On the zero temperature critical point in heavy fermions

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We generalize the scaling theory of heavy fermions for the case the shift exponent describing the critical Néel line is different from the crossover exponent characterizing the coherence line. We obtain the properties of the non-Fermi liquid system at the critical point and in particular the electrical resistivity. We study violation of hyperscaling in the Fermi liquid regime below the coherence line where the properties of heavy fermion systems are described by mean-field exponents.

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I. INTRODUCTION

Many of the physical properties of heavy fermions are determined by the proximity of these systems to a zero temperature critical point \[ (J/W)_c \]. The existence of a zero temperature phase transition, from a phase with long range magnetic order to a non-magnetic state, allows for a formulation of a scaling theory which gives a very satisfactory description of these materials \[ 1 \]. This scaling theory is most easily formulated within the context of the Kondo lattice model which emphasizes the importance of spin fluctuations neglecting charge fluctuations. This model has two relevant energy scales, namely the bandwidth of the large conduction band \( W \), and the interaction \( J \) between the local \( f \)-electrons and the conduction electrons in the wide conduction band. At \( T = 0 \) the relevant quantity is the ratio \( J/W \) and the zero temperature phase transition occurs at \( (J/W)_c \) which is different from zero for \( d \geq d_L \), the lower critical dimension (probably \( d_L = 2 \)). For finite temperatures the phase diagram for \( d > d_L \), where a long range ordered magnetic phase exits at finite temperatures, is shown in Figure 1.

This magnetic phase, generally of the antiferromagnetic type occurs for \( (J/W) < (J/W)_c \) and below a critical Néel line.

For \( (J/W) > (J/W)_c \) and finite temperatures we pointed out the existence of a crossover line, which we identified as the coherence line and marks the onset, with decreasing temperature, of a renormalized Fermi-liquid regime on this non-critical part of the phase diagram \[ 5 \]. This physical interpretation of the crossover line allows to recognize immediately that a system just at the critical point, i.e., with \( (J/W) = (J/W)_c \), should behave as a non-Fermi liquid since it does not cross the coherence line and consequently does not enter the Fermi liquid (FL) region of the phase diagram \[ 5 \]. In this paper we obtain the thermodynamic properties of this critical system. We also discuss and clarify the question of breakdown of hyperscaling below the coherence line.

FIG. 1. The phase diagram of heavy fermions close to the zero temperature critical point (schematic). In the present case the shift exponent of the Néel line \( \psi = 1 \) is different from the crossover exponent \( \nu z > 1 \) characterizing the coherence line \( T_c \). The crossover lines \( T_{c1} \) and \( T_{c2} \) are both governed by the same shift exponent of the Néel line \[ 5 \]. Below \( T_c \) hyperscaling is violated and the thermodynamic properties are governed by mean-field exponents.
II. RENORMALIZATION GROUP EQUATIONS

The scaling results and the equations for the Néel and coherence lines can be obtained considering the expansion of renormalization group (RG) equations close to the zero temperature fixed point at \((J/W)_c\). These expansions have a general character and can be written as,

\[
K_{n+1} = K_c + b^{1/\nu}(K_n - K_c) - (T/J)_n
\]

\[
\left(\frac{T}{J}\right)_{n+1} = b^\nu \left(\frac{T}{J}\right)_{n}
\]

The first equation describes the renormalization of the ratio \(K = (J/W)\) close to the critical point \(K_c = (J/W)_c\) under a change in length scale by a factor \(b\). The exponent \(\nu\) is the correlation length exponent. Eq. 2 describes the renormalization of the temperature close to the zero temperature critical point. Since temperature is a parameter its scaling is controlled by that of the coupling constant \(J\) which at the critical point scales according to \(:= \left(\frac{T}{J}\right)\) (3)

where \(a\) is a constant. Since \(l\) is arbitrary we iterate until

\[
l^{1/\nu}(K - K_c - a(T/J)) = 1
\]

and this length scale defines the correlation length

\[
l = \xi = |K - K_c - a(T/J)|^{-\nu} = |j - j_c(T)|^{-\nu}
\]

where \(j = (J/W)\) and \(j_c(T) = (J/W)_c + a(T/J)\). At this length scale, substituting \(l\) for \(\xi\) in Eq.3, we obtain

\[
K_\xi = K_c + 1 + \frac{a(T/J)}{|j - j_c(T)|^{\nuz}}
\]

The line \(T_c/J = |j - j_c(T)|^{\nuz}\) in the non-critical side of the phase diagram is a crossover line since \(K_\xi\) and consequently all functions of this ratio have different asymptotic behaviors for \(T >> T_c\) and \(T << T_c\). It is governed by the exponent \(\nuz\) and is identified as the coherence line in Figure 1. On the other hand, at the line \(j = j_c(T)\) or \(T_N/J = \frac{1}{\xi}|j - j_c(T = 0)|\) the correlation length diverges and this line can be identified with the Néel line in the critical region, i.e., \((J/W) < (J/W)_c\). If we define a \textit{shift exponent} \(\psi\) through the relation \(T_N/J \propto |j - j_c|^\psi\), the present approach leads to \(\psi = 1\) independent of the crossover exponent \(\nuz\). The critical line vanishes linearly close to the zero temperature critical point which is due to the fact that in Eq.1 the lowest order analytic term in temperature is a linear contribution.

III. GENERAL SCALING RELATIONS

The general recursion relations, Eqs.1 and 2 give rise to the following scaling form for the singular part of the free energy density close to the zero temperature critical point \(T_c\)

\[
f \propto |\delta|^{2-\alpha} f_0[T/T_c]
\]

where \(\delta = j - j_c(T = 0)\) and \(T_c \propto |\delta|^{\nuz}\). In this section we consider the scaling properties of physical quantities in the non-critical side of the phase diagram, i.e., \((J/W) > (J/W)_c\). We shall neglect the temperature dependence of \(j_c(T)\) which is valid as long as \(\nuz > \psi\) and \(T \leq T_c\). At \(T = 0\) , \(f_0(0) = constant\) and the above expression gives the singular part of the ground state free energy density close to the transition (\(\delta = 0\) defines the critical point). In fact at \(T = 0\), Eq.4 defines the critical exponent \(\alpha\). The exponents \(\alpha\), \(\nu\) and \(z\) defined above are associated with the zero temperature unstable fixed point governing the quantum phase transition [4].

The existence of a FL regime for \(T << T_c\), as found experimentally, allows to take a Sommerfeld type of expansion for the scaling function of the free energy density in this temperature range [5]. We get

\[
f \propto |\delta|^{2-\alpha} \{a_1 + a_2(T/T_c)^2 + a_3(T/T_c)^4 + \cdots\}
\]

where \(a_1, a_2\) and \(a_3\) are constants. The specific heat for \(T << T_c\) is then given by

\[
C \propto T \frac{\partial^2 f}{\partial T^2} \propto |\delta|^{2-\alpha-2\nuz} T
\]

which shows the characteristic Fermi-liquid linear temperature dependence with an effective thermal mass \(m_T \propto C/T \propto |\delta|^{2-\alpha-2\nuz}\).

The scaling expression for the free energy can be easily generalized to take into account the presence of a field conjugate to the order parameter, in the present case a staggered field \(H\) and of a uniform magnetic field \(h\), we get

\[
f \propto |\delta|^{2-\alpha} F \left[ \frac{T}{|\delta|^{\beta}}, \frac{H}{|\delta|^{\gamma}}, \frac{h}{|\delta|^\phi_h} \right]
\]

where \(\nu\) and \(z\) are the correlation length and dynamic exponents as before, \(\beta\) and \(\gamma\) are standard critical exponents which describe the critical behavior of the staggered magnetization and susceptibility respectively. \(\phi_h\) is a new exponent associated with the uniform magnetic field \(h\). These exponents obey standard scaling relations as \(\alpha + 2\beta + \gamma = 2\) but the hyperscaling relation is modified due to the quantum character of the critical point and is now given by

\[
2 - \alpha = \nu(d + z)
\]
where $d$ is the dimensionality of the system. From Eq.7 we can deduce the scaling properties of the physical quantities of interest like the low temperature $(T \ll T_c)$ field-dependent uniform magnetization $m(h) \propto \langle \partial f/\partial h \rangle \propto \vert \delta \vert^{2-\alpha-2\phi_h} f_m(h/h_c)$ with $h_c \propto \vert \delta \vert^{\phi_h}$, the uniform, low field, susceptibility, $\chi_0 \propto \partial^2 f/\partial h^2 \propto \vert \delta \vert^{2-\alpha-2\phi_h} f_S(T/T_c)$, etc.

The scaling theory formulated above is completely general and any description of heavy fermions in terms of a zero temperature phase transition should fall within this approach. The crucial question is of course to determine the critical exponents, i.e., the universality class of the zero temperature critical point. It is interesting to point out that for $d = 1 < d_L$, $(J/W)_c = 0$, but there is still a crossover line for $(J/W) > 0$ which is given by $k_B T_c \propto W e^{U/W}$, an expression similar to that of the Kondo line.

IV. GAUSSIAN THEORIES

Eq. 6 for the specific heat in the Fermi liquid regime $(T \ll T_c)$ can be written, using the modified hyperscaling relation, Eq.8, as

$$m_T \propto \gamma = C/T \propto \vert \delta \vert^{\nu(d-z)}$$  

(9)

For dimension $d = 3$ and in the case the dynamic exponent $z = 3$, which is generally associated with ferromagnetic fluctuations, this gives essentially the result $\gamma \propto L\vert \delta \vert$ although the scaling theory cannot reproduce the logarithmic singularity. This logarithmic dependence has the same origin of the paramagnon mass enhancement of a nearly ferromagnetic system obtained originally by Doniach and Engelsberg for the Hubbard model. In this case $\delta = \vert (U/W) - (U/W)_c \vert$ where $U$ is the Coulomb repulsion, $W$ the bandwidth and $(U/W)_c$ is the critical ratio at which the $T = 0$ ferromagnetic transition occurs.

The case of antiferromagnetic fluctuations characterized by $z = 2$ can be considered going to higher order in the Sommerfeld expansion of the free energy. Again using hyperscaling, i.e., Eq.8, we get

$$\gamma = C/T \propto a_2 \vert \delta \vert^{\nu(d-z)} + a_3 \vert \delta \vert^{\nu(d-3z)} T^2$$  

(10)

In three dimensions with $z = 2$ this yields

$$C/T \propto a_2 \vert \delta \vert^{1/2} + a_3 T^2 / \vert \delta \vert^{3/2}$$  

(11)

for $T \ll T_c$ where we took the Gaussian exponent $\nu = 1/2$. This result coincides with that of the Gaussian theory of Millis for a nearly antiferromagnetic Fermi liquid and relies on the validity of the hyperscaling relation, Eq.8, below $T_c$. In fact the exponent $\alpha$ in Gaussian theories of zero temperature magnetic phase transitions is determined by the modified hyperscaling relation i.e. $\alpha = \frac{4-\nu(d-z)}{d}$, where we used the Gaussian value for the correlation length exponent $\nu = 1/2$. Consequently in these theories hyperscaling is automatically satisfied. Notice that for $d + z > 4$ the Gaussian exponent $\alpha < 0$ implying that singularities in these theories are weaker than those obtained in a mean-field approach where the exponent $\alpha$ is fixed at $\alpha = 0$ for $d + z > d_c$. Here $d_c = 4$ is the upper critical dimension for these magnetic transitions. However the mean field exponents $\alpha = 0$ and $\nu = 1/2$ violate hyperscaling above $d_c$, i.e., do not satisfy Eq.8 for $d + z > 4$.

Let us consider again the enhancement of the thermal mass for an itinerant three dimensional nearly ferromagnetic system on the light of the discussion above. In the general scaling approach we have, $m_T \propto \vert \delta \vert^{2-\alpha-2\nu}$, which is obtained without using the hyperscaling relation. Using the mean-field exponents $\alpha = 0$, $\nu = 1/2$ and $z = 3$, which violate Eq.8 for $d = 3$, we get $m_T \propto \vert (U/W) - (U/W)_c \vert^{-1} = \vert \delta \vert^{-1}$.

Since the mean-field susceptibility exponent $\gamma = 1$, we find $\chi_0 \propto \vert \delta \vert^{-1}$ where $\chi_0$ is the limiting, $T \to 0$, uniform susceptibility. Consequently the ratio $\chi_0/m_T = \text{constant independent of (U/W)}$, within mean-field, differently from the paramagnon or Gaussian results which yield $m_T \propto L\vert \delta \vert$, as we saw previously and consequently to a diverging ratio $\chi_0/m_T \propto \vert \delta \vert^0$. It is interesting to compare these different predictions with measurements at low temperatures of the electronic and magnetic Grueneisen parameters of bulk Palladium, the prototype of an itinerant 3 - $d$ nearly ferromagnetic system. These pressure experiments in which the ratio $(U/W)$ is varied due to its dependence on volume $V$ yield

$$\frac{\partial \ln \chi_0/m_T}{\partial \ln V} = +2.8 - 2.2 = +0.6 \pm 0.5$$

This near cancellation implies $\chi_0/m_T \approx \text{constant}$ in good agreement with the mean-field scaling prediction. The analysis above points out that the true critical behavior of the renormalized, nearly magnetic, Fermi liquid below the coherence line is governed by mean-field exponents. Since the existence of paramagnons in nearly ferromagnetic systems is well established we expect that these Gaussian type of fluctuations become important above this line as we shall further discuss.

The breakdown of hyperscaling in the Fermi liquid regime below the coherence line can be more specifically discussed within Hertz approach to quantum magnetic phase transitions. In this case it can be seen how it arises. It is due to the dangerous irrelevant quartic interaction $u_0$ in the field operators, for $d + z > 4$. The scaling form of the singular part of the $T = 0$ free energy can be written in this approach as $f \propto \vert \delta \vert^{\nu(d-z)} F[u_0/\delta]^{d+z}$.
\[ F(x \to 0) \propto 1/x. \] This asymptotic behavior of the scaling function can be easily obtained if we consider the expansion of the free energy in the ordered magnetic phase in terms of the order parameter \( \eta \), i.e., \( f \propto \eta^2 + u_0 \eta^4 + \ldots \) and recall that \( \eta \propto (u_0)^{-1/2} \) in Landau theory \([8]\). Using this result for \( F(x \to 0) \) we get \( f \propto |\delta|^{2(d+z)} / (u_0 |\delta|^{d+z-2}) \) which leads to \( f \propto |\delta|^2 \) and consequently to the mean-field exponent \( \alpha = 0 \) which violates Eq. 8 for \( d + z > 4 \) since \( \nu = 1/2 \).

The argument presented above, for \( T = 0 \), should hold below the coherence line, i.e., for \( T < T_c \). For \( T > T_c \), the interaction \( u_0 \) couples to temperature giving rise to a new effective field \((u_0 T)\) which is now relevant \([6]\). Consequently the previously discussed mechanism for violation of hyperscaling associated with the dangerous irrelevance of \( u_0 \) is inoperative for \( T > T_c \). Then, for \( T > T_c \) and in particular at \( \delta = 0 \), there is no reason to expect violation of hyperscaling and Gaussian fluctuations should become dominant. This is indeed indicated by the experiments in heavy fermion systems as we will discuss.

V. THE HEAVY FERMION FIXED POINT

The analysis of experimental data on heavy fermions below the coherence line, \( T \leq T_c \), has led to the following empirical relations between the exponents associated with the zero temperature fixed point: \( 2 - \alpha = \nu z \) and \( \phi_h = \nu z \)(\([11]\)), where \( \phi_h \) controls the scaling of the uniform magnetic field \([15]\). The former relation clearly violates hyperscaling, Eq. 8, for \( d = 3 \) suggesting that the fixed point governing the physics of heavy fermions when approached from below the coherence temperature \((T < T_c)\) is characterized by classical mean-field exponents at least in \( d = 3 \). As we pointed out above mean-field exponents, contrary to the Gaussian ones, violate hyperscaling above the upper critical dimension \( d_c \), i.e., in the case of quantum transitions, for \( d + z > d_c \)(\([9]\)). Notice that the relations \( 2 - \alpha = \nu z \) and \( \phi_h = \nu z \), which lead to \( 2 - \alpha = \phi_h \) imply that the \( T = 0 \) uniform susceptibility behaves as \( \chi_0 \propto |\delta|^{-\nu z} \) and the thermal mass, \( m_T \propto |\delta|^{-\nu z} \) as the critical point is approached at zero temperature. This behavior of \( \chi_0 \) shows the importance of ferromagnetic fluctuations and the reason for the enhanced uniform susceptibility in heavy fermions. Since the staggered susceptibility also diverges at the transition, the zero temperature critical point which governs the physics of heavy fermions is in fact a multicritical point. This led us to consider that the heavy fermion fixed point is in the universality class of the classical tricritical point with \( \alpha = 1/2, \nu = 1/2, \gamma = 1, \phi_T = \nu z / \phi_h = 1 \) and furthermore with the dynamic exponent \( z \) assuming the value \( z = 3 \). This yields for the coherence line the equation \( T_c \propto |\delta|^{3/2} \). Since \( \nu z > 1 \), this line rises from the \( T = 0 \) axis with zero derivative reducing the Fermi liquid region close to the critical point for \((J/W) > (J/W)_c \). This may be responsible for the ubiquity of non-Fermi liquid behavior in heavy fermions driven to a non-magnetic state by chemical or external pressure \([12,13]\). There are of course other possible scenarios as the nearly antiferromagnetic Fermi liquid \([11]\) with \( z = 2 \) such that \( \nu z = 1 \). For disordered systems one finds \( z = 4 \) associated with diffusive modes \([9]\) and \( \nu = 1/2 \) such that \( \nu z = 2 \).

VI. NON-FERMI LIQUID BEHAVIOR

Fig. 1 shows the phase diagram of heavy fermions for \( \nu z > \psi = 1 \) such that the critical line vanishes linearly close to the \( T = 0 \) critical point as observed \([2]\). It is clear from this figure and our interpretation of the crossover line that a system just at the critical point, i.e., with \((J/W) = (J/W)_c, |\delta| = 0\), does not cross the coherence line and consequently does not enter the Fermi liquid regime. In previous papers \([11]\) we have obtained the properties of such a system under the assumption that \( \psi = \nu z \) which is the content of the so-called generalized scaling hypothesis \([15]\). The experiments however indicate that this may not hold in heavy fermions and in fact \( \nu z > \psi = 1 \)(\([1]\)). We shall then calculate the thermodynamic properties of the system at the critical point for the case \( \nu z \geq \psi \). For this purpose we introduce a set of critical exponents, which we distinguish by a tilde and describe the singularities along the critical Néel line \([13]\). It is clear that these tilde exponents are different from those of the zero temperature critical point. In the renormalization group language this is a consequence that temperature is a relevant field and consequently the finite temperature transitions are governed by another, \( T \neq 0 \), fixed point.

A. Free energy

Let us consider the expression for the free energy density,

\[ f \propto |j - j_c(T)|^{2-\alpha} F[t] \]

where

\[ t = \frac{T/J}{|j - j_c(T)|^{\nu z}} \]

and

\[ j_c(T) - j_c(0) \propto T^{1/\psi} \]

The scaling function \( F[t] \) has the following asymptotic behaviors. When \( t \to 0 \), \( F[0] = constant \). On the other hand for \( j \to j_c(T) \), i.e., for \( t \to \infty \) we should recover the critical behavior of the finite temperature antiferromagnetic transition, then \( F[t \to \infty] \propto t^x \) with \( x = \frac{\nu z}{2-\alpha} \)(\([9]\)). This guarantees that in this limit \( f \propto |j - j_c(T)|^{2-\alpha} A(T) \) or \( f \propto |T - T_N(j)|^{2-\alpha} A(T) \) where
the amplitude $A(T) = T^{\frac{d+z}{z+d}}$. At the critical point, i.e., $(J/W) = (J/W)_c$, we obtain

$$f \propto T^{\frac{d+z}{z+d} + \frac{2-\alpha}{z+d}}$$ (15)

Let us consider some particular cases.

i) $\psi = \nu z$. In this case $f \propto T^{\frac{d+z}{z+d}}$. Using hyperscaling, i.e., Eq. 8, we find a correction to the singular part of the free energy density $\propto T^{\frac{d+z}{z+d}}$ which yields for the specific heat $C/T \propto T^{\frac{d+z}{z+d}}$. In particular for $d = z$ this yields $C/T \propto \ln T$. On the other hand notice that using the empirical mean-field relation $2 - \alpha = \nu z$ we get $f \propto T$ and the specific heat is determined by analytic contributions to the free energy such that $C \propto T$.

ii) $\psi \leq \nu z$. In this case the specific heat is generally given by,

$$C/T \propto \frac{\partial^2 f}{\partial T^2} \propto T^{\frac{(2-\alpha)(\nu z - \psi) + \psi(2\nu z - 2\nu z)}{2\nu z}}$$

Hyperscaling yields

$$C/T \propto \frac{\partial^2 f}{\partial T^2} \propto T^{\frac{(2-\alpha)(\nu z - \psi) + \psi(2\nu z - 2\nu z)}{2\nu z}}$$

for $\psi = 1$, $d = z = 3$ and $\nu z = 3/2$ we get $C/T \propto T^{\frac{3}{2}}$. For $\nu z > 2$, such that $\nu z = 1$, we obtain $C/T \propto T^{1/\nu z}$ as found previously by Millis using a different approach [3]. In every case we should consider an additional temperature independent regular contribution to $C/T$.

The mean-field result is obtained using the empirical relation $2 - \alpha = \nu z$, we get

$$C \propto T^{\frac{(2-\alpha)(\nu z - 1)}{2\nu z}}$$

In the case $\psi = 1$, $\nu z = 3/2$ and $\alpha \approx 0$ we get a stronger than logarithmic divergence for the thermal mass, namely, $C/T \propto T^{-1/3}$. For $\psi = 1$, $\nu z = 2$, which is the disordered case, we get $C/T \propto T^{-\tilde{\nu}_z/2}$. For $\tilde{\nu}_z$ small this gives essentially $C/T \propto \ln T$.

B. Order Parameter Susceptibility

We start with the scaling expression

$$\chi_s \propto |j - j_c(T)|^{-\gamma} F(t)$$

which close to the critical Néel line becomes

$$\chi_s \propto |j - j_c(T)|^{-\gamma} T^{\frac{2-\alpha}{z+d}}$$

For $(J/W) = (J/W)_c$ we get

$$\chi_s \propto T^{-\gamma (\frac{2-\alpha}{z+d}) - \frac{2-\alpha}{z+d}}$$ (16)

C. Uniform Susceptibility

If we take into account that the uniform susceptibility does not diverge at the critical Néel line we get the following expression for $\chi_0$ at $(J/W)_c$,

$$\chi_0 \propto T^{\frac{2-\alpha - 2\phi}{z+d}}$$ (17)

Now using hyperscaling, which is consistent with Gaussian exponents and the relation $\phi_h = \nu z$ [3], we obtain

$$\chi_0 \propto T^{\frac{2-\alpha - 2\phi_h}{z+d}}$$ (18)

which, assuming $z = 3$, leads to a constant or at most a weakly logarithmic divergent susceptibility at $(J/W)_c$. In this case also $dm/dh \propto \ln h$ or constant.

On the other hand taking the dynamic exponent $z = 2$, appropriate for antiferromagnetic fluctuations, yields, $\chi_0 \propto T^{1/2}$, at $\delta = 0$, eventually with an additional, temperature independent, regular contribution. We also get $dm/dh \propto h^{1/2}$ in this case. The above temperature dependence of $\chi_0$ has in fact been observed [3].

Notice that using the mean-field empirical relation $2 - \alpha = \nu z$ and $\nu z = \phi_h$ we find that at the critical point $\chi_0 \propto T^{-1}$ which is the expected mean-field result for a Néel transition occurring at $T = 0$. This singular behavior is quite different from that observed. For the differential uniform susceptibility mean-field yields at the critical point, $\chi_h = dm/dh \propto h^{-1}$.

The behavior of the uniform susceptibility at $|\delta| = 0$ is relevant as concerns the possibility of superconductivity at this point. In fact a $\chi_0$ diverging with temperature at the critical point would appear detrimental for superconductivity since it signals the presence of unscreened net magnetic moments.

D. Nuclear Relaxation Time

The nuclear relaxation time is given by [16]:

$$\frac{1}{T_1} = k_B T \sum_q |A_q|^2 \frac{\text{Im} \chi(q, \omega_N)}{\omega_N}$$ (19)

where $A_q$ is the form factor and $\text{Im} \chi$ is the imaginary part of the wavevector and frequency dependent susceptibility $\chi(q, \omega)$ of the system at the nuclear resonance frequency $\omega_N$. This expression has contribution from all wavevectors $q$. Let us consider the contribution of wavevectors near $q = Q_0$ the wavevector characterizing the ordered antiferromagnetic phase. Using the dynamic scaling hypothesis [17]

$$\text{Im} \chi(q, \omega_N) = \chi_s(T) D(q \xi, \omega \xi^z)$$ (20)

where $\xi$ is the correlation length and $z$ the dynamic exponent and substituting in the previous expression we get [16]
\[
\frac{1}{T_1} = k_B T \int d^2 q \chi_s(T) D(q \xi, \omega \xi^z) \frac{\omega}{\omega_N} 
\]
which yields
\[
\frac{1}{T_1} \propto T \chi_s(T) \xi^{z-d} \tag{22}
\]

Note that the quantity \(\xi^{z-d}\) scales as the thermal mass, i.e., \(\xi^{z-d} \propto m_T \propto C/T\) such that \(\frac{1}{T_1} \propto \chi_s(T) C(T)\). Consequently if \(C/T \propto \ln T\), we expect to find using the previous results for \(\chi_s\) at least a logarithmic diverging linewidth, i.e., \(\frac{1}{T_1} \propto \ln T\) close to the critical point \((\gamma \geq 1)\) [15].

For completeness we give the temperature dependence of the correlation length at the critical point, we get, \(\xi \propto T^{-\psi/\psi+(\delta-\nu)/\nu z}\).

We now return to the phase diagram of Fig.1. The form of the scaling function Eq.12 together with Eqs. 13 and 14 allow to identify two additional crossover lines in this phase diagram, \(T_{c1}\) and \(T_{c2}\), whenever \(\psi \neq \nu z\) [3]. Both these lines are governed by the same exponent \(\psi\) of the critical Néel line [3] [13] [14], i.e., \(T_{c1} \propto |\langle j \rangle|^{\psi}\) and \(T_{c2} \propto |\langle j \rangle|^{\nu z}\). The line to the left of \((J/W)_{c1}, T_{c1}\), marks the onset of the classical regime where the singularities close to the Néel line are described by the tilde exponents. In the non-critical side of the phase diagram the temperature dependence of the physical quantities in the regime between \(T_{c2}\) and the coherence line \((T_c < T < T_{c2})\), are obtained from the expressions deduced before for \(\delta = 0\) and taking \(\psi = \nu z\) since \(\psi\) is the relevant crossover exponent in this temperature interval [3]. In this region we expect hyperscaling to hold and that Gaussian fluctuations become dominant. The thermal mass for example in this temperature region \((T_c < T < T_{c2})\) is given by \(C/T \propto \ln T\), for \(z = 3\).

E. Resistivity at the critical point

The resistivity at the critical point can be easily obtained generalizing previous calculations that consider the scattering of conduction electrons by different type of bosons [13] [20]. The resistivity can be expressed, using the fluctuation-dissipation theorem, in terms of the dynamic susceptibility \(\chi(k, \omega)\) associated with the elementary excitations [13]. In three dimensions it is given by [20]

\[
\rho_{\text{c}} = A \beta \int_0^\infty d\omega \frac{\omega}{e^{\beta \omega} - 1} \frac{1}{1 - e^{-\beta \omega}} \times 
\int_0^{2k_F} dk k^3 |F_k|^2 \Im \chi(k, \omega) \tag{23}
\]

where \(A = \frac{3}{4 \pi k_F^2}, \beta = \frac{1}{k_B T}\) and \(F_k\) is a form factor. A similar expression but with the integral in \(k\) extending in the interval \([0, +\infty]\) and weighted by a factor \(k^2\) rather than \(k^3\) is obtained if the scattering of the electrons by the bosons does not conserve momentum [20]. This may occur in a disordered alloy in which case the boson propagator should be averaged over all wavevectors [21]. In this case there is always a finite residual resistivity. In order to make contact with previous approaches [13] [19] we consider, without loss of generality, an explicit expression for the imaginary part of the dynamic susceptibility at the critical point

\[
\Im \chi(k, \omega) = \chi_{\text{static}} \frac{\tilde{\omega}}{\tilde{\omega}^2 + \Delta^2} \tag{24}
\]

where \(\tilde{\omega} = (\omega/k^l)\), \(\Delta = Dk^m\). Taking \(F_k = C k^p\) and making a double change of variables in the integrals for the resistivity we obtain [20]

\[
\rho \propto T^{2\nu z + 4}\tag{25}
\]

and in the case momentum is not conserved

\[
\Delta \rho_{\text{Alloy}} \propto T^{2\nu z + 4}\tag{26}
\]

where \(z = l + m\) is the dynamic critical exponent. We neglect the k-dependence of the form factor \((p = 0)\) and the temperature dependence of \(\chi_{\text{static}}\). For \(p = 0, l = 1\) and \(z = 3\) this gives the result \(\rho \propto T^4\) obtained by Mathon [19] due to critical paramagnons. In the same case but without momentum conservation one finds at the critical point \(\Delta \rho_{\text{Alloy}} \propto T^3\). For diffusive modes with \(l = 2\) and \(z = 4\) we get, \(\rho \propto T^3/2\) and without momentum conservation \(\Delta \rho_{\text{Alloy}} \propto T^{5/4}\) at the critical point.

The case of antiferro-paramagnons, \(l = 0, z = 2\) yields, \(\rho \propto T^2\) and \(\Delta \rho_{\text{Alloy}} \propto T^2\). In the former case, taking into account the temperature dependence of \(\chi_{\text{static}} = \chi_s \propto T^{-1}\) obtained before, we get a linear temperature dependent resistivity at the critical point as observed [2].

It is interesting to consider the high temperature behavior \((T >> T_c)\) of Eq. 23 for the resistivity away from the critical point. Using the Kramers-Kronig relation we find

\[
\rho \approx A k_B T \int_0^{2k_F} dk k^3 |F_k|^2 \chi_{\text{static}}(k) 
\]

This high temperature linear behavior is characteristic of alloys where a crossover to a \(T^2\) behavior at low temperatures is also observed [22].

VII. DISCUSSION AND CONCLUSIONS

Using an expansion of the renormalization group equations close to the zero temperature fixed point describing the physics of heavy fermions we have obtained a critical line which vanishes linearly close to the zero temperature critical point independently of the crossover exponent \(\nu z\).
We have then generalized our previous scaling approach for the case the shift exponent describing the critical Néel line is different from the crossover exponent of the coherence line. In particular we have obtained the properties of the non-Fermi liquid system at the critical point, i.e., for \((J/W) = (J/W)_c\). In the general case that \(\psi \neq \nu z\), we have to consider two new crossover lines in the phase diagram of Figure 1, both governed by the shift exponent \(\psi\). In the region between \(T_{c1}\) and \(T_{c2}\), which includes the non-Fermi liquid regime above the critical point, the \(\tilde{\psi}\) exponents which describe the singularities along the Néel line appear explicitly in the temperature dependence of the physical quantities besides those associated with the \(T = 0\) fixed point. We have considered different scenarios or universality classes for the \(T = 0\) transition.

Our previous analysis of the pressure dependence of several physical quantities in different heavy fermion systems below the coherence temperature led to the relations \(2 - \alpha = \nu z\) and \(\phi_b = \nu z\). The former equality is inconsistent with hyperscaling implying that in this region of the phase diagram hyperscaling is violated and the physical properties are described by mean-field exponents. Due to the generality of the concept of coherence line this result is also valid for nearly ferromagnetic systems as our analysis of the pressure dependence of the Wilson ratio of \(Pd\) has shown.

While mean-field exponents are appropriate to describe the Fermi liquid regime for \(T < T_c\), above the coherence line and in particular at the critical point hyperscaling is eventually restored. Experimentally this is shown, in the case of heavy fermions, by the saturation of the uniform susceptibility at very low temperatures and the weak divergence of the thermal mass at the critical point. This can be obtained using hyperscaling while according to mean-field \(\chi_0\) is expected to diverge strongly \((\chi_0 \propto T^{-1})\) and \(C/T\) has a stronger than logarithmic divergence. In the context of the phenomenological scaling approach it is not possible to determine \textit{a priori} the validity of hyperscaling in the different regions of the phase diagram. However using Hertz approach to quantum magnetic phase transitions we have discussed how a mechanism for the breakdown of hyperscaling at low temperatures, i.e., below the coherence line, in fact becomes ineffective for \(T > T_c\) and hyperscaling is restored above this line.

We have pointed out the relevance of ferromagnetic fluctuations in heavy fermions as indicated, for example, by the enhanced uniform susceptibility. These fluctuations, which are associated with the dynamic exponent \(z = 3\), may be due to the existence of ferromagnetically correlated planes which in turn order antiferromagnetically. This type of correlations is often found in metamagnetic systems. Our analysis has shown that in the case \(\nu z \neq \psi\) we do not recover in general, for \(z = 3\), the result \(C/T \propto \ln T\) at \(\delta = 0\) but away from the critical point, i.e., for \(T_c < T < T_{c2}\). This region extends down to very low temperatures close to \(\delta = 0\) since \(T_c \propto |\delta|^{3/2}\) making it difficult to identify the critical point. Also away from \(\delta = 0\) and \(T > T_c\), it is easier to obtain a linear temperature dependent resistivity as outlined in section 6.5.

We found that taking the dynamic exponent \(z = 2\) yields the observed temperature dependence of the uniform susceptibility at \(\delta = 0\). In this case, even without considering the existence of \textit{hot lines} in the Fermi surface, we could obtain a linear temperature dependent resistivity at \(\delta = 0\). It is clear from our previous remarks that the case of \(z = 2\) is also the most favorable for the appearance of superconductivity at the critical point.

We hope further experiments are carried out to distinguish between the different scenarios and determine unambiguously the universality class of the zero temperature heavy fermion fixed point. The general scaling theory formulated above provides a powerful tool to accomplish this task.

The scaling theory has proved to be useful to describe the properties of \(Ce\) based heavy fermions. However in the case of \(U\) compounds, probably due to the extension of the \(5 - f\) orbitals, new phenomena appear as the coexistence of long range magnetic order and superconductivity which would be interesting to incorporate in the present approach. On the other hand there is a family of materials based on \textit{Uranium} which exhibit many of the non-Fermi liquid properties discussed above. Whether these systems, which have high residual resistivities, may be described by the ideas discussed here, possibly with different exponents or within a \textit{single impurity} scenario remains to be investigated.

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