We review an approach to the construction and classification of $p$-brane solitons in arbitrary dimensions, with an emphasis on those that arise in toroidally-compactified M-theory. Procedures for constructing the low-energy supergravity limits in arbitrary dimensions, and for studying the supersymmetry properties of the solitons are presented. Wide classes of $p$-brane solutions are obtained, and their properties and classification in terms of bound states and intersections of M-branes are described.
1 Introduction

String theory and its goals have undergone a number of dramatic re-appraisals since it was first introduced. First seen as a candidate for describing the strong interactions [1], it fell into disfavour on account of the fact that its spectrum included a massless spin-2 excitation that was not seen in the hadronic arena. A subsequent revival of interest resulted from the realisation that this spin-2 state should be interpreted not as a hadron but as the graviton [2, 3]. With this change of emphasis, string theory moved to the forefront of attempts to find a framework for describing a quantum theory of gravity, and even more ambitiously, a unified “theory of everything.” Encouraged by the discovery that considerations of anomaly-freedom greatly restricted the possible gauge groups [4], and by the subsequent discovery of the heterotic string, many attempts were to make contact with the phenomenological world, of relatively low energy compared with the Planck-scale unification regime. It is probably fair to say that early claims of the virtual uniqueness and predictive power of the passage to the phenomenological arena have proved to be an exaggeration, and at present the best that can be said is that at least there seem to be ways of embedding the standard model into the theory. After a fallow period during which little further progress was achieved, the subject was revolutionised again in 1995 with far-reaching discoveries about some non-perturbative aspects of string theory. Most notable of these was the observation that by including non-perturbative states in the spectrum of the type IIA string, whose presence is indicated by duality symmetries, the degrees of freedom would be described in the strong-coupling regime not by a ten-dimensional theory, but instead an eleven-dimensional one [5, 6]. In fact this discovery has led to a revival of the fortunes of eleven-dimensional supergravity [7], which, although extensively investigated in the past as a possible candidate for superunification (see, for example, [8]), had long ago been abandoned on account of its apparent inability to yield a realistic four-dimensional low-energy description of the world. It is now viewed as the low-energy limit of some yet to be discovered M-theory, which would provide the more appropriate description of the degrees of freedom of the type IIA string in all except the perturbative weak-coupling regime [9].

Evidence for the ability of $D = 11$ supergravity to describe elements of the spectrum of the type IIA string can be seen by considering the BPS-saturated $p$-brane soliton solutions in the low-energy effective theory from the type IIA string. Included in these are not only electrically-charged string solutions, which can be directly identified with elementary string states in the perturbative spectrum, but also other solutions that are more akin to solitons. The conjectured duality symmetries of the string, together with the fact that these solutions
are BPS saturated, and thus are expected to be protected at the quantum level, lead to
the expectation that they can be identified with non-perturbative quantum states in the
string spectrum. There are, for example, BPS-saturated black hole solutions in the type IIA
low-energy effective theory whose mass spectrum can be shown to coincide precisely with
the spectrum of massive particles coming from the Kaluza-Klein dimensional reduction of
$D = 11$ supergravity on a circle $[3]$. The above, and other, considerations lead to a strong belief that M-theory, and its low-
energy $D = 11$ supergravity limit, are relevant for investigating the type IIA string. A
further surprise occurred when it was argued that M-theory is also relevant for describing
the $E_8 \times E_8$ heterotic string, by making the relatively innocuous-sounding modification of
compactifying it on $S^1/Z_2$ rather than $S^1 [9]$. With these, and other recent developments, the vital rôle of M-theory and its low-energy
$D = 11$ supergravity limit have become established. In what follows, we shall present an
overview of a particular approach to the study of the solitonic $p$-brane spectrum of the
theory, and its toroidal compactifications to dimensions $D \leq 11$. Our emphasis will be
on adopting a general and unified approach, in which solutions in all dimensions can be
studied and classified in a systematic way. We are aided in this aim by the fact that it
is not necessary, for these purposes, to have the complete toroidally-compactified maximal
supergravity theories at our disposal. The reason for this is that the $p$-brane solitons of
interest here are purely bosonic solutions, and the fermionic sectors of the theories need be
considered only insofar as they determine the fractions of supersymmetry that are preserved
by the solutions. It turns out that such questions can be answered very straightforwardly by
reformulating the problem as a bosonic one in $D = 11$ itself, in a way that can then easily
be re-expressed in the lower-dimensional reduced theories without the necessity of explicitly
performing a dimensional reduction of the fermionic sector of $D = 11$ supergravity.

We begin this review in section 2 by giving an explicit construction of the bosonic sectors
all the maximal supergravities in $D \leq 10$ that are obtained by toroidal compactification
of $D = 11$ supergravity. In section 3, we discuss the basic structure of $p$-brane soliton
solutions, and we also introduce the Bogomol’nyi matrix in $D = 11$, and its dimensional
reductions, which can be used to determine the fractions of supersymmetry that are pre-
served by the various solitons. In section 4, we discuss in detail various classes of $p$-brane
solutions in the maximal supergravities in arbitrary dimensions. These include extremal
solutions, which saturate Bogomol’nyi bounds, and non-extremal solutions, where the mass
in general exceeds the Bogomol’nyi bound. Our discussion includes not only the standard
kinds of $p$-brane solutions, but also a rather general analysis of the equations of motion, yield additional solutions that have received less attention in the literature. Included in these are different kinds of non-extremal $p$-brane solutions, and also non-extremal $p$-branes that arise as solutions of certain systems of Toda equations. Section 4 also includes a discussion of an interpretation for certain kinds of $p$-branes as bound states of more fundamental ones. Finally, in section 5, we discuss the dimensional reduction and oxidation of $p$-brane solutions. Topics considered here include the two kinds of dimensional reduction, corresponding to vertical and diagonal descent in a plot of spacetime dimension versus $p$-brane dimension, and the oxidation of lower-dimensional $p$-brane back to eleven dimensions, where some of them acquire a new interpretation as intersections of the fundamental M-branes of M-theory.

2 Maximal supergravities in $D \leq 11$

In this section, we discuss the toroidal dimensional reduction of the bosonic sector of $D = 11$ supergravity, whose Lagrangian takes the form \[ \mathcal{L} = e^{R} - \frac{1}{48} e^{2} \hat{F}_{4}^{2} + \frac{1}{6} \hat{F}_{4} \wedge \hat{F}_{4} \wedge \hat{A}_{3} . \] (2.1)

For brevity, we have written the final term as an 11-form; it is understood that it should be dualised before integrating the Lagrangian over the $D = 11$ spacetime. The subscripts on the potential $A_{3}$ and its field strength $F_{4} = dA_{3}$ indicate the degrees of the differential forms. We shall reduce the theory to $D$ dimensions in a succession of 1-step compactifications on circles. At each stage in the reduction, say from $(D + 1)$ to $D$ dimensions, the metric is reduced according to the standard Kaluza-Klein prescription

\[ ds_{D+1}^{2} = e^{2\alpha \varphi} ds_{D}^{2} + e^{-2(D-2)\alpha \varphi} (dz + A_{1})^{2} , \] (2.2)

where the $D$ dimensional metric, the Kaluza-Klein vector potential $A_{1} = A_{M} dx^{M}$ and the dilatonic scalar $\varphi$ are taken to be independent of the additional coordinate $z$ on the compactifying circle. The constant $\alpha$ is given by $\alpha^{-2} = 2(D-1)(D-2)$, and the parameterisation of the metric is such that a pure Einstein action is reduced again to a pure Einstein action together with canonically-normalised kinetic terms for $F_{2} = dA_{1}$ and $\varphi$:

\[ e R \rightarrow e R - \frac{1}{4} e^{-2(D-1)\alpha \varphi} F_{2}^{2} - \frac{1}{2} e (\partial \varphi)^{2} . \] (2.3)

Gauge potentials are reduced according to the prescription $A_{n}(x, z) = A_{n}(x) + A_{n-1}(x) \wedge dz$, implying that a kinetic term for an $n$-form field strength $F_{n}$ reduces according to the
There is a subtlety here in the expression for the dimensionally-reduced field strength $F_n$, which is most easily seen by working in a vielbein basis, since this facilitates the computation of the inner products in the kinetic terms. From the ansatz for the reduction of the gauge potential we have

$$F_n ightarrow dA_{n-1} + dA_{n-2} \wedge dz = dA_{n-1} - dA_{n-2} \wedge A_1 + dA_{n-2} \wedge (dz + A_1).$$  \hspace{1cm} (2.5)

Thus while it is natural to define the dimensionally-reduced field strength $F_{n-1}$ by

$$F_{n-1} = dA_{n-2},$$

for $F_n$ we should define

$$F_n = dA_{n-1} - dA_{n-2} \wedge A_1,$$

and it is this gauge-invariant field strength that appears on the right-hand side of (2.4). These so-called Chern-Simons modifications to the lower-dimensional field strengths become progressively more complicated as the descent through the dimensions continues.

It is not too difficult now to apply the above reduction procedures iteratively, to construct the $D$-dimensional toroidally-compacted theory from the $D=11$ starting point. It is easy to see that the original eleven-dimensional fields $g_{MN}$ and $A_{MNP}$ will give rise to the following fields in $D$ dimensions,

$$g_{MN} \rightarrow g_{MN}, \quad \phi, \quad A_1^{(i)}, \quad A_0^{(ij)},$$

$$A_3 \rightarrow A_3, \quad A_2^{(i)}, \quad A_1^{(ij)}, \quad A_0^{(ijk)},$$  \hspace{1cm} (2.6)

where the indices $i, j, k$ run over the $11-D$ internal toroidally-compactified dimensions, starting from $i = 1$ for the step from $D = 11$ to $D = 10$. The potentials $A_1^{(i)}$ and $A_0^{(ijk)}$ are automatically antisymmetric in their internal indices, whereas the 0-form potentials $A_0^{(ij)}$ that come from the subsequent dimensional reductions of the Kaluza-Klein vector potentials $A_1^{(i)}$ are defined only for $j > i$. The quantity $\tilde{\phi}$ denotes the $(11-D)$-vector of dilatonic scalar fields coming from the diagonal components of the internal metric.

The Lagrangian for the bosonic $D$-dimensional toroidal compactification of eleven-dimensional supergravity then takes the form [10]

$$\mathcal{L} = eR - \frac{1}{2} e (\tilde{\phi})^2 - \frac{1}{16} e e^{-\phi} F_4^2 - \frac{1}{16} e \sum_i e^{\tilde{a}_i} \phi (F_3^i)^2 - \frac{1}{16} e \sum_{i<j} e^{\tilde{a}_{ij}} \phi (F_2^{ij})^2$$

$$- \frac{1}{16} e \sum_i e^{\tilde{b}_i} \phi (F_2^i)^2 - \frac{1}{16} e \sum_{i<j<k} e^{\tilde{b}_{ijk}} \phi (F_1^{ijk})^2 + \mathcal{L}_{FFA},$$  \hspace{1cm} (2.7)

where the “dilaton vectors” $\tilde{a}, \tilde{a}_i, \tilde{a}_{ij}, \tilde{a}_{ijk}, \tilde{b}_i, \tilde{b}_{ij}$ are constants that characterise the couplings of the dilatonic scalars $\tilde{\phi}$ to the various gauge fields. They are given by [10]
\[ F_{MNPQ} \]

4-form: \[ \tilde{a} = -\tilde{g} , \]

3-forms: \[ \tilde{a}_i = \tilde{f}_i - \tilde{g} , \]

2-forms: \[ \tilde{a}_{ij} = \tilde{f}_{ij} + \tilde{f}_{k} - \tilde{g} , \quad \tilde{b}_{ij} = -\tilde{f}_{ij} , \quad \text{(2.8)} \]

1-forms: \[ \tilde{a}_{ijk} = \tilde{f}_{ijk} + \tilde{f}_{j} + \tilde{f}_{k} - \tilde{g} , \quad \tilde{b}_{ijk} = -\tilde{f}_{ijk} + \tilde{f}_{j} + \tilde{f}_{k} , \]

0-forms: \[ \tilde{a}_{ijk\ell} = \tilde{f}_{ijk\ell} + \tilde{f}_{j} + \tilde{f}_{k} + \tilde{f}_{\ell} - \tilde{g} , \quad \tilde{b}_{ijk\ell} = -\tilde{f}_{ijk\ell} + \tilde{f}_{j} + \tilde{f}_{k} + \tilde{f}_{\ell} , \]

where the vectors \( \tilde{g} \) and \( \tilde{f}_i \) have \((11 - D)\) components in \( D \) dimensions, and are given by

\[ \tilde{g} = 3(s_1, s_2, \ldots, s_{11-D}) , \]
\[ \tilde{f}_i = \left( 0, 0, \ldots, 0, (10 - i)s_i, s_{i+1}, s_{i+2}, \ldots, s_{11-D} \right) , \quad \text{(2.9)} \]

where \( s_i = \sqrt{2/((10 - i)(9 - i))} \). It is easy to see that they satisfy

\[ \tilde{g} \cdot \tilde{g} = 2(11 - D) - 2 , \quad \tilde{g} \cdot \tilde{f}_i = 6D - 2 , \quad \tilde{f}_i \cdot \tilde{f}_j = 2\delta_{ij} + \frac{2}{D} \cdot \frac{2}{D} . \quad \text{(2.10)} \]

We have also included the dilaton vectors \( \tilde{a}_{ijk\ell} \) and \( \tilde{b}_{ijk} \) for “0-form field strengths” in (2.8), although they do not appear in (2.7), because they fit into the same general pattern and they do arise if more general kinds of reduction procedure are carried out [11, 12, 13, 14, 15].

The field strengths are associated with the gauge potentials in the obvious way; for example \( F_4 \) is the field strength for \( A_3 \), \( F_{3}^{(i)} \) is the field strength for \( A_2^{(i)} \), etc. In general, the field strengths appearing in the kinetic terms are not simply the exterior derivatives of their associated potentials, but have Chern-Simons corrections as well, as discussed above.

On the other hand the terms included in \( \mathcal{L}_{F_{FA}} \), which denotes the dimensional reduction of the \( F_4 \wedge F_4 \wedge A_3 \) term in \( D = 11 \), are expressed purely in terms of the potentials and their exterior derivatives. The complete details of all the field strengths, in the notation we are using here, were obtained in [10]. The field strengths are given by

\[ F_4 = \tilde{F}_4 - \gamma^{ij} \tilde{F}_3^i \wedge A_4^j - \frac{1}{2} \gamma^{ik} \gamma^{jl} \tilde{F}_2^{ij} \wedge A_4^k \wedge A_4^l + \frac{1}{6} \gamma^{ij} \gamma^{km} \gamma^{kn} \tilde{F}_1^{ijk} \wedge A_4^k \wedge A_4^m \wedge A_4^n , \]
\[ F_3^i = \gamma^{ji} \tilde{F}_3^j - \gamma^{ji} \gamma^{kl} \tilde{F}_2^{jk} \wedge A_4^l - \frac{1}{2} \gamma^{ij} \gamma^{km} \gamma^{ln} \tilde{F}_1^{ikl} \wedge A_4^m \wedge A_4^n , \]
\[ F_2^{ij} = \gamma^{ki} \gamma^{lj} \tilde{F}_2^{kl} - \gamma^{ki} \gamma^{lj} \gamma^{mn} \tilde{F}_1^{kln} \wedge A_4^n , \quad \text{(2.11)} \]
\[ F_1^{ijk} = \gamma^{li} \gamma^{mj} \gamma^{nk} \tilde{F}_1^{lmn} , \]
\[ F_2^i = \tilde{F}_2^i - \gamma^{ik} \tilde{F}_1^k \wedge A_4^j , \]
\[ F_1^{ij} = \gamma^{kj} \tilde{F}_1^{jk} , \]
where the tilded quantities represent the unmodified pure exterior derivatives of the corresponding potentials, and \( \gamma^{ij} \) is defined by

\[
\gamma^{ij} = [(1 + A_0)^{-1}]^{ij} = \delta^{ij} - A_0^{ij} + A_0^{ik} A_0^{kj} + \cdots.
\]  

(2.12)

Recalling that \( A_0^{ij} \) is defined only for \( j > i \) (and vanishes if \( j \leq i \)), we see that the series terminates after a finite number of terms.

The term \( L_{F,F,A} \) in (2.7) is the dimensional reduction of the \( \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_3 \) term in \( D = 11 \), and is given in lower dimensions by

\[
D = 10 : \quad \frac{1}{2} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_2, \\
D = 9 : \quad \left( -\frac{1}{4} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_1^{ij} - \frac{1}{2} \tilde{F}_3 \wedge \tilde{F}_3 \wedge A_3 \right) \epsilon_{ij}, \\
D = 8 : \quad \left( -\frac{1}{16} \tilde{F}_4 \wedge \tilde{F}_4 A_0^{jk} - \frac{1}{6} \tilde{F}_3 \wedge \tilde{F}_3 \wedge A_5 + \frac{1}{3} \tilde{F}_2 \wedge \tilde{F}_2 \wedge A_3 \right) \epsilon_{ijk}, \\
D = 7 : \quad \left( -\frac{1}{12} \tilde{F}_4 \wedge \tilde{F}_3 A_1^{kl} + \frac{1}{6} \tilde{F}_3 \wedge \tilde{F}_3 \wedge A_7 + \frac{1}{2} \tilde{F}_2 \wedge \tilde{F}_2 \wedge A_2 \right) \epsilon_{ijkl}, \\
D = 6 : \quad \left( -\frac{1}{2} \tilde{F}_4 \wedge \tilde{F}_2 A_0^{klm} + \frac{1}{12} \tilde{F}_3 \wedge \tilde{F}_3 \wedge A_9 + \frac{1}{6} \tilde{F}_2 \wedge \tilde{F}_2 \wedge A_2 \right) \epsilon_{ijklm}, \\
D = 5 : \quad \left( -\frac{1}{48} \tilde{F}_3 \wedge \tilde{F}_2 A_0^{lnm} - \frac{1}{38} \tilde{F}_2 \wedge \tilde{F}_2 \wedge A_1^{lnn} - \frac{1}{12} \tilde{F}_1 \wedge \tilde{F}_1 \wedge A_3 \right) \epsilon_{ijklmn}, \\
D = 4 : \quad \left( -\frac{1}{12} \tilde{F}_2 \wedge \tilde{F}_2 A_0^{mpq} - \frac{1}{2} \tilde{F}_1 \wedge \tilde{F}_1 \wedge A_2 \right) \epsilon_{ijklmnp}, \\
D = 3 : \quad \frac{1}{44} \tilde{F}_1 \wedge \tilde{F}_1 \wedge A_0^{pq} \epsilon_{ijklmnpq}, \\
D = 2 : \quad \frac{1}{192} \tilde{F}_1 \wedge \tilde{F}_1 \wedge A_0^{pqr} \epsilon_{ijklmnpq}.
\]  

In the subsequent sections, we shall be making extensive use of the results presented here, in order to discuss various aspects of \( p \)-brane solitons in toroidally-compactified type II strings.

3 \( p \)-branes and supersymmetry

Extremal \( p \)-branes in various supergravities in different dimensions were constructed in the past [16-25]. Our principle aim in this section will be to explain a procedure for determining the fractions of supersymmetry that are preserved by the various \( p \)-brane solitons that we shall be discussing later. In order to set the stage for this, it is necessary first for us to describe the basic structure of the \( p \)-brane solitons. They arise as solutions to the supergravity theories described by (2.7), where in any given solution only a subset of the bosonic fields will be involved. More specifically, in a \( p \)-brane soliton solution the metric tensor, one or more of the dilatonic scalars, and one or more field strengths are active, where the degrees of the field strengths are either \( p + 2 \) or \( D - p - 2 \). In the former case,
the field strengths carry electric-type charges, whilst in the latter, they carry magnetic-type charges. The form of the metric is

\[ ds^2 = e^{2A} dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B} dy^m dy^m , \]

where \( x^\mu \) are the coordinates on the world-volume for the \( p \)-brane, of dimension \( d = p + 1 \), and \( y^m \) are the remaining \( (D - d) \) coordinates of the \( D \)-dimensional spacetime, which are transverse to the \( p \)-brane worldsheet. It will be convenient for future reference to define the quantity \( \tilde{d} = D - d - 2 \). The functions \( A \) and \( B \) are independent of the world-volume coordinates \( x^\mu \). In the simplest situation, where one considers a single-centre \( p \)-brane solution which can be located at the origin \( y^m = 0 \) without loss of generality, \( A \) and \( B \) will depend only on \( r = \sqrt{y^m y^m} \). These solutions will be sufficient for our present discussion. In this case, we may rewrite the ansatz (3.1) using hyperspherical polar coordinates in the transverse space thus [16]:

\[ ds^2 = e^{2A} dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B} (dr^2 + r^2 d\Omega^2) , \]

where \( d\Omega^2 \) is the metric on the unit \( (\tilde{d} + 1) \)-sphere. In all cases, the \( p \)-brane solutions are such that at large distance the metric approaches the flat metric, the scalars become constant, and the field strengths go to zero.

The charges carried by the field strengths are measured by performing appropriate surface integrals over the \( (\tilde{d} + 1) \)-sphere at infinity. If a field strength \( F \) carries electric charge \( u \), or magnetic charge \( v \), then these are given by [26]

\[ u = \frac{1}{4 \tilde{\omega}_{\tilde{d}+1}} \int_{S^{\tilde{d}+1}} *F , \quad \text{or} \quad v = \frac{1}{4 \tilde{\omega}_{\tilde{d}+1}} \int_{S^{\tilde{d}+1}} F , \]

respectively, where \( \tilde{\omega}_{\tilde{d}+1} \) is the volume of the unit \( (\tilde{d} + 1) \)-sphere. (We are assuming here for simplicity that the dilatonic scalars, which are asymptotically constant at infinity, are chosen to vanish there.)

We are now ready to present a framework for discussing the supersymmetry of the solutions. We shall do this by first describing the situation in \( D = 11 \), and then performing a dimensional reduction to \( D \) dimensions. If an asymptotically-flat solution preserves some fraction of the supersymmetry, there will exist Killing spinors \( \epsilon \) that become asymptotically constant at infinity. From these, global supercharges can be defined. In \( D = 11 \), this supercharge will be given by

\[ Q_\epsilon = \int_{\partial \Sigma_{\tilde{d}+1}} \bar{\epsilon} \Gamma^{MNP} \psi_P d\Sigma_{MN} , \]

respectively.
where $\partial \Sigma_{d+1}$ is the $(\tilde{d}+1)$-sphere of radius $r$ in the transverse space. The anti-commutator of the resulting supercharges is given by

$$\{Q_{\epsilon_1}, Q_{\epsilon_2}\} = \delta_{\epsilon_1} Q_{\epsilon_2} = \int_{\partial \Sigma} N^{AB} d\Sigma_{AB}, \quad (3.5)$$

where $N^{AB} = \bar{\epsilon}_1 \Gamma^{ABC} \delta_{\epsilon_2} \psi_C$. From the transformation rule for the gravitino in $D = 11$ supergravity, we therefore obtain the Nester form

$$N^{AB} = \bar{\epsilon}_1 \Gamma^{ABC} D_C \epsilon_2 + \frac{1}{8} \bar{\epsilon}_1 \Gamma^{C_1 C_2} \epsilon_2 F^{AB}_{C_1 C_2} + \frac{1}{96} \bar{\epsilon}_1 \Gamma^{ABC_1 \cdots C_4} \epsilon_2 F_{C_1 \cdots C_4}. \quad (3.6)$$

Since only the $d \Sigma_{0r}$ component of the $p$-brane spatial volume element contributes in (3.5), we may read off the Bogomol'nyi matrix $\mathcal{M}$ from the integral

$$\frac{1}{\omega_{d+1}} \int_{\partial \Sigma \text{ at } r \to \infty} N^{0r} r^{d+1} d\Omega_{(d+1)} = \epsilon_1^1 \mathcal{M} \epsilon_2, \quad (3.7)$$

where $\omega_{d+1}$ is the volume of the unit $(\tilde{d}+1)$-sphere. If there is an unbroken supersymmetry, then there exists a Killing spinor such that eqn. (3.5) vanishes. In other words, the Bogomol'nyi matrix (3.7) has a zero eigenvalue for each component of the unbroken supersymmetry.

We can now use the Bogomol'nyi matrix to study the supersymmetry of the $p$-brane solutions in $D = 11$ dimensions. There is only one field strength in $D = 11$ supergravity, namely the 4-form, which gives rise to an electrically-charged membrane or a magnetically-charged 5-brane. Note that in the electric case the last term in (3.6) vanishes, whilst the second term vanishes in the magnetic case. Substituting (3.7), we obtain [27]

$$\text{electric : } \mathcal{M} = m \mathbf{1} + u \Gamma_{012},$$

$$\text{magnetic : } \mathcal{M} = m \mathbf{1} + v \Gamma_{12345}. \quad (3.8)$$

where the hats indicate index values in the transverse space, while indices without hats live in the world-brane volume, and $u$ and $v$ denote the electric and magnetic charges defined in (3.3). The parameter $m$ denotes the mass per unit volume of the $p$-brane, which is calculated using the ADM mass formula. It is a measure of the rate at which the metric approaches flatness at infinity, and arises from the spin connection in the first term in (3.6).

We shall postpone a detailed discussion of the supersymmetry of specific solutions until subsequent sections, but we just remark for now that one can easily determine from (3.8) that the eigenvalues of the Bogomol’nyi matrix $\mathcal{M}$ are given by $m \pm u$ or $m \pm v$, with sixteen eigenvalues for each sign choice, and thus half the supersymmetries are preserved if $u = m$ or $v = m$. These correspond to the BPS-saturated membrane or 5-brane in $D = 11$. 

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The above analysis of supersymmetry can easily be generalised to lower dimensions. In fact the Nester form for maximal supergravity in any dimension is just the Kaluza-Klein dimensional reduction of the 11-dimensional expression \((3.6)\). For example, the Nester form upon reduction to type IIA supergravity in \(D = 10\) is given by \([16, 10]\)

\[
N^{AB} = \bar{\epsilon}_1 \Gamma^{ABC} D_C \epsilon_2 + e^{-\frac{3}{2} \phi} \bar{\epsilon}_1 \Gamma_{10} \left( \frac{1}{4} F^{AB} + \frac{1}{8} \Gamma^{ABCD} F_{CD} \right) \epsilon_2 \\
- e^{\frac{1}{2} \phi} \bar{\epsilon}_1 \Gamma_{10} \left( \frac{1}{4} \Gamma^C F^{AB} C + \frac{1}{24} \Gamma^{ABCD} F_{CD} \right) \epsilon_2 \\
+ e^{-\frac{1}{4} \phi} \bar{\epsilon}_1 \Gamma_{10} \left( \frac{1}{8} \Gamma^{CD} F^{AB} CD + \frac{1}{96} \Gamma^{ABCD} F_{CD} \right) \epsilon_2 .
\]

The Nester forms become increasingly complicated as we descend through the dimensions, since more and more antisymmetric tensors are generated. However, for the purpose of studying the supersymmetries of \(p\)-brane solutions, some simplifications can be made. First, note that the dilaton factor for each field strength is precisely the square root of the dilaton factor for the kinetic term of the same field strength that appears in the Lagrangian. In fact all these dilaton factors can be set to unity since the Bogomol’nyi matrix we are considering is defined at \(r = \infty\), and we are taking the dilatons to vanish there. As we showed above, in order to obtain the eigenvalues of a Bogomol’nyi matrix, we do not need to decompose the \(\Gamma\) matrices into world-volume and transverse space factors. Furthermore, we do not need to decompose the 11-dimensional \(\Gamma\) matrices into the product of \(D\)-dimensional spacetime and compactified \((11 - D)\)-dimensional factors. This greatly simplifies the discussion for lower dimensions.

In order to present the general Bogomol’nyi matrix for arbitrary forms and arbitrary dimensions, we first establish a notation for the charges carried by the various field strengths:

\[
F_4 \quad F_3^i \quad F_2^{ij} \quad F_1^{ijk} \quad F_1^i \quad F_1^{ij}
\]

| Electric | u | u \(_i\) | u \(_{ij}\) | u \(_{ijk}\) | p \(_i\) | p \(_{ij}\) |
| Magnetic | v | v \(_i\) | v \(_{ij}\) | v \(_{ijk}\) | q \(_i\) | q \(_{ij}\) |

where the electric \(u\)-type or \(p\)-type charges, and the magnetic \(v\)-type or \(q\)-type charges, are given by \((3.3)\). We then find that the general Bogomol’nyi matrix in \(D\) dimensions is given by \([10]\)

\[
\mathcal{M} = m \mathbb{1} + u \Gamma_{012} + u \Gamma_{01} + \frac{1}{6} u \Gamma_{0ij} + \frac{1}{6} u \Gamma_{ijk} + p \Gamma_{0i} + \frac{1}{2} p \Gamma_{ij} \\
+ v \Gamma_{12345} + v \Gamma_{1234i} + \frac{1}{2} v \Gamma_{123ij} + \frac{1}{2} v \Gamma_{12ijk} + q \Gamma_{123\hat{i}} + \frac{1}{2} q \Gamma_{12\hat{ij}} ,
\]

where the first line contains the contributions for electrically-charged solutions, and the second line contains the contributions for magnetically-charged solutions. For a given degree
n of antisymmetric tensor field strength, only the terms with the corresponding charges, as indicated in (3.10), will occur. As always, the indices 0, 1, ... run over the dimension of the p-brane worldvolume, 1, 2, ... run over the transverse space of the $y^m$ coordinates, and $i, j, ...$ run over the dimensions that were compactified in the Kaluza-Klein reduction from 11 to $D$ dimensions. The mass per unit p-volume $m$ in (3.11) arises from the connection term in the covariant derivative in the Nester form, and it is given by $m = \frac{1}{2} \lim_{r \to \infty} (B' - A') e^{-B r^{d+1}}$.

In the subsequent sections, we shall make use of the Bogomol’nyi matrix constructed above in order to determine the fractions of supersymmetry that are preserved by the various p-brane solutions.

4 p-brane solitons in maximal supergravities

When solving the equations of motion (2.7) for p-brane solutions with a given $p$, only the subset of field strengths whose degrees are either $(p + 2)$ (in the case of electric charges) or $(D - p - 2)$ (in the case of magnetic charges) are involved. Thus the relevant part of the supergravity Lagrangian that describes the p-brane solutions will be of the form

$$\mathcal{L} = eR - \frac{1}{2} e (\partial \tilde{\phi})^2 - \frac{1}{2n!} \sum_{\alpha=1}^N e^{\tilde{c}_\alpha \cdot \tilde{\phi}} F^2_{\alpha},$$

(4.1)

where we suppose that $N$ field strengths $F_{\alpha}$ of degree $(p + 2)$ or $(D - p - 2)$, labelled by $\alpha$, are active. These field strengths, and their associated dilaton vectors $\tilde{c}_\alpha$, are therefore a subset of the ones appearing in (2.7).

4.1 Multi-charge extremal solutions

We begin our discussion of p-brane solitons by considering the case of extremal solutions. We shall make the spherically-symmetric ansatz (3.2) for the metric, while each field strength, carrying an electric or a magnetic charge, will take the form

$$F^\alpha_{m\mu_1...\mu_{n-1}} = \epsilon_{\mu_1...\mu_{n-1}} (C_{\alpha})' \frac{y^m}{r}, \quad \text{or} \quad F^\alpha_{m_1...m_n} = \lambda_\alpha \epsilon_{m_1...m_n} p \frac{y^p}{r^{n+1}},$$

(4.2)

where a prime denotes a derivative with respect to $r$. These two ansätze both preserve the same $SO(1, d - 1) \times SO(D - d)$ subgroup of the original $SO(1, D - 1)$ Lorentz group as does the metric (3.2). Substituting the ansätze (4.2) and (3.2) directly into the equations of motion that follow from the Lagrangian (4.1), we find that $\tilde{\phi}$, $A$ and $B$ satisfy
\[ \vec{\phi}'' + \frac{\tilde{d} + 1}{r} \vec{\phi}' + (dA' + \tilde{d}B')\vec{\phi} = -\frac{1}{2} \epsilon \sum_{\alpha} \vec{c}_\alpha S_\alpha^2, \]

\[ A'' + \frac{\tilde{d} + 1}{r} A' + (dA' + \tilde{d}B')A' = \frac{\tilde{d}}{2(D - 2)} \sum_{\alpha} S_\alpha^2, \]

\[ B'' + \frac{\tilde{d} + 1}{r} B' + (dA' + \tilde{d}B')(B' + \frac{1}{r}) = -\frac{d}{2(D - 2)} \sum_{\alpha} S_\alpha^2, \tag{4.3} \]

\[ d(D - 2)A'^2 + \tilde{d}(dA'' + \tilde{d}B'') - (dA' + \tilde{d}B')^2 - \frac{\tilde{d}}{r} (dA' + \tilde{d}B') + \frac{1}{2} \tilde{d} \vec{\phi}^2 = \frac{1}{2} \tilde{d} \sum_{\alpha} S_\alpha^2, \tag{4.4} \]

where \( \epsilon = 1 \) or \(-1\) for the electric or magnetic ansatz respectively, and the functions \( S_\alpha \) are given by

\[ S_\alpha = \lambda_\alpha e^{-\frac{1}{2} \epsilon \vec{c}_\alpha \cdot \vec{\phi} - \tilde{d} \vec{B} \cdot \vec{\phi} - \tilde{d} - 1}. \tag{4.5} \]

In the electric case, \( \lambda_\alpha \) arises as the integration constant for the function \( C_\alpha \), given by

\[ (e^{C_\alpha})' = \lambda_\alpha e^{\vec{c}_\alpha \cdot \vec{\phi} + dA - \tilde{B}} e^{d - d - d - 1}. \tag{4.6} \]

From (4.3) and (4.3), we see that a natural solution for \( B \) is to take

\[ dA + \tilde{d}B = 0 . \tag{4.7} \]

(We shall return later to the discussion of more general solutions in which this relation is not imposed.) We may also consistently set to zero the \((11 - D - N)\) components of \( \vec{\phi} \) that are orthogonal to the space spanned by the \( N \) dilaton vectors \( \vec{c}_\alpha \). The remaining equations become

\[ \varphi''_\alpha + \frac{\tilde{d} + 1}{r} \varphi'_{\alpha} = -\frac{1}{2} \epsilon \sum_{\beta} M_{\alpha\beta} S_{\beta}^2, \tag{4.8} \]

\[ A'' + \frac{\tilde{d} + 1}{r} A' = \frac{\tilde{d}}{2(D - 2)} \sum_{\alpha} S_\alpha^2, \tag{4.9} \]

\[ d(D - 2)A'^2 + \tilde{d \sum}_{\alpha,\beta} (M^{-1})_{\alpha\beta} \varphi'_{\alpha} \varphi'_{\beta} = \frac{1}{2} \tilde{d} \sum_{\alpha} S_\alpha^2, \tag{4.10} \]

where we have defined \( \varphi_{\alpha} = \vec{c}_{\alpha} \cdot \vec{\phi} \), and \( M_{\alpha\beta} \) is the matrix of dot products of the dilaton vectors \( \vec{c}_{\alpha} \). The remaining equations become

\[ \varphi''_{\alpha} + \frac{\tilde{d} + 1}{r} \varphi'_{\alpha} = -\frac{1}{2} \epsilon \sum_{\beta} M_{\alpha\beta} S_{\beta}^2, \tag{4.11} \]

\[ A'' + \frac{\tilde{d} + 1}{r} A' = \frac{\tilde{d}}{2(D - 2)} \sum_{\alpha} S_\alpha^2, \tag{4.12} \]

\[ d(D - 2)A'^2 + \tilde{d \sum}_{\alpha,\beta} (M^{-1})_{\alpha\beta} \varphi'_{\alpha} \varphi'_{\beta} = \frac{1}{2} \tilde{d} \sum_{\alpha} S_\alpha^2, \tag{4.13} \]

where we have defined \( \varphi_{\alpha} = \vec{c}_{\alpha} \cdot \vec{\phi} \), and \( M_{\alpha\beta} \) is the matrix of dot products of the dilaton vectors

\[ M_{\alpha\beta} = \vec{c}_{\alpha} \cdot \vec{c}_{\beta} . \tag{4.14} \]

(Here we are assuming that \( M_{\alpha\beta} \) is non-singular, and we shall comment on the case when it is singular later.) Note that the number of non-vanishing scalar fields \( \varphi_{\alpha} \) is precisely the
same as the number $N$ of participating field strengths. By acting on (4.7) with $(M^{-1})_{\alpha\beta}$, and comparing with (4.8), we see that it is natural to solve for $A$ by taking

$$A = -\frac{\epsilon \hat{d}}{D - 2} \sum_{\alpha,\beta} (M^{-1})_{\alpha\beta} \varphi_{\alpha} .$$

(4.11)

The equations of motion now reduce to

$$\sum_{\beta} (M^{-1})_{\alpha\beta} (\varphi''_{\beta} + \frac{\hat{d} + 1}{r} \varphi'_{\beta}) = -\frac{1}{2} \epsilon \lambda_{\alpha}^{2} e^{-\epsilon \varphi_{\alpha} + 2dA} r^{-2(d+1)} ,$$

(4.12)

$$d(D - 2)A'^{2} + \frac{\hat{d}}{2} \sum_{\alpha,\beta} (M^{-1})_{\alpha\beta} \varphi'_{\alpha} \varphi'_{\beta} = \frac{1}{2} \hat{d} \sum_{\alpha} \lambda_{\alpha}^{2} e^{-\epsilon \varphi_{\alpha} + 2dA} r^{-2(d+1)} .$$

(4.13)

The solutions are determined completely by the structure of the dot products $M_{\alpha\beta}$ of dilaton vectors $\vec{c}_{\alpha}$ of the corresponding field strengths $F_{\alpha}$. Solutions exist only for $N \leq (11 - D)$.

In general, the solutions of (4.12) and (4.13) are still very complicated. However, we can find simple solutions if we make the ansatz that the quantity $(-\epsilon \varphi_{\alpha} + 2dA)$ appearing in the exponential in $S_{\alpha}^{2}$ is proportional to the quantity $\sum_{\beta} (M^{-1})_{\alpha\beta} \varphi_{\beta}$ appearing on the left-hand side of (4.12). For this to be true, it implies that $M_{\alpha\beta}$ must take the form

$$M_{\alpha\beta} = 4\delta_{\alpha\beta} - \frac{2d\hat{d}}{D - 2} .$$

(4.14)

Note that the coefficient of $\delta_{\alpha\beta}$ can a priori be any constant, but it is fixed to be 4 in maximal supergravity theories, as can be verified by computing the magnitudes of all the dilaton vectors, defined by (2.8). We can now solve (4.12) and (4.13) completely by making the further ansatz that $S_{\alpha} \propto (-\epsilon \varphi'_{\alpha} + 2dA')$. Thus the solutions for the dilaton and p-brane metric are [28]

$$e^{\frac{1}{2} \epsilon \varphi_{\alpha} - dA} = 1 + \frac{\lambda_{\alpha}}{d} r^{-\hat{d}} ,$$

(4.15)

$$ds^{2} = \prod_{\alpha=1}^{N} \left( 1 + \frac{\lambda_{\alpha}}{d} r^{-\hat{d}} \right) \frac{d}{(D-2)} dx^{\mu} dx^{\nu} \eta_{\mu\nu} + \prod_{\alpha=1}^{N} \left( 1 + \frac{\lambda_{\alpha}}{d} r^{-\hat{d}} \right) \frac{d}{(D-2)} (dr^{2} + r^{2} d\Omega^{2}) .$$

(4.16)

Note that the functions $H_{\alpha} \equiv (1 + \frac{\lambda_{\alpha}}{d} r^{-\hat{d}})$ are harmonic on the internal space, and thus we may express the solution more succinctly in terms of these harmonic functions [29, 30],

$$e^{\frac{1}{2} \epsilon \varphi_{\alpha} - dA} = H_{\alpha} ,$$

$$ds^{2} = \prod_{\alpha=1}^{N} H_{\alpha}^{-\frac{d}{(D-2)}} dx^{\mu} dx^{\nu} \eta_{\mu\nu} + \prod_{\alpha=1}^{N} H_{\alpha}^{\frac{d}{(D-2)}} dy^{m} dy^{m} .$$

(4.16)

It is straightforward to see from the ansätze (4.2) that the field strengths are given in the electric or magnetic cases by

$$F_{\alpha} = dH_{\alpha}^{-1} \wedge d^{d}x , \quad \text{or} \quad F_{\alpha} = *(dH_{\alpha}^{-1} \wedge d^{d}x) ,$$

(4.17)
respectively. The extremal metrics given in (4.15) have an horizon at $r = 0$. In general, this coincides with a singularity of the curvature tensor, and also the dilatonic scalars diverge there. In the special cases where the dilatons are finite on the horizon, the curvature is finite there too. (An example of this arises in the four-charge solution in $D = 4$, for which it can easily be verified that the dilatonic scalars are finite on the horizon at $r = 0$. If the charges are set equal, then the dilatonic scalars become constants everywhere, and in fact the solution reduces to the extremal Reissner-Nordstrøm solution.) It should be remarked also that if the matrix $M_{\alpha\beta}$ defined by (4.14) happens to be degenerate the solutions are still given by (4.16), but now it turns out that some linear combination of the dilatonic scalars $\varphi_\alpha$ vanishes, and so there is one fewer scalar degrees of freedom in such cases.

Solutions of the above kind can be found in the maximal supergravity theories in each dimension $D \leq 11$. The values of $p$ for which solutions exist are determined by the degrees of the fields strengths that exist in the particular dimension $D$ in question. The solutions in general carry $N$ independent electric or magnetic charges, which, from (4.17) and (3.3), are easily found to be given by $Q_\alpha = \frac{1}{4} \lambda_\alpha$. The mass per unit $p$-volume can also easily be calculated, and turns out to be given by

$$m = \sum_{\alpha=1}^{N} Q_\alpha .$$

(4.18)

The number $N$ of independent charges that can arise for a given $p$ in a given dimension $D$ depends on two factors. First of all, $N$ is certainly bounded by the number of dilaton vectors $\vec{c}_\alpha$ in the toroidally-compactified theory whose dot products satisfy the necessary relation (4.14). If the field strengths appearing in (2.7) and (4.1) were all simply the exterior derivatives of their associated potentials, then in fact this would be the only criterion determining the numbers of field strengths that could be used in constructing multi-charge solutions. However, as we saw in the previous section, there are Chern-Simons corrections in the expressions for the field strengths, and these imply that the complete system of field equations for the fields in the supergravity theories are much more complicated than at first sight might appear. In particular, Chern-Simons corrections involving a field that is being set to zero in a particular solution can nevertheless impose constraints on the fields that are retained, since one must vary the Lagrangian with respect to all the fields before setting any of them to zero. The complete analysis of all possible $p$-brane solutions is therefore extremely complicated. In practice, a useful strategy for approaching the problem is to proceed first with finding configurations that would be solutions in the absence of the Chern-Simons complications, and then check which of them survives after taking account
of the constraints implied by the setting to zero of the non-participating fields. It is not
certain that one will find all solutions by this means, but at least one will find some of them.
Indeed, it is not clear to us that any completely exhaustive discussion of the solution set
has been given in the literature.

The known multi-charge extremal solutions can be summarised as follows. Using the 4-
form field strength, we can clearly only construct single-charge solutions, since there is only
one such field strength. If it carries an electric charge, we obtain an extremal membrane
solution for each \( D \), while if it carries a magnetic charge, we get an extremal \( (D - 6) \)-brane.

For 3-form field strengths, it turns out that although there will more than one of them
in each dimension \( D \leq 9 \), their dilaton vectors never satisfy the necessary relation (4.14),
and consequently one can only obtain single-charge solutions. Thus we have single-charge
extremal solutions in \( D \leq 10 \), which are strings if the charge is electric, and \( (D - 5) \)-branes
if the charge is magnetic. For the case of solutions using 2-form or 1-form field strengths,
it turns out that multi-charge solutions can arise. The possibilities are summarised in the
following table [11]:

| Dim. | 2-Forms | 1-Forms |
|------|---------|---------|
| \( D = 10 \) | \( N = 1 \) | \( p = 0, 6 \) | |
| \( D = 9 \) | \( N = 2 \) | \( p = 0, 5 \) | \( N = 1 \) | \( p = 6 \) |
| \( D = 8 \) | \( p = 0, 4 \) | \( N = 2 \) | \( p = 5 \) |
| \( D = 7 \) | \( p = 0, 3 \) | \( p = 4 \) |
| \( D = 6 \) | \( p = 0, 2 \) | \( N = 3, 4' \) | \( p = 3 \) |
| \( D = 5 \) | \( N = 3 \) | \( p = 0, 1 \) | \( p = 2 \) |
| \( D = 4 \) | \( N = 4 \) | \( p = 0 \) | \( N = 4, 5, 6, 7 \) | \( p = 1 \) |
| \( D = 3 \) | \( N = 8 \) | \( p = 0 \) |

Table 1: Numbers of charges in multi-scalar \( p \)-brane solutions

Here we list the highest dimensions where \( p \)-brane solutions with the indicated numbers \( N \)
of field strengths first occur. They then occur also at all lower dimensions.

Special cases of the multi-charge solutions arise if all \( N \) charges are set equal, in which
case the harmonic functions \( H_\alpha \) in (4.16) become equal. Under these circumstances, it is
easy to see from (4.16) that all except one combination of the dilatonic scalar will become
zero, and at the same time all the participating field strengths will become equal. The
resulting single-scalar configuration is a solution of the truncated Lagrangian

\[ \mathcal{L} = eR - \frac{1}{2} e (\partial \phi)^2 - \frac{1}{2n!} e^{a \phi} F^2, \]  

and is given by \[ 25 \]

\[ e^{\Delta / 2} = H, \]

\[ ds^2 = H^{-3(D-2)} dx^\mu dx^\nu \eta_{\mu\nu} + H^{3d} (dr^2 + r^2 d\Omega^2), \]  

where \( \Delta = 4/N \) and

\[ a^2 = \Delta - \frac{2dd}{D-2}. \]  

### 4.2 Supersymmetry of the multi-charge \( p \)-brane solitons

Having obtained the extremal multi-charge solutions, we may now apply the formalism developed in the previous section for determining the fractions of supersymmetry that are preserved by them. To do this, it is simply a matter of substituting the appropriate charges, and the expression \[ 4.18 \] for the mass per unit length, into the general expression \[ 3.11 \] for the Bogomol’nyi matrix. Then, an elementary calculation gives the eigenvalues of the Bogomol’nyi matrix, and the fraction of supersymmetry that is preserved is equal to the fraction of the total of 32 eigenvalues that are equal to zero. Some caution has to be exercised in applying this formula, and we shall comment on this further as we proceed.

Let us first consider solutions involving just a single charge \( Q \). In all such cases, we see from \[ 3.11 \] that the form of the corresponding Bogomol’nyi matrix will be

\[ \mathcal{M} = m \mathbb{I} + Q \Gamma, \]  

where \( \Gamma \) represents the particular product of gamma matrices associated with the field strength that carries the charge, as given in \[ 3.11 \]. It is clear that in all cases corresponding to \( p \)-branes with \( p \geq 0 \), the associated product of gamma matrices is hermitean\[1\] and \( \Gamma^2 = \mathbb{I} \). Thus we see from \[ 4.22 \] that \( (\mathcal{M} - m)^2 = Q^2 \), and hence by the Cayley-Hamilton theorem the eigenvalues \( \mu \) of the Bogomol’nyi matrix for single-charge solutions are

\[ \mu = m \pm Q, \]  

\[ ^1 \text{The products of gamma matrices of the types } \Gamma_{ijk} \text{ or } \Gamma_{ij} \text{ are exceptions, since these are anti-hermitean. However, these are associated with electric } (-1) \text{-branes, which are instantons whose existence requires that the timelike coordinate be Euclideanised. There will be an extra factor of } i \text{ coming from the electric charge in such cases, which restores the hermiticity of the Bogomol’nyi matrix.}\]
with 16 eigenvalues for each sign choice. From (4.18), we therefore see that the extremal one-charge $p$-branes all have Bogomol’nyi eigenvalues given by

$$\mu = 2Q \{0_{16}, 1_{16} \} ,$$

(4.24)

where the subscripts on each term indicate their degeneracies. Thus the single-charge extremal $p$-branes all preserve $\frac{1}{2}$ of the supersymmetry.

Turning now to two-charge solutions, it is easiest first to consider a particular example. Let us take the case of a black hole in $D = 9$ carrying two electric charges. There are in total three 2-form field strengths in $D = 9$, namely $F^{12}, F^{1}_2$ and $F^{2}_2$. From the definitions (2.8) for their associated dilaton vectors $\vec{a}_{12}, \vec{b}_1$ and $\vec{b}_2$, it is easy to see that either of the pairs $\{\vec{a}_{12}, \vec{b}_1\}$ or $\{\vec{a}_{12}, \vec{b}_2\}$ satisfies the condition (4.14), whilst the pair $\{\vec{b}_1, \vec{b}_2\}$ does not. Let us take the first case, where the two charges are carried by the field strengths $F^{12}_2$ and $F^{1}_2$. Denoting these charges by $Q_1$ and $Q_2$, we have from (3.11) that the Bogomol’nyi matrix is

$$\mathcal{M} = m I + Q_1 \Gamma_{0\tilde{i}} + Q_2 \Gamma_{0\tilde{j}} ,$$

(4.25)

where $\tilde{i}$ and $\tilde{j}$ denote the index values $i$ associated with the first and the second steps of reduction from $D = 11$ to $D = 9$. Thus we have $(\mathcal{M} - m)^2 = Q_1^2 + Q_2^2 + 2Q_1 Q_2 \Gamma_2$, and, after shifting terms and squaring again, $((\mathcal{M} - m)^2 - Q_1^2 - Q_2^2)^2 = 4Q_1^2 Q_2^2$. This implies that the eigenvalues of the Bogomol’nyi matrix in this case are given by

$$\mu = m \pm Q_1 \pm Q_2 ,$$

(4.26)

where the two $\pm$ signs are independent. It is not hard to see that in all the two-charge $p$-branes, the expression for the eigenvalues of the Bogomol’nyi matrix will be the same. If we now use the expression (4.18) for the mass of the extremal two-charge solution, namely $m = Q_1 + Q_2$, we see that the eigenvalues are

$$\mu = 2 \{0, Q_1, Q_2, Q_1 + Q_2 \} .$$

(4.27)

where each eigenvalue occurs with degeneracy 8. Thus for two-charge extremal solutions with generic values for the charges, $\frac{1}{4}$ of the supersymmetry is preserved. If either charge is set to zero, the situation reduces to the previously-discussed single-charge solution, and $\frac{1}{2}$ of the supersymmetry is preserved in this case.

At this point a word of caution is appropriate. It might seem from the form of (4.27) that a supersymmetry enhancement from $\frac{1}{4}$ to $\frac{1}{2}$ could also be achieved by choosing $Q_2$ to be $-Q_1$. However, this is in fact not the case, and the reason is that in the discussion of
the Bogomol’nyi matrix, and its relation to supersymmetry, it was tacitly assumed that the
class of metrics that were being discussed were free of naked singularities. Provided this is
ture, then zeroes of the Bogomol’nyi matrix are associated with components of unbroken
supersymmetry. However, bearing in mind that the charges $Q_\alpha$ are related to the integration
constants $\lambda_\alpha$ appearing in the metric (4.15) by $Q_\alpha = \frac{1}{4}\lambda_\alpha$, we see that choosing any of the
charges $Q_\alpha$ here to be negative will imply that the metric functions will become singular
for some positive value of $r$, and in fact the curvature tensor will diverge there. Now the
horizon of the extremal $p$-brane lies at $r = 0$, and so it follows that if any of the charges $Q_\alpha$
is negative, there will be naked singularities outside the horizon. Under such circumstances
the validity of the Bogomol’nyi matrix discussion in the previous section breaks down, and
in particular the association between zero eigenvalues and unbroken supersymmetry ceases
to be generally valid. A further illustration of the breakdown of the discussion is provided
by the fact that if either of the charges is chosen to be negative, the Bogomol’nyi matrix
(4.27) will also have negative eigenvalues. This would contradict the fact that, subject to
appropriate regularity assumptions for the metric, its eigenvalues are always non-negative.
The resolution, of course, is that the naked singularities violate the regularity assumptions.

Turning now to 3-charge solutions, it is straightforward to carry out the analogous steps
to those described above, in order to calculate the eigenvalues of the Bogomol’nyi matrix.
Again, it turns out that the expressions for the eigenvalues in terms of the charges $Q_1$, $Q_2$
and $Q_3$ are the same for all cases, and after some algebra we find that they are given by

$$\mu = m \pm Q_1 \pm Q_2 \pm Q_3,$$  \hspace{1cm} (4.28)

where the three $\pm$ signs are independent. Applying this formula to the extremal 3-charge
solutions, for which from (4.18) we have $m = Q_1 + Q_2 + Q_3$, we see that the eigenvalues of
the Bogomol’nyi matrix are

$$\mu = 2 \{0, Q_1, Q_2, Q_3, Q_{12}, Q_{13}, Q_{23}, Q_{123}\},$$  \hspace{1cm} (4.29)

with each eigenvalue occurring with degeneracy 4. Here, we have introduced the notation that $Q_{i\cdots j} \equiv Q_i + \cdots + Q_j$. Thus all generic 3-charge solutions preserve $\frac{1}{8}$ of the

\footnote{It should perhaps be emphasised that there is really nothing special about positive rather than negative
charges here. Our statements are made with respect to a convenient set of conventions that we have chosen,
in which we pick the $p$-brane solutions in which positive charge means positive mass. There are another set of
solutions where negative charge means positive mass. Rather than increase the complexity of all discussions
by having to keep track of both sets of solutions, we have picked just the first set, and consequently there is
an asymmetry between positive and negative charges with respect to this subset of the solutions.}
supersymmetry. If one or more charges are set to zero, the results reduce to those of the previously-discussed two-charge or one-charge solutions. Again, any apparent enhancement of supersymmetry achieved by taking some charges to be negative to get further zeroes in (4.29) is “bogus,” for the same reasons we discussed above.

One might think that the discussion would proceed uneventfully to all $N$-charge solutions for all higher values of $N$. However, starting with $N = 4$ it turns out that the situation becomes a little more complicated. In particular, there are two different kinds of result that can arise for the eigenvalues of the Bogomol’nyi matrix for 4-charge solutions. In the case of 2-form field strengths, only one of these possibilities can be realised, although in fact this possibility itself divides into two sub-categories. We find that the eigenvalues of the Bogomol’nyi matrix for 4-charge 2-form solutions are given by

$$\mu = m \pm Q_1 \pm Q_2 \pm Q_3 \pm Q_4 ,$$

(4.30)

but in this case the ± signs are not all independent, and only eight combinations out of the total of 16 occur in any given case. In fact there are exactly two possibilities for the combinations that occur; either it is the eight cases where there are an even number of minus signs, or it is the other eight cases where there are an odd number of minus signs. It is the details of the charge configurations in a given solution that determine which of the two possibilities is realised for that solution. In the case of the extremal 4-charge solutions, we therefore either obtain the eigenvalues

$$\mu = 2 \{0, Q_{12}, Q_{13}, Q_{14}, Q_{23}, Q_{24}, Q_{34}, Q_{1234}\} ,$$

(4.31)

or else, with the other set of sign choices, we get the eigenvalues

$$\mu = 2 \{Q_1, Q_2, Q_3, Q_4, Q_{234}, Q_{134}, Q_{124}, Q_{123}\} ,$$

(4.32)

each with degeneracy 4. Thus the generic 4-charge solutions using 2-form field strengths again preserve $\frac{1}{8}$ of the supersymmetry, in the first choice of sign combinations. In the second choice, the solution will preserve no supersymmetry at all, even though it is extremal. Both of these possibilities can be realised for all configurations using four 2-form field strengths. As we mentioned above, since the charges enter the field equations quadratically it follows that there is a bifurcation of solutions for each of the participating charges: in one branch positive charge contributes positively to the mass, while in the other branch negative charge contributes positively to the mass. In solutions with $N \leq 3$ charges all $2^N$ branches have the same supersymmetry, but when $N = 4$ eight branches give the eigenvalues (4.31) whilst the other eight give (4.32).
The other possibility for the structure of the Bogomol’nyi eigenvalues for 4-charge solutions can occur only for 1-form field strengths. This is the case denoted by 4, as opposed to $4'$, in table 1. Here, we find that the eigenvalues of the Bogomol’nyi matrix are again given by the expression (4.30), except that now the ± choices are all independent. In this case each eigenvalue therefore occurs with degeneracy 2. Since all the sign combinations occur here, there is no division into two sub-categories in this case. Extremal 4-charge solutions of this kind have eigenvalues

$$\mu = 2 \{ 0, Q_1, Q_2, Q_3, Q_4, Q_{12}, Q_{13}, Q_{14}, Q_{23}, Q_{24}, Q_{34}, Q_{134}, Q_{124}, Q_{123}, Q_{1234} \}$$  (4.33)

and so they preserve $\frac{1}{16}$ of the supersymmetry. In this case there is no non-supersymmetric variant. An example of a 4-charge solution that gives this set of Bogomol’nyi eigenvalues is one using the 1-form field strengths $F_{12}, F_{34}, F_{56}$ and $F_{127}$, whereas the previous eigenvalues (4.31) and (4.32) are achieved using, for example, $F_{12}, F_{15}, F_{123}$ and $F_{1234}$.

For 1-form solutions with 5, 6, 7 or 8 charges we find that again there are two possible sub-categories of eigenvalue structures, one yielding two zero eigenvalues, thus implying that $\frac{1}{16}$ of the supersymmetry is preserved, and the other yielding no zero eigenvalues. Further details can be found in [28].

4.3 Non-extremal $p$-brane solitons

The $p$-brane solitons that we have discussed up until now have been extremal solutions, in which the mass per unit $p$-volume takes its lowest possible value with respect to the charges carried by the field strengths in the solution, while still avoiding the occurrence of naked singularities. In this circumstance, for which a Bogomol’nyi bound is saturated, the solution typically preserves some fraction of the supersymmetry. However, in cases where four or more independent field strengths carry charges, we have seen that there can also exist solutions which, although still extremal, preserve none of the supersymmetry. In this section, we shall discuss more general $p$-brane solutions in which the mass per unit $p$-volume is a further free parameter, independent of the charges. These non-extremal, or “black,” $p$-branes preserve no supersymmetry.

We shall discuss two different kinds of generalisation away from the previous extremal solutions. The first of these, giving what have been called type-2 non-extremal $p$-branes in [31], involves a modification to the ansatz (3.2) for the metric. Specifically, the new ansatz becomes [22]

$$ds^2 = e^{2A} (-e^{2f} dt^2 + dx^i dx^i) + e^{2B} (e^{-2f} dr^2 + r^2 d\Omega^2)$$  (4.34)

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where the new function $f$, like $A$ and $B$, depends only on $r$. The ansatz for the field strengths remains unchanged from the extremal case. After straightforward calculations (see, for example, [32] for the details), one finds that the function $f$ has the following universal solution:

$$e^{2f} = 1 - \frac{k}{r^{\hat{d}}} ,$$  \hspace{1cm} (4.35)

and that the dilaton and metric have the solutions

$$e^{\frac{1}{2} \varphi - dA} = 1 + \frac{k}{r^{\hat{d}}} \sinh^2 \mu_{\alpha} , \quad e^{2f} = 1 - \frac{k}{r^{\hat{d}}} ,$$

$$ds^2 = \prod_{\alpha=1}^{N} \left( 1 + \frac{k}{r^{\hat{d}}} \sinh^2 \mu_{\alpha} \right)^{-\frac{\hat{d}}{D-2}} \left( -e^{2f} dt^2 + dx^i dx^i \right)$$

$$+ \prod_{\alpha=1}^{N} \left( 1 + \frac{k}{r^{\hat{d}}} \sinh^2 \mu_{\alpha} \right)^{\frac{\hat{d}}{D-2}} \left( e^{-2f} dr^2 + r^2 d\Omega^2 \right).$$  \hspace{1cm} (4.36)

The metrics described by (4.36) have an outer event horizon at $r = k^{1/\hat{d}}$ (assuming $k$ is positive), and in general the curvature diverges at $r = 0$. Thus they describe $p$-brane generalisations of black holes, in which the curvature singularity is hidden behind an horizon.

In the limit where $k$ goes to zero, the previous extremal solutions are recovered, in which, in general, the horizon at $r = 0$ coincides with a curvature singularity. The mass per unit volume and the charges for this solution are given by

$$m = k \left( \hat{d} \sum_{\alpha=1}^{N} \sinh^2 \mu_{\alpha} + \hat{d} + 1 \right) , \quad Q_{\alpha} = \frac{1}{2} \hat{d} k \sinh 2\mu_{\alpha} ,$$  \hspace{1cm} (4.37)

where we used the ADM mass formula obtained in [33] for the metric (4.36). Thus the mass $m$ and the $N$ charges $Q_{\alpha}$ are parameterised in terms of the $N+1$ independent constants $k$ and $\mu_{\alpha}$. For non-negative values of $k$, the mass and charges satisfy the bound

$$m - \sum_{\alpha=1}^{N} Q_{\alpha} = \frac{1}{2} \hat{d} k \sum_{\alpha=1}^{N} (e^{-2\mu_{\alpha}} - 1) + k(\hat{d} + 1) \geq \frac{k\hat{d}(\hat{d} - 1)}{\hat{d}} \geq 0 ,$$  \hspace{1cm} (4.38)

which coincides with the Bogomol’nyi bound. In the extremal limit $k \rightarrow 0$ it is saturated, and the solutions become supersymmetric. When the parameters are chosen such that the mass exceeds the bound (4.38), the Bogomol’nyi matrix has only positive eigenvalues, as can be seen explicitly from our formulae (4.23), (4.26), (4.28) and (4.30) in the cases of $N = 1, 2, 3$ and 4 charges.

As in the discussion of the extremal multi-charge solutions in section 4.1, we may again consider the special case where all $N$ charges are set equal. This gives a solution which also solves the reduced single scalar, single field strength system (4.19), with
\[ ds^2 = 
\left((1 + \frac{k}{r^d} \sinh^2 \mu)^{\frac{4d}{\Delta(D-2)}}(-e^{2f} dt^2 + dx^i dx^i) + (1 + \frac{k}{r^d} \sinh^2 \mu)^{\frac{4d}{\Delta(D-2)}}(e^{-2f} dr^2 + r^2 d\Omega^2)\right), \]

where \( \Delta = 4/N \) and \( d \) is given by (4.39). The two free parameters \( k \) and \( \mu \) are related to the charge \( Q \) and the mass per unit \( p \)-volume \( m \) by

\[ Q = \frac{dk}{\sqrt{\Delta}} \sinh 2\mu , \quad m = k\left(\frac{4d}{\Delta} \sinh^2 \mu + \tilde{d} + 1\right). \] (4.40)

There is also another kind of generalisation away from the extremal \( p \)-brane solitons [34], giving rise to what have been called type-1 non-extremal \( p \)-branes in [31]. In this case the metric ansatz (3.2) remains unchanged from its extremal form. The change from the procedure that gives the extremal \( p \)-branes comes as a result of not imposing any further restriction on the various functions in the ansätze for the metric, dilaton and field strengths, but instead constructing the most general solution of the equations of motion. In particular, the relation \( dA + \tilde{d}B = 0 \) for the metric functions in (3.2) is no longer imposed. Accordingly, we begin by defining

\[ X = dA + \tilde{d}B , \quad Y = A + k \sum_{\alpha,\beta} (M^{-1})_{\alpha\beta} \varphi_\alpha, \]

\[ \Phi_\alpha = \epsilon \varphi_\alpha - 2dA . \] (4.41)

It is advantageous to introduce a new radial variable \( \rho = r^{-\tilde{d}} \), in terms of which the equation of motion for \( X \) and \( Y \) turn out to be

\[ \frac{d^2 X}{d\rho^2} + \left(\frac{dX}{d\rho}\right)^2 - \frac{1}{\rho} \frac{dX}{d\rho} = 0 , \quad \frac{d^2 Y}{d\rho^2} + \frac{dX}{d\rho} \frac{dY}{d\rho} = 0 , \] (4.42)

giving the solutions \( e^X = 1 - k^2 \rho^2 \) and \( Y = -(\mu/k) \arctanh(k\rho) \). The further change of radial variable to \( \xi \) defined by \( k \rho = \tanh(k \xi) \) reduces the remaining equations of motion to [34]

\[ \ddot{\Phi}_\alpha = -\frac{32Q^2}{d^2} e^{-\Phi_\alpha} , \] (4.43)

\[ \sum_{\alpha=1}^{N} \left(\frac{32Q^2}{d} e^{-\Phi_\alpha} - \frac{4d}{\tilde{d}} \ddot{\Phi}_\alpha\right) = -16(\tilde{d} + 1)k^2 + 2d(2(D-2) - d\tilde{d}N)\mu^2 , \] (4.44)

provided that the dilaton vectors for the participating field strengths satisfy the relations (4.14). Note that a dot denotes a derivative with respect to the redefined radial variable \( \xi \).
In terms of $\xi$, the solutions for $X$ and $Y$ are
\[ e^{-\frac{1}{2}X} = \cosh k\xi, \quad Y = -\mu \xi. \tag{4.45} \]

The equations (4.43) are $N$ independent Liouville equations for the functions $\Phi_\alpha$, subject to the single first-integral constraint (4.44). This can be re-expressed in terms of the Hamiltonian
\[ H \equiv \sum_{\alpha=1}^{N} \left( \frac{1}{2}p_\alpha^2 - \frac{32Q_\alpha^2}{d^2} e^{-\Phi_\alpha} \right) = \frac{16(\tilde{d} + 1)}{d} k^2 - \frac{2d}{d} \left( 2(D - 2) - d\tilde{d}N \right) \mu^2, \tag{4.46} \]

where $p_\alpha$ is the momentum conjugate to $\Phi_\alpha$. Hamilton’s equations then give (4.43).

The solutions of the Liouville equations (4.43) for $\Phi_\alpha$ imply that
\[ e^{\frac{1}{2}d\Phi_\alpha - dA} = \frac{4Q_\alpha}{d\beta_\alpha} \sinh(\beta_\alpha \xi + \gamma_\alpha), \tag{4.47} \]

where $\beta_\alpha$ and $\gamma_\alpha$ are constants, while (4.44) gives the constraint $\tilde{d} \sum_\alpha \beta_\alpha^2 = -8(\tilde{d} + 1)k^2 + d(2(D - 2) - d\tilde{d}N)\mu^2$. The solutions for the functions $A$ and $B$ in the metric (3.2) can be written as
\[ e^{-2(D-2)A/\tilde{d}} = e^{(2(D-2) - d\tilde{d}N)\mu\xi/\tilde{d}} \prod_{\alpha=1}^{N} \left( \frac{4Q_\alpha}{d\beta_\alpha} \sinh(\beta_\alpha \xi + \gamma_\alpha) \right), \tag{4.48} \]
\[ e^{2(D-2)B/d} = (\cosh(k\xi))^{-2(D-2)/d(\tilde{d})} e^{2(D-2) - d\tilde{d}N)\mu\xi/\tilde{d}} \prod_{\alpha=1}^{N} \left( \frac{4Q_\alpha}{d\beta_\alpha} \sinh(\beta_\alpha \xi + \gamma_\alpha) \right). \]

The solutions have an outer event horizon at $r^{\tilde{d}} = k$ (i.e. at $\xi = \infty$). The mass per unit $p$-volume is given by
\[ m = 2(D - 2)\mu - d\tilde{d}N\mu + \sum_\alpha Q_\alpha \cosh \gamma_\alpha, \tag{4.49} \]

where the integration constants $\gamma_\alpha$ are chosen such that $4Q_\alpha \sinh \gamma_\alpha = \tilde{d} \beta_\alpha$, so that the metric approaches the standard Minkowski metric at infinity.

It is clear that when $p$ is greater than zero, the type-1 non-extremal $p$-brane solutions given by this construction are quite different from the type-2 solutions described by (4.36). In particular, in the type-2 solutions the spacetime metric on the $p$-brane world-volume is no longer Poincaré invariant, owing to the extra $e^{2f}$ factor in front of $dt^2$ in (4.34). By contrast, the metric for the type-1 solutions has the same fully Poincaré invariant form (3.2) as for the extremal $p$-branes. However, in the special case that $p = 0$, it is clear that the two metric ansätze (4.34) and (3.2) are simply related by a coordinate transformation of the radial variable $r$, and so in the special case of black holes, the type-1 non-extremal solutions encompass the type-2 ones.
4.4 Dyonic p-brane solutions

If the spacetime dimension $D$ is even, it is possible that a field strength of degree $n = 1/2D$ can carry both electric and magnetic charge at the same time. In such cases, the possibility of having dyonic p-brane solutions arises. Since we are considering the toroidal compactifications of M-theory, the dimensions in which this might occur are $D = 8, 6$ and $4$.

We shall postpone the discussion of $D = 8$ until the end of this section, and consider $D = 6$ and $D = 4$ first. The dyonic solutions arise in the case where just one of the field strengths, of degree $n = 3$ in $D = 6$ or degree $n = 2$ in $D = 4$, is non-zero, and thus the configurations satisfy the equations of motion from the reduced single-scalar system (4.19). Let us begin by considering the general equations of motion for this system, with the metric ansatz (3.2) and the two field-strength ansätze (4.2) imposed simultaneously, so that $F$ carries both electric and magnetic charge. Introducing the redefined radial coordinate $\xi$, one finds that $X$ and $Y$ have the same solutions as in section 4.3, and the remaining equations of motion can be cast into the form

$$
\ddot{q}_1 = e^{\alpha q_1 + (1-\alpha)q_2} , \quad \ddot{q}_2 = e^{(1-\alpha)q_1 + \alpha q_2} ,
$$

$$
H \equiv \frac{\alpha}{2(2\alpha - 1)}(p_1^2 + p_2^2) + \frac{\alpha - 1}{2\alpha - 1}p_1p_2 - e^{\alpha q_1 - (1-\alpha)q_2} - e^{(1-\alpha)q_1 + \alpha q_2} = 8nk^2 ,
$$

where

$$
A = \frac{1}{4(n-1)}(q_1 + q_2 - 2\log \frac{\lambda_1\lambda_2}{n-1}) , \quad \phi = \frac{a}{2(n-1)}(q_1 - q_2) + \frac{1}{a}\log \frac{\lambda_2}{\lambda_1} ,
$$

the constant $\alpha$ is related to $a$ by

$$
\alpha = \frac{1}{2} + \frac{a^2}{2(n-1)} = \frac{\Delta}{2(n-1)} ,
$$

and $H = H(p_1, p_2, q_1, q_2)$ is the Hamiltonian. Thus Hamilton’s equations $q_i' = \partial H/\partial p_i$ imply that

$$
p_1 = \alpha \dot{q}_1 + (1-\alpha)\dot{q}_2 , \quad p_2 = (1-\alpha)\dot{q}_1 + \alpha \dot{q}_2 ,
$$

while $\dot{p}_i = -\partial H/\partial q_i$ gives precisely the equations of motion (4.50).

As far as we know, the general solution to the equations (4.50) cannot be given in closed form except for two special values of $\alpha$, namely

$$
\alpha = 1 : \quad \ddot{q}_1 = e^{q_1} , \quad \ddot{q}_2 = e^{q_2} ,
$$

$$
\alpha = 2 : \quad \ddot{q}_1 = e^{2q_1 - q_2} , \quad \ddot{q}_2 = e^{2q_2 - q_1} .
$$

\[3\]There is another kind of solution that is sometimes called dyonic, in which two or more field strengths carry charges, some of them electric and the others magnetic. These are not really intrinsically dyonic, since they can be rendered purely electric or purely magnetic by dualisations.
The first case gives two independent Liouville equations, while the second gives the $SL(3, R)$ Toda equations. They correspond to values of $\Delta$ that are allowed in the maximal supergravities in $D = 6$ and $D = 4$ respectively, namely $\Delta = 4$ in each case. Since the allowed values take the form $\Delta = 4/N$, we see that indeed, as stated above, these two solvable cases involve just one field strength.

The dyonic string solution in $D = 6$, where the equations separate as two Liouville equations, is easily found to be

$$e^{-\phi/\sqrt{2} - 2A} = \frac{2Q_m}{\beta_1} \sinh(\beta_1 \xi + \gamma_1),$$

$$e^{\phi/\sqrt{2} - 2A} = \frac{2Q_e}{\beta_2} \sinh(\beta_1 \xi + \gamma_1),$$

(4.55)

with the constraint $\beta_1^2 + \beta_2^2 = 4nk^2$, where $Q_e$ and $Q_m$ are the electric and magnetic charges of the string. The solution has an outer event horizon at $\rho = 1/k$ (i.e. at $\xi = \infty$), and the mass per unit length is

$$m = \sqrt{Q_e^2 + \frac{3}{2} k^2} + \sqrt{Q_m^2 + \frac{3}{2} k^2},$$

(4.56)

where we have chosen $2Q_m \sinh \gamma_1 = \beta_1$ and $2Q_e \sinh \gamma_2 = \beta_2$ so that the solution approaches the standard Minkowski spacetime at infinity, and the dilaton vanishes there. The usual extremal dyonic string is recovered in the limit when $k$ goes to zero. The eigenvalues of the Bogomol’nyi matrix in the case of this dyonic string are

$$\mu = m \pm Q_e \pm Q_m,$$

(4.57)

where the $\pm$ signs are independent, and thus in the extremal limit where

$$m = Q_e + Q_m$$

(4.58)

we have $\mu = 2\{0, Q_e, Q_m, Q_e + Q_m\}$ and the solution preserves $\frac{1}{4}$ of the supersymmetry. As usual, the occurrence of 8 further zero eigenvalues when $Q_m = -Q_e$ does not imply any enhancement of the supersymmetry since the solution then has naked singularities and the Bogomol’nyi analysis becomes invalid.

Before moving on to the dyonic Toda black hole in $D = 4$, we should remark that in addition to the type-1 non-extremal dyonic string obtained above, there is also a more standard type-2 non-extremal solution, where the metric ansatz has the form (4.34), and $B = -A$. The solutions for $\phi$ and $A$ are given by

$$e^{-\phi/\sqrt{2} - 2A} = 1 + \frac{k}{r^2} \sinh^2 \mu_1, \quad e^{\phi/\sqrt{2} - 2A} = 1 + \frac{k}{r^2} \sinh^2 \mu_2,$$

(4.59)
with \( f \) as usual given by (4.33). The mass per unit length and the charges are given in terms of \( k \) and \( \mu_i \) by

\[
m = k(2 \sinh^2 \mu_1 + 2 \sinh^2 \mu_2 + 1), \quad Q_m = k \sinh 2\mu_1, \quad Q_e = k \sinh 2\mu_2. \tag{4.60}
\]

For non-negative values of \( k \), the mass and the charges satisfy the bound

\[
m - Q_e - Q_m = k + k e^{-2\mu_1} + k e^{-2\mu_2} \geq 0.
\]

The bound is saturated in the extremal limit \( k \to 0 \).

Turning now to the dyonic black hole in \( D = 4 \), for which the equations of motion reduce to the \( \alpha = 2 \) case in [4.54], one finds from the general solution of the Toda equation that the solutions for \( \phi \) and \( A \) are given in terms of four arbitrary constants \( c_1, c_2, \mu_1 \) and \( \mu_2 \) by

\[
Q_e^{-4/3} Q_m^{-2/3} e^{-\phi/\sqrt{3}-2A} = \frac{16c_1 e^{\mu_1 \xi}}{\nu_1 (\nu_1 - \nu_2)} - \frac{16c_2 e^{\mu_2 \xi}}{\nu_2 (\nu_1 - \nu_2)} + \frac{16 e^{-(\mu_1 + \mu_2) \xi}}{c_1 c_2 \nu_1 \nu_2},
\]

\[
Q_e^{-4/3} Q_m^{-2/3} e^{\phi/\sqrt{3}-2A} = \frac{16 e^{-\mu_1 \xi}}{c_1 \nu_1 (\nu_1 - \nu_2)} - \frac{16 e^{-\mu_2 \xi}}{c_2 \nu_2 (\nu_1 - \nu_2)} - \frac{16 c_1 c_2 (\mu_1 + \mu_2)}{\nu_1 \nu_2},
\]

where \( \nu_1 = 2\mu_1 + \mu_2 \) and \( \nu_2 = 2\mu_2 + \mu_1 \), together with the constraint \( H = \mu_1^2 + \mu_2^2 + \mu_1 \mu_2 = 16k^2 \).

The extremal limit of the dyonic black hole can be found by taking \( k \) to zero appropriately in the above solution. An easier way of obtaining the extremal solution is by directly re-solving the Toda equations subject to the Hamiltonian constraint [4.51] with \( k = 0 \). The required solution is obtained by making the ansatz that \( e^{-q_2} = e^{-q_1} + \text{const.} \) With this ansatz, it is easy to verify that \((e^{-q_1})'' = (e^{-q_2})'' = 1\), where here a prime denotes a derivative with respect to the redefined radial variable \( \rho = 1/r \). Thus, in terms of the original variables \( \phi \) and \( A \), the solution takes the form

\[
e^{\phi/\sqrt{3}-2A} \equiv T_m = 1 + 4Q_m^{2/3} (Q_e^{2/3} + Q_m^{2/3}) \frac{1}{r} + 8Q_e^{2/3} Q_m^{4/3} \frac{1}{r^2},
\]

\[
e^{-\phi/\sqrt{3}-2A} \equiv T_e = 1 + 4Q_e^{2/3} (Q_e^{2/3} + Q_m^{2/3}) \frac{1}{r} + 8Q_m^{2/3} Q_e^{4/3} \frac{1}{r^2},
\]

where we have chosen certain integration constants so that \( \phi \) and \( A \) approach zero as \( r \) tends to infinity. The metric of the extremal dyonic black hole is given by

\[
 ds^2 = -(T_e T_m)^{-1/2} dt^2 + (T_e T_m)^{1/2} (dr^2 + r^2 d\Omega^2). \tag{4.64}
\]

An interesting feature of this solution is that the mass is given in terms of the electric and magnetic charges by the curious formula

\[
m = \left(Q_e^{2/3} + Q_m^{2/3}\right)^{3/2}. \tag{4.65}
\]
Since the Bogomol’nyi matrix in this case has eigenvalues $\mu = m \pm Q_e \pm Q_m$, it follows that even in this extremal limit, the solution has no supersymmetry (unless $Q_e = 0$ or $Q_m = 0$): It is easily seen that the eigenvalues are strictly positive unless one of the charges vanishes.

The final example of a dyonic solution in toroidally-compactified M-theory arises in the eight-dimensional theory. Here, there the 4-form field strength $F_4$ can carry both electric and magnetic charge, giving rise to a dyonic membrane solution \[37\]. This solution is rather different from the previous ones we have discussed, in that it not only involves the 4-form field strength and a dilatonic scalar, but also the 0-form potential $A_0^{(123)}$. This is clear from the form of the cubic $FFA$ terms in $D = 8$, given in (2.13), which imply that $A_0^{(123)}$ will have $F_{MNPQ} F_{RSTU} \epsilon^{MNPQRTU}$ as a source on the right-hand side of its field equation. When $F_4$ carries both electric and magnetic charge, this source will be non-zero. In fact, the dyonic membrane solution can be obtained by performing a duality rotation on a simple purely electric or purely magnetic membrane solution of the standard kind. In this respect, the situation is quite different from that for the $D = 6$ dyonic string or the $D = 4$ dyonic black hole described above, where the dyonic solutions are not simply related to previously-known purely electric or purely magnetic ones by duality rotations. The forms of the metric, dilaton and axion in the dyonic membrane solution are given by \[37\]:

$$\begin{align*}
ds^2 &= H^{-\frac{1}{2}} (-dt^2 + dx_1^2 + dx_2^2) + H^\frac{1}{2} dy^m dy^m, \\
F_4 &= \frac{1}{2} (\star dH) \cos \delta + \frac{1}{2} dH^{-1} \wedge dt \wedge dx_1 \wedge dz \sin \delta, \\
A_0 + ie^{2\sigma} &= \frac{1}{2} \frac{(1-H) \sin 2\delta + 2iH^{\frac{1}{2}}}{(\sin^2 \delta + H \cos^2 \delta)},
\end{align*}$$

(4.66)

where $\sigma = \phi_1/2 + 3\phi_2/\sqrt{7} + 6\phi_3/\sqrt{21}$, and the two orthogonal combinations of $\phi_1$, $\phi_2$ and $\phi_3$ are zero. The angle $\delta$ parameterises the duality rotation. Since the U-duality symmetry commutes with supersymmetry, the solution preserves the same fraction $\frac{1}{2}$ of the supersymmetry as in the pure electric and pure magnetic cases. This can also be seen from the Bogomol’nyi bound, which is saturated by the solution (4.66) since the mass per unit 2-volume is given by

$$m = \sqrt{Q_e^2 + Q_m^2},$$

(4.67)

where $Q_e = Q \sin \delta$ and $Q_m = Q \cos \delta$, and $Q$ is the purely magnetic charge before the U-duality rotation.
4.5 $SL(N + 1, R)$ Toda solitons

Before leaving the subject of $p$-brane soliton solutions, we shall consider one further class of solutions that arises in toroidally-compactified M-theory. As we have indicated, the conditions that govern whether a particular set of field strengths can be active in a multiple-charge solution are quite stringent. For example, we have seen that if one restricts attention to single-scalar solutions, then in general these can only occur if the constant $\Delta$ defined in (4.21) is of the form $\Delta = 4/N$, where $N$ is an integer.\footnote{In [10], various solutions with other values of $\Delta$, such as $\Delta = 3$ were described. These would be perfectly valid solutions if the field strengths appearing in (2.7) were all simply the exterior derivatives of potentials. However, the Chern-Simons modifications that arise from the dimensional reduction procedure complicate matters considerably, and in particular, in general they rule out such other values of $\Delta$ in the toroidally-compactified supergravity theories.} There is, however, one additional class of exceptional cases where solutions in the toroidally-compactified supergravity theories can arise, which, when the charges are set equal, give values of $\Delta$ other than $4/N$. These occur for solutions using 1-form field strengths, which will be either $(D - 3)$-branes if the field strengths carry magnetic charges, or $(-1)$-branes (i.e. instantons, for which the time coordinate must be Euclideanised) if they carry electric charges.

The solutions of the kind we are discussing here arise if the dilaton vectors $\vec{c}_\alpha$ for a set of $N$ 1-form field strengths satisfy the dot-product relations [10]

$$
M_{\alpha\beta} \equiv \vec{c}_\alpha \cdot \vec{c}_\beta = 4\delta_{\alpha\beta} - 2\delta_{\alpha,\beta+1} - 2\delta_{\alpha,\beta-1},
$$

which is in fact twice the Cartan matrix for $SL(N + 1, R)$. It is straightforward to verify from the expressions given in (2.8) that there are indeed sets of 1-form field strengths whose dilaton vectors satisfy (4.68), namely those of the form $\mathcal{F}^i_{i+1}$. The remarkable thing is that, as can be verified from (2.11), these particular field strengths have no Chern-Simons modifications. Thus we may consider a set of $N$ 1-form field strengths $\mathcal{F}_\alpha \equiv \mathcal{F}^\alpha_{1,\alpha+1}$ whose dilaton vectors satisfy (4.68) and which are given simply by $\mathcal{F}_\alpha = d\chi_\alpha$. In $D$ dimensions, we can clearly have up to $N_{\text{max}} = 10 - D$ such 1-forms. The Lagrangian (2.7) can then be consistently truncated to [38]

$$
\mathcal{L} = eR - \frac{1}{2} e \sum_{\alpha,\beta=1}^{N} (M^{-1})_{\alpha\beta} \partial_\mu \varphi_\alpha \partial_\nu \varphi_\beta - \frac{1}{2} e \sum_{\alpha=1}^{N} e^{-\varphi_\alpha} (\partial \varphi_\alpha)^2.
$$

We proceed by making the standard metric and magnetic field strength ansätze, which in this case is

$$
ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + e^{2B(r)} (dr^2 + r^2 d\theta^2),
$$
\[\chi_{\alpha} = 4Q_{\alpha} \theta . \quad (4.70)\]

Substituting into the equations of motion following from (4.69), we obtain

\[
\varphi''_{\alpha} = -8 \sum_{\beta} M_{\alpha\beta} Q^2_{\beta} e^{-\varphi_{\beta}}, \quad B = \sum_{\alpha,\beta} (M^{-1})_{\alpha\beta} \varphi_{\alpha}, \quad (4.71)
\]

\[
\sum_{\alpha,\beta} (M^{-1})_{\alpha\beta} \varphi'_{\alpha} \varphi'_{\beta} = 16 \sum_{\alpha} Q^2_{\alpha} e^{-\varphi_{\alpha}}, \quad (4.72)
\]

where a prime denotes a derivative with respect to \( \rho = \log r \). Making the redefinition \( \Phi_{\alpha} = -2 \sum_{\beta} (M^{-1})_{\alpha\beta} \varphi_{\beta} \), these equations become

\[
\Phi''_{\alpha} = 16Q^2_{\alpha} \exp(\frac{1}{2} \sum_{\beta} M_{\alpha\beta} \Phi_{\beta}), \quad B = -\frac{1}{2} \sum_{\alpha} \Phi_{\alpha}, \quad (4.73)
\]

The further redefinition \( \Phi_{\alpha} = q_{\alpha} - 4 \sum_{\beta} (M^{-1})_{\alpha\beta} \log(4Q_{\beta}) \) removes the charges from the equations, giving \[38\]

\[
q''_{1} = e^{2q_{1}-q_{2}}, \quad q''_{2} = e^{-q_{1}+2q_{2}-q_{3}}, \quad q''_{3} = e^{-q_{2}+2q_{3}-q_{4}}, \quad \ldots \quad q''_{N} = e^{-q_{N-1}+2q_{N}}. \quad (4.74)
\]

These are precisely the \( SL(N+1, R) \) Toda equations. The solution is subject to the further constraint (4.72), which, in terms of the \( q_{\alpha} \), becomes the constraint that the Hamiltonian

\[
\mathcal{H} = 4 \sum_{\alpha,\beta} (M^{-1})_{\alpha\beta} p_{\alpha} p_{\beta} - \sum_{\alpha} \exp(\frac{1}{2} \sum_{\beta} M_{\alpha\beta} q_{\beta}) \quad (4.75)
\]

for the Toda system (4.74) vanishes.

The general solution to the \( SL(N+1, R) \) Toda equations is presented in an elegant form in \[38, 41\]:

\[
e^{-q_{\alpha}} = \sum_{k_{1}<\ldots<k_{\alpha}} f_{k_{1}} \cdots f_{k_{\alpha}} \Delta^2(k_{1}, \ldots, k_{\alpha}) e^{(\mu_{k_{1}}+\cdots+\mu_{k_{\alpha}})\rho}, \quad (4.76)
\]

where \( \Delta^2(k_{1}, \ldots, k_{\alpha}) = \prod_{k_{i}<k_{j}} (\mu_{k_{i}} - \mu_{k_{j}})^2 \) is the Vandermonde determinant, and \( f_{k} \) and \( \mu_{k} \) are arbitrary constants satisfying

\[
\prod_{k=1}^{N+1} f_{k} = -\Delta^{-2}(1,2,\ldots,N+1), \quad \sum_{k=1}^{N+1} \mu_{k} = 0. \quad (4.77)
\]
The Hamiltonian, which is conserved, takes the value $H = \frac{1}{2} \sum_{k=1}^{N+1} \mu_k^2$.

The solution (4.76) in general involves exponential functions of $\rho$. Furthermore, the vanishing of the Hamiltonian implies that the parameters $\mu_k$, and hence the solutions, will in general be complex. However, there exists a limit, under which all the $\mu_k$ constants vanish, which achieves a vanishing Hamiltonian and real solutions that are finite polynomials in $\rho$. Since we are constructing $(D - 3)$-branes in $D \geq 3$, it follows that we are interested in obtaining solutions to the $SL(N + 1, R)$ Toda equations for $N \leq 7$. When $N = 1$, the Toda system reduces to the Liouville equation, giving rise to the usual single field strength solution that preserves $1/2$ the supersymmetry, namely

$$e^{-q_1} = 1 + 4Q \rho .$$

(4.78)

Note that since there is only a single independent $\mu$ parameter when $N = 1$, which has to be zero by the Hamiltonian constraint, (4.78) is in fact the only solution in this case.

For $N = 2$, we find that the polynomial solution to the $SL(3, R)$ Toda equations (4.74) is

$$e^{-q_1} = a_0 + a_1 \rho + \frac{1}{2} \rho^2 ,$$

$$e^{-q_2} = a_1^2 - a_0 + a_1 \rho + \frac{1}{2} \rho^2 ,$$

(4.79)

where $a_0$ and $a_1$ are constants that are related to the charge parameters $Q_1$ and $Q_2$, on using the boundary condition that the dilatonic scalars, and hence $\Phi_\alpha$, vanish “asymptotically” (i.e. at $\rho = 0$). Thus we have

$$a_0 = \frac{1}{16} Q_1^{-4/3} Q_2^{-2/3} ,$$

$$a_1 = \frac{1}{4} Q_1^{-2/3} Q_2^{-2/3} (Q_1^{2/3} + Q_2^{2/3})^{1/2} ,$$

(4.80)

which implies that the metric is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + T_1 T_2 (dr^2 + r^2 d\theta^2) ,$$

(4.81)

where

$$T_1 = 1 + 4Q_1^{2/3} (Q_1^{2/3} + Q_2^{2/3})^{1/2} \rho + 8Q_1^{4/3} Q_2^{2/3} \rho^2 ,$$

$$T_2 = 1 + 4Q_2^{2/3} (Q_1^{2/3} + Q_2^{2/3})^{1/2} \rho + 8Q_2^{4/3} Q_1^{2/3} \rho^2 ,$$

(4.82)

and $\rho = \log r$. The mass per unit $(D - 3)$-volume is given by

$$m = (Q_1^{2/3} + Q_2^{2/3})^{3/2} .$$

(4.83)

This is the same rather unusual looking mass formula that arises in the $a = \sqrt{3}$ four-dimensional dyonic black hole [36], which we discussed in the previous section, and which
was also associated with a solution of the $SL(3, R)$ Toda equations. For non-vanishing Hamiltonian the black hole is non-extremal, becoming extremal when the Hamiltonian vanishes. The mass formula (4.83) implies that the solution describes a system with negative binding energy, since the total mass of the widely-separated constituents is given by $m_\infty = Q_1 + Q_2$, which is smaller than $m$. The Bogomol’nyi matrix in this case is $M = m \mathbf{1} + Q_1 \Gamma_{1212} + Q_2 \Gamma_{1223}$, and therefore its eigenvalues are
\[ \mu = m \pm \sqrt{Q_1^2 + Q_2^2}. \] (4.84)

It follows from (4.83) that the $\mu$ is strictly positive, and hence the Bogomol’nyi bound is exceeded and there is no supersymmetry, unless either $Q_1$ or $Q_2$ vanishes.

For $N = 3$, we find the following polynomial solution of the $SL(4, R)$ Toda equations:
\begin{align*}
e^{-q_1} &= a_0 + a_1 \rho + a_2 \rho^2 + \frac{1}{6} \rho^3, \\
e^{-q_2} &= a_1^2 - 2a_0 a_2 + (2a_1 a_2 - a_0) \rho + 2a_2 \rho^2 + \frac{2}{3} a_2 \rho^3 + \frac{1}{12} \rho^4, \quad (4.85) \\
e^{-q_3} &= a_0 - 4a_1 a_2 + 8a_2^3 + (4a_2^2 - a_1) \rho + a_2 \rho^2 + \frac{1}{6} \rho^3,
\end{align*}

where the constants $a_0$, $a_1$ and $a_2$ are determined in terms of the charges $Q_1$, $Q_2$ and $Q_3$ by the requirement that the dilatonic scalars vanish at $\rho = 0$. This implies that
\[ e^{q_{\alpha}(0)} = \prod_\beta (4Q_{\beta})^{4(M^{-1})_{\alpha\beta}}, \] (4.86)

and hence
\begin{align*}
a_0 &= \frac{1}{64} Q_1^{-3/2} Q_2^{-1} Q_3^{-1/2}, \\
a_1^2 - 2a_0 a_2 &= \frac{1}{256} Q_1^{-1} Q_2^{-2} Q_3^{-1}, \\
a_0 - 4a_1 a_2 + 8a_2^3 &= \frac{1}{64} Q_1^{-1/2} Q_2^{-1} Q_3^{-3/2}. \quad (4.87)
\end{align*}

The metric is given by (4.70), with
\[ e^{2B} = \prod_\alpha e^{q_{\alpha}(0) - q_\alpha}, \] (4.88)

and hence
\[ m = \frac{a_1}{4a_0} + \frac{2a_1 a_2 - a_0}{4(a_1^2 - 2a_0 a_2)} + \frac{4a_2^2 - a_1}{4(a_0 - 4a_1 a_2 + 8a_2^3)}. \] (4.89)

Thus we find [38] that the mass is given in terms of the charges by the positive root of the sextic
\[
\begin{align*}
& m^6 - (3Q_1^2 + 2Q_1Q_3 + 3Q_3^2 + 3Q_2^2)m^4 - 36\sqrt{Q_1Q_3Q_2}(Q_1 + Q_3)m^3 \\
& + \left[(Q_1 + Q_3)^2(3Q_1^2 - 2Q_1Q_3 + 3Q_3^2) - Q_2^2(21Q_1^2 + 122Q_1Q_3 + 21Q_3^2) + 3Q_2^4\right]m^2 \\
& + 4\sqrt{Q_1Q_3Q_2}(Q_1 + Q_3)(9Q_1^2 - 14Q_1Q_3 + 9Q_3^2 - 18Q_2^2)m \\
& -(Q_1 - Q_3)^2(Q_1 + Q_3)^4 - Q_2^2(3Q_1^4 - 68Q_1^3Q_3 + 114Q_1^2Q_3^2 - 68Q_1Q_3^3 + 3Q_3^4) \\
& - Q_2(3Q_1^2 + 38Q_1Q_3 + 3Q_3^2) - Q_2^6 = 0 .
\end{align*}
\]

There seems to be no way to give an explicit closed-form expression for the mass in terms of the charges. The Bogomol'nyi matrix \( \mathcal{M} = m \mathbb{1} + Q_1\Gamma_{i212} + Q_2\Gamma_{i223} + Q_3\Gamma_{i234} \) has eigenvalues
\[
\mu = m \pm \sqrt{(Q_1 \pm Q_3)^2 + Q_2^2} ,
\]
where the two \( \pm \) signs are independent. For generic values of the charges, \( \mu > 0 \) and the solution has no supersymmetry. If \( Q_2 = 0 \), the solution reduces to the two-charge supersymmetric solution, preserving \( \frac{1}{4} \) of the supersymmetry. In this case, the \( SL(4, R) \) Toda equations reduce to two decoupled Liouville equations.

For higher values of \( N \), the explicit forms of the polynomial solutions to the \( SL(N+1, R) \) Toda equations become increasingly complicated \([8] \). The structure of these polynomials can be summarised as follows. For each \( N \), we find that \( e^{-q_\alpha} \) are polynomials in \( \rho \) of degree \( n_\alpha = \alpha(N + 1 - \alpha) \), \( i.e. \)
\[
\frac{d^{n_\alpha+1}}{d\rho^{n_\alpha+1}}e^{-q_\alpha} = 0 .
\]

After substituting these into the \( SL(N+1, R) \) Toda equations \([1,74] \), we find that there are \( N \) independent parameters, which can be related to the \( N \) charges \( Q_\alpha \) by equation \([1,86] \). The metric is given by \([4,70] \) with \( e^{2B} \) again given by \([1,88] \). The mass is given in terms of the charges by an \( N! \)’th-order polynomial equation. Although it appears not to be possible to give closed-form expressions for the mass in terms of the charges for \( N \geq 3 \), we expect nevertheless that it is less than the sum of the charges, indicating again that they are bound states with negative binding energies. One can see this explicitly in the special case where the charges have the fixed ratio given by
\[
Q_\alpha = aQ\left(\sum_\beta (M^{-1})_{\alpha\beta}\right)^{1/2} = \frac{1}{2}aQ\sqrt{\alpha(N + 1 - \alpha)} ,
\]
where \( a \) is given \( a^2 = \Delta = 24/(N(N + 1)(N + 2)) \). Under these circumstances the solutions reduce to single-scalar solutions, given by
\[
ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + H^{1/\Delta} (dv^2 + v^2 d\theta^2) ,
\]

\[31\]
\[ e^{\alpha \phi /2} = H = 1 + k \log r , \quad \chi = 4 Q \theta , \]  
(4.94)

and have mass
\[ m = \frac{2Q}{a} . \]  
(4.95)

It is easy to verify that this is always larger than the total mass of the widely-separated constituents, \( m_\infty = \sum_\alpha Q_\alpha \). The calculation of the eigenvalues of the Bogomol'nyi matrix becomes increasingly complicated with increasing \( N \). For example, for the \( SL(5, R) \) case we find
\[ \mu = m \pm \sqrt{Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 + 2(\sqrt{Q_1 Q_3})^2 + (Q_1 Q_4)^2 + (Q_2 Q_4)^2} , \]  
(4.96)

whilst for \( SL(6, R) \) we find that \( \mu = m \pm \kappa \), where \( \kappa \) denotes the roots of the quartic equation
\[ \kappa^4 - 2\kappa^2 \alpha - 8\kappa Q_1 Q_3 Q_5 + \beta = 0 , \]

\[ \alpha = Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 + Q_5^2 , \]  
(4.97)

\[ \beta = \alpha^2 - 4((Q_1 Q_3)^2 + (Q_1 Q_4)^2 + (Q_1 Q_5)^2 + (Q_2 Q_4)^2 + (Q_2 Q_5)^2 + (Q_3 Q_5)^2) . \]

For all \( N \), the solutions are non-supersymmetric for generic values of the charges. However, they can be reduced to the previously-known supersymmetric solutions if appropriate charges are set to zero, such that the remaining charges \( Q_\alpha \) have non-adjacent indices. In these cases, the solutions preserve \( 2^{-n} \) of the supersymmetry, where \( n \) is the number of charges remaining.

### 4.6 Fission and fusion bound states of \( p \)-brane solitons

So far, we have obtained a large class of \( p \)-branes solutions in the toroidally-compactified M-theory. It is useful to classify and organise these solutions. One approach is to observe that supersymmetric \( p \)-branes that carry a single electric or magnetic charge, and hence preserve half the supersymmetry, can be interpreted as the constituents from which all the multiply-charged \( p \)-branes can be constructed as bound states. The binding energy can be zero, positive and negative, depending on the specific choice of constituents.

The binding energy of a \( p \)-brane can easily be calculated by comparing its mass with the sum of the masses of its individual constituents when their locations are widely separated. Of course if the binding energy is non-zero, this configuration will not be an exact solution. However, it can be made arbitrarily good by taking the separations to be sufficiently large. Since each individual constituent satisfies the Bogomol'nyi bound, \( i.e. \) its mass is equal to
the charge, it follows that the total mass when the constituents are widely separated is given by

$$m_\infty = \sum_{\alpha} Q_\alpha ,$$

(4.98)

where $Q_\alpha$ are the charges of the individual constituents.

There are various ways to obtain multiply-charged $p$-branes. The simplest way is to act with a U-duality rotation on a singly-charged $p$-brane. The new solution preserves the same fraction of supersymmetry as the original one. Since these solutions contain multiple charges, they can be viewed as bound states. Such bound states usually have positive binding energy. For example, the 8-dimensional dyonic membrane can be obtained by an $SL(2, \mathbb{Z})$ T-duality transformation from a purely electric or purely magnetic membrane, as we discussed in section 4.4. The mass of the dyonic membrane is given by (4.67), which is always smaller than the combined mass of the two widely-separated electrically-charged and magnetically-charged membranes. Thus the dyonic membrane is a bound state of these two basic constituents with positive binding energy. Another simple example is provided by the two string solutions of type IIB supergravity in $D = 10$. One of these uses the NS-NS 3-form field strength, whilst the other uses the R-R 3-form. There is a non-perturbative $SL(2, \mathbb{Z})$ symmetry of the type IIB theory, which rotates between the NS-NS string and the R-R string solutions. Thus one obtains a bound state of the NS-NS string and the R-R string by acting with an $SL(2, \mathbb{Z})$ transformation on either the NS-NS or the R-R string solution [11]. The mass of the bound state is given by $m = \sqrt{Q_{\text{NS-NS}}^2 + Q_{\text{R-R}}^2}$, and hence it has positive binding energy [12]. In general, all the bound states that are obtained by acting with U-duality rotations on singly-charged $p$-branes have positive binding energy.

The multiply-charged solutions we obtained in section 4.1 are of a different type. They are not related to the single-charge solutions of any of their constituents by U-duality rotations. One way to see this is that they preserve a different fraction of the supersymmetry. The masses of these solutions are given by (4.18) and hence they can be viewed as bound states with zero binding energy [13, 29, 30]. For example, the $a = 1, 1/\sqrt{3}$ and 0 black holes in $D = 4$ can be viewed as bound states of two, three or four $a = \sqrt{3}$ black holes [13]. Another example is provided by the dyonic string in $D = 6$, discussed in section 4.4, which can be viewed as a bound state of an electric and a magnetic string [29]. These bound states are in general supersymmetric, although non-supersymmetric solutions can also arise. For example, the $a = 0$ Reissner-Nordstrøm back hole in $D = 4$ can be non-supersymmetric for certain choices of sign of the four constituent charges, which appear quadratically in the equations of motion, but linearly in the supersymmetry transformation...
rules [28, 14], as discussed in section 4.2. For all these solutions, strictly speaking, the term bound state is a misnomer since the binding energy is actually zero. This zero binding energy is consistent with the fact that the charges in the above multi-charge solutions can be located independently; the bound states can be “pulled apart” into constituents that can sit in static equilibrium at any separation. This can be seen from the fact that generalisations of the solution (4.16) exist where the functions $H_\alpha$ can be any harmonic functions [51] on the transverse space $y^m$, implying in particular that each individual charge can be located at any point in the transverse space. We shall discuss such multi-centre solutions further in section 5.1. This type of multi-charge, multi-centre solution was first discussed in [52], where a four-charge supersymmetric black hole in $D = 4$ was “split” into two $a = 1$ two-charge black holes (which themselves can be further split into $a = \sqrt{3}$ black holes).

The third type of bound states are those with negative binding energy. These solutions are non-supersymmetric. Examples are provided by the 4-dimensional dyonic black hole, and by the $SL(N + 1, R)$ solitons in various dimensions, which were discussed in section 4.4 and 4.5 respectively. It is easy to see from the mass formulae for the solutions that these bound states have negative binding energy. Note that the dyonic black hole in section 4.4 reduces to the 4-dimensional Reissner-Nordstrøm black hole when the electric and magnetic charges are equal. Thus a Reissner-Nordstrøm black hole in $D =$ can be viewed either as an inert bound state of four constituents with zero binding energy, or as a “dyonic fission bomb” of negative binding energy, comprised of an electric black hole and a magnetic black hole of equal charges. The two cases are distinguished by the choice of field strengths that carry the charges.

Finally, let us note that in $D = 8$ the only dyonic membrane is the one that can be obtained by acting on the purely electric or purely magnetic membrane with a T-duality $SL(2, R)$ transformation [37]. The resulting dyonic membrane preserves half of the supersymmetry, and is a bound state with positive binding energy. In $D = 6$ and $D = 4$, rotations of this kind cannot be used to convert a solution with purely electric or purely magnetic charges into a dyonic solution; other field strengths will also acquire charges at the same time. In $D = 6$ and $D = 4$, dyonic solutions can be constructed directly, as we saw in section 4.4. In $D = 6$, the dyonic string preserves $\frac{1}{4}$ of the supersymmetry, and is a bound state with zero binding energy. In $D = 4$, the dyonic black hole solution is non-supersymmetric, and is a bound state with negative binding energy.
5 Dimensional reduction and oxidation

In the previous sections, we have extensively discussed classes of \( p \)-brane solitons that arise as solutions in the toroidal compactifications of M-theory. Of course since the toroidally-compactified supergravities are themselves consistent truncations of \( D = 11 \) supergravity, it follows that if higher-dimensional \( p \)-brane solutions are themselves dimensionally reduced, they will give rise to solutions of the lower-dimensional theories. In fact many, but not all, of the lower-dimensional \( p \)-brane solitons can be obtained simply as the dimensional reductions of \( p \)-branes in higher dimensions.

As we shall discuss below, there are two ways of dimensionally reducing a \( p \)-brane solution to give another such solution in one lower dimension. The simpler of the two is “diagonal” reduction, in which one of the spatial coordinates on the \( p \)-brane world-volume is used for the Kaluza-Klein reduction. This removes one dimension from the spacetime and from the world-volume simultaneously, and thus we go from a \( p \)-brane in \( D \) dimensions to a \((p-1)\)-brane in \( D-1 \) dimensions, *i.e.* \( (D,p) \to (D-1,p-1) \). The other procedure is known as “vertical” dimensional reduction, and in this case one of the transverse-space coordinates is used for the Kaluza-Klein reduction, taking us from \( (D,p) \) to \( (D-1,p) \). This is more complicated to implement, because to perform a Kaluza-Klein reduction on a coordinate it is necessary that translations along that direction should be a Killing symmetry, whereas a standard single \( p \)-brane soliton depends isotropically on all the coordinates \( y^m \) of the Cartesian transverse space. It is first necessary to construct a multi-centre \( p \)-brane solution in the higher dimension, with the centres periodically aligned along the chosen coordinate axis, such that as the continuum limit is taken, the solution becomes independent of this coordinate. By means of these two reduction procedures, a given \( p \)-brane in \( D \) dimensions gives rise to a \((p-1)\)-brane and a \( p \)-brane in \( D-1 \) dimensions.

However, whilst it is certainly true, owing to the consistency of the truncations to the lower-dimensional supergravities, that all the lower-dimensional \( p \)-branes will also be solutions in the higher-dimensional theories, not all of them “oxidise” back to simple \( p \)-brane solutions in the higher dimensions. Consequently, there will be other types of solutions of the higher-dimensional supergravities that are not immediately recognisable as \( p \)-branes, which nevertheless, upon dimensional reduction, give rise to lower-dimensional \( p \)-brane solitons of the usual kind. These more complicated higher-dimensional configurations can be interpreted as intersections of various \( p \)-branes, or as “twisted” \( p \)-branes. We shall discuss them further in section 5.2 below, having first described the simpler situation of the diagonal and vertical reduction of \( p \)-branes.
5.1 Diagonal and vertical dimensional reduction

First, let us consider the diagonal dimensional reduction of an \(N\)-charge \(p\)-brane solution in \(D\) dimensions to a \((p-1)\)-brane in \(D-1\) dimensions. Thus we begin with the metric (5.1),

\[
ds^2_D = \prod_{\alpha=1}^{N} H_{\alpha}^{(D-2)} \, dx^\mu dx^\nu \eta_{\mu\nu} + \prod_{\alpha=1}^{N} H_{\alpha}^{(D-2)} (dr^2 + r^2 d\Omega^2) ,
\]

where the \(H_{\alpha}\) are harmonic functions on the transverse space, of the form \(H_{\alpha} = 1 + \lambda_{\alpha} r^{-d}\).

The Kaluza-Klein reduction of this solution to \(D-1\) dimensions is therefore described by the metric \(ds^2_{D-1}\), related to \(ds^2_D\) by

\[
ds^2_D = e^{2\phi} ds^2_{D-1} + e^{-2(D-3)\alpha \phi} dz^2 ,
\]

where \(\alpha = (2(D-2)(D-3))^{-1/2}\) and the coordinate \(z\) is one of the spatial coordinates \(x^i\) in the \(D\)-dimensional \(p\)-brane world-volume. Thus we see that \(e^{2(D-3)\alpha \phi} = \prod_{\alpha} H_{\alpha}^{d/(D-2)}\), and hence the \((D-1)\)-dimensional metric is given by

\[
ds^2_{D-1} = \prod_{\alpha=1}^{N} H_{\alpha}^{(D-3)} dx^\mu dx^\nu \eta_{\mu\nu} + \prod_{\alpha=1}^{N} H_{\alpha}^{(D-3)} (dr^2 + r^2 d\Omega^2) ,
\]

where now the \(\mu\) index ranges over one fewer spatial indices than in (5.1). We see that (5.3) is describing an \(N\)-charge \((p-1)\)-brane solution in \(D-1\) dimensions, with otherwise precisely the same structural form as (5.1). Although we have now acquired one more dilatonic scalar, namely \(\phi\), it is evident from the solutions for the dilatons \(\phi_{\alpha}\) in (4.16) that a certain linear combination of \(\phi_{\alpha}\) and \(\phi\) vanishes, and hence there is no net increase in the number of excited scalars. In fact a careful calculation shows that the excited dilatonic degrees of freedom in the dimensionally-reduced solution are precisely of the same form as in the \(D\)-dimensional solution, where the appropriate changes are made to account for the additional component to the dilaton vectors that is acquired by virtue of the reduction procedure. Note, incidentally, that this description of the diagonal dimensional reduction can easily be extended also to the case of non-extremal \(p\)-brane solutions.

Now let us consider the vertical dimensional reduction process. To do this, we need first to construct more general multi-centre \(p\)-brane solutions in \(D\) dimensions. In fact the way in which the single-centre solutions (4.16) are written already suggests the form of the more general solutions. It is quite straightforward to show that the harmonic functions \(H_{\alpha}\) need not be restricted to be single-centre isotropic functions, and that we still have solutions of the equations if they are taken to have the quite general multi-centre form

\[
H_{\alpha} = 1 + \sum_{i} \frac{k_{i}^{\alpha}}{|\vec{y} - \vec{y}_{i}|^{d}} ,
\]
where we now use the Cartesian coordinates \( y^m \) on the \((D - d)\)-dimensional transverse space, and \( k_\alpha^i \) are arbitrary constants. As a special case we may choose the centres to be distributed periodically along a particular axis, which for simplicity could be the last of the \( y^m \) coordinates, such that in the continuum limit the harmonic functions become independent of this coordinate. The integration over the continuous line of centres reduces the powers in the denominators in the harmonic functions, 

\[
\int_{-\infty}^{\infty} dz (r^2 + z^2)^{-d/2} \sim r^{-d+1},
\]

with the net result that what remains is harmonic with respect to a transverse space of one lower dimension. (Multi-centre extremal static solutions have been constructed in \[45, 46, 47\], and their application for dimensional reduction was considered in \[48, 49\]. A detailed discussion of this, including the cases where the resulting harmonic functions have logarithmic, or even linear, coordinate dependence, can be found in \[50\].) Reading off from (5.2), where now the reduction coordinate is the last of the \( y^m \) variables in \( D \) dimensions, we see that this time 

\[
e^{-2(D-3)\alpha \phi} = \prod_{\alpha} H_\alpha^{d/(D-2)},
\]

where the \( H_\alpha \) are harmonic in the remaining \((D - d - 1)\)-dimensional transverse space. Substituting back into (5.2), we find that the resulting metric in \( D - 1 \) dimensions has the form

\[
ds_{D-1}^2 = \prod_{\alpha=1}^{N} H_\alpha^{(d-1)/(D-3)} dx^\mu dx^\nu \eta_{\mu\nu} + \prod_{\alpha=1}^{N} H_\alpha^{(D-3)/d} dy^m dy^m.
\]

(5.5)

This is precisely of the same structural form as the original \( p \)-brane metric in the higher dimension, except that here the dimension of spacetime has been lowered by removing one of the transverse dimensions, while preserving the dimension of the worldvolume.

### 5.2 Bound states as intersecting \( p \)-branes

In section 4, we obtained large classes of \( p \)-brane solutions. In section 4.6, we discussed a way to classify some of them, by viewing multiply-charged \( p \)-branes as bound states of singly-charged \( p \)-branes, which preserve half the supersymmetry. Another approach is to organise the various solutions using the U-duality group of the theory. In particular, it has been shown that \( p \)-brane solutions form representations of the Weyl group of the U-duality group \[53\]. Since duality symmetry commutes with supersymmetry, each member of such a U-duality multiplet preserves the same fraction of the supersymmetry.

Both of these organisational schemes are applicable in a given, fixed, dimension. A different approach is to interpret lower-dimensional solutions from the viewpoint of the fundamental dimension of the theory, namely \( D = 11 \) in the case of M-theory. In other words, the lower-dimensional solutions can be oxidised, by the inverse of the Kaluza-Klein reduction procedure, to solutions in \( D = 11 \). All maximal supergravities in the lower
dimensions can be obtained by Kaluza-Klein reduction of $D = 11$ supergravity by truncating out the massive modes. Since such a truncation is consistent, it implies that any lower dimensional solution can be oxidised back to a solution of $D = 11$ supergravity. It has been shown that lower-dimensional supersymmetric $p$-branes can be viewed as intersecting M-branes, or boosted or twisted intersecting M-branes in $D = 11$ [51,54-58]. We shall present a few examples to illustrate this.

In the previous subsection, we saw that a $p$-brane can undergo both vertical and diagonal dimensional reduction. Thus, for example, a membrane in $D = 11$ can become a string in $D = 10$ by diagonal dimensional reduction, and then by four steps of vertical reduction it becomes an electrically-charged string in $D = 6$. On the other hand, the 5-brane in $D = 11$ can be vertically reduced to a 5-brane in $D = 10$, and then diagonally reduced to give a magnetically-charged string in $D = 6$. Thus electric strings and magnetic strings in $D = 6$ can be oxidised back to give membranes and 5-branes respectively in $D = 11$. Naturally a dyonic string $D = 6$, which carries both electric and magnetic charges, can also be oxidised to $D = 11$, where it becomes an intersection a membrane and a 5-brane in $D = 11$. The metric of the $D = 11$ solution, obtained by simply reversing the Kaluza-Klein reduction process described by (2.2), is given by

$$ds_{11}^2 = H_e^{-\frac{2}{3}}H_m^{-\frac{1}{3}}(-dt^2 + dx_1^2) + H_e^{-\frac{2}{3}}H_m^{\frac{2}{3}}dz_1^2 + H_e^{\frac{1}{3}}H_m^{\frac{2}{3}}dy^m dy^m$$

$$+ H_e^{\frac{1}{3}}H_m^{-\frac{1}{3}}(dz_2^2 + dz_3^2 + dz_4^2 + dz_5^2),$$

(5.6)

where $H_e$ and $H_m$ are harmonic functions in the 4-dimensional space $y^m$, associated with the electric and magnetic charges respectively, as in (1.17). This solution gives rise to a dyonic string in $D = 6$ when compactified on a 5-torus with coordinates $z_i, i = 1, \ldots, 5$. If the magnetic charge of the dyonic string is set to zero, in which case $H_m = 1$, the solution becomes a configuration describing membranes with world volume coordinates $(t, x_1, z_1)$, whose charges are uniformly distributed over the hyperplane $(z_2, z_3, z_4, z_5)$. On the other hand, if instead the electric charge is set to zero, in which case $H_e = 1$, it describes a line (along the $z_1$ axis) of uniformly distributed 5-branes, with world volume coordinates $(t, x_1, z_2, z_3, z_4, z_5)$. Thus the solution (5.6) describes an interpolation between membranes and 5-branes, and hence is called an intersection of membranes and 5-branes.

Another example is provided by the supersymmetric Reissner-Nordstrøm black hole in $D = 4$. It is a bound state of four basic $a = \sqrt{3}$ constituent black holes. There are various ways of constructing such 4-charge black holes, the set of which form a representation under the Weyl group of $E_7$ [33]. One member of such a multiplet is the solution involving the
field strengths $F_2^{(12)}, F_2^{(34)}$ carrying electric charges, and $F_2^{(13)}$ and $F_2^{(24)}$ carrying magnetic charges. Oxidising to $D = 11$, the metric of this particular 4-charge black hole solution becomes

$$\begin{align*}
    ds_{11}^2 &= \left( \frac{H_3}{H_1 H_2} \right)^{\frac{1}{8}} ds_4^2 + \left( \frac{H_3^2}{H_1^2 H_4} \right)^{\frac{1}{4}} dz_1^2 + \left( \frac{H_3^2}{H_2^2 H_3} \right)^{\frac{1}{2}} dz_2^2 + \left( \frac{H_3^2}{H_2^2 H_4} \right)^{\frac{1}{4}} dz_3^2 \\
    &\quad + \left( \frac{H_1 H_2^2}{H_2^2 H_3} \right)^{\frac{1}{4}} dz_4^2 + \left( \frac{H_1 H_2^2}{H_2^2 H_4} \right)^{\frac{1}{2}} (dz_5^2 + dz_6^2 + dz_7^2),
\end{align*}$$

(5.7)

where $ds_4^2$ is the metric for the 4-charge black hole, given by

$$ds_4^2 = -(H_1 H_2 H_3 H_4)^{-\frac{1}{4}} dt^2 + (H_1 H_2 H_3 H_4)^{\frac{1}{2}} dy^m dy^m.$$

(5.8)

The functions $H_\alpha$ are harmonic in the 3-dimensional transverse space described by the coordinates $y^m$, and the metric (5.8) becomes that of the usual extremal Reissner-Nordstrøm black hole if all the charges, and hence all the $H_\alpha$, are equal. Thus it is easy to see now with the explicit metric (5.7) that this Reissner-Nordstrøm black hole becomes an intersection of two membranes and two 5-branes in $D = 11$.

The above two examples have the feature that the field strengths that are involved in the solutions all come from the dimensional reduction of the 4-form field strength in $D = 11$. The oxidation of these bound-state $p$-branes in lower dimensions give intersections of M-branes in $D = 11$, with the electrically-charged constituents becoming membranes, and the magnetically-charged constituents becoming 5-branes. The complete classification of such intersecting M-branes can be found in [58]. Of course in lower dimensions field strengths can also come from the reduction of the 11-dimensional metric tensor. In these cases, the oxidised metric in $D = 11$ will acquire off-diagonal components. An electrically-charged constituent in the lower dimension will describe a “boost” in $D = 11$, involving an off-diagonal component that mixes time and spatial directions in the world-volume of the M-brane. On the other hand, a magnetically-charged constituent will oxidise to a “twisted” metric in $D = 11$, involving a monopole-like configuration in the transverse space of the M-brane.

So far we described how bound states with zero binding energy can be viewed as intersections of M-branes (with possible boosts or twists) in $D = 11$. As we saw in section 4.6, bound states can also exist that have either positive or negative binding energy. Bound-state $p$-branes with positive binding energy are usually obtained from U-duality rotations of singly-charged constituent $p$-branes. The rotation involves non-linear transformations of axions, which are the dimensional reduction from $D = 11$ of the 3-form potential or the metric. Such a solution has a complicated metric structure in $D = 11$, and cannot be
simply interpreted as an intersection of M-branes. However, these solutions are in the same U-duality multiplet as simple solutions that can be interpreted as intersecting M-branes.

Bound states with negative binding energy can be oxidised straightforwardly, as in the case of the supersymmetric bound states with zero binding energy. We may take the dyonic black hole solution in $D = 4$ as an illustration. Again, we shall only consider the case where the 2-form field strength involved in the solution comes from the 4-form in $D = 11$. For example, let us consider the dyonic black hole where the field strength $F^{(12)}_2$ carries the electric and magnetic charges. The solution is given in section 4.4, and upon oxidation to $D = 11$ the metric becomes

$$
\frac{ds^2_{11}}{\left( T_e T_m \right)^{-\frac{1}{6}} ds^2_4 + \left( T_e T_m \right)^{-\frac{2}{3}} (dz_1^2 + dz_2^2) + \left( T_e T_m \right)^{\frac{1}{3}} (dz_3^2 + \cdots + dz_7^2)},
$$

(5.9)

where $ds^2_4$, $T_e$ and $T_m$ are given by (4.64) and (4.63). Thus when the electric charge $Q_e$ or the magnetic charge $Q_m$ vanishes, the metric reduces to a 5-brane or a membrane respectively. Note that both (5.7) and (5.9) give rise to Reissner-Nordström black holes in $D = 4$; however the former has zero binding energy whilst the latter has negative binding energy. The main difference from the 11-dimensional point of view is that the latter gives rise to a metric with a larger group of symmetries. We see from this example that a membrane can intersect with a 5-brane in different ways in $D = 11$. In one form of intersection there is zero binding energy and the solution reduces to the dyonic string in $D = 6$, while in another there is negative binding energy, and the solution reduces to the dyonic black hole in $D = 4$. A third possibility is for the intersection of the membrane and 5-brane to have positive binding energy, reducing to the dyonic membrane in $D = 8$.
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