ON THE SMOOTHNESS OF SOLUTIONS TO ELLIPTIC EQUATIONS IN DOMAINS WITH HÖLDER BOUNDARY.

I. V. Tsylin

Abstract: The dependence of the smoothness of variational solutions to the first boundary value problems for second order elliptic operators are studied. The results use Sobolev-Slobodetskii and Nikolskii-Besov spaces and their properties. Methods are based on real interpolation technique and generalization of Savaré-Nirenberg difference quotient technique.

1 Introduction

Let \((M, g)\) be a smooth connected compact oriented Riemannian manifold without boundary, \(\Omega \subseteq M\) be a subdomain with a Hölder boundary. The aim of this paper is to study the dependence of the smoothness of the variational solutions to the following equation:

\[ Au = f, \quad u \in \dot{H}^1(\Omega), \]

(1)
on the regularity of the right hand-side \(f \in H^{-1+\varepsilon}(M)\), \(\varepsilon > 0\). By definition, the operator \(A\) is generated by the continuous positive bilinear form \(\Phi\) defined on \(\dot{H}^1(\Omega)\), associated with the differential operation \(A'\), which is locally represented as follows:

\[-\frac{1}{\sqrt{\det g}} \partial_i \left( \sqrt{\det g} a^{ij} \partial_j u \right) + b^i \partial_i u + cu;\]

(2)

where \(a^{ij}, b^i, c\) are sufficiently regular coefficients.

By ellipticity, if the right hand-side belongs to \(L_2(\Omega)\), then the solution of (1) belongs to \(H^2_{loc}(\Omega)\). One cannot replace \(H^2_{loc}(\Omega)\) with \(H^2(\Omega)\) even if the boundary \(\partial\Omega\) is Lipschitz continuous \([4]\). However, this is possible whenever \(\Omega\) is a convex set or \(\partial\Omega \in C^{1,1}\) \([4]\, Theorems 2.2.2.3, 3.2.1.2).\n
Suppose \(\Omega \subseteq \mathbb{R}^d\) is a bounded domain with a Lipschitz boundary, \(A = -\Delta\), and \(\dot{H}^{-1+s}(\Omega)\) is the space of all functions \(v \in H^{1+s}(M)\), such that \(\text{supp}\, v \subseteq \bar{\Omega}\). It was shown by Jerison and Kenig (in \([6]\)) that if \(f \in H^{-1+s}(\Omega), s \in [0, 1/2]\), then the solution \(u \in \dot{H}^{1+s}(\Omega)\). In \([11]\), G. Savaré elaborated a new method to generalize this Proposition to the case of Lipschitz coefficients.

Theorem ([11]). Let \(\Omega \subseteq \mathbb{R}^d\) be a bounded domain, \(\partial\Omega\) be a Lipschitz continuous boundary, \(A\) be generated by \((\mathcal{L})\), and \(a^{ij} \in C^{0,1}(\bar{\Omega})\) be a symmetric positive definite matrix in \(\bar{\Omega}\), \(b^i \equiv 0\), \(c \equiv 0\). Then, the solution of \((\mathcal{L})\) belongs to \(\dot{H}^{1+s}(\Omega), s \in [0, 1/2]\), whenever \(f \in H^{-1+s}(\Omega)\).

In this paper we establish similar results in the situation when both the boundary and the coefficients are Hölder continuous. One of the results is the following (the proof will be given in Section 5).
**Theorem 1.1.** Let $M$ be a $d$–dimensional $C^{1,1}$–smooth compact Riemannian manifold without boundary, a domain $\Omega \subset M$ be such that $\partial \Omega$ is Hölder continuous of order $\gamma_{\Omega} \in (0, 1]$. Moreover, let $A$ be generated by (2), and for some $\varepsilon > 0$ the coefficients $a^{ij}$ and $b^i$ define a symmetric positive definite $C^{0, \gamma_c}(M)$–smooth section of $T^2M$, $L_{\frac{d+\gamma_c}{\gamma_c}}(\Omega)$–section $T \Omega$ respectively, $c \in W^{-1+\gamma+c}(\Omega)$ with $0 < \gamma_c \leq 1$, ($c \in L_{\max\{d, 2+\varepsilon\}}(\Omega)$ if $\gamma_c = 1$) and the form $\Phi$ be positive in $H^1(\Omega)$. Then the operator

$$R : H^{-1+s}(M) \to \tilde{H}^{1+\gamma s}(\Omega), \ s \in [0, \gamma_c/2).$$

solving problem (1) is continuous.

Our method is based on ideas in [12] and [13].

# 2 Terms and concepts

## 2.1 Domain $\Omega$

Further assume that $(M, g)$ is a $C^{1,1}$–smooth connected oriented compact manifold without boundary and that every coordinate mapping acts to $\mathbb{R}^d$ with fixed Euclidean norm $|\cdot|$.

**Definition 2.1.** A non-empty open set $\Omega \subset M$ is called a domain with a Hölder continuous boundary of order $\gamma_{\Omega}$ ($\partial \Omega \in C^{0, \gamma_{\Omega}}$) if there is an atlas $\mathcal{V} = \{(V, \kappa_V)\}$ such that for any $(V, \kappa_V) \in \mathcal{V}$ there exist a unit vector $\xi_V \in \mathbb{R}^d$ and a function $g_V : \xi_V \to \mathbb{R}$ such that $g_V \in C^{0, \gamma_{\Omega}}(\xi_V)$ with $0 < \gamma_{\Omega} \leq 1$, $\kappa_V(V \cap \partial \Omega)$ is a subset of the graph of $g_V$ and the intersection of $\kappa_V(V \cap \partial \Omega)$ with the epigraph of $g_V$ is empty.

## 2.2 Operator $A$

Let us suppose that $a^{ij}$ and $b^i$ in (2) define a symmetric positive definite $C^{0, \gamma_c}(M)$–smooth section $A$ of $T^2M$, and $L_{\frac{d+\gamma_c}{\gamma_c}}(\Omega)$–section $\mathcal{B}$ of $T \Omega$ respectively. We shall denote by $G$ the section of $T^2M$ generated by the Riemannian structure $g$. Since $A$ and $G$ are dependent on $x \in M$, we denote them as $A_x$ and $G_x$. The following conditions are assumed

**A1** There exists a constant $\alpha > 0$ such that

$$\forall x \in M \ \forall \xi \in T^*_x M \Rightarrow \alpha G_x(\xi, \xi) \leq A_x(\xi, \xi);$$

**A2** Section $A$ belongs to $C^{0, \gamma_c}(M)$, $\gamma_c \in (0, 1]$. 

We endow $C^{0, \gamma_c}(M)$ by the following norm:

$$\|A\|_{C^{0, \gamma_c}(M)} \overset{\text{def}}{=} \|A\|_{C(M)} + [A]_{C^{0, \gamma_c}(M)};$$

\[\text{Here } V \text{ is an open subset of } M \text{ and } \kappa_V : V \to \tilde{V} \subset \mathbb{R} \text{ is a diffeomorphism.} \]
where \( \|A\|_{C(M)} = \max_{x \in M} \max_{\xi \in T_x^* M, \xi \neq 0} \frac{A(x, \xi)}{\xi} \), \([A]_{C^{0,\gamma}}(M) = \sum_{U} \max_{ij} \left[ a_{ij}^U \right]_{C^{0,\gamma}(U)} \),
\[
[v]_{C^{0,\gamma}(U)} = \sup_{x,y \in U, x \neq y} \frac{|v(x) - v(y)|}{|x - y|}, \quad v : U \to \mathbb{R},
\]
Here \( U = \{(U, \kappa_U)\} \) is a fixed finite atlas for \( M \), and \( a_{ij}^U \) are coordinates of \( A \) with respect to the maps \( \kappa_U \). It can easily be proved that the convergence in \( C^{0,\gamma}(M) \) is independent of \( U = \{(U, \kappa_U)\} \).

Suppose that
\[
(u, v)_{\hat{H}^1(\Omega)} = \int_{\Omega} G(\nabla u, \nabla v) d\mu, \quad (u, v)_{L^2(\Omega)} = \int_{\Omega} uv d\mu,
\]
are inner products in \( \hat{H}^1(\Omega), L^2(\Omega) \) respectively, and that the measure \( \mu \) is associated with the Riemannian structure \( g \). Let for \( 1 \leq p \leq \infty \)
\[
\|u\|_{L^p(\Omega)} = \left( \int_{\Omega} |u|^p d\mu \right)^{1/p}, \quad \|u\|_{\hat{W}^1_p(\Omega)} = \left( \int_{\Omega} |G(\nabla u, \nabla v)|^{p/2} d\mu \right)^{1/p}, \quad p \in [1, \infty),
\]
\[
\|u\|_{L^\infty(\Omega)} = \sup_{x \in \Omega} |u(x)|, \quad \|u\|_{\hat{W}^1_p(\Omega)} = \sup_{x \in \Omega} |G(\nabla u, \nabla v)|^{1/2}.
\]
Further denote
\[
\|v\|_{\hat{W}^{q-1}_q(\Omega)} = \sup_{u \in \hat{W}^1_p(\Omega), u \neq 0} \frac{|v(u)|}{\|u\|_{\hat{W}^1_p(\Omega)}}, p \in (1, \infty), 1/p + 1/q = 1.
\]

A3 Assume that \( b, c \) are such that the form \( \Phi \) is positive and continuous in \( \hat{H}^1(\Omega) \).

Consider the following representation of \( \Phi \):
\[
\Phi(u, v) = \Phi_0(u, v) + \Phi_r(u, v),
\]
\[
\Phi_0(u, v) = \int_M A(\nabla u, \nabla v) d\mu, \quad \Phi_r(u, v) = \int_{\Omega} b(\nabla u) v d\mu + \int_{\Omega} cuvd\mu.
\]

We consider \( \int_{\Omega} cuvd\mu \) as the action of the functional \( c \) on the product \( uv \). Since A3 is satisfied, this action is well defined. In the same way, one can define \( \tau(f, v) = \int_{\Omega} fvd\mu \).

Then a function \( u \in \hat{H}^1(\Omega) \) is a weak variational solution to (1) with \( f \in H^{-1}(\Omega) \) if and only if
\[
\Phi_0(u, v) + \Phi_r(u, v) = \tau(f, v) \quad \forall v \in \hat{H}^1(\Omega).
\]
By A3 it follows that there exists a unique operator \( \mathcal{A} \) generated by \( \Phi \), such that \( \mathcal{A}u = f \).

The operator \( \mathcal{A} \) has a bounded inverse operator \( \mathcal{R} : H^{-1}(\Omega) \to \hat{H}^1(\Omega) \).

\[2\text{Friedrichs extension is meant } \mathcal{R}.\]
In the fourth section we shall impose additional conditions on the coefficients of \( \mathcal{A} \) (see (26)–(27)).

Let \( g(\cdot, \cdot) \) be the section of \( T^*M \times T^*M \) associated with the structure \( g \) and \( C_{emb} \) be the embedding constant of the continuous embedding \( \dot{H}^1(\Omega) \to L_q(\Omega), 1/p + 1/q = 1/2 \). Then, \( A3 \) holds whenever \( \mathbf{b} \in L_p(\Omega), c \in W_p^{-1}(\Omega) \), and

\[
C_{emb}(\|\mathbf{b}\|_{L_p(\Omega)} + \|c\|_{W_p^{-1}(\Omega)}) < \alpha,
\]

where \( \alpha \) is a constant in \( A1 \) and

\[
\|\mathbf{b}\|_{L_\infty(\Omega)} = \operatorname{ess sup}_{x \in \Omega} |g(\mathbf{b}, \mathbf{b})|^{1/2}, \quad \|\mathbf{b}\|_{L_p(\Omega)} = \left( \int_{\Omega} |g(\mathbf{b}, \mathbf{b})|^{p/2} \right)^{1/p}, \quad p \in [1, \infty).
\]

### 2.3 Weak smoothness

Further we shall use the following notion. Let \( L \) be a (not necessarily compact) manifold, \( E(L) \) be a real Banach space of functions \( f : L \to \mathbb{R} \). Denote

\[
\tilde{E}(\Omega) \overset{\text{def}}{=} \{ u \in E(L) \mid \text{supp} u \subset \bar{\Omega} \}.
\]

#### 2.3.1 Nikol’skii spaces

Let us denote \( v_h(x) = v(x + h) \) for \( v : \mathbb{R}^d \to \mathbb{R} \). We need to recall the definition of Nikol’skii spaces \( N_p^{k, \gamma}(\mathbb{R}^d), \gamma \in (0, 1], k \in \mathbb{Z}_+, p \in [1, \infty] \):

\[
N_p^{k, \gamma}(\mathbb{R}^d) = \left\{ v \in W_p^k(\mathbb{R}^d) \mid \|v\|_{N_p^{k, \gamma}(\mathbb{R}^d)} \overset{\text{def}}{=} \|v\|_{W_p^k(\mathbb{R}^d)} + \|v\|_{N_p^{k, \gamma}(\mathbb{R}^d)} \right\},
\]

\[
[v]_{N_p^{k, \gamma}(\mathbb{R}^d)} \overset{\text{def}}{=} \begin{cases} \max_{|\alpha| = k} \sup_{h \in \mathbb{R}^d, h \neq 0} \frac{\|\partial^\alpha v_h - \partial^\alpha v\|_{L_p(\mathbb{R}^d)}}{|h|^{\gamma}}, & \gamma \in (0, 1) \\ \max_{|\alpha| = k} \sup_{h \in \mathbb{R}^d, h \neq 0} \frac{\|\partial^\alpha v_h - 2\partial^\alpha v_h + \partial^\alpha v\|_{L_p(\mathbb{R}^d)}}{|h|}, & \gamma = 1 \end{cases}.
\]

Here \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d \) are multi-indices, \( \partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} \), \( |\alpha| = \sum_{i=1}^d \alpha_i \). Assume that \( N_p^{k, \gamma}(\mathbb{R}^d) \subset N_p^{k, \gamma}(\mathbb{R}^d) \), \( \gamma \in (0, 1) \), is the space of all functions satisfying the condition\(^3\)

\[
\lim_{|h| \to 0, \infty} \max_{|\alpha| = k} \frac{\|\partial^\alpha v - \partial^\alpha u\|_{L_p(\mathbb{R}^d)}}{|h|^\gamma} = 0.
\]

Let \( \mathcal{U} = \{(U, \kappa_U)\} \) be a certain fixed finite atlas of \( M \) and \( \{\psi_U\} \) be a corresponding smooth partition of unity. We define \( N_p^{k, \gamma}(M) \) (or \( \tilde{N}_p^{k, \gamma}(M) \)) as the space of all functions \( u \in L_p(M) \) such that for any \((U, \kappa_U) \in \mathcal{U}\) the product \( \psi_U \cdot u \) belongs to \( \tilde{N}_p^{k, \gamma}(U) \) (or \( \tilde{N}_p^{k, \gamma}(U) \)), and

\[
\|u\|_{N_p^{k, \gamma}(M)} = \sum_{U} \|\psi_U u\|_{\tilde{N}_p^{k, \gamma}(U)}, \quad \|u\|_{\tilde{N}_p^{k, \gamma}(M)} = \sum_{U} \|\psi_U u\|_{\tilde{N}_p^{k, \gamma}(U)}.
\]

\(^3\)In \([7]\), it is denoted by \( B^{k, \gamma}_{p, \infty}(\mathbb{R}^d) \) (see Proposition 2.6).
By Lemma 4.2 in [9] it follows that the definition of the spaces \( N^{k+\gamma}_p(M) \) is independent of atlas and partition of unity.

Let us consider a compact metric space \((X, \delta)\). For sets \(A, B \subset X\) let
\[
\text{dist}(A, B) \overset{\text{def}}{=} \inf_{x \in A, y \in B} \delta(x, y).
\]

**Proposition 2.2.** For any open set \(\Omega \subset M\) the following embedding holds
\[
\dot{W}^m_p(\Omega) \hookrightarrow \tilde{N}^m_p(\Omega), \quad m = 1, 2, \quad p \in [1, \infty].
\]
Moreover, for any function \(v \in \dot{W}^1_p(\Omega)\), any map \((U, \kappa_U)\) of \(M\), and any open set \(V \Subset \Omega \cap U\) the following implication holds
\[
\forall \varphi \in C(U) \forall h \in \mathbb{R}^d: |h| < \text{dist}(V, \partial(\Omega \cap U)) \Rightarrow \|\varphi(v_h - v)\|_{L^p(V)} \leq \|\varphi\|_{C(\bar{V})} C_V \|v\|_{\dot{W}^1_p(\Omega)} |h|,
\]
\[
\|\varphi(v_h - v)\|_{\tilde{L}^p(V)} \leq \|\varphi\|_{C(\bar{V})} \tilde{C}_V \|v\|_{\tilde{W}^1_p(\Omega)} |h|.
\]
Here \(\text{dist}\) is computed with respect to the metric \(|\cdot|\) in \(\kappa_U(U)\), and partial difference is defined with respect to the linear structure in the image of \(\kappa_U\).

The statement above follows by Proposition IX.3 [2]: if \(v \in \dot{W}^1_p(U')\), \(p \in [1, \infty]\), \(U \Subset U' \subset \mathbb{R}^d\), then for any \(h \in \mathbb{R}^d\), \(|h| < \text{dist}(U, \partial U')\) holds
\[
\|v_h - v\|_{L^p(U')} \leq |h| \|\nabla v\|_{L^p(U')}.
\]
Furthermore, if \(v \in \dot{W}^1_p(\mathbb{R}^d)\), then
\[
\|v_h - v\|_{L^p(\mathbb{R}^d)} \leq |h| \|\nabla v\|_{L^p(\mathbb{R}^d)}. \tag{4}
\]

### 2.3.2 Besov spaces

Let us recall the definition of Besov spaces (following [14]). Propositions 2.4 and 2.5 below follow from the similar propositions for domains with Lipschitz boundaries \(\Omega \subset \mathbb{R}^d\) and the fact that for any simply-connected bounded domains \(V_1, V_2 \subset \mathbb{R}^d\) there is a linear homeomorphism \(K : \hat{H}^m(V_2) \to \hat{H}^m(V_1), m = 1, 2, \ K : L_2(V_2) \to L_2(V_1)\), \(K : u \mapsto u \circ K_0\), generated by \(C^{1,1}\)-diffeomorphism \(K_0 : V_1 \to V_2\). Here \(K_0\) is defined in a larger open set \(V \ni V_1\).

Assume that \(F\) is the Fourier transform, \(M_0 = \{\xi \in \mathbb{R}^d \mid |\xi| \leq 2\}\), and \(M_j = \{\xi \in \mathbb{R}^d \mid 2^j-1 \leq |\xi| \leq 2^{j+1}\}\) for \(j \in \mathbb{N}\), \(\mathcal{S}'\) is the space of distributions of moderate growth.

**Definition 2.3.** Let us define for \(s \in \mathbb{R}, p \in (1, \infty)\) the following spaces
\[
B^s_{p,q}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \mid f = \sum_{j=0}^{\infty} a_j(x); \ \text{supp} F a_j \subset M_j; \right\}
\]
\[
\|\{a_j\}\|_{L^q(L^p)} = \left[ \sum_{j=0}^{\infty} (2^{sj} \|a_j\|_{L^p(\mathbb{R}^d)})^q \right]^{1/q} < \infty, \quad q \in [1, \infty)
\]
\[ B_p^{s_\infty}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \mid f = \sum_{j=0}^{\infty} a_j(x); \text{ supp } F a_j \subset M_j; \|\{a_j\}\|_{L^p(\mathbb{R}^d)} = \sup_{j \in \mathbb{Z}_+} 2^{qs} \|a_j\|_{L^p(\mathbb{R}^d)} < \infty \right\}, \quad q = \infty \]

with the norms

\[ \|f\|_{B_p^{s,q}(\mathbb{R}^d)} = \inf_{f \leq \sum_{j=0}^{\infty} a_j} \|\{a_j\}\|_{L^p(\mathbb{R}^d)}. \]

The space \( B_p^{s,q}(M) \) is defined by a finite atlas \( \mathcal{U} = \{(U, \kappa_U)\} \) and subordinate partition of unity \( \{\psi_U\} \), supp \( \psi_U \subset U \), in the following way. We suppose \( u \in B_p^{s,q}(M) \), if \( \psi_U u \in \tilde{B}_p^{s,q}(U) \); and introduce the norm

\[ \|u\|_{B_p^{s,q}(M)} = \sum_U \|\psi_U u\|_{\tilde{B}_p^{s,q}(U)}. \quad (5) \]

By the interpolation property of \( B_p^{s,q}(\mathbb{R}^d) \), one obtains that the norm is independent on \( \mathcal{U} \) and \( \{\psi_U\} \), up to equivalence.

For \( s \in \mathbb{R}_+ \setminus \mathbb{Z}_+ \), Nikol’skii and Sobolev–Slobodetskii spaces are the special cases of Besov spaces:

\[
N_p^s(M) = B_p^{s_\infty}(M), \quad \tilde{N}_p^s(\Omega) = \tilde{B}_p^{s_\infty}(\Omega), \\
W_p^s(M) = B_p^{s,q}(M), \quad \tilde{W}_p^s(\Omega) = \tilde{B}_p^{s,q}(\Omega),
\]

and for any \( p \in (1, \infty), q \in [1, \infty], s > \epsilon > 0 \), the following chain of embeddings holds

\[ \tilde{B}_p^{s+\epsilon}(\Omega) \hookrightarrow \tilde{B}_p^{s}(\Omega) \hookrightarrow \tilde{B}_p^{s,q}(\Omega) \hookrightarrow \tilde{B}_p^{s,q}(\Omega) \hookrightarrow \tilde{B}_p^{s-\epsilon}(\Omega). \quad (6) \]

Let us denote \( B_{p,q}^s(\Omega) = B_{p,q}^s(M) / \{ u \in B_{p,q}^s(M) \mid u|_{\Omega} \equiv 0 \} \) with the norm

\[ \|v\|_{B_{p,q}^s(\Omega)} \overset{\text{def}}{=} \inf_{u|_{\Omega} = v} \|u\|_{B_{p,q}^s(M)} \]

Consider \((\cdot, \cdot)_{s,q}\) to be the real interpolation functor.

**Proposition 2.4.** For any \( s \in (0, 1), q \in [1, \infty], \) and domain \( \Omega' \subseteq M \) with a Lipschitz boundary, Besov spaces are the results of the following interpolation procedure:

\[ \tilde{B}_{2,q}^{s}(\Omega') = (L_2(\Omega'), H^1(\Omega'))_{s,q}, \quad \tilde{B}_{2,q}^{1+s}(\Omega') = (H^1(\Omega'), H^2(\Omega'))_{s,q}, \]

\[ B_{2,q}^{-s}(\Omega') = (L_2(\Omega'), H^{-1}(\Omega'))_{s,q}; \]

in case of \( \mathbb{R}^d \) the similar relations hold

\[ B_{2,q}^s(\mathbb{R}^d) = (L_2(\mathbb{R}^d), H^1(\mathbb{R}^d))_{s,q}, \quad B_{2,q}^{1+s}(\mathbb{R}^d) = (H^1(\mathbb{R}^d), H^2(\mathbb{R}^d))_{s,q}, \]

\[ B_{2,q}^{-s}(\mathbb{R}^d) = (L_2(\mathbb{R}^d), H^{-1}(\mathbb{R}^d))_{s,q}. \]

\(^4\text{Since } \partial M = \emptyset \in C^{0,1}, \text{ in Propositions } 2.4, 2.5, 2.6 \text{ one can suppose that domain } \Omega' \text{ coincides with } M\)
Proposition 2.5. For any $t, s \in (0,1)$, $0 < s_1 < s_2 < 1$, $q \in [1, \infty]$, and domain $\Omega' \subseteq M$, $\partial \Omega' \in C^{0,1}$, the following relations hold

$$\left( H^1(\Omega'), \tilde{B}^{1+s}_2(\Omega') \right)_{t,2} = \tilde{H}^{1+ts}(\Omega'),$$
$$\left( H^{-1}(\Omega'), B^{-1+s}_2(\Omega') \right)_{t,2} = H^{-1-ts}(\Omega'),$$
$$\left( B^{-s_1}_2(\Omega'), B^{-s_2}_2(\Omega') \right)_{t,2} = H^{-(1-t)s_1-ts_2}(\Omega');$$

and in case of $\mathbb{R}^d$ one has

$$\left( H^1(\mathbb{R}^d), B^{1+s}_2(\mathbb{R}^d) \right)_{t,2} = H^{1+ts}(\mathbb{R}^d),$$
$$\left( H^{-1}(\mathbb{R}^d), B^{-1+s}_2(\mathbb{R}^d) \right)_{t,2} = H^{-1-ts}(\mathbb{R}^d),$$
$$\left( B^{-s_1}_2(\mathbb{R}^d), B^{-s_2}_2(\mathbb{R}^d) \right)_{t,2} = H^{-(1-t)s_1-ts_2}(\mathbb{R}^d).$$

Proposition 2.6 (by [7]). Let $s \in \mathbb{R}_+, p \in (1, \infty)$, $q \in (1, \infty)$, $\Omega \subseteq M$ be a domain with a Lipschitz boundary then

$$\left[ \tilde{N}^s_{p,0}(\Omega) \right]' = B^{-s}_{p',q}(\Omega), \quad \left[ \tilde{B}^s_{p,q}(\Omega) \right]' = B^{-s}_{p',q}(\Omega), \quad 1/p + 1/p' = 1, \ 1/q + 1/q' = 1.$$

Proposition 2.7 (by [11]). Let $E_1$, $E_2$, $F$ be Banach spaces, an embedding $E_1 \hookrightarrow E_0$ be continuous, an operator $T : E_1 \rightarrow F$ be bounded. If there exists a constant $L > 0$ and a number $s \in (0,1)$ such that

$$\|T e\|_F \leq L \|e\|_{E_0}^{1-s} \|e\|_{E_1}^s, \quad \forall e \in E_1,$$

then by continuity one can extend $T$ as an operator from $(E_0, E_1)_{s,1}$ to $F$, and there is a constant $c_s$ depending only on $s$ for which

$$\|T\|_{(E_0, E_1)_{s,1} \rightarrow F} \leq c_s L.$$

3 Estimates of solutions to problem (1)

Definition 3.1. A bilinear form $\zeta$ is called Hölder continuous of order $\gamma \in (0,1]$ in $H \subset \text{dom} \zeta \subset L_1(\Omega)$, $\Omega \subset M$, if there exists an atlas $\mathcal{U}$ such that for any map $(U, \kappa_U) \in \mathcal{U}$ and any function $\varphi \in C^{1,1}(M)$ with supp $\varphi \subset U$ there are constants $C_{U,\varphi}$, $C^H_{\zeta}$ such that

$$\forall u \in H \forall h \in \mathbb{R}^d; \ |h| < \text{dist(supp}\varphi, \partial U), \ z = \varphi(u - u_h) \in H \Rightarrow |\zeta(z)| \leq C_{U,\varphi} C^H_{\zeta} \|u\|_H |h|^{\gamma},$$

where $\varphi u_h$ equals $\varphi(x) \left[ u \circ \kappa_U^{-1} \circ (x + h) \circ \kappa_U \right]$ for $x \in \text{supp} \varphi$, and equals zero if $x \notin \text{supp} \varphi$.

Theorem 3.2. Assume that $u$ is a solution to the equation (1), $\Omega$ has a Hölder boundary of order $\gamma_{\Omega} \in (0,1]$, $A$ satisfies conditions A1–A3, the linear forms $\Phi_r(u, \cdot)$, $\tau(f, \cdot)$ are Hölder continuous of order $\beta_0 \in (0,1]$ in $\tilde{H}^1(\Omega)$. If $\gamma_0 = \min\{\gamma_c, \beta_0\}$, then

$$\|u\|_{\tilde{N}^{\gamma_0+\gamma_{\Omega}/2}(\Omega)} \leq C(A, \Omega, M) \|u\|_{\tilde{H}^1(\Omega)} \left[ \|u\|_{\tilde{H}^1(\Omega)} + C_{\tilde{H}^1(\Omega)} + C_{\Phi_r(u, \cdot)} \right], \quad (7)$$
\[ C(\mathcal{A}, \Omega, M) > 0 \text{ depends only on } \mathcal{A}, \Omega, M, \text{ and } C_{r}^{\hat{H}^{1}(\Omega)}, C_{\Phi_{\ast}(u_{\ast})}^{\hat{H}^{1}(\Omega)} \text{ are the constants in Definition 3.1.} \]

Further, if \( u \in \hat{N}_{2}^{1+\ast}(\Omega), s \in (0,1/2), \) and the forms \( \Phi_{\ast}(u_{\ast}), \tau(f_{\ast}) \) are Hölder continuous of order \( \beta_{s} \in (0,1] \) in \( \hat{N}_{2}^{1+\ast}(\Omega), \gamma_{s} = \min \{ \gamma_{c}, \beta_{s} \}, \) then for the constants \( C_{\tau(f_{\ast})}, C_{\Phi_{\ast}(u_{\ast})} \) in Definition 2.1, one has

\[
\| u \|_{\hat{N}_{2}^{1+\gamma_{s}/2}(\Omega)}^{2} \leq C(\mathcal{A}, \Omega, M) \left[ \| u \|_{\hat{H}^{1}(\Omega)}^{2} + \| u \|_{\hat{N}_{2}^{1+\ast}(\Omega)}^{2} \left( C_{\tau(f_{\ast})}^{\hat{N}_{2}^{1+\ast}(\Omega)} + C_{\Phi_{\ast}(u_{\ast})}^{\hat{N}_{2}^{1+\ast}(\Omega)} \right) \right].
\]

**Proof.** As the \( \hat{N}_{2}(\Omega) \)-norm is independent of the atlas from its definition (up to equivalence), we shall assume that this atlas coincides with atlas \( \mathcal{V} \) in Definition 2.1. Let \( \{ \psi_{\Omega} \} \) be a subordinate (with respect to \( \phi \)) partition of unity, supp \( \psi_{\Omega} \subset V. \) Consider the function \( \phi(h) = |h| + C_{\Omega}|h|^{\gamma_{\Omega}}, h \in \mathbb{R}^{d}. \) Then the functions \( (\psi_{\Omega})_{h \in \phi(h)|_{\Omega}} \) are well defined for \( |h| < \phi^{-1} [\text{dist} (\text{supp} \phi_{\Omega}, \partial V)]/20. \)

Since for the proof it suffices to obtain (7)–(8) with the left-hand side replaced by \( \psi_{\Omega}u, (V, \kappa_{\Omega}) \in \mathcal{V}, \) without the loss of generality, we can assume that a chart \( V \) is fixed and for convenience we shall write \( \psi, \xi \) instead of \( \psi_{\Omega} \) and \( \xi_{\Omega}. \) We estimate the difference

\[
\| \psi \cdot (u - u_{\ast}) \|_{\hat{H}^{1}(V)} \leq \| \psi \cdot (u - u_{\ast}) \|_{\hat{H}^{1}(V)} + \| \psi \cdot (u_{\ast} - u) \|_{\hat{H}^{1}(V)},
\]

where \( u_{\ast_{\pm}} = u_{\mp \phi(h)\xi}. \) One can note that it is possible to rewrite the terms in the right hand-side of (9) as \( \| \psi(u_{\ast} - v) \|_{\hat{H}^{1}(V)}, \) where \( v \) is equal to \( u \) and \( u_{\ast} \) for the first and the second terms respectively. The following inequality holds:

\[
\| \psi(u_{\ast} - v) \|_{\hat{H}^{1}(V)}^{2} = \int_{M} g(x, \nabla [\psi(u_{\ast} - v)])d\mu - \int_{M} g(x, \nabla [\psi(u_{\ast} - v)])d\mu + \int_{M} g(x, \phi(h)\xi, \nabla [\psi(u_{\ast} - v)])d\mu_{\ast_{\mp}} - \int_{M} g(x, \phi(h)\xi, \nabla [\psi(u_{\ast} - v)])d\mu_{\ast_{\mp}} - \int_{M} g(x, \nabla [\psi(u_{\ast} - v)])(d\mu_{\ast_{\mp}} - d\mu),
\]

here \( g(x, \eta) = G_{\omega}(\eta, \eta). \) Due to smoothness of \( M, \) it is evident that

\[
\int_{M} g(x, \phi(h)\xi, \nabla [\psi(u_{\ast} - v)])d\mu_{\ast_{\mp}} \leq C(M, \Omega)\| \psi \|_{C^{1}(M)}\| u \|_{\hat{H}^{1}(\Omega)}^{2} |h|^{\gamma_{\Omega}},
\]

\[
\int_{M} g(x, \nabla [\psi(u_{\ast} - v)])(d\mu_{\ast_{\mp}} - d\mu) \leq C(M, \Omega)\| \psi \|_{C^{1}(M)}\| u \|_{\hat{H}^{1}(\Omega)}^{2} |h|^{\gamma_{\Omega}}.
\]

Thus

\[
\| \psi(u_{\ast} - u) \|_{\hat{H}^{1}(V)}^{2} - \| \psi(u_{\ast} - v) \|_{\hat{H}^{1}(V)}^{2} + C_{M, \Omega, \psi}\| u \|_{\hat{H}^{1}(\Omega)}^{2} |h|^{\gamma_{\Omega}} + C_{M, \Omega, \psi}\| u \|_{\hat{H}^{1}(\Omega)}^{2} |h|^{\gamma_{\Omega}},
\]

\[
\| \psi(u_{\ast} - u) \|_{\hat{H}^{1}(V)}^{2} - \| \psi(u_{\ast} - u_{\ast}) \|_{\hat{H}^{1}(V)}^{2} + C_{M, \Omega, \psi}\| u \|_{\hat{H}^{1}(\Omega)}^{2} |h|^{\gamma_{\Omega}} + C_{M, \Omega, \psi}\| u \|_{\hat{H}^{1}(\Omega)}^{2} |h|^{\gamma_{\Omega}}.
\]

8
Let us define the following operator \[ T^\psi_h u = \psi u_h + (1 - \psi)u, \]
and introduce \( \varphi_1(h) = \phi(h)x, \varphi_2(h) = h + \varphi_1(h) \). Since \( \text{supp}(u - T^\psi_{\varphi_1(h)}u) \subset \overline{\Omega} \) and \( H^1(\Omega) = \tilde{H}^1(\Omega) \) one can obtain that \( (u - T^\psi_{\varphi_1(h)}u) \in \tilde{H}^1(\Omega) \).

Hence, from condition A1 it follows that:
\[
\|\psi(u - u_t)\|_{H^1(\Omega)}^2 \leq \frac{1}{\alpha} \Phi_0 \left( T^\psi_{\varphi_1(h)}u - u, T^\psi_{\varphi_1(h)}u - u \right),
\]
\[
\|\psi(u - (u_t)_{-})\|_{H^1(\Omega)}^2 \leq \frac{1}{\alpha} \Phi_0 \left( T^\psi_{\varphi_2(h)}u - u, T^\psi_{\varphi_2(h)}u - u \right),
\]
\[
\Phi_0 \left( T^\psi_{\varphi_1(h)}u - u, T^\psi_{\varphi_1(h)}u - u \right) = \Phi_0(T^\psi_{\varphi_1(h)}u, T^\psi_{\varphi_1(h)}u) - \Phi_0(u, u) + 2\Phi_0(u, T^\psi_{\varphi_1(h)}u - u). \tag{10}
\]

Since the linear form \( \tau(f, \cdot) \) is Hölder continuous of order \( \beta_s \) in \( \tilde{N}_2^{1+s}(\Omega), s \in (0, 1) \) and in \( \tilde{H}^1(\Omega), s = 0 \), we have
\[
|\tau(f, \psi(u - u_{\varphi_1(h)}))| \leq C_{10} C_{V, \psi} C_{\tau(f, \cdot)} \|u\|_{\tilde{H}^1(\Omega)} |h|^{\gamma_0 \beta_s}, \quad s = 0,
\]
\[
|\tau_1(f, \psi(u - u_{\varphi_1(h)}))| \leq C_{10} C_{V, \psi} C_{\tau_1(f, \cdot)} \|u\|_{\tilde{N}_2^{1+s}(\Omega)} |h|^{\gamma_1 \beta_s}, \quad s \in (0, 1/2).
\]

Similarly, as the form \( \Phi_1(u, \cdot) \) is Hölder continuous of order \( \beta_s \), we obtain
\[
|\Phi_1(u, \psi(u - u_{\varphi_1(h)}))| \leq C_{10} C_{V, \psi} C_{\Phi_1(u, \cdot)} \|u\|_{\tilde{H}^1(\Omega)} |h|^{\gamma_0 \beta_s}, \quad s = 0,
\]
\[
|\Phi_1(u, \psi(u - u_{\varphi_1(h)}))| \leq C_{10} C_{V, \psi} C_{\Phi_1(u, \cdot)} \|u\|_{\tilde{N}_2^{1+s}(\Omega)} |h|^{\gamma_1 \beta_s}, \quad s \in (0, 1/2).
\]

It remains to estimate the terms \( \Phi_0(T^\psi_{\varphi_1(h)}u, T^\psi_{\varphi_2(h)}u) - \Phi_0(u, u) \) in (13). Let us note that the gradient of \( T^\psi_{\varphi_1(h)}u \) equals
\[
\nabla T^\psi_{\varphi_1(h)}u = (\psi \nabla u_{\varphi_1(h)} + (1 - \psi) \nabla u) + (\nabla \psi)(u_{\varphi_1(h)} - u) = T^\psi_{\varphi_1(h)} \nabla u + (\nabla \psi)(u_{\varphi_1(h)} - u).
\]

For \( a(x, \eta) = A_x(\eta, \eta) \) we see that
\[
\Phi_0(T^\psi_{\varphi_1(h)}u, T^\psi_{\varphi_1(h)}u) - \Phi_0(u, u) \leq \int_M a(x, T^\psi_{\varphi_1(h)} \nabla u) + (\nabla \psi)(u_{\varphi_1(h)} - u) d\mu - \int_M a(x, T^\psi_{\varphi_1(h)} \nabla u) d\mu + \int_M a(x, T^\psi_{\varphi_1(h)} \nabla u) d\mu - \int_M a(x, T^\psi_{\varphi_1(h)} \nabla u) d\mu. \tag{13}
\]

Due to condition A2 and the Cauchy inequality we have
\[
a(x, \xi + \eta) - a(x, \xi) \leq (a(x, \eta)a(x, 2\xi + \eta))^{1/2} \leq \|A\|_{C(M)} \|g(x, \eta)\|^{1/2} (2g(x, \xi)^{1/2} + g(x, \eta)^{1/2}).
\]
Thus for (12) the following estimate holds
\[
\int_M a(x, T^\psi_{\varphi(h)} \nabla u + (\nabla \psi)(u_{\varphi(h)} - u))d\mu - \int_M a(x, T^\psi_{\varphi(h)} \nabla u)d\mu \leq C_V\|A\|_{C(M)} \|\psi(u_{\varphi(h)} - u)\|_{L^2(V)} \left(\|\psi(u_{\varphi(h)} - u)\|_{L^2(V)} + 2\|T^\psi_{\varphi(h)} \nabla u\|_{L^2(V)}\right).
\]
From Proposition 2.2 we conclude
\[
\|\psi(u_{\varphi(h)} - u)\|_{L^2(V)} \leq \tilde{C}_V,MC_{\Omega}|h|\|u\|_{\hat{H}^1(\Omega)},
\]
and therefore we can obtain an upper estimate for (12) as $C'_m C_{V,\psi} C_{\Omega}|h|\|u\|_{\hat{H}^1(\Omega)}^2$. Since $a$ is convex, we have
\[
a(x, T^\psi_{\varphi(h)} \nabla u) - a(x, \nabla u) \leq \left[T^\psi_{\varphi(h)} a(x, \nabla u)\right] - a(x, \nabla u) = \psi \left[a(x, \nabla u_{\varphi(h)}) - a(x, \nabla u)\right];
\]
thus,
\[
\int_M a(x, T^\psi_{\varphi(h)} \nabla u) - a(x, \nabla u)d\mu \leq \int_M \psi \left[a(x, \nabla u_{\varphi(h)}) - a(x, \nabla u)\right]d\mu,
\]
so, by Hölder continuity of $A$ we have the following estimate for sum (13):
\[
\int_M \psi \left[a(x, \nabla u_{\varphi(h)}) - a(x, \nabla u)\right]d\mu = \int_M \psi \cdot a(x, \nabla u)d\mu - \int_M \psi \cdot a(x, \nabla u)d\mu \leq \int_M \psi \cdot a(x - \varphi_1(h), \nabla u)d\mu - \int_M \psi \cdot a(x, \nabla u)d\mu + \int_M \psi \cdot a(x - \varphi_1(h), \nabla u)d\mu - \int_M a(x - \varphi_1(h), \nabla u)d\mu \leq \|A\|_{C^{0,\gamma}(M)} C_\Omega C_{V,\psi}|h|\|u\|_{\hat{H}^1(\Omega)}^2.
\]

4 Conditions for Hölder continuity of linear forms

Lemma 4.1. We have the following inequalities
\[
\|u - u_h\|_{N^\gamma_{2,1}(\mathbb{R}^d)} \leq C_{\gamma_1,\gamma_2}|h|^{|\gamma_2 - \gamma_1}|u|_{N^\gamma_{2,1}(\mathbb{R}^d)}, \quad u \in N^{\gamma_1}_{2,1}(\mathbb{R}^d), \quad 0 < \gamma_1 < \gamma_2 < 1; \quad (14)
\]
\[
\|u - u_h\|_{N^{\gamma_{1,1}}_{2,1}(\mathbb{R}^d)} \leq C_{\gamma_1,\gamma_2}|h|^{|\gamma_1 - \gamma_2}|u|_{H^1(\mathbb{R}^d)}, \quad u \in H^1(\mathbb{R}^d), \quad 0 < \gamma_1 < 1; \quad (15)
\]
\[
\|u - u_h\|_{N^{\gamma_1}_{2,1}(\mathbb{R}^d)} \leq \tilde{C}_{\gamma_1}|h|^{|\gamma_1 + \gamma_2}|u|_{N^{\gamma_1}_{2,1}(\mathbb{R}^d)}, \quad u \in N^{\gamma_1 + \gamma_2}_{2,1}(\mathbb{R}^d), \quad 0 < \gamma_1 < \gamma_1 + \gamma_2 \leq 1. \quad (16)
\]

Proof. In fact, from estimate (11) we have
\[
\|u - u_h\|_{H^1(\mathbb{R}^d)} \leq 2\|u\|_{H^1(\mathbb{R}^d)}; \quad (17)
\]
\[
\|u - u_h\|_{L^2(\mathbb{R}^d)} \leq |h|\|u\|_{H^1(\mathbb{R}^d)}; \quad (18)
\]
\[
\|u - u_h\|_{H^1(\mathbb{R}^d)} \leq |h|\|u\|_{H^2(\mathbb{R}^d)}. \quad (19)
\]
By the real interpolation of (17), (18) and from Proposition 2.4 one can obtain
\[ \| u - u_h \|_{N_2^0(\mathbb{R}^d)} = \| u - u_h \|_{(L_2(\mathbb{R}^d), H^1(\mathbb{R}^d))\gamma_1,\infty} \leq C_{\gamma_1,\infty}^{t} \cdot 2^{\gamma_1}|h|^{1-\gamma_1} \| u \|_{H^1(\mathbb{R}^d)}. \]

In the same way, from Propositions 2.3, 2.5 by real interpolation of (15) and (19) we conclude that (16) is true. Indeed, for $0 < \gamma_1 < 1 - \gamma_2$, we have
\[ \| u - u_h \|_{N_2^{1-\gamma_2}(\mathbb{R}^d)} = \| u - u_h \|_{(H^1(\mathbb{R}^d), N_2^0(\mathbb{R}^d))t,\infty} \leq C_{t,s}|h|(1-t)|u|_{(H^2(\mathbb{R}^d), H^1(\mathbb{R}^d))t,\infty} = C_{t,s}|h|^{1-t}\gamma_1 \| u \|_{N_2^{1-\gamma_2}(\Omega)}, \]
where $t = 1 - \gamma_2$, $s = 1 - \frac{\gamma_1}{1-\gamma_2}$. As above, by real interpolation of (18) and (19) in case of $\gamma_1 = 1 - \gamma_2$ we infer that (10) is true. Since $\mathcal{T}_h$ is a linear continuous operator in $N_2^0(\mathbb{R}^d)$, the following holds
\[ \| u - u_h \|_{N_2^1(\mathbb{R}^d)} \leq 2\| u \|_{N_2^0(\mathbb{R}^d)}, \quad (20) \]

thus,
\[ \| u - u_h \|_{N_2^1(\mathbb{R}^d)} \leq C_{\gamma_1,\infty}|h|(1-\gamma_1)|u|_{(N_2^1(\mathbb{R}^d), H^1(\mathbb{R}^d))t,\infty} = C_{\gamma_1,\infty}|h|(1-\gamma_1)|u|_{N_2^{1-\gamma_2}(\Omega)}, \]
here $t = \frac{\gamma_1}{1-\gamma_2}$. \qed

**Corollary 4.2.** For any function $f \in \hat{N}_{2,0}^{1-s}(\Omega)'$, $s \in (0,1)$, the linear form $\tau(f, \cdot)$ is H"older continuous of order $(s+t)$ in the space $\hat{N}_{2,0}^{1+t}(\Omega)$, $s < t \leq 1$, and of order $s$ in the space $\hat{H}^1(\Omega)$, and
\[ C_{\tau(f, \cdot)} = C_{\tau(f, \cdot)}^{\hat{H}^1(\Omega)} = \| f \|_{\hat{N}_{2,0}^{1-s}(\Omega)'}. \]

If $f \in L_2(\Omega)$, the linear form $\tau(f, \cdot)$ is H"older continuous of order $1$ in the space $\hat{H}^1(\Omega)$, and
\[ C_{\tau(f, \cdot)} = \| f \|_{L_2(\Omega)}. \]

**Proof.** Indeed, for an arbitrary map $(U, \kappa_U)$ from the atlas of manifold $M$ and function $\chi \in C^{1,1}(U)$, supp $\chi \subset U$, $t > 0$, using estimate (16), one can obtain:
\[
|\tau(f, \chi(u - u_h))| \leq \| f \|_{\hat{N}_{2,0}^{1-s}(\Omega)'} \| \chi(u - u_h) \|_{\hat{N}_{2,0}^{1-s}(\Omega)} \leq \]
\[
C_M(\| \chi \|_{C^{0,1}(M)} + 1) \| f \|_{\hat{N}_{2,0}^{1-s}(\Omega)'} \| \chi u - (\chi u)_h \|_{\hat{N}_{2,0}^{1-s}(\Omega)} \leq \]
\[
C_M \| f \|_{\hat{N}_{2,0}^{1-s}(\Omega)'} \| u \|_{\hat{N}_{2,0}^{1+s-t}(\Omega)} |h|^{s+t}. \]

Now we must only prove that the form $\tau(f, \cdot)$ is H"older continuous of order $s$. This follows by combining inequality (23) and estimate (15) of Lemma 3.1. \qed

The following embedding Theorem is proved in [1].
Lemma 4.4. Let 

Similarly, from Theorem 4.3 it follows that

and by Hölder’s inequality it follows that

One can replace in the formula above \( \mathbb{R}^d \) with \( M \).

It follows from Theorem 1 in [3] that one can obtain the following

Lemma 4.4. Let \( u \in N_2^\alpha(M) \), \( v \in N_2^\beta(M) \), \( w \in L_2(M) \), \( 0 < \alpha \leq \beta < 1 \). Then for any \( \varepsilon' > 0 \) the products \( uv \), \( vw \) belong to \( N_{s-\varepsilon}(M) \) and \( L_{s-\varepsilon}(M) \) respectively, where

Proof. Since the multiplication and embedding operators

are continuous, for \( u \in N_2^\alpha(M) \), \( v \in N_2^\beta(M) \) it clearly follows that the product \( uv \) belongs to \( N_s^\alpha(M) \), \( s = 1 \). Let us refine the order of summability \( s \). Without loss of generality, we can assume that the supports of all functions \( u \), \( v \) are contained in a subdomain of a fixed chart \( U \). Hence, the shift operator is well defined. Therefore

and by Hölder’s inequality it follows that

where \( p_j, q_j \geq 1, 1/p_j + 1/q_j = 1, j = 1, 2 \). For the boundedness of the right hand-side of (24) as \(|h| \to 0\) it is sufficient to consider the case \( p_1s_1 = 2 \). From Theorem 4.3 for any \( \varepsilon > 0 \), we have the embedding \( N_2^\beta(M) \hookrightarrow L_{q_1s_1-\varepsilon}(M) \), and

Similarly, from Theorem 4.3 it follows that \( N_2^\beta(M) \hookrightarrow N_{q_2s_2}^\alpha(M) \), \( N_2^\alpha(M) \hookrightarrow L_{p_2s_2-\varepsilon}(M) \) and

By setting \( s = \min\{s_1, s_2\} \) we obtain the required. \( \square \)
Lemma 4.5 (see [3]). For any function \( u \in L_2(M), \, v \in H^1(M) \), and \( \varepsilon > 0 \) the product \( uv \) belongs to \( L_s(M) \), \( s = \min\{2 - \varepsilon, \frac{d}{d-1}\} \).

Lemma 4.6. The linear form \( \Phi_t(u, \cdot), \, u \in \tilde{H}^1(\Omega) \), is H"older continuous of order \( \beta_t = \gamma + t \in (0, 1], \, \gamma \in (0, 1], \, t \in [0, 1) \), in Nikolskii space \( \tilde{N}_2^{1+t}(\Omega) \), \( 0 < t < 1 \) (\( \tilde{H}^1(\Omega) \) if \( t = 0 \)) if for some number \( \varepsilon > 0 \) the following conditions hold:

1. Let \( \gamma \in (0, 1) \) then

\[
\begin{align*}
 & b \in L_{\frac{d-1}{\gamma+1}}(\Omega), \quad c \in \left[ \tilde{N}_{\frac{\gamma+1}{\gamma+1}}(\Omega) \right], \\
 & \text{and } C^\gamma_{\Phi_t(u, \cdot)} = C^\gamma_{\Phi_t}(\Omega) \left( \| b \|_{L_{\frac{d-1}{\gamma+1}}(\Omega)} + \| c \|_{\tilde{N}_{\frac{\gamma+1}{\gamma+1}}(\Omega)} \right) \| u \| \tilde{H}^1(\Omega),
\end{align*}
\]

(26)

2. If \( \gamma = 1, \, t = 0 \), then

\[
\begin{align*}
 & b \in L_\infty(\Omega), \quad c \in L_{\max(2+\varepsilon,d)}(\Omega), \\
 & \text{and } C^{1}_{\Phi_t(u, \cdot)} (\Omega) \left( \| b \|_{L_\infty(\Omega)} + \| c \|_{L_{\max(2+\varepsilon,d)}(\Omega)} \right) \| u \| \tilde{H}^1(\Omega).
\end{align*}
\]

Proof. Let us set \( q_b = \frac{d-2}{\gamma+1}, \, 1/q_b + 1/p_b = 1, \, p_c = \frac{d+\varepsilon}{\gamma+1} - \varepsilon \) and consider an arbitrary function \( \chi \in C^{1,1}(U) \), \( \text{supp} \chi \subset U \), \( (U, \kappa_U) \) is a map from the atlas of manifold \( M \). From condition (26) using Lemmas 4.1 and 4.4 one can obtain

\[
\left| \int_{\Omega} b(\nabla v)\chi(u - u_h) d\mu \right| \leq \| b \|_{L_{q_b}(\Omega)} \cdot \| G(\nabla v, \nabla v)^{1/2} \chi(u - u_h) \|_{L_{p_b}(\Omega)} \leq \\
C_U (1 + \| \chi \|_{C^{0,1}(U)}) \| b \|_{L_{q_b}(\Omega)} \| v \| \tilde{H}^1(\Omega) \| \chi u - (\chi u)_h \| \tilde{N}_2^{1-\gamma}(\Omega) \leq \\
C_U \chi \| b \|_{L_{q_b}(\Omega)} \| h \|_{\tilde{H}^1(\Omega)} \| u \| \tilde{N}^t, \]

where \( N^t = \tilde{N}_2^{1+t}(\Omega) \) if \( t \in (0, 1) \), and \( N^0 = \tilde{H}^1(\Omega) \). In analogous way

\[
\left| \int_{\Omega} c\chi(u - u_h) d\mu \right| \leq \| c \|_{\tilde{N}_2^{1-\gamma}(\Omega)} \cdot \| v \chi(u - u_h) \| \tilde{N}_2^{1-\gamma}(\Omega) \leq \\
C_U (1 + \| \chi \|_{C^{0,1}(U)}) \| c \|_{\tilde{N}_2^{1-\gamma}(\Omega)} \| v \| \tilde{H}^1(\Omega) \| \chi u - (\chi u)_h \| \tilde{N}_2^{1-\gamma}(\Omega) \leq \\
C_U \chi \| c \|_{\tilde{N}_2^{1-\gamma}(\Omega)} \| h \|_{\tilde{H}^1(\Omega)} \| u \| \tilde{N}^t.
\]

As above, using Lemma 4.5 and Proposition 2.2 one can conclude that

\[
\left| \int_{\Omega} b(\nabla v)\chi(u - u_h) d\mu \right| \leq \| b \|_{L_\infty(\Omega)} \cdot \| G(\nabla v, \nabla v)^{1/2} \chi(u - u_h) \|_{L_1(\Omega)} \leq \\
C_U (1 + \| \chi \|_{C^{0,1}(U)}) \| b \|_{L_\infty(\Omega)} \| v \| \tilde{H}^1(\Omega) \| \chi u - (\chi u)_h \| L_2(\Omega) \leq \\
C_U \chi \| b \|_{L_\infty(\Omega)} \| h \|_{\tilde{H}^1(\Omega)} \| u \| \tilde{H}^1(\Omega),
\]

\[
13.
\]
and
\[
\left| \int_{\Omega} c v \chi (u - u_h) d\mu \right| \leq \| c \|_{L_{q_c}(\Omega)} \cdot \| v \chi (u - u_h) \|_{L_{p_c}(\Omega)} \leq C_U (1 + \| \chi \|_{C^{0,1}(U)}) \| c \|_{L_{q_c}(\Omega)} \| v \|_{\tilde{H}^1(\Omega)} \| \chi u - (\chi u)_h \|_{L_2(\Omega)} \leq C_U \chi \| c \|_{L_{q_c}(\Omega)} \| h \|_{\tilde{H}^1(\Omega)} \| u \|_{\tilde{H}^1(\Omega)},
\]
where \( 1/q_c + 1/p_c = 1 \).

5 Savaré-type theorems

Taking into account the Propositions from Sections 3 and 4 let us prove the following

**Theorem 5.1.** Assume that \( M \) is a \( C^{1,1} \)-smooth compact Riemannian manifold without boundary, \( \Omega \subset M \) be a subdomain, \( \partial \Omega \in C^{0,\gamma} \), operator \( \mathcal{A} \) satisfies \( A1-A3 \), and for \( \gamma = \gamma_c \in (0, 1) \) and \( \gamma = \gamma_c = 1 \) conditions \((\mathcal{A})\) and \((\mathcal{C})\) respectively, are fulfilled. Then the operator solving the problem \((1)\)

\[ \mathcal{R} : \left( H^{-1}(\Omega), \left( H^{-1}(\Omega), \left[ \tilde{N}^{1-\gamma} \right]_1 \right)_{1/2,1} \right)_{t,2} \rightarrow \tilde{H}^{1+\gamma \gamma_c t/2}(\Omega), \ t \in (0, 1), \ \gamma_c \in (0, 1); \]

\[ \mathcal{R} : \left( H^{-1}(\Omega), (H^{-1}(\Omega), L_2(\Omega))_{1/2,1} \right)_{t,2} \rightarrow \tilde{H}^{1+\gamma \gamma_c t/2}(\Omega), \ t \in (0, 1), \ \gamma_c = 1 \]

is continuous.

**Proof.** Let us use Theorem 3.2 and estimate the constants \( C_{\gamma}(\mathcal{H}^1(\Omega)), C_{\Phi}(\mathcal{H}^1(\Omega)) \) using Corollary 4.2 and Lemma 4.6. Then

\[ \| u \|_{\tilde{N}^{1+\gamma \gamma_c t/2}(\Omega)} \leq C \| u \|_{\tilde{H}^1(\Omega)} \left( \| f \|_{\tilde{H}^1(\Omega)} + \| f \|_{\tilde{H}^1(\Omega)} \right) \leq C \| f \|_{H^{-1}(\Omega)} \| f \|_{\left[ \tilde{N}^{1-\gamma} \right]_1}. \]

On the one hand, it follows from Proposition 2.7 that the operator

\[ \mathcal{R} : \left( H^{-1}(\Omega), \left[ \tilde{N}^{1-\gamma} \right]_1 \right)_{1/2,1} \rightarrow \tilde{N}^{1+\gamma \gamma_c t/2}(\Omega) \subset \tilde{N}^{1+\gamma \gamma_c t/2}(M) \]

is bounded. On the other hand,

\[ \mathcal{R} : H^{-1}(\Omega) \rightarrow \tilde{H}^1(\Omega) \subset H^1(M), \]

therefore, applying Propositions 2.3 and 2.4 one can obtain that \( \mathcal{R} \) is bounded as an operator from the space \( \left( H^{-1}(\Omega), (H^{-1}(\Omega), L_2(\Omega))_{1/2,1} \right)_{t,2} \) to the space

\[ (H^1(M), B^{1+\gamma \gamma_c t/2}_2(M))_{t,2} = H^{1+\gamma \gamma_c t/2}(M), \ t \in (0, 1). \]

\[ \square \]
Now, let us prove Theorem 1.1. Due to the embedding chain (6), for any $\varepsilon > 0$ from (26), there exists $\varepsilon > 0$ such that the following is true:

$$\tilde{N}^{1-\gamma_c}_{\frac{d}{2}\gamma - \varepsilon,0}(\Omega) \hookrightarrow \tilde{N}^{1-\gamma_c-\varepsilon/2}_{\frac{d}{2}\gamma}(\Omega) \hookrightarrow \tilde{W}^{1-\gamma_c-\varepsilon}_{\frac{d}{2}\gamma}(\Omega), \quad \gamma_c \in (0, 1),$$

and one can choose $\varepsilon \to 0$ as $\varepsilon \to 0$. Thus there is a linear bounded operator

$$T : \left( \tilde{W}^{1-\gamma_c-\varepsilon}_{\frac{d}{2}\gamma}(\Omega) \right) \to \left( \tilde{N}^{1-\gamma_c}_{\frac{d}{2}\gamma - \varepsilon,0}(\Omega) \right),$$

and hence the conditions from p. 1 of Lemma 4.6 are fulfilled if $c \in W^{1+\gamma_c+\varepsilon}(\Omega)$. Similarly, from Proposition 2.6 it follows that operator

$$S_{\Omega} : B^{-1+\gamma_c}_{2,1}(M) = [\tilde{N}^{1-\gamma_c}_{2,0}(M)] \to [\tilde{N}^{1-\gamma_c}_{2,0}(\Omega)], \quad \Omega \subset M,$$

is well defined. Using Propositions 2.4 and 2.5 we conclude

$$B^{-1+\gamma_c/2}_{2,1}(M) = (H^{-1}(M), B^{-1+\gamma_c}_{2,1}(M))_{1/2,1} \to (H^{-1}(\Omega), B^{-1+\gamma_c}_{2,1}(\Omega))_{1/2,1},$$

hence, operator

$$\mathcal{R} : \left( H^{-1}(M), B^{-1+\gamma_c/2}_{2,1}(M) \right)_{t,2} = H^{-1+\gamma_c/2}_{t,2}(M) \to \tilde{H}^{1+\gamma_c\gamma/2}_{t,2}(\Omega), \quad t \in (0, 1).$$

is bounded.

We conclude with the following generalization of Theorem 1.1.

**Theorem 5.2.** Let $M$ be a $C^{1,\infty}$-smooth compact Riemannian manifold without boundary, $\Omega \subset M$ be a subdomain, $C^{0,\gamma_0}$, operator $\mathcal{A}$ satisfies to conditions A1–A3, $b \in L_{\frac{d}{2}\gamma - \gamma_0}(\Omega)$, $c \in \left[ \tilde{N}^{1-\gamma_0}_{\frac{d}{2}\gamma - \varepsilon,0}(\Omega) \right]'$, $\gamma_0 \in (0, \gamma_c]$. Then operator $\mathcal{R}$ solving the problem (1) is bounded with respect to the following pairs

$$H^{-1+\frac{2n-1}{2}\gamma_0 + \frac{1}{2}\gamma_0 s}(M) \to \tilde{H}^{1+\frac{2n-1}{2}\gamma_0 - \frac{n-1}{2}\gamma_0 + \frac{n+1}{2}\gamma_0 s}_{\frac{d}{2}\gamma - \gamma_0}(\Omega), \quad s \in (0, 1) \quad (29)$$

$$B^{-1+\frac{2n-1}{2}\gamma_0}_{2,1}(M) \to \tilde{N}^{1+\frac{2n}{d}\gamma_0}_{\frac{d}{2}\gamma - \gamma_0}(\Omega), \quad (30)$$

and $n \in \mathbb{N}$ if $\gamma_0 \leq \gamma_c(1 - \gamma_0/2)$. Otherwise if there exists $N \in \mathbb{N}$ such that

$$\gamma_c \geq \frac{2}{2 - \gamma_0} \frac{2N - \gamma_0^N}{2N} \gamma_0,$$

then we have the boundedness of $\mathcal{R}$ with respect to

$$H^{-1+\frac{2N}{2}\gamma_0 + r_N s}(M) \to \tilde{H}^{1+\frac{2N}{2}\gamma_0 - \frac{N-1}{2}\gamma_0(1-s) + \frac{2\gamma_0}{2}\gamma_0 s}_{\frac{d}{2}\gamma - \gamma_0}(\Omega), \quad s \in (0, 1), \quad (31)$$

$$B^{-1+\frac{2N}{2}\gamma_0}_{2,1}(M) \to \tilde{N}^{\frac{2N+1}{2}\gamma_0}_{\gamma_0}(\Omega), \quad (32)$$

where $r_N = \frac{1}{2} \gamma_0 + \gamma_0 \left( \frac{\gamma_c}{2} - \frac{1}{2-\gamma_0} \frac{2\gamma_0 + 1}{2+\gamma_0} \gamma_0 \right)$.
Proof. It is clear that (29), (31) come from the real interpolation of (30) and (32) respectively. Thus we must only check the boundedness in pairs (30), (32). From the proof of Theorem 1.1 it follows that operator \( R : B_{2,1}^{-1+\gamma_0/2}(M) \to \tilde{N}_{2}^{1+\gamma_0/2}(\Omega) \) is continuous. Therefore the solution of (11) belongs to \( \tilde{N}_{2}^{1+\gamma_0/2}(\Omega) \), moreover
\[
\|u\|_{\tilde{N}_{2}^{1+\gamma_0/2}(\Omega)} \leq C \|f\|_{B_{2,1}^{1+\gamma_0/2}(M)};
\]
and we can apply Theorem 3.2. From Corollary 4.2 and Proposition 4.6 it follows that linear forms \( \tau(f, \cdot), \Phi_r(u, \cdot) \) are Hölder continuous of order \( \beta_s = \gamma_0 + s, s = \gamma_0/2 \) in the space \( \tilde{N}_{2}^{1+\gamma_0/2}(\Omega) \) if \( u \in \tilde{H}^1(\Omega) \) and \( f \in B_{2,1}^{-1+\gamma_0}(M) \). Thus we apply Theorem 3.2 again and for \( \beta_s \leq \gamma_c \) we have
\[
\|u\|^2_{\tilde{N}_{2}^{1+\gamma_0/2}(\Omega)} = \|u\|^2_{\tilde{N}_{2}^{1+\gamma_0/2}(\Omega)} \leq c \left( \|u\|_{\tilde{N}_{2}^{1+\gamma_0/2}(\Omega)} \|f\|_{B_{2,1}^{-1+\gamma_0}(M)} + \|u\|_{\tilde{H}^1(\Omega)} \|f\|_{\tilde{N}_{2}^{1+\gamma_0/2}(\Omega)} \right) \leq C \|f\|_{B_{2,1}^{1+\gamma_0/2}(M)} \|f\|_{B_{2,1}^{-1+\gamma_0}(M)}.
\]
Hence due to Proposition 2.7 there exists a bounded linear extension of \( R \) from
\( (B_{2,1}^{-1+\gamma_0/2}(M), B_{2,1}^{-1+\gamma_0}(M))_{1/2,1} = B_{2,1}^{-1+3\gamma_0/4}(M) \)
to \( \tilde{N}_{2}^{1+\gamma_0/2}(\Omega) \). Thus the following estimate holds
\[
\|u\|^2_{\tilde{N}_{2}^{1+\gamma_0/2}(\Omega)} \leq C \|f\|^2_{B_{2,1}^{-1+\gamma_0/2}(M)};
\]
while \( n \leq N \). Therefore the boundedness of \( R \) in pairs (30) is obtained. To justify (32), let us set \( s + t = \gamma_0 + s, t = \gamma_0 \left( \frac{1}{2} + \cdots + \left( \frac{1}{2} \right)^N \right) \) and use Corollary 4.2 then
\[
\|u\|^2_{\tilde{N}_{2}^{1+\gamma_0/2}(\Omega)} \leq C \|f\|^2_{B_{2,1}^{-1+\gamma_0/2}(M)} \|f\|^2_{B_{2,1}^{-1+\gamma_0/2}(M)}.
\]
Using Proposition 2.7 we conclude (32). \( \square\)

Corollary 5.3. Let the conditions from Theorem 5.2 are fulfilled, \( \gamma_0 = \frac{2 - \gamma_0}{2} \gamma_c \). Then operator
\[
R : \tilde{H}^{1+\gamma_0/2}(M) \to \tilde{H}^{1+\gamma_0/2}(\Omega), \quad s \in [0, 1)
\]
is continuous.

In particular, if \( \gamma_0 = 1 \), A1-A3 hold, \( b \in L_{2^{-\gamma_c}}^{\infty}(\Omega) \), and \( c \in B_{\max\{2+\varepsilon, d\}}^{-1+\gamma_0/2}(\Omega) \), \( \varepsilon > 0 \) then operator
\[
R : \tilde{H}^{-1+\varepsilon}(M) \to \tilde{H}^{1+\varepsilon}(\Omega), \quad t \in [0, \gamma_c/2)
\]
is bounded.

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References

[1] Besov O.V., Ilyin V.P. Integral Representations of Functions and Embedding Theorems, N.Y.: Wiley, 1978.

[2] Brezis H. Analyse fonctionnelle – Théorie et applications, P.: Masson, 1983.

[3] Gol’dman M.L. Imbedding theorems for anisotropic Nikol’skij Besov spaces with moduli of continuity of general form. // Proc. Steklov Inst. Math. 1987 170, 95–116.

[4] Grisvard P. Elliptic Problems in Nonsmooth Domains, L.: Pitman, 1985.

[5] Kato T. Perturbation Theory for Linear Operators. N.Y.: Springer Verlag, 1966.

[6] Jerison D., Kenig C. Boundary value problems on Lipschitz domains // Studies in Partial Differential Equations. 1982. 23. 1–68.

[7] Muramatu T. On the dual of Besov spaces // Publ. Res. Inst. Math. Kyoto Univ. 1976. 12. 123–140.

[8] Nikolskii S.M. Approximation of functions of several variables and embedding theorems. Army Foreign Science And Technology Center Charlottesville VA, 1971.

[9] Nikolskii S.M. Properties of certain classes of functions of several variables on differentiable manifolds // Math. sb. 1953. 33(75):2. 261–326. (in russian)

[10] Rychkov V.S. On restrictions and extensions of the Besov and Triebel–Lizorkin spaces with respect to Lipschitz domains // J. London Math. Soc. 1999. 60. 237–257.

[11] Savaré G. Regularity results for elliptic equations in Lipschitz domains // J. Funct. Anal. 1998. 152. 176–201.

[12] Savaré G., Schimperna G. Domain perturbations estimates for the solutions of second order elliptic equations // J. Math. Pures Appl. 2002. 81(11). 1071–1112.

[13] Stepin A.M., Tsylin I.V. On Boundary Value Problems for Elliptic Operators in the Case of Domains on Manifolds. // Doklady Mathematics. 2015. 92 (1). 428–432.

[14] Triebel H. Theory of function spaces. Reprint of the 1983 Edition. Basel: Birkhäuser. 2010.