Playing with the critical point
An experiment with the Mandelbrot set connectivity

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Abstract

By means of a graphical journey across the Mandelbrot set for the classic quadratic iterator 
\( f(z) : z^2 + q \), we illustrate how connectivity breaks as the seed \( z_0 \) is no longer at the critical point of \( f(z) \). Finally we suggest an attack to the MLC conjecture.

1 Introduction

The Mandelbrot set \( M \) generation relates, in the original experiment, to a given non-linear and one-parameter map \( f(z, q) \) where both \( z, q \in \mathbb{C} \) and \( q \) is the complex parameter ranging over the whole \( \mathbb{C} \). The key concept of this experiment refers back to early Fatou’s and Julia’s ideas on testing Julia set connectivity under perturbations of the parameter \( q \), as well as on their renown theorem of existence of one critical point \( c \) in any basin of attraction and the concept of ‘post-critical’ set\(^1\), revealing so helpful today to explore several properties in the dynamics of complex maps. Let \( z_0 \) be the seed\(^2\) at the critical point of \( f \). Let \( q \in D \subseteq \mathbb{C} \). Then \( f(z, q) \) is iterated per each \( q \). One associates each resulting orbit \( O \) to a connected or disconnected Julia set, if \( O \) is bounded or not respectively, so that \( q \) may belong or not to \( M \). Thus \( M \) can be intended as a chart of Julia sets connectivity, compiled and drawn in the \( q \)–parameter space.

![Figure 1: The Mandelbrot set for \( f(z) : z^2 + q \).](image)

Question 1

What does it happen to \( M \) as \( z_0 \) is perturbed and it lies no longer at \( c \)?

The question here posed was previously introduced in section § 6.3 of [1], where, in the course of related experiments, we noticed that the connectivity of the Mandelbrot set \( M \), for the classic quaternionic iterator \( f(h) : h^2 + q \ (h, q \in \mathbb{H}) \), breaks as the seed \( h_0 \) of each orbit is distinct from the critical point (in this case at the origin). Anyway the interpretation of figures with solid 3-D models for the quaternionic Julia sets looks very hard to evince the question here posed; therefore we moved to the more simply looking environment \( \mathbb{C} \), where figures are displayed quite clearer.

\(^1\)Let \( C(f) \) be the set of critical points for \( f \). It is the closure of all forward orbits \( f_n(c), n \geq 1 \), where \( c \in C(f) \).

\(^2\)Under this term, one indicates the first point of any orbit.
2 MLC conjecture

The still open MLC (standing for ‘Mandelbrot Locally Connected’) conjecture states that the Mandelbrot set $\mathcal{M}$ is locally connected\(^3\), roughly speaking, the set includes no uni-dimensional filaments.

3 The journey

These pages want to enjoy an empirical flavor exclusively, i.e. no mathematics is achieved and any goal or pretension to deep into the theory is disclaimed. We will not dare into the proof of it for sure, because we are not skilled enough for that. This is just a communication collecting a bunch of figures processed by playing with the seed $z_0$. Two journeys are shown: given $R = \Re(q)$, $I = \Im(q)$, we let $q$ range along the real $R \in [-1.6,0]$ and the imaginary segment $I \in [0, -1.6i]$ respectively, where $\mathcal{M}$ is more visible. We noticed that as far as one runs away from the origin ($c = 0$ for $f(z) : z^2 + q$) along both real and imaginary directions, $\mathcal{M}$ loses progressively connectivity in the peripheral regions, while the central cardioid is deformed, shrinking again and again. If the seed is very close to the critical point, disconnectivity is hardly noticed because only the most marginal periphery of $\mathcal{M}$ is notched; but, as one moves farther from $c$, the disconnectivity rate increases, even the cardioid is deformed and more evident figures are displayed.

4 One proposal to attack MLC

We liked to give some (naïve, maybe) impressions: one could try to attack the MLC conjecture by an inverse approach, that is, proving that $\mathcal{M}$ is always disconnected when the seed, for each orbit in the $\mathcal{M}$ experiment for $f(z) : z^2 + q$ is not at the origin, according to the original experiment.

**Question 2** Does the connectivity of $\mathcal{M}$ depend on $z_0$ at the critical point?

If so, one would try to determine, if possible, a sort of disconnectivity rate decreasing to 0 along any path leading to the critical point of the given function $f$. Thus, on one hand, we pull out the evidence, from the next figures, that $\mathcal{M}$ is not connected for seeds not lying at the critical point; on the other hand, from the known figure of $\mathcal{M}$ (see fig. 1), it seems to be locally connected. Perhaps, things may match together.

5 Conclusions

My intention here is just to suggest an idea, an attack to MLC and know if it might work or not or where it might lead somewhere. In some cases, like Jacobi’s for elliptic integrals, the history of mathematics showed that an inverted look-up at the problem helped. Who knows whether the above considerations could actually work? Without addressing to audience, one would never check it. The next images are snapshots from a couple of animations we made, downloadable from our site\(^4\) and showing how $\mathcal{M}$ shifts its shape when the seed $z \neq c$, like in the original experiment.

References

[1] Rosa A., *Methods and applications to display quaternion Julia sets*, Differential Equations and Control Processes, 4, 2005, St. Petersburg University.

\(^3\)If at each point $x \in \mathcal{M}$, every neighborhood of $x$ includes a connected open neighborhood.

\(^4\)http://www.malilla.supereva.it/Mirror/Pages/papers.html
M.1: $R = 0.0, I = -0.1$. Infinitesimal disconnections in the peripheral, bottom region; they seem to not happen inside the central cardioid yet.

M.2: $R = 0.0, I = -0.2$. Besides the peripheral filaments, one notices a weak deformation in the bigger circle here.

M.3: $R = 0.0, I = -0.4$. Zoom out: the central cardioid is seriously deforming.

M.4: $R = 0.0, I = -0.6$. Further deformation.

M.5: The left region of $\mathcal{M}$ shortens into the cardioid.

M.6: $R = 0.0, I = -0.9$. The cardioid shape fades away.

M.7: $R = 0.0, I = -1.0$. The previous region turns into two sub-regions whose boundaries intersect at the origin.

M.8: A blow up of figure 5.

M.9: $R = 0.0, I = -1.05$. The two regions split away.
M.10: $\mathcal{M}$ appears to be totally disconnected now.

M.11: $R = 0.1, I = 0.0$. We start moving along the real segment from 0 to -1.6.

M.12: $R = 0.2, I = 0.0$. The same view as in fig. [fig], but different computation.

M.13: $\mathcal{M}$ is blowing away again!

M.14: For $R = 0.6, I = 0.0$.

M.15: $R = 1.2, I = 0.0$. Speeding deformation up.

M.16: $R = 1.5, I = 0.0$. The shrinking.

M.17: $R = 1.6, I = 0.0$.

M.18: $R = 1.6, I = 0.0$. $\mathcal{M}$ turned into a line.