Strong Frequency Dependence in Over-damped Systems

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Strong frequency dependence is unlikely in diffusive or over-damped systems. When exceptions do occur, such as in the case of stochastic resonance, it signals an interesting underlying phenomenon. We find that such a case appears in the motion of a particle in a diffusive environment under the effect of periodically oscillating retarded force emanating from the boundaries. The amplitude for the expectation value of position has an oscillating frequency dependence, quite unlike a typical resonance. We first present an analysis of the associated Fokker-Planck equation, then report the results of a Monte Carlo simulation of the effect of a periodic perturbation on a totally asymmetric simple exclusion process (TASEP) model with single species. This model is known to exhibit a randomly moving shock profile, dynamics of which is a discrete realization of the Fokker-Planck equation. Comparison of relevant quantities from the two analyses indicate that the same phenomenon is apparent in both systems.

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I. INTRODUCTION

Diffusive systems cannot support resonances in the usual sense, a periodic force cannot build oscillations that grow with each cycle. The frequency response of systems vary monotonously. We present an over-damped system which has an oscillating frequency response due to the presence of a position dependent effective force. Such conditions arise for objects under the influence of retard effects from boundaries an example of which we also provide.

The Fokker-Planck (FP) equation \[ \frac{\partial P}{\partial t} = \frac{\partial}{\partial x} \left[ -\gamma P \frac{\partial V}{\partial x} + F_{\text{ret}} + D \frac{\partial P}{\partial x} \right] \] (1)

where \( \gamma \) is the drift constant relating velocity to force and \( D \) is the coefficient of the diffusion term. The potential energy \( V \) is related to the time-independent force field the particle is in, while \( F_{\text{ret}} \) represents a time and position dependent driving force which will be relevant to our problem. Specifically, we will consider a force which emanates from sinusoidally varying boundary effects, propagating with a constant velocity into the interval of interest, so that the space dependence of the force is also sinusoidal.

Considerable amount of work has been carried out on the behaviour of \( P \), under the effect of various types of potentials. Analytical solutions are usually restricted to simpler forms of potentials. The equation may be treated as a Schrödinger equation with imaginary time, and the time dependent solutions have a relaxational type of behaviour. This is the consequence of the assumption of absence of any inertia (or “memory”) in the system: The time rate of change of \( P \) is described in terms of the value of \( P \) at time \( t \). However, we find “resonance” behavior, in a looser context, as a consequence of Eqn. (1) which corresponds to amplified response at a sequence of frequencies, based on matching of the waveform of the driving force to the interval between the boundaries.

A number of other stochastic systems with damped dynamics do display non-monotous frequency response to sinusoidal drives. In Brownian motors, the particle moves in a (usually ratchet-type) static potential and the sinusoidal force is used to drive the particle over the “easier” barrier. (The asymmetry in the direction may be achieved either by the form of the static potential or the form of the time-dependent forces.) Alternatively, a high-frequency drive may effectively act to average over portions the static potential, on a length scale which depends on the properties of the drive. This results in a “vibrational resonance” with transport properties strongly dependent on parameters associated with the drive. The force may take the form of a superposition of a number of sinusoidal (or rectangular) functions, in which case the non-linear effects may lead to interesting effects at certain combination of frequencies. Besides the models we mention here, the reader is referred to a review by Hänggi and Marchesoni for a detailed discussion of these and other similar systems.

Sinusoidal drive on a damped system has also been studied in relation to “stochastic resonance” This relates basically to the motion of a damped particle in a double well potential on which a random diffusive force as well as the sinusoidal force is acting. It had been demonstrated that there would be an optimal magnitude for the random force which amplifies transitions between the potential minima in synchronism with the periodic drive. Jung and Hänggi solved the corresponding FP equation, demonstrating that the time autocorrelation function of a periodically driven bistable over-damped system sustains undamped oscillations. The term “res-
II. ANALYSIS AND DISCUSSIONS

A. Fokker-Planck Equation

We will first describe our analysis of the FP equation. In particular, we take the time-independent potential $V(x)$ to be given by

$$V(x) = \begin{cases} 0 & \text{if } |x| < \frac{L}{2} \\ V_0(|x| - L/2)^2/x_0^2 & \text{otherwise.} \end{cases}$$

The quadratic structure softens the boundaries at $\pm L/2$ with a range related to $x_0$. We also assume that a time dependent force acts on the particle, synchronously from the two boundaries, but retarded in time (with a propagation speed $v$) in proportion to the distance to the boundaries:

$$F_{ret}(x,t) = F_0 \sin \left[ \frac{\omega (t - L/2 + \frac{x}{v})}{v} \right]$$

$$+ F_0 \sin \left[ \frac{\omega (t - L/2 - \frac{x}{v})}{v} \right]$$

$$= 2F_0 \cos \frac{\omega x}{v} \sin \left[ \frac{\omega (t - L/2 v)}{v} \right]. \quad (3)$$

It can be observed that the retarded force results in a position dependent amplitude in the oscillation. This amplitude is not frequency dependent, however we will show that the magnitude of the response to this force does depend on how the wavelength compares to the size of the system. We write Eqn. [4] in scaled form

$$\frac{\partial P(z, \theta)}{\partial \theta} = \Gamma \frac{\partial}{\partial z} \left[ \frac{1}{p} \frac{d\tilde{V}(z)}{dz} - \epsilon \cos(2\pi z/\lambda) \sin(\theta) \right]$$

$$+ \nabla^2 \frac{\partial^2 P}{\partial z^2} \quad (4)$$

where we have used the dimensionless quantities in Table I with unitless potential:

$$\tilde{V}(z) = \begin{cases} 0 & \text{if } |z| < \frac{1}{2} \\ (|z| - 1/2)^2 L^2/x_0^2 & \text{otherwise.} \end{cases}$$

| Dimensionless Quantities |
|---------------------------|
| $\theta = \omega(t - L/(2v))$ |
| $\Gamma = \gamma V_0/(\omega L^2)$ |
| $z = x/L$ |
| $\epsilon = 2F_0 L/V_0$ |
| $\lambda = 2\pi v/(\omega L)$ |
| $\nabla = D/(\omega L^2)$ |

**TABLE I:** Dimensionless quantities that are used in scaling the Fokker-Planck equation.

The parameter $\lambda$, besides representing the wavelength of the time dependent force relative to the distance variable $z$, is also proportional to the period of oscillation: $\tau = 2\pi/\omega = \lambda L/v$.

We have solved equation [4] numerically for various values of the parameters. The probability density $P(z, \theta)$ was solved on a mesh of 256 points in the $z$ direction, corresponding to the $x$-coordinate values for $|x| < L/2$ plus the two boundary regions which were taken to be $3x_0$ wide each. The equation was integrated numerically in the time variable $\theta$ with step sizes such that $\Delta \theta/(\Delta z)^2 \leq 0.1$. The integration for the period of $\theta = 2\pi$ was repeated 10 times, which was sufficient for convergence, i.e. for obtaining no appreciable change in $P(z, \theta)$.

We then calculate the expectation value of position as a function of $\theta$:

$$\mathbb{E}(\theta) = \int_{-\infty}^{\infty} dz \ z P(z, \theta).$$
The size of this oscillation is parametrized through the fundamental Fourier coefficients:

\[ C = \frac{1}{2\pi} \int_0^{2\pi} \bar{\tau}(\theta) \cos(\theta) d\theta \]

\[ S = \frac{1}{2\pi} \int_0^{2\pi} \bar{\tau}(\theta) \sin(\theta) d\theta. \]  

Fig. 1 displays that the response of the system has strong frequency dependence. The amplitude of the oscillations grow as \( x_0/L \) decreases, when the boundaries become sharper. A wider boundary allows more oscillations (as a function of position) in the probability density, diminishing the variations in \( \tau \). As the boundary region expands, so do the features on the plot, implying longer wavelengths. For values of \( \lambda \) much larger than one, the response monotonically increases to its asymptotic value. Note also that the “in phase” component \( S \) dominates the response for smaller \( x_0/L \). The size of \( \epsilon \) was chosen to obtain a response magnitude comparable to that obtained from the Monte Carlo analysis.

The extrema of the response correspond to matching of the wavelength of the driving force to the effective length of the diffusion region \( L \) plus the boundary region. Fig. 2 displays the probability densities for points A and B in Fig. 1. Note that for \( \theta \sim \pi/4 \) and \( z \sim 0 \), the probability densities increase with \( z \) in both plots (a) and (b), consistent with a sinusoidal drive. However, when an even (odd) number of maxima are present, the position expectation value for these cases become negative (positive), producing oscillations as a function of wavelength. As more and more wavelengths “fit” into the system (smaller \( \lambda \)), the change in the expectation value of the particle position becomes less and less discernible.

### B. TASEP

We now turn to the TASEP system which provides a discrete realization of this phenomenon on a one dimensional lattice. Its dynamics is described by the following transition probabilities in time \( dt \): On the leftmost site \( 0 \rightarrow 1 \) with probability \( \alpha dt \), inside the bulk \( 01 \rightarrow 10 \) with probability \( \beta dt \), and on the rightmost site \( 1 \rightarrow 0 \) with probability \( \beta dt \). These systems have been studied extensively for time-independent boundary rates. Recently, the effects of time dependence has also received interest: Popkov et al. studied a vehicular traffic on highways under periodically changing green and red-lights and Basu et al. also showed a frequency dependent modality on similar transport systems.

The time-independent system can be solved exactly. A first order phase transition line separates the high and low density phases for \( \alpha = \beta < 1/2 \). Along this line the density may be shown to correspond to a superposition of shock profiles which move from one end of the system to the other. These structures are not quantities that are measurable at any instant of time: Occupation statistics of states with a specific number of \( n \) particles lead to a shock profile associated with that \( n \). Fig. 3 shows these profiles for a system with 50 sites, for various values of \( n \). The linear dependence of the position of the profile on particle number is apparent. Fig. 4 displays this dependence.

A perturbation to the boundary conditions leads to a change in the number of particles in the system. This then acts as an effective force which results in the change of the shock position after a delay. Moreover for small values of \( \alpha = \beta \) these shock profiles tend to be evenly distributed within the lattice. In other words, the shock front carries out a random walk in the lattice. Fig. 5
is quite similar in form to been represented by a “free-energy functional” [28] which to the central region of the lattice. This constraint has effectively “pushing” the profile to the right. A similar density at that boundary reduces the particle entry rate, If the shock wave moves too close to the left, increased narily constrained by the time-independent boundaries: algorithm was used for generating r

response to a pulsed change in the particle entry rate

intervals ∆ t

Monte Carlo” method [29], in which events occur at time the ones given in Fig.2 for the FP equation.

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along the lattice justifies the association of this system

waves mechanism as well as damping.) While this motion

with the “retarded force” mechanism of Eqn. 1, the rel-

change appreciably with time, resulting in an approximate relation of the form x_s + n ~ N. Since x_s is determined by interpolation, it in general is not an integer.

The change in the probability distribution in Fig. 4 is linear in response to a pulsed change in the particle entry rate α is shown in Fig. 6. The disturbance caused by an influx of extra particles for a short period of time travels along the lattice, with a damped response. (Note the oscillatory nature of the response indicating a travelling wave mechanism as well as damping.) While this motion along the lattice justifies the association of this system with the “retarded force” mechanism of Eqn. 1 the relatively strong damping leads to results not as “clean” as the ones given in Fig.2 for the FP equation.

We now proceed with our discussion of the details of our computations of the sinusoidally varying drive.

The MC procedure we used is known as the “kinetic Monte Carlo” method [29], in which events occur at time intervals ∆ t = − ln r/Ω where r is a random number uniformly distributed between 0 and 1, and Ω is the sum of rates of all possible transitions in the system. The event which does take place is also selected randomly with a probability proportional to its rate. Transition rates at the boundaries are assumed to be constant within the duration ∆ t, since ∆ t ≪ 2π/ω. The Mersenne Twister algorithm was used for generating r [30, 31]. Time-dependent probability density calculations are carried out over 10^6 Monte Carlo steps (MCS). For a lattice of size N, we define a MCS as N^2 changes in the system.

We have observed that the time-dependent boundary effects decay appreciably inside the lattice. A small lattice is therefore necessary for boundary effects to be seen inside the bulk. Moreover, broadest range of a random
walk is known to take place for smaller values of $\alpha$ and $\beta$. In order to extend the range of the random walk to the full range between the boundaries, $\alpha = \beta = 0.1$ is taken for an anchor point on a lattice of size $N = 50$. For the effect to be prominent, boundary rates were varied with a significant amplitude such as $\alpha = 0.1 + 0.099 \sin(\omega t)$ and $\beta = 0.1 - 0.099 \sin(\omega t)$, comparable to the average values of $\alpha$ and $\beta$.

Note that our time-dependent perturbation drives the system between two values of the boundary parameters corresponding to low and high density phases. This drive should then result in an oscillation of the number of particles within the system, hence of the probability density of the shock position. In Fig. 4 the probability distribution for finding $n$ particles in the system (which is linearly related to shock position as can be seen in Fig. 4) as a function of time, is plotted. The results agree qualitatively with the FP results [32].

We have used compatible values for corresponding parameters in the FP analysis: The exact diffusion constant $\Delta = 2 \alpha(1 - \alpha)/(1 - 2\alpha)$ of the system [33] yields the value $\Delta = 0.225$ for $\alpha = \beta = 0.1$. We have also observed atypical behaviour within two lattice sites of the boundaries in the MC data, which were excluded from plots in Fig. 4. This corresponds to a boundary smoothness of 0.08, which too was used in the FP analysis.

Fundamental components of the system’s response were calculated using the time-dependent probability function $\rho(n, t)$ for the occupation of $n$ sites at time $t$:

$$C = \frac{1}{\tau} \sum_{0}^{\tau} \bar{\rho}(t) \cos(2\pi t/\tau)$$

$$S = \frac{1}{\tau} \sum_{0}^{\tau} \bar{\rho}(t) \sin(2\pi t/\tau)$$

where

$$\bar{\rho}(t) = \sum_{n} \rho(n, t)n.$$  

Oscillations in these components are shown in Fig. 5. These oscillations are not as prominent as those obtained in the FP analysis, and displayed in Fig. 2. We attribute this to the decay of the magnitude of the effective force away from the boundaries, damping the effect at higher frequencies. Time dependence of the boundary rates cause sufficient fluctuations in the TASEP system to cause the appearance of the density distributions that are quite different from those which result from constant boundary conditions. The details of these effects in the TASEP model will be reported separately. Here we have only discussed results relevant to the diffusive motion of the shock profile in this system.

**III. CONCLUSIONS**

We have demonstrated that the response of an overdamped system to a retarded oscillatory force from the boundaries leads to resonant effects in the oscillation amplitudes of statistical quantities such as the average po-
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