ON STABILITY OF NON-DOMINATION UNDER TAKING PRODUCTS

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ABSTRACT. We show that non-domination results for targets that are not dominated by products are stable under Cartesian products.

1. MOTIVATION

If $M$ and $N$ are closed oriented manifolds of the same dimension, we say that $M$ dominates $N$, and we write $M \geq N$, if there is a continuous map $f: M \rightarrow N$ of non-zero degree. The existence of such a dominant map is a property of the homotopy types of $M$ and $N$, and it has been known since the pioneering work of Hopf [11] that for such a map $f$ the pullback $f^*$ is an injection of rational cohomology algebras, and that $f_*$ is virtually surjective on the fundamental group. However, the existence of an injective algebra homomorphism $H^*(N; \mathbb{Q}) \rightarrow H^*(M; \mathbb{Q})$ and of a virtually surjective homomorphism $\pi_1(M) \rightarrow \pi_1(N)$ is usually far from sufficient for $M \geq N$.

Motivated by the work of Gromov [7, 8] in particular, (non-)domination between manifolds has in recent years been studied in several different contexts, using a variety of techniques from topology, geometry, and group theory; see for example [7, 4, 8, 5, 12] and the references given there. An idea due to Thurston [16] and Gromov [7] is to study numerical invariants $I$ of manifolds that are monotone under maps of non-zero degree, so that $M \geq N$ implies $I(M) \geq I(N)$. Then, whenever one can compute or estimate $I$ and prove $I(M) < I(N)$ for some specific manifolds, one concludes that $M$ does not dominate $N$. The simplest example of such an invariant is the cuplength in rational cohomology, which is monotone by the result of Hopf mentioned before. A more subtle monotone invariant – of geometric rather than algebraic origin – is the simplicial volume $\| \cdot \|$ defined by Gromov [7]. In general, monotone invariants are closely connected to functorial semi-norms on homology [8, 6, 15].

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According to Gromov, the simplicial volume has a major deficiency: its lack of multiplicativity. In fact, he proved in [7] that the simplicial volume is approximately multiplicative for Cartesian products, and it is known that it is not strictly multiplicative [3]. However, approximate multiplicativity is not good enough to obtain stable non-domination results. Indeed, suppose that $0 < \|M\| < \|N\|$ for some specific $M$ and $N$. Then $M \not\geq N$, but it is unclear whether the $d$-fold product $M \times d$ may dominate $N \times d$ for some $d \geq 2$, or not. The approximate multiplicativity does not rule out the possibility that, as a function of the number of factors, the simplicial volume of direct products of $M$ might grow faster than that of direct products of $N$, so that the former eventually surpasses the latter.

Invariants that are strictly multiplicative – or strictly additive, like the cuplength – do not have this deficiency: if $I(M) < I(N)$, then $I(M \times d) < I(N \times d)$, so that $M \times d \not\geq N \times d$ for all $d \geq 1$. In this case the non-domination result $M \not\geq N$ is stable under Cartesian products.

Gromov [8] suggested that many manifolds $N$ might have the property that they cannot be dominated by a non-trivial product $M = M_1 \times M_2$. This conjecture has since been verified [12], and there are now lots of examples of manifolds that are known not to be dominated by products [12, 13, 14, 17]. We will see here that in general non-domination results for targets that cannot be dominated by products are stable under Cartesian products. This is interesting in its own right, and also has geometric applications [17].

**Conventions.** Throughout this paper, the word manifold means a connected closed oriented non-empty topological manifold; we denote the rational fundamental class of a manifold $M$ by $[M]$. A product of manifolds is always a non-trivial product, so no factor is a point.

2. Results

Our first result is that for targets that are not dominated by products, the loss of information in taking products discussed in the previous section does not occur.

**Theorem 2.1.** Suppose $M$ and $N$ are $n$-manifolds, and that $N$ is not dominated by a product. Then for any $d \geq 2$ we have $M \times d \geq N \times d$ if and only if $M \geq N$.

In a similar spirit, taking Cartesian products with arbitrary manifolds preserves non-domination for targets that are not dominated by products.

**Theorem 2.2.** Suppose $M$ and $N$ are $n$-manifolds, and that $N$ is not dominated by a product. Then for any manifold $W$, we have $M \times W \geq N \times W$ if and only if $M \geq N$.

Note that $W$ may very well have trivial simplicial volume. Even if one deduces $M \not\geq N$ from $\|M\| < \|N\|$, this theorem shows that multiplying with $W$ preserves non-domination, while killing the simplicial volume if $\|W\| = 0$. 
Finally, controlling the dimensions of the factors in a product, we have the following:

**Theorem 2.3.** Let $N$ be an $n$-manifold that is not dominated by a product. Then there is no manifold $V$ for which the product $N \times V$ can be dominated by a product $P = X_1 \times \ldots \times X_s$ that satisfies $\dim X_j < n$ for all $j \in \{1, \ldots, s\}$.

### 3. Proofs

The proofs of the above theorems all use the following lemma, which is a consequence of Thom’s work [18] on the Steenrod problem.

**Lemma 3.1.** Let $N$ be an $n$-manifold that is not dominated by a product. If $f : M_1 \times M_2 \to N$ is a continuous map, then for all $i \in \{1, \ldots, n-1\}$ the map

$$f_* : H_i(M_1; \mathbb{Q}) \otimes H_{n-i}(M_2; \mathbb{Q}) \to H_n(N; \mathbb{Q})$$

induced by the homological cross-product and $f$ is the zero map.

**Proof.** Because elements of $H_i(M_1; \mathbb{Q}) \otimes H_{n-i}(M_2; \mathbb{Q})$ are finite linear combinations of decomposable elements, and $f_*$ is linear, it suffices to show $f_*(\alpha \otimes \beta) = 0$ for all $\alpha \in H_i(M_1; \mathbb{Q})$ and all $\beta \in H_{n-i}(M_2; \mathbb{Q})$. Again by the linearity of $f_*$, there is no loss of generality in replacing $\alpha$ and $\beta$ by non-zero multiples. Thus we may assume that these are integral homology classes. By Thom’s result [18], after replacing the integral classes $\alpha$ and $\beta$ by suitable non-zero multiples, there are continuous maps $g_j : X_j \to M_j$ defined on manifolds $X_j$ of dimensions $i$ and $n-i$ respectively, such that $(g_1)_*[X_1] = \alpha$ and $(g_2)_*[X_2] = \beta$. It follows that

$$f_*(\alpha \otimes \beta) = (f \circ (g_1 \times g_2))_*[X_1 \times X_2].$$

This must vanish, because otherwise the map $f \circ (g_1 \times g_2) : X_1 \times X_2 \to N$ would have non-zero degree, contradicting the assumption on $N$. \qed

Using Lemma 3.1, we now prove the theorems stated in the previous section.

**Proof of Theorem 2.1.** If $M \geq N$, then clearly $M^{\times d} \geq N^{\times d}$ for all $d \geq 2$. Conversely, suppose that $g : M^{\times d} \to N^{\times d}$ has non-zero degree for some $d \geq 2$. We consider the composition $f = p_1 \circ g$, where $p_1$ is the projection to the first factor. Then $f_*$ is surjective in rational homology. Since we assumed that $N$ is not dominated by a product, Lemma 3.1 tells us that, in degree $n$, the map $f_*$ vanishes on tensor products of homology vector spaces of non-zero degree. It follows that for at least one of the inclusions $i : M \to M^{\times d}$ of a factor of $M^{\times d}$, the composition $f \circ i$ has non-zero degree, and thus $M \geq N$. \qed
Proof of Theorem 2.2. If $M \geq N$, then clearly $M \times W \geq N \times W$ for all manifolds $W$. Conversely, suppose that $f : M \times W \rightarrow N \times W$ has non-zero degree for some $W$. We consider the induced map $f_\ast$ on $H_n(\cdot; \mathbb{Q})$ in terms of the Künneth decompositions of the domain and of the target:

$$f_\ast : H_n(M; \mathbb{Q}) \oplus M_1 \oplus H_n(W; \mathbb{Q}) \rightarrow H_n(N; \mathbb{Q}) \oplus M_2 \oplus H_n(W; \mathbb{Q}) ,$$

where $M_1$ denotes the direct sum of tensor products of homology vector spaces in non-zero degrees.

Since we assumed that $N$ is not dominated by a product, Lemma 3.1 tells us that $f_\ast(M_1)$ is contained in $M_2 \oplus H_n(W; \mathbb{Q})$. If we assume for a contradiction that $M \nless N$, then the same is true for $f_\ast(H_n(M; \mathbb{Q}))$.

Because $f_\ast$ is surjective, we conclude that there is an $a_0 \in H_n(W; \mathbb{Q})$ such that $f_\ast(a_0) = [N] \neq 0$ holds in the quotient vector space

$$Q = H_n(N \times W; \mathbb{Q}) / f_\ast(H_n(M; \mathbb{Q}) \oplus M_1) .$$

Note that $Q$ is of finite, non-zero, dimension.

Now we think of $a_0$ as being in the target of $f_\ast$. By surjectivity of $f_\ast$, the class $a_0$ is in its image, so there exists an $a_1 \in H_n(W; \mathbb{Q})$ satisfying $f_\ast(a_1) = a_0$ in $Q$ (though not necessarily in $H_n(N \times W; \mathbb{Q})$). We proceed inductively to find $a_{i+1} \in H_n(W; \mathbb{Q})$ with the property that $f_\ast(a_{i+1}) = a_i$ in $Q$. The assumptions that $N$ is not dominated by a product, or by $M$, imply at every step that $a_i$ does not vanish in the quotient $Q$.

Since $Q$ is finite-dimensional, there is a minimal $k \in \mathbb{N}$ such that $a_{0}, \ldots, a_k$ are linearly dependent in $Q$. There are then $\lambda_i \in Q$ with $\lambda_k \neq 0$ such that

$$\lambda_k a_k + \ldots + \lambda_0 a_0 = 0 \in Q .$$

We now take the left-hand-side of this equation, considered as an element of $H_n(W; \mathbb{Q}) \subset H_n(M \times W; \mathbb{Q})$, and apply $f_\ast$ to it to obtain

$$\lambda_k a_{k-1} + \ldots + \lambda_1 a_0 + \lambda_0 [N] \in f_\ast(H_n(M; \mathbb{Q}) \oplus M_1) .$$

If $\lambda_0 = 0$, then this contradicts the minimality of $k$. If $\lambda_0 \neq 0$, then we reach the conclusion that in $H_n(N \times W; \mathbb{Q})$ the generator $[N] \in H_n(N; \mathbb{Q})$ is a linear combination of $\lambda_k a_{k-1} + \ldots + \lambda_1 a_0 \in H_n(W; \mathbb{Q})$ and of elements in

$$f_\ast(H_n(M; \mathbb{Q}) \oplus M_1) \subset M_2 \oplus H_n(W; \mathbb{Q}) .$$

This contradicts the Künneth decomposition, and hence proves $M \geq N$.

Proof of Theorem 2.3. Suppose $g : X_1 \times \ldots \times X_s \rightarrow N \times V$ is a continuous map, and consider the composition $f = p_1 \circ g$. The assumptions that $N$ is not dominated by a product and that $\dim X_j < n$ for all $j$ imply, as in the proof of Lemma 3.1, that $f_\ast$ is the zero map in degree $n$. Therefore, $g$ has degree zero.
4. Discussion

4.1. Applications of the cuplength. It is not clear to what extent the assumption that \( N \) is not dominated by a product is necessary in the above theorems. While it is crucial for our proofs, this could be an artefact of our method. Indeed, there are cases of targets \( N \) which are dominated by products, and still one can prove our results for them. We now do this for tori, using the cuplength.

Recall that the cuplength of \( M \), denoted \( \text{cl}(M) \), is the maximal number \( k \) for which there are classes \( \alpha_1, \ldots, \alpha_k \in H^*(M; \mathbb{Q}) \) of positive degrees with the property that \( \alpha_1 \cup \ldots \cup \alpha_k \neq 0 \in H^*(M; \mathbb{Q}) \). This is monotone under maps of non-zero degree by \cite{11}. The compatibility of the Künneth decomposition with the cup product implies

\[
\text{cl}(M \times W) = \text{cl}(M) + \text{cl}(W).
\]

The following is easy and well known.

**Lemma 4.1.** An \( n \)-manifold \( M \) dominates \( T^n \) if and only if there is an injective algebra homomorphism \( H^*(T^n; \mathbb{Q}) \to H^*(M; \mathbb{Q}) \), equivalently, if \( \text{cl}(M) = n \).

So this is a case where the algebraic necessary condition for domination derived from rational cohomology is also sufficient.

Lemma 4.1 combined with (1) tells us that Theorem 2.2 holds for \( N = T^n \). Furthermore, we have:

**Proposition 4.2.** If \( M_1 \) and \( M_2 \) are manifolds of dimensions \( m_1 \) and \( m_2 \) respectively, then \( M_1 \times M_2 \geq T^{m_1 + m_2} \) if and only if \( M_1 \geq T^{m_1} \) and \( M_2 \geq T^{m_2} \).

In particular, Theorem 2.1 also holds for \( N = T^n \).

4.2. Infinite products. Gromov has suggested that some non-domination results should extend to infinite products, following his perspective on infinite products and related topics \cite{1, 9, 2, 10, Section 5}.

By increasing the number \( d \) of factors in \( P^{\times d} \), one would naively end up with a countably infinite product \( P^{\times \infty} \), without any extra structure. A better way of looking at infinite products is probably to pick a (discrete, countable) group \( \Gamma \), and to look at the space \( P^\Gamma = \text{Map}(\Gamma, P) \), equipped with the natural shift action of \( \Gamma \). Now in formulating what \( P^\Gamma \not\geq N^\Gamma \) might mean, one should only consider \( \Gamma \)-equivariant continuous maps between these product spaces.

The main issue is of course that for maps between these infinite-dimensional manifolds there is no naive, geometric, notion of degree. Instead, one should make full use of equivariance and define domination via surjectivity in a suitable homology theory, perhaps without necessarily attempting to define a degree.

\footnote{Hopf did not use cohomology, but formulated the conclusion in terms of the Umkehr map on intersection rings.}
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