Nonsequential positive-operator-valued measurements on entangled mixed states do not always violate a Bell inequality

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We present a local-hidden-variable model for positive-operator-valued measurements (an LHVPOV model) on a class of entangled generalized Werner states. We also show that, in general, if the state $\rho'$ can be obtained from $\rho$ with certainty by local quantum operations without classical communication, then an LHVPOV model for the state $\rho$ implies the existence of such a model for $\rho'$.

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I. INTRODUCTION

It is well known that some quantum states of joint systems are “nonlocal,” meaning that outcomes of measurements performed separately on each subsystem at spacelike separation cannot be reproduced by a local-hidden-variable (LHV) model (see [1] and references contained therein). Such nonlocality can be revealed by a violation of an inequality which any LHV model must satisfy. We call any such inequality a “Bell-type inequality.” More specifically, consider a bipartite state $\rho$ (in this paper, we only consider bipartite states). The two subsystems are spatially separated, one being in the possession of an observer Alice and the other in possession of an observer Bob. If Alice performs a measurement $A$ with an outcome $A_i$ and, at spacelike separation, Bob performs a measurement $B$ with an outcome $B_j$, then an LHV model supposes that the joint probability of getting $A_i$ and $B_j$ is given by

$$\Pr(A_i, B_j|A, B, \rho) = \int d\lambda \omega^\rho(\lambda) \Pr(A_i|A, \lambda) \Pr(B_j|B, \lambda),$$

(1)

where $\omega^\rho(\lambda)$ is some distribution over a space, $\Lambda$, of hidden states $\lambda$. If a Bell-type inequality is violated, then no such model exists.

It is also well known that any entangled pure state will violate some Bell-type inequality and is therefore nonlocal [2,3]. This nonlocality can always be revealed by an appropriate choice of projective measurements to be performed on each subsystem. In light of this, one might conjecture that the same holds true for mixed states, namely that with an appropriate choice of projective measurements, some Bell-type inequality will be violated. The conjecture, however, is false. That it is false was shown by Werner, who wrote down an explicit LHV model for projective measurements performed by Alice and Bob on a class of mixed entangled bipartite states, now known as “Werner states” [4,5] (in fact, he did this before the results of [2,3] were known). The situation became more complicated when Popescu showed that certain of the Werner states (specifically those in $\mathcal{H}_d \otimes \mathcal{H}_d$, where $d \geq 5$) have a “hidden nonlocality” [6]. He showed that if Alice and Bob perform a sequence of measurements consisting of a fixed initial projection onto a two-dimensional subspace followed by a projective measurement (corresponding to a test of the CHSH inequality [7] “within” that subspace) then no LHV model will reproduce the results correctly. (More exactly, no “causal” LHV model can reproduce the results correctly, where “causal” means that the outcome of Alice’s first measurement cannot depend on her choice of which measurement to perform second.) Teufel et al. address the question of classifying different types of nonlocality in some detail [8] (see also [9]). In particular, they demonstrate how some states might only display what they call “deeply hidden nonlocality.” They also give conditions which causal local models have to satisfy that are more involved than that of Eq. (1). Other investigations include [10] and [11].

It is clear from the above that, regarding the relationship between entanglement and nonlocality, the situation is rather more complicated than one might suppose simply from a study of pure states. In considering nonlocality, we have to consider separately the cases in which Alice and Bob can perform positive-operator-valued (POV) measurements on their subsystems and in which they are restricted to projective measurements. We must also consider whether they are allowed sequences of measurements or single measurements only and whether these measurements can be collective, i.e., joint measurements performed on several particle pairs at once, or are restricted to measurements performed separately on each particle pair. In this work, we consider the case in which POV measurements are allowed but Alice and Bob cannot perform sequences of measurements or collective measurements.
A rather natural sounding hypothesis then emerges. It is hinted at by Popescu and raised explicitly by Teufel et al:

**Hypothesis 1** Any entangled quantum state will violate some Bell-type inequality if Alice and Bob can perform single (that is, nonsequential) POV measurements on individual copies of the state.

We show that this hypothesis is false via the construction of an explicit LHV model for POV measurements (an “LHVPOV model”) on a class of generalized Werner states. The model as presented simulates the state \( \lambda \) of an “LHVPOV model” on a class of generalized Werner states. The model as presented simulates the state

\[
\rho = \alpha \frac{2 \rho_{anti}}{d(d-1)} + (1-\alpha) I \frac{I}{d^2},
\]

where

\[
\alpha = \frac{1}{d+1} (d-1)^d - d^d (3d-1).
\]

Here, \( I \) is the identity in \( \mathcal{H}_d \otimes \mathcal{H}_d \) and \( \rho_{anti} \) projects onto the antisymmetric subspace. The state \( \rho \) is entangled if and only if \( \alpha > 1/(1+d) \). With \( \alpha \) defined by Eq. (2), \( \rho \) is entangled for any \( d \geq 2 \). The states originally introduced by Werner were of the form of \( \rho \) but with \( \alpha \) set to \( (d-1)/d \).

We present the model in Sec. II. In Sec. III we show that this model implies the existence of an LHVPOV model for a wide class of other entangled mixed states. Sec. IV concludes.

II. THE MODEL

A. Description

In constructing the model, we take some inspiration from Werner’s original model for projective measurements [1] (it was also inspired by the models of [2] and [3]). The hidden state is a vector in \( d \)-dimensional complex Hilbert space, which we denote by \( | \lambda \rangle \). The distribution of \( | \lambda \rangle \) states, \( \omega(\lambda) \), is invariant under \( U(d) \) rotations. Note that \( | \lambda \rangle \) is a hidden state, not a quantum state; we write it as a ket merely for convenience. A hidden state \( | \lambda \rangle \) defines probabilities for Alice’s and Bob’s measurement outcomes. First, we define rules which work in the case that all POVM elements are proportional to projectors. At the end of this section, we will show that this model implies fairly trivially the existence of a model for all POV measurements. We suppose, then, that Alice performs a measurement \( A_i \), corresponding to a decomposition of the identity \( \sum_i A_i = I \), where \( A_i = x_i P_i \), \( 0 \leq x_i \leq 1 \), and \( P_i \) is a projection operator. Similarly, Bob performs a measurement \( B_j \), where \( B_j = y_j Q_j \).

**Alice.** Restrict attention to those \( A_i \) such that \( \langle \lambda | P_i | \lambda \rangle > 1/d \). Either exactly one of these \( A_i \) will be “accepted” or “rejection” will occur. The probability of \( A_i \) being accepted is given by \( \langle \lambda | A_i | \lambda \rangle \). If \( A_i \) is accepted, the corresponding measurement outcome is obtained. If no \( A_i \) is accepted, then rejection has occurred. In this case, we widen our attention again to the complete set of \( A_i \) and outcome \( i \) obtained with probability \( x_i/d \).

It follows that

\[
\Pr(A_i | A, \lambda) = \langle \lambda | A_i | \lambda \rangle \Theta(\langle \lambda | P_i | \lambda \rangle - 1/d)
\]

\[
+ \left( 1 - \sum_k \langle \lambda | A_k | \lambda \rangle \Theta(\langle \lambda | P_k | \lambda \rangle - 1/d) \right) \frac{x_i}{d},
\]

where \( \Theta \) is the Heaviside step function.

**Bob.** Define

\[
\Pr(B_j | B, \lambda) = \frac{1}{d-1} y_j (1 - \langle \lambda | Q_j | \lambda \rangle).
\]

Substituting Eqs. 1 and 2 into Eq. 1, we get

\[
\Pr(A_i, B_j | A, B, \rho) = \int d\lambda \omega(\lambda) \left[ \langle \lambda | A_i | \lambda \rangle \Theta(\langle \lambda | P_i | \lambda \rangle - 1/d) \right.
\]

\[
+ \left( 1 - \sum_k \langle \lambda | A_k | \lambda \rangle \Theta(\langle \lambda | P_k | \lambda \rangle - 1/d) \right) \frac{x_i}{d}
\]

\[
\times \frac{1}{d-1} y_j (1 - \langle \lambda | Q_j | \lambda \rangle).
\]

We aim to show that this is equal to the quantum prediction: \( \text{Tr} (\rho A_i \otimes B_j) \).

B. Proof that the model works

We define

\[
J_{ij} \equiv x_i y_j \int d\lambda \omega(\lambda) \Theta(\langle \lambda | P_i | \lambda \rangle - 1/d) \langle \lambda | P_i | \lambda \rangle \langle \lambda | Q_j | \lambda \rangle.
\]

We can write Eq. as

\[
\Pr(A_i, B_j | A, B, \rho) = \frac{1}{d-1} \left( -J_{ij} - \frac{1}{d} x_i y_j \int d\lambda \omega(\lambda) \langle \lambda | Q_j | \lambda \rangle \right)
\]

\[
+ \frac{1}{d-1} \left( \frac{x_i}{d} \sum_k J_{kj} \right)
\]

\[
+ \frac{y_j}{d-1} \sum_l \left( J_{jd} + \frac{1}{d} x_i y_l \int d\lambda \omega(\lambda) \langle \lambda | Q_l | \lambda \rangle \right)
\]

\[
- \frac{y_j}{d-1} \sum_l \left( \frac{x_i}{d} \sum_k J_{kl} \right).
\]
\[
\begin{align*}
&= \frac{1}{d^2} x_i y_j + \frac{1}{d-1} \left( -J_{ij} + \frac{x_i}{d} \sum_k J_{kj} \right) \\
&\quad + \frac{1}{d-1} \left( y_j \sum_i J_{ij} - \frac{1}{d} x_i y_j \sum_k J_{kl} \right).
\end{align*}
\]

(8)

We have used the fact that \( \sum_j y_j Q_j = I \).

It remains to calculate \( J_{ij} \). Following Mermin, we write \( |\lambda\rangle = \sum_{\nu=1}^d z_{\nu} |\nu\rangle \), where \( |\nu\rangle \) are an orthonormal basis and \( z_{\nu} = r_{\nu} e^{i\theta_{\nu}} \). Our strategy will be to choose coordinates such that \( J \) is diagonal in \( \nu \), and, again following Mermin, to substitute \( u_{\nu} = r_{\nu}^2 \).

We get

\[
J_{ij} = x_i y_j \int \frac{d\lambda}{\lambda} \omega(\lambda) \Theta(\langle \lambda | P_i | \lambda \rangle - 1/d) \langle \lambda | P_j | \lambda \rangle \langle \lambda | Q_j | \lambda \rangle
\]

\[
= \frac{1}{N} x_i y_j \sum_{\nu=1}^d |\langle q_j | \nu \rangle|^2 \\
\times \int_0^1 du_1 \int_0^1 du_2 \ldots \int_0^1 du_d \delta(u_1 + \cdots + u_d - 1) u_{\nu} \\
= x_i y_j \sum_{\nu=1}^d |\langle q_j | \nu \rangle|^2 J_{\nu},
\]

(9)

where

\[
N = \int_0^1 \ldots \int_0^1 \delta(u_1 + \cdots + u_d - 1),
\]

(10)

\[
Q_j = |q_j \rangle \langle q_j|,
\]

(11)

and

\[
J_{\nu} = \frac{1}{N} \int_0^1 \ldots \int_0^1 \delta(u_1 + \cdots + u_d - 1) u_{\nu}.
\]

(12)

We can use the fact that for \( \nu = 2, \ldots, d \),

\[
J_{\nu} = \frac{1}{d-1} (J_2 + \cdots + J_d) = \frac{J_0 - J_1}{d-1},
\]

(13)

where \( J_0 \) is defined by Eq. (12), setting \( u_0 = 1 \), and

\[
\sum_{\nu=2}^d |\langle q_j | \nu \rangle|^2 = 1 - |\langle q_j | 1 \rangle|^2,
\]

(14)

giving

\[
J_{ij} = x_i y_j \left( J_1 |\langle q_j | 1 \rangle|^2 + \frac{J_0 - J_1}{d-1} (1 - |\langle q_j | 1 \rangle|^2) \right).
\]

(15)

Finally, we have chosen |1\rangle so that \( |1 \rangle \langle 1| = P_1 \), so instead of |1\rangle we now write |p_1\rangle:

\[
J_{ij} = x_i y_j \left( J_1 |\langle q_j | p_1 \rangle|^2 + \frac{J_0 - J_1}{d-1} (1 - |\langle q_j | p_1 \rangle|^2) \right)
\]

\[
= x_i y_j \frac{J_0 - J_1}{d-1} + \alpha x_i y_j |\langle q_j | p_1 \rangle|^2,
\]

(16)

where

\[
\alpha = \frac{d^2 J_1 - dJ_0}{d-1}.
\]

(17)

In calling this quantity \( \alpha \), we are anticipating the fact that it will turn out to be equal to the \( \alpha \) of Eqs. (3) and (8).

Plugging Eq. (16) into Eq. (3), the expression for the correlation predicted by the model, we get, after some algebra and using the facts that \( \sum_i x_i P_i = I \) and \( \sum_j x_j = d \),

\[
\Pr(A_i, B_j | A, B, \rho) = \left( \frac{d-1 + \alpha}{d^2(d-1)} \right) x_i y_j - \frac{\alpha}{d(d-1)} |\langle q_j | p_1 \rangle|^2 x_i y_j.
\]

(18)

It is easy to show that this is in fact equal to the quantum prediction, \( \text{Tr}(\rho A_i \otimes B_j) \), for a generalized Werner state, as defined in Eq. (8) (see, for example, [4,5]). The task now is to find \( \alpha \). To this end, we need to evaluate \( J_0 \) and \( J_1 \). Here we simply state the results:

\[
J_0 = \frac{1}{d} \left( 1 - \frac{1}{d} \right)^{d-1} + \frac{1}{d} \left( 1 - \frac{1}{d} \right)^d,
\]

(19)

\[
J_1 = \left[ \left( \frac{1}{d} \right)^2 + \frac{2}{d} \left( 1 - \frac{1}{d} \right) + \frac{2}{d(d+1)} \left( 1 - \frac{1}{d} \right)^2 \right]
\times \left( 1 - \frac{1}{d} \right)^{d-1}.
\]

(20)

This gives, as promised,

\[
\alpha = \frac{1}{d+1}(d-1)^{d-1}d^{-d}(3d-1).
\]

(21)

There is one thing left to do, which is to show that an LHV model which works when the positive operators are proportional to projectors implies the existence of a model which works for all POV measurements. This follows from the spectral decomposition theorem. Any POVM element, \( A_i \), satisfies \( A_i = A_i^\dagger \) and \( 0 \leq A_i \leq 1 \). It follows that we can write \( A_i = \sum_j c_{ij} P_{ij} \), where the \( c_{ij} \) are real constants such that \( 0 \leq c_{ij} \leq 1 \) and the \( P_{ij} \) are one-dimensional projection operators satisfying \( P_{ij} P_{ij'} = \delta_{ij'} P_{ij} \). If each \( A_i \) is written in this form, then we can regard our observer as performing a more “fine-grained” POV measurement than the one they actually perform, with elements \( c_{ij} P_{ij} \), and our model will make appropriate predictions. If the outcome \( P_{ij} \) is predicted by the model, then we can say that outcome \( A_i \)
is actually obtained. The only remaining wrinkle arises when we consider that we may sometimes have $c_{ij} = c_{ij'}$, where $j \neq j'$. In this case, the spectral decomposition for the operator $A_i$ is not unique. We get around this problem by including in the specification of the LHV model a specification of a map from each such $A_i$ to one of its valid spectral decompositions. The choice of map is arbitrary but must remain fixed for each run of the Bell-type experiment being simulated. (A similar manoeuvre is required in the case of Werner's LHV model for projective measurements on Werner states if we want to be able to predict outcomes for degenerate projective measurements.)

III. EXTENDING THE MODEL

It is interesting to investigate which other entangled states might admit an LHVPOV model. In fact, one can show that, quite generally, an LHVPOV model for the state $\rho_1$ implies the existence of an LHVPOV model for the state $\rho_2$ if

$$\rho_2 = \sum_{ij} M_i \otimes N_j \rho_1 M_i^\dagger \otimes N_j^\dagger,$$

(22)

where $\sum_i M_i^\dagger M_i = I$, $\sum_j N_j^\dagger N_j = I$, and $I$ is the identity. Equivalently, an LHVPOV model for $\rho_1$ implies the existence of an LHVPOV model for $\rho_2$ if $\rho_2$ can be obtained from $\rho_1$ with certainty by local operations (without classical communication). To show this, call the LHVPOV model for $\rho_1$ “model 1.” We aim to define an LHVPOV model (“model 2”) for the state $\rho_2$. We denote probabilities assigned by model 1 by $\Pr^1(\ldots)$ and those assigned by model 2 by $\Pr^2(\ldots)$. Models 1 and 2 will involve the same space of hidden states and the same distribution, $\omega(\lambda)$, over hidden states. We define $\Pr^2(A_i|A, \lambda) = \Pr^1(A_i'|A', \lambda)$, where $A_i' = \sum_k M_i A_k M_k^\dagger$ and $\Pr^2(B_j|B, \lambda) = \Pr^1(B_j'|B', \lambda)$, where $B_j' = \sum_l N_j B_l N_l^\dagger$. The $A_i'$ form a decomposition of the identity and we denote the corresponding measurement by $A'$ (similarly $B_j'$ and $B'$). This ensures that model 2 will make the correct predictions for $\rho_2$ because

$$\int d\lambda \omega(\lambda) \Pr^2(A_i|A, \lambda) \Pr^2(B_j|B, \lambda) = \Tr (A_i' \otimes B_j') \rho_1$$

$$= \sum_{kl} \Tr (M_k A_k M_k^\dagger \otimes N_j B_j N_j^\dagger) \rho_1$$

$$= \sum_{kl} \Tr (A_i \otimes B_j) (M_k \otimes N_j \rho_1 M_k^\dagger \otimes N_j^\dagger)$$

$$= \Tr (A_i \otimes B_j) \rho_2.$$

IV. CONCLUSION

Nonlocality is one of the distinctly nonclassical features of quantum mechanics. In some situations we might view the nonlocality of a quantum state as a resource in much the same way that entanglement is now viewed as a resource. The nonlocality of quantum states thus deserves an investigation paralleling the work done on the quantification and manipulation of entanglement. In addition, we might investigate the relationships between entanglement and nonlocality.

To this end, we have presented a model which simulates arbitrary single POV measurements on single copies of a class of (entangled) generalized Werner states. The hypothesis that any entangled state has nonlocality which can be revealed by single POV measurements on individual copies is thus false. A natural hypothesis which remains unknown is:

Hypothesis 2 Any entangled quantum state can be shown to be nonlocal if arbitrary sequences of POV measurements are allowed on individual copies of the state.

It might be interesting to try to prove this hypothesis false by extending the model above to sequences of measurements.

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