TRAVELING WAVE SOLUTIONS FOR A ONE DIMENSIONAL MODEL OF CELL-TO-CELL ADHESION AND DIFFUSION WITH MONOSTABLE REACTION TERM

LIANZHANG BAO*
College of Mathematics
Jilin University
Changchun, Jilin 130012, China

ZHENGFANG ZHOU
Department of Mathematics
Michigan State University
East Lansing, MI 48824, USA

Abstract. This work is concerned with the properties of the traveling wave solutions of a one dimensional model of cell diffusion and aggregation, incorporating volume filling and cell-to-cell adhesion with net birth term

\[ \rho_t = [D(\rho)\rho_x]_x + g(\rho) \quad t \geq 0, \quad x \in \mathbb{R}, \]

where \( D(\rho) \) may take positive or negative values with different population density \( \rho \) and adhesion coefficient \( \alpha \in [0, 1] \), and the negative one will lead to the ill-posedness of the equation. In all these cases we prove the existence of infinitely many traveling wave solutions, where these solutions are parameterized by their wave speed and monotonically connect the stationary states \( \rho \equiv 0 \) and \( \rho \equiv 1 \).

1. Introduction. Reaction diffusion equations have been applied extensively to population dynamics. By using a random walk approach and assuming that the individuals of a population have the same probability of moving from one point to another, Skellam [26] derived the standard reaction diffusion equation (Fisher-KPP equation)

\[ \rho_t = D\Delta \rho + g(\rho) \]

where \( \rho(r,t) \) is the population density, \( D > 0 \) is a constant and \( g(\rho) \) is the net rate of growth. However, it is now clear that in a number of biology contexts, motility varies with population density, requiring nonlinear diffusion terms. This was first realized in ecology [15, 16, 25] and density-dependent dispersal is now a common feature of spatial modeling in biology. This includes models that are of degenerate reaction diffusion equations [8, 9, 12, 13, 14, 22], as well as reaction diffusion aggregation equations [11, 15, 20, 21].

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*Corresponding author.
There has been considerable interest in formulating continuum models for cell structures generated by cell-to-cell adhesion, the motivation is to facilitate mathematical analysis and efficient numerical processes, such as de novo blood-vessel synthesis (i.e. vasculogenesis) and cancer invasion \cite{2}. Several attempts in this direction were made. For example, Armstrong et al. \cite{4} began with a non-local integro-differential equation in which the kernel is integrated over a given cell-sensing radius, Bao and Zhou \cite{5} used a biased random walk model combined with logistic model, Horstmann et al. \cite{18} used a simple random walk model with the assumption that the control substance depends quadratically on the local density of individual and decays via first order kinetics, and Anguige et al. \cite{3} used a simple random walk model but accounting for adhesion, diffusion and volume filling. For all of these approaches, it turns out that the limiting macroscopic model can be ill-posed which will lead to aggregation in the population pattern.

Our intention in this paper is to extend the one-dimensional continuum model for adhesion/diffusion of biological cells developed by Anguige et al. \cite{3} but using a more reasonable governing equation, different form of transferring probability, and also adding the birth term that is very natural to the cell population with reproduction, which will take the form of the non-linear diffusion aggregation equation

$$\rho_t = [D(\rho)\rho_x]_x + g(\rho) \quad t \geq 0, \quad x \in \mathbb{R},$$

with quadratic coefficient

$$D(\rho) = 3\alpha\rho^2 - 4\alpha\rho + 1,$$

where the birth term $g(\rho)$ satisfies

$$g(\rho) \in C^1([0,1]), \quad g(\rho) > 0 \quad \text{in} \quad (0,1), \quad g(0) = g(1) = 0.$$

1.1. The derivation of the model. Here we consider one species living in a one-dimensional habitat. To derive the model we follow a biased random walk approach plus a diffusion approximation. First we discretize space in a regular manner. Let $h$ be the distance between two successive points of the mesh, and let $\rho(x,t)$ be the population density at the point $x$, time $t$ where $0 \leq \rho(x,t) \leq 1$ with scaled density. During a time period $\tau$ an individual which is at the position $x$ and time $t$ can either:

1: move to the right of $x$ to the point $x + h$, with probability $R(x,t)$,

2: move to the left of $x$ to the point $x - h$, with probability $L(x,t)$, or

3: stay at the position $x$, with probability $N(x,t)$.

Assume that there are no other possibilities of movement, we have

$$N(x,t) + R(x,t) + L(x,t) = 1.$$

We also assume that

$$R(x,t) = (1/2)(1 - \rho(x+h,t))(1 - \alpha\rho(x-h,t)),$$

$$L(x,t) = (1/2)(1 - \rho(x-h,t))(1 - \alpha\rho(x+h,t)).$$

Here the first factor $1 - \rho(x+h,t)$ of $R(x,t)$ models volume filling with a scaled maximal density 1, and the second factor $1 - \alpha\rho(x-h,t)$ is a simple model for adhesion, which assumes that the probability of a given species jumping to the right is reduced by the presence of neighbors to the left. The meaning of $L(x,t)$
is analogous. We can see that $0 \leq R(x,t), L(x,t), R(x,t) + L(x,t) \leq 1$, which is different from the form given in [3]. In their case,

$$
\begin{align*}
R(x,t) &= (1 - \rho(x + h,t))(1 - \alpha \rho(x - h,t))/h^2, \\
L(x,t) &= (1 - \rho(x - h,t))(1 - \alpha \rho(x + h,t))/h^2
\end{align*}
$$

may go to infinity as $h \to 0$, which can not be explained from a probabilistic standpoint.

Using the notations above, we can rewrite the density $\rho(x,t)$ as follows:

$$
\rho(x,t+\tau) = N(x,t)\rho(x,t) + R(x-h,t)\rho(x-h,t) + L(x+h,t)\rho(x+h,t) + \tau g(\rho(x,t))
$$

where $g(\rho(x,t))$ is the net birth rate at $(x,t)$. By using Taylor series, we obtain the following approximation,

$$
\rho(x,t) + \tau \frac{d\rho}{dt} = N(x,t)\rho(x,t) + \{R(x,t)\rho(x,t) - h \frac{d(R\rho)}{dx} + \frac{h^2 d^2 (R\rho)}{2 dx^2}\}
$$

$$
+ \{L(x,t)\rho(x,t) + h \frac{d(L\rho)}{dx} + \frac{h^2 d^2 (L\rho)}{2 dx^2}\} + \tau g(\rho(x,t)).
$$

Then we get

$$
\tau \frac{d\rho}{dt} = \{-h \frac{d(R\rho)}{dx} + \frac{h^2 d^2 (R\rho)}{2 dx^2}\} + \{h \frac{d(L\rho)}{dx} + \frac{h^2 d^2 (L\rho)}{2 dx^2}\} + \tau g(\rho(x,t)).
$$

Set

$$
\beta(x,t) = R(x,t) - L(x,t)
$$

$$
\nu(x,t) = R(x,t) + L(x,t)
$$

we obtain

$$
\tau \frac{d\rho}{dt} = -h \frac{d[(R - L)\rho]}{dx} + \frac{h^2 d^2 [(R + L)\rho]}{2 dx^2} + \tau g(\rho(x,t)).
$$

Now substituting $\beta$ and $\nu$ in the above equation, we obtain

$$
\tau \frac{d\rho}{dt} = (1 - \alpha)h^2(\rho_x \rho)_x + \frac{h^2}{2}[(1 - (1 + \alpha)\rho(x) + \alpha \rho^2(x))\rho(x)]_{xx} + \tau g(\rho(x,t)) + O(h^3).
$$

Using the same diffusion approximation as in [5, 18, 24, 26], and assuming that $h^2/\tau \to C > 0$ as $\tau, h \to 0$, we obtain the following,

$$
\frac{d\rho}{dt} = C[(1 - \alpha)\rho_x \rho]_x + C/2[(1 - 2(1 + \alpha)\rho + 3\alpha \rho^2)\rho_x]_x + g(\rho(x,t)),
$$

$$
= C/2[(1 - 4\alpha \rho + 3\alpha \rho^2)\rho_x]_x + g(\rho(x,t)).
$$

Then, letting $C = 2$ for simplicity, we obtain the equation [1].

We recall from [3] that the equation

$$
\rho_t = [D(\rho)\rho_x]_x \quad t \geq 0, \quad x \in \mathbb{R},
$$

with quadratic coefficient $D(\rho) = 3\alpha \rho^2 - 4\alpha \rho + 1$, is globally well posed if $0 \leq \alpha < \frac{3}{4}$. When $\alpha = 3/4$, the equation [1] degenerates at $\rho = 2/3$, and when $1 \geq \alpha > \frac{3}{4}$, the
equation (4) is ill posed if and only if the initial density profile protrudes into the unstable interval
\[I_\alpha = (\rho^\flat(\alpha), \rho^\sharp(\alpha)) := \left(\frac{2\alpha - \sqrt{\alpha(4\alpha - 3)}}{3\alpha}, \frac{2\alpha + \sqrt{\alpha(4\alpha - 3)}}{3\alpha}\right) \subset \left[\frac{1}{3}, 1\right].\] (5)
Equations (1)-(2) can be seen as an extension of the standard Fisher-KPP equation
\[\rho_t = D \Delta \rho + g(\rho),\] (6)
where \(D > 0\) is constant.
Traveling wave solutions (t.w.s) are very important in reaction diffusion equation, which in ecology correspond to invasions, and in cell biology correspond to the advancing edge of an expanding cell population, such as a growing tumor. We recall that a traveling wave solution is a solution \(\rho(x,t)\) having a constant profile, that is \(\rho(x,t) = \rho(x - ct) = \rho(\tau)\) for some function \(\rho(\tau)\), and the constant \(c\) is the wave speed. In particular, a traveling wave solution connecting the steady states \(\rho = 0\) and \(\rho = 1\) satisfies the boundary value problem
\[\begin{align*}
(D(\rho))' + c\rho' + g(\rho) &= 0, \\
\rho(-\infty) &= 1, \quad \rho(+\infty) = 0.
\end{align*}\] (7) (8)
The first systematic analysis on the existence of traveling wave solution of the standard Fisher-KPP equation appeared in two separate works due to Fisher [11] and Kolmogorov et al. [19]. The main ideas of the methodology introduced by Kolmogorov et al. are still used today.
For the non-linear diffusion equation
\[\rho_t = [D(\rho)\rho_x]_x + g(\rho),\]
where \(D(\rho)\) is a strictly positive function on \([0,1]\) and the kinetic part \(g(\rho)\) is as in the classic Fisher-KPP equation, Hadeler [17] gave the lower bound on \(c\) for the existence of traveling wave solution of front type. In the degenerate case where \(D(0) = 0\) with \(D(\rho) > 0\) \(\forall \rho \in (0,1]\), Sánchez-Garduño and Maini [12, 13] used a dynamical systems approach to prove the existence and nonexistence of the traveling wave and the monotone decreasing property of the traveling wave solution.
In the doubly degenerate Fisher-KPP equations when \(D(0) = D(1) = 0\) with \(D(\rho) > 0\) elsewhere, under less regularity conditions on \(g(\rho)\) and \(D(\rho)\), Malaguti and Marcelli [22] obtained a continuum of traveling wave solutions having wave speed \(c\) greater than a threshold value \(c^*\) and showed the appearance of a sharp-type or finite type profile when \(c = c^*\).
For the diffusion aggregation equation
\[\rho_t = [D(\rho)\rho_x]_x + g(\rho),\]
where \(D(\rho)\) changes its sign once, from positive to negative values, in the interval \(\rho \in [0,1]\), and \(g(\rho)\) is a monostable nonlinear term. Maini et al. [21] proved the existence of infinitely many \(C^1\) traveling wave solutions. These solutions are parameterized by their wave speed and monotonically connect to the stationary states \(\rho = 0\) and \(\rho = 1\). In the degenerate case, i.e., when \(D(0) = 0\) and/or \(D(1) = 0\), sharp profiles appear, corresponding to the minimum wave speed. In 2010 Sánchez-Garduño et al. [14] investigated a family of degenerate negative diffusion equation \((D(\rho) \leq 0)\) with mono-stable reaction term \(g(\rho)\) and studied the one dimensional traveling wave solutions for these equations.
In 2011 Kuzmin and Ruggerini \cite{20} considered a modified aggregation diffusion model with the diffusivity \(D(\rho)\) satisfying the following
\[
D(\rho) > 0 \quad \text{in} \quad (0, \beta) \cup (\gamma, 1), \quad D(\rho) < 0 \quad \text{in} \quad (\beta, \gamma), \quad D'(\gamma) = 0,
\]
and \(g(\rho)\) satisfying the bi-stable condition. They gave necessary and sufficient conditions under which traveling wave solutions of the equation exist and provided an estimate for the minimal speed \(c^*\). In their situation, \(D'(\gamma) = 0\) is a very important condition in proving the existence of \(C^1\) traveling waves. However, in this paper we will see if \(D'(\gamma) > 0\), traveling wave solutions may not differentiable at \(\gamma\) with minimal wave speed, and that is the motivation of the weak traveling wave solution which improves results in \cite{10}.

DiCarlo et al. \cite{6} obtained traveling wave solutions to a nonlinear diffusion equation where \(D(\rho)\) contains a negative region and the nonmonotonic traveling wave is driven by a boundary condition on a semi-infinite domain, whereas in our model the traveling wave is driven by the source term on an infinite domain and has the property of monotonicity.

The structure of our paper is as follows: In section 2, we define the weak traveling wave solution and prove the traveling wave speed is positive and the traveling wave solution is monotone decreasing. This helps us define the inverse function of the traveling solution. In section 3, we prove the existence of \(C^1\) traveling wave solutions when \(0 \leq \alpha < 3/4\). In the complicated case when \(3/4 \leq \alpha \leq 1\), we prove the existence of weak and \(C^1\) traveling wave solutions in Theorem 1.1. For the precise definitions of \(C^1\) traveling wave and weak traveling wave solutions, see Definitions 2.1 and 2.2.

**Theorem 1.1.** Let \(D(\rho)\) and \(g(\rho)\) given functions respectively satisfy \(2\) and \(3\) and \(3/4 \leq \alpha \leq 1\). There exists a value \(c^* > 0\), satisfying
\[
\max\{0, D'(\rho^\flat(\alpha))g(\rho^\flat(\alpha))\} \leq \frac{(c^*)^2}{4}
\]
\[
\leq \max\{ \sup_{s \in (\rho^\flat(\alpha), \rho^\sharp(\alpha))} \frac{D(s)g(s)}{s}, \sup_{s \in (\rho^\flat(\alpha), 1], \rho \neq \rho^\flat(\alpha)} \frac{D(s)g(s)}{s - \rho^\flat(\alpha)} \},
\]
such that the equation \(1\) has

i) no weak traveling wave solution for \(c < c^*\),
ii) a unique (up to space shifts) weak traveling wave solution for \(c = c^*\),
iii) a unique (up to space shifts) \(C^1\) traveling wave solution for \(c > c^*\),

Where \(\rho^\flat(\alpha)\) and \(\rho^\sharp(\alpha)\) are given in \(5\).

**Remark 1.** When \(0 \leq \alpha < 3/4\), we can see the diffusion coefficient \(D(\rho) > 0\) and the existence of the traveling wave solution is already known.

2. **Preliminaries and necessary conditions.** We recall that a traveling wave solution for the equation \(1\) is a solution of the form \(\rho(x, t) = \rho(x - ct)\) for some constant traveling speed \(c\). The equation we have to deal with is changed to
\[
(D(\rho)\rho)' + c\rho' + g(\rho) = 0,
\]
where \('\) stands for differentiation with respect to the wave variable \(\xi = x - ct\). Following \cite{21}, we have the definition of the classical traveling wave solution for \(D(\rho) \in C^3[0, 1]\).
Definition 2.1. When $D(\rho) \in C^1[0, 1]$, a classical traveling wave solution of (1) is a function $\rho \in C^1(a, b)$, with $(a, b) \subseteq \mathbb{R}$, such that $D(\rho)\rho' \in C^1(a, b)$, satisfying the equation (9) in $(a, b)$ and the boundary conditions

$$
\rho(a^+) = 1, \quad \rho(b^-) = 0, \quad \lim_{\xi \to a^+} D(\rho(\xi))\rho'(\xi) = \lim_{\xi \to b^-} D(\rho(\xi))\rho'(\xi) = 0.
$$

(10)

Condition (11) added to the classical boundary condition (10) is motivated by the possible occurrence of sharp type profiles, that is, solutions that reach the equilibria at a finite value $a$ or $b$. However, when the existence interval is the whole real line, then the condition (11) is automatically satisfied and then Definition 2.1 is reduced to the classical one which is the front type traveling wave solution.

However $\rho$ may not be $C^1$ because of the singularity of the equation at $\rho = \rho^*(\alpha), \rho^*(\alpha)$, and we have to give a definition of weak traveling wave solution.

Definition 2.2. A weak traveling wave solution of the equation (1), is a function $\rho \in C(a, b)$ satisfying boundary conditions (10), (11) and for any $t$ with $\rho(t) \neq 0, \rho^*(\alpha), \rho^*(\alpha)$ and 1, then $\rho'(t)$ exists, $\int_a^b g(\rho(s))ds = c$ and

$$
D(\rho)\rho'(t) + cp(t) + \int_a^t g(\rho(s))ds = c.
$$

(12)

Remark 2. Equation (12) comes from

$$
[D(\rho)\rho'(t) + cp(t) + \int_a^t g(\rho(s))ds]' = (D(\rho)\rho'(t))' + cp'(t) + g(\rho(t)) = 0,
$$

and $\lim_{t \to a^+} \rho(t) = 1, \lim_{t \to a^+} D(\rho(t))\rho'(t) = 0$. The condition $\int_a^b g(\rho(s))ds = c$ comes from

$$
\int_a^b g(\rho(s))ds = \int_a^b -(D(\rho)\rho')'ds - c \int_a^b \rho' ds = -c[\rho(b) - \rho(a)] = c,
$$

also from the definition of the weak traveling wave solution and the property of $g(\rho), c > 0$.

When the set $\{t | \rho(t) = 0, \rho^*(\alpha), \rho^*(\alpha), 1\}$ is isolated, then $D(\rho(t))\rho'(t)$ can be extended to a continuous function on $(a, b)$. With this definition (12), if $\rho(t)$ is a weak solution on $(a, b)$ with $-\infty < a < b < \infty$, we can extend $\rho(t)$ to $(-\infty, \infty)$, defined by

$$
\tilde{\rho}(t) = \begin{cases} 
\rho(t), & a < t < b, \\
1, & t \leq a, \\
0, & t \geq b.
\end{cases}
$$

It is easy to check that $\tilde{\rho}(t)$ is a weak traveling wave solution on $(-\infty, \infty)$.

Proposition 1. Let $\rho(t)$ be a weak traveling wave solution to the equation (12).

If $a = -\infty, \rho(\xi) < 1$ for $\xi \in (-\infty, b)$, then $\lim_{\xi \to -\infty} D(\rho(\xi))\rho'(\xi) = 0$; if $b = +\infty, \rho(\xi) > 0$ for $\xi \in (a, +\infty)$, then $\lim_{\xi \to +\infty} D(\rho(\xi))\rho'(\xi) = 0$.

Proof. We only prove the case when $a = -\infty$, the idea for $b = +\infty$ is the same. Assume $a = -\infty$, and let $T := \sup\{\xi : \rho(t) > \rho^*(\alpha) \text{ for all } t \in (-\infty, \xi)\}$, and define

$$
H(\rho) := \int_{\rho^*(\alpha)}^b D(s)g(s)ds, \quad \Phi(\xi) := \frac{1}{2}[D(\rho(\xi))\rho'(\xi)]^2 - H(\rho(\xi))
$$
for \( \rho \in (\rho^*(\alpha), 1) \) and \( \xi \in (-\infty, T) \). Then
\[
\Phi'(\xi) = (D(\rho(\xi))\rho'(\xi))^2/D(\rho(\xi))\rho'(\xi) + D(\rho(\xi))g(\rho(\xi))\rho'(\xi) = -cD(\rho(\xi))(\rho'(\xi))^2.
\]
Since \( D(\rho(\xi)) \geq 0 \), we have \( \Phi'(\xi) \) is monotone decreasing and the limit \( \lim_{\xi \to -\infty} \Phi(\xi) \) exists. On the other hand, we know
\[
\lim_{\xi \to -\infty} H(\rho(\xi)) = \int_{\rho^*(\alpha)}^{1} D(s)g(s)ds \in \mathbb{R}.
\]
Hence, from the definition of \( \Phi(\xi) \), we deduce the existence of the limit
\[
\lim_{\xi \to -\infty} D(\rho(\xi))|\rho'(\xi)| = l \in [0, +\infty].
\]
If \( l > 0 \), note \( \rho(-\infty) = 1 \) and \( D(\rho(\xi)) \geq 0 \) in \(( -\infty, T) \). Then,
\[
\lim_{\xi \to -\infty} |\rho'(\xi)| = k \in (0, \infty],
\]
which is in contradiction with the boundedness of \( \rho \). Therefore, \( l = 0 \).

As a consequence, when \( (a, b) = (-\infty, \infty) \) and \( 0 < \rho(t) < 1 \) on \(( -\infty, \infty) \), the condition \((11)\) is satisfied automatically and Definition \((2.1)\) reduces to the classical front type traveling wave solution.

The diffusivity \( D(\rho) = 3\alpha\rho^2 - 4\alpha\rho + 1 \), and \( D(0), D(1) \neq 0 \) except when \( \alpha = 1 \) and \( D(1) = 0 \). With \( \lim_{\xi \to a^+} D(\rho(\xi))\rho'(\xi) = 0 \) and \( D(\rho(a^+)) = D(1) \neq 0 \), we obtain \( \rho'(a^+) = 0 \) which leads to the front type traveling wave solution, otherwise the solution could be continued in the whole real line with \( C^1 \) regularity. The only possible occurrence of the sharp type traveling wave solution is when \( \alpha = 1 \) and the equation degenerate at \( \rho = 1 \).

We should also remark that if there exists \( a < a_1 \) such that \( \rho(a_1) = 1 \), then \( \rho \equiv 1 \) on \([a, a_1]\). In fact, if \( \rho(\xi) \neq 1 \), then we can find a subinterval \((a_2, a_3) \in (a, a_1)\) such that \( 0 < \rho(\xi) < 1 \) on \((a_2, a_3)\), \( \lim_{\xi \to a_2} \rho(\xi) = \lim_{\xi \to a_3} \rho(\xi) = 1 \). Furthermore we have \( \xi_n \in (a_2, a_3) \), \( \xi_n \to a_3 \) such that \( \rho'(\xi_n) \geq 0 \), which implies that
\[
c\rho(\xi_n) + \int_{a}^{\xi_n} g(\rho(s))ds \leq D(\rho(\xi_n))\rho'(\xi_n) + c\rho(\xi_n) + \int_{a}^{\xi_n} g(\rho(s))ds = c.
\]
Hence
\[
c < c\rho(a_3) + \int_{a}^{a_3} g(\rho(s))ds
\]
\[
= \lim_{\xi_n \to a_3} (c\rho(\xi_n) + \int_{a}^{\xi_n} g(\rho(s))ds)
\]
\[
\leq \lim_{\xi_n \to a_3} [D(\rho(\xi_n))\rho'(\xi_n) + c\rho(\xi_n) + \int_{a}^{\xi_n} g(\rho(s))ds] = c,
\]
which is a contradiction.

From now on, we consider the equation on the minimal interval \((a^*, +\infty)\) (possibly the whole real line) such that \( \rho(\xi) = 1 \) for every \( \xi \leq a^* \).

**Proposition 2.** Let \( \rho(t) \) be the weak traveling wave solution of the equation \((1)\). Then \( \rho(t) \) is strictly decreasing on \((a^*, +\infty)\).

**Proof.** We only prove the monotonicity of the solution when \( 0 < \rho(t) < 1 \) and \( 3/4 \leq \alpha \leq 1 \). For \( 0 \leq \alpha < \frac{3}{4} \), we have \( D(\rho) > 0 \) on \( \rho \in [0, 1] \) and it is not difficult to use classical results to obtain the monotonicity of the traveling wave solutions.
When $\frac{3}{4} \leq \alpha \leq 1$, the diffusivity $D(\rho)$ is negative whenever
\[
\rho \in I_\alpha = (\rho_0(\alpha), \rho_1(\alpha)) = \left(\frac{2\alpha - \sqrt{\alpha(4\alpha - 3)}}{3\alpha}, \frac{2\alpha + \sqrt{\alpha(4\alpha - 3)}}{3\alpha}\right),
\]
and $D(\rho) \geq 0$ otherwise. In the following, we split the proof into four steps.

**Step 1.** Let us prove that

\[
\frac{2\alpha + \sqrt{\alpha(4\alpha - 3)}}{3\alpha} < \rho(t) < 1 \quad \text{and} \quad \rho'(t) < 0 \quad \text{for every} \quad t \in (a^*, T_1), \quad (13)
\]

where

\[
T_1 := \sup\{t : \rho(\xi) > \frac{2\alpha + \sqrt{\alpha(4\alpha - 3)}}{3\alpha} \quad \text{for every} \quad \xi \in (a^*, t)\}.
\]

Suppose to the contrary, there exists $t_1 \in (a^*, T_1)$, such that $\rho'(t_1) \geq 0$ and $\rho(t_1) < 1$. Then from

\[
\frac{d}{d\xi}(D(\rho)\rho')|_{\xi = t_1} = -c\rho'(t_1) - g(\rho(t_1)) < -c\rho'(t_1) \leq 0,
\]

we can get $D(\rho)\rho'(t)$ is decreasing in a neighborhood of $t_1$. With $D(\rho) > 0$, this leads to $\rho'(\xi) > 0$ in $(t_1 - \delta, t_1)$ for some $\delta > 0$. From $\rho(a^*) = 1$, there must be another point with $a^* < t_0 < t_1$ satisfying $\rho'(t_0) = 0$, and $\rho(t_0)$ is the local minimum of $\rho$ on $[a^*, t_1]$. By using the equation (12), we obtain

\[
\rho''(t_0) = -\frac{g(\rho(t_0))}{D(\rho(t_0))} < 0,
\]

a contradiction to the fact $t_0$ is the local minimum. Hence (13) is proven.

**Step 2.** Let us now prove

\[
0 < \rho < \frac{2\alpha - \sqrt{\alpha(4\alpha - 3)}}{3\alpha} \quad \text{and} \quad \rho'(t) < 0 \quad \text{for every} \quad t \in (T_2, +\infty), \quad (14)
\]

where

\[
T_2 := \inf\{t : \rho(\xi) < \frac{2\alpha - \sqrt{\alpha(4\alpha - 3)}}{3\alpha} \quad \text{for every} \quad \xi \in (t, +\infty)\}.
\]

Suppose to the contrary, there exists a value $\xi^* \in (T_2, +\infty)$ such that $\rho'(\xi^*) \geq 0$ and $\frac{2\alpha - \sqrt{\alpha(4\alpha - 3)}}{3\alpha} > \rho(\xi^*) > 0$. Then

\[
\frac{d}{d\xi}(D(\rho)\rho')|_{\xi = \xi^*} = -c\rho'(\xi^*) - g(\rho(\xi^*)) < -c\rho'(\xi^*) \leq 0,
\]

hence $D(\rho(\xi))\rho'(\xi)$ is decreasing in a neighborhood of $\xi^*$. With $D(\rho(\xi)) > 0$, we get $\rho'(\xi) > 0$ in $(\xi^* - \delta, \xi^*)$ for some $\delta > 0$. Let

\[
\bar{\xi} := \inf\{\xi : \rho'(t) \geq 0 \quad \text{for every} \quad t \in (\xi, \xi^*)\},
\]

and we have $\bar{\xi} < +\infty$. Since $\rho$ is increasing on $(\bar{\xi}, \xi^*)$ and $\lim_{t \to +\infty} \rho(t) = 0$, we claim $\rho'(\bar{\xi}) = 0$. Otherwise $\rho'(\bar{\xi}) > 0$, this is a contraction to the definition of $\bar{\xi}$. Now we have the local minimum at $\bar{\xi} \in (T_2, +\infty)$, from the equation (12) we have

\[
D(\rho)\rho'' + D'(\rho)(\rho')^2 + c\rho' + g(\rho) = 0,
\]

and this leads to

\[
\rho''(\bar{\xi}) = -\frac{g(\rho(\bar{\xi}))}{D(\rho(\bar{\xi}))} < 0,
\]
which contradicts to the fact that \( \rho \) has local minimum at \( \xi \).

**Step 3.** We first claim \( \rho'(T_1^+) < 0 \). Otherwise when \( \rho'(T_1^+) > 0 \), there exists \( \xi_1 > T_1 \), with \( \rho'(\xi_1) \geq 0, \rho(\xi_1) > \rho(T_1) \) and \( \xi_2 < T_1 \), with \( \rho(\xi_2) \leq \rho(\xi_1) \) such that

\[
0 < D(\rho(\xi_1))\rho'(\xi_1) - D(\rho(\xi_2))\rho'(\xi_2) + c(\rho(\xi_1) - \rho(\xi_2)) + \int_{\xi_2}^{\xi_1} g(\rho(t))dt = 0,
\]

which is a contradiction. If \( \rho'(T_1^+) = 0 \), we have

\[
\frac{d}{d \xi}(D(\rho)\rho')|_{\xi = T_1^+} = -c\rho'(T_1^+) - g(\rho(T_1^+)) = -g(\rho(T_1^+)) < 0,
\]

hence \( D(\rho(\xi))\rho'(\xi) \) is decreasing in the right neighborhood of \( T_1 \). Since \( D(\rho(T_1)) = 0 \), we get \( D(\rho(\xi)) < 0, \rho'(\xi) > 0 \) or \( D(\rho(\xi)) > 0, \rho'(\xi) < 0 \) in \( (T_1, T_1 + \delta) \) for some \( \delta > 0 \), which contradicts the definition of \( T_1 \).

Next, we prove the result

\[
\rho'(\xi) < 0 \quad \text{for every} \quad \xi \in (T_1, T_2).
\]

If not, there must be one point \( T_1 < \xi_0 < T_2 \) such that \( \rho'(\xi_0) \geq 0 \). We suppose \( \rho(\xi_0) \in (\frac{2a - \sqrt{\alpha(4\alpha - 3)}}{3\alpha}, \frac{2a + \sqrt{\alpha(4\alpha - 3)}}{3\alpha}) \). If \( \rho'(\xi_0) = 0 \), we have

\[
\frac{d}{d \xi}(D(\rho)\rho')|_{\xi = \xi_0} = -c\rho'(\xi_0) - g(\rho(\xi_0)) < -c\rho'(\xi_0) \leq 0,
\]

hence \( D(\rho(\xi))\rho'(\xi) \) is decreasing in a neighborhood of \( \xi_0 \). Since \( D(\rho(\xi_0)) < 0 \), we get \( \rho'(\xi) > 0 \) in \( (\xi_0, \xi_0 + \delta) \) for some \( \delta > 0 \). Let

\[
\eta = \sup\{t : \rho'(\xi) > 0 \quad \text{for every} \quad \xi \in (\xi_0, t)\}.
\]

We can see that \( \eta < +\infty, \rho(\eta) \geq \frac{2a - \sqrt{\alpha(4\alpha - 3)}}{3\alpha} \) and \( \eta \) is a local maximum point. If \( \rho(\eta) \in (\frac{2a - \sqrt{\alpha(4\alpha - 3)}}{3\alpha}, \frac{2a + \sqrt{\alpha(4\alpha - 3)}}{3\alpha}) \), by using the equation (12), we obtain

\[
\rho''(\eta) = -\frac{g(\rho(\eta))}{D(\rho(\eta))] > 0,
\]

which is a contradiction.

In the case \( \rho(\eta) \geq \frac{2a + \sqrt{\alpha(4\alpha - 3)}}{3\alpha} \), there exists \( T_1 < \eta_0 < \eta \) such that \( \rho(\eta_0) = \frac{2a + \sqrt{\alpha(4\alpha - 3)}}{3\alpha} \) and

\[
0 < \int_{T_1}^{\eta_0} (D(\rho)\rho')'ds + \int_{T_1}^{\eta_0} g(\rho)ds = -\int_{T_1}^{\eta_0} \rho'(s)ds = 0,
\]

which is a contradiction.

Now let \( \rho(\xi_0) \in [0, 1] \setminus (\frac{2a - \sqrt{\alpha(4\alpha - 3)}}{3\alpha}, \frac{2a + \sqrt{\alpha(4\alpha - 3)}}{3\alpha}) \) and \( \rho'(\xi_0) = 0 \). We only prove for \( \rho(\xi_0) \in (\frac{2a + \sqrt{\alpha(4\alpha - 3)}}{3\alpha}, 1] \) and the case for \( \rho(\xi_0) \in [0, \frac{2a - \sqrt{\alpha(4\alpha - 3)}}{3\alpha}) \) is analogical. By using the equation (12), we have

\[
0 < D(\rho(\xi_0))\rho'(\xi_0) - D(\rho(T_1))\rho'(T_1) + c(\rho(\xi_0) - \rho(T_1)) + \int_{T_1}^{\xi_0} g(\rho(t))dt = 0.
\]

This is a contradiction.

Let \( \rho'(\xi_0) > 0 \). Because \( \rho'(T_1^+) < 0 \), there must be the minimum point \( T_0 \) such that \( \rho'(T_0) = 0 \) except for \( \rho(T_0) = \frac{2a - \sqrt{\alpha(4\alpha - 3)}}{3\alpha} \). We already proved above
\( \rho'(T_0) = 0 \) is not possible for \( T_1 < T_0 < T_2 \). But if \( \rho(T_0) = \frac{2\alpha - 4\alpha^3}{3\delta} \), by using the equation \([12]\), we get
\[
0 < \int_{T_0}^{T_2} (D(\rho)\rho')' ds + \int_{T_0}^{T_2} g(\rho)ds = -\int_{T_0}^{T_2} \rho'(s)ds = 0,
\]
which is a contradiction.

**Step 4.** We claim that
\[
\lim_{t \to T_1^-} D(\rho(t))\rho'(t) = \lim_{t \to T_2^-} D(\rho(t))\rho'(t) = 0.
\]
The ideas to prove \( \lim_{t \to T_1^-} D(\rho(t))\rho'(t) = 0 \) and \( \lim_{t \to T_2^-} D(\rho(t))\rho'(t) = 0 \) are analogous, and we only prove \( \lim_{t \to T_1^-} D(\rho(t))\rho'(t) = 0 \) in the following.

We first prove
\[
\lim_{t \to T_1^-} D(\rho(t))\rho'(t) = 0,
\]
and the idea for \( \lim_{t \to T_2^-} D(\rho(t))\rho'(t) = 0 \) will be analogous, the proof will be omitted here. For any small \( \delta > 0 \) such that \((T_1 - \delta, T_1) \subset (a^*, T_1), D(\rho(t)) > 0, \rho'(t) < 0, \)
which leads to \( D(\rho(t))\rho'(t) < 0 \) on \((T_1 - \delta, T_1)\). Hence \( \lim_{t \to T_1^-} D(\rho(t))\rho'(t) = L \leq 0 \).

If \( L < 0 \), from the equation \([12]\), we have
\[
\lim_{t \to T_1^-} D(\rho(t))\rho'(t) \leq \lim_{t \to T_1^-} D(\rho(t))\rho'(t) < 0.
\]
This is a contradiction to \( D(\rho(t))\rho'(t) > 0 \), for \( T_2 > t > T_1 \).

From the above four steps and Definition \([2, 2]\) we can conclude that \( \rho'(\xi) < 0 \)
for every \( t \in (a^*, \infty) \), with the exception, at most, of \( \xi = T_1, T_2 \). This is the main reason we introduce the weak traveling wave solution.

Now we let \( \rho(\xi) \) be the solution of the equation \([12]\) and let \( \xi(\rho) : (0, 1) \to (a^*, \infty) \) be the inverse function, whose existence is ensured by the monotonicity of \( \rho(\xi) \). Set \( z(\rho) : D(\rho)\rho'(\xi(\rho)) \) for \( \rho \in (0, \rho'(\alpha)) \cup (\rho'(\alpha), \rho'(\alpha)) \cup (\rho'(\alpha), 1) \). We get the following:
\[
\dot{z}(\rho) = \frac{dz}{d\rho} = (D(\rho)\rho')' = \frac{1}{\rho'(\xi)} = (D(\rho)\rho')' \frac{D(\rho)}{z(\rho)} = -c - \frac{g(\rho)D(\rho)}{z(\rho)}.
\]
In the following section we will discuss the relations between equations \([12]\) and \([15]\).

3. Main results of traveling wave solutions. The first part of this section will
deal with the case \( 0 \leq \alpha < \frac{3}{4} \), then we will use these results in the second part to
prove the existence and nonexistence of the traveling wave solution when \( \frac{3}{4} \leq \alpha \leq 1 \).

3.1. Results for \( 0 \leq \alpha < \frac{3}{4} \). In this case, we obtain that the diffusivity coefficient \( D(\rho) > 1 - \frac{4}{3}\alpha > 0 \). Engler \([7]\) and Hadeler \([17]\) obtained the following result.

**Theorem 3.1.** Let \( D, g \in C^1([0, 1]), 0 < D_0 < D(s) \) for all \( s \in [0, 1], g(s) \) satisfies \([3]\), \( g'(0) > 0 \) and \( c \) is a real constant. Then the equation \([15]\) is solvable with \( z(0^+) = z(1^-) = 0 \) if and only if \( c > c^* \) where
\[
2\sqrt{D(0)g'(0)} \leq c^* \leq 2\sqrt{\frac{\sup_{s \in (0, 1]} D(s)g(s)}{s}}.
\]

In \([21]\) the result is extended to the possible degenerate case.
Theorem 3.2. (Theorem 2 [21]) Let $g(s) \in C[0, 1], D(s) \in C^1[0, 1]$, respectively satisfying [3] and $D(s) > 0$ for all $s \in (0, 1)$ and assume $D(0)g'(0) < \infty$. Then there exists $c^* > 0$ satisfying

$$2\sqrt{D(0)g'(0)} \leq c^* \leq 2\sup_{s \in (0, 1)} \frac{D(s)g(s)}{s}$$

such that the boundary value problem:

$$\begin{cases}
\dot{z}(\rho) = -c - \frac{g(\rho)D(\rho)}{z(\rho)}, & \rho \in (0, 1) \\
z(\rho) < 0 \\
z(0^+) = 0, & z(1^-) = 0
\end{cases}$$

(16)

is solvable if and only if $c \geq c^*$. Moreover, for every $c \geq c^*$, the solution is unique.

Theorem 3.3. (Theorem 3 [21]) Let $g(s) \in C[0, 1], D(s) \in C^1[0, 1]$, respectively satisfying [3] and $D(s) > 0$ for all $s \in (0, 1)$. The existence of a traveling wave solution $\rho(t)$ of the equation [11], with wave speed $c$, satisfying boundary conditions [10] and [11], is equivalent to the solvability of the equation (16), with the same $c$.

3.2. Results for $\frac{3}{4} \leq \alpha \leq 1$. In the model when $\frac{3}{4} \leq \alpha \leq 1$, the diffusivity coefficient will be negative which will make the proof of the existence of the traveling wave complicated, especially at the point $\frac{2\alpha + \alpha(4\alpha - 3)}{3\alpha}$. The derivative of $z(\rho)$ may have possibly two different values which will lead to two possible derivative values of $\rho$ at $T_1$ ($\rho'(T_1) = \frac{z'(\rho)}{D'(\rho)}|_{\rho = 2\alpha + \alpha(4\alpha - 3)}$ by L'Hôpital’s rule). To overcome this difficulty, we will use the idea of weak traveling wave solution from [5] and consider the traveling waves in different intervals: $[0, \frac{2\alpha - \alpha(4\alpha - 3)}{3\alpha}], [\frac{2\alpha - \alpha(4\alpha - 3)}{3\alpha}, \frac{2\alpha + \alpha(4\alpha - 3)}{3\alpha}]$, and $[\frac{2\alpha + \alpha(4\alpha - 3)}{3\alpha}, 1]$. Under certain conditions, we can glue these traveling waves together which will give the existence of the weak traveling wave solution in the whole interval.

Before proving Theorem 3.1, we introduce the following Lemma, which will be used in the proof of Theorem 3.1.

Lemma 3.4. Let $z \in C(0, 1)$ be the solution of the equation [15] for every $c > c^*$.

Under the assumption of Theorem 3.1, the following limits:

$$\lim_{\rho \to \rho^*(\alpha)} \frac{z(\rho)}{\rho - \rho^*(\alpha)} = \lambda_1, \quad \lim_{\rho \to \rho^*(\alpha)} \frac{z(\rho)}{\rho - \rho^*(\alpha)} = \lambda_2,$$

(17)

exists, where

$$\lambda_1 = \frac{1}{2} \left( -c + \sqrt{c^2 - 4D'(\rho^*(\alpha))g(\rho^*(\alpha))} \right),$$

$$\lambda_2 = \frac{1}{2} \left( -c + \sqrt{c^2 - 4D'(\rho^*(\alpha))g(\rho^*(\alpha))} \right).$$

Furthermore,

$$\rho'(T_1) = \lim_{\rho \to \rho^*(\alpha)} \frac{z(\rho)}{D(\rho)} = -\frac{2g(\rho^*(\alpha))}{c + \sqrt{c^2 - 4D'(\rho^*(\alpha))g(\rho^*(\alpha))}},$$

$$\rho'(T_2) = \lim_{\rho \to \rho^*(\alpha)} \frac{z(\rho)}{D(\rho)} = -\frac{2g(\rho^*(\alpha))}{c + \sqrt{c^2 - 4D'(\rho^*(\alpha))g(\rho^*(\alpha))}}.$$

If $c = c^*$, $\lambda_2$ may take two values $\frac{1}{2} \left( -c \pm \sqrt{c^2 - 4D'(\rho^*(\alpha))g(\rho^*(\alpha))} \right)$. 


Proof of Lemma 3.4. First we need prove the existence of \( z'(\rho) \) at \( \rho^*(\alpha) \) and \( \rho^*(\alpha) \). We only prove the result at \( \rho^*(\alpha) \), the result at point \( \rho^*(\alpha) \) is analogical.

Let \( z(\rho) \) be the solution of the equation (15) satisfying \( z(\rho^*(\alpha)^+) = z(1^-) = 0 \). First we prove the existence of the limit

\[
\lambda_2 := \lim_{\rho \to \rho^*(\alpha)^+} \frac{z(\rho)}{\rho - \rho^*(\alpha)}. \tag{18}
\]

Assume by contradiction that

\[
0 > L := \limsup_{\rho \to \rho^*(\alpha)^+} \frac{z(\rho)}{\rho - \rho^*(\alpha)} > \liminf_{\rho \to \rho^*(\alpha)^+} \frac{z(\rho)}{\rho - \rho^*(\alpha)} := l.
\]

Let \( \chi \in (l, L) \) and let \( (\rho_n)_n \) be an decreasing sequence converging to \( \rho^*(\alpha) \) such that

\[
\frac{z(\rho_n)}{\rho_n - \rho^*(\alpha)} = \chi, \quad \text{and} \quad \frac{d}{d\rho} \left( \frac{z(\rho)}{\rho - \rho^*(\alpha)} \right)_{\rho = \rho_n} \leq 0.
\]

Since

\[
\frac{d}{d\rho} \left( \frac{z(\rho)}{\rho - \rho^*(\alpha)} \right)_{\rho = \rho_n} = \frac{1}{\rho - \rho^*(\alpha)} \left( \frac{\dot{z}(\rho)}{\rho - \rho^*(\alpha)} \right)_{\rho = \rho_n} \leq 0,
\]

we have

\[
\dot{z}(\rho_n) = -c - \frac{D(\rho_n)g(\rho_n)}{\chi(\rho_n - \rho^*(\alpha))} \leq \chi.
\]

Passing to the limit as \( n \to +\infty \), since \( \chi < 0 \), we have \( \chi^2 + c\chi + [Dg]'(\rho^*(\alpha)) \leq 0 \). Similarly, we can choose an decreasing sequence \( (\nu_n)_n \) converging to \( \rho^*(\alpha) \), such that

\[
\frac{z(\nu_n)}{\nu_n - \rho^*(\alpha)} = \chi, \quad \text{and} \quad \frac{d}{d\rho} \left( \frac{z(\rho)}{\rho - \rho^*(\alpha)} \right)_{\rho = \nu_n} \geq 0.
\]

we can deduce \( \chi^2 + c\chi + [Dg]'(\rho^*(\alpha)) \geq 0 \). As \( \chi \in (l, L) \) is arbitrary, we conclude that

\[
l = L = \lambda_2 = \frac{1}{2} \left( -c + \sqrt{c^2 - 4D'(\rho^*(\alpha))g(\rho^*(\alpha))} \right).
\]

The same argument can be used to prove that

\[
\lim_{\rho \to \rho^*(\alpha)^-} \frac{z(\rho)}{\rho - \rho^*(\alpha)}
\]

exists and its value is either

\[
\frac{1}{2} \left( -c + \sqrt{c^2 - 4D'(\rho^*(\alpha))g(\rho^*(\alpha))} \right) \quad \text{or} \quad -\frac{1}{2} \left( -c + \sqrt{c^2 - 4D'(\rho^*(\alpha))g(\rho^*(\alpha))} \right).
\]

Given \( c > c^* \), let \( z(\rho) \) and \( z^*(\rho) \) be the solutions of the equation (15) on \([\rho^*(\alpha), 1]\) with \( z(\rho^*(\alpha)) = z(1) = 0 \), respectively, for \( c \) and \( c^* \). Assuming the existence of \( \tilde{\rho} \in [\rho^*(\alpha), 1] \) satisfying \( z^*(\tilde{\rho}) \geq z(\tilde{\rho}) \), from (15) we then have

\[
\dot{z}^*(\tilde{\rho}) = -c^* - \frac{D(\tilde{\rho})g(\tilde{\rho})}{z^*(\tilde{\rho})} > -c - \frac{D(\tilde{\rho})g(\tilde{\rho})}{z(\tilde{\rho})} = \dot{z}(\tilde{\rho}).
\]

This implies the contradictory conclusion \( 0 = z^*(1^-) > z(1^-) = 0 \). Hence \( z^*(\rho) < z(\rho) \) for all \( \rho \in [\rho^*(\alpha), 1] \). With \( \dot{z}^*(\rho^*(\alpha)^+ \leq \dot{z}(\rho^*(\alpha)^+) \), and

\[
\dot{z}^*(\rho^*(\alpha)^+) = \frac{1}{2} \left( \pm \sqrt{(c^*)^2 - 4D'(\rho^*(\alpha)^+)g(\rho^*(\alpha))} - c^* \right).
\]

we obtain \( \lambda_2 = \frac{1}{2} \left( \sqrt{c^2 - 4D'(\rho^*(\alpha)^+)g(\rho^*(\alpha))} - c \right) \). However when \( c = c^* \), \( z^*(\rho^*(\alpha)^+) \) have two possible values.
The idea in proving the form $\lambda_1$ is analogous, however

$$\liminf_{\rho \to \rho^*(\alpha)^-} \frac{z(\rho)}{\rho - \rho^*(\alpha)} := l \geq 0,$$

which will lead to the positivity of $\lambda_1 = \frac{1}{2}(-c + \sqrt{c^2 - 4D'(\rho^*(\alpha))g(\rho^*(\alpha))}).$

When $3/4 < \alpha \leq 1$, $D'(\rho^*(\alpha)) \neq 0$, we obtain

$$\rho'(T_1) = \lim_{\rho \to \rho^*(\alpha)} \frac{z(\rho)}{D(\rho)} = \lim_{\rho \to \rho^*(\alpha)} \frac{z(\rho)}{\rho - \rho^*(\alpha)} \cdot \frac{\rho - \rho^*(\alpha)}{D(\rho)}$$

$$= \frac{1}{2}(\sqrt{c^2 - 4D'(\rho^*(\alpha))g(\rho^*(\alpha)) - c} \cdot \frac{1}{D'(\rho^*(\alpha))}$$

$$= \frac{-4D'(\rho^*(\alpha))g(\rho^*(\alpha))}{2(\sqrt{c^2 - 4D'(\rho^*(\alpha))g(\rho^*(\alpha)) + c}) \cdot \frac{1}{D'(\rho^*(\alpha))}$$

$$= \frac{-2g(\rho^*(\alpha))}{(\sqrt{c^2 - 4D'(\rho^*(\alpha))g(\rho^*(\alpha)) + c}) < 0.$$

Using the similar idea, we also obtain

$$\rho'(T_2) = \lim_{\rho \to \rho^*(\alpha)^+} \frac{z(\rho)}{D(\rho)} = -\frac{2g(\rho^*(\alpha))}{c + \sqrt{c^2 - 4D'(\rho^*(\alpha))g(\rho^*(\alpha))}}.$$

If $\alpha = 3/4$, $\rho^*(\alpha) = \rho^*(\alpha) = 2/3, D(\rho^*(\alpha)) = 0$, and $D'(\rho^*(\alpha)) = 0$. It is natural from continuity that we guess

$$\lim_{\rho \to \rho^*(\alpha)^+} \frac{z(\rho)}{D(\rho)} = -\frac{g(\rho^*(\alpha))}{c}.$$

Next, we establish this argument rigorously. If $D'(\rho^*(\alpha)^+) = 0$, we can get $\dot{z}(\rho^*(\alpha)) (\dot{z}(\rho^*(\alpha)) + c) = 0$. Hence $\dot{z}(\rho^*(\alpha)) = 0$ or $\dot{z}(\rho^*(\alpha)) = -c$. Because $\dot{z}(\rho^*(\alpha)^+ - \dot{z}(\rho^*(\alpha)) = 0$.

By the mean value theorem, there exists a sequence $\{\nu_n\}_n$ such that $\nu_n \to \rho^*(\alpha)^+$ and $\dot{z}(\nu_n) \to 0$ as $n \to \infty$. From the equation (15), we can get

$$\frac{z(\nu_n)}{D(\nu_n)} \to \frac{-\frac{1}{2}g(\rho^*(\alpha))}{c}, \text{ as } n \to \infty. \quad (19)$$

Next, we show that $\lim_{\rho \to \rho^*(\alpha)^+} \frac{z(\rho)}{D(\rho)} = -\frac{g(\rho^*(\alpha))}{c}$. Fix $\epsilon \in (0, \frac{g(\rho^*(\alpha))}{c})$. Since $D(\rho)$ is positive in the right of $\rho^*(\alpha)$, from (15) there exists $n_0$ such that

$$z(\nu_n) \leq (-\frac{g(\rho^*(\alpha))}{c} + \epsilon)D(\nu_n), \text{ n } \geq n_0. \quad (20)$$

Let $\xi_c(\rho) := (-\frac{g(\rho^*(\alpha))}{c} + \epsilon)D(\rho)$. Since $\dot{\xi}(\rho) \to 0$ as $\rho \to \rho^*(\alpha)^+$ and

$$-c + \frac{g(\rho)}{g(\rho^*(\alpha)) - c} \cdot \frac{\rho - \rho^*(\alpha)}{g(\rho^*(\alpha)) - c} > 0, \quad \text{ as } \rho \to \rho^*(\alpha)^+,$$

it is possible to find a $\delta > 0$ such that $-c + \frac{g(\rho)}{g(\rho^*(\alpha)) - c} > \xi_c(\rho)$ for each $\rho \in (\rho^*(\alpha), \rho^*(\alpha) + \delta)$. Let $n_1 \geq n_0$ such that $\nu_{n_1} \in (\rho^*(\alpha), \rho^*(\alpha) + \delta)$. We claim that

$$z(\rho) \leq \xi_c(\rho) \text{ for all } \rho \in (\rho^*(\alpha), \nu_{n_1}).$$

That is,

$$z(\rho) \leq (-\frac{g(\rho^*(\alpha))}{c} + \epsilon)D(\rho), \text{ for } \rho \in (\rho^*(\alpha), \nu_{n_1}). \quad (21)$$
Suppose by contradiction that there exists \( \rho_0 \in (\rho^\alpha(\alpha), \nu_{n_1}) \) such that \( z(\rho_0) > \xi_\alpha(n_0) \). Since \( n_1 \geq n_0 \), from (21) it follows that \( z(\nu_{n_1}) \leq \xi_\alpha(n_{n_1}) \). The mean value theorem concludes that there exists \( \rho_1 \in (\rho_0, \nu_{n_1}) \), such that \( z(\rho_1) > \xi_\alpha(\rho_1) \) and \( \dot{z}(\rho_1) - \xi_\alpha(\rho_1) < 0 \). On the other hand, we have

\[
\dot{z}(\rho_1) = -c + \frac{D(p_1)g(p_1)}{-z(\rho_1)} > -c + \frac{D(p_1)g(p_1)}{(\frac{g(\rho_1)}{\rho_1} + c)D(\rho_1)} = -c + c\frac{g(\rho_1)}{g(\rho_1) - c}\cdot
\]

Since \( \rho_1 \in (\rho_0, \nu_{n_1}) \subset (\rho^\alpha(\alpha), \alpha + \delta) \), we get \( \dot{z}(\rho_1) > -c + c\frac{g(\rho_1)}{g(\rho_1) - c} > \xi_\alpha(\rho_1) \). This is a contradiction.

Similarly, from (15) we can choose \( \tilde{n}_0 \) such that

\[
z(\nu_n) \geq \eta_\nu(\nu_n) := (-\frac{g(\rho^\alpha(\alpha))}{c} - c)D(\nu_n), \quad \text{for } n \geq \tilde{n}_0.
\] (22)

Again, \( \eta_\nu(\rho) \to 0 \) as \( \rho \to \rho^\alpha(\alpha)^+ \) and

\[
-c + c\frac{g(\rho)}{g(\rho^\alpha(\alpha)) + c} \to -c + c\frac{g(\rho^\alpha(\alpha))}{g(\rho^\alpha(\alpha)) + c} < 0, \quad \text{as } \rho \to \rho^\alpha(\alpha)^+.
\]

there exists \( \bar{\delta} > 0 \) such that \( -c + c\frac{g(\rho)}{g(\rho^\alpha(\alpha)) + c} < \eta_\nu(\rho) \) for each \( \rho \in (\rho^\alpha(\alpha), \rho^\alpha(\alpha) + \bar{\delta}) \).

Let \( \tilde{n}_1 \geq \tilde{n}_0 \) such that \( \nu_{n_1} \in (\rho^\alpha(\alpha), \rho^\alpha(\alpha) + \bar{\delta}) \). We prove by contradiction that \( z(\rho) \geq \eta_\nu(\rho) \) for each \( \rho \in (\rho^\alpha(\alpha), \nu_{n_1}) \), that is

\[
z(\rho) \geq (-\frac{g(\rho^\alpha(\alpha))}{c} - c)D(\rho), \quad \text{for } \rho \in (\rho^\alpha(\alpha), \nu_{n_1}).
\] (23)

Suppose there exists \( \tilde{\rho}_0 \in (\rho^\alpha(\alpha), \nu_{n_1}) \) such that \( z(\tilde{\rho}_0) < \eta_\nu(\tilde{\rho}_0) \). Since \( \tilde{n}_1 \geq \tilde{n}_0 \), from (23) it follows that \( z(\nu_{n_1}) \geq \eta_\nu(\nu_{n_1}) \). Therefore, there exists \( \tilde{\rho}_1 \in (\tilde{\rho}_0, \nu_{n_1}) \), such that \( z(\tilde{\rho}_1) < \eta_\nu(\tilde{\rho}_1) \) and \( \dot{z}(\rho_1) - \xi_\alpha(\rho_1) > 0 \). On the other hand we have

\[
\dot{z}(\tilde{\rho}_1) = -c + \frac{D(\tilde{\rho}_1)g(\tilde{\rho}_1)}{-z(\tilde{\rho}_1)} < -c + \frac{D(\tilde{\rho}_1)g(\tilde{\rho}_1)}{(\frac{g(\rho_1)}{\rho_1} + c)D(\rho_1)} = -c + c\frac{g(\tilde{\rho}_1)}{g(\rho_1) - c}.
\]

Since \( \tilde{\rho}_1 \in (\tilde{\rho}_0, \nu_{n_1}) \subset (\rho^\alpha(\alpha), \rho^\alpha(\alpha) + \bar{\delta}) \), we get \( \dot{z}(\tilde{\rho}_1) < -c + c\frac{g(\tilde{\rho}_1)}{g(\rho^\alpha(\alpha)) + c} < \eta_\nu(\tilde{\rho}_1) \), hence a contradiction holds. By the inequalities (21) and (23), we proved that

\[
\lim_{\rho \to \rho^\alpha(\alpha)^+} z(\rho) = \frac{g(\rho^\alpha(\alpha))}{c}D(\rho) = -\frac{g(\rho^\alpha(\alpha))}{c}.
\]

The same argument can be used to prove that \( \lim_{\rho \to \rho^\alpha(\alpha)^-} \frac{z(\rho)}{D(\rho)} \) exists, and its value is \( \frac{1}{2}(\sqrt{c^2 - 4D(p^\alpha(\alpha)^-)}g(\rho^\alpha(\alpha)) - c) \). Also,

\[
\rho'(T_1^+) = \lim_{\rho \to \rho^\alpha(\alpha)^+} z(\rho) = \frac{g(\rho^\alpha(\alpha))}{c + \sqrt{c^2 - 4D(p^\alpha(\alpha)^-)g(\rho^\alpha(\alpha))}}.
\]

\[
\boxed{}
\]

**Proof of Theorem 1.1** The strategy is to construct traveling wave solution for \( a^* < \xi < T_1, T_1 < \xi < T_2 \) and \( T_2 < \xi < +\infty \) such that

- \( \rho(\xi) \in [0, \rho^\alpha(\alpha)) \) for \( t \in (T_2, +\infty) \),
- \( \rho(t) \in [\rho^\alpha(\alpha), \rho^\alpha(\alpha)] \) for \( t \in (T_1, T_2) \),
- \( \rho(t) \in [\rho^\alpha(\alpha), 1] \) for \( t \in (a^*, T_1) \),
- \( \lim_{t \to T_1^+} \rho(t) = \lim_{t \to T_2^-} \rho(t) = \rho^\alpha(\alpha), \lim_{t \to T_1^+} \rho(t) = \lim_{t \to T_2^-} \rho(t) = \rho^\alpha(\alpha), \rho(t) = 0 \),
- \( \lim_{t \to T_1^+} D(\rho(t)) \rho'(t) = \lim_{t \to T_2^-} D(\rho(t)) \rho'(t) = 0 \),
\[ \lim_{t \to T_2^-} D(\rho(t))\rho'(t) = \lim_{t \to T_2^-} D(\rho(t))\rho'(t) = 0. \]

- \[ D(\rho)\rho'(t) + c\rho(t) + \int_t^\infty g(\rho(s))ds = c. \]

If we can find such solutions, then the combined solutions will give us a weak solution.

1) When \( 0 < \rho < \rho^*(\alpha) \), let us consider the equation (15) for \( 0 < \rho < \rho^*(\alpha) \). We have \( D(0)g(0) = D(\rho^*(\alpha))g(\rho^*(\alpha)) = 0, \ z(0) = z(\rho^*(\alpha)) = 0 \). We make a change of variable \( \tilde{z}(s) = z(s/\rho^*(\alpha)) \). From here we can deduce the existence of a threshold value \( c_1^* > 0 \), satisfying the estimate

\[ 0 \leq c_1^* \leq 2\sqrt{\sup_{s \in (0, \rho^*(\alpha)]} D(s)g(s)/s}, \]

such that the equation (15) has a unique negative solution \( z(\rho) \) in \( (0, \rho^*(\alpha)) \) with \( z(0^+) = z(\rho^*(\alpha)^-) = 0 \), if and only if \( c \geq c_1^* \). Consider the following Cauchy problem:

\[
\begin{cases}
\rho' = \frac{z(\rho)}{D(\rho)}, & 0 < \rho < \rho^*(\alpha), \\
\rho(0) = \frac{\rho^*(\alpha)}{2}.
\end{cases}
\tag{24}
\]

We obtain that the unique solution \( \rho(t) \) of problem (24) is a solution of (7) in \((\tau_1, \tau_2)\), with \( \tau_1 \geq -\infty, \tau_2 \in \mathbb{R}, \rho(\tau_1^-) = \rho^*(\alpha), \rho(\tau_2^+) = 0 \). Moreover, when \( D(0) \neq 0 \), we can have the inverse function \( \xi(\rho) \) in \([0, \xi(\rho^*)] \) and \( \xi(\rho^*(\alpha)^-) < +\infty \).

Let us consider the equation (15) for \( \rho^*(\alpha) < \rho < \rho^*(\alpha) \). We make the following change of variable:

\[ \tilde{D}(\rho) := -D(\rho^*(\alpha) + \rho^*(\alpha) - \rho), \quad \tilde{g}(\rho) := g(\rho^*(\alpha) + \rho^*(\alpha) - \rho). \]

Since \( \tilde{D}(\rho)\tilde{g}(\rho) > 0 \) for every \( \rho^*(\alpha) < \rho < \rho^*(\alpha) \),

\[ \tilde{D}(\rho^*(\alpha))\tilde{g}(\rho^*(\alpha)) = \tilde{D}(\rho^*(\alpha))\tilde{g}(\rho^*(\alpha)) = 0, \]

\( \tilde{z}(\rho) = -z[\rho^*(\alpha) + \rho^*(\alpha) - \rho] \leq 0 \). According to Theorem 3.2 when considering the equation \( \tilde{z}(\rho) \) on the interval \([\rho^*(\alpha), \rho^*(\alpha)]\) with \( \tilde{z}(\rho^*(\alpha)) = \tilde{z}(\rho^*(\alpha)) = 0 \), and using a linear transformation similar to the case \( 0 < \rho < \rho^*(\alpha) \), we derive the existence of a threshold \( c_2^* > 0 \) satisfying

\[ 2\sqrt{\tilde{D}(\rho^*(\alpha))\tilde{g}(\rho^*(\alpha))} < c_2^* \leq 2\sqrt{\sup_{s \in [\rho^*(\alpha), \rho^*(\alpha)]} \tilde{D}(s)\tilde{g}(s)/s}, \tag{25} \]

that is

\[ 2\sqrt{\tilde{D}(\rho^*(\alpha))\tilde{g}(\rho^*(\alpha))} < c_2^* \leq 2\sqrt{\sup_{s \in [\rho^*(\alpha), \rho^*(\alpha)]} \frac{\tilde{D}(s)\tilde{g}(s)}{s - \rho^*(\alpha)}}, \tag{26} \]

such that the equation:

\[ \dot{\omega} := -c - \frac{\tilde{D}(\rho)\tilde{g}(\rho)}{\omega}, \quad \rho^*(\alpha) < \rho < \rho^*(\alpha), \tag{27} \]

admits a negative solutions \( \omega(\rho) \), satisfying \( \omega(\rho^*(\alpha)^+) = \omega(\rho^*(\alpha)^-) = 0 \), if and only if \( c > c_2^* \). Putting \( \tilde{z}(\rho) := -\omega(\rho^*(\alpha) + \rho^*(\alpha) - \rho) \), we have

\[ \dot{z}(\rho) = \omega(\rho^*(\alpha) + \rho^*(\alpha) - \rho) = -c - \frac{\tilde{D}(\rho^*(\alpha) + \rho^*(\alpha) - \rho)\tilde{g}(\rho^*(\alpha) + \rho^*(\alpha) - \rho)}{\omega(\rho^*(\alpha) + \rho^*(\alpha) - \rho)/z(\rho)}, \quad \rho^*(\alpha) < \rho < \rho^*(\alpha). \]
Moreover, \( z(\rho^+(\alpha)) = z(\rho^-(\alpha)) = 0 \), and \( z(\rho) > 0 \) for every \( \rho \in (\rho^+(\alpha), \rho^-(\alpha)) \).

We consider the following Cauchy problem:

\[
\begin{cases}
\rho' = \frac{z(\rho)}{D(\rho)}, & \rho^+(\alpha) < \rho < \rho^-(\alpha), \\
\rho(0) = \frac{\rho^+(\alpha) + \rho^-(\alpha)}{2},
\end{cases}
\]

and let \( \rho(t) \) be the unique solution of (28) defined in its maximal existence interval \((t_1, t_2)\), with \(-\infty < t_1 < t_2 < +\infty\). Then we obtain \( \rho(t) \) is a solution of (40) in \((t_1, t_2)\).

When \( \rho^+(\alpha) < \rho < 1 \), using the same method as we prove for the situation \( 0 < \rho < \rho^+(\alpha) \), we can deduce the existence of a threshold value \( c^*_3 > 0 \), satisfying the estimate

\[
\sqrt{D'(\rho^+(\alpha))g(\rho^+(\alpha))} < c^*_3 \leq 2 \sqrt{\sup_{s \in (\rho^+(\alpha), 1)} \frac{D(s)g(s)}{s}},
\]

such that the equation (15) has a unique negative solution \( z(\rho) \) in \((\rho^+(\alpha), 1)\) with \( z(\rho^+(\alpha)) = z(1^-) = 0 \), if and only if \( c \geq c^*_3 \). We just consider the Cauchy problem:

\[
\begin{cases}
\rho' = \frac{z(\rho)}{D(\rho)}, & \rho^+(\alpha) < \rho < 1, \\
\rho(0) = \frac{\rho^+(\alpha) + 1}{2},
\end{cases}
\]

and repeat the same arguments developed for the case when \( 0 < \rho < \rho^+(\alpha) \), then obtain that the unique solution \( \rho(t) \) of Problem (15) is a solution of (40) in \((\tau_3, \tau_4)\), with \( \tau_3 \geq -\infty, \tau_4 \in \mathbb{R}, \rho(\tau_4^-) = \rho^+(\alpha), \rho(\tau_4^+) = 1 \). Moreover, when \( D(1) = 0 \), we have that the inverse function \( \xi(\rho) \) in \([\rho^+(\alpha), 1] \) and \( \xi(1^-) \leq +\infty \).

If we put \( c^* := \max\{c_1^*, c_2^*, c^*_3\} \), and glue the solutions of (24), (28) and (29) by a time-shift, we obtain a continuous function \( \rho(t) \) on some interval \((a^*, \infty)\), which is a decreasing function in \((a^*, \infty)\) satisfying

\[
\rho((a^*)^+) = 1, \quad \rho(+\infty) = 0, \quad \lim_{t \to T_1} \rho(t) = \rho^+(\alpha), \quad \lim_{t \to T_2} \rho(t) = \rho^-(\alpha).
\]

From the construction of \( \rho(t) \), we also have

\[
\lim_{t \to T_1} D(\rho(t))\rho'(t) = \lim_{t \to T_2} D(\rho(t))\rho'(t) = 0.
\]

Note that \( \rho(t) \) is smooth on \((a^*, T_1)\), we have,

\[
D(\rho(t))\rho'(t) + c\rho(t) + \int_{a^*}^t g(\rho(s))ds = \text{constant}, \quad t \in (a^*, T_1].
\]

Taking the limit \( t \to a^* \), we see that the constant must be \( c \).

Similarly

\[
D(\rho(t))\rho'(t) + c\rho(t) + \int_{T_1}^{t} g(\rho(s))ds = 0, \quad t \in [T_1, +\infty).
\]

Take \( t = T_1 \) in both (30) and (31), we see that

\[
\int_{a^*}^{+\infty} g(\rho(s))ds = c.
\]

Furthermore for \( T_1 < t < +\infty \), the equation (30) also holds. In fact, using the equation (31), in this case,

\[
\int_{a^*}^{+\infty} g(\rho(s))ds = \int_{a^*}^{T_1} g(\rho(s))ds + \int_{T_1}^{t} g(\rho(s))ds + \int_{t}^{+\infty} g(\rho(s))ds.
\]
We have

\[ D(\rho)\rho'(t) + c\rho(t) + \int_{a^*}^{\xi} g(\rho(s))ds \]

\[ = D(\rho)\rho'(t) + c\rho(t) - \int_{t}^{+\infty} g(\rho(s))ds + \int_{a^*}^{+\infty} g(\rho(s))ds \]

\[ = 0 + \int_{a^*}^{+\infty} g(\rho(s))ds = \rho. \]

II) Non-existence for \( c < c^* \). We proved in the Proposition that \( \rho'(t) < 0 \) for \( \rho \in (0,1) \) and this implies the existence of the inverse function \( \xi(\rho) \) in \( (0,1) \) especially in \((0,\rho^*(\alpha))\) where \( D(\rho) > 0 \), then we define \( \omega(\rho) := D(\rho)\rho'(\xi(\rho)) \), it can be checked the negative solution of (15) satisfies \( \omega(\rho^*(\alpha)^-) := \omega(0^+) = 0 \). Therefore, by applying Theorem 3.2, we can deduce that \( c \geq c_1^* \).

When \( \rho^*(\alpha) < \rho < \rho^*(\alpha) \) and \( \rho^*(\alpha) < \rho < 1 \), we can also use Theorem 3.2 again to get \( c \geq c_2^* \) and \( c \geq c_3^* \).

Summarizing, \( c \geq c^* \) is a necessary condition for the existence of weak traveling wave solution of the equation (1).

\[ \square \]

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REFERENCES

[1] K. Anguige, Multi-phase Stefan problems for a non-linear one-dimensional model of cell-to-cell adhesion and diffusion, European J. Appl. Math., 21 (2010), 109–136.
[2] K. Anguige, A one-dimensional model for the interaction between cell-to-cell adhesion and chemotactic signalling, European J. Appl. Math., 4 (2011), 291–316.
[3] K. Anguige and C. Schmeiser, A one-dimensional model of cell diffusion and aggregation, incorporating volume filling and cell-to-cell adhesion, J. Math. Biol., 58 (2009), 395–427.
[4] N. Armstrong, K. Painter and J. Sherratt, A continuum approach to modelling cell-cell adhesion, J. Theor. Biol., 243 (2006), 98–113.
[5] L. Bao and Z. Zhou, Traveling wave in backward and forward parabolic equations from population dynamics Discrete Contin. Dyn. Syst. Ser. B, 19 (2014), 1507–1522.
[6] D. Dicarlo, R. Juanes, T. LaForce and T. Witelski, Nonmonotonic traveling wave solutions of infiltration into porous media, Water Resources Research, 44 (2008).
[7] H. Engler, Relations between travelling wave solutions of quasilinear parabolic equations Proc. Amer. Math. Soc., 93 (1985), 297–302.
[8] P. Feng and Z. Zhou, Finite traveling wave solutions in a degenerate cross-diffusion model for bacterial colony Commun. Pure Appl. Anal., 6 (2007), 1145–1165.
[9] Z. Feng and G. Chen, Traveling wave solutions in parametric forms for a diffusion model with a nonlinear rate of growth, Discrete Contin. Dyn. Syst., 24 (2009), 763–780.
[10] L. Ferracuti, C. Marcelli and F. Papalini, Travelling waves in some reaction-diffusion-aggregation models, Adv. Dyn. Syst. Appl., 4 (2009), 19–33.
[11] R. Fisher, The wave of advance of advantageous genes, Ann. Eugen 7 (1937), 353–369.
[12] F. S. Garduño and P. Maini, Existence and uniqueness of a sharp travelling wave in degenerate non-linear diffusion Fisher-KPP equations, J. Math. Biol., 33 (1994), 163–192.
[13] F. S. Garduño and P. Maini, Travelling wave phenomena in some degenerate reaction-diffusion equations J. Diff. Eqns., 117 (1995), 281–319.
[14] F. S. Garduño, P. Maini and J. Velázquez, A non-linear degenerate equation for direct aggregation and travelling wave dynamics Discrete Contin. Dyn. Syst. Ser. B, 13 (2010), 455–487.
[15] W. Gurney and R. Nisbet, The regulation of inhomogeneous population, J. Theor. Biol., 52 (1975), 441–457.
W. Gurney and R. Nisbet, A note on nonlinear population transport, J. Theor. Biol., 56 (1976), 249–251.

K. Hadeler, Travelling fronts and free boundary value problems in Numerical Treatment of Free Boundary Value Problems (eds. Albretch, J., Collatz, L., Hoffman, K. H.), Basel: Birkhauser, 58 (1982), 90–107.

D. Horstmann, K. Painter and H. Othmer, Aggregation under local reinforcement: From lattice to continuum European J. Appl. Math., 15 (2004), 546–576.

A. Kolmogorov, I. Petrovsky and I. Piskounov, Study of the diffusion equation with growth of the quantity of matter and its applications to a biological problem, Applicable mathematics of non-physical phenomena, (eds. Oliveira-Pinto, F., Conolly, B. W.) New York: Wiley, 1982.

M. Kuzmin and S. Ruggerini, Front Propagation in Diffusion-Aggregation Models with Bi-Stable Reaction Discrete Contin. Dyn. Syst. Ser. B, 16 (2011), 819–833.

P. Maini, L. Malaguti, C. Marcelli and S. Matucci, Diffusion-aggregation processes with monostable reaction terms Discrete Contin. Dyn. Syst. Ser. B, 6 (2006), 1175–1189.

L. Malaguti and C. Marcelli, Sharp profiles in degenerate and doubly degenerate Fisher-KPP equations J. Diff. Eqns., 195 (2003), 471–496.

J. Sherratt, On the form of smooth-front travelling waves in a diffusion equation with degenerate nonlinear diffusion Mathematical Modelling of Natural Phenomena, 5 (2010), 64–79.

V. Pandrón, Sobolev regularization of a nonlinear ill-posed parabolic problem as a model for aggregating populations Comm. Partial Diff. Eqns., 23 (1998), 457–486.

N. Shigesada, K. Kawasaki and E. Teramoto, Spatial segregation of interacting species J. Math. Biol., 79 (1979), 83–99.

J. Skellam, Random dispersal in theoretical populations Biometrika, 38 (1951), 196–218.

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E-mail address: lzbao@jlu.edu.cn
E-mail address: zfzhou@msu.edu