INFINITE ENERGY SOLUTIONS FOR THE 
(3+1)-DIMENSIONAL YANG-MILLS EQUATION 
IN LORENZ GAUGE

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Abstract. We prove that the Yang-Mills equation in Lorenz gauge in the 
(3+1)-dimensional case is locally well-posed for data of the gauge potential in 
$H^s$ and the curvature in $H^r$, where $s > \frac{5}{7}$ and $r > -\frac{1}{7}$, respectively. This 
improves a result by Tesfahun [16]. The proof is based on the fundamental 
results of Klainerman-Selberg [6] and on the null structure of most of the 
nonlinear terms detected by Selberg-Tesfahun [14] and Tesfahun [16].

1. Introduction. Let $G$ be the Lie group $SO(n, \mathbb{R})$ (the group of orthogonal ma-
trices of determinant 1) or $SU(n, \mathbb{C})$ (the group of unitary matrices of determinant 
1) and $g$ its Lie algebra $\mathfrak{so}(n, \mathbb{R})$ (the algebra of trace-free skew symmetric ma-
trices) or $\mathfrak{su}(n, \mathbb{C})$ (the algebra of trace-free skew hermitian matrices) with Lie bracket
$[X, Y] = XY - YX$ (the matrix commutator). For given $A_\alpha : \mathbb{R}^{1+n} \to g$ we define 
the curvature $F = F[A]$ by

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta],$$ (1)

where $\alpha, \beta \in \{0, 1, \ldots, n\}$ and $D_\alpha = \partial_\alpha + [A_\alpha, \cdot]$. 

Then the Yang-Mills system is given by

$$D^\alpha F_{\alpha\beta} = 0$$ (2)
in Minkowski space $\mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n$, where $n \geq 3$, with metric $\text{diag}(-1, 1, \ldots, 1)$. 

Greek indices run over $\{0, 1, \ldots, n\}$, Latin indices over $\{1, \ldots, n\}$, and the usual 
summation convention is used. We use the notation $\partial_\mu = \frac{\partial}{\partial x_\mu}$, where we write
$(x^0, x^1, \ldots, x^n) = (t, x^1, \ldots, x^n)$ and also $\partial_0 = \partial_t$.

Setting $\beta = 0$ in (2) we obtain the Gauss-law constraint

$$\partial^\alpha F_{\alpha 0} + [A^\alpha, F_{\alpha 0}] = 0.$$ 

The total energy for YM, at time $t$, is given by

$$\mathcal{E}(t) = \sum_{0 \leq \alpha, \beta \leq n} \int_{\mathbb{R}^n} |F_{\alpha\beta}(t, x)|^2 \, dx,$$

and is conserved for a smooth solution decaying sufficiently fast at spatial infinity, i.e.,

$$\mathcal{E}(t) = \mathcal{E}(0).$$

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The system is gauge invariant. Given a sufficiently smooth function \( U : \mathbb{R}^{1+n} \rightarrow \mathcal{G} \) we define the gauge transformation \( T \) by \( TA_0 = A'_0, T(A_1, ..., A_n) = (A'_1, ..., A'_n) \), where

\[
A_\alpha \mapsto A'_\alpha = U A_\alpha U^{-1} - (\partial_\alpha U) U^{-1}.
\]

It is well-known that if \( (A_0, ...A_n) \) satisfies (1), (2) so does \( (A'_0, ..., A'_n) \).

Hence we may impose a gauge condition. We exclusively study the Lorenz gauge \( \partial^\alpha A_\alpha = 0 \). Other convenient gauges are the Coulomb gauge \( \partial^\alpha A_\alpha = 0 \) and the temporal gauge \( A_0 = 0 \). It is well-known that for the low regularity well-posedness problem for the Yang-Mills equation a null structure for some of the nonlinear terms plays a crucial role. This was first detected by Klainerman and Machedon [4], who proved global well-posedness in the case of three space dimensions in temporal gauge in energy space. The corresponding result in Lorenz gauge, where the Yang-Mills equation can be formulated as a system of nonlinear wave equations, was shown by Selberg and Tesfahun [14], who discovered that also in this case some of the nonlinearities have a null structure. This allows to rely on some of the methods that were previously used for the Maxwell-Dirac equation in [2] and the Maxwell-Klein-Gordon equation in [13]. Tesfahun [16] improved the local well-posedness result to data without finite energy, namely for \( (A(0), (\partial_\alpha A(0))) \in H^s \times H^{s-1} \) and \( (F(0), (\partial_\alpha F(0))) \in H^s \times H^{s-1} \) with \( s > \frac{n}{2} \) and \( r > -\frac{1}{2} \), by discovering an additional partial null structure. Local well-posedness in energy space was also given by Oh [10] using a new gauge, namely the Yang-Mills heat flow. He was also able to show that this solution can be globally extended [11]. Tao [15] showed local well-posedness in \( H^s \times H^{s-1} \) for \( s > \frac{3}{4} \) in temporal gauge, but limited to small data. Tao’s result was generalized to space dimensions \( n \geq 3 \) by the author [12]. In space dimension \( n \) the critical regularity with respect to scaling is \( s = \frac{n}{2} - 1 \). In the case \( n = 4 \) where the energy space is critical Klainerman and Tataru [7] proved small data local well-posedness for a closely related model problem in Coulomb gauge for \( s > 1 \). Klainerman and Selberg [6] treated the local well-posedness problem with minimal regularity for some systems of nonlinear wave equations. Especially, they showed local well-posedness for a model problem related to the Yang-Mills system in the almost critical region, where \( s > \frac{n}{2} - 1 \) and \( n \geq 4 \). Recently the result [7] was significantly improved by Krieger and Tataru [9], who were able to show global well-posedness for data with small energy. In high space dimension \( n \geq 6 \) (and \( n \) even) Krieger and Sterbenz [8] proved global well-posedness for small data in the critical Sobolev space.

In the present paper we consider the local well-posedness problem for large data without finite energy for the Yang-Mills system in Lorenz gauge and space dimension \( n = 3 \). Our main result is local well-posedness for \( s > \frac{5}{4} \) and \( r > -\frac{1}{2} \), where existence holds in \( A \in C^0([0, T], H^s) \cap C^1([0, T], H^{s-1}) \), \( F \in C^0([0, T], H^r) \cap C^1([0, T], H^{r-1}) \) and (existence and) uniqueness in a certain subspace (Theorem 2.1 and Corollary 2.1). This is an improvement of Tesfahun’s result [16]. It is the first local well-posedness result for \( s < \frac{5}{4} \) and holds even for large data. Crucial for this result are on one hand the methods developed in the papers by Selberg-Tesfahun [14] and Tesfahun [16], especially their detection of the null structure in most - unfortunately not all - critical nonlinear terms. On the other hand we rely on the methods by Klainerman and Selberg [6] for a model problem for Yang-Mills, which ignores the gauge condition. We have to modify their solution space appropriately and show that its main features are preserved. We were unable to come
down to the critical value $s = \frac{1}{2}$, which is prevented mainly by one of the nonlinear terms, for which no null structure is known and which leads to the estimate $\text{(29)}$. \cite{14} and \cite{16} used solution spaces of wave-Sobolev type $H^{s,b}$, which are closely related to the Bourgain-Klainerman-Machedon spaces $X^{s,b}$, for which a convenient atlas of bilinear estimates was proven by \cite{1} in dimension $n = 3$. If one uses solution spaces of $H^{s,b}$-type it seems to be impossible to obtain our results, because some of the bilinear estimates which we need simply fail. For details we refer to the remark at the end of the paper. Therefore it is necessary to modify the solution spaces appropriately.

In chapter 2 we recall the reformulation of the Yang-Mills equation as a system of nonlinear wave equations and state our main theorem (Theorem 2.1 and Corollary 2.1). We also fix some notation. Chapter 3 contains the bilinear estimates in wave-Sobolev spaces. Moreover we define the solution spaces and state its fundamental properties. We reduce the local well-posedness problem to a suitable set of nonlinearities in Proposition 3.10, where we completely rely on \cite{6}. In chapter 4 we formulate the Yang-Mills equations in final form - using the whole null structure - and the necessary nonlinear estimates as in \cite{16}. We also review some well-known properties of the standard null forms and the additional one detected in \cite{16}. In (the most voluminous) chapter 4 we prove the multilinear estimates for the nonlinearities.

2. Main results. Expanding (2) in terms of the gauge potentials $\{A_\alpha\}$, we obtain:

\[ \square A_\beta = \partial_\beta \partial^\alpha A_\alpha - [\partial^\alpha A_\alpha, A_\beta] - [A^\alpha, \partial^\alpha A_\beta] - [A^\alpha, F_{\alpha\beta}] . \tag{3} \]

If we now impose the Lorenz gauge condition, the system (3) reduces to the nonlinear wave equation

\[ \square A_\beta = -[A^\alpha, \partial_\alpha A_\beta] - [A^\alpha, F_{\alpha\beta}] . \tag{4} \]

In addition, regardless of the choice of gauge, $F$ satisfies the wave equation

\[ \square F_{\beta\gamma} = -[A^\alpha, \partial_\alpha F_{\beta\gamma}] - \partial^\alpha [A_\alpha, F_{\beta\gamma}] - [A^\alpha, [A_\alpha, F_{\beta\gamma}]] \\
- 2[F_{\alpha\beta}, F_{\gamma\alpha}] . \tag{5} \]

Indeed, this will follow if we apply $D^\alpha$ to the Bianchi identity

\[ D_\alpha F_{\beta\gamma} + D_\beta F_{\gamma\alpha} + D_\gamma F_{\alpha\beta} = 0 \]

and simplify the resulting expression using the commutation identity

\[ D_\alpha D_\beta X - D_\beta D_\alpha X = [F_{\alpha\beta}, X] \]

and (2) (\cite{14}).

Expanding the second and fourth terms in (5), and also imposing the Lorenz gauge, yields

\[ \square F_{\beta\gamma} = -2[A^\alpha, \partial_\alpha F_{\beta\gamma}] + 2[\partial_\beta A^\alpha, \partial_\alpha A_\gamma] - 2[\partial_\beta A^\alpha, \partial_\alpha A_\gamma] \\
+ 2[\partial^\alpha A_\beta, \partial_\alpha A_\gamma] + 2[\partial_\beta A^\alpha, \partial_\gamma A_\alpha] - [A^\alpha, [A_\alpha, F_{\beta\gamma}]] \\
+ 2[F_{\alpha\beta}, [A^\alpha, A_\gamma]] - 2[F_{\alpha\gamma}, [A^\alpha, A_\beta]] - 2[[[A^\alpha, A_\beta], A_\alpha, A_\gamma]] . \tag{6} \]

Note on the other hand by expanding the last term in the right hand side of (4), we obtain

\[ \square A_\beta = -2[A^\alpha, \partial_\alpha A_\beta] + [A^\alpha, \partial_\beta A_\alpha] - [A^\alpha, [A_\alpha, A_\beta]] . \tag{7} \]
We want to solve the system (6)-(7) simultaneously for $A$ and $F$. So to pose the Cauchy problem for this system, we consider initial data for $(A, F)$ at $t = 0$:

$$A(0) = a, \quad \partial_t A(0) = \dot{a}, \quad F(0) = f, \quad \partial_t F(0) = \dot{f}. \quad (8)$$

In fact, the initial data for $F$ can be determined from $(a, \dot{a})$ as follows:

$$\begin{align*}
 f_{ij} &= \partial_i a_j - \partial_j a_i + [a_i, a_j], \\
 f_{0i} &= \dot{a}_i - \partial_i a_0 + [a_0, a_i], \\
 \dot{f}_{ij} &= \partial_i \dot{a}_j - \partial_j \dot{a}_i + [\dot{a}_i, a_j] + [a_i, \dot{a}_j], \\
 \dot{f}_{0i} &= \partial_i \dot{f} + [a^i, f_{0i}] 
\end{align*} \quad (9)$$

where the first three expressions come from (1) whereas the last one comes from (2) with $\beta = i$.

Note that the Lorenz gauge condition $\partial^\alpha A_\alpha = 0$ and (2) with $\beta = 0$ impose the constraints

$$\dot{a}_0 = \partial^i a_i, \quad \partial^i f_{0i} + [a^i, f_{0i}] = 0. \quad (10)$$

Now we formulate our main theorem.

**Theorem 2.1.** Let $n = 3$ and assume that $s$ and $r$ satisfy the following conditions:

$$s > \frac{5}{7}, \quad r > -\frac{1}{7}, \quad r < s < r + 1, \quad 2r - s > -1, \quad 2s - r > \frac{3}{2}, \quad 4s - r > 3, \quad 3s - 2r > 2. \quad$$

Given initial data $(a, \dot{a}) \in H^s \times H^{s-1}, (f, \dot{f}) \in H^r \times H^{r-1}$, there exists a time $T > 0$, $T = T(||a||_{H^r}, ||\dot{a}||_{H^{r-1}}, ||f||_{H^r}, ||\dot{f}||_{H^{r-1}})$, such that the Cauchy problem (6), (7), (8) has a unique solution $A \in F^T_T, F \in G^T_T$ (these spaces are defined in Def. 3.1). This solution has the regularity $A \in C^0([0, T], H^s) \cap C^1([0, T], H^{s-1}), \ F \in C^0([0, T], H^r) \cap C^1([0, T], H^{r-1})$.

**Remark 2.1.** 1. The most natural relation between $s$ and $r$ is $r = s - 1$. This is not allowed in Theorem 2.1. In this case the condition $2r - s > -1$ would force $s > 1$, which would exclude the most interesting range $\frac{5}{7} < s \leq 1$.

2. The assumptions on $s$ and $r$ imply $4s - 3 > r > \frac{s}{2} - \frac{1}{2}$, which can only be fulfilled, if $s > \frac{3}{7}$, and therefore $r > -\frac{1}{7}$. One easily checks that the choice $s = \frac{3}{7} + \epsilon, r = -\frac{1}{7} + \epsilon$ satisfies our assumptions, if $\epsilon > 0$ is small enough.

3. The following conditions are automatically fulfilled

$$3r - 2s > -2, \quad 4r - s > -2,$$

because $3r - 2s = (2r - s) + (r - s) > -1 - 1 = -2$ and $4r - s > (2r - s) + 2r > -1 + 2r > -1 - \frac{2}{7} > -2$.

**Corollary 2.1.** Let $s, r$ fulfill the assumptions of Theorem 2.1. Moreover assume that the initial data fulfill (9) and (10). Given any $(a, \dot{a}) \in H^{r+1} \times H^r$, there exists a time $T > 0$, $T = T(||a||_{H^r}, ||\dot{a}||_{H^{r-1}}, ||f||_{H^r}, ||\dot{f}||_{H^{r-1}})$, such that the solution $(A, F)$ of Theorem 2.1 satisfies the Yang-Mills system (1), (2) with Cauchy data $(a, \dot{a})$ and the Lorenz gauge condition $\partial^\alpha A_\alpha = 0$.

**Proof of the Corollary.** The solution $(A, F)$ does not necessarily fulfill the Lorenz gauge condition and (1), i.e. $F = F[A]$. If however the conditions (9) and (10) are assumed then these properties are satisfied and $(A, F)$ is a solution of the Yang-Mills system (1), (2) with Cauchy data $(a, \dot{a})$. This was shown in [14], Remark 2. ∎
Remark 2.2. 1. Because $s < r + 1$ by assumption the potential $A$ possibly looses some regularity compared to its data, whereas this is not the case for $F$, which is the decisive factor, whereas the regularity of $A$ is of minor interest.

2. If $(a, \dot{a}) \in H^{r+1} \times H^r$, then $(f, \dot{f})$, defined by (9), fulfill $(f, \dot{f}) \in H^r \times H^{r-1}$, as one easily checks.

Let us fix some notation. We denote the Fourier transform with respect to space and time by $\hat{\cdot}$. \( \square = \partial^2_t - \Delta \) is the d’Alembert operator, $a \pm \epsilon := a \pm \epsilon$ for a sufficiently small $\epsilon > 0$, and $\langle \cdot \rangle := (1 + |\cdot|^2)^{\frac{1}{2}}$.

$H^{s,r}$ denotes the $L^r$-based Sobolev space with respect to the space variables and $H^s = H^{s,2}$.

The standard wave-Sobolev spaces $H^{s,b}$ of Bourgain-Klainerman-Machedon type are the completion of the Schwartz space $S(\mathbb{R}^{1+3})$ with norm

$$\|u\|_{H^{s,b}} = \|\langle \xi \rangle^s (|\tau| - |\xi|)^b \hat{u}(\tau, \xi)\|_{L^2_{\tau, \xi}}.$$  

We also define $H^{s,b}_T$ as the space of the restrictions of functions in $H^{s,b}$ to $[0, T] \times \mathbb{R}^3$.

Let $\Lambda^\alpha$, $\Lambda^\alpha_+$ and $\Lambda^\alpha_-$ be the multipliers with symbols

$$\langle \xi \rangle^\alpha, \quad \langle |\tau| + |\xi| \rangle^\alpha, \quad \langle |\tau| - |\xi| \rangle^\alpha.$$  

Similarly let $D^\alpha$, $D^\alpha_+$ and $D^\alpha_-$ be the multipliers with symbols

$$|\xi|^\alpha, \quad (|\tau| + |\xi|)^\alpha, \quad |\tau| - |\xi|)^\alpha,$$  

respectively.

Let $\partial$ denote the collection of space and time derivatives.

If $u, v \in S'$ and $\hat{u}, \hat{v}$ are tempered functions, we write $u \preceq v$ iff $|\hat{u}| \leq |\hat{v}|$, and $\preceq$ means $\leq$ up to a constant. If $u = (u^1, \ldots, u^N)$ and $v = (v^1, \ldots, v^N)$, then $u \preceq v$ (resp. $u \preceq v$) means $u^I \preceq v^I$ (resp. $u^I \preceq v^I$) for $I = 1, \ldots, N$.

3. Preliminaries. The Strichartz type estimates for the wave equation are given in the next proposition.

**Proposition 3.1.** If $n = 3$ and

$$2 \leq q \leq \infty, \quad 2 \leq r < \infty, \quad \frac{1}{q} \leq \frac{1}{2} - \frac{1}{r},$$

then the following estimates hold

$$\|e^{\pm utD}f\|_{L^q_t L^r_x} \lesssim \|f\|_{H^{\frac{3}{2} - \frac{1}{q}, \frac{1}{4}}}$$

and

$$\|u\|_{L^q_t L^r_x} \lesssim \|u\|_{H^{\frac{3}{2} - \frac{1}{q}, \frac{1}{4}}}.$$

**Proof.** This is the Strichartz type estimate, which can be found for e.g. in [3], Prop. 2.1, combined with the transfer principle.

**Corollary 3.1.** If $2 \leq r < \infty$, the following estimate holds:

$$\|u\|_{L^{q, r}_{1+\epsilon}} \lesssim \|u\|_{H^{1-\frac{1}{2} + 2\epsilon, \frac{1}{4}(1-\frac{1}{2}) + \frac{3}{4}}}$$

for $0 < \epsilon \leq \frac{4}{r}$.
Proof. We use the following special case of Prop. 3.1:
\[ \|u\|_{L_t^{2+\epsilon}L_x^{\frac{2(2+\epsilon)}{n}}} \lesssim \|u\|_{H^{1,\frac{1}{2+\epsilon}}} \]
for arbitrary \(\epsilon_1, \epsilon_2 > 0\), where we choose \(q = 2 + \epsilon_1, r = \frac{2(2+\epsilon_1)}{\epsilon_1}\), so that \(\frac{3}{2} - \frac{3}{r} - \frac{q}{r} = \frac{2}{2+\epsilon_1} < 1\). Now we interpolate this inequality with the trivial identity \(\|u\|_{L_t^2L_x^2} = \|u\|_{H^{0,0}}\). We choose the interpolation parameter \(\theta\) by \(\frac{1}{r} = \theta \frac{\epsilon_1}{2(2+\epsilon_1)} + (1-\theta)\frac{1}{2} \iff \theta = (2 + \epsilon_1)(\frac{1}{2} - \frac{1}{r})\) and require moreover \(\frac{1}{2\epsilon} = \frac{\theta}{2\epsilon_1} + \frac{1-\theta}{2}\), so that \(\epsilon_1 = 2\frac{\frac{1}{2} - \frac{1}{r}}{\frac{1}{2} - \frac{1}{r}}\).
This implies \(\theta = (2 + \epsilon_1)(\frac{1}{2} - \frac{1}{r}) = 1 - \frac{2}{r} + \frac{\epsilon_1}{2+\epsilon_1} < 1 - \frac{2}{r} + \frac{\epsilon}{2} \leq 1\) for \(\epsilon \leq \frac{2}{r}\), and \(\frac{\theta}{2} + \theta \epsilon_2 = \frac{1}{2}(1 - \frac{2}{r}) + \frac{\epsilon_1}{2(2+\epsilon)} + \theta \epsilon_2 < \frac{1}{2}(1 - \frac{2}{r}) + \frac{\epsilon}{4}\) for sufficiently small \(\epsilon_2 > 0\). Thus we obtain by interpolation
\[ \|u\|_{L_t^{2+\epsilon}L_x^{\frac{2}{2+\epsilon}}} \lesssim \|u\|_{H^{0,\frac{\theta}{2}}} \lesssim \|u\|_{H^{1-\frac{\theta}{2} + \frac{\epsilon}{4}(1-\frac{\theta}{2}) + \frac{\epsilon_2}{4}}} \]

The following proposition follows from [7], Theorem 5 and the transfer principle.

**Proposition 3.2.** Let \(n \geq 2\), and let \((q, r)\) satisfy:
\[ 2 \leq q \leq \infty, \quad 2 \leq r < \infty, \quad \frac{2}{q} \leq (n-1) \left(\frac{1}{2} - \frac{1}{r}\right) \]
Assume that
\[
0 < \sigma < n - \frac{2n}{r} - \frac{4}{q}, \\
\sigma_1, \sigma_2 < \frac{n}{2} - \frac{n}{r} - \frac{1}{q}, \\
\sigma_1 + \sigma_2 + \sigma = n - \frac{2n}{r} - \frac{2}{q}.
\]
then
\[ \|D^{-\sigma}(uv)\|_{L_t^{\frac{q}{2},r}L_x^{\frac{q}{2},r}} \lesssim \|u\|_{H^{\sigma_1,\frac{1}{2}}} \|v\|_{H^{\sigma_2,\frac{1}{2}}} \]

The following product estimates for wave-Sobolev spaces were proven in [1].

**Proposition 3.3.** For \(s_0, s_1, s_2, b_0, b_1, b_2 \in \mathbb{R}\) and \(u, v \in S(\mathbb{R}^{3+1})\) the estimate
\[ \|uv\|_{H^{-s_0, b_0}} \lesssim \|u\|_{H^{s_1, b_1}} \|v\|_{H^{s_2, b_2}} \]
holds, provided the following conditions are satisfied:
\[
b_0 + b_1 + b_2 > \frac{1}{2}, \quad b_0 + b_1 \geq 0, \quad b_0 + b_2 \geq 0, \quad b_1 + b_2 \geq 0 \\
s_0 + s_1 + s_2 > 2 - (b_0 + b_1 + b_2) \\
s_0 + s_1 + s_2 > \frac{3}{2} - \min(b_0 + b_1, b_0 + b_2, b_1 + b_2) \\
s_0 + s_1 + s_2 > 1 - \min(b_0, b_1, b_2) \\
s_0 + s_1 + s_2 > 1 \\
(s_0 + b_0) + 2s_1 + 2s_2 > \frac{3}{2} \\
2s_0 + (s_1 + b_1) + 2s_2 > \frac{3}{2}
\]
\[
2s_0 + 2s_1 + (s_2 + b_2) > \frac{3}{2}
\]
\[
s_1 + s_2 \geq \max(0, -b_0), \quad s_0 + s_2 \geq \max(0, -b_1), \quad s_0 + s_1 \geq \max(0, -b_2).
\]

The following multiplication law is well-known:

**Proposition 3.4** (Sobolev multiplication law). Let \( n = 3, s_0, s_1, s_2 \in \mathbb{R} \). Assume \( s_0 + s_1 + s_2 > \frac{3}{2}, s_0 + s_1 \geq 0, s_0 + s_2 \geq 0, s_1 + s_2 \geq 0 \). Then the following product estimate holds:

\[
\|uv\|_{H^{-\alpha}} \lesssim \|u\|_{H^{s_1}} \|v\|_{H^{s_2}}.
\]

We now come to the definition of the solution spaces, which are similar to the spaces introduced by [6]. We prepare this by defining the following modification of the standard \( \mathcal{L}^q_L \)-spaces.

**Definition 3.1.** If \( 1 \leq q, r \leq \infty, u \in S' \) and \( \hat{u} \) is a tempered function, set

\[
\|u\|_{\mathcal{L}^q_L S} = \sup \left\{ \int_{\mathbb{R}^{1+n}} |\hat{u}(r, \xi)|v(r, \xi) \, drd\xi : v \in \mathcal{S}, \hat{v} \geq 0, \|v\|_{\mathcal{L}^q_r S} = 1 \right\},
\]

where \( 1 = \frac{1}{q} + \frac{1}{q} \) and \( 1 = \frac{1}{r} + \frac{1}{r} \). Let \( \mathcal{L}^q_L S \) be the corresponding subspace of \( S' \).

This is a translation invariant norm and it only depends on the size of the Fourier transform. Observe that

\[
\|u\|_{\mathcal{L}^q_L S} \leq \|u\|_{\mathcal{L}^q_L L} \quad \text{whenever} \quad \hat{u} \geq 0.
\]

**Definition 3.2.** Our solution space is defined as follows: \( \{(u, v) \in F^s \times G^r\} \),

\[
\|u\|_{F^s} := \|\Lambda^s u\|_{H^{s-1, \frac{1}{2}+s}} + \|\Lambda^{\frac{1}{2}+2s} \Lambda^{-\frac{1}{2}} \Lambda^{-\frac{1}{2}+3s} \Lambda^s u\|_{\mathcal{L}^{14}_L}\]
\[
\|v\|_{G^r} := \|\Lambda^r v\|_{H^{r-1, \frac{1}{2}+r}}
\]

where \( \epsilon > 0 \) is sufficiently small. \( F^s_L \) and \( G^r_L \) denotes the restriction to the time interval \([0, T]\).

This is a Banach space ([6], Prop. 4.2).

Next we recall some fundamental properties of the \( \mathcal{L}^q_L S \)-spaces, which were given by [6], starting with a Hölder-type estimate.

**Proposition 3.5.** Suppose \( \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \) and \( \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \), where the \( q \)'s and \( r \)'s all belong to \([1, \infty]\). Then

\[
\|uv\|_{\mathcal{L}^q_L S} \leq \|u\|_{\mathcal{L}^{q_1}_L S} \|v\|_{\mathcal{L}^{q_2}_L S}.
\]

for all \( v \) with \( \hat{v} \geq 0 \).

**Proof.** [6], Proposition 4.3.

The following duality argument holds.

**Proposition 3.6.** Let \( 1 \leq a, b, q, r \leq \infty, \alpha, \beta \in \mathbb{R} \).

(a) If

\[
\|G\|_{\mathcal{L}^q_L S} \lesssim \|\Lambda^\beta \Lambda^\alpha G\|_{\mathcal{L}^q_r L^q_S}
\]

for all \( G \), then

\[
\|F\|_{\mathcal{L}^q_L L^q} \lesssim \|\Lambda^\alpha \Lambda^\beta F\|_{\mathcal{L}^q_L L^q}\]

for all \( F \).
(b) If (12) holds for all $G$ with $\hat{G} \geq 0$, then
\[ \|F\|_{L_t^q L_x^r} \lesssim \|\Lambda^\alpha \Lambda^\beta F\|_{L_t^q L_x^r}. \]
for all $F$.

Proof. [6], Proposition 4.5. \hfill \Box

The next immediate consequence shows that a Sobolev type embedding also carries over to the $L_t^q L_x^r$-spaces.

**Proposition 3.7.** Let $1 \leq a, b, q, r \leq \infty$. If
\[ \|\Lambda^\alpha \Lambda^\beta u\|_{L_t^q L_x^r} \lesssim \|u\|_{L_t^a L_x^b} \]
for all $u$ with $\hat{u} \geq 0$, then
\[ \|\Lambda^\alpha \Lambda^\beta u\|_{L_t^q L_x^r} \lesssim \|u\|_{L_t^q L_x^q}. \]

Proof. [6], Corollary 4.6. \hfill \Box

**Proposition 3.8.** If $1 < p \leq q \leq 2$, $\frac{q}{q-1} + \frac{1}{q^*} = 1$ and $s = \frac{3}{p} - 2 + \frac{1}{q^*}$, then
\[ \|\Lambda^{-s} \Lambda^{(-\frac{1}{2})^-} u\|_{L_t^q L_x^p} \lesssim \|u\|_{L_t^q L_x^p} \]
for all $u$ with $\hat{u} \geq 0$.

Proof. We adapt the proof of [6], Prop. 4.7 in space dimension $n \geq 4$ to the case $n = 3$. Let $U := \Lambda^{-s} \Lambda^{(-\frac{1}{2})^-} u$. Using the estimate
\[ \|U\|_{L_t^p L_x^d} \lesssim \int \hat{U}(\tau, \xi) d\tau \]
for $\hat{U} \geq 0$ we reduce the claimed estimate to
\[ \int \int (\xi)^{-s} \langle|\tau| - |\xi|\rangle^{-\frac{1}{2}} \hat{u}(\tau, \xi) \hat{f}(\xi) d\tau d\xi \lesssim \|u\|_{L_t^q L_x^p} \|f\|_{L_t^q} \]  \hspace{1cm} (13)
for all $f$ with inverse Fourier transform $\hat{f} \geq 0$. Define
\[ \hat{v}_\pm(\tau, \xi) = (\xi)^{-s} (\tau \mp |\xi|)^{-\frac{1}{2}} \hat{f}(\xi). \]
Then the left hand side of (13) is bounded by
\[ \int \int u(v_+ + v_-) dt dx \lesssim \|u\|_{L_t^p L_x^p} (\|v_+\|_{L_t^p L_x^p} + \|v_-\|_{L_t^p L_x^p}), \]
where $\frac{1}{p} + \frac{1}{p^*} = 1$. It remains to show
\[ \|v_\pm\|_{L_t^p L_x^p} \lesssim \|f\|_{L_t^q}. \]
Defining $\hat{c}(\tau) := (\tau)^{-\frac{1}{2} -} \omega$ we obtain
\[ \hat{v}_\pm(\tau, \xi) = (\xi)^{-s} \hat{c}(\tau \mp |\xi|) \hat{f}(\xi) = (\xi)^{-s} \int c(t)e^{i\xi \mp (|\xi|) t} dt \hat{f}(\xi) \]
\[ = \int c(t)e^{i\xi \tau} (\xi)^{-s} e^{|\xi| t} \hat{f}(\xi) dt, \]
so that
\[ v_\pm(t, x) = c(t) \Lambda^{-s} e^{\pm i\xi t} f(x). \]
Since $c \in L^2(\mathbb{R})$ we obtain for $\frac{1}{2} = \frac{1}{2} - \frac{1}{p^*}$:
\[ \|v_\pm\|_{L_t^p L_x^p} \lesssim \|\Lambda^{-s} e^{\pm i\xi t} f\|_{L_t^p L_x^p} \lesssim \|f\|_{L_t^q}. \]
where we applied Strichartz’ estimate (Prop. 3.1) under the assumption $2 \leq \hat{q} \leq \infty$, $2 \leq p' < \infty \Leftrightarrow 1 < p \leq 2$, $1 \leq q \leq 2$ and \( \frac{2}{q} \leq 2(\frac{1}{p}-\frac{1}{p'}) \Leftrightarrow \frac{3}{q}-1 \leq 2(\frac{1}{p}-\frac{1}{2}) \Leftrightarrow p \leq q \). Here $s = \frac{3}{2} - \frac{3}{p'} - \frac{1}{q} = \frac{3}{2} - 2 + \frac{1}{q}$.

\[ \text{Corollary 3.2.} \quad \text{Under the assumptions of Prop. 3.8:} \]
\[ \|u\|_{L^p_t L^q_x} \lesssim \|\Lambda^s \Lambda^\frac{1}{2} u\|_{L^p_t L^q_x}, \]
\[ \text{where} \quad \frac{1}{p} + \frac{1}{p'} = 1. \]

\[ \text{Proof.} \quad \text{This follows from Prop. 3.8 by use of Prop. 3.6.} \]

\[ \text{Corollary 3.3.} \quad \text{If} \ 1 < q \leq 2, \ s = \frac{2}{q} - 1, \ \text{then} \]
\[ \|\Lambda^{-s} \Lambda^{-\frac{1}{2}} u\|_{L^1_t L^q_x} \lesssim \|u\|_{L^p_t L^q_x} \]
\[ \text{for all} \ u \ \text{with} \ \tilde{u} \geq 0. \]

\[ \text{Proof.} \quad \text{In the special case} \ p = q \ \text{Prop. 3.8 gives} \]
\[ \|\Lambda^{-s} \Lambda^{-\frac{1}{2}} u\|_{L^p_t L^q_x} \lesssim \|u\|_{L^p_t L^q_x} \]
\[ \text{for} \ 1 < q \leq 2, \ s = \frac{3}{q} - 1 = 1. \ \text{Interpolation with the trivial identity} \]
\[ \|u\|_{L^p_t L^q_x} = \|u\|_{L^p_t L^q_x} \text{ gives the result.} \]

\[ \text{Proposition 3.9.} \quad \text{If} \ 1 < q \leq 2, \ \frac{1}{q} + \frac{1}{q'} = 1 \text{ and} \ s = 1 - \frac{2}{q} \text{ the following estimate holds} \]
\[ \|u\|_{L^p_t L^q_x} \lesssim \|\Lambda^s \Lambda^\frac{1}{2} u\|_{L^p_t L^q_x}. \]

\[ \text{Proof.} \quad \text{This follows from Corollary 3.3 by use of Prop. 3.6.} \]

Finally, we formulate the fundamental theorem which allows to reduce the local well-posedness for a system of nonlinear wave equations to suitable estimates for the nonlinearities. It is also essentially contained in the paper by [6].

\[ \text{Proposition 3.10.} \quad \text{Let} \ u_0 \in H^s, \ u_1 \in H^{s-1}, \ v_0 \in H^r, \ v_1 \in H^{r-1} \text{ be given. Assume that} \]
\[ \|\Lambda^{-1} \Lambda^{-\frac{1}{2}} M(u, \partial u, v, \partial v)\|_{F^r} \leq \omega_1(\|u\|_{F^r}, \|v\|_{G^r}), \]
\[ \|\Lambda^{-1} \Lambda^{-\frac{1}{2}} N(u, \partial u, v, \partial v)\|_{G^r} \leq \omega_2(\|u\|_{F^r}, \|v\|_{G^r}), \]

\[ \text{and} \]
\[ \|\Lambda^{-1} \Lambda^{-\frac{1}{2}} (M(u, \partial u, v, \partial v) - M(u', \partial u', \partial v'))\|_{F^r} \]
\[ + \|\Lambda^{-1} \Lambda^{-\frac{1}{2}} (N(u, \partial u, v, \partial v) - N(u', \partial u', v', \partial v'))\|_{G^r} \]
\[ \leq \omega(\|u\|_{F^r}, \|u'\|_{F^r}, \|v\|_{G^r}, \|v'\|_{G^r})(\|u - u'\|_{F^r} + \|v - v'\|_{G^r}), \]
\[ \text{where} \ \omega, \omega_1, \omega_2 \ \text{are continuous functions with} \ \omega(0,0,0,0) = \omega_1(0,0) = \omega_2(0,0) = 0. \]

Then the Cauchy problem
\[ \square u = M(u, \partial u, v, \partial v), \quad \square v = N(u, \partial u, v, \partial v) \]
\[ \text{with data} \]
\[ u(0) = u_0, \ (\partial u)(0) = u_1, \ v(0) = v_0, \ (\partial v)(0) = v_1 \]
is locally well-posed, i.e., there exists $T > 0$, such that there exists a unique solution $u \in F^r_T, \ v \in G^r_T$. 
Proof. This is proved by the contraction mapping principle provided the solution space fulfills suitable assumptions. The case of a single equation $\square u = \mathcal{M}(u, \partial u)$ and the solution space $\mathcal{X}$ given by the norm $\|u\|_{x^2} = \|A_+ u\|_{H^{-1, \frac{1}{2}} + \|\Lambda^\gamma A_+^2 u\|_{L^2_{\gamma}}, \gamma > 0}$ small, was proven by [6], Theorems 5.4 and 5.5, Propositions 5.6 and 5.7. Our case is a straightforward modification of their results. We just remark that the only modification in the case of our solution space is the following estimate in the proof of [6], Prop. 5.6:

$$
\|\Lambda_+^{\frac{1}{2}+2\epsilon} \Lambda^{-\frac{1}{2}} \Lambda_+^{\frac{3}{2}} u\|_{L^1_{\epsilon} L^1_{\epsilon, t}} \lesssim \|\Lambda_+^{\frac{1}{2}+2\epsilon} \Lambda^{-\frac{1}{2}} \Lambda_+^{\frac{3}{2}} u\|_{L^1_{\epsilon} L^1_{\epsilon, t}} \\
\lesssim \|\Lambda_+^{\frac{4}{2}+1+5\epsilon} \Lambda_+^{\frac{1}{2}} u\|_{L^1_{\epsilon} L^1_{\epsilon, t}} \\
\lesssim \|\Lambda_+^{\frac{4}{2}+1+5\epsilon} \Lambda_+^{\frac{1}{2}} u\|_{L^1_{\epsilon} L^1_{\epsilon, t}}. 
$$

The first estimate follows from Corollary 3.2, and the last estimate holds by our assumption $s > \frac{9}{7}$.

4. Reformulation of the problem and null structure. The reformulation of the Yang-Mills equations and the reduction of our main theorem to nonlinear estimates is completely taken over from Tesfahun [16] (cf. also the fundamental paper by Selberg and Tesfahun [14]).

The standard null forms are given by

$$
\left\{ \begin{array}{l}
Q_0(u, v) = \partial_\alpha u \partial^\alpha v = -\partial_k u \partial_k v + \partial_i u \partial^j v, \\
Q_{\alpha \beta}(u, v) = \partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v.
\end{array} \right. \quad (14)
$$

For $g$-valued $u, v$, define a commutator version of null forms by

$$
\left\{ \begin{array}{l}
Q_0[u, v] = [\partial_\alpha u, \partial^\alpha v] = Q_0(u, v) - Q_0(v, u), \\
Q_{\alpha \beta}[u, v] = [\partial_\alpha u, \partial_\beta v] - [\partial_\beta u, \partial_\alpha v] = Q_{\alpha \beta}(u, v) + Q_{\alpha \beta}(v, u).
\end{array} \right. \quad (15)
$$

Note the identity

$$
[\partial_\alpha u, \partial_\beta u] = \frac{1}{2} ([\partial_\alpha u, \partial_\beta u] - [\partial_\beta u, \partial_\alpha u]) = \frac{1}{2} Q_{\alpha \beta}[u, v]. \quad (16)
$$

Define

$$
Q[u, v] = -\frac{1}{2} \varepsilon_{ijk} \varepsilon_{klm} Q_{ij} [R^k u^m, v] - Q_{0i} [R^k u, v] ,
$$

where $\varepsilon_{ijk}$ is the antisymmetric symbol with $\varepsilon_{123} = 1$ and $R_i = \Lambda^{-1} \partial_i$ are the Riesz transforms.

Now we refer to Tesfahun [16], who showed that the system (6), (7) in Lorenz gauge can be written in the following form

$$
\square A_\beta = \mathcal{M}_\beta(A, \partial_\alpha A, F, \partial F), \\
\square F_{\beta \gamma} = \mathcal{N}_{\beta \gamma}(A, \partial_\alpha A, F, \partial F), \quad (18)
$$

where

$$
\mathcal{M}_\beta(A, \partial_\alpha A, F, \partial F) = -2Q[\Lambda^{-1} A, A_\beta] + \sum_{i=1}^{4} \tilde{\Gamma}_i^\beta_i(A, \partial A, F, \partial F) - 2[\Lambda^{-2} A^\alpha, \partial_\alpha A_\beta] - [A^\alpha, [A_\alpha, A_\beta]],
$$

$$
\mathcal{N}_{ij}(A, \partial_\alpha A, F, \partial F) = -2Q[\Lambda^{-1} A, F_{ij}] + 2Q[\Lambda^{-1} \partial_j A, A_i] - 2Q[\Lambda^{-1} \partial_i A, A_j] + 2Q_0[A_i, A_j] + Q_{ij} [A^\alpha, A_\alpha] - 2[\Lambda^{-2} A^\alpha, \partial_\alpha F_{ij}] .
$$
3.10 reduce to proving (we remark, that due to the multilinear character of the divergence-free and curl-free parts and a smoother part is used

Now, looking at the terms in $A, \partial A, F, \partial F$

$N_{0i}(A, \partial A, F, \partial F) = -2Q[\Lambda^{-1} A, F_{0i}] + 2Q[\Lambda^{-1} \partial_i A, A_0] - 2Q_{0j}[A^j, A_i]$

$+ 2Q_0[A_0, A_i] + Q_0[A_0, A_\alpha] - 2[\Lambda^{-2} A_\alpha, \partial_\alpha F_{0i}]

+ 2[\Lambda^{-2} \partial_i A_\alpha, \partial_\alpha A_0] - [A_\alpha, [A_\alpha, F_{0i}]] + 2[F_{0i}, [A_\alpha, A_\alpha]]$

$- 2[F_{0i}, [A_\alpha, A_0]] - 2[[A_\alpha, A_0], [A_\alpha, A_j]],$

where

$\tilde{\Gamma}_0^1(A, \partial A, F, \partial F) = -[A_0, \partial_0 A_0] + [\Lambda^{-1} R_j(\partial_i A_0), \Lambda^{-1} R^j \partial_i (\partial_\beta A_0)],$

$\tilde{\Gamma}_0^2(A, \partial A, F, \partial F) = -\frac{1}{2} \epsilon^{ijk\ell} \left\{ Q_{ij} [\Lambda^{-1} R^\alpha_A a_n, \Lambda^{-1} R^i \partial_\beta A^m] + Q_{ij} [\Lambda^{-1} R^\alpha_0 a_n, \Lambda^{-1} R^i \partial_\beta A^m] \right\},$

$\tilde{\Gamma}_0^3(A, \partial A, F, \partial F) = [\Lambda^{-2} \nabla \times F, \Lambda^{-2} \nabla \times \partial_3 F]$

$- [\Lambda^{-2} \nabla \times F, \Lambda^{-2} \partial_3 \nabla \times (A \times A)]$

$- [\Lambda^{-2} \nabla \times (A \times A), \Lambda^{-2} \nabla \times \partial_3 F]$

$+ [\Lambda^{-2} \nabla \times (A \times A), \Lambda^{-2} \partial_3 \nabla \times (A \times A)],$

$\tilde{\Gamma}_0^4(A, \partial A, F, \partial F) = [A^{cf} + A^{df}, \Lambda^{-2} \partial_3 A] + [\Lambda^{-2} A, \partial_3 A].$

Here $F = (F_1, F_2, F_3)$, where $F_i = \sum_{j<k, j,k \neq i} F_{jk}$, $(\nabla \times A)_i = \epsilon_{ijk} \partial_j A^k$ and

$(A \times B)_k = \epsilon_{ijk} A^l B^l.$

Here especially the splitting of the spatial part $A = (A_1, A_2, A_3)$ of the potential into divergence-free and curl-free parts and a smoother part is used

$A = A^{df} + A^{cf} + (\nabla)^{-2} A,$

where

$A^{df} = (\nabla)^{-2} \nabla \times \nabla \times A,$

$A^{cf} = - (\nabla)^{-2} \nabla (\nabla \cdot A).$
the following estimate for $\tilde{\Gamma}_\beta^1$ and other bilinear terms
\begin{align}
\left\| \Lambda_+^{-1} \Lambda_-^{-1} \tilde{\Gamma}_\beta^1 (A, \partial A) \right\|_{F^s} & \lesssim \| A \|_{F^s} \| A \|_{F^s}, \\
\left\| \Lambda_+^{-1} \Lambda_-^{-1} \Pi (A, \Lambda^{-2} \partial A) \right\|_{F^s} & \lesssim \| A \|_{F^s} \| A \|_{F^s}, \\
\left\| \Lambda_+^{-1} \Lambda_-^{-1} \Pi (\Lambda^{-2} A, \partial A) \right\|_{F^s} & \lesssim \| A \|_{F^s} \| A \|_{F^s}, \\
\left\| \Lambda_+^{-1} \Lambda_-^{-1} (\Lambda^{-1} F, \Lambda^{-1} \partial F) \right\|_{F^s} & \lesssim \| F \|_{G^r} \| F \|_{G^r}, \\
\left\| \Lambda_+^{-1} \Lambda_-^{-1} \Pi (\Lambda^{-1} A, \partial A) \right\|_{G^r} & \lesssim \| A \|_{F^s} \| A \|_{F^s}, \\
\left\| \Lambda_+^{-1} \Lambda_-^{-1} \Pi (\Lambda^{-1} A, \Lambda A) \right\|_{G^r} & \lesssim \| A \|_{F^s} \| A \|_{F^s},
\end{align}
and
2. the following trilinear and quadrilinear estimates:
\begin{align}
\left\| \Lambda_+^{-1} \Lambda_-^{-1} \Pi (\Lambda^{-1} F, \Lambda^{-1} (\partial A A)) \right\|_{F^s} & \lesssim \| F \|_{X^{r, \frac{1}{2}, +}} \| A \|_{F^s} \| A \|_{F^s}, \\
\left\| \Lambda_+^{-1} \Lambda_-^{-1} \Pi (\Lambda^{-1} \partial F, \Lambda^{-1} (AA)) \right\|_{F^s} & \lesssim \| F \|_{X^{r, \frac{1}{2}, +}} \| A \|_{F^s} \| A \|_{F^s}, \\
\left\| \Lambda_+^{-1} \Lambda_-^{-1} \Pi (\Lambda^{-1} (AA), \Lambda^{-1} \partial (AA)) \right\|_{F^s} & \lesssim \| A \|_{F^s} \| A \|_{F^s} \| A \|_{F^s}, \\
\left\| \Lambda_+^{-1} \Lambda_-^{-1} \Pi (A, A, A) \right\|_{F^s} & \lesssim \| A \|_{F^s} \| A \|_{F^s} \| A \|_{F^s}, \\
\left\| \Lambda_+^{-1} \Lambda_-^{-1} \Pi (A, A, F) \right\|_{G^r} & \lesssim \| A \|_{F^s} \| A \|_{F^s} \| F \|_{G^r}, \\
\left\| \Lambda_+^{-1} \Lambda_-^{-1} \Pi (A, A, A) \right\|_{G^r} & \lesssim \| A \|_{F^s} \| A \|_{F^s} \| A \|_{F^s},
\end{align}
where $\Pi (\cdots)$ denotes a multilinear operator in its arguments.

The matrix commutator null forms are linear combinations of the ordinary ones, in view of (15). Since the matrix structure plays no role in the estimates under consideration, we reduce (21)–(25) to estimates of the ordinary null forms for $\mathbb{C}$-valued functions $u$ and $v$ (as in (14)).

The null forms above satisfy the following estimates.

Lemma 4.1. The following estimates hold for $0 \leq \alpha \leq 1$ and $Q = Q_0$, or $Q = Q_1$:
\begin{align}
Q_0(u, v) & \lesssim D^1_- - D^\alpha_+ D^{-\alpha}_+ (D^\alpha_+ u D^2_+ v) + (D^\alpha_+ D^{-\alpha}_+ u) (D^\alpha_+ v) + (D^\alpha_+ u) (D^\alpha_+ D^{-\alpha}_+ v) \quad (38) \\
Q_0(u, v) & \lesssim D^1_- - D^\alpha_+ (D^\alpha_+ u D^2_+ v) + (D^\alpha_+ D^{-\alpha}_+ u) (D^\alpha_+ v) \\
& \quad + (D^\alpha_+ D^{-\alpha}_+ u) (D^\alpha_+ v) + (D^\alpha_+ u) (D^\alpha_+ D^{-\alpha}_+ v) \quad (39) \\
Q(u, v) & \lesssim D^{\frac{3}{2}} D^{-\frac{3}{2}} (D^2_+ u D^2_+ v) + D^{\frac{3}{2}} (D^2_+ D^{\frac{3}{2}}_+ u D^2_+ v) + D^{\frac{3}{2}} (D^2_+ u D^{\frac{3}{2}}_+ D^2_+ v) \quad (40) \\
Q(u, v) & \lesssim D^{\frac{3}{2}} - D^{\frac{3}{2}} - 2\epsilon_+ D^{\frac{3}{2}} - 2\epsilon_+ u D^{\frac{3}{2}} - 2\epsilon_+ v + D^{\frac{3}{2}} - 2\epsilon_+ (D^{\frac{3}{2}} - 2\epsilon_+ u D^{\frac{3}{2}} - 2\epsilon_+ v) \\
& \quad + D^{\frac{3}{2}} (D^2_+ u D^2_+ D^{\frac{3}{2}} - 2\epsilon_+ v) \quad (41)
\end{align}

Proof. (38) is Lemma 7.6 in [6], and (40) follows immediately from [5], Prop. 1. (41) follows by interpolating the estimate for the symbol $q = q(\tau, \xi, \lambda, \eta)$ of [5], Prop. 1 which led to (40) with its trivial bound $q \lesssim (|\tau| + |\xi|)(|\lambda| + |\eta|)$. (39) follows by the fractional Leibniz rule for $\Lambda_\alpha$ from (38).

Next we consider the term $\tilde{\Gamma}_\beta^1$. We may ignore its matrix form and treat
\begin{equation}
\tilde{\Gamma}_\beta^1 (A_0, \partial_k A_0) = -A_0 (\partial_k A_0) + \Lambda^{-1} R_j (\partial_t A_0) \Lambda^{-1} R_j (\partial_t \Lambda_0)
\end{equation}
for $k = 1, 2, 3$ and
\begin{equation}
\tilde{\Gamma}_\beta^1 (A_0, \partial^i A_0) = -A_0 (\partial_0 A_0) + \Lambda^{-1} R_j (\partial_t A_0) \Lambda^{-1} R_j (\partial_t \Lambda_0)
\end{equation}
where we used the Lorenz gauge $\partial_0 A_0 = \partial^i A_i$ in the last line in order to eliminate one time derivative. Thus we have to consider

$$
\Gamma^1(u, v) = -u v + \Lambda^{-1} R_j (\partial_t A_0) \Lambda^{-1} R_l (\partial_l A_t),
$$

where $u = A_0$ and $v = \partial^i A_i$ or $v = \partial_k A_0$.

The proof of the following theorem was essentially given by Tesfahun [16]. In fact the detection of this null structure was the main progress of his paper over Selberg-Tesfahun [14].

**Lemma 4.2.** The following estimate holds:

$$
\Gamma^1(u, v) \lesssim \Gamma^1_1(u, v) + \Lambda^{-2} u v + u(\Lambda^{-2} v),
$$

where

$$
\Gamma^1_1(u, v) = D_{\frac{1}{2}-2\varepsilon} D_{\frac{1}{2}+2\varepsilon} (D_{\frac{1}{2}+2\varepsilon} u D_{\frac{1}{2}+2\varepsilon} v) + D_{\frac{1}{2}-2\varepsilon} (D_{\frac{1}{2}+2\varepsilon} u D_{\frac{1}{2}+2\varepsilon} D_{\frac{1}{2}-2\varepsilon} v)
$$

(44)

$$
\Gamma^1_2(u, v) = D_{\frac{1}{2}+2\varepsilon} D_{\frac{1}{2}-2\varepsilon} (D_{\frac{1}{2}+2\varepsilon} \Lambda^{-1} u D_{\frac{1}{2}+2\varepsilon} \Lambda^{-1} v) + D_{\frac{1}{2}+2\varepsilon} \Lambda^{-1} u D_{\frac{1}{2}+2\varepsilon} \Lambda^{-1} v
$$

(45)

Proof. $\Gamma^1(u, v)$ has the symbol

$$
p(\xi, \eta, \lambda) = -1 + \frac{\langle \xi, \eta \rangle \tau \lambda}{\langle \xi \rangle^2 \langle \eta \rangle^2} = \left( -1 + \frac{\langle \xi, \eta \rangle \langle \xi, \eta \rangle}{\langle \xi \rangle^2 \langle \eta \rangle^2} + \frac{\tau \lambda - \langle \xi, \eta \rangle \langle \xi, \eta \rangle}{\langle \xi \rangle^2 \langle \eta \rangle^2} \right) = I + II
$$

Now we estimate

$$
|I| = \left| \frac{\langle \xi \rangle^2 \langle \eta \rangle^2 \cos^2 \angle(\xi, \eta) - 1}{\langle \xi \rangle^2 \langle \eta \rangle^2} \right|
$$

$$
\leq \left| \frac{\langle \xi \rangle^2 \langle \eta \rangle^2 \cos^2 \angle(\xi, \eta) - 1}{\langle \xi \rangle^2 \langle \eta \rangle^2} \right| + \left| \frac{\langle \xi \rangle^2 \langle \eta \rangle^2 - \langle \xi \rangle^2 \langle \eta \rangle^2}{\langle \xi \rangle^2 \langle \eta \rangle^2} \right|
$$

$$
= \sin^2 \angle(\xi, \eta) + \left| \frac{\langle \xi \rangle^2 \langle \eta \rangle^2 - \langle \xi \rangle^2 \langle \eta \rangle^2}{\langle \xi \rangle^2 \langle \eta \rangle^2} \right|
$$

where $\angle(\xi, \eta)$ denotes the angle between $\xi$ and $\eta$. We have

$$
\left| \frac{\langle \xi \rangle^2 \langle \eta \rangle^2 - \langle \xi \rangle^2 \langle \eta \rangle^2}{\langle \xi \rangle^2 \langle \eta \rangle^2} \right| = \frac{\langle \xi \rangle^2 + \langle \eta \rangle^2 + 1}{\langle \xi \rangle^2 \langle \eta \rangle^2} \leq \frac{1}{\langle \xi \rangle^2} + \frac{1}{\langle \eta \rangle^2}
$$

and

$$
\sin^2 \angle(\xi, \eta) \leq \left| \sin \angle(\xi, \eta) \right|^{1-4\varepsilon} = \left| 1 - \cos \angle(\xi, \eta) \right|^{\frac{1}{2}-2\varepsilon} \left| 1 + \cos \angle(\xi, \eta) \right|^{\frac{1}{2}-2\varepsilon}
$$

$$
\lesssim \frac{\langle \xi \rangle + \langle \eta \rangle}{\langle \xi \rangle^{\frac{1}{2}-2\varepsilon} \langle \eta \rangle^{\frac{1}{2}-2\varepsilon}} \left( \left| \tau \right| - ||\tau|| \frac{1}{2}-2\varepsilon + ||\lambda|| \frac{1}{2}-2\varepsilon + ||\tau + \lambda|| - ||\tau + \lambda|| \frac{1}{2}-2\varepsilon \right)
$$

for $0 \leq \varepsilon \leq \frac{1}{2}$ by [5]. Proof of proposition 1. Thus the operator belonging to the symbol $I$ is controlled by $\Gamma^1_1(u, v) + (\Lambda^{-2} u v + u(\Lambda^{-2} v))$. Moreover

$$
|II| \leq \frac{\left| \tau \lambda - \langle \xi, \eta \rangle \right|}{\langle \xi \rangle \langle \eta \rangle}.
$$

This is the symbol of $Q_0(\Lambda^{-1} u, \Lambda^{-1} v)$, which is controlled by $\Gamma^1_2(u, v)$ by (38). Thus we obtain (42) and using the trivial bound $|I| \lesssim 1$ also (43).
5. **Proof of the nonlinear estimates. Important remark:** We assume in the following that the Fourier transforms of \(u\) and \(v\) are nonnegative. This means no loss of the generality, because the norms involved in the desired estimates do only depend on the size of the Fourier transforms.

**Proof of (25).** We recall (39) for \(\alpha = \epsilon\):

\[
Q_0(u, v) \lesssim D_+^{1-\epsilon} (D_+ u D_+ v) + D_-^{1-\epsilon} (D_- u D_- v)
\]

Thus we have to consider the first and third term, because the last two terms are equivalent by symmetry.

1. For the first term it suffices to show

\[
\|\Lambda^s \Lambda^{s-1} \Lambda^r (D_+ D_- u D_+ v)\|_{H^{s+\frac{1}{2}}}
\]

This follows from

\[
\|uv\|_{H^{s-1+\frac{1}{2}}} \lesssim \|u\|_{H^{s-1+\frac{1}{2}}} \|v\|_{H^{s-1+\frac{1}{2}}},
\]

which is a consequence of Prop. 3.3 under our conditions \(s \geq r\) and \(2s - r > \frac{1}{2}\).

2. For the second term we show

\[
\|\Lambda^s \Lambda^{s-1} \Lambda^r (D_+ D_- u D_+ v)\|_{H^{s+\frac{1}{2}}}
\]

This follows from

\[
\|uv\|_{H^{s-1+\frac{1}{2}}} \lesssim \|u\|_{H^{s-1+\frac{1}{2}}} \|v\|_{H^{s-1+\frac{1}{2}}},
\]

which is a consequence of Prop. 3.3 as in 1. under the same assumptions.

**Proof of (24).** We use (41). Thus we have to show the following estimates and remark that we only have to consider the first and third term, because the last two terms are equivalent by symmetry.

1. For the first term it suffices to show

\[
\|\Lambda^s \Lambda^{s-1} \Lambda^r (D_+ D_- u D_+ v)\|_{H^{s+\frac{1}{2}}}
\]

This follows from

\[
\|uv\|_{H^{s-1+\frac{1}{2}}} \lesssim \|u\|_{H^{s-1+\frac{1}{2}}} \|v\|_{H^{s-1+\frac{1}{2}}},
\]

which is a consequence of Prop. 3.3 with parameters \(s_0 = \frac{1}{2} - r + 2\epsilon, s_1 = s_2 = s - \frac{1}{2} - 2\epsilon, b_0 = 0, b_1 = b_2 = \frac{1}{2} + \epsilon\), so that \(s_0 = s_1 + s_2 > 1\), if \(2s - r > \frac{3}{2}\), and \(s_0 = s_1 + s_2 + s_1 + s_2 > \frac{3}{2}\), if \(4s - r > 3\), which holds under our assumptions.

2. For the second term we show

\[
\|\Lambda^s \Lambda^{s-1} \Lambda^r (D_+ D_- u D_+ v)\|_{H^{s+\frac{1}{2}}}
\]

Using \(\Lambda^s u \lesssim \Lambda^s u\) it suffices to show

\[
\|uv\|_{H^{s-1+\frac{1}{2}}+\frac{1}{2}} \lesssim \|u\|_{H^{s-1+\frac{1}{2}}+\frac{1}{2}} \|v\|_{H^{s-1+\frac{1}{2}}+\frac{1}{2}}.
\]
which is a consequence of Prop. 3.3 as in 1., if $2s - r > \frac{3}{2}$ and $3s - 2r > 2$, which holds under our assumptions. \hfill \square

**Proof of (29).** A. We start with the first part of the $F^s$-norm. As before it is easy to see that we can reduce to

$$\|uv\|_{H^{r-1,-\frac{s}{2}+2r}} \lesssim \|u\|_{H^{r+1,\frac{s}{2}+r}} \|v\|_{H^{r,-\frac{s}{2}+r}}.$$ 

This is a consequence of Prop. 3.3. One easily checks that it can be applied under the conditions $s \leq r+1$, $2r - s > -1$, $4r - s > -2$ and $3r - 2s > -2$, all of which are satisfied under our assumptions.

B. For the second part of the $F^s$-norm we reduce to

$$\|\Lambda^{-1+\epsilon} \Lambda^{-\frac{1}{2}+2r} \Lambda^{-\frac{1}{2}+3s} \Lambda^\frac{1}{2} (uv)\|_{L^q_t L^r_x} \lesssim \|\Lambda^\frac{1}{2} u\|_{H^{r,\frac{s}{2}+r}} \|\Lambda^\frac{1}{2} v\|_{H^{r-1,\frac{s}{2}+r}}.$$ 

and further to

$$\|\Lambda^{-1-\frac{1}{2}+7r} \Lambda^{-\frac{1}{2}-\epsilon} (uv)\|_{L^q_t L^r_x} \lesssim \|\Lambda^\frac{1}{2} u\|_{H^{r,\frac{s}{2}+r}} \|v\|_{H^{r,-\frac{s}{2}+r}}.$$ 

By Prop. 3.9 we obtain

$$\|\Lambda^{-1-\frac{1}{2}+7r} \Lambda^{-\frac{1}{2}-\epsilon} (uv)\|_{L^q_t L^r_x} \lesssim \|\Lambda^\frac{1}{2} + 7r (uv)\|_{L^1_t L^2_x}.$$ 

Next we show for a suitable $r_2$ the estimate

$$\|\Lambda^\frac{1}{2} + 7r (uv)\|_{L^2_x} \lesssim \|u\|_{H^{\frac{1}{2} + 2r_2}} \|v\|_{H^{\frac{1}{2} + 7r} x},$$

which by duality is equivalent to

$$\|uv\|_{H^{\frac{1}{2} - 7r}} \lesssim \|u\|_{H^{\frac{1}{2} - 7r_2}} \|w\|_{H^{\frac{1}{2} - 7r}}.$$ 

Using the fractional Leibniz rule we have to consider two terms:

$$\|(\Lambda^{-\frac{1}{2} - 7r} u) w\|_{L^2} \lesssim \|\Lambda^{-\frac{1}{2} - 7r} u\|_{L^2} \|w\|_{L^\infty} \lesssim \|u\|_{H^{\frac{1}{2} - 7r_2}} \|w\|_{H^{\frac{1}{2} - 7r}},$$

where we choose $\frac{1}{q} = \frac{1}{2} - \frac{1}{r_2} + \frac{7}{3} \epsilon$ and $\frac{1}{r_2} = \frac{1}{7} - \frac{7}{3} \epsilon$, so that by Sobolev $H^{\frac{1}{2} - 7r} \hookrightarrow L^q$, and

$$\|u \Lambda^{-\frac{1}{2} - 7r} w\|_{L^2} \lesssim \|u\|_{L^\infty} \|w\|_{H^{\frac{1}{2} - 7r_2}},$$

where $\frac{1}{q_1} = \frac{1}{2} - \frac{1}{2r_2}$, $\frac{1}{r_1} = \frac{2}{2r_2}$, so that by Sobolev $H^{\frac{1}{2}} \hookrightarrow L^{r_1}$ and $H^{\frac{1}{2} - 7r_2} \hookrightarrow L^{r_1}$. Thus we obtain by Cor. 3.1:

$$\|\Lambda^\frac{1}{2} + 7r (uv)\|_{L^2_t L^r_x} \lesssim \|u\|_{L^2_t H^{\frac{1}{2} - 7r_2}} \|v\|_{L^2_t H^{\frac{1}{2} - 7r}},$$

$$\lesssim \|u\|_{H^{1-\frac{7r}{2}+\frac{1}{2} - 7r_2 + \frac{1}{2} + \epsilon}} \|v\|_{H^{r_0}} \lesssim \|u\|_{H^{r_2 - \frac{1}{2} + \frac{1}{2} + \epsilon}} \|v\|_{H^{r_0}} \lesssim \|u\|_{H^{r_2 + 1, \frac{1}{2} + \epsilon}} \|v\|_{H^{r_0}},$$

for $r > -\frac{1}{7}$.

\hfill \square

**Proof of (26).** A. For the first part of the $F^s$-norm it is sufficient to show

$$\|\Gamma^1 (u, v)\|_{H^{s-1,-\frac{s}{2}+2r}} \lesssim \|u\|_{H^s} \|\Lambda^\frac{1}{2} u\|_{H^{s-2,\frac{s}{2}+r}}.$$ 

(46) for the minimal value $s = \frac{5}{7} +$, because the estimate for any $s > \frac{5}{7}$ follows immediately. We use Lemma 4.2.
a. We first consider $\Gamma^2_1(u,v)$. By (45) it suffices to show the following estimates, all of which are consequences of Proposition 3.3.

\[
\|uv\|_{H^{-\frac{1}{2} - 2s, 0}} \lesssim \|u\|_{H^{s+\frac{1}{2} - 2s, \frac{1}{2} + s}} \|v\|_{H^{s-\frac{1}{2} - 2s, \frac{1}{2} + s}},
\]

\[
\|uv\|_{H^{-\frac{1}{2} - 1, 0 + 2s}} \lesssim \|u\|_{H^{s+\frac{1}{2} - 1, \frac{1}{2} + s}} \|v\|_{H^{s-\frac{1}{2} - 1, \frac{1}{2} + s}},
\]

\[
\|uv\|_{H^{-\frac{1}{2} - 1, 0 + 2s}} \lesssim \|u\|_{H^{s+\frac{1}{2} - 2s, \frac{1}{2} + s}} \|v\|_{H^{s-\frac{1}{2} - 2s, \frac{1}{2} + s}}.
\]

Both estimates follow from Proposition 3.3. The second term is reduced to the following estimate

\[
\|uv\|_{H^{-\frac{1}{2} - 2s, 0}} \lesssim \|\Lambda_2^{1 - 2s} \Lambda_2^{-\frac{1}{2} + 2s} u\|_{F^s} \|v\|_{H^{-\frac{1}{2} - 2s, \frac{1}{2} + s}}.
\]

By the fractional Leibniz rule we have to show the following two estimates:

b1.

\[
\|((\Lambda_2^{1 - 2s} u)v)\|_{H^{-\frac{1}{2} - 2s, 0}} \lesssim \|u\|_{H^{s+1 - 2s, 0}} \|v\|_{H^{s-\frac{1}{2} - 2s, \frac{1}{2} + s}},
\]

which follows from Proposition 3.3.

b2.

\[
\|u(\Lambda_2^{1 - 2s} v)\|_{H^{0, -\frac{1}{2} - 2s}} \lesssim \|u(\Lambda_2^{1 - 2s} v)\|_{L_t^{\frac{14}{12}} L_x^2}
\]

\[
\lesssim \|u\|_{L_t^{\frac{14}{12}} L_x^{\infty}} \|\Lambda_2^{1 - 2s} v\|_{L_t^{\frac{14}{12}} L_x^2}
\]

\[
\lesssim \|\Lambda_2^{1 + \epsilon} u\|_{L_t^{\frac{14}{12}} L_x^{\infty}} \|u\|_{H^{s+\frac{1}{2} - 2s, \frac{1}{2} + s}}
\]

\[
\lesssim \|\Lambda_2^{1 - 2s} \Lambda_2^{-\frac{1}{2} + 2s} u\|_{F^s} \|u\|_{H^{s+\frac{1}{2} - 2s, \frac{1}{2} + s}}.
\] (47)

where we used Cor. 3.7 and Prop. 3.5 in order to replace $L_t^p L_x^q$-norms by $L_t^1 L_x^2$-norms.

c. Consider $(\Lambda^{-2} u)v$ and $u(\Lambda^{-2} v)$. It suffices to show

\[
\|((\Lambda^{-2} u)v)\|_{H^{s+\frac{1}{2} - 1, \frac{1}{2} + s}} \lesssim \|u\|_{H^{s+\frac{1}{2} + s}} \|v\|_{H^{-s-1, \frac{1}{2} + s}}.
\]

\[
\|u(\Lambda^{-2} v)\|_{H^{s+\frac{1}{2} - 1, \frac{1}{2} + s}} \lesssim \|u\|_{H^{s+\frac{1}{2} + s}} \|v\|_{H^{-s-1, \frac{1}{2} + s}}.
\]

Both follow easily from Prop. 3.4 under our assumption $s > \frac{1}{2}$.

d. Let us now consider the case where the frequencies of $u$ or $v$ are $\leq 1$. We use (43) instead of (42). Because $\Gamma^2_1(u, v)$ has already been handled, we only have to consider $uv$. If $u$ has low frequencies we obtain by Prop. 3.4:

\[
\|uv\|_{H^{s+\frac{1}{2} - 1, \frac{1}{2} + s}} \lesssim \|u\|_{H^{s+\frac{1}{2} + s}} \|v\|_{H^{-s-1, \frac{1}{2} + s}}
\]

\[
\lesssim \|u\|_{H^{s+\frac{1}{2} + s}} \|v\|_{H^{-s-1, \frac{1}{2} + s}}.
\]

Similarly we treat the case where $v$ has low frequencies.

B. Now we consider the second part of the $F^s$-norm. We want to show

\[
\|\Lambda_+^{1 - \epsilon} \Lambda_2^{1 - 2s} \Lambda_2^{-\frac{1}{2} + 2s} \Lambda_2^{\frac{1}{2} + 3s} \Gamma^2_1(u, v)\|_{L_t^{\frac{14}{12}} L_x^2} \lesssim \|u\|_{F^s} \|\Lambda_+ v\|_{H^{s-2, \frac{1}{2} + s}}.
\] (48)
a. We first consider $\Gamma^1_2(u,v)$ and use (45).

1. The estimate for the first term on the right hand side reduces to

$$\|\Lambda^{-\frac{3}{2}} u \Lambda^{-\frac{1}{2}} \Lambda^{\frac{1}{2} + 3\epsilon} \|_{L^2_t L^{14}_x} \lesssim \|u\|_{H^{s-1}} \|\Lambda^+ u\|_{H^s},$$

and therefore we only have to prove

$$\|\Lambda^{-\frac{3}{2}} + 3\epsilon (uv)\|_{L^2_t L^{14}_x} \lesssim \|u\|_{H^{s-1}} \|\Lambda^+ u\|_{H^s}. \quad (49)$$

This follows from Proposition 3.2 with parameters $q = \frac{26}{13}, r = 28, \sigma = \frac{11}{14} - 3\epsilon$, $s_1 = \frac{s}{2} + 3\epsilon$ and $s_2 = \frac{s}{2} - \frac{3}{2}$. The claimed estimate follows, because $s_1 < s - \frac{1}{2} - 2\epsilon$ and $s_2 < s - \frac{1}{2} - 2\epsilon$ under our assumption $s > \frac{5}{7}$.

2. The second term is modified as follows:

$$D_+ D^\frac{3}{2} u \Lambda^{-\frac{1}{2}} v = D_+ ^\frac{3}{2} (D_+ ^\frac{3}{2} - \Lambda^{-\frac{1}{2}} u \Lambda^{\frac{1}{2} + 2\epsilon} \Lambda^{-\frac{1}{2}} v)$$

$$+ D_+ ^\frac{3}{2} D^\frac{3}{2} u \Lambda^{-\frac{1}{2}} v.$$
for $s > \frac{7}{2}$, where we used Cor. 3.1.

2.2. We need

$$\|A^{-1-\frac{7}{2}+\tau} A^{-\frac{7}{2}-\epsilon}(uv)\|_{L_t^2 L_x^4} \lesssim \|u\|_{H^{s+\frac{7}{2},0}} \|v\|_{H^{s+\frac{7}{2},2s+\epsilon}}.$$  

We obtain by Proposition 3.9

$$\|A^{-1-\frac{7}{2}+\tau} A^{-\frac{7}{2}-\epsilon}(uv)\|_{L_t^2 L_x^4} \lesssim \|A^{-\frac{7}{2}+\tau}(uv)\|_{L_t^1 L_x^2}.$$  

We now show that

$$\|uv\|_{H^{s+\tau}} \lesssim \|u\|_{H^s} \|v\|_{H^{s+\tau}},$$  

which by duality is equivalent to

$$\|uv\|_{H^{s-\tau}_{-\frac{7}{2}}} \lesssim \|u\|_{H^s} \|v\|_{H^{s+\tau}}.$$  

By the fractional Leibniz rule

$$\|uv\|_{H^{s-\tau}_{-\frac{7}{2}}} \lesssim \|A^{\frac{7}{2}-\tau} uv\|_{L_t^\infty L_x^2} \lesssim \|u\|_{H^s} \|A^{\frac{7}{2}-\tau} v\|_{L_t^\infty L_x^2} \lesssim \|u\|_{H^s} \|A^{\frac{7}{2}-\tau} v\|_{L_t^\infty L_x^2},$$  

where we choose $\frac{1}{7} = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} \epsilon$ and $\frac{1}{7} = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} \epsilon$, so that by Sobolev $H^{\frac{7}{2}+\tau} \hookrightarrow L^q$ and $H^{\frac{7}{2}+\tau} \hookrightarrow L^p$. Thus we obtain

$$\|A^{\frac{7}{2}+\tau}(uv)\|_{L_t^1 L_x^2} \lesssim \|u\|_{L_t^1 H^s} \|v\|_{L_t^1 H^{s+\tau}_{-\frac{7}{2}}} \lesssim \|u\|_{H^{s+\tau}_{-\frac{7}{2}}} \|v\|_{H^{s+\tau}_{-\frac{7}{2}}}.  

for $s > \frac{7}{2}$ by Cor. 3.1.

3. We reduce to

$$\|A^{-1-\frac{7}{2}+\tau} A^{-\frac{7}{2}-\epsilon}(uv)\|_{L_t^2 L_x^4} \lesssim \|u\|_{H^{s+\frac{7}{2},0}} \|v\|_{H^{s+\frac{7}{2},2s+\epsilon}}.$$  

We obtain by Proposition 3.9

$$\|A^{-1-\frac{7}{2}+\tau} A^{-\frac{7}{2}-\epsilon}(uv)\|_{L_t^2 L_x^4} \lesssim \|A^{-\frac{7}{2}+\tau}(uv)\|_{L_t^1 L_x^2}.$$  

Now for suitable $p$ we obtain

$$\|uv\|_{H^{s+\tau}_{-\frac{7}{2}}} \lesssim \|u\|_{H^{s+\tau}_{-\frac{7}{2}}} \|v\|_{H^{\frac{7}{2}}},$$  

which by duality is equivalent to

$$\|uv\|_{H^{s+\tau}_{-\frac{7}{2}}} \lesssim \|u\|_{H^{s+\tau}_{-\frac{7}{2}}} \|v\|_{H^{\frac{7}{2}}}.  

By the fractional Leibniz rule

$$\|uv\|_{H^{s+\tau}_{-\frac{7}{2}}} \lesssim \|A^{\frac{7}{2}} uv\|_{L_t^\infty L_x^2} \lesssim \|A^{\frac{7}{2}} u\|_{L_t^\infty L_x^2} \|v\|_{L_t^\infty L_x^2} \lesssim \|A^{\frac{7}{2}} u\|_{L_t^\infty L_x^2} \|v\|_{L_t^\infty L_x^2},$$  

for $s > \frac{7}{2}$.
where we choose \( \frac{1}{q} = \frac{1}{2} - \frac{7}{12} + \frac{5}{2} \epsilon, \frac{1}{p} = \frac{1}{2} - \frac{5}{12} + \frac{7}{3} \epsilon, \frac{1}{q_1} = \frac{1}{2} - \frac{1}{21} + \frac{7}{3} \epsilon, \frac{1}{p_1} = \frac{1}{2} - \frac{7}{3} \epsilon \), so that by Sobolev \( H^{\frac{2}{3} - 7 \epsilon} \hookrightarrow L^q, H^{\frac{2}{3} + 7 \epsilon} \hookrightarrow L^p \) and \( H^{\frac{2}{3} + 5 \epsilon} \hookrightarrow L^{p_1} \). Thus we obtain

\[
\| \Lambda^{-\frac{1}{2} + 7 \epsilon} (uv) \|_{L_t^1 L_x^2} \lesssim \| uv \|_{L_t^1 H_x^{\frac{3}{2}}} \| v \|_{L_t^1 L_x^\frac{2}{3}} \\
\lesssim \| v \|_{H_x^{\frac{2}{3} + 2 \epsilon + \frac{7}{12} + \frac{5}{2} \epsilon}} \| v \|_{H^{\frac{2}{3} - 2 \epsilon}} \\
\lesssim \| v \|_{H^{\frac{2}{3} - 2 \epsilon}} || v ||_{H^{\frac{2}{3} - 2 \epsilon}} 
\]

for \( s > \frac{5}{12} \) by Cor. 3.1.

b. If the frequencies of \( u \) or \( v \) are \( \geq 1 \), we use (42) and consider \( \Gamma \) using (44).

1. The estimate for the first term on the right hand side of (44) reduces to (49).

2. The second term on the right hand side of (44) reduces to

\[
\| \Lambda^{-\frac{1}{2} - \frac{7}{12} + 5 \epsilon} (uv) \|_{L_t^{14} L_x^{14}} \lesssim \| \Lambda^{-\frac{1}{2} + 5 \epsilon} (\Lambda^{-\frac{1}{2} + 2 \epsilon} u \Lambda^{-\frac{1}{2} + 2 \epsilon} v) \|_{L_t^{14} L_x^{14}} 
\]

We start with Proposition 3.9

\[
\| \Lambda^{-\frac{1}{2} - \frac{7}{12} + 5 \epsilon} (uv) \|_{L_t^{14} L_x^{14}} \lesssim \| \Lambda^{-\frac{1}{2} + 5 \epsilon} (\Lambda^{-\frac{1}{2} + 2 \epsilon} u \Lambda^{-\frac{1}{2} + 2 \epsilon} v) \|_{L_t^{14} L_x^{14}} 
\]

By the fractional Leibniz rule we have to consider two terms.

2.1.

\[
\| u(\Lambda^{-\frac{1}{2} + 5 \epsilon} u) \|_{L_t^{14} L_x^{14}} \lesssim \| u \|_{L_t^{14} L_x^{14}} \| v \|_{L_t^{14} L_x^{14}} \\
\lesssim \| \Lambda^{-\frac{1}{2} + 2 \epsilon} u \|_{L_t^{14} L_x^{14}} \| v \|_{H^{\frac{2}{3} + 2 \epsilon}} \\
\lesssim \| \Lambda^{-\frac{1}{2} + 2 \epsilon} u \|_{L_t^{14} L_x^{14}} \| v \|_{H^{\frac{2}{3} - 2 \epsilon}} \\
\lesssim \| \Lambda^{-\frac{1}{2} + 2 \epsilon} u \|_{L_t^{14} L_x^{14}} \| v \|_{H^{\frac{2}{3} - 2 \epsilon}} 
\]

2.2. We obtain

\[
\| (\Lambda^{-\frac{1}{2} + 5 \epsilon} u) \|_{L_t^{14} L_x^{14}} \lesssim \| \Lambda^{-\frac{1}{2} + 5 \epsilon} u \|_{L_t^{14} L_x^{14}} \| v \|_{L_t^{14} L_x^{14}} \\
\lesssim \| \Lambda^{-\frac{1}{2} + 2 \epsilon} u \|_{L_t^{14} L_x^{14}} \| v \|_{H^{\frac{2}{3} + 2 \epsilon}} \\
\lesssim \| \Lambda^{-\frac{1}{2} + 2 \epsilon} u \|_{L_t^{14} L_x^{14}} \| v \|_{H^{\frac{2}{3} - 2 \epsilon}} 
\]

where \( \frac{1}{q} = \frac{1}{14} - \frac{7}{24} + \frac{5}{2} \epsilon \) and \( \frac{1}{p} = \frac{1}{2} - \frac{1}{12} - \frac{5}{3} \epsilon \), so that by Sobolev \( H^{\frac{2}{3} - \frac{1}{14}} \hookrightarrow H^{\frac{2}{3} + \frac{5}{12} \epsilon} \) and \( H^{\frac{2}{3} - \frac{1}{14}} \hookrightarrow L^r \). The last estimate follows as in 2.1.

3. The last term on the right hand side of (44) requires

\[
\| \Lambda^{-\frac{1}{2} - \frac{7}{12} + 5 \epsilon} uv \|_{L_t^{14} L_x^{14}} \lesssim \| u \|_{H^{\frac{2}{3} - 2 \epsilon}} \| v \|_{H^{\frac{2}{3} - 2 \epsilon}} 
\]

The left hand side is estimated using Prop. 3.9 by \( \| \Lambda^{-\frac{1}{2} + 5 \epsilon} (uv) \|_{L_t^{14} L_x^{14}} \). By the fractional Leibniz rule we have to treat two terms.

3.1. By Sobolev and Cor. 3.1 we obtain

\[
\| (\Lambda^{-\frac{1}{2} + 5 \epsilon} u) v \|_{L_t^{14} L_x^{14}} \lesssim \| \Lambda^{-\frac{1}{2} + 5 \epsilon} u \|_{L_t^{14} L_x^{14}} \| v \|_{L_t^{14} L_x^{14}} \\
\lesssim \| \Lambda^{-\frac{1}{2} + 5 \epsilon} u \|_{H^{\frac{2}{3} + 2 \epsilon}} \| v \|_{H^{\frac{2}{3} - 2 \epsilon}} \\
\lesssim \| u \|_{H^{\frac{2}{3} - 2 \epsilon}} \| v \|_{H^{\frac{2}{3} - 2 \epsilon}} 
\]
using $H^{\frac{1}{2} - 2\epsilon} \rightarrow H^{\frac{1}{2} - \frac{1}{2} - 2\epsilon} \rightarrow L^2$.

3.2. Similarly by Cor. 3.1:

$$\|u(\Lambda^{\frac{1}{2} - 2\epsilon} v)\|_{L^2_x L^p_t} \lesssim \|u\|_{L^2_x L^p_t} \|\Lambda^{\frac{1}{2} - 2\epsilon} v\|_{L^2_x L^p_t} \lesssim \|u\|_{L^2_x H^\frac{1}{2} + 6} \|\Lambda^{\frac{1}{2} - 2\epsilon} v\|_{L^2_x L^p_t} \lesssim \|u\|_{H^{\frac{1}{2} + 6} + \frac{1}{2} + \epsilon} \|\Lambda^{\frac{1}{2} - 2\epsilon} v\|_{L^2_x L^p_t} \lesssim \|u\|_{H^{\frac{1}{2} - 2\epsilon} + 0} \|v\|_{H^{\frac{1}{2} + 6} + \frac{1}{2} + \epsilon}$$

for $s > \frac{5}{7}$.

4. Finally, we consider the case where $u$ and/or $v$ have frequency $\leq 1$. In this case we use (43) instead of (42). Because $\Gamma^2(u, v)$ is already handled we only have to estimate $uv$. Thus it suffices to show

$$\|\Lambda^{-\frac{1}{2} - 2\epsilon} \Lambda^{\frac{1}{2} + \epsilon} (uv)\|_{L^{14}_x L^{14}_t} \lesssim \|u\|_{H^{\frac{1}{2} + \frac{1}{2} + \epsilon}} \|v\|_{H^{\frac{1}{2} - 2\epsilon} + \frac{1}{2} + \epsilon}$$

We crudely estimate the left hand side by

$$\|\Lambda^{-\frac{1}{2} - 2\epsilon} \Lambda^{\frac{1}{2} + \epsilon} (uv)\|_{L^{14}_x L^{14}_t} \lesssim \|u\|_{H^{\frac{1}{2} + \frac{1}{2} + \epsilon}} \|v\|_{H^{\frac{1}{2} - 2\epsilon} + \frac{1}{2} + \epsilon}$$

by use of Proposition 3.2 with parameters $r = 28$, $q = \frac{50}{13}$, where $s_1$ and $s_2$ have to fulfill $s_1 + s_2 \geq \frac{5}{7}$ and $s_1, s_2 < \frac{5}{7} - \frac{\epsilon}{2}$. If $u$ has frequency $\leq 1$ choose $s_1 = \frac{5}{7} - \frac{\epsilon}{2}$, $s_2 = -\frac{\epsilon}{2} + \epsilon < s - 1$. The value of $s_1$ is irrelevant in view of the low frequency assumption. If $v$ has frequency $\leq 1$ choose $s_1 = \frac{5}{7} - \frac{\epsilon}{2}$, $s_2 = -\frac{\epsilon}{2} + \epsilon < s - 1$. The proof of (26) is now complete. \(\square\)

Proof of (21) and (22). We have to prove

$$\|\Lambda^{-\frac{1}{2} - 2\epsilon} Q(u, v)\|_{F^s} \lesssim \|\Lambda u\|_{F^s} \|\Lambda v\|_{H^{\frac{1}{2} - 2\epsilon} + \frac{1}{2} + \epsilon}$$

for $s = \frac{5}{7}$, which immediately implies the case $s > \frac{5}{7}$. We want to use (41).

A. The first part of the $F^s$-norm is handled as follows:

1. For the first term we reduce to the estimate

$$\|uv\|_{H^{\frac{1}{2} - 2\epsilon} + \frac{1}{2} + \epsilon} \lesssim \|u\|_{H^{\frac{1}{2} - 2\epsilon} + \frac{1}{2} + \epsilon} \|v\|_{H^{\frac{1}{2} - 2\epsilon} + \frac{1}{2} + \epsilon}$$

which follows by Prop. 3.3.

2. For the last term we reduce to

$$\|uv\|_{H^{\frac{1}{2} - 2\epsilon} + \frac{1}{2} + \epsilon} \lesssim \|u\|_{H^{\frac{1}{2} - 2\epsilon} + \frac{1}{2} + \epsilon} \|v\|_{H^{\frac{1}{2} - 2\epsilon} + \frac{1}{2} + \epsilon}$$

which also holds by Prop. 3.3.

3. For the second term we want to prove

$$\|uv\|_{H^{\frac{1}{2} - 2\epsilon} + \frac{1}{2} + \epsilon} \lesssim \|\Lambda^{\frac{1}{2} - 2\epsilon} \Lambda^{\frac{1}{2} + 2\epsilon} u\|_{F^s} \|v\|_{H^{\frac{1}{2} - 2\epsilon} + \frac{1}{2} + \epsilon}$$

(54)
Using the fractional Leibniz rule we obtain:

3.1. By Sobolev we have

\[ \| u \Lambda^{s-\frac{1}{2}-2\epsilon} v \|_{H^{0,-\frac{1}{2}+2\epsilon}} \leq \| u \Lambda^{s-\frac{1}{2}-2\epsilon} v \|_{L^{14}_t L^{14}_x} \]
\[ \leq \| u \|_{L^{14}_t L^{14}_x} \| \Lambda^{s-\frac{1}{2}-2\epsilon} v \|_{L^{\infty}_t L^2_x} \]
\[ \leq \| \Lambda^{\frac{1}{2}+\epsilon} u \|_{L^{14}_t L^{14}_x} \| v \|_{H^{0,-\frac{1}{2}+2\epsilon}} \]
\[ \leq \| \Lambda^{\frac{1}{2}-2\epsilon} \Lambda^{-\frac{1}{2}u} \|_{F^r} \| v \|_{H^{0,-\frac{1}{2}+2\epsilon}} \]

3.2.

\[ \| \Lambda^{s-\frac{1}{2}-2\epsilon} u v \|_{H^{0,-\frac{1}{2}+2\epsilon}} \leq \| \Lambda^{\frac{1}{2}+\epsilon} u \|_{L^{14}_t L^{14}_x} \| v \|_{L^{\infty}_t L^2_x} \]
\[ \leq \| \Lambda^{\frac{1}{2}+\epsilon} u \|_{L^{14}_t L^{14}_x} \| v \|_{H^{0,-\frac{1}{2}+2\epsilon}} \]
\[ \leq \| \Lambda^{\frac{1}{2}-2\epsilon} \Lambda^{-\frac{1}{2}u} \|_{F^r} \| v \|_{H^{0,-\frac{1}{2}+2\epsilon}} \]

where \( \frac{1}{q} = \frac{1}{2} - \frac{1}{14} + \frac{3}{2} \epsilon \), \( \frac{1}{r} = \frac{1}{4} - \frac{3}{2} \epsilon \) so that the Sobolev embeddings \( H^{s-\frac{1}{2}-2\epsilon} \hookrightarrow L^q \)

and \( H^{2s-14} \hookrightarrow L^r \) hold.

B. The second part of the \( F^s \)-norm is handled as follows:

1. The first term on the right hand side of (41) requires the estimate

\[ \| \Lambda^{-1} \Lambda^{-1+\epsilon_{\Lambda}^{\frac{1}{2}+2\epsilon}} \Lambda^{-\frac{1}{2}-2\epsilon} (\Lambda^{-\frac{1}{2}+2\epsilon} u \Lambda^{\frac{1}{2}-2\epsilon} v) \|_{L^{14}_t L^{14}_x} \]
\[ \lesssim \| \Lambda^s u \|_{H^{0,-\frac{1}{2}+2\epsilon}} \| \Lambda^s v \|_{H^{0,-\frac{1}{2}+2\epsilon}} \]

which reduces to

\[ \| \Lambda^{-\frac{1}{2}-\frac{7}{2}+3\epsilon_s} (uv) \|_{L^{14}_t L^{14}_x} \lesssim \| u \|_{H^{0,-\frac{1}{2}+2\epsilon}} \| v \|_{H^{0,-\frac{1}{2}+2\epsilon}} \]

This follows by Prop. 3.2 with parameters \( q = \frac{28}{13} \), \( r = 28 \), \( \sigma = \frac{11}{13} - 3 \epsilon \). This requires \( s_1, s_2 < \frac{13}{14} \) and \( s_1 + s_2 \geq \frac{11}{12} \). We choose \( s_1 = \frac{13}{14} - \epsilon < s + \frac{1}{2} - 2\epsilon \),

\( s_2 = \frac{1}{2} + \epsilon < s - \frac{1}{2} - 2\epsilon \).

2. The estimate for the second term reduces to

\[ \| \Lambda^{-\frac{1}{2}+\frac{7}{2}+2\epsilon} \Lambda^{-\frac{1}{2}+2\epsilon} (uv) \|_{L^{14}_t L^{14}_x} \lesssim \| \Lambda^{\frac{1}{2}-2\epsilon} \Lambda^{-\frac{1}{2}+2\epsilon} u \|_{F^r} \| v \|_{H^{0,-\frac{1}{2}+2\epsilon}} \]

where we used \( \Lambda^{2\epsilon} u \lesssim \Lambda^{2\epsilon} u \). By Prop. 3.9 we obtain the following bound for the left hand side: \( \| \Lambda^{\frac{1}{2}+2\epsilon} (uv) \|_{L^{14}_t L^{14}_x} \). Using the fractional Leibniz rule we estimate by Prop. 3.5 and Prop. 3.7:

\[ \| (\Lambda^{\frac{1}{2}+2\epsilon} u) v \|_{L^{14}_t L^{14}_x} \lesssim \| \Lambda^{\frac{1}{2}+2\epsilon} u \|_{L^{14}_t L^{14}_x} \| v \|_{L^{14}_t L^{14}_x} \]
\[ \lesssim \| \Lambda^{\frac{1}{2}-2\epsilon} \Lambda^{-\frac{1}{2}u} \|_{F^r} \| v \|_{H^{0,-\frac{1}{2}+2\epsilon}} \]

where we used Sobolev’s embedding \( H^{s-\frac{1}{2}-2\epsilon} \hookrightarrow L^2 \) for \( s > \frac{1}{7} \). Moreover

\[ \| u \Lambda^{\frac{1}{2}+2\epsilon} v \|_{L^{14}_t L^{14}_x} \lesssim \| u \|_{L^{14}_t L^{14}_x} \| \Lambda^{\frac{1}{2}+2\epsilon} v \|_{L^{14}_t L^{14}_x} \]
\[ \lesssim \| \Lambda^{\frac{1}{2}-2\epsilon} \Lambda^{-\frac{1}{2}+2\epsilon} u \|_{F^r} \| v \|_{H^{0,-\frac{1}{2}+2\epsilon}} \]

3. The last term on the right hand side of (41) is modified as follows:

\[ \Lambda^{\frac{1}{2}-2\epsilon} (\Lambda^{\frac{1}{2}+2\epsilon} u \Lambda^{\frac{1}{2}+2\epsilon} \Lambda^{\frac{1}{2}+2\epsilon} v) = \Lambda^s u \Lambda^{\frac{1}{2}+2\epsilon} \Lambda^{\frac{1}{2}-2\epsilon} v + \Lambda^{\frac{1}{2}+2\epsilon} u \Lambda^s \Lambda^{\frac{1}{2}-2\epsilon} v \]

We first consider the second term and reduce to
\[ \| \Lambda^{-1} - r + 4r \Lambda^{-1} - \epsilon (uv) \|_{L^2_t L^2_x} \lesssim \| u \|_{H^r - \frac{3}{2} + 2r} \| v \|_{H^s - \frac{3}{2} + \epsilon} . \]
This is essentially identical with (50).

Finally we reduce the first term to
\[ \| \Lambda^{-1} - r + 4r \Lambda^{-1} - \epsilon (uv) \|_{L^1_t L^2_x} \lesssim \| u \|_{H^r - \frac{5}{2} + 2r} \| v \|_{H^s - \frac{1}{2} + 2r} . \]
By Prop. 3.9 the left hand side is bounded by \( \| \Lambda^{-1} - r + 4r (uv) \|_{L^1_t L^2_x} \). Next we prove for a suitable \( p \):
\[ \| uv \|_{H^t - \frac{3}{2} + \epsilon} \lesssim \| u \|_{H^r - \frac{3}{2} + \epsilon} \| v \|_{H^s - \frac{3}{2} + \epsilon} , \]
which by duality is equivalent to
\[ \| uv \|_{H^t - \frac{3}{2} + \epsilon} \lesssim \| v \|_{H^r - \frac{3}{2} + \epsilon} \| w \|_{H^s - \frac{3}{2} + \epsilon} . \]
We use the fractional Leibniz rule and obtain
\[ \| (\Lambda^\alpha v) w \|_{L^p} \lesssim \| \Lambda^\alpha v \|_{L^2} \| w \|_{L^p} \lesssim \| v \|_{H^r} \| w \|_{H^t - \frac{3}{2} + \epsilon} , \]
where \( \frac{1}{p} = \frac{1}{2} - \frac{3}{4} + \frac{3}{4} \epsilon \), \( \frac{1}{p} = \frac{6}{7} + \frac{3}{4} \epsilon \) and \( \frac{1}{p} = \frac{1}{2} - \frac{3}{4} \epsilon \), so that by Sobolev \( H^\frac{3}{2} - 4\epsilon \mapsto L^q \).
Moreover
\[ \| v \Lambda^\alpha w \|_{L^p} \lesssim \| v \|_{L^1} \| \Lambda^\alpha w \|_{L^2} , \]
where \( \frac{1}{q} = \frac{1}{2} - \frac{1}{4} + \frac{1}{4} \epsilon \), \( \frac{1}{q} = \frac{3}{2} + \frac{3}{4} \epsilon \), so that \( H^\frac{3}{2} \mapsto L^1 \) and \( H^\frac{3}{2} - 4\epsilon \mapsto H^\frac{3}{2} - r^2 \).
Therefore we obtain by Cor. 3.1:
\[ \| \Lambda^{-1} - r + 4r (uv) \|_{L^1_t L^2_x} \lesssim \| u \|_{L^2_t H^s - \frac{3}{2} + \epsilon} \| v \|_{L^2_t H^s_{\frac{1}{2}}} \]
\[ \lesssim \| u \|_{H^{s - \frac{1}{2} + 2r} - \frac{1}{2} + 2r} \| v \|_{H^{s - \frac{1}{2} + 2r} - \frac{1}{2} + 2r} \]
\[ \lesssim \| u \|_{H^{s - \frac{1}{2} + 2r} - \frac{1}{2} + 2r} \| v \|_{H^{s - \frac{1}{2} + 2r} - \frac{1}{2} + 2r} , \]
for \( s > \frac{3}{7} \), as one easily checks.

**Proof of (23).** We may reduce to
\[ \| \Lambda_-^{-1} \Lambda^s v \|_{H^{\frac{3}{2} - r} + \epsilon} \lesssim \| \Lambda v \|_{F^r} \| \Lambda^s v \|_{H^{s - \frac{1}{2} + \epsilon}} . \]
Now we use (41) and estimate the three terms as follows:
1. The estimate for the first term is reduced to (using the trivial estimate \( \Lambda^s u \lesssim \Lambda^s u \)):
\[ \| uv \|_{H^{\frac{3}{2} - r} + 2r} \lesssim \| u \|_{H^{\frac{3}{2} - r} + 2r} \| v \|_{H^{\frac{3}{2} - r} + 2r} , \]
which follows from Prop. 3.3 using our assumptions \( s > \frac{3}{7} \), \( r > -\frac{1}{4} \).
2. The estimate for the last term reduces to
\[ \| uv \|_{H^{\frac{3}{2} - r} + 2r} \lesssim \| u \|_{H^{\frac{3}{2} - r} + 2r} \| v \|_{H^{\frac{3}{2} - r} + 2r} , \]
which is also a consequence of Prop. 3.3 under our assumption \( 2s - r > \frac{3}{2} \).
3. The second term requires
\[ \| uv \|_{H^{\frac{3}{2} - r} + 2r} \lesssim \| u \|_{H^{\frac{3}{2} - r} + 2r} \| v \|_{H^{\frac{3}{2} - r} + 2r} , \]
We first consider the case \( r \leq \frac{1}{2} - 2\epsilon \). By duality we have to show
\[ \| uv \|_{H^{\frac{3}{2} - r} - 2r} \lesssim \| u \|_{H^{\frac{3}{2} - r} - 2r} \| v \|_{H^{\frac{3}{2} - r} - 2r} . \]
By the fractional Leibniz rule we have to consider two terms.

3.1. \[
\| (\Lambda^{\frac{1}{2}-r-2\epsilon \tau} u) w \|_{H^{0,-\frac{1}{2}-2\epsilon}} \lesssim \| u \|_{H^{r+\frac{1}{2},0}} \| w \|_{H^{\frac{1}{2}-r-2\epsilon,\frac{1}{2}-2\epsilon}},
\]
which follows from Prop. 3.3, because \( s_0 + s_1 + s_2 = -s - r + 1 - 4\epsilon > \frac{7}{4} - \frac{r}{2} > \frac{3}{2} \) under our assumption \( 2s - r > \frac{3}{2} \) and \( r \leq \frac{1}{2} - 2\epsilon \).

3.2. \[
\| u (\Lambda^{\frac{1}{2}-r-2\epsilon \tau} w) \|_{H^{0,-\frac{1}{2}-2\epsilon}} \lesssim \| u \Lambda^{\frac{1}{2}-r-2\epsilon \tau} w \|_{L^1_t L^2_x} \lesssim \| u \|_{L^1_t H^{\frac{1}{2}}_x} \| u \|_{L^1_t H^{\frac{1}{2}-r-2\epsilon,\frac{1}{2}-2\epsilon}} \lesssim \| \Lambda^{\frac{1}{2}-r-2\epsilon \tau} u \|_{F^s} \| w \|_{H^{\frac{1}{2}-r-2\epsilon,\frac{1}{2}-2\epsilon}}.
\]

Now consider the case \( r \geq \frac{1}{2} - 2\epsilon \), which is treated by the fractional Leibniz rule as follows:

\[
\| uv \|_{H^{r-\frac{1}{2}+2\epsilon, -\frac{1}{2}-2\epsilon}} \lesssim \| (\Lambda^{\frac{1}{2}+2\epsilon \tau} u) v \|_{H^{r,0,-\frac{1}{2}-2\epsilon}} + \| u (\Lambda^{\frac{1}{2}+2\epsilon \tau} v) \|_{H^{r,0,-\frac{1}{2}-2\epsilon}} \lesssim \| u \|_{H^{r+\frac{1}{2},0}} \| v \|_{H^{r-\frac{1}{2}+2\epsilon, \frac{1}{2}+2\epsilon}} + \| u \|_{L^1_t L^2_x} \| v \|_{H^{r-\frac{1}{2}+2\epsilon, \frac{1}{2}+2\epsilon}}.
\]

Here we used Prop. 3.3 with parameters \( s_0 = 0, s_1 = -s - r + 1 - 4\epsilon, s_2 = r - \frac{1}{2} - 2\epsilon, b_0 = \frac{1}{2} + 2\epsilon, b_1 = 0, b_2 = \frac{1}{2} + 2\epsilon, \) so that \( s_0 + s_1 + s_2 = -s + \frac{1}{2} - 6\epsilon > 1 \) and \((s_0 + s_1 + s_2) + s_1 + s_2 = -s + r - 8\epsilon > \frac{3}{4} + \frac{3}{2} + 4\epsilon > \frac{3}{2} \), the latter by our assumption \( 2s - r > \frac{3}{2} \) and \( r \leq \frac{1}{2} - 2\epsilon \).

**Proof of (27) and (28).** A. The first part of the \( F^s \)-norm in the case of (27) reduces to the following estimate

\[
\| uv \|_{H^{s-1,\frac{1}{2}+\epsilon}} \lesssim \| u \|_{H^{r,\frac{1}{2}+\epsilon}} \| v \|_{H^{r+1,\frac{1}{2}+\epsilon}},
\]
which easily follows from the Sobolev multiplication law (Prop. 3.4), because \((1 - s) + s + (s + 1) = s + 2 > \frac{3}{2} \) and (28) is treated in the same way.

B. In the case of (27) for the second part of the \( F^s \)-norm we use Prop. 3.9 and Prop. 3.4 to obtain:

\[
\| \Lambda^{\frac{1}{2}+2\epsilon \tau} u \|_{L^1_t H^{\frac{1}{2}+\epsilon}} \lesssim \| \Lambda^{\frac{1}{2}+2\epsilon \tau} u \|_{L^1_t L^2_x} \lesssim \| u \|_{L^2_t H^{\frac{1}{2}+\epsilon}} \| v \|_{L^2_t H^{\frac{1}{2}+\epsilon}} \lesssim \| u \|_{H^{r+2,\frac{1}{2}+\epsilon}} \| v \|_{H^{r+1,\frac{1}{2}+\epsilon}}.
\]

In the same way in the case of (28) we obtain that the left hand side of the previous estimate is bounded by \( \| u \|_{H^{r+2,\frac{1}{2}+\epsilon}} \| v \|_{H^{r+1,\frac{1}{2}+\epsilon}} \) as desired.

**Proof of (30) and (31).** (30) reduces to the following estimate

\[
\| uv \|_{H^{r-1,\frac{1}{2}+\epsilon}} \lesssim \| u \|_{H^{r+2,\frac{1}{2}+\epsilon}} \| v \|_{H^{r-1,\frac{1}{2}+\epsilon}},
\]
which follows from Prop. 3.4, because \( s + 2 > \frac{3}{2} \). Similarly (31) reduces to

\[
\| uv \|_{H^{r-1,\frac{1}{2}+\epsilon}} \lesssim \| u \|_{H^{r+1,\frac{1}{2}+\epsilon}} \| v \|_{H^{r-1,\frac{1}{2}+\epsilon}},
\]
which also holds by Sobolev, where we use our assumption \( 2s - r + 1 > \frac{5}{2} \).

**Proof of (32).** A. The estimate for the first part of the \( F^s \)-norm reduces to

\[
\| uvw \|_{H^{s-1,\frac{1}{2}+2\epsilon}} \lesssim \| u \|_{H^{r+1,\frac{1}{2}+\epsilon}} \| vw \|_{H^{r,0}} \lesssim \| u \|_{H^{r+1,\frac{1}{2}+\epsilon}} \| v \|_{H^{r,\frac{1}{2}+\epsilon}} \| w \|_{H^{r,\frac{1}{2}+\epsilon}},
\]
which follows from Prop. 3.3, where we use $2r - s > -1$ and $r \geq s - 1$ for the first step and $2s - r > \frac{3}{2}$ for the second step.

**B. The estimate for the second part of the $F^s$-norm reduces to**

$$
\|\Lambda^{-\frac{3}{4}+5\epsilon} \Lambda^{-\frac{1}{4}+\epsilon} (uvw)\|_{L^1_t L^2_x} \lesssim \|\Lambda^{-\frac{3}{4}+7\epsilon} (uvw)\|_{L^1_t L^2_x} \lesssim \|uvw\|_{L^1_t L^2_x},
$$

$$
\lesssim \|uv\|_{L^\infty_t L^2_x} \|vw\|_{L^1_t L^2_x} \lesssim \|uv\|_{H^{r-\frac{1}{2}+\epsilon}} \|v\|_{L^{2+} L^\infty_x} \|w\|_{L^{2+} L^\infty_x},
$$

which we obtain for $s > \frac{7}{r}$ and $r > \frac{1}{7}$ by Prop. 3.9, Sobolev and Prop. 3.1, where

$$
\frac{1}{p} = \frac{1}{2} + \frac{1}{7} - \frac{7}{7} \epsilon, \frac{1}{r} = \frac{1}{2} - \frac{2}{7} \epsilon \quad \text{and} \quad \frac{1}{q} = \frac{3}{7} - \frac{7}{7} \epsilon,
$$

so that $H^{\frac{3}{7} - 7\epsilon} \to L^2$ and $H^{\frac{3}{7}} \to L^r$.

**Proof of (33). A.** We may reduce to

$$
\|u \Lambda^{-1} (vw)\|_{H^{r-\frac{1}{2}+2\epsilon}} \lesssim \|u\|_{H^{r-\frac{1}{2}+2\epsilon}} \|\Lambda^{-1} (vw)\|_{H^{r-\frac{1}{2}+2\epsilon}},
$$

by our assumption that $2r - s > -1$, where we use Prop. 3.3 twice.

**B.** For the second part of the $F^s$-norm we use Prop. 3.4 and obtain:

$$
\|\Lambda^{-\frac{3}{4}+5\epsilon} \Lambda^{-\frac{1}{4}+\epsilon} (uw)\|_{L^1_t L^2_x} \lesssim \|\Lambda^{-\frac{3}{4}+7\epsilon} (uw)\|_{L^1_t L^2_x} \lesssim \|uw\|_{H^{r-\frac{1}{2}+\epsilon}} \|v\|_{L^{2+} L^\infty_x} \|w\|_{L^{2+} L^\infty_x}.
$$

We have to estimate $\|\Lambda \hat{v} (vw)\|_{L^1_t L^2_x}$, which by symmetry and the fractional Leibniz rule reduces to $\|\Lambda \hat{v} (vw)\|_{L^1_t L^2_x}$. By Prop. 3.1 we obtain

$$
\|\Lambda \hat{v} (vw)\|_{L^1_t L^2_x} \lesssim \|\Lambda \hat{v} (vw)\|_{L^1_t L^2_x} \lesssim \|v\|_{H^{\frac{1}{2}+\epsilon}} \|w\|_{H^{\frac{1}{2}+\epsilon}} \lesssim \|v\|_{H^{r-\frac{1}{2}+\epsilon}} \|w\|_{H^{r-\frac{1}{2}+\epsilon}}.
$$

**Proof of (35). A.** It suffices to consider the case $s = \frac{5}{7}$. We easily obtain the desired estimate by Prop. 3.3:

$$
\|uvw\|_{H^{r-\frac{1}{2}+2\epsilon}} \lesssim \|u\|_{H^{r-\frac{1}{2}+2\epsilon}} \|vw\|_{H^{r-\frac{1}{2}+2\epsilon}} \lesssim \|u\|_{H^{r-\frac{1}{2}+2\epsilon}} \|v\|_{H^{r-\frac{1}{2}+2\epsilon}} \|w\|_{H^{r-\frac{1}{2}+2\epsilon}}.
$$

**B.** For the second part of the $F^s$-norm we use Prop. 3.9 and obtain

$$
\|\Lambda^{-\frac{3}{4}+5\epsilon} \Lambda^{-\frac{1}{4}+\epsilon} (uvw)\|_{L^1_t L^2_x} \lesssim \|\Lambda^{-\frac{3}{4}+7\epsilon} (uvw)\|_{L^1_t L^2_x} \lesssim \|u\|_{L^\infty_t L^2_x} \|v\|_{L^2 L^\infty_x} \|w\|_{L^{2+} L^\infty_x} \lesssim \|u\|_{H^{r-rac{1}{2}+\epsilon}} \|v\|_{H^{r-rac{1}{2}+\epsilon}} \|w\|_{H^{r-rac{1}{2}+\epsilon}},
$$

where $\frac{1}{p} = \frac{1}{2} + \frac{1}{7} - \frac{7}{7} \epsilon$, $\frac{1}{r} = \frac{1}{2} - \frac{2}{7} \epsilon$, $\frac{1}{q} = \frac{3}{7} - \frac{7}{7} \epsilon$, which implies by Sobolev $H^{\frac{3}{7} - 7\epsilon} \to L^q$, and Prop. 3.1 implies $\|v\|_{L^2 L^\infty_x} \lesssim \|v\|_{H^{\frac{3}{7}+\epsilon}}$.

**Proof of (36).** We obtain by Prop. 3.3:

$$
\|uvw\|_{H^{r-\frac{1}{2}+2\epsilon}} \lesssim \|uvw\|_{H^{r-\frac{1}{2}+2\epsilon}} \lesssim \|u\|_{H^{r-\frac{1}{2}+\epsilon}} \|v\|_{H^{r-\frac{1}{2}+\epsilon}} \|w\|_{H^{r-\frac{1}{2}+\epsilon}}.
$$
Proof of (34). A. For the first part of the $F^s$-norm we have to show
\[ \| \Lambda^{-1}(uv) wz \|_{H^{s-1, \frac{1}{2} + \epsilon}} \lesssim \| u \|_{H^{r, \frac{3}{2} + \epsilon}} \| v \|_{H^{r, \frac{3}{2} + \epsilon}} \| w \|_{H^{r, \frac{3}{2} + \epsilon}} \| z \|_{H^{r, \frac{3}{2} + \epsilon}}. \]
It suffices to consider the minimal value $s = \frac{5}{7}$, which by Proposition 3.3 can be estimated as follows:
\[ \| \Lambda^{-1}(uv) wz \|_{H^{s-1, \frac{1}{2} + \epsilon}} \lesssim \| u \|_{H^{r, \frac{3}{2} + \epsilon}} \| v \|_{H^{r, \frac{3}{2} + \epsilon}} \| w \|_{H^{r, \frac{3}{2} + \epsilon}} \| z \|_{H^{r, \frac{3}{2} + \epsilon}}. \]

B. We obtain by Prop. 3.9
\[ \| \Lambda^{-1-\frac{3}{7} + 5\epsilon} \Lambda^{-\frac{3}{7} + \epsilon}(\Lambda^{-1}(uv) wz)) \|_{L^1_t L^2_x} \lesssim \| \Lambda^{-\frac{3}{7} + 7\epsilon}(uvw) \|_{L^1_t L^2_x} \]
\[ \lesssim \| uv \|_{L^\infty_t L^2_x} \| wz \|_{L^1_t L^2_x} \lesssim \| uv \|_{L^\infty_t H^{s, \frac{3}{2} + \epsilon}} \| wz \|_{L^1_t H^{s, \frac{3}{2} + \epsilon}} \| z \|_{L^1_t L^2_x} \]
\[ \lesssim \| u \|_{H^{r, \frac{3}{2} + \epsilon}} \| v \|_{H^{r, \frac{3}{2} + \epsilon}} \| w \|_{H^{r, \frac{3}{2} + \epsilon}} \| z \|_{H^{r, \frac{3}{2} + \epsilon}}, \]
where $\frac{1}{p} = \frac{1}{2} + \frac{1}{14} + \frac{1}{\epsilon}$, $\frac{1}{q} = \frac{1}{6} + \frac{1}{14}$, $\frac{1}{s} = \frac{7}{8} - \frac{1}{7} \epsilon$, so that by Sobolev $H^{s, \epsilon} \hookrightarrow L^q$, and by Prop. 3.1:
\[ \| wz \|_{L^2_t L^2_x} \lesssim \| w \|_{H^{r, \frac{3}{2} + \epsilon}}. \]

Proof of (37). We have to show
\[ \| uvwz \|_{H^{r-1, \frac{1}{2} + \epsilon}} \lesssim \| u \|_{H^{r, \frac{3}{2} + \epsilon}} \| v \|_{H^{r, \frac{3}{2} + \epsilon}} \| w \|_{H^{r, \frac{3}{2} + \epsilon}} \| z \|_{H^{r, \frac{3}{2} + \epsilon}}. \]
By our assumption $2s - r > \frac{3}{2}$ the left hand side is bounded by the term $\| uvwz \|_{H^{2s - r, \frac{1}{2} + \epsilon}}$. It suffices to prove the remaining estimate for the (minimal) value $s = \frac{5}{7}$. By Proposition 3.3 we obtain
\[ \| uvwz \|_{H^{r-\frac{14}{7}, \frac{1}{2} + \epsilon}} \lesssim \| uv \|_{H^{r, \frac{5}{7} + \frac{1}{7} \epsilon}} \| wz \|_{H^{r, \frac{5}{7} + \frac{1}{7} \epsilon}} \]
\[ \leq \| u \|_{H^{r, \frac{5}{7} + \frac{1}{7} \epsilon}} \| v \|_{H^{r, \frac{5}{7} + \frac{1}{7} \epsilon}} \| w \|_{H^{r, \frac{5}{7} + \frac{1}{7} \epsilon}} \| z \|_{H^{r, \frac{3}{2} + \epsilon}}. \]

Remark 5.1. If one would try to use only $H^{s, \frac{1}{2} + \epsilon}$ spaces as solution spaces as Selberg-Tesfahun [14], by modifying in Definition 3.2 the $F^s$-norm appropriately by cancelling its second term, our proof of (26) would fail, because the estimate
\[ \| uv \|_{H^{s, \frac{1}{2} + \epsilon}} \lesssim \| u \|_{H^{r, \frac{3}{2} + \epsilon}} \| v \|_{H^{r, \frac{3}{2} + \epsilon}} \]
only holds for $s > 1$ (cf. (47)). This easily follows from [1], where necessary and sufficient conditions are given for such an estimate. The same remark applies to the proof of (21) and (22) (cf. (54)). The proof of (23) also fails for $s = \frac{5}{7} + \delta$, $r = -\frac{1}{7} + \delta$ for a sufficient small $\delta > 0$. Tesfahun [16] improved the result of [14] by replacing $H^{s, \frac{1}{2} + \epsilon}$ by $H^{s, \frac{1}{2} + \epsilon}$ and could treat the case $s = \frac{6}{7} + \delta$, $r = -\frac{1}{7} + \delta$, but in order to obtain our result $s = \frac{5}{7} + \delta$, $r = -\frac{1}{7} + \frac{1}{14} + \delta$ a manipulation of the $H^{s, \epsilon}$-spaces seems not to be sufficient.

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