Projective reduction of the discrete Painlevé system of type 
$$(A_2 + A_1)^{(1)}$$

Kenji Kajiwara, Nobutaka Nakazono, and Teruhisa Tsuda

28 September. 2009 Revised: 14 April. 2010

Abstract

We consider the $q$-Painlevé III equation arising from the birational representation of the affine Weyl group of type $(A_2 + A_1)^{(1)}$. We study the reduction of the $q$-Painlevé III equation to the $q$-Painlevé II equation from the viewpoint of affine Weyl group symmetry. In particular, the mechanism of apparent inconsistency between the hypergeometric solutions to both equations is clarified by using factorization of difference operators and the $\tau$ functions.

2000 Mathematics Subject Classification: 34M55, 39A13, 33D15, 33E17
Keywords and Phrases: affine Weyl group, discrete Painlevé equation, hypergeometric function

1 Introduction

The discrete Painlevé equations have been studied actively from various points of view. Together with the Painlevé equations, they are now regarded as one of the most important classes of equations in the theory of integrable systems (see, for example, [6]). Originally, the discrete Painlevé equations had been identified as single second-order equations [1–3, 33, 37] and then were generalized to simultaneous first-order equations. A typical example is the following equation known as a discrete Painlevé II equation [33, 37]:

$$x_{n+1} + x_{n-1} = \frac{(an + b)x_n + c}{1 - x_n^2}, \quad (1.1)$$

where $x_n$ is the dependent variable, $n$ is the independent variable, and $a$, $b$, $c \in \mathbb{C}$ are parameters. By applying the singularity confinement criterion [7], (1.1) is generalized to

$$x_{n+1} + x_{n-1} = \frac{(an + b)x_n + c + (-1)^n d}{1 - x_n^2}, \quad (1.2)$$

where $d$ is a parameter, with its integrability preserved. Introducing the dependent variables $X_n$ and $Y_n$ by

$$X_n = x_{2n}, \quad Y_n = x_{2n-1}, \quad (1.3)$$

then (1.2) can be rewritten as

$$Y_{n+1} + Y_n = \frac{(2an + b)X_n + c + d}{1 - X_n^2}, \quad X_{n+1} + X_n = \frac{(a(2n + 1) + b)Y_{n+1} + c - d}{1 - Y_{n+1}^2}. \quad (1.4)$$

Equation (1.4) is known as a discrete Painlevé III equation since it admits a continuous limit to the Painlevé III equation [5]. Conversely, (1.1) can be recovered from (1.4) by putting $d = 0$ and (1.3). We call this procedure “symmetrization” of (1.4), which comes from the terminology of the
Quispel–Roberts–Thompson (QRT) mapping [34, 35]. After this terminology, (1.4) is sometimes called the “asymmetric” discrete Painlevé II equation, and (1.1) is called the “symmetric” discrete Painlevé III equation [21].

It looks that the symmetrization is a simple specialization of parameters at the level of the equation, but some strange phenomena have been reported as to their particular solutions expressed in terms of hypergeometric functions (hypergeometric solutions). The hypergeometric solutions to (1.1) have been constructed as follows [9, 19]:

**Proposition 1.1** For each \( N \in \mathbb{N} \), let \( \tau^n_N \) be an \( N \times N \) determinant defined by

\[
\tau^n_N = \begin{vmatrix}
H_n & H_{n+2} & \cdots & H_{n+2N-2} \\
H_{n+1} & H_{n+3} & \cdots & H_{n+2N-1} \\
\vdots & \vdots & \ddots & \vdots \\
H_{n+N-1} & H_{n+N+1} & \cdots & H_{n+3N-3}
\end{vmatrix},
\tag{1.5}
\]

where \( H_n \) is a function satisfying the three-term relation:

\[
H_{n+1} - z H_n + n H_{n-1} = 0.
\tag{1.6}
\]

Then,

\[
x_n = \frac{2}{z} \frac{\tau^{n+1}_{N+1} \tau^n_N}{\tau^n_{N+1} \tau^n_N} - 1,
\tag{1.7}
\]

satisfies (1.1) with the parameters

\[
a = \frac{8}{z^2}, \quad b = \frac{4(1 + 2N)}{z^2}, \quad c = -\frac{4(1 + 2N)}{z^2}.
\tag{1.8}
\]

On the other hand, since (1.4) appears as the Bäcklund transformation of the Painlevé V equation [28, 38], its hypergeometric solutions are essentially the same as those to the Painlevé V equation [22, 31]. The explicit form of the hypergeometric solutions to (1.4) are given as follows:

**Proposition 1.2** For each \( N \in \mathbb{N} \), let \( \tau^{n,m}_{N} \) be an \( N \times N \) determinant defined by

\[
\tau^{n,m}_{N} = \begin{vmatrix}
K_n & K_{n+1} & \cdots & K_{n+N-1} \\
K_{n+1} & K_{n+2} & \cdots & K_{n+N} \\
\vdots & \vdots & \ddots & \vdots \\
K_{n+N-1} & K_{n+N} & \cdots & K_{n+2N-2}
\end{vmatrix},
\tag{1.9}
\]

where \( K^n_n \) is a function satisfying

\[
K_{n+1}^m - K^n_n - t K_{n+1}^{m+1} = 0, \quad n K_{n+1}^m - (n + t)K^n_n - (n - m)t K_{n+1}^{m+1} = 0.
\tag{1.10}
\]

Then,

\[
X_n = 2(n + 2N - 1) - 2 \frac{\tau^{n+1,m}_{N+1} \tau^{n,m}_{N}}{\tau^{n+1}_{N+1} \tau^{n,m}_{N}} - 1, \quad Y_n = \frac{2}{t} \frac{\tau^{n-1,m-1}_{N+1} \tau^{n,m+1}_{N}}{\tau^{n-1}_{N+1} \tau^{n,m}_{N}} - 1,
\tag{1.11}
\]

satisfy (1.4) with the parameters

\[
a = -\frac{4}{t}, \quad b = \frac{-4(-m + 2N - 1)}{t}, \quad c = \frac{2(1 + 2N)}{t}, \quad d = \frac{2(2m + 2N - 3)}{t}.
\tag{1.12}
\]
It is obvious that substituting $d = 0$ into the hypergeometric solutions to (1.4) in Proposition 1.2 do not yield those to (1.1) in Proposition 1.1. In particular, we remark the following differences between the two solutions:

(i) the hypergeometric functions are different. Equation (1.6) can be solved by the parabolic cylinder function (Weber function), while (1.10) can be solved by the confluent hypergeometric function. In fact, the former function is expressed as a specialization of the latter, but this specialization is not consistent with the symmetrization;

(ii) structures of the determinant are different. The determinant (1.5) has asymmetry in the shift of index: the shift in the vertical direction is one while that in the horizontal direction is two. On the other hand, the determinant (1.9) is an ordinary Hankel determinant.

We note that similar phenomena have been reported also for some other discrete Painlevé equations [8, 18, 25]. Many integrable systems admit particular solutions expressed in terms of determinants, but such an asymmetric structure of the determinant solutions has been seen only in the hypergeometric solutions to the discrete Painlevé equations. Note here that these phenomena cannot be seen for the algebraic (or rational) solutions. For example, it is known that substituting $d = 0$ into the determinant expression of the rational solutions to (1.4) yields those to (1.1); see [20, 23, 24].

The $\tau$ function is one of the most important objects in the theory of integrable systems and is regarded as carrying the underlying fundamental mathematical structures. Concerning the discrete Painlevé equations, investigation of the $\tau$ functions started [18, 19] through the search for the explicit formulae of the hypergeometric and algebraic solutions. In fact, the above mysterious asymmetric structure has been one motivation of further study.

It is now known that theory of birational representations of affine Weyl groups provides us with an algebraic tool to study the Painlevé systems [27, 29–32]. Moreover, a geometric framework of the two-dimensional Painlevé systems has been presented based on certain rational surfaces [15,39]. Combining these results enables us to study the Painlevé systems effectively. For instance, it played a crucial role in the identification of hypergeometric functions that appear as the particular solutions to the Painlevé systems in Sakai’s classification [12–14].

The purpose of this paper is to clarify the mechanism of the phenomena of hypergeometric solutions from the viewpoint of the affine Weyl group symmetry. We shall take the $q$-Painlevé equation of type $(A_2 + A_1)^{(1)}$ as an example, which is the simplest non-trivial discrete Painlevé system [39]. The key is to formulate the symmetrization in terms of the birational representation of the affine Weyl group, where the discrete Painlevé equation arises from the action of the translational subgroup. In fact, the discrete time evolution of the symmetric case comes from a “half-step” of a translation of the affine Weyl group through a restriction to a certain line in the parameter space. Conversely, we can derive various discrete Painlevé equations from elements of infinite order that are not necessarily translations by taking a projection on a certain subspace of the parameters. We call such a procedure to obtain a “smaller” discrete time evolution of Painlevé type a projective reduction.

This paper is organized as follows: in Section 2, we introduce a $q$-Painlevé III equation and derive a $q$-Painlevé II equation by applying the symmetrization. Then we give a brief review on their hypergeometric solutions. In Section 3, we first introduce the family of Bäcklund transformations of the $q$-Painlevé III equation, which is a birational representation of the affine Weyl group
of type \((A_2 + A_1)^{(1)}\). We next lift the representation on the level of \(\tau\) functions and derive various bilinear equations. We then clarify the mechanism of the inconsistency among the hypergeometric solutions by using this framework. Some concluding remarks are given in Section 4.

**Note.** We use the following conventions of \(q\)-analysis throughout this paper.

\(q\)-Shifted factorials:
\[
(a; q)_k = \prod_{i=1}^{k} (1 - a q^{i-1}).
\]

(1.13)

Basic hypergeometric series [4]:
\[
_{1}\phi_1 \left( \frac{a}{b}; q, z \right) = \sum_{k=0}^{\infty} \frac{(a; q)_k}{(b; q)_k (q; q)_k} (-1)^k q^{k(k-1)/2} z^k.
\]

(1.14)

Jacobi theta function:
\[
\Theta(a; q) = (a; q)_{\infty} (qa^{-1}; q)_{\infty},
\]

(1.15)

Elliptic gamma function:
\[
\Gamma(a; p, q) = \frac{(q^2 a^{-1}; p, q)_{\infty}}{(a; p, q)_{\infty}},
\]

(1.16)

where
\[
(a; p, q)_k = \prod_{i,j=0}^{k-1} (1 - p^i q^j a).
\]

(1.17)

It holds that
\[
\Theta(qa; q) = -a^{-1} \Theta(a; q),
\]

(1.18)

\[
\Gamma(qa; q) = \Theta(a; q) \Gamma(a; q).
\]

(1.19)

## 2 \textit{q-PIII} and \textit{q-PII}

We consider the following system of \(q\)-difference equations [11, 17, 39]:
\[
g_{n+1} = \frac{q^{2N+1} c^2}{f_n g_n} \frac{1 + a_0 q^n f_n}{a_0 q^n + f_n}, \quad f_{n+1} = \frac{q^{2N+1} c^2}{f_n g_{n+1}} \frac{1 + a_2 a_0 q^{n-m} g_{n+1}}{a_2 a_0 q^{n-m} + g_{n+1}},
\]

(2.1)

for the unknown functions \(f_n = f_n(m, N)\) and \(g_n = g_n(m, N)\) and the independent variable \(n \in \mathbb{Z}\). Here \(m, N \in \mathbb{Z}\) and \(a_0, a_2, c, q \in \mathbb{C}^\times\) are parameters. Equation (2.1) has the (extended) affine Weyl group symmetry of type \((A_2 + A_1)^{(1)}\) and is known as a \(q\)-Painlevé III equation (\(q\text{-PIII}\)) since the continuous limit yields the Painlevé III equation. We also consider the following \(q\)-difference equation [25, 36]:
\[
X_{k+1} = \frac{q^{2N+1} c^2}{X_k X_{k-1}} \frac{1 + a_0 q^{k/2} X_k}{a_0 q^{k/2} + X_k},
\]

(2.2)

for the unknown function \(X_k = X_k(N)\) and the independent variable \(k \in \mathbb{Z}\). Equation (2.2) is a \(q\)-Painlevé II equation (\(q\text{-PII}\)) and actually it admits a continuous limit to the Painlevé II equation.
Note that substituting
\[ m = 0, \quad a_2 = q^{1/2}, \]  
and putting
\[ f_k(0, N) = X_{2k}(N), \quad g_k(0, N) = X_{2k-1}(N), \]  
in (2.1) yield (2.2).

We shall briefly review the hypergeometric solutions to \( q\)-P\(_{\text{III}}\) and \( q\)-P\(_{\text{II}}\) following [11, 25] and then compare their structures.

### 2.1 Hypergeometric solutions to \( q\)-P\(_{\text{III}}\)

First, we review the hypergeometric solutions to \( q\)-P\(_{\text{III}}\). For each \( N \in \mathbb{Z}_{\geq 0}, \) let \( \psi_{N}^{n,m} \) be an \( N \times N \) determinant defined by

\[
\psi_{N}^{n,m} = \begin{vmatrix}
F_{n,m} & F_{n+1,m} & \cdots & F_{n+N-1,m} \\
F_{n-1,m} & F_{n,m} & \cdots & F_{n+N-2,m} \\
\vdots & \vdots & \ddots & \vdots \\
F_{n-N+1,m} & F_{n-N+2,m} & \cdots & F_{n,m}
\end{vmatrix}, \quad \psi_{N}^{n,m} = 1, \tag{2.5}
\]

where \( F_{n,m} \) satisfies
\[
F_{n+1,m} - F_{n,m} = -a_0^2 q^{2n} F_{n,m-1},
\]
\[
F_{n,m+1} - F_{n,m} = -a_2^{-2} q^{2m+2} F_{n-1,m}. \tag{2.6}
\]

#### Lemma 2.1 ([11])
\( \psi_{N}^{n,m} \) satisfies the following bilinear difference equations:

\[
a_0^2 q^{2n-2} \psi_{N+1}^{n,m-1} \psi_{N}^{n,m} - q^{2N} \psi_{N+1}^{n,m-1,1} \psi_{N}^{n-1,m} + \psi_{N+1}^{n,1,m-1} \psi_{N}^{n,m} = 0, \tag{2.7}
\]

\[
\psi_{N+1}^{n,m} \psi_{N}^{n,m-1} - q^{2N} \psi_{N+1}^{n,m-1,1} \psi_{N}^{n+1,m} - a_2^2 q^{2m} \psi_{N+1}^{n,m} \psi_{N}^{n,m-1} = 0, \tag{2.8}
\]

\[
\psi_{N+1}^{n,1,m} \psi_{N}^{n,m-1} - \psi_{N+1}^{n,1,m-1,1} \psi_{N}^{n+1,m} + a_2^{-2} q^{2m} \psi_{N+1}^{n,m} \psi_{N}^{n,m-1} = 0, \tag{2.9}
\]

\[
\psi_{N+1}^{n,1,m} \psi_{N}^{n,m} - a_0^2 q^{2n+2} \psi_{N+1}^{n,m} \psi_{N}^{n,m-1} = 0. \tag{2.10}
\]

#### Proposition 2.2 ([11])
The hypergeometric solutions to \( q\)-P\(_{\text{III}}\), (2.1), with \( c = 1 \) are given by

\[
f_n = -a_0 q^n \frac{\psi_{N+1}^{n,1,m} \psi_{N}^{n,m}}{\psi_{N+1}^{n,m} \psi_{N}^{n,m-1}}, \quad g_n = a_0^{-1} a_2 q^{-n-m+1} \frac{\psi_{N+1}^{n,m} \psi_{N+1}^{n,1,m-1}}{\psi_{N+1}^{n,m-1} \psi_{N}^{n,m}}. \tag{2.11}
\]

Proposition 2.2 follows from Lemma 2.1.

#### Remark 2.3
(1) The general solution to (2.6) is given by

\[
F_{n,m} = \frac{A_{n,m}}{(a_2^{-2} q^{2m+2}; q^2)_{\infty}} \Phi_1 \left( \frac{0}{a_2^2 q^{2n-2}; q^2, a_2^2 a_0^2 q^{2n-2} \Theta(a_2^2 q^{2n-2}; q^2)} \right) + \frac{B_{n,m}}{(a_2^2 q^{2m-2}; q^2)_{\infty}} \Theta(a_2^2 q^{2n}; q^2) \Phi_1 \left( \frac{0}{a_2^{-2} q^{2m+4}; q^2, a_0^2 q^{2n+2}} \right), \tag{2.12}
\]

where \( A_{n,m} \) and \( B_{n,m} \) are periodic functions of period one for \( n \) and \( m \), i.e.,

\[
A_{n,m} = A_{n+1,m} = A_{n,m+1}, \quad B_{n,m} = B_{n+1,m} = B_{n,m+1}. \tag{2.13}
\]
Note that \( F_{n,m} \) satisfies the three-term relation with respect to \( n \):
\[
F_{n+1,m} + (a_0^2 q^{2n} - a_2^{-2} q^{2m+2} - 1) F_{n,m} + a_2^{-2} q^{2m+2} F_{n-1,m} = 0. \tag{2.14}
\]

(2) \( \psi_{n,m}^N \) satisfies the discrete Toda equation:
\[
\psi_{n+1,m}^N \psi_{n-1,m}^N - (\psi_{n,m}^N)^2 + \psi_{n+1,m}^N \psi_{n-1,m}^N = 0. \tag{2.15}
\]

In general, (2.15) admits a solution expressed in terms of the Toeplitz type determinant
\[
\psi_{n,m}^N = \det(c_{n-i+j,m})_{i,j=1,...,N} \quad (N > 0), \tag{2.16}
\]

for an arbitrary function \( c_{n,m} \) under the boundary conditions
\[
\psi_{0,m}^N = 1, \quad \psi_{N,m}^N = 0 \quad (N < 0). \tag{2.17}
\]

Since the hypergeometric solutions to \( q\text{-P}_{\Pi} \) satisfy the conditions (2.17), the bilinear equation (2.15) is regarded as to fix the determinant structure of the solutions.

### 2.2 Hypergeometric solutions to \( q\text{-P}_{\Pi} \)

Next, we review the hypergeometric solutions to \( q\text{-P}_{\Pi} \). For each \( N \in \mathbb{Z}_{\geq 0} \), let \( \phi_N^k \) be an \( N \times N \) determinant defined by
\[
\phi_N^k = \begin{vmatrix}
G_k & G_{k+2} & \cdots & G_{k+2N-2} \\
G_{k-1} & G_{k+1} & \cdots & G_{k+2N-4} \\
\vdots & \vdots & \ddots & \vdots \\
G_{k-N+1} & G_{k-N+3} & \cdots & G_{k+N-1}
\end{vmatrix}, \quad \phi_0^k = 1, \tag{2.18}
\]

where \( G_k \) satisfies
\[
G_{k+1} - G_k + \frac{1}{a_0^{-2} q^k} G_{k-1} = 0. \tag{2.19}
\]

**Lemma 2.4 ([25])** \( \phi_N^k \) satisfies the following bilinear difference equations:
\[
a_0^{-2} q^{-k+1} \phi_{N+1}^{k-1} \phi_N^k + q^{-N} \phi_{N+1}^{k-1} \phi_N^k - \phi_{N+1}^k \phi_N^{k-1} = 0, \tag{2.20}
\]
\[
q^N \phi_{N+1}^k \phi_N^{k-2} + a_0^{-2} q^{-k-N} \phi_{N+1}^{k-1} \phi_N^k - \phi_{N+1}^{k-1} \phi_N^{k-2} = 0. \tag{2.21}
\]

**Proposition 2.5 ([25])** The hypergeometric solutions to \( q\text{-P}_{\Pi} \), (2.2), with \( c = 1 \) are given by
\[
X_k = -a_0 q^{k(2N+2)/2} \frac{\phi_{N+1}^k \phi_N^{k-1}}{\phi_{N+1}^k \phi_N^{k-1}}. \tag{2.22}
\]

Proposition 2.5 follows from Lemma 2.4.
Remark 2.6  (1) The general solution to (2.19) is given by

\[ G_k = A_k \Theta(i a_0 q^{(2k+1)/4}; q^{1/2})_1 \varphi_1 \left( 0; -q^{1/2}; q^{1/2}, -i a_0 q^{(3+2k)/4} \right) \]

\[ + B_k \Theta(-i a_0 q^{(2k+1)/4}; q^{1/2})_1 \varphi_1 \left( 0; -q^{1/2}; q^{1/2}, ia_0 q^{(3+2k)/4} \right), \]

where \( A_k \) and \( B_k \) are periodic functions of period one, i.e.,

\[ A_k = A_{k+1}, \quad B_k = B_{k+1}. \]  (2.24)

(2) \( \phi_N^k \) also satisfies the bilinear equation

\[ \phi_{N+1} \phi_{N-1}^{k+1} - \phi_N^k \phi_N^{k+1} + \phi_N^{k+2} \phi_N^{k-1} = 0, \]  (2.25)

which is a variant of the discrete Toda equation. Under the conditions

\[ \phi_0^k = 1, \quad \phi_N^k = 0 \quad (N < 0), \]  (2.26)

(2.25) admits a solution expressed by

\[ \phi_N^k = \det \left( c_{k+2i-j-1} \right)_{i,j=1,...,N} \quad (N > 0), \]  (2.27)

for an arbitrary function \( c_k \). Hence, (2.25) can be regarded as the bilinear equation that fixes the determinant structure of the hypergeometric solutions to \( q\)-P\(_{\Pi} \).

2.3 Comparison of the hypergeometric solutions

By comparing the hypergeometric solution to \( q\)-P\(_{\text{III}} \) with that to \( q\)-P\(_{\Pi} \) (see Propositions 2.2 and 2.5) one may immediately notice that a naive application of the specialization (2.3) to the former does not yield the latter. As analogous to the phenomena seen in Section 1, we find the following differences between the two solutions:

(i) the hypergeometric functions are different. In fact, substituting \( a_2 = q^{1/2} \) into (2.14) and (2.12) do not yield (2.19) and (2.23), respectively;

(ii) the determinant structures are different.

Besides the determinant formula for the hypergeometric solution to \( q\)-P\(_{\Pi} \) in Proposition 2.5, one can also obtain another formula from that to \( q\)-P\(_{\text{III}} \) in Proposition 2.2 through a specialization (2.3). We set

\[ f_n = X_{2n}, \quad g_n = X_{2n-1}, \]  (2.28)

and define \( \hat{G}_k \) as

\[ \hat{G}_k = \frac{1 + (-1)^k}{2} \Theta(a_0^2 q^k; q^2) F_{k-1} + \frac{1 - (-1)^k}{2} \Theta(a_0^2 q^{k+1}; q^2) F_{k+1,0} \]

\[ = \begin{cases} \Theta(a_0^2 q^{2n}; q^2) F_{n-1} & (k = 2n) \\ \Theta(a_0^2 q^{2n}; q^2) F_{n,0} & (k = 2n - 1) \end{cases} \]  (2.29)
with $F_{n,m}$ given in Remark 2.3. The system (2.6) reduces to the equation

$$\hat{G}_{k+1} - \hat{G}_k + \frac{1}{a_0^2 q^2} \hat{G}_{k-1} = 0,$$

which coincides with (2.19). Then we have solutions to $q$-$P_\Pi$:

$$X_{2n} = -a_0 q^{\psi_n,0} \frac{\psi_{n+1}^{n-1} \psi_N^{n,0}}{\psi_{n+1}^{n,0} \psi_N^{n,0}}, \quad X_{2n-1} = a_0^{-1} q^{(2n+3)/2} \frac{\psi_{n+1}^{n,0} \psi_N^{n,1-1}}{\psi_{n+1}^{n,1-1} \psi_N^{n,0}},$$

where

$$\psi_{n,0} = \left( \prod_{i=1}^{N} \frac{1}{\Theta(a_0^2 q^{2n+2i-2}; q^2)} \right) \frac{q^{N(N^2-1)/3}}{(-a_0^2 q^{2n})^{N(N-1)/2}} \begin{pmatrix}
\hat{G}_{2n-1} & \hat{G}_{2n+1} & \cdots & \hat{G}_{2n+2N-3} \\
\hat{G}_{2n-2} & q^{-2} \hat{G}_{2n} & \cdots & q^{-2(N-1)} \hat{G}_{2n+2N-4} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{G}_{2n-2N+1} & q^{-2(N-1)} \hat{G}_{2n-2N+3} & \cdots & q^{-2(N-1)^2} \hat{G}_{2n-1}
\end{pmatrix},$$

$$\psi_{n,0} = \left( \prod_{i=1}^{N} \frac{1}{\Theta(a_0^2 q^{2n+2i-2}; q^2)} \right) \frac{q^{N(N^2-1)/3}}{(-a_0^2 q^{2n})^{N(N-1)/2}} \begin{pmatrix}
\hat{G}_{2n-1} & \hat{G}_{2n+1} & \cdots & \hat{G}_{2n+2N-3} \\
\hat{G}_{2n-2} & q^{-2} \hat{G}_{2n} & \cdots & q^{-2(N-1)} \hat{G}_{2n+2N-4} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{G}_{2n-2N+1} & q^{-2(N-1)} \hat{G}_{2n-2N+3} & \cdots & q^{-2(N-1)^2} \hat{G}_{2n-1}
\end{pmatrix}.$$

We can conform the shift of indices of (2.32) and (2.33) to that of (2.18). Actually, in (2.32) and (2.33), multiplying the $i$-th row by $a_0^{-2} q^{-2n+2i-3}$ and $a_0^{-2} q^{-2n+2i-2}$, respectively. Then adding the $(i-1)$-th row to the $i$-th row, the indices of $\hat{G}$ in the $i$-th row increase by one because of (2.30). Repeating this operation, we obtain

$$\psi_{n,0} = (-1)^{-N(N-1)/2} \left( \prod_{i=1}^{N} \frac{1}{\Theta(a_0^2 q^{2n+2i-2}; q^2)} \right) \frac{q^{-N(N-1)(N-5)/6}}{(-a_0^2 q^{2n})^{N(N-1)/2}} \begin{pmatrix}
\hat{G}_{2n-1} & \hat{G}_{2n+1} & \cdots & \hat{G}_{2n+2N-3} \\
\hat{G}_{2n-2} & \hat{G}_{2n} & \cdots & \hat{G}_{2n+2N-4} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{G}_{2n-2N+1} & \hat{G}_{2n-2N+3} & \cdots & \hat{G}_{2n-N-1}
\end{pmatrix},$$

$$\psi_{n,0} = (-1)^{-N(N-1)/2} \left( \prod_{i=1}^{N} \frac{1}{\Theta(a_0^2 q^{2n+2i-2}; q^2)} \right) \frac{q^{-N(N-1)(N-2)/6}}{(-a_0^2 q^{2n})^{N(N-1)/2}} \begin{pmatrix}
\hat{G}_{2n-1} & \hat{G}_{2n+1} & \cdots & \hat{G}_{2n+2N-3} \\
\hat{G}_{2n-2} & \hat{G}_{2n} & \cdots & \hat{G}_{2n+2N-4} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{G}_{2n-N} & \hat{G}_{2n-N+2} & \cdots & \hat{G}_{2n+N-2}
\end{pmatrix}.$$
Thus, we are led to the following unified expression of (2.31):

\[
X_k = -a_0 q^{(k+2N)/2} \frac{\hat{\Phi}_{N+1,1}^{k+1}}{\hat{\Phi}_{N+1,1}^{k+1}}.
\]  

(2.36)

where

\[
\hat{\Phi}_{N}^{k} = \begin{vmatrix}
\hat{G}_k & \hat{G}_{k+2} & \cdots & \hat{G}_{k+2N-2} \\
\hat{G}_{k-1} & \hat{G}_{k+1} & \cdots & \hat{G}_{k+2N-3} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{G}_{k-N+1} & \hat{G}_{k-N+3} & \cdots & \hat{G}_{k+N-1}
\end{vmatrix},
\]  

(2.37)

and

\[
\hat{G}_k = \hat{A}_k \frac{\Theta(a_0^2 q^k; q^2)}{(g^{-1}; q^2)_{\infty}} {}_1\varphi_1 \left( \begin{array}{c} 0 \\ q^3; q^2, a_0^2 q^{k+3} \end{array} \right) + \hat{A}_{k+1} \frac{\Theta(a_0^2 q^{k+1}; q^2)}{(q; q^2)_{\infty}} {}_1\varphi_1 \left( \begin{array}{c} 0 \\ q^2, a_0^2 q^{k+2} \end{array} \right).
\]  

(2.38)

Here \(\hat{A}_k\) is defined by

\[
\hat{A}_k = A_{k,0} \frac{1 + (-1)^k}{2} + B_{k,0} \frac{1 - (-1)^k}{2} = \begin{cases} A_{n,0} & (k = 2n) \\ B_{n,0} & (k = 2n - 1) \end{cases}
\]  

(2.39)

and is a periodic function of period two, i.e.,

\[
\hat{A}_k = \hat{A}_{k+2}.
\]  

(4.40)

In fact, both (2.23) and (2.38) give the general solution to the same equation (2.19) (or equivalently (2.30)). This fact thus implies the existence of certain identities among the basic hypergeometric series \( {}_1\varphi_1 \) with two different bases \( q^2 \) and \( q^{1/2} \); see Appendix A.

In the rest of this paper, we shall clarify the difference of hypergeometric solutions to \( q-P_{\text{III}} \) and \( q-P_{\text{II}} \) (see Propositions 2.2 and 2.5) by using the underlying symmetry of an affine Weyl group.

Remark 2.7 The correspondence between the rational solutions to \( q-P_{\text{III}} \) (see [10]) and that to \( q-P_{\text{II}} \) (see [25]) are straightforward. It is easily verified that substituting \( a_2 = q^{1/2} \) into the former yields the latter.

3 Projective reduction from \( q-P_{\text{III}} \) to \( q-P_{\text{II}} \)

3.1 Birational representation of \( \hat{W}((A_2 + A_1)^{(1)}) \)

We formulate the family of Bäcklund transformations of \( q-P_{\text{III}} \) as a birational representation of the extended affine Weyl group of type \((A_2 + A_1)^{(1)}\) [11, 17]. We refer to [27] for basic ideas of this formulation.

We define the transformations \( s_i \) \((i = 0, 1, 2)\) and \( \pi \) on the variables \( f_j \) \((j = 0, 1, 2)\) and parameters \( a_k \) \((k = 0, 1, 2)\) by

\[
s_i(a_i) = a_i a_i^{-a_i}, \quad s_i(f_j) = f_j \left( \frac{a_i + f_i}{1 + a_i f_i} \right)^{u_j},
\]  

(3.1)

\[
\pi(a_i) = a_{i+1}, \quad \pi(f_i) = f_{i+1},
\]  

(3.2)
for \( i, j \in \mathbb{Z}/3\mathbb{Z} \). Here the symmetric \( 3 \times 3 \) matrix

\[
A = (a_{ij})_{i,j=0}^2 = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \tag{3.3}
\]

is the Cartan matrix of type \( A_2^{(1)} \), and the skew-symmetric one

\[
U = (u_{ij})_{i,j=0}^2 = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \tag{3.4}
\]

represents an orientation of the corresponding Dynkin diagram. We also define the transformations \( T_i, T_j \), \( i, j \) and \( T_{ij} \) as invariant under the actions of \( Z_i \).

\[
\begin{align*}
T_0(f_i) &= \frac{a_i a_{i+1}(a_{i-1}a_i + a_{i-1}f_i + f_{i-1}f_i)}{f_i (a_i a_{i+1} + a_i f_i + f_{i+1} f_{i+1})}, \\
T_1(f_i) &= \frac{1 + a_i f_i + a_i a_{i+1} f_i f_{i+1}}{a_i a_{i+1} f_{i+1} (1 + a_i f_{i-1} + a_{i-1} a_i f_{i-1} f_i)}, \\
T_2(f_i) &= \frac{1}{f_i},
\end{align*}
\]

for \( i \in \mathbb{Z}/3\mathbb{Z} \).

**Proposition 3.1 ([17])** The group of birational transformations \( \langle s_0, s_1, s_2, \pi, w_0, w_1, r \rangle \) forms the extended affine Weyl group of type \((A_2 + A_1)^{(1)}\), denoted by \( \tilde{W}(A_2 + A_1)^{(1)} \). Namely, the transformations satisfy the fundamental relations

\[
s_i^2 = (s_i s_{i+1})^3 = \pi^3 = 1, \quad \pi s_i = s_{i+1} \pi \quad (i \in \mathbb{Z}/3\mathbb{Z}), \quad w_0^2 = w_1^2 = r^2 = 1, \quad rw_0 = w_1 r, \tag{3.8}
\]

and the actions of \( \tilde{W}(A_2^{(1)}) = \langle s_0, s_1, s_2, \pi \rangle \) and \( \tilde{W}(A_1^{(1)}) = \langle w_0, w_1, r \rangle \) commute with each other.

In general, for a function \( F = F(a_i, f_j) \), we let an element \( w \in \tilde{W}(A_2 + A_1)^{(1)} \) act as \( wF(a_i, f_j) = F(a_i w, f_j w) \), that is, \( w \) acts on the arguments from the right. Note that \( a_0 a_1 a_2 = q \) and \( f_0 f_1 f_2 = q c^2 \) are invariant under the actions of \( \tilde{W}(A_2 + A_1)^{(1)} \) and \( \tilde{W}(A_1^{(1)}) \), respectively. We define the translations \( T_i \) \((i = 1, 2, 3, 4)\) by

\[
T_1 = \pi s_2 s_1, \quad T_2 = s_1 s_2, \quad T_3 = s_2 s_1 \pi, \quad T_4 = rw_0,
\]

whose actions on parameters \( a_i \) \((i = 0, 1, 2)\) and \( c \) are given by

\[
\begin{align*}
T_1 &: (a_0, a_1, a_2, c) \mapsto (qa_0, q^{-1} a_1, a_2, c), \\
T_2 &: (a_0, a_1, a_2, c) \mapsto (a_0, qa_1, q^{-1} a_2, c), \\
T_3 &: (a_0, a_1, a_2, c) \mapsto (q^{-1} a_0, a_1, qa_2, c), \\
T_4 &: (a_0, a_1, a_2, c) \mapsto (a_0, a_1, a_2, qc).
\end{align*}
\tag{3.10}
\]

Note that \( T_i \) \((i = 1, 2, 3, 4)\) commute with each other and \( T_1 T_2 T_3 = 1 \). The action of \( T_1 \) on \( f \)-variables can be expressed as

\[
T_1(f_i) = \frac{qc^2}{f_i f_0} \left( 1 + \frac{a_0 f_0}{a_0 + f_0} \right), \quad T_2(f_i) = \frac{qc^2}{f_0 T_1(f_i)} \left( 1 + \frac{a_0 a_0 T_1(f_i)}{a_2 a_0 + T_1(f_i)} \right).
\tag{3.11}
\]
Or, applying $T_1^n T_2^m T_4^N$ ($n, m, N \in \mathbb{Z}$) on (3.11) and putting

$$f_{i,N}^{n,m} = T_1^n T_2^m T_4^N(f_i) \quad (i = 0, 1, 2),$$

we obtain

$$f_{1,N}^{n+1,m} = \frac{q^{2N+1} c_2}{f_{1,N}^{n,m} f_{0,N}^{n,m}} 1 + a_0 q^n f_{0,N}^{n,m}, \quad f_{0,N}^{n+1,m} = \frac{q^{2N+1} c_2}{f_{0,N}^{n,m} f_{1,N}^{n+1}} 1 + a_2 a_0 q^{n+m} f_{1,N}^{n+1,m},$$

(3.13)

which is equivalent to $q$-PIII, (2.1). Then $T_1$ and $T_i$ $(i = 2, 4)$ are regarded as the time evolution and Bäcklund transformations of $q$-PIII, respectively.

In order to formulate the symmetrization to $q$-PII, it is crucial to introduce the transformation $R_1$ defined by

$$R_1 = \pi^2 s_1,$$

(3.14)

which satisfies

$$R_1^2 = T_1.$$

(3.15)

The actions of $R_1$ are given by

$$R_1 : (a_0, a_1, a_2, c) \mapsto (a_2 a_0, a_0^{-1}, a_1 a_0, c),$$

(3.16)

$$R_1(f_0) = \frac{qc^2}{f_0 f_1} \frac{1 + a_0 f_0}{a_0 + f_0}, \quad R_1(f_1) = f_0,$$

(3.17)

which describe the zig-zag motion around the line $a_2 = q^{1/2}$ on the parameter space. However, if we put $a_2 = q^{1/2}$, then $R_1$ becomes the translation on the line $a_2 = q^{1/2}$ with the step $q^{1/2}$ (see Figure 1). In fact, the actions of $R_1$ are now given by

$$R_1 : (a_0, a_1, c) \mapsto (q^{1/2} a_0, q^{-1/2} a_1, c),$$

(3.18)

$$R_1(f_0) = \frac{qc^2}{f_0 f_1} \frac{1 + a_0 f_0}{a_0 + f_0}, \quad R_1(f_1) = f_0.$$

(3.19)

Applying $R_1^k T_4^N$ on (3.19) and putting

$$f_{i,N}^{k} = R_1^k T_4^N(f_i) \quad (i = 0, 1, 2),$$

(3.20)

we have

$$f_{0,N}^{k+1} = \frac{q^{2N+1} c_2}{f_{0,N}^{k} f_{0,N}^{k-1}} 1 + a_0 q^{k/2} f_{0,N}^{k},$$

(3.21)

which is equivalent to $q$-PII, (2.2). Then $R_1$ and $T_4$ are regarded as the time evolution and the Bäcklund transformation of $q$-PII, respectively.

In general, it is possible to obtain various discrete dynamical systems of Painlevé type from elements of infinite order that are not necessarily translations in the affine Weyl group by taking a projection on an appropriate sublattice of corresponding root lattice. We call such a procedure a projective reduction.

By using the above formulation, we can now explain why the difference of hypergeometric solutions to $q$-PIII and that to $q$-PII occurs.
On the other hand, the three-term relation for (3.24) is exactly the operator in (3.25), thus, since

\[ H_{n+1,0} + a_0^{-4} q^{-4n-3} H_{n,0} \]

is given in Remark 2.3. Then, we obtain from (2.14) with \( a_2 = q^{1/2} \) the three-term relation for \( H_{n,0} \):

\[
H_{n+2,0} + \left( a_0^{-2} q^{-2n-3} + a_0^{-2} q^{-2n-2} - 1 \right) H_{n+1,0} + a_0^{-4} q^{-4n-3} H_{n,0} \\
= \left[ T_{n+2} + \left( a_0^{-2} q^{-2n-3} + a_0^{-2} q^{-2n-2} - 1 \right) T_{n+1} + a_0^{-4} q^{-4n-3} T_{n} \right] H_{0,0} = 0. \tag{3.23}
\]

Figure 1. Action of \( R_1 \) on the parameter space \( a = (a_0, a_1, a_2) \in (\mathbb{C}^*)^3 \) with \( a_0a_1a_2 = q \). Left: generic case. Right: \( a_2 = q^{1/2} \).

### 3.2 Hypergeometric functions

First, we explain about the difference of hypergeometric functions. For convenience, we define the function \( H_{n,m} \) by

\[
H_{n,m} = \Theta(a_0^2 q^{2n+1}; q^2) F_{n+\frac{1}{2},m}, \tag{3.22}
\]

where \( F_{n,m} \) is given in Remark 2.3. Then, we obtain from (2.14) with \( a_2 = q^{1/2} \) the three-term relation for \( H_{n,0} \):

\[
H_{n+1,0} + a_0^{-4} q^{-4n-3} H_{n,0} = 0. \tag{3.23}
\]

Since \( R_1^2 = T_1 \), (3.23) is a fourth order difference equation for \( H_{n,0} = T_1^n(H_{0,0}) \) with respect to \( R_1 \). Moreover, it admits the following factorization into two linear difference operators:

\[
T_{n+2} + \left( a_0^{-2} q^{-2n-3} + a_0^{-2} q^{-2n-2} - 1 \right) T_{n+1} + a_0^{-4} q^{-4n-3} T_{n} = \left( R_1^{n+3} + R_1^{n+2} + a_0^{-2} q^{-2n-2} R_1^{n+1} \right) \left( R_1^{n+1} - R_1^{n} + a_0^{-2} q^{-2n} R_1^{n-1} \right). \tag{3.24}
\]

On the other hand, the three-term relation for \( G_n = R_1^n(G_0) \) (see (2.19)) can be expressed as

\[
\left( R_1^{n+1} - R_1^{n} + a_0^{-2} q^{-n} R_1^{n-1} \right) G_0 = 0. \tag{3.25}
\]

Note that the second factor in the right-hand side of (3.24) is exactly the operator in (3.25), thus, \( G_n \) also satisfies (3.23).

### 3.3 Determinant structure

Next, in order to discuss the difference of determinant structures, we need to introduce the \( \tau \) functions and lift the representation to the Weyl group on the level of \( \tau \) functions \([17, 40]\). We introduce the new variables \( \tau_i \) and \( \overline{\tau}_i \) \( (i \in \mathbb{Z}/3\mathbb{Z}) \) with

\[
f_i = q^{1/3} \overline{c}_{2/3} \frac{\overline{\tau}_{i+1} \tau_{i-1}}{\tau_{i+1} \overline{\tau}_{i-1}}. \tag{3.26}
\]
Proposition 3.2 ([40]) We define the action of \( s_i \) (\( i = 0, 1, 2 \)), \( \pi \), \( w_j \) (\( j = 0, 1 \)), and \( r \) on \( \tau_k \) and \( \tau_k' \) (\( k = 0, 1, 2 \)) by the following formulae:

\[
\begin{align*}
    s_i(\tau_j) &= \frac{u_i\tau_{j+1} \tau_{j-1} + \tau_{j+1} \tau_{j-1}}{u_i^{1/2} \tau_j}, \quad s_i(\tau_j) = \tau_j \quad (i \neq j), \\
    s_i(\tau'_j) &= \frac{v_i\tau_{j+1} \tau_{j-1} + \tau_{j+1} \tau_{j-1}}{v_i^{1/2} \tau'_j}, \quad s_i(\tau'_j) = \tau'_j \quad (i \neq j),
\end{align*}
\]

(3.27)

\[
\begin{align*}
    \pi(\tau_j) &= \tau_{i+1}, \quad \pi(\tau'_j) = \tau'_{i+1},
\end{align*}
\]

(3.28)

\[
\begin{align*}
    w_0(\tau_i) &= \frac{a_{i+1}^{1/3}(\tau_i \tau_{i+1} \tau_{i+2} + u_{i-1} \tau_i \tau_{i+1} \tau_{i+2} + u_{i+1}^{-1} \tau_i \tau_{i+1} \tau_{i+2})}{a_{i+1}^{1/3} \tau_{i+1}}, \\
    w_0(\tau_i) &= \tau_i,
\end{align*}
\]

(3.29)

\[
\begin{align*}
    w_1(\tau_i) &= \frac{a_{i+1}^{1/3}(\tau_i \tau_{i+1} \tau_{i+2} + v_{i-1} \tau_i \tau_{i+1} \tau_{i+2} + v_{i+1}^{-1} \tau_i \tau_{i+1} \tau_{i+2})}{a_{i+1}^{1/3} \tau_{i+1}}, \\
    w_1(\tau_i) &= \tau_i,
\end{align*}
\]

(3.30)

\[
\begin{align*}
    r(\tau_i) &= \tau_i, \quad r(\tau'_i) = \tau'_i,
\end{align*}
\]

(3.31)

with

\[
\begin{align*}
    u_i &= q^{-1/3} c^{-2/3} a_i, \quad v_i = q^{1/3} c^{2/3} a_i,
\end{align*}
\]

(3.32)

where \( i, j \in \mathbb{Z}/3\mathbb{Z} \). Then, \( (s_0, s_1, s_2, \pi, w_0, w_1, r) \) realizes the affine Weyl group \( \tilde{W}((A_2 + A_1)^{(1)}) \).

Figure 2. Configuration of the \( \tau \) functions on the lattice with \( N = 0 \).

Then, we define the \( \tau \) functions \( \tau^m_n \) (\( n, m, N \in \mathbb{Z} \)) by

\[
\tau^m_n = T_1^n T_2^{-m} T_4^N(\tau_1).
\]

(3.33)

We note that \( \tau_0 = \tau_0^{-1,0}, \tau_1 = \tau_0^{0,0}, \tau_2 = \tau_0^{0,1}, \tau_0 = \tau_1^{-1,0}, \tau_1 = \tau_1^{0,0}, \) and \( \tau_2 = \tau_1^{0,1} \).
Proposition 3.3  The action of $\mathcal{W}( (A_2 + A_1)^{(1)} )$ on $\tau_{N}^{m,n}$ is

\[ s_0(\tau_{N}^{m,n}) = \tau_{N}^{n,m-n}, \quad s_1(\tau_{N}^{m,n}) = \tau_{N}^{n-1,m+1}, \quad s_2(\tau_{N}^{m,n}) = \tau_{N}^{n-m,-m}, \quad \pi(\tau_{N}^{m,n}) = \tau_{N}^{n-m,-m+1}, \]

\[ (3.34) \]

For convenience, we put

\[ \alpha_i = a_i^{1/6}, \quad \gamma = c^{1/6}, \quad Q = q^{1/6}. \]

Though it is possible to derive more various bilinear difference equations from Proposition 3.2, we present here only the equations that are directly relevant to $q$-$P_{III}$, (3.11).

Proposition 3.4  The following bilinear equations hold:

\[ \tau_{N+1}^{m,n+1,m+1} = Q^{-3n+3m+2N-2} \gamma^2 \alpha_1^{3} \tau_{N+1}^{n+1,m+1} + Q^{-6n+6m+4N-4} \gamma^4 \alpha_1^{6} \tau_{N+1}^{m,n+1,m+1} = 0, \]

\[ (3.37) \]

\[ \tau_{N+1}^{m,n+1,m} = Q^{-3n+2m+4} \gamma^2 \alpha_0^{3} \tau_{N+1}^{m,n+2,m+1} + Q^{6m+4N+8} \gamma^4 \alpha_0^{6} \tau_{N+1}^{m,n+1,m+1} = 0, \]

\[ (3.38) \]

\[ \tau_{N+1}^{m,n+1,m} = Q^{-3n+3m-2N-4} \gamma^2 \alpha_1^{3} \tau_{N+1}^{n+1,m+1} + Q^{-6n+6m-4N-8} \gamma^4 \alpha_1^{6} \tau_{N+1}^{m,n+1,m+1} = 0, \]

\[ (3.39) \]

\[ \tau_{N+1}^{m,n+1,m+1} + Q^{8m+4m-4} \alpha_0^{4} \tau_{N+1}^{m,n+1,m+1} = Q^{-2n+8m+1} \alpha_0^{-1} \tau_{N+1}^{n+1,m+1} = 0. \]

\[ (3.40) \]

The proof of Proposition 3.4 will be given in Appendix B.1.

As seen below $q$-$P_{III}$, (3.11) or (3.13), can be obtained from the bilinear equations. Noticing that

\[ f_{0,N}^{n,m} = Q^{4N+2} \gamma^4 \tau_{N+1}^{m,n+1,m+1} \tau_{N}^{n+1,m+1}, \quad f_{1,N}^{n,m} = Q^{4N+2} \gamma^4 \tau_{N+1}^{m,n+1,m+1} \tau_{N}^{n+1,m+1}, \quad f_{2,N}^{n,m} = Q^{4N+2} \gamma^4 \tau_{N+1}^{n-1,m+1,m+1}, \]

\[ (3.42) \]

we can rewrite (3.37) and (3.39) as

\[ 1 + Q^{-6n+6m-6} \alpha_1^{6} f_{1,N}^{n,m+1} = Q^{-3n+3m+2N-2} \gamma^2 \alpha_1^{3} \tau_{N+1}^{n+1,m+1}, \]

\[ (3.43) \]

\[ 1 + Q^{6n-6m+6} \alpha_1^{-6} f_{1,N}^{n,m+1} = Q^{3n-3m+2N+4} \gamma^2 \alpha_1^{-3} \tau_{N+1}^{n+1,m+1}, \]

\[ (3.44) \]

respectively. Dividing (3.44) by (3.43), we have

\[ \frac{1 + Q^{6n-6m+6} \alpha_1^{-6} f_{1,N}^{n,m+1}}{1 + Q^{-6n+6m-6} \alpha_1^{6} f_{1,N}^{n,m+1}} = Q^{6n-6m+6} \alpha_1^{-6} \tau_{N+1}^{n+1,m+1,m+1} \tau_{N}^{n+1,m+1,m+1} = Q^{6n-6m+6} \alpha_1^{-6} f_{0,N}^{n,m+1} \]

\[ (3.45) \]

which is equivalent to the second equation of (3.13). Similarly, (3.38) and (3.40) yield the first equation of (3.13).

For the hypergeometric solutions, we relate the $\tau$ functions to the determinants $\psi_{N}^{n,m}$, (2.5), by multiplication of appropriate “gauge” factor. We set

\[ \tau_{N}^{n,m} = (-1)^{N(N+1)/2} Q^{-2n-m} Q^{2N^2+6N} \alpha_0^{-2N^2-6N} \alpha_2^{-2N^2} \left( \Theta(-Q^{-6n} \alpha_0^{-6}; Q^6) \Theta(-Q^{6m} \alpha_2^{-6}; Q^6) \right)^N \]

\[ \times \Gamma(Q^{2n-m+1} \alpha_0^2 \alpha_2; Q, Q) \Gamma(Q^{2n+2m-1} \alpha_0 \alpha_2; Q, Q) \Gamma(Q^{2n-m} \alpha_2^2 \alpha_1; Q, Q) \psi_{N}^{n,m-1}, \]

\[ (3.46) \]
and put $\gamma = 1$. Then the bilinear equations (3.37)–(3.41) can be rewritten as

\begin{align}
\psi_{N+1}^{n,m} \psi_{N}^{n+1,m+1} & - Q^{-12n+12N} \alpha_0 - 12 \psi_{N+1}^{n+1,m} \psi_{N+1}^{n+1,m+1} + Q^{-12n} \alpha_0 - 12 \psi_{N}^{n,m} \psi_{N+1}^{n+1,m+1} = 0, \\
\psi_{N+1}^{n+1,m+1} \psi_{N}^{n+1,m+1} & - Q^{-12n} \psi_{N+1}^{n+1,m+1} \psi_{N+1}^{n+1,m+1} - Q^{12n+12} \alpha_0 - 12 \psi_{N}^{n,m+1} \psi_{N+1}^{n+1,m+1} = 0, \\
\psi_{N+1}^{n+1,m+1} \psi_{N}^{n+1,m+1} & - Q^{12n+12} \alpha_2 - 12 \psi_{N}^{n+1,m+1} \psi_{N+1}^{n+1,m+1} = 0, \\
\psi_{N+1}^{n+1,m+1} \psi_{N}^{n+1,m+1} & - Q^{12n+12} \alpha_2 - 12 \psi_{N+1}^{n+1,m+1} \psi_{N+1}^{n+1,m+1} = 0, \\
\psi_{N+1}^{n,1} \psi_{N}^{n,1} & = (\psi_{N}^{n,1})^2 + \psi_{N+1}^{n,1} \psi_{N+1}^{n,1} = 0,
\end{align}

respectively. Equations (3.47)–(3.50) are equivalent to (2.7)–(2.10). Note that (3.51) is exactly the discrete Toda equation, (2.15), which fixes the determinant structure of the hypergeometric solutions as mentioned in Remark 2.3.

**Remark 3.5** The gauge factor $\tau_{N}^{n,m} / \psi_{N}^{n,m-1}$ in (3.46) is obtained by solving the overdetermined system of the bilinear difference equations with $\gamma = 1$ under the boundary conditions $\tau_{N}^{n,m} = 0$ ($N \in \mathbb{Z}_{\geq 0}$) [26].

Let us consider the bilinear equations for $q$-$P_{H}$. Since we need $R_1$, $\tau_i$, and $\overline{\tau}_i$ ($i \in \mathbb{Z}/3\mathbb{Z}$), the lattice is restricted to the “unit-strip” (see Figure 3). Therefore, we have only to consider $\tau_{N}^{n,0}$ and $\tau_{N}^{n,1}$ ($n, N \in \mathbb{Z}$). We set

$$\tau_{N}^{k_i} = R_1^k T_{4N}^N(\tau_1).$$

(3.52)

Note that

$$\tau_0 = \tau_2^{-2}, \quad \tau_1 = \tau_0, \quad \tau_2 = \tau_0^{-1}, \quad \tau_0 = \tau_1^{-2}, \quad \tau_1 = \tau_0, \quad \tau_2 = \tau_1^{-1}.$$  

(3.53)

In general, it follows that

$$\tau_{N}^{n,0} = \tau_{N}^{2^n}, \quad \tau_{N}^{n,1} = \tau_{N}^{2^{n-1}},$$

(3.54)

and

$$f_{0,N}^{4^n} = Q^{4^{N+2}} \gamma^4 \frac{\tau_{N+1}^{2^n} \tau_{N}^{k_n-1}}{\tau_{N}^{2^n} \tau_{N+1}^{2^n-1}}.$$

(3.55)

Figure 3. The actions of $R_1$ on $\tau_i$ ($i = 0, 1, 2$).

**Proposition 3.6** The following bilinear equations hold:

\begin{align}
Q^{-3k+4N+2} \gamma^2 \alpha_0 - 3 k_{N+1} k_{N-1} \tau_{N+1} \tau_{N-1} & - Q^{-3k+4N+2} \gamma^4 \alpha_0 - 6 k_{N+1}^{-1} \tau_{N+1} \tau_{N}^{-1} = 0, \\
Q^{-3k-4N-2} \gamma^{-2} \alpha_0 - 3 k_{N+1} k_{N-1}^{-1} \tau_{N+1} \tau_{N-1} & - Q^{-3k-4N-2} \gamma^{-4} \alpha_0 - 6 k_{N+1} k_{N-1}^{-1} \tau_{N+1} \tau_{N}^{-1} = 0, \\
\tau_{N+1}^{k_1+1} \tau_{N-1}^{-1} & - Q^{-4N+1} \gamma^2 \alpha_0 \tau_{N+1} \tau_{N}^{-1} - Q^{-k+4N-1} \gamma^4 \alpha_0 - 2 k_{N+1} \tau_{N}^{-1} = 0.
\end{align}

(3.56) (3.57) (3.58)
The proof of Proposition 3.6 will be given in Appendix B.2.

One can obtain $q$-$\Pi$, (3.19), from Proposition 3.6 as follows. Equations (3.56) and (3.57) can be rewritten as

$$1 + Q^{-3k} \alpha_0^{-6} f_{0,N}^k = Q^{(-3k+4N+2)/2} \gamma^2 \alpha_0^{-3} \frac{\Gamma_{N+1}^k}{\Gamma_{N+1}^{k-2}}, \tag{3.59}$$

$$1 + Q^{3k} \alpha_0^6 f_{0,N}^k = Q^{(3k+4N+2)/2} \gamma^2 \alpha_0^3 \frac{\Gamma_{N+1}^k}{\Gamma_{N+1}^{k-2}}. \tag{3.60}$$

Dividing (3.60) by (3.59), we have

$$\frac{1 + Q^{3k} \alpha_0^6 f_{0,N}^k}{1 + Q^{-3k} \alpha_0^{-6} f_{0,N}^k} = Q^{3k} \alpha_0^6 \frac{\Gamma_{N+1}^k}{\Gamma_{N+1}^{k-2}} = Q^{3k-12N-6} \gamma^12 \alpha_0^6 f_{0,N}^k f_{0,N}^{k-1}, \tag{3.61}$$

which is equivalent to (3.21).

For hypergeometric solutions, by putting $\gamma = 1$ and

$$\tau_N^k = (-1)^{N(N-1)/2} Q^{N(N-1)(k+n)} \alpha_0^{2N(N-1)} \frac{\Gamma(Q^{2k+3)/2} \alpha_0^2; Q, Q) \Gamma(Q^{-k}/2 \alpha_0^{-1}; Q, Q) \Gamma(Q^{(-k+3)/2} \alpha_0^{-1}; Q, Q)}{\Theta(Q^{3k+1} \alpha_0^{-6}; Q)^N} \phi_N^k, \tag{3.62}$$

we can rewrite the bilinear equations (3.56), (3.57), and (3.58) as

$$Q^{6N-6k+6} \alpha_0^{-12} \phi_N^{k+1} \phi_{N+1}^{k-2} + Q^{6N} \phi_N^{k-1} \phi_{N+1}^k - \phi_N^{k-1} \phi_N^k = 0, \tag{3.63}$$

$$Q^{6N} \phi_N^{k+1} \phi_{N+1}^{k-2} + Q^{-6N-6k} \alpha_0^{-12} \phi_N^{k-1} \phi_{N+1}^k - \phi_N^{k-1} \phi_N^k = 0, \tag{3.64}$$

$$\phi_N^{k+1} \phi_{N+1}^{k-1} - \phi_N^{k} \phi_{N+1}^k + \phi_N^{k+2} \phi_{N-1}^{k-1} = 0, \tag{3.65}$$

which are equivalent to (2.20), (2.21), and (2.25), respectively. The determinant structure of the hypergeometric solutions is fixed by (3.65) as was explained in Remark 2.6. Therefore, the difference of the determinant structures of the hypergeometric solutions to $q$-$\Pi$ and that to $q$-$\Pi$ originates from the following procedures:

(i) the specialization $a_2 = q^{1/2}$ and the restriction of $\tau$ functions on the “unit-strip”;

(ii) taking the half-step translation $R_1$ instead of $T_1$ as a time evolution.

These result in the difference of the bilinear equations (3.41) (or (3.51)) and (3.58) (or (3.65)), which fix the determinant structure of the hypergeometric solutions.

4 Concluding remarks

In this paper, we have clarified the mechanism that gives rise to the apparent “inconsistency” in the hypergeometric solutions to $q$-$\Pi$ and that to $q$-$\Pi$ by using their underlying affine Weyl group symmetry. In general, it is also possible to explain the inconsistency among the hypergeometric solutions to other symmetric and asymmetric discrete Painlevé equations (see, for example, Propositions 1.1 and 1.2).
Before closing, we demonstrate another example of the projective reductions. Let us consider the following system of difference equations [28]:

\[
Z_n + X_n = \frac{3na + b_1}{Y_n} + t, \quad X_{n+1} + Y_n = \frac{(3n + 1)a + b_2}{Z_n} + t, \quad Y_{n+1} + Z_n = \frac{(3n + 2)a + b_3}{X_{n+1}} + t, \quad (4.1)
\]

where \(X_n, Y_n,\) and \(Z_n\) are the dependent variables, \(n \in \mathbb{Z}\) is the independent variable, and \(a, b_1, b_2, b_3, t \in \mathbb{C}\) are parameters. Equation (4.1) is one of the discrete Painlevé systems of type \(A_3^{(1)}\). Namely, it arises from a Bäcklund transformation of the Painlevé V equation, which describes a translation in a different direction from (1.4). Putting \(b_1 = b_2 = b_3 = b, X_n = x_{3n-1}, Y_n = x_{3n},\) and \(Z_n = x_{3n+1},\) we can reduce (4.1) to

\[
x_{n+1} + x_{n-1} = \frac{an + b}{x_n} + t, \quad (4.2)
\]

which is known as a discrete Painlevé I equation [36]. This reduction from (4.1) to (4.2) is a typical example of the projective reductions other than a symmetrization.

It seems that various projective reductions of the discrete Painlevé systems change the underlying symmetry and yield a number of intriguing problems. One interesting project is to make a list of the hypergeometric functions that appear as the solutions to all the symmetric discrete Painlevé equations in Sakai’s classification [13, 14, 39]. These will be discussed in forthcoming papers [16].

**Acknowledgement.** The authors would like to express their sincere thanks to Prof. M. Noumi for fruitful discussions and valuable suggestions. They acknowledge continuous encouragement by Prof. T. Masuda, Prof. H. Sakai, and Prof. Y. Yamada. They also appreciate the valuable comments from the referees which have improved the quality of this paper. This work has been partially supported by the JSPS Grant-in-Aid for Scientific Research No. 19340039.

**A On the difference equation (2.19)**

In this appendix, we consider the equation (2.19) (or (2.30)):

\[
U_{k+1} - U_k + \frac{1}{a_0^2 q^k} U_{k-1} = 0. \quad (A.1)
\]

Recall that we have obtained two solutions to the equation above, i.e, \(G_k\) (2.23) and \(\hat{G}_k\) (2.38). These are described as follows:

\[
G_k = A_k v_k + B_k w_k, \quad (A.2)
\]

\[
v_k = \Theta(i a_0 q^{(2k+1)/4} ; q^{1/2})_1 \varphi_1 \left( \begin{array}{c} 0 \\ -q^{1/2} ; q^{1/2}, -ia_0 q^{(3+2k)/4} \end{array} \right), \quad (A.3)
\]

\[
w_k = \Theta(-ia_0 q^{(2k+1)/4} ; q^{1/2})_1 \varphi_1 \left( \begin{array}{c} 0 \\ -q^{1/2} ; q^{1/2}, ia_0 q^{(3+2k)/4} \end{array} \right), \quad (A.4)
\]
\[ \hat{G}_k = \hat{A}_k \hat{v}_k + \hat{A}_{k+1} \hat{w}_k, \quad (A.5) \]
\[ \hat{v}_k = \frac{\Theta(a_0^2 q^k; q^2)}{(q^{-1}; q^2)_\infty} \varphi_1 \left( \frac{0}{q^3; q^2, a_0^2 q^{k+3}} \right), \quad (A.6) \]
\[ \hat{w}_k = \frac{\Theta(a_0^2 q^{k+1}; q^2)}{(q; q^2)_\infty} \varphi_1 \left( \frac{0}{q; q^2, a_0^2 q^{k+2}} \right). \quad (A.7) \]

Here \( A_k \) and \( B_k \) are periodic functions of period one, and \( \hat{A}_k \) is that of period two. For an initial value \((U_0, U_1)\) given, the values of \( U_k \) \((k \in \mathbb{Z})\) are determined recursively by \((A.1)\). Since the Casoratians \[ \begin{vmatrix} v_0 & w_0 \\ v_1 & w_1 \end{vmatrix} \] and \[ \begin{vmatrix} \hat{v}_0 & \hat{w}_0 \\ \hat{v}_1 & \hat{w}_1 \end{vmatrix} \] do not vanish for generic values of \( a_0 \) and \( q \), the coefficients of \((A.2)\) and \((A.5)\) are specified by the initial value as

\[ \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} v_0 & w_0 \\ v_1 & w_1 \end{pmatrix}^{-1} \begin{pmatrix} U_0 \\ U_1 \end{pmatrix}, \quad (A.8) \]
\[ \begin{pmatrix} \hat{A}_0 \\ \hat{A}_1 \end{pmatrix} = \begin{pmatrix} \hat{v}_0 & \hat{w}_0 \\ \hat{v}_1 & \hat{w}_1 \end{pmatrix}^{-1} \begin{pmatrix} U_0 \\ U_1 \end{pmatrix}. \quad (A.9) \]

Hence we conclude that \((A.2)\) and \((A.5)\) give two different expressions of the general solution to \((A.1)\).

Next we shall show an identity among the basic hypergeometric series \( _1\varphi_1 \) with two different bases \( q^2 \) and \( q^{1/2} \). It follows from \((A.8)\) and \((A.9)\) that

\[ \begin{pmatrix} \hat{A}_0 \\ \hat{A}_1 \end{pmatrix} = \begin{pmatrix} v_0 \hat{v}_1 - v_1 \hat{w}_0 & w_0 \hat{v}_1 - w_1 \hat{w}_0 \\ \hat{v}_0 v_1 - \hat{v}_1 v_0 & \hat{w}_0 w_1 - \hat{w}_1 w_0 \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}. \quad (A.10) \]

By definition we can express \( v_k \), \( w_k \), \( \hat{v}_k \), and \( \hat{w}_k \) as functions in \( a_0 \), namely,

\[ v_k = v(a_0 q^{k/2}), \quad w_k = w(a_0 q^{k/2}), \quad \hat{v}_k = \hat{v}(a_0 q^{k/2}), \quad \hat{w}_k = \hat{w}(a_0 q^{k/2}). \quad (A.11) \]

Note that \( w(a_0) = v(-a_0) \). Substituting \( A_0 = 0 \) (or \( B_0 = 0 \)) in \((A.10)\) leads to the following formula:

\[ v(a_0 q^n) = y(a_0) \hat{v}(a_0 q^n) + z(a_0) \hat{w}(a_0 q^n), \quad (A.12) \]

where

\[ y(a_0) = \frac{v(a_0) \hat{v}(a_0 q^{1/2}) - v(a_0 q^{1/2}) \hat{v}(a_0)}{\hat{v}(a_0) \hat{v}(a_0 q^{1/2}) - \hat{v}(a_0 q^{1/2}) \hat{v}(a_0)}, \quad z(a_0) = \frac{v(a_0 q^{1/2}) \hat{v}(a_0) - v(a_0) \hat{v}(a_0 q^{1/2})}{\hat{v}(a_0) \hat{v}(a_0 q^{1/2}) - \hat{v}(a_0 q^{1/2}) \hat{v}(a_0)}. \quad (A.13) \]

Also, we have \( z(a_0) = y(a_0 q^{1/2}) \) and \( y(a_0) = z(a_0 q^{1/2}) \) from \((A.12)\) with \( n = 1 \).

### B Derivation of bilinear equations

In this appendix, we derive various bilinear equations for \( \tau \) functions from the birational representations of \( \tilde{W}(A_2 + A_1)^{(1)} \) given in Proposition 3.2.
B.1 Bilinear equations for $q$-$P_{III}$

We use the notations introduced in (3.33) and (3.36). For convenience, we classify the bilinear equations into six types so that any equations which belong to the same type can be transformed into each other by the action of $\tilde{W}((A_2 + A_1)^{(1)})$.

**Proposition B.1 (Type I: Discrete Toda type)** The following bilinear equations hold:

\[
\begin{align*}
\tau_{n+1,m}^{n,m} - \tau_{n-1,m}^{n,m} + Q^{2n-2m+4} \alpha_1^{-4} \alpha_2^{-4} \left( \tau_{n}^{n,m} \right)^2 - Q^{n-2m+1} \alpha_1^{-1} \alpha_2 \tau_{n+1}^{n,m} = 0, \\
\tau_{n+1,m}^{n,m} + Q^{2n+2m+4} \alpha_0^{-4} \alpha_2^{-4} \left( \tau_{n}^{n,m} \right)^2 - Q^{n+2m+1} \alpha_0 \alpha_2 \tau_{n-1}^{n,m} = 0, \\
\tau_{n+1,m}^{n,m} + Q^{2n+4m-4} \alpha_0^{-4} \alpha_1^{-4} \left( \tau_{n}^{n,m} \right)^2 - Q^{n+2m+1} \alpha_0 \alpha_1 \tau_{n+1}^{n,m} = 0.
\end{align*}
\]

Figure 4. Configuration of $\tau$ functions for the bilinear equations of type I. Left: (B.1), center: (B.2), right: (B.3).

**Proof.** Application of $T_4 = rw_0$ on $\tau_0$ yields

\[
T_4(\tau_0) = c^{-2/3} a_0^{1/3} a_1^{-1/3} a_2^{-2/3} \frac{\tau_0 \tau_1}{\tau_1} + c^{2/3} a_0^{1/3} a_1^{2/3} a_2^{-1/3} \frac{\tau_0 \tau_2}{\tau_2} + a_1^{1/3} a_2^{-1/3} \frac{\tau_0 \tau_1 \tau_2}{\tau_1 \tau_2},
\]

which is rearranged as

\[
T_4(\tau_0) - c^{-2/3} a_0^{1/3} a_1^{-1/3} a_2^{-2/3} \frac{\tau_0 \tau_1}{\tau_1} \left( \frac{c^{1/3} a_1^{1/3} a_2^{1/3} \tau_0 \tau_1}{\tau_0 \tau_2} + \frac{\tau_0 \tau_2}{\tau_0 \tau_1} \right) + a_1^{-2/3} a_2^{2/3} \frac{\tau_0^2}{\tau_0} = 0.
\]

Applying $T_2 = s_2 \pi s_1$ and $T_3 = s_2 s_1 \pi$ on $\tau_0$ and $\tau_1$, respectively, we obtain

\[
\begin{align*}
q^{1/6} c^{1/3} a_1^{1/2} \tau_1 T_2(\tau_0) &= q^{1/3} c^{1/3} a_1 \tau_0 \tau_2 + \tau_0 \tau_2, \\
q^{1/6} c^{1/3} a_2^{1/2} \tau_2 T_3(\tau_0) &= q^{1/3} c^{1/3} a_2 \tau_1 \tau_0 + \tau_1 \tau_0.
\end{align*}
\]

Using (B.6) and (B.7), we can rewrite (B.5) as

\[
T_4^2(\tau_0) + a_1^{-2/3} a_2^{2/3} T_4(\tau_0)^2 - a_1^{-1/6} a_2^{1/3} T_4(\tau_0) T_3 T_4(\tau_1) = 0.
\]

Then by applying $T_1^{\pm 1} T_2^m T_3^{n-1}$, $T_1^{T_2^m T_3^{n-1}}$, and $T_1^{T_2^m T_3^{n-1}}$ on (B.8), we obtain (B.1), (B.2), and (B.3), respectively. $\square$
Figure 4 shows the configuration of $\tau$ functions in the bilinear equations. Each bilinear equation takes the form of a linear combination of the three quadratic terms in $\tau$ functions. In the left figure, we mark the first, the second, and the third multiplication of $\tau$ functions of (B.1) with the square, the circle, and the triangle, respectively. In the rest of this paper, we use similar representations as above.

**Proposition B.2 (Type II: Discrete 2d-Toda type)** The following bilinear difference equations hold:

\[
(1 - Q^{-12m} \alpha_2^{-12}) \tau_{n,N+1}^{m,n} \tau_{N-1}^{m,n} + Q^{n-11m} \alpha_0^2 \alpha_2^{11} \tau_{N}^{n+1,m+1} \tau_{N-1}^{n-1,m-1} - Q^{n-2m} \alpha_0^2 \alpha_2^2 \tau_{N}^{m+1} \tau_{N-1}^{m-1} = 0, \tag{B.9}
\]

\[
(1 - Q^{-12n} \alpha_0^{-12}) \tau_{N+1}^{n,m} \tau_{N-1}^{n,m} + Q^{10n+m} \alpha_0^{10} \alpha_2^{-1} \tau_{N}^{n+1,m} \tau_{N-1}^{n-1,m} - Q^{p+m} \alpha_0 \alpha_2^{-1} \tau_{N}^{n+1,m+1} \tau_{N-1}^{n-1,m-1} = 0, \tag{B.10}
\]

\[
(1 - Q^{-12n} \alpha_0^{-12} \alpha_2^{-12}) \tau_{n+1}^{n,m} \tau_{N-1}^{n,m} + Q^{10n-11m} \alpha_0^{10} \alpha_2^{11} \tau_{N}^{n+1,m+1} \tau_{N-1}^{n-1,m} - Q^{n-2m} \alpha_0 \alpha_2^2 \tau_{N}^{m+1} \tau_{N-1}^{m-1} = 0. \tag{B.11}
\]

**Proof.** Equation (B.9) is derived by eliminating $\tau_{l,m,n}$ from (B.1) and (B.2). We obtain (B.10) and (B.11) in a similar manner. \(\square\)

**Proposition B.3 (Type III)** The following bilinear equations hold:

\[
(Q^{3n-8m+4} \alpha_1^{-4} \alpha_2^{-4} - Q^{4+4m} \alpha_0^4 \alpha_2^{-4}) \left( \tau_{n,N}^{m,n} \right)^2 + Q^{n+m} \alpha_0 \alpha_2^{-1} \tau_{N}^{n+1,m+1} \tau_{N-1}^{n-1,m-1} - Q^{n-2m} \alpha_1^{-1} \alpha_2 \tau_{N}^{n+1,m+1} \tau_{N-1}^{n-1,m-1} = 0, \tag{B.12}
\]

\[
(Q^{3n+4m} \alpha_0^4 \alpha_2^{-4} - Q^{8n+4m-4} \alpha_0^{-4} \alpha_2^{-4}) \left( \tau_{N}^{m,n} \right)^2 + Q^{2n+m+1} \alpha_0^{-1} \alpha_1 \tau_{N}^{n+1,m+1} \tau_{N-1}^{n-1,m-1} - Q^{n+m} \alpha_0 \alpha_2^{-1} \tau_{N}^{n+1,m+1} \tau_{N-1}^{n-1,m-1} = 0, \tag{B.13}
\]

\[
(Q^{-8n+4m-4} \alpha_0^4 \alpha_2^{-4} - Q^{8n-8m+4} \alpha_0^{-4} \alpha_2^{-4}) \left( \tau_{N}^{m,n} \right)^2 - Q^{2n+m+1} \alpha_0^{-1} \alpha_1 \tau_{N}^{n+1,m+1} \tau_{N-1}^{n-1,m-1} + Q^{n-2m+1} \alpha_1^{-1} \alpha_2 \tau_{N}^{n+1,m+1} \tau_{N-1}^{n-1,m-1} = 0. \tag{B.14}
\]

**Proof.** We obtain (B.12) by eliminating $\tau_{l,m,n+1} \tau_{l,m,n-1}$ from (B.1) and (B.2). Other equations can be derived in a similar manner. \(\square\)

**Proposition B.4 (Type IV)** The following bilinear equations hold:

\[
Q^{-3n} \alpha_0^{-3} (1 - Q^{-12m} \alpha_2^{-12}) \tau_{n,N}^{m,n+1} \tau_{N}^{n-1,m} = Q^{-3m} \alpha_2^3 (1 - Q^{-12m} \alpha_0^{-12}) \tau_{N}^{n+1,m+1} \tau_{N-1}^{n-1,m-1} + (Q^{-12n} \alpha_2^{-12} - Q^{-12m} \alpha_0^{-12}) \tau_{N}^{n+1,m+1} \tau_{N-1}^{n-1,m-1} = 0. \tag{B.15}
\]
First, we prove (B.16)–(B.18). We rewrite (B.4) as
\[ \tau_{n+1,m} = \frac{Q_{n+1,m+1}^{n+1,m+1}}{\tau_{n+1,m-1}^{n+1,m-1}} - \frac{Q_{n+1,m}^{n+1,m}}{\tau_{n+1,m-1}^{n+1,m-1}} - \frac{Q_{n+1,m+1}^{n+1,m+1}}{\tau_{n+1,m-1}^{n+1,m-1}} = 0, \]

\[ (B.16) \]

By using (B.6), we have from (B.22) that
\[ (B.18) \]

\[ (B.17) \]

\[ (B.19) \]

\[ (B.20) \]

\[ (B.21) \]

**Proof.** Equation (B.15) can be derived by eliminating \( \tau_{n,m}^{n,m} \) from (B.12) and (B.13). \( \square \)

**Proposition B.5 (Type V)** The following bilinear equations hold:
\[ \tau_{n+1,m}^{n,m} - \tau_{n+1,m-1}^{n,m} - \tau_{n+1,m+1}^{n,m} = 0, \]

\[ (B.16) \]

\[ \tau_{n+1,m}^{n,m} - \tau_{n+1,m-1}^{n,m} - \tau_{n+1,m+1}^{n,m} = 0, \]

\[ (B.17) \]

\[ \tau_{n+1,m}^{n,m} - \tau_{n+1,m-1}^{n,m} - \tau_{n+1,m+1}^{n,m} = 0, \]

\[ (B.18) \]

\[ \tau_{n+1,m}^{n,m} - \tau_{n+1,m-1}^{n,m} - \tau_{n+1,m+1}^{n,m} = 0, \]

\[ (B.19) \]

\[ \tau_{n+1,m}^{n,m} - \tau_{n+1,m-1}^{n,m} - \tau_{n+1,m+1}^{n,m} = 0. \]

\[ (B.20) \]

\[ \tau_{n+1,m}^{n,m} - \tau_{n+1,m-1}^{n,m} - \tau_{n+1,m+1}^{n,m} = 0. \]

\[ (B.21) \]

**Proof.** First, we prove (B.16)–(B.18). We rewrite (B.4) as
\[ T_4(\bar{\tau}_0) = c^{2/3} a_0^{-1/3} a_1^{-1/3} a_2^{-2/3} \frac{\bar{\tau}_1^{1/3} c^{1/3} a_1^{1/3} a_2^{1/3} \bar{\tau}_2 + \bar{\tau}_0 \bar{\tau}_2}{\bar{\tau}_2} = 0. \]

\[ (B.22) \]

By using (B.6), we have from (B.22) that
\[ T_4(\bar{\tau}_0) = c^{2/3} a_0^{-1/3} a_1^{-1/3} a_2^{-2/3} \frac{\bar{\tau}_1^{1/3} c^{1/3} a_1^{1/3} a_2^{1/3} \bar{\tau}_2 + \bar{\tau}_0 \bar{\tau}_2}{\bar{\tau}_2} = 0, \]

\[ (B.23) \]

which is equivalent to
\[ T_1^{-1} T_4^2(\tau_1) T_2(\tau_0) = c^{2/3} a_0^{1/3} a_1^{1/3} a_2^{1/3} \frac{\bar{\tau}_1^{1/3} c^{1/3} a_1^{1/3} a_2^{1/3} \bar{\tau}_2 + \bar{\tau}_0 \bar{\tau}_2}{\bar{\tau}_2} = 0, \]

\[ (B.24) \]
Applying Proof. First, we prove (B.27)–(B.29). Equations (B.27), (B.28), and (B.29) can be derived by using (B.7), we have from (B.25) that we obtain (B.16), (B.17), and (B.18) by applying $T_1^{l+1}T_2^{m}T_4^{n-1}$, $T_1^{l+1}T_2^{m}T_4^{n-1}$, and $T_1^{l+1}T_2^{m}T_4^{n-1}$ and $T_1^{l+1}T_2^{m}T_4^{n-1}$ on (B.24), respectively. Next, we prove (B.19)–(B.21). We rewrite (4.4) as

$$T_4(T_0) - a_1^{1/3}a_2^{-1/3}t\tau_2^0 (q^{1/3}c^{2/3}a_2\tau_1^0 + \tau_1^0) = 0. \quad (B.25)$$

By using (B.7), we have from (B.25) that

$$T_4^2(t_0)\tau_1 - c^{1/3}a_0^{1/3}a_1^{1/2}a_2^{1/3}T_3T_4(\tau_1)T_4(\tau_2) - c^{1/3}a_0^{1/3}a_1^{1/2}a_2^{1/3}T_3T_4(\tau_0)T_4(\tau_1) = 0. \quad (B.26)$$

We obtain (B.19), (B.20), and (B.21) by applying $T_1^{l+1}T_2^{m}T_4^{n-1}$, $T_1^{l+1}T_2^{m}T_4^{n-1}$, and $T_1^{l+1}T_2^{m}T_4^{n-1}$ on (B.26), respectively.

**Proposition B.6 (Type VI)** The following bilinear equations hold:

$$\begin{align*}
\tau_{n+1,m}^{n+1,m+1} & = Q^{-3n+3m+2N-2}y^2a_1^{3}a_{n+1}^{m+1} + Q^{-6n+6m+4N-4}y^{1}a_1^{6}r_{n+1}^{m+1} = 0, \quad (B.27) \\
\tau_{n+1,m}^{n+1,m+1} & = Q^{-3n+3m+2N-2}y^2a_1^{3}a_{n+1}^{m+1} + Q^{-6n+6m+4N-4}y^{1}a_1^{6}r_{n+1}^{m+1} = 0, \quad (B.28) \\
\tau_{n+1,m}^{n+1,m+1} & = Q^{-3n+3m+2N-2}y^2a_1^{3}a_{n+1}^{m+1} + Q^{-6n+6m+4N-4}y^{1}a_1^{6}r_{n+1}^{m+1} = 0, \quad (B.29) \\
\tau_{n+1,m}^{n+1,m+1} & = Q^{-3n+3m+2N-2}y^2a_1^{3}a_{n+1}^{m+1} + Q^{-6n+6m+4N-4}y^{1}a_1^{6}r_{n+1}^{m+1} = 0, \quad (B.30) \\
\tau_{n+1,m}^{n+1,m+1} & = Q^{-3n+3m+2N-2}y^2a_1^{3}a_{n+1}^{m+1} + Q^{-6n+6m+4N-4}y^{1}a_1^{6}r_{n+1}^{m+1} = 0, \quad (B.31) \\
\tau_{n+1,m}^{n+1,m+1} & = Q^{-3n+3m+2N-2}y^2a_1^{3}a_{n+1}^{m+1} + Q^{-6n+6m+4N-4}y^{1}a_1^{6}r_{n+1}^{m+1} = 0. \quad (B.32)
\end{align*}$$

**Proof.** First, we prove (B.27)–(B.29). Equations (B.27), (B.28), and (B.29) can be derived by applying $T_1^{l+1}T_2^{m}T_4^{n}$, $T_1^{l+1}T_2^{m}T_4^{n}$, and $T_1^{l+1}T_2^{m}T_4^{n}$ on (B.7), respectively.
Next, we prove (B.30)–(B.32). By applying $T_2$ on $\tau_0$, we obtain

$$q^{-1/6} c^{-1/3} a_1^{1/2} T_2(\tau_0) - q^{-1/3} c^{-2/3} a_1 T_2 \tau_0 = 0.$$  \hfill (B.33)

Equations (B.30), (B.31), and (B.32) can be derived by applying $T_1 T_2^m T_4^n$, $T_1 T_2^m T_4^n \pi$, and $T_1 T_2^m T_4^n \pi^2$ on (B.33), respectively. \hfill \Box

**Remark B.7** The bilinear equations in Proposition 3.4 correspond to (B.27), (B.29), (B.30), (B.32), and (B.3).

### B.2 Bilinear equations for $q$-$P_\Pi$

The bilinear equations for $q$-$P_\Pi$ are derived from the equations in Section B.1. Since the parameter space and $\tau$ functions are restricted, we only have to pick up the bilinear equations that consist of the $\tau$ functions on the “unit-strip,” and to rewrite them in terms of $R_1$ instead of $T_1$ (see Figure 3). Therefore, only the bilinear equations of type V and VI are relevant. We use the notation in (3.52).

**Proposition B.8** The following bilinear equations hold:

1. $r_{N+1}^{k+1} r_{N+1}^{k+2} - Q^{(k-4N+2)/2} r_{N}^{k} - Q^{k+4N-2} r_{N}^{k} = 0$, \hfill (B.34)
2. $r_{N+1}^{k+2} r_{N+1}^{k+1} - Q^{(k+4N+2)/2} r_{N}^{k} - Q^{k-4N-2} r_{N}^{k} = 0$, \hfill (B.35)
3. $Q^{-(3k-4N+4)/2} r_{N+1}^{k+3} r_{N}^{k} - Q^{3k+4N-8} r_{N+1}^{k+3} r_{N}^{k} = 0$, \hfill (B.36)
4. $Q^{-(3k+4N+8)/2} r_{N+1}^{k+3} r_{N}^{k} - Q^{3k-4N-8} r_{N+1}^{k+3} r_{N}^{k} = 0$. \hfill (B.37)

Figure 9. Configuration of $\tau$ functions for the bilinear equations of type VI. Upper left: (B.27), upper center: (B.28), upper right: (B.29) lower left: (B.30), lower center: (B.31), lower right: (B.32).
**Proof.** Noticing (3.53), we obtain from (B.23)
\begin{align}
R_1^{-2}T_2^2(\tau_1)R_1^{-1}(\tau_1) - q^{-5/12}c^{1/3}a_0^{1/6}T_4(\tau_1)R_1^{-3}T_4(\tau_1) - q^{5/6}c^{2/3}a_0^{-1/3}R_1^{-2}T_4(\tau_1)R_1^{-1}T_4(\tau_1) = 0,
\end{align}
from which (B.34) is derived by applying $R_1^{m+3}T_4^{-n-1}$. Similarly, we have
\begin{align}
T_4^2(\tau_1)R_1^{-1}(\tau_1) - q^{1/3}c^{1/3}a_0^{1/6}R_1T_4(\tau_1)R_1^{-2}T_4(\tau_1) - q^{2/3}c^{2/3}a_0^{-1/3}T_4(\tau_1)R_1^{-1}T_4(\tau_1) = 0.
\end{align}
by applying $\pi$ on (B.26). Then we obtain (B.35) by applying $R_1^{m+2}T_4^{-n-1}$ on (B.39). Equation (B.36) is derived by applying $R_1^{m+3}T_4^{-n}$ on
\begin{align}
q^{1/6}c^{1/3}a_0^{-1/2}T_4(\tau_1) - q^{1/3}c^{2/3}a_0^{-1}R_1^{-2}(\tau_1)R_1^{-1}T_4(\tau_1) - R_1^{-2}T_4(\tau_1)R_1^{-1}(\tau_1) = 0,
\end{align}
which follows from (B.6). Finally, we obtain (B.37) by applying $R_1^{m+3}T_4^n$ on
\begin{align}
q^{-1/6}c^{-1/3}a_0^{1/2}T_4(\tau_1)R_1^{-3}(\tau_1) - q^{-1/3}c^{-2/3}a_0^{-1}R_1^{-1}(\tau_1)R_1^{-2}T_4(\tau_1) - R_1^{-1}T_4(\tau_1)R_1^{-2}(\tau_1) = 0,
\end{align}
which is follows from (B.33). □

![Diagram](image_url)

Figure 10. Configuration of \( \tau \) functions for the bilinear equations in Proposition B.8. The figures correspond to (B.34), (B.35), (B.36), and (B.37), respectively, from the left to the right.

**Remark B.9** The bilinear equations in Proposition B.8 correspond to (B.36), (B.37), and (B.34).

**References**

[1] E. Brézin and V.A. Kazakov, Exactly solvable theories of closed strings, Phys. Lett. B 236 (1990) 144-150.

[2] M.R. Douglas and S.H. Shenker, Strings in less than one dimension, Nucl. Phys. B 335 (1990) 635–654.

[3] A.S. Fokas, A.R. Its and A.V. Kitaev, The isomonodromy approach to matrix models in 2D quantum gravity, Comm. Math. Phys. 147 (1992) 395–430.

[4] G. Gasper and M. Rahman, Basic Hypergeometric Series, Encyclopedia of Mathematics and Its Applications 35 (Cambridge University Press, Cambridge, 1990).

[5] B. Grammaticos, F.W. Nijhoff, V. Papageorgiou, A. Ramani and J. Satsuma, Linearization and solutions of the discrete Painlevé III equation, Phys. Lett. A 185 (1994) 446–452.

24
[6] B. Grammaticos and A. Ramani, Discrete Painlevé equations: a review, Lect. Notes Phys. 644 (2004) 245–321.

[7] B. Grammaticos, A. Ramani and V. Papageorgiou, Do integrable mappings have the Painlevé property?, Phys. Rev. Lett. 67 (1991) 1825–1828.

[8] T. Hamamoto, K. Kajiwara and N.S. Witte, Hypergeometric solutions to the $q$-Painlevé equation of type $(A_1 + A_1')^{(1)}$, Int. Math. Res. Not. 2006 (2006) Article ID 84619.

[9] K. Kajiwara, The discrete Painlevé II equation and the classical special functions, in Symmetries and integrability of difference equations, eds. by P. Clarkson and F.W. Nijhoff, London Math. Soc. Lecture Note Ser. 255(Cambridge University Press, Cambridge, 1999) 217–227.

[10] K. Kajiwara, On a $q$-difference Painlevé III equation. II. Rational solutions, J. Nonlin. Math. Phys. 10 (2003) 282–303.

[11] K. Kajiwara and K. Kimura, On a $q$-difference Painlevé III equation. I. Derivation, symmetry and Riccati type solutions, J. Nonlin. Math. Phys. 10 (2003) 86–102.

[12] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada, $10E_9$ solution to the elliptic Painlevé equation, J. Phys. A: Math. Gen. 36 (2003) L263–L272.

[13] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada, Hypergeometric solutions to the $q$-Painlevé equations, Int. Math. Res. Not. 2004 (2004) 2497–2521.

[14] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada, Construction of hypergeometric solutions to the $q$-Painlevé equations, Int. Math. Res. Not. 2005 (2005) 1441–1463.

[15] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada, Point configurations, Cremona transformations and the elliptic difference Painlevé equation, Sémin. Congr. 14 (2006) 169–198.

[16] K. Kajiwara and N. Nakazono, In preparation.

[17] K. Kajiwara, M. Noumi and Y. Yamada, A study on the fourth $q$-Painlevé equation, J. Phys. A: Math. Gen. 34 (2001) 8563–8581.

[18] K. Kajiwara, Y. Ohta and J. Satsuma, Casorati determinant solutions for the discrete Painlevé III equation, J. Math. Phys. 36 (1995) 4162–4174.

[19] K. Kajiwara, Y. Ohta, J. Satsuma, B. Grammaticos and A. Ramani, Casorati determinant solutions for the discrete Painlevé-II equation, J. Phys. A: Math. Gen. 27 (1994) 915–922.

[20] K. Kajiwara, K. Yamamoto and Y. Ohta, Rational solutions for the discrete Painlevé II equation, Phys. Lett. A 232 (1997) 189–199.

[21] M.D. Kruskal, K.M. Tamizhmani, B. Grammaticos and A. Ramani, Asymmetric discrete Painlevé equations, Regul. Chaotic Dyn. 5 (2000) 273–280.

[22] T. Masuda, Classical transcendental solutions of the Painlevé equations and their degeneration, Tohoku Math. J. 56 (2004) 467–490.
[23] T. Masuda, Y. Ohta and K. Kajiwara, Rational solutions to the Painlevé V equation and the
universal characters, RIMS Kokyuroku 1203 (2001) 97–108 (in Japanese).

[24] T. Masuda, Y. Ohta and K. Kajiwara, A determinant formula for a class of rational solutions
of Painlevé V equation, Nagoya J. Math. 168 (2002) 1–25.

[25] S. Nakao, K. Kajiwara and D. Takahashi, Multiplicative dP II and its ultradiscretization, Re-
ports of RIAM Symposium No. 9ME-S2, Kyushu University (1998) 125–130 (in Japanese).

[26] N. Nakazono, In preparation.

[27] M. Noumi, Painlevé equations through symmetry (American Mathematical Society, Provi-
dence, 2004).

[28] Y. Ohta, Self-dual structure of the discrete Painlevé equations, RIMS Kokyuroku 1098 (1999)
130–137 (in Japanese).

[29] K. Okamoto, Studies on the Painlevé equations. III. Second and Fourth Painlevé equation, P II
and P IV, Math. Ann. 275 (1986) 221–255.

[30] K. Okamoto, Studies on the Painlevé equations. I. Sixth Painlevé equation P VI, Ann. Mat.
Pura Appl. 146 (1987) 337–381.

[31] K. Okamoto, Studies on the Painlevé equations. II. Fifth Painlevé equation P V, Japan. J.
Math. 13 (1987) 47–76.

[32] K. Okamoto, Studies on the Painlevé equations. IV. Third Painlevé equation P III, Funcl.
Evcav. 30 (1987) 305–332.

[33] V. Periwal and D. Shevitz, Unitary-matrix models as exactly solvable string theories, Phys.
Rev. Lett. 64 (1990) 1326–1329.

[34] G.R.W. Quispel, J.A.G Roberts and C.J. Thompson, Integrable mappings and soliton equa-
tions, Phys. Lett. A 126 (1988) 419–421.

[35] G.R.W. Quispel, J.A.G Roberts and C.J. Thompson, Integrable mappings and soliton equa-
tions II, Physica D 34 (1989) 183–192.

[36] A. Ramani and B. Grammaticos, Discrete Painlevé equations: coalescences, limits and de-
genaracies, Physica A 228 (1996) 150–159.

[37] A. Ramani, B. Grammaticos and J. Hietarinta, Discrete versions of the Painlevé equations,
Phys. Rev. Lett. 67 (1991) 1829–1832.

[38] A. Ramani, Y. Ohta, J. Satsuma and B. Grammaticos, Self-duality and schlesinger chains for
the asymmetric d-P II and q-P III equations, Comm. Math. Phys. 192 (1998) 67–76,

[39] H. Sakai, Rational surfaces associated with affine root systems and geometry of the Painlevé
equations, Comm. Math. Phys. 220 (2001) 165–229.
[40] T. Tsuda, Tau functions of $q$-Painlevé III and IV equations, Lett. Math. Phys. 75 (2006) 39–47.

K. Kajiwara: Faculty of Mathematics, Kyushu University, 744 Motooka, Fukuoka 819-0395, Japan
E-mail address: kaji@math.kyushu-u.ac.jp

N. Nakazono: Graduate School of Mathematics, Kyushu University, 744 Motooka, Fukuoka 819-0395, Japan
E-mail address: n-nakazono@math.kyushu-u.ac.jp

T. Tsuda: Faculty of Mathematics, Kyushu University, 744 Motooka, Fukuoka 819-0395, Japan
E-mail address: tudateru@math.kyushu-u.ac.jp