Superconducting Superfluids in Neutron Stars

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Abstract: For treatment of the layers below the crust of a neutron star it is useful to employ a relativistic model involving three independently moving constituents, representing superfluid neutrons, superfluid protons, and degenerate negatively charged leptons. A Kalb Ramond type formulation is used here to develop such a model for the specific purpose of application at the semi macroscopic level characterised by length-scales that are long compared with the separation between the highly localised and densely packed proton vortices of the Abrikosov type lattice that carries the main part of the magnetic flux, but that are short compared with the separation between the neutron vortices.

1 Introduction

The purpose of this article is to present a concise overview of a class of macroscopic relativistic superconducting superfluid models developed\cite{1,2} as a generalisation of previous non conducting relativistic superfluid models\cite{3,4,5} with a view to applications concerning the layers below the crust of a neutron star, which are believed to be well described by three constituent superconducting superfluid models of the kind that was introduced (as a charged generalisation of the Andreev - Bashkin model\cite{6}) for a superfluid mixture) by Vardanyan and Sedrakyan\cite{7}, and that has more recently been further developed (though still using a non-relativistic treatment) by Mendell and Lindblom\cite{8}. The three basic ingredients in a description of this kind are, firstly, a condensate of superfluid neutrons, secondly an independently moving – effectively superconducting – condensate of superfluid protons, and thirdly a negatively charged degenerate leptonic constituent (consisting mainly of electrons but also including a significant proportion of muons) that is of “normal”, i.e. non-superfluid, kind. Such a treatment does not
include thermal effects (whose inclusion would involve a fourth constituent representing entropy) but should nevertheless be applicable as a very good first approximation except during a short lived high temperature phase immediately after the birth of the neutron star.

As the relativistic analogue of the kind of phenomenological description introduced in a Newtonian context by Bekarevich and Khalatnikov\cite{9} (as a generalisation of Landau’s original two-constituent model) it will first be shown how to set up a general category of three-constituent perfectly conducting perfect fluid models of the type that is needed, as a preliminary for the more specific developments that follow. This category\cite{2} includes, as a specialisation, the case in which the neutronic and the protonic constituents are both characterised by strictly irrotational behaviour of the kind that is relevant in neutron stars on a “mesoscopic” scale, meaning a scale large compared with that of the underlying microscopic particle description, but small compared with the macroscopic scale separation between the vortex defects within which the superfluid comportment is locally violated. Models of this irrotational kind are just a specially simple limit within the more general category that is needed for the purpose of treating the superconducting superfluid on a “macroscopic” scale, meaning a scale that is large compared with the separation between vortices.

The present discussion will be focussed on an intermediate “semi macroscopic” scale meaning a scale that is small compared with the spacing between the superfluid neutron vortices (which will be rather widely separated due to the relatively low angular velocity of the star, though they contain quite a lot of energy due to their “global” nature) but large compared with the Abrikosov lattice spacing between the “local” proton vortices, which are expected to be much more numerous in order to carry the rather large magnetic fluxes that are thought to be present.

2 Generic category of 3-constituent superconducting superfluid models

In so far as its contribution to the mass density is concerned, the most important of the the three independent constituents under consideration is that of the superfluid neutrons, with number current 4-vector $n_n^\rho$, say. The second contribution is that of the superconducting protons – which make up a small but significant part of the mass density – with number current four-vector $n_p^\rho$. The third constituent is that of the degenerate non-superconducting background of negatively charged leptons – consisting mainly of electrons, but including also a certain proportion of muons at the high densities under consideration – with a corresponding lepton number number current vector $n_e^\rho$ say. This negatively charged “normal” (i.e. non superconducting) constituent contributes only a very small fraction of the mass density, but it nevertheless has a crucially important role, not just because the corresponding unit vector $u^\rho$ defined by

$$n_e^\rho = n_e u^\rho,$$

$$n_e^2 = -n_e^\rho n_e^\rho$$

(characterises the natural reference frame of rigid corotation in an equilibrium configuration, but more generally, in so far as electromagnetic effects are concerned, because in
terms of the electron charge coupling constant $e$ the corresponding total electric current 4-vector will be given by

$$J^\mu = e(n_p^\mu - n_e^\mu).$$  \hspace{1cm} (2)

It will be convenient to express formulae such as this in a condensed notation system based on the use of the summation convention for “chemical” indices represented by capital Latin letters running over the three relevant values, namely $x=n, p, e$. Using this convention, the equation (2) for the electric current density can be rewritten in the concise form

$$J^\rho = e^n n^n_\rho,$$

(3)

where the charges per neutron, proton, and electron are given respectively by $e^n = 0$, $e^p = e$, and $e^e = -e$.

Since each of the three independent currents involved is conserved, it will be possible to use a Kalb-Ramond formulation in which, instead of imposing the three corresponding conservation laws

$$\nabla_\rho n^\rho_x = 0,$$

(4)

as dynamical equations, they will be obtained as identities by postulating that the currents should have the form

$$n^\rho_x = \nabla_\sigma b^{\rho\sigma}_x,$$

(5)

for corresponding antisymmetric gauge bivector fields $b^{\rho\sigma}_x$ which are physically defined only modulo Kalb Ramond gauge transformations of the form

$$b^{\rho\sigma}_x \rightarrow b^{\rho\sigma}_x + \nabla_\nu \theta^{\nu\rho\sigma}_x.$$

(6)

for arbitrary antisymmetric trivector fields $\theta^{\nu\rho\sigma}_x$.

Since our present treatment will be restricted to the conservative limit in which dissipative effects are neglected, the analysis will be assumed to be expressible in terms of a variational principle based on a Lagrangian density, in which as usual the representation of the electromagnetic field requires the introduction of a Maxwellian gauge 1-form $A_\rho$, in terms of which the gauge invariant electromagnetic field tensor is given by

$$F^{\rho\sigma} = 2\nabla_\rho A_\sigma,$$

(7)

(using square brackets to indicate index antisymmetrisation). The implementation of the Kalb Ramond formulation requires that the set of independent currents $n^\rho_x$ be supplemented by the introduction of a corresponding set of vorticity 2-forms $w^{x\rho\sigma}_x$ each of which is characterised both by an algebraic degeneracy condition of the form

$$w^{\mu\nu}_x w^{\rho\sigma}_x = 0,$$

(8)

and by a closure condition of the form

$$\nabla_\nu w^{x\rho\sigma}_x = 0,$$

(9)

so that each such 2-form $w^{\rho\sigma}_x$ is interpretable as a pullback of a prescribed area measure on a two-dimensional base space. This means that in terms of suitably chosen local
vorticity base space coordinates $\chi^1$, $\chi^2$, the corresponding pair of scalar fields $\chi^1_x$, $\chi^2_x$ induced on the four dimensional spacetime background will specify the vorticities by prescriptions of the form $w_{\rho\sigma} = 2\chi^1_x \chi^2_{\sigma} |^{\rho}$. 

In terms of these quantities, a Lagrangian of the appropriate kind will will be expressible in the generic form

$$\mathcal{L} = \Lambda + J^\rho A_\rho + \frac{1}{2} b_\chi^{\rho\sigma} w_\rho^\chi \sigma,$$

which consists of a pair of gauge dependent coupling terms preceded by a first term $\Lambda$ that is a function only of the relevant gauge independent field quantities, which in addition, of course to the spacetime metric $g_{\rho\sigma}$ and the electromagnetic field tensor $F_{\rho\sigma}$ consist of the three independent currents $n_\chi^\rho$ and the three corresponding vorticity 2-forms $w_\rho^\chi \sigma$. This means that its most general infinitesimal variation will be given by an expression of the form

$$\delta \Lambda = \mu_\chi^\rho \delta n_\chi^\rho + \frac{1}{2} \lambda_\chi^{\rho\sigma} \delta w_\rho^\chi \sigma + \frac{1}{8\pi} \mathcal{H}_\chi^{\rho\sigma} \delta F_{\rho\sigma} + \frac{\partial \Lambda}{\partial g_{\rho\sigma}} \delta g_{\rho\sigma},$$

where the coefficients of the metric variations are not independent of the others but must satisfy a Noether type identity of the form

$$\frac{\partial \Lambda}{\partial g_{\rho\sigma}} = \frac{\partial \Lambda}{\partial g_{\sigma\rho}} = \frac{1}{2} \mu_\chi^\rho n_\chi^\rho g_{\rho\sigma} + \frac{1}{2} \lambda_\chi^{\rho\sigma} w_\rho^\chi \sigma + \frac{1}{16\pi} \mathcal{H}_\rho^{\mu\sigma} F_{\mu\nu}.$$

The triplet of covectorial quantities $\mu_\rho^\chi$ is simply interpretable as representing usual 4-momenta (per particle) of the neutrons, protons, and leptons. The triplet of rather less familiar bivectorial coefficients $\lambda_\rho^{\chi\sigma} = -\lambda_\rho^{\chi\sigma}$ in this expansion characterises the macroscopic anisotropy arising respectively from the concentration of energy and tension in mesoscopic vortices of the neutron and proton superfluids, as a consequence of their vorticity quantisation conditions, in the manner discussed in our previous work on the single constituent model. These new four dimensional bivectorial coefficients replace the three dimensional (space) vectorial coefficients introduced for a similar purpose in a more restricted Newtonian framework by Bekarevich and Khalatnikov. Finally the bivectorial coefficient $\mathcal{H}_\rho^{\sigma\tau} = -\mathcal{H}_\rho^{\sigma\tau}$ will be interpretable as an electromagnetic displacement tensor, in terms of which the total electromagnetic field tensor (12) will be given by an expression of the form

$$F_{\rho\sigma} = \mathcal{M}_{\rho\sigma} + 4\pi \mathcal{M}_{\rho\sigma},$$

in which $\mathcal{M}_{\rho\sigma}$ is what can be interpreted as the magnetic polarisation tensor. In the application that we are considering, this – typically dominant – Abrikosov polarisation contribution $4\pi \mathcal{M}_{\rho\sigma}$ is to be thought of as representing the part of the magnetic field confined in the vortices, while the – typically much smaller – remainder $\mathcal{H}_{\rho\sigma}$ represents the average contribution from the field in between the vortices, which can be expected to vanish by the “Meissner effect” in strictly static configurations, but which can be expected to acquire a non zero value in rotating configurations due to the London mechanism that will be discussed below.
In the application of such a generalised Kalb Ramond type variation principle, the
gauge fields $b^\sigma_X$ and $A_\rho$ are to be considered as free variables, but $n^\rho_X$ and $w^\rho_{X\sigma}$ are not.
Each current $n^\rho_X$ is to be considered as fully determined by the corresponding gauge
bivector $b^\sigma_X$ according to the prescription (5) while each vorticity 2 form $w^\rho_{X\sigma}$ is to be
considered as being determined by corresponding freely chosen scalar base coordinate
pullback fields $\chi^X_\sigma$; $\chi^X_\sigma$, which means that the variation of any vorticity 2 form $w^\rho_{X\sigma}$ will
be determined by a corresponding freely chosen displacement vector field $\xi^\rho_X$ according
to a prescription [5] of the form $\delta w^\mu_{\rho\nu} = -2\nabla_\rho(w^\nu_{\rho\sigma}\xi^\sigma_X)$.

Subject to these rules, the “diamond” variational integrand

$$\diamond \mathcal{L} = \|g\|^{-1/2} \delta \left(\|g\|^{1/2} \mathcal{L}\right) = \delta \mathcal{L} + \frac{1}{2} \mathcal{L} g^{\mu\nu} \delta g_{\mu\nu}. \quad (14)$$

needed for the application of the variational principle will be given by

$$\diamond \mathcal{L} = (\nabla_\rho \pi_X^\rho - \frac{1}{2} w^\rho_{X\sigma}) \delta b^\sigma_X + f^X_\rho \xi^\rho_X + \left(J^\rho - \frac{1}{4\pi} \nabla_\sigma \mathcal{H}^\rho_{X\sigma}\right) \delta A_\rho + \frac{1}{2} T^\mu_{\rho\nu} \delta g_{\mu\nu} + \nabla_\sigma \mathcal{R}^\sigma, \quad (15)$$

in which it is useful to allow for the possibility of varying the background spacetime
metric $g_{\rho\sigma}$, not only for the purpose of dealing with cases in which one may be concerned
with General Relativistic gravitational coupling, but even for dealing with cases in
which one is concerned only with a flat Minkowski background, since, as will be made
explicit below, the effect of virtual virtuations with respect to the relevant curved
or flat background can be used for evaluating the relevant “geometric” stress energy
momentum density tensor $T^\rho_{\rho\sigma}$. The coefficients of the current variations are the usual
gauge dependent total momentum covectors given by

$$\pi^X_\rho = \mu^X_\rho + e^X_A_\rho. \quad (16)$$

The coefficients $f^X_\rho$ of the three independent displacement displacement vector fields
$\xi^\rho_X$ will be interpretable as the effective force densities acting on the corresponding
constituent currents. The generic expression for these force densities can be read out
for the neutrons $X=n$, protons and $X=p$ and leptons $X=e$ respectively as

$$f^n_\rho = (n^\sigma_n + \nabla_\nu \lambda^\nu_{n\sigma}) w^\sigma_{n\rho}, \quad (17)$$

$$f^p_\rho = (n^\sigma_p + \nabla_\nu \lambda^\nu_{p\sigma}) w^\sigma_{p\rho}, \quad (18)$$

$$f^e_\rho = (n^\sigma_e + \nabla_\nu \lambda^\nu_{e\sigma}) w^\sigma_{e\rho}. \quad (19)$$

Although it is of no relevance for the application of the variation principle, it can be
noted for the record that the current appearing in the final divergence term of (15) will
be given by

$$\mathcal{R}^\sigma = \pi^X_\rho \delta b^\rho_{X\sigma} - (b^\rho_{X\sigma} + \lambda^\rho_{X\sigma}) w^\sigma_{\rho\nu} \xi^\nu_X + \frac{1}{2} \pi^X_\rho b^\rho_{X\sigma} g^{\mu\nu} \delta g_{\mu\nu} + \frac{1}{4\pi} \mathcal{H}^{\nu\sigma} \delta A_\nu. \quad (20)$$

An entity of much greater practical interest is the corresponding stress momentum
energy density tensor, which can be seen to be given by

$$T^\rho_{\rho\sigma} = n^\rho_X \mu^X_\sigma + \lambda^\rho_{X\nu\sigma} w^\nu_{X\nu\sigma} + \frac{1}{8\pi} \mathcal{H}^{\nu\rho} F_{\nu\sigma} + \Psi g^\rho_{\sigma}. \quad (21)$$
where the generalised pressure function is given by

\[ \Psi = \Lambda - n^\nu \mu^\nu + \rho_\sigma (\nabla_{\rho} \pi_{\sigma} - \frac{1}{2} w_{\rho\sigma}). \]  

(22)

The last term in (15) will evidently drop out when we impose the condition of invariance with respect to infinitesimal variations of the bivectorial gauge potentials \( b^\rho_\sigma \) and \( \rho_\sigma \) is imposed, a requirement which can be seen from (15) to give field equations of the form

\[ w^\rho_\sigma = 2\nabla_{[\rho} \pi_{\sigma]} = 2\nabla_{[\rho} \mu_{\sigma]} + e^\sigma F_{\rho\sigma}, \]  

(23)

which are evidently equivalent to what in other formulations could be considered just as defining relations for the vorticity two-forms. The remaining field equations obtained from (15) will consist of

\[ \nabla_{\sigma} H^{\rho\sigma} = 4\pi J^{\rho}, \]  

(24)

together with the condition that the force density coefficients should all vanish, i.e.

\[ f^x_\rho = 0 \]  

(25)

for each of the three relevant chemical index values \( x=n, p, e. \)

3 The semi-macroscopic application.

To be more specific, we need to specify the scale of application for which the model is intended. At a mesoscopic level, meaning on scales large compared with the dimensions of individual molecules or Cooper type pairs, but small compared with the intervortex spacing, the appropriately specialised model will be of purely fluid type, meaning that the function \( \Lambda \) should not depend on the vorticity forms \( w^\rho_\sigma \) which implies the vanishing of the Bekarevich - Khalatnikov coefficients, i.e. the restriction \( \lambda^\sigma_\rho = 0. \) A further restriction that applies at this mesoscopic level is that for the superfluid constituents, namely the neutrons and the protons, but not for the degenerate lepton constituent, the corresponding vorticities themselves should be zero, i.e. for \( x \neq e \) we should have \( w^X_\rho_\sigma = 0, \) which is the integrability condition for the corresponding momenta to have the form \( 2\pi^n_\rho = \hbar \nabla_\rho \varphi^n, \) \( 2\pi^p_\rho = \hbar \nabla_\rho \varphi^p \) in which the scalars \( \varphi^n \) and \( \varphi^p \) will be interpretable as the phases of underlying bosonic quantum condensates, with periodicity \( 2\pi, \) and in which the preceding factors of \( 2 \) have been inserted to allow for the fact that the relevant bosons will consist not of single neutrons and protons but of Cooper type pairs thereof.

At a much larger, fully macroscopic scale, involving averaging over large numbers of the neutron and proton vortices (that arise as topological defects of the mesoscopic phase fields) the neutronic and protonic constituents will be characterised by not just by non vanishing effective large scale vorticities \( w^n_\rho_\sigma \neq 0 \) and \( w^p_\rho_\sigma \neq 0, \) but also by non vanishing Bekarevich - Khalatnikov coefficients, \( \lambda^\sigma_\rho \neq 0 \) and \( \lambda^p_\rho \neq 0, \) so that only the degenerate electrons still behave in an effectively fluid manner, in accordance with the restriction

\[ \lambda^e_\rho = 0. \]  

(26)
The purpose of the present article is to focus on an intermediate scale that will be referred to as “semi macroscopic” meaning that it deals with averages over scales that are large compared with the spacing between proton vortices, but small compared with the spacing between the neutron vortices, which are expected to be relatively widely spaced in typical circumstances within neutron stars, whose angular velocities are very low as measured by local physical timescales, whereas their magnetic fields are typically rather large. On such “semi macroscopic” scales the behaviour of the neutrons constituent will not just be of strictly fluid type, meaning that it will be characterised by
\[ \lambda_n^{\sigma \rho} = 0, \]  
but it will also be subject to the mesoscopic superfluidity condition
\[ 2\pi_n^\rho = \hbar \nabla_\rho \varphi^n, \]  
and hence
\[ w_n^{\rho \sigma} = 0, \]  
so that the corresponding dynamical equation, i.e. the requirement that the net force density should vanish, will be automatically satisfied everywhere outside the microscopic cores of the neutron superfluid vortices (which are of “global” type, meaning that their energy would diverge logarithmically in the absence of the “infra red” cut off imposed by the presence of neighbouring vortices). However since we are considering scales large compared with the separation distance between the much more numerous proton vortices (which are of “local” type, meaning that their energy density falls off exponentially on a microscopic lengthscale \( \ell \) whose evaluation will be discussed below) the proton constituent will be characterised not only by non vanishing averaged vorticity, \( w_p^{\rho \sigma} \neq 0 \) but also by a non vanishing Bekarevich - Khalatnikov coefficient, \( \lambda_p^{\sigma \rho} \neq 0 \). This means that the dynamical requirement that the corresponding net force density should vanish will provide a rather complicated dynamical equation, expressible in the form
\[ 2n_p^\sigma \nabla_\sigma \mu_p^{\rho} + en_p^\sigma F_{\sigma \rho} + w_p^{\rho \sigma} \nabla_\nu \lambda_p^{\sigma \nu} = 0, \]  
in which the first term is interpretable as the negative of the Joukovski force density due to the “Magnus effect” acting on the proton vortices, the middle term is the Lorentz force density representing the effect of the magnetic field on the passing protons, while the last term (which was absent in the mesoscopic description) represents the extra force density on the fluid due to the effect of the tension of the vortices. As a consequence of its “normality” property, the leptonic constituent is governed by a dynamical equation of the simpler form
\[ 2n_e^\sigma \nabla_\sigma \mu_e^{\rho} + en_e^\sigma F_{\sigma \rho} = 0. \]  
In view of the highly localised nature of the proton vortices, it is reasonable to suppose that their action contribution should be fully determined just by the Abrikosov lattice density of such vortices, and that the only independent contribution from the
electromagnetic field $F_{\rho\sigma}$ should be the part provided by the residual – weaker but much more widely extended – part of the flux outside the vortex tubes, as given by the field $H^{\rho\sigma}$ that is given by the relation (13), on the understanding that the polarisation contribution $4\pi M^{\rho\sigma}$ represents the part of the flux attributable to the the vortex tubes. This implies that that the gauge independent term $\Lambda$ in the action will be decomposable in the form

$$\Lambda = \Lambda_{MV} + \Lambda_F,$$

where the macroscopic contribution $\Lambda_{MV}$ is required to be functionally independent of $F_{\mu\nu}$, so that it is determined just by the vorticity $w^p_{\rho\sigma}$ and the currents $n^p_{\xi}$, while for constancy with the variational definition (11) there will be no loss of generality in taking the remaining, electromagnetic field dependent, contribution to have the simple quadratic form

$$\Lambda_F = \frac{1}{16\pi} H_{\rho\sigma} H^{\sigma\rho}.$$

which, in the absence of the polarisation contribution $4\pi M^{\rho\sigma}$ in (13), would reduce just to the usual action contribution for an electromagnetic field in vacuum.

Despite of being much more densely packed than the neutron vortices, the fact that (unlike the neutron vortices) the proton vortices are exponentially localised within a microscopic confinement radius means that their mutual interactions (unlike those of the long range interacting neutron vortices) should remain entirely negligible even for extremely high magnetic fields, so that their contribution to the action should be simply proportional to their density. This means that it will be possible to make the further decomposition

$$\Lambda_{MV} = \Lambda_M + \Lambda_V,$$

in which $\Lambda_M$ is entirely independent of the vorticity $w^p_{\rho\sigma}$, so that it depends only on the three independent currents $n^p_{\xi}$, while the remainder $\Lambda_V$ is just linearly proportional to the protonic vorticity density, so that it will be expressible in the form

$$\Lambda_V = \lambda_p w^p,$$

where $w^p$ is the protonic vorticity magnitude as defined by

$$w^p = \sqrt{w^p_{\rho\sigma} w^p_{\rho\sigma}/2},$$

and where, like $\Lambda_M$, the coefficient $\lambda_p$ depends only on the currents $n^p_{\xi}$. On the basis of dimensional considerations – which should be valid provided the Ginzburg landau ratio of the London penetration length $\ell$ that will be discussed below to the relevant Pippard correlation length is not too far from the order of unity value that characterises the Bogomol’nyi limit[12] – it can be anticipated that, as is confirmed by more detailed analysis[13, 14, 10] the Bekarevich - Khalatnikov coefficient $\lambda_p$ will have an order of magnitude given in terms of the relevant charged particle mass, which in the neutron star case under consideration is the proton mass $m_p$ (but which in an ordinary metallic superconductor would be the electron mass $m_e$) by $\lambda \approx \hbar n_p/m_p$.

The appropriate form for the polarisation tensor $4\pi M^{\rho\sigma}$ in (13), can be seen by decomposing the vector potential $A_{\rho}$ as the sum of a gauge dependent contribution
proportional to the proton momentum covector $\pi_p$ and a gauge independent remainder $A_\rho$ in the form

$$A_\rho = \frac{1}{e} \pi_p^\rho + A_\rho,$$

from which, by exterior differentiation, one obtains a corresponding decomposition of the form

$$F_{\rho\sigma} = \frac{1}{e} u_{\rho\sigma}^p + H_{\rho\sigma},$$

with

$$H_{\rho\sigma} = 2 \nabla_{[\rho} A_{\sigma]} .$$

This decomposition has the required form (33) provided one makes the identification

$$\mathcal{M}_{\rho\sigma} = \frac{1}{4\pi e} w_{\rho\sigma}^p,$$

for what will be referred to as the Abrikosov polarisation tensor.

### 4 Phenomenological interpretation.

As discussed in more detail in particular cases [2, 10] (correcting earlier work [13, 14] in which it was overhastily assumed to cancel out) the quantity $4\pi M_{\rho\sigma}$ defined by (40) will be interpretable as representing the part of the magnetic flux confined to the proton vortices, whose action contribution will be included in the term $\Lambda_V$ given by (35), while $H_{\rho\sigma}$ accounts for the remainder of the flux, which will be distributed over the region outside the proton vortices, and whose contribution to the action will be given by (33). The covector $A$ will be given by

$$A_\rho = -\frac{1}{e} \mu_p^\rho,$$

so that it will be obtainable from the equation of state function for $\mu_p^\rho$ as derived from $\Lambda_M$ as a linear combination of the form

$$A_\rho = A_n^\rho + A_p^\rho + A_M^\rho,$$

in which the terms are proportional respectively to the neutron 4-momentum, the “normal” reference state unit vector $u^\rho$ (as specified according to (1) by the leptonic current), and the (semi macroscopic) electric current $J^\rho$, so that they will be expressible as

$$A_n^\rho = \frac{1}{e} \alpha_n^p \pi_n^\rho, \quad A_p^\rho = -\frac{\mu^L}{e} u_\rho, \quad A_M^\rho = -4\pi \ell^2 J_\rho,$$

with proportionality factors that depend (just) on the form of the master function specifying $\Lambda_M$ in terms of the relevant currents. In particular the dimensionless parameter $\alpha$ would be zero if there were no entrainment, but in view of the effect originally predicted by Andreev and Bashkin [1] can be expected [11] to be of the order of unity, while the effective London mass parameter $\mu^L$ will have a magnitude that is the same as that of the
relevant charge carriers, in this case protons, to within a factor comparable with unity, from which it differs by an amount that also depends on the entrainment effect. The third parameter \( \ell \) is interpretable as the relevant London penetration lengthscale that characterises the effective thickness of the individual proton vortices of the Abrikosov lattice. If it is assumed, as most authors have done, that the entrainment effect only couples the neutrons and protons but does not significantly involve the leptonic background (so that the a master function \( \Lambda_M \) of semi separable form\[2\] can be used) then it can be estimated that this length scale will be given roughly as a function of the effective London mass \( \mu_L \) and the lepton number density \( n_e \) (which must be very close to the proton number density) by

\[
\ell^2 \simeq \frac{\mu_L}{4\pi e^2 n_e}.
\]  

(44)

It follows from (43) that there will be a corresponding decomposition

\[
\mathcal{H}_{\rho\sigma} = \mathcal{H}^n_{\rho\sigma} + \mathcal{H}^l_{\rho\sigma} + \mathcal{H}^M_{\rho\sigma},
\]

(45)

with

\[
\mathcal{H}^n_{\rho\sigma} = 2\nabla_{[\mu} A^a_{\sigma]}, \quad \mathcal{H}^l_{\rho\sigma} = 2\nabla_{[\mu} A^l_{\sigma]}, \quad \mathcal{H}^M_{\rho\sigma} = 2\nabla_{[\mu} A^M_{\sigma]},
\]

(46)

in which far as the averaged flux is concerned, the main contribution will typically be that of the London field \( \mathcal{H}^l_{\rho\sigma} \), meaning the part attributable to the rotation of the “normal” (i.e. non-superfluid) negatively charged background. (In an ordinary metallic superconductor the analogous London field contribution arises as a well known consequence of rotation of the positively charged ionic background). In the absence of entrainment, the neutron vortex contribution \( \mathcal{H}^n_{\rho\sigma} \) would vanish. However the expectation\[11\] that the entrainment coefficient \( \alpha_{pn} \) will actually be of the order order unity implies that although it can be expected to be extremely small outside the immediate neighbourhood of a neutron vortex core (with a confinement radius of the same microscopic order of magnitude \( \ell \) as that of a proton vortex) the integrated flux arising from neutron vortex contribution \( \mathcal{H}^n_{\rho\sigma} \) can be expected to be comparable with that provided by the more smoothly distributed (unconfined) London contribution \( \mathcal{H}^l_{\rho\sigma} \).

In normal circumstances, the least important term in the sum (45) will be what we shall refer to as the Meissner residue, meaning the residual contribution \( \mathcal{H}^M_{\rho\sigma} \) arising from the semi-macroscopic current \( J^\mu \) if any. On the assumption that the lengthscales characterising variation of the coefficients \( \mu_L \) and \( \ell \) are very long compared with the London penetration lengthscale \( \ell \) itself, it can be seen that – except within the microscopic defects forming the actual vortex cores where the mesoscopic superfluid description breaks down so that \( w^a_{\rho\sigma} \) is locally non zero – the dominant contribution in the source equation (24) will be the part arising from the semi-macroscopic current \( J^\mu \) itself, so that, as a very good approximation, the source equation will reduce to the well known London form

\[
\nabla^\sigma \nabla_\sigma J^\rho \simeq \frac{1}{\ell^2} J^\mu,
\]

(47)

whose homogeneous linear character entails that the only spatially and temporally uniform solution is that for which the current \( J^\mu \) simply vanishes. What this implies
is that after any initial high frequency oscillations that may have been present have had
time to radiate away or otherwise be dissipated, the medium will tend to settle towards
a state in which the current actually is zero outside a very small radius of order $\ell$
surrounding each individual vortex, so that its average $\langle J^\rho \rangle$ over scales large compared
with $\ell$ will also tend to vanish,

$$\langle J^\rho \rangle \simeq 0.$$  \hfill (48)

The same conclusion therefore applies to the corresponding residual Meissner field con-
tribution, $\mathcal{H}_{\rho\sigma}^M$, which will end up in a state such that

$$\langle \mathcal{H}_{\rho\sigma}^M \rangle \simeq 0.$$  \hfill (49)

This last result is interpretable as a generalisation to “type II” (London - Abrikosov)
superconductors of the Meissner effect that was originally observed in laboratory examples
of “type I”, meaning cases in which the Ginzburg Landau ratio of the penetration
lengthscale $\ell$ to the relevant Pippard correlation lengthscale is so small that, instead
of condensing into an Abrikosov vortex lattice, the magnetic flux tends to be entirely
expelled into domains where the superconductivity breaks down. In a type I situation
the polarisation $\mathcal{M}_{\rho\sigma}$ associated with the Abrikosov lattice will be absent, so there will
be no distinction between the total flux $F_{\rho\sigma}$ and the contribution $\mathcal{H}_{\rho\sigma}$ as defined here,
which means that in this experimentally more familiar case, expulsion of $\mathcal{H}_{\mu\nu}$ is equiv-
alent to complete expulsion of $F_{\rho\sigma}$. On the other hand in the type II case, although
there will be the same tendency to expulsion of $\mathcal{H}_{\rho\sigma}$, this will not entail the complete
expulsion of $F_{\rho\sigma}$ because the Abrikosov polarisation contribution $\mathcal{M}_{\mu\nu}$ will still remain.

Both in the type II and – as originally remarked by London – in the type I case,
the tendency for $\mathcal{H}_{\mu\nu}$ to vanish will be partially thwarted in a rotating background, for
which the London contribution $\mathcal{H}_{\rho\sigma}^L$ will still remain even after the residual Meissner
contribution $\mathcal{H}_{\rho\sigma}^M$ has been dissipated in accordance with (49). In the usual laboratory
applications, whether of type I or type II, this London contribution is all that will
remain, but in the neutron star case there will also be the neutron vortex contribution
$\mathcal{H}_{\rho\sigma}^n$. If the gradient of the (weakly density dependant) coefficient $\alpha_p^\rho$ is not entirely
negligible, then as well as having a dominant part that is of magnetic character, may
also include a small contribution to the electric displacement vector $D_\rho$ as defined with
respect to the “normal” leptonic background frame by the decomposition

$$\mathcal{H}_{\rho\sigma} = H_{\rho\sigma} + 2u_\beta D_{\sigma}^\beta, \quad D_\rho = \mathcal{H}_{\rho\sigma} w^\sigma.$$  \hfill (50)

Since the alignment of the neutron momentum covector will not on average be very
different from that of the background frame, it can be seen that what is to be expected
is that under typical equilibrium conditions the macroscopically averaged value of the
first contribution in (45) will be given by

$$\langle \mathcal{H}_{\rho\sigma}^n \rangle \simeq 2u_\mu \langle D_\sigma^\mu \rangle + \langle H_{\rho\sigma}^n \rangle,$$  \hfill (51)

in which the main contribution is the purely magnetic part given by

$$H_{\rho\sigma}^n \simeq \frac{1}{e} \langle \alpha_\rho^p w_{\rho\sigma}^n \rangle.$$  \hfill (52)
while the – typically very small – electric part will be given by
\[
\langle D^\rho_\alpha \rangle \simeq -\frac{1}{e} \langle \mu^n \nabla_\rho \alpha^p_\alpha \rangle ,
\]
(53)
where the relevant neutron Fermi energy parameter is given by \( \mu^n = -u^\sigma \mu_n^\sigma \).

The analogous macroscopic average of the London contribution can be instructively formulated in terms of the “normal” background’s acceleration tensor \( \dot{u}^\rho \) and rotation tensor \( \Omega^\rho_{\sigma \alpha} \) (whose magnitude \( \Omega = \sqrt{\Omega^\rho_{\sigma \alpha} \Omega^\sigma_{\rho \alpha} / 2} \) is the local angular velocity) as defined by
\[
\nabla_{[\rho} u_{\sigma]} = \Omega^\rho_{\sigma \alpha} - u_{[\rho} \dot{u}_{\sigma]} , \quad \dot{u}^\rho = u^\sigma \nabla_\sigma u^\rho .
\]
(54)
The ensuing result (correcting a misplaced factor of 2 in the preceding presentation[2]) is expressible as
\[
\langle H^L_{\rho \sigma} \rangle \simeq 2 u_{[\rho} \langle D^L_{\sigma]} \rangle + \langle H^L_{\rho \sigma} \rangle ,
\]
(55)
in which the main contribution is the magnetic part which can be seen to be given by
\[
\langle H^L_{\rho \sigma} \rangle = -\frac{2}{e} \langle \mu^{L} \Omega^\rho_{\sigma \alpha} \rangle
\]
(56)
As originally observed by London, this is proportional to the angular velocity \( \Omega \), with a proportionality factor \( \mu^L \) that in the neutron star application will be given roughly (but due to the “entrainment” effect not exactly) by the proton mass (whereas in the ordinary metallic superconductors originally envisaged by London it is given by the electron mass). As well as this well known magnetic contribution, there will also be a corresponding, but typically much less important, electric displacement contribution given by
\[
\langle D^L_\rho \rangle \simeq \frac{1}{e} \langle \mu^{L} \dot{u}_\sigma + \nabla_\sigma \mu^L \rangle .
\]
(57)
(This electric displacement field is usually ignored in discussions of laboratory applications, but even in the ideally simplified case of a motionless incompressible sample that is postulated to be strictly homogeneous so that the gradient term would be absent, a small residual electric field of this type would still be needed to balance the effect on the conducting particles – which in that case would be electrons – of the ordinary terrestrial gravitation field.)

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References

[1] B. Carter, “The canonical treatment of heat conduction and superfluidity in relativistic hydrodynamics”, in A random walk in Relativity and Cosmology (Essays in honour of P.C. Vaidya & A.K. Raychaudhuri), ed. N. Dadhich, J. Krishna Rao, J.V. Narlikar, C.V. Visveshwara, pp 48-62 (Wiley Eastern, Bombay, 1985).

[2] B. Carter, D. Langlois, “Relativistic model for superconducting superfluid mixtures”, Nucl. Phys. B531, pp 478-504 (1998) [gr-qc/9806024].

[3] V.V. Lebedev, I.M. Khalatnikov, “Relativistic hydrodynamics of a superfluid”, Sov. Phys. J.E.T.P. 56, pp 923-930 (1982).

[4] B. Carter, I.M. Khalatnikov, “Momentum, Vorticity, and Helicity in Covariant Superfluid Dynamics”, Ann. Phys. 219, pp 243-265 (1992).

[5] B. Carter, D. Langlois, “Kalb-Ramond coupled vortex fibration model for relativistic superfluid dynamics”, Nuclear Physics B 454, 402-424 (1995) [hep-th/9611082].

[6] A.F. Andreev, E.P. Bashkin, “Three velocity hydrodynamics of superfluid solutions” Sov. Phys. J.E.T.P., 42, 164-646 (1975).

[7] G.A. Vardanyan, D.M. Sedrakyan, “Magnetohydrodynamics of superfluid solutions”, Sov. Phys. J.E.T.P. 54, 919-921 (1981).

[8] G. Mendell, L. Lindblom, “The coupling of charged superfluid mixtures to the electromagnetic field”, Ann. Phys. 205, 110-129 (1991).

[9] I.L. Bekarevich and I.M. Khalatnikov, “Phenomenological derivation of the equations of vortex motion in HelII”, Sov. Phys. J.E.T.P. 13, 643 (1961).

[10] B. Carter, R. Prix, D. Langlois, “Energy of Magnetic Vortices in Rotating Superconductor” Phys. Rev. B62 (2000) [cond-mat/9910240].

[11] M.A. Alpar, S.A. Langer, J.A. Sauls, “Rapid postglitch spin-up of the superfluid core in pulsars”, Astroph. J. 282, 533-541 (1984).

[12] B. Carter, D. Langlois, R. Prix, “Bogomol’nyi limit for magnetic vortices in rotating superconductor”, Phys. Rev. B62 (2000) [cond-mat/9910263].

[13] G. Mendel, “Superfluid hydrodynamics in rotating neutron stars. I Nondissipative equations”, Astroph. J 380, pp 515-529 (1991).

[14] G. Mendel, “Magnetohydrodynamics in superconducting-superfluid neutron stars”, Mon. Not. R. Astron. Soc. 296, pp 903-912 (1998) [astro-ph/9702032].

[15] I. M. Khalatnikov, Introduction to the Theory of Superfluidity (Benjamin, New York, 1965).