Derivation of general Maxwell type equations for open quantum systems

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Abstract

Using equations of motion with the anisotropic dissipative term for quantum particle and quantum-mechanical commutation rules, the general Maxwell-type differential equations are derived. The direct modifications of the well-known Maxwell equations due to the medium effects (openness of the system) are discussed.

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I. INTRODUCTION

A Maxwell equations have been derived as a mathematical reflection of the already established empirical facts. In the macroscopic quantum electrodynamics, they have been expressed as a relationships between electric $E^0$ and magnetic $H^0$ field vectors, describing an electromagnetic field. The properties of the medium (matter) are taken into account only by means of material equations. Since the latter is not always possible, such a description is not universal [1].

Using the equations of motion

$$m \ddot{x}_j = F^0_j (x, \dot{x}, t) \quad (1)$$

of the non-relativistic point-particle of mass $m$ with the Cartesian position operator $x_j$, $j = 1, 2, 3$, and the quantum mechanical commutation relations

$$[x_j, x_k] = 0, \quad (2)$$

$$m [x_j, \dot{x}_k] = i\hbar \delta_{jk}, \quad (3)$$

the Lorentz force law ($\epsilon_{jkl}$ is the Levi-Civita antisymmetric tensor)

$$F^0_j = E^0_j + \epsilon_{jkl} \dot{x}_k H^0_l \quad (4)$$

and two Maxwell equations

$$\nabla_j H^0_j = 0, \quad (5)$$

$$\dot{H}^0_j + \epsilon_{jkl} \nabla_k E^0_l = 0 \quad (6)$$

have been derived in Ref. [2]. As explicitly demonstrated, $E^0(x, t)$ and $H^0(x, t)$ are the vector functions of the coordinate $x$ and time $t$ but not the velocities. As shown in Ref. [3], the particle motion in the non-inertial frame with the Coriolis like forces or in the weak gravitational field also satisfies the constraints (1)–(3). The velocity-dependent forces are not limited to electromagnetic ones, and the commutation relations (2) and (3) are also determined by the equations of motion (1)–(3). The generalization of this approach [2, 3] to the relativistic case has been done in Ref. [4]. The connection of the classical equations of motion and Maxwell electromagnetic equations in elegant manner has been explicitly shown in Ref. [5]. Note also the derivation of the Lorentz force law in Ref. [6] based on the commutation rules.
In Refs. [2–6], it has been not taken into account that systems by their nature are open systems [1, 7–11]. In this case, it is necessary to explicitly take into consideration the environmental or medium effects in the equations of motion. In the present paper, we try to solve this problem. In the addition to the Lorenz force $F_j^0$, the motion of the test particle (open system) is under the influence of an additional force $f_j$ induced by physical medium. This force effectively describes the interaction of the moving particle with medium and has the experimental justification.

II. GENERAL CONSIDERATION

Let us consider the general equations of motion

$$m\ddot{x}_j = F_j(x, \dot{x}, t) - \lambda_{jk}(x)\dot{x}_k,$$  \hspace{1cm} (7)

where the motion of the test particle is under the influence of an additional dissipative type force $f_j = -\lambda_{jk}(x)\dot{x}_k$ induced by physical medium. Here, $\lambda_{jk}(x) = \lambda_{(jk)}(x) + \lambda_{[jk]}(x)$ is the general tensor field, where $\lambda_{(jk)}$ and $\lambda_{[jk]}$ are the symmetrized and anti-symmetrized tensors, respectively. Note that these tensors depend on the coordinate $x$. In general, Eq. (7) should also include the fluctuating forces, leading to a stochastic dynamics. For simplicity, we consider the average dynamics of the test particle.

Taking the total time derivative of Eq. (8), we obtain

$$[\dot{x}_j, \dot{x}_k] + [x_j, \ddot{x}_k] = 0.$$ \hspace{1cm} (8)

Substituting (7) in Eq. (8), we find

$$[x_j, F_k] = i\hbar \left(\lambda_{(jk)} + \left\{ \frac{im^2}{\hbar} [\dot{x}_j, \dot{x}_k] - \lambda_{[jk]} \right\} \right) = i\hbar \left(\lambda_{(jk)} - \epsilon_{jkl}H_l\right).$$ \hspace{1cm} (9)

Because the term in curly brackets in Eq. (9) is anti-symmetrized with respect to $j \rightarrow k$ and $k \rightarrow j$, we introduce, without loss of generality, the vector "magnetic" field

$$H_l = -\frac{1}{2}\epsilon_{jkl}\left(\frac{im^2}{\hbar} [\dot{x}_j, \dot{x}_k] - \lambda_{[jk]}\right) = H_l^0 + \frac{1}{2}\epsilon_{jkl}\lambda_{[jk]}.$$ \hspace{1cm} (10)

Expressing $[\dot{x}_j, \dot{x}_k]$ from Eq. (10) and substituting into the Jacobi identity

$$[x_l, [\dot{x}_j, \dot{x}_k]] = 0,$$ \hspace{1cm} (11)
we derive
\[ \epsilon_{jkp}(x_t, H_p) = [x_t, \lambda_{(jk)}] = 0 \rightarrow [x_t, H_p] = 0, \] (12)
which means that \( H_p \) only depends on \( x \) and \( t \). The Jacobi identity
\[ \epsilon_{jkl}(\dot{x}_l, [\dot{x}_j, \dot{x}_k]) = 0 \] (13)
together with Eq. (10) implies
\[ [\dot{x}_l, H_l] = \frac{1}{2} \epsilon_{jkl}[\dot{x}_l, \lambda_{(jk)}] \] (14)
which is equivalent to
\[ \text{div} H = \frac{1}{2} \epsilon_{jkl} \nabla_l \lambda_{(jk)}. \] (15)

One can assign the role of "magnetic" charge density \( \rho_m \) to the term in the right hand side of Eq. (15). So, the "magnetic" charge density \( \rho_m \) is a possible source of the static "magnetic" field by analogy with the electric charge density \( \rho_e \) as a source of the static electric field. Here, the "magnetic" charge ("Dirac monopole" [12]) is related to the heterogeneous antisymmetric tensor field \( \lambda_{(jk)} \). When \( \lambda_{(jk)} \) is a homogeneous field and, correspondingly, \( \nabla_l \lambda_{(jk)} = 0 \), we again obtain Eq. (5). In the magnets, the magnetic charge is related with the magnetization \( \mathbf{I} \): \( \rho_m = -\text{div} \mathbf{I} \) [13]. The well-known Maxwell equations assert that \( \rho_m = 0 \) and there are no other sources of magnetic fields, except electric currents.

Substituting now the vector field
\[ F_j = E_j + \epsilon_{jkl} \dot{x}_k H_l \] (16)
into Eq. (9), we obtain
\[ [x_j, E_k] = \frac{i \hbar}{m} \lambda_{(jk)}. \] (17)
The vector "electric" field
\[ E_j = E_j^0 + \lambda_{(jk)} \dot{x}_k \] (18)
follows from Eqs. (2) and (17). As seen from Eqs. (10) and (18), the vector "electric" and "magnetic" fields contain the individual property of the medium, "resistance" or "conductivity". At \( E^0 = 0 \), Eq. (18) is reduced to the material-type equation (e.g., the Ohmic law in the case of electromagnetic forces). Using Eq. (18), one can derive the Maxwell type equation:
\[ \text{div} \mathbf{E} = \rho_e + \nabla \lambda_{(jk)} \dot{x}_k, \] (19)
where \( \rho_e = \text{div}\mathbf{E}^0 \) is an analog of the "electric" charge density.

Taking total time derivative of the vector "magnetic" field \( \mathbf{H} \)

\[
\frac{\partial H_l}{\partial t} + \dot{x}_m \frac{\partial H_l}{\partial x_m} = \frac{1}{2} \epsilon_{jkl} \dot{x}_m \nabla_m \lambda_{[jk]} - \frac{im^2}{\hbar} \epsilon_{jkl} [\dot{x}_j, \dot{x}_k],
\]

and making some tedious but simple algebra, we obtain the following Maxwell-type equation

\[
\frac{\partial H_l}{\partial t} = -\epsilon_{lkj} \frac{\partial E_j}{\partial x_k} - \frac{1}{m} \left\{ \lambda_{jj} H_l - \lambda_{ql} H_q - \lambda_{[lk]} H_k - \epsilon_{jkl} \lambda_{jp} \lambda_{[pk]} \right\} - \epsilon_{jkl} \dot{x}_p \nabla_k \lambda_{jp} + \frac{1}{2} \left\{ \epsilon_{jkl} \dot{x}_m \nabla_m + \epsilon_{jkl} \dot{x}_l \nabla_p \right\} \lambda_{[jk]},
\]

(21)

This equation is the generalized law of "electromagnetic" induction and more complicated than the corresponding Maxwell equation. In the case of \( \lambda_{jk} \equiv 0 \), we re-derive all results of Refs. [2, 3]. If the \( \lambda_{jk} \) are constants, Eq. (21) is transformed into

\[
\frac{\partial H_l}{\partial t} = -\epsilon_{lkj} \frac{\partial E_j}{\partial x_k} - \frac{1}{m} \left\{ \lambda_{jj} H_l - \lambda_{ql} H_q - \lambda_{[lk]} H_k - \epsilon_{jkl} \lambda_{jp} \lambda_{[pk]} \right\}.
\]

(22)

In particular case of the symmetric tensor field \( \lambda_{jk} = \lambda(x) \delta_{jk} \), Eq. (21) is simplified as

\[
\frac{\partial H_l}{\partial t} = -\epsilon_{lkj} \frac{\partial E_j}{\partial x_k} - \frac{2\lambda}{m} H_l - \epsilon_{jlk} \dot{x}_j \nabla_k \lambda
\]

(23)

or

\[
\frac{\partial H}{\partial t} = -\text{rot}\mathbf{E} - \frac{2\lambda}{m} \mathbf{H} - \dot{x} \times \nabla \lambda
\]

(24)

which is similar to one for the ferromagnetic materials, in which \( \lambda \) is proportional to the magnetic conductivity (inverse to the magnetic viscosity) [13]. At constant \( \lambda \), we have

\[
\frac{\partial \mathbf{H}}{\partial t} = -\text{rot}\mathbf{E} - \frac{2\lambda}{m} \mathbf{H}.
\]

(25)

Taking total time derivative of vector "electric" field \( \mathbf{E} \)

\[
\frac{\partial E_l}{\partial t} + \dot{x}_m \frac{\partial E_l}{\partial x_m} = \frac{\partial E^0_l}{\partial t} + \dot{x}_m \frac{\partial E^0_l}{\partial x_m} + \dot{x}_i \nabla_i \lambda_{(lk)} + \lambda_{(lk)} \ddot{x}_k,
\]

(26)

we obtain the following Maxwell-type equation

\[
\frac{\partial E_l}{\partial t} = \epsilon_{lkj} \frac{\partial H_j}{\partial x_k} - j_l + \dot{x}_i \nabla_i \lambda_{[lk]} \ddot{x}_k + \frac{\lambda_{[lk]}}{m} \left\{ E_k + \epsilon_{ljk} \dot{x}_l H_j - \lambda_{kl} \ddot{x}_l \right\}.
\]

(27)

Here, we employ that

\[
\frac{\partial E^0_l}{\partial t} = \epsilon_{lkj} \frac{\partial H^0_j}{\partial x_k} - j_l,
\]

(28)
where $j$ is an analog of the "electric" current density. Eq. (27) is the generalized "Ampere’s" law.

Thus, in the general case for electromagnetic forces, instead of the Maxwell and material equations, more complicated equations should be used: a closed system of coupled equations of motion (7) [or more general - the quantum Langevin equations or the corresponding quantum diffusion equation by taking also into account the quantum fluctuations] for the charge particles and field equations (15), (19), (21), (27). Note that our definitions for the $H$ and $E$ are different from the standard definitions. For example, such type approach can be used to describe electromagnetic processes in a fully ionized plasma.

III. CONCLUSIONS

Employing an equations of motion for the test quantum particle and quantum-mechanical commutation rules, we derived the Maxwell type differential equations for forces $F_j$ of any nature. Because these equations contain the influence of the medium (openness of the system), they are more complicated than the usual Maxwell equations for electromagnetic forces. As seen, for the strongly inhomogeneous anisotropic systems the effective "magnetic" charge appears in Eq. (15). So, the obtained equations acquire more symmetrical form, where there are "magnetic" and "electric" charges. The "magnetic" charge is related to the inhomogeneous antisymmetric tensor field $\lambda_{[jk]}$. It should be noted that magnetic monopole (magnetic charge) has been sought in some materials possessing strongly anisotropic crystal structure (e.g., the nematic materials) and, accordingly, possessing strongly anisotropic dissipative properties. We also found that the influence of homogeneous and isotropic medium leads to the field equation (25) for the ferromagnetic materials with the magnetic conductivity.

Using the nonrelativistic classical equations of motion, corresponding to Eqs. (7), and the usual commutator-Poisson bracket correspondence, one can deduce the same results for the open classical systems. It is also possible to assume that the forces in nature are united only by the general form of the equations of motion which contain the "electric" and "magnetic" components.
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