Minimization problem associated with an improved Hardy-Sobolev type inequality

Megumi Sano

\textsuperscript{a}\textit{Laboratory of Mathematics, Graduate School of Engineering, Hiroshima University, Higashi-Hiroshima, 739-8527, Japan}

Abstract

We consider the existence and the non-existence of a minimizer of the following minimization problems associated with an improved Hardy-Sobolev type inequality introduced by Ioku \cite{12}.

\[ I_a := \inf_{u \in W^{1,p}_0(B_R) \setminus \{0\}} \frac{\int_{B_R} |\nabla u|^p \, dx}{\left( \int_{B_R} |u|^{p^*(s)} V_a(x) \, dx \right)^{\frac{p}{p^*}}}, \quad \text{where } V_a(x) = \frac{1}{|x|^s \left( 1 - a \left( \frac{|x|}{R} \right)^{\frac{N-p}{p-1}} \right)^\beta} \geq \frac{1}{|x|^s}. \]

Only for radial functions, the minimization problem $I_a$ is equivalent to it associated with the classical Hardy-Sobolev inequality on $\mathbb{R}^N$ via a transformation. First, we summarize various transformations including that transformation and give a viewpoint of such transformations. As an application of this viewpoint, we derive an infinite dimensional form of the classical Sobolev inequality in some sense. Next, without the transformation, we investigate the minimization problems $I_a$ on balls $B_R$. In contrast to the classical results for $a = 0$, we show the existence of non-radial minimizers for the Hardy-Sobolev critical exponent $p^*(s) = \frac{p(N-s)}{N-p}$ on bounded domains. Finally, we give remarks of a different structure between two nonlinear scalings which are equivalent to the usual scaling only for radial functions under some transformations.

Keywords: Hardy-Sobolev inequality, Optimal constant, Extremal function

2010 MSC: 35A23, 35J20, 35A08

---

Email address: smegumi@hiroshima-u.ac.jp (Megumi Sano)
1. Introduction and main results

Let $B_R \subset \mathbb{R}^N$, $1 < p < N$, $0 \leq s \leq p$, $p^*(s) = \frac{N-N-s}{N-p}$ and $p^* = p^*(0)$. Then the classical Hardy-Sobolev inequality:

$$C_{N,p,s} \left( \int_{B_R} \frac{|u|^{p^*(s)}}{|x|^s} \, dx \right)^{\frac{p}{p^*}} \leq \int_{B_R} |\nabla u|^p \, dx \quad (1)$$

holds for all $u \in W^{1,p}_0(B_R)$, where $W^{1,p}_0(B_R)$ is the completion of $C_c^\infty(B_R)$ with respect to the norm $\|\nabla \cdot \|_{L^p(B_R)}$ and $C_{N,p,s}$ is the best constant of (1). In the case where $s = 0$ (resp. $s = p$), the inequality (1) is called the Sobolev (resp. Hardy) inequality. The Hardy-Sobolev inequality (1) is quite fundamental and important since it expresses the embeddings of the Sobolev spaces $W^{1,p}_0$. Furthermore the variational problems and partial differential equations associated with the Hardy-Sobolev inequality (1) are well-studied by many mathematicians so far, see [2], [23], [14], [3], [7], [25], to name a few.

Recently, Ioku [12] showed the following improved Hardy-Sobolev inequality for radial functions via the transformation (5), see §2.

$$C_{N,p,s} \left( \int_{B_R} \frac{|u|^{p^*(s)}}{|x|^s} \, dx \right)^{\frac{p}{p^*}} \leq \int_{B_R} |\nabla u|^p \, dx \quad (2)$$

where $\beta = \beta(s) = \frac{(N-1)p-(p-1)s}{N-p}$. One virtue of (2) is that we can take a limit directly for the improved inequality (2) as $p \nearrow N$, differently from the classical one. Indeed, Ioku [12] showed that the limit of the improved inequality (2) with $s = 0$ is the Alvino inequality [1] which implies the optimal embedding of $W^{1,N}_0(B_R)$ into the Orlicz spaces, and also the limit of the improved inequality (2) with $s = p$ is the critical Hardy inequality which implies the embedding of $W^{1,N}_0(B_R)$ into the Lorentz-Zygmund spaces $L^{\infty,N}(\log L)^{-1}$ which is smaller than the Orlicz space. For some indirect limiting procedures for the classical Hardy-Sobolev inequalities, see [24], [4], [18] and a survey [17]. Based on the transformation (5), the improved inequality (2) on $B_R$ equivalently connects to the classical one (1) on the whole space $\mathbb{R}^N$. This yields that the improved inequality (2) has the scale invariance under a scaling to which the usual scaling is changed via the transformation (5), and there exists a radial minimizer of (2) when $0 \leq s < p$. For more details, see [12] or §2.
In this paper, without the transformation, we investigate the following extended minimization problems $I_a$ for $a \in [0, 1]$ associated with improved Hardy-Sobolev inequalities.

$$I_a := \inf_{u \in W_0^{1,p}(B_R)\setminus \{0\}} \frac{\int_{B_R} |\nabla u|^p \, dx}{\left( \int_{B_R} |u|^{p'(s)} V(x) \, dx \right)^{\frac{p}{p'}}}, \quad \text{where } V_a(x) = \frac{1}{|x|^s \left( 1 - a \left( \frac{|x|}{R} \right)^{\frac{s}{p'}} \right)^{\frac{s-p}{p}}} \geq \frac{1}{|x|^s}.$$  

Note that the potential function $V_a(x)$ also has the boundary singularity when $a = 1$. Due to the boundary singularity, $I_1 = 0$ if $a = 1$ and $s < p$, see Proposition 2 in §3. Therefore we exclude the case where $a = 1$ and $s < p$.

Our main results are as follows.

**Theorem 1.** (i) Let $s = p$. Then for any $a \in [0, 1]$, $I_a = I_{a,\text{rad}} = C_{N,p,p} = \left( \frac{N-p}{p} \right)^p$ and $I_a$ is not attained.

(ii) Let $s = 0$. Then $I_a = I_{a,\text{rad}}(1 - a)^{\frac{s}{N-p}} = C_{N,p,0}(1 - a)^{\frac{s}{N-p}}$ and $I_a$ is not attained for any $a \in [0, 1]$.

(iii) Let $0 < s < p$. Then there exists $a_* \in (A, 1)$ such that $I_a < I_{a,\text{rad}}$ for $a \in (a_*, 1)$, $I_a$ is attained for $a \in (0, a_*)$, $I_a = I_{a,\text{rad}}$ for $a \in [0, a_*)$ and $I_a$ is not attained for $a \in [0, a_*)$, where $A = 1 - \left( \frac{s(p-1)}{p(N-1)} \right)^{\frac{1}{s}} \left[ 1 - \left( \frac{s(p-1)}{p(N-1)} \right) \right]$.

**Remark 1.** Note that $\frac{s(p-1)}{p(N-1)} < A$ and the potential function $V_a(x)$ is not monotone-decreasing with respect to $|x|$ for $a \in \left( \frac{s(p-1)}{p(N-1)}, 1 \right)$. Therefore it seems difficult to reduce the radial setting due to the lack of the rearrangement technique, see the first part in §3. However we can show $I_a = I_{a,\text{rad}}$ even for $a \in \left( \frac{s(p-1)}{p(N-1)}, a_* \right]$ thanks to the special shape of the potential function $V_a(x)$, see the last part of the proof of Theorem 1(iii).

Our minimization problem $I_a$ is related to the following nonlinear elliptic equation with the singular potential $V_a(x) \geq |x|^{-s}$.

$$\begin{cases}
-\text{div} (|\nabla u|^{p-2} \nabla u) = bV_a(x)|u|^{p'-2}u & \text{in } B_R, \\
u = 0 & \text{on } \partial B_R.
\end{cases} \quad (3)$$

The minimizer for $I_a$ is a ground state solution of the Euler-Lagrange equation (3) with a Lagrange multiplier $b$. Since the minimizer of Theorem 1(iii) is a non-radial function, we can observe that the symmetry breaking phenomenon of the
ground states of the elliptic equation (4) occurs when \( f(x) = V_a(x) \), \( a \) is close to 1, \( s \in (0, p) \) and \( p < q = p^*(s) < p^* \).

\[
\begin{cases}
-\text{div} \left( |\nabla u|^{p-2} \nabla u \right) = f(x) |u|^{q-2} u & \text{in } B_R, \\
u = 0 & \text{on } \partial B_R.
\end{cases}
\]  

It is well-known that the symmetry breaking phenomenon of the ground states of (4) occurs when \( f(x) = \left( \frac{|x|}{R} \right)^\alpha \), \( \alpha > 0 \) is sufficiently large and \( p < q < p^* \) (ref. [20]). Theorem 1 (iii) gives another example of the potential function \( f(x) \), which is not monotone-increasing and is not bounded differently from the Hénon potential function \( \left( \frac{|x|}{R} \right)^\alpha \).

In the case where \( a = 0 \), Theorem 1 is the same as the classical result which is the non-existence of the minimizer of the classical Hardy-Sobolev inequality on bounded domains. However, in Theorem 1 (iii), we see that there exists a minimizer of \( I_a \) even on bounded domains. This is different from the classical result. Concretely, we show that \( I_{a,\text{rad}} \) is the concentration level of minimizing sequence of \( I_a \) for \( s \in (0, p) \), where \( I_{a,\text{rad}} \) is a level of \( I_a \) only for radial functions. By concentration-compactness alternative, we can obtain a minimizer of \( I_a \) for \( a \in (a_*, 1) \). Note that the continuous embedding: \( W_0^{1,p}(B_R) \hookrightarrow L^{p^*(s)}(B_R; V_a(x) \, dx) \) related to our problem is not compact due to the existence of a non-compact sequence given by the nonlinear scaling (10) in \( \S 2 \), see also \( \S 4 \). However, the parameter of \( a \) plays a role of lowering the level of \( I_a \) than \( I_{a,\text{rad}} \), and thanks to it, we can remove the possibility of occurring such non-compact behavior.

This paper is organized as follows: In \( \S 2 \) we summarize various transformations including the transformation and give a viewpoint of such transformations by which the classical Hardy-Sobolev inequality equivalently connects to another inequality. As an application of this viewpoint, we derive an infinite dimensional form of the classical Sobolev inequality in some sense. In \( \S 3 \) we prepare several Lemmas and Propositions and show Theorem 1. In \( \S 4 \) we give remarks of a different structure between two nonlinear scalings which are equivalent to the usual scaling only for radial functions under some transformations.

We fix several notations: \( B_R \) or \( B_R^N \) denotes a \( N \)-dimensional ball centered 0 with radius \( R \). As a matter of convenience, we set \( B_0^N = \mathbb{R}^N \) and \( \frac{1}{\omega_0} = 0 \). \( \omega_{N-1} \) denotes an area of the unit sphere \( S^{N-1} \) in \( \mathbb{R}^N \). \( |A| \) denotes the Lebesgue measure of a set \( A \subset \mathbb{R}^N \) and \( X_{\text{rad}} = \{ u \in X \mid u \text{ is radial} \} \). The Schwarz symmetrization \( u^\#: \mathbb{R}^N \rightarrow [0, \infty) \) of \( u \) is given by

\[
u^\#(x) = u^\#(|x|) = \inf \left\{ \tau > 0 : |\{ y \in \mathbb{R}^N : |u(y)| > \tau \}| \leq |B_{|\nu(0)|}| \right\}.
\]
Throughout the paper, if a radial function $u$ is written as $u(x) = \bar{u}(|x|)$ by some function $\bar{u} = \bar{u}(r)$, we write $u(x) = u(|x|)$ with admitting some ambiguity.

2. Various transformations and an infinite dimensional form of the classical Sobolev inequality

First, we explain the transformation introduced by Ioku [12] for readers convenience and give several remarks.

We use the polar coordinates: $\mathbb{R}^N \ni x = r\omega (r \in \mathbb{R}_+, \omega \in \mathbb{S}^{N-1})$. Let $T \in [R, \infty], y \in B^N_T, x \in B^N_R, r = |x|, t = |y|$, and $w \in C^1_c(B_T)$. By using the fundamental solution of $p-$Laplacian with Dirichlet boundary condition, we consider the following transformation

$$u(r\omega) = w(t\omega), \text{ where } r^{-\frac{N-p}{p-1}} - R^{-\frac{N-p}{p-1}} = t^{-\frac{N-p}{p-1}} - T^{-\frac{N-p}{p-1}}. \quad (5)$$

Note that in the case where $T = \infty$ the transformation (5) is founded by Ioku[12]. Then we see that

$$\int_{B_T} |\nabla w|^p \, dy = \int_{S^{N-1}} \int_0^T \left[ \frac{\partial w}{\partial t} + \frac{1}{t} \nabla_{S^{N-1}} w \right]^p t^{N-1} \, dt \, dS_{\omega}$$

$$= \int_{S^{N-1}} \int_0^R \left[ \frac{\partial u}{\partial r} + \frac{1}{r} \frac{dt}{dr} \nabla_{S^{N-1}} u \right]^p \left( \frac{dr}{dt} \right)^{p-1} t^{N-1} \, dr \, dS_{\omega}.$$

Since

$$t \frac{dr}{dt} = \left( \frac{r}{t} \right)^{\frac{N-1}{p-1}}, \quad \frac{dr}{dt} = r^{\frac{N-1}{p-1}} \left( r^{-\frac{N-p}{p-1}} - R^{-\frac{N-p}{p-1}} + T^{-\frac{N-p}{p-1}} \right),$$

we have

$$\int_{B_T} |\nabla w|^p \, dy = \int_{B_R} |L_p u|^p \, dx, \quad (6)$$

where

$$L_p u = \frac{\partial u}{\partial r} \omega + \frac{1}{r} \nabla_{S^{N-1}} u \left[ 1 - a \left( \frac{r}{R} \right)^{\frac{N-p}{p-1}} \right]^{-1}.$$

where $a = 1 - \left( \frac{R}{T} \right)^{\frac{N-p}{p-1}} > 1$ as $T > R$. Note that the differential operator $L_p$ is not $\nabla$ for $T > R$ due to the last term. However, if $u$ and $w$ are radial functions, then we can obtain the equality of two $L^p$ norms of $\nabla$ between $B_T$ and $B_R$ as follows:

$$\int_{B_T} |\nabla w|^p \, dy = \int_{B_R} |\nabla u|^p \, dx \quad \text{if } u \text{ and } w \text{ are radial.} \quad (7)$$
On the other hand, for the Hardy-Sobolev term, we have

$$
\int_{B_T} \frac{|w|^p(x)}{|y|^s} \, dy = \int_{B_T} \frac{|u|^p(x)}{|x|^s \left(1 - a \left(\frac{|x|}{R} \right)^\beta \right)} \, dx,
$$

(8)

where \( \beta = \beta(s) = \frac{(N-1)p-(p-1)s}{N-p} \). Set

$$
I_{a,rad} := \inf_{u \in W^1,\rad,_{0,rad}(B_R \setminus \{0\}) \setminus \{0\}} \left( \frac{\int_{B_R} |\nabla u|^p \, dx}{\left( \int_{B_R} \frac{|u|^{p^*}(x)}{|y|^p} \, dy \right)^{\frac{p}{p-1}}} \right)^{\frac{1}{p^*}}.
$$

From (7) and (8), we observe that the minimization problem \( I_{a,rad} \) can be reduced the classical Hardy-Sobolev minimization problem \( C_{N,p,s} \):

$$
C_{N,p,s} = \inf_{w \in W^1_{\rad,0,rad}(B_R \setminus \{0\}) \setminus \{0\}} \frac{\int_{B_R} |\nabla w|^p \, dy}{\left( \int_{B_R} \frac{|w|^{p^*}(x)}{|y|^p} \, dy \right)^{\frac{p}{p-1}}}.
$$

It is well-known that \( C_{N,p,s} \) is independent of the radius \( T \) and \( C_{N,p,s} \) is attained if and only if \( T = \infty \) and \( 0 \leq s < p \). Moreover its minimizer is the family of

$$
W_A(t) = \lambda^{\frac{N-p}{p^*}} \left( 1 + (\lambda r)^{\frac{p^*}{p^*-1}} \right)^{\frac{N-p}{p^*-1}} \quad \text{for} \; \lambda \in (0, \infty),
$$

see e.g. \[8\]. Therefore we can obtain the following results for \( I_{a,rad} \) based on the transformation (5).

**Proposition 1.** (\[12\]) \( I_{a,rad} \) is independent of \( a \in [0, 1] \) and \( I_{a,rad} = C_{N,p,s} \). And \( I_{a,rad} \) is attained if and only if \( a = 1 \) and \( 0 \leq s < p \). Moreover, the minimizer of \( I_{1,rad} \) is the family of

$$
U^\lambda(r) = \lambda^{\frac{N-p}{p^*}} \left[ 1 + (\lambda r)^{\frac{p^*}{p^*-1}} \left( 1 - \left(\frac{r}{R} \right)^{\frac{p^*}{p^*-1}} \right)^{\frac{p^*}{p^*-1}} \right] \quad \text{for} \; \lambda \in (0, \infty).
$$

**Remark 2** (Scale invariance). It is well-known that thanks to the zero extension, the classical inequality (7) has the scale invariance under the usual scaling for \( \lambda \in [1, \infty) \).

$$
w_A(t \omega) = \begin{cases} 
\lambda^{\frac{N-p}{p^*}} w(\lambda t \omega) & \text{for} \; t \in [0, \frac{T}{\lambda}], \\
0 & \text{for} \; t \in (\frac{T}{\lambda}, T].
\end{cases}
$$

(9)
Note that in the case where $T = \infty$, we can consider any $\lambda \in (0, \infty)$. By the transformation (5), the usual scaling (24) is changed its form to the following scaling for $\lambda \in [1, \infty)$.

$$u^\lambda(r\omega) = \begin{cases} 
\lambda \frac{N-p}{p} u(\tilde{r}\omega) & \text{for } t \in [0, \tilde{R}], \\
0 & \text{for } t \in (\tilde{R}, R), 
\end{cases}$$

where $\tilde{r} = (\lambda r) \left[ 1 + a \left( \lambda \frac{N-p}{p} - 1 \right) \left( \frac{r}{R} \right)^{\frac{N-p}{p-1}} \right]^{-\frac{p-1}{N-p}}, \tilde{R} = R \left( \lambda \frac{N-p}{p-1} (1 - a) + a \right)^{-\frac{p-1}{N-p}}$.

Respectively, in the case where $a = 1$, we can consider any $\lambda \in (0, \infty)$. Obviously from (6) and (8), we see that the improved Hardy-Sobolev inequality:

$$C_{N,p,s} \left( \int_{B_R} \frac{|u|^{p^*(s)}}{|x|^s} \left( 1 - a \left( \frac{|x|}{R} \right)^{\frac{N-p}{p-1}} \right)^\beta dx \right)^{-\frac{1}{p^*(s)}} \leq \int_{B_R} |L_p u|^p dx$$

with the differential operator $L_p$ is invariant under the scaling (10). The scaling in (12) looks different from the scaling (10). However, taking $\lambda \mapsto \lambda^{-\frac{p-1}{N-p}}$, we observe that these scalings are same essentially. Note that $\|\nabla u\|_{L^p(B_R)}$ is not invariant under the scaling (10) for non-radial functions. Therefore, in §3 we investigate our minimization problem $I_a$ without the transformation (5).

**Remark 3.** Actually, the original transformation is given in Theorem 18. in [10] for $u \in W^{1,2}_{0,\text{rad}}(B_1)$, where $B_1, \Omega \subset \mathbb{R}^2$ as follows.

$$w(y) = u(x), \text{ where } G_{\Omega,z}(y) = G_{B_1,0}(x) = -\frac{1}{2\pi} \log |x|$$

and $G_{\Omega,z}(y)$ is the Green function in a domain $\Omega$, which has a singularity at $z \in \Omega$. Therefore, we can observe that the transformation (5) is (11) in the case where $W^{1,p}_{0,\text{rad}}(B_1), p < N$ and $z = 0, B_1 \subset \mathbb{R}^N = \Omega$.

In addition to (11) and (5), there are various transformations by which the classical Hardy-Sobolev type inequality equivalently connects to another inequality for radial functions (ref. [24], [11], [13], [19], [12], [14]). Next, we explain them comprehensively. An unified viewpoint is to connect two typical functions on each world (e.g. fundamental solution of $p$-Laplacian, virtual minimizer of the
Then they obtain the equality of two norms between the subcritical Sobolev space $W_0^1, p (\mathbb{R}^N) (p < N)$ and the embedding of the critical Sobolev space into the Lorentz-Zygmund spaces $p$ by using two fundamental solutions of the Laplacian and weighted $p$–Laplacian: \[ \text{div}(\{|x|^{p-N} |\nabla u|^p - 2 \nabla u\}) \]

\[ u(r) = w(t), \text{ where } t^{-\frac{q}{p-q}} = \log \frac{R}{r}. \] (12)

Then they obtain the equality of two norms between the subcritical Sobolev space $W_0^1, p (\mathbb{R}^N) (p < N)$ and the weighted critical Sobolev space $W_0^{1,p}(B_R^N, |x|^{p-N} dx)$ as follows.

\[ \int_{\mathbb{R}^N} |\nabla w|^p \, dy = \int_{B_R^N} |x|^{p-N} |\nabla u|^p \, dx, \int_{\mathbb{R}^N} |w|^q \, dy = \frac{p-1}{N} \int_{B_R^N} |x|^N \left( \log \frac{R}{r} \right)^{(\beta(x))} \, dx. \]

On the other hand, in [19] and [16], they consider the following transformation (13) by using two fundamental solutions of the Laplacian and weighted critical Sobolev space $W_0^{1,N}(\mathbb{R}^m) (N < m)$ as follows.

\[ u(r) = w(t), \text{ where } t^{-\frac{q}{N}} = \log \frac{R}{r} \text{ is equivalent to } t^{-\frac{m-N}{N}} = \left( \log \frac{R}{r} \right)^{\frac{N-1}{N}}. \] (13)

A different point of the transformation (13) from these transformations (11), (5), (12) is to consider the difference of dimensions on each world. Thanks to the difference of dimensions, we obtain the equality of two norms between the critical Sobolev space $W_0^{1,N}(B_R^N)$ and the higher dimensional subcritical Sobolev space $W_0^{1,N}(\mathbb{R}^m) (N < m)$ as follows.

\[ \int_{\mathbb{R}^m} |\nabla w|^N \, dy = \int_{B_R^N} |\nabla u|^N \, dx, \int_{\mathbb{R}^m} |w|^q \, dy = \frac{\omega_m-1}{\omega_{m-1}(m-N)} \int_{B_R^N} |x|^N \left( \log \frac{R}{r} \right)^{(\gamma(\alpha))} \, dx, \]

where $\gamma(\alpha) = \frac{(m-1)N-(N-1)\alpha}{m-N}$. This gives an equivalence between the critical Hardy inequality and a part of the higher dimensional subcritical Hardy inequality (ref. [19]). Besides, this also gives an relationship between the embedding of the subcritical Sobolev space into the Lorentz spaces for $q > p$:

$W_0^{1,p}(B_R^N) \hookrightarrow L^p, p \hookrightarrow L^p, q \hookrightarrow L^p, \infty$

and the embedding of the critical Sobolev space into the Lorentz-Zygmund spaces for $q > N$:

$W_0^{1,N}(B_R^N) \hookrightarrow L^{\infty,N}(\log L)^{-1} \hookrightarrow L^{\infty,q}(\log L)^{-1+\frac{1}{q}-\frac{1}{p}} \hookrightarrow L^{\infty,\infty}(\log L)^{-1+\frac{1}{p}} = \text{ExpL}^{N}$. 

8
Since almost all transformations are applicable only for radial functions, we can expect the different phenomena from classical results for any functions, for example the existence and the non-existence of a minimizer. In fact, the author in [16] shows the existence of a non-radial minimizer of the inequality associated with the embedding: $W^{1,N}(B^N_R) \hookrightarrow L^q_0((\log L)^{-1+\frac{N}{p}})$. In this paper, we study an analogue of this work [16].

Finally, as an application of this unified viewpoint for the transformations, we derive some infinite dimensional form of the classical Sobolev inequality in a different way from [4] which is a study of the logarithmic Sobolev inequality. In order to consider a limit as the dimension $m \to \infty$, we reduce the dimension $m$ to $N$ by using the following transformation (14) which connects two norms between the Sobolev space $W_0^{1,p}(\mathbb{R}^m)$ and the lower dimensional Sobolev space $W_0^{1,p}(\mathbb{R}^N)$, where $p < N < m$.

$$u(r) = w(t), \text{ where } t^{-\frac{N-p}{m-p}} = r^{-\frac{N-p}{m-p}}.$$ (14)

Then we can see that

$$\int_{\mathbb{R}^m} |\nabla u|^p \, dx = \frac{\omega_{m-1}}{\omega_{N-1}} \left( \frac{m-p}{N-p} \right)^{p-1} \int_{\mathbb{R}^N} |\nabla w|^p \, dy,$$

$$\int_{\mathbb{R}^m} |u|^{mp} \, dx = \frac{\omega_{m-1}}{\omega_{N-1}} \frac{N-p}{m-p} \int_{\mathbb{R}^N} |w|^{mp} \, dy.$$

Therefore the Sobolev inequality (1) for radial functions $u \in W_0^{1,p}(\mathbb{R}^m)$:

$$C_{m,p,0} \left( \int_{\mathbb{R}^m} |u|^{mp} \, dx \right)^{\frac{mp}{m-p}} \leq \int_{\mathbb{R}^m} |\nabla u|^p \, dx$$

is equivalent to the following inequality for radial functions $w \in W_0^{1,p}(\mathbb{R}^N)$.

$$C_{m,p,0} \left( \frac{\omega_{N-1}}{\omega_{m-1}} \right)^{\frac{m}{p}} \left( \frac{N-p}{m-p} \right)^{p-\frac{m}{p}} \left( \int_{\mathbb{R}^N} |w|^{mp} \, dy \right)^{\frac{mp}{m-p}} \leq \int_{\mathbb{R}^N} |\nabla w|^p \, dy.$$ (15)

Since

$$C_{m,p,0} = \pi^{\frac{m}{p}} m \left( \frac{m-p}{p-1} \right)^{p-1} \left( \frac{\Gamma\left(\frac{m}{p}\right)\Gamma(m+1-\frac{m}{p})}{\Gamma(m)\Gamma\left(1+\frac{m}{2}\right)} \right)^{\frac{p}{m}},$$ (Sobolev’s best constant),

$$\omega_{N-1} = \frac{N\pi^N}{\Gamma\left(1+\frac{N}{2}\right)}, \Gamma(t) = \sqrt{2\pi} t^{t-\frac{1}{2}} e^{-t} + o(1) \text{ as } t \to \infty \text{ (Stirling’s formula)},$$
we have

\[
C_{m,p,0} \left( \frac{\omega_{N-1}}{\omega_{m-1}} \right)^{\frac{m}{p}} \left( \frac{N - p}{m - p} \right)^{\frac{p-2}{2}} = \frac{m}{m-p} \frac{(N-p)^p}{(p-1)^{p-1}} \left( \frac{\omega_{N-1} (m-p)}{N-p} \right)^{\frac{m}{p}} \left( \frac{\Gamma\left(\frac{m}{p}\right)\Gamma\left(\frac{p-1}{p}m+1\right)}{\Gamma(m+1)} \right)^{\frac{m}{p}} \left( \frac{(N-p)^{p-1}}{(m+1)^{m+\frac{1}{2}}} e^{-\frac{1}{p-1}} \right)^{\frac{m}{p}} + o(1)
\]

\[
= \left( \frac{N - p}{p} \right)^p + o(1) \quad (m \to \infty).
\]

Hence we can obtain the limit of the left-hand side of (15) as \( m \to \infty \) as follows.

\[
C_{m,p,0} \left( \frac{\omega_{N-1}}{\omega_{m-1}} \right)^{\frac{m}{p}} \left( \frac{N - p}{m - p} \right)^{\frac{p-2}{2}} \left( \frac{1}{(p-1)^{p-1}} \right) \left( \frac{\omega_{N-1} (m-p)}{N-p} \right)^{\frac{m}{p}} \left( \frac{\Gamma\left(\frac{m}{p}\right)\Gamma\left(\frac{p-1}{p}m+1\right)}{\Gamma(m+1)} \right)^{\frac{m}{p}} \left( \frac{(N-p)^{p-1}}{(m+1)^{m+\frac{1}{2}}} e^{-\frac{1}{p-1}} \right)^{\frac{m}{p}} \to \left( \frac{N - p}{p} \right)^p \int_{\mathbb{R}^N} |w|^p dy.
\]

From above calculations, we can observe an interesting new aspect of the classical Hardy inequality, that is an \textit{infinite dimensional form} of the classical Sobolev inequality. And we also see that under the transformation (14), the Hardy inequality on \( W_{1,p}^0(\mathbb{R}^m) \) is equivalent to it on \( W_{1,p}^0(\mathbb{R}^N) \), that is, the Hardy inequality is independent of the dimension in this sense.

3. Proof of Theorem \[1]: the existence and the non-existence of the minimizer

In this section, we prepare several Lemmas and Propositions and show Theorem \[1\].

Note that we can apply the rearrangement technique to our minimization problem \( I_a \) for \( a \in \left[0, \frac{s(p-1)}{p(N-1)}\right] \). More precisely, since the potential function \( V_a(x) \) is radially decreasing on \( B_R \) for \( a \in \left[0, \frac{s(p-1)}{p(N-1)}\right] \), the Pólya-Szego inequality and the Hardy-Littlewood inequality imply that

\[
\frac{\int_{B_R} |\nabla u|^p dx}{\left( \int_{B_R} |u|^{p_0} V_a(x) dx \right)^{\frac{p}{p_0}}} \geq \frac{\int_{B_R} |\nabla u|^p dx}{\left( \int_{B_R} |u|^{p_0} V_a(x) dx \right)^{\frac{p}{p_0}}} \geq I_{a,\text{rad}}
\]
for any \( u \in W_0^{1,p}(B_R) \) and \( a \in [0, \frac{s(p-1)}{p(N-1)}] \). Therefore we have

\[
I_a = I_{a,\text{rad}} = C_{N,p,s} \quad \text{for any } a \in \left[0, \frac{s(p-1)}{p(N-1)}\right].
\] (16)

However, we can not apply the rearrangement technique to \( I_a \) for \( a \in \left(\frac{s(p-1)}{p(N-1)}, 1\right) \). Therefore it seems difficult to reduce the radial setting in general.

First, instead of rearrangement, we use the following lemma by which we can reduce the radial setting when \( s = p \).

**Lemma 1.** Let \( 1 < q < \infty \), \( f = f(x) \) be a radial function on \( B_R \). If there exists \( C > 0 \) such that for any radial functions \( u \in C^{1}_c(B_R) \) the inequality:

\[
C \int_{B_R} |u|^q f(x) \, dx \leq \int_{B_R} |\nabla u|^q \, dx
\] (17)

holds, then for any functions \( w \in C^{1}_c(B_R) \) the inequality:

\[
C \int_{B_R} |w|^q f(x) \, dx \leq \int_{B_R} \left| \nabla W \cdot \frac{x}{|x|} \right|^q \, dx
\] (18)

holds.

**Proof.** For any \( w \in C^{1}_c(B_R) \), define a radial function \( W \) as follows.

\[
W(r) = \left( \omega_{N-1}^{-\frac{1}{q}} \int_{S^{N-1}} |w(r\omega)|^q \, dS_{\omega} \right)^{\frac{1}{q}} \quad (0 \leq r \leq R).
\]

Then we have

\[
|W'(r)| = \omega_{N-1}^{-\frac{1}{q}} \left( \int_{S^{N-1}} |w(r\omega)|^q \, dS_{\omega} \right)^{\frac{1}{q}-1} \int_{S^{N-1}} |w|^{q-1} \left| \frac{\partial w}{\partial r} \right| \, dS_{\omega}
\]

\[
\leq \omega_{N-1}^{-\frac{1}{q}} \left( \int_{S^{N-1}} \left| \frac{\partial w}{\partial r} (r\omega) \right|^q \, dS_{\omega} \right)^{\frac{1}{q}}.
\]

Therefore we have

\[
\int_{B_R} |\nabla W|^q \, dx \leq \int_{B_R} \left| \nabla W \cdot \frac{x}{|x|} \right|^q \, dx,
\] (19)

\[
\int_{B_R} |W|^q f(x) \, dx = \int_{B_R} |w|^q f(x) \, dx.
\] (20)

From (17) for \( W \), (19), and (20), we obtain (18) for any \( w \). \( \square \)
Second, we give a necessary and sufficient condition of the positivity of \( I_a \) for \( a \in [0, 1] \). As we see Proposition 1, \( I_{a, \text{rad}} = C_{N, p, s} > 0 \) for any \( s \in [0, p] \) and any \( a \in [0, 1] \). However, \( I_a \) is not so due to the boundary singularity. This is also mentioned by [12]. For readers convenience, we give a sketch of the proof.

**Proposition 2.** \( I_a = 0 \iff a = 1 \) and \( 0 \leq s < p \).

**Proof.** Let \( a = 1 \). In the similar way to [16], set \( x_\epsilon = (R - 2\epsilon)^2 \) for \( y \in \partial B_R \) and for small \( \epsilon > 0 \). Then we define \( u_\epsilon \) as follows:

\[
\begin{align*}
    u_\epsilon(x) &= \begin{cases} 
    v \left( \frac{|x|}{\epsilon} \right) & \text{if } x \in B(x_\epsilon), \\
    0 & \text{if } x \in B_R \setminus B(x_\epsilon),
    \end{cases}
\end{align*}
\]

where \( v(t) = \begin{cases} 
    1 & \text{if } 0 \leq t \leq \frac{1}{2}, \\
    2(1 - t) & \text{if } \frac{1}{2} < t \leq 1.
    \end{cases} \)

Then we have

\[
\begin{align*}
    \int_{B_R} |\nabla u_\epsilon(x)|^p \, dx &= \epsilon^{N-p} \int_{B_1} |\nabla v(|z|)|^p \, dz = C \epsilon^{N-p}, \\
    \int_{B_R} \frac{|u_\epsilon(x)|^{p^*(s)}}{|x|^\beta} \, dx &\geq C \int_{B_{(x_\epsilon)}(x_\epsilon)} \frac{|u_\epsilon(x)|^{p^*(s)}}{(R - |x|)^\beta} \, dx \geq \frac{C}{(3\epsilon)^\beta} \frac{1}{(\epsilon^\frac{1}{2})^\beta} \int_{B_{\epsilon}(x_\epsilon)} \, dx = C \epsilon^{N-\beta}.
\end{align*}
\]

Hence we see that

\[
I_1 \leq C \epsilon^{N-\beta-(N-\beta)p^*} = C \epsilon^{\frac{N-\beta}{p}} \to 0 \text{ as } \epsilon \to 0 \text{ if } 0 \leq s < p.
\]

Therefore \( I_1 = 0 \) if \( a = 1 \) and \( 0 \leq s < p \). Conversely, we can easily show that \( I_a > 0 \) except for that case. Indeed, if \( s = p \), then \( I_a = I_{a, \text{rad}} > 0 \) for any \( a \in [0, 1] \) from Proposition 1 and Lemma 1. And also, if \( a < 1 \), then there is no boundary singularity. Thus \( I_a > 0 \) for any \( a \in [0, 1) \). Therefore, we obtain the necessary and sufficient condition of the positivity of \( I_a \). \( \Box \)

Third, we show that \( I_a \) is monotone decreasing and continuous with respect to \( a \in [0, 1] \). The potential function \( V_a(x) \) is continuously monotone-increasing with respect to \( a \in [0, 1] \). Thus it is easy to show the monotone-decreasing property of \( I_a \) with respect to \( a \in [0, 1] \) and the continuity of \( I_a \) with respect to \( a \in [0, 1] \). Here, we give a proof of the continuity of \( I_a \) at \( a = 1 \) only.

**Lemma 2.** \( I_a \) is monotone-decreasing and continuous with respect to \( a \in [0, 1] \).
Proposition 3. Let $R < \infty$ and $f : B_R \to \mathbb{R}$ be a nonnegative bounded continuous function with $f \neq 0$. Then

$$S := \inf_{u \in W_{0}^{1,p}(B_R) \setminus \{0\}} \frac{\int_{B_R} |\nabla u|^p \, dx}{\left( \int_{B_R} |u|^\frac{Np}{N-p} f(x) \, dx \right)^\frac{N-p}{N}} = \left( \max_{x \in \overline{B}_R} f(x) \right)^\frac{N-p}{N} C_{N,p,0}$$

and there is no minimizers of the minimization problem $S$.

Proof. Since we easily obtain $S \geq \left( \max_{x \in \overline{B}_R} f(x) \right)^\frac{N-p}{N} C_{N,p,0}$, we shall show $S \leq \left( \max_{x \in \overline{B}_R} f(x) \right)^\frac{N-p}{N} C_{N,p,0}$. Let $z \in \overline{B}_R$ be a maximum point of $f$. For simplicity, we assume that $z \in B_R$. For any $\varepsilon > 0$ there exist $T > 0$ and $v \in C_c^\infty(B_T)$ such that

$$C_{N,p,0} \geq \frac{\int_{B_T} |\nabla v|^p \, dx}{\left( \int_{B_T} |v|^\frac{Np}{N-p} \, dx \right)^\frac{N-p}{N}} = \frac{\varepsilon}{2}.$$
Set $u_0(x) = \lambda \frac{N-p}{p} v(\lambda(x - z))$ for $\lambda > 0$. Then for large $\lambda > 0$ we have

$$
C_{N,p,0} \geq \frac{\int_{B_{x-\frac{1}{2}}(z)} |\nabla u_0|^p \, dx}{\left( \int_{B_{x-\frac{1}{2}}(z)} |u_0|^{N/p} \, dx \right)^{N/p} \frac{N-p}{N}} - \frac{\varepsilon}{2} \geq f(z)^{\frac{N-p}{N}} \frac{\int_{B_{x-\frac{1}{2}}(z)} |\nabla u_0|^p \, dx}{\left( \int_{B_{x-\frac{1}{2}}(z)} |u_0|^{N/p} \, dx \right)^{N/p} \frac{N-p}{N}} - \varepsilon
$$

$$
\geq f(z)^{\frac{N-p}{N}} S - \varepsilon.
$$

Since $\varepsilon$ is arbitrary, we obtain $S \leq \left( \max_{x \in \partial B_R} f(x) \right)^{\frac{N-p}{N}} C_{N,p,0}$. The case where $z \in \partial B_R$ is also showed in the same way. We omit the proof in that case.

On the other hand, the non-attainability of $S$ comes from it of $C_{N,p,0}$. Indeed, if we assume that $v \equiv 0$ is a minimizer of $S$, then

$$
S = \frac{\int_{B_R} |\nabla v|^p \, dx}{\left( \int_{B_R} |v|^{N/p} \, dx \right)^{N/p} \frac{N-p}{N}} \geq \left( \max_{x \in \partial B_R} f(x) \right)^{\frac{N-p}{N}} \frac{\int_{B_R} |\nabla u|^p \, dx}{\left( \int_{B_R} |u|^{N/p} \, dx \right)^{N/p} \frac{N-p}{N}}
$$

where the last inequality comes from the non-attainability of $C_{N,p,0}$. This is a contradiction. \qed

Fifth, we show the concentration level of minimizing sequences of $I_a$ is $I_{a,\text{rad}}$ when $0 < s < p$.

**Lemma 3.** Let $0 < s < p$ and $0 \leq a < 1$. If $I_a < I_{a,\text{rad}} = C_{N,p,s}$, then $I_a$ is attained by a non-radial function.

In order to show Lemma \textbf{3} also in the case where $p \neq 2$, we prepare two Lemmas. Lemma \textbf{5} is concerning with almost everywhere convergence of the gradients of a sequence of solutions. This guarantees to use Lemma \textbf{4} in the proof of Lemma \textbf{3}.

**Lemma 4.** (\cite{6}) For $p \in (0, +\infty)$, let $(g_m)_{m=1}^{\infty} \subset L^p(\Omega, \mu)$ be a sequence of functions on a measurable space $(\Omega, \mu)$ such that

(i) $\|g_m\|_{L^p(\Omega, \mu)} \leq 3C < \infty$ for all $m \in \mathbb{N}$, and

(ii) $g_m(x) \rightarrow g(x)$ $\mu$-a.e. $x \in \Omega$ as $m \rightarrow \infty$. 

...
Then
\[
\lim_{m \to \infty} \left( \|g_m\|_{L^p(\Omega, \mu)} - \|g_m - g\|_{L^p(\Omega, \mu)}^p \right) = \|g\|_{L^p(\Omega, \mu)}^p.
\]

Note that we can apply Lemma 3 to \( \mu(dx) = f(x)dx \), where \( f \) is any nonnegative \( L^1(\Omega) \) function.

**Lemma 5.** ([5] Theorem 2.1.) Let \((u_m)_{m=1}^{\infty} \subset W_0^{1,p}(\Omega)\) be such that, as \( m \to \infty \),
\( u_m \rightharpoonup u \) weakly in \( W_0^{1,p}(\Omega) \) and satisfies
\[
-\Delta_p u_m = g_m + f_m \quad \text{in} \quad \mathcal{D}'(\Omega),
\]
where \( f_m \to 0 \) in \( W_0^{-1,p'}(\Omega) \) and \( g_m \) is bounded in \( \mathcal{M}(\Omega) \), the space of Radon measures on \( \Omega \), i.e.
\[
\|g_m, \phi\| \leq C_K \|\phi\|_{\infty}
\]
for all \( \phi \in \mathcal{D}(\Omega) \) with \( \text{supp} \phi \subseteq K \). Then there exists a subsequence, say \( u_{m_k} \), such that
\[
u_{m_k} \rightharpoonup u \quad \text{in} \quad W_0^{1,p}(\Omega) \quad (\forall \gamma < p).
\]

Before showing Lemma 3, we apply Lemma 5 for a minimizing sequence of \( I_a \). Set
\[
J(u) = \|\nabla u\|_{L^p(B_R)}^{\rho(s)} - I_a^{\rho(s)} \int_{B_R} |u|^{\rho(s)} V_a(x) \, dx \quad \text{for} \quad u \in W_0^{1,p}(B_R).
\]

From the definition of \( I_a \), we see that \( \inf_{u \in W_0^{1,p}(B_R)} J(u) = 0 \). And
\[
J'(u)[\varphi] = \rho(s) \|\nabla u\|_{L^p(B_R)}^{\rho(s)-\rho} \int_{B_R} |\nabla u|^{\rho(s)-2} \nabla u \cdot \nabla \varphi \, dx - I_a^{\rho(s)} \int_{B_R} |u|^{\rho(s)-2} u \varphi V_a(x) \, dx
\]
for \( \varphi \in (W_0^{1,p})^* \). Thus we observe that \( J \in C^1 \left(W_0^{1,p} ; \mathbb{R}\right) \). Let \((u_m)_{m=1}^{\infty} \subset W_0^{1,p}(B_R)\) be a minimizing sequence of \( I_a \) with \( \int_{B_R} |u_m|^{\rho(s)} V_a(x) \, dx = 1 \) for any \( m \in \mathbb{N} \) and \( \|\nabla u_m\|_{L^p(B_R)}^p = I_a + o(1) \) as \( m \to \infty \). By Ekeland’s Variational Principle (see e.g. [22]), there exists \((w_m)_{m=1}^{\infty} \subset W_0^{1,p}(B_R)\) such that
\[
(i) \quad 0 \leq J(w_m) \leq J(u_m) = o(1) \quad (m \to \infty),
(iii) \quad \|J'(w_m)\|_{(W_0^{1,p})^*} = o(1) \quad (m \to \infty),
(iii) \quad \|\nabla (w_m - u_m)\|_{L^p(B_R)} = o(1) \quad (m \to \infty).
\]
From (iii), we see that \((w_m)_{m=1}^{\infty}\) is a minimizing sequence of \(I_a\) with \(\int_{B_R} |w_m|^p \phi(x) \, dx = 1 + o(1)\) and \(\|\nabla w_m\|_{L^p(B_R)} = I_a + o(1)\) as \(m \to \infty\). Let \(w_m \to w\) in \(W_0^{1,p}(B_R)\) as \(m \to \infty\), passing to a subsequence if necessary. From (ii), for any \(\varphi \in W_0^{1,p}(B_R)\) we have

\[
\int_{B_R} |\nabla w_m|^{p-2} \nabla w_m \cdot \nabla \varphi \, dx - I_a \frac{\partial}{\partial p} \|\nabla w_m\|_{L^p(B_R)}^{p-1} \int_{B_R} |w_m|^{p-1} \varphi \, dx = o(1)
\]

which yields that \(w_m\) satisfies

\[
- \text{div}\,(|\nabla w_m|^{p-2} \nabla w_m) = I_a \frac{\partial}{\partial p} \|\nabla w_m\|_{L^p(B_R)}^{p-1} |w_m|^{p-2} w_m \varphi + f_m \quad \text{in } \mathcal{D}'(B_R)
\]

and \(f_m \to 0\) in \(W_0^{-1,p}(B_R)\). From Lemma 4, passing to a subsequence if necessary, we have \(\nabla w_m \to \nabla w\) a.e. in \(B_R\). As a consequence, we can apply Lemma 3 for \(\nabla w\) in the proof of Lemma 3.

**Proof of Lemma 3.** Take a minimizing sequence \((u_m)_{m=1}^{\infty} \subset W_0^{1,p}(B_R)\) of \(I_a\). Without loss of generality, we can assume that

\[
\int_{B_R} |u_m|^p \phi(x) \, dx = 1, \quad \int_{B_R} |\nabla u_m|^p \, dx = I_a + o(1) \quad \text{as } m \to \infty.
\]

Since \((u_m)\) is bounded in \(W_0^{1,p}(B_R)\), passing to a subsequence if necessary, \(u_m \to u\) in \(W_0^{1,p}(B_R)\) as \(m \to \infty\). Replacing \(u_m\) with \(w_m\) (we write \(u_m\) again) and applying Lemma 4, we have

\[
I_a = \int_{B_R} |\nabla u_m|^p \, dx + o(1)
\]

\[
= \int_{B_R} |\nabla (u_m - u)|^p \, dx + \int_{B_R} |\nabla u|^p \, dx + o(1)
\]

\[
\geq I_a \left( \int_{B_R} |u_m - u|^p \phi(x) \, dx \right)^{\frac{p}{p'(s)}} + I_a \left( \int_{B_R} |u|^{p'(s)} \phi(x) \, dx \right)^{\frac{p}{p'(s)}} + o(1)
\]

\[
\geq I_a \left( \int_{B_R} (|u_m - u|^p + |u|^{p'(s)}) \phi(x) \, dx \right)^{\frac{p}{p'(s)}} + o(1)
\]

\[
= I_a \left( \int_{B_R} |u_m|^p \phi(x) \, dx \right)^{\frac{p}{p'(s)}} + o(1) = I_a
\]
which implies that either \( u \equiv 0 \) or \( u_m \to u \neq 0 \) in \( L^{p_v(s)}(B_R; V(x))dx \) holds true from the equality condition of the last inequality and the positivity of \( I_a \) for \( a \in (0, 1) \). We shall show that \( u \neq 0 \). Assume that \( u \equiv 0 \). Then we claim that

\[
I_{a, \text{rad}} \leq \int_{B_R} |\nabla u_m|^p \, dx + o(1). \tag{21}
\]

If the claim \((21)\) is true, then we see that \( I_{a, \text{rad}} \leq I_a \) which contradicts the assumption. Therefore \( u \neq 0 \) which implies that \( u_m \to u \neq 0 \) in \( L^{p_v(s)}(B_R; V(x))dx \). Hence we have

\[
1 = \int_{B_R} |u|^{p_v(s)} V_a(x) \, dx, \quad \int_{B_R} |\nabla u|^p \, dx \leq \liminf_{m \to \infty} \int_{B_R} |\nabla u_m|^p \, dx = I_a.
\]

Thus we can show that \( u \) is a minimizer of \( I_a \). We shall show the claim \((21)\). Since \( u_m \to 0 \) in \( L^r(B_R) \) for any \( r \in [1, p^*(0)] \) and the potential function \( V(x) \) is bounded away from the origin, for any small \( \varepsilon > 0 \) we have

\[
1 = \int_{B_R} |u_m|^{p_v(s)} V_a(x) \, dx = \int_{B_{\frac{a}{\varepsilon}}} |u_m|^{p_v(s)} V_a(x) \, dx + o(1).
\]

Let \( \phi_{\varepsilon} \) be a smooth cut-off function which satisfies the followings:

\[
0 \leq \phi_{\varepsilon} \leq 1, \quad \phi_{\varepsilon} \equiv 1 \text{ on } B_{\frac{a}{\varepsilon}}(0), \quad \text{supp} \phi_{\varepsilon} \subset B_{\varepsilon R}(0), \quad |\nabla \phi_{\varepsilon}| \leq C \varepsilon^{-1}.
\]

Set \( \tilde{u}_m(y) = u_m(x) \) and \( \tilde{\phi}_{\varepsilon}(y) = \phi_{\varepsilon}(x) \), where \( y = \frac{x}{\varepsilon} \). Then we have

\[
1 = \left( \int_{B_{\frac{a}{\varepsilon}}} |u_m|^{p_v(s)} V_a(x) \, dx \right)^{\frac{p}{p_v(s)}} + o(1)
\]

\[
\leq \left( \int_{B_{\varepsilon R}} \frac{|u_m \phi_{\varepsilon}|^{p_v(s)}}{|x|^s \left( 1 - \frac{|d|}{\varepsilon} \right)^{\frac{s}{N-p}} \phi_{\varepsilon}} \, dx \right)^{\frac{p}{p_v(s)}} + o(1)
\]

\[
= \left( \int_{B_{\varepsilon R}} \frac{|\tilde{u}_m \tilde{\phi}_{\varepsilon}|^{p_v(s)}}{|x|^s \left( 1 - \alpha \varepsilon^{- \frac{s}{p-1}} \left( \frac{|d|}{\varepsilon} \right)^{\frac{s}{N-p}} \phi_{\varepsilon}} \, dx \right)^{\frac{p}{p_v(s)}} + o(1) \leq \frac{1}{a \varepsilon^{\frac{s}{p-1}} \phi_{\varepsilon}} \int_{B_R} |\nabla (\tilde{u}_m \tilde{\phi}_{\varepsilon})|^p \, dx + o(1).
\]
We see that \( a \varepsilon^\frac{N-p}{p-1} \leq \frac{s(p-1)}{p(N-1)} \) for small \( \varepsilon \). Since \( I_{a \varepsilon^\frac{N-p}{p-1}} = I_{a, \text{rad}} \) for small \( \varepsilon \) by (16), we have

\[
1 \leq I_{a, \text{rad}} \int_{B_R} |\nabla (u_m \phi_c)|^p \, dx + o(1)
\]

\[
\leq I_{a, \text{rad}}^{-1} \left( \int_{B_R} |\nabla u_m|^p \, dx + C \int_{B_R} |\nabla u_m|^{p-1} |\nabla \phi_c| u_m \phi_c^{p-1} + |u_m|^p |\nabla \phi_c|^p \, dx \right) + o(1)
\]

\[
\leq I_{a, \text{rad}}^{-1} \left( \int_{B_R} |\nabla u_m|^p \, dx + pC \varepsilon^{-1} \|\nabla u_m\|_L^p - \|u_m\|_L^p + C \varepsilon^{-p} \|u_m\|_{L^p} \right) + o(1)
\]

\[
\leq I_{a, \text{rad}}^{-1} \int_{B_R} |\nabla u_m|^p \, dx + o(1) \leq I_{a, \text{rad}}^{-1} \int_{B_R} |\nabla u_m|^p \, dx + o(1).
\]

Therefore we obtain the claim (21). The proof of Lemma 3 is now complete. □

Finally, we give a proof of Theorem 1.

**Proof of Theorem 1**

(i) Let \( s = p \). From Lemma 1 and Proposition 1, we easily obtain \( I_a = I_{a, \text{rad}} = C_{N, p, \varepsilon} = (\frac{N-p}{p})^p \) and the non-attainability of \( I_a \). We omit the proof.

(ii) Let \( s = 0 \). From Proposition 3, we obtain \( I_a = C_{N, 0, \varepsilon} = (\frac{N-p}{p})^p \) and the non-attainability of \( I_a \).

(iii) Let \( 0 < s < p \). Note that \( I_1 = 0 \) by Proposition 2 and \( I_a = I_{a, \text{rad}} = C_{N, p, \varepsilon} \) at least for \( a \in [0, \frac{s(p-1)}{p(N-1)}] \) by (16). Since \( I_a \) is continuous and monotone decreasing with respect to \( a \in [0, 1] \) by Lemma 2, there exists \( a_\ast \in (\frac{s(p-1)}{p(N-1)}, 1) \) such that \( I_a < I_{a, \text{rad}} = C_{N, p, \varepsilon} \) for \( a \in (a_\ast, 1) \) and \( I_a = I_{a, \text{rad}} = C_{N, p, \varepsilon} \) for \( a \in [0, a_\ast] \). Hence \( I_a \) is attained by a non-radial function for \( a \in (a_\ast, 1) \) by Lemma 3. On the other hand, if we assume that there exists a nonnegative minimizer \( u \) of \( I_a \) for \( a < a_\ast \), then we can show that at least, \( u \in C^1(B_R \setminus \{0\}) \) and \( u > 0 \) in \( B_R \setminus \{0\} \) by standard regularity argument and strong maximum principle to the Euler-Lagrange equation (3), see e.g. [9], [15]. Therefore we see that

\[
I_{a, \text{rad}} = I_a = \frac{\int_{B_R} |\nabla u|^p \, dx}{\int_{B_R} \left( \frac{|u|^{p(s)}}{1 - a (\frac{p}{N})^p} \right)^{\frac{p}{p(s)}} \, dx} > \frac{\int_{B_R} |\nabla u|^p \, dx}{\int_{B_R} \left( \frac{|u|^{p(s)}}{1 - a (\frac{p}{N})^p} \right)^{\frac{p}{p(s)}} \, dx} \geq I_{a, \text{rad}}.
\]

This is a contradiction. Therefore \( I_a \) is not attained for \( a \in [0, a_\ast] \).
Finally, we show that \(a_r > A > \frac{s(p-1)}{p(p-1)},\) where \(A\) is defined in Theorem 1(iii). Note that the potential function \(V_a\) is increasing with respect to \(a \in [0, 1]\). Since \(V_a(x)\) has one critical point at \(|x| = R_a := \frac{s(p-1)}{a(p-1)} R\), \(V_a\) is decreasing for \(|x| < R_a\) and increasing for \(|x| > R_a\). Therefore we see that \(R^{-s}(1-a)^{-\beta} = V_a(R) = V_1(R_1) = R^{-s} \left( \frac{s(p-1)}{p(p-1)} \right)^{-s} (1 - \frac{s(p-1)}{p(p-1)})^{-\beta}\) for \(a = A\) by the shape of \(V_a\). Let \(\tilde{R} \in (0, R_1)\) satisfy \(V_a(\tilde{R}) = V(A(R) = V_1(R_1).\) Then we have \(V_a^\#(x) = V_A(x)\) for \(x \in B_{\tilde{R}}.\) Since \(V_a^\#(x)\) is decreasing for \(x \in B_{\tilde{R}} \setminus B_{\tilde{R}},\) \(V_1(x)\) is increasing for \(x \in B_{\tilde{R}} \setminus B_{R_1},\) and \(\tilde{R} < R_1,\) we see that

\[
V_a^\#(x) < V_1(x) \text{ for any } x \in B_R \text{ at least for } a \in [A, A + \epsilon].
\]

Then for any \(u \in W_0^{1,\beta}(B_R)\) we have

\[
\int_{B_R} |\nabla u|^p \, dx \geq \left( \int_{B_R} |u|^\sigma V_{A+\epsilon}(x) \, dx \right)^{p/\sigma} \geq \left( \int_{B_R} |u|^\sigma V_{A+\epsilon}(x) \, dx \right)^{p/\sigma} \geq \left( \int_{B_R} |\nabla u|^p \, dx \right)^{p/\sigma} \geq \int_{B_R} |\nabla u|^p \, dx \geq I_{1,rad}.
\]

Hence we have \(I_{A+\epsilon} \geq I_{1,rad} = C_{N,p,s}\) which implies that \(a_r > A.\) \(\square\)

**Remark 4.** We can also show Theorem 1 for general bounded domains with Lipschitz boundary in the similar way in [16], since we can generalize Proposition 22 to such domains.

4. Appendix

In this section, we give remarks of the following two nonlinear scalings (22), (23) for non-radial functions \(v, w\) on the unit ball \(B_1.\)

\[
v_1(x) = \lambda^{-\frac{N-1}{p}} v(y), \text{ where } y = \left( \frac{|x|}{b} \right)^{\frac{1}{2}} x, \ b \geq 1, \ \lambda \leq 1,
\]

\[
w_1(x) = \lambda^{-\frac{N-1}{p}} w(y), \text{ where } y = \lambda \left[ 1 + a \left( \frac{N-1}{p} - 1 \right) |x|^{\frac{N-1}{p-1}} \right] \left| \frac{x}{|x|^{\frac{N-1}{p-1}}} \right|^{\frac{p-1}{p}} x, \ 0 \leq a \leq 1, \ \lambda \geq 1.
\]
Note that each $y$ in (22), (23) is in $B_1$ thanks to the restriction of the length of $\lambda$. If $b = 1$ or $a = 1$ in (22), (23), then we do not need to restrict the length of $\lambda$. From §2 we see that two scalings (22), (23) are equivalent to the usual scaling:

$$u_\lambda(x) = \lambda^{-p_N} u(y), \text{ where } y = \lambda x, \lambda > 0.$$  \hfill (24)

only for radial functions $u$ by the transformations (13), (5). However each response of each derivative norm $\|\nabla (\cdot)\|_{L^N}, \|\nabla (\cdot)\|_{L^p}$ to each scaling (22), (23) is different for non-radial functions. This is a different structure between the minimization problem $I_a$ in this paper and it in [16], which is related to the embedding:

$$W^{1,N}_0(B_1) \hookrightarrow L^q \left( B_1; \frac{dx}{|x|^{N/(N-1)}} \right) \text{ with } q \geq N. \text{ More precisely, we show the followings.}

**Proposition 4.** Let $b \geq 1$ and $a \in [0,1]$. Then $\{v_m\}_{m=1}^\infty \subset W^{1,N}_0(B_1)$ is unbounded for $v \in C^1_c(B_1)$ with $\nabla_{S^{N-1}} v \equiv 0$. On the other hand, $\{w_m\}_{m=1}^\infty \subset W^{1,p}_0(B_1)$ is bounded for any $w \in C^1_c(B_1)$.

**Proof of Proposition 4.** We use the polar coordinates. Let $r = |x|, t = |y|, \omega \in S^{N-1}$. By the scaling (22), we have

$$t = b^{1-N} r^\lambda \iff r = b^{1-N} t^\lambda.$$

Then we have

$$\frac{dt}{dr} = \lambda \frac{t}{r} \to 0 \quad (\lambda \to 0).$$

Therefore we see that for $\lambda \leq 1$

$$\int_{B_1} |\nabla v_\lambda(x)|^N dx = \int_{B_{2r}} |\nabla v_\lambda(x)|^N dx$$

$$= \int_{S^{N-1}} \int_0^r \left| \frac{\partial v_\lambda}{\partial r} \omega + \frac{1}{r} \nabla_{S^{N-1}} v_\lambda \right|^N r^{N-1} dr dS_\omega$$

$$= \lambda^{1-N} \int_{S^{N-1}} \int_0^1 \left| \frac{\partial v}{\partial t} \omega + \left( \frac{dr}{dt} \right)^{-1} \nabla_{S^{N-1}} v \right|^N \left( \frac{dr}{dt} \right)^{N-1} dt dS_\omega$$

$$= \int_{S^{N-1}} \int_0^1 \left| \frac{\partial v}{\partial t} \omega + \left( \frac{dr}{dt} \right)^{-1} \nabla_{S^{N-1}} v \right|^N t^{N-1} dt dS_\omega$$

$$= \int_{S^{N-1}} \int_0^1 \left| \frac{\partial v}{\partial t} \omega + \frac{1}{\lambda t} \nabla_{S^{N-1}} v \right|^N t^{N-1} dt dS_\omega \to \infty \quad (\lambda \to 0).
Hence we observe that \( \{ w_n \}_{n=1}^{\infty} \subset W_0^{1,N}(B_1) \) is unbounded.

On the other hand, by the scaling (23), we have

\[
t = \lambda r \left[ 1 + a \left( \frac{\lambda^{N-p}}{r^{N-p}} - 1 \right) r^{\frac{N-p}{p-1}} \right]^{-\frac{p-1}{N-p}} = \lambda \left[ r^{\frac{N-p}{p-1}} + a \left( \frac{\lambda^{N-p}}{r^{N-p}} - 1 \right) \right]^{-\frac{p-1}{N-p}}
\]

which is equal to

\[
r^{-\frac{N-p}{p-1}} = \lambda^{\frac{N-p}{p-1}} t^{\frac{N-p}{p-1}} - a(\lambda^{\frac{N-p}{p-1}} - 1).
\]

Then we have

\[
\frac{dt}{dr} = t \left[ 1 + a \left( \frac{\lambda^{N-p}}{r^{N-p}} - 1 \right) r^{\frac{N-p}{p-1}} \right]^{-1}
\]

\[
= t \left[ 1 + \frac{a \left( \frac{\lambda^{N-p}}{r^{N-p}} - 1 \right)}{\lambda^{\frac{N-p}{p-1}} t^{\frac{N-p}{p-1}} - a(\lambda^{\frac{N-p}{p-1}} - 1)} \right]^{-1}
\]

\[
= t \left[ 1 + \frac{a}{\lambda^{\frac{N-p}{p-1}} (t - \lambda^{\frac{N-p}{p-1}} / a)} \right]^{-1} \rightarrow t \left[ 1 + \frac{a}{t^{\frac{N-p}{p-1}} - a} \right]^{-1} (\lambda \to \infty).
\]

Therefore we see that for \( \lambda \geq 1 \)

\[
\int_{B_1} |\nabla w_1(x)|^p \, dx = \int_{B_R} |\nabla w_1(x)|^p \, dx
\]

\[
= \int_{\mathbb{S}^{N-1}} \int_0^R \left| \frac{\partial w_1}{\partial r} \right|^p \left( r - \frac{1}{r} \nabla_{\mathbb{S}^{N-1}} w_1 \right)^p r^{N-1} \, dr \, dS \omega
\]

\[
= \lambda^{N-p} \int_{\mathbb{S}^{N-1}} \int_0^1 \left| \frac{\partial w}{\partial r} \right|^p \left( r - \frac{1}{r} \nabla_{\mathbb{S}^{N-1}} w \right)^p r^{N-1} \, dr \, dS \omega
\]

\[
= \int_{\mathbb{S}^{N-1}} \int_0^1 \left| \frac{\partial w}{\partial t} \right|^p \left( t - \frac{1}{t} \nabla_{\mathbb{S}^{N-1}} w \right)^p t^{N-1} \, dt \, dS \omega
\]

\[
\rightarrow \int_{\mathbb{S}^{N-1}} \int_0^1 \left| \frac{\partial w}{\partial t} \right|^p \left( t - \frac{1}{t} \nabla_{\mathbb{S}^{N-1}} w \right)^p t^{N-1} \, dt \, dS \omega (\lambda \to \infty).
\]

Hence we observe that \( \{ w_m \}_{m=1}^{\infty} \subset W_0^{1,p}(B_1) \) is bounded. \( \square \)
As a consequence of Proposition 4, we can also construct a non-compact non-radial sequence of the embedding: $W_0^{1,p}(B_1) \hookrightarrow L^{p'(x)}(B_1; V_a(x) \, dx)$ which is related to our minimization problem $I_a$. On the other hand, in the view of the scaling, it seems difficult to construct a non-compact non-radial sequence of the embedding: $W_0^{1,N}(B_1) \hookrightarrow L^q\left(B_1; \frac{dx}{|x|^N(\log \frac{1}{|x|})^{\frac{N+1}{N+q}}}\right)$ which is related to the minimization problem in [16]. Therefore the following natural question arises.

Let $N = 2$ and $b > 1$. Consider the orthogonal decomposition: $W_0^{1,2}(B_1) = W_0^{1,2}(B_1) \oplus \left(W_0^{1,2}(B_1)\right)^\perp$. Is the embedding: $\left(W_0^{1,2}(B_1)\right)^\perp \hookrightarrow L^q\left(B_1; \frac{dx}{|x|^N(\log \frac{1}{|x|})^{\frac{N+1}{N+q}}}\right)$ compact?

If the above question is affirmative, we may say that “Non-radial compactness” occurs in some sense. That is an opposite phenomenon of Strauss’s radial compactness (ref. [21]). Thus it may be interesting if the question is affirmative.

Acknowledgment

This work was (partly) supported by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics). The author was supported by JSPS KAKENHI Early-Career Scientists, No. JP19K14568.

References

[1] Alvino, A., \textit{A limit case of the Sobolev inequality in Lorentz spaces}, Rend. Accad. Sci. Fis. Mat. Napoli (4) 44 (1977), 105-112 (1978).

[2] Aubin, T., \textit{Problèmes isopérimétriques et espaces de Sobolev}, J. Differential Geometry 11 (1976), no. 4, 573-598.

[3] Baras, P., Goldstein, J. A., \textit{The heat equation with a singular potential}, Trans. Amer. Math. Soc., 284 (1984), 121-139.

[4] W. Beckner, W., Pearson, M., \textit{On sharp Sobolev embedding and the logarithmic Sobolev inequality}, Bull. London Math. Soc., 30 (1998), 80-84.
[5] Boccardo, L., Murat, F., Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations, Nonlinear Anal. TMA. 19 (1992), 581-597.

[6] H. Brezis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc., 88 (1983), 486-490.

[7] Brezis, H., Vázquez, J. L., Blow-up solutions of some nonlinear elliptic problems, Rev. Mat. Univ. Complut. Madrid 10 (1997), No. 2, 443-469.

[8] Cassani, D., Ruf, B., Tarsi, C., Optimal Sobolev type inequalities in Lorentz spaces, Potential Anal. 39 (2013), no. 3, 265-285.

[9] DiBenedetto, E., $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, Nonlinear Anal. 7 (1983), No. 8, 827-850.

[10] Flucher, M., Extremal functions for the Trudinger-Moser inequality in 2 dimensions, Comment. Math. Helv. 67 (1992), no. 3, 471-497.

[11] Horiuchi, T., Kumlin, P., On the Caffarelli-Kohn-Nirenberg-type inequalities involving critical and supercritical weights, Kyoto J. Math. 52 (2012), no. 4, 661-742.

[12] Ioku, N., Attainability of the best Sobolev constant in a ball, Math. Ann. 375 (2019), no. 1-2, 1-16.

[13] Ioku, N., Ishiwata, M., A note on the scale invariant structure of critical Hardy inequalities, Geometric properties for parabolic and elliptic PDE’s, 97-120, Springer Proc. Math. Stat., 176, Springer, [Cham], 2016.

[14] Lieb, E. H., Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, Ann. of Math. (2), 118 (1983), no. 2, 349-374.

[15] Pucci, P., Serrin, J., The maximum principle, Progress in Nonlinear Differential Equations and their Applications, 73, Birkhauser Verlag, Basel, (2007).

[16] Sano, M., Extremal functions of generalized critical Hardy inequalities, J. Differential Equations 267 (2019), no. 4, 2594-2615.

[17] Sano, M., Two limits on Hardy and Sobolev inequalities, arXiv:1911.04105
[18] Sano, M., Sobukawa, T., Remarks on a limiting case of Hardy type inequalities, ArXiv: 1907.09609.

[19] Sano, M., Takahashi, F., Scale invariance structures of the critical and the subcritical Hardy inequalities and their improvements, Calc. Var. Partial Differential Equations 56 (2017), no. 3, Art. 69, 14 pp.

[20] Smets, D., Willem, M., Su, J., Non-radial ground states for the Hénon equation, Commun. Contemp. Math. 4 (2002), no. 3, 467-480.

[21] Strauss, W. A., Existence of solitary waves in higher dimensions, Comm. Math. Phys. 55 (1977), no. 2, 149-162.

[22] Struwe, M., Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems. Fourth edition, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 34. Springer-Verlag, Berlin, (2008).

[23] Talenti, G., Best constant in Sobolev inequality, Ann. Mat. Pura Appl. (4) 110 (1976), 353-372.

[24] Trudinger, N. S., On imbeddings into Orlicz spaces and some applications, J. Math. Mech. 17 (1967), 473-483.

[25] Vázquez, J. L., Zuazua, E., The Hardy inequality and asymptotic behaviour of the heat equation with an inverse-square potential, J. Funct. Anal., 173 (2000), 103-153.

[26] Zographopoulos, N. B. Existence of extremal functions for a Hardy-Sobolev inequality, J. Funct. Anal. 259 (2010), no. 1, 308-314.