Escape rate of the Brownian motions on hyperbolic spaces

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(Communicated by Masaki KASHIWARA, M.J.A., March 13, 2017)

Abstract: We discuss the escape rate of the Brownian motion on a hyperbolic space. We point out that the escape rate is determined by using the Brownian expression of the radial part and a generalized Kolmogorov’s test for the one dimensional Brownian motion.

Key words: Escape rate; Brownian motion; hyperbolic space.

1. Introduction. Let $H^d$ be the $d$-dimensional hyperbolic space and $M = \{(X_t)_{t \geq 0}, \{P_x\}_{x \in H^d}\}$ the Brownian motion on $H^d$ generated by the half of the Laplace-Beltrami operator. For a fixed point $o \in H^d$, define $P_t = P_o$ and $R_t = d(o, X_t)$, where $d$ is the distance function of $H^d$. In this note, we show

**Theorem 1.1.** Let $g(t)$ be a positive function on $(0, \infty)$ such that for some $t_0 > 0$, $\sqrt{g(t)}$ is nondecreasing and $g(t)/\sqrt{t}$ is bounded for all $t \geq t_0$.

(i) For the function $r_1(t) := (d-1)t/2 + \sqrt{t}g(t)$,

\[ P(\text{there exists } T > 0 \text{ such that } R_t < r_1(t) \text{ for all } t \geq T) = 1 \text{ or } 0 \]

according as

\[ \int_0^\infty \left(1 + g(t)e^{-g(t)/\sqrt{t}}\right) dt < \infty \text{ or } \infty. \]

(ii) For the function $r_2(t) := (d-1)t/2 - \sqrt{t}g(t)$,

\[ P(\text{there exists } T > 0 \text{ such that } R_t > r_2(t) \text{ for all } t \geq T) = 1 \text{ or } 0 \]

according as (1.2) holds.

The function $r_1(t)$ is called an upper rate function for $M$ if the probability in (1.1) is 1. By the same way, the function $r_2(t)$ is called a lower rate function for $M$ if the probability in (1.3) is 1. According to Theorem 1.1, we have for $c > 0$,

- the function $r(t) := (d-1)t/2 + \sqrt{ct\log\log t}$ is an upper rate function for $M$ if and only if $c > 2$;
- the function $r(t) := (d-1)t/2 - \sqrt{ct\log\log t}$ is a lower rate function for $M$ if and only if $c > 2$.

For the Brownian motions on Riemannian manifolds, more generally symmetric diffusion processes generated by regular Dirichlet forms, upper and lower rate functions are given in terms of volume growth rate ([1–4,6,11]). As for the upper rate functions, the results in 2–6,11] are applicable to the Brownian motions on Riemannian manifolds with exponential volume growth rate, as to $M$; however, as for the lower rate functions, the results in 1–3] are not applicable to $M$ because the doubling condition is imposed on the volume growth. Grigor’yan and Hsu [4] also discussed the sharpness of the upper rate functions for $M$ or for the Brownian motion on a model manifold, that is, a spherically symmetric Riemannian manifold with a pole. Using the fact that

\[ \lim_{t \to \infty} \frac{R_t}{t} = \frac{d-1}{2}, \quad \text{P-a.s.} \]

(which follows from (2.2) below), they remarked that the function $r(t) = ct$ is an upper rate function for $M$ if $c > (d-1)/2$, and not if $0 < c < (d-1)/2$. This observation is still valid for the lower rate functions. See also [7] for the result of the law of the iterated logarithm-type to the Brownian motions on model manifolds.

For the proof of Theorem 1.1, we make use of the Brownian expression of the radial part $R_t$ (2.2 below) as in [4,7], together with a generalized version of Kolmogorov’s test for the one dimensional Brownian motion ([9,10]). In fact, the integral in (1.2) is the same with that in this test. The assumption on $g(t)/\sqrt{t}$ will be needed in (2.7) and (2.8) below.

2. Proof of Theorem 1.1. Let $B = \{(B_t)_{t \geq 0}, P\}$ be the one dimensional Brownian motion starting from the origin. Then a generalized
Kolmogorov’s test holds:

**Theorem 2.1** ([9, Theorem 3.1 and Lemma 3.3] and [10, Theorem 2.1]). Under the full conditions of Theorem 1.1,

(2.1) \[ P \left( \text{there exists } T > 0 \text{ such that } |B_t| < \sqrt{\epsilon}g(t) \text{ for all } t \geq T \right) = 1 \text{ or 0} \]

according as (1.2) holds. This assertion is valid even if \(|B_t|\) in the equality above is replaced by \(B_t - B_1\).

By comparison with Kolmogorov’s test (see, e.g., [8, 4.12]), we do not need to assume that \(g(t) \not\to \infty \text{ as } t \to \infty\) in Theorem 2.1.

**Proof of Theorem 1.1.** Recall that \(M = (\{X_t\}_{t \geq 0}, P)\) is the Brownian motion on \(H^d\) starting from a fixed point \(o \in H^d\) and \(R_t = d(o, X_t)\) is the radial part of \(X_t\). Then by [5, Example 3.3.3],

(2.2) \[ R_t = B_t + \frac{d - 1}{2} \int_0^t \coth R_s \, ds. \]

Assume the full conditions of Theorem 1.1. We first discuss the lower bound of \(R_t\). Since coth \(x \geq 1\) for any \(x > 0\), we obtain by (2.2),

(2.3) \[ R_t \geq B_t + \frac{d - 1}{2} t \quad \text{for any } t \geq 0. \]

Hence if the integral in (1.2) is convergent, then the probability in (1.3) is 1 by Kolmogorov’s test. By the same way, if the integral in (1.2) is divergent, then the probability in (1.1) is 0. \(\square\)

We next discuss the upper bound of \(R_t\). Since \(B_t = o(t)\) as \(t \to \infty\), we see by (2.3) that there exists \(c > 0\) such that \(P(A) = 1\) for

(2.4) \[ A := \{ \text{there exists } T_1 > 0 \text{ such that } R_t \geq ct \text{ for all } t \geq T_1 \}. \]

Under the event \(A\),

\[ \coth R_s - 1 = \frac{2}{e^{2R_s} - 1} \leq \frac{2}{e^{2s} - 1} \]

for any \(s \geq T_1\), which implies that for all \(t \geq T_1\),

(2.5) \[ \int_0^t \left( \coth R_s - 1 \right) \, ds \leq \int_0^{T_1} \left( \coth R_s - 1 \right) \, ds + \int_{T_1}^t \left( \coth R_s - 1 \right) \, ds \leq \frac{2}{e^{2s} - 1} \, ds =: C_T. \]

Since there exists an integer valued random variable \(N\) such that

(2.6) \[ R_t = B_t + \frac{d - 1}{2} t + \frac{d - 1}{2} \int_0^t \left( \coth R_s - 1 \right) \, ds \leq B_t + \frac{d - 1}{2} t + \frac{d - 1}{2} C_T \]

for all \(t \geq T_1\).

Assume first that the integral in (1.2) is convergent. Then there exists a positive constant \(c_n\) for each \(n \geq 1\) such that the function \(h_1^{(n)}(t) := g(t) - n/\sqrt{t}\) satisfies

(2.7) \[ \int_{B_1}^{B_2} \left( 1 \vee h_1^{(n)}(t) \right) e^{-h_1^{(n)}(t)^2/2} \, dt \leq c_n \int_{B_1}^{B_2} \left( 1 \vee g(t) \right) e^{-g(t)^2/2} \, dt < \infty. \]

Hence Theorem 2.1 implies that for each \(n \geq 1\),

\[ P \left( \text{there exists } T > 0 \text{ such that } |B_t| < r_1^{(n)}(t) \text{ for all } t \geq T \right) = 1 \]

for \(r_1^{(n)}(t) := \sqrt{h_1^{(n)}(t)}(= \sqrt{\epsilon}g(t) - n)\). In particular, we get \(P(B_1) = 1\) for

\[ B_1 := \{ \text{for each } n \geq 1, \text{ there exists } S_n > 0 \text{ such that } |B_t| < r_1^{(n)}(t) \text{ for all } t \geq S_n \}. \]

Under the event \(A \cap B_1\), since there exists \(T_2 > 0\) for \(N \geq 1\) in (2.5) such that

\[ B_t < r_1^{(N)}(t) = \sqrt{\epsilon}g(t) - N \quad \text{for all } t \geq T_2, \]

we have by (2.6),

\[ R_t < \frac{d - 1}{2} t + \sqrt{\epsilon}g(t) \quad \text{for all } t \geq T_1 \vee T_2. \]

Therefore, the probability in (1.1) is 1.

Assume next that the integral in (1.2) is divergent. Then by the same way as in (2.7), the function \(h_2^{(n)}(t) := g(t) + n/\sqrt{t}\) satisfies for each \(n \geq 1\),

(2.8) \[ \int_{B_1}^{B_2} \left( 1 \vee h_2^{(n)}(t) \right) e^{-h_2^{(n)}(t)^2/2} \, dt = \infty. \]

Hence Theorem 2.1 yields that for each \(n \geq 1\),

\[ P \left( \text{for any } t > 0, \text{ there exists } T \geq t \text{ such that } B_T \leq -r_2^{(n)}(T) = 1 \right) = 1 \]

for each \(n \geq 1\) in (2.6).
for $r_2^{(n)}(t) := \sqrt{t} h_2^{(n)}(t)(= \sqrt{t} g(t) + n)$. In particular, $P(B_2) = 1$ for

$$B_2 := \{ \text{for each } n \geq 1, \text{ there exists } U_n \geq t$$
$$\text{for any } t > 0 \text{ such that } B_{U_n} \leq -r_2^{(n)}(U_n) \}. $$

Under the event $A \cap B_2$, since there exists $T_3 \geq t \vee T_1$ for any $t > 0$ and $N \geq 1$ in (2.5) such that

$$B_{T_3} \leq -r_2^{(N)}(T_3) = -\sqrt{T_3} g(T_3) - N,$$

we have for such $T_3$,

$$R_{T_3} \leq \frac{d - 1}{2} T_3 - \sqrt{T_3} g(T_3)$$

by (2.6). Therefore, the probability in (1.3) is 0. □

Acknowledgements. The author would like to thank Prof. Masayoshi Takeda for his valuable discussion motivating this work and his comment on the draft of this paper. This work was supported in part by the Grant-in-Aid for Scientific Research (C) 26400135.

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