Evaluating moments of length of Pitman partition

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Abstract

The Pitman sampling formula has been intensively studied as a distribution of random partitions. One of the objects of interest is the length $K(= K_{n, \theta, \alpha})$ of a random partition that follows the Pitman sampling formula, where $n \in \mathbb{N}$, $\alpha \in (0, \infty)$ and $\theta > -\alpha$ are parameters. This paper presents asymptotic evaluations of its $r$-th moment $\mathbb{E}[K^r]$ ($r = 1, 2, \ldots$) under two asymptotic regimes. In particular, the goals of this study are to provide a finer approximate evaluation of $\mathbb{E}[K^r]$ as $n \to \infty$ than has previously been developed and to provide an approximate evaluation of $\mathbb{E}[K^r]$ as the parameters $n$ and $\theta$ simultaneously tend to infinity with $\theta/n \to 0$. The results presented in this paper will provide a more accurate understanding of the asymptotic behavior of $K$.

Keywords: Pitman $\alpha$-diversity; Pitman sampling formula; random partition.

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1 Introduction

Let $n$ be a positive integer, and let $c_1, \ldots, c_n$ be the component counts of an integer partition of $n$, which means that the partition has $c_i$ parts of size $i$ ($i = 1, \ldots, n$). Note that $c_1, \ldots, c_n$ satisfy $n = \sum_{i=1}^{n} ic_i$. Define a set $\mathcal{P}_n = \{ (c_1, \ldots, c_n) \in (\mathbb{N} \cup \{0\})^n; \sum_{i=1}^{n} ic_i = n \}$. Let $\alpha \in [0, 1)$, $\theta \in (-\alpha, \infty)$. A $\mathcal{P}_n$-valued random variable $C = (C_1^{\alpha, \theta}, \ldots, C_n^{\alpha, \theta})$ is component counts of a Pitman partition when its distribution is given by

$$
\mathbb{P}(C = (c_1, \ldots, c_n)) = n! \cdot \frac{(\theta)_{k; \alpha}}{(\theta)_n} \prod_{i=1}^{n} \left( \frac{(1 - \alpha)_{i-1}}{i!} \right)^{c_i} \frac{1}{c_i!} \quad ((c_1, \ldots, c_n) \in \mathcal{P}_n),
$$

(1.1)

where $k = \sum_{i=1}^{n} c_i$,

$$(x)_i y = \begin{cases} 1 & (i = 0) \\ x(x+y)(x+2y) \cdots (x+(i-1)y) & (i = 1, 2, \ldots) \\ x > -y, y \geq 0, \\ \end{cases}
$$

and $(x)_i = (x)_{i;1}$ ($i = 0, 1, 2, \ldots; x > -1$). The distribution (1.1), called the Pitman sampling formula, was introduced by Jim Pitman; see, for example, [13, 16]. A special case $\alpha = 0$ of (1.1) is known as the Ewens sampling formula, which was introduced by Ewens [5] in the context of population genetics. Henceforth, unless otherwise mentioned, we consider the case $\alpha \neq 0$. The distribution of the length $K(= K_{n, \theta, \alpha}) = \sum_{i=1}^{n} C_i^{\alpha, \theta}$ of a Pitman partition is given by

$$
\mathbb{P}(K = k) = \frac{c(n, k; \alpha)}{\alpha^k} \frac{(\theta)_{k; \alpha}}{(\theta)_n} \quad (k = 1, \ldots, n),
$$

(1.2)

where $c(n, k, \alpha) = (-1)^{n-k} C(n, k, \alpha)$ and $C(n, k, \alpha)$ is the generalized Stirling number or the C-number of Charalambides and Singh [2]; see, for example, Yamato, Sibuya and Nomachi [22]. This paper discusses an asymptotic property of $K$.

Random partition models have been received considerable attention, not only because they are interesting as mathematical models, but also because they relate to broad scientific fields; see, for example, Crane [3] and Johnson, Kotz and Balakrishnan [11] (Chapter 41, its write-up was provided by S. Tavaré and W.J. Ewens). As a typical distribution of random partition models, the distribution (1.1) has been intensively studied. In particular, the properties of (1.2) have been investigated by Dolera and Favaro [4], Favaro, Feng and Gao [6], Feng and Hoppe [9],

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Pitman [14, 15], Yamato and Sibuya [21], and Yamato, Sibuya and Nomachi [22]. A prominent limit theorem associated with (1.2) is

\[
\frac{K}{n^\alpha} \Rightarrow S_{\alpha, \theta}
\]

as \( n \to \infty \), where \( \Rightarrow \) denotes convergence in distribution and the limit random variable \( S_{\alpha, \theta} \), called the Pitman \( \alpha \)-diversity, is a continuous random variable whose distribution has the density

\[
\frac{\Gamma(\theta + 1)}{\Gamma(\frac{\theta}{\alpha} + 1)} x^{\frac{\theta}{\alpha} - 1} g_{\alpha}(x) \quad (x > 0),
\]

where \( g_{\alpha}(\cdot) \) is the function satisfying

\[
\int_0^\infty x^p g_{\alpha}(x) dx = \frac{\Gamma(p + 1)}{\Gamma(p\alpha + 1)} \quad (p > -1);
\]

see, for example, Pitman [14, 15] or Yamato and Sibuya [21]. Note that \( g_{\alpha}(x) \) can be written as

\[
g_{\alpha}(x) = \frac{1}{\pi\alpha} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i!} \Gamma(i\alpha + 1) x^{i-1} \sin(\pi i\alpha) \quad (x > 0).
\]

A summary of the proof of (1.3) given by Yamato and Sibuya [21] is as follows:

1. For \( r = 1, 2, \ldots \), it holds that

\[
E[K^r] = \sum_{i=0}^{r} (-1)^{r-i} \left(1 + \frac{\theta}{\alpha}\right) R\left(r, i, \frac{\theta}{\alpha}\right) \frac{\Gamma(\theta + i\alpha + n)}{\Gamma(\theta + n)} \frac{\Gamma(\theta + 1)}{\Gamma(\theta + i\alpha + 1)},
\]

where \( R(\cdot, \cdot, \cdot) \) is the weighted Stirling number of the second kind introduced by Carlitz [1].

2. Using expression (1.4), the Stirling formula yields

\[
E\left[\left(\frac{K}{n^\alpha}\right)^r\right] = \left(1 + \frac{\theta}{\alpha}\right) \frac{\Gamma(\theta + 1)}{\Gamma(\theta + r\alpha + 1)} + o(1) \quad (r = 1, 2, \ldots)
\]

as \( n \to \infty \).

3. Hence, (1.3) follows from the method of moments.

In this paper, the moments \( E[K^r] (r = 1, 2, \ldots) \) are investigated in more detail than (1.5). In particular, there are two goals in this paper: the first is to provide an approximate evaluation of \( E[K^r] \) as \( n \to \infty \) that is finer than that of (1.5); the second is to provide an approximate evaluation of \( E[K^r] \) as the parameters \( n \) and \( \theta \) simultaneously tend to infinity with \( \theta/n \to 0 \). The phenomenon described by (1.3) is attractive, and our results will provide a more accurate understanding of (1.3).

Remark 1. There are some known results stronger than (1.3). In particular,

- \( K/n^\alpha \) converges to \( S_{\alpha, \theta} \) almost surely and in \( p \)-th mean for any \( p > 0 \).
- A Berry–Esseen-type theorem holds: When \( \alpha \in (0, 1) \), \( \theta > 5 \), \( \sup_{x \geq 0} |P(K/n^\alpha \leq x) - P(S_{\alpha, \theta} \leq x)| \leq C(\alpha, \theta)/n^\alpha \) holds for \( n \in \mathbb{N} \), where \( C(\alpha, \theta) \) is a constant depending only on \( \alpha \) and \( \theta \).

For details, see Dolera and Favaro [4] and Pitman [16].

2 Asymptotic regime

In this paper, we consider two asymptotic regimes. The first is \( n \to \infty \) with fixed \( \theta \), and the second is

\[
n \to \infty, \theta \to \infty, \frac{\theta}{n} \to 0.
\]

The former \( (n \to \infty \) with fixed \( \theta \) has been frequently considered. When \( \alpha = 0 \), (2.1) has also been considered in some studies; see Remark 2 below. However, when \( \alpha \neq 0 \), (2.1) has not been considered, although it also seems natural. Indeed, in the application of (1.1) to microdata risk assessment by Hoshino [10], the estimates of \( \theta \) take large values (e.g., 523, \ldots, 21298, where \( n = 27320 \)); see Tables 3–6 of [10]. Throughout the paper, we assume that \( \theta > 0 \) when considering the regime (2.1).
Remark 2. When \( \alpha = 0 \), the asymptotic regime in which \( n \) and \( \theta \) simultaneously tend to infinity has been considered by Feng [7] and Tsukuda [18, 19, 20]. In particular, (2.1) is Case D in Section 4 of [7].

Remark 3. For the Pitman sampling formula, the asymptotic regime \( \theta \to \infty \) with fixed \( n \) was considered by Kerov [12]. Moreover, Feng [8] considered the asymptotic regime \( \theta \to \infty \) in studying the Pitman–Yor process, which is closely related to the Pitman sampling formula. For details of the Pitman–Yor process, see, for example, Pitman and Yor [17].

3 Results

3.1 Asymptotic evaluation as \( n \to \infty \) with fixed \( \theta \)

First, we provide an evaluation that is finer than that of (1.5) under the asymptotic regime \( n \to \infty \) with fixed \( \theta \).

Theorem 3.1. Suppose that \( \alpha > 0 \). For \( r = 1, 2, \ldots \), it holds that

\[
\mathbb{E} \left[ \left( \frac{K}{n^\alpha} \right)^r \right] = \left( 1 + \frac{\theta}{\alpha} \right)_r \frac{\Gamma(\theta + 1)}{\Gamma(\theta + r\alpha + 1)} \left[ 1 - \left( \frac{r(r - 1)\alpha}{2} + r\theta \right) \frac{\Gamma(\theta + r\alpha)}{\Gamma(\theta + (r - 1)\alpha + 1) n^\alpha} \right] + O \left( \frac{1}{n^{\alpha + 1}} \right)
\]

as \( n \to \infty \).

Proof. It follows from (1.4) that

\[
\mathbb{E} [K^r] = \sum_{i=0}^{r-2} (-1)^{r-i} \left( 1 + \frac{\theta}{\alpha} \right)_i R \left( r, i, \frac{\theta}{\alpha} \right) \frac{\Gamma(\theta + i\alpha + n)}{\Gamma(\theta + n)} \frac{\Gamma(\theta + 1)}{\Gamma(\theta + (i\alpha + 1) n^\alpha)}
\]

\[
- \left( 1 + \frac{\theta}{\alpha} \right)_{r-1} R \left( r, r-1, \frac{\theta}{\alpha} \right) \frac{\Gamma(\theta + (r - 1)\alpha + n)}{\Gamma(\theta + n)} \frac{\Gamma(\theta + 1)}{\Gamma(\theta + (r - 1)\alpha + 1) n^\alpha}
\]

\[
+ \left( 1 + \frac{\theta}{\alpha} \right)_r R \left( r, r, \frac{\theta}{\alpha} \right) \frac{\Gamma(\theta + r\alpha + n)}{\Gamma(\theta + n)} \frac{\Gamma(\theta + 1)}{\Gamma(\theta + r\alpha + 1)}.
\]

(3.1)

where \( \sum_{i=0}^{r-1} a_i = 0 \) for any sequence \( \{a_1, a_2, \ldots \} \). As

\[
\Gamma(x) = \sqrt{2\pi e^{-x}x^{x-1/2}} \left( 1 + \frac{1}{12x} + O \left( \frac{1}{x^2} \right) \right) \quad (x \to \infty),
\]

(3.2)

it holds that

\[
\frac{\Gamma(\theta + i\alpha + n)}{\Gamma(\theta + n)} = n^\alpha \left[ 1 + \frac{i\alpha\{n(2\theta + i\alpha - 1) + 2\theta^2\}}{2n(n + \theta)} + O \left( \frac{1}{n^2} \right) \right] = n^\alpha \left( 1 + O \left( \frac{1}{n} \right) \right)
\]

for \( i = 0, 1, \ldots \). The first term on the right-hand side of (3.1) is \( O(n^{(r-2)\alpha}) \). The second term on the right-hand side of (3.1) is

\[
- \left( 1 + \frac{\theta}{\alpha} \right)_{r-1} \left\{ \frac{r(r - 1)}{2} + \frac{r\theta}{\alpha} \right\} \frac{\Gamma(\theta + 1)}{\Gamma(\theta + (r - 1)\alpha + 1) n^{(r-1)\alpha}} \left( 1 + O \left( \frac{1}{n} \right) \right)
\]

\[
= - \left( 1 + \frac{\theta}{\alpha} \right)_r \frac{\Gamma(\theta + 1)}{\Gamma(\theta + r\alpha + 1)} \left\{ \frac{r(r - 1)}{2} + \frac{r\theta}{\alpha} \right\} \frac{\Gamma(\theta + 1)}{\Gamma(\theta + (r - 1)\alpha + 1) n^{(r-1)\alpha}} + O \left( n^{(r-1)\alpha-1} \right)
\]

\[
= - \left( 1 + \frac{\theta}{\alpha} \right)_r \frac{\Gamma(\theta + 1)}{\Gamma(\theta + r\alpha + 1)} \left\{ \frac{r(r - 1)}{2} + \frac{r\theta}{\alpha} \right\} \frac{\Gamma(\theta + 1)}{\Gamma(\theta + (r - 1)\alpha + 1) n^{(r-1)\alpha}}
\]

because (3.2) of Carlitz [1] implies that

\[
R \left( r, r - 1, \frac{\theta}{\alpha} \right) = \sum_{i=0}^{r} \left( \frac{\theta}{\alpha} \right)_i S_2(r - i, r - 1) = S_2(r, r - 1) + \frac{r\theta}{\alpha} \frac{r(r - 1)}{2} + \frac{r\theta}{\alpha} \frac{1}{2}.
\]
where \( S_2(\cdot, \cdot) \) is the Stirling number of the second kind. The third term on the right-hand side of (3.1) is
\[
\left(1 + \frac{\theta}{\alpha}\right) \frac{\Gamma(\theta + 1)}{\Gamma(\theta + ra + 1)} n^{ra} + O \left(n^{ra-1}\right).
\]
As \( \alpha \in (0, 1) \), we have that
\[
O(n^{(r-2)\alpha}) + O \left(n^{(r-1)\alpha-1}\right) + O \left(n^{ra-1}\right) = O \left(n^{(r-2)\alpha} + n^{ra-1}\right).
\]
Therefore,
\[
E \left[ \left(\frac{K}{n^\alpha}\right)^r \right] = \left(1 + \frac{\theta}{\alpha}\right) \frac{\Gamma(\theta + 1)}{\Gamma(\theta + ra + 1)}
- \left(1 + \frac{\theta}{\alpha}\right) \frac{\Gamma(\theta + 1)}{\Gamma(\theta + ra + 1)} \left\{ \frac{r(r-1)\alpha}{2} + r\theta \right\} \frac{\Gamma(\theta + ra)}{\Gamma(\theta + (r-1)\alpha + 1) n^\alpha}
+ O \left(\frac{1}{n^{2\alpha}} + \frac{1}{n}\right).
\]
This completes the proof.

**Remark 4.** When \( r > 3 \), as \( \theta > -\alpha \), it holds that
\[
\frac{r(r-1)\alpha}{2} + r\theta \frac{\Gamma(\theta + ra)}{\Gamma(\theta + (r-1)\alpha + 1) n^\alpha} > 0.
\]
This means that almost all moments of \( K/n^\alpha \) are smaller than those of \( S_{\alpha,\theta} \) for large \( n \). In particular, if \( \theta > 0 \), then all moments of \( K/n^\alpha \) are smaller than those of \( S_{\alpha,\theta} \) for large \( n \).

In some cases, correcting some moments improves the quality of an approximation. Thus, a primitive application of Theorem 3.1 is replacing \( K/n^\alpha \) in (1.3) by
\[
\frac{K}{n^\alpha - \frac{\theta \Gamma(\theta + \alpha)}{\Gamma(\theta + 1)}}
\]
whose expectation is
\[
E \left[ \left(\frac{K}{n^\alpha - \frac{\theta \Gamma(\theta + \alpha)}{\Gamma(\theta + 1)}}\right)^r \right] = \frac{\Gamma(\theta + 1)}{\alpha \Gamma(\theta + \alpha)} + O \left(\frac{1}{n^{2\alpha}} + \frac{1}{n}\right).
\]
When \( \theta > 0 \), this correction enlarges \( K/n^\alpha \), and is consistent with Remark 4.

### 3.2 Asymptotic evaluation under (2.1)

Next, under the asymptotic regime of (2.1), we provide a new evaluation.

**Theorem 3.2.** Suppose that \( \alpha > 0 \). For \( r = 1, 2, \ldots \), it holds that
\[
E \left[ \left(\frac{\alpha K}{\theta \left\{\frac{n^\alpha - \theta}{\theta}\right\}^{\alpha - 1}}\right)^r \right] = 1 + O \left(\frac{\theta^{2\alpha}}{n^{2\alpha}} + \frac{\theta}{n} + \frac{1}{\theta}\right)
\]
under the asymptotic regime of (2.1); in particular, for \( r = 1, 2, \ldots \), under the asymptotic regime of (2.1), if
\[
\frac{\theta^{2\alpha+1}}{n^{2\alpha}} \to 0 \quad \text{and} \quad \frac{\theta^2}{n} \to 0,
\]
then
\[
E \left[ \left(\frac{\alpha K}{\theta \left\{\frac{n^\alpha - \theta}{\theta}\right\}^{\alpha - 1}}\right)^r \right] = 1 + r^2 \alpha(1-\alpha) \frac{2\theta}{2\theta} + O \left(\frac{\theta^{2\alpha}}{n^{2\alpha}} + \frac{\theta}{n} \left(\frac{\theta^\alpha}{n^\alpha} + \frac{1}{\theta}\right)\right).
\]
Lemma 3.3. Suppose that \( \alpha > 0 \). For \( r = 1, 2, \ldots, \) it holds that
\[
E \left[ \alpha K \genfrac{[}{]}{0pt}{}{\theta}{\theta(\frac{r}{n})^\alpha} \right] = 1 - r \left( \frac{\theta}{n} \right)^\alpha + r^2 \alpha (1 - \alpha) + O \left( \frac{\theta^2}{n^2} + \frac{\theta}{n} \right)
\]
deriving under the asymptotic regime of (2.1).

Proof. It follows from (1.4) that
\[
E[K^r] = \sum_{i=0}^{r-2} (-1)^{r-i} \left( 1 + \frac{\theta}{\alpha} \right) R \left( r, i, \frac{\theta}{\alpha} \right) \frac{\Gamma(\theta + i \alpha + n) \Gamma(\theta + 1)}{\Gamma(n) \Gamma(\theta + i \alpha + 1)}
\]
\[
- \left( 1 + \frac{\theta}{\alpha} \right)^{r-1} \left\{ \frac{r(r-1)}{2} + \frac{r}{\alpha} \right\} \frac{\Gamma(\theta + (r-1)\alpha + n) \Gamma(\theta + 1)}{\Gamma(n) \Gamma(\theta + (r-1)\alpha + 1)}
\]
\[
+ \left( 1 + \frac{\theta}{\alpha} \right) \frac{\Gamma(\theta + r \alpha + n) \Gamma(\theta + 1)}{\Gamma(n) \Gamma(\theta + r \alpha + 1)}
\]
\[
= \sum_{i=0}^{r-2} (-1)^{r-i} R \left( r, i, \frac{\theta}{\alpha} \right) \frac{\Gamma(\theta + i + 1) \Gamma(\theta + n + i \alpha) \Gamma(\theta + 1)}{\Gamma(n) \Gamma(\theta + 1 + i \alpha)}
\]
\[
- \left\{ \frac{r(r-1)}{2} + \frac{r}{\alpha} \right\} \frac{\Gamma(\theta + r) \Gamma(\theta + n + (r-1)\alpha) \Gamma(\theta + 1)}{\Gamma(n) \Gamma(\theta + (r-1)\alpha + 1)}
\]
\[
+ \frac{\Gamma(\theta + 1 + r) \Gamma(\theta + n + r \alpha) \Gamma(\theta + 1)}{\Gamma(n) \Gamma(\theta + 1 + r \alpha)}
\]
As \( R(r, i, \theta/\alpha) = O(\theta^{r-i}) \) for \( i = 0, 1, \ldots, r - 2 \), which follows from (3.2) of [1], according to Lemma 4.1, the first term is
\[
O \left( \theta^2 \left( \frac{n^\alpha}{\theta} \right)^{r-2} \right) = O \left( \left( \frac{n^\alpha}{\theta} \right)^{r-2} \right)
\]
that stems from the term of \( i = r - 2 \). Moreover, using Lemma 4.1 again, the second term is
\[
- \left( O(1) + \frac{r \theta}{\alpha} \right) \left( \frac{n^\alpha}{\theta} \right)^{r-1} \left\{ 1 + O \left( \frac{\theta}{n} \right) \right\}
\]
\[
= - \left( \frac{n^\alpha}{\theta} \right)^r \left\{ \frac{\theta}{n} \right\} + O \left( \frac{n^{\alpha+1}}{\theta^2 + \frac{n^\alpha}{\theta}} \right)
\]
and the third term is
\[
\left( \frac{n^\alpha}{\theta} \right)^r \left\{ 1 + \frac{r \alpha}{n} + \frac{r^2 \alpha (1 - \alpha)}{2 \theta} + O \left( \frac{1}{\theta^2} + \frac{\theta}{n} \right) \right\}
\]
These formulae yield
\[
E[K^r] = \left( \frac{n^\alpha}{\theta} \right)^r \left\{ 1 - r \left( \frac{\theta}{n} \right) + \frac{r^2 \alpha (1 - \alpha)}{2 \theta} + O \left( \frac{n^{2 \alpha}}{\theta^2} + \frac{1}{n^{\alpha}} + \frac{1}{\theta^2} + \frac{\theta}{n} \right) \right\}
\]
where
\[
O \left( \frac{n^{2 \alpha}}{\theta^2} + \frac{1}{n^{\alpha}} + \frac{1}{\theta^2} + \frac{\theta}{n} \right)
\]
is used. Thus, it holds that
\[
E \left[ \frac{\alpha K \theta}{\theta(\frac{n}{\theta})^\alpha} \right] = 1 - r \left( \frac{\theta}{n} \right) + \frac{r^2 \alpha (1 - \alpha)}{2 \theta} + O \left( \frac{n^{2 \alpha}}{\theta^2} + \frac{1}{n^{\alpha}} + \frac{1}{\theta^2} + \frac{\theta}{n} \right)
\]
This completes the proof.
Using Lemma 3.3, we prove Theorem 3.2.

**Proof of Theorem 3.2.** It follows from

\[
\frac{(\frac{n+\theta}{\theta})^\alpha - 1}{(\frac{n}{\theta})^\alpha} = \left(1 + \frac{\theta}{n}\right)^\alpha - \left(\frac{\theta}{n}\right)^\alpha = 1 - \left(\frac{\theta}{n}\right)^\alpha + O\left(\frac{\theta}{n}\right)
\]

that

\[
\left\{\left(\frac{\theta}{n}\right)^\alpha - \left(\frac{(n+\theta)}{\theta(n+\theta))^\alpha - 1}\right)^r = 1 + r\left(\frac{\theta}{n}\right)^\alpha + O\left(\frac{\theta^2 + \theta}{n^{2\alpha} + \frac{n}{\theta}}\right).
\]

(3.3)

Lemma 3.3 and (3.3) yield

\[
E\left[\left(\frac{\alpha K}{\theta (\frac{n+\theta}{\theta})^\alpha - 1}\right)^r\right] = E\left[\left(\frac{\alpha K}{\theta (n+\theta)^\alpha - 1}\right)^r\right]
\]

\[
= 1 + r\left(\frac{\theta}{n}\right)^\alpha + O\left(\frac{\theta^2}{n^{2\alpha} + \frac{n}{\theta}}\right) + \left\{1 - r\left(\frac{\theta}{n}\right)^\alpha + r^2\frac{(1-\alpha)}{2\theta} + O\left(\frac{\theta^2}{n^{2\alpha} + \frac{n}{\theta}}\right)\right\}
\]

\[
= 1 + r^2\frac{(1-\alpha)}{2\theta} + O\left(\frac{\theta^2}{n^{2\alpha} + \frac{n}{\theta}}\right).
\]

It implies the desired conclusion. This completes the proof. \(\square\)

**Corollary 3.4.** Suppose that \(\alpha > 0\). It holds that

\[
\frac{\alpha K}{\theta (\frac{n+\theta}{\theta})^\alpha - 1} \to^p 1
\]

under the asymptotic regime of (2.1), where \(\to^p\) denotes convergence in probability.

**Proof.** Using Theorem 3.2, the first and second moments of the left-hand side in (3.4) converge to 1. This completes the proof. \(\square\)

This corollary shows that, under the asymptotic regime of (2.1), \(K\) may asymptotically behave as if \(\alpha\) was 0 from the perspective of the following remark.

**Remark 5.** When \(\alpha = 0\), it is known that

\[
\frac{K_{n,\theta,0}}{\theta \log(1 + \frac{n}{\theta})} \to^p 1
\]

under the asymptotic regime of (2.1); see [7, 18]. Hence, it holds that

\[
\frac{\alpha K_{n,\theta,\alpha}}{\theta (\frac{n+\theta}{\theta})^\alpha - 1} \to^p 1.
\]

Corollary 3.4 shows the same limit without the first operation \(\alpha \to +0\). In this sense, (3.4) complements the previous result of (3.5).

### 4 Technical lemma

In this section, we prove the following lemma, which was used in the proof of Lemma 3.3.

**Lemma 4.1.** Under the asymptotic regime of (2.1), it holds that

\[
\frac{\Gamma\left(\frac{\theta}{\alpha} + 1 + i\right)}{\Gamma\left(\frac{\theta}{\alpha} + 1\right)} \Gamma\left(\theta + n + i\alpha\right) \Gamma\left(\theta + 1\right) = \left(\frac{\theta}{\alpha}\right)^i \left\{1 + i\frac{\alpha}{n} + \frac{i^2\alpha(1-\alpha)}{2\theta} + O\left(\frac{\theta^2}{n}\right)\right\}
\]

for \(i = 1, 2, \ldots\).
To prove this assertion, we first prove the following three lemmas.

**Lemma 4.2.** Under the asymptotic regime (2.1), it holds that

$$\frac{\Gamma(\theta + n + i\alpha)}{\Gamma(\theta + n)} = n^{i\alpha} \left( 1 + \frac{i\alpha\theta}{n} + O\left(\frac{1}{n}\right) + O\left(\frac{1}{n^2}\right) \right)$$

for $i = 1, 2, \ldots$.

**Proof.** It follows from (3.2) that

$$\frac{\Gamma(\theta + n + i\alpha)}{\Gamma(\theta + n)} = \frac{e^{-(\theta + n + i\alpha)}(\theta + n + i\alpha)^{\theta + n + i\alpha - 1/2} \left\{ 1 + \frac{1}{12(\theta + i\alpha + n)} + O\left(\frac{1}{(\theta + n)^2}\right) \right\}}{e^{-(\theta + n)}(\theta + n)^{\theta + n - 1/2} \left\{ 1 + \frac{1}{12(\theta + n)} + O\left(\frac{1}{(\theta + n)^2}\right) \right\}}.$$ 

Hence, it holds that

$$\frac{\Gamma(\theta + n + i\alpha)}{\Gamma(\theta + n)} = e^{-i\alpha}(\theta + n)^{i\alpha} \left\{ 1 + \frac{i\alpha}{\theta + n} \left( i\alpha - \frac{1}{2} \right) + O\left(\frac{1}{(\theta + n)^2}\right) \right\} e^{i\alpha}$$

$$\times \left\{ 1 - \frac{i^2\alpha^2}{2(\theta + n)} + O\left(\frac{1}{(\theta + n)^2}\right) \right\} \left\{ 1 + \frac{1}{12(\theta + i\alpha + n)} + O\left(\frac{1}{(\theta + n)^2}\right) \right\}$$

$$\times \left\{ 1 - \frac{1}{12(\theta + n)} + O\left(\frac{1}{(\theta + n)^2}\right) \right\}$$

$$= (\theta + n)^{i\alpha} \left\{ 1 + \frac{i\alpha}{\theta + n} \left( i\alpha - \frac{1}{2} \right) + O\left(\frac{1}{(\theta + n)^2}\right) \right\} \left\{ 1 - i^2\alpha^2/2(\theta + n) + O\left(\frac{1}{(\theta + n)^2}\right) \right\}$$

$$\times \left\{ 1 + \frac{1}{12(\theta + i\alpha + n)} - \frac{1}{12(\theta + n)} + O\left(\frac{1}{(\theta + n)^2}\right) \right\}$$

$$= (\theta + n)^{i\alpha} \left\{ 1 + \frac{i\alpha}{\theta + n} \left( i\alpha - \frac{1}{2} \right) + O\left(\frac{1}{(\theta + n)^2}\right) \right\} \left\{ 1 - i^2\alpha^2/2(\theta + n) + O\left(\frac{1}{(\theta + n)^2}\right) \right\}$$

$$\times \left\{ 1 + O\left(\frac{1}{(\theta + n)^2}\right) \right\}$$

$$= (\theta + n)^{i\alpha} \left\{ 1 + \frac{i\alpha(i\alpha - 1)}{2(\theta + n)} + O\left(\frac{1}{(\theta + n)^2}\right) \right\}.$$ 

From $\theta/n \to 0$ in (2.1), it follows that

$$\frac{\Gamma(\theta + n + i\alpha)}{\Gamma(\theta + n)} = n^{i\alpha} \left( 1 + \frac{\theta}{n} + O\left(\frac{\theta^2}{n^2}\right) \right)$$

$$\times \left\{ 1 + \frac{i\alpha(i\alpha - 1)}{2n} - \frac{\theta}{n} + O\left(\frac{\theta^2}{n^2}\right) \right\} + O\left(\frac{1}{n}\right) + O\left(\frac{1}{n^2}\right) + O\left(\frac{\theta^2}{n^2}\right).$$ 

This completes the proof. 

\[ \square \]
Lemma 4.3. As $\theta \to \infty$,
\[
\frac{\Gamma(\theta + 1)}{\Gamma(\theta + 1 + i\alpha)} = \theta^{-i\alpha} \left\{ 1 - i\alpha(i\alpha + 1) \frac{1}{2\theta} + O\left(\frac{1}{\theta^2}\right) \right\}
\]
for $i = 1, 2, \ldots$.

Proof. Using a similar argument as in the proof of Lemma 4.2, we have
\[
\frac{\Gamma(\theta + 1)}{\Gamma(\theta + 1 + i\alpha)} = (\theta + 1)^{-i\alpha} \left\{ 1 - i\alpha(i\alpha - 1) \frac{1}{2(\theta + 1)} + O\left(\frac{1}{\theta^2}\right) \right\}.
\]
Hence, it holds that
\[
\frac{\Gamma(\theta + 1)}{\Gamma(\theta + 1 + i\alpha)} = \theta^{-i\alpha} \left(1 + \frac{1}{\theta}\right)^{-i\alpha} \left\{ 1 - i\alpha(i\alpha + 1) \left(1 + \frac{1}{\theta}\right)^{-1} \frac{1}{2\theta} + O\left(\frac{1}{\theta^2}\right) \right\}
\]
\[
= \theta^{-i\alpha} \left(1 - \frac{i\alpha}{\theta} + O\left(\frac{1}{\theta^2}\right)\right) \left\{ 1 - \frac{i\alpha(i\alpha - 1)}{2\theta} + O\left(\frac{1}{\theta^2}\right) \right\}
\]
\[
= \theta^{-i\alpha} \left\{ 1 - \frac{i\alpha(i\alpha + 1)}{2\theta} + O\left(\frac{1}{\theta^2}\right) \right\}.
\]
This completes the proof. \qed

Lemma 4.4. As $\theta \to \infty$,
\[
\frac{\Gamma\left(\frac{\theta}{\alpha} + 1 + i\right)}{\Gamma\left(\frac{\theta}{\alpha} + 1\right)} = \left(\frac{\theta}{\alpha}\right)^i \left\{ 1 + \frac{i(i+1)\alpha}{2\theta} + O\left(\frac{1}{\theta^2}\right) \right\}
\]
for $i = 1, 2, \ldots$.

Proof. Using a similar argument as in the proof of Lemma 4.2, we have
\[
\frac{\Gamma\left(\frac{\theta}{\alpha} + 1 + i\right)}{\Gamma\left(\frac{\theta}{\alpha} + 1\right)} = \left(\frac{\theta}{\alpha} + 1\right)^i \left\{ 1 + \frac{i(i-1)\alpha}{2\theta} + O\left(\frac{1}{\theta^2}\right) \right\}.
\]
Hence, it holds that
\[
\frac{\Gamma\left(\frac{\theta}{\alpha} + 1 + i\right)}{\Gamma\left(\frac{\theta}{\alpha} + 1\right)} = \left(\frac{\theta}{\alpha}\right)^i \left(1 + \frac{\alpha\theta}{\theta^\alpha}\right)^i \left\{ 1 + \frac{i(i-1)\alpha}{2\theta} + O\left(\frac{1}{\theta^2}\right) \right\}
\]
\[
= \left(\frac{\theta}{\alpha}\right)^i \left\{ 1 + \frac{i(i+1)\alpha}{2\theta} + O\left(\frac{1}{\theta^2}\right) \right\}.
\]
This completes the proof. \qed

Proof of Lemma 4.1. From Lemmas 4.2, 4.3, and 4.4, it follows that
\[
\frac{\Gamma\left(\frac{\theta}{\alpha} + 1 + i\right)}{\Gamma\left(\frac{\theta}{\alpha} + 1\right)} \frac{\Gamma(\theta + n)}{\Gamma(\theta + n + i\alpha)} \frac{\Gamma(\theta + 1)}{\Gamma(\theta + 1 + i\alpha)}
\]
\[
= \left(\frac{\theta}{\alpha}\right)^i \left(1 + \frac{i\alpha}{\theta} + \frac{\theta^2}{\theta^2}ight)^i \left\{ 1 + \frac{i(i+1)\alpha}{2\theta} + O\left(\frac{1}{\theta^2}\right) \right\}
\]
\[
= \left(\frac{\theta}{\alpha}\right)^i \left\{ 1 + \frac{i\alpha}{\theta} + \frac{i^2\alpha(1-\alpha)}{2\theta} + O\left(\frac{1}{\theta^2} + \frac{\theta^2}{\theta^2}\right) \right\},
\]
where
\[
O\left(\frac{1}{\theta^2} + \frac{1}{\theta^2} + \frac{\theta^2}{\theta^2}\right) = O\left(\frac{1}{\theta^2} + \frac{\theta^2}{\theta^2}\right)
\]
is used. This completes the proof. \qed
5 Concluding remark

Under the asymptotic regime

\[ n \to \infty, \quad \theta \to \infty, \quad \frac{\theta^{2\alpha+1}}{n^{2\alpha}} \to 0, \quad \frac{\theta^2}{n} \to 0, \tag{5.1} \]

Theorem 3.2 yields \( E[Z] \to 0 \) and \( E[Z^2] \to \alpha(1-\alpha) \), where

\[ Z = \sqrt{\theta} \left[ \frac{\alpha K}{\theta \left\{ \left( \frac{n+\theta}{n+\theta} \right)^\alpha - 1 \right\} - 1} \right]. \]

We thus expect that \( Z \) (or asymptotically equivalent quantities to \( Z \)) has a non-degenerate limit distribution under (5.1) or stronger regimes. Deriving asymptotic properties of \( Z \) under such regimes is a possible future direction.

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