Finding Balance-Fair Short Paths in Graphs

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Abstract
The computation of short paths in graphs with edge lengths is a pillar of graph algorithmics and network science. In a more diverse world, however, not every short path is equally valuable. We contribute to a broader view on path finding by injecting a natural fairness aspect. Our fairness notion relates to vertex-colored graphs. Herein, we seek to find short paths in which all colors should appear with roughly the same frequency. Among other results, we prove the introduced problems to be computationally intractable (NP-hard and parameterized hard with respect to the number of colors), while also presenting an encouraging algorithmic result (“fixed-parameter tractability”) related to the length of the sought solution path.

1 Introduction

Travel agency WhataWonderfulWorld offers adventure bus trips from New Orleans to New York through exciting country sides. It guarantees small travel costs (thus addressing both financial as well as environmental demands) combined with maximum variety and balance of impressions. Clearly, such a sustainable travel from a starting point $s$ to an endpoint $t$ can be modeled as finding an $s$-$t$-path in a graph with positive edge lengths. To model maximum variety and balance, the vertices—the places to visit—are colored according to agency-chosen categories, say blue vertices are interesting for bathing, green vertices for hiking, et cetera. To address balance and variety as well as keeping the travel cost small, the quest is to find a short travel path in which every color appears almost equally often, that is, there should be about the same number of visits to bathing spots as to hiking places as to every other type of point of interest. We study the computational complexity of finding such “balance-fair” short paths, encountering both computational hardness for general cases as well as tractability results for practically relevant special cases. A two-color example of a maximally balance-fair path is depicted in Figure 1.

Related work. Path finding in vertex-colored graphs has been a subject of broad and intensive study. Here, we only point to algorithmically motivated work that seems particularly close to our scenario of balance-fair short paths. First of all, finding colorful paths (here meaning that each color appears at most once) is an important algorithmic topic, both in static and in temporal graphs [4, 8]. Cohen et al. [7] analyze the complexity of finding tropical paths (shortest and longest), that is, paths containing at least one vertex of each color. They provide both tractability and intractability results.
Another close (and also broad) area is that of finding resource-constrained shortest paths, where roughly speaking the desired path shall have minimum cost and only a limited consumption of resources. Generally, the problem is NP-hard [16] and it has been extensively studied over the years [12, 17, 24]. Given its intractability, several heuristics have been proposed recently [1]. One of the most recent variants of resource-constrained shortest path problems deals with weighted edge-colored graphs, using the colors as edge labels in order to take into account the matter of path reliability while optimizing path cost [11]. Note, however, that there are no immediate fairness aspects modeled here. Hanaka et al. [15] are somewhat closer to fairness aspects in path finding. They study shortest paths under diversity aspects, meaning that they search for multiple shortest paths that are maximally different from each other; this fits into the recent trend of finding diverse sets of solutions to optimization problems [5, 23].

Finally, we only mention in passing that fairness aspects are currently investigated in all kinds of optimization problems (particularly graph-based ones), including topics such as graph-based data clustering [2, 3, 13, 14], influence maximization [20], matching [6], and graph mining [9, 18].

**Our contributions.** We introduce and study a natural fairness scenario for one of the back-bone problems in network algorithmics: finding short paths. In this way, we also open the floor for several future extensions and variants (see Section 5 for more on that). We consider two problems, δ-Fair Short Path and its special case δ-Fair Shortest Path (see Section 2). Simply put, the latter problem prioritizes shortness over fairness, while the former does not prioritize one over the other. Both turn out to be NP-hard in general. To cope with this computational intractability, we investigate the parameterized complexity of these two problems with respect to the two perhaps most natural parameters: the number of different vertex colors in the input graph and the length of the sought-after solution path. While we prove that δ-Fair Short Path is para-NP-hard for the parameter number of colors, notably δ-Fair Shortest Path turns out to be polynomial-time solvable for a constant number of colors. We also prove that there is little hope for significantly improving the latter result by showing W[1]-hardness. Actually, both hardness results even hold when all edges have unit lengths. For the parameter path length, we prove that δ-Fair Short Path is fixed-parameter tractable using a color-coding-based technique.
2 Preliminaries

For two integers \( n \leq m \), let \([n, m] := \{n, n + 1, \ldots, m\}\). For \( n \in \mathbb{N} \), let \([n] := [1, n]\). We use standard graph-theoretic terminology. In particular, for an undirected graph \( G = (V, E) \) we set \( n := |V| \) and \( m := |E| \). For a vertex \( v \in V \), we denote by \( N_G(v) \) the (open) neighborhood of \( v \) in \( G \). The degree \( \deg_G(v) \) of \( v \) is the number of vertices in the neighborhood of \( v \). A path \( P \) on \( \ell \) vertices is a graph with vertex set \( \{v_1, v_2, \ldots, v_\ell\} \) and edge set \( \{\{v_i, v_{i+1}\} \mid i \in [\ell - 1]\} \). The vertices \( v_1 \) and \( v_\ell \) are called endpoints. Given an edge-length function, the length of the path is the sum of its edge lengths. Let \( G = (V, E) \) be a graph with two vertices \( s \) and \( t \) and let \( w : E \to \mathbb{N} \) be an edge-length function. An \( s \)-path \( P \) is a subgraph of \( G \) whose endpoints are \( s \) and \( t \). We denote by \( \text{dist}_G(s, t) \) the length of a shortest \( s \)-path in \( G \). Whenever clear from context, we may drop the subscript \( G \).

For a graph \( G = (V, E) \), a vertex coloring \( \chi : V \to [c] \), and a color \( i \in [c] \), we denote by \( \chi^i := \{v \in V \mid \chi(v) = i\} \) the set of vertices of color \( i \). For a subgraph \( H \) of \( G \) and a color \( i \in [c] \), we denote by \( \chi^i_H \) the coloring \( \chi \) restricted to the vertices of \( H \), and by \( \chi^i_H \) the set of vertices of color \( i \) in \( H \).

We can now formally define the two problems of interest.

\( \delta \)-Fair Short Path

**Input:** An undirected graph \( G = (V, E) \), a vertex coloring \( \chi : V \to [c] \), an edge-length function \( w : E \to \mathbb{N} \), two vertices \( s, t \in V \), and an integer \( \ell \).

**Question:** Is there an \( s \)-path \( P \) of length at most \( \ell \) such that

\[
\max_{i \in [c]} |\chi^i_P| - \min_{i \in [c]} |\chi^i_P| \leq \delta?
\]

If a path \( P \) satisfies \( \max_{i \in [c]} |\chi^i_P| - \min_{i \in [c]} |\chi^i_P| \leq \delta \), then we call it balance-fair. The following problem is the special case where \( \ell = \text{dist}_G(s, t) \), that is, the sought balance-fair path is required to be a shortest \( s \)-path.

\( \delta \)-Fair Shortest Path

**Input:** An undirected graph \( G = (V, E) \), a vertex coloring \( \chi : V \to [c] \), an edge-length function \( w : E \to \mathbb{N} \), and two vertices \( s, t \in V \).

**Question:** Is there a shortest \( s \)-path \( P \) such that \( \max_{i \in [c]} |\chi^i_P| - \min_{i \in [c]} |\chi^i_P| \leq \delta \)?

We remark that we consider the fairness measure \( \delta \) as part of the problem name and thus as a constant.

Lastly, we recall some basic (parameterized) complexity concepts. A parameterized problem is fixed-parameter tractable if there exists an algorithm solving any instance \((I, \rho)\) (\(I\) is the input instance and \(\rho\) is some parameter) in \( f(\rho) \cdot |I|^{O(1)} \) time, where \( f \) is a (computable) function solely depending on \( \rho \). The class \( \text{XP} \) contains all parameterized problems which can be solved in polynomial time if the parameter \( \rho \) is constant, that is, in \(|I|^{f(\rho)} \) time. To show that a parameterized problem \( L \) is presumably not fixed-parameter tractable, one may use a parameterized reduction from a \( W[1] \)-hard problem to \( L \). A parameterized reduction from a parameterized problem \( L \) to another parameterized problem \( L' \) is a function satisfying the following. There are two computable functions \( f \) and \( g \), such that given an instance \((I, \rho)\) of \( L \), the reduction computes in \( f(\rho) \cdot |I|^{O(1)} \) time an
instance \((I', \rho')\) of \(L'\) such that \(\rho' \leq g(\rho)\) and \((I, \rho)\) is a yes-instance if and only if \((I', \rho')\) is a yes-instance.

3 The Parameter Number of Colors

In this section, we study the computational complexity of our two problems parameterized by the number of colors. First, we look at the more general \(\delta\)- FAIR SHORT PATH. If every vertex has the same color, then we clearly get polynomial-time solvability. But already when there are two colors, the problem becomes NP-hard, as seen by a polynomial-time reduction from the NP-hard HAMILTONIAN PATH problem.

**Proposition 1.** \(\delta\)- FAIR SHORT PATH is NP-hard even with unit edge lengths and only two vertex colors.

**Proof.** We reduce from the well-known NP-hard problem HAMILTONIAN PATH, where one is asked whether a given graph contains a path that visits all vertices. For a given graph \(G = (V, E)\), we construct an equivalent instance \((G' = (V', E'), \chi, w, s, t, \ell)\) of \(\delta\)- FAIR SHORT PATH as follows. The graph \(G'\) contains the graph \(G\) as an induced subgraph. Additionally, it contains the vertices \(s\) and \(t\) and a set \(P = \{v_i \mid i \in [\lceil |V| + \delta - 2\rceil]\}\) of new vertices. The vertices in \(P\) form a path, that is, \(v_i\) and \(v_{i+1}\) are connected by an edge for all \(i \in [\lceil |V| + \delta - 3\rceil]\). To complete the description of \(G'\), we add an edge between \(v_{|V|+\delta-2}\) and \(t\) and the set \(\{s, u\}, \{v_i, u\} \mid u \in V\} of edges. All vertices in \(V\) get one color and all newly introduced vertices receive a second color. All edges have unit lengths and we set \(\ell := 2|V| + \delta - 1\).

Since the construction is clearly computable in polynomial time, it remains to show that the constructed instance is equivalent. To this end, first assume that the graph \(G\) contains a path \(Q\) through all vertices. Let \(a, b \in V\) be the endpoints of \(Q\). Consider the \(s\)-t-path \(R\) in \(G'\) that starts in \(s\), then continues with all vertices in \(V\) in the same order as \(Q\) followed by the vertices in \(P\), and finally ends in \(t\). Since \(R\) contains all \(2|V| + \delta\) vertices, its length is \(2|V| + \delta - 1 = \ell\). Finally, it contains \(|V|\) vertices of the first color and \(|V| + \delta\) vertices of the second color. Thus, it is balance-fair.

For the reverse direction, assume that there is a balance-fair \(s\)-t-path \(R\) in \(G'\). Note that \(R\) contains all \(|V| + \delta\) vertices of the second color (the vertices in \(P \cup \{s, t\}\)). Moreover, it can only contain vertices from \(V\) (vertices of the first color) between \(s\) and \(v_1\). Thus, in order to be balance-fair, it needs to contain a path of at least \((|V| + \delta) - \delta = |V|\) vertices from the original graph \(G\). Thus, \(G\) contains a path through all its vertices, concluding the proof.

The reduction in Proposition 1 makes use of the fact that one may take detours to make the path balance-fair. Therefore, it does not work for the special case \(\delta\)- FAIR SHORTEST PATH. When enforcing the solution to be a shortest \(s\)-t-path, we can show the following.

**Theorem 2.** \(\delta\)- FAIR SHORTEST PATH parameterized by the number \(c\) of colors is solvable in \(O((2n + 1)^{c-1} \cdot m)\) time.

**Proof.** Let \((G = (V, E), \chi, s, t)\) be an instance of \(\delta\)- FAIR SHORTEST PATH with \(c\) colors. We devise a dynamic program with a Boolean table \(T: V \times [-n, n]^{c-1} \rightarrow \{0, 1\}\) storing for each vertex \(v\) and each tuple \((x_1, x_2, \ldots, x_{c-1})\) whether there is a shortest \(s-v\)-path in \(G\) in which the difference between the number of vertices of colors \(c\) and \(i\) is exactly \(x_i\)
for all \( i \in [c - 1] \). We say that such paths respect the tuple \((x_1, x_2, \ldots, x_{c-1})\). The table is computed for all vertices in order of their distances from \( s \). Note that for each edge \( \{u, v\} \) in any shortest \( s-t \)-path \( P \), one of the two endpoints, say \( u \), is always closer to \( s \) and \( \text{dist}(s, v) = \text{dist}(s, u) + w(\{u, v\}) \). Hence, we can assume that the edges are oriented “from \( s \) to \( t \)” and fulfill said equality. Denote by

\[
A := \{(u, v) \in V \times V \mid \{u, v\} \in E \text{ and } \text{dist}_G(s, v) = \text{dist}_G(s, u) + w(\{u, v\})\}
\]

the set of relevant oriented arcs. To compute an entry \( T[v, x_1, x_2, \ldots, x_{c-1}] \), we distinguish whether the color \( i \) of \( v \) is \( c \) or not. If \( i < c \), then we iterate over all incoming arcs \( \{u, v\} \) of \( v \) and compute

\[
T[v, x_1, x_2, \ldots, x_{c-1}] = \bigvee_{(u, v) \in A} T[u, x_1', x_2', \ldots, x_{c-1}'],
\]

wherein \( x_i' = x_i - 1 \) and \( x_j' = x_j \) for all \( j \neq i \). That is, \( T[v, x_1, x_2, \ldots, x_{c-1}] \) is set to true if and only if \( v \) has a neighbor \( u \) for which \( T[u, x_1', x_2', \ldots, x_{c-1}'] \) is true. If \( T[v, x_1, \ldots, x_{c-1}] \) is set to true, then there is a shortest \( s-u \)-path respecting \( (x_1', x_2', \ldots, x_{c-1}') \). Appending \( v \) to this path results in a path respecting \( (x_1, x_2, \ldots, x_{c-1}) \) since \( x_i = x_i' + 1 \). By the definition of \( A \), it also holds that this path is a shortest \( s-v \)-path. In the other direction, assume that there is a shortest \( s-v \)-path \( P \) respecting \( (x_1, x_2, \ldots, x_{c-1}) \) and let \( \chi(v) = i \). Consider the penultimate vertex \( u \) in \( P \) and the subpath \( P' \) from \( s \) to \( u \). Note that since \( P \) and \( P' \) only differ in \( v \), it holds that \( P' \) is a shortest \( s-u \)-path respecting \( (x_1', x_2', \ldots, x_{c-1}') \), where \( x_i' = x_i - 1 \) and \( x_j' = x_j \) for all colors \( j \neq i \). Thus, \( T[u, x_1', x_2', \ldots, x_{c-1}'] \) is set to true and thus by construction also \( T[v, x_1, x_2, \ldots, x_{c-1}] \).

If \( v \) has color \( c \), then we compute

\[
T[v, x_1, x_2, \ldots, x_{c-1}] = \bigvee_{(u, v) \in A} T[u, x_1 + 1, x_2 + 1, \ldots, x_{c-1} + 1],
\]

If such an arc \( \{u, v\} \) exists, then there is a shortest \( s-u \)-path respecting \( (x_1 + 1, x_2 + 1, \ldots, x_{c-1} + 1) \). Appending \( v \) to this path results in a path that has one additional vertex of color \( c \) and hence this path respects the tuple \((x_1, x_2, \ldots, x_{c-1})\). Again, this is also a shortest \( s-v \)-path and thus \( T[v, x_1, x_2, \ldots, x_{c-1}] \) is correctly computed in this case. For the other direction, if a shortest \( s-v \)-path \( P \) respecting \( (x_1, x_2, \ldots, x_{c-1}) \) exists, then the subpath \( P' \) from \( s \) to the penultimate vertex \( u \) in \( P \) shows that \( T[v, x_1, x_2, \ldots, x_{c-1}] \) is correctly set to true in this case.

Once the whole table is computed, we can check whether there is a balance-fair shortest \( s-t \)-path in \( G \) by iterating over all tuples \((x_1, x_2, \ldots, x_{c-1})\) where

\[
\max\{x_1, x_2, \ldots, x_{c-1}, 0\} - \min\{x_1, x_2, \ldots, x_{c-1}, 0\} \leq \delta
\]

and check whether \( T(t, x_1, x_2, \ldots, x_{c-1}) \) is true. If at least one such entry is true, then there is a balance-fair shortest \( s-t \)-path in \( G \) since the shortest path respecting the tuple \((x_1, x_2, \ldots, x_{c-1})\) contains at least \( x + \min\{x_1, x_2, \ldots, x_{c-1}, 0\} \) vertices of each color and at most \( x + \max\{x_1, x_2, \ldots, x_{c-1}, 0\} \) vertices of each color, where \( x \) is the number of vertices of color \( c \). Conversely, if all these values are false, then there is no balance-fair shortest \( s-t \)-path in \( G \).

Lastly, we analyze the running time. Observe that there are \((2n+1)^{c-1}\) table entries for each vertex \( v \) and computing one table entry takes \( O(\deg(v)) \) time. Moreover, computing
whether \( G \) contains a balance-fair shortest \( s \)-\( t \)-path once the table \( T \) is completely filled takes \((2n + 1)^{<1}\) time. Thus, the overall running time is in \( \mathcal{O}(m \cdot (2n + 1)^{<1}) \) by the handshaking lemma.

The next result shows that there is little hope for a significantly better algorithm for \( \delta \)-Fair Shortest Path parameterized by the number \( c \) of colors. We prove that \( \delta \)-Fair Shortest Path is \( \text{W}[1] \)-hard with respect to \( c \) by providing a parameterized reduction from Multicolored Clique: Given a \( k \)-partite undirected graph \( G = (V, E) \) with partitions \( V_1, V_2, \ldots, V_k \), is there a clique on \( k \) vertices in \( G \)? Multicolored Clique is known to be \( \text{W}[1] \)-complete [10].

**Theorem 3.** \( \delta \)-Fair Shortest Path is \( \text{W}[1] \)-hard when parameterized by the number \( c \) of colors.

**Proof.** We reduce from Multicolored Clique parameterized by solution size \( k \). To this end, let \( G = (V, E) \) be the input graph of our instance of Multicolored Clique with partitions \( V_1, V_2, \ldots, V_k \). For the sake of simplicity, we will assume that each \( V_i \) has the same number \( \eta := n/k \) of vertices (this is no restriction since we can simply add isolated vertices) and that \( V_i = \{v_{i1}, v_{i2}, \ldots, v_{i\eta}\} \). Figure 2 gives an example used throughout this proof.

In the following, we will construct a graph \( H \) with two vertices \( s \) and \( t \) that contains a balance-fair shortest \( s \)-\( t \)-path if and only if \( G \) contains a clique of size \( k \). Let us first give an intuitive description of the different pieces. The graph \( H \) will be made mostly from two parts: a vertex-selection gadget for each partition \( V_i \) and an edge-verification gadget for each pair \( V_i \neq V_j \) of partitions. The former (broadly speaking) decides for each \( i \in [k] \) which vertex of \( V_i \) is supposed to be in the clique. The latter verifies that there is an edge between the two chosen vertices of the two respective partitions. Our reduction will use \( k^2 + 1 \) colors: a color \( r_i^\ell \) for each \( i \in [k] \) and each \( \ell \in [0, k - 1] \) and a special color \( q \). The main idea behind the reduction is to use a base-\( k \) encoding of the numbers between 1 and \( \eta \). For each \( i \in [k] \), the \( k \) colors \( r_i^\ell \) with \( \ell \in [0, k - 1] \) represent the \( k \) different digits in this encoding. With this encoding, we can represent a vertex \( v_{ij} \) by a path containing \( k^{\ell - 1} \) vertices of color \( r_i^\ell \), where \( d \) is the \( \ell \)-th least significant digit in

\[ \text{Figure 2: An example instance of MULTICOLORED CLIQUE with } k = 3 \text{ colors and } \eta = 5 \text{ vertices of each color.} \]
the base-$k$ encoding of $j$. This encoding is used within the edge-verification gadgets. In the vertex-selection gadgets, we will represent vertices of $G$ by paths of vertices such that if this path is part of the final balance-fair path, then it can only be a balance-fair path if it contains exactly $k - 1$ paths in edge-selection gadgets that correspond to the respective vertex.

More formally, for each partition $V_i$, we create a vertex-selection gadget as follows (see Figure 3 for an illustration). The gadget consists of $\eta$ vertex-disjoint paths, the endpoints of which are adjacent to two vertices of color $q$. Each of these paths represents one vertex $v_i^j$ and is constructed using the base-$k$ encoding of $j$. To this end, let $\beta := \lceil \log_k \eta \rceil$ be the number of digits required to encode $\eta$. For each $\ell \in [\beta]$, let $d_\ell$ be the $\ell$-th least significant digit in the base-$k$ encoding of $j$. Then, we add $(k - 1) \cdot k^{\ell - 1}$ vertices of each color $r_i^p$ with $p \in [0, k - 1] \setminus \{d_\ell\}$. We call the set of all vertices added for a specific $\ell$ the $\ell$-th level of this gadget. The number of vertices in all levels combined is

$$x := (k - 1)^2 \cdot \sum_{\ell=1}^\beta k^{\ell - 1}.$$

The edge-verification gadget for a pair $V_i, V_i'$ of partitions is again a collection of vertex-disjoint parallel paths whose endpoints are adjacent to two vertices of color $q$ (see Figure 4 for an illustration). Each such path represents an edge between two vertices $v_i^j$ and $v_i'^{j'}$ in $G$. For each $\ell \in [\beta]$, let $d_\ell$ be the $\ell$-th least significant digit in the base-$k$ encoding of $j$ and let $d_\ell'$ be the $\ell$-th least significant digit in the base-$k$ encoding of $j'$. We then add $k^{\ell - 1}$ vertices of color $r_i^{d_\ell}$ and $k^{\ell - 1}$ vertices of color $r_i'^{d_\ell'}$. We again call the set of all vertices added for a specific $\ell$ the $\ell$-th level of this gadget. The number of vertices in all levels combined is

$$y := 2 \cdot \sum_{\ell=1}^\beta k^{\ell - 1}.$$

Next, we place the different gadgets behind one another (in an arbitrary order) where we identify the end of one gadget with the beginning of the next gadget (both vertices have color $q$). Observe that the number of vertices that are not of color $q$ in each shortest
Figure 4: The edge-verification gadget for \( V_1 \) and \( V_2 \) in Figure 2 and a legend providing the names for each color. Each path represents an edge and both incident vertices are listed above the respective path.

path through all the different gadgets is exactly

\[
\begin{align*}
k \cdot x + \left( \frac{k}{2} \right) y &= k \cdot (k - 1)^2 \cdot \sum_{\ell=1}^{\beta} k^{\ell-1} + \left( \frac{k}{2} \right)^2 \cdot \sum_{\ell=1}^{\beta} k^{\ell-1} \\
&= (k^3 - 2k^2 + k + k^2 - k) \cdot \sum_{\ell=1}^{\beta} k^{\ell-1} \\
&= (k^3 - k^2) \cdot \sum_{\ell=1}^{\beta} k^{\ell-1}.
\end{align*}
\]

Hence, in order for such a path to contain the same amount of vertices of each color, each of the \( k^2 \) colors different from \( q \) have to appear \((k - 1) \cdot \sum_{\ell=1}^{\beta} k^{\ell-1} \) times. Moreover, every path passing through all gadgets contains exactly \( k + \left( \frac{k}{2} \right) + 1 \) vertices of color \( q \). To complete the construction, we add a path on \((k - 1) \cdot \sum_{\ell=1}^{\beta} k^{\ell-1} - (k + \left( \frac{k}{2} \right) + 1) \geq 0 \) vertices of color \( q \) to \( H \), connect it to the last vertex in the last gadget, and call the last vertex in this path \( t \) and the first vertex in the first gadget \( s \).

We next show that the reduction runs in polynomial time. Since

\[
\sum_{\ell=1}^{\beta} k^{\ell-1} = \sum_{\ell=1}^{\lfloor \log_k \eta \rfloor} k^{\ell-1} \leq \sum_{\ell=0}^{\lfloor \log_k \eta \rfloor} k^{\ell} \leq k^{1+\log_k \eta} = k^{\eta} = n,
\]

the number of vertices in the constructed instance is polynomial in the number of vertices in \( G \). As the base-\( k \) representation of a number can be computed in polynomial time as well, the whole reduction takes polynomial time.

As the construction uses \( k^2 + 1 \) colors, it remains to show that the constructed instance is a \textit{yes}-instance if and only if the original instance is a \textit{yes}-instance. For the forward direction, suppose that \( G \) contains a clique \( K \) on \(|K| = k \) vertices. Let \( K = \{v_{j_1}^1, v_{j_2}^2, \ldots, v_{j_k}^k\} \). Consider the path \( P \) from \( s \) to \( t \) that contains all vertices of color \( q \), for each \( V_i \) the path representing vertex \( v_i^j \) in the vertex-selection gadget for \( V_i \), and for each pair \( V_i \neq V_j \) the path representing \( \{v_i^j, v_j^i\} \) in the edge-verification gadget for \( V_i \) and \( V_j \). Note that all these edges exist since \( K \) is a clique. We claim that \( P \) contains exactly \((k - 1) \cdot \sum_{\ell=1}^{\beta} k^{\ell-1} \) vertices of each color. First, this is true for the color \( q \) by the construction of the final
path of vertices of this color. Now consider any color \( r_a \neq q \). Since \( r_a \) only appears in the vertex-selection gadget for \( V_i \) and in edge-verification gadgets for \( V_i \) and \( V_j \) for each \( j \neq i \), we will ignore all other gadgets. We will show that \( P \) contains exactly \((k-1) \cdot \ell^{-1}\) vertices of color \( r_a \) in all \( \ell \)-th levels of the considered gadgets combined. To this end, we make a case distinction whether the \( \ell \)-th least significant digit \( d_\ell \) in the base-\( k \) encoding of \( j_i \) is \( a \) or not. If \( d_\ell = a \), then the path \( P_a \) representing \( v_j^i \in K \) in the vertex-selection gadget for \( V_i \) contains no vertices of color \( r_a \) in the \( \ell \)-th level and a path \( P_b \) representing an edge containing \( v_j^i \) in an edge-verification gadget contains by construction \( k^{\ell-1} \) vertices of color \( r_a \) in the \( \ell \)-th level. Conversely, if \( d_\ell \neq a \), then the path \( P_a \) representing \( v_j^i \in K \) in the vertex-selection gadget for \( V_i \) contains \((k-1) \cdot \ell^{-1}\) vertices of color \( r_a \) in the \( \ell \)-th level and a path \( P_b \) representing an edge containing \( v_j^i \) in an edge-verification gadget contains no vertices of color \( r_a \) in the \( \ell \)-th level. Since there are exactly \( k-1 \) edge-verification gadgets for \( V_i \) and some \( V_k \) with \( i' \neq i \), it holds that there are exactly \((k-1) \cdot \sum_{\ell=1}^{\beta} k^{\ell-1}\) vertices of color \( r_a \) in \( P \). Moreover, since we chose \( r_a \) arbitrarily, it holds that \( P \) contains exactly \((k-1) \cdot \sum_{\ell=1}^{\beta} k^{\ell-1}\) vertices of each color. Thus, the resulting instance of \( \delta \)-FAIR SHORTEST PATH is a yes-instance.

For the backward direction, assume that there is a balance-fair shortest \( s-t \)-path \( P \) in \( H \), that is, \( P \) contains exactly \((k-1) \cdot \sum_{\ell=1}^{\beta} k^{\ell-1}\) vertices of each color. Consider the set \( K' = \{v_j^1, v_j^2, \ldots, v_j^k\} \) of vertices corresponding to those paths of the vertex-selection gadgets that are contained in \( P \). We will show that \( K' \) forms a clique in \( G \). To this end, consider any vertex \( v_j^i \in K' \). We claim that \( P \) contains \( k-1 \) paths of the edge-verification gadgets representing edges incident to \( v_j^i \). Note that there are \( \binom{k}{2} \) edge-verification gadgets and we chose \( v_j^i \in K' \) arbitrarily (meaning that the claim holds for all vertices in \( K' \)). Hence, if the claim holds, then \( P \) contains \( \binom{k}{2} \) paths representing edges between \( k \) vertices. This implies that \( K' \) is a (multicolored) clique of size \( k \), concluding the proof. Assume towards a contradiction that the claim does not hold, that is, there are subpaths of \( P \) through edge-verification gadgets that encode edges with an endpoint in \( V_i \) other than \( v_j^i \). Let \( F \) be the set of these endpoints and let \( \ell \) be the smallest number such that the \( \ell \)-th least significant digit \( d \) in the base-\( k \) encoding of \( j_i \) differs from the \( \ell \)-th least significant digit in the base-\( k \) encoding of any \( j' \) with \( v_j^i \in F \). Consider the color \( r_d \) and consider the number of vertices of color \( r_d \) in \( P \). By the minimality of \( \ell \), it holds for all \( \ell' < \ell \) that the \( \ell' \)-th least significant digit of \( j' \) and \( j_i \) are the same for all \( v_j^i \in F \). Hence, if \( d \) is the \( \ell \)-th least significant digit of \( j_i \), then \( P \) contains \((k-1) \cdot k^{\ell'}\) vertices of color \( r_d \) in the \( \ell' \)-th level in the vertex-selection gadget and no vertices of color \( r_d \) in the \( \ell \)-th level of any edge-selection gadget. Moreover, if \( d \) is not the \( \ell \)-th least significant digit of \( j_i \), then \( P \) contains no vertices of color \( r_d \) in the \( \ell' \)-th level in the vertex-selection gadget and \( k^{\ell'} \) vertices of color \( r_d \) in the \( \ell \)-th level of each of the \((k-1)\) edge-selection gadgets for \( V_i \) and some \( V_j \) with \( i' \neq i \). It follows that \( P \) contains \((k-1) \cdot \sum_{a=1}^{k^{\ell-1}} k^{a-1}\) vertices of color \( r_d \) in the first \( \ell \)-th levels in all gadgets. Since the target number for each color is \((k-1) \cdot \sum_{a=1}^{k^{\ell-1}} k^{a-1}\) and each level \( \ell' > \ell \) contains each color a multiple of \( k \) times, it follows that the number of vertices of color \( r_d \) in the \( \ell \)-th level in all gadgets is \((k-1) \cdot k^{\ell-1} \) modulo \( k^\ell \). By the definition of \( d \), there are no vertices of color \( r_d \) in the \( \ell \)-th level of the vertex-selection gadget for \( V_i \). Since each edge-verification gadget contains at most \( k^{\ell-1} \) vertices of color \( r_d \) in the \( \ell \)-th level, it follows that each edge-verification gadget for \( V_i \) and \( V_j \) with \( i' \neq i \) contains exactly \( k^{\ell-1} \) vertices of color \( r_d \) in the \( \ell \)-th level. This contradicts the assumption that the \( \ell \)-th least significant digit in the base-\( k \) encoding of \( j_i \) differs from the \( \ell \)-th least significant digit in
the base-$k$ encoding of some $j'$ with $v_{j'}^i \in F$. This concludes the proof.

\section{The Parameter Path Length}

This section is devoted to proving that $\delta$-\textsc{Fair Short Path} is fixed-parameter tractable with respect to the length $\ell$.

\textbf{Theorem 4.} $\delta$-\textsc{Fair Short Path} is solvable in $(n+m)\log n \cdot (2(\delta+1)e^2)^{\ell+2} \cdot \ell^{O(\log \ell)}$ time.

The algorithm works in two steps. We first guess how many vertices of each color are contained in a solution. We then check whether such a path exists. Details to the former step are encapsulated in the proof of Theorem 4 at the end of this section. We first provide an algorithm for the latter step. Following Mulmuley et al. \cite{20} and Kellerhals et al. \cite{19}, we name this problem \textsc{Exact Colored Path}.

\textsc{Exact Colored Path}.

\textbf{Input:} An undirected graph $G = (V, E)$, a vertex coloring $\chi : V \to [c]$, an edge-length function $\ell : E \to \mathbb{N}$, two vertices $s, t \in V$, and integers $\ell, k_1, k_2, \ldots, k_c$.

\textbf{Question:} Does $G$ contain an $s$-$t$-path $p$ of length at most $\ell$ wherein $|V(P) \cap \chi^i|$ is exactly $k_i$ for each $i \in [c]$?

We will show that \textsc{Exact Colored Path} is fixed-parameter tractable with respect to $k := \sum_{i=1}^{c} k_i$ using the color-coding technique of Alon et al. \cite{4}. The algorithm assigns $k_i$ distinct labels to each color and then checks whether there is an $s$-$t$-path that contains each of these labels exactly once. The checking step can be solved by a simple dynamic programming algorithm.

\textbf{Lemma 5.} If $k_i = 1$ for each $i \in [c]$, then \textsc{Exact Colored Path} can be solved in $\mathcal{O}(2^c \cdot \ell \cdot m)$ time.

\textbf{Proof.} We devise a dynamic-programming table $T : V \times 2^{|c|} \times \{0, 1, \ldots, \ell\} \to \{0, 1\}$ that stores for each vertex $v$, each color subset $S$, and each length $d$, whether $G$ contains an $s$-$v$-path of length exactly $d$ using exactly one vertex of each color in $S$. For $|S| = 1$, we have that $T[v, S, d]$ is true if and only if $v = s$, $S = \{\chi(s)\}$, and $d = 0$. For $|S| > 1$, the entry $T[v, S, d]$ for some $v \in V$ and $d \in [\ell]$ is false if $\chi(v) \notin S$. Otherwise, we have

$$T[v, S, d] = \bigvee_{u \in N(v)} T[u, S \setminus \{\chi(v)\}, d - \ell - w(u, v)].$$

Clearly, if there is an $s$-$v$-path of length $\ell$ using all colors in $S \subseteq [c]$ exactly once, then there has to be a neighbor $u$ of $v$ such that there is an $s$-$u$-path of length $\ell - w(u, v)$ using all colors in $S \setminus \{\chi(v)\}$ exactly once. Finally, there is a colorful $s$-$t$-path of length at most $\ell$ if and only if $T[t, [c], d]$ is true for some $d \leq \ell$. As the value for each table entry $T[v, S, d]$ can be computed in $\mathcal{O}(\deg(v))$ time and there are $n \cdot 2^c \cdot \ell$ table entries, filling the whole table takes $\mathcal{O}(m \cdot 2^c \cdot \ell)$ time by the handshaking lemma. Computing the solution takes $\mathcal{O}(\ell)$ time afterwards.\qed
We will now describe the color-coding algorithm for **Exact Colored Path** in detail. While color-coding was initially described as a randomized technique, one can use perfect hash families [22] to derandomize this technique in most applications. For the sake of brevity, we will only provide the derandomized algorithm.

An \((n, k)\)-perfect hash family \(F\) is a family of functions \([n] \rightarrow [k]\) such that for every set \(S \subseteq [n]\) of size \(k\) there exists a function \(f \in F\) which assigns a unique element from \([k]\) to each element in \(S\).

**Lemma 6 ([22]).** For any \(n, k \geq 1\) one can construct an \((n, k)\)-perfect hash family of size \(e^k k^{O(\log k)} \log n\) in \(e^k k^{O(\log k)} n \log n\) time.

**Lemma 7.** **Exact Colored Path** is solvable in \(\ell(n + m) \log n \cdot (2e)^k \cdot k^{O(\log k)}\) time, where \(k = \sum_{i=1}^c k_i\).

**Proof.** Let \(V := [n]\). We first create an \((n, k)\)-perfect hash family \(F\) of functions \([n] \rightarrow [k]\) in \(e^k k^{O(\log k)} \log n\) time using **Lemma 6**. We say that \(f \in F\) is exact if \(f(u) \neq f(v)\) whenever \(\chi(u) \neq \chi(v)\) and if \(f\) assigns for each \(i \in [c]\) exactly \(k_i\) different labels to the vertices of color \(i\). We ignore any non-exact functions in \(F\). For each remaining function \(f \in F\), we check whether \(G\) contains an \(s\)-\(t\)-path \(P\) of length at most \(\ell\) that is colorful with respect to the labeling \(f\). For this, we use our \(O(2^k \cdot \ell \cdot m)\)-time algorithm for **Colorful** \(s\)-\(t\)-**PATH with Edge Lengths** with \(k\) labels described in **Lemma 5**. The overall running time is then \(\ell(n + m) \log n \cdot (2e)^k \cdot k^{O(\log k)}\).

If the algorithm returns \textbf{yes}, then there is an exact function \(f \in F\) and an \(s\)-\(t\)-path \(P\) of length at most \(\ell\) that contains exactly one vertex of each label. Since \(f\) is exact, \(P\) contains exactly \(k_i\) vertices of each color \(i \in [c]\). Conversely, if there exists an exact \(s\)-\(t\)-path of length at most \(\ell\), then there exists a function in \(f \in F\) that assigns a unique label to each vertex of that path. Hence, this path is colorful with respect to \(f\) and our algorithm returns \textbf{yes}.

With **Lemma 7** at hand we can now prove **Theorem 4**.

**Theorem 4.** **\(\delta\)-Fair Short Path** is solvable in \((n + m) \log n \cdot (2(\delta + 1)c^2)\ell + 2 \cdot k^{O(\log \ell)}\) time.

**Proof.** Our algorithm starts off by guessing\(^1\) the number \(k \leq \ell + 1\) of vertices in a solution path \(P\).

Suppose first that the number \(c\) of colors is at most \(k\). Then we first guess the minimum number \(q\) of vertices of each color in \(P\). By the pigeon hole principle, we have \(k/c - \delta \leq q \leq k/c\). Hence, there are at most \(\delta + 1\) choices for \(q\). We next guess for each \(i \in [c]\) the number \(k_i = |\chi_i^P|\) of vertices of color \(i\) such that \(q \leq k_i \leq q + \delta\) and \(\sum_{i=1}^c k_i = k\). Then, we use **Lemma 7** to check if there is an \(s\)-\(t\)-path of length at most \(\ell\) containing exactly \(k_i\) vertices of color \(i\). Since we brute-force all possible assignments for \(k_i\), the algorithm is correct if \(c \leq k\).

Now suppose that \(c > k\). Then, there exists a color that will not appear on \(P\). Hence, there are at most \(\delta\) vertices of each color in \(P\). We start off by guessing the number \(c' \leq k\) of colors to appear in \(P\). Then, we construct a \((c, c')\)-perfect hash family \(F\) that assigns each color a label from \([c']\).

For each \(f \in F\), let \(\varphi_f : V \rightarrow [c']\) with \(\varphi_f(v) = f(\chi(v))\) be the corresponding vertex labeling function. We guess for each label \(j \in [c']\) the number \(k_j \in [\delta]\)

\(^1\)Whenever we pretend to guess something, we iterate over all possibilities and consider the correct iteration.
of vertices with label $j$ in $P$ such that $\sum_{j=1}^{c'} k_j = k$. Now, we return $\text{yes}$ if and only if there exists a function $f \in \mathcal{F}$ such that there exists an $s$-$t$-path of length at most $\ell$ which contains exactly $k_j$ vertices which are assigned label $j$ by $\varphi_f$. This can be checked in $\ell(n + m) \log n \cdot (2e)^k \cdot k^{O(\log k)}$ time using Lemma 7.

To prove the correctness of our algorithm for the case where $c > k$, assume first that our algorithm returns $\text{yes}$, that is, there are $c', k_j$ for all $j \in [c']$, and a function $f \in \mathcal{F}$ such that there is an exact $s$-$t$-path of length at most $\ell$. Note that the number of vertices of color $i$ is at most the number of vertices with label $f(i)$. Hence, for each color $i$, the path contains at most $\delta$ vertices and is therefore balance-fair. Conversely, suppose that there exists a balance-fair $s$-$t$-path of length at most $\ell$. Then it contains $c' \leq k$ different colors, and for each $j \in [c']$, the path contains $k_j \leq \delta$ vertices of color $j$. Hence, there exists a function $f \in \mathcal{F}$ that assigns a unique label to each of the colors appearing on the path. As the remaining colors do not appear on the path, it contains exactly $k_j$ vertices that are assigned label $j$ by $\varphi_f$ and our algorithm returns $\text{yes}$.

We next analyze the running time for a given guess of $k$. In the case $c \leq k$ we require $(\delta + 1)^{c+1} \leq (\delta + 1)^{k+1}$ calls of Lemma 7. For the case $c > k$ we first need to construct the $(c, c')$-perfect hash families for all $c' \leq k$ in overall $e^k k^{O(\log k)} c \log c$ time. Afterwards, we require $\delta^k e^k k^{O(\log k)}$ calls of Lemma 7. Summing the running times for all $\ell$ possible guesses of $k$ and substituting $k \leq \ell + 1$ yields the final running time.

5 Conclusion

Our study indicated that even very simple cases of $\delta$-Fair Short Path and $\delta$-Fair Shortest Path turn out to be computationally hard. An issue left open in our work is the case of concrete (fine-grained) running-time lower bounds for a constant number of colors based on e.g. the Exponential Time Hypothesis. This would complement our general polynomial-time solvability result (Theorem 2).

We would like to conclude our work with a discussion on other plausible fairness notions and variations to our model. A very common notion is the one of proportional fairness, wherein each color should appear roughly with the same frequency in the solution as in the input graph. We conjecture that all results presented in this work carry over to this fairness notion. An intriguing model variation is to enforce the fairness not only to the path as a whole, but also to each (sufficiently long) subpath of the solution. Lastly, we believe that considering list-colored input graphs, that is, graphs in which each vertex may carry a set of colors, is a natural extension of our scenario.

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