On a stochastic version of Prouse model in fluid dynamics

B. Ferrario
Dipartimento di Matematica - Università di Pavia

F. Flandoli
Dipartimento di Matematica Applicata - Università di Pisa

Abstract

A stochastic version of a modified Navier–Stokes equation (introduced by Prouse) is considered in a 3-dimensional torus. For equation (1), we prove existence and uniqueness of martingale solutions. A different model with the non linearity \( \Phi(u) = \nu |u|^4 u \) is analyzed; for the structure function of this model, some insights towards an expression similar to that obtained by the Kolmogorov 1941 theory of turbulence are presented.

**Key words:** stochastic hydrodynamics, existence and uniqueness of martingale solutions, stationary solutions, structure function in turbulence.

**AMS Subject Classification (2000):** 76F55, 76M35, 76D06, 76D03, 35Q35.

1 Introduction

The three dimensional Navier-Stokes equations are a not yet completely understood mathematical problem, in the sense that there is no proof of uniqueness of solutions in the spaces where existence is proved. This mathematical problem has been investigated since long, also in connection with the analysis of how good are the Navier–Stokes equations to model turbulence. Some attempts have been made to overcome the problem of uniqueness, introducing some modification in the Navier-Stokes equations. In this paper we are concerned with the model proposed by Prouse in [15]. Here, we study a stochastic version of this problem, as explained below. As soon as a stochastic equation is introduced, statistical properties typical of turbulence can be investigated.

We remind that nowadays there are many results on stochastic three dimensional Navier–Stokes equations (see, among the others, [1], [2], [3], [4], [5], [6], [8], [9], [12], [19]): however, the uniqueness problem is not solved also in the stochastic framework.
Let us consider the partial differential equations of Navier-Stokes type

\[
\begin{cases}
du + [-\Delta \Phi(u) + (u \cdot \nabla) u + \nabla p - \nabla \text{div} \Phi(u)] \, dt = G(u) \, dw \\
\text{div} u = 0 \\
u|_{t=0} = u_0
\end{cases}
\]

where, for \( t \geq 0 \) and \( x \in \mathcal{T} \subset \mathbb{R}^3 \), \( u = u(t, x) \) is the velocity vector field, \( p = p(t, x) \) the pressure field; \( \nu > 0 \) the viscosity coefficient. \( G \) is an operator acting on the noise and on the velocity; the vector function \( \Phi : \mathbb{R}^3 \to \mathbb{R}^3 \) is defined as follows:

\[
\Phi(u) = \sigma(|u|)u \quad \text{with} \quad \begin{cases}
\sigma \in C^1([0, \infty)) \\
\sigma(\xi) \geq \nu > 0, \sigma'(\xi) \geq 0 \\
a_1 \xi^{k-1} \leq \sigma(\xi) \leq a_2 \xi^{k-1} \text{ when } \xi > K
\end{cases}
\]

where \( a_2 \geq a_1 > 0 \) and \( b \geq 4 \).

\( \Phi \) describes the nonlinear relationship between the stress tensor and the deformation velocity tensor, as explained in [15]. When this relationship is linear, then \( \Phi(u) = \nu u \) and (1) are the usual Navier–Stokes equations for an homogeneous incompressible viscous fluid with random forcing term; indeed, the first equation becomes

\[
du + [-\nu \Delta u + (u \cdot \nabla) u + \nabla p] \, dt = G(u) \, dw
\]

For problem (1)-(2), in Section 3 we prove a result on existence and uniqueness of martingale solutions (Theorem 6) and on existence of stationary martingale solutions (Theorem 9).

In Section 4 another model with \( \Phi(u) = \nu |u|^4 u \) in (1) is investigated; analysis on existence of martingale solutions and stationary martingale solutions is presented (Theorem 10). Moreover, introducing a scaling transformation suggested by turbulence theory, some insights in the behaviour of the function structure of any order \( p \) are shown (Claim 14).

Preliminaries are in Section 2 auxiliary results are in the two Appendixes.

2 Notations and preliminaries

Let the spatial domain be a torus, i.e. the spatial variable \( x \) belongs to \( \mathcal{T} = [0, L]^3 \) and periodic boundary conditions are assumed. \( L^2 \) is defined as the space of vector fields \( u : \mathcal{T} \to \mathbb{R}^3 \) with \( L^2(\mathcal{T}) \)-components. For every \( \alpha > 0 \) and \( p > 1 \), \( \mathbb{W}^{\alpha,p} \) is the space of fields \( u \in \mathbb{L}^p \) with components in the Sobolev space \( W^{\alpha,p}(\mathcal{T}) \). For \( \alpha < 0 \), \( \mathbb{W}^{\alpha,p} \) is the dual space of \( \mathbb{W}^{-\alpha,p'} \) with \( \frac{1}{p'} + \frac{1}{p} = 1 \). Set \( \mathbb{H}^\alpha = \mathbb{W}^{0,2} \).

We introduce the classical spaces for the Navier–Stokes equations (see, e.g., [17]). \( \mathcal{D}^\infty \) is defined as the space of infinitely differentiable divergence free periodic fields \( u \) on \( \mathcal{T} \), with zero mean (\( \int_\mathcal{T} u(x) dx = 0 \)). Let \( H \) be the closure of \( \mathcal{D}^\infty \) in the \( L^2 \)-topology; it is the space of all fields
Let \( u \in \mathbb{L}^2 \) such that \( \text{div} \ u = 0, \ u \cdot n \) on the boundary is periodic, \( \int_T u(x) \, dx = 0. \) We endow \( H \) with the inner product
\[
\langle u, v \rangle_H = \frac{1}{L^3} \int_T u(x) \cdot v(x) \, dx
\]
and the associated norm \( |\cdot|_H. \)

Let \( V \) (resp. \( D(A) \)) be the closure of \( \mathcal{D}^\infty \) in the \( \mathbb{H}^1 \)-topology (resp. \( \mathbb{H}^2 \)-topology); it is the space of divergence free, zero mean, periodic elements of \( \mathbb{H}^1 \) (resp. of \( \mathbb{H}^2 \)). The spaces \( V \) and \( D(A) \) are dense and compactly embedded in \( H \) (Rellich theorem). Due to the zero mean condition we also have
\[
\int_T |Du(x)|^2 \, dx \geq \lambda \int_T |u(x)|^2 \, dx
\]
for every \( u \in V, \) for some positive constant \( \lambda \) (Poincaré inequality). Here
\[
|Du(x)|^2 = \sum_{i,j=1}^3 (\partial_j u_i(x))^2 \quad \text{(and} \quad \partial_j = \frac{\partial}{\partial x_j}).
\]
So we may endow \( V \) with the inner product
\[
\langle u, v \rangle_V = \sum_{i,j=1}^3 \int_T \partial_j u_i(x) \partial_j v_i(x) \, dx
\]
and the associated norm \( ||\cdot||_V. \)

Let \( A : D(A) \subset H \to H \) be the operator \( Au = -\Delta u \) (componentwise). There is a complete orthonormal system in \( H \) made by the eigenvectors \( h_{k,j} \) of the operator \( A \) (\( Ah_{k,j} = \lambda_{k,j} h_{k,j} \)). Since the spatial domain is the torus, we know the expressions of these eigenvectors with their eigenvalues. Indeed, let \( k = (k_1, k_2, k_3) \) with integer components, i.e. \( k \in \mathbb{Z}^3. \) We denote by \( \mathbb{Z}^3_+ \) the half space of \( \mathbb{Z}^3 \) defined as \( = \{k_1 > 0 \} \cup \{k_1 = 0, k_2 > 0 \} \cup \{k_1 = 0, k_2 = 0, k_3 > 0 \}. \) Then for any \( k \in \mathbb{Z}^3_+, \) there exist two unit vectors \( v_{k,1} \) and \( v_{k,2}, \) orthogonal to each other and belonging to the plane orthogonal to \( k. \) Then the (four sequences of) eigenvectors are
\[
h_{k,1}(x) = \frac{\sqrt{2}}{L^{3/2}} v_{k,1} \cos(\frac{2\pi}{L} k \cdot x), \quad h_{k,2}(x) = \frac{\sqrt{2}}{L^{3/2}} v_{k,2} \cos(\frac{2\pi}{L} k \cdot x)
\]
\[
h_{k,3}(x) = \frac{\sqrt{2}}{L^{3/2}} v_{k,1} \sin(\frac{2\pi}{L} k \cdot x), \quad h_{k,4}(x) = \frac{\sqrt{2}}{L^{3/2}} v_{k,2} \sin(\frac{2\pi}{L} k \cdot x)
\]
with eigenvalues
\[
\lambda_{k,1} = \lambda_{k,2} = \lambda_{k,3} = \lambda_{k,4} = \frac{(2\pi)^2}{L^2} |k|^2
\]
for any \( k \in \mathbb{Z}^3_+. \)
Hence, \( H = \text{span}\{h_{k,j} : j = 1, 2, 3, 4 \text{ and } k \in \mathbb{Z}^3_+\} \) and we set \( H_n = \text{span}\{h_{k,j} : j = 1, 2, 3, 4 \text{ and } k \in \mathbb{Z}^3_+, |k| \leq n\}; \) moreover, we denote by \( \pi_n \) the projection operator from \( H \) (or any subspace, as \( V \) or \( D(A) \)) onto \( H_n. \) The operators \( A \) and \( \pi_n \) commute.
We may take the Poincaré constant \( \lambda \) above equal to \((2\pi)^2/L^2\) (the first eigenvalue of \( A \)). Notice that we have

\[
\langle Au, u \rangle_H = \| u \|_V^2
\]

for every \( u \in D(A) \), so in particular

\[
\langle Au, u \rangle_H \geq \frac{(2\pi)^2}{L^2} |u|^2_H.
\]

Let \( V' \) be the dual of \( V \) with respect to the \( H \)-norm; with proper identifications we have \( V \subset H \subset V' \) with continuous injections, and the scalar product \( \langle \cdot, \cdot \rangle_H \) extends to the dual pairing \( \langle \cdot, \cdot \rangle_{V',V} \) between \( V \) and \( V' \) and to the dual pairing \( \langle \cdot, \cdot \rangle_{L^q,L^{q'}} \) between \( L^q \) and \( L^{q'} \) \((1/q + 1/q' = 1)\).

Let \( B(\cdot, \cdot) : V \times V \to V' \) be the bilinear operator defined as

\[
\langle w, B(u,v) \rangle_{V,V'} = \sum_{i,j=1}^{3} \int_{\mathcal{T}} u_i (\partial_i v_j) w_j \, dx
\]

for every \( u,v,w \in V \). By the incompressibility condition, we have

\[
\langle B(u,v), v \rangle = 0, \quad \langle B(u,v), w \rangle = -\langle B(u,w), v \rangle
\]

Using the latter relationship, by Hölder inequality we estimate

\[
|B(u,u)|_{V'} = \sup_{\|\psi\|_{V} \leq 1} |\langle B(u,u), \psi \rangle| \leq |u|^2_H,
\]

We list here a number of inequalities.

**Lemma 1**

\[
\langle A\Phi(u), u \rangle_H \geq \nu \| u \|_V^2
\]

**Proof.** We have

\[
\langle A\Phi(u), u \rangle_H = \langle \Phi(u), u \rangle_V = \int_{\mathcal{T}} \sum_{i,k=1}^{3} [\partial_k \Phi_i(u)] \partial_k u_i \, dx
\]

The estimate on \( \sum_{i,k=1}^{3} [\partial_k \Phi_i(u)] \partial_k u_i \) comes from [15].

**Lemma 2**

\[
\langle \Phi(u^{(1)}) - \Phi(u^{(2)}), u^{(1)} - u^{(2)} \rangle_H \geq \nu |u^{(1)} - u^{(2)}|^2_H
\]

**Proof.** The proof is by [15]. We rewrite it here, because we shall need it in Section 4.
Set $\sigma(|u|) = \nu + \tilde{\sigma}(|u|)$ with $\tilde{\sigma}' \geq 0$. Then

$$\left[\tilde{\sigma}(|u^{(1)}|)u^{(1)} - \tilde{\sigma}(|u^{(2)}|)u^{(2)}\right] \cdot [u^{(1)} - u^{(2)}]$$

$$= \tilde{\sigma}(|u^{(1)}|)|u^{(1)}|^2 + \tilde{\sigma}(|u^{(2)}|)|u^{(2)}|^2 - \tilde{\sigma}(|u^{(1)}|)u^{(1)} \cdot u^{(2)} - \tilde{\sigma}(|u^{(2)}|)u^{(1)} \cdot u^{(2)}$$

$$\geq \tilde{\sigma}(|u^{(1)}|)|u^{(1)}|^2 + \tilde{\sigma}(|u^{(2)}|)|u^{(2)}|^2$$

$$- \frac{1}{2} \tilde{\sigma}(|u^{(1)}|)|u^{(1)}|^2 - \frac{1}{2} \tilde{\sigma}(|u^{(2)}|)|u^{(2)}|^2 - \frac{1}{2} \tilde{\sigma}(|u^{(1)}|)|u^{(1)}|^2 - \frac{1}{2} \tilde{\sigma}(|u^{(2)}|)|u^{(1)}|^2$$

$$= \frac{1}{2} \tilde{\sigma}(|u^{(1)}|)|u^{(1)}|^2 + \frac{1}{2} \tilde{\sigma}(|u^{(2)}|)|u^{(2)}|^2 - \frac{1}{2} \tilde{\sigma}(|u^{(1)}|)|u^{(2)}|^2 - \frac{1}{2} \tilde{\sigma}(|u^{(2)}|)|u^{(1)}|^2$$

$$= \frac{1}{2} \left(\tilde{\sigma}(|u^{(1)}|) - \tilde{\sigma}(|u^{(2)}|)\right) \left|[u^{(1)}] - [u^{(2)}]\right| \left|[u^{(1)}] + [u^{(2)}]\right|$$

$$\geq 0$$

Hence

$$[\sigma(|u^{(1)}|)u^{(1)} - \sigma(|u^{(2)}|)u^{(2)}] \cdot [u^{(1)} - u^{(2)}]$$

$$= \nu|u^{(1)} - u^{(2)}|^2 + [\tilde{\sigma}(|u^{(1)}|)u^{(1)} - \tilde{\sigma}(|u^{(2)}|)u^{(2)}] \cdot [u^{(1)} - u^{(2)}]$$

$$\geq \nu|u^{(1)} - u^{(2)}|^2$$

Next lemma is crucial to prove uniqueness. Notice that the regularity $u \in L^5(0, T; L^5)$ is needed here. The weak solutions of the Navier–Stokes equations (deterministic or stochastic), which are known to exist, are not proved to have such a regularity; here the modified term $\Phi$ (with $b \geq 4$) plays its role. We remind that Prodi [13] proved uniqueness for the deterministic three dimensional Navier–Stokes equations, if $u \in L^{\frac{2q}{q-1}}(0, T; L^q)$ for some $3 < q \leq \infty$. For $q = 5$ the required regularity is $u \in L^5(0, T; L^5)$ and this implies uniqueness also in the Prouse model (see [15]).

**Lemma 3** If $u \in L^5$ and $v \in H$, then for any $\nu > 0$

$$|\langle B(u, v), A^{-1}\pi_m v \rangle| \leq \frac{\nu}{4} |v|_H^2 + C_B |u|_{L^5}^2 |\pi_m v|_V^2$$

$$|\langle B(v, u), A^{-1}\pi_m v \rangle| \leq \frac{\nu}{4} |v|_H^2 + C_B |u|_{L^5}^2 |\pi_m v|_V^2$$

for some positive constant $C_B$.

**Proof.** In [13], there is a very similar lemma, but with $v$ instead of $\pi_m v$ (here we consider any finite projection operator $\pi_m$). Following the lines of that proof, we get our result. ■

**Properties of $G$**

Let $G : H \to L(H)$ be a mapping with the properties

$$\|G(u)\|_{L^2(H)}^2 \leq \lambda_0 |u|_H^2 + \rho$$

(6)

5
and
\[ \left\| A^{−1/2}(G(v) − G(z)) \right\|_{HS(H)}^2 \leq L_G |v − z|^2_H, \] (7)

Here \( \|T\|_{HS(H)} \) is the Hilbert-Schmidt norm of an operator in \( H \), defined as
\[ \|T\|_{2}^{HS(H)} = \sum_{j=1}^{4} \sum_{k \in \mathbb{Z}^3_+} |T^h_{k,j}|^2_H \]

Now, we project equation (1) onto the space of divergence free vectors fields; both the \( \nabla \)-terms disappear, as when we deal with the Navier–Stokes equations (see, e.g., [16]). Then, we obtain an evolution equation (still formally), which with our notations is
\[ du + [A\Phi(u) + B(u, u)] dt = G(u) dw, \quad u(0) = u_0 \] (8)

From now on, \( \Phi \) will be assumed to satisfy (2) for a given \( b ≥ 4 \).

The rigorous interpretation of this equation will be given in the sequel, but for the time being let us at least write it in weak form
\[ \langle u_t, \psi \rangle_H + \int_0^t \langle \Phi(u_s), A\psi \rangle_{L^1 + 1 \times 1 + b} ds - \int_0^t \langle B(u_s, \psi), u_s \rangle_{L^1 + 1 \times 4} ds \]
\[ = \langle u_0, \psi \rangle_H + \int_0^t \langle G(u_s) dw_s, \psi \rangle_H \] (9)

with \( \psi \in \mathcal{D}^\infty \) and \( 0 < t < \infty \).

We assume that \( w \) is a cylindrical Wiener process in \( H \) (see, e.g., [17]). We can represent it as follows. Suppose we are given a Brownian stochastic basis, i.e. a probability space \( (\mathcal{W}, \mathcal{F}, Q) \), a filtration \( (\mathcal{F}_t)_{t ≥ 0} \) and a sequence \( \{\beta_{k,j}(t)\}_{k,j} \) of independent Brownian motions on \( (\mathcal{W}, \mathcal{F}, (\mathcal{F}_t)_{t ≥ 0}, Q) \). Namely, for \( k \in \mathbb{Z}^3_+ \) and \( j = 1, 2, 3, 4 \), the real valued processes \( \beta_{k,j}(t) \) are independent, adapted to \( (\mathcal{F}_t)_{t ≥ 0} \), continuous for \( t ≥ 0 \) and null at \( t = 0 \), with increments \( \beta_{k,j}(t) - \beta_{k,j}(s) \) that are \( N(0, t - s) \)-distributed and independent of \( \mathcal{F}_s \). Then
\[ w(t) = \sum_{j=1}^{4} \sum_{k \in \mathbb{Z}^3_+} \beta_{k,j}(t) h_{k,j} \] (10)

is a cylindrical Wiener process in \( H \).

The convergence of this series requires proper distributional topologies. The stochastic integral in equation (9) is well defined under the Hilbert-Schmidt assumption made on \( G \) (see [17] for details).
3 Well posedness

3.1 Concepts of solution

Consider the abstract (formal) stochastic evolution equation (8) and its weak formulation over test functions (9). We have

\[ \int_0^t \left| \langle \Phi(u_s), A\psi \rangle \right|_{L^{1+b}} \, ds \leq \int_0^t \left| \Phi(u_s) \right|_{L^{1+b}} \, |A\psi|_{L^{1+b}} \, ds \]

\[ \leq C_1 \int_0^t (1 + |u_s|_{L^{1+b}}) \, ds \]

because

\[ |\Phi(u)|_{L^{1+b}} = \int_T |\Phi(u(x))|^{1+b} 1_{\{|u(x)| \leq K\}} \, dx + \int_T |\Phi(u(x))|^{1+b} 1_{\{|u(x)| > K\}} \, dx \]

\[ \leq K^{1+b} \int_T |\sigma(|u(x)|)|^{1+b} 1_{\{|u(x)| \leq K\}} \, dx + \int_T (a_2|u(x)|^b)^{1+b} \, dx \]

\[ \leq C_2(1 + \int_T |u(x)|^{1+b} \, dx) \] (11)

since \( \sigma \in C^1 \) implies that \( \sigma \) is bounded on \([0, K]\). Then, in equation (9) the term \( \int_0^t \langle \Phi(u_s), A\psi \rangle \, ds \) is well defined for functions \( u \) that live in \( L^{1+b}(0, T; L^{1+b}) \), \( T > 0 \).

Moreover,

\[ \int_0^t \left| \langle B(u_s, \psi), u_s \rangle \right|_{L^{1+b}} \, ds \leq \int_0^t |u_s|_{L^{1+b}}^2 \, \|\psi\|_{V} \, ds \leq C_3 \int_0^t |u_s|_{L^{1+b}}^2 \, ds \] (12)

Hence, in equation (9) the term \( \int_0^t \langle B(u_s, \psi), u_s \rangle \, ds \) is well defined for functions \( u \) that live in \( L^2(0, T; L^b) \).

We conclude, in both cases, that given \( b \geq 4 \) the regularity \( u \in L^{1+b}(0, T; L^{1+b}) \) is enough to define these quantities. Moreover, from now on the duality pairing for these two terms has to be understood in the sense above specified (as written also in equation (9)).

As in the deterministic case, strong continuity of trajectories in \( H \) is an open problem. There will be strong continuity in weaker spaces (like \( W^{-2-\theta, 1+b} \)), and a uniform bound in \( H \). Let \( H_\sigma \) be the space \( H \) with the weak topology.

Since

\[ C([0, T]; W^{-2-\theta, 1+b}) \cap L^\infty(0, T; H) \subset C([0, T]; H_\sigma) \]

then the trajectories of the solutions will be at least weakly continuous in \( H \) (see [10] pg. 263).

Given a separable Banach space \( W' \) (it will be \( W' = W^{-2-\theta, 1+b} \)), let us set

\[ \Omega = C([0, \infty); W') \]
and denote by \((\xi_t)_{t \geq 0}\) the canonical process \((\xi_t(\omega) = \omega_t)\), by \(F\) the Borel \(\sigma\)-algebra in \(\Omega\) and by \(F_t\) the \(\sigma\)-algebra generated by the events \((\xi_s \in A)\) with \(s \in [0, t]\) and \(A \in \mathcal{B}(\mathcal{W}')\).

**Definition 4 (solution to the martingale problem)** Given a probability measure \(\mu_0\) on \(H\), we say that a probability measure \(P\) on \((\Omega, F)\) is a solution of the martingale problem associated to equation (8) with initial law \(\mu_0\) if

\[
[MP1] \quad \text{for every } T > 0 \\
P \left( \sup_{t \in [0, T]} |\xi_t|_H + \int_0^T \|\xi_s\|^2_V \, ds + \int_0^T |\xi_s|_{1+b} \, ds < \infty \right) = 1
\]

\[
[MP2] \quad \text{for every } \psi \in \mathcal{D}^\infty \text{ the process } M^\psi_t \text{ defined } \text{P.-a.s on } (\Omega, F) \text{ as}
\]

\[
M^\psi_t := (\xi_t, \psi)_H - (\xi_0, \psi)_H - \int_0^t (\Phi(\xi_s), A\psi) \, ds + \int_0^t (B(\xi_s, \psi), \xi_s) \, ds
\]

is square integrable and \( (M^\psi_t, F_t, P) \) is a continuous martingale with quadratic variation

\[
[M^\psi_t]_t = \int_0^t |G(\xi_s)\psi|_H^2 \, ds
\]

\[
[MP3] \quad \mu_0 = \Pi_0 P, \text{ where } \Pi_0 \text{ denotes the restriction on } F_0.
\]

**Remark 5** A solution of the martingale problem is also a weak solution. The definition of weak solution is as follows: there exists a Brownian stochastic basis \( (\mathcal{W}, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, Q, (\beta_i(t))_{t \geq 0, k, j}) \) and a \(\mathcal{W}'\)-valued process \(u\) on \((\mathcal{W}, \mathcal{F}, Q)\) such that

\[
[WM1] \quad u \text{ is a continuous adapted process in } \mathcal{W}' \text{ and}
\]

\[
u(., \omega) \in L^\infty(0, T; H) \cap L^2(0, T; V) \cap L^{1+b}(0, T; L^{1+b}) \quad Q\text{-a.s.}
\]

\[
\text{for every } T > 0
\]

\[
[WM2] \quad (9) \text{ is satisfied } Q\text{-a.s.}
\]

\[
[WM3] \quad u(0) \text{ has law } \mu_0.
\]

Finally, in this context, we call strong solution a process \(u\) satisfying the three above properties on any a priori given stochastic basis.
3.2 Main result

**Theorem 6** Let \( \mu \) be a measure on \( H \) such that \( m_p := \int_H |v|_H^p \mu(\text{d}v) < \infty \) for some \( p > 2 \). Then there exists one and only one solution to the martingale problem \( (9) \) with initial condition \( \mu \).

Moreover, two strong solutions on the same Brownian stochastic basis coincide a.s.

**Proof. Step 1** (Galerkin approximations). Let

\[
\left( \mathcal{W}, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, Q, (\beta_{k,j}(t))_{t \geq 0; k,j} \right)
\]

be a Brownian stochastic basis supporting also an \( \mathcal{F}_0 \)-measurable r.v. \( u_0 : W \to H \) with law \( \mu \). For every \( n \), let \( u_0^n := \pi_n u_0 \) and consider the Galerkin system

\[
\begin{align*}
    du^n_t + [A\Phi(u^n_t) + \pi_n B(u^n_t, u^n_t)] dt &= \pi_n G(u^n_t) \text{ d}w_t, \\
    u^n(0) &= u_0^n
\end{align*}
\]  

(13)

obtained by applying the projection operator \( \pi_n \) to both sides of equation \( 8 \). (Notice that \( \pi_n A \Phi(u^n_t) = A \Phi(u^n_t) \).) Equation (13) is a stochastic ordinary equation in the finite-dimensional Hilbert space \( H_n \).

Local existence and uniqueness (on a random time interval) is classical, since the nonlinearities are locally Lipschitz continuous (see, e.g., [14]). Global existence is then a consequence of the a priori estimates given in Appendix 1. There, defined \( \tau^n_R = \inf \{ t \geq 0 : |u^n_t|^2_H = R \} \) we shall prove that, for any \( T > 0 \)

\[
    E \sup_{0 \leq t \leq T} |u^n_{t \wedge \tau^n_R}|_H^p \leq C_1
\]  

(14)

\[
    E \int_0^T \|u^n_{s \wedge \tau^n_R}\|_V^2 1_{\{s < \tau^n_R\}} ds \leq C_2
\]  

(15)

\[
    E \int_0^T \|u^n_{s \wedge \tau^n_R}\|_{L_{1+b}} 1_{\{s < \tau^n_R\}} ds \leq C_3
\]  

(16)

for some positive constants \( C_1 = C_1(p, T, \lambda_0, \rho, m_p) \), \( C_2 = C_2(T, \lambda_0, \rho, m_2) \), \( C_3 = C_3(a_1, C_1, C_2) \), independent of \( n \) and \( R \).

Now, assume first that the initial velocity is bounded: \( |u_0|_H \leq K \). Take \( R > K \); so \( \tau^n_R > 0 \) Q-a.s.. The solution \( u^n_t \) to the Galerkin system \( 13 \) is defined at least in the time interval \( [0, \tau^n_R) \). Since we know from \( 14 \) that

\[
    E \sup_{t \in [0,T]} |u^n_{t \wedge \tau^n_R}|_H^2 \leq \tilde{C}_1
\]

for some constant \( \tilde{C}_1 = C_1^{2/p} \) independent of \( n \) and \( R \), we have

\[
    E \left( 1_{\{\tau^n_R < T\}} |u^n_{T \wedge \tau^n_R}|_H^2 \right) \leq \tilde{C}_1
\]
for $T > 0$ fixed. Moreover

$$Q(\tau^n_R < T) = E1_{\{\tau^n_R < T\}} = \frac{1}{R}E \left(1_{\{\tau^n_R < T\}}|u^n_{T \wedge \tau^n_R}|^2_H\right)$$

because $|u^n_{T \wedge \tau^n_R}|^2_H = R$ on the set $\{\tau^n_R < T\}$. Hence

$$Q(\tau^n_R < T) \leq \frac{\tilde{C}_1}{R}$$

Notice that $\tau^n_R > \tau^n_{\tilde{R}}$ for $\tilde{R} > R$. Therefore, setting $\tau^n_\infty = \sup_{R > K} \tau^n_R$ the process $u^n_t$ is defined for $t \in [0, \tau^n_\infty)$. But we have

$$Q(\tau^n_\infty < T) \leq Q(\tau^n_R < T) \leq \frac{\tilde{C}_1}{R} \quad \forall R$$

Hence

$$Q(\tau^n_\infty < T) = 0$$

and finally we conclude that $u^n_t$ is a solution for $t \in [0, T)$. Since $T$ has been chosen arbitrarily, we conclude that the Galerkin solution is defined on any finite time interval.

For a general initial velocity satisfying the assumption of Theorem 6 we proceed as follows. Let $W_K \in \mathcal{F}$ be defined as $W_K = \{|u_0|_H \leq K\}$; we have $Q(\cup_K W_K) = Q(|u_0|_H < \infty) = 1$. Define $u_{0K}$ as $u_0$ on $W_K$ and 0 otherwise. Let $u^n_{tK}$ be the unique solution to the Galerkin system $(\forall t \geq 0)$ with initial condition $u_{0K}$. If $\tilde{K} > K$, then

$$Q \{W_K \cap \{u^n_{tK} = u^n_{t\tilde{K}} \forall t \geq 0\}\} = Q\{W_K\}$$

We may uniquely define a process $u^n_\infty$ on $W' = \cup_K W_K$ as $u^n_\infty = u^n_{tK}$ on $W_K$. Looking at the Galerkin equation in the integral form, it is clear that $u^n_\infty$ solves the equation on $W'$. But $Q(W') = 1$. Thus we have proved the existence of a global solution to the Galerkin system for any initial velocity with $m_p < \infty$ for some $p > 2$. This solution is a continuous adapted Markov process in $H_\alpha$ (uniqueness holds for the Galerkin problem; it can be checked directly or obtained as a byproduct of next Step 5).

Hence we have proved that, for any $T < \infty$

$$E \sup_{0 \leq t \leq T} |u^n_t|^p_H \leq C_1$$

$$E \int_0^T \|u^n_s\|^2_V ds \leq C_2$$

$$E \int_0^T |u^n_s|^{1+b} ds \leq C_3$$
From these estimates, we also get the following one. Given \( \psi \in D^\infty \) and \( \varepsilon \in (0, 2) \), we have

\[
E \left| \int_0^t \langle \pi_n G(u^n_s) dw_s, \psi \rangle \right|^{2+\varepsilon} \leq \left( E \left| \int_0^t \langle \pi_n G(u^n_s) dw_s, \psi \rangle \right|^{4} \right)^{2+\varepsilon}
\]

\[
= \left( C \left( E \int_0^t |\pi_n G(u^n_s)\psi|^2_H ds \right)^2 \right)^{2+\varepsilon}
\]

by Gaussianity

\[
\leq C |\psi|^{2+\varepsilon}_H \left( \left( E \int_0^t \|G(u^n_s)\|^2_{H^S(H)} ds \right)^2 \right)^{2+\varepsilon}
\]

\[
\leq C |\psi|^{2+\varepsilon}_H E \int_0^t (|u^n_s|^{2+\varepsilon}_H + 1) ds
\]

Here (and in the following) \( C \) denotes different positive constants, independent of \( n \). Taking \( 2 + \varepsilon \leq p \) and bearing in mind (14), we conclude that for any finite \( t \)

\[
\sup_n E \left| \int_0^t \langle \pi_n G(u^n_s) dw_s, \psi \rangle \right|^{2+\varepsilon} < \infty \quad (17)
\]

Here the limitation \( \varepsilon < 2 \) can be easily removed, but in the sequel it will be enough to consider a positive quantity \( \varepsilon \) as small as we want.

**Step 2** (time regularity and reformulation in path space). In view of the time regularity, equation (13) has the form

\[
u^n_t = u^n_0 + I^n_t + J^n_t + K^n_t
\]

where

\[
I^n_t = - \int_0^t A\Phi(u^n_s) ds
\]

\[
J^n_t = - \int_0^t \pi_n B(u^n_s, u^n_s) ds
\]

\[
K^n_t = \int_0^t \pi_n G(u^n_s) dw_s
\]

For the first term we have

\[
\|I^n\|_{W^{1+\frac{1}{2}, \frac{1}{2}}(0, T; W^{-2, 1+\frac{1}{2}})} \leq C \int_0^T |A\Phi(u^n_s)|_{W^{-2, 1+\frac{1}{2}}} ds
\]

\[
\leq C \int_0^T \|\Phi(u^n_s)\|_{L^{1+\frac{1}{2}}} ds
\]

\[
\leq C(T + \int_0^T |u^n_s|_{L^{1+\frac{1}{2}}} ds)
\]
according to (11).

For $J_n$, using (5) we have
\[
\|J_n\|^2_{W^{1,2}(0,T;V')} \leq C \int_0^T |B(u^n_s, u^n_s)|^2_{V'} \, ds \\
\leq C \int_0^T \|u^n_s\|^2_{L^4} \, ds \\
\leq C(T + \int_0^T \|u^n_s\|_{L^{1+b}}^{1+b} \, ds)
\]

Finally, for every $q > 1$, $\alpha \in \left(0, \frac{1}{2}\right)$, $T > 0$, we have (see, e.g., [9])
\[
E\|K^n\|^q_{W^{\alpha,q}(0,T;H)} \leq C E \|\pi_n G(u^n_s)\|^q_{HS(H)} ds
\]
and by (6) and the mean estimates of the previous step we conclude that
\[
E\|K^n\|^p_{W^{\alpha,p}(0,T;H)} \leq \tilde{C} (\text{independent of } n \text{ and } p > 2 \text{ as stated in Theorem 6.})
\]

Therefore, for $\alpha \in \left(0, \frac{1}{2}\right)$
\[
u^n \in W^{1,1+b} \left(0, T; \mathbb{W}^{-2,1+b}\right) + W^{1,2} \left(0, T; V'\right) + W^{\alpha,p} \left(0, T; H\right)
\]
in mean. Notice that $H \subset V' \subset \mathbb{W}^{-2,1+b}$, $W^{1,2}(0,T) \subset W^{\alpha,1+b}(0,T)$ and $W^{\alpha,p}(0,T) \subset W^{\alpha,1+b}(0,T)$.

We conclude that, in mean
\[
u^n \in W^{\alpha,1+b} \left(0, T; \mathbb{W}^{-2,1+b}\right)
\]

Under the embedding $H_n \subset H$, we have that $(u^n_t)_{t \geq 0}$ is a continuous adapted process in $H$, so it defines a measure $P_n$ on $C([0, \infty); H)$, and thus on $(\Omega, F)$.

Actually, $P_n$ is concentrated on $C([0, \infty); H_n)$. For every $\alpha \in \left(0, \frac{1}{2}\right)$, $T > 0$, the above estimates may be rewritten as
\[
E^{P_n} \left[ \sup_{t \in [0,T]} |\xi|^p_{H} + \int_0^T ||\xi_s||^2_{V'} \, ds + \int_0^T ||\xi_s||_{L^{1+b}}^{1+b} \, ds \right] \leq C_4 \left(T, \lambda_0, \rho, m_p, b\right)
\]
and
\[
E^{P_n} \left[ ||\xi||_{W^{\alpha,1+b}(0,T;\mathbb{W}^{-2,1+b})} \right] \leq C_5 \left(T, \lambda_0, \rho, m_p, b\right)
\]
for any $n$.

Relationships (14)-16 may be rewritten in a similar way.

**Step 3** (tightness). Use now Chebyshev inequality and (15), (19). Then, given $\alpha \in \left(0, \frac{1}{2}\right)$, $T > 0$, for every $\varepsilon > 0$ there is a bounded set $B_\varepsilon$ such that
\[
B_\varepsilon \subset L^2 \left(0, T; V\right) \cap W^{\alpha,1+b} \left(0, T; \mathbb{W}^{-2,1+b}\right)
\]
Given \( Q \) converges on \( WC \) measures on \([MP1]\) in the definition of solution to the martingale problem.  

\[
\inf_n P_n (B_\varepsilon) > 1 - \varepsilon 
\]

The space \( L^2(0, T; V) \cap W^{\alpha,1+\frac{b}{p}}(0, T; \mathbb{W}^{-2,1+\frac{b}{p}}) \) is compactly embedded in \( L^{1+\frac{b}{p}}(0, T; H) \) (see, e.g., Theorem 2.1 in \([9]\)). Hence, for every \( \varepsilon > 0 \) there is a compact set \( K_\varepsilon \) such that

\[
K_\varepsilon \subset L^{1+\frac{b}{p}}(0, T; H) \quad \text{and} \quad \inf_n P_n (K_\varepsilon) > 1 - \varepsilon
\]

Now, take any separable Banach space \( \mathbb{W}' \) such that \( \mathbb{W}^{-2,1+\frac{b}{p}} \) is compactly embedded in \( \mathbb{W}' \); e.g. \( \mathbb{W}' = \mathbb{W}^{-2,\theta,1+\frac{b}{p}} \) for some \( \theta > 0 \). Notice that all the spaces \( W^{1,1+\frac{b}{p}}(0, T), W^{1,2}(0, T), W^{\alpha,p}(0, T) \) (for \( \alpha p > 1 \)) are continuously embedded into \( C([0, T]) \). Hence, the space of vectors with the regularity specified by \([13]\) is compactly embedded in \( C([0, T]; \mathbb{W}') \) (see, e.g., Theorem 2.2 in \([9]\)). From the boundedness in the mean of \( J^n \) in \( W^{1,1+\frac{b}{p}}(0, T; \mathbb{W}^{-2,1+\frac{b}{p}}) \) of \( J^n \) in \( W^{1,2}(0, T; V') \) and of the law of the Wiener process in \( W^{\alpha,p}(0, T; H) \) for every \( \alpha \in (\frac{1}{p}, \frac{1}{2}) \), again by Chebyshev inequality and compact embedding we obtain that for every \( \varepsilon > 0 \) there exists a compact set \( K'_\varepsilon \) such that

\[
K'_\varepsilon \subset C([0, T]; \mathbb{W}') \quad \text{and} \quad \inf_n P_n (K'_\varepsilon) > 1 - \varepsilon
\]

Therefore the family of measures \( \{P_n\} \) is tight in \( L^{1+\frac{b}{p}}(0, T; H) \) and in \( C([0, T]; \mathbb{W}') \), with their Borel \( \sigma \)-fields. Hence there exists a probability measure \( P \) on

\[
C([0, T]; \mathbb{W}') \cap L^{1+\frac{b}{p}}(0, T; H)
\]

that is the weak limit in such spaces of a subsequence \( \{P_{n_k}\} \).

**Step 4** (\( P \) is a solution to the martingale problem). From the uniform estimates on \( \{P_{n_k}\} \) in \( L^2(0, T; V), L^\infty(0, T; H) \) and \( L^{1+b}(0, T; \mathbb{L}^{1+b}) \) we may deduce that \( P \) gives probability one to each one of these spaces and has bounds in the mean similar to those uniform of \( P_{n_k} \). This way we have checked property \([MP1]\) in the definition of solution to the martingale problem.

Concerning \([MP3]\), we have \( P_{n_k} \to P \) as weak convergence of probability measures on \( C([0, T]; \mathbb{W}') \); in particular \( \Pi_0 P_{n_k} \to \Pi_0 P \) as probability measures on \( \mathbb{W}' \). But \( \Pi_0 P_{n_k} \) is the law of \( \pi_{n_k} u_0 \), which converges to \( \mu \) since \( \pi_{n_k} u_0 \) converges \( Q \)-a.s. to \( u_0 \). Hence \( \Pi_0 P \) is \( \mu \).

Finally, let us check property \([MP2]\). We proceed as in \([7]\) (Sec. 8.4) or in \([9]\).

Given \( \psi \in \mathcal{D}^\infty \), we have to prove that for every \( t > s \geq 0 \) and every bounded \( F_s \)-measurable random variable \( Z \), we have

\[
E^P \left[ \left( M^\psi_t \right)^2 \right] < \infty
\]

\[
E^P \left[ \left( M^\psi_t - M^\psi_s \right) Z \right] = 0
\]

\[
E^P \left[ \left( \left( M^\psi_t \right)^2 - \varsigma_t \right) - \left( \left( M^\psi_s \right)^2 - \varsigma_s \right) \right] Z = 0
\]
where \( \zeta_t := \int_0^t |G(\xi_s)\psi|^2_H ds \). Defined

\[
M^{\psi,n_k}_t := \langle \xi_t, \pi_{n_k}\psi \rangle_H - \langle \xi_0, \pi_{n_k}\psi \rangle_H - \int_0^t \langle \Phi(\xi_s), \pi_{n_k}A\psi \rangle ds
\]

\[
+ \int_0^t \langle B(\xi_s, \pi_{n_k}\psi), \xi_s \rangle ds ,
\]

for the measure \( P_{n_k} \) we know (see, e.g., [7] Sec 8.4) that \( \left( M^{\psi,n_k}_t, F_t, P_{n_k} \right) \) is a square integrable martingale with quadratic variation

\[
[M^{\psi,n_k}]_t = \zeta^{n_k}_t = \int_0^t |\pi_{n_k}G(\xi_s)\psi|^2_H ds
\]

Thus

\[
E^{P_{n_k}} \left[ \left( M^{\psi,n_k}_t - M^{\psi,n_k}_s \right) Z \right] = 0 \tag{20}
\]

\[
E^{P_{n_k}} \left[ \left( (M^{\psi,n_k}_t)^2 - \zeta^{n_k}_t \right) - \left( (M^{\psi,n_k}_s)^2 - \zeta^{n_k}_s \right) \right] Z = 0 \tag{21}
\]

Moreover, by (17) we know that there exists some \( \varepsilon > 0 \) such that

\[
\sup_k E^{P_{n_k}} \left| M^{\psi,n_k}_t \right|^{2+\varepsilon} < \infty \tag{22}
\]

Now, let us consider the limit as \( k \to \infty \).

We know that \( P_{n_k} \) converges weakly to \( P \); then by Skorohod theorem there exists a stochastic basis \((\tilde{\Omega}, \tilde{F}, \tilde{F}_t, \tilde{P})\) and, on this basis, there exist \( L^{1+\frac{1}{2}}(0,T;H) \cap C([0,T];W') \)-valued random variables \( \tilde{u}, \tilde{u}_{n_k} \) such that \( \tilde{u} \) has the same law of \( u \), \( \tilde{u}_{n_k} \) has the same law of \( u_{n_k} \) and \( \tilde{u}_{n_k} \to \tilde{u} \) \( \tilde{P} \)-a.s. in the \( L^{1+\frac{1}{2}}(0,T;H) \cap C([0,T];W') \)-norm.

Define

\[
\tilde{M}^{\psi,n_k}_t := \langle \tilde{u}_{n_k}, \psi \rangle_H - \langle \tilde{u}_0, \psi \rangle_H - \int_0^t \langle \Phi(\tilde{u}_{n_k}), A\psi \rangle ds
\]

\[
+ \int_0^t \langle B(\tilde{u}_{n_k}, \pi_{n_k}\psi), \tilde{u}_{n_k} \rangle ds
\]

Then (20)- (22) hold true (with the obvious change of notation).

If we prove that \( \tilde{M}^{\psi,n_k}_t \to \tilde{M}^{\psi} \) \( \tilde{P} \)-a.s. as \( k \to \infty \), then by the equiboundedness relationship (22) we obtain that \( \tilde{M}^{\psi,n_k}_t \to \tilde{M}^{\psi} \) in \( L^1(\tilde{\Omega}, \tilde{P}) \) and in \( L^2(\tilde{\Omega}, \tilde{P}) \) and \( \zeta^{n_k}_t \to \zeta_t \) in \( L^1(\tilde{\Omega}, \tilde{P}) \). This concludes our proof. So, we have to prove a \( \tilde{P} \)-a.s. convergence for each term in the definition of \( \tilde{M}^{\psi,n_k}_t \).

It is trivial that \( \tilde{P} \)-a.s.

\[
\langle \tilde{u}_{n_k}, \psi \rangle_H - \langle \tilde{u}_0, \psi \rangle_H \to \langle \tilde{u}, \psi \rangle_H - \langle \tilde{u}_0, \psi \rangle_H
\]
Notice that there appears the scalar product in $H$ and not the duality pairing $\langle \tilde{u}_t, \psi \rangle_{W', W}$, because the limit process $u$ belongs to $C([0,T]; H_x)$ with probability one.

Moreover, there exists a subsequence (we do not write that we consider a subsequence, since we shall pass through subsequences a few times from now on) such that

$$\hat{P} - \text{a.s.} \quad \tilde{u}^{n_k}_s(x) \to \tilde{u}_s(x) \quad \text{for a.e. } (s, x) \in [0, T] \times T$$

We also have, for any $k$

$$\tilde{E} \int_0^T |\tilde{u}^{n_k}_s|_{L^2 \times T}^b \, ds \leq C_3$$

Keeping in mind (11) and (5), it follows that $\Phi(\tilde{u}^{n_k})$ is equibounded in $L^{1+\frac{1}{2}}(\tilde{\Omega} \times [0, T] \times T)$ and $\langle B(\tilde{u}^{n_k}, \pi_n, \psi), \tilde{u}^{n_k} \rangle$ is equibounded in $L^{\frac{1+\frac{1}{2}}{2}}(\tilde{\Omega} \times [0, T])$ respectively. First, we get that $\Phi(\tilde{u}^{n_k}(x)) \to \Phi(\tilde{u}_s(x)) \hat{P}\text{-a.s.}$ and for a.e. $(s, x)$. By the equiboundedness of $\Phi(\tilde{u}^{n_k}(x))$ in $L^{1+\frac{1}{2}}(\tilde{\Omega} \times [0, T] \times T)$ it follows that $\Phi(\tilde{u}^{n_k})$ converges to $\Phi(\tilde{u})$ in $L^1(\tilde{\Omega} \times [0, T] \times T)$; we get

$$\tilde{E} \int_0^T \langle \Phi(\tilde{u}^{n_k}), A\psi \rangle ds = \tilde{E} \int_0^T \langle \Phi(\tilde{u}_s), A\psi \rangle ds$$

Hence a subsequence of $\int_0^T \langle \Phi(\tilde{u}^{n_k}), A\psi \rangle ds$ converges $\hat{P}\text{-a.s.}$

On the other hand, another (sub)subsequence can be extracted so that

$$\hat{P} - \text{a.s.} \quad \tilde{u}^{n_k}_s \to \tilde{u}_s \quad \text{in } L^4 \text{ for a.e. } s$$

Then, by triangle inequality $\langle B(\tilde{u}^{n_k}_s, \pi_n, \psi), \tilde{u}^{n_k}_s \rangle \to \langle B(\tilde{u}_s, \psi), \tilde{u}_s \rangle \hat{P}\text{-a.s.}$ and for a.e. $s$. By the equiboundedness of $\langle B(\tilde{u}^{n_k}_s, \pi_n, \psi), \tilde{u}^{n_k}_s \rangle$ in $L^{\frac{1+\frac{1}{2}}{2}}(\tilde{\Omega} \times [0, T])$, we conclude as above that there exists a subsequence of $\int_0^T \langle B(\tilde{u}^{n_k}_s, \psi), \tilde{u}^{n_k}_s \rangle ds$ converging $\hat{P}\text{-a.s.}$.

Considering the convergence of a suitable subsequence (the last extracted), we get that (20)–(22) in the limit allow to conclude the proof.

**Step 5** (uniqueness). Let $u^{(i)}$, $i = 1, 2$, be two strong solutions on the same Brownian stochastic basis. We are going to prove pathwise uniqueness, which implies uniqueness of martingale solutions.

Let

$$v_t = u^{(1)}_t - u^{(2)}_t, \quad v^m_t = \pi_m v_t$$

$$\theta_t = 2CB \left[ |u^{(1)}_t|^5_{L^5} + |u^{(2)}_t|^5_{L^5} \right] + L_G,$$

with $C_B$ as in Lemma 3 and $L_G$ as in (7).

We have

$$de^{-\int_0^t \theta_s ds} |v^m_t|^2_{V^m} = -\theta_t e^{-\int_0^t \theta_s ds} |v^m_t|^2_{V^m} \, dt + e^{-\int_0^t \theta_s ds} d|v^m_t|^2_{V^m},$$

$$d\tilde{E} \int_0^T |v^m_t|^2_{V^m} \, ds \leq C_3$$

Thus,
By Itô formula, the last differential is
\[
\begin{align*}
    d|v^m|^2 &= -2\langle \pi_m[f(u^{(1)}) - f(u^{(2)})], v^m \rangle_H dt - 2\langle \pi_m[B(u^{(1)}, v) + B(v, u^{(2)})], A^{-1}v^m \rangle_H dt \\
    &\quad + 2\langle \pi_m[G(u^{(1)}) - G(u^{(2)})] dv, A^{-1}v^m \rangle_H + \left\| \pi_m A^{-1/2}[G(u^{(1)}) - G(u^{(2)})] \right\|_{HS(H)}^2 dt
\end{align*}
\]

We estimate some terms as follows. By Lemma 3
\[
2|\langle \pi_m[B(u^{(1)}, v) + B(v, u^{(2)})], A^{-1}v^m \rangle| = 2|\langle B(u^{(1)}, v) + B(v, u^{(2)})], A^{-1}v^m \rangle| \leq \nu|v^m|^2_H + 2C_B \left( |u^{(1)}|_{L^2} + |u^{(2)}|_{L^2} \right) |v^m|^2_H,
\]

By Lemma 2
\[
2\langle \pi_m[f(u^{(1)}) - f(u^{(2)})], v^m \rangle_H = 2\langle \pi_m[u^{(1)} - u^{(2)}], v^m \rangle_H - 2\epsilon^m \geq 2\nu|v^m|^2_H - 2\epsilon^m
\]

where \( \int_0^T \epsilon_t^m dt \leq C(|u^{(1)}|_{L^2(0,T; L^2)}, |u^{(2)}|_{L^2(0,T; L^2)}) \) and \( \lim_{m \to \infty} \int_0^T \epsilon_t^m dt = 0. \)

By (7)
\[
\left\| \pi_m A^{-1/2}[G(u^{(1)}) - G(u^{(2)})] \right\|_{HS(H)}^2 \leq \left\| A^{-1/2}[G(u^{(1)}) - G(u^{(2)})] \right\|_{HS(H)}^2 \leq L_G |v|^2_H,
\]

Now, we integrate in time equation (2) and use the above estimates, obtaining
\[
\begin{align*}
    &e^{-\int_0^T \theta_t ds} |v_t^m|^2_H + \nu \int_0^T e^{-\int_0^s \theta_r ds} (2 |v^m|^2_H - |v_t^m|^2_H) dt \\
    \leq & |v_0^m|^2_H + 2 \int_0^T e^{-\int_0^s \theta_r ds} \epsilon_t^m dt + \int_0^T L_G e^{-\int_0^s \theta_r ds} (|v_t^m|^2_H - |v_t^m|^2_{H^2}) dt \\
    &\quad + 2 \int_0^T \langle \pi_m[G(u_t^{(1)}) - G(u_t^{(2)})] dw_t, A^{-1}v_t^m \rangle_H
\end{align*}
\]

We can take the limit as \( m \to \infty \) in every term. We get
\[
\begin{align*}
    &e^{-\int_0^T \theta_t ds} |v_T^m|^2_H + \nu \int_0^T e^{-\int_0^s \theta_r ds} |v_t|^2_H dt \\
    \leq & |v_0|^2_H + 2 \int_0^T \langle [G(u_t^{(1)}) - G(u_t^{(2)})] dw_t, A^{-1}v_t \rangle_H
\end{align*}
\]

Hence
\[
E \left[ e^{-\int_0^T \theta_t ds} |v_T|^2_H \right] + \nu E \left[ \int_0^T e^{-\int_0^s \theta_r ds} |v_t|^2_H dt \right] \leq E |v_0|^2_H.
\]
When the initial conditions of $u^{(i)}$ coincide, we deduce

$$
\int_0^T e^{-\int_0^T \theta_s ds} |v_t|_H^2 dt = 0
$$

with probability one. Since $\int_0^T \theta_s ds < \infty$ a.s., we have $v = 0$ a.s., as considering $v$ as a measurable function of $t$ with values in $H$. This implies that with probability one $u^{(1)} = u^{(2)}$, where the equality holds in $L^\infty(0,T; H)$. ■

### 3.3 Markov and Feller property

**Lemma 7** Let $u^n_0, u_0$ be initial data satisfying the assumption of Theorem 6 and let $(u^n_t)_{t \geq 0}$ and $(u_t)_{t \geq 0}$ be the corresponding strong solutions on the same given Brownian stochastic basis. If $E|u^n_0 - u_0|_V^2 \to 0$, then for every $T > 0$, $(u^n_t)_{t \geq 0}$ converges to $(u_t)_{t \geq 0}$ in probability on $[0,T] \times \Omega$ in the topology of $H$, and $u^n_T$ converges to $u_T$ in probability on $\Omega$ in the topology of $V'$.

**Proof.** We proceed as in the previous Step 5 to get the following estimate

$$
E \left[ e^{-\int_0^T \theta_s ds} |u^n_T - u_T|_{V'}^2 \right] + \nu E \left[ \int_0^T e^{-\int_0^T \theta_s ds} |u^n_t - u_t|_H^2 dt \right] \leq E |u^n_0 - u_0|_V^2,
$$

where

$$
\theta_t = 2CB |u^n_t|_L^5 + |u_t|_L^5 + L_G
$$

Since $\int_0^T \theta_s ds < \infty$ with probability one, we get the result. ■

**Theorem 8** The strong solutions of equation (8) on a given Brownian stochastic basis define a Markov process in $H$ with the Feller property in $V'$.

**Proof.** Denote by $u(t; y)$ the solution at time $t$ which started at time 0 from $y$. Given $t > 0$ the dynamics $y \mapsto u(t; y)$ is uniquely defined in $H$; hence the Markov property is inherited by $u$ from the Galerkin approximations $u^n$.

The process solution enjoys the Feller property if

$$
Eg(u(t; z)) \to Eg(u(t; y)) \quad \text{as } z \to y \text{ in } V'
$$

for any $t \geq 0, g \in C_b(V')$. For this it is enough the convergence in probability: $u(t; z) \to u(t; y)$ as $z \to y$ in $V'$. But, as in Lemma 7 (now the initial data are deterministic), we know that

$$
E \left[ e^{-\int_0^T \theta_s ds} |u(t; z) - u(t; y)|_{V'}^2 \right] \leq |z - y|_{V'}^2
$$

Then, we conclude as before that $|u(t; z) - u(t; y)|_{V'} \to 0$ in probability as $|z - y|_{V'} \to 0$. ■
3.4 Stationary solutions

As in [9], existence of stationary solutions is obtained in the limit, showing first that the Galerkin problem has at least one stationary solution. Our result is the following

**Theorem 9** Assume that \(2\nu \frac{(2\pi)^2}{L^2} > \lambda_0\). Then equation (8) has a stationary solution.

**Proof.** Let us consider

\[
du_t^n + [A\Phi(u_t^n) + \pi_n B(u_t^n, u_t^n)] dt = \pi_n G(u_t^n) dw_t, \quad u_0^n = 0
\]

We use estimates from Appendix 1. By (29), using \(\|u\|_V \geq \frac{2\pi}{L}|u|_H\) we get

\[
\frac{d}{dt}E|u_t^n|_H^p + \nu \frac{(2\pi)^2}{L^2}E|u_t^n|_H^p \leq \frac{1}{2}p(p-1)\left[\lambda_0 E|u_t^n|_H^p + \rho E|u_t^n|_H^{p-2}\right]
\]

\[
\leq \frac{1}{2}p(p-1)(\lambda_0 + \varepsilon)E|u_t^n|_H^p + C(\varepsilon, p, \rho)
\]

for some positive \(\varepsilon\).

If \(2\nu \frac{(2\pi)^2}{L^2} > \lambda_0\), then there exist \(p > 2\) and \(\varepsilon > 0\) such that \(p\nu \frac{(2\pi)^2}{L^2} > \frac{1}{2}p(p-1)(\lambda_0 + \varepsilon)\). Therefore there exists \(a > 0\) such that

\[
\frac{d}{dt}E|u_t^n|_H^p + aE|u_t^n|_H^p \leq C(\varepsilon, p, \rho) \quad \text{with} \quad u_0^n = 0;
\]

by Gronwall Lemma we get

\[
E|u_t^n|_H^p \leq C_6 \quad \forall t \geq 0, \forall n \geq 1
\]

Hence, the family of random variables \(\{u_t^n\}_{t\geq0}\) is tight in \(H_n\). Notice that the Galerkin problem is Feller in \(H_n\). Then, by the Krylov-Bogoliubov method we get that there exists a stationary solution (whose law we denote by \(\mu_n\)) for the Galerkin equation.

Now, consider the Galerkin problem with initial velocity of law \(\mu_n\) and denotes the law of the solution by \(P_n\) (a probability measure on \(C([0, \infty); \mathcal{W}')\). We have

\[
E^{P_n}[\xi_0|_H^p] \leq C_6 \quad \forall n \geq 1
\]

The corresponding solution \(P_n\) is a stationary process in \(H_n\), i.e.

\[
P_n\left\{\xi_t + \int_r^t A\Phi(\xi_s)ds + \int_r^t \pi_n B(\xi_s, \xi_s)ds = \xi_r + \int_r^t \pi_n G(\xi_s)dw_s\right\} = 1
\]

and

\[
P_n(\xi_t) = P_n(\xi_r)
\]

for any \(0 \leq r \leq t < \infty\).
Now we proceed as in [9]. Endow $L^{1+1/2}(0, \infty; H)$ with the distance
\[
d_{1+1/2}(u,v) = \sum_{k=1}^{\infty} \frac{1}{2^k} \left( |u - v|_{L^{1+1/2}(0,k;H)} \wedge 1 \right)
\]
and $C([0, \infty]; W')$ with the distance
\[
d_{\infty}(u,v) = \sum_{k=1}^{\infty} \frac{1}{2^k} \left( |u - v|_{C([0,k];W')} \wedge 1 \right)
\]
The convergence with respect to $d_{1+1/2} + d_{\infty}$ is equivalent with the convergence on every finite time interval. We come back to the bounds (14'), (15'), (16') and (19), to notice that they hold true because they depend only on $E|u_0|^{p}$. Thus we get tightness on every finite interval; we pass to the limit for a subsequence and get the limit process $P$ which is stationary, since the $P_n$ are so. It can be shown as before that $P$ is a martingale solution to equation (8).

Defined the Markov semigroup $P_t$ acting on the space of Borel bounded functions $B_b(H)$ as $P_t \phi(y) = E\phi(u(t;y))$, we get that the law $\mu$ of this stationary solution is an invariant measure, in the sense that
\[
\int P_t \phi d\mu = \int \phi d\mu
\]
for any $\phi \in B_b(H)$ and $t \geq 0$.

4 The case $\Phi(u) = \nu|u|^4u$

Instead of (2), let us assume that
\[
\Phi(u) = \nu|u|^4u
\]
This corresponds to the case $b = 5$ with the nonlinearity acting everywhere. The interest in this model will be explained in Subsection 4.2.

We can analyze this model as done in the previous section, with few changes. Mainly, the solution will live in $L^{5}(0, T; H) \cap L^{6}(0, T; X)$, where $X$ is the closure of $D^\infty$ w.r.t. the norm
\[
|u|_X = \left( \int_T \left\{ |u(x)|^4 |\nabla u(x)|^2 + 4|u(x)|^2 \sum_{i=1}^{3} [u(x) \cdot \partial_i u(x)]^2 \right\} dx \right)^{1/6}
\]
Notice that the term $\int_0^t \langle \Phi(u_s), A\psi \rangle_H ds$ in the equation is well defined, since
\[
\int_0^t |\langle \Phi(u_s), A\psi \rangle| ds \leq |A\psi|_{L^6} \int_0^t |\Phi(u_s)|_{L^6} ds = |A\psi|_{L^6} \nu \int_0^t |u_s|^5_{L^6} ds
\]
The last integral is well defined for functions $u \in L^5(0, T; L^6)$. But, if $u \in X$, then $u \in L^6$ by Theorem [15] in Appendix 2.

We have the following result
Theorem 10 Let $\Phi(u) = \nu|u|^4u$.
Let $\mu$ be a measure on $H$ such that $m_p := \int_H |v|^p \mu(dv) < \infty$ for some $p > 2$.
Then there exists at least one solution to the martingale problem (9) with initial condition $\mu$, assuming that condition $[MP1]$ in Definition 5 is replaced with

\[ P \left( \sup_{t \in [0,T]} |\xi_t|_H + \int_0^T \|\xi_s\|_X^6 ds < \infty \right) = 1 \]

Moreover, if $2\nu C_X > \lambda_0$ (with $|u|^6_X \geq C_X|u|^6_H$), then there exists a stationary solution.

Proof. Let us check step by step how our previous proof (for $b = 5$) can be adapted to handle this model.

Step 1 Instead of Lemma (1), we use

\[ \langle A\Phi(u), u \rangle_H = \nu|u|^6 \]

Hence if we apply Itô formula (for $p \geq 2$) to $|u^n_{s \wedge \tau_R}|_H^p$, we get (14); however (15) is replaced by

\[ E\int_0^T \|u^n_{s \wedge \tau_R}\|_X^6 1_{\{s < \tau_R\}} ds \leq C_2' \]

(16) is a consequence of the latter relationship, since $X \subset L^6$.

Step 2 The estimates are still valid:

\[ \sup_n E|u^n|_{W^{\alpha, \frac{6}{p}}(0,T;W^{-2, \frac{6}{p}})} < \infty \quad \text{for } 0 < \alpha < \frac{1}{2} \]

and

\[ \sup_n E|u^n|_{L^6(0,T;X)} < \infty \]

Step 3 What we need is a compact embedding, which is given in Theorem 12. Thus, by Theorem 2.1 in [9] the space $L^6(0,T;X) \cap W^{\alpha, \frac{6}{p}}(0,T;W^{-2, \frac{6}{p}})$ is compactly embedded in $L^{\frac{6}{p}}(0,T;L^6)$. Therefore the family of measures $\{P_n\}$ is tight in $L^{\frac{6}{p}}(0,T;\mathbb{L}^6)$ and in $C([0,T];W')$, chosen $W'$ such that $W^{-2, \frac{6}{p}}$ is compactly embedded in $W'$.

Step 4 The remaining part of the proof for the existence holds true.

Step 5 As far as the uniqueness is concerned, Lemma 2 has to be replaced with

\[ \langle \Phi(u^{(1)}), u^{(1)} - u^{(2)} \rangle_H \geq 0 \]

This comes from its proof, when we put $\sigma = \tilde{\sigma}$. The above inequality is not enough to get uniqueness. Other estimates failed to be useful so far and uniqueness is an open problem.

Stationary martingale solutions. We consider the sequence of Galerkin solutions $\{u^n\}_{n \geq 1}$, all with zero initial velocity. From the estimates in the Appendix 1, we get

\[ \frac{d}{dt} E|u^n_t|^p_H + p\nu E|u^n_t|^p_H - \frac{1}{2} p(p-1)\lambda_0 E|u^n_t|^p_H + \frac{1}{2} p(p-1)\rho E|u^n_t|^p_H - 2 \]

20
By the embeddings $X \subset L^6 \cap H$, we get $|u|^6_X \geq C_X |u|^6_H$; thus
\[
\frac{d}{dt} E[|u|^p_H] + p\nu C_X E[|u|^p_{H}] \leq \frac{1}{2}p(p-1)\lambda_0 E[|u|^p] + \frac{1}{2}\frac{1}{2}(p-1)\rho E[|u|^p_{H}]
\]

Using that $|u|^2_H \leq |u|^6_H + \frac{2}{3\sqrt{3}}$, we obtain by easy computations that
\[
\frac{d}{dt} E[|u|^p_{H}] + p\nu C_X E[|u|^p_{H}] \leq \frac{1}{2}p(p-1)(\lambda_0 + \varepsilon) E[|u|^p] + C(\varepsilon, p, \rho, C_X, \nu)
\]
for any $\varepsilon > 0$ (with the latter constant $C$ being a suitable positive constant). If $2\nu C_X > \lambda_0$, we conclude as before by Gronwall Lemma that there exists $p > 2$ such that
\[
E[|u|^p_{H}] \leq C_7 \quad \forall t \geq 0, \forall n \geq 1
\]
From now on, the proof proceeds as in the previous case. ■

4.1 Scaling

Let us start with an heuristic digression. We recall that in the Kolmogorov 1941 theory of turbulence (see, e.g., Sect. 6.3.1 in [10] dealing with the deterministic equations and [11] dealing with the stochastic equations), one believes that the following equality in law is approximatively true
\[
u(r + \lambda x) - u(r) \overset{\text{law}}{=} \lambda^{1/3} [u(r + x) - u(r)]
\]
for any $r, x \in \mathbb{R}^3$ and for $\lambda$ in some range of small positive real numbers.
(In the whole section, $u(x)$, without the time variable, denotes a stationary solution.)
This implies $\lambda^{-1/3} [u(\lambda x) - u(0)] \overset{\text{law}}{=} u(x) - u(0)$.
According to this result, we are interested in the scaled velocity $u_\lambda$, defined by the following scaling transformation
\[
u_\lambda(t, x) := \lambda^{-1/3} u \left( \lambda^{2/3} t, \lambda x \right)
\]
for $\lambda \in (0, 1)$; hence the function $u_\lambda(t, x)$ is defined for $x \in \left[0, \frac{L}{\lambda}\right]^3$.
We assume that $u = u(t, x)$ solves in the torus $[0, L]^3$ the modified Navier–Stokes equation (with $\Phi(u) = \nu |u|^4 u$), with additive noise
\[
du + [-\nabla \Phi(u) + (u \cdot \nabla)u + \nabla q] dt = \sum_{(k,j) \in \Lambda} \sigma_{k,j} d\beta_{k,j}(t)h_{k,j}
\]
(with respect to [10], there are the coefficients $\sigma_{k,j}$; some of them may vanish and therefore we denote by $\Lambda$ the set for the summation on $\sigma_{k,j} \neq 0$. Condition (6) is satisfied if $\sum_{(k,j) \in \Lambda} |\sigma_{k,j}|^2 < \infty$.)
The scaled velocity satisfies an equation very similar to this one.
Proposition 11 We have

\[ du_\lambda + [-\Delta \Phi (u_\lambda) + (u_\lambda \cdot \nabla) u_\lambda + \nabla q_\lambda] \, dt = \sum_{(k,j) \in \Lambda} \sigma_{k,j} d\beta_{k,j}^\lambda (t) h_{k,j}^\lambda \]

where \( q_\lambda \) is a suitable function, \( h_{k,j}^\lambda (x) = h_{k,j}(\lambda x) \) and the processes

\[ \beta_{k,j}^\lambda (t) := \lambda^{-1/3} \beta_{k,j} \left( \lambda^{2/3} t \right) \]

are independent standard Brownian motions.

Proof. The rigorous proof has to be performed at the level of the integral weak formulation of the equation and it is tedious and elementary. We just point out the main (somewhat heuristic) arguments behind it. We have

\[ \frac{\partial u_\lambda}{\partial t} \bigg|_{(t,x)} = \lambda^{1/3} \left. \frac{\partial u}{\partial t} \right|_{(\lambda^{2/3} t, \lambda x)} \]

\[ (u_\lambda \cdot \nabla) u_\lambda \big|_{(t,x)} = \lambda^{1/3} (u \cdot \nabla) u \big|_{(\lambda^{2/3} t, \lambda x)} \]

\[ \frac{\partial \beta_{k,j}^\lambda}{\partial t} \bigg|_{(t)} \sim \frac{\beta_{k,j}^\lambda (t + dt) - \beta_{k,j}^\lambda (t)}{dt} \]

\[ = \lambda^{-1/3} \beta_{k,j} \left( \lambda^{2/3} t + \lambda^{2/3} dt \right) - \beta_{k,j} \left( \lambda^{2/3} t \right) \]

\[ = \lambda^{1/3} \beta_{k,j} \left( \lambda^{2/3} t + \lambda^{2/3} dt \right) - \beta_{k,j} \left( \lambda^{2/3} t \right) \]

\[ \sim \lambda^{1/3} \left. \frac{\partial \beta_{k,j}}{\partial t} \right|_{(\lambda^{2/3} t)} \]

\[ \Delta \Phi (u_\lambda) \big|_{(t,x)} = \lambda^{1/3} \Delta \Phi (u) \big|_{(\lambda^{2/3} t, \lambda x)} \]

because

\[ D_1 |u_\lambda|^4 u_\lambda = \lambda^{-5/3} D_1 \left[ |u \left( \lambda^{2/3} t, \lambda x \right) |^2 \, u \left( \lambda^{2/3} t, \lambda x \right) \right] \]

\[ = \lambda^{-5/3} u \left( \lambda^{2/3} t, \lambda x \right)^2 \left[ u \left( \lambda^{2/3} t, \lambda x \right) \right]^2 \left[ D_1 \left[ u \left( \lambda^{2/3} t, \lambda x \right) \right] \right] \]

\[ + \lambda^{-5/3} \left[ u \left( \lambda^{2/3} t, \lambda x \right) \right]^2 D_1 \left[ u \left( \lambda^{2/3} t, \lambda x \right) \right] \]

\[ = \lambda \cdot \lambda^{-5/3} \left[ u^2 |u|^2 u \cdot (D_1 u) + |u|^2 \, (D_1 u) \right] \bigg|_{(\lambda^{2/3} t, \lambda x)} \]

\[ = \lambda \cdot \lambda^{-5/3} D_1 \left[ |u|^4 u \right] \bigg|_{(\lambda^{2/3} t, \lambda x)} \]
and then
\[ \triangle |u_\lambda|^4 u_\lambda |_{(t,x)} = \lambda^2 \cdot \lambda^{-5/3} \triangle \left[ |u|^4 u \right]_{(\lambda^{2/3} t, \lambda x)} \]

Now, for any \( r \in \mathbb{R}^3 \), define the (space) translation operator \( \hat{r} \) as \((\hat{r}V)(x) = V(x + r)\), to be understood as an identity in the distributional sense. We say that a process \( V \) is spatially homogeneous if all the space increments \( \delta V(x,h) = V(x+h) - V(x) \) are statistically invariant with respect to the translation operator \( \hat{r} \): \( V(x + h) - V(x) \overset{\text{in law}}{=} V(x + h + r) - V(x + r) \).

In the same way, we say that a process \( V \) is isotropic if the law of all the space increments \( \delta V(x,h) \) do not change under simultaneous rotation \( \theta \) of the space variables and of the vector \( V \). Since the space variable lives in a torus, only rotations of \( \mathbb{R}^3 \) which leave the torus invariant are allowed.

It is easy to check when the Wiener process on the r.h.s. of our equation enjoys these statistical invariances. Indeed
\[ E[w(t, x + h) - w(t, x)]^2 = t \sum_{(k,j) \in \Lambda} |\sigma_{k,j}|^2 |h_{k,j}(x + h) - h_{k,j}(x)|^2 \]

If for any \( k \) with \((k,j) \in \Lambda\), each coefficient \( \sigma_{k,j} \) in front of \( \cos(\frac{2\pi}{L} k \cdot x) \) is equal, in absolute value, to a coefficient in front of \( \sin(\frac{2\pi}{L} k \cdot x) \), then the second moment of the space increment is equal to \( t \sum_{j=1,2} |\sigma_{k,j}|^2 |e^{i\frac{2\pi}{L} k \cdot h} - 1|^2 \) and therefore depends only on \( h \). This implies that \( w \) is spatially homogeneous (considering the space variables in \( \mathbb{R}^3/[0,L]^3 \)). On the other hand,
\[ E[\theta w(t, x + \theta h) - \theta w(t, x)]^2 = t \sum_{(k,j) \in \Lambda} |\sigma_{k,j}|^2 |h_{k,j}(\theta x + \theta h) - h_{k,j}(\theta x)|^2 \]

But \( h_{k,j}(\theta x + \theta h) - h_{k,j}(\theta x) = h_{\theta^{-1}k,j}(x + h) - h_{\theta^{-1}k,j}(x) \). Then we can consider only rotations \( \theta \) such that \((k,j) \in \Lambda \iff (\theta k, j) \in \Lambda \) and in these cases \( w \) is isotropic if \( |\sigma_{\theta k,j}| = |\sigma_{k,j}| \) for all \((k,j) \in \Lambda\).

**Corollary 12** Let \( \Lambda \) be such that the process \( \sum_{(k,j) \in \Lambda} \sigma_{k,j} h_{k,j}(x) \) is spatially homogeneous and isotropic. For \( \Phi(u) = \nu |u|^4 u \), consider the equation
\[ du + [\Delta \Phi(u) + (u \cdot \nabla) u + \nabla q] \, dt = \sum_{(k,j) \in \Lambda} \sigma_{k,j} d\beta_{k,j}(t) h_{k,j}(x) \quad x \in [0,L]^3 \]  \hspace{1cm} (24)

with initial velocity spatially homogeneous and isotropic, satisfying the assumptions of Theorem 11 and \( \sum_{(k,j) \in \Lambda} |\sigma_{k,j}|^2 < \infty \). Then there exists a solution \( u \) spatially homogeneous and isotropic for any \( t \geq 0 \). For any \( \psi \in C^\infty \), we have
\[ E\left[ |\langle u(t, \lambda e) - u(t, 0), \psi \rangle|^p \right] = \lambda^{p/3} E\left[ |\langle u_\lambda(\lambda^{-2/3} t, e) - u_\lambda(\lambda^{-2/3} t, 0), \psi \rangle|^p \right] \]  \hspace{1cm} (25)
where $u_\lambda(t,x)$ is spatially homogeneous and isotropic, and solves the equation

$$
\begin{align*}
  du_\lambda + [-\Delta \Phi(u_\lambda) + (u_\lambda \cdot \nabla)u_\lambda + \nabla q_\lambda] \, dt &= \sum_{(k,j) \in \Lambda} \sigma_{k,j} d\beta^\lambda_{k,j}(t) h_{k,j}(\lambda x) \\
  x &\in [0, \frac{1}{\lambda}]^3
\end{align*}
$$

with initial velocity $u_\lambda(0,\cdot) = \lambda^{-1/3} u(0,\cdot)$. 

Remark 13 The statistical invariance for the solution is obtained from the same property of the Galerkin approximations, as done in a similar context in [18]. Indeed, we construct a solution as limit of a Galerkin subsequence. But the statistical invariance for the Galerkin processes $u_n$ (for any $n$) is easy to show, since for any $n$ the finite-dimensional problem has a unique solution. Notice that in our case we can trivially consider a vanishing initial velocity.

Now, let us consider the structure function of order $p$ ($p = 1, 2, \ldots$) with respect to a stationary solution $u$ of the modified Navier–Stokes equation (24):

$$
S_p(\lambda) := E\left[|u(\lambda e) - u(0)|^p\right]
$$

($e$ is a unitary vector in $\mathbb{R}^3$ and $\lambda \in (0, 1)$). Similarly, we can work with the longitudinal structure function.

We point out that Theorem 10 provides the existence of a stationary solution leaving in $X$, but this is not enough to define the velocity in every point of the torus. We would need to analyze the regularity of stationary solutions, but we decide to postpone the study of existence of more regular stationary solutions to future work (it would be enough to have the law of $u$ supported by the space $C^0(T)$).

According to the previous Corollary, for the structure function we get that

$$
S_p(\lambda) = \lambda^{p/3} E\left[|u_\lambda(e) - u_\lambda(0)|^p\right]
$$

(27)

Kolmogorov 1941 theory states (see, e.g., [10], Sect. 6.3.1) that

$$
S_p(\lambda) = C_p \varepsilon^{p/3} \lambda^{p/3}
$$

(28)

where $C_p$ are dimensionless and $\varepsilon$ is the mean energy dissipation rate. For $p = 2$, (28) is the so called two-thirds law of turbulence, which is supported by experimental results. For $p = 3$, (28) is the four-fifths law of turbulence ($C_3 = -\frac{4}{5}$), deduced from the assumptions of homogeneity, isotropy and finiteness of the energy dissipation. For $p > 3$, (28) is not confirmed by experimental data and its truthfulness is questionable.

According to the above Proposition and Corollary, we shall provide a relationship similar to (28) for our model (24).

Keeping in mind (27), we investigate the behaviour of $E\left[|u_\lambda(e) - u_\lambda(0)|^p\right]$ in order to get insights on the structure function. First, we remark that $\beta_{k,j}(t)$ and $\beta^\lambda_{k,j}(t)$ are unitary Brownian motions. Thus, the random forces in equations
and (26) are the same in law; what changes is the dimension of the torus and correspondingly the eigenvectors $h_{k,j}$. No other terms in the equation (when projected onto $H$) change with the scaling. It is important to point out that this property is not true for the usual Navier–Stokes equation or for the Prouse model introduced at the beginning; namely, after the scaling the viscous term $\nu \Delta u$ becomes $\lambda^{-4/3}\nu \Delta u_\lambda$ (and then the two problems for $u$ and $u_\lambda$ are very different, because the scaled viscosity $\nu_\lambda = \lambda^{-4/3}\nu$ explodes as $\lambda \to 0$).

Since the coefficients $\sigma_{k,j}$ in the noise do not change with the scaling transformation, the mean energy introduced in a unit of volume per unit of time by the stochastic forcing term is independent of $\lambda$ and is equal to $\frac{1}{2\pi} \sum_{(k,j) \in \Lambda} |\sigma_{k,j}|^2$. When $\lambda \to 0$, the size of the domain becomes bigger and bigger but the unitary energy does not change.

If, as in turbulence theory, we assume that during the motion there is energy transfer from large scales to small scales with a universal cascade mechanism depending only on the unit volume energy, then we would conclude that any stationary state is independent of $\lambda$. Hence $E[|u_\lambda(e) - u_\lambda(0)|^p] = k_p$ for any $\lambda$. Coming back to (25), we conclude that

$$S_p(\lambda) = k_p \lambda^{p/3}$$

for any $0 < \lambda < 1$ and $p \in \mathbb{N}$.

Summing up, we have the following result, providing a result on the structure function (of any order $p$) under two assumptions. The first assumption is technical and can be removed as soon as we are able to prove existence of regular stationary solutions. On the other hand, the second assumption on energy cascade has to be considered as an hypothesis quite hard to justify rigorously (as it is for fluids modeled by the Navier–Stokes equations).

**Claim 14** Let us assume that system (24) has a stationary solution $u$, which at any fixed time has law supported by the space $C^0(T)$. Let us further assume that there is energy transfer from large scales to small scales with a universal cascade mechanism depending only on the unit volume energy.

Then for the structure function, given any $0 < \lambda < 1$ and $p \in \mathbb{N}$ we have

$$S_p(\lambda) = k_p \lambda^{p/3}$$

for some constant $k_p$ independent of $\lambda$.

## 5 Appendix 1: a priori estimates

We present the estimates on the Galerkin approximations; these are quite standard (see, e.g., [9] and [15]). Besides the usual estimates (33), (34), we need also (37) to prove uniqueness.

Let $(u^n_t)_{t \geq 0}$ be a continuous adapted solution of equation (18). Let

$$\tau^n_R = \inf \{ t \geq 0 : |u^n_t|^2_H = R \}$$
We have
\[ u^n_{t \wedge \tau_R} = u^n_0 + \int_0^{t \wedge \tau_R} [-A \Phi(u^n_s) - \pi_n B(u^n_s, u^n_n)] ds + \int_0^{t \wedge \tau_R} \pi_n G(u^n_s) dw(s) \]
\[ = u^n_0 + \int_0^t [-A \Phi(u^n_{s \wedge \tau_R}) - \pi_n B(u^n_{s \wedge \tau_R}, u^n_{s \wedge \tau_R})] 1_{\{s < \tau_R\}} ds \]
\[ + \int_0^t 1_{\{s < \tau_R\}} \pi_n G(u^n_{s \wedge \tau_R}) dw(s) \]

For \( p \geq 2 \) apply Itô formula to \( |u^n_{t \wedge \tau_R}|_H^p \):
\[ d|u^n_{t \wedge \tau_R}|_H^p \leq |u^n_{t \wedge \tau_R}|_H^{p-2} \langle u^n_{t \wedge \tau_R}, du^n_{t \wedge \tau_R} \rangle_H \\
+ \frac{1}{2}(p-1)|u^n_{t \wedge \tau_R}|_H^{p-2} \| \pi_n G(u^n_{t \wedge \tau_R}) \|_{HS(H)}^2 1_{\{t < \tau_R\}} dt \]

Then, integrating in time, we have
\[ |u^n_{t \wedge \tau_R}|_H^p \leq |u^n_0|_H^p \\
+ p \int_0^t |u^n_{s \wedge \tau_R}|_H^{p-2} \langle -A \Phi(u^n_{s \wedge \tau_R}), \pi_n B(u^n_{s \wedge \tau_R}, u^n_{s \wedge \tau_R}) \rangle_H 1_{\{s < \tau_R\}} ds \\
+ p \int_0^t |u^n_{s \wedge \tau_R}|_H^{p-2} \langle u^n_{s \wedge \tau_R}, \pi_n G(u^n_{s \wedge \tau_R}) \rangle_H 1_{\{s < \tau_R\}} ds \\
+ \frac{1}{2}p(p-1) \int_0^t |u^n_{s \wedge \tau_R}|_H^{p-2} \| \pi_n G(u^n_{s \wedge \tau_R}) \|_{HS(H)}^2 1_{\{s < \tau_R\}} ds \]

Then, by Lemma 1 and 3
\[ |u^n_{t \wedge \tau_R}|_H^p + \nu \int_0^t |u^n_{s \wedge \tau_R}|_H^{p-2} \| u^n_{s \wedge \tau_R} \|_V^2 1_{\{s < \tau_R\}} ds \]
\[ \leq |u^n_0|_H^p + p \int_0^t |u^n_{s \wedge \tau_R}|_H^{p-2} \langle u^n_{s \wedge \tau_R}, \pi_n G(u^n_{s \wedge \tau_R}) \rangle_H 1_{\{s < \tau_R\}} ds \\
+ \frac{1}{2}p(p-1) \int_0^t |u^n_{s \wedge \tau_R}|_H^{p-2} \| G(u^n_{s \wedge \tau_R}) \|_{HS(H)}^2 1_{\{s < \tau_R\}} ds \]

By assumption (6)
\[ |u^n_{t \wedge \tau_R}|_H^p + \nu \int_0^t |u^n_{s \wedge \tau_R}|_H^{p-2} \| u^n_{s \wedge \tau_R} \|_V^2 1_{\{s < \tau_R\}} ds \]
\[ \leq |u^n_0|_H^p + p \int_0^t |u^n_{s \wedge \tau_R}|_H^{p-2} \| \pi_n G(u^n_{s \wedge \tau_R}) \|_{HS(H)}^2 1_{\{s < \tau_R\}} ds \\
+ \frac{1}{2}p(p-1) \int_0^t |u^n_{s \wedge \tau_R}|_H^{p-2} \left( \lambda_0 |u^n_{s \wedge \tau_R}|_H^2 + \rho \right) 1_{\{s < \tau_R\}} ds \]

where
\[ \widetilde{M}_t^n = \int_0^t |u^n_{s \wedge \tau_R}|_H^{p-2} \langle u^n_{s \wedge \tau_R}, 1_{\{s < \tau_R\}} \pi_n G(u^n_{s \wedge \tau_R}) \rangle_H \]

26
is a square integrable martingale. Therefore

$$
\sup_{t \in [0, r]} \left| u^n_{t, \tau^n_R} \right|^p_H \leq \left| u^n_0 \right|^p_H + p \sup_{t \in [0, r]} \left| \tilde{M}^n_t \right| + \frac{1}{2} p(p - 1)(\lambda_0 + \rho) \int_0^r \left| u^n_{s, \tau^n_R} \right|^p_H 1_{\{ s < \tau^n_R \}} ds + \frac{1}{2} p(p - 1) \rho r
$$

By Burkholder-Davis-Gundy inequality, we estimate the supremum of the martingale $\tilde{M}^n_t$; for some constant $C > 0$ we have

$$
pE \sup_{0 \leq t \leq r} \left| \int_0^t \left| u^n_{s, \tau^n_R} \right|^p_H \langle u^n_{s, \tau^n_R}, 1_{\{ s < \tau^n_R \}} \rangle_H \right| G(u^n_{s, \tau^n_R}) dw(s) \right| 

\leq C p E \left( \int_0^r \left| u^n_{s, \tau^n_R} \right|^{2p - 2} \left\| G(u^n_{s, \tau^n_R}) \right\|_{H^2(H)}^2 \right) \left\{ 1_{\{ s < \tau^n_R \}} \right\} ds \right)^{1/2}
$$

Then by assumption (6)

$$
pE \sup_{t \in [0, r]} \left| \tilde{M}^n_t \right| 

\leq E \left[ \sup_{0 \leq t \leq r} \left| u^n_{t, \tau^n_R} \right|^p_H \left( p - 2 \right) (\lambda_0 + \rho) \left\{ 1_{\{ s < \tau^n_R \}} \right\} ds \right]^{1/2}

\leq \left( \sup_{0 \leq t \leq r} \left| u^n_{t, \tau^n_R} \right|^p_H \right)^{1/2} \left( E \left( p - 2 \right)^2 (\lambda_0 + \rho) \left\{ 1_{\{ s < \tau^n_R \}} \right\} ds \right) \left\{ 1_{\{ s < \tau^n_R \}} \right\} ds \right)^{1/2}

\leq \frac{1}{2} E \sup_{0 \leq t \leq r} \left| u^n_{t, \tau^n_R} \right|^p_H + \frac{1}{2} C^2 p^2 (\lambda_0 + \rho) E \int_0^r \left| u^n_{s, \tau^n_R} \right| H 1_{\{ s < \tau^n_R \}} ds + \frac{1}{2} C^2 p^2 \rho r
$$

By Gronwall lemma, for any $r > 0$ we have

$$
E \sup_{0 \leq t \leq r} \left| u^n_{t, \tau^n_R} \right|^p_H \leq C_1
$$

for some positive constant $C_1 = C_1(p, T, \lambda_0, \rho, m_p)$ independent of $n$ and $R$. Here $m_p = E|u^n_0|^p_H$. Notice that $E\left| u^n_0 \right|^p_H \leq m_p$.
Coming back to (29), with similar arguments we also obtain
\[
E \int_0^T |u_{s \wedge \tau_R}^n|_{H}^{p-2} \|u_{s \wedge \tau_R}^n\|_H^2 1_{\{s < \tau_R\}} ds \leq C \quad \forall n, R
\]
for a new positive constant \( C \) depending on \( m_p, p, \lambda_0, \rho, T \) but not on \( n, R \). For \( p = 2 \) we have
\[
E \int_0^T \|u_{s \wedge \tau_R}^n\|_H^2 1_{\{s < \tau_R\}} ds \leq C_2
\]
for some positive constant \( C_2 = C_2(T, \lambda_0, \rho, m_2) \) independent of \( n \) and \( R \).

For the last estimate, we proceed as follows. First, from (2) we have that
\[
\int \Phi(u, u)_{H} \leq a_1 |u|_{L_2}^{1+b} - a_1 K^{1+b} |T|
\]
(35)
Moreover
\[
2|\langle B(u, u), A^{-1}u \rangle| \leq 2|u|^2_{L_2} |u|_{V'} \leq |u|^4_{L_2} + |u|^2_{H'} \leq a_1 |u|_{L_{2+b}}^{1+b} + C |u|_{H}^2
\]
(36)
Apply Itô formula to \( |u_{s \wedge \tau_R}^n|_{V'}^2 = \langle u_{s \wedge \tau_R}^n, A^{-1} u_{s \wedge \tau_R}^n \rangle_H \) and get
\[
|u_{s \wedge \tau_R}^n|_{V'}^2 = |u_0|_{V'}^2 - 2 \int_0^t \langle \Phi(u_{s \wedge \tau_R}^n), u_{s \wedge \tau_R}^n \rangle_{H} 1_{\{s < \tau_R\}} ds
\]
\[-2 \int_0^t \langle \pi_n B(u_{s \wedge \tau_R}^n, A^{-1} u_{s \wedge \tau_R}^n) H 1_{\{s < \tau_R\}} ds
\]
\[+ M^n + \int_0^t \pi_n A^{-1/2} G(u_{s \wedge \tau_R}^n) \|_{HS(H)}^2 1_{\{s < \tau_R\}} ds \]
where
\[M^n = 2 \int_0^t \langle u_{s \wedge \tau_R}^n, 1_{\{s < \tau_R\}} \pi_n A^{-1} G(u_{s \wedge \tau_R}^n) dw(s) \rangle_{H}
\]
is a square integrable martingale.

Use (35) and (36); then
\[
E|u_{T \wedge \tau_R}^n|_{V'}^2 + a_1 E \int_0^T |u_{s \wedge \tau_R}^n|_{L_{2+b}}^{1+b} 1_{\{s < \tau_R\}} ds
\]
\[\leq E|u_0|_{V'}^2 + CTE \sup_{0 \leq t \leq T} |u_{t \wedge \tau_R}^n|_{H}^2
\]
\[+ C \lambda_0 E \int_0^T |u_{s \wedge \tau_R}^n|_{H}^2 1_{\{s < \tau_R\}} ds + C(1 + \rho)T
\]
According to (33) (for \( p = 2 \)), we conclude that

\[
E \int_0^T |u^n_{x \wedge \tau^n_R}|^{1+b} 1_{\{s < \tau^n_R\}} ds \leq C_3
\]  

(37)

for some positive constant \( C_3 = C_3(T, \lambda_0, a_1, C_1, C_2) \).

6 Appendix 2: a compactness result

Let \( X \) be the closure of \( D^\infty \) w.r.t. the norm

\[
|u|_X := \left( \int_T \{ |u(x)|^4 |\nabla u(x)|^2 + 4|u(x)|^2 \sum_{i=1}^{3} |u(x) \cdot \partial_i u(x)|^2 \} dx \right)^{1/6}
\]

**Theorem 15** \( X \subset L^6 \cap H \) and the immersion is compact.

**Proof.** First observe that for smooth fields \( u \) we have

\[
\partial_i (|u|^2 u) = |u|^2 \partial_i u + 2u \cdot \partial_i u
\]

Hence

\[
\| |u|^2 u \|_V^2 \leq C \int_T |u|^4 |\nabla u|^2 dx \leq C |u|^6_X
\]

and thus by Poincaré inequality

\[
|u|^6_{L^6} = \| |u|^2 u \|_L^2 \leq \| |u|^2 u \|_V^2 \leq C' |u|^6_X
\]

This proves that the closure of \( D^\infty \) with respect to the \( L^6 \)-norm is a space bigger than its closure with respect to the \( X \)-norm; hence \( X \subset L^6 \cap H \).

Moreover, if \( \{u_n\} \) is a bounded sequence in \( X \), it is bounded in \( L^6 \) and \( \{ \| |u_n|^2 u_n \|_V^2 \} \) is also bounded. By Rellich Theorem, the sequence \( \{ |u_n|^2 u_n \} \) is relatively compact in \( L^2 \) and so there exists a subsequence \( \{ |u_{n_k}|^2 u_{n_k} \} \) converging strongly in \( L^2 \) to some field \( \xi \); we also have that \( \{ u_{n_k} \} \) converges weakly in \( L^6 \) to some field \( u \). The strong convergence implies in particular that

\[
|u_{n_k}|^6_{L^6} = \| |u_{n_k}|^2 u_{n_k} \|_{L^2} \rightarrow |\xi|^2_{L^2}.
\]

Thus, if we prove that \( |\xi|^2_{L^2} = |u|^6_{L^6} \), then from the weak convergence of \( \{u_n\} \) to \( u \) in \( L^6 \) and the convergence of norms \( |u_{n_k}|^6_{L^6} \rightarrow |u|^6_{L^6} \), we deduce that \( \{ u_{n_k} \} \) converges strongly to \( u \) in \( L^6 \) and the proof of the compact embedding will be complete.

So it remains to show that \( |\xi|^2_{L^2} = |u|^6_{L^6} \). Let us introduce the function

\[
v(x) = \begin{cases} \frac{\xi(x)}{|\xi(x)|} & \text{if } \xi(x) \neq 0 \\ 0 & \text{if } \xi(x) = 0 \end{cases}
\]
Let us prove that there is a subsequence \( \{u_{n_k}'\} \) such that
\[
 u_{n_k}' \to v \quad \text{a.s. on } T. \tag{38}
\]
This implies \( v = u \) (the a.s. limit and the \( L^6 \) weak limit must coincide, since by Vitali theorem there is strong convergence in any \( L^p \) with \( p < 6 \)). Since
\[
|v(x)|^6 = |\xi(x)|^2 \quad \text{where } \xi(x) \neq 0,
\]
we have
\[
|\xi|_{L^2}^2 = |v|_{L^6}^6 = |u|_{L^6}^6,
\]
as we want.

Thus it remains to prove (38). The strong convergence above implies that there is a subsequence \( \{|u_{n_k}'|^2 u_{n_k}'\} \) that converges to \( \xi \) a.s. on \( T \). Let \( x \in T \) be such that
\[
|u_{n_k}'(x)|^2 u_{n_k}'(x) \to \xi(x).
\]
Taking the norm in \( \mathbb{R}^3 \), this implies that
\[
|u_{n_k}'(x)|^3 \to |\xi(x)|,
\]
hence
\[
|u_{n_k}'(x)| \to |\xi(x)|^{1/3}.
\]
If \( \xi(x) = 0 \), this implies
\[
u_{n_k}'(x) \to 0,
\]
as we want in (38). If \( \xi(x) \neq 0 \), this implies
\[
|u_{n_k}'(x)| \neq 0 \quad \text{eventually}
\]
and
\[
u_{n_k}'(x) = \frac{|u_{n_k}'(x)|^2 u_{n_k}'(x)}{|u_{n_k}'(x)|^2} \to \frac{\xi(x)}{|\xi(x)|^{2/3}}.
\]
Thus (38) is true. The proof is complete.  

References

[1] Bensoussan A. (1995). Stochastic Navier-Stokes equations, *Acta Appl. Math.* 38, no. 3, 267–304.

[2] Brzeźniak Z., Capiński M., Flandoli F. (1992). Stochastic Navier-Stokes equations with multiplicative noise, *Stochastic Anal. Appl.* 10, no. 5, 523–532.

[3] Brzeźniak Z., Peszat S. (2000). Strong local and global solutions for stochastic Navier-Stokes equations, *Infinite dimensional stochastic analysis (Amsterdam, 1999)*, 85–98, Verh. Afd. Natuurkd. 1. Reeks. K. Ned. Akad. Wet., 52, R. Neth. Acad. Arts Sci., Amsterdam.

[4] Capiński M., Cutland N. (1995). *Nonstandard Methods in Stochastic Fluid Mechanics*, World Scientific, Singapore.

[5] Crauel H., Flandoli F. (1995). Dissipativity of three-dimensional stochastic Navier-Stokes equation, *Seminar on Stochastic Analysis, Random Fields and Applications* (Ascona, 1993), 67–76, Progr. Probab., 36, Birkhäuser, Basel.

[6] Da Prato G., Debussche A. (2003). Ergodicity for the 3D stochastic Navier-Stokes equations, *J. Math. Pures Appl.* (9) 82, no. 8, 877–947.

[7] Da Prato G., Zabczyk J. (1992). *Stochastic Equations in Infinite Dimensions*, Encyclopedia of Mathematics and its Applications, 44; Cambridge University Press, Cambridge.
[8] Flandoli F. (1997). Irreducibility of the 3D stochastic Navier-Stokes equation, *J. Funct. Anal.* 149 (1), 160–177.

[9] Flandoli F., Gątarek D. (1995). Martingale and stationary solutions for stochastic Navier–Stokes equations, *Probab. Theory Related Fields* 102, no. 3, 367-391.

[10] Frisch U. (1995). *Turbulence. The legacy of A. N. Kolmogorov*, Cambridge University Press, Cambridge.

[11] Kupiainen A. (2000). *Lessons for turbulence*, Alon N. (ed.) et al., GAFA 2000 (Tel Aviv, 1999). Geom. Funct. Anal., Special Volume, Part I, 316–333. Birkhäuser, Basel.

[12] Mikulevicius R., Rozovskii B. L. (2004). Stochastic Navier-Stokes equations for turbulent flows, *SIAM J. Math. Anal.* 35, no. 5, 1250–1310.

[13] Prodi G. (1959). Un teorema di unicità per le equazioni di Navier–Stokes, *Ann. Mat. Pura Appl.* (4) 48, 173-182.

[14] Protter P. (2004). *Stochastic integration and differential equations*, 2nd ed. Springer, Berlin.

[15] Prouse G. (1991). On a Navier-Stokes type equation, *Nonlinear analysis*, 289-305, Sc. Norm. Super. di Pisa Quaderni, Scuola Norm. Sup., Pisa.

[16] Temam R. (1984). *Navier-Stokes Equations, Theory and Numerical Analysis*, 3rd ed. North-Holland, Amsterdam.

[17] Temam R. (1983). *Navier-Stokes Equations and Nonlinear Functional Analysis*, SIAM, Philadelphia.

[18] Vishik M. J., Fursikov A. V. (1978). Translationally homogeneous statistical solutions and individual solutions with infinite energy of a system of Navier-Stokes equations, *Siberian Math. J.* 19, no. 5, 710–729.

[19] Vishik M. J., Fursikov A. V. (1988). *Mathematical Problems in Statistical Hydromechanics*, Kluwer, Boston.