GROUP TOPOLOGIES ON AUTOMORPHISM GROUPS OF HOMOGENEOUS STRUCTURES

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Abstract. We classify all group topologies coarser than the topology of stabilizers of finite sets in the case of automorphism groups of countable free-homogeneous structures, Urysohn space and Urysohn sphere.

1. Introduction

Minimality. A topological group \((G, \tau)\) consists of a group \((G, \cdot)\) and a topology \(\tau\) on \(G\) such that the map \(\rho : G \times G \rightarrow G\) where \(\rho(g, h) = gh^{-1}\) is jointly continuous.

Definition 1.1. A Hausdorff topological group \(G\) is called minimal if every bijective continuous homomorphism from \(G\) to another Hausdorff topological group is a homeomorphism. The group \(G\) is totally minimal if every continuous surjective homomorphism to a Hausdorff topological group is open.

In fact \((G, \tau)\) is minimal if and only if \(G\) does not admit a strictly coarser Hausdorff group topology than \(\tau\). Furthermore, it is also clear that every totally minimal group is minimal.

The notion of minimality for topological groups was introduced as back as 1971 as a generalization of compactness. In fact it is easy to see that any compact Hausdorff topological group is minimal. For more information about minimality, we refer the reader to the survey by Dikranjan and Megrelishvili [6].

Given a group \(G\) of permutations of some set \(\Omega\) and \(A \subseteq \Omega\), let \(G_A = \{g \in G \mid \forall a \in A, \ ga = a\}\). Let \([\Omega]^{<\omega}\) be the set of all finite subsets of \(\Omega\). The collection \(\{G_A \mid A \in [\Omega]^{<\omega}\}\) is a base of neighbourhoods at the identity of a group topology which we call the standard topology and denote by \(\tau_{st}\). More generally for each \(G\)-invariant \(X \subseteq \Omega\) there is an associated group topology \(\tau_X\) generated by \(\{G_A \mid A \in [X]^{<\omega}\}\).

One of the earliest results on minimality due to Gaughan [9] states that \((S_\infty, \tau_{st})\) is totally minimal where \(S_\infty\) denotes the group of all permutations of a countable set \(\Omega\).

Given a countable first order structure \(\mathcal{M}\) with universe \(M\), the automorphism group of \(\mathcal{M}\) is a \(\tau_{st}\)-closed subgroup of \(S_\infty = S(M)\) and vice versa: any closed subgroup of \(S(M)\) is the automorphism group of some countable structure on \(M\). The interplay between the dynamical properties of \(\text{Aut}(\mathcal{M})\) and the logical and combinatorial properties of \(\mathcal{M}\) has been widely studied in the literature, beginning with the characterization due to Engeler, Ryll-Nardzewski, Svenonius and others of oligomorphic subgroups of \(S_\infty\) as the automorphism groups of \(\omega\)-categorical countable structures. Recall that an
oligomorphic group is a closed subgroup of $S_\infty$ whose diagonal action on $M^n$ has finitely many orbits, for each $n \in \mathbb{N}$.

In this context $\tau_{st}$ is often referred to in the literature as the point-wise convergence topology, in implicit reference to the discrete metric on $M$. When discussing isometry groups it will be important for us distinguishing between $\tau_{st}$ as above and the point-wise convergence topology relative to the metric in question which we will denote by $\tau_m$, so we will avoid this practice.

In light of the above the following is thus a natural question, already asked in [6]

**Problem 1.** Let $\mathcal{M}$ be a countable $\omega$-categorical (\(\omega\)-saturated, sufficiently nice) first order structure and $G = \text{Aut}(\mathcal{M})$. When is $(G, \tau_{st})$ (totally) minimal?

A deep result in this direction appeared in recent work by Ben Yaacov and Tsankov [3], where the authors show that automorphism groups of countable $\omega$-categorical, stable continuous structures are totally minimal with respect to the point-wise convergence topology. This specializes to the result that the automorphism groups of classical $\omega$-categorical stable structures are totally minimal with respect to $\tau_{st}$.

Not all oligomorphic groups are minimal with respect to $\tau_{st}$. As pointed out in [3], an example of this is $\text{Aut}(\mathbb{Q}, <)$ (see Theorem 10.6 for a generalization). However even in those cases it is possible to formulate the following more general question:

**Problem 2.** Let $\mathcal{M}$ be a countable $\omega$-categorical (or sufficiently nice) first order structure and $G = \text{Aut}(\mathcal{M})$. Describe the lattice of all Hausdorff group topologies on $G$ coarser than $\tau_{st}$.

This work was mainly motivated by [3] and is meant as a preliminary exploration of Problems 1 and 2 in the classical setting outside the stability constraint.

In its broadest lines the strategy followed by [3] goes back to [20], where it was shown by Uspenskij that the isometry group of the Urysohn sphere is totally minimal with the point-wise convergence topology. Both proofs rely on the assumption that the group in question is Roelcke precompact and use a well behaved independence relation among (small) subsets of the structure to endow the Roelcke precompletion of the group with a topological semigroup structure. Information on the topological quotients of the original group is then recovered from the latter via the functoriality of Roelcke compactification and Ellis lemma. Recall that a topological group $(G, \tau)$ is Roelcke precompact if for any neighbourhood $W$ of 1 there exists a finite $F \subseteq G$ such that $WFW = G$. For closed subgroups of $S_{\infty}$ this is equivalent to being oligomorphic.

In contrast, our methods for obtaining (partial) minimality results are completely elementary. There are drawbacks to this lack of sophistication: for instance, we are not able to recover the result in [3] for classical structures. On the other hand we do not rely on assumptions of Roelcke pre-compactness (except for certain residual assumptions in some cases). In particular, we are able to answer in the positive the question about the minimality of the isometry group of the (unbounded) Urysohn space posed in [20] (Theorem C). It is worth emphasizing that in some cases we manage to obtain complete classifications of continuous homomorphic images of topological groups which are neither Roelcke precompact nor separable (see Theorem D).

A section by section summary with our main results can be found below.

**Free amalgamation and one basedness.** Section 3 provides a simple technical criterion (Lemma 3.10) of (relative) minimality for $\tau_{st}$ from which more concrete applications are derived in Section 4.

Recall that the free amalgam of two relational structures $A, B$ over a common substructure $C$ is the structure resulting from taking unrelated copies of $A$ and $B$ and then gluing together the two copies of $C$ without adding any extra relations. A free amalgamation class $K$ is a collection of finite structures closed under substructures and free amalgams. Associated with any such $K$ there is a unique Fraïssé limit: a countable structure in which every $A \in K$ embeds and which is ultra-homogeneous, i.e., any finite partial isomorphism extends to an automorphism of the structure.

**Theorem A.** Let $\mathcal{M}$ be the Fraïssé limit of a free amalgamation class in a countable relational structure. Let $G = \text{Aut}(\mathcal{M})$. Then any group topology $\tau \subseteq \tau_{st}$ on $G$ is of the form $\tau^X_{st}$, where $X \subseteq M$. 
is some $G$-invariant set. In particular, if the action of $G$ on $M$ is transitive, then $(G, \tau_{st})$ is totally minimal.

Simple structures (i.e. theories) occupy an important place in classification theory. We refer the reader to [18], [22] and [12] for the definition of simple theories, forking and canonical bases. We say that a simple theory $T$ is one-based if $Cb(a/A) \subseteq bdd(a)$ for any hyperimaginary element $a$ and small subset $A$ of the monster model.

**Theorem B.** Let $M$ be a simple, $\omega$-saturated countable structure with elimination of hyperimaginaries, locally finite algebraic closure and weak elimination of imaginaries. Assume furthermore that $Th(M)$ is one-based. Let $G = Aut(M)$. Then

1. If $G$ acts transitively on $M$, then $(G, \tau_{st})$ is minimal.
2. If all singletons are algebraically closed, then any group topology $\tau$ on $G$ coarser than $\tau_{st}$ is of the form $\tau^*_a$ for some $G$-invariant $X \subseteq M$.

By an independence relation is usually meant some ternary relation $\perp$ on (some) collection of sets of parameters of a structure such that $A \perp_C B$ captures the intuitive idea that $B$ does not contain any information about $A$ not already contained in $C$. The paradigmatic example is that of forking independence. The connections between the existence of an independence relations on a homogeneous structure satisfying certain axioms and the properties of the automorphism group goes back to [19] (see also [7]). Of particular relevance to us is the freedom axiom, explored in detail in [5]. We explain Theorems A and B in terms of the existence of an independence relation satisfying certain sets of axioms. The roles played by stationarity and the freedom axiom in A are replaced by one basedness and the independence property respectively in B.

**Generalized universal metric spaces.** Urysohn universal space $U_0\omega$ is a homogeneous space that contains all separable metric spaces due to Urysohn. It is both $\omega$-universal, i.e. it contains any finite metric space as a subspace and $\omega$-homogeneous, i.e. any partial isometry between finite subspaces of $U$ extends to some global isometry. Associated with the class of metric spaces with diameter at most 1 there is an object with similar properties $U_{[0,1]}$, known as the Urysohn sphere. The isometry groups Isom($U_0\omega$) and Isom($U_{[0,1]}$) endowed with the point-wise convergence topology $\tau_m$ (‘m’ is for ‘metric’) are Polish groups whose algebraic and dynamical properties have been widely studied. It is known, for example, that any Polish group is isomorphic to a closed subgroup of Isom($U_0\omega$) and that Isom($U_{[0,1]}$) is extremely amenable.

It is shown in [20] that any continuous quotient of $(\text{Isom}(U_{[0,1]}), \tau_m)$ is either trivial or a homeomorphism. Theorem C below extends this result to Isom($U_{\omega}$).

We work in the framework of generalized metric spaces introduced by Conant in [5]. A distance monoid is an abelian monoid endowed with a compatible linear order (see Subsection 5 for more details). Given a distance monoid $\mathcal{R} = (\mathcal{R}, 0, \oplus, \leq)$ an $\mathcal{R}$-metric space is a set $X$ endowed with a map $d: X^2 \to \mathcal{R}$ satisfying the obvious generalization of the axioms for a metric space. In our terminology an $\mathcal{R}$-Urysohn space $\mathcal{U}$ will be an $\mathcal{R}$-metric space satisfying the obvious generalization of $\omega$-homogeneity and $\omega$-universality to this setting, ignoring any separability and cardinality considerations. If $\mathcal{R} \vdash \forall x \neq 0 \exists y \neq 0 y \oplus y \leq x$, then the collection $\{N_u(\epsilon) \mid u \in \mathcal{U}, \epsilon \in \mathcal{R} \setminus \{0\}\}$, where $N_u(\epsilon) = \{g \in G \mid d(gu, u) \leq \epsilon\}$ generates a group topology on the isometry group of $\mathcal{U}$. For plain metric spaces the result is the point-wise convergence topology so we keep the notation $\tau_m$ in the general case.

We say that a distance monoid $\mathcal{R}$ as above is archimedean if for any $r, s \in \mathcal{R} \setminus \{0\}$ there exists some $m \in \mathbb{N}$ such that $m \cdot s := s \oplus s \oplus \cdots \oplus s$ ($m$ times) $\geq r$.

**Theorem C.** Let $\mathcal{R} = (\mathcal{R}, 0, \leq, \oplus)$ be an archimedean distance monoid, $\mathcal{U}$ a $\mathcal{R}$-Urysohn space, $G = \text{Isom}(\mathcal{U})$ and let $\tau_0$ be either:

- $\tau_m$ in case for any $r \in \mathcal{R} \setminus \{0\}$ there exists $s \in \mathcal{R} \setminus \{0\}$ with $s \oplus s \leq r$; or,
- $\tau_{st}$ otherwise.

Then $\tau_0$ is the coarsest non-trivial group topology on $G$ coarser than the stabilizer topology $\tau_{st}$. In particular, $(G, \tau_0)$ is totally minimal.
Given some $S \subseteq \mathbb{R}$ closed under addition and $b \in S_{>0} \cup \{\infty\}$, we let $S_b$ be the distance monoid given by the tuple \( \{ (r \in S | 0 < r \leq b), 0, \leq, +b \} \), where $x +_b y = \min\{x + y, b\}$.

**Theorem D.** Let $S$ be a dense subgroup of $\mathbb{R}$ and $b \in S_{>0} \cup \{\infty\}$, $\mathcal{U}$ an $S_b$-Urysohn space and $G = \text{Isom}(\mathcal{U})$. Then there are exactly 4 group topologies on $G$ coarser than $\tau_{st}$:

\[ \tau_{st} \supseteq \tau_{0+,0} \supseteq \tau_m \supseteq \{\emptyset, G\} \]

where $\tau_{0+,0}$ is the topology generated at the identity by the collection

\[ \{ \{g \in G | d(gu,v) \leq d(u,v) \} | u,v \in \mathcal{U}, d(u,v) > 0\} \].

In Section 8 we describe a general family of group topologies on the isometry group of a $\mathcal{R}$-Urysohn space $\mathcal{U}$ that includes all the topologies involved in the two results above. Theorems C and D, can be seen as evidence for the much more general conjecture that these are in fact all group topologies coarser than $\tau_{st}$ on $\text{Isom}(\mathcal{U})$.

**Algebraic minimality: the Zariski topology.** Given a group $G$ the Zariski topology $\tau_Z$, is generated by the subbase consisting of the sets \( \{ x \in G | x^a g_1 x^b g_2 \cdots x^n g_n \neq 1 \} \), where $n \in \mathbb{N}$, $g_1, \ldots, g_n \in G$, and $a_1, \ldots, a_n \in \{-1, 1\}$. According to the result of Gaughan in [9] for the group $S_m$ the Zariski topology $\tau_Z$ and $\tau_{st}$ coincide. In Section 9 we investigate the following general question.

**Question 3.** For which (sufficiently homogeneous) structures is it true that $\tau_Z = \tau_{st}$. For which of them is the Zariski topology a group topology?

First we provide a variety of Fraïssé limits for which the question above has a negative answer. In all cases this follows via Lemma 9.5 from the property that over $\text{Aut}(\mathcal{M})$ of non-trivial equations in one variable have meager sets of solutions. The latter is in turn from the criterion formulated in Lemma 9.3, according to which the conclusion holds whenever any $\alpha \in \text{Aut}(\mathcal{M}) \setminus \{1\}$ is what we call strongly unbound (Definition 9.1). Intuitively, the latter means that points ‘largely displaced’ by $\alpha$ are in some sense dense in $M$. Theorem E below collects some miscellaneous results found in Corollary 9.14 (for 1.), 9.19 (for 2.), 9.21 (for 3.) and Corollary 9.25 (for 4.) below.

**Theorem E.** The Zariski topology $\tau_Z$ on $\text{Aut}(\mathcal{M})$ is not a group topology if $\mathcal{M} = \text{Flim} (\mathcal{K})$ for a Fraïssé class $\mathcal{K}$ in a relational language $\mathcal{L}$ in each of the following cases:

1. $\mathcal{K}$ is a non-trivial free amalgamation class and the action of $G$ on $M$ is transitive;
2. $\mathcal{M}$ is the rational Urysohn space;
3. $\mathcal{M}$ is the random tournament;
4. $\mathcal{K}$ is of the form $\mathcal{K}_1 \otimes \mathcal{K}_2$ (see Definition 9.23) for strong amalgamation classes $\mathcal{K}_1$ and $\mathcal{K}_2$ where:
   - $\mathcal{K}_1$ is non-trivial and either it is as in 2. or the action of $\text{Aut}(\text{Flim} (\mathcal{K}_1))$ on the set $M^2 \setminus \{(a, a)\}_{a \in M}$ is transitive.
   - $\text{Flim} (\mathcal{K}_2)$ is the countable dense meet tree, the cyclic tournament $S(2)$ or $(\mathcal{Q}, <)$.

Here we say that $\mathcal{K}$ is trivial if the equality type of a tuple from $M$ determines its type or, equivalently, if $\text{Aut}(\mathcal{M})$ is the full symmetric group. Additionally in Corollary 9.29 we can prove the following:

**Theorem F.** Suppose $\mathcal{M}_\eta$ is the Hrushovski generic structure that is obtained from a pre-dimension function with the coefficient $\eta \in (0, 1]$. Then the Zariski topology for $\text{Aut}(\mathcal{M}_\eta)$ is not a group topology.

On the flip side there is the following positive result:

**Theorem G.** The Zariski topology $\tau_Z$ on $\text{Aut}(\mathcal{M})$ is a group topology in case $\mathcal{M}$ is one of the following:

- Some reduct of $(\mathcal{Q}, <)$;
- A countable dense meet-tree or the lexicographically ordered dense meet-tree, in which case $\tau_Z = \tau_{st}$;
- The cyclic tournament $S(2)$.
Topologies and partial types. Finally in Section 10, we present a natural variant of ideas of [20] and [3] in the context of automorphism groups of first order structures. Given a structure $M$ with group of automorphisms $G$, we describe a semi-group of partial types $R^p(M)$ containing $G$ consisting of partial types and show that any idempotent in $R^p(M)$ which is invariant under the involution and the action of $G$ can be associated to a group topology on $G$ coarser than $\tau_{sp}$.

2. Review of some classical constructions of homogeneous structures

2.1. Fraïssé construction. Let us briefly review the Fraïssé construction method in a relational language. For a more detailed and general introduction see Chapter 6 in [11].

Let $\mathcal{L}$ be a relational signature and $K$ be a countable class of finite $\mathcal{L}$-structures closed under isomorphisms. Suppose $A, B \in K$ by $A \subseteq B$ we mean $A$ is an $\mathcal{L}$-substructure of $B$. We say $K$ is a Fraïssé class if it satisfies the following properties:

- (HP) It is closed under substructures;
- (JEP) For any $A, B \in K$ there is $C$ in $K$ such that $A, B \subseteq C$;
- (AP) Given $A_1, A_2, B \in K$ and isometric embeddings $g_1 : B \rightarrow A_1$, $i = 1, 2$ there exists $C \in K$ and isometric embeddings $h_1 : A_1 \rightarrow C$ such that $h_1 \circ g_1 = h_2 \circ g_2$.

We say that a Fraïssé class $K$ has strong amalgamation if in (AP) we might assume that $h_1(A_1) \cap h_2(A_2) = h_1(B)$.

According to a theorem of Fraïssé for any Fraïssé class $K$ there is a unique countable structure $M$ called the Fraïssé limit of $K$ and denoted by $\text{Flim}(K)$, such that:

- $M$ is ultrahomogeneous, i.e. every finite partial isomorphism between substructures of $M$ extends to an automorphism of $M$;
- $\text{Age}(M)$, the collection of all finite substructures of $M$, coincides with $K$.

Classical examples of Fraïssé limit structures are $(\mathbb{Q}, <)$ and the random graph. If $\mathcal{L}$ is empty, then $K$ is the class of finite sets and $\text{Flim}(K)$ an infinite countable set. More in general, we say that $K$ is trivial if the equality type of a finite tuple of elements from $M$ determines its type (equivalently, if $\text{Aut}(M)$ is the full permutation group of $M$).

Suppose $A, B$ and $C$ are structures in some relational language $\mathcal{L}$ with $A \subseteq B, C$. By the free-amalgam of $B$ and $C$ over $A$, denoted by $B \otimes_A C$, we mean the structure with domain $B \coprod_A C$ in which a relation holds if and only if it already did in either $B$ or $C$.

By a free amalgamation class we mean a class $K$ of finite structures in a relational language satisfying (HP) and such that $B \otimes_A C \in K$ for any $A, B, C \in K$ such that $A \subseteq B, C$. Note this is automatically a Fraïssé class with strong amalgamation. We write $B \downarrow_A^{fr} C$ if and only if the structure generated by $ABC$ is isomorphic (with the right identifications) with the free amalgam $B \otimes_A C$. If $B \downarrow_\emptyset^{fr} C$ we write $B \downarrow^{fr} C$ and say $B$ and $C$ are free from each other.

2.2. Hrushovski’s pre-dimension construction. Originally Hrushovski’s pre-dimension construction was introduced as a means of producing countable strongly minimal structures which are not field-like or vector-space like. There are many variants of the method, but to fix notation, we consider the following basic case and later focus on a version that produces $\omega$-categorical structures. We refer readers to [21], [1] and [7] for most of the properties that are mentioned here about Hrushovski constructions and some of their variations.

Suppose $s \geq 2$ and $\eta \in (0,1]$. We work with the class $\mathcal{C}$ of finite $s$-uniform hypergraphs, that is, structures in a language with a single $s$-ary relation symbol $R(x_1, \ldots, x_s)$ whose interpretation is invariant under permutation of coordinates and satisfies $R(x_1, \ldots, x_s) \rightarrow \bigwedge_{i<j}(x_i \neq x_j)$.

To each $B \in \mathcal{C}$ we assign the predimension

$$\delta(B) = |B| - \eta |R[B]|;$$

where $R[B]$ denotes the set of hyperedges on $B$. For $A \subseteq B$, we define $A \leq B$ iff for all $A \subseteq B' \subseteq B$ we have $\delta(A) \leq \delta(B')$, and let $C_{\eta} := \{ B \in \mathcal{C} | \emptyset \subseteq B \}$. The following is standard
Lemma 2.1. Suppose $A, B \subseteq C \in \mathcal{C}_\eta$. Then:

1. $\delta(AB) \leq \delta(A) + \delta(B) - \delta(A \cap B)$;
2. If $A \leq B$ and $X \subseteq B$, then $A \cap X \leq X$;
3. If $A \leq B \leq C$, then $A \leq C$.

If $A, B \subseteq C \in \mathcal{C}_\eta$ then we define $\delta(A/B) = \delta(AB) - \delta(B)$. Note that this is equal to $|A|B| - \eta[R[AB]\setminus R[B]|].$ Then $B \leq AB$ if $\delta(A'/B) > 0$ for all $A' \subseteq A$. Moreover, if $N$ is an infinite $\mathcal{L}$-structure such that $A \subseteq N$, we write $A \leq N$ whenever $A \leq B$ for every finite substructure $B$ of $N$ that contains $A$. One can show $\mathcal{C}_\eta$ has the $\leq$-free amalgamation property (cf. Lemma 4.8 in [1]), by which we mean free amalgamation with $\leq$ inclusions. An analogue of Fraïssé’s theorem holds in this situation:

Proposition 2.2. There is a unique countable structure $\mathcal{M}^\eta$, up to isomorphism, satisfying:

1. The set of all finite substructures of $\mathcal{M}^\eta$, up to isomorphism, is precisely $\mathcal{C}_\eta$;
2. $\mathcal{M}^\eta = \bigcup_{i \in \omega} A_i$ where $(A_i : i \in \omega)$ is a chain of $\leq$-closed finite sets;
3. If $A \leq \mathcal{M}^\eta$ and $A \leq B \in \mathcal{C}_\eta$, then there is an embedding $f : B \rightarrow \mathcal{M}^\eta$ with $f|_A = \text{id}_A$ and $f(B) \leq \mathcal{M}^\eta$.

The structure $\mathcal{M}^\eta$, that is obtained in the above proposition, is called the *Hrushovski generic* structure.

2.2.1. $\omega$-categorical case. Here we briefly discuss a variation on the Hrushovski’s pre-dimension construction method as a way to generate $\omega$-categorical structures. The original version of this is used to provide a counterexample to Lachlan’s conjecture, where it is used to construct a stable $\omega$-categorical pseudolocale (see Section 5 in [21]). Here we follow similar setting used Section 5.2. in [7].

Suppose $\eta = \frac{m}{n} \in (0,1]$ where $\gcd(m,n) = 1$. Consider the same setting of the previous subsection for $\mathcal{L}$ and $\mathcal{C}_\eta$. Choosing an unbounded convex increasing function $f : \mathbb{R}^0 \rightarrow \mathbb{R}^0$ which is “good” enough one can consider

$$C^f_\eta := \{A \in \mathcal{C}_\eta | \delta(X) \geq f(|X|) \ \forall X \subseteq A\}$$

where $(C^f_\eta, \leq_d)$ has the $\leq_d$-free amalgamation property and $\leq_d$ is defined as follows: $A \leq_d B$ when $\delta(A'/A) > 0$, for each $A \subseteq A' \subseteq B$. In this case we have an associated countable *generic structure* $\mathcal{M}^f_\eta$ which is $\omega$-categorical.

Remark 2.3. As a good function we can take some piecewise smooth $f$ where its right derivative $f'$ satisfies $f'(x) \leq 1/x$ and is non-increasing, for $x \geq 1$. The latter condition implies that $f(x+y) \leq f(x) + y f'(x)$ (for $y \geq 0$). It can be shown that under these conditions, $C^f_\eta$ has the free $\leq_d$-amalgamation property.

We assume that $f$ is a good function. We will assume that $f(0) = 0$ and $f(1) > 0$, and in this case the $\leq$-closure of empty set is empty. We shall also assume that $f(1) = n$ and one can show $\text{Aut} (\mathcal{M}^f_\eta)$ acts transitively on $M^f_\eta$. See examples 5.11 and 5.12. in Section 5.2. [7] for details.

3. A relative minimality criterion for $\tau_{st}$

Given a topological group $(G, \tau)$ and $g \in G$ we denote by $\mathcal{N}_\tau(g)$ the filter of neighbourhoods of $g$ in $\tau$. Since $\mathcal{N}_\tau(g) = g\mathcal{N}_\tau(1) = \mathcal{N}_\tau(1)g$ for any $g \in G$, any group topology $\tau$ is uniquely determined by $\mathcal{N}_\tau(1)$. Given a filter $\mathcal{V}$ on $G$ at $1$ such that

- For every $U \in \mathcal{V}$ there is $V \in \mathcal{V}$ such that $V^{-1} \subseteq U$;
- For every $U \in \mathcal{V}$ there is $V \in \mathcal{V}$ such that $VV \subseteq U$;
- $U^* \in \mathcal{V}$ for every $U \in \mathcal{V}$ and $g \in G$;

then there is a unique group topology $\tau$ on $G$ such that $\mathcal{V} = \mathcal{N}_\tau(1)$. Given a family $\mathcal{Y}$ of subsets of $G$ containing $1$, we say that $\mathcal{Y}$ generates a group topology $\tau$ at the identity if $\mathcal{Y}$ generates $\mathcal{N}_\tau(1)$ as a filter.
Given a set $X$ we let $[X]^{<\omega}$ stand for the collection of all finite subsets of $X$. Our setting consists of an infinite set $\Omega$ and some $G \subseteq S(\Omega)$, where $S(\Omega)$ is the group of permutations of $\Omega$. It is easy to see using the criterion above that the collection $\{G_A\mid A \in [\Omega]^{<\omega}\}$ is a base of neighbourhoods of the identity of a unique group topology $\tau_{st}$, which we will refer to as the standard topology. We are mainly interested in the case in which $\Omega$ is countable, in which case $S(\Omega)$, abbreviated as $S_{\infty}$, is a Polish group.

By a closure operator on $[\Omega]^{<\omega}$ we mean a map $cl : [\Omega]^{<\omega} \to [\Omega]^{<\omega}$ that preserves inclusion and satisfies $A \subseteq cl(A) = cl(cl(A))$ for each $A \in [\Omega]^{<\omega}$. There is a bijective correspondence between ($G$-equivariant) closure operators $cl$ and ($G$-invariant) families $X \subseteq [\Omega]^{<\omega}$ closed under intersections. Each $X$ gives a closure operator $cl(\cdot)$ by taking as $cl(A)$ the smallest set in $X$ containing $A$. In the opposite direction we associate $cl$ with the class of $cl$-closed sets: $X = \{A \in [\Omega]^{<\omega}\mid cl(A) = A\}$.

Given tuples $A, B, C$ of elements from $\Omega$ we write $A \equiv^G B$ if there exists some $g \in G$ such that $gA = B$ and given an additional $C$ we write $A \equiv_G^C B$ if there is $g \in G_C$ such that $gA = B$. Given $A \subseteq \Omega$ we let $acl^G(A)$ stand for the union of all elements of $[\Omega]^{<\omega}$ whose orbit under $G_A$ is finite. We say $acl^G(\cdot)$ is locally finite if $acl^G(A)$ is finite whenever $A$ is. In that case the restriction of $acl^G$ to $[\Omega]^{<\omega}$ is a closure operator on $[\Omega]^{<\omega}$. We write $X^{G} = \{A \in [\Omega]^{<\omega}\mid acl^G(A) = A\}$ and we say that $acl^G$ is trivial if $X^{G} = [\Omega]^{<\omega}$.

Given a family $X$ of subsets of a set $\Omega$, denote by $(X)$ the collection of all tuples of elements whose coordinates enumerate some member of $X$. As is customary, the same letter will be used to refer to either a tuple or the corresponding set depending on the context. In particular we might use an expression such as $BC$ to denote the union of the ranges of $B$ and $C$.

Let $G$ be the group of automorphisms of some structure $M$ with universe $M$. Recall that if $M$ is $\omega$-saturated, then for finite $A$ we have that $acl^G(A)$ coincides with the algebraic closure of $A$. If $M$ is $\omega$-saturated and countable, then in particular it is $\omega$-homogeneous, i.e. $A \equiv^G B \iff tp(A) = tp(B)$ (alt. $A \equiv B$) for any $A, B \in [M]^{<\omega}$. One says $M$ is ultra-homogeneous if the stronger equivalence $A \equiv^G B \iff A \equiv B$ holds for any $A, B \in [M]^{<\omega}$.

**Lemma 3.1.** Let $G$ be a group of permutations of a set $\Omega$ for which $acl^G(-)$ is locally finite. Suppose we are given some $G$-invariant $X \subseteq \Omega$ and another group topology $\tau^* \subseteq \tau^X$ such that for some constant $K \in \mathbb{N}$ the following property holds:

$(\circ)$ For any $A, B \in X^{G}$ and $U \in \mathcal{N}_{\tau^*}(1)$ there exists $U' \in \mathcal{N}_{\tau^*}(1)$ such that $((G_A \cap U)G_B)^K = G_{A \cap U} \cap U'$.

Then any group topology $\tau \subseteq \tau_{st}$ must satisfy at least one of the following two conditions:

1. Given $x \in X$ there exists $W \in \mathcal{N}_{\tau^*}(1)$ such that $gx \in acl^G(x)$, for each $g \in W$; or,

2. There exists some $G$-invariant $X' \subseteq X$ such that for all $W \in \mathcal{N}_{\tau^*}(1)$ there is $U' \in \mathcal{N}_{\tau^*}(1)$ and $U'' \in \mathcal{N}_{\tau^*}(1)$ such that $U' \cap U'' \subseteq W$.

**Proof.** Assume the first alternative does not hold. Then there is $x_0 \in X$ such that for any $W \in \mathcal{N}_{\tau^*}(1)$ there exists $g \in W$ such that $g(x_0) \notin acl^G(x_0)$. Let $X' = X \setminus G \cdot x_0$. Our goal is to show point 2., that is, that any neighbourhood $W$ of 1 in $\tau$ is also a neighbourhood of the identity in any topology containing $\tau^*$ and $\tau^X$.

**Observation 3.2.** For any $a \in G \cdot x_0$, any finite $B \subseteq \Omega$ and any $W \in \mathcal{N}_{\tau^*}(1)$ there exists some $g \in W$ such that $ga \notin B$.

**Proof.** Suppose the condition above fails for some $a, B,$ and $W$. By Neumann’s lemma there exists some $h \in G_a$ such that $h(B) \cap B \subseteq acl^G(a)$. This means that any $g$ in $W \cap W^{-1} \in \mathcal{N}_{\tau^*}(1)$ must take $a$ to a point in $acl^G(a)$, a contradiction.

The following observation follows from $(\circ)$ by an induction argument.

**Observation 3.3.** There is a function $\mu : \mathbb{N} \to \mathbb{N}$ such that given any finite collection $\{B_j\}_{j=1}^r \subseteq X^G$, $U \in \mathcal{N}_{\tau^*}(1)$ and $W \subseteq G$ containing $U \cap \bigcup_{j=1}^r G_{B_j}$ there exists $U' \in \mathcal{N}_{\tau^*}(1)$ such that $G_{\cap_{j=1}^r B_j} \cap U' \subseteq W^{\mu(r)}$. 


Fix some arbitrary $W \in \mathcal{N}_\tau(1)$. Pick $W_0 = W_0^{-1} \in \mathcal{N}_\tau(1)$ such that $W_0^{2K} \subseteq W$. Since $\tau \subseteq \tau_\kappa$, there exists some finite $A \subseteq X$ such that $G_A \subseteq W_0$. By local finiteness we may assume $A = acl^G(A)$. Let $\{a_j\}_{j=1}^r := A \cap (G \cdot x_0)$.

Pick $W_1 = W_1^{-1} \in \mathcal{N}_\tau(1)$ such that $W_1^{\mu(r)} \subseteq W_0$, where $\mu$ is the function given by Observation 3.3. Let $B \subseteq \Omega$ be a finite subset such that $G_B \subseteq W_1$. We may assume again $B \in \mathcal{X}^G$. By Observation 3.2 for any $1 \leq j \leq r$ there exists some $g_j \in W_1$ such that $g_ja_j \notin B$ or, equivalently, $a_j \notin B_j := g_j^{-1}B$.

Notice that $G_B = G_B^g \subseteq W_1^{r+1}$.

Let $C = \bigcap_{j=1}^r B_j$. According to 3.3 (for $U = G$) there is $U' \in \mathcal{N}_{\tau'}(1)$ such that $G_C \subseteq U' \subset (W_1^{r+1})_{\mu(r)} \subseteq W_0$. A final direct application of $(\circ)$ yields some $U'' \in \mathcal{N}_{\tau'}(1)$ such that

$$U'' \cap G_{C \cap A} \subseteq (G_C G_A)^K \subseteq W_0^{2K} \subseteq W.$$

By construction $C \cap A \subseteq X'$ so we are done. \hfill \Box

Here is another instance of the same idea.

**Lemma 3.4.** Let $G$ be a group of permutations of a set $\Omega$, $\{X_j\}_{j \in J}$ some collection of $G$-invariant subsets of $\Omega$ and $Z = \bigcap_{j \in J} X_j$. Assume that $acl^G(x) = x$ for any $x \in \Omega$ and that there exists $K > 0$ such that for any finite $A, B \subseteq \Omega$ we have $(G_A G_B)^K = G_{A \cap B}$. Then $(t_{st})^Z_\tau = \bigcap_{j \in J} t_{st}^X J$.

**Proof.** Let $\tau_0 = \bigcap_{j \in J} t_{st}^X J$. The inclusion $(t_{st})^Z_\tau \subseteq \tau_0$ is clear. Take now any $W \in \mathcal{N}_{\tau_0}(1)$. Fix $j_0 \in J$. Since $W \in t_{st}^X j_0$, there exists some finite $A \subseteq X_{j_0}$ such that $G_A \subseteq W$. Let $\{a_j\}_{j=1}^r := A \setminus Z$. Just as in Observation 3.3 one can show by induction:

**Claim.** There exists a function $\mu : \mathbb{N} \to \mathbb{N}$ such that for any finite collection $\{B_l\}_{l=1}^r \subseteq [\Omega]^{<\omega}$ and any $V \subseteq G$ containing $G_{B_l}$ for all $1 \leq l \leq r$ we have $G_{\bigcap_{l=1}^r B_l} \subseteq V^{\mu(r)}$.

Pick $W_0 = W_0^{-1} \in \mathcal{N}_{\tau_0}(1)$ such that $W_0^{\mu(r+1)} \subseteq W$. For each $1 \leq l \leq r$ choose some $j_l \in J$ such that $a_l \notin X_{j_l}$ and some finite $B_l \subseteq X_{j_l}$ such that $G_{B_l} \subseteq W_0$. The Claim and the choice of $W_0$ implies $G_C \subseteq W$ where $C = A \cap \bigcap_{l=1}^r B_l$. Since $C \subseteq Z$ we are done. \hfill \Box

**Lemma 3.5.** Let $G$ be the automorphism group of some structure $M$ endowed with a $G$-invariant locally finite closure operator $cl(\cdot)$ on $M$ and a group topology $\tau$ coarser than $\tau_{st}$. Assume that the action of $G$ is transitive and there is some $W \in \mathcal{N}_\tau(1)$ and $a \in M$ such that $g a \in cl(a)$, for each $g \in W$. Then either $\tau$ is not Hausdorff or $\tau = \tau_{st}$.

**Proof.** Notice that by the transitivity of the action of $G$ on $M$ and continuity of the inverse operation for every $a \in M$ there are $W_a, W_a \in \mathcal{N}_\tau(1)$ such that $f(a) \in cl(a)$ for any $f \in W_a$ and $g^{-1}(a) \in cl(a)$ for any $g \in W_a$. For a finite tuple $A$ in $M$ we write $W_A = \bigcap_{a \in A} W_a$. Given $a, b \in M$, we say that $a \sim b$ if $a \in cl(b)$ and $b \in cl(a)$. This is clearly an equivalence relation. If we let $W'_a = W_a \cap \bigcap_{z \in cl(b)} U_z$, then any $f \in W'_a$ must preserve the class $[a] \subseteq M/\sim$ set-wise, that is $W'_a \subseteq G_{[a]}$.

For any $V \in \mathcal{N}_\tau(1)$ and any finite $\sim$-closed $A \subseteq M$ consider the set

$$Y^A_V = \{f : A \to A \mid \exists g \in V \; g \mid_A = f, \forall a \in A \; g([a]) = [a]\}.$$

Notice that this set is finite, and that given $\sim$-closed $A \subseteq B \subseteq M$ and $f \in Y^B_V$ we have $f \mid_A \in Y^A_V$. Invariance should be clear from the fact that $A$ is $\sim$-closed and the definition of $Y^A_V$.

**Claim 3.6.** Either $Y^A_V = \{id_A\}$ for some $V \in \mathcal{N}_\tau(1)$ and $\sim$-closed $A$ or there exists $f \in G$ such that for all $\sim$-closed $A \subseteq M$ and all $V \in \mathcal{N}_\tau(1)$ we have $f \mid_A \in Y^A_V$.

**Proof.** Notice that this assumption the closure is locally finite. If the first alternative is not the case, then from Observation 3.3 and König’s lemma follows that there is a function $f : M \to M$ such that $f \mid_A \in Y^A_V$ for any $\sim$-closed $A$ and $V \in \mathcal{N}_\tau(1)$. The fact that $f \mid_A$ is a type-preserving bijection of $A$ for any such $A$ implies $f \in G$. \hfill \Box
If the first possibility in Claim 3.6 holds true, then \( G_A \) contains \( W_A^1 \cap V \) and is thus a neighbourhood of the identity in \( \tau \), which implies that \( \tau = \tau_{st} \). We claim that if the second possibility is satisfied the resulting \( f \in G \setminus 1 \) satisfies \( f \in \bigcap_{V \in N_\tau(1)} V \), so \( \tau \) is not Hausdorff. Given any \( V \in N_\tau(1) \), the closure in \( \tau_{st} \) of any symmetric \( W \in N_\tau(1) \cap \tau_{st} \) satisfying \( W^2 \subset V \) is itself contained in \( V \). Hence, \( N_\tau(1) \) admits a basis consisting entirely of \( \tau_{st} \)-closed neighbourhoods of the identity. It is thus enough to show that \( f \) belongs to the closure of \( V \) in \( \tau_{st} \) for any \( V \in N_\tau(1) \), which is immediate from the definition of \( Y^A \).

The following ubiquitous observation is crucial for the application of the results above. We provide a proof for the sake of completeness.

**Lemma 3.7.** Let \( G \) be a group of permutations of a set \( \Omega \) and \( A, B \) tuples of elements from \( \Omega \) for which there is a chain \( A = A_0, B_0, \ldots, A_n = g(A) \) such that \( A_i B_i \cong G A_{i+1} B_i \cong AB \) for \( 0 \leq i < n \). Then \( g \in (G_A G_B)^n G_A \).

**Proof.** The proof is by induction on \( n \). In the base case \( n = 0 \) we have \( A = g(A) \) that is \( g \in G_A \). Assume now \( n > 0 \). Since \( AB_0 \cong AB \), there exists \( h \in G_A \) such that \( h(B_0) = B \). Now \( A_1 B_0 \cong AB \) implies \( h(A_1)B = h(A_1)h(B_0) \cong AB \), which implies that there exists \( h'(B) \in G_B \) such that \( h'(h(A_1)) = A \). Applying induction to the sequence \( \{B'_1, B'_2, \ldots, B'_n\} \) given by \( B'_i = h'h(B_{i+1}) \) yields that \( h'hg \in (G_A G_B)^{n-1} G_A \), from which it follows that \( g \in (G_A G_B)^n G_A \) as desired.

**Definition 3.8.** Suppose we are given a group \( G \) of permutations of a set \( \Omega \), and \( \mathcal{X} \) a \( G \)-invariant family of subsets of \( \Omega \) closed under intersection. We say \( \mathcal{X} \) has the \( n \)-zigzag property (with respect to the action of \( G \)) if for every \( A, B \in (\mathcal{X}) \) and any \( A' \) with \( A \cong_{G \setminus A \setminus B} A' \) there are \( A_0, \ldots, A_n \) and \( B_0, \ldots, B_{n-1} \) such that

1. \( A_0 := A \) and \( A_n = A' \);
2. \( A_i B_i \cong G A_{i+1} B_i \cong AB \) for \( 0 \leq i \leq n - 1 \).

We will refer to the sequence \( A_0, B_0, A_1, \ldots, A_n \) above as a \((n, B)\)-zigzag path from \( A \) to \( A' \).

**Observation 3.9.** Given an \( n \)-zigzag path as above it is easy to show by induction that if we write \( C = A \cap B \) then \( C \subseteq A_i B_i \cong C A_{i+1} B_i \cong C \) \( AB \) for all \( 0 \leq i \leq n - 1 \). In particular, \( A_i \cap B_i = A_{i+1} \cap B_i = C \).

Notice that for fixed \( A, B \) and \( n \) the existence of a \((n, B)\)-zigzag path from \( A \) to \( A' \) depends only on the orbit of \( A' \) under \( G_A \).

**Proposition 3.10.** Suppose \( M \) is a countable first order structure and \( G = \text{Aut}(M) \). Assume \( acl^G(\overline{M}) \) is locally finite and the corresponding \( \mathcal{X}^G \) has the \( n \)-zigzag property for some \( n \). Then:

1. If the action of \( G \) on \( M \) is transitive, then \( (G, \tau_{st}) \) is minimal.
2. If \( acl^G(x) = x \) for any \( x \in M \), then any group topology \( \tau \subseteq \tau_{st} \) is of the form \( \tau^X_{st} \) for some \( G \)-invariant \( X \subseteq M \).

**Proof.** Let us show 1. first. Let \( \tau \) be a group topology on \( G \) coarser than \( \tau_{st} \). By Lemma 3.7 it is possible to apply 3.1 with \( \tau^* = \{\emptyset, G\} \). If the first alternative of 3.1 holds, then by Lemma 3.5 either \( \tau \) is not Hausdorff or \( \tau = \tau_{st} \). Since by assumption the only invariant subsets of \( M \) are \( \emptyset \) and \( M \), the second alternative implies that \( \tau = \{\emptyset, G\} \).

Let us now show 2. Let \( \tau \) be a group topology on \( G \) coarser than \( \tau_{st} \). By Lemma 3.4 there exists some unique minimal \( G \)-invariant set \( X \) such that \( \tau \subseteq \tau^X_{st} \). Apply Lemma 3.1 with \( \tau^* = \{\emptyset, G\} \). The second alternative produces some \( G \)-invariant \( X' \subseteq X \) such that \( \tau \subseteq \tau^{X'}_{st} \), in contradiction with the choice of \( X \). Since we assume \( acl^G \) to be trivial, the first alternative implies \( \tau = \tau^X_{st} \). \( \square \)
4. Minimality and independence

4.1. Independence. Throughout this section we work in the following setting: $\Omega$ is a set, $G$ is a permutation group of $\Omega$, $\text{cl}(-)$ a $G$-equivariant closure operator on $[\Omega]^{<\omega}$ and $\mathcal{X} = \{\text{cl}(A) \mid A \in [\Omega]^{<\omega}\}$ the associated family of closed sets. Our goal is to derive concrete applications from the results of the previous section to the case where $\Omega$ is the underlying set of a first order structure $\mathcal{M}$ and $G = \text{Aut}(\mathcal{M})$.

Definition 4.1. Given $\text{cl}(-)$ and $\mathcal{X}$ as above and a ternary relation $\downarrow$ between members of $[\Omega]^{<\omega}$ we say that $(\text{cl}, \downarrow)$ (alternatively, $(\mathcal{X}, \downarrow)$) is a compatible pair if for all $A, B, C, D \in [\Omega]^{<\omega}$ the following properties are satisfied:

- (compatibility) $A \downarrow_C B$ if and only if $A \downarrow_{\text{cl}(C)} B$ and only if $\text{cl}(AC) \downarrow C \text{cl}(BC)$.
- (invariance) If $g \in G$ and $A \downarrow_B C$ then $gA \downarrow_{gB} gC$.
- (weak monotonicity) If $A \downarrow_B CD$ or $AD \downarrow_B C$ then $A \downarrow_B C$.
- (anti-reflexivity) If $A \downarrow_C B$, then $A \cap B \subseteq \text{cl}(C)$.

We write $A \downarrow B$ as an abbreviation of $A \downarrow_\emptyset B$.

Definition 4.2. We define some additional properties for a compatible pair $(\mathcal{X}, \downarrow)$:

- (transitivity) If $A \downarrow_B C$ and $A \downarrow_{BC} D$, then $A \downarrow_B CD$.
- (symmetry) If $A \downarrow_B C$ then $C \downarrow_A A$.
- (existence) For any $A, B, C$ there is $g \in G$ such that $gA \downarrow_B C$.
- (independence) Suppose we are given $A, B_1, B_2, C_1, C_2 \in \mathcal{X}$ such that $B_1 \downarrow_A B_2$, $A \subseteq B_i$ and $C_i \downarrow_A B_i$ for $i = 1, 2$ and $C_1 \equiv_B C_2$. Then there exists $D \in \mathcal{X}$ such that $D \equiv_B C_i$ for $i = 1, 2$ and $D \downarrow_A B_1 B_2$.
- (stationarity) If $B \in \mathcal{X}$ and $A_i \downarrow_B C$ for $i = 1, 2$, then $A_1 \equiv_B A_2$ implies $A_1 \equiv_B C A_2$.

Additionally, we consider

- (freedom) $\mathcal{X} = [\Omega]^{<\omega}$ and moreover if $A \downarrow_C B$ and $C \cap AB \subseteq D \subseteq C$, then $A \downarrow_D B$.
- (one-basedness) $A \downarrow_{A \cap B} B$ for every $A, B \in \mathcal{X}$.

The one-basedness property admits the following generalization:

Definition 4.3. Given $k \geq 1$, we say that $(\mathcal{X}, \downarrow)$ satisfies $k$-narrowness if for any $A \in [\Omega]^{<\omega}$ and any $C, A_0, A_1, \ldots, A_k$ in $\mathcal{X}$ the conditions

- $A_i \cap A_{i+1} = C$, for each $0 \leq i < k - 1$;
- $A_{i+1} \downarrow_{A_i} A_{i-1} \ldots A_0$, for each $1 \leq i \leq k - 1$;

imply that $A_0 \downarrow_C A_k$ (notice that for $k = 1$ we recover the one basedness property).

Lemma 4.4. Let $(\mathcal{X}, \downarrow)$ be a compatible pair that satisfies existence. Then

1. If it satisfies freedom or one-basedness, then for any $A, B \in \mathcal{X}$ there is $A' \in \mathcal{X}$ such that $A' \equiv_B A$, $A' \cap A = A \cap B$ and $A \downarrow_{A \cap B} A'$.

2. If it satisfies transitivity, symmetry and 2m-narrowness, then for any $A, B \in \mathcal{X}$ there is $A' \in \mathcal{X}$ such that an $(m, B)$-zigzag path from $A$ to $A'$ exists, $A' \cap A = A' \cap B$ and $A \downarrow_{A \cap B} A'$.

Proof. Existence yields $A' \in \mathcal{X}$ such that $A' \equiv_B A$ and $A' \downarrow_B A$. Anti-reflexivity implies that $A' \cap A \subseteq B$, i.e. $A' \cap A \subseteq A \cap B$. On the other hand $A' \equiv_B A$ implies $A \cap B = A' \cap B$.

If we assume the freedom axiom, then $A' \downarrow_{A \cap B} A$ follows from $A' \downarrow_B A$ and $B \cap (A' \cup A) = (B \cap A') \cup (B \cap A) = B \cap A$. Alternatively, the same conclusion follows directly from one-basedness.

Let $C = A \cap B$. For 2. construct sequences $B_0 = B, B_1, \ldots, B_{m-1}$ and $A_0 = A, A_1, \ldots, A_m$ as follows. Assuming we have already taken $(A_i, B_i)_{i=0}^k$, existence provides $A_{k+1} \equiv_B A_k$ with $A_{k+1} \downarrow_{B_k} A_0 B_0 \ldots A_k B_k$. If $k < m$ we can use the same argument to choose $B_{k+1} \equiv_{A_{k+1}} B_k$ with $B_{k+1} \downarrow_{A_k} A_0 B_0 \ldots A_k B_{k+1}$. It is clear that this yields an $(m, B)$-zigzag path from $A$ to $A_m$.

By transitivity we have $A_j \downarrow_{B_{j-1}} A_l$ for any $0 \leq l \leq j - 1$ so that $A_j \cap A_l = A_j \cap B_{j-1}$. Since $A_j \cap B_{j-1} = C$ and $C \subseteq A_j \cap A_l$ by 3.9 we conclude that $A_j \cap A_l = C$. Arguing in a similar manner one
can show that \( A_j \cap B_l = C \) for any \( 0 \leq j \leq m \) and \( 0 \leq l \leq m - 1 \). This establishes that the sequence \( A_0, B_0, \ldots B_{m-1}, A_m \) satisfies the first property of the condition in the definition of 2m-narrowness, while the second follows by transitivity and construction. If we let \( A' = A_m \) we then get \( A' \downarrow C, A \) and \( A \downarrow C, A' \) by symmetry, while the sequence above is an \((m, B)\)-zigzag path from \( A \) to \( A' \).

**Lemma 4.5.** Let \((\mathfrak{X}, \sqsubseteq)\) be a compatible pair satisfying symmetry existence and transitivity and assume that for any \( A, B \in \mathfrak{X} \) there exists an \((m, B)\)-zigzag path from \( A \) to some \( A_1 \) such that \( A_1 \downarrow A \). Then

1. If stationarity holds, then \( \mathfrak{X} \) has the 2m-zigzag property;
2. If independence holds, then \( \mathfrak{X} \) has the 4m-zigzag property.

**Proof.** Let \( A, A', B \in \mathfrak{X} \) with \( A' \cong_{A \cap B} A \). Let \( C := A \cap B \). In both cases using the assumption we start by choosing \( A_1 \in \mathfrak{X} \) for which there is an \( m \)-zigzag path from \( A \) to \( A_1 \) and \( A_1 \downarrow C, A \).

Consider case 1. first. By extension there is \( A_2 \) such that \( A_2 \cong_A A_1 \) and \( A_2 \downarrow_A A' A \). The first implies the existence of an \((m, B)\)-zigzag path from \( A \) to \( A_2 \). The second, together with \( A_2 \downarrow C, A \) implies \( A_2 \downarrow C, A' A \) by right transitivity. By weak monotonicity we get \( A_2 \downarrow C, A' \) and by symmetry \( A_2 \downarrow C, A \) and \( A' \downarrow C, A_2 \). Stationarity yields \( A \cong C, A' \). Thus, there is also an \((m, B')\)-zigzag path from \( A_2 \) to \( A' \), where \( A' B' \cong_G AB \) and combining both paths we get a \((2m, B)\)-zigzag path from \( A \) to \( A' \).

We move on to case 2. By invariance and existence there is \( A'_1 \) such that \( A'_1 \cong_G A_1 A \) (so that by invariance \( A'_1 \downarrow C, A' \) and \( A'_1 \downarrow C, A' A_1 \)). Transitivity and monotonicity then imply \( A'_1 \downarrow A_1 \).

Independence applied to the tuple \( C, A_1, A'_1, A, A' \) in place of the \( A, B_1, B_2, C_1, C_2 \) of the definition implies the existence of some \( D \) such that \( DA_1 \cong_G AA_1 \) and \( DA'_1 \cong_G AA_1 \). This witnesses the existence of a \((4m, B)\)-zigzag path from \( A \) to \( A' \). Notice that symmetry is required in order to get \( A' \downarrow C, A'_1 \).

**Theorem (A).** Let \( \mathcal{M} \) be the Fraïssé limit of a free amalgamation class in a countable relational structure. Let \( G = \text{Aut}(\mathcal{M}) \). Then any group topology \( \tau \subseteq \tau_{st} \) on \( G \) is of the form \( \tau^X \), where \( X \subseteq M \) is some \( G \)-invariant set. In particular, if the action of \( G \) on \( M \) is transitive, then \( (G, \tau_{st}) \) is totally minimal.

**Proof.** If we let \( \mathfrak{X} = [M]^\omega \) where \( M \) is the underlying set of \( \mathcal{M} \) and \( \downarrow = \downarrow_{fr} \), then part 1. of Lemma 4.4 and part 1. of Lemma 4.5 apply to the pair \((\mathfrak{X}, \sqsubseteq)\). Together, they imply \( \mathfrak{X} \) has the 2-zigzag property with respect to the action of \( G \). The result then follows from an application of Proposition 3.10.

**Theorem (B).** Let \( \mathcal{M} \) be a simple, \( \omega \)-saturated countable structure with elimination of hyperimaginaries, locally finite algebraic closure and weak elimination of imaginaries. Assume furthermore that Th(\( \mathcal{M} \)) is one-based. Let \( G = \text{Aut}(\mathcal{M}) \). Then

1. If \( G \) acts transitively on \( M \), then \( (G, \tau_{st}) \) is minimal.
2. If all singletons are algebraically closed, then any group topology \( \tau \) on \( G \) coarser than \( \tau_{st} \) is of the form \( \tau^X \) for some \( G \)-invariant \( X \subseteq M \).

**Proof.** As cl we take the algebraic closure acl and \( \downarrow \) the forking independence. We claim part 1. of Lemma 4.4 and part 2. of Lemma 4.5 both apply to \((\mathfrak{X}, \sqsubseteq)\).

The pair clearly satisfies invariance, weak monotonicity, transitivity and symmetry. Existence follows from the fact that \( M \) is \( \omega \)-saturated, so it is left to check one-basedness and independence in sense of Definition 4.2.

Take \( A, B \in \mathfrak{X} \). The fact that the theory is one-based in the sense of simplicity theory and has elimination of hyperimaginaries implies \( A \downarrow_{\text{acl}^v(A) \cap \text{acl}^v(B)} B \). The relation \( A \downarrow_{ACB} B \) follows then from weak elimination of imaginaries.

Lastly, elimination of hyperimaginaries and weak elimination of imaginaries imply that the type of a tuple over a finite real closed set determines its Lascar strong type over that same set. Hence, Kim
and Pillay’s independence theorem [13] (see also Chapter 2.3 and Theorem 2.3.1 in [12]) translates into abstract independence (amalgamation of types) for $(\text{acl}, \downarrow)$.

It is known that simple one-based $\omega$-categorical structures have elimination of hyperimaginaries. This follows from the fact that $\omega$-categorical theories are small and simple one-based theories admit finite coding. See section 6 and Proposition 6.1.21. in [22] for definitions and details. For stable theories the notion of being $k$-ample (for some $k \geq 1$) generalizes the negation of one-basedness. See [8] for details. In the absence of algebraic closure being not $k$-ample translates into $(\text{acl}, \downarrow^f)$ being $k$-narrow where $\downarrow^f$ is the forking independence. From an argument similar to the one in the two theorems above we can deduce the following result:

**Theorem 4.6.** Let $\mathcal{M}$ be a countable $\omega$-saturated stable structure such that $\text{Th}(\mathcal{M})$ has trivial algebraic closure, weak elimination of imaginaries and is not $k$-ample for some $k \geq 1$. Then any group topology on $G = \text{Aut}(\mathcal{M})$ coarser than $\tau_{\text{st}}$ is of the form $\tau_X^\mathcal{M}$ for some $G$-invariant $X \subseteq M$.

**Example:** total minimality is not preserved under taking open finite index subgroups. Consider the relational language $\mathcal{L}_1 = \{E^{(2)}, P^{(1)}\}$ and let $\mathcal{K}_1$ be the class of all finite $\mathcal{L}_1$-structures in which $E$ is interpreted as the edge relation of a bipartite graph with with edges only between the domain of the unary predicate $P$ and its complement. Consider also the class $\mathcal{K}_2$ in the language $\mathcal{L}_2 = \{E^{(2)}, F^{(2)}\}$ consisting of all finite $\mathcal{L}_2$-structures in which $F$ is interpreted as an equivalence relation with at most 2 classes and $E$ as the edge relation of a bipartite graph with edges only among vertices that belong to distinct $E$-classes.

Let $\mathcal{M}_1 = \text{Flim}(\mathcal{K}_1)$ and $G_1 = \text{Aut}(\mathcal{M}_1)$. Clearly $\mathcal{M}_2$ is a reduct of $\mathcal{M}_1$, so that $G_1 \preceq G_2$ and in fact $[G_2 : G_1] = 2$. It is easy to check that $\mathcal{K}_1$ has free amalgamation and then by Theorem A there are exactly two group topologies on $G_1$ strictly coarser than $\tau_{\text{st}}$, namely $\tau_{\text{st}}^{P(\mathcal{M}_1)}$ and $\tau_{\text{st}}^{\mathcal{M}_1}$. Notice that both are Hausdorff, since no automorphism of $\mathcal{M}_1$ can fix $P(\mathcal{M}_1)$ or its complement (given any two points $a, b$, there exists $c$ in $P$ (resp $\neg P$) such that $\text{tp}(c,a) \neq \text{tp}(c,b)$) so $(G_1, \tau_{\text{st}})$ is not minimal.

In this case we have an additional non-Hausdorff group topology, $\tau^* = \{\emptyset, G_1\}$. Apply Lemma 3.1 to conclude that any group topology on $G_1$ strictly contained in $\tau_{\text{st}}$ is contained in $\tau^*$.

On the other hand, it follows from Theorem B that $(G_2, \tau_{\text{st}})$ is minimal.

### 4.2. Simple non-modular Hrushovski structures.

In Subsection 2.2 we discussed in more detail some instances of the Hrushovski construction, in particular the $\omega$-categorical version (see section 5.2. in [7] for more). Here is a brief reminder of the setting:

Choosing an unbounded convex function $f$ which is “good” enough, one can consider $\mathcal{C}_f^n$, a subclass of $\mathcal{C}_n$, with the free amalgamation property where the limit structure $\mathcal{M}_f^n$ is $\omega$-categorical and such that $\text{Aut}(\mathcal{M}_f^n)$ acts transitively on $M_f^n \setminus \{\emptyset\}$ (underlying set of $\mathcal{M}_f^n$).

It is shown in Lemma 5.7 in [7] that there is an independence relation defined for the class of $\preceq$-closed subsets of $M_f^n$ that satisfy all the properties of part 1. in Lemma 4.5. Then using Proposition 3.10 we conclude the following.

**Corollary 4.7.** Let $\mathcal{M}_f^n$ be an $\omega$-categorical Hrushovski generic structure such that $G := \text{Aut}(\mathcal{M}_f^n)$ acts transitively on $M_f^n \setminus \{\emptyset\}$. Then $(G, \tau_{\text{st}})$ is a minimal topological group.

### 5. Generalized Urysohn spaces

We start by recalling some notions from [4]. A *distance magma* $\mathcal{R} = (R, \leq, \oplus, 0)$ is a set $R$ endowed with a linear order $\leq$ and an operation $\oplus$ such that the following axioms are satisfied:

- $\forall s \ s \oplus 0 = s$;
- $\forall s, r \ r \leq r \oplus s$;
- $\forall s, t \ s \oplus t = t \oplus s$;
- $\forall s, s', t \ s \leq s' \rightarrow s \oplus t \leq s' \oplus t$. 

In the presence of such a magma, we can define a *distance space*.

---

Note: This text provides a rich source of information on group topologies, foundational independence relations, and Hrushovski structures, with a focus on algebraic closure and minimality. Theorems and examples are referenced to aid understanding, and the section on simple non-modular structures highlights the complexity and flexibility of these constructions. The final section on generalized Urysohn spaces introduces the concept for further exploration.
When referring to a monoid $\mathcal{R}$, unless anything to the contrary is said, it will be implicit in the notation that $\mathcal{R}$ is its underlying set, and so forth. We say that $\mathcal{R}$ is a distance monoid if additionally $\oplus$ satisfies associativity, i.e.:

- $\forall r, s, t \quad r \oplus (s \oplus t) = (r \oplus s) \oplus t$.

Given some additively closed subset $S$ of some ordered abelian group $(\Lambda, +, \leq)$ and $b \in \{S, 0, \infty\}$ the structure $\mathcal{S}_b = \{r \in S | 0 \leq r \leq b\}$ is a distance monoid, where $x + y = \min\{b, x + y\}$ for $b \in S$ and $x + \infty = x + y$. We write $\mathcal{S}$ for $\mathcal{S}_\infty$ and $\mathcal{Q}_b = \mathcal{S}_b$ in case $S = \mathcal{Q}^{\geq 0}$. We will refer to any distance monoid $\mathcal{R}$ of the form $\mathcal{S}_b$ as basic. If additionally $S$ is a subgroup of $\Lambda$ with no minimal element then we will say $\mathcal{R}$ is standard. When talking about a standard distance monoid we may use the symbols $+$ (as opposed to $\oplus$) and $-$ to refer to the operations in the ambient group $\Lambda$ without explicitly referencing $\Lambda$. Notice that in the case of basic archimedean distance monoids we can always assume $\Lambda = \mathbb{R}$.

Given $m \in \mathbb{N}$ and $r \in \mathbb{R}$ we will write $m \cdot r$ for the $\oplus$ addition of $r$ with itself $m$ times. Given two elements $r, s \in \mathcal{R}$, we write $r \sim s$ if there exists some positive integer $n$ such that $n \cdot r \geq s$ and $n \cdot s \geq r$. We refer to the $\sim$-class $[r]$ of $r$ as its archimedean class. A distance monoid with a single archimedean class of non-zero elements will be called archimedean. We write $[r] < [t]$ if $r' < t'$ for all $r' \sim r$ and $t' \sim t$.

Fix a distance magma $\mathcal{R} := (R, \leq, \oplus)$. An $\mathcal{R}$-metric space $(X, d)$ consists of a set $X$ together with a map $d : X^2 \to R$ such that for all $x, y, z \in X$:

1. $d(x, y) = 0 \iff x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq d(x, y) + d(y, z)$.

Notice that if $(R, \leq)$ is a substructure of $(\mathbb{R}^{\geq 0}, \leq)$ and $r \oplus s \leq r + s$ for all $r, s \in \mathcal{R}$, then an $\mathcal{R}$-metric spaces are just a particular class of metric spaces. In particular, this holds for standard archimedean distance monoids.

An isometric embedding of an $\mathcal{R}$-metric spaces $(X, d)$ into another $(X', d')$ is a map $f : (X, d) \to (X', d')$ such that $d'(f(x), f(y)) = d(x, y)$, for each $x, y \in X$. A surjective isometric embedding is called an isometry. Given an $\mathcal{R}$-metric space $(X, d)$, we let $\text{Isom}(X, d)$ stand for the group of isometries from $(X, d)$ to itself. We will use the symbol $\cong$ to denote the existence of an isometry between two tuples in $\mathcal{R}$-metric spaces.

In the same spirit, given finite tuples $A = (a_i)_{i=1}^k$ and $A' = (a'_i)_{i=1}^k$ inside an $\mathcal{R}$-metric space we will write $A \cong_{B} A'$ if there is a partial isometry fixing $B$ and sending each $a_i$ to $a'_i$, that is, if for any $1 \leq i \leq k$ and $b \in B$ we have $d(a_i, b) = d(a'_i, b)$ and $d(a_i, a_j) = d(a'_i, a'_j)$ for distinct $i$ and $j$.

We will say that an $\mathcal{R}$-metric space $U$ is an $\mathcal{R}$-Urysohn space if it satisfies:

1. **(U)** Any finite $\mathcal{R}$-metric space embeds in $U$; and,
2. **(H)** Any isometry between finite subspaces of $U$ extends to an isometry of $U$.

The following strengthening of (U) is implied by the conjunction of (U) and (H) and equivalent to it under the assumption that $U$ is countable.

**EP** For any finite $\mathcal{R}$-metric space $B$ and $A \subseteq B$ any isometric embedding $h : A \to U$ extends to some $\bar{h} : B \to U$.

Assume $(A, d_A)$ and $(B, d_B)$ are two finite $\mathcal{R}$-metric spaces where $C := A \cap B \neq \emptyset$ and let $D$ be the disjoint union of $A$ and $B$ over $C$. Let $\bar{d} : D^2 \to R$ be given as follows:

- $\bar{d}$ restricts to $d_A$ and $d_B$ on $A \times A$ and $B \times B$; respectively,
- $\bar{d}(a, b) = \min\{d(a, c) \oplus d(c, b) | c \in C\}$ for any $a \in A \setminus C$ and any $b \in B \setminus C$.

It can be shown that if $\mathcal{R}$ is a distance monoid, then the $(D, \bar{d})$ above is itself an $\mathcal{R}$-metric space, which we will denote as $\mathcal{A} \otimes_{C} B$.

This implies that the class $\mathcal{K}$ of all finite $\mathcal{R}$-metric spaces has the joint embedding and amalgamation properties. See 2.7 in [4] for a more precise result (here we are only interested in $S = R$). Therefore if
\( \mathcal{R} \) is countable, then \( \mathcal{K} \) determines a unique countable Fréchet limit structure \( \mathcal{U}_\mathcal{R} = \text{Flin} (\mathcal{K}) \). This is a countable \( \mathcal{R} \)-metric space satisfying property (EP) above and thus an \( \mathcal{R} \)-Urysohn space (see Theorem 2.7.7 in [4]). An object satisfying the two properties above might exist even if \( \mathcal{R} \) is not countable. The classical Urysohn space and Urysohn sphere are examples of this for \( \mathcal{R} = (\mathbb{R}^{\geq 0}, 0, \leq, +) \) and 
\( \mathcal{R} = (\{0, 1\}, 0, \leq, +, 1) \) respectively.

Given finite sets \( A, B \) of an \( \mathcal{R} \)-metric space \( (X, d) \) we define \( \text{diam}(A) := \max\{d(a, a') \mid a, a' \in A\} \) and 
\[ d(A, B) := \min\{d(a, b) \mid a \in A, b \in B\}. \]

Given finite subsets \( A, B, C \) of \( X \) such that \( C \subset A \cap B \) we say that \( A \downarrow_C B \) if the subspace \( A \cup B \) of isomorphic to \( A \otimes_C B \). We generalize this notation to the case in which \( C \) is not a common subsets of \( A \) and \( B \) by letting \( A \downarrow_C B \) if and only if \( AC \downarrow_C BC \).

### 6. Isometry groups of archimedean Urysohn spaces

The goal of this section is to prove Theorem \( C \) of the introduction. We start with three preliminary lemmas in the following general setting: \( \mathcal{R} = (R, 0, \leq, \oplus) \) is a distance monoid and \( \mathcal{U} \) an \( \mathcal{R} \)-Urysohn space.

**Lemma 6.1.** Suppose \( A \) and \( B \) are finite subsets of \( \mathcal{U} \) and \( r \in R \) such that 
\[ \text{diam}(A) \leq r \leq 2 \cdot d(A, B). \]
Then there is \( A' \subseteq \mathcal{U} \) such that \( A' \equiv_B A \) and \( d(a, a') = r \) for all \( a \in A \) and \( a' \in A' \).

**Proof.** Consider the set \( D = A' \bigcup B \) which is the amalgamated union of two copies \( A', A'' \) of \( A \) over \( B \). We define an \( R \)-valued distance function on \( D \) as follows. On \( A'B \) and \( A''B \) the distance between two points equals the distance between the corresponding pair in \( \mathcal{U} \), while we set \( d(a', a'') = r \) for any \( a' \in A' \) and \( a'' \in A'' \). In order to show that the resulting function satisfies the triangle inequality it suffices to check triples \( \{u, v, w\} \) with \( u \in A' \), \( v \in B \) and \( w \in A'' \). We have \( d(u, w) = r \leq 2 \cdot d(A, B) \leq d(u, v) \oplus d(v, w) \). Without loss of generality assume \( d(u, v) \leq d(v, w) \). In that case \( d(v, w) \leq \text{diam}(A) \oplus d(v, u) \leq r \oplus d(v, u) = d(u, w) \oplus d(u, v) \). Then the result follows from (EP). \( \square \)

**Lemma 6.2.** Suppose we are given finite \( A, B, C \subseteq \mathcal{U} \) with \( A, B \neq \emptyset \) and \( d(A, B) \neq 0 \). Then for each \( n \in \mathbb{N} \) there is \( g \in G_B(G_AG_B)^n \) such that \( d(C, g(A)) \geq (2n + 1) \cdot d(A, B) \).

**Proof.** Take \( A' \equiv_B A \) with \( A' \downarrow_B C \). Construct a sequence \( A_i, B_i \), for \( i \geq 0 \) as follows. We start by taking \( A_0 = A' \) and \( B_0 = B \). For any \( 0 \leq i < n \) let \( C_i = (CB_iA_j)_{j \leq i} \) and take \( B_{i+1} \equiv_{A_i} B_i \) with \( B_{i+1} \downarrow_{A_i} C_i \). Then, take \( D_i = (CB_{i+1}A_j)_{j \leq i} \) and let \( A_{i+1} \equiv_{B_{i+1}} A_i \) with \( A_{i+1} \downarrow_{B_{i+1}} D_i \). The independence \( A' \downarrow_B C \) implies \( d(A', C) \geq d(A, B) \). From Lemma 3.7 we know \( A_n \) is of the form \( g(A) \) for some \( g \in G_B(G_AG_B)^n \). Since by construction \( A_iB_{i+1} \equiv A_iB_i \equiv AB \), independence implies that for any \( c \in C_i \) we have \( d(c, B_{i+1}) \geq d(c, A_i) \oplus d(A, B) \). Similarly, for any \( c \in D_i \) we have \( d(c, A_{i+1}) \geq d(c, A_i) \oplus d(A, B) \). The result follows by an easy induction argument. \( \square \)

**Lemma 6.3.** Let \( A = \bigcup_{i=1}^k A_i \subseteq \mathcal{U} \) be a finite set and \( r \in R \setminus \{0\} \). Assume \( B_i \subseteq \mathcal{U} \) is a finite set such that \( d(A_i, B_i) \geq r \) for all \( 1 \leq i \leq k \). Then there is a finite \( C \subseteq \mathcal{U} \) such that \( d(A, C) \geq r \) and \( G_C \subset (\bigcup_{i=2}^k G_{B_i})^{2k-1} \). More precisely \( C \) is the translate of \( B_1 \) by an element of \( G_{B_2}G_{B_3} \ldots G_{B_k} \).

**Proof.** The proof for a general \( k \) follows by a simple induction argument from case \( k = 2 \), whose proof we now present. Take \( C \) such that \( C \equiv_{A_1B_2} B_1 \) and \( C \downarrow_{A_1B_2} A_2 \). Since \( C \equiv_{A_1} B_1 \) we have \( d(A_1, C) = d(A_1, B_1) \geq r \). We claim that \( d(C, A_2) \geq r \). Indeed, take any \( c \in C \) and \( a \in A_2 \). There exists \( e \in A_1B_2 \) such that \( d(c, a) = d(c, e) \oplus d(e, a) \). There are two possibilities. If \( e \in B_2 \), then \( d(c, a) \geq d(e, a) \geq r \). In the choice of \( A_2 \) and \( B_2 \). If \( e \in A_1 \) then \( d(c, a) \geq d(c, e) \geq r \). \( \square \)

Given an \( \mathcal{R} \)-metric space \( (X, d) \), a point \( x \in X \) and \( \epsilon \in R \setminus \{0\} \) let \( N_x(\epsilon) := \{ g \in \text{Isom}(X, d) \mid d(gx, x) \leq \epsilon \} \). The following claim is easy to check. See Lemmas 8.2 and 8.6 below for a more detailed explanation.

**Claim 6.4.** Suppose a distance monoid \( \mathcal{R} \) has the property that for any \( r \in R \setminus \{0\} \) there exists \( s \in R \setminus \{0\} \) with \( s+s \leq r \). Then for any \( \mathcal{R} \)-metric space \( (X, d) \) the collection \( \{ N_x(\epsilon) \mid x \in X, \epsilon \in R \setminus \{0\} \} \) generates a Hausdorff group topology on \( G = \text{Isom}(X, d) \) at the identity.
We denote the topology above by $\tau_m$. For metric spaces this is just the usual point-wise convergence topology on $G \subseteq X^X$. The following theorem generalizes Uspenski’s minimality result for the isometry group of the Urysohn sphere.

**Theorem (C).** Let $R = (R, 0, \leq, \oplus)$ be an archimedean distance monoid, $U$ a $R$-Urysohn space, $G = \text{Isom}(U)$ and let $\tau_0$ be either:

- $\tau_m$ in case for any $r \in R \setminus \{0\}$ there exists $s \in R \setminus \{0\}$ with $s \oplus s \leq r$; or,
- $\tau_{st}$ otherwise.

Then $\tau_0$ is the coarsest non-trivial group topology on $G$ coarser than the stabilizer topology $\tau_{st}$. In particular, $(G, \tau_0)$ is totally minimal.

**Proof.** Suppose $\tau_0$ does not satisfy the conclusion of the theorem and let $\tau$ be a group topology that is coarser than $\tau_{st}$ but not finer $\tau_0$. This implies that there is $s \in R \setminus \{0\}$ such that $N_v(s)$ is not a $\tau$-neighbourhood of $1$ for any $v \in U$ in $\tau$.

**Lemma 6.5.** Given $t \in R$ with $2t \leq s$, a neighbourhood $V$ of $1$ in $\tau$, as well as $a \in U$, $k \in \mathbb{N}$ and $b_1, \ldots, b_k \in U$ there is $g \in V$ such that $ga \notin B_{b_i}(t)$, for each $1 \leq i \leq k$, i.e. $d(ga, \{b_i\}_{i=1}^k) > t$.

**Proof.** Assume the conclusion fails. Take $h \in G_a$ such that $h(B)$ and $B$ are independent over $a$ where $B = \{b_1, \ldots, b_k\}$ and consider $V \cap V^{h^{-1}}$. Take $g \in V \cap V^{h^{-1}}$. There are $1 \leq i, j \leq k$ such that $ga \in B_{b_i}(t) \cap B_{b_j}(t)$. This implies $d(b_i, b_j) \leq 2t \leq s$. Independence of $B$ and $h(B)$ over $a$ implies that either $d(b_i, a) \leq t$ or $d(hb_j, a) \leq t$. As $ga \in B_{b_i}(t) \cap B_{b_j}(t)$, either of the two cases implies $d(ga, a) \leq 2t \leq s$ and hence $V \cap V^{h^{-1}} \subseteq N_v(s)$ which is a contradiction. □

**Lemma 6.6.** Given $t \in R$ with $2t \leq s$, $V \in \mathcal{N}_r(1)$ and $A = \{a_1, a_2, \ldots, a_k\}$ a finite subset of $U$ there is a finite subset $C$ of $U$ such that $G_C \subseteq V$ and $d(A, C) > t$.

**Proof.** Consider $W$ be a neighbourhood of $1$ in $\tau$ with $W = W^{-1}$ such that $W^{6k-3} \subseteq V$. Let $C_0$ be a finite subset of $U$ such that $G_{C_0} \subseteq W$. By Claim 6.5 there is $g_i \in W$ such that $d(g_i a_i, C_0) > t$ for each $1 \leq i \leq k$. Then $d(g_i^{-1}(C_0), a_i) > t$ for each $i$ and by applying Lemma 6.3 with $B_i = g_i^{-1}(C_0)$ and $A_i = \{a_i\}$, we find a finite subset $C$ such that $G_C \subseteq \bigcup_i G_{B_i}^{2k-1} \subseteq W^{6k-3} \subseteq V$ as $G_{B_i} \subseteq W^3$. □

**Lemma 6.7.** For any $V \in \mathcal{N}_r(1)$, any finite $C \subseteq U$ and $r \in R \setminus \{0\}$ there is a finite $D$ with $d(D, C) \geq r$ and $G_D \subseteq V$.

**Proof.** Recall that $s$ is fixed before Lemma 6.5. We claim that there exists $t_0 \in R$ such that $2t_0 \leq s$ and for any $r \in R$ there is $m \in \mathbb{N}$ such that $mt_0 \geq r$ for any $t_0 > t_0$. Indeed, there is $t > 0$ with $2t \leq s$, in which case we can take $t_0 = t$, or else $2t > s$ for all $t > 0$ and we can take $t_0 = s$.

Fix now $V \in \mathcal{N}_r(1)$ and $r \in R$ and let $m$ be as above. Let $k = \lceil \frac{m-1}{2}\rceil$. Choose $W \in \mathcal{N}_r(1)$ with $W = W^{-1}$ and $W^{2m+5} \subseteq V$.

By Lemma 6.6 applied to $C$ and $W$ there are $A$ and $B$ with $d(A, B) > t$ such that $G_A, G_B \subseteq W$. Lemma 6.2 then implies the existence of $g \in G_B(G_A G_B)^k \subseteq W^{2k+1} \subseteq W^{m+2}$ such that $d(C, g(A)) \geq m \cdot d(A, B) \geq r$, which in turn implies $G_{g(A)} = g G_A g^{-1} \subseteq W^{2m+5}$. Take $D = A$. □

We are now ready to finish the proof of Theorem C. Pick any neighbourhood $W$ of $1$ in $\tau$ with $W = W^{-1}$ and $g \in G \setminus \{1\}$ where $g \notin W^4$. Since $W$ is a neighbourhood of identity in $\tau$ it must contain $G_A$ for some finite subset $A$ of $U$. Lemma 6.7 implies the existence of some finite $B \subset U$ such that $G_B \subseteq W$ and $d(A, g(A), B) \geq \text{diam}(A g(A))$. By Lemma 6.1 there is an isomorphic copy $A'$ of $A$ over $B$ and $s \in R$ such that $d(a, a') = s$ for all $a' \in A'$ and $a \in A g(A)$.

In particular, $A \cong A', g(A)$, which implies there is $B'$ such that $g(A) B' \cong A' B'$. By Lemma 3.7 the chain $A, B, A', B', g(A)$ witnesses $g \in (G_A G_B)^2 \subseteq W^4$, contradicting the choice of $W$. □
7. Group topologies on Isom(\mathcal{U}) coarser than \(\tau_{st}\)

In the light of Theorem C one might conjecture there is a gap between \(\tau_{st}\) and the point-wise convergence topology \(\tau_m\) in those cases in which the latter exists. This turns out to be false.

Fix a distance monoid \(\mathcal{R}\), an \(\mathcal{R}\)-metric space \((X,d)\) and let \(G = \text{Isom}(X,d)\). For any distinct \(x,y \in X\) write

\[
N^{sp}_{x,y} := \{g \in G \mid d(gx,y) \leq d(x,y)\}.
\]

The following is easy to check; see Lemmas 8.2 and 8.6 of the following section.

Claim 7.1. Suppose that \(\mathcal{R}\) has the property that for any \(r \in R \setminus \{0\}\) there exists \(s,s' \in R \setminus \{0\}\) with \(s \oplus s' = r\). Then the collection \(\{N^{sp}_{x,y} \mid x,y \in X, \ d(x,y) > 0\}\) generates a Hausdorff group topology \(\tau_{0+0}\) on \(G = \text{Isom}(X,d)\) at the identity. Moreover, if \((X,d)\) is an \(\mathcal{R}\)-Urysohn space, then the inclusions \(\tau_m \subset \tau_{0+0} \subset \tau_{st}\) are proper.

Due to the following obstruction the topology \(\tau_{0+0}\) is not eliminated by an application of Lemma 3.1 to the pair \((\tau_m,\tau_{st})\). Take for instance \(\mathcal{U} = \mathcal{U}_3\) and two disjoint sets \(A, B \subset \mathcal{U}_3\) of size \(k \geq 1\) that lie entirely on a common line (i.e. any triangle spanned by three points in \(AB\) is degenerate) and alternate on said line. Let us say that the “leftmost” point is \(\alpha \in A\) and the “rightmost” point is \(\beta \in B\). Then for any chain \(s_1 = A, s_2 = B, s_3 = A_1, s_4 = B_1, \ldots, s_{3k-1} = B_{k-1}\) for which \(A,B \subset A_1B_1\) and \(A_{i+1}B_i \cong A_iB_{i+1}\) for all \(0 \leq i \leq k-2\) we have the constraint \(d(\alpha,\beta') \leq d(\alpha,\beta)\), where \(\beta\) and \(\beta'\) are components of the same index in \(B\) and \(B_k\) respectively. The main content of Theorem D is the existence of a gap between \(\tau_{0+0}\) and \(\tau_m\) (Proposition 7.13), which involves a series of small technical intermediate Lemmas collected in subsections 7.1 and 7.2. In contrast, the existence of a gap between \(\tau_{st}\) and \(\tau_{0+0}\) (Lemma 7.19) is a direct consequence of Lemma 3.1.

The Lemmas in Subsection 7.1 highlight different aspects of the obstruction mentioned above. In particular, Lemma 7.3 can be read as saying that this is in fact the only obstruction for the assumptions of Lemma 3.1 to hold for the pair \((\tau_m,\tau_{st})\).

Subsection 7.2 gathers Lemmas allowing one to move downwards: we are given a group topology \(\tau\) and we know that there exists \(W \in N_r(1)\) such that all \(y \in W\) preserves a certain property and we want to replace it with \(W' \in N_r(1)\) such that all \(g \in W'\) preserve some different (stronger) property.

7.1. Point alignment. For future reference we state Lemma 7.3 and other lemmas in this section in greater generality than required by Theorem D. The reader might as well take \(\mathcal{R} = \mathcal{S}_0\) where \(S\) is some dense subgroup of \((R,0,\leq,+)\) and \(b \in S \supset R \cup \{\infty\}\). In the definitions below \(\mathcal{R} = \langle R,0,\leq,\rangle\) is a distance monoid and \((X,d)\) an \(\mathcal{R}\)-metric space.

Given \(\epsilon \in \mathcal{R}\) we say that an unordered triple \(r_1,r_2,r_3 \in R\) is \(\epsilon\)-flexible if \(r_i \oplus \epsilon \leq r_j \oplus r_k\) where \(r_i = \max\{r_1,r_2,r_3\}\) and \(\{j,k\} = \{1,2,3\} \setminus \{i\}\). We say that it is strongly \(\epsilon\)-flexible if moreover for any \(r_j'\) and \(r_k'\) such that \(r_j' \oplus \epsilon \geq r_j\) and \(r_k' \oplus \epsilon \geq r_k\) we have \(r_i \leq \min\{r_j' \oplus r_k',r_j \oplus r_k'\}\).

A 0-\(\epsilon\)-flexible triple will be called simply triangular. We say that a triangular triple is (strongly) flexible if it is (strongly) \(\epsilon\)-flexible for some \(\epsilon \in R \setminus \{0\}\).

We say that an unordered triple of points \(u,v,w\) in \(X\) is \((\epsilon,\epsilon)\)-flexible if \(d(u,v),d(u,w),d(v,w)\) is \((\epsilon,\epsilon)\)-flexible. We say it is tight if it is not \(\epsilon\)-flexible for any \(\epsilon > 0\). We say that an ordered triple of points \((u_1,u_2,u_3)\) is \(\epsilon\)-general if the triple \((d(u_1,u_2),d(u_2,u_3),d(u_2,u_3))\) is \((\epsilon,\epsilon)\)-flexible.

Observation 7.2. If \(\mathcal{R}\) is a basic distance monoid, \(\mathcal{U}\) an \(\mathcal{R}\)-Urysohn space and \(u,v,w \in \mathcal{U}\) such that \(u \downarrow v, w\) and in the ambient group \(d(u,v) + d(v,w) < \sup R \in R \cup \{\infty\}\) holds, then \((u,v,w)\) is aligned. If \(\mathcal{R}\) is standard then the opposite implication is also true: the triple of points \((u,v,w)\) is aligned only if \(u \downarrow v, w\).


Given two finite subsets $A, B \subset X$ and $r \in R^*$ we say that $B$ \textit{r-cuts} $A$ if there exists $a, a' \in A$ with $d(a, a') \leq r$ such that $B \cap [a, a') \neq \emptyset$. We say that $B$ \textit{cuts} $A$ if it $r$-cuts $A$ for some $r \in R$. We say that $B$ is in \textit{$(\epsilon, r)$-general position} relative to $A$ if for any distinct $a_1, a_2 \in A$ and any $b \in B$ the triple $a_1, a_2, b$ is in $(\epsilon, r)$-general position. Notice that in particular this implies that $B$ does not cut $A$.

The following Lemma is the main source of motivation of the definitions above.

**Lemma 7.3.** Let $\mathcal{R}$ be any distance monoid, $\mathcal{U}$ an $\mathcal{R}$-Urysohn space, $G = \text{Isom}(\mathcal{U})$ and $A, B$ finite subsets of $\mathcal{U}$ such that $B$ is in $\epsilon$-general position relative to $A$. Then $G_A G_B G_A \supseteq \bigcap_{\alpha \in A} N_{\alpha}(\epsilon)$. In particular, $G_A G_B G_A \in \mathcal{N}_{\tau_m}(1)$ in case $\tau_m$ exists (See Claim 6.4).

**Proof.** By virtue of Lemma 3.7 the result can be rephrased as follows. Given any finite metric space $(D, \bar{d})$ whose underlying set consists of $A$ and an isometric copy of $A'$ with the property that $\bar{d}(a', a) \leq \epsilon$ for conjugate points $a \in A, a' \in A'$ the extension of $\bar{d}$ to $(D \bigcup B)^2$ given by $d(a, b) = \bar{d}(a', b) = d(a, b)$ for $a, a' \in A$ and $b \in B$ defines an $\mathcal{R}$-metric space. It suffices to check the triangular inequality for triples of points of the form $(a_1, a_2, b)$. On the one hand for any triples of points $a_1, a_2, b$ where $a_1, a_2 \in A$ and $b \in B$ we have:

$$d(a_1, a_2') \leq \epsilon + d(a_1, a_2) \leq d(a_1, b) + d(b, a_2) = \bar{d}(a_1, b) + \bar{d}(b, a_2),$$

where the second inequality comes from $\epsilon$-flexibility of $\{a_1, a_2, b\}$. On the other hand $\bar{d}(a_1, a_2') + \epsilon \geq d(a_1, a_2)$ so strong $\epsilon$-flexibility of $\{a_1, a_2, b\}$ yields:

$$\bar{d}(a_1, b) = d(a_1, b) \leq \bar{d}(a_1, a_2') + \bar{d}(a_2, b) = \bar{d}(a_1, a_2') + \bar{d}(a_2', b).$$

We say that $\mathcal{R}$ has no gaps if for any $r < s$ there exists $\epsilon \in R \setminus \{0\}$ such that $r + \epsilon \leq s$.

**Lemma 7.4.** Let $\mathcal{R}$ be a distance monoid with no gaps and $A, B, C$ finite subsets of some $\mathcal{R}$-metric space $(X, d)$ satisfying $A \perp_B C$, where $B = B_1 \cup B_2$. If $A$ r-cuts $C$, then at least one of the following holds:

- $B_1$ r-cuts $C$;
- $A$ r-cuts $B_2$.

**Proof.** Pick $a \in A$ and $c_1, c_2$ such that $d(c_1, c_2) \leq r$ and $a \in [c_1, c_2]$. For $i = 1, 2$ there exists $b_i \in B$ such that $d(a, c_i) = d(a, b_i) + d(b_i, c_i)$. We claim $b_1, b_2 \in [c_1, c_2]$. Otherwise for some $\epsilon > 0$ and $i \in \{1, 2\}$ we have:

$$d(c_1, c_2) + \epsilon \leq d(c_1, b_i) + d(b_i, c_3-i) \leq d(c_1, b_i) + d(b_i, a) + d(a, c_3-i) = d(c_1, b_i) + d(a, c_3-i) = d(c_1, a) + d(a, c_2),$$

contradicting $a \in [c_1, c_2]$. We claim that $a \in [b_1, b_2]$ and $d(b_1, b_2) \leq d(c_1, c_2)$. On the one hand, if $a \notin [b_1, b_2]$, then for some $\epsilon > 0$ we have:

$$d(c_1, c_2) + \epsilon \leq d(c_1, b_1) + d(b_1, b_2) + \epsilon \leq d(c_1, b_1) + d(b_1, a) + d(a, b_2) \leq d(c_1, a) + d(a, c_2),$$

contradicting $a \in [c_1, c_2]$. On the other hand $d(b_1, b_2) \leq d(c_1, c_2)$, since otherwise

$$d(c_1, c_2) + \epsilon \leq d(b_1, b_2) \leq d(b_1, a) + d(a, b_2) \leq d(c_1, a) + d(a, c_2)$$

for some $\epsilon > 0$, since $\mathcal{R}$ has no gaps. So in case $\{b_1, b_2\} \subseteq B_2$, then the second alternative in the statement holds, while if $\{b_1, b_2\} \cap B_1 \neq \emptyset$, then the first one must hold.

**Corollary 7.5.** Let $\mathcal{R}$ be a distance monoid with no gaps and $\mathcal{U}$ an $\mathcal{R}$-Urysohn space. Let $r \in R$ and $A_j, B_j$, $1 \leq j \leq k$ be finite subsets of $\mathcal{U}$ such that $A_j$ does not r-cut $B_j$ for any $1 \leq j \leq k$. Then there exists $g \in G_{B_k} G_{B_{k-1}} \cdots G_{B_2}$ such that $A := \bigcup_{1 \leq j \leq k} A_j$ does not r-cut $gB_1$. 
Proof. The argument is analogous to the one in the proof of Claim 6.3. We will restrict to the case \( k = 2 \), since the general case can be deduced from it by an easy induction argument. Take \( B'_1 \cong A_1, B_2 \) such that \( B'_1 \downarrow A_1, B_2 \). If \( A_2 \) cuts \( B'_1 \) then by Lemma 7.4 we have \( A_1 \) cuts \( B'_1 \) or \( A_2 \) cuts \( B_2 \). Both alternatives are ruled out by our assumptions on \( A_1, B_2 \) and the fact that \( B'_1 \cong A_1, B_1 \). \( \square 

Lemma 7.6. Let \( R \) be a standard distance monoid and suppose we are given a triangular triple \( r_1, r_2, r_3 \in R \setminus \{0\} \) with \( r_1 \leq r_2 \leq r_3 \) as well as \( \epsilon_1, \epsilon_2, \epsilon_3, \delta \in R \) such that:

- \( 2 \cdot \epsilon_i \leq r_1 \) for every \( 1 \leq i \leq 3 \);
- \( \epsilon_3 \leq \max \{ \epsilon_1, \epsilon_2 \} \);
- \( \delta \leq \max \{ \epsilon_1, \epsilon_2 \} - \epsilon_3 \).

Then the triple \( r_i \oplus \epsilon_i, 1 \leq i \leq 3 \) is strongly \( \delta \)-flexible.

Proof. On the one hand:

\[ r_3 \oplus \epsilon_3 \oplus \delta \leq r_1 \oplus r_2 \oplus \epsilon_1 \oplus \epsilon_2 \leq r_1 \oplus r_2 \oplus \epsilon_1 \oplus \epsilon_2 = (r_1 \oplus \epsilon_1) \oplus (r_2 \oplus \epsilon_2), \]

where we use the fact that \( \epsilon_3 \oplus \delta \leq \epsilon_1 \oplus \epsilon_2 \). On the other hand, for \( i \in \{1, 2\} \) we have:

\[ r_i \oplus \epsilon_i \oplus \delta \leq r_3 \oplus 2 \max \{ \epsilon_1, \epsilon_2 \} \leq r_3 \oplus r_{3-i} \leq (r_3 \oplus \epsilon_3) \oplus (r_{3-i} \oplus \epsilon_{3-i}), \]

This shows that the triple is \( \delta \)-flexible. Moreover, for \( i \in \{1, 2\} \) we have:

\[ (r_i + \epsilon_i - \delta) + (r_{3-i} + \epsilon_{3-i}) \geq r_1 + r_2 + \max \{ \epsilon_1, \epsilon_2 \} - \delta \geq r_3 + \epsilon_3 \]

which implies that \( r'_i \oplus (r_{3-i} \oplus \epsilon_{3-i}) \geq r_3 \oplus \epsilon_3 \) for any \( r'_i \) for which \( r_i \leq r'_i \). Likewise for \( i \in \{1, 2\} \) we have:

\[ (r_3 - \delta) + (r_i \oplus \epsilon_i) \geq r_3 - \delta + r_i \geq r_3 - \delta + 2 \max \{ \epsilon_1, \epsilon_2 \} \geq r_3 - \delta + (\delta + \max \{ \epsilon_1, \epsilon_2 \}) \geq r_{3-i} + \epsilon_{3-i} \]

so the tuple \( r_1 \oplus \epsilon_1, r_2 \oplus \epsilon_2, r_3 \oplus \epsilon_3 \) is strongly \( \delta \)-flexible. \( \square 

Lemma 7.7. Let \( R \) be a standard distance monoid with no gaps, no minimal positive element and such that for any \( r \in R \) there exists \( s \in R \) such that \( s \oplus s \leq r \). Let \( U \) an \( R \)-Urysohn space and \( A, B \subset U \) finite sets such that \( B \) does not cut \( A \). Then there exist \( A' \cong_B A \) such that \( A' \) is in general position relative to \( A \).

Proof. Fix \( \epsilon \in R \setminus \{0\} \) such that \( d(a, a') \oplus \epsilon \leq d(a, b) \oplus d(b, a') \) for all \( a, a' \in A \) and \( b \in B \) (we include the case \( a = a' \)). Fix some symmetric injective function \( f : A \times A \to (0, \epsilon) \subset R \) such that:

- \( 2 \max \{ (im(f)) \} \leq \min \{ d(a, a') \mid a, a' \in A, a \neq a' \} \);
- \( d(a_1, a_2) < d(a_1', a_2') \) implies \( f(a_1, a_2) > f(a_1', a_2') \) for \( a_1, a_2, a_1', a_2' \in A \).

Claim 7.8. There exists \( h \in G_B \) such that \( d(a, ha') = d(a, a') \oplus f(a, a') \), for all \( a, a' \in A \).

Let us first show how to prove the Lemma using the claim above. We need to show that for any \( a_1, a_2, a_3 \in A \), \( a_1 \neq a_2 \), the triple \( a_1, a_2, ha_3 \) is strongly \( \delta \)-flexible, where \( \delta \) is the minimum between the minimum value of \( f \) and the smallest absolute difference between two values in the image of \( f \).

If \( a_3 = a_i \) for some \( i = 1, 2 \), let’s say \( a_3 = a_1 \), then strong \( \delta \)-flexibility of \( \{a_1, a_2, ha_3\} \) follows from \( f(a_1, a_2) + \delta \leq f(a_1, a_3) = d(a_1, ha_3), d(a_2, ha_1) = d(a_2, a_1) \oplus f(a_1, a_2) \) and \( f(a_1, a_1) \leq d(a_1, a_2) \).

If \( a_1, a_2, a_3 \) are all different we apply Lemma 7.6 to the three distances between \( a_1, a_2, a_3 \), which we name in increasing order \( r_1 \leq r_2 \leq r_3 \). Here we take \( \epsilon_i = 0 \) exactly for one value of \( i \) for which \( r_i := d(a_1, a_2) \), while for \( i = 1, 2 \) if \( r_3 = d(a_3, a_3) \), we take \( \epsilon_j = f(a_3, a_3) \).

The first condition in Lemma 7.6 follows immediately from the first property of \( f \). On the other hand, our second condition on \( f \) guarantees that whichever element in \( \{\epsilon_1, \epsilon_2\} \) is non-zero must be also larger than \( \epsilon_3 \) so the second assumption of Lemma 7.6 also holds. The third one follows from the choice of \( \delta \).

It follows from 7.6 for any \( a_1, a_2, a_3 \in A \), \( a_1 \neq a_2 \), the triple \( a_1, a_2, ha_3 \) is strongly flexible. This in turn implies that \( hA \) is in general position relative to \( A \).
Let us now prove the Claim. For \(i = 1, 2\) let \(A' = \{a^i \mid a \in A\}\) be a copy of \(A\) and \(D := A^1 \coprod A^2 \coprod B\). It suffices to check that the function \(\bar{d} : D^2 \to R\) given by:

- \(\bar{d}(a^i_1, a^i_2) = d(a_1, a_2)\) for \(i = 1, 2\) and \(a_1, a_2 \in A\);
- \(\bar{d}(a^i, b) = d(a, b)\) for \(i = 1, 2, a \in A\) and \(b \in B\);
- \(\bar{d}(a^i_1, a^i_2) = d(a_1, a_2) \oplus f(a_1, a_2)\) for any \(a_1, a_2 \in A\);

satisfies the triangle inequality. For triples of points contained in \(A^1 \cup A^2\) this is part of the conclusion of Lemma 7.6. All that is left to check is the triangle inequality for triples of the form \((a^1_1, a^2_2, b)\), where \(a_1, a_2 \in A\) and \(b \in B\). We have \(\bar{d}(a^1_1, a^2_2) \leq d(a_1, a_2) \oplus \epsilon \leq d(a_1, b) \oplus d(b, a_2) = \bar{d}(a^1_1, b) \oplus \bar{d}(b, a^2_2)\) by the choice of \(\epsilon\) and the fact that \(im(f) \subseteq (0, \epsilon)\), while the remaining inequalities are straightforward from the inequality \(d(a_1, a_2) \leq \bar{d}(a^1_1, a^2_2)\).

\[\square\]

**Lemma 7.9.** Let \(\mathcal{R}\) be a standard distance monoid and \(\mathcal{U}\) an \(\mathcal{R}\)-Urysohn space. Assume we are given \(r \in \mathcal{R}\) and finite \(A, B \subseteq \mathcal{U}\) such that \(B\) does not \(r\)-cut \(A\). Take \(A'\) such that \(AB \cong A'B\) and \(A' \downarrow_B A\). Then \(A'\) does not behave \((r \circ r)\)-cut with \(A\).

**Proof.** As usual, let \(a'\) stand for the conjugate of any given \(a \in A\) in \(A'\). Aiming for contradiction, suppose that there exist \(a_1, a_2, a_3 \in A\) such that \(d(a_1, a_2) \leq r \oplus r\) and \(a_3 \in [a_1, a_2]\). This implies, in particular, that \(d(a_1, a_2) < \sup R\).

Notice that by independence for all \(a, \hat{a} \in A\), we have \(d(a, \hat{a}) \geq d(a, \hat{a}')\), with equality only if \(B \cap [a, \hat{a}] \neq \emptyset\) or \(d(a, \hat{a}) = \sup R\) (in the bounded case).

Since \(d(a_1, a_2) = d(a_1, a_3') + d(a_3', a_2)\), as \(d(a_1, a_2) < \sup R\), the triangle inequality implies \(d(a_j, a_3') = d(a_j, a_3)\) for \(j = 1, 2\). By the previous paragraph the intersections \(B \cap [a_1, a_3]\) and \(B \cap [a_2, a_3]\) are both non-empty. However, \(d(a_i, a_3) \leq r\) for at least one \(i \in \{1, 2\}\): a contradiction. \[\square\]

### 7.2. Downward lemmas

Throughout this subsection \(\mathcal{R}\) will be a standard distance monoid, \(\mathcal{U}\) an \(\mathcal{R}\)-Urysohn space and \(G = \text{Isom}(\mathcal{U})\).

**Lemma 7.10.** For any finite \(a, c \in \mathcal{U}\) and finite \(B \subseteq \mathcal{U}\) we have \(GB \subseteq N^c_{a,c}\) if and only if \(B \cap [a,c] \neq \emptyset\).

**Proof.** We may assume that \(d(a, c) < \sup R\), since otherwise both the condition on the right and that on the left are trivially satisfied.

For the only if direction assume \(B \cap [a,c] = \emptyset\) and choose \(h \in GB\) such that \(ha \downarrow_B c\). We then have \(d(ha, c) = \min\{d(a, b) \oplus d(b, c) \mid b \in B\} < d(a, c)\), so that \(h \in GB \setminus N^c_{a,c} \neq \emptyset\). The opposite direction is a direct consequence of the triangle inequality together with the existence for \(b \in B\) such that \(d(a,c) = d(a, b) \oplus d(b, c)\).

\[\square\]

**Lemma 7.11.** Let \(\tau\) be a non-trivial group topology on \(G\) strictly coarser than \(\tau st\) and assume that there exists \(W \in \mathcal{N}_\tau(1)\) and distinct points \(a,b_1,b_2,\ldots, b_k \in \mathcal{U}\) such that \(W \subseteq \bigcup_{1 \leq j \leq k} N^c_{a,b_j}\) and \(d(a,b_j) + d(a,b_j) \in R\) for all \(1 \leq j \leq k\). Then there exists \(1 \leq j_0 \leq k\) such that \(N^c_{a,b_{j_0}} \in \mathcal{N}_{\tau st}(1)\).

**Proof.** Assume the conclusion does not hold. This means \(U \not\subseteq N^c_{c,d}\) for any \(U \in \mathcal{N}_{\tau st}(1)\) and any \(c,d \in \mathcal{U}\) with \(d(c,d) = d(a,b_j)\) for \(1 \leq j \leq k\). Notice also that since \(\tau\) is not trivial and coarser than \(\tau st\) by Theorem C it must be finer than \(\tau m\) so that \(N_{\tau m}(c) \cap U \in \mathcal{N}_{\tau st}(1)\) for any \(x \in \mathcal{U}\) and \(c > 0\).

Let \(B = \{b_j\}_{j=1}^k\) and take \(h \in G_a\) such that \(h^{-1}B \downarrow a\). Let \(W' = W \cap W^h \subseteq \bigcup_{j=1}^k N^c_{a,b_j}\) \(\cap \bigcup_{j=1}^k N_{a,h^{-1}b_j}\). For any \(g \in W'\) we have \(g \in N^c_{a,b_j}\) for some \(1 \leq j \leq k\). Combining this with the assumption on \(d(a,b_j)\), \(1 \leq l \leq k\), independence and the triangular inequality we get for any \(1 \leq i \leq k\):

\[
d(a,b_i) + d(h^{-1}b_j) = d(a,b_i) \oplus d(h^{-1}b_j) = d(b_i, h^{-1}b_j) \leq d(ga,b_i) \oplus d(ga,h^{-1}b_j) \subseteq d(ga,b_i) + d(ga,h^{-1}b_j)
\]

which implies that \(d(ga,b_i) \geq d(a,b_i)\).

Pick some \(U \in \mathcal{N}_{\tau st}(1)\) such that \(U^k \subseteq W'\). We construct a sequence of elements \(g_1, g_2, \ldots, g_k \in U\) and \(\epsilon_1 \geq \epsilon_2 \geq \ldots \epsilon_k \in R \setminus \{0\}\) in the inductive fashion described below. We use the notation \(\bar{g}_j = g_jg_{j-1}\ldots g_1\) for \(1 \leq j \leq k\).
To start with, we choose $g_1 \in U \setminus N_{a,b}^{sp}$. Fix $\epsilon_1 > 0$ such that $(k - 1)\epsilon_1 < d(g_1a, b_1) - d(a, b_1)$. Now, suppose that for some $1 \leq j \leq k$ the elements $g_1, g_2, \ldots, g_j$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_j$ have already been chosen.

We distinguish two cases. If $d(\bar{g}_ja, b_{j+1}) > d(a, b_{j+1})$, then let $g_{j+1} = 1$.

Otherwise, the choice of $W'$ together with the fact that $\bar{g}_j \in U^j \subseteq W'$ implies that $d(\bar{g}_ja, b_{j+1}) = d(a, b_{j+1})$. Since by the second observation in the first paragraph we have $U^r := U \cap N_a(\epsilon_j) \in \mathcal{N}_r(1)$, it follows from the first observation that same paragraph that we can choose $g_{j+1} = U \cap N_{\bar{g}_ja, b_{j+1}}$. Notice that in both cases we get $d(\bar{g}_ja, b_{j+1}) > d(a, b_{j+1})$. Finally, we choose $\epsilon_{j+1} \in (0, \epsilon_j)$ such that $(k - j - 1)\epsilon_{j+1} < d(\bar{g}_{j+1}a, b_{j+1}) - d(a, b_{j+1})$.

We claim that $\bar{g}_k \notin N_{a,b}^{sp}$ for any $1 \leq j \leq k$. Since $\bar{g}_k \in U^k \subseteq W$, this contradicts the initial hypothesis. For $j = k$ this has already been shown. For $j < k$ we have
\[
d(\bar{g}_ja, b_j) > d(\bar{g}_ja, b_j) - \sum_{l=j}^{k-1} d(\bar{g}_la, \bar{g}_{l+1}a) \geq
\]
\[
d(\bar{g}_ja, b_j) - \sum_{l=j}^{k-1} \epsilon_l > d(\bar{g}_ja, b_j) - (k - j)\epsilon_j > d(a, b_j)
\]
\[\square\]

Lemma 7.12. Let $\tau$ be a group topology on $G$. Suppose we are given $W \in \mathcal{N}_r(1)$ and points $a, b_1, c_1, b_2, c_2, \ldots, b_k, c_k \in U$ such that $d(b_i, c_i) + d(b_i, c_i) \in R^1$ for all $1 \leq i \leq k$ and for all $g \in W$ there is some $1 \leq j \leq k$ such that $ga \in [b_j, c_j]$. Assume $k' \leq k$ is such that $a \in [b_j, c_j]$ precisely for $1 \leq j \leq k'$. Then
\[
\bigcup_{1 \leq j \leq k'} (N_{a,b}^{sp} \cap N_{a,c}^{sp}) \in \mathcal{N}_r(1).
\]
Proof. Let $(b'_i, c'_i)_{1 \leq i \leq k}$ be an isometric copy of $(b_i, c_i)_{1 \leq i \leq k}$ independent from the latter over $a$. Pick \(g \in G_a\) such that $gb_i = b'_i$ and $gc_i = c'_i$ and consider $W' = W^g^{-1} \cap W$.

Let $h$ be any element in $W'$. We need to show that there exists some $1 \leq i \leq k'$ such that $h \in N_{a,b}^{sp} \cap N_{a,c}^{sp}$. We know there exist $1 \leq i, j \leq k$ such that $ha \in [b_i, c_i] \cap [b'_j, c'_j]$. Let $\lambda_i := d(a, b_i) = d(a, b'_j)$, $\mu_i := d(a, c_i) = d(a, c'_j)$ and let also $\lambda_i := d(ha, b_i)$, $\mu_i := d(ha, c_i)$ and $\lambda'_j := d(ha, b'_j)$, $\mu'_j := d(ha, c'_j)$ for $1 \leq i \leq k'$.

Our assumption on $h$ translates into equations
\[
(1) \quad \lambda_i + \mu_i = d(b_i, c_i) \leq \lambda_j + \mu_j, \quad \lambda'_j + \mu'_j = d(b'_j, c'_j) = d(b_j, c_j) \leq \lambda_j + \mu_j.
\]
while independence of $(b_i, c_i)$ and $(b'_j, c'_j)$ over $a$ (and the assumption on the distances $d(b_i, c_i)$) implies
\[
(2) \quad d(b_i, b'_j) = \lambda_i + \lambda_j, \quad d(c_i, c'_j) = \mu_i + \mu_j.
\]
By the triangular inequality $\lambda_i + \lambda'_j \geq d(b_i, b'_j)$ and $\mu_i + \mu'_j \geq d(c_i, c'_j)$. Putting this together with 1 and 2 yields:
\[
(3) \quad \lambda_i + \lambda_j + \mu_i + \mu_j \geq \lambda_i + \lambda_j + \mu_i + \mu_j \geq d(b_i, b'_j) + d(c_i, c'_j) = \lambda_i + \lambda_j + \mu_i + \mu_j.
\]
Thus, we must have $\lambda_i + \lambda'_j = d(b_i, b'_j)$ and $\mu_i + \mu'_j = d(c_i, c'_j)$ and similarly $\lambda_i + \lambda'_j = d(b_j, c_j)$ and $\mu_i + \mu'_j = d(c_i, c'_j)$. Using this we get:
\[
\lambda_i - \mu_i = (\lambda_i + \lambda'_j) - (\mu_i + \mu'_j) = d(b_i, b'_j) - d(c_i, c'_j) = (\lambda_i + \lambda_j) - (\mu_i + \mu_j) = \lambda_i - \mu_i.
\]
Notice that here $d(c_i, b'_j) = \mu_i + \lambda_j$ follows from $c_i \downarrow_a b'_j$. Since all inequalities involved in 3 are equalities, so must be those involved in 1, so that $\lambda_i + \mu_i = d(b_i, c_i) = \lambda_i + \mu_i$, which implies $i \leq k'$. Together with the previous equation it also gives $\lambda_i = \lambda_i$ and $\mu_i = \mu_i$. Thus any $h \in W'$ belongs to \[
\bigcup_{1 \leq j \leq k'} (N_{a,b}^{sp} \cap N_{a,c}^{sp}),\]
as needed.  

\[1\] In the sense of $+ \neq \ominus$
7.3. Proof of Theorem D.

Proposition 7.13. Let \( \mathcal{R} \) be a standard archimedean distance monoid with no least positive element. Let \( \mathcal{U} \) be an \( \mathcal{R} \)-Urysohn space and \( G = \text{Isom}(\mathcal{U}) \). Then any group topology \( \tau \) strictly coarser than \( \tau_{0+,0} \) is coarser than \( \tau_m \).

Proof. Fix some group topology \( \tau \) on \( G \) coarser than \( \tau_{0+,0} \). Denote by \( \Delta \) the collection of all \( r \in R \setminus \{0\} \) such that \( N_{a,v}^{sp} \in \mathcal{N}_r(1) \) for some (equivalently, any) pair \( u,v \in \mathcal{U}^2 \) with \( d(u,v) = r \).

Let \( \Gamma \) be the collection of all \( r \in R \) such that there exist \( a \in \mathcal{U}, W \in \mathcal{N}_r(1) \) and some finite \( B \subset \mathcal{U} \) such that \( \{ga\} \) \( r \)-cuts \( B \), for each \( g \in W \).

The fact that \( \Delta \) is upper-closed, i.e., that \( s \leq r \) and \( s \in \Delta \) implies \( r \in \Delta \), follows from the fact that \( \mathcal{R} \) is closed under taking positive differences and the following observation:

Observation 7.14. Let \( \tau \) a group topology on \( G = \text{Isom}(\mathcal{U}) \) where \( \mathcal{U} \) is an \( \mathcal{R} \)-Urysohn space. Suppose that we are given \( u,v,w \in \mathcal{U} \) with \( d(u,v) = d(v,w) \). Then \( N_{a,v}^{sp} \subseteq N_{a,w}^{sp} \).

Proof. Indeed, if \( g \in G \) is such that \( d(gu,v) \leq d(u,v) \) it follows that \( d(gu,w) \leq d(gu,v) + d(v,w) \leq d(u,v) + d(v,w) = d(u,w) \).

On the other hand \( \Gamma \) is upper-closed by definition. Notice as well that \( \Delta = R \setminus \{0\} \) implies \( \tau = \tau_{0+,0} \).

Lemmas 7.11 and 7.12 come into play through the following lemma (notice \( s+s \in \Gamma \) and \( s+s \in \Delta \), rather than \( s \oplus s \)).

Lemma 7.15. If \( s+s+s+s+R \) and \( s+s \in \Gamma \) then \( s \in \Delta \).

Proof. Let \( a \in \mathcal{U}, B \subset \mathcal{U} \) be an enumeration of all the pairs of points in \( B \) at distance at most \( s+s \). If \( s \notin \Delta \), then \( ga \in \bigcup_{j \leq j \leq N} \{b_j,c_j\} \), for each \( g \in W \). Lemma 7.12 tells us that \( \bigcup_{1 \leq j \leq k'} (N_{a,b_j}^{sp} \cap N_{a,c_j}^{sp}) \in \mathcal{N}_r(1) \) where (up to reindexing) \( a \in \{b_j,c_j\} \) if and only if \( j \leq k' \). We may assume that \( d(a,b_j) \leq d(a,c_j) \), so that \( d(a,b_j) \leq s \). By Lemma 7.11, there exists some \( 1 \leq j_0 \leq k' \) such that \( N_{a,b_j}^{sp} \in \mathcal{N}_r(1) \) and we are done.

Lemma 7.16. If \( r \notin \Gamma \), then for any \( U \in \mathcal{N}_r(1) \) and finite subsets \( A,B \subset \mathcal{U} \) there is \( h \in U \) such that \( hA \) does not \( r \)-cut \( B \).

Proof. Take \( A \) and \( B \) as above, write \( A = \{a_j\}_{j=1}^k \) and pick some \( U_0 = U_0^{-1} U_0 \in \mathcal{N}_r(1) \) such that \( U_0^{-1} \subseteq U \). Let \( C \in U_0 \) be finite such that \( e \in C \subseteq U \). We may assume \( B \subseteq C \). Assume that \( r \notin \Gamma \). This implies that for each \( 1 \leq j \leq k \) there exists some \( h_j \in U_0 \) such that \( \{a_j\} \) does not \( r \)-cut \( h_j C \). Let \( C_j = h_j B \). By Lemma 7.5 there exists \( g \in G_{C_j} G_{C_{j-1}} \cdots G_{C_2} \subseteq \mathcal{N}_r(1) \) such that \( A \) does not \( r \)-cut \( gC_j = gh_j B \). Equivalently, \( A' := gh_j^{-1} g^{-1} A \) does not \( r \)-cut \( B \).

Remark 7.17. In the previous proof we are using only the fact that \( \tau \) is coarser than \( \tau_{st} \) (as opposed to \( \tau_{0+,0} \)).

Lemma 7.18. \( \Delta \subseteq \Gamma \).

Proof. Assume for the sake of contradiction that \( r \in \Delta \setminus \Gamma \) for some \( r \). On the one hand, given \( u,v \in \mathcal{U} \) with \( d(u,v) = r \) we have \( N_{u,v}^{sp} \in \mathcal{N}_r(1) \), so there exists some \( W \in \mathcal{N}_r(1) \) such that \( W^3 \subseteq N_{u,v}^{sp} \). Pick some \( A \subseteq U \) such that \( G_A \subseteq W \).

On the other hand, by Lemma 7.16 there must be some \( g \in W \) such that \( gA \) does not cut \( \{u,v\} \). By Lemma 7.10 this implies that \( G_{gA} \subseteq N_{u,v}^{sp} \). However \( G_{gA} \subseteq W^3 \subseteq N_{u,v}^{sp} \), a contradiction.

We are now ready to finish the proof of Proposition 7.13. Let \( S = \{s \in R \mid s+s \in R\} \). Observation 7.14, Lemmas 7.18 and 7.15, together with the fact that \( \mathcal{R} \) is archimedean imply that either \( \Gamma = \Delta = R \setminus \{0\} \) or \( \Gamma \cap S = \emptyset \). The former implies \( \tau = \tau_{0+,0} \) so from now on assume \( \Gamma \cap S = \emptyset \).
We need to show that $U \in \mathcal{N}_{r_0}(1)$ for any arbitrary $U \in \mathcal{N}_{r}(1)$. Pick $U_0 \in \mathcal{N}_{r}(1)$ such that $U_0^{17} \subseteq U$ and $A \subseteq U_0$ such that $G_A \subseteq U_0$. By Lemma 7.16, there exists some $g \in U_0$ such that $A' := gA$ does not $s$-cut $A$ for any $s \in S$. By Lemma 7.9, there exists $A_1$ such that $A_1 \cong A'$, $A$ such that $A_1$ does not cut $A$. Lemma 7.7 implies the existence of $A_2$ such that $A_2 \cong A_1$, $A$ such that and $A_2$ is in general position relative to $A$.

Notice that $G_A \subseteq G_A^{-1} \subseteq U_0^{13}$ and $G_A \subseteq G_A G_A A' \subseteq U_0^{17}$. Likewise, $G_{A_2} \subseteq U_0^{15}$ and thus $G_A G_{A_2} G_{A} \subseteq U_0^{17} \subseteq U$. On the other hand, by Lemma 7.3 we have $G_A G_{A_2} G_A \in \mathcal{N}_{r}(1)$, hence $U \in \mathcal{N}_{r_0}(1)$ and we are done.

\[ \square \]

**Lemma 7.19.** Let $\mathcal{R}$ be a standard distance monoid, $U$ an $\mathcal{R}$-Urysohn space and $G = \text{Isom}(U)$. Then, given finite sets $A_1, A_2 \subseteq \mathcal{R}$ and $V \in \mathcal{N}_{r_0+0}$, there exists $W \in \mathcal{N}_{r_0+0}$ such that $G_{A_1} \cap A_2 \cap V \subseteq (G_{A_2} \cap W)(G_{A_1} \cap W)(G_{A_2} \cap W)$.

**Proof.** We may assume $W = N_{\mathcal{R}}^{{\mathcal{R}}}$ for some finite $B \subseteq \mathcal{R}$ containing $A_1 \cup A_2$, where $N_{\mathcal{R}}^{{\mathcal{R}}} = \bigcap_{b,b' \in B} N_{b,b'}^{{\mathcal{R}}}$.

Given $c, c' \in \mathcal{R}$ with $d(c, c') \neq \sup R$ denote by $\tilde{N}_{c,c'}^{{\mathcal{R}}} = \{ g \in G \mid d(gc, c') = d(c, c') \}$. It is easy to see that $\tilde{N}_{c,c'}^{{\mathcal{R}}} \subseteq \tau_{r_0+0}$, since $N_{c,c'} \cap N_{c',c''} \subseteq \tilde{N}_{c,c'}^{{\mathcal{R}}}$ for any $c''$ such that $d(c', c'') = d(c', c) + d(c, c'')$. Since we also know that $\tau_{r_0+0}$ is a standard space, it follows that $V(e) = \bigcap_{(b,b') \in X} \tilde{N}_{b,b'} \cap NB(e) \in \mathcal{N}_{r_0+0}$ for any $e \in R \setminus \{0\}$, where $X = \{(b,b') \in B^2 \mid b \neq b', d(b, b') \neq \sup R\}$.

Fix some $\epsilon > 0$ smaller than the distance between two distinct points of $B$. We will check that $V = V(e)$ satisfies the desired property.

Fix $g \in V \cap G_{A_1 \cup A_2}$. We let $C = A_1 \cap A_2$, $D_1 = A_1 \setminus C$ and $E = B \setminus (A_1 \cup A_2)$. Write $D_1'' = gD_1$, $E'' = gE$ and so on. The proof of the Lemma reduces to that of the following claim:

Let $F = (B \cup B'') \bigcup D_1 \bigcup E'$, where where $D_1$, $E'$ as two copies of $D_1$ and $E$ as abstract sets. As usual, given $d \in D_1$ we use $d', d''$ to denote the elements (entries) of $D_1'$ and $D_1''$ respectively corresponding to $d$ and similarly for $e \in E$.

In order to prove the Lemma it suffices to show that the following claim:

**Claim 7.20.** The function $\tilde{d} : F \times F$ given as follows satisfies the triangle inequality:

1. $\tilde{d}(f, f) = 0$ for any $f \in F$;
2. $\tilde{d}_{(B \cup B'') \setminus B''} = \tilde{d}_{(B \cup B'') \setminus B''}$;
3. $\tilde{d}(b_1, b_2) = \tilde{d}(b_1', b_2'') = \tilde{d}(b_1, b_2)$ for any distinct $b_1, b_2 \in D_1 \cup E$.
4. $\tilde{d}(b_1, b_2) = \tilde{d}(b_1, b_2)$ for $b_1, b_2 \in D_1 \cup E$.
5. $\tilde{d}(b, b) = \tilde{d}(b', b'') = \epsilon$ for any $b \in D_1 \cup E$.

Indeed, if the claim holds then we can embed a copy of $(F, \tilde{d})$ isometrically in $\mathcal{R}$ over $BB''$. The chain $CD_1D_2E \cong CD_1'D_2'E' \cong CD_1''D_2''E'' \cong CD_1'\hat{D}_2'E''$ then witness the desired inclusion in a fashion to the proof of 3.7. Our conditions imply any $b_1$ mapping $CD_1D_2E$ to $CD_1'D_2'E'$ is in $G_{A_1} \cap \hat{N}_B$, any $h_2$ mapping $h_1^{-1}(CD_1D_2E') = B$ to $h_1^{-1}(CD_1'D_2'E')$ is in $G_{A_1} \cap \hat{N}_B$ and $h_2^{-1}h_1^{-1}g \in G_{A_1} \cap \hat{N}_B$.

As for the proof of the claim, we begin by noticing that the triangle inequality is automatically satisfied by triples of distinct points contained in $B \cup B''$. By points 3., 4. the same is true for triples contained in $B \cup D_1 \cup E'$ and $B'' \cup D_1 \cup E'$ which do not contain pairs of points of the form $\{b, b'\}$ or $\{b'', b''\}$ for $b \in D_1 \cup E$.

If our triple is of the form $\{b, b', c\}$ with $b \in D_1 \cup E$, $c \in B \cup D_1 \cup E'$, then by 3., 4. and 5., and the choice of $\epsilon$ yields $\tilde{d}(b, b') = \epsilon < \tilde{d}(b, c) = \tilde{d}(b', c)$, from which the triangle inequality easily follows. The same argument works for triples of the form $\{b', b'', c\}$ with $c \in D_1 \cup E' \cup B''$, $b \in D_1 \cup E$. So we are left with triples of the form $\{b_1, b_2, b_3\}$ for $b_1, b_3 \in B$, $b_2 \in D_1 \cup E'$.

If $b_1 = b_3 \neq b_2$, then the triangle inequality can be established by a similar argument to that of the previous paragraph using the fact that $g \in N_B(e)$ and $\epsilon < \inf_{b_1 \neq b_2, b_1 \neq b_2} \tilde{d}(b_1, b_2)$ instead of 5..
If \( b_1 = b_2 \neq b_3 \), then the argument is similar. We have

\[
\bar{d}(b_1, b'_2) = \epsilon < \bar{d}(b'_2, b''_3) = d(b_2, b_3) = d(b_1, b_3) \leq d(b_1, b'_2) + \epsilon = \bar{d}(b_1, b''_3) + \bar{d}(b_1, b'_2),
\]

where in the last inequality we are using the fact that \( g \in \bigcap_{b \in B} N_b(\epsilon) \) and \( b''_3 = g b_3 \). If \( b_2 = b_3 \neq b_1 \) the argument is entirely analogous.

If \( b_1 = b_2 = b_3 \) we have \( \bar{d}(b_1, b'_2) \leq \epsilon = \bar{d}(b_1, b'_2) = d(b'_2, b''_3) \) and the result follows as well.

Finally we have the case in which \( b_1, b_2, b_3 \) are all distinct. First of all, we have:

\[
\bar{d}(b_1, b'_2) \leq \bar{d}(b_1, b_3) \leq \bar{d}(b_1, b_2) + \bar{d}(b_2, b_3) = \bar{d}(b_1, b'_2) + \bar{d}(b'_2, b''_3).
\]

If the maximum distance between two points in \( \{b_1, b_2, b_3\} \) is witnessed by \( d(b_1, b_3) \), then we also have

\[
\bar{d}(b_1, b'_2) = d(b_1, b_2) \leq d(b_1, b_3) \leq \bar{d}(b_1, b'_2) + \epsilon \leq d(b_1, b'_2) + d(b_2, b_3) = \bar{d}(b_1, b'_2) + \bar{d}(b'_2, b''_3)
\]

and the same argument yields \( \bar{d}(b'_2, b''_3) \leq \bar{d}(b'_2, b_1) + \bar{d}(b_1, b'_2) \).

We are left with the situation in which the maximum distance between two points in \( \{b_1, b_2, b_3\} \) is witnessed by \( d(b_1, b_2) \) or \( d(b_2, b_3) \). We might as well assume it is witnessed by \( d(b_1, b_2) \), since the other sub-case is entirely analogous.

One possibility is that \( \bar{d}(b_1, b_3) < \sup R \), in which using \( g \in \bar{N}^{sp} \) we get:

\[
\bar{d}(b_1, b'_2) = \bar{d}(b_1, b_2) \leq d(b_1, b_3) + \bar{d}(b_2, b_3) = \bar{d}(b_1, b'_2) + \bar{d}(b'_2, b''_3)
\]

This suffices to settle the triangle inequality in this case, as we also have:

\[
\bar{d}(b'_2, b''_3) = \bar{d}(b_2, b_3) \leq d(b_1, b_2) = \bar{d}(b_1, b'_2).
\]

The other possibility left is that \( d(b_1, b_2) = d(b_1, b_3) = \sup R \). The triangle inequality is then clear:

\[
\bar{d}(b'_1, b''_3) \leq \bar{d}(b'_1, b''_3) = d(b_1, b_2) = d(b_1, b_3) \leq \bar{d}(b'_1, b''_3) + \epsilon \leq \bar{d}(b'_1, b''_3) + \bar{d}(b_2, b_3) = \bar{d}(b'_1, b''_3) + \bar{d}(b'_2, b''_3),
\]

and

\[
\bar{d}(b'_2, b''_3) = d(b_2, b_3) \leq d(b_1, b_2) = \bar{d}(b_1, b'_2).
\]

This concludes the proof of the Claim and with it the proof of the Lemma.

\[\square\]

**Theorem (D).** Let \( \mathcal{R} \) be a standard archimedean distance monoid with no minimal positive element, \( \mathcal{U} \) an \( \mathcal{R} \)-Urysohn space and \( G = \text{Isom}(\mathcal{U}) \). Then there are exactly 4 group topologies on \( G \) coarser than \( \tau_{st} \)

\[\tau_{st} \supseteq \tau_{0+0} \supseteq \tau_m \supseteq \{\emptyset, G\}.\]

**Proof.** The fact that \( \{\emptyset, G\} \) is the only group topology strictly coarser than \( \tau_m \) is a particular case of Theorem C and the fact that there is no group topology \( \tau \) with \( \tau_{0+0} \supseteq \tau \supseteq \tau_m \) is the content of Proposition 7.13.

The fact that any group topology strictly coarser than \( \tau_{st} \) is coarser than \( \tau_{0+0} \) follows from a combination Lemma 3.1 with Lemma 7.19. We are applying Lemma 3.1 with \( \tau^* = \tau_{0+0} \). The first alternative in its conclusion leads to \( \tau = \tau_{st} \), while in the second \( X' = \emptyset \), since in this case the action is transitive.

\[\square\]

We will explain the system behind the notation \( \tau_{0+0} \) in the next section. As the reader might have guessed, the true identity of \( \tau_{st} \) and \( \tau_m \) will be \( \tau_{0,0} \) and \( \tau_{0+0} \).
8. Parametrizing topologies of isometry groups of generalized Urysohn spaces

We borrow the following construction from Conant [4]. Let \( \mathcal{R} \) be a distance monoid. By an end segment of \( \mathcal{R} \) we mean a subset \( \alpha \subset \mathcal{R} \) with the property that \( t \in \alpha \) whenever \( s \in \alpha \) for some \( s \leq t \). Let \( \mathcal{R}^* \) be either the collection of end segments of \( \mathcal{R} \) in case \( \mathcal{R} \) has no maximal element or the collection of non-empty end segments in case \( \mathcal{R} \) has a maximal element. There is a natural order \( \leq^* \) on the set \( \mathcal{R}^* \) given by \( \alpha \leq^* \beta \) if and only if \( \beta \subset \alpha \). One can endow \( \mathcal{R}^* \) the operation \( +^* \), defined as \( \alpha +^* \beta = \inf_{\mathcal{R}^*} \{ r + s \mid r \in \alpha, s \in \beta \} \) (see [4][2.6.4., 2.6.5.]). This gives \( \mathcal{R}^* \) the structure of a distance monoid \( \mathcal{R}^* \).

The natural embedding \( \nu \) from \( \mathcal{R} \) into \( \mathcal{R}^* \) sending \( r \in \mathcal{R} \) to \( \{ s \in \mathcal{R} \mid s \geq r \} \) respects the linear order and the operations embedding both sides: \( \nu(s \oplus t) = \nu(s) \oplus^* \nu(t) \). From now on we identify \( \mathcal{R} \) with \( \nu(\mathcal{R}) \) and write \( \oplus \) instead of \( \oplus^* \). If \( \mathcal{R} \) contains no minimal element greater than \( r \in \mathcal{R} \), then we denote the successor \( \{ s \mid s > r \} \in \mathcal{R}^* \) as \( r^+ \). Of particular interest for us will be \( 0^+ \). Notice that provided \( 0^+ \) exists, the condition \( \forall r \in \mathcal{R} \setminus \{0\} \exists s \in \mathcal{R} \setminus \{0\} s \oplus r \leq s \) is equivalent to \( 0^+ \oplus 0^+ = 0^+ \).

Let \( U \) be an \( \mathcal{R} \)-Urysohn space and \( G \) its group of isometries. By an ideal of \( \mathcal{R} \) we mean a non-empty closed subset of \( \mathcal{R} \) closed under addition and such that \( s \leq r \in \mathcal{R} \) implies \( s \in \mathcal{R} \). Given an ideal \( \mu \) of \( \mathcal{R} \) let:

\[
G_{\mu}^b := \{ g \in G \mid \exists u \in U \, d(\mu u, u) \in \mu \}.
\]

**Observation 8.1.** If \( \mu \) is an ideal of \( \mathcal{R} \) then \( G_{\mu}^b \) is a normal subgroup of \( G \). If \( \mathcal{R} \) is countable or archimedean then \( G_{\mu}^b = G_{\mu'}^b \) only if \( \mu = \mu' \).

This is true in other situations as well, but we will skip the discussion at this point.

Given \( u, v \in U \) and \( r \in \mathcal{R} \), let \( N_{u,v}(r) = \{ g \in G \mid d(\mu u, v) \leq r \} \). Given a function \( f : \mathcal{R} \rightarrow \mathcal{R}^* \), let \( S_f = \{ N_{u,v}(r) \mid r \in f(d(u,v)) \} \) and \( \mu_f = \{ r \in \mathcal{R} \mid \forall s \in \mathcal{R} \, s \oplus r \leq f(s) \} \).

Let also

\[
T(r) = \{ (s,t) \in \mathcal{R}^2 \mid r \leq s \oplus t, s \leq r \oplus t, t \leq r \oplus s \}.
\]

**Lemma 8.2.** Let \( f : \mathcal{R} \rightarrow \mathcal{R}^* \) be a function such that:

(a) \( f(d) \geq d \), for each \( d \in \mathcal{R} \);

(b) \( f(d) \geq \inf_{\mathcal{R}^*} \{ f(s) \oplus f(t) \mid (s,t) \in T(d) \} \), for each \( d \in \mathcal{R} \).

Then \( S_f \) generates the base of neighbourhoods of the identity of a group topology \( \tau_f \). Moreover, \( \mu_f \) is a normal ideal and the closure of 1 in \( \tau_f \) coincides with the group \( G_{\mu_f}^b \).

**Proof.** To begin with observe that \( N_{u,v}(r)^{-1} = N_{u,v}(r) \), all sets \( N_{u,v}(r) \) contain the identity map and the collection \( S_f \) is invariant under conjugation. Take now any two points \( u, v \in U \). Let \( d = d(u, v) \). Since \( r \geq f(d) \geq \inf_{\mathcal{R}^*} \{ f(s) \oplus f(t) \mid (s,t) \in T(d) \} \), there must be \( (s,t) \in T(d) \) such that \( f(s) \oplus f(t) \leq r \). Since \( (s,t) \in T(d) \), there must exist some point \( u \in U \) such that \( d(u,u) = s \) and \( d(u,v) = t \). Given any \( h, g \in W := N_{u,w}(s) \cap N_{u,v}(t) \) we have \( d(gu, w) \leq s, d(hw, v) \leq t \). Hence \( d(hgu, v) \leq d(hgu, hw) \oplus d(hw, v) \leq s \oplus t \leq r \). We conclude \( W \subseteq N_{u,v}(r) \) and we are done.

If \( r, r' \in \mu_f \) then for all \( g \in \mathcal{R} \) and \( (s,t) \in T(q) \) we have \( f(s) \oplus f(t) \geq s \oplus t \oplus r \oplus r' \geq q \oplus r \oplus r' \). This implies that \( f(q) = \inf_{\mathcal{R}^*} \{ f(s) \oplus f(t) \mid (s,t) \in T(q) \} \geq q \oplus r \oplus r' \). Thus \( r \oplus r' \in \mu_f \). Hence, \( \mu_f \) is an ideal of \( \mathcal{R} \). Let \( K \) be the closure of 1 in \( \tau_f \), i.e., \( K = \bigcap_{W \in \mathcal{S}_f} W \). Clearly \( G_{\mu_f}^b \subseteq K \). For the opposite inclusion consider \( g \notin G_{\mu_f}^b \); there exists some \( u \in U \) such that \( r = d(gu, u) \notin \mu_f \), which means that there exists \( s \in \mathcal{R} \) such that \( s \oplus r > f(s) \). Universality of \( U \) implies the existence of \( v \in U \) such that \( d(gu, v) = s \oplus r \), so that \( g \notin N_{u,v}(r) \in S_f \).

An additional condition is needed to ensure the faithfulness of the parametrization \( f \mapsto \tau_f \).

**Definition 8.3.** We say that \( f : \mathcal{R} \rightarrow \mathcal{R}^* \) is an \( \mathcal{R} \)-modulus of continuity if it satisfies Conditions (a) and (b) of Lemma 8.2 together with the following:

(c) \( f(d) \leq \inf_{\mathcal{R}^*} \{ s \oplus f(t) \mid (s,t) \in T(d) \} \), for each \( d \in \mathcal{R} \).
Notice that this implies the inequalities in (b) and (c) are actually equalities.

**Lemma 8.4.** If \( f, g : R \to R^* \) are \( \mathcal{R} \)-moduli of continuity, then \( \tau_f \subseteq \tau_g \) if and only if \( g(r) \leq f(r) \) for all \( r \in R \).

This is an easy consequence of Condition (c) together with the following fact:

**Lemma 8.5.** Suppose we are given \( v, w \in \mathcal{U} \), \( r \in R \) and a finite collection \( X \subseteq \mathcal{U}^2 \times R \), where \((u, u', s) \in X \) implies \( d(u, u') \leq s \). Then \( \bigcap_{(u, u', s) \in X} N_{u, u'}(s) \subseteq N_{v, w}(r) \) if and only if there exists \((u, u', s) \in X \) such that
\[
\begin{align*}
&\text{• } d(v, u) \oplus d(u, u') \oplus d(u', v) \leq r \quad \text{or} \\
&\text{• } d(v, u) \oplus d(u, u') \oplus d(u', w) \leq r.
\end{align*}
\]

*Proof.* The ‘if’ part is clear. For the only part, assume that neither of the two cases above holds. We want to show that \( \bigcap_{(u, u', s) \in X} N_{u, u'}(s) \nsubseteq N_{v, w}(r) \). We may assume there is a finite set \( Y = \{u_1\}_{i=1}^q \) such that \( v = u_1, w = u_2 \) (assume without loss of generality that \( v \neq w \)) and \( X \) contains exactly one triple \((u_i, u_j, s_{i,j})\) for any \( 1 \leq j \leq q \) and that \( s_{i,j} = s_{j,i} \). We construct a new finite \( \mathcal{R} \)-metric space as follows. As the underlying set \( Z \) we take the disjoint union of two copies \( Y_j = \{u_j^1\}_{i=1}^q, j = 1, 2 \) of \( Y \) after identifying \( u_j^1 \) and \( u_j^2 \) in case \( s_{i,j} = 0 \). Consider the map \( \tilde{d} : Y \times Y \to \mathcal{R} \) given by \( \tilde{d}(u_j^1, u_k^2) = d(u_j, u_k) \) (abbreviated as \( d_{j,k} \)) and \( \tilde{d}(u_j^1, u_j^2) = \min\{d_{j,j} \oplus s_{j,k} \oplus d_{j,k} | 1 \leq j, k' \leq q\} \). (Abbreviated as \( s_{j,k} \)). We claim that \((Z, \tilde{d})\) is an \( \mathcal{R} \)-metric space. Since our starting assumption translates as \( s_{1,2} > r \), this will witness \( \bigcap_{(u, u', s) \in X} N_{u, u'}(s) \nsubseteq N_{v, w}(r) \). By symmetry, all we need to check is \( s_{i,j} \leq s_{i,j} \oplus s_{j,j} \) as well as the symmetric inequality for all \( 1 \leq i, j, j' \leq q \). This follows easily from the definition and the inequality \( d_{i,q} \oplus d_{i,j} \geq d_{i,j} \).

Given any distance monoid \( \mathcal{R} \), let \( Id(\mathcal{R}^*) \) stand for the collection of all idempotents of \( \mathcal{R}^* \). The following claim follows easily from the definitions and the fact that \( \alpha \oplus \beta = \inf_{\mathcal{R}} \{s \oplus t | s \in \alpha, t \in \beta\} \)

**Lemma 8.6.** Let \( g : R \to Id(\mathcal{R}^*) \) satisfy the following properties:

(i) \( g \) is constant on any archimedean class;

(ii) \( g \) is non-increasing;

(iii) If \([r] < [s]\) and \( g(s) < g(r) \), then \( g(r) \leq s \);

(iv) For any \( r \in R \) there exist \( s, t \in [r] \) such that \( s \oplus t = r \).

Then the function \( \tilde{g} : R \to R^* \) given by \( \tilde{g}(r) = r \oplus g(r) \) is an \( \mathcal{R} \)-modulus of continuity.

*Proof.* Let us check condition (c) first. If it does not hold, then there exists \( r \in R \) such that \( r \oplus g(r) > s \oplus t \oplus g(t) \) for some \( (s, t) \in T(r) \). Since \( s \oplus t \geq r \) this implies \( g(t) < g(r) \) and thus \([t] > [r]\), since \( g \) is non-increasing and constant on archimedean classes. Since \((s, t) \in T(r)\), this implies in turn \( [s] = [t] \). By condition (iii) it also implies \( g(r) \leq t \). We cannot have \([g(r)] < [t]\) so necessarily \([g(r)] = [t] > [r]\). Notice that in general if an archimedean class \([q]\) contains an idempotent \( q_0 \), then \( q_0 = \max[q] \) and \( q_0 \oplus p = q_0 \) for any \( p \leq q_0 \). It follows that \( t = g(r) \) and both left and right-hand side of the inequality we started with are actually equal to \( g(r) \); a contradiction.

Condition (b) can be proved using (iv) in a similar way.

We will refer to any map \( g : R \to Id(\mathcal{R}^*) \) as above as an \( \mathcal{R} \)-ladder of idempotents.

*Example.* Given any \( \alpha \in Id(\mathcal{R}^*) \) it is easy to check that the function \( g \) with constant value \( \alpha \) satisfies the definition above. If \( \alpha = 0 \), then \( \tau_g = \tau_{\alpha} \). If \( \alpha = 0^+ \), then \( \tau_{\alpha} \) coincides with the generalized pointwise convergence topology \( \tau_m \). The system of generating sets at the identity given this way is larger than the one in the definitions above, but it can be easily checked the extra generators are redundant. Notice that \( \tau_m \) is Hausdorff if \( \inf \{im(g) | g \in G\} \in \{0, 0^+\} \). This is an only if in case \( \mathcal{R} \) is countable or \( \mathcal{U} \) the completion of a countable Urysohn space.
If $\mathcal{R}$ has only finitely many archimedean classes, we can think of a ladder as the non-increasing sequence $\sigma$ of its values on $R/\sim$ and accordingly write $\tau_\sigma$ instead of $\tau_0$. In particular, in the archimedean case we can write $\tau_{0,0}$ for $\tau_{st}$ and $\tau_{0^+,0^+}$ for $\tau_m$. Notice that the condition in Claim 7.1 is precisely what one need in order to get (iv)

**Example.** The archimedeanity assumption of Theorem C is essential. Let $(\Lambda,0,+,<)$ be the abelian group $\mathbb{Q} \times \mathbb{Q}$ equipped with the lexicographical order. Let $\mathcal{R} = (R,0,\leq,\oplus)$ be the distance monoid given by the restriction of $(\Lambda,0,+,<)$ to the non-negative part of $\Lambda$. Let $\alpha \in R^*$ be the upper closed set $\{(a,b) \in \Lambda^{\geq 0} | a > 0\}$. The four idempotents of $\mathcal{R}^*$ are $0 < 0^+ < \alpha < \emptyset$. Here we have ladders $(0,0,0)$, $(0^+,0,0)$ and $(0^+,0^+,0^+)$ whose associated topologies give analogues to $\tau_{st}$, $\tau_{0^+,0}$ and $\tau_m$ in the archimedean case.

But we get also $(0^+,0^+,0)$, witnessing the failure of Lemma 7.13, as well as $(\alpha,\alpha,0^+)$ and $(\alpha,\alpha,0)$, witnessing the fact that $\tau_m$ is not minimal. Notice that by condition (iii) there aren’t any other ladders whose minimal value is either 0 or $0^+$. Indeed, $g(0) = 0 = \max R^*$ implies $g([r]) = \emptyset$, for each $r \in R$, while $g((0,\alpha) \cap R) < \alpha$ implies also $g(0) = \alpha$.

**Lemma 8.7.** Assume $\mathcal{R} = S$ where $S = \Lambda^{\geq 0}$ for some ordered abelian group $(\Lambda,0,\leq,+)$. Then any modulus of continuity comes from some $\mathcal{R}$-ladder of idempotents.

**Proof.** Consider any $\mathcal{R}$-modulus of continuity $f : R \to R^*$ and let $g : R \to R^*$ be given by $g(r) = f(r) - r$. Notice that since $R$ is closed under differences and $f(r) \geq r$ the right hand side is a well defined element of $R^*$. Given any $t',t \in R$ with $t \leq t'$. Property (c) implies that $f(t') \leq f(t) + (t' - t)$ from which it follows that $g(t') \leq g(t)$. The same property applied to $(t',t' - t) \in T(t)$ also yields $g(t) \leq g(t') + (t' - t)$.

We now claim that $\text{im}(g) \subseteq \text{Id}(\mathcal{R}^*)$. Property (b) yields:

$$g(r) \geq \inf_{R^*} \{f(s) + f(t) - r | (s,t) \in T(r)\} = \inf_{R^*} \{g(s) + s + g(t) + t - r | (s,t) \in T(r)\}.$$  

It thus suffices to show that $g(s) + g(t) + ((s + t) - r) \geq g(r) + g(r)$, for each $(s,t) \in T(r)$. This is clear in case $s,t \leq r$, since $g(r)$ is non-increasing and $s + t - r \geq 0$. If $s$ or $t$ are larger than $r$, then the result follows from inequalities $g(t) + (t - r) \geq g(r)$, $g(s) + (s - r) \geq g(r)$. Since $g(r + r) \leq r + g(r) \leq g(r) + g(r) = g(r)$ we conclude that $g$ is constant on archimedean classes. The second property of the definition follows easily from (c).

**Remark 8.8.** In general not all moduli of continuity need come from a ladder. Take $a \in \mathbb{R}^+ \setminus \mathbb{Q}$ and consider $R = \mathbb{Q}^{\geq 0} + \mathbb{Q}^{\geq 0}a$. The sum and order inherited from $\mathbb{R}$ make $R$ into a distance monoid. Let $f : R \to R^*$ evaluate to $r$ on any $r \notin \mathbb{Q} \cap R$ and to $r^+$ on any $r \in \mathbb{Q} \cap R$. It is easy to check that $f$ is a $\mathcal{R}$-modulus of continuity.

**Question 4.** Does some distance monoid $\mathcal{R}$ admitting moduli of continuity whose range contains non-idempotent elements?

**Problem 5.** Classify the collection of $\mathcal{R}$-moduli of continuity associated with an arbitrary $\mathcal{R}$.

An alternative (and in the long run better way) of thinking about moduli of continuity is in terms of parameters of a generalized version of bi-Katetov maps as described in [20], or types of pairs of copies of $\mathcal{U}$. Any function $f : R \to R^*$ satisfying (c) can be associated to a $G$-invariant bi-Katetov map that assigns distance $f(r)$ to any pair $u',v''$ where $u'$ and $v''$ are copies of $u \in \mathcal{U}$ and $v \in \mathcal{U}$ respectively in the two copies of $\mathcal{U}$. Condition (b) on the other hand states that the type is idempotent. So the following conjecture seems natural from that point of view as well.

**Conjecture 6.** Given any distance monoid $\mathcal{R}$ and any $\mathcal{R}$-Urysohn space $\mathcal{U}$ any group topology on $\text{Isom}(\mathcal{U})$ strictly coarser than $\tau_{st}$ is of the form $\tau_f$ for some $\mathcal{R}$-modulus of continuity $f$. 


9. Zariski topology

Given a group $G$, an equation (inequality) over $G$ in one variable $x$ is an expression of the form $w(x, \alpha) = 1$ ($w(x, \alpha) \neq 1$), where $w$ is a term over $x \cup \alpha$ in the language of groups with inversion. We can think of $w$ as an expression of the form

$$a_0x^{e_0}a_1 \cdots x^{e_m-1}a_m;$$

where $j_l \in \{1, \ldots, r\}$ and $e_l \in \{1, -1\}$ for $0 \leq l \leq m - 1$. This represents an element of the group $G \ast \langle x \rangle$, where $\langle x \rangle$ is the cyclic free group over $x$. It is easy to check that if $w$ and $w'$ correspond to the same group element then the equations $w = 1$ and $w' = 1$ have the same set of solutions. Hence, one can always assume that the above word is reduced, i.e. $\alpha_l \neq 1$ whenever $e_l + e_{l+1} = 0$. We say that an equation is trivial if $w$ represents the trivial element in $G \ast \langle x \rangle$.

A system of equations (inequalities) is just the conjunction of finitely many equations (inequalities). It can be checked that the collection of all sets of solutions of systems of inequalities over $F$ or the induction step, assume $f$ strongly unbounded and $w$ reduced, i.e. $w \neq 1$ whenever $e_l + e_{l+1} = 0$. We say that an equation is trivial if $w$ represents the trivial element in $G \ast \langle x \rangle$.

Definition 9.1. Let $\alpha \in \text{Aut}(\mathcal{M})$. We say $\alpha$ is strongly unbounded if for every finite subset $A$ of $M$ and $b \in M \setminus \text{acl}(A)$, there is a realization $c \models \text{tp}(b/A)$ such that $\alpha(c) \notin \text{acl}(cA)$.

Remark 9.2. Notice that being strongly unbounded is a strictly weaker notion than moving maximally in the sense of [9] (and almost moving maximally in [7]).

The following is a generalization of a classical argument for finding embeddings of free groups into automorphism groups of $\omega$-categorical structures (see [14], Prop. 4.2.3).

Lemma 9.3. Suppose $\mathcal{M}$ is a countable $\omega$-saturated first order structure in which $\text{acl}$ is locally finite and put $G := \text{Aut}(\mathcal{M})$. Assume $\alpha$ is a finite tuple of automorphisms of $\mathcal{M}$ where $\alpha_i$ is either 1 or strongly unbounded and $w(x, \alpha) := a_0x^{e_0}a_1 \cdots x^{e_m-1}a_m$ a reduced word in one variable. Then the set of solutions of the equation $w = 1$ is meager in $(G, \tau_{st})$.

Proof. As remarked above, the set of solutions of $w(x, \alpha) = 1$ is closed in any group topology. We want to show it has empty interior. Aiming for contradiction suppose that is not the case. Up to performing a change of variable of the form $x \mapsto x\gamma$ we can assume that there is a finite subset $B$ such that $w(G_{B}, \alpha) = 1$.

We will construct inductively a chain of elementary maps $\text{id}_B = f_{m+1} \subseteq f_m \subseteq \cdots \subseteq f_0$ together with $a \in M$ and $c_k := \alpha_k f_k^e \alpha_{k+1} \cdots f_m^e \alpha_{m+1}(a)$ for $0 \leq k \leq m + 1$ with the property that:

$$c_k \notin \text{acl}(\text{dom}(f_k^{e_k-1}));$$

for $1 \leq k \leq m + 1$ and $c_0 \neq a$. This finishes the proof. Indeed, given any extension $\beta \in G_{0}$, clearly $\beta \in G_B$ but $w(\beta, \alpha)(a) = c_0 \neq a$.

We start by choosing any $a \in \alpha_{m+1}^{-1}(M \setminus \text{acl}(B))$ so that $c_{m+1} = \alpha_{m+1}(a) \notin \text{acl}(B) = \text{acl}(\text{dom}(f_{m+1}))$. For the induction step, assume $f_{k+1}$ has been successfully constructed for some $2 \leq k \leq m$. We want to extend it to a map $f_{k}$ satisfying $(\dagger)$. Let $D_k = \text{dom}(f_k^{e_k})$ and $q(x, y) = \text{tp}(c_{k+1}, D_k)$. Let $q(x) := q(x, D_k)$ and $p'(x) := q(x, f_k^{e_k}(D_k))$. For any realization $e \models p'(x)$ the map $g'_e$ defined by $g'_e = f_k^{e_k} \cup \{(c_{k+1}, e)\}$ is elementary by construction. Our goal is thus to show that for some such $e$ if we let $f_k = g_e$ then the resulting $c_k \in M$ satisfies both $(\dagger)$ and $c_k \neq a$. Notice that by the induction hypothesis $c_{k+1} \notin \text{acl}(D_k)$, i.e. $p(x)$ is non-algebraic and hence so is $p'(x)$. Since by assumption $\mathcal{M}$ is $\omega$-saturated, $p'(x)$ has infinitely many realizations in $M$. There are two different scenarios to consider.
If $\epsilon_k = \epsilon_{k-1}$, then take $e \models p'$ with $e \notin \{a\} \cup \alpha_{k-1}^{-1}(\text{acl}(D_k c_{k+1}))$. This is possible by the observation of the last paragraph and the local finiteness of $\text{acl-}$. Taking $f_k := g_e$ we obtain:

$$c_k = \alpha_k(f_k^{e_k}(c_{k+1})) = \alpha_k(e) \notin \text{acl}(D_k c_{k+1}) = \text{acl}(\text{dom}(f_k^{e_k})) = \text{acl}(\text{dom}(g_e^{e_k})) = \text{acl}(\text{dom}(f_k^{e_k})).$$

Consider now the case $\epsilon_k = -\epsilon_{k-1}$. Since $w$ is reduced, this implies that $\alpha_k \neq 1$ and thus, by assumption, that $\alpha_k$ is unbounded. The type $p'(x)$ is non-algebraic with parameters in $D' := f_{k+1}^{e_{k+1}}(D_k)$. Unboundedness implies there exists a realization $e$ of $p'(x)$ such that $\alpha_k(e) \notin \text{acl}(D'e)$. But $D'e = \text{im} f_k^{e_k} = \text{dom} f_k^{e_k-1}$, hence condition (i) follows for $c_k = \alpha_k(e)$ as well. In the last step all we have to do is to choose $e \models p'$ such that $\alpha_0(e) \neq a$. This is clearly possible by the fact that $p'(x)$ is non-algebraic. 

Remark 9.4. The actual sufficient condition given by the proof is that no two occurrences of opposite sign of $x$ in $w$ are separated by non-strongly unbounded element from the group. In particular, words involving only positive powers of $x$ have always meager sets of solutions.

Lemma 9.5. Suppose $M$ is a countable first order structure such that the solution sets of all non-trivial equations of the form $w(x, \alpha)$ are meager in $G = \text{Aut}(M)$ with respect to the standard topology. Then $\tau_Z$ is not a group topology for $G$.

Proof. Indeed, fix $\alpha \in G$ and consider the equation $z \alpha^{-1} = 1$ in $G$, where $\alpha \in G$. Now, suppose we are given two systems of inequalities in one variable:

$$\Sigma(x, \beta) \neq 1;$$
$$\Pi(y, \beta) \neq 1;$$

where $\beta = (\beta_1, \ldots, \beta_k) \in G^k$ is the tuple of parameters appearing in the two systems, i.e., the non-trivial elements of $G$ appearing in the corresponding normal forms. Consider the system of inequalities:

$$\{\Sigma(x, \beta) \neq 1\} \cup \{\Pi'(x, \beta') \neq 1\};$$

where $\Pi'(x, \beta')$ is the system obtained from $\Pi(y, \beta) \neq 1$ by replacing $y$ with $x^{-1} \alpha$ (the substitution corresponds to an automorphism of $G * \langle x \rangle$, so this is still a non-trivial system of equations) and $\beta'$ the updated superset of parameters. Given a solution $x_0$ of $\Sigma(x, \beta) \neq 1$ and $\Pi'(x, \beta') \neq 1$ the pair $(x_0, x_0^{-1} \alpha)$ belongs to the neighbourhood defined by the systems $\Sigma \neq 1$ and $\Pi \neq 1$ but their product satisfies the initial equation $z \alpha^{-1} = 1$. Note that in a topological group any finite conjugation of group action is continuous and the pre-image of a nowhere dense set is nowhere dense. Hence the conclusion above finishes the proof.

Combining Lemma 9.3 and Lemma 9.5 one gets the following:

Corollary 9.6. Suppose $M$ is a countable homogeneous first order structure in which algebraic closure is locally finite. Assume all non-trivial automorphims of $M$ are strongly unbounded. Then $\tau_Z$ is not a group topology for $\text{Aut}(M)$.

There is another consequence of the meagerness of solution sets of equations worth mentioning. We start with the observation that the multivariate case follows from the univariate case.

Lemma 9.7. Let $(G, \tau)$ be a non-meager Polish group. If the set of solutions of any non-trivial equality in one variable with parameters in $G$ is meager in $G$ then the same holds for non-trivial equalities with parameters in any number of variables.

Proof. Let $w(x, \alpha) = 1$ be the equation in question, where $x = (x_0, x_1, \ldots, x_k) = (x_0, y)$ by induction. For each value of $y_0 := (y_0^0, \ldots, y_0^k)$ consider the term in $x_0$ obtained by replacing each $x_j$ by the element $x_j^y$ for $j \geq 1$ in $w(x, \alpha)$ and then merging together all consecutive constants. If all the resulting products that lay between two consecutive occurrences of $x_0^y$ with opposite exponents are non-zero then the resulting expression is already reduced and is a non-trivial inequality in $x_0$ (which without loss of generality appeared in the original expression). Therefore for such $y_0$ comeagery many
values of $x_0$ satisfy the equation, by the single variable case. Now, the condition above can be expressed as a system of finitely many non-trivial inequalities in the variable $y$ and hence holds for comeagerly many values of $y$ by the induction hypothesis. 

Using the Baire Category Theorem one can derive the following corollary:

**Corollary 9.8.** Let $(G, \tau)$ be a non-meager Polish group such that the set of solutions of any non-trivial equation in one variable with parameters in $G$ is meager. Then for any countable subgroup $A \subseteq G$ there exists some free group $F \leq G$ over a countable base such that $(A, F) \cong A * F$.

9.1. **Fraïssé constructions.** Suppose $M$ is a countable first order structure. Let $G = \text{Aut}(M)$ and for any $\alpha \in G$ define $\text{Supp}(\alpha) := \{m \in M \mid \alpha(m) \neq m\}$.

Recall the setting from Subsection 2.1; namely $\mathfrak{L}$ is a relational signature and $\mathcal{K}$ is a class of finite $\mathfrak{L}$-structures. Suppose $A, B, C$ are $\mathfrak{L}$-structures with $A, B \subseteq C$. Let $B' \subseteq B \setminus A$. By $\Delta^C(B'; A)$ we mean the set of all positive Boolean combinations of all $\phi(b, a)$ where $b \subseteq B'$, $a \subseteq A$ and $\phi$ is an atomic $\mathfrak{L}$-formula.

9.1.1. **Free amalgamation classes.** For a definition of free amalgamation classes see Subsection 2.1. First, we remind the reader the following fact about strong amalgamation classes (hence also about free amalgamation classes).

**Fact 9.9.** Suppose $\mathcal{K}$ is a Fraïssé class with strong amalgamation. Then $\text{acl}_M(A) = A$ for all $A \subseteq M$ where $M = \text{Flim}(\mathcal{K})$.

From Corollary 2.10 in [15] one can easily conclude the following.

**Fact 9.10.** Suppose $\mathcal{K}$ is a Fraïssé class with the free amalgamation property and $M = \text{Flim}(\mathcal{K})$. Assume $\text{Aut}(M)$ is transitive and $\text{Aut}(M) \neq S_\infty$. Then every non-trivial automorphism of $\mathcal{M}$ where $\mathcal{M} = \text{Flim}(\mathcal{K})$ is strongly unbounded.

Here we introduce a more general setting for Fraïssé classes that include the setting of [15]. Then Lemma 9.12 is a mild generalisation of Corollary 2.10 in [15] which we give a complete proof here.

**Definition 9.11.** Suppose $\mathcal{K}$ is a Fraïssé class. We say $\mathcal{K}$ is non-discrete (ND) if there is $m \in \mathbb{N}$ (where $\min \{n_R : R^n \in \mathfrak{L}\} \leq m - 2 \leq \max \{n_R : R^n \in \mathfrak{L}\}$) such that for every $A \in \mathcal{K}$ with $|A| \geq m$, and $a_1, a_2 \in A$ with $a_1 \neq a_2$ and $A' \subset A \setminus \{a_1, a_2\}$ with $|A'| = m - 2$ there is $B \in \mathcal{K}$ such that

1. $B = A \coprod \{b\}$;
2. There is $\phi(ba_1, s) \in \Delta^B(ba_1, A')$ such that $B \models \phi(ba_1, s)$ but $B \models \neg \phi(ba_2, s')$ for all $s' \subseteq A$ where $s' \cong s$.

We call $B$ a non-discrete one-point extension of $A$ of the form $a_1 \triangleright_A^\alpha a_2$. Occasionally, we use $m$-ND when we want to specify the cardinality of $|A|$.

Notice that in the definition above if $B \subseteq \text{Flim}(\mathcal{K})$, $\alpha \in \text{Aut}(\text{Flim}(\mathcal{K}))$ where $\alpha^{\pm 1}(a_1) = a_2$ and $\alpha$ fixes $A$ setwise, then $d \in \text{Supp}(\alpha)$.

**Lemma 9.12.** Suppose $\mathcal{K}$ is an ND Fraïssé class with the free amalgamation property. Then every non-trivial automorphism of $\mathcal{M}$ where $\mathcal{M} = \text{Flim}(\mathcal{K})$ is strongly unbounded.

**Lemma 9.13.** Suppose $\alpha$ is an automorphism of a first order structure $\mathcal{M}$ where the algebraic closure in $\mathcal{M}$ is trivial. Assume for every finite subset $A$ of $\mathcal{M}$ and $b \in \mathcal{M} \setminus A$ there is a realisation $c$ of $\text{tp}(b/A)$ such that $c \in \text{Supp}(\alpha)$. Then $\alpha$ is strongly unbounded.

**Proof.** Given a finite subset $A$ of $\mathcal{M}$ and $b \in \mathcal{M} \setminus A$, we prove the set $T_b := \{r \in \mathcal{M} \mid r \in \text{Supp}(\alpha), r \models \text{tp}(b/A)\}$ is infinite. Assuming $T_b$ is infinite, because the algebraic closure is trivial, there is a realisation $c \in T_b$ such that $\alpha(c) \notin cA$. This shows $T_b$ is strongly unbounded.

Now we show $T_b$ is infinite. Put $p_0 = \text{tp}(b/A)$. By the assumption there is $r_0 \models p_0$ where $r_0 \in \text{Supp}(\alpha)$. Now consider $p_1 := \text{tp}(s_1/r_0A)$ where $s_1 \models p$ and $s_1 \neq r_0A$ (and such $s_1$ exists because $r_0 \notin A = \text{acl}(A)$). Let $r_1 \models p_1$ where $r_1 \in \text{Supp}(\alpha)$ again using the assumption. Clearly $r_1 \models p_0$. 

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Inductively, we can build $p_i$’s and find realisation $r_i$’s for $i \in \mathbb{N}$ which $r_i \models p_j$ and $r_i \in \operatorname{Supp}(\alpha)$ when $i \geq j$. Hence we have infinitely many realisation of $p_0$ in $\operatorname{Supp}(\alpha)$ and hence $T_b$ is infinite.

By applying Corollary 9.6, we conclude the following.

**Corollary 9.14.** Assume $\mathcal{K}$ is a non-trivial free amalgamation class. Let $\mathcal{M} = \operatorname{Flim}(\mathcal{K})$ and $G = \operatorname{Aut}(\mathcal{M})$ and assume $G$ acts transitively on $\mathcal{M}$. Then $\tau_Z$ is not a group topology.

**Proof.** Since $\mathcal{K}$ is non-trivial there exists some $m \geq 1$ such that the action $G$ on $\mathcal{M}$ is $(m - 1)$-transitive (let us say every action is 0-transitive) but not $m$-transitive. Our second hypothesis implies that in fact $m \geq 2$.

This implies that any substructure of size at most $m - 1$ in $\mathcal{K}$ is essentially a pure set in the sense that the type of a tuple enumerating all of its members without repetitions does not depend on the order, but that this pure set of size $m - 1$ extends to some $A \in \mathcal{K}$ which this is not the case anymore.\(^1\)

Consider now any $B \in \mathcal{K}$ with $B \geq m - 1$ and any $b_1, b_2 \in B$. Pick $B' \subset B$ with $|B'| = m - 1$ where $b_1 \in B', b_2 \notin B'$ and let $A = B' \cup \{c\}$ be a non-pure extension of $B'$. Then $A \otimes_{B'} B \in \mathcal{K}$ is a non-discrete extension of $B$ of the form $b_1 \triangleleft_{B'}^j b_2$, since no relation holds for a tuple containing both $b_2$ and $c$. From Lemma 9.13 follows that $\mathcal{K}$ is an ND Fraïssé class. The final conclusion then follows from Corollary 9.6.

\(\square\)

**Remark 9.15.** The assumption of ND in Lemma 9.12 is a necessary condition in order to conclude that every non-trivial automorphisms of a Fraïssé limits of free amalgamation classes is strongly unbounded. Notice that by the result of Gaughan in [9], for the Fraïssé class all finite (in empty signature) $\tau_Z = \tau_{st}$ and hence $\tau_Z$ is a group topology on $S_\infty$.

9.1.2. **Rational Urysohn spaces.** Consider the distance monoids $\mathcal{Q} = (\mathbb{Q}^{>0}, +, \leq, 0)$ and $\mathcal{Q}_b = (\mathbb{Q} \cap [0, b], +_b, \leq, 0)$ for $b \in \mathbb{Q}^{>0}$ where $+_b$ is addition truncated at $b$. Let $\mathcal{U}_\mathcal{Q}$ and $\mathcal{U}_{\mathcal{Q}_b}$ be the corresponding Urysohn space respectively – see Section 5. They are precisely the classical rational Urysohn space and rational Urysohn $b$-spheres (or sometimes bounded rational Urysohn space). Here we prove $\tau_Z$ for the automorphism groups of $\mathcal{U}_\mathcal{Q}$ and $\mathcal{U}_{\mathcal{Q}_b}$ are not group topologies.

We briefly discuss how rational Urysohn space and rational Urysohn spheres are constructed in first order logic as Fraïssé limits. Fix $\mathcal{R} \in \{\mathcal{Q}, \mathcal{Q}_b \mid b \in \mathbb{Q}^{>0}\}$. Let $\mathcal{L}$ be the first-order language with a binary relation $R_\mathcal{Q}(x, y)$ for each $q \in \mathcal{R}$. A metric space $(A, d)$ with $\mathcal{R}$-rational distances is an $\mathcal{L}$-structure in the following manner: for $x, y \in A$ and $q \in \mathcal{R}$ we have $R_q(x, y)$ if $d(x, y) \leq q$. Let $\mathcal{C}_\mathcal{R}$ be the class of all finite metric spaces with $\mathcal{R}$-rational distances as $\mathcal{L}$-structures.

**Proposition 9.16.** The class $\mathcal{C}_\mathcal{R}$ where $\mathcal{R} \in \{\mathcal{Q}, \mathcal{Q}_b \mid b \in \mathbb{Q}^{>0}\}$ has the amalgamation property.

Let $\mathcal{U}_\mathcal{R}$ be the corresponding Fraïssé limit. On easy fact is the following:

**Lemma 9.17.** The class $\mathcal{C}_\mathcal{R}$ where $\mathcal{R} \in \{\mathcal{Q}, \mathcal{Q}_r \mid r \in \mathbb{Q}^{>0}\}$ is non-discrete.

**Proof.** We prove $\mathcal{C}_\mathcal{R}$ is 2-ND. Suppose $A \in \mathcal{C}_\mathcal{R}$ where $|A| \geq 2$. Let $a_1, a_2 \in A$ be two elements which we want to separate by a one-point extension. Let $q = d(a_1, a_2)$ and consider $B = \{a_1, a_2, b\}$ to be an $\mathcal{L}$-structure with $d(a_1, a_2) = q$ and $d(a_1, b) = \frac{q}{2}$ and $\frac{q}{2} + \epsilon$ where $\epsilon \in (0, \frac{q}{2})$. It is easy to check $B$ is a metric space with rational distance and its diameter is $q$ hence $B \in \mathcal{C}_\mathcal{R}$. Note that $B$ is a one-point non-discrete extension of $a_1 a_2$ that separates $a_1$ and $a_2$. Now the amalgamation of $A$ and $B$ over $a_1 a_2$ is the one-point extension of $A$ which we are looking for.

**Lemma 9.18.** Non-trivial automorphisms of $\mathcal{U}_\mathcal{Q}$ and $\mathcal{U}_{\mathcal{Q}_b}$ for $b \in \mathbb{Q}^{>0}$ are strongly unbounded.

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\(^1\)We may assume no relation holds for a constant tuple. No positive atomic formula can hold for all tuples of distinct elements of size $\geq 2$ because of free amalgamation.
Proof. By Lemma 9.17 the class $\mathcal{C}_R$ is 2-ND when $R \in \{Q, Q_b \mid b \in \mathbb{Q}^{>0}\}$. Let $\alpha$ be a non-trivial automorphism of $U_R$. One can easily check that the algebraic closure is trivial in $U_R$. Hence based on Lemma 9.13 it is enough to show for a given finite subset $A$ of $U_R$ and $r \in U_R \setminus A$ there is a realization of $p := \text{tp}(r/A)$ in $\text{Supp}(\alpha)$.

First note that $\text{Supp}(\alpha)$ is infinite and hence one can find distinct $s, s_1 \in U_R$ such that $s_1 = \alpha(s)$ and $ss_1 \cap A = \emptyset$. Consider the metric space $ss_1 \setminus A$. If $d(r, s) \neq d(r, s_1)$ then $r$ separates $s$ and $s_1$ and hence $r \in \text{Supp}(\alpha)$. Assume now $d(r, s) = d(r, s_1)$. Consider a one-point extension $C = \text{Ass}_s \cup c$ of $\text{Ass}_s$ with the following properties:

1. $d(c, a) = d(c, s) = l$ for all $a \in A$ where $l \in \mathcal{R}$ and $\frac{\text{diam}(\text{Ass}_s)}{2} < l < \text{diam}(U_R)$;
2. $d(c, s_1) = l'$ where $l' \in \mathcal{R}$, $l' > l$ and $l' - l \leq \min\{d(s, s_1), d(s_1, a) \mid a \in A\}$.

It is easy to check $C \in \mathcal{C}_R$ and any realization of $\text{tp}(c/\text{Ass}_s)$ is in $\text{Supp}(\alpha)$. Since there are infinitely many realizations of $\text{tp}(c/\text{Ass}_s)$ we can assume $c, \alpha(c) \notin Ar$. Rename $c$ and $\alpha(c)$ to $s$ and $s_1$, respectively. Now looking at $D_1 := sA$ and $D_2 := rA$ we can amalgamate them over $A$ if the following distance between $r$ and $s$ in $D_1D_2$ holds:

$$\max\{|l - d(r, a)| \mid a \in A\} \leq d(s, r) \leq \min\{l + d(r, a), \text{diam}(U_R) \mid a \in A\}.$$ 

Let $l_1 := \max\{|l - d(r, a)| \mid a \in A\}$ and $l_2 := \min\{l + \mu, \text{diam}(U_R)\}$. Clearly $l_1 < l_2$. In $D_1D_2$ assigning $d(r, s) = t$ for $t \in (l_1, l_2) \cap \mathcal{R}$ satisfies the triangle inequality.

Choose $t \in (l_1, l_2) \cap \mathcal{R}$ and let $D_1D_2$ be the structure such that $d(r, s) = t$. With abuse of notation call one of its isomorphic copies over $A$ again $D_1D_2$ and $b$ be the element that has the same type as $p$. If $d(r, s) \neq d(r, s_1)$ we are done.

Suppose again $d(r, s) = d(r, s_1) = t$. We claim the following structure is a metric structure: Let $E = c\text{Ass}_s$ where $\text{tp}(c/A) = p$ and assign $d(c, s_1) = t$ and $d(c, s) = t'$ where $t < t'$, $t' \in (l_1, l_2) \cap \mathcal{R}$. In order to have $c\text{Ass}_s$ as a metric space we only need to have the triangle inequality for $(c, s, s_1)$ and for that we choose $t' \in (l_1, l_2) \cap \mathcal{R}$ in such a way that $t' - t < d(s, s_1)$. That is always possible and hence we conclude every non-trivial automorphism is strongly unbounded. \hfill \Box

Now by applying Corollary 9.6 we conclude

Corollary 9.19. The Zariski topology $\tau_Z$ for $\text{Aut}(U_Q)$ and $\text{Aut}(U_{Q_b})$ for $b \in \mathbb{Q}^{>0}$ is not a group topology.

9.1.3. Random tournament. A tournament is a digraph where there is exactly one edge between every pair of vertices. According to Lachlan’s classification there are only three countable homogeneous tournaments up to isomorphism: the dense linear order on the rational numbers $\mathbb{Q}$, the dense local order $S(2)$, and the tournament $T^\infty$ that is universal for the set of all finite tournaments. It is easy to check that the classes of finite substructure of homogeneous tournaments (also homogeneous digraphs) are 2-ND. Moreover, the algebraic closure in all three cases is trivial. One can show the following in this case:

Lemma 9.20. Given a non-trivial automorphism $\alpha \in \text{Aut}(T^\infty)$ and any finite subset $A$ of $T^\infty$ and $b \in T^\infty \setminus A$ there is a realization $c \in \text{tp}(b/A)$ such that $c \in \text{Supp}(\alpha)$.

Proof. One can easily show $\text{Supp}(\alpha)$ is infinite when $\alpha$ is non-trivial. Then consider $t \in \text{Supp}(\alpha)$ where $t, \alpha(t) \notin A$. Let $B = A \cup \{t, \alpha(t)\}$ and consider the tournament $C = B \cup \{c\}$ where $c \notin B$, and $c$ and $t$ has the same qf-type over $A$ and moreover $c$ relates to $t$ and $\alpha(t)$ with an opposite direction. Then $C$ is a tournament and $B \subseteq C$ and by the theorem of Fraïssé one can find a copy of $C$ in $M$ over $B$. With abuse of notation denote it again by $C$. Then $c \in \text{Supp}(\alpha)$. \hfill \Box

Then from Lemma 9.13 and Corollary 9.6 we conclude

Corollary 9.21. The Zariski topology $\tau_Z$ in $\text{Aut}(T^\infty)$ is not a group topology.

However, it is not hard to see that if $\mathcal{M} \in \{Q, S(2)\}$ then there are non-trivial automorphisms of $\mathcal{M}$ which are not strongly unbounded.
9.2. Products of Fraïssé classes. Given two distinct elements \(a, b\) of an \(\omega\)-categorical structure \(M\) such that \(tp(a) = tp(b)\) and any \(k \geq 1\) denote by \(\Delta_{a,b}^k(x)\) the formula over \(x = (x_i)_{i=1}^k\) stating that \(tp(x/a) \neq tp(x/b)\) and \(x \cap \{a, b\} = \emptyset\). Notice that for any \(\alpha \in \text{Aut}(M)\) if \(b = \alpha(a)\) then at least one component of any realisation of \(\Delta_{a,b}^k(x)\) must belong to \(\text{Supp}(\alpha)\). We denote by \(\Delta_{a,b}^{k,L'}\) the result of calculating \(\Delta\) in \(M \models \varphi\), where \(L' \subset L\). We say that a Fraïssé class \(K\) in a relational signature \(L\) is discriminating if for each pair of distinct elements \(a, b \in M = \text{Flim}(K)\) there exists \(k \geq 1\) such that the formula \(\Delta_{a,b}^k(x)\) is non-algebraic.

**Observation 9.22.** Let \(K\) be a non-trivial Fraïssé class in a finite relational signature and \(M = \text{Flim}(K)\). Then \(K\) is discriminating provided one of the following holds:

- \(M\) has trivial algebraic closure and the action of \(\text{Aut}(M)\) on \(M^2 \setminus \{(a, a)\}_{a \in M}\) is transitive; or,
- \(K\) has free amalgamation and \(\text{Aut}(M)\) acts transitively on \(M\).

**Proof.** For the first part, take some \(R^{(k)} \in L\) holding for some \(k\)-tuples but not for others. This implies the existence of \(a, b \in M^k\) (with pairwise distinct coordinates) differing only in one coordinate \(a_i \neq b_i\) and such that \(tp(a) \neq tp(b)\). This implies \(\Delta_{a,b}^{k-1}(x)\) is non-algebraic. By our transitivity condition, it follows that \(\Delta_{a,b}^{k-1}(x)\) is non-algebraic for any distinct \(c, d \in M\). The second part follows from the proof of Corollary 9.14. □

We say \(K\) is dense if for any distinct \(a, b \in M = \text{Flim}(K)\) and any non-algebraic 1-type \(p\) over finitely many parameters of \(M\) isolated by a formula \(\varphi(x)\) the formula \(\Delta_{a,b}^k(x) \wedge p(x)\) is not algebraic.

**Definition 9.23.** Given two Fraïssé classes \(K_1\) and \(K_2\) over finite relational languages \(L_1\) and \(L_2\), respectively, define \(K_1 \otimes K_2\) to be the class of \(L\)-structures \(A\) where \(L = L_1 \coprod L_2\) and \(A \models_{L_1} K_1\) and \(A \models_{L_2} K_2\).

**Lemma 9.24.** For \(i = 1, 2\) let \(K_i\) be a Fraïssé class over a finite relational language \(L_i\) such that \(K = K_1 \otimes K_2\) is a Fraïssé class. Assume that:

- \(K_1\) is discriminating;
- \(K_2\) is dense;

and let \(M = \text{Flim}(K)\). Then any non-trivial element of \(G = \text{Aut}(M)\) is strongly unbounded.

**Proof.** Take \(\alpha \in G \setminus \{1\}\) and some non-algebraic formula \(\varphi(z, a)\) in one variable over a finite tuple \(a\) of parameters which isolates a non-algebraic type. We can write \(\varphi = \phi_1 \wedge \phi_2\), where \(\phi_1\) is a quantifier free formula in the language \(L_1\). Suppose that \(\alpha(c) = c'\) for distinct \(c, c' \in M\). The fact that \(K_1\) is discriminating implies there is some \(k\) such that the formula \(\Delta_{a,c, c'}^{k,L_1}(x)\) is non-algebraic. In particular, it can be realized in \(M \models_{L_1}\) by some tuple \(d\) disjoint from \(a\). Since the \(L_2\) formula \(\varphi_2(x, a)\) is non-algebraic, it is possible to find such \(d\) in \(M\) with the property that all of its entries satisfy \(\varphi_2(z, a)\). On the other hand, \(\alpha(d_i) \neq d_i\) for some \(d_i\). Density of \(K_2\) then implies \(\varphi_2(z, a) \wedge \Delta_{d, d', d''}^{1,L_2}(z)\) is not algebraic. Therefore, neither is \(\varphi(x) \wedge \Delta_{d, d', d''}^{1,L_2}(z)\) (again, by quantifier elimination and the definition of \(K_1 \otimes K_2\)). This implies that \(\varphi(M) \cap \text{Supp}(\alpha)\) is non-empty. □

We collect below a handful of particular cases of Lemma 9.24.

**Corollary 9.25.** Let \(K_1\) and \(K_2\) be two Fraïssé classes with strong amalgamation and \(K = K_1 \otimes K_2\). Assume \(K_1\) is non-trivial and satisfies one of the following:

- The action of \(\text{Aut}(\text{Flim}(K_1))\) on the set \(M^2 \setminus \{(a, a)\}_{a \in M}\) is transitive;
- \(K_1\) has free amalgamation and the action of \(\text{Aut}(\text{Flim}(K_1))\) on \(\text{Flim}(K_1)\) is transitive.

Assume also \(\text{Flim}(K_2)\) one of the following:

- \((\mathbb{Q}, <)\);
- The countable dense meet tree;
- The cyclic tournament \(S(2)\).

Then the solution set of any non-trivial equation with parameters in \(G\) is meager.
9.3. Hrushovski’s pre-dimension construction. Recall the setting in subsection 2.2. Suppose $s \geq 2$ and $\eta \in (0,1]$. Let $C_{\eta} := \{B \in C \mid \emptyset \subseteq B \}$ and $M^0$ be the countable structure that one obtains from Proposition 2.2.

Suppose $A$ is a finite subset of $M^0$. Using the pre-dimension function $\delta$ one can define the dimension of $A$ as $d(A) := \delta (cl(A))$, where $cl(A)$ is the smallest $\leq$-closed finite subset of $M^0$ that contains $A$. Given $b \in M^0$ and $A$ a finite subset of $M^0$, we denote $d(b/A)$ for $d(bA) - d(A)$. From part (2) of Lemma 2.1 and part (2) Proposition 2.2 it follows that $cl(A)$ is well-defined. Similarly to Lemma 9.3 we prove the following

Lemma 9.26. Suppose $\alpha$ is a finite tuple of automorphisms of $G = Aut(M^0)$. Then the set of solutions of a non-trivial equation $w(x,\alpha) := \alpha_0 x^{\alpha_0} \alpha_1 \cdots x^m \alpha_{m+1} = 1$ is meager in $G$.

In order to prove Lemma 9.26 we recall some facts about non-trivial automorphisms of $M^0$. Recall that $\alpha \in Aut(M^0)$ is gcl-bounded if there exists a finite subset $B$ of $M^0$ such that $m \in gcl(mB)$ for all $m \in M^0$ where $gcl(X) := \{m \in M^0 \mid d(m/X) = 0\}$. One can define an independence notion $\perp^d$ between finite subsets of $M^0$ using the dimension function; namely $A \perp^d B$ if $d(A/B) = d(A/BC)$ where $A, B$ and $C$ are finite subsets of $M^0$. It turns out $\perp^d$ is indeed the forking-independence in $M^0$ and for simplicity we denote it by $\perp$ in $M^0$ and remove the superscript $d$. From Lemma 3.2.27 and Theorem 3.2.29 in [10] follows:

Proposition 9.27. For every non-trivial automorphism $\alpha \in Aut(M^0)$ and $X, Y \subseteq C_{\eta}$ where $X \subseteq Y$ and $Y \cap gcl(X) = X$, there is $Y'$ where $tp(Y'/X) = tp(Y/X)$ and $Y' \perp \alpha(Y')$.

One can modify the definition of strongly unbounded to strongly gcl-unbounded such that for an automorphism of a structure that gcl is well-defined. Namely $\alpha$ is strongly gcl-unbounded if for every finite set $A$ and $b \in M \setminus gcl(A)$ there is a realization $c \in M$ of $tp(b/A)$ where $\alpha(c) \notin gcl(cA)$. Proposition 9.27 implies immediately the following.

Fact 9.28. All non-trivial automorphisms of $M^0$ are strongly gcl-unbounded.

It has to be remarked that Proposition 9.27 is proving something stronger than just that non-trivial automorphisms are strongly gcl-unbounded.

Proof of Lemma 9.26. Let $G = Aut(M^0)$. We want to show the set of solutions of a non-trivial equation $w(x,\alpha) = \alpha_0 x^{\alpha_0} \alpha_1 \cdots x^m \alpha_{m+1} = 1$ has empty interior in $G$. From Fact 9.28 all the non-trivial automorphisms are strongly gcl-unbounded.

Essentially the same arguments of the proof of Lemma 9.3 works and we only need to replace acl by gcl and apply Fact 9.28 when $\alpha_i$’s are non-trivial.

In order to show the starting point of the argument we provide some details and leave the rest (avoiding a repetition). We follow closely the proof of Lemma 9.3. Aiming for contradiction suppose that is not the case. Again up to performing a change of variable of the form $x \mapsto x^\gamma$ we can assume that there is a finite $\leq$-closed subset $B$ such that $G_B \subseteq w(G,\alpha)$. We will construct inductively a chain of partial isomorphisms $id_B = f_1 \subseteq f_2 \subseteq \cdots \subseteq f_0$ together with $a \in M^0$ with the property that for $0 \leq k \leq m$ + 1

$$c_k := \alpha_k f_k^e \alpha_{k+1} \cdots f_k^m \alpha_{m+1}(a) \notin gcl(dom(f_k^e));$$

where $c_k \neq a$ when $k \neq m + 1$.

The starting point is choosing $a$ any element in $M^0 \setminus gcl(B)$. We have two possibilities: If $\alpha_{m+1}$ is 1, then let $c_{m+1} = a$. Suppose $\alpha_{m+1}$ is strongly gcl-unbounded. Using Fact 9.28 and Proposition 9.27 set $c_{m+1} := \alpha_{m+1}(c)$ where $c \models tp(a/B)$ and $\alpha_{m+1}(c) \notin gcl(cB)$ and rename $c$ to $a$. Assume now we have successfully constructed $f_k$. Then the same arguments of the proof of Lemma 9.3 work only replacing acl by gcl and apply Fact 9.28 when $\alpha_i$’s are non-trivial.

Then from Lemma 9.26 and Theorem E follows

Corollary 9.29. The Zariski topology $\tau_Z$ for $Aut(M^0)$ is not a group topology.
9.4. Some cases when the Zariski topology is a group topology. By a family of generalized intervals in \( \mathcal{M} \) we mean a \( G \)-equivariant collection of data consisting of a collections \( \mathcal{I} \) of infinite subsets of \( \mathcal{M} \) and a map \( \lambda \) assigning to each \( I \in \mathcal{I} \) a set of 2 elements of \( \mathcal{M} \) with the following properties, where we write \( I^* = M \setminus (I \cup \lambda(I)) \):

(a) for \( I, J \in \mathcal{I} \), if \( \lambda(I) \subset J \) then either \( I \subset J \) or \( I^* \subset J \). The same holds with \( J^* \) in place of \( J \).

(b) for each \( I \in \mathcal{I} \) and every \( K \in \{I, I^*\} \) there exists some \( \alpha_K \in Aut(\mathcal{M}) \) such that \( \text{Supp}(\alpha_K) = K \);

(c) for any distinct \( x, y \in M \) there exists some \( I \in \mathcal{I} \) such that \( x \in I \) and \( y \notin I \);

(d) for any \( I \in \mathcal{I} \) and \( x \in I \) there exist \( I', I'', I_1, I_2 \in \mathcal{I} \) and contained in \( I \) with the following properties.

(i) \( x \in I'' \subset I' \);

(ii) \( I_1, I''_1, I_2 \) are disjoint;

(iii) \( \lambda(I') \cap I_i \neq \emptyset \) for \( i = 1, 2 \).

Given a structure \( \mathcal{M}, x \in M \) and \( I \subseteq M \) we write \( [x : I] = \{g \in Aut(\mathcal{M}) \mid gx \in I\} \).

Lemma 9.30. Let \( \mathcal{M} \) be a structure and \( G = Aut(\mathcal{M}) \). For any family of generalized intervals \( \mathcal{I} \) in \( \mathcal{M} \) the collection \( \{(x, I) \mid x \in I \in \mathcal{I}\} \) forms a sub-base of neighbourhoods of 1 for the Zariski topology on \( G \) and the latter is a group topology.

Proof. Given \( I_1, I_2 \in \mathcal{I} \) let \( \Lambda_{I_1, I_2} \) be the set of solutions in \( G \) of the inequality \( u_{I_1, I_2}(x) = [\alpha_{I_1}, \alpha_{I_2}] \neq 1 \). Clearly \( g^{-1}I_1 \cap I_2 \neq \emptyset \) for any \( g \in \Lambda_{I_1, I_2} \).

Let \( \Gamma_{I_1, I_2} \) be the intersection of all sets of the form \( \Lambda_{J_1, J_2} \), where \( J_i \in \{I_i, I_i^*\} \). Observe that if \( g \in \Lambda_{I_1, I_2} \) then both \( g^{-1}\lambda(I_1) \cap I_2 \) and \( g^{-1}\lambda(I_1) \cap I_2^* \) must be non-empty by (a).

We will now show that every set of the form \( [x : I] \) for \( x \in M \) and \( I \in \mathcal{I} \) is a neighbourhood of the identity in the Zariski topology. Let \( x \in M \) and \( I \in \mathcal{I} \) with \( x \in I \). Choose \( I', I'', I_1, I_2 \) as in (d) and consider the set \( \Theta = \Lambda_{I', I_1} \cap \Lambda_{I'', I_2} \cap \Gamma_{I', I''} \). Clearly \( 1 \in \Theta \). Now let \( g \in \Theta \). Since \( g \in \Lambda_{I', I_1} \cap \Lambda_{I'', I_2} \) we must have \( g^{-1}\lambda(I') \cap I_1 \neq \emptyset \) for \( i = 1, 2 \) and thus \( g^{-1}\lambda(I') \subseteq I_1 \cup I_2 \). Since \( (I_1 \cup I_2) \cap I'' = \emptyset \), then by (a) either \( g^{-1}I' \) contains \( I'' \) or has empty intersection with \( I'' \). Since \( g \in \Gamma_{I', I''} \), the former must be the case, which implies that \( x \in g^{-1}I' \), i.e. \( gx \in I' \subset I \). Hence \( \Theta \subseteq [x : I] \).

All that is left to show is that the neighbourhoods \( [x : I] \) for \( I \in \mathcal{I} \) form a basis of a Hausdorff group topology. Given \( x \) and \( I \in \mathcal{I} \), let \( I', I'', I_1, I_2 \) be as given by (d). Suppose that \( \lambda(I') = \{a_1, a_2\} \) with \( a_i \in I_i \). Since \( I' \cap I'' = \emptyset \), the argument above implies that for any \( h \in \{a_1 : I_1 \cap a_2 : I_2\} \) we have \( h(I') \subset I \). So \( \{(a_1 : I_1 \cap a_2 : I_2) : [x : I] \subseteq [x : I] \} \) and thus multiplication is continuous. Continuity of inversion can be checked in a similar way. Hausdorffness is straightforward from property (c). □

Definition 9.31. A tree is a partial order \( (T, \leq) \) for each \( t \in T \) the set \( \{s \in T \mid s \leq t\} \) is a linear order.

We say \( (T, \leq) \) is a meet tree if for every \( t_1, t_2 \in T \) the set \( \{s \in T \mid s \leq t_1, t_2\} \) has a greatest element which we denote it by \( \text{meet}(t_1, t_2) \). We say a meet tree \( (T, \leq) \) is dense if

- for any \( t \) the set \( \{s \in T \mid s \leq t\} \) is dense and has no first element;
- every point \( t \) is a meet of infinitely many pairs.

Theorem 9.32. The Zariski topology \( \tau_Z \) on \( Aut(\mathcal{M}) \) is a group topology in the following cases:

- Any non-trivial reduct of \( (\mathbb{Q}, <) \).
- The cyclic tournament \( S(2) \).
- The countable dense meet tree. In this case \( \tau_Z = \tau_{st} \).

Proof. In the case of a non-trivial reduct of \( (\mathbb{Q}, <) \) the collection \( \mathcal{I} \) consists of all bounded intervals in the structure and \( \lambda(I) \) are the two endpoints of \( I \).

For \( S(2) \) we can take as \( \mathcal{I} \) the collection of all the sets of the form \( \Delta_{a,b} = \{x \mid x \neq a \wedge x \neq b \land \neg E(a, x) \leftrightarrow E(x, b)\} \), where \( E(a, b) \). Notice that this consists of the union of an interval \((a, b)\) on the circle with its antipodal interval, which lacks a corresponding closed interval in the structure. Here we let \( \lambda(\Delta_{a,b}) = \{a, b\} \). This can be described as the union of two intervals on opposite sides of the circle.
For the dense meet tree as $I$ we take the collection of all sets the form $I(a, b) = \{a < \text{meet}(b, x) < b\}$ where $a < b$ and we let $\lambda(I(a, b)) = \{a, b\}$. Take incomparable $b_1, b_2$ and let $c = \text{meet}(b_1, b_2)$. Take $a < b$. Then $I(a, b_1) \cap I(a, b_2) = \{x \in M | a < x < c\}$. Note that $W := \{c : I(a, b_1) \cap I(a, b_2)\} \in \mathcal{N}_Z(1)$. Let now $g \in W \cap W^{-1}$. Since $g \in W$ the previous discussion implies that $gc \leq c$ and since $g^{-1} \in W$ we know that $g^{-1}c \leq c$, that is $c \leq gc$ so in fact $gc = c$. It follows that $W \cap W^{-1} \subseteq G_c$ and thus $\tau_Z = \tau_{st}$.

9.5. $\alpha$-minimality. Let $G$ be a group. The intersection of all Hausdorff topological groups structures on $G$ is called the Markov topology, denoted by $\tau_M$. The topology $\tau_M$ is always $T_1$ but not necessarily a group topology.

We say that a group $G$ is $\alpha$-minimal if $(G, \tau_M)$ is a topological group. Notice that if $\tau_Z$ is a group topology, then $\tau_M = \tau_Z$.

**Question 7.** For which (sufficiently homogeneous) structures $\mathcal{M}$ is $\text{Aut}(\mathcal{M})$ $\alpha$-minimal?

## 10. Topologies and types

Let $\mathcal{M}$ be a first order structure and $T = \text{Th}(\mathcal{M})$. Consider two tuples of variables $x = (x_m)_{m \in M}$ and $y = (y_m)_{m \in M}$ indexed by the elements of $M$. Given some finite tuple $a = (a_1, a_2, \ldots, a_k) \subseteq M$ we write $x_a$ in lieu of $(x_{a_1}, x_{a_2}, \ldots, x_{a_k})$. Let $p_M(x) = \text{tp}(M)$, where variable $x_m$ is made to correspond with $m \in M$. Let $R(\mathcal{M})$ stand for the collection of all $T$-complete types in variables $x, y$ containing $p_M(x) \cup p_M(y)$ and let $R^{pa}(\mathcal{M})$ for the collection of partial types in variables $x, y$ in $T$ containing $p_M(x) \cup p_M(y)$ (i.e., of all closed subsets of $R(\mathcal{M})$ in the logic topology). Here we assume types are deduction closed. Given any partial type $p(x, y)$ we will denote the deduction closure of $p(x, y) \cup p_M(x) \cup p_M(y)$ in $T$ as $\langle p \rangle$. The set $R^{pa}(\mathcal{M})$ can be endowed with the so-called logic topology, which we denote by $\tau_L$, generated by neighbourhoods of the form $\{\phi \in R(\mathcal{M}) | \phi \in p\}$, where $\phi$ is any formula in $(x, y)$. The result is Stone space.

Given $p^1, p^2 \in R^{pa}(\mathcal{M})$ we let $(p^1 \ast p^2)(x, y) \in R^{pa}(\mathcal{M})$ denote the collection of all formulas $\psi(x, y)$ such that there exist $\phi^1(x, y) \in p^1(x, y), i = 1, 2$ such that

$$\phi^1(x, z) \land \phi^2(z, y) \vdash \psi(x, y).$$

It can be checked that $\ast$ endows $R^{pa}(\mathcal{M})$ with a semigroup structure. If we let $0 = \langle \emptyset \rangle \in R^{pa}$ then clearly $p \ast 0 = 0$ for any $p \in R^{pa}$. We write $p \leq q$ for $p \vdash q$.

Given $p \in R^{pa}$, let $\bar{p} \in R^{pa}$ be defined by $\theta(x, y) \in \bar{p} \iff \theta(y, x) \in p$. Every $g \in \text{Aut}(\mathcal{M})$ is associated to some $\iota(g) = \{x_{ym} = y_{ym} | m \in M\} \in R^{pa}$. It can be easily checked that $\iota$ is a continuous homomorphic embedding of $(G, \tau_{st})$ into $(R^{pa}(\mathcal{M}), \tau_L)$ whose image is contained in $R(\mathcal{M})$. From now on we will write simply $g$ instead of $\iota(g)$. Notice that $p^2 := g^{-1} \ast p \ast g = \{\phi(x_a, y_b) | \phi(x_{g^a}, y_{g^b}) \in p\}$ for any $p \in R^{pa}$ and $g \in G$. Notice that $\ast$ is a continuous map $R^{pa}(\mathcal{M}) \times R^{pa}(\mathcal{M}) \to R^{pa}(\mathcal{M})$ and $p \mapsto \bar{p}$ continuous with respect to $\tau_L$. For the first, notice that given $p_1, p_2 \in R^{pa}(\mathcal{M})$ and a formula $\phi(x, y)$ with $p_1 \ast p_2 \in N_{\phi(x, y)}$, the definition of $\ast$ together with compactness implies the existence of $\phi_1(x, z) \in p_1$ and $\phi_2(z, y) \in p_2$ such that $T \cup \{\phi_1(x, z), \phi_2(z, y)\} \vdash \phi(x, y)$, which implies that $N_{\phi_1} \ast N_{\phi_2} \subseteq N_\phi$.

**Definition 10.1.** Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure and $G = \text{Aut}(\mathcal{M})$. We say that $q \in R^{pa}$ is an invariant idempotent if the following conditions are satisfied:

1. $1_G \leq q$;
2. $q = \bar{q}$;
3. $q \ast q = q$; and,
4. $q = q^q$ for any $g \in G$.

Notice that assumption 1. implies $q = 1 \ast q \leq q \ast q$, so that item 3. could be replaced by the a priori weaker condition $q \ast q \leq q$. 

Lemma 10.2. Given any structure $M$ the following statements hold, where $G = \text{Aut}(M)$:
1. Any invariant idempotent $q \in R^\text{pa}(M)$ the family $N_q$ forms a basis of neighbourhoods of a (unique) group topology $\tau_q$ on $G$.
2. The closure of 1 in $\tau$ coincides with the collection of all $g \in G$ such that $g \leq q$.
3. Given invariant idempotents $p, q \in R^\text{pa}(M)$ such that $p \leq q$ we have $\tau_p \supseteq \tau_q$ and then the implication from right to left holds as well if $M$ is countable and $\omega$-saturated.

Proof. On the one hand for any $\phi(x,y) \in q$ we have:
$$N^-_{\phi(x,y)} = \{ g \in G \mid M \models \phi(g^{-1}a,b) \} = \{ g \in G \mid M \models \phi(a,gb) \} = N_{\phi(y,x)} \subseteq N_q = N_q.$$ On the other hand, the condition $q * q = q$ is equivalent to the following: for any finite $A$ and $B$ there is $C \subseteq M$ and formulas $\psi(x,A,z_C), \psi'(z_C,y_B) \in q$ such that modulo $T$ we have:

$$p_M(x) \cup p_M(y) \cup p_M(z) \cup \{ \psi(x,A,z_C) \land \psi'(z_C,y_B) \} \models \phi(x,y).$$

Let $N = N_{\psi(x,A,y_C) \land \psi(x,C,y_B)}$. Given $h, g \in N$ we have $M \models \psi(gA,C) \land \psi'(hC,B)$. Formulas are of course $h$ invariant, hence $M \models \psi(hgA,hC)$. Likewise, $hgA \models p_A$ and $hC \models p_C$ and thus by 4 we conclude that $M \models \phi(hgA,B)$ and therefore $hg \in N_\phi$. This settles part 1. Part 2 follows easily from the fact that $\iota(g)$ is a complete type for $g \in G$ and left to the reader. As for 3., the implication from left to right is trivial. Assume now $M$ is $\omega$-saturated and we are given $p, q$ such that $p \not\leq q$. Then there exists some $\phi(x,a,\bar{y}) \in q$ for $\bar{a} \in [M]^{<\omega}$ such that $p \not\models \phi$. This implies there exists some $g \in G$ such that $M \models \psi(ga,a)$, for each $\psi(x,y) \in p$ but $M \models \lnot\phi(ga,a)$.

Remark 10.3. In particular, $1 \in G \in R^\text{pa}$ is an invariant idempotent. The associated topology $\tau_1$ is just the standard topology. It can be checked by inspection that all topologies on automorphism groups that feature in this paper are of the form $\tau_q$ for some invariant idempotent $q$.

Question 8. Let $M$ be a countable $\omega$-categorical (homogeneous) structure. Is it true that any group topology on $\text{Aut}(M)$ is of the form $\tau_q$ for some invariant idempotent $q \in R^\text{pa}$?

10.1. Non-minimality in the trivial $\text{acl}$ case. Fix some structure $M$ in a finite relational language in which $\text{acl}$ is trivial, i.e. $\text{acl}(A) = A$ for any finite $A \subseteq M$. Consider the type $q_{\text{in}f} \in R^\text{pa}(M)$ consisting of all the formulas of the form $\phi_{A,B}(x,y) \in \text{tp}(A,B)$, where $A \cap B = \emptyset$. Notice that $q_{\text{in}f}$ is clearly invariant under the action of $\text{Aut}(M)$ on $x_M$ and $y_M$.

Definition 10.4. We say that $M$ has the separation property if for any two disjoint finite tuples $a, b \in [M]^{<\omega}$ there exists $c \in [M]^{<\omega}$ disjoint with both $a$ and $b$ such that $\text{tp}^{x,z}(a,c) \cup \text{tp}^{y,z}(c,b) \models \text{tp}^{x,y}(a,b)$.

Lemma 10.5. The type $q_{\text{in}f}$ is an invariant idempotent in $R^\text{pa}(M)$ if and only if $M$ has the separation property. Moreover, $q_{\text{in}f} \not

Proof. Properties 1., 2., and 4. are immediate from the definition. For property 2., all we need to check is that $q * q \leq q$ as remarked after Definition 10.1, but this is precisely the content of the separation property, as in its definition $\text{tp}^{x,z}(a,c) \cup \text{tp}^{y,z}(c,b) \models \text{tp}^{x,y}(a,b)$ we have $\text{tp}^{x,y}(a,c) \cup \text{tp}^{x,y}(c,b) \subseteq q_{\text{in}f}$ and thus $\text{tp}^{x,y}(a,b) \subseteq q_{\text{in}f}$ for the arbitrary fragment $\text{tp}^{x,y}(a,b) \subseteq q_{\text{in}f}$ we started with.

If $q_{\text{in}f} = q_1$, then for any $b \in M$ there must be some finite $A \subseteq M \setminus \{ b \}$ such that $\text{tp}^{x,A,y_b}(A,b) \models y_b = x_b$, which can only be the case if $b \in \text{acl}(A)$. The final claim then follows from last point of Lemma 10.2.
Distal theories are a particular class of NIP theories introduced in [16]. One main feature is the following fact (Theorem 21 in [17]):

**Fact 10.6.** Let $T$ be distal. Then for any formula $\phi(x,y)$ there is a formula $\theta(x,z)$ such that for any $\text{tp}(\varphi(a/C))$ over a finite set of parameters $C$ there is a tuple $d \subset C$ such that $\theta(a,d)$ holds, and $\theta(x,d) \vdash \text{tp}(\varphi(a/C))$, i.e $\theta(x,y) \cup \text{tp}(\varphi(d,C)) \vdash \text{tp}(\varphi(x/C))$, where $|y| = |d|$.

**Lemma 10.7.** Let $\mathcal{M}$ be any distal Fraïssé limit in a finite relational language with trivial algebraic closure. Then $\mathcal{M}$ has the separation property.

**Proof.** Consider any two disjoint finite tuples $a, b \in M$. Since $\mathcal{M}$ has quantifier elimination, there exists some formula $\phi(x,y)$ such that for any $C \subset M$ the full type $\text{tp}(\varphi(a/C))$ is equivalent to the $\phi$-type $\text{tp}(\varphi(a/C))$ ($|a| = |x|$). Let $\theta(x,z)$ be the formula provided by Fact 10.6 and let $s = |z|$. Take a sequence $b_{-s}, b_{-s+1}, \ldots, b_0 = b, b_1, \ldots, b_s$ of instances of $\text{tp}(\varphi(b/a))$ indiscernible over $a$, where $b_i$ and $b_j$ are disjoint for $i \neq j$. Let $C = b_{-s}b_{-s+1} \ldots b_s$ and $d$ be the tuple obtained from applying 10.6 to $\text{tp}(\varphi(a/C))$. Let $J$ be the set of indices $j \in \{-s, -s + 1, \ldots, s\}$ such that $d \cap b_j \neq \emptyset$. Now, there must be some $j_0 \in \{-s, -s + 1, \ldots, s\} \setminus J$ and some order preserving bijection $\phi : J \cup \{j_0\} \rightarrow J' \subseteq \mathbb{Z}$ sending $j_0$ to 0. Since $(b_i)_j$ is indiscernible, the fact that $\text{tp}(a(b_1)_{j_0})$ isolates $\text{tp}(a(b_1)_{j_0})$ implies that $\text{tp}(a(b_1)_{j_0})$ isolates $\text{tp}(a(b_1)_{j_0})$, so that the tuple $C = (b_1)_{j_0}J' \setminus \{0\}$ witnesses the separation property for the pair $(a,b)$. \hfill \Box

**Corollary 10.8.** Let $\mathcal{M}$ be any distal Fraïssé limit in a finite relational language with trivial algebraic closure. Then the type $\text{qinf}$ defines a group topology on $G = \text{Aut}(\mathcal{M})$ strictly coarser than $\tau_d$.

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