Periodic Solutions and KAM Tori for the Spatial Maxwell Restricted $N + 1$-Body Problem with Manev Potential

Mauricio Ascencio$^1$ · Claudio Vidal$^2$

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Abstract
We consider the motion of an infinitesimal mass under the Newtonian attraction of $N$ point masses forming a ring plus a central body where a Manev potential $(-1/r + e/r^2, e \in \mathbb{R})$, is applied to the central body. More precisely, the bodies are arranged in a planar ring configuration. This configuration consists of $N - 1$ primaries of equal mass $m$ located at the vertices of a regular polygon that is rotating on its own plane about its center of mass with a constant angular velocity $\omega$. Another primary of mass $m_0 = \beta m$ ($\beta > 0$ parameter) is placed at the center of the ring. Moreover, we assume that the central body may be an ellipsoid, or a radiation source, which introduces a new parameter $e$. The existence and stability of periodic solutions of the spatial Maxwell restricted $N + 1$-body problem is obtained using averaging theory. The determination of KAM 3-tori encasing some of the linearly stable periodic solutions is proved. The planar case is moreover considered.

Keywords Spatial Manev restricted $(N + 1)$-body problem · Periodic solutions · KAM-tori

Mathematics Subject Classification 70F10 · 70F15 · 70H12 · 37J40 · 34C25

Abbreviations
Manev MR($N + 1$)BP : Maxwell restricted ($N + 1$)–body problem with Manev potential
1 Introduction and statement of the problem

The Manev potential was introduced by Newton in his work Philosophiae Naturalis Principia Mathematica (Book I, Article IX, Proposition XLIV, Theorem XIV, Corollary 2). The reason was the impossibility to explain the Moon apsidal motion within the framework of the inverse-square force law. Manev (1924) in [9] proposed a similar corrective term in order to maintain dynamical astronomy within the framework of classical mechanics and offering at the same time equally good justification of the observed phenomena as in relativity theory. In the last years, this quasi-homogeneous potentials have been studied. Other types of quasi-homogeneous potential were studied, for example, in [8], in addition to the Manev type.

In this work we consider a spatial Maxwell restricted $N + 1$-body problem with Manev potential which was studied previously in [4]. Here, $n = N − 1$ bodies $\mathcal{P}_i$, $i = 1, \ldots, n$ of mass $m$ are in an $n$-gon, these primaries are known as peripherals. Other primary $\mathcal{P}_0$ of mass $m_0 = \beta m$ is located in the center of the $n$-gon, known as central body. Assume that the gravitational attraction of the primary $\mathcal{P}_0$ is generated by a Manev potential $\left(−1/r + e/r^2\right)$, with parameter $e$. We emphasize that the parameter $e \in \mathbb{R}$ models several problems, for instance, where the central body of the ring is no longer spherical, but an ellipsoid of revolution (spheroid). According to [4, 5] the parameter $e$ is associated to oblateness, with natural bodies like planets, the spheroid is oblate $e < 0$, but we can also think of artificial bodies and assume them to be prolate, in that case, $e > 0$. We can also think of the central body to be a radiation source, repulsive if $e > 0$ and attractive if $e < 0$, and then, the effect of the radiation can be modeled in a similar way to the oblate ellipsoid (see, for instance, [6]). The gravitational attraction due to $\mathcal{P}_i$, $i = 1, \ldots, n$ is Newtonian $\left(−1/r\right)$. In an inertial reference system the peripheral bodies move in a circular orbit around $\mathcal{P}_0$ with angular velocity $\omega$. Let $Oxyz$ be Cartesian coordinates in a rotating reference frame with origin at the central body $\mathcal{P}_0$ where the $Ox$ axis coincides with the line joining the primaries $\mathcal{P}_0$ and $\mathcal{P}_i$. We choose the units of distance, mass and time such that the distance between two consecutive peripherals is one and $Gm = 1$, where $G$ is the Gravitational constant. Then, the coordinates of the primaries $\mathcal{P}_i$, $i = 0, \ldots, n$ in our synodic reference frame are, respectively, $\left(0, 0, 0\right)$, $\left(x_i, y_i, 0\right)$, $i = 1, \ldots, n$. According to [4, 5], the condition to keep the peripherals on their circular orbit of radius $1/\rho$ and angular velocity $\omega$ is that $\omega^2 = \Delta$, where

\[
\Delta = \Delta(\beta, e) = \rho(\Lambda + \beta \rho^2 - 2\beta e \rho^3),
\]

with $\rho = 2 \sin(\pi/n)$ and $\Lambda = \sum_{i=2}^{n} \frac{\sin^2(\pi/i)}{\sin((i-1)\pi/n)}$. The function $\Delta$ has to be positive. Thus, the parameter $e$ must satisfy the following sharp bound,

\[
e < e_{\text{sup}} := \frac{\Lambda + \beta \rho^2}{2\beta \rho^3},
\]

for each fixed $\beta > 0$.

The equations of motion of a small body $\mathbf{P}$ with infinitesimal mass in a rotating coordinate system $Oxyz$ for the spatial Maxwell restricted $N + 1$-body problem with
Manev potential (in short Manev MR\((N + 1)\)BP) (see Fig. 1), are given by the following differential equations:

\[
\begin{align*}
\ddot{x} - 2\dot{y} &= \Omega_x, \\
\ddot{y} + 2\dot{x} &= \Omega_y, \\
\ddot{z} &= \Omega_z,
\end{align*}
\]

where the function \(\Omega\) is defined by

\[
\Omega = \Omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1}{\Delta} \left[ \beta \left( \frac{1}{r_0} - \frac{e}{r_0^2} \right) + \sum_{i=1}^{n} \frac{1}{r_i} \right],
\]

with

\[
\begin{align*}
\Delta &= \sqrt{x^2 + y^2 + z^2} \\
r_0 &= \sqrt{x^2 + y^2 + z^2} \\
r_i &= \sqrt{(x - x_i)^2 + (y - y_i)^2 + z^2}, \quad i = 1, \ldots, n.
\end{align*}
\]

See [4, 5] for details. Thus, the Hamiltonian function for the Spatial Manev MR\((N + 1)\)BP is given by

\[
H = \frac{1}{2}(X^2 + Y^2 + Z^2) + yX - xY - V,
\]

which is autonomous and has three degrees of freedom, with \(X = \dot{x} - y\), \(Y = \dot{y} + x\) and \(Z = \dot{z}\), also \(V\) is the potential, given by

\[
V = \frac{1}{\Delta} \left[ \beta \left( \frac{1}{r_0} - \frac{e}{r_0^2} \right) + \sum_{i=1}^{n} \frac{1}{r_i} \right].
\]

Dynamical aspects of the case \(n = 2\) were given in [3]. For the planar case, a particular numerical study \((n = 7)\) is made on the amount of equilibria and the bifurcations that depend on the Manev parameter in [2]. Other studies about the existence of equilibria and permitted region of motion can be found in [4, 5]. Also in [5] it was studied the existence of some symmetric periodic solutions in the planar case using numerical methods.

The purpose of this article is to obtain an analytical view of the spatial Manev MR\((N + 1)\)BP. More precisely, we search for different families of periodic solutions.
with averaging theories of Hamiltonian systems [10, 11, 14]. We also achieve the existence of KAM tori of dimension three close to some of the previous stable periodic solutions. For this we consider convenient situations of the parameter $\epsilon$ and $\beta$.

In our approach, we consider the formulation of the problem associated to the Hamiltonian (1.6) using perturbations motivated by [11], where the authors conveniently introduce a small parameter ($\epsilon$) by some re-scaling variables and parameters. These new coordinates received the name of the Hill type and the Comet type, respectively. The first coordinates for studying the behaviour of orbits that are very close to the primary ones, while the second variables are useful to study those orbits that are very far from them. Additionally, we will scale the Manev parameter $e$ as a function of $\epsilon$ (only for the Hill type). By doing this, perturbations are obtained from the Kepler problem ($H_{kep} = \frac{1}{2}(X^2 + Y^2 + Z^2) - \frac{1}{\sqrt{x^2+y^2+z^2}}$) and the Coriolis effect ($H_{cor} = yX - xY$), respectively. Then, we will use different sets of Poincaré–Delau-nay variables that are appropriate to find circular periodic solutions or close to circular ones using average theory through Reeb’s theorem [14] for Hamiltonian systems. With the purpose of finding KAM tori we will introduce appropriate action-angle variables for each case. After that, we will use the Han–Li–Yi’s Theorem [7] (Reeb’s and Han–Li–Yi’s theorems, can also be seen in [12, 13]). These techniques were used by Meyer et al., for example, in [13], where the authors studied the restricted three body problem. The first perturbed term that they get when introducing Hill-type coordinates is the Corioliss effect, which differs from our work (we have a Coriollis effect plus another term). When looking for KAM tori, unlike the study in [13], we will not need higher order terms in $\epsilon$, only the first and second terms of the Hamiltonian. Meyer et al. in [12] studied several models, which can be written as a perturbation of the Coriolis effect where the first perturbed term is the Kepler problem, demonstrating that there are periodic solutions close to large radius circles and close to the equatorial plane of large period. Furthermore, they proved the existence of KAM tori in the planar case and in the spatial comet case of the restricted three-body problem, the radiation pressure problem, and the rotating double material segment problem.

In order to obtain the results of this work, we have organized it as follows. In Sect. 2 we show the main results. Section 3 considers different scenarios which consist in writing our problem as a perturbation of an integrable Hamiltonian problem, namely, the Kepler problem. The existence of different families of near circular periodic solutions in a neighbourhood of the central body, and close to $xy$-plane, and their linear stability is proven. In Sect. 4, we apply the KAM theory, specifically the Han–Li–Yi Theorem, to guarantee the existence of KAM 3-tori around the periodic solutions found in Sect. 3. Similarly, in Sects. 5 and 6, we prove the existence of periodic solutions and KAM 3-tori, respectively. Additionally, in Sects. 7 and 8, we perform a similar study in the invariant $xy$-plane, for orbits close to the central body, that is, we find periodic solutions and KAM 2-tori. It should be noted that the results obtained do not depend on the ratio of the mass of the central body to the mass of one of the remaining bodies ($\beta$) or on the number of peripherals ($n$). Finally, in Sect. 1, we introduce Reeb and Han–Li–Yi theorems, necessary in the proof of our main results.
2 Statements of the Main Results About Periodic Solutions and KAM Tori for the Manev MR\((N + 1)\)BP

2.1 Spatial Case

Let a be the length of the semi-major axis of an elliptic or circular orbit of the Kepler problem. We have the following results for periodic solutions in a neighbourhood of the central body, close to circular and close to the equatorial plane.

**Theorem 2.1** Assume that \( e = e_0 e^p \) (with \( e_0 \neq 0 \)), \( p \in \mathbb{N} \) and \( p > 4 \), \( a^{5/2} \beta^{1/3} + e_0 (\rho^3 \beta + \rho \Lambda)^{1/3} \delta_2 \neq 0 \), with \( \delta_2 \in \{0, 1\} \) and \( \varepsilon \) small parameter. There exist two families of near circular (with radius \( \left( \frac{\beta}{\rho^3 \beta + \rho \Lambda} \right)^{1/3} a \varepsilon^2 \)) linearly stable periodic solutions of the spatial restricted \( N + 1 \)-body problem with Manev potential which encircle the central body for all values \( \beta \) of the mass ratio parameter. These families are one-parameter with period close to \( 2\pi \varepsilon^3 \). They are close to the xy-plane and the orbits are very close to the central body.

The existence of the KAM 3-tori around the periodic solutions found in Theorem 2.1 is given by the following result.

**Theorem 2.2** Assume the same hypothesis that in Theorem 2.1 and \( \delta_1 = \delta_2 = 1 \). The near-circular periodic solutions of the spatial restricted \( N + 1 \)-body problem with Manev potential around the central body are enclosed by invariant KAM 3-tori for small enough \( \varepsilon \). The excluded measure for the existence of quasi-periodic invariant tori is of order \( O(\varepsilon^6) \).

Let \( K \) the third component of the angular moment. A result that applies to our study, with respect to the existence of periodic solutions far away from the primaries (close to infinity) is the following.

**Theorem 2.3** Assume \( \varepsilon \) a small parameter. There are two families of near circular (with radius \( \left( \frac{\Delta}{\rho^3 \beta + \rho \Lambda} \right)^{1/3} a \varepsilon^{-2} \)) linearly stable periodic solutions of the spatial restricted \( N + 1 \)-body problem with Manev potential that are very far from the primaries, but that enclose them, for all values \( \beta \) of the mass ratio parameter and for all values of Manev parameter \( e \). These families are one-parameter with period close to \( 2\pi \varepsilon^{-1} \). They are close to the xy-plane and the orbits tend to infinity (in the sense, that the radius of the circular orbit close the continued periodic solution, tends to infinity).

The existence of the KAM 3-tori around the periodic solutions found in Theorem 2.3 is given by the following result.

**Theorem 2.4** The near-circular periodic solutions of the spatial restricted \( N + 1 \)-body problem with Manev potential far away of the primaries are enclosed by invariant KAM 3-tori for small enough \( \varepsilon \). The excluded measure for the existence of quasi–periodic invariant tori is of order \( O(\varepsilon^6) \).
2.2 Planar Case

It is clear that the problem associated with the Hamiltonian function in (1.6) possesses a subproblem which is invariant, namely, the planar case $z = Z = 0$. This subproblem, shortly, will be called Planar Manev MR$(N + 1)$BP. Let $a$ be as before.

**Theorem 2.5** Assume that $e = e_0 e^p$ (with $e_0 \neq 0$), $p \in \mathbb{N}$ and $p \geq 3$, $a^{5/2} \beta^{1/3} \delta_1 + e_0 (\rho^2 \beta + \rho \Lambda)^{1/3} \delta_2 \neq 0$, with $\delta_1, \delta_2 \in \{0, 1\}$, $\delta_1^2 + \delta_2^2 \neq 0$ and $\varepsilon$ small parameter.

There exist two families of near circular (with radius $\varepsilon^2 (\frac{\beta}{\rho^2 \beta + \rho \Lambda})^{1/3} a$) linearly stable periodic solutions of the planar restricted $N + 1$-body problem with Manev potential which encircle the central body for all values $\beta$ of the mass ratio parameter. These families are one-parameter with period $2\pi \varepsilon^3$. The orbits are very close to the central body.

**Theorem 2.6** Assume the same hypothesis that in Theorem 2.5. The near-circular periodic solutions of the planar restricted $N + 1$-body problem with Manev potential around of the central body are enclosed by invariant KAM 2-tori for small enough $\varepsilon$. The excluded measure for the existence of quasi-periodic invariant tori is of order $O(\varepsilon^a)$, where $\alpha = 1$ if $\delta_1 = 0$ and $\delta_2 = 1$, $\alpha = 2$ if $\delta_1 = \delta_2 = 1$ and $\alpha = 3$ if $\delta_1 = 1$ and $\delta_2 = 0$.

3 Proof of Theorem 2.1

Let $q = (x, y, z)$ and $p = (X, Y, Z)$. In order to prove Theorem 2.1 we introduce the scale parameter $\varepsilon$ and the change of variables

$$
Q = \varepsilon^{-2} \mu^{-1/3} q, \quad P = \varepsilon \mu^{-1/3} p,
$$

with $\mu = \frac{\beta}{\rho^2 \beta + \rho \Lambda}$, which is a $\varepsilon^{-1} \mu^{-2/3}$-symplectic change of variables. After re-scaling the time by $t = \varepsilon^3 \tau$, the Hamiltonian (1.6) function becomes

$$
\mathcal{H} = \mathcal{H}(Q, P) = \varepsilon^2 \mu^{-2/3} H(\varepsilon^2 \mu^{1/3} Q, \varepsilon^{-1} \mu^{1/3} P).
$$

Our next step is to write $e = e_0 \varepsilon^p$, with $p \in \mathbb{N}$, $p \geq 3$, this scaling that we do with the parameter of Manev $e$, is assuming that $\varepsilon$ is a parameter close to 0. Then we develop $\mathcal{H}$ in Taylor a series around $\varepsilon = 0$. Eliminating the constant term $\frac{\varepsilon^2 n}{\rho^2 \beta + \rho \Lambda}$, we arrive to

$$
\mathcal{H}(Q, P, \varepsilon) = \mathcal{H}_0(Q, P) + \varepsilon^a \mathcal{H}_1(Q, P) + O(\varepsilon^{a+1}),
$$

with $a = 1, 2$ or 3, and
\[ \mathcal{H}_0 = \frac{||P||^2}{2} - \frac{1}{||Q||}, \quad \mathcal{H}_1 = -\delta_1 Q^T K P + \delta_2 \frac{C}{||Q||^2}, \quad (3.4) \]

with \( K = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), \( C = e_0 \mu^{-1/3} \), and where \( \delta_i \in \{0, 1\} \), \( i = 1, 2 \), but \( \delta_1^2 + \delta_2^2 \neq 0 \).

The values of \( \delta_1 \) and \( \delta_2 \) depend on the value of \( p \) that we choose. For all \( p \in \mathbb{N} \), with \( p \geq 3 \), \( \mathcal{H}_0 \) is the Kepler problem. \( \mathcal{H}_1 \) can be written in three different ways. If \( p = 3 \) or \( p = 4 \), then \( \delta_1 = 0, \delta_2 = 1 \) and \( \alpha = 1 \) and \( \alpha = 2 \), respectively. If \( p = 5 \), then \( \delta_1 = \delta_2 = 1 \) and \( \alpha = 3 \). If \( p > 5 \), then \( \delta_1 = 1, \delta_2 = 0 \) and \( \alpha = 3 \).

In the following we use the Delaunay variables for studying the periodic orbits of the Hamiltonian system associated to the Hamiltonian (3.3) (see [15] for more details). The Hamiltonian (3.3) in the Delaunay coordinates is given by

\[ \mathcal{H} = -\frac{1}{2L^2} + \varepsilon^a \mathcal{P}(l, g, h, L, G, K) + O(\varepsilon^{a+1}) \]

where \( l, g, h, L, G, K \) are the classical Delaunay variables and

\[
\begin{align*}
    l &= E - \varepsilon \sin E & L &= \sqrt{a}, \\
    g &= \theta - f = \omega & G &= \sqrt{a(1 - \varepsilon^2)}, \\
    h &= \nu = \Omega & K &= G \cos \Omega,
\end{align*}
\]

where \((a, \varepsilon, \iota, \omega, \Omega)\) are the classic Keplerian elements of elliptic orbit. \( \theta \) is the latitude argument, \( f \) is the true anomaly and \( E \) the eccentric anomaly and both are auxiliary quantities defined by the relations

\[
\sqrt{1 - \varepsilon^2} = G/L, \quad r = a(1 - \varepsilon \cos E), \quad l = E - \varepsilon \sin E
\]

with

\[
\sin f = \frac{a \sqrt{1 - \varepsilon^2} \sin E}{r}, \quad \cos f = \frac{a(\cos E - \varepsilon)}{r}
\]

and \( \varepsilon \) is the eccentricity of the unperturbed elliptic orbit.

We note that the perturbation \( \mathcal{P}(l, g, h, L, G, K) \) is from the representation of \( \mathcal{H}_1 \), but before getting to this, we introduce the polar variables. The Hamiltonian (3.3) in these coordinates is given by

\[ \mathcal{H} = \frac{1}{2} \left( R^2 + \frac{1}{r^2} \right) - \frac{1}{r} + \varepsilon^a \left( -\delta_1 K + \delta_2 \frac{C}{r^2} \right) + O(\varepsilon^{a+1}). \quad (3.6) \]

Now we can write \( \mathcal{P} \) in the mixed Polar-Delaunay variables, that is, \( \mathcal{P} = -\delta_1 K + \delta_2 \frac{C}{L^3(-1 + \varepsilon \cos(E))^2} \). So,

\[ \mathcal{H} = -\frac{1}{2L^2} + \varepsilon^a \mathcal{P} + O(\varepsilon^{a+1}) = -\frac{1}{2L^2} + \varepsilon^a \left( -\delta_1 K + \delta_2 \frac{C}{L^3(-1 + \varepsilon \cos(E))^2} \right) + O(\varepsilon^{a+1}). \quad (3.7) \]
Now, we introduce the following Poincaré-Delaunay coordinates (see details, for example, in [1] or [15])

\begin{align}
Q_1 &= l + g \pm h, \quad P_1 = L, \\
Q_2 &= \mp \sqrt{2(L - G)} \sin(g \pm h), \quad P_2 = \pm \sqrt{2(L - G)} \cos(g \pm h), \\
Q_3 &= \mp \sqrt{2(G \mp K)} \sin(h), \quad P_3 = \sqrt{2(G \mp K)} \cos(h),
\end{align}

(3.8)

where the upper sign applies for \( K > 0 \) (prograde motions), whereas the lower sign is used when \( K < 0 \) (retrograde motions). Note that the Delaunay coordinates are defined for \( 0 < |K| \leq G \leq L \), where in the Poincaré-Delaunay variables \( L = P_1 \), \( G = P_1 - \frac{1}{2}(Q_2^2 + P_2^2) \) and \( |K| = G - \frac{1}{2}(Q_3^2 + P_3^2) \). We will use these coordinates in the proof of Theorem 2.1.

**Proof of Theorem 2.1** The function \( \mathcal{P}(E, g, L, G, K) \) given by (3.7) is

\[
-\delta_1 K + \delta_2 \frac{C}{L^4(-1 + \epsilon \cos(E))^2}.
\]

Its averaged function \( \bar{\mathcal{P}} \) with respect the mean anomaly \( l \) is

\[
\bar{\mathcal{P}} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{P}(l, g, L, G, K)dl = -K\delta_1 + \delta_2 \frac{C}{G^3}.
\]

We remark that the map \( \bar{\mathcal{P}} \) only depends on the three action variables \( L, G, K \). The differential equations with respect to the mean anomaly are

\[
\begin{align*}
\frac{dg}{dl} &= \frac{\partial \mathcal{P}}{\partial G} = -\frac{C\delta_2}{G^3L^3}, \\
\frac{dh}{dl} &= \frac{\partial \mathcal{P}}{\partial K} = -\delta_1, \\
\frac{dG}{dl} &= -\frac{\partial \mathcal{P}}{\partial q}_g = 0, \\
\frac{dK}{dl} &= -\frac{\partial \mathcal{P}}{\partial h} = 0.
\end{align*}
\]

(3.9)

Note that in these coordinates we cannot apply Reeb’s Theorem 9.1. We can solve this problem using the Poincaré-Delaunay coordinates (prograde) as in (3.8). The averaged function in these coordinates is given by

\[
\bar{\mathcal{P}} = \frac{1}{2}\delta_1 (2P_1 - P_2^2 - P_3^2 - Q_2^2 - Q_3^2) + \frac{C\delta_2}{P_1^3 \left( P_1 - \frac{P_3^2}{2} - \frac{Q_2^2}{2} \right)}.
\]

(3.10)

The differential equations with respect to the mean anomaly are now
Here $P_1 = 1/\sqrt{-2k^*} + O(\epsilon)$, where $k^* < 0$ is a fixed value of the energy from the Hamiltonian (3.3). Taking into account that $p > 4$, that is $\delta_1 = 1$, and $C \geq 0$, the only equilibrium solution of the system (3.11) is $(Q_2^1, Q_3^1, P_2^0, P_3^0) = (0, 0, 0, 0)$. The Jacobian matrix associated with the system (3.11) is

$$A = \begin{pmatrix}
0 & 0 & \delta_1 + \frac{C\delta_0}{P_1^1} & 0 \\
0 & 0 & 0 & \delta_1 \\
-\delta_1 - \frac{C\delta_0}{P_1^1} & 0 & 0 & 0 \\
0 & -\delta_1 & 0 & 0 
\end{pmatrix}.$$  

The Jacobian of the last matrix is equal to $\delta_1^2(p_1^5\delta_1 + C\delta_0)^2/P_1^{10} \neq 0$, under the hypothesis of Theorem 2.1. By Reeb’s Theorem, for $\epsilon$ sufficiently small, there exists a $2\pi$-periodic solution $\gamma_\epsilon(l) = (P_1 + O(\epsilon), Q_2(l, \epsilon), Q_3(l, \epsilon), P_1(l, \epsilon), P_2(l, \epsilon), P_3(l, \epsilon))$ such that $\gamma_\epsilon(0)$ tends to $(P_1, 0, 0, 0, 0, 0)$ when $\epsilon$ tends to 0. Considering the relationship of the original variables with the new variables given by (3.1) and the reparametrization in time, we have that in the original variables, the periodic solutions are almost circular with radius close to $\epsilon^2 \mu^{1/3}(P_1)^2$ and period close to $T = 2\pi \epsilon^2$.

In order to study the type of stability of the previous periodic solutions, the non-trivial characteristic multipliers are of the form $1 \pm \epsilon^a \lambda_1 T + O(\epsilon^{a+1})$, where $\lambda_1$, $\tilde{\lambda}_1$, $\lambda_2$, and $\tilde{\lambda}_2$ are the eigenvalues of the matrix $A$: $\lambda_1 = i\delta_1$ and $\lambda_2 = i\frac{P_1^1\delta_1 + C\delta_0}{P_1^1}$. According to Theorem 9.2, in the appendix, the corresponding periodic orbits are linearly stable.

With this we have found one of the two families of periodic orbits. For the other family, are used Poincaré–Delanuay coordinates (3.8) for retrograde motions, the procedure is analogous. Thus, with this we conclude the proof. \(\square\)

**Remark 3.1** To guarantee that the periodic solutions given in Theorem 2.1 are singularity-free, it is necessary the condition $P_1^1 > a_0$, where $a_0$ is the equatorial radius of the central body in the appropriate units for coordinates $(Q, P)$.

See Fig. 2 for some numerical simulation of Theorem 2.1.
Proof of Theorem 2.2

First, we write the Hamiltonian function (3.7) in the variables (3.8). Now, we introduce action-angle variables defined through the relations

\[
Q_2 = \sqrt{2I_1} \cos \theta_1, \quad P_2 = \sqrt{2I_1} \sin \theta_1, \\
Q_3 = \sqrt{2I_2} \cos \theta_2, \quad P_3 = \sqrt{2I_2} \sin \theta_2.
\]  

Using the coordinates (4.1), the Hamiltonian (3.7), averaged respect to the angular variable \(l\), assumes the form

\[
\mathcal{H}_\epsilon = -\frac{1}{2P_1^2} + \epsilon^a \left( \delta_1 (-P_1 + I_1 + I_2) - \frac{C\delta_2}{(I_1 - P_1)P_1^3} \right) + O(\epsilon^{a+1}).
\]  

Note that, replacing \(P_1 = L\) by \(1/\sqrt{-2h}\), removing constant terms and rescaling time by dividing by \(\epsilon^a\), we obtain

\[
\tilde{\mathcal{H}}_\epsilon = (I_1 + I_2)\delta_1 + \frac{C\delta_2}{L^3(L - I_1)} + O(\epsilon).
\]

But, in order to apply Han–Li–Yi’s Theorem 9.3, we incorporate in (4.2) the terms associated with the action \(L\) dropped in the process of normalization and undo the time scaling. So we get the Hamiltonian (3.3) in the local coordinates that have been introduced through the process. Specifically, we arrive at

\[
\mathcal{H}_\epsilon = h_0(L) + \epsilon^a h_1(L,I) + \epsilon^{a+1},
\]

Fig. 2  a Circular solution of the Kepler problem (\(\epsilon = 0\)) with initial condition (1, 0, 0, 1, 0) contained in the \(xy\)-plane with radius \(a = 1\) and period \(T = 2\pi\). b Periodic solution associated to the Hamiltonian (3.3) with \(\delta_1 = \delta_2 = C = 1\) and \(\epsilon = 10^{-4}\) close to \(xy\)-plane, obtained by Theorem 2.1
where
\[ h_0 = -\frac{1}{2L^2} \quad \text{and} \quad h_1 = \delta_1(-L + I_1 + I_2) + \frac{C\delta_2}{(L - I_1)L^3}. \]

From Theorem 2.1, \( \alpha = 3 \). Now, we are in a position to apply Han–Li–Yi’s Theorem 9.3 taking \( n = 3, a = 1, m_1 = 3, n_0 = 1, n_1 = 3, I^0 = \bar{I}^0 = L, \bar{I}^1 = (I_1, I_2), \) and \( \bar{I}^0 = (I_1, I_2) \). In this case, the vector of frequencies has dimension 3 and is given by
\[ \Omega(L, I_1, I_2) = \left( L^{-3}, \delta_1 + \frac{C\delta_2}{(I_1 - L)2L^3}, \delta_1 \right). \]

The 3 × 4 matrix whose columns are \( \Omega(L, I_1, I_2) \), \( \partial \Omega/\partial L \), \( \partial \Omega/\partial I_1 \) and \( \partial \Omega/\partial I_2 \) reads as
\[
M_\Omega = \begin{pmatrix}
L^{-3} & -\frac{3L^{-4}}{I_1 - L)^2L^3} & 0 & 0 \\
\delta_1 & \frac{C\delta_2}{(I_1 - L)^2L^3} & \frac{2C\delta_2}{L^3(l - l)^3} & 0 \\
\delta_1 & 0 & 0 & 0
\end{pmatrix}
\]
and has rank three only if \( \delta_1 = \delta_2 = 1 \). Additionally, since \( s = 1 \) and \( b = \sum_{i=1}^{n} m_i(n_i - n_{i-1}) = 6 \), according to Han–Li–Yi’s Theorem, the excluding measure for the existence of quasi–periodic invariant tori is of order \( O(\epsilon^b) \), which in our case is \( O(\epsilon^6) \). This completes the proof. \( \square \)

5 Proof of Theorem 2.3

In order to prove Theorem 2.3 we introduce the scale parameter \( \epsilon \) and the change of variables by
\[
Q = \epsilon^2 \bar{\mu}^{-1/3} q, \quad P = \epsilon^{-1} \bar{\mu}^{-1/3} p, \tag{5.1}
\]
where \( \bar{\mu} = \frac{n+\beta}{\Delta} \), which is a \( \epsilon \bar{\mu}^{-2/3} \)-symplectic change of variables. So, the Hamiltonian (1.6), becomes
\[
\mathcal{H} = \mathcal{H}(Q, P) = \epsilon \bar{\mu}^{-2/3} \mathcal{H}(\epsilon^{-2} \bar{\mu}^{-1/3} Q, \epsilon \bar{\mu}^{1/3} P). \tag{5.2}
\]

We develop \( \mathcal{H} \) in a Taylor series around \( \epsilon = 0 \) obtaining
\[
\mathcal{H}(Q, P, \epsilon) = \mathcal{H}_0(Q, P) + \epsilon^3 \mathcal{H}_1(Q, P) + \epsilon^5 \mathcal{H}_2(Q) + O(\epsilon^7), \tag{5.3}
\]
where,
\[
\mathcal{H}_0 = -Q^T K P, \quad \mathcal{H}_1 = \frac{||P||^2}{2} - \frac{1}{||Q||}, \quad \mathcal{H}_2 = \frac{c_1}{||Q||^2}, \tag{5.4}
\]
with \( c_1 = \frac{e^2 \tilde{\mu}}{\Delta} \). The Hamiltonian (1.6) can be rewritten as the Hamiltonian (5.3), which we call of Comet type, that is, the infinitesimal particle is far away of the primaries in the spatial restricted \( N + 1 \)-body problem with Manev potential.

In [12], the authors studied different models, including the spatial restricted \( N \)-body problem and the spatial circular restricted three-body problem. Like us, they introduced a smaller parameter \( \epsilon \), that is, they introduced comet type variables. The resulting Hamiltonian coincides with ours, in the sense that \( H_0 \) and \( H_1 \) are the same. Although the higher order terms do not match, they are not considered. Then Theorem 4.1 in [12] is applied to the Hamiltonian (5.3), therefore the proof of the theorem follows.

6 Proof of Theorem 2.4

In [12] several models were studied, all of them coincide in the unperturbed Hamiltonian and in the first perturbed term of the Hamiltonian (5.3) of our study. To prove the existence of KAM tori, higher order terms have to be considered. However, none of these terms coincide with the one we will study. We will use similar ideas studied in [12] for our proof.

Let us consider the periodic solutions found in Theorem 2.3, associated to the Hamiltonian (5.3). We consider Polar–nodal coordinates which are a set of symplectic variables \((r, \theta, \nu, R, \Theta, K)\), where \( r \) stands for the radial distance from the origin to the particle, \( \theta \) represents the argument of latitude, \( \nu \) accounts for the right ascension of the node, whereas \( R \) is the conjugate momentum of \( r \). Additionally, \( \Theta = ||\Theta|| \) is the magnitude of the angular momentum vector, and \( K \) is the third component of \( \Theta \), so \( 0 \leq |K| \leq \Theta \leq L \). Thus, the spatial Delaunay coordinates are given by \((l, g, \nu, L, \Theta, K)\). Now, we introduce Poincaré-like coordinates related to polar-nodal coordinates

\[
\begin{align*}
Q_1 &= \sqrt{2(\Theta \mp K)} \sin \theta, & P_1 &= \sqrt{2(\Theta \mp K)} \cos \theta, \\
Q_2 &= r, & P_2 &= R.
\end{align*}
\]

The inverse of (6.1) is given by

\[
\begin{align*}
\Theta &= \frac{1}{2}(Q_1^2 + P_1^2 \pm 2K), & \theta &= \pm \tan^{-1} \left( \frac{Q_1}{P_1} \right), \\
r &= Q_2, & R &= P_2.
\end{align*}
\]

Using (6.2), the Hamiltonian (5.3) in terms of the polar-nodal coordinates is given by

\[
\mathcal{H} = -K + \epsilon^3 \left( \frac{1}{2} \left( R^2 + \frac{\Theta^2}{r^2} \right) - \frac{1}{r} \right) + \epsilon^5 \frac{c_1}{r^2} + O(\epsilon^7).
\]

While in the Poincaré-like variables the Hamiltonian \( \mathcal{H} \) is given by
\[ \mathcal{H} = -K + \varepsilon^3 \left( \frac{P_2^2}{2} + \frac{(P_1^2 + Q_1^2 \pm 2K)^2}{8Q_2^2} - \frac{1}{Q_2} \right) + \varepsilon^5 \frac{c_1}{Q_2^2} + O(\varepsilon^7). \]  
(6.4)

Note that the Hamiltonians (6.3) and (6.4) are already in their normal form. We consider \( K \) as constant, eliminating it and dividing the Hamiltonian (6.4) by \( /u_1D700^3 \), we obtain

\[ \tilde{\mathcal{H}}_\varepsilon = \frac{P_2^2}{2} + \frac{(P_1^2 + Q_1^2 \pm 2K)^2}{8Q_2^2} - \frac{1}{Q_2} + \varepsilon^3 \frac{c_1}{Q_2^2} + O(\varepsilon^4), \]  
(6.5)

where the unperturbed Hamiltonian associated with (6.5) is

\[ \tilde{\mathcal{H}} = \frac{P_2^2}{2} + \frac{(P_1^2 + Q_1^2 \pm 2K)^2}{8Q_2^2} - \frac{1}{Q_2}. \]  
(6.6)

Thus, we linearize \( \tilde{\mathcal{H}} \) around the equilibrium point \((Q_1^0, Q_2^0, P_1^0, P_2^0)\), shifting the origin of the coordinates to the equilibrium point. This is attained by the linear change

\[
Q_1 = \varepsilon \tilde{Q}_1 + Q_1^0, \quad Q_2 = \varepsilon \tilde{Q}_2 + Q_2^0, \quad P_1 = \varepsilon \tilde{P}_1 + P_1^0, \quad P_2 = \varepsilon \tilde{P}_2 + P_2^0.
\]  
(6.7)

Since the trajectory is of equatorial type, then \( Q_1^0 = P_1^0 = 0 \). It is clear that \( P_2^0 = 0 \) and \( Q_2^0 = K^2 \). Then the equilibrium in the variables (6.7) is \((0, K^2, 0, 0)\). After applying the linear change to \( \tilde{\mathcal{H}} \) and multiplying by \( \varepsilon^{-2} \), we expand the result in powers of \( \varepsilon \), taking only the terms independent of \( \varepsilon \).

We apply the linear change (6.7) to the Hamiltonian (6.5). This time, since we are taking into account the terms of order two in \( \varepsilon \), in the Poincaré-like coordinates the equilibrium point is no longer \((0, K^2, 0, 0)\). Indeed, it is given by

\[
(Q_1^0, Q_2^0, P_1^0, P_2^0) = (0, K^2 + 2c_1\varepsilon^2, 0, 0).
\]

The above implies that, when including terms of order \( \varepsilon^2 \) in (6.5), the estimate of the radii of the near-circular, near-coplanar solutions is improved. That is, they continue to be circular and coplanar solutions with radii \( K^2 + 2c_1\varepsilon \).

Now, we shift the origin to the equilibrium and scale variables with the multiplier \( \varepsilon^{-2} \). The resulting Hamiltonian is expanded in powers of \( \varepsilon \). After dropping constant terms, we get

\[ \tilde{\mathcal{H}}_\varepsilon = \sum_{j=2}^{9} \varepsilon^{j-2} \tilde{\mathcal{H}}_j + O(\varepsilon^8), \]  
(6.8)

where each \( \tilde{\mathcal{H}}_j \) is a homogeneous polynomial in \( \tilde{P}_i \) and \( \tilde{Q}_i \), with \( i = 1, 2 \) of degree \( j \). In particular
\[
\tilde{H}_2 = \pm \frac{K^4 - 4c_1K^2\epsilon^2 + 12c_1\epsilon^4}{K^7} \tilde{P}_2^2 \pm \frac{K^4 - 4c_1K^2\epsilon^2 + 12c_1\epsilon^4}{2K^7} \tilde{Q}_1^2 + \frac{1}{2} \tilde{P}_2^2 + \frac{1}{2K^{10}} \tilde{Q}_2^2.
\]

(6.9)

The next step is the passage to action-angle coordinates and the removal of the angles. This can be achieved by introducing the symplectic transformation

\[
\begin{align*}
\tilde{Q}_1 &= \sqrt{2I_1} \sin \theta_1, \\
\tilde{Q}_2 &= (\pm K)^{5/2} \sqrt{2I_2} \sqrt{\frac{1}{K^4 - 6c_1K^2\epsilon^2 + 24c_1^2\epsilon^4}} \sin \theta_2, \\
\tilde{P}_1 &= \sqrt{2I_1} \cos \theta_1, \\
\tilde{P}_2 &= (\pm K)^{-5/2} \sqrt{2I_2} \sqrt{K^4 - 6c_1K^2\epsilon^2 + 24c_1^2\epsilon^4} \cos \theta_2,
\end{align*}
\]

(6.10)

which converts \(\tilde{H}_2\) into

\[
\tilde{H}_2 = \pm \frac{K^4 - 4c_1K^2\epsilon^2 + 12c_1\epsilon^4}{K^7} I_1 \pm \frac{\sqrt{K^4 - 6c_1K^2\epsilon^2 + 24c_1^2\epsilon^4}}{K^5} I_2.
\]

The Hamiltonians with \(i > 2\) are finite Fourier series in \(\theta_1\) and \(\theta_2\) whose coefficients are polynomials in \(\sqrt{I_1}\) and \(\sqrt{I_2}\).

Using Lie transformations (see [11]), we will construct a normal form in order to eliminate the angles \(\theta_1\) and \(\theta_2\). The transformed Hamiltonian yields

\[
\tilde{H}_\epsilon = \pm \frac{1}{K^3}(I_1 + I_2) - \frac{\epsilon^2}{4K^6}(8c_1^2 \pm 4c_1(4I_1 + 3I_2)K + (14I_1^2 + 36I_1I_2 + 21I_2^2)K^2) + O(\epsilon^3).
\]

(6.11)

In order to apply the Han–Li–Yi Theorem, we multiply the Hamiltonian by \(\epsilon^3\) and incorporate the unperturbed part of the initial Hamiltonian and grouping the terms factorising by powers of \(\epsilon\), that is,

\[
H = -K \pm \frac{\epsilon^3}{K^3}(I_1 + I_2) - \frac{\epsilon^5}{4K^6}(8c_1^2 \pm 4c_1(4I_1 + 3I_2)K + (14I_1^2 + 36I_1I_2 + 21I_2^2)K^2) + O(\epsilon^6).
\]

(6.12)

**Proof of Theorem 2.4** Using the notations of Han–Li–Yi’s Theorem, the Hamiltonian (6.12) is rewritten as follows

\[
H = h_0(K) + \epsilon^3 h_1(K, I_1, I_2) + \epsilon^5 h_2(K, I_1, I_2),
\]

(6.13)

where

\[
h_0 = -K, \quad h_1 = \pm \frac{1}{K^3}(I_1 + I_2) \quad \text{and} \quad h_2 = -\frac{1}{4K^6}(8c_1^2 \pm 4c_1(4I_1 + 3I_2)K + (14I_1^2 + 36I_1I_2 + 21I_2^2)K^2).
\]

Now, we are in a position to apply Han–Li–Yi’s Theorem, taking \(n = 3\), \(a = 2\), \(m_1 = 3\), \(m_2 = 5\), \(n_0 = 1\), \(n_1 = n_2 = 3\), \(P_0 = P_0 = (K)\), \(P_1 = P_2 = (K, I_1, I_2)\) and \(\tilde{P}_1 = \tilde{P}_2 = (I_1, I_2)\). The frequency vector is given by
The $5 \times 4$ matrix whose columns are \( \Omega(K, I_1, I_2) \), \( \partial\Omega/\partial K \), \( \partial\Omega/\partial I_1 \) and \( \partial\Omega/\partial I_2 \) reads as

\[
M_\Omega = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
\pm K^{-3} & \mp 3 K^{-4} & 0 & 0 \\
\pm K^{-3} & \mp 3 K^{-4} & 0 & 0 \\
-\frac{\pm 4 c_1 + (7 I_1 + 9 I_2) K}{2 K^5} & \frac{\mp (\pm 4 c_1 + (7 I_1 + 9 I_2) K)}{2 K^5} & -7 K^{-4} & -9 K^{-4} \\
-\frac{\pm 3 (2 c_1 + 7 I_1 + 7 I_2) K}{2 K^5} & \frac{\mp (\pm 3 (2 c_1 + 7 I_1 + 7 I_2) K)}{2 K^5} & -9 K^{-4} & -\frac{21}{2} K^{-4}
\end{pmatrix},
\]

which clearly has rank 4 (note that here we choose the integer \( s = 1 \) of the main Han–Li–Yi Theorem). Additionally, since \( s = 1 \) and \( b = \sum_{i=1}^{a} m_i (n_i - n_{i-1}) = 6 \). According to Han–Li–Yi’s Theorem the excluding measure for the existence of quasi–periodic invariant tori is of order \( O(e^b) \), which in our case is \( O(e^6) \). This completes the proof. \( \blacksquare \)

\section{Proof of Theorem 2.5}

\textbf{Proof} In the planar case, the function \( \mathcal{P} \) assumes the form

\[
\mathcal{P} = -\delta_1 G + \delta_2 \frac{C}{L^4(1 - \epsilon \cos(E))^2}.
\]

It’s averaged function \( \bar{\mathcal{P}} \) with respect to the mean anomaly \( l \) is

\[
\bar{\mathcal{P}} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{P}(l, g, L, G) dl = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{P}(E, g, L, G)(1 - \epsilon \cos E) dE = -G \delta_1 - \delta_2 \frac{C}{GL^3}.
\]

We remark that the map \( \bar{\mathcal{P}} \) only depends on the two action variables, \( L \) and \( G \). The differential equations with respect to the mean anomaly are

\[
\begin{align*}
\frac{dg}{dl} &= \frac{\partial \bar{\mathcal{P}}}{\partial G} = -\delta_1 - \frac{C \delta_2}{L^3 G^2}, \\
\frac{dG}{dl} &= -\frac{\partial \bar{\mathcal{P}}}{\partial g} = 0.
\end{align*}
\]

Note that in these coordinates we cannot apply Reeb’s Theorem 9.1. We can solve this problem by using Poincaré–Delaunay coordinates similar to those given in (3.8). In this sense, now, we consider the set of Poincaré–Delaunay variables given by
where the upper sign applies for $G > 0$ (prograde motions), whereas the lower sign is used when $G < 0$ (retrograde motions). In what follows, we will assume $G > 0$ (the study with $G < 0$ is analogous). We point out that the domain of the Poincaré-Delaunay variables when $G > 0$ is the set $D^+ = \{(Q_1, Q_2, P_1, P_2) : P_1 > 0, Q_1 \in [0, 2\pi), P_1 > \frac{1}{2}(Q_2^2 + P_2^2)\}$ (in the case $G < 0$, the domain is $D^- = \{(Q_1, Q_2, P_1, P_2) : P_1 > 0, Q_1 \in [0, 2\pi), P_1 < \frac{1}{2}(Q_2^2 + P_2^2)\}$).

Now, the average function is given by

$$\bar{\mathcal{P}} = \delta_1 \left(-L + \frac{1}{2}(Q_2^2 + P_2^2)\right) + \frac{2C\delta_2}{L^3(2L - P_2^2 - Q_2^2)}.$$  

The differential equations with respect to the mean anomaly are

$$\begin{align*}
\frac{dQ_2}{dl} &= \frac{\partial \bar{\mathcal{P}}}{\partial P_2} = \delta_1 P_2 + \delta_2 \frac{4CP_2^2P_1^2}{(2P_1 - Q_2^2 - P_2^2)^2}, \\
\frac{dP_2}{dl} &= -\frac{\partial \bar{\mathcal{P}}}{\partial Q_2} = -\delta_1 Q_2 - \delta_2 \frac{4CP_2^2Q_2}{(2P_1 - Q_2^2 - P_2^2)^2},
\end{align*}$$  

where $P_1 = 1/\sqrt{2k^* + O(\varepsilon)}$ and $k^* < 0$ is a fixed value of the energy from the Hamiltonian (3.3). The only equilibrium solution of the system (7.4) is $(Q_2^0, P_2^0) = (0, 0).$ The Jacobian matrix associated with the system (7.4) is

$$A = \begin{pmatrix}
0 & \delta_1 + \frac{C\delta_2}{P_1^2} \\
-\delta_1 - \frac{C\delta_2}{P_1^2} & 0
\end{pmatrix}.$$  

The Jacobian of the last matrix is equal to $(P_1^5\delta_1 + C\delta_2)^2/P_1^{10} \neq 0.$ By Reeb’s Theorem, for $\varepsilon$ sufficiently small, there exists a $2\pi$–periodic solution $\gamma_\varepsilon(l) = (P_1 + O(\varepsilon), Q_2(l, \varepsilon), P_1(l, \varepsilon), P_2(l, \varepsilon))$ such that $\gamma_\varepsilon(0)$ tends to $(P_1, 0, 0, 0)$ when $\varepsilon$ tends to $0.$

In order to study the type of stability of the previous periodic solutions, the non-trivial characteristic multipliers are of the form $1 \pm \varepsilon^a \lambda_1 T + O(\varepsilon^{a+1}),$ where $\lambda_1$ and $\bar{\lambda}_1 = -\lambda_1$ are the eigenvalues of the matrix $A.$ $\lambda_1 = i \left(\delta_1 + \frac{C\delta_2}{P_1^2}\right)$ and, according to Theorem 9.2, in the appendix, the corresponding periodic orbits are linearly stable.

8 Proof of Theorem 2.6

Proof of Theorem 2.6 First, in the planar case, the Hamiltonian function $\mathcal{H}$ assumes the form

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We write the Hamiltonian (8.1) in the variables (7.2). Now, we introduce action-angle variables defined through the relations

\[ Q_2 = \sqrt{2I} \cos \theta, \quad P_2 = \sqrt{2I} \sin \theta. \quad (8.2) \]

The Hamiltonian (8.1), averaged respect to the mean anomaly, assumes the form

\[ \mathcal{H}_\varepsilon = -\frac{1}{2P_1^2} + \varepsilon^a \left(-\delta_1(-L+I) + \frac{C\delta_2}{(I-L)L^3}\right) + O(\varepsilon^{a+1}). \quad (8.3) \]

Note that, replacing \( P_1 = L \) by \( 1/\sqrt{-2h} \), removing constant terms and rescaling time by dividing by \( \varepsilon^a \), we obtain

\[ \tilde{\mathcal{H}}_\varepsilon = I\delta_1 + \frac{C\delta_2}{L^3(I-L)} + O(\varepsilon). \]

In order to apply Han–Li–Yi’s Theorem, we incorporate in (8.3) the terms associated with the action \( L \) dropped in the process of normalization and undo the time scalings, so we get Hamiltonian (3.3) in the local coordinates that have been introduced through the process. Specifically, we arrive at

\[ \mathcal{H}_\varepsilon = h_0(L) + \varepsilon^a h_1(L,I) + \varepsilon^{a+1}, \quad (8.4) \]

where

\[ h_0 = -\frac{1}{2L^2} \quad \text{and} \quad h_1 = \delta_1(I-L) - \frac{C\delta_2}{(I-L)L^3}. \]

Now, we are in a position to apply Han–Li–Yi’s Theorem taking \( n = 2, a = 1, m_1 = \alpha, n_0 = 1, n_1 = 2, I^n_0 = P_0 = L, P_1 = (L,I), \) and \( P_1^0 = I \). In this case, the vector of frequencies has dimension 2 and is given by

\[ \Omega(L,I) = \left( L^{-3}, \delta_1 + \frac{C\delta_2}{(I-L)^2L^3} \right). \]

The \( 2 \times 3 \) matrix whose columns are \( \Omega(L,I) \), \( \partial \Omega / \partial L \) and \( \partial \Omega / \partial I \) reads as

\[ M_\Omega = \begin{pmatrix} L^{-3} & -3L^{-4} & 0 \\ \delta_1 + \frac{C\delta_2}{(I-L)^2L^3} & \frac{C(-3L+5\alpha)\delta_2}{(I-L)^3L^4} & 2C\delta_2 \\ \frac{C(-3\alpha+L)\delta_2}{(I-L)^3L^4} & \frac{2C\delta_2}{L^3(-L+\alpha)} & \end{pmatrix}. \]

\( M_\Omega \) has rank two. Note that we choose the integer \( s = 1 \) and also \( b = \sum_{i=1}^a m_i(n_i - n_{i-1}) = \alpha \). According to Han–Li–Yi’s Theorem, the excluding measure for the existence of quasi-periodic invariant tori is of order \( O(\varepsilon^b) \), which in our case is \( O(\varepsilon^a) \). This completes the proof. \( \square \)
9 Conclusions

We have studied a spatial R(N + 1)BP where the gravitational attraction of the central body is given by a Manev potential \((-1/r + e/r^2)\) with \(e \neq 0\) and the other \(N - 1\) masses are equal with Newtonian potential \((-1/r)\), called spatial Manev MR(N + 1) BP. The problem depends on three parameters, the ratio of the mass of the central body to the mass of one of two other bodies, \(\beta\), the number of the peripherals, \(n\) and the Manev perturbation, \(e (e \neq 0)\). When \(e\) is negative, we say that the potential associated with the central body is attractive and when \(e\) is positive, we say that the potential is repulsive. In the present work we have focused on the existence and stability of families of the periodic orbits around of the central body and far away of the primaries, but that enclose them, using averaging theory. On the other hand, we have guaranteed the existence of quasi-periodic orbits that enclose the periodic orbits found, using the Han–Li–Yi’ Theorem.

There are two families of periodic solutions around the central body, close to circular solutions of the Kepler problem, close to the \(xy\)-plane and close to the central body. There are two families of stable periodic solutions, close to circular solutions of the Coriolis problem, close to the \(xy\)-plane and far away of the primaries, but that encloses them.

With respect to quasi-periodic orbits. There are KAM 3-tori that enclose the periodic solutions described above.

In the Hill type: The excluded measure for the existence of quasi-periodic invariant tori is of order \(O(e^4)\). In the Comet type: The excluded measure for the existence of quasi-periodic invariant tori is of order \(O(e^6)\).

The results obtained are very similar in the Planar Manev MR(N + 1)BP case (\(z = z = 0\)).

Appendix

Let \((M, \Omega)\) be a symplectic manifold of dimension \(2n\), \(\mathcal{H}_0 : M \to \mathbb{R}\) a smooth Hamiltonian which defines a Hamiltonian vector field \(\mathbf{Y}_0 = (d\mathcal{H}_0)^\#\) with symplectic flow \(\varphi^\eta_{0}\). Let \(I \subset \mathbb{R}\) be an interval such that each \(h \in I\) is a regular value of \(\mathcal{H}_0\) and \(\mathcal{N}_0(h) = \mathcal{H}_0^{-1}(h)\) is a compact connected circle bundle over a base space \(\mathcal{B}(h)\) with projection \(\pi : \mathcal{N}_0(h) \to \mathcal{B}(h)\). So, this is the setting of regular reduction theory. Assume that all the solutions of \(\mathbf{Y}_0\) in \(\mathcal{N}_0(h)\) are periodic and have periods smoothly depending only on the value of the Hamiltonian; i.e., the period is a smooth function \(T = T(h)\). Let \(\epsilon\) be a small parameter, \(\mathcal{H}_\epsilon : M \to \mathbb{R}\) be smooth, \(\mathcal{H}_\epsilon = \mathcal{H}_0 + \epsilon \mathcal{H}_1\), \(\mathbf{Y}_\epsilon = \mathbf{Y}_0 + \epsilon \mathbf{Y}_1 = d\mathcal{H}_\epsilon^\#\), \(\mathcal{N}(h) = \mathcal{H}^{-1}(h)\), \(\pi : \mathcal{N}(h) \to \mathcal{B}(h)\) the projection, and \(\varphi^\eta_{\epsilon}\) be the flow defined by \(\mathbf{Y}_\epsilon\).

Let the average of \(\mathcal{H}_1\) be

\[
\mathcal{H} = \frac{1}{T} \int_0^T \mathcal{H}_1(\varphi^\eta_{0})dt.
\]
Theorem 9.1 (Reeb’s Theorem) If $\mathcal{H}$ has a non-degenerate critical point at $\pi(p) = \hat{p} \in B(h)$ with $p \in N_0(h)$, then there are smooth functions $p(\epsilon)$ and $T(\epsilon)$ for $\epsilon$ small with $p(0) = p$, $T(0) = T$, and $p(\epsilon) \in N_\epsilon$, and the solution of $Y_\epsilon$ through $p(\epsilon)$ is $T(\epsilon)$–periodic. In addition, if the characteristic exponents of the critical point $\hat{p}$ (that is, the eigenvalues of the matrix $A = 3D^2\mathcal{H}(\hat{p})$) are $\lambda_1, \lambda_2, \ldots, \lambda_{2n-2}$, then the characteristic multipliers of the periodic solution through $p(\epsilon)$ are

$$1, 1, 1 + \epsilon \lambda_1 T + O(\epsilon^2), 1 + \epsilon \lambda_2 T + O(\epsilon^2), \ldots, 1 + \epsilon \lambda_{2n-2} T + O(\epsilon^2).$$

Theorem 9.2 Let $p$ and $\hat{p}$ as in the previous theorem. If one or more of the characteristic exponents $\lambda_j$ is real or has nonzero real part, then the periodic solution through $p(\epsilon)$ is unstable. If the matrix $A$ is strongly stable, then the periodic solution through $p(\epsilon)$ is elliptic, i.e., linearly stable.

Consider a Hamiltonian system of the form

$$\mathcal{H}_\epsilon(I, \varphi, \epsilon) = h_0(I^{n_0}) + \epsilon^{m_1} h_1(I^{n_1}) + \ldots + \epsilon^{m_a} h_a(I^{n_a}) + \epsilon^{m_a+1} p(I, \varphi, \epsilon),$$

where $(I, \varphi) \in \mathbb{R}^n \times \mathbb{T}^n$ are action-angle coordinates with the standard symplectic structure $dI \wedge d\varphi$, and $\epsilon > 0$ is a sufficiently small parameter. Hamiltonian $\mathcal{H}_\epsilon$ is real analytic, and the parameters $a, m, n_i$ ($i = 0, 1, \ldots, a$) and $m_j$ ($j = 1, 2, \ldots, a$) are positive integers satisfying $n_0 \leq n_1 \leq \ldots n_a = n$, $m_1 \leq m_2 \ldots \leq m_a = m$, $I^{n_i} = (I_1, \ldots, I_{n_i})$, for $i = 1, 2, \ldots, a$, and $p$ depends on $\epsilon$ smoothly.

Hamiltonian $\mathcal{H}_\epsilon(I, \varphi, \epsilon)$ is taken in a bounded closed region $Z \times \mathbb{T}^n \subset \mathbb{R}^n \times \mathbb{T}^n$. For each $\epsilon$ the integrable part of $\mathcal{H}_\epsilon$,

$$X_\epsilon(I) = h_0(I^{n_0}) + \epsilon^{m_1} h_1(I^{n_1}) + \ldots + \epsilon^{m_a} h_a(I^{n_a}),$$

admits a family of invariant $n - \text{tori}$ $T_\zeta = \{ \zeta \} \times \mathbb{T}^n$, with linear flows $\{ x_0 + \omega^\zeta(\zeta)t \}$, where, for each $\zeta \in Z$, $\omega^\zeta(\zeta) = \nabla X_\zeta(\zeta)$ is the frequency vector of the $n - \text{torus}$ $T_\zeta$ and $\nabla$ is the gradient operator. When $\omega^\zeta(\zeta)$ is nonresonant, the $n - \text{torus}$ $T_\zeta$ becomes quasi-periodic with slow and fast frequencies of different scales. We refer to the integrable part $X_\epsilon$ and its associated tori $\{ T_\zeta \}$ as the intermediate Hamiltonian and intermediate tori, respectively.

Let $\tilde{I}^n = (I_{n-i+1}, \ldots, I_n)$, $i = 0, 1, \ldots, a$ (where $n_{-1} = 0$, hence $\tilde{I}^{n_0} = I^{n_0}$), and define

$$\Omega = (\nabla_{\tilde{I}^{n_0}} h_0(I^{n_0}), \ldots, \nabla_{\tilde{I}^{n_a}} h_a(I^{n_a}))),$$

such that, for each $i = 0, 1, \ldots, a$, $\nabla_{\tilde{I}^i}$ denotes the gradient with respect to $\tilde{I}^i$.

We assume the following high-order degeneracy-removing condition of Bruno–Rüssman type (so named by Han, Li and Yi), giving credit to Bruno and Rüssman, who provided weak conditions on the frequencies guaranteeing the persistence of invariant tori, the so-called (A) condition: there is a positive integer $s$ such that

$$\text{Rank}\{ \partial^\alpha \Omega(I) : 0 \leq |\alpha| \leq s \} = n, \quad \forall I \in Z.$$

For the usual case of a nearly integrable Hamiltonian system of the type...
\[ \mathcal{H}_\varepsilon(I, \varphi, \varepsilon) = X(I) + \varepsilon p(I, \varphi, \varepsilon), \quad (I, \varphi) \in Z \times \mathbb{T}^n \subset \mathbb{R}^n \times \mathbb{T}^n. \]  \tag{9.2}

Condition (A) given above generalises the classical Kolmogorov non-degenerate condition that \( \partial \Omega(I) \) be nonsingular over \( Z \), where \( \Omega(I) = \nabla X(I) \); Bruno’s non-degenerate condition the Rank\( \{ \Omega(I), \partial \Omega \} = n \), \( \forall I \in Z \); and the weakest non-degenerate condition guaranteeing such persistence provided by Rüssman, that \( \omega(Z) \) should not lie in any \( (n - 1) \)-dimensional subspace. Rüssman condition is equivalent to condition (A) for systems like (9.2). However, Bruno or Rüssman conditions do not apply to Hamiltonian (9.1), as it is too degenerate.

The following theorem gives the right setting in which one can ensure the persistence of KAM tori for a Hamiltonian like (9.1).

**Theorem 9.3** (Han, Li and Yi’s Theorem) Assume the condition (A), and let \( \delta \) with \( 0 < \delta < 1/5 \) be given. Then there exists an \( \varepsilon_0 > 0 \) and a family of Cantor sets \( Z_\varepsilon \subset Z \), \( 0 < \varepsilon < \varepsilon_0 \), with \( |Z \setminus Z_\varepsilon| = O(\varepsilon^{\delta/\varepsilon}) \), such that \( \zeta \in Z_\varepsilon \) corresponds to a real analytic, invariant, quasi-periodic \( n \)–torus \( \bar{T}_\varepsilon \) of Hamiltonian (9.1), which is slightly deformed from the intermediate \( n \)–torus \( T_\varepsilon \). Moreover, the family \( \{ \bar{T}_\zeta \ : \ \zeta \in Z_\varepsilon, 0 < \varepsilon < \varepsilon_0 \} \) varies Whitney smoothly.

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