Classicization and the relativistic correction to the Yukawa potential

Qi Chen* and Kaixun Tu†
Department of Physics, Tsinghua University, Beijing 100084, China

Qing Wang‡
Department of Physics, Tsinghua University, Beijing 100084, China and Center for High Energy Physics, Tsinghua University, Beijing 100084, China

Abstract

The relativistic correction for the Yukawa potential needs to be considered in self-interacting dark matter, high velocity nucleon-nucleon scattering, neutron star binaries and non-static quark pairs. To avoid tedious calculations in non-relativistic effective field theory, we propose a new classicization scenario focusing on the correspondence between quantum states and classical observables, which both describe the physical systems. We classicize the Yukawa interaction through this method and get the corresponding relativistic classical theory, from which we derive the effective potential to the next-to-leading order of the relativistic correction.

*Electronic address: chenq20@mails.tsinghua.edu.cn
†Electronic address: tkx19@mails.tsinghua.edu.cn
‡Electronic address: wangq@mail.tsinghua.edu.cn
I. INTRODUCTION

Yukawa theory is widely used to describe the interaction between fermions through a scalar mediator, such as the interaction of nucleons. In particular, the fast moving nucleon-nucleon scattering calls a Yukawa potential with relativistic corrections\[^1\]. In dark phenomenology, self-interacting dark matter (SIDM) seems to be a better explanation for observed galactic structures than collisionless models. This is especially so for dark matter cores in low-mass galaxies, which can be understood via Yukawa interaction between cold dark matter particles\[^2\]. If dark matter travels in high speed, such as two dark particles coming very close to each other, then the relativistic corrections of Yukawa theory should be considered. In a neutron star binary system, each neutron star can create axions that act as light scalar mediators and undergo Yukawa interaction\[^3\]–^5\]. If the two neutron stars come close to each other, then the high speed of them causes relativistic corrections of Yukawa theory. Furthermore, if we consider charmonium moving in quark-gluon plasma, it is also important to include relativistic corrections for the Yukawa-like interaction term\[^12\]. Usually, we use effective field theory approaching non-relativistic effective field theory (NREFT) from the full Lagrangian. However, NREFT is sometimes a little tedious, and the subjects we are interested in may be classical like neutron stars, which need a relativistic classical theory of Yukawa model. Therefore, we seek a fully classical Yukawa interaction that includes relativistic corrections. What’s more, our classicization procedure could also help people get the classical limit of the quantum gravity\[^13\]. This is because in loop quantum gravity, gravity has been quantized analogying with lattice QCD, and the quantum states are not built on the traditional fock space with creation-annihilation operators, instead quantum gravity uses the spin network basis which are defined on the coordinate of real spacetime not in the momentum-space\[^14\].

There are several popular procedures for constructing a classical theory from a given quantum field theory. Such methods can be roughly classified into two categories. One is the $\hbar \to 0$ scenario proposed by Planck and Einstein separately to research the low frequency and high temperature asymptotic behavior of blackbody radiation. In this case, the energy spectrum converges to the classical equipartition theorem. The other is the $N \to \infty$ scenario introduced by Bohr in his correspondence principle, in which the classical theory emerges from the quantized one when the size of the system goes to infinity. (Strictly speaking, the $N \to \infty$ limit can be mathematically viewed as a special case of the $\hbar \to 0$ scenario. However, their underlying physical situations are quite different\[^15\].) In quantum field theory, there is another method. One can replace the field in the quantized Lagrangian with solutions of the equation of motion\[^16, 17\]. However, none of them can explain the classical world, though they give some equations looking formally like classical theory.

For the $\hbar \to 0$ scenario, we may consider such a classicization procedure with a path integral. For example, in a path integral, each path $x(t)$ beginning at $t_1$ and ending at $t_2$ is associated with a weight function $\exp[iS(x; t_1, t_2)/\hbar]$. If we take the limit $\hbar \to 0$, only the path extreme action $\delta S(x; t_1, t_2) = 0$ can coherently survive. According to the stationary action principle, the remaining path is the solution of the classical equation of motion. This means we have written the classical theory from the quantized one\[^18\]. Some researchers
prefer embedding $\hbar$ into a geometric quantization to investigate the different mathematical structures between classical and quantum theory[19–21]. In these cases, we hope that a quantum system described by a Hilbert space and operator algebras that act irreducibly on the Hilbert space can return to the symplectic manifolds in classical mechanics when we impose the limit $\hbar \to 0$. However, the $\hbar \to 0$ scenario does not always work well in path integral or geometric classicization. Classical behavior in quantum field theory can only be seen from special observables and states[15, 21]. For example, fermion dynamics classicization via the $\hbar \to 0$ limit is physically unsound. The Grassmann numbers in quantum field theory break the Poisson structure in the $\hbar \to 0$ limit[22, 23]. Another serious problem is that we may not even be able to take the limit $\hbar \to 0$ in some cases. Applying such a limit to quantum field theory may not be a good approximation because of the infinite degrees of freedom. To illustrate the potential problem, we focus on scalar field theory for convenience but without loss of generality. The action of a spinless field can be written as

$$S = \int dt \int dx \left[ \frac{1}{2} \left( \frac{\partial \phi(x,t)}{\partial t} \right)^2 - \frac{1}{2} c^2 \left( \frac{\partial \phi(x,t)}{\partial x} \right)^2 \right] - \frac{1}{2} \left( \frac{mc^2}{\hbar} \right)^2 \phi^2(x,t)$$

This can be regularized through a lattice, and the weight function of the path integral becomes

$$\exp \left( \frac{iS}{\hbar} \right) = \exp \left\{ \frac{i}{\hbar} \int dt \sum_n \left[ \frac{1}{2} \left( \frac{\partial \phi_n(t)}{\partial t} \right)^2 - \frac{1}{2} c^2 \left( \frac{\phi_{n+1}(t) - \phi_n(t)}{a} \right)^2 \right] - \frac{1}{2} \left( \frac{mc^2}{\hbar} \right)^2 \phi^2_n(t) \right\}$$

which presents $\hbar$ in the denominator with numerator $a$. It is worth noting that the $\hbar \to 0$ limit does not truly take the dimensional constant $\hbar$ to 0. Instead, we take limiting cases of parameters associated with $\hbar$, such as taking the high temperature limit (putting $\hbar \nu/kT$ into 0 by taking $T \to \infty$) to recover the classical theory of blackbody radiation. However, in our problem, $a$ is an infinitesimal parameter that avoids the need to take the $\hbar \to 0$ limit. The $\hbar \to 0$ limit therefore raises ambiguities in such situations.

In this work, we propose a classicization scenario to avoid the $\hbar \to 0$ limit. A quantum system(including quantum field theory) is described in terms of states in Hilbert space while classical theory is described by classical fields or classical physical observables. The differences between these two descriptions encourage us to research the corresponding relations in both sides. Coherent states in a quantum system correspond most closely to classical theory. However, coherent states are usually presented as eigenstates of creation and annihilation operators. In other words, coherent states are presented in Fock space, where it is hard to investigate the relations between quantum states and classical fields or observables. To explain the structures of coherent states precisely, we derive them in a new procedure where we treat the coherent states in a representative space spanned by the eigenstates of quantum field operators. We take Yukawa theory as a research model. States that look like Gaussian wave packets can be interpreted as quantum states corresponding to classical fields. If we evaluate Gaussian wave packet-like states in a representative space spanned by eigenstates of quantized fields, we discover that the evolution of quantum states under a specific Hamiltonian coincides with the Hamilton equation of motion in classical field theory. This surprising coincidence implies that we can treat a function in the wave packet
as a classical canonical variable, and the correspondence emerges from this interpretation. Consequently, we conclude that the classicization is finished.

Once we write down the classical and relativistically invariant Lagrangian or Hamiltonian, which has two separate components for fermionic particles and a bosonic field, we can derive the equation of motion from these two components. The solution for the bosonic part describes the interaction between two fermions. To consider the relativistic correction to arbitrary order of $\frac{v}{c}$, we just need to expand the relativistic bosonic field induced by a moving charged particle to any order we want. The procedure is just like the relativistic correction in electrodynamics[24]. Using this together with the fermion Lagrangian, we can get the relativistically corrected Yukawa potential to any order. However, the solution to an equation of motion in Yukawa interaction is much more difficult than in electrodynamics because the retarded potential in electrodynamics only depends on one trajectory point, while in Yukawa theory, the retarded potential depends on all trajectory points in the past light cone because of the mass term.

This paper is organized as follows: In Sec. II, we first review classical and quantum electrodynamics and propose a classicization conjecture based on these two theories. We then use the conjecture in Yukawa theory to reproduce the Yukawa potential. Finally, we systematically classicize Yukawa theory through correspondence between quantum states and classical observables. In Sec. III, we solve the equation of motion for a bosonic field and consider the relativistic corrections to the next-to-leading order. Section IV presents discussion and conclusions.

II. CLASSICIZATION OF QUANTUM FIELD THEORY

In this section, we introduce classicization of Yukawa interaction to indicate how we can write the classical action or Lagrangian from a given quantum field, and suggest what happens when we revert to classical theory from quantum field theory.

We focus on quantum electrodynamics (QED) first because both quantum and classical theory are well established. Through classicization of QED, we explain the physical and mathematical meaning obtained from the procedure II A. Then, we investigate classical Yukawa theory analogized to QED classicization. To confirm the validity of the analogy or conjecture, we calculate the Lagrangians for both fermionic particles and bosonic fields to the lowest order, which reproduces the Yukawa potential and Newton’s second law of motion II B. Finally, we systematically demonstrate why the conjecture is true. We use the fermionic field to describe the matter field and the bosonic field to represent the interaction between matter fields. In classical field theory, we prefer to use the particle concept rather than the fermionic field, which requires some procedure to reduce the quantum field theory to the particle version. We treat this procedure systematically in the final part of the section II C.
A. QED classicization

QED is the most precise and best known theory for which calculation agrees with experiment. For example, the theoretical prediction of an anomalous magnetic moment agrees remarkably well with experimental data. We also have a good understanding of both the quantized and classical theories of electromagnetism. Therefore, we can compare QED with the classical theory to understand classicization. We write the QED Lagrangian first:

\[ \mathcal{L}_{\text{QED}} = i \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - e \bar{\psi} \gamma^\mu \psi A_\mu \]  

(1)

For classical electrodynamics (ignoring the magnetic moment of the particle), the action can be written as

\[ S^1_{\text{QED}} = -mc^2 ds - e \int A_\mu dx^\mu \]

\[ S^2_{\text{QED}} = \int d^4x \left( -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - ej^\mu A_\mu \right) \]  

(2)

where \( ds = \sqrt{1 - v^2/c^2} dt \) is the Lorentz invariant spacetime interval and \( J^\mu \) is the electric current density. The first equation is the action for a charged particle in an electromagnetic field while the second represents the evolution of the electromagnetic field for a given source. It should be noted that we cannot combine \( S^1_{\text{QED}} \) and \( S^2_{\text{QED}} \) directly into a total action such as \( S_{\text{QED}} = S^1_{\text{QED}} + S^2_{\text{QED}} \) in classical theory. The quantity \( A_\mu \) is the external electromagnetic potential in \( S^1_{\text{QED}} \). If \( A_\mu \) is the total electromagnetic potential, it will diverge at the position of the point-like charged particle. Furthermore, the corresponding Lagrangians for \( S^1_{\text{QED}} \) and \( S^2_{\text{QED}} \) separately are:

\[ L^1_{\text{QED}} = -m \sqrt{1 - v^2/c^2} - eA_0 + eA \cdot v/c \]

\[ L^2_{\text{QED}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - ej^\mu A_\mu \]  

(3)

The equation of motion for each Lagrangian or Lagrangian density yields the Lorentz force law and the Maxwell equations:

\[ \frac{d}{dt} \left( \frac{mv}{\sqrt{1 - v^2/c^2}} \right) = e \left( E + v/c \times B \right) \]

\[ \partial_\mu F^{\mu\nu} = ej^\nu \]  

(4)

Therefore, comparing \( L^1_{\text{QED}} \) and \( L^2_{\text{QED}} \) with \( L_{\text{QED}} \), we can draw a conjecture for writing down the classical Lagrangian for a given quantum field and reverting to classical theory.

**Conjecture**: For a bosonic field, we just copy the form of a quantum field Lagrangian. For a point particle, we have the following correspondence:

\[ i \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi \rightarrow -m \sqrt{1 - v^2/c^2} \]  

(5)

Let us illustrate more about the conjecture[32]. In quantum field theory, fermion is described by fermionic field. For example, electron and quark are described by spinor field \( \psi \). But
in classical theory, fermion like electron is viewed as a massive point instead of fields. This difference suggests us that there could be a correspondence between field scenario and mass point scenario. From the classical and quantum electrodynamics, we see such a correspondence. Thus, we propose a conjecture from this indication. And in the following discussion, we prove this conjecture in Yukawa theory.

B. Towards classicization of Yukawa theory

With the correspondence between quantum and classical theory, we try to classicize Yukawa theory in this section. Using a procedure similar to that in II A, we start with the Yukawa interaction Lagrangian in quantum field theory and use the conjecture to write the classical Yukawa Lagrangian. We expect the first order of our results to reproduce Newton’s second law of motion and the Yukawa potential. Yukawa interaction has a more simplified Lagrangian:

\[
L_{\text{Yukawa}} = i\hbar \bar{\psi} \gamma^\mu \partial_\mu \psi - m_1 c^2 \bar{\psi} \psi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \left( \frac{m_0 c^2}{\hbar} \right)^2 \phi^2 - \hbar \frac{1}{2} c^2 g \phi \bar{\psi} \psi \]

where \( m_1 \) and \( m_0 \) are the fermion and boson masses respectively, and \( \partial_\mu \equiv c \frac{\partial}{\partial x^\mu} \). Guided by the conjecture in II A, we can infer the following correspondence:

\[
i\hbar \bar{\psi} \gamma^\mu \partial_\mu \psi - (m_1 c^2 + \hbar \frac{1}{2} c^2 g \phi) \bar{\psi} \psi \rightarrow -(m_1 c^2 + \hbar \frac{1}{2} c^2 g \phi) \sqrt{1 - \dot{v}^2/c^2} \]

Consequently, the classical Yukawa interaction action yields

\[
S^1_{\text{Yukawa}} = - \int \left( m_1 c^2 + \hbar \frac{1}{2} c^2 g \phi \right) ds \]

\[
S^2_{\text{Yukawa}} = \int d^4x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \left( \frac{m_0 c^2}{\hbar} \right)^2 \phi^2 - \hbar \frac{1}{2} c^2 g j \phi \right) \]

where \( j \) is the classical external source. It is worth noting that there is a scalar current in both quantum and classical theory in addition to the vector current. Classical Yukawa interaction contributes a mass correction for a particle acted on by a force, which is different from electrodynamics. We tend to consider the equation of motion for two actions. If we consider only a point source created by another Yukawa-type charged particle, the equation of motion yields

\[
- \ddot{\phi} + c^2 \nabla_1^2 \phi - \left( \frac{m_0 c^2}{\hbar} \right)^2 \phi = \hbar \frac{1}{2} c^2 g \sqrt{1 - \dot{r}_2^2(t_1)/c^2} \delta(r_1 - r_2(t_1)) \]

\[
\frac{d}{dt} \left[ \left( m_1 c^2 + \hbar \frac{1}{2} c^2 g \phi \right) \frac{\dot{r}_1/c^2}{\sqrt{1 - \dot{r}_1^2/c^2}} \right] = -\hbar \frac{1}{2} c^2 g \sqrt{1 - \dot{r}_1^2/c^2} \nabla_1 \phi \]

where \( r_2(t_1) \) is the trajectory of the source and \( r_1 \) is the position vector of the forced particle. The equation of motion for a bosonic field is obviously a wave equation with a nonlinear
source term \( g \sqrt{1 - \dot{r}_2^2(t_1)/c^2} \delta(r_1 - r_2(t_1)) \). For such a wave equation, it is convenient to write the solution as a Green function:

\[
\phi(r_1, t_1) = \int dt_2 h^2 c^2 g \sqrt{1 - \dot{r}_2^2(t_2)/c^2} G(r_1, r_2(t_2); t_1, t_2)
\]

(11)

where \( G(r_1, r_2; t_1, t_2) \) is the Green function:

\[
-\ddot{G}(r_1, r_2; t_1, t_2) + c^2 \nabla_1^2 G(r_1, r_2; t_1, t_2) - \left( \frac{m_0 c^2}{\hbar} \right)^2 G(r_1, r_2; t_1, t_2)
\]

\[
= \delta(r_1 - r_2) \delta(t_1 - t_2)
\]

(12)

Obviously, \( G(r_1, r_2; t_1, t_2) \) is the Green function of a massive scalar field, for which the solution is just the propagator for a massive scalar field:

\[
G(r_1, r_2; t_1, t_2) = \frac{i}{\hbar^2} \int \frac{d^3p}{2p^0(2\pi)^3} \left[ e^{ip(x-y)} - e^{-ip(x-y)} \right]
\]

(13)

where \( p^0 = \sqrt{c^2 p^2 + m_0^2 c^4} \). Because we consider the time-like region and the Green function depends only on \( r_1 - r_2 \), we can just evaluate \( G(0, t') \) to get the exact solution through a Lorentz boost. Hence, without loss of generality, we focus on \( G(0, t') \) first:

\[
G(0, t') = \frac{m_0}{4\pi t' \hbar c} J_1 \left( \frac{m_0 c^2 t'}{\hbar} \right) + \frac{m_0^2 c}{4\hbar^2} \delta \left( \frac{m_0 c^2 t'}{\hbar} \right) Y_1 \left( \frac{m_0 c^2 t'}{\hbar} \right)
\]

(14)

where \( t' = \sqrt{t^2 - r^2/c^2} \) and \( r = |r_1 - r_2| \) is the distance between the source and the particle acted on by the force. The general solution of the Green function (15) is the boosted \( G(0, t') \), where \( t' \) is replaced by the spacetime interval \( \sqrt{(t_2 - t_1)^2 - (r_1 - r_2)^2 / c^2} \):

\[
G(r_1, r_2; t_1, t_2) = \frac{m_0}{4\pi \hbar c \sqrt{(t_2 - t_1)^2 - (r_1 - r_2)^2 / c^2}} J_1 \left( \frac{m_0 c^2 \sqrt{(t_2 - t_1)^2 - (r_1 - r_2)^2 / c^2}}{\hbar} \right)
\]

\[
+ \frac{m_0^2 c}{4\hbar^2} \delta \left( \frac{m_0 c^2 \sqrt{(t_2 - t_1)^2 - (r_1 - r_2)^2 / c^2}}{\hbar} \right) Y_1 \left( \frac{m_0 c^2 \sqrt{(t_2 - t_1)^2 - (r_1 - r_2)^2 / c^2}}{\hbar} \right)
\]

(15)

The quantities \( J_1 \) and \( Y_1 \) are Bessel functions of the first and second kind. Note that the term proportional to \( Y_1 \) is canceled by the boundary term in integration by parts. Therefore, we can focus on the first term only. Substituting the Green function (15) into the solution for the equation of motion (11), we find that

\[
\phi(r_1, t_1) = \int_{-\infty}^{t_1} dt_2 h^2 c^2 g \sqrt{1 - \dot{r}_2^2(t_2)/c^2} \frac{m_0}{4\pi \hbar c \sqrt{(t_2 - t_1)^2 - (r_1 - r_2)^2 / c^2}} J_1 \left( \frac{m_0 c^2 \sqrt{(t_2 - t_1)^2 - (r_1 - r_2)^2 / c^2}}{\hbar} \right)
\]

(16)
To check the validity of the correspondence in Yukawa theory, we consider a special case where \( r_2(t_2) \) is time-dependent, \( r_2(t_2) = r_2 \). If we keep the lowest order of \( 1/c \) and use a variable transformation, the solution for the equation of motion is

\[
\phi(r_1, t_1) = \int_{-\infty}^{t_1} dt_2 \frac{\hbar c^3 g}{4\pi \hbar c \sqrt{(t_2 - t_1)^2 - (r_1 - r_2)^2 / c^2}} J_1 \left( \frac{m_0 c^2 \sqrt{(t_2 - t_1)^2 - (r_1 - r_2)^2 / c^2}}{\hbar} \right)
\]

\[
= -\frac{\hbar c^{-\frac{3}{2}} g}{4\pi |r_1 - r_2|} \exp \left( -\frac{m_0 c^2}{\hbar c} |r_1 - r_2| \right)
\]

(17)

We should emphasize again that \( r_2 \) is a fixed constant representing the position of the source particle.

Substituting the solution for the equation of motion into (10) and considering also the lowest order, we can get:

\[
m_1 \frac{d}{dt} \dot{r}_1 = -\hbar c^{\frac{3}{2}} g \nabla_1 \phi
\]

(18)

According to Newtonian second law of motion, the force acting on particle is \( F = -g \nabla_1 \phi \), then the corresponding potential is just Yukawa potential describing interaction between nucleons[25]:

\[
V(r) = -\frac{\hbar c g^2}{4\pi r} e^{-\frac{m_0 c^2}{\hbar c} r}
\]

(19)

We now can conclude the conjecture may be valid because it reproduces the Yukawa potential and the equation of motion for a particle under a force in a Yukawa potential. However, the physical insight in that conjecture is not as clear. In the next section, we derive the classical theory more carefully and look for a systematic method that reveals the physical meaning of the conjecture.

**C. Systematic approach to classicization**

In quantum field theory, particles are viewed as excitations of a field, which is a little different from classical field theory. In the latter, we prefer to write the Lagrangian in separate parts, one for point-like particles and the other for fields. Therefore, to analyze classical field theory, we split the Yukawa Lagrangian into two parts:

\[
\mathcal{W}_1 = i\hbar \bar{\psi} \gamma^\mu \partial_\mu \psi - m_1 c^2 \bar{\psi} \psi - \frac{\hbar c^{\frac{3}{2}} g}{4\pi} \phi \bar{\psi} \psi
\]

\[
\mathcal{W}_2 = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \left( \frac{m_0 c^2}{\hbar} \right)^2 \phi^2 - \hbar c^{\frac{3}{2}} g \phi j
\]

(20)

where \( j \) is interpreted as a classical external source and can be derived from quantum theory, which will be presented later. The first Lagrangian \( \mathcal{W}_1 \) describes a massive fermionic field interacting with a bosonic field, which is like the classical Lagrangian for a particle in a Yukawa-type field. The second part \( \mathcal{W}_2 \) is nearly the Lagrangian for a free scalar field except for a classical external source. In the next two subsections, we will systematically prove the correspondence between classical and quantum theory.
1. The quantum state corresponding to the classical field

In quantum field theory, we use the quantum states in Hilbert space to describe the evolution of the system, while preferring to choose the field $\phi(x, t)$ itself and its conjugate momentum $\pi(x, t)$ in classical field theory. Thus, if we want to classicize a quantum field theory and revert, we need to figure out the relation between these two different evolution frameworks. Quantum field theory has more degrees of freedom than classical theory and can include more complex physical phenomena. This means there are some states that cannot be governed by classical theory, such as squeezed states, and we do not consider such states in this paper. We focus on states that can work in both quantum and classical theory. Therefore, if we find a quantum state $|\psi(t)\rangle$ that corresponds to the classical fields $\phi(x, t)$ and $\pi(x, t)$, then the time evolution of these classical fields can be derived via the time evolution of the quantum state $|\psi(t)\rangle$. In this part, we focus on the states associated with the classical field that will help us show that $\mathcal{W}_2$ truly yields classical Yukawa field theory.

Let us recall that the vacuum state $|\Omega\rangle$ corresponds to the classical field consisting of $\phi(x, t) = 0$ and $\pi(x, t) = 0$. We can use the Gaussian ansatz to write the vacuum state wave function:

$$\langle \phi | \Omega \rangle = N \exp \left\{ -\frac{1}{2} \int d^3x d^3y \mathcal{E}(x, y) \phi(x) \phi(y) \right\}$$

where $|\phi\rangle$ is the eigenstate of the operator $\hat{\phi}(x)$ with the eigenvalue $\phi(x)$, and $\mathcal{E}(x, y)$ is the kernel determined by the definition of the annihilation operator:

$$\mathcal{E}(x, y) = \frac{1}{\hbar^3} \int \frac{d^3p}{(2\pi)^3} e^{i \frac{p}{\hbar^3}(x-y)} E_p$$

We focus on the vacuum state wave function that looks like a Gaussian wave packet with the center located at $\phi(x) = 0$. This wave function can be generalized to a form in which the center of the wave packet is located at $\phi(x) = \phi_{\text{class}}(x)$. We can write the general Gaussian wave ansatz on the basis of this generalization:

$$\langle \phi | \varphi \rangle = N \exp \left\{ -\frac{1}{2} \int d^3x d^3y \mathcal{E}(x, y) [\phi(x) - f(x)] [\phi(y) - f(y)] \right\}$$

where $N$ is the normalization coefficient, $f(x)$ is an arbitrary function and $f(x)$ is the expected center location for a non-vacuum state wave packet. For a classical field theory to describe quantum states, its phase space or configuration space should be a closed subspace of the quantized theory. This means that all the states have the classical correspondence should still have such correspondence during quantum evolution. Therefore, if a classical field truly corresponds to a quantum state with the wave function represented in (23), then the wave function should maintain a Gaussian ansatz during its evolution:

$$\langle \phi | \varphi(t) \rangle = N(t) \exp \left\{ -\frac{1}{2} \int d^3x d^3y \mathcal{E}(x, y) [\phi(x) - f(x, t)] [\phi(y) - f(y, t)] \right\}$$
where $|\varphi(t)\rangle$ is a time-dependent state defined by the time evolution operator $|\varphi(t)\rangle = e^{-iHt}|\varphi\rangle$ and $H$ is the field Hamiltonian:

$$\hat{H} = \int d^3x \left[ \frac{1}{2} \hat{\pi}^2 + \frac{c^2}{2} (\nabla \hat{\phi})^2 + \frac{1}{2} \left( \frac{m_0 c^2}{\hbar} \right)^2 \hat{\phi}^2 + \hbar^2 c^2 g \hat{\phi} \right] + \Lambda \quad (25)$$

where $\Lambda$ is the vacuum energy. Note that all states or wave functions are governed by the Schrodinger equation, which indicates that we can determine the normalization coefficient $N(t)$ and the center function $f(x,t)$. Furthermore, the consistency of the Schrodinger equation and general Gaussian wave function will prove our ansatz. The calculation is a little tedious, so the details are presented in Appendix A. We just show the final results for the normalization coefficient $N(t)$ and center function $f(x,t)$:

$$N(t) = N_0 \exp \left\{ -\frac{i}{4} \int d^3x d^3y \mathcal{E}(x,y) f(x,t)f(y,t) - \frac{1}{2} \hbar^{-\frac{3}{2}} c^2 g \int_0^t d\tau \int d^3y j(y,\tau)f(y,\tau) \right\} \quad (26)$$

$$f(x,t) = f_1(x,t) + \frac{i}{\hbar} \int d^3y \mathcal{E}_{-1}(x,y) f_2(y,t) \quad (27)$$

where $\mathcal{E}_{-1}(x,y)$ is a newly defined kernel function presented in Appendix A. $f_1(x,t)$ and $f_2(x,t)$ are both real functions that are the solutions of the following equations:

$$\dot{f}_1(x,t) = f_2(x,t)$$

$$\dot{f}_2(x,t) = c^2 \nabla^2 f_1(x,t) - \left( \frac{m_0 c^2}{\hbar} \right)^2 f_1(x,t) - \hbar^{-\frac{3}{2}} c^2 g j(x,t) \quad (28)$$

In order to see the physical meaning of $f_1(x,t)$ and $f_2(x,t)$ more explicitly, we insert (27) back into (24):

$$\langle \phi | \varphi(t) \rangle \sim \exp \left\{ -\frac{1}{2} \int d^3x d^3y \mathcal{E}(x,y) [\phi(x) - f_1(x,t)][\phi(y) - f_1(y,t)] \right\} \times \exp \left\{ \frac{i}{\hbar} \int d^3x f_2(x,t) \phi(x) \right\} \quad (29)$$

Since $f_1(x,t)$ and $f_2(x,t)$ (as well as $\mathcal{E}(x,y)$ and $\phi(x)$) are both real, then the first line in (29) indicates that the wave function $\langle \phi | \varphi(t) \rangle$ is a Gaussian wave packet located around $f_1(x,t)$. Recall that in quantum mechanics, the usual wave packets have another term $\propto e^{ip\cdot x}$ which coincides with the second line in (29). Such coincidence suggests us to interpret $f_2(x,t)$ in (29) as the conjugate momentum. Further more, we will prove that the probability distribution of the wave function in $\phi$-representation $|\langle \phi | \varphi(t) \rangle|^2$ has a peak at $\phi(x) = f_1(x,t)$, while the probability distribution of the wave function in $\pi$-representation $|\langle \pi | \varphi(t) \rangle|^2$ has a peak at $\pi(x) = f_2(x,t)$ which confirms our interpretation.

Therefore, the quantum state $|\varphi(t)\rangle$ is a Gaussian wave packet that corresponds to the classical fields with $\phi_{\text{class}}(x,t) \equiv f_1(x,t)$ and $\pi_{\text{class}}(x,t) \equiv f_2(x,t)$. Replacing $f_1(x,t)$ and
And the Lagrangian corresponding to (28), we can get the equations of motion for the classical fields:

\[ \dot{\phi}_{\text{class}}(\mathbf{x}, t) = \pi_{\text{class}}(\mathbf{x}, t) \]
\[ \dot{\pi}_{\text{class}}(\mathbf{x}, t) = c^2 \nabla^2 \phi_{\text{class}}(\mathbf{x}, t) - \left( \frac{m_0 c^2}{\hbar} \right)^2 \phi_{\text{class}}(\mathbf{x}, t) - \hbar \frac{c^2}{2} g \phi_{\text{class}, j}(\mathbf{x}, t) \] (30)

And (30) can be viewed as the Hamiltonian canonical equations for classical fields, then the corresponding Hamiltonian can be written as:

\[ H_{\text{class}} = \int d^3x \left[ \frac{1}{2} \pi_{\text{class}}^2 + \frac{c^2}{2} (\nabla \phi_{\text{class}})^2 + \frac{1}{2} \left( \frac{m_0 c^2}{\hbar} \right)^2 \phi_{\text{class}}^2 + \hbar \frac{c^2}{2} g \phi_{\text{class}, j} \right] \] (31)

And the Lagrangian corresponding to (31) is exactly the second part of (20).

Let’s analyze the property of the wave function more carefully. Replacing \( f_1(\mathbf{x}, t) \) and \( f_2(\mathbf{x}, t) \) with \( \phi_{\text{class}}(\mathbf{x}, t) \) and \( \pi_{\text{class}}(\mathbf{x}, t) \), we can rewrite the time-dependent wave function (24) as

\[ \langle \phi | \varphi(t) \rangle = \mathcal{N}'(t) \exp \left\{ -\frac{1}{2} \int d^3x d^3y \mathcal{E}(\mathbf{x}, \mathbf{y}) [\phi(\mathbf{x}) - \phi_{\text{class}}(\mathbf{x}, t)] [\phi(\mathbf{y}) - \phi_{\text{class}}(\mathbf{y}, t)] \right\} \times \exp \left\{ \frac{i}{\hbar} \int d^3x \pi_{\text{class}}(\mathbf{x}, t) \phi(\mathbf{x}) \right\} \] (32)

where

\[ \mathcal{N}'(t) = \mathcal{N}(t) \exp \left\{ \frac{1}{2\hbar^2} \int d^3x d^3y \mathcal{E}_{-1}(\mathbf{x}, \mathbf{y}) \pi_{\text{class}}(\mathbf{x}, t) \pi_{\text{class}}(\mathbf{y}, t) - \frac{i}{\hbar} \int d^3x \pi_{\text{class}}(\mathbf{x}, t) \phi_{\text{class}}(\mathbf{x}, t) \right\} \] (33)

Because the normalization coefficient \( \mathcal{N}'(t) \) does not contain the variable \( \phi(\mathbf{x}) \), the terms containing \( \phi \) in the probability distribution obey the relation

\[ | \langle \phi | \varphi(t) \rangle |^2 \sim \exp \left\{ - \int d^3x d^3y \mathcal{E}(\mathbf{x}, \mathbf{y}) [\phi(\mathbf{x}) - \phi_{\text{class}}(\mathbf{x}, t)] [\phi(\mathbf{y}) - \phi_{\text{class}}(\mathbf{y}, t)] \right\} \] (34)

To confirm that the wave packet is localized at \( \phi_{\text{class}}(\mathbf{x}, t) \) and \( \phi_{\text{class}}(\mathbf{y}, t) \), we treat the probability distribution (34) carefully. For convenience, we define the functional \( F[\phi] \) as

\[ F[\phi] \equiv \int d^3x d^3y \mathcal{E}(\mathbf{x}, \mathbf{y}) \Delta \phi(\mathbf{x}) \Delta \phi(\mathbf{y}) \] (35)

where \( \Delta \phi(\mathbf{x}) = \phi(\mathbf{x}) - \phi_{\text{class}}(\mathbf{x}, t) \) and \( \Delta \phi(\mathbf{y}) = \phi(\mathbf{y}) - \phi_{\text{class}}(\mathbf{y}, t) \). We consider the Fourier transformation for \( \phi(\mathbf{x}, t) \):

\[ \Delta \phi(\mathbf{x}) = \int d^3k \phi(\mathbf{k}) e^{i \mathbf{k} \cdot \mathbf{x}} \] (36)

In the Fourier space, \( F[\phi] \) can be expressed in terms of \( \phi(\mathbf{k}) \):

\[ F[\phi] = \hbar c \int d^3k \sqrt{k^2 + m^2c^2} |\phi(\mathbf{k})|^2 \geq 0 \] (37)
The non-negative property of \( F[\phi] \) shows that \( F[\phi(x)] \) reaches its minimum when \( \Delta \phi(x) = \phi(x) - \phi_{\text{class}}(x, t) = 0 \). This means that \( |\langle \phi|\varphi(t)\rangle|^2 \sim e^{-F[\phi]} \) decreases as \( \phi \) goes away from \( \phi_{\text{class}} \). Furthermore, \( \mathcal{E}(x, y) \propto e^{-m|x-y|} \rightarrow 0 \) when \( x - y \rightarrow \infty \), so the entanglement between quantum states with a large spatial separation becomes weak. The state localizes somewhere when viewed from a macro perspective. Therefore, we can conclude that the wave function is a kind of wave packet centered around the classical field \( \phi_{\text{class}}(x, t) \) without changing its shape, which is exactly the meaning of coherent.

A Gaussian wave packet is usually centered at some position in real or momentum space. However, the expression for \( \phi \) is a wave packet centered around the classical field \( \phi_{\text{class}} \). The remaining problem is whether the wave function is centered around \( \pi_{\text{class}}(x, t) \) in the \( \pi \) representation. Thus, we need to investigate the wave function in the \( \pi \) representation. The full calculation is shown in Appendix (B), and we show only the wave function here:

\[
\langle \pi|\varphi(t)\rangle \sim \exp \left\{-\frac{1}{2\hbar^2} \int d^3x d^3y \mathcal{E}(x, y) [\pi(x) - \pi_{\text{class}}(x, t)] [\pi(y) - \pi_{\text{class}}(y, t)] \right\}
\]  

(38)

Obviously, \( \langle \pi|\varphi(t)\rangle \) is a wave packet centered around the classical field conjugate \( \pi_{\text{class}} \). Therefore, the quantum state \( |\varphi(t)\rangle \) is a Gaussian wave function that truly corresponds to the classical field consisting of \( \phi_{\text{class}}(x, t) \) and \( \pi_{\text{class}}(x, t) \). The equation for the center function coincides with the equation of motion for classical fields. With these two facts, we can conclude that the evolution of the classical fields \( \phi(x, t) \) and \( \pi(x, t) \) is governed by the second line of (32), and the Lagrangian is consistent with what we have derived from the conjecture.

It is desired to discuss more about the structure of quantum state (32) which is actually coherent states in a representative space spanned by the eigenstates of quantum field operators. Usually, we would like to introduce the quantum superposition principle through the double slit interference phenomenon analogy to Young’s double slit interference in optics. However, this may have some ambiguities in quantum field theory if we naively view the classical superposition as some results from the quantum superposition principle. To show this, we can denote \( \phi_{\text{class}} \) as the superposition of two classical fields \( \phi_1 \) and \( \phi_2 \), then we can write down the wave function for \( \phi_{\text{class}} \) in the \( \phi \) representation:

\[
\langle \phi|\varphi_{1+2}(t)\rangle = \mathcal{N}(t) \exp \left\{-\frac{1}{2} \int d^3x d^3y \mathcal{E}(x, y) [\phi(x) - \phi_1(x, t) - \phi_2(x, t)] [\phi(y) - \phi_1(y, t) - \phi_2(y, t)] \right\} \\
\times \exp \left\{\frac{i}{\hbar} \int d^3x [\pi_1(x, t) + \pi_2(x, t)] \phi(x) \right\}
\]

(39)

And the wave function for \( \phi_1 \) and \( \phi_2 \) are separately described by (32). From the explicit expression for wave function of \( \phi_{\text{class}} \) in \( \phi \) representation, it is obvious that the wave function for classical field superposition is not the wave function superposition for two classical fields.

\[
\langle \phi|\varphi_{1+2}(t)\rangle \neq \langle \phi|\varphi_1(t)\rangle + \langle \phi|\varphi_2(t)\rangle
\]  

(40)
Therefore, we can not naively view the classical superposition as some macro version of quantum superposition.

Now we can consider the plain waves from the perspective of particle or, in other words, the perspective of quantum state. For example, two plain waves with the same frequency and amplitude can interfere with each other at somewhere as we tune their phase to some proper values. This phenomenon can be illustrated by the classical field theory. What we want to emphasis is that with the wave function we have calculated above from the point of quantum field theory, we can also reproduce the same interference pattern here. More precisely, denote \( \phi_1(\mathbf{x}, t) \) and \( \phi_2(\mathbf{x}, t) \) as two classical fields corresponding to plain wave and interfere with each other at somewhere with \( \phi_1(\mathbf{x}, t) + \phi_2(\mathbf{x}, t) = 0 \). According to the wave function for these two field (39), we can find that the wave function coincident with the vacuum wave function at the crossover regime as \( \phi_1(\mathbf{x}, t) + \phi_2(\mathbf{x}, t) = 0 \). This is what we expect from the calculation of classical interference. But we recalculate this obvious result from the perspective of quantum field theory.

2. Wave packet and particle

In this part, we show how to derive the classical Lagrangian for a particle in an external field from \( \mathcal{W}_1 \) in (20). There are no fermionic fields in classical theory, which means reverting to classical theory from quantum field theory requires a method to reduce the fermionic field to the classical Lagrangian. The evolution of systems in quantum field theory is through quantum states, which implies that a wave packet with a special shape corresponds to a particle located somewhere. In the following, we demonstrate the relation between wave packets and particles.

We work in the Heisenberg picture, where the time dependence of operators is determined by the Hamiltonian including a classical external field \( \phi(\mathbf{x}, t) \) but no other interaction term. The equation of motion for a fermionic field in the Heisenberg picture can be derived from the quantized Lagrangian or Hamiltonian:

\[
\frac{i\hbar}{\partial t} \hat{\psi}(\mathbf{x}, t) = -i\hbar \mathbf{c} \cdot \nabla \hat{\psi}(\mathbf{x}, t) + (mc^2 + \frac{1}{2}c^2g\phi)\beta \hat{\psi}(\mathbf{x}, t)
\] (41)

where

\[
\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}
\]

Note that variables in the equation of motion are still operators. In classical theory, all physical quantities are \( c \)-numbers, not operators. In full quantum theory, the state space is the full Hilbert space, while in classical theory there is no particle creation or annihilation, and only one particle state evolves. We thus need to reduce the full Hilbert space to a subspace consisting of only the one particle state where the operators in the equation of motion revert to the \( c \)-number.

The wave function for one particle state \( |\psi\rangle \) (not necessarily the eigenstate of the field operator) is defined as

\[
\psi(\mathbf{x}) = \langle \Omega | \hat{\psi}(\mathbf{x}, 0) |\psi\rangle
\] (42)
Then, the time-dependent wave function is
\[
\psi(x, t) = \langle \Omega | \hat{\psi}(x, 0) | \psi(t) \rangle = \langle \Omega | \hat{\psi}_a(x, 0) e^{-iHt} | \psi \rangle = \langle \Omega | \hat{\psi}_a(x, t) | \psi \rangle
\]  
(43)
In one particle state \( | \psi \rangle \) and vacuum state \( | \Omega \rangle \) on both sides of the equation of motion (41), we get a wave function version of the Dirac equation with a classical field \( \phi(x, t) \):
\[
\frac{i\hbar}{\partial t} \psi(x, t) = c\alpha \cdot \hat{p}\psi(x, t) + (mc^2 + \hbar^2 c^2 g\phi)\beta\psi(x, t)
\]  
(44)
For convenience, we assume the external classical field is static, \( \phi(x, t) = \phi(x) \), and denote the energy of the state as \( E \). Then, the equation of motion yields
\[
c\alpha \cdot \hat{p}\psi(x, t) + (mc^2 + \hbar^2 c^2 g\phi)\beta\psi(x, t) = E\psi(x, t)
\]  
(45)
We write the four-component wave function \( \psi(x, t) \) as
\[
\psi(x, t) = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}
\]  
(46)
where \( \varphi \) and \( \chi \) are two-component wave functions. Then the equation of motion can be written as
\[
c\sigma \cdot \hat{p}\chi(x, t) = (E - mc^2 - \hbar^2 c^2 g\phi)\varphi(x, t)
\]  
\[
c\sigma \cdot \hat{p}\varphi(x, t) = (E + mc^2 + \hbar^2 c^2 g\phi)\chi(x, t)
\]  
(47)
For normal particles, \( E > 0 \), then \( \chi(x, t) = \frac{-c\sigma \cdot \hat{p}}{E + mc^2 + \hbar^2 c^2 g\phi} \varphi(x, t) \sim \frac{\hbar}{c} \varphi(x, t) \), which means \( \varphi(x, t) \) is much larger than \( \chi(x, t) \), so we may just pay attention on \( \varphi(x, t) \) and ignore \( \chi(x, t) \) in the following discussion. After annihilating \( \chi(x, t) \) in (47), we can get the equation of \( \varphi(x, t) \):
\[
\frac{\hbar}{E/c^2 + m + \hbar^2 c^2 g\phi/c^2} [i\nabla \phi \cdot \hat{p} - \sigma \cdot (\nabla \phi \times \hat{p})] \varphi(x, t) + [(mc^2 + \hbar^2 c^2 g\phi)^2 + c^2 \hat{p}^2] \varphi(x, t) = E^2 \varphi(x, t)
\]  
(48)
The term \( \nabla \phi \cdot \hat{p} \) is not Hermitian and represents the interaction with antiparticles. Thus, this term does not have classical correspondence. The term \( \sigma \cdot (\nabla \phi \times \hat{p}) \) is the interaction arising from particle spin, just like the interaction between a magnetic moment and magnetic field in electrodynamics. These two terms are both on the order of \( \hbar \), so we can ignore them to first order. We also require the wave function \( \varphi \) of the one particle quantum state to be a wave packet centered around position \( x \) and momentum \( p \). Then, according to (48), we can get the Hamiltonian of a classical particle in an external field:
\[
H = E = \sqrt{c^2 \hat{p}^2 + (mc^2 + \hbar^2 c^2 g\phi)^2}
\]  
(49)
The corresponding Lagrangian is
\[
L_{\text{Yukawa}} = -\int dt (mc^2 + \hbar^2 c^2 g\phi) \sqrt{1 - \frac{v^2}{c^2}}
\]  
(50)
which is the Lagrangian for a particle in a Yukawa-type external field. This has a structure similar to that of the classical electromagnetic Lagrangian \( L_{\text{QED}}^1 \) in (3). This Lagrangian is consistent with the one in (8), which is obtained through the conjecture.
III. RELATIVISTIC CORRECTION OF THE YUKAWA POTENTIAL

In this section, we will introduce the relativistic correction for the Yukawa potential. Starting from the general Yukawa interaction Lagrangian obtained from classicization, we will take the velocity into consideration using essentially the same method in the relativistic correction as for the Coulomb potential [24].

We first review the relativistic correction in QED. The classical Lagrangian for a charged particle in an external electromagnetic field can be written as

$$L_1 = -mc^2 \sqrt{1 - \frac{\dot{r}_1^2}{c^2}} - eA^0 + \frac{e}{c} \dot{r}_1 \cdot A$$  \hspace{1cm} (51)

where \( m \) is the particle mass and \( r_1 \) is the position vector of the charged particle. We can extend the Lagrangian to include creation of an electromagnetic field by a moving charged particle. We can use the Lienard-Wiechert potential to describe the potential created by a moving charged particle:

$$A^0 = \frac{e_2}{|r_2 - r_1| + \dot{r}_2 \cdot (r_2 - r_1)/c}$$  \hspace{1cm} (52)

$$A = \frac{e_2}{c |r_2 - r_1| + \dot{r}_2 \cdot (r_2 - r_1)/c}$$

where \( r_2 \) and \( e_2 \) are the position vector and charge of the source.

Lorentz or Galilean invariance constrains the field Lagrangian to be local, which means that the Lagrangian can only depend on one time variable. Therefore, we need to express the Lagrangian in terms of a time parameter of a particle under a force:

$$r_{ret}(t) = r - \frac{\dot{r}_2 \cdot (r_2 - r_1)}{c} + \frac{r}{2c^2} \times \left[ \dot{r}_2^2 + \dot{r}_2 \cdot (r_2 - r_1) + \frac{(\dot{r}_2 \cdot (r_2 - r_1))^2}{r^2} \right]$$  \hspace{1cm} (53)

where \( r_{ret} \) is the retarded distance between the source and particle under a force and \( r = |r_2 - r_1| \) is the real distance between these two particles. All the variables above only depend on the time \( t \). Hence, the Lagrangian for a particle in an external electromagnetic field created by another charged particle is

$$L = -m_1c^2 \sqrt{1 - \frac{\dot{r}_2^2}{c^2}} - m_0c^2 \sqrt{1 - \frac{\dot{r}_2^2}{c^2}} - \frac{e_1e_2}{r} + \frac{e_1e_2}{2c^2} \left[ \dot{r}_1 \cdot \dot{r}_2 + \dot{r}_1 \cdot (r_2 - r_1) \dot{r}_2 \cdot (r_2 - r_1) \right]$$  \hspace{1cm} (54)

We can write the relativistic correction for Yukawa theory analogized to the Darwin correction for QED. We follow the same procedure as in QED and start from the action derived from classicization of Yukawa theory:

$$S_2 = \int d^4x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \left( \frac{m_0c^2}{\hbar} \right)^2 \phi^2 - \hbar c \frac{g}{2} \sqrt{1 - \frac{\dot{r}_2^2(t_1)}{c^2}} \delta(r_1 - r_2(t_1)) \phi \right)$$  \hspace{1cm} (55)
where \( \mathbf{r}_2(t) \) is the trajectory of the source and \( x = (t_1, \mathbf{r}_1) \). Denoting \( \kappa = 1/c \) and \( \tau = c(t_2 - t_1) \), we can rewrite the solution for the equation of motion calculated in (16) as

\[
\phi (\mathbf{r}_1, t_1) = \frac{m_0 \hbar c^2}{4\pi \hbar c} \int dy \frac{d\tau}{dy} \sqrt{1 - \kappa^2 \mathbf{r}_2^2(t_1 + \kappa \tau)} \frac{J_1 \left( \frac{m_0 c \Phi}{\hbar} \right)}{y} \tag{56}
\]

where \( y = \sqrt{\tau^2 - (\mathbf{r}_1(t_1) - \mathbf{r}_2(t_1 + \kappa \tau))^2} \) and its inverse \( \tau = \sqrt{y^2 + (\mathbf{r}_1(t_1) - \mathbf{r}_2(t_1 + \kappa \tau))^2} \). We have dropped the term \( \propto Y_1 \) as we have explained in \( \Pi B \).

If we consider the relativistic correction only to \( \mathcal{O}(1/c^2) \), then \( \tau \) can be approximated as

\[
\tau = \sqrt{\frac{y^2 + r^2 + \mathbf{r}_2 \cdot \mathbf{r} \kappa + \frac{1}{2} \frac{1}{\sqrt{y^2 + r^2}} \left[ r_2^2 (y^2 + r^2) + (\mathbf{r}_2 \cdot \mathbf{r})^2 - r_2 \cdot \mathbf{r} (y^2 + r^2) \right]}{\kappa^2 + \mathcal{O}(1/c^3)}}
\]

where \( \mathbf{r} = \mathbf{r}_1(t_1) - \mathbf{r}_2(t_1) \). The Lorentz factor can be expanded in terms of \( \kappa \):

\[
\sqrt{1 - \kappa^2 \mathbf{r}_2^2(t_1 + \kappa \tau)} = 1 - \frac{1}{2} r_2^2(t_1) \kappa^2 + \mathcal{O}(1/c^3)
\]

Therefore, the solution for the equation of motion containing the relativistic correction to order \( \mathcal{O}(1/c^2) \) yields

\[
\phi (\mathbf{r}_1, t_1) = \frac{\hbar c^2}{4\pi c^2} \left[ - \frac{e^{-m_0 c^2 \Phi}}{r} + \frac{\mathbf{r}_2 \cdot \mathbf{r} e^{-m_0 c^2 \Phi}}{2c^2} - \frac{(\mathbf{r}_2 \cdot \mathbf{r})^2}{2c^2} \left( \frac{1}{r} + \frac{m_0 c^2}{\hbar c} \right) e^{-m_0 c^2 \Phi} \right] \tag{59}
\]

It should be noticed that the classical field \( \phi(\mathbf{r}_1) \) created by a particle with the trajectory \( \mathbf{r}_2(t_2) \) depend on \( \mathbf{r}_1 \) only by the relative coordinate \( \mathbf{r}_1 - \mathbf{r}_2 \), which is the manifestation of translation symmetry. In other words, the classical field \( \phi(\mathbf{r}_1) \) created by a particle with the trajectory \( \mathbf{r}_2(t_2) \) is invariant if we shift both the source particle at \( \mathbf{r}_2(t_2) \) and the field \( \mathbf{r}_1 \) a same distance \( \mathbf{r}_0 \).

In the following classicization procedure, we calculate the Lagrangian for a charged particle in a Yukawa-type field created by another moving source. The Lagrangian for a Yukawa-type field consists of two terms and is similar to a charged particle in electromagnetic field. One is the kinetic term for a charged particle and the other is the current-field interaction:

\[
L_1 = -m_1 \sqrt{1 - \dot{\mathbf{r}}_1^2(t_1)/c^2} - \frac{\hbar^2 c^3}{2 \pi} g \phi \sqrt{1 - \dot{\mathbf{r}}_1^2(t_1)/c^2} \tag{60}
\]

where \( m_1 \) is the mass of the charged particle and \( \phi \) is the Yukawa field created by a source, which is actually the solution for the equation of motion calculated in (59). Therefore, the Lagrangian for a particle in an external Yukawa field created by another charged particle is

\[
\begin{align*}
L &= -m_1 \sqrt{1 - \dot{\mathbf{r}}_1^2/c^2} - m_2 \sqrt{1 - \dot{\mathbf{r}}_2^2/c^2} \\
&\quad + \frac{\hbar c g e^{-m_0 c^2 \Phi}}{4\pi} \frac{r}{\mathbf{r}} + \frac{1}{c^2} \left[ \frac{\hbar c^2 \mathbf{r}_1 \cdot \dot{\mathbf{r}}_2}{8\pi} - \frac{\hbar c^2 r_2^2 + \dot{\mathbf{r}}_2^2}{8\pi} \right] \frac{1}{c} \left( \frac{m_0 c^2}{\hbar c} \right) \frac{(\dot{\mathbf{r}}_2 \cdot \mathbf{r}) (\dot{\mathbf{r}}_1 \cdot \mathbf{r})}{r^2} e^{-m_0 c^2 \Phi} \tag{61}
\end{align*}
\]
where $m_2$ is the mass of the particle with the trajectory $r_2(t)$. We recall the definition of a Lagrangian: $L = T - V$. Hence, we can read off the potential between two moving Yukawa charged particles:

$$V = -\frac{\hbar c g^2}{4\pi} \frac{e^{m_0 c^2 r}}{r} - \frac{1}{c^2} \frac{\hbar c g^2}{8\pi} \left[ \dot{r}_1 \cdot \dot{r}_2 - \dot{r}_1^2 + \dot{r}_2^2 - \left( \frac{1}{r} + \frac{m_0 c^2}{\hbar c} \right) \frac{(\dot{r}_2 \cdot r) (\dot{r}_1 \cdot r)}{r} \right] e^{-\frac{m_0 c^2 r}{\hbar c}}$$

The first term is the Yukawa potential describing interaction between nucleons[25]. The other terms arise from the relativistic correction and depend on the velocity of both the source and forced particle. If the two particles have the same mass $m_1 = m_2$, then in the center-of-mass frame we have $\dot{r}_1 = -\dot{r}_2$, which reproduces almost the same form of the Yukawa potential in [27], which is derived from non-relativistic Yukawa potential theory up to some matching factors. We can impose another limit, $m_0 \to 0$, which reduces the Darwin correction for QED in (54) except for a squared total momentum term. The difference between the massless limits in Yukawa theory and QED is that the single particle Lagrangian in an external Yukawa field is not consistent with QED. Let us recall the Lagrangian for a charged particle in an external electromagnetic field is $L = -\frac{m_1}{\gamma^2} \sqrt{1 - \dot{r}_1^2/c^2} - eA^0 + eA \cdot \dot{r}_1/c$, while that for the Yukawa interaction is $L = -\frac{m_1}{\gamma^2} \sqrt{1 - \dot{r}_1^2/c^2} - g\phi \sqrt{1 - \dot{r}_1^2/c^2}$. Obviously, the two Lagrangians have different structures, thus they may have different effective potential even the mass of the scalar is zero.

**IV. DISCUSSION AND CONCLUSION**

In this paper, we try to find the quantum states which have classical field correspondence, and focus on the evolution of such states, through the Schrodinger equation we can evaluate the evolution equations for the classical fields. This is different from the usual $\hbar \to 0$ or $N \to \infty$ procedure in that we do not take any limit, which avoids the ambiguity caused by such limits as we have reviewed in introduction. Starting from the Lagrangian of quantum field theory, we can write down the wave packet for a quantum state in the representative space spanned by field eigenstates, which reveals that the wave function is Gaussian. The time evolution of a wave packet induce the classical Hamiltonian equation of motion. We classicized Yukawa theory to show the physical meaning of our classicization method. Meanwhile, we obtained classical Yukawa theory but preserved relativistic invariance. Then, we calculated the next-to-leading order correction $O(v/c)$, which may be used for interaction between nucleons[12, 28] and dark matter phenomenology. Specifically, if dark matter particles move at a velocity comparable to the speed of light, it is necessary to consider the relativistic correction for the Yukawa potential between two particles[2, 6–11].

We should also mention that the key point of our method is the quantum states corresponding to the classical observables. In quantum theory, we use states to describe the system. But, in classical theory, we take physical observables such as momentum and coordinate or classical field to describe the system. However, not all of quantum states can correspond to classical observables. For example, the state which has two $\delta$-function peak may can not correspond to classical theory. Nevertheless, we can list some natural requirements of the
quantum states which have classical correspondence. First, the vacuum states in quantum theory corresponds to the classical field configuration $\phi_{\text{class}} = 0, \pi_{\text{class}} = 0$. Second, if some quantum states corresponds to classical field configuration $\phi_{\text{class}} = \phi_0, \pi_{\text{class}} = \pi_0$ (Actually, as we have shown in II C, $\phi_0$ and $\pi_0$ is the center location of the wave packet for quantum states.), then the probability of getting $\phi_0, \pi_0$ is the largest after measuring $\phi_{\text{class}}, \pi_{\text{class}}$ of the quantum state, while the probability of getting classical field far away from $\phi_0, \pi_0$ is much more smaller, just like the property of the wave packet that describes a classical particle with definite location and momentum. Finally, the quantum states which have classical fields correspondence will maintain to have classical fields correspondence during the evolution of time, it will not evolve into the quantum state which do not have classical fields correspondence. Then by analyzing the structure of the vacuum state in $\phi/\pi$-representation and combing the constraints above, we can find out the quantum states corresponding to the classical fields. The derivation in II C shows that the states which can correspond to classical theory are the Gaussian-like wave-packets. Once we find out correspondence between quantum states and the classical fields, we can trace the states evolution in quantum theory by Schrodinger equation to research the classical observables’ evolution. As we can see in II C the corresponding classical evolution equation is governed by the Lagrangian in second line of (20). In principle, the classicization procedure above is a general method, not just for Yukawa theory. Although it is not straightforward to write the quantum states in field representation for QED owing to gauge invariance, it should be noted that if we fix the gauge, the classicization procedure from Yukawa theory can be suitable for QED still.

It should be noted that our relativistic correction almost reproduces the results derived from NREFT[27]. Fortunately, we do not need to calculate some coefficients that need to be evaluated by matching the physical processes between the full theory and NREFT. Nevertheless, there are some differences between the two relativistic corrections. First, we ignored the terms associated with the particle spin and gradient of $\phi$ in (48), which contribute to $\delta(r)$ and $\mathbf{L} \cdot \mathbf{S}$ in NREFT. Second, we did not consider the radiation of bosonic fields, and we do not have the imaginary term in our potential. These two differences can be understood and canceled through careful calculation. There is still an additional term $\propto (\dot{\mathbf{r}}_1 \cdot \mathbf{r}) (\dot{\mathbf{r}}_2 \cdot \mathbf{r}) / r^2$ in our effective potential energy comparing with the one derived from NREFT[27]. But our result is consistent with another earlier work[29] which also has the term $\propto (\dot{\mathbf{r}}_1 \cdot \mathbf{r}) (\dot{\mathbf{r}}_2 \cdot \mathbf{r}) / r^2$. We should also note that in non-relativistic QED, the results derived from NREFT[30, 31] is agree with the Darwin correction[24]. So it is confusing that our effective potential has an different term compared with the result from Yukawa NREFT[27], even though we use the same method as Darwin[24]. In fact, there is a term $\propto (\dot{\mathbf{r}}_1 \cdot \mathbf{r}) (\dot{\mathbf{r}}_2 \cdot \mathbf{r}) / r^2$ in our results while in their outcome, such term should be changed to $\propto i(\mathbf{p} \cdot \mathbf{k})/k^2$ which is purely imaginary. In principle, the imaginary term represents the correction from the radiation. Therefore it seems a little bit subtle that the effective potential of Yukawa NREFT has a imaginary part while the effective potential of QED NREFT does not have imaginary part as such two methods do not consider radiation contribution both. In momentum space, the imaginary potential is roughly $(\mathbf{p} \cdot \mathbf{k})/k^2$. Besides, it is interesting to note that, if the term $(\mathbf{p} \cdot \mathbf{k})/k^2$ is changed into its square $((\mathbf{p} \cdot \mathbf{k})/k^2)^2$, the corresponding potential in position space will no longer have imaginary part but with a term
\( (\dot{r}_1 \cdot r) (\dot{r}_2 \cdot r) / r^2 \) instead, which will be consistent with our result.

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Appendix A: Calculation for \( N(t) \) and \( f(x, t) \)

In this section, we consider the evolution of a given quantum state in the Schrodinger picture. A general state that can revert to classical field theory looks like a coherent state with a Gaussian wave function(23). To continue our calculation, we first define two new kernel functions \( \mathcal{E}_{-1}(x, y) \) and \( \mathcal{E}_2(x, y) \):

\[
\mathcal{E}_{-1}(x, y) \equiv \int \frac{d^3 p}{(2\pi)^3} \frac{1}{E_p} e^{ip(x-y)} \tag{A1}
\]

\[
\mathcal{E}_2(y, z) \equiv \int d^3 x \mathcal{E}(x, y) \mathcal{E}(x, z) = \int \frac{d^3 p}{(2\pi)^3} e^{ip(y-z)} E_p^2 \tag{A2}
\]

The kernel \( \mathcal{E}_2 \) acts as the Hamiltonian for any arbitrary function \( g(x) \):

\[
\int d^3 x \mathcal{E}_2(x, y) g(x) = -\nabla^2 g(y) + m^2 g(y) \tag{A3}
\]

The following are the relations between the three kernel functions \( \mathcal{E}_{-1}, \mathcal{E} \) and \( \mathcal{E}_2 \):

\[
\int d^3 x \mathcal{E}(x, y) \mathcal{E}_{-1}(x, z) = \delta(y - z) \tag{A4}
\]

\[
\int d^3 x \mathcal{E}_2(x, y) \mathcal{E}_{-1}(x, z) = \mathcal{E}(y, z) \tag{A5}
\]

It is convenient to calculate the Schrodinger equation in the \( \phi \) representation. In this representation, the momentum operator acts as a gradient functional operator. For a general state in this representation, we have

\[
\hat{\pi}^2 \langle \phi | \varphi \rangle = \mathcal{E}(x, x) \langle \phi | \varphi \rangle - \int d^3 y \mathcal{E}(x, y)[\phi(y) - f(y)] \int d^3 z \mathcal{E}(x, z)[\phi(z) - f(z)] \langle \phi | \varphi \rangle \tag{A6}
\]

Considering the property (A2), we find that

\[
\int d^3 x \hat{\pi}^2 \langle \phi | \varphi \rangle = \int d^3 x \mathcal{E}(x, x) \langle \phi | \varphi \rangle - \int d^3 y \int d^3 z [\phi(y) - f(y)][\phi(z) - f(z)] \mathcal{E}_2(y, z) \langle \phi | \varphi \rangle \tag{A7}
\]
Then, we let the Hamiltonian act on a given quantum state (23) and consider the property (A3):

$$
\hat{H} \langle \phi | \varphi \rangle = \int d^3x \left\{ \left[ \frac{1}{2} \hat{\pi}^2 + \frac{1}{2} \left( \nabla \phi \right)^2 + \frac{1}{2} m_0^2 \hat{\phi}^2 + g \hat{\phi} \right] + C \right\} \langle \phi | \varphi \rangle
$$

$$
\left. \frac{1}{2} \int d^3y \int d^3z f(y) f(z) \hat{E}_2(y, z) + \int d^3y \int d^3z f(y) \phi(z) \hat{E}_2(y, z) \right. \left. + g \int d^3x \phi(x) j(x, t) + \frac{1}{2} \int d^3x \hat{E}(x, x) + C \right\} \langle \phi | \varphi \rangle
\tag{A8}
$$

Recall that the time evolutions of the state |\varphi\rangle and wave function are separate:

$$
|\varphi(t)\rangle = e^{-iHt} |\varphi(t = 0)\rangle
\tag{A9}
$$

$$
\langle \phi | \varphi(t) \rangle = \mathcal{N}(t) \exp \left\{ \frac{1}{2} \int d^3x d^3y \hat{E}(x, y) [\phi(x) - f(x, t)][\phi(y) - f(y, t)] \right\}
\tag{A10}
$$

The time-dependent normalization coefficient arises from the evolution of the wave function, which suggests that we can write the time-dependent coefficient as \( \mathcal{N}(t) = \mathcal{N}_0 e^{F(f, t)} \), where \( F(f, t) \) depends on the center position function \( f \) and time \( t \). We can consider the Schrodinger equation to calculate the explicit form of \( f \) and \( F \) in the quantum state. For the time derivative term, we have

$$
\frac{\partial}{\partial t} \langle \phi | \varphi(t) \rangle = \frac{\partial}{\partial t} \left[ \mathcal{N}_0 e^{F(f, t)} \exp \left\{ \frac{1}{2} \int d^3x d^3y \hat{E}(x, y) [\phi(x) - f(x, t)][\phi(y) - f(y, t)] \right\} \right]
$$

$$
= \left\{ \int d^3x d^3y \hat{E}(x, y) f(x, t)[\phi(y) - f(y, t)] + \frac{\partial}{\partial t} F(f, t) \right\} \langle \phi | \varphi(t) \rangle
\tag{A11}
$$

Hence, with both sides of the Schrodinger equation (A8) and (A11), the full time-dependent Schrodinger equation can be expressed as

$$
i \left\{ \int d^3x d^3y \hat{E}(x, y) \hat{f}(x, t)[\phi(y) - f(y, t)] + \frac{\partial}{\partial t} F(f, t) \right\} \langle \phi | \varphi \rangle
$$

$$
= \left. \left[ \frac{1}{2} \int d^3y \int d^3z f(y, t) f(z, t) \hat{E}_2(y, z) + \int d^3y \int d^3z f(y, t) \phi(z) \hat{E}_2(y, z) \right. \right. \left. + g \int d^3x \phi(x) j(x, t) + \frac{1}{2} \int d^3x \hat{E}(x, x) + \Lambda \right] \langle \phi | \varphi \rangle
\tag{A12}
$$

The Schrodinger equation is valid for any arbitrary \( \phi \), which constrains the correspondence between both sides of (A12):

$$
i \int d^3x \hat{E}(x, y) \hat{f}(x, t) = \int d^3x f(x, t) \hat{E}_2(x, y) + gj(y, t)
\tag{A13}
$$

$$
i \int d^3x d^3y \hat{E}(x, y) \hat{f}(x, t) f(y, t) - \frac{1}{2} \frac{\partial}{\partial t} F(f, t) = \frac{1}{2} \int d^3y \int d^3z f(y, t) f(z, t) \hat{E}_2(y, z)
\tag{A14}
$$
\[ \Lambda = -\frac{1}{2} \int d^3 \mathbf{x} \mathcal{E}(\mathbf{x}, \mathbf{x}) \]  

(A15)

For convenience, we separate \( f \) into real and imaginary parts:

\[ f(\mathbf{x}, t) = f_1(\mathbf{x}, t) + i \int d^3 \mathbf{y} \mathcal{E}_{-1}(\mathbf{x}, \mathbf{y}) f_2(\mathbf{y}, t) \]  

(A16)

where \( f_1 \) and \( f_2 \) are both real. Substituting (A16) into (A13), we obtain

\[ i \int d^3 \mathbf{x} \mathcal{E}(\mathbf{x}, \mathbf{y}) \dot{f}_1(\mathbf{x}, t) - \dot{f}_2(\mathbf{y}, t) = i \int d^3 \mathbf{z} \mathcal{E}(\mathbf{y}, \mathbf{z}) f_2(\mathbf{z}, t) - \nabla^2 f_1(\mathbf{y}, t) + m_0^2 f_1(\mathbf{y}, t) + g j(\mathbf{y}, t) \]  

(A17)

where we have used the relation (A3). Comparing the real and imaginary parts of both sides in (A17), we find the relation between the real part \( f_1 \) and imaginary part \( f_2 \):

\[ -\dot{f}_2(\mathbf{y}, t) = -\nabla^2 f_1(\mathbf{y}, t) + m_0^2 f_1(\mathbf{y}, t) + g j(\mathbf{y}, t) \]

\[ \dot{f}_1(\mathbf{x}, t) = f_2(\mathbf{x}, t) \]  

(A18)

We can then write the time derivative of the center position function \( \dot{f}(\mathbf{x}, t) \) in terms of the real part \( f_1(\mathbf{x}, t) \) and imaginary part \( f_2(\mathbf{x}, t) \):

\[ \dot{f}(\mathbf{x}, t) = f_2(\mathbf{x}, t) - i \int d^3 \mathbf{z} [\mathcal{E}(\mathbf{x}, \mathbf{z}) f_1(\mathbf{z}, t) + g \mathcal{E}_{-1}(\mathbf{x}, \mathbf{z}) j(\mathbf{z}, t)] \]  

(A19)

Finally, substituting (A19) and (A16) into (A14), we can work out the explicit form of \( F(f, t) \) in terms of the center function \( f(\mathbf{x}, t) \) and classical external source \( j(\mathbf{x}, t) \):

\[ F(f, t) = -\frac{i}{4} \int d^3 \mathbf{x} d^3 \mathbf{y} \mathcal{E}(\mathbf{x}, \mathbf{y}) f(\mathbf{x}, t) f(\mathbf{y}, t) - \frac{1}{2} g \int_{t_0}^{t} d\tau \int d^3 \mathbf{y} j(\mathbf{y}, \tau) f(\mathbf{y}, \tau) \]  

(A20)

where \( t_0 \) is an arbitrary initial time. The normalization coefficient yields

\[ \mathcal{N}(t) = \mathcal{N}_0 \exp \left\{ -\frac{i}{4} \int d^3 \mathbf{x} d^3 \mathbf{y} \mathcal{E}(\mathbf{x}, \mathbf{y}) f(\mathbf{x}, t) f(\mathbf{y}, t) - \frac{1}{2} g \int_{t_0}^{t} d\tau \int d^3 \mathbf{y} j(\mathbf{y}, \tau) f(\mathbf{y}, \tau) \right\} \]  

(A21)

**Appendix B: Wave function in the \( \pi \) representation**

To demonstrate whether the wave function of the quantum state is localized around \( \pi_{\text{class}} \), we can calculate the Gaussian wave function in the \( \pi \) representation. The eigenstate and eigenvalue of the conjugate momentum operator \( \hat{\pi}(\mathbf{x}) \) are \( |\pi\rangle \) and \( \pi(\mathbf{x}) \), which means \( \hat{\pi}(\mathbf{x}) |\pi\rangle = \pi(\mathbf{x}) |\pi\rangle \). The commutation relation of \( \hat{\pi}(\mathbf{x}) \) and \( \hat{\phi}(\mathbf{x}) \) yields

\[ \langle \phi | \pi \rangle = C \exp \left\{ i \int d^3 \mathbf{x} \pi(\mathbf{x}) \phi(\mathbf{x}) \right\} \]  

(B1)
Using completeness, we can write the wave function for state $|\varphi(t)\rangle$ in the $\pi$ representation:

$$
\langle \pi | \varphi(t) \rangle = \int D\phi \langle \pi | \phi \rangle \langle \phi | \varphi(t) \rangle = N'(t)C \exp \left\{ i \int d^3x [\pi_{\text{class}}(x, t) - \pi(x)] \phi_{\text{class}}(x, t) \right\} \times \int D\phi \exp \left\{ -\frac{1}{2} \int d^3x d^3y E(x, y) \phi(x) \phi(y) + i \int d^3x [\pi_{\text{class}}(x, t) - \pi(x)] \phi(x) \right\}
$$

(B2)

To finish the integral, we define an orthogonal matrix $T(x, x')$ to diagonalize the matrix $E(x, y)$:

$$
\int d^3x d^3y T(x, x') E(x, y) T(y, y') = \delta(x' - y')
$$

(B3)

Considering the normalization condition (A4) for the kernel function $E(x, y)$, we can explicitly write the auxiliary matrix $T(x, x')$:

$$
T(x, x') \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{E_p}} e^{ip \cdot (x-y)}
$$

(B4)

To finish diagonalizing with the auxiliary matrix $T(x, x')$, we need to change the integration variable $\phi$ to $\phi'$ via

$$
\phi(x) = \int d^3x' T(x, x') \phi'(x')
$$

(B5)

and change the measure of the integral to

$$
D\phi(x) = J D\phi'(x')
$$

(B6)

where $J = |\det[T(x, x')]|$ is the determinant of the auxiliary matrix. Then, the integral in
(B2) can be finished:
\[
\int \mathcal{D}\phi \exp \left\{ -\frac{1}{2} \int d^3x d^3y \mathcal{E}(x, y) \phi(x) \phi(y) + i \int d^3x [\pi_{\text{class}}(x, t) - \pi(x)] \phi(x) \right\}
\]
\[
= J \int \mathcal{D}\phi' \exp \left\{ -\frac{1}{2} \int d^3x' [\pi_{\text{class}}(x', t) - \pi(x')] \phi'(x') \right\}
\]
\[
= J \prod_{x'} \int \mathcal{D}\phi'(x') \exp \left\{ -\frac{1}{2} \int d^3x' [\pi_{\text{class}}(x', t) - \pi(x')] \phi'(x') \right\}
\]
\[
= J \prod_{x'} 2 \int_0^\infty \mathcal{D}\phi'(x') \exp \left\{ -\frac{1}{2} \int d^3x' [\pi_{\text{class}}(x', t) - \pi(x')] \phi'(x') \right\}
\]
\[
= J C' \exp \left\{ -\frac{1}{2} \int d^3x' \int d^3y [\pi_{\text{class}}(x', t) - \pi(x')] \int d^3y' [\pi_{\text{class}}(y, t) - \pi(y)] \right\}
\]
(B7)

where \(C' = \prod_{x'} (\sqrt{\pi}/\sqrt{2} d^3x')\) is a constant. Furthermore, substituting (B7) into (B2), we can work out the integral:
\[
\langle \pi|\varphi(t)\rangle = J N'(t) C' \exp \left\{ i \int d^3x [\pi_{\text{class}}(x, t) - \pi(x)] \phi_{\text{class}}(x, t) \right\}
\]
\[
\times \exp \left\{ -\frac{1}{2} \int d^3x' \int d^3y [\pi_{\text{class}}(x', t) - \pi(x')] \int d^3y' [\pi_{\text{class}}(y, t) - \pi(y)] \right\}
\]
\[
= J N''(t) C'' \exp \left\{ -i \int d^3x \pi(x) \phi_{\text{class}}(x, t) \right\}
\]
\[
\times \exp \left\{ -\frac{1}{2} \int d^3x d^3y \left( \int d^3x' \mathcal{T}(x, x') \mathcal{T}(y, x') \right) [\pi_{\text{class}}(x, t) - \pi(x)] [\pi_{\text{class}}(y, t) - \pi(y)] \right\}
\]
(B8)

where \(N'' = N'(t) \exp \left[i \int d^3x \pi_{\text{class}}(x, t) \phi_{\text{class}}(x, t) \right]\) is the new normalization coefficient.

Note that the integral of the auxiliary matrix can be simplified further in terms of the kernel function \(\mathcal{E}_1(x, y)\):
\[
\int d^3x' \mathcal{T}(x, x') \mathcal{T}(y, x') = \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_p} e^{ip(x-y)} = \mathcal{E}_1(x, y)
\]
(B9)
Therefore, the final expression of the wave function in the $\pi$ representation is

\[
\langle \pi | \varphi(t) \rangle = J \mathcal{N}''(t) C'' \exp \left\{ -i \int d^3 x \pi(x) \phi_{\text{class}}(x, t) \right\} \times \exp \left\{ -\frac{1}{2} \int d^3 x d^3 y E_{-1}(x, y)[\pi(x) - \pi_{\text{class}}(x, t)][\pi(y) - \pi_{\text{class}}(y, t)] \right\}
\]  

(B10)

Therefore, the wave function is a Gaussian wave packet with the center located around $\pi(x)$ and $\pi(y)$.

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The full conjecture for QED can be written as:

\[-i \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi + e \bar{\psi} \gamma^\mu \psi A_\mu = (-m + eu^\mu A_\mu) \sqrt{1 - v^2}\]

where \(u^\mu\) is the four-velocity \(u^\mu = (\gamma c, \gamma v)\) and \(\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\) is the Lorentz factor. And we can see that the conjecture looks like the same as Yukawa but a different current. As there is vector current in QED while scalar current in Yukawa. The exact correspondence containing
current is model dependent. Thus, to show the general correspondence, we just present the kinetic term in our paper.