Application of Weak Signal Denoising Based on Improved Wavelet Threshold

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Abstract: In this Study, the application of wavelet threshold denoising in weak signal detection of underwater targets under complex sea conditions is discussed. Firstly, the basic principle and steps of wavelet threshold denoising method based on Mallat decomposition are analyzed. Then a new threshold function is proposed to overcome the shortcomings of traditional threshold functions. The denoising effects of wavelet and wavelet packet threshold denoising methods under hard threshold, soft threshold and new threshold functions are simulated, and the signal-to-noise ratio (SNR) and mean square error (MSE) after denoising are taken as evaluation indexes. The results show that the improved threshold denoising method outperforms the traditional threshold denoising methods.

1. Introduction

The background noise of marine environment has a serious impact on the detection of underwater target signal. The generation, processing and transmission of underwater target physical field signal are inevitably disturbed by noise. Especially in the detection of weak target signal, the noise elimination is the key issue [1].

For the wavelet transform has the ability to concentrate the signal energy on a few wavelet coefficients [2], the effect of white noise on the signal can be well removed by using the wavelet transform. There are three methods commonly used wavelet denoising: Mallat's denoising algorithm based on the wavelet transform modulus maxima technique [3]; Xu's spatial correlation denoising algorithm [4]; Donoho's threshold denoising algorithm [5]. Among them, the threshold denoising method was proposed by D.L. Donoho in 1995. For this method can get the best estimation in Besov space, and no other linear estimation can achieve the same estimation effect, the wavelet threshold denoising method has attracted wide attention of scholars at home and abroad. However, the soft threshold and hard threshold functions used in this method still have some problems to be improved.

The paper 1 proposed an improved wavelet threshold method and subsequently verified its de-noising performance through simulations and actual applications, using SNR and MSE as the evaluation indexes in the simulations [6]. The paper 2 proposed a new wavelet denoising algorithm, the proposed method overcomes the deviation between the estimated wavelet coefficient and the noisy signal coefficient, and has a better regulation and continuity [7].

Wavelet packet transform further decomposes the high frequency part of the signal on the basis of the wavelet transform and improves the processing ability of the signal. In this paper, the basic principle and implementation steps of wavelet threshold denoising method are analyzed. In view of the...
shortcomings of traditional soft and hard thresholding methods, a new threshold function is proposed, and the denoising effects of wavelet and wavelet packet threshold denoising methods are simulated and analyzed.

2. Mallat decomposition algorithm

2.1 The Wavelet Transform

The continuous signal $z(t)$ is expanded on the wavelet basis to generate the continuous wavelet transform (CWT). The expression of CWT is:

$$ WT_t(a,b) = z(t), \varphi_{a,b}(t) = a^{-1/2} \int_R z(t) \varphi(\frac{t-b}{a}) dt $$

(1)

The $\varphi(t)$ is the wavelet basis function, A family of wavelet functions can be obtained by scaling and translating the wavelet basis function. Expressed as:

$$ \varphi_{a,b}(t) = a^{-1/2} \varphi(\frac{t-b}{a}) $$

(2)

2.2 The wavelet transform of the Mallat algorithm

Mallat decomposition algorithm is a fast wavelet transform algorithm based on multi-resolution analysis. Each decomposition produces high frequency detail component $d_j(k)$ and low frequency approximation component $c_j(k)$. The transformation process is equivalent to using a set of high-pass and low-pass filters to decompose time series signals step by step.

Firstly, the wavelet basis function which is closest to the signal is selected, then the level $J$ of the wavelet decomposition is determined, $J \leq \log(2gN)$ and the $J$ is natural number. The multiresolution decomposition formulas are as follows:

$$ c_{j,k} = \sum h_j(m-2k) c_{j-1,m} $$

(3)

$$ d_{j,k} = \sum h_j(m-2k) c_{j-1,m} $$

(4)

$h_j(n) = < \phi_{j,0}(k), \phi_{j-1,n}(k) >$ is equivalent to a low-pass filter bank, and the approximation coefficient $c_{j,k}$ is decomposed. $h_j(n) = < \phi_{j,a}(k), \phi_{j-1,n}(k) >$ is equivalent to a high-pass filter bank, and the detail coefficients $d_{j,k}$ is decomposed. $\phi$ and $\varphi$ are the corresponding scaling function and wavelet basis function respectively.

2.3 Reconstruction of Wavelet

After signal $z(t)$ is processed by wavelet transform, it can be reconstructed to obtain useful signals:

$$ x(t) = C^{-1}_o \int_o^\infty \int a \frac{1}{\sqrt{a}} WT_t(a,b) \varphi(\frac{t-b}{a}) db $$

(5)

The $C_o = \int_R \frac{|\psi(w)|^2}{|w|} dw < \infty$, $\psi(w)$ is the Fourier transform of the basis function $\varphi(t)$.

The reconstruction of discrete wavelet can be expressed as:

$$ x(t) = C_N(t) + \sum_{j=1}^N D_j(t) $$

(6)

In formula, $c_{j,(t)}$ represents the reconstruction information from the approximation coefficients of layer $N$, and $d_{j,(t)}$ represents the reconstruction information from the approximation detail coefficients of different scales.
3. The Selection of Threshold

3.1 Donoho threshold function
At present, the hard and soft thresholding denoising method proposed by Donoho is the most widely used. Donoho hard threshold processing is to compare the absolute value of signal transformation with the threshold value. The hard thresholding is carried out as follows:

\[
\hat{w}_{j,k} = \begin{cases} 
  w_{j,k}, & |w_{j,k}| \geq \lambda \\
  0, & |w_{j,k}| < \lambda 
\end{cases}
\]

(7)

In the case of soft thresholding the coefficients are carried as follows:

\[
\hat{w}_{j,k} = \begin{cases} 
  \text{sign}(w_{j,k}) \cdot |w_{j,k}| - \lambda, & |w_{j,k}| \geq \lambda \\
  0, & |w_{j,k}| < \lambda 
\end{cases}
\]

(8)

Where sign(x) is +1 if x is positive and -1 if x is negative, \(w_{j,k}\) is the coefficients, \(\lambda\) is the threshold.

The disadvantage of Donoho thresholding method is that it uses a fixed threshold to process the wavelet coefficients, which will distort the weak feature components contained in the original signal, thus causing the distortion of the reconstructed signal. In signal processing, many weak feature components are important. If the noise is removed indiscriminately, the reconstructed signal cannot accurately reflect the characteristics of the signal.

3.2 Construction of New Threshold Function
The core of wavelet thresholding denoising method is the threshold processing of wavelet coefficients. Although Donoho's hard threshold and soft threshold functions have achieved good results, they also have some shortcomings. The hard thresholding function is discontinuous at \(\pm \lambda\), and the signal reconstructed by \(\hat{w}_{k}^{n,j}\) may produce some oscillations. Although the soft threshold function has good overall continuity, when the wavelet coefficients are large, there is always a constant deviation \(\lambda\) between \(\hat{w}_{k}^{n,j}\) and \(w_{k}^{n,j}\), which will bring inevitable errors to the reconstructed signal. But it is not necessarily the best to reduce the deviation to zero \([8]\). In order to overcome the shortcomings of hard and soft threshold function, a new threshold function is constructed in this paper.

\[
\hat{w}_{k}^{n,j} = \begin{cases} 
  \text{sign}(w_{k}^{n,j}) \cdot \left[(w_{k}^{n,j})^2 - \frac{\lambda^2}{1 + \exp(\|w_{k}^{n,j}\|/N)}\right]^{1/2}, & |w_{k}^{n,j}| \geq \lambda \\
  0, & |w_{k}^{n,j}| < \lambda 
\end{cases}
\]

(9)

In the formula, \(N\) is a variable parameter and positive integer. The schematic diagram of the new threshold function at \(N=2\) and \(N=12\) is shown in Figure 1.

![Figure 1. New threshold function diagram](image)

In Figure 1, the solid line represents the hard threshold, the dot dash line represents the soft threshold, and the dotted line represents the new threshold function. When \(N=2\), the dotted line is closer to the hard threshold line. Through mathematical analysis of the new threshold function, it is not difficult to get that it takes \(\hat{w}_{k}^{n,j} = w_{k}^{n,j}\) as asymptote, with the increase of \(w_{k}^{n,j}\), \(\hat{w}_{k}^{n,j} \rightarrow w_{k}^{n,j}\), it
overcomes the shortcomings of the constant deviation between the soft thresholds $w_{kn}^j$ and $\hat{w}_{kn}^j$. When $|w_{kn}^j| \geq \lambda$, the new threshold function is high-order differentiable, which is convenient for various mathematical processing. The new threshold function is between the soft threshold and hard threshold, and closer to the hard thresholds, which is closer to the real coefficients, but not completely equal. At the same time, the smaller the N is, the faster the $\hat{w}_{kn}^j$ converges to $w_{kn}^j$. As long as the parameter N is adjusted properly, better denoising effect can be achieved. It can be seen that the new threshold function is a flexible choice between soft and hard threshold. The practical and effective threshold function can be obtained by selecting N.

4. Experiments simulation

4.1 The settings of experiment simulation

Four typical signals (Blocks, Bumps, Heavy sine and Doppler) used by Donoho are selected as experimental signal, and the simulation results of Doppler signal are given. Wavelet and wavelet packet threshold denoising methods are used to denoise noisy signal. Hard threshold, soft threshold and new threshold functions are selected for the threshold functions respectively, and the denoising effects of various methods are compared. The same assumption is made for both wavelet and wavelet packet denoising methods: assuming the signal-to-noise ratio of input signal is 2dB and the additive noise is zero-mean Gauss white noise, sym4 wavelet is selected as the wavelet basis, the scale of wavelet and wavelet packet decomposition is 4, and the threshold is fixed threshold (sqtwolog threshold), that is $\lambda = \sqrt{2 \ln N}$, N is the length of the signal. In MATLAB, the selection of the optimal wavelet packet basis is realized by the entropy criteria and threshold parameters, and the selection of the entropy criteria depends on different signal analysis requirements.

The signal-to-noise ratio (SNR) and mean square error (MSE) after denoising are selected as the evaluation indexes of denoising effect.

$$\text{SNR} = 10 \log \frac{\sum s^2(n)}{\sum (s(n) - s_o(n))^2}$$

$$\text{MSE} = \frac{1}{N} \sum [s(n) - s_o(n)]^2$$

In the formula, $s_o(n)$ is the original signal and $s(n)$ is the denoised signal. Obviously, the bigger the SNR is and the smaller the MSE is, the better the denoising effect is.

The original Doppler signal and the noisy signal are shown in figure 2.

![Figure 2. The original signal and the noisy signal](image)

The results of wavelet denoising using three threshold functions are shown in Figure 3, and the corresponding results of wavelet packet denoising are shown in Figure 4.
The comparison of SNR and MSE after denoising is shown in Table 1.

| Denoising algorithm       | Hard threshold | Soft threshold | New threshold function |
|---------------------------|----------------|----------------|-----------------------|
| Wavelet denoising        | SNR/dB         | 13.5859        | 12.8300               |
|                           | MSE            | 0.1750         | 0.2083                |
| Wavelet Packet Denoising | SNR/dB         | 14.0456        | 13.6101               |
|                           | MSE            | 0.1574         | 0.1740                |

Figure 3. The result of wavelet denoising

Figure 4. The result of denoising by wavelet packet
4.2 The analysis of the denoising performance
From figure 3, figure 4 and table 1, we can see that the denoising effect of the wavelet packet threshold denoising method is better than that of the wavelet threshold denoising method when the same threshold function is used. But the effect is not obvious. The reason is that the high-frequency part of the signal is also decomposed by multi-scale wavelet packet transform, which is not available in wavelet transform. At the same time, the simulation results also show that the denoising effect of using hard threshold is better than that of soft threshold, because there is always a constant deviation between $\hat{w}_k^{n,j}$ and $w_k^{n,j}$ in soft thresholding processing, which brings some errors to reconstructed signal. The new threshold function proposed in this paper has better denoising effect than the traditional soft and hard thresholding method, the SNR of the new threshold function is about 2 dB higher than that of hard threshold function, and about 3 dB higher than that of soft threshold function. As can be seen from Table 1, the value of parameter N has little influence on the denoising effect.

5. Conclusion
In this paper, the application of wavelet threshold denoising method in weak signal detection is studied. A new threshold function is proposed to overcome the shortcomings of traditional soft and hard thresholding method. The simulation results show that the threshold denoising using wavelet packet transform has advantages over using wavelet transform, and is more conducive to restoring the high frequency information of the original signal. The denoising effect of the new threshold function is better than that of the traditional soft and hard thresholding method. However, how to design the best threshold function according to the characteristics of different types of signal remains to be further studied.

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