COARSE-GRAINING OPEN MARKOV PROCESSES

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Abstract. Coarse-graining is a standard method of extracting a simple Markov process from a more complicated one by identifying states. Here we extend coarse-graining to open Markov processes. An 'open' Markov process is one where probability can flow in or out of certain states called ‘inputs’ and ‘outputs’. One can build up an ordinary Markov process from smaller open pieces in two basic ways: composition, where we identify the outputs of one open Markov process with the inputs of another, and tensoring, where we set two open Markov processes side by side. In previous work, Fong, Pollard and the first author showed that these constructions make open Markov processes into the morphisms of a symmetric monoidal category. Here we go further by constructing a symmetric monoidal double category where the 2-morphisms are ways of coarse-graining open Markov processes. We also extend the already known ‘black-boxing’ functor from the category of open Markov processes to our double category. Black-boxing sends any open Markov process to the linear relation between input and output data that holds in steady states, including nonequilibrium steady states where there is a nonzero flow of probability through the process. To extend black-boxing to a functor between double categories, we need to prove that black-boxing is compatible with coarse-graining.
1. Introduction

An ‘open’ Markov process is one in which probability can flow in or out of certain states designated as ‘inputs’ and ‘outputs’:

Open Markov processes can be seen as morphisms in a category, since we can compose two open Markov processes by identifying the outputs of the first with the inputs of the second. We can also ‘tensor’ two open Markov processes by setting them side by side. These operations allow us to build a Markov process from smaller open parts—or conversely, to analyze the behavior of a potentially large, complex Markov process in terms of its parts. Research along these lines is part of a broader initiative to study open systems with the help of category theory [4, 15, 22].

In this paper we continue the study of open Markov processes and extend it to include coarse-graining. Coarse-graining is a widely studied method of simplifying a Markov process by mapping its set of states onto some smaller set in a manner that respects, or at least approximately respects, the dynamics [1, 8]. Here we introduce coarse-graining for open Markov processes. Since open Markov processes are already morphisms in a category, and coarse-graining maps one open Markov process to another, it is natural to treat coarse-graining as a ‘morphism between morphisms’, or ‘2-morphisms’.

We can do this using a structure known as a ‘double category’. Double categories were first introduced by Ehresmann [12, 13], and have long been used in topology and other branches of pure mathematics [9, 10]. More recently they have been used to study open dynamical systems [20] and open discrete-time Markov chains [21]. So, it should not be surprising that they are also useful for open Markov processes.

A 2-morphism in a double category looks like this:

While a mere category has only objects and morphisms, here we have a few more types of entities. We call $A, B, C$ and $D$ ‘objects’, $f$ and $g$ ‘vertical 1-morphisms’, $M$ and $N$ ‘horizontal 1-cells’, and $a$ a ‘2-morphism’. The barred arrows for horizontal 1-cells help us distinguish them from vertical 1-morphisms even when we write the latter horizontally, e.g., $f: A \rightarrow C$. We can compose vertical 1-morphisms to get new vertical 1-morphisms, much as in a category. Similarly we can compose horizontal 1-cells. We can compose the 2-morphisms in two ways: horizontally by setting squares side by side, and vertically by setting one on top of the other. In an ordinary ‘strict’ double category all these forms of composition are associative. In a ‘pseudo’ double category, horizontal 1-cells compose in a weakly associative manner: that is, the associative law holds only up to an invertible 2-morphism, called the ‘associator’. This is just a quick sketch of the idea, so for the full definitions one should turn elsewhere, for example the works of Grandis and Paré [16, 17].
The only Markov processes we study are continuous-time Markov processes with a finite set of states. Thus, we construct a double category $\operatorname{CoarseMark}$ with:

(i) finite sets as objects,
(ii) open Markov processes as horizontal 1-cells,
(iii) bijections between finite sets as vertical 1-morphisms, and
(iv) coarse-grainings of open Markov processes as 2-morphisms.

Composition of open Markov processes is only weakly associative, so this is a pseudo double category.

The plan of the paper is as follows. In Section 2 we define open Markov processes, discuss some of their basic properties, and establish some notation used throughout the rest of the paper. In Section 3 we introduce coarse-graining first for Markov processes and then open Markov processes. In Section 4 we introduce a category $\operatorname{CoarseMark}$ whose objects are Markov processes and whose morphisms are coarse-grainings. In Section 5 we go further and construct the double category $\operatorname{CoarseMark}$ described above. We prove this is a ‘symmetric monoidal’ double category, in the sense of Shulman [23]. This implies that we can not only compose open Markov processes but also ‘tensor’ them, which has the effect of setting them side by side. For example, if we compose this open Markov process:
but if we tensor them we obtain this:

The key new feature of an open Markov process is that probability can flow in and out at both its inputs or its outputs. To describe these modified dynamics, Fong, Pollard and the first author \cite{4} introduced a generalization of the usual master equation for Markov processes, called the ‘open master equation’. In this equation the probabilities at input and output states are arbitrary specified functions of time, while the probabilities at other states obey the usual master equation. As a result, the probabilities are not necessarily normalized. We interpret this by saying probability can flow either in or out at both the input and the output states.

If we fix constant probabilities at the inputs and outputs, there typically exist solutions of the open master equation with these boundary conditions that are constant as a function of time. These are called ‘steady states’. Typically they are nonequilibrium steady states, meaning that there is a nonzero flow of probabilities at the inputs and outputs. For example, probability can flow through an open Markov process at a constant rate in a nonequilibrium steady state.

Fong, Pollard and the first author studied the relation between probabilities and flows at the input and output states that holds in nonequilibrium steady states. They called the process of extracting this relation from an open Markov process ‘black-boxing’, and they proved that black-boxing is compatible with composition and tensoring. This result can be summarized by saying that black-boxing is a symmetric monoidal functor. (More precisely, they proved this result for open Markov processes obeying a detailed balance condition \cite{4}. Then a subset of these authors generalized this result to arbitrary open Markov processes, and even further, to certain nonlinear open dynamical systems \cite{5}.)

In Section 6 we show that black-boxing is compatible with coarse-graining. To make this idea precise, we prove that black-boxing gives a map from $\text{CoarseMark}$ to another double category, called $\text{LinRel}$, which has:

(i) finite-dimensional real vector spaces as objects,
(ii) linear relations as horizontal 1-cells,
(iii) linear maps as vertical 1-morphisms, and
(iv) linear maps between linear relations as 2-morphisms.

Here a ‘linear relation’ from a vector space $V$ to a vector space $W$ is simply a linear subspace of $V \oplus W$. Since linear relations are linear subspaces, we can define linear maps between them, and these are our 2-morphisms. This double category $\text{LinRel}$ becomes
symmetric monoidal using direct sum as the ‘tensor product’, but unlike \( \text{CoarseMark} \) it is strict: that is, composition of linear relations is associative.

Maps between symmetric monoidal double categories are called ‘symmetric monoidal double functors’ \([11]\). Our main result, Thm. \( 25 \), says that black-boxing gives a symmetric monoidal double functor

\[
\blacksquare : \text{CoarseMark} \to \mathbb{LinRel}.
\]

The hardest part is to show that black-boxing preserves composition of horizontal 1-cells: that is, black-boxing a composite of open Markov processes gives the composite of their black-boxings. Luckily, for this we can adapt an earlier argument for black-boxing open dynamical systems \([5]\). Thus, the new content of this result concerns the vertical 1-morphisms and especially the 2-morphisms, which describe coarse-grainings.

An alternative approach to studying 2-morphisms uses ‘bicategories’ rather than double categories \([7, 24]\). In Section 7 we use a result of Shulman \([23]\) to construct symmetric monoidal bicategories \( \text{CoarseMark} \) and \( \mathbb{LinRel} \) from the symmetric monoidal double categories \( \text{CoarseMark} \) and \( \mathbb{LinRel} \). We conjecture that the black-boxing double functor determines a functor between these symmetric monoidal bicategories. However, double categories seem to be a simpler framework for coarse-graining open Markov processes.

It is worth comparing some related work. Fong, Pollard and the first author have constructed a symmetric monoidal category where the morphisms are open Markov processes \([4]\). Like us, they only consider Markov processes where time is continuous and the set of states is finite. However, they formalized such Markov processes in a slightly different way than we do here: they defined a Markov process to be a directed multigraph where each edge is assigned a positive number called its ‘rate constant’. In other words, they defined it to be a diagram

\[
\begin{array}{ccc}
(0, \infty) & \xrightarrow{r} & E \\
& \xleftarrow{s} & X
\end{array}
\]

where \( X \) is a finite set of vertices or ‘states’, \( E \) is a finite set of edges or ‘transitions’ between states, the functions \( s, t: E \to S \) give the source and target of each edge, and \( r: E \to (0, \infty) \) gives the rate constant of each edge. They explained how from this data one can extract a matrix of real numbers \( (H_{ij})_{i,j \in X} \) called the ‘Hamiltonian’ of the Markov process, with two familiar properties:

(i) \( H_{ij} \geq 0 \) if \( i \neq j \),
(ii) \( \sum_i H_{ij} = 0 \).

A matrix with these properties is called ‘infinitesimal stochastic’, since these conditions are equivalent to \( \exp(tH) \) being stochastic for all \( t \geq 0 \). In our work we skip the directed multigraphs and work directly with the Hamiltonians. Thus, we define a Markov process to be a finite set \( X \) together with an infinitesimal stochastic matrix \( (H_{ij})_{i,j \in X} \). This allows us to work more directly with the Hamiltonian and the all-important ‘master equation’

\[
\frac{dp(t)}{dt} = Hp(t)
\]

which describes the evolution of a time-dependent probability distribution \( p(t): X \to \mathbb{R} \).

Clerc, Humphrey and Panangaden have constructed a bicategory \([21]\) with finite sets as objects, ‘open discrete labeled Markov processes’ as morphisms, and ‘simulations’ as 2-morphisms. In their framework, ‘open’ has a similar meaning as it does in works listed above. These open discrete labeled Markov processes are also equipped with a set of ‘actions’ which represent interactions between the Markov process and the environment, such as an outside entity acting on a stochastic system. A ‘simulation’ is then a function
between the state spaces that map the inputs, outputs and set of actions of one open discrete labeled Markov process to the inputs, outputs and set of actions of another.

Another compositional framework for Markov processes is given by de Francesco Albasini, Sabadini and Walters [18] in which they construct an algebra of ‘Markov automata’. A Markov automaton is a family of matrices with non-negative real coefficients that is indexed by elements of a binary product of sets, where one set represents a set of ‘signals on the left interface’ of the Markov automata and the other set analogously for the right interface.

**Notation and Terminology.** Following Shulman, we use ‘double category’ to mean ‘pseudo double category’, and use ‘strict double category’ to mean a double category for which horizontal composition is strictly associative and unital. (In older literature, ‘double category’ often refers to a strict double category.)

It is common to use blackboard bold for the first letter of the name of a double category, and we do so here. Ordinary categories are written in roman font, while bicategories are written in boldface. Thus, three main players in this paper are a category $\text{CoarseMark}$, a double category $\text{CoarseMark}$ and a bicategory $\text{CoarseMark}$, all closely related.

2. Open Markov processes

Before explaining open Markov processes we should recall a bit about Markov processes. As mentioned in the Introduction, we use ‘Markov process’ as a short term for ‘continuous-time Markov process with a finite set of states’, and we identify any such Markov process with the infinitesimal stochastic matrix appearing in its master equation. We make this precise with a bit of terminology that is useful throughout the paper.

Given finite sets $X$ and $Y$, we call a function $T: Y \times X \to \mathbb{R}$ a ‘matrix’, and we write the image of $(i, j) \in Y \times X$ as $T_{ij}$. This is only a matrix in the traditional sense if we choose an ordering on $X$ and $Y$, but nothing we do shall ever require such an ordering, so the abuse of language is harmless.

Similarly, we call a function $v: X \to \mathbb{R}$ a ‘vector’ and call its values $v_i$ for $i \in X$ its ‘components’. We define a ‘probability distribution’ on $X$ to be a vector $p: X \to \mathbb{R}$ whose components are nonnegative and sum to 1. As usual, we use $\mathbb{R}^X$ to denote the vector space of functions $v: X \to \mathbb{R}$. We identify a matrix $H: X \times X \to \mathbb{R}$ with the linear operator from $\mathbb{R}^X$ to $\mathbb{R}^Y$ that sends any vector $v \in \mathbb{R}^X$ to the vector $Tv \in \mathbb{R}^Y$ with components $(Tv)_i = \sum_{j \in X} T_{ij} v_j$.

**Definition 1.** A matrix $H: X \times X \to \mathbb{R}$ is said to be infinitesimal stochastic if

(i) $H_{ij} \geq 0$ for $i \neq j$ and
(ii) $\sum_{j \in X} H_{ij} = 0$ for each $i \in X$.

The reason for being interested in infinitesimal stochastic matrices is that when exponentiated they give stochastic matrices.

**Definition 2.** A matrix $T: Y \times X \to \mathbb{R}$ is stochastic if it maps probability distributions on $X$ to probability distributions on $Y$: that is, if $p \in \mathbb{R}^X$ is a probability distribution then so is $Tp \in \mathbb{R}^Y$.

It is well known [2] that a matrix $H: X \times X \to \mathbb{R}$ is infinitesimal stochastic if and only if the matrix

$$\exp(tH) = \sum_{n=0}^{\infty} \frac{(tH)^n}{n!}$$
is stochastic for all $t \geq 0$. Thus, given an infinitesimal stochastic matrix $H$, we can apply the operators $\exp(tH) : \mathbb{R}^X \to \mathbb{R}^X$ to any probability distribution $p \in \mathbb{R}^X$ and get a probability distribution

$$p(t) = \exp(tH)p$$

for any time $t \geq 0$. It is easy to check that these probability distributions obey the master equation

$$\frac{dp(t)}{dt} = H p(t).$$

Moreover, any solution of the master equation arises this way [2].

All the material so far is standard. We now turn to open Markov processes and the open master equation, which are a bit newer. As mentioned in the Introduction, we take a slightly different approach than in our previous work [4]:

**Definition 3.** We define a Markov process to be a pair $(X, H)$ where $X$ is a finite set and $H : X \times X \to \mathbb{R}$ is an infinitesimal stochastic matrix. We also call $H$ a Markov process on the set $X$.

**Definition 4.** We define an open Markov process to consist of finite sets $X$, $S$ and $T$ and functions

$$i : S \to X$$

$$o : X \to T$$

together with a Markov process $(X, H)$. We call the $S$ the set of inputs and $T$ the set of outputs.

In general, a diagram of this shape in any category:

$$X$$

$$\begin{array}{c}
S \\
i \nearrow \searrow o
\end{array}$$

$$T$$

is called a cospan. The objects $S$ and $T$ are called the feet, the object $X$ is called the apex, and the morphisms $i$ and $o$ are called the legs. We use FinSet to stand for the category of finite sets and functions. Thus, an open Markov process is a cospan in FinSet together with a Markov process on its apex. We often abbreviate an open Markov process as

$$S \to (X, H) \leftarrow T.$$
We write $f: X \rightarrow Y$ as morphisms. In other words, $f: X \rightarrow Y$ assigns to any state in $X$ the total inflow at that state. Similarly, we define $o_*(O): R \rightarrow R^X$ by

$$o_*(O)x(t) = \sum_{[t, o(t) = x]} Ox(t).$$

With this notation, the **open master equation** is

$$\frac{dp(t)}{dt} = Hp(t) + i_*(I(t)) - o_*(O(t)).$$

This says that for any state $x \in X$ the time derivative of the probability $p_x(t)$ takes into account not only the usual term from the master equation, but also the sum of all inflows at $x \in X$ such that $i(x) = x$, minus the sum of outflows at $t \in T$ such that $o(t) = x$.

If the inflows and outflows are constant in time, a solution of the open master equation where $p: R \rightarrow R^X$ is also constant in time is called a **steady state**. More formally:

**Definition 5.** Given an open Markov process $S: (X, H) \leftarrow T$ together with $I \in R^S$ and $O \in R^T$, a **steady state** with inflows $I$ and outflows $O$ is an element $p \in R^X$ such that

$$Hp + i_*(I) - o_*(O) = 0.$$ 

Given $p \in R^X$, we can define vectors $i^*p \in R^S$ and $o^*p \in R^T$ by

$$(i^*p)_s = (p \circ i)_s,$$

and

$$(o^*p)_t = (p \circ o)_t.$$ 

We call these **input probabilities** and **output probabilities**, respectively.

**Definition 6.** Given an open Markov process $S: (X, H) \rightarrow T$ we define its **black-boxing** to be the set

$$\mathcal{B}(S: (X, H) \rightarrow T) \subseteq R^S \oplus R^S \oplus R^T \oplus R^T$$

consisting of all 4-tuples $(i^*p, I, o^*p, O)$ where $p \in R^X$ is some steady state with inflows $I \in R^S$ and outflows $O \in R^T$.

Thus, black-boxing records the relation between input probabilities, inflows, output probabilities and outflows that holds in steady state. This is the ‘externally observable steady state behavior’ of the open Markov process.

### 3. Coarse-graining

To understand coarse-graining it will be useful to think of functions between finite sets as a special case of stochastic matrices, which in turn are a special case of matrices.

Starting with the most general concept here, there is a category $\text{Mat}(R)$ having finite sets as objects, where a morphism from $X$ to $Y$ is a matrix $H: Y \times X \rightarrow R$, or equivalently, a linear operator $H: R^X \rightarrow R^Y$. Composition in $\text{Mat}(R)$ is the usual composition of linear operators. We use wiggly arrows for morphisms in $\text{Mat}(R)$: thus, $f: X \rightsquigarrow Y$ will be our shorthand for a matrix $H: Y \times X \rightarrow R$ or linear operator $H: R^X \rightarrow R^Y$.

At the other extreme, there is a category $\text{FinSet}$ having finite sets as objects and functions as morphisms. We write $f: X \rightarrow Y$ with a straight arrow for a function between finite sets. But straight arrows can be seen as a special case of wiggly arrows. For any finite set
we shall show that given a Markov process $X$, the vector space $\mathbb{R}^X$ has a basis $\{e_j\}_{j \in X}$ where $e_j$ is the function that equals 1 at $j$ and zero elsewhere. Any function $f: X \rightarrow Y$ between finite sets gives a linear operator from $\mathbb{R}^X$ to $\mathbb{R}^Y$ that sends each basis element $e_j$ of $\mathbb{R}^X$ to the basis element $e_{f(j)}$ of $\mathbb{R}^Y$. Composition of functions then becomes a special case of composition in Mat($\mathbb{R}$). In short, FinSet is a subcategory of Mat($\mathbb{R}$).

Thanks to this, we can compose two straight arrows and get a straight arrow, compose two wiggly arrows and get a wiggly arrow, and compose a straight arrow with a wiggly arrow in either order and get a wiggly arrow.

An intermediate case is a stochastic matrix $s: X \rightarrow Y$. This is a linear operator $s: \mathbb{R}^X \rightarrow \mathbb{R}^Y$ that maps probability distributions to probability distributions. Equivalently, a matrix $s: Y \times X \rightarrow \mathbb{R}$ is stochastic if and only if

(i) $s_{ij} \geq 0$ for all $i \in Y$, $j \in X$,

(ii) $\sum_{i \in X} s_{ij} = 1$ for all $j \in X$.

If we think of $s_{ij}$ as the probability for $j \in X$ to be mapped to $i \in I$, these conditions make intuitive sense. Since stochastic matrices are those that preserve probability distributions, the composite of stochastic matrices is stochastic. There is thus a category FinStoch with finite sets as objects and stochastic matrices as morphisms [6]. Since we do not wish to burden the reader with yet another style of arrow for morphisms in FinStoch, we use wiggly arrows for these.

An important special case of a stochastic matrix is one coming from a function $f: X \rightarrow Y$. This gives a stochastic matrix where the probabilities $s_{ij}$ are all 0 or 1. This makes FinSet into a subcategory of FinStoch, which in turn is a subcategory of Mat($\mathbb{R}$):

$$\text{FinSet} \subset \text{FinStoch} \subset \text{Mat}(\mathbb{R}).$$

With these prerequisites out of the way, we can turn to coarse-graining. There are various ways to approximate a Markov process by another Markov process on a smaller set, all of which can be considered forms of coarse-graining. Coarse-graining behaves in a specially nice way for a ‘lumpable’ Markov process [8]:

**Definition 7.** Given a surjection $p: X \rightarrow Y$ of finite sets and a Markov process $H$ on $X$, we say $H$ is **lumpable** with respect to $p$ if $pHe_j = pHe_{j'}$ for all $j, j' \in X$ such that $p(j) = p(j')$.

The surjection $p: X \rightarrow Y$ defines a partition on $X$ where two states $j, j' \in X$ lie in the same block of the partition if and only if $p(j) = p(j')$. The elements of $Y$ correspond to these blocks. The lumpability condition says that $pH: X \rightsquigarrow Y$ agrees on basis vectors $e_j$ and $e_{j'}$ whenever $j$ and $j'$ are in the same block.

Our goal is to take a Markov process $H$ on $X$ and a surjection $p: X \rightarrow Y$ and create a Markov process on $Y$. We describe a general procedure doing this, but it depends on an arbitrary choice unless $H$ is lumpable with respect to $p$. In this special case, the choice turns out not to matter.

What is this choice? It is a choice of a ‘stochastic section’ for $p$:

**Definition 8.** Given a function $p: X \rightarrow Y$ between finite sets, a **stochastic section** for $p$ is a stochastic matrix $s: Y \rightsquigarrow X$ such that $ps = 1_Y$.

It is easy to check that a stochastic section for $p$ exists iff $p$ is a surjection. In Thm. 15 we shall show that given a Markov process $H$ on $X$ and a surjection $p: X \rightarrow Y$, any stochastic section $s: Y \rightsquigarrow X$ gives a Markov process on $Y$, namely

$$H' = pHs.$$
In Thm. 17 we further show that $H$ is lumpable with respect to $p$ iff this new Markov process $H'$ is independent of $s$. These results are probably well-known to experts, but we give proofs below for completeness.

We thus make the following definition:

**Definition 9.** Given Markov processes $(X, H)$ and $(Y, H')$, a **coarse-graining** from $(X, H)$ to $(Y, H')$ is a surjection $p: X \rightarrow Y$ with a stochastic section $s: Y \rightsquigarrow X$ such that

$$H' = pHs.$$  

In this game, experts call the matrix corresponding to $p$ the **collector matrix**, and they call $s$ the **distributor matrix** [8]. The names help clarify what is going on. The collector matrix, coming from the surjection $p: X \rightarrow Y$, typically maps many states of $X$ to each state of $Y$. The distributor matrix, the stochastic section $s: Y \rightsquigarrow X$, typically maps each state in $Y$ to a linear combination of many states in $X$. Thus, $H' = pHs$ distributes each state of $Y$, applies $H$, and then collects the results.

In Section 4 we construct a category with Markov processes as objects and coarse-grainings between these as morphisms. However, we can also define coarse-graining for **open** Markov processes:

**Definition 10.** A **coarse-graining** from the open Markov process $S \rightarrow (X, H) \leftarrow T$ to the open Markov process $S' \rightarrow (X', H') \leftarrow T'$ is a triple of functions $f: S \rightarrow S'$, $p: X \rightarrow X'$, $g: T \rightarrow T'$ making this diagram commute:

$$
\begin{array}{c}
S \\
\downarrow f \\
S' \\
\downarrow \\
X \\
\downarrow p \\
X' \\
\downarrow \\
T \\
\downarrow g \\
T' \\
\end{array}
$$

**Definition 10.** A **coarse-graining** from the open Markov process $S \rightarrow (X, H) \leftarrow T$ to the open Markov process $S' \rightarrow (X', H') \leftarrow T'$ is a triple of functions $f: S \rightarrow S'$, $p: X \rightarrow X'$, $g: T \rightarrow T'$ making this diagram commute:

$$
\begin{array}{c}
S \\
\downarrow f \\
S' \\
\downarrow \\
(X, H) \\
\downarrow (p, s) \\
(X', H') \\
\downarrow \\
T \\
\downarrow g \\
T' \\
\end{array}
$$

**Definition 10.** A **coarse-graining** from the open Markov process $S \rightarrow (X, H) \leftarrow T$ to the open Markov process $S' \rightarrow (X', H') \leftarrow T'$ is a triple of functions $f: S \rightarrow S'$, $p: X \rightarrow X'$, $g: T \rightarrow T'$ making this diagram commute:

$$
\begin{array}{c}
S \\
\downarrow f \\
S' \\
\downarrow \\
(X, H) \\
\downarrow (p, s) \\
(X', H') \\
\downarrow \\
T \\
\downarrow g \\
T' \\
\end{array}
$$

**Definition 10.** A **coarse-graining** from the open Markov process $S \rightarrow (X, H) \leftarrow T$ to the open Markov process $S' \rightarrow (X', H') \leftarrow T'$ is a triple of functions $f: S \rightarrow S'$, $p: X \rightarrow X'$, $g: T \rightarrow T'$ making this diagram commute:

$$
\begin{array}{c}
S \\
\downarrow f \\
S' \\
\downarrow \\
(X, H) \\
\downarrow (p, s) \\
(X', H') \\
\downarrow \\
T \\
\downarrow g \\
T' \\
\end{array}
$$

We often abbreviate a coarse-graining between open Markov processes as

Recall that the existence of a stochastic section for $p$ forces $p$ to be a surjection.
As an example, consider the following open Markov process:

![Open Markov process diagram]

This is a way of drawing an open Markov process with state space

\[ S = \{x_1, v_1, v_2, w_1, w_2, u_1\} \]

and infinitesimal stochastic matrix \( H : S \times S \to \mathbb{R} \) given as follows:

\[
H = \begin{pmatrix}
-5 & 0 & 0 & 0 & 0 \\
5 & -16 & 0 & 0 & 0 \\
0 & 8 & -6 & 4 & 0 \\
0 & 8 & 0 & -10 & 0 \\
0 & 0 & 6 & 6 & 0
\end{pmatrix}
\]

Let \( S' = \{x, v, w, u\} \). We can define a surjection \( p : S \to S' \) where each element of \( S \) goes to the element of \( S' \) named by the same letter. It is easy to check that \( H \) is lumpable with respect to \( p \). We can choose a stochastic section \( s : S' \rightsquigarrow S \) of \( p \) given as follows:

\[
s = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 / 3 & 0 \\
0 & 0 & 2 / 3 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

The infinitesimal stochastic matrix \( H' = pHs \) is given by

\[
pHs = \begin{pmatrix}
-5 & 0 & 0 & 0 \\
5 & -16 & 0 & 0 \\
0 & 16 & -6 & 0 \\
0 & 0 & 6 & 0
\end{pmatrix}
\]

By Thm. 17, since \( H \) is lumpable with respect to \( p \), this matrix \( pHs \) is independent of our choice of a stochastic section \( s \). In this example one can check explicitly that given any other stochastic section \( s' : S' \rightsquigarrow S \) we have \( pHs = pHs' \).

So far we have described a coarse-graining of the Markov process \((S, H)\), but together with identity functions on the inputs \( X \) and \( Y \) this defines a coarse-graining of open Markov processes, going from the above open Markov process to the one drawn as follows:

![Coarse-grained open Markov process diagram]

In Section 5 we construct a double category \textbf{CoarseMark} with open Markov processes as horizontal 1-cells and coarse-grainings of the above sort as 2-morphisms. This double category is our main object of study. First, however, we should prove the results mentioned above. For this it is helpful to introduce a few standard concepts:
Definition 11. A 1-parameter semigroup of operators is a collection of linear operators $U(t): V \rightarrow V$ on a vector space $V$, one for each $t \in [0, \infty)$, such that

(i) $U(0) = 1$

(ii) $U(s + t) = U(s)U(t)$ for all $s, t \in [0, \infty)$. If $V$ is finite-dimensional we say the collection $U(t)$ is continuous if $t \mapsto U(t)v$ is continuous for each $v \in V$.

Definition 12. Let $X$ be a finite set. A Markov semigroup is a continuous 1-parameter semigroup $U(t): \mathbb{R}^X \rightarrow \mathbb{R}^X$ such that $U(t)$ is stochastic for each $t \in [0, \infty)$.

Using our wiggly arrow notation we can write the operators in a Markov semigroup as

Let $p$ be a function between finite sets with a stochastic section $s: Y \rightsquigarrow X$, and let $H: X \rightsquigarrow X$ be an infinitesimal stochastic operator. Then $H' = pHs: Y \rightsquigarrow Y$ is also infinitesimal stochastic.

Proof. Let $U(t) = p\exp(tH)s$. As $H$ is infinitesimal stochastic, by Lemma 13 we have that $\exp(tH)s$ is stochastic for all $t \geq 0$. Since FinSet is a subcategory of FinStoch we can also treat $p$ as a stochastic matrix, and $s$ is stochastic by assumption. Thus, $U(t) = p\exp(tH)s$ is stochastic for all $t = 0$. Differentiating, we conclude that

$$\frac{d}{dt}U(t)\bigg|_{t=0} = \frac{d}{dt}p\exp(tH)s\bigg|_{t=0} = p\exp(tH)Hs\bigg|_{t=0} = pHs$$

is infinitesimal stochastic by Lemma 14. □

Lemma 13. Let $X$ be a finite set and $U(t): X \rightsquigarrow X$ a Markov semigroup. Then $U(t) = \exp(tH)$ for a unique infinitesimal stochastic operator $H: X \rightsquigarrow X$, which is given by

$$ Hv = \left. \frac{d}{dt}U(t)v \right|_{t=0}$$

for all $v \in \mathbb{R}^X$. Conversely, given an infinitesimal stochastic operator $H$, then $\exp(tH) = U(t)$ is a Markov semigroup.

Proof. This is well-known [2, Thm. 17]. □

Lemma 14. Let $U(t): Y \rightsquigarrow Y$ be a differentiable family of stochastic matrices defined for $t \in [0, \infty)$ and having $U(0) = 1$. Then $\frac{d}{dt}U(t)\big|_{t=0}$ is infinitesimal stochastic.

Proof. Let $H = \frac{d}{dt}U(t)\big|_{t=0} = \lim_{t \to 0} (U(t) - 1)/t$. As $U(t)$ is stochastic, its entries are nonnegative and the column sum of any particular column is 1. Then the column sum of any particular column of $U(t) - 1$ will be 0 with the off-diagonal entries being nonnegative. Thus $U(t) - 1$ is infinitesimal stochastic for all $t \geq 0$, as is $(U(t) - 1)/t$, from which it follows that $\lim_{t \to 0} (U(t) - U(0))/t = H$ is infinitesimal stochastic. □

Theorem 15. Let $p: X \rightarrow Y$ be a surjection between finite sets with a stochastic section $s: Y \rightsquigarrow X$, and let $H: X \rightsquigarrow X$ be an infinitesimal stochastic operator. Then $H' = pHs: Y \rightsquigarrow Y$ is also infinitesimal stochastic.

Proof. Let $U(t) = p\exp(tH)s$. As $H$ is infinitesimal stochastic, by Lemma 13 we have that $\exp(tH)s$ is stochastic for all $t \geq 0$. Since FinSet is a subcategory of FinStoch we can also treat $p$ as a stochastic matrix, and $s$ is stochastic by assumption. Thus, $U(t) = p\exp(tH)s$ is stochastic for all $t = 0$. Differentiating, we conclude that

$$\frac{d}{dt}U(t)\bigg|_{t=0} = \frac{d}{dt}p\exp(tH)s\bigg|_{t=0} = p\exp(tH)Hs\bigg|_{t=0} = pHs$$

is infinitesimal stochastic by Lemma 14. □

Lemma 16. Let $p: X \rightarrow Y$ be a surjection of finite sets and let $H$ be a Markov process on $X$. Then $H$ is lumpable with respect to $p$ if and only if there exists a linear operator $H': Y \rightsquigarrow Y$ such that $pH = H'p$.

Proof. Suppose that $H$ is lumpable. Define $H': Y \rightsquigarrow Y$ on basis vectors $e_i \in \mathbb{R}^Y$ by setting

$$ H' e_i = pH e_j$$

for any $j$ with $p(j) = i$. Note that $H'$ is well-defined: since $p$ is a surjection such $j$ exists, and since $H$ is lumpable $H'$ is independent of the choice of such $j$. Next, note that for any $j \in X$, if we choose $i$ with $p(j) = i$ we have $pH e_j = H' e_i = H' p e_j$. Since the vectors $e_j$ form a basis for $\mathbb{R}^X$, it follows that $pH = H'p$. □
Conversely, suppose that there exists a linear operator $H' : Y \rightsquigarrow Y$ such that $pH = H'p$. For any $j \in X$ with $p(j) = i$ we have $pHe_j = H'pHe_j = H'e_i$. Thus, if $j, j'$ have $p(j) = p(j')$ we have $pHe_j = pHe_{j'}$, which says that $H$ is lumpable with respect to $p$.

**Theorem 17.** Let $p : X \to Y$ be a surjection and let $H$ be a Markov process on $X$. Then $H$ is lumpable with respect to $p$ if and only if $pH : Y \rightsquigarrow Y$ is independent of the choice of stochastic section $s$ for $p$.

**Proof.** Let $p : X \to Y$ be a surjection and let $H : X \rightsquigarrow X$ be a Markov process on $X$. If $H$ is lumpable with respect to $p$, then by Lemma 16 there exists a matrix $H' : Y \rightsquigarrow Y$ such that $pH = H'p$. Choose such an matrix; then for any stochastic section $s$ for $p$ we have $pHs = H'ps = H'$.

It follows that $pHs$ is independent of $s$.

Conversely, suppose that $pHs : Y \rightsquigarrow Y$ is independent of the choice of stochastic section $s : Y \rightsquigarrow X$. Such a stochastic section is simply an arbitrary linear operator that maps each $e_i \in \mathbb{R}^Y$ to a probability distribution on $X$ supported on the set \{ $j \in X : p(j) = i$ \}. Thus, for any $j, j' \in X$ with $p(j) = p(j') = i$, we can find stochastic sections $s, s' : Y \rightsquigarrow X$ such that $s(e_i) = e_j$ and $s'(e_i) = e_{j'}$. Since $pHs = pHs'$, we have $pHe_j = pHs(e_i) = pHs'(e_i) = pHe_{j'}$.

This is precisely the definition of $H$ being lumpable with respect to $p$. \qed

4. A category of Markov processes and coarse-grainings

In this section we define a category CoarseMark whose objects are Markov processes and whose morphisms are ways of coarse-graining Markov processes. This is a warmup for the double category that we construct in the next section.

**Theorem 18.** There is a category CoarseMark where:

(i) An object is a Markov process: that is, a pair $(X, H)$ where $X$ is a finite set and $H : X \rightsquigarrow X$ is an infinitesimal stochastic operator.

(ii) A morphism from $(X, H)$ to $(X', H')$ is a coarse-graining: that is, a pair $(p, s)$ where $p : X \to X'$ is a function and $s : X' \rightsquigarrow X$ is a stochastic section of $p$ such that:

$$H' = pHs.$$

(iii) Given morphisms $(p_1, s_1) : (X_1, H_1) \to (X_2, H_2)$ and $(p_2, s_2) : (X_2, H_2) \to (X_3, H_3)$, their composite is given by the pair $(p_2 p_1, s_1 s_2) : (X_1, H_1) \to (X_3, H_3)$.

**Proof.** That the composite of two morphisms $(p_1, s_1)$ and $(p_2, s_2)$ is really a morphism is a straightforward calculation. Namely, if $p_1 : X_1 \to X_2$ and $p_2 : X_2 \to X_3$ are functions with stochastic sections $s_1 : X_2 \rightsquigarrow X_1$ and $s_2 : X_3 \rightsquigarrow X_2$, respectively, then $s_1 s_2 : X_3 \rightsquigarrow X_1$ is a stochastic section of $p_2 p_1 : X_1 \to X_3$, and since

$$H_2 = p_1 H_1 s_1, \quad H_3 = p_2 H_2 s_2$$

we have

$$H_3 = p_2 H_2 s_2 = p_2 (p_1 H_1 s_1) s_2 = (p_2 p_1) H_1 (s_1 s_2).$$

Given an object $(X, H)$, the identity morphism is given by $(\text{id}_X, \text{id}_X)$. Since functions compose associatively and so do stochastic maps, composition in CoarseMark is associative. Since the unit laws for composition hold for functions and stochastic maps, they hold in CoarseMark as well. \qed
We can make the category CoarseMark into a symmetric monoidal category by taking the tensor product of two objects \((X_1, H_1)\) and \((X_2, H_2)\) to be \((X_1 + X_2, H_1 \oplus H_2)\) where \(X_1 + X_2\) is the disjoint union of the sets \(X_1\) and \(X_2\) and \(H_1 \oplus H_2\) is the direct sum of the matrices \(H_1\) and \(H_2\). The unit object is then given by \((\emptyset, 0)\) where 0 denotes the \(0 \times 0\) matrix. The associator, braiding, and left and right unitors come from their counterparts in the symmetric monoidal categories \((\text{FinSet}, +)\) and \((\text{FinVect}, \otimes)\).

**Proposition 19.** The category \(\text{CoarseMark}\) is symmetric monoidal.

**Proof.** One can check this directly, but this result, as well as the previous one, follows from Thm. 29, because \(\text{CoarseMark}\) is just the hom-category \(\text{hom}_{\text{CoarseMark}}(\emptyset, \emptyset)\) in the symmetric monoidal bicategory \(\text{CoarseMark}\) that we discuss in Section 7. The reason is that any Markov process \((X, H)\) can be seen as an open Markov process \(\emptyset \to (X, H) \leftarrow \emptyset\).

\[\square\]

5. A double category of open Markov processes and coarse-grainings

In this section we construct a symmetric monoidal double category \(\text{CoarseMark}\) with open Markov processes as horizontal 1-cells and coarse-grainings as 2-morphisms. Symmetric monoidal double categories were introduced by Shulman [23] and applied to various examples from engineering by the second author [11]. Since the definition is rather long, we urge the reader to those papers rather than recalling it here.

In more detail, the pieces of the double category \(\text{CoarseMark}\) work as follows:

(i) An object is a finite set.
(ii) A horizontal 1-cell is an open Markov process \(S \to (X, H) \leftarrow T\). Recall that an open Markov process from \(S\) to \(T\) is a cospan in \(\text{FinSet}\):

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & X' \\
\downarrow & & \downarrow \\
S & \xleftarrow{f} & T \\
\end{array}
\]

with a Markov process \(H\) on \(X\).

(iii) A vertical 1-morphism \(f: S \to S'\) is a bijection between finite sets.
(iv) A 2-morphism from the open Markov process \(S \to (X, H) \leftarrow T\) to the open Markov process \(S' \to (X', H') \leftarrow T'\) is a coarse-graining:

\[
\begin{array}{ccc}
S & \longrightarrow & (X, H) \leftarrow T \\
\downarrow & (p, s) & \downarrow \\
S' & \longrightarrow & (X', H') \leftarrow T'. \\
\end{array}
\]

where \(f\) and \(g\) are bijections. Recall that a coarse-graining of open Markov processes is a triple of functions \(f: S \to S', p: X \to X', g: T \to T'\) making this
COARSE-GRAINING OPEN MARKOV PROCESSES

diagram commute:

\[
\begin{array}{ccc}
S & \rightarrow & X \\
\downarrow f & & \downarrow p \\
S' & \rightarrow & X'
\end{array}
\quad
\begin{array}{ccc}
& & T \\
& & \downarrow g \\
& & T'
\end{array}
\]

together with a stochastic section \( s: X' \xrightarrow{\sim} X \) of \( p \) such that \( H' = pHs \). Here however we additionally require that \( f \) and \( s \) be bijections.

In this section, we heavily make use of being able to think of a map \( H: X \times X \rightarrow \mathbb{R} \) as a matrix, as explained in Sec. 2. Given two horizontal 1-cells whose target and source coincide:

\[
\begin{array}{ccc}
(X, H) & \rightarrow & (Y, G) \\
\downarrow & & \downarrow \\
S & \rightarrow & T
\end{array}
\]

their composite is given by taking the pushout in FinSet and then ‘cascading’ the infinitesimal stochastic matrices \( H \) and \( G \). Namely, if \( H: X \times X \rightarrow \mathbb{R} \) and \( G: Y \times Y \rightarrow \mathbb{R} \) are infinitesimal stochastic matrices and we have a span of finite sets given by \( i: T \rightarrow X \) and \( o: T \rightarrow Y \), then we can first obtain \( X +_T Y \) by pushing out over \( T \), and then we can obtain a map \( H \odot G: (X +_T Y) \times (X +_T Y) \rightarrow \mathbb{R} \) which is again infinitesimal stochastic: denoting the image of \((x, y)\) as \( f_{xy} \), if \( x \) is not in the image of \( i: T \rightarrow X \), we then have that \( h_{xy} \geq 0 \) for every \( y \) in \( X +_T Y \) and \( \sum_{y \in X +_T Y} f_{yx} = 0 \) since \( H \) is infinitesimal stochastic, and likewise for any \( y \in Y \) that is not in the image of \( o: T \rightarrow Y \). If \( x = i(t) \) for some \( t \in T \), then we ‘overlay’ the columns \( h_{yt} \) and \( g_{yt} \) by adding entries that get identified in the pushout. This results in a column of length \( |X +_T Y| \) that again sums to 0 since every column in \( H \) and \( G \) sum to 0 to begin with. That every non-diagonal entry is non-negative follows from every non-diagonal entry of \( H \) and \( G \) being non-negative.

\[
\begin{array}{ccc}
(X +_T Y, H \odot G) & \rightarrow & \\
\downarrow & & \\
S & \rightarrow & U
\end{array}
\]

This composition is weakly associative as it involves taking pushouts in FinSet. The symmetric monoidal structure on objects is given by the symmetric monoidal structure of \((\text{FinSet}, +, 0)\).

**Theorem 20.** There exists a double category \( \text{CoarseMark} \) as defined above.

**Proof.** The objects, vertical 1-morphisms, horizontal 1-cells and 2-morphisms are given precisely as above. Let \( \text{CoarseMark}_0 \) denote the ‘category of objects’, consisting of finite sets and bijections, and let \( \text{CoarseMark}_1 \) denote the ‘category of arrows’, consisting of
open Markov processes and coarse-grainings

\[
\begin{array}{c}
S \\ f \\ \downarrow \\
\downarrow \\
S' \\
\end{array}
\begin{array}{c}
\rightarrow \ (X, H) \\
(p, s) \\
\uparrow g \\
\leftarrow T' \ \\
\uparrow \\
\rightarrow \\
\end{array}
\begin{array}{c}
\rightarrow \ (X', H') \\
\downarrow \\
\downarrow \\
\leftarrow T' \\
\end{array}
\]

where \( f \) and \( g \) are bijections.

We have functors

\[
U : \text{CoarseMark}_0 \to \text{CoarseMark}_1
\]

\[
S, T : \text{CoarseMark}_1 \to \text{CoarseMark}_0
\]

\[
\otimes : \text{CoarseMark}_1 \times \text{CoarseMark}_0 \to \text{CoarseMark}_1
\]

where if \( Z \) is a finite set, to avoid notational confusion with the functor \( S \), then \( U(Z) \) is given by

\[
\begin{array}{c}
(Z, 0_Z) \\
\downarrow \\
Z \\
\end{array}
\begin{array}{c}
\leftarrow \\
\uparrow f \\
Z' \\
\end{array}
\]

and for a surjection \( f : Z \to Z' \), \( U(f) \) is given by

\[
\begin{array}{c}
Z \\
\downarrow \\
(Z, 0_Z) \\
\end{array}
\begin{array}{c}
\leftarrow \\
\uparrow f \circ f^{-1} \\
Z' \\
\end{array}
\begin{array}{c}
\rightarrow \\
\uparrow f \\
Z' \\
\end{array}
\]

Note that we can choose the inverse \( f^{-1} : Z' \to Z \) as our stochastic section for \( f \) because we are assuming \( f \) is a bijection. This is why we demand that the vertical 1-morphisms be bijections.

The functor \( \otimes \) is for composition of horizontal 1-cells where the pullback is taken over the source and target functors \( S \) and \( T \), which do the obvious things. We also have that

\[
S(U(s)) = s = T(U(s))
\]

and if \( (X, H) \) and \( (Y, G) \) denote the apices of two composable open Markov processes, then

\[
S((Y, G) \otimes (X, H)) = S(X, H)
\]

\[
T((Y, G) \otimes (X, H)) = T(Y, G)
\]

Lastly, we have three natural isomorphisms

\[
\alpha : ((X, H) \otimes (Y, G)) \otimes (Z, I) \to (X, H) \otimes ((Y, G) \otimes (Z, I))
\]

\[
\lambda : U_i \otimes (X, H) \to (X, H)
\]
\[ \rho : (X, H) \odot U \rightarrow (X, H) \]

which are based on the corresponding natural isomorphisms of the double category 
\( \text{Csp}(\text{FinSet}) \) of finite sets, functions, cospans of finite sets and maps of cospans. These 
natural isomorphisms satisfy the triangle and pentagon equations, and with regards to the 
source and target functors \( S \) and \( T \), we have that \( S(\alpha), S(\lambda), S(\rho), T(\alpha), T(\lambda) \) and \( T(\rho) \) are 
all identities.

Next we check the interchange law. In this and subsequent calculations we write a 
2-morphism as

\[
\begin{array}{c}
S \rightarrow (X, H) \leftarrow T \\
\downarrow^f \downarrow \downarrow^p \downarrow \downarrow^g \\
S' \rightarrow (X', pHs) \leftarrow T'.
\end{array}
\]

rather than

\[
\begin{array}{c}
S \rightarrow (X, H) \leftarrow T \\
\downarrow^f \downarrow \downarrow^{(p, s)} \downarrow \downarrow^g \\
S' \rightarrow (X', H') \leftarrow T'.
\end{array}
\]

to slightly lighten the demands on the reader; while technically imprecise this notation 
should be sufficiently clear. Given four 2-morphisms

\[
\begin{array}{c}
\begin{array}{c}
S \rightarrow (X, H) \leftarrow T \\
\downarrow^f \downarrow \downarrow^p \downarrow \downarrow^g \\
S' \rightarrow (X', pHs) \leftarrow T'.
\end{array} & \begin{array}{c}
T \rightarrow (Y, G) \leftarrow U \\
\downarrow^g \downarrow \downarrow^q \downarrow \downarrow^h \\
T' \rightarrow (Y', qGt) \leftarrow U'.
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
S' \rightarrow (X', pHs) \leftarrow T' \\
\downarrow^{f'} \downarrow \downarrow^{p'} \downarrow \downarrow^{g'} \\
S'' \rightarrow (X'', pHss') \leftarrow T''.
\end{array} & \begin{array}{c}
T' \rightarrow (Y', qGt) \leftarrow U' \\
\downarrow^{g'} \downarrow \downarrow^{q'} \downarrow \downarrow^{h'} \\
T'' \rightarrow (Y'', q'Gt') \leftarrow U''.
\end{array}
\end{array}
\]
if we first horizontally compose these, we get

\[
S \quad \xrightarrow{f} \quad (X + T Y, (pH) \odot (qGt)) \quad \xleftarrow{h} \quad U
\]

\[
S' \quad \xrightarrow{f'} \quad (X' + T' Y', (p'Hs) \odot (q'G't')) \quad \xleftarrow{h'} \quad U'
\]

and then composing these vertically gives

\[
S \quad \xrightarrow{f \circ f} \quad (X + T Y, (pH) \odot (qGt)) \quad \xleftarrow{h \circ h} \quad U
\]

\[
S'' \quad \xrightarrow{f' \circ f} \quad (X'' + T'' Y'', (p'Hs') \odot (q'G't'')) \quad \xleftarrow{h' \circ h} \quad U''
\]

If we first compose vertically, this gives

\[
S \quad \xrightarrow{f' \circ f} \quad (X, (p'Hs') \odot (q'G't'')) \quad \xleftarrow{h' \circ h} \quad U
\]
and then composing these horizontally gives

\[
\begin{align*}
S \xrightarrow{f' \circ f} (X + T, Y, H \circ G) & \xleftarrow{h' \circ h} U \\
S'' \xrightarrow{(p' + g') \circ (q' \circ q)} (X'' + T', Y'', (p'Hss') \circ (q'G)) & \xleftarrow{(p'Hss') \circ (q'G)} U''
\end{align*}
\]

The only aesthetic difference between these two is the surjection in the middle, namely \((p' + g') \circ (p + q)\) and \((p' \circ p) + (g' \circ q)\) but these are in fact the same map, and so the interchange law holds. □

Next we give \textbf{CoarseMark} a symmetric monoidal structure. We call the tensor product ‘addition’. Given two objects \(S, S'\) we define their sum \(S + S'\) to be their disjoint union (or coproduct). We can similarly add vertical morphisms, since given bijections \(f : S \to T\) and \(f' : S' \to T'\) there is a natural surjection \(f + f' : S + S' \to T + T'\). Given two horizontal 1-cells,

\[
\begin{align*}
(X_1, H_1) & \xleftarrow{f_1} S_1 \xrightarrow{p_1} T_1 \\
(X_2, H_2) & \xleftarrow{f_2} S_2 \xrightarrow{p_2} T_2
\end{align*}
\]

we can add them to obtain

\[
\begin{align*}
(X_1 + X_2, H_1 \oplus H_2) & \xleftarrow{f_1 + f_2} S_1 + S_2 \xrightarrow{p_1 + p_2} T_1 + T_2
\end{align*}
\]

where, as in the symmetric monoidal category \textbf{CoarseMark}, \(H_1 \oplus H_2 : X_1 + X_2 \to X_1 + X_2\) is the infinitesimal stochastic operator with \(H_1\) and \(H_2\) as blocks. In other words, the maps \(H_1 : X_1 \times X_1 \to \mathbb{R}\) and \(H_2 : X_2 \times X_2 \to \mathbb{R}\) give rise to a map \(H_1 \oplus H_2 : (X_1 + X_2) \times (X_1 + X_2) \to \mathbb{R}\). The unit morphism is given by \(\emptyset \to (0, 0) \leftarrow \emptyset\) where here \(0\) denotes the map \(! : \emptyset \to \emptyset\).

We can also add two 2-morphisms in an obvious way. Given two 2-morphisms

\[
\begin{align*}
S_1 \xrightarrow{f_1} (X_1, H_1) & \xleftarrow{p_1} T_1 \\
S_2 \xrightarrow{f_2} (X_2, H_2) & \xleftarrow{p_2} T_2
\end{align*}
\]

and

\[
\begin{align*}
S_1' \xrightarrow{f_1'} (X_1', p_1 H_1 s_1) & \xleftarrow{p_1} T_1' \\
S_2' \xrightarrow{f_2'} (X_2', p_2 H_2 s_2) & \xleftarrow{p_2} T_2'
\end{align*}
\]
adding these results in

\[ S_1 + S_2 \rightarrow (X_1 + X_2, H_1 \oplus H_2) \leftarrow T_1 + T_2 \]

\[ f_1 + f_2 \]

\[ S_1' + S_2' \rightarrow (X_1' + X_2', (p_1H_1s_1) \oplus (p_2H_2s_2)) \leftarrow T_1' + T_2' \]

and the unit 2-morphism is given by

\[ \emptyset \rightarrow (0, 0 \oplus 0) \leftarrow \emptyset \]

where \( 0 \oplus 0 \) is also the empty matrix. The identity morphism on an object \( S \) is given by \( S \rightarrow (S, 0_3) \leftarrow S \) where \( 0_3 \) is the \( S \times S \) matrix of 0’s and identity 2-morphisms are given by taking the functions \( (f, p, g) \) to all be identities.

**Theorem 21.** The double category \( \text{CoarseMark} \) is symmetric monoidal.

**Proof.** First we note that both the category of objects \( \text{CoarseMark}_0 \) and category of arrows \( \text{CoarseMark}_1 \) are symmetric monoidal categories. The monoidal unit for the category of objects is given by \( \emptyset \) and the monoidal unit for the category of arrows is given by

\[ (0, 0_0) \]

where \( 0_0 \) is the map \( 0_0 : \emptyset \rightarrow \mathbb{R} \). For two horizontal 1-cells \( (X, H) \) and \( (Y, G) \) we have that \( S((X, H) \otimes (Y, G)) = S(X, H) \otimes S(Y, G) \) and \( T((X, H) \otimes (Y, G)) = T(X, H) \otimes T(Y, G) \), and the functors \( S \) and \( T \) also preserve the associativity and unit constraints. There are a fair number of commuting diagrams to check in the definition of symmetric monoidal double category [23] and many of them use two globular 2-isomorphisms, one of which says how the composition of horizontal 1-cells interacts with the tensor product in the category of arrows and another that says how the functor \( U \) relates the tensor product of the category of objects to the tensor product in the category of arrows. We show how to obtain these two globular 2-isomorphisms and then check a few diagrams in the definition.
For horizontal 1-cells \((X_1, H_1), (X_2, H_2), (Y_1, G_1)\) and \((Y_2, G_2)\) given by

\[
\begin{align*}
(X_1, H_1) & \xrightarrow{S_1} (Y_1, G_1) \\
(X_2, H_2) & \xrightarrow{S_2} (Y_2, G_2)
\end{align*}
\]

we have that

\[
(X_1, H_1) \circ (Y_1, G_1) = (X_1 + T_1 Y_1, H_1 \circ G_1)
\]

and

\[
(X_2, H_2) \circ (Y_2, G_2) = (X_2 + T_2 Y_2, H_2 \circ G_2)
\]

are given, respectively, by

\[
\begin{align*}
(X_1 + T_1 Y_1, H_1 \circ G_1) & \xrightarrow{S_1+U_1} (X_1 + T_1 Y_1, H_1 \circ G_1) \\
(X_2 + T_2 Y_2, H_2 \circ G_2) & \xrightarrow{S_2+U_2} (X_2 + T_2 Y_2, H_2 \circ G_2)
\end{align*}
\]

and \((X_1, H_1) \otimes (X_2, H_2) = (X_1 + X_2, H_1 \oplus H_2)\) and \((Y_1, G_1) \otimes (Y_2, G_2) = (Y_1 + Y_2, G_1 \oplus G_2)\) are given, respectively, by

\[
\begin{align*}
(X_1 + X_2, H_1 \oplus H_2) & \xrightarrow{S_1+S_2} (X_1 + X_2, H_1 \oplus H_2) \\
(Y_1 + Y_2, G_1 \oplus G_2) & \xrightarrow{S_1+S_2} (Y_1 + Y_2, G_1 \oplus G_2)
\end{align*}
\]

Then we see that \(((X_1, H_1) \circ (Y_1, G_1)) \otimes ((X_2, H_2) \circ (Y_2, G_2))\) is given by

\[
((X_1 + T_1 Y_1) + (X_2 + T_2 Y_2), (H_1 \circ G_1) \oplus (H_2 \circ G_2))
\]

\[
\begin{align*}
(X_1 + X_2, H_1 \oplus H_2) & \xrightarrow{S_1+S_2} (X_1 + X_2, H_1 \oplus H_2) \\
(Y_1 + Y_2, G_1 \oplus G_2) & \xrightarrow{S_1+S_2} (Y_1 + Y_2, G_1 \oplus G_2)
\end{align*}
\]

and \(((X_1, H_1) \otimes (X_2, H_2)) \circ ((Y_1, G_1) \otimes (Y_2, G_2))\) is given by

\[
((X_1 + X_2) + T_1 T_2 (Y_1 + Y_2), (H_1 \oplus H_2) \otimes (G_1 \oplus G_2))
\]

\[
\begin{align*}
(X_1 + X_2, H_1 \oplus H_2) & \xrightarrow{S_1+S_2} (X_1 + X_2, H_1 \oplus H_2) \\
(Y_1 + Y_2, G_1 \oplus G_2) & \xrightarrow{S_1+S_2} (Y_1 + Y_2, G_1 \oplus G_2)
\end{align*}
\]

The required globular 2-isomorphism

\[
\chi: ((X_1, H_1) \circ (Y_1, G_1)) \otimes ((X_2, H_2) \circ (Y_2, G_2)) \to ((X_1, H_1) \otimes (X_2, H_2)) \circ ((Y_1, G_1) \otimes (Y_2, G_2))
\]
is then given by \((\text{id}, \hat{\chi}, \text{id})\) where \(\hat{\chi}\) is the bijection
\[
\hat{\chi} : (X_1 +_{T_1} Y_1) + (X_2 +_{T_2} Y_2) \rightarrow (X_1 + X_2) +_{T_1+T_2} (Y_1 + Y_2)
\]
in FinSet obtained from taking the colimit of the diagram

\[
\begin{array}{cccc}
X_1 & Y_1 & X_2 & Y_2 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
S_1 & T_1 & U_1 & S_2 \\
& & & \downarrow \\
S + T & S + T & \\
\end{array}
\]

in two different ways. If we denote the domain and codomain of the above \(\hat{\chi}\) as \(\hat{\chi} : N \rightarrow N'\), then \((H_1 \circ G_1) \oplus (H_2 \circ G_2) : N \rightsquigarrow N'\) and the bijection \(\hat{\chi}\) satisfies the equation
\[
\hat{\chi}((H_1 \circ G_1) \oplus (H_2 \circ G_2))\hat{\chi}^{-1} = (H_1 \oplus H_2) \circ (G_1 \oplus G_2) : N' \rightsquigarrow N'.
\]

For the other globular 2-isomorphism, if \(S\) and \(T\) are finite sets, then \(U_{S+T}\) is given by
\[
\begin{array}{c}
(S + T, 0_{S+T}) \\
\downarrow \\
S + T \\
\end{array}
\]

and \(U_S \otimes U_T\) is given by
\[
\begin{array}{c}
(S + T, 0_S \oplus 0_T) \\
\downarrow \\
S + T \\
\end{array}
\]

and so we have a globular 2-isomorphism
\[
\mu : U_{S+T} \rightarrow U_S \otimes U_T
\]
given simply by \((\text{id}, \text{id}, \text{id})\), as the matrices \(0_{S+T}\) and \(0_S \oplus 0_T\) are precisely the same.

Let us check the first diagram given in the definition [23]. The first diagram which must commute is given by

\[
\begin{array}{cccc}
((X_1, H_1) \circ ((Y_1, G_1) \circ (Z_1, I_1))) \circ ((X_2, H_2) \circ ((Y_2, G_2) \circ (Z_2, I_2))) \\
\downarrow \chi \\
((X_1, H_1) \circ (X_2, H_2)) \circ (((Y_1, G_1) \circ (Z_1, I_1)) \circ ((Y_2, G_2) \circ (Z_2, I_2))) \\
\downarrow \text{id} \circ \chi \\
((X_1, H_1) \circ (X_2, H_2)) \circ (((Y_1, G_1) \circ (Z_2, I_2)) \circ ((Y_2, G_2) \circ (Z_1, I_1))) \\
\downarrow \sigma \\
(((X_1, H_1) \circ (X_2, H_2)) \circ ((Y_1, G_1) \circ (Z_2, I_2)) \circ ((Y_2, G_2) \circ (Z_1, I_1))) \\
\downarrow \chi \circ \text{id} \\
(((X_1, H_1) \circ (Y_1, G_1)) \circ ((X_2, H_2) \circ (Y_2, G_2)) \circ ((Z_1, I_1) \circ (Z_2, I_2))) \\
\downarrow \sigma \circ \sigma \\
((X_1, H_1) \circ ((Y_1, G_1) \circ (Z_1, I_1))) \circ ((X_2, H_2) \circ ((Y_2, G_2) \circ (Z_2, I_2)))
\end{array}
\]
where the top and the bottom are to be thought of as coinciding. In other words, starting in the middle of the diagram and going up is the same as starting in the middle of the diagram and going down. For notational purposes, let us define the following horizontal 1-cells:

\[
S_1(\mathbf{X}_1, H_1) = \mathbf{Y}_1 \\
T_1(\mathbf{Y}_1, G_1) = \mathbf{Z}_1 \\
U_1(\mathbf{Z}_1, I_1) = \mathbf{V}_1
\]

\[
S_2(\mathbf{X}_2, H_2) = \mathbf{Y}_2 \\
T_2(\mathbf{Y}_2, G_2) = \mathbf{Z}_2 \\
U_2(\mathbf{Z}_2, I_2) = \mathbf{V}_2
\]

If we ignore the infinitesimal stochastic operators \(H_j, G_j, I_j\) for a moment, the above diagram becomes

\[
S_1 + S_2 \rightarrow (X_1 + T_1 (Y_1 + U_1 Z_1)) + (X_2 + T_2 (Y_2 + U_2 Z_2)) \leftarrow V_1 + V_2
\]

\[
S_1 + S_2 \rightarrow ((X_1 + T_1 Y_1) + U_1 Z_1) + ((X_2 + T_2 Y_2) + U_2 Z_2) \leftarrow V_1 + V_2
\]

\[
S_1 + S_2 \rightarrow ((X_1 + T_1 Y_1) + T_2 (X_2 + Y_2)) + U_1 (Z_1 + Z_2) \leftarrow V_1 + V_2
\]

\[
S_1 + S_2 \rightarrow ((X_1 + X_2) + T_1 (Y_1 + Y_2)) + U_1 (Z_1 + Z_2) \leftarrow V_1 + V_2
\]

\[
S_1 + S_2 \rightarrow ((X_1 + X_2) + T_1 (Y_1 + Y_2)) + U_2 (Z_1 + Z_2) \leftarrow V_1 + V_2
\]

\[
S_1 + S_2 \rightarrow ((X_1 + X_1) + T_1 (Y_1 + U_1 Z_1)) + (Y_2 + U_2 Z_2) \leftarrow V_1 + V_2
\]

\[
S_1 + S_2 \rightarrow ((X_1 + T_1 (Y_1 + U_1 Z_1)) + (X_2 + T_2 (Y_2 + U_2 Z_2)) \leftarrow V_1 + V_2
\]

The vertical 1-morphisms on the left and right are identities, the vertical 1-morphisms in the center and the horizontal 1-morphisms are universal maps, and the top cospan is the same as the bottom cospan making a bracelet-like figure in which all faces commute. The operators \(H_j, G_j, I_j\) together with the above universal maps \(\phi_j\) make the following diagram
Another requirement is that the braiding of the category of arrows be a transformation of double categories, meaning that the following diagram commutes.

This amounts to the following diagram commuting in FinSet:
which follows from the universal property of a colimit of a digram taken in various different ways. The operators $H_j, G_j$ are then related to each other by the diagram

\[
(H_1 \odot G_1) \oplus (H_2 \odot G_2) \xrightarrow{(\psi_1 \circ \cdot \circ \psi_1^{-1})} (H_2 \odot G_2) \oplus (H_1 \odot G_1)
\]

\[
(H_1 \oplus H_2) \odot (G_1 \oplus G_2) \xrightarrow{(\psi_1 \circ \cdot \circ \psi_1^{-1})} (H_2 \oplus H_1) \odot (G_2 \oplus G_1)
\]

For finite sets $S$ and $T$, we have the following diagram

\[
(S + T, 0_{S+T}) \xrightarrow{\mu} (S, 0_S) \otimes (T, 0_T)
\]

\[
U_{S,T} \quad \beta_{U_{S,T}}
\]

\[
(T + S, 0_{T+S}) \xrightarrow{\mu} (T, 0_T) \otimes (S, 0_S)
\]

which clearly commutes. The other diagrams can be shown to commute in a similar way. □

6. Black-boxing for open Markov processes

In a previous work of Pollard and the first author [5], a black-boxing functor $\boxtimes: \text{Dynam} \to \text{SemiAlgRel}$ is constructed where Dynam is a category of finite sets and ‘open dynamical systems’ and SemiAlgRel is a category of finite-dimensional real vector spaces and relations defined by polynomials and inequalities. Roughly speaking, black-boxing an open dynamical system means that only the steady state behavior of the system at the inputs and outputs is observed, as if the open dynamical system were placed inside of a black box and we were unable to see what is going on inside, only what went in and what went out.

A special case of an open dynamical system is an open Markov process as defined in this paper. Thus, we could restrict the black-boxing functor $\boxtimes: \text{Dynam} \to \text{SemiAlgRel}$ to a category Mark with finite sets as objects and open Markov processes as morphisms. Since the steady state behavior of a Markov process is linear, we would get a functor $\boxtimes: \text{Mark} \to \text{LinRel}$ where LinRel is the category of finite-dimensional real vector spaces and linear relations. However, instead of doing this, we will go further and define black-boxing on the double category $\text{CoarseMark}$. This will exhibit the relation between black-boxing and coarse-graining.

To do this, we promote $\text{LinRel}$ to a double category $\text{LinRel}$ with:

(i) finite-dimensional real vector spaces as objects,
(ii) linear maps as vertical 1-morphisms,
(iii) linear relations as horizontal 1-cells,
(iv) linear maps between linear relations as 2-morphisms:

\[ X_1 \xrightarrow{R \subseteq X_1 \oplus X_2} X_2 \]

\[ f \quad \sqsubset \alpha \quad g \]

\[ Y_1 \xrightarrow{S \subseteq Y_1 \oplus Y_2} Y_2. \]

The last item requires some explanation. Since we can compose linear relations, and linear operators are a special case of linear relations, we can form the composites \( g \circ R \) and \( S \circ f \). These are both linear relations from \( X_1 \) to \( Y_2 \), hence linear subspaces of \( X_1 \oplus Y_2 \). In the above diagram, \( \alpha \) stands for a linear map from the subspace \( g \circ R \) to the subspace \( S \circ f \). In what follows we abbreviate this as

\[ \alpha: g \circ R \Rightarrow S \circ f \]

Note that horizontal 1-cells are given by linear relations which compose associatively, so this double category is in fact a strict double category.

**Theorem 22.** There exists a strict double category \( \text{LinRel} \) as described above.

**Proof.** The category of objects is given by real finite-dimensional real vector spaces and linear maps; this is just \( \text{FinVect}_\mathbb{R} \), which is symmetric monoidal. The category of arrows is given by linear relations and linear maps as above. Composition of horizontal 1-cells is given by composing linear relations and composing 2-morphisms is given by

\[ X_1 \xrightarrow{R \subseteq X_1 \oplus X_2} X_2 \]

\[ f \quad \sqsubset \alpha \quad g \]

\[ Y_1 \xrightarrow{S \subseteq Y_1 \oplus Y_2} Y_2 \]

\[ f' \quad \sqsubset \beta \quad g' \]

\[ Z_1 \xrightarrow{T \subseteq Z_1 \oplus Z_2} Z_2 \]

where \( gR \) is a linear subspace of \( X_1 \oplus Y_2 \) and likewise for \( Sf \subseteq X_1 \oplus Y_2 \). Explicitly,

\[ gR = \{ (x_1, y_2): (x_1, x_2) \in R \text{ and } g(x_2) = y_2 \} \subseteq X_1 \oplus Y_2 \]

and

\[ Sf = \{ (x_1, y_2): f(x_1) = y_1 \text{ and } (y_1, y_2) \in S \} \subseteq X_1 \oplus Y_2. \]
The linear map $\alpha: gR \Rightarrow S f$ is then a linear map between these linear subspaces, and similarly, we have a linear map $\beta: g'S \Rightarrow T f'$. We then have $g'gR \Rightarrow g'S f \Rightarrow T f'f$ so that $\beta \alpha: g'gR \Rightarrow T f''f$.

If we have four composable 2-morphisms as in the following diagram

If we first compose horizontally, we get
and then composing vertically gives

\[
\begin{array}{c}
X_1 \xrightarrow{R' \subseteq X_1 \oplus X_3} X_3 \\
\downarrow f' f \\
Z_1 \xrightarrow{T' \subseteq Z_1 \oplus Z_3} Z_3
\end{array}
\]

If we compose vertically first, we obtain

\[
\begin{array}{c}
X_1 \xrightarrow{R \subseteq X_1 \oplus X_3} X_2 \xrightarrow{R' \subseteq X_2 \oplus X_3} X_3 \\
\downarrow f' f \\
Z_1 \xrightarrow{T \subseteq Z_1 \oplus Z_3} Z_2 \xrightarrow{T' \subseteq Z_2 \oplus Z_3} Z_3
\end{array}
\]

and then composing horizontally gives

\[
\begin{array}{c}
X_1 \xrightarrow{R' \subseteq X_1 \oplus X_3} X_3 \\
\downarrow f' f \\
Z_1 \xrightarrow{T' \subseteq Z_1 \oplus Z_3} Z_3
\end{array}
\]

where, as usual, only the 2-morphism in the middle appears different. For the first linear map in the composite \((\beta' \alpha') \circ (\beta \alpha)\), we have that \(\beta \alpha: g' \circ g R \Rightarrow T' \circ f' f\) and \(\beta' \alpha': h' \circ h R' \Rightarrow T' \circ g' \circ g\) for the second linear map. We then have that \(h' \circ h R' \Rightarrow T' \circ g' \circ g R \Rightarrow T' \circ f' f\). Comparing this with \((\beta' \circ \beta) (\alpha' \circ \alpha)\), we have that \(\alpha': h R' \Rightarrow S' \circ g\) so then \(h R' \Rightarrow S' \circ g R\) and similarly \(S' \circ g R \Rightarrow S' \circ f\) so that \(\alpha' \circ \alpha: h R' \Rightarrow S' \circ f\). Likewise, \(\beta' \circ \beta: h' \circ S' \circ f \Rightarrow T' \circ f' f\). Then we have that \(h' \circ h R' \Rightarrow T' \circ g' \circ g R \Rightarrow T' \circ f' f\) so that \((\beta' \circ \beta) (\alpha' \circ \alpha): h' \circ h R' \Rightarrow T' \circ f' f\).

The source and target functors \(S, T: \text{LinRel}_0 \to \text{LinRel}_1\) are clear. The unit functor \(U: \text{LinRel}_0 \to \text{LinRel}_0\) sends a finite-dimensional real vector space \(X\) to the identity map.
id_X and a linear map f : X \to Y to the square

These structure functors satisfy the equations

\[ S(U_X) = X = T(U_X) \]

\[ S(R_2 R_1) = R_1 \]

\[ T(R_2 R_1) = R_2 \]

and we have natural isomorphisms

\[ \alpha : (R_1 R_2) R_3 \Rightarrow R_1 (R_2 R_3) \]

\[ \lambda : \text{id}_Y R_1 \Rightarrow R_1 \]

\[ \rho : R_1 \text{id}_X \Rightarrow R_1 \]

such that the coherence axioms of a monoidal category are satisfied and such that S(\alpha), S(\lambda), S(\rho), T(\alpha), T(\lambda) and T(\rho) are all identities.

\[ \square \]

**Theorem 23.** The strict double category \( \mathbb{LinRel} \) is symmetric monoidal.

**Proof.** The category of objects \( \mathbb{LinRel}_0 \) is simply \( \text{FinVect}_\mathbb{R} \) which is symmetric monoidal under binary direct sums. The category of arrows is also symmetric monoidal; given two linear relations \( R_1 \subseteq X_1 \oplus Y_1 \) and \( R_2 \subseteq X_2 \oplus Y_2 \), we can add these to obtain \( R_1 \oplus R_2 \subseteq X_1 \oplus X_2 \oplus Y_1 \oplus Y_2 \). Given two 2-morphisms
we can add them to obtain

Given four horizontal 1-cells

we have that

so then

We also have that

so that

which gives the globular 2-isomorphism \( \chi: (R_2 \oplus S_2)(R_1 \oplus S_1) \Rightarrow (R_2 R_1) \oplus (S_2 S_1) \).
The other globular 2-isomorphism $\mu: U_{X\oplus Y} \Rightarrow U_X \oplus U_Y$ is given simply by

$$
\begin{array}{ccc}
X \oplus Y & \xrightarrow{\text{id}_{X\oplus Y}} & X \oplus Y \\
\downarrow 1 & \swarrow \mu & \downarrow 1 \\
X \oplus Y & \xrightarrow{id_X \oplus id_Y} & X \oplus Y
\end{array}
$$

All of the commutative diagrams in the definition [23] are straightforward. For example, the first one is given by

$$
\begin{align*}
X_1 \oplus X_2 \oplus X_3 & \rightarrow (R_1 R_2) \oplus ((S_1 S_2) \oplus (T_1 T_2)) \leftrightarrow Z_1 \oplus Z_2 \oplus Z_3 \\
X_1 \oplus X_2 \oplus X_3 & \rightarrow (R_1 R_2) \oplus ((S_1 \oplus T_1) \oplus (S_2 \oplus T_2)) \leftrightarrow Z_1 \oplus Z_2 \oplus Z_3 \\
X_1 \oplus X_2 \oplus X_3 & \rightarrow (R_1 \oplus (S_1 \oplus T_1))(R_2 \oplus (S_2 \oplus T_2)) \leftrightarrow Z_1 \oplus Z_2 \oplus Z_3 \\
X_1 \oplus X_2 \oplus X_3 & \rightarrow ((R_1 \oplus S_1) \oplus T_1)((R_2 \oplus S_2) \oplus T_2) \leftrightarrow Z_1 \oplus Z_2 \oplus Z_3 \\
X_1 \oplus X_2 \oplus X_3 & \rightarrow ((R_1 \oplus S_1)(R_2 \oplus S_2)) \oplus (T_1 T_2) \leftrightarrow Z_1 \oplus Z_2 \oplus Z_3 \\
X_1 \oplus X_2 \oplus X_3 & \rightarrow ((R_1 R_2) \oplus (S_1 S_2)) \oplus (T_1 T_2) \leftrightarrow Z_1 \oplus Z_2 \oplus Z_3
\end{align*}
$$

where again, as in the proof of Theorem 21, this diagram is to be thought of as 3-dimensional with the top and bottom horizontal 1-cells coinciding. The last few diagrams that we show here express that the braiding of the category of arrows is a transformation.
of double categories:

\[
\begin{array}{cccc}
X_1 \oplus X_2 & \rightarrow & (S_1 \oplus R_1)(S_2 \oplus R_2) & \leftarrow & Y_1 \oplus Y_2 \\
\uparrow & \swarrow & \uparrow & \searrow & \\
X_1 \oplus X_2 & \rightarrow & (S_1 S_2) \oplus (R_1 R_2) & \leftarrow & Y_1 \oplus Y_2 \\
\uparrow & \swarrow & \uparrow & \searrow & \\
X_1 \oplus X_2 & \rightarrow & (R_1 R_2) \oplus (S_1 S_2) & \leftarrow & Y_1 \oplus Y_2 \\
\uparrow & \swarrow & \uparrow & \searrow & \\
X_1 \oplus X_2 & \rightarrow & (S_1 \oplus R_1)(S_2 \oplus R_2) & \leftarrow & Y_1 \oplus Y_2 \\
\uparrow & \swarrow & \uparrow & \searrow & \\
X_1 \oplus X_2 & \rightarrow & (S_1 \oplus R_1)(S_2 \oplus R_2) & \leftarrow & Y_1 \oplus Y_2 \\
\end{array}
\]

It follows that \( \text{LinRel} \) is symmetric monoidal. □

This sets the stage for how our double functor \( \Box : \text{CoarseMark} \to \text{LinRel} \) will be defined. We start by making the following definitions:

(i) For a finite set \( X \), we define \( \Box(X) \) to be the vector space \( R^X \oplus R^X \).

(ii) For a bijection \( b : S \to S' \), we define \( \Box(b) : R^S \oplus R^S \to R^{S'} \oplus R^{S'} \) to be the linear map that sends basis elements of \( R^S \) to basis elements of \( R^{S'} \) via \( b \).

(iii) For an open Markov process \( S \leftarrow (X, H) \to T \), we define \( \Box((S \leftarrow (X, H) \to T)) \) as in Def. 6.

What remains to be done is to define how \( \Box \) acts on 2-morphisms of \( \text{CoarseMark} \), namely coarse-grainings. Let \( (M, H) : X \to Y \) be an open Markov process and let \( (M', pHs) : X' \to Y' \) be a coarse-graining of the first open Markov process given by a triple of functions \( (f, p, g) \) with \( f, g \) bijections together with a stochastic section \( s : M' \rightsquigarrow M \) of \( p \). We have

\[
\Box((M, H) : X \to Y) = (i^*(v), I_1, o^*(v), O_1) \subseteq R^X \oplus R^X \oplus R^Y \oplus R^Y
\]
and
\[ \mathbf{\square}(M, pHs): X' \rightarrow Y' = (i'^*(v'), I_2, o'^*(v'), O_2) \subset \mathbb{R}^{X'} \oplus \mathbb{R}^{X'} \oplus \mathbb{R}^{Y'} \oplus \mathbb{R}^{Y'}. \]

The probabilities of the original open Markov process are given by a vector \( v \in \mathbb{R}^S \).
To obtain probabilities for the lumped open Markov process, we push \( v \) forward along the surjection \( p: S \rightarrow S' \) obtaining a vector \( v' = p_* v \). This then defines the input probabilities \( i'^*(v') \) and output probabilities \( o'^*(v') \) of the coarse-grained open Markov process. We likewise push forward the inflows and outflows \( I_1 \) and \( O_1 \) along the surjections \( f \) and \( g \) to obtain inflows and outflows \( I_2 = f_* (I_1) \) and \( O_2 = g_* (O_1) \) on the coarse-grained open Markov process.

We thus have a linear map between linear subspaces
\[ \mathbf{\square}: \mathbb{R}^{X} \oplus \mathbb{R}^{X} \oplus \mathbb{R}^{Y} \oplus \mathbb{R}^{Y} \rightarrow \mathbb{R}^{X'} \oplus \mathbb{R}^{X'} \oplus \mathbb{R}^{Y'} \oplus \mathbb{R}^{Y'}. \]
defined by
\[ (i^*(v), I_1, o^*(v), O_1) \mapsto (i'^*(v'), f_* (I_1), o'^*(v'), g_* (O_1)) \]
where \( v \in \mathbb{R}^S \) is a steady state of the original open Markov process and \( v' \in \mathbb{R}^{S'} \) is a steady state of the coarse-grained open Markov process.

The following result is a special case of a result by Pollard and the first author on black-boxing open dynamical systems [5]. To make this paper self-contained we adapt the proof to the case at hand:

**Proposition 24.** The black-boxing of a composite of two open Markov processes equals the composite of their black-boxings.

**Proof.** Consider composable open Markov processeses Mark given by
\[ X \xrightarrow{i} (S, H) \xleftarrow{o} Y \]
and
\[ Y \xrightarrow{i'} (S', G) \xleftarrow{o'} Z. \]
To compose these, we form the pushout
\[
\begin{array}{ccc}
X & \xrightarrow{i} & M \\
\downarrow & & \downarrow j \\
Y & \xrightarrow{o} & N \\
\downarrow & & \downarrow j' \\
Z & \xleftarrow{o'} & \end{array}
\]
Then their composite is
\[ X \xrightarrow{j^*} (M +_Y N, H \circ G) \xleftarrow{j'^* o'} Z \]
where
\[ H \circ G = j_* \circ H \circ j^* + j'_* \circ G \circ j'^*. \]

To prove that \( \mathbf{\square} \) preserves composition, we first show that
\[ \mathbf{\square}(M +_Y N, H \circ G) \subseteq \mathbf{\square}(N, G) \mathbf{\square}(M, H) \]
Thus, given
\[ (i^*(v), I, o^*(v), O) \in \mathbf{\square}(M, H), \quad (i'^*(v'), I', o'^*(v'), O') \in \mathbf{\square}(N, G) \]
with
\[ o^*(v) = i'^*(v'), \quad O = I' \]
we need to prove that
\[(i^*(v), I, o^*(v'), O') \in \square(M + Y N, H \odot G).\]
To do this, it suffices to find probabilities \(w \in \mathbb{R}^{M+Y N}\) such that
\[(i^*(v), I, o^*(v'), O') = ((ji)^*(w), I, (j'o')^*(w), O')\]
and \(w\) is a steady state of \((M + Y N, H \odot G)\) with inflows \(I\) and outflows \(O'\).
Since \(o^*(v) = i^*(v')\), this diagram commutes:

\[
\begin{array}{c}
\mathbb{R} \\
\downarrow q \\
M + Y N \\
\downarrow j \\
M \\
\downarrow o \\
Y \\
\end{array}
\]
\[
\begin{array}{c}
\mathbb{R} \\
\downarrow p' \\
N \\
\downarrow p \\
Y \\
\end{array}
\]

so by the universal property of the pushout there is a unique map \(w: M + Y N \rightarrow \mathbb{R}\) such that this commutes:

\[
\begin{array}{c}
\mathbb{R} \\
\downarrow q \\
M + Y N \\
\downarrow j \\
M \\
\downarrow o \\
Y \\
\end{array}
\]
\[
\begin{array}{c}
\mathbb{R} \\
\downarrow p' \\
\downarrow p \\
\end{array}
\]

This simply says that because the probabilities \(v\) and \(v'\) agree on the ‘overlap’ of our two open Markov processes, we can find a probability \(w\) for the composite system that restricts to \(v\) on \(M\) and \(v'\) on \(N\).

We now prove that \(w\) is a steady state of the composite open Markov process with inflows \(I\) and outflows \(O'\):

\[(H \odot G)(w) + (ji)_*(I) - (j'o')_* (O') = 0. \tag{2}\]

To do this we use the fact that \(v\) is a steady state of \((M, H): X \rightarrow Y\) with inflows \(I\) and outflows \(O\):

\[H(v) + i_* (I) - o_* (O) = 0 \tag{3}\]

and \(v'\) is a steady state of \((N, G): Y \rightarrow Z\) with inflows \(I'\) and outflows \(O'\):

\[G(v') + i'_* (I') - o'_* (O') = 0. \tag{4}\]

We push forward Eq. (3) along \(j\), push forward Eq. (4) along \(j'\), and sum them:

\[j_*(H(v)) + (ji)_*(I) - (jo)_*(O) + j'_*(G(v')) + (j'o')_* (O') = 0.\]

Since \(O = I'\) and \(jo = j'i'\), two terms cancel, leaving us with

\[j_*(H(v)) + (ji)_*(I) + j'_*(G(v')) - (j'o')_* (O') = 0.\]
Next we combine the terms involving the infinitesimal stochastic operators $H$ and $G$, with the help of Eq. (1) and the definition of $H \odot G$:

$$j_\ast(H(v)) + j_\ast(G(v')) = j_\ast(H(q \circ j)) + j_\ast(G(q \circ j')) = (j_\ast H \circ j' + j_\ast G \circ j'')(w) = (H \odot G)(w). \tag{5}$$

This leaves us with

$$(H \odot G)(w) + (ji)_\ast(I) - (j o')(O') = 0$$

which is Eq. (2), precisely what we needed to show.

To finish showing that $\blacksquare$ is a functor, we need to show that

$$\blacksquare(M +_Y N, H \odot G) \subseteq \blacksquare(N, G) \blacksquare(M, H).$$

So, suppose we have

$$((ji')'(w), I, (j' o')'(w), O') \in \blacksquare(M +_Y N, H \odot G).$$

We need to show

$$((ji')'(w), I, (j' o')'(w), O') = (i''(v), I, o''(v'), O') \tag{6}$$

where

$$(i''(v), I, o''(v), O) \in \blacksquare(M, H), \quad (i'''(v'), I', o'''(v'), O') \in \blacksquare(N, G)$$

and

$$o''(v) = i'''(v'), \quad O = I'.$$

To do this, we begin by choosing

$$v = j'(w), \quad v' = j''(w).$$

This ensures that Eq. (6) holds, and since $jo = j' i'$, it also ensures that

$$o''(v) = (jo)'(w) = (j' i')''(w) = i'''(v').$$

So, to finish the job, we only need to find an element $O = I' \in \mathbb{R}^Y$ such that $v$ is a steady state of $(M, H)$ with inflows $I$ and outflows $O$ and $v'$ is a steady state of $(N, G)$ with inflows $I'$ and outflows $O'$. Of course, we are given the fact that $w$ is a steady state of $(M +_Y N, H \odot G)$ with inflows $I$ and outflows $O'$.

In short, we are given Eq. (2), and we want to find $O = I'$ such that Eqs. (3) and (4) hold. Thanks to our choices of $v$ and $v'$, we can use Eq. (5) and rewrite Eq. (2) as

$$j_\ast(H(v) + i_\ast(I)) + j_\ast(G(v') - o_\ast(O')) = 0. \tag{7}$$

Eqs. (3) and (4) say that

$$H(v) + i_\ast(I) - o_\ast(O) = 0 \tag{8}$$

$$G(v') + i_\ast(I') - o_\ast(O') = 0.$$
is a pushout. Applying the ‘free vector space on a finite set’ functor, which preserves colimits, this implies that

![Diagram](image)

is a pushout in the category of vector spaces. Since a pushout is formed by taking first a coproduct and then a coequalizer, this implies that

\[ \begin{array}{ccc}
\mathbb{R}^M & \xrightarrow{(\alpha,0)} & \mathbb{R}^M \oplus \mathbb{R}^N \\
(0,\delta) & \xrightarrow{\alpha + \delta} & \mathbb{R}^{M+N}
\end{array} \]

is a coequalizer. Thus, the kernel of \( j_* + j'_* \) is the image of \((\alpha, 0) - (0, \delta)\). Eq. (7) says precisely that

\[ (H(v) + i_*(I), G(v') - o'_*(O')) \in \ker(j_* + j'_*). \]

Thus, it is in the image of \( \alpha_* - \delta' \). In other words, there exists some element \( O = I' \in \mathbb{R}^Y \) such that

\[ (H(v) + i_*(I), G(v') - o'_*(O')) = (\alpha_*(O), -\delta'(I')). \]

This says that Eqs. (3) and (4) hold, as desired.

Finally, we need to check that \( \boxdot \) is symmetric monoidal. But this is a straightforward calculation, so we leave it to the reader. □

The main result of this paper is the following theorem, which says how coarse-graining interacts with black-boxing:

**Theorem 25.** There exists a symmetric monoidal double functor \( \boxdot : \text{CoarseMark} \to \text{LinRel} \) with the following properties:

1. Objects: For a finite set \( X \), \( \boxdot(S) \) is the vector space \( \mathbb{R}^S \oplus \mathbb{R}^S \).
2. Vertical 1-morphisms: For a bijection \( b : S \to S' \), \( \boxdot(b) : \mathbb{R}^S \oplus \mathbb{R}^S \to \mathbb{R}^{S'} \oplus \mathbb{R}^{S'} \) is the linear map that sends basis elements of \( \mathbb{R}^S \) to basis elements of \( \mathbb{R}^{S'} \).
3. Horizontal 1-cells: For an open Markov process \( S \leftarrow (X, H) \to T \), we define \( \boxdot(S \leftarrow (X, H) \to T) \) as in Def. 6. Namely:
   \[ \boxdot(S \leftarrow (X, H) \to T) = \{(\iota^*(v), I, \alpha^*(v), O) : H(v) + i_*(I) - i_*(O) = 0 \text{ for some } I, v, O\} \]
4. 2-morphisms: For a coarse-graining given by a triple of functions \( f, p, g \) with \( f, g \) bijections together with a stochastic section \( s : M' \twoheadrightarrow M \) of \( p \), \( \boxdot(f, p, g, s) \) is the linear map between linear subspaces determined by linear relations.
   \[ (\iota^*(v), I_1, \alpha^*(v), O_1) \mapsto (i^*(v'), f_*(I_1), \alpha^*(v'), g_*(O_1)) \]

**Proof.** We denote the categories of objects and arrows as \( \text{CoarseMark}_0 \) and \( \text{CoarseMark}_1 \) respectively, and likewise for \( \text{LinRel} \). We have symmetric monoidal functors \( \boxdot_0 : \text{CoarseMark}_0 \to \text{LinRel}_0 \) and \( \boxdot_1 : \text{CoarseMark}_1 \to \text{LinRel}_1 \) between the categories of objects and the categories of arrows, respectively. That the first functor \( \boxdot_0 \) is symmetric monoidal is clear, as the functor is defined on objects by \( \boxdot_0(S) = \mathbb{R}^S \oplus \mathbb{R}^S \) for a finite set \( S \), and on morphisms by \( \boxdot_0(f) : \mathbb{R}^S \oplus \mathbb{R}^S \to \mathbb{R}^{S'} \oplus \mathbb{R}^{S'} \) for a bijection \( f : S \to S' \). These
are just objects and morphisms in the symmetric monoidal category \( \text{FinVect}_\mathbb{R} \). The functor \( \llbracket 1 \rrbracket \) is defined on objects by

\[
(X \to (M, H) \leftarrow Y) \mapsto \llbracket (M, H) \rrbracket : R^X \oplus R^X \rightsquigarrow R^Y \oplus R^Y
\]

where \( \llbracket (M, H) \rrbracket : R^X \oplus R^X \rightsquigarrow R^Y \oplus R^Y \) is the linear relation determined by the linear subspace

\[
\llbracket (M, H) \rrbracket : X \to Y = \{(i^*(v), 1, o^*(v), O) : H(v) + i_1(I) - i_{\ell}(O) = 0\}
\]

of \( R^X \oplus R^X \oplus R^Y \oplus R^Y \) consisting of possible tuples \((i^*(v), 1, o^*(v), O)\) representing possible input probabilities, inflows, output probabilities and outflows in a steady state. The functor \( \llbracket 1 \rrbracket \) is also a symmetric monoidal functor.

The functor \( \llbracket : \text{CoarseMark} \to \text{LinRel} \) is a symmetric monoidal double functor. To see this, first note that the following diagrams commute.

\[
\begin{array}{ccc}
\text{CoarseMark}_1 & \llbracket & \text{LinRel}_1 \\
\downarrow & & \downarrow \\
\text{CoarseMark}_0 & \llbracket & \text{LinRel}_0
\end{array}
\]

\[
\begin{array}{ccc}
\text{CoarseMark}_1 & \llbracket & \text{LinRel}_1 \\
\downarrow & & \downarrow \\
\text{CoarseMark}_0 & \llbracket & \text{LinRel}_0
\end{array}
\]

We have natural isomorphisms \( \varepsilon : \{0\} \to \mathbb{R}^0 \oplus \mathbb{R}^0 \) and \( \mu_{S,S'} : R^S \oplus R^S \oplus R^{S'} \oplus R^{S'} \to R^{S+S'} \oplus R^{S+S'} \) for any two finite sets \( S \) and \( S' \). These isomorphisms make the following diagrams commute:

\[
\begin{array}{ccc}
(R^S \oplus R^S) \oplus (R^{S'} \oplus R^{S'}) & \llbracket & (R^S \oplus R^S) \oplus ((R^{S'} \oplus R^{S'}) \oplus (R^{S''} \oplus R^{S''}))) \\
\downarrow \mu_{S+S''} & & \downarrow \mu_{S+S''} \\
(R^{S+S'} \oplus R^{S+S'}) \oplus (R^{S''} \oplus R^{S''}) & \llbracket & (R^S \oplus R^S) \oplus (R^{S'+S'} \oplus R^{S''} \oplus R^{S''})
\end{array}
\]

\[
\begin{array}{ccc}
(R^S \oplus R^S) \oplus \{0\} & \llbracket & R^S \oplus R^S \\
\downarrow 1 \oplus \varepsilon & & \downarrow \varepsilon \oplus 1 \\
(R^S \oplus R^S) \oplus (R^0 \oplus R^0) & \llbracket & (R^S \oplus R^S) \oplus (R^0 \oplus R^0) \oplus (R^0 \oplus R^0)
\end{array}
\]

\[
\begin{array}{ccc}
\{0\} \oplus (R^S \oplus R^S) & \llbracket & (R^S \oplus R^S) \oplus (R^0 \oplus R^0) \\
\downarrow \varepsilon \oplus 1 & & \downarrow 1 \oplus \varepsilon \\
(R^S \oplus R^S) \oplus (R^0 \oplus R^0) & \llbracket & (R^S \oplus R^S) \oplus (R^0 \oplus R^0) \oplus (R^0 \oplus R^0)
\end{array}
\]
We also have a natural isomorphism \( \delta: U_{1, \text{inRel}} \to \boxempty(U_{1, \text{coarseMark}}) \) where \( U_{1, \text{inRel}} \) is given by \( !: \{0\} \to \{0\} \) and \( \boxempty(U_{1, \text{coarseMark}}) \) is given by \( !: \mathbb{R}^0 \oplus \mathbb{R}^0 \to \mathbb{R}^0 \oplus \mathbb{R}^0 \). For two open Markov processes \( S \leftarrow (X, H) \to T \) and \( S' \leftarrow (Y, G) \to T' \), we have a natural isomorphism

\[ \nu_{X,Y}: \boxempty((S \leftarrow (X, H) \to T) \otimes (S \leftarrow (Y, G) \to T) \to \boxempty(S + S' \leftarrow (X, H) \otimes (Y, G) \to T + T') \]

where

\[ \boxempty(S \leftarrow (X, H) \to T) = \{(i^*(v), I, o^*(v), O): H(v) + i^*(I) - o^*(O) = 0\} \]

and

\[ \boxempty(S' \leftarrow (Y, G) \to T') = \{(i'^*(v'), I', o'^*(v'), O'): G(v') + i'^*(I') - o'^*(O') = 0\} \]

and the black-boxing of the tensor product of these horizontal 1-cells is given by
These two natural isomorphisms make the following diagrams commute:

\[
\begin{align*}
\square (S + S') & \leftrightarrow (X, H) \otimes (Y, G) \to T + T' = \\
\{ (i' + i')(v + v'), I + I', (o' + o')(v + v'), O + O' : (H \otimes G)(v + v') + (i' + i')(I + I') - (i' + i')(O + O') = 0 \}.
\end{align*}
\]

It follows that \( \square : \text{CoarseMark} \to \text{LinRel} \) is a symmetric monoidal double functor. \( \square \)

7. A bicategory of open Markov processes and coarse-grainings

In a previous section, we constructed a symmetric monoidal double category \( \text{CoarseMark} \) consisting of:

(i) Finite sets as objects.
(ii) Surjectons as vertical 1-morphisms.
(iii) Open Markov processes as horizontal 1-cells.
(iv) Lumpings of open Markov processes as 2-morphisms.

Using the following result of Shulman [23], we can obtain a symmetric monoidal bicategory \( \text{CoarseMark} \) consisting of:

(i) Finite sets as objects.
(ii) Open Markov processes as morphisms.
(iii) Lumpings of open Markov processes as 2-morphisms.

provided that the symmetric monoidal double category \( \text{CoarseMark} \) is ‘isofibrant’. This bicategory arises as the ‘horizontal bicategory’ of the double category \( \text{CoarseMark} \).

**Definition 26.** Let \( \mathcal{D} \) be a double category. Then the **horizontal bicategory** of \( \mathcal{D} \), which we denote as \( H(\mathcal{D}) \), is the bicategory consisting of objects of \( \mathcal{D} \), 1-morphisms that are horizontal 1-cells of \( \mathcal{D} \) and 2-morphisms that are globular 2-morphisms.

**Theorem 27** (Shulman). Let \( \mathcal{D} \) be an isofibrant symmetric monoidal double category. Then \( H(\mathcal{D}) \) is a symmetric monoidal bicategory, where \( H(\mathcal{D}) \) is the horizontal bicategory of \( \mathcal{D} \).

**Theorem 28.** The symmetric monoidal double category \( \text{CoarseMark} \) is isofibrant.
Proof. To show that CoarseMark is isofibrant, we need to show that every vertical 1-isomorphism has both a companion and a conjoint. Given a vertical 1-isomorphism \( f: S \to S' \), then the companion of \( f \), which we denote as \( \hat{f} \), is a horizontal 1-cell \( \hat{f}: S \to S' \).

\[ (S', 0_{S'}) \xrightarrow{\hat{f}} S \]

\[ \xleftarrow{id} \]

\[ S \]

\[ S' \]

\[ \xrightarrow{id} \]

\[ \xrightarrow{id} \]

\[ S 

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\]
All unlabeled arrows are identities. A conjoint of \( f: S \rightarrow S' \) is given by \( \tilde{f}: S' \rightarrow S \)

\[
\begin{array}{ccc}
(S', 0_{S'}) & \xrightarrow{id} & S' \\
\downarrow & \downarrow & \downarrow \\
S & \xrightarrow{f} & S
\end{array}
\]

together with the two 2-morphisms

\[
\begin{array}{ccc}
S' & \xrightarrow{f} & (S', 0_{S'}) \\
\downarrow & \downarrow & \downarrow \\
S & \xrightarrow{id} & S
\end{array}
\quad
\begin{array}{ccc}
S & \xrightarrow{id} & (S, 0_{S}) \\
\downarrow & \downarrow & \downarrow \\
S & \xrightarrow{f} & S
\end{array}
\]

that satisfy analogous equations as the two above.

\[\square\]

**Theorem 29.** \textbf{CoarseMark} is a symmetric monoidal bicategory.

\[\textsf{Proof.}\] This follows immediately from Shulman’s theorem; \textbf{CoarseMark} is an isofibrant symmetric monoidal double category and so we obtain the symmetric monoidal bicategory \textbf{CoarseMark} as the horizontal bicategory of \textbf{CoarseMark}.

\[\square\]

We can also obtain a symmetric monoidal bicategory \textbf{LinRel} from the symmetric monoidal double category \textbf{LinRel} if \textbf{LinRel} is isofibrant.

**Theorem 30.** The symmetric monoidal double category \textbf{LinRel} is isofibrant.

\[\textsf{Proof.}\] Let \( f: X \rightarrow Y \) be a linear isomorphism between finite-dimensional real vector spaces. Define \( \hat{f} \) to be the linear relation given by the linear isomorphism \( f \) and define 2-morphisms as

\[
\begin{array}{ccc}
X & \xrightarrow{\hat{f}} & Y \\
\downarrow & \downarrow & \downarrow \\
Y & \xrightarrow{1} & Y
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{1} & X \\
\downarrow & \downarrow & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

where the 2-morphisms \( \alpha_f \) and \( f\alpha \) are identity 2-morphisms. These two 2-morphisms and \( \hat{f} \) satisfy the required equations, and the conjoint of \( f \) is given by reversing the direction of \( \hat{f} \), which is just \( f^{-1}: Y \rightarrow X \). It follows that \textbf{LinRel} is fibrant.

\[\square\]

**Theorem 31.** There exists a symmetric monoidal bicategory \textbf{LinRel} consisting of:

(i) Finite-dimensional real vector spaces as objects.
(ii) Linear relations as morphisms.
(iii) Linear maps between linear relations as 2-morphisms.
Proof. Apply Shulman’s result, Thm. 27, to the isofibrant symmetric monoidal double category $\text{LinRel}$ to obtain the symmetric monoidal bicategory $\text{LinRel}$ as the horizontal edge bicategory of $\text{LinRel}$. □

Thus we have symmetric monoidal bicategories $\text{CoarseMark}$ and $\text{LinRel}$, both of which come from discarding the vertical 1-morphisms of the symmetric monoidal double categories $\text{CoarseMark}$ and $\text{LinRel}$, respectively. Morally, we should be able to do something similar to the symmetric monoidal double functor $\blacksquare : \text{CoarseMark} \to \text{LinRel}$ to obtain a symmetric monoidal functor of bicategories $\blacksquare : \text{CoarseMark} \to \text{LinRel}$.

Conjecture 32. There exists a symmetric monoidal functor $\blacksquare : \text{CoarseMark} \to \text{LinRel}$ that maps:

(i) any finite set $X$ to the finite-dimensional real vector space $\blacksquare(X) = \mathbb{R}^X \oplus \mathbb{R}^X$.

(ii) any open Markov processes $(M, H) : X \to Y$ to the linear relation given by the linear subspace $\blacksquare(M, H) = \{(i^v(I), I, o^v(v), O) : H(v) + i_v(I) - i_v(O) = 0 \} \subset \mathbb{R}^X \oplus \mathbb{R}^X \oplus \mathbb{R}^Y \oplus \mathbb{R}^Y$.

(iii) Any coarse-graining of an open Markov process $(M, H) : X \to Y$ given by a surjection $p : M \to M'$ together with a stochastic section $s : M' \twoheadrightarrow M$ to the linear map between linear subspaces given by pushing forward the probabilities along the surjection $p$:

$\blacksquare(i^v(I), I, o^v(v), O) \mapsto (i^v(p_v(I), I, o^v(p_v(I)), O) \subset \mathbb{R}^X \oplus \mathbb{R}^X \oplus \mathbb{R}^Y \oplus \mathbb{R}^Y$.

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References

[1] D. Andrieux, Bounding the coarse graining error in hidden Markov dynamics, Applied Mathematics Letters 25 (2012) pp 1734-1739. Available as arXiv:1104.1025. (Referred to on page 2.)

[2] J. C. Baez and D. Biamonte, Quantum Techniques for Stochastic Mechanics, to be published by World Scientific. Draft available as arXiv:1209.3632. (Referred to on page 6, 7, 12.)

[3] J. C. Baez and B. Fong, A compositional framework for passive linear networks. Available as arXiv:1504.05625. (Referred to on page.)

[4] J. C. Baez, B. Fong and B. Pollard, A compositional framework for Markov processes, Jour. Math. Phys. 57 (2016), 033301. Available as arXiv:1508.06448. (Referred to on page 2, 4, 5, 7.)

[5] J. C. Baez and B. Pollard, A compositional framework for reaction networks, Rev. Math. Phys. 29 (2017), 1750028. Available as arXiv:1704.02051. (Referred to on page 4, 5, 25, 33.)

[6] J. C. Baez, T. Fritz, A Bayesian characterization of relative entropy, Theory and Applications of Categories 29 (2014) No.16, pp 421-456. Available as arXiv:1402.3067. (Referred to on page 9.)

[7] J. Bénabou, Introduction to Bicategories, Lecture Notes in Mathematics 47, Springer, Berlin, 1967, pp. 1–77. (Referred to on page 5.)

[8] P. Buchholz, Exact and ordinary lumpability in finite Markov chains. Applied Probability Trust 31 (1994), 59–75. Available at http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.45.1632. (Referred to on page 2, 9, 10.)

[9] R. Brown and C. B. Spencer, Double groupoids and crossed modules, Cah. Top. Géom. Diff. 17 (1976), 343–362. (Referred to on page 2.)

[10] R. Brown, K. Hardie, H. Kamps and T. Porter, The homotopy double groupoid of a Hausdorff space, Th. Appl. Categ. 10 (2002), 71–93. (Referred to on page 2.)

[11] K. Courser, A bicategory of decorated cospans, Th. Appl. Categ. 32 (2017), 995–1027. Available as arXiv:1605.08100. (Referred to on page 5, 14.)

[12] C. Ehresmann, Catégories structurées III: Quintettes et applications covariantes, Cah. Top. Géom. Diff. 5 (1963), 1–22. (Referred to on page 2.)

[13] C. Ehresmann, Catégories et Structures, Dunod, Paris, 1965. (Referred to on page 2.)
[14] B. Fong, Decorated cospans, *Th. Appl. Categ.* 30 (2015), 1096–1120. Available as arXiv:1502.00872. (Referred to on page 43.)

[15] B. Fong, *The Algebra of Open and Interconnected Systems*, Ph.D. thesis, University of Oxford, 2016. Available as arXiv:1609.05382. (Referred to on page 43.)

[16] M. Grandis and R. Paré, Limits in double categories, *Cah. Top. Géom. Diff.* 40 (1999), 162–220. (Referred to on page 43.)

[17] M. Grandis and R. Paré, Adjoints for double categories, *Cah. Top. Géom. Diff.* 45 (2004), 193–240. (Referred to on page 43.)

[18] L. de Francesco Albasini, N. Sabadini and R. F. C. Walters, The compositional construction of Markov processes, *Appl. Cat. Str.* 19 (2011), 425–437. Available as arXiv:0901.2434. (Referred to on page 43.)

[19] T. Leinster, Basic bicategories, Available as arXiv:math/9810017. (Referred to on page 43.)

[20] E. Lerman and D. Spivak, An algebra of open continuous time dynamical systems and networks. Available as arXiv:1602.01017. (Referred to on page 43.)

[21] F. Clerc, H. Humphrey and P. Panangaden, Bicategories of Markov processes, to appear. (Referred to on page 43.)

[22] B. S. Pollard, *Open Markov Processes and Reaction Networks*, Ph.D. thesis, University of California at Riverside, 2017. Available as arXiv:1709.09743. (Referred to on page 43.)

[23] M. Shulman, Constructing symmetric monoidal bicategories. Available as arXiv:1004.0993. (Referred to on page 43.)

[24] M. Stay, Compact closed bicategories. *Th. Appl. Categ.* 31 (2016), 755–798. Available as arXiv:1301.1053. (Referred to on page 43.)