Pricing Power Exchange Options with Hawkes Jump Diffusion Processes

Puneet Pasricha and Anubha Goel*

Department of Mathematics
Indian Institute of Technology Delhi
Hauz Khas, Delhi, 110016, India

(Communicated by Biao Luo)

Abstract. In this article, we propose a jump diffusion framework to price the power exchange options. We model the price dynamics of assets using a Hawkes jump diffusion model with common factors to describe the correlated jump risk and clustering of asset price jumps. In the proposed model, the jumps, reflecting common systematic risk and idiosyncratic risk, are modeled by self-exciting Hawkes process with exponential decay. A pricing formula for valuation of power exchange option is obtained following the measure-change technique. Existing models in the literature are shown to be special cases of the proposed model. Finally, sensitivity analysis is given to illustrate the effect of jump risk and jump clustering on option prices. We observe that jump clustering significantly effects the option prices.

1. Introduction. In this article, we consider the valuation of European power exchange options. These options are generalization of power options (Tompkins, (2000) [31]), and Fischer and Margrabe exchange options (Fischer, (1978) [13]; Margrabe, (1978) [24]). Margrabe (1978) [24] investigated the exchange options and their valuation. The exchange options are derivative products that allow the holder of the option to exchange an asset for another on the maturity of the option. Fischer (1978) [13] also studied the valuation of exchange options considering the scenario when the exercise price is the same as the price of an un-traded asset. Tompkins (1999) [31] discussed power options and their applications to hedge nonlinear risks. Power options and exchange options have many practical and useful applications. Even though the power exchange option is a generalization of these two options, power exchange options are not much explored in the literature.

Blenman and Clark (2005) [6], first time explored power exchange options as a generalization of power options and exchange options. They obtained a closed form expression for the value of power exchange options assuming that geometric Brownian motions govern the asset price dynamics. Because of the tractability, diffusion models are extensively used for pricing the derivatives. But, these models have serious limitations. As suggested by various numerical evidences (Bakshi, Cao, & Chen, (1997) [3]; Bates, (1996), (2000) [4], [5]; Eraker, (2004) [10]; Eraker, Johannes, & Polson, (2003) [11]; Pan, (2002) [28]), the random arrival of new

2010 Mathematics Subject Classification. Primary: 60G51; Secondary: 60G55.

Key words and phrases. Jump clustering, Hawkes process, power exchange options, jump diffusion, change of measure.

* Corresponding author: Anubha Goel.
information in the market causes sudden jumps in the asset prices which diffusion models are incapable of capturing. To overcome this limitation, several models are proposed in the literature to price derivative securities. For example, Kou (2002) [17] proposed a double exponential jump diffusion model to price the European options. Cai and Kou (2011) [7] studied the valuation of options where mixed exponential jump diffusion process governs the asset prices.

In the context of power exchange options, Wang (2016) [32] considered jump diffusion processes divided into systematic and idiosyncratic components to capture the correlated jump risk and obtained a closed form solution for the price of power exchange options. Further, Wang et al. (2017) [33] proposed a pricing model to obtain the price of power exchange options with counterparty risk under jump diffusion dynamics. Recently, Pasricha and Goel (2019) [29] proposed a model to obtain the price of vulnerable European power exchange option in a reduced form framework of credit risk.

The propensity of price jumps to cluster over several days has been recently documented in the financial markets (Yu (2004) [34]; Carr and Wu (2004) [8]; Maheu and McCurty (2004) [22]). Their empirical analysis indicates that financial markets strongly self-excite over time, i.e., a jump in the market increases the probability of the future jumps, resulting in self-exciting jump clustering. This phenomenon is more prevalent during the periods of distress. Aït-Sahalia et al. (2015) [2] argued that “what makes financial crisis take place is typically not the initial jump, but the amplification that occurs subsequently over hours or days, and the fact that other markets become affected as well”. They proposed a reduced-form model for asset returns that can capture jump clustering in time and across assets, or financial contagion, in line with the behavior of the data. Several articles (e.g., Lux & Marchesi, (2000) [19]; Mandelbrot, (1963) [23]) have observed that large moves in financial markets tend to be followed by additional large movements. Hence, all these studies imply that jumps tend to cluster in financial markets.

As a result, Poisson jump diffusion model, due to independent increments property, is inappropriate to address the clustering of jumps observed in the financial assets. These empirical observations motivated the development of models that are capable of capturing the clustering of jumps and hence it turns our attention to the so-called self-exciting processes. For instance, Errais et al. (2010) [12] observed using real data on defaults that the default clustering could be addressed more consistently using self-exciting Hawkes processes originally proposed by Hawkes (1971a,b) [14, 15]. Hawkes processes, also known as self-exciting processes, can reproduce jump clusters since the intensity of jump increases with the occurrence of jumps and consequently, the probability of arrival of future jumps increases. Thus, Hawkes processes are suitable for modeling clustered jumps. The self-exciting point processes are now widely applied in various fields (in seismology (Adamopoulos (1976) [1]; Ogata (1981) [26], (1988) [27]), in credit risk (Errais et al. (2010) [12], Dassios and Zhao (2011) [9])). For instance, Meyer et al. (2012) [25] introduced a self-exciting spatiotemporal point process to predict the incidence of invasive meningococcal disease which can be transmitted between infected humans and sometimes forms epidemics.

Recently, Ma & Xu (2016) [21] proposed a structural based approach and derived an explicit expression for the value of the firm and default correlation by modeling the firm’s value using Hawkes jump diffusion process. The Hawkes jump diffusion model is different from the pure jump Hawkes process (Hawkes, 1971) by adding
a continuous Brownian component. They showed that the default correlation obtained from Hawkes jump diffusion is larger than that derived from the Poisson jump diffusion model and thus Hawkes jump diffusion model has an advantage over classical jump diffusion structural models. As a typical example of its application in derivatives pricing, Ma et al. (2017) [20] presented a model for obtaining price valuation of vulnerable European options using a model with self-exciting Hawkes processes that allow for clustered jumps. Recently, Liu and Zhu (2019) [18] presented an analytical approach for pricing variance swaps with discrete sampling times when the underlying asset follows a Hawkes jump diffusion process characterized with both stochastic volatility and clustered jumps. These articles brought new insights into modeling the asset price dynamics by considering jump clustering.

Motivated by the increased popularity of the Hawkes processes in quantitative finance, we propose a model in which the asset price dynamics are governed by Hawkes jump diffusion process. The proposed model is an extension to the models by Xingchun Wang (2016) [32] and Blenman and Clark (2005) [6] and thus provide a significant contribution to the existing literature. In this model, the jumps in the asset prices are characterized by Hawkes processes also called self-exciting processes. In the Hawkes process, the occurrence of a jump increases the intensity and hence the probability of arrival of next jumps. Consequently, these processes are suitable to address the jump clustering in financial assets. We use the Hawkes jump diffusion process to model the dynamics of both the assets. Specifically, the jumps in assets prices are described by an asset specific Hawkes process to capture idiosyncratic clustering of jumps growing out of idiosyncratic shock and a common Hawkes process to capture jump clusters resulting from systematic shocks. As a result, the assets’ prices are correlated in continuous part by Brownian motion and in jump part by common Hawkes process. We obtain an explicit expression for the price of European power exchange option. Sensitivity analysis is given to illustrate the effect of jump risk and jump clustering on option prices.

The structure of this paper is as follows. Section 2 describes the proposed model for asset price dynamics followed by the derivation of explicit pricing formula of power exchange options in Section 3. Section 4 compares the proposed model with classical models. Section 5 gives the sensitivity analysis to illustrate the effect of jump clustering on option prices. Section 6 concludes the paper.

2. Model setup. Assume that the probability space $(\Omega, F, Q)$ models the uncertainty in the economy. Let $E$ denotes the expectation with respect to the risk neutral measure $Q$. We adopt Hawkes jump diffusion processes to describe the dynamics of two underlying assets $P_1$ and $P_2$, where two types of jumps, idiosyncratic and systematic, are modeled by two independent Hawkes processes. Under the risk neutral measure $Q$, the asset price dynamics are given as

\[ \frac{dP_{i,t}}{P_{i,t-}} = r dt + \sigma_i dW_{i,t} + \sigma_i \left[ \sum_{j=1}^{N_{i,t}+N_{i,t}} Z_{i,j} - \beta_i \int_0^t \lambda_{i,s} \, ds - \beta_i \int_0^t \lambda_{i,s} \, ds \right] \]

where $r$ is the risk free interest rate, $\sigma_i, i = 1, 2$ are the volatilities of the underlying assets. $W_{1,t}, W_{2,t}$ are two correlated Brownian motions with correlation coefficient $\rho$. The i.i.d. random variables $\{Z_{i,j}\}_{j \in \mathbb{N}}$ with support in $(-1, \infty)$ and $E[Z_{i,j}] = \beta_i$, for a review on self-exciting point process and applications in different fields, one can refer Reinhart (2018) [30], Hawkes (2018) [16].
represent the magnitudes of jumps. The variables $\lambda_t$, $\lambda_{1,t}$ and $\lambda_{2,t}$ represent the intensities of independent Hawkes processes defined as follows:

\[ N_t = \sum_{j=1}^{\infty} I(T_j \leq t) \]  \hspace{1cm} (2)

\[ N_{i,t} = \sum_{j=1}^{\infty} I(T_{i,j} \leq t), \ i = 1, 2 \]  \hspace{1cm} (3)

where $\{T_j\}_{j \in \mathbb{N}} > 0$ and $\{T_{i,j}\}_{j \in \mathbb{N}} > 0$ are the jump times of $N_t$ and $N_{i,t}$, $i = 1, 2$, respectively. The sequence $\{T_j\}_{j \in \mathbb{N}} > 0$ represents the sequence of times of systematic shocks and the sequence $\{T_{i,j}\}_{j \in \mathbb{N}} > 0$ represents the times of idiosyncratic shocks for $i$th asset, $i = 1, 2$. Assume that the intensity processes of Hawkes processes $N_t$, $N_{i,t}$, $i = 1, 2$ are, respectively, given by

\[ \lambda_t = \lambda_0 + \int_0^t \theta e^{-\delta (t-s)} dN_s = \lambda_0 + \theta \sum_{T_j \leq t} e^{-\delta (t-T_j)} \]  \hspace{1cm} (4)

\[ \lambda_{i,t} = \lambda_{i,0} + \int_0^t \theta_i e^{-\delta_i (t-s)} dN_{i,s} = \lambda_{i,0} + \theta_i \sum_{T_{i,j} \leq t} e^{-\delta_i (t-T_{i,j})} \]  \hspace{1cm} (5)

where $\lambda_0, \theta, \lambda_{i,0}, \theta_i, \delta_i$ are positive constants. Here, we have considered a special case of self-exciting Hawkes process with exciting function being an exponential decay function.

In this article, the Hawkes process $N_t$, is considered as the common jump process. It represents the jumps resulting from systematic shocks that affects all the assets in the financial market such as macroeconomic events or financial crisis. The processes $N_{i,t}$, $i = 1, 2$ reflect idiosyncratic jumps, i.e. the jumps resulting from the firm-specific events, such as investment losses or some bad news. The proposed model implies that if a systematic shock occurs in the financial market, then the intensity process $\lambda_t$ of common Hawkes process will increase immediately by $\theta$ and then this increment decays exponentially at rate $\delta$. In other words, when a systematic shock occurs, it increases the probability of the occurrence of future systematic shocks, thus resulting in jump clustering. Further, larger $\theta$ (smaller $\delta$) implies more clustering of jumps. In Poisson jump models, due to independent increments, this self exciting behavior is not observed, i.e. occurrence of a jump doesn’t stimulate the future jumps. Thus, Hawkes processes (self exciting processes) are appropriate to model the jump clustering phenomena. However, the process reduces to Poisson process with constant intensity $\lambda_0$ when the variable $\theta$ goes to zero. The same explanation is valid for the idiosyncratic Hawkes processes.

Using Equations (1), (4) and (5), we have

\[
\frac{dP_{i,t}}{P_{i,t-}} = \left[ -\beta_i \left( \lambda_0 + \lambda_{i,0} + \theta_i \sum_{T_{i,j} \leq t} e^{-\delta_i (t-T_{i,j})} + \theta \sum_{T_j \leq t} e^{-\delta (t-T_j)} \right) + \sigma_i dW_{i,t} + d \left( \sum_{j=1}^{N_i+N_{i,t}} Z_{i,j} \right) \right] dt
\]  \hspace{1cm} (6)
3. Valuation of power exchange options. In this subsection, we derive the formula for value of a power exchange option assuming that the jump amplitude $Z_{i,j}$ follows log-normal distribution. A power exchange option is an European option to exchange the power value $\gamma_1 P_{\alpha_1}$ of one asset to the power value $\gamma_2 P_{\alpha_2}$ of another asset. Let $C^*$ denote price of the power exchange option, which is given by discounted expected payoff of option on maturity date. Then, we have

$$C^* = e^{-rT}E[(\gamma_1 P_{\alpha_1}(T) - \gamma_2 P_{\alpha_2}(T))^+]$$

The following proposition gives the analytic expressions for the price dynamics of the underlying assets.

**Proposition 3.1.**

$$X_{i,t} = P_{i,0} \exp \left\{ \sigma_i W_{i,t} + (r - \beta_i \lambda_0 + \beta_i \lambda_{i,0} - \frac{1}{2} \sigma_i^2) t + \sum_{j=1}^{N_i} \frac{\beta_i \theta}{\delta} \left( e^{-\delta(t-T_j)} - 1 \right) - \sum_{j=1}^{N_i} \frac{\beta_i \theta_i}{\delta_i} \left( e^{-\delta_i(t-T_{i,j})} - 1 \right) \right\}$$

and

$$J_{i,t} = \prod_{j=1}^{N_i+N_{i,t}} (1 + Z_{i,j})$$

where the product over null set is assumed to be 1. Then

$$P_{i,t} = X_{i,t} J_{i,t}$$

**Proof.** For $i = 1, 2$, we know that $X_{i,t}$ are continuous stochastic processes. Applying Ito’s formula, we obtain

$$dX_{i,t} = \left[ r - \beta_i (\lambda_0 + \lambda_{i,0} + \theta_i \sum_{T_{i,j} \leq t} e^{-\delta_i(t-T_{i,j})} + \theta \sum_{T_j \leq t} e^{-\delta(t-T_j)} \right] X_{i,t} dt + \sigma_i X_{i,t} dW_{i,t}$$

(9)

Besides,

$$dJ_{i,t} = J_{i,t-} d \left( \sum_{j=1}^{N_i+N_{i,t}} Z_{i,j} \right)$$

(10)

Since $J_i$ is a pure jump process and $X_i$ is continuous stochastic process, then using Ito’s formula for semi-martingales, we have

$$dP_{i,t} = d(X_{i,t} J_{i,t})$$

$$= X_{i,t-} dJ_{i,t} + dX_{i,t} J_{i,t-} + d[X_i, J_i](t)$$

$$= X_{i,t-} dJ_{i,t} + dX_{i,t} J_{i,t-}$$

(11)

Substituting Equations (9) and (10) into (11), we get the desired result.

**Proposition 3.2.** Let $\mathcal{F}_{i,t} = \sigma\{N_s \cup N_{i,s} : 0 \leq s \leq t\}, i = 1, 2$ and $\mathcal{F}_t = \mathcal{F}_{1,t} \cup \mathcal{F}_{2,t}$. Assume that the amplitude of jumps of each asset follow a log-normal distribution; specifically $\log(1 + Z_{i,j}) \sim N(a_i, b_i), j = 1, 2, \ldots$ and let $\beta_i = e^{a_i + \frac{1}{2} b_i^2} - 1$. Because $Z_{i,j}$ and $W_{i,T}$ are independent, then given information $\mathcal{F}_{i,T}$, i.e., say $N_T = n_i, N_{i,T} = n_i, i = 1, 2$, the value of European exchange option at time 0 is given by
\[ C^* = \gamma_1 e^{-rT + M_2 + \frac{1}{2} \Sigma_2} \left[ e^{M_1 + \bar{\rho} \sqrt{\Sigma_1} \sqrt{\Sigma_2} + \frac{1}{2} \Sigma_1 N(d_1) - \frac{\gamma_2}{\gamma_1} N(d_2) \right] \] (12)

where for \( i = 1, 2 \), we have

\[ K_i = \hat{n} + n_i \]

\[ x_i = r - \beta_i \lambda_0 - \beta_i \lambda_i \theta - \frac{1}{2} \sigma_i^2 \]

\[ Y_i = \sum_{j=1}^{n_i} \frac{\beta_i \theta_j}{\delta_j} (e^{-\delta_j (T_j - 1)} - 1) + \sum_{j=1}^{n_i} \frac{\beta_i \theta_j}{\delta_j} (e^{-\delta_j (T_j - 1)} - 1) \]

\[ M_1 = \ln \left( \frac{P_{1i}^{\alpha_i}(0)}{P_{2i}^{\alpha_i}(0)} \right) + (\alpha_1 x_1 - \alpha_2 x_2) T + \alpha_1 Y_1 - \alpha_2 Y_2 + \alpha_1 K_1 a_1 - \alpha_2 K_2 a_2 \] (13)

\[ M_2 = \ln \left( \frac{P_{2i}^{\alpha_i}(0)}{P_{2i}^{\alpha_i}(0)} \right) + \alpha_2 x_2 T + \alpha_2 Y_2 + \alpha_2 K_2 a_2 \] (14)

\[ \Sigma_1 = \left[ \sigma_1^2 \sigma_1^2 + \alpha_2^2 \alpha_2^2 \right] T + \sigma_1^2 K_1 b_1 + \alpha_2^2 K_2 b_2 - 2 \sigma_1 \sigma_2 \rho T \] (15)

\[ \Sigma_2 = \alpha_2^2 \sigma_2^2 T + \alpha_2^2 K_2 b_2 \] (16)

\[ d_1 = \frac{M_1 + \bar{\rho} \sqrt{\Sigma_1} \sqrt{\Sigma_2} + \Sigma_1 - \ln \left( \frac{\gamma_2}{\gamma_1} \right)}{\sqrt{\Sigma_1}} \] (17)

\[ d_2 = \frac{M_1 + \bar{\rho} \sqrt{\Sigma_1} \sqrt{\Sigma_2} - \ln \left( \frac{\gamma_2}{\gamma_1} \right)}{\sqrt{\Sigma_1}} \] (18)

**Proof.** From (8), we have

\[ \ln(P_{2i}^{\alpha_i}(T)) = \ln(P_{2i}^{\alpha_i}(0)) + \alpha_2 x_2 T + \alpha_2 \sigma_2 W_{2,i} + \alpha_2 Y_2 + \alpha_2 \sum_{j=1}^{K_2} \ln(1 + Z_{2,j}) \]

\[ \ln \left( \frac{P_{1i}^{\alpha_i}(T)}{P_{2i}^{\alpha_i}(T)} \right) = \ln \left( \frac{P_{1i}^{\alpha_i}(0)}{P_{2i}^{\alpha_i}(0)} \right) + (\alpha_1 x_1 - \alpha_2 x_2) T + \alpha_1 \sigma_1 W_{1,i} + \alpha_2 \sigma_2 W_{2,i} + \alpha_1 Y_1 - \alpha_2 Y_2 + \alpha_2 \sum_{j=1}^{K_1} \ln(1 + Z_{1,j}) - \alpha_2 \sum_{j=1}^{K_2} \ln(1 + Z_{2,j}) \]

The above expressions show that \( \ln \left( \frac{P_{1i}^{\alpha_i}(T)}{P_{2i}^{\alpha_i}(T)} \right) \) and \( \ln(P_{2i}^{\alpha_i}(T)) \) are two normal random variables with means given by \( M_1 \) and \( M_2 \) and variances \( \Sigma_1 \) and \( \Sigma_2 \) respectively given in Equations (13) to (16). The covariance between these two normal random variables is given by

\[ \text{Cov} \left( \ln(P_{2i}^{\alpha_i}(T)), \ln \left( \frac{P_{1i}^{\alpha_i}(T)}{P_{2i}^{\alpha_i}(T)} \right) \right) = \alpha_1 \sigma_1 \alpha_2 \sigma_2 \rho T - \alpha_2^2 \sigma_2^2 T - \alpha_2^2 K_2 b_2 \]

Hence, we can rewrite \( \ln(P_{2i}^{\alpha_i}(T)) \) and \( \ln(P_{1i}^{\alpha_i}(T)) \) in the following form

\[ \ln(P_{2i}^{\alpha_i}(T)) = M_2 + \sqrt{\Sigma_2} \epsilon_2 \] (19)

\[ \ln \left( \frac{P_{1i}^{\alpha_i}(T)}{P_{2i}^{\alpha_i}(T)} \right) = M_1 + \sqrt{\Sigma_1} \epsilon_1 \] (20)
where $\epsilon_1$ and $\epsilon_2$ are normally distributed with correlation coefficient
\[
\tilde{\rho} = \frac{\alpha_1 \sigma_1 \alpha_2 \sigma_2 \rho T - \alpha_2^2 \sigma_2^2 T - \alpha_2^2 K_2 b_2}{\sqrt{\Sigma_1} \sqrt{\Sigma_2}}.
\]

Based on the expression of $\ln \left( \frac{P^{\alpha_1}(T)}{P^{\alpha_2}(T)} \right)$ and $\ln(P^{\alpha_2}(T))$ in Equations (19) and (20), we have
\[
E \left[ \left( \gamma_1 P^{\alpha_1}(T) - \gamma_2 P^{\alpha_2}(T) \right)^+ \right] = \gamma_1 E \left[ P^{\alpha_2}(T) \left( \frac{P^{\alpha_1}(T)}{P^{\alpha_2}(T)} - \frac{\gamma_2}{\gamma_1} \right)^+ \right]
\]
\[
= \gamma_1 E[P^{\alpha_2}(T)] E \left[ \left( \frac{P^{\alpha_2}(T)}{E[P^{\alpha_2}(T)]} \right)^+ \right]
\]
\[
= \gamma_1 E[P^{\alpha_2}(T)] E^{(\alpha_2)} \left[ \left( \frac{P^{\alpha_1}(T)}{P^{\alpha_2}(T)} - \frac{\gamma_2}{\gamma_1} \right)^+ \right], \tag{21}
\]
where the expression operator $E^{(\alpha_2)}$ is related to the measure $Q^{(\alpha_2)}$, which is equivalent to $Q$ with the following Radon-Nikodym derivative,
\[
\frac{dQ^{(\alpha_2)}}{dQ} = \frac{P^{\alpha_2}(T)}{E[P^{\alpha_2}(T)]} \tag{22}
\]

From Equation (19) and the assumption that $\epsilon_2 \sim N(0, 1)$, we can easily obtain the following.
\[
E[P^{\alpha_2}(T)] = E[e^{M_2+\sqrt{\Sigma_2} \epsilon_2}] = e^{M_2+\frac{1}{2} \Sigma_2}
\]

Now, we simplify the last term in Equation (21). Since $\epsilon_1$ and $\epsilon_2$ are normally distributed with correlation coefficient $\tilde{\rho} = \frac{\alpha_1 \sigma_1 \alpha_2 \sigma_2 \rho T - \alpha_2^2 \sigma_2^2 T - \alpha_2^2 K_2 b_2}{\sqrt{\Sigma_1} \sqrt{\Sigma_2}}$, it holds that under $Q^{(\alpha_2)}$, the variable $\epsilon_1$ is still a normal random variable. The mean of $\epsilon_1$ under $Q^{(\alpha_2)}$ is $\tilde{\rho} \sqrt{\Sigma_2}$ and standard deviation is 1. Suppose $\epsilon$ follows standard normal distribution under $Q^{(\alpha_2)}$, which implies
\[
E^{(\alpha_2)} \left[ \left( \frac{P^{\alpha_1}(T)}{P^{\alpha_2}(T)} - \frac{\gamma_2}{\gamma_1} \right)^+ \right]
\]
\[
= E^{(\alpha_2)} \left[ \left( e^{M_1+\sqrt{\Sigma_1} \tilde{\epsilon}_1} - \frac{\gamma_2}{\gamma_1} \right)^+ \right]
\]
\[
= \left[ \left. e^{M_1+\tilde{\rho} \sqrt{\Sigma_1} \sqrt{\Sigma_2} + \sqrt{\Sigma_1} \tilde{\epsilon}_1} - \frac{\gamma_2}{\gamma_1} \right| \tilde{\epsilon}_1 \geq \frac{\gamma_2}{\Sigma_1} \right] \right]
\]
\[
- \frac{\gamma_2}{\gamma_1} E^{(\alpha_2)} \left[ I \left( e^{M_1+\tilde{\rho} \sqrt{\Sigma_1} \sqrt{\Sigma_2} + \sqrt{\Sigma_1} \tilde{\epsilon}_1} \leq \frac{\gamma_2}{\Sigma_1} \right) \right]
\]
\[
= e^{M_1+\tilde{\rho} \sqrt{\Sigma_1} \sqrt{\Sigma_2} + \frac{1}{2} \Sigma_1} N(d_1) - \frac{\gamma_2}{\gamma_1} N(d_2)
\]
where $d_1$ and $d_2$ are given in Equations (17) and (18) and $I(A)$ is an indicator function, i.e., $I(A) = 1$ if $A$ is true and 0 if $A$ is false. Hence, the value of power exchange option using Equation (21), is given by
\[
C^* = \gamma_1 e^{-rT+M_2+\frac{1}{2} \Sigma_2} \left[ e^{M_1+\tilde{\rho} \sqrt{\Sigma_1} \sqrt{\Sigma_2} + \frac{1}{2} \Sigma_1} N(d_1) - \frac{\gamma_2}{\gamma_1} N(d_2) \right]
\]
From Proposition 3.2, we observe that the analytical solution to completely
determine the price dynamics is not possible. Thus in order to estimate the option
prices, we have to rely on the simulations. Firstly, the paths of Hawkes processes are
simulating using thinning algorithm (Ogata (1981) [26]). Then, the option prices
are estimated by taking sample means of the prices over the simulated paths. Thin-
ning algorithm for \( N_t \) is shown below. Same algorithm is applied to obtain paths
of \( N_{n,t} \), \( n = 1, 2 \).

**Thinning Algorithm**

1. Set \( \lambda \leftarrow \lambda_0, n \leftarrow 1 \).
2. Generate a random variable \( U \) with standard uniform distribution, i.e., \( U \sim U(0, 1) \). Let \( s \leftarrow -\ln \frac{U}{\lambda} \). If \( s \leq T \), then \( t_1 \leftarrow s \); else go to the Step 6.
3. Set \( n \leftarrow n + 1, \lambda \leftarrow \lambda_{n-1} + \theta \). If \( s \leq T \), then go to the Step 4; else go to the
Step 6.
4. Generate \( U \sim U(0, 1) \), and let \( s \leftarrow -\ln \frac{U}{\lambda} \).
5. Generate \( V \sim U(0, 1) \). If \( V \leq \frac{\lambda s}{\theta} \), a new event occurs at time
\( s \), i.e., \( t_n \leftarrow s \), and then return to Step 3; else return to Step 4.
6. Stop. The times of the occurrence of the events are \( t_1, t_2, \ldots, t_n \).

4. The proposed model as a generalization of classical models. In this
section, we discuss that the proposed model can be considered as an extension to
the already existing models in the literature to price power exchange options.

4.1. Xingchun Wang (2016) [32]. Assuming \( \theta = \theta_1 = \theta_2 = 0 \) in the proposed
model, the Hawkes processes terms reduces to the Poisson Processes and hence the
proposed model reduces to the model by Xingchun Wang (2016). This can be shown
as follows:

For \( i = 1, 2 \), we have

\[
K_i = \tilde{n} + n_i
\]

\[
x_i = r - \beta_i \lambda_0 - \beta_i \lambda_{i,0} - \frac{1}{2} \sigma_i^2
\]

\[
M_1 = \ln \left( \frac{P_1^{\alpha_1}(0)}{P_2^{\alpha_2}(0)} \right) + (\alpha_1 x_1 - \alpha_2 x_2)T + \alpha_1 K_1 a_1 - \alpha_2 K_2 a_2
\]

\[
M_2 = \ln(P_2^{\alpha_2}(0)) + \alpha_2 x_2 T + \alpha_2 K_2 a_2
\]

\[
V_1 = [\alpha_1^2 \sigma_1^2 + \alpha_2^2 \sigma_2^2]T + \alpha_1^2 K_1 b_1 + \alpha_2^2 K_2 b_2 - 2\alpha_1 \alpha_2 \sigma_1 \sigma_2 \rho T
\]

\[
V_2 = \alpha_2^2 \sigma_2^2 T + \alpha_2^2 K_2 b_2
\]

\[
d_1 = \frac{M_1(n, n_1, n_2) + \tilde{\rho} \sqrt{\Sigma_1} \sqrt{\Sigma_2} + \Sigma_1 - \ln \left( \frac{\Sigma_1}{\Sigma_2} \right)}{\sqrt{\Sigma_1}}
\]

\[
d_2 = \frac{M_1(n, n_1, n_2) + \tilde{\rho} \sqrt{\Sigma_1} \sqrt{\Sigma_2} - \ln \left( \frac{\Sigma_1}{\Sigma_2} \right)}{\sqrt{\Sigma_1}}
\]

and the value of European power exchange option given in Equation (12) with
variables given above. The difference with the proposed model is the absence of
terms \( Y_i \).

4.2. Blenman and Clark (2005) [6]. Assuming \( \theta = \theta_1 = \theta_2 = 0 \) and \( \lambda_0 = \lambda_{1,0} = \lambda_{2,0} = 0 \) in the proposed model, we have no jump risk and no clustering of jumps.
Hence, the proposed model reduces to the model by Blenman and Clark (2005).
For $i = 1, 2$, we have

$$x_i = r - \frac{1}{2} \sigma_i^2$$

$$M_1 = \ln \left( \frac{P_{1}^{\alpha_1}(0)}{P_{2}^{\alpha_2}(0)} \right) + (\alpha_1 x_1 - \alpha_2 x_2) T$$

$$M_2 = \ln(P_{2}^{\alpha_2}(0)) + \alpha_2 x_2 T$$

$$V_1 = [\alpha_1^2 \sigma_1^2 + \alpha_2^2 \sigma_2^2] T - 2 \alpha_1 \sigma_1 \alpha_2 \sigma_2 \rho T$$

$$V_2 = \alpha_2^2 \sigma_2^2 T$$

$$d_1 = \frac{M_1 + \bar{\rho} \sqrt{\Sigma_1} \sqrt{\Sigma_2} + \Sigma_1 - \ln \left( \frac{\gamma_2}{\gamma_1} \right)}{\sqrt{\Sigma_1}}$$

$$d_2 = \frac{M_1 + \bar{\rho} \sqrt{\Sigma_1} \sqrt{\Sigma_2} - \ln \left( \frac{\gamma_2}{\gamma_1} \right)}{\sqrt{\Sigma_1}}$$

and the value of European power exchange option given in Equation (12) with variables given above. The resulting expression is same as that obtained in Blenman and Clark (2005) [6].

5. **Numerical results.** In this section, we perform sensitivity analysis in order to explore the impact of jump clustering on option prices, when the prices of both assets are affected by common risk in both continuous process and jump process components. Also, a comparative analysis between the proposed model and models proposed in the literature is given to illustrate the impact of jump clustering. We let $\lambda_0 = \lambda_{1,0} = \lambda_{2,0} = 1$, $\theta = \theta_1 = \theta_2 = 1$, $\delta = \delta_1 = \delta_2 = 2$, i.e., half of jumps results from systematic factors and the other half grows out of idiosyncratic factors. For consistency of the expected number of jumps in unit time for systematic jumps and idiosyncratic jumps [21], the intensities of Poisson processes are taken to be $\frac{\lambda_0}{1 - \bar{\rho}}$ and $\frac{\lambda_{1,0}}{1 - \bar{\rho}}$, respectively. The parameters values in the base case are listed in Table 1 and are chosen according to the previous literature. For performing the numerical calculations, i.e., for sensitivity analysis and comparison purpose, value of only one parameter is changed at one time while the value of the other parameters are kept same as in the base case.

![Figure 1. Option prices against time to maturity](image_url)
Table 1. Values of the Parameters in the Base Case

| Parameters | Values | Parameters | Values |
|------------|--------|------------|--------|
| $S_1(0)$   | 10     | $S_2(0)$   | 10     |
| $\sigma_1$| 0.2    | $\sigma_2$| 0.2    |
| $a_1$      | 0      | $a_2$      | 0      |
| $b_1$      | 0.01   | $b_2$      | 0.01   |
| $\lambda_{1,0}$ | 1 | $\lambda_{2,0}$ | 1 |
| $\theta_1$ | 1      | $\theta_2$ | 1      |
| $\delta_1$ | 2      | $\delta_2$ | 2      |
| $\lambda_0$ | 1 | $\alpha_1$ | 1      |
| $\theta$  | 1      | $\alpha_2$ | 1      |
| $\delta$  | 2      | $\eta_1$  | 1      |
| $r$       | 0.02   | $\eta_2$  | 1      |

Figure 2. Option prices against correlation coefficient when $T=1.5$

Figure 1 plots option prices against time to maturity. From Figure 1, we observe that option prices increases with increase in time to maturity irrespective of the model applied. This agree with the fact that option prices are increasing function of time to maturity. We also observe that the distances between prices of options in models with jump components and without jump components become large with the increase in time to maturity. This increase in gap with time indicates that the jump risk becomes more pronounced with the life of the option. Also, we observe that the jump risk increases option prices whereas on the other hand jump clustering decreases it.

The impacts of correlation coefficient $\rho$ are shown in Figure 2 for the case when $T=1.5$. We observe that the option prices decrease with increase in the value of correlation coefficient. This observation is because of the fact that when $\rho$ is less than zero, it is more likely that the values $P_1$ and $P_2$ may move in the opposite directions and hence the distance between $P_1$ and $P_2$ may increase at the maturity. Similarly, if $\rho > 0$, the prices are more likely to move in same direction and hence option value is lesser than the prices when $\rho < 0$. From Figure 2, we conclude that the correlation coefficient is an important factor in pricing power exchange options. We also observe that the difference between the prices obtained from the three models is not much when $\rho$ goes from -.1 to 0. However this difference becomes more pronounce as $\rho$ varies from 0 to 1. Specifically, when $\rho > 0.5$, the difference
in option prices derived from jump diffusion model and diffusion models widens rapidly.

Figure 3. Option prices against the parameters of common Hawkes process

Figure 3 show the sensitivity of the option prices to $\lambda_0, \theta, \delta$ for maturity time $T = 1, 1.5, 2$. These are the parameters of the systematic jumps modeled by common Hawkes process. As presented in Figure 3(a), the option prices increases with increase in the initial intensity $\lambda_0$. The reason of this behavior is that due to larger common jump risk, it is more likely that $P_1$ and $P_2$ jump at the same time, and the possibility that two assets change in opposite directions increases the distance of the two asset values at the maturity and makes the option more valuable. Further, the larger parameter $\theta(\delta)$ is, the higher (lower) jump clustering risk is. Therefore,
from Figures 3(c) and 3(b), we conclude that jump clustering will lower the option prices, that is, the higher jump clustering risk is, the less the option prices are.

Figures 4 and 5 describe the effect of parameters of asset specific Hawkes processes, i.e., $\lambda_{1,0}, \lambda_{1,0}$. A higher value of $\lambda_{1,0}$ will affect $S_1$ strongly and thus increasing the chances of a call option to expire in the money, as a result increasing the value of the option. On the other hand, a higher value of $\lambda_{2,0}$ increases the value of $S_2$ and hence, chances to mature in the money is less. Intuitively, the asset specific jump intensities $\lambda_{1,0}$ ($\lambda_{2,0}$) have an impact on the asset prices $S_1$ ($S_2$) only and hence the expected payoff of the option behaves in the observed way. We
can observe that, jump clustering in asset $S_1$, decreases the options prices whereas option prices doesn’t change much with respect to jump clustering in asset $S_2$.

Figures 6 gives the comparison of option prices against initial intensities ($\lambda_0, \lambda_{1,0}$, $\lambda_{2,0}$), decay rates ($\delta, \delta_1, \delta_2$) and jump size ($\theta, \theta_1, \theta_2$). The option price increases when $\lambda_0, \lambda_{1,0}$ and $\lambda_{2,0}$ increases but the increase with respect to $\lambda_{2,0}$ is at lower rate as compared to the increase with respect to $\lambda_0, \lambda_{1,0}$. Similarly, the option prices increases (decreases) with increase in the values of decay rates (jump sizes) but the rate of increase or decrease is lower as for asset 2 as compared to that of common Hawkes process or asset 1.
Finally, Figure 7 gives the sensitivity of the option prices with respect to the parameters (mean and variance) of jump sizes in the asset price dynamics.

6. Conclusion. In this article, we proposed a jump diffusion model based on Hawkes processes for valuation of power exchange options. The price dynamics of two assets are modeled by Hawkes jump diffusion process with a common jump process to introduce correlation in jump components. The proposed model addresses the issue of jump clustering that is observed in financial market. It is shown that the proposed model is an extension to the models by Wang (2016) [32] and Blenman & Clark (2005) [6]. We studied the sensitivity analysis and comparative analysis of the proposed model. We observed that the jump clustering reduces the
power exchange option prices whereas jump risk increases it as compared to the classical diffusion model. Further, the sensitivity of the option prices are illustrated with respect to different parameters in the model.

In the proposed model, we have considered exponential decay function in the intensity process of the Hawkes process, which makes it a Markov process. The possible future extension could be to consider the non-Markovian case and other generalizations of the Hawkes process proposed in the literature [9].

REFERENCES

[1] L. Adamopoulos, Cluster models for earthquakes: Regional comparisons, J. of the Internat. Assoc. for Math. Geology, 8 (1976), 463–475.
[2] Y. Aiat-Sahalia, J. Cacho-Diaz and R. J. Laeven, Modeling financial contagion using mutually exciting jump processes, J. Financial Economics, 117 (2015), 585–606.
[3] G. Bakshi, C. Cao and Z. Chen, Empirical performance of alternative option pricing models, The Journal of Finance, 52 (1997), 2003–2049.
[4] D. S. Bates, Jumps and stochastic volatility: Exchange rate processes implicit in deutsche mark options, The Review of Financial Studies, 9 (1996), 69–107.
[5] D. S. Bates, Post-’87 crash fears in the S&P 500 futures option market, J. Econometrics, 94 (2000), 181–238.
[6] L. P. Blenman and S. P. Clark, Power exchange options, Finance Research Letters, 2 (2005), 97–106.
[7] N. Cai and S. G. Kou, Option pricing under a mixed-exponential jump diffusion model, Management Science, 57 (2011), 2067–2081.
[8] P. Carr and L. Wu, Time-changed lévy processes and option pricing, J. Financial Economics, 71 (2004), 113–141.
[9] A. Dassios and H. Zhao, A dynamic contagion process, Adv. in Appl. Probab., 43 (2011), 814–846.
[10] B. Eraker, Do stock prices and volatility jump? Reconciling evidence from spot and option prices, J. Finance, 59 (2004), 1367–1403.
[11] B. Eraker, M. Johannes and N. Polson, The impact of jumps in volatility and returns, J. Finance, 58 (2003), 1269–1300.
[12] E. Errais, K. Giesecke and L. R. Goldberg, Affine point processes and portfolio credit risk, SIAM J. Financial Math., 1 (2010), 642–665.
[13] S. Fischer, Call option pricing when the exercise price is uncertain, and the valuation of index bonds, J. Finance, 33 (1978), 169–176.
[14] A. G. Hawkes, Point spectra of some mutually exciting point processes, J. Roy. Statist. Soc. Ser. B, 33 (1971), 438–443.
[15] A. G. Hawkes, Spectra of some self-exciting and mutually exciting point processes, Biometrika, 58 (1971), 83–90.
[16] A. G. Hawkes, Hawkes processes and their applications to finance: A review, Quant. Finance, 18 (2018), 193–198.
[17] S. G. Kou, A jump-diffusion model for option pricing, Management Science, 48 (2002), 1086–1101.
[18] W. Liu and S.-P. Zhu, Pricing variance swaps under the Hawkes jump-diffusion process, J. Futures Markets, 39 (2019).
[19] T. Lux and M. Marchesi, Volatility clustering in financial markets: A microsimulation of interacting agents, Int. J. Theor. Appl. Finance, 3 (2000), 675–702.
[20] Y. Ma, K. Shrestha and W. Xu, Pricing vulnerable options with jump clustering, J. Futures Markets, 37 (2017), 1155–1178.
[21] Y. Ma and W. Xu, Structural credit risk modelling with Hawkes jump diffusion processes, J. Comput. Appl. Math., 303 (2016), 69–80.
[22] J. M. Maheu and T. H. McCurdy, News arrival, jump dynamics, and volatility components for individual stock returns, J. Finance, 59 (2004), 755–793.
[23] B. Mandelbrot, The variation of certain speculative prices, J. Business, 36 (1963), 394–419.
[24] W. Margrabe, The value of an option to exchange one asset for another, J. Finance, 33 (1978), 177–186.
[25] S. Meyer, J. Elias and M. Höhle, A space–time conditional intensity model for invasive meningococcal disease occurrence, Biometrics, 68 (2012), 607–616.
[26] Y. Ogata, On Lewis’ simulation method for point processes, IEEE Transactions on Information Theory, 27 (1981), 25–31.
[27] Y. Ogata, Statistical models for earthquake occurrences and residual analysis for point processes, J. Amer. Statistical Association, 83 (1988), 9–27.
[28] J. Pan, The jump-risk premia implicit in options: Evidence from an integrated time-series study, J. of Financial Economics, 63 (2002), 3–50.
[29] P. Pasricha and A. Goel, Pricing vulnerable power exchange options in an intensity based framework, J. Comput. Appl. Math., 355 (2019), 106–115.
[30] A. Reinhart, A review of self-exciting spatio-temporal point processes and their applications, Statist. Sci., 33 (2018), 330–333.
[31] R. Tompkins, Power options: hedging nonlinear risks, J. Risk, 2 (2000), 29–45.
[32] X. Wang, Pricing power exchange options with correlated jump risk, Finance Research Letters, 19 (2016), 90–97.
[33] X. Wang, S. Song and Y. Wang, The valuation of power exchange options with counterparty risk and jump risk, J. Futures Markets, 37 (2017), 499–521.
[34] J. Yu, Empirical characteristic function estimation and its applications, Econometric Rev., 23 (2004), 93–123.
E-mail address: paarichapuneet5@gmail.com
E-mail address: anubha.goel1@gmail.com