A non-uniform extension of Baranyai’s Theorem

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Abstract

A celebrated theorem of Baranyai states that when \( k \) divides \( n \), the family \( K^k_n \) of all \( k \)-subsets of an \( n \)-element set can be partitioned into perfect matchings. In other words, \( K^k_n \) is 1-factorable. In this paper, we determine all \( n, k \), such that the family \( K^{\leq k}_n \) consisting of subsets of \( [n] \) of size up to \( k \) is 1-factorable, and thus extend Baranyai’s Theorem to the non-uniform setting. In particular, our result implies that for fixed \( k \) and sufficiently large \( n \), \( K^{\leq k}_n \) is 1-factorable if and only if \( n \equiv 0 \) or \( -1 \pmod k \).

1 Introduction

A hypergraph is a system of subsets of finite sets. Formally, a hypergraph \( H = (V, E) \) consists of a vertex set \( V \), and an edge set \( E \) which is a family of non-empty subsets of \( V \). A \( k \)-uniform hypergraph is a hypergraph such that all its edges are of size \( k \). A \( \ell \)-factor of a hypergraph \( H \) is a spanning sub-hypergraph \( H' \) in which every vertex is contained in \( \ell \) edges. We say that \( H \) has a \( \ell \)-factorization if its edge set can be partitioned into \( \ell \)-factors. A hypergraph \( H \) is said to be \( \ell \)-factorable if it admits a \( \ell \)-factorization. There have been extensive research on 1-factorization of graphs (see [1, 10, 11, 16, 17, 19, 21, 23, 24, 25, 26] and the resolution of the 1-factorization conjecture [13]).

We denote the complete \( k \)-uniform hypergraph on \( n \) vertices by \( K^k_n \). Clearly, a necessary condition for \( K^k_n \) to be 1-factorable is \( k \mid n \). It turns out that this is also sufficient for \( k = 2 \) (folklore) and \( k = 3 \) (proved by Peltesohn [22] in 1936). The sufficiency for general \( k \) was eventually established by Baranyai [7] in 1975 as follows.

**Theorem 1.1** (Baranyai [7]). For any positive integers \( k, n \) such that \( k \) divides \( n \), the complete \( k \)-uniform hypergraph \( K^k_n \) can be decomposed into \( \binom{n}{k} k_n = \binom{n-1}{k-1} \) 1-factors.
His proof was based on an ingenious use of the Max-Flow Min-Cut Theorem. Generalizations and extensions of Baranyai’s Theorem can be found in [3, 4, 5, 6, 8, 9, 18].

In this paper, we mainly consider the non-uniform hypergraph $K_{n,k}$, whose vertex set is $[n] = \{1, \ldots, n\}$ and edge set consists of all the non-empty subsets of $[n]$ of size up to $k$, denoted by $\binom{[n]}{\leq k}$. We show that when $k$ is fixed and $n$ is sufficiently large, a necessary and sufficient condition for such hypergraph to be 1-factorable is that $n$ is congruent to 0 or $-1$ modulo $k$. Our result is actually much more precise.

**Theorem 1.2.** For positive integers $n,k$ such that $k < n/2$, $K_{n,k}^\leq k$ is 1-factorable if and only if one of the two following conditions is met:

(i) $n \equiv 0 \pmod{k}$ and $n \geq k(k - 2)$,

(ii) $n \equiv -1 \pmod{k}$ and $n \geq k\left\lceil \frac{k}{2} \right\rceil - 1 - 1$.

The $k \geq n/2$ case can be reduced to the previous range by the following equivalence.

**Theorem 1.3.** For positive integers $n,k$ such that $n/2 \leq k \leq n - 1$, $K_n^{\leq k}$ is 1-factorable if and only if $K_n^{\leq n-k-1}$ is 1-factorable.

Theorems 1.2 and 1.3 together provide a complete characterization of all $n,k$ such that $K_n^{\leq k}$ has a 1-factorization.

The rest of this paper is organized as follows: in the next section, using the Max-Flow Min-Cut Theorem, we show that the 1-factorization problem is equivalent to finding non-negative integer solutions to a system of linear equations given by the partitions of $[n]$ into parts of size up to $k$. Section 3 determines when this equivalent problem is feasible. Section 4 discusses an extension of Theorem 1.2 to other families of subsets obtained from taking the union of multiple levels of the hypercube. The last section contains some concluding remarks and open problems.

## 2 A reduction using network flow

In this section, we reduce the 1-factorization problem of $K_n^{\leq k}$ to finding non-negative integer solutions to a system of linear equations (as we should see soon, both problems in fact are equivalent). Throughout this section, let $n, k$ be two fixed positive integers with $n \geq k$, and let $\mathcal{L}$ be a set consisting of $k$ and some positive integers in $\{1, 2, \ldots, k - 1\}$. We denote by $\binom{[n]}{\mathcal{L}}$ the family of subsets of $[n]$ whose size is an element of $\mathcal{L}$. We will prove a more general reduction for the 1-factorization of $\binom{[n]}{\mathcal{L}}$ which holds for any $\mathcal{L}$ (the case when $\mathcal{L} = [k]$ corresponds to our main results).

For given $n, k$ and $\mathcal{L}$, an $(n, \mathcal{L})$-type, or simply a type is a vector $\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ in $\mathbb{Z}_{\geq 0}^k$ such that $\sum_{j \in \mathcal{L}} j \cdot \lambda_j = n$ and $\lambda_j = 0$ for every $j \in [k]\setminus \mathcal{L}$. If $\mathcal{L} = [k]$, then we also call it an $(n,k)$-type. Let $|\vec{\lambda}| = \sum_{j=1}^k \lambda_k$. We say that $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ is a $\vec{\lambda}$-partition of $[n]$, if $A_i$’s are pairwise disjoint subsets of $[n]$ such that $\bigcup_{i=1}^k A_i = [n]$, and for every $1 \leq j \leq k$, there are $\lambda_j$ subsets $A_i$’s of size $j$.

For given $n, k$, we now define a matrix $A_\mathcal{L}$, which we will soon show to be closely related to the 1-factorization of $\binom{[n]}{\mathcal{L}}$. Let $A_\mathcal{L}$ be the matrix with $k$ columns composed by all admissible
(n, \mathcal{L})$-types $\bar{\lambda}$ as follows:

$$A_{\mathcal{L}} = \begin{pmatrix} \bar{\lambda} \\ \bar{\lambda}' \\ \vdots \end{pmatrix}$$

When $\mathcal{L}$ is clear from the context, we will simply write it for $A$. Below we give an example of $A_{\mathcal{L}}$.

**Example 2.1.** For $n = 7$, $k = 3$ and $\mathcal{L} = [3]$, the matrix $A$ is defined as follows. For example, the third row corresponds to $\bar{\lambda} = (2, 1, 1)$ which is a $(7, 3)$-type because $7 = (2, 1, 1) \cdot (1, 2, 3)^T$.

$$A_{\mathcal{L}} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \\ 4 & 0 & 1 \\ 1 & 3 & 0 \\ 3 & 2 & 0 \\ 5 & 1 & 0 \\ 7 & 0 & 0 \end{pmatrix}$$

For given $n, k$ and $\mathcal{L}$, suppose the hypergraph $(\binom{n}{\mathcal{L}})$ can be decomposed into 1-factors $\mathcal{A}_1, \cdots, \mathcal{A}_m$. Denote the number of $\mathcal{A}_j$’s that are $\bar{\lambda}$-partitions of $[n]$ by $x_{\bar{\lambda}}$. Then the total number of $i$-subsets of $[n]$ appeared in $\{\mathcal{A}_1, \cdots, \mathcal{A}_m\}$ is equal to the $i$-th coordinate of $(A_{\mathcal{L}})^T \bar{x}$, where $\bar{x} = (x_{\bar{\lambda}})$ is a column vector indexed by all the $(n, \mathcal{L})$-types $\bar{\lambda}$. Observe that $(\binom{n}{\mathcal{L}})$ contains exactly $(\binom{n}{i})$ subsets of size $i$ for each $i \in \mathcal{L}$. Thus by setting $b_{\mathcal{L}} = (b_1, \cdots, b_k)^T$ with $b_i = (\binom{n}{i})$ for $i \in \mathcal{L}$ and $b_i = 0$ for $i \in [k]\setminus\mathcal{L}$, we immediately have

$$(A_{\mathcal{L}})^T \bar{x} = b_{\mathcal{L}}. \quad (1)$$

Therefore the system \(\text{(I)}\) having a non-negative integer solution (meaning that all $x_{\bar{\lambda}}$ are non-negative integers) is a necessary condition for $(\binom{n}{\mathcal{L}})$ to be 1-factorable.

Our next theorem shows that this is indeed sufficient. We remark that for Baranyai’s Theorem (corresponding to $n, k$ and $\mathcal{L} = \{k\}$ with $k \mid n$), a non-negative integer solution to the corresponding system exists trivially because the only type involved is $(0, \cdots, 0, n/k)$ and $\frac{n}{k} \mid \binom{n}{k}$.

**Theorem 2.2.** Given positive integers $n, k$ and a set $\mathcal{L}$ of positive integers with $n \geq k$ and $k \in \mathcal{L} \subseteq [k]$, the hypergraph $(\binom{n}{\mathcal{L}})$ is 1-factorable if and only if the system of linear equations \(\text{(I)}\) associated with it has a non-negative integer solution.

We will imitate Baranyai’s ideas and construct a flow network to prove Theorem 2.2. Here we give a brief review of the definition of flow network and the statement of the Max-Flow Min-Cut Theorem of Ford and Fulkerson (1956), to facilitate our later discussion. A network is a finite digraph $D = (V, E)$ together with two distinguished vertices called the source $s$ and the sink $t$, and a capacity function $\kappa : E(D) \to \mathbb{R} \geq 0$ which associates a non-negative real number $\kappa(a)$ to each arc $a \in E(D)$. The source $s$ must be the tail of every arc containing $s$, and the sink $t$ must be the head of every arc containing $t$. We further assume that $D$ does not contain any arc of the form

\[^1\text{It is easy to see that we must have } m = \sum_{j \in \mathcal{L}} \binom{n-1}{j-1}.\]
\(a = (v,v)\) for a vertex \(v \in V\). A flow on \(D\) is a function \(f : E(D) \to \mathbb{R}_{\geq 0}\) which assigns to each arc \(a \in E(D)\) a non-negative real number \(f(a)\) such that

1. (Capacity Constraint) \(0 \leq f(a) \leq \kappa(a)\) for all arcs \(a \in E(D)\);
2. (Conservation of Flow) for each vertex \(v \in V \setminus \{s,t\}\), we have
   \[
   \sum_{a \in E(D): v \text{ is the head of } a} f(a) = \sum_{a \in E(D): v \text{ is the tail of } a} f(a).
   \]

The value of the flow \(f\), denoted by \(|f|\), is the sum of \(f(a)\) over all arcs \(a\) leaving \(s\). A cut \((S,T)\) is a partition of \(V(D) = S \cup T\) such that \(s \in S\) and \(t \in T\). The capacity of a cut \((S,T)\), denoted by \(c(S,T)\), is the sum of the capacities of the arcs which has tail in \(S\) and head in \(T\), that is

\[
c(S,T) = \sum_{xy \in E(D): x \in S, y \in T} \kappa(xy).
\]

**Theorem 2.3** (The Max-Flow Min-Cut Theorem). Given a flow network \(D = (V,E)\), the maximum value of a flow on \(D\) is equal to the minimum capacity over all cuts in \(D\).

We will also utilize the Integral Flow Theorem by Dantzig and Fulkerson [14].

**Theorem 2.4** (The Integral Flow Theorem). If \(D = (V,E)\) is a network in which every arc has integral capacity, then there exists a maximum flow \(f\) on \(D\) such that for each \(a \in E(D)\), \(f(a)\) is an integer.

Now we are ready to prove Theorem 2.2. In the coming proofs, for any integers \(a, b\), the binomial coefficient \(\binom{a}{b}\) is interpreted as zero whenever \(a < 0\), \(b < 0\) or \(a < b\). In particular, we have \(\binom{a}{b} = 1\) if \(b = 0\) and \(\binom{0}{b} = 0\) otherwise. In this way, \(\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b}\) holds for any integers \(a, b\).

**Proof of Theorem 2.2**. Throughout this proof, \(n, k\) and \(\mathcal{L}\) are fixed. From our previous discussions, it suffices to show that if the system \((\mathbf{I})\) has a non-negative integer solution \(\tilde{x} = (x_{\tilde{x}})\) where \(\tilde{x}\) is over all \((n,\mathcal{L})\)-types, then \(\binom{[n]}{\mathcal{L}}\) is 1-factorable.

For a given \((n,\mathcal{L})\)-type \(\tilde{x}\), we slightly extend the definition of a \(\tilde{\lambda}\)-partition of \([n]\) to partitions of \([\ell]\) for any \(0 \leq \ell \leq n\). A partition \(\mathcal{A} = \{A_1, A_2, \ldots, A_t\}\) of \([\ell]\) with \(t = |\tilde{x}|\) is called a \(\tilde{\lambda}\)-partition of \([\ell]\), if for every \(j \in \mathcal{L}\), we assign the label, which we call potential value, \(j\) to exactly \(\lambda_j\) subsets \(A_i\). We point out here that repetitions are allowed only for empty sets. Let

\[
M = \sum_{j \in \mathcal{L}} \binom{n-1}{j-1}.
\]

We will prove the following statement by induction on \(\ell\): for any \(0 \leq \ell \leq n\), there exists a collection of \(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_M\) of partitions of \([\ell]\) such that all of the following hold:

1. each set \(S\) appeared in each partition is associated with a potential value \(j \in \mathcal{L}\) with \(j \geq |S|\),
2. for each \((n,\mathcal{L})\)-type \(\tilde{x}\), there are exactly \(x_{\tilde{x}}\) partitions \(\mathcal{A}_i\) that are \(\tilde{x}\)-partitions, and
for each $S \subseteq [\ell]$ and each $j \in \mathcal{L}$ with $j \geq |S|$, $S$ occurs $\binom{n-\ell}{j-|S|}$ times with potential value $j$ as subsets in the partitions $\mathcal{A}_1, \ldots, \mathcal{A}_M$.

Observe that when $\ell = n$, the third property ensures that every set $S$ in $\binom{[n]}{\ell}$ appears exactly once (only with potential value $j = |S|$), which would provide a 1-factorization of $\binom{[n]}{\ell}$. It would be helpful to view this inductive proof as an evolution where each set (say with potential value $j$) in the partitions grows from an empty set to a set of size $j$ gradually.

Now we start the proof. For the base case when $\ell = 0$, the existence of $\{\mathcal{A}_1, \ldots, \mathcal{A}_M\}$ is given by the non-negative integer solution $\vec{x}$ of the system (1). This is because for each $(n, \mathcal{L})$-type $\vec{\lambda}$ we could construct $x_{\vec{\lambda}}$ partitions formed by taking $|\vec{\lambda}|$ empty sets, and assigning a potential value $j \in \mathcal{L}$ to $\lambda_j$ of them. Note that the total number of partitions is indeed

$$\sum_{\vec{\lambda}} x_{\vec{\lambda}} = \binom{1,2,\ldots,k}{n} \cdot (A_L)^T \vec{x} = \binom{1,2,\ldots,k}{n} \cdot \vec{b}_L = M.$$ 

Now for the inductive step, assume that the statement holds for some $0 \leq \ell \leq n - 1$. We construct the following network.

Let $s$ be the source and $t$ be the sink. Each partition $\mathcal{A}_i$ defines a vertex and we add an arc from the source $s$ to each $\mathcal{A}_i$. For each subset $S \subseteq [\ell]$, in the network we create vertices $S(j)$ for all $j \in \mathcal{L}$ with $j \geq |S|$, where $S(j)$ stands for the set $S$ with potential value $j$. We add an arc from each $S(j)$ to the sink $t$. If $S$ occurs as a subset with potential value $j$ in the partition $\mathcal{A}_i$, then we add to the network an arc from $\mathcal{A}_i$ to $S(j)$. Next we define the capacity function $\kappa$ as follows:

$$\kappa(s, \mathcal{A}_i) = 1, \quad \kappa(\mathcal{A}_i, S(j)) = +\infty, \quad \text{and} \quad \kappa(S(j), t) = \binom{n-\ell-1}{j-1-|S|}.$$ 

Then we define a flow $f$ on this network as follows:

$$f(s, \mathcal{A}_i) = 1, \quad f(\mathcal{A}_i, S(j)) = \frac{j-|S|}{n-\ell}, \quad \text{and} \quad f(S(j), t) = \binom{n-\ell-1}{j-1-|S|}.$$ 

Let us check that $f$ is indeed a flow. It is easy to check that $0 \leq f(a) \leq \kappa(a)$ for every arc $a$ in this network. To see that $f$ satisfies the conservation of flow, we consider the vertices $\mathcal{A}_i$ and $S(j)$ separately:
1. For each \( \tilde{\lambda} \)-partition \( \mathcal{A}_i \) of \( [\ell] \) with \( \tilde{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_k) \), the total value of flow leaving \( \mathcal{A}_i \) is
\[
\sum_{S \in \mathcal{A}_i} j - |S| \cdot \frac{n - \ell}{n - \ell} = \frac{\sum_{j \in L} j \cdot \lambda_j - \sum_{S \in \mathcal{A}_i} |S|}{n - \ell} = \frac{n - \ell}{n - \ell} = 1 = f(s, \mathcal{A}_i);
\]

2. For each \( S^{(j)} \), by the inductive hypothesis, it appears in the partitions \( \{\mathcal{A}_1, \ldots, \mathcal{A}_M\} \) for exactly \( \binom{n - \ell}{j - 1} \) times. So we have the total value of flow entering \( S^{(j)} \) is
\[
\left( \frac{n - \ell}{j - |S|} \right) \cdot \frac{j - |S|}{n - \ell} = \left( \frac{n - \ell - 1}{j - 1 - |S|} \right) = f(S^{(j)}, t).
\]

Since \( f(a) = \kappa(a) \) for all the edges \( a \) leaving the source \( s \), the so-defined \( f \) must be a maximum flow. By Theorem 2.4, there is an integral flow \( f^* \) of the same maximum value. Therefore for each \( \mathcal{A}_i \), we have \( f^*(s, \mathcal{A}_i) = 1 \), and consequently, there is a unique arc \( \mathcal{A}_i \rightarrow S^{(j)} \) with \( f^*(\mathcal{A}_i, S^{(j)}) = 1 \). As for each vertex \( S^{(j)} \), we have \( f^*(S^{(j)}, t) = \binom{n - \ell - 1}{j - 1 - |S|} \), and by the conservation of flow, there are exactly \( \binom{n - \ell - 1}{j - 1 - |S|} \) arcs directed to \( S^{(j)} \) with \( f^*(S^{(j)}, t) = \binom{n - \ell - 1}{j - 1 - |S|} = 0 \), each arc \( a \) directed to \( S^{(j)} \) must have \( f^*(a) = 0 \), hence it is impossible for any of the vertices \( \mathcal{A}_i \) to have the unique arc \( \mathcal{A}_i \rightarrow S^{(j)} \) with \( f^*(\mathcal{A}_i, S^{(j)}) = 1 \).

Finally, we use \( f^* \) to construct a desired collection \( \mathcal{A}_1', \mathcal{A}_2', \ldots, \mathcal{A}_M' \) of partitions of \( [\ell + 1] \). As mentioned above, every \( \mathcal{A}_i \) has a unique \( S^{(j)} \) such that \( f^*(\mathcal{A}_i, S^{(j)}) = 1 \), where \( j > |S| \) by the above discussion. Let \( S' = S \cup \{\ell + 1\} \), and update \( \mathcal{A}_i \) by replacing \( S \) with \( S' \) and assigning to \( S' \) the same potential value \( j \). Note that we have \( j \geq |S'| \). By definition, the new partition \( \mathcal{A}_i' \) is still a \( \tilde{\lambda} \)-partition of \( [\ell + 1] \). So the first and second properties for the new partitions \( \mathcal{A}_1', \ldots, \mathcal{A}_M' \) are satisfied. Since there are \( \binom{n - \ell - 1}{j - 1 - |S|} \) many arcs directed to \( S^{(j)} \), the new set \( S' = S \cup \{\ell + 1\} \) with potential value \( j \) (i.e. \( S'^{(j)} \)) is contained in exactly \( \binom{n - \ell - 1}{j - 1 - |S|} = \binom{n - \ell - 1}{j - 1 - |S|} \) new partitions \( \mathcal{A}_i' \). For those \( S \subset [\ell + 1] \) not containing the element \( \ell \), by induction they occur
\[
\binom{n - \ell}{j - |S|} - \binom{n - \ell - 1}{j - 1 - |S|} = \binom{n - \ell + 1}{j - |S|}
\]
times with potential value \( j \) in the new partitions \( \mathcal{A}_1', \ldots, \mathcal{A}_M' \). This proves the third property for the new partitions \( \mathcal{A}_1', \ldots, \mathcal{A}_M' \). Hence the statement holds for \( \ell + 1 \) and the proof is completed.

3 Finding non-negative integer solutions

After establishing Theorem 2.2 to prove Theorem 1.2, it remains to determine for which \( n, k, \ell \), the system (for \( L = [k] \)) of linear equations (II) has a non-negative integer solution. For convenience, in this section the system (II) without specified \( L \) (or \( A \) and \( b \) without subscripts) always means the case \( L = [k] \). In the next two subsections we discuss the necessary and sufficient conditions respectively, and we prove Theorems 1.2 and 1.3 in the last subsection.
3.1 Necessary condition

The following lemma shows that it is necessary for \( n \) to come from certain congruence classes modulo \( k \).

**Lemma 3.1.** For any \( n, k \) satisfying \( 2 \leq k < \frac{n}{2} \), if \( n \not\equiv 0, -1 \pmod{k} \), then the system (1) does not have a non-negative real solution.

It turns out that unlike Baranyai’s Theorem, the congruence condition in Lemma 3.1 alone is not enough to guarantee a 1-factorization. For example, one could show that \( K_{18} \leq 6 \) is not 1-factorable even though \( 18 \equiv 0 \pmod{6} \). More generally, if \( n \) is not very large compared to \( k \), even if it is from those congruence classes, it is still possible that the system (1) does not have a non-negative integer solution. In these cases, there are not even non-negative real solutions.

**Lemma 3.2.** For every positive integer \( k \geq 2 \),

(i) Suppose \( n = j \cdot k + k \), where \( j \) is a non-negative integer. If \( 2 \leq j \leq k - 4 \), then the system (1) has no non-negative solution.

(ii) Suppose \( n = j \cdot k + k - 1 \), where \( j \) is a non-negative integer. If \( 2 \leq j \leq \lceil \frac{k}{2} \rceil - 3 \), then the system (1) has no non-negative solution.

The proofs of Lemma 3.1 and 3.2 use Farkas’ Lemma. In order to show that \( A^T \bar{x} = \bar{b} \) has no non-negative real solution, we will construct a hyperplane separating the convex cone formed by the column vectors of the matrix \( A^T \) and the vector \( \bar{b} \). Here we say a vector \( \bar{x} \geq 0 \) if each of its coordinates is non-negative.

**Lemma 3.3** (Farkas [15]). Let \( P \in \mathbb{R}^{m \times n} \) and \( \bar{b} \in \mathbb{R}^n \). Then exactly one of the following two assertions is true:

1. There exists a vector \( \bar{x} \in \mathbb{R}^m \) such that \( P^T \bar{x} = \bar{b} \) and \( \bar{x} \geq 0 \).
2. There exists a vector \( \bar{y} \in \mathbb{R}^n \) such that \( P \bar{y} \geq \bar{0} \) and \( \bar{b}^T \bar{y} < 0 \).

Below we give proofs to both Lemmas 3.1 and 3.2.

**Proof of Lemma 3.1** Let \( n = j \cdot k + r \), \( 0 < r < k - 1 \). We first consider the case when \( j \geq 3 \). Let

\[
\bar{y}^T = \left( \frac{j}{r-1}, \ldots, \frac{j}{2}, \frac{j-1}{2}, \ldots, \frac{j-1}{2}, -1 \right).
\]

By Lemma 3.3 we just need to prove \( A\bar{y} \geq \bar{0} \) and \( \bar{b}^T \bar{y} < 0 \).

Take an arbitrary row vector of \( A \). By definition we know that it is of the form \( \bar{\lambda} = (\lambda_1, \ldots, \lambda_k) \) such that \( \sum_{i=1}^{k} i \cdot \lambda_i = n \) and all \( \lambda_i \)'s are non-negative integers. Since \( n = jk + r \), we have \( \lambda_k \leq j \). Suppose \( \lambda_k = j \), then \( \sum_{i=1}^{k-1} i \cdot \lambda_i = n - kj = r > 0 \) and thus for some index \( 1 \leq s \leq r \), \( \lambda_s \) must be strictly positive. Either we have at least two such \( \lambda_a \) to be positive, or \( \lambda_r = 1 \). In either case, \( \sum_{i=1}^{r} \lambda_i y_i \geq j \). This already gives

\[
\sum_{i=1}^{k} \lambda_i y_i \geq \left( \sum_{i=1}^{r} \lambda_i y_i \right) + \lambda_k y_k \geq j - j = 0.
\]
Now we may assume that \( \lambda_k \leq j-1 \). Similar as before, if at least two of \( \lambda_1, \cdots, \lambda_{r-1} \), or \( \lambda_r \) itself is strictly positive, then \( \sum_{i=1}^{k} \lambda_i y_i \) is non-negative. Otherwise suppose \( \lambda_r = 0 \) and \( \lambda_1 + \cdots + \lambda_{r-1} \leq 1 \), then
\[
\sum_{i \neq r, 1 \leq i \leq k-1} i \cdot \lambda_i = n - k \lambda_k \geq jk + r - (j - 1)k = k + r.
\]
Since
\[
\sum_{i \neq r, 1 \leq i \leq k-1} i \cdot \lambda_i \leq (k - 1) \sum_{i \neq r, 1 \leq i \leq k-1} \lambda_i,
\]
we immediately have \( \sum_{i \neq r, 1 \leq i \leq k-1} \lambda_i \geq 2 \). Therefore we also have
\[
\sum_{i=1}^{k} \lambda_i y_i \geq \frac{j - 1}{2} \cdot \left( \sum_{i \neq r, 1 \leq i \leq k-1} \lambda_i \right) - \lambda_k \geq (j - 1) - (j - 1) = 0.
\]

Next we prove \( \vec{gT}_r \vec{b} < 0 \). Recall that \( b_i = \binom{n}{i} \). So it suffices to verify that for \( 1 \leq r \leq k - 2 \) and \( n = jk + r \) with \( j \geq 3 \),
\[
\sum_{i=1}^{r-1} \frac{j}{2} \binom{n}{i} + j \binom{n}{r} + \sum_{i=r+1}^{k-1} \frac{j - 1}{2} \binom{n}{i} < \binom{n}{k}. \tag{2}
\]

Note that
\[
\binom{n}{k - i + 1} = \frac{j - 1}{2} \binom{n}{k - i} + \frac{j - 1}{2} \binom{n}{k - i} + \frac{r + 1 + j(i - 1)}{k - (i - 1)} \binom{n}{k - i}.
\]

By substituting one of the \( \binom{n}{k-i} \) by the above identity for \( i + 1 \) and repeat this process, we obtain
\[
\binom{n}{k} = \sum_{i=1}^{k-1} \frac{j - 1}{2} \binom{n}{k - i} + \frac{j - 1}{2} \binom{n}{k - i} + \sum_{i=1}^{k-1} \frac{j - 1}{2} \binom{n}{k - i} - \frac{r + 1 + j(i - 1)}{k - (i - 1)} \binom{n}{k - i}. \tag{3}
\]

We compare the coefficients of \( \binom{n}{k-i} \) in this expression with the left hand side of (2). Note that
\( 1 \leq r \leq k - 2 \), we discuss the following cases according to the value of \( j \).

1. For \( j \geq 6 \), using that for \( i \geq k - r \geq 2 \), \( \left( \frac{j - 1}{2} \right)^i \geq \left( \frac{k - 1}{2} \right)^2 > j \),
\[
\binom{n}{k} > \sum_{i=1}^{k-1} \left( \frac{j - 1}{2} \right)^i \binom{n}{k - i} > \sum_{i=1}^{k-r-1} \frac{j - 1}{2} \binom{n}{k - i} + \sum_{i=k-r}^{k-1} j \binom{n}{k - i}.
\]

2. For \( j = 5 \), note that \( \left( \frac{j - 1}{2} \right)^3 - j > 0 \), we could establish the same inequality when \( r \leq k - 3 \). It suffices to check the case \( r = k - 2 \) and compare the coefficients of \( \binom{n}{k-2} \). Here we also involve the last sum on the right hand side of (3). Note that in (3) the coefficients of \( \binom{n}{k-2} \) add up to be \( 2^2 + 2 \cdot \frac{k - 2 + 1 + 5(2 - 1)}{k - (2 - 1)} > 5 = j \). It completes the proof of the \( j = 5 \) case.
3. For \( j = 4 \), since \((\frac{j-1}{2})^4 - j > 0\), like in Case 2, it suffices to check the cases \( r = k - 2 \) and \( r = k - 3 \). For \( r = k - 3 \), the coefficient of \( \binom{n}{k-3} \) from (3) is \((\frac{j-1}{2})^3 \cdot \frac{(k-3)(k-1)+j(3-1)}{k-(3-1)}\), which is greater than \( j \). For \( r = k - 2 \), the coefficient of \( \binom{n}{k-3} \) from (3) is \((\frac{j-1}{2})^3 \cdot \frac{(k-2)(k-1)+j(3-1)}{k-(3-1)}\), greater than \( j \). The coefficient of \( \binom{n}{k-2} \) from (3) is \((\frac{j-1}{2})^2 \cdot \frac{(k-2)(k-1)+j(2-1)}{k-(2-1)}\), combined with the surplus term \( \frac{(k-2)+1}{k} \cdot \binom{n}{k-1} \geq \binom{n}{k-2} \cdot \binom{n}{k-2} \), which is not hard to check that \( \frac{9}{4} + \frac{3}{2} \cdot \frac{3}{k} + k \) is greater than \( j = 4 \) for all \( k \).

4. When \( j = 3 \), we compare the identity (3) with inequality (2), note that what we need to prove is

\[
\binom{n}{1} + \sum_{i=1}^{k-1} \frac{r+1+3(i-1)}{k-i} \binom{n}{k-i} > 2 \binom{n}{r} + \sum_{i=k-r+1}^{k-1} \frac{1}{2} \binom{n}{k-i}. \tag{4}
\]

Note that \( \binom{n}{k-i} / \binom{n}{k-i-1} = (n-k+i+1)/(k-i) > 2 \), therefore the right hand side is at most \( 2 \binom{n}{r} + \binom{n}{r-1} \), while the left hand side is at least

\[
\frac{r+1}{k} \binom{n}{k-1} + \frac{r+4}{k-1} \binom{n}{k-2} + \frac{r+7}{k-2} \binom{n}{k-3}.
\]

Note that

\[
\frac{n}{r+2} = \frac{(n-k+2) \cdots (n-r+1)}{(r+1) \cdots (k-1)} \geq \frac{n-k+2}{r+1} \geq \frac{k}{r+1}.
\]

For \( r = k - 2 \), \( \frac{r+4}{k-2} \binom{n}{k-2} = \frac{k+2}{k-2} \binom{n}{k-2} > \binom{n}{k-2} \), and for \( r \leq k - 3 \),

\[
\frac{n}{r+2} = \frac{(n-k+3) \cdots (n-r+1)}{(r+1) \cdots (k-2)} \geq \frac{n-k+3}{r+1} \geq \frac{k-1}{r+4}.
\]

Finally, when \( r = k - 2 \), \( \frac{r+7}{k-3} \binom{n}{k-3} = \frac{k+5}{k-3} \binom{n}{k-3} > \binom{n}{k-3} \), and for \( r \leq k - 3 \),

\[
\frac{n}{r-1} = \frac{(n-k+4) \cdots (n-r+1)}{r \cdots (k-3)} \geq \frac{n-k+4}{r} \geq \frac{k-2}{r+7}.
\]

These three inequalities immediately imply inequality (4).

For \( j = 2 \), we will use a different hyperplane when applying Farkas’ Lemma. Now we have \( n = 2k + r, 1 \leq r \leq k - 2 \). We define \( \vec{y} \) as follows. We always take \( y_1 = \cdots = y_{r-1} = 1, y_r = 2 \), and \( y_k = -1 \). For \( r \not\equiv k \pmod{2} \), we take \( y_{r+1} = \cdots = y_{\lceil (r+k)/2 \rceil} = 1 \), and for \( r \equiv k \pmod{2} \), we take \( y_{r+1} = \cdots = y_{\lfloor (r+k)/2 \rfloor} = 1/2 \). The remaining \( y_i \)'s are set to be 0.

First we prove \( \sum_{i=1}^{k} \lambda_i y_i \geq 0 \). This is obviously true if \( \lambda_k = 0 \). Suppose \( \lambda_k = 2 \), then \( \sum_{i=1}^{k-1} i \lambda_i = r \). In this case we either have \( \sum_{i=1}^{r-1} \lambda_i = 2 \) or \( \lambda_r = 1 \), both implying \( \sum_{i=1}^{k} \lambda_i y_i \geq 0 \). The remaining case is \( \lambda_k = 1 \) and we have \( \sum_{i=1}^{k-1} i \lambda_i = k + r \). If there exists \( i \leq r \) with \( \lambda_i > 1 \) then we are already done. Now suppose \( \lambda_1 = \cdots = \lambda_r = 0 \), we have \( \sum_{i=r+1}^{k-1} i \lambda_i = k + r \). Either (in the case when \( k \) and \( r \) have the same parity) we have \( \lambda_{(k+r)/2} = 2 \), or there exists some \( i < (k+r)/2 \) such that \( \lambda_i = 1 \). By our choice of \( \vec{y} \), it is not hard to see that once again \( \sum_{i=1}^{k} \lambda_i y_i \geq 0 \).
Finally let us prove that \( \bar{y}^T \bar{b} < 0 \). For \( r \equiv k \mod 2 \), \( 1 \leq r \leq k - 2 \) and \( n = 2k + r \), we need to show
\[
\left( \sum_{i=1}^{(k+r)/2-1} \binom{n}{i} \right) + \frac{1}{2} \binom{n}{(k+r)/2} + \binom{n}{r} < \binom{n}{k} \tag{5}
\]
For \( i \leq (k+r)/2 \) we have
\[
\binom{n}{i-1}/\binom{n}{i} = \frac{i}{n-i+1} \leq \frac{(k+r)/2}{(3k+r)/2+1} \leq \frac{k-1}{2k}.
\]
Also \( \binom{n}{i} \leq \binom{n}{(k+r)/2-1} \), thus the left hand side of (5) is at most
\[
\left( \frac{k-1}{2k} + \left( \frac{k-1}{2k} \right)^2 + \cdots \right) \left( \binom{n}{(k+r)/2} \right) + \frac{1}{2} \binom{n}{(k+r)/2} + \frac{k-1}{2k} \binom{n}{(k+r)/2} 
\leq \frac{k-1}{k+1} \binom{n}{(k+r)/2} + \frac{2k-1}{2k} \binom{n}{(k+r)/2} = \frac{4k^2 - k - 1}{2k^2 + 2k} \binom{n}{(k+r)/2} \leq \binom{n}{k}.
\]
The last inequality follows from
\[
\binom{n}{k}/\binom{n}{(k+r)/2} = \frac{(n-k+1) \cdots (n-(k+r)/2)}{(k+r)/2 + 1} \cdots k \geq \frac{n-(k+r)/2}{(k+r)/2+1} = \frac{3k+r}{k+r+2} \geq \frac{3k+(k-2)}{k+(k-2)+2} = \frac{2k-1}{k}.
\]
For the case \( r \not\equiv k \mod 2 \), in this case we have \( r \leq k - 3 \), we need to show
\[
\binom{n}{r} + \sum_{i=1}^{(k+r-1)/2} \binom{n}{i} < \binom{n}{k} \tag{6}
\]
For \( i \leq (k+r+1)/2 \),
\[
\binom{n}{i-1}/\binom{n}{i} = \frac{i}{n-i+1} \leq \frac{k+r+1}{3k+r+1} \leq \frac{k+(k-3) + 1}{3k+(k-3)+1} = \frac{k-1}{2k-1}.
\]
Also \( \binom{n}{i} \leq \binom{n}{(k+r-1)/2} \leq \frac{k+1}{2k-1} \binom{n}{(k+r-1)/2} \). So the left hand side of (6) is at most
\[
\left( \frac{k-1}{2k-1} + \left( \frac{k-1}{2k-1} \right)^2 + \cdots \right) \left( \binom{n}{(k+r+1)/2} \right) + \binom{n}{r} 
\leq \frac{k-1}{k} \binom{n}{(k+r+1)/2} + \frac{k-1}{2k-1} \binom{n}{(k+r+1)/2} \leq \binom{n}{k}.
\]
The last inequality follows from
\[
\binom{n}{k}/\binom{n}{(k+r+1)/2} = \frac{(n-k+1) \cdots (n-(k+r+1)/2)}{(k+r+1)/2 \cdots k} \geq \frac{n-(k+r+1)/2}{(k+r+3)/2 \cdots k} = \frac{3k+r-1}{k+r+3} \geq \frac{3k+(k-3)-1}{k+(k-3)+3} = \frac{2k-2}{k} \geq \frac{k-1}{2k-1}.
\]
Proof of Lemma 3.2  (i) We first consider the case \( n = jk + k, 1 \leq j \leq k - 4 \). Let
\[
\vec{y}^T = (j + 1, \ldots, j + 1, j/2, \ldots, j/2, -1, 0).
\]
Then once again from Farkas’ Lemma, we just need to verify \( A\vec{y} \geq 0 \) and \( \vec{y}^T \vec{y} < 0 \).

For each row vector \( \vec{\lambda} = (\lambda_1, \cdots, \lambda_k) \) of \( A \), we know \( \sum_{i=1}^{k} i \lambda_i = n \). Since \( n = jk + k = (j + 1)(k - 1) + j + 1 \) and \( 1 \leq j \leq k - 4 \), we know that \( \lambda_k + \lambda_{k-1} \leq j + 1 \) and \( \lambda_k + \lambda_{k-1} \leq j \) if \( \sum_{i=j+2}^{k} \lambda_i > 0 \). We discuss these two cases separately:

1. When \( \sum_{i=j+2}^{k-2} \lambda_i = 0 \). If we also have \( \sum_{i=1}^{j+1} \lambda_i = 0 \), then \( \vec{\lambda} = (0, \ldots, 0, j + 1) \) and trivially \( \vec{\lambda}^T \vec{y} \geq 0 \). Now suppose \( \sum_{i=1}^{j+1} \lambda_i \geq 1 \), this case is also trivial since \( \lambda_k + \lambda_{k-1} \leq j + 1 \).

2. When \( \sum_{i=j+2}^{k-2} \lambda_i > 0 \). If \( \sum_{i=j+2}^{k-2} \lambda_i \geq 2 \), observe that \( \lambda_k + \lambda_{k-1} \leq j \) and we also have \( \vec{\lambda}^T \vec{y} \geq 2 \cdot (j/2) - j \geq 0 \). Thus we just need to consider the case \( \sum_{i=j+2}^{k-2} \lambda_i = 1 \). If \( \sum_{i=1}^{j+1} \lambda_i = 0 \) then \( (k-1)\lambda_{k-1} + k\lambda_k \in \{jk+2, (j+1)k-(j+2)\} \), which implies that \( \lambda_{k-1} \in \{j+2, \cdots, k-2\} \), contradicting \( \lambda_k + \lambda_{k-1} \leq j \). Therefore \( \sum_{i=1}^{j+1} \lambda_i \geq 1 \), and this implies \( \vec{\lambda}^T \vec{y} \geq (j + 1) - j > 0 \).

To apply Farkas’ Lemma, we will also need to show that \( \vec{b}^T \vec{y} < 0 \). We know \( b_i = \binom{n}{i} \) in the system (10). First of all, similar as before,
\[
\binom{n}{k-i+1} = \frac{j}{2} \binom{n}{k-i} + \frac{j}{2} \binom{n}{k-i} + \frac{1 + (j+1)(i-1)}{k-(i-1)} \binom{n}{k-i}.
\]
Thus by repeatedly applying this identity, we have
\[
\binom{n}{k-1} = \sum_{i=2}^{k-1} \left( \frac{j}{2} \right)^{i-1} \binom{n}{k-i} + \frac{j}{2} \binom{n}{1} + \sum_{i=2}^{k-1} \left( \frac{j}{2} \right)^{i-2} \frac{1 + (j+1)(i-1)}{k-(i-1)} \binom{n}{k-i}.
\] (7)

What we need to prove is
\[
\binom{n}{k-1} > \sum_{i=2}^{k-1} \frac{j}{2} \binom{n}{k-i} + \sum_{i=k-j-1}^{k-1} \frac{j+2}{2} \binom{n}{k-i}.
\] (8)

Notice that \( j + 1 \leq k - 3 \) since \( 2 \leq j \leq k - 4 \), we divide our discussion into several cases based on the value of \( j \).

1. For \( j \geq 5 \), we have \( \left( \frac{j}{2} \right)^2 - j - 1 > 0 \) and thus
\[
\binom{n}{k-1} = \sum_{i=2}^{k-1} \left( \frac{j}{2} \right)^{i-1} \binom{n}{k-i} + \frac{j}{2} \binom{n}{1} + \sum_{i=2}^{k-1} \left( \frac{j}{2} \right)^{i-2} \frac{1 + (j+1)(i-1)}{k-(i-1)} \binom{n}{k-i} > \sum_{i=2}^{k-1} \left( \frac{j}{2} \right)^{i-1} \binom{n}{k-i} > \sum_{i=2}^{k-1} \frac{j}{2} \binom{n}{k-i} + \sum_{i=k-j-1}^{k-1} \frac{j+2}{2} \binom{n}{k-i}.
\]
2. For \( j = 4 \), we have \( \left( \frac{n}{2} \right)^3 - j - 1 > 0 \). It is easy to verify the inequality \( \mathbb{8} \) for \( j + 1 \leq k - 4 \). So it remains to check the case when \( j + 1 = k - 3 \) (i.e. \( k = j + 4 = 8 \) and \( n = jk + k = 40 \)). We have \( \bar{g}^T = (5, 5, 5, 5, 2, -1, 0) \) and \( \bar{b}^T = \left( \begin{pmatrix} 40 \\ 8 \end{pmatrix}, \ldots, \begin{pmatrix} 40 \\ 1 \end{pmatrix} \right) \) when \( k = j + 4 = 8 \) and \( n = jk + k = 40 \). We have \( \bar{b}^T \bar{y} < 0 \) by calculation.

3. For \( j = 3 \), note that \( \left( \frac{n}{2} \right)^4 - j - 1 > 0 \), it is easy to check that the inequality \( \mathbb{8} \) when \( j + 1 \leq k - 5 \). So we just need to check the case that \( j + 1 = k - 3 \) and \( j + 1 = k - 4 \). We have \( \bar{g}^T = (4, 4, 4, 4, 1.5, -1, 0) \) and \( \bar{b}^T = \left( \begin{pmatrix} 28 \\ 8 \end{pmatrix}, \ldots, \begin{pmatrix} 29 \\ 1 \end{pmatrix} \right) \) when \( k = j + 4 = 7 \) and \( n = jk + k = 28 \). We have \( \bar{b}^T \bar{y} < 0 \) by calculation. Similarly, we have \( \bar{g}^T = (4, 4, 4, 4, 1.5, 1.5, -1, 0) \) and \( \bar{b}^T = \left( \begin{pmatrix} 32 \\ 8 \end{pmatrix}, \ldots, \begin{pmatrix} 32 \\ 1 \end{pmatrix} \right) \) when \( k = j + 5 = 8 \) and \( n = jk + k = 32 \). Both cases have \( \bar{b}^T \bar{y} < 0 \) by calculation.

4. For \( j = 2 \), we compare the identity \( \mathbb{7} \) with the inequality \( \mathbb{8} \), after canceling some terms, what we need to prove is
\[
\binom{n}{1} + \sum_{i=2}^{k-1} \frac{1+3(i-1)}{k-(i-1)} \binom{n}{k-i} > 2 \left( \binom{n}{3} + \binom{n}{2} + \binom{n}{1} \right).
\]  

Note that \( \frac{1+3(i-1)}{k-(i-1)} \) is increasing in \( i \). Furthermore setting \( i = k - 3 \), we have \( \frac{1+3(i-1)}{k-(i-1)} = \frac{3k-11}{4k-3} \). If \( 3k-11 \geq 2 \), i.e. \( k \geq 7 \), we have proved the inequality \( \mathbb{9} \). Since \( j \leq k - 4 \), we also have \( k \geq 6 \). Thus we just need to check the case that \( j = 2 \) and \( k = 6 \). We have \( \bar{g}^T = (3, 3, 3, 1, -1, 0) \) and \( \bar{b}^T = \left( \begin{pmatrix} 18 \\ 6 \end{pmatrix}, \ldots, \begin{pmatrix} 18 \\ 1 \end{pmatrix} \right) \). It is easy to check that \( \bar{b}^T \bar{y} < 0 \) by calculation.

(ii) Next we consider the case \( n = j \cdot k + k - 1, 1 \leq j \leq \left[ \frac{k}{2} \right] - 3 \). Let
\[
\bar{g}^T = (j + 1, \ldots, j + 1, j/2, \ldots, j/2, -1, j, -1).
\]
Then we just need to prove \( A\bar{y} \geq 0 \) and \( \bar{b}^T \bar{y} < 0 \). We start by checking \( A\bar{y} \geq 0 \).

For each row vector \( \bar{x} = (\lambda_1, \ldots, \lambda_k) \) of \( A \), by definition we have \( \sum_{i=1}^{k} i \lambda_i = n \). Since \( n = j \cdot k + k - 1 = (j + 1) \cdot (k - 2) + 2j + 1 \) and \( 1 \leq j \leq \left[ \frac{k}{2} \right] - 3 \), we have \( \lambda_k + \lambda_{k-2} \leq j + 1 \) and \( \lambda_k + \lambda_{k-2} \leq j \) if \( \sum_{i=2j+2}^{k-3} \lambda_i + \lambda_{k-1} > 0 \). Then we consider the following two cases:

1. When \( \sum_{i=2j+2}^{k-3} \lambda_i = 0 \), if \( \sum_{i=2j+2}^{k-3} \lambda_i = 0 \), note that the equation \( (k-2)\lambda_{k-2} + k\lambda_k = jk + k - 1 \) has no non-negative integer solution. Therefore the only possibility is \( \bar{x} = (0, \ldots, 0, 1, j) \), which gives \( \bar{x}\bar{y} \geq 0 \) trivially. The case \( \sum_{i=2j+2}^{k-3} \lambda_i = 1 \) is also trivial since \( \lambda_k + \lambda_{k-2} \leq j + 1 \).

2. Now suppose \( \sum_{i=2j+2}^{k-3} \lambda_i = 1 \). If \( \sum_{i=2j+2}^{k-3} \lambda_i = 1 \), then \( \lambda_k + \lambda_{k-2} = j \), and thus \( \bar{x}\bar{y} \geq 0 \). So we just need to consider the case \( \sum_{i=2j+2}^{k-3} \lambda_i = 1 \). To prove \( \bar{x}\bar{y} \geq 0 \), we just need to show \( (\sum_{i=1}^{2j+1} \lambda_i) + \lambda_{k-1} \geq 1 \). This is true since the inequality \( n - (k-3) \leq (k-2)\lambda_{k-2} + k\lambda_k \leq n - (2j+2) \) has no non-negative integer solution for \( 1 \leq j \leq \left[ \frac{k}{2} \right] - 3 \).
Next we will prove that $\bar{b}^T \bar{y} < 0$. Recall that $b_i = \binom{n}{i}$ in the system \( \mathbf{II} \). It is easy to check that

$$
\binom{n}{k-i+1} = \frac{j}{2} \binom{n}{k-i} + \frac{j}{2} \binom{n}{k-i} + \frac{(j+1)(i-1)}{k-(i-1)} \binom{n}{k-i}.
$$

Thus we have

$$
\binom{n}{k-2} = \sum_{i=3}^{k-1} \left( \frac{j}{2} \right)^{i-2} \binom{n}{k-i} + \sum_{i=k-2j-1}^{k-1} \frac{j+2}{2} \binom{n}{k-i}. \tag{10}
$$

Observe that $\binom{n}{k} = j \binom{n}{k-1}$ when $n = jk + (k - 1)$, so what we need to prove is

$$
\binom{n}{k-2} > \sum_{i=3}^{k-1} \frac{j}{2} \binom{n}{k-i} + \sum_{i=k-2j-1}^{k-1} \frac{j+2}{2} \binom{n}{k-i}. \tag{11}
$$

Note that $2j + 1 \leq k - 4$ since $2 \leq j \leq \lceil \frac{k}{2} \rceil - 3$, we discuss the following cases according to the value of $j$.

1. For $j \geq 5$, we have $\left( \frac{j}{2} \right)^2 - j - 1 > 0$, therefore

$$
\binom{n}{k-2} > \sum_{i=3}^{k-1} \frac{j}{2} \binom{n}{k-i} + \sum_{i=k-2j-1}^{k-1} \frac{j+2}{2} \binom{n}{k-i}.
$$

2. For $j = 4$, we have $\left( \frac{j}{2} \right)^3 - j - 1 > 0$, it is easy to check that the inequality \( \mathbf{II} \) similarly with Case 1 when $2j + 1 \leq k - 5$. So we just need to check the case that $2j + 1 = k - 4$, which gives $k = 2j + 5 = 13$ and $n = jk + k - 1 = 64$. We have $\bar{g}^T = (5, 5, 5, 5, 5, 5, 5, 2, -1, 4, -1)$ and $\bar{b}^T = (\binom{64}{1}, \ldots, \binom{64}{1})$. By calculation $\bar{b}^T \bar{g} < 0$.

3. For $j = 3$, we have $\left( \frac{j}{2} \right)^4 - j - 1 > 0$. it is easy to check that the inequality \( \mathbf{II} \) similarly with case 1 when $2j + 1 \leq k - 6$. So we just need to check the case that $2j + 1 = k - 4$ and $2j + 1 = k - 5$. We have $\bar{g}^T = (4, 4, 4, 4, 4, 4, 1, 5, -1, 3, -1)$ and $\bar{b}^T = (\binom{13}{1}, \ldots, \binom{13}{1})$ when $k = 2j + 5 = 11$ and $n = jk + k - 1 = 43$. Similarly, we have $\bar{g}^T = (4, 4, 4, 4, 4, 4, 1, 5, 1, 3, -1)$ and $\bar{b}^T = (\binom{17}{1}, \ldots, \binom{17}{1})$ when $k = 2j + 6 = 12$ and $n = jk + k - 1 = 47$. In both cases, calculations give $\bar{b}^T \bar{g} < 0$.

4. For $j = 2$, we compare the identity \( \mathbf{II} \) with the inequality \( \mathbf{II} \) and note that it suffices to prove

$$
\binom{n}{1} + \sum_{i=3}^{k-1} \frac{3(i-1)}{k-(i-1)} \binom{n}{k-i} > 2 \sum_{i=1}^{5} \binom{n}{i}. \tag{12}
$$

13
Note that \(\frac{3(i-1)}{k-(i-1)}\) is monotone increasing in \(i\). So we could focus our attention on the coefficient of the term \(\binom{n}{i}\). Note that when \(i = k - 5\), \(\frac{3(i-1)}{k-(i-1)} = \frac{3k-18}{6}\). If \(\frac{3k-18}{6} \geq 2\), i.e. \(k \geq 10\), the inequality (12) is obviously true. Since \(j \leq \left\lceil \frac{k}{2} \right\rceil - 3\), we also have \(k \geq 9\). Thus it remains to check the case that \(j = 2\) and \(k = 9\). We have \(\vec{y}' = (3, 3, 3, 3, 1, -1, 2, -1)\) and \(\vec{b}' = ((26), \ldots, (26))\). It is easy to verify \(\vec{b}'^T \vec{y}' < 0\) by calculation. \(\square\)

### 3.2 Sufficient condition

In the previous subsection, we have found necessary conditions for the system (1) to have a non-negative integer solution. The following two lemmas below show that these necessary conditions are indeed sufficient.

**Lemma 3.4.** Suppose \(n > 2k\) and \(n = j \cdot k + k\), where \(j\) is a non-negative integer such that \(j \geq k - 3\), then the system (1) has a non-negative integer solution.

**Proof.** Suppose \(n = jk + k\) and \(j \geq k - 3\). So \(n\) is at least \(k^2 - 2k\) and divisible by \(k\). Below we construct an explicit non-negative integer solution to the system (1). First, we consider the case when \(n \geq k^2 - k\). For each \(i \in \{1, \ldots, k-1\}\), we consider \(\vec{\lambda}_i = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{Z}_{\geq 0}^k\) such that

\[
\lambda_i = \frac{k}{\gcd(k, i)}, \quad \lambda_k = \frac{n}{k} - \frac{i}{\gcd(k, i)}, \quad \text{and} \quad \lambda_s = 0 \text{ for } s \notin \{i, k\}.
\]

It is easy to check that \(\sum_{i=1}^k \ell \lambda_i = n\). Also note that

\[
\lambda_k = n/k - i/\gcd(k, i) \geq n/k - i \geq n/k - (k - 1) \geq 0.
\]

So \(\vec{\lambda}_i\) is an \((n, k)\)-type and thus appears as a row of the matrix \(A\). We will assign \(\binom{n}{i}/(k/\gcd(k, i))\) to \(x_{\vec{\lambda}_i}\), but first we explain why \(\binom{n}{i}/(k/\gcd(k, i))\) is an integer. For every prime \(p\), if the \(p\)-adic valuation of \(n\) is \(x\) and that of \(i\) is \(y\), then the \(p\)-adic valuation of \(n/\gcd(n, i)\) equals \(\max\{x - y, 0\}\). On the other hand, Kummer’s Theorem [20] says that for any prime \(p\), the \(p\)-adic valuation of \(\binom{n}{i}\) is equal to the number of carries when \(n - i\) is added to \(i\) in base \(p\). This quantity is obviously at least \(\max\{x - y, 0\}\). Consequently \(n/\gcd(n, i)\) always divides \(\binom{n}{i}\). Since \(n\) is a multiple of \(k\), it is easy to see that \(k/\gcd(k, i)\) divides \(n/\gcd(n, i)\). This completes the proof of our claim.

Once we have chosen the value of \(x_{\vec{\lambda}_i}\) for the aforementioned \(\vec{\lambda}_i\) for \(i \in \{1, \ldots, k-1\}\), the first \(k - 1\) equations of the system (1) have already been taken care of. We just use the type \(\vec{\lambda}_k = (\lambda_1, \ldots, \lambda_k)\) with \(\lambda_k = n/k\) and \(\lambda_s = 0\) for \(s \neq k\) to cover whatever is left for \(b_k = \binom{n}{k}\). Recall that \(\binom{n}{k-i-1}/\binom{n}{k-i} \leq k/(n - k + 1)\) for \(i \geq 0\), and \(n \geq k^2 - k\), therefore the accumulated value \(\gamma_{n,k}\) of the last equation of the system (1) equals

\[
\sum_{i=1}^{k-1} \frac{n/k - i/\gcd(k, i)}{k/\gcd(k, i)} \cdot \binom{n}{i} \leq \sum_{i=1}^{k-1} \frac{n}{2k - 1} \cdot \binom{n}{i} \cdot \left(\frac{k}{2} + \frac{k}{n-k+1} + \frac{k^2}{2(n-k+1)} + \cdots\right) \binom{n}{k} \leq \frac{n-2}{2(n-2k+1)} \binom{n}{k} \leq \frac{k^2 - k - 2}{2(k^2 - 3k + 1)} \binom{n}{k} \leq \frac{n}{k} = b_k
\]
if $k \geq 4$. For $k = 2$ the left hand side is $(n/2 - 1)n/2$, which is always less than $\binom{n}{2}$. For $k = 3$, the left hand side is $(n - 3)n/9 + (n - 6)\binom{n}{2}/9$, which is also less than $\binom{n}{2}$. It shows the number $x_{\lambda_k} = (\binom{n}{k} - \gamma_{n,k})/(n/k)$ of the type $\tilde{\lambda}_k$ needed is non-negative (it should be mentioned that the divisibility of $n/k$ to $\binom{n}{k} - \gamma_{n,k}$ is automatic by the setting). This proves the case when $n \geq k^2 - k$.

For the remaining case $n = k^2 - 2k$, we define the same $\tilde{\lambda}_k$ and repeat the same assignment $x_{\lambda_i}$ as the $n \geq k^2 - k$ case for $i \in \{1, \cdots, k - 3\}$. We then let $x_{\lambda_k} = \binom{n}{k-2}$ for the $(n,k)$-type $\tilde{\lambda}$ with $\lambda_{k-2} = 1$ and $\lambda_{k-1} = k - 2$. As $\binom{n}{k-2}/\binom{n}{k-1} = 1/(k-2)$, these together take care of the first $k - 1$ equations of the system (1). Finally, using the type $\tilde{\lambda}_k$ and by the same estimation as the the $n \geq k^2 - k$ case, we can cover the level $k$ precisely. This also gives a non-negative integer solution to the system (1).

Lemma 3.5. Suppose $n > 2k$ and $n = j \cdot k + k - 1$, where $j$ is a non-negative such that $j \geq \lceil \frac{k}{2} \rceil - 2$, then the system (1) has a non-negative integer solution.

Proof. Suppose $n = jk + (k - 1)$ for $j \geq \lceil k/2 \rceil - 2$. As $n > 2k$, in fact we have $n \geq 3k - 1$. First we consider the case when $k$ is even. Then we have $n \geq k^2/2 - k - 1$ and $k$ divides $n + 1$. Now instead of finding solutions to $A^T \vec{x} = \vec{b}$, we consider the $n + 1$ case and the new system $(A')^T \vec{y} = \vec{b}'$. Here $A'$ is the matrix defined for the $(n+1, \mathcal{L})$ case with $\mathcal{L} = \{2, 4, \cdots, k\}$, and $\vec{b}'$ is a vector in $\mathbb{R}^k$ such that $b_i = \binom{n+1}{i}$ for even $i$, and 0 for odd $i$. This system of linear equations has a non-negative integer solution by taking the type $\tilde{\lambda}_k = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ with

$$\lambda_i = \frac{k}{\gcd(k,i)}, \quad \lambda_k = \frac{n + 1}{k} - \frac{i}{\gcd(k,i)}, \quad \text{and} \quad \lambda_s = 0 \text{ for } s \notin \{i, k\}.$$ 

and assigning $\binom{n+1}{i}/(k/\gcd(k,i))$ to $y_{\lambda_i}$, for $i = 2, 4, \cdots, k - 2$ and then using the type $\tilde{\lambda}_k = (\lambda_1', \lambda_2', \cdots, \lambda_k')$ such that $\lambda'_k = (n+1)/k$ and $\lambda'_s = 0$ for each $s \neq k$ to cover whatever is left for $b_k$. Similar as before, all defined $y_{\lambda_i}$ are integers. Furthermore, since we only do it for even $i$, $k$ is also even and $n + 1 \geq k(k-2)/2$, we always have

$$\frac{n+1}{k} - \frac{i}{\gcd(k,i)} \geq \frac{n+1}{k} - \frac{i}{2} \geq \frac{n+1}{k} - \frac{k-2}{2} \geq 0.$$ (13)

It only remains to show that the type $\tilde{\lambda}_k$ appears for a non-negative number of times. This can be estimated as follows (in a similar way as in the proof of Lemma 3.4)

$$\sum_{i \in \{2, 4, \cdots, k-2\}} \frac{(n+1)/k - i/\gcd(k,i)}{k/\gcd(k,i)} \cdot \binom{n}{i} \leq \frac{n-3}{2k} \cdot \left( \frac{k}{n-k+2} \right)^2 \left( \frac{k}{n-k+2} \right)^3 \cdots \left( \frac{k}{n-k+2} \right) \cdot \binom{n+1}{k} \leq \frac{n+1}{k} \cdot \left( \frac{k}{n-k+2} \right)^2 \cdot \binom{n+1}{k} = \frac{k(n-3)}{(n-k+2)^2} \cdot \binom{n+1}{k} \leq \binom{n+1}{k} = b_k,$$ (14)
where we have \((\binom{n+1}{k+1})/\binom{n+1}{k-1} \leq k/(n-k+2) \leq 1/2\) for \(i \geq 0\) and \(k(n-3) \leq (n-k+2)^2\) under the fact that \(n \geq 3k-1\). So this indeed gives a non-negative integer solution to \((A')^T \vec{y} = \vec{b}'\). Therefore by Theorem 2.2 if we let \(F\) consist of all the subsets of \([n+1]\) of even size up to \(k\), i.e., \(F = \binom{[n+1]}{2,4,\ldots,k}\), then this family can be decomposed into 1-factors. We delete the element \(n+1\) from the unique subset containing it in each 1-factor. It is not hard to see that this immediately gives a 1-factorization of \(K_{n+1}^k\), and thus the system \((\text{I})\) has a non-negative integer solution (under the conditions of Lemma 3.5) when \(k\) is even.

For the remaining proof, we assume that \(k\) is odd. In this case, we have \(n = jk + k - 1\) with \(j \geq [k/2] - 2 = (k - 3)/2\). So we may assume that

\[
n = (k^2 - k - 2)/2 + tk
\]

for some integer \(t \geq 0\).

It is not hard to see that the above proof for the even \(k\) case still works for the odd \(k\) case (i.e., applied to \(\binom{[n+1]}{1,3,\ldots,k-2,k}\)) whenever the corresponding form for \((\text{I3})\) holds. However, for odd \(k\) and \(i \in \{1, 3, \ldots, k - 2\}\), we cannot ensure \(\gcd(k, i) \geq 2\), which was used in \((\text{I3})\) for the even \(k\) case. As to infer that \(\frac{n+1}{k} - \frac{i}{\gcd(k,i)} \geq \frac{n+1}{k} - i \geq 0\) holds for all \(i \in \{1, 3, \ldots, k - 2\}\), this approach requires an additional condition \(n + 1 \geq k(k - 2)\) or equivalently \(t \geq (k-3)/2\). Therefore, from now on we may further assume that

\[
0 \leq t \leq (k-5)/2.
\]

Since \(n = k(k+2t-1)/2 - 1\) and \((k+2t-1)/2 \leq k-3\), if we view \((k+2t-1)/2\) as a new parameter \((k'\) is odd or even, we can always conclude that \(\binom{[n]}{k+2t-1/2} = \binom{[n]}{k'}\) is 1-factorable. So it suffices to show that the family consisting of subsets of \([n]\) of size between \((k+2t+1)/2\) and \(k\), denoted by \(\binom{[n]}{(k+2t+1)/2,\ldots,k-1,k}\), is 1-factorable.

The rest of the proof will be divided into two cases: when \(t = (k-5)/2\) and when \(0 \leq t \leq (k-7)/2\). First we consider the case when \(t = (k-5)/2\). In this case, we have \(n = (k^2-k-2)/2+tk = k^2-3k-1,\) and \((k+2t+1)/2 = k-2\). So it suffices to show \(\binom{[n]}{k-2,k-1,k}\) is 1-factorable. We will use the following three \((n, \{k-2, k-1, k\})\)-types (all unspecified coordinates \(\lambda_i\) are 0 by default):

\[
A: \quad \lambda_{k-2} = (k+1)/2, \quad \text{and} \quad \lambda_k = (k-5)/2.
\]

\[
B: \quad \lambda_{k-2} = (k-1)/2, \quad \lambda_{k-1} = 2, \quad \text{and} \quad \lambda_k = (k-7)/2.
\]

\[
C: \quad \lambda_{k-1} = 1, \quad \text{and} \quad \lambda_k = k-4.
\]

Note that \(\binom{n}{k}/\binom{n}{k-1} = (n-k+1)/k = k-4\) and type \(C\) has the same ratio for level \(k-1\) and \(k\). So if we assume that type \(A\) is used \(a\) times and type \(B\) is used \(b\) times, then we need to find a non-negative integer solution to the following system of equations:

\[
\frac{k+1}{2}a + \frac{k-1}{2}b = \begin{pmatrix} n \\ k-2 \end{pmatrix} = \begin{pmatrix} k^2 - 3k - 1 \\ k - 2 \end{pmatrix},
\]

\(^2\text{Theorem 2.2 states that for any set }\mathcal{L}\text{ of distinct positive integers, the family }\binom{[n]}{\mathcal{L}}\text{ is 1-factorable if and only if the corresponding system }\text{(I)}\text{ for }\mathcal{L}\text{ has a non-negative integer solution. We are aware that we are looking for non-negative integer solutions for the (corresponding) system }\text{(I)},\text{ however for convenience of presentation, we should mention and identify both settings.}\)
\[
\left( \frac{k - 5}{2} a + \frac{k - 7}{2} b \right) / 2b = k - 4.
\]

Solving it gives that
\[
a = \frac{(k - 3)/2}{(k^2 - 3k - 1)/3} \left( \frac{k^2 - 3k - 1}{k - 2} \right) \quad \text{and} \quad b = \frac{(k - 5)/2}{k^2 - 3k - 1} \left( \frac{k^2 - 3k - 1}{k - 2} \right).
\]

Note that \( \frac{k^2 - 3k - 1}{3} \) divides \( \frac{k^2 - 3k - 1}{\gcd(3,k-2)} \), and \( \frac{k^2 - 3k - 1}{\gcd(3,k-2)} = \frac{k^2 - 3k - 1}{\gcd(k^2 - 3k - 1,k-2)} \) divides \( (k^2 - 3k - 1) \); also \( k \) is odd, so we can see that \( a \) is a non-negative integer. For \( b \), note that when \( k \equiv 2 \pmod{3} \), \( k^2 - 3k - 1 = \frac{k^2 - 3k - 1}{\gcd(3,k-2)} \) divides \( (k^2 - 3k - 1) \), while when \( k \equiv 2 \pmod{3} \) \((k-5)/6 \) is an integer. Therefore \( b \) is also an non-negative integer. Finally by considering the level \( k - 1 \), we can use the type C for \( c \) times, where
\[
c = \left( \frac{n}{k - 1} \right) - 2b = \left( \frac{n}{k - 1} \right) - \frac{k - 5}{k^2 - 3k - 1} \left( \frac{n}{k - 2} \right) \geq 0.
\]

The above analysis gives a non-negative integer solution to the corresponding system \( [\] for \( L = \{k - 2, k - 1, k\} \) and thus the proof for the case \( t = (k - 5)/2 \) is complete.

Now we consider the last case when \( 0 \leq t \leq (k - 7)/2 \). For this case, we have \( k \geq 7 \) and we want to show that \( \binom{n}{L} \) is 1-factorable for \( L = \{(k+2t+1)/2, \ldots, k-1, k\} \). To show that, we will use the following \( (n,L) \)-types labelled by
\[
R, S_0, S_1, S_2, \ldots, S_{\frac{k-5}{2}-t}, T_1, T_2, \ldots, T_{\frac{k-5}{2}-t}.
\]

Each of them is a vector \( \vec{\lambda} = (\lambda_1, \cdots, \lambda_k) \) defined as follows (here all the unspecified coordinates \( \lambda_i \) are equal to 0): for each \( 1 \leq i \leq (k-5)/2 - t \),
\[
R: \quad \lambda_{k-1} = 1, \quad \text{and} \quad \lambda_k = (k-3)/2 + t,
\]
\[
S_0: \quad \lambda_{k-2} = (k+1)/2, \quad \text{and} \quad \lambda_k = t,
\]
\[
S_i: \quad \lambda_{k-2-i} = 1, \quad \lambda_{k-2} = (k-1)/2 - i, \quad \lambda_{k-1} = i, \quad \text{and} \quad \lambda_k = t, \quad \text{and}
\]
\[
T_i: \quad \lambda_{k-2-i} = 2, \quad \lambda_{k-2} = (k-3)/2 - i, \quad \lambda_{k-1} = 0, \quad \text{and} \quad \lambda_k = t + i.
\]

It is not hard to verify that each such \( \vec{\lambda} \) satisfies \( \sum_{\ell \in L} \ell \lambda_\ell = n \) so all of them are indeed \( (n,L) \)-types. The assumption \( t \leq (k - 7)/2 \) guarantees that some types other than \( R \) and \( S_0 \) are used. Let us assume that we use type \( R \) for \( x \) times, type \( S_i \) for \( a_i \) times for each \( i \in \{0,1, \cdots, (k-5)/2 - t\} \), and type \( T_i \) for \( b_i \) times for each \( i \in \{1, \cdots, (k-5)/2 - t\} \). Note that only the type \( R \) corresponds to 1-factors of size \( (k-1)/2 + t \), while the other types correspond to 1-factors of size \( (k+1)/2 + t \).

If we let \( y = a_0 + a_1 \cdots + a_{(k-5)/2-t} + b_1 + \cdots + b_{(k-5)/2-t} \), then we have
\[
\left( \frac{k - 1}{2} + t \right) x + \left( \frac{k + 1}{2} + t \right) y = \binom{n}{(k+1)/2 + t} + \cdots + \binom{n}{k}, \quad (14)
\]
and
\[
x + y = \binom{n - 1}{(k-1)/2 + t} + \cdots + \binom{n - 1}{k-1} \quad (15).
\]

17
The first equation is derived from double counting the total number of subsets in \( \binom{n}{(k+2t+1)/2, \ldots, k-1, k} \), while the second equation follows by double counting the number of 1-factors in the 1-factorization of \( \binom{n}{(k+2t+1)/2, \ldots, k-1, k} \). Solving (14) and (15) (see Appendix A for a detailed proof), we can obtain the precise values of \( x \) and \( y \) and show that both \( x \) and \( y \) are non-negative integers, where

\[
a_0 + \sum_{i=1}^{(k-5)/2-t} (a_i + b_i) = y = \sum_{i=0}^{(k-5)/2-t} k - 2 + 2t + i((k-1)/2 + t) \binom{n}{k-2-i}. \tag{16}
\]

After fixing the number of times \( x \) for type \( \mathcal{R} \) and the total number of times \( y \) for all other types, our next step is to use the types \( S_1, \ldots, S_{(k-5)/2-t}, T_1, \ldots, T_{(k-5)/2-t} \) to fully occupy the levels from \((k + 2t + 1)/2 \) to \( k - 3 \). This give rise to the following equations:

\[
a_i + 2b_i = \binom{n}{k-2-i} \quad \text{for each } i \in \{1, \ldots, (k-5)/2-t\} \tag{17}
\]

Also note that \( \binom{n}{k}/\binom{n}{k-1} = (k-3)/2 + t \) and the type \( \mathcal{R} \) maintains this ratio. Hence, we also need to guarantee that the contributions of other types except \( \mathcal{R} \) to level \( k-1 \) and \( k \) are at an \( 1: (k^2/2 + t) \) ratio. This leads to the following equality

\[
ty + \sum_{i=1}^{(k-5)/2-t} ib_i - \left( \frac{k-3}{2} + t \right) \sum_{i=1}^{(k-5)/2-t} ia_i = 0. \tag{18}
\]

It turns out that to find a non-negative integer solution to the corresponding system (11) for \( \mathcal{L} = \{(k + 2t + 1)/2, \ldots, k-1, k\} \), it will suffice to find a non-negative integer solution \( a_0, a_1, \ldots, a_{(k-5)/2-t}, b_1, \ldots, b_{(k-5)/2-t} \) to the system of \((k-1)/2 - t \) equations formed by (16), (17) and (18). To solve the latter system, we could simply take

\[
b_i = \left\lfloor \frac{n}{k-2-i} \right\rfloor / 2 \quad \text{and} \quad a_i = \left\lfloor \frac{n}{k-2-i} \right\rfloor - 2b_i \quad \text{for each } i = 2, \ldots, (k-5)/2 - t,
\]

and leave the other three variables \( a_0, a_1, b_1 \) to be uniquely determined by the three equations (16), (17) for \( i = 1 \) and (18). A proof for showing the above assertions will be fully provided in Appendix A. We have now completed the proof of Lemma 3.3.

### 3.3 Proofs of Theorems 1.2 and 1.3

With all the preparations above, finally we are ready to address our main theorems.

**Proof of Theorem 1.2** For \( k < n/2 \), suppose \( n, k \) satisfy one of the two conditions. By Lemmas 3.4 and 3.5, the system (11) has a non-negative integer solution. Theorem 2.2 immediately tells us that for such \( n, k \), \( K_n^\leq k \) is 1-factorable.

Now suppose \( K_n^\leq k \) is 1-factorable. Again using Theorem 2.2, it is necessary that the system (11) has a non-negative integer solution. By Lemma 3.4, \( n \) must be congruent to 0 or \(-1\) mod \( k \). Now apply Lemma 3.2, we know that one of the two conditions in the statement of Theorem 1.2 must be met and this completes our proof.
As of now, we have completely characterized all the \( n, k \) in the range \( k < n/2 \) such that \( K_n^{≤ k} \) is 1-factorable. The range \( k ≥ n/2 \) could be tackled in a similar fashion by applying Farkas’ Lemma and Theorem 2.2. However, the statement of Theorem 1.3 already suggests that there is a very simple reduction to the \( k < n/2 \) range, as demonstrated below.

**Proof of Theorem 1.3**: We first show that for \( k ≥ n/2 \), if \( K_n^{≤ n−k−1} \) is 1-factorable, then \( K_n^{≤ k} \) is also 1-factorable. Note that in this range, we always have \( n−k ≤ k \). Take an arbitrary 1-factorization \( M_1, \cdots, M_t \) of \( K_n^{≤ n−k−1} \). For every subset \( S \) of \([n]\) of size between \( n−k \) and \( k \), we just pair \( S \) with its complement. This gives \( \frac{1}{2} \sum_{i=n−k}^{n} \binom{n}{i} \) 1-factors, which together with \( M_1, \cdots, M_t \) form a 1-factorization of \( K_n^{≤ k} \).

Next we prove the opposite direction. Suppose \( k ≥ n/2 \) and \( K_n^{≤ k} \) can be decomposed into 1-factors \( M_1, \cdots, M_s \). For every \( k \)-set \( S \), it must appear in some \( M_i \). Suppose \( M_i \) consists of the subset \( S \), together with \( \ell \) other subsets \( T_1, \cdots, T_\ell \). Then \( |T_1 ∪ \cdots ∪ T_\ell| = n− |S| = n− k ≤ k \). So \( T_1 ∪ \cdots ∪ T_\ell \) must also appear in some \( M_j \) (possibly \( j = i \) if \( \ell = 1 \)). We move those \( T_1, \cdots, T_\ell \) from \( M_i \) to \( M_j \), and also move \( T_1 ∪ \cdots ∪ T_\ell \) from \( M_j \) to \( M_i \), to obtain a new 1-factorization of \( K_n^{≤ k} \). Now \( S \) is paired with \( \overline{S} \). We repeat this process for every \( k \)-set \( S \), and apply the same operation for sets of size between \( n/2 \) and \( k−1 \) as well. At the end of this process, we end up with a 1-factorization of \( K_n^{≤ k} \) such that for each \( n−k ≤ i ≤ k \), every \( i \)-set is paired with its complement. Removing these 1-factors gives an 1-factorization of \( K_n^{≤ n−k−1} \).

**4 Extensions to other unions of levels of hypercube**

Given a set \( \mathcal{L} \) of distinct positive integers, recall that we denote by \( \binom{[n]}{\mathcal{L}} \) the family of subsets of \([n]\) whose size is an element of \( \mathcal{L} \). Can we find a necessary and sufficient condition for \( \binom{[n]}{\mathcal{L}} \) to be 1-factorable, when \( n \) is sufficiently large? Our Theorem 1.2 answers this question for \( \mathcal{L} = \{1, \cdots, k\} \), showing that the condition needed is simply \( n ≡ 0, −1 \) (mod \( k \)). Does there exist such a neat sufficient and necessary condition for general \( \mathcal{L} \)? In this section we establish a number of results in this direction. For the proofs below, we always let \( k \) be the maximum element of \( \mathcal{L} \).

**Theorem 4.1.** When \( n \) is sufficiently large and divisible by \( k \), \( \binom{[n]}{\mathcal{L}} \) is always 1-factorable.

**Proof.** Suppose \( \mathcal{L} = \{\ell_1, \cdots, \ell_t\} \) with \( \ell_1 < \cdots < \ell_t = k \), \( n = jk \) and \( j \) is sufficient large. We present a non-negative integer solution to the system \((1)\) with \( b_i = \binom{n}{i} \) for \( i \in \mathcal{L} \) and 0 otherwise, along the same line of the proof of Lemma 3.3 and the conclusion follows from Theorem 2.2.

For each \( 1 ≤ i ≤ t−1 \), we take \( \lambda_i ∈ \mathbb{Z}_{≥0}^k \) such that

\[
λ_{\ell_i} = \frac{k}{\gcd(k, \ell_i)}, \quad λ_k = \frac{n}{k} − \frac{\ell_i}{\gcd(k, \ell_i)}, \quad λ_s = 0 \quad \text{for} \quad s ∉ \{\ell_i, k\}.
\]

Similarly as before, it is not hard to see that \( \lambda \) is a \((n, \mathcal{L})\)-type for sufficiently large \( n \) divisible by \( k \). Now we just let \( x_\lambda = \binom{n}{\ell_i}/(k/\gcd(k, \ell_i)) \) to take care of \( b_{\ell_i} \) for each \( i = 1, \cdots, t−1 \). Finally we use the type \( \lambda = (0, \cdots, 0, n/k) \) to take care of the remainder for \( b_k \). The number of such types is non-negative follows similarly as before.

Our next result generalizes Lemma 3.4.
Theorem 4.2. For sufficiently large $n$, if $\binom{n}{i}$ is 1-factorable, then $n \equiv 0$ or $–1 \pmod{k}$.

Proof. Suppose $n = jk + r$ for $1 \leq r \leq k – 2$. We use Farkas’ Lemma and take $\vec{y} \in \mathbb{R}^k$ such that $y_k = -1$, $y_r = j$ and $y_s = j/2$ for $s \not\in \{k, r\}$. First we explain why $A_L\vec{y} \geq 0$. Take a row $\vec{\lambda}$ of $A_L$ by our definition of $A_L$, we have $\sum_{i=1}^k i\lambda_i = n$ and $\lambda_i = 0$ for those $i \not\in L$. Note that $n = jk + r < (j + 1)k$, we have $\lambda_k \leq j$ and $\sum_{i=1}^{k-1} i\lambda_i = r$ or $\sum_{i=1}^{k-1} i\lambda_i \geq r + k$. These imply that $\lambda_k = 1$ or $\sum_{i=1}^{k-1} \lambda_i \geq 2$. In either case we have

$$\sum_{i=1}^k i\lambda_i y_i = \left(\sum_{i=1}^{k-1} \lambda_i y_i\right) + \lambda_k y_k \geq j - j = 0.$$ 

Next we show that $\vec{y} \cdot \vec{b}_L < 0$. Equivalently, we need to prove

$$\sum_{i \in L \setminus \{k\}} \frac{j}{2} \binom{n}{i} + \frac{j}{r} \binom{n}{r} < \binom{n}{k}.$$ 

We instead prove a stronger inequality

$$\sum_{i=1}^{k-1} \frac{j}{2} \binom{n}{i} + \frac{j}{r} \binom{n}{r} < \binom{n}{k}. \tag{19}$$

Note that for $i \geq 1$,

$$\binom{n}{k-i} / \binom{n}{k-i+1} = \frac{k-i+1}{n-k+i} = \frac{k-i+1}{(j-1)k+i+r} \leq \frac{k}{(j-1)k+r+1} \leq \frac{1}{j-1}. $$

Also we have $\binom{n}{r} \leq \binom{n}{k-2}$. Therefore the left hand side of (19) is at most

$$\left(\frac{1}{j-1} + \frac{1}{(j-1)^2} + \cdots\right) \cdot \frac{j}{2} \binom{n}{k} + \frac{j}{r} \cdot \frac{1}{(j-1)^2} \cdot \binom{n}{k} < \binom{j}{2(j-2)} \binom{n}{k} < \binom{n}{k}$$

for $j \geq 5$. Since we assume that $n$ is sufficiently large, this completes the proof. \hfill $\Box$

When $n \equiv -1 \pmod{k}$, we can further prove the following.

Theorem 4.3. Suppose $n$ is sufficient large and $n = jk + k – 1$ for some positive integer $k$. If $\binom{n}{L}$ is 1-factorable, then $L$ must contain the element $k – 1$.

Proof. We prove the contra-positive: suppose $k – 1 \not\in L = \{\ell_1, \cdots, \ell_t\}$ with $\ell_1 < \cdots < \ell_t = k$, then $\binom{n}{L}$ is not 1-factorable. Once again we would like to use Farkas’ Lemma to show that $A_L^T \vec{x} = \vec{b}_L$ has no non-negative solution. Take the vector $\vec{y} \in \mathbb{R}^k$ such that

$$y_{\ell_i} = \frac{j}{2} \text{ for } i = 1, \cdots, t - 1, \quad y_k = -1.$$
The rest of the coordinates of $\vec{y}$ are set to be zero. First we show that $A_L \vec{y} \geq 0$. Take a row $\vec{\lambda} = (\lambda_1, \cdots, \lambda_k)$, recall that $\lambda_i = 0$ for $i \notin L$. So $\sum_{i=1}^{t} \ell_i \lambda_i = n = jk + (k-1)$. Therefore $\lambda_k \leq j$ and $\sum_{i=1}^{t-1} \lambda_i \geq 1$. If $\sum_{i=1}^{t-1} \lambda_i \geq 2$, then

$$\sum_{i=1}^{k} \lambda_i y_i \geq \frac{j}{2} \cdot \left( \sum_{i=1}^{t-1} \lambda_i \right) - \lambda_k \geq j - j = 0.$$ 

If $\sum_{i=1}^{t-1} \lambda_i = 1$, then $n$ is congruent to $\ell_i$ modulo $k$, for some $1 \leq i \leq t - 1$, this is not possible since $k - 1 \notin L$ and thus all such $\ell_i$ are at most $k - 2$, but we have $n \equiv k - 1 \pmod{k}$.

Next we prove $b_L \cdot \vec{y} < 0$. Since $k - 1 \notin L$, it suffices to prove

$$\sum_{i=1}^{k} i \binom{n}{i} < \binom{n}{k}.$$ 

(20)

Note that for every $i \geq 1$,

$$\frac{n}{k-i} / \frac{n}{k-i+1} = \frac{k-i+1}{n-k} = \frac{k-i+1}{jk+i-1} \leq \frac{1}{j}.$$ 

Therefore the left hand side of inequality (20) is at most

$$\frac{j}{2} \left( \frac{1}{j^2} + \frac{1}{j^3} + \cdots \right) \binom{n}{k} < \frac{1}{2(j-1)} \binom{n}{k} \leq \binom{n}{k},$$

as long as $j \geq 2$. 

We make the following conjecture based on the above results.

**Conjecture 4.4.** Given a set of distinct positive integers $L$ whose largest element is $k$. Depending on the choice of $L$, exactly one of the following statements must be true:

(i) For sufficiently large $n$, $\binom{n}{L}$ is 1-factorable if and only if $n \equiv 0 \pmod{k}$,

(ii) For sufficiently large $n$, $\binom{n}{L}$ is 1-factorable if and only if $n \equiv 0, -1 \pmod{k}$.

When $k - 1 \in L$, it seems that both scenarios are possible. For example, $L = \{k - 1, k\}$ satisfies (ii), since for $n = jk - 1$, we could use the type $\vec{\lambda} = (0, \cdots, 0, 1, j - 1)$ exactly $\binom{n}{k-1}$ times, and Theorem 4.1 takes care of the $n = jk$ case. On the other hand, if we take $k = 4$ and $L = \{2, 3, 4\}$, then for $n = 4j - 1$, applying Farkas’ Lemma with the vector $\vec{y} = (-1/2, j - 1, -1)^T$ shows that $\binom{n}{L}$ is not 1-factorable, and thus $L = \{2, 3, 4\}$ corresponds to case (i) in Conjecture 4.1. It would be great if some criteria on those $L$ satisfying $k - 1 \in L$ could be found to tell whether $\binom{n}{L}$ is 1-factorable for sufficiently large $n \equiv -1 \pmod{k}$.

### 5 Concluding Remarks

In this paper, we determine all the pairs of positive integers $(n, k)$ such that $\binom{n}{L}$ is 1-factorable. We include a few open questions and possible future projects below.
Our method might be extendable to the scenario when repetition of subsets are allowed in the hypergraph. In [2], Bahmanian show that the existence of a symmetric Latin cube is equivalent to the existence of a partition into parallel classes of some non-uniform set system. In particular, he determined all $n$ such that $(\binom{n}{1}) \cup 3(\binom{n}{2}) \cup 2(\binom{n}{3})$ is 1-factorable. Here every pair appears three times and every triple shows up twice. Would it be possible to determine for which $n$ and sequence $\{m_i\}_{i=1,\ldots,n}$ of non-negative integers, $\cup_{i=1}^{n} m_i(\binom{n}{i})$ is 1-factorable?

A famous conjecture of Chvátal [12] says that any for any given hereditary family $\mathcal{F}$ (i.e. $A \in \mathcal{F}$ and $B \subset A$ implies $B \in \mathcal{F}$), the maximum size of its intersecting subfamily is attained by taking all sets containing the most popular element. If we construct a graph $G_\mathcal{F}$ such that $V(G_\mathcal{F}) = \mathcal{F}$, and two vertices $S, T \in \mathcal{F}$ are adjacent if and only if $S \cap T = \emptyset$, then Chvátal’s conjecture is equivalent to showing $\alpha(G_\mathcal{F}) = \max_x |\mathcal{F}_x|$, where we denote by $\mathcal{F}_x$ the family of subsets containing $x$. Observe that when $\mathcal{F}$ is regular, i.e. every element in the ground set occurs for an equal number of times, $\mathcal{F}$ is 1-factorable if and only if $\chi(G_\mathcal{F}) = \max_x |\mathcal{F}_x|$, which is a stronger bound and immediately implies $\alpha(G_\mathcal{F}) = \max_x |\mathcal{F}_x|$. Inspired by Theorem 1.2, one may wonder whether there is a nice characterization of all the regular hereditary families $\mathcal{F}$ that are 1-factorable.

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Appendix A  Finding a non-negative integer solution in the proof of Lemma 3.5

We consider the case when \( k \) is odd, \( n = (k^2 - k - 2)/2 + tk \) and \( 0 \leq t \leq (k - 7)/2 \). In particular \( k \geq 7 \). Our aim here is to provide detailed arguments to the proof described for this case of Lemma 3.5, as to find a non-negative integer solution to the corresponding system \((1)\) for \( \mathcal{L} = \{(k + 2t + 1)/2, \ldots, k - 1, k\} \).

First let us determine \( x \) and \( y \) as non-negative integers. From \((14)\) and \((15)\),

\[
y = \sum_{i=(k+1)/2+t}^{k} \left( \binom{n}{i} - \left( \frac{k-1}{2} + t \right) \cdot \binom{n-1}{i-1} \right)
\]

\[
= \left( \sum_{i=(k+1)/2+t}^{k-2} \frac{n-i((k-1)/2+t)}{n} \cdot \binom{n}{i} \right) + \frac{(k-3)/2+t}{n} \binom{n}{k-1} - \frac{1}{n} \binom{n}{k}
\]

\[
= \sum_{i=(k+1)/2+t}^{k-2} \frac{n-i((k-1)/2+t)}{n} \cdot \binom{n}{i}
\]

\[
= \frac{(k-5)/2-t}{n} \sum_{i=0}^{k-5/2-t} \frac{k-2+2t+i((k-1)/2+t)}{n} \cdot \binom{n}{k-2-i}.
\]  \( (21) \)

From the first equality, we know that \( y \) is an integer, and the last form clearly indicates that \( y \) is non-negative. From \((14)\) and \((15)\), using \( n = k(\frac{k+1}{2}+t) - (k+1) \) and \( \frac{k+1}{2}+t \leq k+1 \), we also have

\[
x = \sum_{i=(k+1)/2+t}^{k} \left( \left( \frac{k+1}{2} + t \right) \cdot \binom{n-1}{i-1} - \binom{n}{i} \right)
\]

\[
= \sum_{i=(k+1)/2+t}^{k} \frac{i((k+1)/2+t) - n}{n} \cdot \binom{n}{i}
\]

\[
= \frac{1}{n} \sum_{i=(k+1)/2+t}^{k} \left( (k+1) - (k-i)(\frac{k+1}{2}+t) \right) \cdot \binom{n}{i}
\]

\[
\geq \frac{1}{n} \sum_{i=(k+1)/2+t}^{k-1} \left( (k+1) \cdot \binom{n}{i+1} - (k-i)(\frac{k+1}{2}+t) \cdot \binom{n}{i} \right)
\]

\[
\geq \frac{k+1}{n} \sum_{i=(k+1)/2+t}^{k-1} \frac{n-i-(k-i)(i+1)}{i+1} \cdot \binom{n}{i} \geq 0,
\]
where the first equality shows \( x \) is an integer and the last inequality follows from the fact that 
\[ n - i - (k - i)(i + 1) = n - (k - i)i - k \geq \frac{k^2 - k - 2}{4} - k \geq 0 \] holds for \( k \geq 7 \).

Recall the system of \((k-1)/2 - t\) equations formed by \([16],[17]\) and \([18]\), and that to find a non-negative integer solution \(a_0,a_1,\cdots,a_{(k-5)/2-t},b_1,\cdots,b_{(k-5)/2-t}\) to this system, we take for each \(i \in \{2,3,\cdots,(k-5)/2-t\}\),

\[
 b_i = \left\lfloor \frac{n}{k - 2 - i} \right\rfloor /2 \quad \text{and} \quad a_i = \left( \frac{n}{k - 2 - i} \right) - 2b_i \in \{0,1\}. \tag{22}
\]

It remains to find non-negative integers \(a_0,a_1,b_1\) for the system. Consider \(A = \sum_{i=1}^{(k-5)/2-t} ia_i\) and \(B = \sum_{i=1}^{(k-5)/2-t} ib_i\). Rewriting \([18]\), we obtain

\[
 B = \left( \frac{k - 3}{2} + t \right) A - ty. \tag{23}
\]

Now we multiply \(i\) to \([17]\), and sum over all defined \(i\), we have

\[
 A + 2B = \sum_{i=1}^{(k-5)/2-t} i \left( \frac{n}{k - 2 - i} \right). \tag{24}
\]

Plugging in the expression \([23]\) on \(B\) to \([24]\), we can solve \(A\) as follows:

\[
 (k - 2 + 2t)A = 2ty + \sum_{i=1}^{(k-5)/2-t} i \left( \frac{n}{k - 2 - i} \right)
 = \sum_{i=0}^{(k-5)/2-t} \frac{(k - 2 + 2t)(2t + i(t + \frac{k + 1}{2}))}{n} \left( \frac{n}{k - 2 - i} \right) \tag{25}
 = (k - 2 + 2t) \sum_{i=0}^{(k-5)/2-t} 2t + i \left( t + \frac{k + 1}{2} \right) \left( \frac{n}{k - 2 - i} \right)
 = (k - 2 + 2t) \sum_{i=0}^{(k-5)/2-t} \left( n - (k - 2 - i) \left( \frac{k + 1}{2} + t \right) \right) \left( \frac{n}{k - 2 - i} \right)
 = (k - 2 + 2t) \sum_{i=0}^{(k-5)/2-t} \left( \left( \frac{n}{k - 2 - i} \right) - \left( \frac{k + 1}{2} + t \right) \left( \frac{n}{k - 3 - i} \right) \right).
\]

This shows that \(A\) is an non-negative integer (where the third last form shows the non-negativity). Using \([23]\), \(B\) is also an integer.

We now show \(a_1,b_1,a_0\) are non-negative integers in order. First we determine \(a_1\) from \(A\). If \(\frac{k-5}{2} - t = 1\), then clearly \(a_1 = A\) is a non-negative integer. Assume \(\frac{k-5}{2} - t \geq 2\), which implies that \(k \geq 9\). By \([22]\), we have \(a_i \in \{0,1\}\) for \(2 \leq i \leq (k-5)/2 - t\). Using \([25]\) that
\[
A = \sum_{i=0}^{(k-5)/2-t} \frac{2t+i(t+(k+1)/2)}{n} \binom{n}{k-2-i}, \text{ we then get }
\]

\[
a_1 = A - \sum_{i=2}^{(k-5)/2-t} ia_i = \sum_{i=2}^{(k-5)/2-t} \left( 2t + i(t + \frac{k+1}{2}) \right) \frac{\binom{n}{k-2-i}}{n} - \sum_{i=2}^{(k-5)/2-t} ia_i
\]

\[
\geq \sum_{i=2}^{(k-5)/2-t} \left[ \left( 2t + i(t + \frac{k+1}{2}) \right) \frac{\binom{n}{k-2-i}}{n} - i \right] \geq \sum_{i=2}^{(k-5)/2-t} \left( \frac{k+1}{2} - 1 \right) i \geq 0,
\]

as desired. Since \(B\) is an integer, it is easy to see that \(b_1 = B - \sum_{i=2}^{(k-5)/2-t} ib_i\) is also an integer. We also have \(a_1 + 2b_1 = \binom{n}{k-2-1}\) from (17), which gives

\[
b_1 = \frac{1}{2} \left( \left( \binom{n}{k-3} - a_1 \right) \geq \frac{1}{2} \left( \left( \binom{n}{k-3} - A \right) \right) \right).
\]

Using (25), \(\binom{n}{k-2}/\binom{n}{k-3} = \frac{n-k+3}{n-k-1}\), and \(\binom{n}{k-3-i}/\binom{n}{k-3-i-1} \leq \frac{n-k+3}{n-k+4}\) for \(i \geq 0\), we have

\[
\binom{n}{k-3} - A = \left( 1 - \frac{2t(n-k+3)}{n(k-2)} - \frac{k+1}{2} + 3t \right) \binom{n}{k-3} - \sum_{i=2}^{(k-5)/2-t} \frac{2t + \left( \frac{k+1}{2} + t \right) i}{n} \binom{n}{k-2-i}
\]

\[
= \frac{k-3-2t}{k-2} \binom{n}{k-3} - \sum_{i=2}^{(k-5)/2-t} 2t + \left( \frac{k+1}{2} + t \right) i \binom{n}{k-2-i}
\]

\[
\geq \frac{k-3-2t}{k-2} \binom{n}{k-3} - \sum_{i=2}^{(k-5)/2-t} \frac{k-4k-5}{4n} \binom{n}{k-2-i}
\]

\[
\geq \frac{k-3-2t}{k-2} \binom{n}{k-3} - \frac{k^2-4k-5}{4n} \left( \frac{k-3}{n-k+4} + \left( \frac{k-3}{n-k+4} \right)^2 + \cdots \right) \binom{n}{k-3}
\]

\[
\geq \frac{k-3-2t}{k-2} \binom{n}{k-3} - \frac{k^2-4k-5}{4n} \cdot \frac{k-3}{n-2k+7} \binom{n}{k-3}
\]

\[
\geq \left( \frac{4}{k-2} - \frac{1}{k-2} \frac{k^2-8k+15}{k^2-5k+12} \right) \binom{n}{k-3} \geq \frac{3}{k-2} \binom{n}{k-3} \geq 0,
\]

where the third last inequality uses that \(t \leq (k-7)/2\) and \(n \geq (k^2 - k - 2)/2\). From the analysis above, we have shown that \(b_1\) is a non-negative integer.

Recall that \(y = a_0 + \sum_{i=1}^{(k-5)/2-t} (a_i + b_i)\). It is easy to check that \(\sum_{i=1}^{(k-5)/2-t} a_i \leq \sum_{i=1}^{(k-5)/2-t} ia_i = A\) and \(b_i \leq \frac{1}{2} \binom{n}{k-2-i}\) for \(2 \leq i \leq (k-5)/2 - t\). In addition, we have \(b_1 = \frac{1}{2} \left( \binom{n}{k-3} - a_1 \right)\) and
Recall from (21) that

\[ a_1 \geq A - \sum_{i=2}^{(k-5)/2-t} i. \]  

Thus we have

\[ a_0 = y - \sum_{i=1}^{(k-5)/2-t} a_i - \sum_{i=1}^{(k-5)/2-t} b_i \]

\[ \geq y - A - \frac{1}{2} \sum_{i=2}^{(k-5)/2-t} \left( \binom{n}{k-2-i} - \frac{1}{2} \left( \binom{n}{k-3} - \left( A - \sum_{i=2}^{(k-5)/2-t} i \right) \right) \right) \]

\[ = y - \frac{1}{2} A - \frac{1}{2} \sum_{i=1}^{(k-5)/2-t} \left( \binom{n}{k-2-i} - \frac{(k-1)/2-t}{4} (k-7)/2-t \right). \]

Recall from (21) that

\[ y = \sum_{i=0}^{(k-5)/2-t} \frac{k-2+2t+i(t+(k-1)/2)}{n} \binom{n}{k-2-i}, \tag{26} \]

and from (25) that \( A = \sum_{i=0}^{(k-5)/2-t} \frac{2t+i(t+(k+1)/2)}{n} \binom{n}{k-2-i} \). Using \( ((k-1)/2-t)((k-7)/2-t) \leq (k-1)(k-7)/4 \leq n/2 \) and \( \binom{n}{k-2-i}/\binom{n}{k-2-i} \leq \frac{k-2}{n-k+3} \) for \( i \geq 0 \), we can derive that (note that \( 2k - 4 - n < 0 \))

\[ a_0 \geq \left( \sum_{i=1}^{(k-5)/2-t} \frac{2k-4+2t+i(\frac{k-3}{2}+t)-n}{2n} \binom{n}{k-2-i} \right) + \frac{k-2+t}{n} \binom{n}{k-2} - \frac{n}{8} \]

\[ \geq \left( \sum_{i=1}^{(k-5)/2-t} \frac{2k-4-n}{2n} \binom{n}{k-2-i} \right) + \frac{k-2}{n} \binom{n}{k-2} - \frac{n}{8} \]

\[ \geq - \frac{2k-4+4}{2n} \binom{n}{k-2-i} \left( \frac{k-2}{n-k+3} + \left( \frac{k-2}{n-k+3} \right)^2 \right) \binom{n}{k-2} + \frac{k-2}{n} \binom{n}{k-2} - \frac{n}{8} \]

\[ = \frac{k-2}{n} \binom{n}{k-2} - \frac{n-2k+4}{2n} \frac{k-2}{n-2k+5} \binom{n}{k-2} - \frac{n}{8} \]

\[ \geq \frac{k-2}{2n} \binom{n}{k-2} - \frac{n}{8} \geq \frac{1}{2} \binom{n-1}{k-3} - \frac{n}{8} \geq 0. \]

This completes the proof for finding a non-negative integer solution

\[ S = \{ a_0, a_1, \cdots, a_{(k-5)/2-t}, b_1, \cdots, b_{(k-5)/2-t} \} \]

to the system formed by (16), (17) and (18).

Lastly, we show that the above non-negative integer solution \( S \) together with the non-negative integer \( x \) solved from (14) and (15), along with the corresponding \((n, \mathcal{L})\)-types, give a non-negative integer solution to the corresponding system \( \mathcal{H} \) for \( \mathcal{L} = \{(k+2t+1)/2, \cdots, k-1, k\} \). By (17), the contribution of these types to level \( k-2-i \) satisfies

\[ a_i + 2b_i = \binom{n}{k-2-i} \quad \text{for each} \quad i \in \{1, \cdots, (k-5)/2-t\}. \tag{27} \]
Thus, it is enough to check that the contribution of these types to each level \( j \in \{k - 2, k - 1, k\} \) equals \((n_j)\). We observe that as \( S \) satisfies (14), (15) and (18), it in fact suffices to verify that the contribution of these types to level \( k - 2 \) equals \((n_{k-2})\), that is, we need to show the following

\[
\frac{k + 1}{2} a_0 + \sum_{i=1}^{(k-5)/2-t} \left( \frac{k - 1}{2} - i \right) a_i + \sum_{i=1}^{(k-5)/2-t} \left( \frac{k - 3}{2} - i \right) b_i = \left( \frac{n}{k - 2} \right). \tag{28}
\]

We claim that the above equation (28) is a linear combination of (18), (26), and (27) for all \( i \in \{1, \cdots, (k - 5)/2 - t\} \), with coefficients

\[
\frac{1}{k - 2 + 2t}, \quad \frac{n}{k - 2 + 2t}, \quad \text{and} \quad -\frac{k - 2 + 2t + i((k - 1)/2 + t)}{k - 2 + 2t} \quad \text{for all} \quad i \in \{1, \cdots, \frac{k - 5}{2} - t\},
\]

respectively. It is easy to see that the right side hand of the above linear combination equals \((n_{k-2})\). For the left side hand of this linear combination, a careful calculation, using \( t + n = \frac{k + 1}{2}(k - 2 + 2t) \) and \( y = a_0 + \sum_{i=1}^{(k-5)/2-t}(a_i + b_i) \), shows that it equals

\[
\frac{(t + n)y}{k - 2 + 2t} + \frac{1}{k - 2 + 2t} \sum_{i=1}^{(k-5)/2-t} \left[ i - 2 \left( k - 2 + 2t + i\left( \frac{k - 1}{2} + t \right) \right) \right] b_i
\]

\[
-\frac{1}{k - 2 + 2t} \sum_{i=1}^{(k-5)/2-t} \left[ \left( \frac{k - 3}{2} + t \right) i + \left( k - 2 + 2t + i\left( \frac{k - 1}{2} + t \right) \right) \right] a_i
\]

\[
= \frac{k + 1}{2} \left( a_0 + \sum_{i=1}^{(k-5)/2-t} (a_i + b_i) \right) - \sum_{i=1}^{(k-5)/2-t} (i + 2)b_i - \sum_{i=1}^{(k-5)/2-t} (i + 1)a_i
\]

\[
= \frac{k + 1}{2} a_0 + \sum_{i=1}^{(k-5)/2-t} \left( \frac{k - 1}{2} - i \right) a_i + \sum_{i=1}^{(k-5)/2-t} \left( \frac{k - 3}{2} - i \right) b_i,
\]

which is the left side hand of (28). This proves our claim, completing the proof of this appendix. □