TWISTED COHOMOLOGICAL EQUATIONS
FOR TRANSLATION FLOWS

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ABSTRACT. We prove by methods of harmonic analysis a result on existence of solutions for twisted cohomological equations on translation surfaces with loss of derivatives at most 3+ in Sobolev spaces. As a consequence we prove that product translation flows on (3-dimensional) translation manifolds which are products of a (higher genus) translation surface with a (flat) circle are stable in the sense of A. Katok. In turn, our result on product flows implies a stability result of time-τ maps of translation flows on translation surfaces.

1. INTRODUCTION

The first result on solutions of the cohomological equation for a parabolic non-homogeneous (but “locally homogeneous”) smooth flow was given by the author in [F97] in the case of translation flows (and their smooth time changes) on higher genus translation surfaces, by method of harmonic analysis based on the theory of boundary behavior of holomorphic functions.

Since then refined versions of that result have been proved by (dynamical) renormalization methods based on “spectral gap” (and hyperbolicity) properties of the Rauzy-Veech-Zorich cocycle [MMY05], [MY16], of the Kontsevich–Zorich cocycle over the Teichmüller flow [F07] or, more recently, of the transfer operator of a pseudo-Anosov map on appropriate anisotropic Banach space of currents [FGL].

The renormalization approach has the immediate advantage of a refined control on the regularity loss and of more explicit conditions of Diophantine type on the dynamics, and in particular applies to self-similar translation flows or Interval Exchange Transformations. It also gives a direct approach to results for almost all translation surfaces, while an extension to almost all directions for any given translation surface had to wait for the work of J. Chaika and A. Eskin [CE] based on the breakthrough of A. Eskin, M. Mirzakhani [EM], A. Eskin, M. Mirzakhani and A. Mohammadi [EMM] and S. Filip [Fi].

In this paper we apply a twisted version of the arguments of [F97] to the solution of the twisted cohomological equation for translation flows and derive results on the cohomological equation for 3-dimensional “translation flows” on products of...
a higher genus translation surface with a circle. For these problems no renormalization approach is available at the moment, although steps in that direction have been made in the work of A. Bufetov and B. Solomyak [BS14], [BS18a], [BS18b], [BS18c], [BS19] and of the author [F19], who have introduced twisted versions of the Rauzy–Veech–Zorich and Kontsevich–Zorich cocycles, respectively, and proved “spectral gap” results for them.

For any translation surface $(M, h)$ (a pair of a Riemann surface $M$ and an Abelian differential $h$ on $M$) let $H^s(M)$ denote the scale of (weighted) Sobolev spaces (introduced in [F97], and recalled below in §2). For the horizontal translation flow $\phi_R^f$ on $M$ (of generator the horizontal vector field $S$) and for any $\sigma \in \mathbb{R}$, let $\mathcal{H}^s_{R, \sigma} \subset H^{-s}_{R}(M)$ denote the space of $(S + \iota \sigma)$-invariant distributions, that is, the subspace

$$I^s_{h, \sigma} := \{ D \in H^{-s}_h(M) | (S + \iota \sigma)D = 0 \in H^{-r-1}_h(M) \}.$$

For all $\theta \in \mathbb{T}$, let $h_\theta := e^{-i\theta}h$ denote the rotated Abelian differential and let $S_\theta$ denote the generator of the horizontal translation flow on $(M, h_\theta)$.

We prove the following results.

**Theorem 1.1.** For any translation surface $(M, h)$, for almost all $\theta \in \mathbb{T}$ and for almost all $\sigma \in \mathbb{R}$ (with respect to the Lebesgue measure) the following holds. For all $f \in H^s_h(M)$ with $s > 3$, satisfying the distributional conditions $D(f) = 0$ for all $(S_\theta + \iota \sigma)$-invariant distributions $D \in H^{-s}_h(M)$, the twisted cohomological equation $(S_\theta + \iota \sigma)u = f$ has a solution $u \in H^r_h(M)$ for all $r < s - 3$, and there exists a constant $C_{r,s}(\theta, \sigma) > 0$ such that

$$|u|_r \leq C_{r,s}(\theta, \sigma)|f|_s.$$

In other words, the theory of the twisted cohomological equation of translation flows is analogous, for Lebesgue almost all twisting parameters, to the untwisted theory of the cohomological equation for translation flows.

**Remark 1.2.** In the untwisted case the optimal loss of regularity of solutions of the cohomological equation is known to be $1+$ for $L^2$ Sobolev norms, for almost all translation flows with respect to any $\text{SL}(2, \mathbb{R})$ invariant measure under the hypothesis of hyperbolicity of the KZ cocycle [F07]. Marmi and Yoccoz [MY16] proved a similar, but slightly weaker, statement for Hölder norms. For self-similar translation flows, the loss of $1+$ derivatives for Hölder norms should follow from the recent work of Fauve, Gouëzel and Lanneau [FGL], although spaces with fractional exponents are not explicitly considered in their paper.

It is natural to conjecture that the optimal loss of derivatives is $1+$ also in the twisted case, and it seems plausible that the whole argument of [F07] would carry over under the (equivalent?) hypotheses of hyperbolicity of the twisted cocycles introduced in [BS18c] and [F19]. At the moment the only known results on such twisted exponents are upper bounds (in particular that the top exponent is $< 1$) [BS18c], [BS19], [F19] but no lower bounds are known.

A result on the existence of solutions of the cohomological equation for twisted horocycle flows was recently proved by L. Flaminio, the author and J. Tanis [FFT16].
who were motivated by applications to the cohomological equation for horocycle time-$\tau$ maps (see also [Fa12]) and to deviation of ergodic averages for twisted horocycle integrals and horocycle time-$\tau$ maps. Twisted nilflows are still nilflows so the theory of twisted cohomological equations in the nilpotent case is covered by the general results of L. Flaminio and the author [FlaFo07]. As for results on deviation of ergodic averages for nilflows, they are related to bounds on Weyl sums for polynomials. The Heisenberg (and the general step 2) case are better understood by renormalization methods (see for instance [FaFo06]), while the higher step case is not renormalizable, hence harder (see for instance [GT12], [FlaFo14]). Results on twisted ergodic integrals of translation flows and applications to effective weak mixing were recently proved by the author [F19].

For all $(s, v) \in \mathbb{R}^+ \times \mathbb{N}$, let $H^{s,v}(M \times \mathbb{T})$ denote the $L^2$ Sobolev space on $M \times \mathbb{T}$ with respect to the invariant volume form $\omega_h \wedge d\phi$ and the vector fields $S$, $T$, and $\partial / \partial \phi$: for all $s, v \geq 0$, we define

$$H^{s,v}(M \times \mathbb{T}) := \{ f \in L^2(M \times \mathbb{T}, d\operatorname{vol}) | \sum_{i+j \leq s} \sum_{\ell \leq v} \| S^i T^j \partial^\ell f \|_0 < +\infty \};$$

the space $H^{-s,-v}_h(M \times \mathbb{T})$ is defined as the dual space of $H^{s,v}(M \times \mathbb{T})$.

The space $L^2(M \times \mathbb{T}, d\operatorname{vol})$ of the product manifold with respect to the invariant volume form $\omega_h \wedge d\phi$ decomposes as a direct sum of the eigenspaces $\{ H^0_n | n \in \mathbb{Z} \}$ of the circle action:

$$L^2(M \times \mathbb{T}, d\operatorname{vol}) = \bigoplus_{n \in \mathbb{Z}} H^0_n.$$

Let now $X_{\theta,c} = S_\theta + c \frac{\partial}{\partial \phi}$ denote a translation vector field on $M \times \mathbb{T}$, and let $\mathcal{J}^{s,v}_{h,c} \subset \mathcal{H}^{s,-v}_h(M \times \mathbb{T})$ denote the space of $X_{\theta,c}$-invariant distributions. The subspace of $X_{\theta,c}$-invariant distributions in $\mathcal{J}^{s,v}_{h,c}$ supported on the Sobolev subspace of $H_h^{s,v} \subset H^0_n$ has finite and non-zero dimension, uniformly bounded with respect to $n \in \mathbb{N}$. It follows that the space $\mathcal{J}^{s,v}_{h,c}$ has countable dimension.

**Theorem 1.3.** For any translation surface $(M, h)$, for almost all $\theta \in \mathbb{T}$ and for almost all $c \in \mathbb{R}$ (with respect to the Lebesgue measure) the following holds. For all $f \in H^1_h(M \times \mathbb{T})$ with $s > 3$ and $v > 2$, satisfying the distributional conditions

$$D(f) = 0$$

for all $X_{\theta,c}$-invariant distributions $D \in \mathcal{J}^{s,v}_{h,c} \subset H^{s,-v}_h(M \times \mathbb{T})$, the cohomological equation $X_{\theta,c}u = f$ has a solution $u \in H^{s,\mu}_h(M \times \mathbb{T})$ for all $r < s - 3$ and $\mu < v - 2$, and there exists a constant $C^{(\mu,v)}(\theta,c) > 0$ such that

$$\| u \|_{s,\mu} \leq C^{(\mu,v)}(\theta,c) \| f \|_{s,v}.$$
hyperbolic surfaces [FlaFo03] and Heisenberg nilflows [FlaFo06]. However, there has been recent progress (although conditional) on “Ruelle asymptotics” and deviation of ergodic averages for horocycle flows for negatively curved metrics on surfaces [Ad] (see also [FG18]), hence a proof of smooth stability (at least in low regularity) for such flows seems within reach of current methods for the analysis for the transfer operator of hyperbolic flows, along the lines of the work of P. Giulietti and C. Liverani [GL] for Anosov flows of tori.

By a well-known argument we can derive from our result on the cohomological equation for the product flows a result on the cohomological equation for the time-τ maps of translation flows. Let \( \Phi^\tau_\theta \) denote the time-τ map of the horizontal translation flow of the Abelian differential \( h_\theta \) on \( M \). For all \( s \geq 0 \), let \( \mathcal{D}^{s}_{h,\theta,\tau} \subset H^{-s}(M) \) denote the space of \( \Phi^\tau_\theta \)-invariant distributions, that is, the space of all distributions in \( H^{-s}(M) \) which vanish on the subspace \( \{ u \circ \Phi^\tau_\theta - u \mid u \in H^s_h(M) \} \cap H^s_h(M) \subset H^s_h(M) \).

We have the following result:

**Corollary 1.4.** For any translation surface \( (M,h) \), for almost all \( \theta \in \mathbb{T} \) and for almost all \( T \in \mathbb{R} \) (with respect to the Lebesgue measure) the following holds. For all \( f \in H^s_h(M) \) with \( s > 3 \), satisfying the distributional conditions \( \mathcal{D}(f) = 0 \) for all \( \Phi^\tau_\theta \)-invariant distributions \( \mathcal{D} \in \mathcal{D}^{s}_{h,\theta,\tau} \subset H^{-s}(M) \), the cohomological equation \( u \circ \Phi^\tau_\theta - u = f \) has a solution \( u \in H^s_h(M) \), for all \( r < s - 3 \), and there exists a constant \( C_{r,s}(\theta,\tau) > 0 \) such that
\[
|u|_r \leq C_{r,s}(\theta,\tau)|f|_s.
\]

The paper is organized as follows. In section 2 we recall basic facts of analysis on translation surfaces as developed by the author in [F97] and [F07]. In section 3 we introduce a twisted version of the Beurling-type isometry of the \( L^2 \) space of a translation surface defined in [F97] (see also [F02]). Section 4 recalls results from the theory of boundary behavior of Cauchy integrals of finite measures on the circle and applications to the spectral theory of general unitary operators on Hilbert spaces, following [F97]. In section 5 we prove the core result about the existence of solutions of the cohomological equation, by the following the presentation given in [F07] of the original argument of [F97], generalized to the twisted case. Subsection 5.1 is devoted to the core result about existence of distributional solutions, subsection 5.2 to finiteness result for the spaces of twisted invariant distributions and, finally, subsection 5.3 to the proof of the main results on existence of smooth solutions for the twisted cohomological equations, the product flows and the time-τ maps.

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This section gathers basic results on the flat Laplacian of a translation surface, following [F97], §2 and §3, and [F07], §2.

Let $\Sigma_h := \{p_1, \ldots, p_\sigma\} \subset M_h$ be the set of zeros of the holomorphic Abelian differential $h$ on a Riemann surface $M$, of even orders $(k_1, \ldots, k_\sigma)$ respectively with $k_1 + \cdots + k_\sigma = 2g - 2$. Let $R_h := |h|$ be the flat metric with cone singularities at $\Sigma_h$ induced by the Abelian differential $h$ on $M$ and let $\omega_h$ denote its area form. With respect to a holomorphic local coordinate $z = x + iy$ at a regular point, the Abelian differential $h$ has the form $h = \phi(z)dz$, where $\phi$ is a locally defined holomorphic function, and, consequently,

$$R_h = |\phi(z)|(dx^2 + dy^2)^{1/2}, \quad \omega_h = |\phi(z)|^2 \, dx \wedge dy.$$  \hfill (1)

The metric $R_h$ is flat, degenerate at the finite set $\Sigma_h$ of zeroes of $h$ and has trivial holonomy, hence $h$ induces a structure of translation surface on $M$.

The weighted $L^2$ space is the standard space $L^2_h(M) := L^2(M, \omega_h)$ with respect to the area element $\omega_h$ of the metric $R_h$. Hence the weighted $L^2$ norm $| \cdot |_0$ are induced by the hermitian product $\langle \cdot, \cdot \rangle_h$ defined as follows: for all functions $u,v \in L^2_h(M)$,

$$\langle u, v \rangle_h := \int_M u \bar{v} \omega_h. \hfill (2)$$

Let $\mathcal{F}_h$ be the horizontal foliation, $\mathcal{F}_{-h}$ be the vertical foliation for the holomorphic Abelian differential $h$ on $M$. The foliations $\mathcal{F}_h$ and $\mathcal{F}_{-h}$ are measured foliations (in the Thurston’s sense): $\mathcal{F}_h$ is the foliation given by the equation $\text{Im} h = 0$ endowed with the invariant transverse measure $|\text{Im} h|$, $\mathcal{F}_{-h}$ is the foliation given by the equation $\text{Re} h = 0$ endowed with the invariant transverse measure $|\text{Re} h|$. Since the metric $R_h$ is flat with trivial holonomy, there exist commuting vector fields $S_h$ and $T_h$ on $M \setminus \Sigma_h$ such that

1. The frame $\{S_h, T_h\}$ is a parallel orthonormal frame with respect to the metric $R_h$ for the restriction of the tangent bundle $TM$ to the complement $M \setminus \Sigma_h$ of the set of cone points;
2. the vector field $S_h$ is tangent to the horizontal foliation $\mathcal{F}_h$, the vector field $T_h$ is tangent to the vertical foliation $\mathcal{F}_{-h}$ on $M \setminus \Sigma_h$ [F97], [F07].

In the following we will often drop the dependence of the vector fields $S_h, T_h$ on the Abelian differential in order to simplify the notation. The symbols $\mathcal{L}_S$, $\mathcal{L}_T$ denote the Lie derivatives, and $i_S$, $i_T$ the contraction operators with respect to the vector field $S$, $T$ on $M \setminus \Sigma_h$. We have:

1. $\mathcal{L}_S \omega_h = \mathcal{L}_T \omega_h = 0$ on $M \setminus \Sigma_h$, that is, the area form $\omega_h$ is invariant with respect to the flows generated by $S$ and $T$;
2. $i_S \omega_h = \text{Re} h$ and $i_T \omega_h = \text{Im} h$, hence the 1-forms $\eta_S := i_S \omega_h$, $\eta_T := -i_T \omega_h$ are smooth and closed on $M$ and $\omega_h = \eta_T \wedge \eta_S$. 

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2. Analysis on translation surfaces

This section gathers basic results on the flat Laplacian of a translation surface, following [F97], §2 and §3, and [F07], §2.

The weighted differential $h$ has the form $h = \phi(z)dz$, where $\phi$ is a locally defined holomorphic function, and, consequently,

$$R_h = |\phi(z)|(dx^2 + dy^2)^{1/2}, \quad \omega_h = |\phi(z)|^2 \, dx \wedge dy.$$  \hfill (1)

The metric $R_h$ is flat, degenerate at the finite set $\Sigma_h$ of zeroes of $h$ and has trivial holonomy, hence $h$ induces a structure of translation surface on $M$.

The weighted $L^2$ space is the standard space $L^2_h(M) := L^2(M, \omega_h)$ with respect to the area element $\omega_h$ of the metric $R_h$. Hence the weighted $L^2$ norm $| \cdot |_0$ are induced by the hermitian product $\langle \cdot, \cdot \rangle_h$ defined as follows: for all functions $u,v \in L^2_h(M)$,

$$\langle u, v \rangle_h := \int_M u \bar{v} \omega_h. \hfill (2)$$

Let $\mathcal{F}_h$ be the horizontal foliation, $\mathcal{F}_{-h}$ be the vertical foliation for the holomorphic Abelian differential $h$ on $M$. The foliations $\mathcal{F}_h$ and $\mathcal{F}_{-h}$ are measured foliations (in the Thurston’s sense): $\mathcal{F}_h$ is the foliation given by the equation $\text{Im} h = 0$ endowed with the invariant transverse measure $|\text{Im} h|$, $\mathcal{F}_{-h}$ is the foliation given by the equation $\text{Re} h = 0$ endowed with the invariant transverse measure $|\text{Re} h|$. Since the metric $R_h$ is flat with trivial holonomy, there exist commuting vector fields $S_h$ and $T_h$ on $M \setminus \Sigma_h$ such that

1. The frame $\{S_h, T_h\}$ is a parallel orthonormal frame with respect to the metric $R_h$ for the restriction of the tangent bundle $TM$ to the complement $M \setminus \Sigma_h$ of the set of cone points;
2. the vector field $S_h$ is tangent to the horizontal foliation $\mathcal{F}_h$, the vector field $T_h$ is tangent to the vertical foliation $\mathcal{F}_{-h}$ on $M \setminus \Sigma_h$ [F97], [F07].

In the following we will often drop the dependence of the vector fields $S_h, T_h$ on the Abelian differential in order to simplify the notation. The symbols $\mathcal{L}_S$, $\mathcal{L}_T$ denote the Lie derivatives, and $i_S$, $i_T$ the contraction operators with respect to the vector field $S$, $T$ on $M \setminus \Sigma_h$. We have:

1. $\mathcal{L}_S \omega_h = \mathcal{L}_T \omega_h = 0$ on $M \setminus \Sigma_h$, that is, the area form $\omega_h$ is invariant with respect to the flows generated by $S$ and $T$;
2. $i_S \omega_h = \text{Re} h$ and $i_T \omega_h = \text{Im} h$, hence the 1-forms $\eta_S := i_S \omega_h$, $\eta_T := -i_T \omega_h$ are smooth and closed on $M$ and $\omega_h = \eta_T \wedge \eta_S$. 

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It follows from the area-preserving property (1) that the vector field \( S, T \) are anti-symmetric as densely defined operators on \( L^2_h(M) \), that is, for all functions \( u, v \in C_0^\infty(M \setminus \Sigma_h) \), (see \([F97], (2.5)\)),

\[
\langle Su, v \rangle_h = -\langle u, Sv \rangle_h , \quad \text{respectively} \quad \langle Tu, v \rangle_h = -\langle u, Tv \rangle_h .
\]

In fact, by Nelson’s criterion \([Nel59]\), Lemma 3.10, the anti-symmetric operators \( S, T \) commute as densely defined operators on \( L^2_h(M) \).

The weighted Sobolev norms \( | \cdot |_k \), with integer exponent \( k > 0 \), are the euclidean norms, introduced in \([F97]\), induced by the hermitian product defined as follows: for all functions \( u, v \in L^2_h(M) \),

\[
\langle u, v \rangle_k := \frac{1}{2} \sum_{i+j \leq k} \langle S^i T^j u, S^i T^j v \rangle_h + \langle T^i S^j u, T^i S^j v \rangle_h .
\]

The weighted Sobolev norms with integer exponent \( -k < 0 \) are defined to be the dual norms. The weighted Sobolev space \( H^k_h(M) \), with integer exponent \( k \in \mathbb{Z} \), is the Hilbert space obtained as the completion with respect to the norm \( | \cdot |_k \) of the maximal common invariant domain

\[
H^\infty_h(M) := \bigcap_{i,j \in \mathbb{N}} D(\tilde{S}^i \tilde{T}^j) \cap D(\tilde{T}^i \tilde{S}^j) .
\]

of the closures \( \tilde{S}, \tilde{T} \) of the essentially skew-adjoint operators \( S, T \) on \( L^2_h(M) \). The weighted Sobolev space \( H^{-k}_h(M) \) is isomorphic to the dual space of the Hilbert space \( H^k_h(M) \), for all \( k \in \mathbb{Z} \).

Since the vector fields \( S, T \) commute as operators on \( C_0^\infty(M \setminus \Sigma_h) \), the following weak commutation identity holds on \( M \).

**Lemma 2.1.** (\([F97], \text{Lemma 3.1})\) For all functions \( u, v \in H^1_h(M) \),

\[
\langle Su, Tv \rangle_h = \langle Tu, Sv \rangle_h .
\]

By the anti-symmetry property (3) and the commutativity property (6), the frame \( \{S, T\} \) yields an essentially skew-adjoint action of the Lie algebra \( \mathbb{R}^2 \) on the Hilbert space \( L^2_h(M) \) with common domain \( H^1_h(M) \).

If \( \Sigma_h \neq \emptyset \), the (flat) Riemannian manifold \( (M \setminus \Sigma_h, R_h) \) is not complete, hence its Laplacian \( \Delta_h \) is not essentially self-adjoint on \( C_0^\infty(M \setminus \Sigma_h) \). By a theorem of Nelson \([Nel59], \text{§9})\), this is equivalent to the non-integrability of the action of \( \mathbb{R}^2 \) as a Lie algebra (to an action of \( \mathbb{R}^2 \) as a Lie group).

Following \([F97]\), the Fourier analysis on the flat surface \((M, h)\) will be based on a canonical self-adjoint extension \( \Delta_h^C \) of the Laplacian \( \Delta_h \), called the Friedrichs extension, which is uniquely determined by the Dirichlet hermitian form \( \Omega : H^1_h(M) \times H^1_h(M) \to \mathbb{C} \). We recall that, for all \( u, v \in H^1_h(M) \),

\[
\Omega(u, v) := \langle Su, Sv \rangle_h + \langle Tu, Tv \rangle_h .
\]

**Theorem 2.2.** (\([F97], \text{Th. 2.3})\) The hermitian form \( \Omega \) on \( L^2_h(M) \) has the following spectral properties:
We then recall the definition of the Friedrichs (fractional) weighted Sobolev norms introduced in [F07], §2.2.

(1) \( \mathcal{Q} \) is positive semi-definite and the set \( \text{EV}(\mathcal{Q}) \) of its eigenvalues is a discrete subset of \([0, +\infty)\):

(2) Each eigenvalue has finite multiplicity, in particular \( 0 \in \text{EV}(\mathcal{Q}) \) is simple and the kernel of \( \mathcal{Q} \) consists only of constant functions;

(3) The space \( L^2_\mathcal{Q}(M) \) splits as the orthogonal sum of the eigenspaces. In addition, all eigenfunctions are \( C^\infty \) (real analytic) on \( M \).

The Weyl asymptotics holds for the eigenvalue spectrum of the Dirichlet form. For any \( \Lambda > 0 \), let \( N_h(\Lambda) := \text{card}\{\lambda \in \text{EV}(\mathcal{Q}) / \lambda \leq \Lambda\} \), where each eigenvalue \( \lambda \in \text{EV}(\mathcal{Q}) \) is counted according to its multiplicity.

**Theorem 2.3.** ([F97]), Th. 2.5 There exists a constant \( C > 0 \) such that

(8) \[ \lim_{\Lambda \to +\infty} \frac{N_h(\Lambda)}{\Lambda} = \text{vol}(M, R_h). \]

Let \( \partial_\pm := S_h \pm \imath T_h \) (with \( \imath = \sqrt{-1} \)) be the Cauchy-Riemann operators induced by the holomorphic Abelian differential \( h \) on \( M \), introduced in [F97], §3. Let \( \mathcal{M}^\pm_h \subset L^2_\mathcal{Q}(M) \) be the subspaces of meromorphic, respectively anti-meromorphic functions (with poles at \( \Sigma_h \)). By the Riemann-Roch theorem, the subspaces \( \mathcal{M}^\pm_h \) have the same complex dimension equal to the genus \( g \geq 1 \) of the Riemann surface \( M \). In addition, \( \mathcal{M}^+_h \cap \mathcal{M}^-_h = \mathbb{C} \), hence

(9) \[ H_h := (\mathcal{M}^+_h)^\perp \oplus (\mathcal{M}^-_h)^\perp = \{u \in L^2_\mathcal{Q}(M) \mid \int_M u \omega_h = 0\}. \]

Let \( H^1_h := H_h \cap L^2_\mathcal{Q}(M) \). By Theorem 2.2 the restriction of the hermitian form to \( H^1_h \) is positive definite, hence it induces a norm. By the Poincaré inequality (see [F97], Lemma 2.2 or [F02], Lemma 6.9), the Hilbert space \( (H^1_h, \mathcal{Q}) \) is isomorphic to the Hilbert space \( (H^1_h, \langle \cdot, \cdot \rangle_1) \).

**Proposition 2.4.** ([F97], Prop. 3.2) The Cauchy-Riemann operators \( \partial_\pm \) are closable operators on the common domain \( C^\infty_0(M \setminus \Sigma_h) \subset L^2_\mathcal{Q}(M) \) and their closures (denote by the same symbols) have the following properties:

1. the domains \( D(\partial_\pm) = H^1_h(M) \) and the kernels \( N(\partial_\pm) = \mathbb{C} \);
2. the ranges \( \overline{\text{Ran}(\partial_\pm)} := (\mathcal{M}^\pm_h)^\perp \) are closed in \( L^2_\mathcal{Q}(M) \);
3. the operators \( \partial_\pm : (H^1_h, \mathcal{Q}) \to (R^\pm, \langle \cdot, \cdot \rangle_h) \) are isometric.

Let \( \mathcal{E} = \{e_n \mid n \in \mathbb{N}\} \subset H^1_h(M) \cap C^\infty(M) \) be an orthonormal basis of the Hilbert space \( L^2_\mathcal{Q}(M) \) of eigenfunctions of the Dirichlet form (7) and let \( \lambda : \mathbb{N} \to \mathbb{R}^+ \cup \{0\} \) be the corresponding sequence of eigenvalues:

(10) \[ \lambda_n := \langle \mathcal{Q}(e_n, e_n), \quad \text{for each } n \in \mathbb{N}. \]

We then recall the definition of the Friedrichs (fractional) weighted Sobolev norms and spaces introduced in [F07], §2.2.
**Definition 2.5.**  
(i) The Friedrichs (fractional) weighted Sobolev norm \( \| \cdot \|_s \) of order \( s \geq 0 \) is the norm induced by the hermitian product defined as follows: for all \( u, v \in L^2_h(M) \),
\[
(u, v)_s := \sum_{n \in \mathbb{N}} (1 + \lambda_n)^s \langle u, e_n \rangle_h \langle e_n, v \rangle_h ;
\]
(ii) the Friedrichs weighted Sobolev space \( \tilde{H}^s_h(M) \) of order \( s \geq 0 \) is the Hilbert space
\[
\tilde{H}^s_h(M) := \{ u \in L^2_h(M) / \sum_{n \in \mathbb{N}} (1 + \lambda_n)^s |\langle u, e_n \rangle_h|^2 < +\infty \}
\]
endowed with the hermitian product given by (11);
(iii) the Friedrichs weighted Sobolev space \( \bar{H}^{-s}_h(M) \) of order \( -s < 0 \) is the dual space of the Hilbert space \( \tilde{H}^s_h(M) \).

As stated in [F07], Lemma 2.6, the family of Friedrichs (fractional) weighted Sobolev spaces is a holomorphic interpolation family in the sense of Lions-Magenes [LM], Chap. 1, endowed with the canonical interpolation norm.

The family \( \{H^s_h(M)\}_{s \in \mathbb{R}} \) of fractional weighted Sobolev spaces will be defined as follows. Let \( [s] \in \mathbb{N} \) denote the integer part and \( \{s\} \in [0, 1) \) the fractional part of any real number \( s \geq 0 \).

**Definition 2.6.** ([F07], Def. 2.7)  
(i) The fractional weighted Sobolev norm \( \| \cdot \|_s \) of order \( s \geq 0 \) is the euclidean norm induced by the hermitian product defined as follows: for all functions \( u, v \in H^\infty_h(M) \),
\[
(u, v)_s := \frac{1}{2} \sum_{i+j \leq |s|} \langle S^i T^j u, S^i T^j v \rangle_{|s|} + \langle T^i S^j u, T^i S^j v \rangle_{|s|} .
\]
(ii) The fractional weighted Sobolev norm \( \| \cdot \|_{-s} \) of order \( -s < 0 \) is defined as the dual norm of the weighted Sobolev norm \( \| \cdot \|_s \).
(iii) The fractional weighted Sobolev space \( H^s_h(M) \) of order \( s \in \mathbb{R} \) is defined as the completion with respect to the norm \( \| \cdot \|_s \) of the maximal common invariant domain \( H^\infty_h(M) \).

It can be proved that the weighted Sobolev space \( H^{-s}_h(M) \) is isomorphic to the dual space of the Hilbert space \( H^s_h(M) \), for all \( s \in \mathbb{R} \).

The definition of the fractional weighted Sobolev norms is motivated by the following basic result.

**Lemma 2.7.** ([F07], Lemma 2.9) For all \( s \geq 0 \), the restrictions of the Cauchy-Riemann operators \( \partial^\pm_h : H^s_h(M) \to L^2_h(M) \) to the subspaces \( H^{s+1}_h(M) \subset H^s_h(M) \) yield bounded operators
\[\partial^\pm_s : H^{s+1}_h(M) \to H^s_h(M)\]
(which do not extend to operators \( H^{s+1}_h(M) \to \tilde{H}^s_h(M) \) unless \( M \) is the torus). On the other hand, the Laplace operator
\[\Delta_h = \partial^+_h \partial^-_h = \partial^-_h \partial^+_h : H^2_h(M) \to L^2_h(M)\]
yields a bounded operator \( \tilde{\Delta}_k : \tilde{H}^{s+2}_h(M) \rightarrow \tilde{H}^{s}_h(M) \), defined as the restriction of the Friedrichs extension \( \Delta^F_h : \tilde{H}^{2}_h(M) \rightarrow L^2_h(M) \).

We do not know whether the fractional weighted Sobolev spaces form a holomorphic interpolation family. However, the fractional weighted Sobolev norms do satisfy interpolation inequalities (see [F07], Lemma 2.10 and Corollary 2.26). A detailed comparison between Friedrichs weighted Sobolev norms and weighted Sobolev norms and the corresponding weighted Sobolev spaces is carried out in [F07], §2. In particular, we have the following result.

Let \( H^{s}(M), \ s \in \mathbb{R} \), denote a family of standard Sobolev spaces on the compact manifold \( M \) (defined with respect to a Riemannian metric).

**Lemma 2.8.** ([F07], Lemma 2.11) The following continuous embedding and isomorphisms of Banach spaces hold:

1. \( H^s(M) \subset \tilde{H}^s_q(M) \equiv \tilde{H}^s_q(M), \) for \( 0 \leq s < 1 \);
2. \( H^s(M) \equiv \tilde{H}^s_q(M) \equiv \tilde{H}^s_q(M), \) for \( s = 1 \);
3. \( \tilde{H}^s_q(M) \subset \tilde{H}^s_q(M) \subset H^s(M), \) for \( s > 1 \).

For \( s \in [0,1] \), the space \( H^s(M) \) is dense in \( \tilde{H}^s_q(M) \) and, for \( s > 1 \), the closure of \( \tilde{H}^s_q(M) \) in \( \tilde{H}^s_q(M) \) or \( H^s(M) \) has finite codimension.

We also have the following a sharp version of Lemma 4.2 of [F07]:

**Theorem 2.9.** ([F07], Lemma 2.5 and Corollary 2.25) For each \( k \in \mathbb{Z}^+ \) there exists a constant \( C_k > 1 \) such that, for any holomorphic Abelian differential \( h \) on \( M \) and for all \( u \in \tilde{H}^k_q(M) \),

\[
C_k^{-1} |u|_k \leq \|u\|_k \leq C_k |u|_k.
\]

For any \( 0 < r < s \) there exists constants \( C_r > 0 \) and \( C_{rs} > 0 \) such that, for all \( u \in \tilde{H}^r_q(M) \), the following inequalities hold:

\[
C_r^{-1} \|u\|_r \leq |u|_r \leq C_{rs} \|u\|_s.
\]

## 3. The Twisted Beurling Transform

For every \( \sigma \in \mathbb{R} \) and for every Abelian differential \( h \), we introduce a family of partial isometries \( U_{h,\sigma} \) (of Beurling transform type), defined on a finite codimensional subspace of \( L^2_h(M) := L^2(M, \omega_h) \), which generalized the partial isometry \( U_q = U_{h,0} \) (for \( q = h^2 \)) first introduced in [F97], §3, in the study of the cohomological equation for translation flows.

The partial isometry \( U_{h,\sigma} \) is extended in an arbitrary way to a unitary operator \( U_{f,\sigma} \) on the whole space \( L^2_h(M) \). Resolvent estimates for \( U_{f,\sigma} \) will appear to be related with a priori estimates for the twisted cohomological equations for translation flows on \( (M,h) \). Consequently, we derive our results on twisted cohomological equation from basic estimates on the limiting behavior as \( z \rightarrow \partial D \) of the resolvent \( \mathcal{R}_U(z) := (U - z)^{-1} \), defined on the unit disc \( D \subset \mathbb{C} \), of a unitary operator \( U \) on a general Hilbert space. Such estimates, established in [F97], are based on fundamental facts of classical harmonic analysis, in particular of Fatou’s theory on
the boundary behavior of holomorphic functions. The results obtained are then specialized to the case of the unitary operator \( U := U_{1, \sigma} \).

Let \( h \) be a holomorphic Abelian differential on a Riemann surface \( M \) of genus \( g \geq 2 \). Let \( \{ S, T \} \) be the orthonormal frame for \( TM \setminus \Sigma \) introduced in §2. We recall that the 1-forms \( \eta_S = i_S \omega_h \) and \( \eta_T = i_T \omega_h \) are closed and describe the horizontal, resp. vertical, foliation of \( \omega \) on \( M \). It is possible to associate to \( h \) a one-parameter family of measured foliations parametrized by \( \theta \in \mathbb{T} := \mathbb{R}/2\pi \mathbb{Z} \) in the following way: let \( h_\theta := e^{-i\theta} h \) and let \( \mathcal{F}_\theta \) be the horizontal foliation of the Abelian differential \( h_\theta \), i.e. the foliation defined by the closed 1-form

\[
\operatorname{Im} h_\theta = \{ e^{-i\theta} (\eta_T + i\eta_S) - e^{i\theta} (\eta_T - i\eta_S) \} / 2i .
\]

The foliation \( \mathcal{F}_\theta \) can also be obtained by integrating the dual vector field

\[
S_\theta := (\cos \theta) S + (\sin \theta) T = \{ e^{-i\theta} (S + iT) + e^{i\theta} (S - iT) \} / 2 ,
\]

which corresponds to the rotation of the vector field \( S \) by an angle \( \theta \in \mathbb{T} \) in the positive direction.

In the following we will denote by \( \partial^{\pm}_{h, \sigma} \) the Cauchy-Riemann operators \( S \pm iT \) respectively. The twisted Cauchy-Riemann operators

\[
\partial^{\pm}_{h, \sigma} := (S + i\sigma) \pm iT = \partial^{\pm}_h + i\sigma ,
\]

will play a crucial role. For all \( \theta \in \mathbb{T} \), let \( \sigma_\theta := \sigma \cos \theta \). We remark that, for every \( \theta \in \mathbb{T} \), we have

\[
S_\theta + i\sigma_\theta = \{ e^{-i\theta} \partial^{\pm}_{h, \sigma} + e^{i\theta} \partial^{\pm}_{h, \sigma} \} / 2 ,
\]

hence we have the formal factorization

\[
S_\theta + i\sigma_\theta = \frac{e^{-i\theta}}{2} \left( (\partial^{\pm}_{h, \sigma}) (\partial^{\mp}_{h, \sigma})^{-1} + e^{2i\theta} \right) \partial^{\mp}_{h, \sigma}
\]

\[
= \frac{e^{i\theta}}{2} \left( (\partial^{\mp}_{h, \sigma}) (\partial^{\pm}_{h, \sigma})^{-1} + e^{-2i\theta} \right) \partial^{\pm}_{h, \sigma} .
\]

Let \( Q_{h, \sigma} \) denote the bilinear form defined, for all \( u, v \in H^1_h(M) \), as follows:

\[
Q_{h, \sigma}(u, v) := \langle (S + i\sigma) u, (S + i\sigma) v \rangle_h + \langle Tu, Tv \rangle_h .
\]

Let \( K_{h, \sigma} \subset H^1_h(M) \cap C^\infty(M \setminus \Sigma) \) denote the finite-dimensional subspace

\[
K_{h, \sigma} := \{ u \in H^1_h(M) \cap C^\infty(M \setminus \Sigma) | (S + i\sigma) u = Tu = 0 \} .
\]

**Lemma 3.1.** The twisted bilinear form \( Q_{h, \sigma} \) induces a norm on \( K_{h, \sigma} \). In fact, for all \( \sigma \in \mathbb{R} \), there exists a constant \( C_h > 1 \) such that, for all \( u \in K_{h, \sigma} \)

\[
C_h^{-1} Q_{h, 0}(u, u) \leq Q_{h, \sigma}(u, u) \leq C_h (1 + \sigma^2) \left( Q_{h, 0}(u, u) + | \int_M u \omega_h | \right) .
\]

**Proof.** Since translation flows are area-preserving, hence symmetric on their common domain, we have, for all \( u \in H^1_h(M) \),

\[
Q_{h, \sigma}(u, u) := \langle (S + i\sigma) u, (S + i\sigma) u \rangle_h + \|Tu\|_{L^2_h(M)}^2
\]

\[
= \|Su\|_{L^2_h(M)}^2 + 2i\sigma \langle u, Su \rangle_h + \sigma^2 \| u \|_{L^2_h(M)}^2 + \|Tu\|_{L^2_h(M)}^2 .
\]
By the Cauchy-Schwarz inequality, we have

\[ |\langle u, Su \rangle_h| \leq \|u\|_{L^2_h(M)} \|Su\|_{L^2_h(M)} \leq \frac{\|Su\|_{L^2_h(M)} + \|u\|_{L^2_h(M)}^2}{2}. \]

It follows that

\[ \|Su\|_{L^2_h(M)}^2 + 2\sigma \langle u, Su \rangle_h \leq (1 + |\sigma|)\|Su\|_{L^2_h(M)}^2 + |\sigma|\|u\|_{L^2_h(M)}^2, \]

hence we derive that

\[ Q_{h,\sigma}(u, u) \leq (1 + |\sigma|)Q_{h,0}(u, u) + (\sigma^2 + |\sigma|)\|u\|_{L^2_h(M)}^2. \]

By the Poincaré inequality there exists a constant \(C_h > 0\) such that, for all \(u \in H^1_h(M)\), we have

\[ Q_{h,\sigma}(u, u) \leq (1 + |\sigma|)Q_{h,0}(u, u) + (\sigma^2 + |\sigma|)\|u\|_{L^2_h(M)}^2 \leq [(1 + |\sigma|) + C(\sigma^2 + |\sigma|)]Q_{h,0}(u, u) + (\sigma^2 + |\sigma|)\int_M \omega_h. \]

The upper bound in the statement is therefore proved.

To prove the lower bound, we proceed as follows. By the definition of \(Q_{h,\sigma}\), for the splitting \(u = v + \bar{u} \in \mathbb{C}^+ \oplus \mathbb{C} \subset H^1_h(M)\) we have

(21) \[ Q_{h,\sigma}(u, u) = Q_{h,\sigma}(v, v) + \sigma^2 \bar{u}^2, \]

hence without loss of generality we can reduce the argument to functions \(u \in H^1_h(M)\) of zero average. By the compact embedding \(H^1_h(M) \rightarrow L^2(M)\) we derive that there exists a constant \(c_h > 0\) such that, for all \(u \in H^1_h(M)\) we have

\[ \|u\|_{L^2_h(M)}^2 \geq c_h^2 Q_{h,0}(u, u) \geq c_h^2 \|Su\|_{L^2_h(M)}^2. \]

It follows then by formula (20) that, for \(|\sigma| \geq 2c_h^{-1}\) we have

\[ Q_{h,\sigma}(u, u) \geq Q_{h,0}(u, u) + |\sigma|\|u\|_{L^2_h(M)}^2 (|\sigma|\|u\|_{L^2_h(M)} - 2\|Su\|_{L^2_h(M)}) \geq Q_{h,0}(u, u). \]

It remains to prove the bound for \(|\sigma| \leq 2c_h^{-1}\). Let us then assume by contradiction that for all \(n \in \mathbb{N}\) there exist a bounded sequence \((\sigma_n)\) and a sequence \(u_n \in K^\perp_{h,\sigma_n} \subset H^1_h(M)\) of zero average such that

\[ Q_{h,0}(u_n, u_n) \geq nQ_{h,\sigma_n}(u_n, u_n). \]

After normalizing, it is not restrictive to assume that \(Q_{h,0}(u_n, u_n) = 1\), for all \(n \in \mathbb{N}\), hence \(Q_{h,\sigma_n}(u_n, u_n) \rightarrow 0\). By the Poincaré inequality, it follows that after passing to a subsequence we can assume that \(u_n \rightarrow u\) in \(L^2(M)\) and \(u\) has zero average, as well as that \(\sigma_n \rightarrow \sigma \in \mathbb{R}\). Let \(\Phi^\perp_S\) and \(\Phi^\perp_R\) denote, respectively, the horizontal and
the vertical flow. By assumption, since
\[
\|e^{\sigma t}u_n \circ \Phi_S^t - u_n\|_{L^2_h(M)} = \| \int_0^t (S + \iota \sigma_n) u_n \circ \Phi_S^s \, ds \|_{L^2_h(M)} \\
\leq \int_0^t \| (S + \iota \sigma_n) u_n \circ \Phi_S^s \|_{L^2_h(M)} \, ds \leq t Q_{h,\sigma_n}^{1/2}(u_n, u_n) \to 0
\]
\[
\|u_n \circ \Phi_T^t - u_n\|_{L^2_h(M)} = \| \int_0^t T u_n \circ \Phi_T^s \, ds \|_{L^2_h(M)} \\
\leq \int_0^t \| T u_n \circ \Phi_T^s \|_{L^2_h(M)} \, ds \leq t Q_{h,\sigma_n}^{1/2}(u_n, u_n) \to 0,
\]
it follows that the limit function \( u \in K_{h,\sigma}^+ \subset L^2_h(M) \) is a zero-average eigenfunction of eigenvalue \(-\iota \sigma\) for the flow \( \Phi_S^t \) and it is invariant for the flow \( \Phi_T^t \). It follows in particular that \( u \in C^\infty(M \setminus \Sigma) \cap H^1_h(M) \) which implies that \( u \) is constant on all minimal components of the flow \( \Phi_T^t \) and \( \Phi_T^t \)-invariant on cylindrical component. In particular \( u \in K_{h,\sigma} \), hence \( u = 0 \). However, from \( u_n \to 0 \) in \( L^2_h(M) \) and \( \sigma_n \to \sigma \), from the identity in formula (20) we then derive
\[
0 = \lim_{n \to \infty} Q_{h,\sigma_n}(u_n, u_n) = \lim_{n \to \infty} Q_{h,0}(u_n, u_n) = 1,
\]
a contradiction. We have thus proved that there exists \( C_h > 1 \) such that, for all \( u \in K_{h,\sigma}^+ \cap H^1_h(M) \),
\[
C_h^{-1} Q_{h,0}(u, u) \leq Q_{h,\sigma}(u, u).
\]
\[\square\]

The twisted Cauchy-Riemann operators \( \partial_{h,\sigma}^{\pm} \) on \( L^2_h(M) \) will be described in the following Proposition.

**Proposition 3.2.** The Cauchy-Riemann operators \( \partial_{h,\sigma}^{\pm} \) are closable operators on \( C^\infty(M \setminus \Sigma) \subset L^2_h(M) \) and their closures (denoted by the same symbols) have the following properties:

(i) \( D(\partial_{h,\sigma}^{\pm}) = H^1_h(M) \) and \( N(\partial_{h,\sigma}^{\pm}) = K_{h,\sigma} \subset H^1_h(M) \).

(ii) The kernels \( M_{\Sigma}^{\pm}(\sigma) \subset L^2_h(M) \) of the adjoint operators \( (\partial_{h,\sigma}^{\pm})^* \) have finite dimensions \( d^{\pm}(\sigma) \) and there exists \( d_h \in \mathbb{N} \) such that
\[
d^{+}(\sigma) = d^{-}(-\sigma) = d_h, \quad \text{for all } \sigma \in \mathbb{R}.
\]

(iii) the adjoints \( (\partial_{h,\sigma}^{\pm})^* \) of \( \partial_{h,\sigma}^{\pm} \) are extensions of \( -\partial_{h,\sigma}^{\mp} \), and we have closed ranges
\[
R_{h,\sigma}^{\pm} := \text{Ran} \left( \partial_{h,\sigma}^{\pm} \right) = \left[ M_{\Sigma}^{\pm}(\sigma) \right]^{\perp}.
\]

(iv) The operators \( \partial_{h,\sigma}^{\pm} : (K_{h,\sigma} \cap H^1_h(M), Q_{h,\sigma}) \to (R_{h,\sigma}^{\pm}, (\cdot, \cdot)_h) \) are isometric.

**Proof.** If \( u, v \in H^1_h(M) \), Lemma 2.1 implies the following identity:
\[
\langle \partial_{h,\sigma}^{\pm} u, \partial_{h,\sigma}^{\mp} v \rangle_h = \langle (S + \iota \sigma) u, (S + \iota \sigma) v \rangle_h + \langle T u, T v \rangle_h \\
\pm \iota \left( \langle T u, (S + \iota \sigma) v \rangle_h - \langle (S + \iota \sigma) u, T v \rangle_h \right) = Q_{h,\sigma}(u, v).
\]
It follows immediately that the operators $\partial^\pm_{h,\sigma}$ are closed with domain $D(\partial^\pm_{h,\sigma}) = H^1_h(M)$ and that their kernels are both equal to $K_{h,\sigma} \subset H^1_h(M)$.

Since, for all $u, v \in H^1_h(M)$, we have

$$\langle \partial^\pm_{h,\sigma} u, v \rangle_h = \langle [(S+i\sigma) \pm iT]u, v \rangle_h = -\langle u, [(S+i\sigma) \mp iT]v \rangle_h = -\langle u, \partial^\mp_{h,\sigma} v \rangle_h,$$

the adjoint $(\partial^\pm_{h,\sigma})^*$ of $\partial^\pm_{h,\sigma}$ is an extension of $-\partial^\mp_{h,\sigma}$. By the theory of elliptic partial differential equations, the distributional kernels $M^\pm_h(\sigma)$ of $\partial^\pm_{h,\sigma}$ in $L^2_h(M)$ are finite dimensional subspaces of $C^\infty(M \setminus \Sigma)$, which depend continuously on $\sigma \in \mathbb{R}$, hence have constant dimension. Since by complex conjugation we have $\overline{\partial^\pm_{h,\sigma}} = \partial^\mp_{h,-\sigma}$, it follows that $M^\pm_h(\sigma) = M^\mp_h(-\sigma)$, hence the dimension $d^\pm(\sigma) = d^-(\sigma)$ is constant over $\sigma \in \mathbb{R}$.

The formula for the range $R^\pm_{h,\sigma}$ follows from a general fact of Hilbert space theory, as soon as we have proved that the range is closed. It follows from Lemma 3.1 that $R^\pm_{h,\sigma}$ are closed. In fact, the subspaces $R^\pm_{h,\sigma}$ coincide with the range of restrictions of the operators $\partial^\pm_{h,\sigma}$ to the subspace $K_{h,\sigma} \cap H^1_h(M)$. By Lemma 3.1 these restrictions have closed range.

Finally, $(iv)$ is a direct consequence of the identity in formula (22). \[ \square \]

The results just proved in Proposition 3.2 in particular $(iv)$, allow us to give a precise meaning to the formal factorization (18), by introducing a family of unitary operators on $L^2_h(M)$, which, as it will be seen, contains a great deal of information about the properties of the differential operator $S_\theta + i\sigma (\theta \in \mathbb{T})$ defined in formula (17). Let $U_{h,\sigma} : R^-_{h,\sigma} \rightarrow R^+_{h,\sigma}$ be defined as

$$U_{h,\sigma} := (\partial^+_{h,\sigma})(\partial^-_{h,\sigma})^{-1}.$$  \hspace{1cm} (23)

It is an immediate consequence of the assertion $(iv)$ of Proposition 3.2 that $U_{h,\sigma}$ is a partial isometry. Thus, we extend in the natural way the domain of definition of $U_{h,\sigma}$ as follows. Let

$$J : M^+_h(\sigma) \rightarrow M^-_h(\sigma)$$  \hspace{1cm} (24)

be an isometric operator, with respect to the euclidean structures induced on $M^+_h(\sigma)$ and $M^-_h(\sigma)$ by the Hilbert space $L^2_h(M)$. The existence of $J$ is a consequence of the fact that the deficiency subspaces $M^+_h(\sigma)$ and $M^-_h(\sigma)$ are isomorphic finite dimensional vector spaces of the same complex dimension (equal to the genus $g$ of the surface $M$). In fact, there exists a whole family of operators $J$ as required, parametrized by the Lie group $U(g, \mathbb{C})$. Let $\pi^-_{h,\sigma} : L^2_h(M) \rightarrow R^-_{h,\sigma}$ be the orthogonal projections. We recall that $R^\pm_{h,\sigma}$ are the orthogonal complements of $M^\pm_h(\sigma)$ respectively (Proposition 3.2). Once an isometric operator $J$ as in formula (24) is fixed, the partial isometry $U_{h,\sigma}$, associated with the holomorphic Abelian differential $h$ on $M$ and $\sigma \in \mathbb{R}$ as in formula (23), will be extended to a unitary operator $U_{J,\sigma}$ on the whole $L^2_h(M)$ by the formula

$$U_{J,\sigma}(u) := U_{h,\sigma} \pi^-_{h,\sigma}(u) + J(I - \pi^-_{h,\sigma})(u), \quad \text{for all} \ u \in L^2_h(M)$$
A priori estimates for \( W \) we recall below general results on the boundary behavior of Cauchy integrals of valued (the dependence of the unitary operator \( U_{J,\sigma} \) on the Abelian differential is omitted in the notation for convenience).

The following version of the formal identities \([13]\) holds on \( H^1_h(M) \):

\[
S_\theta + i\sigma_\theta = \frac{e^{-i\theta}}{2} (U_{J,\sigma} + e^{2i\theta}) \partial_{h,\sigma}^- = \frac{e^{i\theta}}{2} (U_{J,\sigma}^{-1} + e^{-2i\theta}) \partial_{h,\sigma}^+ .
\]

A priori estimates for \( S_\theta \) are related by formula \((25)\) to estimates for the resolvent \( R_{J,\sigma}(z) := (U_{J,\sigma} - zI)^{-1} \) of any of the operators \( U_{J,\sigma} \), as \( z \to \mathbb{T} \) non-tangentially. Since these are unitary operators, their spectrum is contained in the unit circle \( \{ z \in \mathbb{C} \mid |z| = 1 \} \). As a consequence, the resolvent \( R_{J,\sigma}(z) \) is a well defined operator-valued holomorphic function on the unit disk \( D := \{ z \in \mathbb{C} \mid |z| < 1 \} \). In addition, by the spectral theorem for unitary operators, it is given by a Cauchy integral on \( \partial D \) of the spectral measure associated with \( U_{J,\sigma} \).

\[ \text{4. Spectral Theory of Unitary Operators} \]

Let \( U : \mathcal{H} \to \mathcal{H} \) is any unitary operator on a (separable) Hilbert space \( \mathcal{H} \). By the spectral theorem \([Yo]\), XI.4, its resolvent \( R_U(z) := (U - zI)^{-1} \) can be represented as a Cauchy integral of the spectral family, as follows. For any \( u, v \in \mathcal{H} \),

\[
(\mathcal{R}_U(z)u, v)_{\mathcal{H}} = \int_0^{2\pi} (z - e^{it})^{-1} d(E_U(t)u, v)_{\mathcal{H}} , \quad \text{for all } z \in D ,
\]

where \((\cdot, \cdot)_{\mathcal{H}}\) denotes the inner product in \( \mathcal{H} \) and \( \{E_U(t)\}_{0 \leq t \leq 2\pi} \) denotes the spectral family associated with the unitary operator \( U \) on \( \mathcal{H} \). Our approach from \([F97]\) is based on the fundamental property of holomorphic functions on \( D \), which can be represented as Cauchy integrals on \( \partial D \) of complex measures, of having non-tangential boundary values almost everywhere.

We recall below general results on the boundary behavior of Cauchy integrals of complex measures. The modern theory of these singular integrals, based on the Lebesgue integral, was initiated by P. Fatou in his thesis \([F1]\). The results gathered below were originally obtained by F. Riesz \([Rz]\), V. Smirnov \([Sm]\), G. H. Hardy and J. E. Littlewood \([HL]\). The arguments, given in \([F97]\), §3, follow the approach of A. Zygmund \([Zy]\), VII.9, based on real variables methods, and is taken from the books of W. Rudin \([Rd]\) and E. M. Stein and G. Weiss \([SW]\).

Let \( \mu \) be a complex Borel measure (of finite total mass) on \( \partial D \). The Cauchy integral of \( \mu \) is the holomorphic function \( I_\mu \) on \( D \) defined as

\[
I_\mu(z) := \int_0^{2\pi} (z - e^{it})^{-1} d\mu(t) , \quad \text{for all } z \in D .
\]

**Lemma 4.1.** \([F97], \text{Lemma } 3.3 \text{ A}) \text{ The non-tangential limit}

\[
I_\mu(z) \to I_\mu^*(\theta) , \quad \text{as } z \to e^{i\theta} ,
\]

exists almost everywhere with respect to the (normalized) Lebesgue measure \( L \) on the circle \( \mathbb{T} \). In addition, there exists a constant \( C > 0 \) such that the following weak type estimate holds:

\[
L\{ \theta \in \mathbb{T} \mid |I_\mu^*(\theta)| > t \} \leq \frac{C}{t} \| \mu \| , \quad \text{for all } t > 0
\]
where \( \| \mu \| \) denotes the total mass of the measure \( \mu \).

Lemma 4.1 is not enough for our purposes, since it gives no information concerning
the behavior of Cauchy integrals (26) as the convergence to the limiting boundary
values takes place. The necessary estimates are given below, following \[F97\], in
terms of non-tangential maximal functions, the definition of which we recall below
following \[Rd\], §11.18.

For \( 0 < \alpha < 1 \), we define the non-tangential approach region \( \Omega_\alpha \) to be the cone
over \( D(0, \alpha) \) of vertex \( z = 1 \), that is, the union of the disk \( D(0, \alpha) \) and the line
segments from the point \( z = 1 \) to the points of \( \Omega_\alpha \). Rotated copies of \( \Omega_\alpha \), having
vertex at \( e^{i\theta} \), will be denoted by \( \Omega_\alpha(\theta) \).

For any complex function \( \Phi \) on the unit disk \( D \) and \( 0 < \alpha < 1 \), its non-tangential
maximal function \( N_\alpha(\Phi) \) is defined on \( \mathbb{T} \) as

\[
N_\alpha(\Phi)(\theta) := \sup\{ |\Phi(z)| : z \in \Omega_\alpha(\theta) \}.
\]

We would like to complete Lemma 4.1 with estimates on the non-tangential max-
imal function \( N_\alpha(I_\mu) \) of the Cauchy integral in formula (26). This can be
accomplished by a standard argument of basic Hardy space theory.

For the convenience of the reader, we will recall the definition of Hardy spaces
\( H^p(D) \) on the unit disk \( D \) \[Rd\], §§17.6-7. Let \( \Phi \) be a complex function on \( D \). For
\( 0 < r < 1 \), we define the functions \( \Phi_r \) on \( \mathbb{T} \) by the formula

\[
\Phi_r(\theta) := \Phi(re^{i\theta}), \quad \text{for all } \theta \in \mathbb{T},
\]

and, for \( 0 < p \leq \infty \), we define

\[
\| \Phi \|_p = \sup\{ |\Phi_r|_p : 0 \leq r < 1 \},
\]

where \( | \cdot |_p \) denotes the \( L^p \) norm on \( \mathbb{T} \) with respect to the Lebesgue measure. The
Hardy space \( H^p(D) \) is defined to be the space of holomorphic functions \( \Phi \) on the
unit disk \( D \) such that \( \| \Phi \|_p < \infty \).

**Lemma 4.2.** \([F97]\), Lemma 3.3B) The holomorphic function \( I_\mu \), given as a Cauchy
integral (see formula (26)) of a Borel complex measure \( \mu \) on \( \partial D \), belongs to the
Hardy spaces \( H^p(D) \), for any \( 0 < p < 1 \). (Consequently, it admits non-tangential
limit almost everywhere on \( \partial D \).) In addition, its non-tangential maximal function
\( N_\alpha(I_\mu) \) belongs to \( L^p(\mathbb{T}, \mathcal{L}) \), for any \( 0 < p < 1 \) and for all \( \alpha < 1 \), and there exist
constants \( A_\alpha, A_{\alpha,p} > 0 \), with \( A_{\alpha,p} \to \infty \) as \( p \to 1 \), such that the following estimates
hold:

\[
|N_\alpha(I_\mu)|_p \leq A_\alpha \| I_\mu \|_p \leq A_{\alpha,p} \| \mu \|,
\]

where \( \| \mu \| \) denotes the total mass of the measure \( \mu \).

The general harmonic analysis Lemmas 4.1 and 4.2 are then applied via the spectral
theorem to the resolvent of an arbitrary unitary operator on a Hilbert space. The
abstract Hilbert space result which we obtain in this way will then be applied to the
unitary operators \( U_{f,\sigma}, U_{f,\sigma}^{-1} \) introduced in §3

**Corollary 4.3.** \([F97]\), Corollary 3.4) Let \( \mathcal{R}_u(z) : \mathcal{H} \to \mathcal{H} \), \( z \in D \), denote the re-
solvent of a unitary operator \( U : \mathcal{H} \to \mathcal{H} \) on a Hilbert space \( \mathcal{H} \). Then, for any \( u \),
v ∈ ℍ, the holomorphic functions Φ(u,v)(·) := (ℜ(·)u,v)_{3ξ} belong to the Hardy spaces \( H^p(D) \), for any \( 0 < p < 1 \). Consequently, they admit non-tangential limit almost everywhere on \( \partial D \). Furthermore, their non-tangential maximal functions \( N_α(u,v) \) belong to \( L^p(\mathbb{T}, \mathcal{L}) \), for any \( 0 < p < 1 \) and for all \( α < 1 \), and there exist constants \( A_α, A_{α,p} > 0 \), with \( A_{α,p} \to \infty \) as \( p \to 1 \), such that the following estimates hold:

\[
|N_α(u,v)|_p \leq A_α\|\Phi(u,v)\|_p \leq A_{α,p}\|u\|_{3ξ}\|v\|_{3ξ},
\]

where \( \| \cdot \|_{3ξ} \) denotes the Hilbert space norm.

5. Solutions of the Twisted Cohomological Equation

In this section we adapt to the twisted cohomological equation the streamlined version [F07] of the main argument of [F97] (Theorem 4.1) given with the goal of establishing the sharpest bound on the loss of Sobolev regularity within the reach of the methods of [F97].

5.1. Distributional solutions. We derive results on distributional solutions of the twisted cohomological equation from the harmonic analysis results of §4 about the boundary behavior of the resolvent of a unitary operator.

**Definition 5.1.** Let \( h \) be an Abelian differential and let \( σ \in \mathbb{R} \). A distribution \( u ∈ \tilde{H}^{-s}_h(M) \) will be called a (distributional) solution of the cohomological equation \((S + iσ)u = f\) for a given function \( f ∈ L^κ_s(M) \) if

\[
\langle u, (S + iσ)v \rangle = -\langle f, v \rangle, \quad \text{for all } v ∈ H^{κ+1}_h(M) \cap \tilde{H}^s(M).
\]

Let \( h_θ := e^{-iθ}h \) be its rotation and let \( σ_θ := σ \cos θ \). Let \( \{S_θ\} \) denote the one-parameter family of rotated vector fields introduced in formula (17):

\[
S_θ := \{e^{-iθ}(S + iT) + e^iθ(S - iT)\}/2.
\]

For all \( s ∈ \mathbb{R} \), let \( \mathcal{C}_s^0(M) \subset H^s_h(M) \) the subspace of distributions vanishing on constant functions.

**Theorem 5.2.** Let \( h \) be an Abelian differential on \( M \) with minimal vertical foliation. Let \( r > 2 \) and \( p ∈ (0,1) \) be such that \( rp > 2 \). For any \( σ ∈ \mathbb{R} \), there exists a bounded linear operator

\[
\mathcal{U}_σ : \mathcal{C}_s^{-1}(M) → L^p(\mathbb{T}, \tilde{H}^{-r}_h(M))
\]

such that the following holds. For any \( σ ∈ \mathbb{R} \) and any \( f ∈ H^{-1}(M) \) there exists a full measure subset \( \mathcal{F}_r(σ, f) \subset \mathbb{T} \) such that \( u := \mathcal{U}_σ(f)(θ) \in \tilde{H}^{-r}_h(M) \) is a distributional solution of the cohomological equation \((S_θ + iσ_θ)u = f\), for all \( θ ∈ \mathcal{F}_r(f, σ) \). In addition, there exists a constant \( B_h := B_h(p,r) > 0 \) such that, for all \( f ∈ \mathcal{C}^{-1}_s(M) \), vanishing on constant functions,

\[
\|\mathcal{U}_σ(f)\|_p := \left( \int_\mathbb{T} \|\mathcal{U}_σ(f)(θ)\|_{L^p_r}^p \, dθ \right)^{1/p} \leq B_h\|f\|_{-1}.
\]

The above theorem is a consequence of the following estimate:
Lemma 5.3. Let $h$ be an Abelian differential on $M$ with minimal vertical foliation. Let $r > 2$ and $p \in (0, 1)$ be such that $rp > 2$. For any $\sigma \in \mathbb{R}$ and any $f \in H_r^0(M)$, vanishing on constant functions, there exists a measurable function $A_{h, \sigma}(f) := A_{h, \sigma}(f, p, r) \in L^p(\mathbb{T}, \mathcal{L})$ such that, for all $v \in H^r_0(M)$ we have
\begin{equation}
|\langle f, v \rangle| \leq A_{h, \sigma}(f, \theta) \| (S_{\theta} + i \sigma \theta) v \|_r.
\end{equation}

In addition, the following bound for the $L^p$ norm of the function $A_{h, \sigma}(f)$ holds. There exists a constant $B_h := B_h(p, r) > 0$ such that, for every $\sigma \in \mathbb{R}$ and for every $f \in H^r_0(M)$, vanishing on constant functions, we have
\begin{equation}
|A_{h, \sigma}(f)|_p \leq B_h \| f \|_{-1}.
\end{equation}

Proof. We recall the formulas (25):
\begin{equation}
S_{\theta} + i \sigma \theta = \frac{e^{-i \theta}}{2} \left( U_{1, \sigma} + e^{2i \theta} \right) \partial_{h, \sigma}^- = \frac{e^{i \theta}}{2} \left( U_{1, \sigma} + e^{-2i \theta} \right) \partial_{h, \sigma}^+.
\end{equation}
The proof of estimate (28) is going to be based on properties of the resolvent of the operator $U_{1, \sigma}$. In fact, the proof of (28) is based on the results, summarized in §4, concerning the non-tangential boundary behavior of the resolvent of a unitary operator on a Hilbert space, applied to the operators $U_{1, \sigma}$, $U_{1, \sigma}^{-1}$ on $L^2(M)$. The Fourier analysis of [F97], §2, also plays a relevant role through Lemma 2.9 and the Weyl’s asymptotic formula (Theorem 2.3).

Since the vertical foliation of $h$ is minimal, it follows that all $T$-invariant functions in the space $H^1_0(M)$ are constant, hence the common kernel of the twisted Cauchy-Riemann operators $K_{h, \sigma} \subset H^1_0(M)$ coincides with the subspace of constant functions.

Following [F97], Prop. 4.6A, or [F02], Lemma 7.3, we prove that there exists a constant $C_h > 0$ such that the following holds.

For any $\sigma \in \mathbb{R}$ and for any distribution $f \in H^1_0(M)$ there exist (weak) solutions $F_{\sigma}^\pm \in L^2_0(M)$ of the equations $\partial_{h, \sigma}^\pm F_{\sigma}^\pm = f$ such that
\begin{equation}
|F_{\sigma}^\pm|_0 \leq C_h \| f \|_{-1}.
\end{equation}

In fact, the maps given by
\begin{equation}
\partial_{h, \sigma}^\pm v \to - \langle f, v \rangle, \quad \text{for all } v \in H^1_0(M),
\end{equation}
are bounded linear functionals on the (closed) ranges $R_{h, \sigma}^\pm \subset L^2_0(M)$ of the twisted Cauchy-Riemann operator $\partial_{h, \sigma}^\pm : H^1_0(M) \to L^2_0(M)$. In fact, the functionals are well-defined since by assumption $K_{h, \sigma} = \mathbb{C}$ and $f$ vanishes on constant function, and they are bounded since, by Lemma 3.1 and Proposition 3.2, there exists a constant $C_h > 0$ such that, for any $\sigma \in \mathbb{R}$ and for any $v \in H^1_0(M)$ of zero average,
\begin{equation}
|\langle f, v \rangle| \leq \| f \|_{-1} |v|_1 \leq C_h \| f \|_{-1} Q_{h, 0}(v)
\leq C_h \| f \|_{-1} Q_{h, \sigma}(v) = C_h \| f \|_{-1} \| \partial_{h, \sigma}^\pm v \|_0.
\end{equation}

Let $Q_{\sigma}^\pm$ be the unique linear extension of the linear map (32) to $L^2(M)$ which vanishes on the orthogonal complement of $R_{h, \sigma}^\pm$ in $L^2(M)$. By (33), the functionals $Q_{\sigma}^\pm$
are bounded on $L_h^2(M)$ with norm
\[
\|\Phi^\pm_\sigma\| \leq C_h \|f\|_{-1}.
\]
By the Riesz representation theorem, there exist two (unique) functions $F^\pm_\sigma \in L_h^2(M)$ such that
\[
\langle v, F^\pm_\sigma \rangle_h = \Phi^\pm_\sigma(v), \quad \text{for all } v \in L_h^2(M).
\]
The functions $F^\pm_\sigma$ are by construction (weak) solutions of the twisted Cauchy–Riemann equations $\partial^\pm_{h,\sigma} F^\pm_\sigma = f$ satisfying the required bound (31).

The identities (30) immediately imply that
\[
\langle \partial^\pm_{h,\sigma} v, F^\pm_\sigma \rangle_h = 2e^{\mp i\theta} \langle R^\pm_{J,\sigma}(z)(S_\theta + i\sigma_\theta)v, F^\pm_\sigma \rangle_h
\]
\[
- (z + e^{\mp i\theta})\langle R^\pm_{J,\sigma}(z)\partial^\pm_{h} v, F^\pm_\sigma \rangle_h,
\]
where $R^\pm_{J,\sigma}(z)$ and $R^-_{J,\sigma}(z)$ denote the resolvents of the unitary operators $U_{J,\sigma}$ and $U^\dagger_{J,\sigma}$ respectively, which yield holomorphic families of bounded operators on the unit disk $D \subset \mathbb{C}$.

Let $r > 2$ and let $p \in (0, 1)$ be such that $pr > 2$. Let $\mathcal{E} = \{e_k\}_{k \in \mathbb{N}}$ be the orthonormal Fourier basis of the Hilbert space $L_h^2(M)$ described in §2. By Corollary 4.3 all holomorphic functions
\[
R^\pm_{h,\sigma,k}(z) := \langle R^\pm_{J,\sigma}(z)e_k, F^\pm_\sigma \rangle_h, \quad k \in \mathbb{N},
\]
belong to the Hardy space $H^p(D)$, for any $0 < p < 1$. The corresponding non-tangential maximal functions $N^\pm_{h,\sigma,k}$ (over cones of arbitrary fixed aperture $0 < \alpha < 1$) belong to the space $L^p(\mathbb{T},\mathcal{L})$ and for all $0 < p < 1$ there exists a constant $A_{\alpha,p} > 0$ such that, for any Abelian differential $h$ on $M$, for every $\sigma \in \mathbb{R}$ and $k \in \mathbb{N}$, the following inequalities hold:
\[
|N^\pm_{h,\sigma,k}|_p \leq A_{\alpha,p}|e_k|_0|F^\pm_\sigma|_0 = A_{\alpha,p}|F^\pm_\sigma|_0 \leq A_{\alpha,p}C_h \|f\|_{-1}.
\]
Let $\lambda_k \in \mathbb{N}$ be the sequence of the eigenvalues of the Dirichlet form $\Omega := \Omega_{h,0}$ introduced in §2. Let $w \in H_h^2(M)$. We have
\[
\langle R^\pm_{J,\sigma}(z)w, F^\pm_\sigma \rangle_h = \sum_{k=0}^{\infty} \langle w, e_k \rangle_h R^\pm_{h,\sigma,k}(z),
\]
hence, by the Cauchy–Schwarz inequality,
\[
|\langle R^\pm_{J,\sigma}(z)w, F^\pm_\sigma \rangle_h| \leq \left( \sum_{k=0}^{\infty} \frac{|R^\pm_{h,\sigma,k}(z)|^2}{(1 + \lambda_k)^r} \right)^{1/2} \|w\|_r.
\]
Let $N^\pm_{h,\sigma}(\theta)$ be the functions defined as
\[
N^\pm_{h,\sigma}(\theta) := \left( \sum_{k=0}^{\infty} \frac{|R^\pm_{h,\sigma,k}(\theta)|^2}{(1 + \lambda_k)^r} \right)^{1/2}.
\]
Let $N_{h,\sigma}^\pm (w)$ denote the non-tangential maximal function for the holomorphic function $\langle R_{j,\sigma}^\pm (z)w, F_{\sigma}^\pm \rangle$.

By formulas (38) and (39), it follows that, for all $\theta \in \mathbb{T}$ and all functions $w \in \tilde{H}_h^p(M)$, we have

$$N_{h,\sigma}^\pm (w)(\theta) \leq N_{h,\sigma}^\pm (\theta) \|w\|_r .$$

The functions $N_{h,\sigma}^\pm \in L^p(\mathbb{T}, \mathcal{L})$ for any $0 < p < 1$. In fact, by formula (36) and (following a suggestion of Stephen Semmes) by the ‘triangular inequality’ for the space $L^{p/2}$ with $0 < p < 1$, we have

$$|N_{h,\sigma}^\pm|^p \leq (A_{\alpha,\beta}C_h) \left( \sum_{k=0}^{\infty} \frac{1}{(1 + \lambda_k)^{pr/2}} \right) \|f\|^p_{-1} < +\infty .$$

The series in formula (41) is convergent by the Weyl asymptotics (Theorem 2.3) since $pr/2 > 1$. Let then

$$B_h(p,r) := (A_{\alpha,\beta}C_h) \left( \sum_{k=0}^{\infty} \frac{1}{(1 + \lambda_k)^{pr/2}} \right)^{1/p} .$$

By taking the non-tangential limit as $z \to -e^{r+2\theta}$ in the identity (34), formula (40) implies that, for all $\theta \in \mathbb{T}$ such that $N_{h,\sigma}^\pm(\pi + 2\theta) < +\infty$,

$$|\langle \partial_{\rho,\sigma} v, F_{\sigma}^\pm \rangle| \leq N_{h,\sigma}^\pm(\pi + 2\theta) \| (S_\theta + i\sigma_\theta)v \|_r ,$$

hence the required estimates (28) and (29) are proved with the choice of the function $A_{h,\sigma}(f,\theta) := N_{h,\sigma}^\pm(\pi - 2\theta)$ or $A_{h,\sigma}(f,\theta) := N_{h,\sigma}^\pm(\pi + 2\theta)$, for all $\theta \in \mathbb{T}$.

Proof of Theorem 5.2 By the estimate (28) of Lemma 5.3 the linear map given by

$$(S_\theta + i\sigma_\theta)v \to - (f,v) , \quad \text{for all } v \in \tilde{H}_h^{r+1}(M) ,$$

is well defined and extends by continuity to the closure of the range $\tilde{R}_h(\theta)$ of the linear operator $S_\theta + i\sigma_\theta$ in $\tilde{H}_h^r(M)$. Let $\mathcal{U}_\sigma(f)(\theta)$ be the extension uniquely defined by the condition that $\mathcal{U}_\sigma(f)(\theta)$ vanishes on the orthogonal complement of $\tilde{R}_h(\theta)$ in $\tilde{H}_h^r(M)$. By construction, for almost all $\theta \in \mathbb{T}$ the linear functional $u := \mathcal{U}_\sigma(f)(\theta) \in \tilde{H}_h^{-r}(M)$ yields a distributional solution of the cohomological equation $(S_\theta + i\sigma_\theta)u = f$ whose norm satisfies the bound

$$\|\mathcal{U}_\sigma(f)(\theta)\|_{-r} \leq A_{h,\sigma}(f,\theta) .$$

By (29) the $L^p$ norm of the measurable function $\mathcal{U}_\sigma(f) : \mathbb{T} \to \tilde{H}_h^{-r}(M)$ satisfies the required estimate

$$\|\mathcal{U}_\sigma(f)\|_p := \left( \int_{\mathbb{T}} \|\mathcal{U}_\sigma(f)(\theta)\|_{-r}^p d\theta \right)^{1/p} \leq B_h \|f\|_{-1} .$$

Theorem 5.4. Let $h$ be an Abelian differential with minimal vertical foliation. For any $r > 2$ and $p \in (0, 1)$ such that $pr > 2$, there exists a constant $C_{h,p,r} > 0$ such that, for all zero-average functions $f \in \tilde{H}_h^{-r-1}(M)$, for all $\sigma \in \mathbb{R}$ and for Lebesgue almost
all $\theta \in \mathbb{T}$, the twisted cohomological equation $(S_\theta + i\sigma_\theta)u = f$ has a distributional solution $u_\theta \in \tilde{H}^{r-\eps}_h(M)$ satisfying the following estimate:

$$
(43) \quad \left( \int_T \|u_\theta\|^p_{r-\eps} d\theta \right)^{1/p} \leq C_{h,p,r} \|f\|_{r-1}.
$$

Proof. Let $E = \{e_k\}_{k \in \mathbb{N}}$ be the orthonormal Fourier basis of the Hilbert space $L^2_h(M)$ described in §2. Let $r > 2$ and $p \in (0, 1)$ be such that $pr > 2$. By Theorem 5.2, for any $k \in \mathbb{N} \setminus \{0\}$ there exists a function with distributional values $u_k : L^p(T, \mathbb{H}^{r-\eps}_h(M))$ such that the following holds. There exists a constant $C_{h,r} := C_h(p, r) > 0$ such that

$$
(44) \quad \left( \int_T \|u_k(\theta)\|^p_{r-\eps} d\theta \right)^{1/p} \leq C_{h,r} \|e_k\|_{r-\eps} \leq C_{h,r} (1 + \lambda_k)^{-1/2}.
$$

In addition, for any $k \in \mathbb{N} \setminus \{0\}$, there exists a full measure set $F_k(\sigma) \subset \mathbb{T}$ such that, for all $\theta \in F_k(\sigma)$, the distribution $u := u_k(\theta) \in \tilde{H}^{r-\eps}_h(M)$ is a (distributional) solution of the cohomological equation $(S_\theta + i\sigma_\theta)u = e_k$.

Any function $f \in \tilde{H}^{r-1}_h(M)$ of zero average has a Fourier decomposition in $L^2_h(M)$:

$$
f = \sum_{k \in \mathbb{N} \setminus \{0\}} \langle f, e_k \rangle_h e_k.
$$

A (formal) solution of the cohomological equation $(S_\theta + i\sigma_\theta)u = f$ is therefore given by the series

$$
(45) \quad u_\theta := \sum_{k \in \mathbb{N} \setminus \{0\}} \langle f, e_k \rangle_h u_k(\theta).
$$

By the triangular inequality in $\tilde{H}^{r-\eps}_h(M)$ and by Hölder inequality, we have

$$
\|u_\theta\|_{r-\eps} \leq \left( \sum_{k \in \mathbb{N} \setminus \{0\}} \|u_k(\theta)\|^2_{r-\eps} \right)^{1/2} \leq \left( \sum_{k \in \mathbb{N} \setminus \{0\}} \frac{1}{(1 + \lambda_k)^{pr/2}} \right) \|f\|_{r-1},
$$

hence by the ‘triangular inequality’ for $L^p$ spaces (with $0 < p < 1$) and by the estimate (44),

$$
(46) \quad \int_T \|u_\theta\|^p_{r-\eps} d\theta \leq C_{h,r}^p \left( \sum_{k \in \mathbb{N} \setminus \{0\}} \frac{1}{(1 + \lambda_k)^{pr/2}} \right) \|f\|^p_{r-1}.
$$

Since $pr/2 > 1$ the series in (46) is convergent, hence $u_\theta \in \tilde{H}^{r-\eps}_h(M)$ is a solution of the equation $(S_\theta + i\sigma_\theta)u = f$ which satisfies the required bound (43). \qed

5.2. Twisted invariant distributions and basic currents. In this section we describe the structure of the space of obstructions to the existence of solutions of the twisted cohomological equation $(S + i\sigma)u = f$.

Definition 5.5. For all $r > 0$, let $\mathcal{Y}^{r}_{h,\sigma} \subset H^{-r}_h(M)$ denote the space of distributions invariant for the twisted Lie derivative operator $S + i\sigma$, that is, the space

$$
\mathcal{Y}^{r}_{h,\sigma} := \{D \in H^{-r}_h(M) | (S + i\sigma)D = 0\}.
$$
Twisted invariant distributions are in one-to-one correspondence with twisted basic currents. We introduce Sobolev space of 1-forms and of 1-dimensional currents.

**Definition 5.6.** For all $r > 0$, the weighted Sobolev space of 1-forms $W^r_h(M)$ is defined as follows:

$$W^r_h(M) := \{ \alpha \in L^2(M, T^*M) \mid (t_\Sigma \alpha, t_\mathcal{T} \alpha) \in H^r_h(M) \times H^r_h(M) \}.$$  

The weighted Sobolev space of 1-currents $W^{-r}_h(M)$ is defined as the dual spaces of the weighted Sobolev space of 1-forms $W^r_h(M)$.

Twisted basic currents are defined as follows:

**Definition 5.7.** For all $r > 0$, let $\mathcal{B}^r_{h,\sigma} \subset H^{-r}_h(M)$ denote the space of twisted basic currents, that is, the space

$$\mathcal{B}^r_{h,\sigma} := \{ C \in W^{-r}_h(M) \mid (\mathcal{L}_S + i\sigma) C = t_\Sigma C = 0 \}.$$  

[Here $\mathcal{L}_S$ denotes the Lie derivative operator on currents in the direction of the vector field $S$ on $M \setminus \Sigma_h$.]

The notions of twisted invariant distributions and twisted basic currents are related (for the untwisted case see [F02], Lemmas 6.5 and 6.6, or [F07], Lemma 3.14):

**Lemma 5.8.** A 1-dimensional current $C \in \mathcal{B}^r_{h,\sigma}$ if and only if the distribution $C \wedge \text{Re}(h) \in \mathcal{I}^r_{h,\sigma}$. In addition, the map

$$\mathcal{D}_h : C \to -C \wedge \text{Re}(h)$$

is a bijection from the space $\mathcal{B}^r_{h,\sigma}$ onto the space $\mathcal{I}^r_{h,\sigma}$.

**Proof.** The map $\mathcal{D}_h$ is well-defined since for any $C \in \mathcal{B}^r_{h,\sigma}$ we have

$$(\mathcal{L}_S + i\sigma)[C \wedge \text{Re}(h)] = [\mathcal{L}_S + i\sigma]C] \wedge \text{Re}(h) = 0.$$  

The inverse map is the map $\mathcal{B}_h : \mathcal{I}^r_{h,\sigma} \to \mathcal{B}^r_{h,\sigma}$ defined as

$$\mathcal{B}_h(D) = t_\Sigma D,$$  

for all $D \in \mathcal{I}^r_{h,\sigma}$.

[Here $t_\Sigma$ denotes the contraction operator with respect to the vector field $S$ on $M \setminus \Sigma_h$, which maps distributions, as degree 2 currents, to degree 1 currents].

The map $\mathcal{B}_h$ is well-defined since $t_\Sigma \circ t_\Sigma = 0$, and

$$(\mathcal{L}_S + i\sigma) \circ t_\Sigma = t_\Sigma \circ (\mathcal{L}_S + i\sigma).$$

It follows that if $D \in \mathcal{I}^r_{h,\sigma}$ then $C = t_\Sigma D \in \mathcal{B}^r_{h,\sigma}$ since

$$(\mathcal{L}_S + i\sigma)C = t_\Sigma \circ (\mathcal{L}_S + i\sigma)D = 0$$  

and

$$t_\Sigma C = t_\Sigma(t_\Sigma D) = 0.$$  

Finally, the map $\mathcal{B}_h$ is the inverse of the map $\mathcal{D}_h$. In fact, since $t_\Sigma C = 0$ (as $C$ is basic) and $D \wedge \text{Re}(h) = 0$ (as a current of degree 3), and $t_\Sigma \text{Re}(h) = 1$, we have

$$(\mathcal{B}_h \circ \mathcal{D}_h)(C) = -t_\Sigma(C \wedge \text{Re}(h)) = -t_\Sigma C \wedge \text{Re}(h)$$(48)$$;$$

$$(\mathcal{D}_h \circ \mathcal{B}_h)(D) = -(t_\Sigma D \wedge \text{Re}(h)) = -t_\Sigma(D \wedge \text{Re}(h)) + (D \wedge t_\Sigma \text{Re}(h)) = D.$$  

The argument is complete. □
We introduce the twisted exterior differential $d_{h,\sigma}$, defined on 1-forms as
\[
d_{h,\sigma}\alpha := d\alpha + i\sigma \text{Re}(h) \wedge \alpha, \quad \text{for all } \alpha \in W^1_h(M).
\]
The twisted exterior differential extends to currents by duality.

**Definition 5.9.** A current $C \in W^{-s}_h(M)$ is $d_{h,\sigma}$-closed if
\[
d_{h,\sigma}C = dC + i\sigma (\text{Re}(h) \wedge C) = 0.
\]

**Lemma 5.10.** A current $C \in \mathcal{B}^s_{h,\sigma}$ if and only if $i\Sigma C = 0$ and $C$ is $d_{h,\sigma}$-closed.

**Proof.** If $C \in \mathcal{B}^s_{h,\sigma}$, then $i\Sigma C = 0$ by definition, and
\[
i\Sigma d_{h,\sigma}C = i\Sigma [dC + i\sigma (\text{Re}(h) \wedge C)] = \mathcal{L}_S C + i\sigma C = 0,
\]
so that $i\Sigma d_{h,\sigma}C = 0$, which implies $d_{h,\sigma}C = 0$, as $d_{h,\sigma}C$ is a current of degree 2 (and dimension 0) and the contraction operator $i\Sigma$ is surjective onto 2-forms. Conversely, if $i\Sigma C = 0$ and $d_{h,\sigma}C = 0$, then by the above formula $\mathcal{L}_S C + i\sigma C = 0$, hence $C \in \mathcal{B}^s_{h,\sigma}$, thereby completing the argument. \qed

By the de Rham theorem for twisted cohomology, it is possible to attach a twisted cohomology class to any $d_{h,\sigma}$-closed current.

**Definition 5.11.** Let $\Omega^s(D)$ denote the space of all smooth differential forms on a domain $D \subset M$, with compact support on $D$. Let $\eta$ be a real closed smooth 1-form on $D$ and let $d_\eta$ denote the twisted exterior derivative defined as
\[
d_\eta\alpha = d\alpha + i\eta \wedge \alpha, \quad \text{for all } \alpha \in \Omega^s(M).
\]
The twisted cohomology (with complex coefficients) $H^s_{\eta}(D, \mathbb{C})$ is the cohomology of the differential complex $(\Omega^s(D), d_\eta)$.

For every Abelian differential $h$ on $M$ and $\sigma \in \mathbb{R}$, let us adopt the notation
\[
H^1_{h,\sigma}(M \setminus \Sigma_h, \mathbb{C}) := H^1_{\sigma \text{Re}(h)}(M \setminus \Sigma_h, \mathbb{C}).
\]

**Lemma 5.12.** For every $r > 0$, the exists a cohomology map $j_r : \mathcal{B}^r_{h,\sigma} \to H^1_{h,\sigma}(M, \mathbb{C})$ such that $j_r(C)$ is the twisted cohomology class of the twisted basic current $C \in \mathcal{B}^r_{h,\sigma}$.

**Proof.** A current $C \in W^{-r}_h(M)$ does not in general extend to a linear functional on $C^\infty(M)$, hence is not a current on the compact surface $M$. However, since $C^\infty_0(M \setminus \Sigma_h) \subset W^{-r}_h(M)$ for all $r > 0$, it follows from the de Rham theorem for the twisted cohomology that current $C \in W^{-r}_h(M)$ such that $d_{h,\sigma}C = 0$ in $W^{-r+1}_h(M)$ has a well-defined twisted cohomology class $[C] \in H^1_{h,\sigma}(M \setminus \Sigma_h, \mathbb{C})$. \qed

The structure of the space of basic currents with vanishing cohomology class, with respect to the filtration induced by weighted Sobolev spaces with integer exponent, was described in [F02], §7 (see also [F07], §3.3, with respect to the filtration induced by weighted Sobolev spaces with general real exponent). We extend below these results to the space of twisted basic currents.
Let $\delta_r : \mathcal{B}_{h,\sigma}^r \to \mathcal{B}_{h,\sigma}^{r-1}$ be the linear maps defined as follows (see [F02], formula (7.18') and [F07], formulas (3.61) and (3.62) for the untwisted case):

$$\delta_r(C) := (d_{h,\sigma} \circ \iota_{r})(C) = -d_{h,\sigma}\left(\frac{C \wedge \text{Re}(h)}{\omega_h}\right), \quad \text{for } C \in \mathcal{B}_{h,\sigma}^r.$$  

Indeed, it can be proved by Lemma 5.8 and by the definition of the weighted Sobolev spaces $H_h^r(M)$ and $W_h^r(M)$ that the above formula (49) defines, for all $r > 0$, bounded linear maps $\delta_r : \mathcal{B}_{h,\sigma}^r \to \mathcal{B}_{h,\sigma}^{r-1}$.

Let $\mathcal{K}_{h,\sigma}^r \subset \mathcal{Y}_{h,\sigma}^r \subset H_h^{r-1}(M)$ denote the subspace of distributions which are twisted $S$-invariant and $T$-invariant, that is,

$$\mathcal{K}_{h,\sigma}^r := \{D \in H_h^{r-1}(M) | (S + i\sigma)D = TD = 0\}.$$  

Let $\iota_r : \mathcal{K}_{h,\sigma}^r \to \mathcal{B}_{h,\sigma}^r$ denote the restriction to $\mathcal{K}_{h,\sigma}^r$ of the inverse of the map $\mathcal{D}^h : \mathcal{Y}_{h,\sigma}^r \to \mathcal{B}_{h,\sigma}^r$ (see Lemma 5.8), that is, the map defined as

$$\iota_r(D) := \iota SD, \quad \text{for all } D \in \mathcal{K}_{h,\sigma}^r.$$  

**Theorem 5.13.** For all $r > 0$ there exist exact sequences

$$0 \to \mathcal{K}_{h,\sigma}^{r-1} \xrightarrow{\delta_{r-1}} \mathcal{B}_{h,\sigma}^{r-1} \xrightarrow{\delta_r} \mathcal{B}_{h,\sigma}^r \xrightarrow{j_r} H_h^1(M \setminus \Sigma_h, \mathbb{C}) \to 0.$$  

**Proof.** The map $\iota_r : \mathcal{K}_{h,\sigma}^r \to \mathcal{B}_{h,\sigma}^r$ is by definition injective, since the contraction operator is surjective onto the space of functions (0-forms).

The identity $\text{Im}(\iota_r) = \ker(\delta_r)$ holds since by Lemma 5.8 $C \in \mathcal{B}_{h,\sigma}^r$ if and only if $C = \iota SD$ with $D \in \mathcal{Y}_{h,\sigma}^r$ and in addition

$$\delta_r(\iota SD) = d_{h,\sigma}(\iota T SD) = \iota T(S + i\sigma)D - \iota SD(TD) = -\iota SD(TD),$$

hence $\delta_r(\iota SD) = 0$ if and only if $TD = 0$ (since $TD$ has degree 2 and the contraction is surjective onto the space of functions (0-forms).

The identity $\text{Im}(\delta_r) = \ker(\delta_r)$ holds by the following argument. Let $C' \in \mathcal{B}_{h,\sigma}^s$ be a current such that $[C'] = 0 \in H_h^1(M \setminus \Sigma_h, \mathbb{C})$, hence there exists a current $U$ of degree 0 (and dimension 2) such that $C' = d_{h,\sigma}U$. Let $C = U \wedge \text{Im}(h)$. By definition we have $C' = \delta_r(C)$. We claim that $C = U \wedge \text{Im}(h) \in \mathcal{B}_{h,\sigma}^{r+1}$. In fact, by definition $\iota SD(U \wedge \text{Im}(h)) = 0$, and since $\iota SD(U) = C'$, we have

$$\left(\mathcal{L}_S + i\sigma\right)(U \wedge \text{Im}(h)) = (\mathcal{L}_S + i\sigma)(U) \wedge \text{Im}(h) = \iota SD(U) \wedge \text{Im}(h) = \iota SD(U) + \text{Im}(h) = 0.$$  

The argument is thus complete.  

The above theorem and Lemma 5.8 imply the following finiteness result:

**Corollary 5.14.** For any Abelian differential $h$ on $M$, for all $\sigma \in \mathbb{R}$ and for all $s \geq 0$, the spaces $\mathcal{Y}_{h,\sigma}^s$ of twisted invariant distributions for the operator $S + i\sigma \text{Re}h$ and the corresponding space of $\mathcal{B}_{h,\sigma}^s$ of twisted basic currents have finite dimension.

We conclude the section by proving a lower bound on the dimensions of the spaces of twisted invariant distributions.
Corollary 5.15. Let \( h \) be an Abelian differential with minimal vertical foliation. For all \( \theta \in \mathbb{T} \), let \( h_{\theta} := e^{-i\theta} h \) be the rotated Abelian differential and let \( \sigma_{\theta} := \sigma_{h_{\theta}} \). For any \( r > 2 \) and for almost all \( \theta \in \mathbb{T} \), the subspace \( j_{r}(B_{h_{\theta}, \sigma_{h_{\theta}}}^{r}) \cap H^{1}_{h_{\theta}, \sigma_{h_{\theta}}}(M, \mathbb{C}) \) has codimension at most equal to one in \( H^{1}_{h_{\theta}, \sigma_{h_{\theta}}}(M, \mathbb{C}) \).

Proof. Let \( h_{\theta} := e^{-i\theta} h \) be the rotated Abelian differential, let \( \sigma_{\theta} := \sigma_{h_{\theta}} \) and let \( \alpha \) be any twisted closed 1-form, that is, a 1-form such that \( d_{h_{\theta}, \sigma_{h_{\theta}}} \alpha := d\alpha + i\sigma_{h_{\theta}} \operatorname{Re}(h_{\theta}) \wedge \alpha = 0 \).

Let \( \alpha := f \operatorname{Re}(h_{\theta}) + g \operatorname{Im}(h_{\theta}) \) and assume that \( f \in \mathcal{H}_{h_{\theta}}^{r-1}(M) \) with \( r > 2 \) and that \( f \) is orthogonal to constant functions. Then by Theorem 5.4 it follows that the cohomological equation

\[
(S_{\theta} + i\sigma_{\theta})u = f
\]

has a distributional solution \( u \in \mathcal{H}_{h_{\theta}}^{r}(M) \) for almost all \( \theta \in \mathbb{T} \). Let \( C \) denote the current of degree 1 (and dimension 1) uniquely determined by the formula

\[
d_{h_{\theta}, \sigma_{h_{\theta}}} u := (d + i\sigma_{h_{\theta}} \operatorname{Re}(h_{\theta})) u = \alpha + C.
\]

It is clear from the definition that \( C \) is closed with respect to the twisted differential \( d_{h_{\theta}, \sigma_{h_{\theta}}} \), that is, \( d_{h_{\theta}, \sigma_{h_{\theta}}} C = 0 \), and in addition \( \iota_{S_{\theta}} C = 0 \), hence, by Lemma 5.10 the current \( C \) is a twisted basic current \( d_{h_{\theta}, \sigma_{h_{\theta}}}-\text{cohomologous to the 1-form} \alpha \).

Finally, it can be proved that for all \( \sigma \in \mathbb{R} \) all cohomology classes in \( H^{1}_{h_{\theta}, \sigma}(M, \mathbb{C}) \) can be represented by \( d_{h_{\theta}, \sigma}-\text{closed 1-forms} \alpha \in W^{r}_{h}(M) \) with \( r > 1 \).

\[\square\]

5.3. Smooth solutions. In this section we prove our main result on existence of smooth solutions of the twisted cohomological equation for translation flows, which holds for any Abelian differential in almost all directions, and derive as a corollary our result on cohomological equations for product translation flows.

Lemma 5.16. Let \( h \) be an Abelian differential with minimal vertical foliation. For every \( s > r \geq 0 \) such that \( s - r > 3 \) there exists \( p \in (0, 1) \) such that for every \( \sigma \in \mathbb{R} \) there exists a function \( A_{h, \sigma} := A_{h, \sigma}(p, r, s) \in L^{p}(\mathbb{T}, \mathcal{L}) \) such that the following holds. For almost all \( \theta \in \mathbb{T} \) and for all zero average functions \( v \in H^{r+1}_{h}(M) \), we have

\[
|v|_{r} \leq A_{h, \sigma}(\theta) \| (S_{\theta} + i\sigma_{\theta})v \|_{s},
\]

and there exists a constant \( B_{h} := B_{h}(p, r, s) > 0 \) such that, for all \( \sigma \in \mathbb{R} \), we have

\[
|A_{h, \sigma}|_{p} \leq B_{h}.
\]

Proof. Let \( \mathcal{E} := (e_{k})_{k \in \mathbb{N}} \) denote the orthonormal Fourier basis of the space \( L^{2}_{h}(M) \) of eigenvalues of the Friedrichs extension of the flat Laplacian, described in \( \S 2 \). Let \( \alpha > 2 \) and let \( p \in (0, 1) \) be such that \( \alpha p > 2 \). By Lemma 5.3, for all \( k \in \mathbb{N} \setminus \{0\} \) there exists a function \( A^{(k)}_{h, \sigma} := A^{(k)}_{h, \sigma}(p, \alpha) \in L^{p}(\mathbb{T}, \mathcal{L}) \) such that, for all \( v \in H^{r+1}_{h}(M) \) of zero average we have

\[
|\langle v, e_{k} \rangle| \leq A^{(k)}_{h, \sigma}(\theta) \| (S_{\theta} + i\sigma_{\theta})v \|_{\alpha}.
\]
In addition, there exists a constant $B_h := B_h(p, \alpha)$ such that
\[
|A_{h,\alpha}^{(k)}|_p \leq B_h \|e_k\|_{-1} = B_h(1 + \lambda_k)^{-1/2}.
\]
Let $\beta > 1$ such that $(\beta + 1)p > 2$. It follows that, for any $v \in H^{\alpha+1}_h(M)$ of zero average we have
\[
\|v\|_{-\beta} \leq \left( \sum_{k \in \mathbb{N} \setminus \{0\}} (1 + \lambda_k)^{-\beta} A_{h,\sigma}^{(k)}(\theta)^2 \right)^{1/2} \|(S_\theta + i\sigma_\theta)v\|_{\alpha}.
\]
Let then $A_{h,\sigma} := A_{h,\sigma}(p, \alpha, \beta)$ denote the function defined, for $\theta \in \mathbb{T}$, as follows:
\[
A_{h,\sigma}(\theta) := \left( \sum_{k \in \mathbb{N} \setminus \{0\}} (1 + \lambda_k)^{-\beta} A_{h,\sigma}^{(k)}(\theta)^2 \right)^{1/2}.
\]
By the triangular inequality for the space $L^{p/2}$ (with $p/2 < 1$) we have
\[
|A_{h,\sigma}|_p^p = \left| \sum_{k \in \mathbb{N} \setminus \{0\}} (1 + \lambda_k)^{-\beta} A_{h,\sigma}^{(k)} \right|_{p/2}^p \leq C_h \sum_{k \in \mathbb{N} \setminus \{0\}} (1 + \lambda_k)^{-(\beta+1)p/2}.
\]

By the Weyl asymptotics, the series on the RHS of the above formula is convergent as soon as $(\beta + 1)p > 2$. Let then $v \in H^{\alpha+3}_h(M)$ so that $\Delta_h v \in H^{\alpha}_h(M)$ and we have
\[
\|v\|_{-\beta+2} = \| (I - \Delta^F_h) v \|_{-\tau} \leq A_{h,\sigma}(\theta) \|(S_\theta + i\sigma_\theta)(I - \Delta^F_h)v\|_{\alpha} = A_{h,\sigma}(\theta) \| (I - \Delta_h)(S_\theta + i\sigma_\theta)v\|_{\alpha+2} = A_{h,\sigma}(\theta) \| (S_\theta + i\sigma_\theta)v\|_{\alpha+2}.
\]
By the interpolation inequality for the Friedrichs norms and by Lemma 2.8 for every $\rho \in [0, 1)$, whenever $\alpha \rho > 2$, $(\beta + 1)p > 2$ and $\rho + \beta \leq 2$, we have
\[
\|v\|_{\rho} = \|v\|_{\rho} \leq A_{h,\sigma}(\theta) \|(S_\theta + i\sigma_\theta)v\|_{\alpha+\beta+p} \leq A_{h,\sigma}(\theta) \|(S_\theta + i\sigma_\theta)v\|_{\alpha+\beta+p}.
\]
Finally, for all $r \geq 0$, by applying the above bound to all functions $S_iT^j v$ and $T^iS^j$, for all $i, j \leq [r]$, we finally have that there exists a constant $C_r > 0$ such that
\[
|v|_{r} \leq C_r A_{h,\sigma}(\theta) \|(S_\theta + i\sigma_\theta)v\|_{\alpha+\beta+r}.
\]
Since, given $s > r \geq 0$ with $s - r > 3$ it is always possible to find $\alpha > 2$, $\beta > 1$ and $p \in (0, 1)$ such that
\[
s = \alpha + \beta, \quad \alpha p > 2, \quad (\beta + 1)p > 1 \quad \text{and} \quad \{r\} + \beta \leq 2,
\]
the bound in formula (51) follows immediately from that in the above formula (52), hence the argument is complete.

\[\square\]

**Theorem 5.17.** Let $h$ be an Abelian differential with minimal vertical foliation. For any $s > r \geq 0$ such that $s - r > 3$, there exists $p \in (0, 1)$ and a constant $C_{r,s} > 0$ such that the following holds. For any $\sigma \in \mathbb{R}$ and for almost all $\theta \in \mathbb{T}$, for any $f \in H^{\alpha}_h(M)$ of zero average such that $\mathcal{D}(f) = 0$ for all twisted invariant distributions $\mathcal{D} \in \mathcal{D}_h^\alpha$,
the cohomological equation \((S_\theta + i\sigma_\theta)u = f\) has a zero-average solution \(U_\theta(f) \in H^*_h(M)\) satisfying the following estimate:

\[
\left( \int_T |U_\theta(f)|^p d\theta \right)^{1/p} \leq C_{r,s} |f|_s .
\]

(53)

**Proof:** It follows from the a priori bound of Lemma 5.16 that, for all \(\sigma \in \mathbb{R}\) and for almost all \(\theta \in \mathbb{T}\), the subspace

\[
\{ f \in \mathcal{H}^s_h(M) | f \in (S_\theta + i\sigma_\theta)[\mathcal{H}^s_h(M)] \}
\]

is closed in \(\mathcal{H}^s_h(M)\), hence it coincides with the kernel of the subspace \(\mathcal{I}^{-s}_{h_0, \sigma_0} \cap \mathcal{I}^{-s}(M)\) of all twisted invariant distributions vanishing on constant functions. In addition, it follows by continuity that, there exists \(p \in (0,1)\) and a function \(A_{h,\sigma} \in L^p(\mathbb{T}, \mathcal{L})\) such that, for all \(f \in \mathcal{H}^s_h(M) \cap \text{Ker}(\mathcal{I}^{-s}_{h_0, \sigma_0})\) the unique zero-average solution \(U_\theta(f) \in H^s_h(M)\) of the cohomological equation \((S_\theta + i\sigma_\theta)u = f\) satisfies the bound

\[
|U_\theta(f)|_r \leq A_{h,\sigma}(\theta)|f|_s .
\]

From the above inequality and the bounds on the \(L^p\) norm of the function \(A_{h,\sigma}\) established in Lemma 5.16 it follows immediately that

\[
\left( \int_T |U_\theta(f)|^p d\theta \right)^{1/p} \leq |A_{h,\sigma}|_p |f|_s \leq B_h |f|_s .
\]

The proof of the theorem is therefore complete.

\[\square\]

**Proof of Theorem 1.1** The condition that \(h\) has a minimal vertical foliation it is not restrictive since the statement is rotation invariant, and any Abelian differential has a minimal direction \([Ma], [AG]\).

If the function \(f \in H^s_h(M)\) is constant, then for \(\sigma \neq 0\) the constant functions \(u = -if/\sigma\) is a solution (which is unique in \(L^2_h(M)\) for almost all \(\theta \in \mathbb{T}\). For \(\sigma = 0\), there is no solution unless \(f = 0\), in which case the solution is the zero constant. The argument is therefore reduced to the case of functions of zero average.

By Theorem 5.17 for any Abelian differential \(h\) with minimal vertical foliation, the twisted cohomological equation \((S_\theta + i\sigma \cos \theta)u = f\) can be solved with Sobolev bounds for all \(f \in H^s_h(M)\) of zero-average in the kernel of all twisted invariant distributions, for all \(\sigma \in \mathbb{R}\) and for almost all \(\theta \in \mathbb{T}\). Let then \(\mathcal{F} \subset \mathbb{T} \times \mathbb{R}\) be the set of \((\theta, \sigma) \in \mathbb{T} \times \mathbb{R}\) such that the twisted cohomological equation \((S_\theta + i\sigma)u = f\) can be solved with Sobolev bounds for all \(f \in H^s_h(M)\) of zero average in the kernel of all twisted invariant distributions. Since the map \((\theta, \sigma) \to (\theta, \sigma \cos \theta)\) from \(\mathbb{T} \times \mathbb{R}\) into itself is absolutely continuous, it follows from Theorem 5.17 that the set \(\mathcal{F}\) has full Lebesgue measure. Finally, the statement of the theorem follows by Fubini’s theorem.

\[\square\]
For all \( s, \nu \in \mathbb{N} \), let \( H^{s,\nu}_h(M \times \mathbb{T}) \) denote the \( L^2 \) Sobolev space on \( M \times \mathbb{T} \) with respect to the invariant volume form \( \omega_h \wedge d\phi \) and the vector fields \( S, T \), and \( \partial/\partial \phi \):

\[
H^{s,\nu}_h(M \times \mathbb{T}) := \{ f \in L^2(M \times \mathbb{T}, d\text{vol}) \mid \sum_{i+j\leq s} \sum_{\ell \leq \nu} \| S^i T^j \partial^\ell f/\partial \phi^\ell \|_0 < +\infty \};
\]

the space \( H^{-s,-\nu}_h(M \times \mathbb{T}) \) is defined as the dual space of \( H^{s,\nu}_h(M \times \mathbb{T}) \).

Let now \( V_{\theta,c} = S_\theta + c \cos \theta \partial/\partial \theta \) denote a translation vector field on \( M \times \mathbb{T} \), and let \( \mathcal{H}_{h,c} \) denote the space of \( V_{\theta,c} \) invariant distributions.

The space \( L^2(M \times \mathbb{T}, d\text{vol}) \) of the product manifold with respect to the invariant volume form \( \omega_h \wedge d\phi \) decomposes as a direct sum of the eigenspaces \( \{ H^{s,\nu}_h \}_{n \in \mathbb{Z}} \) of the circle action:

\[
L^2(M \times \mathbb{T}, d\text{vol}) = \bigoplus_{n \in \mathbb{Z}} H^0_{n}.
\]

**Corollary 5.18.** Let \( h \) be an Abelian differential with minimal vertical foliation. Let \( s > r \geq 0 \) be such that \( s - r > 3 \) and let \( \nu > 2 \) and \( \mu < \nu - 2 \). For all \( c \in \mathbb{R} \) and for almost all \( \theta \in \mathbb{T} \) there exists a constant \( C^{(s,\nu)}_{r} (\theta, c) > 0 \) such that the following holds. For any \( f \in H^{s,\nu}_h(M \times \mathbb{T}) \) such that \( D(f) = 0 \) for all \( V_{\theta,c} \)-invariant distributions \( D \in \mathcal{H}_{h,c} \subset H^{-s,-\nu}_h(M \times \mathbb{T}) \), the cohomological equation \( V_{\theta,c} u = f \) has a solution \( u := \mathcal{U}_\theta(f) \in H^{r,\mu}_h(M) \) satisfying the following estimate:

\[
|u(f)|_{r,\mu} \leq C^{(s,\nu)}_{r} (\theta, c) |f|_{s,\nu}.
\]

**Proof:** By the Fourier decomposition with respect to the circle action, the argument is reduced to proving the existence of solutions for the cohomological equations

\[
(S_\theta + 2\pi \nu \cos \theta) u_n = f_n.
\]

For \( n = 0 \) the above equation reduces to the cohomological equation for the translation flow on \( M \), so that the result already follows from [Fo]. For every \( n \in \mathbb{N} \setminus \{0\} \), let \( \sigma^{c,n} := 2\pi n c \in \mathbb{R} \). The (finite dimensional) space \( \mathcal{H}_{h,c}^{c,n} \subset H^{-s,-\nu}_h(M) \) of twisted \( (S_\theta + i\sigma^{c,n}) \)-invariant distributions embeds as a subspace of the space \( \mathcal{H}_{h,c} \subset H^{-s,-\nu}_h(M \times \mathbb{T}) \) of \( V_{\theta,c} \)-invariant distributions, for all \( n \in \mathbb{N} \), by the formula

\[
D \left( \sum_{n \in \mathbb{Z}} f_n e^{2\pi n \phi} \right) = D(f_n).
\]

By Theorem 5.17 there exist constants \( C_{r,s} > 0 \) and \( p \in (0, 1) \) and, for every \( \varepsilon > 0 \), there exists a full measure set \( \mathcal{F}_{c,n}(\varepsilon) \subset \mathbb{T} \) of measure at least \( 1 - \varepsilon/n^2 \), such that for all \( \theta \in \mathcal{F}_{c,n}(\varepsilon) \), for every \( f_n \in \mathcal{H}_{h,c}^{c,n} \) there exists a solution \( u_n \in H^{r,\mu}_h(M) \cap H^0_n \) of the cohomological equation (55) which satisfies the Sobolev estimate

\[
|u_n|_r \leq C_{r,s} \varepsilon^{-1/p} n^{2/p} |f_n|_{s}.
\]

In fact, the above claim follows immediately from Theorem 5.17. From the claim it follows that for all functions \( f \in H^{r,\nu}_h(M \times \mathbb{T}) \) with \( \nu > 2/p \), such that \( f_n \in \mathcal{H}_{h,c}^{c,n} \) for all \( n \neq 0 \), for all \( \theta \in \mathcal{F}_c(\varepsilon) := \cup_{n \neq 0} \mathcal{F}_{c,n}(\varepsilon) \) the function
Proof of Theorem 1.3 The space \( \mathcal{H}_{\mu}^s \) of \( V_{\theta,c} \)-invariant distributions is generated by the union of subspaces \( \mathcal{H}_{\mu}^{s,\sigma} \) over all \( n \in \mathbb{Z} \). The statement of the theorem then follows from Corollary 5.18 by Fubini’s theorem.

Proof of Corollary 1.2 For any \( \phi_0 \in \mathbb{T} \), let \( M_{\phi_0} = M \times \{ \phi_0 \} \subset M \times \mathbb{T} \). The return map of the flow of the vector field \( X_{\theta,c} \) to the transverse surface \( M_{\phi_0} \subset M \) is smoothly conjugate to the time-1/c map \( \Phi_{\theta,c}^{1/c} \) of the translation flow generated by the horizontal vector field \( S_{\theta} \) on \( M \). Since the return time function is constant (equal to 1), it is possible to derive results on the cohomological equation for the return (Poincaré) map (the time-1/c map) from results on the cohomological equation for the flow. In fact, the procedure is as follows. Let \( \Phi_{\theta,c}^{E} \) denote the flow of the vector field \( X_{\theta,c} \) on \( M \times \mathbb{T} \). Let \( \chi \in C^\infty(\mathbb{T}) \) be a smooth function with integral equal to 1 supported on a closed interval \( I \subset \mathbb{T} \setminus \{ \phi_0 \} \). Let \( F(f) \in H_{h}^{s,\infty}(M \times \mathbb{T}) \) be the function defined as follows:

\[
F(f) \circ \Phi_{\theta,c}^{t}(x,\phi_0) = \begin{cases} f(x)\chi(t), & \text{for } t \in I, \\ 0, & \text{for } t \notin I. \end{cases}
\]

Let \( f \in H_{h}^{1}(M) \) and let us assume that the cohomological equation \( X_{\theta,c}U = F(f) \) has a solution \( U \in H_{h}^{1,\mu}(M \times \mathbb{T}) \). Then the restriction \( u = U|_{M_{\phi_0}} \) is a solution of the cohomological equation \( u \circ \Phi_{\theta,c}^{1/c} - u = f \) for the time-1/c map. In fact, for all \( x \in M \), we have

\[
u \Phi_{\theta,c}^{1/c} - u = \int_{0}^{1/c} X_{\theta,c}U \circ \Phi_{\theta,c}^{t}dt = \int_{0}^{1/c} f\chi(\phi_0 + ct)dt = f(x)\]

By the Sobolev trace theorem, for any \( \mu > 1/2 \), the restriction \( U|_{M_{\phi_0}} \) of a function \( U \in H_{h}^{1,\mu}(M \times \mathbb{T}) \) is a function \( u \in H_{h}^{1}(M) \) and there exists \( C_{\mu} > 0 \) such that

\[
|u|_{r} \leq C_{\mu} \|U\|_{H_{h}^{1,\mu}(M \times \mathbb{T})}.
\]

The result then follows from Theorem 1.3. In fact, for every \( X_{\theta,c} \)-invariant distribution \( D \in \mathcal{H}_{\mu}^{s,\infty} \subset H_{h}^{s,\infty}(M \times \mathbb{T}) \) we define the distribution \( D_{M} \in H_{h}^{s,\infty}(M) \) as

\[
D_{M}(f) := D(F(f)).
\]

By Theorem 1.3 it follows that, for \( D_{M}(f) = 0 \) for all \( D \in \mathcal{H}_{\mu}^{s,\infty} \), then there exists a solution \( U \in H_{h}^{1,\mu}(M \times \mathbb{T}) \) of the cohomological equation \( X_{\theta,c}U = F(f) \), hence a solution \( u = U|_{M} \in H_{h}^{1}(M) \) of the equation \( u \circ \Phi_{\theta,c}^{1/c} - u = f \). Finally, we have that for all \( u \in H_{h}^{1}(M) \) such that \( u \circ \Phi_{\theta,c}^{1/c} - u \in H_{h}^{1}(M) \) we have

\[
D_{M}(u \circ \Phi_{\theta,c}^{1/c} - u) = D(F(u \circ \Phi_{\theta,c}^{1/c} - u)) = D(F(u) \circ \Phi_{\theta,c}^{1/c} - F(u)) = 0,
\]

since \( D \in \mathcal{H}_{\mu}^{s,\infty} \) is by assumption \( X_{\theta,c} \)-invariant.
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