On a gap in the proof of the generalised quantum Stein's lemma and its consequences for the reversibility of quantum resources

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Based on

• On composite quantum hypothesis testing
  B., Brandão, Hirche
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• On a gap in the proof of the generalised quantum Stein's lemma and its consequences for the reversibility of quantum resources
  B., Brandão, Gour, Lami, Plenio, Regula, Tomamichel
  arXiv:2205.02813 (2022)
Outline

• Quantum hypothesis testing
• Quantum resource theories
• Asymptotic reversibility?
• Proof techniques
• Conclusion
Quantum hypothesis testing
Symmetric quantum hypothesis testing

- Two sequences $\rho_n, \sigma_n$ on $H \otimes^2$, discriminate them with two outcome POVM $\{M_n, (1 - M_n)\}$
- Two types of errors:
  \[ \alpha^n(M_n) := Tr[\rho_n(1 - M_n)] \text{ Type 1} \quad \text{and} \quad \beta^n(M_n) := Tr[\sigma_n M_n] \text{ Type 2} \]
- Asymptotic independent and identically distributed (IID) for $\rho_n = \rho \otimes^n, \sigma_n = \sigma \otimes^n$
- Symmetric setting

\[ \xi(\rho \otimes^n, \sigma \otimes^n) := \inf_{0 \leq M_n \leq 1} \frac{\alpha^n(M_n)}{2} + \frac{\beta^n(M_n)}{2} \]

gives quantum Chernoff bound [Audenaert et al., PRL 07]

\[ \xi(\rho, \sigma) := \lim_{n \to \infty} \frac{1}{n} \log \xi(\rho \otimes^n, \sigma \otimes^n) = -\log \min_{0 \leq s \leq 1} Tr[\rho^s \sigma^{1-s}] \]
Asymmetric quantum hypothesis testing

• Asymptotic IID  $\rho_n = \rho^\otimes n, \sigma_n = \sigma^\otimes n$ asymmetric setting
  
  $$\beta_\varepsilon(\rho^\otimes n, \sigma^\otimes n) := \inf_{0 \leq M_n \leq 1} \{\beta^n(M_n): \alpha^n(M_n) \leq \varepsilon\}$$

  gives quantum Stein’s lemma [Hiai & Petz, CMP 91]

  $$\beta(\rho, \sigma) := \lim_{\varepsilon \to 0} \lim_{n \to \infty} -\frac{\log \beta_\varepsilon(\rho^\otimes n, \sigma^\otimes n)}{n} = D(\rho||\sigma) := Tr[\rho \log \rho - \log \sigma]$$

• Fundamental tasks in quantum statistics, underlying much of quantum information theory

• What about composite hypotheses? That is,

  $$\rho^\otimes n \text{ with } \rho \in T \quad \text{versus} \quad \sigma^\otimes n \text{ with } \sigma \in S?$$
Composite hypothesis testing

- Asymptotic IID $\rho^{\otimes n}$ with $\rho \in T$ versus $\sigma^{\otimes n}$ with $\sigma \in S$, asymmetric setting
  \[
  \beta_\varepsilon(T^n, S^n) := \inf_{0 \leq M_n \leq 1} \{ \sup_{\sigma \in S} \text{Tr}[M_n \sigma^{\otimes n}] : \sup_{\rho \in T} \text{Tr}[(1 - M_n)\rho^{\otimes n}] \leq \varepsilon \}
  \]
- Thought-after characterization
  \[
  \beta(T, S) := \lim_{\varepsilon \to 0} \lim_{n \to \infty} -\frac{\log \beta_\varepsilon(T^n, S^n)}{n} \quad ?
  \]
- For $\rho \in T, \sigma \in S$ pairwise commuting, composite Stein’s lemma [Levitan & Merhav, IEEE 02]
  \[
  \beta(T, S) = \inf_{P \in T} \inf_{Q \in S} \beta(P, Q) = \inf_{P \in T} \inf_{Q \in S} D_{KL}(P || Q) \quad \text{with} \quad D_{KL}(P || Q) := \sum_x p_x \log \frac{p_x}{q_x}
  \]
  for eigendistributions $P, Q$ in common eigenbasis of $\rho, \sigma$
- What about fully quantum version?
Composite quantum hypothesis testing

- Partial results for special cases:

  [Hayashi, JPA 02], [Bjelaković et al., CMP 05], [Brandão & Plenio, CMP 10], [Hayashi & Tomamichel, JMP 16], etc.

- **Composite quantum Stein’s lemma** for $T, S$ convex [B. et al., CMP 21]

\[
\beta(T, S) = \lim_{n \to \infty} \frac{1}{n} \inf_{\rho \in T} \inf_{\mu \in \text{Meas}(S)} D\left(\rho^\otimes n \| \int \sigma^\otimes n d\mu(\sigma)\right) \neq \inf_{\rho \in T} \inf_{\sigma \in S} D(\rho \| \sigma)
\]

in general,

see also [Mosonyi et al., arXiv 21], as one does not have the quantum entropy inequality

\[
D\left(\rho^\otimes n \| \int \sigma^\otimes n d\mu(\sigma)\right) \geq n \cdot \inf_{\sigma \in S} D(\rho \| \sigma)
\]

- Nevertheless, various examples of interest do become single-letter anyway
Quantum resource theories
Resource theory of entanglement

- All also works for general resource theories (under suitable axiom set)
- Free states are separable states on \( H_{AB} := H_A \otimes H_B \), that is, convex hull of product states
  \[
  S_{A:B} := \text{conv}\{|\psi_A\rangle\langle\psi_A| \otimes |\phi\rangle\langle\phi|_B: |\psi\rangle_A \in H_A, |\phi\rangle_B \in H_B\}
  \]
  + all other states are entangled (i.e., resourceful)
- Unit is ebit \( \Phi_{AB} := |\Phi\rangle\langle\Phi|_AB \) with \( |\Phi\rangle_{AB} := \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB}) \)
- Entanglement measure: \( R_S(\rho_{AB}) := \{s \geq 0: \frac{\rho_{AB} + s\sigma_{AB}}{1+s} \in S_{A:B}\} \) global resource robustness
- Free operations for transformations \( \rho_{AB} \to \omega_{AB} \)? The largest meaningful such set is given by \( \delta \)-non-entangling operations
  \[
  NE_{\delta(AB \to A'B')} := \{\Lambda \in CPTP(AB \to A'B'): R_S(\Lambda(\sigma_{AB})) \leq \delta \ \forall \sigma_{AB} \in S_{A:B}\}
  \]
Asymptotic resource theory of entanglement

• Distillable entanglement under asymptotically non-entangling operations (ANE):

\[ E_D^{ANE}(\rho) := \sup_{(k_n),(\delta_n)} \{ \liminf_{n \to \infty} \frac{k_n}{n} : \lim_{n \to \infty} \min_{\Lambda \in NE\delta_n} ||\Lambda(\rho^\otimes n) - \Phi^\otimes k_n||_1 = 0, \lim_{n \to \infty} \delta_n = 0 \} \]

• Entanglement cost under ANE:

\[ E_C^{ANE}(\rho) := \inf_{(k_n),(\delta_n)} \{ \limsup_{n \to \infty} \frac{k_n}{n} : \lim_{n \to \infty} \min_{\Lambda \in NE\delta_n} ||\Lambda(\Phi^\otimes k_n) - \rho^\otimes n||_1 = 0, \lim_{n \to \infty} \delta_n = 0 \} \]

• Asymptotic transformation rate \( \rho_{AB} \to \omega_{AB} \) under ANE:

\[ R^{ANE}(\rho \to \omega) := \sup_{(k_n),(\delta_n)} \{ \liminf_{n \to \infty} \frac{k_n}{n} : \lim_{n \to \infty} \min_{\Lambda \in NE\delta_n} ||\Lambda(\rho^\otimes n) - \omega^\otimes k_n||_1 = 0, \lim_{n \to \infty} \delta_n = 0 \} \]
• Asymptotic reversibility?
Asymptotic characterization of entanglement

- Asymptotically reversible under ANE?
  \[ R_{ANE}^{\omega} (\rho \rightarrow \omega) \cdot R_{ANE}^{\omega} (\omega \rightarrow \rho) = 1 \quad \text{or in other words} \quad E_{D,\omega}^{ANE} (\rho) = E_{C,\omega}^{ANE} (\rho)? \]

- Entanglement cost [Brandão & Plenio, CMP 10], [Datta, IEEE 09]
  \[ E_{C,\omega}^{ANE} (\rho) = \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\sigma \in S^n} D_{\max} (\rho \otimes^n |\sigma^n) = \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\sigma \in S^n} D (\rho \otimes^n |\sigma^n) \neq \min_{\sigma \in S} D (\rho |\sigma) \]

- Distillable entanglement [Brandão & Plenio, CMP 10]
  \[ E_{D,\omega}^{ANE} (\rho) = \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log \beta_{\epsilon} (\rho \otimes n, S^n) \]
  for the hypothesis testing \[ \beta_{\epsilon} (\rho \otimes n, S^n) := \inf_{0 \leq M_n \leq 1} \{ \sup_{\sigma^n \in S^n} Tr [M_n \sigma^n]: Tr [(1 - M_n) \rho \otimes n] \leq \epsilon \} \]

- Composite quantum hypothesis testing question
  \[ -\frac{1}{n} \log \beta_{\epsilon} (\rho \otimes n, S^n) \rightarrow \min_{\sigma \in S^n} D (\rho \otimes^n |\sigma^n)? \]
Reduction to hypothesis testing

- Question if $E_D^{ANE}(\rho) = E_C^{ANE}(\rho)$ reduces to composite quantum hypothesis question

$$-\frac{1}{n} \log \beta_\varepsilon(\rho, S^n) \rightarrow \frac{1}{n} \min_{\sigma^n \in S^n} D(\rho^\otimes n || \sigma^n)?$$

- Converse direction by standard arguments [Brandão & Plenio, CMP 10]:

$$\lim_{\varepsilon \rightarrow 0, n \rightarrow \infty} -\frac{1}{n} \log \beta_\varepsilon(\rho, S^n) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\sigma^n \in S^n} D(\rho^\otimes n || \sigma^n)$$

- [B. et al., arXiv 22] recently found that achievability direction “ $\geq$ ” remains open

- Setting: $T^n = \{\rho^\otimes n\}$ singleton, but separable set $S^n \equiv S_{A^n:B^n} \equiv S(A_1 \cdots A_n:B_1 \cdots B_n)$ is not IID and could be entangled across different $A_i$'s and $B_i$'s, resp.

- Results from [B. et al., CMP 21] do not directly apply!
What can be shown?

• Pseudo-entanglement theory:

\[ \bar{S}_{A^n:B^n} := \text{conv}\{\otimes_{j=1}^{n} \sigma_{A_jB_j}^{(j)} : \sigma_{A_jB_j}^{(j)} \in S_{A_jB_j} \ \forall j\} \]

separable across the partition \(A_1 : \cdots : A_n : B_1 : \cdots : B_n\)

and combination of [Brandão et al., IEEE 20], [B. et al., CMP 21] gives

\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \beta_{\varepsilon} (\rho \otimes^n, \bar{S}^n) = \lim_{n \to \infty} \frac{1}{n} \min_{\sigma^n \in S^n} D (\rho \otimes^n \| \sigma^n)
\]

• Pseudo-entanglement in blocks \(A^k := A_1 \cdots A_k, B^k := B_1 \cdots B_k\) with \(\bar{S}_{A^n:B^n}^k\) [B. et al., arXiv 22]

\[
\lim_{k \to \infty} \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{nk} \min_{\sigma^n \in S^n,k} \log \beta_{\varepsilon} (\rho \otimes^{nk}, \sigma^{nk}) = \lim_{n \to \infty} \frac{1}{n} \min_{\sigma^n \in S^n} D (\rho \otimes^n \| \sigma^n)
\]

• Remains open if \(\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \min_{\sigma^n \in S^n} \log \beta_{\varepsilon} (\rho \otimes^n, \sigma^n) \geq \lim_{n \to \infty} \frac{1}{n} \min_{\sigma^n \in S^n} D (\rho \otimes^n \| \sigma^n) ?\)
• Proof techniques: Universal hypothesis tests

1) via Petz-Rényi divergences?
2) via measured divergence?
3) via max-relative entropy?
1) Universal hypothesis tests via Petz-Rényi divergences

- Sion minimax + Audenaert inequality for $s \in (0,1)$ gives [Audenaert et al., CMP 08]
  $$-\frac{1}{n}\log \beta(\rho^{\otimes n}, S^n) = -\frac{1}{n} \sup_{\sigma^n \in S^n} \inf_{0 \leq M_n \leq 1} \log Tr[M_n \rho^{\otimes n}] \geq \frac{1}{n} \inf_{\sigma^n \in S^n} D_s(\rho^{\otimes n} || \sigma^n) - \frac{1}{n} \cdot s \cdot \frac{1}{1-s} \log \frac{1}{\epsilon}$$
  for the additive $D_s(\rho || \sigma) \equiv \frac{1}{s-1} \log Tr[\rho^s \sigma^{1-s}]$ with $\lim_{s \to 1} D_s(\rho || \sigma) = D(\rho || \sigma)$

- Single-letter: de Finetti, take limits
  (i) $n \to \infty$
  (ii) $\epsilon \to 0$
  (iii) $s \to 1$
  in order [B. et al., CMP 21]

- Generally, with information variance $V(\rho || \sigma) \equiv Tr[\rho (\log \rho - \log \sigma - D(\rho || \sigma))^2]$ to bound
  $$\frac{1}{n} |D_s(\rho^{\otimes n} || \sigma^n) - D(\rho^{\otimes n} || \sigma^n)| \leq \frac{s - 1}{2} \cdot \frac{V(\rho^{\otimes n} || \sigma^n)}{n} + O(\frac{(s - 1)^2}{n})$$

- [Brandão & Plenio, CMP 10] claimed that $V(\rho^{\otimes n} || \sigma^n) \leq o(2^{-n})$, but already
  $$V(\rho^{\otimes n} || \sigma^{\otimes n}) = n \cdot V(\rho || \sigma) \not\in o(2^{-n}) \quad \rightarrow \text{Remains open: de Finetti / Schur-Weyl duality?}$$
2) Universal hypothesis tests via measured divergence

- Measured relative entropy [Donald, CMP 86] with [Brandão et al., IEEE 20]

\[ D_M(\rho || \sigma) := \sup_M D_{KL}(M(\rho) \| M(\sigma)) \text{ with } \inf_{\rho \in T, \sigma \in S} D_M(\rho || \sigma) = \sup_M \inf_{\rho \in T, \sigma \in S} D_{KL}(M(\rho) || M(\sigma)) \]

- (i) measure, (ii) apply classical composite hypothesis result, (iii) use asymptotic achievability of measured relative entropy for \( \rho^n, \sigma^n \) permutation invariant [B. et al., CMP 21]

\[ \frac{1}{n} D_M(\rho^n || \sigma^n) \rightarrow \frac{1}{n} D(\rho^n || \sigma^n) \text{ for } n \rightarrow \infty \]

- Gives pseudo-entanglement theory and pseudo-entanglement in blocks [B. et al., arXiv 22]

- Remains open: entanglement theory. Alternatively, one has [Brandão et al., IEEE 20]

\[ \lim_{n \to \infty} \frac{1}{n \min_{\sigma^n \in S^n}} D_{MSEP}(\rho \otimes^n || \sigma^n) \rightarrow \lim_{n \to \infty} \frac{1}{n \min_{\sigma^n \in S^n}} D(\rho \otimes^n || \sigma^n) ? \]
3) Universal hypothesis tests via max-relative entropy

- For $\varepsilon \in (0,1)$ we have [Anshu et al., JMP 19]

$$-\frac{1}{n} \log \sup_{\sigma^n \in \mathcal{S}} \beta_{\varepsilon}(\rho \otimes^n, \sigma^n) \geq \frac{1}{n} \min_{\sigma^n \in \mathcal{S}} D_{\max}^{\sqrt{1-\varepsilon}}(\rho \otimes^n || \sigma^n) - \frac{1}{n} \log \frac{1}{\varepsilon}$$

- Previously mentioned asymptotic equipartition property (AEP) for max-relative entropy

$$\lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \min_{\sigma^n \in \mathcal{S}} D_\delta(\rho \otimes^n || \sigma^n) = \lim_{n \to \infty} \frac{1}{n} \min_{\sigma^n \in \mathcal{S}} D(\rho \otimes^n || \sigma^n)$$

not enough as no strong converse!

- Similar open problems:
  - Quantum channel AEP [Gour & Winter, PRL 19]
  - Strong converse channel discrimination & channel capacities [Fang et al., arXiv 21] [Bergh et al., arXiv 21]
  - Stronger entropy accumulation [Metger et al., arXiv 22]
• Conclusion
Outlook

• Question if $E_{D}^{ANE}(\rho) = E_{C}^{ANE}(\rho)$ reduces to composite quantum hypothesis question

$$-\frac{1}{n} \log \beta_{\epsilon}^{n}(\rho, S^{n}) \to \frac{1}{n} \min_{\sigma^{n} \in S^{n}} D(\rho^{\otimes n}||\sigma^{n})?$$

This remains open.

• Take step back, classical version of non-IID problem? Not clear, cf. [Mosonyi et al., arXiv 21]

• Composite hypothesis testing will hold for some resource theories under suitable axiom set, but must be shown “manually” every time – and so far, we only have single-letter solutions

• Reversibility of resource theories? If I had to guess, reversibility does not hold in general

• Hint for resource theory of entanglement: [Lami & Regula, arXiv 21]
Lami & Regula arXiv:2111.02438

• Title: No second law of entanglement manipulation after all

• Recall:
  - $\delta$-non-entangling operations $NE_{\delta(AB\rightarrow A'B')} = \{\Lambda \in CPTP(AB \rightarrow A'B') : R_S(\Lambda(\sigma_{AB})) \leq \delta \ \forall \sigma_{AB} \in S_{A:B}\}$
  - with global resource robustness $R_S(\rho_{AB}) = \{s \geq 0 : \frac{\rho_{AB} + s\sigma_{AB}}{1+s} \in S_{A:B}\}$

• Replace $R_S(\rho_{AB})$ with resource robustness

  $$\bar{R}_S(\rho_{AB}) := \{s \geq 0 : \frac{\rho_{AB} + s\sigma_{AB}}{1+s} \in S_{A:B}, \sigma_{AB} \in S_{A:B}\} \geq R_S(\rho_{AB})$$

  and correspondingly

  $$\overline{NE}_{\delta(AB\rightarrow A'B')} := \{\Lambda \in CPTP(AB \rightarrow A'B') : \bar{R}_S(\Lambda(\sigma_{AB})) \leq \delta \ \forall \sigma_{AB} \in S_{AB}\}$$

• Main result: there exists quantum state $\rho$ with $E_D^{ANE}(\rho) < E_C^{ANE}(\rho)$
Thank you!

- B., Brandão, Hirche: CMP 385, 55 (2021)
- B., Brandão, Gour, Lami, Plenio, Regula, Tomamichel: arXiv:2205.02813 (2022)
- Audenaert, Nussbaum, Szkola, Verstraete: CMP 279, 251 (2008)
- Brandão, Harrow, Lee, Peres: IEEE 66, 5037 (2020)
- Mosonyi, Szilágyi, Weiner: arXiv:2011.04645 (2021)
- Lami & Regula: arXiv:2111.02438 (2021)
- Bergh, Datta, Salzmann, Wilde: arXiv:2206.08350 (2022)