On the Fekete–Szegö Type Functionals for Close-to-Convex Functions

Katarzyna Trąbka-Wieclaw *©, Paweł Zaprawa, Magdalena Gregorczyk© and Andrzej Rysak©
Mechanical Engineering Faculty, Lublin University of Technology, ul. Nadbystrzycka 36, 20-618 Lublin, Poland; p.zaprawa@pollub.pl (P.Z.); m.gregorczyk@pollub.pl (M.G.); a.rysak@pollub.pl (A.R.)
* Correspondence: k.trabka@pollub.pl
Received: 28 October 2019; Accepted: 7 December 2019; Published: 10 December 2019

Abstract: In this paper, we consider two functionals of the Fekete–Szegö type
\[ \Theta_f(\mu) = a_4 - \mu a_2 a_3 \]
and
\[ \Phi_f(\mu) = a_2 a_4 - \mu a_3^2 \]
for a real number \( \mu \) and for an analytic function
\[ f(z) = z + a_2 z^2 + a_3 z^3 + \ldots, \]
\(|z| < 1\). This type of research was initiated by Hayami and Owa in 2010. They obtained results for functions satisfying one of the conditions \( \text{Re}\{f(z)/z\} > \alpha \) or \( \text{Re}\{f'(z)\} > \alpha, \alpha \in [0, 1) \). Similar
estimates were also derived for univalent starlike functions and for univalent convex functions. We discuss \( \Theta_f(\mu) \) and \( \Phi_f(\mu) \) for close-to-convex functions such that \( f'(z) = h(z)/(1 - z)^2 \), where \( h \)
is an analytic function with a positive real part. Many coefficient problems, among others estimating
of \( \Theta_f(\mu) \), \( \Phi_f(\mu) \) or the Hankel determinants for close-to-convex functions or univalent functions, are
not solved yet. Our results broaden the scope of theoretical results connected with these functionals
defined for different subclasses of analytic univalent functions.

Keywords: coefficient problem; close-to-convex function; Fekete–Szegö functional; functional of
Fekete–Szegö type

1. Introduction
Let \( \mathcal{A} \) be the family of all functions analytic in \( \Delta = \{z \in \mathbb{C} : |z| < 1\} \) having the power
series expansion:
\[ f(z) = z + a_2 z^2 + a_3 z^3 + \ldots, \quad (1) \]
and let \( S^* \) denote the class of univalent starlike functions in \( \mathcal{A} \) (for the definitions and properties of \( S^* \)
and other classes, see [1]). For a given real argument \( \beta \in (-\pi/2, \pi/2) \) and a given function \( g \in S^* \),
a function \( f \in \mathcal{A} \) is called close-to-convex with argument \( \beta \) with respect to \( g \) if:
\[ \text{Re}\left\{ \frac{e^{i\beta} f'(z)}{g(z)} \right\} > 0, \quad z \in \Delta. \]
Let \( \mathcal{C}_\beta(g) \) be the class of all such functions. Moreover, let:
\[ \mathcal{C}_\beta = \bigcup_{g \in S^*} \mathcal{C}_\beta(g). \]
Let \( \mathcal{C} \) denote the family of all close-to-convex functions (see [2,3]). It is obvious that:
\[ \mathcal{C} = \bigcup_{\beta \in (-\pi/2, \pi/2)} \mathcal{C}_\beta. \]
All functions in \( \mathcal{C} \) are univalent.
In this paper, we consider the class \( C_0(k) \), where \( k \) is the Koebe function:
\[
k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \ldots.
\] (2)

The class \( C_0(k) \) is sometimes denoted by \( \mathcal{CR}^+ \). Such functions have a well known geometrical meaning. Namely, for each function \( f \) in this class, the set \( f(\Delta) \) is a domain such that \( \{ w + t : t \geq 0 \} \subset f(\Delta) \) for every \( w \in f(\Delta) \). Such functions \( f \) are convex in the positive direction of the real axis.

For a function \( f \) analytic in \( \Delta \) of the form (1), we define two functionals for a fixed real \( \mu \):
\[
\Theta_f(\mu) = a_4 - \mu a_2 a_3
\] (3)
and:
\[
\Phi_f(\mu) = a_2 a_4 - \mu a_3^2.
\] (4)

The functionals \( \Theta_f(\mu) \) and \( \Phi_f(\mu) \) are the generalizations of two well known expressions: \( a_4 - a_2 a_3 \) and \( a_2 a_4 - a_3^2 \). Both functionals are symmetric, or invariant, under rotations. The first one is a particular case of the generalized Zalcman functional. It was investigated, among others, by Ma [4] and Efraimidis and Vukotić [5]. The second functional is known as the second Hankel determinant, and it was studied in many papers. The investigation of Hankel determinants for analytic functions was started by Pommerenke (see [6,7]) and continued by many mathematicians in various classes of univalent functions (see, for example, [8–16]). The functional \( \Phi_f(\mu) \) was first studied by Hayami and Owa [17]. They discussed an even more general functional \( a_4 a_{n+2} - \mu a_{n+1}^2 \) for the classes \( Q(a) \) and \( \mathcal{R}(a) \), \( a \in [0,1) \), of functions \( f \in \mathcal{A} \) such that \( \Re \{ f(z)/z \} > a \) and \( \Re \{ f'(z) \} > a \), respectively. The functionals \( \Phi_f(\mu) \) and \( \Theta_f(\mu) \) for the classes \( S^* \) and \( \mathcal{K} \) of starlike and convex functions, respectively, were discussed in [18].

It is worth pointing out a particularly interesting property of \( \Phi_f(\mu) \). The sharp estimates of this functional are often symmetric with respect to a certain point. It was shown in [18] that such points for \( S^* \) and \( \mathcal{K} \) are \( 8/9 \) and one, respectively. We have:
\[
|\Phi_f(\mu)| \leq \max\{ |9\mu - 8|, 1 \} \quad \text{for } S^*
\] (5)
and:
\[
|\Phi_f(\mu)| \leq \max\{ |\mu - 1|, 1/8 \} \quad \text{for } \mathcal{K}.
\]

A similar situation occurs for \( Q(1/2) \) and for the class \( C_0(h) \), where \( h(z) = z/(1 - z^2) \); this point is \( 1/2 \) (see [17,19]). This situation appears even in the class \( \mathcal{T} \) of typically real functions, which do not necessarily have to be univalent (see [19]).

In this work, we derive bounds of \( \Theta_f(\mu) \) and \( \Phi_f(\mu) \) for functions in \( C_0(k) \).

2. Preliminary Results

Let \( \mathcal{P} \) denote the class of all analytic functions \( h \) with a positive real part in \( \Delta \) satisfying the normalization condition \( h(0) = 1 \). Let \( h \in \mathcal{P} \) have the Taylor series expansion:
\[
h(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots.
\] (6)

We shall need here three results. The first one is known as Caratheodory’s lemma (see, for example, ref. [1]). The second one is due to Libera and Zlotkiewicz ([20,21]), and the third one is the result of Hayami and Owa.

**Lemma 1** ([1]). If \( h \in \mathcal{P} \) is given by (6), then the sharp inequality \( |p_n| \leq 2 \) holds for \( n \geq 1 \).
Lemma 2 ([20,21]). Let $h$ be given by (6) and $p_1$ be a given real number, $p_1 \in [-2, 2]$. Then, $h \in \mathcal{P}$ if and only if:

$$2p_2 = p_1^2 + (4 - p_1^2)x$$

and:

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1x - (4 - p_1^2)p_1x^2 + 2(4 - p_1^2)(1 - |x|^2)y$$

for some complex numbers $x, y$ such that $|x| \leq 1, |y| \leq 1$.

Lemma 3 ([17]). If $h \in \mathcal{P}$ is given by (6), then:

$$|p_3 - \mu p_1 p_2| \leq \max\{2, |2 - 4\mu|\}.$$  

The next lemma is an improvement of Lemma 3 for $\mu \in [1/2, 1]$.

Lemma 4 ([22]). If $h \in \mathcal{P}$ is given by (6) and $\mu \in [1/2, 1]$, then:

$$|p_3 - \mu p_1 p_2| \leq \begin{cases} \frac{1}{4} \mu^2 p_3 - \frac{1}{2} \mu (2 - \mu) p^2 + 2, & p \in [0, 2/(2 - \mu)] , \\ (3 - 2\mu)p - (1 - \mu)p^3, & p \in [2/(2 - \mu), 2] , \end{cases}$$  

where $p = |p_1|$. The inequality is sharp.

The following lemma was proven by Lecko (see Corollary 2.3 in [23]).

Lemma 5 ([23]). If $h \in \mathcal{P}$ is given by (6), then:

$$|p_{n+1} + 2p_n + p_{n-1}| \leq 2(2 + \text{Re}\{p_1\}).$$  

We have proven the next lemma.

Lemma 6. If $h \in \mathcal{P}$ is given by (6), then:

$$|p_1 p_3 - p_2^2| \leq 4 - |p_1|^2.$$  

The inequality is sharp.

Proof. By Lemma 2,

$$4(p_1 p_3 - p_2^2) = (4 - p_1^2) \left[ 2p_1(1 - |x|^2)y - 4x^2 \right].$$

Applying the invariance of $|p_1 p_3 - p_2^2|$ under rotation, we can assume that $p_1$ is a non-negative real number. Writing $r = |x| \in [0, 1]$ and $p = p_1 \in [0, 2]$, we get by the triangle inequality and the assumption $|y| \leq 1:

$$4|p_1 p_3 - p_2^2| \leq (4 - p^2)\left[2p(1 - r^2) + 4r^2\right] = (4 - p^2)\left[2p + (4 - 2p)r^2\right] \leq 4(4 - p^2),$$

which gives the desired bound. The equality (9) holds for:

$$h(z) = \left(1 - \frac{p}{2}\right) \frac{1 + z^2}{1 - z^2} + \frac{p}{2} \frac{1 + z}{1 - z} = 1 + pz + 2z^2 + pz^3 + \ldots ,$$  

which means that there is equality in (9) for rotations of (10).  \(\Box\)
The next lemma is a special case of more general results due to Choi et al. [24] (see also [9]). Let \( \mathcal{X} = \{ z \in \mathbb{C} : |z| \leq 1 \} \). Define:

\[
Y(a, b, c) = \max_{z \in \mathcal{X}} \left( |a + bz + cz^2| + 1 - |z|^2 \right), \quad a, b, c \in \mathbb{R}.
\]

**Lemma 7.** If \( ac < 0 \), then:

\[
Y(a, b, c) = \begin{cases} 
1 + |a| + \frac{b^2}{4(1 + |c|)}, & |b| < 2(1 + |c|) \text{ and } b^2 < -4a(1 - c^2)/c, \\
1 - |a| + \frac{b^2}{4(1 - |c|)}, & |b| < 2(1 - |c|) \text{ and } b^2 \geq -4a(1 - c^2)/c, \\
R(a, b, c), & \text{otherwise,}
\end{cases}
\]

where:

\[
R(a, b, c) = \begin{cases} 
|a| + |b| - |c|, & |ab| \geq |c| \left( |b| + 4|a| \right), \\
-|a| + |b| + |c|, & |ab| \leq |c| \left( |b| - 4|a| \right), \\
(|c| + |a|) \sqrt{1 - \frac{b^2}{4ac}}, & \text{otherwise.}
\end{cases}
\]

If \( ac \geq 0 \), then:

\[
Y(a, b, c) = \begin{cases} 
|a| + |b| + |c|, & |b| \geq 2(1 - |c|), \\
1 + |a| + \frac{b^2}{4(1 - |c|)}, & |b| < 2(1 - |c|).
\end{cases}
\]

Applying the correspondence between the functions in \( C_0(k) \) and \( \mathcal{P} \):

\[
(1 - z)^2 f'(z) = h(z), \quad f \in C_0(k), \quad h \in \mathcal{P}
\]

and Expansions (1) and (6) we get:

\[
2a_2 = 2 + p_1, \quad 3a_3 = 3 + 2p_1 + p_2, \quad 4a_4 = 4 + 3p_1 + 2p_2 + p_3.
\]

Moreover, by Lemma 1, \( \text{Re}\{a_2\} \geq 0 \) with equality if and only if \( p_1 = -2 \). The equality is possible only for the function \( h(z) = \frac{1}{1 + z^2} \in \mathcal{P} \), and then, \( f(z) = \frac{1}{2} \log \frac{1 + z}{1 - z} \in C_0(k) \).

Hence, we can express \( \Theta_{f}(\mu) \) and \( \Phi_{f}(\mu) \) for \( f \in C_0(k) \) as coefficients of a corresponding function \( h \in \mathcal{P} \) in the following way:

\[
\Theta_{f}(\mu) = \frac{1}{4} p_3 + \left( \frac{1}{2} - \frac{3}{8} \mu \right) p_2 + \left( \frac{3}{4} - \frac{7}{8} \mu \right) p_1 - \frac{1}{8} \mu p_1 p_2 - \frac{1}{3} \mu p_1^2 + 1 - \mu
\]

and:

\[
\Phi_{f}(\mu) = \frac{1}{8} p_1 p_3 - \frac{1}{8} \mu p_2^2 + \frac{1}{4} p_3 + \left( \frac{1}{4} - \frac{1}{2} \mu \right) p_1 p_2 + \left( \frac{1}{2} - \frac{7}{3} \mu \right) p_2 + \left( \frac{3}{8} - \frac{4}{9} \mu \right) p_1^2 + \left( \frac{5}{4} - \frac{4}{9} \mu \right) p_1 + 1 - \mu.
\]

3. **Example**

Let us consider the function:

\[
F(z) = \frac{1}{2} (1 - \alpha) \log \frac{1 + z}{1 - z} + \alpha \frac{z}{(1 - z)^2}, \quad \alpha \in [0, 1],
\]

(15)
which has the following Taylor series expansion:

\[ F(z) = (1 - \alpha) \left( z + \frac{1}{2}z^3 + \ldots \right) + \alpha \left( z + 2z^2 + 3z^3 + 4z^4 + \ldots \right) \]

\[ = z + 2\alpha z^2 + \frac{1}{2}(1 + 8\alpha)z^3 + 4\alpha z^4 + \ldots . \]

Since:

\[ (1 - z)^2 F'(z) = (1 - \alpha) \frac{1 - z}{1 + z} + \alpha \frac{1 + z}{1 - z} \in \mathcal{P}, \]

so \( F \in \mathcal{C}_0(k) \). Moreover,

\[ F(\Delta) = C \setminus \left\{ x \pm i(1 - \alpha)\frac{\pi}{4} : x \le \frac{1}{2}(1 - \alpha) \ln \frac{1 - \alpha}{\alpha} - 1 \right\} . \]

For \( F \), we have:

\[ \Theta_F(\mu) = \frac{2}{3} \left[ -8\mu \alpha^2 + (6 - \mu)\alpha \right] \]

and:

\[ \Phi_F(\mu) = \frac{1}{9} \left[ 8(9 - 8\mu)\alpha^2 - 16\mu \alpha - \mu \right] . \]

For \( \mu < 0 \), we have: \( \Theta_F(\mu) \le 4 - 6\mu \) and \( \Phi_F(\mu) \le 8 - 9\mu \). We find the estimation of \( \Theta_F(\mu) \) and \( \Phi_F(\mu) \) for \( \mu \ge 0 \).

Let us denote:

\[ f(\alpha) = \frac{2}{3} \left[ -8\mu \alpha^2 + (6 - \mu)\alpha \right] \quad \text{and} \quad g(\alpha) = \frac{1}{9} \left[ 8(9 - 8\mu)\alpha^2 - 16\mu \alpha - \mu \right] . \]

The critical point \( a_0 = (6 - \mu)/(16\mu) \) of \( f(\alpha) \) is in \((0, 1)\) if \( \mu \in (6/17, 6) \). Hence,

\[ |\Theta_F(\mu)| \le \max\{|f(a_0)|, |f(1)|, |f(0)|\} = \max\left\{ \left| \frac{(6 - \mu)^2}{48\mu} \right|, |4 - 6\mu|, 0 \right\} \]

for \( \mu \in (6/17, 6) \) and:

\[ |\Theta_F(\mu)| \le \max\{|f(1)|, |f(0)|\} = |4 - 6\mu| \quad \text{for} \quad \mu \in [0, 6/17] \cup [6, \infty) . \]

Similarly, the critical point \( a_1 = \mu/(9 - 8\mu) \) of \( g(\alpha) \) is in \((0, 1)\) if \( \mu \in (0, 1) \). Hence,

\[ |\Phi_F(\mu)| \le \max\{|g(a_1)|, |g(1)|, |g(0)|\} = \max\left\{ \left| \frac{\mu}{8\mu - 9} \right|, |8 - 9\mu|, \left| -\frac{\mu}{9} \right| \right\} \]

for \( \mu \in (0, 1) \) and:

\[ |\Phi_F(\mu)| \le \max\{|g(1)|, |g(0)|\} = |9\mu - 8| \quad \text{for} \quad \mu \in \{0\} \cup [1, \infty) . \]

Finally, for a function \( F \) given by (15), we obtain:

\[ \Theta_F(\mu) \le \begin{cases} 4 - 6\mu, & \mu \le 6/17 = 0.352 \ldots \\ \frac{(6 - \mu)^2}{48\mu}, & 6/17 \le \mu \le 6(15 + 16\sqrt{2})/287 = 0.786 \ldots \\ 6\mu - 4, & \mu \ge 6(15 + 16\sqrt{2})/287 \end{cases} \]

and:

\[ \Phi_F(\mu) \le \begin{cases} 8 - 9\mu, & \mu \le (73 - \sqrt{145})/72 = 0.846 \ldots \\ \frac{\mu}{9 - 8\mu}, & (73 - \sqrt{145})/72 \le \mu \le 1 \\ 9\mu - 8, & \mu \ge 1 . \end{cases} \]
4. Bounds of $|\Theta(\mu)|$ for the Class $C_0(k)$

In the main theorem of this section, we establish the sharp bounds of $|\Theta(\mu)|$ for the class $C_0(k)$. The proof is divided into six lemmas. The first one is a particular case of the result obtained in \cite{22} (Theorem 3.1 or Theorem 3.3 in \cite{22}), and the second one is obvious.

**Lemma 8.** Let $f \in C_0(k)$. Then, $|\Theta_f(1)| = |a_4 - a_2a_3| \leq 2$. The result is sharp.

**Lemma 9.** Let $f \in C_0(k)$ and $\mu \leq 0$. Then, $|\Theta_f(\mu)| \leq 4 - 6\mu$. The result is sharp.

**Lemma 10.** Let $f \in C_0(k)$ and $\mu > 1$. Then, $|\Theta_f(\mu)| \leq 6\mu - 4$. The result is sharp.

**Proof.** From (13), we can write $\Theta_f(\mu)$ as follows:

$$\Theta_f(\mu) = \frac{1}{4} (p_3 - \frac{2}{3} \mu p_1 p_2) + \left(\frac{1}{2} - \frac{1}{3} \mu \right) p_2 - \frac{1}{2} \mu p_1^2 + \left(\frac{3}{4} - \frac{7}{6} \mu \right) p_1 + 1 - \mu.$$  

If $\mu \geq 3/2$, then, taking into account Lemmas 1 and 3, we get:

$$|\Theta_f(\mu)| \leq \frac{1}{4} \left(\frac{3}{2} \mu - 2\right) + 2 \left(\frac{1}{3} \mu - \frac{1}{2}\right) + \frac{3}{4} \mu + 2 \left(\frac{7}{6} \mu - \frac{3}{4}\right) + \mu - 1 = 6\mu - 4.$$  

If $\mu \in (1,3/2)$, then we have:

$$\Theta_f(\mu) = (3 - 2\mu)(a_4 - a_2a_3) + (2\mu - 2)(a_4 - \frac{3}{2}a_2a_3).$$

Now, from Lemma 8 and the first part of this proof (i.e., $|a_4 - \frac{3}{2}a_2a_3| \leq 5$), we obtain:

$$|\Theta_f(\mu)| \leq 2(3 - 2\mu) + 5(2\mu - 2) = 6\mu - 4.$$  

It is clear that $\Theta_f(\mu) = 4 - 6\mu$ only when $p_1 = p_2 = p_3 = 2$, which means that this equality holds only for the Koebe function (2). In other words, the Koebe function is the extremal function for $\mu > 1$. \qed

Taking into account (13) and Lemma 2, we can write $\Theta_f(\mu)$ as follows:

$$\Theta_f(\mu) = 1 + \frac{1}{16} p_1^3 + \frac{1}{4} p_1^2 + \frac{3}{4} p_1 - \frac{1}{18} \mu p_1^3 - \frac{1}{2} \mu p_1^2 - \frac{7}{6} \mu p_1 - \mu + \left[\frac{3}{8} p_1 + \frac{1}{4} - \frac{1}{12} \mu (2 + p_1)\right] (4 - p_1)^2 x - \frac{1}{16} (4 - p_1^2) p_1 x^2 + \frac{1}{8} (4 - p_1^2) (1 - |x|^2) y.$$  

From the above formula, we can obtain bounds of $|\Theta_f(\mu)|$, while $\mu \in (0,1)$ and $f \in C_0(k)$, but only with an additional assumption that $a_2$ is a positive real number. The assumption of Lemma 2 enforces that $p_1 \in [-2,2]$. Notice that if $p_1 = 2$, then $f(z) = k(z)$ given by (2), and we have:

$$\Theta_f(\mu) = 4 - 6\mu.$$  

If $p_1 = -2$, then $f(z) = \frac{1}{2} \log \frac{1 + z^2}{1 - z^2} = z + \frac{1}{2} z^3 + \frac{1}{9} z^5 + \ldots$ is in $C_0(k)$, and so:

$$\Theta_f(\mu) = 0.$$  

To shorten notation, we write $p$ instead of $p_1$. One can observe that $\Theta_f(\mu)$ can be written as:

$$8 \Theta_f(\mu) = (4 - p^2) \left[ a + bx + cx^2 + (1 - |x|^2)y \right],$$  

(18)
where:

\[
a = \frac{48(1-\mu) + 4(9-14\mu)p + 12(1-2\mu)p^2 + (3-4\mu)p^3}{6(4-p^2)},
\]

\[
b = (2+p)(1-\frac{2}{3}\mu),
\]

\[
c = -\frac{1}{2}p.
\]

From (18), the triangle inequality, \(|y| \leq 1\), and Lemma 2, we get:

\[
8 |\Theta_f(\mu)| \leq (4-p^2) \left[|a + bx + cx^2| + 1 - |x|^2\right],
\]

where \(a, b,\) and \(c\) are given by (19).

Lemma 11. Let \(f \in C_0(k), a_2\) be a real number, \(a_2 \in [0, 2]\) and \(\mu \in (0, 1/3]\). Then, \(|\Theta_f(\mu)| \leq 4 - 6\mu\). The result is sharp.

Proof. For \(\mu = 1/3\), we have (20) with:

\[
a = \frac{5p^2 + 2p + 48}{18(2-p)}, \quad b = \frac{7(2+p)}{9}, \quad c = -\frac{p}{2}.
\]

We use Lemma 7. Clearly, \(ac < 0\) for \(p \in (0, 2)\). Note that the inequality \(|b| < 2(1 + |c|)\) from the first case of Lemma 7 is equivalent to the obviously true inequality:

\[
7(2+p) < 18(1+p/2).
\]

The inequality \(b^2 < -4a(1-c^2)/c\), which can be written as:

\[
\frac{4(2+p)(p^2 + 20p - 108)}{81p} < 0,
\]

holds for all \(p \in (0, 2)\). Hence, for \(p \in (0, 2)\), we have:

\[
Y(a, b, c) = 1 + |a| + \frac{b^2}{4(1 + |c|)} = \frac{2(238 - 36p - p^2)}{81(2-p)}.
\]

For \(p \in (-2, 0]\), we have \(ac \geq 0\), and the inequality \(|b| < 2(1 - |c|)\) from the last case of Lemma 7 is equivalent to (21). Therefore, \(Y(a, b, c)\) is also given by (22).

Thus, from (20) for \(\mu = 1/3\), Lemma 7, (16) and (17), we obtain:

\[
|\Theta_f(1/3)| \leq g(p),
\]

where \(g(p) = (2+p)(238 - 36p - p^2)/324\) and \(p \in [-2, 2]\) according to the assumption. The function \(g\) is increasing for \(p \in [-2, 2]\); therefore:

\[
|\Theta_f(1/3)| \leq g(2) = 2.
\]

Moreover, we have by the triangle inequality:

\[
|\Theta_f(\mu)| = (1-3\mu)a_4 + 3\mu(a_4 - \frac{1}{3}a_2a_3), \quad \mu \in (0, 1/3).
\]

From Lemma 9 and from (24), we get:

\[
|\Theta_f(\mu)| \leq 4(1-3\mu) + 2 \cdot 3\mu = 4 - 6\mu.
\]
and the proof is complete. Equality holds for the Koebe function (2).

Let us denote:

\[ p_0 = 2(\sqrt{103} - 10)/3 = 0.099 \ldots , \]
\[ K = 16(103\sqrt{103} - 910)/2187 = 0.9901 \ldots . \]  

(25)

\textbf{Lemma 12.} Let \( f \in C_0(k) \), \( a_2 \) be a real number, \( a_2 \in [0, 2] \), and \( K \) be given by (25). Then, \(|\Theta_f(2/3)| \leq K\). The result is sharp.

\textbf{Proof.} For \( \mu = 2/3 \), we have (20) with:

\[ a = \frac{12 - p}{18}, \quad b = \frac{5(2 + p)}{9}, \quad c = -\frac{p}{2}. \]

We use Lemma 7. Clearly, \( ac < 0 \) for \( p \in (0, 2] \). First, note that the inequality \(|b| < 2(1 + |c|)\) is equivalent to the obviously true inequality:

\[ 5(2 + p) < 18(1 + p/2). \]  

(26)

The inequality \( b^2 < -4a(1 - c^2)/c \), which is equivalent to:

\[ \frac{8(2 + p)(2p^2 + 22p - 27)}{81p} < 0, \]

holds for \( p \in \left(0, \frac{5\sqrt{7} - 11}{2}\right) \). For \( p \in \left(0, \frac{5\sqrt{7} - 11}{2}\right) \), we have:

\[ Y(a, b, c) = 1 + |a| + \frac{b^2}{4(1 + |c|)} = \frac{8(p + 20)}{81}, \]

(27)

so from (20) for \( \mu = 2/3 \) and Lemma 7, we obtain:

\[ |\Theta_f(2/3)| \leq (4 - p^2)(p + 20)/81. \]  

(28)

From Lemma 7, the inequality system consists of \(|b| < 2(1 + |c|)\), and \( b^2 \geq -4a(1 - c^2)/c \) is contradictory, because the first inequality gives \( p < 4/7 \), while the second one yields \( p \geq (5\sqrt{7} - 11)/2 \).

Now, consider the third case of Lemma 7. Let \( p \in \left[\frac{5\sqrt{7} - 11}{2}, 2\right] \). The inequality \(|ab| \geq |c|(|b| + 4|a|)\) is equivalent to \(60 - 128p - 16p^2 \geq 0\), and it is not satisfied for any \( p \in \left[\frac{5\sqrt{7} - 11}{2}, 2\right] \). The inequality \(|ab| \leq |c|(|b| - 4|a|)\), which can be written as \(30 + 44p - 17p^2 \leq 0\), is also not satisfied for any \( p \in \left[\frac{5\sqrt{7} - 11}{2}, 2\right] \). Thus, for \( p \in \left[\frac{5\sqrt{7} - 11}{2}, 2\right] \), we have:

\[ Y(a, b, c) = (|c| + |a|) \sqrt{1 - \frac{b^2}{4ac}} = \frac{4(2p + 3)}{27} \sqrt{\frac{(2p + 25)(2p + 1)}{(12 - p)p}}. \]  

(29)

From (20) for \( \mu = 2/3 \) and Lemma 7, we obtain:

\[ |\Theta_f(2/3)| \leq \frac{(4 - p^2)(2p + 3)}{54} \sqrt{\frac{(2p + 25)(2p + 1)}{(12 - p)p}}. \]  

(30)
For $p \in [-2, 0]$, we have $ac \geq 0$, and the inequality $|b| < 2(1 - |c|)$ from the last case of Lemma 7 is equivalent to the inequality in (26).

Thus, $Y(a, b, c)$ is given by (27). Finally, from (16), (28) and (30), we obtain:

$$|\Theta_f(2/3)| \leq g(p),$$

where:

$$g(p) = \begin{cases} 
\frac{1}{81} (4 - p^2)(p + 20), & p \in [-2, (5\sqrt{7} - 11)/2) \\
\frac{1}{54} (4 - p^2)(2p + 3)^{\sqrt{12p + 25}} & p \in [(5\sqrt{7} - 11)/2, 2].
\end{cases}$$

Now, let us consider the function $g$ for $p \in [(5\sqrt{7} - 11)/2, 2]$. We have:

$$g'(p) = \frac{M(p)}{54(12 - p)^2p^2} \sqrt{(12 - p)p}$$

where $M(p) = 24p^6 - 52p^5 - 3802p^4 - 4801p^3 + 4242p^2 + 1500p - 1800$ and:

$$M(p) = 24p^6 - 52p^5 - 3802p^4 - 4801p^3 + 4242p^2 + 1500p - 1800$$

for $p \in (1, 2]$. Hence, $g'(p) < 0$ for $p \in [(5\sqrt{7} - 11)/2, 2]$.

Taking the above into account, one can check that the function $g$ is increasing for $p \in [-2, p_0)$ and is decreasing for $p \in (p_0, 2]$, where $p_0$ is given by (25). Therefore,

$$|\Theta_f(2/3)| \leq g(p_0) = 16(103\sqrt{103} - 910)/2187 = 0.9901 \ldots,$$

so we have the desired result. ☐

**Lemma 13.** Let $f \in C_0(k)$, $a_2$ be a real number, and $a_2 \in [0, 2]$.

1. If $\mu \in (1/3, 2/3)$, then $|\Theta f(\mu)| < 3 - 3\mu$.
2. If $\mu \in (2/3, 1)$, then $|\Theta f(\mu)| < 3\mu - 1$.

**Proof.** We have:

$$|\Theta f(\mu)| = |(2 - 3\mu)(a_4 - \frac{1}{2}a_2a_3) + (3\mu - 1)(a_4 - \frac{2}{3}a_2a_3)|, \quad \mu \in (1/3, 2/3).$$

From Lemmas 11 and 12, and the triangle inequality, we get the first part of Lemma 13, i.e.,

$$|\Theta f(\mu)| \leq 2(2 - 3\mu) + K \cdot (3\mu - 1) = 2(2 - 3\mu) + 1 \cdot (3\mu - 1) = 3 - 3\mu.$$

Since:

$$|\Theta f(\mu)| = |(3 - 3\mu)(a_4 - \frac{2}{3}a_2a_3) + (3\mu - 2)(a_4 - a_2a_3)|, \quad \mu \in (2/3, 1),$$

from Lemma 12, Lemma 8, and the triangle inequality, we get the second part of Lemma 13, i.e.,

$$|\Theta f(\mu)| \leq 1 \cdot (3 - 3\mu) + 2(3\mu - 2) = 3\mu - 1 < 1 \cdot (3 - 3\mu) + 2(3\mu - 2) = 3\mu - 1.$$

☐

The results presented in Lemmas 8–13 can be collected as follows.
Theorem 1. Let \( f \in C_0(k), a_2 \) be a real number, and \( a_2 \in [0,2] \). Then:

\[
|\Theta_f(\mu)| \leq \begin{cases}
4 - 6\mu, & \mu \leq 1/3, \\
3 - 3\mu, & \mu \in (1/3,2/3), \\
K, & \mu = 2/3, \\
3\mu - 1, & \mu \in (2/3,1), \\
6\mu - 4, & \mu \geq 1,
\end{cases}
\]

where \( K \) is given by (25). The results are sharp for \( \mu \leq 1/3, \mu = 2/3, \) and \( \mu \geq 1 \). The equality holds for the Koebe function (2) in the first and the last case. The assumption \( a_2 \in [0,2] \) is not necessary for \( \mu \leq 0 \) and \( \mu \geq 1 \).

5. Bounds of \( |\Phi_f(\mu)| \) for the Class \( C_0(k) \)

At the beginning of this section, we will quote the well known theorem of Marjono and Thomas [14].

Theorem 2 ([14]). If \( f \in C_0(k) \), then:

\[
|\Phi_f(1)| = |a_2a_4 - a_3^2| \leq 1.
\]

Now, we shall prove the bound for \( \mu \geq 1 \).

**Theorem 3.** Let \( f \in C_0(k) \) and \( \mu \geq 1 \). Then, \( |\Phi_f(\mu)| \leq 9\mu - 8 \). The result is sharp.

**Proof.** Rearranging the components in (14):

\[
\Phi_f(\mu) = \frac{1}{\delta} (p_1p_3 - p^2) - \left(\frac{\mu}{\delta} - \frac{1}{2}\right) p_2^2 + \left(\frac{\mu}{\delta} - \frac{1}{2}\right) (p_3 - p_1p_2) - \left(\frac{\mu}{\delta} - \frac{1}{4}\right) p_1^2 + (\frac{3}{8} - \frac{\mu}{\delta}) p_2 - (\frac{3}{8} - \frac{\mu}{\delta}) p_1 - (\mu - 1),
\]

and writing \( p \) instead of \( |p_1| \), by Lemmas 1, 3, and 6, for \( \mu \geq 9/8 \), we obtain:

\[
|\Phi_f(\mu)| \leq \frac{1}{\delta} \left(4 - p^2\right) + \left(\frac{\mu}{\delta} - \frac{1}{2}\right) + \left(\frac{\mu}{\delta} - \frac{1}{2}\right) p + (\frac{3}{8} - \frac{\mu}{\delta}) p + (\mu - 1)
\]

\[
= \left(\frac{3}{8} - \frac{\mu}{\delta}\right) p^2 + \left(\frac{3}{8} - \frac{\mu}{\delta}\right) p + (\frac{25}{8} - \frac{\mu}{2})
\]

\[
\leq 9\mu - 8.
\]

If \( \mu \in (1,9/8) \), then:

\[
\Phi_f(\mu) = (9 - 8\mu) \left(a_2a_4 - a_3^2\right) + (8\mu - 8) \left(a_2a_4 - \frac{9}{6}a_3^2\right).
\]

From the previous part of this proof \( |a_2a_4 - \frac{9}{6}a_3^2| \leq \frac{17}{8} \) and from Theorem 2, after using the triangle inequality, we get:

\[
|\Phi_f(\mu)| \leq (9 - 8\mu) \cdot 1 + (8\mu - 8) \cdot \frac{17}{8} = 9\mu - 8.
\]

It is easy to verify that for the Koebe function (2), we have \( \Phi_3(\mu) = 8 - 9\mu \), so the derived estimate is sharp. \( \Box \)

In the next step, we shall prove that the Koebe function (2) is the extremal function for \( \mu \leq 63/92 \).

**Theorem 4.** Let \( f \in C_0(k) \) and \( \mu \leq 63/92 \). Then, \( |\Phi_f(\mu)| \leq 8 - 9\mu \). The result is sharp.
Proof. At the beginning, let us discuss the case \( \mu = 63/92 \). From (14), it follows that:

\[
184 \Phi_f \left( \frac{63}{92} \right) = 14(p_1p_3 - p_2^2) + 9p_1p_3 + 20(p_3 - \frac{1}{2}p_1p_2) \\
+ 4(p_3 + 2p_2 + p_1) + 22p_3 + 58p_1 + 13p_1^2 + 58.
\]

Now, applying Lemmas 1 and 4 for \( \mu = 1/2 \), Lemma 5 (remembering that \( 2(2 + \text{Re}p_1) \leq 2(2 + |p_1|) \)), Lemma 6, and the triangle inequality and writing \( p \) instead of \(|p_1|\), we obtain:

\[
184 | \Phi_f \left( \frac{63}{92} \right) | \leq 14(4 - p^2) + 18p + 20h(p) + 8(2 + p) + 44 + 58p + 13p^2 + 58,
\]

where:

\[
h(p) = \begin{cases} 
\frac{1}{2}p^3 - \frac{3}{8}p^2 + 2, & p \in [0, 4/3] \\
2p - \frac{1}{4}p^3, & p \in [4/3, 2].
\end{cases}
\]

Hence,

\[
184 | \Phi_f \left( \frac{63}{92} \right) | \leq H(p),
\]

where:

\[
H(p) = \begin{cases} 
\frac{5}{4}p^3 - \frac{17}{2}p^2 + 84p + 214, & p \in [0, 4/3] \\
-10p^3 - p^2 + 124p + 174, & p \in [4/3, 2].
\end{cases}
\]

| \Phi_f \left( \frac{63}{92} \right) | \leq H(2) = \frac{338}{184} = 8 - 9 \cdot \frac{63}{92}.

If \( \mu \in (0, 63/92) \), then:

\[
\Phi_f(\mu) = (1 - \frac{92}{63}\mu)a_2a_4 + \frac{92}{63}\mu \left( a_2a_4 - \frac{63}{92}a_3^2 \right).
\]

From the previous part of this proof and the bound \(|a_n| \leq n\) valid for all functions in \( C_0(k) \),

\[
| \Phi_f(\mu) | \leq (1 - \frac{92}{63}\mu) \cdot 8 + \frac{92}{63}\mu \cdot \frac{338}{184} = 8 - 9\mu.
\]

Equality holds for the Koebe function. \( \square \)

It is worth adding that the function \( H \) given by (31) is decreasing for \( p > 2 \), so the choice \( \mu = 63/92 \) is important.

Now, we will find the exact bound of \( \Phi_f(\mu) \) for \( \mu \) close to one. Namely, we will discuss the case \( \mu \in [\mu_0, 1] \), where:

\[
\mu_0 = 18/19 = 0.947\ldots.
\]

In this result, we need in addition that the coefficient \( a_2 \) should be real and \( a_2 \in [0, 2] \). From (12), we get \( p = p_1 \in [-2, 2] \). In the proof, we are going to apply Lemma 7.

Taking into account (14) and Lemma 2, we can write \( \Phi_f(\mu) \) as follows:

\[
144 \Phi_f(\mu) = A_0 + A_1x + A_2x^2 + B(1 - |x|^2)y,
\]

where:
Theorem 5. Equality holds for the function \( F \) given by (15).

Proof of Lemma 14. If \( p = -2 \) and \( p = 2 \), then \( f(z) = \frac{1}{2} \log \frac{1+z}{1-z} \) and \( f(z) = \frac{z}{1-z^2} \), respectively, so:

\[
\Phi_f(\mu) = -\mu/9 \quad \text{and} \quad \Phi_f(\mu) = 8 - 9\mu. \tag{33}
\]

We will show that these values are less than or equal to the real bound of \(|\Phi_f(\mu)|\) for all \( f \in C_0(k) \).

Now and on, we assume that \( p \in (-2, 2) \). Taking into account (14) and Lemma 2, by the triangle inequality and the assumption \(|y| \leq 1\), we get:

\[
|\Phi_f(\mu)| \leq \frac{1}{16} (4 - p^2)(2 + p) \left| a + bx + cx^2 \right| + 1 - |x|^2, \tag{34}
\]

where:

\[
a = \frac{1}{9(4 - p^2)(2 + p)} \left[ \frac{1}{2} (9 - 8\mu)p^4 + (27 - 32\mu)p^3 + 2(45 - 56\mu)p^2 + 12(15 - 16\mu)p + 144(1 - \mu) \right],
\]

\[
b = \frac{1}{9(2 + p)} \left[ (9 - 8\mu)p^2 + 4(9 - 8\mu)p + 12(3 - 4\mu) \right], \tag{35}
\]

\[
c = -\frac{1}{16} (9 - 8\mu)p + 16\mu.
\]

Now, we are ready to establish the main theorem of this section.

**Theorem 5.** Let \( f \in C_0(k), a_2 \) be a real number, \( a_2 \in [0, 2] \), and \( \mu \in [\mu_0, 1] \), where \( \mu_0 = 18/19 \). Then:

\[
|\Phi_f(\mu)| \leq \frac{\mu}{9 - 8\mu}. \tag{36}
\]

Equality holds for the function \( F \) given by (15).

In the proof of this theorem, we will need the two lemmas that follow. We assume that \( a, b, \) and \( c \) are given by (35).

**Lemma 14.** If \((p, \mu) \in (-2, 2) \times [\mu_0, 1]\) are such that \( a \leq 0 \), then (36) holds.

**Lemma 15.** If \((p, \mu) \in (-2, 2) \times [\mu_0, 1]\) are such that \( a > 0 \), then the following inequalities hold:

\[
b < 0, \ |b| \geq 2(1 - |c|), \ b^2 \geq -4a(1 - c^2)/c, \ |ab| \leq |c|(|b| - 4|a|) .
\]

**Proof of Lemma 14.** At the beginning, observe that if \((p, \mu) \in (-2, 2) \times [\mu_0, 1]\), then:

\[
c = -\frac{1}{18} (9p + 8(2 - p)\mu) \leq -\frac{1}{18} \left[ 9p + 8(2 - p) \cdot \frac{18}{19} \right] = -\frac{1}{18}(3p + 32) < 0. \tag{37}
\]

According to Lemma 7 from (34), we obtain:

\[
|\Phi_f(\mu)| \leq \frac{1}{18} (4 - p^2)(2 + p) \cdot Y(a, b, c),
\]
where:

\[
Y(a,b,c) = \begin{cases} 
-a + |b| - c, & |b| \geq 2(1 + c), \\
1 - a + \frac{b^2}{4(1 + c)}, & |b| < 2(1 + c).
\end{cases}
\]

If \(|b| < 2(1 + c)|\), then from (34), we get:

\[
144|\Phi_f(\mu)| \leq 9(4 - p^2)(2 + p) - \frac{1}{2}\left[(9 - 8\mu)p^4 + (27 - 32\mu)p^3 + 2(45 - 56\mu)p^2 + 12(15 - 16\mu)p + 144(1 - \mu)\right] + \frac{\left[(9 - 8\mu)p^2 + 4(9 - 8\mu)p + 12(3 - 4\mu)\right]^2}{2(9 - 8\mu)}.
\]

Because the right hand side of this inequality is constant and equal to \(144\mu/(9 - 8\mu)\); hence, \(|\Phi_f(\mu)| \leq \mu/(9 - 8\mu)\).

If \(|b| \geq 2(1 + c)|\), then:

\[
|\Phi_f(\mu)| \leq \begin{cases} 
\frac{1}{16}(4 - p^2)(2 + p)(-a + b - c), & b \geq 0, \\
\frac{1}{16}(4 - p^2)(2 + p)(-a - b - c), & b \leq 0.
\end{cases}
\] (38)

The first expression in (38) is equal to:

\[
-\frac{1}{144}\left[-2(9 - 8\mu)p^4 - 8(9 - 8\mu)p^3 - 24(3 - 4\mu)p^2 + 64\mu p + 16\mu\right] = -\frac{1}{72}\left[(9 - 8\mu)p^2(p + 2)^2 - 16\mu(p + 1)^2 + 8\mu\right].
\]

Substituting \(q = p + 1\), \(q \in (-1, 3)\), we obtain:

\[
W_1(q) = -\frac{1}{72}\left[(9 - 8\mu)q^4 - 18q^2 + 9\right] = -\frac{1}{72}\left[(3 - 2\sqrt{2}\mu)q^2 - 3\right] \cdot \left[(3 + 2\sqrt{2}\mu)q^2 - 3\right].
\]

Hence, the maximum value of \(W_1(q)|\) is achieved for:

\[
q^*_2 = \frac{1}{2}\left(\frac{3}{3 - 2\sqrt{2}\mu} + \frac{3}{3 + 2\sqrt{2}\mu}\right) = \frac{9}{9 - 8\mu}.
\]

This value is equal to \(W_1(q^*_2) = \mu/(9 - 8\mu)|\).

The second expression in (38) is equal to:

\[
W_2(p) = \frac{1}{144}\left[-(9 - 8\mu)p^2 - 4(9 - 10\mu)p - 2(18 - 25\mu)\right],
\]

so:

\[
W_2(p) \leq W_2\left(\frac{2(10\mu - 9)}{9 - 8\mu}\right) = \frac{\mu}{9 - 8\mu}.
\]

It is easy to check that for \(p_+ = q^*_2 - 1 = 3/\sqrt{9 - 8\mu} - 1\) and \(p_{**} = 2(10\mu - 9)/(9 - 8\mu)\), we have \(b = 2(1 + c)|\) and \(b = -2(1 + c)|\), respectively. This means that the maximum value of \(|\Phi_f(\mu)|\) for \(|b| \geq 2(1 + c)|\) is obtained if \(|b| = 2(1 + c)|\). \(\square\)

**Proof of Lemma 15**. Let \((p, \mu) \in (-2, 2) \times [\mu_0, 1]\). At the beginning, we want to constrain the range of variability of \(p\) to some subset of \((-2, 2)\) for which \(a > 0\).

From (35) for \(a = 0\), we have:

\[
\frac{1}{2}(9 - 8\mu)p^4 + (27 - 32\mu)p^3 + 2(45 - 56\mu)p^2 + 12(15 - 16\mu)p + 144(1 - \mu) = 0,
\]
which is equivalent to:

\[ 9(p^2 + 2p + 8)(2 + p)^2 - 8(p^2 + 4p + 6)^2\mu = 0. \]

If \( p = 0, \mu = 0 \), then from (35), \( a = 2 \). Hence, points for which \( a > 0 \) lie below the curve \( a = 0 \). For the function \( M(p) = 9(p^2 + 2p + 8)(2 + p)^2/8(p^2 + 4p + 6)^2, \) \( p \in (-2, 2) \), there is:

\[ M'(p) = \frac{9(2 + p)}{4(p^2 + 4p + 6)} \cdot (p^3 + 2p^2 - 10p - 4). \]

Consequently, \( M(p) \) is an increasing function if \( p \in (p_0, 2) \) for \( p_0 = -0.376 \ldots \), where \( p_0 \) is the only solution of \( M'(p) = 0 \) in \((-2, 2)\). Since \( M(-1) < \mu_0 \) and \( M(2/3) < \mu_0 \), then \( M(p) < \mu_0 \) for \( p \in (-2, -1) \cup [2/3, 2) \). This means that \( a > 0 \) and \( \mu \in [\mu_0, 1) \) hold for \( p \in I, I \subset (-1, 2/3) \) (in other words, if \( a > 0 \) and \( \mu \in [\mu_0, 1) \), then \(-1 < p < 2/3\).

I. Since \( \mu \in [\mu_0, 1) \) and:

\[ b = \frac{1}{9} (9 - 8\mu) (2 + p) - \frac{16\mu}{9(2 + p)} \]

as a function of \( p \in (-1, 2/3) \), is increasing, it is enough to estimate this expression taking \( p = 2/3 \) as a limit value. Therefore,

\[ b < \frac{2}{27} (36 - 41\mu) < 0. \]

II. The inequality \(-b \geq 2(1 + c)\) can be written as \((8\mu - 9)p + 20\mu - 18 \geq 0\). For \( \mu \in [\mu_0, 1) \) and \( p \in (-1, 2/3) \),

\[ (8\mu - 9)p + 20\mu - 18 > \frac{76}{9} \left( \mu - \frac{18}{19} \right) \geq 0. \]

III. With the notation \( W = b^2 + 4a(1 - c^2)/c \) and:

\[ g(p, \mu) = (9 - 8\mu) \left[ (16\mu - 9)p^3 + 18(4\mu - 3)p^2 + 4(25\mu - 27)p \right] - 8(32\mu^2 - 117\mu + 81), \]

we can write:

\[ W = \frac{8 g(p, \mu)}{9(2 + p)^2 \left[ (9 - 8\mu)p + 16\mu \right]} \]

We shall prove that \( g(p, \mu) \geq 0 \) for \( \mu \in [\mu_0, 1) \) and \( p \in (-1, 2/3) \). We have:

\[ \frac{\partial g}{\partial p}(p, \mu) = (9 - 8\mu) \left[ 3(16\mu - 9)p^2 + 36(4\mu - 3)p + 4(25\mu - 27) \right]. \]

For \( \mu \in [\mu_0, 1) \), we obtain:

\[ \frac{\partial g}{\partial p}(-1, \mu) = (4\mu - 27)(9 - 8\mu) < 0, \]

and:

\[ \frac{\partial g}{\partial p}(2/5, \mu) = 4 \left( 1033\mu - 972 \right) (9 - 8\mu)/25 > 0. \]

This means that:

\[ \min \{ g(p, \mu) : p \in (-1, 2/3), \mu \in [\mu_0, 1) \} = \min \{ g(p, \mu) : p \in (-1, 2/5), \mu \in [\mu_0, 1) \}. \]
Theorem 6. Let $f$ Symmetry 2019

By Theorems 4 and 5,

Proof. µ in (33), are less than or equal to

For this reason, the maximum value of

From Lemma 14, we know that if

$h$ and $\Phi f(\mu)$ are positive for $\mu \in [\mu_0, 1]$, we can write:

In this way, we have proven that $b^2 + 4a(1 - c^2)/c \geq 0$.

IV. Let us denote $V = c(b + 4a) + ab$ and:

We have

The function $h$ of a variable $\mu$ increases for $\mu \in [\mu_0, 1]$. Indeed, for a fixed $p \in (-1, 2/3)$,

and is greater than zero. Finally,

so $h$, as well as $V$ are positive for $\mu \in [\mu_0, 1]$ and $p \in (-1, 2/3)$. □

Proof of Theorem 4. From Lemma 14, we know that if $a \leq 0$ and $\mu \in [\mu_0, 1]$, then (36) holds. Assume now that $a > 0$ and $\mu \in [\mu_0, 1]$. By Lemmas 7 and 15, and Formula (37),

This expression is the same as in the second line in (38), and it takes the maximum value $\mu/(9 - 8\mu)$ for $p = p_{**} = 2(10\mu - 9)/(9 - 8\mu)$. Observe that the function $\mu \in [\mu_0, 1] \ni \mu \mapsto 2(10\mu - 9)/(9 - 8\mu)$ increases. Hence, $2/3 \leq p_{**} \leq 2$, so $p_{**}$ is not less than $2/3$. For this reason, the maximum value of $|\Phi f(\mu)|$ is equal to $\mu/(9 - 8\mu)$, but this value is obtained if $a \leq 0$.

It is easy to check that both values of $\Phi f(\mu)$ for $f(z) = \frac{1}{2}\log \frac{1+e}{1-e}$ and $f(z) = k(z)$, which are given in (33), are less than or equal to $\mu/(9 - 8\mu)$. This completes the proof. □

Theorem 6. Let $f \in C_0(k), \mu \in [63/92, 18/19]$, and $a_2$ be a real number, $a_2 \in [0, 2]$. Then:

$|\Phi f(\mu)| \leq (396 - 361\mu)/81$.

Proof. By Theorems 4 and 5, $|\Phi f(63/92)| \leq 169/92$ and $|\Phi f(18/19)| \leq 2/3$. Putting $\alpha = 4(414 - 437\mu)/459$, we can write:

$\Phi f(\mu) = \alpha(a_2a_4 - \frac{63}{72}a_2^2) + (1 - \alpha)(a_2a_4 - \frac{18}{72}a_2^2)$.
Applying the triangle inequality, we obtain our claim. □

The results presented in Theorems 2–6 can be collected as follows.

**Corollary 1.** Let \( f \in C_0(k) \) be given by (1), \( a_2 \) be a real number, and \( a_2 \in [0, 2] \). Then:

\[
|\Phi_f(\mu)| \leq \begin{cases} 
8 - 9\mu, & \mu \leq 63/92, \\
(396 - 361\mu)/81, & \mu \in [63/92, 18/19], \\
\mu, & \mu \in [18/19, 1], \\
9\mu - 8, & \mu \geq 1,
\end{cases}
\]

The results are sharp for \( \mu \leq 63/92 \) and \( \mu \geq 18/19 \). The equality holds for the Koebe function (2) in the first and the last case. The function \( F \) given by (15) is an extremal function when \( \mu \in [18/19, 1] \).

Observe that for \( \mu \in (18/19, 1) \), we have \( \mu/(9 - 8\mu) < 1 \), so the sharp bound for \( C_0(k) \) is less than the sharp bound for \( S^* \) given by (5).

6. **Concluding Remarks**

In this paper, we estimated two functionals \( \Theta_f(\mu) = a_4 - \mu a_2 a_3 \) and \( \Phi_f(\mu) = a_2 a_4 - \mu a_3^2 \) for the family \( C_0(k) \), where \( \mu \) is a real number. This family is a subset of the class \( C \) of all close-to-convex functions.

The results presented above broaden our knowledge about the behavior of the coefficient functionals defined for functions not only in \( C \), but also generally in the class \( S \) of univalent functions. Unfortunately, there are no good estimates of the discussed functionals in the whole classes \( C \) and \( S \). It seems that further research on the classes of the type \( C_0(f) \), where \( f \) is different from \( k \), may result in obtaining some conclusions about \( S \).

In our opinion, the most important problem to be solved now is the estimating of the second Hankel determinant, or in other words \( \Phi_f(1) \) for \( f \in S \). Even in the class \( C_0 \), the exact bound is unknown. It is only known that for \( C_0 \), there is \( |a_2 a_4 - a_3^2| < 1.242 \ldots \) (see [25]). On the other hand, the conjecture posed by Thomas [26] about 30 years ago that \( |a_n a_{n+2} - a_{n+1}^2| \leq 1 \) for \( S \) and \( n \geq 2 \) was disproven. This means that there are functions in \( S \) for which \( |a_n a_{n+2} - a_{n+1}^2| > 1 \). Finding (even non-sharp) estimates of \( \Phi_f(1) \) for \( f \in S \) remains an interesting open problem.

**Author Contributions:** All authors contributed equally to this work.

**Funding:** The project/research was financed in the framework of the project Lublin University of Technology-Regional Excellence Initiative, funded by the Polish Ministry of Science and Higher Education (Contract No. 030/RID/2018/19).

**Conflicts of Interest:** The authors declare no conflict of interest.

**References**

1. Duren, P.L. *Univalent Functions*; Springer: New York, NY, USA, 1983.
2. Goodman, A.W.; Saff, E.B. On the definition of close-to-convex function. *Int. J. Math. Math. Sci.* 1978, 1, 125–132.
3. Kaplan, W. Close to convex schlicht functions. *Mich. Math. J.* 1952, 1, 169–185.
4. Ma, W. Generalized Zalcman conjecture for starlike and typically real functions. *J. Math. Anal. Appl.* 1999, 234, 328–339.
5. Efraimidis, I.; Vukotić, D. Applications of LivinGston-Type Inequalities to the Generalized Zalcman Functional. *arxiv* 2017, arxiv:1611.00682v3.
6. Pommerenke, C. On the coefficients and Hankel determinants of univalent functions. *J. Lond. Math. Soc.* 1966, 41, 111–122.
7. Pommerenke, C. On the Hankel determinants of univalent functions. *Mathematika* 1967, 14, 108–112.
8. Cho, N.E.; Kowalczyk, B.; Kwon, O.S.; Lecko, A.; Sim, Y.J. Some coefficient inequalities related to the Hankel determinant for strongly starlike functions of order alpha. *J. Math. Inequal.* 2017, 11, 429–439.
9. Cho, N.E.; Kowalczyk, B.; Kwon, O.S.; Lecko, A.; Sim, Y.J. The bounds of some determinants for starlike functions of order alpha. *Bull. Malays. Math. Sci. Soc.* 2018, 41, 523–535.
10. Hayman, W.K. On the second Hankel determinant of mean univalent functions. *Proc. Lond. Math. Soc.* 1968, 18, 77–94.
11. Janteng, A.; Halim, S.A.; Darus, M. Hankel determinant for starlike and convex functions. *Int. J. Math. Anal.* 2007, 1, 619–625.
12. Krishna, D.V.; RamReddy, T. Hankel determinant for starlike and convex functions of order alpha. *Tbil. Math. J.* 2012, 5, 65–76.
13. Lee, S.K.; Ravichandran, V.; Supramaniam, S. Bounds for the second Hankel determinant of certain univalent functions. *J. Inequal. Appl.* 2013, 2013, 281.
14. Marjono, M.; Thomas, D.K. The second Hankel determinant of functions convex in one direction. *Int. J. Math. Anal.* 2016, 10, 423–428.
15. Noor, K.I. On the Hankel determinant problem for strongly close-to-convex functions. *J. Nat. Geom.* 1997, 11, 29–34.
16. Zaprawa, P. Second Hankel determinants for the class of typically real functions. *Abstr. Appl. Anal.* 2016, 2016, 3792367.
17. Hayami, T.; Owa, S. Generalized Hankel determinant for certain classes. *Int. J. Math. Anal.* 2010, 52, 2573–2585.
18. Zaprawa, P. On the Fekete–Szegö type functionals for starlike and convex functions. *Turk. J. Math.* 2018, 42, 537–547.
19. Zaprawa, P. On the Fekete–Szegö type functionals for functions which are convex in the direction of the imaginary axis. *C. R. Math. Acad. Sci. Paris* 2020.
20. Libera, R.J.; Złotkiewicz, E.J. Coefficients bounds for the inverse of a function with derivative in $P$. *Proc. Am. Math. Soc.* 1983, 87, 251–257.
21. Libera, R.J.; Złotkiewicz, E.J. Early coefficients of the inverse of a regular convex function. *Proc. Am. Math. Soc.* 1982, 85, 225–230.
22. Trąbka-Więtchaw, K.; Zaprawa, P. On the coefficient problem for close-to-convex functions. *Turk. J. Math.* 2018, 42, 2809–2818.
23. Lecko, A. On coefficient inequalities in the Caratheodory class of functions. *Ann. Pol. Math.* 2000, 75, 59–67.
24. Choi, J.H.; Kim, Y.C.; Sugawa, T. A general approach to the Fekete–Szegö problem. *J. Math. Soc. Jpn.* 2007, 59, 707–727.
25. Răducanu, D.; Zaprawa, P. Second Hankel determinant for close-to-convex functions. *C. R. Math. Acad. Sci. Paris* 2017, 355, 1063–1071.
26. Thomas, D.K. Bazilevič functions with logarithmic growth. In *New Trends in Geometric Function Theory and Application*; Parvatham, R., Ponnusamy, S., Eds.; World Scientific Publishing Company: Singapore; New Jersey, NJ, USA; London, UK; Hong Kong, China, 1991; pp. 146–158.

© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).