Quick Streaming Algorithms for Maximization of Monotone Submodular Functions in Linear Time

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Abstract
We consider the problem of monotone, submodular maximization over a ground set of size $n$ subject to cardinality constraint $k$. For this problem, we introduce streaming algorithms with linear query complexity and linear number of arithmetic operations; these algorithms are the first deterministic algorithms for submodular maximization that require a linear number of arithmetic operations. Specifically, for any $c \geq 1, \varepsilon > 0$, we propose a single-pass, deterministic streaming algorithm with ratio $\left(1/(4c) - \varepsilon\right)$, query complexity $\lceil n/c \rceil + c$, memory complexity $O(ck \log(k) \log(1/\varepsilon))$, and $O(n)$ total running time. As $k \to \infty$, the ratio converges to $\left(1 - 1/e\right)/(c+1)$.

In addition, we propose a deterministic, multi-pass streaming algorithm with $O(1/\varepsilon)$ passes that achieves ratio $1 - 1/e - \varepsilon$ in $O(n/\varepsilon)$ queries, $O(k \log(k))$ memory, and $O(n)$ time. We prove a lower bound that implies no constant-factor approximation exists using $o(n)$ queries, even if queries to infeasible sets are allowed. An experimental analysis demonstrates that our algorithms require fewer queries (often substantially less than $n$) to achieve better objective value than the current state-of-the-art algorithms.

1 Introduction

A nonnegative, set function $f : 2^\mathcal{U} \to \mathbb{R}^+$, where ground set $\mathcal{U}$ is of size $n$, is submodular if for all $S \subseteq T \subseteq \mathcal{U}, u \in \mathcal{U}\setminus T$, $f(T \cup \{u\}) - f(T) \leq f(S \cup \{u\}) - f(S)$. Intuitively, submodularity captures a natural diminishing returns property that arises in many machine learning applications, such as viral marketing [16], network monitoring [18], video summarization [22], and MAP Inference for Determinantal Point Processes [11]. An important and well-studied NP-hard optimization problem in this context is submodular maximization subject to a cardinality constraint (SMCC): $\arg \max_{|S| \leq k} f(S)$, where the cardinality constraint $k$ is an input parameter and the function $f$ is submodular. In this work,

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we study SMCC under the assumption the objective function $f$ is monotone ($f(A) \leq f(B)$ if $A \subseteq B$) as well as submodular.

The applications viral marketing and video summarization illustrate how the ground sets of many applications have grown massive in recent years: online social networks such as Facebook exceed one billion users, and more than 300 hours of video were uploaded to YouTube every minute in 2019. In the value query model, an algorithm is provided with an oracle to $f$, which when queried with set $S$ returns $f(S)$. In this model, the best ratio for SMCC is $1 - 1/e \approx 0.632$ [23] which is achieved by a simple greedy procedure [24] in $O(kn)$ time. In this work, we will consider both the query complexity of an algorithm and also the total runtime of the algorithm in terms of arithmetic operations (considering the oracle query as an $O(1)$ operation).

If $k = \Omega(n)$, the standard greedy algorithm has query complexity and runtime $\Omega(n^2)$, which is too large to be practical on modern instances. Therefore, there has been an effort to design faster algorithms for SMCC, initiated by Badanidiyuru and Vondrák [1]. Several randomized approximation algorithms [20, 3, 8] have been designed that require $O(n)$ time, independent of $k$. The current state-of-the-art in query complexity is the stochastic greedy algorithm (LTL) of Mirzasoleiman et al. [20], which achieves approximation ratio $1 - 1/e - \varepsilon$ in expectation in $n \log(1/\varepsilon)$ time. Despite the success of linear-time randomized approximation algorithms, no deterministic algorithm with constant ratio has been developed that requires $o(n \log k)$ runtime for SMCC, although a deterministic algorithm does exist with $O(n)$ query complexity [5].

Another difficulty with modern instances is the storage of large ground sets in main memory. To address this difficulty, streaming algorithms have been developed by Gomes and Krause [12], Badanidiyuru et al. [2], Chakrabarti and Kale [5], among many others (see Section 1.2 below). In this work, we adopt the common assumption that a single element of the ground set may be stored in $O(1)$ space. A streaming algorithm in this context has memory complexity that is nearly linear in $k$ and has at most a logarithmic dependence on $n$; the algorithm may access the ground set only through one or more sequential passes in which elements of the ground set arrive one element at a time in an arbitrary order.

1.1 Contributions

In this work, we propose the first approximation algorithms for SMCC that have linear runtime in the size $n$ of the ground set. These algorithms are streaming algorithms. Our contributions may be summarized in the following theorems.

**Theorem 1.** Let $c \geq 1$ be an integer, and let $\varepsilon > 0$. There exists a deterministic, single-pass streaming algorithm that makes at most $[n/c] + c$ queries, has memory complexity $O(ck \log(k) \log(1/\varepsilon))$ has approximation ratio at least $1/(4c) - \varepsilon$ for SMCC, and the ratio converges to $(1 - 1/e)/(c + 1)$ as $k \to \infty$. Further, the total runtime of the algorithm is $O(n)$.

Theorem 1 is proven by employing one of three related algorithms based
Table 1: State-of-the-art algorithms for SMCC in terms of streaming or query complexity.

| Reference | Passes | Ratio | Memory | Queries | Runtime |
|-----------|--------|-------|--------|---------|---------|
| LTL [20]  | $k$    | $1 - 1/e - \varepsilon$ | $O(n)$ | $n \log(1/e)$ | $O(n)$ |
| P-Pass [25] | $O(1/e)$ | $1 - 1/e - \varepsilon$ | $O(k \log(k)/e)$ | $O(n \log(k)/\varepsilon^2)$ | $O(n \log k)$ |
| SieveStream++ [15] | 1 | $1/2 - \varepsilon$ | $O(k/e)$ | $O(n \log(k)/\varepsilon)$ | $O(n \log k)$ |
| C&K [5]   | 1     | $1/4$ | $O(k)$ | $2n$ | $O(n \log k)$ |
| Theorem 1, $c \geq 1$ | 1     | $1/(4c) - \varepsilon$ | $O(c \log(k) \log(1/e))$ | $[n/e] + c$ | $O(n)$ |
| Theorem 3 | $O(1/e)$ | $1 - 1/e - \varepsilon$ | $O(k \log(k))$ | $O(n/e)$ | $O(n)$ |

on the value of $k$: QuickSingleton, if $k = 1$, QuickStream, if $1 < k < 8c/e$, and QuickStreamLargeK, if $k \geq 8c/e$. We describe and analyze QuickStream in Section 2 and the other two algorithms in Appendix C. To the best of our knowledge, these are the first algorithms that obtain a constant ratio with high probability and make fewer than $n$ queries (if $c \geq 2$).

While it is clear that at least $n$ queries are required for any constant factor if the algorithm is only allowed to query feasible sets (consider $k = 1$), our algorithms bypass this restriction. Our next result is a lower bound on the number of queries required to obtain a constant-factor approximation. Theorem 2 is proven in Appendix B and holds even if $k = 1$ in the instances of SMCC.

**Theorem 2.** Let $c \geq 2$ be an integer, and let $\varepsilon > 0$. Any (randomized) approximation algorithm for SMCC with ratio $1/c + \varepsilon$ for SMCC with probability $\delta$ requires at least $\lceil \delta n/(c - 1) \rceil$ oracle queries.

Theorem 2 implies no constant-factor approximation exists with $o(n)$ queries to the oracle. Another consequence of Theorem 2 is that any algorithm with ratio $(1/2 + \varepsilon)$ with probability greater than $1 - 1/n$ requires at least $n$ queries.

Finally, we show how any constant factor ratio can be deterministically improved to nearly $1 - 1/e$ in a multi-pass streaming algorithm with a linear number of queries, which results in the following theorem.

**Theorem 3.** There exists a deterministic, multi-pass streaming algorithm for SMCC that achieves approximation ratio $1 - 1/e - \varepsilon$, makes $O(n/e)$ oracle queries, requires $O(1/e)$ passes over the ground set, and requires $O(k \log k)$ memory. Further, the total runtime of the algorithm is $O(n)$.

Theorem 3 is proven in Section 3 using a deterministic, multi-pass streaming algorithm BoostRatio (Alg. 2) in conjunction with any algorithm provided by Theorem 1. Altogether, this is the first deterministic algorithm, streaming or otherwise, to obtain nearly the optimal ratio for SMCC in linear query complexity.

Table 1 shows how our algorithms compare to state-of-the-art streaming algorithms designed for SMCC and the linear-time stochastic greedy algorithm (LTL) of Mirzasoleiman et al. [20].
Finally, an empirical evaluation of our single-pass algorithm \textsc{QuickStream}_c shows that if \textsc{QuickStream}_c is supplemented with a linear-time post-processing procedure (which does not compromise any of the theoretical guarantees of the algorithm), it empirically exceeds the objective value of the state-of-the-art single-pass streaming algorithm \textsc{SieveStream}++ \cite{Kazemi15} and the non-streaming LTL algorithm, while using fewer queries than either algorithm.

1.2 Related Work

The literature studying SMCC is vast, so we only discuss algorithms for SMCC with monotone objective and cardinality constraint in this section. Streaming algorithms for more generalized constraints and submodular but not necessarily monotone functions include the works of Chekuri et al. \cite{Chekuri10}, Mirzasoleiman et al. \cite{Mirzasoleiman15}, Mirzasoleiman et al. \cite{Mirzasoleiman16}, and Feldman et al. \cite{Feldman16}, among others.

Approximation Algorithms in Linear Time Although LTL makes fewer than \(n\) queries if \(\varepsilon > 1/e\), its ratio holds only in expectation: LTL returns a poor solution with constant probability if \(k = O(1)\). We refer the reader to Hassidim and Singer \cite{Hassidim10} for discussion and further analysis of the ratio of LTL; also, in Section 4, we empirically explore the behavior of LTL for large values of \(\varepsilon\).

In addition to LTL, two other randomized approximation algorithms with linear query complexity have been developed. The algorithm of Buchbinder et al. \cite{Buchbinder16} achieves ratio \(1/e - \varepsilon\) in \(O(n \log(1/\varepsilon)/\varepsilon^2)\) queries. Very recently, the randomized, parallelizable algorithm of Fahrbach et al. \cite{Fahrbach16} obtains ratio \(1 - 1/\varepsilon\) in expectation with query complexity \(O(n \log(1/\varepsilon)/\varepsilon^3)\). In contrast to our algorithms, none of these algorithms are streaming algorithms or are deterministic.

Single-Pass Streaming Algorithms Chakrabarti and Kale \cite{Chakrabarti14} provided the first single-pass streaming algorithm for SMCC; they designed a \((1/4)\)-approximation with one pass, \(2n\) total queries, and \(O(k)\) memory. However, this algorithm, if implemented to use a priority queue, requires \(\Omega(n \log k)\) arithmetic operations. Badanidiyuru et al. \cite{Badanidiyuru13} improved the ratio for a single-pass algorithm to \(1/2 - \varepsilon\) in \(O(k \log(k)/\varepsilon)\) memory, and \(O(n \log(k)/\varepsilon)\) total queries. Kazemi et al. \cite{Kazemi15} have provided the single pass \(1/2 - \varepsilon\) approximation \textsc{SieveStream}++, which improves the algorithm of Badanidiyuru et al. \cite{Badanidiyuru13} to have memory complexity of \(O(k/\varepsilon)\) as indicated in Table 1. The current state-of-the-art, single-pass algorithm is \textsc{SieveStream}++, which is empirically compared to our algorithms in Section 4. Finally, Feldman et al. \cite{Feldman17} recently showed that any one-pass streaming algorithm with approximation guarantee of \(1/2 + \varepsilon\) must essentially store all elements of the stream.

Multi-Pass Streaming Algorithms The first multi-pass streaming algorithm for SMCC has been given by Gomes and Krause \cite{Gomes16}, which obtains
Algorithm 1 For each $c \geq 1$, a single-pass algorithm with approximation ratio $(1/(4c) - \varepsilon)$ if $k \geq 2$, query complexity $\lfloor n/c \rfloor + c$, and memory complexity $O(ck \log(k) \log(1/\varepsilon))$.

1: procedure $\text{QuickStream}_c(f, k, \varepsilon)$

2: Input: oracle $f$, cardinality constraint $k$, $\varepsilon > 0$

3: $A \leftarrow \emptyset$, $A' \leftarrow \emptyset$, $C \leftarrow \emptyset$, $\ell \leftarrow \lceil \log_2(1/(4\varepsilon)) \rceil + 3$

4: for element $e$ received do

5: $C \leftarrow C \cup \{e\}$

6: if $|C| = c$ or stream has ended then

7: if $f(A \cup C) - f(A) \geq f(A)/k$ then

8: $A \leftarrow A \cup C$

9: if $|A| > 2c\ell(k + 1) \log_2(k)$ then

10: $A \leftarrow \{c\ell(k + 1) \log_2(k) \text{ elements most recently added to } A\}$

11: $C \leftarrow \emptyset$

12: $A' \leftarrow \{ck \text{ elements most recently added to } A\}$

13: Partition $A'$ arbitrarily into at most $c$ sets of size at most $k$. Return the set of the partition with highest $f$ value.

value $\text{OPT}/2 - k\varepsilon$ using $O(k)$ memory and $O(B/\varepsilon)$ passes, where $f$ is upper bounded by $B$. Norouzi-Fard et al. [25] designed a multi-pass algorithm $\text{P-Pass}$ that obtains ratio $1 - 1/e - \varepsilon$ in $O(1/\varepsilon)$ passes, $O(k \log(k)/\varepsilon)$ memory, $O(n \log(k)/\varepsilon^2)$ queries. This is a generalization of the multi-pass algorithm of McGregor and Vu [19] for the maximum coverage problem. The current state-of-the-art, multi-pass algorithm is $\text{P-Pass}$, which is empirically compared to our algorithms in Section 4.

Additional related work on online algorithms is described in Appendix A.

2 The QuickStream$_c$ Algorithm

For each $c \geq 1$, the algorithm QuickStream$_c$ is a single-pass, deterministic streaming algorithm. The parameter $c$ is the number of elements buffered before the algorithm processes them together and determines the approximation ratio, query complexity, and memory complexity of the algorithm. To handle the case that $k = 1$ and obtain better ratios if $k \geq 8c/e$, we provide two related algorithms in Appendix C.

The algorithm QuickStream$_c$ maintains a set $A$, initially empty. We refer to the sets of size at most $c$ of elements processed together as blocks of size $c$. When a new block $C$ is received, the algorithm makes one query of $f(A \cup C)$. If $f(A \cup C) - f(A) \geq f(A)/k$, the block $C$ is added to $A$; otherwise, it is discarded. If the size $|A|$ exceeds $2c\ell(k + 1) \log_2(k)$, elements are deleted from $A$. At the end of the stream, the algorithm partitions the last $ck$ elements added to $A$ into $c$ pieces of size at most $k$ and return the one with highest $f$ value. Pseudocode is given in Alg. 1.

Below, we prove the following theorem.
**Theorem 4.** Let $c \geq 1$, $\varepsilon \geq 0$, and let $(f, k)$ be an instance of SMCC with $k \geq 2$. The solution $S$ returned by $\text{QuickStream}_c$ satisfies

$$f(S) \geq (1/(4c) - \varepsilon) \text{OPT},$$

where OPT is the optimal solution value on this instance. Further, $\text{QuickStream}_c$ makes at most $\lceil n/c \rceil + c$ queries and has memory complexity $O(ck \log(k) \log(1/\varepsilon))$.

We remark that using the the value $f(A)$ of a potentially infeasible set $A$ is a unique feature of our algorithms; we are unaware of any other algorithms that employ the function value of infeasible sets, except for the ones in Feldman et al. [10] for a multi-player model that lies between the offline and streaming model. The use of infeasible sets is necessary to obtain a constant ratio with fewer than $n$ queries. Furthermore, while many streaming algorithms use a threshold-based approach (e.g. [2, 25, 15]), usually these algorithms employ multiple fixed thresholds that are based upon $\log(k)/\varepsilon$ many guesses for OPT, in contrast to our single, variable threshold of $f(A)/k$.

### 2.1 Theoretical Analysis

**Proof of Theorem 4.** The query complexity and memory complexity of $\text{QuickStream}_c$ are clear from the limit on the size of $A$, the choice of $\ell$, and the fact that one query is required every $c$ elements together with $c$ queries at the termination of the stream. The rest of the proof establishes the ratio of $\text{QuickStream}_c$.

First, we argue it is sufficient to prove the ratio in the case $c = 1$. Let $N = \{C_1, \ldots, C_m\}$, where each $C_i$ is the $i$-th block of at most $c$ elements of $U$ considered for addition to $A$ on line 7. Define monotone, submodular function $g : 2^N \to \mathbb{R}^+$ by $g(S) = f(\bigcup_{C \in S} C)$. Observe that if we omit lines 12 and 13, the behavior of $\text{QuickStream}_c$ on instance $(f, k)$ is equivalent to $\text{QuickStream}_1$ run on instance $(g, k)$ of SMCC; further, $\arg \max_{\|S\| \leq k} g(S) \geq \arg \max_{\|S\| \leq k} f(S)$.

Let $S$ be the solution returned by $\text{QuickStream}_1$ on instance $(g, k)$. Then the value of $A'$ at termination of $\text{QuickStream}_c$ is $A' = \bigcup_{C \in S} C$. Let $\{D_1, \ldots, D_c\}$ be the partition of $A$ on line 13 of Alg. 1. Then by submodularity of $f$

$$g(S) = f(A') \leq \sum_{i=1}^{c} f(D_i) \leq c \max_{1 \leq i \leq c} f(D_i).$$

Since $\text{QuickStream}_c$ returns $\arg \max_{1 \leq i \leq c} f(D_i)$, it suffices to show that $\text{QuickStream}_1$ has approximation ratio $(1/4 - \varepsilon)$.

For the rest of the proof, we let $c = 1$. We require the following claim, which follows directly from the inequality $\log x \geq 1 - 1/x$ for $x > 0$.

**Claim 1.** For $y \geq 1$, if $i \geq (k + 1) \log y$, then $(1 + 1/k)^i \geq y$.

Throughout the proof, let $A_i$ denote the value of $A$ at the beginning of the $i$-th iteration of the for loop; let $A_{n+1}$ be the value of $A$ after the for loop completes. Also, let $A^* = \bigcup_{i \leq n+1} A_i$, and let $e_i$ denote the element received at the beginning of iteration $i$. We refer to line numbers of the pseudocode Alg.
1. First, we show the value of \( f(A) \) does not decrease between iterations of the for loop, despite the possibility of deletions from \( A \).

**Lemma 1.** For any \( 1 \leq i \leq n \), it holds that \( f(A_i) \leq f(A_{i+1}) \).

**Proof.** If no deletion is made during iteration \( i \) of the for loop, then any change in \( f(A) \) is clearly nonnegative. So suppose deletion of set \( B \) from \( A \) occurs on line 10 of Alg. 1 during this iteration. Observe that \( A_{i+1} = (A_i \setminus B) \cup \{e_i\} \), because the deletion is triggered by the addition of \( e_i \) to \( A_i \). In addition, at some iteration \( j < i \) of the for loop, it holds that \( A_j = B \). From the beginning of iteration \( j \) to the beginning of iteration \( i \), there have been \( \ell(k+1) \log_2(k) - 1 \geq (\ell - 1)(k+1) \log_2(k) \) additions and no deletions to \( A \), which add precisely the elements in \( (A_i \setminus A_j) \).

It holds that

\[
\begin{align*}
f(A_i) &= f(A_i \setminus A_j) \geq f(B) - f(A_j) \\
&\geq \left( 1 + \frac{1}{k^{\ell-1}} \right) \cdot \frac{\ell(k+1)}{k} \cdot f(A_j) - f(A_j) \\
&\geq (\ell - 1)(k+1) \log_2(k) \cdot f(A_j)
\end{align*}
\]

where inequality (a) follows from submodularity and nonnegativity of \( f \), inequality (b) follows from the fact that each addition from \( B \) to \( A \) increases the value of \( f(A) \) by a factor of at least \((1 + 1/k)\), and inequality (c) follows from Claim 1. Therefore

\[
f(A_i) \leq f(A_i \setminus A_j) + f(A_j) \leq \left( 1 + \frac{1}{k^{\ell-1}} \right) f(A_i \setminus A_j).
\]

Next,

\[
\begin{align*}
f((A_i \setminus A_j) \cup \{e_i\}) - f(A_i \setminus A_j) \geq f(A_i \cup \{e_i\}) - f(A_i) \geq f(A_i) / k \geq f(A_i \setminus A_j) / k,
\end{align*}
\]

where inequality (d) follows from submodularity, and inequality (e) is by the condition to add \( e_i \) to \( A_i \) on line 7. Finally, using Inequalities (1) and (2) as indicated below, we have

\[
\begin{align*}
f(A_{i+1}) &= f(A_i \setminus A_j \cup \{e_i\}) 
\quad \text{By (2)} \quad \left( 1 + \frac{1}{k} \right) f(A_i \setminus A_j) 
\quad \text{By (1)} \quad \geq \frac{1 + \frac{1}{k^{\ell-1}}}{k^{\ell-1}} \cdot f(A_i) \geq f(A_i),
\end{align*}
\]

where the last inequality follows since \( k \geq 2 \) and \( \ell \geq 3 \). \( \square \)

Next, we bound the total value of \( f(A) \) lost from deletion throughout the run of the algorithm.

**Lemma 2.** \( f(A^*) \leq \left( 1 + \frac{1}{k^{\ell-1}} \right) f(A_{n+1}) \).

**Proof.** Observe that \( A^* \setminus A_{n+1} \) may be written as the union of pairwise disjoint sets, each of which is size \( \ell(k+1) \log_2(k) + 1 \) and was deleted on line 10 of Alg. 1. Suppose there were \( m \) sets deleted from \( A \); write \( A^* \setminus A_{n+1} = \{B^i : 1 \leq i \leq m\} \), where each \( B^i \) is deleted on line 10, ordered such that \( i < j \) implies \( B^i \) was deleted after \( B^j \) (the reverse order in which they were deleted); finally, let \( B^0 = A_{n+1} \).
Claim 2. Let $0 \leq i \leq m$. Then $f(B^i) \geq k^i f(B^{i+1})$.

Proof. Let $B^i, B^{i+1} \in B$. There are at least $\ell(k+1) \log k + 1$ elements added to $A$ and exactly one deletion event during the period between starting when $A = B^{i+1}$ until $A = B^i$. Moreover, each addition except possibly one (corresponding to the deletion event) increases $f(A)$ by a factor of at least $1 + 1/k$. Hence, by Lemma 1 and Claim 1, $f(B^i) \geq k^i f(B^{i+1})$. \hfill \Box

By Claim 2, for any $0 \leq i \leq m$, $f(A_{n+1}) \geq k^i f(B^i)$. Thus,

$$f(A^*) \leq f(A^* \setminus A_{n+1}) + f(A_{n+1}) \leq \sum_{i=0}^{m} f(B^i)$$

(Submodularity, Nonnegativity of $f$)

$$\leq f(A_{n+1}) \sum_{i=0}^{\infty} k^{-i} \quad \text{(Claim 2)}$$

$$= f(A_{n+1}) \left( \frac{1}{1 - k^{-1}} \right) \quad \text{(Sum of geometric series)}$$

Next, we bound the value of $\text{OPT}$ in terms of $f(A_{n+1})$.

Lemma 3. $\left( 2 + \frac{1}{k^2-1} \right) f(A_{n+1}) \geq \text{OPT}.$

Proof. Let $O \subseteq U$ be an optimal solution of size $k$ to SMCC; for each $o \in O$, let $i(o)$ be the iteration in which $o$ was processed. Then

$$f(O) - f(A^*) \leq f(O \cup A^*) - f(A^*) \quad \text{(Monotonicity of $f$)}$$

$$\leq \sum_{o \in O \setminus A^*} f(A^* + o) - f(A^*) \quad \text{(Submodularity of $f$)}$$

$$\leq \sum_{o \in O \setminus A^*} f(A_{i(o)}) / k \quad \text{(Submodularity of $f$, Line 7 of Alg. 1)}$$

$$\leq \sum_{o \in O \setminus A^*} f(A_{n+1}) / k \leq f(A_{n+1}). \quad \text{(Lemma 1, $|O| \leq k$)}$$

From here, the result follows from Lemma 2. \hfill \Box

Recall that $\text{QuickStream}_1$ returns the set $A'$, the last $k$ elements added to $A$. Lemma 4 shows that $2f(A') \geq f(A_{n+1})$.

Lemma 4. $f(A_{n+1}) \leq 2f(A').$
Proof. If \( |A_{n+1}| \leq k \), \( f(A') \geq f(A_{n+1}) \) by monotonicity, and the lemma holds. Therefore, suppose \( |A_{n+1}| > k \). Let \( A' = \{a'_1, \ldots, a'_k\} \), in the order these elements were added to \( A \). Let \( A'_i = \{a'_1, \ldots, a'_i\} \), \( A'_0 = \emptyset \). Then

\[
f(A') \geq f(A_{n+1}) - f(A_{n+1} \setminus A')
= \sum_{i=1}^{k} f((A_{n+1} \setminus A') \cup A'_{i-1} + a'_i) - f((A_{n+1} \setminus A') \cup A'_{i-1})
\geq \sum_{i=1}^{k} \frac{f((A_{n+1} \setminus A') \cup A'_{i-1})}{k} \quad \text{(Line 7 of Alg. 1)}
\geq \sum_{i=1}^{k} \frac{f(A_{n+1} \setminus A')}{k} = f(A_{n+1} \setminus A').
\]

(Monotonicity of \( f \))

Thus \( f(A_{n+1}) \leq f(A_{n+1} \setminus A') + f(A') \leq 2f(A') \). \( \square \)

Since \( k \geq 2 \), Lemmas 3 and 4 show that the set \( A' \) of \( \text{QuickStream}_1 \) satisfies \( f(A') \geq \left( \frac{1}{4+2/(k^2-1)} \right) \text{OPT} \). By the choice of \( \ell \), \( f(A') \geq (1/4 - \varepsilon) \text{OPT} \).

\( \square \)

2.2 Post-Processing: \( \text{QuickStream}_{c++} \)

In this section, we describe a simple post-processing procedure to improve the objective value obtained by \( \text{QuickStream}_c \). At the termination of the stream, \( \text{QuickStream}_c \) stores a set \( A \) of size \( O(k \log k) \) from which the set \( A' \) and solution are extracted, on which the worst-case approximation ratio is proven in the previous section. However, the set \( A \) may be regarded as a filtered ground set of size \( O(k \log k) \leq n \), upon which any algorithm may be run to extract a solution. As long as the post-processing algorithm has query complexity and runtime \( O(n) \), Theorem 4 still holds for the resulting single-pass streaming algorithm with post-processing. This modification of \( \text{QuickStream} \) is termed \( \text{QuickStream}_{c++} \).

We remark that the condition of Line 7 of \( \text{QuickStream}_c \) may be changed to the following condition:

\[
f(A \cup C) - f(A) \geq \delta f(A)/k,
\]
for input parameter \( \delta > 0 \). In this case, it is not difficult to extend the analysis in the previous section to show that the algorithm achieves ratio

\[
[c(1 + \delta)(1 + 1/\delta)]^{-1},
\]
in memory \( O(k \log k) \) and the same query complexity and runtime. This ratio is optimized for \( \delta = 1 \), but when using post-processing with \( \text{QuickStream}_{c++} \), smaller values of \( \delta \) result in larger sets \( A \), although still bounded in \( O(k \log k) \leq n \). Thus, we found in our empirical evaluation in Section 4 that setting \( \delta = c/10 \) for \( \text{QuickStream}_{c++} \) yields very good empirical results.
Algorithm 2: A procedure to boost to from constant ratio $\alpha$ to ratio $1 - \epsilon^{-1+\epsilon}$ in $O(1/\epsilon)$ passes, 1 query per element per pass, and $O(k)$ memory.

1: procedure BOOSTRATIO$(f, k, \alpha, \Gamma, \epsilon)$
2: $\textbf{Input:}$ evaluation oracle $f : 2^N \rightarrow \mathbb{R}^+$, constraint $k$, constant $\alpha$, value $\Gamma$ such that $\Gamma \leq \text{OPT} \leq \Gamma/\alpha$, and $0 < \epsilon < 1$.
3: $\tau \leftarrow \Gamma/(\alpha k)$, $A \leftarrow \emptyset$.
4: while $\tau \geq (1 - \epsilon)\Gamma/(4k)$ do
5: $\tau \leftarrow \tau(1 - \epsilon)$
6: for $n \in \mathbb{N}$ do
7: if $f(A + n) - f(A) \geq \tau$ then
8: $A \leftarrow A + n$
9: if $|A| = k$ then
10: return $A$
11: return $A$

3 Multi-Pass Streaming Algorithm to Boost Constant Ratio to $1 - 1/e - \epsilon$

In this section, we describe BOOSTRATIO (Alg. 2), which given any $\alpha$-approximation $A$ for SMCC can boost the ratio to $1 - \epsilon^{-1+\epsilon} \geq 1 - 1/e - \epsilon$ using the output of $A$. Theorem 5 is proven in Appendix D.

Theorem 5. Let $0 < \epsilon < 1$. Suppose a deterministic $\alpha$-approximation $A$ exists for SMCC. Then algorithm BOOSTRATIO is a multi-pass streaming algorithm that when applied to the solution of $A$ yields a solution within factor $1 - \epsilon^{-1+\epsilon} \geq 1 - 1/e - \epsilon$ of optimal in at most $n(\log(4/\alpha)/\epsilon + 1)$ queries, $\log(4/\alpha)/\epsilon + 1$ passes, and $O(k)$ memory.

When the algorithm $A$ is any algorithm provided by Theorem 1, this establishes Theorem 3.

As input, the algorithm BOOSTRATIO takes an instance $(f, k)$ of SMCC, an approximate solution value $\Gamma$, and accuracy parameter $\epsilon > 0$. On the instance $(f, k)$, it must hold that $\Gamma \leq \text{OPT} \leq \Gamma/\alpha$, where OPT is the value of an optimal solution. The algorithm works by making one pass (line 6) through the ground set for each threshold value $\tau$, during which any element with marginal gain at least $\tau$ to $A$ is added to $A$ (lines 7–8). The maximum and minimum values of $\tau$ are determined by $\Gamma, \alpha,$ and $k$: initially $\tau = \Gamma/(\alpha k)$, and the algorithm terminates if $\tau < (1 - \epsilon)\Gamma/(4k)$; each iteration of the while loop, $\tau$ is decreased by a factor of $(1 - \epsilon)$. The set $A$ is initially empty; if $|A| = k$, the algorithm terminates and returns $A$; otherwise, at most $O(\log(1/\alpha)/\epsilon)$ passes are made until the minimum threshold value is reached.

Intuitively, the $1 - 1/e - \epsilon$ ratio is achieved since the $\alpha$-approximate solution $\Gamma$ allows the algorithm to approximate the value for $\tau$ of $\text{OPT}/k$ in a constant number of guesses. Once this threshold has been reached, only $\log(1/4)/\epsilon$ more values of $\tau$ are needed to achieve the desired ratio. While BOOSTRATIO may be
used with any $\alpha$-approximation, if it is used with \textsc{QuickStream}_1, the resulting algorithm is the first linear-time, deterministic, $(1 - 1/e - \varepsilon)$-approximation for \textsc{SMCC}, which is a multi-pass streaming algorithm.

4 Empirical Evaluation

In this section, we demonstrate that the objective value achieved empirically by our algorithm \textsc{QuickStream}_c++ beats that of the state-of-the-art algorithms \textsc{LTL}, \textsc{SieveStream}++, and \textsc{C&K}, while using the fewest queries and only a single pass. The multi-pass algorithm \textsc{QS+BR} (\textsc{QuickStream}_1 followed by \textsc{BoostRatio}) achieved mean objective value better than 0.99 of the standard \textsc{Greedy} value across all instances tested.

4.1 Methodology

\textbf{Algorithms} Our algorithms are compared to the following methods: \textsc{Greedy}, the standard greedy algorithm analyzed by Nemhauser et al. [24], \textsc{LTL} [20], \textsc{SieveStream}++, \textsc{P-Pass} [25], and \textsc{C&K} [5], as described in Section 1. Randomized algorithms were averaged over 10 independent runs and the shaded regions in plots correspond to one standard deviation. Any algorithm with an accuracy parameter $\varepsilon$ is run with $\varepsilon = 0.1$ unless otherwise specified.

We evaluate our algorithm \textsc{QuickStream}_c++ for various values of $c$. The post-processing procedure run on $A$ is taken to be our linear time \textsc{BoostRatio} and we set parameter $\delta = c/10$ (see Section 2.2 for the definition of $\delta$). We also evaluate our multi-pass algorithm \textsc{QS+BR}.

\textbf{Applications} We evaluate all of the algorithms on two applications of \textsc{SMCC}: the first is maximum coverage on a graph: for each set of vertices $S$, the value of $f(S)$ is the number of vertices adjacent to the set $S$. The second application is the revenue maximization problem on a social network [13], a variant of influence maximization. For detailed specification of these applications, see Appendix E. We evaluate on a variety of network technologies from the Stanford Large Network Dataset Collection [17], including ego-Facebook ($n = 4039$) and web-Google ($n = 875713$), among others listed in Appendix E. Values of $k$ evaluated include small values ($k \leq 1000$) and large values ($k = \Omega(n)$).

4.2 Results: Single-Pass Algorithms

In Fig. 1, representative results are shown in comparison with single-pass. Results were qualitatively similar across applications and datasets; additional results are shown in Appendix E.

\textbf{Objective Value} For small $k$ ($k \leq 1000$), the mean objective value (normalized by the standard \textsc{Greedy} value) obtained by each single-pass algorithm across all instances is as follows: \textsc{QuickStream}_1++ 0.99; \textsc{QuickStream}_4++
Figure 1: Evaluation of single-pass streaming algorithms on web-Google ($n = 875713$), in terms of objective value normalized by the standard greedy value, total number of queries, and the maximum memory used by each algorithm normalized by $k$. The legend shown in (a) applies to all subfigures.

0.95; C&K 0.93; SieveStream++ 0.87; QuickStream16++ 0.84. On the instances with large $k$ ($k \leq 0.1n$), the means are: QuickStream1++ 0.99; QuickStream4++ 0.94; SieveStream++ 0.89; QuickStream16++ 0.88.

Queries In terms of queries, QuickStream$_c$++ required roughly $n/c$ queries for small $k$; the next smallest was C&K, which required $2n$ queries, followed by SieveStream++, which started at more than $10n$ queries and increased logarithmically with $k$. For large $k$, the queries of QuickStream$_c$++ increased due to the $O(n)$ post-processing step which depends on $k$, but always remained less than $2n$.

The algorithm C&K, while very efficient in terms of queries, was unable to run in a reasonable timeframe on our larger instances. Most of the algorithms we evaluate (including both of our algorithms) use a marginal gain query of sets that only increase in size, which yields an optimized implementation for the maximum cover application. However, C&K cannot be implemented with this optimization and requires the full $O(n)$ oracle query; thus, on some instances we were able to run the standard greedy algorithm but not C&K. This illustrates the fact that the oracle query complexity only constitutes partial information about the runtime of the algorithm.
Figure 2: Evaluation of our algorithms compared with the multi-pass P-Pass and non-streaming algorithm LTL. We compare the objective value (normalized by the standard Greedy objective value) and total queries on web-Google for the maximum cover application for both small and large $k$ values. The large $k$ values are given as a fraction of the number of nodes in the network. The legend shown in (a) applies to all subfigures.
Memory As shown in Figs. 1(c) and 1(f), the memory usage of the algorithms remained at most a constant times $k$; for $\text{QuickStream}_{c}^{++}$, this constant decreased as $k$ increased, and with large enough $k$, the algorithms used less memory than $\text{SieveStream}^{++}$. In terms of memory, C&K is optimal both theoretically and in practice, as it stores only $k$ elements.

4.3 Results: Multi-Pass and Non-Streaming Algorithms

In Fig. 2, we show results of our algorithms $\text{QuickStream}_{c}^{++}$ and $\text{QS+BR}$, in comparison with the multi-pass P-Pass algorithm and the non-streaming LTL algorithm on web-Google. Surprisingly, our single-pass algorithm $\text{QuickStream}_{1}^{++}$ beats the objective values of both P-Pass and LTL, as it obtained 0.99 of the standard greedy value on average across all instances (both small and large $k$). The only algorithm with better objective value than $\text{QuickStream}_{1}^{++}$ is our multipass $\text{QS+BR}$. The algorithm $\text{QuickStream}_{4}^{++}$ exceeded the objective value of LTL despite using $1/8$ of the queries.
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A Related Work

Online Algorithms A more restrictive streaming model is the preemptive, online model proposed by Buchbinder et al. [4]. In this setting, the algorithm receives elements one by one in an arbitrary order and must maintain a competitive solution with respect to the optimal solution on elements seen so far; the algorithm is allowed to discard elements that were previously chosen into the solution and must maintain a feasible solution (a set of size at most \( k \)).

Buchbinder et al. [4] described a deterministic \( 1/4 \)-competitive algorithm in this model that requires \( O(kn) \) queries. Chan et al. [6] improved the competitive ratio to \( O(kn) \) queries; their ratio converges to \( \approx 0.318 \) as \( k \to \infty \). They also show that the ratio of \( 0.318 \) is optimal in this online model. Our algorithms are not online in this sense, since they maintain an infeasible set of size \( O(k \log k) \) rather than a feasible set of size \( k \) and if \( c > 1 \), QuickStream requires additional processing at termination of the stream. However, QuickStreamLargeK requires no processing at the end of the stream and does maintain a competitive ratio that converges to \( \approx 0.316/c \).

B Proof of Theorem 2

For convenience, we restate Theorem 2 here.

**Theorem.** Let \( c \geq 2 \) be an integer, and let \( \varepsilon > 0 \). Any (randomized) approximation algorithm for SMCC with ratio \( 1/c + \varepsilon \) for SMCC with probability \( \delta \) requires at least \( \lceil \delta n/(c-1) \rceil \) oracle queries.

**Proof.** We prove the theorem for instances of SMCC with \( k = 1 \) (for which the optimal solution may be found with \( n \) queries). Let \( c \in \mathbb{N}, c \geq 2 \), and let \( 0 < \varepsilon < 1 \). Let \( n \in \mathbb{N} \), and let \( U_n = \{0, 1, \ldots, n-1\} \). Define \( f : 2^{U_n} \to \mathbb{R}^+ \) by \( f(A) = \min\{|A|, c\} \), for \( A \subseteq U_n \). Next, we define a function \( g \) that is hard to distinguish from \( f \): pick \( a \in U_n \) uniformly randomly. Let \( g(A) = f(A) \) if \( a \notin A \), and \( g(A) = c \) otherwise. Clearly, both \( f \) and \( g \) are monotone and submodular.

Now, consider queries to \( f \) and \( g \) of a set \( A \subseteq U_n \). These queries can only distinguish between \( f \) and \( g \) if \( |A| \leq c - 1 \) and \( a \in A \); in any other case, the values of \( f(A) \) and \( g(A) \) are equal. Fix any \( A \) with \( |A| \leq c - 1 \). Then \( a \in A \) with probability at most \( (c-1)/n \). Therefore, to distinguish between \( f \) and \( g \) with probability at least \( \delta \) requires at least \( \lceil \delta n/(c-1) \rceil \) queries.

Since any approximation algorithm with ratio at least \( 1/c + \varepsilon \) with probability \( \delta \) would distinguish between \( f, g \) with probability \( \delta \), since the maximum singleton of \( f \) has value 1, while \( g(a) = c \), the theorem is proven.

\( \Box \)
C Variants of QuickStream\textsubscript{c}

In this section, we describe algorithms that are similar in design to QuickStream\textsubscript{c}. In Section C.1, we describe a modification to QuickStream\textsubscript{c} that can only improve the objective value at the expense of \( c \) additional queries and \( O(ck) \) additional memory. In Section C.2, we describe QuickSingleton\textsubscript{c}, designed for the case \( k = 1 \). Finally, in Section C.3, we describe QuickStreamLargeK\textsubscript{c}, designed to have an improved ratio for \( k \geq 8c/e \).

Observe that Theorem 1 is a direct consequence of Theorems 4, 6, and 7.

C.1 The QuickStream\textsubscript{c++} Modification

In this section, we describe how QuickStream\textsubscript{c} may be augmented to improve its solution quality. This modification requires at most \( c \) additional queries to the oracle for \( f \); however, it also requires \( \lceil n/c \rceil \log k \) comparisons between real numbers, which are not required in QuickStream\textsubscript{c}.

The modification is simple. The algorithm QuickStream\textsubscript{c++} works as QuickStream; in addition, it keeps track of the \( k \) blocks of size \( c \) with highest \( f \) value seen so far. At termination of the algorithm, these \( ck \) extra elements are arbitrarily partitioned into sets of size \( k \); the partition with the best \( f \) value is compared to the best partition of the last \( ck \) elements added to \( A' \), and the best of the two is returned.

When a new block \( C \) of at most \( c \) elements is received, it requires \( \log(k) \) comparisons to determine if the value \( f(C) \) is in the top \( k \) of blocks seen so far, given that the objective values of these blocks are maintained in a sorted list.

C.2 The QuickSingleton\textsubscript{c} Algorithm

In this section, we describe the algorithm QuickSingleton\textsubscript{c}, a deterministic, single-pass algorithm that has guarantees summarized in the following theorem. Full pseudocode is given in Alg. 3. After receipt of \( c \) elements stored in buffer \( C \), the algorithm evaluates \( f(C) \) and replaces \( A \) with \( C \) if \( f(C) > f(A) \). At termination, the maximum singleton in \( A \) is returned.

**Theorem 6.** The algorithm QuickSingleton\textsubscript{c} is a deterministic, single-pass algorithm with ratio \( 1/c \) if \( k = 1 \), query complexity \( \lceil n/c \rceil + c \), and memory complexity \( O(c) \).

**Proof.** Suppose \( k = 1 \). Observe that at termination of the algorithm any singleton \( u \in U \) satisfies \( f(u) \leq f(A) \). Further, at termination of the stream, the element \( a \) in \( A \) maximizing \( f \) is returned. Let \( b \) be an optimal singleton; by submodularity and the fact \( |A| \leq c, cf(a) \geq f(A) \geq f(b) \).

Memory complexity and query complexity are clear. \( \square \)
Algorithm 3 For each $c \geq 1$, a single-pass algorithm with approximation ratio $1/c$ for SMCC if $k = 1$. The query complexity is $\lceil n/c \rceil + c$, memory complexity is $O(c)$.

1: procedure $\text{QuickSingleton}_c(f,k)$
2: Input: oracle $f$, cardinality constraint $k$
3: $A \leftarrow \emptyset$, $C \leftarrow \emptyset$
4: for element $e$ received do
5:     $C \leftarrow C + e$
6:     if $|C| = c$ or stream has ended then
7:         if $f(C) > f(A)$ then
8:             $A \leftarrow C$
9:         $C \leftarrow \emptyset$
10: return $\arg\max_{a \in A} f(a)$

C.3 The QuickStreamLargeKc Algorithm

In this section, we describe algorithms, parameterized by $c$, that require $\lceil n/c \rceil$ queries, have $O(ck \log(k))$ memory complexity, and have ratio that converges to $(1 - 1/e - (2c)/(ke) - c^2/(k^2e))$ as $k \to \infty$. However, for small $k$, these algorithms may not have any approximation ratio. We refer to these algorithms as QuickStreamLargeKc.

Full pseudocode for QuickStreamLargeKc is given in Alg. 4. The main differences with QuickStreamc are 1) a block $C$ is added to $A$ only if the gain exceeds $cf(A)/k$ rather than $f(A)/k$ as in QuickStreamc; (2) $A'$ keeps only the last $k$ elements added, rather than the last $k$ blocks; hence, there is no need to partition $A'$ at the end of the algorithm. Instead, the set $A'$ is simply returned. The rest of the section proves the following theorem.

Theorem 7. The algorithm QuickStreamLargeKc is a single-pass, deterministic streaming algorithm with approximation ratio

$$\left(\frac{1}{1 + c + 1/(k^3 - 1)}\right) (1 - 1/e - (2c)/(ke) - c^2/(k^2e)),$$

if $k \geq 8c/e$, query complexity $\lceil n/c \rceil$, and memory complexity $O(ck \log(k))$.

Proof. In addition to Claim 1 above, we need the following elementary fact about the number $e$:

Claim 3. For any real number $x > 0$, $(1 + 1/x)^x < e < (1 + 1/x)^{x+1}$.

We will actually show that QuickStreamLargeK maintains a competitive ratio with respect to the optimal solution on the elements seen thus far; suppose $m$ blocks have been received, let $C_i$ denote the $i$-th block of elements processed on line 7. Let $\text{OPT}_{\mathcal{X}}$ denote the optimal solution to SMCC with input $(f|_{\mathcal{X}}, k)$, where $\mathcal{X} = \bigcup_{i=1}^{m} C_i \subseteq \mathcal{U}$. Let $A_i$ denote the value of set $A$ immediately before processing the $i$-th block $C_i$, and let $A_{m+1}$ denote the value of $A$ after processing all blocks. Finally, let $A^*$ denote $\bigcup_{i=1}^{m+1} A_i$. 

20
Algorithm 4 For each $c \geq 1$, a single-pass algorithm with approximation ratio
\[
\left(\frac{1}{1+c+\frac{1}{(k-1)}}\right) \left(1 - \frac{1}{e} - \frac{2c}{(ke)} - \frac{c^2}{(k^2e)}\right)
\]
if $k \geq 8c/e$. The query complexity is $\lceil n/c \rceil$.

1: procedure QUICKSTREAMLARGEK$_e(f,k)$
2: Input: oracle $f$, cardinality constraint $k$
3: $A \leftarrow \emptyset$, $A' \leftarrow \emptyset$, $C \leftarrow \emptyset$, $j \leftarrow 0$
4: for element $e$ received do
5: $C \leftarrow C + e$
6: if $|C| = c$ or stream has ended then
7: if $f(A \cup C) - f(A) \geq cf(A)/k$ then
8: $A \leftarrow A \cup C$
9: $j \leftarrow j + 1$
10: if $j > 6(k+1) \log_2(k)$ then
11: $A \leftarrow \{3(k+1) \log_2(k)\}$ blocks most recently added to $A$
12: $j \leftarrow 3(k+1) \log_2(k)$
13: $C \leftarrow \emptyset$
14: $A' \leftarrow \{k$ elements most recently added to $A\}$
15: return $A'$

The following two lemmas have exactly analogous proofs to Lemmas 1 and 2 by replacing blocks for elements, 2 for $\ell$, and noting that $(1 + c/k) \geq (1 + 1/k)$. We provide the proofs for completeness.

Lemma 5. Suppose $k > 1$; let $1 \leq i \leq m$. Then $f(A_i) \leq f(A_{i+1})$.

Proof. If no deletion is made during the processing of block $C_i$, then the change in $f(A)$ is clearly nonnegative. So suppose deletion of set $B$ from $A$ occurs on line 11 during this iteration. Observe that $A_{i+1} = (A_i \setminus B) \cup C_i$, because the deletion is triggered by the addition of block $C_i$ to $A_i$. In addition, at some iteration $j < i$ of the for loop, it holds that $A_j = B$. From the beginning of iteration $j$ to the beginning of iteration $i$ there have been $3(k+1) \log_2(k) - 1 \geq 2(k+1) \log_2(k)$ additions of blocks and no deletions to $A$, which add precisely the elements in $(A_i \setminus A_j)$.

It holds that
\[
f(A_i \setminus A_j) \geq f(A_i) - f(A_j) \geq \left(1 + \frac{1}{k}\right)^{2(k+1) \log k} \cdot f(A_j) - f(A_j) \geq (k^2 - 1)f(A_j),
\]
where inequality (a) follows from submodularity and nonnegativity of $f$, inequality (b) follows from the fact that each addition from $A_j$ to $A_i$ increases the value of $f(A)$ by a factor of at least $(1 + 1/k)$, and inequality (c) follows from Claim 1. Therefore
\[
f(A_i) \leq f(A_i \setminus A_j) + f(A_j) \leq \left(1 + \frac{1}{k^2 - 1}\right) f(A_i \setminus A_j).
\]
Next,

\[
f((A_i \setminus A_j) \cup C_i) - f(A_i \setminus A_j) \geq f(A_i \cup C_i) - f(A_i) \geq \frac{f(A_i)}{k} \geq \frac{f(A_i \setminus A_j)}{k},
\]

where inequality (d) follows from submodularity, and inequality (e) is by the condition to add \( C_i \) to \( A_i \) on line 7. Finally, using Inequalities (3) and (4) as indicated below, we have

\[
f(A_{i+1}) = f(A_i \setminus A_j \cup C_i) \geq \left(1 + \frac{1}{k}\right) f(A_i \setminus A_j) \geq \frac{1 + \frac{1}{k}}{1 + \frac{1}{k^2-1}} \cdot f(A_i) \geq f(A_i),
\]

where the last inequality follows since \( k \geq 2 \).

\[\square\]

**Lemma 6.**

\[f(A^*) \leq \left(1 + \frac{1}{k^3-1}\right) f(A_m+1).\]

**Proof.** Observe that \( A^* \setminus A_{m+1} \) may be written as the union of pairwise disjoint sets, each of which is size \( 3c(k+1) \log_2(k)+1 \) and was deleted on line 11 of Alg. 4. Suppose there were \( l \) sets deleted from \( A \); write \( A^* \setminus A_{m+1} = \{B^i : 1 \leq i \leq l\} \), where each \( B^i \) is deleted on line 10, ordered such that \( i < j \) implies \( B^i \) was deleted after \( B^j \) (the reverse order in which they were deleted); finally, let \( B^0 = A_{m+1} \).

**Claim 4.** Let \( 0 \leq i \leq l \). Then \( f(B^i) \geq k^3 f(B^{i+1}) \).

**Proof.** Let \( B^i, B^{i+1} \in B \). There are at least \( 3(k+1) \log k + 1 \) blocks added to \( A \) and exactly one deletion event during the period between starting when \( A = B^{i+1} \) until \( A = B^i \). Moreover, each addition except possibly one (corresponding to the deletion event) increases \( f(A) \) by a factor of at least \( 1 + 1/k \). Hence, by Lemma 5 and Claim 1, \( f(B^i) \geq k^3 f(B^{i+1}) \). \[\square\]

By Claim 4, for any \( 0 \leq i \leq l \), \( f(A_{m+1}) \geq k^3 f(B^i) \). Thus, by submodularity and nonnegativity of \( f \) and the sum of a geometric series,

\[
f(A^*) \leq f(A^* \setminus A_{m+1}) + f(A_{m+1}) \leq \sum_{i=0}^{m} f(B^i)
\]

\[
\leq f(A_{m+1}) \sum_{i=0}^{\infty} k^{-3i}
\]

\[
= f(A_{m+1}) \left(\frac{1}{1 - k^{-3}}\right).\]  

The next lemma shows that \( f(A_{m+1}) \) has a significant fraction of the optimal value.

**Lemma 7.** \( \left(1 + c + \frac{1}{k^3-1}\right) f(A_{m+1}) \geq \text{OPT}_N \).
Proof. Let $O \subseteq \mathcal{N}$ be an optimal solution to of size $k$ to SMCC. Let $C_o$ denote the block containing $o \in O$ that is considered for addition into $A$. Then by monotonicity and submodularity of $f$, the fact that if block $C_i$ is not added to $A$, $f(A \cup C_i) - f(A_i) < cf(A_i)/k$, and by Lemma 5, we have

$$f(O) - f(A^*) \leq f(O \cup A^*) - f(A^*)$$

$$\leq \sum_{o \in O \setminus A^*} f(A^* \cup \{o\}) - f(A^*)$$

$$\leq \sum_{o \in O \setminus A^*} f(A_o \cup \{o\}) - f(A_o)$$

$$\leq \sum_{o \in O \setminus A^*} f(A_o \cup C_o) - f(A_o)$$

$$\leq \sum_{o \in O \setminus A^*} \frac{cf(A_o)}{k}$$

$$\leq \sum_{o \in O \setminus A^*} \frac{cf(A_{m+1})}{k} \leq cf(A_{m+1}).$$

From here, the result follows from Lemma 6. \qed

Recall that QuickStreamLargeK$_c$ returns the set $A'$, the last $k$ elements added to $A$. The last portion of the proof shows that $f(A')$ is a large fraction of the value of $f(A_{m+1})$; this part of the proof departs from the proof of Theorem 4 above.

Lemma 8. Let $A'$ have its value after processing block $C_m$. Then

$$f(A_{m+1}) \leq \left(\frac{e}{e - (1 + c/k)^2}\right) f(A').$$

Proof. If $|A_{m+1}| \leq k$, $A' = A_{m+1}$, and the lemma holds. Suppose $|A_{m+1}| > k$. Let $A' = \{a'_1, \ldots, a'_k\}$, in the order these elements were added to $A_{m+1}$. Let $A'_i = \{a'_1, \ldots, a'_i\}$, $A'_0 = \emptyset$. Observe that by the condition on the marginal gain the addition of each block to $A$,

$$f(A_{m+1}) \geq (1 + c/k)^{[k/c]} f(A_{m+1} \setminus A') \geq \frac{e}{(1 + c/k)^2} f(A_{m+1} \setminus A'),$$

by Claim 3. Hence, by submodularity and nonnegativity of $f$,

$$f(A') \geq f(A_{m+1}) - f(A_{m+1} \setminus A') \geq \left(\frac{e}{(1 + c/k)^2} - 1\right) f(A_{m+1} \setminus A'). \quad (5)$$

From (5), we have

$$f(A_{m+1}) \leq f(A_{m+1} \setminus A') + f(A') \leq \left(\frac{e}{(1 + c/k)^2} - 1\right)^{-1} + 1) f(A')$$

$$= \left(\frac{e}{e - (1 + c/k)^2}\right) f(A'). \quad \Box$$
Since \( k \geq 8c/e \), Lemmas 7 and 8 show that the set \( A' \) of QuickStreamLargeK, maintains \( f(A') \geq \left( \frac{1}{1+1/k} \right) (1 - \frac{1}{e} - (2c^2)/(k^2 e)) \OPT' \).

### D Omitted Proofs from Section 3

**Proof of Theorem 5.** Suppose \( 0 < \varepsilon < 1 \). Let \( (f, k) \) be an instance of SMCC. The algorithm is to first run \( A \) to obtain set \( A' \). Next, BOOSTRATIO is called with parameters \( (f, k, \alpha, f(A'), \varepsilon) \). Observe that the initial value of the threshold \( \tau \) in the while loop is at least \((1-\varepsilon)\OPT/k\) and the final value of \( \tau \) is at most \( \OPT/(4k) \).

Consider the case that at termination \( |A| < k \). Then by the last iteration of the while loop, submodularity and monotonicity of \( f \),

\[
 f(O) - f(A) \leq f(O \cup A) - f(A) \leq \sum_{o \in O \setminus A} f(A \cup \{o\}) - f(A) \\
\leq \sum_{o \in O \setminus A} \frac{\Gamma}{4k} \leq \OPT/4,
\]

from which \( f(A) \geq 3\OPT/4 \geq (1 - e^{-1+\varepsilon})\OPT \).

Next, consider the case that at termination \( |A| = k \). Let \( A_i = \{a_1, a_2, \ldots, a_i\} \), ordered by the addition of elements to \( A \), and let \( A_0 = \emptyset \).

**Claim 5.** Let \( i \in \{0, \ldots, k-1\} \). Then

\[
f(A_{i+1}) - f(A_i) \geq \frac{(1-\varepsilon)}{k} (\OPT - f(A_i))
\]

**Proof.** Let \( i \in \{0, \ldots, k-1\} \). First, suppose \( a_{i+1} \) is added to \( A_i \) during an iteration with \( \tau \geq (1-\varepsilon)\OPT/k \). In this case, \( f(A_{i+1}) - f(A_i) \geq (1-\varepsilon)\OPT/k \geq \frac{(1-\varepsilon)}{k} (\OPT - f(A_j)) \).

Next, suppose \( a_{i+1} \) is added to \( A_i \) during an iteration with \( \tau < (1-\varepsilon)\OPT/k \). Consider the set \( O \setminus A_i \); in the previous iteration of the while loop, no element of \( O \setminus A_i \) is added to \( A_i \); hence, by submodularity, for all \( o \in O \setminus A_i \), \( f(A_i + o) - f(A_i) < \tau/(1-\varepsilon) \). Therefore,

\[
f(A_{i+1}) - f(A_i) \geq \tau \geq \frac{(1-\varepsilon)}{k} \sum_{o \in O \setminus A_i} f(A_i \cup \{o\}) - f(A_i) \\
\geq \frac{(1-\varepsilon)}{k} (f(O \cup A_i) - f(A_i)) \\
\geq \frac{(1-\varepsilon)}{k} (\OPT - f(A_i)).\]

From Claim 5, standard arguments show the \( f(A_k) \geq \OPT (1 - e^{-1+\varepsilon}) \geq \OPT(1 - 1/e - \varepsilon) \).

For the query complexity, observe that the for loop of BOOSTRATIO makes at most \( n \) queries, and the while loop requires \( \log(\alpha/4)/\log(1-\varepsilon) + 1 \leq \log(4/\alpha)/\varepsilon + 1 \) iterations.
E Additional Empirical Evaluation

E.1 Applications and Datasets

The maximum cover objective is defined as follows. Suppose $G = (V, E)$ is a graph. For any set $S \subseteq V$, let $S^I$ be the set of all vertices incident with any edge incident with a vertex in $S$. Then, define

$$f(S) = |S^I|.$$  

This objective is monotone and submodular.

The revenue maximization application uses the concave graph model introduced in Hartline et al. [13]. Given a social network $G = (V, E)$ with nonnegative edge weights, each user $u \in V$ is associated with a non-negative, concave function $f_u : \mathbb{R}^+ \rightarrow \mathbb{R}^+$. In Hartline et al. [13], optimal marketing strategies are defined, for which each user $u \in V$ has an associated revenue function $R_u(S)$, which depends on the set $S$ of players who have bought the item. Thus, the total revenue from set $S$ is

$$f(S) = \sum_{u \in V} R_u(S).$$

For this evaluation, we choose $R_u(S) = \left(\sum_{v \in S} w_{uv}\right)^{\alpha_u}$ where $\alpha_u$ is chosen independently for each $u$ uniformly in $(0, 1)$. The revenue maximization objective $f$ is monotone and submodular.

Network topologies are used from Stanford Large Network Dataset Collection [17]: ca-Astro ($n = 18772$), a collaboration network of Arxiv Astro Physics; ego-Facebook ($n = 4039$); and as-Skitter ($n = 1696415$).

E.2 Additional Results

Additional results from the maxcover application are shown in Fig. 3. Results from the revenue maximization application are shown in Figs. 4 and 5. These results are qualitatively similar to the results from maximum coverage discussed in Section 4.
Figure 3: Additional empirical results on the maxcover application on as-Skitter.

Figure 4: Additional empirical results for the revenue maximization application on soc-Facebook.
Figure 5: Additional empirical results for the revenue maximization application on ca-AstroPh.