THE STABILIZER OF A COLUMN IN A MATRIX GROUP
OVER A POLYNOMIAL RING

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Abstract. An original non-standard approach to describing the structure of
a column stabilizer in a group of \( n \times n \) matrices over a polynomial ring or
a Laurent polynomial ring of \( n \) variables is presented. The stabilizer is de-
scribed as an extension of a subgroup of a rather simple structure using the
\((n - 1) \times (n - 1)\) matrix group of congruence type over the corresponding ring
of \( n - 1 \) variables. In this paper, we consider cases where \( n \leq 3 \). For \( n = 2 \), the
stabilizer is defined as a one-parameter subgroup, and the proof is carried out
by direct calculation. The case \( n = 3 \) is nontrivial; the approach mentioned
above is applied to it. Corollaries are given to the results obtained. In partic-
ular, we prove that for the stabilizer in the question, it is not generated by its
a finite subset together with the so-called tame stabilizer of the given column.
We are going to study the cases when \( n \geq 4 \) in a forthcoming paper. Note
that a number of key subgroups of the groups of automorphisms of groups are
defined as column stabilizers in matrix groups. For example, this describes
the subgroup \( \text{IAut}(M_r) \) of automorphisms that are identical modulo a com-
mutant of a free metabelian group \( M_r \) of rank \( r \). This approach demonstrates
the parallelism of theories of groups of automorphisms of groups and matrix
groups that exists for a number of well-known groups. This allows us to use
the results on matrix groups to describe automorphism groups. In this work,
the classical theorems of Suslin, Cohn, as well as Bachmuth and Mochizuki
are used.

Key words: matrix group over a ring, elementary matrices, stabilizer of
a column, ring of polynomials, ring of Laurent polynomials, residue, free
metabelian group, automorphism group.

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Introduction

In the group theory, matrix methods have been used by a number of authors to
produce new interesting results on endomorphisms and automorphisms of groups.
Birman [1] has given a matrix characterization of automorphisms of a free group
\( F_r \) of rank \( r \) with basis \( \{f_1, \ldots, f_r\} \) among arbitrary endomorphisms (the ”inverse

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function theorem”) as follows. For an endomorphism \( \phi \) define the matrix \( J_\phi = (d_j \phi(f_i)), 1 \leq i, j \leq r \) (the ”Jacobian matrix” of \( \phi \)), where \( d_j \) denotes partial Fox derivation (with respect to \( f_j \)) in the free group ring \( \mathbb{Z}[F_r] \) (see [2] or [3]). Then \( \phi \) is an automorphism if and only if the matrix \( J_\phi \) is invertible.

Bachmuth [4] has obtained an inverse function theorem of the same kind on replacing the Jacobian matrix \( J_\phi \) by its image \( \bar{J}_\phi \) over the abelianized group ring \( \mathbb{Z}[F_r/F_1^r] \). Thus he established a matrix characterization of automorphisms of a free metabelian group \( M_r \). Umirbaev [5] has generalized Birman’s result to primitive systems of free groups, Roman’kov [6], [7] and Timoshenko [8] have characterized primitive systems of free metabelian groups. By definition, primitive system is a system of elements of a relatively free group that can be a part of some basis of this group.

For any commutative associative ring \( K \) with identity, an \( r \times r \) elementary matrix (transvection) \( t_{ij}(a) \) over \( K \) is a matrix of the form \( E + aE_{ij} \) where \( i \neq j, a \in K \), \( E_{ij} \) is the \( r \times r \) matrix whose \((ij)\) component is 1 and all other components are zero. As usual, \( E \) denotes the identity matrix. Let \( \text{SL}(r, K) \) be the group of all the \( r \times r \) matrices of determinant 1 whose entries are elements of \( K \), and let \( \text{E}(r, K) \) be the subgroup of \( \text{SL}(r, K) \) generated by the elementary matrices. By \( \Lambda^K_{nk} = \langle a_1, ..., a_k, a_{k+1}, ..., a_{n+1} \rangle \) we denote a mixed polynomial ring over \( K \).

In particular, \( \Lambda^K_{nn} = \langle a_1, ..., a_n \rangle \) is the polynomial ring and \( \Lambda^K_{n0} = \langle a_1^{\pm 1}, ..., a_n^{\pm 1} \rangle \) is the Laurent polynomial ring in \( n \) variables over \( K \).

Then the famous Suslin’s Stability theorem [9] implies that for any \( r \geq 3 \) and any ring \( \Lambda^F_{nk} \) where \( F \) is an arbitrary field, \( \text{SL}(r, \Lambda^n_F) = \text{E}(r, \Lambda^n_F) \).

By \( \text{GE}(r, K) \) we denote the subgroup of \( \text{GL}(r, K) \) generated by \( \text{E}(r, K) \) and all diagonal matrices. It follows that for any \( r \geq 3 \) and any ring \( \Lambda^F_{nk} \), \( \text{GL}(r, \Lambda^n_F) = \text{GE}(r, \Lambda^n_F) \).

In contrast, \( \text{GL}(2, \Lambda^n_{nk}) \) has a number of specific properties. In [10], Cohn proved that

\[
\begin{pmatrix}
1 + a_1a_2 & a_1^2 \\
-a_1^2 & 1 - a_1a_2
\end{pmatrix} \in \text{GL}(2, \Lambda^n_{22}) \setminus \text{GE}(2, \Lambda^n_{22}).
\]

In [11], Bachmuth and Mochizuki proved that if \( n \geq 2 \) then

\[
\text{GL}(2, \Lambda^n_{00}) \neq \text{GE}(2, \Lambda^n_{00}).
\]

Let \( M_r \) be the free metabelian group of rank \( r \) with basis \( \{x_1, ..., x_r\} \), and \( A_r = M_r/M'_r \) be the abelianization of \( M_r \), the free abelian group with the corresponding basis \( \{a_1, ..., a_r\} \). The group ring \( \mathbb{Z}[A_r] \) can be considered as the Laurent polynomial ring \( \Lambda_{r0} \).

For any group \( G \), \( \text{IAut}(G) \) denotes the subgroup of the automorphism group \( \text{Aut}(G) \) consisting of all automorphisms that induce the identity map on the abelianization \( G_{ab} = G/G' \). In the similar way the subsemigroup \( \text{IEnd}(G) \) of the endomorphism semigroup \( \text{End}(G) \) is defined too.

In [4], Bachmuth introduced the following embedding:

\[
\beta : \text{IAut}(M_r) \to \text{GL}(r, \Lambda^n_{10}), \beta : \phi \mapsto \bar{J}_\phi, \phi \in \text{IAut}(M_r).
\]

This embedding is called Bachmuth’s embedding.
The image $\beta(\text{IAut}(M_r))$ in $\text{GL}_r(\Lambda^2_{\mathbb{Z}_0})$ consists of all matrices $A$ such that

$$A\bar{a} = \bar{a}$$

for $\bar{a}$ as given.

In other words, $\text{IAut}(M_r) = \text{Stab}_{\text{GL}(r, \Lambda^2_{\mathbb{Z}_0})}(\bar{a}_r)$ (the stabilizer of $\bar{a}_r$ in $\text{GL}(r, \Lambda^2_{\mathbb{Z}_0})$).

Thus, this is an example showing that key subgroups can act as column stabilizers in matrix groups. In [12], Shpilrain obtained a matrix characterization of IA-endomorphisms with non-trivial fixed points ('eigenvectors') which, although is similar to the corresponding well-known characterization in linear algebra, also reveals a subtle difference. All these and some other results show a wonderful parallelism between the theory of automorphisms and endomorphisms of a free (or free metabelian) group and the theory of linear operators in vector space.

The main goal of this paper is to present an original non-standard approach to the description of column stabilizers in matrix groups over rings. We consider matrix groups over polynomial rings $\Lambda^k_{\mathbb{Z}_n}$ and matrix groups over Laurent polynomial rings $\Lambda^k_{\mathbb{Z}_0}$. In both cases $K$ is an arbitrary commutative domain with identity element. For simplicity, we formulate some statements only in the following important cases: $K = \mathbb{Z}$ or $\mathbb{F}$, where $\mathbb{F}$ is an arbitrary field.

In this paper we consider only cases of $n \times n$ matrices for $n \leq 3$. The case $n = 3$ is the least non-trivial in the subject. In the forthcoming paper we'll extend our method to the description of column stabilizers for the cases $n \geq 4$. We also restrict ourselves to considering stabilizers of columns of a certain type – either columns of variables for rings of polynomials, or columns with components of the form "a variable minus 1" for Laurent polynomials.

For the case $n = 2$, we give an exhaustive description of the stabilizer of a column as a one-parameter subgroup. For $n = 3$ we describe a stabilizer of a column as an extension of a subgroup with a simple structure by a specific group of congruence type of $2 \times 2$ matrices over ring on 2 variables. The idea of such description was originated in [13] and [14]. Such a description was successfully used in [14] to prove that every automorphism of $M_r$, $r \geq 4$, is induced by an automorphism of $F_r$, i.e., is tame. Also this description was used in [7] to prove that $M_3$ contains primitive elements that are not images of primitive elements of $F_3$.

At the last Section 3 we derive a number of corollaries of the obtained results about stabilizers of columns in the case $n = 3$.

Remark 1. The column stabilizer in a $n \times n$ matrix group over a field can be described as follows. Having included the stabilized vector as the last element of the basis of the corresponding linear space, we get each of the stabilizer matrices in the half-expanded form when the last column is of the form

$$\begin{pmatrix}
0 \\
\vdots \\
0 \\
1
\end{pmatrix}.$$ 

The stabilizer consists of all matrices of the such form. It has as a homomorphic image the corresponding group of $(n-1) \times (n-1)$ matrices with the kernel of obvious structure. A similar description for matrix group over a ring is possible if at least one component
of the stabilized column is invertible. See [2.4] below. If \( c_3 \) is invertible one has a homomorphism as above. Our approach is useful for other cases.

1. Preliminaries

Let \( K \) be an arbitrary commutative associative domain with identity. For any \( n \in \mathbb{N} \), let \( \Lambda^K_n \) denotes the polynomial ring \( \Lambda^K_{nn} \) or the Laurent polynomial ring \( \Lambda^K_{nn0} \). Let \( \Delta^K_n \) stays for \( \text{id}(a_1, ..., a_n) \) of \( \Lambda^K_{nn} \) or for \( \text{id}(a_1 - 1, ..., a_n - 1) \) of \( \Lambda^K_{nn} \) (the augmentation ideal of \( \Lambda^K_n \)). Denote \( c_i = a_i \) in the case of \( \Lambda^K_n = \Lambda^K_{nn} \), and \( c_i = a_i - 1 \) in the case of \( \Lambda^K_{nn} \). Further in the paper, we’ll omit \( K \) for brevity and simply write \( \Lambda_n \).

Each element \( g \in \Lambda_k, k \leq n \), has for every \( t \geq 1 \) the unique expression of the form

\[
g = \sum_{i=0}^{t} g_i c_k^i,
\]

where \( g_i \in \Lambda_{k-1} \) for \( i = 0, ..., t - 1 \), and \( g_t \in \Lambda_k \). Since every ring \( \Lambda_k \) embeds into a field of fractions, we can consider a \( \Lambda_k \)-submodule \( \Lambda_k^{-} = \Lambda_k + c_k^{-1} \Lambda_k \), and each element of \( \Lambda_k^{-} \) has for each \( t \geq 0 \) the unique expression of the form

\[
g = \sum_{i=-1}^{t} g_i c_k^i,
\]

where \( g_i \in \Lambda_{k-1} \) for \( i = -1, ..., t - 1 \), and \( g_t \in \Lambda_k \).

Denote

\[
\bar{c}_n = \begin{pmatrix} c_1 \\ c_2 \\ ... \\ c_n \end{pmatrix}.
\]

All along the paper we assume \( n = 3 \) (with one short exception for \( n = 2 \) at the beginning of the next Section 2). Let \( G = \text{Stab}(\bar{c}_3) \) be the subgroup of \( \text{GL}(3, \Lambda_3) \) consisting of all matrices \( g \) such that

\[
g \bar{c}_3 = \bar{c}_3,
\]

in other words, \( G \) is the stabilizer of the column \( \bar{c}_3 \) in the group \( \text{GL}(3, \Lambda_3) \). We’ll show how to construct an explicit matrix group \( H \leq \text{GL}(2, \Lambda_2) \) and a homomorphism \( \rho \) of \( G \) onto \( H \) for which \( \ker(\rho) \) is well understood. In other words, we’ll describe \( G \) as an extension \( \ker(\rho) \) by \( \text{im}(\rho) \) with explicitly described factors. We’ll give a number of applications of these results.

2. On the stabilizer of a column in \( \text{GL}(3, \Lambda_3) \)

Before considering the case of \( 3 \times 3 \) matrices, we show how the stabilizer of the vector \( \bar{c}_2 \) is arranged in the group of \( 2 \times 2 \) matrices over \( \Lambda_2 \).

**Proposition 2.** In \( \text{GL}(2, \Lambda_2) \),

\[
\text{Stab}(\bar{c}_2) = \left\{ \begin{pmatrix} 1 + ac_1c_2 & -ac_1^2 \\ ac_2^2 & 1 - ac_1c_2 \end{pmatrix} \right\},
\]

where \( a \in \Lambda_2 \).
Proof. Obviously, every matrix $A$ in $M(2, \Lambda_2)$ such that $Ac_2 = \bar{c}_2$ has the form

$$
(2.2) \quad \begin{pmatrix}
1 + ac_2 & -ac_1 \\
bc_2 & 1 - bc_1
\end{pmatrix},
$$

A matrix of the form \((2.2)\) is invertible if and only if its determinant is 1. By direct computation we obtain that this happens if and only if this matrix has the form \((2.1)\). □

Now $c_1, c_2, c_3$ are three pairwise non-associated prime elements of $\Lambda_3$ such that each element $g \in \Lambda_3$ can be uniquely expressed in the form

$$
(2.3) \quad g = \sum_{i=0}^{2} g_i c_i^3,
$$

where $g_0, g_1 \in \Lambda_2$ and $g_2 \in \Lambda_3$. Let $G$ is the stabilizer of the column $\bar{c}_3$ in the group $GL(3, \Lambda_3)$. Denote

$$
(2.4) \quad C = \begin{pmatrix}
1 & 0 & c_1 \\
0 & 1 & c_2 \\
0 & 0 & c_3
\end{pmatrix}.
$$

For $A = (a_{ij}) \in G$ we have the following equality

$$
(2.5) \quad C^{-1}AC = \begin{pmatrix}
1 + a_{11} - a_{31}c_1c_3^{-1} & a_{12} - a_{32}c_1c_3^{-1} & 0 \\
a_{21} - a_{31}c_2c_3^{-1} & 1 + a_{22} - a_{32}c_2c_3^{-1} & 0 \\
a_{31}c_3^{-1} & a_{32}c_3^{-1} & 1
\end{pmatrix} \in GL(2, \Lambda_3^{(-)}).
$$

Then we have homomorphism

$$
(2.7) \quad \theta : G \rightarrow GL(2, \Lambda_3^{(-)}), \quad \theta : A \mapsto R(A).
$$

Using \((2.2)\), we obtain a decomposition of the form

$$
(2.8) \quad R = R(A) = E + R_2c_3^2 + R_1c_3 + R_0 + R_{-1}c_3^{-1},
$$

where

$$
(2.9) \quad R_1, R_0, R_{-1} \in M_2(\Lambda_2), R_2 \in M_2(\Lambda_3).
$$

We put

$$
(2.10) \quad X = \begin{pmatrix}
c_1c_2 & -c_1^2 \\
c_2^2 & -c_1c_2
\end{pmatrix}.
$$

Theorem 3. In the above notation, there exist elements $\alpha, \beta, \gamma, \delta \in \Lambda_2$ such that

$$
(2.11) \quad R_{-1} = \alpha X, R_0 X = \beta X, X R_0 = \gamma X, X R_1 X = \delta X.
$$

Proof. Since $A \in G$, we have $a_{31}c_1 + a_{32}c_2 + a_{33}c_3 = c_3$. Then $(a_{31})_0c_1 + (a_{32})_0 = 0$, and so $(a_{31})_0 = -\alpha c_2, (a_{32})_0 = \alpha c_1$ for some $\alpha \in \Lambda_2$. It follows, that $R_{-1} = \alpha X$.

Note that $T = E - c_3c_2 + c_2c_1 \in G$ and $R(T) = E + Xc_3^{-1}$. It follows that $R(A)R(T)$ has the $(-1)$-component $R_0 X$ and so $R_0 X = \beta X, \beta \in \Lambda_2$. Similarly, $R(T)R(A)$ has the $(-1)$-component $XR_0$, hence $XR_0 = \gamma X, \gamma, \beta \in \Lambda_2$. At last, $R(T)R(A)R(T)$ has the $(-1)$-component $XR_1 X$, hence $XR_1 X = \delta X, \delta \in \Lambda_2$. □
Thus, we can associate the elements \( \alpha, \beta, \gamma, \delta \in \Lambda_2 \) with the matrix \( A \in G \). These elements are called residues of \( A \) and of \( R \) with respect to \( c_3 \). Now we give explicit formulas for the residues in terms of elements of the matrices \( R_i, i = 1, 0, -1 \). These formulas are obtained by direct computations.

\[
(2.12) \quad \alpha = -(a_{31})_0c_2^{-1} = (a_{32})_0c_1^{-1}, \quad \beta = -(a_{31})_1c_1 - (a_{32})_1c_2, \quad \gamma = (a_{11})_0 - (a_{21})_0c_1c_2^{-1},
\]

\[
\delta = -a_{21})_1c_2^{-1} + (a_{12})_1c_2^{-1} + ((a_{11})_1 - (a_{21})_1)c_1c_2.
\]

**Theorem 4.** The map

\[
(2.13) \quad \rho : G \to \text{GL}_2(\Lambda_2), \ A \mapsto \begin{pmatrix} 1 + \beta & \alpha \\ \delta & 1 + \gamma \end{pmatrix}
\]

is a homomorphism.

**Proof.** Let \( A' \in G \) and let

\[
(2.14) \quad R' = R(A') = E + R_2c_2^2 + R_1c_3 + R_0 + R_{-1}c_1^{-1},
\]

be decomposition of the form \((2.8)\). Let \( \alpha', \beta', \gamma', \delta' \) be the residues of \( A' \) and of \( R' \) with respect to \( c_3 \).

Then

\[
(2.15) \quad \tilde{R} = R(AA') = E + \tilde{R}_2c_2^2 + \tilde{R}_1c_3 + \tilde{R}_0 + \tilde{R}_{-1}c_1^{-1},
\]

be decomposition of the form \((2.8)\). Here \( \tilde{R}_{-1} = (E + R_0)R_{-1} + R_{-1}(E + R_0) = (\alpha + \alpha' \beta + \alpha' \delta + \alpha \gamma')X \). Hence the corresponding residue is

\[
(2.16) \quad \tilde{\alpha} = \alpha + \alpha' \beta + \alpha' \delta + \alpha \gamma'.
\]

Further, \( \tilde{R}_0 = E + R_0 + R'_0 + R_0R'_0 + R_1R'_{-1} + R_{-1}R'_1 \). Hence

\[
(2.17) \quad \tilde{R}_0X = (1 + \beta + \beta' + \beta' \delta + \alpha \delta')X.
\]

Hence

\[
(2.18) \quad \tilde{\beta} = 1 + \beta + \beta' + \beta' \delta + \alpha \delta'
\]

and

\[
(2.19) \quad \tilde{\gamma} = 1 + \gamma + \gamma' + \gamma \gamma' + \delta \alpha'.
\]

Then \( \tilde{R}_1 = R_1 + R'_1 + R_1R'_0 + R_0R'_1 + R_0R_{-1} + R_{-1}R'_2 \). Hence

\[
(2.20) \quad \tilde{\delta} = 1 + \delta + \delta' + \delta \delta' + \gamma \delta'.
\]

Consequently,

\[
(2.21) \quad \begin{pmatrix} 1 + \tilde{\beta} & \tilde{\alpha} \\ \tilde{\delta} & 1 + \tilde{\gamma} \end{pmatrix} \begin{pmatrix} 1 + \beta & \alpha \\ \delta & 1 + \gamma \end{pmatrix} = \begin{pmatrix} 1 + \beta' & \alpha' \\ \delta' & 1 + \gamma' \end{pmatrix},
\]

equivalent to \( \rho(AA') = \rho(A) \rho(A') \). \( \square \)

Now we are to compute \( \text{im}(\rho) = \rho(G) \). Let \( \text{GL}(2, \Lambda_2, \Delta_2) \) denote the congruence subgroup of \( \text{GL}(2, \Lambda_2) \) with respect to the augmentation ideal \( \Delta_2 \) of \( \Lambda_2 \). We denote by \( \text{GL}(2, \Lambda_2, \Delta_2, \Delta_2^2) \) the subgroup of \( \text{GL}(2, \Lambda_2) \) consisting of the matrices corresponding to the following inclusion scheme:

\[
(2.22) \quad \begin{pmatrix} 1 + \Delta_2 & \Lambda_2 \\ \Delta_2^2 & 1 + \Delta_2 \end{pmatrix}.
\]
\textbf{Theorem 5.} \textit{Im(\(\rho\)) = GL(2, \Lambda_2, \Delta_2, \Delta_2^2)\).}

\textit{Proof.} Let
\begin{equation}
B = \begin{pmatrix}
1 + \beta & \alpha \\
\delta & 1 + \gamma
\end{pmatrix}
\end{equation}

be an invertible matrix corresponding to the inclusion scheme (2.22). Then we have the following decompositions:
\begin{equation}
\beta = \beta_1 c_1 + \beta_2 c_2, \gamma = \gamma_1 c_1 + \gamma_2 c_2, \delta = \delta_1 c_1^2 + \delta_1 c_1 c_2 + \delta_2 c_2^2,
\end{equation}
where \(\beta_1, \ldots, \delta_2 \in \Lambda_2\).

First we define a matrix that stabilizes the column \(\bar{e}\) such that \(\rho(C) = B\) subject to its invertibility.
\begin{equation}
C = \begin{pmatrix}
1 + \gamma_2 c_2 + \delta_1 c_3 & -\gamma_2 c_1 + \delta_2 c_3 & -\delta_1 c_1 - \delta_2 c_2 \\
-\gamma_1 c_2 - \delta_1 c_3 & 1 + \gamma_1 c_1 - \delta_1 c_2 & \delta_1 c_1 + \delta_2 c_2 \\
-\alpha c_2 - \beta_1 c_3 & \alpha c_1 - \beta_2 c_3 & 1 + \beta_1 c_1 + \beta_2 c_2
\end{pmatrix}.
\end{equation}

Obviously, \(C\bar{e} = \bar{c}\). By direct computation we obtain that
\begin{equation}
\det(C) = \det(B) + r,
\end{equation}
where
\begin{equation}
r = \delta_1 c_3 + \gamma_1 \delta_1 c_1 c_3 - \delta_1 c_3 - \delta_1 \delta_2 c_3^2 - \gamma_2 \delta_2 c_2 c_3 +
\beta_2 \delta_1 c_2 c_3 - \beta_1 \delta_1 c_2 c_3 - \beta_1 \delta_2 c_2 c_3 + \gamma_1 \delta_2 c_2 c_3 - \gamma_2 \delta_1 c_1 c_3 + \delta_1 \delta_2 c_3^2 + \beta_2 \delta_1 c_1 c_3.
\end{equation}

Suppose, that \(B \in \text{GL}(2, \Lambda_1)\), then \(\beta_2, \gamma_2, \delta_1, \delta_2 = 0\,\text{, hence} \, r = 0\,\text{, and} \, C \in G\). Similarly we obtain, that \(C \in G\) if \(B\) does not depend of \(c_1\). Since \(B = B_1 B'\) where \(B_1\) does not depend of \(c_2\) and \(B'\) lies in the congruence subgroup with respect to \(c_2\), i.e.,
\begin{equation}
B' \in \begin{pmatrix}
1 + \Lambda_2 c_2 & \Lambda_2 c_2 \\
\Lambda_2 c_2 & 1 + \Lambda_2 c_2
\end{pmatrix}.
\end{equation}

Both matrices, \(B_1\) and \(B'\) are invertible, and \(B_1 \in \text{im}(G)\). There are decompositions (2.19) in which \(\beta_1, \gamma_1, \delta_{11} = 0\). Note that transvection \(t = t_{21}((-\delta_1 - \delta_1')c_1 c_2\text{ lies in } B\) and has a preimage in \(G\). Then
\begin{equation}
B'' = B't \in \begin{pmatrix}
1 + \Lambda_2 c_2 & \Lambda_2 c_2 \\
\Lambda_2 c_2 & 1 + \Lambda_2 c_2
\end{pmatrix}.
\end{equation}

The elements of \(B''\) are decompositions (2.12) such that \(\beta_1, \gamma_1, \delta_{12}, \delta_{12}, \delta_{11} = 0\). The corresponding matrix \(C''\) defined in the form (2.23) is invertible because its determinant is equal to \(\det(B'')\). Hence \(B'' \in \text{im}(\rho)\), and \(B \in \text{im}(\rho)\). \(\square\)

Then \(G\) is an extension of \(\ker(\rho)\), that is described by formulas (2.12), by \(\text{im}(\rho)\), that is consisting of all invertible matrices corresponding to (2.22). By the way, we note, that \(\ker(\rho)\) contains the subgroup \(H\) of all matrices in \(G\) of the form
\begin{equation}
\begin{pmatrix}
1 + \Lambda_3 c_3 & \Lambda_3 c_3 & \Lambda_3 c_3 \\
\Lambda_3 c_3 & 1 + \Lambda_3 c_3 & \Lambda_3 c_3 \\
\Lambda_3 c_3 & \Lambda_3 c_3 & 1 + \Lambda_3 c_3
\end{pmatrix}.
\end{equation}

The quotient \(\text{im}(\rho)/H\) is easily understood.
3. On the tame stabilizer of a column in $\text{GL}(3, \Lambda_3)$.

In general, the stabilizer $G$ of $c_3$ in $\text{GL}(n, K)$ for any commutative associative ring $K$ with identity contains each matrix of the form
\[(3.1) \quad T_{i,j,k}(a) = E + ac_kE_{ij} - ac_jE_{ik}, \text{ for } i \neq j, k; j < k; a \in K.
\]

Also, given the Proposition 2, $G$ contains each matrix of the form
\[(3.2) \quad S_{i,j}(a) = E + ac_ic_je_{ii} - ac_i^2e_{ij} + ac_j^2e_{ji} - ac_iE_{jj}, \text{ for } i < j, a \in K\]
(see (2.1)).

We denote by $G_t$ (the tame stabilizer) the subgroup of $G$ generated by all matrices $T_{i,j,k}(a)$ and $S_{i,j}(a)$ defined by (3.1) and (3.2), respectively. A question arises: Does $G_t$ coincide with $G$? For $n = 2$, the answer ”Yes” is obvious by Proposition 2.

We’ll show below that the answer for $n = 3$ is ”No”.

Now, let $G \leq \text{GL}(3, \Lambda_3)$ be the stabilizer of $\vec{c}_3$ and let $G_t \leq G$ be the corresponding tame stabilizer. As above, $\Lambda_3$ denotes $\Lambda_{30}^3, \Lambda_{30}^2,$ or $\Lambda_{30}^1$.

We exlude Laurent polynomial rings $\Lambda_{33}^3$ over a field. The following results show a connection between $G_t$ and $\text{GE}(2, \Lambda_2)$, that allows to show that $G_t$ is small with respect to $G$.

**Proposition 6.** $\text{im}(G_t) \leq \text{GL}(2, \Lambda_2)$.

**Proof.** If the matrix $A = (a_{ij}) \in G$ has the form $E + A'$, and all rows of the matrix $A'$ are zero except for one row, then $\rho(A)$ lies in the subgroup $\text{GE}(2, \Lambda_2)$. Indeed, formulas (2.12) show that in this case $\alpha = 0$ or $\delta = 0$. Then $\rho(A)$ is a triangular matrix. But every triangular matrix lies obviously in $\text{GE}(2, \Lambda_2)$. This proves the statement for any matrix $A = T_{i,j,k}(a)$.

By formulas (2.12) for any matrix $S_{i,j}(a)$, one has $\alpha = 0$, and we conclude as above. \(\Box\)

**Theorem 7.** Let $G$ be the stabilizer of the column $\vec{c}_3$ in $\text{GL}(3, \Lambda_{33}^3)$. Then for every finite subset $L \subseteq G$
\[(3.3) \quad \text{gp}(L, G_t) \neq G.
\]

**Proof.** By Bachmuth and Mochizuki result [11], if $n \geq 2$ then $\text{GL}(2, \Lambda_n^z)$ can not be generated by any finite subset together with the subgroup $\text{GE}(2, \Lambda_n^z)$.

Hence,
\[(3.4) \quad \text{gp}(\text{GE}(2, \Lambda_{22}^z), \rho(L)) \neq \text{GL}(2, \Lambda_{22}^z).
\]

Then there is a matrix $A$ that belongs to the difference between the two sides of (3.3). We’ll show that there is a similar matrix with elements corresponding to the scheme (2.22). To prove this assertion, we define the image $E + A_0$ of $A$ that lies in $\text{GL}(2, \mathbb{Z})$ under specialization homomorphism $\text{GL}(2, \Lambda_{22}^z) \rightarrow \text{GL}(2, \mathbb{Z})$ defined by the map $c_i \mapsto 1, i = 1, 2$.

In other words, $A_0$ is the 0 part of $A$ under the decomposition form (2.23). Then $E + A_0 \in \text{GE}(2, \mathbb{Z})$. We multiply $A$ by $(E + A_0)^{-1}$ and get new matrix $\hat{A}$ with the same property. Suppose, that its (21) component $\hat{a}_{21} = q_1c_1 + q_2c_2 + q_3$, while $q_1, q_2 \in \mathbb{Z}, q_3 \in \Delta_{22}^z$ does not lie in $\Delta_{22}^z$. This means that $q_1 \neq 0$ or $q_2 \neq 0$. Then we multiply $\hat{A}$ by $t_{21}(-q_1c_1 - q_2c_2)$ and obtain matrix $\hat{A}$ that lies in the difference the two sides of (3.3) and corresponds to the scheme (2.22). Thus $\hat{A} \in \text{im}(\rho)$ but has no preimages in $\text{gp}(\text{GE}(2, \Lambda_{22}^z), \rho(L))$. \(\Box\)
Conclusion

The main results of this paper were obtained for matrix groups over polynomial rings under fairly rigorous assumptions regarding a stabilized vector. The similar results can be obtained for other rings. For example, Theorem 1 can be proved for any polynomial ring over a commutative associative ring with identity $K$ over one variable $x$ for the corresponding stabilized vector $c = \begin{pmatrix} k_1 \\ k_2 \\ x \end{pmatrix}$, where $k_1, k_2$ are arbitrary elements of $K$. The main advantage of the proposed method is the fact that we move by the homomorphism $\rho$ from matrices over the module $\Lambda_3 + c_3^{-1}\Lambda_3$ to matrices over $\Lambda_2$. This process can be considered as an elimination of the residue $c_3^{-1}$. Such moving in a number of cases allows the using of the corresponding induction. This approach also demonstrates the parallelism of theories of groups of automorphisms of groups and matrix groups that exists for a number of well-known groups. This allows us to use the results on matrix groups to describe automorphism groups.

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