Shape stability of pasta phases: Lasagna case

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The stability of periodically placed slabs (lasagna phase) is examined by exact geometrical methods for the first time. It appears the slabs are stable against any perturbation modes for the whole range of volume fraction occupied by the slab. The calculations are done in the framework of the compressible liquid drop model but obtained results are universal and do not depend on model parameters like surface tension or charge density. The results shows the transition to other pasta shapes requires crossing the finite energy barrier.

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I. INTRODUCTION

The very long structures appearing in neutron star matter, called pasta phases, are a commonly accepted phenomenon. Following the seminal work [1] the pasta phases have been studied in many different manners. Many approaches are based on the compressible liquid drop model (CLDM) where the system is described by two homogeneous phases separated by a sharp boundary with non-zero surface tension [2–6]. The competition between the Coulomb and surface energy leads to different shapes which are usually described in the Wigner-Seitz approximation, where the geometry of the phase is imposed at the beginning. The spherical and cylindrical cells are not the correct unit cells as they are not able to fill the whole space by periodic placement. Structures obeying periodicity, in the form of gyroid and diamond-like shapes, were introduced in [7, 8]. However, they must be treated only as an approximation of a true shape, because they do not satisfy the necessary condition for the cell energy extremum. It was shown in [9] that the condition relates the mean curvature \( H(x) \) of the cluster surface and electrostatic potential \( \Phi(x) \)

\[
2\sigma H(x) = C + \Delta \rho \Phi(x),
\]

where \( \sigma \) and \( \Delta \rho \) are surface tension and charge difference between the cluster and its surroundings and \( C \) is a gauge dependent constant. It means the surface curvature depends on the potential distribution and is not constant in general. In fact, the phases considered in [7] represent surfaces with \( H = 0 \) and as such they cannot be the true solution of Eq. (1). In works [7, 8] it was also shown that they have larger energy than the flat slabs. Nevertheless, the energy difference is not large, so it seems that such triple-periodic structures are likely.

Recent analysis of pasta phases, including the periodicity, based on quantum or classical Molecular Dynamics [10–14] and Time-Dependent Hartree-Fock [15, 16] or Hartree-Fock with twist-averaged boundary conditions [17] have shown the existence of triply-periodic in the form of gyroid, diamond, twisted spaghetti and waffles. The general motivation of our work is to confirm the existence of such non-trivial periodic structures also in the framework of CLDM. Therefore, we propose to examine the possibility of transition from lasagna phase to other shapes by its shape deformation. The inspection of various perturbation modes could indicate to what kind of structures the lasagna is going to transform. Such consideration corresponds to the stability analysis of the lasagna phase. The answer to the stability question is not obvious. Though the contribution from surface energy for flat slab is always positive, the Coulomb energy is not and could destabilize the lasagna.

In the work [9] an overall discussion of the rigorous treatment of periodically placed proton clusters of any shape was presented. The stable cluster surface should satisfy not only the Eq. (1), which represents the necessary condition for the extremum, but also the condition for the minimum coming from the inspection of second variation of the energy functional. The second order analysis, in limited sense, was already carried for cylinders and balls in the [5], where particular deformation mode was examined in the isolated Wigner-Seitz cell. Periodically placed cylinders and balls were considered in [6] and later in [19], but we need to be aware that these structures do not represent the proper minimum determined by the Eq. (1), their shape was assumed a priori. Up to now, the only known true periodic solution of Eq. (1) is lasagna (the boundaries of slabs coincide with the equipotential surfaces) and thus the stability analysis of these structure may be carried out.

Partial stability analysis of lasagna phase was done in by Pethick and Potekhin in the context of elastic properties of the phase in the work [19]. In order to determine the elasticity coefficient, they considered one particular deformation mode and moreover the expression for the energy was valid only in the limit of wavelength going to infinity. Such analysis corresponds to the stability consideration, however, being limited to the only one type of deformation. Here, we present general, unconstrained stability analysis for any kind of deformation with finite wavelength, what, to the authors’ knowledge, has never been carried out.
Deformations of this kind may be expressed in terms of Fourier series on the \( y, z \)-plane. The deformations on each slab face are independent, so we get two series for each face located at \( x_1 \) and \( x_2 \). For further calculations it is convenient to introduce expansion based on complex amplitudes \( \alpha_{mk}^j \), where \( m \) and \( k \) are the mode indices and the superscript \( j \) corresponds to the face number

\[
e^j(y, z) = \sum_{m,k=-\infty}^\infty \alpha_{mk}^j \exp(iK_{0mk} \cdot x) ,
\]

where we introduce 3-dimensional discrete wave vector

\[
K_{nmk} = \left( \frac{2\pi n}{a}, \frac{2\pi m}{b}, \frac{2\pi k}{c} \right)
\]

and \( x = (x, y, z) \). The \( m \) and \( k \) indices take integer values from \(-\infty\) to \(+\infty\) except the case when both of them equal zero, which is indicated by an apostrophe in the sum sign. The lack of \((0,0)-mode\) comes from the volume conservation condition, Eq. \([3]\). Since the deformation function \( e^j(y, z) \) must be real it means that the complex amplitudes should fulfill the following relations

\[
(\alpha_{mk}^j)^*=\alpha_{-m,-k}^j.
\]

It is more natural to introduce the cosine \( \epsilon_{mk,C} \cos(\frac{\pi m}{a} + \frac{\pi k}{c}) \) and sine modes \( \epsilon_{mk,S} \sin(\frac{\pi m}{a} + \frac{\pi k}{c}) \) in the expansion Eq. \(4\). Then the relation between complex and real amplitudes is

\[
e_{mk,C}^j = \alpha_{mk}^j + \alpha_{-m,-k}^j ;
\]

\[
e_{mk,S}^j = i(\alpha_{mk}^j - \alpha_{-m,-k}^j) .
\]

In the Fig. 2 an example of the slab deformation with real amplitudes taking the following values \( \epsilon_{10,S}^j = \epsilon_{10,C}^j \neq 0 \) and \( \epsilon_{10,C} = \epsilon_{10,S} = 0 \) is shown. Below we present and discuss the subsequent contribution to the total energy variation \( \delta^2 \dot{\epsilon} \), Eq. \(2\). In terms of complex amplitudes the surface energy contribution to the 2nd order variation takes simple form

\[
\delta^2 \dot{\epsilon}_S = \frac{1}{2V_c} \int_{\partial_\mathcal{P}} (\sigma((\nabla \epsilon)^2 - B^2 \epsilon^2) + \Delta \rho \left( \partial_n \Phi_0 \epsilon^2 + \dot{\delta}_\Phi \epsilon \right)) \, dS,
\]

where \( B^2 = \kappa_1^2 + \kappa_2^2 = 0 \) is the sum of squared principal curvatures, \( \partial_n \Phi_0 \) is the normal derivative of unperturbed potential, \( \delta_\Phi \) is the first order perturbation of the potential caused by the surface deformation \( \epsilon \). One must remember that here we consider only the deformation which preserves the volume of the cluster, that means

\[
\int_{\partial_\mathcal{P}} \epsilon \, dS = 0 .
\]

It is always positive, as \( B^2 = 0 \) for the slab and due to relations \(6\) we get \( \alpha_{mk}^j \alpha_{-m,-k}^j = |\alpha_{mk}^j|^2 > 0 \). That is an important fact, because in many cases the surface energy acts as a destabilizer, like, for example, \( \delta^2 \dot{\epsilon}_S < 0 \) in the case of Rayleigh-Plateau instability.

The contribution coming from the electrostatic interaction includes two terms. The first one, determined by the normal derivative of potential \( \Phi_0 \), is given by

\[
\delta^2 \dot{\epsilon}_{\text{norm}} = \frac{\Delta \rho}{2V_c} \int_{\partial_\mathcal{P}} \partial_n \Phi_0 \epsilon^2 \, dS
\]

\[
= \frac{\Delta \rho}{2a} \sum_{j=1}^{\infty} \sum_{mk} \partial_n \Phi_0(x_j) \alpha_{mk}^j \alpha_{-m,-k}^j .
\]
The normal derivatives at the slab faces are $\partial_n \Phi_0(x_1) = \partial_n \Phi_0(x_1) = -2\pi(1 - w)wa\Delta \rho$. So, finally one gets

$$\delta^2 \varepsilon_{\text{norm}} = -\pi\Delta \rho^2 w(1 - w)\sum_{j=1}^{2} \sum_{m,k} \epsilon_{m,k} \alpha_{j,m,-k}^j$$

which is always negative. The second term of Coulomb interaction part is associated with perturbation of the potential $\delta \Phi$

$$\delta^2 \varepsilon_{\Phi} = \frac{\Delta \rho}{2V_c} \int_{\partial S} \delta \Phi \epsilon \, dS,$$  

where $\delta \Phi$ is the first order potential perturbation calculated thanks to the periodic Green function

$$\delta \Phi(\mathbf{x}) = \Delta \rho \int_{\partial S} G_P(\mathbf{x}, x') \epsilon(x') \, dS'. \quad (13)$$

For the three-dimensional unit cell with sizes $a, b, c$ the periodic Green function can be expressed as the sum over discrete modes numbered by three indices $n, m, k$

$$G_P(\mathbf{x}, x') = \frac{4\pi}{abc} \sum_{n,m,k=-\infty}^{+\infty} \exp(i K_{nmk} \cdot (\mathbf{x} - \mathbf{x}')) \frac{K_{nmk}^2}{K_{nmk}^2}.$$  

The prime sign in the summation means that $n, m, k$ cannot vanish simultaneously, similarly as in the Eq. (3). Such Green function fulfills the Poisson equation in the unit cell with periodic boundary conditions (20) (we use CGS units)

$$\nabla^2 G_P(\mathbf{x}, x') = -4\pi \left( \delta(\mathbf{x} - \mathbf{x}') - \frac{1}{abc} \right) \quad (15)$$

and the absence of $(0,0,0)$ mode in the expansion (14) is a consequence of the unit cell neutrality. By joining

eq \text{Eqs.}(4,13,14) we obtain the change in the energy with respect to the potential variation

$$\delta^2 \varepsilon_{\Phi} = 2\pi \Delta \rho^2 \sum_{i,j=1}^{2} \sum_{m,k} \alpha_{m,k} \alpha_{m,-k} F(\xi_{ij}, \chi_{mk}), \quad (16)$$

where $\xi_{ij}$ corresponds to the difference between the faces location

$$\xi_{ij} = \frac{x_i - x_j}{a} \quad (17)$$

and $\chi_{mk}$ is the norm of the dimensionless wave vector $\chi_{mk} = \frac{2\pi ma}{b}, \frac{2\pi ka}{c}$

$$\chi_{mk} = aK_{nmk} = \sqrt{\left( \frac{2\pi ma}{b} \right)^2 + \left( \frac{2\pi ka}{c} \right)^2}. \quad (18)$$

The vector $\chi_{mk}$ describes the manner in which the face is vibrating - the indices $m$ and $k$ determine the number of wavelengths being placed in the face sizes $b$ and $c$. The function $F(\xi, \chi)$ comes from the summation over the $n$ index. As the $n$-numbered modes do not depend on $x$, the summation over $n$ can be done separately and defines a function $F$

$$F(\xi, \chi) = \sum_{n=-\infty}^{+\infty} e^{2\pi i n \xi} \frac{1}{(2\pi n)^2 + \chi^2}. \quad (19)$$

The function is positive for any $\xi$ and $\chi$. In some special cases the function simplifies to

$$F(0, \chi) = \frac{\coth \left( \frac{\chi}{2} \right)}{2\chi}, \quad (21)$$

$$F(\frac{1}{2}, \chi) = \frac{1}{2\chi \sinh \left( \frac{\chi}{2} \right)}. \quad (22)$$

Taking together all energy terms we get the total energy variation expressed in terms of mode amplitudes $\alpha_{mk}$

$$\delta^2 \varepsilon = \sum_{i,j=1}^{2} \sum_{m,k} \left\{ \frac{\sigma}{2\alpha^2} \chi_{mk}^2 \delta_{ij} - \pi \Delta \rho^2 w(1 - w)\delta_{ij} + 2\pi \Delta \rho^2 F(\xi_{ij}, \chi_{mk}) \right\} \alpha_{m,k} \alpha_{m,-k}.$$  

As the $\delta^2 \varepsilon$ is the quadratic form of the amplitudes $\alpha_{mk}$ the competition between the terms in curly brackets decides about stability of the face surface. There are three characteristic terms: one from the surface energy being positive and two from Coulomb interactions: first is negative and the second is always positive. It seems the stability consideration depends on the values of surface tension $\sigma$ or charge contrast $\Delta \rho$ but it appears that these
parameters may be removed from our analysis. One of
the conditions for the minimum of the energy for unit cell
is the virial theorem \[9\]. The theorem takes the form of
relation between the surface and Coulomb energy of
the cell. For the cell with high symmetry, it has simple form
\[ E_S = 2E_{Coul} \],
from which we may get the relation between \( \sigma \) and \( \Delta \rho \)
\[ \sigma = \frac{1}{6} \pi a^3 \Delta \rho^2 (w - 1)^2 w^2 . \]
Finally the total energy variation may be written as the
quadratic form of \( \epsilon \)
\[ \delta^2 \hat{\epsilon} = \pi \Delta \rho^2 \sum_{i,j=1}^{2} \sum_{m,k} \hat{A}^{ij}_{mk} \alpha_{i,m}^{\prime} \alpha_{i,m,-k}^{\prime} . \]
with its coefficients \( \hat{A}^{ij}_{mk} \) given by
\[ \hat{A}^{ij}_{mk} = \{( \frac{1}{12} w^2 (1 - w)^2 \hat{\chi}_{m,k} - w(1 - w) \} \delta_{ij}
+ 2 F(\xi_{ij}, \chi_{m,k}) \} \]
The obtained general expression, Eq. \[20\] for the second
order variation of the total energy allows us to determine
whether the slab is stable with respect to any deforma-
tions preserving its volume.
It is worth noting that \( \hat{A}^{ij}_{mk} \) coefficients, which decide
about stability, does not depend on the strength of inter-
actions being determined by surface tension \( \sigma \) and charge
contrast \( \Delta \rho \). In this way, we obtained an interesting re-
sult that the stability of pasta depends only on the geo-
metry of phase and mode under consideration and not
on the details of strong or electromagnetic interactions.
Before general discussion of the stability of a single
slab we show how the above results work in the stability
analysis for particular class modes.

III. AN EXAMPLE OF STABILITY ANALYSIS

Let’s consider the simplest surface perturbation con-
sisting of combination of sine and cosine modes going
along \( y \) axis for each face, which corresponds to \( m = \pm 1 \)
and \( k = 0 \). Then, for the \( i \)-th face, the only non-vanishing
complex amplitudes \( \alpha_{i,m}^{\prime} \) are:
\[ \alpha_{i,10}^{\prime} = \frac{1}{2} (\epsilon_{i,1}^C - i \epsilon_{i,2}^S) , \]
\[ \alpha_{i,-10}^{\prime} = \frac{1}{2} (\epsilon_{i,1}^C + i \epsilon_{i,2}^S) , \]
where we introduced the real amplitudes \( \epsilon_{i,1}^S \) and \( \epsilon_{i,1}^C \) cor-
responding the functions \( \sin(\frac{2\pi y}{b}) \) and \( \cos(\frac{2\pi y}{b}) \). The
deformation \( \epsilon \) for \( i \)-th face is then a function of \( y \) only
\[ \epsilon^i(y) = \epsilon_{i,1}^C \cos\left(\frac{2\pi y}{b}\right) + \epsilon_{i,2}^S \sin\left(\frac{2\pi y}{b}\right) . \]
It is convenient to introduce the vector \( \epsilon \) built of deforma-
tion amplitudes
\[ \epsilon = (\epsilon_{1,1}^C, \epsilon_{1,2}^C, \epsilon_{2,1}^S, \epsilon_{2,2}^S) . \]

Then the total energy variation for such deformation may
be written down in the matrix form
\[ \delta^2 \hat{\epsilon} = \pi \Delta \rho^2 \epsilon \hat{M} \epsilon^T , \]
where the dimensionless matrix \( \hat{M} \) is
\[ \hat{M} = \begin{pmatrix} A & B & 0 & 0 \\ B & A & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & B & A \end{pmatrix} \]
and its elements are:
\[ A = \frac{1}{24} (1 - w)^2 w^2 \chi^2 - \frac{1}{2} w(1 - w) + F(0, \chi) , \]
\[ B = F(w, \chi) , \]
where \( w \) is the volume fraction and \( \chi \) determines the
wavelength of mode in comparison to the cell size \( \chi = 2\pi a/b \). The inspection of eigenvalues and eigenvectors of
\( \hat{M} \) allows for a complete stability analysis for the defor-
ma tion we have chosen. The matrix \( \hat{M} \) given by Eq. \[33\]
possesses the two-fold degenerated two eigenvalues. Fur-
ther, we call the \( \hat{M} \)-eigenvalues \( \lambda_i, i = 1, \ldots , 4 \) as stability
function for our concrete form of \( \hat{M} \) the eigenvalues and
their eigenvectors are
\[ \lambda_{1,3} = \frac{1}{24} (1 - w)^2 w^2 \chi^2 - \frac{1}{2} w(1 - w) + F(0, \chi) - F(w, \chi) \]
\[ \epsilon_1 = (-1, 1, 0, 0) , \quad \epsilon_3 = (0, 0, -1, 1) \]
\[ \lambda_{2,4} = \frac{1}{24} (1 - w)^2 w^2 \chi^2 - \frac{1}{2} w(1 - w) + F(0, \chi) + F(w, \chi) \]
\[ \epsilon_2 = (1, 1, 0, 0) , \quad \epsilon_4 = (0, 0, 1, 1) . \]
The stability functions \( \lambda_i \) do not depend on the details of
interactions \( \sigma, \Delta \rho \) but only on the geometry of our sys-
tem which is described by volume fraction \( w \) and mode
wavelength \( \chi \). In Fig. 3 the stability functions are plotted
in the \( w, \chi \) parameter space for their eigenvectors. These
vectors represent two classes of modes. We may call them
as snaky and hourglass-shaped modes. The snaky modes
occur when the deformations on both faces are in phase
\( \epsilon_1 \) and \( \epsilon_3 \) and hourglass modes occur when the deforma-
tions on faces are in antiphase \( \epsilon_2 \) and \( \epsilon_4 \). As we may notice,
for all values of volume fraction and mode wavelengths
the stability functions are positive, that means, that for
both classes of modes the slab is stable. However the
stability is not too strong. Careful inspection of \( \lambda_1 \) and
\( \lambda_2 \) shows that those functions go to 0 when \( \chi \) and \( w \napproach to some specific values
\[ \lambda_1 \mid_{\chi \to 0} \to 0 \quad \text{for any } w \]
and
\[ \lambda_2 \mid_{\chi \to \infty} \to 0 \quad \text{for } w \to 0 \text{ or } 1 . \]
It means that snaky modes become unstable in the limit of very long waves regardless slab thickness, whereas the hourglass modes become unstable for very thin slab and very short waves. To sum up, we may say the slab becomes asymptotically unstable for very long mode or for very short mode when the slab becomes very thin in comparison to unit cell width. One should note that, the case when $w \approx 0$ or $w \approx 1$ must be treated with caution because in reality the cluster surface has finite thickness and for very thin slab the validity of liquid drop model could be questioned.

IV. ELASTIC PROPERTIES

Nuclear pastas share their elastic properties with liquid crystals. If the wavelength of the snaky mode is very large in comparison to the cell size, it corresponds to the so-called splay deformation of the liquid crystal. It allows to determine elastic constant $K_1$ for the lasagna phase what was shown by Pethick et al. in [19]. By definition, the constant $K_1$, relates the deformation energy with the transverse derivative of the deformation field when the mode wavelength becomes very large. In our notation the relation takes the form

$$\delta^2 \tilde{\varepsilon} = \frac{K_1}{2} \langle (\partial^2_y \varepsilon)^2 \rangle ,$$

where the bracket $\langle .. \rangle$ means the average over the slab surface. Taking the definition of $K_1$ in limit of the very long mode, we get

$$K_1 = 4a^4 \lim_{\chi \to 0} \frac{\delta^2 \tilde{\varepsilon}}{\chi^4 \epsilon^2} , \quad (38)$$

(here $\epsilon$ denote the mode amplitude only). The deformation energy for the snaky mode is

$$\delta\tilde{\varepsilon}_{\text{snaky}} = \pi \Delta \rho^2 \left( \frac{1}{12} (1-w)^2 w^2 \chi^2 - (1-w)w ight) + 2 \left( F(0,\chi) - F(w,\chi) \right) \epsilon^2 . \quad (39)$$

Applying the limit in Eq. (38) we get

$$K_1 = \frac{1}{180} \pi a^4 \Delta \rho^2 (w-1)^2 w^2 (1 + 2w - 2w^2) , \quad (40)$$

what exactly corresponds to the result of [19] if the $\Delta \rho$ is replaced by the unperturbed Coulomb energy of the cell $\varepsilon_{C,0} = \frac{1}{\pi} \pi a^2 \Delta \rho^2 (1-w)^2 w^2$. In comparison to [19] our approach represents an improvement. Pethick et al. used a summation for the deformation energy which is in fact divergent. Then, the authors treated the divergent series as an asymptotic expansion with respect to $\chi^{-1}$ and finally they obtained correct result. In our approach we avoid any divergences.
V. GENERAL DISCUSSION OF STABILITY

In the Section III the stability analysis for the simplest slab deformation was carried out. The full stability analysis would require the inclusion of modes for all multiplicities \( m \) and \( k \). Writing down the matrix \( \hat{M} \) for all modes it appears to take the block-diagonal form

\[
\hat{M} = \begin{pmatrix}
\ddots & A & B & 0 & 0 \\
A & B & 0 & 0 \\
0 & B & A & 0 \\
0 & 0 & A & B \\
0 & 0 & B & A \\
\ddots & & & & 
\end{pmatrix},
\]

(41)

where in \( m k \)-th position we get the same matrix as in Eq. (33) with \( A \) and \( B \) elements taking the same form like in Eq. (35) but with the replacement \( \chi \rightarrow \chi_{km} \). Such a form of \( \hat{M} \) means that the modes with different multiplicity \( m, k \) and the cosine-like and sine-like modes do not couple. So, taking fixed \( m, k \) one may repeat the discussion from the Section III and finally conclude that the slab is stable for any mode keeping in mind the asymptotic cases, described by the Eqs. (36,37).

VI. CONCLUSIONS

In this work, by use of analytical methods, we have shown that proton clusters having the form of slabs placed periodically in space are stable for all values of volume fraction occupied by the cluster. It is a compelling result. All works based on CLDM (see for example [1,4,6,7]), show that different shapes of pasta are preferred for different values of \( w \). Here, we have shown that lasagna phase is stable in the whole range of \( w \). One must remember that our analysis means the lasagna phase represents merely a local minimum. The transition to another geometry is not totally blocked, but requires finite size deformation in order to exceed the energy barrier. It may be interpreted that, at least for some range of volume fraction the lasagna phase is metastable. That could be quite interesting for the dynamics of pasta appearance during the neutron star formation.

Our analysis is based on small deformations so it cannot state what range of \( w \) the lasagna represents global minimum of the cell energy. First, the global minimum statement requires the knowledge of exact solutions of Eq. (1) for other kind of shape than flat slab. So far, we have not known such solutions in the CLDM approach. The seeking of them mark out the direction of further research of pasta by differential geometry methods.

We are also conscious that the approach, based on the CLDM, has its limitations and inclusion of such effects like finite thickness of the cluster surface or the temperature fluctuations could change the final conclusion concerning lasagna phase stability.

[1] D. G. Ravenhall, C. J. Pethick, and J. R. Wilson, Phys. Rev. Lett. 50, 2066 (1983).
[2] G. Baym, H. A. Bethe, and C. Pethick, Nucl. Phys. A 175, 225 (1971).
[3] M. Hashimoto, H. Seki, and M. Yamada, Prog. Theor. Phys. 71, 320 (1984).
[4] K. Oyamatsu, Nucl. Phys. A 561, 431 (1993).
[5] K. Iida, G. Watanabe, and K. Sato, Prog. Theor. Phys. 106, 551 (2001); Erratum: Prog. Theor. Phys. 110, 847 (2003).
[6] R. D. Williams and S. E. Koonin, Nucl. Phys. A 435, 844 (1985).
[7] K. Nakazato, K. Oyamatsu, and S. Yamada, Phys. Rev. Lett. 103, 132501 (2009).
[8] K. Nakazato, K. Iida and K. Oyamatsu, Phys. Rev. C 83, 065811 (2011).
[9] S. Kubis and W. Wójcik, Phys. Rev. C 94, 065805 (2016).
[10] G. Watanabe, T. Maruyama, K. Sato, K. Yasuoka, and T. Ebisuzaki, Phys. Rev. Lett. 94, 031101 (2005).
[11] A. S. Schneider, C. J. Horowitz, J. Hughto, and D. K. Berry, Phys. Rev. C 88, 065807 (2013).
[12] A. S. Schneider, D. K. Berry, C. M. Briggs, M. E. Caplan, and C. J. Horowitz, Phys. Rev. C 90, 055805 (2014).
[13] D. K. Berry, M. E. Caplan, C. J. Horowitz, G. Huber, and A. S. Schneider, Phys. Rev. C 94, 055801 (2016).
[14] P. N. Alcain, P. A. Giménez Molinelli, and C. O. Dorso, Phys. Rev. C 90, 065803 (2014).
[15] B. Schuetrumpf, M. A. Klett, K. Iida, J. Maruhn, K. Mecke, and P. G. Reinhard, Phys. Rev. C 87, 055805 (2013).
[16] B. Schuetrumpf, M. A. Klett, K. Iida, G. E. Schroeder-Turk, J. A. Maruhn, K. Mecke, and P.-G. Reinhard, Phys. Rev. C 91, 025801 (2015).
[17] B. Schuetrumpf and W. Nazarewicz, Phys. Rev. C 92, 045806 (2015).
[18] C. O. Dorso, P. A. Gimenez Molinelli, and J. A. Lopez, Phys. Rev. C 86, 055805 (2012).
[19] C. J. Pethick and A. Y. Potekhin, Phys. Lett. B 427, 7 (1998).
[20] S. L. Marshall, J. Phys.: Cond. Matt., 12, 4575 (2000); S. Tyagi, Phys. Rev. E 70, 066703 (2004).
[21] I.S. Gradshteyn and I.M. Ryzhik Table of Integrals, Series, and Products, New York, Academic Press (1994).