HEREDITARY PROPERTIES OF CO-KÄHLER MANIFOLDS

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Abstract. We show how certain topological properties of co-Kähler manifolds derive from those of the Kähler manifolds which construct them. We go beyond Betti number results and describe the cohomology algebra structure of co-Kähler manifolds. As a consequence, we prove that co-Kähler manifolds satisfy the Toral Rank Conjecture: \( \dim(H^r(M; \mathbb{Q})) \geq 2^r \), for any \( r \)-torus \( T^r \) which acts almost freely on \( M \). Finally, we show that the existence of parallel forms on a co-Kähler manifold reduces the computation of cohomology from the de Rham complex to certain amenable sub-cdga’s defined by geometrically natural operators derived from the co-Kähler structure.

In memory of Sergio Console

1. Introduction

Co-Kähler manifolds may be thought of as odd-dimensional versions of Kähler manifolds and various structure theorems explicitly display how the former are constructed from the latter (see [19, 3]). In this paper, we take the point of view that topological and geometric properties of co-Kähler manifolds are inherited from those of the Kähler manifolds that construct them. We shall see this in both topological and geometric contexts. First, let us recall some basic definitions.

Let \((M^{2n+1}, J, \xi, \eta, g)\) be an almost contact metric manifold given by the conditions

\( J^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(JX, JY) = g(X, Y) - \eta(X)\eta(Y), \)

for vector fields \( X \) and \( Y \), \( I \) the identity transformation on \( TM \) and \( g \) a Riemannian metric. Here, \( \xi \) is a vector field as well, \( \eta \) is a 1-form and \( J \) is a tensor of type \((1, 1)\). A local \( J \)-basis for \( TM \), \( \{X_1, \ldots, X_n, JX_1, \ldots, JX_n, \xi\} \), may be found with \( \eta(X_i) = 0 \) for \( i = 1, \ldots, n \). The fundamental 2-form on \( M \) is given by

\[ \omega(X, Y) = g(JX, Y), \]
and if \( \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \eta\} \) is a local 1-form basis dual to the local \( J \)-basis, then

\[
\omega = \sum_{i=1}^{n} \alpha_i \wedge \beta_i.
\]

Note that \( i_\xi \omega = 0 \).

**Definition 1.1.** The geometric structure \((M^{2n+1}, J, \xi, \eta, g)\) is a **co-Kähler** structure on \( M \) if

\[
[J, J] + 2 d\eta \otimes \xi = 0 \quad \text{and} \quad d\omega = 0 = d\eta
\]

or, equivalently, \( J \) is parallel with respect to the metric \( g \).

Two crucial facts about co-Kähler manifolds are contained in the following lemma. For a direct proof of these facts, see [3].

**Lemma 1.2.** On a co-Kähler manifold, the vector field \( \xi \) is Killing and parallel. Furthermore, the 1-form \( \eta \) is parallel and harmonic.

Lemma 1.2 is a key point in Theorem 1.4 below. In fact, in [19] it is shown that we can replace \( \eta \) by a harmonic integral form \( \eta_\theta \) with dual parallel vector field \( \xi_\theta \) and associated metric \( g_\theta \), (1, 1)-tensor \( J_\theta \) and closed 2-form \( \omega_\theta \) with \( i_{\xi_\theta} \omega_\theta = 0 \). Then we have the following result of H. Li.

**Theorem 1.3** ([19]). With the structure \((M^{2n+1}, J_\theta, \xi_\theta, \eta_\theta, g_\theta)\), there is a compact Kähler manifold \((K, h)\) and a Hermitian isometry \(\psi: K \to K\) such that \( M \) is diffeomorphic to the mapping torus

\[
K_\psi = \frac{K \times [0,1]}{(x,0) \sim (\psi(x),1)}
\]

with associated fibre bundle \( K \to M = K_\psi \to S^1 \).

In [3], the following refinement of Li’s result is proved:

**Theorem 1.4** ([3], Theorem 3.3). Let \((M^{2n+1}, J, \xi, \eta, g)\) be a compact co-Kähler manifold with integral structure and mapping torus bundle \( K \to M \to S^1 \). Then \( M \) splits as \( M \cong S^1 \times_{\mathbb{Z}_m} K \), where \( S^1 \times K \to M \) is a finite cover with structure group \( \mathbb{Z}_m \) acting diagonally and by translations on the first factor. Moreover, \( M \) fibres over the circle \( S^1/\mathbb{Z}_m \) with finite structure group.

The first true study of the topological properties of co-Kähler manifolds was made in [7] where the focus was on things such as Betti numbers and a modified Lefschetz property. The two results above allow us to say something about the fundamental group and, moreover, to display the higher homotopy groups as those of the constituent Kähler manifold \( K \) (groups which, of course, are generally unknown as well). Nevertheless, the principle (which gives rise to the paper’s title) remains that topological qualities of a co-Kähler manifold are intimately tied up with those of the Kähler manifold that constructs it. In this paper, we shall explore this principle
in several ways. We begin by examining the cohomology algebra of a co-Kähler manifold and its effect on the manifold’s rational homotopy structure. We then will consider the structure of the minimal models (in the sense of Sullivan) of co-Kähler manifolds in terms of the decompositions given in Theorem 1.3 and Theorem 1.4. In Section 3, we go beyond algebraic considerations in showing that co-Kähler manifolds satisfy the so-called Toral Rank Conjecture. This theorem strongly connects the geometry of the co-Kähler manifold to the size of its cohomology. Finally, in Section 4, we use work of Verbitsky [23] and the geometric structure of co-Kähler manifolds to give a completely new decomposition of the cohomology of a co-Kähler manifold in terms of the basic cohomology of the associated transversally Kähler characteristic foliation. This leads to a new, simpler proof of the “Lefschetz” property of [7].

We have written this paper for an audience of geometers who may not be experts in rational homotopy. Therefore, we have included a substantial review of basic facts in the subject. The two main references for this material are [12, 13].

2. The Lefschetz Property and Associated Algebraic Models

2.1. Cohomology Algebra Structure. Using Theorem 1.4, the following description of the cohomology of a compact co-Kähler manifold was obtained in [3].

**Theorem 2.1** ([3], Theorem 4.3). If \((M^{2n+1}, J, \xi, \eta, g)\) is a compact co-Kähler manifold with integral structure and splitting \(M \cong K \times \mathbb{Z}_m S^1\), then

\[
H^*(M; \mathbb{R}) \cong H^*(K; \mathbb{R})^G \otimes H^*(S^1; \mathbb{R})
\]

as commutative graded algebras, where \(G = \mathbb{Z}_m\). Hence, the Betti numbers of \(M\) satisfy:

(i) \(b_s(M) = \overline{b_s}(K) + \overline{b}_{s-1}(K)\), where \(\overline{b_s}(K)\) denotes the dimension of \(G\)-invariant cohomology \(H^s(K; \mathbb{R})^G\);

(ii) \(b_1(M) \leq b_2(M) \leq \ldots \leq b_n(M) = b_{n+1}(M)\).

In order to study cohomological properties of co-Kähler manifolds, we recall the notion of cohomologically Kählerian differential graded algebra.

**Definition 2.2.** Let \((A, d)\) be a commutative differential graded algebra (cdga) of cohomological dimension \(2n\), whose cohomology algebra satisfies Poincaré duality. The cdga \((A, d)\) is called **cohomologically Kählerian** if there exists a closed element \(\phi \in A^2\) such that the map

\[
\mathcal{L}^p : H^p(A) \to H^{2n-p}(A), \quad [\sigma] \mapsto [\phi]^{n-p} : [\sigma]
\]

is an isomorphism for every \(0 \leq p \leq n\). Note that we include the case where \((A, d)\) has \(d = 0\) and we then refer to \(A\) as a commutative graded algebra (cga).
Clearly, if $K$ is a Kähler manifold, the cohomology algebra $H^*(K; \mathbb{R})$ is cohomologically Kähler, where $\mathbb{k}$ can be $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. Note that there exist examples of non-Kähler manifolds whose de Rham algebra is cohomologically Kähler (see for instance [13]).

Let $(M^{2n+1}, J, \xi, \eta, g)$ be a compact co-Kähler manifold with integral structure and mapping torus bundle $K \to M \to S^1$ and consider the cohomology algebra $H^*(K; \mathbb{R})$ of the Kähler manifold $K$. The finite group $G \cong \mathbb{Z}_m$ acts on $H^*(K; \mathbb{R})$ and, according to Theorem 2.1, the cohomology algebra of $M$ is the product of the invariant part of the cohomology of $K$ and the cohomology of $S^1$. We now show that the invariant part of $H^*(K; \mathbb{R})$ is a cohomologically Kählerian algebra.

**Proposition 2.3.** The cga $H^*(K; \mathbb{R})^G$ is cohomologically Kählerian.

**Proof.** In [3, Lemma 4.2], it is proved that $H^*(K; \mathbb{R})^G$ contains a $G$-invariant element $\phi$ of degree 2 which behaves like a symplectic form. Such an element is the pullback of the Kähler form in $K \times S^1$ under the inclusion $K \hookrightarrow K \times S^1$.

In order to see that $H^*(K; \mathbb{R})^G$ is cohomologically Kähler, we must show, further, that the Lefschetz map $H^p(K; \mathbb{R})^G \to H^{2n-p}(K; \mathbb{R})^G$ is an isomorphism for every $0 \leq p \leq n$. We check injectivity first. Since $H^*(K; \mathbb{R})$ is cohomologically Kählerian, the multiplicity by $\phi^{n-p}$ is injective on the whole space $H^p(K; \mathbb{R})$ and remains injective when restricted to $H^p(K; \mathbb{R})^G$. For surjectivity, take $\tau \in H^{2n-p}(K; \mathbb{R})^G$. Again, since $H^*(K; \mathbb{R})$ is cohomologically Kählerian, there exists $\tau' \in H^p(K; \mathbb{R})^G$ such that $\tau = \tau' \wedge \phi^{n-p}$. We must show that $\tau' \in H^p(K; \mathbb{R})^G$. We have

$$\tau' \wedge \phi^{n-p} = \tau = g(\tau) = g(\tau' \wedge \phi^{n-p}) = g(\tau') \wedge g(\phi^{n-p}) = g(\tau') \wedge \phi^{n-p},$$

so $(\tau' - g(\tau')) \wedge \phi^{n-p} = 0$. But $H^*(K; \mathbb{R})$ is cohomologically Kählerian, so the multiplication by $\phi^{n-p}$ is injective, and therefore $g(\tau') = \tau'$ and $\tau' \in H^p(K; \mathbb{R})^G$. \hfill $\Box$

This immediately gives another old result about the Betti numbers of co-Kähler manifolds [7, Theorem 11].

**Corollary 2.4.** Let $(M^{2n+1}, J, \xi, \eta, g)$ be a compact co-Kähler manifold. Then, for $0 \leq i \leq n$, the differences $b_{2i+1}(M) - b_{2i}(M)$ are even integers and non-negative if $0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$.

**Proof.** By Theorem 2.1, once we replace the co-Kähler structure on $M$ with an integral one, we obtain $b_i(M) = b_i(K) + b_{i-1}(K)$, where $M$ sits in the mapping torus fibration $K \to M \to S^1$, with $K$ a $2n$-dimensional Kähler manifold and $b_i(K) = \dim H^i(K; \mathbb{R})^G$. We proved in Proposition 2.3 that $H^*(K; \mathbb{R})^G$ is cohomologically Kähler, and this implies that $b_{2i+1}(K)$ is even for $0 \leq i \leq n - 1$. Therefore,

$$b_{2i+1}(M) - b_{2i}(M) = b_{2i+1}(K) + b_{2i}(K) - b_{2i}(K) - b_{2i-1}(K) = b_{2i+1}(K) - b_{2i-1}(K),$$

which is an even number and non-negative if $0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$. \hfill $\Box$
Now, according to Theorem 1.3, the choice of an integral co-Kähler structure \((J_\theta, \xi_\theta, \eta_\theta, g_\theta)\) on \(M\) produces a mapping torus bundle \(K \to M \to S^1\). By Theorem 1.4, this gives in turn a finite cover \(K \times S^1 \to M\) with deck group \(G \cong \mathbb{Z}_m\). We then have the following fundamental data for a co-Kähler manifold.

**Definition 2.5.** Let \((M, J, \xi, \eta, g)\) be a compact co-Kähler manifold and choose an integral structure \((J_\theta, \xi_\theta, \eta_\theta, g_\theta)\) on \(M\). The data \((K, G)\) of the mapping torus bundle \(K \to M \to S^1\) and the finite \(G\)-cover \(K \times S^1 \to M\) form a presentation of \((M, J, \xi, \eta, g)\).

Clearly the presentation of \((M, J, \xi, \eta, g)\) depends on the choice of an integral co-Kähler structure on \(M\). Nevertheless, according to Theorem 2.1,

\[
H^\ast(M; \mathbb{Q}) \cong H^\ast(K; \mathbb{Q})^G \otimes H^\ast(S^1; \mathbb{Q})
\]

and therefore, if we are given two different presentations

\[
K_1 \to M \to S^1, \quad K_1 \times S^1 \overset{G_1}{\to} M
\]

and

\[
K_2 \to M \to S^1, \quad K_2 \times S^1 \overset{G_2}{\to} M,
\]

then we must have

\[
H^\ast(K_1; \mathbb{Q})^G_1 \cong H^\ast(K_2; \mathbb{Q})^G_2.
\]

Let \((M, J, \xi, \eta, g)\) be a compact co-Kähler manifold and fix a presentation \((K, G)\). The finite cyclic group \(G\) acts on the product \(K \times S^1\) and, according to Theorem 1.4, \(G\) acts by translations on \(S^1\), so the \(G\) action is free. But the action need not remain free when we restrict it to \(K\). Therefore, the quotient space \(K/G\) need not be a manifold. However, since \(G\) is a finite, cyclic group, \(K/G\) is a Kähler orbifold which is canonically associated to the presentation of the co-Kähler manifold \(M\). Indeed, \(G\) acts by Hermitian isometries on \(K\), so the Kähler structure is preserved under the \(G\)-action, and passes to the quotient. Since \(G\) is finite, we have \(H^\ast(K/G; \mathbb{Q}) \cong H^\ast(K; \mathbb{Q})^G\), so the rational cohomology of the quotient \(K/G\) is computed by the invariant rational cohomology of \(K\). In view of [2], the cohomology of \(M\) contains information about the cohomology of the Kähler orbifold \(K/G\). Such a Kähler orbifold \(K/G\) is associated to the chosen presentation \(K \to M \to S^1\) of the co-Kähler manifold \((M, J, \xi, \eta, g)\), but since all possible presentations yield diffeomorphic \(M\), the corresponding orbifolds have the same rational cohomology.

2.2. **Rational Homotopy Structure.** In fact, the algebra splitting of Theorem 2.1 tells us much more about the structure of the co-Kähler manifold \(M\). For this we need to recall some notions from Rational Homotopy Theory. The reader is referred to [12, Chapters 2 and 3] and [2] for details and proofs of the statements that follow.

A commutative graded algebra (cga) over a field of characteristic zero \(k\), \(A\), is called free graded commutative if \(A\) is the quotient of \(TV\), the tensor
algebra on the graded vector space \( V \), by the bilateral ideal generated by the elements \( a \otimes b - (-1)^{|a||b|} b \otimes a \), where \( a \) and \( b \) are homogeneous elements of \( A \). As an algebra, \( A \) is the tensor product of the symmetric algebra on \( V^{\text{even}} \) with the exterior algebra on \( V^{\text{odd}} \):

\[
A = \text{Symmetric}(V^{\text{even}}) \otimes \text{Exterior}(V^{\text{odd}}).
\]

We denote the free commutative graded algebra on the graded vector space \( V \) by \( \wedge V \). Note that this notation refers to a free commutative graded algebra and not necessarily to an exterior algebra alone. We usually write \( \wedge V = \wedge(x_i) \), where \( x_i \) is a homogeneous basis of \( V \). Clearly the cohomology of a cdga is a commutative graded algebra. A morphism of cdga’s inducing an isomorphism in cohomology will be called a quasi-isomorphism. A Sullivan cdga is a cdga \((\wedge V, d)\) whose underlying algebra is free commutative, with \( V = \{ V^n \}, n \geq 1 \), and such that \( V \) admits a basis \( x_\alpha \) indexed by a well-ordered set such that \( d(x_\alpha) \in \wedge(x_\beta)_{\beta < \alpha} \). A (Sullivan) minimal cdga is a Sullivan cdga \((\wedge V, d)\) satisfying the additional property that \( d(V) \subset \wedge^{\geq 2} V \).

Minimal cdga’s play an important role because they are tractable models for “all” other cdga’s. (For the path-connected non-simply-connected case of the following result, see [17, Chapter 6] or, from a functorial viewpoint, [2], especially Chapters 7 and 12.)

**Theorem 2.6 (Existence and Uniqueness of the Minimal Model).**

Let \((A, d)\) be a cdga over \( k \) satisfying \( H^0(A, d) = k \), where \( k \) is \( \mathbb{R} \) or \( \mathbb{Q} \) and \( \dim(H^p(A, d)) < \infty \) for all \( p \). Then,

1. There is a quasi-isomorphism \( \varphi: (\wedge V, d) \to (A, d) \), where \( (\wedge V, d) \) is a minimal cdga.
2. The minimal cdga \((\wedge V, d)\) is unique in the following sense: If \((\wedge W, d)\) is a minimal cdga and \( \psi: (\wedge W, d) \to (A, d) \) is a quasi-isomorphism, then there is an isomorphism \( f: (\wedge V, d) \to (\wedge W, d) \) such that \( \psi \circ f \) is homotopic (see [12]) to \( \varphi \).

The cdga \((\wedge V, d)\) is then called the minimal model of \((A, d)\).

The connection between this type of algebra and topology is via the de Rham cdga of differential forms on the manifold \( M \), \((\Omega(M), d)\), when \( k \) is \( \mathbb{R} \) and Sullivan’s rational polynomial forms on \( M \) (thought of as a simplicial complex, say), \((A_{PL}(M), d)\), when \( k \) is \( \mathbb{Q} \). Applying Theorem 2.6 to these cdga’s produces a minimal model of the space \( M \) denoted by \( \varphi: \mathcal{M}_M = (\wedge V, d) \to A \), where we let \( A \) stand for either the de Rham or Sullivan algebras. We shall not distinguish the minimal models depending on the field because the context will always be clear. The minimal model thus provides a special type of cdga associated to a space. Note that the condition \( H^0(A, d) = k \) in Theorem 2.6 means that any path-connected space has a minimal model. There are two key facts that make minimal cdga’s an important tool.

**Lemma 2.7.**
(1) If \( f : (\Lambda V, d) \to (\Lambda Z, d) \) is a quasi-isomorphism between minimal \( cdga \)'s, then \( f \) is an isomorphism.

(2) For a Sullivan \( cdga \) \( (\Lambda V, d) \), a \( cdga \) quasi-isomorphism \( f : (A, d) \to (B, d) \) and a \( cdga \) morphism \( \varphi : (\Lambda V, d) \to (B, d) \), there is a \( cdga \) morphism \( \psi : (\Lambda V, d) \to (A, d) \) such that \( f \circ \psi \) is homotopic (see \([12]\)) to \( \varphi \).

Here is one application. Say that the spaces \( X \) and \( Y \) have the same rational homotopy type if there is a finite chain of maps \( X \to Y_1 \leftarrow Y_2 \to \cdots \to Y \) such that each induced map in rational cohomology is an isomorphism. If we consider the \( cdga \) morphisms

\[
\mathcal{M}_{Y_1} \to A_{PL}(Y_1) \to A_{PL}(X) \leftarrow \mathcal{M}_X
\]

and apply (2) of Lemma 2.7, we obtain a \( cdga \) morphism \( \mathcal{M}_{Y_1} \to \mathcal{M}_X \) which is a quasi-isomorphism (since the other morphisms are). By (1) of Lemma 2.7, we then have \( \mathcal{M}_{Y_1} \cong \mathcal{M}_X \). We carry on this process through the chain of maps to get \( \mathcal{M}_Y \cong \mathcal{M}_X \).

**Proposition 2.8.** If \( X \) and \( Y \) have the same rational homotopy type, then their minimal models are isomorphic. Moreover, if \( X \) and \( Y \) are nilpotent spaces (e.g. simply connected), then the converse is true.

The second statement follows from the existence of spatial rationalizations coming from homotopical localization theory. In general, these do not exist for non-nilpotent spaces. This is important to note because compact co-Kähler manifolds are rarely nilpotent spaces (they are never simply connected of course). So, in the case of non-nilpotent spaces such as typical co-Kähler manifolds, it is the isomorphism class of the minimal model that really represents some sort of rational type. Of course, everything we have said applies to models over \( \mathbb{R} \) as well.

Some minimal models are even more special; they are isomorphic to the minimal models associated to the cohomology algebra (considered as a \( cdga \) with zero differential). Spaces with this property are called formal. Lemma 2.7 implies that there is the following equivalent definition.

**Definition 2.9.** A space \( X \), with minimal model \((\Lambda V, d)\), is called formal if there is a quasi-isomorphism

\[
\theta : (\Lambda V, d) \to (H^*(X; \mathbb{Q}), 0).
\]

**Remark 2.10.** We can also define a \( cdga \) \((A, d)\) to be formal if there is a chain of quasi-isomorphisms

\[
(A, d) \leftarrow (B_1, d_1) \to \cdots (B_k, d_k) \to (H^*(A), 0).
\]
We can take the minimal models of \((A,d)\), the minimal models of the \((B_i,d_i)\) and the minimal models of the morphisms and apply Lemma 2.7 to see that this is equivalent to Definition 2.9.

The last piece of Rational Homotopy Theory that we shall need is the notion of an equivariant minimal model. Let \(\Gamma\) be a finite group. A \(\Gamma\)-cdga is a cdga on which the group \(\Gamma\) acts by a homomorphism \(\Gamma \rightarrow \text{aut}_{cdga}(A,d)\).

**Definition 2.11.** A \(\Gamma\)-cdga \((A,d_A)\) is called \(\Gamma\)-minimal if \((A,d_A) = (\wedge V,d)\) with

1. \(d(V) \subset \wedge^{\geq 2}(V)\);
2. Each \(V^n\) is a \(\Gamma\)-module (i.e. this gives a \(\Gamma\)-structure to \(\wedge V\));
3. \(d\) is \(\Gamma\)-equivariant: \(d(ga) = gd(a)\);
4. \(V\) admits a filtration by sub \(\Gamma\)-spaces
   \[
   0 \subset V(0) \subset V(1) \subset \cdots \subset V(n) \subset \cdots \subset V = \bigcup_n V(n),
   \]
   with \(d(V(n)) \subset (\wedge V(n-1))\).

Generalizing the non-equivariant case, we have the following. Note that, while all proofs (e.g. [13, Theorem 3.26]) of this result assume \(H^1(A,d_A) = 0\), this is for convenience only. In the same way that the ordinary minimal model can be constructed for general path-connected spaces (see Theorem 2.6) by a limiting process, we can also construct an equivariant model.

**Theorem 2.12.** Let \((A,d_A)\) be a \(\Gamma\)-cdga. Suppose that \(H^0(A,d_A) = k\), where \(k = \mathbb{R}\), or \(k = \mathbb{Q}\). Then there exists a \(\Gamma\)-minimal algebra \((\wedge V,d)\) and a \(\Gamma\)-equivariant quasi-isomorphism \(\varphi: (\wedge V,d) \rightarrow (A,d_A)\). The \(\Gamma\)-minimal algebra \((\wedge V,d)\) is called the \(\Gamma\)-minimal model of the \(\Gamma\)-cdga \((A,d_A)\), and it is unique up to \(\Gamma\)-isomorphism.

Suppose a finite group \(\Gamma\) acts on the space \(X\). If the \(\Gamma\)-equivariant minimal model of \(X\), \((\wedge V,d)\), is equivariantly isomorphic to the \(\Gamma\)-equivariant minimal model of \(H^*(X;\mathbb{k})\), then we say that \((X,\Gamma)\) is \(\Gamma\)-formal. It can be shown that a formal \(\Gamma\)-space is \(\Gamma\)-formal [22]. That is, if \(X\) is a formal space with an action of a finite group \(\Gamma\), then the equivariant minimal model can be constructed from the action of \(\Gamma\) on \(H^*(X;\mathbb{k})\). Moreover, in this situation, we can show that the minimal model of \(X/\Gamma\) is the minimal model of \(H^*(X/\Gamma;\mathbb{k})\), so that \(X/\Gamma\) is formal. To see this, let \(\phi: (\wedge W,d) \rightarrow (\wedge V,d)^\Gamma\) be the minimal model of \((\wedge V,d)^\Gamma\). By computing the invariant part of cohomology, we know that \((\wedge W,d)\) is the minimal model of \(X/\Gamma\) (see [13, Corollary 3.29]). Now consider the commutative diagram below, where the right square comes from the inclusion of invariant elements and the equivariant formality quasi-isomorphism \(\theta\), and the left square comes from lifting the composition \((\wedge W,d) \xrightarrow{\phi} (\wedge V,d)^\Gamma \xrightarrow{\theta^\Gamma} H^*(X;\mathbb{k})^\Gamma\) through the isomorphism...
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\[ H^*(X/\Gamma; k) \cong H^*(X; k)^\Gamma. \]

\[
\begin{array}{ccc}
(W, d) & \phi & (V, d) \\
\downarrow \quad \downarrow & \phi^\Gamma & \downarrow \theta \\
H^*(X/\Gamma; k) & \cong & H^*(X; k)^\Gamma \\
\end{array}
\]

Then, since \( \theta \) is a quasi-isomorphism, so is \( \theta^\Gamma \). But then the lift \((W, d) \to H^*(X/\Gamma; k)\) is also a quasi-isomorphism. Hence, \( X/\Gamma \) is formal if \( X \) is.

Let \((M, J, \xi, \eta, g)\) be a compact co-Kähler manifold and let \((K, G)\) be a presentation. Let \((M_K, d)\) denote the minimal model of \( K \). Then its invariant part is a rational model for the space \( K/G \). A main result of \cite{10} states that compact Kähler manifolds are formal. This means, among other things, that the minimal model of a Kähler manifold \( K \) is determined by its rational cohomology algebra \( H^*(K; \mathbb{Q}) \). (Also, note that formality does not depend on the field \( k \).) Since \( K \) is a formal space, so is \( K/G \), and hence its minimal model can be computed from the cohomology algebra \( H^*(K/G; \mathbb{Q}) \cong H^*(K; \mathbb{Q})^G \). Furthermore, co-Kähler manifolds are also formal (see \cite{7}), so the rational minimal model of \( M \) can be constructed directly from its rational cohomology, which in view of Theorem \( 2.1 \) is isomorphic to \( H^*(K; \mathbb{Q})^G \otimes H^*(S^1; \mathbb{Q}) \), for any presentation \((K, G)\) of \( M \). Putting this together, we obtain the following result.

**Theorem 2.13.** Let \( M \) be a compact co-Kähler manifold and let \((K, G)\) be a presentation. Then the minimal model of \( M \) has the following cdga splitting:

\[ \mathcal{M}_M \cong \mathcal{M}_{K/G} \otimes \mathcal{M}_{S^1}. \]

**Proof.** Because all spaces are formal, we have the following diagram:

\[
\begin{array}{ccc}
\mathcal{M}_M & \phi & \mathcal{M}_{K/G} \otimes \mathcal{M}_{S^1} \\
\downarrow \theta_M & & \downarrow \theta_{K/G} \otimes \theta_{S^1} \\
H^*(M; \mathbb{Q}) & \cong & H^*(K; \mathbb{Q})^G \otimes H^*(S^1; \mathbb{Q}),
\end{array}
\]

where the top arrow comes from Lemma \( 2.7 \). Also by Lemma \( 2.7 \) we see that \( \phi \) is an isomorphism. \( \square \)

Theorem \( 2.13 \) is quite interesting, in the following sense. Since a compact co-Kähler manifold \((M, J, \xi, \eta, g)\) is never simply connected, when its fundamental group \( \pi_1(M) \) is not nilpotent, or acts non-nilpotently on higher homotopy groups (see \cite{13}), the minimal model of \( M \) does not give, in general, information about the usual rational homotopy structure of \( M \). For instance, we don’t see things such as rational homotopy groups and Whitehead products. However, the isomorphism class of a minimal model is always an invariant attached to any space, so Theorem \( 2.13 \) says that inside the minimal model of \( M \) we can see the “auxiliary” Kähler orbifold \( K/G \) (i.e. its
minimal model). So the minimal model provides a new type of (geometric) information that is non-classical.

**Example 2.14.** Here is another description of the Chinea-de León-Marrero example contained in [7]. Consider the torus $T^2$ with its standard Kähler structure and let $\phi: T^2 \to T^2$ be the holomorphic isometry covered by the linear transformation $A: \mathbb{R}^2 \to \mathbb{R}^2$,

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The Betti numbers of the mapping torus $T^2_\phi$ are easily computed to be the following:

- $b_0(T^2_\phi) = b_3(T^2_\phi) = 1$;
- $b_1(T^2_\phi) = 1$, generated by the volume form of the circle $S^1$;
- $b_2(T^2_\phi) = 1$, generated by the Kähler class of the torus $T^2$.

The minimal model of $T^2_\phi$ is

$$\left( \wedge(t, u, v), |t| = 1, |u| = 2, |v| = 3, dv = u^2 \right),$$

which is isomorphic to the minimal model of $S^2 \times S^1$. The automorphism $\phi$ of $T^2$ has order 4 and $M$ can be seen as the quotient of $T^2 \times S^1$ by the $\mathbb{Z}_4$-action given by

$$(x, y, z) \mapsto (y, -x, z + 1/4).$$

Now consider the quotient $T^2/G$. When we think of $T^2$ as the square $[0, 1] \times [0, 1]$ with the sides identified, the action of $G$ on $T^2$ is a rotation of $\pi/2$ around the center of the square. There are therefore 2 fixed points, $(0, 0)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$. Using the Riemann-Hurwitz formula, one sees that the quotient $T^2/G$ is a compact surface of genus 0, hence topologically a sphere $S^2$.

**Example 2.15.** Take two copies $(\mathbb{C}P^1_i, \varnothing_i)$, $i = 1, 2$, of $\mathbb{C}P^1$ with its standard Kähler structure, and consider the manifold $K = \mathbb{C}P^1 \times \mathbb{C}P^1$ endowed with Kähler structure $\varnothing = \varnothing_1 + \varnothing_2$. Let $\phi: K \to K$ denote the map $\phi(p, q) = (q, p)$. Then $\phi$ is a holomorphic isometry of $K$. The rational cohomology of the co-Kähler manifold $N = K_{\phi}$ is:

- $H^1(K_{\phi}; \mathbb{Q}) = \langle [u] \rangle$, generated by the class of the circle $S^1$;
- $H^2(K_{\phi}; \mathbb{Q}) = \langle [\varnothing] \rangle$, generated by the Kähler class of $K$;
- $H^3(K_{\phi}; \mathbb{Q}) = \langle [\varnothing \wedge u] \rangle$;
- $H^4(K_{\phi}; \mathbb{Q}) = \langle [\varnothing^2] \rangle$.

The minimal model of $K_{\phi}$ is

$$\left( \wedge(t, u, v), |t| = 1, |u| = 2, |v| = 5, dv = u^3 \right),$$

which is isomorphic to the minimal model of $\mathbb{C}P^2 \times S^1$. The automorphism $\phi$ of $K$ has order 2 and $M$ can be seen as the quotient of $K \times S^1$ by the $\mathbb{Z}_2$-action given by

$$(p, q, t) \mapsto (q, p, t + 1/2).$$
It is not hard to see that the quotient $K/Z_2$ is smooth, and isomorphic (as algebraic varieties) to $\mathbb{C}P^2$. Indeed, let $D = (p, p) \subset K$ be the diagonal; the Segre map gives an embedding $\iota: K \to \mathbb{C}P^3$ which realizes $K$ as a smooth quadric $Q$. The projection from $Q$ to a plane $\pi \subset \mathbb{C}P^3$ is a $2 : 1$ cover, branched over the conic $C \subset \pi$ which is the image under the projection of $\iota(D)$. Therefore, the quotient $Q/Z^2$ is precisely $\pi \cong \mathbb{C}P^2$.

**Remark 2.16.** It is worth pointing out here that in neither of the two examples does the minimal model compute typical rational homotopy information (beyond cohomology) about the corresponding Kähler mapping torus. Indeed, in the first case, $T^2_{\phi}$ is an aspherical manifold, as can be seen directly from the long exact sequence of homotopy groups of the fibration $T^2 \to T^2_{\phi} \to S^1$, but the minimal model of $T^2_{\phi}$ has generators in degree 2 and 3. In the second case, by the same method one sees that $\pi_2(K_{\phi}) = \mathbb{Z} \oplus \mathbb{Z}$, but the minimal model of $K_{\phi}$ has only one generator in degree 2. In both cases, the reason for this apparent mis-match is that neither $T^2_{\phi}$ nor $K_{\phi}$ are nilpotent spaces.

### 3. Toral Rank of Co-Kähler Manifolds

The **Toral Rank Conjecture** (TRC), due to Halperin [16], has been a very influential and motivating problem in the development of Rational Homotopy Theory. In this section we show that a co-Kähler manifold satisfies the conjecture. This is again an instance of our principle that co-Kähler manifolds inherit properties from their constituent Kähler manifolds since the TRC has long been known in the Kähler case (see [1, 20]). Before we state the conjecture, recall that a compact Lie group $G$ (continuously) acts almost freely on a space $X$ if all isotropy groups are finite. The **toral rank** of a space $X$, $\text{rk}(X)$, is the dimension of the largest torus that can act almost freely on $X$.

**Conjecture 3.1** (Toral Rank Conjecture). *If the toral rank of a space $X$ is $r$, then*

$$\dim H^*(X; \mathbb{Q}) \geq 2^r.$$  

The notation $\dim V$ means (total) dimension of $V$ as a rational graded vector space. Our methods allow us to establish this conjecture for a large class of spaces, which (strictly) contains co-Kählerian manifolds. Furthermore we obtain a strong form of the conjecture; namely, we will show that, for our class of spaces, the rational cohomology algebra actually contains a "cohomological $r$-torus". Note that toral rank is a homeomorphism invariant, but is not a homotopy invariant. This suggests that we are getting at deeper topological qualities of co-Kähler manifolds than Betti numbers or even the full algebra structure of cohomology. We begin with some terminology.
Definition 3.2 (Property B). Say that a graded algebra $H$ has Property B if, for any negative-degree derivation $\theta$ of $H$, we have
\[
\theta(H^1) = 0 \implies \theta(H) = 0.
\]
We say that a space $X$ has Property B if its (rational) cohomology algebra has Property B.

For example, any simply connected space whose rational cohomology algebra does not admit a non-zero, negative-degree derivation has Property B. Also, it is known that any cohomologically Kählerian space has Property B. This fact is due to Blanchard [5, Th.II.1.2], and this accounts for our choice of the letter B here. Of course, since the property is intrinsic to the cohomology algebra, any space with the same cohomology algebra has Property B. A main result about Property B spaces is the following (see for instance [13, Proposition 4.40, Theorem 4.36]).

Proposition 3.3. Suppose $F \to E \to X$ is a fibration such that $F$ satisfies Property B and $X$ is simply connected. In the Leray-Serre spectral sequence, if $d_2(H^1(F; \mathbb{Q})) = 0$, then the spectral sequence collapses and $H^*(E; \mathbb{Q}) \cong H^*(X; \mathbb{Q}) \otimes H^*(F; \mathbb{Q})$ as $H^*(X; \mathbb{Q})$-modules. In particular, if $F$ is cohomologically Kählerian and $d_2(H^1(F; \mathbb{Q})) = 0$, then the spectral sequence collapses.

Now suppose that we have an action $T^r \times X \to X$, of an $r$-torus on a space $X$. Recall that we say the action is homologically injective if the orbit map $T^r \to X$ of the action induces an injection $H_1(T^r; \mathbb{Q}) \to H_1(X; \mathbb{Q})$ on first rational homology groups. This property of actions has been extensively studied (see, e.g. [8]). In [1], Allday and Puppe show that a cohomologically Kählerian space satisfies Conjecture 3.1. (In [20], this can be extended to spaces of Lefschetz type.) Distilling their argument a little reveals that it is really Property B that is the key, and not the cohomologically Kählerian structure, as such. In the following result, we extend the Allday-Puppe result by relaxing their hypothesis. Nonetheless, the basic argument, which we repeat here for the convenience of the reader, remains that of [1, Th.2.2].

Theorem 3.4. Let $X$ be a space that satisfies Property B above. If an $r$-torus $T^r$ acts almost freely on $X$, then the action is homologically injective, and we have
\[
H^*(X; \mathbb{Q}) \cong H^*(X_{T^r}; \mathbb{Q}) \otimes H^*(T^r; \mathbb{Q})
\]
as graded algebras, where $X_{T^r} = ET^r \times_{T^r} X$ is the Borel construction. In particular, we have
\[
\dim H^*(X; \mathbb{Q}) \geq \dim H^*(T^r; \mathbb{Q}) = 2^r,
\]
and thus $X$ satisfies Conjecture 3.1.

Proof. Suppose that a torus $\mathbb{T} = T^r$ acts almost freely on the space $X$ for some $r$. Let $ET \to B\mathbb{T}$ be a universal principal $\mathbb{T}$–bundle and let $X_{\mathbb{T}} = (X \times ET)/\mathbb{T}$ be the Borel construction. Let $\{E^p,q\}$ be the rational
cohomology Leray-Serre spectral sequence of $X \to X_T \to BT$ and let $s$ be the rank of the linear map

$$d_2: E_2^{0,1} = H^1(X; \mathbb{Q}) \to E_2^{2,0} = H^2(BT; \mathbb{Q}).$$

Now, we can choose a basis for $H^*(BT; \mathbb{Q})$ so that $H^*(BT; \mathbb{Q}) \cong \mathbb{Q}[a_1, \ldots, a_r]$ with $|a_i| = 2$ for $i = 1, \ldots, r$ and $d_2(y_i) = a_i$, $i = 1, \ldots, s$ for $y_1, \ldots, y_s \in H^1(X; \mathbb{Q})$. Since $d_2$ is a derivation, we obtain

$$d_2(y_1 \cdots y_{i+1}) = \sum_{\ell=1}^{j+1} \pm a_i \otimes y_1 \cdots \hat{y}_i \cdots y_{i+1}.$$ 

By induction, using the algebraic independence of the $a_j$, we see that $y := y_1 \cdots y_s$ must also be non-zero.

Suppose that $s < r$. By duality, the Hurewicz theorem and the fact that elements of $\pi_1(T)$ are realizable by homomorphisms from $S^1$, we can obtain a sub-torus $S \subseteq T$ which realizes the subalgebra $\langle a_1, \ldots, a_s \rangle$. Now, every sub-torus of a torus has a complement, so let $K \subseteq T$ be such that $T = S \times K$. In particular, $\dim(K) = r - s$. We then see that $K$ is the sub-torus such that the ideal generated by the $a_i$, $i = 1, \ldots, s$ is the kernel of the projection in cohomology:

$$(a_1, \ldots, a_s) = \ker(H^*(BT; \mathbb{Q}) \to H^*(B(K); \mathbb{Q})).$$

We now restrict the action of $T$ on $X$ to $K$ and note that it is also almost free. If we form the Borel fibration for the $K$ action, then the Leray-Serre spectral sequence for $X_T$ pulls back to that for $X_K$. Then, because $\text{Im}((d_2)_T) \subseteq \ker(H^*(BT; \mathbb{Q}) \to H^*(B(K); \mathbb{Q}))$, we have $(d_2)_K = 0$ on $H^1(X; \mathbb{Q})$. But because $X$ satisfies Property B, Proposition 3.3 guarantees that the spectral sequence collapses. However, this implies that $H^*(B(K); \mathbb{Q}) \to H^*(X_K; \mathbb{Q})$ is injective and the Borel fixed point theorem then says that the fixed point set $X_K$ is non-empty, contradicting the fact that $K$ acts almost freely. Hence, $K$ is trivial and $r = s$.

Thus we have $y_1, \ldots, y_r \in H^*(X; \mathbb{Q})$ which generate an exterior algebra. In fact, stepping back in the Barratt-Puppe sequence to the fibration $T \to X \to X_T$, we see that $\langle y_1, \ldots, y_r \rangle$ maps onto $H^*(T; \mathbb{Q})$. Therefore this spectral sequence collapses and $H^*(X; \mathbb{Q}) \cong H^*(X_T; \mathbb{Q}) \otimes H^*(T; \mathbb{Q})$. Thus, $\dim(H^*(X; \mathbb{Q})) \geq \dim(H^*(T; \mathbb{Q})) = 2^r$. \hfill $\square$

Next, we show that the class of graded algebras that satisfy Property B is closed under tensor products.

**Proposition 3.5.** If $H$ and $G$ are graded algebras that satisfy Property B, then so too $H \otimes G$ satisfies Property B.

**Proof.** Suppose $H$ and $G$ have Property B, and that $\theta: H \otimes G \to H \otimes G$ is a negative degree derivation that vanishes on $(H \otimes G)^1 = H^1 \otimes 1 + 1 \otimes G^1$. We wish to show that $\theta$ must be zero.

First, we show that $\theta$ vanishes on $H$. For suppose that $\theta(H) \neq 0$, and let $k \geq 0$ be the smallest integer for which $\theta(H) \cap H \otimes G^k \neq 0$. Take
any $\chi \in H$, and write $\theta(\chi) = \theta_k(\chi) + \theta_{k+1}(\chi)$, with $\theta_k(\chi) \in H \otimes G^k$ and $\theta_{k+1}(\chi) \in I(G^{\geq k+1})$, the ideal of $H \otimes G$ generated by elements of $G$ of degree $k + 1$ or greater. Further, suppose that we have a basis $\{g^i\}$ of $G^k$. Then we may write $\theta_k(\chi) = \sum_i \theta_k^i(\chi) \otimes g^i$. This defines linear maps $\theta_k^i : H \to H$, of negative degree — in fact of degree equal to $|\theta| - k$. It is straightforward to check that each $\theta_k^i$ is a derivation of $H$, so $\theta_k^i(\chi) = 0$, for each $\chi \in H^1$, by the assumption that $H$ has Property B. But this implies that we have $\theta(H) \subseteq H \otimes G^{\geq k+1}$, which contradicts our assumption on $k$. Therefore, we must have $\theta(H) = 0$. The same argument, with $H$ and $G$ interchanged, gives that $\theta$ must vanish on $G$. Hence, $\theta = 0$ and $H \otimes G$ has Property B. \hfill \square

Corollary 3.6. If $M$ is a compact co-Kähler manifold, then it satisfies the Toral Rank Conjecture.

Proof. By Theorem 2.1, $H^*(M; \mathbb{R}) = H^*(K; \mathbb{R})^G \otimes H^*(S^1; \mathbb{R})$ and by Proposition 2.3, $H^*(K; \mathbb{R})^G$ is Kählerian and has Property B. Clearly $H^*(S^1; \mathbb{R})$ has Property B for degree reasons. Hence, by Proposition 3.5, $H^*(M; \mathbb{R})$ has Property B. Now apply Theorem 3.4. \hfill \square

This result points out again that properties of co-Kähler manifolds often derive from properties of the constituent Kähler manifold. Also note that, by [3], a co-Kähler manifold always has toral rank at least equal to one. Note that we also have the following result, where the 1 is added to account for the $S^1$ factor in cohomology.

Corollary 3.7. Let $(M, J, \xi, \eta, g)$ be a compact co-Kähler manifold with presentation $(K, G)$ so that $M = (K \times S^1)/G$. Then

$$\text{rk}(M) \leq \tilde{\alpha}_1(K) + 1,$$

where $\tilde{\alpha}_1(K)$ is the maximal number of algebraically independent elements in $H^1(K; \mathbb{Q})$ which are fixed by the induced $G$-action on $H^1(K; \mathbb{Q})$.

By the Myers-Steenrod theorem [21], the isometry group $\text{Isom}(M, g)$ of a compact Riemannian manifold is a compact Lie group. As a consequence, it is observed in [3] that, when $(M, J, \xi, \eta, g)$ is compact co-Kähler, the closure of the Reeb flow in $\text{Isom}(M, g)$ is a compact torus $T$, which acts almost freely on $M$. Therefore $M$ is endowed with an almost free torus action.

Corollary 3.8. Let $(M, J, \xi, \eta, g)$ be a compact co-Kähler manifold and assume that $M = (K \times S^1)/G$ for a Kähler manifold $K$. Let $T \subset \text{Isom}(M, g)$ be the closure of the Reeb flow in the isometry group of $M$. Then

$$\dim(T) \leq \tilde{\alpha}_1(K) + 1.$$

When $b_1(M) = 1$, then $\dim(T) = 1$ and the Reeb flow generates a homologically injective circle action on $M$.

Proof. The torus $T$ acts almost freely on $M$, hence $\dim(T) \leq \text{rk}(M)$. By Corollary 3.7, we get $\dim(T) \leq \text{rk}(M) \leq \tilde{\alpha}_1(K) + 1$. When $b_1(M) = 1$, we have $H^1(K; \mathbb{Q})^G = 0$ by Theorem 2.1, so the group $G$ fixes no element on
$H^1(K;\mathbb{Q})$. Therefore $\tilde{\alpha}_1(K) = 0$ and $\text{rk}(M) \leq 1$. Notice that $\dim(T) \geq 1$, since the flow of the Reeb vector field $\xi$ generates at least a circle in $\text{Isom}(M,g)$. Therefore we get $1 \leq \dim(T) \leq \text{rk}(M) \leq 1$, and $T = S^1$, hence $\xi$ generates a circle action, which is homologically injective by the argument given in [3, Section 2]. □

Examples 3.9.

(1) As already observed, any cohomologically Kählerian space satisfies Property B.

(2) Any algebra generated in degree 1 (tautologically) satisfies Property B. In particular, this remark applies to $H^*(T^r;\mathbb{Q})$ for each $r \geq 1$. Note that these algebras are cohomologically Kählerian only for even $r$.

(3) Therefore, if $H$ is cohomologically Kählerian and $G$ is generated in degree 1, then the tensor product $H \otimes G$ satisfies Property B.

(4) Suppose that $H$ is a finite-dimensional complete intersection, i.e., generated by even-degree generators with ideal of relations generated by a (maximal length) regular sequence. Another long-standing, open conjecture due to Halperin is that any such algebra does not admit any non-zero negative-degree derivation. This conjecture has been established in many cases. If $H$ is any such algebra for which this conjecture is true, then tensor products of the form $H \otimes G$, with $G$ generated in degree 1 (or, more generally, any algebra with Property B) satisfy Property B.

Note that these examples include many that are neither cohomologically Kählerian, nor finite-dimensional complete intersections.

Remark 3.10. If $X$ and $Y$ are spaces that satisfy Property B, then by Theorem 3.4 each satisfies Conjecture 3.1, and by Proposition 3.5 and once again Theorem 3.4 their product $X \times Y$ also satisfies Conjecture 3.1. In this way, we are able to generate spaces that are products, and that satisfy Conjecture 3.1. It is worth emphasizing that, in general, the toral rank—the maximum rank of a torus that may act almost freely—may not behave well with respect to products. In [18], an example is given of a product $X \times Y$ that admits a free circle action, and yet neither $X$ nor $Y$ admit an almost-free circle action. Generally, therefore, the toral rank does not behave in a “sub additive” way with respect to products. This means, in particular, that as yet there is no a priori reason to conclude $X \times Y$ satisfies Conjecture 3.1, simply because $X$ and $Y$ do.

4. Parallel forms and quasi-isomorphisms on co-Kähler manifolds

In [23], Verbitsky shows that, in case a smooth Riemannian manifold has a parallel form, one can define a derivation of the de Rham algebra whose kernel is quasi-isomorphic to the manifold’s real cohomology algebra.
In this section we will use this construction in the context of co-Kähler manifolds, where the 1-form \( \eta \) is parallel. Once again, we shall see that some topological properties of co-Kähler manifolds may be derived from corresponding properties of Kähler manifolds. This can be interpreted as a geometric incarnation of our hereditary principle.

Let \( M \) be a smooth manifold and let \( \Omega^*(M; \mathbb{R}) \) be the (real) de Rham algebra. A linear map \( f \in \text{End}(\Omega^*(M; \mathbb{R})) \) has degree \( |f| \) if \( f: \Omega^k(M; \mathbb{R}) \to \Omega^{k+|f|}(M; \mathbb{R}) \). Every linear map \( f: \Omega^1(M; \mathbb{R}) \to \Omega^{1+|f|}(M; \mathbb{R}) \) can be extended to a graded derivation \( \rho_f \) of \( \Omega^*(M; \mathbb{R}) \) by imposing the Leibniz rule, i.e.

\[
\rho_f|_{\Omega^0(M; \mathbb{R})} = 0 \\
\rho_f|_{\Omega^1(M; \mathbb{R})} = f \\
(3) \quad \rho_f(\alpha \wedge \beta) = \rho_f(\alpha) \wedge \beta + (-1)^{|\alpha||f|} \alpha \wedge \rho_f(\beta).
\]

where \( \alpha, \beta \in \Omega^*(M; \mathbb{R}) \) and \( |\alpha| \) is the degree of \( \alpha \). (While this apparently well-known fact is used in \[23\], it is not proved there. See \[15\], Lemma 4.3 for a proof.) Given two linear operators \( f, \tilde{f} \in \text{End}(\Omega^*(M; \mathbb{R})) \), their supercommutator is defined as

\[
\{f, \tilde{f}\} = f \circ \tilde{f} - (-1)^{|f||\tilde{f}|} \tilde{f} \circ f.
\]

Let \( (M, g) \) be a smooth Riemannian manifold and let \( \eta \in \Omega^k(M; \mathbb{R}) \) be a \( k \)-form. Define a linear map \( \tilde{\eta}: \Omega^1(M; \mathbb{R}) \to \Omega^{k-1}(M; \mathbb{R}) \), with \( |\tilde{\eta}| = k - 2 \), by

\[
\tilde{\eta}(\nu) = \iota_{\nu\#} \eta
\]

where \( \# : T^*M \to TM \) is the isomorphism given by the metric. Denote by \( \rho_\eta: \Omega^*(M; \mathbb{R}) \to \Omega^{*+k-2}(M; \mathbb{R}) \) the corresponding derivation. Define the linear operator \( d_\eta: \Omega^*(M; \mathbb{R}) \to \Omega^{*+k-1}(M; \mathbb{R}) \) as

\[
d_\eta = \{d, \rho_\eta\}.
\]

Since \( d_\eta \) is the supercommutator of two graded derivations, one sees easily that it is itself a graded derivation of degree \( k - 1 \) and that it supercommutes with \( d \). As a consequence, \( \ker(d_\eta) \subset \Omega^*(M; \mathbb{R}) \) is a differential subalgebra and has the structure of a cdga. In \[23\], Verbitsky proves following:

**Theorem 4.1.** Let \( (M, g, \eta) \) be a compact Riemannian manifold equipped with a parallel form \( \eta \). Then the natural embedding

\[
(\ker(d_\eta), d) \hookrightarrow (\Omega^*(M; \mathbb{R}), d)
\]

is a quasi-isomorphism.

Let \( (M, g, \eta) \) be a Riemannian manifold equipped with a parallel form \( \eta \). Theorem 4.1 says that we can recover the cohomology of \( M \) by considering the subalgebra of forms \( \nu \) which are annihilated by \( d_\eta \), i.e. those for which \( d_\eta(\nu) = 0 \). This allows one to greatly simplify, in many cases, the computation of the de Rham cohomology of this kind of manifold.
Recall from Lemma 1.2 that the 1-form \( \eta \) is parallel on a co-Kähler manifold. According to Verbitsky’s construction, there is a derivation \( d_\eta \) of \((\Omega^*(M; \mathbb{R}), d)\) described explicitly as follows.

**Lemma 4.2.** Let \((M, J, \eta, \xi, g)\) be a co-Kähler manifold. Then \( d_\eta = L_\xi \), where \( L_\xi \) denotes the Lie derivative in the direction of the vector field \( \xi \).

**Proof.** Denote by \( \bar{\eta} : \Omega^*(M; \mathbb{R}) \to \Omega^*(M; \mathbb{R}) \) the operator which acts on 1-forms as \( \bar{\eta}(\nu) = \nu_\# \eta \). Since \( |\bar{\eta}| = -1 \), we have \( d_\eta = \{ d, \rho_\eta \} = d \circ \rho_\eta + \rho_\eta \circ d \), and \(|d_\eta| = 0 \). To prove the lemma, by [15], it is enough to consider the action of \( d_\eta \) on 0- and 1-forms. Now, according to the formulas in (3) extending \( \bar{\eta} \) to a derivation \( \rho_\eta \), on 1-forms we have \( \rho_\eta = \bar{\eta} \) and

\[
\bar{\eta}(\nu) = \nu_\# \eta = \eta(\nu^\#) = g(\xi, \nu^\#) = \nu(\xi) = \nu_\xi. 
\]

Note that this identifies \( \bar{\eta} = \nu_\xi \) which is already a derivation, so \( \rho_\eta = \nu_\xi \). Hence, \( (d \circ \eta)(\nu) = d \nu_\xi \nu \) and, on the other hand, \( (\bar{\eta} \circ d)(\nu) = \nu_\xi(d\nu) \). By Cartan’s magic formula, we obtain

\[
d_\eta(\nu) = (d \circ \bar{\eta})(\nu) + (\bar{\eta} \circ d)(\nu) = d\nu_\xi \nu + \nu_\xi(d\nu) = L_\xi(\nu).
\]

Thus \( d_\eta = L_\xi \) on 1-forms. On a 0-form (i.e. a function) \( f \), we have

\[
d_\eta(f) = \rho_\eta(df) = \bar{\eta}(df) = df(\xi) = \xi(f) = L_\xi(f)
\]

by the calculation above. Since \( d_\eta \) and \( L_\xi \) are graded derivations of the de Rham algebra which agree on 0-forms and 1-forms, the result follows. \( \square \)

Let us consider the following graded differential subalgebra \((\Omega^*_\eta(M), d)\) of \((\Omega^*(M; \mathbb{R}), d)\) given by

\[
\Omega^*_\eta(M) = \{ \nu \in \Omega^*(M; \mathbb{R}) \mid L_\xi(\nu) = 0 \}.
\]

As a consequence of Theorem 4.1 we obtain the following result.

**Corollary 4.3.** On a compact co-Kähler manifold, the natural inclusion

\[
(\Omega^*_\eta(M), d) \hookrightarrow (\Omega^*(M; \mathbb{R}), d)
\]

is a cdga quasi-isomorphism and

\[
H^*(M; \mathbb{R}) \cong H^*_\eta(M),
\]

where \( H^*_\eta(M) \) is the cohomology of \((\Omega^*_\eta(M), d)\).

We shall use the cdga \( \Omega^*_\eta(M) \) to give an alternative proof of the Lefschetz property and of formality for co-Kähler manifolds in the hereditary framework of the rest of the paper.

Let \((M, J, \xi, \eta, g)\) be a compact co-Kähler manifold. In [7], the authors defined a Lefschetz map on harmonic forms and proved that it is an isomorphism. This is, of course, different from the Kähler context, where the Lefschetz map can be defined directly on all forms and depends only on the underlying symplectic structure, not on the metric. On forms, the Lefschetz map is \( L^{n-p} : \Omega^p(M; \mathbb{R}) \to \Omega^{2n+1-p}(M, \mathbb{R}) \), given by

\[
(4) \quad \alpha \mapsto \omega^{n-p+1} \wedge \iota_\xi \alpha + \omega^{n-p} \wedge \eta \wedge \alpha
\]
One sees immediately that the Lefschetz map does not commute with $d$ (as happens in the Kähler case), hence does not descend to a map in cohomology. However, by restricting the Lefschetz map to the cdga $\Omega^*_\eta(M)$, we are able to descend to cohomology.

**Proposition 4.4.** The Lefschetz map $[\mathcal{L}]$ restricts to
$$\mathcal{L}^{n-p} : \Omega^p_\eta(M) \to \Omega^{2n+1-p}_\eta(M)$$
for $0 \leq p \leq n$ and commutes with $d$ on $(\Omega^p_\eta(M), d)$. Hence, $\mathcal{L}$ descends to cohomology $H^*_\eta(M) \cong H^*(M; \mathbb{R})$.

**Proof.** We first show that if $\alpha \in \Omega^p_\eta(M)$, then $\mathcal{L}^{n-p}(\alpha) \in \Omega^{2n+1-p}_\eta(M)$.
$$L_\xi(L^{n-p}(\alpha)) = L_\xi(\omega^{n-p+1} \wedge t_\xi \alpha + \omega^{n-p} \wedge \eta \wedge \alpha) = \omega^{n-p+1} \wedge L_\xi(t_\xi \alpha) =$$
$$= \omega^{n-p+1} \wedge t_\xi d_\xi \alpha = -\omega^{n-p+1} \wedge t_\xi d_\xi \alpha = 0,$$
where we have used the facts that the Lie derivative $L_\xi$ is a derivation, $L_\xi = t_\xi d_\xi + d_\xi t_\xi$ (Cartan’s Magic formula), $t_\xi t_\xi = 0$ and $L_\xi \omega = L_\xi \eta = L_\xi \alpha = 0$. In order to see that $\mathcal{L}$ descends to cohomology, we will show that it (super)commutes with $d$. (Note, we are simply saying that the Lefschetz map applied to a cocycle is a cocycle and applied to a coboundary is a coboundary, thus giving an induced map on cohomology.) For $\alpha \in \Omega^p_\eta(M)$, we have
$$\mathcal{L}^{n-p}(d\alpha) = \omega^{n-p+1} \wedge t_\xi d_\xi \alpha + \omega^{n-p} \wedge \eta \wedge d\alpha =$$
$$= -\omega^{n-p+1} \wedge d_\xi \alpha - d(\omega^{n-p} \wedge \eta \wedge \alpha) =$$
$$= -d(\omega^{n-p+1} \wedge t_\xi \alpha + \omega^{n-p} \wedge \eta \wedge \alpha) =$$
$$= -d(\mathcal{L}^{n-p} \alpha). \quad \Box$$

Consider the following two subalgebras of $\Omega^*_\eta(M)$:
$$\Omega^p_1(M) = \{ \alpha \in \Omega^p_\eta(M) \mid t_\xi \alpha = 0 \}, \quad \Omega^p_2(M) = \mathbb{Q} \oplus \{ \alpha \in \Omega^p_\eta(M) \mid \eta \wedge \alpha = 0 \}.$$

**Lemma 4.5.** $\Omega^p_i(M) = \Omega^p_i(M) \oplus \Omega^p_i(M)$ for all $p > 0$ and $\Omega^*_\eta(M)$ is a differential subalgebra of $\Omega^*_\eta(M)$, $i = 1, 2$.

**Proof.** Given any $\alpha \in \Omega^p_i(M)$, we can write tautologically
$$\alpha = (\alpha - \eta \wedge t_\xi \alpha) + \eta \wedge t_\xi \alpha =: \alpha_1 + \alpha_2. \quad (5)$$
Since $\eta(\xi) = 1$, we see immediately that $t_\xi \alpha_1 = 0$, so $\alpha_1 \in \Omega^p_1(M)$. Clearly $\alpha_2 \in \Omega^p_2(M)$. Now suppose that $\alpha \in \Omega^p_1(M) \cap \Omega^p_2(M)$. Then $\eta \wedge \alpha = 0$ and hence, by applying $t_\xi$, we get $0 = \alpha - \eta \wedge t_\xi \alpha = \alpha$, which gives $\alpha = 0$.

Now, if $\alpha \in \Omega^p_2(M)$, then $L_\xi \alpha = d_\xi \alpha + t_\xi d_\alpha = 0$, so $t_\xi d_\alpha = -d_\xi \alpha$. If $\alpha \in \Omega^p_1(M)$, then we also have $t_\xi (d\alpha) = -d_\xi \alpha = 0$ since $\alpha \in \Omega^p_1(M)$. Hence $d : \Omega^p_1(M) \to \Omega^{p+1}_1(M)$.

Finally, suppose $\alpha \in \Omega^p_2(M)$. Then, since $\eta$ is closed, we have $\eta \wedge d\alpha = -d(\eta \wedge \alpha) = 0$. Hence $d : \Omega^p_2(M) \to \Omega^{p+1}_2(M). \quad \Box$
As a consequence, the cohomology $H^p_\eta(M)$ of the cdga $\Omega^*_\eta(M)$ can be written as

$$H^p_\eta(M) \cong H^p_1(M) \oplus H^p_2(M),$$

where $H^p_i(M) = H^p(\Omega^*_i(M))$, $i = 1, 2$. Now consider a form $\alpha \in \Omega^p_\eta(M)$. Applying the derivation $\iota_\xi$ to the equation $\eta \wedge \alpha = 0$, we obtain $\alpha = \eta \wedge \iota_\xi \alpha$, where clearly $\iota_\xi \alpha \in \Omega^{p-1}_1(M)$. This tells us that $\Omega^p_\eta(M) = \eta \wedge \Omega^{p-1}_1(M)$ and, since $d\eta = 0$, we have a differential splitting

$$\Omega^p_\eta(M) = \Omega^p_1(M) \oplus \eta \wedge \Omega^{p-1}_1(M).$$

From this, we immediately deduce

**Corollary 4.6.** The cohomology $H^p_\eta(M)$ of $\Omega^*_\eta(M)$ splits as

$$H^p_\eta(M) = H^p_1(M) \oplus [\eta] \wedge H^p_1(M)$$

This corollary shows that the cohomology of $\Omega^*_\eta(M)$ only depends on the cohomology of the cdga $\Omega^*_\eta(M)$.

Let us now consider the characteristic foliation $\mathcal{F}_\xi$ on a compact co-Kähler manifold $(M, J, \xi, \eta, g)$ given by $(\mathcal{F}_\xi)_x = \langle \xi_x \rangle$ for every $x \in M$. Such a foliation is Riemannian and transversally Kähler. Indeed, at every point $x \in M$, the orthogonal space to $\xi$ is endowed with a Kähler structure given by $(J, g, \omega)$, and all these data vary smoothly with $x$.

Recall that, given a foliation $\mathcal{F}$ on a compact manifold $M$, the basic cohomology is defined as the cohomology of the complex $\Omega^*(M, \mathcal{F})$, where

$$\Omega^p(M, \mathcal{F}) = \{ \alpha \in \Omega^p(M) \mid 1_X \alpha = 1_X d\alpha = 0 \ \forall X \in \mathfrak{X}(\mathcal{F}) \}$$

and $\mathfrak{X}(\mathcal{F})$ denotes the subalgebra of vector fields tangent to $\mathcal{F}$. In our case, we have the following.

**Lemma 4.7.** Let $(M, J, \xi, \eta, g)$ be a compact co-Kähler manifold and let $\mathcal{F}_\xi$ be the characteristic foliation. Then $\Omega^*_1(M) = \Omega^*(M, \mathcal{F}_\xi)$.

**Proof.** This is clear, since

$$\alpha \in \Omega^*_1(M) \iff L_\xi \alpha = \iota_\xi \alpha = 0 \iff \iota_\xi d\alpha = \iota_\xi \alpha = 0 \iff \alpha \in \Omega^p(M, \mathcal{F}_\xi).$$

\[ \square \]

**Corollary 4.8.** On a compact co-Kähler manifold $M$, $H^*_\eta(M) \cong H^*(M, \mathcal{F}_\xi)$ and

$$H^*(M; \mathbb{R}) \cong H^*_\eta(M) = H^*(M, \mathcal{F}_\xi) \oplus [\eta] \wedge H^{*1}(M, \mathcal{F}_\xi)$$

**Theorem 4.9.** Let $(M, J, \xi, \eta, g)$ be a compact co-Kähler manifold. Then the Lefschetz map

$$L^{n-p} : H^p(M; \mathbb{R}) \cong H^p_\eta(M) \to H^{2n+1-p}_\eta(M) \cong H^{2n+1-p}(M; \mathbb{R}),$$

$$\alpha \mapsto \omega^{n-p+1} \wedge \iota_\xi \alpha + \omega^{n-p} \wedge \eta \wedge \alpha$$

is an isomorphism for $0 \leq p \leq n$. 
Proof. First note that, by Poincaré duality, it is sufficient to show that \( \mathcal{L}^{n-p} \) has zero kernel. Now, by Corollary 4.3 on a compact co-Kähler manifold we have an isomorphism \( H^p_\eta(M) \cong \mathcal{H}^p(M) \). In particular, Corollary 4.3 tells us that the (harmonic) cohomology of \( M \) can be computed as a cylinder on the basic cohomology of the characteristic foliation. Since the latter is transversally Kähler, in view of [11], the map \( H \) on the basic cohomology of the characteristic foliation. Since the latter is transversally Kähler, in view of [11], the map \( H^p(M, \mathcal{F}_\xi) \to H^{2n-p}(M, \mathcal{F}_\xi) \) given by multiplication with the Kähler form \( \omega^{n-p} \) is an isomorphism for \( p \leq n \). Again by Corollary 4.3, the corresponding map \( H^p_\eta(M) \to H^{2n-p}_\eta(M) \) is also an isomorphism.

Now consider the Lefschetz map \( \mathcal{L}^{n-p} : H^p_\eta(M) \to H^{2n-p}_\eta(M) \) given by (4). Decompose any \( \alpha \in H^p_\eta(M) \) as \( \alpha = \alpha_1 + \alpha_2 \) according to (5) so that \( \iota_\xi \alpha_1 = 0 \) and \( \alpha_2 = \eta \wedge \iota_\xi \alpha \). We shall show that the Lefschetz map is non-zero on both \( \alpha_1 \) and \( \alpha_2 \) with \( \mathcal{L}^{n-p}((\alpha_1) \in \eta \wedge H^{2n-p}_1(M) \) and \( \mathcal{L}^{n-p}((\alpha_2) \in \mathcal{H}^{2n-p}_1(M) \). Then, because these sub-algebras are complementary, we will have \( \mathcal{L}^{n-p}(\alpha) \neq 0 \) for all \( \alpha \neq 0 \).

For \( \alpha_1 \in H^p_1(M) \cong H^p(M, \mathcal{F}_\xi) \), because \( \iota_\xi \alpha_1 = 0 \), the first term in the Lefschetz map definition applied to \( \alpha_1 \) vanishes. Hence, we get that \( \omega^{n-p} \wedge \alpha_1 \neq 0 \) in \( H^{2n-p}(M, \mathcal{F}_\xi) \) and, in view of Corollary 4.3, this implies that \( \omega^{n-p} \wedge \eta \wedge \alpha_1 \) is non-zero in \( \eta \wedge H^{2n-p}_1(M) \subseteq H^{2n+1-p}_\eta(M) \).

Because \( \alpha_2 = \eta \wedge \iota_\xi \alpha \), we see that the second term in the Lefschetz map definition applied to \( \alpha_2 \) vanishes. Now, \( \iota_\xi \alpha_2 \in H^{p-1}_\eta(M) \cong H^{p-1}(M, \mathcal{F}_\xi) \), so \( \omega^{n-p+1} \wedge \iota_\xi \alpha_2 \neq 0 \) in \( H^{2n-p+1}(M, \mathcal{F}_\xi) \cong H^{2n-p+1}_1(M) \). Therefore, when \( p \geq 1 \),

\[
\mathcal{L}^{n-p}(\alpha) = \omega^{n-p+1} \wedge \iota_\xi \alpha + \omega^{n-p} \wedge \eta \wedge \alpha \\
= \omega^{n-p+1} \wedge \iota_\xi \alpha_2 + \omega^{n-p} \wedge \eta \wedge \alpha_1 \\
\neq 0,
\]

so \( \mathcal{L}^{n-p} \) has zero kernel and is thus an isomorphism on cohomology. Furthermore, when \( p = 0 \), we get

\[
\mathcal{L}^n(1) = \omega^n \wedge \eta \neq 0,
\]
since \( \omega^n \wedge \eta \) is a volume form by assumption and, hence, cannot be exact. □

Since \( H^p_\eta(M) \cong \mathcal{H}^p(M) \) (harmonic forms) on a compact co-Kähler manifold, we obtain

**Corollary 4.10.** Let \( (M, J, \xi, \eta, g) \) be a compact co-Kähler manifold. Then the Lefschetz map \( \mathcal{L}^{n-p} : \mathcal{H}^p(M) \to \mathcal{H}^{2n-p}(M) \) is an isomorphism for \( 0 \leq p \leq n \).

In [9] the authors prove that the minimal model \( \mathcal{M}_{M, \mathcal{F}} \) of the basic forms \( \Omega^*(M, \mathcal{F}) \) of a transversally Kähler foliation \( \mathcal{F} \) on a compact manifold is formal. We would like to use our characterization (in a slightly different form) of the cohomology of a compact co-Kähler manifold to give an alternative proof of formality as well as a new description of the minimal model.
of a co-Kähler manifold. Note that Corollary 4.8 may be phrased as the following.

**Corollary 4.11.** On a compact co-Kähler manifold $M$, $H^*_1(M) \cong H^*(M, F_\xi)$ and

$$H^*(M; \mathbb{R}) \cong H^*_\eta(M) = H^*(M, F_\xi) \otimes \wedge([\eta]).$$

Furthermore, the splitting $\Omega^*_\eta(M) = \Omega^1_\eta(M) \oplus \eta \wedge \Omega^{p-1}_1(M)$ (for each $p$) may be written as

$$\Omega^*_\eta(M) = \Omega^1_\eta(M) \otimes \wedge(\eta).$$

Using this description, we can now see the transversally Kähler structure reflected in the minimal model of $M$.

**Proposition 4.12.** Let $(M, J, \xi, \eta, g)$ be a compact co-Kähler manifold. Then $M$ is formal in the sense of Sullivan and the minimal model splits as a tensor product of cdga’s

$$M_M \cong M_{M,F} \otimes \wedge(\eta, d = 0).$$

**Proof.** We use the formality of $M_{M,F}$ (by [9]), $\wedge(\eta, d = 0)$ and the identification $\Omega^*(M, F) \cong \Omega^1_1(M)$ to obtain the following diagram.

Here, the quasi-isomorphism $\rho$ is obtained from Lemma 2.7 applied to the bottom part of the diagram. By composition, we then obtain $\theta$ and we see it is a quasi-isomorphism. Hence, $M$ is formal and, again by Lemma 2.7, the quasi-isomorphism $\rho$ is an isomorphism.

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