EXISTENCE OF NONNEGATIVE SOLUTIONS FOR FRACTIONAL SCHRÖDINGER EQUATIONS WITH NEUMANN CONDITION

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Abstract. In this paper we study a Neumann problem for the fractional Laplacian, namely
\[
\begin{cases}
\varepsilon^{2s}(-\Delta)^s u + u = f(u) & \text{in } \Omega \\
N_s u = 0 & \text{in } \mathbb{R}^N \setminus \Omega
\end{cases}
\] (0.1)
where \(\Omega \subset \mathbb{R}^N\) is a smooth bounded domain, \(N > 2s, s \in (0,1), \varepsilon > 0\) is a parameter and \(N_s\) is the nonlocal normal derivative introduced by Dipierro, Ros-Oton, and Valdinoci. We establish the existence of a nonnegative, non-constant small energy solution \(u_\varepsilon\), and we use the Moser-Nash iteration procedure to show that \(u_\varepsilon \in L^\infty(\Omega)\).

1. Introduction

In this paper, we study a Neumann elliptic problem for an equation driven by the fractional Laplacian. More precisely, we consider the problem
\[
\begin{cases}
\varepsilon^{2s}(-\Delta)^s u + u = f(u) & \text{in } \Omega \\
N_s u = 0 & \text{in } \mathbb{R}^N \setminus \Omega
\end{cases}
\] (1.1)
where \(\Omega \subset \mathbb{R}^N\) is a smooth bounded domain, \(N > 2s, s \in (0,1), \varepsilon > 0\) is a parameter and \(N_s u\) is the nonlocal normal derivative defined by
\[
N_s u(x) = C_{N,s} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad x \in \mathbb{R}^N \setminus \Omega.
\] (1.2)
where \(C_{N,s}\) is the normalization constant of the fractional Laplacian, defined for smooth functions by
\[
(-\Delta)^s \phi(x) = C_{N,s} \int_{\mathbb{R}^N} \frac{\phi(x) - \phi(y)}{|x - y|^{N+2s}} dy,
\]
with both integrals being understood in the principle value sense. One advantage of the present approach is that the integration by parts formulas
\[
\int_{\Omega} \Delta u = \int_{\partial \Omega} \partial_\nu u \quad \text{and} \quad \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} v(-\Delta u) + \int_{\Omega} v \partial_\nu u
\]
2020 Mathematics Subject Classification. 35R11, 35A01, 35B45.

Key words and phrases. fractional operators, Neumann problem, variational methods, a priori estimates.
are substituted, respectively, by
\[ \int_{\Omega} (-\Delta)^s u = - \int_{\Omega^c} \mathcal{N}_s u(x) \]
and
\[ \frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N} \setminus (\Omega^c)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dxdy = \int_{\Omega} v(-\Delta)^s u + \int_{\Omega^c} v\mathcal{N}_s u, \]
where \( \Omega^c = \mathbb{R}^N \setminus \Omega \) and \((\Omega)^2 = \Omega \times \Omega \). For further details on the fractional Neumann derivative \( \mathcal{N}_s u \), see Dipierro, Ros-Oton, and Valdinoci [10], where this concept was introduced.

This type of boundary problem for the fractional Laplacian has a probabilistic interpretation: if a particle has gone to \( x \in \mathbb{R}^N \setminus \Omega \), then it may come back to any point \( y \in \Omega \), the probability of jumping from \( x \) to \( y \) being proportional to \( |x - y|^{-N-2s} \). So, it generalizes the classical Neumann conditions for elliptic (or parabolic) differential equations since, as \( s \to 1 \), then \( \mathcal{N}_s u = 0 \) turns into the classical Neumann condition. For more details, see [10] and also [11, 12].

Du et al. introduced volume constraints for a general class of nonlocal diffusion problems on a bounded domain in \( \mathbb{R}^N \) via a nonlocal vector calculus. If we rewrite (1.2) using that vector calculus, then a modified version of \( \mathcal{N}_s u = 0 \) can be considered as a particular case of the volume constraints defined by them.

Neumann problems for the fractional Laplacian and other nonlocal operators were introduced in [4, 5, 8, 9]. All these generalizations to nonlocal operators recover the classical Neumann problem as a limit case, and most also have clear probabilistic interpretations. In Dipierro et al. [10, Section 7], the authors compared all these models with the one considered here.

The case \( f(t) = |t|^{p-1}t \) with \( 1 < p < \frac{N+2s}{N-2s} \), which is known as the singularly perturbed Neumann problem, was studied by Guoyuan Chen in [7]. The author established the existence of non-negative small energy solutions and investigated their integrability in \( \mathbb{R}^N \).

When \( s = 1 \), the problem (1.1) reduces to the Laplacian case, considered in the classical paper by Lin, Ni, and Takagi [14], which studies the existence of solutions to the semilinear Neumann boundary problem
\[ \begin{cases} \varepsilon^2 (-\Delta) u + u = g(u) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial \Omega \end{cases} \tag{1.3} \]
where \( \nu \) denotes the outer normal to \( \partial \Omega \) and \( g(t) \) is a suitable nonnegative nonlinearity on \( \mathbb{R} \) vanishing for \( t \leq 0 \), growing superlinearly at infinity. It was shown that, if \( \varepsilon \) is small enough, there exists a positive smooth solution \( u_\varepsilon \) that satisfies \( J_\varepsilon(u_\varepsilon) \leq C \varepsilon^{\frac{N-2s}{N+2s}} \), where \( C \) is a positive constant independent of \( \varepsilon \) and \( J_\varepsilon \) is the energy functional of problem (1.3).

Stinga-Volzone [20] extended the results in [14] to the square root of the Laplacian, obtaining similar results. More precisely, they considered problem
\[ \begin{cases} \varepsilon (-\Delta)^{\frac{1}{2}} u + u = g(u) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial \Omega \end{cases} \tag{1.4} \]
for the nonlinearity
\[ g(t) = \begin{cases} 
    t^p & \text{if } t \geq 0, \\
    0 & \text{if } t \leq 0,
\end{cases} \quad (1.5) \]
with \(1 < p < \frac{N+1}{N-1}\).

Recently, Haige Ni, Aliang Xia, and Xiogjun Zheng [17] studied the problem
\[
\varepsilon^{2s}(-\Delta)^s u + u = g(u) \quad \text{in } \Omega \\
\frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial \Omega \\
u > 0 \text{ in } \Omega \quad (1.6)
\]
where \(g\) satisfies (1.5) and \(s \in (0, s_0)\), with \(s_0 \geq \frac{1}{2}\). The authors used the extension technique to obtain the existence of nonnegative solutions for \(\varepsilon\) small enough and \(L^\infty\)-estimates to show that they are bounded. In their paper, they considered the spectral fractional Laplacian, which differs from its integral form, see [16, 19]. By applying the Mountain Pass Theorem of Ambrosetti and Rabinowitz, they proved the existence of nonconstant solutions of (1.6) provided \(\varepsilon\) is small. They also studied regularity and the Harnack inequality in the same paper.

Here, we study problem (1.1) considering the normal derivative defined by Dipierro, Ros-Oton, and Valdinoci in [10]. We suppose that the continuous nonlinearity \(f\) satisfies the following conditions.

\((f_1)\) \(f(t) = 0\) for \(t < 0\), and \(f(t) > 0\) for \(t > 0\);
\((f_2)\) \(\lim_{t \to 0^+} \frac{f(t)}{t} = 0\), and \(\lim_{t \to \infty} \frac{f(t)}{t^{p-1}} = 0\) for some \(2 < p < \frac{2N}{N-2s} = 2^*_s\);
\((f_3)\) \(\lim_{t \to \infty} \frac{f(t)}{t} = +\infty\);
\((f_4)\) There exist \(\theta > 2\) and \(a_3 \geq 0\), such that
\[ 0 < \theta F(t) \leq tf(t), \quad \forall t \geq a_3, \]
where \(F(t)\) denotes the primitive of \(f\).
\((f_5)\) \(\alpha := \inf \left\{ \frac{t^2}{t^2 - F(t)} ; t \in \text{ Fix}(f) \right\} > 0\), where \(\text{ Fix}(f) = \{ t > 0 ; f(t) = t \} \).

Condition \((f_5)\) permits us to discard constant solutions.

**Remark 1.1.** It follows from \((f_1)\) and \((f_2)\) that, for any fixed \(\eta > 0\) (or any fixed \(C_\eta > 0\)), there exists a constant \(C_\eta\) (respectively, \(\eta > 0\)) such that
\[ |f(t)| \leq \eta t + C_\eta t^{p-1}, \quad \forall t \geq 0 \quad (1.7) \]
and analogously, denoting \(F(t) = \int_0^t f(s)ds\) we have
\[ |F(t)| \leq \eta t^2 + C_\eta t^p \leq C(t^2 + t^p), \quad \forall t \geq 0 \quad (1.8) \]
for any \(2 < p < 2^*_s = \frac{2N}{N-2s} \).
Our first result is the following.

**Theorem 1.** Assume \((f_1)-(f_5)\). Then, for \(\varepsilon\) sufficiently small, there exists a non-constant, nonnegative solution of\((1.1)\) satisfying

\[
I_\varepsilon(u_\varepsilon) \leq C\varepsilon^N
\]

where \(C > 0\) depends only on \(\Omega\) and \(f\).

We use the Mountain Pass Theorem of Ambrosetti and Rabinowitz to prove this result, see [18, 21]. The main difficulties arise from the degeneracy of the operator and also from the geometry of the problem.

We also prove the following result.

**Theorem 2.** Suppose \(0 < s < 1\), \((f_1)-(f_3)\) holds. If \(u_\varepsilon\) is a solution to problem \((1.1)\) with \(\varepsilon > 0\) small enough, then \(u_\varepsilon \in L^\infty(\Omega)\).

We prove Theorem 2 by using Moser-Nash’s iteration method (see [13]), which has been used to study uniform bounds for fractional elliptic problems, see [6, 1, 2, 15, 3, 22].

### 2. Variational formulation

Problem \((1.1)\) has a variational structure. More precisely, consider

\[
\langle u, v \rangle_{\varepsilon,s} := \frac{C_{N,s} \varepsilon^{2s}}{2} \int_{\mathbb{R}^N \setminus (\Omega^c)^2 \setminus \Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy + \int_{\Omega} uv \, dx \tag{2.1}
\]

where \(\Omega^c = \mathbb{R}^N \setminus \Omega\) and \((\Omega)^2 = \Omega \times \Omega\). The space

\[
H^s_\varepsilon(\Omega) := \{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable and } \langle u, u \rangle_{\varepsilon,s} < \infty \}
\]

is a Hilbert space with the norm \(\|u\|_{H^s_\varepsilon(\Omega)} = \langle u, u \rangle_{\varepsilon,s}^{1/2}\), see [10] for details.

**Remark 2.1.** Note that constant functions are contained in \(H^s_\varepsilon(\Omega)\), see [7]. Moreover, for all \(u \in H^s_\varepsilon(\Omega)\), we have that \(u|_{\Omega} \in H^s(\Omega)\). Using the compact embedding \(H^s(\Omega) \hookrightarrow L^q(\Omega)\) for \(q \in \left(1, \frac{2N}{N-2s}\right)\), we conclude that the embedding

\[
H^s_\varepsilon(\Omega) \hookrightarrow L^q(\Omega), \text{ for all } 1 < q < \frac{2N}{N-2s}.
\]

is compact. So, if \((u_n)\) is bounded sequence in \(H^s_\varepsilon(\Omega)\), then \(u_n|_{\Omega}\) has a convergence subsequence in \(L^q(\Omega)\).

More precisely, considering the Sobolev constant,

\[
S = \inf_{u \in H^s_\varepsilon(\Omega), u \neq 0} \left( \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}} \left( \int_{\Omega} |u(x)|^2 \, dx \right)^{\frac{1}{2s}} \tag{2.2}
\]

we have the following Sobolev inequality:
Lemma 3. Let $\Omega \subset \mathbb{R}^N$ bounded and $\varepsilon > 0$. Then

$$
\left( \frac{1}{\Omega} \int_{\Omega} |u|^2^s \, dx \right)^{\frac{1}{s}} \leq S^2 \varepsilon^{-2s} \|u\|^2_{H^s(\Omega)}, \quad \forall u \in H^s_s(\Omega).
$$

(2.3)

where $S$ is the Sobolev constant defined in (2.2).

Proof. For any fixed $u \in H^s_s(\Omega)$, consider the function $v_\varepsilon(x) = u(\varepsilon x)$ defined in $\Omega_\varepsilon = \{ x \in \mathbb{R}^N : \varepsilon x \in \Omega \}$. It follows from the Sobolev inequality that

$$
\|u\|^2_{H^s_s(\Omega)} = \frac{C_{N,s}\varepsilon^{2s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \int_{\Omega} |u(x)|^2 \, dx
$$

$$
= \frac{C_{N,s}\varepsilon^{N+2s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(\varepsilon y)|^2}{|x-\varepsilon y|^{N+2s}} \, dx \, dy + \varepsilon^N \int_{\Omega} |u(\varepsilon x)|^2 \, dx
$$

$$
= \varepsilon^N \left[ \frac{C_{N,s}\varepsilon^{N+2s}}{2} \int_{\Omega} \int_{\Omega} \frac{|v_\varepsilon(x) - v_\varepsilon(y)|^2}{|x-\varepsilon y|^{N+2s}} \, dx \, dy + \int_{\Omega} |v_\varepsilon(x)|^2 \, dx \right]
$$

$$
\leq \frac{\varepsilon^N}{S^2} \left( \frac{1}{\Omega_\varepsilon} \int_{\Omega_\varepsilon} |v_\varepsilon(x)|^2^s \, dx \right)^{\frac{1}{s}} = \frac{\varepsilon^N}{S^2} \left( \frac{1}{\Omega} \int_{\Omega} |u(x)|^2^s \, dx \right)^{\frac{1}{s}}.
$$

Since

$$
N \left( 1 - \frac{2}{2s} \right) = N \left( 1 - \frac{N - 2s}{N} \right) = 2s,
$$

we obtain

$$
\left( \frac{1}{\Omega} \int_{\Omega} |u(x)|^2 \, dx \right)^{\frac{1}{s}} \leq S^2 \varepsilon^{-2s} \|u\|^2_{H^s_s(\Omega)}.
$$

Definition 2.1. We say that $u \in H^s_s(\Omega)$ is a weak solution of (1.1) if

$$
\frac{C_{N,s}\varepsilon^{2s}}{2} \int_{\Omega^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dx \, dy + \int_{\Omega} uv - \int_{\Omega} f(u) v \, dx = 0
$$

for all $v \in H^s_s(\Omega)$.

For all $u, v \in C^2(\mathbb{R}^N) \cap H^s_s(\Omega)$, it follows from a direct computation that

$$
\frac{C_{N,s}}{2} \int_{\Omega^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dx \, dy = \int_{\Omega} v(-\Delta)^s u \, dx + \int_{\Omega^c} v \mathcal{N}_s u \, dx,
$$

what yields

$$
\int_{\Omega} (\varepsilon^{2s}(-\Delta)^s u + u - f(u)) v \, dx + \varepsilon^{2s} \int_{\Omega^c} v \mathcal{N}_s u \, dx = 0.
$$

Thus, for $x \in \mathbb{R}^N \setminus \Omega$,

$$
\int_{\Omega^c} v(x) \left( C_{N,s} \int_{\Omega} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy \right) \, dx = 0,
$$

meaning that we have, weakly, $\mathcal{N}_s u = 0$. 

Let us define, for all $u \in H^s_0(\Omega)$,

$$I_\varepsilon(u) = \frac{C_{N,s,\varepsilon}}{4} \int_{\mathbb{R}^N \setminus (\Omega^c)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy + \frac{1}{2} \int_{\Omega} |u|^2 \, dx - \int_{\Omega} F(u) \, dx.$$ 

As an easy consequence of Remark (1.1) and of the above discussion, we have that the functional $I_\varepsilon$ is well-defined and $I_\varepsilon \in C^1(H^s_0(\Omega), \mathbb{R})$.

The derivative of the functional $I_\varepsilon$ is given by

$$I_\varepsilon'(u) \cdot v = \frac{C_{N,s,\varepsilon}}{2} \int_{\mathbb{R}^N \setminus (\Omega^c)^2} \frac{(u(x) - u(y))(v(x) - u(y))}{|x - y|^{N+2s}} \, dx \, dy + \int_{\Omega} uv \, dx - \int_{\Omega} f(u) v \, dy.$$ 

Therefore, critical points of $I_\varepsilon$ are weak solutions of (1.1).

3. Proof of Theorem 1

With arguments similar to that of Lin, Ni, and Takagi [14], we prove Theorem 1, which is a consequence of the following lemmas.

**Lemma 4.** There exist $\rho, \delta > 0$ such that $I_\varepsilon|_{S} \geq \delta > 0$ for all $u \in S$, where

$$S = \{ u \in H^s_0(\Omega) : \|u\|_{H^s_0} = \rho \}.$$

**Proof.** Maintaining the notation of Remark (1.1), the Sobolev embedding yields

$$I_\varepsilon(u) = \frac{C_{N,s,\varepsilon}}{4} \int_{\mathbb{R}^N \setminus (\Omega^c)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy + \frac{1}{2} \int_{\Omega} |u|^2 \, dx - \int_{\Omega} F(u) \, dx \geq \frac{1}{2} \|u\|_{H^s_0(\Omega)}^2 - \eta \int_{\Omega} |u|^2 \, dy - C_\eta \int_{\Omega} |u|^p \, dy = \frac{1}{2} \|u\|_{H^s_0(\Omega)}^2 - \eta \|u\|_{H^s_0(\Omega)}^p - C_\eta \|u\|_p^p$$

$$\geq \left( \frac{1}{2} - \eta \right) \|u\|_{H^s_0(\Omega)}^2 - S^2 \varepsilon^{-2s} \|u\|_{H^s_0(\Omega)}^p.$$ 

Taking $0 < \eta < \frac{1}{2}$, denote by $a = \frac{1}{2} - \eta$ and $A > 0 = S^2 \varepsilon^{-2s}$. So we obtain

$$I_\varepsilon(u) \geq a \|u\|_{H^s_0(\Omega)}^2 - A \|u\|_{H^s_0(\Omega)}^p,$$

for all $u \in H^s_0(\Omega)$

$$\geq \|u\|_{H^s_0(\Omega)}^2 \left( a - A \|u\|_{H^s_0(\Omega)}^{p-2} \right).$$

Since $p \in \left( 2, \frac{2N}{N-2s} \right)$, for $\rho \leq \left( \frac{a}{A} \right)^{\frac{2}{p-2}}$ we have

$$I_\varepsilon(u) \geq \rho^2 (a - Ap^{p-2}) > 0,$$

for all $\|u\|_{H^s_0(\Omega)} = \rho$. \hfill \Box

**Lemma 5.** If (f1)-(f4) hold, then $I_\varepsilon$ satisfies the Palais-Smale condition.

**Proof.** Let $(u_n)$ be a (PS)-sequence for $I_\varepsilon$ in $H^s_0(\Omega)$. Thus,

$$I_\varepsilon(u_n) \leq k_0 \quad \text{and} \quad I_\varepsilon'(u_n) \to 0.$$
Consequently, there exists \( n_0 \in \mathbb{N} \) such that

\[
\left| \|u_n\|_{H^s}^2 - \int_{\Omega} f(u_n) u_n \, dx \right| = |I'_s(u_n) \cdot u_n| \leq \|u_n\|_{H^s}^2, \quad \forall n \geq n_0.
\]

It follows from condition \((f_4)\) the existence of \( \theta > 2 \) and \( a_3 > 0 \) such that \( 0 < \theta F(t) \leq tf(t), \quad \forall t \geq a_3 \).

So, we obtain

\[
\frac{1}{2} \|u_n\|_{H^s}^2 - k_0 \leq \int_{\Omega} F(u_n) \, dx \leq \frac{1}{\theta} \int_{\{x \in \Omega; u_n \geq a_3\}} f(u_n) u_n \, dx + \int_{\{x \in \Omega; u_n \leq a_3\}} F(u_n) \, dx
\]

\[
\leq \frac{1}{\theta} \left( \|u_n\|_{H^2(\Omega)}^2 + \|u_n\|_{H^2(\Omega)}^2 \right) + \int_{\{x \in \Omega; u_n \leq a_3\}} F(u_n) \, dx
\]

\[
\leq \frac{1}{\theta} \|u_n\|_{H^2(\Omega)}^2 + \frac{1}{\theta} \|u_n\|_{H^2(\Omega)}^2 + A_1,
\]

where \( A_1 = |\Omega| \left( \max_{0 \leq t \leq a_3} F(t) \right) < \infty. \)

Therefore, for all \( n \geq n_0, \)

\[
\left( \frac{1}{2} - \frac{1}{\theta} \right) \|u_n\|_{H^2(\Omega)}^2 \leq \|u_n\|_{H^2(\Omega)}^2 + k_1.
\]

Since \( \theta > 2 \), it follows that \((u_n)\) is bounded in \( H^s(\Omega) \).

Thus, for a subsequence

\[
u_n \to u \quad \text{in} \quad H^s(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in} \quad L^q(\Omega),
\]

for all \( q \in \left( 1, \frac{2N}{N-2s} \right) \).

Condition \((f_2)\) allows us to conclude that \((u_n)\) converges weakly and

\[
\int_{\Omega} (f(u_n) - f(u))(u_n - u) \, dx \to 0 \quad \text{as} \quad n \to \infty. \tag{3.1}
\]

Combining \((3.1)\) with the identity

\[
(I'_s(u_n) - I'_s(u)) \cdot (u_n - u) = \|u_n - u\|_{H^s(\Omega)}^2 + \int_{\Omega} (f(u_n) - f(u))(u_n - u) \, dx,
\]

it follows that

\[
\lim_{n \to \infty} \|u_n - u\|_{H^s(\Omega)}^2 = \lim_{n \to \infty} (I'_s(u_n) - I'_s(u)) \cdot (u_n - u) = 0,
\]

that is, \( u_n \to u \) in \( H^s(\Omega) \).

\[\square\]

From now on, without loss of generality, we assume that \( 0 \in \Omega \). For any \( \varepsilon > 0 \) such that \( B_{\varepsilon}(0) \subset \Omega \), following Lin, Ni, and Takagi [14] we define

\[
\phi_\varepsilon(x) = \begin{cases} 
\varepsilon^{-N} \left( 1 - \frac{|x|}{\varepsilon} \right) & \text{if} \ |x| \leq \varepsilon, \\
0 & \text{if} \ |x| \geq \varepsilon.
\end{cases}
\]
According to Chen [7, Lemma 3.4], we have that, for \( \varepsilon > 0 \) small, \( \phi_\varepsilon \in H^s_\varepsilon(\Omega) \) and the following estimate is valid

\[
\| \phi_\varepsilon \|_{H^2_\varepsilon(\Omega)}^2 \leq \frac{C}{\varepsilon^N}, \quad \text{where} \quad C = C(N, s, \Omega). \tag{3.2}
\]

Moreover, we have (see [14, Equation 2.11])

\[
\int_\Omega |\phi_\varepsilon(x)|^q \, dx = K_q \varepsilon^{(1-q)N}, \quad \text{with} \quad K_q = N \Omega \int_0^1 (1 - \rho)^q \rho^{N-1} \, d\rho. \tag{3.3}
\]

The following lemmas are adaptations of results in Lin, Ni e Takagi [14].

**Lemma 6.** There exists a unique \( \sigma \in (0, 1) \) such that

\[
\int_{\Omega_\sigma} |\phi_\varepsilon(x)|^2 \, dx = \frac{1}{2} \int_{\Omega} |\phi_\varepsilon(x)|^2 \, dx
\]

where \( \Omega_\sigma = \{ x \in \Omega : \phi_\varepsilon(x) > \sigma \varepsilon^{-N} \} \).

**Proof.** In fact, note that if \( \sigma \in (0, 1) \) and \( \phi_\varepsilon(x) > \sigma \varepsilon^{-N} \), then \( |x| < (1 - \sigma)\varepsilon \).

Thus

\[
\int_{\Omega_\sigma} |\phi_\varepsilon(x)|^2 \, dx = \int_{B_{(1-\sigma)\varepsilon}(0)} \varepsilon^{-2N} \left( 1 - \frac{|x|}{\varepsilon} \right)^2 \, dx = \frac{1}{\varepsilon^{2N+2}} \int_{B_{(1-\sigma)\varepsilon}(0)} (\varepsilon - |x|)^2 \, dx
\]

\[
= \frac{N \Omega \varepsilon^N}{\varepsilon^{2N+2}} \int_0^{(1-\sigma)\varepsilon} (\varepsilon - r)^2 r^{N-1} \, dr
\]

\[
= \frac{N \Omega \varepsilon^N (1 - \sigma)^N}{\varepsilon^N} \left[ \frac{1}{N} - \frac{2(1 - \sigma)}{N + 1} + \frac{(1 - \sigma)^2}{N + 2} \right].
\]

On the other hand, taking \( q = 2 \) in (3.3), we obtain

\[
\int_\Omega |\phi_\varepsilon|^2 \, dx = \frac{N \Omega \varepsilon^N}{\varepsilon^N} \left[ \frac{1}{N} - \frac{2}{N + 1} + \frac{1}{N + 2} \right].
\]

Thus, we conclude the claim just by taking \( \sigma \in (0, 1) \) such that

\[
(1 - \sigma)^N \left[ \frac{1}{N} - \frac{2(1 - \sigma)}{N + 1} + \frac{(1 - \sigma)^2}{N + 2} \right] = \left[ \frac{1}{N} - \frac{2}{N + 1} + \frac{1}{N + 2} \right]. \quad \square
\]

Now, consider the function \( g : [0, \infty) \to \mathbb{R} \) defined by

\[
g(t) = I_\varepsilon(t\phi_\varepsilon) = \frac{t^2}{2} \| \phi_\varepsilon \|^2_{H^2_\varepsilon(\Omega)} - \int_\Omega F(t\phi_\varepsilon) \, dx. \tag{3.4}
\]

**Lemma 7.** There exist \( t_1, t_2 \in [0, \infty) \) with \( 0 < t_1 < t_2 \) such that

(i) \( g'(t) < 0 \) if \( t > t_1 \);

(ii) \( g(t) < 0 \) if \( t \geq t_2 \).
Proof. Taking the derivative in (3.4) and applying estimate (3.2), we obtain
\[ g'(t) = t\|\phi\|_{H^2(\Omega)}^2 - \int\limits_{\Omega} f(t\phi)\phi\,dx \leq \frac{tC}{\varepsilon N} - \int\limits_{\Omega} f(t\phi)\phi\,dx. \] (3.5)
Note that condition (f3) implies that, for any $R > 0$, there exists $M_R > 0$ such that for all $\xi \geq M_R$, we have
\[ f(\xi) \geq R\xi. \] (3.6)
Denote $\Omega_1 = \left\{ x \in \Omega : \phi(x) > \frac{M_R}{t} \right\}$.
Keeping in mind Lemma 6, note that $\Omega_0 \subset \Omega_1$ for $t > \frac{M_R\varepsilon^N}{\sigma}$. Since $f(t) > 0$, for such $t$, it follows from (3.6) that
\[ \int\limits_{\Omega} f(t\phi(x))\phi(x)\,dx \geq \int\limits_{\Omega_1} f(t\phi(x))\phi(x)\,dx \geq \int\limits_{\Omega_1} R\phi(x)\phi(x)\,dx \geq Rt \int\limits_{\Omega} (\phi(x))^2\,dx. \]
Substituting into (3.5) and applying Lemma 6, we obtain
\[ g'(t) \leq \frac{Ct}{\varepsilon N} - Rt \int\limits_{\Omega} (\phi(x))^2\,dx = \frac{Ct}{\varepsilon N} - \frac{Rt}{2} \int\limits_{\Omega} (\phi(x))^2\,dx = t\varepsilon^{-N} \left( C - \frac{K_2R}{2} \right). \]
where $K_2$ was defined in (3.3). So, for $R_1 > \frac{2C}{K_2}$, we have
\[ g'(t) < 0 \quad \text{for any } \quad t > \frac{M_R\varepsilon^N}{\sigma} = t_1. \]
In order to prove (ii), note that (f1) and (3.6) imply that, for any $\xi \geq M_R$, we have
\[ F(\xi) = \int_0^\xi f(\tau)d\tau = \int_0^{M_R} f(\tau)d\tau + \int_{M_R}^\xi f(\tau)d\tau \geq \int_{M_R}^\xi R\tau d\tau = \frac{R\xi^2}{2} - m_R \]
where $m_R = \frac{M_R^2R}{2}$. Applying again (3.3), we obtain
\[ g(t) = \frac{t^2}{2}\|\phi\|_{H^2(\Omega)}^2 - \int\limits_{\Omega} F(t\phi)\,dx \]
\[ \leq \frac{t^2C}{2\varepsilon N} - \frac{RK_2t^2}{2\varepsilon^2} + m_R|\Omega| \]
\[ = \frac{t^2}{2\varepsilon^N} (C - RK_2) + m_R|\Omega|. \]
Taking $R_2 > \frac{C}{K_2}$, it follows that
\[ g(t) < 0 \quad \text{for all } \quad t > 0 \quad \text{such that } \quad t^2 > \frac{2m_r|\Omega|\varepsilon^N}{R_2K_2 - C}. \]
In order to have \( t_2 > t_1 \), we take \( t_2 \) satisfying
\[
\frac{M_R\varepsilon^N}{\sigma} < t_2 < \frac{2m_R|\Omega|\varepsilon^N}{R_2K_2 - C}.
\]
We are done. \( \square \)

**Lemma 8.** For all \( \varepsilon > 0 \) sufficiently small, there exists a nonnegative function \( \phi \in H^s_0(\Omega) \) and \( t_0 > 0 \) such that \( I_\varepsilon(t_0\phi) = 0 \). Moreover, there is \( C = C(N, s, \Omega) > 0 \)
\[
I_\varepsilon(t\phi) \leq C\varepsilon^N \quad \text{for all } t.
\]

**Proof.** According to Lemma 4, we have \( g(t) > 0 \) for \( t \) sufficiently small. Lemma 7 and \((f_1)\) imply that \( g(t) \geq 0 \) for \( 0 < t < t_1 \). Thus, by substituting (3.2) into (3.7), we obtain
\[
\max_{t \geq 0} g(t) = \max_{0 \leq t \leq t_1} \left\{ \frac{2t^2}{2\varepsilon^N} - \frac{\int_{\Omega} F(t\phi)dx}{2\varepsilon^N} \right\} \leq \max_{0 \leq t \leq t_1} \frac{Ct^2}{2\varepsilon^N} = \frac{Ct_1^2}{2\varepsilon^N}.
\]
Since \( t_1 = \frac{M_R\varepsilon^N}{\sigma} \), we have
\[
I_\varepsilon(t\phi) = g(t) \leq \max_{t \geq 0} g(t) = \frac{CM_2^2\varepsilon^{2N}}{2\varepsilon^N\sigma^2} = C_1\varepsilon^N
\]
for a positive constant \( C_1 \). The existence of \( t_0 > t_1 \) also follows from Lemma 7. \( \square \)

**Theorem 1.** Assume \((f_1)-(f_5)\). Then, for \( \varepsilon \) is sufficiently small, there exists a non-constant, nonnegative solution of (1.1) satisfying
\[
I_\varepsilon(u_\varepsilon) \leq C\varepsilon^N
\]
where \( C > 0 \) depends only on \( \Omega \) and \( f \).

**Proof.** Choose \( t_2 \) as in Lemma 7 and define \( e = t_2\phi_\varepsilon \in H^s_0 \). The geometry of the Mountain Pass Theorem was obtained in Lemmas 4 and 7, while the (PS)-condition was proved in Lemma 5. Considering
\[
\Gamma = \{ \gamma \in C([0, 1]; H^s_0(\Omega)); \quad \gamma(0) = 0 \quad \text{and} \quad \gamma(1) = e \},
\]
the value
\[
c_\varepsilon := \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I_\varepsilon(\gamma(t)) \geq \delta > 0
\]
is a critical value of \( I_\varepsilon \). Therefore, there exists \( u_\varepsilon \in H^s_0(\Omega) \) such that,
\[
I_\varepsilon(u_\varepsilon) = c_\varepsilon \quad \text{and} \quad I_\varepsilon'(u_\varepsilon) = 0.
\]
In particular, Lemma 8 implies that
\[
I_\varepsilon(u_\varepsilon) = c_\varepsilon \leq \max_{0 \leq t \leq t_2} I_\varepsilon(t\phi_\varepsilon) \leq C\varepsilon^N.
\]
Observe that, if \( u = \mu \) is a solution to our problem (1.1), then
\[
f(\mu) = \mu.
\]
It follows from condition \((f_5)\) that
\[
I_\varepsilon(\mu) = \frac{\mu^2}{2} - \int_{\Omega} F(\mu)dx = \left(\frac{\mu^2}{2} - F(\mu)\right)|\Omega| \geq \alpha|\Omega| > 0.
\]
Thus, for \(\varepsilon < \left(\frac{\alpha|\Omega|}{C_\varepsilon\varepsilon}\right)^{\frac{1}{4}}\) we obtain
\[
I_\varepsilon(u_\varepsilon) \leq C\varepsilon^N < \alpha|\Omega| = I_\varepsilon(\mu),
\]
meaning that, for \(\varepsilon > 0\) sufficiently small, \(u_\varepsilon\) cannot be constant, and therefore, is nontrivial.

Finally, condition \((f_1)\) implies that \(f(u_\varepsilon) = 0\) if \(x \in \{x \in \Omega; u_\varepsilon \leq 0\}\). Thus, denoting for \(u_\varepsilon^- = \max\{-u_\varepsilon, 0\}\) we have
\[
\int_{\Omega} f(u_\varepsilon^-)u_\varepsilon^- dx = \int_{\{x \in \Omega; u_\varepsilon^+ > 0\}} f(u_\varepsilon^-)u_\varepsilon^- dx + \int_{\{x \in \Omega; u_\varepsilon^- \leq 0\}} f(u_\varepsilon^-)u_\varepsilon^- dx = 0.
\]
Therefore,
\[
0 = I'_\varepsilon(u_\varepsilon^-) \cdot u_\varepsilon^- = \frac{C_{N,s}2s}{2} \int \int_{\mathbb{R}^{2N}\backslash(\Omega)^2} \frac{(u_\varepsilon^-(x) - u_\varepsilon^-(y))(u_\varepsilon^-(x) - u_\varepsilon^-(y))}{|x - y|^{N+2s}} dxdy + \int_{\Omega} |u_\varepsilon^-|^2 dx.
\]
Now, the inequality \((\xi - \eta)(\xi^- - \eta^-) \geq |\xi^- - \eta^-|^2\) guarantees that
\[
\frac{C_{N,s}2s}{2} \int \int_{\mathbb{R}^{2N}\backslash(\Omega)^2} \frac{|u_\varepsilon^-(x) - u_\varepsilon^-(y)|^2}{|x - y|^{N+2s}} dxdy + \int_{\Omega} |u_\varepsilon^-|^2 dx = 0,
\]
proving that \(u_\varepsilon^- \equiv 0\), that is, \(u_\varepsilon \geq 0\).

\[\square\]

**Corollary 9.** Assume conditions \((f_1)\)-\((f_5)\) with \(a_3 = 0\). If \(u_\varepsilon\) is a solution of (1.1), then there exists a constant \(K_0 > 0\) such that
\[
\|u_\varepsilon\|^2_{H^s} = \int_{\Omega} f(u_\varepsilon)u_\varepsilon dx \leq K_0\varepsilon^N.
\]

**Proof.** Since \(I'_\varepsilon(u_\varepsilon^-) \cdot u_\varepsilon^- = 0\), we have
\[
\frac{C_{N,s}2s}{2} \int \int_{\mathbb{R}^{2N}\backslash(\Omega)^2} \frac{|u_\varepsilon^-(x) - u_\varepsilon^-(y)|^2}{|x - y|^{N+2s}} dxdy + \int_{\Omega} |u_\varepsilon^-|^2 dx = \int_{\Omega} f(u_\varepsilon)u_\varepsilon dx,
\]
that is,
\[
\|u_\varepsilon\|^2_{H^s(\Omega)} = \int_{\Omega} f(u_\varepsilon)u_\varepsilon dx.
\]

Theorem 1 and \((f_i)\) yield
\[
C\varepsilon^N \geq I_\varepsilon(u_\varepsilon) = \frac{1}{2}\|u_\varepsilon\|^2_{H^s(\Omega)} - \int_{\Omega} F(u_\varepsilon)dx \geq \frac{1}{2}\|u_\varepsilon\|^2_{H^s(\Omega)} - \frac{1}{\theta} \int_{\Omega} f(u_\varepsilon)u_\varepsilon dx = \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_\varepsilon\|^2_{H^s(\Omega)}
\]
Since $\theta > 2$, we obtain that
\[ \|u_\varepsilon\|_{H^2_0(\Omega)}^2 \leq K_0 \varepsilon^N. \]

4. Proof of Theorem 2

**Theorem 2.** Suppose $0 < s < 1$, $(f_1)$-$(f_3)$ holds. If $u_\varepsilon$ is a solution to problem (1.1) with $\varepsilon > 0$ small enough, then $u_\varepsilon \in L^\infty(\Omega)$.

**Proof.** In order to simplify the notation, we denote $u = u_\varepsilon$ a solution of (1.1) for $\varepsilon > 0$ sufficiently small. Theorem 1 guarantees that $u \geq 0$. Given $\alpha > 1$ and $M > 0$, consider the functions $u_M = \min\{u, M\}$ and
\[ g_{\alpha,M}(t) = t \left(\min\{t, M\}\right)^{\alpha-1} = \begin{cases} t^\alpha, & \text{if } t \leq M \\ tM^{\alpha-1}, & \text{if } t > M. \end{cases} \]

Since $g_{\alpha,M}$ is Lipschitz continuous and increasing, we conclude that $g_{\alpha,M}(u) \in H^s(\Omega)$, for all $u \in H^s(\Omega)$.

Thus,
\[ I'_{\varepsilon}(u) \cdot g_{\alpha,M}(u) = 0 \]
that is,
\[ \frac{C_{N,s} \varepsilon^{2s}}{2} \int_{\mathbb{R}^N \setminus (\Omega^c)} \frac{(u(x) - u(y)) (g_{\alpha,M}(u)(x) - g_{\alpha,M}(u)(y))}{|x-y|^{N+2s}} \, dx \, dy + \int_{\Omega} u g_{\alpha,M}(u) \, dx = \int_{\Omega} f(u) g_{\alpha,M}(u) \, dx. \]  

(4.1)

We define the function,
\[ G_{\alpha,M}(t) = \int_{0}^{t} \left( g'_{\alpha,M}(\tau) \right)^{\frac{1}{2}} \, d\tau. \]

A direct calculation shows that
\[ G_{\alpha,M}(t) \geq \frac{2}{\alpha + 1} t \left(\min\{t, M\}\right)^{\frac{\alpha-1}{2}}, \quad \text{for all } t \in \mathbb{R}. \]  

(4.2)

Moreover,
\[ |G_{\alpha,M}(a) - G_{\alpha,M}(b)|^2 \leq (g_{\alpha,M}(a) - g_{\alpha,M}(b)) (a - b), \quad \forall a, b \in \mathbb{R}. \]  

(4.3)

It follows from (4.3) that
\[ [G_{\alpha,M}(u)]_{s,2}^2 := \frac{C_{N,s} \varepsilon^{2s}}{2} \int_{\Omega} \int_{\Omega} \frac{|G_{\alpha,M}(u(x)) - G_{\alpha,M}(u(y))|^2}{|x-y|^{N+2s}} \, dx \, dy \]
\[ \leq \frac{C_{N,s} \varepsilon^{2s}}{2} \int_{\mathbb{R}^N \setminus (\Omega^c)} \frac{(u(x) - u(y)) (g_{\alpha,M}(u)(x) - g_{\alpha,M}(u)(y))}{|x-y|^{N+2s}} \, dx \, dy. \]
Lemma 3 (i.e., the Sobolev inequality) and (4.1) yield
\[ S^{-2\varepsilon^{2\alpha}} \left( \int_{\Omega} |G_{\alpha,M}(u)|^{2^*} \, dx \right)^{\frac{2}{2^*}} \leq |G_{\alpha,M}(u)|^2 + \int_{\Omega} |G_{\alpha,M}(u)|^2 \, dx \]
\[ \leq \int_{\Omega} f(u) g_{\alpha,M}(u) \, dx = \int_{\Omega} f(u) u^{\alpha-1} \, dx. \]

Combining (4.2) with the last inequality, we obtain
\[ S^{-2\varepsilon^{2\alpha}} \left( \int_{\Omega} u^{\frac{\alpha-1}{2}} |2^*_s| \, dx \right)^{\frac{2}{2^*}} \leq S^{-2\varepsilon^{2\alpha}} \left( \int_{\Omega} |G_{\alpha,M}(u)|^{2^*} \, dx \right)^{\frac{2}{2^*}} \]
\[ \leq \int_{\Omega} f(u) u^{\alpha-1} \, dx. \]  (4.4)

According to Remark 1.1, for any fixed \( C_\eta > 0 \), there exists \( \eta > 0 \) such that
\[ |f(t)| \leq \eta|t|^{2^* - 1} + C_\eta |t|^{2^* - 1}, \quad \forall t \in \mathbb{R}. \]  (4.5)
Applying (4.5) and the Hölder inequality, we estimate the right-hand of (4.4).
\[ \int_{\Omega} f(u) u^{\alpha-1} \, dx \leq \eta \int_{\Omega} u^2 u_M^{\alpha-1} \, dx + C_0 \int_{\Omega} u^{2^* - 2} u_M^{\alpha-1} \, dx \]
\[ = \eta \int_{\Omega} u^2 u_M^{\alpha-1} \, dx + C_0 \int_{\Omega} u^{2^* - 2} u_M^{\alpha-1} \, dx \]
\[ \leq \eta \int_{\Omega} u^2 u_M^{\alpha-1} \, dx + C_0 \int_{\Omega}^{\alpha,\alpha} \left( \int_{\Omega} u_M^{\alpha-1} \, dx \right)^{\frac{\alpha}{\alpha-1}}. \]
Choosing \( C_\eta > 0 \) small enough such that
\[ C_\eta \int_{\Omega} \left( \int_{\Omega} u^{\alpha-1} \right) \leq S^{-2\varepsilon^{2\alpha}} \left( \frac{2}{\alpha + 1} \right)^2, \]
we conclude that
\[ \int_{\Omega} f(u) u^{\alpha-1} \, dx \leq \eta \int_{\Omega} u^2 u_M^{\alpha-1} + \frac{S^{-2\varepsilon^{2\alpha}}}{2} \left( \frac{2}{\alpha + 1} \right) \left( \int_{\Omega} u_M^{\alpha-1} \, dx \right)^{\frac{\alpha}{\alpha-1}}. \]
Combining with (4.4) yields
\[ \frac{S^{-2\varepsilon^{2\alpha}}}{2} \left( \frac{2}{\alpha + 1} \right) \left( \int_{\Omega} u_M^{\alpha-1} \, dx \right)^{\frac{\alpha}{\alpha-1}} \leq \eta \int_{\Omega} u^2 u_M^{\alpha-1} \, dx. \]
Thus, for a positive constant \( C \),
\[ \left( \int_{\Omega} u_M^{\frac{\alpha-1}{2^*}} \, dx \right)^{\frac{2}{2^*}} \leq C(\alpha + 1)^2 \int_{\Omega} u^2 u_M^{\alpha-1} \, dx. \]
Making \( M \to \infty \), Fatou’s lemma and the dominate convergence theorem yield
\[ \|u\|_{2^*(\frac{\alpha+1}{\alpha-1})} \leq C(\alpha + 1)^2 \|u\|_{2(\frac{\alpha+1}{\alpha-1})}. \]
and taking $\beta = \frac{\alpha + 1}{2}$, we obtain

$$
\|u\|_{2^\ast \beta}^{2\beta} \leq C\beta^2 \|u\|_{2\beta}^{2\beta}.
$$

(4.6)

Now, choose $K > 1$ such that $C\beta^2 \leq Ke^{\sqrt{\beta}}$. Then, (4.6) can be written as

$$
\|u\|_{2^\ast \beta}^{\beta} \leq K e^{\sqrt{\beta}} \|u\|_{2\beta}^{\beta}.
$$

Thus,

$$
\|u\|_{2^\ast \beta} \leq K e^{\frac{1}{\sqrt{\beta}}} \|u\|_{2\beta}^{\frac{1}{\beta}}, \quad \text{for all } \beta > 0.
$$

(4.7)

Consider the sequence defined by

$$
\beta_1 = 1, \quad \beta_{n+1} = \left(\frac{2n}{2}\right) \beta_n \quad \text{for } n = N = \{1, 2, \ldots\}.
$$

Since $\frac{\beta_{n+1}}{\beta_n} = \frac{2}{2} < 1$, the series

$$
\sum_{n=0}^{\infty} \frac{1}{\beta_n} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{\sqrt{\beta_n}}
$$

(4.8)

are both convergent.

Using the sequence $(\beta_n)$ in (4.7) and iterating we obtain

$$
\|u\|_{2^\ast \beta_2} \leq K e^{\frac{1}{\sqrt{\beta_2}}} \|u\|_{2\beta_2} = K e^{\frac{1}{\sqrt{\beta_2}}} e^{\frac{1}{2\sqrt{\beta_2}}} \|u\|_2.
$$

Proceeding repeatedly, yields

$$
\|u\|_{2^\ast \beta_n} \leq K \left(\sum_{i=0}^{n} \frac{1}{\beta_i}\right) e^{\left(\sum_{i=0}^{n} \frac{1}{\sqrt{\beta_i}}\right)} \|u\|_2.
$$

Making $n \to \infty$, we conclude that, for positive constants $\gamma_1$ and $\gamma_2$

$$
\|u\|_{\infty} \leq K^{\gamma_1} e^{\gamma_2} \|u\|_2 < \infty,
$$

that is, $u \in L^\infty(\Omega)$.

**Acknowledgements:** The authors thank Gilberto A. Pereira for useful conversations.

**References**

[1] C. Alves and O. Miyagaki. Existence and concentration of solution for a class of fractional elliptic equation in $\mathbb{R}^n$ via penalization method. *Calc. Var. Partial Differential Equations*, 55(03):19 pp., 2016.

[2] V. Ambrosio. Periodic solutions for a pseudo-relativistic Schrödinger equation. *Nonlinear Anal.*, 120:262–284, 2015.

[3] V. Ambrosio. Multiplicity of positive solutions for a class of fractional Schrödinger equations via penalization method. *Ann. Mat. Pura Appl. (4)*, 196:2043–2062, 06 2017.

[4] G. Barles, E. Chasseigne, C. Georgelin, and E. Jakobsen. On Neumann type problems for nonlocal equations set in a half space. *Trans. Amer. Math. Soc.*, 366(9):4873–4917, 2014.

[5] G. Barles, C. Georgelin, and E. Jakobsen. On Neumann and oblique derivatives boundary conditions for nonlocal elliptic equations. *J. Differential Equations*, 256(4):1368–1394, 2014.

[6] L. Brasco and E. Parini. The second eigenvalue of the fractional p-Laplacian. *Adv. Calc. Var.*, 9(4):323–355, 2016.
[7] G. Chen. Singularly perturbed Neumann problem for fractional Schrödinger equations. Sci. China Math., 61(4):695–708, 2018.
[8] C. Cortazar, M. Elgueta, J. D. Rossi, and N. Wolanski. Boundary fluxes for nonlocal diffusion. J. Differential Equations, 234(2):360–390, 2007.
[9] C. Cortazar, M. Elgueta, J. D. Rossi, and N. Wolanski. How to approximate the heat equation with Neumann boundary conditions by nonlocal diffusion problems. Arch. Ration. Mech. Anal., 187(1):137–156, 2008.
[10] S. Dipierro, X. Ros-Oton, and E. Valdinoci. Nonlocal problems with Neumann boundary conditions. Rev. Mat. Iberoam., 33(2):377–416, 2017.
[11] Q. Du, M. Gunzburger, and R. Lehoucq. Analysis and approximation of nonlocal diffusion problems with volume constraints. SIAM Rev., 54:667–696, 4 2012.
[12] Q. Du, M. Gunzburger, R. Lehoucq, and K. Zhou. A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws. Math. Models Methods Appl. Sci., 23:493–540, 3 2013.
[13] D. Gilbarg and N. S. Trudinger. Elliptic Partial Differential Equations of Second Order, volume 224 of Classics in Mathematics. 2001.
[14] C.-S. Lin, W.-M. Ni, and I. Takagi. Large amplitude stationary solutions to a chemotaxis system. J. Differential Equations, 72(1):1–27, 1988.
[15] E. Montefusco, B. Pellacci, and G. Verzini. Fractional diffusion with Neumann boundary conditions: The logistic equation. Discrete and Continuous Dynamical Systems - Series B, 18, 08 2012.
[16] R. Musina and A. I. Nazarov. On fractional laplacians. Comm. Partial Differential Equations, 39(9):1780–1790, 2014.
[17] H. Ni, A. Xia, and X. Zheng. Existence of positive solutions for nonlinear fractional Neumann elliptic equations. Differ. Equ. Appl., 10(1):115–129, 2018.
[18] R. Servadei and E. Valdinoci. Variational methods for non-local operators of elliptic type. Discrete Contin. Dyn. Syst., 33(5):2105–2137, 2013.
[19] R. Servadei and E. Valdinoci. On the spectrum of two different fractional operators. Proc. Roy. Soc. Edinburgh Sect. A, 144(4):831–855, 2014.
[20] P. Stinga and B. Volzone. Fractional semilinear Neumann problems arising from a fractional Keller-Segel model. Calc. Var. Partial Differential Equations, 54(1):1009–1042, 2015.
[21] M. Willem. Minimax theorems, volume 24 of Progress in Nonlinear Differential Equations and their Applications. Birkhäuser Boston, Inc., Boston, MA, 1996.
[22] A. Xia and J. Yang. Regularity of nonlinear equations for fractional Laplacian. Proc. Amer. Math. Soc., 141:2665–2672, 08 2013.

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