of $k$. Since $\mathbb{E}[f(B_1)] = 1$, this yields the desired conclusion.

This “transfer of mass” between levels is achieved by carefully perturbing the underlying Brownian motion. The Hessian condition (2) ensures that $f$ behaves predictably under small perturbations. The primary difficulty is to perform the perturbations without changing the measure of the underlying Brownian motion too much. For this purpose, we will employ an appropriate Itô process, and Girsanov’s change of measure theorem will be essential.

**Related work.** Our use of random measures and stochastic calculus to study the geometry of Gaussian space is certainly closely related to the works Eld13a, Eld13b. On the other hand, the
idea to study functionals using an “optimal” adapted coupling to Brownian motion (see Section 2) comes from the viewpoint of stochastic control theory [Fol85, Leh13] and its geometric applications [Leh13]. Other variational perspectives appear in the work [BD98] and in Borell’s papers [Bor00, Bor02] where one of his primary goals is their use in proving functional inequalities. An important distinction between our work and some previous ones involves our use of second-order methods. Specifically, we study the effect of perturbations on the optimal drift.

Finally, we should mention two vast bodies of work closely related to our study: Markov diffusions and semigroup methods (see, e.g., [BGL14]), as well as the theory of optimal transportation. For the latter topic, one might consult [Vil03, Ch. 9] for an excellent review of the literature related to functional inequalities.

1.2 Talagrand’s conjecture for the discrete cube

Talagrand [Tal89] posed the following conjecture which is a generalization of Conjecture 1.1. Let \( n \geq 1 \) and \( t \geq 0 \) be given. Consider the probability measure on the set \( \{-1, 1\}^n \) given by

\[
\mu_t := \frac{1 - e^{-t}}{2} \delta_{-1} + \frac{1 + e^{-t}}{2} \delta_1.
\]

Denote \( \mu^n_t \) the corresponding product measure on \( \{-1, 1\}^n \), and put \( \mu = \mu_\infty \) so that \( \mu^n \) is the uniform measure on \( \{-1, 1\}^n \). Let \( L^2(\mu^n_t) = L^2(\{-1, 1\}^n, \mu^n_t) \) denote the Hilbert space of real-valued functions \( f : \{-1, 1\}^n \to \mathbb{R} \).

Consider the operator \( T_t : L^2(\mu^n) \to L^2(\mu^n) \) given by convolution with an \( e^{-t} \)-biased measure, i.e.

\[
T_t f = f * \mu^n_t,
\]

where one uses the natural multiplicative group structure on \( \{-1, 1\}^n \).

As in the Gaussian case, this operator admits a hypercontractive estimate [Bon70, Bec75, Gro75]: \( T_t \) is a contraction from \( L^p(\mu^n) \) to \( L^q(\mu^n) \) for \( 1 < p \leq q \) and \( t \geq \frac{1}{2} \log \frac{q-1}{p-1} \).

**Conjecture 1.5** ([Tal89]). For every \( t > 0 \), there exists a function \( \varphi_t : [1, \infty) \to [1, \infty) \) with \( \lim_{\alpha \to \infty} \varphi_t(\alpha) = \infty \) such that for every \( f : \{-1, 1\}^n \to \mathbb{R}_+ \) and any \( \alpha > 1 \),

\[
\mu^n \left( \{ x \in \{-1, 1\}^n : T_t f(x) > \alpha \| f \|_1 \} \right) \leq \frac{1}{\alpha \varphi_t(\alpha)}.
\]

It is a straightforward observation that Conjecture 1.5 implies Conjecture 1.1 with \( \psi_t = \varphi_t \). This is proved by embedding Gaussian space (approximately) into a sequence of discrete cubes of growing dimension via the central limit theorem; we refer to the discussion in [BBB+13]. At present, Conjecture 1.5 is open for any value of \( t > 0 \).

In his original paper [Tal89], Talagrand did provide a proof of a related inequality for the averaged operator \( A = \int_0^t T_t dt \). Specifically, there is a constant \( C > 0 \) such that for all \( \alpha > e^3 \),

\[
\mu^n \left( \{ x : Af(x) \geq \alpha \| f \|_1 \} \right) \leq \frac{C \log \log \alpha}{\log \alpha}.
\]

His proof makes clever use of \( \approx \log \alpha \) invocations of the aforementioned hypercontractive inequality.

In Section 4 we discuss the possibility and difficulty of extending our approach to Conjecture 1.5. In fact, the discretely inclined reader might gain some intuition from first reading the proof of the log-Sobolev inequality for \( \{-1, 1\}^n \) presented there.
2 Entropy, energy, and the Föllmer drift

Fix $n \geq 1$ and consider $\mathbb{R}^n$ with the equipped with the standard Euclidean structures $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, and the Gaussian measure $\gamma_n$ defined by

$$d\gamma_n(x) = \frac{1}{\sqrt{(2\pi)^n}} \exp\left(-\frac{\|x\|^2}{2}\right).$$

We now lay out the basic objects of our study and prove some preliminary properties. In the next section, we begin with an informal discussion highlighting a stochastic calculus approach to the geometry of Gaussian space. This is followed by a broad outline of our arguments. The formal preliminaries begin in Section 2.2 and the main theorem is proved in Section 2.3 save for the core technical lemma of the paper to which Section 3 is devoted.

2.1 Overview and proof sketch

Suppose now that $f : \mathbb{R}^n \to \mathbb{R}_+$ has continuous second-order partial derivatives and $\int f \, d\gamma_n = 1$. Recall that, given $\alpha > 0$, we are interested in showing that $P(f(B_1) \in [\alpha, 2\alpha]) \ll 1/\alpha$ as $\alpha \to \infty$, where $\{B_t\}$ is a Brownian motion with $B_0 = 0$. Since $f$ could be concentrated on a set of very small measure, this would necessitate the study of events of very small probability. Instead, we will restrict our attention to the interesting parts of the space by changing the measure of the Brownian motion so that $B_1$ has law $f \, d\gamma_n$.

To this end, we define an Itô process $\{W_t\}$ by the stochastic differential equation

$$W_0 = 0, \quad dW_t = dB_t + v_t \, dt$$

for some predictable drift process $\{v_t\} \in [0,1]$ such that $B_1 + \int_0^1 u_t \, dt$ has law $f \, d\gamma_n$. Among all such drifts, we will define $\{v_t\}$ to be the one that minimizes the quantity

$$\frac{1}{2} \int_0^1 E \|u_t\|^2 \, dt.$$  (5)

It is quite beneficial to think of $\{v_t\}$ as the minimum-energy adapted coupling between $d\gamma_n$ and $f \, d\gamma_n$. Furthermore, one can connect this energy to the entropy of $f$:

$$H_{\gamma_n}(f) = \frac{1}{2} \int_0^1 E \|v_t\|^2 \, dt,$$  (5)

where $H_{\gamma_n}(f) := \int f \log f \, d\gamma_n$ denotes the relative entropy of $f$ with respect to $\gamma_n$. As one might expect, this optimality property of $v_t$ implies that $\{v_t\}$ is a martingale with respect to $\{\mathcal{F}_t\}$, a fact that will be central in our study. In particular, the martingale property will imply that the behavior of $\{W_t\}$ at small times must have echoes that reverberate to time 1.

As we will see below, one can compute explicitly

$$v_t = \nabla \log P_{1-t} f(W_t).$$  (6)

6
This has a straightforward geometric interpretation. Consider the relative density
\[
\phi_t(x) = f(x)e^{-\|x-W_t\|^2/(1-t)},
\]
and let \(\bar{\phi}_t(x)\) be the normalization of \(\phi_t(x)\) such that \(\bar{\phi}_t(x)\) \(dx\) is a probability density. Then,
\[
v_t = (1-t)^{-1} \left( \int x\bar{\phi}_t(x)\,dx - W_t \right)
\]
is the vector pointing from \(W_t\) to the center of mass of \(f\) with respect to a Gaussian distribution of variance \(1-t\) centered at \(W_t\). The scaling by \((1-t)^{-1}\) stands to reason: The fact that \(W_1 \sim f\,d\gamma_n\) means that as \(t\) approaches 1, if \(W_t\) is far from the “bulk” of \(f\), the desperation of the drift increases.

It is possible to show (e.g., using the tools of the next section) that equation (6) implies that for every \(t \in [0,1]\),
\[
\mathbb{E} \|v_t\|^2 = \int \frac{\|\nabla P_{1-t}f\|^2}{P_{1-t}f} \,d\gamma_n.
\] (7)
The latter quantity is the Fisher information of \(P_{1-t}f\) (see [BGL14, Ch. II.5]). Thus \(v_t\) reflects the geometry of \(f\) seen from many “granularities.” Given our discussion so far, it is difficult to avoid stating Lehec’s elegant proof [Leh13] of the Gaussian log-Sobolev inequality:
\[
H_{\gamma_n}(f) \geq \frac{1}{2} \int_0^1 \mathbb{E} \|v_t\|^2 \,dt \leq \frac{1}{2} \mathbb{E} \|v_1\|^2 \leq \frac{1}{2} \int \frac{\|\nabla f\|^2}{f} \,d\gamma_n,
\] (8)
where the only inequality is an immediate consequence of the fact that \(v_t\) is a martingale.

**Changes of measure and gradient ascent.** Recall that our goal is to bound \(\mathbb{P}(f(B_1) \in [\alpha,2\alpha]) \ll 1/\alpha\). To this end, we will study the Doob martingale \(P_{1-t}f(B_t)\). As just argued, it will be beneficial to consider instead the process \(P_{1-t}f(W_t)\). At least intuitively (and formally justified in the next section), our change of measure was helpful: Since \(W_1\) has law \(f\,d\gamma_n\), we have
\[
\mathbb{P}(f(W_1) \in [\alpha,2\alpha]) \approx \alpha \cdot \mathbb{P}(f(B_1) \in [\alpha,2\alpha]).
\]

Thus it suffices to prove simply that \(\mathbb{P}(f(W_1) \in [\alpha,2\alpha]) \to 0\) as \(\alpha \to \infty\). Now the story comes together, as Itô’s formula will tell us that our process \(P_{1-t}f(W_t)\) can be related directly to the drift \(\{v_t\}\): For all \(t \in [0,1]\),
\[
P_{1-t}f(W_t) = \exp \left( \int_0^t \langle v_s, dB_s \rangle + \frac{1}{2} \int_0^t \|v_s\|^2 \,ds \right).
\]

As alluded to in the introduction, we will bound \(\mathbb{P}(f(W_1) \in [\alpha,2\alpha])\) by showing that if the former is large, then so is \(\mathbb{P} \left( f(W_1) \in [2^k\alpha, 2^{k+1}\alpha] \right)\) for many values of \(k\). This will be accomplished by perturbing the process \(\{W_t\}\) to achieve these larger values, and then arguing that the measure of \(\{W_t\}\) is relatively insensitive to such perturbations. The perturbed processes are essentially of the following form. For fixed \(\delta > 0\), they are given by the stochastic differential equation
\[
dW_t^\delta = dB_t + (1+\delta)v_t \,dt.
\]
Girsanov’s theorem will tell us that there is a measure $Q_\delta$ under which $W^\delta_t$ is a Brownian motion, and furthermore that

$$\frac{dQ_\delta}{dQ} = \exp\left(-\delta \int_0^1 \langle v_t, dB_t \rangle - \left(\delta + \frac{\delta^2}{2}\right) \int_0^1 \|v_t\|^2 dt\right).$$

This expresses the relative probability of Brownian motion having the sample path $\{W^\delta_t : t \in [0, 1]\}$ vs. the sample path $\{W_t : t \in [0, 1]\}$. From a high-level perspective, the most important factor here is $\exp\left(-\delta \int_0^1 \|v_t\|^2 dt\right)$.

On the other hand, this loss in measure will be compensated by an increase in the value $f$. Here we employ the Hessian condition (2) to conclude that since $W^\delta_1 = W_1 + \delta \int_0^1 v_t dt$,

$$f(W^\delta_1) \geq f(W_1) \exp\left(\delta \left\langle v_1, \int_0^1 v_t dt \right\rangle - \beta \delta^2 \left\| \int_0^1 v_t dt \right\|^2\right),$$

where we have used the fact that $v_1 = \nabla \log f(W_1)$ from (6).

In order to accomplish our goal, we need that the loss of measure is almost exactly compensated for by the increase in the value of $f$. Ignoring another low-order term (which requires $\delta$ to be small), this necessitates that

$$\exp\left(-\delta \int_0^1 \|v_t\|^2 dt\right) \approx \exp\left(\delta \left\langle v_1, \int_0^1 v_t dt \right\rangle\right).$$

Now we use the martingale property of $v_t$. It implies immediately that

$$\mathbb{E}\left[\left\langle v_1, \int_0^1 v_t dt \right\rangle\right] = \int_0^1 \mathbb{E}\|v_t\|^2 dt.$$

Thus the last issue we need to address is the concentration of $\left\langle v_1, \int_0^1 v_t dt \right\rangle$ and how it interacts with the many details and lower-order terms that we have glossed over. Controlling this presents the bulk of the technical difficulties in the proof to come.

### 2.2 Formal preliminaries

We fix a non-negative function $f : \mathbb{R}^n \to \mathbb{R}_+$ with continuous partial derivatives of second order. Moreover, we fix a measurable sample space $(\Omega, \Sigma)$ which we assume to be rich enough to support an $n$-dimensional Brownian motion.

Let $\{W_t : t \in [0, 1]\}$ be a process adapted to a filtration $\{F_t\}$ and let $Q$ be a measure over the sample space $(\Omega, \Sigma)$ such that $W_t$ is a standard $n$-dimensional Brownian motion with respect to $Q$. Define for all $0 \leq t \leq 1$,

$$M_t = P_{1-t}f(W_t).$$

Recall that the heat semigroup satisfies

$$\partial_t P_{1-t}f = -\frac{1}{2} \Delta P_{1-t}f, \quad \forall 0 < t < 1,$$
and that for all $t < 1$, $P_{1-t} f$ has continuous derivatives of all orders. This allows us to apply Itô’s formula (see, e.g., [Øks03]) in order to calculate

$$
   dM_t = d(P_{1-t}f(W_t)) = \partial_t P_{1-t}f(W_t)dt + \langle \nabla(P_{1-t}f)(W_t), dW_t \rangle + \frac{1}{2} \Delta(P_{1-t}f)(W_t) dt
   = \langle \nabla(P_{1-t}f)(W_t), dW_t \rangle
   = M_t \langle v_t, dW_t \rangle,
$$

where we define

$$
   v_t := \frac{\nabla(P_{1-t}f)(W_t)}{M_t} = \frac{\nabla(P_{1-t}f)(W_t)}{P_{1-t}f(W_t)} = \nabla \log P_{1-t}f(W_t).
$$

Moreover, by definition of the operator $P_{1-t}$ we have $M_t = \mathbb{E}_Q[M_1 | \mathcal{F}_t]$, so that $M_t$ is a martingale under $Q$.

Next, we construct a measure $P$ on $(\Omega, \Sigma)$ using the equation

$$
   P(A) = \mathbb{E}_Q[1_A M_1] \quad (11)
$$

for every measurable $A \subset \Omega$. We can also formally understand this definition as $dP \, dQ = M_1$.

We define an $\mathcal{F}_t$-adapted process $B_t$ by the equation

$$
   B_t = W_t - \int_0^t \langle v_s, dW_s \rangle.
$$

In other words, the process $B_t$ is defined by the equations

$$
   B_0 = 0, \quad dW_t = dB_t + v_t \, dt. \quad (12)
$$

The existence of this process is justified under the following theorem which immediately follows as a special case of Theorem 2 and Lemma 3 in [Leh13].

**Theorem 2.1.** The process $\{B_t : t \in [0, 1]\}$ is well-defined. Moreover, this process is an $\mathcal{F}_t$-Brownian motion under the measure $P$. Furthermore, the following assertions hold.

i) $W_1$ has the law $fd\gamma_n$ under the measure $P$.

ii) Almost surely in $P$, $\int_0^t v_s \, ds$ is defined for all $0 \leq t \leq 1$.

iii) $\int_0^1 \|v_t\|^2 \, dt < \infty$ almost surely in $P$.

iv) $\mathbb{E}_P[\int_0^1 \|v_t\|^2 \, dt] = 2 H_{\gamma_n}(f)$.

Next, fix $\tau \in [0, 1]$, and recall that $M_t$ is a martingale. Using equation (11), we learn that for all $A \in \mathcal{F}_\tau$,

$$
   P(A) = \mathbb{E}_Q[\mathbb{E}_Q[1_A M_1 | \mathcal{F}_\tau]] = \mathbb{E}_Q[1_A M_\tau].
$$

It follows that $\{W_t : t \in [0, \tau]\}$ has the law of a Brownian motion under the measure $\frac{1}{M_\tau} \, dP$ and, furthermore, that for any $0 \leq s \leq \tau$, the process $\{W_t - W_s : t \in [s, \tau]\}$ has the law of a Brownian motion under measure $\frac{M_s}{M_\tau} \, dP$. Thus, we also have that

$$
   P(A | \mathcal{F}_s) = \mathbb{E}_Q \left[ 1_A \frac{M_\tau}{M_s} \bigg| \mathcal{F}_s \right]
$$

for all $A \in \mathcal{F}_\tau$. The next fact will be crucial (and is also observed in [Leh13], in somewhat greater generality).
**Fact 2.2.** The process \( \{v_t : t \in [0,1]\} \) is a martingale under the measure \( P \).

To see this, fix some \( 0 \leq s \leq t \leq 1 \). Define \( \sigma_t = \nabla P_{1-t}f(W_t) = P_{1-t}(\nabla f)(W_t) \) (recalling that \( f \) is twice-differentiable). Since \( W_t \) is a \( Q \)-Brownian motion, we have

\[
\mathbb{E}_Q[\sigma_t \mid \mathcal{F}_s] = \mathbb{E}_Q[P_{1-t}(\nabla f)(W_t) \mid \mathcal{F}_s] = P_{1-s}(\nabla f)(W_s) = \sigma_s.
\]

This yields

\[
\mathbb{E}_P[v_t \mid \mathcal{F}_s] = \mathbb{E}_P\left[\frac{\sigma_t}{M_t} \mid \mathcal{F}_s\right] = \mathbb{E}_Q\left[\frac{\sigma_t}{M_s} \mid \mathcal{F}_s\right] \stackrel{(14)}{=} \frac{\sigma_s}{M_s} = v_s,
\]

which establishes the fact.

Finally, using Itô’s formula, equation (9) becomes

\[
d \log M_t = \langle v_t, dW_t \rangle - \frac{1}{2} \| v_t \|^2 dt,
\]

yielding the representation

\[
P_{1-t}f(W_t) = M_t = \exp \left( \int_0^t \langle v_s, dW_s \rangle - \frac{1}{2} \int_0^t \| v_s \|^2 ds \right)
\]

\[
= \exp \left( \int_0^t \langle v_s, dB_s \rangle + \frac{1}{2} \int_0^t \| v_s \|^2 ds \right).
\]

A combination of (11) with the last equation finally gives

\[
dQ = \exp \left( - \int_0^1 \langle v_t, dB_t \rangle - \frac{1}{2} \int_0^1 \| v_t \|^2 dt \right) dP = \frac{1}{M_1} dP = \frac{1}{f(W_1)} dP.
\]

In the next section, all probabilities and expectations are taken by default with respect to \( P \), the law under which the process \( \{B_t\} \) is a Brownian motion. When other measures are used, we will use the notations \( \mathbb{P}_Q \) and \( \mathbb{E}_Q \).

### 2.3 Proof of the Main Theorem

Consider a measurable function \( f : \mathbb{R}^n \to \mathbb{R}_+ \) with \( \int f \gamma_n = 1 \) and such that for some \( \beta > 0 \) and all \( x \in \mathbb{R}^n \),

\[
\nabla^2 \log f(x) \succeq -\beta.
\]

We will use the processes and measures defined in Section 2.2 (which depend on \( f \)).

Assume that a bound of the form

\[
\gamma_n \left( \{ z \in \mathbb{R}^n : f(z) \in [\alpha, e\alpha] \} \right) \leq \frac{\psi(\alpha)}{\alpha}
\]

(18)
holds for all $\alpha \geq 1$ and some monotonically non-increasing function $\psi : [1, \infty) \to [1, \infty)$. In that case, we have

$$
\gamma_n \left( \left\{ z \in \mathbb{R}^n : f(z) \geq \alpha \right\} \right) = \sum_{k=0}^{\infty} \gamma_n \left( \left\{ z \in \mathbb{R}^n : f(z) \in [e^{k \alpha}, e^{k+1} \alpha] \right\} \right)
\leq \frac{1}{\alpha} \sum_{k=0}^{\infty} e^{-k \psi(e^k \alpha)} \leq \frac{e}{e-1} \frac{\psi(\alpha)}{\alpha}.
$$

Thus it suffices to prove a bound of the form (18) in Theorem 1.2.

Now recall that, under the measure $Q$, the process $W_t$ is a Brownian motion. Thus for $\alpha > 1$,

$$
\gamma_n \left( \left\{ z \in \mathbb{R}^n : f(z) \in [\alpha, e\alpha] \right\} \right) = \mathbb{P}_Q \left( f(W_1) \in [\alpha, e\alpha] \right)
= \mathbb{E}_Q \left[ \mathbb{1}_{f(W_1) \in [\alpha, e\alpha]} \right]
\leq \frac{1}{\alpha} \mathbb{P} \left( f(W_1) \in [\alpha, e\alpha] \right).
$$

Fix $\alpha > 1$ and for $y \geq 0$, define

$$
q(y) = \mathbb{P}(\log f(W_1) \in [\log \alpha, \log \alpha + y]).
$$

In light of (19), to prove Theorem 1.2 it suffices to argue that for some constant $C > 0$ and $\alpha \geq e^3$, 

$$
q(1) \leq C \frac{\beta (\log \alpha)^4}{\sqrt{\log \alpha}}.
$$

In Section 3 we prove the following main technical lemma. The basic idea is that, under some technical conditions, we have “$q(2y) \geq 2 q(y)$” as long as $y \leq \log_2 \sqrt{\log \alpha}$. After stating the lemma, we employ it to prove (21) along the following line of argument: If $q(1) \gg (\log \alpha)^{-1/2}$, then by repeatedly doubling, we will contradict the fact that $q(\infty) \leq 1$.

**Lemma 2.3** (Expansion of level sets). For $\alpha \geq x^3$, the following holds. Set $\varepsilon := \frac{1}{32 \log_2 \log \alpha}$. Let $y \geq 0$ be given subject to the following constraints:

i) $\varepsilon q(y) \geq \frac{32 e^2 \beta \log \log \alpha}{\sqrt{\log \alpha}},$

ii) $y \leq \frac{\varepsilon (q(y) \log \alpha)}{3 \sqrt{\beta}}.$

Then we have

$$
q(y') \geq (2 - 12 \varepsilon) q(y)
$$

for the value

$$
y' = y \left( 2 + \frac{32 e^2 \beta \log \log \alpha}{\varepsilon q(y) \sqrt{\log \alpha}} \right) + 5 \log \log \alpha + 3.
$$

For now, we indicate how this lemma allows us to complete the proof of the main theorem.
Proof of Theorem 1.2. Recall that $\alpha \geq e^3$. Let $y_0 := 5 \log \log \alpha + 3$, and assume, for the sake of contradiction, that

$$q(y_0) \geq \frac{9e^{11} \beta y_0^2}{\varepsilon^2 \sqrt{\log \alpha}}.$$  \hfill (23)

If this is not satisfied, then

$$q(1) \leq q(y_0) \leq \frac{O(\beta)(\log \log \alpha)^4}{\sqrt{\log \alpha}},$$

and we are done.

For $k \geq 0$, put

$$y_{k+1} := y_k \left(2 + \frac{32e^2 \sqrt{\beta \log \log \alpha}}{\varepsilon q(y_k) \sqrt{\log \alpha}}\right) + y_0.$$

Let $k^* = \min(2 \log_2 \log \alpha, k')$ where $k' \in \mathbb{N}$ is the smallest value such that condition (ii) of the lemma fails to hold for $y = y_{k'+1}$. Observe that for $k \leq k^* \leq 2 \log_2 \log \alpha$, we have

$$(2 - 12 \varepsilon)^k = 2^k (1 - 6 \varepsilon)^k \geq 2^k (1 - 6 \varepsilon k) \geq 2^{k-1}. \hfill (24)$$

Using this in conjunction with (24), we see that if $k \leq k^* + 1$, then

$$q(y_k) \geq 2^{k-1} q(y_0). \hfill (25)$$

We will argue that

$$k^* \geq \log_2 \sqrt{\log \alpha}. \hfill (26)$$

In this case, we will contradict the fact that $q(\cdot)$ is a probability:

$$q(y_{k^*+1}) \overset{25}{\geq} \sqrt{\log \alpha} \cdot q(y_0) \overset{23}{>} 1,$$

meaning that (23) cannot hold.

Use (25) along with the fact that condition (i) is verified (see (23)) to conclude that for $k \leq k^*$,

$$y_{k+1} \leq y_k \left(2 + 2^{-k+1}\right) + y_0 = y_k \left(2 + 2^{-k+1} + \frac{y_0}{y_k}\right).$$

Setting $a_k = \log_2 y_k$ and using the fact that $\log_2 (1 + x) \leq 3x/2$, this yields the family of inequalities

$$a_{k+1} \leq a_k + 1 + \frac{3}{2} 2^{-k+1} + 2^{-a_k} y_0,$$

valid for all $k \leq k^*$.

In particular, using the fact that $a_0 = \log_2 y_0$, one concludes immediately that for $k \leq k^*$, we have

$$a_{k+1} \leq k + 5 + \log_2 y_0. \hfill (27)$$

Using again (25), the following inequality implies that condition (ii) is valid for $y = y_{k+1}$:

$$\log_2 y_{k+1} \leq \log_2 (e) \log_2 (3 \sqrt{\beta}) + 1 + \frac{1}{2} k \log_2 \log \alpha + \frac{k}{2} + \frac{1}{2} \log_2 q(y_0).$$

Plugging in (27) and rearranging, the preceding inequality is implied by

$$k \leq 2 \log_2 (e) - 2 \log_2 (3 \sqrt{\beta}) + \log_2 \log \alpha - 10 - 2 \log_2 y_0 + \log_2 q(y_0).$$

Recalling our assumption (23), we see that condition (ii) is valid for $k \leq -1 + \log_2 \sqrt{\log \alpha}$ and thus

$$k^* \geq \log_2 \sqrt{\log \alpha} \text{ as desired, verifying (26).} \hfill \Box$$
3 Anti-concentration of temperature

Our goal is now to prove Lemma 2.3. Section 3.1 sets up an associated family of stochastic processes. In Section 3.2, we provide some preliminary estimates, and in Section 3.3 we complete the proof of Lemma 2.3.

3.1 The perturbations

We couple our process \( W_t \) with a family of Itô processes. Fix \( \alpha \geq e^3 \) and define a stopping time

\[
T = 1 \land \inf \left\{ t : \int_0^t \|v_s\|^2 \, ds \geq 2 \log \alpha \right\} \land \inf \left\{ t : \left| \int_0^t \langle v_s, dB_s \rangle \right| \geq 4 \sqrt{\log \alpha \log \log \alpha} \right\}.
\]

For \( \delta \in [0, 1] \), we define \( \{X^\delta_t : t \in [0, 1]\} \) by

\[
X^\delta_0 = 0, \quad dX^\delta_t = dB_t + (1 + \delta 1_{\{t \leq T\}}) v_t \, dt.
\]

Note that due to Theorem 2.1(ii), a unique solution exists, thus \( X^\delta_t \) is well-defined.

Next, we would like to argue that Girsanov’s formula (see, e.g., [LS11, Chapter 6]) applies so that \( \{X^\delta_t : t \in [0, 1]\} \) has the law of a Brownian motion under the change of measure

\[
dQ_\delta = \exp \left( -\int_0^1 (1 + \delta 1_{\{t \leq T\}}) \langle v_t, dB_t \rangle - \frac{1}{2} \int_0^1 (1 + \delta 1_{\{t \leq T\}})^2 \|v_t\|^2 \, dt \right) dP. \tag{28}
\]

To see this, we first notice that by definition of the stopping time \( T \), almost surely

\[
\int_0^1 \|\delta 1_{\{t \leq T\}} v_t\|^2 \, dt \leq 2 \delta^2 \log \alpha.
\]

It follows that

\[
\mathbb{E}_Q \left[ \exp \left( \frac{1}{2} \int_0^1 \|\delta 1_{\{t \leq T\}} v_t\|^2 \, dt \right) \right] < \infty.
\]

In other words, Novikov’s condition holds over the measure \( Q \) for the drift \( \delta 1_{\{t \leq T\}} v_t \), so Girsanov’s formula is valid. In particular, \( \{X^\delta_t : t \in [0, 1]\} \) has the law of a Brownian motion under the change of measure

\[
dQ_\delta = \exp \left( -\int_0^1 \delta 1_{\{t \leq T\}} \langle v_t, dW_t \rangle - \frac{1}{2} \int_0^1 \delta^2 1_{\{t \leq T\}} \|v_t\|^2 \, dt \right) dQ.
\]

Combining this with the change of measure formula (16) yields (28).

From assumption (17) (which comes from (2) in Theorem 1.2), it follows that for all \( z, u \in \mathbb{R}^n \),

\[
f(z + u) \geq f(z) \exp \left( \langle u, \nabla \log f(z) \rangle - \beta \|u\|^2 \right).
\]

Combining this with (10) and fact that \( X^\delta_t = W_1 + \delta \int_0^t v_t \, dt \) yields

\[
f(X^\delta_t) \geq f(W_1) \exp \left( \delta \left( \int_0^T v_t \, dt \right) - \beta \delta^2 \left\| \int_0^T v_t \, dt \right\|^2 \right). \tag{29}
\]
Finally, recalling (16) and (28), we have the expression
\[
f(W_1) \frac{dQ_\delta}{dP} = \frac{dQ_\delta}{dQ} = \exp \left( -\delta \int_0^T \langle v_t, dB_t \rangle - \left( \delta + \frac{\delta^2}{2} \right) \int_0^T \|v_t\|^2 dt \right).
\] (30)

### 3.2 Gradients, stopping times, and the change of measure

For \( \lambda, \gamma \geq 0 \), we define the following two events:
\[
\mathcal{E}_\lambda = \left\{ \int_0^T \langle v_1 - v_t, v_t \rangle dt \leq -\lambda \right\},
\]
\[
\mathcal{B}_\gamma = \left\{ \int_0^T \langle v_t, dB_t \rangle \geq \gamma \sqrt{\log \alpha} \right\}.
\]

The next two lemmas bound the probabilities of these “bad” events. Additionally, the next lemma provides a key estimate on the concentration of the quantity \( f(X_1^\delta) \frac{dQ_\delta}{dP} \).

**Lemma 3.1.** For every \( \lambda \geq 0 \), we have
\[
P(\mathcal{E}_\lambda) \leq \frac{4e^{2\sqrt{\beta \log \alpha \log \log \alpha}}}{\lambda}.
\]

Furthermore, for any event \( A \) with \( q = P(A) > 0 \) and any \( \delta > 0 \), we have
\[
\mathbb{E} \left[ f(X_1^\delta) \frac{dQ_\delta}{dP} 1_A \right] \geq qK^{-1} \left( 1 - 6 \max \left\{ \frac{K - 1}{q}, \sqrt{\frac{K - 1}{q}} \right\} \right),
\] (31)

where \( K := \exp(3\beta\delta^2 \log \alpha) \).

**Proof.** From (29), we derive the pointwise inequality
\[
f(X_1^\delta) \frac{dQ_\delta}{dP} = f(W_1) \frac{f(X_1^\delta) \frac{dQ_\delta}{dP}}{f(W_1)} \geq \exp \left( \delta \int_0^T \langle v_1 - v_t, v_t \rangle dt - \beta \delta^2 \left\| \int_0^T v_t dt \right\|^2 - \delta \int_0^T \langle v_t, dB_t \rangle - \frac{2\delta + \delta^2}{2} \int_0^T \|v_t\|^2 dt \right)
\]
\[
= \exp \left( \delta \int_0^T \langle v_1 - v_t, v_t \rangle dt - \delta \int_0^T \langle v_t, dB_t \rangle \right) \exp \left( -\beta \delta^2 \left\| \int_0^T v_t dt \right\|^2 - \frac{\delta^2}{2} \int_0^T \|v_t\|^2 dt \right).
\] (32)

By definition of the stopping time \( T \), Jensen’s inequality yields
\[
\left\| \int_0^T v_t dt \right\|^2 \leq \int_0^T \|v_t\|^2 dt \leq 2 \log \alpha,
\]

\[14\]
and we have the bound\[\left| \int_0^T \langle v_t, dB_t \rangle \right| \leq 4 \sqrt{\log \alpha \log \log \alpha}.\]

Hence the second factor in (33) is always at least \[C(\beta, \delta)^{-1},\]

where \[C(\beta, \delta) := \exp \left(3\beta \delta^2 \log \alpha + 4\delta \sqrt{\log \alpha \log \log \alpha} \right),\]

recalling that \(\alpha \geq e^3\) and \(\beta \geq 1\). Now fix \(\delta := \frac{1}{4\sqrt{\beta \log \alpha \log \log \alpha}}\) (34)

so that \(C(\beta, \delta) \leq e^2\).

Taking expectations and using the fact that \(1 = \mathbb{E}[f(B_1)] = \mathbb{E}\left[f(X_1^\delta \frac{dQ_1}{dP})\right]\), we conclude

\[\mathbb{E} \left[ \exp \left( \delta \int_0^T \langle v_1 - v_t, v_t \rangle dt \right) \right] \leq \mathbb{E} \left[ f(X_1^\delta) \frac{dQ_1}{dP} C(\beta, \delta) \right] \leq e^2. \tag{35}\]

Now we claim that

\[Z = \int_0^T \langle v_1 - v_t, v_t \rangle dt\]

satisfies \(\mathbb{E}[Z] = 0\). To this end, we use the fact that \(\{v_t\}\) is a martingale (Fact 2.2): For \(0 \leq t_0 \leq t \leq 1\),

\[\mathbb{E} [v_1 | \mathcal{F}_{t_0}] = \mathbb{E} [v_t | \mathcal{F}_{t_0}], \tag{36}\]

where \(\{\mathcal{F}_t\}\) is the filtration underlying the Brownian motion \(B_t\).

Therefore,

\[
\begin{align*}
\mathbb{E} \left[ \int_0^T \langle v_1 - v_t, v_t \rangle dt \right] &= \mathbb{E} \left[ \int_0^1 \langle v_1 - v_t, v_t \rangle dt \right] - \mathbb{E} \left[ \int_T^1 \langle v_1 - v_t, v_t \rangle dt \right] \\
&= -\mathbb{E} \left[ \int_T^1 \langle v_1 - v_t, v_t \rangle dt \right] \\
&= -\mathbb{E} \left( \mathbb{E} \left[ \int_T^1 \langle v_1 - v_t, v_t \rangle dt | T \right] \right) \\
&= -\mathbb{E}(0) = 0. \tag{37}\end{align*}
\]

Let \(p = p(\lambda) = \mathbb{P}(Z \leq -\lambda)\). An application of Jensen’s inequality yields

\[
\mathbb{E} [e^{\delta Z} | Z > -\lambda] \geq e^{\mathbb{E}[\delta Z | Z > -\lambda]} \geq \exp \left( \delta \lambda \frac{p}{1-p} \right).
\]

Combining (35) with the preceding inequality, we learn that

\[e^2 \geq C(\beta, \delta) \geq \mathbb{E}[e^{\delta Z}] \geq (1-p) \exp \left( \delta \lambda \frac{p}{1-p} \right),\]

15
which implies
\[ p \leq \frac{(1-p) \log(e^2/(1-p))}{\delta \lambda} \leq \frac{e^2}{\delta \lambda}. \]
This completes the proof of the first claim of the lemma.

For the second claim, we take \( \delta > 0 \) to be arbitrary. Consider that the second factor in (32) is always at least \( C'(\beta, \delta)^{-1} \), where
\[ C'(\beta, \delta) := \exp(3\beta^2 \log \alpha). \]
Let \( Z' = Z - \int_0^T \langle v_t, dB_t \rangle \). Taking expectations again in (32) yields \( \mathbb{E}[e^{\delta Z'}] \leq C'(\beta, \delta) \). Since we have already shown that \( \mathbb{E}[Z] = 0 \), we have \( \mathbb{E}[Z'] = 0 \) as well since \( T \) is a bounded stopping time and the process \( \{v_t\} \) is predictable.

Consider now any event \( A \) and let \( q = P(A) > 0 \). We denote \( y = \mathbb{E}[\delta Z' | A] \). Using the preceding bound and convexity yields
\[ C'(\beta, \delta) \geq \mathbb{E}[e^{\delta Z'}] = q\mathbb{E}[e^{\delta Z'} | A] + (1-q)\mathbb{E}[e^{\delta Z'} | \bar{A}] \]
\[ \geq qe^{\mathbb{E}[\delta Z' | A]} + (1-q)e^{\mathbb{E}[\delta Z' | \bar{A}]} \]
\[ = qe^{-y} + (1-q)e^{1-qy}. \]
where \( y := -\mathbb{E}[\delta Z' | A] \). The preceding inequality is equivalent to
\[ q(e^{-y} + y - 1) + (1-q) \left( e^{\frac{-y}{1-q}} - \frac{q}{1-q}y - 1 \right) \leq C'(\beta, \delta) - 1. \]
And using the fact that \( e^a \geq 1 + a \), this implies that the second term in the left hand side is non-negative. Consequently,
\[ e^{-y} + y - 1 \leq \frac{C'(\beta, \delta) - 1}{q}. \]
Finally, we observe the inequality \( e^{-y} + y - 1 \geq \min(y, y^2)/6 \) valid for all \( y \in \mathbb{R} \). It follows that
\[ \mathbb{E}[\delta Z' | A] \geq -6 \max \left( \frac{C'(\beta, \delta) - 1}{q}, \sqrt{\frac{C'(\beta, \delta) - 1}{q}} \right). \tag{38} \]
Now multiplying through by \( 1_A \) in (33), and taking expectations, we see that
\[ \mathbb{E} \left[ f(X_1^\delta) \frac{dQ_\delta}{dP} 1_A \right] \geq C'(\beta, \delta)^{-1}\mathbb{E}[e^{\delta Z'} 1_A] \]
\[ = qC'(\beta, \delta)^{-1}\mathbb{E}[e^{\delta Z'} | A] \]
\[ \geq qC'(\beta, \delta)^{-1}(1 + \mathbb{E}[\delta Z' | A]) \]
\[ \geq qC'(\beta, \delta)^{-1} \left( 1 - 6 \max \left( \frac{C'(\beta, \delta) - 1}{q}, \sqrt{\frac{C'(\beta, \delta) - 1}{q}} \right) \right), \]
completing the proof. \( \square \)
Lemma 3.2. For every $\gamma \geq 0$, we have

$$\mathbb{P}(\mathcal{B}_\gamma) \leq e^{-\gamma^2/4}.$$ 

Proof. Consider the quadratic variation process

$$V(t) = \int_0^t \|v_s\|^2 ds.$$ 

According to the theorem of Dambis and Dubins-Schwartz (see, e.g., [RY99, Chapter V, Theorem 1.10]), the process

$$S(t) := \int_0^{V^{-1}(t)} \langle v_s, dB_s \rangle$$ 

is a Brownian motion up to the stopping time $\tau = V(1)$.

Using Doob’s theorem (e.g., [RY99, Chapter II, Theorem 1.7]) and a standard Gaussian tail estimate, we have

$$\mathbb{P}(F_\gamma) \leq e^{-\gamma^2/4},$$

where

$$F_\gamma := \left\{ \max_{t \in [0, 2 \log \alpha]} |S(t)| \geq \gamma \sqrt{\log \alpha} \right\}.$$ 

By definition of the stopping time $T$, we have $V(T) \leq 2 \log \alpha$, thus

$$\mathbb{P}\left(|S(T)| \geq \gamma \sqrt{\log \alpha}\right) \leq \mathbb{P}(F_\gamma),$$

completing the proof in light of (39). \hfill $\square$

3.3 Expansion of the level sets

We repeat the statement of Lemma 2.3 here for the convenience of the reader.

Lemma 3.3. For $\alpha \geq x^3$, the following holds. Set $\varepsilon := \frac{1}{32 \log_2 \log \alpha}$. Let $y \geq 0$ be given subject to the following constraints:

i) $\varepsilon q(y) \geq \frac{32e^2 \sqrt{2 \log \log \alpha}}{\sqrt{\log \alpha}},$

ii) $y \leq \frac{\varepsilon \sqrt{q(y) \log \alpha}}{3\sqrt{3}}.$

Then we have

$$q(y') \geq (2 - 12\varepsilon)q(y)$$

for the value

$$y' = y \left(2 + \frac{32e^2 \sqrt{2 \log \log \alpha}}{\varepsilon q(y) \sqrt{\log \alpha}}\right) + 5 \log \log \alpha + 3.$$
Proof. Define the event $A_y = \{ \log f(W_1) \in [\log \alpha, \log \alpha + y] \}$ and put

$$G := A_y \setminus \{ E_\lambda \cup B_\gamma \}$$

where $\lambda = \frac{4e^2 \sqrt[3]{\log \alpha \log \log \alpha}}{\varepsilon q(y)}$ and $\gamma = 4 \sqrt{\log \log \alpha}$. An application of Lemma 3.1 ensures that $\mathbb{P}(E_\lambda) \leq \varepsilon q(y)$ and an application of Lemma 3.2 together with condition (i) ensures that $\mathbb{P}(B_\gamma) \leq \frac{1}{(\log \alpha)^{\frac{1}{4}}} \leq \varepsilon q(y)$.

Recall that $\mathbb{P}(A_y) = q(y)$ by definition. A union bound yields

$$\mathbb{P}(G) \geq (1 - 2\varepsilon) q(y). \tag{41}$$

Now observe that when $A_y$ occurs, $\text{(15)}$ implies

$$\log \alpha \leq \log f(W_1) = \int_0^1 \langle v_t, dB_t \rangle + \frac{1}{2} \int_0^1 \| v_t \|^2 \, dt. \tag{42}$$

By definition, $B_\gamma$ and $T < 1 \implies \int_0^T \| v_t \|^2 \, dt = 2 \log \alpha$.

Combining this with $\text{(42)}$, we see that conditioned on $A_y \setminus B_\gamma$, we have

$$\int_0^T \| v_t \|^2 \, dt \geq 2 \log \alpha - \gamma \sqrt{\log \alpha}.$$ 

We conclude that

$$G \text{ holds } \implies \int_0^T \langle v_1, v_t \rangle \, dt \geq 2 \log \alpha - \gamma \sqrt{\log \alpha} - \lambda. \tag{43}$$

Now we define

$$\delta := \frac{y}{2 \log \alpha - 4 \lambda}. \tag{44}$$

Recall that condition (i) implies $4 \lambda \leq \log \alpha$ and therefore $\delta \leq \frac{y}{\log \alpha}$. Together with condition (ii), this yields

$$\delta \leq \frac{\varepsilon \sqrt{q(y)}}{3 \sqrt{\beta \log \alpha}}. \tag{45}$$

Using $\text{(43)}$ together with the gradient estimate $\text{(29)}$, we see that, if $G$ holds then

$$f(X_1^\delta) \geq f(W_1) \exp \left( \delta \left( \langle v_1, \int_0^T v_t \, dt \rangle - \beta \delta^2 \left\| \int_0^T v_t \, dt \right\|^2 \right) \right) \geq \alpha \exp \left( 2\delta (\log \alpha - \gamma \sqrt{\log \alpha - \lambda/2}) - 2\beta \delta^2 \log \alpha \right).$$

Since $\text{(45)}$ implies that $\delta \leq (2\beta \log \alpha)^{-1/2}$, we conclude that $2\beta \delta^2 \log \alpha \leq \delta \leq \delta \lambda$. Together with the fact that $\gamma \sqrt{\log \alpha} \leq \lambda$, this finally gives

$$G \text{ holds } \implies f(X_1^\delta) \geq \alpha \exp (2\delta (\log \alpha - 2\lambda)), \implies \log f(X_1^\delta) - \log \alpha \geq 2\delta \log \alpha - 4\delta \lambda. \tag{46}$$
On the other hand,

\[ 1 = \mathbb{E}[f(B_1)] - \mathbb{E} \left[ f(X_{\delta}^t) \frac{dQ_\delta}{dP} \right] \]

(28)

\[ = \mathbb{E} \left[ f(X_{\delta}^t) \exp \left( -\delta \int_0^T \langle v_t, dB_t \rangle - \frac{2\delta + \delta^2}{2} \int_0^T \|v_t\|^2 \right) \right]. \]

(30)

Using the definition of the stopping time \( T \), we have

\[ \mathbb{E} \left[ f(X_{\delta}^t) f(W_1) \right] \leq \exp \left( 2\delta \log \alpha + \delta^2 \log \alpha + 4\delta \sqrt{\log \alpha \log \log \alpha} \right). \]

Since (45) gives us \( \delta \leq 1/\sqrt{4\log \alpha \log \log \alpha} \), an application of Markov’s inequality implies that for any \( \theta > 0 \),

\[ P \left( f(X_{\delta}^t) > \theta \alpha^2 f(W_1) \right) \leq \frac{e^3}{\theta^3}. \]

(44)

By choosing \( \theta = e^3/\varepsilon q(y) \) and recalling that \( \log f(W_1) \leq \log \alpha + y \) whenever \( A_y \) holds, we get

\[ P \left( A_y \text{ and } \left\{ \log f(X_{\delta}^t) > \log \alpha + y + 2\delta \log \alpha + 3 - \log(\varepsilon q(y)) \right\} \right) \leq \varepsilon q(y). \]

(47)

So if we put

\[ I := [\log \alpha + y, \log \alpha + y + 2\delta \log \alpha + 3 - \log(\varepsilon q(y))], \]

then employing (46) with (44) and (47) and using a union bound yields

\[ P \left( \log f(X_{\delta}^t) \in I \right) \geq P(G) - \varepsilon q(y) \geq (1 - 3\varepsilon)q(y). \]

Using this lower bound, we may apply Lemma 3.1, inequality (31), which tells us that

\[ \mathbb{E} \left[ f(X_{\delta}^t) \frac{dQ_\delta}{dP} 1_{\{\log f(X_{\delta}^t) \in I\}} \right] \]

\[ \geq (1 - 3\varepsilon)q(y)K^{-1} \left( 1 - 6 \max \left\{ \frac{K - 1}{(1 - 3\varepsilon)q(y)}, \sqrt{\frac{K - 1}{(1 - 3\varepsilon)q(y)}} \right\} \right), \]

(48)

where \( K := \exp(3\beta \delta^2 \log \alpha) \).

Now, (45) and the fact that \( \varepsilon \leq 1/32 \) by assumption, yields

\[ \delta \leq \varepsilon \sqrt{\frac{(1 - 3\varepsilon)q(y)}{3\beta \log \alpha}}. \]

We infer that \( K \leq \exp \left( \varepsilon^2(1 - 3\varepsilon)q(y) \right) \leq 1 + \frac{3}{2}\varepsilon^2(1 - 3\varepsilon)q(y) \) which, in turn, gives

\[ \frac{K - 1}{(1 - 3\varepsilon)q(y)} \leq \frac{3}{2} \varepsilon^2. \]
Plugging this into (48) and using again that $\varepsilon \leq 1/32$, we arrive at

$$
\mathbb{E} \left[ f(X^t_1) \frac{dQ^t_1}{dP} 1_{\{\log f(X^t_1) \in I\}} \right] \geq (1 - 3\varepsilon) q(y)(1 - \varepsilon^2 q(y))(1 - 8\varepsilon) \\
\geq q(y)(1 - 12\varepsilon).
$$

(49)

Next, we observe that

$$
\mathbb{E} \left[ f(X^t_1) \frac{dQ^t_1}{dP} 1_{\{\log f(X^t_1) \in I\}} \right] = \mathbb{E}[f(B_1) 1_{\{\log f(B_1) \in I\}}] = \mathbb{P}(\log f(W_1) \in I).
$$

Thus by setting $y' = y + 2\delta \log \alpha + 3 - \log(\varepsilon q(y))$, equation (49) finally becomes

$$
q(y') = \mathbb{P}(W_1 \in [\log \alpha, \log \alpha + y]) + \mathbb{P}(W_1 \in [\log \alpha + y, \log \alpha + y']) \geq q(y) (2 - 12\varepsilon).
$$

(50)

Thanks to condition (i) and the fact that $\alpha \geq e^3$ and $\varepsilon \leq 1/32$, we have $-\log(\varepsilon q(y)) \leq 5 \log \log \alpha$.

Finally, observe that

$$
2\delta \log \alpha \overset{\text{(44)}}{=} y \left( \frac{1}{1 - \frac{2\lambda}{\log \alpha}} \right) \overset{\text{(i)}}{\leq} y \left( 1 + \frac{32\varepsilon^2 \sqrt{\beta} \log \log \alpha}{\varepsilon q(y) \sqrt{\log \alpha}} \right),
$$

which yields

$$
y' \leq y \left( 2 + \frac{32\varepsilon^2 \sqrt{\beta} \log \log \alpha}{\varepsilon q(y) \sqrt{\log \alpha}} \right) + 5 \log \log \alpha + 3.
$$

This fact together with (50) completes the proof.

4 Remarks on the discrete cube

It would be quite interesting to extend Corollary 1.4 to other product spaces. In light of Section 1.2 and Conjecture 1.5, the discrete cube \{-1,1\}^n is a prominent example. Many aspects of our proof extend to this setting. If one wants to replace Brownian motion by the standard random walk on \{-1,1\}^n, there are some natural analogs of the process \{W_t\}. To illustrate this, we first offer a proof of the log-Sobolev inequality in \{-1,1\}^n along the lines of [8].

4.1 Log-Sobolev inequalities

Equip \{-1,1\}^n with the uniform measure $\mu$. Consider $f : \{-1,1\}^n \to \mathbb{R}_+$ with $\int f \, d\mu = 1$ and denote the relative entropy $H_\mu(f) := \int f \log f \, d\mu$. For $i \in \{1, \ldots, n\}$, we define

$$(\partial_i f)(x) = \frac{f(x \mid x_i = 1) - f(x \mid x_i = -1)}{2},$$

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where $f(x \mid x_i = b)$ denotes $f(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n)$.

Now consider a random variable $B = (b_1, \ldots, b_n)$ with law $\mu$. We will give a way of sampling a random variable $W = (w_1, \ldots, w_n)$ with density $fd\mu$. Suppose that $w_1, \ldots, w_t$ have been chosen. We define the quantity

$$v_t = \frac{\mathbb{E}[\partial_{t+1}f(B) \mid b_1 = w_1, \ldots, b_t = w_t]}{\mathbb{E}[f(B) \mid b_1 = w_1, \ldots, b_t = w_t]}.$$

Then we define the next bit of $W$ by

$$w_{t+1} = \begin{cases} +1 & \text{with prob. } \frac{1+v_t}{2} \\ -1 & \text{with prob. } \frac{1-v_t}{2}. \end{cases} \quad (51)$$

Here, we have sampled $w_{t+1}$ according to the density $fd\mu$ conditioned on the choices $w_1, \ldots, w_t$. Thus $W$ has law $fd\mu$.

One should verify the equality

$$M_t := \mathbb{E}[f(B) \mid b_1 = w_1, \ldots, b_t = w_t] = \prod_{i=1}^{t} (1 + v_{i-1}w_i). \quad (52)$$

This formula implies that $1/M_n$ is precisely the change of measure under which $(w_1, \ldots, w_t)$ has the law of $(b_1, \ldots, b_t)$. For the sake of analysis, define

$$v^i_t = \frac{\mathbb{E}[\partial_if(B) \mid b_1 = w_1, \ldots, b_t = w_t]}{\mathbb{E}[f(B) \mid b_1 = w_1, \ldots, b_t = w_t]}.$$

Observe that $v^{t+1}_t = v_t$ for all $t \in \{0, 1, \ldots, n-1\}$, and for any fixed $i \in \{1, \ldots, n\}$, the process $M^i_t = \mathbb{E}[\partial_if(B) \mid b_1, \ldots, b_t]$ is a Doob martingale. Therefore the change of measure formula (52) implies that $\{v^i_t\}$ is a martingale (one could also verify this directly from the definition).

Now we may write

$$H_\mu(f) = \mathbb{E}[\log f(W)]$$

$$\leq \sum_{t=1}^{n} \mathbb{E}[\log(1 + v_{t-1}w_t)]$$

$$\leq \sum_{t=1}^{n} \mathbb{E}[v_{t-1}w_t]$$

$$\leq \sum_{t=1}^{n} \mathbb{E}[(v_{t-1})^2] = \sum_{t=1}^{n} \mathbb{E}[(v^i_{t-1})^2] \leq \sum_{t=1}^{n} \mathbb{E}[(v^i_t)^2],$$

where in the final inequality, we have used that $\{v^i_t\}$ is a martingale.

Using the fact that $1/M_n = 1/f(W)$ is the change of measure under which $W$ has the law $\mu$, we have

$$\mathbb{E}[(v^i_t)^2] = \mathbb{E}\left[\frac{\partial_if(W))^2}{f(W)^2}\right] = \mathbb{E}\left[\frac{\partial_if(B))^2}{f(B)}\right].$$

Combining this with the preceding inequality finally yields

$$H_\mu(f) \leq \sum_{i=1}^{n} \int \frac{(\partial_i f)^2}{f} d\mu, \quad (53)$$

21
which is the claimed log-Sobolev inequality.

The astute reader might observe that, in the discrete setting, (53) is called the modified log-Sobolev inequality (see [BT06]), and unlike in the continuous case, can actually be weaker than the usual log-Sobolev inequality:

$$H_\mu(f) \leq 2 \sum_{i=1}^{n} \int (\partial_i \sqrt{f})^2 d\mu. \tag{54}$$

To recover (54) (with constant 4 instead of 2), one should proceed as follows. For the sake of analysis, define the function $f_i(x) = \frac{1}{2} (f(x \mid x_i = 1) + f(x \mid x_i = -1))$, and the value

$$\hat{v}^i_t = \frac{\mathbb{E}[\partial_i f_i(B) \mid b_1 = w_1, \ldots, b_t = w_t]}{\mathbb{E}[f_i(B) \mid b_1 = w_1, \ldots, b_t = w_t]},$$

and note that $v_t = \hat{v}_t^{t+1}$ for all $t \in \{0, 1, \ldots, n - 1\}$.

Next, fix $i \in \{1, \ldots, n\}$, and observe that if $\hat{M}_t^i = \mathbb{E}[f_i(B) \mid b_1 = w_1, \ldots, b_t = w_t]$, then $1/\hat{M}_t^i$ is the change of measure that gives $(w_1, \ldots, w_{i-1}, b_i, w_{i+1}, \ldots, w_n)$ the law of $B$. Thus $\{\hat{v}_t^i\}$ is also a martingale, since it does not depend on the value of the $i$th coordinate.

The proof now proceeds exactly as before to arrive at the inequality

$$H_\mu(f) \leq \sum_{t=1}^{n} \mathbb{E}[(\hat{v}_t^n)^2] \leq \sum_{t=1}^{n} \int (\frac{\partial_i f_i}{f_t})^2 d\mu \leq 4 \sum_{t=1}^{n} \int (\partial_i \sqrt{f})^2 d\mu,$$

where the final inequality uses the simple numerical fact valid for all $a, b \geq 0$:

$$(a - b)^2 \leq 2(\sqrt{a} - \sqrt{b})^2(a + b).$$

We remark that this method seems quite powerful. As an example, a similar line of argument recovers the Lee-Yau log-Sobolev inequality in the symmetric group [LY98]. We discuss this and related issues in a forthcoming manuscript [EL14].

### 4.2 A different family of perturbations

There is a natural analog of $\{X_t^\delta\}$ in the discrete setting. Fix $\delta \in [0, 1]$. Recalling the definition of $W$ from (51) in the preceding section, one might define a coupled random variable $X^\delta = (x_1^\delta, \ldots, x_n^\delta) \in \{-1, 1\}^n$ by assigning

$$x_t^\delta = \begin{cases} w_t & \text{with prob. } 1 - \delta |v_t| \\ \text{sign}(v_t) & \text{with prob. } \delta |v_t| \end{cases}$$

Here, the additional randomness is taken to be independent of that generating $W$.

A significant difficulty over the continuous world is that, while

$$\|\mathbb{E}[X^\delta - W]\|^2 = 4\delta^2 \sum_{t=0}^{n-1} \mathbb{E} |v_t|^2,$$

one has

$$\mathbb{E} \|X^\delta - W\|^2 = 4\delta \sum_{t=0}^{n-1} \mathbb{E} |v_t|.$$
The discrete structure forces the perturbed vector to be quite far from $W$, making it hard to control the value $f(X^\delta)$.

A possible remedy is to consider a different kind of perturbation. Let us move back to the Gaussian setting. We define a new family of processes: Given $\delta > 0$, consider

$$Y^\delta_0 = 0, \quad dY^\delta_t = dB_t + (1 + \delta)\nabla \log P_1 f(Y^\delta_t) \quad \text{for } t \in [0, 1].$$

Note that $\{Y^0_t\}$ has the same law as $\{W_t\}$ (recalling Section 2.2). If one could successfully analyze these processes in the continuous setting, it might provide the proper stage for extension to the discrete cube.

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Proof. The proof is a simple application of the fact that a mixture of log-convex densities is log-convex (see, e.g., [MOA11, p.649]). Observe that for any $y \in \mathbb{R}^n$, the function
\[
x \rightarrow \frac{|x|^2}{2t} + \log \left( P_t(\delta_y)(x) \right)
\]
is convex (here, $\delta_y$ denotes a Dirac mass supported on $\{y\}$). We now apply the aforementioned fact to conclude that for any integrable function $g : \mathbb{R}^n \to [0, \infty)$, the function
\[
x \rightarrow \frac{|x|^2}{2t} + \log \left( \int_{\mathbb{R}^n} g(y) \left( P_t(\delta_y)(x) \right) dy \right)
\]
must also be convex. In other words, the function
\[
x \rightarrow \frac{|x|^2}{2t} + \log P_t(g)
\]
is convex. We conclude that
\[
\nabla^2 \log P_t(g) \succeq -\nabla^2 \left( \frac{|x|^2}{2t} \right) = -\frac{1}{t} \text{Id}.
\]