SYK model with quadratic perturbations: the route to a non-Fermi-liquid.

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We study stability of the SYK4 model with a large but finite number of fermions \( N \) with respect to a perturbation, quadratic in fermionic operators. We develop analytic perturbation theory in the amplitude of the SYK2 perturbation and demonstrate stability of the SYK4 infra-red asymptotic behavior characterized by a Green function \( G(\tau) \propto 1/\tau^{3/2} \), with respect to weak perturbation. This result is supported by exact numerical diagonalization. Our results open the way to build a theory of non-Fermi-liquid states of strongly interacting fermions.

The plenty of available data on various strongly correlated electronic materials \([1,2]\) calls for the development of a general theory of non-Fermi-liquid ground state(s) of an interacting many-body fermionic system. Still, no general theoretical scheme leading to such a behavior in the zero-temperature limit is known (for a recent review see Ref. \([3]\)). Mathematically, complexity of the problem is due to the absence of any general method to calculate non-Gaussian functional integrals which appear in the theory of strongly interacting fermions.

A new and fresh view on this old problem is provided by the recently proposed \([4-6]\) Sachdev-Ye-Kitaev (SYK) model of interacting fermions. It has attracted a lot of attention recently as a possible boundary theory of a two-dimensional gravitational bulk \([5,7,8]\). Original SYK model contains \( N \gg 1 \) Majorana fermions, with the Hamiltonian consisting of a sum of all possible 4-fermion terms with random matrix elements \( J_{ijkl} \sim J/N^{3/2} \) (note that the free (quadratic) term is missing in the SYK Hamiltonian). This model can be considered as a nonlinear generalization of usual random-matrix Hamiltonians \([9]\). Furthermore, SYK4 models with arbitrary even \( q = 2k \) were introduced and studied \([8]\). These models provide the most straightforward way to enhance the role of interaction between fermions, avoiding formation of any simply ordered structures which lead - usually, but not always \([3]\) - to a breakdown of some evident symmetry of the Hamiltonian.

The SYK model is analytically tractable in the large-\( N \) limit and shows two different types of asymptotic behavior for the fermionic Green function \( G(\tau) \). In the intermediate time range \( 1/J < \tau < t_c \), with \( t_c \sim N/J \), the self-consistent approximation for interaction self-energy is valid and \( G(\tau) \propto \tau^{-1/2} \). For even larger times \( \tau \gg t_c \), it was found in Ref. \([10]\) that fluctuations beyond the self-consistent treatment change the behavior of the Green function to \( G(\tau) \propto \tau^{-3/2} \) (we treat exponentially large ergodic time-scale \( \propto 2^{N/2} \) as being infinite). Both these types of behavior are crucially different from the standard Fermi-liquid scaling \( G(\tau) \propto 1/\tau \) which corresponds to the pole structure of the Green function in the energy representation. In other terms, low-energy excitations of the SYK model are not described by any kind of quasi-particles.

For the reasons described above, the SYK model seems to be a very promising starting point to approach a theory of non-Fermi-liquid ground state. Few problems arise, however: i) the absence of a quadratic term in the Hamiltonian makes pure SYK Hamiltonian unrealistic for electronic systems; ii) original SYK model contains Majorana fermions, which are quite scarce in Nature (see however few relevant proposals in Refs.\([11,13]\)); iii) most interesting properties of a non-Fermi-liquid state are those related to transport phenomena, while SYK is a random-matrix-type model without spatial coordinates. Quite a number of recent publications address the issues listed above \([14-17]\). In particular, generalization of the SYK model for complex fermions was developed in Refs. \([15,17]\). A sequence of SYK "quantum dots" connected by weak (quadratic) tunnelling was considered in Refs. \([16,17]\), making it possible to define and study transport quantities like resistance, thermal resistance, etc; see also very recent extensive study in the same direction \([18]\).

However, all (known to us) studies of stability of SYK behavior w.r.t. quadratic perturbations, indicate its runaway instability. As it was shown in Refs. \([14,16,18]\) in the framework of the self-consistent approximation, the scaling dimension of the SYK2 perturbation is negative when estimated within the conformal limit, corresponding to the time-scales \( 1/J \ll \tau \ll t_c \). The papers \([12,18]\) demonstrate an interesting non-Fermi-liquid behavior in the intermediate temperature region \( T^* < T \ll J \), but still obtain Fermi-liquid behavior in the lowest \( T \) range below \( T^*\).

In the present Letter we reconsider the problem of the SYK4 stability w.r.t. quadratic perturbations, going beyond the saddle-point approximation. We study fermionic Green function in the region \( \tau \gg t_c \) by means of perturbation theory in the amplitude of SYK2 terms, using the infra-red asymptotic solution \([8,10]\) as a starting point. We show analytically that a weak SYK2 perturbation does not change the \( G(\tau) \propto 1/\tau^{3/2} \) asymptotics of the Green function, but simply renormalizes the coefficient. This result proves the existence of a domain of stability, with a non-zero area in the parameter space.
of Hamiltonians, where a non-Fermi-liquid is realized as a ground-state. We also perform numerical analysis of the Green function of the mixed SYK$_4$ + SYK$_2$ model to support our analytic study.

The model and basic equations. We consider the model defined by the following Hamiltonian

$$H = \frac{1}{4!} \sum_{i,j,k,l} J_{i,j,k,l} \chi_i \chi_j \chi_k \chi_l + \frac{i}{2!} \sum_{i,j} \Gamma_{i,j} \chi_i \chi_j$$

(1)

where $\chi_i$ are Majorana fermions and all indices run from 1 to $N$. The matrix elements $J_{i,j,k,l}$ and $\Gamma_{i,j}$ are fully antisymmetric and independent random Gaussian variables with zero mean and the variances $\langle J_{i,j,k,l}^2 \rangle = \frac{2J^2}{N}$, $\langle \Gamma_{i,j}^2 \rangle = \frac{\Gamma^2}{N}$. The functional integral representation of this theory is described by the action $S = -\frac{N}{2} (S_1 + S_2)$ with two contributions $[8, 10]$:

$$S_1 = \text{tr} \log (\partial_\tau - \Sigma_{\tau'}) + \int d\tau d\tau' \left( \frac{J^2}{4} G_{\tau'\tau}' - \Sigma_{\tau'\tau'} G_{\tau'\tau}' \right),$$

and

$$S_2 = \int d\tau d\tau' \frac{\Gamma^2}{2} G_{\tau'\tau}^2. \quad (2)$$

In the limit $N \gg 1$ the mean-field analysis is appropriate and the corresponding saddle-point equations read

$$\partial_\tau G_{\tau'\tau} - \int d\tau'' \Sigma_{\tau''\tau'} G_{\tau''\tau'} = \delta(\tau - \tau'), \quad (3)$$

$$\Sigma_{\tau'\tau'} = J^2 G_{\tau'\tau}' + \Gamma^2 G_{\tau'\tau}. \quad (4)$$

We are going to study corrections to the SYK model Green function $G(\tau)$ assuming dimensionless parameter $\beta = \Gamma/J$ to be small. Within applicability range of the saddle-point Eqs. (3, 4), the scaling dimension of the perturbation is negative, $\Delta_{\beta} = -\frac{1}{2}$. As a result, $G_{\text{SYK}}^0(\tau) \propto (J\tau)^{-1/2}$ (the mean-field solution at $\beta = 0$) is unstable w.r.t. the perturbation: at $\tau \gtrsim \tau^* \sim 1/J^2$ it is replaced by the usual Fermi-liquid behavior $G(\tau) \propto (\beta \tau)^{-1}$. On the other hand, at sufficiently long times $t \gg t_c$, soft re-parametrization modes $[5, 6, 8]$ become relevant and the Green function of the pure SYK$_4$ model acquires different scaling $[10]$

$$G(\tau) = \frac{1}{4\pi^{3/4}} \frac{1}{\sqrt{t_c/\tau}} \left( \frac{t_c}{\tau} \right)^{3/4} = \frac{\Gamma^4(\frac{1}{4})}{\sqrt{2MJ^2\pi^{5/4}}} \left( \frac{M}{\tau} \right)^{3/4} \quad (5)$$

For the reasons that become clear soon, we introduced new notation $M = \pi t_c \Gamma^4(\frac{1}{4})$, where $\Gamma(x)$ is the Euler gamma-function.

For sufficiently weak perturbation $\beta \ll 1/\sqrt{N}$, the crossover timescale $\tau^*$ becomes larger than $t_c$ and loses its relevance: the analysis of the SYK solution stability should now be performed using the asymptotic behavior $[5]$ as a starting point. Before we develop this analysis, a brief reminder on the origin of the result $[5]$ is in order.

The saddle-point solution of Eqs. (3, 4) at $\Gamma = 0$ is invariant w.r.t. reparametrization of time, $\tau \rightarrow f(\tau)$, which is an approximate symmetry of the full action $S_1$, see $[10]$. Fluctuations around the saddle-point can be accounted for by a kind of "sigma-model" defined on the manifold of functions $\phi(\tau)$, defined via relation $df/dt = e^{\phi(t)}$. This field theory has a very simple action $[10]$:

$$S_\phi = \frac{M}{2} \int (\phi')^2 dt. \quad (6)$$

Asymptotic behavior of the Fermionic Green function $G_{\tau'\tau} = \frac{1}{N} \sum_i \chi_i(\tau) \chi_i(\tau')$ can then be obtained by the averaging of the functional

$$G_{\tau'\tau}[\phi(\tau)] = \frac{1}{\sqrt{2J^2\pi}} \text{sgn}(\tau' - \tau') \frac{e^{\phi(\tau)/4}e^{\phi(\tau')/4}}{\int e^{\phi(\tilde{\tau})} d\tilde{\tau}^{1/2}}, \quad (7)$$

with the action (6). For actual calculations of functional integrals like Eq.(7) with the action (6), we follow Refs. $[10, 19]$ where a very useful reduction to the Liouville quantum mechanics $[20–22]$ was employed.

There are various results in the literature $[8, 10]$ concerning determination of the important parameter $M = M(N, J)$ which enter the action (6). We prefer to employ the relation between results for the full Density of States of the SYK$_4$ model obtained i) via asymptotic low-energy theory $[19]$ (expressed in terms of $M$), and ii) by the method of generalized orthogonal polynomials $[23]$. Comparison of two approaches yields

$$M = \frac{m(N)N}{32\sqrt{2J}} \quad \text{where} \quad \lim_{N \to \infty} m(N) = 1. \quad (8)$$

Note that convergence of $m(N)$ upon increase of $N$ is very slow; in particular, $m(32) \approx 0.54$. Note also numerical factor $\sim 0.02$ in the RHS of Eq.(5), which makes $M$ much smaller than $N/J$. Fortunately, the actual time-scale which enters Green functions $G(\tau)$ is $t_c \approx 54M \approx 1.2N m(N)/J$; it will be important below for the comparison with numerical data at large but finite $N = 32$.

Perturbation theory. First-order correction to the Green function $G(\tau)$ due to the quadratic term $S_2$ in the action can be found (see Supplementary Material for more details) in a straightforward way as follows (notation $\langle \cdot \rangle_0$ means the average over $\phi$ field with the action $S_\phi$, see Eq. (6)):

$$\delta G(\tau) = -\langle G_{\tau,0}[\phi]S_2[\phi] \rangle_0 + \langle G_{\tau,0} \rangle_0 \langle S_2 \rangle_0 \quad (9)$$

Substituting here Eq. (2), we find that the first term of Eq. (9) contains an average (over $\phi(\tau)$ fluctuations) of the product of three functionals like (7), with the time arguments $0, \tau$ and $\tau_1, \tau_2$, where further integration over $\tau_1, \tau_2$ is implied. Functional integration over $\phi(\tau)$ with the action (6) should be performed separately in 6 different time regions with the following order of time arguments:
The Green function $G(\epsilon)$ of the SYK$_{2+}\text{SYK}_4$ model can be numerically studied with two complementary approaches: via exact diagonalization at finite $N$ and directly in the limit of $N \to \infty$ via solution of the mean-field equations Eqs. 3 and 4. Within numerical analysis below, we put $J = 1$. For exact diagonalization, we consider the Hamiltonian of Eq. 4 for $N = 32$ fermions for the range of $\beta$ and average over hundreds of disorder realizations (we employ representation of Majorana algebra, used in Ref. 20). This results into red, blue and brown curves on the Fig. 1. The most interesting regime is realized at small $\epsilon \leq 1/t_c$ (that is, below the minimum of the function $G(\epsilon)$). Characteristic time-scale $t_c \approx 20$ for $N = 32$. Unfortunately, the region $\epsilon \leq 1/t_c$ is limited from below by the many-body energy scale $\epsilon = \epsilon_{MB}$ (defined as an energy, counted from the ground-state, where the many-body level spacing $\delta \epsilon$ becomes comparable to $\epsilon$ itself.) For $N = 32$ the corresponding cut-off $\epsilon_{MB} \approx 2 \times 10^{-3}$ determines the left edge on the Fig. 1. The respective energy interval $\epsilon_{MB} \ll \epsilon \ll \beta^{-1}$ is not really large enough to admit for the predicted asymptotic behaviour $G(\epsilon) \propto \epsilon^{1/2}$: compare with the dotted line which is evaluated according to theoretical prediction of Eq. (22) of the Ref. 10. However, qualitatively at $\beta = 0$ we find the expected behaviour. At small $\beta \ll 1$ the corresponding part of $G(\epsilon)$ dependence shifts up in the log-log plot (see blue curve for $\beta = 0.01 \ll 1/N$) without change of behavior as function of $\epsilon$, in agreement with our analytical result 12. However, at slightly larger values of $\beta \sim 1/N$ (brown curve) the Green function $G(\epsilon)$ saturates at low energies, in agreement with the Fermi-liquid behaviour.

At even larger values of $\beta \geq 1/\sqrt{N}$, the asymptotic region with $G(\epsilon) \propto \sqrt{\epsilon}$ disappears completely, and Green function can be approximated by the solution of mean-field Eqs. 3 and 4. Interaction term is then important at higher energies $\epsilon \geq \epsilon_{MB} = \beta^2 J > 1/t_c$ only. The green and cyan curves show the numerical solutions to these equations. Note that mean-field solutions differ in the region $\epsilon \gg 1/t_c$ from the analytical result given by Eq.(22) of Ref. 10 evaluated at $N = 32$ (dotted line in Fig. 1). These deviations appear since the asymptotic region $1/t_c = 0.05 \ll \epsilon \ll 1$ is apparently not wide enough. Thus, finite-size effects are detrimental for all analytically available asymptotics in this problem even for relatively large system of $N = 32$ fermions. Although available system size is on border-line of emergence of respective asymptotic regions, we believe that the results of ED and mean-field studies are consistent with our analytical estimates. In particular, low-$\epsilon$ limits of $G(\epsilon)$ demonstrated by cyan and green lines are in agreement with analytic result for pure SYK$_2$ theory, $G(\epsilon \to 0) = \Gamma/\epsilon$.

Conclusions. Schematically, our results for the zero-temperature phase diagram of the combined SYK$_4$ - SYK$_2$ model are shown in the Fig. 2. We emphasize somewhat unusual scaling limit of large $N$ that is employed here. Namely, we consider $N \gg 1$ as some finite number, but we neglect exponentially small many-
temperatures Lyapunov exponent drops to zero as \( T > T_a \) finite positive temperature \( T \), the transition is of the same kind as we found at zero temperature for small \( \beta \sim b/N \).

It would be very interesting to study similar Non-Fermi-Liquid - Fermi-Liquid transition in a chain (or lattice) of SYK-like "quantum dots". Note that for such extended models there is no issue with finiteness of the non-zero many-body level spacing \( \epsilon_{MB} \) and the problem of the NFL-FL phase transition can be formulated in the strict sense. As was already mentioned in the Introduction, the transition of that kind was studied in Refs. [17, 18] at non-zero temperatures within a \( N \to \infty \) limit. Another approach was developed in Ref. [20] where an effect of the SYK-like interaction upon the properties of a random-hopping chain was investigated via numerical analysis of the level statistics. The authors of [20] found the MBL-type transition between fully localized and ergodic ground states at rather \textit{low and decreasing with} \( N \) ratio \( J/\Gamma \) of the SYK coupling to quadratic coupling; thus it seems to be qualitatively different from the transition we found for a single SYK system.

Finally, we would like to mention an interesting physical problem which may bear some resemblance with SYK2 - SYK4 model considered here. It is well known that strong potential disorder suppresses superconducting transition with non-trivial (d-wave or p-wave) pairing, due to random mixing of electron states between different sectors of the Fermi-surface. However, mean-square magnitude of the (random-sign) Cooper interaction amplitude survives impurity scattering. It means that electron states with energies exactly at the Fermi-energy constitute a kind of SYK-type model with a random 4-fermion interaction. Potentially, this interaction may occur to be strong enough to lead to a non-Fermi-liquid ground state without formation of any order parameter.

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\[ G(\epsilon) \]

**FIG. 2:** Sketch of the Green function \( G(\epsilon) \) in log-log scale, in the limit of \( N \gg 1 \) for several values of \( b = \beta N \) ordered as \( b_1 < b_2 < b_c < b_3 < b_4 \), with critical \( b_c \) of the order unity. NFL-FL transition occurs between blue (2) and grey (3) lines. Light-blue line (4) corresponds to high energies above \( \epsilon_B \gg 1/t_c \) only.

\[ \beta \epsilon_{c}/t_1 \]

\[ \epsilon \]

\[ \epsilon^{1/2} \]

\[ \epsilon^{-1/2} \]

\[ 1/t_c \]

\[ \epsilon \beta \]

body level spacing \( \epsilon_{MB} \sim 2^{-N/2} \). Then our results demonstrate the presence of a \textit{phase transition} between fully chaotic non-Fermi-liquid ground state realized at \( b \equiv \beta N < b_c \), and Fermi-liquid ground state existing at \( b > b_c \), with \( b_c \sim 1 \).

Recently, chaotic-integrable transition for SYK model with quadratic perturbation was studied in Ref. [25]. The authors have shown (judging it by Lyapunov exponent of the 4-point out-of-time-order correlation function) that for several values of \( \beta \sim O(N^0) \), there exists a finite positive temperature \( T = T_c(\beta) \) such that at \( T > T_c(\beta) \) the system behaves chaotically, while at lower temperatures Lyapunov exponent drops to zero as it is expected for Fermi-liquid state with quasiparticle-based classification of eigenstates. We believe that this transition is of the same kind as we found at zero temperature for small \( \beta \sim b/N \).

\[ 1 \quad \text{for several values of} \quad \beta \sim O(N^0) \]

\[ J/\Gamma \]

\[ \epsilon_{MB} \sim 2^{-N/2} \]

\[ \epsilon_{c}/t_1 \]

\[ \epsilon \]

\[ \epsilon^{1/2} \]

\[ \epsilon^{-1/2} \]

\[ 1/t_c \]

\[ \epsilon \beta \]

\[ \beta \]

\[ \epsilon_{MB} \]

\[ \beta \epsilon_{c}/t_1 \]

\[ \epsilon \]

\[ \epsilon^{1/2} \]

\[ \epsilon^{-1/2} \]

\[ 1/t_c \]

\[ \epsilon \beta \]

\[ \beta \]

\[ \epsilon_{MB} \sim 2^{-N/2} \]

\[ \epsilon_{c}/t_1 \]

\[ \epsilon \]

\[ \epsilon^{1/2} \]

\[ \epsilon^{-1/2} \]

\[ 1/t_c \]

\[ \epsilon \beta \]

\[ \beta \]

\[ \epsilon_{MB} \sim 2^{-N/2} \]

\[ \epsilon_{c}/t_1 \]

\[ \epsilon \]

\[ \epsilon^{1/2} \]

\[ \epsilon^{-1/2} \]

\[ 1/t_c \]

\[ \epsilon \beta \]

\[ \beta \]

\[ \epsilon_{MB} \sim 2^{-N/2} \]

\[ \epsilon_{c}/t_1 \]

\[ \epsilon \]

\[ \epsilon^{1/2} \]

\[ \epsilon^{-1/2} \]

\[ 1/t_c \]

\[ \epsilon \beta \]

\[ \beta \]

\[ \epsilon_{MB} \sim 2^{-N/2} \]

\[ \epsilon_{c}/t_1 \]

\[ \epsilon \]

\[ \epsilon^{1/2} \]

\[ \epsilon^{-1/2} \]

\[ 1/t_c \]

\[ \epsilon \beta \]

\[ \beta \]

\[ \epsilon_{MB} \sim 2^{-N/2} \]

\[ \epsilon_{c}/t_1 \]

\[ \epsilon \]

\[ \epsilon^{1/2} \]

\[ \epsilon^{-1/2} \]

\[ 1/t_c \]

\[ \epsilon \beta \]

\[ \beta \]

\[ \epsilon_{MB} \sim 2^{-N/2} \]

\[ \epsilon_{c}/t_1 \]

\[ \epsilon \]
Therefore Eq. (15) with Green function: objects like can be conveniently parametrised in terms of the field \( \phi \) re-parametrization symmetry of the mean-field solution in the scaling limit. The functions defined on this manifold \( f \) is described by the “sigma-model” action over the manifold of monotonic functions \( \tau \), then we will put quantum mechanics. Necessary to calculate the corrections to the Green function generated by the perturbation of the SYK model with \( \Delta = \frac{1}{q} \) and \( b = (\frac{1}{2} - \Delta) \frac{\tan(\pi \Delta)}{\pi} \). To simplify the notation in the following, we will consider averaging of the objects like \( G \). The average Green function will come up as a specific result at \( n = \Delta \).

Below in Sec. II we rederive some results from Ref. [10] in a slightly different way; namely, we show how to reduce evaluation of the Green function \( (14) \) with the action \( (13) \) to the calculation of matrix elements of the Liouville quantum mechanics.

In Sec. III we evaluate correlation functions of the products of various powers of Green functions \( G \) which are necessary to calculate the corrections to the Green function generated by the perturbation of the SYK type:

\[
\delta(G_{\tau_1, \tau_2}^{n/\Delta}) = -\langle G_{\tau_1, \tau_2}^{n/\Delta} S_{int} \rangle + \langle G_{\tau_1, \tau_2}^{n/\Delta} \rangle \langle S_{int} \rangle. 
\]

\[
S_{int} = -\frac{\Delta N \Gamma^2}{2m} \int d\tau_1 d\tau_2 G_{\tau_1, \tau_2}^{m/\Delta} \tau_1, \tau_2. 
\]

Finally we will be interested in the case \( m = \frac{1}{3} \) and \( n = \frac{1}{3}, \Delta = \frac{1}{4} \).

To simplify further formulae, we switch to dimensionless time units \( t = \frac{\tau}{\Gamma M} \) and introduce new notation for the Green function:

\[
g_n(t, t') = \left( \frac{1}{\Gamma(2n)} \frac{b^n}{(2M)^{2n}} \right)^{-1} G_{\tau_1, \tau_2}^{n/\Delta}(\tau, \tau'). 
\]

Therefore Eq. (15) with \( S_{int} \) from Eq. (16) can now be rewritten in the form

\[
\delta(G_{\tau_1, \tau_2}^{n/\Delta}) = \frac{N \Gamma^2 \Delta}{m} \frac{b^{(n+m)}}{\Gamma(2n) \Gamma(2m) (2M)^{(2(n+m)-2)}} \int_{t_{3}>t_{4}} dt_{3} dt_{4} \left[ (g_n(t_1, t_2) g_m(t_3, t_4)) - \langle g_n(t_1, t_2) \rangle \langle g_m(t_3, t_4) \rangle \right] 
\]
To evaluate expression \( \langle \phi_t(t_1, t_2) \phi_{m}(t_3, t_4) \rangle \) one has to calculate the average. The difficulty of this calculation is due to the absence of Wick contraction rules, so all possible time orderings have to be considered explicitly. Symmetry of Green function allows us to fix relations \( t_1 > t_2 \) and \( t_3 > t_4 \), leaving 6 possible time orderings: 1) \( t_1 > t_2 > t_3 > t_4 \), 2) \( t_1 > t_3 > t_2 > t_4 \), 3) \( t_3 > t_1 > t_2 > t_4 \), 4) \( t_1 > t_3 > t_4 > t_2 \), 5) \( t_3 > t_1 > t_4 > t_2 \) and 6) \( t_3 > t_4 > t_1 > t_2 \). The orderings 1 and 6 are trivial \( \[10] \):

\[
\langle \phi_t(t_1, t_2) \phi_{m}(t_3, t_4) \rangle = \langle \phi_{t}(t_1, t_2) \rangle \langle \phi_{m}(t_3, t_4) \rangle
\] (19)

In Sec. III we present evaluation of the average values corresponding to the remaining four variants of time ordering. In the remaining Secs. IV - VI we combine various contributions in the long-time limit and derive the final result.

II. AVERAGING VARIOUS POWERS OF THE GREEN FUNCTION

First of all, we rederive some results from Ref. \[10\] in a slightly different way. We start from the formula for average power of the Green function

\[
\langle G^n/\Delta(\tau, \tau') \rangle = \int D\phi b^n \frac{e^{n\phi(\tau)} e^{n\phi(\tau')}}{|f^{\tau}_{\tau'} e^{\phi(\tau')} d\tau|^{2n}} f(\phi')^2 d\tau
\] (20)

Switching to dimensionless time, we write

\[
\langle G^n/\Delta(t, t') \rangle = \int D\phi \frac{b^n}{(2M)^{2n} |f^t_{t'} e^{\phi(t')} dt|^2} e^{-\frac{1}{2} t f(\phi')^2 dt}
\] (21)

Using identity \( \frac{1}{2\pi} = \int_0^\infty \frac{a^{n-1}}{\Gamma(2n)} e^{-\alpha a} da \) one can rewrite above expression as follows:

\[
\langle G^n(t, t') \rangle = \int_0^\infty \frac{\alpha^{2n-1} d\alpha}{\Gamma(2n)} \int D\phi \frac{b^n}{(2M)^{2n} e^{\alpha\phi(t)} e^{\phi(t')} e^{-\frac{1}{2} f(\phi')^2 dt - \alpha f^t_{t'} e^{\phi(t')} dt}}
\] (22)

Functional integral over \( \phi(t) \) can be interpreted as a quantum mechanical amplitude and evaluated explicitly. There is a technical problem however: the field \( \phi(t) \) in Eq. \[22\] can be shifted by a constant: \( \phi(t) \rightarrow \phi(t) + \phi_0 \), producing a divergent integral. In the calculation provided in Ref. \[10\] this zero mode appeared as an infinite multiplicative constant, coming from divergent integration over parameter \( \alpha \). This divergence was argued \[10\] to be irrelevant since it is related to the symmetry of the action. Slightly different formulation of the same approach is to put \( \alpha = 1 \) instead of integration over \( \alpha \) since it is related to the symmetry of the action. Slightly different formulation of the same approach is to put \( \alpha = 1 \) instead of integration over \( \alpha \).

Therefore in our further calculations we will follow the approach of Ref. \[10\] which is simpler in implementation.

Rewriting formulae in terms of \( g_n(t, t') \), we find

\[
\langle g_n(t, t') \rangle = \int_0^\infty \frac{\alpha^{2n-1} d\alpha}{\Gamma(2n)} \int d\phi_1 \langle \phi_0 | e^{\alpha\phi} U_\alpha(t, t') e^{\alpha\phi} | \phi_1 \rangle,
\] (23)

where \( U_\alpha(\tau, \tau') \) is the evolution operator corresponding to the Liouville’s Hamiltonian \( H = -\frac{1}{4} \delta^2_{\phi} + \alpha e^\phi \). It can be written as

\[
U_\alpha(t, t') = \int_0^\infty \frac{dk}{2\pi} e^{-k^2(t-t')|k, \alpha \rangle \langle k, \alpha |}
\] (24)

with eigenstates

\[
\langle \phi | k, \alpha \rangle = \frac{2}{\Gamma(2ik)} K_{2ik}(2\sqrt{\alpha e^\phi}).
\] (25)

It is more convenient to work with Mellin transformed eigenfunctions

\[
\frac{2}{\Gamma(2ik)} K_{2ik}(2\sqrt{x}) = \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(p - ik) \Gamma(p + ik)}{\Gamma(2ik)} x^{\frac{p}{2} - \frac{1}{2}} dp.
\] (26)

We now introduce the ”matrix element” \( G(p, k) \) as follows:

\[
G(p, k) = \frac{\Gamma(p - ik) \Gamma(p + ik)}{\Gamma(2ik)}.
\] (27)
Then Eq. (23) becomes

\[
\langle g_n(t, t') \rangle = \int_0^\infty \frac{dk}{2\pi} e^{-k^2(t-t')} \int_0^\infty \alpha^{2n-1} d\alpha \int_{-\infty}^\infty d\phi_1 e^{\alpha \phi_1} e^{\alpha \phi_1} \langle \phi_0 | k, \alpha \rangle \langle k, \alpha | \phi_1 \rangle = \int_0^\infty \frac{dk}{2\pi} e^{-k^2(t-t')} \int_0^\infty \alpha^{2n-1} d\alpha \int_{-\infty}^\infty d\phi_1 e^{\alpha \phi_1} e^{\alpha \phi_1} \int_{-\infty}^\infty \frac{dp_1}{2\pi i} \frac{dp_1}{2\pi i} G(p_1, k)G(p_1, -k) e^{-p_1 \alpha} e^{-p_1 \alpha} e^{-p_2 \alpha} e^{-p_2 \alpha} = \int_0^\infty \frac{dk}{2\pi} e^{-k^2(t-t')} G(n, k)G(n, -k) \tag{28}
\]

In the limit of \( t \gg 1 \) we find

\[
\langle g_n(t, t') \rangle = \int_0^\infty \frac{dk}{2\pi} e^{-k^2(t-t')} \Gamma^4(\Delta n)(4k^2) = \frac{\Gamma^4(n)}{2\sqrt{\pi}(t-t')^{2}} \tag{29}
\]

which coincides with the result of Ref. [10].

### III. AVERAGING THE PRODUCTS OF VARIOUS POWERS OF GREEN FUNCTION.

We now turn to the calculation of the averages of the type \( \langle g^n(t_1, t_2)g^m(t_3, t_4) \rangle \). Following the same steps as in the Sec. II we obtain

\[
\langle g_n(t_1, t_2)g_m(t_3, t_4) \rangle = \int_0^\infty \alpha^{2n-1} d\alpha \int_0^\infty \beta^{2m-1} d\beta \int D\phi e^{\alpha \phi_1} e^{\beta \phi_2} e^{m \phi_3} e^{m \phi_4} e^{-\frac{1}{2} \int (\phi')^2 dt - \alpha \int t_1^3 e^{\phi(t)} dt - \beta \int t_2^3 e^{\phi(t)} dt} \tag{30}
\]

with shorthand notation \( \phi_i = \phi(t_i) \). Like in Sec. II, we interpret the functional integral over \( \phi(t) \) as a quantum-mechanical amplitude. It is convenient to fix the "gauge" by setting \( \alpha \rightarrow 1 \), to simplify calculations. The result of averaging depends crucially on the specific time ordering (see discussion in Sec. I). We present here details of the calculation for the cases 2 and 3. Results for the cases 4 and 5 can be obtained in similar way, so we will provide the results only.

#### A. Time-ordering 2: \( t_1 > t_3 > t_2 > t_4 \)

Quantum mechanical representation of the problem corresponds to the free particle motion at times \( t < t_4 \). In the range of times \( t_4 < t < t_2 \) the exponential potential \( e^{\phi} \) with the magnitude equal to \( \beta \) is turned on, so the evolution during this time interval is described by \( U_\beta(t_2, t_4) \) (see Sec. II for the definition of \( U(t, t') \)). Next, in the time region between \( t_2 \) and \( t_3 \), the evolution is governed by \( U_{1+\beta}(t_3, t_2) \), and between \( t_1 \) and \( t_4 \) it is given by \( U_1(t_1, t_3) \). Finally, at \( t > t_1 \) the particle is free again.

Then quantum-mechanical average is of the following form (hereafter we use Roman subscripts to denote specific time ordering, which is the 2nd one currently):

\[
\langle g_n(t_1, t_2)g_m(t_3, t_4) \rangle_{II} = \int_0^\infty \beta^{2m-1} d\beta \int d\phi_1 d\phi_2 d\phi_3 d\phi_4 e^{\alpha \phi_1} e^{\beta \phi_2} e^{m \phi_3} e^{m \phi_4} \langle \phi_1 | U_1(t_1, t_3) | \phi_3 \rangle \langle \phi_3 | U_{1+\beta}(t_3, t_2) | \phi_2 \rangle \langle \phi_2 | U_\beta(t_2, t_4) | \phi_4 \rangle \tag{31}
\]

Using explicit representation for \( U \) we find

\[
\langle g_n(t_1, t_2)g_m(t_3, t_4) \rangle_{II} = \int_0^\infty \frac{dk_1}{2\pi} \frac{dk_2}{2\pi} \frac{dk_3}{2\pi} e^{-k_1^2 t_1^2} e^{-k_2^2 t_2^2} e^{-k_3^2 t_4^2} \int_0^\infty \beta^{2m-1} d\beta \int d\phi_1 d\phi_2 d\phi_3 d\phi_4 e^{\alpha \phi_1} e^{\beta \phi_2} e^{m \phi_3} e^{m \phi_4} \langle \phi_1 | \rho_{k_1, 1} = k_1 | \phi_3 \rangle \langle \phi_3 | k_2, 1 + \beta \rangle | k_2, 1 + \beta | \phi_2 \rangle \langle \phi_2 | k_3, \beta | k_3, \beta | \phi_4 \rangle \tag{32}
\]

The integrations over \( \phi \) are factorized. Integrations over \( \phi_1 \) and \( \phi_4 \) are trivial:

\[
\int d\phi_1 e^{\alpha \phi_1} \langle \phi_1 | k_1, 1 \rangle = \int d\phi_1 e^{\alpha \phi_1} \int_{-i\infty}^{c+i\infty} G(p, k_1) e^{-p \phi_1} \frac{dp}{2\pi i} = G(n, k_1) \tag{33}
\]

\[
\int d\phi_4 e^{m \phi_4} | k_3, \beta | \phi_4 \rangle = \int d\phi_4 e^{m \phi_4} \int_{-i\infty}^{c+i\infty} G(p, -k_3) e^{-p \phi_4} \beta^{-p} \frac{dp}{2\pi i} = G(m, -k_3) \beta^{-m} \tag{34}
\]
Integrations over $\phi_3$ and $\phi_2$ are more involved. Integrating over $\phi_3$ we find

$$
\int d\phi_3 e^{i\phi_3} \beta^m (k_1,1)\langle \phi_3 | k_2,1 + \beta \rangle = \int d\phi_3 e^{i\phi_3} \beta^m \int_{c-i\infty}^{c+i\infty} G(p_1,-k_1)G(p_2,k_2)e^{-\phi_3(p_2+p_1)}(1 + \beta)^{-p}\frac{dp_1 dp_2}{2\pi i} = \\
\beta^m \int_{c-i\infty}^{c+i\infty} G(m-p,-k_1)G(p,k_2)(1 + \beta)^{-p}\frac{dp}{2\pi i} = \\
(1 + \beta)^{ik_2} \beta^m \frac{\Gamma(2ik_1)}{\Gamma(2ik_2)} G(m + ik_2,-k_1)G(m - ik_2,-k_1)\mathbf{F}(m + ik_2 - ik_1,m + ik_2 + ik_1,2m,-\beta) = \\
(1 + \beta)^{ik_2} \frac{\Gamma(2ik_1)}{\Gamma(2ik_2)} G(m - ik_2,-k_1) \int_{c_m+i\infty}^{c_m+i\infty} G(p + ik_2,k) \frac{\Gamma(m-p)}{\Gamma(m+p)} \beta^p \frac{dp}{2\pi i}.
$$

(35)

with $\mathbf{F}(a,b,c,z)$ for the normalized Hypergeometric function, $c > 0$ and $c_m \in (0,m)$. Integration over $\phi_2$ can be performed in the similar manner. We note a useful identity:

$$
\int d\phi^m e^{i\phi} \langle k_L,\alpha | \phi | k_R,\alpha + \beta \rangle = \int d\phi^m e^{i\phi} \langle -k_R,\alpha + \beta | \phi | -k_L,\alpha \rangle \\
(1 + \beta)^{ik_2} \frac{\Gamma(2ik_L)}{\Gamma(2ik_R)} G(m - ik_2,-k_L) \int_{c_m-i\infty}^{c_m+i\infty} G(p + ik_2,k_L) \frac{\Gamma(m-p)}{\Gamma(m+p)} \beta^p \frac{dp}{2\pi i}.
$$

(36)

With this identity, we find

$$
\langle g_n(t_1,t_2)g_m(t_3,t_4) \rangle_{I,I} = \int_0^\infty \frac{dik_1 dik_2 dik_3}{(2\pi)^3} e^{-ik_1^2} e^{-ik_2^2} e^{-ik_3^2} \int_0^\infty \beta^{-1}G(n,k_1)G(m,-k_3)d\beta \\
(1 + \beta)^{ik_2} \frac{\Gamma(2ik_1)}{\Gamma(2ik_2)} G(m - ik_2,-k_1) \int_{c_m-i\infty}^{c_m+i\infty} G(p + ik_2,k_1) \frac{\Gamma(m-p)}{\Gamma(m+p)} \beta^p \frac{dp}{2\pi i} \\
\beta^{ik_2}(1 + \beta)^{-ik_2} \frac{\Gamma(-2ik_3)}{\Gamma(-2ik_2)} G(n + ik_2,k_3) \int_{c_m-i\infty}^{c_m+i\infty} G(q - ik_2,-k_3) \frac{\Gamma(n-q)}{\Gamma(n+q)} \beta^{-q} \frac{dq}{2\pi i}.
$$

(37)

The last integration over $\beta$ gives:

$$
\langle g_n(t_1,t_2)g_m(t_3,t_4) \rangle_{I,I} = \int_0^\infty \frac{dik_1 dik_2 dik_3}{(2\pi)^3} e^{-ik_1^2} e^{-ik_2^2} e^{-ik_3^2} \int_0^\infty G(n,k_1)G(m,-k_3) \frac{\Gamma(2ik_1)}{\Gamma(2ik_2)} G(n + ik_2,k_3)G(m - ik_2,-k_1) \\
\int_{\min(m,n)-i\infty}^{c_{\min(m,n)}+i\infty} G(p + ik_2,k_1) \frac{\Gamma(m-p)}{\Gamma(m+p)} G(p,-k_3) \frac{\Gamma(n-p - ik_2)}{\Gamma(n+p + ik_2)} \frac{dp}{2\pi i}.
$$

(38)

B. Time-ordering 3: $t_3 > t_1 > t_2 > t_4$

Following the same steps as above we come to

$$
\langle g_n(t_1,t_2)g_m(t_3,t_4) \rangle_{I,I} = \int_0^\infty \beta^{2m-1}d\beta \int d\phi_1 d\phi_2 d\phi_3 d\phi_4 e^{n\phi_1} e^{n\phi_2} e^{n\phi_3} e^{n\phi_4} \\
\langle \phi_3 | U_\beta(t_3,t_1) | \phi_1 \rangle \langle \phi_1 | U_{1+\beta}(t_1,t_2) | \phi_2 \rangle \langle \phi_2 | U_\beta(t_2,t_4) | \phi_4 \rangle.
$$

(39)

With explicit expression for $U$, we find

$$
\langle g_n(t_1,t_2)g_m(t_3,t_4) \rangle_{I,I} = \int_0^\infty \frac{dik_1 dik_2 dik_3}{(2\pi)^3} e^{-ik_1^2} e^{-ik_2^2} e^{-ik_3^2} \int_0^\infty \beta^{2m-1}d\beta \\
\int d\phi_1 d\phi_2 d\phi_3 d\phi_4 e^{n\phi_1} e^{n\phi_2} e^{n\phi_3} e^{n\phi_4} \langle \phi_3 | k_1, \beta \rangle \langle k_1, \beta | \phi_1 \rangle \langle \phi_1 | k_2, 1 + \beta \rangle \langle k_2, 1 + \beta | \phi_2 \rangle \langle \phi_2 | k_3, \beta \rangle \langle k_3, \beta | \phi_4 \rangle.
$$

(40)
Using the identity in Eq. (36) we integrate over $\phi$:

$$
\langle g_n(t_1, t_2)g_m(t_3, t_4) \rangle_{III} = \int_0^\infty \frac{dk_1dk_2dk_3}{(2\pi)^3} e^{-k_1^2t_{3,1}}e^{-k_2^2t_{1,2}}e^{-k_3^2t_{2,4}} \int_0^\infty \frac{d\beta}{2\pi} d\beta G(m, k_1)G(m, -k_3)

(1 + \frac{1}{\beta})^{-ik_2} \frac{\Gamma(-2ik_3)}{\Gamma(-2ik_2)} G(n + ik_2, k_3) \int^{\infty+i\infty}_{c_\alpha-i\infty} \frac{\Gamma(n-q)}{\Gamma(n+q)} \frac{1}{\Gamma(1+\frac{1}{\beta})} dq \int^{\infty+i\infty}_{c_\alpha-i\infty} \frac{\Gamma(n-p)}{\Gamma(n+p)} \frac{1}{\Gamma(1+\frac{1}{\beta})} dp

(41)
$$

Finally, $\beta$-integration gives

$$
\langle g_n(t_1, t_2)g_m(t_3, t_4) \rangle_{III} = \int_0^\infty \frac{dk_1dk_2dk_3}{(2\pi)^3} e^{-k_1^2t_{3,1}}e^{-k_2^2t_{1,2}}e^{-k_3^2t_{2,4}} G(m, k_1)G(m, -k_3) \frac{\Gamma(-2ik_3)}{\Gamma(-2ik_2)} \frac{\Gamma(2ik_1)}{\Gamma(2ik_2)} G(n - ik_2, -k_1)G(n + ik_2, k_3) \int^{\infty+i\infty}_{c_\alpha-i\infty} \frac{G(-p - ik_2, -k_3)}{\Gamma(n+p)} \frac{dp}{2\pi i}

(42)
$$

In fact, for time-ordering 3 we can go even further and calculate one of the momentum integrals analytically: $p$-integration gives the momentum conservation law $2\pi\delta(k_1 - k_3)$ and as a result:

$$
\langle g_n(t_1, t_2)g_m(t_3, t_4) \rangle_{III} = \int_0^\infty \frac{dk_1dk_2}{(2\pi)^2} e^{-k_1^2t_{3,1}}e^{-k_2^2t_{1,2}}e^{-k_3^2t_{2,4}} G(m, k_1)G(m, -k_1)G(n - ik_1, -k_2)G(n + ik_1, k_2)

(43)
$$

C. Results for time-orderings 4 and 5

For the orderings 4 and 5 we provide only the results:

$$
\langle g_n(t_1, t_2)g_m(t_3, t_4) \rangle_{IV} = \int_0^\infty \frac{dk_1dk_2}{(2\pi)^2} e^{-k_1^2t_{3,1}}e^{-k_2^2t_{1,2}}e^{-k_3^2t_{2,4}} G(n, k_1)G(n, -k_1)G(m - ik_1, -k_2)G(m + ik_1, k_2)

(44)
$$

$$
\langle g_n(t_1, t_2)g_m(t_3, t_4) \rangle_V = \int_0^\infty \frac{dk_1dk_2dk_3}{(2\pi)^3} e^{-k_1^2t_{3,1}}e^{-k_2^2t_{1,2}}e^{-k_3^2t_{2,4}} G(m, k_1)G(n, -k_3) \frac{\Gamma(2ik_1)}{\Gamma(2ik_2)} \frac{\Gamma(-2ik_3)}{\Gamma(-2ik_2)} G(n + ik_2, k_3)G(n - ik_2, -k_1)

\int^{\infty+i\infty}_{c_{\min(m,n)}-i\infty} \frac{G(p + ik_2, k_1)}{\Gamma(n+p)} \frac{\Gamma(n-p)}{\Gamma(n-p-k_3)} \frac{G(m - p - ik_2)}{\Gamma(m+p+ik_2)} \frac{dp}{2\pi i}

(45)
$$

IV. CANCELLATION OF INFRA-RED SINGULARITIES FOR 3RD AND 4TH TIME ORDERINGS

We need to calculate integrals like the one indicated in Eq.(9) of the main text:

$$
f(t_1 - t_2) = \int_{t_3 < t_4} \langle g_n(t_1, t_2)g_m(t_3, t_4) \rangle - \langle g_n(t_1, t_2) \rangle \langle g_m(t_3, t_4) \rangle

(46)
$$

We introduce the following notations for the integrands corresponding to different variants of the time ordering:

$$
f_i(t_1 - t_2) = \int_{T_i} dt_3dt_4 \langle g_n(t_1, t_2)g_m(t_3, t_4) \rangle_i
$$

$$
f_Z = \int_{t_3 < t_4, t_2 < t_4, t_3 < t_1} \langle g_n(t_1, t_2) \rangle \langle g_m(t_3, t_4) \rangle

(47)
$$

Here $T_i$ is the area of integration which satisfies the $ith$ order of times. Using these functions we can write: $f(t) = \sum_{i=II}^{V} f_i(t) - f_Z(t)$. The correction to the Green function can be expressed via $f(t)$ as it is present in Eq.(10) of
Fortunately, these divergencies cancel each other. To demonstrate with fact, we write these function explicitly

\[
 f_{III}(t_1 - t_2) = \delta_{III}(g_n(t_1, t_2)) = \int_{-\infty}^{t_1} dt_3 \int_{t_1}^{\infty} dt_4 \langle g_n(t_1, t_2)g_n(t_3, t_4) \rangle_{III} = \\
 \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} \frac{e^{-k_1^2(t_1-t_2)} - e^{-k_2^2(t_1-t_2)}}{k_1^2 - k_2^2} G(m, k_1)G(n + ik_1, k_2)G(n - ik_1, -k_2)G(m, -k_1) 
\]

(48)

and

\[
 f_{IV}(t_1 - t_2) = \delta_{IV}(g_n(t_1, t_2)) = \int_{t_1}^{t_2} dt_3 \int_{t_2}^{t_3} dt_4 \langle g_n(t_1, t_2)g_n(t_3, t_4) \rangle_{IV} = \\
 \int_0^\infty \int_0^\infty \frac{dk_1}{2\pi} \frac{dk_2}{2\pi} e^{-k_1^2(t_1-t_2)} - e^{-k_2^2(t_1-t_2)} (1 + (k_1^2 - k_2^2)(t_1 - t_2)) \\
 G(n, k_1)G(m + ik_1, k_2)G(m - ik_1, -k_2)G(n, -k_1) 
\]

(49)

Finally,

\[
 f_Z(t_1 - t_2) = \delta_Z(g_n(t_1, t_2)) = \int_{t_3 > t_4, t_4 < t_5, t_5 > t_2} dt_3 dt_4 \langle g_n(t_1, t_2) \rangle \langle g_n(t_3, t_4) \rangle = \\
 \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} e^{-k_1^2(t_1-t_2)} (1 + k_2^2(t_1-t_2)) \\
 G(n, k_1)G(n, -k_1)G(m, k_2)G(m, -k_2) 
\]

(50)

It is convenient to split \( f_Z \) in two parts:

\[
 f_{Z,III}(t) = \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} \frac{e^{-k_1^2t}}{k_1^2} G(m, k_1)G(n, -k_1)G(n + ik_1, k_2)G(n - ik_1, -k_2)G(m, k_2)G(m, -k_2) 
\]

(51)

\[
 f_{Z,IV}(t) = \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} \frac{e^{-k_2^2t}}{k_2^2} G(n, k_1)G(n, -k_1)G(m, k_2)G(m, -k_2) 
\]

(52)

The following combinations are free from divergencies upon integration: \( \tilde{f}_{III}(t) = f_{III}(t) - f_{Z,III}(t) \) and \( \tilde{f}_{IV}(t) = f_{IV}(t) - f_{Z,IV}(t) \). In the next Section, we evaluate the asymptotic behaviour of the result of this integration.

**V. CONTRIBUTION FROM THE REGIONS III AND IV.**

In this Section we calculate contributions to the Green function correction coming from the time orderings 3 and 4. They can be represented explicitly as some coefficient multiplying \( \langle g_n(t) \rangle \). To calculate it, we find the asymptotic behavior of \( \tilde{f}_{III}(t) \) and \( \tilde{f}_{IV}(t) \) in the limit of long time \( t \). We start from \( \tilde{f}_{III}(t) \). We use here the fact that for \( t \gg 1 \) one has \( k_2 \ll 1 \) and \( k_1 \sim 1 \):

\[
 \tilde{f}_{III}(t) = \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} \frac{e^{-k_1^2t}}{k_1^2} G(m, k_1)G(m, -k_1)G(n + ik_1, k_2)G(n - ik_1, -k_2) - G(n, k_2)G(n, -k_2)G(m, k_1)G(m, -k_1) \\
 \Gamma(n)^4 \frac{2\sqrt{\pi}t^2}{\int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} \frac{e^{-k_1^2t}}{k_1^2} G(m, k_1)G(m, -k_1)G(n + ik_1, k_2)G(n - ik_1, -k_2)G(n, k_2)G(n, -k_2) = C_{III}(n, m) \langle g_n(t) \rangle 
\]

(53)

To evaluate the contribution of the 4th time ordering it is convenient to split it into two parts. The first one is

\[
 f_{IV,1}(t) = \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} \frac{e^{-k_1^2t} - e^{-k_2^2t}}{(k_1^2 - k_2^2)^2} G(n, k_1)G(n, -k_1)G(m + ik_1, k_2)G(m - ik_1, -k_2)G(n, k_2)G(n, -k_2) = \\
 \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} \frac{e^{-k_1^2t} - e^{-k_2^2t}}{(k_1^2 - k_2^2)^2} G(m + ik_1, k_2)G(m - ik_1, -k_2)G(n, k_2)G(n, -k_2) \\
 \frac{\Gamma^2(n + ik_1)\Gamma^2(n - ik_1)\Gamma(m + ik_1 + ik_2)\Gamma(m + ik_1 - ik_2)\Gamma(m - ik_1 + ik_2)\Gamma(m - ik_1 - ik_2)}{\Gamma(2ik_1)\Gamma(-2ik_1)\Gamma(-2ik_2)\Gamma(-2ik_2) 
\]

(54)
We symmetrize it over interchange of $k_{1,2}$:

\[
f_{IV,1}(t) = \frac{1}{2} \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} e^{-k_1^2t} - e^{-k_2^2t} \frac{\Gamma^2(n + ik_1)\Gamma^2(n - ik_1) - \Gamma^2(n + ik_2)\Gamma^2(n - ik_2)}{\Gamma(-2ik_1)\Gamma(2ik_1)} G(m + ik_1, k_2)G(m - ik_1, -k_2).
\]

As a result:

\[
f_{IV,1}(t) = \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} e^{-k_1^2t} \frac{\Gamma^2(n + ik_1)\Gamma^2(n - ik_1) - \Gamma^2(n + ik_2)\Gamma^2(n - ik_2)}{\Gamma(-2ik_1)\Gamma(2ik_1)} G(m + ik_1, k_2)G(m - ik_1, -k_2) \approx \frac{\Gamma(n)^4}{2\sqrt{\pi}t^2} \int_0^\infty \frac{dk_1}{2\pi} \frac{\Gamma^2(n + ik_1)\Gamma^2(n - ik_1) - \Gamma^4(n)}{k_1^4\Gamma(n)^4} G(m, -k_1)G(m, k_1) = C_{IV,1}(n, m g_n(t))
\]

To calculate the remaining terms from the 4th time ordering, we need to consider the following expression

\[
f_{IV,U}(t) = \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} e^{-k_1^2t} G(n, k_1)G(n, -k_1)G(m, k_2)G(m, -k_2) \quad (57)
\]

Let us evaluate $f_{IV,2} - f_{IV,U}$:

\[
f_{IV,2}(t) - f_{IV,U}(t) = \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} e^{-k_1^2t} G(n, k_1)G(n, -k_1)(G(m + ik_1, k_2)G(m - ik_1, -k_2) - G(m, k_2)G(m, -k_2)) \quad (58)
\]

For $t \gg 1$ one has $k_1 \ll 1$ and

\[
f_{IV,2}(t) - f_{IV,U}(t) = - \frac{\Gamma^4(n)}{2\sqrt{\pi}t^2} \int_0^\infty \frac{dk_2}{2\pi} e^{-k_1^2t} \frac{3\Gamma^2(-ik_2 + m)\Gamma^2(ik_2 + m)(\psi'(m - ik_2) + \psi'(m + ik_2))\sinh(2k_2\pi)}{\pi k_2} = C_{IV,1}(n, m g_n(t)), (59)
\]

where $\psi$ is digamma function. The next step is to evaluate

\[
f_{IV,U}(t) - f_{Z,IV}(t) = \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} e^{-k_1^2t} (k_1^2 - k_2^2) G(n, k_1)G(n, -k_1)G(m, k_2)G(m, -k_2) \quad (60)
\]

In this integral $k_1 \ll 1$ and

\[
f_{IV,U}(t) - f_{Z,IV}(t) \approx \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} e^{-k_1^2t} (k_1^2 - k_2^2) \frac{\Gamma^4(n)}{k_1^4k_2^4} G(m, k_2)G(m, -k_2) \quad (61)
\]

To calculate the asymptotics, we add and subtract the following expression (below we will see that it is equal to zero):

\[
\delta = \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} e^{-k_1^2t} 4k_1^4k_2^4 \Gamma^4(n)(m)G(m, k_2)G(m, -k_2) - \Gamma^4(m)4k_2^2.
\]

This gives:

\[
f_{IV,U}(t) - f_{Z,IV}(t) - \delta = \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} e^{-k_1^2t} 4k_1^4k_2^4 \Gamma^4(n)(m)G(m, k_2)G(m, -k_2) - \Gamma^4(m)4k_2^2 \quad (63)
\]

Here we can expand in $k_1$ and obtain

\[
f_{IV,U}(t) - f_{Z,IV}(t) - \delta = \frac{\Gamma^4(n)}{2\sqrt{\pi}t^2} \int_0^\infty \frac{dk_2}{2\pi} \frac{3}{2k_2^2} (G(m, k_2)G(m, -k_2) - \Gamma^4(m)4k_2^2) \equiv C_{IV,2}(n, m g_n(t)) \quad (64)
\]
The last step to do is the calculation of $\delta$:

$$\delta = 16t\Gamma^4(n)\Gamma^4(m) \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} e^{-k_1^2t} k_1^4 8t\Gamma^4(n)\Gamma^4(m) \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} e^{-k_2^2t} k_2^4$$

$$8\Gamma^4(n)\Gamma^4(m) \frac{\partial^2}{\partial t^2} \int_0^\infty \frac{dk_1}{2\pi} \int_0^\infty \frac{dk_2}{2\pi} e^{-k_1^2t} k_1^4 - e^{-k_2^2t} k_2^4 = 0$$

Finally we have:

$$\bar{f}_{IV}(t) \equiv f_{IV}(t) - f_Z(t) = (C_{III}(n, m) + C_{IV,1}(n, m) + C_{IV,2}(n, m)) \langle g_n(t) \rangle$$

where coefficients $C_i(n, m)$ are defined in Eqs.\(53,59,64\).

VI. CONTRIBUTIONS FROM REGIONS II AND V AND THE FINAL RESULT

The time regions II and V provides equal corrections to the Green function, so we will consider the region II only. Here the correction to the Green function is

$$f_{II}(t) = \int_0^t dt_3 \int_{-\infty}^0 (g_n(t, 0)g_m(t_3, t_4)) dt_3 dt_4 =$$

$$\int_0^\infty \frac{dk_1dk_2dk_3}{(2\pi)^3} e^{-k_1^2t} - e^{-k_2^2t} G(n, k_1)G(m, -k_3)G(n + ik_2, k_3)G(m - ik_2, -k_1)$$

$$\Gamma(2ik_2)\Gamma(-2ik_2) \times G^{2,4}_{4,4}(1, 1)$$

Here $G^{2,4}_{4,4}$ is Meijer G-function. In the limit $t \to \infty$ we can obtain the following asymptotic formula for this function:

$$f_{II}(t) = \langle g_n(t) \rangle (C_{II,1}(n, m) + C_{II,2}(n, m))$$

where coefficients $C_i(n, m)$ are given by

$$C_{II,1}(n, m) = \int_0^\infty \frac{dk_3}{k_2k_3^4\Gamma(n)^2} \Gamma(-ik_2 + m)^2 \Gamma(m + ik_3)\Gamma(n + ik_2 - ik_3)\Gamma(n + ik_2 + ik_3)$$

$$\times G^{2,4}_{4,4}(1, 1)$$

$$C_{II,2}(n, m) = \int_0^\infty \frac{dk_3}{k_1k_3\pi^4\Gamma(n)} \Gamma(m - ik_1)\Gamma(m + ik_3)\Gamma(n - ik_1)\Gamma(m - ik_3)\Gamma(n + ik_1)$$

$$\times G^{2,4}_{4,4}(1, 1)$$

We combine now Eqs.\(53,60,65,69\) to obtain the complete result for the relative correction to the Green function:

$$\frac{\delta \langle G^2 \rangle}{\langle G^2 \rangle} = \frac{Nt^2\Delta}{\frac{m}{\Gamma(2m)(2M)^{2m-2}}} [2C_{III}(n, m) + C_{IV,1}(n, m) + C_{IV,2}(n, m) + 2(C_{II,1}(n, m) + C_{II,2}(n, m))]$$

Now we set $n = \frac{1}{3}$, $m = \frac{1}{2}$ and $\Delta = \frac{1}{2}$ in the above Eq.\(70\) and obtain the result for the first order correction to the Green function of the SY$^2$K$_4$ model in presence of SYK$_2$ perturbation, as it is presented in Eq.\(11\) of the main text.