RESIDUE FORMULAS FOR LOGARITHMIC FOLIATIONS AND APPLICATIONS

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ABSTRACT. In this work we prove a Baum-Bott type formula for non-compact complex manifold of the form $\tilde{X} = X - D$, where $X$ is a complex compact manifold and $D$ is a normal crossing divisor on $X$. As applications, we provide a Poincaré-Hopf type Theorem and an optimal description for a smooth hypersurface $D$ invariant by an one-dimensional foliation $\mathcal{F}$ on $\mathbb{P}^n$ satisfying $\text{Sing}(\mathcal{F}) \subset \subset D$.

1. Introduction

In [4] P. Baum and R. Bott developed a work about residues of singularities of holomorphic foliations on complex manifolds. In the case of one-dimensional holomorphic foliation $\mathcal{F}$, with isolate singularities, on an $n$-dimensional complex compact manifold $X$ we have the following classical Baum-Bott formula:

$$\int_X c_n(T_X - T_\mathcal{F}) = \sum_{p \in \text{Sing}(\mathcal{F})} \mu_p(\mathcal{F}), \quad (Baum-Bott \text{ formula})$$

where the $\mu_p(\mathcal{F})$ are the Milnor number of $\mathcal{F}$ in $p$. Baum-Bott formula is a generalization (for holomorphic vector fields) of the Poincaré-Hopf Theorem

$$\int_X c_n(T_X) = \sum_{p \in \text{Sing}(\mathcal{F})} \mu_p(\mathcal{F}),$$

where $\mathcal{F}$ is a foliation induced by a global holomorphic vector field, with isolated singularities, on $X$.

In this work we provide a Baum-Bott type formula for non-compact complex manifold of the form $\tilde{X} = X - D$, where $X$ is a complex compact manifold and $D$ is an analytic divisor contained in $X$ invariant by an one-dimensional holomorphic foliation $\mathcal{F}$ which is called by logarithmic foliation along $D$. As an application, we obtained a Poincaré-Hopf type Theorem for these non-compact manifolds. Furthermore, for logarithmic foliations on projective spaces, we prove a necessary and sufficient conditions for all singularities of the foliation occur in an analytic invariant hypersurface.

We prove the following result.

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Theorem 1. Let $\tilde{X}$ be an $n$-dimensional complex manifold such that $\tilde{X} = X - D$, where $X$ is an $n$-dimensional complex compact manifold and $D$ is a smooth hypersurface on $X$. Let $F$ be a foliation of dimension one on $X$, with isolated singularities and logarithmic along $D$. Suppose that $\text{Ind}_{\log D, p}(F) = 0$, for all $p \in \text{Sing}(F) \cap D$. Then
\[ \int_X c_n(T_X(-\log D) - T_F) = \sum_{p \in \text{Sing}(F) \cap (\tilde{X} \setminus D)} \mu_p(F). \]

Here, $\text{Ind}_{\log D, p}(F)$ denotes the logarithmic index of $F$ on $p$, see section 2.4.

The classical Gauss-Bonnet theorem for a complex compact manifold $X$, proved by S. Chern in [8], says us that
\[ \int_X c_n(T_X) = \chi(X). \]

The following version of Gauss-Bonnet formula for non-compact manifolds was initially proposed by S. Iitaka [13] and proved by Y. Norimatsu [17], R. Silvotti [20] and P. Aluffi [2]:

**Theorem** (Norimatsu-Silvotti-Aluffi). Let $\tilde{X}$ be an $n$-dimensional complex manifold such that $\tilde{X} = X - D$, where $X$ is an $n$-dimensional complex compact manifold and $D$ is a normally crossing hypersurface on $X$. Then
\[ \int_X c_n(T_X(-\log D)) = \chi(\tilde{X}), \]

where $\chi(\tilde{X})$ denotes the Euler characteristic given by
\[ \chi(\tilde{X}) = \sum_{i=1}^n \dim H^i_c(\tilde{X}, \mathbb{C}). \]

X. Liao in [16] has provided more general formulas in terms of Chern-Schwartz-MacPherson class of $\tilde{X}$.

In the Section 4, we consider the case where $D$ is a normal crossing hypersurface and we prove the following Baum-Bott type formula:

**Theorem 2.** Let $\tilde{X}$ be an $n$-dimensional complex manifold such that $\tilde{X} = X - D$, where $X$ is a $n$-dimensional complex compact manifold, $D$ is a normally crossing hypersurface on $X$. Let $F$ be a foliation on $X$ of dimension one, with isolated singularities (non-degnerates) and logarithmic along $D$. Then,
\[ \int_X c_n(T_X(-\log D) - T_F) = \sum_{p \in \text{Sing}(F) \cap (\tilde{X})} \mu_p(F). \]

As a consequence of Theorem 2 and Norimatsu-Silvotti-Aluffi Theorem we obtain the following Poincaré-Hopf type Theorem.
Corollary 1. Let \( \tilde{X} \) be an \( n \)-dimensional complex manifold such that \( \tilde{X} = X - D \), where \( X \) is an \( n \)-dimensional complex compact manifold, \( D \) is a reduced normal crossing hypersurface on \( X \). Let \( \mathcal{F} \) be a foliation on \( X \) of dimension one given by a global holomorphic vector field, with isolated singularities (non-degenerates) and logarithmic along \( D \). Then

\[
\chi(\tilde{X}) = \sum_{p \in \text{Sing}(\mathcal{F}) \cap \tilde{X}} \mu_p(\mathcal{F}),
\]

where \( \mu_p(\mathcal{F}) \) denotes the Milnor number of \( \mathcal{F} \) on \( p \).

Finally, in the Section 6, we prove a complete characterization in order that an invariant hypersurface contains all the singularities of the projective foliation.

Theorem 3. Let \( D \subset \mathbb{P}^n \) be a smooth and irreducible hypersurface and let \( \mathcal{F} \) be a foliation of dimension one on \( \mathbb{P}^n \), with isolated singularities (non-degenerates) and logarithmic along \( D \). Then, the following properties hold

1. If \( n \) is odd, then:
   - (a) \# (Sing(\mathcal{F}) \cap \mathbb{P}^n \setminus D) > 0 \iff \deg(D) < \deg(\mathcal{F}) + 1;
   - (b) \# (Sing(\mathcal{F}) \cap \mathbb{P}^n \setminus D) = 0 \iff \deg(D) = \deg(\mathcal{F}) + 1.

2. If \( n \) is even, then:
   - (a) \# (Sing(\mathcal{F}) \cap \mathbb{P}^n \setminus D) > 0 \iff \begin{cases} \deg(D) \neq \deg(\mathcal{F}) + 1 \\ or \\ \deg(D) = \deg(\mathcal{F}) + 1, \text{ with } \deg(\mathcal{F}) \neq 0 \end{cases}
   - (b) \# (Sing(\mathcal{F}) \cap \mathbb{P}^n \setminus D) = 0 \iff \deg(D) = 1 \text{ and } \deg(\mathcal{F}) = 0.

3. In general, we have the formula

\[
\# (\text{Sing}(\mathcal{F}) \cap \mathbb{P}^n \setminus D) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (\deg(D) - 1)^i \deg(\mathcal{F})^{n-i}.
\]

Observe that if \( n \) is odd, then \( \text{Sing}(\mathcal{F}) \subseteq D \) if and only if the Soares’s bound for the Poincaré problem is achieved \([21]\).

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2. Preliminaries

2.1. Logarithmics forms and logarithmics vector fields. Let \( X \) an \( n \)-dimensional complex manifold and \( D \) a reduced hypersurface on \( X \). Given a meromorphic \( q \)-form \( \omega \) on \( X \), we say that \( \omega \) is a logarithmic \( q \)-form along \( D \) at \( x \in X \) if the following conditions occurs:

(i) \( \omega \) is holomorphic on \( X - D \);

(ii) If \( h = 0 \) is a reduced equation of \( D \), locally at \( x \), then \( h \omega \) and \( h d\omega \) are holomorphic.

Denoting by \( \Omega^q_X,\omega(x) (\text{log} D) \) the set of germs of logarithmic \( q \)-form along \( D \) at \( x \), we define the following coherent sheaf of \( \mathcal{O}_X \)-modules

\[
\Omega^q_X(\text{log} D) := \bigcup_{x \in X} \Omega^q_{X,x}(\text{log} D),
\]

which is called by sheaf of logarithmic \( q \)-forms along \( D \). See [9], [14] and [18] for details.

Now, given \( x \in X \), let \( v \in T_{X,x} \) be germ at \( x \) of a holomorphic vector field on \( X \). We say that \( v \) is a logarithmic vector field along of \( D \) at \( x \), if \( v \) satisfies the following condition: if \( h = 0 \) is a equation of \( D \), locally at \( x \), then the derivation \( v(h) \) belongs to the ideal \( \langle h_x \rangle \mathcal{O}_{X,x} \). Denoting by \( T_{X,x}(\text{− log} D) \) the set of germs of logarithmic vector field along of \( D \) at \( x \), we define the following coherent sheaf of \( \mathcal{O}_X \)-modules

\[
T_X(\text{− log} D) := \bigcup_{x \in X} T_{X,x}(\text{− log} D),
\]

which is called by sheaf of logarithmic vector fields along \( D \).

It is known that \( \Omega^1_X(\text{log} D) \) and \( T_X(\text{− log} D) \) is always a reflexive sheaf, see [18] for more details. If \( D \) is an analytic hypersurface with normal crossing singularities, the sheaves \( \Omega^1_X(\text{log} D) \) and \( T_X(\text{− log} D) \) are locally free, furthermore, the Poincaré residue map

\[
\text{Res} : \Omega^1_X(\text{log} D) \longrightarrow \mathcal{O}_D \cong \bigoplus_{i=1}^N \mathcal{O}_{D_i}
\]

give the following exact sequence of sheaves on \( X \)

\[
0 \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X(\text{log} D) \xrightarrow{\text{Res}} \bigoplus_{i=1}^N \mathcal{O}_{D_i} \longrightarrow 0,
\]

(4)

where \( \Omega^1_X \) is the sheaf of holomorphics 1-forms on \( X \) and \( D_1, \ldots, D_N \) are the irreducible components of \( D \).

Now, if \( D \) is such that \( \text{cod}(\text{Sing}(D)) > 2 \) then there exist the following exact sequence of sheaves on \( X \) (see V. I. Dolgachev [10]):

\[
0 \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X(\text{log} D) \longrightarrow \mathcal{O}_D \longrightarrow 0,
\]

(5)
On the projective space \( \mathbb{P}^n \), if \( D \) is a smooth hypersurface, then there exist the following exact sequence of sheaves (see E. Angeline \[3\]):

\[
0 \longrightarrow T_{\mathbb{P}^n}(-\log D) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(k) \longrightarrow 0,
\]

where \( k \) is the degree of \( D \).

### 2.2. Singular one-dimensional holomorphic foliations.

**Definition 2.1.** Let \( X \) be a connected complex manifold. An one-dimensional holomorphic foliation is given by the following data:

\( i \) an open covering \( \mathcal{U} = \{ U_\alpha \} \) of \( X \);

\( ii \) for each \( U_\alpha \) an holomorphic vector field \( \zeta_\alpha \);

\( iii \) for every non-empty intersection, \( U_\alpha \cap U_\beta \neq \emptyset \), a holomorphic function \( f_{\alpha\beta} \in \mathcal{O}^*_X(U_\alpha \cap U_\beta) \);

such that \( \zeta_\alpha = f_{\alpha\beta} \zeta_\beta \) in \( U_\alpha \cap U_\beta \) and \( f_{\alpha\beta}f_{\beta\gamma} = f_{\alpha\gamma} \) in \( U_\alpha \cap U_\beta \cap U_\gamma \).

We denote by \( K_\mathcal{F} \) the line bundle defined by the cocycle \( \{ f_{\alpha\beta} \} \in H^1(X, \mathcal{O}^*) \).

Thus, a one-dimensional holomorphic foliation \( \mathcal{F} \) on \( X \) induces a global holomorphic section \( \zeta_\mathcal{F} \in H^0(X, T_X \otimes K_\mathcal{F}) \).

The line bundle \( T_\mathcal{F} := (K_\mathcal{F})^* \hookrightarrow T_X \) is called the *tangente bundle* of \( \mathcal{F} \). The singular set of \( \mathcal{F} \) is \( \text{Sing}(\mathcal{F}) = \{ \zeta_\mathcal{F} = 0 \} \). We will assume that \( \text{cod}(\text{Sing}(\mathcal{F})) \geq 2 \).

**Definition 2.2.** Let \( V \) an analytic subspace of a complex manifold \( X \). We say that \( V \) is invariant by a foliation \( \mathcal{F} \) if \( T_\mathcal{F}|_V \subset (\Omega^1_V)^* \). If \( V \) is a hypersurface we say that \( \mathcal{F} \) is *logarithmic along* \( V \).

**Definition 2.3.** A foliation on a complex projective space \( \mathbb{P}^n \) is called by *projective foliation*. Let \( \mathcal{F} \) be a projective foliation with tangent bundle \( T_\mathcal{F} = \mathcal{O}_{\mathbb{P}^n}(r) \). The integer number \( d := r + 1 \) is called by the *degree* of \( \mathcal{F} \).

### 2.3. The GSV-Index.

X. Gomez-Mont, J. Sead and A. Verjovsky \[12\] introduced the *GSV-index* for a holomorphic vector field over an analytic hypersurface, with isolated singularities, on a complex manifold, generalizing the (classical) Poincaré-Hopf index. The concept of GSV-index was extended to holomorphic vector field on more general contexts. For example, J. Seade and T. Suwa in \[19\], defined the GSV-index for holomorphic vector field on analytic subvariety type isolated complete intersection singularity. J.-P. Brasselet, J. Seade and T. Suwa in \[5\], extended the notion of GSV-index for vector fields defined in certain types of analytical subvariety with non-isolated singularities.

In \[11\] X. Gomez-Mont defined the *homological index* of holomorphic vector field on an analytic hypersurface with isolated singularities, which coincides with GSV-index. There is also the *virtual index*, introduced by D. Lehmann, M. Soares and T. Suwa \[15\], that via Chern-Weil theory can be interpreted as the GSV-index.
M. Brunella [7] also present the GSV-index for foliations on complex surfaces by a different approach.

Let \( X \) be an \( n \)-dimensional complex manifold, \( D \) a isolated hypersurface singularity on \( X \) and let \( \mathcal{F} \) be a foliation on \( X \) of dimension one, with isolated singularities. Suppose \( \mathcal{F} \) logarithmic along \( D \), i.e., the analytic hypersurface \( D \) is invariant by each holomorphic vector field that is a local representative of \( \mathcal{F} \). The GSV-index of \( \mathcal{F} \) in \( x \in D \) will be denoted by \( \text{GSV}(\mathcal{F}, D, x) \).

For definition and details on the GSV-index we refer to [6] and [22].

2.4. The Logarithmic Index. Recently, A. G. Aleksandrov introduced in [1] the notion of logarithmic index for logarithmic a vector field. Let \( \mathcal{F} \) be an one-dimensional holomorphic foliation on \( X \) with isolated singularities and logarithmic along \( D \). Fixed a point \( x \in X \), let \( v \in T_X(-\log D)|_U \) a germ of vector field on \((U, x)\) tangent to \( \mathcal{F} \). The interior multiplication \( i_v \) induces the complex of logarithmic differential forms

\[
0 \longrightarrow \Omega^0_{X,x}(\log D) \xrightarrow{i_v} \Omega^{-1}_{X,x}(\log D) \xrightarrow{i_v} \cdots \xrightarrow{i_v} \Omega^1_{X,x}(\log D) \xrightarrow{i_v} \mathcal{O}_{n,x}.
\]

Since all singularities of \( v \) are isolated, the \( i_v \)-homology groups of the complex \( \Omega^*_{X}(\log D) \) are finite-dimensional vector spaces (see [1]). Thus, the Euler characteristic

\[
\chi(\Omega^*_{X}(\log D), i_v) = \sum_{i=0}^{n} (-1)^i \dim H_i(\Omega^*_{X,x}(\log D), i_v).
\]

of the complex of logarithmic differential forms is well defined. Since this number does not depend on local representative \( v \) of the foliation \( \mathcal{F} \) at the point \( x \), we define the logarithmic index of \( \mathcal{F} \) at the point \( x \) by

\[
\text{Ind}_{\log D,x}(\mathcal{F}) := \chi(\Omega^*_{X}(\log D), i_v).
\]

It follows from the definition that \( \text{Ind}_{\log D,x}(\mathcal{F}) = 0 \) for all \( x \in X - \text{Sing}(\mathcal{F}) \).

We have the following important property (see [1]):

**Proposition 2.4.** [1] Let \( X, D \) and \( \mathcal{F} \) be as described above. Then, for each \( x \in \text{Sing}(\mathcal{F}) \cap D \) we have

\[
\text{Ind}_{\log D,x}(\mathcal{F}) = \mu_x(\mathcal{F}) - \text{GSV}(\mathcal{F}, D, x)
\]

where \( \mu_x(v) \) and \( \text{GSV}(v, D, x) \) denote, respectively, the Milnor number and GSV index of \( v \).

**Remark 2.5.** If \( x \in \text{Sing}(\mathcal{F}) \cap D_{\text{reg}} \), we obtain

\[
\text{Ind}_{\log D,x}(\mathcal{F}) = \mu_x(\mathcal{F}) - \mu_x(\mathcal{F}|_{D_{\text{reg}}}),
\]

since, in this case, the GSV index of \( \mathcal{F} \) in \( x \) coincides with the Milnor number of \( \mathcal{F}|_{D_{\text{reg}}} \) in \( x \). In particular,

\[
\text{Ind}_{\log D,x}(\mathcal{F}) = 0,
\]
whenever $x$ is a non-degenerate singularity of $\mathcal{F}$.

3. Proof of Theorem [\textit{\ref{thm:main}}]

To prove the Theorem [\textit{\ref{thm:main}}] we will firstly prove the following result.

**Theorem 3.1.** Let $X$ be an $n$-dimensional complex compact manifold and $\mathcal{D}$ a smooth hypersurface on $X$. Then for all line bundle $L$ on $X$ we have

$$\int_X c_n(T_X(-\log \mathcal{D}) - L) = \int_X c_n(T_X - L) - \int_{\mathcal{D}} c_{n-1}(T_X - [\mathcal{D}] - L).$$

**Proof.** By using properties of Chern class we get

$$\int_X c_n(T_X(-\log \mathcal{D}) - L) = \sum_{j=0}^{n} \int_X c_{n-j}(T_X(-\log \mathcal{D})) c_1(L)^j = \sum_{j=0}^{n} (-1)^{n-j} \int_X c_{n-j}(\Omega^1_X(\log \mathcal{D})) c_1(L)^j.$$

On the one hand, since $\mathcal{D}$ is smooth we can use the exact sequence \[(\ref{eq:exact_sequence})\] to obtain

$$c_i(\Omega^1_X(\log \mathcal{D})) = \sum_{k=0}^{i} c_{i-k}(\Omega^1_X) c_k(\mathcal{O}_\mathcal{D}), \forall i \in \{1, \ldots, n\}.$$

On the other hand, the Chern classes of $\mathcal{O}_\mathcal{D}$ are

$$c_k(\mathcal{O}_\mathcal{D}) = c_k(\mathcal{O}_X - \mathcal{O}(-\mathcal{D})) = c_1([\mathcal{D}])^k, \quad k = 1, \ldots, n.$$

Thus,

$$\int_X c_n(T_X(-\log \mathcal{D}) - L) = \sum_{j=0}^{n} (-1)^{n-j} \int_X \left[ \sum_{k=0}^{n-j} c_{n-j-k}(\Omega^1_X) c_1([\mathcal{D}])^k \right] c_1(L)^j.$$

Now, we split this sum in two parts as follows:

$$\sum_{j=0}^{n} (-1)^{n-j} \int_X \left[ \sum_{k=0}^{n-j} c_{n-j-k}(\Omega^1_X) c_1([\mathcal{D}])^k \right] c_1(L)^j = \sum_{j=0}^{n} (-1)^{n-j} \int_X \left[ \sum_{k=1}^{n-j} c_{n-j-k}(\Omega^1_X) c_1([\mathcal{D}])^k \right] c_1(L)^j + \sum_{j=0}^{n} (-1)^{n-j} \int_X c_{n-j}(\Omega^1_X) c_1(L)^j.$$
In the first part appears all terms with \( k \geq 1 \) and in second one part are the terms with \( k = 0 \). By using the Poincaré duality, we compute the first part as follow:

\[
\sum_{j=0}^{n-1} (-1)^{n-j} \int_X \left[ \sum_{k=1}^{n-j} c_{n-j-k}(\Omega_X^1) c_1([\mathcal{D}])^k \right] c_1(L^*)^j = \\
= \sum_{j=0}^{n-1} (-1)^{n-j} \int_X \left[ \sum_{k=1}^{n-j} c_{n-j-k}(\Omega_X^1) c_1([\mathcal{D}])^{k-1} \right] c_1(L^*)^j = \\
= -\sum_{j=0}^{n-1} \int_D \left[ \sum_{k=1}^{n-j} (-1)^{n-j-k} c_{n-j-k}(\Omega_X^1)(-1)^{k-1} c_1([\mathcal{D}])^{k-1} \right] c_1(L^*)^j = \\
= -\sum_{j=0}^{n-1} \int_D c_{n-1-j}(T_X - [\mathcal{D}]) c_1(L^*)^j = \\
= -\int_D c_{n-1}(T_X - [\mathcal{D}] - L).
\]

Now, by basics proprieties of Chern classes we compute the second sum as follow:

\[
\sum_{j=0}^{n} (-1)^{n-j} \int_X c_{n-j}(\Omega_X^1) c_1(L^*)^j = \sum_{j=0}^{n} (-1)^{n-j} \int_X (-1)^{n-j} c_{n-j}(T_X) c_1(L^*)^j = \\
= \sum_{j=0}^{n} \int_X c_{n-j}(T_X) c_1(L^*)^j = \\
= \int_X c_n(T_X - L).
\]

Finally, we conclude that

\[
\sum_{j=0}^{n} (-1)^{n-j} \int_X \left[ \sum_{k=0}^{n-j} c_{n-j-k}(\Omega_X^1) c_1([\mathcal{D}])^k \right] c_1(L^*)^j = -\int_D c_{n-1}(T_X - [\mathcal{D}] - L) + \int_X c_n(T_X - L).
\]

and this proves the result. □

Now, we will prove the Theorem 3.1

Proof. Since \( \mathcal{D} \) is smooth, we can invoke the formula (1) of the Theorem 3.1 to obtain the following equality

\[
\int_X c_n(T_X(-\log \mathcal{D}) - T_{\mathcal{F}}) = \int_X c_n(T_X - T_{\mathcal{F}}) - \int_X c_{n-1}(T_X - [\mathcal{D}] - T_{\mathcal{F}}).
\]

By hypothesis, the one-dimensional foliation \( \mathcal{F} \) is logarithmic along \( \mathcal{D} \) and has only isolated singularities, then it follows from \([22]\) that the top Chern number of
restriction \( (T_X - [D] - T_\mathcal{F})|_D \) coincides with the sum of GSV-Index of \( \mathcal{F} \) along \( D \). That is

\[
\int_D c_n-1(T_X - [D] - T_\mathcal{F}) = \sum_{p \in \text{Sing}(\mathcal{F}) \cap D} GSV(\mathcal{F}, D, p),
\]

Hence

\[
\int_X c_n(T_X - \log D - T_\mathcal{F}) = \int_X c_n(T_X - T_\mathcal{F}) - \sum_{p \in \text{Sing}(\mathcal{F}) \cap D} GSV(\mathcal{F}, D, p)
\]

\[
= \sum_{p \in \text{Sing}(\mathcal{F})} \mu_p(\mathcal{F}) - \sum_{p \in \text{Sing}(\mathcal{F}) \cap D} GSV(\mathcal{F}, D, p),
\]

where in the last step we are using the Baum-Bott classical formula (1). Now, since \( \text{Ind}_{\log D, p}(\mathcal{F}) = 0 \) for all \( p \in \text{Sing}(\mathcal{F}) \cap D \), by Proposition 2.4 we get the following relation

\[
\text{GSV}(\mathcal{F}, D, p) = \mu_p(\mathcal{F}), \forall p \in \text{Sing}(\mathcal{F}) \cap D.
\]

Therefore, we obtain

\[
\sum_{p \in \text{Sing}(\mathcal{F})} \mu_p(\mathcal{F}) = \sum_{p \in \text{Sing}(\mathcal{F}) \cap (X \setminus D)} \mu_p(\mathcal{F}),
\]

and the desired formula is proved. \( \square \)

4. Proof of Theorem 2

In this section we will consider \( D = D_1 \cup \ldots \cup D_N \) an analytic hypersurface on \( X \), with normal crossing singularities. Fixing an irreducible component, say \( D_N \), we define

\[
\mathcal{D}_N := \bigcup_{j=1}^{N-1} D_j \quad \text{and} \quad \mathcal{D}_N|D_N := \bigcup_{j=1}^{N-1} D_j \cap D_N.
\]

We note that \( \mathcal{D}_N|D_N \) is an analytic hypersurface on \( D_N \) with normal crossings singularities and \( N - 1 \) irreducible components. We will use the following multiple index notation: for each multi-index \( J = (j_1, \ldots, j_N) \) and \( J' = (j'_1, \ldots, j'_{N-1}) \), with \( 1 \leq j_i, j'_i \leq n \), we denote

\[
c_1(D)^J = c_1([D_1])^{j_1} \cdots c_1([D_N])^{j_N},
\]

\[
c_1(\mathcal{D}_N)^{J'} = c_1([D_1])^{j'_1} \cdots c_1([D_{N-1}])^{j'_{N-1}}.
\]

Lemma 4.1. In the above conditions, for each \( i = 1, \ldots, n \), we have

\[
c_i(\Omega_X^1(\log D)) = \sum_{k=0}^{n} c_{i-k}(\Omega_X^1) c_1(D)^J.
\]

Proof. Since \( D \) is an analytic hypersurface with normal crossing singularities, the Poincaré residue map

\[
\text{Res} : \Omega_X^1(\log D) \longrightarrow \mathcal{O}_D \cong \bigoplus_{i=1}^{N} \mathcal{O}_{D_i}
\]
induces the following exact sequence

$$0 \rightarrow \Omega^1_X \rightarrow \Omega^1_X(\log D) \xrightarrow{\text{Res}} \bigoplus_{i=1}^N \mathcal{O}_{D_i} \rightarrow 0.$$ 

By using this exact sequence, we get

$$c_i(\Omega^1_X(\log D)) = \sum_{k=0}^i c_{i-k}(\Omega^1_X)c_k\left(\bigoplus_{i=1}^N \mathcal{O}_{D_i}\right)$$

$$= \sum_{k=0}^i c_{i-k}(\Omega^1_X) \left( \sum_{j_1+\ldots+j_N=k} c_{j_1}(\mathcal{O}_{D_1}) \ldots c_{j_N}(\mathcal{O}_{D_N}) \right)$$

$$= \sum_{k=0}^i c_{i-k}(\Omega^1_X) \left( \sum_{j_1+\ldots+j_N=k} c_1([D_1])^{j_1} \ldots c_1([D_N])^{j_N} \right),$$

where in last equality we use the following relations

$$c_i(\mathcal{O}_{D_j}) = c_1([D_j])^i, \quad i = 1, \ldots, n,$$

which can be obtained of \( \mathcal{O} \). \( \square \)

**Lemma 4.2.** In the above conditions, for each \( i = 1, \ldots, n-1 \),

$$c_i(\Omega^1_X)|_{D_N} = c_i(\Omega^1_{D_N}) - c_{i-1}(\Omega^1_{D_N})c_1([D_N])_{|D_N}.$$

**Proof.** It follows from by taking the total Chern class in the exact sequence

$$0 \rightarrow T_{D_N} \rightarrow T_X|_{D_N} \rightarrow [D_N]|_{D_N} \rightarrow 0.$$

**Lemma 4.3.** In the above conditions, if \( L \) is a holomorphic line bundle on \( X \), then the following relations hold:

(10) \[ \int_X c_n(T_X(-\log D) - L) = \sum_{j=0}^n \sum_{k=0}^{n-j} \sum_{|J|=k} \int_X (\text{Res})c_{n-j-k}(\Omega^1_X)c_1(D)^j c_1(L^*)^j. \]

In particular,

(11) \[ \int_X c_n(T_X(-\log \hat{D}_N) - L) = \sum_{j=0}^n \sum_{k=0}^{n-j} \sum_{|J'|=k} \int_X (\text{Res})c_{n-j-k}(\Omega^1_X)c_1(\hat{D}_N)^{j'} c_1(L^*)^{j'}. \]

and

(12) \[ \int_{D_N} c_{n-1}(T_{D_N}(-\log ([D_N]|D_N)) - L|_{D_N}) = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1-j} \sum_{|J'|=k} \int_{D_N} (\text{Res})c_{n-1-j-k}(\Omega^1_{D_N})c_1(\hat{D}_N)^{j'} c_1(L^*)^{j}. \]
Proof. By basics proprieties of Chern classes, we get
\[
\int_X c_n(T_X(-\log D) - L) = \int_X \sum_{j=0}^n c_{n-j}(T_X(-\log D))c_1(L^*)^j
\]
\[
= \int_X \sum_{j=0}^n (-1)^{n-j}c_{n-j}(\Omega_X^1(\log D))c_1(L^*)^j.
\]
By Lemma 4.1 we get
\[
c_{n-j}(\Omega_X^1(\log D)) = \sum_{k=0}^{n-j} \sum_{|J|=k} c_{n-j-k}(\Omega_X^1) c_1(D)^J c_1(L^*)^j.
\]
Substituting this, we obtain (10). The relation (11) is obtained by taking \(D = \hat{D}_N\) in relation (10). Analogously, applying the relation (11), we can obtain (12) by taking \(X = D_N\) as a complex manifold of dimension \(n-1\) and \(D = D_N|D_N\) as an analytic subvariety of \(D_N\) with normal crossings. \(\square\)

**Proposition 4.4.** In the above conditions, if \(L\) is a holomorphic line bundle on \(X\), then
\[
\int_X c_n(T_X(-\log D) - L) = \int_X c_n(T_X(-\log(D_N) - L) - \int_{D_N} c_{n-1}(T_{D_N}(-\log(D_N|D_N) - L|D_N).\]

Proof. By Lemma 4.3, it is sufficient to show that the following equality occurs
\[
\sum_{j=0}^n \sum_{k=0}^{n-j} \sum_{|J|=k} \int_X (-1)^{n-j}c_{n-j-k}(\Omega_X^1) c_1(D)^J c_1(L^*)^j =
\]
\[
\sum_{j=0}^n \sum_{k=0}^{n-j} \sum_{|J'|=k} \int_X (-1)^{n-j}c_{n-j-k}(\Omega_X^1) c_1(D)^{J'} c_1(L^*)^j =
\]
\[
- \sum_{j=0}^{n-1} \sum_{k=0}^{n-1-j} \sum_{|J'|=k} \int_{D_N} (-1)^{n-1-j}c_{n-1-j-k}(\Omega_{D_N}^1) c_1(D)^{J'} c_1(L^*)^j.
\]
Indeed, we can decompose the sum on the left hand side into the terms with \(k = 0\) and those with \(k \geq 1\) as follows:
\[
\sum_{j=0}^n \sum_{k=0}^{n-j} \sum_{|J|=k} \int_X (-1)^{n-j}c_{n-j-k}(\Omega_X^1) c_1(D)^J c_1(L^*)^j =
\]
\[
= \sum_{j=0}^n \int_X (-1)^{n-j}c_{n-j}(\Omega_X^1) c_1(L^*)^j + \sum_{j=0}^{n-1} \sum_{k=1}^{n-j} \sum_{|J|=k} \int_X (-1)^{n-j}c_{n-j-k}(\Omega_X^1) c_1(D)^J c_1(L^*)^j.
\]
By using the fact that we obtain:

$$\sum_{j=0}^{n-1} \sum_{k=1}^{n-j} \sum_{|J|=k} \int_X (-1)^{n-j} c_{n-j-k}(\Omega_X^1 c_1(D))^j c_1(L^*)^j =$$

$$\sum_{j=0}^{n-1} \sum_{k=1}^{n-j} \sum_{|J|=k} \int_X (-1)^{n-j} c_{n-j-k}(\Omega_X^1 c_1(D))^j c_1(L^*)^j +$$

$$+ \sum_{j=0}^{n-1} \sum_{k=1}^{n-j} \sum_{|J|=k} \sum_{j_N \geq 1} \int (\Omega_X^1 c_1([D_L])^{j_1} \ldots c_1([D_N])^{j_N-1} c_1(L^*)^j.$$

By using the fact that $c_1([D_N])$ is Poincaré dual to the fundamental class of $D_N$, we obtain:

$$\sum_{j=0}^{n-1} \sum_{k=1}^{n-j} \sum_{|J|=k} \int_X (-1)^{n-j} c_{n-j-k}(\Omega_X^1 c_1(D))^j c_1(L^*)^j =$$

$$\sum_{j=0}^{n-1} \sum_{k=1}^{n-j} \sum_{|J|=k} \int_X (-1)^{n-j} c_{n-j-k}(\Omega_X^1 c_1(D))^j c_1(L^*)^j +$$

$$+ \sum_{j=0}^{n-1} \sum_{k=1}^{n-j} \sum_{|J|=k} \sum_{j_N \geq 1} \int (\Omega_X^1 c_1([D_L])^{j_1} \ldots c_1([D_N])^{j_N-1} c_1(L^*)^j.$$

Now, using the relation of Lemma 1.2, we get

$$\sum_{j=0}^{n-1} \sum_{k=1}^{n-j} \sum_{|J|=k} \int_D (-1)^{n-j} c_{n-j-k}(\Omega_X^1 c_1([D_L])^{j_1} \ldots c_1([D_N])^{j_N-1} c_1(L^*)^j =$$

$$= \sum_{j=0}^{n-1} \sum_{k=1}^{n-j} \sum_{|J|=k} \int_D (-1)^{n-j} c_{n-j-k}(\Omega_X^1 c_1([D_L])^{j_1} \ldots c_1([D_N])^{j_N-1} c_1(L^*)^j -$$

$$- \sum_{j=0}^{n-1} \sum_{k=1}^{n-j} \sum_{|J|=k} \sum_{j_N \geq 1} \int_D (-1)^{n-j} c_{n-j-k}(\Omega_X^1 c_1([D_L])^{j_1} \ldots c_1([D_N])^{j_N-1} c_1(L^*)^j =$$

$$= \sum_{j=0}^{n-1} \int_D (-1)^{n-j} c_{n-j-1}(\Omega_X^1 c_1(L^*)^j + \sum_{j=0}^{n-1} \sum_{k=1}^{n-j-1} \sum_{|J|=k} \int_D (-1)^{n-j} c_{n-j-k}(\Omega_X^1 c_1(D)^j c_1(L^*)^j =$$

$$= - \sum_{j=0}^{n-1} \sum_{k=0}^{n-j-1} \sum_{|J|=k} \int_D (-1)^{n-1-j} c_{n-1-j-k}(\Omega_X^1 c_1(D)^j c_1(L^*)^j.$$

Hence,
We will prove by induction on the number of irreducible components of $D$. Indeed, if the number of irreducible component of $D$ is 1, then $D$ is smooth. By hypothesis, the singularities of $\mathcal{F}$ are non-degenerate, and thus the theorem follows from Theorem 1.

Let us suppose that for every analytic hypersurface on $X$, satisfying the hypothesis of theorem and having $N - 1$ irreducible components, the formula (3) holds. Let $D$ be an analytic hypersurface on $X$ with $N$ irreducible components, satisfying the hypotheses of the theorem. We will prove that the formula (3) is true for $D$.

We know that $D_N$ is an analytic hypersurface on $X$ and $D_N|D_N$ is an analytic hypersurface on $D_N$, both with normal crossing singularities and having exactly $N - 1$ irreducible components. Moreover, $\mathcal{F}$ and its restriction $\mathcal{F}|D_N$ on $D_N$ are

\[
\sum_{j=0}^{n-1} \sum_{k=0}^{n-j} \sum_{|J|=k}^{n-j} \int_X (-1)^{n-j} c_{n-j-k}(\Omega^1_X) c_1(D)^j c_1(L)^j =
\]

\[
\sum_{j=0}^{n-1} \sum_{k=0}^{n-j} \sum_{|J|=k}^{n-j} \int_X (-1)^{n-j} c_{n-j-k}(\Omega^1_X) c_1(\tilde{D}_N)^j c_1(L)^j - 
\]

\[
- \sum_{j=0}^{n-1} \sum_{k=0}^{n-j} \sum_{|J|=k}^{n-j} \int_{D_N} (-1)^{n-j} c_{n-j-k}(\Omega^1_{D_N}) c_1(\tilde{D}_N)^j c_1(L)^j =
\]

\[
= \sum_{j=0}^{n} \int_X (-1)^{n-j} c_{n-j}(\Omega^1_X) c_1(L)^j + \sum_{j=0}^{n-1} \sum_{k=0}^{n-j} \sum_{|J|=k}^{n-j} \int_X (-1)^{n-j} c_{n-j-k}(\Omega^1_X) c_1(\tilde{D}_N)^j c_1(L)^j - 
\]

\[
- \sum_{j=0}^{n-1} \sum_{k=0}^{n-j} \sum_{|J|=k}^{n-j} \int_{D_N} (-1)^{n-j} c_{n-j-k}(\Omega^1_{D_N}) c_1(\tilde{D}_N)^j c_1(L)^j =
\]

\[
= \sum_{j=0}^{n} \int_X (-1)^{n-j} c_{n-j}(\Omega^1_X) c_1(L)^j + \sum_{j=0}^{n-1} \sum_{k=0}^{n-j} \sum_{|J|=k}^{n-j} \int_X (-1)^{n-j} c_{n-j-k}(\Omega^1_X) c_1(\tilde{D}_N)^j c_1(L)^j - 
\]

\[
- \sum_{j=0}^{n-1} \sum_{k=0}^{n-j} \sum_{|J|=k}^{n-j} \int_{D_N} (-1)^{n-j} c_{n-j-k}(\Omega^1_{D_N}) c_1(\tilde{D}_N)^j c_1(L)^j.
\]

Now, we will prove the Theorem 2:

Proof. We will prove by induction on the number of irreducible components of $D$. Indeed, if the number of irreducible component of $D$ is 1, then $D$ is smooth. By hypothesis, the singularities of $\mathcal{F}$ are non-degenerate, and thus the theorem follows from Theorem 1.
logarithmic along $D_N$ and $\hat{D}_N|D_N$, respectively. Thus, we can use the induction hypothesis and we obtain

$$\sum_{p \in \text{Sing}(\mathcal{F}) \cap (X \setminus \hat{D}_N)} \mu_p(\mathcal{F}) = \int_X c_n(T_X(- \log \hat{D}_N) - T_{\mathcal{F}})$$

and

$$\sum_{p \in \text{Sing}(\mathcal{F}) \cap [D_N \setminus (\hat{D}_N|D_N)]} \mu_p(\mathcal{F}) = \int_{D_N} c_n - (T_D_N(- \log (\hat{D}_N|D_N)) - T_{\mathcal{F}|D_N}).$$

By using the following identity

$$X - D = (X - \hat{D}_N) - [D_N - (\hat{D}_N \cap D_N)],$$

we get

$$\sum_{p \in \text{Sing}(\mathcal{F}) \cap (X \setminus \hat{D}_N)} \mu_p(\mathcal{F}) = \sum_{p \in \text{Sing}(\mathcal{F}) \cap (X \setminus \hat{D}_N)} \mu_p(\mathcal{F}) - \sum_{p \in \text{Sing}(\mathcal{F}) \cap [D_N \setminus (\hat{D}_N|D_N)]} \mu_p(\mathcal{F}).$$

Therefore, by (14) and (15), we get

$$\sum_{p \in \text{Sing}(\mathcal{F}) \cap (X \setminus \hat{D}_N)} \mu_p(\mathcal{F}) = \int_X c_n(T_X(- \log \hat{D}_N) - T_{\mathcal{F}}) - \int_{D_N} c_n - (T_D_N(- \log (\hat{D}_N|D_N)) - T_{\mathcal{F}|D_N}),$$

and we obtain the desired equality by applying the Proposition 4.4. Thus, we prove that the formula (3) is true for $D$ and the proof of the theorem follows by induction.

□

5. Application: A Poincaré-Hopf Type Theorem

In this section we will prove a Poincaré-Hopf type Theorem for non-compact complex manifolds. More precisely, we prove the following:

**Corollary 1** Let $\hat{X}$ be an $n$-dimensional complex manifold such that $\hat{X} = X - D$, where $X$ is an $n$-dimensional complex compact manifold, $D$ is a reduced normal crossing hypersurface on $X$. Let $\mathcal{F}$ be a foliation on $X$ of dimension one given by a global holomorphic vector field, with isolated singularities (non-degenerates) and logarithmic along $D$. Then

$$\chi(\hat{X}) = \sum_{p \in \text{Sing}(\mathcal{F}) \cap \hat{X}} \mu_p(\mathcal{F}),$$

where $\mu_p(\mathcal{F})$ denotes the Milnor number of $\mathcal{F}$ on $p$.

**Proof.** On the one hand, it follows from Norimatsu-Silvotti-Aluffi Theorem that

$$\int_X c_n(T_X(- \log D)) = \chi(\hat{X}).$$

On the other hand, since $D$ is a normal crossing hypersurface, it follows from Theorem 2 that

$$\int_X c_n(T_X(- \log D)) = \sum_{p \in \text{Sing}(\mathcal{F}) \cap \hat{X}} \mu_p(\mathcal{F}).$$
6. Application to one-dimensional projective foliations

In this section we give an optimal description for a smooth hypersurface $D$ invariant by an one-dimensional foliation $\mathcal{F}$ on $\mathbb{P}^n$ satisfying $\text{Sing}(\mathcal{F}) \subseteq D$. More precisely, we will prove the Theorem 3.

Firstly, we need some preliminary results.

**Lemma 6.1.** Let $f(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \left( \frac{n+1}{n-i-j} \right) x^j y^i$, with $n \in \mathbb{N}$.

(i) If $x \neq y$, then $f(x,y) = \frac{(1+x)^{n+1} - (1+y)^{n+1}}{x-y}$.

(ii) If $x = y$, then $f(x,y) = (n+1)(1+x)^n$

**Proof.** Developing the summation in the following triangular format

$$
f(x,y) = \binom{n+1}{n} x + \binom{n+1}{n-2} x^2 + \ldots + \binom{n+1}{0} x^n + \binom{n+1}{n-1} y + \binom{n+1}{n-2} yx + \ldots + \binom{n+1}{0} yx^{n-1} + \binom{n+1}{n-2} y^2 + \ldots + \binom{n+1}{0} y^2 x^{n-2} + \ldots + \binom{n+1}{0} y^n.
$$

We can put in evidence the common factor in each columns and we get the following forma:

$$
f(x,y) = \sum_{k=0}^{n} \left[ \binom{n+1}{n-k} \left( \sum_{j=0}^{k} x^{k-j} y^j \right) \right].
$$

(i) Suppose $x \neq y$. Since

$$x^k + x^{k-1} y + \ldots + xy^{k-1} + y^k = \frac{x^{k+1} - y^{k+1}}{x-y}, \quad 0 \leq k \leq n,$$
we obtain

\[ f(x, y) = \sum_{k=0}^{n} \left( \frac{n+1}{n-k} \right) \left( \frac{x^{k+1} - y^{k+1}}{x-y} \right) \]

\[ = \frac{1}{x-y} \left[ \sum_{k=0}^{n} \left( \frac{n+1}{n-k} \right) x^{k+1} - \sum_{k=0}^{n} \left( \frac{n+1}{n-k} \right) y^{k+1} \right] \]

\[ = \frac{(1+x)^{n+1} - (1+y)^{n+1}}{x-y}, \]

where in the last equality we have used the binomial theorem.

(ii) Consider the case where \( x = y \). By equality (16) we have

\[ f(x, y) = \left( \frac{n+1}{n} \right) + \left( \frac{n+1}{n-1} \right) 2x + \left( \frac{n+1}{n-1} \right) 3x^2 + \ldots + \left( \frac{n+1}{0} \right)(n+1)x^n. \]

Hence, by using the binomial theorem we obtain

\[ f(x, y) = \frac{d}{dx} (1 + x)^{n+1} \]

\[ = (n+1)(1+x)^n. \]

\[ \square \]

**Lemma 6.2.** Let \( k, d \) and \( n \) natural numbers, with \( k \geq 1, \ d \geq 0 \) and \( n \geq 2 \). Consider the natural number \( \delta(k, d, n) \) defined by the relation

\[ \delta(k, d, n) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \left( \frac{n+1}{n-i-j} \right) (-k)^j (d-1)^i. \]

Then \( \delta(k, d, n) \) satisfies the following conditions:

(i) If \( n \) is odd, then:

(a) \( \delta(k, d, n) > 0 \iff k < d + 1; \)

(b) \( \delta(k, d, n) = 0 \iff k = d + 1; \)

(c) \( \delta(k, d, n) < 0 \iff k > d + 1. \)

(ii) If \( n \) is even, then \( \delta(k, d, n) \geq 0 \) and moreover:
(a) \( \delta(k, d, n) > 0 \iff \begin{cases} k \neq d + 1 \\
 or \\
 k = d + 1, \text{ with } d \neq 0 \end{cases} \)

(b) \( \delta(k, d, n) = 0 \iff k = 1 \text{ and } d = 0. \)

(iii) \( \delta(k, d, n) = \sum_{i=0}^{n} (-1)^i (k - 1)^i d^{n-i}. \)

Proof. In order to prove the present lemma, we can consider \( x = -k \) and \( y = d - 1 \) in \( f(x, y) \) of the lemma 6.1. Hence, we obtain

\[
\delta(k, d, n) = \frac{(1 - k)^{n+1} - d^{n+1}}{-k - d + 1}, \text{ if } k \neq 1 \text{ or } d \neq 0
\]

and

\( \delta(k, d, n) = 0, \text{ if } k = 1 \text{ and } d = 0. \)

The proof of items (i) and (ii) can readily be obtained by the study of sign of the expression

\[
\frac{(1 - k)^{n+1} - d^{n+1}}{-k - d + 1}
\]

and also using the relation (17).

Now, let us consider the summation \( \sum_{i=0}^{n} (-1)^i (k - 1)^i d^{n-i}. \) We have:

\[
\sum_{i=0}^{n} (-1)^i (k - 1)^i d^{n-i} = d^n \left[ \sum_{i=0}^{n} (-1)^i (k - 1)^i d^{-i} \right]
\]

\[
= d^n \left[ \sum_{i=0}^{n} \left( \frac{(-1)(k - 1)}{d} \right)^i \right].
\]

By the property

\[
\forall a \in \mathbb{Z}, \quad 1 + a + a^2 + \ldots + a^n = \frac{1 - a^{n+1}}{1 - a}
\]

we get

\[
\sum_{i=0}^{n} (-1)^i (k - 1)^i d^{n-i} = \frac{(k - 1)^{n+1} - d^{n+1}}{-k - d + 1}.
\]

Hence, this proves the equality of item (iii). \( \square \)
Lemma 6.3. Let $D \subset \mathbb{P}^n$ a smooth and irreducible hypersurface of degree $k$. Then, for $l = 1, \ldots, n$, we obtain

$$c_l(T_{\mathbb{P}^n}(-\log D)) = \left[ \sum_{j=0}^{l-1} \binom{n+1}{l-j} (-1)^j k^j \right] c_1(\mathcal{O}_{\mathbb{P}^n}(1))^l. \tag{18}$$

Proof. The formula (18) can be obtained by considering the recursion

$$c_j(T_{\mathbb{P}^n}(-\log D)) = \left( \frac{n+1}{j+1} \right) c_1(\mathcal{O}_{\mathbb{P}^n}(1))^{j+1} - c_j(T_{\mathbb{P}^n}(-\log D))(k c_1(\mathcal{O}_{\mathbb{P}^n}(1))),$$

for $j = 0, \ldots, n-1$, which can be obtained considering the exact sequence (11):

$$0 \rightarrow T_{\mathbb{P}^n}(-\log D) \rightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \rightarrow \mathcal{O}_{\mathbb{P}^n}(k) \rightarrow 0.$$

Now, we will prove the Theorem 3:

Proof. Let $\deg(D) = k$ and $\deg(\mathcal{F}) = d$. On the one hand, we have

$$\int_{\mathbb{P}^n} c_n(T_{\mathbb{P}^n}(-\log D) - T_{\mathcal{F}}) = \sum_{i=0}^{n} \int_{\mathbb{P}^n} c_{n-i}(T_{\mathbb{P}^n}(-\log D)) c_1(T_{\mathcal{F}})^i.$$

Now, by using the formula (18) in each $c_{n-i}(T_{\mathbb{P}^n}(-\log D))$, in the summation above, we obtain

$$\int_{\mathbb{P}^n} c_n(T_{\mathbb{P}^n}(-\log D) - T_{\mathcal{F}}) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{n+1}{n-i-j} (-1)^j k^j \int_{\mathbb{P}^n} c_1(\mathcal{O}_{\mathbb{P}^n}(1))^{n-i} c_1(T_{\mathcal{F}})^i.$$

On the other hand, the tangent bundle $T_{\mathcal{F}}$ of foliation on $\mathbb{P}^n$ is such that $T_{\mathcal{F}} = \mathcal{O}_{\mathbb{P}^n}(1-d)$. Therefore, we obtain $c_1(T_{\mathcal{F}}^*_{\mathbb{P}^n}) = (d-1) c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. Hence,

$$\int_{\mathbb{P}^n} c_n(T_{\mathbb{P}^n}(-\log D) - T_{\mathcal{F}}) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{n+1}{n-i-j} (-1)^j k^j (d-1)^i \int_{\mathbb{P}^n} c_1(\mathcal{O}_{\mathbb{P}^n}(1))^n$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{n+1}{n-i-j} (-k)^j (d-1)^i,$$

where in the last equality we have used the fact that $\int_{\mathbb{P}^n} c_1(\mathcal{O}_{\mathbb{P}^n}(1))^n = 1$.

By hypothesis, the singularities of $\mathcal{F}$ are non-degenerates. Then, the number $\# [\operatorname{Sing}(\mathcal{F}) \cap \mathbb{P}^n \backslash D]$ corresponds to the sum of the numbers of Milnor of the singular points of $\mathcal{F}$ in $\mathbb{P}^n \backslash D$. Moreover, for all $p \in \operatorname{Sing}(\mathcal{F}) \cap D_{\text{reg}}$ we have $\operatorname{Ind}_{\log D,p}(\mathcal{F}) = 0$, since the singularities are non-degenerates. Thus, it follows from Theorem 1 that

$$\# [\operatorname{Sing}(\mathcal{F}) \cap \mathbb{P}^n \backslash D] = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{n+1}{n-i-j} (-k)^j (d-1)^i.$$
Now, the conclusion of proof can readily be obtained by the signal study of $\delta(k, d, n)$ that was done in Lemma 6.2.

Particularly, the items (1b) and (2b) of Theorem 3 characterize the situations in which all the singularities of $\mathcal{F}$ occur in the invariant hypersurface $\mathcal{D}$. We will present optimal examples.

**Example 6.4.** Let $\mathcal{F}$ be the foliation on $\mathbb{P}^3$ induced by the polynomial vector field

$$v = (-z_1^{k-1} - z_2^{k-1} - z_3^{k-1}) \frac{\partial}{\partial z_0} + (z_0^{k-1} - z_2^{k-1} - z_3^{k-1}) \frac{\partial}{\partial z_1} +$$

$$+ (z_0^{k-1} + z_1^{k-1} - z_3^{k-1}) \frac{\partial}{\partial z_2} + (z_0^{k-1} + z_1^{k-1} + z_2^{k-1}) \frac{\partial}{\partial z_3}.$$  

The hypersurface $\mathcal{D} = \{z_0^{k} + z_1^{k} + z_2^{k} + z_3^{k} = 0\}$ is invariant by $\mathcal{F}$. It is not difficult to see that $\text{Sing}(\mathcal{F}) \subset \mathcal{D}$. Note that $\deg(\mathcal{D}) = k$ and $\deg(\mathcal{F}) = k - 1$, according to item (1.b) of the Theorem 3.

**Example 6.5.** Consider the foliation $\mathcal{F}$ induced by the vector field $v = \partial/\partial z_0$. For each $1 \leq i \leq n$, the hypersurface $\mathcal{D}_i = \{z_i = 0\}$ is invariant by $\mathcal{F}$. Moreover, that for all $i = 1, \ldots, n$, we have

$$\text{Sing}(\mathcal{F}) = \{(1 : 0 : \ldots : 0)\} \subset \mathcal{D}_i.$$

Note that we have $\deg(\mathcal{F}) = 0$ and $\deg(\mathcal{D}_i) = 1$, for all $i$. Therefore, if we consider $n$ even, we are in the case of item (2b) of Theorem 3.

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