Putting an Edge to the Poisson Bracket

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Abstract

We consider a general formalism for treating a Hamiltonian (canonical) field theory with a spatial boundary. In this formalism essentially all functionals are differentiable from the very beginning and hence no improvement terms are needed. We introduce a new Poisson bracket which differs from the usual “bulk” Poisson bracket with a boundary term and show that the Jacobi identity is satisfied. The result is geometrized on an abstract world volume manifold. The method is suitable for studying systems with a spatial edge like the ones often considered in Chern-Simons theory and General Relativity. Finally, we discuss how the boundary terms may be related to the time ordering when quantizing.

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1 Introduction

Seen in the light of the renewed interest for theories where the edge plays a prominent role, cf. Maldacena’s Conjecture [1], Carlip’s and Strominger’s different approaches for the microscopic counting of states on the (inner or outer) edge of the world [2], ’t Hooft’s and Susskind’s principle of holography [3], but also Chern-Simon theories [4] and General Relativity [5, 6] in general, there is strikingly few papers devoted to develop the general formalism for Hamiltonian (canonical) field theory in the presence of a spatial boundary. Here we are thinking of the fact that the usual equal-time Poisson bracket

\[
\{ F(t), G(t) \}_0 = \int_\Sigma d^d x \frac{\delta F(t)}{\delta \phi^A(x, t)} \omega^{AB} \frac{\delta G(t)}{\delta \phi^B(x, t)} \tag{1.1}
\]

fails to satisfy the Jacobi identity

\[
\sum \text{cycl. } F, G, H \{ F(t), \{ G(t), H(t) \} \} = 0 , \tag{1.2}
\]

when space \( \Sigma \) has a boundary \( \partial \Sigma \neq \emptyset \), at least if we apply the usual Euler-Lagrange formula for the functional derivatives in (1.1). (We shall show below how to ensure the differentiability of the functionals by using the notion of higher functional derivatives, so that the above violation is a fully legitimate problem to raise.) The failure of the Jacobi identity can be seen even in the most simple toy examples which have a non-trivial boundary. An equivalent manifestation of this fact is that functional derivatives cease to commute when a spatial boundary is present [7].

The most common example of the above phenomenon is the usual \( d \)-dimensional flat space \( \Sigma = \mathbb{R}^d \). Here the spatial infinity \( |x| = \infty \) constitute a boundary for the space. This statement can be made precise by a so-called one-point compactification.

An often used cure is to impose conditions on the dynamical fields \( \phi^A(x, t) \) at the boundary which are consistent with the time-evolution. However, that approach may exclude interesting topological questions, such as solitonic field configurations. Our main goal in this paper is to see how far we can get without imposing boundary conditions.

On the other hand, to calm the reader who perhaps finds these facts strange, let us mention that if “there is no boundary” (for instance, think of a torus, or equivalently periodic boundary conditions, or even vanishing boundary conditions), integrations by part inside the spatial integral does not produce boundary contributions, and the Jacobi identity for the above “bulk” Poisson bracket can be demonstrated after some straightforward manipulations.

The paper is organized as follows: In next Subsection 1.2, we present a new Poisson bracket. Thereafter, we give some further introductional remarks about differentiability and improvement terms. In Section 2 we give a manifest formulation of the new Poisson bracket, discuss the higher Euler-Lagrange derivatives and develop a generator method (Section 3) which in particular is suitable of handling the arithmetic manipulations involved in the proof of the Jacobi identity. The proof itself is postponed to an Appendix B. After that we turn to the questions that naturally arise with the existence of the new Poisson Bracket. Can it be given a geometrically covariant form (Section 4)? How does the boundary affect the Hamiltonian dynamics (Section 5)? To answer the last question, we have included a technical Section 5 to explain some supplementary formalism. Finally, we discuss the role of time order in connection with boundary terms (Section 6).
\section*{1.1 Notation}

The \( \phi^A(x, t), A = 1, \ldots, 2N, \) denote the (bosonic) coordinate and momenta fields of the phase space. (Generalizations to fermionic variables are straightforward.) The non-degenerate symplectic structure \( \omega^{AB} \) is for simplicity taken to be ultra-local and constant. This vast simplification is the most interesting case for applications and already contain as we shall see an interesting structure worth analyzing by itself. Needless to say that a general field transformation \( \phi \rightarrow \phi' \) will violate such assumption. A manifestly covariant formalism under general field transformation is out of the scope of the present work. Also we assume, to avoid technicalities, that the \( d \)-dimensional space \( \Sigma \) can be covered by a single coordinate patch with a flat measure. (We shall relax these assumptions in Section 4.) Furthermore, in agreement with the spirit of the Hamiltonian (canonical) formalism, we shall assume that the functionals \( F(t) \) do not contain time derivatives \( (\partial_t)^k \phi^A(x, t) \) of the dynamical field variables \( \phi^A(x, t) \). So if one for instance is interested in a (total) time derivative \( G(t) = \partial_t F(t) \), where \( F(t) \) contains no time derivatives, one should study \( F(t) \) instead of \( G(t) \), etc. Finally, we assume that there is no temporal boundaries. This being said, time \( t \) plays no active role, and we can suppress the time variable \( t \) in what follows.

\section*{1.2 New Poisson Bracket}

As mentioned in the introduction the bulk Poisson bracket \((1.1)\) does not satisfy the Jacobi identity. (A purist would perhaps then claim that \((1.1)\) does not qualify for being called a Poisson bracket at all! However, we shall continue to call it a Poisson bracket.) Knowing that the failure of the Poisson bracket \((1.1)\) is at most a total derivative term, it is natural to speculate whether one can add a boundary contribution \( B(F, G) \) to this bulk Poisson bracket,

\[
\{F, G\} = \{F, G\}_{(0)} + B(F, G),
\]

so that the Jacobi identity is satisfied even in the presence of a boundary. In fact, this is so. We find that the following boundary term

\[
B(F, G) = \sum_{k \neq 0} \int_\Sigma d^d x \, \partial^k \left[ \frac{\delta F}{\delta \phi^A(x)} \omega^{AB} \frac{\delta G}{\delta \phi^B(x)} \right] - (F \leftrightarrow G)
\]

does the job. We have employed a multi-index notation: For instance the index

\[
k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d \backslash \{(0, \ldots, 0)\}, \quad \mathbb{N}_0 \equiv \{0, 1, 2, \ldots\},
\]

runs over the \( d \)-dimensional non-negative integers (except the origo), and

\[
\partial^k = \partial_1^{k_1} \cdots \partial_d^{k_d}, \quad \partial_i \equiv \frac{\partial}{\partial x^i}.
\]

(The main features of the construction are already present in the dimension \( d = 1 \) case. A first-time reader will not miss the essential points by treating the multi-index \( k \) as an integer, \( i.e. \) letting \( d = 1 \).) More importantly, in a perhaps conceptionally dangerous – but in practice convenient – notation, the

\[
\frac{\delta F}{\delta \phi^A(x)}(x), \quad k \in \mathbb{N}_0^d,
\]

denote the higher functional derivatives of \( F \) of order \( k \). Here

\[
\frac{\delta F}{\delta \phi^A(x)}(x) \equiv \frac{\delta F}{\delta \phi^A(x)}(x)
\]

\[(1.8)\]
is the usual functional derivative. In general, the higher functional derivatives are required to satisfy

\[ \delta F = \int_{\Sigma} d^d x \sum_{k=0}^{\infty} \partial^k \left[ \frac{\delta F}{\delta \phi^A(x)} \delta \phi^A(x) \right] \]  

(1.9)

for arbitrary infinitesimal variations \( \phi^A(x) \rightarrow \phi^A(x) + \delta \phi^A(x) \). In particular, the variations \( \delta \phi^A(x) \) are not restricted at the boundary.

A quick estimate shows that if the entries contain spatial derivatives \( \partial_i \phi^A(x) \) of the dynamical field variables \( \phi^A(x) \) to order \( N \), then the full Poisson Bracket (1.3) contains spatial derivatives up to order \( 3N \). Hence the algebra \( A_{N=0} \) of functionals with no spatial derivatives closes on the new Poisson bracket. However, physical interesting theories usually have functionals with up to \( N = 1 \) spatial derivative, \( i.e. \) they belong to the class \( A_{N=1} \). This class \( A_{N=1} \) of functionals does not close on the new nor on the bulk Poisson bracket, and this is the main reason why we are force into summing over the index lattice (1.5).

The idea of adding surface contribution is far from new. In a seminal work Regge and Teitelboim [5] emphasized the importance of having a boundary term in the action of canonical general relativity. However, they did strangely enough not add surface contributions to the Poisson bracket. Lewis, Marsden, Montgomery and Ratiu [8] considered a truncated version of (1.4) where only the terms with \( |k| = 1 \) are present. Because they didn’t add the \( |k| > 1 \) terms, they needed to impose additional boundary conditions to ensure the Jacobi identity. The first successful attempt to remedy this was conducted in 1993 by Soloviev [9]. In our notation, his solution [9, formula(3.4)] reads

\[ \{ F, G \} = \sum_{k, \ell=0}^{\infty} \int_{\Sigma} d^d x \partial^k+\ell \left[ \frac{\delta F}{\delta \phi^A(x)} \omega_{AB} \frac{\delta G}{\delta \phi^B(x)} \right] . \]  

(1.10)

It is easy see that his bracket is different from our solution (1.4). It would be interesting to know whether his bracket supports a manifest formulation (see Subsection 2.2 below), or more generally, if it is independent of the representative for the higher functional derivatives. After the first preprint of this paper appeared, Soloviev has made a comparison [10] of the two solutions (1.4) and (1.10).

We shall in this paper concentrate fully on the solution (1.4).

### 1.3 Review of Differentiability and Improvement Terms

The classical point of view [5] on the problem with the Jacobi identity has been to view this as not so much a problem of the Poisson bracket itself, but rather that functional derivatives in general in the case of a non-trivial boundary \( \partial \Sigma \neq \emptyset \) are ill-defined when the functional, say \( F \), depends on the spatial derivatives \( \partial_i \phi^A(x) \) of the dynamical field variables \( \phi^A(x) \). In this case there does not always exist functions \( f_A(x) \), such that the change in the functional \( F \) is fully described by

\[ \delta F = \int_{\Sigma} d^d x \ f_A(x) \ \delta \phi^A(x) , \]  

(1.11)

for an arbitrary infinitesimal variation \( \delta \phi^A(x) \). (Of course in the affirmative case, we usually call \( f_A(x) \) the functional derivatives of \( F \).)

**Example.** Consider an interval \( \Sigma = [a, b] \) and the toy functional

\[ F = \int_{a}^{b} dx \ \partial^2 \phi(x) = \partial \phi(x)|_{x=b} . \]  

(1.12)
The variation $\delta F$ can be identically rewritten as

$$\delta F = \int_{\Sigma} (\delta \Sigma(x, b) - \delta \Sigma(x, a)) \partial \delta \phi(x)$$  \hspace{1cm} (1.13)

To bring $\delta F$ of the form (1.11), one is tempted to do an integration by part. But this does not help us, partly because of the total derivative term does not vanish on the boundary. (Remember that we do not want to impose boundary conditions on the field $\phi$, cf. the Introduction.) In fact, it is not apriori clear what should be meant (viewed separately) by any of the $2 \times 2$ terms arising from such an integration by part. So the traditional functional derivative is ill-defined. These difficulties should be compared with the ease that the same variation $\delta F$ is described by the higher functional derivatives (1.9)

$$\frac{\delta F}{\delta \phi^{(k)}(x)} = \delta^2_k.$$  \hspace{1cm} (1.14)

This is the general picture: A traditional functional derivative is often ill-defined or have a very singular behavior at the boundary. (And at this point we haven’t even touched the problems of building up the Poisson bracket (1.11) out of two highly singular functional derivatives, i.e. multiplying two delta distributions together, both of which typically have support on the boundary. We should also mention that authors for such reasons often additionally require the functional derivatives $f_A$ in Eq. (1.11) to be continuous functions.) In any case, this makes the traditional definition (1.11) not very suitable for systems with a boundary.

Let us mention an important algebra $A_0$ of functionals, that are differentiable in this traditional sense (1.11), namely those functionals that do not depend on the spatial derivatives $\partial_i \phi^A(x)$ of the dynamical field variables $\phi^A(x)$. They form an algebra under the Poisson bracket. (In this algebra $A_0$ the bulk Poisson bracket (1.1) and the full Poisson bracket (1.3) coincide.)

The traditional cure in case of an ill-defined derivative, is to improve the functional $F \sim F_{\text{impr}}$ with a boundary term, a so-called improvement term, so that the derivative becomes smooth and well-defined. The drawback is of course that we are then studying a different functional than we originally started out with! Typically, one meets the following preparation of an observable in the literature: A function $f(x) = f(\partial^k \phi(x), x)$ is smeared with a “test function” $\eta(x)$ into a functional of the type $F[\eta] = \int_{\Sigma} \delta^d x \; f(x) \; \eta(x)$. It is then improved $F[\eta] \sim F_{\text{impr}}[\eta] \in A_0$ so that it belongs to the above mentioned class $A_0$ by recasting all the spatial derivatives to hit the test function.

We will bypass all this, i.e. the bottle-neck of (1.11), by using more functions (the higher functional derivatives) in the description of an arbitrary variation $\delta F$. The format of (1.9) is so broad, that it essentially covers all interesting functionals, which do not contain time-derivatives, cf. the discussion in Subsection 1.1. (One could of course give meaning to differentiation of a functional with temporal derivatives simply by brute force extending the multi-index $k$ in (1.9) from $d$ dimensions to $d+1$ dimensions. Although relevant for so-called covariant formulations (covariant in the sense that time and space are treated on equal footing), this is not in the line of the Hamiltonian theories, and hence not something we will pursue in this paper.)

2 General Formalism
2.1 Partial Derivatives of a Functional

Let us describe the higher partial derivatives of a functional $F$. (They are not to be confused with the usual higher partial derivatives of a function, although they are related.) In fact, they are objects, given the suggestive notation

$$\frac{\partial F}{\partial \phi^{A(k)}(x)}$$

that satisfies

$$\delta F = \int_{\Sigma} d^d x \sum_{k=0}^{\infty} \frac{\partial F}{\partial \phi^{A(k)}(x)} \partial^k \delta \phi^{A}(x) ,$$

for arbitrary variations $\delta \phi^{A}(x)$. If this notation (2.1) in the future leads to ambiguities, we will specify whether we mean partial differentiation wrt. a function or a functional. Usually the context will exclude one of the possibilities. In fact, in this article we will often use the notation

$$P_{A(k)} f(x) \equiv \frac{\partial f(x)}{\partial \phi^{A(k)}(x)} .$$

for the usual (higher) partial derivative for a function $f(x) = f(\partial^k \phi(x), x)$.

2.2 Manifest Formulation

The set of the higher functional (and the partial) derivatives may not be uniquely defined, so one may worry that the full Poisson bracket given by (1.3) and (1.4) depends on the choice of the representatives for the higher functional derivatives. The answer is that it is independent. This follows from the manifest formulation given below.

We begin by giving a more useful definition of the (usual) functional derivatives (1.8) than the traditional definition, cf. Eq. (1.11). The differential $\delta F = \delta F[\phi, \delta \phi]$ of $F$ is assumed to split

$$\delta F = dF + \partial F$$

into a bulk integral

$$dF = dF[\phi, \delta \phi] = \int_{\Sigma} d^d x \frac{\delta F}{\delta \phi^{A}(x)} \delta \phi^{A}(x)$$

and a boundary integral

$$\partial F = \partial F[\phi, \delta \phi] = \sum_{i=1}^{d} \int_{\Sigma} d^d x \partial_i \tilde{F}^i_{A} \delta \phi^{A}(x) ,$$

where $\tilde{F}^i_{A}$ in general can be differential operators acting on $\delta \phi^{A}$. That the integral (2.6) is a boundary integral is justified by the divergence theorem. If we furthermore require the (usual) functional derivative to be a continuous function, this function together with the above split (2.4) are uniquely defined. (We stress that the uniqueness of the (usual) functional derivative is jeopardized if for instance it contains a delta distribution with support on the boundary. This is ruled out by requiring continuity. See also the uniqueness discussion in the next Section.)

The bulk Poisson bracket $\{\cdot, \cdot\}_{(0)}$ is now well-defined by the Eq. (1.1), and the full Poisson bracket Eq. (1.3) differs from this by a boundary term

$$B(F,G) = \partial F \left[ \phi, \delta \phi = \{\phi, G\}_{(0)} \right] - (F \leftrightarrow G) .$$

(2.7)
At this point the reader can merely take $\delta \phi = \{\phi, G\}_{(0)}$ as being a convenient short hand notation for

$$\delta \phi^A(x) = \omega^{AB} \frac{\delta G}{\delta \phi^B(x)} . \quad (2.8)$$

There is two obvious ansatzes for the differential operator $\hat{F}_A^i$. Either with the spatial derivatives ordered to the right of some coefficient functions (also called the normal order of $x$ and $\partial$). Or vice-versa. In the former case

$$\hat{F}_A^i = \sum_{k=0}^\infty \frac{\partial^k F}{\partial \phi^A(k)} \partial^k , \quad (2.9)$$

we call the coefficient functions for the higher partial vector derivatives. The name “vector” refers to the $i$-index. In the latter case

$$\hat{F}_A^i = \sum_{k=0}^\infty \partial^k \frac{\delta^i F}{\delta \phi^A(k)} . \quad (2.10)$$

the coefficient functions are called the higher functional vector derivatives.

### 2.3 Uniqueness of the Higher Derivatives

Up to now, we have only characterized the higher functional (partial, vector) derivatives of a functional $F$ in a descriptive manner. The question of existence yields rather mild conditions, that we shall not be concerned about. The issue of uniqueness is a much more interesting question. The ambiguity in the choice is most clearly displayed via the higher partial vector derivatives:

**Uniqueness Theorem.** Assume that there is given a sequence of continuous functions $f_A$ and $f_{iA}^{(k)}$ that all vanish identically with the exception of a finite number and such that for an arbitrary variation $\delta \phi$

$$0 \equiv \int_\Sigma d^d x\ f_A(x)\ \delta \phi^A(x) + \sum_{i=1}^d \int_\Sigma d^d x\ \partial_i \sum_{k=0}^\infty f_{iA}^{(k)}(x)\ \partial^k \delta \phi^A(x) . \quad (2.11)$$

Then $f_A \equiv 0$ in the entire space $\Sigma$ and the (higher) partial vector derivatives $f_{A}^{(k)}$ are tangential to the boundary $\partial \Sigma$. In detail,

$$\forall x \in \partial \Sigma : \ f_{A}^{(k)}(x) \equiv (f_{A}^{(1)}(x), \ldots, f_{A}^{d}(x)) \in T_x(\partial \Sigma) . \quad (2.12)$$

That the first function $f_A \equiv 0$ vanishes is just a restatement of the uniqueness of the (usual) functional derivative, while the boundary condition reflects that the higher partial vector derivatives may be modified with a vector field that doesn’t locally carry a boundary flux.

### 2.4 Relations among the Different Kinds of Higher Derivatives

In order to get an idea of how ambiguous the other variational discriptions are, let us give some maps between the mentioned choices of the higher functional (partial, vector) derivatives. We start with a bijective correspondance between the two scalar definitions (1.9) and (2.2) of the higher derivatives. If there is given a sequence of the higher partial derivatives satisfying the definition (2.2), then

$$\frac{\delta F}{\delta \phi^A(k)(x)} = \sum_{m \geq k} \left( \begin{array}{c} m \\ k \end{array} \right) (-\partial)^{m-k} \frac{\partial F}{\partial \phi^A(m)(x)} , \quad \left( \begin{array}{c} m \\ k \end{array} \right) = \left( \begin{array}{c} m_1 \\ k_1 \end{array} \right) \ldots \left( \begin{array}{c} m_d \\ k_d \end{array} \right) . \quad (2.13)$$
satisfies the definition (1.9). On the other hand, starting from the point of view of the higher functional derivatives, we get a solution to the higher partial derivatives by
\[
\frac{\partial F}{\partial \phi^A(k)(x)} = \sum_{m \geq k} \binom{m}{k} \partial^{m-k} \frac{\delta F}{\delta \phi^A(m)(x)}. \tag{2.14}
\]
The variational descriptions (1.9) and (2.2) coincide because of the x-pointwise identity
\[
\sum_{k=0}^{\infty} \frac{\partial F}{\partial \phi^A(k)(x)} \partial^k \delta \phi^A(x) = \sum_{k=0}^{\infty} \partial^k \left[ \frac{\delta F}{\delta \phi^A(k)(x)} \delta \phi^A(x) \right]. \tag{2.15}
\]
For a proof, see the equation (A.2) in the Appendix. Similarly, one may transform bijectively between the two vectorial definitions by use of the corresponding relations
\[
\frac{\delta^i F}{\delta \phi^A(k)(x)} = \sum_{m \geq k} \binom{m}{k} (-\partial)^{m-k} \frac{\partial^i F}{\partial \phi^A(m)(x)}, \quad \frac{\partial F}{\delta \phi^A(k)(x)} = \sum_{m \geq k} \binom{m}{k} \partial^{m-k} \frac{\delta^i F}{\delta \phi^A(m)(x)}. \tag{2.16}
\]
Not surprisingly, as the vectorial definitions (2.9) and (2.10) carry the most indices, they have the greatest flexibility in representing a solution. We may convert from the higher vector to the higher scalar derivatives via the formulas (for \(k \neq 0\)):
\[
\frac{\partial F}{\partial \phi^A(k)(x)} = \sum_{i \in I(k)} \frac{\partial^i F}{\delta \phi^A(k-e_i)(x)} + \sum_{i=1}^d \frac{\partial^i F}{\delta \phi^A(k)(x)}, \quad \frac{\delta F}{\delta \phi^A(k)(x)} = \sum_{i \in I(k)} \frac{\delta^i F}{\delta \phi^A(k-e_i)(x)}. \tag{2.17}
\]
Here \(e_i \equiv (0, \ldots, 0, 1, 0, \ldots, 0)\) is the \(i\)’th unit vector in the index lattice and \(I(k) \equiv \{ i = 1, \ldots, d \mid k_i \neq 0 \}\). Going from the higher scalar to the higher vector derivatives is not unique. A natural choice is for the functional derivatives are
\[
\frac{\delta^i F}{\delta \phi^A(k)(x)} = \frac{1}{|I(k+e_i)|} \frac{\delta F}{\delta \phi^A(k+e_i)(x)}. \tag{2.18}
\]
We leave out the corresponding relation between the partial derivatives to carry on with our main application: Local field theories.

### 2.5 Local Field Theory

Let us restrict ourselves to local field theories, i.e. all functionals can be expressed as an integral
\[
F = \int_{\Sigma} d^d x \ f(x), \quad f(x) \equiv f \left( \partial^k \phi(x), x \right), \tag{2.19}
\]
where \(k\) runs over a finite subset of \(\mathbb{N}_0^d\). Note that we have allowed for explicit \(x\)-dependence in \(f\). It essentially costs no extra work, and it becomes important later on. We shall postpone the analysis of functionals that depends on external space-points to a later section (Section 3), partly because it would be notational inconvenient to address those now.

As mentioned before the (higher) partial derivatives of \(F\) need not be unique. Our strategy will be to “lower” the definition from the level of integrals to the level of integral kernels. In other words, this means that if a functional has different integral kernel representations, this may lead
to different definitions of the higher derivatives. For instance, at the case at hand, \textit{i.e.} of a \textit{local} functional (2.19) with a given integral kernel \( f \), there is one natural candidate
\[
\frac{\partial F}{\partial \phi^{(k)}}(x) = P_{A(k)} f(x) ,
\]
that fits the relation (2.2). So strictly speaking, the (higher) partial derivatives are really (higher) partial derivatives of the kernels, although we will not indicate this explicitly in the notation. In the same fashion the distinguished candidate for the functional derivatives becomes the higher Euler-Lagrange derivatives:
\[
\frac{\delta F}{\delta \phi^{(k)}}(x) = E_{A(k)} f(x) \equiv \sum_{m \geq k} \binom{m}{k} (-\partial)^{m-k} P_{A(m)} f(x) .
\]
For a mathematical textbook on higher Euler-Lagrange derivative, see for instance Olver \[11, \text{p.365-367}]\. Note that the \( m \)-summation in (2.21) terminates after finite many terms in case of a local functional, so that we do not have to worry about convergence of the sum. Let us simply use (2.20) (and (2.21)) as the working definitions for the local functionals. After all, our primary goal is to prove the Jacobi identity for the full Poisson Bracket, and this does not depend on the choice of the representatives for the higher derivatives. It is evident from the natural solution (2.20) (and (2.21)) of the (higher) derivatives, that in practice all the local functionals that one encounters in physics are differentiable.

### 2.6 Ultra-local Poisson Bracket

Having restricted ourselves to the ultra-local case, let us for each pair of local functional
\[
F = \int_{\Sigma} d^l x \ f(x) \quad \text{and} \quad G = \int_{\Sigma} d^l x \ g(x)
\]
define a \( x \)-pointwise version
\[
\{ f, g \}(x) \equiv \{ f, g \}_{(0)}(x) + B(f, g)(x)
\]
of the Poisson bracket
\[
\{ F, G \} = \int_{\Sigma} d^l x \ \{ f, g \}(x) = \{ F, G \}_{(0)} + B(F, G) .
\]
Namely define
\[
\begin{align*}
\{ f, g \}_{(0)}(x) & \equiv E_{A(0)} f(x) \ \omega^{AB} E_{B(0)} g(x) \\
B(f, g)(x) & \equiv \sum_{k \neq 0} \partial^k \left[ E_{A(k)} f(x) \ \omega^{AB} E_{B(0)} g(x) \right] - (f \leftrightarrow g) .
\end{align*}
\]
This means that the full \( x \)-pointwise Poisson bracket reads
\[
\begin{align*}
\{ f, g \}(x) & = \sum_{k = 0}^{\infty} \partial^k \left[ E_{A(k)} f(x) \ \omega^{AB} E_{B(0)} g(x) \right] - \frac{1}{2} \{ f, g \}_{(0)}(x) - (f \leftrightarrow g) \\
& = \sum_{k = 0}^{\infty} P_{A(k)} f(x) \ \omega^{AB} \partial^k E_{B(0)} g(x) - \frac{1}{2} \{ f, g \}_{(0)}(x) - (f \leftrightarrow g) .
\end{align*}
\]
The last equality in (2.26) follows from equation (2.13). We can now conduct our analysis \( x \)-pointwisely. In the next Section 3 we shall suppress the space point \( x \in \Sigma \).
3 Generator Methods

3.1 Heisenberg Algebra

Due to the quite heavy combinatorics involved in the proof of the Jacobi identity, it is useful to map
the above problem into a simpler and in fact well-studied object, namely the Heisenberg algebra.
Although the actual proof is presented in the Appendix, we find the central idea, while perhaps
not entirely original, is quite important, so we will present it here. For a recent exposition of Fock
space methods for variational systems, see also [12].

Let us study the interplay between the partial and the spatial derivatives. The higher partial
derivatives $P_{A(k)}$ commute among each other, but they do not commute with the (total) spatial
derivatives

$$\partial_i = \phi^{A(k+e_i)} P_{A(k)} + \phi^{\text{explicit}}, \quad (3.1)$$

where $e_i \equiv (0, \ldots, 0, 1, 0, \ldots, 0)$ is the $i$’th unit vector in the index lattice. More precisely, we have

$$P_{A(k)} \partial^n = \sum_{m=0}^{\min(k,n)} \binom{n}{m} \partial^{n-m} P_{A(k-m)} \cdot \quad (3.2)$$

The main idea is to simulate this complicated disentanglement formula with the help of a Heisenberg
algebra. Let us introduce abstract (bosonic) algebra elements $Y^i_A$ that obey the following Heisenberg
algebra commutator relations

$$[Y^i_A, \partial_j] = \delta^i_j, \quad [Y^i_A, Y^j_B] = 0, \quad [\partial_i, \partial_j] = 0 \cdot \quad (3.3)$$

The third equation is not a definition, but is a well-known consequence of (3.1). It is a remarkable
fact that we can mimic the non-commuting behavior of formula (3.2) by formally writing the
higher partial derivatives as a product

$$P_{A(k)} \equiv P_A \frac{Y^k_A}{k!} \quad (3.4)$$

of the $Y^i_A$ algebra elements and what is basically reduced to be a passive spectator in what follows,

$$P_A \equiv \frac{\partial}{\partial \phi^A} \cdot \quad (3.5)$$

We take $P_A$ to commute with everything:

$$[P_A, Y^i_B] = 0, \quad [P_A, \partial_i] = 0, \quad [P_A, P_B] = 0 \cdot \quad (3.6)$$

Above we have adapted the following multi-index conventions:

$$Y^k_A = (Y^1_A)^{k_1} \cdots (Y^d_A)^{k_d}, \quad k! = k_1! \cdots k_d! \cdot \quad (3.7)$$

That the Heisenberg algebra (3.3) with the formal assignment (3.4) really reproduces (3.2) is proven
in the Appendix A, see equation (A.1). The proof becomes very simple once we adapt the generator
techniques of the next section.
3.2 Generator Methods

As a second computational improvement, it is useful to hide the integer indices inside generating functions which depend on continuous parameters \( q_i \), i.e. we shall sum up in generalized Fourier series. We implement this program for the (higher) partial and the (higher) Euler-Lagrange derivatives, respectively, as follows

\[
P_A(q) \equiv P_{A(k)} q^k = P_A e^{q_Y} ,
E_A(q) \equiv E_{A(k)} q^k = \exp [(q - \partial) Y_A] P_A : = \exp \left[ -\partial \frac{\partial}{\partial q} \right] P_A(q) .
\]

From now on we will implicitly imply summation over repeated multi-indices \( k \in \mathbb{N}_0^d \). In fact, we may view the multi-indices sums as running over the entire \( d \)-dimensional integer lattice \( k \in \mathbb{Z}^d \) by simply declaring that objects like

\[
Y^k_A , \, \partial^k , \, P_{A(k)} , \, E_{A(k)},
\]

are zero if \( k \) is outside the original non-negative \( d \)-dimensional quadrant \( \mathbb{N}_0^d \). The next-to-last equality in (3.8) follows from the mere definition of the (higher) Euler-Lagrange derivatives (2.21), once we have declared the following normal ordering prescription:

\[
: Y^j_A \partial_j : = \partial_j Y^j_A = \partial_j Y^j_A .
\]

3.3 Fourier Transform

As a third computational improvement, let us Fourier transform the variables \( q_i \) to variables \( y^i \).

\[
P_A(y) \equiv \int d^d q \ e^{-qy} P_A(q) = P_A \delta(Y_A - y) ,
E_A(y) \equiv \int d^d q \ e^{-qy} E_A(q) = e^{-\partial y} P_A(y) = P_A \delta(Y_A) e^{-\partial y} .
\]

With the theory developed so far, we have reached the second objective of this paper (the first objective being to give the form of the full Poisson bracket (1.3)), namely achieved a formalism that are capable of giving a short proof of the Jacobi identity. For the proof itself, see the Appendix B.

Note that because the Poisson bracket is independent of the actual choice of representatives for the higher functional (partial) derivatives, it implies no limitation that we use the natural choice (2.21) and (2.20). We will now turn to the question of geometrizing the Poisson bracket.

4 Abstract Manifolds

In this section we formulate the results obtained so far in a geometrically covariant manner independent of the choice of coordinates. More precisely, the construction is generalized from a \( d \)-dimensional subset \( \Sigma \subseteq \mathbb{R}^d \) (where the space and the chart are identified) to an abstract \( d \)-dimensional manifold \( \Sigma \) with spatial covariant derivatives \( D_i \) and a \( d \)-dimensional volume density \( \rho \). In other words, the spatial derivatives \( \partial_i \) are replaced by covariant derivatives (let us indicate this in an oversimplified way as \( D_i = \partial_i + \Gamma_{ik} \)), and the trivial measure \( d^d x \) is replaced by \( \rho d^d x \). We will assume that \( D_i = D_i(x) \) and \( \rho = \rho(x) \) do not depend on the dynamical fields \( \phi^A(x, t) \) nor on time \( t \). We do not assume that the volume density is covariantly preserved, i.e. that \( D_i \rho \equiv (\partial_i - \Gamma_{ik})\rho = 0 \).
In passing from derivatives $\partial_i$ to covariant derivatives $D_i$, we face the main complication compared to the flat case. In general, the spatial covariant derivatives do not commute, when the curvature is non-vanishing. We have

$$[D_i, D_j] = [D_i^{\text{explicit}}, D_j^{\text{explicit}}] ,$$

where the (total) covariant derivative $D_i$ is given as

$$D_i = \phi^{A(i+K)} P_{A(K)} + D_i^{\text{explicit}} .$$

The index structure of the first term in (4.2) will be explained below. In the general case of non-vanishing curvature, one can proceed by declaring that the functionals of the theory depend on ordered tuples of covariant derivatives $\phi^{A(K)}(x) = D^K \phi^A(x)$ of the dynamical fields $\phi^A(x)$, rather than only unordered sets of derivatives $\phi^{A(k)}(x)$. An ordered tuple $K$ is of the form

$$K = (k_1, \ldots, k_{|K|}) \in \{1, \ldots, d\} \times \ldots \times \{1, \ldots, d\} ,$$

where $d$ is the space dimension, i.e. the dimension of $\Sigma$. We have given a resumé of the calculus of ordered tuples in the Appendix C. All formulas carries over to the curved case in essentially the same format. However, there are some notable differences that we now stress. The description of the higher derivatives (2.2), (1.9), (2.13) and (2.14) are replaced with

$$\delta F = \int_\Sigma \rho(x) d^d x \sum_{K=0}^{\infty} D^K \delta \phi^A(x)$$

$$\frac{\delta F}{\delta \phi^{A(K)}(x)} = \sum_{M \geq K} (-D)^{M+K} \frac{\partial F}{\partial \phi^{A(M)}(x)} ,$$

$$\frac{\partial F}{\partial \phi^{A(K)}(x)} = \sum_{M \geq K} D^{M+K} \frac{\delta F}{\delta \phi^{A(M)}(x)} .$$

For details concerning the notation in (4.4), see definitions in Appendix C.1. The (higher) functional and partial derivatives inherits tensor properties, if $F$ is covariant. So the formulas are covariant. The bulk and the boundary term of the Poisson bracket reads

$$\{F, G\}_{(0)} = \int_\Sigma \rho(x) d^d x \frac{\delta F}{\delta \phi^A(x)} \omega^{AB} \frac{\delta G}{\delta \phi^B(x)}$$

$$B(F, G) = \sum_{K \neq 0} \int_\Sigma \rho(x) d^d x \delta^K \left[ \phi^{A(K)}(x) \omega^{AB} \frac{\delta G}{\delta \phi^B(x)} \right] - (F \leftrightarrow G) .$$

Note the apparent asymmetry between the two last formulas in (4.4) with a transposition $t$ of the order of covariant derivatives in the third equation. As a rule of thumb one may say that the spatial derivatives $(-\partial)^k$ in the flat metric formulation becomes $(-D)^{K^t}$ in the covariant formulation. This generic feature carries over to the generator formalism (3.8):

$$P_A(q) \equiv P_{A(K)} q^K = P_A e^{q Y_A} = \exp \left[ D \cdot \frac{\partial}{\partial q} \right] E_A(q) ,$$

$$E_A(q) \equiv E_{A(K)} q^K = \exp \left[ (-D)^t \cdot \frac{\partial}{\partial q} \right] P_A(q) .$$

Let us note that the $q_i$’s (besides commuting with everything else) does not commute among themselves. More precisely, they are freely generated. This is necessary in order not to loose
information about the operator ordering when passing to the generating functions. Also the Fourier transform \( q^i \rightarrow y_i \), cf. Section 3.3, can be given sense in the non-commutative case.

The replacement of the disentanglement formula (3.2) becomes

\[
P_{A(K)} D^N = \sum_{M=0}^{M \leq K, M \leq N} D^{N+M} P_{A(-M+K)} .
\]

We are able to define contravariant elements \( Y_{A(K)} \), such that

\[
P_{A(K)} \equiv P_A \frac{Y_{A(K)}}{|K|!} .
\]

The Heisenberg algebra (3.3) is traded for

\[
[Y_{A(K)}, D_i] = |K| Y_{A(-i+K)} .
\]

Remarkably, even in this non-commutative case, the exponentiated version can be recasted into the following simple form

\[
e^{q^i Y} e^{D_i y} = e^{(q+D)^t y} e^{q^i Y} .
\]

We shall have more to say about this construction at the end of Appendix C. The main point is that the proof of the covariant Jacobi identity can be demonstrated in almost exactly the same way as in the flat metric case, cf. Appendix D.

Note that for non-zero curvature \( [D_i, D_j] \phi^A(x, t) \neq 0 \), the actual field value \( \phi^A(x, t) \) is apparently not well-defined, i.e. if one tries to sum up the change in \( \phi^A(x, t) \) along a closed loop, one obtain a non-zero result. This is worse than a global obstruction. Perhaps it should be called a local obstruction. (A similar situation occurs, say, in bosonic string theory with the Polyakov action when the worldsheet metric has a non-zero curvature. This also leads to problems in locally assigning values to the target space fields.) The problem seems less formidable in the context of the Feynman path integral, where we only assign field values along one path at the time. But it is a genuine challenge for the operator formalism. One way of making sense out of this would be to declare the decendent fields \( \phi^{A(K)}(x, t) \) to be independent fields living in a non-commutative jet-bundle. In any case, we feel that it would be too hasty a priori to draw conclusions in general, and we leave it to the future to appropriately implement non-zero curvature in specific physical theories.

5 Supplementary Formalism

Until now, we have only discussed functionals \( F \) with no external space dependence, i.e. all space-variables are integrated out. However for physical applications, we would like to conduct manipulations directly on the integral kernels rather than the integrals. For instance, to give sense to the fundamental equal-time relations

\[
\{ \phi^A(x, t), \phi^B(y, t) \} = \omega^{AB}(x, y) = \omega^{AB} \delta_\Sigma(x, y) .
\]

The plan for the rest of this article are

- to treat the Dirac delta distributions (and the derivatives thereof) in the presence of a boundary. Distributions is a vast subject in their own right, and we will here only give a heuristic treatment.
• to extend the definition of the higher functional derivatives to more general types of local functionals.

• to analyse the implications for the Hamiltonian dynamics.

While parts of this section is standard material, it is reviewed for continuity and to fix notation.

5.1 Embedded Approach

Having a geometrically covariant formulation at our disposal enables us to reduce the discussion to a single chart. We can slice up space in smaller regions; thereby producing unphysical double-sided boundaries (unphysical domain-walls), and we can hence consider space within such a smaller region $\Sigma$ covered by a single chart. The local geometric data about the physical space $\Sigma$ is stored in the volume density $\rho$ and the covariant derivatives $D_i$.

Furthermore, we will assume that $\Sigma$ takes place inside a bounded region of the chart $\mathbb{R}^d$, i.e. that it can be placed inside a large ball in $\mathbb{R}^d$. Note that we are not placing any restriction on the distances in the physical space $\Sigma$; only on the distances in the chart. Or perhaps we should say: in the choice of the chart. For instance, if space $\Sigma = \mathbb{R}^d$ is the ordinary flat space, one should map flat space into a bounded region $\tilde{\Sigma}$ of the chart $\mathbb{R}^d$ using a non-trivial $\rho$ and $D_i$. In this case the spatial infinity is truly the boundary of the region $\tilde{\Sigma}$. The perspective will be that of a typical Penrose diagram: “There is always room for something beyond spatial infinity.” The motivation for the above assumption is deeply founded in the theory of distributions, cf. below.

This being said, we will adapt the usual practice of identifying the space $\Sigma \subseteq \mathbb{R}^d$ with a region of the chart.

5.2 Regularized Characteristic Functions

Let us consider the characteristic function

$$1_\Sigma(x) = \begin{cases} 
1 & \text{if } x \in \Sigma \\
0 & \text{otherwise},
\end{cases} \quad (5.2)$$

for the space $\Sigma \subseteq \mathbb{R}^d$ as a limit of a smooth function $\chi_\epsilon(x)$, where $0 < \chi_\epsilon(x) \rightarrow 1_\Sigma(x)$ for $\epsilon \rightarrow 0^+$. We regard $\chi_\epsilon = \chi_\epsilon(x)$ as independent of the dynamical fields $\phi^A(x)$ and as a scalar under coordinate changes $x \rightarrow x'$. The actual implementation of the regularization $\chi_\epsilon(x)$ should not matter, so one might as well choose a convenient form. One could for instance do as follows. Let $d_\Sigma(x)$ denote the signed distance from $x$ to the boundary $\partial \Sigma \subseteq \mathbb{R}^d$ as measured in the chart $\mathbb{R}^d$. The signed distance $d_\Sigma(x) > 0$ is positive if $x \in \Sigma^\circ$ belongs to the interior and it is negative $d_\Sigma(x) < 0$ if $x \in (\mathbb{R}^d \setminus \Sigma^\circ)$ belongs to the exterior. Then we could implement $\chi_\epsilon(x)$ as

$$\chi_\epsilon(x) = \left(1 + \exp \left[ -\frac{d_\Sigma(x)}{\epsilon} \right] \right)^{-1}. \quad (5.3)$$

(This looks horrible in other coordinates, so a geometrically minded reader might prefer to substitute the chart $\mathbb{R}^d$ with an abstract unphysical embedding manifold. We shall not explore this point of view further in this paper.)
Next, we extend \( \rho \) and \( D_i \) smoothly (and arbitrarily) to the unphysical sector \( \mathbb{R}^d \setminus \Sigma \). It may happen that \( \rho \) or \( D_i \) themselves are singular at the boundary \( \partial \Sigma \). In that case one should consider smooth regularized functions \( \rho_\epsilon \) or \( D_i^\epsilon \), that in the limit \( \epsilon \to 0 \) reproduces \( \rho \) and \( D_i \). Then an integral over \( \Sigma \) should be thought of as a limit
\[
\int_\Sigma \rho(x) \, d^d x \, f(x) = \lim_{\epsilon \to 0} \int \chi_\epsilon(x) \rho_\epsilon(x) \, d^d x \, f_\epsilon(x) .
\] (5.4)
where the integrand \( f(x) \) also should be smoothly (and arbitrarily) extended to the unphysical sector \( \mathbb{R}^d \setminus \Sigma \). (In case of more than one region, \( \chi_\epsilon(x) \) should be a differentiable partition of the unity.) From now on we will not write nor question the limit \( \epsilon \to 0 \), but merely take for granted that this \( \epsilon \)-prescription makes sense.

5.3 Dirac Delta Distributions

Throughout the paper, the Dirac delta distribution \( \delta(x - z) \) refers to the full \( \mathbb{R}^d \)-chart, while \( \delta_\Sigma(x, z) \) refers to the physical space \( \Sigma \). The physical Dirac delta distribution \( \delta_\Sigma(x, z) \) is characterized by the property:
\[
\forall \eta : \int_{\Sigma \times \Sigma} \rho(x) \, d^d x \, \rho(z) \, d^d z \, \eta(x, z) \, \delta_\Sigma(x, z) = \int_{\Sigma} \rho(x) \, d^d x \, \eta(x, x) .
\] (5.5)

We can realize the physical Dirac delta distribution \( \delta_\Sigma(x, z) \) in terms of the unphysical Dirac delta distribution \( \delta(x - z) \) as
\[
\delta_\Sigma(x, z) = \frac{\delta(x - z)}{\chi_\epsilon(x) \rho(x)} .
\] (5.6)

The main idea behind demanding that \( \Sigma \) should occupy a bounded region of the chart \( \mathbb{R}^d \), is that we can then perform formal integrations by part on the unphysical Dirac delta distributions \( \delta(x - z) \). This is so because we can consider all test functions as having a bounded support in the chart \( \mathbb{R}^d \). “Test functions” should here be read in the broad sense of the word that in particular includes functions \( f(\phi^K(x), x) \) of the dynamical fields \( \phi^K(x) \).

On the other hand, integration by part of the physical Dirac delta distribution \( \delta_\Sigma(x, z) \) will in general lead to boundary contributions at the physical boundary \( \partial \Sigma \). The detailed form can be inferred from the above relation (5.6). The benefit of this procedure, is that we do not have to postulate peculiar rules for the physical delta distribution. They may simply be derived from (5.6).

The above is the key observation in our analysis of distributions. Mathematicians have always (and presumably for good reasons) considered test functions in \( \mathbb{R}^d \) as having compact support. We observe that if space \( \Sigma \), which itself could be unbounded, fills a bounded region of the chart \( \mathbb{R}^d \), we can without touching the above principle, still probe boundary issues at the physical boundary \( \partial \Sigma \).

From (5.4) and (5.6) it also becomes clear that the study of a non-trivial boundary \( \partial \Sigma \) and the study of a non-trivial volume density \( \rho \) are intimately related. With the spatial integration interpreted as (5.4), we may define the adjoint \( D_i^\dagger \) of \( D_i \) by formal integration by part in the chart \( \mathbb{R}^d \). It becomes
\[
D_i^\dagger = -\frac{1}{\chi_\epsilon \rho} \tilde{D}_i \chi_\epsilon \rho(\cdot) .
\] (5.7)
(The arrow over \( D_i \) indicates that the derivative \( D_i \) acts all the way to the right.)
5.4 Space of Functionals

Consider now a function depending on variables \(z_{(1)}, \ldots, z_{(r)}\),

\[
f(z_{(1)}, \ldots, z_{(r)}) = f \left( D^{K_{(1)}} \phi(z_{(1)}), z_{(1)}, \ldots, D^{K_{(r)}} \phi(z_{(r)}), z_{(r)} \right),
\]

(5.8)

where \(K_{(1)}, \ldots, K_{(r)}\) are multi-indices. For convenience we shall often use the compact notation \(z \equiv (z_{(1)}, \ldots, z_{(r)})\) if there is several space points \(z_{(i)}\). We shall restrict ourselves to the space \(A\) of functionals \(F\) that can be expressed as a \(s\)-fold multiple integral over \(\Sigma\)

\[
F(z) = \int_{\Sigma \times \ldots \times \Sigma} \rho d^{d}z_{(1)} \ldots \rho d^{d}z_{(s)} f(z),
\]

(5.9)

for some \(s \in \mathbb{N}\). (It is implicitly understood that the \(z_{(i)}\)'s which are integrated out on the right hand side, do not enter the argument list on the left hand side.) Furthermore, this space \(A\) is a \(\mathcal{C}\)-vector space. It is stable under multiplication whenever defined. (Recall that the product of distributions need not be well-defined.) It is closed under integrating out external variables, or identifying external variables, say \(z_{(i)} = z_{(j)}\). We shall see below that it is also closed under the full Poisson bracket.

5.5 Suitable Form of Functional

Consider a local functional \(F(z) = \int d^{d}x_{(1)} \ldots d^{d}x_{(r)} f(x, z)\) with external dependence \(z\). The typical integrand \(f(x, z)\) consists of

1. The Dirac delta distributions \(\delta(x-y)\) and \(\delta_{\Sigma}(x,y)\); The regularized characteristic functions \(\chi_{p}^{\rho}(x)\) in some power \(p \in \mathbb{R}\).

2. Smooth test functions with bounded support. This in particular includes smooth functions \(g(\phi^{(K)}(x), x)\) of the dynamical fields \(\phi^{A(K)}(x)\) and the smooth volume density \(\rho(x)\).

3. Derivatives \(D_{i}\) acting on the various factors of the integrand mentioned under point 1 – 2.

The above listed objects appears in two versions:

A. an external type, if it depends on non-integrated external variables.

B. an internal type, if it (at least partially) depends on integrated internal variables.

An integral (kernel) is declared to be on suitable form if all internal derivatives (B3) act on type B2 objects. In other words that the more singular type (B1) objects are not hit by the internal derivatives (B3).

A functional (kernel) \(f(x, z)\) is not well-defined if one cannot obtain a suitable form by pure algebraic manipulations. In practice, this means

- after formal internal integration by part,
- after use of the Leibnitz rule and linearity.
- after use of the identity \((g(x) - g(y)) \delta(x-y) = 0\),
- and after use of the identity \((D(x) + D(y)) \delta(x-y) = 0\)
for the (partially) internal variables.

Just from the freedom to perform a formal integration by part on the internal differentiation \((B3)\) (or choosing not to do so, respectively), there is \(2^n\) ways of writing down an functional, where \(n\) is the number of internal differentiations \((B3)\). In practice, in all interesting functionals, every internal variable \(x_i\) appear at least once in the argument list of an internal singular object (i.e. of type \(B1\)). As a consequence, in these cases, at most one of the above mentioned \(2^n\) choices leads to a suitable form.

Needless to say that if one also integrate out the external \(z\)-variables \(F(z)\) without yielding enough room for the smearing type 2 objects, the result may not be well-defined.

### 5.6 Higher Partial Derivatives

Consider a functional \(F(z) \in A\) in the algebra \(A\). Assume that the functional (kernel) is of suitable form. Let us now define the higher partial derivatives as

\[
\frac{\partial F(z)}{\partial \phi^A(K)(x)} \equiv \int_{\Sigma} \rho d^d z(1) \cdots \rho d^d z(s) \sum_{i=1}^r \delta_{\Sigma}(x, z(i)) \ P^{(z(i))}_{A(K)} f(z). \tag{5.10}
\]

In case of a function

\[
F(z) = F\left(D^K \phi(z), z\right), \tag{5.11}
\]

this reduces to

\[
\frac{\partial F(z)}{\partial \phi^A(x)} = \sum_{i=1}^r \delta_{\Sigma}(x, z(i)) \ P^{(z(i))}_{A(K)} F(z). \tag{5.12}
\]

We can formally extend the application range of the above equation (5.12) to include functionals \(F(z)\) as well, by implicitly assuming that the internal delta distributions automatically are placed inside the integration symbol. Then (5.12) becomes a convenient shorthand notation for (5.10).

### 5.7 Higher Functional Derivatives

The general definition (4.4) for a functional of suitable form yields

\[
\frac{\delta F(z)}{\delta \phi^A(K)(x)} = E^{(x)}_{A(K)} \sum_{i=1}^r \left[ \delta_{\Sigma}(x, z(i)) \ F\left(z(1), \ldots, z(i-1), x, z(i+1), \ldots\right) \right] = \sum_{i=1}^r \sum_{M \geq K} \left( -D_{(x)} \right)^{M^t + K^t} \delta_{\Sigma}(x, z(i)) \ P^{(z(i))}_{A(M)} F(z). \tag{5.13}
\]

The above derivatives of a delta distribution may be resolved in two different ways:

- **By inner evaluation:** The derivatives leave the delta distribution \(\delta_{\Sigma}(x, z(i))\) via the \(z(i)\)-leg. If there is enough internal integrations inside the functional \(F\), one may resolve the delta distributions by integration, thereby prolonging the derivative to an object inside the functional \(F\). If all terms are to be resolved this way, this means that all the \(z\)-variables have to be internal.

- **By outer evaluation:** The derivatives leave the delta distribution \(\delta_{\Sigma}(x, z(i))\) via the \(x\)-leg. We await an external integration over the \(x\)-variable to evaluate the derivative of the delta distribution by formal integration by part. Let us stress the fact, that if one rely on the latter
As the most important example, we mention the two methods yields the same result. This distinction is important if one is to conduct further partial differentiations wrt. to the dynamical fields $\phi^A(x)$ on the functional (kernel). However, if no further differentiations are performed, the two methods yields the same result.

As the most important example, we mention

$$\frac{\partial \phi^B(y)}{\partial \phi^A(x)} = \frac{\delta \phi^B(y)}{\delta \phi^A(x)} = \delta^B_A \delta(x,y).$$

Note that the above definitions (5.12) and (5.13) guarantee the linearity and the Leibnitz' rule of the (higher) partial and functional derivatives. We also find that two (higher) partial derivatives commute. One may show in the case of a vanishing boundary $\partial \Sigma = \emptyset$, that the usual functional derivatives commute. In the case of a non-trivial boundary $\partial \Sigma \neq \emptyset$, the higher functional derivatives (as well as the usual functional derivatives) do not commute in general.

### 5.8 Induced Functional Derivative

Finally, one may define a induced functional derivative from the perspective of the embedding manifold, i.e. the chart $\mathbb{R}^d$:

$$\delta F(z) = \int d^d x \frac{\delta [\chi^\iota(x) \rho(x) F(z)]}{\delta \phi^A(x)} \delta \phi^A(x),$$

for an arbitrary variation $\delta \phi^A(x)$. This is so, because $\Sigma \subseteq \mathbb{R}^d$ is bounded inside the chart $\mathbb{R}^d$, so integration by part yields no boundary contributions at $|x| = \infty$. The induced functional derivative makes sense, even though there appears coinciding space points, because $\chi^\iota(x) \rho(x)$ does not depend on the dynamical fields $\phi^A(x)$. We can write it constructively as

$$\frac{\delta [\chi^\iota(x) \rho(x) F(z)]}{\delta \phi^A(x)} = \sum_{i=1}^r \sum_{M \geq \emptyset} (-D(x))^{M^T} (\chi^\iota(x) \rho(x)) \frac{\delta F(z)}{\delta \phi^A(M^T)}(x).$$

It is related to the higher functional derivatives via

$$\frac{\delta [\chi^\iota(x) \rho(x) F(z)]}{\delta \phi^A(x)} = (-D(x))^{M^T} (\chi^\iota(x) \rho(x)) \frac{\delta F(z)}{\delta \phi^A(M^T)}(x).$$

This induced functional derivative has the remarkable property of commuting with the spatial derivatives

$$D_i(z^{(\iota)}) \frac{\delta [\chi^\iota(x) \rho(x) F(z)]}{\delta \phi^A(x)} = \frac{\delta [\chi^\iota(x) \rho(x) D_i(z^{(\iota)}) F(z)]}{\delta \phi^A(x)}.$$
5.9 Annihilation Principle

As mentioned before, integration without smearing can produce ill-defined terms. However it can be very cumbersome to a priori discard all the bad terms of an expression. We shall therefore formally allow ill-defined terms to appear by giving a prescription, that consistently identify them and put them to zero. This is done by defining a little more restrictive version of the above so-called suitable form. Notation: For simplicity, we will assume from now on that the covariant derivatives commute, i.e. that the curvature vanishes. Assume further from now on that the volume density $\rho$ is covariantly preserved, $D_i \rho = 0$. A typical functional (kernel) consists of

1. Dirac delta distributions $\delta(x-y)$.
2. Regularized characteristic functions $\chi_\epsilon(x)$. (Integral powers $\chi_\epsilon^n(x)$, $n \geq 2$ should be considered as a $n$-fold product of elementary $\chi_\epsilon(x).$)
3. Negative powers $\chi_\epsilon^{-p}(x)$, $p \geq 0$, of the regularized characteristic function.
4. Smooth test functions with bounded support. This includes smooth functions $g(\phi^{(K)}(x), x)$ of the dynamical fields $\phi^{(K)}(x)$ and the smooth volume density $\rho(x)$.
5. Derivatives $D_i$ acting on various factors of the integrand mentioned under point 1 – 4.

We may assume by use of the Leibnitz rule and breaking the integral $F(z)$ into several terms if necessary, that all derivatives ($A5 + B5$) only act on one elementary object under point 1, 2 and 4. (Here the letters $A$ and $B$ refers to the notation introduced in Subsection 5.5, while the numbers 1 – 5 are those defined above in this subsection.)

A functional $F(z)$ of the above atomic type is declared to be identical zero if one cannot by algebraic means, cf. above, obtain a form where all internal derivatives ($B5$) acts on type $B4$ objects. (Or in other words, the more singular type of objects $B1 – B3$ are not hit by the derivatives.)

6 Hamiltonian Edge Dynamics

In this section we discuss implications of the full Poisson bracket for the Hamiltonian dynamics. We first have to extend the definition (4.5) of the full Poisson bracket to more general functionals with external dependence. As a first principle for writing down the more general Poisson bracket, we shall demand that integrations $\int_\Sigma \rho d^d z(i)$ commute with the Poisson bracket $\{\cdot , \cdot \}$, that is

$$\int_\Sigma \rho d^d z(i) \{F(z), G(w)\} = \{\int_\Sigma \rho d^d z(i) F(z), G(w)\}.$$ (6.1)

This principle leads naturally (modulo the action of the annihilation principle) to what we shall call the solid Poisson bracket. We shall later see that it can be recasted into a so-called floating Poisson bracket, that (at a superficial level) takes different shape on different types of functionals. However, one may show by applying the annihilation principle that no actual differences take place.

6.1 Solid Poisson Bracket

Using the extrapolation of the formulas in the previous sections, the full Poisson bracket becomes

$$\{F(z), G(w)\} = \{F(z), G(w)\}_{(0)} + B(F(z), G(w)),$$ (6.2)
where

\[
\{ F(z), G(w) \}_{(0)} = \int_\Sigma \rho(x) d^d x \frac{\delta F(z)}{\delta \phi^A(x)} \omega^{AB} \frac{\delta G(w)}{\delta \phi^B(x)} \\
= \int_\Sigma \int_\Sigma \rho(x) d^d x \rho(y) d^d y \frac{\delta F(z)}{\delta \phi^A(x)} \omega^{AB}(x, y) \frac{\delta G(w)}{\delta \phi^B(y)},
\]

\[
B(F(z), G(w)) = \sum_{K \neq \emptyset} \int_\Sigma \rho(x) d^d x \ D^K(x) \left[ \frac{\delta F(z)}{\delta \phi^A(K)} \omega^{AB} \frac{\delta G(w)}{\delta \phi^B(x)} \right] - (F(z) \leftrightarrow G(w))
\]

Alternatively, we may write the full Poisson bracket as

\[
\{ F(z), G(w) \} = \int_\Sigma \int_\Sigma \rho(x) d^d x \rho(y) d^d y \frac{\partial F(z)}{\partial \phi^A(x)} \omega^{A(M)B(N)}(x, y) \frac{\partial G(w)}{\partial \phi^B(y)},
\]

where the symplectic kernel \( \omega^{A(M)B(N)}(x, y) \) reads

\[
\omega^{A(M)B(N)}(x, y) = \omega^{AB} \left[ \left( -D^M_{(x)} \right)^{(N)} \left( -D^N_{(y)} \right)^{(M)} \right. \\
+ \left. \left( -D^N_{(x)} \right)^{(M)} \left( -D^M_{(y)} \right)^{(N)} \right. \\
+ \left. \left( -D^M_{(x)} \right)^{(N)} \left( -D^N_{(y)} \right)^{(M)} \right] \delta_{\Sigma}(x, y)
\]

In the case where (at least) one of the \( M \) and \( N \) are \( \emptyset \), the symplectic kernel can neatly be written as

\[
M = \emptyset \lor N = \emptyset \quad \Rightarrow \quad \omega^{A(M)B(N)}(x, y) = \omega^{AB} \left( D^M_{(x)} D^N_{(y)} \right) \delta_{\Sigma}(x, y).
\]

Note on the other hand, that the case \( M = \emptyset \lor N = \emptyset \) is also the maximal case to make fully sense out of the expression \( D^M_{(x)} D^N_{(y)} \delta_{\Sigma}(x, y) \) without employing the annihilation principle. Beyond that case, i.e. if \( M \neq \emptyset \land N \neq \emptyset \), there is no escape ways left open for the \( \chi^{-1}_\epsilon \)-function (via formal integrations by part of the derivatives). It is sandwiched inside the delta distribution between derivatives. It should be merged with a \( \chi_\epsilon \)-function outside, whose mere existence on the other hand prohibits that a suitable (and hence a well-defined) form can be reached by means of integration by part.

### 6.2 Hamilton Equations of Motion

Consider the Hamilton equations of motion

\[
\frac{d}{dt} F(z) = - \{ H, F(z) \} + \frac{\partial}{\partial z} F(z).
\]

Let the Poisson bracket be the solid Poisson bracket (6.3). And \( H = \int_\Sigma \rho(x) d^d x \ H(x) \) be a local Hamiltonian. Then the Hamilton equations of motion for the fundamental fields read

\[
\frac{d}{dt} \phi^A = \{ \phi^A, H \} = \frac{\omega^{AB}}{\chi_\epsilon} E_{B(0)}(\chi_\epsilon H).
\]
We also get
\begin{equation}
\frac{d}{dt} \phi^A(K) = \{ \phi^A(K), H \} = \frac{\omega^{AB}}{\chi_\epsilon} \left( D^K E_{B(0)} (\chi_\epsilon H) - \left[ D^K, \chi_\epsilon \right] E_{B(0)} H \right). \tag{6.9}
\end{equation}

On the other hand, a spatial differentiation of (6.8) yields
\begin{equation}
D^K \frac{d}{dt} \phi^A = D^K \{ \phi^A, H \} = \omega^{AB} D^K \frac{1}{\chi_\epsilon} E_{B(0)} (\chi_\epsilon H). \tag{6.10}
\end{equation}

Superficially, the spatial differentiations do not commute with the time derivatives; compare (6.9) and (6.10). But we shall see that this is an illusion. First of all, it is easy to localize potential problems to the boundary \( \partial \Sigma \). Away from the boundary, the two expressions (6.9) and (6.10) fully agree. However, being interested in the boundary dynamics, this does not quite satisfy us. Let us check that they also agree on the boundary. Clearly, the expressions are singular at the boundary, so the only way to extract meaningful information is to prepare both the expressions with a general test function \( \eta_A(x) \). It is not difficult to see that the annihilation principle sweeps away any differences between the two smeared expressions:
\begin{equation}
\int_\Sigma \rho(x) d^d x \eta_A(x) \left[ \frac{d}{dt}, D^K \right] \phi^A(x) = 0. \tag{6.11}
\end{equation}

### 6.3 Floating Poisson bracket

The floating Poisson bracket is defined via the induced functional derivatives
\begin{equation}
\{ F(z), G(w) \} = \int_\Sigma \rho d^d x \left[ \frac{\delta}{\delta \phi^A(x)} \right] \frac{\delta [\chi_\epsilon(x) F(z)]}{\delta \phi^B(x)} \frac{\delta [\chi_\epsilon(x) G(w)]}{\delta \phi^B(x)} \omega^{AB} \chi_\epsilon(x).
\end{equation}

In the second equality, we used the identity (5.17). Let us check that the floating Poisson bracket (6.12) becomes the full Poisson bracket (4.5), by use of the annihilation principle when \( F \) and \( G \) are both local functionals with no external dependence. This is perhaps best seen from the second right hand side of (6.12). The idea is now to obtain a restrictive suitable form by recasting all the derivatives onto smooth objects. In the case at hand, the higher functional derivatives are both smooth functions of \( x \). One may deduce that the required form can only be obtained for terms \((M,N)\) when at least one of the indices \( M,N \) are \( \emptyset \). This reproduces precisely the full Poisson bracket (4.5). Technically speaking, for more general functionals \( F(z) \) and \( G(w) \), the above truncation takes place on parts of the functionals where both functionals are evaluated by the inner method. If at least one of them is evaluated by the outer method, then no truncation is carried out (although a truncation may take place at a later integration). This is particular the case for two functions \( F(z) \) and \( G(w) \).

The advantage of the floating formulation is at least two-fold: First of all, the full Poisson bracket can be written in a more compact manner. The second reason is that the floating Poisson bracket manifestly commutes with the spatial derivatives, cf. (5.13). This is quite useful. For instance, consider as previously, the Hamilton equations of motions, but this time with the floating bracket as the Poisson bracket. We get as before
\begin{equation}
\frac{d}{dt} \phi^A = \{ \phi^A, H \} = \frac{\omega^{AB}}{\chi_\epsilon} E_{B(0)} (\chi_\epsilon H). \tag{6.13}
\end{equation}
However, this time the spatial derivatives commute manifestly with the time evolution
\[
\frac{d}{dt}D^K\phi^A = \{\phi^A(K), H\} - D^K\{\phi^A, H\} = 0 .
\]  
(6.14)

At the end of the day, it always boils down to the full Poisson bracket (1.5), when all the variables are integrated out (and the annihilation principle applied). We voluntarily throw in a lot of formal zeroes in the floating Poisson bracket dressed up in the above “divergent-looking” disguise. This gambit enables us to write the full Poisson bracket in a very compact manner.

As perhaps the most important point, let us note that the Hamilton equations of motions (6.13) follows from extremizing, in the sense of (6.15), the following natural action:
\[
S = \int dt \int_\Sigma \rho d^d x \left[ \frac{1}{2} \phi^A(x,t)\omega_{AB}\dot{\phi}^A(x,t) - H(x,t) \right] .
\]  
(6.15)

One may define, a more general floating $\alpha$-bracket, for $\alpha \geq 1$, where
\[
\{F(z), G(w)\}_{(\alpha)} = \int_\Sigma \rho d^d x \chi^{-1}_\epsilon(x) \delta[\chi^\alpha_\epsilon(x)F(z)] \omega^{AB} \chi^{-1}_\epsilon(x) \frac{\delta[\chi^\alpha_\epsilon(x)G(w)]}{\delta \phi(x)}
\]  
\[
= \int_\Sigma \rho d^d x \frac{(-D)^{M_\epsilon} \chi^\alpha_\epsilon(x)}{\chi_\epsilon(x)} \frac{\delta F(z)}{\delta \phi^{(M)}(x)} \omega^{AB} \frac{(-D)^{N_\epsilon} \chi^\alpha_\epsilon(x)}{\chi_\epsilon(x)} \frac{\delta G(w)}{\delta \phi^{(N)}(x)} .
\]  
(6.16)

In this paper, we will merely view the floating ($\alpha > 1$)-brackets as a curiosity, which nevertheless has some relevance when addressing time-order issues, see next Section 7. It coincide in the inner sector with the above floating ($\alpha = 1$)-bracket. And it satisfies the Jacobi identity. We give an independent proof of the important the $\alpha = 1$ case in the Appendix E, and leave the general case $\alpha > 1$ to the reader. The $\alpha$-factor slows down the $\epsilon$-convergence process, but viewed as an isolated issue, it does not jeopardize the convergence.

7 Time Order and Quantization

One might suspect that the boundary terms are related to the time order prescriptions. We can give some heuristic arguments which points in that direction. Consider first the following totally ordered, antisymmetric and transitive time order prescription for two spacetime points $(x(1),t(1))$ and $(x(2),t(2))$ in unphysical space-time $\mathbb{R}^d \times \mathbb{R}$.
\[
(x(1),t(1)) \prec (x(2),t(2)) \iff (x(1),x(2) \in \Sigma \wedge t(1) < t(2)) \lor (x(1) \notin \Sigma \wedge x(2) \in \Sigma) \wedge (x(1),x(2) \notin \Sigma \wedge |d_\Sigma(x(1))| > |d_\Sigma(x(2))|) .
\]  
(7.17)

This prescription has as a consequence, that the boundary $\partial \Sigma$ is assigned to the infinite past, so there effectively is no spatial boundary. (In a similar manner, one may link it with the infinite future.) As we shall see below, the equal-time relation $\sim$ plays a crucial role, so let us define it properly:
\[
(x(1),t(1)) \sim (x(2),t(2)) \iff (x(1),x(2) \in \Sigma \wedge t(1) = t(2)) \lor (x(1),x(2) \notin \Sigma \wedge |d_\Sigma(x(1))| = |d_\Sigma(x(2))|) .
\]  
(7.18)
We can now give the time order prescription $T_\Sigma$ for $n$ operators $\hat{F}_i = \hat{F}_i(x, t_i)$, where $i = 1, \ldots, n$.

$$T_\Sigma \left[ \hat{F}_n \ldots \hat{F}_1 \right] = \hat{F}_{(\pi(n))} \ldots \hat{F}_{(\pi(1))}$$  \hspace{1cm} (7.19)

where $\pi \in S_n$ is the unique permutation $\{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$, that satisfies

$$(x_{(n)}, t_{(n)}) \geq \ldots \geq (x_{(1)}, t_{(1)}),$$  \hspace{1cm} \forall i = 1, \ldots, n - 1 : \pi(i+1) < \pi(i) \Rightarrow (x_{(\pi(i+1))}, t_{(\pi(i+1))}) \not\sim (x_{(\pi(i))}, t_{(\pi(i))}).$$  \hspace{1cm} (7.20)

Let us now consider an equal-time slice $t = t(0)$, i.e. in the traditional sense of the word “equal-time”, as we did in the previous sections. We shall suppress the time coordinate in the following. We have

$$T_\Sigma[\hat{F}(x), \hat{G}(y)] = \left[ 1_{\Sigma}(x) 1_{\Sigma}(y) + 1_{[0]} \left( |d_{\Sigma}(x(1))| - |d_{\Sigma}(x(2))| \right) (1 - 1_{\Sigma}(x)) (1 - 1_{\Sigma}(y)) \right] [\hat{F}(x), \hat{G}(y)]$$

$$\approx 1_{\Sigma}(x) 1_{\Sigma}(y) [\hat{F}(x), \hat{G}(y)].$$  \hspace{1cm} (7.21)

In the wavy equality $\approx$, we neglected a contribution from a spatial hypersurface of dimension $d - 1$, and hence of Lebesgue measure zero. Let us regularized this as

$$T^\beta_\Sigma[\hat{F}(x), \hat{G}(y)] = \chi^\beta(x) \chi^\beta(y) [\hat{F}(x), \hat{G}(y)] \hspace{1cm} (7.22)$$

for some positive power $\beta > 0$. When time ordering the spatial derivatives we get

$$D^M_{(x)} D^N_{(y)} T^\beta_\Sigma[\hat{F}(x), \hat{G}(y)] = \frac{1}{\chi^\beta(x)} \frac{1}{\chi^\beta(y)} T^\beta_\Sigma \left[ D^M_{(x)} \left( \chi^\beta(x) \hat{F}(x) \right), D^N_{(y)} \left( \chi^\beta(y) \hat{G}(y) \right) \right].$$  \hspace{1cm} (7.23)

(This equation should be understood as follows: On the left hand side the time order $T^\beta_\Sigma$ is everywhere present while we form the quotient of differences for the derivative $D_i$. In particular, it is present before we actually take the infinitesimal limit to produce the derivatives $D_i$. On the right hand side the time order $T^\beta_\Sigma$ does not recognize how the spatial derivatives earlier were produced. $T^\beta_\Sigma$ only sees the result: an operator depending on one space point.) This should be compared to the corresponding property of the floating ($\alpha > 1$)-Poisson bracket

$$D^M_{(x)} D^N_{(y)} \left\{ \hat{F}(x), \hat{G}(y) \right\}_{(\alpha)} = \frac{1}{\chi^\alpha(x)} \frac{1}{\chi^\alpha(y)} \left\{ D^M_{(x)} \left( \chi^{\alpha-1}(x) \hat{F}(x) \right), D^N_{(y)} \left( \chi^{\alpha-1}(y) \hat{G}(y) \right) \right\}_{(\alpha)}.$$  \hspace{1cm} (7.24)

This carries some evidence, that we should translate the floating ($\alpha > 1$)-Poisson bracket into the commutator with the above peculiar time order prescription, with $\beta = \alpha - 1$:

$$\frac{1}{i\hbar} T^{\beta=\alpha-1}_\Sigma[\hat{F}(x), \hat{G}(y)] \leftrightarrow \left\{ F(x), G(y) \right\}_{(\alpha)}.$$

(7.25)

Although the exact value of $\alpha > 1$ should be taken with a grain of salt, let us compare this behavior with the behavior of the floating ($\alpha = 1$)-Poisson bracket:

$$D^M_{(x)} D^N_{(y)} \{ F(x), G(y) \} = \left\{ D^M_{(x)} F(x), D^N_{(y)} G(y) \right\}.$$  \hspace{1cm} (7.26)

This corresponds to the commutator with the usual time order prescription. It is also interesting to compare with the corresponding property of the bulk Poisson bracket, cf. (5.19), although it of
course does not satisfy the Jacobi identity and is therefore not expected to play any leading role at
the level of quantization:

\[
\frac{1}{\chi(x)} \frac{1}{\chi(y)} D^M_x D^N_y \{\chi(x) F(x), \chi(y) G(y)\}(0) = \left\{ D^M_x F(x), D^N_y G(y) \right\}(0).
\] (7.27)

So we have here presented two physically different, but both consistent, time orderings. One gov-
erned by the floating ($\alpha > 1$)-Poisson bracket, but with some of the “equal-time” surfaces wrapped
up along the boundary $\partial \Sigma$. And another system governed by the floating ($\alpha = 1$)-Poisson bracket
with the boundary $\partial \Sigma$ being a true spatial boundary for the system. Although the above analysis
clearly may be criticized wrt. 1) the order of various limits taken, 2) the omission of the role played
by the annihilation principle, and 3) its disregards of further ordering issues (like $*$-product), it
tends to confirm the importance of the boundary terms of the Poisson bracket, and that they
should not be discarded in a full treatment of a quantum field theory with a spatial boundary.

8 Conclusions

In this article we have

- Reviewed the higher functional derivatives and the extended notion of differentiability of
  functionals.

- Shown a new way to add a boundary contribution to the usual “bulk” Poisson bracket, so
  that the Jacobi identity is satisfied.

- Given a manifest formulation of this new Poisson bracket.

- Geometrized the Poisson bracket to an abstract world volume manifold.

- Reviewed an embedded framework to treat Dirac delta distributions $\delta_{\Sigma}(x, y)$ in the presence
  of a boundary $\partial \Sigma$.

- Introduced an annihilation principle and a floating Poisson bracket.

- Given an action principle for Hamiltonian systems with a spatial boundaries.

- Discussed the relation between the boundary terms in the Poisson bracket and the choice of
time order in a heuristic manner.

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A Various Identities

First of all, let us prove that the Heisenberg algebra reproduces the algebra (3.2) of partial and
spatial derivatives. This follows from

\[
\sum_{k,n \geq 0} \frac{q^k y^n}{n!} P_{A(k)} \partial^n = P_A e^{\partial Y} e^{\partial q} = P_A e^{\partial q} e^{\partial Y} e^{[q Y, \partial q]} = e^{\partial q} P_A(q) e^{\partial q}
\]
Similary, we mention Next, let us check the identity (2.15). The proof goes as follows:

\[ k \]

It is worth pointing out the case

\[ \text{Consider three functions } f, g, h \text{, we have} \]

\[ \text{where we have introduced a shorthand notation for the following five terms} \]

\[ \text{The Jacobi identity, containing 30 } T_r \text{-terms, now follows from the fact that} \]

\[ T_2(f, g, h) = T_2(h, g, f) \]
\[ T_1(f, g, h) = T_3(h, g, f) \]
\[ T_4(f, g, h) = T_5(h, g, f) \]
The first equation is trivial and the next two equations follow by rewriting in terms of Fourier transforms

\[
T_1(f, g, h) = P_A(q_A) f \omega^{AB} \exp \left[ \frac{\partial}{\partial q_A} - \frac{\partial}{\partial q_B} \right] \left[ P_B(q_B) E_C(q_C) g \omega^{CD} E_D(q_D) h \right] \bigg|_{q=0}
\]

\[
= \int d^d y P_A(y_A) f \omega^{AB} e^{\partial(y_A - y_B)} \left[ P_B(y_B) E_C(y_C) g \omega^{CD} E_D(y_D) h \right]
\]

\[
= \int d^d y P_A(y_A) f \omega^{AB} e^{\partial(y_A - y_B)} \left[ e^{-\partial y_C} P_B(y_B + y_C) P_C(y_C) g \omega^{CD} E_D(y_D) h \right]
\]

\[
= \int d^d y P_A(y_A) f \omega^{AB} e^{\partial(y_A - y_B)} P_B(y_B) P_C(y_C) g \omega^{CD} e^{\partial(y_A + y_C - y_B)} E_D(y_D) h ,
\]

\[
T_3(f, g, h) = \exp \left[ \frac{\partial}{\partial q_A} \right] E_A(q_A) f \omega^{AB} P_B(q_B) \exp \left[ \frac{\partial}{\partial q_D} \right] E_C(q_C) g \omega^{CD} P_D(q_D) h \bigg|_{q=0}
\]

\[
= \int d^d y e^{\partial y_B} E_A(y_A) f \omega^{AB} P_B(y_B) e^{\partial y_D} E_C(y_C) g \omega^{CD} P_D(y_D) h
\]

\[
= \int d^d y e^{\partial y_B} E_A(y_A) f \omega^{AB} e^{\partial(y_D - y_C)} P_B(y_B + y_C - y_D) P_C(y_C) g \omega^{CD} P_D(y_D) h
\]

\[
= \int d^d y e^{\partial(y_B - y_C)} E_A(y_A) f \omega^{AB} e^{\partial(y_D - y_C)} P_B(y_B) P_C(y_C) g \omega^{CD} P_D(y_D) h ,
\]

\[
T_4(f, g, h) = \exp \left[ -\frac{\partial}{\partial q_B} \right] E_A(q_A) f \omega^{AB} P_B(q_B) E_C(q_C) g \omega^{CD} E_D(q_D) h \bigg|_{q=0}
\]

\[
= \int d^d y e^{\partial y_B} E_A(y_A) f \omega^{AB} P_B(y_B) e^{\partial y_C} P_B(y_B + y_C) P_C(y_C) g \omega^{CD} E_D(y_D) h
\]

\[
= \int d^d y e^{\partial(y_B - y_C)} E_A(y_A) f \omega^{AB} e^{\partial y_C} P_B(y_B) P_C(y_C) g \omega^{CD} E_D(y_D) h ,
\]

\[
T_5(f, g, h) = E_A(q_A) f \omega^{AB} \exp \left[ \partial q_B \right] \left[ P_B(q_B) E_C(q_C) g \omega^{CD} E_D(q_D) h \right] \bigg|_{q=0}
\]

\[
= \int d^d y E_A(q_A) f \omega^{AB} e^{\partial y_B} \left[ P_B(y_B) E_C(y_C) g \omega^{CD} E_D(y_D) h \right]
\]

\[
= \int d^d y E_A(q_A) f \omega^{AB} e^{\partial y_B} \left[ e^{-\partial y_C} P_B(y_B + y_C) P_C(y_C) g \omega^{CD} E_D(y_D) h \right]
\]

\[
= \int d^d y E_A(q_A) f \omega^{AB} e^{\partial y_B} P_B(y_B) P_C(y_C) g \omega^{CD} e^{\partial(y_C - y_B)} E_D(y_D) h .
\]

We have used the following shorthand notation for the integration measure

\[
d^d y = d^d y_A d^d y_B d^d y_C d^d y_D ,
\]

and we have performed the following change of integration variables

\[
\begin{pmatrix}
  y_A \\
  y_B \\
  y_C \\
  y_D
\end{pmatrix} = \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 1 & * \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
  y_A \\
  y_B \\
  y_C \\
  y_D
\end{pmatrix} ,
\]

which has Jacobian equal to 1.

\[\square\]
C Calculus of Words

To be self-contained, we will here give a short treatment of the calculus with ordered index-structure, merely giving the main definitions and formulas.

C.1 Words

An ordered tuple $K$ (or a positive word) takes the form

$$K = (k_1, \ldots, k_{|K|}) \in \{1, \ldots, d\}^{|K|}. \quad (C.1)$$

Here we have employed a $d$-letter alphabet $\{1, \ldots, d\}$ and $|K| \in \mathbb{N}_0$ denotes the length of $K$. The transposed word is $K^t = (k_{|K|}, \ldots, k_1)$. We define the (non-commutative, associative) sum of two tuples $K = (k_1, \ldots, k_{|K|})$ and $L = (\ell_1, \ldots, \ell_{|L|})$ as the concatenation

$$K + L \equiv (k_1, \ldots, k_{|K|}, \ell_1, \ldots, \ell_{|L|}). \quad (C.2)$$

Obviously, the empty index set $\emptyset$ is the neutral element. Formally, we can define negative words

$$-K = -(k_{|K|}) - \ldots - (k_1).$$

Moreover $-(K + L) = -L - K$. We define a left subtraction $-K + L$ as the unique solution to $K + (-K + L) = L$. If the left subtraction $-K + L$ is a positive word, we say that $K \leq L$. Clearly, $(K + L)^t = L^t + K^t$.

There exists another partial order between two arbitrary positive words $K$ and $L$. Namely, define that $K$ precede (or is equal to) $L$, written $K \preceq L$, if we can obtain $K$ from $L$ by deleting some (possible no or all) elements in $L$. Said in a mathematical precise manner, there exists a (strongly) increasing index function $\pi : \{1, \ldots, |K|\} \to \{1, \ldots, |L|\}$, called an (orderpreserving) embedding, such that

$$k_1 = \ell_{\pi(1)}, \ldots, k_{|K|} = \ell_{\pi(|K|)}, \quad (C.3)$$

In the affirmative case, we define the subtraction $L \div K$ as the tuple of deleted entries. More precisely, in this case there exists a unique (strongly) increasing index function $\pi^e : \{1, \ldots, |L| - |K|\} \to \{1, \ldots, |L|\}$, called the complementary embedding,

$$\pi^e : \{1, \ldots, |L| - |K|\} \to \{1, \ldots, |L|\}, \quad (C.4)$$

such that the images of $\pi$ and $\pi^e$ are disjoint, i.e.

$$\pi(\{1, \ldots, |K|\}) \cap \pi^e(\{1, \ldots, |L| - |K|\}) = \emptyset, \quad \pi^e(1) < \ldots < \pi^e(|L| - |K|). \quad (C.5)$$

Then the subtraction $L \div K$ is defined as

$$L \div K = (\ell_{\pi^e(1)}, \ldots, \ell_{\pi^e(|L| - |K|)}). \quad (C.6)$$

We stress that the embedding $\pi$ is not necessary unique. Therefore $L \div K$ depends on the embedding $\pi$. For instance, in the entanglement formula (4.7), one should sum over all possible embeddings. A closed expression for the degeneracy $d(K \preceq L)$ of imbeddings $\pi$ is not known to the author. By definition $d(\emptyset \preceq L) = 1$. Note that $K \preceq L$ implies $L \div K \preceq L$. In the affirmative case the pair $K$ and $L \div K$ is called \[3\] an unshuffle of $L$. There exist $2^{|L|}$ unshuffles of $L$. Furthermore,

$$(L \div K)^t = L^t \div K^t,$$
\[ K \leq L \leq N \quad \Rightarrow \quad N \div L = (N \div K) \div (L \div K). \quad (C.7) \]

A shuffle \( K \# M \) of two positive words \( K \) and \( M \) is defined as the opposite of an unshuffle in the sense that it is a solution \( X \) to \( X \div K = M \). Clearly, the number of shuffles for fixed \( K \) and \( M \) is
\[ \binom{|K|+|M|}{|M|}. \]

The number of solutions \( X \) to \( L \div X = M \) for fixed positive \( M, L \) with \( M \leq L \) is \( d(M \leq L) \).

### C.2 Alphabets of Operators

A \( d \)-letter alphabet of operators (or more generally, of associative abstract algebra elements), is just \( d \) operators \( A = (A_1, \ldots, A_d) \). The sum and the product (i.e. usually the operator composition) of two alphabets are defined letterwise
\[ A + B = (A_1 + B_1, \ldots, A_d + B_d), \quad AB = (A_1B_1, \ldots, A_dB_d), \quad (C.8) \]
respectively.

### C.3 Words of Operators

If we have an alphabet of operators \( A = (A_1, \ldots, A_d) \) we can form words of operators
\[ A^K = A_{k_1} \cdots A_{k_{|K|}}. \quad (C.9) \]

We invoke the convention that \( A_K = A^{(K^t)} \) denotes the transposed word. The empty word operator \( A^\emptyset = 1 \) is the identity. Operators for non-positive words, i.e. for words containing negative letters, are declared to be zero. Concatenation leads to a non-commutative product
\[ A^N A^M = A^{N+M} \quad (C.10) \]

between words (and \( A_N A_M = A_{M+N} \) for the transposed). It coincides with the letterwise multiplication. But this is not the only associative product of words. Shuffling leads to a commutative \(*\)-product
\[ \frac{A^K}{|K|!} \ * \ \frac{A^L}{|L|!} = \sum_{N=K \# L} \frac{A^N}{|N|!}, \quad \left( A^K \ * 1 = A^K \right), \quad (C.11) \]

where the sum is over possible shuffles \( K \# L \). The definition of the \(*\)-product is extended by \( \mathcal{C}\)-bilinearity. The concatenation product and the \(*\)-product coincide for commutative alphabets. The following binomial relation is a consequence of the special features of unshuffles, cf. (C.8) and (C.9),
\[ [A_i, B_j] = 0 \quad \Rightarrow \quad (A + B)^N = \sum_{M=\emptyset}^{M \leq N} A^M B^{N-M}. \quad (C.12) \]

We can also define a non-commutative, associative sum by the following binomial expression
\[ \frac{(A \# B)^N}{|N|!} = \sum_{K,L \geq \emptyset}^{K+L \leq N} \frac{A^K}{|K|!} \frac{B^L}{|L|!}. \quad (C.13) \]

This sum “\( \# \)” and the usual sum “+” do not coincide in general, not even for commutative alphabets. But see the equation (C.22) below for further relationship.
Note that the definition \((C.11)\) makes no use of the algebra multiplication. It only needs a vector space with a basis of vectors \(A^K\) indexed by words \(K\). The definition of the \#-sum \((C.13)\) needs an algebra of words \(A^K\), but it is irrelevant whether \(A^K\) is a composite object of more elementary letters or not. The same remark could be made about formulas \((C.10)\) and \((C.12)\), that in the minimalistic interpretation becomes definitions. Even in the case where there exists a letterwise algebra multiplication, we will often use the binomial formula \((C.12)\), also known as a convolution, which makes sense even for mutually non-commuting alphabets.

We have that 
\[
-(A \# B) = (-A) \# (-B) \quad \text{and} \quad (A^N)^t = A^{(N^t)} \quad \text{and} \quad (B^N)^t = B^{(N^t)}.
\]

\((C.14)\)

Sometimes we will also need to define
\[
\frac{(A \# B)^N}{|N|!} \equiv \sum_{K, L \geq \emptyset}^{K + L = N} \frac{A^{(K^t)} B^L}{|K|! |L|!}.
\]

\(\text{etc.}\)

C.4 Functions of Operators

For an analytic function \(f(x) = \sum_{n=0}^{\infty} a_n x^n\), we define \(f(A) = \sum_{N=0}^{\infty} a_{|N|} A^N\). In particular, the exponential of an alphabet is
\[
\exp(A) = \sum_{N=0}^{\infty} \frac{A^N}{|N|!}.
\]

\((C.15)\)

We have that
\[
[A_i, A_j] = 0 \Rightarrow \exp(A) = \exp(A_1) \ldots \exp(A_d) ,
\]
\[
[A_i, B_j] = 0 \Rightarrow \exp(A + B) = \exp(A) \exp(B).
\]

\((C.16)\)

Also we have the important orthogonality relation
\[
((-A)^t + A)^N \equiv \sum_{M=0}^{M \leq N} (-A)^{(M^t)} A^{N-M} = \delta_{N, \emptyset} \equiv 0^N.
\]

\((C.17)\)

This property leads to the vital relation \(e^A e^{-A} = 1\) even for a non-commutative alphabet. Similary, we have \((-A)^t \# A)^N = 0^N\).

In practice, we only use \(f(AB)\) for a product of two alphabets. (Dummy indices usually come in pairs.) It is convenient to define a “dot product” notation
\[
f(A \cdot B) = \sum_{N=0}^{\infty} a_{|N|} A^N B^{(N^t)} ,
\]

\((C.18)\)

implementing a transposition of one of the alphabets. Also \(f((-A)^t \cdot B) \equiv f((-A)B)\). We have the following inversion relation for the concatenation product
\[
[A_i, B_j] = 0 \Rightarrow \exp \left[ (-A) \cdot B \right] \exp(A \cdot B) = 1.
\]

\((C.19)\)

This should be compared with the inversion relation for the the mixed case of an implicitly written concatenation product and a ∗-product
\[
[A_i, B_j] = 0 \Rightarrow \exp \left[ (-A)^t \cdot B \right] B^* \exp(A \cdot B) = 1.
\]

\((C.20)\)
We have the following distributive laws
\[
\begin{align*}
[A_i, B_j] &= 0 \\
[A_i, C_j] &= 0 \\
[B_i, C_j] &= 0
\end{align*}
\Rightarrow \begin{cases} 
\exp \left[ (-A)^t \cdot (B \# C) \right] = \exp((-A)^t \cdot B) \exp((-A)^t \cdot C) \\
\exp [A \cdot (B \# C)] = \exp(A \cdot C) \exp(A \cdot B) \\
\exp [(A + B) \cdot C] = \exp(A \cdot C) *_C \exp(B \cdot C) \\
= \exp(B \cdot C) *_C \exp(A \cdot C)
\end{cases} .
\] (C.21)

Note the reversed order in the second equation. The sum “#” and the usual sum “+” coincide loosely speaking in average. More precisely,
\[
[A_i, A_j] = 0 \Rightarrow \exp [A \cdot (B \# C)] = \exp [A \cdot (B + C)] .
\] (C.22)

### C.5 Trace and Fourier Analysis

We define a trace on the vector space of words, which is constructed from two mutually commuting freely generated alphabets \( A \) and \( B \):
\[
\int d^d A \  d^d B \ e^{(-A)^t \cdot B} \ A^N \ B_M \equiv \text{Tr}(A^N \ B_M) = |N|! \ \delta_M^N ,
\] (C.23)

and extend by \( \mathcal{C} \)-bilinearity. As the first equality suggests, we will sometimes use a suggestive notation for the trace borrowed from the Fourier analysis in the usual commutative case. One can take this analogy quite far. We do not give any meaning to the position of the measure \( d^d A \  d^d B \), \textit{i.e.} it is taken to commute with everything. A theoretically perhaps more convenient form is
\[
\int d^d A \  d^d B \ \exp \left[ (-A - A')^t \cdot (B \# B') \right] = 1 .
\] (C.24)

From here it follows trivially that the integration measure \( d^d A \  d^d B \) is translation invariant under \( A \to A + A' \), \( B \to B \# B' \) (but not under \( B \to B' \# B!) \).

### C.6 Differentiation of Words

Consider the differential alphabet \( \frac{\partial}{\partial A} = (\frac{\partial}{\partial A_1}, \ldots, \frac{\partial}{\partial A_d}) \) of freely generated associative algebra elements \( A_i \). Let us define differentiation at the level of words ( \textit{i.e.} not letterwise), as
\[
\frac{1}{|K|!} \frac{\partial}{\partial A^K} \left[ A^L \right] = \sum_{\pi: K \leq L} A^{L \# K}, \quad \left( \frac{\partial}{\partial A^0} [1] = 1 \right) ,
\] (C.25)

where the sum is over possible embeddings \( \pi \). Extend the definition by \( \mathcal{C} \)-bilinearity. One of the main motivations behind this definition is to implement the Taylor formula
\[
[A_i, B_j] = 0 \Rightarrow \exp \left[ B \cdot \frac{\partial}{\partial A} \right] f(A) = f(B + A) .
\] (C.26)

The composition of derivatives is described by the \( * \)-product
\[
\frac{1}{|K|!} \frac{\partial}{\partial A^K} \frac{1}{|L|!} \frac{\partial}{\partial A^L} = \sum_{N=K \# L} \frac{1}{|N|!} \frac{\partial}{\partial A^N} \equiv \frac{1}{|K|!} \frac{\partial}{\partial A^K} * \frac{1}{|L|!} \frac{\partial}{\partial A^L} .
\] (C.27)

As consequences, the derivatives are associative and commutative wrt. composition. They enjoy the following properties
\[
[A_i, B_j] = 0 \Rightarrow \exp \left[ (-B)^t \cdot \frac{\partial}{\partial A} \right] \exp \left[ B \cdot \frac{\partial}{\partial A} \right] = 1 ,
\]

29
\[ [A_i, C_j] = 0 \]
\[ [B_i, C_j] = 0 \]
\[ \Rightarrow \exp \left[ (B + C) \cdot \frac{\partial}{\partial A} \right] = \exp \left[ B \cdot \frac{\partial}{\partial A} \right] \exp \left[ C \cdot \frac{\partial}{\partial A} \right]. \tag{C.28} \]

We can of course implement the concatenation product for the differential. However, in practical calculations it plays no role. The derivatives \tag{C.25} do not satisfy the Leibnitz’ rule.

The are other kinds of differential alphabets. In our case, we have the covariant derivatives \( D_i \) that act on words \( (-Y)_K \) according to
\[ [D_i, (-Y)_K] = \begin{cases} |K| \cdot (-Y)^{(i)+K} & \text{if } (i) \leq K, \\ 0 & \text{otherwise}. \end{cases} \tag{C.29} \]

(One can consider the words \( (-Y)_K \) as originating from an associative algebra alphabet \( (-Y^1, \ldots, -Y^d) \) that behaves non-associatively wrt. the \( D_i \)’s, but it is unnecessary.) The covariant derivatives satisfy Leibnitz’ rule on functions:
\[ D^K(fg) = \sum_{M=0}^{M=K} D^M f \cdot D^{K-M}g. \tag{C.30} \]

This can be recasted into the Taylor-like form \( e^{D^y(fg)} = e^{D^yf} \ast_y e^{D^yg} \).

**D Proof of the Jacobi Identity (Non-Commutative Case)**

We have
\[
\{ f, \{ g, h \} \} = P_{A(A)} f \omega^{AB} D^A E_{B(0)} \{ g, h \}_{(0)} + D^B E_{A(0)} f \omega^{AB} P_{B(B)} \left[ D^D E_{C(0)} g \omega^{CD} P_{D(D)} h \right.
\]
\[ + P_{C(C)} g \omega^{CD} D^C E_{D(0)} h - \{ g, h \}_{(0)} \left] - E_{A(0)} f \omega^{AB} E_{B(0)} \{ g, h \}_{(0)} \right)
\]
\[ = T_1(f, g, h) + T_2(f, g, h) + T_3(f, g, h) - T_4(f, g, h) - T_5(f, g, h) - (g \leftrightarrow h), \tag{D.1} \]

where we have introduced a shorthand notation for the following five terms
\[
T_1(f, g, h) = P_{A(A)} f \omega^{AB} D^A (-D)^{B'} \left[ P_{B(B)} E_{C(0)} g \omega^{CD} E_{D(0)} h \right]
\]
\[ T_2(f, g, h) = D^B E_{A(0)} f \omega^{AB} P_{B(B)} P_{C(C)} g \omega^{CD} D^C E_{D(0)} h, \]
\[ T_3(f, g, h) = D^B E_{A(0)} f \omega^{AB} P_{B(B)} D^D E_{C(0)} g \omega^{CD} P_{D(D)} h, \]
\[ T_4(f, g, h) = D^B E_{A(0)} f \omega^{AB} P_{B(B)} E_{C(0)} g \omega^{CD} E_{D(0)} h, \]
\[ T_5(f, g, h) = E_{A(0)} f \omega^{AB} (-D)^{B'} \left[ P_{B(B)} E_{C(0)} g \omega^{CD} E_{D(0)} h \right]. \tag{D.2} \]

Here we have chosen to use the same index symbol \( A, B, C \) and \( D \) to label the indices of the fields \( \phi \) and the words. It should not lead to any ambiguities, and it hopefully becomes easier to grasp the index structure. The Jacobi identity, containing 30 \( T_i \)-terms, now follows from the fact that
\[ T_2(f, g, h) = T_2(h, g, f), \quad T_1(f, g, h) = T_3(h, g, f), \quad T_4(f, g, h) = T_5(h, g, f). \tag{D.3} \]

The first equation is trivial and the next two equations follows by rewriting in terms of Fourier transforms
\[
T_1(f, g, h) = P_A f \omega^{AB} \exp \left[ D \cdot \frac{\partial}{\partial q_A} \right] \exp \left[ (-D)^{B'} \cdot \frac{\partial}{\partial q_B} \right] \left[ P_B E_{C(0)} g \omega^{CD} E_{D(h)} \right]_{q=0}
\]
We have performed the following type of translation of the integration variables

\[ y' = yB \# yC \quad \text{or} \quad y' = yB \# (-yD) \# yC. \]  

Note that after the shift of integration variables the \( y \)-alphabets do no longer mutually commute. However, one may convince oneself that the integrations can be unwound, and we can consistently declare them to mutually commute also in the new variables. Finally to prove (D.3), one should relabel dummy variables \( ABCD \rightarrow DCBA \).

\[ \square \]

### E Proof of the Jacobi Identity (Floating Type)

We now turn to the proof of the Jacobi identity for the floating Poisson bracket, cf. Eq. (6.12). Consider the local functionals of Subsection 2.3. We assume that \( D_{ij} = 0 \). Suppressing the integrations, we have

\[
\{ f, \{ g, h \} \} = \frac{E_{A(0)}(\chi_{\epsilon} f)}{\chi_{\epsilon}} \omega^{AB} E_{B(0)} \left[ \frac{E_{C(0)}(\chi_{\epsilon} g)}{\chi_{\epsilon}} \omega^{CD} E_{D(0)}(\chi_{\epsilon} h) \right]
\]
Here we have applied the following shorthand notation

\[ T(f, g, h) = \frac{E_{A(0)}(X_t f)}{X_t} \omega^{AB} (-D)^B \left[ P_{B(B)} E_{C(0)}(X_t g) \omega^{CD} \frac{E_{D(0)}(X_t h)}{X_t} \right], \]

where

\[
\begin{align*}
T(f, g, h) &\equiv \frac{E_{A(0)}(X_t f)}{X_t} \omega^{AB} (-D)^B \left[ P_{B(B)} E_{C(0)}(X_t g) \omega^{CD} \frac{E_{D(0)}(X_t h)}{X_t} \right] \\
&= \frac{E_{A(0)}(X_t f)}{X_t} \omega^{AB} \exp \left[ (-D)^B \cdot \frac{\partial}{\partial q_B} \right] \left[ P_{B(B)} E_{C(0)}(X_t g) \omega^{CD} \frac{E_{D(0)}(X_t h)}{X_t} \right]_{q=0} \\
&\sim T_D e^{\frac{1}{2} (-D)^{y_C} \cdot X_t f} \omega^{AB} \left[ P_{B(B)} e^{\frac{1}{2} (-D)^{y_C} \cdot X_t g} \omega^{CD} \frac{E_{D(0)}(X_t h)}{X_t} \right] \\
&= T_D e^{\frac{1}{2} (-D)^{y_C} \cdot X_t f} \omega^{AB} \left[ P_{B(B)} e^{\frac{1}{2} (-D)^{y_C} \cdot X_t g} \omega^{CD} \frac{E_{D(0)}(X_t h)}{X_t} \right] \\
&\times T_B e^{\frac{1}{2} (-D)^{y_C} \cdot X_t g} \omega^{AB} \left[ P_{B(B)} e^{\frac{1}{2} (-D)^{y_C} \cdot X_t f} \omega^{CD} \frac{E_{D(0)}(X_t h)}{X_t} \right].
\end{align*}
\]

The \( \sim \) indicates that the equality holds up to total derivative terms. They are unphysical terms living far away from the bounded physical region \( \Sigma \), and therefore vanishing. In the last step we substituted \( y_B = y_B \# y_C \). The annihilation principle will not change the fact, that the Jacobi identity is fulfilled, because all annihilated terms appear in pairs with opposite sign.

\[ \square \]

Let us now turn to more general functionals \( F(u), G(v) \) and \( H(w) \). We have

\[
\{ F(u), \{ G(v), H(w) \} \} = \int \rho(x) d^4 x \rho(y) d^4 y \ F^B(x) \left( \frac{\delta}{\delta \phi^B(x)} \right) \left( X_t(x) \frac{\delta [X_t(y) G(v)]}{\delta \phi^C(y)} \right) H^C(y),
\]

where we have applied the following shorthand notation

\[
F^B(x) = \frac{\delta [X_t(x) F(u)]}{\delta \phi^A(x)} \omega^{AB}, \quad H^C(y) = \frac{\omega^{CD} \delta [X_t(y) H(w)]}{X_t(y) \frac{\delta \phi^D(y)}{\delta \phi^C(y)}},
\]

and

\[
T(F, G, H) = F^B(x) (-D(x))^{B^*} \left[ X_t(x) \frac{\partial}{\partial \phi^B(x)} \frac{\delta [X_t(y) G(v)]}{\delta \phi^C(y)} \right] H^C(y)
\]

\[
= F^B(x) (-D(x))^{B^*} \left[ X_t(x) \frac{\partial}{\partial \phi^B(x)} \sum_{j=1}^r (-D(y))^{B^*} \frac{\delta (y - v(j))}{\rho(y)} \phi^{(v(j))}(Y) \right] H^C(y)
\]

\[
= T_1(F, G, H) + T_2(F, G, H),
\]

where the last equality will be explained below. We distinguish between the so-called inner \( j \)-terms \( j = 1, \ldots, s \), where the spatial \( D(y) \)-differentiation are applied on the \( P_{B}\)-derivatives of the \( G \)-functional before the partial derivative \( P_B \), and on the other hand the so-called outer \( j \)-terms,
where the order is the opposite. For each inner $j$-term, we may write $G(v) = \int \rho(v_{(j)}) d^{d}v_{(j)} g_{j}(v)$. Together, the diagonal piece of inner $j$-terms becomes

\[
T_{1}(F,G,H) \equiv F^{B}(x) \left[ (-D(x))^{(j)} \sum_{j=1}^{s} \frac{\delta(x-v_{(j)})}{\rho(x)} P_{B(B)}^{(v_{(j)})} (-D(v_{(j)}))^{C} P_{C(C)}^{(v_{(j)})} g_{j}(v)_{v_{(j)}=y} \right] H^{C}(y)
\]

\[
= F^{B}(x) \left[ (-D(x))^{(j)} \sum_{j=1}^{s} \frac{\delta(x-y)}{\rho(y)} P_{B(B)}^{(v_{(j)})} (-D(v_{(j)}))^{C} P_{C(C)}^{(v_{(j)})} g_{j}(v)_{v_{(j)}=y} \right] H^{C}(y),
\]

(E.6)

The $y$-integration may be explicitly performed in the diagonal $T_{1}$-term:

\[
\int \rho(y) d^{d}y T_{1}(F,G,H) = F^{B}(x) (-D(x))^{B} \left[ \sum_{j=1}^{s} P_{B(B)}^{(v_{(j)})} (-D(v_{(j)}))^{C} P_{C(C)}^{(v_{(j)})} g_{j}(v)_{v_{(j)}=x} \right] H^{C}(x).
\]

(E.7)

It may now be treated similarly to the local case discussed in equation (E.2). The rest of the terms appearing in (E.5) can be organized so that they are manifestly symmetric in $F$ and $H$, and hence do not effectively contribute to the Jacobi identity:

\[
T_{2}(F,G,H) \equiv F^{B}(x) \left[ (-D(x))^{B} (-D(y))^{C} \sum_{i,j} \frac{\delta(x-v_{(i)}) \delta(y-v_{(j)})}{\rho(x) \rho(y)} P_{B(B)}^{(v_{(i)})} P_{C(C)}^{(v_{(j)})} G(v) \right] H^{C}(y).
\]

(E.8)

(The prime $'$ indicates that the above inner diagonal $j$-terms should not be included in the $T_{2}$-sum.)

\[\square\]

F Realization of Derivatives

In this section we will like to enode in an alternative manner the information about which part of an expression that are hit by a derivative. Instead of the usual practice of indicating this with arbitrarily many arrows, we define a linear chain of operators containing the same information. Consider some expression, where upon the derivatives $D_{i}$ act. Notation: For simplicity, we assume that the covariant derivatives commute, i.e. that the curvature vanishes. First of all, we assign names $\alpha$ to the derivatives $D_{i} \sim D_{(\alpha)i}$ so that we may distinguish them. For all derivatives $D_{(\alpha)i}$, let us introduce a scalar elements $a_{(\alpha)}$ and a vector element $b_{(\alpha)}^{i}$ that obey

\[
a_{(\alpha)}a_{(\alpha)} = 1, \quad b_{(\alpha)}^{i}a_{(\alpha)} = 0, \quad b_{(\alpha)}^{i}a_{(\alpha)} = -b_{(\alpha)}^{i}, \quad a_{(\alpha)}b_{(\alpha)}^{i} = b_{(\alpha)}^{i}.
\]

(F.9)

The notions scalar and vector refer to their properties under coordinate transformations. We have the following commutator relations

\[
[a_{(\alpha)}, a_{(\beta)}] = 0, \quad [b_{(\alpha)}^{i}, b_{(\beta)}^{j}] = 0, \quad [a_{(\alpha)}, b_{(\alpha)}^{i}]_{+} = 0 , \quad [a_{(\alpha)}, b_{(\beta)}^{j}] = 2\delta_{\alpha\beta} b_{(\beta)}^{j}.
\]

(F.10)

Furthermore, we introduce a Berezin-like integration $\int db_{(\alpha)i}$. In the following, $f$ and $f'$ denote functions which do not depend on $a_{(\alpha)}$ and $b_{(\alpha)}^{i}$, but possibly on other $a$’s and $b$’s. We will assign the value

\[
\int db_{(\alpha)i} b_{(\alpha)}^{i} f = \delta_{\xi}^{i} f, \quad \int db_{(\alpha)i} f = 0.
\]

(F.11)
We emphasize that we have not defined the integral $\int db^{(a)}i$ if there are $a^{(a)}$ elements present in the integrand. Let us also declare

$$[b^{(a)}i, f] = 0, \quad [f, f'] = 0, \quad [f, a^{(a)} f' a^{(a)}] = 0,$$  \hspace{1cm} (F.12)

Now we are ready to define the transition from derivatives to $a$’s and $b$’s. The derivative $D^{(a)}i f$ may now be written as

$$D^{(a)}i f \rightarrow \int db^{(a)}i a^{(a)} f a^{(a)} = \int db^{(a)}i [a^{(a)}, f] a^{(a)} .$$  \hspace{1cm} (F.13)

This on the other hand is evaluated by declaring

$$- [a^{(a)}, f] \equiv [f, a^{(a)}] = \sum_i b^{(a)}i D_i f .$$  \hspace{1cm} (F.14)

In this way we may imagine an expression with an arbitrary even (non-negative) number of $a^{(a)}$’s. For instance, one may check that

$$\int db^{(a)}i f a^{(a)} g a^{(a)} f' a^{(a)} g' a^{(a)} f'' = f f' f'' D_i (g g') .$$  \hspace{1cm} (F.15)

Here we see that the $a$’s plays the role of “start and stop signs” for the action of the derivatives. The Leibnitz rule is implemented via the assertion (F.14). An integration by part of an expression may be performed by cyclicly moving an $a^{(a)}$ from one end of the expression to the other end under an additional change of sign. Let now $s$ and $s'$ be two singular expression, cf. point $B1$ in Subsection 5.5. Some of the consequences of the annihilation principle may now be stated as

$$s(x^{(a)}) [a^{(a)}, s'(x^{(a)})] = 0 , \quad s(x^{(a)}) [a^{(a)}, \delta(x^{(a)} - y)] s'(y) = 0 .$$  \hspace{1cm} (F.16)

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