REGULARITY OF SYMBOLIC POWERS OF SQUARE-FREE MONOMIAL IDEALS

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ABSTRACT. We study the regularity of symbolic powers of square-free monomial ideals. We prove that if $I = I_{\Delta}$ is the Stanley-Reisner ideal of a simplicial complex $\Delta$, then $\text{reg}(I^{(n)}) \leq \delta(n - 1) + b$ for all $n \geq 1$, where $\delta = \lim_{n \to \infty} \frac{\text{reg}(I^{(n)})}{n}$, and $b = \max\{\text{reg}(I_{\Gamma}) \mid \Gamma$ is a subcomplex of $\Delta$ with $\mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta)\}$. This bound is sharp for any $n$. When $I = I(G)$ is the edge ideal of a simple graph $G$, we obtain a general linear upper bound $\text{reg}(I^{(n)}) \leq 2n + \text{order-match}(G) - 1$, where order-match($G$) is the ordered matching number of $G$.

INTRODUCTION

Throughout the paper, let $K$ be a field and $R = K[x_1, \ldots, x_r]$ the polynomial ring of $r$ variables $x_1, \ldots, x_r$ with $r \geq 1$. Let $I$ be a homogeneous ideal of $R$. Then the $n$-th symbolic power of $I$ is defined by

$$I^{(n)} = \bigcap_{p \in \text{Min}(I)} I^n R_p \cap R,$$

where Min($I$) is as usual the set of minimal associated prime ideals of $I$.

Cutkosky, Herzog, Trung [5], and independently Kodiyalam [21], proved that the function $\text{reg}(I^n)$ is a linear function in $n$ for $n \gg 0$. The similar result for symbolic powers is not true even when $I$ is a square-free monomial ideal (see e.g. [8, Theorem 5.15]) except for the case $\dim(R/I) \leq 2$ (see [19]).

If $I$ is a square-free monomial ideal, Hoa and the second author (see [18, Theorem 4.9]) proved that the limit

$$\delta(I) = \lim_{n \to \infty} \frac{\text{reg}(I^{(n)})}{n},$$

(1)

does exist, in fact the limit exists for arbitrary monomial ideals (see [8]). Moreover, $\text{reg}(I^{(n)}) < \delta(I)n + \dim(R/I) + 1$ for all $n \geq 1$. This bound is obvious not sharp for every $n$ (see Corollary [2,4]). There have been many recent results which establish

2020 Mathematics Subject Classification. 13D45, 05C90, 05E40, 05E45.

Key words and phrases. Castelnuovo-Mumford regularity, symbolic power, edge ideal, matching.
sharp bounds for $\text{reg}(I^{(n)})$ in the case $I$ is the edge ideal of a simple graph (see e.g. [11, 13, 14, 20]).

The aim of this paper is to find sharp bounds for $\text{reg}(I^{(n)})$, for a square-free monomial ideal $I$, in terms of combinatorial data from its associated simplicial complexes and hypergraphs.

For a simplicial complex $\Delta$ on the set $V = \{1, \ldots, r\}$, the Stanley-Reisner ideal of $\Delta$ is defined by

$$I_\Delta = \left( \prod_{i \in \tau} x_i \mid \tau \subseteq V \text{ and } \tau \notin \Delta \right) \subseteq R.$$ Let us denote by $F(\Delta)$ the set of all facets of $\Delta$.

The first main result of the paper is the following theorem.

**Theorem 2.3.** Let $\Delta$ be a simplicial complex. Then,

$$\text{reg}(I_\Delta^{(n)}) \leq \delta(I_\Delta)(n - 1) + b,$$

for all $n \geq 1$,

where $b = \max \{\text{reg}(I_\Gamma) \mid \Gamma$ is a subcomplex of $\Delta$ with $F(\Gamma) \subseteq F(\Delta)\}$.

This bound is sharp for every $n$ (see Example 2.7). It is worth mentioning that the number $\delta(I_\Delta)$, which is determined by Equation (1), may be not an integer and even bigger than $\text{reg}(I_\Delta)$ (see [8, Lemma 5.14 and Theorem 5.15]).

For a simple hypergraph $\mathcal{H} = (V, E)$ with vertex set $V = \{1, \ldots, r\}$, the edge ideal of $\mathcal{H}$ is defined by

$$I(\mathcal{H}) = \left( \prod_{e \in E} x_i \mid e \in E \right) \subseteq R.$$ Let $\mathcal{H}^*$ be the simple hypergraph corresponding to the Alexander duality $I(\mathcal{H})^*$ of $I(\mathcal{H})$. Let $\epsilon(\mathcal{H}^*)$ be the minimum number of cardinality of edgewise dominant sets of $\mathcal{H}^*$, this concept was introduced by Dao and Schweig [7].

Then second main result of the paper is the following theorem.

**Theorem 2.6.** Let $\mathcal{H}$ be a simple hypergraph. Then,

$$\text{reg}(I(\mathcal{H})^{(n)}) \leq \delta(I(\mathcal{H}))(n - 1) + |V(\mathcal{H})| - \epsilon(\mathcal{H}^*),$$

for all $n \geq 1$.

A hypergraph is a graph if every edge has exactly two vertices. For a graph $G$, a linear lower bound for $\text{reg}(I(G)^{(n)})$ is given in [14]:

$$\text{reg}(I(G)^{(n)}) \geq 2n + \nu(G) - 1,$$
where $\nu(G)$ is the induced matching number of $G$. Note that this lower bound is also valid for ordinary powers (see [2, Theorem 4.5]).

On the upper bounds, Fakhari (see [13, Conjecture 1.3]) conjectured that

$$\text{reg}(I(G)^{(n)}) \leq 2n + \text{reg}(I(G)) - 2,$$

This conjecture, it may be the best bound up to now of our knowledge.

By using Theorem 2.3, we obtain a general linear upper bound for $\text{reg}(I(G)^{(n)})$ in terms of the ordered matching number of $G$, although it is weaker than the one in this conjecture, it provides us a sharp bound. Note that this result also settles the question (2) of Fakhari in [12].

**Theorem 3.4.** Let $G$ be a graph. Then,

$$\text{reg}(I(G)^{(n)}) \leq 2n + \text{order-match}(G) - 1,$$

for all $n \geq 1$,

where order-match($G$) is the ordered matching number of $G$.

Let us explain the idea to prove Theorems 2.3 and 2.6 as follows. Let $i \geq 0$ such that $\text{reg}(R/I^{(n)}) = a_i(R/I^{(n)}) + i$.

The first key point is to prove that $a_i(R/I^{(n)}) \leq \delta(I)(n - 1)$. Assume that $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$ such that

$$H^i_m(R/I^{(n)})_\alpha \neq 0, \text{ and } a_i(R/I^{(n)}) = |\alpha|,$$

where $m = (x_1, \ldots, x_r)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_r$. We reduce to the case $\alpha \in \mathbb{N}^r$. In order to bound $|\alpha|$, we use Takayama’s formula (see Lemma 1.4) to compute $H^i_m(R/I^{(n)})_\alpha$, which allows us to search for $\alpha$ in a polytope in $\mathbb{R}^r$, so that we can get the desired bound of $|\alpha|$ via theory of convex polytopes (see Theorem 2.2).

The second key point is to bound the index $i$ by using the regularity of a Stanley-Reisner ideal in terms of the vanishing of reduced homology of simplicial complexes which derived from Hochster’s formula about the Hilbert series of the local cohomology module of Stanley-Reisner ideals (see Lemma 1.2).

Our paper is structured as follows. In the next section, we collect notations and terminology used in the paper, and recall a few auxiliary results. In Section 2, we prove Theorems 2.3 and 2.6. In the last section, we prove Theorem 3.4.

1. **Preliminaries**

We shall follow standard notations and terminology from usual texts in the research area (cf. [9, 16, 22]). For simplicity, we denote the set $\{1, \ldots, r\}$ by $[r]$. 


1.1. **Regularity and projective dimension.** Through out this paper, let $K$ be a field, and let $R = K[x_1, \ldots, x_r]$ be a standard graded polynomial ring of $r$ variables over $K$. The object of our work is the Castelnuovo-Mumford regularity of graded modules and ideals over $R$. This invariant can be defined via either the minimal free resolutions or the local cohomology modules.

Let $M$ be a nonzero finitely generated graded $R$-module and let

$$
0 \to \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_p,j(M)} \to \cdots \to \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_0,j(M)} \to 0
$$

be the minimal free resolution of $M$. The *Castelnuovo–Mumford regularity* (or regularity for short) of $M$ is defined by

$$\operatorname{reg}(M) = \max \{ j - i \mid \beta_{i,j}(M) \neq 0 \},$$

and the *projective dimension* of $M$ is the length of this resolution

$$\operatorname{pd}(M) = p.$$

Let us denote by $d(M)$ the maximal degree of a minimal homogeneous generator of $M$. The definition of the regularity implies

$$d(M) \leq \operatorname{reg}(M).$$

For any nonzero proper homogeneous ideal $I$ of $R$, by looking at the minimal free resolution, it is easy to see that $\operatorname{reg}(I) = \operatorname{reg}(R/I) + 1$, so we shall work with $\operatorname{reg}(I)$ and $\operatorname{reg}(R/I)$ interchangeably.

The regularity of $M$ can also be computed via the local cohomology modules of $M$. For $i = 0, \ldots, \dim(M)$, we define the $a_i$-invariant of $M$ as follows

$$a_i(M) = \max \{ t \mid H^i_m(M)_t \neq 0 \},$$

where $H^i_m(M)$ is the $i$-th local cohomology module of $M$ with the support $m = (x_1, \ldots, x_r)$ (with the convention $\max \emptyset = -\infty$). Then,

$$\operatorname{reg}(M) = \max \{ a_i(M) + i \mid i = 0, \ldots, \dim(M) \},$$

and

$$\operatorname{pd}(M) = r - \min \{ i \mid H^i_m(M) \neq 0 \}.$$

For example, since $\dim(R/m) = 0$ and $H^0_m(R/m) = R/m$, we have

$$\operatorname{reg}(m) = \operatorname{reg}(R/m) + 1 = a_0(R/m) + 1 = \max \{ i \mid (R/m)_i \neq 0 \} + 1 = 1.$$

**Remark 1.1.** As usual we shall make the convention that $\operatorname{reg}(M) = -\infty$ if $M = 0$. 

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1.2. Simplicial complexes and Stanley-Reisner ideals. A simplicial complex $\Delta$ over a finite set $V$ is a collection of subsets of $V$ such that if $F \in \Delta$ and $G \subseteq F$ then $G \in \Delta$. Elements of $\Delta$ are called faces. Maximal faces (with respect to inclusion) are called facets. For $F \in \Delta$, the dimension of $F$ is defined to be $\dim F = |F| - 1$. The empty set, $\emptyset$, is the unique face of dimension $-1$, as long as $\Delta$ is not the void complex $\emptyset$ consisting of no subsets of $V$. If every facet of $\Delta$ has the same cardinality, then $\Delta$ is called a pure complex. The dimension of $\Delta$ is $\dim \Delta = \max \{ \dim F \mid F \in \Delta \}$. The link of $F$ inside $\Delta$ is its subcomplex: 

$$\text{lk}_\Delta(F) = \{ H \in \Delta \mid H \cup F \in \Delta \text{ and } H \cap F = \emptyset \}.$$ 

Every element in a face of $\Delta$ is called a vertex of $\Delta$. Let us denote $V(\Delta)$ to be the set of vertices of $\Delta$. If there is a vertex, say $j$, such that $\{j\} \cup F \in \Delta$ for every $F \in \Delta$, then $\Delta$ is called a cone over $j$. It is well-known that if $\Delta$ is a cone, then it is an acyclic complex. A complex is called a simplex if it contains all subsets of its vertices, and thus a simplex is a cone over every its vertex.

For a subset $\tau = \{j_1, \ldots, j_i\}$ of $[r]$, denote $x^\tau = x_{j_1} \cdots x_{j_i}$. Let $\Delta$ be a simplicial complex over the set $V = \{1, \ldots, r\}$. The Stanley-Reisner ideal of $\Delta$ is defined to be the squarefree monomial ideal

$$I_\Delta = (x^\tau \mid \tau \subseteq [r] \text{ and } \tau \notin \Delta) \text{ in } R = K[x_1, \ldots, x_r]$$

and the Stanley-Reisner ring of $\Delta$ to be the quotient ring $k[\Delta] = R/I_\Delta$. This provides a bridge between combinatorics and commutative algebra (see [22, 26]).

Note that if $I$ is a square-free monomial ideal, then it is a Stanley-Reisner ideal of the simplicial complex $\Delta(I) = \{ \tau \subseteq [r] \mid x^\tau \notin I \}$. When $I$ is a monomial ideal (maybe not square-free) we also use $\Delta(I)$ to denote the simplicial complex corresponding to the square-free monomial ideal $\sqrt{I}$.

The regularity of a square-free monomial ideal can compute via the vanishing of reduced homology of simplicial complexes. From Hochster’s formula on the Hilbert series of the local cohomology module $H^i_m(I_\Delta)$ (see [22, Corollary 13.16]), one has

Lemma 1.2. For a simplicial complex $\Delta$, we have

$$\text{reg}(I_\Delta) = \max \{ d \mid \tilde{H}_{d-1}(\text{lk}_\Delta(\sigma); K) \neq 0, \text{ for some } \sigma \in \Delta \}. $$

The Alexander dual of $\Delta$, denoted by $\Delta^*$, is the simplicial complex over $V$ with faces

$$\Delta^* = \{V \setminus \tau \mid \tau \notin \Delta\}.$$ 

Notice that $(\Delta^*)^* = \Delta$. If $I = I_\Delta$ then we shall denote the Stanley-Reisner ideal of the Alexander dual $\Delta^*$ by $I^*$. It is a well-known result of Terai [28] (or see [22]...
Theorem 5.59]) that the regularity of a squarefree monomial ideal can be related to the projective dimension of its Alexander dual.

**Lemma 1.3.** Let $I \subseteq R$ be a square-free monomial ideal. Then,

$$\text{reg}(I) = \text{pd}(R/I^*)$$

Let $\mathcal{F}(\Delta)$ denote the set of all facets of $\Delta$. We say that $\Delta$ is generated by $\mathcal{F}(\Delta)$ and write $\Delta = \langle \mathcal{F}(\Delta) \rangle$. Note that $I_{\Delta}$ has the minimal primary decomposition (see [22, Theorem 1.7]):

$$I_{\Delta} = \bigcap_{F \in \mathcal{F}(\Delta)} (x_i | i \notin F),$$

and therefore the $n$-th symbolic power of $I_{\Delta}$ is

$$I_{\Delta}^{(n)} = \bigcap_{F \in \mathcal{F}(\Delta)} (x_i | i \notin F)^n.$$

We next describe a formula to compute the local cohomology modules of monomial ideals. Let $I$ be a non-zero monomial ideal. Since $R/I$ is an $N^r$-graded algebra, $H^i_m(R/I)$ is an $\mathbb{Z}^r$-graded module over $R/I$ for every $i$. For each degree $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$, in order to compute $\dim_K H^i_m(R/I)_{\alpha}$ we use a formula given by Takayama [27, Theorem 2.2] which is a generalization of Hochster’s formula for the case $I$ is square-free [26, Theorem 4.1].

Set $G_{\alpha} = \{ i | \alpha_i < 0 \}$. For a subset $F \subseteq [r]$, we set $R_F = R[x_i^{-1} | i \in F \cup G_{\alpha}]$. Define the simplicial complex $\Delta_{\alpha}(I)$ by

$$(2) \quad \Delta_{\alpha}(I) = \{ F \subseteq [r] \setminus G_{\alpha} | x^\alpha \notin IR_F \}.$$ 

**Lemma 1.4.** [27, Theorem 2.2] $\dim_K H^i_m(R/I)_{\alpha} = \dim_K \tilde{H}^{i-|G_{\alpha}|-1}_{|G_{\alpha}|} (\Delta_{\alpha}(I); K)$.

The following result of Minh and Trung is very useful to compute $\Delta_{\alpha}(I^{(n)}_{\Delta})$, which allows us to investigate $\text{reg}(I^{(n)}_{\Delta})$ by using the theory of convex polyhedra.

**Lemma 1.5.** [23, Lemma 1.3] Let $\Delta$ be a simplicial complex and $\alpha \in \mathbb{N}^r$. Then,

$$\mathcal{F}(\Delta_{\alpha}(I^{(n)}_{\Delta})) = \left\{ F \in \mathcal{F}(\Delta) \mid \sum_{i \notin F} \alpha_i \leq n - 1 \right\}.$$ 

This lemma can be generalized a little bit as follows.

**Lemma 1.6.** [19, Lemma 1.3] Let $\Delta$ be a simplicial complex and $\alpha \in \mathbb{Z}^r$. Then,

$$\mathcal{F}(\Delta_{\alpha}(I^{(n)}_{\Delta})) = \left\{ F \in \mathcal{F}(|\text{lk}_\Delta(G_{\alpha})|) \mid \sum_{i \notin F \cup G_{\alpha}} \alpha_i \leq n - 1 \right\}.$$
1.3. Hypergraphs. Let $V$ be a finite set. A simple hypergraph $\mathcal{H}$ with vertex set $V$ consists of a set of subsets of $V$, called the edges of $\mathcal{H}$, with the property that no edge contains another. We use the symbols $V(\mathcal{H})$ and $E(\mathcal{H})$ to denote the vertex set and the edge set of $\mathcal{H}$, respectively.

In this paper we assume that all hypergraphs are simple unless otherwise specified.

In the hypergraph $\mathcal{H}$, an edge is trivial if it contains only one element, a vertex is isolated if it is not appearing in any edge, a vertex is a neighbor of another one if they are in some edge.

A hypergraph $\mathcal{H}'$ is a subhypergraph of $\mathcal{H}$ if $V(\mathcal{H}') \subseteq V(\mathcal{H})$ and $E(\mathcal{H}') \subseteq E(\mathcal{H})$. For an edge $e$ of $\mathcal{H}$, we define $\mathcal{H} \setminus e$ to be the hypergraph obtained by deleting $e$ from the edge set of $\mathcal{H}$. For a subset $S \subseteq V(\mathcal{H})$, we define $\mathcal{H} \setminus S$ to be the hypergraph obtained from $\mathcal{H}$ by deleting the vertices in $S$ and all edges containing any of those vertices.

A set $S \subseteq E(\mathcal{H})$ is called an edgewise dominant set of $\mathcal{H}$ if every non-isolated vertex of $\mathcal{H}$ not contained in some edge of $S$ or contained in a trivial edge has a neighbor contained in some edge of $S$. Define,

$$\epsilon(\mathcal{H}) = \min\{|S| \mid S \text{ is edgewise dominant}\}.$$

For a hypergraph $\mathcal{H}$ with $V(\mathcal{H}) \subseteq [r]$, we associate to the hypergraph $\mathcal{H}$ a square-free monomial ideal

$$I(\mathcal{H}) = (x^e \mid e \in E(\mathcal{H})) \subseteq R,$$

which is called the edge ideal of $\mathcal{H}$.

Notice that if $I$ is a square-free monomial ideal, then $I$ is an edge ideal of a hypergraph with the edge set uniquely determined by the generators of $I$.

Let $\mathcal{H}^*$ be the simple hypergraph corresponding to the Alexander duality $I(\mathcal{H})^*$ of $I(\mathcal{H})$. We will determine the edge set of $\mathcal{H}^*$, it turns out that $E(\mathcal{H}^*)$ is the set of all minimal vertex covers of $\mathcal{H}$. A vertex cover in a hypergraph is a set of vertices, such that every edge of the hypergraph contains at least one vertex of that set. It is an extension of the notion of vertex cover in a graph. A vertex cover $S$ is called minimal if no proper subset of $S$ is a vertex cover. From the minimal primary decomposition (see [22, Definition 1.35 and Proposition 1.37]):

$$I(\mathcal{H}^*) = \bigcap_{e \in E(\mathcal{H})} (x_i \mid i \in e),$$

it follows that $E(\mathcal{H}^*)$ is just the set of minimal vertex covers of $\mathcal{H}$. Thus,

$$I(\mathcal{H}^*) = (x^\tau \mid \tau \text{ is a minimal vertex cover of } \mathcal{H}).$$

In the sequel, we need the following result of Dao and Schweig [7, Theorem 3.2].
Lemma 1.7. Let \( \mathcal{H} \) be a hypergraph. Then, \( \text{pd}(R/I(\mathcal{H})) \leq |V(\mathcal{H})| - \epsilon(\mathcal{H}) \).

1.4. Matchings in a graph. Let \( G \) be a graph. A matching in \( G \) is a subgraph consisting of pairwise disjoint edges. If this subgraph is an induced subgraph, then the matching is called an induced matching. A matching of \( G \) is maximal if it is maximal with respect to inclusion. The matching number of \( G \), denoted by \( \text{match}(G) \), is the maximum size of a matching in \( G \); and the induced matching number of \( G \), denoted by \( \nu(G) \), is the maximum size of an induced matching in \( G \).

An independent set in \( G \) is a set of vertices no two of which are adjacent to each other. An independent set in \( G \) is maximal (with respect to set inclusion) if the set cannot be extended to a larger independent set. Let \( \Delta(G) \) denote the set of all independent sets of \( G \). Then, \( \Delta(G) \) is a simplicial complex, called the independence complex of \( G \). It is well-known that \( I(G) = I_{\Delta(G)} \).

According to Constantinescu and Varbaro [3], we say that a matching \( M = \{\{u_i, v_i\} \mid i = 1, \ldots, s\} \) is an ordered matching if:

1. \( \{u_1, \ldots, u_s\} \in \Delta(G) \),
2. \( \{u_i, v_j\} \in E(G) \) implies \( i \leq j \).

The ordered matching number of \( G \), denoted by \( \text{order-match}(G) \), is the maximum size of an ordered matching in \( G \).

The following result gives a lower bound for \( \text{reg}(I(G)(n)) \) in terms of the induced matching number \( \nu(G) \).

Lemma 1.8. [14, Theorem 4.6] Let \( G \) be a graph. Then,
\[
\text{reg}(I(G)(n)) \geq 2n + \nu(G) - 1, \quad \text{for all } n \geq 1.
\]

1.5. Convex polyhedra. The theory of convex polyhedra plays a key role in our study.

For a vector \( \alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r \), we set \( |\alpha| := \alpha_1 + \cdots + \alpha_r \) and for a nonempty bounded closed subset \( S \) of \( \mathbb{R}^r \) we set
\[
\delta(S) := \max\{|\alpha| \mid \alpha \in S\}.
\]

Let \( \Delta \) be a simplicial complex over \([r]\). In general, \( \text{reg}(I(\Delta)(n)) \) is not a linear function in \( n \) for \( n \gg 0 \) (see e.g. [8, Theorem 5.15]), but a quasi-linear function as in the following result.

Lemma 1.9. [18, Theorem 4.9] There exist positive integers \( N, n_0 \) and rational numbers \( a, b_0, \ldots, b_{N-1} < \dim(R/I(\Delta)) + 1 \) such that
\[
\text{reg}(I(\Delta)(n)) = an + b_k, \quad \text{for all } n \geq n_0 \text{ and } n \equiv k \mod N, \quad \text{where } 0 \leq k \leq N - 1.
\]
Moreover, \( \text{reg}(I_{\Delta}^{(n)}) < an + \dim(R/I_{\Delta}) + 1 \) for all \( n \geq 1 \).

By virtue of this result, we define

\[
\delta(I_{\Delta}) = a = \lim_{n \to \infty} \frac{\text{reg}(I_{\Delta}^{(n)})}{n}.
\]

In order to compute this invariant we can use the geometric interpretation of it by means of symbolic polyhedra defined in [4, 8]. Let \( \mathcal{SP}(I_{\Delta}) \) be the convex polyhedron in \( \mathbb{R}^r \) defined by the following system of linear inequalities:

\[
\begin{cases}
\sum_{i \in F} x_i \geq 1 & \text{for } F \in \mathcal{F}(\Delta), \\
x_1 \geq 0, \ldots, x_r \geq 0,
\end{cases}
\]

which is called the \textit{symbolic polyhedron} of \( I_{\Delta} \). Then, \( \mathcal{SP}(I_{\Delta}) \) is a convex polyhedron in \( \mathbb{R}^r \). By [8, Theorem 3.6] we have

\[
\delta(I_{\Delta}) = \max\{\|v\| \mid v \text{ is a vertex of } \mathcal{SP}(I_{\Delta})\}.
\]

Now assume that

\[
H_{m}^i(I_{\Delta}^{(n)})|_{\alpha} \neq 0
\]

for some \( 0 \leq i \leq \dim(R/I_{\Delta}) \) and \( \alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r \).

By Lemma [1, 4] we have

\[
\dim_K \tilde{H}_{i-1}(\Delta_{\alpha}(I_{\Delta}^{(n)}); K) = \dim_K H_{m}^i(R/I_{\Delta}^{(n)})|_{\alpha} \neq 0.
\]

In particular, \( \Delta_{\alpha}(I_{\Delta}^{(n)}) \) is not acyclic.

Suppose that \( \mathcal{F}(\Delta) = \{F_1, \ldots, F_t\} \) for \( t \geq 1 \). By Lemma [1, 5] we may assume that

\[
\mathcal{F}(\Delta_{\alpha}(I_{\Delta}^{(n)})) = \{F_1, \ldots, F_s\}, \text{ where } 1 \leq s \leq t.
\]

For each integer \( m \geq 1 \), let \( \mathcal{P}_m \) be the convex polyhedron of \( \mathbb{R}^r \) defined by:

\[
\begin{cases}
\sum_{i \in F_j} x_i \leq m - 1 & \text{for } j = 1, \ldots, s, \\
\sum_{i \in F_j} x_i \geq m & \text{for } j = s + 1, \ldots, t, \\
x_1 \geq 0, \ldots, x_r \geq 0.
\end{cases}
\]

Then, \( \alpha \in \mathcal{P}_n \). Moreover, by Lemma [1, 5] one has

\[
\Delta_{\beta}(I_{\Delta}^{(m)}) = \langle F_1, \ldots, F_s \rangle = \Delta_{\alpha}(I_{\Delta}^{(m)}) \text{ whenever } \beta \in \mathcal{P}_m \cap \mathbb{N}^r.
\]

Note also that for such a vector \( \beta \), by Formula (7) we have

\[
\dim_K \tilde{H}_{i-1}(\Delta_{\beta}(I_{\Delta}^{(m)}); K) = \dim_K \tilde{H}_{i-1}(\Delta_{\alpha}(I_{\Delta}^{(m)}); K) \neq 0.
\]
Together with Lemma 1.4 this fact yields

(8) \[ H_m^i(R/I^{(m)}_\Delta)_\beta \neq 0. \]

In order to investigate the convex polyhedron \( P_m \) we also consider the convex polyhedron \( C_m \) in \( \mathbb{R}^r \) defined by:

\[
\begin{align*}
\sum_{i \in F_j} x_i & \leq m \quad \text{for } j = 1, \ldots, s, \\
\sum_{i \in F_j} x_i & \geq m \quad \text{for } j = s + 1, \ldots, t, \\
x_1 & \geq 0, \ldots, x_r \geq 0.
\end{align*}
\]

Note that \( C_m = mC_1 \) for all \( m \geq 1 \), where \( mC_1 = \{my \mid y \in C_1\} \).

By the same way as in the proof of [15, Lemma 2.1] we obtain the following lemma.

**Lemma 1.10.** \( C_1 \) is a polytope with \( \dim C_1 = r \).

The next lemma gives an upper bound for \( \delta(C_1) \).

**Lemma 1.11.** \( \delta(C_1) \leq \delta(I_\Delta) \).

**Proof.** Since \( C_1 \) is a polytope with \( \dim C_1 = r \) by Lemma 1.10, \( \delta(C_1) = |\gamma| \) for some vertex \( \gamma \) of \( C_1 \). By [25, Formula (23) in Page 104] we imply that \( \gamma \) is the unique solution of a system of linear equations of the form

(10) \[
\begin{align*}
\sum_{i \in F_j} x_i & = 1 \quad \text{for } j \in S_1, \\
x_j & = 0 \quad \text{for } j \in S_2,
\end{align*}
\]

where \( S_1 \subseteq [t] \) and \( S_2 \subseteq [r] \) such that \( |S_1| + |S_2| = r \). By using Cramer’s rule to get \( \gamma \), we conclude that \( \gamma \) is a rational vector. In particular, there is a positive integer, say \( p \), such that \( p\gamma \in \mathbb{N}^r \). Note that \( C_p = pC_1 \), so \( p\gamma \in C_p \cap \mathbb{N}^r \).

For every \( j \geq 1 \), let \( y = j\gamma + \alpha \). Then, \( y \in \mathbb{N}^r \) and \( |y| = \delta(C_1)jp + |\alpha| \). On the other hand, by using the fact that \( j\gamma \in C_{jp} \), we can check that

\[
\begin{align*}
\sum_{i \in F_j} y_i & \leq jp + n - 1 \quad \text{for } j = 1, \ldots, s, \\
\sum_{i \in F_j} y_i & \geq jp + n \quad \text{for } j = s + 1, \ldots, t,
\end{align*}
\]

and so \( y \in P_{jp+n} \cap \mathbb{N}^r \).

Together with Equation (8), we deduce that \( H_m^i(R/I^{(jp+n)}_\Delta)_y \neq 0 \), and therefore

\[ \text{reg}(R/I^{(jp+n)}_\Delta) \geq |y| + i = \delta(C_1)jp + |\alpha| + i. \]
Combining with Lemma 1.9, this inequality yields
\[ \delta(C_1)jp + |\alpha| + i < \delta(I_\Delta)(jp + n) + \dim(R/I_\Delta). \]
Since this inequality valid for any positive integer \( j \), it forces \( \delta(C_1) \leq \delta(I_\Delta). \] □

2. Regularity of symbolic powers of ideals

In this section we will prove the upper bound for \( \reg(I_\Delta^{(n)}) \). First we start with the following fact.

**Lemma 2.1.** Let \( \sigma \subseteq [r] \) with \( \sigma \neq [r] \), \( S = K[x_i \mid i \notin \sigma] \) and \( J = IR_\sigma \cap S \). Then,
\[ \reg(J^{(n)}) \leq \reg(I^{(n)}) \] for all \( n \geq 1 \).
In particular, \( \delta(J) \leq \delta(I) \).

**Proof.** We may assume that \( S = K[x_1, \ldots, x_s] \) for some \( 1 \leq s \leq r \). Let \( i \) be an index and \( \alpha \) a vector in \( \mathbb{Z}^s \) such that
\[ H^i_n(S/J^{(n)})_\alpha \neq 0 \text{ and } \reg(S/J^{(n)}) = |\alpha| + i, \]
where \( n = (x_1, \ldots, x_s) \) is the homogeneous maximal ideal of \( S \).

Let \( \beta = (\alpha_1, \ldots, \alpha_s, -1, \ldots, -1) \in \mathbb{Z}^r \) so that \( G_\beta = G_\alpha \cup \{s + 1, \ldots, r\} \). By Formula (2) we deduce that
\[ \Delta_\alpha(J^{(n)}) = \Delta_\beta(I^{(n)}). \] (11)

By Lemma 1.4,
\[ \dim_K H^i_n(S/J^{(n)})_\alpha = \dim_K \tilde{H}_{i-|G_\alpha|-1}(\Delta_\alpha(J^{(n)}); K), \]
and thus \( \tilde{H}_{i-|G_\alpha|-1}(\Delta_\alpha(J^{(n)}); K) \neq 0 \). Together with Equation (11), it yields
\[ \tilde{H}_{i-|G_\alpha|-1}(\Delta_\beta(I^{(n)}); K) \neq 0 \] by Lemma 1.4 again, it gives \( H^{i+(r-s)}_m(R/I^{(n)})_\beta \neq 0 \) since \( |G_\beta| = |G_\alpha| + (r - s) \). Therefore,
\[ \reg(R/I^{(n)}) \geq |\beta| + i + (r - s) = |\alpha| + i = \reg(S/J^{(n)}), \]
it follows that \( \reg(J^{(n)}) \leq \reg(I^{(n)}) \).

Finally, together this inequality with Lemma 1.9 we have
\[ \delta(J) = \lim_{n \to \infty} \frac{\reg(J^{(n)})}{n} \leq \lim_{n \to \infty} \frac{\reg(I^{(n)})}{n} = \delta(I), \]
and the lemma follows. □
Theorem 2.2. Let \( I \) be a square-free monomial ideal. Then, for all \( i \geq 0 \) we have

\[
a_i(R/I^{(n)}) \leq \delta(I)(n-1).
\]

Proof. If \( n = 1 \), the theorem follows from Hochster’s formula on the Hilbert series of the local cohomology module \( H^i_m(R/I) \) (see [26, Theorem 4.1]).

We may assume that \( n \geq 2 \). If \( a_i(R/I^{(n)}) = -\infty \), the theorem is obvious, so that we also assume that \( a_i(R/I^{(n)}) \neq -\infty \).

Suppose \( \alpha \in \mathbb{Z}^r \) such that

\[
H^i_m(R/I^{(n)})_{\alpha} \neq 0 \text{ and } a_i(R/I^{(n)}) = |\alpha|.
\]

By Lemma 1.4 we have

\[
\dim_K \tilde{H}_{i-|G_{\alpha}|}(\Delta_{\alpha}(I^{(n)}); K) = \dim_K H^i_m(R/I^{(n)})_{\alpha} \neq 0.
\]

In particular, \( \Delta_{\alpha}(I^{(n)}) \) is not acyclic.

If \( G_{\alpha} = [r] \), then \( a_i(R/I^{(n)}) = |\alpha| \leq 0 \), and hence the theorem holds in this case.

We therefore assume that \( G_{\alpha} = \{m+1, \ldots, r\} \) for \( 1 \leq m \leq r \). Let \( S = K[x_1, \ldots, x_m] \) and \( J = IR_{G_{\alpha}} \cap S \).

Let \( \alpha' = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m \). By using Formula (2), we have

\[
\Delta_{\alpha'}(J^{(n)}) = \Delta_{\alpha}(I^{(n)}).
\]

Together with (12), it gives \( \tilde{H}_{i-|G_{\alpha}|}(\Delta_{\alpha'}(J^{(n)}); K) \neq 0 \). By Lemma 1.4 we get

\[
H^i_n(S/J^{(n)})_{\alpha'} \neq 0,
\]

where \( n = (x_1, \ldots, x_m) \) is the homogeneous maximal ideal of \( S \).

Let \( \Delta \) be the simplicial complex over \([m]\) corresponding to the square-free monomial ideal \( J \). Assume that \( \mathcal{F}(\Delta) = \{F_1, \ldots, F_t\} \).

By Lemma 1.5 we may assume that \( \mathcal{F}(\Delta_{\alpha'}(J^{(n)})) = \{F_1, \ldots, F_s\} \) for \( 1 \leq s \leq t \).

Let

\[
\beta = (\beta_1, \ldots, \beta_m) = \frac{1}{n-1} \alpha' \in \mathbb{R}^m.
\]

By Lemma 1.5 again, we deduce that

\[
\begin{align*}
\sum_{i \in F_j} \beta_i &= \frac{1}{n-1} \sum_{i \in F_j} \alpha_i \leq 1 & \text{for } j = 1, \ldots, s, \\
\sum_{i \notin F_j} \beta_i &= \frac{1}{n-1} \sum_{i \notin F_j} \alpha_i \geq \frac{n}{n-1} > 1 & \text{for } j = s+1, \ldots, t.
\end{align*}
\]
It follows that $\beta \in C_1$, where $C_1$ is a polyhedron in $\mathbb{R}^m$ defined by
\[
\begin{align*}
\sum_{i \in F_j} x_i &\leq 1 & \text{for } j = 1, \ldots, s, \\
\sum_{i \not\in F_j} x_i &\geq 1 & \text{for } j = s + 1, \ldots, t, \\
x_1 &\geq 0, \ldots, x_m \geq 0.
\end{align*}
\]
By Lemma 1.10, $C_1$ is a polytope in $\mathbb{R}^m$.

Hence $|\beta| \leq \delta(C_1)$, and hence $|\alpha'| = (n-1)|\beta| \leq \delta(C_1)(n-1)$. Observe that $\alpha_j < 0$ for all $j \in G_\alpha = \{m+1, \ldots, r\}$, so
\begin{equation}
(14) \quad a_i(R/I(I^{(n)})) = |\alpha| = |\alpha'| + (\alpha_{m+1} + \cdots + \alpha_r) \leq |\alpha'| \leq \delta(C_1)(n-1).
\end{equation}

On the other hand, by Lemmas 1.11 and 2.1 we deduce that
\[\delta(C_1) \leq \delta(J) \leq \delta(I).\]

Together with Formula (14), it yields $a_i(R/I(I^{(n)})) \leq \delta(I)(n-1)$, and the proof of the theorem is complete. $\square$

We are now in position to prove the main result of the paper.

**Theorem 2.3.** Let $\Delta$ be a simplicial complex. Then,
\[\text{reg}(I_\Delta^{(n)}) \leq \delta(I_\Delta)(n-1) + b, \text{ for all } n \geq 1,\]
where $b = \max\{\text{reg}(I_\Gamma) \mid \Gamma \text{ is a subcomplex of } \Delta \text{ with } \mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta)\}$.

**Proof.** For simplicity, we put $I = I_\Delta$. Let $i \in \{0, \ldots, \dim(R/I)\}$ and $\alpha \in \mathbb{Z}^r$ such that
\[H_m^i(R/I(I^{(n)}))_{\alpha} \neq 0, \text{ and } \text{reg}(R/I(I^{(n)})) = a_i(R/I(I^{(n)})) + i = |\alpha| + i.\]

By Lemma 1.4, we have
\begin{equation}
(15) \quad \dim_K \tilde{H}_{i-|G_\alpha|-1}(\Delta_\alpha(I(n)); K) = \dim_K H_m^i(R/I(I^{(n)}))_{\alpha} \neq 0.
\end{equation}

In particular, $\Delta_\alpha(I(n))$ is not acyclic.

If $G_\alpha = \{r\}$, then $\Delta_\alpha(I(n))$ is either $\{0\}$ or a void complex. Because it is not acyclic, $\Delta_\alpha(I(n)) = \{0\}$. By Formula (15) we deduce that $i = |G_\alpha| = r$, and hence $\dim R/I = r$. It means that $I = 0$, so $I^{(n)} = 0$ as well. Therefore, $\text{reg}(I^{(n)}) = -\infty$, and the theorem holds in this case.

We may assume that $G_\alpha = \{m+1, \ldots, r\}$ for some $1 \leq m \leq r$. Let $S = K[x_1, \ldots, x_m]$ and $J = I_{G_\alpha} \cap S$.

Let $\alpha' = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$. By using Formula (2), we have
\begin{equation}
(16) \quad \Delta_{\alpha'}(J^{(n)}) = \Delta_\alpha(I^{(n)}).
\end{equation}
Together with (15), it gives
\[ \tilde{H}_{i-|G_\alpha|-1}(\Delta_{\alpha'}(J^{(n)}); K) \neq 0. \]
By Lemma 1.4 we get
\[ H_{n-|G_\alpha|}(S/J^{(n)})_{\alpha'} \neq 0, \]
where \( n = (x_1, \ldots, x_m) \) is the homogeneous maximal ideal of \( S \). In particular,
\[ |\alpha'| \leq a_{i-|G_\alpha|}(S/J^{(n)}). \]
Together with Lemma 2.1 and Theorem 2.2, it yields
\[ |\alpha'| \leq \delta(J)(n - 1) \leq \delta(I)(n - 1). \]
Therefore,
\[ \text{reg}(I^{(n)}) = | \alpha | + i = | \alpha' | + \sum_{j=m+1}^{r} \alpha_j + i \leq | \alpha' | + i - |G_\alpha| \leq \delta(I)(n - 1) + i - |G_\alpha|. \]
It remains to prove that \( i - |G_\alpha| \leq b \). By Lemma 1.6, we have
\[ \Delta_{\alpha'}(J^{(n)}) = \Delta_{\alpha}(I^{(n)}) = \left\{ F \in \mathcal{F}(\Delta(G_\alpha)) \mid \sum_{j \notin F \cup G_\alpha} \alpha_j \leq n - 1 \right\}. \]
It follows that there is a simplicial complex \( \Gamma \) with \( \mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta) \) such that
\[ \Delta_{\alpha'}(J^{(n)}) = \text{lk}_\Gamma(G_\alpha). \]
Since \( \tilde{H}_{i-|G_\alpha|-1}(\text{lk}_\Gamma(G_\alpha); K) \neq 0 \), by Lemma 1.2 we have \( i - |G_\alpha| \leq \text{reg}(I_\Gamma) \leq b \), and then proof of the theorem is complete. \( \square \)

As a direct consequence of Theorem 2.3, we have a simple bound. Namely,

**Corollary 2.4.** Let \( I \) be a square-free monomial ideal. Then,
\[ \text{reg}(I^{(n)}) \leq \delta(I)(n - 1) + \dim(R/I) + 1, \quad \text{for all } n \geq 1. \]

**Proof.** Let \( \Delta \) be the simplicial complex corresponding to the square-free ideal \( I \). For every subcomplex \( \Gamma \) of \( \Delta \) we have \( \dim \Gamma \leq \dim \Delta \). It follows from Lemma 1.2 that
\[ \text{reg}(I_\Gamma) \leq \dim(R/I_\Gamma) + 1 \leq \dim(R/I_\Delta) + 1. \]
Therefore, \( b = \max\{\text{reg}(I_\Gamma) \mid \mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta)\} \leq \dim(R/I_\Delta) + 1. \) Now the corollary follows from Theorem 2.3. \( \square \)

We next reformulate the theorem 2.3 for a square-free monomial ideal arising from a hypergraph.
**Theorem 2.5.** Let $H$ be a hypergraph. Then, for all $n \geq 1$, we have

$$\text{reg}(I(H)^{(n)}) \leq \delta(I(H))(n - 1) + b,$$

where $b = \max\{\text{pd}(R/I(H')) \mid H' \text{ is a subhypergraph of } H^* \text{ with } E(H') \subseteq E(H^*)\}$.

**Proof.** Let $\Delta$ be the corresponding simplicial complex of the square-free monomial ideal $I(H)$. Assume that $F(\Delta) = \{F_1, \ldots, F_p\}$. Since $I(H) = \bigcap_{j=1}^{p} (x_i \mid i /\in F_j)$, so that $E(H^*) = \{C_1, \ldots, C_p\}$, where $C_j = [r] \setminus F_j$ for all $j = 1, \ldots, p$.

Let $\Gamma$ be a subcomplex of $\Delta$ with $F(\Gamma) \subseteq F(\Delta)$. We may assume that $F(\Gamma) = \{F_1, \ldots, F_k\}$ for $1 \leq k \leq p$. Then, we have $I^*_\Gamma = I(H')$ where $H'$ is the subhypergraph of $H^*$ with $E(H') = \{C_1, \ldots, C_k\}$.

By Lemma 1.3 we have $\text{reg}(I^*_\Gamma) = \text{pd}(R/I^*_\Gamma) = \text{pd}(R/I(H'))$, and therefore the theorem follows from Theorem 2.3. $\square$

The next theorem is the second main result of the paper. It bounds the regularity of symbolic powers of a square-free monomial ideal via the combinatorial properties of the associated hypergraph.

**Theorem 2.6.** Let $H$ be a simple hypergraph. Then,

$$\text{reg}(I(H)^{(n)}) \leq \delta(I(H))(n - 1) + |V(H)| - \epsilon(H^*), \text{ for all } n \geq 1.$$

**Proof.** By Theorem 2.5 it suffices to show that

$$\text{pd}(R/I(G)) \leq |V(H)| - \epsilon(H^*)$$

for every hypergraph $G$ with $E(G) \subseteq E(H^*)$. By Lemma 1.7 it suffices to prove that

$$|V(G)| - \epsilon(G) \leq |V(H^*)| - \epsilon(H^*).$$

In order to prove this inequality, without loss of generality we may assume that $H^*$ has no both trivial edges and isolated vertices.

Let $S$ be an edgewise-dominant set of $G$ such that $|S| = \epsilon(G)$. For each vertex $v \in V(H^*) \setminus V(G)$, we take an edge of $H^*$ containing $v$, and denote this edge by $F(v)$. Then,

$$S' = S \cup \{F(v) \mid v \in V(H^*) \setminus V(G)\}$$

is an edgewise-dominant set of $H^*$. It follows that

$$\epsilon(H^*) \leq |S'| \leq |S| + |V(H^*) \setminus V(G)| = |S| + |V(H^*)| - |V(G)|,$$

and therefore $|V(G)| - \epsilon(G) \leq |V(H^*)| - \epsilon(H^*),$ as required. $\square$
The following example shows that the bound in Theorem 2.3 is sharp at every $n$ for the class of matroid complexes. Recall that a simplicial complex $\Delta$ is called a matroid complex if for every subset $\sigma$ of $V(\Delta)$, the simplicial complex $\Delta[\sigma]$ is pure (see e.g. [20, Chapter 3]). Here, $\Delta[\sigma]$ is the restriction of $\Delta$ to $\sigma$ and defined by $\Delta[\sigma] = \{ \tau \mid \tau \in \Delta \text{ and } \tau \subseteq \sigma \}$.

**Example 2.7.** Let $\Delta$ be a matroid complex that is not a cone. Then,
\[
\text{reg}(I_{\Delta}(n)) = \delta(I_\Delta)(n-1) + b, \quad \text{for all } n \geq 1,
\]
where $b = \max\{\text{reg}(I_\Gamma) \mid \Gamma \text{ is a subcomplex of } \Delta \text{ with } \mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta)\}$.

**Proof.** Let $I = I_\Delta$ and $s = \dim(R/I_\Delta)$. By [24, Theorem 4.5], for all $n \geq 1$ we have:
\[
\text{reg}(I(n)) = d(I)(n-1) + s + 1.
\]
It implies that
\[
\lim_{n \to \infty} \frac{\text{reg}(I(n))}{n} = d(I),
\]
so $\delta(I) = d(I)$. It remains to show that $b = s + 1$.

Together the fact $\delta(I) = d(I)$ with Theorem 2.3, we get $s + 1 \leq b$. On the other hand, by the same argument as in the proof of Corollary 2.4, we obtain $b \leq s + 1$. Hence, $b = s + 1$, as required. \square

We conclude this section with a remark on lower bounds.

**Remark 2.8.** Let $I$ be a square-free monomial ideal. By [8, Lemma 4.2(ii)] we deduce that $d(I)n \leq d(I^{(n)})$, and therefore
\[
\text{reg}(I^{(n)}) \geq d(I)n, \quad \text{for all } n \geq 1.
\]
In general, $d(I) < \delta(I)$ (see e.g. [8, Lemma 5.14]), so that the bound is not optimal.

On the other hand, by Lemma 1.9, there is a number $b$ such that
\[
\text{reg}(I^{(n)}) \geq \delta(I)n + b, \quad \text{for all } n \geq 1.
\]
The natural question is to find a good bound for $b$.

3. Applications

In this section we will apply Theorem 2.3 to the regularity of symbolic powers of the edge ideal of a graph. We start with a result which allows us to bound the number $b$ in Theorem 2.3 by choosing a suitable numerical function, it is of independent interest.
Theorem 3.1. Let $\Delta$ be a simplicial complex over $[r]$ and let
\[ \text{Simp}(\Delta) = \{ \text{lk}_\Delta(\sigma) \mid \sigma \in \Delta \}. \]
Assume that $f : \text{Simp}(\Delta) \to \mathbb{N}$ is a function which satisfies the following properties:
1. If $\Lambda \in \text{Simp}(\Delta)$ is a simplex, then $f(\Lambda) = 0$.
2. For every $\Lambda \in \text{Simp}(\Delta)$ and every $v \in V(\Lambda)$ such that $\Lambda$ is not a cone over $v$, $f(\text{lk}_\Lambda(v)) + 1 < f(\Lambda)$.

Then, for every subcomplex $\Gamma$ of $\Delta$ with $\mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta)$ we have $\text{reg}(I_\Gamma) \leq f(\Delta) + 1$.

Proof. For a subset $S$ of $[r]$ we set $p_S = (x_i \mid i \in S) \subseteq R$. In order to facilitate an induction argument on the number of vertices of $\Delta$ we prove the following assertion:
\[ (17) \quad \text{reg}(p_S + I_\Gamma) \leq f(\Delta) + 1, \quad \text{for every } S \subseteq [r], \]
where all simplicial complexes is considered over $[r]$.

Indeed, if $|V(\Delta)| \leq 1$, then $\Delta$ is a simplex. In this case, the assertion is obvious.

Assume that $|V(\Delta)| \geq 2$. If $\Delta$ is a simplex, the assertion holds, so we assume that $\Delta$ is not a simplex. We now prove by backward induction on $|S|$. If $|S| = r$, then $p_S + I_\Gamma = (x_1, \ldots, x_r)$.

In this case $\text{reg}(p_S + I_\Gamma) = 1$, and so the assertion holds.

Assume that $|S| < r$. If $p_S + I_\Gamma$ is a prime, i.e. it is generated by variables, then $\text{reg}(p_S + I_\Gamma) = 1$, and then the assertion holds.

Assume that $p_S + I_\Gamma$ is not a prime. Then, there is a variable, say $x_v$ with $v \in [r]$, such that $x_v$ appears in some monomial generator of $p_S + I_\Gamma$ of order at least 2 and $v \not\in S$. Note that if $u$ is not a vertex of $\Gamma$ then $x_u$ is a monomial generator of $I_\Gamma$, and if $\Gamma$ is a cone over some vertex $w$ then $x_w$ does not appear in any monomial generator of $I_\Gamma$. It implies that $v$ is a vertex of $\Gamma$ and $\Gamma$ is not a cone over $v$. In particular, $\Delta$ is not a cone over $v$.

Since
\[ (p_S + I_\Gamma) + (x_v) = p_{S \cup \{v\}} + I_\Gamma, \quad \text{and} \quad (p_S + I_\Gamma) : (x_v) = p_S + I_{\Gamma'}, \]
where $\Gamma'$ is a subcomplex of $\Gamma$ with $\mathcal{F}(\Gamma') = \{ F \in \mathcal{F}(\Gamma) \mid v \in F \}$, by [6, Lemma 2.10] we have
\[ (18) \quad \text{reg}(p_S + I_\Gamma) \leq \max\{ \text{reg}(p_{S \cup \{v\}} + I_\Gamma), \text{reg}(p_S + I_{\Gamma'}) + 1 \}. \]

By the backward induction hypothesis, we have
\[ (19) \quad \text{reg}(p_{S \cup \{v\}} + I_\Gamma) \leq f(\Delta) + 1. \]
We now claim that
\[(20) \text{reg}(p_S + I_{\Gamma'}) \leq f(\Delta).\]
Indeed, if \(p_S + I_{\Gamma'}\) is prime, then \(\text{reg}(p_S + I_{\Gamma'}) = 1\). As \(\Delta\) is not a cone over \(v\), by the definition of \(f\) we have \(f(\Delta) \geq f(lk_{\Delta}(v)) + 1 \geq 1\), and the claim holds in this case.

Assume that \(p_S + I_{\Gamma'}\) is not a prime. Observe that
\[I_{\Gamma''} = (x_v) + I_{\Gamma'},\]
where \(\Gamma'' = lk_{\Gamma'}(v)\) and this simplicial complex is considered over \([r]\). Since variable \(x_v\) does not appear in any generator of \(I_{\Gamma'}\), hence \(\text{reg}(I_{\Gamma''}) = \text{reg}(I_{\Gamma'})\).

On the other hand, by the induction hypothesis, we have
\[\text{reg}(I_{\Gamma''}) = \text{reg}(lk_{\Gamma'}(v)) \leq f(lk_{\Delta}(v)) + 1.\]
It follows that
\[\text{reg}(p_S + I_{\Gamma'}) \leq \text{reg}(I_{\Gamma'}) = \text{reg}(I_{\Gamma''}) \leq f(lk_{\Delta}(v)) + 1.\]
Together with the inequality \(f(lk_{\Delta}(v)) + 1 \leq f(\Delta)\), it yields \(\text{reg}(p_S + I_{\Gamma'}) \leq f(\Delta)\), as claimed.

By combining three Inequalities (18)-(20), we obtain \(\text{reg}(p_S + I_{\Gamma'}) \leq f(\Delta) + 1\), and so the inequality (17) is proved. The lemma now follows from the assertion by taking \(S = \emptyset\), and the proof is complete. □

We now reformulate the theorem 3.1 for graphs. A graph \(G\) is called trivial if it has no edges. For a subset \(S\) of \(V(G)\), the closed neighborhood of the set \(S\) in \(G\) is the set \(N_G[S] = S \cup \{v \in V(G) \mid v\text{ is a neighbor of some vertex in } S\}\). For a vertex \(v\) of \(G\), we write \(N_G[v]\) stands for \(N_G[\{v\}]\). Recall that \(\Delta(G)\) is the set of independent sets of \(G\), which is a simplicial complex and \(I(G) = I_{\Delta(G)}\).

**Corollary 3.2.** Let \(G\) be a graph and let \(\mathcal{I}_G = \{G \setminus N_G[S] \mid S \in \Delta(G)\}\). Assume that \(f: \mathcal{I}_G \to \mathbb{N}\) is a function which satisfies the following properties:

1. \(f(H) = 0\) if \(H\) is trivial.
2. For every \(H\) and every non-isolated vertex \(v\) of \(H\), \(f(H \setminus N_H[v]) + 1 \leq f(H)\).

Then, for every subcomplex \(\Gamma\) of \(\Delta(G)\) with \(\mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta(G))\) we have
\[\text{reg}(I_{\Gamma}) \leq f(G) + 1.\]

**Proof.** First we note that, for every graph \(H\) and every \(S \in \Delta(H)\) we have
\[\Delta(H \setminus N_H[S]) = lk_{\Delta(H)}(S).\]
It implies that
\[\text{Simp}(\Delta(G)) = \{\Delta(H) \mid H \in \mathcal{I}_G\}.\]
Therefore, we can define a function $g: \text{Simp}(\Delta(G)) \to \mathbb{N}$, by sending $\Delta(H)$ to $f(H)$ for all $H \in \mathcal{I}_G$.

Note that for every graph $H$, we have $\Delta(H)$ is a simplex if and only if $H$ is trivial; and $\Delta(H)$ is a cone over a vertex $v$ if and only if $v$ is an isolated vertex of $H$. Together with the definition of the function $g$, it shows that $g$ satisfies all conditions of Theorem 3.1, and therefore by this theorem we obtain $\text{reg}(I_\Gamma) \leq g(\Delta(G)) + 1 = f(G) + 1$, as required. □

The theorem when applying to an edge ideal of a graph has the following form.

**Lemma 3.3.** Let $G$ be a graph. Then,

$$\text{reg}(I(G)^{(n)}) \leq 2(n-1) + b,$$

where $b = \max\{\text{reg}(I_\Gamma) \mid \Gamma$ is a subcomplex of $\Delta(G)$ with $\mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta(G))\}$.

**Proof.** Since $I(G) = I_{\Delta(G)}$ and $\delta(I(G)) = 2$ by [8, Example 4.4], therefore the lemma follows from Theorem 2.3. □

We are now in position to prove the main result of this section.

**Theorem 3.4.** Let $G$ be a graph. Then,

$$\text{reg}(I(G)^{(n)}) \leq 2n + \text{order-match}(G) - 1,$$

for all $n \geq 1$.

**Proof.** By Lemma 3.3, it remains to show that $\text{reg}(I_\Gamma) \leq \text{order-match}(G) + 1$, for every subcomplex $\Gamma$ of $\Delta(G)$ with $\mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta(G))$.

Consider the function $f: \mathcal{I}_G \to \mathbb{N}$ defined by

$$f(H) = \begin{cases} 
0 & \text{if } H \text{ is trivial,} \\
n \text{order-match}(H) & \text{otherwise.}
\end{cases}$$

For every non-isolated vertex $v$ of $H$, we have $f(H \setminus N_H[v]) + 1 \leq f(H)$ by [10, Lemma 2.1], hence $f$ satisfies all conditions of Corollary 3.2, so that by this corollary

$$\text{reg}(I_\Gamma) \leq f(G) + 1 = \text{order-match}(G) + 1,$$

and the theorem follows. □

**Remark 3.5.** Let $G$ be a graph with $\text{order-match}(G) = \nu(G)$. Then,

$$\text{reg}(I(G)^{(n)}) = 2n + \nu(G) - 1,$$

for all $n \geq 1$.

Indeed, for every positive integer $n$, the lower bound $\text{reg}(I(G)^{(n)}) \geq 2n + \nu(G) - 1$ comes from Lemma 1.8 and the upper bound follows from Theorem 3.4 because $\text{order-match}(G) = \nu(G)$. 

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As a consequence, we quickly recover the main result of Fakhari in [12], which says that the equality holds when \( G \) is a *Cameron-Walker* graph, where a graph \( G \) is called Cameron-Walker if \( \nu(G) = \text{match}(G) \) (see e.g. [17]). For such a graph \( G \), \( \text{order-match}(G) = \nu(G) \) since \( \nu(G) \leq \text{order-match}(G) \leq \text{match}(G) \).

**Acknowledgment.** We are supported by Project ICRTM.02.2021.02 of the International Centre for Research and Postgraduate Training in Mathematics (ICRTM), Institute of Mathematics, VAST.

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