Improved Space efficient algorithms for BFS, DFS and applications

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Abstract. Recent work by Elmasry et al. (STACS 2015) and Asano et al. (ISAAC 2014), reconsidered classical fundamental graph algorithms focusing on improving the space complexity. Elmasry et al. gave, among others, implementations of breadth first search (BFS) and depth first search (DFS) in a graph on \(n\) vertices and \(m\) edges, taking \(O(m + n)\) time using \(O(n)\) and \(O(n \log \log n)\) bits of space respectively improving the naive \(O(n \log n)\) bits implementation. We continue this line of work focusing on space.

Our first result is a simple data structure that can maintain any subset \(S\) of a universe of \(n\) elements using \(n + o(n)\) bits and support in constant time, apart from the standard insert, delete and membership queries, the operation \(\text{findany}\) that finds and returns any element of the set (or outputs that the set is empty). Using this we give a BFS implementation that takes \(O(m + n)\) time using at most \(2n + o(n)\) bits. Later, we further improve the space requirement of BFS to at most \(\frac{585}{585}n + o(n)\) bits albeit with a slight increase in running time to \(O(m \log n f(n))\) time where \(f(n)\) is any extremely slow growing function of \(n\). These improve the space by a constant factor from earlier representations.

We demonstrate the use of our data structure by developing another data structure using it that can represent a sequence of \(n\) non-negative integers \(x_1, x_2, \ldots, x_n\) using at most \(\sum_{i=1}^{n} x_i + 2n + o(\sum_{i=1}^{n} x_i + n)\) bits and, in constant time, determine whether the \(i\)-th element is 0 or decrement it otherwise. We use this data structure to output the vertices of a

- directed acyclic graph in topological sorted order in \(O(m + n)\) time and \(O(m + n)\) bits, and
- graph with degeneracy \(d\) in degeneracy order in \(O(nd)\) time using \(O(nd)\) bits.

We also discuss an algorithm for finding a minimum weight spanning tree of a weighted undirected graph using at most \(n + o(n)\) bits. For DFS, we have two kinds of results. Specifically,

- we provide an implementation for DFS that takes \(O(m + n)\) time using \(O(m + n)\) bits. This partially answers at least for sparse graphs, a question in [3] that asked whether DFS can be performed in \(O(m + n)\) time and \(O(n)\) bits. Using the DFS algorithm and other careful

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1 We use \(\log\) to denote logarithm to the base 2.
implementations, we can determine cut vertices, bridges and maximal 2-connected components among others, in \(O(m + n)\) time and \(O(m + n)\) bits; earlier linear time algorithms for these problems used \(\Omega(n \lg n)\) bits of space. These space efficient implementations are obtained using a space efficient implementation of chain decomposition and the unary encoding of the degrees of the vertices of the graph, and some succinct data structures.

- we present space efficient implementations for finding strongly connected components of a directed graph, to output the vertices of a directed acyclic graph in a topologically sorted manner, and to find a sparse biconnected subgraph of a biconnected graph using \(O(n)\) bits. These improve the space required for earlier implementations from \(\Omega(n \lg n)\) bits.

1 Introduction

Motivated by the rapid growth of huge data set ("big data"), algorithms that utilize space efficiently are becoming increasingly important than ever before. One other reason for the importance of space efficient algorithm is the proliferation of specialized handheld devices and embedded systems that have a limited supply of memory. As a consequence, algorithms that are oblivious to space constraint are not desired in such scenario. Even if mobile devices and embedded systems are designed with large supply of memory, it might be useful to restrict the number of write operations specifically for two reasons. One being writing into flash memory is a costly operation in terms of speed and time, and, secondly, it reduces the longevity of such memory. Keeping all these constraints in mind, it makes sense to consider algorithms that do not modify the input and use only a limited amount of work space. In this paper, we focus on the space requirement for implementing some fundamental graph algorithms in such settings.

1.1 Our results and organization of the paper

Asano et al. [3], in a recent paper, show that DFS of a directed or undirected graph on \(n\) vertices and \(m\) edges can be performed using \(n + o(n)\) bits and (an unspecified) polynomial time. Using \(2n + o(n)\) bits, they can bring down the running time to \(O(mn)\) time, and using a larger \(O(n)\) bits, their running time is \(O(m \lg n)\). In a similar vein,

- we show in Section 3 that the vertices of a directed or undirected graph can be listed in BFS order using \(1.585n + o(n)\) bits and \(O(m f(n) \lg n)\) time where \(f(n)\) is any (extremely slow-growing) function of \(n\) i.e. \(\lg^* n\) (the \(o\) term in the space is a function of \(f(n)\)), while the runtime can be brought down to the optimal \(O(m + n)\) time using \(2n + o(n)\) bits.

En route to this algorithm, we develop in Section 2,

- a data structure that maintains a set of elements from a universe of size \(n\), say \([1..n]\) using \(n + o(n)\) bits to support, apart from insert, search and delete
operations, the operation \textit{findany} of finding an arbitrary element of the set, and returning its value all in constant time. It can also output all elements of the set in no particular order in $O(k + 1)$ time where $k$ is the number of elements currently belonging to the set$^2$.

Our structure gives an explicit implementation, albeit for a weaker set of operations than that of Elmasry et al. [19] whose space requirement was $cn + o(n)$ bits for an unspecified constant $c > 2$; furthermore, our structure is simple and is sufficient to implement BFS space efficiently, improving by a constant factor of their BFS implementation keeping the running time same.

We could support the \textit{findany} operation by keeping track of one of the elements, but once that element is deleted, we need to find another element to answer a subsequent \textit{findany} query. This is easy to support in constant time if we have the elements stored in a linked list which takes $O(n \lg n)$ bits, or if we have a dynamic rank-select structure [25] where each operation takes $O(\frac{\lg n}{\lg \lg n})$ time.

In the same section we improve the space for BFS further at the cost of slightly increased runtime. We also provide a similar tradeoff for the minimum spanning tree problem. Our algorithm takes $n + O(n/f(n))$ bits and $O(m \lg n f(n))$ time, for any function $f(n)$ such that $1 \leq f(n) \leq n$. While this algorithm is similar in spirit to that of Elmasry et al. which works in $O(m \lg n)$ time using $O(n)$ bits or $O(m + n \lg n)$ time using $O(n \lg(2 + \frac{m}{n}))$ bits, we work out the constants in the higher order term for space, and improve them slightly though with a slight degradation in time.

- Using our data structure, in Section 4 we develop another data structure to represent a sequence $x_1, x_2, \ldots, x_n$ of $n$ integers using $m + 2n + o(m + n)$ bits where $m = \sum_{i=1}^{n} x_i$. In this, we can determine whether the $i$-th element is 0 and if not, decrement it, all in constant time. In contrast, the data structure claimed (without proof) in [19] can even change (not just decrement) or access the elements, but in constant amortized time. However, their structure requires an $O(\lg n)$ limit on the $x_i$ values while we pose no such restriction.

Using this data structure in Section 4,
- we determine whether a given directed graph is acyclic and give an implementation of topological sort of the graph if it is in $O(m + n)$ time and $O(m + n)$ bits of space. This improves an earlier bound of $O(m + n)$ time and $O(n \lg \lg n)$ space [19], and is more space efficient for sparse directed graphs (that includes those directed graphs whose underlying undirected graph is planar or has bounded treewidth or degeneracy).
- A graph has a degeneracy $d$ if every induced subgraph of the graph has a vertex with degree at most $d$ (for example, planar graphs have degeneracy 5, and trees have degeneracy 1). An ordering $v_1, v_2, \ldots, v_n$ of the vertices in such a graph is a degenerate order if for any $i$, the $i$-th vertex has degree at most $d$ among vertices $v_{i+1}, v_{i+2}, \ldots, v_n$. There are algorithms [20, 9] that can find the degeneracy order in $O(m + n)$ time.

$^2$ Since our initial submission to COCOON, Hagerup and Kamber [24] have reported a structure with $n + o(n)$ bits for the data structure.
using $O(n)$ words. We show that, given a $d$, we can output the vertices of a $d$-degenerate graph in $O(m + n)$ time using $O(m + n)$ bits of space in the degeneracy order. We can even detect if the graph is $d$-degenerate in the process. As $m$ is $O(nd)$, we have an $O(nd)$ bits algorithm which is more space efficient if $d$ is $o(\lg n)$ (this is the case, for example, in planar graphs or trees).

- For DFS, we have two kinds of results improving on the result of Asano et al. [3] who showed that DFS in a directed or undirected graph can be performed in $O(m \lg n)$ time and $O(n)$ bits of space, and of Elmasry et al. [19] who improved the time to $O(m \lg \lg n)$ time still using $O(n)$ bits of space.
- In Section 5, we first show that for sparse graphs (graphs where $m = O(n)$), we can perform DFS in linear time using $O(m + n)$ (i.e. $O(n)$ in sparse graphs) bits. Building on top of this encoding and other observations, we show how to efficiently compute the chain decomposition of a connected undirected graph. This lets us perform a variety of applications of DFS (including testing 2-vertex and 2-edge connectivity, finding cut vertices and edges, maximal 2-connected components and (open) ear decompositions) in the same time and space. Our algorithms for these applications improve the space requirement of all the previous algorithms from $\Theta(n \lg n)$ bits to $O(m + n)$ bits, preserving the same linear runtime.
- Section 6 talks about applications of DFS using $O(n)$ bits. Using $O(n)$ bits of space, we show that
  * we can compute the strongly connected components of a directed graph in $O(m \lg n \lg \lg n)$ time,
  * we can output the vertices of a directed acyclic graph in a topologically sorted fashion in $O(m \lg \lg n)$ time, and
  * we can find a sparse spanning biconnected subgraph of a biconnected undirected graph in $O(m \lg \lg n)$ time.

### 1.2 Model of Computation

We assume that the input graph is given in a read-only memory (and so cannot be modified). If an algorithm must do some outputting, this is done on a separate write-only memory. When something is written to this memory, the information can not be read or rewritten again. So the input is “read only” and the output is “write only”. In addition to the input and the output media, a limited random-access workspace is available. The data on this workspace is manipulated wordwise as on the standard word RAM, where the machine consists of words of size $w$ in $O(\lg n)$ bits and any logical, arithmetic, and bitwise operations involving a constant number of words take a constant amount of time. We count space in terms of the number of bits used by the algorithms in workspace. This model is called the register input model and it was introduced by Frederickson [21] while studying some problems related to sorting and selection.

We assume that the input graphs are represented using the standard adjacency list throughout the paper. For the algorithms in Section 5 we require that the input graph must be represented using the standard adjacency list along
with cross pointers, i.e. for undirected graphs given a vertex \( u \) and the position in its list of a neighbor \( v \) of \( u \), there is a pointer to the position of \( u \) in the list of \( v \). When we work with directed graphs, we assume that the graphs are represented as in and out adjacency lists i.e. given a vertex \( u \), we have a list of out-neighbors and in-neighbors of \( u \). We then augment these two lists for every vertex with cross pointers, i.e. for each \((u, v) \in E\), given \( u \) and the position of \( v \) in out-neighbors of \( u \), there is a pointer to the position of \( u \) in in-neighbors of \( v \). This representation was used by Elmasry et al. \[19\]. When discussing graph algorithms below, we always use \( n \) and \( m \) to denote the number of vertices and the number of edges respectively, in the input graph.

1.3 Related Work

In computational complexity theory, the constant work-space model is represented by the complexity class \text{LOGSPACE} \[2\]. There are several algorithmic results for this class, most celebrated being Reingold’s method for checking reachability between two vertices in an undirected graph \[31\]. Barnes et al. gave a sub-linear space algorithm for directed graph reachability \[8\]. Recent work has focused on space requirement in special classes of graphs like planar and H-minor free graphs \[12, 4\]. In the algorithms literature, where the focus is also on improving time, a large amount of research has been devoted to memory constrained algorithms, even as early as in the 1980s \[27\]. Early work on this focused on the selection problem \[27, 21, 28\], but more recently on computational geometry problems \[5, 7, 16\] and graph algorithms \[19, 3, 6\]. Regarding the data structure we develop to support \text{findany} operation, Elmasry et al.[Lemma 2.1,\[19\]] state a data structure (without proof) that supports all the operations i.e. insert, search, delete and findany (they call it \text{some id}) among others, in constant time. But their data structure takes \( O(n) \) bits of space where the constant in the \( O \) term is not explicitly stated. Our data structure, on the other hand, is probably simpler and takes just \( n + o(n) \) bits of space.\(^3\)

1.4 Preliminaries

We will use the following well-known lemma:

**Lemma 1.** A sequence of \( n \) integers in the range \( \{1, \ldots, c\} \) where \( c \) is a constant, can be represented using \( n \lg c + o(n) \) bits where the \( i \)-th integer can be accessed or modified in constant time.

**Proof.** We divide the sequence of \( n \) elements into contiguous blocks of size \( d \lg n \) each, for some constant \( d \) such that \( d \lg n \) is an integer and \( d \lg c < 1 \). We represent each block using \( \lceil d \lg n \lg c \rceil \) bits so that the \( i \)-th element of the block can be accessed or modified in constant time using standard RAM operations. Overall space used is \( (n/(d \lg n))(\lceil d \lg n \lg c \rceil) \leq (n/(d \lg n))(d \lg n \lg c + 1) \leq n \lg c + n/(d \lg n) = n \lg c + o(n) \) bits.

\(^3\)Hagerup and Kammer \[24\] have recently reported a structure with \( n + o(n) \) bits for the data structure supporting the same set of operations.
We also need the following theorem.

**Theorem 1.** [14, 26, 23] We can store a bitstring $O$ of length $n$ with additional $o(n)$ bits such that rank and select operations (defined below) can be supported in $O(1)$ time. Such a structure can also be constructed from the given bitstring in $O(n)$ time.

Here the rank and select operations are defined as following:

- $\text{rank}_a(O, i) =$ number of occurrences of $a \in \{0, 1\}$ in $O[1, i]$, for $1 \leq i \leq n$;
- $\text{select}_a(O, i) =$ position in $O$ of the $i$th occurrence of $a \in \{0, 1\}$.

## 2 Maintaining dictionaries under findany operation

We consider the data structure problem of maintaining a set $S$ of elements from $\{1, 2, \ldots, n\}$ to support the following operations in constant time.

- **insert** $(i)$: Insert element $i$ into the set.
- **search** $(i)$: Determine whether the element $i$ is in the set.
- **delete** $(i)$: Delete the element $i$ from the set if it exists in the set.
- **findany**: Find any element from the set and return its value. If the set is empty, return a NIL value.

It is trivial to support the first three operations in constant time using $n$ bits. Our main result in this section is that the **findany** operation can also be supported in constant time using $o(n)$ additional bits.

**Theorem 2.** A set of elements from a universe of size $n$ can be maintained using $n + o(n)$ bits to support insert, delete, search and findany operations in constant time. We can also output all elements of the set (in no particular order) in $O(k + 1)$ time where $k$ is the number of elements in the set.

**Proof.** Let $S$ be the characteristic bit vector of the set having $n$ bits. We follow a two level blocking structure of $S$, as in the case of succinct structures supporting rank and select [14, 26]. However, as $S$ is ‘dynamic’ (in that bit values can change due to insert and delete), we need more auxiliary information. In the discussion below, sometimes we omit floors and ceilings to keep the discussion simple, but they should be clear from the context.

We divide the bit vector $S$ into $n/\lg^2 n$ blocks of consecutive $\lg^2 n$ bits each, and divide each such block into up to $2\lg n$ small blocks of size $\lceil \lg n/2 \rceil$ bits each. We refer to the small blocks explicitly as small blocks, and by blocks we refer to the (big) blocks of size $\lg^2 n$ bits. We call a block (big or small) non-empty if it contains at least a 1. We maintain the non-empty (big) blocks, and the non-empty small blocks within each (big) block in linked lists (not necessarily in order). Within a small block, we find the first 1 or the next 1 by a table lookup. We provide the specific details below.

First, we maintain an array $\text{number}$ indicating the number of 1s in each block, i.e. $\text{number}[i]$ gives the number of 1s in the $i$-th block of $S$. It takes $O(n \lg \lg n/\lg^2 n)$ bits as each block can have at most $\lg^2 n$ elements of the given...
set. Then we maintain a queue (say implemented in a space efficient resizable array \([11]\)) block-queue having the block numbers that have a 1 bit, and new block numbers are added to the list as and when new blocks get 1. It can have at most \(\frac{n}{\log^2 n}\) elements and so has \(O(n/\log^2 n)\) indices taking totally \(O(n/\log n)\) bits. In addition, every element in block-queue has a pointer to another queue of small block numbers of that block that have an element of \(S\). Each such queue has at most \(2\log n\) elements each of size at most \(2\log \log n\) bits each (for the small block index). Thus the queue block-queue along with the queues of small block indices takes \(O(n/\log n)\) indices taking totally \(O(n/\log n)\) bits. In addition, every element in block-queue has a pointer to another queue of small block numbers of that block that have an element of \(S\). Each such queue has at most \(2\log n\) elements each of size at most \(2\log \log n\) bits each (for the small block index). Thus the queue block-queue along with the queues of small block indices takes \(O(n/\log n)\) indices taking totally \(O(n/\log n)\) bits. We also maintain an array, block-array, of size \(\frac{n}{\log n}\) where block-array[i] points to the position of block \(i\) in block-queue if it exists, and is a NIL pointer otherwise and array, small-block-array, of size \(2n/\log n\) where small-block-array[i] points to the position of the subblock \(i\) in its block’s queue if its block was present in block-queue, and is a NIL pointer otherwise. So, block-array takes \(n/\log n\) bits and small-block-array takes \(2n/\log n\) bits.

We also maintain a global table \(T\) precomputed that stores for every bitstring of size \(\lceil\log n/2\rceil\), and a position \(i\), the position of the first 1 bit after the \(i\)-th position. If there is no ‘next 1’, then the answer stored is \(-1\) indicating a NIL value. The table takes \(O(\sqrt{n}(\log \log n)^2)\) bits. This concludes the description of the data structure that takes \(n+O(n/\log n)\) bits.

Now we explain how to support each of the required operations. Membership is the easiest, as it is a static operation, just look at the \(i\)-th bit of \(S\) and answer accordingly. In what follows, when we say the ‘corresponding bit or pointer’, we mean the bit or the pointer corresponding to the block or the small block corresponding to an element (inserted or deleted) which can be determined in constant time from the index of the element. To insert an element \(i\), first determine from the table \(T\), whether there is a 1 in the corresponding small block (before the element is inserted), set the \(i\)-th bit of \(S\) to 1, and increment the corresponding value in number. If the corresponding pointer of block-array was NIL, then insert the block index to block-queue at the end of the queue, and add the small block corresponding to the \(i\)-th bit into the queue corresponding to the index of the block in block-queue, and update the corresponding pointers of block-array and small-block-array. If the corresponding bit of block-array was not NIL (the big block already had an element), and if the small block did not have an element before (as determined using \(T\)), then find the position of the block index in block-queue from block-array, and insert the small block index into the queue of that block at the end of the queue. Update the corresponding pointer of small-block-array.

To support the delete operation, set the \(i\)-th bit of \(S\) to 0 (if it was already 0, then there is nothing more to do) and decrement the corresponding number in number. Determine from the table \(T\) if the small block of \(i\) has a 1 (after the \(i\)-th bit has been set to 0). If not, then find the index of the small block from the arrays block-array and small-block-array and delete that index from the block’s queue from block-queue. If the corresponding number in number remains more than 0, then there is nothing more to do. If the number becomes 0, then find the corresponding block index in block-queue from the array block-array, and delete
that block (along with its queue that will have only one small block) from block-queue. Update the pointers in block-array and small-block-array respectively. As we don’t maintain any order in the queues in block-queue, if we delete an intermediate element from the queue, we can always replace that element by the last element in the queue updating the pointers appropriately.

To support the findany operation, we go to the tail of the queue block-queue, if it is NIL, we report that there is no element in the set, and return the NIL value. Otherwise, go to the block at the tail of block-queue, and get the first (non-empty) small block number from the queue, and find the first element in the small block from the table $T$, and return the index of the element.

To output the elements of the set, we traverse the list block-queue and the queues of each element of block-queue, and for each small block in the queues, we find the next 1 in constant time using the table $T$ and output the index.

We can generalize to maintain a collection of more than one disjoint subsets of the given universe to support the insert, delete, membership and findany operations. In this case, insert, delete and findany operations should come with a set index (to be searched, inserted or deleted). Specifically, we show the following.

**Theorem 3.** A collection of $c$ disjoint sets that partition the universe of size $n$ can be maintained using $n \lg c + o(n)$ bits to support insert, delete, search and findany operations in constant time, where $c$ is a fixed constant. We can also output all elements of any given set (in no particular order) in $O(k + 1)$ time where $k$ is the number of elements in the set.

**Proof.** The higher order term is for representing the (generalized) characteristic vector $S$ where $S[i]$ is set to the number (index) of the set where the element is present. From Lemma 1, $S$ can be represented using $n \lg c + o(n)$ bits so that the $i$-th value can be retrieved or set in constant time. The rest of the data structures and the algorithms are as in the proof of Theorem 2, we have a copy of such structures for each of the $c$ sets.

3 Breadth First Search

Following the observations of [19], the space efficient implementation follows using the data structure of Theorem 3. We explain the details for completeness. Our goal is to output the vertices of the graph in the BFS order. We start as in the textbook BFS by coloring all vertices white. The algorithm grows the search starting at a vertex $s$, making it grey and adding it to a queue. Then the algorithm repeatedly removes the first element of the queue, and adds all its white neighbors at the end of the queue (coloring them grey), coloring the element black after removing it from the queue. As the queue can store up to $O(n)$ elements, the space for the queue can be $O(n \lg n)$ bits. To reduce the space to $O(n)$ bits, the two crucial observations on the properties of BFS are that: (i) elements in the queue are only from two consecutive levels of the BFS tree, and that the (iii) elements belonging to the same level can be processed in any order,
but elements of the lower level must be processed before processing elements of the higher level.

The algorithm maintains four colors: white, grey0, grey1 and black, and represents the vertices with each of these colors as sets \( W, S_0, S_1 \) and \( B \) respectively using the data structure of Theorem 3. It starts with initializing \( S_0 \) (grey 0) to \( s \), \( S_1 \) and \( B \) as empty sets and \( W \) to contain all other vertices. Then it processes the elements in each set \( S_0 \) and \( S_1 \) switching between the two until both sets are empty. As we process an element from \( S_i \), we add its white neighbors to \( S_{i+1 \mod 2} \) and delete it from \( S_i \) and add it to \( B \). When \( S_0 \) and \( S_1 \) become empty, we scan the \( W \) array to find the next white vertex and start a fresh BFS again from that vertex. As insert, delete, membership and findany operations take constant time, and we are maintaining four sets, we have from Theorem 3,

**Theorem 4.** Given a directed or undirected graph, its vertices can be output in a BFS order starting at a vertex using \( 2n + o(n) \) bits in \( O(m + n) \) time.

Note that it is sufficient to build findany structures only on sets \( S_0 \) and \( S_1 \) to efficiently find grey vertices.

### 3.1 Improving the space to \( n \log 3 + o(n) \) bits

There are several ways to implement BFS using just two of the three colors used in the standard BFS [15], but the space restriction, hence our inability to maintain the standard queue, provides challenges.

We give a 3 color implementation overloading grey and black vertices, i.e. we use one color to represent grey and black vertices. Grey vertices remain grey even after processing. This poses the challenge of separating the grey vertices from the black ones correctly before exploring. We will have three colors, one (color 2) for the unexplored vertices and two colors (0 and 1) for those explored including those currently being explored. The two colors indicate the parity of the level (the distance from the starting vertex) of the explored vertices. Thus the starting vertex \( s \) is colored 0 to mark that its distance from \( s \) is of even length and every other vertex is colored 2 to mark them as unexplored (or white). We will have these values stored in the representation of Lemma 1 using \( 1.585n + o(n) \) bits and we call this as the color array. The algorithm repeatedly scans this array and in the \( i \)-th scan, it changes all the 2 neighbors of \( i \mod 2 \) to \( i + 1 \mod 2 \). The exploration (of the connected component) stops when in two consecutive scans of the list, no 2 neighbor is found. Each scan list takes \( O(m) \) time and at most \( n + 2 \) scans of the list are performed resulting in an \( O(mn) \) time algorithm.

The \( O(m) \) time for each scan of the previous algorithm is because while looking for vertices labelled 0 that are supposed to be ‘grey’, we might cross over spurious vertices labelled 0 that are ‘black’ (in the normal BFS coloring). To improve the runtime further, we maintain two queues \( Q_0 \) and \( Q_1 \) each storing up to \( n/\log^2 n \) values of the grey 0 and grey 1 vertices. We also store two boolean variables, \( overflow-Q0, overflow-Q1 \), initialized to 0 and to be set to 1 when more elements are to be added to these queues (but they don’t have room). Now the algorithm proceeds in a similar fashion as the previous algorithm except
that, along with marking corresponding vertices 0 or 1 in the color array, it also
inserts them into the appropriate queues. i.e. when it expands vertices from $Q_0$
($Q_1$), it inserts their (white) neighbors colored 2 to $Q_1$ ($Q_0$ respectively) apart
from setting their color entries to 1 (0 respectively). When it runs out of space in
any of these queues to insert the new elements (as we have limited only $n/\lg^2 n$
values in each of the queues), it continues to make the changes (i.e. 2 to 1 or
2 to 0) in the color array directly without adding those vertices to the queue,
but set the corresponding overflow bit. Now instead of scanning the color array
for vertices labelled 0 or 1, we traverse the appropriate queues spending time
proportional to the sum of the degree of the vertices in the level. If the overflow
bit in the corresponding queue is 0, then we simply move on to the next queue
and continue. Otherwise, we switch to our previous algorithm and scan the array
appropriately changing the colors of their white neighbors and adding them to
the appropriate queue if possible. It is easy to see that this method correctly
explores all the vertices of the graph using $1.585n + o(n)$ bits.

To analyse the runtime, notice that as long as the overflow bit of a queue is
0, we spend time proportional the number of neighbors of the vertices in that
level, and we spend $O(m)$ time otherwise. When an overflow bit is 1, then the
number of nodes in the level is at least $n/\lg^2 n$ and this can not happen for more
than $\lg^2 n$ levels where we spend $O(m)$ time each. Hence, the total runtime is
$O(m \lg^2 n)$ proving the following.

**Theorem 5.** Given a directed or undirected graph, its vertices can be output in
a BFS order starting at a vertex using $1.585n + o(n)$ bits and in $O(m \lg^2 n)$ time.

By making the sizes of the two queues to $O(n/(f(n) \lg n))$ for any (slow
growing) function $f(n)$, we obtain\(^4\)

**Theorem 6.** Given a directed or undirected graph, its vertices can be output
in a BFS order starting at a vertex using $1.585n + O(n/f(n))$ bits and in $O(m f(n) \lg n)$ time
where $f(n)$ is any slow-growing function of $n$.

We do not know whether we can reduce the space further (to possibly $n + o(n)$
bits) while still maintaining the runtime to $O(m \lg^c n)$ for some constant $c$ or
even $O(mn)$. However, in the next section we provide such an algorithm for
MST.

### 3.2 Minimum Spanning Tree

In this section, we prove the following.

**Theorem 7.** A minimum spanning forest of a given undirected weighted graph,
where the weights of any edge can be represented in $O(\lg n)$ bits, can be found
using $n + O(n/f(n))$ bits and in $O(m \lg n f(n))$ time, for any function $f(n)$ such that
$1 \leq f(n) \leq n$.

\(^4\) Hagerup and Kammer [24] in their recent paper obtain a better time bound using
the same space.
Proof. Our algorithm is inspired by the MST algorithm of [19], but we work out the constants carefully. We provide a space efficient implementation of the Prim’s algorithm to find a minimum spanning tree. Prim’s algorithm starts with initializing a set $S$ with a vertex $s$. For every vertex $v$ not in $S$, it finds and maintains $d[v] = \min\{w(v,x) : x \in S\}$ and $\pi[v] = x$ where $w(v,x)$ is the minimum among $\{w(v,y) : y \in S\}$. Then it repeatedly deletes the vertex with the smallest $d$ value from $V - S$ adding it to $S$. Then the $d$ values are updated by looking at the neighbors of the newly added vertex. The space for $d$ values can take up to $O(n \lg n)$ bits. To reduce the space to $O(n)$ bits, we find and keep, in $O(n)$ time, the set $M$ of the smallest $n/(f(n) \lg n)$ values among the $d$ values of the elements of $V \setminus S$ in a binary heap. This takes $O(n/f(n))$ bits and $O(n)$ time. We maintain the set $S$ in a bit vector taking $n$ bits. We maintain the indices of $M$ in a balanced binary search tree, and each node (index $v$) has a pointer to its position in the heap of the $d$ values, and also stores the index $\pi[v]$. Thus we can think of $M$ as consisting of triples $(v,d[v],\pi[v])$ where $d[v]$ is actually a pointer to $d[v]$ in the heap. The storage for $M$ takes $O(n/f(n))$ bits. We also find and store the max value of $M$ in a variable $\text{Max}$ that also has the vertex label that achieves the maximum.

Now we execute Prim’s algorithm by repeatedly deleting elements only from $M$ and updating (decreasing) values in $M$ until $M$ becomes empty. In particular, while updating the values we check if the new value is larger than the variable $\text{Max}$. In such cases, we don’t do anything. Otherwise, we insert the new value in $M$ and delete the current vertex realizing the maximum value and we proceed further till $M$ becomes empty. Then (for $f(n) \lg n$ times) we find the next smallest $n/(\lg n f(n))$ values from $V \setminus M \setminus S$ and continue the process. Finding the $d$ values of every element in $L = V \setminus S \setminus M$ requires $O(m)$ time (for finding the minimum among all edges incident with vertices in $S$), and finding the smallest $n/f(n) \lg n$ values among them take $O(n)$ time. These steps are repeated $O(f(n) \lg n)$ times resulting in the overall runtime of $O(m f(n) \lg n)$. Note that the set $M$ can be found using $O(n/f(n))$ bits without necessarily storing all the $d$ values, but only up to $2n/f(n) \lg n$ values at a time. Membership queries in $S$ take a constant time, while membership in $M$ takes $O(\lg n)$ time. Insertions and deletions in $M$ take $O(\lg n)$ time.

In the heap, $n - 1$ deletemins and up to $m$ decrease key operations are executed which take $O((m+n+\lg n) \lg n)$ time by using a binary heap. Note that more sophisticated (for example, Fibonacci heap) implementations are unnecessary as the other operations dominate the running time. \qed

4 Applications of Findany dictionary

In what follows we use our findany data structure of Section 2 to develop a data structure as below.

Theorem 8. Let $x_1, x_2, \ldots, x_n$ be a sequence of non-negative integers, and let $m = \sum_{i=1}^{n} x_i$. Then the sequence can be represented using at most $m + 2n +$
\( o(m+n) \) bits such that we can determine whether the \( i \)-th element of the sequence is 0 and decrement it otherwise, in constant time.

Proof. Our first attempt is to encode each integer in unary, delimited by a separate bit, to take \( m + n \) bits. A select structure, as in Theorem 1, can help us access the corresponding elements. However, decrementing them involves changing this bitstring and so we need a dynamic version of Theorem 1 that has a \( O(\lg n / \lg \lg n) \) runtime for each of the operations [30].

Now we show how to use our findany structure of Theorem 3 to obtain a linear time algorithm. We maintain conceptually separate findany structures for each \( x_i \) using \( x_i + o(x_i) \) bits. Each of them stores a subset of \( \{1, 2, \ldots, x_i\} \) and is initialized to the full universe set. The total space is \( O(m+n) + o(m+n) \) bits. The findany structures of all the integers are concatenated into a single array of bits, separated by a delimiter (say the symbol 2). And we build a select structure for the delimiter (2) in \( O(m+n) \) time and using \( o(m+n) \) bits of extra space using a generalization of Theorem 1 for 3 symbols (see, for example [23]) so that we can identify the findany structure of vertex \( i \) in constant time by navigating to the \( i \)-th delimiter symbol. Note that though this bitstring will change in the course of an operation, the delimiter symbols aren’t modified, and the length of the string doesn’t change and so a static select structure suffices. Now decrementing \( x_i \) amounts to performing the findany operation on \( x_i \)’s structure (that actually tests whether \( x_i = 0 \)) and deleting the element if any, that is output from the findany operation. \( \square \)

Using the data structure we just developed, we show the following theorems,

**Theorem 9.** Given a directed acyclic graph \( G \), its vertices can be output in topologically sorted order using \( O(m+n) \) time using \( m + 3n + o(n+m) \) bits of space. The algorithm can also detect if \( G \) is not acyclic.

Proof. A standard algorithm repeatedly outputs a vertex with indegree zero and deletes that vertex along with its outgoing edges, until there are no more vertices. To implement this, we maintain first the set \( Z \) of indegree 0 vertices in the datastructure of Theorem 2 to support findany operation in constant time. This takes \( O(m+n) \) time and \( n + o(n) \) bits. We also represent the indegree sequence of the vertices using the data structure of Theorem 8. The algorithm repeatedly finds any element from \( Z \), outputs and deletes it from \( Z \). Then it decrements the indegree of its out-neighbors, and includes them in \( Z \) if any of them has become 0 (that can be determined by another findany structure) in the process. If \( Z \) becomes empty even before all elements are output (that can be checked using a counter or a bit vector), then at some intermediate stage of the algorithm, we did not encounter a vertex with indegree zero which means that the graph is not acyclic.

**Theorem 10.** Given a \( d \)-degenerate graph \( G \), its vertices can be output in \( d \)-degenerate order using \( m + 3n + o(m+n) \) bits and \( O(m+n) \) time. The algorithm can also detect if the given graph is not \( d \)-degenerate.
Proof. As in the topological sort algorithm, we maintain the set $Z$ of vertices whose degree in the entire graph is at most $d$, using our findany data structure. This takes $O(m + n)$ time and $n + o(n)$ bits. Then we represent the degree sequence of the vertices using the data structure of Theorem 8 except we subtract $d$ from each of them. I.e. $x_i = \max\{0, d_i - d\}$ where $d_i$ is the degree of the $i$-th vertex. The algorithm repeatedly finds any element from $Z$, outputs and deletes it from $Z$. Then it decrements the degree of its neighbors, and includes them in $Z$ if any of them has become 0. If $Z$ becomes empty even before all elements are output (that can be checked using a counter or a bit vector), then at some intermediate stage of the algorithm, we did not encounter a vertex with degree less than $d$, which means that the graph is not $d$-degenerate.

5 DFS and its applications using $O(m + n)$ bits

5.1 DFS

In what follows we describe how to solve DFS in $O(n + m)$ time and $O(n + m)$ bits of space and this is better (in terms of time and space) than Theorem 16 for sparse graphs. Recall that, our input graphs $G = (V,E)$ are represented using the standard adjacency list along with cross pointers, i.e. for undirected graphs given a vertex $u$ and the position in its list of a neighbor $v$ of $u$, there is a pointer to the position of $u$ in the list of $v$. In case of directed graphs, for every vertex $u$, we have a list of out-neighbors of $u$ and a list of in-neighbours of $u$. And, finally we augment these two lists for every vertex with cross pointers, i.e. for each $(u,v) \in E$, given $u$ and the position of $v$ in out-neighbors of $u$, there is a pointer to the position of $u$ in in-neighbors of $v$.

We describe our algorithm for directed graphs and mention at the end the changes required for undirected graphs. Central to our algorithm is an encoding of the out-degrees of the vertices in unary. Let the vertex set $V = \{1, 2, \cdots, n\}$. The unary degree sequence encoding $O$ of the directed graph $G$ has $n$ 0s to represent the $n$ vertices and each 0 is followed by a number of 1s equal to the out-degree of that vertex. Clearly $O$ uses $n + m$ bits and can be obtained from the out-neighbors of each vertex in $O(m + n)$ time. We use another bit string $E$ of the same length where every bit is initialized to 0. The bit string of $E$ corresponding to a position of 1 in the bitstring $O$ will correspond to an edge in the graph, and $E$ will be used to help us backtrack when the DFS has finished exploring a vertex. In particular, $E$ will mark the tree edges of the DFS as we build the DFS tree. Once DFS finishes traversing the whole graph, the number of ones in the $E$ array is exactly the number of tree edges (more precisely $n$ minus the number of connected components of the graph). We also have the color array, say $C$, having entries from $\{\text{white, gray, black}\}$ with the usual meaning i.e. each vertex is initially white meaning unexplored, is grayed when DFS discovers for the first time, and is colored black when it is finished i.e. its adjacency list has been checked completely. We can represent $C$ using Lemma 1 in $n \log 3 + o(n)$ bits so that individual entries can be accessed or modified in constant time. On $O$, we will have a static rank-select data structure that uses additional $o(m + n)$
bits using Theorem 1. So overall we have used $2m + (\lg 3 + 2)n + o(m + n)$ bits. Suppose $v_j$ is a child of $v_i$ in the final DFS tree. As suggested by Asano et al. [3], we can think of the DFS procedure as performing the following two steps repeatedly until all the vertices are explored. First step takes place when DFS discovers a vertex $v_j$ for the first time, and as a result $v_j$’s color changes to gray from white. We call this phase as forward step. And, when DFS completes exploring $v_j$ i.e. the subtree rooted at $v_j$ in the DFS tree, it needs to perform two tasks subsequently. First, it backtracks to its parent $v_i$, and then finds in $v_i$’s list the next white neighbor to explore. The later part is almost similar to the forward step described before. But, we call the first part alone as backtrack step. In what follows, we describe how to implement each step in detail.

We start our DFS with the starting vertex, say $r$, changing its color to gray in the color array $C$. Then, as in the usual DFS algorithm, we scan the out-adjacency list of $r$, and find the first white neighbor, say $v$, to make it gray. As the edge $(r,v)$ has been added to the DFS tree, we mark the position corresponding to the edge $(r,v)$ in $E$ (suppose $v$ is the $k$-th entry in $r$’s out-neighbor list, we mark the $k$-th 1 following the $r$-th 0) to $1^5$. We continue the process with the new vertex making it gray until we encounter a vertex $w$ that has no white out-neighbors. At this point, we will color the vertex $w$ black, and we need to backtrack.

To find the vertex to backtrack, we do the following. We go to $w$’s in-neighbor list to find a gray vertex which is its parent. For each gray vertex $t$ in $w$’s in-neighbor list, we follow the cross pointers to reach $w$ in $t$’s out-adjacency list and check its corresponding entry $(t,w)$ in $E$ array (using select operation to find $w$ after $t$-th 0). Observe that, among all these gray in-neighbors of $w$, only one edge out of them to $w$ will be marked in $E$ as this is the edge that DFS traversed while going in the forward direction to $w$. So once we find an in-neighbor $t$ such that the position corresponding to $(t,w)$ in $E$ is marked and $t$ is gray, we know that $w$’s parent is $t$ in the DFS tree. Also the cross pointer puts us in the position of $w$ in $t$’s out-neighbor list, and we start from that position to find the next white vertex to explore DFS. So the only extra computation we do is to spend time proportionate to the degree of each black vertex (to find its parent to backtrack) and so overall there is an extra overhead of $O(m)$ time. The navigation we do to determine the tree edges are on $O$ which is a static array, and so from Theorem 1, all these operations can be performed in constant time. Thus we have

**Theorem 11.** A DFS traversal of a directed graph $G$ can be performed in $O(n + m)$ time using $(2m + (\lg 3 + 2)n) + o(m + n)$ bits.

For undirected graphs, first observe that the unary degree sequence encoding $O$ takes $2m + n$ bits as each edge appears twice. As $E$ also takes $2m + n$ bits, overall we require $(4m + (\lg 3 + 2)n) + o(m + n)$ bits of space. As for DFS, observe that the forward step, as defined before, can be implemented in exact same manner. It is crucial to mention one subtle point that, while marking an edge $(v_i, v_j)$, we don’t mark its other entry i.e. $(v_j, v_i)$. So when DFS finishes, for

\[5\] Note that to implement this step, we only require the `select_0` operation in Theorem 1.
Tree edges exactly one of the two entries will be marked one in $E$ array whereas for back edges both entries are marked zero. Backtracking step is now little easier as we don’t have to switch between two lists. We essentially follow the same steps in adjacency list to check for a vertex $t$ in $w$’s list such that $t$ is gray and the corresponding entry for the edge $(t,w)$ is marked in $E$. Once found, we start with the next white vertex. Hence,

**Theorem 12.** A DFS traversal of an undirected graph $G$ can be performed in $O(n + m)$ time using $(4m + (\log 3 + 2)n) + o(m + n)$ bits.

We can decrease the space slightly by observing that, we are not really using the third color black. More specifically, we can continue to keep a vertex gray even after its subtree has been explored. As we only explore white vertices always and never expand gray or black, the correctness follows immediately. This gives us the following.

**Theorem 13.** A DFS traversal of a directed graph $G$ can be performed in $O(n + m)$ time using $(2m + 3n) + o(m + n)$ bits. For undirected graphs, the space required is $(4m + 3n) + o(m + n)$ bits.

### 5.2 Applications of DFS

One of the classical applications of DFS is to determine, in a connected undirected graph, all the cut vertices and bridges which are defined as, respectively, the vertices and edges whose removal results in a disconnected graph. Tarjan [33, 34] presented the first linear-time algorithms for these problems, which are based on computing what are called ‘low’ values for each vertex. This is the algorithm dealt with in textbooks [15, 1]. Since then several linear time algorithms have been proposed (See [33, 34, 10, 22, 32] and references in [32]). All these algorithms take $O(m + n)$ time and $O(n)$ words or $O(n \log n)$ bits of space. Here we give an $O(m + n)$ bits implementation using Schmidt [32]’s chain decomposition based algorithm to prove the following.

**Theorem 14.** Given a connected undirected graph $G$, in $O(m + n)$ time and $O(m + n)$ bits of space we can determine whether $G$ is 2-vertex (and/or edge) connected. If not, in the same amount of time and space, we can compute all the bridges and cut vertices of the graph. Also, within same time and space bound, we can output 2-vertex (and edge) connected components.

We briefly recall Schimdt’s algorithm and its main ingredient of chain decomposition. The algorithm first performs a depth first search on $G$. Let $r$ be the root of the DFS tree $T$. DFS assigns an index to every vertex $v$ i.e. the time vertex $v$ is discovered for the first time (discovery time) during DFS. Call it depth-first-index ($DFI(v)$). Imagine that the the back edges are directed away from $r$ and the tree edges are directed towards $r$. The algorithm decomposes the graph into a set of paths and cycles called chains as follows. See Figure 1 for an illustration.
First we mark all the vertices as unvisited. Then we visit every vertex starting at \( r \) in increasing order of DFI, and do the following. For every back edge \( e \) that originates at \( v \), we traverse a directed cycle or a path. This begins with \( v \) and the back edge \( e \) and proceeds along the tree towards the root and stops at the first visited vertex or the root. During this step, we mark every encountered vertex visited. This forms the first chain. Then we proceed with the next back edge at \( v \), if any, or move towards the next \( v \) in increasing DFI order and continue the process. Let \( D \) be the collection of all such cycles and paths. Notice that, the cardinality of this set is exactly the same as the number of back edges in the DFS tree as each back edge contributes to one cycle or a path. Also as initially every vertex is unvisited, the first chain would be a cycle as it would end in the starting vertex. Schmidt proved the following theorem.

**Theorem 15.** [32] Let \( D \) be a chain decomposition of a connected graph \( G(V, E) \). Then \( G \) is 2-edge-connected if and if the chains in \( D \) partition \( E \). Also, \( G \) is 2-vertex-connected if and if \( \delta(G) \geq 2 \) (where \( \delta(G) \) denotes the minimum degree of \( G \)) and \( D_1 \) is the only cycle in the set \( D \) where \( D_1 \) is the first chain in the decomposition. An edge \( e \) in \( G \) is bridge if and if \( e \) is not contained in any chain in \( D \). A vertex \( v \) in \( G \) is a cut vertex if and if \( v \) is the first vertex of a cycle in \( D \setminus D_1 \).

The algorithm (the tests in Theorems 15) can be implemented easily in \( O(m + n) \) time using \( O(m + n) \) words as we can store the DFIs and entire chain decomposition \( D \). To reduce the space to \( O(m + n) \) bits, we first perform a depth first search of the graph \( G \) (as mentioned in Theorem 12) and recall that at the end of the DFS procedure, we have the color array \( C \) with all colors black and the array \( E \) which encodes the DFS tree. Here for a tree edge \((i, j)\) where
i is closer to the root, the position corresponding to the edge \((i,j)\) is marked 1 and that corresponding to \((j,i)\) is marked 0, and the backedges are marked 0.

To implement the chain decomposition, we do not have space to store the chains or the DFS indices. To handle the latter (DFI), we (re)run DFS and then use Schmidt’s algorithm along with DFS in an interleaved way. Towards the end, we recolor all the vertices to white. To handle the former, we use two more arrays, one to mark the vertices visited in the chain decomposition, called visited and another array \(M\), to mark the edges visited during the chain decomposition. The array \(M\) has size \((n + 2m)\) bits, and it has the same initial structure as \(E\) i.e. 0’s separated by 1’s where 0’s denote edges and 1’s denote vertices. The details of forming the chain decomposition and finding all cut vertices, bridges and maximal 2-connected components using these arrays \(O\) (original outdegree encoding), \(E\) (the DFS tree), \(C\) (color array), visited and \(M\) (to mark edges) are explained below.

**Proof of Theorem 14.** We explain here the details of forming the chain decomposition and finding all cut vertices, bridges and maximal 2-connected components using these arrays \(O\) (original outdegree encoding), \(E\) (the DFS tree), \(C\) (color array), arrays visited and \(M\) (to mark edges).

We start at the root vertex \(r\), and using the array \(E\), find the first ‘back edge’ (non-tree edge) \((r,x)\) to \(r\). This can be found by going to the \(r\)-th 0 in \(O\) and then to the corresponding position in \(E\) that represents the vertex \(r\) (note that \(E\) has a lot more zeroes, and so we should get to the corresponding 0 of \(r\) in \(E\) by first getting to the corresponding position in \(O\)). If \(O\) has 1s after the corresponding 0, then we look for the first 0 after the corresponding position in \(E\) to find the back edge (as all the tree edges are marked 1). We mark \(r\) and \(x\) visited (if they were unvisited before) and mark both copies of the edge \((r,x)\) (unlike what we do in the forward step of DFS) using the cross pointer in \(M\). Now to obtain the chain, we need to follow the tree edges from \(x\). We use the ‘backtracking’ procedure we used earlier for DFS. We look for an (the only) edge marked 1 in \(E\) out of the edges incident on \(x\) by scanning the adjacency list, and that gives the parent \(y\) of \(x\) (Here is where we use the fact we only mark one copy of the edge as we explore the DFS tree.).

We continue after marking \(y\) visited, and the edge \((x,y)\) (both copies) in \(M\) until we reach \(r\) or a visited vertex when we complete the chain. Now we continue from where we left of in \(r\)’s neighborhood to look for the next back edge and continue this process. Once we are done with back edges incident on \(r\), we need to proceed to the next vertex in DFS order. As we have not stored the Depth First Indices, we essentially (re)run the DFS using the color array \(C\). For this, we flush out the color array to make every vertex white again. Note that we don’t make any changes to array \(E\) and \(O\) respectively. As this DFS procedure is deterministic, it will follow exactly the same sequence of paths like before, ultimately leading to the same DFS tree structure, and note that, this structure is already saved in array \(E\).

Clearly, the amount of space taken is \(O(m + n)\) bits. To analyze the runtime, note that, we first perform a DFS traversal which takes linear time. At the second
step, we basically perform one more round of DFS. As a visited node is never
explored (using the visited array), the overall runtime is \(O(m + n)\).

Edge connectivity (Theorem 15) can easily be checked using the array \(M\)
once we have the chain decomposition. The bridges are the edges marked 0 in
the array \(M\). Cut vertices can be obtained and listed out if and when we
reach the starting vertex while forming a chain, except at the first chain (if
exists).

To output the 2-vertex-connected components (i.e., the maximal 2-connected
subgraphs) of \(G\) we keep one more array of size \(n\) bits to mark which vertices are
cut vertices. Now, we perform a DFS starting at each cut vertex treating the cut
vertices as leaves and we output the visited vertices. Each such set of vertices
forms a 2-connected component. This also gives the so-called block-cut trees of
\(G\). In a similar fashion we can output maximal 2-edge-connected subgraphs of
\(G\) using the bridges marked in the array \(M\), and skipping those edges when
performing DFS.

\[\square\]

6 DFS and its applications using \(O(n)\) bits

6.1 Directed Graphs

Classical applications of DFS in directed graphs (see [15]) are to find strongly
connected components of a directed graph, and to do a topological sort of a
directed acyclic graph. We show here that while topological sort can be solved
using the same \(O(n)\) bits and \(O(m \log \log n)\) time (as for DFS), strongly con-
ected components of a directed graph can be obtained using \(O(n)\) bits and
\(O(m \log n \log \log n)\) time. These algorithms use the space efficient DFS version [19]
as blackbox, which is stated below (and is obtained by putting \(t(n) = O(\log \log n)\)
in Theorem 3.3 of [19]).

**Theorem 16.** [19] A DFS of a directed or an undirected graph on \(n\) vertices
and \(m\) edges can be performed in \(O(m \log \log n)\) time using \(O(n)\) bits of space.

**Strongly Connected Components** There is a classical two pass algorithm
(see [17] or [15]) for computing the Strongly Connected Components (SCC) of a
given directed graph \(G\) which works as follows. In the first step, it runs a DFS on
\(G^R\), the reverse graph of \(G\). In the second pass, it runs the connected component
algorithm using DFS in \(G\) but it processes the vertices in the decreasing order
of the finishing time from the first pass.

We can obtain \(G^R\) by switching the role of in and out adjacency lists present
in the input representation. As we can not remember the vertex ordering from
the first pass due to space restriction, we process them in batches of size \(\frac{n}{\log n}\)
in the reverse order i.e. we run a full DFS in \(G^R\) to obtain and store the last
\(\frac{n}{\log n}\) vertices in an array \(A\) as they are the ones which have the highest set
of finishing numbers in decreasing order. I.e. we maintain \(A\) as a queue of size \(\frac{n}{\log n}\)
and as and when a new element is finished, it is added to the queue and the
element with the earliest finish time at the other end of the queue is deleted. Now, we pick the vertices from $A$ one by one in the order from the queue with the latest finish time and start a fresh DFS in $G$ to compute the connected components and output all the vertices reachable as a SCC. The output vertices are marked in a bitmap so that we don’t output them again. Once we are done with all the vertices in $A$, we restart the DFS from the beginning and produce the next chunk of $\frac{n}{\lg n}$ vertices by remembering the last vertex produced in the previous step and stop as soon as we hit that boundary vertex. Then we repeat the connected component algorithm from this chunk of vertices and continue this way. It is clear that the algorithm produces the SCCs correctly. As we are calling the DFS algorithm $O(n)$ times, total time taken by this algorithm is $O(m \lg n \lg \lg n)$ with $O(n)$ bits of space. Hence, we have the following,

**Theorem 17.** Given a directed graph $G$ on $n$ vertices and $m$ edges, we can output the strongly connected components of $G$ in $O(m \lg n \lg \lg n)$ time and $O(n)$ bits of space.

**Topological Sort** The standard algorithm for computing topological sort [15] outputs the vertices of a DFS in reverse order. If we can keep track of the DFS numbers, then reversing is an easy task. While working in space restricted setting (with $o(n \lg n)$ bits), this is a challenge as we don’t have space to keep track of the DFS order. We can do as we did in the strongly connected components algorithm in the last section, by storing and outputting vertices in batches of $n/\lg n$ resulting in an $O(m \lg n \lg \lg n)$ time algorithm.

Elmasry et al.[19] showed that, the vertices of a DAG $G$ can be output in the order of a topological sort within the time and space bounds of a DFS in $G$ plus an additional $O(n \lg \lg \lg n)$ bits. As they also showed how to perform DFS in $O(m + n)$ time and $O(n \lg \lg n)$ bits, overall their algorithm takes $O(m + n)$ time and $O(n \lg \lg n)$ bits to compute a topological sorting of $G$. Their main idea is to maintain enough information about a DFS to resume it in the middle and apply this repeatedly to reverse small chunks of its output, produced in reverse order, one by one.

We observe that, instead of storing information to restart DFS and produce the reverse order, we simply work with the reverse graph itself (which can be obtained from the input representation by switching the role of in and out adjacency lists) and do a DFS in the reverse graph and output vertices as they are finished (or blackened) i.e. in the increasing order of finishing time. Note that the reverse graph is also a DAG. Now, if $(i, j)$ is an edge in the DAG $G$, then then $(j, i)$ is an edge in the reverse graph and $i$ will become black before $j$ while the algorithm performs DFS in the reverse graph. Hence, $i$ will be placed before $j$ in the correct topological sorted order. Thus we have

**Theorem 18.** Given a DAG $G$ on $n$ vertices and $m$ edges, if the black vertices of the DFS of $G$ can be output using $s(n)$ space and $t(n)$ time, then its vertices can be output in topologically sorted order using $s(n)$ space and $t(n)$ time assuming that the input representation has both the in and out adjacency list of the graph.
Corollary 1. Given a DAG \( G \) on \( n \) vertices and \( m \) edges, its vertices can be output in topologically sorted order using \( O(m \log \log n) \) time and \( O(n) \) bits.

6.2 Undirected Graphs

Finding a sparse biconnected subgraph of a biconnected graph. The problem of finding a \( k \)-connected spanning subgraph with the minimum number of edges of a \( k \)-connected graph is known to be NP-hard for any \( k \geq 2 \). But the complexity of the problem decreases drastically if all we want is to produce a “sparse” \( k \)-connected spanning subgraph, i.e. one with \( O(n) \) edges. Nagamochi and Ibaraki [29] gave a linear time algorithm which produces a \( k \)-connected spanning subgraph with at most \( kn - \frac{k(k+1)}{2} \) edges. Later, Cheriyan et al. [13] gave another linear time algorithm for \( k = 2 \) and 3 that produced a 2-connected spanning subgraph with at most \( 2n - 2 \) edges, and a 3-connected subgraph with at most \( 3n - 3 \) edges. Later, Elmasry [18], gave another linear time algorithm using DFS. We show here that this algorithm can be implemented using \( O(n) \) bits after we construct a DFS.

We briefly describe Elmasry’s algorithm. Let \( DFI(v) \) denote the index (integer) that represents the time at which vertex \( v \) is first discovered by DFS. Let \( low(v) \) be the smallest \( DFI \) value among the \( DFI \) values of vertices \( w \) such that \((v,w)\) is a back edge. (Note that this quantity is different from the “lowpoint” variable used in Tarjan’s [33] classical biconnectivity algorithm.) The algorithm maintains, along with all the the tree edges, \( DFI \) and \( low \) values, along with the back edge that realizes it (if any), for every vertex. As the root of the DFS tree doesn’t have any back edge and, as the underlying graph is 2-connected, root has only one child \( v \) so that there is no back edge emanating from \( v \) as well. Thus we get at most \( n - 2 \) back edges along with \( n - 1 \) tree edges, giving a subgraph with at most \( 2n - 3 \) edges. Elmasry [18] proved that the resulting graph is indeed a spanning 2-connected subgraph of \( G \). It is easy to see that the space requirement for the algorithm is \( O(n) \) words. We show how to perform the same task in \( O(n) \) bits, albeit with slightly more time. We assume that the given undirected graph \( G \) is 2-connected. The aim is to output the tree and backedges of the solution as we perform the DFS. Let \( \{v_1, v_2, \ldots, v_n\} \) be the vertices of the graph.

Our algorithm makes two passes over the input graph. In the first pass, it performs a DFS to output all the necessary back edges and in the second pass, it performs one more DFS to output the relevant tree edges. In the first pass, the algorithm performs a DFS with the usual color array and compressed stack (as in Theorem 16) along with one more array of \( n \) bits, which we call \( DBE \) (for Deepest Back Edge) array, which is initialized to all zero. \( DBE[i] \) is set to 1 if and only if the algorithm has found and output the the deepest back edge emanating from vertex \( v_i \). So, in this pass, whenever a white vertex \( v_i \) becomes grey (i.e. \( v_i \) is visited for the first time), we scan \( v_i \)’s adjacency list to mark, for every white neighbor \( v_j \), \( DBE[j] \) to 1 if and only if it was 0 before. The correctness of this step follows from the fact that as we are visiting vertices in DFS order, and if \( DBE[j] \) is 0, then vertex \( v_j \) is not adjacent to any of the
vertices we have visited so far, and as it is adjacent to \( v_j \), the deepest back edge emanating from \( v_j \) is \((v_i, v_j)\). Hence we output this edge and move on to the next neighbor and eventually with the next step of DFS. Note that some of the \( v_i \)'s neighbors who may eventually become children of \( v_i \) in the DFS tree, may spuriously output the tree edge as a backedge, we deal with this in the next pass. In the second pass, we output only tree edges. To do so, we start a fresh DFS and as we go along a new tree edge everytime, we output the edge as a part of the solution, except for those tree edges we had already outputted in the first pass, as we pointed out. To deal with this, all we have to do, before outputting an edge out of \( v_j \) (which is a child of \( v_i \) in the DFS tree), is to check if \( v_j \) has any back edge by checking whether we encountered a grey vertex while exploring \( v_j \). If yes, we continue to output the edge from \( v_j \) to its parent as the tree edge. Otherwise, \( v_j \) didn’t have any backedge, and the only tree edge out of it would have been output in the first pass while exploring its parent in the first pass, so we do not output the tree edge. This completes the description of the algorithm. As we performed just two DFSs, we have from Theorem 16,

**Theorem 19.** Given an undirected biconnected graph \( G \) on \( n \) vertices and \( m \) edges, we can output the edges of a sparse spanning 2-connected subgraph of \( G \) in \( O(m \log \log n) \) time and \( O(n) \) bits of space.

### 7 Conclusions and Open Problems

We have provided several implementations of BFS focusing on optimizing space without much degradation in time. In particular with \( 2n + o(n) \) bits we get an optimal linear time algorithm whereas squeezing space further gives an algorithm with running time \( O(m f(n) \log n) \) where \( f(n) \) can be any (extremely slow-growing) function of \( n \). One can immediately obtain similar time-space tradeoffs for natural applications of BFS including testing whether a graph is bipartite or to obtain all connected components of a graph.

For the MST problem, we could reduce the space further to \( n + o(n) \) bits. It is an interesting question whether we can perform BFS using \( n + o(n) \) bits with a runtime of \( O(m \log^c n) \) for some constant \( c \) or even \( O(mn) \).

En route, we developed a data structure that can maintain a subset of a universe of \( n \) elements and support \( \text{findany} \) operation, besides the standard dictionary operations, in constant time. We gave another application of this to find a topological sort of a DAG or the degeneracy order of a degenerate graph in \( O(m + n) \) time and \( O(m + n) \) bits of space. We believe that there will be other interesting applications of this data structure.

We have given efficient \( O(n) \) bits space implementations of topological sort and finding strongly connected components of a directed graph in time almost proportional to the time to compute DFS using \( O(n) \) bits. We also presented an \( O(m + n) \) time and \( O(m + n) \) space DFS traversal method using the unary degree sequence of the graph and some succinct rank-select data structures. For a large class of graphs including planar, bounded degree and bounded treewidth graphs,
this gives an $O(n)$ bits and $O(m+n)$ time DFS algorithm, partially answering an open question in [3]. It is still an interesting open problem whether DFS can be performed using $O(m+n)$ time and $O(n)$ bits (even in dense graphs).

By rerunning DFS and with some more bookkeeping, we can compute the chain decomposition of an undirected graph space efficiently. We showed that we can obtain cut vertices, bridges and maximal 2-connected components using the chain decomposition in the same time and space. Another nice application [10] of the chain decomposition is to obtain what is called an $s-t$ numbering of a 2-connected graph. This is simply a (directed) acyclic orientation of the undirected graph with two given vertices $s$ and $t$ such that $s$ has no incoming edges and $t$ has no outgoing edges. We do not how to compute an $s-t$ numbering of a 2-connected graph using $O(n)$ or even $O(m+n)$ bits.

Also we developed an $O(n)$ bits algorithm to find a sparse biconnected subgraph of a 2-connected graph, but we do not know whether we can test for 2-connectivity using $O(n)$ bits. These are interesting open problems.

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