MESON-PHOTON TRANSITION FORM FACTORS

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Abstract

We report results on the $\pi$-$\gamma$ transition form factor obtained within the hard scattering approach including transverse momentum effects and Sudakov corrections. The results clearly favor distribution amplitudes close to the asymptotic form, $\sim x_1 x_2$, and disfavor distribution amplitudes which are strongly concentrated in the end-point regions. This observation is backed by information on the elastic form factor of the pion and on its valence quark distribution function. Applications of our approach to the $\eta$-$\gamma$ and $\eta'$-$\gamma$ transition form factors are discussed as well. Combining the form factor data with the two-photon decay widths, we determine the $\eta$ and the $\eta'$ decay constants and the $\eta$-$\eta'$ mixing angle.

12.38.Bx, 13.40.Gp, 13.40.Hq, 14.40.Aq

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I. INTRODUCTION

Hadronic form factors at large momentum transfer provide information on the constituents the hadrons are made of and on the dynamics controlling their interactions. Therefore, the form factors always found much interest and many papers, both theoretical and experimental ones, are devoted to them. Recently a new perturbative approach has been proposed by Botts, Li and Sterman [1–3] which allows to calculate the large momentum behavior of form factors. In this new approach, which one may term the modified hard scattering approach (HSA), the transverse degrees of freedom as well as gluonic radiative corrections—condensed in a Sudakov factor—are taken into account in contrast to the standard perturbative approach [4]. Applications of the modified HSA to the pion’s and nucleon’s form factor [2,3,5,6] revealed that the perturbative contributions to these form factors can reliably and self-consistently (in the sense that the bulk of the contributions is accumulated in regions where the strong coupling $\alpha_s$ is sufficiently small) be calculated. It turned out, however, that the perturbative contributions are too small as compared with the data. Responsible for that discrepancy might be omitted contributions from higher order perturbation theory and/or from higher Fock states. Another and perhaps the most important source of additional contributions to the form factors are the overlap of the soft wave functions for the initial and final state hadrons [7]. All these contributions are inherent to the standard as well as to the modified HSA but, with very few exceptions, have not been considered as yet.

In this paper we are going to apply the modified HSA to pseudoscalar meson-photon transition form factors for which data in the few GeV region is now available [8–10]. These transition form factors are exceptional cases since there is no overlap contribution in contrast to the elastic hadronic form factors and, to lowest order, they are QED processes. QCD should only provide corrections of the order of $10 - 20\%$ and higher Fock state contributions are suppressed by additional powers of $\alpha_s/Q^2$. For these reasons one may expect the modified HSA to work for momentum transfer $Q$ larger than about 1 GeV. The low $Q$
behavior of the transition form factors has been investigated in a Salpeter model in [11]. Input to calculations within the modified HSA are the hadronic wave functions which contain the long-distance physics and are not calculable at present. However, the pion wave function required for the calculation of the $\pi$-$\gamma$ transition form factor, $F_{\pi\gamma}(Q^2)$, is rather well constrained. A reliable quantitative estimate of the $\pi$-$\gamma$ transition form factor can therefore be made, as we will discuss in Sec. II. We extend this calculation in Sec. III to the $\eta$-$\gamma$ and $\eta'$-$\gamma$ form factors and determine the decay constants and the mixing angle for pseudoscalar mesons. Sec. IV is devoted to a discussion of the elastic form factor of the pion and to its valence quark distribution function. It will turn out that the same pion wave function as is used in the calculation of the $\pi$-$\gamma$ transition form factor, also leads to a reasonable description of the valence quark distribution function and the elastic pion form factor provided the overlap contribution is also taken into account in the latter case. We recall that the Drell-Yan-West overlap formula [7] is the starting point of the derivation of the hard scattering formula for the elastic pion form factor. The paper terminates with a few concluding remarks (Sec. V).

II. THE $\pi$-$\gamma$ TRANSITION FORM FACTOR

Adapting the modified HSA to the case of $\pi$-$\gamma$ transitions we write the corresponding form factor as

$$F_{\pi\gamma}(Q^2) = \int dx_1 \frac{d^2b_1}{4\pi} \hat{\Psi}_0(x_1, -\vec{b}_1) \hat{T}_H(x_1, \vec{b}_1, Q) \exp \left[-S(x_1, b_1, Q)\right],$$

(2.1)

where $\vec{b}_1$ is the quark-antiquark separation in the transverse configuration space. $x_1$ and $x_2 = 1 - x_1$ denote the usual momentum fractions the quark and the antiquark carry, respectively. $\hat{T}_H$ is the Fourier transform of the momentum space hard scattering amplitude to be calculated from the Feynman diagrams shown in Fig. I. Neglecting the quark masses and the mass of the pion, the hard scattering amplitude $T_H$ in momentum space reads

$$T_H(x_1, \vec{k}_\perp, Q) = 2\sqrt{6} C_\pi \left\{ \frac{1}{x_2 Q^2 + k_\perp^2} + \frac{1}{x_1 Q^2 + k_\perp^2} \right\},$$

(2.2)
where the charge factor $C_\pi$ is $(e_u^2 - e_d^2)/\sqrt{2}$. $e_i$ denotes the charge of quark $i$ in units of the elementary charge. The Fourier transform of this amplitude reads

$$
\hat{T}_H(x_1, \vec{b}_1, Q) = \frac{2\sqrt{6}}{\pi} C_\pi K_0(\sqrt{x_2 Q} b_1),
$$

(2.3)

where $K_0$ is the modified Bessel function of order zero. Strictly speaking, (2.3) represents twice the Fourier transform of the first term in (2.2). Because of the symmetry of the pion wave function under the replacements $x_1 \leftrightarrow x_2$, which appears as a consequence of charge conjugation invariance, the second term leads to the same contribution as the first term after integration over the transverse separation $\vec{b}_1$. The Sudakov exponent $S$ in (2.1) comprising the gluonic radiative corrections, is given by

$$
S(x_1, Q, b_1) = s(x_1, Q, b_1) + s(x_2, Q, b_1) - \frac{4}{\beta_0} \ln \frac{\ln(t/\Lambda_{QCD})}{\ln(1/b_1 \Lambda_{QCD})},
$$

(2.4)

where a Sudakov function $s$ appears for each quark line entering the hard scattering amplitude. The last term in (2.4) arises from the application of the renormalization group equation ($\beta_0 = 11 - 2/3 n_f$). A value of 200 MeV for $\Lambda_{QCD}$ is used throughout and $t$ is taken to be the largest mass scale appearing in $T_H$, i.e., $t = \max(\sqrt{x_2 Q}, 1/b_1)$. For small $b_1$ there is no suppression from the Sudakov factor; as $b_1$ increases the Sudakov factor decreases, reaching zero at $b_1 = 1/\Lambda_{QCD}$. For even larger $b_1$ the Sudakov factor is set to zero. The Sudakov function $s$ is explicitly given in [1,2].

The quantity $\hat{\Psi}_0$ appearing in (2.1) represents the soft part of the transverse configuration space pion wave function, i.e., the full wave function with the perturbative tail removed from it. Following [3] we write the wave function as

$$
\hat{\Psi}_0(x_1, \vec{b}_1) = \frac{f_\pi}{2\sqrt{6}} \phi(x_1) \hat{\Sigma}(\sqrt{x_1 x_2} b_1).
$$

(2.5)

It is subject to the auxiliary conditions

$$
\hat{\Sigma}(0) = 4\pi, \quad \int_0^1 dx_1 \phi(x_1) = 1.
$$

(2.6)

The wave function does not factorize in $x_1$ and $b_1$, but in accord with the basic properties of the HSA [12,13] the $b_1$-dependence rather appears in the combination $\sqrt{x_1 x_2} b_1$. The transverse part of the wave function is assumed to be a simple Gaussian.
\[ \hat{\Sigma}(\sqrt{x_1 x_2} b_1) = 4\pi \exp \left( -x_1 x_2 b_1^2/4a^2 \right). \] 

(2.7)

In [12] it is shown how (2.7) is related to the equal time harmonic oscillator wave function. More complicated forms than (2.7) (e.g., a two-humped shape of the momentum space wave function) are proposed in [13] on the basis of dispersion relations and duality. At large transverse momentum, however, the soft momentum space wave function should behave like a Gaussian [13]. The examination of a number of examples corroborates our expectation that forms of \( \hat{\Sigma} \) other than (2.7) will not change the results and the conclusions presented in our paper markedly.

The two free parameters contained in our ansatz, namely the value of the wave function at the origin and the parameter \( a \) controlling the transverse size, are well fixed by the decay processes \( \pi^+ \rightarrow \mu^+ \nu_\mu \) and \( \pi^0 \rightarrow \gamma \gamma \) providing the relations [12]

\[ \int_0^1 dx_1 \hat{\Psi}_0(x_1, \vec{b}_1 = 0) = \frac{2\pi f_\pi}{\sqrt{6}}, \quad \int dx_1 d^2 b_1 \hat{\Psi}_0(x_1, \vec{b}_1) = \frac{\sqrt{6}}{f_\pi}, \] 

(2.8)

where \( f_\pi(= 130.7 \text{ MeV}) \) is the usual pion decay constant. The first relation is automatically satisfied by our ansatz (2.5), whereas the second relation is used to fix the parameter \( a \). For the distribution amplitude, \( \phi \), we use the asymptotic form

\[ \phi_{\text{AS}}(x_1) = 6 x_1 x_2 \] 

(2.9)

and alternatively, as a representative of strongly end-point concentrated distribution amplitudes, a form proposed by Chernyak and Zhitnitsky [14]

\[ \phi_{\text{CZ}}(x_1) = 30 x_1 x_2 (x_1 - x_2)^2. \] 

(2.10)

It can be shown (see [14] and references therein) that the distribution amplitudes are subject to evolution and can be expanded over Gegenbauer polynomials which are the eigenfunctions of the evolution equation for mesons

\[ \phi(x_1) = \phi_{\text{AS}}(x_1) \left[ 1 + \sum_{n=2}^{\infty} B_n \left( \frac{\alpha_s(\mu_F)}{\alpha_s(\mu_0)} \right)^{n-2} C_n^{3/2}(x_1 - x_2) \right]. \] 

(2.11)
\( \mu_F \) is a scale of order \( Q \) at which soft and hard physics factorizes and \( \mu_0 \) is a typical hadronic scale of order 1 GeV. Charge conjugation invariance requires the odd \( n \) expansion coefficients \( B_n \) to vanish. Since the \( \gamma_n \) are positive fractional numbers any distribution amplitude evolves into \( \phi_{AS}(x) \) asymptotically, i.e., for \( \ln(Q/\mu_0) \to \infty \). The asymptotic distribution amplitude itself shows no evolution. In the representation (2.11) the CZ distribution amplitude is given by \( B_2 = 2/3 \) and \( B_n = 0 \) for \( n > 2 \). Its evolution can safely be ignored in the very limited range of momentum transfer we are interested in.

In the standard HSA the distribution amplitude (2.10) leads to a prediction for the elastic pion form factor in apparent agreement with experiment at the expense, however, of the dominance of contributions from the end-point regions, \( x_1 \to 0 \) or 1, where the use of the perturbative QCD is unjustified as has been pointed out by several authors [15,16]. It is controversial whether or not (2.10) is supported by QCD sum rules. We do not want to enter that discussion but consider (2.10) as an example whose significance is given by its frequent use.

From the \( \pi^0 \to \gamma\gamma \) constraint (2.8) we find for the transverse size parameter \( a \) the values 861 MeV for the asymptotic wave function and 673 MeV for the CZ wave function. These values for \( a \) correspond to the following characteristic properties of the pion’s valence Fock state: 0.21 (0.32) fm for the radius, 367 (350) MeV for the root mean square transverse momentum and 0.25 (0.32) for the probability when the asymptotic (CZ) wave function is used. The small radius of the valence Fock state has a lot of implications in hard processes [17].

Numerical results for the transition form factor \( F_{\pi\gamma}(Q^2) \) obtained from (2.1) are displayed in Fig. 2. We emphasize that there is no free parameter to be adjusted once the wave function is chosen. Obviously the results obtained from the asymptotic wave function are in very good agreement with the CELLO data [11] as is also indicated by a \( \chi^2 \) value of 6.2 for the five data points; there is not much room left for contributions from higher order perturbative QCD and/or from higher Fock states. The results obtained from the CZ wave function overshoot the data significantly (\( \chi^2 = 44.5 \)). Thus the comparison of both the results with the data on the \( \pi-\gamma \) transition form factor evidently ends in favor of the asymptotic wave function.
function. Of course, the limited quality and quantity of the data allows mild modifications of the asymptotic wave functions without worsening the agreement between theory and experiment considerably. For example, if we follow Brodsky et al. [12] and multiply (2.9) by the exponential \( \exp(-a^2 m_q^2 / x_1 x_2) \) where the parameter \( m_q \) represents a constituent quark mass of, say, 330 MeV, we find similarly good results from this modified asymptotic wave function (termed BHL subsequently) as from (2.9) itself. On the other hand, strongly end-point concentrated wave functions are clearly disfavored. Admixtures of the second Gegenbauer polynomial enhancing the end-point regions, must be small; already a value of 0.1 for the expansion coefficient \( B_2 \) deteriorates the agreement with the experimental data substantially \( (\chi^2 = 10) \).

The standard HSA [4,18] predicts for the \( \pi-\gamma \) transition form factor

\[
F_{\pi\gamma}(Q^2) = \frac{\sqrt{2}}{3} f_\pi \langle x_1^{-1} \rangle Q^{-2}.
\]  

The bracket term denotes the \( x_1^{-1} \) moment of the distribution amplitude. This moment receives the value 3 and 5 for the asymptotic and the CZ distribution amplitude respectively. Obviously, the standard HSA, while exact at large \( Q \), fails to describe the data in the few GeV region. It does not provide the substantial \( Q \)-dependence the data for \( Q^2 F_{\pi\gamma} \) exhibits. This is to be contrasted with the modified HSA in which the QCD corrections, condensed in the Sudakov factor, and the transverse degrees of freedom provide the required \( Q \)-dependence. Asymptotically, i.e., for \( \ln(Q/\mu_0) \to \infty \), the Sudakov factor damps any contribution except those from configurations with small quark-antiquark separation and, as the limiting behavior, the QCD prediction \[15,19\] \( F_{\pi\gamma} \to \sqrt{2} f_\pi Q^{-2} \) emerges.

The \( Q \)-dependence of the form factor can be parameterized as

\[
F_{\pi\gamma}(Q^2) = A/(1 + Q^2/s_0)
\]  

where for the constant \( s_0 \) a value of about 0.6 – 0.7 GeV\(^2\) is required by the data. In vector meson dominance models \( s_0 \) is to be identified with the square of the \( \rho \)-meson mass (0.59 GeV\(^2\)). Brodsky and Lepage [18] propose the formula (2.13) as an interpolation between the two limits, \( F_{\pi\gamma}(Q^2=0) = (2\sqrt{2} \pi^2 f_\pi)^{-1} \) known from current algebra and the above
mentioned asymptotic behavior \(\sqrt{2} f_\pi Q^{-2}\), hence \(s_0 = 4\pi^2 f_\pi^2 = 0.67\, \text{GeV}^2\). We stress that this interpolation formula is not derived within the standard HSA. In QCD sum rule analyses \(s_0\) is related to the pion’s duality interval \([20]\). Our results respect \((2.13)\) approximately and \(s_0\) represents the net effect of the suppressions due to both the Sudakov factor and the \(b_1\)-dependence of the wave function and of \(\hat{T}_H\).

We close the discussion of the \(\pi\)-\(\gamma\) transition form factor with a reference to a paper by Szczepaniak and Williams \([21]\) in which soft, nonperturbative corrections to the standard HSA are estimated. These corrections bear resemblance to the intrinsic transverse separation effects considered by us.

III. THE \(\eta\)-\(\gamma\) AND \(\eta'\)-\(\gamma\) TRANSITION FORM FACTORS

We are now going to generalize \((2.1)\) to the cases of \(\eta\)-\(\gamma\) and \(\eta'\)-\(\gamma\) transitions. We start with the SU(3) basis states, \(\eta_8\) and \(\eta_1\), and employ the usual mixing scheme

\[
\begin{align*}
| \eta \rangle &= \cos \vartheta_P | \eta_8 \rangle - \sin \vartheta_P | \eta_1 \rangle \\
| \eta' \rangle &= \sin \vartheta_P | \eta_8 \rangle + \cos \vartheta_P | \eta_1 \rangle.
\end{align*}
\]

(3.1)

Insertion of this scheme into the \(\eta\)-\(\gamma\) and \(\eta'\)-\(\gamma\) matrix elements of the electromagnetic current leads to relations between the physical transition form factors and the \(\eta_8\)-\(\gamma\) and the \(\eta_1\)-\(\gamma\) ones. The latter form factors can be calculated analogously to the \(\pi\)-\(\gamma\) case. For the \(\eta_8\) and \(\eta_1\) wave functions we use the same ansatz as for the pion \((i = 1, 8)\)

\[
\hat{\Psi}_i(x_1, \vec{b}_1) = \frac{f_i}{2\sqrt{6}} \phi_i(x_1) \Sigma_i(\sqrt{x_1 x_2} b_1) \quad (3.2)
\]

and assume that, except of the decay constants \(f_i\), the two wave functions are identical to the asymptotic pion wave function, \((2.7)\) and \((2.9)\). In the hard scattering amplitude \((2.2)\) the charge factor of the pion is to be replaced by either

\[
C_8 = \left( e_u^2 + e_d^2 - 2 e_s^2 \right) / \sqrt{6}
\]

or

\[
8
\]
\[ C_1 = \left( e_u^2 + e_d^2 + e_s^2 \right) / \sqrt{3}. \] (3.4)

Furthermore we correct for the rather large masses of the \( \eta_i \)-mesons \((m_8 = 566 \text{ MeV}, m_1 = 947 \text{ MeV}, \text{ see } [22])\). Hence, in the transverse separation space, the hard scattering amplitude reads

\[ \hat{T}_H(x_1, \vec{b}_1, Q) = \frac{2\sqrt{6}C_i}{\pi}K_0(\sqrt{x_2Q^2 + x_1x_2m_i^2} \cdot \vec{b}_1). \] (3.5)

Inserting (3.2) and (3.3) into (2.1) we can compute the \( F_{\eta_8;\gamma} \) form factors and, then, using the mixing scheme (3.1), the \( \eta \text{-}\gamma \) and \( \eta' \text{-}\gamma \) transition form factors. However, we also need for this calculation the values of the decay constants and the mixing angle. Since these parameters are not known with sufficient accuracy we will change our attitude and, encouraged by the success of the modified HSA in the \( \pi \text{-}\gamma \) case, try to determine them. Admittedly, additional information is required for this task since the \( \eta \text{-}\gamma \) and \( \eta' \text{-}\gamma \) transition form factors do not suffice to fix the three parameters; for any value of the mixing angle a reasonable fit to the data is obtained. The necessary extra information is provided by the two-photon decays of the \( \eta \) and \( \eta' \). Adapting the PCAC result for the \( \pi^0 \to \gamma\gamma \) decay to the \( \eta \) and \( \eta' \) case with proper mixing at the amplitude level, one finds for the decay widths

\[ \Gamma(\eta \to \gamma\gamma) = \frac{9\alpha^2}{16\pi^3}m_\eta^3 \left[ \frac{C_8}{f_8} \cos \vartheta_P - \frac{C_1}{f_1} \sin \vartheta_P \right]^2 \]
\[ \Gamma(\eta' \to \gamma\gamma) = \frac{9\alpha^2}{16\pi^3}m_{\eta'}^3 \left[ \frac{C_8}{f_8} \sin \vartheta_P + \frac{C_1}{f_1} \cos \vartheta_P \right]^2 \] (3.6)

The experimental values for the decay widths are [23]:

\[ \Gamma(\eta \to \gamma\gamma) = 0.510 \pm 0.026 \text{ keV} \quad \Gamma(\eta' \to \gamma\gamma) = 4.26 \pm 0.19 \text{ keV} \] (3.7)

For obvious reasons we have only quoted the PDG average of the two-photon measurements of the \( \eta \to \gamma\gamma \) width. The value of \( 0.324 \pm 0.046 \text{ keV} \) obtained from the Primakoff production measurement [24] will not be used in our analysis.

We evaluate the three parameters, \( f_1, f_8 \) and \( \vartheta_P \), through a combined least square fit to the data on the form factors and the decay widths. The parameters acquire the following values:
\[ f_1 = 145 \pm 3 \text{ MeV}, \quad f_8 = 136 \pm 10 \text{ MeV}, \quad \vartheta_P = -18^\circ \pm 2^\circ \]  \quad (3.8)

and the \( \chi^2 \) value is 14.8 for the 18 data points. The values of the parameters and their errors are only correct provided the \( \eta \) and \( \eta' \) wave functions are at least approximately given by (3.2), (2.7) and (2.8). Since \( f_1 \) and \( f_8 \) have rather similar values nonet symmetry of the wave functions holds approximately. The decay constants of the physical mesons are:

\[ f_\eta = 175 \pm 10 \text{ MeV}; \quad f_{\eta'} = 95 \pm 6 \text{ MeV}. \]  \quad (3.9)

The quality of the fit can be judged from Fig. 3 where fit and data \([8–10]\) for the transition form factors are shown. As expected from the very good value of the \( \chi^2 \) the agreement between theory and experiment is excellent. The computed values for the decay widths are \( \Gamma (\eta \rightarrow \gamma \gamma) = 0.5 \text{ keV} \) and \( \Gamma (\eta' \rightarrow \gamma \gamma) = 4.17 \text{ keV} \).

Our value for the mixing angle is compatible with other results. For instance, a value of \(-20^\circ \pm 4^\circ \) is obtained from chiral perturbation theory \([22]\). An analysis of two-meson final states produced in proton-antiproton annihilations seems to provide a similar value for the mixing angle \((-17.3^\circ \pm 1.8^\circ) \) \([23]\). Gilman and Kauffman \([26]\) found from a phenomenological analysis of various decay processes and of \( \pi^-p \) scattering that a value of about \(-20^\circ \) is favored. From a similar analysis but under inclusion of constituent quark mass effects Bramon and Scadron \([27]\) obtained \( \vartheta_P = -14^\circ \pm 2^\circ \).

From chiral perturbation theory Gasser and Leutwyler \([22]\) predicted a value of \(170 \pm 7 \text{ MeV} \) for the \( \eta_8 \) decay constant whereas Donoghue et al. \([28]\) found 163 MeV. While both the values are compatible within the uncertainties of the chiral perturbation theory, they are larger than our result. In order to see whether or not such large value is definitively excluded in our approach we repeat the combined fit, keeping \( f_8 \) at the value of 163 MeV. The resulting fit is not as good as the precedent fit but still of acceptable quality, the value of \( \chi^2 \) is 20.7. It provides: \( f_1 = 143 \pm 3 \text{ MeV}, \vartheta_P = -21^\circ \pm 1^\circ \) and hence \( f_\eta = 203 \pm 2 \text{ MeV}, f_{\eta'} = 76 \pm 5 \text{ MeV} \). The results for the form factors are almost as good as before. The resulting decay widths are \( \Gamma (\eta \rightarrow \gamma \gamma) = 0.47 \text{ keV} \) and \( \Gamma (\eta' \rightarrow \gamma \gamma) = 4.20 \text{ keV} \), i. e., imposing larger values upon \( f_8 \) forces the two-photon decay width of the \( \eta \) towards smaller values closer to the result
of the Primakoff experiment \cite{24}. In view of the experimental uncertainties in the $\eta \to \gamma\gamma$ and of the moderate difference in the $\chi^2$ we cannot exclude the possibility of a $f_8$ as large as 163 MeV although a value around 140 MeV is favored from our analysis. In contrast to $f_8$ the other two parameters, $f_1$ and $\vartheta$ , are tightly constrained. We stress that the quoted errors are only those obtained in the statistical analysis. By no means they reflect the full uncertainties of the parameters which are rather represented by the differences between the two sets of parameters. In \cite{9,10} the $\eta$ and $\eta'$ decay constants have been determined from fitting pole formulæ analogous to (2.13) to the transition form factor data. The values obtained for the decay constants differ from our ones considerably. Mixing is not taken into account in these analyses.

It is often speculated upon a gluon admixture to the $\eta_1$

$$\eta_1 = \left[ u\bar{u} + d\bar{d} + s\bar{s} + \epsilon G \right] / \sqrt{3 + \epsilon^2}. \quad (3.10)$$

Allowing for that component which only changes the singlet charge factor (3.4) through the normalization, and repeating the fits to the data, we do not find any evidence for a sizeable gluon admixture to the $\eta_1$ ($\epsilon = 0 \pm 0.2$).

Finally we would like to comment on another mixing scheme frequently discussed in the literature (see, for instance, \cite{26,27,29}), in which one uses quark states, $u\bar{u} + d\bar{d}$ and $s\bar{s}$, as the basis. Along the same lines as discussed above one may calculate the transition form factors using this mixing scheme. However, since we do not know well enough the masses and the wave functions of the quark basis states we refrain from carrying out that analysis.

**IV. DO WE KNOW THE PION’S WAVE FUNCTION?**

In this section we are going to put together the available information upon the pion’s wave function. For this purpose we call to mind previous analyses of the elastic form factor of the pion and of its valence quark distribution function and discuss the results of these analyses in the light of our observations made on the $\pi-\gamma$ transition form factor.

Within the modified HSA the pion form factor is to be calculated from the relation \cite{4,8,5}
\[ F_{\pi}^{\text{pert}}(Q^2) = \int_0^1 dx_1 dy_1 \frac{1}{(4\pi)^2} \int_{-\infty}^{\infty} d^2b_1 d^2b_2 \hat{\Psi}_0^*(y_1, \vec{b}_2) \hat{T}_H(x_1, y_1, Q, \vec{b}_1, \vec{b}_2, t) \hat{\Psi}_0(x_1, -\vec{b}_1) \times \exp\left[-S(x_1, Q, b_1, t) - S(y_1, Q, b_2, t)\right] \]  

(4.1)

where the hard scattering amplitude \( \hat{T}_H \) is given by the product of two zeroth order Bessel functions arising from the gluon and the quark propagators in the elementary Feynman diagrams

\[ \hat{T}_H(x_1, y_1, Q, \vec{b}_1, \vec{b}_2) = \frac{4\alpha_s(t)C_F}{\pi} K_0(\sqrt{x_1y_1Qb_2}) K_0(\sqrt{x_1Q|\vec{b}_1 + \vec{b}_2|}) \]  

(4.2)

\( C_F(= 4/3) \) is the color factor and the Sudakov exponents \( S \) are defined in (2.4). \( t \), the largest mass scale appearing in the hard scattering amplitude, is now given by \( \max(\sqrt{x_1y_1Q, 1/b_1, 1/b_2}) \). Equation (4.1) implies two angle integrations, say the integration over the direction of \( \vec{b}_2 \) which simply provides a factor of \( 2\pi \), and the integration over the relative angle \( \phi \) between \( \vec{b}_1 \) and \( \vec{b}_2 \). The latter integration can be carried out analytically by means of Graf’s theorem

\[ \frac{1}{2\pi} \int d\phi \ K_0(c|\vec{b}_1 + \vec{b}_2|) = \theta(b_1 - b_2)K_0(c b_1)I_0(c b_2) + \theta(b_2 - b_1)K_0(c b_2)I_0(c b_1), \]  

(4.3)

where \( \theta \) is the usual step function and \( I_0(x) = J_0(ix) \); \( J_0 \) is the Bessel function of order zero. Thus a four dimensional integration remains to be carried out numerically.

The perturbative contributions to the elastic form factor obtained from both the wave functions, the asymptotic and the CZ one, are compared to the data \[30\] in Fig. 4. Both the predictions, while theoretically self-consistent for momentum transfers larger than about 2 GeV, fall short of the data. One may wonder about the reason for the failure of the modified HSA in the case of the elastic form factor while it works so well for the \( \pi-\gamma \) transition form factor. At this point we remind the reader that the elastic form factor also gets a contribution from the overlap of the initial and final state soft wave functions \( \hat{\Psi}_0 \) \[7\]. A contribution of this type does not exist for the transition form factor. Formally the perturbative contribution to the elastic form factor represents the overlap of the large transverse momentum \( (k_\perp) \) tails of the wave functions while the overlap of the soft parts of the wave
functions is assumed to be negligible small at large $Q$. Since the soft wave functions are explicitly used in applications of the modified HSA it is natural, even obligatory, to examine the validity of that presumption. If the overlap of the soft wave functions turns out to be of substantial magnitude it must be taken into account in the calculation of the elastic form factor for consistency! According to [7] the overlap of the soft transverse configuration space wave functions reads

$$F^{\text{soft}}_{\pi}(Q^2) = \frac{1}{4\pi} \int dx_1 \int d^2b_1 \exp \left[ -(1 - x_1)\vec{b}_1 \cdot \vec{q}_\perp \right] |\bar{\Psi}_0(x_1, \vec{b}_1)|^2,$$

(4.4)

where $Q^2 = \vec{q}_\perp^2$. As the inspection of (4.4) reveals only the end-point region $1 \geq x_1 \geq 1 - 1/(Q\langle b_1^2 \rangle^{1/2})$ contributes at large $Q$, i. e., only configurations where the photon interacts with a parton carrying almost the entire momentum of the pion. Higher Fock states provide similar overlap terms. Obviously with an increasing number of partons sharing the pion’s momentum it becomes less likely that one parton carries the full momentum of the pion. Therefore higher Fock state overlap terms are strongly suppressed at large $Q$.

Consider for example the $q\bar{q}g$ Fock state and assume that the corresponding wave function is a suitable 3-particle generalization of (2.5), (2.7) and (2.9) where the asymptotic form of the $q\bar{q}g$ distribution amplitude is $\sim x_1x_2x_3^2$ ($x_3$ refers to the gluon) [31]. It is easy to convince oneself that the $q\bar{q}g$ overlap term behaves as $1/Q^{12}$ at large $Q$; above $Q \simeq 2\text{GeV}$ it can be neglected to any degree of accuracy.

Soft contributions calculated from (4.4) are shown in Fig. 4. The results obtained from the asymptotic wave function obviously have the right magnitude to fill the gap between the corresponding perturbative results and the data. The broad flat maximum of the overlap contribution simulates a $Q$-dependence of the form factor which, in the few GeV region, resembles the $Q$-dependence predicted by the dimensional counting rules. For the CZ wave function the overlap contribution exceeds the data significantly; the maximum value of $Q^2F_{\pi}$

\[1\] We refrain from showing the sums of soft and perturbative contributions because some double-counting might be implied.
amounts to $2.1 \text{ GeV}^2$ and is located at $Q = 7.4 \text{ GeV}$. We consider this result as a serious failure of the CZ wave function.

At large $Q$ the overlap contributions from our wave functions are suppressed by $1/Q^2$ as compared to the perturbative contribution. At which value of momentum transfer the transition from the dominance of soft to hard contributions actually happens depends on the end-point behavior of the wave function sensitively. Already a moderate modification of the wave function may shift that value considerably. For the asymptotic wave function the transition takes place at the rather large value of about $10 \text{ GeV}$ while for the BHL wave function the additional exponential effectuates a sharper decrease of the soft contribution and the hard contribution takes the control at about $5 \text{ GeV}$ (see Fig. [4]). However, below $3 \text{ GeV}$ the contributions from both the wave functions do not differ much. They also provide similar results for the $\pi-\gamma$ form factor as we already mentioned.

Large overlap contributions have also been observed by other authors [5,15,32,33]. Thus the small size of the perturbative contribution finds a comforting explanation a fact which has already been pointed out by Isgur and Llewellyn-Smith [15]. Were that contribution (including higher order perturbative corrections) much larger the existing large overlap contributions would be inexplicable. This conclusion seems to be compatible with calculations of the one-loop corrections to the perturbative contribution [34,35].

The structure function of the pion or rather its parton distribution functions $q_\nu(x)$ offer another possibility to test our wave functions against data. As has been discussed in [12] and [36] the parton distribution functions are determined by the Fock state wave functions. Each Fock state contributes through the modulus squared of its wave function integrated over transverse momenta up to $Q$ and over all fractions $x_i$ except those pertaining to partons of the type $\nu$. Obviously the valence Fock state wave function only feeds the valence quark distribution function $q^V_d = q^V_{\bar{u}}$. Since each Fock state contributes to the distribution functions positively, the inequality

$$q^V_d(x, Q) \geq \int dx_1 \int^Q d^2 k_\perp \frac{d^2 k_\perp}{16\pi^3} |\Psi(x_1, \vec{k}_\perp)|^2 \delta(x_1 - x) \quad (4.5)$$

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holds. In order to calculate the $Q$ dependence of the valence quark distribution function we need to know the full pion wave function, its perturbative tail is responsible for the evolution behavior. Yet the soft wave function (2.3) provides the bulk of the valence Fock state contribution to the distribution function and, consequently, in the few GeV region this contribution should respect the inequality (4.5). Hence

$$q_d^V(x) \geq \frac{\pi^2}{3} f^2 \frac{a^2}{x(1-x)} \frac{\phi^2(x)}{x}$$. \hspace{1cm} (4.6)

Since, on the average, higher Fock state partons possess smaller values of $x$ than partons from the valence Fock state the inequality should become an equality for $x \to 1$. For other values of $x$ many other Fock states contribute to the distribution function in general, a fact which has to be contrasted with exclusive reactions to which only the valence Fock state contributes at large $Q$.

In Fig. 5 the results for the valence quark structure function obtained from the asymptotic and from the CZ wave functions are shown and compared to the parameterization

$$x q_d^V(x) = A x^{\delta_1} (1 - x)^{\delta_2} \hspace{1cm} (4.7)$$

The constant $A$ is determined by the requirement $\int_0^1 q_d^V(x) \, dx = 1$. Sutton et al. [37] analysed the recent data on the Drell-Yan process $\pi^- p \to \mu^+ \mu^- X$ and determined the values of the other two parameters; they quote $\delta_1 = 0.64 \pm 0.03$ and (the averaged value) $\delta_2 = 1.11 \pm 0.06$ at $Q = 2$ GeV. In Fig. 5 the parameterization of [37] is displayed as a band indicating the uncertainties of it. The width of the band does not take into account the uncertainties induced by theoretical assumptions underlying that analysis (e. g., a $K$ factor). In particular near $x = 1$ the width of the band may be underestimated since the parameterization of the proton structure function, used as input in that analysis, is an extrapolation for $x \gtrsim 0.75$, i. e., it is not supported by data.

The comparison of the parameterization (4.7) with the predictions obtained from our wave functions clearly reveals that the asymptotic wave function respects the inequality (4.5). The little excess around $x = 0.9$ is perhaps a consequence of the above mentioned extra
uncertainties in the parameterization (4.7) and/or of small deviations of the wave function from the asymptotic form. On the other hand, the CZ wave function exceeds the result of [37] dramatically at large $x$. The above-mentioned uncertainties can not account for that. Thus, the CZ wave function fails again and is, therefore, to be rejected. This conclusion has already been drawn by Huang et al. [36]. However, our analysis of the $\pi$-$\gamma$ transition form factor, a quantity which has not been considered in [36], strengthens the evidence for the asymptotic wave function and against the CZ one.

Frequently the pion form factor is calculated via the overlap formula (4.4) using a wave function normalized to unity (e.g., [32,33]). It is customarily asserted that such a wave function describes the binding of constituent quarks in a pion. While such a wave function may provide an overlap contribution to the pion form factor in fair agreement with the data at all $Q^2$ and may also respect the $\pi \to \mu \nu$ constraint it is hard to see how a conflict with the data for the other three reactions we are considering, can be avoided. In order to illustrate the difficulties arising from a wave function normalized to unity, let us look to the asymptotic wave function again. Duplication of the parameter $a$ normalizes the wave function to unity and then the following results are obtained: the contribution to the valence quark distribution function is four times bigger than that one shown in Fig. 4, the $\pi$-$\gamma$ form factor becomes too large ($\chi^2 = 49$) and the $\pi \to \gamma \gamma$ constraint is badly violated. The pion form factor on the other hand is well described. We finally note that the BHL wave function (normalized to unity) leads to results for the elastic form factor which are very similar to those presented in [32]. Yet the deficiencies in the other reactions remain.

**V. CONCLUSIONS**

We summarize our findings about reactions involving pions as the only hadrons: The asymptotic wave function as the only phenomenological input leads to a successful description of four exclusive processes and is compatible with the band on the pion’s valence quark distribution function. The four processes are, on the one side, the leptonic decay of the
charged pion and the two-photon decay of the uncharged pion fixing the parameters of the wave function and, on the other side, the $\pi$-$\gamma$ transition form factor as well as the pion’s elastic form factor which are calculated in the few GeV region within the modified HSA. This success nicely demonstrates the universality of the pion wave function, i.e., its process independence. Of course, the present quality of the form factor data does not pin down the form of the wave function precisely. Wave functions close to the asymptotic form are compatible with the data likewise. The BHL form is an example of such a wave function. On the other hand, the CZ wave function as well as other strongly end-point concentrated wave functions are clearly in conflict with the data and should therefore be discarded. The use of such wave functions in the analyses of other exclusive reactions, e.g., $\gamma\gamma \rightarrow \pi\pi$ or $B \rightarrow \pi\pi$, is unjustified and likely leads to overestimates of the perturbative contributions.

We also analyzed the $\eta$-$\gamma$ and $\eta'$-$\gamma$ transition form factors within the modified HSA. Assuming for the SU(3) basis states the same asymptotic wave functions as for the pion, we determined the decay constants and the mixing angle from a combined fit to the data on form factors and decay widths and found: $f_\eta = 175$ MeV, $f_\eta' = 95$ MeV and $\vartheta_P = -18^\circ$. These values correspond to $f_8 = 145$ MeV which is smaller than the chiral perturbation theory prediction. Keeping $f_8$ at the predicted value of 163 MeV, we found: $f_\eta = 203$ MeV, $f_\eta' = 76$ MeV and $\vartheta_P = -21^\circ$. Owing to the experimental uncertainties in the $\eta \rightarrow \gamma\gamma$ and of the moderate difference in the $\chi^2$ we cannot exclude the second set of parameters although the first set is favored from our analysis.

The parameters are determined under the assumption that the asymptotic wave function is close to reality. Since our values of the parameters are fairly compatible with those found in other analyses, we are tempted to conclude that the $\eta_8$ and $\eta_1$ wave functions we use, are indeed approximately correct.
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FIGURES

FIG. 1. The basic diagrams for the meson-photon transition form factor.

FIG. 2. The $\pi$-$\gamma$ transition form factor vs. $Q^2$. The solid (dashed) line represents the prediction obtained with the modified HSA using the asymptotic (CZ) wave function. The dotted line represents the results of the standard HSA (for the asymptotic wave function). Data are taken from [10].

FIG. 3. The $\eta$-$\gamma$ and $\eta'$-$\gamma$ transition form factors vs. $Q^2$. The solid lines represent the predictions obtained from the modified HSA using the asymptotic wave function. Data are taken from PLUTO [8] (■), TPC/2γ [9] (●) and CELLO [10] (○).

FIG. 4. The elastic form factor of the pion vs. $Q^2$. The dash-dotted (dash-dot-dotted) line represents the perturbative contributions obtained from (4.1) using the asymptotic (CZ) wave function. The solid (dashed) line represents the overlap contribution obtained from the soft part of the asymptotic (CZ) wave function. For comparison the overlap contribution obtained from the BHL wave function is also shown (dotted line). Data are taken from [30].

FIG. 5. The valence quark distribution function $xq^V_d(x)$ at $Q^2 \simeq 5$ GeV$^2$. The shadowed band represents the parameterization by [37]. For other symbols refer to Fig. 2. The bar and the arrow indicate the range $x$ in which an extrapolation for the proton structure function is used.
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Fig. 5