The trigonal construction in the ramified case

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Abstract
To every double cover ramified in two points of a general trigonal curve of genus \(g\), one can associate an étale double cover of a tetragonal curve of genus \(g + 1\). We show that the corresponding Prym varieties are canonically isomorphic as principally polarized abelian varieties. This extends Recillas’ trigonal construction to covers ramified in two points.

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1 INTRODUCTION

Let \(\mathcal{R}^{tr}_g\) denote the moduli space of non-trivial étale double coverings of smooth trigonal curves of genus \(g\) and \(\mathcal{M}^{tet}_{g-1,0}\), the open set of the moduli space of tetragonal curves of genus \(g - 1\) consisting of tetragonal curves whose fibres of the 4:1 map have at least one étale point. The classical trigonal construction due to Recillas [8] gives a canonical isomorphism

\[ \mathcal{R}^{tr}_g \rightarrow \mathcal{M}^{tet}_{g-1,0} \]

such that for every covering \(\tilde{C} \rightarrow C\) of \(\mathcal{R}^{tr}_g\) the Prym variety is isomorphic to the Jacobian of its image in \(\mathcal{M}^{tet}_{g-1,0}\) as principally polarized abelian varieties.

The aim of this paper is to show that a similar statement is valid in the case of double covers over trigonal curves with two ramification points. If \(f : \tilde{C} \rightarrow C\) is a double cover of smooth curves ramified exactly at two points, the Prym variety of the cover, which we denote by \(P(f)\) or \(P(\tilde{C}/C)\), is a principally polarized abelian variety (ppav) ([7]). Apart from the étale case, it is the only way to obtain a ppav from a covering between curves.\(^{\dagger}\) In the sequel, a \textit{ramified double cover} will always mean a double covering ramified at exactly two points. We denote by \(Rb^{tr}_g\) the moduli space of

\(^{\dagger}\) With the exception of non-cyclic triple coverings over a genus 2 curve, whose Prym variety is also a ppav.
ramified double covers \( f : \tilde{C} \to C \) of smooth trigonal covers \( h : C \to \mathbb{P}^1 \) with \( g \) the genus of \( C \) and the additional property that the branch locus of \( f \) is disjoint from the ramification locus of \( h \).

We call an element \( \tilde{C} \xrightarrow{f} C \xrightarrow{h} \mathbb{P}^1 \) of \( Rb_{g, sp}^{tr} \) special if the branch locus of \( f \) is contained in a fibre of \( h \) and general otherwise. Let \( Rb_{g, sp}^{tr} \) denote the closed subset of \( Rb_{g, sp}^{tr} \) consisting of special coverings and \( Rb_{g, gen}^{tr} \) its complement consisting of general coverings. Moreover, let \( M_{tet}^{g, 1} \) denote the subspace of the moduli space of smooth tetragonal curves of genus \( g \) as defined in Section 3 and let \( R_{tet}^{g} \) the moduli space of étale double covers of smooth tetragonal curves of genus \( g \). Then our main theorem is (Theorems 4.3 and 5.1),

**Theorem 1.1.**

(a) There is a canonical isomorphism

\[
Rb_{g, sp}^{tr} \to M_{tet}^{g, 1}.
\]

If \( \tilde{C} \xrightarrow{f} C \xrightarrow{h} \mathbb{P}^1 \) is an element of \( Rb_{g, sp}^{tr} \) and \( X' \) the corresponding smooth tetragonal cover, we get an isomorphism of principally polarized abelian varieties

\[
P(f) \xrightarrow{\cong} JX'.
\]

(b) There is a canonical map

\[
Rb_{g, gen}^{tr} \to R_{tet}^{g+1}.
\]

If \( \tilde{C} \xrightarrow{f} C \xrightarrow{h} \mathbb{P}^1 \) is an element of \( Rb_{g, gen}^{tr} \) and \( \pi : Y \to X \) the corresponding étale double cover, then the principally polarized abelian varieties \((P(f), \Xi_f)\) and \((P(\pi), \Xi_\pi)\) are canonically isomorphic.

Furthermore, in case (b) of the theorem the image of the map is contained in the subspace \( R_{tet}^{g+1, 2} \), the locus of étale double coverings over tetragonal curves \( X \), such that the 4:1 map \( k : X \to \mathbb{P}^1 \) has exactly two fibres consisting of two simple ramification points (Proposition 5.2). Note that \( Rb_{g, gen}^{tr} \) and \( R_{tet}^{g+1, 2} \) are of the same dimension (see Remark 5.3). We do not know the exact image of the map between these moduli spaces nor whether it is generically injective.

One could also investigate the effect of applying the tetragonal construction to the double coverings ramified in two points. Unfortunately, the picture is not so clear as in the case of the trigonal construction. For instance, if one starts with a double covering \( f : \tilde{C} \to C \) ramified at two points, over a tetragonal curve \( C \) of genus \( g \) and assume that the branching points lie on the different fibers of the 4:1 map \( C \to \mathbb{P}^4 \), then the tetragonal construction produces two other étale double coverings over curves of genus \( g + 2 \). Since the tetragonal construction is not closed in the locus of ramified coverings over tetragonal curves, its meaning is not evident. That is the reason why this construction is not included here.

During the preparation of this paper we came across the article [6], where it is claimed that the Prym variety of a double covering of a trigonal curve ramified at two points is isomorphic as ppav to the Jacobian of certain tetragonal curve \( X \). The author uses a Galois-theoretic approach to construct a tower of curves, which links the curves in the double covering and \( X \). The referee took care of reporting on Dalalyan’s papers [5, 6], and apart from a substantial flaw, namely showing
that the obtained curves are connected, he concludes that his proofs are correct. Most likely this approach is equivalent to ours, but we still think it is worthwhile to give a more modern treatment of the question. So, we are not claiming that the isomorphisms between the ppav’s appearing in Theorem 1.1 were unknown, however, our methods are different and we provide a clear picture in terms of moduli spaces.

In Section 2, we define for every $\tilde{C} \rightarrow C \rightarrow \mathbb{P}^1$ the corresponding covering $Y \rightarrow X$ with $X$ tetragonal and work out its geometric properties. In Section 3, we recall a special case of Donagi’s extension of the trigonal construction which is used in Section 4 for the proof of part (a) of the theorem. Finally, in Section 5, we give the proof of part (b). Throughout the paper, we work over an algebraically closed field of characteristic zero.

## 2 DOUBLE COVERS OF TRIGONAL COVERS

Let $C$ be a smooth trigonal curve of genus $g \geq 3$ with trigonal cover $h : C \rightarrow \mathbb{P}^1$. According to Hurwitz formula the ramification divisor $R_h$ of $h$ is of degree $2g + 4$. Let $f : \tilde{C} \rightarrow C$ be a double cover branched over 2 points $p_1, p_2$ of $C$. We assume that $p_1$ and $p_2$ are disjoint from $R_h$. Let $C^{(3)}$ and $\tilde{C}^{(3)}$ the third symmetric products of $C$ and $\tilde{C}$, respectively. Let $\mathbb{P}^1 \simeq g_3^1 \hookrightarrow C^{(3)}$ be the natural embedding of the trigonal linear system. We define the variety $Y$ by the following left hand cartesian diagram:

\[
\begin{array}{ccc}
Y & \rightarrow & \tilde{C}^{(3)} \\
\downarrow{s:1} & & \downarrow{s:1} \\
g_3^1 & \rightarrow & C^{(3)}
\end{array}
\]

The fibre of $\tilde{k}$ over a point $a \in \mathbb{P}^1$ consists of the eight sections $s$ of $f$ over $a$:

\[ s : h^{-1}(a) \rightarrow f^{-1}h^{-1}(a) \quad \text{with} \quad f \circ s = \text{id}. \]

(In [4] Donagi denotes $Y$ by $h_\ast \tilde{C}$, since considering $\tilde{C}$ as a local system on $C$, this is just the push forward local system on $\mathbb{P}^1$.) Let denote $\iota$ the involution on $\tilde{C}$ associated to $f$. There are two structures on $Y$, an involution denoted also by $\iota$:

\[ \iota : Y \rightarrow Y, \quad q_1 + q_1 + q_3 \mapsto \iota(q_1) + \iota(q_2) + \iota(q_3), \]

and an equivalence relation: two sections

\[ s_1, s_2 : h^{-1}(a) \rightarrow f^{-1}h^{-1}(a) \]

are equivalent $s_1 \sim s_2$ if they differ by an even number of changes $q \mapsto \iota(q)$. The orientation cover of $h \circ f$ is the quotient $O := O(h \circ f) = Y / \sim$ by this equivalence relation. It is a branched double cover over $\mathbb{P}^1$. For $n = 3$, we have the cartesian diagram (see [4, Section 2.1 and Lemma 2.1]):
with \( X := Y/t \).

The following lemma is easy to check.

**Lemma 2.1.** The involution \( \iota : Y \to Y \) is fixed-point free.

Identifying the \( \mathbb{P}^1 \) of diagram (2.2) with the image of \( h \), we have according to [4, Lemma 2.3]:

**Lemma 2.2.** The cover \( O \to \mathbb{P}^1 \) is branched exactly at \( h(p_1) \) and \( h(p_2) \).

Here, a branch point is a point over which the fibre of \( O \to \mathbb{P}^1 \) consists of 1 point, so the fibre might be a singular point of \( O \). Note that \( O \to \mathbb{P}^1 \) is of degree 2, so Lemma 2.2 implies that \( O \) is connected.

**Proposition 2.3.** Suppose the branch locus of \( f \) and the ramification locus of \( h \) are disjoint. Then the curve \( Y \) is connected.

**Proof.** It was proven in [9] that \( Y \) consists of two connected components when \( f \) is \( \text{étale} \). In that case, all the elements of the monodromy group of \( \tilde{k} \) are given by even permutations. The existence of ramification points for \( f \) adds also odd permutations to the monodromy group, and taking into account that \( O \) is connected, this implies that the curve \( Y \) is connected. \( \square \)

Recall that an element \( \tilde{C} \to C \to \mathbb{P}^1 \) of \( Rb^r_g \) is **special** if the branch locus of \( f \) is contained in a fibre of \( h \) and it is **general** otherwise.

**Proposition 2.4.** Suppose the branch locus of \( f \) and the ramification locus of \( h \) are disjoint.

(1) If \( h \circ f \) is general, then the curve \( Y \) is smooth;
(2) If \( h \circ f \) is special, the curve \( Y \) is smooth apart from two nodes.

**Proof.** The proof is very similar to the proof of [9, Proposition on p. 107] where Welters proves that \( Y \) is smooth when \( f \) is an \( \text{étale} \) double cover of a \( d \)-gonal curve (with some assumptions on the ramification of \( h \)). In particular, \( Y \) is smooth for \( d = 3 \). As we will see, the main difference lies in the fact that, if a fibre of \( h \) contains both branch points \( p_1 \) and \( p_2 \) of \( f \), the curve \( Y \) acquires two nodes; otherwise, \( Y \) is smooth.

Let \( D_i \) be the fibre of \( h \) passing through \( p_i \), we denote by the same letter the divisor on \( C \) as well as the corresponding point of \( C^{(3)} \). Of course, \( D_1 = D_2 \) if \( h \circ f \) is special. Let \( \tilde{D}_i \) denote the points of \( Y \) defined by \( D_i \) for \( j = 1, \ldots, \nu_i \) (\( \nu_i = 2 \), respectively \( \nu_i = 4 \), if \( h \circ f \) is special, respectively general). The same proof as in [9] works for all points of \( Y \setminus (\cup_{i,j} \tilde{D}_i) \). We shall show that \( Y \) is smooth at the points \( \tilde{D}_i \) if \( h \circ f \) is general and nodal if it is special.
So, let $D$ be one of the points $D_i \in C^{(3)}$ and $\tilde{D} \in \tilde{C}^{(3)}$ a point above it. Suppose

$$D = p + q + r \quad \text{and} \quad \tilde{D} = \tilde{p} + \tilde{q} + \tilde{r},$$

where $p$ is one of the branch points of $f$, so $f^*(p) = 2\tilde{p}$. The Zariski tangent spaces yield the cartesian diagram

$$
\begin{array}{ccc}
T_Y(\tilde{D}) & \rightarrow & T_{\xi_0}(\tilde{D}) \\
\downarrow{df} & & \downarrow{df^{(0)}} \\
T_{g_3'}(D) & \rightarrow & T_{\epsilon_0}(D)
\end{array}
$$

(2.3)

where $j : g_3^1 \hookrightarrow C^{(3)}$ denotes the inclusion map. The curve $Y$ will be smooth at $\tilde{D}$ if and only if $\dim T_Y(\tilde{D}) = 1$. It is easy to see ([9, p. 104]) that this is the case if and only if

$$T_{C^{(3)}}(D) = djT_{g_3^1}(D) + df^{(3)}T_{\tilde{C}^{(3)}}(\tilde{D}).$$

(2.4)

According to deformation theory, the lower right triangle of diagram (2.3) is given by

$$
\begin{array}{ccc}
H^0(\mathcal{O}_D(\tilde{D})) & \rightarrow & H^0(\mathcal{O}_D(D)) \\
\downarrow{\beta} & & \\
H^0(\mathcal{O}_C(D))/H^0(\mathcal{O}_C) & \rightarrow & H^0(\mathcal{O}_D(D))
\end{array}
$$

(2.5)

where $\alpha$ is the canonical map. In order to define the map $\beta$ precisely, consider $\mathcal{O}_C(D)$ (respectively $\mathcal{O}_{\tilde{C}}(\tilde{D})$) as a subsheaf of the rational function field $R_C$ of the curve $C$ (respectively, $R_{\tilde{C}}$ of the curve $\tilde{C}$) by putting for each point $p \in C$ and similarly for each $\tilde{p} \in \tilde{C}$,

$$\mathcal{O}_C(D)_p := m_{C,p}^{-\nu_p(D)} \quad \text{and} \quad \mathcal{O}_{\tilde{C}}(\tilde{D})_{\tilde{p}} := m_{\tilde{C},\tilde{p}}^{-\nu_{\tilde{p}}(\tilde{D})},$$

where $m_{C,p}$ respectively $m_{\tilde{C},\tilde{p}}$ is the maximal ideal in $\mathcal{O}_{C,p}$, respectively $\mathcal{O}_{\tilde{C},\tilde{p}}$. Translating diagram (2.5) to $\mathcal{O}_C$ and $\mathcal{O}_{\tilde{C}}$ gives the diagram

$$
\begin{array}{ccc}
m_{C,p}^{-1}/\mathcal{O}_{C,p} \oplus m_{C,q}^{-1}/\mathcal{O}_{C,q} \oplus m_{C,r}^{-1}/\mathcal{O}_{C,r} & \rightarrow & m_{\tilde{C},\tilde{p}}^{-1}/\mathcal{O}_{\tilde{C},\tilde{p}} \oplus m_{\tilde{C},\tilde{q}}^{-1}/\mathcal{O}_{\tilde{C},\tilde{q}} \oplus m_{\tilde{C},\tilde{r}}^{-1}/\mathcal{O}_{\tilde{C},\tilde{r}} \\
\downarrow{\gamma} & & \\
H^0(\mathcal{O}_C(D))/H^0(\mathcal{O}_C) & \rightarrow & m_{C,p}^{-1}/\mathcal{O}_{C,p} \oplus m_{C,q}^{-1}/\mathcal{O}_{C,q} \oplus m_{C,r}^{-1}/\mathcal{O}_{C,r}
\end{array}
$$

where $\beta$ is given by the transposition of the natural map

$$\beta^* : \Omega_{C,p}^{1}/m_{C,p}^{1} \oplus \Omega_{C,q}^{1}/m_{C,q}^{1} \oplus \Omega_{C,r}^{1}/m_{C,r}^{1} \rightarrow \Omega_{\tilde{C},\tilde{p}}^{1}/m_{\tilde{C},\tilde{p}}^{1} \oplus \Omega_{\tilde{C},\tilde{q}}^{1}/m_{\tilde{C},\tilde{q}}^{1} \oplus \Omega_{\tilde{C},\tilde{r}}^{1}/m_{\tilde{C},\tilde{r}}^{1}$$

induced by $f$. Let $t_p, t_q, t_r$ be local parameters around $p, q$ and $r$, respectively. Then $t_{\tilde{p}} := t^{\nu_p(p)} = t^2$ is a local parameter around $\tilde{p}$, since $f$ has ramification index 2 at $\tilde{p}$. We claim that $\beta^*$ restricted
to the summand $\Omega^1_{C,p}/m_{C,p}^1$ is not injective, hence $\beta$ restricted to $\Omega^1_{\tilde{C},p}/m_{\tilde{C},p}^1$ cannot be surjective. Indeed, if $dt_p$ is a generator of the one-dimensional space $\Omega^1_{C,p}/m_{C,p}^1$, then
\[
\beta^*(dt_p) = dt_p = d(t_p^2) = 2t_p \, dt_p
\]
vanishes modulo $m_{C,p}^1$, therefore the restriction is zero. In particular, the restriction of $\beta$ to $\Omega^1_{\tilde{C},\tilde{p}}/m_{\tilde{C},\tilde{p}}^1$ is zero, since the target is one-dimensional.

(1) Suppose $h \circ f$ is general. In this case $f$ is étale at $\tilde{q}$ and $\tilde{r}$. Let $t_{\tilde{q}}$ and $t_{\tilde{r}}$ local parameters around $\tilde{q}$ and $\tilde{r}$. We have that
\[
\beta(t_{\tilde{p}}) = 0, \quad \beta(t_{\tilde{q}}) = t_{\tilde{q}}, \quad \beta(t_{\tilde{r}}) = t_{\tilde{r}}.
\]
Then $\beta$ is surjective onto the summands $m_{C,q}^{-1}/\mathcal{O}_{C,q}$ and $m_{C,r}^{-1}/\mathcal{O}_{C,r}$, but not onto the first one, where $\beta$ vanishes.

The curve $Y$ is smooth at $\tilde{D}$ if and only if the composition of $\alpha$ with the cokernel of $\beta$ is surjective, that is, if and only if the map
\[
H^0(\mathcal{O}_C(D))/H^0(\mathcal{O}_C) \to m_{C,p}^{-1}/\mathcal{O}_{C,p} \tag{2.6}
\]
is surjective. As a vector space, the target is generated by $t_p^{-1}$. Let $\psi \in H^0(\mathcal{O}_C(D))$ be a section with corresponding divisor $\tilde{D} = \text{div} \, \psi + D$. Its image $\tilde{\psi}$ in $m_{C,p}^{-1}/\mathcal{O}_{C,p}$ can be written as $\tilde{\psi} = c_k t^{-k}$ with $c_k \neq 0$. So the map (2.6) is surjective if such function exists with $k = 1$. As $\nu_p(D) = \nu_p(\psi) + \nu_p(D) = -1 + 1 = 0$, it is enough to take a divisor $\tilde{D}$ with $\nu_p(D) = 0$, that is, which does not contain $p$. Therefore, $Y$ is smooth at $\tilde{D}$.

(2) Suppose $h \circ f$ is special. In this case, we can assume $p = p_1$ and $q = p_2$ are the branch points of $f$. Then
\[
\beta(t_{\tilde{p}}) = 0, \quad \beta(t_{\tilde{q}}) = 0, \quad \beta(t_{\tilde{r}}) = t_{\tilde{r}}
\]
and $Y$ is smooth at $\tilde{D}$ if and only if the map
\[
H^0(\mathcal{O}_C(D))/H^0(\mathcal{O}_C) \to m_{C,p}^{-1}/\mathcal{O}_{C,p} \oplus m_{C,q}^{-1}/\mathcal{O}_{C,q} \tag{2.7}
\]
is surjective. Now the target space has as basis the vectors $(t_p^{-1}, 0), (0, t_q^{-1})$, so for the surjectivity one requires the existence of a section $\psi \in H^0(\mathcal{O}_C(D))$ such that its corresponding divisor $\tilde{D}$ satisfies
\[
\nu_p(D) = \nu_p(\psi) + \nu_p(D) = -1 + 1 = 0, \quad \nu_q(D) = \nu_q(\psi) + \nu_q(D) = 0 + 1 = 1,
\]
that is, a divisor $\tilde{D}$ in the fibre of $h$ containing $q$ but not $p$, which gives a contradiction since the linear series $g_3^1$ defining $h : C \to \mathbb{P}^1$ is base-point-free.

In consequence, $Y$ has two singularities at the divisors $\tilde{D}_1$ and $\tilde{D}_2$ over $D = \tilde{p}_1 + \tilde{p}_2 + \tilde{r}$, that is, a divisor $\tilde{D}$ in the fibre of $h$ containing $q$ but not $p$, which gives a contradiction since the linear series $g_3^1 \cong \mathbb{P}^1 \to C^{(3)}$ is given by $t \mapsto (t, t, t)$. This map corresponds to the ring homomorphism
On the other hand, the map \( \overline{C}^{(3)} \to C^{(3)} \) can be locally given by \( (w_1, w_2, w_3) \mapsto (w_1^2, w_2^2, w_3) \), so at the level of rings this map is \( \mathbb{C}[z_1, z_2, z_3] \to \mathbb{C}[w_1, w_2, w_3], z_i \mapsto w_i^2 \). Then the local description for the fibered product \( Y \) is the tensor product

\[
\mathbb{C}[t] \otimes_{\mathbb{C}[z_1, z_2, z_3]} \mathbb{C}[w_1, w_2, w_3] = \mathbb{C}[w_1, w_2, w_3, t] = \mathbb{C}[w_1, w_2]
\]

which shows that the singularities \( D_1 \) and \( D_2 \) are nodal.

\[ \square \]

**Proposition 2.5.** Assume \( h \circ f \) is a general, then \( g_Y = 2g + 1 \).

**Proof.** There are two types of ramification of \( \bar{k} : Y \to \mathbb{P}^1 \) over a branch point \( a \) of \( \bar{k} \). Either \( a \) is a branch point of \( h \) of ramification index \( i = 1 \) or \( 2 \). Then \( a \) is a branch point for \( \bar{k} \) with ramification of type \( (2, 2, 1, 1, 1) \) if \( i = 1 \), and of type \( (3, 3, 1, 1) \) if \( i = 2 \). Or \( a = h(p_\nu) \) for \( \nu = 1 \) or \( 2 \). Then \( a \) is a branch point for \( h \) with ramification of type \( (2, 2, 2, 2) \). Since \( |R_h| = 2g + 4 \) and \( p_1 \) and \( p_2 \) are disjoint from \( R_h \), this gives for the ramification divisor \( R_{\bar{k}} \) of \( \bar{k} \):

\[
|R_{\bar{k}}| = 2(2g + 4) + 2 \cdot 4 = 4g + 16
\]

and the result follows from the Hurwitz formula.

\[ \square \]

We will see in Section 4 that in the case of special coverings, \( Y \) has arithmetic genus \( 2g + 1 \); more precisely, it is the union of two curves of genus \( g \) intersecting in two points.

### 3 | DONAGI’S EXTENSION OF THE TRIGONAL CONSTRUCTION

The usual trigonal construction for étale double covers of smooth trigonal curves is the following theorem, due to Recillas [8]. We recall it for étale double covers \( f \) of trigonal curves \( C' \) of genus \( g + 1 \) instead of genus \( g \), since we need it in this case.

Let \( \mathcal{R}^{lr}_{g + 1} \) denote the moduli space of étale double covers of smooth trigonal curves of genus \( g + 1 \), so it consists of triples \( (C, \eta, g_3^1) \), where \( \eta \in \text{Pic}^0(C)[2] \setminus \{0\} \) (which defines the double covering over \( C \)) and \( g_3^1 \) is a linear series on \( C \). Let \( \mathcal{C}^1_{g,4} \) denote the moduli space of pairs \( (C, g_4^1) \), with \( C \) a smooth curve of genus \( g \) together with a \( g_4^1 \), and \( \mathcal{M}^{tet}_{g,0} \subset \mathcal{C}^1_{g,4} \) denote the open subspace of tetragonal curves \( X \) of genus \( g \) with the property that above each point of \( \mathbb{P}^1 \) there is at least one étale point of \( X \). In the sequel, trigonal and tetragonal curves are understood as a pair of a curve with its linear series. In [8] Recillas showed

**Theorem 3.1.** The trigonal construction gives an isomorphism

\[
T^0 : \mathcal{R}^{lr}_{g + 1} \to \mathcal{M}^{tet}_{g,0}.
\]

Moreover, for each \([h \circ f] \in \mathcal{R}^{lr}_{g + 1}\), \( T^0 \) induces an isomorphism of principally polarized abelian varieties

\[
\text{Pr}(f) \simeq J(T^0(h \circ f)).
\]
Remark 3.2. Recall that curves of genus $g \geq 5$ (respectively, $g > 9$ and not elliptic–hyperelliptic) admit at most one $g^1_3$ (respectively, at most one $g^1_4$) and they form a closed subset in the moduli of smooth curves $M_g$. So, for high genus one can identify the moduli space $M^\text{tet}_{g,0}$ with its corresponding closed subspace in $M_g$ and something similar holds for $R^\text{tr}_{g+1}$. On the other hand, any curve of genus $g \leq 4$ (respectively, $g \leq 6$) possesses a $g^1_3$ (respectively, $g^1_4$) and in this case the isomorphism $T_0$ sends triples $(C, \eta, g^1_3)$ to pairs $(X, g^1_4)$.

In [4] Donagi extended the map $T^0_0$ to the partial compactification consisting of admissible double covers of trigonal curves of genus $g+1$, whose Prym variety is an element in $A_g$, the moduli space of principally polarized abelian varieties of dimension $g$. Recall that an admissible double cover of a connected stable curve $C'$ is a double cover $\pi : \tilde{C}' \to C'$ with $\tilde{C}'$ stable such that

- the nodes of $\tilde{C}'$ map under $\pi$ exactly to the nodes of $C'$;
- away from the nodes $\pi$ is an étale double cover.

Let $\overline{R}^\text{tr}_{g+1}$ denote the moduli space of admissible\(^\dagger\) double covers of curves of arithmetic genus $g+1$, whose Prym variety is in $A_g$ and let $M^\text{tet}_g$ denote the moduli space of smooth tetragonal curves of genus $g$. Donagi showed in [4, Theorem 2.9],

**Theorem 3.3.** The trigonal construction gives an isomorphism

$$T : \overline{R}^\text{tr}_{g+1} \to M^\text{tet}_g.$$ 

Moreover, he showed that if $\tilde{C}' \to C' \to \mathbb{P}^1$ corresponds to the tetragonal curve $k' : X' \to \mathbb{P}^1$ and $p$ is a point of $\mathbb{P}^1$, then the following types of fibres over $p$ of $f', k'$ and $h'$ correspond to each other,

1. $k'$ and $h'$ are étale, $f'$ is étale;
2. $k'$ and $h'$ have simple ramification points, $f'$ is étale;
3. $k'$ and $h'$ each have a ramification point of index 2, $f'$ is étale;
4. $k'$ has two simple ramification points, the fibre of $h'$ consists of a simple node and a smooth point, $f'$ is admissible;
5. $k'$ has a ramification point of index 3, the fibre of $h'$ consists of a node and at exactly one branch of it $h'$ is ramified, $f'$ is admissible.

In particular, if $h'$ admits nodes of types (4) or (5), the covering $f'$ is admissible. By [2], its Prym variety, defined again as the component of the norm map containing 0, is a principally polarized abelian variety of dimension $g$, where $g+1$ is the arithmetic genus of $C'$.

The *trigonal construction* is formally the same as in the smooth case. Let $\tilde{C}' \to C' \to \mathbb{P}^1$ be an element of $\overline{R}^\text{tr}_{g+1}$, and define $Y'$ in the same way as we defined $Y$ in diagram (2.1). The involution of $\tilde{C}'$ induces an involution on $Y'$ of which $X'$ is the quotient. (Donagi defines $X'$ as follows: The restriction $f'^0 : \tilde{C}'^0 \to C'^0$ of $f'$ to the smooth parts can be considered as a local system on $C'^0$. Its push-forward $h'_s \tilde{C}'^0$ to $\mathbb{P}^1$ admits an involution $i$. Then $X'$ is defined as the completion of the quotient $h'_s \tilde{C}'^0 / (i)$.)

\(^\dagger\) In Donagi’s work these coverings are called “allowable.” Often the coverings with this property are also referred to as “admissible.” We use this notion.
Conversely, the inverse $T^{-1}$ of $T$ is given as follows. Given $k' : X' \to \mathbb{P}^1$ in $\mathcal{M}_{g,1}^{\mathrm{et}}$, we denote by $k'_0 : X'_0 \to \mathbb{P}^1_0$ its restriction to the open part consisting of fibres of types (1), (2) and (3). Let $\mathcal{C}'$ denote the closure of $S^2_{\mathbb{P}^1}X'_0$, the relative second symmetric product of $X'_0$ over $\mathbb{P}^1$. Any fibre over a point $p \in \mathbb{P}^1$ consists of all unordered pairs in $k'^{-1}(p)$ and $\mathcal{C}'$ admits an obvious involution of which $\mathcal{C}'$ is the quotient. Moreover, the cover $\tilde{C}' \to \mathbb{P}^1$ factorizes via maps $f' : \tilde{C}' \to \mathcal{C}$ and $h' : \mathcal{C} \to \mathbb{P}^1$. The tower $\tilde{C}' f' \to \mathcal{C} h' \to \mathbb{P}^1$ is then an element of $\mathcal{R}^{\mathrm{tr}}_{g+1}$.

The same proof as in the smooth case gives an isomorphism

$$\Pr(f') \simeq J(T(h' \circ f')) \quad (3.1)$$

In the next section, we need the following corollary. Let

$$S^{\mathrm{tr}}_{g+1} \subset \mathcal{R}^{\mathrm{tr}}_{g+1}$$

the locally closed subset of $\mathcal{R}^{\mathrm{tr}}_{g+1}$ consisting of all elements of type (4), that is, all admissible double covers $f' : \tilde{C}' \to \mathcal{C}'$ of trigonal curves $\mathcal{C}'$ of arithmetic genus $g + 1$ with $\mathcal{C}'$ admitting exactly one node, such that the fibre of $h'$ containing the node contains also a smooth point. Let $\mathcal{M}_{g,1}^{\mathrm{et}}$ be the corresponding subset of tetragonal covers $g' : X' \to \mathbb{P}^1$ with exactly one fibre consisting of two simple ramification points and otherwise only fibres of types (1), (2), and (3). So, we get

**Corollary 3.4.** Restricting the trigonal construction to $S^{\mathrm{tr}}_{g+1}$ gives an isomorphism

$$T : S^{\mathrm{tr}}_{g+1} \to \mathcal{M}^{\mathrm{et}}_{g,1}$$

inducing for every element of $S^{\mathrm{tr}}_{g+1}$ the isomorphism (3.1).

**4 | PROOF OF THE MAIN THEOREM FOR SPECIAL RAMIFIED COVERS**

Let $C$ be a smooth trigonal curve of genus $g$ with trigonal cover $h : C \to \mathbb{P}^1$ and let $f : \tilde{C} \to C$ be a special ramified double cover, that is, the two branch points of $f$ are contained in a fibre of $h$. Suppose $p_1, p_2 \in C$ are the branch points of $f$ and $q_1, q_2 \in \tilde{C}$ the corresponding ramification points. Let $Rb^{\mathrm{tr}}_{g,sp}$ denote the closed subset of the variety $Rb^{\mathrm{tr}}_g$ consisting of special covers.

Define new curves $\tilde{C}'$ and $C'$ by identifying the points:

$$\tilde{C}' := \tilde{C} / (q_1 \sim q_2) \quad \text{and} \quad C' := C / (p_1 \sim p_2).$$

The covering $f$ induces a covering $f' : \tilde{C}' \to C'$ which clearly is an admissible double cover. Similarly, $h$ induces a trigonal cover $h' : \mathcal{C}' \to \mathbb{P}^1$, such that the tower $\tilde{C}' f' \to \mathcal{C}' h' \to \mathbb{P}^1$ is an element of $S^{\mathrm{tr}}_{g+1}$. So, we get a map

$$n : Rb^{\mathrm{tr}}_{g,sp} \to S^{\mathrm{tr}}_{g+1}. $$
Lemma 4.1. The map \( n : R_{g,sp}^{tr} \to S_{g+1}^{tr} \) is an isomorphism.

Proof. Normalization gives the inverse map. \( \square \)

Lemma 4.2. Let \( \tilde{C} \xrightarrow{f} C \xrightarrow{h} \mathbb{P}^1 \) be an element of \( R_{g,sp}^{tr} \) and \( \tilde{C}' \xrightarrow{f'} C' \xrightarrow{h'} \mathbb{P}^1 \) the corresponding element of \( S_{g+1}^{tr} \). Normalization induces the following isomorphism of principally polarized abelian varieties

\[
N : P(f') \to P(f).
\]

Proof. According to the proof of [2, Proposition 3.5], the norm map induces the following commutative diagram with exact rows and columns

\[
\begin{array}{ccc}
0 & \to & \mathbb{Z}/2 \\
\downarrow & & \downarrow \\
0 & \to & P(f') \times \mathbb{Z}/2 \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{C}' \\
\downarrow Nm & & \downarrow Nm \\
0 & \to & \mathcal{C} \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

\[
\begin{array}{ccc}
0 & \to & \mathcal{J}\mathcal{C}' \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{J}\mathcal{C} \\
\downarrow Nm & & \downarrow Nm \\
0 & \to & \mathcal{J}\mathcal{C}' \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

This gives the assertion. \( \square \)

Combining Corollary 3.4 with Lemma 4.1 and Equation (3.1) with Lemma 4.2 gives the following theorem:

Theorem 4.3. The map

\[
T \circ n : R_{g,sp}^{tr} \to \mathcal{M}^{tet}_{g,1}
\]

is an isomorphism.

Moreover, if \( \tilde{C} \xrightarrow{f} C \xrightarrow{h} \mathbb{P}^1 \) is an element of \( R_{g,sp}^{tr} \) and \( X' \) is the corresponding (smooth) tetragonal cover, then we have an isomorphism of principally polarized abelian varieties

\[
P(f) \to \mathcal{J}X'.
\]

Remark 4.4.

(i) In [4], Donagi outlines also explicitly the inverse map \( T^{-1} \). Since the inverse of the map \( n \) is clear, this gives an explicit version of the inverse map of Theorem 4.3.
(ii) One can express $JX'$ as the Prym variety of an étale double cover of $X'$. In fact, the double cover $Y' \to X'$ which occurred in Corollary 3.4 is trivial. Hence, $JX' = P(Y'/X')$.

Finally, we shall describe the covering $Y \to X$ with $X$ and $Y$ defined as in Section 2. As a consequence, it shows that $P(Y/X)$ is an abelian variety of the right dimension.

Let $D := p_1 + p_2 + r$ be the unique divisor in $g_3^1$ containing the two branch points of $f$. Denote by

$$\tilde{k} := k \circ \pi : Y \to \mathbb{P}^1$$

the $8 : 1$ covering of diagram (2.2). If $r_1$ and $r_2 = \iota(r_1)$ are the two points of $f^{-1}(r)$, then

$$D_1 := q_1 + q_2 + r_1 \in Y \quad \text{and} \quad D_2 := q_1 + q_2 + r_2 \in Y$$

are exactly the two points of $\tilde{k}^{-1}(h(p_1))$.

**Lemma 4.5.** The curve $Y$ consists of two smooth irreducible components

$$Y = Y_1 \cup Y_2$$

with

$$Y_2 = \iota(Y_1), \quad Y_1 \cap Y_2 = \{D_1, D_2\}, \quad \text{and} \quad \iota(D_1) = D_2.$$

**Proof.** According to Lemma 2.2 the cover $\omega : O \to \mathbb{P}^1$ is branched exactly at $h(p_1) = h(p_2)$. Since $\mathbb{P}^1$ is simply connected, this implies that $O$ consists of two components $O = O_1 \cup O_2$ with $O_i \cong \mathbb{P}^1$ intersecting in the point lying over $h(p_1)$.

It follows that $Y = \psi^{-1}(O)$ consists of at least two irreducible components. But it cannot consist of more than two components, since the open dense part $\tilde{k}^{-1}(\mathbb{P}^1 \setminus h(p_1))$ is isomorphic to an open dense part of the cover $Y' \to X'$ of Remark 4.4, which consists of exactly 2 components $Y_1$ and $Y_2$.

Clearly, $Y_1$ and $Y_2$ are smooth with $Y_1 \cap Y_2 = \{D_1, D_2\}$ and $\iota(Y_1) = Y_2$. \qed

**Corollary 4.6.** The double cover

$$Y \to X := Y/\iota$$

is a Wirtinger cover in the sense of [4, Example 1.9,(I)]. In particular, its Prym variety $P(Y/X)$ is an abelian variety.

**Proof.** This follows from the fact that $\iota : Y \to Y$ is fixed-point free by Lemma 2.1 and $\iota(D_1) = D_2$. \qed

**Lemma 4.7.** We have

$$p_a(Y) = 2g + 1, \quad p_a(X) = g + 1,$$

and hence $\dim P(Y/X) = p_a(Y) - p_a(X) = g$. 

Proof. For \(i = 1\) and \(2\) the map \(\psi_i := \psi|Y_i : Y_i \to O_i = \mathbb{P}^1\) is a fourfold cover, where we identify \(O_i\) with \(\mathbb{P}^1\) via the map \(\omega\). The map \(\psi_i\) can be considered as the map \(\tilde{k}_i := \tilde{k}|Y_i : Y_i \to \mathbb{P}^1\).

We then have

\[
p_a(Y) = 2g(Y_i) + 1
\]

and it suffices to show that \(g(Y_i) = g\). Then also \(p_a(X) = g + 1\), since \(X\) admits exactly 1 node and \(Y_i\) is the normalization of \(X\).

There are two types of ramification of \(\tilde{k}_i : Y_i \to \mathbb{P}^1\) for a branch point \(a\) of \(\tilde{k}_i\). Either \(a\) is a branch point of \(h\) of ramification index \(i = 1\) or \(2\). Then \(a\) is a branch point for \(\tilde{k}_i\) with ramification of type \((2, 1, 1)\) if \(i = 1\) and \((3, 1)\) if \(i = 2\).

Alternatively, \(a = h(p_1) = h(p_2)\). Then \(a\) is a branch point for \(\tilde{k}_i\) with ramification of type \((2, 2)\), that is, \(\tilde{k}_i^{-1}(a) = \{2\tilde{D}_1, 2\tilde{D}_2\}\).

Since \(|R_h| = 2g + 4\) and \(p_1\) and \(p_2\) are disjoint from \(R_h\), this gives for the ramification divisor \(R_{\tilde{k}_i}\) of \(\tilde{k}_i\):

\[
|R_{\tilde{k}_i}| = 2g + 4 + 2 = 2g + 6.
\]

and the Hurwitz formula gives \(g(Y_i) = g\). \(\square\)

5 \ THE MAIN THEOREM FOR GENERAL COVERS

Let \(C\) be a smooth trigonal curve of genus \(g\) with trigonal cover \(h : C \to \mathbb{P}^1\) and let \(f : \tilde{C} \to C\) be a general ramified double cover branched over two points \(p_1, p_2 \in C\), that is, \(p_1\) and \(p_2\) do not lie in a fibre of \(h\) and are disjoint from the ramification locus of \(h\). Let \(Y\) be the curve defined by diagram (2.1). According to Propositions 2.3, 2.4 and 2.5, \(Y\) is smooth and irreducible of genus \(2g + 1\). According to Proposition 2.1, it admits a fixed-point free involution \(\iota\). As above, let \(\pi : Y \to X\) denote the corresponding étale double cover with diagram (2.2). Let \((P(\pi), \Xi_\pi)\) denote the corresponding principally polarized Prym variety of dimension \(g\). Let \(R_{btr}^{g, \text{gen}}\) and \(R_{\text{tet}}^{g+1}\) the moduli spaces as defined in the introduction.

**Theorem 5.1.** There is a canonical map

\[
R_{btr}^{g, \text{gen}} \to R_{\text{tet}}^{g+1}.
\]

If \(\tilde{C} \xrightarrow{f} C \xrightarrow{h} \mathbb{P}^1\) is an element of \(R_{btr}^{g, \text{gen}}\) and \(\pi : Y \to X\) the corresponding étale double cover, then the principally polarized abelian varieties \((P(f), \Xi_f)\) and \((P(\pi), \Xi_\pi)\) are canonically isomorphic.

**Proof.** **Step 1:** Choose a point \(y_0 \in Y\) and let \(\overline{a} = \overline{a}_{y_0} : (\tilde{C}^{(3)}) \to J\tilde{C}\) and \(\alpha = \alpha_{f(y_0)} : C^{(3)} \to JC\) be the corresponding Abel–Jacobi maps. Then the following diagram is commutative:

\[
\begin{array}{ccc}
Y^{(3)} & \xrightarrow{\overline{\alpha}} & \tilde{C}^{(3)} \\
\downarrow{f(y_0)} & & \downarrow{\text{Nm}_{f}} \\
C^{(3)} & \xrightarrow{\alpha} & JC
\end{array}
\]  \(\text{(5.1)}\)
This shows that
\[ \overline{\alpha}(Y) \subset \ker \text{Nm}_f = \text{P}(f), \]

since \( \alpha \) maps \( f^{(3)}(Y) = g_3^1 = \text{P}^1 \) to 0 in JC. Denote by \( \varphi \) the restriction \( \overline{\alpha}|Y \) as a map into \( \text{P}(f) \),

\[ \varphi := \overline{\alpha}|Y : Y \to \text{P}(f). \]

For any \( y = \sum_{i=1}^{3} c_i \in Y \), we have

\[ (\varphi + \varphi t)(y) = \mathcal{O}_Y(y + t(y) - 2y_0) \]
\[ = \mathcal{O}_Y\left(\sum_{i=1}^{3} c_i + \sum_{i=1}^{3} t(c_i) - 2y_0\right) = \mathcal{O}_Y(f^* g_3^1 - 2y_0). \]

But this is a constant in \( \text{P}(f) \). So, replacing \( \alpha \) by a suitable translate, we may assume that we have

\[ \varphi t = -\varphi. \]

According to the universal property for Prym varieties (valid also for ramified double covers, see [3, 12.5.1]) the map \( \varphi \) factorizes via the Abel–Prym map \( \psi : Y \to \text{P}(\pi) \), that is, the following diagram is commutative:

\begin{center}
\begin{tikzcd}
Y \arrow{r}{\varphi} \arrow{d}{\psi} & \text{P}(f) \\
\text{P}(\pi) \arrow{r}{-\varphi_0} & \text{P}(f)
\end{tikzcd}
\end{center}

where \( \varphi(y) = \mathcal{O}_{\overline{C}}(y - y_0) \), with \( y, y_0 \) viewed as divisors on \( \overline{C} \), so \( \varphi(y_0) = 0 \in \text{P}(f) \) and \( \psi(y) = \mathcal{O}_Y((y - t) - (y_0 - t_0)) \). If \( z \in \text{P}(\pi) \), then \( z = (1 - t)z' \) for some \( z' \in JY \). Writing \( z' \) for any divisor representing \( z' \) and then considering it as a divisor on \( \overline{C} \), we have \( \overline{\varphi}(z) = \mathcal{O}_{\overline{C}}(z') \).

**Step 2:** We claim that for the proof of Theorem 5.1 it suffices to show that

\[ \varphi_*[Y] = \frac{2}{(g-1)!} \wedge^{g-1} [\Xi_f] \quad \text{in} \quad H^{2g-2}(\text{P}(f), \mathbb{Z}). \quad (5.2) \]

**Proof.** Note first that \( \varphi(Y) \) generates \( \text{P}(f) \) as an abelian variety. This implies that \( \overline{\varphi} \) is an isogeny, \( \text{P}(\pi) \) and \( \text{P}(f) \) being of the same dimension. According to [3, Welters’ criterion 12.2.2], we have

\[ \psi_*[Y] = \frac{2}{(g-1)!} \wedge^{g-1} [\Xi_\pi] \quad \text{in} \quad H^{2g-2}(\text{P}(\pi), \mathbb{Z}). \]

Since we assume (5.2), the commutativity of the diagram gives

\[ \overline{\varphi}_*(\wedge^{g-1}[\Xi_\pi]) = \wedge^{g-1}[\Xi_f]. \]

But then [3, Lemma 12.2.3] implies that \( \overline{\varphi} \) is an isomorphism of polarized abelian varieties. \( \square \)
**Step 3:** Let \( \iota_{P(f)} : P(f) \hookrightarrow J \overline{C} \) denote the canonical embedding. Then

\[
\tilde{\iota}_{P(f)*} \tilde{\alpha}_n[Y] = \frac{8}{(g-1)!} \wedge^{g-1} [\Xi_f].
\]

The proof applies the formula of Macdonald for the class of the curve \( g_3^1 \) in the variety \( C(3) \) in \( H^4(C(3), \mathbb{Z}) \) (see [1, Lemma 8.3.2]):

\[
[g_3^1] = \sum_{k=0}^2 \left( \frac{2-g}{k} \right) \eta^k \cdot \frac{\wedge^{2-k} [\alpha^* \Theta]}{(2-k)!} \quad \text{in} \quad H^4(C(3), \mathbb{Z}),
\]

where \( \eta^k \in H^{2k}(C(3), \mathbb{Z}) \) denotes the fundamental class of \( C^{(3-k)} \) in \( C(3) \) under the embedding \( \sum_{k=1}^{3-k} p_i \to \sum_{k=1}^{3-k} p_i + kp \) with some fixed point \( p \in C \). Similarly, we define the classes \( \tilde{\eta}^k \in H^{2k}(\overline{C}(3), \mathbb{Z}) \). It is well known that \( \eta^k = \wedge^k \eta^1 \) and \( \tilde{\eta}^k = \wedge^k \tilde{\eta}^1 \) and both are related by

\[
f^{(3)*} \eta^k = 2^k \tilde{\eta}^k.
\]

Applying this to Macdolnald’s formula, we get with the commutativity of diagram (5.1)

\[
[Y] = f^{(3)*}[g_3^1] = \sum_{k=0}^2 \left( \frac{2-g}{k} \right) 2^k \eta^k \cdot \frac{\tilde{\alpha}^* \text{Nm}_f^* \wedge^{2-k} [\Theta]}{(2-k)!} \quad \text{in} \quad H^4(\overline{C}(3), \mathbb{Z}).
\]

(5.3)

Now, Poincaré’s formula [3, 11.2.1] in this case says

\[
\tilde{\alpha}^*_n \tilde{\eta}^k = \frac{\wedge^{2g+k-3} [\Theta]}{(2g + k - 3)!}.
\]

Applying \( \tilde{\alpha}_n \) to (5.3), Poincaré’s formula and the projection formula give

\[
\tilde{\alpha}_n[Y] = \sum_{k=0}^2 \left( \frac{2-g}{k} \right) 2^k \frac{\wedge^{2g+k-3} [\Theta]}{(2g + k - 3)!} \cdot \frac{\text{Nm}_f^* \wedge^{2-k} [\Theta]}{(2-k)!} \quad \text{in} \quad H^{4g-2}(J \overline{C}, \mathbb{Z}).
\]

(5.4)

According to [3, Proposition 12.3.4], we have

\[
2[\Theta] = \text{Nm}_f^* [\Theta] + \tilde{\iota}^*_{P(f)} [\Xi_f].
\]

Applying this and the binomial formula to (5.4), we get

\[
\tilde{\alpha}_n[Y] = 2^{3-2g} \sum_{k=0}^2 \left( \frac{2-g}{k} \right) \sum_{j=0}^{2g+k-3} \binom{2g+k-3}{j} \frac{\text{Nm}_f^* \wedge^{j+2-k} [\Theta]}{(2-k)!} \cdot \frac{\tilde{\iota}^*_{P(f)} \wedge^{2g+k-3-j} [\Xi_f]}{(2g + k - 3)!}
\]

(5.5)

Now, we claim that

\[
\tilde{\iota}^*_P(f)^* \text{Nm}_f^* \wedge^n \frac{[\Theta]}{n!} = \begin{cases} 2^{2g} & \text{if } n = g, \\ 0 & \text{if } 0 \leq n \leq g - 1. \end{cases}
\]
For this consider the composed map
\[ \hat{\iota}_{P(f)} \circ \text{Nm}_f^* : H^{2n}(JC, \mathbb{Z}) \to H^{2n}(J\tilde{C}, \mathbb{Z}) \to H^{2n-2g}(P(f), \mathbb{Z}). \]

By degree reasons, \( \hat{\iota}_{P(f)} \circ \text{Nm}_f^* \equiv 0 \) for \( n \neq g \). For \( n = g \) note that \( \hat{\iota}_{P(f)} \) is multiplication by 2 on \( P(f) \). So we get, denoting by \([0]\) the class of the point 0 in \( JC \),
\[ \hat{\iota}_{P(f)} \circ \text{Nm}_f^* \bigwedge^g[\Theta] = \hat{\iota}_{P(f)}[\Theta] = \text{deg}(2P(f)) = 2^g. \]
Inserting this into Equation (5.5), we get
\[ \hat{\iota}_{P(f)} \circ \alpha^*[Y] = 2^{3-2g} \sum_{k=0}^2 \binom{2-g}{k} \binom{2g+k-3}{g+k-2} \frac{2^g g!}{(2-k)!} \cdot \bigwedge^{g-1}[\Xi_f] \]
\[ = \frac{2^3}{(g-1)!} \bigwedge^{g-1}[\Xi_f] \sum_{k=0}^2 \binom{2-g}{k} \binom{g}{2-k}, \]
which is also the coefficient of \( x^2 \) in the product \((1 + x)^{2-g}(1 + x)^g\).

Step 4: Now, \( \alpha^*[Y] \subset P(f) \) and \( \hat{\iota}_{P(f)} \) restricted to \( P(f) \) is multiplication by 2, so
\[ \hat{\iota}_{P(f)} \circ \alpha^*[Y] = 2\hat{\iota}_{P(f)} \circ \alpha^*[Y] = 4\alpha^*[Y] = \frac{8}{(g-1)!} \bigwedge^{g-1}[\Xi_f] \quad \text{in} \quad H^{2g-2}(P(f), \mathbb{Z}). \]

Since \( H^{2g-2}(P(f), \mathbb{Z}) \) is torsion free, we obtain Equation (5.2). This completes the proof of Theorem 5.1. \(\square\)

**Proposition 5.2.** Let \( R_{\text{tet}}^{g+1,2} \) be the locus of double coverings \([Y \to X] \in R_{\text{tet}}^{g+1,2}\) such that the tetragonal covers \( k : X \to \mathbb{P}^1 \) have exactly two fibres consisting of two simple ramification points and otherwise only fibres containing at least one smooth point of \( X \). Then the image of the canonical map of Theorem 5.1 is contained in \( R_{\text{tet}}^{g+1,2} \).

**Proof.** According to the proof of Proposition 2.5 the fibres over \( h(p_\nu) \), the images of the 2 branch points of \( f \), are the only ones of type \((2, 2)\) in the corresponding tetragonal cover \( k : X \to \mathbb{P}^1 \). \(\square\)

**Remark 5.3.** According to Theorem 5.1 and Proposition 5.2 the trigonal construction in the general ramified case is given by the map
\[ R_{h_{g,gen}}^\text{tr} \to R_{g+1,2}^\text{tet}. \]

Both moduli spaces are of dimension \( 2g + 3 \). It would be interesting to compute the exact image as well as the degree of this map. We hope to come back to this question.
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