STATISTICS ON SMALL GRAPHS

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Abstract. We create the unlabeled or vertex-labeled graphs with up to 10 edges and up to 10 vertices and classify them by a set of standard properties: directed or not, vertex-labeled or not, connectivity, presence of isolated vertices, presence of multiedges and presence of loops. We present tables of how many graphs exist in these categories.

1. Classifications

A finite graph on \( V \) vertices with \( E \) edges may be classified by some properties, which it either does have or does not:

- Each edge in a directed graph has one of two orientations. Edges in unoriented graphs do not have orientations. We reserve the tag \( d \) for the directed and \( \neg d \) for the undirected graphs. The adjacency matrices of undirected graphs are symmetric.

- Graphs may have at least one loop (loops are defined as edges that start and end at the same vertex), or may be loopless. The adjacency matrices of loopless graphs have zero trace. We reserve the tag \( l \) for the graphs with at least one loop and the tag \( \neg l \) for the loopless graphs.

- A multiedge is a collection of two or more edges having identical endpoints \([4, D7]\). This implies that in a directed graph two edges of opposite sense do not yet establish a multiedge. We reserve the tag \( m \) for the graphs with at least one multiedge and \( \neg m \) for the others.

- A undirected graph is connected if one can walk from any vertex to any other vertex of the graph along edges. A directed graph is (weakly) connected if replacing each arc with an undirected edge (defining the underlying graph \([4, D24]\)) reduces to a connected undirected graph. This implies that for the sake of weak connectivity it is not required that all arcs are traversed along their orientation to walk from one vertex to the other. We reserve the tag \( c \) for the graphs which are (weakly) connected and \( \neg c \) for the others.

A directed graph is strongly connected if one can walk from any vertex to any other vertex of the graph along edges in the directions demanded by their orientation. We reserve the tag \( \mathcal{C} \) for the digraphs which are strongly connected and \( \neg \mathcal{C} \) for the others.

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• An isolated vertex is a vertex with no edge to any other vertex (so all its edges are loops). There is a loose relation with connectivity, because an isolated vertex in a graph with two or more vertices means the graph is disconnected. (There are disconnected graphs without isolated vertices...where each component contains at least two vertices.) We reserve the tag $i$ for the graphs which have at least one isolated vertex and $-i$ for the others. Therefore all graphs with $V = 1$ are getting the $i$ tag.

There are no graphs with the following combinations of tags

• d-Cci A directed, weakly connected graph with at least one isolated vertex has only this vertex (because with two or more vertices the graph could not be connected), and therefore the graph must also be strongly connected. So the $-C$ contradicts the other tags.

• dC-c If the directed graph is strongly connected, it is also weakly connected, so the $-c$ tag contradicts the $C$ tag.

There are some non-interesting cases, which are not tabulated explicitly:

• There is the case with the tags $dCci$: A directed strongly-connected graph with isolated vertices has only one vertex, so a table with these graphs counts at most 1 graph for any number of edges (which all are loops).

• Similarly there is the case with the tags $-dci$: An undirected connected graph with isolated vertices has only one vertex, so a table with these graphs counts at most 1 graph for any number of edges (where all edges are loops).

There are many other characterizations of graphs concerning cycles, paths, diameters, transitivity and so on which are not dealt with here.
Tables 1–32 collect the statistics of directed graphs; Tables 33–56 collect the statistics of undirected graphs. Rows and columns are sorted along the number $E$ of edges and along the number $V$ of vertices. There are always successive tables referring to the unlabeled graphs and referring to the vertex-labeled graphs. (The latter count is obtained by weighting each unlabeled graph by the number of distinct adjacency matrices that are created by row-column permutations of the adjacency matrix. This weight is a divisor of the order of the permutation group on $V$ elements [5, Thm 15.2].)

2.1. Directed Graphs.
### Table 3. d-Cc-i-m-l unlabeled

| E \ V | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|       | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
|       | 1   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
|       | 2   | 0   | 0   | 3   | 0   | 0   | 0   | 0   | 0   | 0   |
|       | 3   | 0   | 0   | 3   | 8   | 0   | 0   | 0   | 0   | 0   |
|       | 4   | 0   | 0   | 2   | 21  | 27  | 0   | 0   | 0   | 0   |
|       | 5   | 0   | 0   | 0   | 33  | 107 | 91  | 0   | 0   | 0   |
|       | 6   | 0   | 0   | 0   | 31  | 319 | 581 | 350 | 0   | 0   |
|       | 7   | 0   | 0   | 0   | 16  | 609 | 2422| 3023|     |     |
|       | 8   | 0   | 0   | 0   | 5   | 887 | 7529|     |     |     |
|       | 9   | 0   | 0   | 0   | 2   | 912 |     |     |     |     |
|       | 10  | 0   | 0   | 0   | 0   |     |     |     |     |     |

### Table 4. d-Cc-i-m-l vertex-labeled

| E \ V | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|       | 0   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |
|       | 1   | 0   | 2   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
|       | 2   | 0   | 0   | 12  | 0   | 0   | 0   | 0   | 0   | 0   |
|       | 3   | 0   | 0   | 18  | 128 | 0   | 0   | 0   | 0   | 0   |
|       | 4   | 0   | 0   | 6   | 426 | 2000| 0   | 0   | 0   | 0   |
|       | 5   | 0   | 0   | 0   | 684 | 11080| 41472| 0   | 0   | 0   |
|       | 6   | 0   | 0   | 0   | 604 | 33160| 337800| 1075648| 0   | 0   |
|       | 7   | 0   | 0   | 0   | 300 | 67040| 1529520| 11967984|     |     |
|       | 8   | 0   | 0   | 0   | 78  | 96610| 4954230|     |     |     |
|       | 9   | 0   | 0   | 0   | 8   | 101580|     |     |     |     |
|       | 10  | 0   | 0   | 0   | 0   |     |     |     |     |     |

### Table 5. d-C-ci-m-l unlabeled

| E \ V | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|       | 0   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |
|       | 1   | 0   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |
|       | 2   | 0   | 0   | 1   | 4   | 5   | 5   | 5   | 5   | 5   |
|       | 3   | 0   | 0   | 0   | 4   | 13  | 16  | 17  | 17  | 17  |
|       | 4   | 0   | 0   | 0   | 4   | 27  | 61  | 76  | 79  | 80  |
|       | 5   | 0   | 0   | 0   | 1   | 38  | 154 | 288 | 346 | 361 |
|       | 6   | 0   | 0   | 0   | 1   | 48  | 379 | 1043| 1637| 1894|
|       | 7   | 0   | 0   | 0   | 0   | 38  | 707 | 3242|     |     |
|       | 8   | 0   | 0   | 0   | 0   | 27  | 1155|     |     |     |
|       | 9   | 0   | 0   | 0   | 0   | 13  |     |     |     |     |
|       | 10  | 0   | 0   | 0   | 0   |     |     |     |     |     |
| \( E \setminus V \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 6 | 12 | 20 | 30 | 42 | 56 | 72 | 90 |
| 2 | 0 | 0 | 3 | 54 | 190 | 435 | 861 | 1540 | 2556 | 4005 |
| 3 | 0 | 0 | 0 | 80 | 900 | 3940 | 11480 | 27720 | 59640 | 117480 |
| 4 | 0 | 0 | 0 | 60 | 2325 | 21945 | 106890 | 365610 | 1028790 | 2555190 |
| 5 | 0 | 0 | 0 | 24 | 3900 | 81264 | 699468 | 3628296 | 13870584 |
| 6 | 0 | 0 | 0 | 4 | 4610 | 218720 | 3374770 | 27446524 | 148477308 |
| 7 | 0 | 0 | 0 | 0 | 3960 | 453240 | 12650400 |
| 8 | 0 | 0 | 0 | 0 | 2475 | 748395 |
| 9 | 0 | 0 | 0 | 0 | 1100 |
| 10 | 0 | 0 | 0 | 0 | 0 |

Table 6. \( d-C-ci-m-1 \) vertex-labeled

| \( E \setminus V \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 4 | 7 | 1 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 7 | 35 | 42 | 7 | 1 | 0 |
| 6 | 0 | 0 | 0 | 12 | 101 | 271 | 234 | 48 | 7 |
| 7 | 0 | 0 | 0 | 16 | 230 | 1057 | 1848 |
| 8 | 0 | 0 | 0 | 24 | 462 | 3285 |
| 9 | 0 | 0 | 0 | 30 | 855 |
| 10 | 0 | 0 | 0 | 41 |

Table 7. \( d-C-ci-m-1 \) unlabeled

| \( E \setminus V \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 24 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 72 | 720 | 360 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 132 | 3520 | 22560 | 20160 | 6720 | 0 |
| 6 | 0 | 0 | 0 | 210 | 10100 | 154650 | 806400 | 974400 | 604800 |
| 7 | 0 | 0 | 0 | 312 | 23120 | 630360 | 7141008 |
| 8 | 0 | 0 | 0 | 441 | 46970 | 1991325 |
| 9 | 0 | 0 | 0 | 600 | 88280 |
| 10 | 0 | 0 | 0 | 792 |

Table 8. \( d-C-ci-m-1 \) vertex-labeled
### Table 9. \( d-\text{Cc-im-l} \) unlabeled

| \( E \setminus V \) | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-------------------|----|----|----|----|----|----|----|----|----|----|
| 0                 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 1                 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 2                 | 0  | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 3                 | 0  | 1  | 4  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 4                 | 0  | 1  | 16 | 18 | 0  | 0  | 0  | 0  | 0  | 0  |
| 5                 | 0  | 1  | 30 | 109| 80 | 0  | 0  | 0  | 0  | 0  |
| 6                 | 0  | 1  | 53 | 391| 694| 367| 0  | 0  | 0  | 0  |
| 7                 | 0  | 1  | 77 | 1042| 3574| 4207| 1708| 0  | 0  | 0  |
| 8                 | 0  | 1  | 116| 2402| 14093| 29082| 0  | 0  | 0  | 0  |
| 9                 | 0  | 1  | 156| 5001 | 46144| 0  | 0  | 0  | 0  | 0  |
| 10                | 0  | 1  | 215| 9737| 0  | 0  | 0  | 0  | 0  | 0  |

### Table 10. \( d-\text{Cc-im-l} \) vertex-labeled

| \( E \setminus V \) | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-------------------|----|----|----|----|----|----|----|----|----|----|
| 0                 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 1                 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 2                 | 0  | 2  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 3                 | 0  | 2  | 24| 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 4                 | 0  | 2  | 90 | 384| 0  | 0  | 0  | 0  | 0  | 0  |
| 5                 | 0  | 2  | 180| 2472| 8000| 0  | 0  | 0  | 0  | 0  |
| 6                 | 0  | 2  | 300| 8960| 75400| 207360| 0  | 0  | 0  | 0  |
| 7                 | 0  | 2  | 462| 24324| 405160| 2648880| 6453888| 0  | 0  | 0  |
| 8                 | 0  | 2  | 672| 56322| 1623440| 19251960| 0  | 0  | 0  | 0  |
| 9                 | 0  | 2  | 936| 118168| 5394560| 0  | 0  | 0  | 0  | 0  |
| 10                | 0  | 2  | 1260| 230760| 0  | 0  | 0  | 0  | 0  | 0  |

### Table 11. \( d-\text{C-cim-l} \) unlabeled

| \( E \setminus V \) | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-------------------|----|----|----|----|----|----|----|----|----|----|
| 0                 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 1                 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 2                 | 0  | 0  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 3                 | 0  | 0  | 2  | 6  | 7  | 7  | 7  | 7  | 7  | 7  |
| 4                 | 0  | 0  | 3  | 20 | 42 | 49 | 50 | 50 | 50 | 50 |
| 5                 | 0  | 0  | 3  | 41 | 158| 273| 315| 322| 323| 0  |
| 6                 | 0  | 0  | 4  | 82 | 506| 1302| 1940| 2174| 2222| 0  |
| 7                 | 0  | 0  | 4  | 132| 1330| 5174| 10439| 0  | 0  | 0  |
| 8                 | 0  | 0  | 5  | 222| 3213| 18293| 0  | 0  | 0  | 0  |
| 9                 | 0  | 0  | 5  | 335| 7097| 0  | 0  | 0  | 0  | 0  |
| 10                | 0  | 0  | 6  | 511| 0  | 0  | 0  | 0  | 0  | 0  |
Table 12. d-C-cim-1 vertex-labeled

| E \ V | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0     | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 1     | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 2     | 0   | 0   | 6   | 12  | 20  | 30  | 42  | 56  | 72  |     |
| 3     | 0   | 0   | 12  | 120 | 400 | 900 | 1764| 3136| 5184|     |
| 4     | 0   | 0   | 15  | 414 | 3290| 13155|37065|87836|186660|    |
| 5     | 0   | 0   | 18  | 948 | 15480|113190|499926|1634976|4483296|    |
| 6     | 0   | 0   | 21  | 1802| 52720|667375|4685387|22082536|80250072|   |
| 7     | 0   | 0   | 24  | 3120| 147320|3031920|33055848|   |     |   |
| 8     | 0   | 0   | 27  | 5094| 362655|11463930|   |     |   |   |
| 9     | 0   | 0   | 30  | 7948| 818780|   |     |   |   |   |
| 10    | 0   | 0   | 33  | 11946|     |     |     |     |     |     |

Table 13. d-C-c-i-ml unlabeled

| E \ V | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0     | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 1     | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 2     | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 3     | 0   | 0   | 0   | 2   | 0   | 0   | 0   | 0   | 0   | 0   |
| 4     | 0   | 0   | 0   | 7   | 13  | 2   | 0   | 0   | 0   | 0   |
| 5     | 0   | 0   | 0   | 7   | 52  | 70  | 13  | 2   | 0   |     |
| 6     | 0   | 0   | 0   | 6   | 106 | 373 | 362 | 82  | 13  |     |
| 7     | 0   | 0   | 0   | 2   | 137 | 1092| 2392|     |     |     |
| 8     | 0   | 0   | 0   | 1   | 125 | 2262|     |     |     |     |
| 9     | 0   | 0   | 0   | 0   | 83  |     |     |     |     |     |
| 10    | 0   | 0   | 0   | 0   |     |     |     |     |     |     |

Table 14. d-C-c-i-ml vertex-labeled

| E \ V | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0     | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 1     | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 2     | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 3     | 0   | 0   | 0   | 48  | 0   | 0   | 0   | 0   | 0   | 0   |
| 4     | 0   | 0   | 0   | 120 | 1200| 720 | 0   | 0   | 0   | 0   |
| 5     | 0   | 0   | 0   | 132 | 5000| 34560|35280|13440|0   |     |
| 6     | 0   | 0   | 0   | 78  | 10100|202920|1164240|1579200|1088640|    |
| 7     | 0   | 0   | 0   | 24  | 12750|630360|8919176|     |     |     |
| 8     | 0   | 0   | 0   | 3   | 10940|1314405|     |     |     |     |
| 9     | 0   | 0   | 0   | 0   | 6570|     |     |     |     |     |
| 10    | 0   | 0   | 0   | 0   |     |     |     |     |     |     |
Table 15. d-Cc-i-ml unlabeled

| E\V | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 2   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 3   | 0   | 1   | 7   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 4   | 0   | 0   | 16  | 26  | 0   | 0   | 0   | 0   | 0   | 0   |
| 5   | 0   | 0   | 16  | 111 | 107 | 0   | 0   | 0   | 0   | 0   |
| 6   | 0   | 0   | 7   | 262 | 702 | 458 | 0   | 0   | 0   | 0   |
| 7   | 0   | 0   | 2   | 372 | 2663| 4251| 2058| 0   | 0   | 0   |
| 8   | 0   | 0   | 0   | 361 | 6936| 22925| 0   | 0   | 0   | 0   |
| 9   | 0   | 0   | 0   | 240 | 13442| 0   | 0   | 0   | 0   | 0   |
| 10  | 0   | 0   | 0   | 115 | 0   | 0   | 0   | 0   | 0   | 0   |

Table 16. d-Cc-i-ml vertex-labeled

| E\V | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 2   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 3   | 0   | 2   | 36  | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 4   | 0   | 0   | 90  | 512 | 0   | 0   | 0   | 0   | 0   | 0   |
| 5   | 0   | 0   | 84  | 2472| 10000| 0   | 0   | 0   | 0   | 0   |
| 6   | 0   | 0   | 36  | 5804| 75400| 248832| 0   | 0   | 0   | 0   |
| 7   | 0   | 0   | 6   | 8352| 296600| 2648880| 7529536| 0   | 0   | 0   |
| 8   | 0   | 0   | 0   | 7986| 787600| 15073560| 0   | 0   | 0   | 0   |
| 9   | 0   | 0   | 0   | 5212| 1542450| 0   | 0   | 0   | 0   | 0   |
| 10  | 0   | 0   | 0   | 2304| 0   | 0   | 0   | 0   | 0   | 0   |
### Table 17. d-C-ci-ml unlabeled

| $E \setminus V$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----------------|---|---|---|---|---|---|---|---|---|----|
| 0               | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 1               | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  |
| 2               | 0 | 1 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4  |
| 3               | 0 | 0 | 6 | 17 | 20 | 20 | 20 | 20 | 20 | 20 |
| 4               | 0 | 0 | 3 | 35 | 83 | 100 | 103 | 103 | 103 | 103 |
| 5               | 0 | 0 | 1 | 46 | 236 | 457 | 548 | 565 | 568 |   |
| 6               | 0 | 0 | 0 | 40 | 504 | 1659 | 2756 | 3210 | 3313 |   |
| 7               | 0 | 0 | 0 | 25 | 833 | 4986 | 12171 |   |   |   |
| 8               | 0 | 0 | 0 | 10 | 1064 | 12330 |   |   |   |   |
| 9               | 0 | 0 | 0 | 3  | 1084 |   |   |   |   |   |
| 10              | 0 | 0 | 0 | 1  |   |   |   |   |   |   |

### Table 18. d-C-ci-ml vertex-labeled

| $E \setminus V$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----------------|---|---|---|---|---|---|---|---|---|----|
| 0               | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 1               | 0 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 9  |
| 2               | 0 | 1 | 21 | 54 | 110 | 195 | 315 | 476 | 684 |   |
| 3               | 0 | 0 | 28 | 292 | 1160 | 3080 | 6944 | 13944 | 25680 |   |
| 4               | 0 | 0 | 15 | 693 | 6605 | 30780 | 99946 | 268086 | 634950 |   |
| 5               | 0 | 0 | 3  | 948 | 22626 | 199926 | 1020936 | 3791256 | 11630052 |   |
| 6               | 0 | 0 | 0 | 830 | 52720 | 902265 | 7573790 | 40926732 | 167212668 |   |
| 7               | 0 | 0 | 0 | 480 | 89990 | 3031920 | 42473544 |   |   |   |
| 8               | 0 | 0 | 0 | 180 | 117425 | 7978770 |   |   |   |   |
| 9               | 0 | 0 | 0 | 40 | 119900 |   |   |   |   |   |
| 10              | 0 | 0 | 0 | 4  |   |   |   |   |   |   |

### Table 19. d-C-ci-iml unlabeled

| $E \setminus V$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----------------|---|---|---|---|---|---|---|---|---|----|
| 0               | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 1               | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 2               | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 3               | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 4               | 0 | 0 | 0 | 6 | 0 | 0 | 0 | 0 | 0 | 0  |
| 5               | 0 | 0 | 0 | 34 | 46 | 6 | 0 | 0 | 0 | 0  |
| 6               | 0 | 0 | 0 | 107 | 347 | 314 | 52 | 6 | 0 | 0  |
| 7               | 0 | 0 | 0 | 250 | 1473 | 2869 | 1995 |   |   |   |
| 8               | 0 | 0 | 0 | 527 | 4731 | 15676 |   |   |   |   |
| 9               | 0 | 0 | 0 | 994 | 12883 |   |   |   |   |   |
| 10              | 0 | 0 | 0 | 1797 |   |   |   |   |   |   |
| $E \setminus V$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----------------|---|---|---|---|---|---|---|---|---|---|----|
| 0              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 1              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 2              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 3              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 4              | 0 | 0 | 0 | 0 | 144 | 0 | 0 | 0 | 0 | 0 | 0  |
| 5              | 0 | 0 | 0 | 768 | 4800 | 2880 | 0 | 0 | 0 | 0 | 0  |
| 6              | 0 | 0 | 0 | 2340 | 37000 | 180000 | 176400 | 67200 | 0 | 0 | 0  |
| 7              | 0 | 0 | 0 | 5568 | 159600 | 1750320 | 7514640 | 0 | 0 | 0 | 0  |
| 8              | 0 | 0 | 0 | 11634 | 518350 | 9908640 | 0 | 0 | 0 | 0 | 0  |
| 9              | 0 | 0 | 0 | 22368 | 1427320 | 0 | 0 | 0 | 0 | 0 | 0  |
| 10             | 0 | 0 | 0 | 40392 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |

Table 20. $d$-$C^c$-$iml$ vertex-labeled

| $E \setminus V$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----------------|---|---|---|---|---|---|---|---|---|---|----|
| 0              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 1              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 2              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 3              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 4              | 0 | 0 | 0 | 0 | 19 | 0 | 0 | 0 | 0 | 0 | 0  |
| 5              | 0 | 0 | 0 | 98 | 94 | 0 | 0 | 0 | 0 | 0 | 0  |
| 6              | 0 | 0 | 0 | 286 | 761 | 479 | 0 | 0 | 0 | 0 | 0  |
| 7              | 0 | 0 | 0 | 645 | 3522 | 5398 | 2480 | 0 | 0 | 0 | 0  |
| 8              | 0 | 0 | 0 | 1290 | 12111 | 34960 | 36619 | 0 | 0 | 0 | 0  |
| 9              | 0 | 0 | 0 | 2372 | 34847 | 167682 | 0 | 0 | 0 | 0 | 0  |
| 10             | 0 | 0 | 0 | 4110 | 89361 | 0 | 0 | 0 | 0 | 0 | 0  |

Table 21. $d$-$Cc$-$iml$ unlabeled

| $E \setminus V$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----------------|---|---|---|---|---|---|---|---|---|---|----|
| 0              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 1              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 2              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 3              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 4              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 5              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 6              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 7              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 8              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 9              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 10             | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |

Table 22. $d$-$Cc$-$iml$ vertex-labeled
| $E \setminus V$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----------------|---|---|---|---|---|---|---|---|---|----|
| 0              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 1              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 2              | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  |
| 3              | 0 | 2 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8  |
| 4              | 0 | 3 | 27 | 55 | 63 | 63 | 63 | 63 | 63 | 63  |
| 5              | 0 | 3 | 55 | 224 | 402 | 468 | 468 | 468 | 468 | 468  |
| 6              | 0 | 4 | 97 | 671 | 1956 | 3051 | 3444 | 3508 | 3516 |  |  |
| 7              | 0 | 4 | 154 | 1661 | 7607 | 17024 | 23868 |  |  |  |  |
| 8              | 0 | 5 | 235 | 3670 | 25207 | 81289 |  |  |  |  |  |
| 9              | 0 | 5 | 342 | 7505 | 74029 |  |  |  |  |  |  |
| 10             | 0 | 6 | 483 | 14483 |  |  |  |  |  |  |  |

**Table 23.** d-C-ciml unlabeled

| $E \setminus V$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|----------------|---|---|---|---|---|---|---|---|---|
| 0              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2              | 0 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 3              | 0 | 4 | 45 | 112 | 225 | 396 | 637 | 960 | 1377 |
| 4              | 0 | 5 | 150 | 1042 | 3815 | 9831 | 21784 | 43268 | 79101 |
| 5              | 0 | 6 | 315 | 4744 | 33825 | 142386 | 442715 | 1157920 | 2696625 |
| 6              | 0 | 7 | 553 | 14864 | 191335 | 1339211 | 6175162 | 21778968 | 64759989 |
| 7              | 0 | 8 | 894 | 37768 | 805875 | 9075456 | 62861757 |  |  |
| 8              | 0 | 9 | 1368 | 84739 | 2785270 | 48311766 |  |  |  |  |
| 9              | 0 | 10 | 2005 | 175140 | 8398505 |  |  |  |  |  |
| 10             | 0 | 11 | 2838 | 340402 |  |  |  |  |  |  |

**Table 24.** d-C-ciml vertex-labeled
| E \ V | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   |
|-------|------|------|------|------|------|------|------|------|------|------|
| 0     | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    |
| 1     | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    |
| 2     | 0    | 1    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    |
| 3     | 0    | 0    | 1    | 0    | 0    | 0    | 0    | 0    | 0    | 0    |
| 4     | 0    | 0    | 2    | 1    | 0    | 0    | 0    | 0    | 0    | 0    |
| 5     | 0    | 0    | 1    | 4    | 1    | 0    | 0    | 0    | 0    | 0    |
| 6     | 0    | 0    | 1    | 16   | 7    | 1    | 0    | 0    | 0    | 0    |
| 7     | 0    | 0    | 0    | 22   | 58   | 10   | 1    | 0    | 0    | 0    |
| 8     | 0    | 0    | 0    | 22   | 240  | 165  | 0    | 0    | 0    | 0    |
| 9     | 0    | 0    | 0    | 11   | 565  | 0    | 0    | 0    | 0    | 0    |
| 10    | 0    | 0    | 0    | 5    | 0    | 0    | 0    | 0    | 0    | 0    |

Table 25. dCc-i-m-1 unlabeled

| E \ V | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   |
|-------|------|------|------|------|------|------|------|------|------|------|
| 0     | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    |
| 1     | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    |
| 2     | 0    | 1    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    |
| 3     | 0    | 0    | 2    | 0    | 0    | 0    | 0    | 0    | 0    | 0    |
| 4     | 0    | 0    | 9    | 6    | 0    | 0    | 0    | 0    | 0    | 0    |
| 5     | 0    | 0    | 6    | 84   | 24   | 0    | 0    | 0    | 0    | 0    |
| 6     | 0    | 0    | 1    | 316  | 720  | 120  | 0    | 0    | 0    | 0    |
| 7     | 0    | 0    | 0    | 492  | 6440 | 6480 | 720  | 0    | 0    | 0    |
| 8     | 0    | 0    | 0    | 417  | 26875| 107850| 0    | 0    | 0    | 0    |
| 9     | 0    | 0    | 0    | 212  | 65280| 0    | 0    | 0    | 0    | 0    |
| 10    | 0    | 0    | 0    | 66   | 0    | 0    | 0    | 0    | 0    | 0    |

Table 26. dCc-i-m-1 vertex-labeled

| E \ V | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   |
|-------|------|------|------|------|------|------|------|------|------|------|
| 0     | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    |
| 1     | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    |
| 2     | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    |
| 3     | 0    | 0    | 1    | 0    | 0    | 0    | 0    | 0    | 0    | 0    |
| 4     | 0    | 0    | 2    | 1    | 0    | 0    | 0    | 0    | 0    | 0    |
| 5     | 0    | 0    | 8    | 1    | 0    | 0    | 0    | 0    | 0    | 0    |
| 6     | 0    | 0    | 3    | 25   | 21   | 1    | 0    | 0    | 0    | 0    |
| 7     | 0    | 0    | 3    | 51   | 140  | 40   | 1    | 0    | 0    | 0    |
| 8     | 0    | 0    | 4    | 101  | 565  | 525  | 69   | 0    | 0    | 0    |
| 9     | 0    | 0    | 4    | 174  | 1731 | 3719 | 0    | 0    | 0    | 0    |
| 10    | 0    | 0    | 5    | 290  | 4602 | 0    | 0    | 0    | 0    | 0    |

Table 27. dCc-im-1 unlabeled
### Table 28. $dCc-im-1$ vertex-labeled

| $E \setminus V$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-----------------|----|----|----|----|----|----|----|----|----|----|
| 0               | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 1               | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 2               | 2  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 3               | 3  | 2  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 4               | 4  | 3  | 6  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 5               | 5  | 4  | 48 | 24 | 0  | 0  | 0  | 0  | 0  | 0  |
| 6               | 6  | 5  | 140| 480| 120| 0  | 0  | 0  | 0  | 0  |
| 7               | 7  | 6  | 306| 3276| 4680| 720| 0  | 0  | 0  | 0  |
| 8               | 8  | 7  | 588| 13230| 61040| 47880| 0  | 0  | 0  | 0  |
| 9               | 9  | 8  | 1036| 41024| 437320| 0  | 0  | 0  | 0  | 0  |
| 10              | 10 | 9  | 1710| 109152| 0  | 0  | 0  | 0  | 0  | 0  |

### Table 29. $dCc-i-ml$ unlabeled

| $E \setminus V$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-----------------|----|----|----|----|----|----|----|----|----|----|
| 0               | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 1               | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 2               | 2  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 3               | 3  | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 4               | 4  | 1  | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 5               | 5  | 0  | 6  | 1  | 0  | 0  | 0  | 0  | 0  | 0  |
| 6               | 6  | 0  | 9  | 17 | 1  | 0  | 0  | 0  | 0  | 0  |
| 7               | 7  | 0  | 6  | 78 | 34 | 1  | 0  | 0  | 0  | 0  |
| 8               | 8  | 0  | 2  | 185| 346| 60 | 0  | 0  | 0  | 0  |
| 9               | 9  | 0  | 1  | 259| 1775| 0  | 0  | 0  | 0  | 0  |
| 10              | 10 | 0  | 0  | 252| 0  | 0  | 0  | 0  | 0  | 0  |

### Table 30. $dCc-i-ml$ vertex-labeled

| $E \setminus V$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-----------------|----|----|----|----|----|----|----|----|----|----|
| 0               | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 1               | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 2               | 2  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 3               | 3  | 2  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 4               | 4  | 1  | 6  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 5               | 5  | 0  | 33 | 24 | 0  | 0  | 0  | 0  | 0  | 0  |
| 6               | 6  | 0  | 47 | 372| 120| 0  | 0  | 0  | 0  | 0  |
| 7               | 7  | 0  | 30 | 1792| 3840| 720| 0  | 0  | 0  | 0  |
| 8               | 8  | 0  | 9  | 4206| 39640| 40680| 0  | 0  | 0  | 0  |
| 9               | 9  | 0  | 1  | 5968| 206095| 0  | 0  | 0  | 0  | 0  |
| 10              | 10 | 0  | 0  | 5634| 0  | 0  | 0  | 0  | 0  | 0  |
\begin{table}
\centering
\begin{tabular}{|c|cccccccccc|}
\hline
$E \backslash V$ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 8 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 16 & 38 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 0 & 25 & 151 & 110 & 6 & 0 & 0 & 0 & 0 & 0 \\
8 & 0 & 40 & 431 & 898 & 250 & 7 & 0 & 0 & 0 & 0 \\
9 & 0 & 56 & 1040 & 4475 & 3665 & 0 & 0 & 0 & 0 & 0 \\
10 & 0 & 80 & 2252 & 17039 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{tabular}
\caption{dCc-iml unlabeled}
\end{table}

\begin{table}
\centering
\begin{tabular}{|c|cccccccccc|}
\hline
$E \backslash V$ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 16 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 30 & 225 & 120 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 0 & 50 & 897 & 2592 & 720 & 0 & 0 & 0 & 0 & 0 \\
8 & 0 & 77 & 2562 & 21196 & 29400 & 5040 & 0 & 0 & 0 & 0 \\
9 & 0 & 112 & 6190 & 106336 & 431360 & 0 & 0 & 0 & 0 & 0 \\
10 & 0 & 156 & 13437 & 405552 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{tabular}
\caption{dCc-iml vertex-labeled}
\end{table}
### SMALL UNLABELED GRAPHS

#### Table 33. \(-d-c-i-m-l\) unlabeled

| $E \setminus V$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0               | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 1               | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 2               | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   |
| 3               | 0   | 0   | 0   | 1   | 0   | 1   | 0   | 0   | 0   | 0   |
| 4               | 0   | 0   | 0   | 1   | 3   | 1   | 1   | 0   | 0   | 0   |
| 5               | 0   | 0   | 0   | 0   | 3   | 6   | 3   | 1   | 0   | 0   |
| 6               | 0   | 0   | 0   | 0   | 2   | 9   | 15  | 7   | 0   | 0   |
| 7               | 0   | 0   | 0   | 0   | 0   | 1   | 8   | 0   | 0   | 0   |
| 8               | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 9               | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 10              | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |

#### Table 34. \(-d-c-i-m-l\) vertex-labeled

| $E \setminus V$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0               | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 1               | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 2               | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 3               | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 4               | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 5               | 0   | 0   | 0   | 0   | 3   | 30  | 15  | 0   | 0   | 0   |
| 6               | 0   | 0   | 0   | 0   | 0   | 285 | 4410| 5880| 3780| 0   |
| 7               | 0   | 0   | 0   | 0   | 0   | 100 | 6797| 71078|116550| 0   |
| 8               | 0   | 0   | 0   | 0   | 0   | 15  | 5460| 0   | 0   | 0   |
| 9               | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   |
| 10              | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |

#### Table 35. \(-dc-i-m-l\) unlabeled

| $E \setminus V$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0               | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 1               | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 2               | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 3               | 0   | 0   | 1   | 2   | 0   | 0   | 0   | 0   | 0   | 0   |
| 4               | 0   | 0   | 0   | 2   | 3   | 0   | 0   | 0   | 0   | 0   |
| 5               | 0   | 0   | 0   | 1   | 5   | 6   | 0   | 0   | 0   | 0   |
| 6               | 0   | 0   | 0   | 1   | 5   | 13  | 11  | 0   | 0   | 0   |
| 7               | 0   | 0   | 0   | 4   | 19  | 33  | 0   | 0   | 0   | 0   |
| 8               | 0   | 0   | 0   | 2   | 22  | 0   | 0   | 0   | 0   | 0   |
| 9               | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   |
| 10              | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |

#### 2.2. Undirected Graphs.
Table 36. \(-dc-i-m-1\) vertex-labeled

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|----|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 2 | 0 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 3 | 0 | 0 | 1 | 16 | 0 | 0 | 0 | 0 | 0 | 0  |
| 4 | 0 | 0 | 0 | 15 | 125 | 0 | 0 | 0 | 0 | 0  |
| 5 | 0 | 0 | 0 | 6 | 222 | 1296 | 0 | 0 | 0 | 0  |
| 6 | 0 | 0 | 0 | 1 | 205 | 3660 | 16807 | 0 | 0 | 0  |
| 7 | 0 | 0 | 0 | 0 | 120 | 5700 | 68295 | 0 | 0 | 0  |
| 8 | 0 | 0 | 0 | 0 | 45 | 6165 | 0 | 0 | 0 | 0  |
| 9 | 0 | 0 | 0 | 0 | 10 | 0 | 0 | 0 | 0 | 0  |
| 10| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |

Table 37. \(-dci-m-1\) unlabeled

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|----|
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  |
| 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  |
| 2 | 0 | 0 | 0 | 1 | 2 | 2 | 2 | 2 | 2 | 2  |
| 3 | 0 | 0 | 0 | 1 | 3 | 4 | 5 | 5 | 5 | 5  |
| 4 | 0 | 0 | 0 | 0 | 2 | 6 | 9 | 10 | 11 | 11 |
| 5 | 0 | 0 | 0 | 0 | 1 | 6 | 15 | 21 | 24 | 0  |
| 6 | 0 | 0 | 0 | 0 | 1 | 6 | 21 | 41 | 56 | 0  |
| 7 | 0 | 0 | 0 | 0 | 0 | 4 | 24 | 0 | 0 | 0  |
| 8 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0  |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 10| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |

Table 38. \(-dci-m-1\) vertex-labeled

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|----|
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  |
| 1 | 0 | 0 | 3 | 6 | 10 | 15 | 21 | 28 | 36 | 45 |
| 2 | 0 | 0 | 0 | 12 | 45 | 105 | 210 | 378 | 630 | 990 |
| 3 | 0 | 0 | 0 | 4 | 90 | 440 | 1330 | 3276 | 7140 | 14190 |
| 4 | 0 | 0 | 0 | 0 | 75 | 1035 | 5670 | 20370 | 58905 | 148995 |
| 5 | 0 | 0 | 0 | 0 | 30 | 1422 | 15939 | 92400 | 373212 | 0  |
| 6 | 0 | 0 | 0 | 0 | 5 | 1245 | 30660 | 305662 | 1831242 | 0  |
| 7 | 0 | 0 | 0 | 0 | 0 | 720 | 42525 | 0 | 0 | 0  |
| 8 | 0 | 0 | 0 | 0 | 0 | 270 | 0 | 0 | 0 | 0  |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 10| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| $E \setminus V$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----------------|---|---|---|---|---|---|---|---|---|---|
| 0              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 1              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 2              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 3              | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0  |
| 4              | 0 | 0 | 0 | 2 | 2 | 1 | 0 | 0 | 0 | 0  |
| 5              | 0 | 0 | 0 | 2 | 6 | 8 | 2 | 1 | 0 | 0  |
| 6              | 0 | 0 | 0 | 3 | 10| 25| 21| 9 | 2 | 0  |
| 7              | 0 | 0 | 0 | 3 | 16| 53| 80|   |   |   |
| 8              | 0 | 0 | 0 | 4 | 23| 102|   |   |   |   |
| 9              | 0 | 0 | 0 | 4 | 32|   |   |   |   |   |
| 10             | 0 | 0 | 0 | 5|   |   |   |   |   |   |

Table 39. -d-c-im-l unlabeled

| $E \setminus V$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----------------|---|---|---|---|---|---|---|---|---|---|
| 0              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 1              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 2              | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 3              | 0 | 0 | 0 | 6 | 0 | 0 | 0 | 0 | 0 | 0  |
| 4              | 0 | 0 | 0 | 9 | 90| 45| 0 | 0 | 0 | 0  |
| 5              | 0 | 0 | 0 | 12| 220|1410|1260|420|0 | 0  |
| 6              | 0 | 0 | 0 | 15| 400|4875|25200|30450|18900|   |
| 7              | 0 | 0 | 0 | 18| 650|11700|113232|   |   |   |
| 8              | 0 | 0 | 0 | 21| 980|24045|   |   |   |   |
| 9              | 0 | 0 | 0 | 24| 1400|   |   |   |   |   |
| 10             | 0 | 0 | 0 | 27|   |   |   |   |   |   |

Table 40. -d-c-im-l vertex-labeled
| $E \setminus V$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|--------------|---|---|---|---|---|---|---|---|---|----|
| 0            | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 1            | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 2            | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 3            | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 4            | 0 | 1 | 3 | 3 | 0 | 0 | 0 | 0 | 0 | 0  |
| 5            | 0 | 1 | 4 | 10 | 6 | 0 | 0 | 0 | 0 | 0  |
| 6            | 0 | 1 | 6 | 21 | 29 | 16 | 0 | 0 | 0 | 0  |
| 7            | 0 | 1 | 7 | 37 | 81 | 91 | 37 | 0 | 0 | 0  |
| 8            | 0 | 1 | 9 | 61 | 191 | 326 | 0 | 0 | 0 | 0  |
| 9            | 0 | 1 | 11 | 95 | 395 | 0 | 0 | 0 | 0 | 0  |
| 10           | 0 | 1 | 13 | 141 | 0 | 0 | 0 | 0 | 0 | 0  |

**Table 41.** $-dc$-$im$-$l$ unlabeled

| $E \setminus V$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|--------------|---|---|---|---|---|---|---|---|---|----|
| 0            | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 1            | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 2            | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 3            | 0 | 1 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 4            | 0 | 1 | 12 | 48 | 0 | 0 | 0 | 0 | 0 | 0  |
| 5            | 0 | 1 | 18 | 156 | 500 | 0 | 0 | 0 | 0 | 0  |
| 6            | 0 | 1 | 25 | 340 | 2360 | 6480 | 0 | 0 | 0 | 0  |
| 7            | 0 | 1 | 33 | 636 | 7060 | 41400 | 100842 | 0 | 0 | 0  |
| 8            | 0 | 1 | 42 | 1092 | 17290 | 162120 | 0 | 0 | 0 | 0  |
| 9            | 0 | 1 | 52 | 1764 | 37740 | 0 | 0 | 0 | 0 | 0  |
| 10           | 0 | 1 | 63 | 2718 | 0 | 0 | 0 | 0 | 0 | 0  |

**Table 42.** $-dc$-$im$-$l$ vertex-labeled

| $E \setminus V$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|--------------|---|---|---|---|---|---|---|---|---|----|
| 0            | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 1            | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 2            | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  |
| 3            | 0 | 0 | 1 | 2 | 3 | 3 | 3 | 3 | 3 | 3  |
| 4            | 0 | 0 | 1 | 4 | 9 | 11 | 12 | 12 | 12 | 12 |
| 5            | 0 | 0 | 1 | 5 | 17 | 29 | 37 | 39 | 40 | 0  |
| 6            | 0 | 0 | 1 | 7 | 31 | 70 | 111 | 132 | 141 | 0 |
| 7            | 0 | 0 | 1 | 8 | 48 | 145 | 289 | 0 | 0 | 0  |
| 8            | 0 | 0 | 1 | 10 | 75 | 289 | 0 | 0 | 0 | 0  |
| 9            | 0 | 0 | 1 | 12 | 111 | 0 | 0 | 0 | 0 | 0  |
| 10           | 0 | 0 | 1 | 14 | 0 | 0 | 0 | 0 | 0 | 0  |

**Table 43.** $-d$-$cim$-$l$ unlabeled
### Table 44. \(-d\)-cim-1 vertex-labeled

| \(E \setminus V\) | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0                 | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 1                 | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 2                 | 0   | 0   | 3   | 6   | 10  | 15  | 21  | 28  | 36  | 45  |
| 3                 | 0   | 0   | 3   | 3   | 10   | 22   | 44   | 78   | 129   | 196   | 2025   |
| 4                 | 0   | 0   | 3   | 54     | 45   | 1650 | 4641 | 10990 | 23346   | 45585   |
| 5                 | 0   | 0   | 3   | 78     | 1030   | 7215 | 31521 | 102676 | 281016   |
| 6                 | 0   | 0   | 3   | 106   | 2035   | 22400 | 150766 | 700378   | 2529696   |
| 7                 | 0   | 0   | 3   | 138   | 3610   | 56745 | 557676 |
| 8                 | 0   | 0   | 3   | 174   | 5995   | 127170 |
| 9                 | 0   | 0   | 3   | 214   | 9470   |
| 10                | 0   | 0   | 3   | 258   |

### Table 45. \(-d\)-c-i-ml unlabeled

| \(E \setminus V\) | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0                 | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 1                 | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 2                 | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 3                 | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   |
| 4                 | 0   | 0   | 0   | 2   | 3   | 1   | 0   | 0   | 0   | 0   |
| 5                 | 0   | 0   | 0   | 1   | 7   | 10   | 3   | 1   | 0   |
| 6                 | 0   | 0   | 0   | 1   | 8   | 28   | 28   | 11   | 3   |
| 7                 | 0   | 0   | 0   | 0   | 6   | 42   | 91   |
| 8                 | 0   | 0   | 0   | 0   | 3   | 48   |
| 9                 | 0   | 0   | 0   | 0   | 1   |
| 10                | 0   | 0   | 0   | 0   |

### Table 46. \(-d\)-c-i-ml vertex-labeled

| \(E \setminus V\) | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0                 | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 1                 | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 2                 | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 3                 | 0   | 0   | 0   | 12  | 0   | 0   | 0   | 0   | 0   | 0   |
| 4                 | 0   | 0   | 0   | 18   | 150   | 90   | 0   | 0   | 0   | 0   |
| 5                 | 0   | 0   | 0   | 12   | 350   | 2205  | 2205  | 840   | 0   |
| 6                 | 0   | 0   | 0   | 3   | 400   | 6960  | 37485  | 49980  | 34020  |
| 7                 | 0   | 0   | 0   | 0   | 250   | 11700 | 151214  |
| 8                 | 0   | 0   | 0   | 0   | 80   | 12330  |
| 9                 | 0   | 0   | 0   | 0   | 10   |
| 10                | 0   | 0   | 0   | 0   |

Table 44. \(-d\)-cim-1 vertex-labeled

Table 45. \(-d\)-c-i-ml unlabeled

Table 46. \(-d\)-c-i-ml vertex-labeled
| $E \setminus V$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0               | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 1               | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 2               | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 3               | 0   | 1   | 2   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 4               | 0   | 0   | 3   | 4   | 0   | 0   | 0   | 0   | 0   | 0   |
| 5               | 0   | 0   | 2   | 10  | 9   | 0   | 0   | 0   | 0   | 0   |
| 6               | 0   | 0   | 1   | 12  | 30  | 20  | 0   | 0   | 0   | 0   |
| 7               | 0   | 0   | 0   | 10  | 57  | 93  | 48  | 0   | 0   | 0   |
| 8               | 0   | 0   | 0   | 5   | 73  | 240 | 0   | 0   | 0   | 0   |
| 9               | 0   | 0   | 0   | 2   | 67  | 0   | 0   | 0   | 0   | 0   |
| 10              | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   |

Table 47. -dc-i-ml unlabeled

| $E \setminus V$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0               | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 1               | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 2               | 0   | 2   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 3               | 0   | 1   | 9   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 4               | 0   | 0   | 12  | 64  | 0   | 0   | 0   | 0   | 0   | 0   |
| 5               | 0   | 0   | 6   | 156 | 625 | 0   | 0   | 0   | 0   | 0   |
| 6               | 0   | 0   | 1   | 178 | 2360| 7776| 0   | 0   | 0   | 0   |
| 7               | 0   | 0   | 0   | 116 | 4495| 41400| 117649| 0   | 0   | 0   |
| 8               | 0   | 0   | 0   | 45  | 5495| 115020| 0   | 0   | 0   | 0   |
| 9               | 0   | 0   | 0   | 10  | 4710| 0   | 0   | 0   | 0   | 0   |
| 10              | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   |

Table 48. -dc-i-ml vertex-labeled
### Table 49. -d-ci-ml unlabeled

| $E \setminus V$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-----------------|----|----|----|----|----|----|----|----|----|----|
| 0               | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 1               | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 2               | 2  | 2  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  |
| 3               | 3  | 3  | 7  | 9  | 9  | 9  | 9  | 9  | 9  | 9  |
| 4               | 4  | 4  | 1  | 9  | 20 | 25 | 27 | 27 | 27 | 27 |
| 5               | 5  | 5  | 0  | 0  | 6  | 30 | 58 | 74 | 79 | 81 |
| 6               | 6  | 6  | 0  | 0  | 0  | 3  | 32 | 104| 183| 226|
| 7               | 7  | 7  | 0  | 0  | 0  | 1  | 27 | 149| 381|    |
| 8               | 8  | 8  | 0  | 0  | 0  | 0  | 16 | 175|    |    |
| 9               | 9  | 9  | 0  | 0  | 0  | 0  | 7  |    |    |    |
| 10              | 10 | 10 | 0  | 0  | 0  | 0  |    |    |    |    |

### Table 50. -d-ci-ml vertex-labeled

| $E \setminus V$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-----------------|----|----|----|----|----|----|----|----|----|----|
| 0               | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 1               | 1  | 0  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
| 2               | 2  | 2  | 3  | 12 | 30 | 60 | 105| 168| 252| 360|
| 3               | 3  | 3  | 0  | 10 | 35 | 875| 1946|3864| 7050|12045|
| 4               | 4  | 0  | 1  | 113| 1005|4530|14490|38430|90090|192060|
| 5               | 5  | 0  | 0  | 78 | 1776|15141|75726|277872|844767|
| 6               | 6  | 0  | 0  | 28 | 2035|34523|284991|1521072|6163248|
| 7               | 7  | 0  | 0  | 4  | 1570|56745|798897|
| 8               | 8  | 0  | 0  | 0  | 815 |69705|
| 9               | 9  | 0  | 0  | 0  | 275 |    |
| 10              | 10 | 0  | 0  | 0  | 0  |    |    |

### Table 51. -d-c-ml unlabeled

| $E \setminus V$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-----------------|----|----|----|----|----|----|----|----|----|----|
| 0               | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 1               | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 2               | 2  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 3               | 3  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 4               | 4  | 0  | 0  | 3  | 0  | 0  | 0  | 0  | 0  | 0  |
| 5               | 5  | 0  | 0  | 11 | 10 | 3  | 0  | 0  | 0  | 0  |
| 6               | 6  | 0  | 0  | 27 | 51 | 44 | 11 | 3  | 0  | 0  |
| 7               | 7  | 0  | 0  | 51 | 157| 236| 153|
| 8               | 8  | 0  | 0  | 93 | 386| 850|
| 9               | 9  | 0  | 0  | 150| 838|
| 10              | 10 | 0  | 0  | 241|    |    |    |    |    |    |
**Table 52.** -d-c-iml vertex-labeled

| $E\setminus V$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|--------------|----|----|----|----|----|----|----|----|----|----|
| 0            | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 1            | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 2            | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 3            | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 4            | 0  | 36 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 5            | 0  | 0  | 144| 600| 360| 0  | 0  | 0  | 0  | 0  |
| 6            | 0  | 0  | 360| 3250|11925|11025|4200|0  | 0  | 0  |
| 7            | 0  | 0  | 738|10650|76635|257985|0  | 0  | 0  | 0  |
| 8            | 0  | 0  | 1365|27650|308385|0  | 0  | 0  | 0  | 0  |
| 9            | 0  | 0  | 2352|62940|0  | 0  | 0  | 0  | 0  | 0  |
| 10           | 0  | 0  | 3834|0  | 0  | 0  | 0  | 0  | 0  | 0  |

**Table 53.** -dc-iml unlabeled

| $E\setminus V$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|--------------|----|----|----|----|----|----|----|----|----|----|
| 0            | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 1            | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 2            | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 3            | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 4            | 0  | 5  | 5  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 5            | 0  | 8  | 19 | 13 | 0  | 0  | 0  | 0  | 0  | 0  |
| 6            | 0  | 11 | 45 | 70 | 35 | 0  | 0  | 0  | 0  | 0  |
| 7            | 0  | 15 | 87 | 227| 245| 95 | 0  | 0  | 0  | 0  |
| 8            | 0  | 19 | 153| 579|1029| 840|0  | 0  | 0  | 0  |
| 9            | 0  | 24 | 252|1302|3346| 0  | 0  | 0  | 0  | 0  |
| 10           | 0  | 29 | 394| 2681|0  | 0  | 0  | 0  | 0  | 0  |

**Table 54.** -dc-iml vertex-labeled

| $E\setminus V$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|--------------|----|----|----|----|----|----|----|----|----|----|
| 0            | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 1            | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 2            | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 3            | 0  | 4  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 4            | 0  | 9  | 27 | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 5            | 0  | 14 | 102| 256| 0  | 0  | 0  | 0  | 0  | 0  |
| 6            | 0  | 20 | 240|1420|3125| 0  | 0  | 0  | 0  | 0  |
| 7            | 0  | 27 | 471|4688 |23535|46656|0  | 0  | 0  | 0  |
| 8            | 0  | 35 | 840|12250|102900|453096|0  | 0  | 0  | 0  |
| 9            | 0  | 44 | 1400|28080|345730|0  | 0  | 0  | 0  | 0  |
| 10           | 0  | 54 | 2214|58914|0  | 0  | 0  | 0  | 0  | 0  |
| $E \setminus V$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-----------------|----|----|----|----|----|----|----|----|----|----|
| 0               | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 1               | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 2               | 0  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
| 3               | 0  | 4  | 27 | 64 | 125| 216| 343| 512| 729| 1000|
| 4               | 0  | 5  | 69 | 358| 1190|2946|6349|12356|22239|37630|
| 5               | 0  | 6  | 123| 1104|6275|23796|70315|177920|404109|
| 6               | 0  | 7  | 193| 2554|22585|130286|543837|1813568|5197044|
| 7               | 0  | 8  | 285| 5102|64340|538614|3165841|
| 8               | 0  | 9  | 402| 9363|158520|1829799|
| 9               | 0  | 10 | 547| 16176|354905|
| 10              | 0  | 11 | 723| 26626|

Table 55. -d-ciml unlabeled

| $E \setminus V$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-----------------|----|----|----|----|----|----|----|----|----|----|
| 0               | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 1               | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 2               | 0  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
| 3               | 0  | 4  | 27 | 64 | 125| 216| 343| 512| 729| 1000|
| 4               | 0  | 5  | 69 | 358| 1190|2946|6349|12356|22239|37630|
| 5               | 0  | 6  | 123| 1104|6275|23796|70315|177920|404109|
| 6               | 0  | 7  | 193| 2554|22585|130286|543837|1813568|5197044|
| 7               | 0  | 8  | 285| 5102|64340|538614|3165841|
| 8               | 0  | 9  | 402| 9363|158520|1829799|
| 9               | 0  | 10 | 547| 16176|354905|
| 10              | 0  | 11 | 723| 26626|

Table 56. -d-ciml vertex-labeled
3. Accumulated Marginal statistics

Adding the contents of one or more of the previous arrays defines the union of their graph sets, and regards some of the properties as irrelevant in these tables. If we look on the flags as defining a hypertable along five or six axes, these sums are the marginal sums; they create the Tables in Section 3.

The properties that are not taken into account while counting the graphs are either replaced by the filler .* or not tagged at all, using regular expressions of the usual programming languages as the tags.

Example 1. -d.*-m-l flags graphs that are undirected, have any type of isolated vertices or connectivity, but have no multiedges or loops.

Example 2. -dc flags graphs that are undirected and connected, but have any type of isolated vertices, multiedges or loops.

3.1. Undirected Graphs. Tables 57–70 summarize statistics of undirected graphs.
| $E \setminus V$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----------------|---|---|---|---|---|---|---|---|---|---|
| 0              | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1              | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 2              | 2 | 4 | 6 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| 3              | 3 | 6 | 14 | 20 | 22 | 23 | 23 | 23 | 23 | 23 |
| 4              | 4 | 9 | 28 | 53 | 69 | 76 | 78 | 79 | 79 | 79 |
| 5              | 5 | 12 | 52 | 125 | 198 | 245 | 264 | 271 | 273 | 273 |
| 6              | 6 | 16 | 93 | 287 | 550 | 782 | 915 | 973 | 993 | 993 |
| 7              | 7 | 20 | 152 | 606 | 1441 | 2392 | 3111 | 3111 | 3111 | 3111 |
| 8              | 8 | 25 | 242 | 1226 | 3611 | 7118 | 7118 | 7118 | 7118 | 7118 |
| 9              | 9 | 30 | 370 | 2358 | 8608 | 8608 | 8608 | 8608 | 8608 | 8608 |
| 10             | 10 | 36 | 546 | 4356 | 4356 | 4356 | 4356 | 4356 | 4356 | 4356 |

Table 59. (-d) unlabeled. The number of undirected graphs allowing loops and multiedges [10, A290428]. Sum of Table 63 and Table 67.

| $E \setminus V$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|----------------|---|---|---|---|---|---|---|---|---|
| 0              | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1              | 1 | 3 | 6 | 10 | 15 | 21 | 28 | 36 | 45 |
| 2              | 2 | 6 | 21 | 55 | 120 | 231 | 406 | 666 | 1035 |
| 3              | 3 | 10 | 56 | 220 | 680 | 1771 | 4060 | 8436 | 16215 |
| 4              | 4 | 15 | 126 | 715 | 3060 | 10626 | 31465 | 82251 | 194580 |
| 5              | 5 | 21 | 252 | 2002 | 11628 | 53130 | 201376 | 658008 | 1906884 |
| 6              | 6 | 28 | 462 | 5005 | 38760 | 230230 | 4060 | 8436 | 16215 |
| 7              | 7 | 36 | 792 | 11440 | 116280 | 888030 | 5379616 | 15890700 | 15890700 |
| 8              | 8 | 45 | 1287 | 24310 | 319770 | 3108105 | 3108105 | 3108105 | 3108105 |
| 9              | 9 | 55 | 2002 | 48620 | 817190 | 817190 | 817190 | 817190 | 817190 |
| 10             | 10 | 66 | 3003 | 92378 | 92378 | 92378 | 92378 | 92378 | 92378 |

Table 60. (-d) vertex-labeled [10, A098568]. Sum of Table 64 and Table 68.

| $E \setminus V$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----------------|---|---|---|---|---|---|---|---|---|---|
| 0              | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1              | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2              | 2 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3              | 3 | 0 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 4              | 4 | 0 | 1 | 3 | 5 | 3 | 3 | 3 | 3 | 3 |
| 5              | 5 | 0 | 1 | 4 | 11 | 11 | 6 | 6 | 6 | 6 |
| 6              | 6 | 0 | 1 | 6 | 22 | 34 | 29 | 29 | 29 | 29 |
| 7              | 7 | 0 | 1 | 7 | 37 | 85 | 110 | 70 | 70 | 70 |
| 8              | 8 | 0 | 1 | 9 | 61 | 193 | 348 | 339 | 185 | 47 |
| 9              | 9 | 0 | 1 | 11 | 95 | 396 | 969 | 1318 | 1067 | 479 |
| 10             | 10 | 0 | 1 | 13 | 141 | 771 | 2445 | 4457 | 4940 | 3294 |

Table 61. (-dc.*-l unlabeled. Undirected loopless connected multigraphs with $E$ edges and $V$ vertices [10, A191646]. With the exception of the 1 at $E = 0, V = 1$, the sum of Table 35 and Table 41.
Table 62. \((-\text{dc}.*-1)\) vertex-labeled [10, A290776].

| $E \setminus V$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0              | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 1              | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 2              | 0   | 1   | 3   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 3              | 0   | 1   | 7   | 16  | 0   | 0   | 0   | 0   | 0   | 0   |
| 4              | 0   | 1   | 12  | 63  | 125 | 0   | 0   | 0   | 0   | 0   |
| 5              | 0   | 1   | 18  | 162 | 722 | 1296| 0   | 0   | 0   | 0   |
| 6              | 0   | 1   | 25  | 341 | 2565| 10140| 16807| 0   | 0   | 0   |
| 7              | 0   | 1   | 33  | 636 | 7180| 47100| 169137| 0   | 0   | 0   |
| 8              | 0   | 1   | 42  | 1092| 17335| 168285| 0   | 0   | 0   | 0   |
| 9              | 0   | 1   | 52  | 1764| 37750| 0   | 0   | 0   | 0   | 0   |
| 10             | 0   | 1   | 63  | 2718| 0   | 0   | 0   | 0   | 0   | 0   |

Table 63. \((-\text{dc}.)\) unlabeled. Row sums in [10, A007719].

| $E \setminus V$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0              | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 1              | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 2              | 1   | 2   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 3              | 1   | 4   | 4   | 2   | 0   | 0   | 0   | 0   | 0   | 0   |
| 4              | 1   | 6   | 11  | 9   | 3   | 0   | 0   | 0   | 0   | 0   |
| 5              | 1   | 9   | 25  | 34  | 20  | 6   | 0   | 0   | 0   | 0   |
| 6              | 1   | 12  | 52  | 104 | 99  | 49  | 11  | 0   | 0   | 0   |
| 7              | 1   | 16  | 94  | 274 | 387 | 298 | 118| 0   | 0   | 0   |
| 8              | 1   | 20  | 162 | 645 | 1295| 1428| 0   | 0   | 0   | 0   |
| 9              | 1   | 25  | 263 | 1399| 3809| 0   | 0   | 0   | 0   | 0   |
| 10             | 1   | 30  | 407 | 2823| 0   | 0   | 0   | 0   | 0   | 0   |

Table 64. \((-\text{dc})\) vertex-labeled.
Table 65. (*-d-m-l) unlabeled. Simple graphs with $E$ edges and $V$ vertices \[10, A008406\][11, vol. 4, Tables 2.2–2.2g]. Sum of tables 57, 81–84 and contributions by more than 5 components.

| $E \backslash V$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 |
|----------------|----|----|----|----|----|----|----|----|----|----|----|----|
| 0              | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    |    |    |
| 1              | 0  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    |    |
| 2              | 0  | 0  | 1  | 2  | 2  | 2  | 2  | 2  | 2  |    |    |    |
| 3              | 0  | 0  | 1  | 3  | 4  | 5  | 5  | 5  | 5  | 5  |    |    |
| 4              | 0  | 0  | 0  | 2  | 6  | 9  | 10 | 11 | 11 | 11 | 11 |    |
| 5              | 0  | 0  | 0  | 1  | 6  | 15 | 21 | 24 | 25 | 26 | 26 |    |
| 6              | 0  | 0  | 0  | 1  | 6  | 21 | 41 | 56 | 63 | 66 | 67 |    |
| 7              | 0  | 0  | 0  | 0  | 4  | 24 | 65 | 115| 148| 165| 172|    |
| 8              | 0  | 0  | 0  | 0  | 2  | 24 | 97 | 221| 345| 428| 467|    |
| 9              | 0  | 0  | 0  | 0  | 1  | 21 | 131| 402| 771| 1103|1305|1405|
| 10             | 0  | 0  | 0  | 0  | 1  | 15 | 148| 663|1637|2769|3664|4191|

Table 66. (*-d-m-l) vertex-labeled \[10, A084546\].

| $E \backslash V$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |    |    |
|----------------|----|----|----|----|----|----|----|----|----|----|----|----|
| 0              | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    |    |    |    |
| 1              | 0  | 1  | 3  | 6  | 10 | 15 | 21 | 28 | 36 | 45 |    |    |
| 2              | 0  | 0  | 3  | 15 | 45 | 105| 210|378 |630 |990 |    |    |
| 3              | 0  | 0  | 1  | 20 | 120| 455|1330|3276|7140|14190|    |    |
| 4              | 0  | 0  | 0  | 1  | 20 | 120| 455|1330|3276|7140|14190|    |
| 5              | 0  | 0  | 0  | 2  | 20 | 120| 455|1330|3276|7140|14190|    |
| 6              | 0  | 0  | 0  | 1  | 210|1365|5985|20475|58905|    |    |
| 7              | 0  | 0  | 0  | 0  | 6  | 120| 252|3003 |30349|98280|376992|    |
| 8              | 0  | 0  | 0  | 1  | 210|5005|54264|376740|    |    |
| 9              | 0  | 0  | 0  |    |    |    |    |    |    |    |    |    |

Table 67. (*-d-c) unlabeled. See \[10, A007717\] for the limit $V \to \infty$. 

| $E \backslash V$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|----------------|----|----|----|----|----|----|----|----|----|----|
| 0              | 0  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    |    |
| 1              | 0  | 1  | 2  | 2  | 2  | 2  | 2  | 2  | 2  |    |
| 2              | 0  | 2  | 5  | 7  | 7  | 7  | 7  | 7  | 7  | 7  |
| 3              | 0  | 2  | 10 | 18 | 22 | 23 | 23 | 23 | 23 | 23 |
| 4              | 0  | 3  | 17 | 44 | 66 | 76 | 78 | 79 | 79 | 79 |
| 5              | 0  | 3  | 27 | 91 | 178| 239|264 |271 |273 |    |
| 6              | 0  | 4  | 41 | 183| 451|733 |904 |973 |993 |    |
| 7              | 0  | 4  | 58 | 332|1054|2094|2993|    |    |    |
| 8              | 0  | 5  | 80 | 581|2316|5690|    |    |    |    |
| 9              | 0  | 5  | 107| 959|4799|    |    |    |    |    |
| 10             | 0  | 6  | 139| 1533|    |    |    |    |    |    |
Table 68. \((-d-c)\) vertex-labeled.  

| $E \setminus V$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---------------|---|---|---|---|---|---|---|---|---|
| 0             | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1             | 0 | 2 | 6 | 10 | 15 | 21 | 28 | 36 | 45 |
| 2             | 0 | 3 | 18 | 55 | 120 | 231 | 406 | 666 | 1035 |
| 3             | 0 | 4 | 40 | 204 | 680 | 1771 | 4060 | 8436 | 16215 |
| 4             | 0 | 5 | 75 | 188 | 2935 | 10626 | 31465 | 82251 | 194580 |
| 5             | 0 | 6 | 126 | 1428 | 10281 | 51834 | 201376 | 658008 | 1906884 |
| 6             | 0 | 7 | 196 | 3066 | 30710 | 212314 | 1090761 | 4496388 | 15890700 |
| 7             | 0 | 8 | 288 | 6000 | 81070 | 752874 | 5092830 |
| 8             | 0 | 9 | 405 | 10923 | 194040 | 2371704 |
| 9             | 0 | 10 | 550 | 18766 | 429000 |
| 10            | 0 | 11 | 726 | 30745 |

Table 69. \((-d.\bullet-1)\) unlabeled. Undirected loopless multigraphs with $E$ edges and $V$ vertices [10, A192517]. Sum of tables 61, 85–88 and so forth.  

| $E \setminus V$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---------------|---|---|---|---|---|---|---|---|---|---|
| 0             | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1             | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2             | 0 | 1 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 3             | 0 | 1 | 3 | 6 | 7 | 8 | 8 | 8 | 8 | 8 |
| 4             | 0 | 1 | 4 | 11 | 17 | 21 | 22 | 23 | 23 | 23 |
| 5             | 0 | 1 | 5 | 18 | 35 | 52 | 60 | 64 | 65 |
| 6             | 0 | 1 | 7 | 32 | 76 | 132 | 173 | 197 | 206 |
| 7             | 0 | 1 | 8 | 48 | 149 | 313 | 471 |
| 8             | 0 | 1 | 10 | 75 | 291 | 741 |
| 9             | 0 | 1 | 12 | 111 | 539 |
| 10            | 0 | 1 | 14 | 160 |

Table 70. \((-d.\bullet-1)\) vertex-labeled.  

| $E \setminus V$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---------------|---|---|---|---|---|---|---|---|---|---|
| 0             | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1             | 0 | 1 | 3 | 6 | 10 | 15 | 21 | 28 | 36 | 45 |
| 2             | 0 | 1 | 6 | 21 | 55 | 120 | 231 | 406 | 666 | 1035 |
| 3             | 0 | 1 | 10 | 56 | 220 | 680 | 1771 | 4060 | 8436 | 16215 |
| 4             | 0 | 1 | 15 | 126 | 715 | 3060 | 10626 | 31465 | 82251 | 194580 |
| 5             | 0 | 1 | 21 | 252 | 2002 | 11628 | 53130 | 201376 | 658008 |
| 6             | 0 | 1 | 28 | 462 | 5005 | 38760 | 230230 | 1107568 | 4496388 |
| 7             | 0 | 1 | 36 | 792 | 11440 | 116280 | 888030 |
| 8             | 0 | 1 | 45 | 1287 | 24310 | 319770 |
| 9             | 0 | 1 | 55 | 2002 | 48620 |
| 10            | 0 | 1 | 66 | 3003 |
### Small Unlabeled Graphs

#### Table 71. (d.*Cc-i) unlabeled. The number of connected directed multigraphs with loops and no isolated vertex, with $E$ arcs and $V$ vertices [10, A139621].

| $E \backslash V$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0               | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 1               | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 2               | 0   | 4   | 3   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 3               | 0   | 8   | 15  | 8   | 0   | 0   | 0   | 0   | 0   | 0   |
| 4               | 0   | 16  | 57  | 66  | 27  | 0   | 0   | 0   | 0   | 0   |
| 5               | 0   | 25  | 163 | 353 | 295 | 91  | 0   | 0   | 0   | 0   |
| 6               | 0   | 40  | 419 | 1504| 2203| 1407| 350 | 0   | 0   | 0   |
| 7               | 0   | 56  | 932 | 5302| 12382| 13372| 6790|
| 8               | 0   | 80  | 1940| 16549| 58237| 96456|
| 9               | 0   | 105 | 3743| 46566| 237904|
| 10              | 0   | 140 | 6867| 121111|

#### Table 72. (d.*Cc-i) vertex-labeled

3.2. **Directed Graphs.** Tables 71–80 summarize statistics of oriented/directed graphs.
Table 73. (dC) unlabeled. The number of strongly connected directed multigraphs with loops and no vertex of degree zero, with n arcs and k vertices [10, A139622].

| E \ V | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------|---|---|---|---|---|---|---|---|---|----|
| 0     | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 1     | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 2     | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 3     | 3 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 4     | 4 | 6 | 4 | 1 | 0 | 0 | 0 | 0 | 0 | 0  |
| 5     | 5 | 10 | 9 | 6 | 1 | 0 | 0 | 0 | 0 | 0  |
| 6     | 6 | 19 | 73 | 59 | 9 | 1 | 0 | 0 | 0 | 0  |
| 7     | 7 | 28 | 208 | 350 | 138 | 12 | 1 | 0 | 0 | 0  |
| 8     | 8 | 44 | 534 | 1670 | 1361 | 301 | 0 | 0 | 0 | 0  |
| 9     | 9 | 60 | 1215 | 6476 | 9724 | 0 | 0 | 0 | 0 | 0  |
| 10    | 10 | 85 | 2542 | 21898 | 0 | 0 | 0 | 0 | 0 | 0  |

Table 74. (dC) vertex-labeled.

| E \ V | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------|---|---|---|---|---|---|---|---|---|----|
| 0     | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 1     | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 2     | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 3     | 3 | 4 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| 4     | 4 | 10 | 21 | 6 | 0 | 0 | 0 | 0 | 0 | 0  |
| 5     | 5 | 20 | 111 | 132 | 24 | 0 | 0 | 0 | 0 | 0  |
| 6     | 6 | 35 | 413 | 1288 | 960 | 120 | 0 | 0 | 0 | 0  |
| 7     | 7 | 56 | 1233 | 8152 | 15680 | 7920 | 720 | 0 | 0 | 0  |
| 8     | 8 | 84 | 3159 | 39049 | 156955 | 201450 | 0 | 0 | 0 | 0  |
| 9     | 9 | 120 | 7227 | 153540 | 1140055 | 0 | 0 | 0 | 0 | 0  |
| 10    | 10 | 165 | 15147 | 520404 | 0 | 0 | 0 | 0 | 0 | 0  |

Table 75. (d.Cc.\*m-1) unlabeled. The number of weakly connected directed graphs without multiedges or loops [10, A054733,A283753]. The undirected variants are in Table 57.
### Table 76. \((d \ast (d \ast c) \ast m)\) vertex-labeled. \([10, A062735]\)

| \(E \setminus V\) | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|------------------|----|----|----|----|----|----|----|----|----|----|
| 0                | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 1                | 0  | 2  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 2                | 0  | 1  | 12 | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 3                | 0  | 0  | 20 | 128| 0  | 0  | 0  | 0  | 0  | 0  |
| 4                | 0  | 0  | 15 | 432| 2000| 0  | 0  | 0  | 0  | 0  |
| 5                | 0  | 0  | 6  | 768| 11104| 41472| 0  | 0  | 0  | 0  |
| 6                | 0  | 0  | 1  | 920| 33880| 337920| 1075648| 0  | 0  | 0  |
| 7                | 0  | 0  | 0  | 792| 73480| 1536000| 11968704| 0  | 0  | 0  |
| 8                | 0  | 0  | 0  | 495| 123485| 5062080| 0  | 0  | 0  | 0  |
| 9                | 0  | 0  | 0  | 220| 166860| 0  | 0  | 0  | 0  | 0  |
| 10               | 0  | 0  | 0  | 66 | 0  | 0  | 0  | 0  | 0  | 0  |

### Table 77. \((d \ast (d \ast c) \ast m)\) unlabeled. The number of strongly connected directed graphs without loops or multiedges. Strongly connected variant of Table 75. With the exception of the 1 at \(E = 0, V = 1\) this is the same as Table 25.

| \(E \setminus V\) | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|------------------|----|----|----|----|----|----|----|----|----|----|
| 0                | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 1                | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 2                | 0  | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 3                | 0  | 0  | 2  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 4                | 0  | 0  | 9  | 6  | 0  | 0  | 0  | 0  | 0  | 0  |
| 5                | 0  | 0  | 6  | 84 | 24 | 0  | 0  | 0  | 0  | 0  |
| 6                | 0  | 0  | 1  | 316| 720| 120| 0  | 0  | 0  | 0  |
| 7                | 0  | 0  | 0  | 492| 6440| 6480| 720| 0  | 0  | 0  |
| 8                | 0  | 0  | 0  | 417| 26875| 107850| 0  | 0  | 0  | 0  |
| 9                | 0  | 0  | 0  | 212| 65280| 0  | 0  | 0  | 0  | 0  |
| 10               | 0  | 0  | 0  | 66 | 0  | 0  | 0  | 0  | 0  | 0  |

### Table 78. \((d \ast (d \ast c) \ast m)\) vertex-labeled. With the exception of the 1 at \(E = 0, V = 1\) this is the same as Table 26.
Table 79. (d) unlabeled. The number of directed graphs allowing loops and multiedges [10, A138107].

| $E \setminus V$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----------------|---|---|---|---|---|---|---|---|---|----|
| 0               | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  |
| 1               | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2  |
| 2               | 1 | 6 | 10| 11| 11| 11| 11| 11| 11| 11 |
| 3               | 1 | 10| 31| 47| 51| 52| 52| 52| 52| 52 |
| 4               | 1 | 19| 90 |98 |269|291|295|296|296|296 |
| 5               | 1 | 28| 222|713|1270|1596|1697|1719|1723|10 |
| 6               | 1 | 44| 520|2423|5776|8838|10425|10922|11033|10 |
| 7               | 1 | 60|1090|7388|24032|46384|63419|10 |
| 8               | 1 | 85|2180|21003|93067|230848|10 |
| 9               | 1 |110|4090|55433|333948|10 |
| 10              | 1 |146|7356|137944|10 |

Table 80. (d) vertex-labeled. The number of labeled directed graphs allowing loops and multiedges [10, A214398].

| $E \setminus V$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|---|---|---|---|---|---|---|---|
| 0               | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1               | 1 | 4 | 9 | 16| 25 |36 |49 |64 |
| 2               | 1 |10 |45 |136|325 |666|1225|2080|
| 3               | 1 |20 |165 |816|2925|8436|20825|45760|
| 4               | 1 |35 |495 |3876|20475|82251|270725|766480|
| 5               | 1 |56 |1287|15504|118755|658008|2869685|10424128|
| 6               | 1 |84 |3003|54264|593775|4496388|25827165|119877472|
| 7               | 1 |120|6435|170544|2629575|26978328|202927725|10 |
| 8               | 1 |165|12870|490314|10518300|145008513|10 |
| 9               | 1 |220|24310|1307504|38567100|10 |
| 10              | 1 |286|43758|3268760|10 |
4. CONNECTED MULTIGRAPHS UP TO 7 VERTICES

4.1. **Algorithm.** The columns of the undirected connected multigraphs in Table 61 have rational ordinary generating functions. To compute them, we first classify each multigraph by the number of edges and vertices of the underlying simple graph—in as many ways as counted in Table 57—and then distribute the edges of the multigraph over the edges of the underlying graph using Polya’s counting method to deal with the symmetry of the simple graphs.

The process is illustrated in Section 4.2 for $V = 4$ vertices. Explicit intermediate results are tracked in the files $G.V.E.txt$ in the ancillary directory for $V = 2–7$. Each of these files contains the contributing underlying simple graphs with $V$ vertices and $E$ edges. The file starts with $V$ and $E$ printed in the first line. Then each graph is represented by

1. a canonical adjacency matrix (binary, symmetric and traceless),
2. the label as in Section 1 followed by the multiplicity of the graph as if one would create all vertex-labeled graphs by permuting rows and columns (i.e. $V!$ divided by the order of the automorphism group),
3. the cycle index multinomial. This could also be derived from the table of symmetry groups in [11, Vol. 1, Sec. 7, Table 8].

The minimum number of edges for connected simple graphs is $E \geq V - 1$ (sparsest, trees on $V$ vertices), and the maximum number is $E \leq \binom{V}{2}$ (complete graph on $V$ vertices). Summing over all multinomials over the underlying graphs constitutes the generating function by a finite sum of rational polynomials ([10, A001349] terms).

4.2. **4 vertices.** The ordinary generating function for the number of connected multigraphs on 4 vertices is derived by adding the contributions of the 6 distinct geometries of the underlying connected simple graph.

We consider connected multigraphs with 4 vertices and $E$ edges. The multigraph thus has at least one (unoriented) edge attached to edge vertex, so all degrees are $\geq 1$. Loops are not allowed; the vertices are not labeled.

If all multiedges are replaced by a single edge, the underlying simple graph has one of 6 shapes [3]:

1. The linear chain with 3 edges.
2. A triangle with an edge to a lone vertex of degree 1 (4 edges).
3. The star graph with 3 edges.
4. The quadrangle (cycle of 4 edges).
5. A quadrangle with a single diagonal, total of 5 edges.
6. The complete graph on 4 vertices with 6 edges.

**Linear Chain.** We wish to distribute $E \geq 3$ edges over the 3 edges of the simple graph of the linear chain. This can be done by putting any number of $k \geq 1$ edges in the middle, and distributing the remaining $n - k$ edges over the two edges connected to two endpoints.

Due to the left-right symmetry of the graph, the distribution of the $n - k$ edges can only be done in $\lfloor (n - k)/2 \rfloor$ ways. The total number of graphs of this kind with multiedges is

\[
\sum_{k=1}^{E-2} \lfloor \frac{E-k}{2} \rfloor = 0, 0, 0, 1, 2, 4, 6, 9, 12, 16, 20, 25, \ldots (E \geq 0)
\]
with generating function [10, A002620]

\[ g_1(x) = \sum_{E \geq 0} a_1(E) x^E = \frac{x^3}{(1 + x)(1 - x)^3}. \]

The generating function is the product of \( t_1(x) \) representing the number of ways of placing \( n \) vertices at the middle edge, by the factor \( x^2/[(1 + x)(1 - x)^2] \). The latter factor is obtained by considering the symmetry of the cyclic group \( C_2 \) that swaps the edges that inhabit the first and last edges of the underlying simple graph without generating a new graph. The cycle index of the group is [2]

\[ Z(C_2) = (t_1^2 + t_2^2)/2, \]

where the associated generating functions are the number of ways of placing \( n \) edges without imposing symmetry on any of them:

\[ t_i(x) = \begin{cases} 0, 1, 1, 1, 1, 1 & , \ i \geq 1. \end{cases} \]

So the latter factor can be written as [10, A004526]

\[ \frac{t_1(x)^2 + t_2(x^2)}{2} = \frac{x^2}{(1 + x)(1 - x)^2}. \]

Triangle. The contribution from the triangular graph is the number of ways of placing \( 2 \leq k \leq n - 2 \) edges on the edge to the lone vertex and the triangle edge opposite to it, and then distributing the residual \( n - k \) edges to the remaining two edges under the symmetry constraint of the group \( C_2 \) that swaps the other two edges:

\[ \sum_{k=2}^{E-2} (k - 1)[\frac{E-k}{2}] = 0, 0, 0, 0, 1, 3, 7, 13, 22, 34, 50, 70, 95, 125, 161, 190, 225, 265, 310, 361, 416, 475, 540, 610, 685, 765, 850, 941, 1030, 1125, 1225, 1330, 1441, 1556, 1675, 1800, 1925, 2056, 2191, 2330, 2475, 2625, 2780, 2941, 3105, 3270, 3441, 3616, 3795, 3975, 4160, 4349, 4545, 4736, 4900, 5085, 5270, 5461, 5656, 5850, 6050, 6255, 6461, 6669, 6875, 7080, 7289, 7486, 7680, 7875, 8075, 8275, 8479, 8680, 8880, 9085, 9289, 9480, 9675, 9860, 10055, 10245, 10433, 10621, 10809, 11005, 11200, 11395, 11590, 11781, 11976, 12165, 12349, 12525, 12700, 12875, 13050, 13220, 13391, 13556, 13715, 13870, 14026, 14181, 14335, 14485, 14630, 14776, 14911, 15045, 15178, 15305, 15425, 15546, 15660, 15779, 15896, 16015, 16128, 16245, 16360, 16476, 16591, 16705, 16818, 16930, 17041, 17156, 17265, 17376, 17485, 17596, 17701, 17805, 17908, 18005, 18102, 18198, 18288, 18376, 18461, 18545, 18626, 18705, 18782, 18857, 18932, 19005, 19076, 19145, 19212, 19278, 19345, 19408, 19470, 19531, 19590, 19648, 19705, 19756, 19805, 19848, 19885, 19918, 19945, 19968, 19985, 20005, 20020, 20032, 20036, 20032, 20020, 20005, 19976, 19936, 19885, 19825, 19759, 19685, 19604, 19515, 19418, 19315, 19206, 19090, 18965, 18834, 18695, 18545, 18385, 18215, 18032, 17835, 17624, 17395, 17150, 16885, 16596, 16285, 15959, 15615, 15253, 14870, 14470, 14049, 13598, 13125, 12625, 12105, 11565, 11005, 10425, 9825, 9205, 8565, 7895, 7195, 6465, 5705, 4915, 4105, 3275, 2425, 1555, 675, 195, 0 \]

See [10, A002623]

\[ g_2(x) = \frac{x^4}{(1 + x)(1 - x)^4}. \]

This generating function is the product of \( x^2/(1 - x)^2 \)—contribution of two cycles of length 1, fixed points under the symmetry—by \( x^2/[(1 + x)(1 - x)^2] \), where again the latter is (5), the number of ways of distributing \( E \) edges symmetrically over two edges of the simple graph.

Star Graph. The contribution from the star graph is the number of ways of partitioning \( E \) into 3 positive integers [10, A009905],

\[ \rightarrow 0, 0, 0, 1, 1, 2, 3, 4, 5, 7, 8, 10, 12, 14, 16(E \geq 0), \]

\[ g_3(x) = \frac{x^3}{(1 + x)(1 - x)^2(1 - x^3)}. \]

Alternatively this expression is obtained if we consider the symmetry group of order 6 of the underlying simple graph, which can be generated by (i) the group \( C_3 \) of the triangle combined with (ii) the mirror symmetry along a diagonal.

```
with(group):
g := permgroup(3, [[[1, 2, 3]], [[2, 3]]]) ;
for i in elements(g) do
    print(i) ;
```
end do;
The cycle index obtained with this Maple code is \[ Z(S_3) = \frac{t_3^3 + 3t_1t_2 + 2t_3^4}{6} \] (10).

Insertion of (4) gives (9).

Square. The symmetry group of the square is the Dihedral Group of order 8 which essentially is generated by rotation by 90 degrees or flips along the horizontal or vertical axes or diagonals.

with(group):
g := permgroup(4, [[[1, 2, 3, 4]], [[1, 3]]]);
for i in elements(g) do
    print(i);
end do;
The cycle index is \[ Z(D_8) = \frac{t_4^4 + 2t_4^2 + 2t_2^2t_1 + 3t_2^2}{8} \] (11).

The enumeration theorem turns this into the generating function [10, A005232]

\[ g_4(x) = \frac{x^4(1 - x + x^2)}{(1 + x^2)(1 + x)^2(1 - x)^4} \] (12).

\[ \mapsto 0, 0, 0, 0, 1, 1, 3, 4, 8, 10, 16, 20, 29, 35, 47(E \geq 0) \] (13).

Square with Diagonal. In the square with a diagonal edge, the diagonal stays inert under the symmetry operations, and contributes a factor \( t_1(x) \) to the generating function. The symmetry group of the four other edges allows a flip along any of the two diagonals and generates a symmetry group of order 4:

with(group):
g := permgroup(4, [[[2, 4], [1, 3]], [[1, 4], [2, 3]]]);
for i in elements(g) do
    print(i);
end do;
The cycle index is \[ Z(C_2 \times C_2) = \frac{t_4^4 + 3t_2^4}{4} \] (14).

Insertion of (4) into the enumeration theorem yields \( x^4(1 - x + x^2)/[(1 + x)^2(1 - x)^4] \) [10, A053307], and convolved with the inert factor

\[ g_5(x) = \frac{x^5(1 - x + x^2)}{(1 + x)^2(1 - x)^5} \] (15).

This expands to

\[ \mapsto 0, 0, 0, 0, 1, 2, 6, 11, 22, 36, 60, 90, 135, 190, 266, 357, 476(E \geq 0) \] (16).
Complete Graph. The cycle index of the complete graph \(K_4\) is \([2]\):

\[
Z(S_4) = \frac{t_1^6 + 9t_1^2t_2^2 + 8t_1^2t_3 + 6t_2t_4}{24}.
\]

Insertion of (4) into the enumeration theorem yields

\[
g_6(x) = \frac{x^6(1 - x + x^2 + x^4 + x^6 - x^7 + x^8)}{(1 - x)^6(1 + x)^2(1 + x^2)(1 + x + x^2)^2},
\]

with \([10, \ A003082]\)

\[
(20) \quad \sum_{i=1}^{6} g_i(x) = \frac{x^3(-x^{10} + x^9 + 2x^7 - x^6 + x^5 - 3x^4 + x^2 + x + 2)}{(x - 1)^6(1 + x)^2(1 + x^2)(1 + x + x^2)^2},
\]

which expands to \([10, A290778]\)

\[
(21) \quad \mapsto 0, 0, 0, 2, 5, 11, 22, 37, 61, 95, 141, 203, 288, 393, 531, 704, 918, 1180, 1504(\geq 0, V = 4)
\]

4.3. **Up to 7 vertices.** The generating function for the number of connected loopless multigraphs on 2 vertices is

\[
(22) \quad \frac{x}{1 - x} \mapsto 0, 1, 1, 1, 1, (\geq 0, V = 2).
\]

The generating function for the number of connected loopless multigraphs on 3 vertices is \([10, A253186]\)

\[
(23) \quad \frac{(x^3 - x - 1)x^2}{(-1 + x)^3(x + 1)(x^2 + x + 1)} \mapsto 0, 0, 1, 2, 3, 4, 6, 7, 9, 11, 13, 15, 18, 20, 23(\geq 0, V = 3).
\]

On 5 vertices

\[
(24) \quad \frac{x^4p_5(x)}{(-1 + x)^{10}(x^2 + x + 1)^3(x + 1)^3(x^2 - x + 1)(x^4 + x^3 + x^2 + x + 1)^2(x^2 + 1)^2} \mapsto 0, 0, 0, 0, 3, 11, 34, 85, 193, 396, 771, 1411, 2490, 4221(\geq 0, V = 5),
\]

where

\[
(25) \quad p_5 = 3 + 5x + 12x^2 + 17x^3 + 26x^4 + 27x^5 + 35x^6 + 28x^7 + 38x^8 + 30x^9 + 39x^{10} + 37x^{11} + 34x^{12} + 24x^{13} + 15x^{14} + 3x^{15} - 7x^{16} - 9x^{17} + 4x^{20}
\]

\[
+ 5x^{22} + 3x^{23} - 8x^{18} - x^{19} + 6x^{21} - 2x^{24} - 2x^{25} - 2x^{26} - x^{27} + x^{29}.
\]

On 6 vertices

\[
(26) \quad \frac{x^5p_6}{(-1 + x)^{15}(x + 1)^6(x^2 + x + 1)^3(x^2 - x + 1)(x^4 + x^3 + x^2 + x + 1)^3} \mapsto 0, 0, 0, 0, 6, 29, 110, 348, 969, 2445, 5746, 12736, 26843, 54256, 105669(\geq 0, V = 6),
\]
where

\[(27) \quad p_5 \equiv 6 + 11x + 35x^2 + 70x^3 + 134x^4 + 217x^5 + 348x^6 + 533x^7 + 726x^8 + 1038x^9 + 1290x^{10} + 1629x^{11} + 1810x^{12} + 2040x^{13} + 1976x^{14} + 1984x^{15} + 1696x^{16} + 1542x^{17} + 1206x^{18} + 1050x^{19} + 787x^{20} + 636x^{21} + 474x^{22} + 273x^{23} + 169x^{24} - 11x^{25} - 31x^{26} - 97x^{27} - 44x^{28} - 8x^{29} + 33x^{30} + 63x^{31} + 32x^{32} + 38x^{33} - 17x^{34} - 14x^{35} - 31x^{36} - 8x^{37} - 5x^{38} + 8x^{39} + 11x^{40} + 4x^{41} + 3x^{42} - 4x^{43} - 3x^{45} + x^{47}.
\]

On 7 vertices

\[(28) \quad \frac{x^6 p_7(x)}{(-1 + x)^{21} (x^4 + x^3 + x^2 + x + 1)^4 (x^2 + x + 1)^4 (x + 1)^3} \times \frac{1}{(x^4 - x^2 + 1) (x^4 - x^3 + x^2 - x + 1) (x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)^3} \equiv 0, 0, 0, 0, 0, 11, 70, 339, 1318, 4457, 13572, 38201, 100622, 251078, \ldots \quad (E \geq 0, V = 7),\]

where

\[(29) \quad p_7(x) \equiv -11 - 48x - 188x^2 - 570x^3 - 1526x^4 - 3675x^5 - 8284x^6 - 17431x^7 - 35005x^8 - 66742x^9 - 121908x^{10} - 213342x^{11} - 359515x^{12} - 583522x^{13} - 916091x^{14} - 1391716x^{15} - 2051981x^{16} - 2938963x^{17} - 4097420x^{18} - 5564508x^{19} - 7373793x^{20} - 9539279x^{21} - 12063528x^{22} - 14919997x^{23} - 18064473x^{24} - 21418776x^{25} - 24890827x^{26} - 28355984x^{27} - 31688266x^{28} - 34742272x^{29} - 37387611x^{30} - 39493274x^{31} - 4063946x^{32} - 41723196x^{33} - 40973187x^{34} - 39511812x^{35} - 37405689x^{36} - 34764514x^{37} - 31705308x^{38} - 2837226x^{39} - 24898844x^{40} - 21424940x^{41} - 18060699x^{42} - 14913079x^{43} - 12050303x^{44} - 9525196x^{45} - 7357519x^{46} - 5550815x^{47} - 4085547x^{48} - 2932089x^{49} - 2048825x^{50} - 1393454x^{51} - 920594x^{52} - 590477x^{53} - 366935x^{54} - 220705x^{55} - 128024x^{56} - 71511x^{57} - 37993x^{58} - 18392x^{59} - 8318x^{60} - 2668x^{61} + 247x^{62} + 1501x^{63} + 1827x^{64} + 1523x^{65} + 980x^{66} + 357x^{67} + 99x^{68} - 369x^{70} - 387x^{71} - 247x^{72} - 23x^{73} + 152x^{74} + 230x^{75} + 205x^{76} + 118x^{77} + 15x^{78} - 61x^{79} - 88x^{80} - 74x^{81} - 33x^{82} + 3x^{83} + 26x^{84} + 28x^{85} + 19x^{86} + 5x^{87} - 4x^{88} - 7x^{89} - 5x^{90} - 9x^{91} + 9x^{92} + 2x^{93}.
\]

As a cross-check on these numbers we note that using a weight \( t_i(x) = x \) instead of (4) just counts the underlying simple graphs; it computes the generating functions down columns of Table 57.

5. Multisets of Connected Graphs

If a table of connected graphs as a function of vertex count and edge count is known, the Multiset Transformation generates tables of disconnected graphs with fixed number of components. The calculation involves creating an intermediate multiset of edge-vertex pairs of the components, and looking up a product of multiset coefficients as a function of the number of connected graphs that support the
pairs. The technique is demonstrated for simple (undirected, unlabeled, loopless) graphs and for undirected, unlabeled loopless graphs allowing multiedges.

5.1. The Multiset Coefficient. The concept of the multiset is based on the concept of the set (a collection of objects, only one object of a given type), but allows to put more than one object of a type into the collection [9].

Definition 1. A Multiset is a collection of objects with some individual count (object of type \(i\) appearing \(f_i\) times in the collection). The objects have no order in the collection.

The number of ways of assembling a multiset with \(m\) objects plugged from a set of \(n\) different objects is a variant of Pascal’s triangle of binomial coefficients,

\[
P(n, m) = \binom{n + m - 1}{m}.
\]

The equation may be illustrated for small orders \(m\):

- If there is only \(n = 1\) type of objects, the multiset has only one choice: it contains \(m\) replicates of the unique object. \(P(1, m) = 1\).
- If the multiset contains \(m = 1\) object, it contains one object of \(n\) candidates. \(P(n, m) = n\).
- If the multiset contains \(m = 2\) objects, it contains either the same type of object twice (\(n\) choices), or two different objects (\(\binom{n}{2}\) choices), so \(P(n, 2) = n + \binom{n}{2} = \frac{n(n+1)}{2}\).
- If the multiset contains \(m = 3\) objects, it either contains the same type of object thrice (\(n\) choices), or one type of object once and another type of object twice (\(n(n-1)\) choices), or two different types of objects (\(\binom{n}{3}\) choices); so \(P(n, 3) = n + n(n-1) + \binom{n}{3} = \frac{n(n+1)(n+2)}{3}\).
- If the multiset contains \(m = 4\) objects, we consider all five partitions of \(m\), namely \(4^1, 1^3, 2^1, 1^22^1, 1^4\): it either contains the same type of object 4 times (\(n\) choices), or one type of object once and another type of object thrice (\(n(n-1)\) choices), or two pairs of objects (\(\binom{n}{2}\) choices), or two different objects and one pair of objects (\(\binom{n}{2}(n-2)\) choices), or four different types of objects (\(\binom{n}{4}\) choices); so \(P(n, 4) = n + n(n-1) + \binom{n}{2} + \binom{n}{2}(n-2) + \binom{n}{4} = \frac{n(n+1)(n+2)(n+3)}{3}\).

Proof. Formula (30) can be rephrased with [7, (3.8)] (setting \(r = m - 1\), \(k = j\), \(n = 1\) there) or with [8, (1.11)]:

\[
\binom{n + m - 1}{m} = \sum_{j=1}^{m} \binom{n}{j} \binom{m-1}{j-1}.
\]

The first factor on the right hand side indicates that in a first step one can create a set of \(j\) distinct objects out of \(n\) in \(\binom{n}{j}\) ways. Consider that set sorted by some lexicographic order. Then the factor \(\binom{m-1}{j-1}\) counts in how many ways one can insert separators in the multiset of the same lexicographic ordering to select switch-over from one type of object to the next one. \(\square\)

Example 3. To create multisets of \(m = 4\) objects given \(n\) distinct objects, selecting \(j = 1\) type of object gives the multiset \(o_1o_1o_1o_1\) (no separator), \(\binom{3}{0}\) = 1 choices; selecting \(j = 2\) types of objects gives \(o_1o_1o_1|o_2\) or \(o_1o_1|o_2o_2\) or \(o_1|o_2o_2o_2\) with \(\binom{3}{1}\) =
3 positions of the separator; selecting \( j = 3 \) types of objects gives \( o_1o_1|o_2|o_3 \) or \( o_1|o_2|o_2|o_3 \) or \( o_1|o_2o_2|o_3 \) with \( \binom{3}{3} = 3 \) positions of the 2 separators; or selecting \( j = 4 \) types of objects gives \( o_1|o_2|o_3|o_4 \) with \( \binom{3}{3} = 1 \) positions of the 3 separators.

5.2. Multiset Transform of An Integer Sequence. The Multiset Transform deals with the question: if the objects of type \( i \) have some weight (expressed as a positive integer), in how many ways can we assemble a multiset of the objects with some prescribed total weight? The total weight is the usual arithmetic sum of the weights of the objects.

Example 4. (Money Exchange Problem) In how different ways can you combine coins (weight 5 for type nickels, weight 10 for type dimes and weight 25 for type quarter...) for a wallet worth 200 (2 dollars)?

Example 5. In how different ways can you fill a bag of 10 kg (weight 100) with apples of 100 g (weight 1) and oranges of 200 g (weight 2)?

The Multiset Transform computes the number \( T_{n,k} \) of multisets containing \( k \) objects, drawn from a set of objects of which there are \( T_{n,1} \) of some additive weight \( n \). Given \( T_{0,1} = 1 \) and an integer sequence \( T_{n,1} \) for the number of objects in the weight class \( n \), the \( T_{n,k} \) of total weight \( n \) are calculated recursively by

\[
T_{n,k} = \sum_{f_1n_1 + f_2n_2 + \ldots + f_kn_k = n} \prod_{i=1}^{k} P(T_{n_i,1}, f_i),
\]

where the sum is over the partitions of \( n \) into parts \( n_i \) which occur with frequencies \( f_i \).

Remark 1. The row sums \( \sum_{k=1}^{n} T_{n,k} \) are obtained by the Euler Transform of the sequence \( T_{n,1} \) [1, (25)].

Example 6. Given two types of nickels (2 types of weight 5, tin and copper), one type of dime (weight 10), and one type of quarter (weight 25), the integer sequence \( T_{n,1} \) is \( (1), 0, 0, 0, 0, 2, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 25 \).

Example 7. Given three types of apples (brown, yellow and red, each of weight 1), one type of banana (weight 2), and one type melon (weight 4), the integer sequence is \( (1), 3, 1, 0, 1, 0, 0, 0 \ldots \). The Multiset Transform generates the triangular table

| \( n \backslash k \) | 1 | 2 | 3 | 4 | 5 | 6 |
|------------------|---|---|---|---|---|---|
| 1                | 3 |
| 2                | 1 6 |
| 3                | 0 3 10 |
| 4                | 1 1 6 15 |
| 5                | 0 3 3 10 21 |
| 6                | 0 1 7 6 15 28 |
| 7                | 0 0 3 13 10 21 36 |
| 8                | 0 1 1 7 21 15 28 45 |

The row sums in the table are 3, 7, 13, 23, 37, 57, 83, 118... There are \( T_{3,2} = 3 \) ways of generating a weight of 3 with two objects (a banana and any of the three types of apples). There are \( T_{3,3} = 10 \) ways of generating a weight of 3 with three objects [three apples (bbb), (yy), (rrr), (byy), (brr), (bby), (yrr), (bbr), (yry), (bry)]. There is \( T_{6,2} = 1 \) way to generate a weight of 6 with two objects (a banana and a melon).
Example 8. If there is one type of object of each weight, \( T_{n,1} = 1 \), the Multiset Transform generates the partition numbers \([10, A008284]\), and the row sums are the partition numbers \([10, A000041]\).

5.3. Graphs specified by number of components. If the sequence \( T_{n,1} \) enumerates connected graphs of type \( n \) (where the weight \( n \) is either the vertex count or the edge count), one fundamental way of generating a multiset is putting \( k \) of them side by side and considering them a graph with \( k \) components. Vertex or edge number are additive, as required.

Example 9. If \( T_{n,1} \) denotes connected graphs with \( n \) nodes \([10, A001349]\), the Multiset Transform counts graphs with \( k \) components \([10, A201922]\).

Example 10. If \( T_{n,1} \) denotes connected graphs with \( n \) edges \([10, A002905]\), the Multiset Transform counts graphs with \( n \) edges and \( k \) components \([10, A275421]\).

Example 11. If \( T_{n,1} \) denotes trees with \( n \) nodes \([10, A000055]\), the Multiset Transform counts forests with \( k \) trees \([10, A095133]\).

Example 12. If \( T_{n,1} \) denotes rooted trees with \( n \) nodes \([10, A000081]\), the Multiset Transform counts rooted forests with \( k \) trees \([10, A033185]\).

Example 13. If \( T_{n,1} \) denotes connected regular graphs with \( n \) nodes \([10, A005177]\), the Multiset Transform counts regular graphs with \( k \) components \([10, A275420]\). In the case of cubic graphs the transform pair is \([10, A002851]\) and \([10, A275744]\).

6. Graphs specified by number of edges, vertices and components

6.1. Union of connected graphs. Let \( G(E, V, k) \) be the number of graphs with \( E \) edges, \( V \) vertices and \( k \) components. \( G(E, V, 1) \) is the number of connected graphs with \( E \) edges and \( V \) vertices. The other properties like whether the graphs are labeled, may contain loops or multiedges, are not classified here, but assumed to be fixed while composing graphs with \( k \) components from connected graphs. (A multiset of labeled connected graphs is a disconnected labeled graph; a multiset of connected oriented graphs is a disconnected oriented graph; and so on.) The unified graph is the multiset union of connected graphs \( G_1 \) which individually have \( e_i \) edges and \( v_i \) vertices:

\[
G_k(E, V) = \bigcup_{i=1}^{k} G_1(e_i, v_i),
\]

where both the number of edges and the number of vertices are additive:

\[
E = \sum_{i=1}^{k} e_i; \quad V = \sum_{i=1}^{k} v_i.
\]

Summation over a set of the variables creates marginal sums: \( G(., V, k) = \sum_{E \geq 0} G(E, V, k) \) are the graphs with \( V \) vertices and \( k \) components. \( G(E, ., k) = \sum_{V \geq 1} G(E, V, k) \) is the number of graphs with \( E \) edges and \( k \) components. \( G(E, V, .) = \sum_{k \geq 1} G(E, V, k) \) is the number of graphs with \( E \) edges and \( V \) vertices.
6.2. Correlated Multiset Transforms. The particular case we explore here is that constructing a disconnected graph from connected components means building a multiset of connected graphs, where the number of edges and also the number of vertices are such an additive weight.

The examples of Section 5.3 illustrated how \(G(E,.,k)\) is the Multiset Transform of \(G(E,1)\) \([10, A076864,A275421,A191970]\) and \(G(.,V,k)\) is the Multiset Transform of \(G(1,V)\) \([10, A054924,A275420,A281446]\). The aim of this paper is to demonstrate a similar technique for \(G(E,V,k)\) assuming \(G(E,1)\) is known.

Each graph which is a component contributing to \(G(E,V,k)\) has a specific pair \((e_i,v_i)\) of edge and vertex count; the union of these graphs is a multiset of such pairs—which means in the multiset of graphs contributing to \(G(E,V,k)\), each pair may occur more than once, and each pair may represent (in the sense of the weights above) more than one graph because there may be more than one distinct connected graph for one pair of \(E\) and \(V\).

The calculation starts by constructing all weak compositions of \(E\) into \(k\) parts \(e_i\), and all weak compositions of \(V\) into \(k\) parts \(v_i\) of pairs \((e_i,v_i)\) compatible with the requirement (34). This defines a two-dimensional \((E^{k-1}) \times (V^{k-1})\) outer product matrix with multisets \([13]\).

**Example 14.** If \(E = 2\) and \(V = 3\) and \(k = 3\), the compositions are \(2 = 2+0+0 = 0+2+0 = 0+0+2 = 1+1+0 = 1+0+1 = 0+1+1, 3 = 3+0+0 = 0+3+0 = 2+1+0 = 1+2+0 = 2+0+1 = 2+1+0 = \ldots\), and the matrix contains multisets with \(k\) pairs:

| \(\sum e_i\) \(\sum v_i\) | \(3+0+0\) | \(0+3+0\) | \(2+1+0\) | \(\ldots\) | \(1+1+1\) |
|---------------------------|-------------|-------------|-------------|-------------|-------------|
| \(2+0+0\)                | (2,3)(0,0)(0,0) | (2,0)(0,3)(0,0) | \ldots | (2,1)(0,1)(0,1) |
| \(0+2+0\)                | (0,3)(2,0)(0,0) | (0,0)(2,3)(0,0) | \ldots | (0,1)(2,1)(0,1) |
| \(0+0+2\)                | \ldots | \ldots | \ldots | \ldots |
| \(1+1+0\)                | \ldots | \ldots | \ldots | \ldots |
| \(0+1+1\)                | (0,3)(1,0)(1,0) | (0,0)(1,3)(1,0) | \ldots | (0,1)(1,1)(1,1) |

Each element of the matrix is a multiset \((e_1,v_1)(e_2,v_2)\ldots(e_k,v_k)\) of pairs obtained by interleaving the \(e_i\) and \(v_i\) components of the compositions. At that point we realize that

1. if any of the \(v_i\) is zero, the method of selecting such a null-graph into the disconnected graph would not fulfill the requirement of being \(k\)-connected. So actually only the compositions (not the weak compositions) of \(V\) need to be considered as table columns.

2. because the decompositions \((e_1,v_1),(e_2,v_2)\ldots\) are candidates for multiset compositions, the order of the pairs does not matter. In the example, \((2,1)(0,1)(0,1)\) and \((0,1),(2,1)(0,1)\) are the same scheme of selecting connected graphs. So we may sort for example each \(k\)-set by the \(e_i\) member of the pair without loss of samples, which means, we may build the table by just considering weak partitions (not all weak compositions) of \(E\) into parts \(e_i\) as table rows.

**Example 15.** Continuing Example 14 above, the table reduces to

| \(\sum e_i\) \(\sum v_i\) | \(1+1+1\) |
|---------------------------|-------------|
| \(2+0+0\)                | (2,1)(0,1)(0,1) |
| \(1+1+0\)                | (1,1)(1,1)(0,1) |
Table 81. $G(E,V,2)$. Simple graphs with 2 components and a total of $E$ edges and $V$ vertices. See [10, A274934] for column sums, [10, A274937] for the diagonal.

The $(2,1)(0,1)(0,1)$ entry indicates to take one connected graph with 2 edges and one vertex (obviously a double-loop) and two graphs without edges and one vertex (single points). The $(1,1)(1,1)(0,1)$ entry indicates to take 2 graphs with an edge and a vertex (2 points, each with a loop) and a graph without edges and a single vertex (a single point).

The number of $k$-compositions of graphs then is the sum over all unique remaining multisets in the table, akin to Equation (32):

\[
G(E,V,k) = \sum_{(e_i,v_i)} \prod_{i} P(G(e_i,v_i,1), f_i)
\]

where $f_i$ is the frequency (number of occurrences) of the pair $(e_i,v_i)$ in the multiset.

6.3. Simple Graphs. The most important example considers undirected, unlabeled graphs without multiedges or loops. Table 57 shows the base information $G(E,V,1)$ which is fed into the formula to compute the tables $G(E,V,k)$ 81–84 for $k \geq 1$. (These tables differ from Steinbach’s tables [11, Vol. 4, Table 1.2a] because our components may be/have isolated vertices.)

The arithmetic sum over these tables $k \geq 1$ yields Table 65.

| $E \setminus V$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|-----------------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| 0               | 1 | 1 |   |   |   |   |   |   |   |    |    |    |    |    |
| 1               | 1 | 3 | 2 |   |   |   |   |   |   |    |    |    |    |    |
| 2               | 0 | 0 | 0 | 2 |   |   |   |   |   |    |    |    |    |    |
| 3               | 0 | 0 | 0 | 0 | 3 |   |   |   |   |    |    |    |    |    |
| 4               | 0 | 0 | 0 | 0 | 3 | 6 |   |   |   |    |    |    |    |    |
| 5               | 0 | 0 | 0 | 0 | 1 | 8 | 11 |   |   |    |    |    |    |    |
| 6               | 0 | 0 | 0 | 0 | 1 | 7 | 22 | 23 |   |    |    |    |    |    |
| 7               | 0 | 0 | 0 | 0 | 0 | 5 | 27 | 58 | 46 |    |    |    |    |    |
| 8               | 0 | 0 | 0 | 0 | 0 | 2 | 28 | 101 | 157 | 99 |    |    |    |    |
| 9               | 0 | 0 | 0 | 0 | 0 | 1 | 23 | 142 | 358 | 426 | 216 |    |    |    |
| 10              | 0 | 0 | 0 | 0 | 0 | 1 | 15 | 161 | 660 | 1233 | 1166 | 488 |    |    |
| 11              | 0 | 0 | 0 | 0 | 0 | 0 | 10 | 156 | 1010 | 2873 | 4163 | 3206 | 1121 |    |
| 12              | 0 | 0 | 0 | 0 | 0 | 5 | 138 | 1356 | 5705 | 11987 | 13847 | 8892 | 2644 |    |
| 13              | 0 | 0 | 0 | 0 | 0 | 2 | 101 | 1613 | 9985 | 29652 | 48071 | 45505 | 24743 |    |
Table 82. $G(E, V, 3)$. Simple graphs with 3 components and a total of $E$ edges and $V$ vertices. Column sums are in column 3 of [10, A201922] $\mapsto$ 1, 1, 3, 9, 32, 154, 1065, 12513, 276114, 12021725 $\ldots$.

| $E \backslash V$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0               | 0   | 0   | 1   |     |     |     |     |     |     |     |     |     |     |     |     |
| 1               | 0   | 0   | 0   | 1   |     |     |     |     |     |     |     |     |     |     |     |
| 2               | 0   | 0   | 0   | 0   | 2   |     |     |     |     |     |     |     |     |     |     |
| 3               | 0   | 0   | 0   | 0   | 1   | 4   |     |     |     |     |     |     |     |     |     |
| 4               | 0   | 0   | 0   | 0   | 0   | 3   | 8   |     |     |     |     |     |     |     |     |
| 5               | 0   | 0   | 0   | 0   | 0   | 1   | 9   | 15  |     |     |     |     |     |     |     |
| 6               | 0   | 0   | 0   | 0   | 0   | 1   | 7   | 26  | 32  |     |     |     |     |     |     |
| 7               | 0   | 0   | 0   | 0   | 0   | 0   | 5   | 29  | 71  | 66  |     |     |     |     |     |
| 8               | 0   | 0   | 0   | 0   | 0   | 0   | 2   | 29  | 112 | 196 | 143 |     |     |     |     |
| 9               | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 23  | 150 | 406 | 539 | 315 |     |     |     |
| 10              | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 15  | 164 | 706 | 1417| 1486| 710 |     |     |
| 11              | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 10  | 157 | 1044| 3110| 4834| 4105| 1631|     |
| 12              | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 5   | 139 | 1376| 5951| 13102|16193|11408|     |
| 13              | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 2   | 101 | 1626| 10202|31198|52966|53519|     |

Table 83. $G(E, V, 4)$. Simple graphs with 4 components and a total of $E$ edges and $V$ vertices.
Table 84. $G(E, V, 5)$. Simple graphs with 5 components and a total of $E$ edges and $V$ vertices.

| $E \backslash V$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 |
|----------------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 0              | 0  | 0  | 0  | 0  | 1  |    |    |    |    |    |    |    |    |    |    |
| 1              | 0  | 0  | 0  | 0  | 0  | 1  |    |    |    |    |    |    |    |    |    |
| 2              | 0  | 0  | 0  | 0  | 0  | 0  | 2  |    |    |    |    |    |    |    |    |
| 3              | 0  | 0  | 0  | 0  | 0  | 0  | 1  | 4  |    |    |    |    |    |    |    |
| 4              | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 3  | 8  |    |    |    |    |    |    |
| 5              | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 1  | 9  | 16 |    |    |    |    |    |
| 6              | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 1  | 7  | 26 | 33 |    |    |    |    |
| 7              | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 5  | 29 | 72 | 69 |    |    |    |
| 8              | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 2  | 29 | 112 | 199 | 149 |    |    |
| 9              | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 1  | 23 | 150 | 408 | 549 | 330 |    |
| 10             | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 1  | 15 | 164 | 707 | 1426 | 1516 | 742 |
| 11             | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 10 | 157 | 1044 | 3117 | 4874 | 4193 |
| 12             | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 5  | 139 | 1376 | 5954 | 13142 | 16343 |    |
| 13             | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 2  | 101 | 1626 | 10203 | 31230 | 53170 |    |
Table 85. Undirected loopless multigraphs with 2 components and a total of $E$ edges and $V$ vertices.

| $E \setminus V$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-----------------|---|---|---|---|---|---|---|---|---|----|----|----|
| 0               | 0 | 1 |   |   |   |   |   |   |   |    |    |    |
| 1               | 0 | 0 | 0 | 1 |   |   |   |   |   |    |    |    |
| 2               | 0 | 0 | 1 | 2 |   |   |   |   |   |    |    |    |
| 3               | 0 | 0 | 1 | 3 | 3 |   |   |   |   |    |    |    |
| 4               | 0 | 0 | 1 | 5 | 8 | 6 |   |   |   |    |    |    |
| 5               | 0 | 0 | 1 | 6 |17 |20 |11 |   |   |    |    |    |
| 6               | 0 | 0 | 1 | 9 |32 |58 |52 |23 |   |    |    |    |
| 7               | 0 | 0 | 1 |10 |53 |135|185|132|46 |    |    |    |
| 8               | 0 | 0 | 1 |13 |84 |290|548|586|344|99  |    |    |
| 9               | 0 | 0 | 1 |15 |127|565|1441|2108|1829|900 |216 |    |
| 10              | 0 | 0 | 1 |18 |184|1055|3456|6696|7884|5680|2834|488 |
| 11              | 0 | 0 | 1 |20 |259|1859|7774|19288|29633|28718|17546|6811|
| 12              | 0 | 0 | 1 |24 |359|3178|16578|51799|126013|102743|54469|

Table 86. Undirected loopless multigraphs with 3 components and a total of $E$ edges and $V$ vertices.

| $E \setminus V$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-----------------|---|---|---|---|---|---|---|---|---|----|----|----|
| 0               | 0 | 0 | 1 |   |   |   |   |   |   |    |    |    |
| 1               | 0 | 0 | 0 | 1 |   |   |   |   |   |    |    |    |
| 2               | 0 | 0 | 0 | 1 | 2 |   |   |   |   |    |    |    |
| 3               | 0 | 0 | 0 | 1 | 3 | 4 |   |   |   |    |    |    |
| 4               | 0 | 0 | 0 | 1 | 5 | 9 | 7 |   |   |    |    |    |
| 5               | 0 | 0 | 0 | 1 | 6 |19 |23 |14 |   |    |    |    |
| 6               | 0 | 0 | 0 | 1 | 9 |35 |65 |62 |29 |    |    |    |
| 7               | 0 | 0 | 0 | 1 |10 |57 |148|214|159|60  |    |    |
| 8               | 0 | 0 | 0 | 1 |13 |89 |313|614|681|421 |128 |    |
| 9               | 0 | 0 | 0 | 1 |15 |134|601|1577|2374|2148|1104|284 |
| 10              | 0 | 0 | 0 | 1 |18 |192|1110|3711|7353|8938|6683|3389|
| 11              | 0 | 0 | 0 | 1 |20 |269|1938|8225|20752|32692|32639|20712|
| 12              | 0 | 0 | 0 | 1 |24 |371|3289|17332|54847|108802|139316|117082|

6.4. Loopless connected Multigraphs. Another application of the algorithm is to construct [10, A192517] from [10, A191646]. The base information of $G(E, V, k = 1)$ is this time in Table 61, and the Multiset Transformations creates Tables 85–88 and so on ($k \geq 2$). The sum over all $k \geq 1$ converges to Table 69.

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Table 87. Undirected loopless multigraphs with 4 components and a total of $E$ edges and $V$ vertices.

\begin{center}
\begin{tabular}{|l|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
$E \backslash V$ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
\hline
0 & 0 & 0 & 0 & 1 &  &  &  &  &  &  &  &  &  \\
1 & 0 & 0 & 0 & 0 & 1 &  &  &  &  &  &  &  &  \\
2 & 0 & 0 & 0 & 0 & 1 & 2 &  &  &  &  &  &  &  \\
3 & 0 & 0 & 0 & 0 & 1 & 3 & 4 &  &  &  &  &  &  \\
4 & 0 & 0 & 0 & 0 & 1 & 5 & 9 & 8 &  &  &  &  &  \\
5 & 0 & 0 & 0 & 0 & 1 & 6 & 19 & 24 & 15 &  &  &  &  \\
6 & 0 & 0 & 0 & 0 & 1 & 9 & 35 & 67 & 65 & 32 &  &  &  \\
7 & 0 & 0 & 0 & 0 & 1 & 10 & 57 & 151 & 221 & 169 & 66 &  \\
8 & 0 & 0 & 0 & 0 & 1 & 13 & 89 & 318 & 628 & 711 & 449 & 143 &  \\
9 & 0 & 0 & 0 & 0 & 1 & 15 & 134 & 607 & 1603 & 2445 & 2248 & 1185 & 315 &  \\
10 & 0 & 0 & 0 & 0 & 1 & 18 & 192 & 1119 & 3754 & 7506 & 9227 & 7025 & 3608 &  \\
11 & 0 & 0 & 0 & 0 & 1 & 20 & 269 & 1949 & 5849 & 21049 & 33426 & 33790 & 21799 &  \\
12 & 0 & 0 & 0 & 0 & 1 & 24 & 371 & 3304 & 17437 & 55396 & 110485 & 142723 & 121385 &  \\
\hline
\end{tabular}
\end{center}

Table 88. Undirected loopless multigraphs with 5 components and a total of $E$ edges and $V$ vertices.

\begin{center}
\begin{tabular}{|l|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
$E \backslash V$ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 &  &  &  &  &  &  &  &  &  \\
1 & 0 & 0 & 0 & 0 & 0 & 1 &  &  &  &  &  &  &  &  \\
2 & 0 & 0 & 0 & 0 & 0 & 1 & 2 &  &  &  &  &  &  &  \\
3 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 4 &  &  &  &  &  &  \\
4 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & 9 & 8 &  &  &  &  &  \\
5 & 0 & 0 & 0 & 0 & 0 & 1 & 6 & 19 & 24 & 16 &  &  &  &  \\
6 & 0 & 0 & 0 & 0 & 0 & 1 & 9 & 35 & 67 & 65 & 33 &  &  &  \\
7 & 0 & 0 & 0 & 0 & 0 & 1 & 10 & 57 & 151 & 223 & 172 & 69 &  \\
8 & 0 & 0 & 0 & 0 & 0 & 1 & 13 & 89 & 318 & 631 & 718 & 459 & 149 &  \\
9 & 0 & 0 & 0 & 0 & 0 & 1 & 15 & 134 & 607 & 1608 & 2459 & 2278 & 1213 & 330 &  \\
10 & 0 & 0 & 0 & 0 & 0 & 1 & 18 & 192 & 1119 & 3761 & 7533 & 9299 & 7126 & 3690 &  \\
11 & 0 & 0 & 0 & 0 & 0 & 1 & 20 & 269 & 1949 & 8304 & 21095 & 33584 & 34084 & 22146 &  \\
12 & 0 & 0 & 0 & 0 & 0 & 1 & 24 & 371 & 3304 & 17450 & 55472 & 110799 & 143481 & 122562 &  \\
\hline
\end{tabular}
\end{center}

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