THE WITTEN GENUS AND EQUIVARIANT ELLIPTIC COHOMOLOGY

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Abstract. We construct a Thom class in complex equivariant elliptic cohomology extending the equivariant Witten genus. This gives a new proof of the rigidity of the Witten genus, which exhibits a close relationship to recent work on non-equivariant orientations of elliptic spectra.

1. Introduction

Let \( T \) denote the circle group. If \( X \) is a \( T \)-space, let \( X_T \) denote the Borel construction \( E_T \times_T X \). Let \( H_T \) denote Borel \( T \)-equivariant ordinary cohomology with complex coefficients. Choose a generator of the complex character group of \( T \) and so an isomorphism \( H_T(\text{pt}) \cong \mathbb{C}[z] \).

Suppose that \( \tau \) is a complex number with positive imaginary part. Consider the lattice \( \Lambda = 2\pi i \mathbb{Z} + 2\pi i \tau \mathbb{Z} \), and let \( \mathbb{C} = \mathbb{C}/\Lambda \) be the associated elliptic curve. Grojnowski ([Gro94]; see also [Ros99, GKV95]) has defined a \( T \)-equivariant elliptic cohomology functor \( E \) from compact \( T \)-spaces to sheaves of algebras over \( \mathcal{O}_C \), equipped with a canonical isomorphism \( E(X)_0 \cong H_T(X) \otimes_{\mathbb{C}[z]} \mathcal{O}_C \).

The parameter \( \tau \) determines a sigma function \( \sigma = \sigma(z, \tau) \); it is an odd holomorphic function of \( z \) which vanishes to first order at the points of \( \Lambda \); see §5.1. Its Taylor expansion at the origin determines a Thom class \( \phi(V)_T \in H_T(V, V_0) \otimes_{\mathbb{C}[z]} \mathcal{O}_C \) for any oriented \( T \)-vector bundle \( V \), which we call the “equivariant \( \sigma \)-orientation”.

The associated genus is the Witten genus ([AHS98]; see also §5.1). If \( W \) is a virtual spin \( T \)-vector bundle over \( X \), then there are genuine spin \( T \)-vector bundles \( T \) and \( V \) such that \( W = V - T \), and we may define \( E(V - T) \) to be the invertible sheaf

\[
E(W) = E(T)^{-1} \otimes_{E(X)} E(V),
\]

as the right-hand-side is independent of \( V \) and \( T \) up to canonical isomorphism (see §3).

We recall that if \( W \) is a virtual spin vector bundle over \( X \), then there is a characteristic class \( \frac{\nu}{2}(W) \in H^4(X; \mathbb{Z}) \), twice which is the first Pontrjagin class. Our main result is the construction of a Thom class in \( E \)-theory.

Theorem 1.1. Suppose that \( T \) acts non-trivially on a compact smooth manifold \( M \). Suppose that \( W \) is an virtual spin \( T \)-vector bundle over \( M \) such that

\[
\begin{align*}
  w_2(W_T) &= 0, \\
  \frac{\nu}{2}(W_T) &= 0.
\end{align*}
\]

Then the invertible sheaf of \( E(X) \)-modules \( E(W) \) has a global section \( \Sigma \), whose stalk at the origin is the equivariant \( \sigma \)-orientation.
In the Theorem we have used the fact that the map \( V \mapsto V_T \), which sends a \( T \)-vector bundle to its Borel construction, defines a homomorphism \( KO_T(X) \to KO(X_T) \). We have also used the fact that the Whitney sum formula implies that the second Stiefel-Whitney class is well-defined for oriented virtual vector bundles, and the class \( w_2 \) is well-defined for virtual bundles with \( w_2 = 0 \). More precisely, we have maps of infinite loop spaces

\[
\begin{array}{ccc}
BSpin & \longrightarrow & BSO \\
\downarrow & \ & \downarrow w_2 \\
K(\mathbb{Z}, 2) & \longrightarrow & K(\mathbb{Z}, 4),
\end{array}
\]

in which the row is a fibration.

Now consider the case that \( W = V - T \), where \( T \) is the tangent space of \( M \). If \( T \) is spin so that Proposition \( \ref{prop:spin} \) defines \( E(-T) \), then the Pontrjagin-Thom construction provides a map of sheaves

\[
E(V - T) \xrightarrow{\zeta} E(-T) \xrightarrow{PT} E(*) \cong \mathcal{O}_C,
\]

where \( \zeta \) is the relative zero section. The class \( PT\zeta^*(\Sigma) \) is the equivariant Witten genus of \( M \) twisted by \( V \) (see \( \S 3.1 \)). Since \( PT\zeta^*(\Sigma) \) is a global section of \( \mathcal{O}_C \), the twisted Witten genus is “rigid”:

**Corollary 1.3.** If \( T \) is the tangent bundle of \( M \), the characteristic classes of \( V - T \) satisfy \( \{1, 2, \ldots\} \), and \( T \) (and so also \( V \)) is spin, then the equivariant Witten genus of \( M \) twisted by \( V \) is constant.

This result was discovered by Witten, who gave a physical proof in \( \text{[Wit87]} \). The first mathematical proofs were given by Taubes and Bott-Taubes \( \text{[Tau89, BT89]} \). Subsequently Kefeng Liu gave a shorter proof and found many new cases \( \text{[Liu94, Liu95a, Liu95b, GL96, Liu96a, Liu96b]} \).

Besides giving an appealing relationship with equivariant elliptic cohomology, our construction has the virtue of axiomatizing the properties of the \( \sigma \)-function on which it depends. We construct global sections of \( E(W) \) for a class of theta functions described in \( \text{[Bre83]} \). Theorem \( \text{[1.1]} \) is a special case of Theorem \( \text{[6.1]} \).

We were led to the theta functions of \( \text{[1]} \) by the work of Hopkins, Strickland, and the first author. A theta function such as we consider descends to a “\( \Sigma \) structure” on the elliptic curve \( C \), in the sense of \( \text{[Bre83]} \). Ando-Hopkins-Strickland have shown that, if \( E \) is a (non-equivariant) elliptic cohomology theory with elliptic curve \( C \), then maps of ring spectra \( MO(\mathbb{R}) \to E \) are given by \( \Sigma \) structures on the ideal sheaf \( \mathcal{I}_C(0) \) of the origin in the formal group \( C \) of \( C \).

The Weierstrass \( \sigma \)-function given in \( \text{[5.1]} \) descends to the unique \( \Sigma \) structure on the ideal sheaf \( \mathcal{I}_C(0) \), and so gives a \( \Sigma \) structure on \( \mathcal{I}_C(0) \) by restriction. It follows that the \( \sigma \)-orientation is the \text{unique natural orientation} from \( MO(\mathbb{R}) \) to elliptic spectra. Theorem \( \text{[1.3]} \) establishes a fundamental relationship between these results and the rigidity theorems for equivariant elliptic genera. It seems reasonable to hope that a (rational) equivariant analogue of the methods of \( \text{[AHS98]} \) will produce a functorial equivariant \( MO(\mathbb{R}) \) orientation to the rational equivariant elliptic spectra of Greenlees, Hopkins, and Rosu \( \text{[3HR99]} \), without the elaborate calculations given below. We will return to that problem in another paper.

This paper was also inspired by the work of Haynes Miller and Ioanid Rosu \( \text{[Mil81, Ros99]} \). Miller proposed, before even Grojnowski’s functor was available, that an uncompleted equivariant elliptic cohomology should offer a proof of the rigidity theorems. Rosu and the first author observed that the rigidity theorems follow from the existence of a global section of the Thom sheaf in Grojnowski’s theory. Rosu constructed a Thom class whose value at the origin gives the Ochanine genus. In fact, he showed that the “transfer” argument of Bott and Taubes is the essential ingredient in the construction of the Thom section. The rigidity of the Ochanine genus and Rosu’s analysis require only that the bundles in question be spin bundles of even rank. This paper represents a first attempt to understand the Thom classes whose rigidity require the restrictions \( \text{[1.2]} \).

In fact Theorems \( \text{[1.1]} \) and \( \text{[5.1]} \) really only require

\[
\begin{align*}
W_2(W_T) &= 0 \\
p_1(W_T) &= 0,
\end{align*}
\]

as one can see by investigating the use of these conditions in \( \text{[8]} \): the vanishing of \( \frac{w_2}{4}(W_T) \) is used only in rational expressions. Indeed Liu \( \text{[Liu96b]} \) has shown that Corollary \( \text{[1.3]} \) holds without any condition on
We write $\pi$ for the covering map
\[ \mathbb{C} \xrightarrow{\pi} C. \]
If $V$ is an open set in a complex analytic variety, then we write $\mathcal{O}_V$ for the sheaf of holomorphic functions on $V$, and $\mathcal{M}_V$ for the sheaf of meromorphic functions.

Both $\mathbb{C}$ and $C$ are abelian topological groups. If $G$ is an abelian topological group and $g \in G$, then we write $\tau_g$ for the translation map; and if $V \subset G$ is an open set, then we write
\[ V - g \overset{\text{def}}{=} \tau_{-g}(V). \]

**Definition 2.1.** An open set $U$ of $C$ is small if $\pi^{-1}U$ is a disjoint union of connected components $V$ such that $\pi|_V : V \to U$ is an isomorphism.

If $U$ is small and $V$ is a component of $\pi^{-1}U$, then the covering map induces an isomorphism
\[ (\pi|_V)_* \mathcal{O}_V \cong \mathcal{O}_U. \]

In particular, if $U$ contains the origin of $C$, then there is a unique component $V$ of $\pi^{-1}U$ containing 0. This determines a $\mathbb{C}[z]$-algebra structure on $\mathcal{O}_U$, and a $\mathbb{C}[z, z^{-1}]$-algebra structure on $\mathcal{O}_U|_{\mathbb{C} \setminus \{0\}}$.

**2.2. Adapted open cover of an elliptic curve.** Now suppose that $X$ is a compact $T$-manifold. If $a$ is a point of $C$, then we define
\[ X^a = \begin{cases} X^T[k] & \text{if } a \text{ is of order exactly } k \text{ in } C \\ X^T & \text{otherwise}. \end{cases} \]

Let $N \geq 1$ be an integer.

**Definition 2.2.** A point $a \in C$ is special to level $N$ for $X$ if $X^{Na} \neq X^T$.

If $V$ is a $T$-bundle over a compact $T$-space $X$, then it is convenient to consider a few additional points to be special. For each component $F$ of $X^T$, there are integers $m_j$ and an isomorphism of real $T$-vector bundles
\[ V|_F \cong V(0) + \bigoplus V(m_j)^\mathbb{R}. \]

Here $V(0)$ is the summand of $V|_F$ on which $T$ acts trivially, $V(m_j)$ is a complex vector bundle on which $z \in T$ acts by fiberwise multiplication by the complex number $z^{m_j}$, and if $W$ is a complex vector bundle we write $W^\mathbb{R}$ for the underlying real vector bundle. The $m_j$ are called exponents or rotation numbers of $V$ at $F$. Let $V^+$ denote the one-point compactification of $V$. 

$w_2(W_T)$. The vanishing of $w_2(W_T)$ seems to be necessary for Theorem 1.1 and the full conditions (1,2) will certainly be required for any natural equivariant $MO(8)$ orientation to the category of equivariant elliptic spectra.

This can already be seen in §5 where we investigate the situation in the case that $W$ is a virtual complex representation of $T$ (so $M$ is a point). In that case too only the condition $p_1(W_T) = 0$ is required to construct a global section of $E(W)$. However, we do choose a generator $z$ of the character group of $T$. The section of $E(W)$ we construct is independent of this choice only when $w_2(W_T) = 0$.

Our formulation leads to proofs of many of the rigidity theorems in the literature, including the other genera considered in [Vit87, BT89] and the twisted loop group genera of [Liu94]. In the interest of brevity, we shall return to these issues at another time.
Definition 2.3. A point $a$ in $C$ is special to level $N$ for $V$ if it is special for $V^+$ or if for some component $F$ of $X^T$ there is a rotation number $m_j$ of $V$ such that $m_jN_a = 0$.

In any case, for fixed $N$ the set of points which are special to level $N$ is a finite subset of the torsion subgroup of $C$. Mostly we shall fix an $N$, and say simply that $a$ is special.

Definition 2.4. An indexed open cover $\{U_a\}_{a \in C}$ of $C$ is adapted to $X$ or $V$ (to level $N$) if it satisfies the following.

1) $a$ is contained in $U_a$ for all $a \in C$.
2) If $a$ is special and $a \neq b$, then $a \notin U_a \cap U_b$.
3) If $a$ and $b$ are both special and $a \neq b$, then the intersection $U_a \cap U_b$ is empty.
4) If $b$ is ordinary, then $U_a \cap U_b$ is non-empty for at most one special $a$.
5) Each $U_a$ is small.

Lemma 2.5. $C$ has an adapted open cover, and any two adapted open covers have a common refinement. □

2.3. Equivariant cohomology. We write $H_T$ for Borel $T$-equivariant ordinary cohomology with complex coefficients: so if $X$ is a $T$-space then

$$H^*_T(X) = H^*(X_T; \mathbb{C}).$$

We choose a generator of the character group of $T$, and write $z$ for the resulting generator of $H^2(BT; \mathbb{Z})$; this gives a generator of $H^2_T(\ast)$ which we also call $z$. We shall often write $H_T$ for the ring $H^*_T(\ast) \cong \mathbb{C}[z]$. Moreover we shall consider $H_T$ to be a subring of the ring $O_C(\mathbb{C})$ of global analytic functions on $\mathbb{C}$, by considering $z: \mathbb{C} \to \mathbb{C}$ to be the identity map.

We recall [Qui71] that $H_T$ satisfies a localization theorem.

Theorem 2.6. If $X$ has the homotopy type of a finite $T$-CW complex (e.g. if $X$ is a compact $T$-manifold), then the natural map

$$H_T(X) \to H_T(X_T)$$

induces an isomorphism

$$H_T(X) \otimes_{\mathbb{C}[z]} \mathbb{C}[z, z^{-1}] \cong H_T(X_T) \otimes_{\mathbb{C}[z]} \mathbb{C}[z, z^{-1}].$$

□

2.4. Elliptic cohomology of a space. The equivariant elliptic cohomology of $X$ is a sheaf of $O_C$-algebras over $C$. We shall describe it as follows. Let $\{U_a\}_{a \in C}$ be an adapted open cover of $C$. We first describe sheaves $E(X)_a$ over $U_a$ for each $a$, and then we assemble these into a sheaf $E(X)$ over all of $C$.

For each $a \in C$, we define an sheaf of $O_{C(U_a)}$-algebras by the formula

$$E(X)_a(U) = H_T(X^a) \otimes_{O_C(U_a)} O_C(U - a) \quad (2.7)$$

for $U \subset U_a$. Here $O_C(U - a)$ is a $\mathbb{C}[z]$-algebra via the $\mathbb{C}[z]$-structure on $O_C|_{U_a - a}$. The ring $O_C(U)$ acts by the formula

$$g \cdot (x \otimes y) = x \otimes y\tau^*_a g.$$

If $a \neq b$ and $U_a \cap U_b$ is not empty, then by the definition (2.4) of an adapted cover, at least one of $U_a$ and $U_b$, suppose $U_b$, contains no special point. In particular we have $X^b = X^T$ and so an isomorphism of $\mathbb{C}[z]$-algebras

$$H_T(X^b) \cong H^*_T(X^b) \otimes_{\mathbb{C}[z]} \mathbb{C}[z]. \quad (2.8)$$

Lemma 2.9. If $U \subset U_a \cap U_b$, and $b$ is not special, then the inclusion

$$i: X^b \to X^a$$

induces an isomorphism

$$H_T(X^a) \otimes_{\mathbb{C}[z]} O_C(U - a) \xrightarrow{i^* \otimes 1} H_T(X^b) \otimes_{\mathbb{C}[z]} O_C(U - a).$$
Proof. If \( a \) is not special, then \( X^a = X^b \) and the result is obvious. If \( a \) is special, then it is not contained in \( U \) (by the definition of an adapted cover), and so 0 is not contained in \( U - a \). In particular, \( z \) is a unit in \( \mathcal{O}_C(U - a) \). The localization theorem (2.6) gives the result. 

We then define an isomorphism
\[ \phi_{ab} : E(X)_a|_{U_a \cap U_b} \cong E(X)_b|_{U_a \cap U_b} \]
of sheaves over \( U_a \cap U_b \) as the composition (for \( U \subset U_a \cap U_b \))
\[
\begin{align*}
H_T(X^a) \otimes \mathcal{O}_C(U - a) &\xrightarrow{\tau_b \otimes 1} H_T(X^b) \otimes \mathcal{O}_C(U - a) \\
&\xrightarrow{\cong} H^*(X^b) \otimes \mathcal{O}_C(U - a) \\
&\xrightarrow{1 \otimes \tau_{a-b}} H^*(X^b) \otimes \mathcal{O}_C(U - b) \\
&\xrightarrow{\cong} H_T(X^b) \otimes \mathcal{O}_C(U - b). 
\end{align*}
\]
The first map is an isomorphism by Lemma 2.9. The rest of the isomorphisms are tautologies, although some use the isomorphism (2.8).

The cocycle condition
\[ \phi_{bc} \phi_{ab} = \phi_{ac} \]
needs to be checked only when two of \( a, b, c \) are not special; and in that case it follows easily from the equation
\[ \tau_{c-b} \tau_{a-b} = \tau_{c-a}. \]
We shall write \( E(X) \) for the resulting sheaf over \( C \). One then has the following. It was certainly known to Grojnowski [Gro94], but for a detailed account the reader may wish to consult [Ros99].

**Proposition 2.11.** The sheaf \( E(X) \) is a sheaf of analytic \( \mathcal{O}_C \)-algebras, which is independent up to canonical isomorphism of the choice of adapted open cover.

Note that the construction (2.7) implies that there is a canonical isomorphism
\[ E(X)_a \cong H_T(X^a) \otimes_{\mathbb{C}[z]} \mathcal{O}_C \]
of stalks at \( a \), as promised in the introduction.

### 3. Elliptic cohomology of vector bundles

#### 3.1. Orientations.

Let \( HP \) denote even periodic cohomology, that is
\[ HP^*(X) = H^*(X; \mathbb{C}[u, u^{-1}]), \]
where \(|u| = -2\). A power series
\[ f(z) = z + \text{ higher terms } \in HP^2(BT) \cong \mathbb{C}[z] \]
satisfying
\[ f(-z) = -f(z) \]
determines an orientation
\[ \phi : MSO \to HP, \]
characterized by the property that if \( L \) is a complex line bundle, then its Euler class is \( f(c_1 L) \).

If \( V \) is a vector bundle, we write \( \phi(V) \in HP(V) = HP(X^V) \) for the resulting Thom class. It is multiplicative, in the sense that
\[ \phi(V \oplus W) = \phi(V) \wedge \phi(W) \]
under the isomorphism
\[ (X \times Y)^{V \oplus W} \cong X^V \wedge Y^W. \]
3.2. **Equivariant characteristic classes.** If $c$ is a characteristic class of vector bundles and $V$ is a $T$-vector bundle over a space $T$-space $X$, then we write

$$c(V)_T \overset{\text{def}}{=} c(V_T),$$

provided that makes sense. We refer to $c(-)_T$ as the “equivariant characteristic class” associated to $c$. For example, if $V$ is oriented, then so is $V_T$, and so

$$V \mapsto w_2(V)_T$$

is a characteristic class of oriented $T$-vector bundles.

If the class $c$ is additive, then we may similarly define $c(W)_T$ when $W$ is a virtual vector bundle. For example, if $W$ is an oriented virtual $T$-bundle then we may write

$$W = V - V'$$

where $V$ and $V'$ are oriented $T$-vector bundles; the quantity

$$w_2(W)_T = w_2(V)_T - w_2(V')_T;$$

is easily seen to be independent of the choice of oriented $V$ and $V'$. Similarly, if $W$ is any virtual complex vector bundle, then

$$c_1(W)_T = c_1(V)_T - c_1(V')_T$$

if

$$W = V - V'$$

in $K_T(X)$.

The equivariant Thom isomorphism provides another example. If

$$MSO \xrightarrow{\phi} HP$$

is an orientation and

$$W = V - V'$$

where $V$ and $V'$ are oriented $T$-vector bundles, then we write

$$\phi(W)_T \in HP_T(X^W) \cong HP_T(X^V) \otimes_{HP_T(X)} HP_T(X^{V'})^{-1}$$

for the resulting Thom class; and this is independent of the choice of oriented $V$ and $V'$.

We will use the notation $e_\phi$ for the *equivariant* Euler class associated to $\phi$: so if $V$ is an oriented $T$-vector bundle over $X$ then

$$e_\phi(V) = \zeta^*(\phi(V)_T) \in HP_T(X),$$

where

$$\zeta : X \to X^V$$

is the zero section.

3.3. **Analytic orientations.** Let

$$\phi : MSO \to HP$$

be an orientation determined by a power series $f$ as in §3.1.

**Definition 3.2.** The orientation $\phi$ is **analytic** if $f$ is contained in the subring $O_{C_0} \subset \mathbb{C}[z]$ of germs of holomorphic functions at 0; equivalently, if there is a neighborhood $U$ of 0 in $\mathbb{C}$ on which the power series $f$ converges to a holomorphic function.

The basic fact about analytic orientations is the following.
Proposition 3.3 (Ros99). If \( \phi \) is analytic and \( V \) is an oriented \( \mathbb{T} \)-vector bundle over a compact \( \mathbb{T} \)-space \( X \), then the equivariant Euler class \( e_\phi \) associated to \( \phi \) satisfies

\[
e_\phi(V) \in H^*_\mathbb{T}(X) \otimes \mathcal{O}_{\mathbb{C}^0}.
\]

If \( \Phi \) denotes the standard Thom isomorphism, then

\[
\phi(V)_\mathbb{T} = \frac{e_\phi(V)}{e_\Phi(V)} \Phi(V)_\mathbb{T},
\]

and the ratio of Euler classes is a unit in \( H^*_\mathbb{T}(X) \otimes \mathcal{O}_{\mathbb{C}^0} \). Of course multiplication by \( \Phi(V)_\mathbb{T} \) induces an isomorphism

\[
H^*_\mathbb{T}(X) \cong H^*_\mathbb{T}(V),
\]

and so we have the following

Corollary 3.4. There is a neighborhood \( U \) of the origin in \( \mathbb{C} \) such that

\[
\phi(V)_\mathbb{T} \in H^*_\mathbb{T}(V) \otimes \mathcal{O}_C(U),
\]

and such that multiplication by this class induces an isomorphism of sheaves

\[
H^*_\mathbb{T}(X) \otimes \mathcal{O}_{\mathbb{C}^0} \xrightarrow{\phi} H^*_\mathbb{T}(V) \otimes \mathcal{O}_{\mathbb{C}^0}.
\]

In other words, for every open set \( U' \subseteq U \), multiplication by \( \phi(V)_\mathbb{T} \) induces an isomorphism

\[
H^*_\mathbb{T}(X) \otimes \mathcal{O}_C(U') \xrightarrow{\phi} H^*_\mathbb{T}(V) \otimes \mathcal{O}_C(U').
\]

Definition 3.5. Let \( V \) be a \( \mathbb{T} \)-vector bundle over a compact \( \mathbb{T} \)-manifold \( X \), and let \( \phi \) be a multiplicative analytic orientation. An indexed open cover \( \{U_a\}_{a \in C} \) of \( C \) is adapted to the pair \((V, \phi)\) to level \( N \) if it is adapted to \( V \) to level \( N \) (see Definition 2.4), and if for every point \( a \in C \), the equivariant Thom class \( \phi(V^a) \) induces an isomorphism

\[
H^*_\mathbb{T}(X) \otimes \mathcal{O}_{U_a-a} \xrightarrow{\phi} H^*_\mathbb{T}(V^a) \otimes \mathcal{O}_{U_a-a}.
\]

3.4. Cohomology of the Thom space. Now suppose that \( V \) is a \( \mathbb{T} \)-vector bundle over \( X \). Let \( E(V) \) denote the reduced equivariant elliptic cohomology of the one-point compactification of \( V \). In the case that \( V \) is spin, we give an explicit cocycle which exhibits \( E(V) \) as an invertible sheaf of \( E(X) \)-modules. The main tool which makes this possible is the following.

Lemma 3.6. Let \( V \) be a Spin \( \mathbb{T} \)-vector bundle over \( X \). For all \( n \), the fixed bundle \( V^\mathbb{T}[n] \) over \( X^\mathbb{T}[n] \) is orientable (that is, it has a Thom isomorphism in ordinary cohomology with integer coefficients).

Proof. If the rank \( V \) is even, this is Lemma 10.1 in [BT89]. If the rank \( V \) is odd, then we may apply that lemma to \( V' = V \oplus \mathbb{R} \).

Suppose then that \( V \) is a Spin \( \mathbb{T} \)-vector bundle over \( X \). Let \( \phi \) be a multiplicative analytic orientation. Let \( e \) be the associated (equivariant) Euler class. We define a class \([\phi, V] \in H^1(C; E(X)^*)\) as follows. Choose an open cover adapted to the pair \((V, \phi)\). Suppose that \( a, b \) are two points of \( C \), such that \( U_a \cap U_b \) is non-empty; we may suppose that \( b \) is not special. Suppose that \( U \subseteq U_a \cap U_b \). Recall that there is an isomorphism (2.11)

\[
E(X)(U) \cong H^*_\mathbb{T}(X^b) \otimes \mathcal{O}_C(U - a).
\]

For each special point \( a \) of \( C \), choose an orientation on \( V^a \) and on \( V \); we may do so by Lemma 3.6. This gives equivariant euler classes \( e(V^a) \) for \( a \in C \).
Lemma 3.7. There is a unit \( e(a, b) \in E(X)(U) \) such that \( e(V^b) e(a, b) = e(V^a|_X) \). Moreover we have

\[
e(a, a) = 1
\]

for all \( a \), and

\[
e(a, b) e(b, c) = e(a, c)
\]

whenever that makes sense.

Proof. Without loss of generality we may suppose that \( V^b = V^T \). If in addition \( V^a = V^T \), then \( e(a, b) = 1 \). Otherwise, we have \( a \not\in U \) so \( 0 \not\in U - a \), and so \( z \) is a unit in \( O_C(U - a) \). On each component \( F \) of \( X^b \), there are integers \( m_j \not= 0 \) and complex vector bundles \( V(m_j) \) over \( F \) such that

\[
V^a|_F = V^b|_F \oplus \bigoplus_{m_j} V(m_j)^R.
\]

Here \( T \) acts on \( V(m_j) \) fiberwise by the character \( u \mapsto u^{m_j} \).

Let \( f(z) = z + \text{higher terms} \) be the characteristic series of the orientation \( \phi \), so that

\[
e(L) = f(c_1 L)
\]

for \( L \) a complex line bundle. Let \( x_{j,1}, \ldots, x_{j,d_j} \) be the roots of the total Chern class of \( V(m_j) \). Since \( F \) is compact, the \( x_{i,j} \) are nilpotent. It suffices to check that \( e(a, b)|_F \) is a unit modulo nilpotents in \( H(F) \otimes O_C(U - a) \). We have

\[
e(a, b)|_F = \prod_j \prod_i f(m_jz + x_{j,i})
\]

\[
= \prod_j \prod_i f(m_jx)
\]

\[
= (\prod_j m_j^{d_j}) z^{\sum d_j} (1 + \text{higher terms in } z),
\]

where the equivalences are modulo nilpotents. The result follows, since the \( m_j \) are non-zero and \( z \) is a unit in \( O_C(U - a) \).

The cocycle condition (3.8) is easy, because as usual the equation needs only to be verified when at most one of \( a \), \( b \), and \( c \) is special. \( \square \)

Let \( [\phi, V] \in H^1(C; E(X)^\times) \) be the cohomology class defined by the \( e(a, b) \). Let \( E(X)^{[\phi, V]} \) denote the resulting invertible sheaf of \( E(X) \)-modules over \( C \). Explicitly, the sheaf \( E(X)^{[\phi, V]} \) is assembled from the sheaves

\[
E(X)_a(U) = H_T(X^a) \otimes_{C[z]} O_C(U - a)
\]

over \( U_a \), using the sequence of isomorphisms

\[
E_a(X)(U) \cong H_T(X^a) \otimes_{C[z]} O_C(U - a)
\]

\[
\xrightarrow{\imath^* \otimes 1_b} H_T(X^b) \otimes_{C[z]} O_C(U - a)
\]

\[
\xrightarrow{e(a, b) \cdot} H_T(X^b) \otimes_{C[z]} O_C(U - a)
\]

\[
\cong H^*(X^b) \otimes_{C} O_C(U - a)
\]

\[
\cong H^*_T(X^b) \otimes_{C} O_C(U - b)
\]

\[
\cong E_b(X)(U)
\]

(3.9)
as transition functions to assemble a sheaf over \( C \).

It will be important in \( \S 6 \) that we allowed the orientation of \( V^T \) to vary with the special point \( a \). In fact, the resulting sheaf \( E(X)^{[\phi,V]} \) is independent of the choices, up to canonical isomorphism.

**Proposition 3.10.** If \( V \) is a spin \( T \)-bundle, then the Thom isomorphism \( \phi \) induces an isomorphism
\[
E(X)^{[\phi,V]} \cong E(V)
\]
of sheaves of \( E(X) \)-modules.

**Proof.** Choose a cover adapted to \( X \) and the pair \( (V,\phi) \). Suppose that \( U \subset U_a \cap U_b, a \) is special, and \( b \) is not. Then the diagram
\[
\begin{array}{ccc}
H_T(X^a) \otimes O_C(U - a) & \xrightarrow{\phi} & H_T(V^a) \otimes O_C(U - a) \\
\downarrow e(a,b)i^* & & \downarrow i^* \\
H_T(X^b) \otimes O_C(U - a) & \xrightarrow{\phi} & H_T(V^b) \otimes O_C(U - a) \\
\downarrow 1 \otimes \tau_{a-b} & & \downarrow 1 \otimes \tau_{a-b}
\end{array}
\]
commutes (all the arrows are isomorphisms). The left column describes the sheaf \( E(X)^{[\phi,V]} \), while the right column describes \( E(V) \).

Now suppose that \( W \) is a virtual \( T \)-bundle. We may write
\[
W = V - T
\]
with \( V \) and \( T \) genuine \( T \)-bundles of even rank. We also clearly have
\[
W = (V + 3T) - 4T,
\]
and it is easy to check that
\[
\begin{align*}
w_1(4T)_T &= 0 \\
w_2(4T)_T &= 0 \\
w_1(V + 3T)_T &= w_1(W)_T \\
w_2(V + 3T)_T &= w_2(W)_T.
\end{align*}
\]

Thus if \( W \) is a virtual spin bundle, then we may require that \( V \) and \( T \) are spin bundles of even rank. If
\[
W = V - T = V' - T'
\]
with \( V, V', T, \) and \( T' \) spin bundles of even rank, then
\[
V = V' \pm D
\]
where \( D \) is a spin bundle. It follows that we may define \( E(W) \) to be the invertible sheaf
\[
E(W) = E(T)^{-1} \otimes_{E(X)} E(V).
\]
As above, we can as construct a class \([\phi,V - T] \in H^1(C,E(X)^x)\), equipped with a canonical isomorphism
\[
E(W) \cong E(X)^{[\phi,V - T]}.
\]
4. Theta functions

The orientations in this paper will arise as the Taylor expansions of certain meromorphic \( \theta \) functions. The \( \theta \) functions we consider are a special case of those considered in [Mum70, Ch. I], except that Mumford considers only holomorphic functions.

Let \( \Lambda \subset \mathbb{C} \) be a lattice. By a theta function for \( \Lambda \) we mean a meromorphic function \( \theta: \mathbb{C} \to \mathbb{C} \) with the following properties. First, there is an integer \( N \) such that the zeroes and poles of \( \theta \) are contained in \( N^{-1} \Lambda \). Second, there are functions

\[
\gamma: \Lambda \to \mathbb{C} \\
c: \Lambda \to \{\pm 1\}
\]

such that, for all \( \lambda \in \Lambda \) and \( z \in \mathbb{C} \), we have

\[
\begin{align*}
\theta(-z) &= -\theta(z) \\
\theta(0) &= 0 \\
\theta'(0) &\neq 0 \\
\theta(z + \lambda) &= c(\lambda)e^{\gamma(\lambda)(z + \frac{\lambda}{2})}\theta(z).
\end{align*}
\]

Using (4.1d) in various ways to compare \( \theta(z + \lambda + \lambda') \) with \( \theta(z) \), one finds that \( \gamma \) is necessarily linear, and that

\[
\frac{c(\lambda + \lambda')}{c(\lambda)c(\lambda')} = e^{\frac{1}{2}(\gamma(\lambda)\lambda' - \lambda\gamma(\lambda'))}
\]

for \( \lambda, \lambda' \in \Lambda \). It follows that \( \gamma \) must satisfy the “period relation”

\[
\gamma(\lambda)\lambda' - \lambda\gamma(\lambda') \in 2\pi i\mathbb{Z}.
\]

The linearity of \( \gamma \) gives the useful formula

\[
\theta(z + \ell \lambda) = c(\ell \lambda)e^{\gamma(\lambda)(\ell z + \ell^2 \frac{\lambda}{2})}\theta(z).
\]

**Remark 4.3.** Equation (4.1d) means that \( \theta \) is a meromorphic section of the line bundle

\[
\mathcal{L}(\gamma, c) = \mathbb{C} \times \mathbb{C}/(z, v) \sim (z + \lambda, ve(\lambda)e^{\gamma(\lambda)(z + \frac{\lambda}{2})}), \lambda \in \Lambda
\]

over \( \mathbb{C}/\Lambda \), so the theta functions considered here are a special case of [Mum70, Ch. I].

The equations (4.1a–4.1c) imply that the Taylor expansion \( \hat{\theta} \) of \( \theta \) at the origin determines a multiplicative analytic orientation (see Definition 3.2)

\[
\phi: \text{MSO} \to HP.
\]

The genus associated to this orientation is

\[
M \mapsto \int_M \prod_{j=1}^d \frac{x_j}{\theta(x_j)},
\]

where \( M \) is an oriented manifold of dimension \( 2d \), and the \( x_j^2 \) are the roots of the total Pontrjagin class of \( M \).

5. Examples

5.1. The Witten genus. One example of a theta function satisfying (4.1) is the Weierstrass \( \sigma \)-function. We describe a variant associated to the Witten genus.

Let \( \sigma \) denote the expression

\[
\sigma = (u^{\frac{1}{2}} - u^{-\frac{1}{2}}) \prod_{n \geq 1} \frac{(1 - qu^n)(1 - q^n u^{-1})}{(1 - q^n)^2}.
\]
This may be considered as an element of \( \mathbb{Z}[q][u^{\pm 1}] \) which is a holomorphic function of \( (u^{\frac{1}{2}}, q) \in \mathbb{C}^\times \times D \), where \( D = \{ q \in \mathbb{C} | 0 < |q| < 1 \} \). Let \( \mathfrak{h} = \{ \tau \in \mathbb{C} | \text{Im } \tau > 0 \} \) be the open upper half plane. We may consider \( \sigma \) as a holomorphic function of \( (z, \tau) \in \mathbb{C} \times \mathfrak{h} \) by setting

\[
\begin{align*}
  u^{\frac{1}{2}} &= e^z \\
  q &= e^{2\pi i \tau}.
\end{align*}
\]

For our purposes, it is sufficient to fix \( \tau \), and consider \( \sigma \) as a function of \( z \) alone. From the formula (5.1) it follows that \( \sigma(z) \) is a theta function of the form (4.1) for the lattice \( \Lambda = 2\pi i \mathbb{Z} + 2\pi i \tau \mathbb{Z} \), with

\[
\gamma(2\pi ia + 2\pi ib\tau) = -b
\]

and

\[
c(\lambda) = \begin{cases} 1 & \lambda \in 2\Lambda \\ -1 & \text{otherwise.} \end{cases}
\]

If \( V \) is a complex vector bundle, let \( rV = \text{rank } V - V \) denote the associated virtual bundle of rank 0. Let \( S^k V \) denote the \( k \) symmetric power of \( V \), and let

\[
\text{Sym}_k(V) = \sum_{k \geq 0} t^k S^k V.
\]

This extends to an operation \( K(X) \to (1 + tK(X)[t])^\times \) because of the formula

\[
\text{Sym}_k(V \oplus W) = \text{Sym}_k V \cdot \text{Sym}_k W.
\]

The genus associated to the (Taylor series expansion of)

\[
a(x) = e^{\frac{x}{2}} - e^{-\frac{x}{2}}
\]

is called the \( \hat{A} \)-genus. Formulae (4.5) and (5.1) show that the genus associated to \( \sigma \) is

\[
M \mapsto \hat{A}(M; \bigotimes_{n \geq 1} \text{Sym}_n(rTM^C))
\]

(this is explained in [AHS98]), which is equivalent to formula (27) in [Wit87].

Remark 5.2. Let \( C \) be the complex elliptic curve \( \mathbb{C}/\Lambda \). The equations (4.1) in this case are descent data for the “\( \Sigma \)-structure” on the ideal sheaf \( \mathcal{I}_C(0) \) of the origin in \( C \), in the sense of Breen [Bre83]; see particularly §3.12.

Remark 5.3. The sigma function (5.1) is related to the classical Weierstrass sigma function \( \sigma_{\text{Weierstrass}} \) in for example [Sil86] by the formula

\[
\sigma_{\text{Weierstrass}}(z) = e^{az^2} \sigma(z),
\]

where \( a \) is a constant. It is not hard to check that these define the same invariants of \( MO(8) \)-manifolds. The \( \sigma \)-function (5.1) also arises as the \( p \)-adic \( \sigma \)-function of the Tate curve [MT91].

5.2. The Ochanine genus. Let \( C \cong \mathbb{C}/\Lambda \) be an elliptic curve, and let \( p \) be a point of exact order 2 of \( C \). Let \( r_1 \) and \( r_2 \) be the other two non-zero points of order two, and let \( P, R_1, R_2 \) be representatives of these points, such that \( R_1 + R_2 = P \). There is a meromorphic function \( f \) on \( C \) with divisor

\[
\text{div } f = (0) + (p) - (r_1) - (r_2);
\]

\( f \) is uniquely determined by requiring that its pull-back \( s \) to \( \mathbb{C} \) have a Taylor series expansion of the form

\[
s(z) = z + o(z^2).
\]

In fact in this case \( s \) is given by the formula

\[
s(z) = \frac{\sigma(z)\sigma(-R_1)\sigma(-R_2)\sigma(z-P)}{\sigma(z-R_1)\sigma(z-R_2)\sigma(-P)}. \tag{5.4}
\]
It is odd, and the resulting genus is the elliptic genus of Ochanine [Och87] for the pair consisting of the lattice \( \Lambda \) and the point \( p \). It is customary to consider \( s \) as a theta function for the lattice
\[
\Lambda' = \{ \lambda + nP | \lambda \in \Lambda, n \in \mathbb{Z} \}.
\]
The function \( s \) is not quite periodic with respect to this lattice, but satisfies
\[
s(z + \lambda + nP) = (-1)^n s(z)
\]
for \( \lambda \in \Lambda \). Thus
\[
\gamma(\lambda + nP) = 0 \quad \text{and} \quad c(\lambda + nP) = (-1)^n.
\]
Explicitly, if \( \Lambda = 2\pi i \mathbb{Z} + 4\pi i \tau \mathbb{Z} \) and \( P = 2\pi i \tau \), then using (5.1) in (5.4) gives the formula
\[
s(z) = -2 \frac{1 - u}{1 + u} \prod_{n \geq 1} \frac{(1 - q^n u)(1 - q^n u^{-1})(1 + q^n)^2}{(1 + q^n u)(1 + q^n u^{-1})(1 - q^n)^2}.
\]
Theorem 6.1 for this function is due to Rosu [Ros99].

6. The equivariant Thom class

We fix a meromorphic function \( \theta \) satisfying the conditions of §4. We recall (4.4) that the Taylor expansion \( \hat{\theta} \) of \( \theta \) at 0 determines a multiplicative analytic orientation
\[
\text{MSO} \to HP,
\]
and we write \( \phi \) for the resulting Thom isomorphism.

Suppose that \( X \) is a compact smooth \( T \)-manifold, and that \( W \) is a virtual oriented \( T \)-bundle over \( X \). The Thom isomorphism \( \phi \) gives a generator \( \phi(W)_T \) of \( H_T(W) \), and so a generator of
\[
H_T(W) \otimes \mathcal{O}_{C_0}
\]
which we shall also denote \( \phi(W)_T \).

**Theorem 6.1.** If \( W \) is an oriented \( T \)-vector bundle over \( X \), and either
1. \( w_2(W) = 0 \) and the function \( \gamma \) of (4.1d) is identically zero; or
2. the equations (1.2) hold,
then the invertible sheaf of \( E(X) \)-modules \( E(W) \) has a global section \( \Theta \), such that
\[
\Theta_0 = \phi(W)_T
\]
under the isomorphism (2.12)
\[
H_T(W) \otimes \mathcal{O}_{C_0} \cong E(W)_0.
\]
The proof will occupy the rest of this section. To give it we fix a cover of \( C \) adapted to level \( N \) to \((V, \phi)\) and \((T, \phi)\), where \( N \) is large enough that the zeroes and poles of \( \theta \) are contained in \( N^{-1} \Lambda \). We also write
\[
W = V - T,
\]
where \( V \) and \( T \) are spin \( T \)-vector bundles of even rank, as in §3.

Let us indicate precisely what it is we must construct. Since \( \phi \) is multiplicative, we have
\[
\phi(W)_T = \phi(T)_T^{-1} \otimes \phi(V)_T.
\]
Proposition 3.10 shows that it is equivalent to construct a global section of the sheaf \( E(X)^{[\phi, V - T]} \), whose value in
\[
E(X)^{[\phi, V - T]}_0 \cong H_T(X) \otimes \mathcal{O}_C(U_0)
\]
is 1. The formula (3.3) for this sheaf shows that such a global section is assembled from sections \( \Theta_a \in E(X)_a(U_a) \) which satisfy the formula
\[
\Theta_b = \tau_{b-a}^* (e(a, b)i^* \Theta_a).
\]
on $U \subset U_a \cap U_b$. Because we are using an adapted open cover, it will suffice in (6.2) to suppose that $b$ is not special.

In order to give the formula for $\Theta$, we introduce some notation and results in section 6.1. The construction of $\Theta$ using these results begins in § 6.2.

6.1. Notations and lemmas. Suppose that $a$ is a special point of order $n$. The structure of the formula for $\Theta_a$ depends on the parity of $n$, but the two cases share many components. Let $h = n/2$. In the following, terms which involve $h$ are simply absent in the case that $n$ is odd.

Let $P$ be a component of the submanifold $X^{[n]}$ of points fixed by $T[n]$. Let $F \subset P$ be a component of $X^T$. We have decompositions of real $T$-vector bundles

$$T|_F = T_0 \oplus T_h \oplus \bigoplus_{0 < r < h} T_r,$$

$$V|_F = V_0 \oplus V_h \oplus \bigoplus_{0 < r < h} V_r.$$

For $0 < r < h$, $T_r$ and $V_r$ are complex vector bundles over $P$ on which $\zeta \in T[n]$ acts by multiplication by $\zeta^r$. The bundles $T_0$ and $V_0$ are the summands of $T$ and $V$ on which $T[n]$ acts trivially; and if $n$ is even then $T_h$ and $V_h$ are the summands on which $T[n]$ acts by the sign representation.

Over the submanifold $F$ there are decompositions

$$T_0|_F = T(0) \oplus \bigoplus_{0 \neq m_j \equiv 0} T(m_j)^R$$

$$T_h|_F = \bigoplus_{m_j \equiv h} T(m_j)^R$$

$$T_r|_F = \bigoplus_{m_j \equiv r} T(m_j)^R$$

$$V_0|_F = V(0) \oplus \bigoplus_{0 \neq m_j' \equiv 0} V(m_j')^R$$

$$V_h|_F = \bigoplus_{m_j \equiv h} V(m_j)^R$$

$$V_r|_F = \bigoplus_{m_j' \equiv r} T(m_j')^R$$

the equivalences are modulo $n$. For $m \neq 0$, $T(m)$ and $V(m)$ are complex vector bundles on which $z \in \mathbb{T}$ acts by fiberwise by multiplication by $z^m$. Since the real representations $\mathbb{C}(m)^R$ and $\mathbb{C}(-m)^R$ of $\mathbb{T}$ are isomorphic, we are free to choose the signs of the $m_j$ and $m_j'$ which are not congruent to 0 or $h$ modulo $n$ so that there are integers $\ell_j, r_j, \ell_j'$, and $r_j'$ satisfying

$$m_j = n \ell_j + r_j$$

$$m_j' = n \ell_j' + r_j'.$$

with $0 < r_j, r_j' < h$.

We must choose orientations for $T(0)$, $T_0$, and $T_h$; our choice will depend on the parity of $n$.

If $n$ is odd, then the complex structures on $T_0 \oplus T_h$ and the orientation of $T$ itself determine an orientation for $T_0$. We choose the orientation of $T(0)$ and the signs of the $m_j \equiv 0$ so that (6.4) respects the orientations on both sides. In this case let $\delta = 0$.

If $n$ is even, then we simply choose an orientation for $T_0$; this induces one on $T_h$ so that (6.3) respects the orientations. We choose the orientation of $T(0)$ and the signs of the $m_j \equiv 0, h$ so that

$$(T_0 + T_h)|_F = T(0) \oplus \bigoplus_{m_j \equiv 0, h} T(m_j)^R.$$

is compatible with the orientations on both sides. For those $j$ such that $m_j \equiv h$, we define $\ell_j$ so that

$$m_j = n \ell_j + h.$$
In other words, \( \varepsilon \)

**Proof.** This is Lemma 9.3 in [BT89].

The quantity \( \varepsilon \) will be given in § (1.2) holds will be given in the proof of the first lemma. The next three lemmas are trivial in the case of the translation formula (4.1d) for \( \theta \) over \( X \) in particular, this quantity is constant on \( P \).

Lemma 6.7. If \( n \) is odd, then the complex \( \mathbb{T} \)-line bundle \( \mathcal{V} \) has an \( n \)th root \( \mathcal{V}^{1/n} \) over \( P \), with the property that

\[ S(i^* c_1(\mathcal{V}^{1/n})_\mathbb{T}) = S(H). \]
Lemma 6.10. If \( n \) is even, then the complex \( \mathbb{T} \)-line bundle \( V \) has an \( h^{th} \) root \( V^{1/h} \) over \( P \), with the property that

\[
S\left(\frac{1}{2}i^*c_1(V^{1/h})_{\mathbb{T}}\right) = S(H + \frac{1}{2}c_1(V_h - T_h))_{\mathbb{T}}.
\]

6.2. Construction of \( \Theta \): ordinary points. First we give a formula for \( \Theta_b \) when \( b \) is ordinary. Taking \( a = 0 \) and \( \Theta_0 = 1 \) in (6.2) gives the formula

\[
\Theta_b = \tau_b^* (e(0, b)) = \tau_b^* e(0, b).
\]

A priori, this formula determines \( \Theta_b \) only on open subsets \( U \) of \( U_b \setminus \{b\} \) and only on those \( U_b \) such that \( U_b \cap U_0 \) is non-empty, but in fact it determines \( \Theta_b \) on all of \( U_b \) for any \( b \). With our notations, we have

\[
e(0, b) = \prod_j \prod_{i=1}^{d_j} \theta(x_{j,i} + m_j^* b).
\]

If \( U_b \cap U_0 \) is non-empty, or if \( U_b \cap U_a \) is empty for all special points \( a \), then we orient \( T(0) \) and \( V(0) \) by choosing the \( m_j \) and \( m_j^* \) to be positive. Otherwise, there is a unique \( a \) such that \( U_a \cap U_b \) is non-empty, and we follow the procedure described above to orient \( T(0) \) and choose signs for the \( m_j \).

Lemma 6.13. As a function of \( z \) we have

\[
e(0, b)(z + \lambda) = e(0, b)(z).
\]

In other words, the formula (6.12) defines a global section of \( H_{T}(X^b) \otimes M_C \).

The proof is given in §8.

Lemma 6.14. The formula (6.11) defines an element of \( E(X)_b(U_b) \).

Proof. What must be shown is that \( \tau_b^* e(0, b) \) has no pole at the origin. Suppose it does; we shall show that \( b \) is a special point. Choose a lift of \( b \) to \( C \); we may call it \( b \) in view of Lemma 6.13. We have

\[
\tau_b^* e(0, b) = \prod_j \prod_{i=1}^{d_j} \theta(x_{j,i} + m_j^* b).
\]

If this has a pole at the origin then \( \theta \) has a zero at \( m_j b \) for some \( m_j \) or a pole at \( m_j^* b \) for some \( m_j^* \). Let us take the first case for definiteness. By assumption, the zeros and poles of \( \theta \) are contained in \( N^{-1} \Lambda \), so

\[
m_j b \in N^{-1} \Lambda
\]

and

\[
N m_j b \in \Lambda.
\]

By Definition 2.3, \( b \) is a special point. \( \square \)

6.3. Construction of \( \Theta \): special points. If \( a \) is special, then as usual \( U \subset U_a \cap U_b \) is non-empty only if \( b \) is ordinary. Combining the two equations (6.2) and (6.11) gives

\[
\tau_b^* e(0, b) = \tau_{a-b}^* (e(a, b)) = \tau_b^* e(0, b).
\]

or equivalently

\[
e(a, b)^{-1} \tau_b^* e(0, b) = i^* \Theta_a.
\]

Special points of odd order. In this section we give the formula for \( \Theta_a \), supposing that \( n \) is odd.

For \( 0 < r < h \), let \( Q_r \) be the exponential characteristic class for complex vector bundles defined by the formula

\[
Q_r(L) = \theta(c_1 L + ra).
\]

Let \( \Theta_a \in E(X)(U_a) \) be given by the formula

\[
\Theta_a = e S(\alpha a) S(c_1(V^{1/n})_{\mathbb{T}}) \prod_{0 < r < h} \frac{Q_r(V_r)}{Q_r(T_r)}
\]

(17)
Proposition 6.18. If \( n \) is odd, then the class \( \Theta_a \) in \((6.17)\) satisfies the equation \((6.16)\).

Proof. We have

\[
e(0, b) = \frac{\prod_j \prod_{i=1}^{d_j} \theta(x'_{j,i} + m'_j z)}{\prod_j \prod_{i=1}^{d_j} \theta(x_{j,i} + m_j z)}.
\]

The iterated transformation formula \((4.2)\) gives

\[
\tau^*a \cdot e(0, b) = \frac{\prod_j \prod_{d'_{j,i}} \theta(x'_{j,i} + m'_j z + m'_j a)}{\prod_j \prod_{d_{j,i}} \theta(x_{j,i} + m_j z + m_j a)}
\]

\[
= \epsilon S(aG + H) \prod_{0 \leq r < h} \Pi_r,
\]

where

\[
\Pi_r = \frac{\prod_{m'_j \equiv r (n)} \prod_{i=1}^{d'_j} \theta(x'_{j,i} + m'_j z + ra)}{\prod_{m_j \equiv r (n)} \prod_{i=1}^{d_j} \theta(x_{j,i} + m_j z + ra)}
\]

for \( 0 \leq r < h \).

In view of Lemmas 6.8 and 6.9, it remains to observe that

\[
\Pi_0 = \epsilon(a, b),
\]

while for \( 0 < r < h \) we have

\[
\Pi_r = i^* \left( \frac{Q_r(V_r)}{Q_r(T_r)} \right).
\]

Special points of even order. In this section we give the formula for \( \Theta_a \), supposing that \( n \) is even.

Once again, for \( 0 < r < h \), let \( Q_r \) be the power series

\[
Q_r(x) = \theta(x + ra).
\]

For all \( r \) these give characteristic classes for complex vector bundles.

Now, however, for \( r = h \), set

\[
Q_h(x) = S(-\frac{1}{2} x) \theta(x + \lambda).
\]

The formulae

\[
\theta(-x) = -\theta(x)
\]

\[
\theta(x + \lambda) = c(\lambda) S(x + \frac{\lambda}{\frac{1}{2}}) \theta(x)
\]

imply the following.

Lemma 6.22. The power series \( Q_h \) satisfies

\[
Q_h(-x) = -c(\lambda)Q_h(x),
\]

and so defines a characteristic class of oriented even real vector bundles.

Let \( \Theta_a \in E(X)_a(U_a) \) be given by the formula

\[
\Theta_a = cS(a\alpha)S(\frac{1}{2} c_1(V^{1/h})_T) \prod_{0 < r < h} \frac{Q_r(V_r)}{Q_r(T_r)}.
\]

We have used Lemma 6.22 to ensure that the class \( Q_h(V_h)/Q_h(T_h) \) is well-defined. The proof of Theorem 6.1 is completed by the following.

Proposition 6.24. If \( n \) is even, then the class \( \Theta_a \) in \((6.23)\) satisfies the equation \((6.16)\).
The result follows from Lemmas 6.8, 6.10, and the equations
\[ \tau^*_a e(0, b) = c(\lambda(\sum d'_j f'_j - \sum d_j f_j)) S(aG + H) \prod_{0 \leq r \leq h} \Pi_r. \]

The argument is similar to the proof in the odd case, Proposition 6.18. Once again we have
\[ E V, W \text{ virtual complex representation of } T \]

of the Thom isomorphism (3.1) (and so of its associated euler class), together with the construction (3.9) of
\[ E T \text{ ring of Laurent polynomials} \]

This is usually denoted\[ R[T] \approx \mathbb{Z}[z, z^{-1}]. \]

If \( f \) is a Laurent polynomial in \( z \), we shall write \( V(f) \) for the associated virtual complex representation of \( T \).

If \( n \) is a natural number, let \( C[n] \) be the subgroup of \( C \) consisting of points of order \( n \). Let \( \mathcal{I}(C[n]) \subset O_C \) be the sheaf of ideals consisting of germs of holomorphic functions which vanish at \( C[n] \); it is a holomorphic line bundle over \( C \). If \( C[n] \) is regarded as a divisor on \( C \), then \( \mathcal{I}(C[n]) \) coincides with the line bundle which is usually denoted \( O_C(-C[n]) \).

Proposition 7.1. The line bundles \( E(V) \) for \( V \in R[T] \) are determined up to isomorphism by the following.
\[ E(V(0)) = O_C \]
\[ E(V(z^n)) \cong \mathcal{I}(C[n]) \]

and
\[ E(V + W) \cong E(V) \otimes_{O_C} E(W). \]

for \( V, W \in R[T] \).

Proof. The first part is clear, since \( E(V(0)) = E(*) \). The third part follows from the multiplicative property of the Thom isomorphism (3.1) (and so of its associated euler class), together with the construction (3.9) of
\[ E(X)^{[a,V]} \cong E(V). \]

For (6.2), let \( V = V(z^n) \), and consider the multiplicative orientation \( \phi \) given by the \( \sigma \) function. A point \( a \in C \) is special if and only if \( na = 0 \), and for such \( a \) we have \( V^a = V \). For ordinary \( b \) we have \( V^b = 0 \). It follows that
\[ e(a, b) = \sigma(nz), \]

a function which vanishes at precisely the points \( w \in C \) such that \( nw \in \Lambda \). Thus in the gluing (6.3), a trivialization of \( E(V)_a \) corresponds to a section of \( E(V)_b \) which vanishes at \( a \); this is a description in terms of cocycles of the ideal sheaf \( \mathcal{I}(C[n]) \). The isomorphism
\[ E(X)^{[a,V]} \cong E(V) \]

of Proposition 6.10 gives the result. \( \square \)
If \( f = \sum d_j z^{m_j} \), let \( D(f) \) be the divisor
\[
D(f) = -\sum d_j C[m_j]
\]
(The subgroup \( C[m_j] \) is considered as a divisor and the sum is as divisors). Proposition 7.1 says that there is an isomorphism
\[
E(V(f)) \cong \mathcal{O}_C(D(f)).
\]
The degree of this line bundle is
\[
\deg E(V(f)) = -\sum d_j m_j^2. \tag{7.3}
\]

**Corollary 7.4.** If \( V \in R[\mathbb{T}] \) is a complex virtual representation of \( \mathbb{T} \), then
\[
p_1(V)_{\mathbb{T}} = -z^2 \cdot \deg E(V) \in H^4(B\mathbb{T}; \mathbb{Z}).
\]

The application of Theorem 1.1 to complex representations of \( \mathbb{T} \) is as follows.

**Proposition 7.5.** The line bundle \( E(V) \) associated to a virtual representation \( V \) of \( \mathbb{T} \) is trivial precisely when \( p_1(V)_{\mathbb{T}} = 0 \). If
\[
f = \sum d_j z^{m_j}
\]
and \( p_1(V(f))_{\mathbb{T}} = 0 \), then the meromorphic function
\[
\prod_j \sigma(m_j z)^{d_j} \tag{7.6}
\]
defines a holomorphic trivialization of \( \mathcal{O}_C(D(f)) \cong E(V(f)) \). If in addition \( w_2(V)_{\mathbb{T}} = 0 \), it is precisely the trivialization given by Theorem 1.1.

**Proof.** Let
\[
D = \sum n_P(P)
\]
be a divisor on \( C \). Recall that the line bundle \( \mathcal{O}_C(D) \) associated to \( D \) is trivial precisely when
\[
\deg D = \sum n_P = 0 \tag{7.7a}
\]
\[
\sum C n_P(P) = 0; \tag{7.7b}
\]
the second sum is taken in the group structure of the elliptic curve \( C \). For the divisors we are considering, the condition (7.7b) is always satisfied, since as a group
\[
C[n] \cong (\mathbb{Z}/n)^2,
\]
and
\[
\sum_{g \in (\mathbb{Z}/n)^2} g = 0.
\]
It follows that \( E(V) \) is trivial precisely when \( \deg E(V) = 0 \), that is when \( p_1(V)_{\mathbb{T}} = 0 \).

It is well-known and easy to check that the product of sigma functions (7.6) descends to a trivialization of \( \mathcal{O}_C(D(f)) \), and it is also easy to check that it coincides with the trivialization of Theorem 1.1. \( \square \)

Proposition 7.5 raises the question of the role of the condition (1.2a) on the equivariant second Stiefel-Whitney class. One way to say this is as follows. Let
\[
f = \sum d_j z^{m_j}
\]
as in the Proposition, and let \( g \) be the associated trivialization of \( \mathcal{O}_C(D(f)) \): as a function on \( \mathbb{C} \),
\[
g(z) = \prod_j \sigma(m_j z)^{d_j}.
\]
Let $\iota : C \to C$ be the involution
\[ \iota(P) = -P. \]

The equality of divisors
\[ \iota^* D(f) = D(f) \]
gives a canonical isomorphism
\[ \iota^* \mathcal{O}_C(D(f)) \cong \mathcal{O}_C(D(f)), \]
of line bundles over $C$, and it is natural to ask whether
\[ \iota^* g = g \quad (7.8) \]
under this isomorphism. As a function on $C$, this amounts to asking whether
\[ g(-z) = g(z). \]

The function $g$ is uniquely determined among trivializations of $\mathcal{O}_C(D(f))$ by the equation
\[ g'(0) = \sum d_j m_j, \]
and the parity of this quantity is the second Stiefel-Whitney class
\[ w_2(V)_T = g'(0) z \in H^2(B\mathbb{T}; \mathbb{Z}/2). \]

Thus
\[ \iota^* g = g \]
precisely when $w_2(V)_T = 0$. This means, for example, that the trivialization $g$ is independent of the choice of generator $z$ of the character group of $\mathbb{T}$.

8. Consequences of the characteristic class restrictions

In this section we prove Lemmas 6.8, 6.9, and 6.10. We retain the set-up and notations of the previous section.

First let us work out explicitly some consequences of the characteristic class restrictions (1.2).

Lemma 8.1. If equations (1.2) hold, then we have
\[ \sum d_j m_j \equiv \sum d'_j m'_j \mod 2 \quad (8.2) \]
\[ \sum d_j m_j^2 = \sum d'_j m'_j^2 \quad (8.3) \]
\[ \sum_j m_j \sum_{i=1}^d x_{j,i} = \sum_j m'_j \sum_{i=1}^{d'} x'_{j,i}. \quad (8.4) \]

Proof. Introduce formal roots $y_{0,j}$ and $y'_{0,j}$ so that
\[ 1 + p_1 T(0) + \ldots = \prod (1 - y_{0,i}^2) \]
\[ 1 + p_1 V(0) + \ldots = \prod (1 - y'_{0,i}^2). \]

The equation (1.2b)
\[ \frac{p_1}{2}(V - T)|_{M^c} = 0 \]
implies that
\[ \frac{p_1}{2}(V|_{M^c} - T|_{M^c}) = 0. \]

On the other hand, this class is given by half the degree-four component of
\[ \frac{\prod (1 - y'_{0,i}^2) \prod_j \prod_{i=1}^{d'} (1 - (x'_{j,i}^2 + 2m'_{j,i} z + m'_{j} z^2))}{\prod (1 - y_{0,i}^2) \prod_j \prod_{i=1}^{d} (1 - (x_{j,i}^2 + 2m_{j,i} z + m_{j} z^2))} \]
Examining the coefficient of $z$ gives (8.4), and examining the coefficient of $z^2$ gives (8.3). Equation (8.2) follows from the equation $w_2(T - V)_\tau = 0$ by a similar argument.

**Lemma 8.5.** If equations (1.2) hold, and $n$ is odd, then

$$\sum_{0 < r < h} r(c_1T_r - c_1V_r) \equiv 0 \quad (n).$$

If $n$ is even, then

$$\frac{1}{h} \left( \sum_{0 < r < h} r(c_1T_r - c_1V_r) \right) \equiv w_2(V_h - T_h) \quad (\text{mod} \ 2).$$

**Proof.** We treat the case that $n$ is even; the case that $n$ is odd is similar. Let $z_n = z|_{BT[n]}$. Introduce formal roots $y_{r,j}$ and $y'_{r,j}$ so that

$$1 + c_1T_r + \ldots = \prod_{0 < r < h} (1 + y_{r,j}) \quad 0 < r < h$$
$$1 + c_1V_r + \ldots = \prod_{0 < r < h} (1 + y'_{r,j}) \quad 0 < r < h$$
$$1 + p_1T_r + \ldots = \prod_{r = 0, h} (1 - y^2_{r,j}) \quad r = 0, h$$
$$1 + p_1V_r + \ldots = \prod_{r = 0, h} (1 - y'^2_{r,j}) \quad r = 0, h.$$

Then $\frac{w_2}{T}(T - V)|_{BT[n] \times MT[n]}$ is given by minus half the degree-four component of

$$\prod_{0 < r < h} \prod_{i=1}^{d_r} (1 - (y_{r,i}^2 + 2ry_{r,i}z_n + r^2z_n^2))$$
$$\prod_{0 < r < h} \prod_{i=1}^{d'_r} (1 - (y'_{r,i}^2 + 2ry'_{r,i}z_n + r^2z_n^2)).$$

The coefficient of $z_n$ is

$$\sum_{0 < r < h} r(c_1T_r - c_1V_r) + h \sum_{j} y_{h,j} - y'_{h,j}.$$

The characteristic class restriction (1.2) implies that this quantity is zero; the claims of the lemma follow.

**Proof of Lemma 6.1.** We have

$$e(0, b)(z + \lambda) = \frac{\prod_{j} \prod_{i=1}^{d_j} \theta(x_{j,i} + m_{j}z + m_{j}\lambda)}{\prod_{j} \prod_{i=1}^{d'_j} \theta(x_{j,i} + m_{j}z + m_{j}\lambda)}$$
$$= e(\lambda \sum_{j} d_{j}'m_{j}' - d_{j}m_{j})) 
S(\sum_{j} m_{j}'(d_{j}'m_{j}'z + \sum x_{j,i}'(i)) - \sum_{j} m_{j}(d_{j}m_{j}z + \sum x_{j,i})) 
S(\frac{1}{2} \sum_{j} d_{j}'m_{j}'^2 - \sum_{j} d_{j}m_{j}^2)) 
eq e(0, b)(z).$$

The third equation uses Lemma 8.1.

**Proof of Lemma 6.8.** Recall that

$$G = \sum_{j} \frac{n}{2}(d_{j}'\ell_{j}'^2 - d_{j}\ell_{j}^2) + d_{j}'\ell_{j}'r_{j}' - d_{j}\ell_{j}r_{j}.$$

Substituting the equation

$$m_{j} = n\ell_{j} + r_{j}$$
(and similarly for \( m'_j \)) into equation (3.3) gives
\[
0 = \sum_j d'_j m'_j^2 - d_j m_j^2
= \sum_j d'_j (n^2 \ell_j^2 + 2n\ell_j r'_j + r_j'^2) - \sum_j d_j (n^2 \ell_j^2 + 2n\ell_j r_j + r_j'^2).
\]
It follows that
\[
2nG = - \sum_r (e'_r - e_r)r^2,
\]
and that this quantity is divisible by \( n \).

**Proof of Lemma 6.4.** The first Chern class of the bundle
\[
\mathcal{V} = \prod_{0 < r < h} \det(V_r)^{-r} \det(T_r)^r
\]
is
\[
c_1 \mathcal{V} = - \sum_r r(c_1 V_r - c_1 T_r).
\]
The restriction of \( \mathcal{V} \) to \( M^r \) has Chern class
\[
c_1(\mathcal{V}|_{M^r})_T = - \sum_j r'_j (d'_j m'_j z + \sum_{i=1}^{d'_j} x'_{j,i}) + \sum_j r_j (d_j m_j z + \sum_{i=1}^{d_j} x_{j,i}).
\]
Adding to this zero in the form (from (8.4) and (8.3))
\[
\sum_j m'_j (d'_j m'_j + \sum_{i=1}^{d'_j} x'_{j,i}) - \sum_j m_j (d_j m_j + \sum_{i=1}^{d_j} x_{j,i})
\]
gives
\[
c_1(\mathcal{V}|_{M^r})_T = \sum_j (m'_j - r'_j)(d'_j m'_j z + \sum_{i=1}^{d'_j} x'_{j,i}) - \sum_j (m_j - r_j)(d_j m_j z + \sum_{i=1}^{d_j} x_{j,i})
= n \sum_j \ell'_j (d'_j m'_j z + \sum_{i=1}^{d'_j} x'_{j,i}) - n \sum_j \ell_j (d'_j m'_j z + \sum_{i=1}^{d'_j} x'_{j,i})
= n H.
\]

**Proof of Lemma 6.10.** First observe that if
\[
\mathcal{V} = \prod_{0 < r < h} \det(V_r)^{-r} \det(T_r)^r,
\]
then Lemma 8.3 shows that \( \mathcal{V} \) has an \( h \)-th root, with the property that
\[
c_1 \mathcal{V}^{1/h}_{\mod 2} = w_2(V_h - T_h).
\]
The restriction of \( \mathcal{V} \) to \( M^r \) has Chern class
\[
c_1(\mathcal{V}|_{M^r})_T = - \sum_{0 < r'_j < h} r'_j (d'_j m'_j z + \sum_{i=1}^{d'_j} x'_{j,i}) + \sum_{0 < r_j < h} r_j (d_j m_j z + \sum_{i=1}^{d_j} x_{j,i}).
\]
Adding to this zero in the form (from (8.4) and (8.3))
\[
\sum_j m'_j (d'_j m'_j + \sum_{i=1}^{d'_j} x'_{j,i}) - \sum_j m_j (d_j m_j + \sum_{i=1}^{d_j} x_{j,i})
\]
gives
\[ c_1(\mathcal{V}|_{M^c}) = nH \]
\[ + \sum_{m_j \equiv h} m_j'(d_j'm_j'z + \sum_{i=1}^{d_j'} x'_{j,i}) - \sum_{m_j \equiv h} n\ell_j'(d_j'm_j'z + \sum_{i} x'_{j,i}) \]
\[ - \sum_{m_j \equiv h} m_j(d_j m_jz + \sum_{i=1}^{d_j} x_{j,i}) + \sum_{m_j \equiv h} n\ell_j(d_j m_jz + \sum_{i} x_{j,i}) \]
\[ = nH \]
\[ + \hbar \sum_{i=1}^{d_j'} (d_j'm_j'z + \sum_{i=1}^{d_j} x'_{j,i}) - nH \sum_{m_j \equiv h} (d_j m_jz + \sum_{i=1}^{d_j} x_{j,i}). \]

\[ \square \]

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