Abstract

In a previous paper ([14]), the author was able to show that the volumes of certain hyperbolic semi-adequate links can be bounded above and below in terms of two diagrammatic quantities: the twist number and the number of special tangles in a semi-adequate diagram. It was also shown that, for semi-adequate plat closures of a family of braids, these bounds can often be expressed in terms of a single stable coefficient of the colored Jones polynomial, thus showing that these links satisfy a Coarse Volume Conjecture. In this paper, we shift our focus to a different family of links. Specifically, we show that a family of semi-adequate closed braids fits into the framework mentioned above. Restricting attention to closed three-braids, we are also able to show that the volume can be bounded in terms of the parameter $s$ from the Schreier normal form of the three-braid.

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1 Introduction

An important goal in the study of knots and links is to strengthen the connections among geometric link invariants, quantum link invariants, and diagrammatic/combinatorial data associated to link diagrams. A more specific goal is to find conditions under which data extracted from a certain link diagram or data extracted from a quantum polynomial link invariant can be used to bound the volume of the link complement.

From a combinatorial and quantum perspective, a rich family of links to study is the family of semi-adequate (A- and B-adequate) links. By reflecting the diagram if needed, we may focus our attention on the collection of A-adequate links. To an A-adequate link diagram we can associate a loop-free graph. This graph, which can be cellularly embedded in the Turaev surface associated to the link diagram, is intimately related to the Jones polynomial of the link (a quantum link invariant). For example, the Jones polynomial of the link can be recovered from the Bollobás-Riordan polynomial of the cellularly embedded graph ([4]). Additionally, Dasbach and Lin ([5]) have shown that the absolute value of the penultimate coefficient of the colored Jones polynomial stabilizes to a value that is independent of \( n > 1 \) and only depends on a reduction of the cellularly embedded graph (viewed as an abstract graph).

From a geometric perspective, a rich family of links to study is the family of hyperbolic links ([25]). Such links are characterized by having complements that admit a unique hyperbolic metric ([19], [22]). By this uniqueness the volume, \( \text{vol}(S^3 \setminus K) \), of the complement of a hyperbolic link \( K \) is a geometric link invariant.

Acting as a bridge between quantum and geometric invariants of links, the Volume Conjecture of Kashaev, Murakami, and Murakami ([7], [20]) predicts that the (asymptotic behavior of the) colored Jones polynomial can be used to determine the volume of a hyperbolic link complement. While it has been verified for a handful of hyperbolic links, the conjecture remains open in general.

Recently, Futer, Kalfagianni, and Purcell have related the colored Jones polynomial of an A-adequate link to both the geometry of essential surfaces in the link complement and to bounds on the hyperbolic volume of the link complement. (See [10] for the complete work or [9] for a survey of results.) As an example, it was shown that, for sufficiently twisted negative (positive using their conventions) braid closures and for a large family of Montesinos links, the volume of the link complement can be bounded above and below in terms of the twist number of an A-adequate link diagram. For the class of alternating links, similar two-sided bounds were found by Lackenby.
(18), with the upper bound later improved by Agol and D. Thurston (18, Appendix) and the lower bound later improved by Agol, Storm, and W. Thurston (11). In addition to being expressed in terms of the twist number, the volume of a number of families of links has also been bounded above and below in terms of coefficients of the colored Jones polynomial ([3], [6], [10], [11], [12], [13], [24]). From this point of view (that is, by using the coefficients of the colored Jones polynomial to bound the volume), we can say that these families of links satisfy a Coarse Volume Conjecture ([10], Section 10.4).

The main result of the previous paper of the author (14) is given below. (Note that the definitions of connected, prime, A-adequate, twist region, and twist number, as well as more details concerning the two-edge loop condition and special tangles, will be provided in Section 2.1.) Let $t(D)$ denote the twist number of $D(K)$, let $st(D)$ denote the number of special tangles in $D(K)$, let $v_8 = 3.6638...$ denote the volume of a regular ideal octahedron, and let $v_3 = 1.0149...$ denote the volume of a regular ideal tetrahedron.

**Theorem 1.1** ([14], Main Theorem). Let $D(K)$ be a connected, prime, A-adequate link diagram that satisfies the two-edge loop condition and contains $t(D) \geq 2$ twist regions. Then the link $K$ is hyperbolic and the complement $S^3 \setminus K$ satisfies the following volume bounds:

$$\frac{1}{3}v_8 \cdot (t(D) - st(D)) \leq \text{vol}(S^3 \setminus K) < 10v_3 \cdot (t(D) - 1),$$

where $t(D) \geq st(D)$. If $t(D) = st(D)$, then $D(K)$ is alternating and the lower bound of $\frac{v_8}{2} \cdot (t(D) - 2)$ from Theorem 2.2 of [7] may be used.

In this paper, we show that a family of closed braids satisfies the above theorem and, more generally, fits into the framework of [8], [10] and [14]. By studying the particular structure of these closed braids, we show that the lower bound on volume provided by Theorem 1.1 can often be improved. The main results of this paper are summarized below. (For the definition of non-essential wandering circle, see Section 3.3.)

**Theorem 1.2.** Let $D(K)$ denote the closure of a certain $n$-braid diagram. Then $K$ is a hyperbolic link. In the case that $n = 3$, we get the following volume bounds:

$$\frac{v_8}{2} \cdot (t(D) - 1) - \frac{v_8}{2} \leq \text{vol}(S^3 \setminus K) < 10v_3 \cdot (t(D) - 1).$$

Suppose, in the case that $n \geq 4$, that the all-A state contains $m$ non-essential wandering circles. Then we get the following volume bounds:

$$\frac{v_8}{2} \cdot (t(D) - 1) - \frac{v_8}{2} \cdot (2(n + m) - 5) \leq \text{vol}(S^3 \setminus K) < 10v_3 \cdot (t(D) - 1).$$
Furthermore, we are able to translate the above volume bounds so that they may be expressed in terms of a single stable coefficient, $\beta'_K$, of the colored Jones polynomial. (For more information about the colored Jones polynomial of an A-adequate link, see Section 3.7.)

**Theorem 1.3.** Let $D(K)$ denote the closure of a certain $n$-braid. Then $K$ is a hyperbolic link. In the case that $n = 3$, we get the following volume bounds:

$$v_8 \cdot (|\beta'_K| - 1) \leq \text{vol}(S^3 \setminus K) < 20v_3 \cdot (|\beta'_K| - 1) + 10v_3.$$ 

Suppose, in the case that $n \geq 4$, that the all-A state contains $m$ non-essential wandering circles. Then we get the following volume bounds:

$$v_8 \cdot (|\beta'_K| - 1) \leq \text{vol}(S^3 \setminus K) < 20v_3 \cdot (|\beta'_K| - 1) + 10v_3 \cdot (2(n + m) - 5).$$

Additionally, for the closed 3-braids under consideration, we are able to bound the volume in terms of the parameter $s$ from the Schreier normal form of 3-braid. (For more information about the Schreier normal form of a closed 3-braid and the parameter $s$, see Section 4.1.)

**Theorem 1.4.** Let $D(K)$ denote the closure of a certain 3-braid. Then the link $K$ is hyperbolic and the complement $S^3 \setminus K$ satisfies the following volume bounds:

$$v_8 \cdot (s - 1) \leq \text{vol}(S^3 \setminus K) < 4v_8 \cdot s.$$ 

**Remark 1.1.** It should be noted that volume bounds for closed 3-braids in terms of the parameter $s$ were previously found by Futer, Kalfagianni, and Purcell in [13]. By using the results of their more recent work in [8] and [10], we are often able to improve the lower bound on volume given in [13].

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**2 Preliminaries**

The first two subsections, presenting definitions and the background on volume, are very similar to that found in an earlier paper ([14]) of the author. A reader familiar with this work may choose to proceed to Section 2.3.
Figure 1: A schematic depiction of non-prime link diagram $D(K)$. The solid boxes, marked $D_1$ and $D_2$, contain the remainder of the link diagram. Each such box is assumed to contain at least one crossing of $D(K)$. The simple closed curve, $C$, exhibits the non-primeness of the diagram.

Figure 2: A crossing neighborhood of a link diagram (middle), along with its A-resolution (right) and B-resolution (left).

2.1 Definitions

For the remainder of this paper, we will let $D(K) \subseteq S^2$ denote a diagram of a link $K \subseteq S^3$. We call $D(K)$ connected if the projection graph formed by replacing all crossings of $D(K)$ with 4-valent vertices is path-connected. We call $D(K)$ prime if there does not exist a simple closed curve in the projection sphere that both intersects $D(K)$ twice transversely (away from the crossings) and also contains crossings on both sides. For a schematic depiction of a non-prime diagram, see Figure 1.

To smooth a crossing of the link diagram $D(K)$, we may either A-resolve or B-resolve this crossing according to Figure 2. Note that interchanging overcrossings and undercrossings (called reflecting the link diagram) interchanges A-resolutions and B-resolutions. By A-resolving each crossing of $D(K)$ we form the all-A state of $D(K)$, which we will denote by $H_A$. The all-A state $H_A$ consists of a disjoint collection of all-A circles and a disjoint collection of dotted line segments, called A-segments, which are used record the locations of crossing resolutions. We will adopt the convention throughout this paper that any unlabeled segments are assumed to be A-segments.

Definition 2.1. A link diagram $D(K)$ is called A-adequate if $H_A$ does not contain any A-segments that join an all-A circle to itself. A link diagram is called semi-adequate if either it or its reflection is A-adequate. A link $K$ is called A-adequate if it has a diagram that is A-adequate and is called semi-adequate if it has a diagram that is semi-adequate.

Remark 2.1. While we will focus exclusively on A-adequate links, our results can easily be extended to semi-adequate links by reflecting the link diagram $D(K)$ and obtaining the corresponding results for B-adequate links.

Definition 2.2. From $H_A$ we may form the all-A graph, denoted $G_A$, by contracting the all-A circles to vertices and reinterpreting the A-segments as edges. From this graph we can form the
Figure 3: A link diagram $D(K)$, its all-A resolution $H_A$, its all-A surface $S_A$, its all-A graph $G_A$, and its reduced all-A graph $G'_A$.

reduced all-A graph, denoted $G'_A$, by replacing all multi-edges with a single edge. Said another way, we reduce the graph by removing all redundant parallel edges that join the same pair of vertices.

For an example of a diagram $D(K)$, its all-A resolution $H_A$, its all-A graph $G_A$, and its reduced all-A graph $G'_A$, see Figure 3.

**Notation:** Since reducing $G_A$ to form $G'_A$ leaves the vertices unchanged, then let $v(G'_A)$ denote the number of vertices of either $G'_A$ or $G_A$. Let $e(G_A)$ denote the number of edges of $G_A$ and let $e(G'_A)$ denote the number of edges of $G'_A$. Let $-\chi(G'_A) = e(G'_A) - v(G'_A)$ denote the negative Euler characteristic of $G'_A$.

**Remark 2.2.** Note that $v(G'_A)$ is the same as the number of all-A circles in $H_A$ and that $e(G_A)$ is the same as the number of A-segments in $H_A$. From a graphical perspective, A-adequacy of $D(K)$ can equivalently be defined by the condition that $G_A$ (or $G'_A$) contains no one-edge loops that connect a vertex to itself.

**Definition 2.3.** Define a twist region of $D(K)$ to be a longest possible string of crossings built from exactly two strands of $D(K)$. Denote the number of twist regions in $D(K)$ by $t(D)$ and call $t(D)$ the twist number of $D(K)$. See Figure 4 for an example.

Notice that it is potentially possible for a twist region to consist of a single crossing.

**Definition 2.4.** If a given twist region contains two or more crossings, then the A-resolution of this twist region will consist of portions of two all-A circles that are either:

1. joined by a path of A-segments and (small) all-A circles, or
2. joined by a string of parallel A-segments.
Figure 4: A link diagram with \( t(D) = 3 \) twist regions.

Figure 5: Long and short resolutions of a twist region of \( D(K) \).

Call a resolution in which (1) occurs a long resolution and call a resolution in which (2) occurs a short resolution. See Figure 5 for depictions of long and short resolutions. We will call a twist region long if its A-resolution is long and short if its A-resolution is short.

The following definition, which is very important, describes a key property of the link diagrams that will be considered in this work.

Definition 2.5. A link diagram \( D(K) \) satisfies the two-edge loop condition (TELC) if, whenever two all-A circles in \( H_A \) share a pair of A-segments, these segments correspond to crossings from the same short twist region of \( D(K) \).

Recall that the volume bounds from Theorem 1.1 are expressed in terms of two pieces of diagrammatic data, the first of which is the twist number \( t(D) \). The second piece of diagrammatic data, a count of specific types of alternating tangles in the diagram \( D(K) \), is defined below.

Definition 2.6. Call an alternating tangle in \( D(K) \) a special tangle if, up to planar isotopy, it consists of exactly one of the following:

1. a tangle sum, \( T(1,s) \), of a vertical one-crossing twist region and a vertical short twist region
2. a tangle sum, \( T(s_1,s_2) \), of two vertical short twist regions
3. a tangle sum, \( T(l,s) \), of a horizontal long twist region and a vertical short twist region

The values of \( l, s, s_1, s_2 \geq 1 \) denote the number of crossings in the relevant twist regions described above. To look for such tangles in \( D(K) \subseteq S^2 \), we look for simple closed curves in the projection sphere that intersect \( D(K) \) exactly four times (away from the crossings) and that contain a special tangle on one side of the curve. Equivalently, special tangles of \( D(K) \) can be found in the all-A state \( H_A \) by looking for all-A circles that are incident to A-segments from the resolution of a pair of twist regions from the tangle sums mentioned above. We will call these all-A circles special circles of \( H_A \). See Figure 6 for depictions of special tangles and special circles. Let \( st(D) \) denote the number of special tangles in \( D(K) \) (or, equivalently, the number of special circles in \( H_A \)).
The advantage to looking for special circles in $H_A$, as opposed to looking for special tangles in $D(K)$, is that special circles are necessarily disjoint. Special tangles, on the other hand, can share one or both twist regions with another special tangle.

2.2 Volume Bounds

The previous section provided the definitions of the terms involved in the statement of Theorem 1.1. We would now like to provide the mathematical background surrounding this theorem. First, to consider the hyperbolic volume of the link complement, we need to be able to ensure that our links are hyperbolic. The following result, due to Futer, Kalfagianni, and Purcell ([8]), does exactly this.

**Proposition 2.1** ([8]). Let $D(K)$ be a connected, prime, $A$-adequate link diagram that satisfies the TELC and contains $t(D) \geq 2$ twist regions. Then $K$ is hyperbolic.

From the all-$A$ state $H_A$ of the link diagram $D(K)$ we form the all-$A$ surface of $D(K)$, denoted $S_A$, by associating a disk to each all-$A$ circle and a half-twisted band to each $A$-segment, doing this in such a way that the boundary of $S_A$ is the link $K$ and so that inner disks lie above outer disks. See the top right side of Figure 3 for an example of an all-$A$ surface $S_A$. Note that $S_A$ can be properly embedded in the link complement $S^3 \setminus K$.

Recall that a surface $S$ is called *essential* in a 3-manifold $M$ if the boundary of a regular neighbor-
hood of $S$ is both incompressible and boundary-incompressible in $M$. (See [15] for more details.) As shown by Ozawa ([21]), the $A$-adequacy of a connected link diagram $D(K)$ implies that the all-$A$ surface $S_A$ is essential in the link complement $S^3 \setminus K$.

For a connected $A$-adequate link diagram $D(K)$ with essential all-$A$ surface $S_A$ in $S^3 \setminus K$, we form the 3-manifold $M_A = (S^3 \setminus K) \setminus S_A$ by cutting the the link complement along the all-$A$ surface. As proved in Lemma 2.4 of [10], the 3-manifold $M_A$ is homeomorphic to a handlebody and therefore contains no essential tori. By the annulus version of the Jaco-Shalen-Johannson Decomposition Theorem ([16], [17]) and by Lemma 4.1 of [10], we can cut $M_A$ along annuli (disjoint from the parabolic locus) to produce the following types of pieces:

(1) I-bundles over subsurfaces of $S_A$
(2) solid tori
(3) guts (which are the hyperbolic pieces with totally geodesic boundary)

Let $\chi_-(\text{guts}(M_A)) = \max\{-\chi(\text{guts}(M_A)), 0\}$. With the lower bound provided by Agol, Storm, and W. Thurston ([1]) and the upper bound provided by Agol and D. Thurston ([18], Appendix), we get the following volume bounds for the hyperbolic link complement.

**Proposition 2.2 ([1], [18]).** Let $D(K)$ be a connected, prime, $A$-adequate diagram of a hyperbolic link $K$. Then:

$$v_8 \cdot \chi_-(\text{guts}(M_A)) \leq \text{vol}(S^3 \setminus K) < 10v_3 \cdot (t(D) - 1),$$

where $v_8 = 3.6638\ldots$ denotes the volume of a regular ideal octahedron and $v_3 = 1.0149\ldots$ denotes the volume of a regular ideal tetrahedron.

**Assumption:** We will assume throughout this work that guts($M_A$) is nonempty. Otherwise, we would have that $\chi_-(\text{guts}(M_A)) = 0$ and, consequently, our lower bounds on volume would not be valuable.

By Remark 5.15 and Corollary 5.19 of [10], we get the following result:

**Theorem 2.1.** Let $D(K)$ be a connected, prime, $A$-adequate link diagram that satisfies the TELC. Then:

$$-\chi(G_A') = \chi_-(\text{guts}(M_A)).$$

By combining the above results together, we get the following theorem.

**Theorem 2.2.** Let $D(K)$ be a connected, prime, $A$-adequate link diagram that satisfies the TELC and contains $t(D) \geq 2$ twist regions. Then $K$ is hyperbolic and:

$$-v_8 \cdot \chi(G_A') = v_8 \cdot \chi_-(\text{guts}(M_A)) \leq \text{vol}(S^3 \setminus K) < 10v_3 \cdot (t(D) - 1).$$
Proof. By combining Proposition 2.1 with Proposition 2.2 and Theorem 2.1 we get the desired results.

2.3 The Braid Group and Braid Closure

Recall that the braid group on \( n \) strings, called the \( n \)-braid group for short, has Artin presentation:

\[
B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \text{ and } \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle,
\]

where the first relation (sometimes called far commutativity) occurs whenever \(|i - j| \geq 2\) and \(1 \leq i < j \leq n - 1\), and where the second relation occurs whenever \(1 \leq i \leq n - 2\).

As a special case, the \( 3 \)-braid group has presentation:

\[
B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle.
\]

The braid group and its generators can also be represented geometrically. See Figure 7. The multiplication operation in the braid group is concatenation. Furthermore, we will adopt the convention that the braid word, read from left to right, is visually represented by stacking the braid generators of Figure 7 vertically (reading the braid word from the top down).

Let \( \tilde{\beta} \) denote the closure of the \( n \)-braid \( \beta \). To form the braid closure means that, for all braid string positions \( 1 \leq i \leq n \), we identify the top of the string in the \( i \)th position with the bottom of string in the \( i \)th position. The closure of a braid can be schematically represented as in Figure 8.

**Definition 2.7.** Call a subword \( \gamma \) of \( \beta \in B_n \) (cyclically) induced if the letters of \( \gamma \) are all (cyclically) adjacent and appear in the same order as in the full braid word \( \beta \).

As an example, given the braid word:

\[
\beta = \sigma_1^3 \sigma_2^{-3} \sigma_1^2 \sigma_3^{-2} \sigma_2 \sigma_3 = \sigma_1^3 \sigma_2^{-1} (\sigma_2^{-2} \sigma_1^2 \sigma_3^{-1}) \sigma_3^{-1} \sigma_2 \sigma_3,
\]

the subword \( \gamma = \sigma_2^{-2} \sigma_1^2 \sigma_3^{-1} \) is an induced subword of \( \beta \) but the subword \( \gamma' = \sigma_2^{-2} \sigma_1^2 \sigma_2 \) is **not** an induced subword of \( \beta \).
2.4 A-Adequacy for Closed 3-Braids

Because conjugate braids have isotopic braid closures and because our focus will ultimately be on links that are braid closures, we will usually work within conjugacy classes of braids. Furthermore, note that cyclic permutation is a special case of conjugation.

Definition 2.8. As in [24], a braid \( \beta = \sigma_{m_1}^{r_1} \sigma_{m_2}^{r_2} \cdots \sigma_{m_l}^{r_l} \in B_n \) is called \textit{cyclically reduced into syllables} if the following hold:

1. \( r_i \neq 0 \) for all \( i \)
2. there are no occurrences of induced subwords of the form \( \sigma_i \sigma_i^{-1} \) or \( \sigma_i^{-1} \sigma_i \) for any \( i \) (looking up to cyclic permutation)
3. \( m_i \neq m_{i+1} \) for all \( i \) (modulo \( l \))

Definition 2.9. Let \( \beta = \sigma_{i_1}^{r_{i_1}} \sigma_{j_1}^{r_{j_1}} \cdots \sigma_{i_l}^{r_{i_l}} \sigma_{j_l}^{r_{j_l}} \in B_3 \) denote a 3-braid that has been cyclically reduced into syllables, where \( \{i, j\} = \{1, 2\} \). As was done in [24], form the \textit{exponent vector} \( (r_1, r_2, \ldots, r_3) \).

Definition 2.10. Call a braid \( \beta \in B_n \) \textit{positive} if all of the exponents \( r_i \) are positive and \textit{negative} if all of the exponents \( r_i \) are negative.

As indicated in the introduction, the family of A-adequate links is a rich family of links about which much can be said. Consequently, it will be useful to be able to determine when a given closed braid is A-adequate. In what follows, we present Stoimenow’s ([24]) classification of A-adequate closed
Proposition 2.3 ([24], Lemma 6.1). Let $D(K) = \hat{\beta}$ denote the closure of a 3-braid:

$$\beta = \sigma_i^{r_1} \sigma_j^{r_2} \cdots \sigma_i^{r_{2l-1}} \sigma_j^{r_{2l}},$$

where $\{i, j\} = \{1, 2\}$ and where $\beta$ has been cyclically reduced into syllables. Furthermore, assume that $l \geq 2$ (which says that the exponent vector has length at least four). Then $D(K)$ is A-adequate if and only if either:

1. $\beta$ is positive, or
2. $\beta$ does not contain $\sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1} = \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1}$ as a cyclically induced subword and $\beta$ also has the property that all positive entries of the exponent vector are cyclically isolated (meaning they must be adjacent to negative entries on both sides).

Remark 2.3. The condition that positive entries of the exponent vector be cyclically isolated can equivalently be phrased as the condition that positive syllables in the braid word (of the form $\sigma_i^p$ for $p > 0$) must be cyclically adjacent to negative syllables (of the form $\sigma_j^{n'}$ for $n' < 0$) on both sides, where $\{i, j\} = \{1, 2\}$.

Proof of Proposition. ($\Rightarrow$) We shall proceed by contraposition.

Case 1: Suppose $\beta$ is not positive and contains $\sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1} = \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1}$ as a cyclically induced subword. From the portion of the all-A state $H_A$ corresponding to this subword, it can be seen that $D(K)$ is not A-adequate. (See Figure 9.)

Case 2: Suppose $\beta$ is not positive and there exist cyclically adjacent positive syllables in the braid word. Since $\beta$ is not positive, then there must exist at least one negative syllable in the braid word. Choose a pair of cyclically adjacent positive syllables that are immediately followed by a negative syllable. Hence, $\beta$ must contain an induced subword of the form $\sigma_i^{p_1} \sigma_j^{p_2} \sigma_i^{n'}$, where $p_1, p_2 > 0$ are positive integers, where $n' < 0$ is a negative integer, and where $\{i, j\} = \{1, 2\}$. From the portion of the all-A state $H_A$ corresponding to this subword, it can be seen that $D(K)$ is not A-adequate. (See Figure 10.)
Figure 10: The subdiagram of a closed 3-braid diagram corresponding to a subword $\sigma_1^{p_1} \sigma_2^{p_2} \sigma_1^{n'}$, where the $p_i$ are positive integers and $n'$ is a negative integer (left), and the corresponding portion of the all-A state that exhibits the non-A-adequacy of $D(K)$ (right).

Figure 11: A positive closed 3-braid diagram (left) and the corresponding all-A state that exhibits the A-adequacy of $D(K)$ (right).
Figure 12: Potential non-A-adequacy coming from $\sigma_1^{-1}$. From the perspective of the all-A state $H_A$, the vertical segment $s$ coming from $\sigma_1^{-1}$ is assumed to join a state circle $C$ to itself (left). Also depicted are the case where $\sigma_1^{-1}$ is followed by another copy of $\sigma_1^{-1}$ (center) and the case where $\sigma_1^{-1}$ is followed a positive syllable $\sigma_p^2$, which must then be followed by the letter $\sigma_1^{-1}$ (right).

(⇐) We now show that Condition (1) and Condition (2) of the proposition each imply that $D(K)$ is A-adequate.

Case 1: Assume $D(K)$ is a positive closed 3-braid. From the corresponding all-A state $H_A$ it can be seen that $D(K)$ is A-adequate. (See Figure 11)

Case 2: Assume $\beta$ does not contain $\sigma_1^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_1^{-1}$ as a cyclically induced subword and assume $\beta$ has the property that all positive entries of the exponent vector are cyclically isolated (which implies that $\beta$ cannot be positive). For a contradiction, suppose $D(K)$ is not A-adequate.

Subcase 1: Suppose non-A-adequacy comes from the occurrence of $\sigma_1^{-1}$ (or, symmetrically, $\sigma_2^{-1}$) in the braid word. Since the argument is very similar, we will not consider the symmetric case. Having non-A-adequacy in this setting means that when we A-resolve the crossing of $D(K)$ corresponding to $\sigma_1^{-1}$, the corresponding vertical A-segment of $H_A$, call it $s$, will connect an all-A circle, call it $C$, to itself. (See the left side of Figure 12) Let us now consider what other letters can surround $\sigma_1^{-1}$ in the braid word. First, notice that $\sigma_1^{-1}$ cannot be surrounded by other copies of $\sigma_1^{-1}$ on either side, as this will contradict how $C$ must close up. (See the center of Figure 12) Second, $\sigma_1^{-1}$ cannot be surrounded on both sides by $\sigma_2^{-1}$, as this would imply the existence of a forbidden subword $\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}$, a contradiction. Thus, it must be the case that $\sigma_1^{-1}$ is surrounded on at least one side by a positive syllable $\sigma_p^2$. Furthermore, since the exponent vector is assumed to have length at least four and since positive entries are assumed to be cyclically isolated, then the next adjacent letter after the positive syllable $\sigma_p^2$ must be $\sigma_1^{-1}$. But then it can be seen that the A-resolution of the subword $\sigma_1^{-1}\sigma_p^2\sigma_1^{-1}$ contradicts how $C$ must close up. (See the right side of Figure 12)

Subcase 2: Suppose non-A-adequacy comes from the occurrence of a positive syllable $\sigma_1^p$ (or, symmetrically, $\sigma_2^p$) in the braid word. Since the argument is very similar, we will not consider the symmetric case. Having non-A-adequacy in this setting means that when we A-resolve the twist region of $D(K)$ corresponding to $\sigma_1^p$, the corresponding horizontal A-segments will connect a state circle, call it $C$, to itself. Since the exponent vector is assumed to have length at least four and since positive entries are assumed to be cyclically isolated, then $\sigma_1^p$ must be surrounded by the same
Figure 13: Potential non-A-adequacy coming from a positive syllable $\sigma_1^p$. From the perspective of the all-A state $H_A$, the horizontal A-segments coming from $\sigma_1^p$ are assumed to join a state circle $C$ to itself.

letter $\sigma_2^{-1}$ on both sides. But then it can be seen that the A-resolution of the subword $\sigma_2^{-1} \sigma_1^p \sigma_2^{-1}$ contradicts how $C$ must close up. (See Figure 13.)

2.5 State Circles of A-Adequate Closed 3-Braids

The goal of this section is to classify the possible types of all-A circles that can arise in the all-A state of an A-adequate closed 3-braid.

Proposition 2.4. Let $\beta = \sigma_i^{r_i} \sigma_j^{r_j} \cdots \sigma_i^{r_{i1}} \sigma_j^{r_{j1}}$ denote a 3-braid that has been cyclically reduced into syllables, where $l \geq 2$ and $\{i,j\} = \{1,2\}$. Assume that $D(K) = \hat{\beta}$ is an A-adequate closed 3-braid. Then we may categorize the all-A circles of $H_A$ into the following types:

(1) small inner circles that come from negative exponents $r_i \leq -2$ in the braid word $\beta$

(2) medium inner circles that come from cyclically isolated positive syllables in the braid word $\beta$

(3) wandering circles whose wandering arises from adjacent negative syllables in the braid word $\beta$

(4) nonwandering circles that come from the case when a given generator $\sigma_i$ occurs with only positive exponents in the braid word $\beta$

For depictions of the types of circles mentioned above (or portions of the types of circles mentioned above), see Figure 14.

Remark 2.4. From the 3-braid perspective, it will be important to note that the left half of Figure 5 depicts the A-resolution of a negative syllable (excluding a portion of a third trivial braid string). Similarly, the right half of Figure 5 depicts the A-resolution of a positive syllable (excluding a portion of a third trivial braid string).

Proof of Proposition. Recall that A-adequate closed 3-braids $D(K) = \hat{\beta}$ were classified by Proposition 2.3 into two main types.
Case 1: Suppose $\beta$ is positive. Then each all-A circle in the all-A state $H_A$ of $D(K)$ is a nonwandering circle. (See Figure 11.)

Case 2: Suppose $\beta$ does not contain $\sigma_{2}^{-1} \sigma_{1}^{2} \sigma_{2}^{-1} \sigma_{1}^{2}$ as a cyclically induced subword and suppose $\beta$ also has the property that all positive syllables are cyclically isolated. Since positive syllables of $\beta$ are cyclically isolated, then (using cyclic permutation if needed) we may decompose $\beta$ as $\beta = N_1$ (in the case that $\beta$ is a negative braid) or $\beta = P_1 N_1 \cdots P_t N_t$, where $P_i$ denotes a positive syllable of $\beta$ and $N_i$ denotes a maximal length negative induced subword of $\beta$.

Small Inner Circles: Let $\sigma_{i}^n \sigma_{j}^n$ denote a negative syllable. Then, except for the additional vertical trivial braid string portion to the right or left (depending on whether $i = 1$ or $i = 2$), the A-resolution of this syllable will look like the left side of Figure 14 if $n = -2$ and look like the left side of Figure 5 in general. In particular, having $n \leq -2$ is equivalent to the existence of small inner circles.

Wandering Circles: Let $\sigma_{i}^{n_1} \sigma_{j}^{n_2} = \sigma_{i}^{-1} \sigma_{i}^{-1} \sigma_{j}^{-1} \sigma_{j}^{-1}$ denote a pair of adjacent negative syllables, where $\{i, j\} = \{1, 2\}$. Then the corresponding portion of $H_A$ will resemble the right side of Figure 14 (except that the long resolutions may possibly consist of longer paths of A-segments and small inner circles). The key feature of this figure is the fact that we see a portion of a wandering circle, where the wandering behavior corresponds to the existence of the $\sigma_{i}^{-1} \sigma_{j}^{-1}$ induced subword.

Medium Inner Circles and Nonwandering Circles: Let $\sigma_{i}^{-1} \sigma_{j}^{p} \sigma_{i}^{-1}$ denote a positive syllable that is cyclically surrounded by negative letters, where $\{i, j\} = \{1, 2\}$. Then the corresponding portion of $H_A$ will resemble the center of Figure 14 In particular, the existence of a (cyclically isolated) positive syllable corresponds to the existence of a medium inner circle. Furthermore, a generator $\sigma_1$ or $\sigma_2$ occurring with only positive exponents (again, see the center of Figure 14) will correspond to a nonwandering circle.

Since the A-resolutions of all portions of the closure of $\beta = N_1$ and $\beta = P_1 N_1 \cdots P_t N_t$ have
been considered locally and since gluing such portions together joins wandering and potential nonwandering circle portions together, then we have the desired result.

\[ \square \]

3 Volume Bounds for A-Adequate Closed Braids

3.1 Primeness, Connectedness, and Twistedness for Closed Braid Diagrams

Assume that an \( n \)-braid \( \beta \in B_n \) has been cyclically reduced into syllables and let \( D(K) = \hat{\beta} \). The main goal of this section is to find sufficient conditions for the diagram \( D(K) \) to be prime.

**Definition 3.1.** Call two induced subwords \( \gamma_1 \) and \( \gamma_2 \) of a given braid word \( \beta \) disjoint if the subwords share no common letters when they are viewed as part of \( \beta \).

As an example, given the braid word:

\[ \beta = \sigma_1^2 \sigma_2^{-3} \sigma_1^2 \sigma_3^{-2} \sigma_2 \sigma_3 = \sigma_1^2 (\sigma_1 \sigma_2^{-3} \sigma_1) \sigma_1 (\sigma_3^{-2} \sigma_2) \sigma_3, \]

the subwords \( \gamma_1 = \sigma_1 \sigma_2^{-3} \sigma_1 \) and \( \gamma_2 = \sigma_3^{-2} \sigma_2 \) are disjoint induced subwords of \( \beta \).

**Definition 3.2.** Call a subword \( \gamma \in B_n \) of a braid word \( \beta \in B_n \) complete if it contains, at some point, each of the generators \( \sigma_1, \ldots, \sigma_{n-1} \) of \( B_n \).

It is important to note that the generators \( \sigma_1, \ldots, \sigma_{n-1} \) need not occur in any particular order and that repetition of some or all of these generators is allowed.

**Proposition 3.1.** Let \( \beta \in B_n \) be cyclically reduced into syllables. If \( \beta \) contains two disjoint induced complete subwords \( \gamma_1 \) and \( \gamma_2 \), then \( D(K) = \hat{\beta} \) is a prime link diagram.

**Remark 3.1.** Note that, in the case when \( n = 3 \), the condition that \( \beta \) contains two disjoint induced complete subwords is equivalent to the condition that \( \sigma_1 \) and \( \sigma_2 \) each occur at least twice nontrivially in the braid word \( \beta \) is equivalent to the condition that the exponent vector of \( \beta \) has length at least four.

**Proof of Proposition.** Let \( \gamma_1 \) and \( \gamma_2 \) be two disjoint induced complete subwords of \( \beta \in B_n \). See Figure 15 for a schematic depiction of this situation. Let \( C \) be a simple closed curve that intersects \( D(K) = \hat{\beta} \) exactly twice (away from the crossings). We need to show that \( C \) cannot contain crossings of \( D(K) \) on both sides. Note that we may view a closed braid diagram as lying in an annular region of the plane. (See Figure 8.)

Suppose \( C \) contains a point \( p \) that lives outside of this annular region. Since \( C \) only intersects \( D(K) \) twice and must start and end at \( p \), then it must be the case that \( C \) intersects \( D(K) \) twice
in braid string position 1 or twice in braid string position $n$. In either case, it is impossible for such a closed curve $C$ to contain crossings on both sides. This is because $C$ would need to intersect $D(K)$ more than twice to be able to close up and surround crossings on both sides, and this would contradict the assumption that $C$ intersects $D(K)$ exactly twice.

Suppose $C$ contains a point $p$ between braid string positions $i$ and $i + 1$ for some $1 \leq i \leq n - 1$. See Figure 16. Since $C$ intersects $D(K)$ exactly twice and since $C$ is a simple closed curve, then it must be the case that $C$ either intersects $D(K)$ twice in braid string position $i$ or twice in braid string position $i + 1$. Since $\beta$ contains two disjoint induced complete subwords, then the generator $\sigma_i$, the generator $\sigma_{i-1}$ (if it exists), and the generator $\sigma_{i+1}$ (if it exists) must each occur at least twice, once cyclically before $p$ and once cyclically after $p$. Since $\sigma_{i-1}$ and $\sigma_{i+1}$ occur both before and after $p$, then it is impossible for $C$ to both close up and contain crossings on both sides. This is because $C$ is forced to cross a braid string position before encountering $\sigma_i^\pm$ (cyclically above and below), and because the occurrences of $\sigma_{i-1}^\pm$ and $\sigma_{i+1}^\pm$ (cyclically above and below) block $C$ from being able to close up in a way that will contain crossings on both sides.

The assumptions about subwords of $\beta$ in Proposition 3.1 also force $D(K)$ to have other desired properties, as seen in the proposition below.

**Proposition 3.2.** Let $\beta \in B_n$ be cyclically reduced into syllables. If $\beta$ contains two disjoint induced complete subwords $\gamma_1$ and $\gamma_2$, then $D(K) = \hat{\beta}$ is a connected link diagram with $t(D) \geq 2(n - 1)$ twist regions.
Proof. Recall that $D(K)$ is connected if and only if the projection graph of $D(K)$ is path-connected. Since $\beta \in B_n$ contains a complete subword, then each generator of $B_n$ (each of which corresponds to a crossing of $D(K)$ between adjacent braid string positions) must occur at least once. This fact implies that the closed $n$-braid diagram $D(K)$ must be connected. (See Figure 8.)

Since $\beta$ contains two disjoint complete subwords, then it must be that each of the generators $\sigma_1, \ldots, \sigma_{n-1}$ of $B_n$ must occur at least twice in (two distinct syllables of) the braid word. Since the syllables of the cyclically reduced braid word $\beta$ correspond to the twist regions of $D(K)$, then we have that $t(D) \geq 2(n - 1)$.

Because the majority of the braids considered in this paper will satisfy the same set of assumptions, we make the following definition.

Definition 3.3. Call a braid $\beta \in B_n$ nice if the following hold:

1. $\beta$ is cyclically reduced into syllables.
2. $\beta$ contains two disjoint induced complete subwords $\gamma_1$ and $\gamma_2$. 

Figure 16: A schematic depiction of the fact that a simple closed curve $C$ intersecting $D(K)$ twice and containing a point $p$ (between braid string positions $i$ and $i+1$) cannot contain crossings on both sides. Each box in the figure represents the eventual occurrence of the generator $\sigma_i$, the generator $\sigma_{i-1}$ (if it exists), and the generator $\sigma_{i+1}$ (if it exists). No assumptions are made about the order in which these generators appear or the frequency with which these generators appear.
3.2 The Foundational Theorem for Closed Braids

**Theorem 3.1.** Let \( D(K) = \hat{\beta} \) denote the closure of a nice \( n \)-braid \( \beta \) of the form:

\[
\beta = \sigma_{m_1}^{r_1} \cdots \sigma_{m_l}^{r_l} \in B_n,
\]

where \( 1 \leq m_1, \ldots, m_l \leq n - 1 \). If \( \beta \) satisfies the conditions:

(1) all negative exponents \( r_i < 0 \) in \( \beta \) satisfy the stronger requirement that \( r_i \leq -3 \), and

(2) when \( r_i > 0 \), we have that \( r_{i-1} < 0 \) and \( r_{i+1} < 0 \) and that either \( m_{i-1} = m_{i+1} = m_i + 1 \) or \( m_{i-1} = m_{i+1} = m_i - 1 \),

then \( D(K) \) is a connected, prime, \( A \)-adequate link diagram that satisfies the TELC and contains \( t(D) \geq 2(n - 1) \) twist regions. Furthermore, we have that \( K \) is a hyperbolic link.

**Remark 3.2.** Condition (2) above says that positive syllables in the letter \( \sigma_{m_i} \) are cyclically adjacent to negative syllables in the same adjacent letter (either \( \sigma_{m_i+1} \) or \( \sigma_{m_i-1} \)). This condition extends to \( n \)-braids Stoimenow’s 3-braid condition (see Proposition 2.3) that positive entries in the exponent vector are cyclically isolated. Condition (2) also implies that \( \beta \) cannot be a positive braid. Furthermore, Condition (1) above trivially satisfies Stoimenow’s condition that the 3-braid does not contain the induced subword \( \sigma_{-1} \sigma_{-1} \sigma_{-1} \). Hence, to conclude \( A \)-adequacy (among other things) for braids with \( n \geq 3 \) strings, we generalize one of Stoimenow’s two conditions and make an assumption stronger than the other condition.

**Definition 3.4.** Since \( \beta \in B_n \) is cyclically reduced into syllables, then the positive syllables \( \sigma_{m_i}^p \) of \( \beta \) (where \( p > 0 \)) correspond to what we will call positive twist regions of \( D(K) \) and the negative syllables \( \sigma_{m_i}^{-n'} \) of \( \beta \) (where \( n' < 0 \)) correspond to what we will call negative twist regions of \( D(K) \).

**Proof of Theorem.** From Proposition 3.1 and Proposition 3.2 we have already seen that \( D(K) \) is connected and prime with \( t(D) \geq 2(n - 1) \) twist regions. Furthermore, once it is shown that \( D(K) \) is \( A \)-adequate and satisfies the TELC, then the conclusion that \( K \) is hyperbolic will follow from Proposition 2.1.

To prove that \( D(K) \) is \( A \)-adequate, we show that no \( A \)-segment of the all-\( A \) state \( H_A \) joins an all-\( A \) circle to itself. Note that positive syllables in \( \beta \) (positive twist regions of \( D(K) \)) \( A \)-resolve to give horizontal segments and that negative syllables in \( \beta \) (negative twist regions of \( D(K) \)) \( A \)-resolve to give vertical segments. Suppose, for a contradiction, that an \( A \)-segment joins an all-\( A \) circle to itself.

**Case 1:** Suppose the \( A \)-segment is a vertical segment. Condition (1), that negative exponents are at least three in absolute value, implies that it is impossible for a vertical \( A \)-segment to join an all-\( A \) circle to itself. This is because all vertical \( A \)-segments either join distinct small inner circles or join a small inner circle to a medium or a wandering circle. (See the left side of Figure 5.)
Case 2: Suppose the A-segment is a horizontal segment. Condition (2), that positive syllables are cyclically surrounded by negative syllables in the same adjacent generator, implies that it is impossible for a horizontal A-segment to join an all-A circle to itself. This is because all horizontal A-segments join a medium inner circle to either a wandering circle or a nonwandering circle. (See Figure 13)

We will now show that $D(K)$ satisfies the TELC. Let $C_1$ and $C_2$ be two distinct all-A circles that share a pair of distinct A-segments, call them $s_1$ and $s_2$.

Case 1: Suppose one of $s_1$ and $s_2$ is a horizontal segment. By Conditions (1) and (2), it is impossible for the other segment to be a vertical segment. This is because, as in Figure 13, one of $C_1$ and $C_2$ must be a medium inner circle. Let us say that $C_1$ is a medium inner circle. This circle is adjacent, via horizontal A-segments, to the second circle $C_2$, which will either be a wandering circle or a nonwandering circle. The circle $C_1$ is also adjacent to small inner circles above and below. Therefore, since a small inner circle is neither wandering nor nonwandering, then the second segment cannot be vertical. Thus, if one of the segments $s_1$ and $s_2$ is horizontal, then the other segment must also be horizontal. Since the horizontal segments incident to the medium inner circle $C_1$ necessarily belong to the same short twist region of $D(K)$, then the TELC is satisfied.

Case 2: Suppose both $s_1$ and $s_2$ are vertical segments. Since all vertical A-segments either join distinct small inner circles or join a small inner circle to a medium inner circle or a wandering circle, then one of $C_1$ and $C_2$ must be a small inner circle. But, by construction, it is impossible for a small inner circle to share more than one A-segment with another circle. Therefore, this case cannot occur.

3.3 State Circles of A-Adequate Closed $n$-Braids (where $n \geq 4$)

The goal of this section is to classify the possible types of all-A circles that can arise in the all-A state of an A-adequate closed $n$-braid, where $n \geq 4$ and where the braid is of the type described in Theorem 3.3

Proposition 3.3. Let $D(K) = \hat{\beta}$ denote the closure of a nice $n$-braid $\beta$ of the form:

$$\beta = \sigma_{m_1}^{r_1} \cdots \sigma_{m_l}^{r_l} \in B_n,$$

where $1 \leq m_1, \ldots, m_l \leq n-1$. Furthermore, assume that $\beta$ satisfies the assumptions of Theorem 3.1 and assume that $n \geq 4$. View $D(K)$ as lying in an annular region of the plane. Then we may categorize the all-A circles of $H_A$ into the following types:

1. small inner circles that come from negative exponents $r_i \leq -2$ in the braid word $\beta$

2. medium inner circles that come from cyclically isolated positive syllables in the braid word $\beta$
(3a) essential wandering circles that are essential in the annulus and have wandering that arises from adjacent negative syllables in the braid word $\beta$

(3b) non-essential wandering circles that are non-essential (contractible) in the annulus and have wandering that arises from adjacent negative syllables in the braid word $\beta$

(4) nonwandering circles that come from the cases when $\sigma_1$ or $\sigma_{n-1}$ or an adjacent pair of generators $\sigma_i$ and $\sigma_{i+1}$ for $2 \leq i \leq n-3$ occur with only positive exponents in the braid word $\beta$.

Remark 3.3. Note that, as compared with the $n = 3$ case of Section 2.5, we now have that a new type of wandering circle, called a non-essential wandering circle, is possible. Also note that neither the pair of generators $\sigma_1$ and $\sigma_2$ nor the pair of generators $\sigma_{n-2}$ and $\sigma_{n-1}$ can occur with only positive exponents. This is due to Condition (2) of Theorem 3.1 (the condition that positive syllables are cyclically surrounded by negative syllables in the same adjacent generator).

Proof of Proposition. With some exceptions, this proof is similar to the proof of Proposition 2.4. Recall that, as noted in Remark 3.2, $\beta$ cannot be positive. Because positive syllables of $\beta$ are cyclically isolated, we may decompose $\beta$ as $\beta = N_1 \cdots N_t$, where $P_i$ denotes a positive syllable of $\beta$ and $N_i$ denotes a maximal length negative induced subword of $\beta$.

Small Inner Circles: Let $\sigma_i^m$ denote a negative syllable. Then, except for the additional $n-2$ vertical trivial braid string portions to the right and/or left of the twist region, the A-resolution of this syllable will look like the left side of Figure 14 if $n' = 2$ and look like the left side of Figure 5 in general. In particular, having $n' \leq -2$ is equivalent to the existence of small inner circles.

Wandering Circles: Let $\sigma_i^{n_1}\sigma_j^{n_2} = \sigma_i^{n_1+1}(\sigma_i^{-1}\sigma_j^{-1})\sigma_j^{n_2+1}$ denote a pair of adjacent negative syllables.

Case 1: Suppose $\{i, j\} = \{m_q, m_q+1\}$. Then the pair of adjacent negative syllables involve adjacent letters (generators of $B_n$). Consequently, the corresponding portion of $H_A$ will resemble the right side of Figure 14 (except that the long resolutions may possibly consist of longer paths of A-segments and small inner circles). The key feature of this figure is the fact that we see a portion of a wandering circle, where the wandering behavior corresponds to the existence of the $\sigma_i^{-1}\sigma_j^{-1}$ induced subword.

Case 2: Suppose $\{i, j\} = \{m_q, m_{q+r}\}$, where $r \geq 2$. In this case, the pair of adjacent negative syllables involve far commuting generators of $B_n$. Then, except for the additional $n-4$ vertical trivial braid string portions around the two negative twist regions, we get that the A-resolution will look like two copies of the left side of Figure 5. Thus, we return to a case that has already been considered.

Medium Inner Circles and Nonwandering Circles: Let $\sigma_i^{-1}\sigma_j^p\sigma_i^{-1}$ denote a positive syllable of $\beta$ that is cyclically surrounded by negative letters, where $\{i, j\} = \{m_q, m_{q+1}\}$. Note that the letters
involved in this induced subword are adjacent generators of $B_n$. Consequently, by adding $n - 3$ trivial braid string portions, the corresponding portion of $H_A$ will resemble the center of Figure 14. In particular, the existence of an isolated positive syllable corresponds to the existence of a medium inner circle. Furthermore, a nonwandering circle will occur in $H_A$ precisely when $\sigma_1$ or $\sigma_{n-1}$ or an adjacent pair of generators $\sigma_i$ and $\sigma_{i+1}$ for $2 \leq i \leq n - 3$ occur with only positive exponents.

Note that we may classify the wandering circles in the annulus into essential and nonessential circles. Since the $A$-resolutions of all portions of the closure of $\beta = N_1$ and $\beta = P_1 N_1 \cdots P_t N_t$ have been considered locally and since gluing such portions together joins wandering and potential nonwandering circle portions together, then we have the desired result.

\[\square\]

### 3.4 Computation of $-\chi(G'_A)$

**Definition 3.5.** We call an all-$A$ circle of $H_A$ an other circle (OC) if it is not a small inner circle.

To find lower bounds on volume for the closed braids considered in this paper, we need the following lemma from [14].

**Lemma 3.1 ([14]).** Let $D(K)$ be a connected $A$-adequate link diagram that satisfies the TELC. Then we have that:

$$-\chi(G'_A) = t(D) - \# \{\text{OCs}\}.$$  

**Notation:** Let $t^+(D)$ denote the number of positive (short) twist regions in $D(K)$ and $t^-(D)$ denote the number of negative (long) twist regions in $D(K)$.

To compute $-\chi(G'_A)$, we will consider the cases $n = 3$ and $n \geq 4$ separately. This is because it is only in the case that $n \geq 4$ that non-essential wandering circles can exist.

**Case 1:** Suppose $n = 3$.

**Lemma 3.2.** Let $D(K) = \hat{\beta}$ denote the closure of a nice 3-braid $\beta \in B_3$. Furthermore, assume that the positive syllables of $\beta$ are cyclically isolated. Then the all-$A$ state, $H_A$, of $D(K)$ satisfies precisely one of the following two properties:

1. $H_A$ contains exactly one nonwandering circle and no wandering circles.
2. $H_A$ contains exactly one wandering circle and no nonwandering circles.

**Proof.** Either $\beta$ is alternating or $\beta$ is nonalternating.

Suppose $\beta$ is alternating. Then one of the braid generators must always occur with positive exponents and the other generator must always occur with negative exponents. Hence, by Proposition 3.3, since a generator occurs with only positive exponents, then there will be a nonwandering
circle. Since only one generator occurs with only positive exponents, then there is only one such nonwandering circle. Also, since adjacent negative syllables cannot occur in $\beta$, then wandering circles cannot occur in $H_A$.

Suppose $\beta$ is nonalternating. Since positive syllables are cyclically isolated, then the nonalternating behavior of $\beta$ must come from a pair of adjacent negative syllables. This implies the existence of a wandering circle in $H_A$ and prevents the existence of a nonwandering circle (since both generators occur once with negative exponent). Finally, since a wandering circle (which is always essential in the case that $n = 3$) “uses up” a braid string from the braid closure and since a wandering circle must wander from braid string position 1 to braid string position 3 and back before closing up, then the existence of a second such wandering circle is impossible.

**Theorem 3.2.** Let $D(K) = \hat{\beta}$, where $\beta \in B_3$ satisfies the assumptions of Theorem 3.1. Then:

$$-\chi(G'_A) = t^-(D) - 1 \geq \frac{1}{2} \cdot (t(D) - 1) - \frac{1}{2}.$$  

*Proof.* Since Theorem 3.1 guarantees that $D(K)$ is connected, $A$-adequate, and satisfies the TELC, then by Lemma 3.1 we have that:

$$-\chi(G'_A) = t(D) - \#\{OCs\}.$$  

By Proposition 2.4, we have classified the types of all-A circles. By Condition (2) of Theorem 3.1 (that positive syllables are cyclically isolated) and by inspecting the A-resolution of a positive twist region (see Figure 13), we see that the number of medium inner circles in $H_A$ is exactly the number of positive twist regions in $D(K)$, which we have denoted by $t^+(D)$. By Lemma 3.2, we know that the total number of wandering circles and nonwandering circles is one. Again using the assumption that positive syllables are cyclically isolated, we have that at least half of the twist regions in $D(K)$ must be negative twist regions. Therefore, using all of what was said above (in order) gives:

$$-\chi(G'_A) = t(D) - \#\{OCs\}$$
$$= t(D) - \#\{medium inner circles\} - \#\{wandering circles\}$$
$$= t(D) - t^+(D) - 1$$
$$= t^-(D) - 1$$
$$\geq \frac{t(D)}{2} - 1$$
$$= \frac{1}{2} \cdot (t(D) - 1) - \frac{1}{2}.$$  

$\square$
**Case 2**: Suppose $n \geq 4$.

**Theorem 3.3.** For $n \geq 4$, let $D(K) = \hat{\beta}$, where $\beta \in B_n$ satisfies the assumptions of Theorem 3.1. Let $m$ denote the number of non-essential wandering circles in the all-$A$ state $H_A$ of $D(K)$. Then:

$$-\chi(G'_A) \geq t^-(D) - (n + m - 2) \geq \frac{1}{2} \cdot (t(D) - 1) - \frac{1}{2} \cdot (2(n + m) - 5).$$

**Proof.** By Lemma 3.1 we have that $-\chi(G'_A) = t(D) - \# \{\text{OCs}\}$. By Proposition 3.3, we have classified the types of all-$A$ circles. As discussed in the proof of Theorem 3.2, the number of medium inner circles in $H_A$ is exactly the number of positive twist regions in $D(K)$ and at least half of the twist regions in $D(K)$ must be negative twist regions.

Since Condition (2) of Theorem 3.1 implies that $\beta$ is not a positive braid, then there must be at least one negative syllable in the braid word. Also, note that both essential wandering circles and nonwandering circles “use up” a braid string from the braid closure. The previous two facts imply that:

$$\# \{\text{essential wandering circles}\} + \# \{\text{nonwandering circles}\} \leq n - 2.$$

Therefore, using all of what was said above gives:

$$-\chi(G'_A) = t(D) - \# \{\text{OCs}\}$$

$$= t(D) - \# \{\text{medium inner circles}\} - \# \{\text{non-essential wandering circles}\}$$

$$- \# \{\text{essential wandering circles}\} - \# \{\text{nonwandering circles}\}$$

$$\geq t(D) - t^+(D) - m - (n - 2)$$

$$= t^-(D) - (n + m - 2)$$

$$\geq \frac{t(D)}{2} - (n + m - 2)$$

$$= \frac{1}{2} \cdot (t(D) - 1) - \frac{1}{2} \cdot (2(n + m) - 5).$$

☐

**Remark 3.4.** Recall that, by Proposition 3.2, we have that $t(D) \geq 2(n - 1)$. Since at least half of the twist regions of $D(K)$ are negative, then:

$$t^-(D) \geq \frac{t(D)}{2} \geq n - 1,$$

which implies that:

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\[-\chi(G'_A) \geq t^-(D) - (n + m - 2) \geq 1 - m.\]

Note in particular that, for \(m = 0\), we are able to conclude that the lower bound, \(-v_8 \cdot \chi(G'_A)\), on volume given by Theorem 2.2 will be positive.

### 3.5 Applying the Main Theorem to Closed Braids

In this section, we apply the Main Theorem of [14] (Theorem 1.1) to obtain volume bounds for the family of closed braids considered in Theorem 3.1.

**Proposition 3.4.** Applying Theorem 1.1 to the closed braids of Theorem 3.1, we get that:

\[
\frac{v_8}{3} \cdot (t(D) - 1) \leq \text{vol}(S^3 \setminus K) < 10v_3 \cdot (t(D) - 1)
\]

in the case that \(n = 3\) and get that:

\[
\frac{v_8}{3} (t(D) - 1) - \frac{v_8}{3} \leq \text{vol}(S^3 \setminus K) < 10v_3 \cdot (t(D) - 1)
\]

in the case that \(n \geq 4\).

**Proof.** Recall that special circles in the all-A state \(H_A\) are other circles that are incident to A-segments from one of three possible pairs of twist region resolutions. (See Definition 2.6 and Figure 6.) Given the assumptions of Theorem 3.1, the all-A state circles of \(H_A\) have been classified by Proposition 2.4 and Proposition 3.3. Furthermore, we claim that medium circles, essential wandering circles, and non-essential wandering circles (which can only exist if the number of braid strings is \(n \geq 4\)) must all be incident to A-segments from three or more distinct twist region resolutions. To see why this is true for medium inner circles, see the center of Figure 14. To see why this is true for wandering circles, note that wandering circles must wander at least twice (wandering and wandering back) to be able to return to the same braid string position and close up. Therefore, the image on the right side of Figure 14 occurs at least twice per wandering circle and the claim can now seen to be true for wandering circles. Thus, we have that medium inner circles and wandering circles cannot be special circles. It is possible, however, for a nonwandering circle to be a special circle. Given the classification of nonwandering circles in Proposition 2.4 and Proposition 3.3, we see that there are two main cases to consider.

**Case 1:** Suppose that \(\sigma_1\) (or, symmetrically, \(\sigma_{n-1}\)) occurs in the braid word with only positive exponents. Then braid string 1 (or, symmetrically, braid string \(n\)) is a nonwandering circle. If this is the case, to be a special circle, the generator \(\sigma_1\) (or, symmetrically, \(\sigma_{n-1}\)) must appear exactly twice (as a positive syllable) in the braid word.

**Case 2:** Suppose that an adjacent pair of generators \(\sigma_i\) and \(\sigma_{i+1}\) for \(2 \leq i \leq n - 3\) occur with only positive exponents. Then braid string \(i + 1\) is a nonwandering circle. To be a special circle,
this braid string must be incident to A-segments from exactly two (positive) twist regions of $D(K)$ (which are positive syllables in $\beta$). Since $\beta$ is nice, then each generator must occur at least twice nontrivially in the braid word. Therefore, since we need to have two positive syllables in $\sigma_1$ and two positive syllables in $\sigma_{i+1}$, then we get a total of at least four positive twist region resolutions incident to braid string $i+1$. This prevents the nonwandering circle (braid string $i+1$) from being a special circle.

To summarize, we have shown that at most two special circles (namely braid string 1 and braid string $n$) are possible, so $st(D) \leq 2$. In the case that $n = 3$, Lemma 3.2 gives that there can only be at most one nonwandering circle and, therefore, at most one special circle. Thus $st(D) \leq 1$ when $n = 3$. Notice that the conclusions of Theorem 3.1 allow us to apply Theorem 1.1. Applying Theorem 1.1 in the case that $n = 3$, we get that:

$$\frac{v_8}{3} \cdot (t(D) - 1) \leq \text{vol}(S^3 \setminus K) < 10v_3 \cdot (t(D) - 1).$$

Applying Theorem 1.1 in the case that $n \geq 4$, we get that:

$$\frac{v_8}{3} \cdot (t(D) - 1) - \frac{v_8}{3} \leq \text{vol}(S^3 \setminus K) < 10v_3 \cdot (t(D) - 1).$$

\[\square\]

### 3.6 Volume Bounds in Terms of $t(D)$ (and $t^-(D)$)

We will now use our study of A-adequate closed braids to obtain bounds on volume. Additionally, we will compare the lower bounds on volume we find to those that were found in Proposition 3.4 by applying Theorem 1.1. In doing this, we show that studying the specific structure of the closed braids considered in this paper often provides sharper lower bounds on volume.

**Theorem 3.4.** Let $D(K) = \beta$ denote the closure of a nice $n$-braid of the form:

$$\beta = \sigma_{m_1}^{r_1} \cdots \sigma_{m_l}^{r_l},$$

where $1 \leq m_1, \ldots, m_l \leq n - 1$. If $\beta$ satisfies the conditions:

1. all negative exponents $r_i < 0$ in $\beta$ satisfy the stronger requirement that $r_i \leq -3$, and
2. when $r_i > 0$, we have that $r_{i-1} < 0$ and $r_{i+1} < 0$ and that either $m_{i-1} = m_{i+1} = m_i + 1$ or $m_{i-1} = m_{i+1} = m_i - 1$,

then $D(K)$ is a connected, prime, A-adequate link diagram that satisfies the TELC and contains $t(D) \geq 2(n - 1)$ twist regions. Furthermore, we have that $K$ is a hyperbolic link. In the case that $n = 3$, we get the following volume bounds:

$$\frac{v_8}{2} \cdot (t(D) - 1) - \frac{v_8}{2} \leq v_8 \cdot (t^-(D) - 1) \leq \text{vol}(S^3 \setminus K) < 10v_3 \cdot (t(D) - 1).$$
Suppose, in the case that \( n \geq 4 \), that the all-A state \( H_A \) contains \( m \) non-essential wandering circles. Then we get the following volume bounds:

\[
\frac{v_8}{2} \cdot (t(D) - 1) - \frac{v_8}{2} \cdot (2(n + m) - 5) \leq v_8 \cdot (t^-(D) - (n + m - 2)) \\
\leq \text{vol}(S^3 \setminus K) < 10v_3 \cdot (t(D) - 1).
\]

Proof. By combining the results of Theorem 3.1, Theorem 2.2, Theorem 3.2, and Theorem 3.3, we get the desired results.

Let us now compare the lower bounds of Proposition 3.4 to those of Theorem 3.4. We would like to determine exactly when the lower bounds given by our study of A-adequate closed braids are an improvement over those that result from applying Theorem 1.1 (using Proposition 3.4).

In the case that \( n = 3 \), we have that:

\[
\frac{v_8}{2} \cdot (t(D) - 1) - \frac{v_8}{2} \geq \frac{v_8}{3} \cdot (t(D) - 1)
\]

is equivalent to the condition that \( t(D) \geq 4 = 2(3 - 1) = 2(n - 1) \). But this condition is always satisfied because the assumption that \( \beta \) is nice allows us to use Proposition 3.2. Therefore, the lower bound found in Theorem 3.4 is always the same or sharper than the lower bound provided by Proposition 3.4.

In the case that \( n \geq 4 \), we have that:

\[
\frac{v_8}{2} \cdot (t(D) - 1) - \frac{v_8}{2} \cdot (2(n + m) - 5) \geq \frac{v_8}{3} \cdot (t(D) - 1) - \frac{v_8}{3}
\]

is equivalent to the condition that \( t(D) \geq 6(n + m) - 16 \). Therefore, the lower bound found in Theorem 3.4 is the same or sharper than the lower bound provided by Proposition 3.4 effectively when the number of twist regions is large compared to the sum, \( n + m \), of the number of braid strings and number of non-essential wandering circles.

Remark 3.5. It is also worth mentioning that, at the cost of keeping track of both twist regions and negative twist regions, there is further potential to improve the lower bounds on volume by using the lower bounds of Theorem 3.4 that are expressed in terms of \( t^-(D) \). This improvement will be especially noticeable when \( t^-(D) - t^+(D) \geq 0 \) is large.
### 3.7 A-Adequacy and the Colored Jones Polynomial

Denote the $n^{th}$ colored Jones polynomial of a link $K$ by:

$$J^K_n(t) = \alpha_n t^{m_n} + \beta_n t^{m_n-1} + \cdots + \beta'_n t^{r_n+1} + \alpha'_n t^{r_n},$$

where $n \in \mathbb{N}$ and where the degree of each monomial summand decreases from left to right.

The following result, due to Dasbach and Lin, relates the penultimate coefficient of the colored Jones polynomial of an A-adequate link to the reduced all-A graph associated to an A-adequate diagram of the link.

**Theorem 3.5** ([5]). Let $D(K)$ be a connected A-adequate link diagram. Then $|\beta'_n|$ is independent of $n$ for $n > 1$. Specifically, for $n > 1$, we have that:

$$|\beta'_K| := |\beta'_n| = 1 - \chi(G'_A). \quad (1)$$

This result has been and will be key in translating from volume bounds in terms of the twist number to volume bounds in terms of coefficients of the colored Jones polynomial.

### 3.8 Volume Bounds in Terms of the Colored Jones Polynomial

To conclude our study of volume bounds for hyperbolic A-adequate closed braids, we will translate our diagrammatic volume bounds (in terms of $t(D)$) to volume bounds in terms of the penultimate coefficient $\beta'_K$ of the colored Jones polynomial.

**Theorem 3.6.** Let $D(K) = \tilde{\beta}$ denote the closure of a nice $n$-braid of the form:

$$\beta = \sigma^{r_1}_{m_1} \cdots \sigma^{r_l}_{m_l},$$

where $1 \leq m_1, \ldots, m_l \leq n - 1$. If $\beta$ satisfies the conditions:

1. all negative exponents $r_i < 0$ in $\beta$ satisfy the stronger requirement that $r_i \leq -3$, and
2. when $r_i > 0$, we have that $r_{i-1} < 0$ and $r_{i+1} < 0$ and that either $m_{i-1} = m_{i+1} = m_i + 1$ or $m_{i-1} = m_{i+1} = m_i - 1$,

then $D(K)$ is a connected, prime, A-adequate link diagram that satisfies the TELC and contains $t(D) \geq 2(n - 1)$ twist regions. Furthermore, we have that $K$ is a hyperbolic link. In the case that $n = 3$, we get the following volume bounds:

$$v_8 \cdot (|\beta'_K| - 1) \leq \text{vol}(S^3 \setminus K) < 20v_3 \cdot (|\beta'_K| - 1) + 10v_3.$$
Suppose, in the case that \( n \geq 4 \), that the all-A state \( H_A \) contains \( m \) non-essential wandering circles. Then we get the following volume bounds:

\[
v_8 \cdot (|\beta'_K| - 1) \leq \text{vol}(S^3 \setminus K) < 20v_3 \cdot (|\beta'_K| - 1) + 10v_3 \cdot (2(n + m) - 5).\]

**Proof.** The first conclusions of the theorem follow from Theorem \textbf{3.1}. Combining Theorem \textbf{3.5} with Theorem \textbf{2.2} we get that (for all \( n \geq 3 \)):

\[
v_8 \cdot (|\beta'_K| - 1) = -v_8 \cdot \chi(G'_A) \leq \text{vol}(S^3 \setminus K).
\]

Consider the case when \( n = 3 \). Combining Theorem \textbf{3.5} with Theorem \textbf{3.2} gives:

\[
|\beta'_K| = 1 - \chi(G'_A) \geq 1 + \frac{1}{2} \cdot (t(D) - 1) - \frac{1}{2} \cdot \frac{t(D)}{2},
\]

which implies that:

\[
t(D) - 1 \leq 2 \cdot (|\beta'_K| - 1) + 1.
\]

Applying this inequality to Theorem \textbf{2.2} we get:

\[
\text{vol}(S^3 \setminus K) < 10v_3 \cdot (t(D) - 1) \leq 20v_3 \cdot (|\beta'_K| - 1) + 10v_3.
\]

This gives the first desired set of volume bounds.

Now consider the case when \( n \geq 4 \). Combining Theorem \textbf{3.5} with Theorem \textbf{3.3} gives:

\[
|\beta'_K| = 1 - \chi(G'_A) \geq 1 + \frac{1}{2} \cdot (t(D) - 1) - \frac{1}{2} \cdot \frac{t(D) - 2(n + m) + 6}{2},
\]

which implies that:

\[
t(D) - 1 \leq 2 \cdot (|\beta'_K| - 1) + 2(n + m) - 5.
\]

Applying this inequality to Theorem \textbf{2.2} we get:

\[
\text{vol}(S^3 \setminus K) < 10v_3 \cdot (t(D) - 1) \leq 20v_3 \cdot (|\beta'_K| - 1) + 10v_3 \cdot (2(n + m) - 5).
\]

This gives the second desired set of volume bounds.
4 Volume Bounds for A-Adequate Closed 3-Braids in Terms of the Schreier Normal Form

4.1 The Schreier Normal Form for 3-Braids

A useful development in the history of 3-braids was the solution to the Conjugacy Problem (and, as a corollary, the Word Problem) for the 3-braid group ([23]). As it turns out, there is an explicit algorithm that produces from an arbitrary 3-braid word $\beta$ a conjugate 3-braid word $\beta'$ which is called the Schreier normal form of $\beta$. This new braid word $\beta'$ is the unique representative of the conjugacy class of $\beta$. See below (or see Section 7.1 of [2]) for a modern exposition of the algorithm.

Given such an algorithm, Birman and Menasco ([2]) gave a complete classification of links that can be represented as 3-braid closures. To be more specific they showed that, up to an explicit list of exceptions, each 3-braid closure comes from a single conjugacy class. Hence, up to some exceptions, the normal form of the braid $\beta$ determines the unique link type of the closed braid $\hat{\beta}$.

Because it will be needed in what follows, we now present the algorithm (adapted from Section 7.1 of [2]) that, given a 3-braid $\beta \in B_3$, produces the Schreier normal form $\beta'$ of $\beta$.

**Remark 4.1.** It is important to note that cyclic permutation (a special case of conjugacy) may be needed during the steps of this algorithm.

**Schreier Normal Form Algorithm:**

1. Let $\beta \in B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$ be cyclically reduced into syllables.

   Introduce new variables $x = (\sigma_1\sigma_2\sigma_1)^{-1}$ and $y = \sigma_1\sigma_2$. Thus we have that:
   
   $\bullet \sigma_1 = y^2x$
   $\bullet \sigma_2 = xy^2$
   $\bullet \sigma_1^{-1} = xy$
   $\bullet \sigma_2^{-1} = yx$

   Possibly using cyclic permutation, rewrite $\beta$ as a cyclically reduced word that is positive in $x$ and $y$.

2. Introduce $C = x^{-2} = y^3 \in Z(B_3)$, where $Z(B_3)$ denotes the center of the 3-braid group $B_3$.

   By using these relations and commutativity of $C$ as much as possible, group all powers of $C$ at the beginning of the braid word and reduce the exponents of $x$ and $y$ as much as possible.

   Rewrite $\beta$ as:
\[ \beta = C^k \eta, \]

where \( k \in \mathbb{Z} \) and where \( \eta \) is a subword whose syllables in \( x \) have exponent at most one and whose syllables in \( y \) have exponent at most two. To be more precise, by possibly using cyclic permutation:

\[
\eta = \begin{cases} 
(xy)^{p_1} (xy^2)^{q_1} \cdots (xy)^{p_s} (xy^2)^{q_s} & \text{for some } s, p_i, q_i \geq 1 \\
(xy)^p (xy^2)^q & \text{for some } p \geq 1 \\
y & \text{for some } q \geq 1 \\
x \\
1
\end{cases}
\]

(3) Possibly using cyclic permutation and the commutativity of \( C \), rewrite \( \beta \) back in terms of \( \sigma_1 \) and \( \sigma_2 \) as \( \beta' = C^k \eta' \), where:

\[
\eta' = \begin{cases} 
\sigma_1^{-p_1} \sigma_2 \cdots \sigma_1^{-p_s} \sigma_2^{q_s} & \text{for some } s, p_i, q_i \geq 1 \\
\sigma_1^p & \text{for some } p \in \mathbb{Z} \\
\sigma_1 \sigma_2 \\
\sigma_1 \sigma_2 \sigma_1 \\
\sigma_1 \sigma_2 \sigma_1 \sigma_2
\end{cases}
\]

**Definition 4.1.** We call \( \beta' \in B_3 \) the Schreier normal form of \( \beta \in B_3 \). The braid word \( \beta' \) is the unique representative of the conjugacy class of \( \beta \).

**Definition 4.2.** Following [13], we will call a braid \( \beta \) generic if it has Schreier normal form

\[ \beta' = C^k \sigma_1^{-p_1} \sigma_2 \cdots \sigma_1^{-p_s} \sigma_2^{q_s}. \]

### 4.2 Hyperbolicity for Closed 3-Braids and Volume Bounds

Using the Schreier normal form of a 3-braid, Futer, Kalfagianni, and Purcell ([13]) classified the hyperbolic 3-braid closures. Furthermore, given such a hyperbolic closed 3-braid, they gave two-sided bounds on the volume of the link complement, expressing the volume in terms of the parameter \( s \) from the Schreier normal form of the 3-braid. Because we will compare the lower bounds on volume of this paper to those of [13], we give precise statements of the relevant results from [13] below.

**Proposition 4.1** ([13], Theorem 5.5). Let \( D(K) = \tilde{\beta} \) denote the closure of a 3-braid \( \beta \in B_3 \). Then \( K \) is hyperbolic if and only if:

(1) \( \beta \) is generic, and

(2) \( \beta \) is not conjugate to \( \sigma_1^p \sigma_2^q \) for any integers \( p \) and \( q \).
Proposition 4.2 ([13], Theorem 5.6). Let $D(K) = \hat{\beta}$ denote the closure of a 3-braid $\beta \in B_3$ and let $\beta'$ denote the Schreier normal form of $\beta$. Then, assuming that $K$ is hyperbolic, we have that:

$$4v_3 \cdot s - 276.6 < \text{vol}(S^3 - K) < 4v_8 \cdot s.$$ 

By using the more recent machinery built by the same authors in [10], we will often obtain a sharper lower bound on volume.

4.3 Volume Bounds in Terms of the Schreier Normal Form

In this section, we study the Schreier normal form of the 3-braids $\beta$ given in Theorem 3.1

Theorem 4.1. Let $D(K) = \hat{\beta}$ denote the closure of a nice 3-braid $\beta = \sigma_i^{r_1} \sigma_j^{r_2} \cdots \sigma_i^{r_2} \sigma_j^{r_2} \in B_3$, where $\{i, j\} = \{1, 2\}$. Assume that $\beta$ satisfies the conditions:

1. all negative exponents $r_i < 0$ in $\beta$ satisfy the stronger requirement that $r_i \leq -3$.
2. positive syllables are cyclically isolated.

Then $\beta$ is generic with Schreier normal form $\beta' = C^k \sigma_i^{-p_1} \sigma_2^{q_1} \cdots \sigma_i^{-p_s} \sigma_2^{q_s}$, where $k \in \mathbb{Z}$ and where $s, p_i, q_i \geq 1$. Furthermore, we are able to express the parameters $k$ and $s$ of the Schreier normal form $\beta'$ in terms of the original braid $\beta$ as follows:

(a) $k = -\# \{\text{induced products } \sigma_2^{n_2} \sigma_1^{n_1} \text{ of negative syllables of } \beta, \text{ where } n_1, n_2 \leq -3\}$.
(b) $s = t^- (D) = \# \{\text{negative syllables in } \beta\}$.

Note that, when looking for the induced products $\sigma_2^{n_2} \sigma_1^{n_1}$ of negative syllables of $\beta$ to find $k$, one must look cyclically in the braid word. As an example computation, consider the 3-braid:

$$\beta = \sigma_1^3 \sigma_2^{-3} \sigma_1^{-5} \sigma_2^{-3}.$$ 

Applying the Schreier Normal Form Algorithm, we get:

\[
\begin{align*}
\beta &= \sigma_1^3 \sigma_2^{-3} \sigma_1^{-5} \sigma_2^{-3} \\
&= (y^2 x)^3 (y x)^3 (x y)^3 (y x)^3 \\
&= y^2 (x y^2)^2 (x y)^3 (x^2 y) (x y^3) (x y^2) (x y)^2 x \\
&= y^2 (x y^2)^2 (x y)^3 (C^{-1}) y (x y)^3 (x y^2) (x y)^2 x \\
&= C^{-1} y^2 (x y^2)^2 (x y)^2 (x y^2) (x y)^3 (x y^2) (x y)^2 x \\
&= C^{-1} (x y)^2 (x y^2)^3 (x y)^2 (x y)^2 (x y^2) (x y^2) (x y)^2 \\
&= C^{-1} (x y)^2 (x y^2)^3 (x y)^2 (x y^2) (x y)^3 (x y^2) \\
&= C^{-1} \sigma_1^{-2} \sigma_2 \sigma_1^{-2} \sigma_2 \sigma_1^{-3} \sigma_2 \\
&= \beta',
\end{align*}
\]
where \( \cong \) denotes that either cyclic permutation or the fact that \( C \in Z(B_3) \) has been used. Thus, we see that:

\[
\begin{align*}
    k &= -1 = -\# \{\text{induced products } \sigma_2^{n_2}\sigma_1^{n_1} \text{ of negative syllables of } \beta, \text{ where } n_1, n_2 \leq -3\} \\
    \text{and:} \\
    s &= 3 = \# \{\text{negative syllables in } \beta\}.
\end{align*}
\]

**Remark 4.2.** As a special case of the above theorem, notice that if \( \beta = \sigma_i^{p_1}\sigma_j^{p_1} \cdots \sigma_i^{p_l}\sigma_j^{p_l} \) is an alternating 3-braid where \( \{i, j\} = \{1, 2\} \), then \( k = 0 \) and \( s = l \).

Deferring the proof of the above theorem to the next section, we now relate \( s \) to the colored Jones polynomial of a hyperbolic A-adequate closed 3-braid from Theorem 3.1. In particular, we relate \( s \) to the penultimate coefficient \( \beta_K' \) of the colored Jones polynomial.

**Corollary 4.1.** Given the assumptions of Theorem 4.1, we have that:

\[
    s = t^-(D) = |\beta_K'|.
\]

Thus, \( t^-(D) \) and \( s \) are actually link invariants.

**Proof.** To begin, note that the assumptions of Theorem 4.1 are the same as those of Theorem 3.1 because Condition (2) of Theorem 3.1 can be expressed more simply for 3-braids. Theorem 3.2 also relies on the assumptions of Theorem 3.1. Using the conclusions of Theorem 3.1, we can apply Theorem 3.5, Theorem 3.2, and Theorem 4.1 respectively to get that:

\[
    |\beta_K'| = 1 - \chi(G_A') \\
    = 1 + (t^-(D) - 1) \\
    = t^-(D) \\
    = s.
\]

Since the colored Jones polynomial and therefore its coefficients are link invariants, then we can conclude that \( t^-(D) \) and \( s \) are link invariants.

\( \square \)

**Theorem 4.2.** Let \( D(K) = \hat{\beta} \) denote the closure of a nice 3-braid \( \beta \), which we denote by \( \beta = \sigma_i^{r_1}\sigma_j^{r_2} \cdots \sigma_i^{r_{2l-1}}\sigma_j^{r_{2l}}, \) where \( \{i, j\} = \{1, 2\} \). Assume that \( \beta \) satisfies the conditions:

1. all negative exponents \( r_i < 0 \) in \( \beta \) satisfy the stronger requirement that \( r_i \leq -3 \).
(2) positive syllables are cyclically isolated.

Then $D(K)$ is a connected, prime, $A$-adequate link diagram that satisfies the TELC and contains $t(D) \geq 2$ twist regions. Furthermore, we have that $K$ is a hyperbolic link and $\beta$ is generic. Moreover, we get the following volume bounds:

$$v_8 \cdot (s - 1) \leq \text{vol}(S^3 \setminus K) < 4v_8 \cdot s.$$ 

Proof. By combining Theorem \ref{thm:hyperbolic} Proposition \ref{prop:lower_bound} Corollary \ref{cor:hyperbolicity} and Proposition \ref{prop:beta}, we get the desired result.

\[ \square \]

Remark 4.3. Comparing the lower bound on volume of Theorem \ref{prop:beta} to that of Proposition \ref{prop:lower_bound}, we get that $v_8 \cdot (s - 1) \geq 4v_3 \cdot s - 276.6$ is equivalent to the condition that $s \leq \frac{276.6 - v_8}{4v_3 - v_8} < 689$. Therefore, the lower bound found in Theorem \ref{thm:volume_bound} is sharper than the lower bound provided by Proposition \ref{prop:beta} unless the parameter $s$ from the Schreier normal form is very large.

4.4 The Proof of Theorem \ref{thm:volume_bound}

Proof. By Theorem \ref{thm:hyperbolic} we have that $K$ is hyperbolic. By Proposition \ref{prop:lower_bound} this implies that $\beta$ is generic. Consider the following two cases.

Case 1: Suppose $\beta$ is a negative braid. Then, possibly using cyclic permutation, we may write $\beta$ as $\beta = \sigma_1^{n_1} \sigma_2^{n_2} \cdots \sigma_1^{n_{2m-1}} \sigma_2^{n_{2m}}$, where $n_i \leq -3$ and the fact that $\beta$ is nice forces the condition that $m \geq 2$. Applying the Schreier Normal Form Algorithm, we get:

$$\beta = \sigma_1^{n_1} \sigma_2^{n_2} \cdots \sigma_1^{n_{2m-1}} \sigma_2^{n_{2m}}$$

$$= (xy)^{-n_1} (yx)^{-n_2} \cdots (xy)^{-n_{2m-1}} (yx)^{-n_{2m}}$$

$$= (xy)^{-n_1} (x^2)(yx)^{-n_2} \cdots (x^2)y(xy)^{-n_{2m-1} - 2}(x^2)(xy)^{-n_{2m-1} x}$$

$$= (xy)^{-n_1} (x^2)(yx)^{-n_2} \cdots (C^{-1})y(xy)^{-n_{2m-1} - 2}(x^2)(xy)^{-n_{2m-1} x}$$

$$\cong (C^{-1})^{m-1} (xy)^{-n_1} (x^2)(yx)^{-n_2} \cdots (x^2)(xy)^{-n_{2m-1} - 2}(x^2)(xy)^{-n_{2m-1} x}$$

$$\cong (C^{-1})^{m-1} (x)(y(xy)^{-n_1} (x^2)(yx)^{-n_2} \cdots (x^2)(xy)^{-n_{2m-1} - 2}(x^2)(xy)^{-n_{2m-1} x}$$

$$= (C^{-1})^{m} (xy)^{-n_1} (x^2)(yx)^{-n_2} \cdots (x^2)(xy)^{-n_{2m-1} - 2}(x^2)(xy)^{-n_{2m-1} x}$$

$$\cong (C^{-1})^{m} (x)(y(xy)^{-n_1} (x^2)(yx)^{-n_2} \cdots (x^2)(xy)^{-n_{2m-1} - 2}(x^2)(xy)^{-n_{2m-1} x}$$

$$= (C^{-1})^{m} (xy)^{-n_1} (x^2)(yx)^{-n_2} \cdots (x^2)(xy)^{-n_{2m-1} - 2}(x^2)(xy)^{-n_{2m-1} x}$$

$$= (C^{-1})^{m} \sigma_1^{n_1+2} \sigma_2 \sigma_2^{n_2+2} \cdots \sigma_2 \sigma_1^{n_{2m-1}+2} \sigma_2 \sigma_1^{n_{2m}+2} \sigma_2$$

$$= \beta'.$$
where \( \sim \) denotes that cyclic permutation or the fact that \( C \in Z(B_3) \) has been used. Thus, we see that:

\[
k = -m = -\#\{\text{induced products } \sigma_2^{n_2} \sigma_1^{n_1} \text{ of negative syllables of } \beta, \text{ where } n_1, n_2 \leq -3\}
\]

and:

\[
s = 2m = \# \{\text{negative syllables in } \beta\}.
\]

Recall that, when looking for the induced products \( \sigma_2^{n_2} \sigma_1^{n_1} \) of negative syllables of \( \beta \) to find \( k \), one must look cyclically in the braid word.

**Case 2**: Suppose \( \beta \) is not a negative braid. Then, possibly using cyclic permutation, we may write \( \beta = P_1 N_1 \cdots P_t N_t \), where \( P_i \) denotes a positive syllable of \( \beta \) and where \( N_i \) denotes a maximal length negative induced subword of \( \beta \). Note that this decomposition arises as a result of the fact that positive syllables are assumed to be cyclically isolated.

Our strategy for the proof of this case is to apply the Schreier Normal Form Algorithm to the subwords \( P_i N_i \), show that the theorem is locally satisfied for the \( P_i N_i \), show that cyclic permutation is never needed in applying the algorithm to the \( P_i N_i \), and to finally show that juxtaposing the subwords \( P_i N_i \) to form \( \beta \) allows the local conclusions of the theorem for the \( P_i N_i \) to combine to give the global conclusion of the theorem for \( \beta \).

To begin, we list the possible subwords \( P_i N_i \). For each subword, we consider the two possible subtypes. Let \( p > 0 \) be a positive exponent and let the \( n_i \leq -3 \) be negative exponents.

**List of Types of Induced Subwords** \( P_i N_i \) **of** \( \beta $$

1. \( \sigma_2^p \sigma_1^{n_1} \)
2. \( \sigma_1^p \sigma_2^{n_1} \)
3. \( \sigma_2^p \sigma_1^{n_1} \sigma_2^{n_2} \)
4. \( \sigma_1^p \sigma_2^{n_1} \sigma_1^{n_2} \)
5. \( \sigma_2^p \sigma_1^{n_1} \sigma_2^{n_2} \sigma_1^{n_3} \)
6. \( \sigma_1^p \sigma_2^{n_1} \sigma_1^{n_2} \sigma_2^{n_3} \)
7. \( \sigma_2^p \sigma_1^{n_1} \sigma_2^{n_2} \cdots \sigma_1^{n_{2m-1}} \sigma_2^{n_{2m}}, \text{ where } m \geq 2 \)
8. \( \sigma_1^p \sigma_2^{n_1} \sigma_1^{n_2} \cdots \sigma_2^{n_{2m-1}} \sigma_1^{n_{2m}}, \text{ where } m \geq 2 \)
9. \( \sigma_2^p \sigma_1^{n_1} \sigma_2^{n_2} \cdots \sigma_1^{n_{2m-1}} \sigma_2^{n_{2m}} \sigma_1^{n_{2m+1}}, \text{ where } m \geq 2 \)
10. \( \sigma_1^p \sigma_2^{n_1} \sigma_1^{n_2} \cdots \sigma_2^{n_{2m-1}} \sigma_1^{n_{2m}} \sigma_2^{n_{2m+1}}, \text{ where } m \geq 2 \)
We now apply the Schreier Normal Form Algorithm to each (sub)type of induced subword above, implicitly showing along the way that cyclic permutation is never used during the application of the algorithm. Let \( k_i \) denote the exponent of \( C \) in the normal form for the subword \( P_i N_i \). We also explicitly show that:

\[
k_i = -\# \{ \text{induced products } \sigma_2^{n_j} \sigma_1^{n_{j+1}} \text{ of negative syllables of } P_i N_i, \text{ where } n_j, n_{j+1} \leq -3 \}.
\]

The Algorithm for Type (1a):

\[
P_i N_i = \sigma_2^p \sigma_1^{n_1} = (xy^2)^p(xy)^{-n_1} = \sigma_2^p \sigma_1^{n_1}
\]

Thus we get that:

\[
k_i = 0 = -\# \{ \text{induced products } \sigma_2^{n_j} \sigma_1^{n_{j+1}} \text{ of negative syllables of } P_i N_i, \text{ where } n_j, n_{j+1} \leq -3 \}.
\]

The Algorithm for Type (1b):

\[
P_i N_i = \sigma_1^p \sigma_2^{n_1} = (y^2 x)^p(y x)^{-n_1} = y^2(xy^2)^{p-1}(xy)^{-n_1}x = y^2 \sigma_2^{p-1} \sigma_1^{n_1} x
\]

Thus we get that:

\[
k_i = 0 = -\# \{ \text{induced products } \sigma_2^{n_j} \sigma_1^{n_{j+1}} \text{ of negative syllables of } P_i N_i, \text{ where } n_j, n_{j+1} \leq -3 \}.
\]

The Algorithm for Type (2a):

\[
P_i N_i = \sigma_2^p \sigma_1^{n_1} \sigma_2^{n_2} = (xy^2)^p(xy)^{-n_1}(yx)^{-n_2} = (xy^2)^p(xy)^{-n_1-1}(xy^2)(xy)^{-n_2-1}x = \sigma_2^p \sigma_1^{n_1+1} \sigma_2 \sigma_1^{n_2+1} x
\]

Thus we get that:

\[
k_i = 0 = -\# \{ \text{induced products } \sigma_2^{n_j} \sigma_1^{n_{j+1}} \text{ of negative syllables of } P_i N_i, \text{ where } n_j, n_{j+1} \leq -3 \}.
\]
The Algorithm for Type (2b):

\[ P_1N_i = \sigma_2^p \sigma_1^{n_1} \sigma_1^{n_2} \]
\[ = (y^2x)^p (yx)^{-n_1} (xy)^{-n_2} \]
\[ = y^2 (xy^2)^{p-1} (xy)^{-n_1} (x^2) y (xy)^{-n_2-1} \]
\[ = y^2 (xy^2)^{p-1} (xy)^{-n_1} (C^{-1}) y (xy)^{-n_2-1} \]
\[ \equiv C^{-1} y^2 (xy^2)^{p-1} (xy)^{-n_1-1} (xy^2) (xy)^{-n_2-1} \]
\[ = C^{-1} y^2 \sigma_2^{p-1} \sigma_1^{n_1+1} \sigma_2 \sigma_1^{n_2+1} \]

Thus we get that:

\[ k_i = -1 = -\# \{ \text{induced products } \sigma_2^{n_j} \sigma_1^{n_{j+1}} \text{ of negative syllables of } P_1N_i, \text{ where } n_j, n_{j+1} \leq -3 \}. \]

The Algorithm for Type (3a):

\[ P_1N_i = \sigma_2^p \sigma_1^{n_1} \sigma_2^{n_2} \sigma_1^{n_3} \]
\[ = (xy^2)^p (xy)^{-n_1} (xy^2)^{-n_2} (xy)^{-n_3} \]
\[ = (xy^2)^p (xy)^{-n_1} (xy^2) (xy)^{-n_2-1} (x^2) y (xy)^{-n_3-1} \]
\[ = (xy^2)^p (xy)^{-n_1} (xy^2) (xy)^{-n_2-1} (C^{-1}) y (xy)^{-n_3-1} \]
\[ \equiv C^{-1} (xy^2)^p (xy)^{-n_1-1} (xy^2) (xy)^{-n_2-2} (xy^2) (xy)^{-n_3-1} \]
\[ = C^{-1} \sigma_2^p \sigma_1^{n_1+1} \sigma_2 \sigma_1^{n_2+2} \sigma_2 \sigma_1^{n_3+1} \]

Thus we get that:

\[ k_i = -1 = -\# \{ \text{induced products } \sigma_2^{n_j} \sigma_1^{n_{j+1}} \text{ of negative syllables of } P_1N_i, \text{ where } n_j, n_{j+1} \leq -3 \}. \]

The Algorithm for Type (3b):

\[ P_1N_i = \sigma_1^p \sigma_2^{n_1} \sigma_1^{n_2} \sigma_2^{n_3} \]
\[ = (y^2x)^p (yx)^{-n_1} (xy)^{-n_2} (yx)^{-n_3} \]
\[ = y^2 (xy^2)^{p-1} (xy)^{-n_1} (x^2) y (xy)^{-n_2-2} (xy^2) (xy)^{-n_3-1} x \]
\[ = y^2 (xy^2)^{p-1} (xy)^{-n_1} (C^{-1}) y (xy)^{-n_2-2} (xy^2) (xy)^{-n_3-1} x \]
\[ \equiv C^{-1} y^2 (xy^2)^{p-1} (xy)^{-n_1-1} (xy^2) (xy)^{-n_2-2} (xy^2) (xy)^{-n_3-1} x \]
\[ = C^{-1} y^2 \sigma_2^{p-1} \sigma_1^{n_1+1} \sigma_2 \sigma_1^{n_2+2} \sigma_2 \sigma_1^{n_3+1} x \]

Thus we get that:
Thus we get that:

The Algorithm for Type (4a):

\[ P_i N_i = \sigma_2^p \sigma_1^{n_1} \sigma_2^{n_2} \cdots \sigma_1^{n_{2m-1}} \sigma_2^{n_{2m}} \]
\[ = (xy^2)^p (xy)^{-n_1} (yx)^{-n_2} \cdots (yx)^{-n_{2m-1}} (xy)^{-n_{2m}} \]
\[ = (xy^2)^p (xy)^{-n_1} (xy)^{-n_2-1} (x^2) y \cdots (x^2) y (xy)^{-n_{2m-1}-2} (xy^2) (xy)^{-n_{2m-1}} x \]
\[ = (xy^2)^p (xy)^{-n_1-1} (xy)^{-n_2-1} (C^{-1}) y \cdots (C^{-1}) y (xy)^{-n_{2m-1}-2} (xy^2) (xy)^{-n_{2m-1}} x \]
\[ \cong (C^{-1})^m (xy^2)^p (xy)^{-n_1-1} (xy)^{-n_2-2} (xy^2) \cdots \]
\[ \cdots (xy^2) (xy)^{-n_{2m-1}-2} (xy^2) (xy)^{-n_{2m-1}} x \]
\[ = (C^{-1})^m \sigma_2^{p-1} \sigma_1^{n_1+1} \sigma_2 \sigma_1^{n_2+2} \cdots \sigma_2 \sigma_1^{n_{2m-1}+2} \sigma_2 \sigma_1^{n_{2m}+1} x \]

Thus we get that:

\[ k_i = -(m-1) = -\# \{ \text{induced products } \sigma_2^{n_j} \sigma_1^{n_{j+1}} \text{ of negative syllables of } P_i N_i, \text{ where } n_j, n_{j+1} \leq -3 \} . \]

The Algorithm for Type (4b):

\[ P_i N_i = \sigma_2^p \sigma_1^{n_1} \sigma_2^{n_2} \cdots \sigma_2^{n_{2m-1}} \sigma_1^{n_{2m}} \]
\[ = (y^2 x)^p (yx)^{-n_1} (xy)^{-n_2} \cdots (yx)^{-n_{2m-1}} (xy)^{-n_{2m}} \]
\[ = y^2 (xy^2)^p^{-1} (xy)^{-n_1} (xy)^{-n_2-2} (xy^2) \cdots (xy^2) (xy)^{-n_{2m-1}-1} (x^2) y (xy)^{-n_{2m-1}} \]
\[ = y^2 (xy^2)^p^{-1} (xy)^{-n_1} (C^{-1}) y (xy)^{-n_2-2} (xy^2) \cdots (xy^2) (xy)^{-n_{2m-1}-1} (C^{-1}) y (xy)^{-n_{2m-1}} \]
\[ \cong (C^{-1})^m y^2 (xy^2)^p^{-1} (xy)^{-n_1-1} (xy^2) (xy)^{-n_2-2} (xy^2) \cdots \]
\[ \cdots (xy^2) (xy)^{-n_{2m-1}-2} (xy^2) (xy)^{-n_{2m-1}} \]
\[ = (C^{-1})^m y^2 \sigma_2^{p-1} \sigma_1^{n_1+1} \sigma_2 \sigma_1^{n_2+2} \cdots \sigma_2 \sigma_1^{n_{2m-1}+2} \sigma_2 \sigma_1^{n_{2m}+1} \]

Thus we get that:

\[ k_i = -m = -\# \{ \text{induced products } \sigma_2^{n_j} \sigma_1^{n_{j+1}} \text{ of negative syllables of } P_i N_i, \text{ where } n_j, n_{j+1} \leq -3 \} . \]
The Algorithm for Type (5a):

\[ P_i N_i = \sigma_i^p \sigma_i^{n_1} \sigma_i^{n_2} \cdots \sigma_i^{n_{2m-1}} \sigma_i^{n_{2m}} \sigma_i^{n_{2m+1}} \]
\[ = (xy^2)^p(xy)^{-n_1}(yx)^{-n_2} \cdots (x^2)(xy)^{-n_{2m-1}}(y^2)(xy)^{-n_{2m+1}} \]
\[ = (xy^2)^p(xy)^{-n_1}(x^2)(xy)^{-n_2-1}(x^2)y \cdots \]
\[ \cdots (x^2)(y^2)(xy)^{-n_{2m-1}}(x^2)(xy)^{-n_{2m-2}}(x^2)y \cdots \]
\[ = (xy^2)^p(xy)^{-n_1}(x^2)(xy)^{-n_2-1}(C^{-1})y \cdots \]
\[ \cdots (x^2)(y^2)(xy)^{-n_{2m-1}}(x^2)(xy)^{-n_{2m-2}}(x^2)y \cdots \]
\[ = (C^{-1})^m(x^2)^p(xy)^{-n_1}(x^2)(xy)^{-n_2-2}(x^2) \cdots \]
\[ \cdots (x^2)(xy)^{-n_{2m-1}}(x^2)(xy)^{-n_{2m-2}}(x^2)y \cdots \]

Thus we get that:

\[ k_i = -m - \# \{ \text{induced products } \sigma_i^{n_j} \sigma_i^{n_{j+1}} \text{ of negative syllables of } P_i N_i, \text{ where } n_j, n_{j+1} \leq -3 \}. \]

The Algorithm for Type (5b):

\[ P_i N_i = \sigma_i^p \sigma_i^{n_1} \sigma_i^{n_2} \cdots \sigma_i^{n_{2m-1}} \sigma_i^{n_{2m}} \sigma_i^{n_{2m+1}} \]
\[ = (y^2x)^p(yx)^{-n_1}(yx)^{-n_2} \cdots (y^2)(xy)^{-n_{2m-1}}(yx)^{-n_{2m+1}} \]
\[ = y^2(x^2)^p-1(xy)^{-n_1}(x^2)^2(y^2) \cdots \]
\[ \cdots (x^2)(xy)^{-n_{2m-1}}(x^2)(xy)^{-n_{2m-2}}(x^2)^2(y^2) \cdots \]
\[ = y^2(x^2)^p-1(xy)^{-n_1}(C^{-1})y^2 \cdots \]
\[ \cdots (xy^2)^{-n_{2m-1}}(xy^2)^{-n_{2m-2}}(xy^2)^2(y^2) \cdots \]
\[ = (C^{-1})^m y^2(xy^2)^p-1(xy)^{-n_1-1}(xy^2)^2(xy)^{-n_{2m-1}}(xy^2)^{-n_{2m-2}}(xy^2)^2 \cdots \]
\[ \cdots (xy^2)^{-n_{2m-1}}(xy^2)^{-n_{2m-2}}(xy^2)^2(y^2) \cdots \]
\[ = (C^{-1})^m y^2(xy^2)^p-1 \sigma_i^{n_1+1} \sigma_i^{n_2+2} \cdots \sigma_i^{n_{2m-1}+2} \sigma_i^{n_{2m+2}} \sigma_i^{n_{2m+1}+1} \]

Thus we get that:

\[ k_i = -m = - \# \{ \text{induced products } \sigma_i^{n_j} \sigma_i^{n_{j+1}} \text{ of negative syllables of } P_i N_i, \text{ where } n_j, n_{j+1} \leq -3 \}. \]

It is very important to notice that, in all the the cases above, the application of the Schreier Normal Form Algorithm to each of the induced subword types \( P_i N_i \) does not ever require cyclic permutation. This is very important because the goal is to juxtapose the normal forms of the induced subwords \( P_i N_i \) and claim that this gives, after some minor modifications, the normal form of the full braid word \( \beta \). To summarize, below is the list of normal forms of the induced subwords \( P_i N_i \) of \( \beta \) (as computed above).

List of Normal Forms of Induced Subwords \( P_i N_i \) of \( \beta \):
(1a) $\sigma_2^p \sigma_1^{n_1}$
(1b) $y^2 \sigma_2^{p-1} \sigma_1^{n_1} x$
(2a) $\sigma_2^p \sigma_1^{n_1+1} \sigma_2 \sigma_1^{n_2+1}$
(2b) $C^{-1} y^2 \sigma_2^{p-1} \sigma_1^{n_1+1} \sigma_2 \sigma_1^{n_2+1}$
(3a) $C^{-1} \sigma_2^p \sigma_1^{n_1+1} \sigma_2 \sigma_1^{n_2+2} \sigma_2 \sigma_1^{n_3+1}$
(3b) $C^{-1} y^2 \sigma_2^{p-1} \sigma_1^{n_1+1} \sigma_2 \sigma_1^{n_2+2} \sigma_2 \sigma_1^{n_3-1}$
(4a) $(C^{-1})^m \sigma_2^p \sigma_1^{n_1+1} \sigma_2 \sigma_1^{n_2+2} \sigma_2 \cdots \sigma_2 \sigma_1^{n_2m-1+2} \sigma_2 \sigma_1^{n_2m+1} x$, where $m \geq 2$
(4b) $(C^{-1})^m y^2 \sigma_2^{p-1} \sigma_1^{n_1+1} \sigma_2 \sigma_1^{n_2+2} \sigma_2 \cdots \sigma_2 \sigma_1^{n_2m-1+2} \sigma_2 \sigma_1^{n_2m+1}$, where $m \geq 2$
(5a) $(C^{-1})^m \sigma_2^p \sigma_1^{n_1+1} \sigma_2 \sigma_1^{n_2+2} \sigma_2 \cdots \sigma_2 \sigma_1^{n_2m-1+2} \sigma_2 \sigma_1^{n_2m+1+1}$, where $m \geq 2$
(5b) $(C^{-1})^m y^2 \sigma_2^{p-1} \sigma_1^{n_1+1} \sigma_2 \sigma_1^{n_2+2} \sigma_2 \cdots \sigma_2 \sigma_1^{n_2m-1+2} \sigma_2 \sigma_1^{n_2m+2} \sigma_2 \sigma_1^{n_2m+1+1}$, where $m \geq 2$

Let us now consider which types of induced subwords $P_{i+1} N_{i+1}$ can (cyclically) follow a given induced subword $P_i N_i$. Since $\beta$ is assumed to be cyclically reduced into syllables and since the the induced subwords $P_i N_i$ contain unbroken syllables of $\beta$, then we get that:

- Induced subwords of Types (1), (3), and (5) can be followed by any of the five subwords, so long as they are of the same subtype.
- Induced subwords of Types (2) and (4) can be followed by any of the five subwords, so long as they are of different subtype.

See the “List of Types of Induced Subwords $P_i N_i$ of $\beta$” to verify these claims. As an example, we have that an induced subword of Type (1a) can be followed by an induced subword of Type (2a) but can not be followed by a subword of Type (2b). On the other hand, an induced subword of Type (2a) can be followed by an induced subword of Type (5b) but can not be followed by a subword of Type (5a).

Recall that $C \in Z(B_3)$ commutes with the generators of $B_3$. Thus, when juxtaposing the normal form of $P_i N_i$ with the normal form of $P_{i+1} N_{i+1}$, we may move the factor $C^{k_{i+1}}$ out of the way, moving it from between the normal forms to the beginning of the normal form of $P_i N_i$. This fact will be utilized below.

Looking now at the “List of Normal Forms of Induced Subwords $P_i N_i$ of $\beta$”, notice that half of the normal forms may potentially contain the variable $x$ at the end of the word and half of the normal forms may contain the expression $y^2 \sigma_2^{p-1}$ at the beginning of the word (immediately after the $C^{k_i}$ term that will be moved out of the way). We can now see that juxtaposing the normal form of $P_i N_i$ with that of $P_{i+1} N_{i+1}$ either:

(1) involves neither $x$ at the end of the normal form of $P_i N_i$ nor $y^2$ at the beginning of the normal form of $P_{i+1} N_{i+1}$, or
(2) involves (after moving $C^{k_{i+1}}$ out of the way) both an $x$ at the end of the normal form of $P_i N_i$ and a $y^2 \sigma_2^{p-1}$ at the beginning of the normal form of $P_{i+1} N_{i+1}$.
Consequently, upon juxtaposing the normal forms of all of the induced subwords together, we see that all factors \( x \) and \( y^2 \sigma_2^{p-1} \) in the list of normal forms combine to form
\[
xy^2 \sigma_2^{p-1} = \sigma_2 \sigma_2^{p-1} = \sigma_2^p.
\]

Note, in particular, that juxtaposing the normal form of \( P_i N_i \) with that of \( P_{i+1} N_{i+1} \) does not create any new nontrivial powers of \( C \). Therefore, since \( C \) commutes with the generators of \( B_3 \), then we may group all \( C^{k_i} \) terms together at the beginning of the normal form braid word. We also have that juxtaposing the normal form of \( P_i N_i \) with that of \( P_{i+1} N_{i+1} \) does not create any induced products \( \sigma^{n_j}_i \sigma^{n_{j+1}}_i \) of negative syllables. Thus, we can add the local number \(-k_i\) of induced products \( \sigma^{n_j}_i \sigma^{n_{j+1}}_i \) of negative syllables in the \( P_i N_i \) together to get the global number of such subwords. With this information, we are now able to conclude that:

\[
\begin{align*}
    k &= \sum_{i=1}^t k_i \\
    &= \sum_{i=1}^t -\# \{ \text{induced products } \sigma^{n_j}_i \sigma^{n_{j+1}}_i \text{ of negative syllables of } P_i N_i, \text{ where } n_j, n_{j+1} \leq -3 \} \\
    &= -\# \{ \text{induced products } \sigma^{n_1}_2 \sigma^{n_2}_1 \text{ of negative syllables of } \beta, \text{ where } n_1, n_2 \leq -3 \}.
\end{align*}
\]

This gives the first desired result, that the parameter \( k \) from the Schreier normal form \( \beta' \) can be seen in the original braid \( \beta \). To conclude the proof, we need to relate the global parameter \( s \) from the Schreier normal form to local versions of the parameter \( s \). Let \( s_i \) denote the local version of the parameter \( s \), which comes from the normal form for the subword \( P_i N_i \) and will be more precisely defined below.

Look again at the “List of Normal Forms of Induced Subwords \( P_i N_i \) of \( \beta \)”. Recall that juxtaposing the normal forms to create a braid word groups together the \( x \) and \( y^2 \) factors in the normal forms in such a way that they are absorbed into a trivial or positive syllable \( \sigma^{p-1}_2 \) to form \( \sigma^p_2 \). Also recall that we will collect together all individual powers of \( C \) from each \( P_i N_i \) normal form and use commutativity to form a single power of \( C \) at the beginning of the normal form braid word. Thus, after juxtaposition of the normal forms of the \( P_i N_i \), what results is a braid word that looks like \( C^k W_1 \cdots W_t \), where \( k \) is an integer and \( W_i \) is an alternating word that is positive in \( \sigma_2 \), negative in \( \sigma_1 \), begins with a \( \sigma_2 \) syllable, and ends with a \( \sigma_1 \) syllable. By cyclic permutation and the commutativity of \( C \), we get that this braid word is equivalent to the generic normal form of \( \beta \).

Given an alternating subword \( W_i = \sigma^{p_1}_{2,i} \sigma^{n_1}_{1,i} \cdots \sigma^{p_{n_2}}_{2,i} \sigma^{n_{n_2}}_{1,i} \) as described above, we define \( s_i = q \). To provide an example, for the normal form:
\[
C^{-1} \sigma_2^{p_{n_1+1}} \sigma^{n_2+2}_2 \sigma_2 \sigma^{n_3+2}_1 \sigma_2 \sigma^{n_4+1}_1 x
\]

of Type (4a), we have that \( s_i = 4 \). Consider the product:
\[ W_i W_{i+1} = (\sigma_2^{p_1,i} \sigma_1^{n_1,i} \cdots \sigma_2^{p_q,i} \sigma_1^{n_q,i}) \cdot (\sigma_2^{p_1,i+1} \sigma_1^{n_1,i+1} \cdots \sigma_2^{p_q,i+1} \sigma_1^{n_q,i+1}). \]

Note that the subword \([\sigma_1^{n_1,i} \cdots \sigma_2^{p_q,i} \sigma_1^{n_q,i} \sigma_2^{p_1,i+1}]\) looks like the alternating part of a generic braid word. From this perspective of (cyclically) borrowing the first syllable of the next subword, the local parameter \(s_i\) makes sense as being the local version of the global parameter \(s\).

Returning to the above application of the Schreier Normal Form Algorithm to each subtype of induced subword \(P_i N_i\), it can be seen that the number of negative syllables in \(P_i N_i\) is equal to \(s_i\) for the normal form of \(P_i N_i\). Therefore, combining this fact with what we have learned above about how the juxtaposition works, we are now able to conclude that:

\[
    t^{-}(D) = \# \{\text{negative syllables in } \beta\} \\
    = \sum_{i=1}^{t} \# \{\text{negative syllables in } P_i N_i\} \\
    = \sum_{i=1}^{t} s_i \\
    = s.
\]

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\square
\]

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