New conserved quantities for \( N = 1 \) SKdV, the supersymmetric cohomology and \( W \)-algebras

S. Andrea\(^a\), A. Restuccia\(^b\), A. Sotomayor \(^c\)*

November 26, 2009

\(^a\)Departamento de Matemáticas,
\(^b\)Departamento de Física
Universidad Simón Bolívar
Venezuela
\(^c\)Departamento Matemáticas
Universidad de Antofagasta
Chile
e-mail: sandrea@usb.ve, arestu@usb.ve, sotomayo81@yahoo.es

Abstract

An infinite sequence of new non-local conserved quantities for the \( N = 1 \) Super KdV (SKdV) equation is obtained. The sequence is constructed, via a Gardner trasformation, from a new conserved quantity of the Super Gardner equation. The SUSY generator defines a nilpotent operator of the space of all conserved quantities into itself. On the ring of \( C^\infty \) superfields the local conserved quantities are closed but not exact. However on the ring of \( C^\infty_{NL,1} \) superfields, an extension of the \( C^\infty \) ring, they become exact and equal to the SUSY transformed of the subset of odd non-local conserved quantities of the appropriate weight. The remaining odd non-local ones generate closed geometrical objects which become exact when the ring is extended to the \( C^\infty_{NL,2} \) superfields, and equal to the SUSY transformation of the new even non-local conserved quantities we have obtained. These ones fit exactly in the SUSY cohomology of the already known conserved quantities. Finally we show that the algebra of conserved quantities of SKdV is a \( W \)-algebra extension of the superconformal algebra.

Keywords: supersymmetry, supersymmetric models, integrable systems, conservation laws.
Pacs: 11.30.Pb, 12.60.Jv, 02.30.Ik, 11.30.-j

*On leave of absence from UNEXPO, Caracas, Venezuela.
1 Introduction

Supersymmetric integrable systems are an interesting scenario for the analysis of ADS/CFT correspondence [1], in particular in relation to $N = 4$ Super Yang-Mills models. Super conformal algebras are also realized in Super KdV equations (SKdV) [2, 3, 4]. SKdV equations are also directly related to supersymmetric quantum mechanics. In fact, the whole SKdV hierarchy arises from the asymptotic expansion of the Green’s function of the Super heat operator [5].

One of the main properties of the integrable systems is the presence of an infinite sequence of conserved quantities. For $N = 1$ SKdV an infinite sequence of local conserved quantities was found in [2]. It was then observed, by analyzing the symmetries of SKdV, the existence of odd non-local conserved quantities [6, 7]. In [8] they were obtained from a Lax formulation of the Super KdV hierarchy and generated from the super residue of a fractional power of the Lax operator.

In [9] all these odd non-local conserved quantities were obtained from a single conserved quantity of the SUSY Gardner equation (SG), which was introduced in [2], see also [10]. So far only two conserved quantities of the SG equation are known: the one generating all the local conserved quantities of SKdV, of even parity and of dimension 1, and the above mentioned one of odd parity and of dimension $\frac{1}{2}$.

Conformal symmetry is a very powerful symmetry. It has remarkable applications in String Theory and in the study of critical phenomena in statistical mechanics. $W$-algebras are extensions of the conformal symmetry and also play an important role. There is an interesting and direct relation between the Virasoro algebra and the algebra defined in terms of the Poisson bracket in KdV [11]. The same relation holds between the superconformal algebra and the Poisson bracket algebra in Super KdV [3]. The Poisson bracket of conserved quantities is also a conserved quantity. In particular, the local conserved quantities commutes under the Poisson bracket with any other conserved quantity. It is then the center of the algebra of conserved quantities. The non trivial part of the algebra arises from the Poisson bracket of non-local conserved quantities. The structure of this algebra of conserved quantities was till now unknown.

In this paper we introduce a new infinite sequence of non-local conserved quantities of $N = 1$ SKdV. We construct them via a Gardner map [12, 13], from a new conserved quantity of SG. It is non-local and it has even parity and dimension 1. We then introduce the SUSY cohomology in the space of conserved quantities and obtain the relation between all the conserved quantities whether local, non-local, odd, or even. In this sense it is natural to introduce the new even conserved quantities. The SUSY cohomology we introduce here is used for the $N = 1$ KdV system, however several of the arguments are general, beyond SKdV, and we expect they will be useful in the analysis of other integrable systems. We also obtain the Poisson bracket of several non-local conserved quantities, in particular using the results of the Susy cohomology. The algebra of conserved quantities is a $W$-
algebra extension of the superconformal algebra. This is an interesting result since it was conjectured in [8] that the algebra of conserved quantities was linear on the generators.

In section 2 we present basic definitions, in sections 3 and 4 the new conserved quantities, in section 5 the SUSY cohomology and in section 6 we evaluate the Poisson bracket of several conserved quantities.

## 2 Basic facts

A first step in order to analyze \( N = 1 \) SKdV and its known local and odd non-local conserved quantities is to consider the ring of polynomials with one odd generator \( a_1 \) and a superderivation \( D \) defined by \( Da_n = a_{n+1} \) \((n \in \mathbb{N})\). The elements \( a_n \) have the same parity as the positive integer \( n \) and satisfy \( a_n a_n = \pm a_{n_2} a_{n_1} \), with a minus sign only in the case when \( n_1 \) and \( n_2 \) are odd. On the products \( D \) acts following the rule \( D(a_n a_n) = (D a_n) a_n + (-1)^{n_1} a_{n_1} D a_{n_2} \). The explicit algebraic presentation is given by the ring \( \mathcal{A} \) of elements of the form \( b(a_1, a_2, \ldots) \), with \( b \) any polynomial, and \( D = a_2 \frac{\partial}{\partial a_3} + a_3 \frac{\partial}{\partial a_2} + \cdots \)

A second step to complete the construction is to extend the ring \( \mathcal{A} \) with a new set of generators of the form \( \lambda_n = D^{-1} h_n \), with \( h_n \) being the non trivial integrands of the known even local conserved quantities [9]. An element of the new ring \( \tilde{\mathcal{A}} \) is a polynomial of the form \( \tilde{b}(a_1, a_2, \ldots, \lambda_1, \lambda_3, \ldots) \) and the corresponding superderivation is given by \( \tilde{D} = D + P \) with \( P = h_1 \frac{\partial}{\partial \lambda_1} + h_3 \frac{\partial}{\partial \lambda_3} + h_5 \frac{\partial}{\partial \lambda_5} + \cdots \)

This algebraic presentation is directly connected to the analytical one by the substitution \( a_1 \leftrightarrow \Phi \), where \( \Phi \) is a superfield, that is, a function \( \Phi : \mathbb{R} \rightarrow \Lambda \), with \( \Lambda \) a Grassmann algebra one of whose generators \( \theta \) has been singled out. In this setting \( D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x} \) and \( \Phi(x, \theta) = \xi(x) + \theta u(x) \), with \( t \) an implicit variable. The parity of the generator \( a_1 \) requires \( \xi \) to be odd and \( u \) to be even for each \( x \in \mathbb{R} \).

The covariant derivative \( D \) has the property \( D^2 = \frac{\partial}{\partial x} \) acting on differentiable superfields, and the generator of SUSY transformations \( Q = -\frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x} \) satisfies \( DQ + QD = 0 \).

In order to realize “integration” the ring of infinitely differentiable superfields \( C^\infty(\mathbb{R}, \Lambda) \) must be restricted. We then introduce the ring of Schwartz superfields, that is,

\[
C_\Lambda^\infty(\mathbb{R}, \Lambda) = \left\{ \Phi \in C^\infty(\mathbb{R}, \Lambda) / \lim_{x \to \pm \infty} x^p \frac{\partial^q}{\partial x^q} \Phi = 0 \right\},
\]

for every \( p, q \geq 0 \). The “superintegration” in this space is well defined and we write \( \int \Phi dx d\theta = \int_{-\infty}^{\infty} u dx \). It is natural to introduce the space of integrable superfields \( C_1^\infty(\mathbb{R}, \Lambda) = \{ \Phi \in C^\infty(\mathbb{R}, \Lambda) / \frac{\partial}{\partial \theta} \Phi \in C_1^\infty(\mathbb{R}, \Lambda) \} \) and clearly \( C_1^\infty(\mathbb{R}, \Lambda) \subset C_\Lambda^\infty(\mathbb{R}, \Lambda) \).

The next ring to be considered allows us to deal with the known odd non-local conserved quantities. We introduce then the ring of non-local superfields \( C_{NL,1}^\infty(\mathbb{R}, \Lambda) = \{ \Phi \in C^\infty(\mathbb{R}, \Lambda) / D \Phi \in C_1^\infty(\mathbb{R}, \Lambda) \} \). We have \( C_1^\infty(\mathbb{R}, \Lambda) \subset C_{NL,1}^\infty(\mathbb{R}, \Lambda) \subset C_\Lambda^\infty(\mathbb{R}, \Lambda) \). If \( \Psi = U + \theta V \in C_{NL,1}^\infty(\mathbb{R}, \Lambda) \), we take, as in the case of integrable superfields \( \int \Psi dx d\theta = \int_{-\infty}^{\infty} u dx \).
\[ \int_{-\infty}^{\infty} Vdx. \]

In conclusion, for a given superfield to be integrable it is sufficient that the application of \( \frac{\partial}{\partial \theta} = D|_{\theta=0} \) to it gives a Schwartz superfield.

The crucial fact we use in the formulation in the next sections is that \( C^\infty_1(\mathbb{R}, \Lambda) \) is an ideal of \( C^\infty_{NL,1}(\mathbb{R}, \Lambda) \).

We recall that a candidate for a conserved quantity (for example in the \( C^\infty_1(\mathbb{R}, \Lambda) \) ring) associated to a given partial differential equation (PDE)

\[ \Phi_t = k(\Phi, D\Phi, D^2\Phi, \ldots) \quad (1) \]

may be presented by \( H = \int h(\Phi, D\Phi, D^2\Phi, \ldots) dx d\theta \) \( (k, h \in C^\infty_1(\mathbb{R}, \Lambda)) \). Then, \( H \) is a conserved quantity if \( H_t = 0 \), which is equivalent to having \( \frac{d}{ds}|_{s=0} H(\Phi + s\Psi) = 0 \) whenever \( \Psi(x, t) = \frac{d}{dt}\Phi(x, t) = k(\Phi(x, t), D\Phi(x, t), \ldots) \). In terms of functional derivatives a necessary and sufficient condition for \( H \) to be a conserved quantity of (1) is that \( \delta h = Dg \) for some \( g \in C^\infty_1(\mathbb{R}, \Lambda) \), where \( \delta \) is the functional derivative in the direction defined by \( k(\Phi, D\Phi, D^2\Phi, \ldots) \). This condition will be used in section 4.

To describe the known local conserved quantities the last setting is sufficient and the ring \( \tilde{A} \) may be reduced by considering only polynomials of the form \( b(a_1, a_2, \ldots) \).

For the study of the known odd non-local conserved quantities of SuperKdV we must consider the complete ring \( \tilde{A} \). For example, the fourth odd non-local known conserved quantity is given by

\[ \int \left\{ \frac{1}{24}(D^{-1}\Phi)^4 - \frac{1}{2}(D\Phi)^2 + (D^{-1}\Phi)D^{-1}(\Phi D\Phi) \right\} dx d\theta, \quad (2) \]

and the terms \( D^{-1}\Phi \) and \( D^{-1}(\Phi D\Phi) \) clearly belong to the ring extension \( C^\infty_{NL,1} \) mentioned before, where \( \Phi = h_1 \) and \( \Phi D\Phi = h_3 \). We put \( a_0 \equiv \lambda_1 = D^{-1}\Phi \) in accordance with the notation of [9]. The formal analysis of this type of conserved quantities is similar to the local ones but in this case the ring \( C^\infty_{NL,1}(\mathbb{R}, \Lambda) \) plays a fundamental role.

To obtain the known local and non-local conserved quantities we start with the pair of SuperKdV and Super Gardner equations given by

\[ \Phi_t = D^6\Phi + 3D^2(\Phi D\Phi), \quad (3) \]

and

\[ \chi_t = D^6\chi + 3D^2(\chi D\chi) - 3\epsilon^2(D\chi)D^2(\chi D\chi), \quad (4) \]

with \( \epsilon \) a formal parameter and \( \chi = \sigma + \theta w \) another odd superfield [2].

(3) and (4) are connected by the Super Gardner map given by

\[ \Phi = \chi + \epsilon D^2\chi - \epsilon^2\chi D\chi. \quad (5) \]

It gives

\[ \Phi_t - D^6\Phi - 3D^2(\Phi D\Phi) = [1 + \epsilon D^2 - \epsilon^2(D\chi + \chi D)] \cdot \{\chi_t - D^6\chi - 3D^2(\chi D\chi) + 3\epsilon^2(D\chi)D^2(\chi D\chi)\}, \quad (6) \]
whenever $\Phi$ and $\chi$ are related by (5). We recall that (5) sends solutions of (4) into (3). The inverse is also true if we restrict the space of possible solutions of (4) to formal series $\chi = \sum_{n=0}^{\infty} a_n(\Phi, D\Phi, \ldots)\epsilon^n$. The conserved quantity $\int \chi dxd\theta$ of (4) induces the infinitely many known local conserved quantities for (3). In the language of [9] the right members of (3), (4), (5) are denoted by $h, l, r$ respectively, with $h \in \mathcal{A}$ and $l, r \in \mathcal{A}[\epsilon]$. Condition (6) is equivalent to $h \circ r = r' l$ and is necessary and sufficient for (5) to map solutions of Super Gardner to solutions of SuperKdV.

In [9] it was shown that $\int \exp(\epsilon D^{-1} \chi) dxd\theta$ is also a conserved quantity for (4). It induces the known infinite sequence of odd non-local conserved quantities of SKdV.

### 3 Infinite sequence of new non-local conserved quantities of SKdV

We will introduce in this section a new non-local conserved quantity of the Super Gardner equation. From it, following the previous section, we may obtain an infinite set of new non-local conserved quantities of the SKdV. There are two already known conserved quantities of the Super Gardner equation. The first one [2] provides the infinite set of local conserved quantities of SKdV equation. The other one [9] give rise to the infinite set of odd non-local conserved quantities of the SKdV, originally found in [6] and also obtained from the Lax operator in [8]. These two conserved quantities of Super Gardner equation exhaust all the known conserved quantities of SKdV equation. We will now introduce a new infinite set of even non-local conserved quantities of the SKdV.

The quantity $H_G$,:

$$\begin{align*}
H_G &= \int \left\{ D^{-1} \left[ \frac{\exp(\epsilon D^{-1} \chi) + \exp(-\epsilon D^{-1} \chi) - 2}{2\epsilon^2} \right] + \right.
\left. \frac{1}{2} \left[ \frac{\exp(\epsilon D^{-1} \chi) - 1}{\epsilon} \right] D^{-1} \left[ \frac{\exp(-\epsilon D^{-1} \chi) - 1}{\epsilon} \right] \right\} dxd\theta
\end{align*}$$

(7)

where $\chi \in C^\infty$, exists and is conserved by every solution of the Super Gardner equation. It has the same parity as the local conserved quantities and opposite to the already known non-local conserved quantities of SKdV. When we apply the inverse Gardner transformation we obtain an infinite set of well defined even non-local conserved quantities of SKdV.

The terms

$$D^{-1} \left[ \frac{\exp(\epsilon D^{-1} \chi) + \exp(-\epsilon D^{-1} \chi) - 2}{2\epsilon^2} \right]$$

and

$$D^{-1} \left[ \frac{\exp(-\epsilon D^{-1} \chi) - 1}{\epsilon} \right]$$

do not belong to $C^\infty_{NL,1}$, but to $C^\infty_{NL,2} = \{ \Phi \in C^\infty(\mathbb{R}, \Lambda) / D^2\Phi \in C^\infty_1(\mathbb{R}, \Lambda) \}$. 

5
Although each term is not integrable the complete integrand belongs to $C^\infty_I$.

We notice that (7) may be rewritten as

$$H_G = \int \frac{1}{2} D^{-1} \left\{ D \left[ \frac{\exp(\epsilon D^{-1} \chi) - 1}{\epsilon} \right] D^{-1} \left[ \frac{\exp(-\epsilon D^{-1} \chi) - 1}{\epsilon} \right] \right\} dx d\theta. \quad (8)$$

We may then perform the $\theta$ integration and obtain

$$H_G = \int_{-\infty}^{\infty} \frac{1}{2} \left\{ D \left[ \frac{\exp(\epsilon D^{-1} \chi) - 1}{\epsilon} \right] D^{-1} \left[ \frac{\exp(-\epsilon D^{-1} \chi) - 1}{\epsilon} \right] \right\} \big|_{\theta=0} dx \quad (9)$$

which under the assumption $\chi \in C^\infty_1$ is a well defined integral. The first two new conserved quantities of Super KdV which may be obtained from the inverse Gardner transformation are

$$H^{NL}_1 = \int -\frac{1}{2} D^{-1} (\Phi D^{-2} \Phi) dx d\theta, \quad (10)$$

$$H^{NL}_3 = \int \left[ -\frac{1}{2} D^{-1} (\Phi D^2 \Phi) + D^{-1} (D^2 \Phi \cdot D \cdot D^{-2} \Phi) + \frac{1}{24} D^{-1} (D^{-1} \Phi)^4 - \frac{1}{6} D^{-2} \Phi (D^{-1} \Phi)^3 + \frac{1}{8} (D^{-1} \Phi)^2 D^{-1} (D^{-1} \Phi)^2 \right] dx d\theta \quad (11)$$

which exactly agree with the two non-local conserved quantities obtained in [14] by the supersymmetric recursive gradient procedure. The proofs of existence in [14] for the recursive gradient procedure were only given for local conserved quantities, there are no proofs in the literature for the recursive gradient procedure involving non-local quantities. Hence the existence of the infinite set of conserved quantities is not guaranteed from that approach.

(11) may be rewritten using (8) as

$$H^{NL}_3 = \int D^{-1} \left[ \Phi \left( -\frac{1}{2} D^2 \Phi + D \Phi \cdot D^{-2} \Phi - \frac{1}{2} D^{-2} \Phi \cdot (D^{-1} \Phi)^2 + \frac{1}{4} D^{-1} \Phi \cdot D^{-1} (D^{-1} \Phi)^2 \right) \right] dx d\theta, \quad (12)$$

where the integrand is manifestly in $C^\infty_I$.

In the next section we prove the claims of existence and conservation of $H_G$ and consequently of the new set of infinite conserved quantities of SKdV.

## 4 Conservation of $H_G$ under the Super Gardner flow

We will prove in this section the following:

**Theorem 1** $H_G$ given by (7) is a conserved quantity for Super Gardner equation (4). The coefficients of $D^{-1} \left[ \frac{\exp(\epsilon D^{-1} \chi) + \exp(-\epsilon D^{-1} \chi) - 2}{2\epsilon^2} + \frac{1}{2} \frac{\exp(\epsilon D^{-1} \chi) - 1}{\epsilon} \right] D^{-1} \left[ \frac{\exp(-\epsilon D^{-1} \chi) - 1}{\epsilon} \right]$ induce an infinite sequence of new non-local conserved quantities for SUSY KdV equation (3).
Proof of Theorem 1 We take here that $D^{-1}\chi = a_0, \chi = a_1, D\chi = a_2, \ldots$

We will show that the quantity \( [7] \), written in a equivalent form by

$$G_1^{NL} \equiv H_G = -\frac{1}{2} \int D^{-1} \left( (Df) (D^{-1}g) \right) dx d\theta,$$

(13)

is a conserved quantity for the Super Gardner equation given by \([4]\), with $f = \frac{1}{e} (\exp^{\epsilon a_0} - 1)$ and $g = \frac{1}{e} (1 - \exp^{-\epsilon a_0}).$

Since $Df = \exp^{\epsilon a_0} a_1 \in C_1^\infty (R, \Lambda)$ while $D^{-1}g \in C_{NL,1}^\infty$ the integrand is in $D^{-1}C_1^\infty (R, \Lambda).$

Consequently $\frac{\partial}{\partial \epsilon} D^{-1}C_1^\infty (R, \Lambda) \subset C_{NL}^\infty (R, \Lambda)$, proving that $G_1^{NL}$ is given by a well-defined integral. This agrees exactly with the comment below formula (9).

To prove that the above quantity is conserved by the Super Gardner equation, it is sufficient to see that $\delta (D^{-1} ((Df) (D^{-1}g))) = D\Omega,$ with $\Omega \in C_1^\infty (R, \Lambda),$ where $\delta$ was defined in section 2.

It holds

$$\delta (D^{-1} ((Df) (D^{-1}g))) = D^{-1} ((D\delta f) (D^{-1}g) + (Df) (D^{-1}\delta g)).$$

We then have, applying $\delta$ to $f$, the following:

$$\delta f = \sum_0^\infty \epsilon^n \frac{d}{ds} \bigg|_{s=0} (D^{-1}\chi + sD^{-1}\Gamma\chi)^{n+1} = \exp^{\epsilon D^{-1}\chi} (D^{-1}\Gamma\chi) = \exp^{\epsilon a_0} (w - \epsilon^2 v),$$

where we are using for the Super Gardner equation the notation

$$\frac{\partial}{\partial t} \chi = \Gamma\chi = (a_7 + 3a_1a_4 + 3a_2a_3) - \epsilon^2 (3a_2^2a_3 + 3a_1a_2a_4) = D(w - \epsilon^2 v),$$

with $w = a_6 + 3a_2^2 - 3a_1a_3$ and $v = 2a_2^2 - 3a_1a_2a_3.$ A direct computation shows that $\exp^{\epsilon a_0} (w - \epsilon^2 v) = D(Q(\epsilon))$ in which $Q(\epsilon) = \exp^{\epsilon a_0} (F_0 + \epsilon F_1 + \epsilon^2 F_2),$ with $F_0 = a_5 + 3a_1a_2, F_1 = a_1a_4 - a_2a_3, F_2 = -2a_1a_2^2.$

Noting that $g$ is just $f$ with $\epsilon$ replaced by $-\epsilon$ we have $\delta g = D(Q(-\epsilon)).$

Then,

$$\delta (D^{-1} ((Df) (D^{-1}g))) = D^{-1} \left( (D^2 Q(\epsilon)(D^{-1}g) + (Df)Q(-\epsilon)) \right) =$$

$$= D^{-1} \left( D^2 (Q(\epsilon)D^{-1}g) + (Df)Q(-\epsilon) + (Dg)Q(\epsilon) \right),$$

using that $Dg$ and $Q(\epsilon)$ anticommute because their coefficients in the $\epsilon$ expansion are odd superfields.

Then finally,

$$(Df)Q(-\epsilon) = (\exp^{\epsilon a_0} a_1) \exp^{-\epsilon a_0} (F_0 - \epsilon F_1 + \epsilon^2 F_2)$$

$$(Dg)Q(\epsilon) = (\exp^{-\epsilon a_0} a_1) \exp^{\epsilon a_0} (F_0 + \epsilon F_1 + \epsilon^2 F_2)$$
and $(Df)Q(-\epsilon) + (Dg)Q(\epsilon) = 2a_1a_5 = D^2(2a_1a_3)$.

It follows that

\[
\delta (D^{-1} ((Df)(D^{-1}g))) = D(2a_1a_3 + Q(\epsilon)D^{-1}g).
\]

The coefficients of $D^{-1}g$ are in $C_{NL,1}^{\infty}(\mathbb{R},\Lambda)$ but they are multiplied by the coefficients of $Q(\epsilon)$ which are in $C_1^{\infty}(\mathbb{R},\Lambda)$. Therefore $\delta (D^{-1} ((Df)(D^{-1}g)))$ is of the form $D\Omega$ with $\Omega \in C_1^{\infty}(\mathbb{R},\Lambda)$ proving that $G^{NL}_1$ is a conserved quantity of the Super Gardner equation.

The fact that (13) determines an infinite sequence of new non-local conserved quantities for Susy KdV is straightforward.

This completes the proof of the theorem.

5 The SUSY cohomology on the space of conserved quantities

The invariance under supersymmetry of SKdV equations implies that the SUSY transformations of conserved quantities are also conserved quantities. That is, if $H = \int h, h \in C_1^{\infty}$, is conserved under the SKdV flow then

\[
\delta_Q H := \int Qh
\]

is also a conserved quantity.

The operation $\delta_Q$ acting on functionals of the above form is well defined since under the change, leaving $H$ invariant,

\[
h \rightarrow h + Dg
\]

with $g \in C_1^{\infty}$, we have

\[
Qh \rightarrow Qh + QDg = Qh + D(-Qg)
\]

where $Qg \in C_1^{\infty}$.

$\delta_Q$ is a superderivation satisfying $\delta_Q\delta_Q = 0$. In fact,

\[
\delta_Q\delta_Q H = \int Q^2h = -\partial_\theta h|_{-\infty}^{\infty} = 0
\]

since $h \in C_1^{\infty}$.

For the local conserved quantities of SKdV, which we denote $H_{2n+1}(\Phi), n = 0, 1, \ldots$, we have

\[
\delta_Q H_{2n+1}(\Phi) = 0, n = 0, 1, \ldots
\]

(14)

where the index $2n + 1$ denotes the dimension of $H_{2n+1}$.

If we consider the ring $C_1^{\infty}$ of superfields, $H_{2n+1}$ is closed but not exact. However if we extend the ring to the superfields $C_{NL,1}^{\infty}, C_{1}^{\infty} \subset C_{NL,1}^{\infty}$, then $H_{2n+1}$ becomes exact and it is
expressed in terms of $\delta Q H_{2n+\frac{1}{2}}, n = 0, 1, \ldots$ where $H_{n+\frac{1}{2}}, n = 0, 1, \ldots$ denote the odd non-local conserved quantities of SKdV [6, 8, 9], they have dimension $n + \frac{1}{2}$. The remaining $H_{2n+\frac{1}{2}}, n = 0, 1, \ldots$ plus a polynomial of lower dimensional conserved quantities is closed but not exact in $C_{NL,1}^\infty$, however if we extend the ring of superfields to $C_{NL,2}^\infty$ they become exact and equal to $\delta Q H_{2n+\frac{1}{2}}, n = 0, 1, \ldots$ plus the even non-local conserved quantities we have introduced in the previous section. They have dimension $2n + 1$. To obtain the exact relation between them we use the conserved quantities of the Super Gardner equation.

We denote them $G_1, G_{NL}^{\frac{1}{2}}$ and $G_{1,2}^{NL}$. We have

$$G_1 = \int \chi = \sum_{n=0}^{\infty} \epsilon^{2n} H_{2n+1}$$

(15)

$$G_{\frac{1}{2}}^{NL} = \int \frac{\exp(\epsilon^{-1}\chi) - 1}{\epsilon} = \sum_{n=0}^{\infty} \epsilon^n H_{n+\frac{1}{2}}^{NL}$$

(16)

$$H_G \equiv G_1^{NL} = \int \left\{ D^{-1} \left[ \frac{\exp(\epsilon^{-1}\chi) + \exp(-\epsilon^{-1}\chi)}{2\epsilon} - 1 \right] \right\} + \frac{1}{2} \left[ \frac{\exp(\epsilon^{-1}\chi) - 1}{\epsilon} \right] D^{-1} \left[ \frac{\exp(-\epsilon^{-1}\chi) - 1}{\epsilon} \right] = \sum_{n=0}^{\infty} \epsilon^n H_{n+1}^{NL}$$

(17)

where $\chi \in C_1^\infty$.

The odd powers of $\epsilon$, in (17), do not provide new conserved quantities of SKdV. For example

$$H_{2}^{NL} = \frac{1}{2} H_{2}^{NL} H_{\frac{1}{2}}^{NL}.$$  

(18)

We then have

$$\delta Q G_1 = \int Q \chi = \chi|^{+\infty}_{-\infty} = 0$$

hence we obtain (14).

We also have

$$\delta_Q G_{\frac{1}{2}}^{NL} = \frac{\exp(\epsilon G_1) - 1}{\epsilon} = G_1 + \frac{1}{2} \epsilon G_1^2 + \cdots$$

$$= \sum_{n=0}^{\infty} \epsilon^{2n} H_{2n+1} + \frac{1}{2} \epsilon (\sum_{n=0}^{\infty} \epsilon^{2n} H_{2n+1})^2 + \cdots$$

(19)

from which we obtain the relation between the odd non-local and local conserved quantities. In particular we get

$$\delta_Q H_{\frac{1}{2}}^{NL} = H_1,$$  

(20)

and

$$\delta_Q H_{\frac{1}{2}}^{NL} = \frac{1}{2} H_1^2 = \delta_Q \left( \frac{1}{2} H_1 H_{\frac{1}{2}}^{NL} \right)$$

that is

$$\delta_Q \left( H_{\frac{1}{2}}^{NL} - \frac{1}{2} H_1 H_{\frac{1}{2}}^{NL} \right) = 0.$$  

(21)
This is the generic situation, from (19), $H_{2n+1}, n = 0, 1, \ldots$ is expressed as an exact quantity in terms of $\delta_Q [H_{2n+\frac{1}{2}} + \Sigma \text{products of lower dimensional conserved quantities}]$ while $[H_{2n+\frac{1}{2}} + \Sigma \text{products of lower dimensional conserved quantities}]$ is closed in the ring $C_{NL,1}^\infty$. If we extend the ring of superfields to $C_{NL,2}^\infty$, then the closed quantity becomes exact and expressed in terms of $H_{2n+1}$, $n = 0, 1, \ldots$. The integrand of $H_{2n+1}$ is expressed in terms of superfields in $C_{NL,2}^\infty$, with the property that the whole integrand belongs to $C_{I}^\infty$. In this particular case

$$H_{\frac{1}{2}}^{NL} = \int \left[ D^{-1} \left( \frac{1}{2} (D^{-1} \Phi)^2 \right) - \frac{1}{2} D^{-1} \Phi \cdot D^{-1} (D^{-1} \Phi) \right]$$

(22)

each term in the integrand belongs to $C_{NL,2}^\infty$, it is not integrable but the combination is in $C_{I}^\infty$. This expression is in terms of $D^{-1}h$ where $h$ are the integrands of previously known conserved quantities $H_{n+\frac{1}{2}}, H_{2n+1}$. In this particular case

$$H_{\frac{1}{2}}^{NL} = \int \frac{1}{2} (D^{-1} \Phi)^2,$$

$$H_1 = \int \Phi,$$

$$H_1^{NL} = \int D^{-1} \Phi.$$

This is also a generic property of $H_{2n+1}$, $n = 0, 1, \ldots$ and as we already knew of $H_{n+\frac{1}{2}}, n = 0, 1, \ldots$ whose integrands may be expressed in terms of polynomials in $D^{-1}h$ where $h$ are the integrands of the local conserved quantities $H_{2n+1}$.

From (22) we have

$$\delta_Q H_1^{NL} = H_{\frac{1}{2}}^{NL} - \frac{1}{2} H_1 H_{\frac{1}{2}}^{NL}$$

that is, the closed quantity becomes exact in $C_{NL,2}^\infty$. Similar relations are obtained from (17) for higher dimensional conserved quantities. The general formula is

$$\sum_{n=0} \epsilon^n \delta_Q H_{n+1}^{NL} = \sum_{n=0} \epsilon^{2n} H_{2n+\frac{3}{2}} - \frac{1}{2 \epsilon} \left[ \exp \left( \sum_{n=0} \epsilon^{2n+1} H_{2n+1} \right) - 1 \right] \left[ \sum_{n=0} (-\epsilon)^n H_{n+\frac{1}{2}} \right]$$

(23)

In order to obtain (23) we used the non trivial property

$$\exp^{\epsilon D^{-1} \chi} = \sum_{n=0} \epsilon^n p_n$$

$$\exp^{-\epsilon D^{-1} \chi} = \sum_{n=0} (-\epsilon)^n p_n + D \sum_{n=0} \epsilon^n \Psi_n,$$

for certain odd superfields $\Psi_n \in C_{I}^\infty (\mathbb{R}, \Lambda)$. 

10
We then have the following relations between the conserved quantities of SKdV equation:

\[
\begin{array}{cccccc}
H_1 & H_3 & H_5 & H_7 & \cdots \\
H^{NL}_{1/2} & H^{NL}_{3/2} & H^{NL}_{5/2} & H^{NL}_{7/2} & \cdots \\
H^{NL}_1 & H^{NL}_3 & H^{NL}_5 & H^{NL}_7 & \cdots \\
\end{array}
\]

where the arrow denotes the action of \(\delta_Q\), up to lower dimensional conserved quantities as explained previously.

The new conserved quantities \(H^{NL}_1, H^{NL}_3, \ldots\) fit then exactly in the SUSY cohomology of the previously known conserved quantities.

6 On the Poisson bracket of conserved quantities

The superconformal algebra in terms of Poisson brackets is

\[
\begin{align*}
\{L_n, L_m\} &= (n-m) L_{n+m} + \frac{1}{12} cn (n^2 - 1) \delta_{n+m,0} \\
\{L_n, G_r\} &= \left(\frac{1}{2} n - r\right) G_{n+r} \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{1}{3} c \left( r^2 - \frac{1}{4} \right) \delta_{r+s,0}
\end{align*}
\]

where \(L_n\) are the Virasoro generators and \(G_r\) have odd parity.

If the fields \(u(x)\) and \(\xi(x)\) defined on \(S^1\) are expanded in Fourier series with coefficients \(L_n, G_r\)

\[
\begin{align*}
u(x) &= \frac{6}{c} \sum_{n=-\infty}^{+\infty} \exp(-inx) L_n - \frac{1}{4}, \\
\xi(x) &= \frac{3}{c} \exp\left(-\frac{1}{4}i\pi\right) \sum_{n=-\infty}^{+\infty} \exp(-irx) G_r
\end{align*}
\]

then, the superfield \(\Phi(x, \theta) = \xi(x) + \theta u(x)\) satisfies the algebra

\[
\{\Phi(x_1, \theta_1), \Phi(x_2, \theta_2)\} = P_1 \Delta,
\]

(25)
see [2], where \( P_1 = D_1^5 + 3\Phi D_1 + (D_1\Phi)D_1 + 2(D_1^2\Phi) \) and \( \Delta = \delta(x_1 - x_2)(\theta_1 - \theta_2) \),

and the Super KdV equation may be rewritten as

\[
\partial_t \Phi = \{\Phi, H\},
\]

(26)

where \( H = \frac{1}{2} \int \Phi D\Phi \).

The superconformal algebra is then equivalent to the algebra (25), in terms of Poisson brackets. We now consider the Poisson bracket of conserved quantities of (26). It turns out that the Poisson bracket of a local conserved quantity \( H_{2n+1} \) with any other conserved quantity is zero. However the Poisson bracket of non-local conserved quantities is non-trivial and define a W-algebra. We now use results of the previous section to evaluate the Poisson bracket of several conserved quantities.

We obtain,

\[
\begin{aligned}
\left\{ H^{NL}_{\frac{3}{2}}, H_{2n+1} \right\} &= \delta Q H_{2n+1} = 0, \\
\left\{ H^{NL}_{\frac{1}{2}}, H^{NL}_{n+\frac{1}{2}} \right\} &= \delta Q H_{n+\frac{1}{2}}, \\
\left\{ H^{NL}_{\frac{1}{2}}, H^{NL}_{n+1} \right\} &= \delta Q H_{n+1}.
\end{aligned}
\]

(27-29)

For example,

\[
\begin{aligned}
\left\{ H^{NL}_{\frac{1}{2}}, H^{NL}_{\frac{3}{2}} \right\} &= H_1, \\
\left\{ H^{NL}_{\frac{1}{2}}, H^{NL}_{\frac{1}{2}} \right\} &= \frac{1}{2} H_1^2, \\
\left\{ H^{NL}_{\frac{1}{2}}, H^{NL}_{1} \right\} &= -\frac{1}{2} H_1 H^{NL}_{\frac{1}{2}} + H^{NL}_{\frac{3}{2}}.
\end{aligned}
\]

(30-32)

We notice that the nonlinear term on the right hand member of (30) are missing in [8].

Using the final result of the previous section we may obtain all the Poisson brackets \( \left\{ H^{NL}_{\frac{1}{2}}, H^{NL}_{n+1} \right\} \).

Besides the Poisson bracket between \( H_{\frac{1}{2}} \) and the other conserved quantities, which arise as a consequence of the previous section, we get

\[
\begin{aligned}
\left\{ H^{NL}_{\frac{3}{2}}, H^{NL}_{\frac{3}{2}} \right\} &= -H_3 + \frac{1}{3}(H_1)^3, \\
\left\{ H^{NL}_{1}, H^{NL}_{1} \right\} &= 0.
\end{aligned}
\]

(33-34)

The Poisson brackets of conserved quantities is also a conserved quantity. It defines an algebra. We have show then that this algebra, involves non linear terms which characterize extensions of the superconformal algebra.
7 Conclusions

We found a new infinite sequence of non-local conserved quantities of $N = 1$ SkdV equations. They have even parity and dimension $2n + 1, n = 0, 1, \ldots$. We introduced the SUSY cohomology in the space of conserved quantities: local, odd non-local and even non-local. We found all the cohomological relations between them. We obtained the Poisson bracket between several non-local conserved quantities and showed that the algebra of conserved quantities is a $W$-algebra extension of the superconformal algebra.

Although we consider $N = 1$ SkdV, we expect the SUSY cohomological arguments to be valid in general for SUSY integrable systems.

References

[1] J. Maldacena, “The Large N Limit of Superconformal Field Theories and Supergravity”, Adv. Theor. Math. Phys. 2:231-252, (1998).

[2] P. Mathieu, “Supersymmetric extension of the Korteweg-de Vries equation”, J. Math. Phys. 29, 2499 (1988).

[3] P. Mathieu, “Superconformal algebra and supersymmetric Korteweg-de Vries equation”, Phys. Lett. B 203, 287 (1988).

[4] Yu. I. Manin and A. O. Radul, “A supersymmetric extension of the Kadomtsev-Petviashvili hierarchy”, Commun. Math. Phys. 98, 65 (1985).

[5] S. Andrea, A. Restuccia, A. Sotomayor, “Supersymmetric exact sequence, heat Kernel and super KdV hierarchy”, J. Math. Phys. 45, 1715 (2004).

[6] P. H. M. Kersten, “Higher order supersymmetries and fermionic conservation laws of the supersymmetric extension of the KdV and the mKdV equation”, Phys. Lett. A 134, 25 (1988).

[7] P. Kersten, I. Krasil’shchik, A. Verbovetsky, “(Non) local Hamiltonian and symplectic structures, recursions, and hierarchies: a new approach and applications to the N=1 supersymmetric KdV equation”, J. Phys. A: Math. Gen. 37, 5003-5014 (2004).

[8] P. Dargis and P. Mathieu, “Nonlocal conservation laws for supersymmetric KdV equations”, Phys. Lett. A 176, 67-74 (1993).

[9] S. Andrea, A. Restuccia, A. Sotomayor, “The Gardner Category and Non-local Conservation Laws for N=1 SuperKdV”, J. Math. Phys. 46, 103517 (2005).

[10] S. Andrea, A. Restuccia, A. Sotomayor, “An Operator Valued Extension of the Super KdV Equations”, J. Math. Phys. 42, 2625 (2001).
[11] J. L. Gervais and A. Neveu, Nucl. Phys. B 209, 125 (1982).

[12] R. M. Miura, C. S. Gardner, and M. D. Kruskal, “Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion”, J. Math. Phys. 9, 1204 (1968).

[13] A. V. Kiselev, “Algebraic properties of Gardner’s deformations for integrable systems”, Theoret. Math. Phys. 151:(1), 963-976, (2007)

[14] S. Andrea, A. Restuccia, A. Sotomayor, “New non-local SUSY conserved laws from a recursive gradient algorithm”, arXiv:0705.4436.