ON HOLOMORPHIC FUNCTIONS WITH CLUSTER SETS OF
FINITE LINEAR MEASURE

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Abstract We prove that if \( f \) is a holomorphic function on the open unit disc in \( \mathbb{C} \) whose cluster set \( C(f) \) has finite linear measure and is such that \( \mathfrak{C} \backslash C(f) \) has finitely many components, then the derivative \( f' \) belongs to the Hardy space \( H^1 \).

1. Introduction and the result

Let \( f \) be a complex valued function defined on the open unit disc \( \Delta \subset \mathbb{C} \). The cluster set \( C(f) \) of \( f \) is defined as the set of all \( w \in \mathbb{C} \) such that \( w = \lim_{n \to \infty} f(z_n) \) where \( z_n \in \Delta \), \( \lim_{n \to \infty} |z_n| = 1 \). If \( f \) is bounded and continuous then \( C(f) \) is a compact connected set.

If \( E \subset \mathbb{C} \) then the linear measure \( \Lambda(E) \) of \( E \) is the one dimensional Hausdorff measure of \( E \), that is,

\[
\Lambda(E) = \lim_{\varepsilon \to 0} \inf_{(D_n)} \sum_n \text{diam} D_n
\]

where the infimum is taken over all systems of discs with \( \text{diam} D_n < \varepsilon \) that cover \( E \). If \( E \) is a rectifiable Jordan arc then \( \Lambda(E) \) equals the arclength of \( E \).

Let \( f \) be a nonconstant holomorphic function on \( \Delta \).

Theorem 1.1 [D, p.42] The function \( f \) extends continuously through \( \overline{\Delta} \) and \( f \) is absolutely continuous on \( b\Delta \) if and only if \( f' \) belongs to Hardy space \( H^1 \), that is, if and only if

\[
\sup_{0 < r < 1} \int_0^{2\pi} |f'(re^{i\theta})|d\theta < \infty.
\]

This happens if \( f \) is of bounded variation on \( b\Delta \).

A special case when this happens is

Theorem 1.2 [P2, p.320] Let \( f \) be a biholomorphic map from \( \Delta \) onto a domain \( D \). Then \( f' \in H^1 \) if and only if \( \Lambda(bD) < \infty \). In particular, if \( D \) is bounded by a rectifiable simple closed curve then \( f' \in H^1 \).

We want to find more general conditions for a holomorphic function \( f \) on \( \Delta \) which imply that \( f' \in H^1 \). Looking at Theorem 1.2 it is a natural question whether for a holomorphic function \( f \) on \( \Delta \) the assumption \( \Lambda(C(f)) < \infty \) implies that \( f' \in H^1 \). In the case when \( \Lambda(C(f)) < \infty \) we have the following result of H. Alexander and Ch. Pommerenke
\textbf{Theorem 1.3 [A, P1]} Let $f$ be a holomorphic function on $\Delta$ such that $\Lambda(C(f)) < \infty$. Then $f$ extends continuously through $\overline{\Delta}$.

However, as shown by D. Gnuschke-Hauschild [G-H, p.592], the condition that $\Lambda(C(f)) < \infty$ does not necessarily imply that $f' \in H^1$.

If $f$ is a nonconstant holomorphic function on $\Delta$ then $f(\Delta)$ is a nonempty open set; if $\Lambda(C(f)) < \infty$ then $C(f)$ is a compact connected set with empty interior so $\mathcal{C}\setminus C(f)$ is an open set having at least one bounded component. In the example of Gnuschke-Hauschild $\mathcal{C}\setminus C(f)$ has infinitely many components. The principal result of the present paper is that the situation is different when $\mathcal{C}\setminus C(f)$ has only finitely many components:

\textbf{Theorem 1.4} Let $f$ be a holomorphic function on $\Delta$ such that $\Lambda(C(f)) < \infty$ and such that $\mathcal{C}\setminus C(f)$ has finitely many components. Then $f' \in H^1$.

2. Proof of Theorem 1.4, Part 1

Let $f$ be as in Theorem 1.4. With no loss of generality assume that $f$ is not a constant. By Theorem 1.3, $f$ extends continuously through $\overline{\Delta}$. We denote the extension with the same letter $f$. By Theorem 1.1 we have to show that there is a constant $M < \infty$ such that

$$\sum_{j=1}^{\ell} |f(e^{i\theta_j}) - f(e^{i\theta_{j-1}})| < M$$

whenever $0 \leq \theta_0 < \theta_1 < \cdots < \theta_{\ell} \leq 2\pi$.

As Gnuschke-Hauschild did in [G-H] we decompose $\Delta$ into subsets on which $f$ has simpler behavior. If $V$ is a component of the open set $f(\Delta) \setminus C(f)$ then $V$ is an open connected set whose boundary is contained in $C(f) = f(b\Delta)$ and hence it is a component of $\mathcal{C}\setminus C(f)$. By our assumption $\mathcal{C}\setminus C(f)$ has finitely many components so it follows that $f(\Delta) \setminus C(f)$ has finitely many components. Denote these components by $U_1, U_2, \cdots U_m$. Since $C(f)$ is connected the domains $U_j$ are simply connected.

The set $E = (f(\Delta))^{-1}(C(f)) = \{z \in \Delta : f(z) \in C(f)\} = \{z \in \Delta : f(z) \in f(b\Delta)\}$ is a closed subset of $\Delta$. Let $G_k$, $k = 1, 2, \cdots$ be the components of the open set $\Delta \setminus E$. Then, as shown in [G-H], each domain $G_k$ is simply connected and $f(G_k)$ equals one of the domains $U_1, \cdots, U_m$, moreover $f|G_k: G_k \rightarrow f(G_k)$ is a proper map, that is, if $\varphi_k: \Delta \rightarrow G_k$ and $\psi_k: \Delta \rightarrow f(G_k)$ are biholomorphic maps then $g_k = \psi_k^{-1} \circ f \circ \varphi_k$ is a proper holomorphic map from $\Delta$ to $\Delta$, that is, a finite Blaschke product whose multiplicity we denote by $\nu_k$.

We show that the number of components $G_k$ is finite. To see this, assume the opposite, that there are one of the components $U$ among $U_1, \cdots, U_m$ and a sequence of components $G_k$, $k \in \mathbb{N}$, such that for each $k \in \mathbb{N}$, $f(G_k) = U$. Let $w \in U$. Clearly $w \notin C(f)$. For each $k$ there is a $z_k \in G_k$ such that $f(z_k) = w$. Since $G_k$ are pairwise disjoint, $z_k$ is an injective sequence. We claim that $|z_k| \rightarrow 1$ as $k \rightarrow \infty$. If not, passing to a subsequence if necessary, we may assume that $z_k \rightarrow z \in \Delta$ and thus $f(z_k) = w$ for all $k$ which is not possible since $f$ is nonconstant. Thus, $|z_k| \rightarrow 1$. But since $f(z_k) = w$ for all $k$ it follows that $w \in C(f)$, a contradiction. Thus, the number of components $G_k$ is finite, denote them by $G_1, G_2, \cdots G_n$. Obviously $n \geq m$. 


The cluster set $C(f)$ is a plane continuum. Since $\Lambda(C(f)) < \infty$ it follows that $C(f)$ is locally connected [CC, Lemma 2, p.49]. Since $\mathcal{C} \setminus C(f)$ has finitely many components, Thorhorst’s theorem [CC, p.44, W, p.113] implies that each component of $\mathcal{C} \setminus C(f)$ has locally connected boundary. Thus, $bU_j$ is locally connected for each $j$, $1 \leq j \leq m$.

For each $G_j$ there is some $U_k$ such that $f(G_j) = U_k$ and such that $f|G_j: G_j \rightarrow U_k$ is a proper map. Since $bU_k$ is locally connected, [CC, Lemma 1, p.46] implies that $bG_j$ is locally connected. Thus, for each $j$, $1 \leq j \leq n$, the domain $G_j$ is simply connected and has locally connected boundary.

For each $j$, $1 \leq j \leq n$, let $\psi_j: \Delta \rightarrow G_j$ be a biholomorphic map. Since $bG_j$ is locally connected, the map $\psi_j$ extends continuously through $\overline{\Delta}$ and $\psi_j(b\Delta) = bG_j$ [P2, p. 279]. In particular, for each $a, b \in bG_j$, $a \neq b$, there is an arc $L$, contained in $G_j$ except for its endpoints $a$ and $b$.

Notice that the set $E = (f|\Delta)^{-1}(C(f))$ has no interior: If $E$ contains a nonempty open set then, since $f$ is open, $f(E)$ contains a nonempty open set. Since $f(E) \subset C(f)$ this is not possible since $C(f)$, being of finite linear measure, has no interior. Since $E$ has no interior and since $E \cup \bigcup_{j=1}^{n} G_j = \Delta$ it follows that $\cup_{j=1}^{n} G_j$ is dense in $\Delta$. Thus, $\cup_{j=1}^{n} \overline{G_j} = \overline{\Delta}$ so $b\Delta \subset \cup_{j=1}^{n} bG_j$.

3. Some estimates

Assume that $D$ is a domain such that $\Lambda(bD) < \infty$ and that $\Phi: \Delta \rightarrow D$ is a biholomorphic map. We know that $\Phi$ extends continuously through $\overline{\Delta}$. The proof of [Pom., Th.10.1, p.321] shows that

$$\int_{0}^{2\pi} |\Phi'(re^{i\theta})|d\theta \leq \pi \Lambda(bD) \quad (0 < r < 1).$$

Since $r \int_{0}^{2\pi} |\Phi'(re^{i\theta})|d\theta$ equals the length of the curve $\{\theta \mapsto \Phi(re^{i\theta}) : 0 \leq \theta \leq 2\pi\}$ it follows that given $\theta_j$,

$$\theta_0 < \theta_1 < \cdots < \theta_\ell \leq \theta_0 + 2\pi$$

(3.2)

we have $\sum_{j=1}^{\ell} |\Phi(re^{i\theta_j}) - \Phi(re^{i\theta_j-1})| \leq r \int_{0}^{2\pi} |\Phi'(re^{i\theta})|d\theta \leq r\pi \Lambda(bD)$, which, as $r \rightarrow 1$, gives

$$\sum_{j=1}^{\ell} |\Phi(e^{i\theta_j}) - \Phi(e^{i\theta_j-1})| \leq \pi \Lambda(bD)$$

(3.3)

whenever $\theta_j$ satisfy (3.2).

Assume now that $D$ is a simply connected domain with $\Lambda(bD) < \infty$ and that $g: \Delta \rightarrow D$ is a proper holomorphic map. If $\Phi: \Delta \rightarrow D$ is a biholomorphic map then $\Phi^{-1} \circ g$ is a proper holomorphic map from $\Delta$ to $\Delta$, a finite Blaschke product $B$, so $g = \Phi \circ B$ where the multiplicity of the map $g$ equals the multiplicity of $B$.

If $\nu$ is the multiplicity of $B$ then, as $\zeta$ runs around $b\Delta$ once, $B(\zeta)$ runs around $b\Delta$ $\nu$ times so if $\theta_j$ satisfy (3.2) then there are $\tau_j$, $\tau_0 < \tau_1 < \cdots < \tau_\ell \leq \tau_0 + \nu 2\pi$ such that $B(e^{i\theta_j}) = e^{i\tau_j}$ ($1 \leq j \leq \ell$). The preceding discussion now implies that $\sum_{j=1}^{\ell} |g(e^{i\theta_j}) - g(e^{i\theta_j-1})| = \sum_{j=1}^{\ell} |\Phi(e^{i\tau_j}) - \Phi(e^{i\tau_{j-1}})| \leq \nu \pi \Lambda(bD)$. This gives
Proposition 3.1 Let $D$ be a simply connected domain such that $\Lambda(bD) < \infty$ and let $g: \Delta \rightarrow D$ be a proper holomorphic map of multiplicity $\nu$. Then $g$ extends continuously through $\overline{\Delta}$. If $\theta_0 < \theta_1 < \cdots < \theta_\ell \leq \theta_0 + 2\pi$ then

$$
\sum_{j=1}^{\ell} |g(e^{i\theta_j}) - g(e^{i\theta_{j-1}})| \leq \nu \pi \Lambda(bD)
$$

4. Proof of Theorem 1.4, Part 2

We shall show that

$$
b\Delta \text{ can be written as a finite union of pairwise disjoint semiopen arcs}
$$

$$L_1, L_2, \cdots, L_\mu, \text{ each being of the form } \{e^{i\theta} : \alpha \leq \theta < \beta\} \text{ such that each}
$$

$$L_k \text{ is contained in } bG_\sigma \text{ for some } \sigma, 1 \leq \sigma \leq n.
$$

Assume for a moment that we have done this. With no loss of generality assume that the initial point $e^{i\alpha}$ of $L_1$ is 1.

Let $A_j, 1 \leq j \leq \nu$, be points on $b\Delta$. We say that the $\nu$–tuple $(A_1, A_2, \cdots, A_\nu)$ is ordered positively if $A_j = e^{i\theta_j}$ where $\theta_1 < \theta_2 < \cdots < \theta_\nu < \theta_1 + 2\pi$.

To proceed we need the following

Proposition 4.1 Let $G \subset \Delta$ be a simply connected domain with locally connected boundary and let $\Phi: \Delta \rightarrow G$ be a biholomorphic map (that extends continuously through $\overline{\Delta}$ as $bG$ is locally connected). Suppose that $L$ is a closed arc in $b\Delta$ such that $L \subset bG$ and assume that $A_j \in L, 1 \leq j \leq \nu$, are such that the $\nu$–tuple $(A_1, A_2, \cdots, A_\nu)$ is ordered positively. For each $j, 1 \leq j \leq \nu$, let $a_j \in b\Delta$ be such that $\Phi(a_j) = A_j (1 \leq j \leq \nu)$. Then the $\nu$–tuple $(a_1, a_2, \cdots, a_\nu)$ is ordered positively.

Remark. Note that we do not assume that Int$L$ is an open subset of $bG$. Note also that since $\Phi(b\Delta) = bG$ it follows that for each $j, 1 \leq j \leq \nu$, there is an $a_j \in b\Delta$ which satisfies $\Phi(a_j) = A_j$. Note that this $a_j$ is not necessarily unique.

Proof. It is enough to prove the special case of Proposition 4.1 when $\nu = 3$ as the general case will then follow by using this special case inductively. So let $\nu = 3$ and let $A_j, a_j, 1 \leq j \leq 3$, be as in Proposition 4.1. Let $\lambda \subset b\Delta$ be the arc obtained by sliding $a_1$ to $a_3$ along $b\Delta$ in positive direction, that is, if $a_1 = e^{i\omega_1}$ and $a_3 = e^{i\omega_3}$ where $\omega_1 < \omega_3 < \omega_1 + 2\pi$, then $\lambda = \{e^{i\omega} : \omega_1 < \omega < \omega_3\}$. To see that $(a_1, a_2, a_3)$ is ordered positively we must show that $a_2 \in \lambda$.

Let $\ell$ be the arc in $\Delta$ consisting of the segment joining $a_3$ with 0 and the segment joining 0 with $a_1$ and let $\Omega$ be the domain bounded by $\lambda \cup \ell$. Orient $b\Omega = \lambda \cup \ell$ in the positive direction. In particular, $\ell$ has $a_3$ as the initial point and $a_1$ as the final point. The arc $\mathcal{L} = \Phi(\ell)$ is contained in $\Delta$ except for its endpoints $A_3 = \Phi(a_3)$ and $A_1 = \Phi(a_1)$; we keep the orientation from $\ell$ so $A_3$ is the initial point and $A_1$ is the final point of $\mathcal{L}$.

Since the endpoints of $\mathcal{L}$ belong to $b\Delta$, $\Delta \setminus \mathcal{L}$ has two components $D_1$ and $D_2$ where $D_1$ is bounded by $\mathcal{L}$ and by the arc $\Sigma \subset b\Delta$ obtained by sliding $A_1$ to $A_3$ along $b\Delta$ in positive
direction. The arc $\Sigma$ oriented in this direction together with the arc $\mathcal{L}$ oriented as above form the positively oriented boundary of $D_1$. Since $\Phi : \Delta \to \Phi(\Delta) = G$ is a biholomorphic map, $\Phi(\Omega)$ must be contained either in $D_1$ or in $D_2$. Along $\ell$, $\Omega$ lies to the left of $\ell$ with respect to the above orientation of $\ell$ and since $\Phi$ is conformal it follows that along $\mathcal{L}$, $\Phi(\Omega)$ lies to the left of $L$. This implies that $\Phi(\Omega) \subset D_1$. If $a_2 \in b\Delta \setminus \lambda$ then $a_2$ is in the closure of $\Delta \setminus \Omega$, which, by the continuity of $\Phi$ on $\overline{\Sigma}$, implies that $\Phi(a_2) = A_2$ is in the closure of $D_2$ which is impossible since $A_2$ is an interior point of $\Sigma$. This completes the proof of Proposition 4.1.

Recall that for each $\sigma$, $1 \leq \sigma \leq n$, $\nu_\sigma$ is the multiplicity of the proper map $f|G_\sigma : G_\sigma \to f(G_\sigma)$. Let $V = \max_{1 \leq \sigma \leq n} \nu_\sigma$.

Now, let
\[ 0 \leq \theta_0 < \theta_1 < \cdots < \theta_\ell \leq 2\pi. \tag{4.2} \]

Fix $k$, $1 \leq k \leq \mu$, and consider those points $e^{i\theta_j}$ that belong to $L_k$, suppose that these points are $e^{i\theta_s}, e^{i\theta_{s+1}}, \ldots, e^{i\theta_t}$. Let $\sigma$, $1 \leq \sigma \leq n$, be such that $L_k \subset bG_\sigma$. By Proposition 4.1 there are $\omega_j$, $\omega_s < \omega_{s+1} < \cdots < \omega_t < \omega_s + 2\pi$, such that
\[ \psi_\sigma(e^{i\omega_p}) = e^{i\theta_p} \quad (s \leq p \leq t) \]
and hence by Proposition 3.1
\[
\sum_{j=s+1}^{t} |f(e^{i\theta_j}) - f(e^{i\theta_{j-1}})| = \sum_{j=s+1}^{t} |f(\psi_\sigma(e^{i\omega_j})) - f(\psi_\sigma(e^{i\omega_{j-1}}))| \leq \nu_\sigma \pi \Lambda(b(f(G_\sigma))) \leq V \pi \Lambda(C(f)).
\]

We repeat this for every $k$, $1 \leq k \leq \mu$. In the sum
\[
\sum_{j=1}^{\ell} |f(e^{i\theta_j}) - f(e^{i\theta_{j-1}})| \tag{4.3}
\]
there are terms as above whose total sum does not exceed $\mu V \pi \Lambda(C(f))$. However, in the sum (4.3) there are also terms $|f(e^{i\theta_j}) - f(e^{i\theta_{j-1}})|$ where $e^{i\theta_{j-1}}$ is the last point in $L_k$ and $e^{i\theta_j}$ is the first point in $L_{k+1}$. Each such term does not exceed $\text{diam}(C(f))$ and there are $\mu - 1$ such terms. Thus, the sum (4.3) is bounded above by $\mu V \pi \Lambda(C(f)) + (\mu - 1)\text{diam}(C(f))$ whenever $\theta_j$ satisfy (4.2). Consequently, $f|(b\Delta)$ has bounded variation which was to be proved.

It remains to prove (4.1).

5. Proof of Theorem 1.4, Part 3

We now want to study how domains $G_j$ ”touch” $b\Delta$, that is we want to look at $(b\Delta) \cap \overline{G_j} = (b\Delta) \cap bG_j$. Fix $j$, $1 \leq j \leq n$. Consider the set $\Sigma = \overline{G_j} \cap b\Delta = (bG_j) \cap b\Delta$. This is a closed subset of $b\Delta$. Its complement $(b\Delta) \setminus \Sigma$ is an open subset of $b\Delta$. We show that it has finitely many components. To see this, let $\lambda \subset b\Delta$ be a component of $b\Delta \setminus \Sigma$
Thus, $\lambda$ is an open arc on $b\Delta$ disjoint from $\overline{G_j}$ whose endpoints belong to $\overline{G_j}$. As observed in Section 2 there is an arc $L$ contained in $G_j$ except for its endpoints which coincide with endpoints of $\lambda$. Since $b\Delta = \bigcup_{i=1}^{n} bG_i \cap b\Delta$ it follows that there are a $k$, $1 \leq k \leq n$, $k \neq j$, and a point $a \in \lambda \cap (bG_k)$. Notice that $bG_k$ meets no other component of $(b\Delta) \setminus \Sigma$. Indeed, if it does, there is an arc $L_1$ contained in $G_k$ except its endpoints $a$ and $b \in \lambda_1$ where $\lambda_1$ is a component of $b\Delta \setminus \Sigma$ different from $\lambda$. However, such an $L_1$ would have to intersect $L$ which is impossible since $L \subset G_j$ and $L_1 \subset G_k$ and $G_j \cap G_k = \emptyset$. Thus, for each component $\lambda$ of $b\Delta \setminus \Sigma$ there is a $G_k$ such that $\lambda \cap bG_k \neq \emptyset$ and such that $bG_k$ meets no other component of $b\Delta \setminus \Sigma$. Obviously all domains $G_k$ so obtained are pairwise disjoint. Since there are only $n$ of them it follows that $b\Delta \setminus \Sigma$ has finitely many components.

Thus, for each $j$, $1 \leq j \leq n$, the set $bG_j \cap b\Delta$, if nonempty is a union of finitely many pairwise disjoint closed sets each of which is either a point or a closed arc. (The only trivial noninteresting exception is when there is just one such $G_j$, $G_1 = \Delta$ when $bG_1 = b\Delta$.) Since $\bigcup_{j=1}^{n} (bG_j \cap b\Delta) = b\Delta$ it follows that the family of all closed arcs so obtained for all $j$, $1 \leq j \leq n$, covers $b\Delta$. It is easy to see that this implies (4.1). The proof of Theorem 1.4 is complete.

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