ZARISKI MULTIPLES ASSOCIATED WITH QUARTIC CURVES

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Abstract. We investigate Zariski multiples of plane curves $Z_1, \ldots, Z_N$ such that each $Z_i$ is a union of a smooth quartic curve, some of its bitangents, and some of its 4-tangent conics. We show that, for plane curves of this type, the deformation types are equal to the homeomorphism types, and that the number of deformation types grows as $O(d^{62})$ when the degree $d$ of the plane curves tends to infinity.

1. Introduction

By a plane curve, we mean a reduced, possibly reducible, complex projective plane curve. We say that two plane curves $C$ and $C'$ have the same combinatorial type if there exist tubular neighbourhoods $T(C)$ of $C$ and $T(C')$ of $C'$ such that $(T(C), C)$ and $(T(C'), C')$ are homeomorphic, whereas we say that $C$ and $C'$ have the same homeomorphism type if $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are homeomorphic.

A Zariski $N$-ple is a set of plane curves $\{C_1, \ldots, C_N\}$ such that the curves $C_i$ have the same combinatorial type, but their homeomorphism types are pairwise different. The notion of Zariski $N$-ples was introduced by Artal-Bartolo [1] in reviving a classical example of 6-cuspidal curves of degree 6 due to Zariski. Since then, this notion has been studied by many people from various points of view. Some of the tools that have been used in this investigation are: Alexander polynomials, characteristic varieties, fundamental groups of complements, topological invariants of branched coverings, and so on. See the survey paper [3]. Recently, Bannai et al. [4, 5, 6] have investigated Zariski $N$-ples such that each member is a union of a smooth quartic curve and some of its bitangents.

In this paper, we introduce 4-tangent conics of a smooth quartic curve, and consider Zariski $N$-ples $Z_1, \ldots, Z_N$ such that each $Z_i$ is a union of a smooth quartic curve, some of its bitangents, and some of its 4-tangent conics.

Let $Q$ be a smooth quartic curve. A bitangent of $Q$ is a line whose intersection multiplicity with $Q$ is even at each intersection point. It is well known that every smooth quartic curve has exactly 28 bitangents. We say that a bitangent $\tilde{l}$ of $Q$ is ordinary if $\tilde{l}$ is tangent to $Q$ at distinct 2 points. A smooth conic $\tilde{c} \subset \mathbb{P}^2$ is called a 4-tangent conic of $Q$ if $\tilde{c}$ is tangent to $Q$ at 4 distinct points. Every smooth quartic curve has 63 one-dimensional connected families of 4-tangent conics (see Theorem 4.1).

Definition 1.1. Let $m$ and $n$ be non-negative integers such that $m \leq 28$. We say that a plane curve $Z$ is a $Q^{(m,n)}$-curve if $Z$ is of the form

$$Z = Q + \tilde{l}_1 + \cdots + \tilde{l}_m + \tilde{c}_1 + \cdots + \tilde{c}_n,$$

2020 Mathematics Subject Classification. 14H50, 14J26.
This work was supported by JSPS KAKENHI Grant Number 20H01798 and 20K20879.
where $Q$ is a smooth quartic curve, $\bar{l}_1, \ldots, \bar{l}_m$ are distinct bitangents of $Q$, and $\bar{c}_1, \ldots, \bar{c}_n$ are distinct 4-tangent conics of $Q$, and they satisfy that 

(i) the bitangents $\bar{l}_1, \ldots, \bar{l}_m$ are ordinary, 
(ii) the intersection of any three of $Q, \bar{l}_1, \ldots, \bar{l}_m, \bar{c}_1, \ldots, \bar{c}_n$ is empty, and 
(iii) the intersection of any two of $\bar{l}_1, \ldots, \bar{l}_m, \bar{c}_1, \ldots, \bar{c}_n$ is transverse.

It is obvious that any two $Q^{(m,n)}$-curves have the same combinatorial type. We construct a non-singular variety $Z^{(m,n)}$ parameterizing all $Q^{(m,n)}$-curves in Section 5.

**Definition 1.2.** We say that two $Q^{(m,n)}$-curves have the same deformation type if they belong to the same connected component of the parameter space $Z^{(m,n)}$.

It is obvious that $Q^{(m,n)}$-curves of the same deformation type have the same homeomorphism type. Our main results are the following:

**Theorem 1.3.** If two $Q^{(m,n)}$-curves have the same homeomorphism type, then they have the same deformation type.

We put

(1.2) \[ d^{(m,n)} := \binom{28}{m} \binom{n+62}{62}, \]

which grows as $O(n^{62})$ when $n \to \infty$.

**Theorem 1.4.** The number $N^{(m,n)}$ of deformation types of $Q^{(m,n)}$-curves satisfies

(1.3) \[ d^{(m,n)}/1451520 \leq N^{(m,n)} \leq d^{(m,n)}. \]

The main ingredient of the proof of these results is the monodromy argument of Harris [9] (see Theorem 3.1). This argument converts the problem of enumerating deformation types of $Q^{(m,n)}$-curves to an easy combinatorial problem of counting orbits of an action of the Weyl group $W(E_7)$ on a certain finite set. In Tables 1.1, 1.2, 1.3, we give a list of $N^{(m,n)}$ for some $(m, n)$. See Section 5.2 for more detail.

To each $Q^{(m,n)}$-curve $Z$, we associate a discrete invariant $g(Z)$, which we call an intersection graph. This invariant is similar to the splitting graph defined in [19].

Note that each of the bitangents $\bar{l}_1, \ldots, \bar{l}_m$ and a 4-tangent conics $\bar{c}_1, \ldots, \bar{c}_n$ of the smooth quartic curve $Q \subset Z$ splits by the double covering $Y \to \mathbb{P}^2$ branched along $Q$. This data $g(Z)$ describes how the irreducible components of these pull-backs intersect on $Y$. See Section 8 for the precise definition. In Tables, we also present the number $G$ of non-isomorphic intersection graphs obtained from $Q^{(m,n)}$-curves. When $n = 0$, the intersection graph $g(Z)$ is the two-graph studied in [5], in which Bannai and Ohno studied $Q^{(m,0)}$-curves for $m \leq 6$, and enumerated their homeomorphism types that can be distinguished by the two-graphs. See Sections 9.1 and 9.2 for the details.

The 4-tangent conics of a smooth quartic curve $Q$ are related to the 2-torsion points of the Jacobian of $Q$ (see Remark 4.3). A similar idea applied to plane cubic curves enabled us to construct in [15] certain equisingular families of plane curves with many connected components. In [15], it was also shown that these connected components cannot be distinguished by the fundamental groups of the complements, because they are all abelian. Then it was shown in [8] and [18] that
the homeomorphism types of distinct connected components of these families can be distinguished by the invariant called *linking numbers*.

The embedding topology of reducible plane curves whose irreducible components are tangent to each other was also investigated by Artal Bartolo, Cogolludo, and Tokunaga in [2] from the view point of dihedral covering of the plane branched along the curve. In this case, the complement can have a non-abelian fundamental group. See [2, Corollary 1].

It would be an interesting problem to study the fundamental groups of the complements of \(\mathcal{Q}(m,n)\)-curves, and their related invariants such as linking numbers and/or (non-)existence of finite coverings of the plane with prescribed Galois groups.

Via the cyclic covering of the plane of degree 4 branched along a smooth quartic curve, the geometry of \(\mathcal{Q}(m,n)\)-curves is related to the geometry of \(K3\) surfaces. By considering the double covering branched along a singular sextic curve and employing Torelli theorem for complex \(K3\) surfaces, we have investigated in [16] Zariski \(N\)-ples of plane curves of degree 6 with only simple singularities. We expect that a similar idea can be applied to Zariski \(N\)-ples associated with singular quartic curves.

This paper is organized as follows. In Section 2, we introduce cyclic coverings \(X_u \to Y_u \to \mathbb{P}^2\) branched over a smooth quartic curve \(Q_u\), and fix some notation. In Section 3, we recall the result of Harris [9]. In Section 4, we construct the family of 4-tangent conics. In Section 5, we construct the space \(\mathcal{Z}(m,n)\) parameterizing all \(\mathcal{Q}(m,n)\)-curves, and prove Theorems 1.3 and 1.4. In Section 6, we further study the family of 4-tangent conics in detail. The geometry of the \(K3\) surface \(X_u\) is closely investigated. In Section 7, we study the configurations of lifts in \(Y_u\) of bitangents.
and 4-tangent conics, and we define the intersection graph \( g(Z) \) in Section 8. In Section 9, we examine some examples for small \( m \) and \( n \).

**Acknowledgement.** Thanks are due to Professor Shinzo Bannai and Professor Taketo Shirane for discussions and comments. The author also thanks the anonymous referee for his/her valuable comments on the first version of this paper.

2. Coverings of \( \mathbb{P}^2 \)

For a positive integer \( d \), we put

\[ \Gamma(d) := H^0(\mathbb{P}^2, \mathcal{O}(d)). \]

Let \( \mathcal{U} \) denote the space of smooth quartic curves, which is a Zariski open subset of \( \mathbb{P}(\Gamma(4)) \). Let \( u \) be a point of \( \mathcal{U} \). We denote by \( Q_u \subset \mathbb{P}^2 \) the smooth quartic curve corresponding to the point \( u \). We consider the following branched coverings:

\[ \gamma_u : X_u \xrightarrow{\eta_u} Y_u \xrightarrow{\pi_u} \mathbb{P}^2, \]

where \( \pi_u \) is the double covering of \( \mathbb{P}^2 \) branched along \( Q_u \), \( \eta_u \) is the double covering of \( Y_u \) branched along \( \pi_u^{-1}(Q_u) \), and \( \gamma_u = \pi_u \circ \eta_u \) is the cyclic covering of degree 4 of \( \mathbb{P}^2 \) branched along \( Q_u \). We put

\[ SY_u := H^2(Y_u, \mathbb{Z}), \]

which is a unimodular lattice of rank 8 with the cup-product \( \langle , \rangle \). Let \( h_u \in SY_u \) be the class of the pull-back of a line on \( \mathbb{P}^2 \) by \( \pi_u \). It is well known that \( Y_u \) is a del Pezzo surface of degree 2 with the anti-canonical class \( h_u \). (See [7, Chapters 6 and 8] about del Pezzo surfaces.) On the other hand, the surface \( X_u \) is a K3 surface. Let \( \langle , \rangle_X \) denote the cup product of \( H^2(X_u, \mathbb{Z}) \), and let \( \bar{h}_u \) be the class \( \eta_u^*(h_u) \). Then \( \bar{h}_u \) is an ample class of degree \( \langle \bar{h}_u, \bar{h}_u \rangle_X = 4 \).

It is classically known that every smooth quartic curve \( Q_u \) has exactly 28 bitangents. Moreover, if \( u \) is general in \( \mathcal{U} \), all bitangents \( l \) of \( Q_u \) are ordinary, that is, \( l \) is tangent to \( Q_u \) at distinct two points.

**Definition 2.1.** A reduced conic \( \tilde{c} \subset \mathbb{P}^2 \) is called a splitting conic of \( Q_u \) if the intersection multiplicity of \( Q_u \) and \( \tilde{c} \) is even at each intersection point.

A smooth conic \( \tilde{c} \) is splitting if and only if \( \pi_u^{-1}(\tilde{c}) \subset Y_u \) has two irreducible components. A singular reduced conic \( \tilde{c} \) is splitting if and only if \( \tilde{c} \) is a union of two distinct bitangents.

It is easy to see that a smooth conic \( \tilde{c} = \{g = 0\} \) defined by \( g \in \Gamma(2) \) is a splitting conic of \( Q_u = \{\varphi = 0\} \) defined by \( \varphi \in \Gamma(4) \) if and only if there exist polynomials \( f \in \Gamma(2) \) and \( q \in \Gamma(2) \) such that \( \varphi = fg + q^2 \). By an easy dimension counting, we see the following:

**Lemma 2.2.** Suppose that \( u \) is general in \( \mathcal{U} \). Let \( \tilde{c} \subset \mathbb{P}^2 \) be a smooth splitting conic of \( Q_u \). Then the intersection multiplicities of \( Q_u \) and \( \tilde{c} \) are either \((2, 2, 2, 2)\) or \((4, 2, 2)\). \( \square \)

**Definition 2.3.** A smooth splitting conic \( \tilde{c} \subset \mathbb{P}^2 \) of \( Q_u \) is called a \(4\)-tangent conic (resp. a \(3\)-tangent conic) of \( Q_u \) if the intersection multiplicities of \( Q_u \) and \( \tilde{c} \) are \((2, 2, 2, 2)\) (resp. \((4, 2, 2)\)).

The following is easy to verify. The results are summarized in Table 2.1.
Table 2.1. Pull-backs of bitangents and 4-tangent conics

Proposition 2.4. (1) Let \( \bar{l} \) be an ordinary bitangent of \( Q_u \). Then \( \pi_u^*(\bar{l}) \) is a union of two smooth rational curves \( l \) and \( l' \) on \( Y_u \) with self-intersection \(-1\) that intersect at two points transversely, and \( \gamma_u^*(\bar{l}) \) is a union of two smooth rational curves \( \bar{l} \) and \( \bar{l}' \) on \( X_u \) with self-intersection \(-2\) that intersect at two points with intersection multiplicities \((2,2)\).

(2) Let \( \bar{c} \) be a 4-tangent conic of \( Q_u \). Then \( \pi_u^*(\bar{c}) \) is a union of two smooth rational curves \( c \) and \( c' \) on \( Y_u \) with self-intersection \(0\) that intersect at four points transversely, and \( \gamma_u^*(\bar{c}) \) is a union of two smooth elliptic curves \( \bar{c} \) and \( \bar{c}' \) on \( X_u \) with self-intersection \(0\) that intersect at four points with intersection multiplicities \((2,2,2,2)\).

Definition 2.5. A curve \( l \) on \( Y_u \) is called a \( Y \)-lift of a bitangent \( \bar{l} \) of \( Q_u \) if \( \pi_u \) maps \( l \) to \( \bar{l} \) isomorphically. We also say that a curve \( c \) on \( Y_u \) is a \( Y \)-lift of a splitting conic \( \bar{c} \) of \( Q_u \) if \( \pi_u \) maps \( c \) to \( \bar{c} \) isomorphically.

3. Monodromy

Let \( u \) be a point of \( \mathcal{U} \). It is well known that the lattice \( SY_u = H^2(Y_u, \mathbb{Z}) \) is isomorphic to the lattice of rank 8 whose Gram matrix is the diagonal matrix \( \text{diag}(1, -1, \ldots, -1) \), and that the orthogonal complement

\[ \Sigma_u := (\mathbb{Z}h_u \hookrightarrow SY_u)^\perp \]

of the ample class \( h_u \) in \( SY_u \) is isomorphic to the negative-definite root lattice of type \( E_7 \). The deck transformation

\[ \iota_u : Y_u \to Y_u \]

of \( \pi_u : Y_u \to \mathbb{P}^2 \) acts on \( \Sigma_u \) as \(-1\). Note that the group \( O(\Sigma_u) \) of isometries of \( \Sigma_u \) is equal to the Weyl group \( W(E_7) \), which is of order 2903040. Hence there exists an injective homomorphism

\[ O(SY_u, h_u) := \{ g \in O(SY_u) \mid h_u^g = h_u \} \hookrightarrow W(E_7). \]

It is easy to check that the action on \( \Sigma_u \) of each of the standard generators of \( W(E_7) \) lifts to an isometry of \( SY_u \) that fixes \( h_u \). Hence the homomorphism (3.1)
is in fact an isomorphism. The family of lattices \( \{ SY_u \mid u \in U \} \) forms a locally constant system

\[ SY \to U. \]

Let \( b \) be a general point of \( U \), which will serve as a base point of \( U \). The monodromy action of \( \pi_1(U, b) \) on the lattice \( SY_b \) preserves \( h_b \in SY_b \).

**Theorem 3.1** (Harris [9]). The monodromy homomorphism

\[ \pi_1(U, b) \to O(SY_b, h_b) \cong W(E_7) \]

is surjective.

The original statement in [9] is not on the monodromy action on the lattice \( SY_b \), but on the Galois group \( W(E_7) / \{ \pm 1 \} \cong \text{GO}_6(F_2) \) of bitangents of \( Q_b \). Moreover the proof in [9] is via the proof of a similar result on cubic surfaces with \( E_7 \) replaced by \( E_6 \). Hence we give a direct and simple proof of Theorem 3.1 below.

For the proof, we prepare some more notation, which will be used throughout this paper. We denote by \( L_u \) the set of bitangents of \( Q_u \), and \( L_u \) the set of \( Y \)-lifts of bitangents of \( Q_u \). Let \( \Sigma^\vee_u \) denote the dual lattice of \( \Sigma_u \). By identifying \( l \in L_u \) with its class \( [l] \in SY_u \), we have an identification

\[ L_u \cong \{ v \in SY_u \mid \langle v, h_u \rangle_Y = 1, \langle v, v \rangle_Y = -1 \} \]

where the second bijection is obtained by the orthogonal projection \( SY_u \to \Sigma^\vee_u \).

We put \( SY_u := SY_u / \langle \iota_u \rangle \), and consider the commutative diagram

\[ \begin{array}{ccc}
L_u & \to & SY_u \\
\downarrow & & \downarrow \\
\overline{L}_u & \to & \overline{SY}_u,
\end{array} \]

where vertical arrows are quotient maps by the involution \( \iota_u \). Since the action of \( \pi_1(U, b) \) on \( SY_b \) commutes with \( \iota_b \), we have a monodromy action of \( \pi_1(U, b) \) on \( \overline{SY_b} \).

Thus we obtain a diagram

\[ \begin{array}{ccc}
\mathcal{L} & \to & SY \\
\downarrow & & \downarrow \\
\overline{\mathcal{L}} & \to & \overline{SY}
\end{array} \]

of locally constant systems over \( U \) parameterizing the diagram (3.4) over \( U \), where vertical arrows are quotient maps by the family of involutions

\[ \iota_U := \{ \iota_u \mid u \in U \}. \]

Note that \( \overline{\mathcal{L}} \) is the space parameterizing all bitangents of smooth quartic curves.

**Proof of Theorem 3.1.** Let \( L_u^{[7]} \) (resp. \( L_u^{[7]} \)) be the set of non-ordered 7-tuples \( \{ l_1, \ldots, l_7 \} \) (resp. ordered 7-tuples \( [l_1, \ldots, l_7] \)) of elements \( l_1, \ldots, l_7 \in L_u \) such that \( \langle l_i, l_j \rangle_Y = 0 \) for \( i \neq j \). By (3.3), we can enumerate all elements of \( L_u^{[7]} \). It turns out that \( |L_u^{[7]}| = 576 \), and hence

\[ |L_u^{[7]}| = 576 \cdot 7! = 2903040 = |W(E_7)|. \]
(See also Remark 6.6.) For 7-tuples $\lambda = [l_1, \ldots, l_7]$ and $\lambda' = [l'_1, \ldots, l'_7]$ in $L_u^{[7]}$, there exists a unique isometry $g_{\lambda, \lambda'} \in O(SY_u \otimes \mathbb{Q})$ such that

$$g_{\lambda, \lambda'}(h_u) = h_u, \quad g_{\lambda, \lambda'}(l_i) = l'_i \quad (i = 1, \ldots, 7).$$

It is enough show that, when $u = b$, these elements $g_{\lambda, \lambda'}$ are contained in the image of the monodromy (3.2). Indeed, by (3.6), this claim implies that these isometries $g_{\lambda, \lambda'}$ constitute the whole group $O(SY_u, h_b) \cong W(E_7)$. To prove this claim, it is enough to show that $\pi_1(\mathcal{U}, b)$ acts on $L_u^{[7]}$ transitively by the monodromy, or equivalently, to show that the total space $\mathcal{L}^{[7]}$ of the locally constant system

$$\mathcal{L}^{[7]} \to \mathcal{U}$$

obtained from the family $\{L_u^{[7]} | u \in \mathcal{U}\}$ is connected.

Let $\lambda = [l_1, \ldots, l_7]$ be a point of $\mathcal{L}^{[7]}$ over $u \in \mathcal{U}$. Contracting the $(-1)$-curves $l_1, \ldots, l_7$, we obtain a birational morphism

$$bl_\lambda : Y_u \to P_\lambda$$

to a projective plane $P_\lambda$. We put $\beta_\lambda := [bl_\lambda(l_1), \ldots, bl_\lambda(l_7)]$. Conversely, we fix a projective plane $P$, and let $P^{[7]}$ denote the set of ordered 7-tuples $[p_1, \ldots, p_7]$ of distinct points of $P$. For a general point $\beta = [p_1, \ldots, p_7]$ of $P^{[7]}$, let

$$bl^\beta : Y^\beta \to P$$

be the blowing-up at the points $p_1, \ldots, p_7$. Then $Y^\beta$ is a del Pezzo surface of degree 2, and the complete linear system of the anti-canonical divisor on $Y^\beta$ gives a double covering $Y^\beta \to \mathbb{P}^2$ branched along a smooth quartic curve $Q^\beta$ such that each of the 7 exceptional curves over $p_1, \ldots, p_7$ is a $Y$-lift of a bitangent of $Q^\beta$. Hence there exist a point $\lambda \in \mathcal{L}^{[7]}$ and an isomorphism $P_\lambda \cong P$ that maps $\beta_\lambda$ to $\beta$.

We put

$$\mathcal{I} := \left\{ (\lambda, \gamma, \beta) \left| \begin{array} {l} \lambda \in \mathcal{L}^{[7]}, \beta \in P^{[7]}, \text{ and } \gamma \text{ is an isomorphism} \\ P_\lambda \cong P \text{ that maps } \beta_\lambda \text{ to } \beta \end{array} \right. \right\}.$$  

Then the projection $\mathcal{I} \to \mathcal{L}^{[7]}$ is surjective with fibers isomorphic to $PGL_3(\mathbb{C})$, whereas the projection $\mathcal{I} \to P^{[7]}$ is dominant with fibers isomorphic to $PGL_3(\mathbb{C})$. Since $P^{[7]}$ and $PGL_3(\mathbb{C})$ are connected, we see that $\mathcal{L}^{[7]}$ is connected. \hfill $\square$

4. FAMILY OF 4-TANGENT CONICS

In this section, we construct a space $\overline{\mathcal{C}}_u$ parameterizing all 4-tangent conics of smooth quartic curves.

Let $\overline{\mathcal{C}}_u$ denote the set of 4-tangent conics of $Q_u$, and let $C_u$ be the set of $Y$-lifts of 4-tangent conics of $Q_u$. We put

$$F_u := \{ v \in SY_u \left| \langle v, h_u \rangle_Y = 2, \langle v, v \rangle_Y = 0 \right\} \cong \{ v \in \Sigma_u \left| \langle v, v \rangle_Y = -2 \right\},$$

where the second bijection is given by the orthogonal projection $SY_u \to \Sigma_u$. As was shown in Table 2.1, we have $\{c\} \in F_u$ for any $c \in C_u$. Note that $|F_u| = 126$, the number of roots of the root lattice $\Sigma_u$ of type $E_7$. We put

$$\overline{F}_u := F_u / \langle t_u \rangle \subset SY_u.$$
Then we have a commutative diagram

\[
\begin{array}{ccc}
C_u & \xrightarrow{\Phi_u} & F_u \\
\downarrow & & \downarrow \\
\overline{C}_u & \xrightarrow{\overline{\Phi}_u} & \overline{F}_u \\
\end{array}
\]

(4.2)

where \(\Phi_u: C_u \to F_u\) is given by \(c \mapsto [c] \in SY_u\), and the vertical arrows are quotient by the involution \(\iota_u: Y_u \to Y_u\). We have locally constant systems \(F \to U\) and \(\overline{F} \to \overline{U}\) obtained from the families \(\{F_u \mid u \in U\}\) and \(\{\overline{F}_u \mid u \in U\}\).

**Theorem 4.1.** There exists a commutative diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\Phi_U} & F \\
\downarrow & & \downarrow \\
\overline{C} & \xrightarrow{\overline{\Phi}_U} & \overline{F} \\
\end{array}
\]

(4.3)

of morphisms over \(U\) that parameterizes the diagrams (4.2) over \(U\). The morphisms \(\Phi_U: C \to F\) and \(\overline{\Phi}_U: \overline{C} \to \overline{F}\) are smooth and surjective, and every fiber of them is a Zariski open subset of \(\mathbb{P}^1\).

For the proof, we use the double covering \(\eta_u: X_u \to Y_u\) of \(Y_u\) by the \(K3\) surface \(X_u\). We consider the Néron-Severi lattice

\[SX_u := H^2(X_u, \mathbb{Z}) \cap H^{1,1}(X_u)\]

with the intersection form \(\langle , \rangle_X\). Then \(\eta_u\) induces a embedding of the lattice

\[\eta_u^*: SY_u(2) \hookrightarrow SX_u,\]

where \(SY_u(2)\) is the lattice obtained from \(SY_u\) by multiplying the intersection form by 2. Let \(j_u: X_u \to X_u\) be a generator of the cyclic group \(\text{Gal}(X_u/\mathbb{P}^2)\) of order 4. Then \(\eta_u: X_u \to Y_u\) is the quotient morphism by \(j_u^2\). Hence \(j_u^2\) acts on the image of \(\eta_u^*: SY_u(2) \hookrightarrow SX_u\) trivially.

**Proof of Theorem 4.1.** Note that the family of involutions \(\iota_U = \{\iota_u \mid u \in U\}\) acts on \(F\) over \(U\) without fixed points. Hence, if the parameterizing space \(\Phi_U: C \to F\) of \(\Phi_u: C_u \to F_u\) is constructed, then \(\overline{\Phi}_U: \overline{C} \to \overline{F}\) is constructed as a quotient of \(\Phi_U: C \to F\) by \(\iota_U\).

Let \(u\) be an arbitrary point of \(U\), and let \(v\) be an element of \(F_u \subset SY_u\). We put \(\tilde{v} := \eta_u^*(v) \in SX_u\). We can easily confirm that there exist exactly 6 pairs \(\{l_i, l'_i\}\) \((i = 1, \ldots, 6)\) of \(Y\)-lifts of bitangents of \(Q_u\) such that \(\langle l_i, l'_i \rangle_Y = 1\) and \(v = [l_i] + [l'_i]\), and that these 12 curves \(l_1, l'_1, \ldots, l_6, l'_6\) are distinct. Hence the complete linear system on the \(K3\) surface \(X_u\) corresponding to \(\tilde{v} \in SX_u\) has no fixed components. The class \(\tilde{v}\) is primitive in \(SX_u\) with \(\langle \tilde{v}, \tilde{v} \rangle_X = 0\) and \(\langle \tilde{h}_u, \tilde{v} \rangle_X = 4\). Therefore there exists an elliptic fibration on \(X_u\) such that the class of a fiber is equal to \(\tilde{v}\). We denote this elliptic fibration by \(\phi_v: X_u \to \mathbb{P}^1\).

If \(c \in C_u\), then \(\eta_u^*(c)\) is an elliptic curve, and hence \(\eta_u^*(c)\) is a smooth fiber of an elliptic fibration of \(\phi_v: X_u \to \mathbb{P}^1\), where \(v = [c]\). Conversely, suppose that \((u, v) \in F\), and let \(f\) be a smooth fiber of the elliptic fibration \(\phi_v: X_u \to \mathbb{P}^1\). We denote by \(a \subset \mathbb{P}^2\) the plane curve \(\gamma_u(f)\) with the reduced structure. Let \(d\) be the degree of \(a\), and \(\delta\) the mapping degree of \(\gamma_u|f: f \to a\). Since \(\langle \tilde{h}_u, f \rangle_X = 4\) and \(\gamma_u: X_u \to \mathbb{P}^2\) is Galois, we have \((d, \delta) = (1, 4), (2, 2),\) or \((4, 1)\). If \((d, \delta) = (1, 4)\), then \(f = \gamma_u^{-1}(a)\) is invariant under the action of \(\text{Gal}(X_u/\mathbb{P}^2) = \langle j_u \rangle\), and hence the class \([f] \in SX_u\) is a non-zero multiple of \(\tilde{h}_u\), which contradicts \(\langle f, f \rangle_X = 0\).
If \((d, \delta) = (4, 1)\), then \(f, j_u(f), j_2^2(f), j_3^3(f)\) are distinct curves that intersect over the points of \(a \cap Q_u\). On the other hand, since \([f] = \eta^*_u(v) \in \Im \eta^*_u\), we have \(j_2^2([f]) = [f]\). This contradicts \((f, f)_X = 0\). Hence \((d, \delta) = (2, 2)\), and we see that \(a\) is a smooth splitting conic. Note that \(a\) is 4-tangent, because otherwise \(f\) would be singular. Thus we have proved that \(c \mapsto \eta^*_u(c)\) gives a bijection from \(C_u\) to the union of the sets of smooth fibers of elliptic fibrations \(\phi_v : X_u \to \mathbb{P}^1\), where \(v\) runs through \(F_u\).

Let \(X \to U\) be the universal family of \(\{ X_u \mid u \in U \}\), and let \(\pi_F : F \times_U X \to F\) be the pull-back of \(X \to U\) by \(F \to U\). Let \(\mathcal{M}\) be a line bundle on \(F \times_U X\) such that the class \([\mathcal{M}]_{|X_u} \in S\pi_u\) of the line bundle \(\mathcal{M}|_{X_u}\) on \(\pi_F^{-1}(u, v) = X_u\) is equal to \(v \in F_u\). Then \(\pi_{F*,} \mathcal{M} \to F\) is a vector bundle of rank \(2\). The fiber over \((u, v) \in F\) of the \(\mathbb{P}^1\)-bundle \(\mathbb{P}(\pi_{F*,} \mathcal{M}) \to F\) is identified with the base curve of the elliptic fibration \(\phi_v : X_u \to \mathbb{P}^1\). We can construct \(C\) as the open subset of \(\mathbb{P}(\pi_{F*,} \mathcal{M})\) consisting of non-critical points of \(\phi_v : X_u \to \mathbb{P}^1\). □

The non-singular varieties \(C\) and \(\overline{C}\) parameterize all pairs \((u, c)\) and \((u, \overline{c})\), respectively, where \(u \in U\) and \(c \in C_u\), \(\overline{c} \in \overline{C_u}\). Since \(\Phi_u\) and \(\overline{\Phi}_u\) have connected fibers, we can regard \(F_u\) as the set of connected families of \(Y\)-lifts of 4-tangent conics of \(Q_u\), and \(\overline{F_u}\) as the set of connected families of 4-tangent conics. The following observation obtained in the proof of Theorem 4.1 will be used in the next section.

**Proposition 4.2.** Every connected family \([c] \in F_u\) of \(Y\)-lifts of splitting conics is a pencil with no base points. □

**Remark 4.3.** A line section \(\Lambda_u\) of \(Q_u \subset \mathbb{P}^2\) is a canonical class of the genus 3 curve \(Q_u\). Let \(\text{Pic}^0(Q_u)\) be the Picard group of line bundles of degree 0 of \(Q_u\), and let \(\text{Pic}^0(Q_u)[2]\) be the subgroup of 2-torsion points of \(\text{Pic}^0(Q_u)\). For a 4-tangent conic \(\overline{c}\) of \(Q_u\), let \(\Theta_u(\overline{c})\) be the reduced part of the divisor \(\overline{c} \cap Q_u\) of \(Q_u\). Then the class of the divisor \(\Theta_u(\overline{c}) - \Lambda_u\) of degree 0 is a point of \(\text{Pic}^0(Q_u)[2] - \{0\}\), and this correspondence gives a bijection \(\overline{\Phi}_u \cong \text{Pic}^0(Q_u)[2] - \{0\}\).

## 5. Proof of the main results

In this section, we construct the space \(Z^{(m, n)}\) parameterizing all \(Q^{(m, n)}\)-curves, and prove Theorems 1.3 and 1.4.

### 5.1. Deformation types

We fix some notation. For a set \(A\), let \(S^k(A)\) denote the symmetric product \(A^k / \mathbb{S}_k\), where \(A^k = A \times \cdots \times A\) (\(k\) times), and let \(S^0_k(A)\) denote the complement in \(S^k(A)\) of the image of the big diagonal in \(A^k\).

For a morphism \(A \to U\), let \(S^k(A)\) denote the symmetric product \(A^k / \mathbb{S}_k\), where \(A^k := A \times_U \cdots \times_U A\) (\(k\) times), and let \(S^0_k(A)\) denote the complement in \(S^k(A)\) of the image of the big diagonal in \(A^k\). Note that, if \(A\) is smooth over \(U\) with relative dimension 1, then \(S^k(A)\) is smooth over \(U\) with relative dimension \(k\).

Recall that \(\overline{Z} \to U\) and \(\overline{C} \to U\) are the spaces parameterizing all bitangents and all 4-tangent conics of smooth quartic curves, respectively. We put

\[ Z^{(m, n)} := S^m_0(\overline{C}) \times_U S^n_0(\overline{C}), \]

which is the space parameterizing all curves \(Z\), where \(Z\) is a union of a smooth quartic curve \(Q\), \(m\) distinct bitangents of \(Q\), and \(n\) distinct 4-tangent conics of \(Q\). Now we can construct the parameter space

\[ \varphi : Z^{(m, n)} \to U \]
of \( Q^{(m,n)} \)-curves as the open subvariety of \( Z^{(m,n)} \) consisting of points corresponding to plane curves \( Z' \) satisfying conditions (i), (ii), (iii) in Definition 1.1. For a point \( \zeta \in Z^{(m,n)} \), we denote by \( Z_\zeta \) the \( Q^{(m,n)} \)-curve corresponding to \( \zeta \).

For \( u \in \mathcal{U} \), we put
\[
P_u^{(m,n)} := S_0^m(\mathcal{T}_u) \times S^n(\mathcal{F}_u) \subset S_0^m(S\mathcal{Y}_u) \times S^n(S\mathcal{Y}_u).
\]
The size of \( P_u^{(m,n)} \) is equal to \( d^{(m,n)} \) defined by (1.2). Then we obtain a finite étale covering
\[
\rho: \mathcal{P}^{(m,n)} := S_0^m(\mathcal{T}) \times_{\mathcal{U}} S^n(\mathcal{F}) \to \mathcal{U}
\]
of degree \( d^{(m,n)} \) parameterizing the family \( \{ P_u^{(m,n)} \mid u \in \mathcal{U} \} \). Using \( \mathcal{U}: \mathcal{C} \to \mathcal{F} \) in

Theorem 4.1, we have a morphism \( \theta': \mathcal{Z}^{(m,n)} \to \mathcal{P}^{(m,n)} \). Restricting \( \theta' \) to the open subvariety \( \mathcal{Z}^{(m,n)} \subset \mathcal{Z}^{(m,n)} \), we obtain a morphism
\[
\theta: \mathcal{Z}^{(m,n)} \to \mathcal{P}^{(m,n)},
\]
which maps \( \zeta \in \mathcal{Z}^{(m,n)} \) to
\[
(5.1) \quad \theta(\zeta) := (\{l_1, \ldots, l_m\}, [[e_1], \ldots, [e_n]]) \in \mathcal{P}^{(m,n)}_{\mathcal{Z}(\zeta)},
\]
where \( Z_\zeta \) has the irreducible components as in (5.4). Thus we obtain the following commutative diagram.
\[
\begin{array}{ccc}
\mathcal{Z}^{(m,n)} & \xrightarrow{\theta} & \mathcal{P}^{(m,n)} \\
\cong & \downarrow \rho & \\
\mathcal{U} & \xrightarrow{\theta} & \mathcal{P}^{(m,n)}
\end{array}
\]

To investigate the image of \( \theta \), we put
\[
\mathcal{U}' := \left\{ u \in \mathcal{U} \quad \text{every bitangent of } Q_u \text{ is ordinary, and their union has only ordinary nodes as its singularities} \right\},
\]
which is a Zariski open dense subset of \( \mathcal{U} \).

**Lemma 5.1.** The morphism \( \theta: \mathcal{Z}^{(m,n)} \to \mathcal{P}^{(m,n)} \) is smooth with each non-empty fiber being an irreducible variety of dimension \( n \). The image of \( \theta \) contains \( \rho^{-1}(\mathcal{U}') \). In particular, the morphism \( \theta \) is dominant.

**Proof.** Since \( \mathcal{U}: \mathcal{C} \to \mathcal{F} \) is smooth and surjective with each fiber being a Zariski open subset of \( \mathbb{P}^1 \), the morphism \( \theta': \mathcal{Z}^{(m,n)} \to \mathcal{P}^{(m,n)} \) is smooth and surjective with each fiber being an irreducible variety of dimension \( n \). Suppose that \( u \in \mathcal{U}' \), and let \( p := (\{l_1, \ldots, l_m\}, [[e_1], \ldots, [e_n]]) \) be a point of \( P_u^{(m,n)} \). By Proposition 4.2 and Bertini’s theorem, if we choose each 4-tangent conic \( c_j \) in the connected family \([c_j] \in \mathcal{F}_u \) generally, the curve \( Q_u + \sum l_i + \sum c_j \) satisfies conditions (ii) and (iii) in Definition 1.1. Hence \( \theta^{-1}(p) = \theta'^{-1}(p) \cap \mathcal{Z}^{(m,n)} \) is non-empty.

**Proof of Theorem 1.4.** By Lemma 5.1, the connected components of \( \mathcal{Z}^{(m,n)} \) are in bijective correspondence with the connected components of \( \mathcal{P}^{(m,n)} \), and hence with the \( \pi_1(\mathcal{U}, b) \)-orbits in \( P_b^{(m,n)} \). By Theorem 3.1, the number \( N^{(m,n)} \) of \( \pi_1(\mathcal{U}, b) \)-orbits in \( P_b^{(m,n)} \) satisfies (1.3), because \( |W(\mathbb{C})/\{\pm 1\}| = 1451520 \). □
5.2. Computation of $N^{(m,n)}$. Recall that $\Sigma_b$ is a negative-definite root lattice of type $E_7$. Let $\Sigma$ be the negative-definite root lattice of type $E_7$ with the standard basis, and let $\Sigma'$ be its dual. According to (3.3) and (4.1), we define the subsets

$$\mathcal{T} := \{ v \in \Sigma' | \langle v, v \rangle = -3/2 \}/\{\pm 1\}$$

of $\Sigma' := \Sigma/\{\pm 1\}$, and

$$\mathcal{F} := \{ v \in \Sigma | \langle v, v \rangle = -2 \}/\{\pm 1\}$$

of $\Sigma := \Sigma/\{\pm 1\}$. We then put

$$P^{(m,n)} := S_0^m(\mathcal{L}) \times S^n(\mathcal{F}).$$

The group $W(E_7)$ is generated by seven standard reflections. The permutations on $\mathcal{L}$ and on $\mathcal{F}$ induced by these generators are easily calculated. Hence the permutations on $P^{(m,n)}$ induced by these generators are also calculated. Thus we can compute the orbit decomposition of $P^{(m,n)}$ by $W(E_7)$, and obtain the number $N^{(m,n)}$ of deformation types of $Q^{(m,n)}$-curves.

Example 5.2. The size $d^{(4,0)}$ of $P^{(4,0)}$ is 20475. The group $W(E_7)$ decomposes this set into three orbits of sizes 315, 5040, 15120. Hence $N^{(4,0)} = 3$.

Example 5.3. The size $d^{(0,4)}$ of $P^{(0,4)}$ is 720720. The group $W(E_7)$ decomposes this set into 30 orbits as follows:

$$720720 = 63 + 945 \times 3 + 1008 \times 2 + 1890 + 2016 + 3780 \times 2 + 5040 \times 2 + 10080 +$$

$$+11340 + 15120 \times 5 + 22680 + 30240 \times 5 + 45360 \times 2 + 90720 + 120960 \times 2.$$

Hence $N^{(0,4)} = 30$.

Example 5.4. The size $d^{(2,2)}$ of $P^{(2,2)}$ is 762048. The group $W(E_7)$ decomposes this set into 23 orbits as follows:

$$762048 = 378 + 1890 + 3780 \times 3 + 6048 + 7560 \times 2 + 12096 \times 2 + 15120 + 22680 +$$

$$+30240 \times 3 + 45360 \times 2 + 60480 \times 4 + 120960 \times 2.$$

Hence $N^{(2,2)} = 23$.

Remark 5.5. For the computation, we used GAP [20], which is good at computations of permutation groups.

5.3. Real quartic curves.

Definition 5.6. Note that $H^2(\mathbb{P}^2, \mathbb{Z}) \cong \mathbb{Z}$ has a canonical generator, that is, the class of a line. Let $C$ and $C'$ be plane curves with the same homeomorphism type. A homeomorphism $\sigma : (\mathbb{P}^2, C) \overset{\sim}{\longrightarrow} (\mathbb{P}^2, C')$ is said to be orientation-preserving (resp. orientation-reversing) if the action of $\sigma$ on $H^2(\mathbb{P}^2, \mathbb{Z})$ is the identity (resp. the multiplication by $-1$).

Example 5.7. Suppose that $Q^{(m,n)}$-curves $Z_\zeta$ and $Z_{\zeta'}$ are of the same deformation type. Let $\alpha : I \to Z^{(m,n)}$ be a path from $\zeta$ to $\zeta'$, where $I := [0,1] \subset \mathbb{R}$. By the parallel transport along $\alpha$, we obtain a homeomorphism $\alpha_* : (\mathbb{P}^2, Z_\zeta) \overset{\sim}{\longrightarrow} (\mathbb{P}^2, Z_{\zeta'})$. It is obvious that $\alpha_*$ is orientation-preserving.

Proposition 5.8. Every $Q^{(m,n)}$-curve $Z_\zeta$ admits an orientation-reversing self-homeomorphism $(\mathbb{P}^2, Z_\zeta) \overset{\sim}{\longrightarrow} (\mathbb{P}^2, Z_\zeta)$.
For the proof of Proposition 5.8, we use a classical result on real quartic curves. We give a structure of the $\mathbb{R}$-scheme to $\mathbb{P}^2$. We denote by $\Gamma_\mathbb{R}(d)$ the space of homogeneous polynomials of degree $d$ on $\mathbb{P}^2$ with real coefficients, and consider the real projective space $\mathbb{P}_+ (\Gamma_\mathbb{R}(4))$ as a closed subset of $\mathbb{P}_+ (\Gamma(4))$. We then put
\[ U_\mathbb{R} := \mathcal{U} \cap \mathbb{P}_+ (\Gamma_\mathbb{R}(4)). \]
The topological types of smooth real quartic curves are classified by Zeuthen and Klein, and the result is summarized in [12, Theorem 1.7]. Using this result, we obtain the following:

**Theorem 5.9** (Zeuthen (1873) and Klein (1876)). There exists a unique connected component $U_{\mathbb{R},4}$ of $U_\mathbb{R}$ consisting of points $u \in U_\mathbb{R}$ such that the real plane curve $Q_u(\mathbb{R})$ is a union of $4$ ovals. If $u \in U_{\mathbb{R},4}$, then the ovals in $Q_u(\mathbb{R})$ are pairwise non-nested, and each bitangent of $Q_u$ is defined over $\mathbb{R}$. □

**Remark 5.10.** For beautiful pictures of real plane quartic curves with real $28$ bitangents, see [11] and [14, Section 10.5]. These pictures are in fact defined over $\mathbb{Q}$, and were obtained by the theory of Mordell-Weil lattices.

For an algebraic variety $V$ defined over $\mathbb{R}$, we denote by $H^*(V, \mathbb{Z})$ the cohomology ring of the topological space $V(\mathbb{C})$ of $\mathbb{C}$-valued points of $V$, by
\[ \tau_V : V(\mathbb{C}) \cong V(\mathbb{C}) \]
the self-homeomorphism of $V(\mathbb{C})$ obtained by the complex conjugation, and by $V_C$ the variety $V \otimes \mathbb{R} \mathbb{C}$ defined over $\mathbb{C}$. Let $S$ be an algebraic surface defined over $\mathbb{R}$, and let $C$ be a reduced irreducible curve on $S_\mathbb{C}$. Then there exists a unique reduced irreducible curve $C'$ on $S_\mathbb{C}$ such that $\tau_S$ induces an orientation-reversing homeomorphism $C(\mathbb{C}) \cong C'(\mathbb{C})$. We denote this curve $C'$ by $\tau_S[C]$. Then we have
\[ [\tau_S[C]] = -\tau_S^5([C]) \]
in $H^2(S, \mathbb{Z})$. If $C$ is also defined over $\mathbb{R}$, then $\tau_S[C] = C$, and hence $\tau_S^5([C]) = -[C]$. If $H^2(S, \mathbb{Z})$ is generated by classes of curves defined over $\mathbb{R}$, then $\tau_S$ acts on $H^2(S, \mathbb{Z})$ as the multiplication by $-1$, and hence, for any curve $C$ (not necessarily defined over $\mathbb{R}$), we have $[\tau_S[C]] = [C]$ in $H^2(S, \mathbb{Z})$.

**Lemma 5.11.** Let $r$ be a point of $U_{\mathbb{R},4}$. If $\tilde{c}$ is a $4$-tangent conic of $Q_r$, then the $4$-tangent conic $\tau_{\mathbb{P}^2}[\tilde{c}]$ of $\tau_{\mathbb{P}^2}[Q_r] = Q_r$ is in the same connected family as $\tilde{c}$.

**Proof.** Note that, for $\varphi \in \Gamma_\mathbb{R}(4)$ and $x \in \mathbb{P}^2(\mathbb{R})$, the sign of $\varphi(x)$ is well-defined, because $\lambda^4 > 0$ for any $\lambda \in \mathbb{R}^\times$. We choose a defining equation $\varphi \in \Gamma_\mathbb{R}(4)$ of $Q_r$ in such a way that $\varphi(x) > 0$ for any point $x$ of $\mathbb{P}^2(\mathbb{R})$ in the outside of the ovals of $Q_r(\mathbb{R})$. We let $Y_r$ be defined over $\mathbb{R}$ by $w^2 = \varphi$, and consider the self-homeomorphism $\tau_Y : Y_r(\mathbb{C}) \cong Y_r(\mathbb{C})$ given by the complex conjugation. For any bitangent $l$ of $Q_r$, each of its $Y$-lifts $\tilde{l}$ satisfies $\tau_Y[l] = \tilde{l}$, because $\varphi(x) \geq 0$ for any point $x$ of $l(\mathbb{R})$. Since the classes of these curves $l$ span $SY_r = H^2(Y_r, \mathbb{Z})$, we see that $\tau_Y$ acts on $SY_r$ as the multiplication by $-1$. Therefore, for any curve $C$ on $Y_r$, we have $[\tau_Y[C]] = [C]$. In particular, if $c \subset Y_r$ is a $Y$-lift of $\tilde{c}$, then $\tau_Y[c]$ is a $Y$-lift of the $4$-tangent conic $\tau_{\mathbb{P}^2}[\tilde{c}]$. Then $[\tau_Y[c]] = [c]$ in $SY_r$ implies that $\tau_{\mathbb{P}^2}[\tilde{c}]$ and $\tilde{c}$ belong to the same connected family of $4$-tangent conics. □

**Proof of Proposition 5.8.** Since $U_{\mathbb{R},4}$ is open in $\mathbb{P}_+ (\Gamma_\mathbb{R}(4))$, it follows that $U_{\mathbb{R},4}$ is Zariski dense in $\mathcal{U}$, and hence there exists a point $r \in U_{\mathbb{R},4} \cap \mathcal{U}$. By Lemma 5.1, we
see that $\varpi^{-1}(r)$ intersects every connected component of $Z^{(m,n)}$. Let $\xi$ be a point of $\varpi^{-1}(r)$ that belongs to the same connected component as $\zeta$, and let

$$Z_\xi = Q_r + \tilde{l}_1 + \cdots + \tilde{l}_m + \tilde{c}_1 + \cdots + \tilde{c}_n$$

be the decomposition of $Z_\xi$ into irreducible components. Remark that $Q_r$ and all of its bitangents are defined over $\mathbb{R}$ by the definition of $U_{\mathbb{R},4}$ (see Theorem 5.9). We choose a path $\alpha : I \to Z^{(m,n)}$ from $\zeta$ to $\xi$. Then we obtain an orientation-preserving homeomorphism

$$\alpha_* : (\mathbb{P}^2, Z_\zeta) \xrightarrow{\sim} (\mathbb{P}^2, Z_\xi).$$

For simplicity, we write $\tau$ instead of $\tau_{\mathbb{R},2}$. We have an orientation-reversing homeomorphism

$$\tau : (\mathbb{P}^2, Z_\zeta) \xrightarrow{\sim} (\mathbb{P}^2, \tau[Z_\zeta])$$

obtained by the complex conjugation. Since

$$\tau[Z_\zeta] = Q_r + \tilde{l}_1 + \cdots + \tilde{l}_m + \tau[\tilde{c}_1] + \cdots + \tau[\tilde{c}_n],$$

and, for $j = 1, \ldots, n$, the 4-tangent conic $\tau[\tilde{c}_j]$ of $\tau[Q_r] = Q_\tau$ belongs to the same connected family as $\tilde{c}_j$ by Lemma 5.11, we see that $Q^{(m,n)}$-curves $\tau[Z_\zeta]$ and $Z_\zeta$ have the same deformation type, and we have an orientation-preserving homeomorphism

$$\beta_* : (\mathbb{P}^2, \tau[Z_\zeta]) \xrightarrow{\sim} (\mathbb{P}^2, Z_\zeta).$$

Composing $\alpha_*, \tau, \beta_*$ and $\alpha_*^{-1}$, we obtain an orientation-reversing self-homeomorphism of $(\mathbb{P}^2, Z_\zeta)$. $\square$

5.4. Homeomorphism types. Let $\zeta$ be a point of $Z^{(m,n)}$ such that $Z_\zeta$ has the decomposition

$$Z = Q + \tilde{l}_1 + \cdots + \tilde{l}_m + \tilde{c}_1 + \cdots + \tilde{c}_n.$$

We consider another point $\zeta' \in Z^{(m,n)}$ with the decomposition

$$Z_{\zeta'} = Q_{u'} + \tilde{l}_1 + \cdots + \tilde{l}_m + \tilde{c}_1 + \cdots + \tilde{c}_n.$$

Recall that the involution $\iota_u$ of $Y_u$ acts on the orthogonal complement $\Sigma_u$ of $h_u \in SY_u$, as $-1$. Let $g : SY_u \xrightarrow{\sim} SY_{u'}$ be an isometry of lattices. Suppose that $g$ maps $h_u$ to $h_{u'}$. Then we have $g \circ \iota_u = \iota_{u'} \circ g$. Moreover, by definitions (3.3) and (4.1), the isometry $g$ maps $L_u$ to $L_{u'}$ and $F_u$ to $F_{u'}$. Therefore $g$ induces a bijection $P_u^{(m,n)} \xrightarrow{\sim} P_{u'}^{(m,n)}$.

Theorem 1.3 is an immediate consequence of the following:

**Theorem 5.12.** The following are equivalent:

(i) $\zeta$ and $\zeta'$ belong to the same connected component of $Z^{(m,n)}$,

(ii) $\theta(\zeta)$ and $\theta(\zeta')$ belong to the same connected component of $P^{(m,n)}$,

(iii) there exists an isometry $g : SY_u \xrightarrow{\sim} SY_{u'}$ of lattices that maps $h_u$ to $h_{u'}$ and such that the induced bijection $P_u^{(m,n)} \xrightarrow{\sim} P_{u'}^{(m,n)}$ maps $\theta(\zeta)$ to $\theta(\zeta')$, and

(iv) there exists a homeomorphism $(\mathbb{P}^2, Z_\zeta) \xrightarrow{\sim} (\mathbb{P}^2, Z_{\zeta'})$.

**Proof.** By Proposition 5.8, condition (iv) is equivalent to the following:

(iv)' there exists an orientation-preserving homeomorphism $(\mathbb{P}^2, Z_\zeta) \xrightarrow{\sim} (\mathbb{P}^2, Z_{\zeta'})$.

We will show that (i), (ii), (iii) and (iv)' are equivalent. The implication (i)$\iff$(ii) follows from Lemma 5.1, and (i)$\implies$(iv)' follows from Example 5.7.

We show (iv)' $\implies$ (iii). Suppose that $\sigma : (\mathbb{P}^2, Z_\zeta) \xrightarrow{\sim} (\mathbb{P}^2, Z_{\zeta'})$ is an orientation-preserving homeomorphism. We can assume, after renumbering the curves, that $\sigma$
induces homeomorphisms $\tilde{I}_i \overset{\sim}{\to} \tilde{I}'_i$ and $\tilde{c}_j \overset{\sim}{\to} \tilde{c}'_j$ that preserve the orientation. We have a homeomorphism $\sigma_Y : Y_u \overset{\sim}{\to} Y_{u'}$ that covers $\sigma$, and $\sigma_Y$ induces an isometry $\sigma_{SY} : SY_u \overset{\sim}{\to} SY_{u'}$, which maps $h_u$ to $h_{u'}$. If $l_i \subset Y_u$ is a $Y$-lift of a bitangent $\tilde{l}_i \subset Z_\zeta$, there exists a $Y$-lift $l'_i \subset Y_{u'}$ of the bitangent $\tilde{l}_i \subset Z_\zeta$ such that $\sigma_Y l_i \overset{\sim}{\to} l'_i$ preserving the orientation. In particular, we have $\sigma_{SY}([l_i]) = [l'_i]$. The same holds for a $Y$-lift $c_j \subset Y_u$ of a 4-tangent conic $\tilde{c}_j \subset Z_\zeta$. Hence the bijection $P_{u}(m,n) \overset{\sim}{\to} P_{u'}(m,n)$ induced by the isometry $\sigma_{SY}$ maps $\theta(\zeta)$ to $\theta(\zeta')$. Thus (iii) holds.

We show (iii) $\implies$ (ii). Suppose that (iii) holds. We choose a path $\beta : I \to U$ from $u$ to the base-point $b$ and a path $\beta' : I \to U$ from $u'$ to $b$, and consider the isometries

$$\beta_* : SY_u \overset{\sim}{\to} SY_b, \quad \beta'_* : SY_{u'} \overset{\sim}{\to} SY_b$$

obtained by the parallel transports along $\beta$ and $\beta'$. Note that $\beta'_* (h_u) = h_b$ and $\beta'_* (h_{u'}) = h_b$. Hence $\beta'_* \circ g \circ \beta^{-1}$ is an element of $O(SY_b, h_b)$. By Theorem 3.1, there exists a loop $\alpha : I \to U$ with the base point $b$ such that

$$\alpha_* = \beta'_* \circ g \circ \beta^{-1}$$

Therefore the isometry $g : SY_u \overset{\sim}{\to} SY_{u'}$ is equal to the parallel transport $\gamma_*$ along the path $\gamma := \beta'^{-1} \alpha \beta$ from $u$ to $u'$. Let

$$\tilde{\gamma} : I \to P^{(m,n)}$$

be the lift of $\gamma$ such that $\tilde{\gamma}(0) = \theta(\zeta)$. Since $g = \gamma_*$ maps $\theta(\zeta)$ to $\theta(\zeta')$, we see that $\tilde{\gamma}(1) = \theta(\zeta')$. Therefore $\theta(\zeta)$ and $\theta(\zeta')$ are in the same connected component of $P^{(m,n)}$. \hfill \Box

6. Geometry of the $K3$ surface $X_u$

We investigate the connected families of 4-tangent conics more closely for a general point $u \in U$. Our main result of this section is as follows.

**Theorem 6.1.** Suppose that $u \in U$ is general. Then each connected family of 4-tangent conics $\tilde{c}$ of $Q_u$ is parameterized by a rational curve minus $12 + 6$ points. A member $\tilde{c}$ of this family becomes a 3-tangent conic at each of 12 punctured points, and $\tilde{c}$ degenerates into a union of two distinct bitangents at each of the remaining 6 punctured points.

For the proof, we add the following easy result to Proposition 2.4.

**Proposition 6.2.** Let $\tilde{c}$ be a 3-tangent conic of $Q_u$. Then $\gamma^*_u(\tilde{c})$ is a union of two one-nodal rational curves. \hfill \Box

Recall that the double covering $\eta_u : X_u \to Y_u$ induces a primitive embedding of lattices $\eta^*_u : SY_u(2) \hookrightarrow SX_u$.

**Proposition 6.3.** If $u \in U$ is general, then $\eta^*_u$ is an isomorphism.

**Proof.** Kondo [10] studied the moduli of genus-3 curves by considering the periods of $K3$ surfaces $X$ that are cyclic covers of $\mathbb{P}^2$ of degree 4 branched along quartic curves $Q \subset \mathbb{P}^2$. Let $j$ denote the generator of $\text{Gal}(X/\mathbb{P}^2) \cong \mu_4$ that acts on $H^{2,0}(X)$ as $\sqrt{-1}$. Kondo exhibits an action of the cyclic group $\mu_4$ on the $K3$ lattice $L := \mathbb{L}^2 \oplus U^{\oplus 3}$
that is obtained by a marking $H^2(X,\mathbb{Z}) \cong \mathbb{L}$. Let $L_S$ and $L_T$ be the kernel of $j^* - 1$ and of $j^* + 1$ on $L$, respectively. Then $L_S$ is of rank 8, and, via the marking, equal to the image of the pull-back of $H^2(Y,\mathbb{Z})(2)$ by the double covering $X \to Y := X/\langle j^* \rangle$. The period $H^{2,0}(X)$ is a point of $\mathbb{P}^s(V_{\sqrt{-1}})$, where $V_{\sqrt{-1}}$ is the kernel of $j^* - \sqrt{-1}$ on $L_T \otimes \mathbb{C}$. We have $\dim \mathbb{P}^s(V_{\sqrt{-1}}) = 6$. The result of [10] implies that, when $Q$ varies, the point $H^{2,0}(X)$ of $\mathbb{P}^s(V_{\sqrt{-1}})$ sweeps an open subset of $\mathbb{P}^s(V_{\sqrt{-1}})$.

We fix a marking $H^2(X_u,\mathbb{Z}) \cong \mathbb{L}$. Since $u \in \mathcal{U}$ is general, the period $H^{2,0}(X_u)$ is general in $\mathbb{P}^s(V_{\sqrt{-1}})$. Since $L_T \otimes \mathbb{C} = V_{\sqrt{-1}} \oplus V_{\sqrt{-1}}$, the minimal $\mathbb{Z}$-submodule $M$ of $L$ such that $M \otimes \mathbb{C}$ contains $H^{2,0}(X_u)$ is equal to $L_T$, and hence its orthogonal complement $M^\perp = SX_u$ is equal to $L_S = \eta^*_u(SY_u(2))$. □

Let $\text{Rats}(X_u)$ denote the set of rational curves on $X_u$, and $\text{Ells}(X_u)$ the set of elliptic fibrations on $X_u$.

**Proposition 6.4.** Suppose that $u \in \mathcal{U}$ is general. Then $\text{Rats}(X_u)$ is equal to the set $\tilde{L}_u := \{ \eta^*_u(l) \mid l \in L_u \}$ of 56 smooth rational curves on $X_u$.

This proposition is proved by Proposition 6.3 and [13, Proposition 98]. See also [13, Remark 99]. We give a proof, however, because the argument is also used in the proof of Proposition 6.5 below. Recall from the proof of Theorem 4.1 that, for $v \in F_u$, there exists an elliptic fibration $\phi_v : X_u \to \mathbb{P}^1$ such that the class of a fiber of $\phi_v$ is $\eta^*_u(v)$.

**Proposition 6.5.** Suppose that $u \in \mathcal{U}$ is general. Then $v \mapsto \phi_v$ gives a bijection $F_u \cong \text{Ells}(X_u)$. Each fibration $\phi_v$ has no section. The singular fibers of $\phi_v$ consist of 6 fibers of type $I_2$ and 12 fibers of type $I_1$.

**Proof of Propositions 6.4 and 6.5.** The space 

$$\{ v \in SX_u \otimes \mathbb{R} \mid \langle v, v \rangle_X > 0 \}$$

has two connected components. Let $\mathcal{P}_u$ be the connected component containing the ample class $h_u$. For a vector $v \in SX_u \otimes \mathbb{R}$ with $\langle v, v \rangle_X < 0$, let $[v]^\perp$ be the hyperplane of $SX_u \otimes \mathbb{R}$ defined by $\langle x, v \rangle_X = 0$, and we put $(v)^\perp := [v]^\perp \cap \mathcal{P}_u$. We then put

$$N_u := \{ v \in \mathcal{P}_u \mid \langle v, \Gamma \rangle_X \geq 0 \text{ for all curves } \Gamma \subset X_u \}.$$ 

It is well known that $N_u$ is equal to

$$\{ v \in \mathcal{P}_u \mid \langle v, \Gamma \rangle_X \geq 0 \text{ for all } \Gamma \in \text{Rats}(X_u) \}.$$

and that each $\Gamma \in \text{Rats}(X_u)$ defines a wall of the cone $N_u$, that is, $(\Gamma)^\perp \cap N_u$ contains a non-empty open subset of $(\Gamma)^\perp$. Let $\overline{N}_u$ be the closure of $N_u$ in $SX_u \otimes \mathbb{R}$. For the proof of Proposition 6.4, it is enough to show that $\overline{N}_u$ is equal to

$$\overline{N}_u' := \{ v \in SX_u \otimes \mathbb{R} \mid \langle v, l \rangle_X \geq 0 \text{ for all } l \in \tilde{L}_u \}.$$ 

A face of the cone $\overline{N}_u'$ is a closed subset $F$ of $\overline{N}_u'$ of the form $F = V \cap \overline{N}_u'$, where $V$ is an intersection of some of the hyperplanes $[l]^\perp (l \in \tilde{L}_u)$ such that $F$ contains a non-empty open subset of $V$. We say that $V$ is the supporting linear subspace of the face $F$, and put $\dim F := \dim V$. A ray is a 1-dimensional face. For the proof of $\overline{N}_u = \overline{N}_u'$, it is enough to show that all rays of $\overline{N}_u'$ are contained in $\overline{N}_u$. We can calculate all the faces $F$ of $\overline{N}_u'$ by descending induction on $d := \dim F$ using linear
programming method (see [17, Section 2.2]). The result is as follows. Suppose that \(d \geq 2\). Then a linear subspace
\[
V = \left[\tilde{l}_1\right]^\perp \cap \cdots \cap \left[\tilde{l}_k\right]^\perp
\]
with \(\tilde{l}_1, \ldots, \tilde{l}_k \in \tilde{L}_u\) is the supporting linear subspace of a face \(F\) with \(\dim F = d\) if and only if \(k = 8 - d\) and \(\tilde{l}_1, \ldots, \tilde{l}_k\) are disjoint from each other, that is, their dual graph is the Dynkin diagram of type \((8 - d)A_1\). Suppose that \(d = 1\). Then a linear subspace \(V\) as (6.1) is the supporting linear subspace of a ray \(F\) if and only if one of the following holds:

\[(7A_1) \quad k = 7 \text{ and the dual graph of } \tilde{l}_1, \ldots, \tilde{l}_7 \text{ is the Dynkin diagram of type } 7A_1.\]

In this case, \(F\) is generated by a vector \(v \in SX_u\) with \(\langle \hat{h}_u, v \rangle_X = 6\) and \(\langle v, v \rangle_X = 2\). There exist exactly 576 rays of this type.

\[(6A_1) \quad k = 12 \text{ and the dual graph of } \tilde{l}_1, \ldots, \tilde{l}_{12} \text{ is the Dynkin diagram of type } 6\tilde{A}_1,\]

where \(A_1\) is \(\square\) \(\square\) \(\square\). In this case, \(F\) is generated by a primitive vector \(\check{v}\) with \(\langle \hat{h}_u, \check{v} \rangle_X = 4\) and \(\langle \check{v}, \check{v} \rangle_X = 0\). There exist exactly 126 rays of this type, and these generators \(\check{v}\) are equal to \(\eta^*_u(v)\) for some \(v \in F_u\).

In Table 6.1, the numbers of faces of \(\overline{\mathcal{N}}_u\) are given.

Suppose that there exists a ray \(F\) of \(\overline{\mathcal{N}}_u\) not contained in \(\overline{\mathcal{N}}_u\). Then the generating class \(v \in SX_u\) of \(F\) given above is effective but not nef. Let \(D\) be an effective divisor of \(X_u\) such that \([D] = v\). Then \(D\) contains a smooth rational curve \(\Gamma\) with \(\langle \Gamma, v \rangle_X < 0\) as an irreducible component. Since \(\hat{h}_u\) is ample, the \((-2)\)-vector \(r = [\Gamma]\) satisfies \(\langle \hat{h}_u, r \rangle_X < \langle \hat{h}_u, v \rangle_X \leq 6\). We make the set of all \((-2)\)-vectors \(r' \in SX_u\) with \(\langle \hat{h}_u, r' \rangle_X = 1, \ldots, 5\), and confirm that this set is equal to the set of classes of \(\tilde{L}_u\). In particular, it contains no element \(r'\) satisfying \(\langle r', v \rangle_X < 0\). This contradiction shows \(\overline{\mathcal{N}}_u = \mathcal{N}_u\) and \(Rats(X_u) = \tilde{L}_u\) is proved.

It is well known that there exists a bijection between \(Rats(X_u)\) and the set of rays contained in \(\overline{\mathcal{N}}_u \cap \partial \overline{\mathcal{F}}_u\). Hence we have \(|Rats(X_u)| = 126\), and \(v \mapsto \phi_v\) gives a bijection from \(F_u\) to \(Rats(X_u)\). Therefore, as was shown in the proof of Theorem 4.1, every fiber \(f\) of any elliptic fibration \(\phi_v\) is a double cover of a splitting conic of \(Q_u\). The class of \(f\) is equal to \(\eta^*_u(v)\). Since no element \(\tilde{l} \in Rats(X_u)\) satisfies \(\langle f, \tilde{l} \rangle_X = 1\), the fiber \(\phi_v\) has no section. Since the dual graph of the set of \(\tilde{l} \in Rats(X_u)\) with \(\langle f, \tilde{l} \rangle_X = 0\) is of type \(6\tilde{A}_1\), the fiber \(\phi_v\) has exactly 6 reducible fibers, each of which is either of type \(I_2\) or of type \(III\). If \(\tilde{l}_i, \tilde{l}_j \in Rats(X_u)\) are in the same fiber of \(\phi_v\), then they satisfy \(\langle \tilde{l}_i, \tilde{l}_j \rangle_X = 2\) and hence \(\tilde{l}_i := \gamma_u(\tilde{l}_i)\) and \(\tilde{l}_j := \gamma_u(\tilde{l}_j)\) are distinct bitangents of \(Q_u\) by Table 2.1. Since \(u \in U\) is general, the intersection point of \(\tilde{l}_i\) and \(\tilde{l}_j\) is not on \(Q_u\). Hence every reducible fiber of \(\phi_v\) is of type \(I_2\). The irreducible singular fibers are either of type \(I_1\) or of type \(II\). By Lemma 2.2 and Proposition 6.2, we see that all irreducible singular fibers must be of type \(I_1\). Calculating the Euler number, we conclude that the number of singular fibers of type \(I_1\) is 12. \(\square\)
Remark 6.6. The set of 576 rays of type $7A_1$ is in bijective correspondence with the set $L_u^{(7)}$ in the proof of Theorem 3.1. A ray $F$ of type $7A_1$ corresponds to a $7$-tuple $\{l_1, \ldots, l_7\} \in L_u^{(7)}$ as follows. The generator $v$ of $F$ with $(v, v)_X = 2$ is the class of the pull-back of a line of a plane $\mathbb{P}$ by the double covering $X_u \to Y_u \to \mathbb{P}$, where $Y_u \to \mathbb{P}$ is the blowing down of the $(-1)$-curves $l_1, \ldots, l_7$.

Proof of Theorem 6.1. In fact, the proof was already given in the last paragraph of the proof of Proposition 6.5. □

7. Configurations of $Y$-lifts

Throughout this section, let $u$ be a general point of $\mathcal{U}$.

7.1. Lemmas on quartic polynomials. Let $[d_1, \ldots, d_m]$ be a list of positive integers satisfying $d_1 + \cdots + d_m = 4$. We put

$$\Gamma(d_1, \ldots, d_m : 2) := \Gamma(d_1) \times \cdots \times \Gamma(d_m) \times \Gamma(2),$$

and denote by $\psi_{[d_1, \ldots, d_m]} : \Gamma(d_1, \ldots, d_m : 2) \to \Gamma(4)$ the morphism

$$(f_1, \ldots, f_m, q) \mapsto f_1 \cdots f_m + q^2.$$

Lemma 7.1. The morphism $\psi_{[d_1, \ldots, d_m]}$ is dominant.

Proof. It is enough to show that $\psi_{[1, 1, 1, 1]}$ is dominant, and then, it suffices to find a point $P$ of $\Gamma(1, 1, 1, 1 : 2)$ at which the differential of $\psi := \psi_{[1, 1, 1, 1]}$ is of rank $\dim \Gamma(4) = 15$. By choosing points $P$ randomly and calculating the rank of $d_P \psi$, we can easily find such a point. □

Definition 7.2. For $[d_1, \ldots, d_m]$ with $d_1 + \cdots + d_m = 4$, we have an open dense subset $\mathcal{V}_{[d_1, \ldots, d_m]} \subset \Gamma(d_1, \ldots, d_m : 2)$ and a dominant morphism

$$\Psi_{[d_1, \ldots, d_m]} : \mathcal{V}_{[d_1, \ldots, d_m]} \to \mathcal{U}$$

such that, for $p = (f_1, \ldots, f_m, q) \in \mathcal{V}_{[d_1, \ldots, d_m]}$, the quartic curve corresponding $\Psi_{[d_1, \ldots, d_m]}(p)$ is defined by $f_1 \cdots f_m + q^2 = 0$.

Lemma 7.3. If $Q_u$ is defined by $f + q^2 = 0$ with $f \in \Gamma(4)$ and $q \in \Gamma(2)$, then $Y_u$ has a divisor that is mapped isomorphically to the divisor $\{ f = 0 \}$ of $\mathbb{P}^2$.

Proof. The surface $Y_u$ is defined by $w^2 = f + q^2$, where $w$ is a new variable, and hence contains a divisor defined by $f = w - q = 0$. It is obvious that $\pi_u$ maps this divisor to the divisor $\{ f = 0 \}$ of $\mathbb{P}^2$ isomorphically. □

7.2. Triangles of bitangents. Recall that $L_u$ is the set of $Y$-lifts $l$ of bitangents $\bar{l} \in \mathcal{T}_u$ of $Q_u$.

Definition 7.4. A triangle on $Y_u$ is a subset $\{l_1, l_2, l_3\}$ of $L_u$ such that $\langle l_1, l_2 \rangle_Y = \langle l_2, l_3 \rangle_Y = \langle l_3, l_1 \rangle_Y = 1$. A liftble triangle of bitangents of $Q_u$ is a subset $\{\bar{l}_1, \bar{l}_2, \bar{l}_3\}$ of $\mathcal{T}_u$ that is the image of a triangle on $Y_u$ by $\pi_u$.

Let $\bar{l}_1, \bar{l}_2, \bar{l}_3$ be bitangents of $Q_u$. We choose $Y$-lifts $l_1, l_2, l_3 \in L_u$ of $\bar{l}_1, \bar{l}_2, \bar{l}_3$ in such a way that $\langle l_1, l_2 \rangle_Y = \langle l_2, l_3 \rangle_Y = \langle l_3, l_1 \rangle_Y = 1$. Then $\{\bar{l}_1, \bar{l}_2, \bar{l}_3\}$ is liftble if and only if $\langle l_1, l_2 \rangle_Y = \langle l_2, l_3 \rangle_Y = \langle l_3, l_1 \rangle_Y = 1$. Let $T_u$ be the set of triangles on $Y_u$. We have calculated $L_u \subset SY_u$ explicitly. Using this data, we enumerate $T_u$, and see that $|T_u| = 2520$. Let

$$\mathcal{T}_u := T_u / \langle t_u \rangle$$
be the set of liftable triangles of bitangents of $Q_u$.

**Corollary 7.5.** There exist exactly $|T_u| = 1260$ liftable triangles. $\square$

By Theorem 3.1, we obtain the following:

**Proposition 7.6.** By the monodromy, $\pi_1(\mathcal{U}, b)$ acts transitively on $T_b$ and hence on $\overline{T}_b$. $\square$

**Proposition 7.7.** Let $\bar{l}_1, \bar{l}_2, \bar{l}_3$ be bitangents of $Q_u$. Suppose that $\bar{l}_i$ is defined by $f_i = 0$ for $i = 1, \ldots, 3$, where $f_i \in \Gamma(1)$. Then $\{\bar{l}_1, \bar{l}_2, \bar{l}_3\}$ is liftable if and only if there exist polynomials $f_4 \in \Gamma(1)$ and $q \in \Gamma(2)$ such that $Q_u$ is defined by $f_1 f_2 f_3 f_4 + q^2 = 0$.

**Proof.** The if-part follows from Lemma 7.3. Let $\bar{\tau}: \overline{T} \to \mathcal{U}$ be the finite étale covering obtained from the family $\{\overline{T}_u | u \in \mathcal{U}\}$. Then $\overline{T}$ is irreducible by Proposition 7.6. Let $p := (f'_1, \ldots, f'_4, q')$ be a point of $\mathcal{V}_{[1,1,1,1]}$, and we put

$$u' := \Psi_{[1,1,1,1]}(p) \in \mathcal{U}$$

where $\mathcal{V}_{[1,1,1,1]}$ and $\Psi_{[1,1,1,1]}$ are given in Definition 7.2. Let $\bar{l}'_i \subset \mathbb{P}^2$ be the line $\{f'_i = 0\}$. By the if-part, we have $\{\bar{l}'_1, \bar{l}'_2, \bar{l}'_3\} \in \overline{T}_{u'}$. By $p \mapsto \{\bar{l}'_1, \bar{l}'_2, \bar{l}'_3\}$, we obtain a morphism $\Psi_{\mathcal{V}}: \mathcal{V}_{[1,1,1,1]} \to \overline{T}$. Since $\bar{\tau} \circ \Psi_{\mathcal{V}} = \Psi_{[1,1,1,1]}$, $\bar{\tau}$ is étale, $\overline{T}$ is irreducible, and $\Psi_{[1,1,1,1]}$ is dominant, we conclude that $\Psi_{\mathcal{V}}$ is dominant. Since $u \in \mathcal{U}$ is general, we obtain the proof. $\square$

**Corollary 7.8.** There exists a set $\overline{\mathcal{R}}_u$ consisting of 315 subsets $\{\bar{l}_1, \bar{l}_2, \bar{l}_3, \bar{l}_4\} \subset \overline{T}_u$ of size 4 with the following properties: a subset $\{\bar{l}_1, \bar{l}_2, \bar{l}_3\} \subset \overline{T}_u$ of size 3 is liftable if and only if there exists an element $\{\bar{l}_1, \bar{l}_2, \bar{l}_3, \bar{l}_4\} \in \overline{\mathcal{R}}_u$ containing $\{\bar{l}_1, \bar{l}_2, \bar{l}_3\}$. $\square$

7.3. **Pairs of splitting conics.** Recall that $F_u \subset SY_u$ is the set of classes $[c]$ of $Y$-lifts $c$ of 4-tangent conics $\bar{c}$ of $Q_u$, and that $\overline{F}_u = F_u / \langle u_n \rangle$ is regarded as the set of connected families of 4-tangent conics of $Q_u$, or equivalently as the set of connected families of splitting conics of $Q_u$. For a splitting conic $\bar{c}$, let $[\bar{c}] \in \overline{F}_u$ denote the connected family containing $\bar{c}$. By Theorem 3.1, we obtain the following:

**Proposition 7.9.** By the monodromy, $\pi_1(\mathcal{U}, b)$ acts transitively on $F_b$ and hence on $\overline{F}_b$. $\square$

**Definition 7.10.** Let $\bar{c}$ be a splitting conic of $Q_u$. We say that a decomposition $\pi_u^*(\bar{c}) = c + c'$ is normal if each of $c$ and $c'$ is a $Y$-lift of $\bar{c}$.

Note that, if $\bar{c}$ is smooth, then the decomposition $\pi_u^*(\bar{c}) = c + c'$ is normal, whereas if $\bar{c}$ is a sum of two bitangents $\bar{l} + \bar{l}'$, then $\pi_u^*(\bar{c}) = c + c'$ being normal means that $c = l + l'$ with $\langle l, l' \rangle_Y = 1$.

**Definition 7.11.** Let $\bar{c}_1$ and $\bar{c}_2$ be splitting conics of $Q_u$, and let $\pi_u^*(\bar{c}_1) = c_1 + c'_1$ and $\pi_u^*(\bar{c}_2) = c_2 + c'_2$ be the normal decompositions. We put

$$I(\bar{c}_1, \bar{c}_2) := \begin{bmatrix} \langle c_1, c_2 \rangle_Y & \langle c_1, c'_2 \rangle_Y \\ \langle c'_1, c_2 \rangle_Y & \langle c'_1, c'_2 \rangle_Y \end{bmatrix}.$$

Since we can make switchings $c_1 \leftrightarrow c'_1$ and $c_2 \leftrightarrow c'_2$, the matrix $I(\bar{c}_1, \bar{c}_2)$ is well-defined only up to the transpositions of the two rows and of the two columns.
We have calculated $F_u \subset S_u$ explicitly. Using this data, we see that the matrix $I([\bar{c}_1], [\bar{c}_2])$ is one of the following:

$$I_A := \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix},$$

$$I_B := \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix},$$

$$I_C := \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$ 

**Proposition 7.12.** Let $\bar{c}_1 = \{g_1 = 0\}$ and $\bar{c}_2 = \{g_2 = 0\}$ be splitting conics of $Q_u$. Consider the following conditions:

(i) $[\bar{c}_1] = [\bar{c}_2]$, that is, $\bar{c}_1$ and $\bar{c}_2$ belong to the same connected family.

(ii) The matrix $I([\bar{c}_1], [\bar{c}_2])$ is equal to $I_A$.

(iii) There exists a polynomial $q \in \Gamma(2)$ such that $Q_u$ is defined by $g_1g_2 + q^2 = 0$. Then we have (iii) $\implies$ (ii) $\iff$ (i). If (i) holds and $\bar{c}_1$ and $\bar{c}_2$ are general in the connected family $[\bar{c}_1] = [\bar{c}_2] \in \overline{F}_u$ of splitting conics, then (iii) holds.

**Proof.** The implication (i) $\implies$ (ii) follows immediately from Table 2.1, and the implication (ii) $\implies$ (ii) follows from Lemma 7.3. Suppose that (ii) holds. Let $\pi_u^*(\bar{c}_1) = c_1 + c'_1$ and $\pi_u^*(\bar{c}_2) = c_2 + c'_2$ be the normal decompositions. Interchanging $c_1$ and $c_2$ if necessary, we can assume that $\langle c_1, c_2 \rangle = 0$. We put $f_1 := \eta_u^*(c_1)$ and $f_2 := \eta_u^*(c_2)$. Note that $f_1$ is a fiber of the elliptic fibration $\phi_1 \in \text{Ells}(X_u)$ corresponding to the class $[\bar{c}_1] \in F_u$ of $c_1$ by $F_u \cong \text{Ells}(X_u)$. Since $(f_1, f_2)_X = 2(c_1, c_2)_Y = 0$, we conclude that $f_2$ is a fiber of $\phi_1$, that is, the elliptic fibration corresponding to $[\bar{c}_2] \in F_u \cong \text{Ells}(X_u)$ is equal to $\phi_1$. Therefore $\bar{c}_1$ and $\bar{c}_2$ belong to the same connected family of splitting conics, and (i) holds. Thus (iii) $\implies$ (ii) $\iff$ (i) is proved.

Suppose that $[\bar{c}_1] = [\bar{c}_2]$. Let $\sigma : \overline{F} \to U$ be the finite étale covering defined by the family $\{F_u \mid u \in U\}$. By Proposition 7.9, we see that $\overline{F}$ is irreducible. Let $p := (g_1, g_2, q)$ be a point of $V_{[2, 2]}$, and we put $u' := \Psi_{[2, 2]}(p) \in U$:

$$Q_{u'} = \{g'_1g'_2 + q'^2 = 0\}.$$ 

Let $\bar{c}'_i$ be the splitting conic $\{g'_i = 0\}$ of $Q_{u'}$ for $i = 1, 2$. By the implication (iii) $\implies$ (i), we have $[\bar{c}'_1] = [\bar{c}'_2]$ in $T_{u'}$. By $p \mapsto [\bar{c}'_1]$, we obtain a morphism $\Psi_{\overline{F}} : V_{[2, 2]} \to \overline{F}$. By the same argument as in the proof of Proposition 7.7, we see that $\Psi_{\overline{F}}$ is dominant. Since $u$ is general in $U$, the point $(u, [\bar{c}_1]) = (u, [\bar{c}_2])$ is general in $\overline{F}$ and the fiber $W$ of $\Psi_{\overline{F}}$ over $(u, [\bar{c}_1])$ is of dimension $\dim \Gamma(2, 2, 2) - \dim U = 18 - 14 = 4$.

Let $S := \{\bar{c}(t) \mid t \in \mathbb{P}^1\}$ be the connected family of splitting conics containing $\bar{c}_1$ and $\bar{c}_2$. If $(g'_1, g'_2, q')$ is a point of the fiber $W$, then we have two members $\bar{c}'_1 = \{g'_1 = 0\}$ and $\bar{c}'_2 = \{g'_2 = 0\}$ of $S$, and thus we have a morphism $W \to \mathbb{P}^1 \times \mathbb{P}^1$, where $\mathbb{P}^1$ is the base curve of the family $S$. If two points $(g'_1, g'_2, q')$ and $(g''_1, g''_2, q'')$ of $W$ are mapped to the same point of $\mathbb{P}^1 \times \mathbb{P}^1$, then there exist scalars $\lambda_1, \lambda_2 \in \mathbb{C}^*$ such that $g''_1 = \lambda_1g'_1$ and $g''_2 = \lambda_2g'_2$. By the dimension reason, we see that $W \to \mathbb{P}^1 \times \mathbb{P}^1$ is dominant. Hence, if $\bar{c}_1 = \{g_1 = 0\}$ and $\bar{c}_2 = \{g_2 = 0\}$ are general members of the family $S$, there exists a polynomial $q \in \Gamma(2)$ such that $(g_1, g_2, q) \in W$, that is, $Q_u$ is defined by $g_1g_2 + q^2 = 0$. $\square$
The following two propositions are confirmed by direct computation.

**Proposition 7.13.** Among the 1953 non-ordered pairs \( \{ [\bar{c}_1], [\bar{c}_2] \} \) of distinct elements \( [\bar{c}_1], [\bar{c}_2] \) of \( \mathcal{T}_u \), exactly 945 pairs satisfy \( I([\bar{c}_1], [\bar{c}_2]) = I_B \), and the remaining 1008 pairs satisfy \( I([\bar{c}_1], [\bar{c}_2]) = I_C \). When \( u = b \), these two sets of pairs are the orbits of the monodromy action of \( \pi_1(\mathcal{U}, b) \) on the set of non-ordered pairs of elements of \( \mathcal{T}_b \).

Recall that each connected family \( c \in F_u \) of \( Y \)-lifts of splitting conics contains exactly 6 reducible members, and the irreducible components \( l, l' \) of a reducible member satisfy \( (l, l')_Y = 1 \). We have a surjective map
\[
\{ (l, l') \mid l, l' \in L_u, (l, l')_Y = 1 \} \to F_u
\]
defined by \( (l, l') \mapsto [l] + [l'] \). Each fiber of size 6. The following gives how the cases \( I([\bar{c}_1], [\bar{c}_2]) = I_B \) and \( I([\bar{c}_1], [\bar{c}_2]) = I_C \) are distinguished.

**Proposition 7.14.** Let \( [c_1] \) and \( [c_2] \) be elements of \( F_u \), and let \( [\bar{c}_1] \) and \( [\bar{c}_2] \) be their images by \( F_u \to \mathcal{T}_u \). Then \( I([\bar{c}_1], [\bar{c}_2]) = I_B \) holds if and only if there exists a triangle \( \{l_1, l_2, l_3\} \) on \( Y_u \) such that \( [c_1] = [l_1] + [l_3] \) and \( [c_2] = [l_2] + [l_3] \).

7.4. **Pairs of a bitangent and a splitting conic.** Let \( l \) be a bitangent of \( Q_u \) with \( \pi_u^a(\bar{l}) = l + l' \), and let \( \bar{c} \) be a splitting conic of \( Q_u \) with the normal decomposition \( \pi_u^a(\bar{c}) = c + c' \). We put
\[
J(\bar{l}, [\bar{c}]) := \begin{bmatrix} 
(\bar{l}, c)_Y & (\bar{l}, c')_Y \\
(\bar{l}', c)_Y & (\bar{l}', c')_Y 
\end{bmatrix}.
\]
The matrix \( J(\bar{l}, [\bar{c}]) \) is one of the following:
\[
J_\alpha := \begin{bmatrix} 
0 & 2 \\
2 & 0 
\end{bmatrix} \text{ or } \begin{bmatrix} 
2 & 0 \\
0 & 2 
\end{bmatrix},
\]
\[
J_\beta := \begin{bmatrix} 
1 & 1 \\
1 & 1 
\end{bmatrix}.
\]
By direct computation, we confirm the following:

**Proposition 7.15.** Let \( \bar{l} \) be a bitangent of \( Q_u \), and \( \bar{c} \) a splitting conic of \( Q_u \). Then \( J(\bar{l}, [\bar{c}]) \) is equal to \( J_\alpha \) if and only if the connected family \( [\bar{c}] \in \mathcal{T}_u \) of splitting conics has a singular member containing \( \bar{l} \) as an irreducible component.

When \( u = b \), the monodromy action of \( \pi_1(\mathcal{U}, b) \) acts on the set of pairs \( (\bar{l}, [\bar{c}]) \in \mathcal{T}_b \times \mathcal{T}_b \) with \( J(\bar{l}, [\bar{c}]) = J_\alpha \) transitively, and the set of pairs \( (\bar{l}, [\bar{c}]) \) with \( J(\bar{l}, [\bar{c}]) = J_\beta \) also transitively.

8. **Intersection graph**

**Definition 8.1.** An intersection graph is a pentad \( (V_l, V_c, T, E_{\bar{c}\bar{c}}, E_{\bar{l}\bar{c}}) \) such that
- \( V_l \) and \( V_c \) are finite sets,
- \( T \) is a subset of \( S_0^0(V_l) \),
- \( E_{\bar{c}\bar{c}} \) is a map \( S^2(V_c) \to \{A, B, C\} \), and
- \( E_{\bar{l}\bar{c}} \) is a map \( V_l \times V_c \to \{\alpha, \beta\} \).

Two intersection graphs \( (V_{l'}, V_{c'}, T, E_{\bar{c}'\bar{c}'}, E_{\bar{l}'\bar{c}'}) \) and \( (V_{l''}, V_{c''}, T', E_{\bar{c}''\bar{c}''}, E_{\bar{l}''\bar{c}''}) \) are isomorphic if there exists a pair of bijections \( V_l \cong V_{l''} \) and \( V_c \cong V_{c''} \) that induces \( T \cong T' \), \( E_{\bar{c}\bar{c}} \cong E_{\bar{c}''\bar{c}''} \), and \( E_{\bar{l}\bar{c}} \cong E_{\bar{l}''\bar{c}''} \).
Table 9.1. The orbit decomposition for $(m, n) = (6, 0)$

| $i$ | $|\alpha_i|$ | $|T|$ | $a_0$ | $a_1$ | $a_2$ |
|-----|--------------|------|------|------|------|
| 1   | 2016         | 0    | 0    | 0    | 0    |
| 2   | 1008         | 0    | 0    | 0    | 0    |
| 3   | 30240        | 4    | 0    | 0    | 6    |
| 4   | 60480        | 6    | 0    | 6    | 9    |
| 5   | 22680        | 8    | 2    | 10   | 16   |
| 6   | 181440       | 8    | 2    | 14   | 12   |
| 7   | 5040         | 8    | 4    | 12   | 12   |
| 8   | 12096        | 10   | 0    | 30   | 15   |
| 9   | 60480        | 10   | 2    | 24   | 19   |
| 10  | 1260         | 12   | 6    | 30   | 30   |

**Definition 8.2.** For a $Q^{(m,n)}$-curve $Z$ as in (5.4), we define an intersection graph $g(Z) := (V_l, V_e, T, E_{\bar{c}c}, E_{\bar{t}t})$ by the following:

- $V_l$ is $\{\bar{t}_1, \ldots, \bar{t}_m\}$ and $V_e$ is $\{\bar{c}_1, \ldots, \bar{c}_n\}$,
- $T$ is the set of liftable triangles $\{\bar{t}_i, \bar{t}_j, \bar{t}_k\} \subset \{\bar{t}_1, \ldots, \bar{t}_m\}$,
- $E_{\bar{c}c}(\bar{c}_i, \bar{c}_j)$ is the type of the matrix $I([\bar{c}_i], [\bar{c}_j])$ defined in Section 7.3, and
- $E_{\bar{t}t}(\bar{t}_i, \bar{t}_j)$ is the type of the matrix $J(\bar{t}_i, [\bar{c}_j])$ defined in Section 7.4.

**Remark 8.3.** By Proposition 7.12, the relation $\bar{c}_i \sim \bar{c}_j \iff E_{\bar{c}c}(\bar{c}_i, \bar{c}_j) = A$ is an equivalence relation on $V_{\bar{c}}$, and the functions $E_{\bar{e}e}$ and $E_{\bar{t}t}$ are compatible with this equivalence relation.

**Remark 8.4.** When $n = 0$, the intersection graph equal to the two-graph in $[5]$.

It is obvious that, if $\zeta$ and $\zeta'$ are in the same connected component of $Z' \cap \bar{c}_j$ are isomorphic. The converse is not true in general, as examples in the next section show.

9. **Examples**

9.1. **The case $(m, n) = (6, 0)$**. We have $|P_b^{(6,0)}| = 376740$. The action of $W(E_7)$ decomposes $P_b^{(6,0)}$ into orbits as in Table 9.1. For each orbit $\alpha_i \subset P_b^{(6,0)}$, we choose a point $\zeta \in \alpha_i$ and indicate the following data of the intersection graph $g(Z_\zeta)$ of $Z_\zeta = Q_\zeta + \bar{t}_1 + \cdots + \bar{t}_6$: $|T| = k$ is the number of the liftable triangles $t_1, \ldots, t_k$ in $\{\bar{t}_1, \ldots, \bar{t}_6\}$, and $a_\nu$ is the number of pairs $\{t_i, t_j\}$ of liftable triangles such that $|t_i \cap t_j| = \nu$. The orbit $\alpha_1$ and $\alpha_2$ cannot be distinguished by the two-graph $(V_l, T)$, but they belong to different $W(E_7)$-orbits, and hence the corresponding $Q^{(6,0)}$-curves are of different homeomorphism types.

9.2. **The case $n = 0$**. We continue to consider the case where $n = 0$. From the two-graph $g = (V_l, T)$, we can construct a graph $\bar{g}$ whose set of vertices is $T$ and whose edge connecting $\bar{t}_i, \bar{t}_j \in T$ has weight $|t_i \cap t_j|$. If the graphs $\bar{g}$ and $\bar{g}'$ are not isomorphic as graphs with weighted edges, then the two-graphs $g$ and $g'$ are not isomorphic. Using this method, we prove the following:
Table 9.2. The orbit decomposition for \((m, n) = (0, 3)\)

| \(i\) | Edge labels | Orbit sizes |
|-------|-------------|-------------|
| 1     | AAA         | 63          |
| 2     | ABB         | 1890        |
| 3     | ACC         | 2016        |
| 4     | BBB         | 3780 + 315  |
| 5     | BBC         | 15120       |
| 6     | BCC         | 15120       |
| 7     | CCC         | 5040 + 336  |

Table 9.3. The orbit decomposition for \((m, n) = (2, 2)\)

| \(i\) | \(E_{\overline{c}}\) | \(E_{\overline{c}\overline{c}}\) | Orbit sizes |
|-------|----------------|----------------|-------------|
| 1     | \([\alpha, \alpha], [\alpha, \alpha]\) | A             | 3780 + 378  |
| 2     | \([\alpha, \alpha], [\alpha, \alpha]\) | B             | 3780 + 1890 |
| 3     | \([\alpha, \alpha], [\alpha, \alpha]\) | C             | 15120       |
| 4     | \([\alpha, \alpha], [\alpha, \beta]\) | B             | 60480       |
| 5     | \([\alpha, \alpha], [\alpha, \beta]\) | C             | 60480 + 12096 |
| 6     | \([\alpha, \beta], [\alpha, \beta]\) | A             | 12096       |
| 7     | \([\alpha, \beta], [\alpha, \beta]\) | B             | 30240       |
| 8     | \([\alpha, \beta], [\alpha, \beta]\) | C             | 60480       |
| 9     | \([\alpha, \beta], [\alpha, \beta]\) | B             | 45360 + 7560 |
| 10    | \([\beta, \beta], [\alpha, \beta]\) | C             | 30240 + 6048 |
| 11    | \([\beta, \beta], [\alpha, \beta]\) | B             | 120960      |
| 12    | \([\beta, \beta], [\alpha, \beta]\) | C             | 120960      |
| 13    | \([\beta, \beta], [\alpha, \beta]\) | B             | 22680 + 3780 |
| 14    | \([\beta, \beta], [\alpha, \beta]\) | C             | 45360       |
| 15    | \([\beta, \beta], [\beta, \beta]\) | A             | 7560        |
| 16    | \([\beta, \beta], [\beta, \beta]\) | B             | 22680 + 3780 |
| 17    | \([\beta, \beta], [\beta, \beta]\) | C             | 45360       |

**Proposition 9.1.** Except for the two orbits \(o_1\) and \(o_2\) in the case \(m = 6\) described in Section 9.1, all \(W(E_7)\)-orbits of \(P_b^{(m,0)}\) are distinguished by their two-graphs. □

**Example 9.2.** Let \(o'_1\) and \(o'_2\) be the orbits in \(P_b^{(22,0)}\) containing 22-tuples obtained by taking the complement in \(L_b\) of 6-tuples in the orbits \(o_1 \subset P_b^{(6,0)}\) and \(o_2 \subset P_b^{(6,0)}\) above, respectively. Let \(g'_1\) and \(g'_2\) be the two-graphs of \(o'_1\) and \(o'_2\). We have \(|T| = 600\) for both \(g'_1\) and \(g'_2\). The associated graphs \(\tilde{g}'_1\) and \(\tilde{g}'_2\) with weighted edges are not isomorphic. The graph \(\tilde{g}'_1\) has exactly 8203640 triples \(\{t_\lambda, t_\mu, t_\nu\}\) of lifttable triangles with weight \(|t_\lambda \cap t_\mu| = |t_\mu \cap t_\nu| = |t_\nu \cap t_\lambda| = 0\), whereas the number of such triples in \(\tilde{g}'_2\) is 8203760.

9.3. **The case \((m, n) = (0, 3)\).** By Remark 8.3, the three edges of the graph \((V_\overline{c}, E_{\overline{c}\overline{c}})\) are labelled as in the second column of Table 9.2. The set \(P_b^{(0,3)}\) of size 43680 is decomposed into nine \(W(E_7)\)-orbits with sizes given in the third column of Table 9.2.
9.4. The case \((m, n) = (2, 2)\). There exist 17 intersection graphs indicated in Table 9.3, where \(E \bar{c} \bar{c}\) is shown by the type of \(I(\bar{c}_1, [\bar{c}_2])\), and
\[
E_{\bar{c}\bar{c}} := \left[ \left[J(I_1, [\bar{c}_1]), J(I_1, [\bar{c}_2])\right], \left[J(I_2, [\bar{c}_1]), J(I_2, [\bar{c}_2])\right] \right].
\]
The set \(P_b^{(2, 2)}\) of size 762048 is decomposed into 23 orbits by the action of \(W(E_7)\), and their sizes are given in the 4th column of Table 9.3.

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