Quantum geometry of elliptic Calabi-Yau manifolds

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Abstract

We study the quantum geometry of the class of Calabi-Yau threefolds, which are elliptic fibrations over a two-dimensional toric base. A holomorphic anomaly equation for the topological string free energy is proposed, which is iterative in the genus expansion as well as in the curve classes in the base. \textit{T}-duality on the fibre implies that the topological string free energy also captures the BPS-invariants of $D4$-branes wrapping the elliptic fibre and a class in the base. We verify this proposal by explicit computation of the BPS invariants of 3 $D4$-branes on the rational elliptic surface.
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1 Introduction

Topological string theory on local Calabi-Yau manifolds has been a remarkable success story. It counts the open and closed instantons corrections to topological numbers, which can be seen as an extension from classical geometry to quantum geometry. By now we can solve it in very different ways, namely by localisation, by direct integration of the holomorphic anomaly equations, by the topological vertex \[1\] or by the matrix model techniques in the remodeled B-model \[11\]. The system gives deep insights in the interplay between large N gauge theory/string theory duality, mirror duality, the theory of modular forms and knot theory and is by geometric engineering \[39\] intimately related to the construction of effective \( N = 2 \) and \( N = 1 \) rigid supersymmetric theories in four dimension.

On global Calabi-Yau manifolds, which give rise to \( N = 2 \) and \( N = 1 \) effective supergravity theories in four dimensions, the situation is less understood. Direct integration extends the theory of modular objects to the Calabi-Yau spaces and establishes that closed topological string amplitudes can be written as polynomial in modular objects, but the boundary conditions for the integration are differently than in the local case not completely known. As an example, on the quintic surface the closed topological string can be solved up to genus 51 \[35\].

In \[18\] mirror symmetry was made local in the decompactification limit of Calabi-Yau threefolds. Here we want to do the opposite and study how the quantum geometry extends from the local to the global case, when a class of local Calabi-Yau geometries is canonically compactified by an elliptic Calabi-Yau fibration with projection \( \pi : M \to B \). This easy class of local to global pairs, will be described to a large extend by complete intersections in explicit toric realizations. As we review in section 2 if the elliptic fibration has only \( I_1 \) fibres the classical cohomology of \( M \) is completely determined by the classical intersection of the base \( B \) and the number of sections, which depends on the Mordell Weyl group of the elliptic family.

The decisive question to which extend this holds for the quantum geometry is addressed in section 3 using mirror symmetry. The instanton numbers are counted by (quasi)-modular forms of congruence subgroups of \( \text{SL}(2, \mathbb{Z}) \) capturing curves with a fixed degree in the base for all degrees in the fibre. The weights of the forms depend
on the genus and the base class. This structure has been discovered for elliptically fibred surfaces in [11] and for elliptically fibred threefolds in [42]. We establish here a holomorphic anomaly equation (3.9) based on the non-holomorphic modular completion of the quasimodular forms which is iterative in the genus, as in [8], and also in the base classes generalizing [60, 32].

Our construction can be viewed also as a step to a better understanding of periods and instanton corrections in F-theory compactifications and a preliminary study using the data of [57, 40, 43] reveals that the structure at the relevant genera \( g = 0, 1 \) extends.

A holomorphic anomaly equation is also known to appear for generating functions of BPS invariants of higher dimensional \( D \)-branes, in particular \( D4 \)-branes on a surface [61, 50, 2, 54]. Interestingly, on elliptic Calabi-Yau fibrations, double T-duality on the elliptic fibre (or Fourier-Mukai transform) [60, 1, 1, 7] transforms \( D2 \)-branes wrapped on base classes into \( D4 \)-branes which also wrap the elliptic fibre and vice versa. The \( D4 \)-brane holomorphic anomaly is therefore related to the one of Gromov-Witten theory for these geometries. Moreover, the mirror periods provide predictions for \( D4 \)-brane BPS invariants which correspond to those of (small) black holes in supergravity.

We discuss higher dimensional branes on Calabi-Yau elliptic fibrations in sections 5 and 6. We compare the predictions from the periods for \( D4 \)-brane BPS invariants with existing methods in the literature for the computation of small charge BPS invariants [23, 24, 20, 17, 10, 53, 55]. The predictions of the periods are in many cases compatible with these methods. We leave a more precise study of \( D4 \)-brane BPS states on general elliptic fibrations to future work.

Section 6 specializes to the elliptic fibration over the Hirzebruch surface \( \mathbb{F}_1 \). The periods of its mirror geometry provide the BPS invariants of \( D4 \)-branes on the rational elliptic surface (also known as \( \frac{1}{2}K_3 \)) as proposed originally by Minahan et al. [60]. We revisit and extend the verification of this proposal for \( \leq 3 \) \( D4 \)-branes using algebraic-geometric techniques [25, 62, 64, 26, 52, 54, 56].

Acknowledgements
We would like to thank Babak Haghighat for useful discussions. Also we would like to thank Marco Rauch for collaboration in an initial state of the project. Part of the work
of J. M. was carried out as a postdoc of the Institute de Physique Théorique of the CEA Saclay. A. K. and T.W. are grateful to acknowledge support by the DFG to the project KL2271/1-1. T.W. is supported by the Deutsche Telekom Stiftung. We would thank Murad Alim and Emanuel Scheidegger for informing us about their related work.

2 Classical geometry of elliptic fibred Calabi-Yau spaces

In this section we study the classical geometry of elliptically fibered Calabi-Yau three manifolds $M$ with base $B$ and projection map $\pi : M \to B$. Such elliptic fibrations might be described locally by a Weierstrass form

$$y^2 = 4x^3 - xw^4g_2(u) - g_3(u)w^6,$$

(2.1)

where $u$ are coordinates on the base $B$. A global description can be defined by an embedding as a hypersurface or complete intersection in an ambient space $W$. Explicitly we consider cases, which allow a representation as a hypersurface or complete intersection in a toric ambient space. We restrict our attention to the case where the fiber degenerations are only of Kodaira type $I_1$, which means that the discriminant $\Delta = g_3^2 - 27g_2^3$ of (2.1) has only simple zeros on $B$, which are not simultaneously zeros of $g_2$ and $g_3$. Of course this is not enough to address immediately phenomenological interesting models in F-theory. However we note that these examples have a particular large number of complex moduli. Adjusting the latter and blowing up the singularities, not necessarily torically, is a more local operation, i.e. at least of co-dimension one in the base, which can be addressed in a second step.

2.1 The classical geometrical data of elliptic fibrations

Denote by $\mathbb{P}^2(w_1, \ldots, w_r)$ a (weighted) projective bundle $W$ over the base $B$. We consider four choices of weights $(w_1, \ldots, w_r) = \{(1, 2, 3), (1, 1, 2), (1, 1, 1), (1, 1, 1, 1)\}$ leading to three hypersurfaces and one complete intersection. In the case of rational elliptic surfaces these fibers lead to $E_8, E_7, E_6$, and $D_5$ del Pezzo surfaces, named so as the cohomology lattice of the surface has the intersection form of the corresponding Cartan-matrix. We keep the names for the fibration types.
Let us discuss the first case. This leads canonically to an embedding with a single section, however most of the discussion below applies to the other cases with minor modifications. Denote by $\alpha = O(1)$ the line bundle on $W$ induced by the hyperplane class of the projective fibre and $K = -c_1$ the canonical bundle of the base.

The coordinates $w, x, y$ are sections of $O(1), O(1)^2 \otimes K^{-2}$ and $O(1)^3 \otimes K^{-3}$ while $g_2$ and $g_3$ are section of $K^{-4}$ and $K^{-6}$ respectively so that (2.1) is a section of $O(1)^6 \otimes K^{-6}$. The corresponding divisors $w = 0, x = 0, y = 0$ have no intersection, i.e. $\alpha(\alpha + c_1)(\alpha + c_1) = 0$ in the cohomology ring of $W$ and

$$\alpha(\alpha + c_1) = 0$$

(2.2)

in the cohomology ring of $M$. Let us assume that the discriminant $\Delta$ vanishes for generic complex moduli only to first order in the coordinates of $B$ at locii, which are not simultaneously zeros of $g_2$ and $g_3$. In this case its class must satisfy

$$[\Delta] = c_1(B) = -K$$

(2.3)

to obey the Calabi-Yau condition and the fiber over the vanishing locus of the discriminant is of Kodaira type $I_1$. For this generic fibration, the properties of $M$ depend only on the properties of $B$.

For example using the adjunction formula and the relation (2.2) to reduce to linear terms in $\alpha$ allows to write the total Chern class as

$$C = \left(1 + \sum_{i=1}^{n-1} c_i\right) \frac{(1 + \alpha)(1 + w_2\alpha + w_2c_1)(1 + w_3\alpha + w_3c_1)}{1 + d\alpha + dc_1}.$$  

(2.4)

The Chern forms $C_k$ of $M$ are the coefficients in the formal expansion of (2.4) of the degree $k$ in terms of $\alpha$ and the monomials of the Chern forms $c_i$ of base $B$. The formulas (2.2) and (2.4) apply for all projectivisations.

For $n = 2$ one gets from table 1 by integrating over the fibre in all cases $\chi(M) = 12 \int_B c_1$ and $\mathbb{P}^1$ is the only admissible base. Similar for $n = 3$ one gets for the different projectivisations $\chi(M) = -60 \int_B c_1^2$, $\chi(M) = -36 \int_B c_1^2$, $\chi(M) = -24 \int_B c_1^2$ and $\chi(M) = -16 \int_B c_1^2$.

\footnote{In the $D5$ complete intersection case $d_1 = d_2 = 2$. One has to add a factor $(1 + \alpha + c_1)$ in the numerator and a factor $(1 + 2\alpha + 2c_1)$ in the denominator.}
The following discussion extends to all dimensions. For the sake of brevity we specialize to Calabi-Yau threefolds. Let $K_i$ denote the Kähler cone and $C_i$ the classes of the curves in the cone dual of the two dimensional base. Let $K_iK_j = c_{ij}$ be the intersection form on the base. We expand the canonical class of the base as

$$K = -c_1 = -\sum_i a^i K_i = -\sum_i a_i C^i, \tag{2.5}$$

with $a_i$ and $a^i$ in $\mathbb{Z}$. We denote by $K_a$ the divisors of the total space of the elliptic fibration and distinguish between $K_e$ the divisor dual to the elliptic fibre curve and $K_i, i = 1, \ldots, b$, which are $\pi^*(C^i)$

$$K_e^3 = a \int_B c_1^2, \quad K_e^2 K_i = aa_i, \quad K_e K_i K_j = ac_{ij}. \tag{2.6}$$

Here $a$ denotes the number of sections, see Tab.1. The intersection with the second Chern class of the total space can be calculated using table 1 as

$$\int_M c_2 J_e = \begin{cases} \int_B (11c_1^2 + c_2) & E_8, \\ 2 \int_B (5c_1^2 + c_2) & E_7, \\ 3 \int_B (3c_1^2 + c_2) & E_6, \\ 4 \int_B (2c_1^2 + c_2) & D_5, \end{cases} \tag{2.7}$$

$$\int_M c_2 J_i = 12a_i.$$

Here we denoted by $J_i$ the basis of harmonic $(1,1)$ forms dual to the $K_i$. 

Table 1: Chern classes $C_i$ of regular elliptic Calabi-Yau manifolds. Integrating $\alpha$ over the fibre yields a factor $a = \prod_i a_i, i.e. the number of sections 1, 2, 3, 4 for the three fibrations in turn.
Let us note two properties about the intersection numbers. They can be proved using the properties of almost Fano bases $B$ and $\beta_{12}$. For the first define the matrix

$$C_e = \begin{pmatrix} \int_B c_1^2 & a_1, \ldots, a_b \\ a_1 & \ddots & c_{ij} \\ \vdots & \ddots & \vdots \\ a_b & \cdots & c_{ab} \end{pmatrix},$$

then

$$\det(C_e) = 0. \quad (2.8)$$

A further property concerns a decoupling limit between base and fibre in the Kähler moduli space. Generally we can make a linear change in the basis of Mori vectors, which results in corresponding linear change in dual spaces of the Kähler moduli and the divisors

$$\tilde{l}_i = m_{ij}l_j, \quad \tilde{t}_i = m_{ij}t_j. \quad (2.10)$$

To realize a decoupling between the base and the fibre we want to find a not necessarily integer basis change, which eliminates the couplings $\tilde{K}_e^3\tilde{K}_i$ and leaves the couplings $\tilde{K}_e\tilde{K}_i\tilde{K}_j$ invariant. It follows from (2.5, 2.12) and the obvious transformation of the triple intersections that there is a unique solution

$$m = \begin{pmatrix} 1 & a_1^2 & \ldots & a_b^2 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}, \quad (2.11)$$

such that

$$\tilde{K}_e^3 = a(\int_B c_1^2 - \frac{3}{2}a_ia^i + \frac{3}{4}c_{ij}a^ia^j)$$

$$\tilde{K}_e^2\tilde{K}_i = 0$$

$$\tilde{K}_e\tilde{K}_i\tilde{K}_j = ac_{ij}. \quad (2.12)$$

As we have seen the classical topological data of the total space of the elliptic fibration follows from simple properties of the fibre and the topology of the base. We want to extend these result in the next section to the quantum cohomology of the elliptic fibration. We focus on the Calabi-Yau threefold case, where the instanton contributions to the quantum cohomology is richest. To actually calculate quantum cohomology we need an explicit realisation of a class of examples, which we discuss in the next subsection.
2.2 Realizations in toric ambient spaces

To have such a concrete algebraic realization we use hypersurfaces or complete intersections in toric ambient spaces.

Possible toric bases $B$ leading to the above described elliptic fibrations with only $I_1$ singularities of the Calabi-Yau $d$-fold are defined by reflexive polyhedra $\Delta_B$ in $d-1$ dimensions [6], as was observed in [40]. For the threefold case one has the following possibilities of 2-dimensional polyhedra.

![Figure 1: These are the 16 reflexive polyhedra $\Delta_B$ in two dimensions, which build 11 dual pairs $(\Delta_B, \Delta_B^*)$. Polyhedron $k$ is dual to polyhedron $17-k$ for $k = 1, \ldots, 5$. The polyhedra 6, \ldots, 11 are selfdual.](image)

The toric ambient spaces, which allow for smooth Calabi-Yau hypersurfaces as section of the canonical bundle, can be described by pairs of reflexive polyhedra $(\Delta, \Delta^*)$. Together with a complete star triangulation of $\Delta$, they define a complex family of Calabi-Yau threefolds. The mirror family is given by exchanging the role of $\Delta$ and $\Delta^*$. A complete triangulation divides $\Delta$ in simplices of volume 1. In a star triangulation all simplices contain the unique inner of the reflexive polyhedron. Let us give first two examples for toric smooth ambient spaces in which the canonical hypersurface leads to the $E_8$ elliptic fibration over $\mathbb{P}^2$ and over the Hirzebruch surface $\mathbb{F}_1$. The polyhedron
for the $E_8$ elliptic fibration over $\mathbb{P}^2$ with $\chi = -540$ is given by the following data

$$
\begin{array}{cccccc|ccc}
D_0 & 1 & 0 & 0 & 0 & 0 & -6 & 0 & 0 \\
D_1 & 1 & 1 & 0 & -2 & -3 & 0 & 1 & 0 \\
D_2 & 1 & 0 & 1 & -2 & -3 & 0 & 1 & 0 \\
D_3 & 1 & -1 & -1 & -2 & -3 & 0 & 1 & 0 \\
D_4 & 1 & 0 & 0 & -2 & -3 & 1 & -3 & 0 \\
D_5 & 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\
D_y & 1 & 0 & 0 & 0 & 1 & 3 & 0 & 0 \\
\end{array}
$$

(2.13)

Here we give the relevant points $\nu_i$ of the four dimensional convex reflexive polyhedron $\Delta$ embedded into a hyperplane in a five dimensional space and the linear relations $l^{(i)}$. This model has an unique star triangulation, see (2.21), with the intersection ring

$$
\mathcal{R} = 9J_e^3 + 3J_e^2J_1 + J_eJ_1^2.
$$

(2.14)

as follows from (2.12) with $a = 1$ The evaluation of $c_2$ on the basis of the Kähler cone is follows from (2.7) as $\int_M c_2J_e = 102$ and $\int_M c_2J_1 = 36$.

The polyhedron for the $E_8$ elliptic fibration over $\mathbb{F}_1$ with $\chi = -480$ reads

$$
\begin{array}{cccccc|ccc}
\nu_i & l^{(e)} & l^{(1)} & l^{(2)} & l^{(e)} + l^{(2)} & l^{(1)} + l^{(2)} & -l^{(2)} \\
D_0 & 1 & 0 & 0 & 0 & -6 & 0 & 0 \\
D_1 & 1 & 1 & 0 & -2 & -3 & 0 & 1 & 0 \\
D_2 & 1 & 0 & 1 & -2 & -3 & 0 & 1 & 0 \\
D_3 & 1 & -1 & -1 & -2 & -3 & 0 & 1 & 0 \\
D_4 & 1 & 0 & -1 & -2 & -3 & 0 & 1 & 0 \\
D_5 & 1 & 0 & 0 & -2 & -3 & 1 & -2 & -1 \\
D_x & 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\
D_y & 1 & 0 & 0 & 0 & 1 & 3 & 0 & 0 \\
\end{array}
$$

(2.15)

This example shows that there are two Calabi-Yau phases possible over $\mathbb{F}_1$, which are related by flopping a $\mathbb{P}^1$ represented by $l^{(2)}$. This transforms the half K3 to a del Pezzo eight surface, which can be shrunken to a point. In the first phase, the triangulation is described by (2.21) the intersection ring and $\int_M c_2J_i$ follows by (2.12, 2.7) as

$$
\mathcal{R} = 8J_e^3 + 3J_e^2J_1 + J_eJ_1^2 + 2J_eJ_2 + J_1J_2J_3.
$$

(2.16)

and $\int_M c_2J_e = 92$, $\int_M c_2J_1 = 36$ and $\int_M c_2J_3 = 24$. For the second phase we flop the $\mathbb{P}^1$ that corresponds to the Mori cone element $l^{(2)}$. Generally if we flop the curve $C$ this changes the triple intersection of the divisors $K_iK_jK_k$ [16] by

$$
\Delta_{ijk} = -(C \cdot K_i)(C \cdot K_j)(C \cdot K_k).
$$

(2.17)
Now the intersection of the curves $C_i$ which correspond to the mori cone vector $l^{(i)}$ with the toric divisors $D_k$ is given by $(C_i \cdot D_k) = l_k^{(i)}$. On the other hand the $\mathcal{K}_k$ are combinations of $D_k$ restricted to the hypersurface so that $(\mathcal{K}_k \cdot C_i) = \delta_i^k$.

In addition one has to change the basis in order to maintain positive intersection numbers\(^2\) $\tilde{l}^{(e)} = l^{(e)} + l^{(2)}$, $\tilde{l}^{(1)} = l^{(1)} + l^{(2)}$ and $\tilde{l}^{(2)} = -l^{(2)}$. For the $(1,1)$ forms $J_i$, which transform dual to the curves, we get then the intersection ring in the new basis of the Kähler cone

$$
\mathcal{R} = 8\tilde{J}_e^3 + 3\tilde{J}_e^2\tilde{J}_1 + \tilde{J}_e\tilde{J}_2 + 9\tilde{J}_e\tilde{J}_2 + 3\tilde{J}_e\tilde{J}_1\tilde{J}_2 + \tilde{J}_1\tilde{J}_2 + 9\tilde{J}_e\tilde{J}_2 + 3\tilde{J}_1\tilde{J}_2 + 9\tilde{J}_2^3. \quad (2.18)
$$

The intersections with $c_2$ are not affected by the flop, only the basis change has to be taken into account. In the second phase the triangulation of the base is given in the middle of figure 2 and the triangulation of $\Delta$ is specified by (2.20). In this phase a $E_8$ del Pezzo surface can be shrunken to get to the elliptic fibration over $\mathbb{P}^2$. This identifies the classes of the latter example as $J_e = \tilde{J}_2$, $J_1 = \tilde{J}_1$, while the divisor dual to $\tilde{J}_e^3$ is shrunken.

![Diagram](image)

Figure 2: The base triangulation for the flop in second example and the blowdown of an $E_8$ del Pezzo surface

With $\Delta_B$ the toric polyhedron for the base and specifying by

$$\{(e_1, e_2)\} = \{(-2, -3), (-1, -2), (-1, -1)\}$$

the toric data for the $E_8$, $E_7$, $E_6$ fibre respectively it is easy to see that all toric hypersurface

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\(^2\)This is one criterion that holds in a simplicial Kähler cone. The full specification is that $\int_C J > 0$, $\int_D J \wedge J > 0$ and $\int_M J \wedge J \wedge J > 0$ for $J$ in the Kähler cone and $C$, $D$ curves and divisors. E.g. if the latter is simplicial and generated by $J_i$ then $J = \sum d_i J_i$ with $d_i > 0$. 

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with the required fibration have the general form of the polyhedron $\Delta$.

$$
\begin{array}{cccc|ccc}
D_0 & \nu_i & 0 & 0 & 0 & 0 & \sum_i e_i -1 & 0 & \cdots & 0 \\
D_1 & 1 & e_1 & e_2 & 0 & * & \cdots & * \\
\vdots & 1 & \Delta_B & \vdots & \vdots & * & \cdots & * \\
D_r & 1 & e_1 & e_2 & 0 & * & \cdots & * \\
D_x & 1 & 0 & 0 & 1 & 0 & -e_1 & 0 & \cdots & 0 \\
D_y & 1 & 0 & 0 & 0 & 1 & -e_2 & 0 & \cdots & 0 \\
\end{array}
$$

(2.19)

We note that the fibre elliptic curve is realized in a two dimensional toric variety, which can be defined also by a reflexive 2 dimensional polyhedron $\Delta_F$. It is embedded into $\Delta$ so that the inner of $\Delta_F$ is also the origin of $\Delta$. Its corners are

$$\{(0,0,e_1,e_2),(0,0,1,0),(0,0,0,1)\}.$$ 

The $E_6$, $E_7$ and $E_8$ fibre types correspond to the polyhedra in figure 1 with numbers 1, 4 and 10. To check the latter equivalence requires an change of coordinates in $SL(2,\mathbb{Z})$.

The dual reflexive polyhedron $\Delta^*$ contains $\Delta_F^*$, embedding likewise in the coordinate plane spanned the 3rd and 4th axis.

A triangulation of $\Delta_B$ as in figure 1 or 2 lifts in an universal way to a star triangulation of $\Delta$ as follows. To set the conventions denote by $(\nu^B_i,e_1,e_2)$ the points of the embedded base polyhedron $\Delta_B$ and label them as the points of $\Delta_B$ starting with the positive x-axis, which points to the right in the figures, and label points of $\Delta_B$ counter clockwise from 1, $\ldots$, $r$. The inner point in $\Delta_B$, $(0,0,e_1,e_2)$ is labelled $z$. The two remaining points of $\Delta$: $(0,0,1,0)$ and $(0,0,0,1)$ are labelled by $x$ and $y$.

Denote the k-th d-dimensional simplex in $\Delta_B$ by the labels of its vertices, i.e.

$$\text{sim}_{k}^{(d)} := (\lambda_{i}^{k}, \ldots, \lambda_{d+1}^{k})$$

and in particular denote the outer edges of $\Delta_B$ by

$$\{ed_k|k = 1, \ldots, r\} := \{(1,2), \ldots, (r,1)\}.$$ 

Any triangulation of $\Delta_B$ is lifted to a star triangulation of $\Delta$, which is spanned by the simplices containing beside the inner point $(0,0,0,0)$ of $\Delta$ the points with the labels

$$\text{Tr}_\Delta = \{(\text{sim}_k^{(2)},x), (\text{sim}_k^{(2)},y)|k = 1, \ldots, p\} \cup \{(ed_k,x,y)|k = 1, \ldots, r\}. \quad (2.20)$$
In particular for star triangulations of $\Delta_B$ one has

$$\text{Tr}_\Delta = \{(ed_k, z, x), (ed_k, z, y), (ed_k, x, y)| k = 1, \ldots r\}$$ (2.21)

and generators of the Mori cone for the elliptic phase contain the Mori cone generators $l^{(1)}, \ldots, l^{(b)}$, which correspond to a star triangulation of the base polyhedron, which is the one in figure 1. We list here the mori cones first seven case

| $\Delta_B$ | $\nu_i^B$ | 1(1) | 2(2) | 3(2) | 4(3) | 5(3) | 6(3) | 7(4) |
|------------|-----------|------|------|------|------|------|------|------|
| $\nu_i^B$  |           | l^{(1)} | l^{(1)} | l^{(2)} | l^{(1)} | l^{(2)} | l^{(3)} | l^{(1)} | l^{(2)} | l^{(3)} | l^{(3)} | l^{(4)} | l^{(5)} | l^{(6)} |
| $z$        |           | -3    | -2   | -2   | -1   | 0    | -2   | -1   | -1   | 0    | -1   | -1   | -1   | -1   | -1   |
| 1          |           | 1     | 1    | 0    | 0    | 1    | -1   | 0    | 1    | 0    | 0    | -1   | 1    | 0    | 0    |
| 2          |           | 1     | 0    | 1    | 1    | 1    | -1   | 1    | 0    | 1    | -1   | 1    | 0    | 0    | 0    |
| 3          |           | 1     | 1    | 0    | 1    | -1   | -2   | 1    | 0    | 1    | -1   | 0    | 0    | 1    | 0    |
| 4          |           | 0     | 1    | 0    | 1    | 0    | 1    | 0    | 1    | -2   | 0    | 0    | 1    | -1   | 1    |
| 5          |           | 1     | 0    | 0    | 0    | 0    | 0    | 1    | 0    | 0    | 0    | 1    | -1   | 1    | 1    |
| 6          |           | -     | -    | 1    | -    | 4    | 3    | 17   | 1    | 0    | 0    | 1    | -1   | 1    | 1    |

The remaining 9 cases are given in the appendix A. We indicate in the brackets behind the model the number of Kähler moduli. If the latter is smaller then the number of Mori generators the Mori and the dual Kähler cone are non simplicial. This is the case for the models 7,9 and for 11-16. In the last column we list the number of extra triangulations. The corresponding phases involve non-star triangulations of $\Delta$ and can be reached by flops. By the rules discussed above we can find the intersection ring and the mori cone in phases related by flop. We understand also the blowing down of one model. Non reflexivity posses a slight technical difficulty in providing the data for the calculation of the instantons. The fastest way to get the data for all cases is to provide for the models 15 and 16 a simplicial Kähler cone and reach all other cases by flop and blowdowns. We will do this in the appendix A.

### 3 Quantum geometry of elliptic fibrations

From the data provided in the last section, namely the Mori cone and the intersection numbers, follow differential equations as well as particular solutions, which allow to calculate the instanton numbers as established mathematically for genus zero by

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3Except for 13 which is available on request.
Givental, Lian and Yau. These can be calculated very efficiently using the program described in [31]. In the cases at hand one can evaluate the genus one data using the genus zero results, the holomorphic anomaly equation for the Ray Singer Torsion, boundary conditions provided by the evaluation of \( \int_M J_i c_2 \) and the behaviour of the discriminant at the conifold to evaluate the elliptic instantons.

The higher genus curves are less systematically studied on compact 3-folds. However if the total space of the elliptic fibration over a base class is a contractable rational surface, one can shrink the latter and obtain a local model on which the modular structure of higher genus amplitudes have been intensively studied. The explicit data suggest that that this structure is maintained for all classes in the base.

We summarize in the next subsection the strategy to obtain the instanton data and based on the results we propose a general form of the instantons corrected amplitude in terms of modular forms coming from the elliptic geometry of the fibre and a simple and general holomorphic anomaly formula, which govern the all genus instanton corrected amplitudes for the above discussed class of models.

In the following subsection we use the B-model to prove some aspects of the proposed statements. This can establish the A-model results for genus 0 and 1, since mirror symmetry is proven and the B-model techniques apply. Higher genus B-model calculations have been first extended to compact multi-moduli Calabi-Yau manifolds in [29].

### 3.1 Quantum cohomology, modularity and the anomaly equations

The basic object, the instanton corrected triple intersections \( C_{abc}(q^\beta) \) are due to special geometry all derivable from the holomorphic prepotential, which reads at the point of maximal unipotent monodromy [13][31]

\[
F^{(0)} = (X^0)^2 \left[ \frac{-\kappa_{abc} t^a t^b t^c}{3!} + A_{ab} t^a t^b + c_a t^a + \frac{\zeta(3)}{2(2\pi i)^3} + \sum_{\beta \in H_2(M,\mathbb{Z})} n_{(0)}^\beta \text{Li}_3(q^\beta) \right]
\]  

(3.1)

where \( q_\beta = \exp(2\pi i \sum_{a=1}^{h_2} \beta_a t^a) \), \( c_a = \frac{1}{24} \int_M c_2 \omega_a \) and \( \chi \) is the Euler number of \( M \). By \( \omega_a, a = 1, \ldots, h_2(M) \), we denote harmonic \((1,1)\), which form a basis of the Kähler cone and the complexified Kähler parameter \( t^a = \int_{C^\beta} (i\omega + b) \), where \( C_\beta \) is a curve class in the
Mori cone dual to the Kähler cone and \( b \) is the Neveu-Schwarz (1, 1)-form \( b \)-field. The real coefficients \( A_{ab} \) are not completely fixed. They are unphysical in the sense that \( K(t, \bar{t}) \) and \( C_{abc}(q) \) do not depend on them. The upper index \( (0) \) on the \( F^{(0)} \) indicates the genus of the instanton contributions. The triple couplings receive only contributions of genus 0. The classical topological data provide us at the point of maximal unipotent monodromy with the \( B \)-model period integrals \( \Pi = (F_g, X_g) = (\int_{B^I} \Omega, \int_{A_I} \Omega) T \) over an integral symplectic basis of 3-cycles \( (A_I, B^I), I = 0, \ldots, h_{21}(W) \). This is achieved by matching the \( b_3(W) \) solutions to the Picard-Fuchs equation with various powers of \( \log(z_a) \sim t^a \), with the expected form of the \( A \)-model period vector

\[
\Pi = X^0 \left( \begin{array}{c} 2F^{(0)} - t^a \partial_{t^a} F^{(0)} \\ \partial_{t^a} F^{(0)} \\ 1 \\ t^a \end{array} \right) = X^0 \left( \begin{array}{c} \kappa_{abc} t^a t^b t^c \frac{\zeta(3)}{3!} + c_a t^a - i \chi \frac{\zeta(3)}{(2\pi i)^3} + 2f(q) - t^a \partial_{t^a} f(q) \\ -\kappa_{abc} t^b t^c + A_{ab} t^b + c_a + \partial_{t^a} f(q) \\ 1 \\ t^a \end{array} \right),
\]

where the lower case indices run from \( a = 1, \ldots, h_{21}(W) = h_{11}(M) \).

One can define a generating function for the free energy in terms of a genus expansion in the coupling \( g_s \)

\[
F(g_s, q) = \sum_{g=0}^{\infty} q^{2g-2} F^{(g)}(q),
\]

where the upper index \( F^{(g)}(q) \) indicates as before the genus.

According to the split of the cohomology \( H_2(M, \mathbb{Z}) \) into the base and the fibre cohomology, we define \( q_B^\beta = \prod_{k=1}^{b_2(B)} \exp(2\pi i \int_{\beta^k} i\omega + b) \), where now by a slight abuse of notation \( \beta \in H_2(B, \mathbb{Z}) \) and \( q = \exp(2\pi i \int f i\omega + b) \), where \( f \) is the curve representing the fibre. Now we define the following objects

\[
F^{(g)}_\beta(q) = \text{Coeff}(F^{(g)}(q), q_B^\beta).
\]

We have the following universal sectors

\[
F^{(0)}_0(q) = \left( \int_B c_1^2 \right) \frac{t^3}{3!} + \chi \frac{\zeta(3)}{2(2\pi i)^3} - \chi \sum_{n=1}^{\infty} \text{Li}_3(q^n),
\]

\[
F^{(1)}_0(q) = \left( \int_B c_2 \right) \frac{t^4}{24} \text{Li}_1(q), \quad F^{(g>1)}_0(q) = (-1)^g \frac{\chi}{2} \frac{|B_{2g}B_{2g-2}|}{2g(2g-2)(2g-2)!}.
\]
We note that it follows from the expression for $F^{(0)}_0(q)$ that

$$C_{\tau\tau\tau} = \int_B c_1^2 + \frac{X}{2} \zeta(-3) - \frac{X}{2} \zeta(-3) E_4(q).$$  \hfill (3.7)

The $F^{(g)}_\beta(q)$ have distinguished modular properties, which we describe now. We note that the general form $F^{(g)}_\beta(q)$ is as follows

$$F^{(g)}_\beta = \left(\frac{q^{\frac{1}{24}}}{\eta}\right)^{12} \sum_i a_i \beta^i P_{2g+6} + 6 \sum_i a_i \beta^i - 2 (E_2, E_4, E_6)$$ \hfill (3.8)

with $P_{2g+6} = (E_2, E_4, E_6)$ a (quasi)-modular form of weight $2g+6 \sum_i a_i \beta^i - 2$ \cite{38}.

For the sectors $\beta > 0$, which describe non-trivial dependence on the Kähler class of the base, we have the following recursion condition

$$\frac{\partial F^{(g)}_\beta(q)}{\partial E_2} = \frac{1}{24} \sum_{h=0}^g \sum_{\beta' + \beta'' = \beta} (\beta' \cdot \beta'') F^{(h)}_\beta F^{(g-h)}_\beta + \frac{1}{24} \beta \cdot (\beta - K_B) F^{(g-1)}_\beta.$$  \hfill (3.9)

For the other types of elliptic fibrations $E_7$, $E_6$, & $D_5$, the right-hand side is divided by $a = 2, 3 \& 4$ respectively. Eq. (3.9) generalizes a similar equation due to \cite{32}, to arbitrary classes in the base and types of fibres. In particular if one restricts on elliptic fibrations over the blow up of $\mathbb{P}^2$ to a Hirzebruch surface $B = F_1$ to the rational fibre class in the base (3.9) becomes the equation of \cite{32} counting curves of higher genus on the $E_8$, $E_7$, $E_6$, & $D_5$ del Pezzo surfaces. The form (3.8) and its relation to \cite{32} has been observed in \cite{42} for the Hirzebruchsurface $F_0$ as base. A derivation of the equation (3.9) is given in section \ref{sec:4}.

### 3.2 The B-model approach to elliptically fibred Calabi-Yau spaces

In this section we assume some familiarity with the formalism developed in \cite{30} and concentrate on features relevant and common to the B-model geometry of elliptic fibrations and how they emerge from the topological data of the A-model discussed in section \ref{sec:2}.

The vectors $l^{(i)}$ are the generators of the Mori cone, i.e. the cone dual to the Kähler cone. As such they reflect classical properties of the Kähler moduli space and the classical intersection numbers, like the Euler number and the evaluation of $\int_M c_2 \omega_a$ on the basis of Kähler forms on the elliptic fibration.
On the other hand the differential operators
\[
\left( \prod_{l(r)>0} \partial_{a(l)}^{r(r)} - \prod_{l(r)<0} \partial_{a(l)}^{r(r)} \right) \tilde{\Pi} = 0,
\]
(3.10)
annihilate the periods \( \tilde{\Pi} = \frac{1}{a_0} \Pi \) of the mirror \( W \). Here the \( a_i \) are the coefficients of the monomials in the equation defining \( W \). They are related to the natural large complex structure variables of \( W \) by
\[
z_r = (-1)^l \prod_i a^r_i.
\]
(3.11)

Note that \( \Pi \) is well defined on \( W \), while \( \tilde{\Pi} \) is not an invariant definition of periods on \( W \). However by commuting out \( a_0^{-1} \) one can rewrite the equations (3.10) so that they annihilate \( \Pi \). Further they can be expressed in the independent complex variables \( z_r \), using the gauge condition \( \theta_{a_i} = \sum_r l^k_r \theta_{z_r} \), where \( \theta_x = \frac{d}{dx} \) denotes the log derivative. The equations (3.10) reflect symmetries of the holomorphic \((3,0)\) form and every positive \( l \) in the Mori cone (3.10) leads a differential operator annihilating \( \Pi \). The operators obtained in this way are contained in the left differential ideal annihilating \( \Pi \), but they do not generate this ideal. There is however a factorisation procedure, basically factoring polynomials \( P(\theta) \) to the left, that leads in our examples to a finite set of generators which determines linear combinations of periods as their solutions. It is referred to as a complete set of Picard-Fuchs operators. In this way properties of the instanton corrected moduli space of \( M \), often called the quantum Kähler moduli space are intimately related to the \( l^{(i)} \) and below we will relate some of it properties to the topology of \( M \).

In particular the mori generator \( l^{(e)} \) determines to a large extent the geometry of the elliptic fibre modulus. As one sees from (2.19) the mixing between the base and the fibre is encoded in the \( z \) row of \( l^{(i)} \), \( i = 1, \ldots, h_{11}(B) \) and \( l^{(e)} \) in (2.19). Let us call this the \( z \)-component of \( l^{(i)} \) and the corresponding variable \( a_z \).

Following the procedure described above one obtains after factorizing from \( l^{(e)} \) a second order generator Picard Fuchs operator. For the fibrations types introduced before it is given by
\[
L^k_e = \theta_{e} (\theta_{e} - \sum_i a^{i}_e \theta_{i}) - D^K
\]
(3.12)
where \( k = E_8, E_7, E_6, D_5 \) refers to the fibration type and \( \mathcal{D}^K \) contains the dependence on the type

\[
\begin{align*}
\mathcal{D}^{E_8} &= 12(6\theta_e - 1)(6\theta_e - 5)z_e, \\
\mathcal{D}^{E_7} &= 4(4\theta_e - 1)(4\theta_e - 3)z_e, \\
\mathcal{D}^{E_6} &= 3(3\theta_e - 1)(3\theta_e - 2)z_e, \\
\mathcal{D}^{D_5} &= 4(2\theta_e - 1)^2z_e.
\end{align*}
\tag{3.13}
\]

Formally setting \( \theta_i = 0 \) corresponds to the large base limit. Then the equation (3.12) becomes the Picard-Fuchs operator, which annihilates the periods over the standard holomorphic differential on the corresponding family of elliptic curves.

In limit of large fibre one gets as local model the total space of the canonical line bundle \( \mathcal{O}(K_B) \rightarrow B \) over the Fano base \( B \). Local mirror symmetry associates to such noncompact Calabi-Yau manifolds a genus one curve with a meromorphic 1-form \( \lambda \) that is the limit of the holomorphic \((3,0)\)-form. The local Picard-Fuchs system \( \mathcal{L}_B \) annihilating the periods \( \Pi_{loc} \) of \( \lambda \) can be obtained as a limit of the compact Picard-Fuchs system for \( l^{(i)}, i = 1, \ldots, h_{11}(B) \) by formally setting \( \theta_e = 0 \). It follows directly from (3.10), since the Mori generators of the base have vanishing first entry and commuting out \( a_{b}^{-1} \) becomes trivial. Differently then for the elliptic curve of the fibre these Picard-Fuchs operators do not annihilate the periods over holomorphic differential one form of the elliptic curve, which are \( \frac{1}{az_{e}}\Pi_{loc} \). Given the local Picard-Fuchs system the dependence on \( \theta_e \) can be restored by replacing \( \theta_{a_{z}} \) by \( \theta_{e} - \sum_{i} a_{i}\theta_{i} \) instead of \(-\sum_{i} a_{i}\theta_{i} \). Since \( l^{(i)} \) is negative \( \theta_{e} \) appears in \( \mathcal{L}_{b}^{i} \) only multiplied by at least one explicit \( z_{b}^{i} \) factor.

There are important conclusions that follow already from the general form of the Picard-Fuchs system. To see them it is convenient to rescale \( x_{e} = c_{k}z_{e} \), where \( c_{E_8} = 432, c_{E_7} = 64, c_{E_6} = 27, c_{D_5} = 16 \). It is often useful to also rescale the \( z_{i} \) and call them \( x_{i} \).

The effect of this is that the symbols of the Picard-Fuchs system become the same for all fiber types. From this we can conclude that for all fibre types the Yukawa-couplings and the discriminants are identical in the rescaled variables.

The second conclusion is that the Picard-Fuchs equation of the compact Calabi-Yau is invariant under the \( \mathbb{Z}_2 \) variable transformation

\[
\begin{align*}
x_{e} &\to (1 - x_{e}), \\
x_{i} &\to \left( -\frac{x_{e}}{1 - x_{e}} \right)^{a_{i}} x_{i}.
\end{align*}
\tag{3.14}
\]
This means that there is always a $\mathbb{Z}_2$ involution acting on the moduli space parametrized by $(x_e, x_i)$, which must be divided out to obtain the truly independent values of the parameters.

Another consequence of this statement is that the discriminants $\Delta_i(x_j)$ of the base Picard-Fuchs system determine the discriminant locus of the global system apart from $\Delta(x_e)$ components. The former contains always a conifold component $\Delta_c(x_j)$ and only that one, if there are no points on the edges of the 2d polyhedron. Points on the edges correspond to $SU(2)$ or $SU(3)$ gauge symmetry enhancement discriminants which contain only $x_i$ variables dual to K"ahler classes, whose $a^i = 0$. They are therefore invariant under (3.14). Moreover the lowest order term in the conifold discriminant is a constant and highest terms are weighted monomials of degree $\chi(B)$ with weights for the $x_i$ $a^i$ or 1 if $a^i = 0$. It follows by (3.14) that the transformed conifold discriminant $\Delta'_c(x_j) \sim (1 - x_e)^{\chi(B)} + O(x_i)$.

### 3.2.1 Examples: elliptic fibrations over $\mathbb{P}^2$ and $\mathbb{F}_1$

Let us demonstrate the above general statements with a couple of examples. We discuss the $E8$ elliptic fibration with base $\mathbb{P}^2$ and with base $\mathbb{F}_1$.

For the first example the Mori vectors are given as

$$l^{(e)} = (-6, 3, 2, 1, 0, 0, 0),$$
$$l^{(2)} = (0, 0, 0, -3, 1, 1, 1).$$

(3.15)

Form this we can derive the following set of Picard Fuchs equations, where we denote $\theta_i = z_i \partial_{z_i}$.

$$L_1 = \theta_e(\theta_e - 3\theta_2) - 12z_e(6\theta_e - 5)(6\theta_e - 5),$$
$$L_2 = \theta_2^3 - z_2(\theta_e - 3\theta_2)(\theta_e - 3\theta_2 - 1)(\theta_e - 3\theta_2 - 2).$$

(3.16)

The Yukawa couplings for this example read as follows, where we use $z_1 = \frac{x_1}{432}$, $z_2 = \frac{x_2}{27}$ and the discriminants $\Delta_1 = 1 - 3x_1 + 3x_1^2 - x_1^3 - x_1^3x_2$ and $\Delta_2 = 1 + x_2$.
The deeper origin of the appearance of modular forms is the monodromy group of the Calabi-Yau. Ref. \cite{14} explains that in the large volume limit of $X_{18}(11169)$, the monodromy group reduces to an $SL_2(\mathbb{Z})$ monodromy group. This section recalls the appearance of this modular group and how it generalizes to other elliptic fibrations. The moduli space of $X_{18}(11169)$ with the degeneration loci is portrayed in Fig. 3.

We continue by recalling the monodromy for the model in \cite{14} adapted to our
Figure 3: The moduli space for the elliptic fibration Calabi-Yau space over \( \mathbb{P}^2 \).

discussion. The fundamental solution is given by:

\[
w_0(x, y) = \sum_{m,n=0}^{\infty} \frac{(18n + 6m)!}{(9n + 3m)! (6n + 2m)! (n!)^3 m!} x^{3n+m} y^m \\
= \sum_{k=0}^{\infty} \frac{(6k)!}{k! (2k)! (3k)!} x^k U_k(y).
\]
with
\[ U_\nu(y) = y^\nu \sum_{n=0}^\infty \frac{\nu!}{(n!)^3 \Gamma(\nu - 3n + 1)} y^{-3n} \]  
\[ = y^\nu \sum_{n=0}^\infty \frac{\Gamma(3n - \nu)}{\Gamma(-\nu) (n!)^3} y^{-3n}, \]
which is a finite polynomial for positive integers \( \nu \), since \( \Gamma(\nu - 3n + 1) = \infty \) for sufficiently large \( n \). The translation to the parameters in \([14]\) is \( (x, y) = (18 \psi, -3 \phi) \).

The natural coordinates obtained from toric methods are \( z_1 = xy \) and \( z_2 = y^{-3} \). Note that the second line (3.20) makes manifest the presence of the elliptic curve in the geometry. For this regime of the parameters one can easily find logarithmic solutions by taking derivates to \( k \) and \( n \) \([30]\):
\[ 2\pi i w_e^{(1)}(x, y) = \log(xy) w_0 + \ldots \]  
\[ 2\pi i w_1^{(1)}(x, y) = -3 \log(y) w_0 + \ldots, \]
The periods are defined by \( \tau = w_e^{(1)}/w_0 \) and \( t_1 = w_1^{(1)}/w_0 \) and \( q = e^{2\pi i \tau}, q_1 = e^{2\pi i t_1} \).

The two monodromies which generate the modular group are:
\[ M_0 : (x, y) \rightarrow (e^{2\pi i x}, y), \ x \text{ small, } y \text{ large}, \]
\[ M_\infty : (x, y) \rightarrow (e^{2\pi i x}, y), \ x \text{ large, } y \text{ large}. \]
The monodromy around \( x = 0 \) follows directly from (3.22), it acts as:
\[ M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \]  
(3.23)
on \((w_e^{(1)}, w_0)^T\). To determine the action on the periods of \( M_\infty \), we need to analytically continue \( w_0 \) and \( w_1^{(1)} \) to large \( x \). To this end, we write \( w_0 \) as a Barnes integral:
\[ w_0(x, y) = \frac{1}{2\pi i} \int_C ds \frac{\Gamma(-s) \Gamma(6s + 1)}{\Gamma(2s + 1) \Gamma(3s + 1)} e^{\pi is} x^s U_s(y) \]  
(3.24)
where \( C \) is the vertical line from \(-i\infty - \varepsilon \) to \( i\infty - \varepsilon \). For small \( |x| \) the contour can be deformed to the right giving back the expression in (3.20). For large \( |x| \) one instead obtains the expansion:
\[ w_0(x, y) = \frac{1}{6\pi^2} \sum_{r=1,5} \sin(\pi r/3) \sum_{k=0}^\infty a_r(k) (-x)^{-k-\frac{r}{6}} U_{-r-\delta}(y), \]  
(3.25)
with
\[ a_r(k) = (-1)^r \frac{\Gamma(k + r/6) \Gamma(2k + r/3) \Gamma(3k + r/2)}{\Gamma(6k + r)}. \]

The logarithmic solution \( w^{(1)}_e \) is given similarly by:
\[
w^{(1)}_e(x, y) = \frac{1}{2\pi i} \int_C ds \frac{\Gamma(-s) \Gamma(6s + 1) \Gamma(s + 1)}{\Gamma(2s + 1) \Gamma(3s + 1)} e^{2\pi i s} x^s U_s(y),
\]
\[
= \frac{1}{6\pi^2 i} \sum_{r=1,5} e^{-\pi ir/6} \cos(\pi r/6) \sum_{k=0}^{\infty} a_r(k) (-x)^{-k-\frac{r}{6}} U_{-k-r/6}(y).
\]

To determine the action of \( M_\infty \), we define the basis \( f_r(x, y) = \sum_{n=0}^\infty a_r(k) (-x)^{-k-\frac{r}{6}} U_{-k-r/6}(y) \) for \( r = 1, 5 \), and the matrix \( A \) which relates to the bases \((w^{(1)}_e, w^{(1)}_1)^T = A (f_1, f_5)^T\).

Clearly, \( M_\infty \) acts diagonally on the \( f_r \): \( T = \text{diag}(\alpha^{-1}, \alpha^{-5}) \) with \( \alpha = e^{2\pi i/6} \), which gives for \( M_\infty \)
\[ M_\infty = ATA^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \in SL_2(\mathbb{Z}). \]

This gives for the monodromy around the conifold locus:
\[ M_1 = M_0 M_\infty^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \]

The generator \( S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) of \( SL_2(\mathbb{Z}) \) corresponds to \( M_0 M_\infty^{-1} \).

The large volume limit is such that \( r = q^{3/2} q_1 \to 0 \). We see that \( M_0 \) and \( M_\infty \) map small \( r \) to small \( r \). The monodromies act on \( r \) by \( M_0 r = -r, \quad M_\infty r = r \).

Thus we have established an action of \( SL_2(\mathbb{Z}) \) on the boundary of the moduli space.

The above analysis can be extended straightforwardly to the other types of fibrations using the expansions \((3.22)\). The matrix \( M_0 \) is for all fibre types the same. We find that \( M_\infty = \begin{pmatrix} 1 - a & -1 \\ a & 1 \end{pmatrix} \in \Gamma_0(a) \) for \( a = 2, 3 \) and 4 corresponding to the fibre types \( E_7, E_6 \) and \( D_5 \). Note that \( M_\infty \) has order 4 and 3 for \( a = 2 \) and 3 respectively, while the order is infinite for \( a = 4 \). Generalization to other base surfaces \( B \) is also straightforward. In case of multiple 2-cycles in the base, it is natural to define parameters for each base class: \( r_i = q^{a_i/2} q_i, \ i = 1, \ldots, b_2(B) \). This is precisely the change of parameters given by \((2.11)\). These transform as:
\[ M_0 r_i = (-)^{a_i} r_i, \quad M_\infty r_i = r_i. \]
4 Derivation of the holomorphic anomaly equation

4.1 The elliptic fibration over $\mathbb{F}_1$

In the following we try to derive the holomorphic anomaly equation at genus zero by adapting the proof which appeared in ref. [32] for a similar geometry. We start by studying the Picard-Fuchs operator associated to the elliptic fiber $X_6[1, 2, 3]$ only. Denoting by $\theta_e = x_e \partial_{x_e}$ the Picard-Fuchs operator can be written as

$$\mathcal{L} = \theta_e^2 - 12x(6\theta_e + 5)(6\theta_e + 1). \quad (4.1)$$

One can immediately write down two solutions as power series expansions around $x_e = 0$. They are given by

$$\phi(x_e) = \sum_{n \geq 0} a_n x_e^n, \quad \tilde{\phi}(x_e) = \log(x_e) \phi(x_e) + \sum_{n \geq 0} b_n x_e^n, \quad (4.2)$$

with

$$a_n = \frac{(6n)!}{(3n)!(2n)!n!}, \quad b_n = a_n(6\psi(1 + 6n) - 3\psi(1 + 3n) - 2\psi(1 + 2n) - \psi(1 + n)), \quad (4.3)$$

where $\psi(z)$ denotes the digamma function. The mirror map is thus given by

$$2\pi i \tau = \frac{\tilde{\phi}(x_e)}{\phi(x_e)}. \quad (4.4)$$

Using standard techniques from the Gauss-Schwarz theory for the Picard-Fuchs equation (cf. [47]) one observes

$$j(\tau) = \frac{1}{x_e(1 - 432x_e)}, \quad (4.5)$$

which can be inverted to yield

$$x_e(\tau) = \frac{1}{864}(1 - \sqrt{1 - 1728/j(\tau)}) = q - 312q^2 + \mathcal{O}(q^3). \quad (4.6)$$

Further, the polynomial solution $\phi(x_e)$ can be expressed in terms of modular forms as

$$\phi(x_e) = \binom{5}{6} \binom{1}{6} 432 x_e = \sqrt[4]{E_4(\tau)}, \quad (4.7)$$
from which one can conclude that
\[
E_4(\tau) = \phi^4(x_e),
\]
\[
E_6(\tau) = \phi^6(x_e)(1 - 864x_e),
\]
\[
\Delta(\tau) = \phi^{12}(x_e)x_e(1 - 432x_e),
\]
\[
\frac{1}{2\pi i} \frac{dx_e}{d\tau} = \phi^2(x_e)x_e(1 - 432x_e).
\]

(4.8)

Let us now examine the periods of the mirror geometry \( Y \) in the limit that the fiber \( F \) of the Hirzebruch surface becomes small. Due to the special structure of the Picard-Fuchs system which is found in eq. (3.19) the first three period integrals in the notation of [32] read

\[
w_0(x_e, y, 0) = \phi(x_e),
\]
\[
w^{(1)}_1(x_e, y, 0) = \tilde{\phi}(x_e),
\]
\[
w^{(1)}_2(x_e, y, 0) = \log(y)\phi(x_e) + \xi(x_e) + \sum_{m \geq 1} (\mathcal{L}_m \phi(x_e))y^m,
\]

(4.9)

with

\[
\xi(x_e) = \sum_{n \geq 0} a_n(\psi(1 + n) - \psi(1))x_e^n,
\]

(4.10)

and

\[
\mathcal{L}_m = \frac{(-)^m}{m(m!)} \prod_{k=1}^{m} (\theta_{x_e} - k + 1).
\]

(4.11)

This can be obtained by applying the Frobenius method to derive the period integrals, see e.g. [31]. The mirror map reads

\[
2\pi it_i = \frac{w^{(1)}_i(x_e, y, 0)}{w_0(x_e, y, 0)}, \quad i = 1, 2.
\]

(4.12)

Comparing this with our previous discussion about the Picard-Fuchs operator of the elliptic fiber we see that for \( t_1 = \tau \) there is nothing left to discuss. Hence, let’s study the mirror map associated to \( t_2 = t \). We observe that by formally inverting, the inverse mirror map can be determined iteratively through the relation

\[
y(q, p) = p\zeta e^{-\sum_{m \geq 1} c_m(x_e)y^m},
\]

(4.13)

where \( \zeta = e^{-\xi(x_e)/\phi(x_e)} \) and

\[
c_m(x_e) = \frac{\mathcal{L}_m \phi(x_e)}{\phi(x_e)}.
\]

(4.14)

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Using eq. (4.8) $c_1(x_e)$ is given by

\[
c_1(x_e) = -\frac{1}{12} (f_1 - 2) - \frac{f_1}{12} \frac{E_2(\tau)}{\phi^2(x_e)}
= -\frac{1}{\phi^6} \frac{f_1}{12} (E_2 E_4 - E_6),
\]  

(4.15)

where we introduced $f_1 = (1 - 432x_e)^{-1}$. In order to obtain the other $c_m(x_e)$ one uses

\[
\begin{align*}
\theta_{x_e} f_1 &= f_1 (f_1 - 1), \\
\theta_{x_e} \left( \frac{E_2}{\phi^2} \right) &= -\frac{1}{\phi^8} \frac{f_1}{12} \left( E_2^2 E_4 - 2E_2 E_6 + E_4^2 \right), \\
\theta_{x_e} \left( \frac{E_6}{\phi^6} \right) &= -\frac{1}{\phi^{12}} \frac{f_1}{12} \left( 6E_4^3 - 6E_6^2 \right),
\end{align*}
\]

(4.16)

and finds the following kind of structure. One can show inductively that

\[
c_m(x_e) = \frac{1}{\phi^{6m}} \left( \frac{f_1}{12} \right)^m Q_{6m}(E_2, E_4, E_6),
\]

(4.17)

where $Q_{6m}$ is a quasi-homogeneous polynomial of degree $6m$ and type $(2, 4, 6)$, i.e.

\[
Q_{6m}(\lambda^2 x_e, \lambda^4 y, \lambda^6 z) = \lambda^{6m} Q_{6m}(x_e, y, z).
\]

Also by induction, it follows from (4.15) and (4.16) that $Q_{6m}$ is linear in $E_2$. This allows to write a second structure which is analogous to the one appearing in ref. 32 and given by

\[
c_m(x_e) = B_m \frac{E_2}{\phi^2} + D_m,
\]

(4.18)

where the coefficients $B_m, D_m$ obey the following recursion relation

\[
\begin{align*}
B_{m+1} &= -\frac{m}{(m+1)^2} \left[ (\theta_{x_e} - m)B_m + D_1 B_m - B_1 D_m \right], \\
D_{m+1} &= -\frac{m}{(m+1)^2} \left[ (\theta_{x_e} - m)D_m - D_1 D_m + B_1 B_m \right],
\end{align*}
\]

(4.19)

with $B_1 = -\frac{f_1}{12}$ and $D_1 = -\frac{1}{12} (f_1 - 2)$. A formal solution to the recursion relation (4.19) can be given by

\[
\begin{align*}
B_m &= -\frac{f_m}{f_1} \frac{1}{12}, \\
D_m &= \frac{1}{f_1} \left[ \frac{(m + 1)^2}{m} f_{m+1} + (\theta_{x_e} - m - \frac{1}{12} (f_1 - 2)) f_m \right],
\end{align*}
\]

(4.20)
where we define $f_m$ to be

$$f_m(x_e) = \tilde{\phi}(x_e) \mathcal{L}_m \phi(x_e) - \phi(x_e) \mathcal{L}_m \tilde{\phi}(x_e).$$  \hfill (4.21)

Due to the relations (4.16) we conclude, that the $f_m$ as well as $B_m$ and $D_m$ are polynomials in $f_1$. Since $f_1$ is a rational function of $x_e$, it transforms well under modular transformations. Therefore modular invariance is broken only by the $E_2$ term in $c_m$. We express this via the partial derivative of $c_m$

$$\frac{\partial c_m(x_e)}{\partial E_2} = -\frac{1}{12} \frac{f_m(x_e)}{\phi^2(x_e)}. \hfill (4.22)$$

In order to prove the holomorphic anomaly equation (3.9) one first shows using the general results about the period integrals in [31] that the instanton part of the prepotential can be expressed by the functions $f_m(x_e)$. A tedious calculation reveals

$$\frac{1}{2\pi i} \frac{\partial}{\partial t} F^{(0)}(\tau, t) = \sum_{m \geq 1} \frac{f_m(x_e)}{\phi^2(x_e)} y_m. \hfill (4.23)$$

Using the inverse function theorem and eqs. (4.22), (4.13) yields

$$\frac{\partial y}{\partial E_2} = \frac{1}{12} \left( \frac{1}{2\pi i} \frac{\partial y}{\partial t} \right) \left( \frac{1}{2\pi i} \frac{\partial F^{(0)}}{\partial t} \right). \hfill (4.24)$$

Now, we have

$$\frac{\partial}{\partial E_2} \left( \frac{1}{2\pi i} \frac{\partial F^{(0)}}{\partial t} \right) = \frac{1}{12} \left( \frac{\partial^2 F^{(0)}}{\partial (2\pi i t)^2} \right) \left( \frac{1}{2\pi i} \frac{\partial F^{(0)}}{\partial t} \right), \hfill (4.25)$$

which implies that up to a constant term in $p$ one arrives at

$$\frac{\partial F^{(0)}}{\partial E_2} = \frac{1}{24} \left( \frac{1}{2\pi i} \frac{\partial F^{(0)}}{\partial t} \right)^2. \hfill (4.26)$$

By definition of $F_n^{(0)}$, Eq. (3.4), we have $\frac{1}{2\pi i} \frac{\partial}{\partial t} F^{(0)}(\tau, t) = \sum_{m \geq 1} m F_m^{(0)} p^m$ and hence obtain by resummation

$$\frac{\partial F_n^{(0)}}{\partial E_2} = \frac{1}{24} \sum_{s=1}^{n-1} s(n - s) F_s^{(0)} F_{n-s}^{(0)}. \hfill (4.27)$$
This almost completes the derivation of (3.9). We still need to determine the explicit form of $F_n^{(0)}$. To achieve this we proceed inductively. Using (4.8), (4.23) and (4.13) one obtains

\[ F_1^{(0)} = \frac{\zeta f_1}{\phi^2} = q^{\frac{1}{2}} \frac{E_4}{\eta^{12}}, \]  

(4.28)

Employing the structure (4.17) one can evaluate (4.23) and calculate that

\[ F_n^{(0)} = \frac{\zeta^n f_1^n}{\phi^{6n}} P_{6n-2}(E_2, E_4, E_6), \]

\[ = \left( \frac{\zeta f_1}{\phi^2} \right)^n \frac{1}{\phi^{4n}} P_{6n-2}(E_2, E_4, E_6), \]

\[ = \frac{q^n}{\eta^{12n}} P_{6n-2}(E_2, E_4, E_6), \]

(4.29)

where $P_{6n-2}$ is of weight $6n - 2$ and is decomposed out of (parts of) $Q_m$’s. This establishes a derivation of the holomorphic anomaly equation (3.9) at genus zero for the elliptic fibration over Hirzebruch surface $\mathbb{F}_1$ with large fibre class. We collect some results for the other fibre types in appendix B.

### 4.2 Derivation from BCOV

The last section provided a derivation of the anomaly equation (3.9) for genus 0 from the mirror geometry. More fundamental is a derivation purely within the context of moduli spaces of maps from Riemann surfaces to a Calabi-Yau manifold. This is the approach taken by BCOV [8] to derive holomorphic anomaly equations for genus $g$ $n$-point correlation function with $2g - 2 + n > 0$. The correlation functions are given by covariant derivatives to the free energies $F^{(g)}$: $C^{(g)}_{i_1 i_2 \ldots i_n} = D_{i_1} \ldots D_{i_n} F^g$, with $D_i$ covariant derivatives of for sections of the bundle $\mathcal{L}^{2-2g} \otimes \text{Sym}^n T$, with $T$ the tangent bundle of the coupling constant moduli space, and $\mathcal{L}$ a line bundle over this space whose Chern class corresponds to $G_{ij}$. 

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The holomorphic anomaly equation reads for the $n$-point functions

$$\partial_\lambda C_{i_1\ldots i_n}^{(g)} = \frac{1}{2} \bar{C}_{ijk} e^{2K} G^{ij} G^{k\bar{k}} C_{j\bar{k}i_1\ldots i_n}^{(g-1)} +$$

$$+ \frac{1}{2} \bar{C}_{ijk} e^{2K} G^{ij} G^{k\bar{k}} \sum_{r=0}^{g} \sum_{s=0}^{n} \frac{1}{s!(n-s)!} \sum_{\sigma \in S_n} F^{(r)}_{j_1\ldots j_{\sigma(1)}\ldots i_{\sigma(s)}} C_{k_{\bar{\bar{\sigma}}(s+1)}\ldots\sigma(n)}^{(g-r)}$$

$$- (2g-2+n-1) \sum_{s=1}^{n} G_{i_1\ldots i_{s-1}i_{s+1}\ldots i_n} C_{i_1\ldots i_s}^{(g)}.$$  \hspace{1cm} (4.30)

This equation can be summarized in terms of the generating function:

$$F(\lambda, x^i; t^i) = \sum_{g=0}^{\infty} \sum_{n=0}^{\infty} \lambda^{2g-2} \frac{1}{n!} C_{i_1\ldots i_n}^{(g)} x^{i_1} \ldots x^{i_n} + \left(\frac{\chi}{24} - 1\right) \log \lambda.$$ \hspace{1cm} (4.31)

Contrary to [8], we take the terms with $2g-2+n \leq 0$ as given by $D_1 \ldots D_n F^{(g)}$ instead of setting them to 0. Eq. (4.30) implies that $F$ satisfies

$$\partial_\lambda \exp(F) = \left[ \frac{\lambda^2}{2} \bar{F}_{ijk} e^{2K} G^{ij} G^{k\bar{k}} \frac{\partial^2}{\partial x^i \partial x^k} - G_{ij} x^j \left( \lambda \frac{\partial}{\partial \lambda} + x^k \frac{\partial}{\partial x^k} \right) \right] \exp(F).$$ \hspace{1cm} (4.32)

To relate (4.32) to the holomorphic anomaly Eq. (3.9) for this geometry, we split the $t^i$ into a fibre parameter $\tau$ and base parameters $t^i$. Then we write $F(\lambda, x; \tau, t)$ as a Fourier expansion instead of a Taylor expansion in $x^i$:

$$F(\lambda, x; \tau, t) = \sum_{g=0}^{\infty} \lambda^{2g-2} F^{(g)}_{\beta}(\tau) f^{(g)}_{\beta}(x^i, t^i) e^{2\pi i \beta x^i} p^\beta + \left(\frac{\chi}{24} - 1\right) \log \lambda,$$ \hspace{1cm} (4.33)

with $p^\beta = e^{2\pi i \beta x^i}$, and $f^{(g)}_{\beta}(x^i, t^i)$ are functions such that $D_1 F_{|x=0} = \partial_{x^i} F_{|x=0}$ and $f^{(0)}_{\beta}(0, t^i) = 1$. In the large volume limit, the covariant derivatives $D_1$ become flat derivatives $\frac{\partial}{\partial \tau}$ and thus $f^{(0)}_{\beta}(x^i, t^i) \rightarrow 1$. Therefore, to deduce (3.9) from (4.32) we can set $x^i = 0$ and replace the $\partial_{x^i}$ by $\partial_{\tau}$.

Eq. (3.9) follows now by considering $\frac{1}{2\pi i} \partial_\tau \exp(F)$ on the right hand side of (4.32).

As discussed earlier, all $\bar{\tau}$ dependence arises from completing the weight 2 Eisenstein series: $\hat{E}_2(\bar{\tau}) = E_2(\tau) - \frac{3}{\pi \tau^2}$, which gives:

$$\frac{\partial}{\partial E_2} = \frac{4\pi^2 \tau^2}{3} \frac{\partial}{2\pi i \partial \bar{\tau}}.$$ \hspace{1cm} (4.34)

We first discuss how the right-hand side of (3.9) can be derived from Eq. (4.32) for the geometry $X_{18}(11169)$. We use the basis (2.12), and choose as parameters the “base”
parameter \( t = b + iJ \) (which is related to \( r \) of Subsec. 3.3 by \( r = e^{2\pi it} \)) and the fibre parameter \( \tau = \tau_1 + i\tau_2 \). We are interested in the large volume limit \( \tau \to i\infty, t \to i\infty \) in such a way that \( J \gg \tau_2 \). In this limit, the Kähler potential is well approximated by the polynomial form:

\[
K \approx -\log\left( \frac{4}{3} \tilde{K}_{ijk} J^i J^j J^k \right) = -\log\left( \frac{4}{3} (\alpha \tau_2^3 + 3\tau_2 J^2) \right) \tag{4.35}
\]

with \( \alpha = \tilde{K}_e^3 \) (2.12). This gives for the metric:

\[
\begin{pmatrix}
G_{\tau\tau} & G_{t\tau} \\
G_{\tau t} & G_{tt}
\end{pmatrix} \approx \begin{pmatrix}
\frac{1}{4\tau_2^2} & \frac{\alpha \tau_2}{3J^2} \\
\frac{\alpha \tau_2}{3J^2} & \frac{1}{2J^2}
\end{pmatrix},
\]

which gives for the matrix \( e^K G^{ij} \):

\[
e^K G^{-1} \approx \begin{pmatrix}
-\frac{2\alpha \tau_2^2}{3J^2} & \frac{1}{4\tau_2^2} \\
\frac{1}{4\tau_2^2} & -\frac{\alpha \tau_2}{3J^2}
\end{pmatrix} \tag{4.36}
\]

Thus in the limit \( J \to \infty \), one finds that only \( e^K G^{tt} \approx \frac{1}{2\tau_2} \) does not vanish. Therefore, \( \bar{C}_{ijk} e^{2K} G^{ij} G^{kk} \frac{\partial^2}{\partial x^i \partial x^k} \approx \frac{1}{4\tau_2^2} \frac{1}{4\tau_2^2} \frac{1}{4\tau_2^2} \frac{1}{4\tau_2^2} \). Using (4.34), this shows that (4.32) reduces to:

\[
\frac{\partial}{\partial E_2} \exp(F) = \frac{\lambda^2}{24} \left( p \frac{\partial}{\partial p} \right)^2 \exp(F). \tag{4.37}
\]

Expansion of both sides in \( p \) and taking the \( p^n \) coefficient gives a holomorphic anomaly equation as (3.9) for \( g = 0 \). It also gives the correct (3.9) for \( g > 0 \) except for the appearance of \( K_B \). We believe that a more thorough analysis of the covariant derivatives will explain this term. Assuming the form \( f^{(g)}_\beta(x, t) \to 1 + x^2 \beta \cdot K_B + \ldots \) would give the shift in (3.9).

The derivation is very similar for the other types of fibres discussed in Section 2. The right hand side of Eq. (4.37) is simply divided by \( a \), in agreement with [32].

5 T-duality on the fibre

One can perform two T-dualities around the circles of the elliptic fibre. Due to the freedom in choosing the circles, this leads to an \( SL_2(\mathbb{Z}) \) (or a congruence subgroup) group of dualities mapping IIA branes to IIA branes. This duality group is equal to the

\[\text{The factor } \frac{1}{16\pi^2} \text{ appears due to a factor } -2\pi i \text{ between the moduli in [8] and ours.}\]
modular subgroup of the monodromy group which leave invariant the $F_g$’s discussed in Sec. 3.3.

Let $D_{2f/\beta}$ be a $D2$-brane wrapped either on the elliptic fibre $f$ or on a class $\beta$ in the base. Moreover, we denote by $D_{4f}$ a D4-brane wrapped around the base and $D_{4\beta}$ is a D4-brane wrapped around the cycle $\beta$ in the base and the fibre $f$. The double T-duality on both circles of the elliptic fibre transforms pairs of D-brane charges heuristically in the following way:

\[
\begin{pmatrix}
D_6 \\
D_{4f}
\end{pmatrix} = \gamma \begin{pmatrix}
\tilde{D}_6 \\
\tilde{D}_{4f}
\end{pmatrix},
\]

\[
\begin{pmatrix}
D_{4\beta} \\
D_{2\beta}
\end{pmatrix} = \gamma \begin{pmatrix}
\tilde{D}_{4\beta} \\
\tilde{D}_{2\beta}
\end{pmatrix},
\]

\[
\begin{pmatrix}
D_{2f} \\
D_{0}
\end{pmatrix} = \gamma \begin{pmatrix}
\tilde{D}_{2f} \\
\tilde{D}_0
\end{pmatrix},
\]

with $\gamma$ in $SL_2(\mathbb{Z})$ or a congruence subgroup. See for more a more formal treatment of T-duality on Calabi-Yau’s [3, 4]. T-duality is not valid for every choice of the Kähler parameter. One way to see this is that the BPS invariants of $D2$ branes do not depend on the choice of the Kähler moduli but those of $D4$ and $D6$ branes do through wall-crossing. The choice where the two are related by T-duality is sufficiently close to the class of the elliptic fibre, this is called a suitable polarization in the literature [36]. Sufficiently close means that no wall is crossed between the fibre class and the suitable polarization.

The equality of invariants of $D0$ branes and $D2$ branes wrapping the fibre can be easily verified. The BPS invariant of D0 branes is known to be equal to the Euler number [44]:

\[
\Omega((0, 0, 0, n), X) = -\chi(X).
\] (5.2)

One can verify in for example [30] that these equal the BPS invariants of D2 branes wrapping the $E_8$ elliptic fibre of $X$. If the modular group is a congruence subgroup of level $n$ then only the BPS index corresponding to $0 \mod n$ $D2$ branes wrapping the fibre equals [5,2].

Our interest is in the $D4$-branes which can be obtained from $D2_\gamma$ with $\gamma = \beta + nf$ by T-duality. These $D4$-branes wrap classes in the bases times the fibre, and have
D0 brane charge $n$. D4-branes on Calabi-Yau manifolds correspond to black holes in 4-dimensional space-time and are well studied [48], in particular M-theory relates the degrees of freedom of D4-brane black holes to those of a $\mathcal{N} = (4, 0)$ CFT with left and right central charges:

$$c_L = P^2 + \frac{1}{2} c_2 \cdot P, \quad c_R = P^3 + c_2 \cdot P,$$

with $P$ the 4-cycle wrapped by the D4-brane, and $c_2$ the second Chern class of the Calabi-Yau. Typically, the number of 2-cycles in the D4-brane is larger than the number of 2-cycles in the Calabi-Yau.

In the following, we will use the notation of [50]. The homology class $P$ gives naturally rise to a quadratic form $D_{ab} = d_{abc} P^c$ which has signature $(1, b_2(X) - 1)$. Let $\Lambda$ be the lattice $\mathbb{Z}^{b_2}$ with quadratic form $D_{ab}$. The dual lattice with quadratic form $D_{ab}^{-1}$ is denoted by $\Lambda^*$. The Kähler modulus $J$ gives the projection of a vector $k \in \Lambda$ to the positive definite subspace of $\Lambda \otimes \mathbb{R}$:

$$k_+ = \frac{k \cdot J}{J^2} J,$$

with $J^2 = d_{abc} P^a J^b J^c$.

The supergravity partition function of D4-branes takes generically the following form [9, 50]:

$$Z_P(C, \tau; t) = \sum_{Q_0, Q} \hat{\Omega}(\Gamma; t) (-1)^{P \cdot Q} \times e \left( -\bar{\tau} \hat{Q}_0 + \tau (Q - B)_+^2 / 2 + \bar{\tau} (Q - B)_-^2 / 2 + C \cdot (Q - B/2) \right).$$

with $t = B + iJ$ the complexified Kähler modulus and $\hat{Q}_0 = -Q_0 + \frac{1}{2} Q^2$. $Z_P(C, \tau; t)$ transforms as a modular form of weight $(\frac{1}{2}, -\frac{3}{2})$. The invariants $\hat{\Omega}(\Gamma; t)$ are rational invariants and related to the integer invariants $\Omega(\Gamma; t)$ by the multi-cover formula:

$$\hat{\Omega}(\Gamma; t) = \sum_{m|\Gamma} \frac{\Omega(\Gamma/m; t)}{m^2}$$

(5.6)

Note that the multi cover contributions come here with a factor $m^{-2}$, whereas in Gromov-Witten theory they are multiplied with $m^{-3}$. The invariants $\hat{\Omega}(\Gamma; t)$ are related to the Euler number of the appropriate moduli space $\mathcal{M}_t(\Gamma)$ by:

$$\Omega(\Gamma; t) = (-1)^{\dim c \mathcal{M}_t(\Gamma)} \chi(\mathcal{M}_t(\Gamma)).$$
If the $B$-field decouples from the stability condition, $Z_P(C, \tau; t)$ allows a theta function decomposition:

$$Z_P(C, \tau; t) = \sum_{\mu} h_{P,\mu}(\tau) \Theta_{P,\mu}(\tau, C, B),$$  \hspace{1cm} (5.8)

and $h_{P,\mu}(\tau)$ is a vector valued modular form of weight $-1 - b_2(X)/2$ given by:

$$h_{P,\mu}(\tau) = \sum_{Q_0} \Omega_P(\hat{Q}_0) q^{\hat{Q}_0}$$  \hspace{1cm} (5.9)

with $\hat{Q}_0 = -Q_0 + \frac{1}{2}\mu^2$. This symmetry is also present in the MSW conformal field theory which arises in the near horizon geometry of a single center $D4$-brane black hole [9]. We refer to [50] for a discussion of the relation between the supergravity partition and the CFT partition function. In terms of the central charges of $c_{L/R}$ of this conformal field theory, $h_{P,\mu}(\tau)$ typically takes the form:

$$h_{P,\mu}(\tau) = \frac{f_{P,\mu}(\tau)}{\eta(\tau)^{c_R}}$$  \hspace{1cm} (5.10)

with $f_{P,\mu}(\tau)$ a vector-valued modular form of weight $-1 - b_2(X)/2 + c_R/2$. Precisely this structure is also found for the genus 0 amplitudes obtained from the mirror periods, see Eq. (3.8) combined with Eq. (2.7). The prediction from the mirror periods is obtained from the large base limit, and corresponds to $h_{P,0}(\tau)$.

The triple intersection $P^3$ vanishes for the $D4$-branes obtained by T-duality from the periods. These are therefore not large black holes, but we nevertheless obtain detailed knowledge about the spectrum of “small” black holes using mirror symmetry. For small $D0$ and $D4$-brane charge, the BPS invariants can be computed either from the microscopic $D$-brane perspective or the supergravity context [23, 24, 20, 17, 10, 53, 55]. For example from the microscopic point of view, the moduli space of a single $D4$-brane is given by projective space $\mathbb{P}^n$. Using index theorems one can compute that $n = \frac{1}{6}P^3 + \frac{1}{12}c_2 \cdot P - 1$ [48]. Therefore, the first coefficient of $h_{P,0}(\tau)$ is expected to be

$$\Omega_P(-\frac{1}{24}c_R) = \frac{1}{6}P^3 + \frac{1}{12}c_2 \cdot P.$$  \hspace{1cm} (5.11)

The second coefficient corresponds to adding a unit of (anti) $D0$-brane charge. Now the linear system for the divisor of the $D4$-brane is constrained to pass through the
This gives with Eq. (5.2) [23]:

\[ \Omega_P(1 - \frac{1}{24} c_R) \cong \chi(X)(\frac{1}{12} c_2 \cdot P - 1). \] \tag{5.12}

Here we have written a “\( \cong \)” instead of “\( = \)” since if \( 1 - \frac{1}{24} c_R \geq 0 \) gravitational degrees of freedom might start contributing which are less well understood.

Continuing with two units of \( D0 \) charge, one finds:

\[ \Omega_P(2 - \frac{1}{24} c_R) \cong \frac{1}{2} \chi(X)(\chi(X) + 5)(\frac{1}{12} c_2 \cdot P - 2). \] \tag{5.13}

One can in principle continue along these lines, which becomes increasingly elaborate since

- effects of \( D2 \)-branes become important,
- single center black holes contribute for \( \hat{Q}_0 > 0 \),
- the index might depend on the background moduli \( t \).

We now briefly explain which bound states appear in the supergravity picture for small \( D0/4 \)-brane charge. The first terms in the \( q \)-expansion cannot correspond to single center black holes since \( \hat{Q}_0 < 0 \). The first terms correspond to bound states of \( D6 \) and \( \bar{D}6 \)-branes [20]. If \( P \) is an irreducible cycle (it cannot be written as \( P = P_1 + P_2 \) with \( P_1 \) and \( P_2 \) effective classes) then the charges \( \Gamma_1 \) and \( \Gamma_2 \) of the constituents are

\[ \Gamma_1 = (1, P, \frac{1}{2} P^2 - \frac{c_2}{24}, \frac{1}{6} P^3 + \frac{c_2 P}{24}), \quad \Gamma_2 = (-1, 0, \frac{c_2}{24}, 0), \] \tag{5.14}

The index of a 2-center bound state is given by:

\[ \langle \Gamma_1, \Gamma_2 \rangle \Omega(\Gamma_1) \Omega(\Gamma_2), \]

with \( \langle \Gamma_1, \Gamma_2 \rangle = -P^0_1 Q_{0,2} + P_1 \cdot Q_2 - P_2 \cdot Q^1 + P^0_2 Q_{0,1} \) the symplectic inner product. Since the constituents are single \( D6 \)-branes with a non-zero flux, their index is \( \Omega(\Gamma_i; t) = 1 \). Therefore, \( \Omega_P(\frac{1}{24} c_R) = \langle \Gamma_1, \Gamma_2 \rangle = \frac{1}{6} P^3 + \frac{1}{12} c_2 \cdot P \), which reproduces Eq. (5.11).

One can continue in a similar fashion with adding other constituents to compute indices with higher charge. For example, BPS states with charge \( \Gamma = (0, 2P, 0, \frac{1}{3} P^3 + \frac{c_2 P}{12}) \) corresponds to \( \Gamma_1 \) as in (5.14) and

\[ \Gamma_2 = (-1, P, -\frac{1}{2} P^2 + \frac{c_2}{24}, \frac{1}{6} P^3 + \frac{c_2 P}{24}), \] \tag{5.15}
One obtains then $\Omega_{2P}(-\frac{1}{24}c_R) = \frac{8}{6}P^3 + \frac{2}{12}c_2 \cdot P$. Similarly, one could also add $\bar{D}0$ charges, and find the right hand sides of Eqs. (5.11) to (5.13) with $P$ replaced by $2P$.

**Example:** $X_{18}(9,6,1,1,1)$

We now consider the periods for $X_{18}(9,6,1,1,1)$, i.e. an elliptic fibration over $\mathbb{P}^2$ and compare with the above discussion. This Calabi-Yau has a 2-dimensional Kähler cone, and lends itself well to studies of D4-branes. We consider D4-branes wrapping the divisor whose Poincaré dual is the hyperplane class $H$ of the base surface $\mathbb{P}^2$. The number of wrappings is denoted by $r$.

The genus 0 Gromov-Witten invariants are well-studied [14, 30]. Adjusting for the different power in the multi-cover formula, one obtains the following predictions for $h_nH(\tau)$:

$$h_{H}(\tau) = \frac{31E_4^4 + 113E_4E_6^2}{48\eta(\tau)^36} = q^{-3/2}(3-1080q+143770q^2+204071184q^3+\ldots),$$

$$h_{2H}(\tau) = -\frac{196319E_4E_6^5 - 755906E_4^2E_6^3 - 208991E_4E_6^7}{221184\eta(\tau)^{72}} - \frac{1}{24}E_2h_{H}(\tau)^2 + \frac{1}{8}h_{H}(2\tau),$$

$$h_{3H}(\tau) = q^{-9/2}(27-17280q+5051970q^2+\ldots) + \frac{1}{9}h_{H}(3\tau).$$

We want to compare this to the expressions derived above from the point of view of D4-branes. For $r = 1$, we have

$$\Omega(\Gamma; J) = \frac{1}{12}c_2 \cdot H = 3,$$

(5.16)

in agreement with the first coefficient of $h_{H}(\tau)$. The second term in the $q$-expansion corresponds to

$$\Omega(1, \frac{1}{2}H, -1) = \chi(X)(\frac{1}{12}c_2 \cdot P - 1) = 1080,$$

(5.17)

which is also in agreement with the periods. For two $\bar{D}0$ branes we find a small discrepancy, one finds:

$$\frac{1}{2}(\frac{1}{12}c_2 \cdot P - 2)\chi(X)(\chi(X) + 5) = 144450.$$

(5.18)
This is an access of $1080 = -2\chi(X)$ states compared to the 3rd coefficient in $h_1(\tau)$. This number is very suggestive of a bound state picture, possibly involving $D2$ branes. Since $\hat{Q}_0 > 0$ one could argue that these states are due to intrinsic gravitational degrees of freedom, but it seems actually a rather generic feature if we consider other elliptic fibrations (e.g. over $\mathbb{F}_1$).

For $r = 2$, also the first two coefficients of the spectrum match with the $D4$-brane indices, and the 3rd differs by $-6\chi(X)$. Something non-trivial happens for $r = 3$. We leave an interpretation of these indices from multi-center solutions for a future publication, and continue with the example of the local elliptic surface [60].

6 BPS invariants of the rational elliptic surface

This section continues with the comparison of the $D4$- and $D2$-brane spectra for $E_8$ elliptic fibration over the Hirzebruch surface $\mathbb{F}_1$ which was first addressed by Refs. [60, 65]. Let $\sigma : \mathbb{F}_1 \to X$ be the embedding of $\mathbb{F}_1$ into the Calabi-Yau. The surface $\mathbb{F}_1$ is itself a fibration $\pi : \mathbb{F}_1 \to C \cong \mathbb{P}^1$ with fibre $f \cong \mathbb{P}^1$, with intersections $C^2 = -1$, $C \cdot f = 1$ and $f^2 = 0$. The Kähler cone of $X$ is spanned by the elliptic fibre class $J_1$, and the classes $J_2 = \sigma_*(C + f)$ and $J_3 = \sigma_*(f)$. The Calabi-Yau intersections and Chern classes are given by (2.16).

A few predictions from the periods for the $D4$-brane partition functions are:

\[
\begin{align*}
    h_C(\tau) &= \frac{E_4(\tau)}{\eta(\tau)^{12}} = q^{-1/2}(1 + 252q + \ldots), \\
    h_f(\tau) &= \frac{2E_4(\tau)E_6(\tau)}{\eta(\tau)^{24}} \\
    &= -2q^{-1} + 480 + 282888q + \cdots, \\
    h_{2C}(\tau) &= \frac{E_2(\tau)E_4(\tau)^2 + 2E_4(\tau)E_6(\tau)}{24\eta(\tau)^{24}} + \frac{1}{8}h_C(2\tau) \\
    &= -9252q - 673760q^2 + \cdots + \frac{1}{4}h_C(2\tau), \\
    h_{3C}(\tau) &= \frac{54E_2^2E_4^3 + 216E_2E_4^2E_6 + 109E_4^4 + 197E_4E_6^2}{15552\eta^{36}} + \frac{2}{27}h_C(3\tau) \\
    &= 848628q^{3/2} + 115243155q^{5/2} + \cdots + \frac{1}{9}h_C(3\tau).
\end{align*}
\]

Since $c_2(X) \cdot f = 24$, explicit expressions in terms of modular forms for the divisors
\( h_{C+nf}(\tau) \) become rather lengthy. Interestingly, one finds that for this class the first coefficients (checked up to \( n = 12 \)), are given by \( 1 + 2n \) in agreement with Eq. (5.11). Moreover, the second and third coefficients are respectively given by \( \chi(X) \left( \frac{1}{12} c_2 \cdot P - 1 \right) \) and \( \frac{1}{2} \chi (\chi + 9) \left( \frac{1}{12} c_2 \cdot P - 2 \right) \) as long as the corresponding \( \hat{Q}_0 \) is.

Another interesting class are \( r \) D4 branes wrapped on the divisor \( C \), which is however not an ample divisor since \( C = J_2 - J_3 \). The Euler number of this divisor is \( c_2 \cdot C = 12 \), it is in fact the rational elliptic surface \( \mathbb{F}_9 \), which is the 9-point blow-up of the projective plane \( \mathbb{P}^2 \), or equivalently, the 8-point blow-up of \( \mathbb{F}_1 \).

For \( r \) D4 branes we have \( P = rC \). Eq. (2.16) shows that the quadratic form \( D_{abc} P^c \) restricted to \( J_1 \) and \( J_3 \) is:

\[
\mathbf{r} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}
\]

The other 8 independent classes of \( H^2(P, \mathbb{Z}) \) are not “visible” to the computation based on periods, since these 2-cycles of \( P \) do not pull back to 2-cycles of \( X \). We continue by confirming the expressions found from the periods with a computation of the Euler numbers of the moduli spaces of semi-stable sheaves as in Refs. [65, 60]. The algebraic computations are more naturally performed in terms of Poincaré polynomials, and thus give more refined information about the moduli space [56]. Moreover, the 8 independent classes which are not visible from the Calabi-Yau point of view, can be distinguished from this perspective.

One might wonder whether the extra parameter appearing with the Poincaré polynomial is related to the higher genus expansion of topological strings. However, the refined information of the genus expansion captures is different. Roughly speaking, the D2-brane moduli space is a torus fibration over a base manifold [28]. The genus expansion captures the cohomology of the torus, whereas the D4-brane moduli space gives naturally the cohomology of the total moduli space. For \( r = 1 \), Ref. [32] argues that the torus fibration is also present for moduli spaces of rank 1 sheaves on \( \mathbb{F}_9 \), but it is non-trivial to continue this to higher rank. Another approach to verify the Fourier-Mukai transform at a refined level is consider the refined topological string partition function with parameters \( \epsilon_1 \) and \( \epsilon_2 \), and then take the Nekrasov-Shatashvili limit \( \epsilon_1 = 0, \epsilon_2 \ll 1 \) instead of the topological string limit \( \epsilon_1 = -\epsilon_2 = g_s \).

The structure described in Sec. 5 for D4-brane partition functions simplifies when
one specializes to a (local) surface. The charge vector $\Gamma$ becomes $(r, c_1, c_2)$ with $r$ the ranks and $c_i$ the Chern characters of the sheaf. Other frequently occurring quantities are the determinant $\Delta = \frac{1}{r}(c_2 - \frac{r - 1}{2r}c_1^2)$, and $\mu = c_1/r \in H^2(S, \mathbb{Q})$. In terms of the Poincaré polynomial $p(M, w) = \sum_{i=0}^{\text{dim}_c(M)} b_i(M) w^i$ of the moduli space $M$, the (refined) BPS invariant reads:

$$\Omega(\Gamma, w; J) := \frac{w^{-\text{dim}_c(M_J(\Gamma))}}{w - w^{-1}} p(M_J(\Gamma), w),$$

In the case of surfaces, a formula is available for the dimension of the moduli space:

$$\text{dim}_c M_J(\Gamma) = 2r^2 \Delta - r^2 \chi(O_S) + 1.$$

One can verify that the Poincaré polynomials computed later in this section are in agreement with this formula.

The rational invariant corresponding to $\Omega(\Gamma, w; J)$ is [52]:

$$\tilde{\Omega}(\Gamma, w; J) = \sum_{m | \Gamma} \Omega(\Gamma/m, -(-w)^m; J) \quad (6.3)$$

The numerical BPS invariant $\Omega(\Gamma; J)$ follows from the $\Omega(\Gamma, w; J)$ by:

$$\Omega(\Gamma; J) = \lim_{w \to -1} (w - w^{-1}) \Omega(\Gamma, w; J), \quad (6.4)$$

and similarly for the rational invariants $\tilde{\Omega}(\Gamma; J)$.

The generating function [5.5] becomes for a complex surface $S$:

$$Z_r(\rho, z, \tau; S, J) = \sum_{c_1,c_2} \tilde{\Omega}(\Gamma, w; J) (-1)^{c_1 \cdot K_S}$$

$$\times q^{\frac{\Delta(\Gamma)}{2r}} \frac{1}{2^r (c_1 + rK_S/2)^2 \cdot \frac{1}{2} \left( c_1 + rK_S/2 \right)^2} e^{2\pi i \rho \cdot (c_1 + rK_S/2) / q}, \quad (6.5)$$

with $\rho \in H^2(S, \mathbb{C})$, $w = e^{2\pi iz}$ and $q = e^{2\pi i \tau}$. Twisting by a line bundle leads to an isomorphism of moduli spaces. It is therefore sufficient to determine $\Omega(\Gamma, w; J)$ only for $c_1 \mod r$, and it moreover implies that $Z_r(\rho, z, \tau; S, J)$ allows a theta function decomposition as in [5.8]:

$$Z_r(\rho, z, \tau; S, J) = \sum_{\mu \in \Lambda^*/\Lambda} h_{r,\mu}(z, \tau; S, J) \Theta_{r,\mu}(\rho, \tau; S), \quad (6.6)$$

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with
\[ h_{r,\mu}(z, \tau; S, J) = \sum_{c_2} \bar{\Omega}(\Gamma, w; J) q^{r\Delta(\Gamma)-r\chi(S)_{24}}, \] (6.7)
and
\[ \Theta_{r,\mu}(\rho, \tau; S) = \sum_{k} (-1)^{rK S} q^{k^2/2r} e^{2\pi i \rho \cdot k}. \]

Note that \( \Theta_{r,\mu}(\rho, \tau; S) \) depends on \( J \) through \( k \pm \) and does not depend on \( z \).

The generating function of the numerical invariants \( \Omega(\Gamma; J) \) follows simply from Eq. (6.4):
\[ Z_r(\rho, \tau; S, J) = \lim_{z \to 1} \frac{1}{2} (w - w^{-1}) Z_r(z, \rho, \tau; S, J). \] (6.8)

Physical arguments imply that this function transforms as a multivariable Jacobi form of weight \( \left( \frac{1}{2}, \frac{-3}{2} \right) \) \[61, 49\] with a non-trivial multiplier system. For rank \( > 1 \) this is only correct after the addition of a suitable non-holomorphic term \[61, 60\].

This section verifies the agreement of the BPS invariants obtained from the periods and vector bundles for \( h_{r,c_1}(z, \tau; \mathbb{F}_9, J_{m,n}) \) for \( r \leq 3 \). The results for \( r \leq 2 \) are due to Göttsche \[25\] and Yoshioka \[65\]. The computations apply notions and techniques from algebraic geometry as Gieseker stability, Harder-Narasimhan filtrations and the blow-up formula. We refer to \[52, 56\] for further references and details. The most crucial difference between the computations for \( \mathbb{F}_9 \) and those for Hirzebruch surfaces in \[52, 56\] is that the lattice arising from \( H^2(\mathbb{F}_9, \mathbb{Z}) \) is now 10 dimensional. We continue therefore with giving a detailed description of different bases of \( H^2(\mathbb{F}_9, \mathbb{Z}) \), gluing vectors and theta functions.

### 6.1 The lattice \( H^2(\mathbb{F}_9, \mathbb{Z}) \)

The second cohomology \( H^2(\mathbb{F}_9, \mathbb{Z}) \) gives naturally rise to a unimodular basis, it is in fact the unique unimodular lattice with signature \( (1,9) \), which we denote by \( \Lambda_{1,9} \). For this paper 3 different bases (\( C, D \) and \( E \)) of \( \Lambda_{1,9} \) are useful. The first basis is the geometric basis \( C \), which keeps manifest that \( \mathbb{F}_9 \) is the 9-point blow-up of the projective plane \( \mathbb{P}^2 \). The basis vectors of \( C \) are \( H \) (the hyperplane class of \( \mathbb{P}^2 \)) and \( c_i \) (the exceptional divisors of the blow-up). \[^5\] The quadratic form is \( \text{diag}(1, -1, \ldots, -1) \). The canonical

\[^5\] We will use in general boldface to parametrize vectors.
class $K_9$ of $\mathbb{F}_9$ is given in terms of this basis by:

$$K_9 = -3H + \sum_{i=1}^{9} c_i. \quad (6.9)$$

One can easily verify that $K_9^2 = 0$. Note that $-K_9$ is numerically effective but not ample.

The second basis $D$ parametrizes $\Lambda_{1,9}$ as a gluing of the two non-unimodular lattices $A$ and $D$. The basis $D$ is given in terms of $C$ by:

$$a_1 = -K_9, \quad a_2 = H - c_9,$$

$$d_i = c_i - c_{i+1}, \quad 1 \leq i \leq 7,$$

$$d_8 = -H + c_7 + c_8 + c_9. \quad (6.10)$$

The $a_i$ are basis elements of $A$ and $d_i$ of $D$. Since $A$ and $D$ are not unimodular, integral lattice elements of $C$ do not correspond to integral elements of $D$. For example, $c_9$ is given by

$$c_9 = \frac{1}{2} \left( a_1 + a_2 + \sum_{i=1}^{6} id_i + 3d_7 + 4d_8 \right). \quad (6.11)$$

The other $c_i$ are easily determined using $c_9$. The quadratic form $Q_A$ of the lattice $A$ is:

$$Q_A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad (6.12)$$

and $Q_D$ of the lattice $D$ is minus the $D_8$ Cartan matrix:

$$Q_D = -Q_{D_8} = -\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \quad (6.13)$$

Gluing of $A$ and $D$ to obtain $\Lambda_{1,9}$ corresponds to an isomorphism between $A^*/A$ and $D^*/D$. This isomorphism is given by 4 gluing vectors $g_i$, since the discriminants
of $A$ and $D$ are equal to 4. We choose them to be:

$$
\begin{align*}
    g_0 &= 0, \\
    g_1 &= \frac{1}{2}(1, 0, 1, 0, 1, 0, 0, 0, 1), \\
    g_2 &= \frac{1}{2}(0, 1, 0, 0, 0, 0, 0, 1, 1), \\
    g_3 &= \frac{1}{2}(1, 1, 1, 0, 1, 0, 1, 0, 0).
\end{align*}
$$

Theta functions which sum over $D$ will play an essential role later in this section. The theta functions $\Theta_{rD, \mu}(\tau)$ are defined by:

$$\Theta_{rD, \mu}(\tau) = \sum_{k=\mu \mod rZ} q^{k^2/r}.$$  \hfill (6.14)

Such sums converge rather slowly. Therefore, we also give their expression in terms of unary theta functions $\theta_i(\tau) = \theta_i(0, \tau)$ (defined in Appendix C). For $r = 1$ and the glue vectors $g_i$ one has:

$$
\begin{align*}
    \Theta_{D, g_0}(\tau) &= \frac{1}{2} (\theta_3(\tau)^8 + \theta_4(\tau)^8), \\
    \Theta_{D, g_1}(\tau) &= \frac{1}{2} \theta_2(\tau)^8, \\
    \Theta_{D, g_2}(\tau) &= \frac{1}{2} (\theta_3(\tau)^8 - \theta_4(\tau)^8), \\
    \Theta_{D, g_3}(\tau) &= \frac{1}{2} \theta_2(\tau)^8.
\end{align*}
$$

For $r = 2$, the $\mu$ in the $\Theta_{2D, \mu}(\tau)$ take values in $D/2D$. The $2^8$ elements are naturally grouped in 6 classes with multiplicities 1, 56, 140, 1, 56 and 2 depending on the corresponding theta function $\Theta_{2D, \mu}(\tau)$. We choose as representative for each class:

$$
\begin{align*}
    d_0 &= 0, \\
    d_1 &= (1, 0, 0, 0, 0, 0, 0, 0), \\
    d_2 &= (1, 0, 1, 0, 0, 0, 0, 0), \\
    d_3 &= (0, 0, 0, 0, 0, 0, 1, 1), \\
    d_4 &= (1, 0, 1, 0, 1, 0, 0, 0), \\
    d_5 &= (1, 0, 1, 0, 1, 0, 1, 0).
\end{align*}
$$

Elements $\mu \in g_i + D/2D$ fall similarly in conjugacy classes corresponding to their theta functions. We let $m_{i,j}$ denote the number of elements in the class represented by $g_i + d_j$. The non-vanishing $m_{i,j}$ are given in Table 2.
Table 2: The number of elements $m_{i,j}$ in $g_i + D/2D$ with equal theta functions $\Theta_{2D,g_i+d_j}(\tau)$.

| $m_{i,j}$ | 0    | 1    | 2    | 3    | 4    | 5    |
|-----------|------|------|------|------|------|------|
| 0         | 1    | 56   | 140  | 1    | 56   | 2    |
| 1         | 128  |      | 128  |     |      |      |
| 2         | 16   | 112  | 112  | 16   |      |      |
| 3         | 128  | 128  |      |      |      |      |

The corresponding theta functions are given by:

$$
\begin{align*}
\Theta_{2D,g_0}(\tau) &= \frac{1}{2} \left( \theta_3(2\tau)^8 + \theta_4(2\tau)^8 \right), \\
\Theta_{2D,g_1}(\tau) &= \frac{1}{16} \left( \theta_3(\tau)^8 - \theta_4(\tau)^8 \right) - \frac{1}{2} \theta_2(2\tau)^6 \theta_3(2\tau)^2, \\
\Theta_{2D,g_2}(\tau) &= \frac{1}{2} \theta_2(\tau)^8, \\
\Theta_{2D,g_3}(\tau) &= \frac{1}{2} \left( \theta_3(2\tau)^8 - \theta_4(2\tau)^8 \right), \\
\Theta_{2D,g_4}(\tau) &= \frac{1}{2} \theta_2(2\tau)^6 \theta_3(2\tau)^2, \\
\Theta_{2D,g_5}(\tau) &= \frac{1}{2} \theta_2(2\tau)^8,
\end{align*}
$$

(6.15)

For $g_1$:

$$
\begin{align*}
\Theta_{2D,g_1}(\tau) &= \frac{1}{8} \theta_2(\tau)^4 \left( \theta_3(2\tau)^4 - \frac{1}{2} \theta_4(2\tau)^4 \right), \\
\Theta_{2D,g_1+d_3}(\tau) &= \Theta_{2D,g_2}(\tau),
\end{align*}
$$

For $g_2$:

$$
\begin{align*}
\Theta_{2D,g_2}(\tau) &= \frac{1}{4} \theta_2(\tau)^2 \theta_3(2\tau)^6, \\
\Theta_{2D,g_2+d_1}(\tau) &= \frac{1}{16} \theta_2(\tau)^6 \theta_3(2\tau)^2, \\
\Theta_{2D,g_2+d_2}(\tau) &= \frac{1}{16} \theta_2(\tau)^6 \left( \theta_3(2\tau)^2 - \theta_4(\tau)^2 \right), \\
\Theta_{2D,g_2+d_4}(\tau) &= \frac{1}{4} \theta_2(2\tau)^6 \theta_2(\tau)^2,
\end{align*}
$$

(6.16)

For $g_3$:

$$
\begin{align*}
\Theta_{2D,g_3}(\tau) &= \Theta_{2D,g_1}(\tau), \\
\Theta_{2D,g_3+d_3}(\tau) &= \Theta_{2D,d_1+d_4}(\tau),
\end{align*}
$$
The third basis is basis $\mathbf{E}$ corresponding to the representation of $\Lambda_{1,9}$ as the direct sum of the two lattices $B$ and $E$, whose basis vectors $\mathbf{b}_i$ and $\mathbf{e}_i$ are:

$$
\begin{align*}
\mathbf{b}_1 &= -K_9, & \mathbf{b}_2 &= c_9, \\
\mathbf{e}_i &= c_i - c_{i+1}, & 1 \leq i \leq 7, \\
\mathbf{e}_8 &= -H + c_6 + c_7 + c_8.
\end{align*}
$$

(6.17)

The element $H$ of basis $\mathbf{C}$ is in terms of this basis: $H = (3, 3, 3, 6, 9, 12, 15, 10, 5, 2)$. The intersection numbers for $\mathbf{b}_i$ are $\mathbf{b}_2^1 = 0$, $\mathbf{b}_2^2 = -1$ and $\mathbf{b}_1 \cdot \mathbf{b}_2 = 1$. The quadratic form $Q_E$ for $E$ is minus the $E_8$ Cartan matrix, which is given by:

$$
\begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 2
\end{pmatrix},
$$

(6.18)

The 256 elements in $E/2E$ fall in 3 inequivalent Weil orbits, and orbits with vectors of length 0, 2 and 4 with multiplicities $m_0 = 1$, $m_1 = 120$ and $m_2 = 135$ respectively. We choose as representatives:

$$
\begin{align*}
\mathbf{e}_0 &= \mathbf{0}, \\
\mathbf{e}_1 &= (1, 0, 0, 0, 0, 0, 0, 0), \\
\mathbf{e}_2 &= (1, 0, 1, 0, 0, 0, 0, 0).
\end{align*}
$$

The corresponding theta functions $\Theta_{rE_8,e_0}$ are for $r = 1, 2$:

$$
\begin{align*}
\Theta_{E_8,e_0}(\tau) &= E_4(\tau), \\
\Theta_{2E_8,e_0}(\tau) &= E_4(2\tau), \\
\Theta_{2E_8,e_1}(\tau) &= \frac{1}{240} \left( E_4(\tau/2) - E_4(\tau/2 + 1/2) \right), \\
\Theta_{2E_8,e_2}(\tau) &= \frac{1}{15} \left( E_4(\tau) - E_4(2\tau) \right).
\end{align*}
$$
6.2 BPS invariants for $r \leq 3$

**Rank 1**
The results from the periods for $h_C(\tau)$ is (6.1):

$$h_C(\tau) = \frac{E_4(\tau)}{\eta(\tau)^{12}}.$$  \hfill (6.19)

This can easily be verified with the results for sheaves on surfaces. The result for $r = 1$ and a surface $S$ is [25]:

$$h_{1,c_1}(z, \tau; S) = \frac{i}{\theta_1(2z, \tau) \eta(\tau)^{b_2(S)-1}}$$ \hfill (6.20)

The dependence on $J$ can be omitted for $r = 1$ since all rank 1 sheaves are stable. If we specialize to $S = \mathbb{F}_9$, take the limit $w \to -1$, and sum over all $c_1 \in E$ one obtains Eq. (6.19).

**Rank 2**
The prediction by the periods for $r = 2$ is given by $h_{2C}(\tau)$ in (6.1). This is a sum over all BPS invariants for $c_1 \cdot a_i = 0$, $i = 1, 2$. In order to verify this result, it is useful to decompose $h_{2C}(\tau)$ according to the three conjugacy classes of $E/2E$: $h_{2C}(\tau) = \sum_{i=0,1,2} m_i h_{2,e_i}(\tau) \Theta_{2E_8,e_i}(\tau)$. One obtains [60]:

$$h_{2,e_0}(\tau) = \frac{1}{24 \eta(\tau)^{24}} \left[ E_2(\tau) \Theta_{2E_8,e_0}(\tau) + \left( \theta_3(\tau)^4 \theta_4(\tau)^4 - \frac{1}{8} \theta_2(\tau)^8 \right) \left( \theta_3(\tau)^4 + \theta_4(\tau)^4 \right) \right],$$

$$h_{2,e_1}(\tau) = \frac{1}{24 \eta(\tau)^{24}} \left[ E_2(\tau) \Theta_{2E_8,e_1}(\tau) - \frac{1}{8} E_4(\tau) \theta_2(\tau)^4 \right],$$  \hfill (6.21)

$$h_{2,e_2}(\tau) = \frac{1}{24 \eta(\tau)^{24}} \left[ E_2(\tau) \Theta_{2E_8,e_2}(\tau) - \frac{1}{8} \theta_2(\tau)^8 \left( \theta_3(\tau)^4 + \theta_4(\tau)^4 \right) \right].$$

Verification of these expressions is much more elaborate then for $r = 1$. We will use the approach of [62, 63, 65]. The main issues are:

- determination of the BPS invariants for a polarization close to the class $a_2$ (a suitable polarization)

- wall-crossing from the suitable polarization to $J = -K_9 = a_1$.  

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These issues are dealt with for the Hirzebruch surfaces \([62, 63]\), and for \(\mathbb{F}_9\) in \([65]\). The main difficulty for \(\mathbb{F}_9\) compared to the Hirzebruch surfaces is that the class \(f\) and \(K_9\) span the lattice \(A\), which is related to \(\Lambda_{1,9}\) by a non-trivial gluing with the lattice \(D\).

Before turning to the explicit expressions, we briefly outline the computation; we refer for more details about the used techniques to \([56]\). The polarization \(J\) is parametrized by \(J_{m,n} = ma_1 + na_2\). In order to determine the BPS invariants for the suitable polarization \(J_{\varepsilon,1}\), view \(\mathbb{F}_9\) as the 8-point blow-up of the Hirzebruch surface \(\mathbb{F}_1\): \(\phi: \mathbb{F}_9 \to \mathbb{F}_1\). We choose to perform this blow-up for the polarization \(J_{\varepsilon,1} = f\), with \(f\) the fibre class of the Hirzebruch surface. The pull back of this class to \(\mathbb{F}_9\) is \(\phi^*f = J_{0,1}\).

The generating function of the BPS invariants for this choice takes a relatively simple form: it either vanishes or equals a product of eta and theta functions \([63, 56]\) depending on the Chern classes. This function represents the sheaves whose restriction to the rational curve \(a_2\) is semi-stable. The generating function \(h_{r,c_1}(z, \tau; \mathbb{F}_9, J_{0,1})\) is therefore this product formula multiplied by the factors due to blowing-up the 8 points. To obtain the BPS invariants from this function, one has to change \(J_{0,1}\) to \(J_{\varepsilon,1}\) and subtract the contribution due to sheaves which became (Gieseker) unstable due to this change \([56]\). Consequently, we can determine the BPS invariants for any other choice of \(J\) by the wall-crossing formula \([64, 64, 37]\). In particular, we determine the invariants for \(J_{1,0} = -K_9\) and change to the basis \(E\) in order to compare with the expression from the periods.

We continue with determining the BPS invariants for \(J = J_{0,1}\). The BPS invariants vanish for \(c_1 \cdot a_2 = 1 \mod 2\):

\[
h_{2,c_1}(z, \tau; \mathbb{F}_9, J_{\varepsilon,1}) = 0, \quad c_1 \cdot a_2 = 1 \mod 2. \quad (6.22)
\]

Since BPS invariants depend on \(c_1 \mod 2\Lambda_{1,9}\), we distinguish further \(c_1 \cdot a_2 = 0 \mod 4\) and \(c_1 \cdot a_2 = 2 \mod 4\). For these cases, we continue as in \([56]\) using the (extended) Harder-Narasimhan filtration. A sheaf \(F\) which is unstable for \(J_{\varepsilon,1}\) but semi-stable for \(J_{0,1}\), can be described as a HN-filtration of length 2 whose quotients we denote by \(E_i\), \(i = 1, 2\). If we parametrize the first Chern class of \(E_2\) by \(k = (k_A, k_D)\), then the
discriminant $\Delta(F)$ is by:

$$2\Delta(F) = \Delta(E_1) + \Delta(E_2) - \frac{1}{4}(2k_A - c_1|A)^2 - \frac{1}{4}(2k_D - c_1|D)^2.$$  \hfill (6.23)

The choice of $k_D$ does not have any effect on the stability of $F$ as long as $J$ is spanned by $J_{0,1}$ and $1,0$. Therefore \[6.23\] shows that the sum over $k_D$ gives rise to the theta functions $\Theta_{2D_8, \mu}(\tau)$. The condition for semi-stability for $J_{0,1}$ but unstable for $J_{\epsilon,1}$ implies $(c_1(E_1) - c_1(E_2)) \cdot a_2 = 0$. This combined with $c_1 \cdot a_2 = 0 \mod 4$ gives for $c_1(E_i) = 0 \mod 2$, which shows that $c_1(E_i) = g_j \mod 2\Lambda_{1,0}$ only for $j = 0, 2$. One obtains after a detailed analysis for $c_1 \cdot a_2 = 0 \mod 4:

$$h_{2,c_1}(z, \tau; F_9, J_{\epsilon,1}) = \frac{-i \eta(\tau)}{\theta_1(2z, \tau)^2 \theta_1(4z, \tau)} \prod_{i=1}^{8} B_{2,\ell_i}(z, \tau)$$

\begin{align*}
&+ \left( \frac{w^4((\frac{1}{2}g_0 - \frac{1}{4}c_1) \cdot a_1)}{1 - w^4} - \frac{1}{2} \delta_0,((\frac{1}{2}g_0 - \frac{1}{4}c_1) \cdot a_1) \right) \Theta_{2D_8, c_1 - 2g_0}(\tau) h_{1,0}(z, \tau)^2 \\
&+ \left( \frac{w^4((\frac{1}{2}g_2 - \frac{1}{4}c_1) \cdot a_1)}{1 - w^4} - \frac{1}{2} \delta_0,((\frac{1}{2}g_2 - \frac{1}{4}c_1) \cdot a_1) \right) \Theta_{2D_8, c_1 - 2g_2}(\tau) h_{1,0}(z, \tau)^2,
\end{align*}

where $\{\lambda\} = \lambda - \lfloor \lambda \rfloor$ and $\ell_i = c_1 \cdot c_i$. The right hand side on the first line correspond to the sheaves whose restriction to $a_2$ are semi-stable. The functions $B_{2,\ell}(z, \tau) = \sum_{n \in \mathbb{Z} + \ell/2} q^n w^n / \eta(\tau)^2$ are due to the blow-up formula \[64, 27, 45, 50]. The second and third line are the subtractions due to sheaves which are unstable for $J_{\epsilon,1}$.

Similarly one obtains for $c_1 \cdot a_2 = 2 \mod 4$:

$$h_{2,c_1}(z, \tau; J_{\epsilon,1}) = \frac{-i \eta(\tau)}{\theta_1(2z, \tau)^2 \theta_1(4z, \tau)} \prod_{i=1}^{8} B_{2,\ell_i}(z, \tau)$$

\begin{align*}
&+ \left( \frac{w^4((\frac{1}{2}g_1 - \frac{1}{4}c_1) \cdot a_1)}{1 - w^4} - \frac{1}{2} \delta_0,((\frac{1}{2}g_1 - \frac{1}{4}c_1) \cdot a_1) \right) \Theta_{2D_8, c_1 - 2g_1}(\tau) h_{1,0}(z, \tau)^2 \\
&+ \left( \frac{w^4((\frac{1}{2}g_3 - \frac{1}{4}c_1) \cdot a_1)}{1 - w^4} - \frac{1}{2} \delta_0,((\frac{1}{2}g_3 - \frac{1}{4}c_1) \cdot a_1) \right) \Theta_{2D_8, c_1 - 2g_3}(\tau) h_{1,0}(z, \tau)^2.
\end{align*}

What remains is to change the polarization $J$ from $J_{\epsilon,1}$ to $J_{1,0}$ and determine the change of the invariants using wall-crossing formulas. For $J = (m, n, 0) \in A \oplus D$, we

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obtain the following expression:

\[
h_{2,c_1}(z, \tau; J_{m,n}) = \frac{-i \eta(\tau)}{\theta_1(2z, \tau)^2 \theta_1(4z, \tau)} \prod_{i=1}^{s} B_{2,\ell_i}(z, \tau) \\
+ \sum_{j=0, \ldots, 3} h_{2,c_1-2g_j}(z, \tau; J_{m,n}) \Theta_{2D,c_1-2g_j}(\tau).
\]  

(6.26)

with

\[
h_{2,c_1}^A(z, \tau; J_{m,n}) = h_{2,c_1}^A(z, \tau; J_{\varepsilon,1}) + \frac{1}{2} \sum_{(a_1,a_2) \in A + c_1} \frac{1}{2} \left( \text{sgn}(a_1n + a_2m) - \text{sgn}(a_1 + a_2\varepsilon) \right) \times (w^{4a_2} - w^{-4a_2}) q^{-4a_1a_2} h_{1,0}(z, \tau)^2,
\]

The functions \( h_{2,c_1}^A(z, \tau; J_{\varepsilon,1}) \) are rational functions in \( w \) multiplied by \( h_{1,0}(z, \tau)^2 \) which can easily be read off from Eq. (6.24). For \( J = J_{1,0} \) the functions can be expressed in terms of modular functions.

Table 3 presents the BPS invariants for \( J = J_{1,0} \). As expected, the Euler numbers are indeed in agreement with the predictions (6.21). One can also verify that for increasing \( c_2 \), the Betti numbers asymptote to those of \( r = 1 \) or equivalently the Hilbert scheme of points of \( \mathbb{F}_9 \).

| \( c_1 \) | \( c_2 \) | \( b_0 \) | \( b_2 \) | \( b_4 \) | \( b_6 \) | \( b_8 \) | \( b_{10} \) | \( b_{12} \) | \( b_{14} \) | \( b_{16} \) | \( \chi \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \( e_0 \) | 2 | 1 | 10 | 55 | | | | | | | 132 |
| 3 | 1 | 11 | 76 | 396 | 1356 | | | | | | 3680 |
| 4 | 1 | 11 | 78 | 428 | 1969 | 7449 | 20124 | | | | 60120 |
| 5 | 1 | 11 | 78 | 430 | 2012 | 8316 | 30506 | 95498 | 221132 | 715968 |
| \( e_1 \) | 1 | 1 | 9 | | | | | | | | 20 |
| 2 | 1 | 11 | 75 | 309 | | | | | | | 792 |
| 3 | 1 | 11 | 78 | 426 | 1843 | 5525 | | | | | 15768 |
| 4 | 1 | 11 | 78 | 430 | 2010 | 8150 | 27777 | 68967 | | | 214848 |
| \( e_2 \) | 1 | 1 | 1 | | | | | | | | 2 |
| 2 | 1 | 11 | 60 | | | | | | | | 144 |
| 3 | 1 | 11 | 78 | 404 | 1386 | | | | | | 3760 |
| 4 | 1 | 11 | 78 | 430 | 1981 | 7495 | 20244 | | | | 60480 |

Table 3: The Betti numbers \( b_n \) (with \( n \leq \dim_{\mathbb{C}} \mathcal{M} \)) and Euler numbers \( \chi \) of the moduli spaces of semi-stable sheaves on \( \mathbb{F}_9 \) with \( r = 2, c_1 = e_i, \) and \( 1 \leq c_2 \leq 4 \) for \( J = J_{1,0} \).
We define the functions \( h_{2,c_1}^A(z,\tau) := h_{2,c_1}^A(z,\tau;J_{1,0}) \), which only depend on \( c_1|_A = \alpha_1 a_1 + \alpha_2 a_2 \) with \( \alpha_1,\alpha_2 \in 0, \frac{1}{2}, 1, \frac{3}{2} \). One finds for \( \alpha_2 = 0 \mod 4 \):

\[
\frac{h_{2,c_1}^A(z,\tau)}{h_{1,0}(z,\tau)^2} = -\frac{1}{8\pi i} \frac{\partial}{\partial z} \ln \left( \theta_1(4\tau,4z+2\alpha_1) \theta_1(4\tau,4z-2\alpha_1) \right), \tag{6.27}
\]

and for \( \alpha_2 \neq 0 \mod 4 \) using \( [C.7] \):

\[
\frac{h_{2,c_1}^A(z,\tau)}{h_{1,0}(z,\tau)^2} = \frac{i}{2} \frac{q^{-\alpha_1\alpha_2} \eta(4\tau)^3}{\theta_1(4\tau,2\alpha_2\tau)} \left( \frac{w^{-2\alpha_2} \theta_1(4\tau,4z+2(\alpha_1-\alpha_2)\tau)}{\theta_1(4\tau,4z+2\alpha_1\tau)} - \frac{w^{2\alpha_2} \theta_1(4\tau,-4z+2(\alpha_1-\alpha_2)\tau)}{\theta_1(4\tau,4z+2\alpha_1\tau)} \right). \tag{6.28}
\]

To prove the agreement of the Euler numbers with the periods, we specialize to \( w = -1 \). Let \( D_k = \frac{1}{2\pi i} \frac{\partial}{\partial \tau} - \frac{k}{12} E_2(\tau) \) be the differential operator which maps weight \( k \) modular forms to modular forms of weight \( k+2 \). Then one can write \( h_{2,c_1}(\tau;J_{1,0}) \) as:

\[
h_{2,c_1}(\tau;J_{1,0}) = \frac{1}{\eta(\tau)^{24}} \left( \frac{1}{2} \delta_{c_1,a_2,0} D_4(\theta_3(2\tau)^m \theta_2(2\tau)^{8-m}) \right)
+ \sum_{i=0,...,3} f_{c_1-2g_j}(\tau) \Theta_{2D,c_1-2g_j}(\tau),
\]

with

\[
\begin{align*}
\frac{f_{A,0}(\tau)}{\tau} &= \frac{1}{4} \theta_3(2\tau)^4 + \frac{1}{24} E_2(\tau), \\
\frac{f_{A,1}(\tau)}{\tau} &= \frac{1}{8} \theta_2(2\tau) \theta_3(2\tau)^3, \\
\frac{f_{A,2}(\tau)}{\tau} &= \frac{1}{2} \theta_2(2\tau)^3 \theta_3(2\tau), \\
\frac{f_{A,3}(\tau)}{\tau} &= \frac{1}{12} \theta_2(2\tau)^4 - \frac{1}{24} \theta_3(2\tau)^4 + \frac{1}{24} E_2(\tau), \\
\frac{f_{A,0}(\tau)}{\tau} &= \frac{1}{24} \theta_2(2\tau)^4 - \frac{1}{12} \theta_3(2\tau)^4, \\
\frac{f_{A,1}(\tau)}{\tau} &= -\frac{1}{8} \theta_2(2\tau)^4.
\end{align*}
\]

If \( c_1|_B = 0 \), this reproduces the functions in \([60, 65]\) depending on whether the classes in lattice \( E \) are even or odd.

**Modularity**

Electric-magnetic duality of \( \mathcal{N} = 4 \) \( U(r) \) Yang-Mills theory implies modular properties for its partition function \([61]\). Determination of the modular properties gives therefore insight about the quantum realization of electric-magnetic duality.

The expression in Eq. \((6.26)\) does not transform as a modular form for generic choices of \( J \). However, using the theory of indefinite theta functions \([66]\), the functions can be completed to a function \( \hat{h}_{2,c_1}(z,\tau;J) \) which does transform as a modular form.
Interestingly, Eq. (6.29) shows that $h_{2,c_1}(z, \tau; J)$ becomes a quasi-modular form for $\lim_{J \to J_{1,0}} \tilde{h}_{2,c_1}(z, \tau; J)$, i.e. it can be expressed in terms of modular forms and Eisenstein series of weight 2. In some cases it becomes even a true modular form. This is due to the special form of $Q_A$.

The transition from mock modular to quasi-modular can be made precise. Due to the gluing vectors, the function $f_{2,c_1}(z, \tau; J) = h_{2,c_1}(z, \tau; J)/h_{1,c_1}(z, \tau)^2$ takes the form:

$$f_{2,c_1}(z, \tau; J_{m,n}) = \sum_{\mu} f_{2,(c_1-2\mu)A}(z, \tau; J_{m,n}) \Theta_{2D,(2\mu-c_1)D}(\tau) + \delta_{c_1, a_2, 0} \frac{i \eta(\tau)^3}{\theta_1(\tau, 4z)} \theta_3(2\tau, 2z)^k \theta_2(2\tau, 2z)^{8-k},$$

(6.30)

where $k$ is the number of $c_1 \cdot c = 1 \mod 2$ for $1 \leq i \leq 8$.

The completed generating function $\tilde{f}_{2,c_1}(z, \tau; J)$ is a slight generalization of Eq. (22) in Ref [54].

$$\tilde{f}_{2,c_1}(\tau; J) = f_{2,c_1}(\tau; J) + \sum_{e \in c_1, \mathbb{Z}} \left( \frac{K_9 \cdot J}{4\pi \sqrt{J^2 y}} e^{-\pi y e^2_t} - \frac{1}{4} K_9 \cdot c \sgn(c \cdot J) \beta_{1}^{2} (c^2 y) \right) (-1)^{K_9 \cdot c} q^{-c^2 / 4},$$

(6.31)

We parametrize $J$ by $a_0 + t a_1$, and carefully study the limit $t \to 0$ (this should correspond to $R \to \infty$ in [60]). In this limit, $J$ approaches $-K_9$. Moreover, $J \cdot K_9 = t$ and $J^2 = t(2 - t)$. If one parametrizes $c$ by $(n_0, n_1, c_\perp)$, only terms with $n_1 = 0$ contribute to the sum in the limit $t \to 0$. Therefore the term with $\beta_{\frac{1}{2}}$ does not contribute to the anomaly. After a Poisson resummation on $n_0$, one finds that the limit is finite and given by

$$\tilde{f}_{2,c_1}(\tau; J_{1,0}) = f_{2,c_1}(\tau; J_{1,0}) + \frac{\delta_{c_1, a_1, 0}}{4\pi \Im \tau} \sum_{e \in c_1, \mathbb{Z}} q^{-e^2_t / 4}.$$  

(6.32)

This reproduces the holomorphic anomaly equation discussed in Section ?? for the periods. Thus the non-holomorphic dependence of $D4$-brane partition functions is consistent with the one of topological strings when the systems can be related via $T$-duality. Note that for $c_1 \cdot a_1 = 1 \mod 2$, the non-holomorphic dependence of $f_{2,c_1}(\tau; J)$ vanishes in the limit $J \to J_{1,0}$, in agreement with (6.29).

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6Here we have used the equation for $\beta^{2}_{\frac{1}{2}}(x)$ in terms of $\beta^{2}_{\frac{1}{2}}(x)$ and $e^{-\pi x}$. 

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Rank 3
Similarly as for $r = 2$, Ref. [60] also decomposes $h_{3C}(\tau)$ into different Weyl orbits. We will restrict in the following to the $e_0 = 0$ orbit in $E/3E$ since the expressions become rather lengthy. In order to present $h_{3,e_0}(\tau)$, define

$$b_{3,\ell}(\tau) = \sum_{m,n \in \mathbb{Z} + \ell/3} q^{m^2+n^2+mn}. \quad (6.33)$$

Then $h_{3,e_0}(\tau)$ is given by [60]:

$$h_{3,e_0}(\tau) = \frac{1}{2592 \eta^{36}} \left[ (51b_{3,0}^{12} - 184b_{3,0}^9b_{3,1}^3 + 336b_{3,0}^6b_{3,1}^6 + 288b_{3,0}^3b_{3,1}^9 + 32b_{3,1}^{12}) 
+E_2b_{3,0} \left( 36b_{3,0}^9 - 112b_{3,0}^6b_{3,1}^3 + 32b_{3,0}^6b_{3,1}^3 - 64b_{3,1}^6 \right) 
+E_2^2b_{3,0}^2 \left( 9b_{3,0}^6 - 16b_{3,0}^3b_{3,1}^3 + 16b_{3,0}^6 \right) \right] \quad (6.34)$$

In order to verify this expression, we extend the analysis for $r = 2$ to $r = 3$. For $c_1 \cdot a_2 = \pm 1 \mod 3$ the BPS invariants vanish for a suitable polarization:

$$h_{3,c_1}(z, \tau; J_{e_1}) = 0. \quad (6.35)$$

The HN-filtrations for the sheaves which are unstable for $J_{e_1}$ but semi-stable for $J_{e_1}$ have length 2 or 3. From those of length 2, one obtains rational functions in $w$ multiplied by $h_{1,0}(z, \tau) h_{2,\mu}(z, \tau) \Theta_{2D_8,\mu}(\tau)$, with $\mu = 0, a_2, d_i$ and $d_i + a_2$. The theta function arising from the sum over the $D_8$ lattice is more involved for filtrations of length 3. Instead of a direct sum, a “twisted” sum of 2 $D_8$-lattices appears; we will denote this lattice by $D_8^t$:

$$\Theta_{2D_8^t,\mu_1,\mu_2}(\tau) = \sum_{k_i \in D_8 + \mu_i, i=1,2} q^{k_1^2+k_2^2+k_1 \cdot k_2} \quad (6.36)$$

$$= \sum_i m_i \Theta_{2D_8,\mu_1+\mu_2+d_i}(\tau) \Theta_{2D_8,\mu_1-\mu_2+d_i}(3\tau) \quad (6.37)$$

where $m_i$ are the multiplicities of the theta characteristics $\mu_1 + \mu_2 + d_i$, thus for $\mu_1 + \mu_2 \in D$, $i = 1, \ldots, 6$, and for $\mu_1 + \mu_2 \in D/2$, $i = 1, \ldots, 4$. For numerical computations the second line is considerably faster than the first line. We obtain after
a careful analysis:

\[ h_{3,0}(z, \tau; J_{e,1}) = \frac{i \eta(\tau)^3}{\theta_1(2z, \tau)^2 \theta_1(4z, \tau)^2 \theta_1(6z, \tau)} B_{3,0}(z, \tau)^8 + 2 \left( \frac{1}{1 - w^{12}} - \frac{1}{2} \right) h_{1,0}(z, \tau) \sum_{i=0,3} h_{2,(0,0,d_i)}(z, \tau; J_{e,1}) \Theta_{2D_8,d_i}(3\tau) \]

\[ + 2 \left( \frac{1}{1 - w^6} - \frac{1}{2} \right) h_{1,0}(z, \tau) \sum_{i=0,3} h_{2,(0,1,d_i)}(z, \tau; J_{e,1}) \Theta_{2D_8,d_i}(3\tau) \]

\[ + 2 \left( \frac{1}{1 - w^6} - \frac{1}{2} \right) h_{1,0}(z, \tau) \sum_{i=0,3} m_{0,1} h_{2,(0,0,d_i)}(z, \tau; J_{e,1}) \Theta_{2D_8,d_i}(3\tau) \]

\[ + 2 \left( \frac{1}{1 - w^6} - \frac{1}{2} \right) h_{1,0}(z, \tau) \sum_{i=0,1,2,4,5} m_{2,i} h_{2,g_2+d_i}(z, \tau; J_{e,1}) \Theta_{2D_8,g_2+d_i}(3\tau) \]

\[ - \left( \frac{1 + w^{12}}{(1 - w^8)(1 - w^{12})} - \frac{1}{1 - w^{12}} + \frac{1}{6} \right) \Theta_{2D_8',0,0}(\tau) h_{1,0}(z, \tau)^3 \]

\[ - 2 \left( \frac{w^6}{(1 - w^4)(1 - w^{12})} - \frac{w^6}{1 - w^{12}} \right) \Theta_{2D_8,g_2,0}(\tau) h_{1,0}(z, \tau)^3 \]

\[ - \left( \frac{w^{12} + w^{16}}{(1 - w^8)(1 - w^{12})} \right) \Theta_{2D_8',0,0}(\tau) h_{1,0}(z, \tau)^3 \]

The functions due to the blowing-up of 8 points are now given by

\[ B_{3,k}(z, \tau) = \sum_{m,n \in \mathbb{Z}^2} q^{m^2+n^2+mn} w^{4m+2n} / \eta(\tau)^3. \]

We have used in (6.38) that \( h_{2,c_1}(z, \tau; J_{m,n}) \) only depends on the conjugacy class of \( c_1 \) in \( D/2D \), and moreover that \( h_{2,c_1}(z, \tau; J_{m,n}) = h_{2,c_1'}(z, \tau; J_{m,n}) \) if \( c_1 = (0,0,d_i) \) and \( c_1' = (0,1,d_i) \) for \( i = 1, 2, 4, 5 \) (but not for \( i = 0, 3 \)) and \( c_1 = (0,0,d_i) + g_2 \) and \( c_1' = (0,1,d_i) + g_2 \).

Having determined \( h_{3,0}(z, \tau; J_{e,1}) \), what rests is to perform the wall-crossing from \( J_{e,1} \) to \( J_{1,0} \). To this end we define:

\[ h_{3,c_1}^A(z, \tau; J) = \sum_{a=c_1|A \mod 2A} \frac{1}{2} (\text{sgn}(a_1 n + a_2 m) - \text{sgn}(a_1 + a_2 \varepsilon)) \]

\[ \left( w^{6a_2} - w^{-6a_2} \right) q^{-3a_1 a_2} h_{2,(a,c_1|D)}(z, \tau; J_{[a_1],[a_2]}) h_{1,0}(z, \tau), \quad (6.39) \]

with \( a = (a_1, a_2) \). Then \( h_{3,0}(z, \tau; J) \) is given by [51, 52]:

\[ h_{3,0}(z, \tau; J) = h_{3,0}(z, \tau; J_{e,1}) + \sum_{a \in 2A/A} m_{i,j} h_{3,a+g_i+d_j}^A(z, \tau; J) \Theta_{2D_8,g_i+d_j}(3\tau). \]

The Betti numbers for \( J = J_{1,0} \) and small \( c_2 \) are presented in Table 4 and indeed agree with the Euler numbers computed from the periods.
Table 4: The Betti numbers $b_n$ (with $n \leq \dim \mathcal{M}$) and the Euler number $\chi$ of the moduli spaces of semi-stable sheaves on $\mathbb{P}_9$ with $r = 3$, $c_1 = 0$, and $3 \leq c_2 \leq 5$ for $J = J_{1,\varepsilon}$.

One might wonder how to derive the modular properties $h_{3,0}(z, \tau; J)$. The completion takes in general a very complicated form due to the quadratic condition on the lattice points [52]. One can show however that for $J = J_{1,0}$ the quadratic condition disappears from the generating function due to a special symmetry of the lattice $A$, and therefore one again obtains quasi-modular forms at this point.

A Toric data for the elliptic hypersurfaces

Here we collect the toric data necessary to treat all models discussed. We list the Mori cones in the star triangulation for the bases of model 8-15 of figure 1.

We thank S. Zwegers for providing this argument.
The simplicial Mori cone for the model 15 and 16 occur e.g. for the triangulation depicted here

![Triangulations](image)

Figure 4: Nonstar triangulations of the basis of model 15 and 16, which lead to simplicial Kähler cone for the Calabi-Yau space
For the model 15 the moricone reads
\[ l^{(c)} = (-6, 0, 0, 0, 1, -1, 1, 0, 0, 2, 3), \quad l^{(1)} = (0, -2, 1, 0, 0, 0, 0, 1, 0, 0, 0) \]
\[ l^{(2)} = (0, 1, -1, 1, 0, 0, 0, 0, -1, 0, 0), \quad l^{(3)} = (0, 0, 1, -2, 1, 0, 0, 0, 0, 0, 0), \]
\[ l^{(4)} = (0, 0, 0, 1, -1, 1, 0, 0, 0, -1, 0, 0), \quad l^{(5)} = (0, 0, 0, 0, 0, -1, 0, 1, -1, 1, 0, 0), \]
\[ l^{(6)} = (0, 0, 0, 0, 0, 1, -2, 1, 0, 0, 0) \]
This yields the intersection numbers
\[ \mathcal{R} = 4J^3_e + 2J^2_eJ_2 + 4J^2_eJ_3 + J_eJ_2J_3 + 2J_eJ_2J_4 + 2J_eJ_3J_4 + J_eJ^2_4 + 2J^2_eJ_5 + J_eJ_2J_5 + 2J_eJ_3J_5 + J_eJ_4J_5 + 6J^2_eJ_6 + 2J_eJ_2J_6 + 4J_eJ_3J_6 + J_2J_3J_6 + 2J^2_eJ_6 + 3J_eJ_4J_6 + J_2J_4J_6 + 2J_3J_4J_6 + J^2_4J_6 + 2J_eJ_5J_6 + J_2J_5J_6 + 2J_3J_5J_6 + J_4J_5J_6 + 6J^2_eJ_6 + 2J_2J^2_e + 4J^3_eJ_6 + 3J_4J^2_6 + 2J^2_5J_6 + 6J^3_6 + 5J^2_7 + 2J_eJ_2J_7 + 4J_eJ_3J_7 + J_2J_3J_7 + 2J^2_3J_7 + J_3J_4J_7 + J_2J_4J_7 + 2J_3J_4J_7 + J^2_4J_7 + 2J_eJ_5J_7 + J_2J_5J_7 + 2J_3J_5J_7 + J_4J_5J_7 + 6J_6J_7 + 2J_2J_6J_7 + 4J_3J_6J_7 + J_4J_6J_7 + 3J_4J_6J_7 + 2J_5J_6J_7 + 6J^2_eJ_7 + 5J_eJ^2_7 + 2J^2_2J_7 + 4J^3_eJ_7 + 3J_4J^2_7 + 2J_5J^2_7 + 6J_6J^2_7 + 5J^2_7 \]
and the evaluation of \( c_2 \) on the basis \( J_i \)
\[ c_2J_e = 52, \quad c_2J_1 = 24, \quad c_2J_2 = 48, \quad c_2J_3 = 36, \quad c_2J_4 = 24, \quad c_2J_5 = 72, \quad c_2J_6 = 62. \]
The same data for the model 16
\[ l^{(c)} = (-6, 0, 0, 0, 1, -1, 1, 0, 0, 0, 2, 3), \quad l^{(1)} = (0, -2, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0), \]
\[ l^{(2)} = (0, 1, -2, 1, 0, 0, 0, 0, 0, 0, 0, 0), \quad l^{(3)} = (0, 0, 0, -1, 1, 0, 0, -1, 0, 0, 1, 0, 0), \]
\[ l^{(4)} = (0, 0, 0, 1, -2, 1, 0, 0, 0, 0, 0, 0, 0), \quad l^{(5)} = (0, 0, 0, 0, 1, -2, 1, 0, 0, 0, 0, 0, 0), \]
\[ l^{(6)} = (0, 0, 0, 0, 0, 1, -2, 1, 0, 0, 0, 0, 0), \quad l^{(7)} = (0, 1, 0, 0, 0, 0, 0, 0, 1, -1, -1, 0, 0) \]
and the intersection by
\[ \mathcal{R} = 3J^3_e + 4J^2_eJ_2 + 2J_eJ^2_2 + 2J^2_eJ_3 + J_eJ_2J_3 + 6J^2_eJ_4 + 4J_eJ_2J_4 + 2J^2_eJ_4 + 2J_eJ_3J_4 + J_2J_3J_4 + 6J^2_eJ_5 + 2J_2J^2_e + 4J_4J^2_5 + 6J^4_5 + 5J^2_eJ_5 + 4J_eJ_2J_5 + 2J^2_eJ_5 + J_4J_5J_5 + 6J_6J_5 + 4J_2J_3J_5 + 2J_3J_4J_5 + 6J^2_4J_5 + 5J_eJ^2_5 + 4J_2J_4J_5 + 2J^2_5J_5 + 6J_4J^2_5 + 5J^2_5 + 4J^2_eJ_6 + 4J_eJ_2J_6 + 2J^2_eJ_6 + 2J_eJ_3J_6 + J_2J_3J_6 + 6J_4J_4J_6 + 4J_2J_4J_6 + J_3J_4J_6 + 6J^2_4J_6 + 5J^2_5J_6 + 4J_eJ^2_6 + 4J_2J^2_6 + 2J_3J^2_6 + \]
53
\[6J_4J_6^2 + 5J_5J_6^2 + 4J_7^3 + 3J_e^2J_7 + 2J_eJ_2J_7 + J_eJ_3J_7 + 3J_eJ_4J_7 + 2J_2J_4J_7 + J_3J_4J_7 + \]
\[3J_e^2J_7 + 3J_eJ_5J_7 + 2J_2J_5J_7 + J_3J_5J_7 + 3J_4J_5J_7 + 3J_5J_5J_7 + 3J_eJ_6J_7 + 2J_2J_6J_7 + J_3J_6J_7 + \]
\[3J_4J_6J_7 + 3J_5J_6J_7 + 3J_eJ_7^2 + J_eJ_7^2 + J_4J_7^2 + J_5J_7^2 + J_6J_7^2 + 6J_e^2J_8 + 4J_eJ_2J_8 + \]
\[2J_eJ_3J_8 + 6J_eJ_4J_8 + 4J_2J_4J_8 + 2J_3J_4J_8 + 6J_4J_5J_8 + 6J_eJ_5J_8 + 4J_2J_5J_8 + 2J_3J_5J_8 + \]
\[6J_4J_5J_8 + 6J_5J_5J_8 + 6J_eJ_6J_8 + 4J_2J_6J_8 + 2J_3J_6J_8 + 6J_4J_6J_8 + 6J_eJ_6J_8 + 6J_5J_6J_8 + 6J_6^2J_8 + 3\]
\[J_eJ_7J_8 + 3J_4J_7J_8 + 3J_5J_7J_8 + 3J_6J_7J_8 + 6J_eJ_8^2 + 6J_4J_8^2 + 6J_5J_8^2 + 6J_6J_8^2\]

(A.6)

and the evaluation on \(c_2\) is

\[c_2J_e = 42, \quad c_2J_1 = 48, \quad c_2J_2 = 24, \quad c_2J_3 = 72, \quad (A.7)\]

\[c_2J_4 = 62, \quad c_2J_5 = 52, \quad c_2J_6 = 36, \quad c_2J_7 = 72.\]

**B  Results for the other fibre types with \(\mathbb{F}_1\) base**

We give some results of the periods for the different fibre types with base \(\mathbb{F}_1\). The corresponding Picard-Fuchs operators read \(^{47}\)

\[\mathcal{L}_{E7} = \theta^2 - 4z(4\theta + 3)(4\theta + 1),\]
\[\mathcal{L}_{E6} = \theta^2 - 3z(3\theta + 2)(3\theta + 1)\]
\[\mathcal{L}_{D5} = \theta^2 - 4z(2\theta + 1)^2\]  \hspace{1cm}  (B.1)

The solutions read as follows

\[\phi_{E7} = \sum_{n \geq 0} \frac{(4n)!}{(n!)^2(2n)!} z^n = 2F_1\left(\frac{3}{4}, \frac{1}{4}, 1, 64z\right),\]
\[\phi_{E6} = \sum_{n \geq 0} \frac{(3n)!}{(n!)^3} z^n = 2F_1\left(\frac{2}{3}, \frac{1}{3}, 1, 27z\right),\]
\[\phi_{D5} = \sum_{n \geq 0} \frac{(2n)!^2}{(n!)^4} z^n = 2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, 16z\right),\]  \hspace{1cm}  (B.2)

with:

\[2F_1(a, b, c; x) = \sum_{n = 0}^{\infty} \frac{(a)_n (b)_n x^n}{(c)_n n!},\]  \hspace{1cm}  (B.3)

where \((a)_n = a(a + 1) \ldots (a + n - 1)\) denotes the Pochhammer symbol.
The $j$-functions read for these read
\[
1728 j_{E7} = \frac{(1 + 192z)^3}{z(1 - 64z)^2} \\
1728 j_{E6} = \frac{(1 + 216z)^3}{z(1 - 27z)^3} \\
1728 j_{D5} = \frac{(1 + 244z + 256z^2)}{z(-1 + 16z)^4}
\]
We collect the expressions for the solutions in terms of modular forms
\[
\phi_{E7}(z(q)) = 1 + 24q + 24q^2 + 96q^3 + \cdots = -E_2(\tau) + 2E_2(2\tau) \\
\phi_{E6}(z(q)) = 1 + 6q + 6q^3 + \cdots = \sum_{m,n\in\mathbb{Z}} q^{m^2+n^2+mn} = \theta_2(\tau)\theta_2(3\tau) + \theta_3(\tau)\theta_3(3\tau) \\
\phi_{D5}(z(q)) = 1 + 4q + 4q^2 + \cdots = \theta_3(2\tau)^2
\]
Following analogous steps presented in section 3.2 one can again proof the holomorphic anomaly equation for genus 0.

C Modular functions

This appendix lists various modular functions, which appear in the generating functions in the main text. Define $q := e^{2\pi i\tau}$, $w := e^{2\pi iz}$, with $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$. The Dedekind eta and Jacobi theta functions are defined by:
\[
\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \\
\theta_1(z, \tau) := i \sum_{r \in \mathbb{Z} + \frac{1}{2}} (-1)^{r-\frac{1}{2}} q^{\frac{r^2}{2}} w^r, \\
\theta_2(z, \tau) := \sum_{r \in \mathbb{Z} + \frac{1}{2}} q^{r^2/2} w^r, \\
\theta_3(z, \tau) := \sum_{n \in \mathbb{Z}} q^{n^2/2} w^n.
\]

We define the indefinite theta function $F(\tau, u, v)$ for $0 < -\text{Im} u / \text{Im} \tau < 1$ and $0 < \text{Im} v / \text{Im} \tau < 1$ [26]
\[
F(\tau, u, v) = \sum_{n\geq0, m>0} q^{mn} e^{2\pi i un + 2\pi i vm} - \sum_{n>0, m\geq0} q^{mn} e^{-2\pi i un - 2\pi i vm} \\
= \sum_{n\geq0, m>0} q^{nm} e^{2\pi i un + 2\pi i vm} - \sum_{n<0, m\leq0} q^{nm} e^{2\pi i un + 2\pi i vm}.
\]
Analytic extension of this function gives:

\[ F(\tau, u, v) = -\frac{\eta(\tau)^3 \theta_1(\tau, u + v)}{\theta_1(\tau, u) \theta_1(\tau, v)}. \]

(C.8)

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