BTZ black hole from Poisson-Lie T-dualizable sigma models with spectators

A. Eghbali, L. Mehran-nia and A. Rezaei-Aghdam

Department of Physics, Faculty of Basic Sciences, Azarbaijan Shahid Madani University, 53714-161, Tabriz, Iran

Abstract

The non-Abelian T-dualization of the BTZ black hole is discussed in detail by using the Poisson-Lie T-duality in the presence of spectators. We explicitly construct a dual pair of sigma models related by Poisson-Lie symmetry. The original model is built on a $2 + 1$-dimensional manifold $\mathcal{M} \cong O \times G$, where $G$ as a two-dimensional real non-Abelian Lie group acts freely on $\mathcal{M}$, while $O$ is the orbit of $G$ in $\mathcal{M}$. The findings of our study show that the original model indeed is canonically equivalent to the $SL(2,\mathbb{R})$ Wess-Zumino-Witten (WZW) model for a given value of the background parameters. Moreover, by a convenient coordinate transformation we show that this model describes a string propagating in a spacetime with the BTZ black hole metric in such a way that a new family of the solutions to low energy string theory with the BTZ black hole vacuum metric, constant dilaton field and a new torsion potential is found. The dual model is built on a $2 + 1$-dimensional target manifold $\tilde{\mathcal{M}}$ with two-dimensional real Abelian Lie group $\tilde{G}$ acting freely on it. We further show that the dual model yields a three-dimensional charged black string for which the mass $M$ and axion charge $Q$ per unit length are calculated. After that, the structure and asymptotic nature of the dual space-time including the horizon and singularity are determined.

Keywords: BTZ black hole, Sigma model, String duality, Poisson-Lie symmetry, Charged black string
1 Introduction

2+1-dimensional black hole solution with a negative cosmological constant, mass, angular momentum and charge was, first, found by Banados, Teitelboim and Zanelli (BTZ) [1]. The study of 2+1-dimensional solutions have received a lot of attention, since the near horizon geometry of these solutions serves as a worthwhile model to investigate some conceptual questions of AdS/CFT correspondence [2], [3]. The BTZ black hole is asymptotically anti-de Sitter rather than asymptotically flat, and has no curvature singularity at the origin. A slight modification of this black hole solution yields an exact solution to string theory [4].

One of the most interesting properties of string theories, or more specially, two-dimensional non-linear sigma models is that a certain class of these models admits the action of duality transformations which change the different spacetime geometries but leave unchanged the physics of such theories at the classical level. Witten had shown [5] that two-dimensional black hole could be obtained by gauging a one-dimensional subgroup of $SL(2,\mathbb{R})$. Then, it was shown [6] that a simple extension of Witten’s construction yields a three-dimensional charged black string such that this solution is characterized by three parameters the mass, axion charge per unit length, and a constant $k$ [6]. After all, it was shown [4] that the BTZ black hole is, under the standard Abelian T-duality, equivalent to the charged black string solution discussed in Ref. [6]. In the present paper, we obtain this result by making use of the approach of the Poisson-Lie T-duality with spectators [7], [8], [9]. The Poisson-Lie T-duality is a generalization of Abelian [10] and traditional non-Abelian dualities [11] that proposed by Klimcik and Severa [7]. It deals with sigma models based on two Lie groups which form a Drinfeld double and the duality transformation exchanges their roles. This duality is a canonical transformation and two sigma models related by Poisson-Lie duality are equivalent at the classical level [12]. A generalization of the Poisson-Lie T-duality transformations from manifolds to supermanifolds has been also carried out in Ref. [13] (see, also, [14]). One of the most interesting applications of the Poisson-Lie T-duality transformations is to the WZW models [15], [16]. Up to now, only few examples of the Poisson-Lie symmetric sigma models have been treated at the quantum level [15], [17]. Furthermore, the Poisson-Lie symmetry in the WZW models based on the Lie supergroups have recently studied in Refs. [18] and [19]. We also refer the reader to the literatures [20]. In [15] it has been shown that the duality relates the $SL(2,\mathbb{R})$ WZW model to a constrained sigma model defined on the $SL(2,\mathbb{R})$ group space. Here in this paper we obtain a new non-Abelian T-dual background for the $SL(2,\mathbb{R})$ WZW model so that it is constructed out on a 2 + 1-dimensional manifold. Moreover, by using the Poisson-Lie T-duality with spectators we find a new family of the solutions to low energy string theory with the BTZ black hole vacuum metric, constant dilaton field and a new torsion potential. The findings of our study show that the dual model yields a three-dimensional charged black string. This solution is stationary and is characterized by three parameters: the mass $M$ and axion charge $Q$ per unit length, and a radius $l$ related to the derivative of the asymptotic value of the dilaton field. In this way, we show that the non-Abelian T-duality transformation (here as the Poisson-Lie T-duality on a semi-Abelian double) relates a solution with no horizon and no curvature singularity (the BTZ vacuum solution) to a solution with a single horizon and a curvature singularity (the charged black string).

To set up conventions and notations, as well as to make the paper self-contained, we review the Poisson-Lie T-dual sigma models construction in the presence of spectators in Section 2. In Section 3
we discuss in detail the Abelian T-dualization of the BTZ black hole solutions by using the approach of
the Poisson-Lie T-duality with spectators. Our main result is derived in Section 4: we explicitly
construct a dual pair of sigma models related by Poisson-Lie symmetry on 2 + 1-dimensional manifolds
\( M \) and \( \tilde{M} \) such that the group parts of \( M \) and \( \tilde{M} \) are two-dimensional real non-Abelian and Abelian
Lie groups, respectively. In subsection 4.1, by a convenient coordinate transformation we show that
the original model describes a string propagating in a spacetime with BTZ black hole vacuum metric.
Moreover, the canonical equivalence of the original model to the \( SL(2,\mathbb{R}) \) WZW model is discussed
in subsection 4.1. In subsection 4.2, first the dual model construction is given. Then it is shown
that the dual model yields a three-dimensional charged black string for which the mass \( M \) and axion
charge \( Q \) per unit length are calculated. Investigation of the structure and asymptotic nature of the
dual space-time including the horizon and singularity are discussed at the end of Section 4. Some
concluding remarks are given in the last Section.

2 A review of Poisson-Lie T-duality with spectators

We start this section by recalling the main features of the Poisson-Lie T-duality and introducing
redefinitions to make the duality transformations more symmetrical. According to [7], the Poisson-Lie
duality is based on the concepts of the Drinfeld double which is simply a Lie group \( D \), such that the
Lie algebra \( D \) of this Lie group as a vector space can be decomposed into a pair of maximally isotropic
subalgebras \( \mathcal{G} \) and \( \tilde{\mathcal{G}} \) of the Lie groups \( G \) and \( \tilde{G} \), respectively. We take the sets \( T_a, a = 1, \ldots, \text{dim}(G) \)
and \( \tilde{T}^a \) as the basis of the Lie algebras \( G \) and \( \tilde{G} \), respectively. They satisfy the commutation relations

\[
[T_a, T_b] = f_{ab}^c T_c, \quad [\tilde{T}^a, \tilde{T}^b] = \tilde{f}^{ab}_c \tilde{T}^c,
\]

\[
[T_a, \tilde{T}^b] = \tilde{f}^{bc}_a T_c + f_{ca}^b \tilde{T}^c. \tag{2.1}
\]

In addition to (2.1), there is an inner product \( < . , . > \) on \( D \) with the various generators obeying

\[
<T_a, \tilde{T}^b> = \delta^b_a, \quad <T_a, T_b> = <\tilde{T}^a, \tilde{T}^b> = 0. \tag{2.2}
\]

In what follows we shall investigate the Poisson-Lie T-duality transformations with spectators [7, 9]
of non-linear sigma models given by the action

\[
S = \frac{1}{2} \int_\Sigma d\sigma^+ d\sigma^- \left[ G_{\Upsilon \Lambda}(\Phi) + B_{\Upsilon \Lambda}(\Phi) \right] \partial_+ \Phi^\Upsilon \partial_- \Phi^\Lambda, \tag{2.3}
\]

where \( \sigma^\pm = \tau \pm \sigma \) are the standard light-cone variables on the worldsheets \( \Sigma \). \( G_{\Upsilon \Lambda} \) and \( B_{\Upsilon \Lambda} \) are components of the metric and antisymmetric tensor field \( B \) on a manifold \( \mathcal{M} \). The functions \( \Phi^\Upsilon : \Sigma \rightarrow \mathbb{R}, \Upsilon = 1, \ldots, \text{dim}(\mathcal{M}) \) are obtained by the composition \( \Phi^\Upsilon = X^\Upsilon \circ \Phi \) of a map \( \Phi : \Sigma \rightarrow \mathcal{M} \) and components of a coordinate map \( X \) on a chart of \( \mathcal{M} \).

Let us now consider a \( d \)-dimensional manifold \( \mathcal{M} \) and some coordinates \( \Phi^\Upsilon = \{ x^\mu, y^\alpha \} \) on it, where
\( x^\mu, \mu = 1, \ldots, \text{dim}(G) \) are the coordinates of Lie group \( G \) that act freely from right on \( \mathcal{M} \). \( y^\alpha \) with
\( \alpha = 1, \ldots, d - \text{dim}(G) \) are the coordinates labeling the orbit \( O \) of \( G \) in the target space \( \mathcal{M} \). We
note that the coordinates \( y^\alpha \) do not participate in the Poisson-Lie T-duality transformations and are
therefore called spectators \[9\]. We also introduce the components of the right invariant one-forms
\[\partial_\pm gg^{-1} = R_\pm^a = \partial_\pm x^\mu R_\mu^a\] and for notational convenience we will also use \(R_\pm^a = \partial_\pm y^a\). Then the
original sigma model on the manifold \(M \approx O \times G\) is defined by the following action
\[
S = \frac{1}{2} \int d\sigma^+ d\sigma^- \left[ E^+_{AB}(g, y^\alpha) R_+^A R_-^B, \right.
\]
\[
= \frac{1}{2} \int d\sigma^+ d\sigma^- \left[ \mathcal{E}_{ab}(g, y^\alpha) R_+^a R_-^b + \Phi_{a\beta}^{(1)}(g, y^\alpha) R_\pm^a \partial_\pm y^\beta \right.
\]
\[
+ \Phi_{ab}^{(2)}(g, y^\alpha) \partial_\pm y^\alpha R_\pm^b + \Phi_{\alpha\beta}(y^\alpha) \partial_\pm y^\alpha \partial_\pm y^\beta \right],
\]
(2.4)
where the index \(A = \{a, \alpha\}\). As it is seen the couplings \(\mathcal{E}_{ab}, \Phi_{a\beta}^{(1)}, \Phi_{ab}^{(2)}\) and \(\Phi_{\alpha\beta}\) may depend on all
variables \(x^\mu\) and \(y^\alpha\). The background matrix \(E^+_{AB}(g, y^\alpha)\) is defined as
\[
E^+_{AB}(g, y^\alpha) = A_A^C(g) E^+_{CD}(g, y^\alpha) A_B^D(g),
\]
(2.5)
where \[7\]
\[
E^+_{AB}(g, y^\alpha) = \left( A(g) + E^+(e, y^\alpha) B(g) \right)^{-1} A_A^C \left( E^+_{CD}(e, y^\alpha) \left( A^{-1} \right)_B^D \right)(g),
\]
(2.6)
with \(e\) being the unit element of \(G\). Here, the matrix \(E^+(e, y^\alpha)\) may be function of the spectator
variables \(y^\alpha\) only and is defined in terms of the new couplings \(E^+_{0ab}, \Phi_{a\beta}^{(1)}, \Phi_{ab}^{(2)}\) and \(\Phi_{\alpha\beta}\) so that it
is read
\[
E^+_{AB}(e, y^\alpha) = \left( \begin{array}{cc} E^+_{0ab}(e, y^\alpha) & \Phi_{a\beta}^{(1)}(e, y^\alpha) \\ \Phi_{ab}^{(2)}(e, y^\alpha) & \Phi_{\alpha\beta}(y^\alpha) \end{array} \right).
\]
(2.7)
In addition, the matrices \(A(g)\) and \(B(g)\) appearing in relations (2.5) and (2.6) are given by\[3\]
\[
A(g) = \left( \begin{array}{cc} a(g) & 0 \\ 0 & Id \end{array} \right), \quad B(g) = \left( \begin{array}{cc} b(g) & 0 \\ 0 & 0 \end{array} \right),
\]
(2.8)
where the submatrices \(a(g)\) and \(b(g)\) associated with the Lie group \(G\) are constructed using
\[
g^{-1} T_a g = a^c_a(g) T_c, \quad g^{-1} T^a_e = b^{ec}_a(g) T_c + (a^{-1})_e^a(g) \tilde{T}^e,
\]
(2.9)
and for later use we also need to consider the following definition
\[
\Pi^{ab}(g) = b^{ac}_a(g) (a^{-1})^b_c(g).
\]
(2.10)
Thus, using the relations (2.5)-(2.8) together with (2.10) one can obtain the backgrounds appearing

\[3\]Here \(Id\) means the identity matrix.
G form a pair of maximally isotropic subalgebras of the Lie algebra and define the matrix \( \tilde{\Pi} \) (which is said to be dual to (2.4) in the sense of the Poisson-Lie T-duality if the Lie algebras (2.4) and (2.17) correspond to Poisson-Lie T-dual sigma models [7].

Notice that \( \tilde{\Pi} \) (2.4) in the action (2.4). They are then given in matrix notation by

\[
\begin{align*}
\Phi^{(1)}(g, y^\alpha) &= \Phi^{(2)}(g, y^\alpha) = (2.11) \\
\Phi(y^\alpha) &= F(y^\alpha) - F^{(2)}(e, y^\alpha) \Pi(g) E^{+}(g, y^\alpha) E_{0}^{-1}(e, y^\alpha) F^{+}(e, y^\alpha). \\
\end{align*}
\]

As we shall see below, one can construct another sigma model (denoted as usual with tilded symbols) which is said to be dual to (2.4) in the sense of the Poisson-Lie T-duality if the Lie algebras \( \mathcal{G} \) and \( \tilde{\mathcal{G}} \) form a pair of maximally isotropic subalgebras of the Lie algebra \( \mathcal{D} \) [7]. Similarly to (2.7) one can define the matrix \( \tilde{E}^{+}(\tilde{e}, y^\alpha) \) and then obtain the relation between both the matrices \( E^{+}(e, y^\alpha) \) and \( \tilde{E}^{+}(\tilde{e}, y^\alpha) \) as follows [7]

\[
\tilde{E}^{+}(\tilde{e}, y^\alpha) = (A + E^{+}(e, y^\alpha) B)^{-1} \left( B + E^{+}(e, y^\alpha) A \right),
\]

in which

\[
A = \begin{pmatrix} 0 & 0 \\ 0 & Id \end{pmatrix}, \quad B = \begin{pmatrix} Id & 0 \\ 0 & 0 \end{pmatrix}.
\]

Now, the dual sigma model on the manifold \( \tilde{M} \approx O \times \tilde{\mathcal{G}} \) is, in the coordinate base \( \{\tilde{x}^\mu, y^\alpha\} \), given by the following action

\[
\tilde{S} = \frac{1}{2} \int d\sigma^{+} d\sigma^{-} \tilde{\Pi}^{+AB}(\tilde{g}, y^\alpha) \tilde{R}_{+A} \tilde{R}_{-B} = \frac{1}{2} \int d\sigma^{+} d\sigma^{-} \left[ \tilde{E}^{+ab}(\tilde{g}, y^\alpha) \tilde{R}_{+a} \tilde{R}_{-b} + \tilde{\Phi}^{(1)\alpha}(\tilde{g}, y^\alpha) \tilde{R}_{+a} \partial_{-} y^{\beta} \right. \\
\left. + \tilde{\Phi}^{(2)\alpha}(\tilde{g}, y^\alpha) \partial_{+} y^{\alpha} \tilde{R}_{-b} + \tilde{\Phi}_{\alpha \beta}(y^\alpha) \partial_{+} y^{\alpha} \partial_{-} y^{\beta} \right],
\]

where \( \tilde{R}_{+a} = \partial_{+} \tilde{x}^\mu \tilde{R}_{\mu a} \) are the components of the right invariant one-forms on the Lie group \( \tilde{\mathcal{G}} \).

The coupling matrices of the dual sigma model are also determined in a completely analogous way as (2.7) and (2.11)-(2.14) such that by using (2.15) one relates them to those of the original one by [7, 8, 9, 20]

\[
\begin{align*}
\tilde{E}^{+}(\tilde{g}, y^\alpha) &= \left( E_{0}^{+}(e, y^\alpha) + \tilde{\Pi}(\tilde{g}) \right)^{-1}, \\
\tilde{\Phi}^{(1)}(\tilde{g}, y^\alpha) &= \tilde{E}^{+}(\tilde{g}, y^\alpha) F^{+}(e, y^\alpha), \\
\tilde{\Phi}^{(2)}(\tilde{g}, y^\alpha) &= -F^{+}(e, y^\alpha) \tilde{E}^{+}(\tilde{g}, y^\alpha), \\
\tilde{\Phi}(y^\alpha) &= F(y^\alpha) - F^{+}(e, y^\alpha) \tilde{E}^{+}(\tilde{g}, y^\alpha) F^{+}(e, y^\alpha). \\
\end{align*}
\]

Notice that \( \tilde{\Pi}(\tilde{g}) \) is defined as (2.10) by replacing untilded quantities by tilded ones. Hence, the actions (2.4) and (2.17) correspond to Poisson-Lie T-dual sigma models [7].
3 The Abelian T-dualization of the BTZ black hole solutions

In this section we give the Abelian T-dualization of the BTZ black hole solutions [1] by making use of the approach of the Poisson-Lie T-duality with the spectators reviewed in the previous section. Let us now begin this section by reviewing the BTZ black hole solutions [1]. The BTZ black holes are 2 + 1-dimensional solutions of Einstein’s equations with a negative cosmological constant, $\Lambda$,

$$R_{\tau\Delta} + (\Lambda - \frac{1}{2}R) G_{\tau\Delta} = 0, \quad \Lambda < 0,$$  \hspace{1cm} (3.1)

where $R_{\tau\Delta}$ and $R$ are the respective Ricci tensor and scalar curvature. The line element for the black hole solutions is given by

$$ds^2 = \left( M - \frac{r^2}{l^2} \right) dt^2 - J dtd\phi + r^2 d\phi^2 + \left( \frac{r^2}{l^2} - M + \frac{J^2}{4r^2} \right)^{-1} dr^2, \quad 0 \leq \phi < 2\pi$$  \hspace{1cm} (3.2)

where the radius $l$ is related to the cosmological constant by $l = (-\Lambda)^{-1/2}$. The constants of motion $M$ and $J$ are the mass and angular momentum of the BTZ black hole, respectively. They are appeared due to the time translation symmetry and rotational symmetry of the metric, corresponding to the killing vectors $\partial/\partial t$ and $\partial/\partial \phi$, respectively. The line element (3.2) describes a black hole solution with outer and inner horizons at $r = r_+$ and $r = r_-$, respectively,

$$r_{\pm} = l \left( \frac{M}{2} \right)^{1/2} \left\{ 1 \pm \left( 1 - \left( \frac{J}{Ml} \right)^2 \right)^{1/2} \right\}^{1/2},$$ \hspace{1cm} (3.3)

where the mass and angular momentum are related to $r_{\pm}$ by

$$M = \frac{r_+^2 + r_-^2}{l^2}, \quad J = \frac{2r_+r_-}{l}.$$

The region $r_+ < r < M^{1/2}l$ defines an ergosphere, in which the asymptotic timelike Killing field $\partial/\partial t$ becomes spacelike. The solutions with $-1 < M < 0$, $J = 0$ describe point particle sources with naked conical singularities at $r = 0$. The metric with $M = -1$, $J = 0$ may be recognized as that of ordinary anti-de Sitter space; it is separated by a mass gap from the $M = 0$, $J = 0$. The vacuum state which is regarded as empty space, is obtained by letting the horizon size go to zero. This amounts to letting $M \to 0$, which requires $J \to 0$. We have to notice that the metric for the $M = J = 0$ black hole is not the same as $AdS_3$ metric which has negative mass $M = -1$. Locally they are equivalent since there is locally only one constant curvature metric in three dimensions. However they are inequivalent globally.

In Ref. [4], first the BTZ black hole solutions have been considered in the context of the low energy approximation, then as the exact conformal field theory. In three dimensions, the low energy string effective action is given by

$$S_{\text{eff}} = \int d^3\Phi \sqrt{-G} e^{-2\phi} \left[ R + 4(\nabla\phi)^2 - \frac{1}{12} H_{\tau\Delta\Xi} H^{\tau\Delta\Xi} - 4\Lambda \right],$$ \hspace{1cm} (3.4)

where $G = \det G_{\tau\Delta}$. $H_{\tau\Delta\Xi}$ are the components of the torsion of the antisymmetric field $B$ and are
defined by $H_{\gamma\Delta\Xi} = \partial_\gamma B_{\Delta\Xi} + \partial_\Delta B_{\Xi\gamma} + \partial_\Xi B_{\gamma\Delta}$, while $\phi$ is the dilaton field. The equations of motion which follow from this action are [21]

\begin{align}
0 &= R_{\gamma\Delta} - \frac{1}{4} H_{\gamma\Delta\Xi} H_{\Xi\Delta}^{\alpha\beta} + 2 \nabla_{\gamma} \nabla_{\Delta} \phi, \\
0 &= -\frac{1}{2} \nabla_{\Xi} H_{\Xi\Delta\gamma} + H_{\Xi\Delta} \nabla_{\Xi} \phi, \\
0 &= R + 4 \nabla^2 \phi - 4(\nabla \phi)^2 - \frac{1}{12} H_{\gamma\Delta\Xi} H_{\Xi\Delta\gamma} - 4 \Lambda.
\end{align}

(3.5)

These equations are known as the vanishing of the one-loop beta-functions equations. Notice that the string effective action is connected to the two-dimensional non-linear sigma model through these equations [21].

In order to obtain an exact solution to string theory, one must modify the BTZ black hole solutions by adding an antisymmetric tensor field $H_{\gamma\Delta\Xi}$ proportional to the volume form $\epsilon_{\gamma\Delta\Xi}$. It has been shown [4] that any solution to three-dimensional general relativity with negative cosmological constant is a solution to low energy string theory with $\phi = 0, H_{\gamma\Delta\Xi} = 2 \epsilon_{\gamma\Delta\Xi}/l$ and $\Lambda = -1/l^2$. In particular it was shown [4] that the two parameter family of black holes (3.2) along with

\begin{align}
B_{\varphi t} &= \frac{r^2}{l}, \\
\phi &= 0,
\end{align}

(3.6)

satisfy the equations of motion (3.5). Then, it was obtained [4] the dual of this solution by Buscher's duality transformations [10]. Abelian duality is a well known symmetry of string theory that maps any solution of the low energy string equations of motion (3.5) with a translational symmetry to another solution.

As a concluding remark for this section we first explicitly reobtain Buscher’s duality transformations [10] using the approach of the Poisson-Lie T-duality with the spectators. Then, in this way, we obtain the Abelian T-dualization of the BTZ black hole solutions given by (3.2). In this case, the Lie groups $G$ and $\tilde{G}$ are Abelian. Therefore, both structures $\Pi_{ab}(g)$ and $\tilde{\Pi}^{ab}(\tilde{g})$ are vanished. Consequently, the background matrices $F_{+AB}(g, y^\alpha)$ and $\tilde{F}^{+AB}(\tilde{g}, y^\alpha)$ are equaled with $E^+_{AB}(e, y^\alpha)$ and $\tilde{E}^{+AB}(\tilde{e}, y^\alpha)$, respectively. Thus, the action (2.4) in the presence of the dilaton field $\phi$ is written as

\begin{equation}
S = \frac{1}{2} \int d\sigma^+ d\sigma^- \left[ E^+_{AB}(e, y^\alpha) \delta^A_R \delta^B_\Lambda \partial_+ \Phi^R \partial_- \Phi^\Lambda - \frac{1}{4} R^{(2)} \phi \right],
\end{equation}

(3.7)

where $R^{(2)}$ is two-dimensional scalar curvature in the worldsheet. Comparing the sigma model (3.7) and (2.23) we find that $E^+_{AB}(e, y^\alpha) \delta^A_R \delta^B_\Lambda = G_{\gamma\Delta} + B_{\gamma\Delta}$. Let us assume that the sigma model (3.7) has an Abelian isometry represented by a translation in a coordinate $x^0$ in the target space. Notice that in the coordinates $\Phi^R = \{x^0, x^\mu\}, \mu = 1, \ldots, d - 1$ adapted to the isometry, the metric, torsion and dilaton field are independent of $x^0$. If we choose the matrix $E^+_{AB}(e, y^\alpha)$ as

\begin{equation}
E^+_{AB}(e, y^\alpha) = \begin{pmatrix}
G_{\alpha\alpha} & G_{\alpha\beta} + B_{\alpha\beta} \\
G_{\beta\alpha} + B_{\beta\alpha} & G_{\beta\beta} + B_{\beta\beta}
\end{pmatrix},
\end{equation}

(3.8)

then from relations (2.15) and (2.16) one immediately gets the following Buscher's duality transfor-
\[
\left( \begin{array}{cc}
\tilde{G}_{00} & \tilde{G}_{0i} + \tilde{B}_{0i} \\
\tilde{G}_{i0} & \tilde{G}_{ij} + \tilde{B}_{ij}
\end{array} \right) = \left( \begin{array}{cc}
\frac{1}{G_{00}} & \frac{G_{0i} + B_{0i}}{G_{00}} \\
\frac{G_{i0} + B_{i0}}{G_{00}} & G_{ij} + B_{ij} - \frac{G_{0i} + B_{0i}}{G_{00}} \left( G_{0j} + B_{0j} \right)
\end{array} \right).
\]

(3.9)

It has been shown \([10, 22]\) that for preserving conformal invariance, at least to one-loop order, the dilaton field has to transform as

\[
\tilde{\phi} = \phi - \frac{1}{2} \ln G_{00}.
\]

(3.10)

This shift of the dilaton field is due to the duality transformation receiving corrections from the jacobian that comes from integrating out the gauge fields \([10, 22]\). In the Poisson-Lie T-duality case, dilaton shifts in both models have been obtained by quantum considerations based on a regularization of a functional determinant in a path integral formulation of Poisson-Lie duality by incorporating spectator fields \([8]\) (see, also, \([23]\))

\[
\phi = \phi_0(y^\alpha) + \frac{1}{2} \ln (\det E^+(g, y^\alpha)) - \frac{1}{2} \ln (\det E_0^+(e, y^\alpha)),
\]

(3.11)

\[
\tilde{\phi} = \phi_0(y^\alpha) + \frac{1}{2} \ln (\det \tilde{E}^+(\tilde{g}, y^\alpha)).
\]

(3.12)

Now one can use the above approach to obtain the Abelian T-dual solutions with the BTZ black hole solutions. Here the target space \(M \approx O \times G\) is defined by the coordinates \(\{\varphi, r, t\}\), where the Lie group \(G\) should be considered to be \(U(1)\). It is more convenient to write the action of sigma model corresponding to the BTZ metric \((3.2)\) and the solutions \((3.6)\). It is then read to be in the following form

\[
S = \frac{1}{2} \int d\sigma^+ d\sigma^\prime \left[ r^2 \partial_+ \varphi \partial_- \varphi + \left( \frac{r^2}{l^2} - M + \frac{J^2}{4r^2} \right)^{-1} \partial_+ r \partial_- r 
\right.
\]

\[
+ \left( M - \frac{r^2}{l^2} \right) \partial_+ t \partial_- t - \left( \frac{J}{2} - \frac{r^2}{l} \right) \partial_+ \varphi \partial_- t - \left( \frac{J}{2} + \frac{r^2}{l} \right) \partial_+ t \partial_- \varphi \right].
\]

(3.13)

On the other hand, since the background described by the action \((3.13)\) depend on the \(r\) coordinate only, from the Poisson-Lie T-duality standpoint the Lie group \(G\) can not be parametrized by this coordinate. However, we assume that the \(G\) is parametrized by the \(\varphi\) coordinate while the coordinates of orbit \(O\) are represented by the \(r\) and \(t\). Comparing the action \((3.13)\) and \((3.7)\) the matrix \(E_{AB}^+(e, y^\alpha)\) is derived as

\[
E_{AB}^+(e, y^\alpha) = \left( \begin{array}{cc}
r^2 & 0 \\
0 & \left( \frac{r^2}{l^2} - M + \frac{J^2}{4r^2} \right)^{-1} \\
-\frac{J}{2} - \frac{r^2}{l} & 0 \\
-\frac{J}{2} + \frac{r^2}{l} & M - \frac{J^2}{4r^2}
\end{array} \right).
\]

(3.14)

Considering \((3.14)\) in the form \((2.7)\) and using the fact that \(G = U(1)\) we find that \(E_{0ab}^+(e, y^\alpha) = r^2\)
and

\[ F_{a\beta}^{(1)} = \left( 0, -\frac{1}{2} + \frac{r^2}{l^2} \right), \quad F_{a\beta}^{(2)} = \left( \frac{1}{l^2}, 0 \right), \quad F_{a\beta} = \left( \frac{\hat{r}^2}{l^2} - M + \frac{J^2}{4r^2}, 0 \right). \tag{3.15} \]

The dual target space \( \hat{\mathcal{M}} \approx O \times \hat{G} \) is defined by the coordinates \( \{\hat{\varphi}, r, t\} \). Obviously, in the standard Abelian case \( D = U(1) \times U(1) \), i.e., \( \hat{G} \) is also identical to \( U(1) \) and is parametrized by the \( \hat{\varphi} \) coordinate. Finally, utilizing the equations (2.18)-(2.21) together with (2.17) the dual sigma model is read off to be of the form

\[ \hat{S} = \frac{1}{2} \int d\sigma^+ d\sigma^- \left[ \frac{1}{r^2} \partial_+ \hat{\varphi} \partial_- \hat{\varphi} + \left( \frac{r^2}{l^2} - M + \frac{J^2}{4r^2} \right)^{-1} \partial_+ r \partial_- r \right. \\
+ \left. (M - \frac{J^2}{4r^2}) \partial_+ t \partial_- t + \left( -\frac{J}{2r^2} + \frac{1}{r} \right) \partial_+ \hat{\varphi} \partial_- t + \left( \frac{J}{2r^2} + \frac{1}{r} \right) \partial_+ t \partial_- \hat{\varphi} - \frac{1}{4} R^{(2)} \hat{\varphi} \right], \tag{3.16} \]

where the new dilaton \( \hat{\varphi} \) at the last term of the action can be calculated by (3.10) to be of the form

\[ \hat{\varphi} = -\ln r. \tag{3.17} \]

In this way the dual geometry is given by

\[ d\hat{s}^2 = (M - \frac{J^2}{4r^2}) dt^2 + \left( \frac{r^2}{l^2} - M + \frac{J^2}{4r^2} \right)^{-1} dr^2 + \frac{2}{l} dt d\hat{\varphi} + \frac{1}{r^2} d\hat{\varphi}^2, \tag{3.18} \]

\[ \hat{B} = -\frac{J}{2r^2} d\hat{\varphi} \wedge dt. \tag{3.19} \]

The solutions (3.17)-(3.19) have been obtained in Ref. [4] by using Buscher’s duality transformations. The three-dimensional charged black string solutions can be obtained from the dual solutions (3.17)-(3.19) by making the coordinate transformation \( t = l(r^2 - r^2 )^{-\frac{1}{2}} (\hat{x} - \hat{t}), \quad \hat{\varphi} = l(r^2 - r^2 )^{-\frac{1}{2}} (\hat{r}^2 \hat{t} - r^2 \hat{x}) \) and \( r^2 = l\hat{r} \) as follows [4]

\[ d\hat{s}^2 = -(1 - \frac{M}{\hat{r}}) d\hat{t}^2 + (1 - \frac{Q^2}{M\hat{r}}) d\hat{x}^2 + (1 - \frac{M}{\hat{r}})^{-1} (1 - \frac{Q^2}{M\hat{r}})^{-1} \frac{\hat{l}^2 d\hat{r}^2}{4\hat{r}^2}, \tag{3.20} \]

\[ \hat{B} = \frac{Q}{\hat{r}} d\hat{x} \wedge d\hat{t}, \tag{3.21} \]

\[ \hat{\varphi} = -\frac{1}{2} \ln(l\hat{r}), \tag{3.22} \]

where \( M = r^2 / l \) and \( Q = J/2 \) are the respective the mass and charge of the black string. For large \( \hat{r} \) the metric is asymptotically flat. Thus, it is important to notice that the Abelian T-duality transformation changes the asymptotic behavior from \( AdS_3 \) to flat. In addition, as it can be seen from the above results, for asymptotically flat solutions of three-dimensional low energy string theory, the duality of conserved quantities defined on asymptotic region is given in such a way that a mass is unchanged, while an axion charge and an angular momentum are interchanged each other [4]. In the next section we give the main goal of the paper. We will study the non-Abelian T-dualization of the BTZ black hole solution in such a way that we shall show that the dual model describes a new three-dimensional charged black string.
4 The BTZ black hole vacuum solution from Poisson-Lie T-dual sigma models with the spectators

In this section we explicitly construct a pair of Poisson-Lie T-dual sigma models on 2 + 1-dimensional manifolds \( M \) and \( \tilde{M} \) as the target spaces. The original model is built on the manifold \( M \approx O \times G \), where \( G = A_2 \) as a two-dimensional real non-Abelian Lie group acts freely on it while \( O \) is the orbit of \( G \) in \( M \) with one spectator \( y^\alpha = \{ y \} \). The target space of the dual model is the manifold \( \tilde{M} \approx O \times \tilde{G} \) with two-dimensional real Abelian Lie group \( \tilde{G} = 2A_1 \) acting freely on it. We shall show the original model is canonically equivalent to the \( SL(2, \mathbb{R}) \) WZW model for a given value of the background parameters. In particular, by a convenient coordinate transformation we show that the model describes a string propagating in a spacetime with the BTZ black hole vacuum metric.

4.1 The original model as the \( SL(2, \mathbb{R}) \) WZW model and BTZ black hole vacuum solution

As mentioned above the original model is built on a 2 + 1-dimensional manifold \( M \approx O \times G \) in which \( G \) is two-dimensional real non-Abelian Lie group whose Lie algebra is denoted by \( A_2 \). According to action (2.4) to construct the original model we need the components of right invariant one-forms \( R_{\pm}^a \) on the Lie group \( A_2 \). To this end, we parametrize an element of \( A_2 \) as

\[
g = e^{x_1 T_1} e^{x_2 T_2},
\]

where \( x_1 \) and \( x_2 \) are the coordinates of the Lie group \( A_2 \). Then \( R_{\pm}^a \)'s are derived to be of the form

\[
R_{\pm}^1 = \partial_{\pm} x_1, \quad R_{\pm}^2 = e^{x_1} \partial_{\pm} x_2.
\]

We note that since the dual Lie group have been considered to be Abelian, hence, by using (2.9) and (2.10) it follows that the \( \Pi_{ab}(g) \) is vanished. In addition to (4.3), we need to determine the couplings \( E_{ab}^+(g, y^\alpha) \), \( \Phi_{a\beta}^{(1)}(g, y^\alpha) \) and \( \Phi_{ab}^{(2)}(g, y^\alpha) \). By a convenient choice of the matrices \( E_{0_{ab}}^+(e, y^\alpha) \), \( F_{a\beta}^{(1)}(e, y^\alpha) \) and \( F_{ab}^{(2)}(e, y^\alpha) \) as

\[
E_{0_{ab}}^+(e, y^\alpha) = \begin{pmatrix} 0 & \frac{1}{2} e^{-2y} \\ \frac{1}{2} e^{-2y} & 0 \end{pmatrix}, \quad F_{a\beta}^{(1)}(e, y^\alpha) = \begin{pmatrix} 0 & 0 \\ 0 & -e^{-2y} \end{pmatrix}, \quad F_{ab}^{(2)}(e, y^\alpha) = \begin{pmatrix} 0 & -e^{-2y} \\ 0 & 0 \end{pmatrix},
\]

and \( F_{a\beta}(y^\alpha) = k \) (\( k \) is a non-zero real constant), and then with the help of relations (2.11)-(2.14) one can get the required couplings. Finally by inserting these into action (2.4) the original sigma model
is found to be of the form
\[ S = \frac{1}{2} \int d\sigma^+ d\sigma^- \left[ k \partial_+ y \partial_- y + \frac{1}{2} e^{x_1-2y} (\partial_+ x_1 \partial_- x_2 + \partial_+ x_2 \partial_- x_1) - e^{x_1-2y} (\partial_+ x_2 \partial_- y - \partial_+ y \partial_- x_2) \right]. \] (4.5)

By identifying action (4.5) with the sigma model of the form (2.3) one can read off the background matrix. The space-time metric and the antisymmetric tensor field corresponding to the action (4.5) can be written as
\[ ds^2 = k dy^2 + e^{x_1-2y} dx_1 dx_2, \] (4.6)
\[ B = -e^{x_1-2y} dx_2 \wedge dy. \] (4.7)

For the metric (4.6) one can find that \( R_{\Upsilon \Lambda} = -\left(\frac{2}{k}\right) G_{\Upsilon \Lambda} \) and then \( R = -6/k. \) This shows that the metric (4.6) with \( k > 0 \) describes an anti-de Sitter space while for \( k < 0 \) we have a de Sitter space. Considering antisymmetric tensor field \( B \) of (4.7) one quickly finds that the only non-zero component of \( H \) is \( H_{x_1 x_2 y} = -e^{x_1-2y}, \) then, it follows that \( H_{\Upsilon \Delta} H^{\Upsilon \Delta} = -24/k. \) Inserting the above results in the vanishing of the one-loop beta-functions equations (3.5), the conformal invariance conditions up to one-loop order are satisfied with \( \Lambda = -1/k \) and a constant dilaton field which satisfies the equation (3.11).

On the one hand, action (4.5) can be simplified by making the following new coordinates
\[ e^{x_1} = \theta_+, \quad x_2 = \theta_-, \quad y = \gamma, \] (4.8)
then, by making use of the integrating by parts over the last two terms of the action (4.9) we get
\[ S = \frac{1}{2} \int d\sigma^+ d\sigma^- \left[ k \partial_+ \gamma \partial_- \gamma + e^{-2\gamma} \partial_+ \theta_+ \partial_- \theta_- \right]. \] (4.10)

By setting \( k = 1 \) the above action precisely becomes the \( SL(2, \mathbb{R}) \) WZW model [15]. On the other hand, to better understand of action (4.5) we diagonalize the metric obtained by this action. Let
\[ e^{x_1} = \frac{1}{l}(t - l\varphi), \quad x_2 = -l(t + l\varphi), \quad e^y = \frac{l}{r}, \] (4.11)

\[ \text{Under the transformation (4.8) the action (4.5) turns into} \]
\[ \tilde{S} = \frac{1}{2} \int d\sigma^+ d\sigma^- \left[ k \partial_+ \gamma \partial_- \gamma + \frac{1}{2} e^{-2\gamma} (\partial_+ \theta_+ \partial_- \theta_- + \partial_+ \theta_- \partial_- \theta_+) - \theta_+ e^{-2\gamma} (\partial_+ \theta_+ \partial_- \gamma - \partial_+ \gamma \partial_- \theta_+) \right]. \] (4.9)

By calculating the momentums corresponding to the actions (4.5) and (4.9) \( (P_\Lambda \text{ and } \tilde{P}_\Lambda, \text{ respectively}) \) and by making use of the transformation (4.8) we get the transformation between momentums. Then, by considering the basic equal-time Poisson brackets for the pair of canonical variables \((\Phi^\Upsilon, P_\Lambda)\) one can show that the equal-time Poisson brackets for the pair of canonical variables \((\tilde{\Phi}^\Upsilon, \tilde{P}_\Lambda)\) are also preserved. Furthermore, we can obtain the Hamiltonians corresponding to the actions (4.5) and (4.9), and then show that they are identical together, that is, under the transformation (4.8) one goes from \( \mathcal{H}(\tilde{\Phi}) \) to \( \mathcal{H}(\Phi) \) and vice versa, hence, proving that the transformation (4.8) is indeed a canonical transformation.
where the radius $l$ introduced in Section 3. Then the action (4.12) turns into

$$S = \frac{1}{2} \int d\sigma^+ d\sigma^- \left[ -\frac{r^2}{l^2} \partial_+ t \partial_- t + r^2 \partial_+ \varphi \partial_- \varphi + \frac{l^2}{r^2} \partial_+ r \partial_- r 
- \frac{r}{l^2} (t - l\varphi)(\partial_+ r \partial_- t - \partial_+ t \partial_- r) - \frac{r}{l} (t - l\varphi)(\partial_+ \varphi \partial_- r - \partial_+ r \partial_- \varphi) \right],$$  \hspace{1cm} (4.12)

where we have, here, set $k = l^2$. This describes a string propagating in a space-time with the BTZ black hole vacuum metric with $M = J = 0$

$$ds^2 = -\frac{r^2}{l^2} dt^2 + r^2 d\varphi^2 + \frac{l^2}{r^2} dr^2,$$  \hspace{1cm} (4.13)

and an antisymmetric tensor field

$$B = -\frac{r}{l} (t - l\varphi) (d\varphi \wedge dr + \frac{1}{l} dt \wedge dr).$$  \hspace{1cm} (4.14)

Since the black hole metric with $M = J = 0$ is locally equivalent to the $AdS_3$ metric, so, for the metric (4.13) one immediately finds that $R_{\Upsilon \Lambda} = -(2/l^2) G_{\Upsilon \Lambda}$ and $R = -6/l^2$. The only non-zero component of antisymmetric field strength corresponding to the $B$-field (4.14) is $H_{\tau \Delta \Xi} = 2r/l$; consequently $H_{\tau \Delta \Xi} H^{\tau \Delta \Xi} = -24/l^2$. Putting these pieces together, one verifies the equations (3.5) with $\Lambda = -1/l^2$ and a constant dilaton field. Therefore, it follows that the conformal invariance of actions (4.5) and (4.12) is, under the transformation (4.11), preserved. Notice that the metric (4.13) has no horizon and no curvature singularity. Indeed, this solution is everywhere regular including $r = 0$. In fact, this was expected since generally the (anti-)de Sitter space-time is a maximally symmetric space with a constant curvature. The resulting space-time is completely nonsingular. Consider now two killing vectors $\partial/\partial t$ and $\partial/\partial \varphi$ corresponding to the time translational and the rotational isometries of the metric (4.13), respectively. The killing field $\partial/\partial t$ becomes null at $r = 0$ and it is timelike for the whole range $r > 0$, while the killing field $\partial/\partial \varphi$ is everywhere spacelike except for $r = 0$. The global structure associated with the metric (4.13) including the Kruskal and Penrose diagrams has been discussed in Ref. [25].

4.2 The dual model as a three-dimensional charged black string

As mentioned at the first of this section, the dual model is constructed on a $2+1$-dimensional manifold $\tilde{M} \approx O \times \tilde{G}$ with two-dimensional Abelian Lie group $\mathbf{G} = 2A_1$ acting freely on it. In the same way to construct out the dual sigma model we parametrize the corresponding Lie group (Abelian Lie group $2A_1$) with coordinates $\tilde{x}^\mu = \{\tilde{x}_1, \tilde{x}_2\}$ so that its elements are defined as (4.12) by replacing untilded quantities by tilded ones. Hence the components of the right invariant one-forms on the Lie group $2A_1$ are simply calculated to be $\tilde{R}_{\pm a} = \partial_{\pm} \tilde{x}_a$. Utilizing relation (2.10) for untilded quantities we get

$$\tilde{\Pi}^{ab}(\tilde{g}) = \begin{pmatrix} 0 & -\tilde{x}_2 \\ \tilde{x}_2 & 0 \end{pmatrix}.$$

(4.15)
Inserting (4.14) and (4.15) into equations (2.18)-(2.21) the dual couplings are obtained to be of the form
\[
\tilde{E}^{ab} = \begin{pmatrix}
0 & \frac{1}{x_2 + \frac{1}{2}e^{-2y}} \\
-\frac{1}{x_2 + \frac{1}{2}e^{-2y}} & 0
\end{pmatrix}, \quad \tilde{F}^{(1)a}_{\beta} = \begin{pmatrix}
\frac{e^{-2y}}{x_2 + \frac{1}{2}e^{-2y}} \\
0
\end{pmatrix}, \quad \tilde{F}^{(2)b}_{\alpha} = \begin{pmatrix}
\frac{e^{-2y}}{x_2 + \frac{1}{2}e^{-2y}} & 0
\end{pmatrix},
\]
and \(\tilde{\Phi}_{a\beta} = k\). Putting these pieces together into (2.17), the action of dual model is worked out to be
\[
\tilde{S} = \frac{1}{2} \int d\sigma^+ d\sigma^- \left\{ k \, \partial_+ y \, \partial_- y + \frac{1}{\Delta} \left[ \left( \frac{1}{2}e^{-2y} - \tilde{x}_2 \right) \partial_+ \tilde{x}_1 \, \partial_- \tilde{x}_2 + \left( \frac{1}{2}e^{-2y} + \tilde{x}_2 \right) \partial_+ \tilde{x}_2 \, \partial_- \tilde{x}_1 \\
- e^{-2y} \left( \frac{1}{2}e^{-2y} - \tilde{x}_2 \right) \partial_+ \tilde{x}_1 \, \partial_- y - e^{-2y} \left( \frac{1}{2}e^{-2y} + \tilde{x}_2 \right) \partial_+ y \, \partial_- \tilde{x}_1 \right] \right\},
\]
where \(\Delta = \frac{1}{4} e^{-4y} - \tilde{x}_2^2\). Comparing the above action with the sigma model action of the form (2.23), the corresponding metric and tensor field \(\tilde{B}\) take the following forms
\[
ds^2 = k \, dy^2 + \frac{e^{-2y}}{\Delta} \left( d\tilde{x}_1 \, d\tilde{x}_2 - e^{-2y} \, d\tilde{x}_1 \, dy \right),
\]
\[
\tilde{B} = -\frac{\tilde{x}_2}{\Delta} \left( d\tilde{x}_1 \wedge d\tilde{x}_2 - e^{-2y} \, d\tilde{x}_1 \wedge dy \right).
\]
As mentioned in Section 2 the spectator fields do not participate in the Poisson-Lie T-duality transformations. Therefore, as it can be seen from the metrics (4.16) and (4.18) this duality changes the group part of the metrics, while leaving the \(y\) component invariant. The metric components (4.18) are ill defined at the regions \(\tilde{x}_2 = \frac{1}{2} e^{-2y}\) and \(\tilde{x}_2 = -\frac{1}{2} e^{-2y}\). We can test whether there are true singularities by calculating the scalar curvature, which is
\[
\tilde{R} = -\frac{2(11 e^{-4y} + 28 \tilde{x}_2 e^{-2y} + 12 \tilde{x}_2^2)}{k(e^{-2y} - 2 \tilde{x}_2)^2}.
\]
Thus \(\tilde{x}_2 = \frac{1}{2} e^{-2y}\) is a true curvature singularity, while the difficult at the \(\tilde{x}_2 = -\frac{1}{2} e^{-2y}\) can be removed by an appropriate change of coordinates. As shown in subsection 4.1 the original model (4.5) is canonically equivalent to the \(SL(2, \mathbb{R})\) WZW model. Therefore, it should be canformally invariant. To check the conformal invariance of the dual model (4.17) we look at vanishing of the one-loop beta-functions equations (3.5). Given a \(\tilde{B}\)-field with (4.19) we find that the only non-zero component of \(\tilde{H}\) is \(\tilde{H}_{\tilde{x}_1 \tilde{x}_2 y} = -4 e^{-2y} / (e^{-2y} - 2 \tilde{x}_2)^2\). Thus, the first two equations of (3.5) are satisfied by the new dilatonic field
\[
\tilde{\phi} = a + \frac{1}{2} \ln \left( \frac{2 \tilde{x}_2 + e^{-2y}}{2 \tilde{x}_2 - e^{-2y}} \right),
\]
where \(a\) is an arbitrary constant. This additive constant plays an important role, as we will see later. Thus, the dilatonic contribution in (3.5) vanishes if the cosmological constant of the dual theory does leave invariant, that is, \(\Lambda = -1/k\). It is also important to note that the dilaton field (4.21) is well behaved for the ranges \(\tilde{x}_2 + \frac{1}{2} e^{-2y} < 0\) and \(\tilde{x}_2 - \frac{1}{2} e^{-2y} > 0\).

As shown, the metric (4.13) of the original model is the BTZ black hole vacuum solution which is locally equivalent to the \(AdS_3\) metric. To continue, we shall show that the dual solution represents
a three-dimensional charged black string which is stationary and asymptotically flat. In order to enhancing and clarifying the structure of the dual space-time, horizon, singularity and also determining the asymptotic nature of one, we first write the solutions (4.18), (4.19) and (4.21) in the coordinate base \( \{t,x,r\} \). Furthermore, since we want to discuss the dual solution to the BTZ black hole vacuum solution, we must here consider \( k = l^2 \). In the following, we separately discuss the solutions for the ranges \( \tilde{x}_2 + \frac{1}{2}e^{-2y} < 0 \) and \( \tilde{x}_2 - \frac{1}{2}e^{-2y} > 0 \).

- **The solution corresponding to the range \( \tilde{x}_2 + \frac{1}{2}e^{-2y} < 0 \)**

  In this case we consider \( \tilde{x}_2 + \frac{1}{2}e^{-2y} = -e^T \) for \( T \in \mathbb{R} \). Then, we introduce the following coordinate transformation

\[
\tilde{x}_1 = U + \frac{l^2}{2}(W + e^W), \quad \tilde{x}_2 = -e^T(1 + \frac{e^{-W}}{2}), \quad \tilde{y} = \frac{1}{2}(W - T),
\]

for \( U, W \in \mathbb{R} \). Under the above transformation, the dual metric (4.18), the \( \tilde{B} \)-field (4.19) and the dilaton field (4.21) are, respectively, turn into

\[
d\tilde{s}^2 = \frac{l^2}{4} dW^2 + \frac{l^2}{4} dT^2 + \frac{1}{e^W + 1} dTdU,
\]

\[
\tilde{B} = \left[ 1 - \frac{1}{2(e^W + 1)} \right] dU \wedge dT + \frac{l^2}{2}(e^W + \frac{1}{2}) dW \wedge dT,
\]

\[
\tilde{\phi} = a + \frac{1}{2} \ln \left( \frac{e^W}{e^W + 1} \right),
\]

(4.22)

Notice that there is no singularity for the metric (4.22). In fact, this was expected since the solutions (4.18), (4.19) and (4.21) are, in this case, defined only for the range \( \tilde{x}_2 + \frac{1}{2}e^{-2y} < 0 \). As explained above, the true singularity of the metric (4.18) occurs at \( \tilde{x}_2 = \frac{1}{2}e^{-2y} \), which this region is located out of the range \( \tilde{x}_2 + \frac{1}{2}e^{-2y} < 0 \). Let us now consider the transformation \( e^W = 1/(\hat{r} - 1) \) so that it requires that \( 1 < \hat{r} < \infty \). In addition to, we follow the following linear transformation

\[
T = -\frac{2}{l}(\hat{t} + \frac{\hat{x}}{\sqrt{3}}), \quad U = l(\hat{t} - \frac{\hat{x}}{\sqrt{3}}),
\]

(4.26)

for \( \hat{t}, \hat{x} \in \mathbb{R} \). By applying the above transformations to the solutions (4.22), (4.24) and (4.25), one obtains the forms of the dual space-time metric, antisymmetric field strength and dilaton field in new coordinate base \( \{\hat{t}, \hat{x}, \hat{r}\} \) as

\[
d\hat{s}^2 = -(1 - \frac{2}{\hat{r}}) d\hat{t}^2 + (1 - \frac{2}{3\hat{r}}) d\hat{x}^2 + \frac{2}{\sqrt{3}} d\hat{t} d\hat{x} + (1 - \frac{1}{\hat{r}})^{-2} \frac{l^2 d\hat{r}^2}{4\hat{r}^2},
\]

(4.27)

\[
\tilde{H}_{\hat{t}\hat{x}} = \frac{2}{\sqrt{3} \hat{r}^2},
\]

(4.28)

\[
\tilde{\phi} = a - \frac{1}{2} \ln \hat{r}.
\]

(4.29)

We note that this solution is valid only for the range \( \tilde{x}_2 + \frac{1}{2}e^{-2y} < 0 \) or \( 1 < \hat{r} < \infty \).

- **The solution corresponding to the range \( \tilde{x}_2 - \frac{1}{2}e^{-2y} > 0 \)**
For this case, we have assumed that $\tilde{x}_2 - \frac{1}{2} e^{-2y} = e^T - e^{-2y}$ for which $T + 2y > 0$. Analogously, by introducing the following transformation

$$\tilde{x}_1 = U - \frac{l^2}{2}(e^W - W), \quad \tilde{x}_2 = e^T (1 - \frac{e^{-W}}{2}), \quad \tilde{y} = \frac{1}{2}(W - T),$$

we obtain

$$d\tilde{s}^2 = \frac{l^2}{4} dW^2 + \frac{l^2}{4} dT^2 - \frac{1}{e^W - 1} dTdU,$$  (4.31)

$$\tilde{B} = [1 + \frac{1}{2(e^W - 1)}] dU \wedge dT - \frac{l^2}{2} (e^W - \frac{1}{2}) dW \wedge dT,$$  (4.32)

$$\tilde{\phi} = a + \frac{1}{2} \ln \left( \frac{e^W}{e^W - 1} \right),$$  (4.33)

where $e^W > 1$, i.e., $W > 0$. We now define the transformation $e^W = 1/(1 - \hat{r})$ so that it requires that $0 < \hat{r} < 1$. Applying this transformation and also using the linear transformation (4.26), one can show that the solutions (4.31)-(4.33) turn into the same form of the solution given by (4.27)-(4.29), that is, the solutions corresponding to both the valid ranges $\tilde{x}_2 + \frac{1}{2} e^{-2y} < 0$ and $\tilde{x}_2 - \frac{1}{2} e^{-2y} > 0$ can be expressed as the solution given by (4.27)-(4.29) only with $0 < \hat{r} < \infty$.

One can simply check that the solution (4.27)-(4.29) does satisfy the equations (3.5) with $\tilde{\Lambda} = -1/l^2$. By considering this solution for the whole space-time, one sees that the metric components (4.27) are ill behaved at $\hat{r} = 0$ and $\hat{r} = 1$. By looking at the scalar curvature, which is $\tilde{R} = 2(4\hat{r} - 7)/l^2 \hat{r}^2$, we find that $\hat{r} = 0$ is a curvature singularity. Note that the singularity at $\hat{r} = 0$ corresponds to the same true singularity at the region $\tilde{x}_2 = \frac{1}{2} e^{-2y}$ which mentioned above. We will see that $\hat{r} = 1$ is also an event horizon. The cross term appeared in the metric is constant and thus for large $\hat{r}$ the metric is asymptotically flat. However, the solution represents a stationary black string.

We wish to express the constant parameter $a$ and the radius $l$ in terms of the physical mass and axion charge per unit length of the black string. To calculate the mass we first need to have the asymptotic behavior of the solution given by (4.27)-(4.29). To this end, we set

$$\hat{r} = \frac{2l}{3} e^{2\rho r},$$  (4.34)

in equations (4.27), (4.28) and (4.29) and only drop the hat sign from on the $\hat{t}$ and $\hat{x}$ coordinates. Notice that it is not possible to similarly to the $t$ and $x$ coordinates fix the overall scaling of the $\hat{r}$ coordinate as $\hat{r}$ goes to infinity, since the metric asymptotically approaches $l^2 d\hat{r}^2/4\hat{r}^2$. Therefore, for large $r$ the black string solution (4.27)-(4.29) approaches the following asymptotic solution

$$d\tilde{s}_n^2 = -dt^2 + dx^2 + d\rho^2,$$

$$\tilde{\phi} = -\rho/l + \frac{1}{2} \ln \left( \frac{3}{2l} \right), \quad \tilde{H} = 0,$$  (4.35)

where we have set $r = e^{2\rho l}$. The solution (4.35) is a flat solution for equations (3.5). As it can be seen from (4.35), the radius $l$ is related to the derivative of the asymptotic value of the dilaton field. Now, in order to calculate the mass per unit length of the string we use the Arnowitt-Deser-Misner
(ADM) procedure which was been applied for obtaining the mass and charge of three-dimensional black strings \[6, 26\] (see, also, \[5\]). In this way, to calculate the mass one needs to perturb the metric as \(G_{\Upsilon \Lambda} = \eta_{\Upsilon \Lambda} + \gamma_{\Upsilon \Lambda}\). Then, by integrating the time-time component of the linearized form of the metric and dilaton contributions in equations \(3.5\) over a spacelike surface, the total mass is derived to be of the form \[6, 26\]:

\[
\mathcal{M}_{\text{tot}} = \frac{1}{2} \oint e^{-2\phi} \left( \partial^i \gamma_{ij} - \partial_i \gamma - 2 \gamma_{ij} \partial^i \phi \right) dS^i,
\]

where \(i, j\) run over spatial indices and \(\gamma\) is the trace of the spatial components of \(\gamma_{\Upsilon \Lambda}\), i.e., \(\gamma = \gamma^i_i\). Finally, in three dimensions, the axion charge per unit length associated with the field strength \(H\) is given by \[6, 26\]

\[
Q = \frac{1}{2} \oint e^{-2\phi} * H,
\]

where * denotes the Hodge dual. Thus, by considering the metric \(\eta_{\Upsilon \Lambda}\) corresponding to the line element of equation \(4.35\) and by using the specific form of \(\gamma_{\Upsilon \Lambda}\) for the solution \(4.27\) we carry out the above procedure. For the black string solution \(4.27\)-\(4.29\), the mass and axion charge per unit length measuring at the \(\rho = \infty\) end are therefore

\[
\begin{align*}
M &= \frac{3}{2l} e^{-2a}, \\
Q &= \frac{2}{\sqrt{3} l} e^{-2a}.
\end{align*}
\]

Clearly, \(M = \frac{3\sqrt{3}}{4} Q\) and thus \(M > Q\). Using these results, the final expression for the black string solution is obtained to be

\[
\begin{align*}
\tilde{d}s^2 &= -(1 - \frac{2M}{r}) dt^2 + (1 - \frac{2M}{3r}) dx^2 + \frac{2}{\sqrt{3}} dt dx + (1 - \frac{M}{r})^{-2} \frac{l^2 dr^2}{4r^2}, \\
\tilde{H}_{rtx} &= \frac{3Q}{2r^2}, \\
\tilde{\phi} &= -\frac{1}{2} \ln r + \frac{1}{2} \ln \left( \frac{3}{2l} \right).
\end{align*}
\]

The metric \(4.40\) possesses a single horizon at \(r = M\) and a curvature singularity at \(r = 0\), since the scalar curvature is \(\tilde{R} = |2M(4r - 7M)|/l^2 r^2\). The metric \(4.40\) appears to be somewhat analogous to the extremal limit \(|Q| = M\) of the charged black string \(5.20\) \[9\]. However there is one important difference which that is the appearance of the cross term in the metric. The metric \(4.40\) also possesses two independent Killing vectors \(\partial/\partial t\) and \(\partial/\partial t + \beta \partial/\partial x\) for \(\beta \neq 0\).

\begin{itemize}
  \item The Killing vector \(\partial/\partial t\) with the norm \(\tilde{G}_{tt} = -1 + 2M/r\) becomes null at \(r = 2M\), which lies outside the event horizon \(r = M\). It is a time translational at infinity and becomes timelike for \(r > 2M\), while becomes spacelike for \(M \leq r < 2M\) and also inside the event horizon.
  \item The Killing vector \(\partial/\partial t + \beta \partial/\partial x\) becomes timelike for the ranges \(r < [2M(\beta - \sqrt{3})]/[3(\beta - 1/\sqrt{3})]\).
\end{itemize}

\(^5\)Note that the exponential contribution of dilaton field in action \(3.3\) has been appeared as \(e^{-2\phi}\). Therefore, the formula of the total mass \(4.36\) differs from those in Ref. \[9\].
when $\beta \in (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$ which lies inside the event horizon, and $r > [2M(\beta - \sqrt{3})]/[3(\beta - 1/\sqrt{3})]$ when $\beta \in (-\sqrt{3}, 1/\sqrt{3}) - \{0\}$. It is also timelike at infinity when $\beta \in (-\sqrt{3}, 1/\sqrt{3}) - \{0\}$. For $\beta = -\sqrt{3}$ it becomes null, and for $\beta = \sqrt{3}$ it stays everywhere spacelike.

As mentioned above, our black string solution is asymptotically flat. Thus, the non-Abelian T-duality transformation changes the asymptotic behavior from $AdS_3$ to flat. One question that is in our mind is that although the black string metric (4.40) is asymptotically flat, its near horizon geometry is of the $AdS_3$ type, that is, in the limit $r \to M$, one gets $\tilde{R} \to -6/l^2$ and $\tilde{R}_{\tau\chi} \to -2/l^2 \tilde{G}_{\tau\chi}$. The study of near horizon behavior is important, because according to Strominger’s proposal [27] one asserts that the statistical entropy of any black hole whose near horizon geometry contains an $AdS_3$ factor can be calculated by using the statistical counting of microstates of the BTZ black hole. Lastly, we have to note the fact that the charged black string solution (4.40)-(4.42) with $l = 1$ is a dual solution for the $SL(2, \mathbb{R})$ WZW model.

5 Discussion and Conclusion

We have reviewed aspects of Poisson-Lie T-duality in the presence of spectator fields. We have reobtained Buscher’s duality transformations from the Poisson-Lie T-duality transformations in the presence of spectators. Then using this approach we have studied the Abelian T-dualization of the BTZ black hole solutions given by (3.2) in such a way that our solutions are in agreement with those of Ref. [4]. We have explicitly constructed a dual pair of sigma models related by Poisson-Lie symmetry so that the original model has built on a 2 + 1-dimensional manifold $\mathcal{M} \approx O \times G$ in which $G$ is a two-dimensional real non-Abelian Lie group that acts freely on $\mathcal{M}$ and $O$ is the orbit of $G$ in $\mathcal{M}$. The metric of the model depends on a non-zero real constant parameter $k$ so that it describes an anti-de Sitter space with $k > 0$ while for $k < 0$ we have a de Sitter space. Then we have shown that the original model indeed is canonically equivalent to the $SL(2, \mathbb{R})$ WZW model for a given value of the background parameters. Therefore, we have shown that the Poisson-Lie T-duality relates the $SL(2, \mathbb{R})$ WZW model to a sigma model defined on a 2+1-dimensional manifold $\mathcal{M}$. In addition, by a convenient coordinate transformation we have shown that the original model describes a string propagating in a spacetime with the BTZ black hole vacuum metric. In this way we have found a new family of the solutions to low energy string theory with the BTZ black hole vacuum metric, constant dilaton field and a new torsion potential. The dual model has built on a 2 + 1-dimensional target manifold $\tilde{\mathcal{M}}$ with two-dimensional real Abelian Lie group $\tilde{G}$ acting freely on it. We have shown that the dual model yields a new three-dimensional charged black string which is stationary and asymptotically flat. In this way, the non-Abelian T-duality transformation has related a solution with no horizon and no curvature singularity to a solution with a single horizon and a curvature singularity. In addition to, it changes the asymptotic behavior of solutions from $AdS_3$ to flat. According to our findings, it seems that under the non-Abelian T-duality transformation the mass and charge have not restored from the dual model to the original one. We have thus found a new family of solutions in the abstract is rather too strong. We have also investigated the effect of Poisson-Lie T-duality on the singularities of spaces of T-dual models and have shown that this duality takes those singular regions to regular regions as was the case with the 2D black holes [28] and 3D black strings [6], [26], [29].

Nevertheless, in order to address the question of the non-Abelian T-dualization of the BTZ black

16
hole solutions one has to show that the BTZ black hole metric has sufficient number of independent Killing vectors. Then the isometry subgroup of the metric can be taken as one of the subgroups of the Drinfeld double. In order to satisfy the dualizability conditions the other subgroup must be chosen Abelian. The isometry groups of metrics can be used for construction of non-Abelian T-dual backgrounds. In the present case, to construct the dualizable metrics by the Poisson-Lie T-duality one needs a three-dimensional subalgebra of Killing vectors that generates group of isometries which acts freely and transitively on the three-dimensional manifold $\mathcal{M}$ where the BTZ metric is defined. Thus one can construct several non-Abelian T-dual backgrounds for the BTZ metric. We intend to address this problem in the future.

Acknowledgments: We would like to thank the referee for comments that helped us improve this issue. A. Eghbali is especially grateful to S. Hoseinzadeh for sharing some of his insights with him. This work has been supported by the research vice chancellor of Azarbaijan Shahid Madani University of Iran under research fund No. 95.537.

References

[1] M. Banados, C. Teitelboim, and J. Zanelli, *Black hole in three-dimensional spacetime*, Phys. Rev. Lett. 69 (1992) 1849.

[2] S. Carlip, *Conformal field theory, (2 + 1)-dimensional gravity and the BTZ black hole*, Class. Quantum Gravity 22 (2005) R85-R123.

[3] E. Witten, *Three-dimensional gravity revisited*, arXiv:0706.3359 [hep-th].

[4] G. Horowitz and D. Welch, *String theory formulation of the three-dimensional black hole*, Phys. Rev. Lett. 71 (1993) 328.

[5] E. Witten, *On string theory and black holes*, Phys. Rev. D 44 (1991) 314.

[6] J. Horne and G. Horowitz, *Exact black string solutions in three dimensions*, Nucl. Phys. B 368 (1992) 444, arXiv:hep-th/9108001.

[7] C. Klimcik and P. Severa, *Dual non-Abelian duality and the Drinfeld double*, Phys. Lett. B 351 (1995) 455, arXiv:hep-th/9502122. C. Klimcik, *Poisson-Lie T-duality*, Nucl. Phys. (Proc. Suppl.) B 46 (1996) 116, arXiv:hep-th/9509095.

[8] E. Tyurin and R. von Unge, *Poisson-Lie T-duality: the path-integral derivation*, Phys. Lett. B 382 (1996) 233, arXiv:hep-th/9512025.

[9] K. Sfetsos, *Poisson-Lie T-duality and supersymmetry*, Nucl. Phys. (Proc. Suppl.) 56 B (1997) 302, arXiv:hep-th/9611199.

[10] T. Buscher, *A symmetry of the string background field equations*, Phys. Lett. B 194 (1987) 59; T. Buscher, *Path-integral derivation of quantum duality in nonlinear sigma-models*, Phys. Lett. B 201 (1988) 466.
[11] M. Rocek and E. Verlinde, *Duality, quotients, and currents*, Nucl. Phys. B 373 (1992) 630; X. C. de la Ossa and F. Quevedo, *Duality symmetries from non-abelian isometries in string theory*, Nucl. Phys. B 403 (1993) 377; A. Giveon and M. Rocek, *On nonabelian duality*, Nucl. Phys. B 421 (1994) 173; A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*, Phys. Rep. 244 (1994) 77; S. Elitzur, A. Giveon, E. Rabinovici, A. Schwimmer and G. Veneziano, *Remarks on non-Abelian duality*, Nucl. Phys. B 435, (1995) 147. E. Alvarez, L. Alvarez-Gaume and Y. Lozano, *An introduction to T-duality in string theory*, Nucl. Phys. (Proc. Suppl.) B 41 (1995) 1;

[12] K. Sfetsos, *Canonical equivalence of non-isometric sigma-models and Poisson-Lie T-duality*, Nucl. Phys. B 517 (1998) 549, arXiv:hep-th/9710163.

[13] A. Eghbali and A. Rezaei-Aghdam, *Poisson-Lie T-dual sigma models on supermanifolds*, J. High Energy Phys. 09 (2009) 094, arXiv:0901.1592 [hep-th].

[14] A. Eghbali and A. Rezaei-Aghdam, *String cosmology from Poisson-Lie T-dual sigma models on supermanifolds*, J. High Energy Phys. 01 (2012) 151, arXiv:1107.2041 [hep-th].

[15] A. Alekseev, C. Klimčík and A. Tseytlin, *Quantum Poisson-Lie T-duality and WZNW model*, Nucl. Phys. B 458 (1996) 430, arXiv:hep-th/9509123.

[16] A. Eghbali and A. Rezaei-Aghdam, *Poisson-Lie symmetry and D-branes in WZW model on the Heisenberg Lie group H_4*, Nucl. Phys. B 899 (2015) 165, arXiv:1506.06233 [hep-th].

[17] M. A. Lledo and V. S. Varadarajan, *SU(2) Poisson-Lie T duality*, Lett. Math. Phys. 45 (1998) 247, arXiv:hep-th/9803175.

[18] A. Eghbali and A. Rezaei-Aghdam, *Super Poisson-Lie symmetry of the GL(1|1) WZNW model and worldsheet boundary conditions*, Nucl. Phys. B 866 (2013) 26, arXiv:1207.2304 [hep-th].

[19] A. Eghbali and A. Rezaei-Aghdam, *WZW models as mutual super Poisson-Lie T-dual sigma models*, J. High Energy Phys. 07 (2013) 134, arXiv:1303.4069 [hep-th].

[20] K. Sfetsos, *Poisson-Lie T-duality beyond the classical level and the renormalization group*, Phys. Lett. B 432 (1998), arXiv:hep-th/9803019. K. Sfetsos, *Duality-invariant class of two-dimensional field theories*, Nucl. Phys. B 561 (1999) 316, arXiv:hep-th/9904188.

[21] C. G. Callan, D. Friedan, E. Martinec and M. J. Perry, *Strings in background fields*, Nucl. Phys. B 262 (1985) 593.

[22] T. Buscher, *Quantum corrections and extended supersymmetry in new \( \sigma \)-models*, Phys. Lett. B 159 (1985) 127.

[23] A. Bossard and N. Mohammedi, *Poisson-Lie duality in the string effective action*, Nucl. Phys. B 619 (2001) 128, arXiv:hep-th/0106211

[24] L. Hlavaty and L. Snobl, *Classification of Poisson-Lie T-dual models with two-dimensional targets*, Mod. Phys. Lett. A 17 (2002) 429-434, arXiv:hep-th/0110139.
[25] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, \textit{Geometry of the }$2+1$\textit{ black hole}, Phys. Rev. D 48 (1993) 1506.

[26] J. Horne, G. Horowitz and A. Steif, \textit{An equivalence between momentum and charge in string theory}, Phys. Rev. Lett. 68 (1992) 568, arXiv:hep-th/9110065.

[27] A. Strominger, \textit{Black hole entropy from near-horizon microstates}, J. High Energy Phys. 02 (1998) 009, arXiv:hep-th/9712251.

[28] R. Dijkgraaf, E. Verlinde and H. Verlinde, \textit{String propagation in black hole geometry}, Nucl. Phys. B 371 (1992) 269.

[29] P. Ginsparg and F. Quevedo, \textit{Strings on curved space-times: black holes, torsion, and duality}, Nucl. Phys. B 385 (1992) 527.