Slow-roll reconstruction for running spectral index

Qing-Guo Huang

School of physics, Korea Institute for Advanced Study,
207-43, Cheongryangri-Dong, Dongdaemun-Gu,
Seoul 130-722, Korea

huangqg@kias.re.kr

ABSTRACT

We reconstruct from the WMAP three-year data the slow-roll parameters in the standard slow-roll inflation model and the noncommutative inflation model. We also investigate the evolution of these slow-roll parameters. Requiring that slow-roll inflation lasts more than 20 e-folds after CMB scales leave the horizon, we find that the potential at the last stage of inflation takes the form $V(\phi) = V_0 (1 + \frac{\eta_c}{2} \frac{\eta_c}{M_p^2}(\phi - \phi_c)^2)$, where $\eta_c$ is a constant. A natural mechanism to end inflation at $\phi = \phi_c$ is the hybrid type inflation.
1 Introduction

Inflationary models [1] not only explain the large-scale homogeneity and isotropy of the universe, but also provide a natural mechanism to generate the observed magnitude of inhomogeneity. During the period of inflation, quantum fluctuations are generated within the Hubble horizon, which then stretch outside the horizon to become classical. In the subsequent deceleration phase after inflation these frozen fluctuations re-enter the horizon, and seed the matter and radiation density fluctuations observed in the universe.

There are two notable problems in cosmology. One is how to extract the amplitude of the power spectrum, including its scale dependence, from observational data. The other is to understand the origin of the inflationary models from the fundamental theory. In the last decade hundreds of inflationary models have been proposed: see for example [2]. However we still can not pick out a realistic model among them. With the development of cosmological observations, such as WMAP [3] and Sloan Digital Sky Survey [4] etc., we believe that now it is time to ask what the observations imply for the above problems. A useful method to try to answer this question is the slow-roll reconstruction, which has been investigated for more than ten years [5–8]. Unfortunately, till now no suggestive implications have been worked out for the inflaton potential. In this paper we try to improve this status.

Now the ΛCDM model is still an excellent fit to the WMAP three-year data. A nearly scale-invariant, adiabatic primordial power spectrum generated during inflation can be taken as the seeds for the anisotropy of CMB. Even though a red power spectrum for the curvature perturbations is a nice fit to the WMAP data, a running spectral index slightly improves the fit [3]. The values of the spectral index and its running and the upper bound on the tensor-scalar ratio are respectively [3]

\[
    n_s = 1.21^{+0.13}_{-0.16}, \quad \alpha_s = -0.102^{+0.050}_{-0.043}, \quad r \leq 1.5, \quad (1.1)
\]

at \( k = 0.002 \text{Mpc}^{-1} \). Combining WMAP with SDSS, a more stringent constraint on the tensor perturbations \( r \leq 0.67 \) is obtained. The running is allowed not only in the WMAP data, but also in combination with other CMB and/or large scale structure information, such as 2dFGRS [9] and the Sloan Digital Sky Survey [4]. Further analysis of a possible running spectral index is discussed in [10]. Many theoretical explorations of WMAP results are carried out recently in [11–18].

The WMAP data also implies that a red power spectrum \( (n_s < 1) \) at \( k = 0.05 \text{Mpc}^{-1} \) is favored and the running of the spectral index is not required at more than the 95% confidence level. Thus not only the spectral index runs, but the running is also scale-
dependent. In [14], the author suggested a new parameter, the running of running, which is defined as

$$\beta_s = \frac{d\alpha_s}{d\ln k}.$$  

(1.2)

to characterize the scale-dependent running. The value of $\beta_s$ is estimated at the linear approximation level to be

$$\beta_s \simeq \frac{\Delta\alpha_s}{\Delta \ln k} \simeq \frac{0 - (-0.102)}{\ln(0.05/0.002)} \simeq 0.0318.$$  

(1.3)

A running of running with the order of magnitude $10^{-2}$ is expected. In [17], the authors also pointed out that the running of the spectral index should not be a constant. In fact, nowadays data cannot provide the running of running at a significantly statistical confidence level. If we consider the constraint from the running of running, the reconstruction is just a sub-set for the case without this constraint.

We use the Horizon Flow Functions [5, 6, 19] to compute the spectral index and its running and running of running. One of the advantages of this approach is that it directly translates between the slow-roll parameters and the running spectral index. A further virtue of this approach is that when the slow-roll hierarchy is truncated at finite order, the truncation is preserved by the evolution equations.

In [3] the scale $0.002 \, \text{Mpc}^{-1}$ is used, which is close to the observable horizon. However, this scale is not directly related to a fundamental scale with the viewpoint of fundamental theory. It is interesting for us to study the evolution of the reconstruction at different scale. On the other hand, for the slow-roll approximation, the location on the inflationary potential corresponding to the observed perturbations is usually quantified by the number of e-folds before the end of inflation as [20]

$$N(k) = -\ln \frac{k}{a_0 H_0} + \frac{1}{3} \ln \frac{\rho_{reh}}{\rho_{\text{end}}} + \frac{1}{4} \ln \frac{\rho_{eq}}{\rho_{reh}} + \ln \sqrt{\frac{8\pi V_k}{3M_p^2 H_{eq}}} + \ln 219\Omega_0 h.$$  

(1.4)

A plausible estimation in the second paper of [20] requires that the slow-roll inflation lasts at least 20 e-folds after the observable Universe leaves the horizon.

In this paper, we only fit the central value of the observational parameters, which clearly implies that the the evolution of the slow-roll parameters should be very important for the reconstructions. In order to get more precise results, we should use Monte Carlo Markov Chain analysis. We hope one can do it in the future. Our paper is organized as follows. In section 2, we calculate the spectral index and its running and running of running in standard slow-roll inflationary models and reconstruct the slow-roll parameters. In section 3, we compute the modification of running spectral index in noncommutative
inflationary models and reconstruct the slow-roll parameters and the noncommutative parameter. The evolution of the slow-roll parameters are also investigated in section 2 and 3. In section 4, we suggest a tentative potential for inflaton at the last stage of inflation. Section 5 contains some concluding remarks.

## 2 Spectral index in usual inflation model

In this paper, we only focus on the inflationary models governed by single inflaton field \( \phi \). In FRW universe, Friedmann equation and the equation of motion for the homogeneous inflaton \( \phi \) are respectively

\[
H^2 = \frac{1}{3 M_p^2} \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right),
\]

\[
\ddot{\phi} = -3H \dot{\phi} - V',
\]

where \( M_p \) is the reduced Planck mass scale, \( V(\phi) \) is the potential of inflaton. When \( \dot{\phi}^2 \ll V \) and \( |\ddot{\phi}| \ll 3H|\dot{\phi}| \), inflaton slowly rolls down its potential. We define for convenience the slow-roll parameters as

\[
\epsilon \equiv \frac{3\dot{\phi}^2}{2} \left( \frac{V}{\dot{\phi}^2} \right)^{-1} = 2 M_p^2 \left( \frac{H'}{H} \right)^2,
\]

\[
\eta \equiv -\frac{\dddot{\phi}}{\ddot{\phi}} = 2 M_p^2 \frac{H''}{H},
\]

where prime denotes the derivative with respect to \( \phi \). The slow-roll conditions become

\[
\epsilon \ll 1, \quad \text{and} \quad |\eta| \ll 3.
\]

In the whole paper, we assume, without loss of generality, \( \dot{\phi} < 0 \), so that \( H'/H > 0 \). The number of e-folds \( N \) before the end of inflation is given by

\[
\frac{dN}{d\phi} = -\frac{H}{\dot{\phi}} = \frac{1}{\sqrt{2\epsilon} M_p}.
\]

The slow-roll parameter \( \epsilon \) determines how fast the inflaton field evolves. If \( \epsilon \ll 1 \), inflaton rolls down its potential very slowly.

Expand around an exact solution to first order, the amplitudes of the scalar and tensor power spectra are respectively [2, 5, 21–23]

\[
\Delta^2_R = (1 - (2C + 1)\epsilon + C\eta)^2 \frac{H^2/M_p^2}{8\pi^2\epsilon},
\]

\[
\Delta^2_T = (1 - (C + 1)\epsilon)^2 \frac{H^2/M_p^2}{\pi^2/2}.
\]
where \( C = -2 + \ln 2 + \gamma \simeq -0.73 \) and \( \gamma \) is the Euler constant originating in the expansion of the gamma function. The tensor-scalar ratio is

\[
r = \frac{\Delta_T^2}{\Delta_R^2} = 16\epsilon(1 + 2C(\epsilon - \eta)). \tag{2.9}
\]

Taking into account the slow-roll condition, we find

\[
\frac{d}{d \ln k} = -M_p \sqrt{2\epsilon} \frac{d}{d \phi}. \tag{2.10}
\]

The evolution of the slow-roll parameters are given by

\[
\frac{d\epsilon}{d \ln k} = \frac{2\epsilon^2 - 2\epsilon\eta}{1 - \epsilon}, \tag{2.11}
\]
\[
\frac{d\eta}{d \ln k} = \frac{-\xi + \epsilon\eta}{1 - \epsilon}, \tag{2.12}
\]
\[
\frac{d\xi}{d \ln k} = \frac{2\epsilon\xi - \eta\xi - \zeta}{1 - \epsilon}, \tag{2.13}
\]
\[
\frac{d\zeta}{d \ln k} = \frac{-\theta - 2\eta\zeta + 3\epsilon\zeta}{1 - \epsilon}, \tag{2.14}
\]

where

\[
\xi \equiv 4M_p^4 \frac{H'H^3}{H^2}, \tag{2.15}
\]
\[
\zeta \equiv 8M_p^6 \frac{H'^2H^4}{H^3}, \tag{2.16}
\]
\[
\theta \equiv 16M_p^8 \frac{H'^3H^5}{H^4}, \tag{2.17}
\]

and \( H^{(n)} = d^n H/d\phi^n \). The above equations for the evolution of the slow-roll parameters preserve the slow-roll hierarchy truncations. For example, if \( \xi = \zeta = \theta = ... = 0 \), \( d\xi/d \ln k = d\zeta/d \ln k = ... = 0 \).

Thus the spectral index and its running and running of running respectively take the form

\[
n_s \equiv \frac{d \ln \Delta_T^2}{d \ln k} = 1 - 4\epsilon + 2\eta - 2C\xi - 8(C + 1)\epsilon^2 + (6 + 10C)\epsilon\eta, \tag{2.18}
\]
\[
\alpha_s \equiv \frac{dn_s}{d \ln k} = -2\xi - 8\epsilon^2 + 10\epsilon\eta - (8 + 14C)\epsilon\xi + 2C\eta\xi + 2C\zeta, \tag{2.19}
\]
\[
\beta_s \equiv \frac{d\alpha_s}{d \ln k} = -14\epsilon\xi + 2\eta\xi - 2C\xi^2 + 2\zeta - 2C\theta
- 32\epsilon^3 + 62\epsilon^2\eta - 20\epsilon\eta^2 - (46 + 56C)\epsilon^2\xi + (26 + 48C)\epsilon\eta\xi
- 2C\eta^2\xi + (10 + 20C)\epsilon\zeta - 6C\eta\zeta. \tag{2.20}
\]
We will use (2.9), (2.18), (2.19) and (2.20) to reconstruct the slow-roll parameters for the running spectral index. In order to describe the evolution of the slow-roll parameters, we introduce a new number of e-folds $n$ which is related to the value of the scale factor by

$$a(n) = a_{0.002}e^n,$$  
(2.21)

where $a_{0.002}$ is the scale factor at the time when the comoving fluctuation mode with wave number $k = 0.002\text{Mpc}^{-1}$ crossed the Hubble horizon during inflation. Using eq. (2.10), (2.11) and $dn = -dN$, we find

$$(1 - \epsilon)\frac{d}{d\ln k} = \frac{d}{dn}.$$  
(2.22)

For $k = 0.05\text{Mpc}^{-1}$, $n_{0.05} \simeq \ln(0.05/0.002) = 3.2$. In the following, we will reconstruct the slow-roll parameters with different slow-roll hierarchy truncations.

2.1 $\xi = \zeta = \theta = 0$

In the first case we set $\xi = \zeta = \theta = 0$. There are two free parameters $\epsilon$ and $\eta$ to fit the two observed parameters $n_s$ and $\alpha_s$. Solving eq. (2.18), we find

$$\eta = \frac{s + 4\epsilon + 8(1 + C)\epsilon^2}{2 + (6 + 10C)\epsilon},$$  
(2.23)

where $s = n_s - 1$. The running of the spectral index becomes

$$\alpha_s = \frac{\epsilon(5s + 12\epsilon + 16\epsilon^2)}{1 + (3 + 5C)\epsilon},$$  
(2.24)

which is positive for the blue spectrum ($n_s > 1$) if $\epsilon < -1/(3 + 5C) \simeq 1.5$. Slow-roll condition requires $\epsilon \ll 1$ and the running of spectral index is positive for blue tilt power spectrum. Thus two slow-roll parameters $\epsilon$ and $\eta$ can not provide a negative running of the spectral index with large absolute value.

2.2 $\zeta = \theta = 0$

Here we take the third slow-roll parameter $\xi$ into account, while $\zeta$ and $\theta$ are still set to zero. First we use three slow-roll parameters $\epsilon$, $\eta$ and $\xi$ to fit the spectral index and its running. Taking the spectral index and its running as input, we show the constraint on these slow-roll parameters and corresponding running of running in fig.1. They can be used to fit the spectral index and its running. However they are roughly evaluated at the same order of magnitude. We do not reach a reliable truncation of the slow-roll hierarchy and higher order parameters should be taken into account.
Figure 1: Ignoring $\zeta$ and $\theta$, the constraints on the slow-roll parameters and the corresponding running of running are showed for $n_s = 1.21$ and $\alpha_s = -0.102$, where $r \simeq 16\epsilon < 1.5$.

The evolution of slow-roll parameters are governed by eq. (2.11), (2.12), (2.13) and (2.14). Solving these equations, we figure out how the slow-roll parameters and spectral index and its running evolve. Since there are only two constraints ($n_s$ and $\alpha_s$), but three free parameters, we can not fix the value of $\epsilon$, $\eta$ and $\xi$ at $k = 0.002\text{Mpc}^{-1}$ (or, $n = 0$). We scan the corresponding value along the line in fig. 1. Here we pick out three points in fig. 1 and show the evolution of the slow-roll parameters and the spectral index and its running respectively in fig. 2. Similar behaviors for them are obtained for the other values of the slow-roll parameters at $k = 0.002\text{Mpc}^{-1}$. We find from fig. 2 that a large enough number of e-folds after the CMB scales leave the horizon can not be achieved. In [16,17], the authors also pointed out that the running of spectral index cannot be obtained if we only consider these three slow-roll parameters and require that the slow-roll inflation lasts not shorter than 30 e-folds after the CMB scales leave the horizon.

2.3 $\theta = 0$

In this case we take $\zeta$ into account. In [7] the authors also suggested this slow-roll parameter $\zeta$ for a running spectral index. Since there are four free parameters, we also consider the constraint from the running of running. Our results are shown in fig. 3.

A premature truncation of the slow-roll hierarchy appears if $\zeta \sim \mathcal{O}(10^{-3})$. For example, for $\zeta < 10^{-2}$, constraints on the other slow-roll parameters are respectively $\epsilon \in [0, 0.002]$, $\eta \in [0.075, 0.078]$ and $\xi \simeq 0.042$. Now the tensor-scalar ratio is smaller than 0.04.

We also investigate the evolution of the slow-roll parameters and the spectral index and its running. The similar behaviors of these parameters are obtained for taking different values of slow-roll parameters at $k = 0.002\text{Mpc}^{-1}$ along the line in fig. 3. For example,
we take $\epsilon = 0.02$, $\eta = 0.112$, $\xi = 0.047$ and $\zeta = 0.014$ and the evolution of the parameters are showed in fig. 4. An interesting result is that all of the slow-roll parameters go to zero within roughly 15 e-folds after $k = 0.002\text{Mpc}^{-1}$ leaves the horizon, except $\eta$ which approaches to a positive constant $\eta_c$. Since $\eta_c$ is still much less than 3, the slow-roll inflation will not end. How to end inflation will be discussed in section 4.

We need to make one point clear here. If we only consider the spectral index and the running to reconstruct the slow-roll parameters, the constraints is looser, since there are four parameters, but only two constraints. Taking the fourth slow-roll parameter for a nice fit to the running of running, we find that a large enough number of e-folds is obtained.
Figure 3: Ignoring $\theta$ only, constraints on the slow-roll parameters are showed for $n_s = 1.21$, $\alpha_s = -0.102$ and $\beta_s = 0.0318$, where $r \simeq 16\epsilon \leq 1.5$.

2.4 Summary

To summarize, the slow-roll reconstruction shows that one more parameter is needed if we want to fit the observational parameters, e.g. three slow-roll parameters is needed to fit the spectral index and its running and four slow-roll parameters are required if we also want to fit the running of running. Including the fourth slow-roll parameter $\zeta$, a tentative truncation of the slow-roll hierarchy is obtained for a small value of $\epsilon$; otherwise, we should take care of the contribution from higher order slow-roll parameters when performing a fit to the spectral index and its running. We also need to keep in mind that we still have not built such an inflationary model from a fundamental theory.

3 Spectral index in noncommutative inflation model

There is not a realistic inflationary model providing a negative running of the spectral index with large enough absolute value. One question is what is the origin of the running spectral index with the viewpoint of fundamental theories. In [13,14,24–28], the authors proposed that the inflationary models in noncommutative spacetime open a window to improve the fit to the running spectral index for the typical inflationary models. The other relevant trans-Planckian physics [29] and non-minimal coupled inflation model [30] have also been discussed widely.

In this section, we compute the amplitude of primordial scalar and tensor power spectra in noncommutative inflation model and discuss the truncation of the slow-roll hierarchy in order to fit the running spectral index.
3.1 Spectral index in noncommutative inflation model

Noncommutative spacetime naturally emerges in string theory [31], which implies a new uncertainty relation

$$\Delta t_p \Delta x_p \geq l_s^2,$$

where $t_p$ and $x_p$ are the physical time and space, $l_s$ is the uncertainty length scale, or string scale in string theory. To make this paper self-consistent, we briefly review the calculation of the primordial power spectrum [27] for noncommutative inflationary models. The spacetime noncommutative effects are encoded in a new product among functions, namely the star product, replacing the usual algebra product. Since the evolution of the background and inflaton is homogeneous, the standard classical cosmological equations does not change.

To make the uncertainty relation in (3.1) clearer in FRW background, we introduce a new time coordinate $\tau$ as

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2 = a^{-2}(\tau)d\tau^2 - a^2(\tau)d\vec{x}^2.$$
Now the uncertainty relationship (3.1) for the coordinates in the above metric becomes
\[ \Delta \tau \Delta x \geq l_s^2. \tag{3.3} \]

The star product can be explicitly defined as
\[ f(\tau, x) \ast g(\tau, x) = e^{-i\frac{2}{\hbar^2}(\partial_\tau \partial_\tau' - \partial_x \partial_x')} f(\tau, x) g(\tau', x') |_{\tau'=\tau, x'=x}. \tag{3.4} \]

Because the comoving curvature perturbation \( \mathcal{R} \) not only depends on time, but also depends on the position in the space, the equation of motion for \( \mathcal{R} \) is modified by the noncommutative effects,
\[ u''_k + \left( k^2 - \frac{z''_k}{z_k} \right) u_k = 0, \tag{3.5} \]
where
\[ z''_k(\tilde{\eta}) = z'' y^2_k(\tilde{\eta}), \quad y^2_k = (\beta^+_k \beta^-_k)^{\frac{1}{2}}, \tag{3.6} \]
\[ \frac{d\tilde{\eta}}{d\tau} = \left( \frac{\beta^-_k}{\beta^+_k} \right)^{\frac{1}{2}}, \quad \beta^\pm_k = \frac{1}{2}(a^{\pm 2}(\tau + l_s^2 k) + a^{\pm 2}(\tau - l_s^2 k)), \]
\[ z = a\dot{\phi}/H, \quad \mathcal{R}_k(\tilde{\eta}) = u_k(\tilde{\eta})/z_k(\tilde{\eta}) \]
is the Fourier modes of \( \mathcal{R} \) in momentum space and the prime in (3.5) denotes the derivative with respect to the modified conformal time \( \tilde{\eta} \). The deviation from the commutative case encodes in \( \beta^\pm_k \) and the corrections from the noncommutative effects can be parameterized by \( \frac{Hk}{aM_s^2} \). After a lengthy but straightforward calculation, we get
\[ \frac{z''_k}{z_k} = 2(aH)^2 \left( 1 + \epsilon - \frac{3}{2} \eta - 2\mu + \mathcal{O}(\xi, \epsilon^2, \eta, \eta^2) \right), \tag{3.7} \]
\[ aH \simeq -\frac{1}{\eta}(1 + \epsilon + \mu), \]
where
\[ \mu = \frac{H^2 k^2}{a^2 M_s^4} \tag{3.8} \]
is the noncommutative parameter and \( M_s = l_s^{-1} \) is the noncommutative mass scale or string mass scale. Solving eq. (3.3) yields the amplitude of the scalar comoving curvature fluctuations in noncommutative spacetime
\[ \Delta^2_\mathcal{R} = \frac{k^3}{2\pi^2} |\mathcal{R}_k(\tilde{\eta})|^2 = (1 - (2C + 1)\epsilon + C\eta^2) \frac{H^2/M_s^2}{8\pi^2\epsilon}(1 + \mu)^{-4}, \tag{3.9} \]
where \( H \) takes the value when the fluctuation mode \( k \) crosses the Hubble radius \( (z''_k/z_k = k^2) \) and \( k \) is the comoving Fourier mode. Using (3.8) and (2.10), we obtain
\[ \frac{d\mu}{d \ln k} = \frac{-4\epsilon \mu}{1 - \epsilon}. \tag{3.10} \]
The spectral index and its running and running of running are respectively

\[ n_s = 1 - 4\epsilon + 2\eta - 2C\xi - 8(C + 1)e^2 + (6 + 10C)\epsilon\eta + 16(\epsilon + \epsilon^2)\mu, \]

\[ \sigma_s = -2\xi - 8\epsilon^2 + 10\epsilon\eta - (8 + 14C)\epsilon\xi + 2C\eta\xi + 2C\zeta - 32(\epsilon^2 + \epsilon\eta)\mu, \]

\[ \beta_s = -14\epsilon\xi + 2\eta\xi - 2C\xi^2 + 2\zeta - 2C\theta - 32\epsilon^3 + 62\epsilon^2\eta - 20\epsilon\eta^2 - (46 + 56C)\epsilon^2\xi + (26 + 48C)\epsilon\eta\xi - 2C\eta^2\xi + (10 + 20C)\epsilon\xi - 6C\eta\zeta + 160\epsilon\eta^2\mu + 64\epsilon^2\eta\mu + 32\epsilon\xi\mu. \]

Similarly, the amplitude of the tensor perturbations and the tensor-scalar ratio are respectively given by

\[ \Delta^2_T = (1 - (C + 1)\epsilon)^2 \frac{H^2/M_p^2}{\pi^2/2} (1 + \mu)^{-4}, \]

\[ r = \frac{\Delta^2_T}{\Delta^2_R} = 16\epsilon(1 + 2C(\epsilon - \eta)). \]

When \( \mu = 0 \) or \( H \ll M_s \), all of the above results in noncommutative inflation are just the same as those in the standard slow-roll inflation model.

### 3.2 Slow-roll reconstruction and slow-roll hierarchy in noncommutative inflation model

In this subsection, we reconstruct the slow-roll parameters and the noncommutative parameter \( \mu \) by fitting the running spectral index from WMAP three-year data.

First, we consider the case with \( \xi = \zeta = \theta = 0 \). Now there are three free parameters: \( \epsilon, \eta \) and \( \mu \). Fitting the spectral index and its running and running of running, we find \( \epsilon = 0.077, \eta = 0.072, \mu = 0.3 \) and the tensor-scalar ratio is 1.23. At first sight, this result seems trivial, because we use three free parameters to fit three observed parameters. But the non-trivial thing is that the slow-roll parameters and \( \mu \) are evaluated within the reasonable range. The evolution of these parameters are figured out in fig. 5. After fluctuation mode \( k = 0.002\text{Mpc}^{-1} \) stretched outside the Hubble horizon, both \( \epsilon \) and \( \mu \) approach zero; however, \( \eta \) goes to a positive constant \( \eta_c \) which is much smaller than 3. The mechanism to end inflation will be discussed in section 4.

Second, we consider the case where only \( \zeta \) and \( \theta \) are set to zero. The constraints on the slow-roll parameters and \( \mu \) to fit the spectral index, its running and running of
Figure 5: Setting $\xi = \zeta = \theta = 0$, the evolution of the slow-roll parameters and spectral index and its running are showed.

running are shown in fig. 6. We are also interested in the evolution of the slow-roll parameters and spectral index and its running. We scan the corresponding value of the slow-roll parameters and $\mu$ in fig. 6. If we require that slow-roll inflation lasts more than 20 e-folds after CMB scales leave the horizon, the constraints on these parameters are $\epsilon \in [0.076, 0.094]$, $\eta \in [0.044, 0.078]$, $\xi \in [-0.039, 0.005]$ and $\mu \in [0.284, 0.352]$ at $k = 0.002\text{Mpc}^{-1}$. Within the permitted region for these parameters at $k = 0.002\text{Mpc}^{-1}$, for example, we take $\epsilon = 0.081$, $\eta = 0.062$, $\xi = -0.01$ and $\mu = 0.318$, and the evolution of the slow-roll parameters and spectral index and its running are shown in fig. 7. Similar behavior for $\epsilon$, $\eta$, $\xi$ and $\mu$ is obtained for the other points along the lines in fig. 6. As a universal result, the slow-roll parameters go to zero, except $\eta$ which approaches a positive value $\eta_c$.

To summarize, noncommutative inflation can provide a running spectral to fit WMAP three-year data and the slow-roll hierarchy is obtained. A feature for the noncommutative inflation model is that a large amplitude of the tensor perturbations is required ($r \geq 1.22$).
Figure 6: Ignoring $\zeta$ and $\theta$, the constraints on the slow-parameters and $\mu$ are showed fitting to $n_s = 1.21$, $\alpha_s = -0.102$ and $\beta_s = 0.0318$.

4 The last stage of inflation

In section 2 and 3, requiring slow-roll inflation lasts more than twenty e-folds after CMB scales leave the horizon, we reconstructed the slow-roll parameters for standard inflation model and noncommutative inflation respectively. Our numerical results show that the slow-roll parameters approach zero in the last stage of inflation, except $\eta$ which goes to a positive constant $\eta_c$. Assume that the value of inflaton and Hubble parameter are respectively $\eta_c$ and $H_c$ at the end of inflation, eq. (2.4) can be written as

$$\eta_c = 2M^2 \frac{1}{H} \frac{d^2 H}{d\phi^2}. \quad (4.1)$$

Since Hubble parameter $H$ and the value of $\phi$ monotonically decrease, eq. (4.1) becomes

$$\frac{dH}{d\phi} = \left( \frac{\eta_c}{2M^2} (H^2 - H_c^2) \right)^{1/2}. \quad (4.2)$$

The solution of above equation is given by

$$H = H_c \cosh \left( \sqrt{\frac{\eta_c}{2}} \frac{\phi - \phi_c}{M_p} \right) \quad \text{for} \quad \phi > \phi_c. \quad (4.3)$$

Since $\epsilon \to 0$ and then $\dot{\phi} \to 0$ in the last stage of inflation, the asymptotical behavior of the potential $V(\phi)$ takes the form

$$V(\phi) \to 3M^2 H^2 = V_0 \cosh^2 \left( \sqrt{\frac{\eta_c}{2}} \frac{\phi - \phi_c}{M_p} \right), \quad (4.4)$$

where $V_0 = 3M^2 H_c^2$. This potential can be trusted only when $\phi \to \phi_c$ and then we expand this potential around $\phi_c$ as

$$V(\phi) \approx V_0 \left( 1 + \frac{\eta_c (\phi - \phi_c)^2}{2M^2} \right). \quad (4.5)$$
Figure 7: Setting $\zeta = \theta = 0$, the evolution of the slow-roll parameters and spectral index and its running are showed.

We need to stress that this is only the potential dominated the last stage of inflation. We can not figure out a simple function of the potential for the stage with running spectral index.

Since $\eta_c \ll 3$, inflation does not end by violating the slow-roll conditions. As we know, hybrid inflation provides a natural mechanism to end this kind of inflation [32]. A tentative potential for hybrid inflation can be

$$V(\phi, \varphi) = V_0 \left( 1 + \frac{\eta_c (\phi - \phi_c)^2}{M_p^2} \right) + \frac{1}{2} g (\phi^2 - \phi_c^2) \varphi^2 + \frac{\lambda}{4} \varphi^4 , \quad (4.6)$$

where both $g$ and $\lambda$ are positive constant. During the period of inflation, $\phi > \phi_c$ and $\varphi$ stays at its global minimum $\varphi = 0$. When inflaton $\phi$ goes to $\phi_c$, the end of inflation is triggered by $\varphi$.

5 Conclusion

In this paper, the slow-roll parameters are reconstructed in the standard inflationary models and the noncommutative inflationary models. Requiring that slow-roll inflation
lasts more than 20 e-folds after CMB scales leave the horizon, we get a stringent constraint on the slow-roll parameters in order to fit the running spectral index: the fourth slow-roll parameter is needed for usual inflationary models, or a large tensor-scalar ratio with $r \geq 1.22$ is needed for noncommutative inflationary models. Thus if $r \ll 1$, the standard inflationary model is favored; but the noncommutative seems better in the case with $r \geq 1$. In both cases, the potential of the inflaton in the last stage of inflation should take the form of eq. (4.3) and hybrid type inflation provides a natural mechanism to end the inflation. Combining with SDSS, the tensor-scaler ratio should be less than 0.67. If so, the noncommutative inflationary model can not help us to get a negative running of the spectral index with large absolute value.

However the running spectral is still not strongly favored by observations. We hope Planck or further data from WMAP can tell us whether the spectral index does run or not. If it really runs, the observations can tell us more details about inflation.

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