It is known that the inhomogeneous quantum group $IGL_{q,r}(2)$ can be constructed as a quotient of the multiparameter $q$–deformation of $GL(3)$. We show that a similar result holds for the inhomogeneous Jordanian deformation and exhibit its Hopf structure.

1 Introduction

It is well–known [1] that, analogous to the classical group–theoretical method, the $q$–deformation of $IGL(2)$ can be constructed by factoring out a certain two–sided Hopf ideal from the multiparameter $q$–deformation of $GL(3)$. This is an interesting procedure, allowing, for example, the construction of a differential calculus on the quantum plane by a reduction of the differential calculus on the quantum group. In this paper, we apply the same construction to the Jordanian deformation. The multiparameter Jordanian deformation of $GL(3)$ is first produced by a contraction from the corresponding $q$–deformation and this is then used to construct the inhomogeneous group by factorisation. The Hopf–structure of $IGL_{J}(2)$ is given explicitly and we show that it is possible to derive from this a coaction of a modified version of $GL_{J}(2)$ on the Jordanian quantum plane.

Note: In this paper, we denote $q$–deformed structures using the (multiparameter) subscript $Q$ and structures that have been contracted to the Jordanian form are written with a subscript $J$ (e.g $GL_{Q}(3)$ and $GL_{J}(3)$).

2 The R–matrix for $GL_{Q}(2)$

Following Aschieri and Castellani [1], the $R$–matrix for $GL_{Q}(3)$ (where $Q = \{r,s,p,q\}$) can be written as

$$R_{Q}(3) = \begin{pmatrix} \Lambda & S^{-1} & r \\ S & \Lambda & \\ r^{-1} & r & R_{Q}(2) \end{pmatrix}$$

(1)
where \( S = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \), \( A = \begin{pmatrix} r - r^{-1} & 0 \\ 0 & r - r^{-1} \end{pmatrix} \) and

\[
R_Q(2) = \begin{pmatrix} r & s \\ r^{-1} - r & s^{-1} \end{pmatrix}
\]  \( (2) \)

The matrix indices of \( R_Q(3) \) run, in order, through the set (11), (12), (13), (21), (31), (22), (23), (32), (33). This numbering system is chosen to clearly show the embedding of the \( R_Q(2) \) matrix in the \( R_Q(3) \) matrix which, in turn, allows the Hopf structure of larger algebra to be analysed in terms of the simpler one. The Hopf structure of \( GL_Q(3) \) is given by the \( RTT \) relations with \( T \)–matrix

\[
T = \begin{pmatrix} f & \theta & \phi \\ x & a & b \\ y & c & d \end{pmatrix}
\]  \( (3) \)

and the multiparameter inhomogeneous \( q \)–deformation \( IGL_Q(2) \) is the quantum homogeneous space

\[
IGL_Q(2) = GL_Q(3)/H
\]  \( (4) \)

where \( H \) is the two–sided Hopf ideal generated by the \( T \)–matrix elements \( \{ \theta, \phi \} \).

### 3 The Contraction Procedure

The \( R \)–matrix of the Jordanian (or \( h \)–deformation) can be viewed as a singular limit of a similarity transformation on the \( q \)–deformation \( R \)–matrix \( \mathbb{R} \). Let \( g(\eta) \) be a matrix dependent on a contraction parameter \( \eta \) which is itself a function of one of the deformation parameters of the \( q \)–deformed algebra. This can be used to define a transformed \( q \)–deformed \( R \)–matrix

\[
\tilde{R}_J = (g^{-1} \otimes g^{-1}) R_Q(g \otimes g)
\]  \( (5) \)

The \( R \)–matrix of the Jordanian deformation is then obtained by taking a limiting value of the parameter \( \eta \). Even though the contraction parameter \( \eta \) is undefined in this limit, the new \( R \)–matrix is finite and gives rise to a new quantum group structure through the \( RTT \)–relations. For example, in the contraction process which takes \( GL_q(2) \) to \( GL_h(2) \), the contraction matrix is

\[
g(\eta) = \begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix}
\]  \( (6) \)

where \( \eta = \frac{h}{1-q} \) with \( h \) a new free parameter.

It has been shown by Alishahiha \( \text{[3]} \) that, in the extension of this procedure to the construction of \( GL_J(3) \), there are essentially two choices of contraction matrix.
The first has been used in a number of papers, e.g. by Quesne and takes the form

\[ G' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \eta & 0 & 1 \end{pmatrix} \]  

There is, however, a second choice (also mentioned in but not pursued there since it gives trivial results for the single–parameter \( q \)–deformation)

\[ G = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \]  

where \( g \) is the \( 2 \times 2 \) contraction matrix

\[ g(\eta) = \begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix} \]  

with \( \eta = \frac{r}{1-q} \). In this present work, we take \( G \) as our contraction matrix because, unlike the matrix \( G' \), after contraction it allows a non–trivial embedding of \( R_J(2) \) in \( R_J(3) \) in a manner similar to the \( q \)–deformed case. It is then possible to perform the quotient construction for the inhomogeneous quantum group.

If the similarity transformation is made using the matrix \( G \), we obtain

\[ R_J(3) = \lim_{r \to 1} \begin{pmatrix} g^{-1}S^{-1}g \\ \Lambda \\ g^{-1}g \end{pmatrix} \begin{pmatrix} g^{-1} \otimes g^{-1} \end{pmatrix} R_J(g \otimes g) \]  

\[ = \begin{pmatrix} 1 & K^{-1} \\ K & R_J(2) \end{pmatrix} \]  

where \( K \) is the matrix \( \begin{pmatrix} p & 0 \\ k & p \end{pmatrix} \) and \( R_J(2) \) is the \( R \)–matrix for the multiparameter Jordanian deformation of \( GL(2) \)

\[ \begin{pmatrix} 1 & 1 \\ -m & 0 & 1 \\ mn & n & -n & 1 \end{pmatrix} \]  

The free parameters \( \{m, n, k\} \) appear as limits in the contraction process while the parameter \( \{p\} \) survives the contraction process. The result is a four parameter Jordanian deformation of \( GL(3) \).
4 Multiparameter Jordanian Deformation of GL(3)

We denote the $T$–matrix for the Jordanian deformation by

$$
T = \begin{pmatrix} f & \theta & \phi \\ x & a & b \\ y & c & d \end{pmatrix} = \begin{pmatrix} f & \Theta \\ X & T \end{pmatrix}
$$

(13)

where $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, and $\Theta = (\theta, \phi)$.

4.1 Algebra Relations

The algebra structure of the quantum group is obtained through the $RTT$–procedure. For the commutation relations between the elements of the matrix $T$, we have the usual relations between the generators of the multiparametric Jordanian deformation of $GL(2)$:

$$
\begin{align*}
[a, b] & = nb^2 \\
[b, d] & = -m\delta^2 \\
[b, c] & = -mba - nbd \\
[c, d] & = n(d^2 - \delta)
\end{align*}
$$

(14)

where $\delta$ is the quantum determinant of the submatrix $T$

$$
\delta = ad - bc - nbd
$$

(15)

with commutation relation

$$
[\delta, a] = (m - n)\delta b \\
[\delta, b] = 0 \\
[\delta, c] = (m - n)(\delta d - a\delta) \\
[\delta, d] = (n - m)\delta b
$$

(16)

Thus $\delta$ is not central in the (sub–Hopf) algebra generated by elements of $T$ unless $m = n$.

The relations between $T$ and $f$ are given by

$$
\begin{align*}
[a, f] & = \frac{k}{p} fb \\
[b, f] & = 0 \\
[c, f] & = \frac{k}{p} (fd - af) \\
[d, f] & = -\frac{k}{p} bf
\end{align*}
$$

(17)

those between $T$ and $X$ are

$$
\begin{align*}
[a, x]_p & = kxb \\
[c, x]_p & = kxd + max \\
[a, y]_p & = kyb - max \\
[b, y]_p & = -mbx \\
[c, y]_p & = kyd + ncx - nay - mnax \\
\delta x & = p^2 x\delta \\
\delta y & = p^2 y\delta + (n - m)\delta x
\end{align*}
$$

(18)

while those between $f$ and $X$ give

$$
\begin{align*}
[f, x]_p & = 0 \\
[f, y]_p & = -kxf
\end{align*}
$$

(19)
The commutation relations between the elements of $X$ are the usual relations for the Jordanian quantum plane $C_J(2)$:

$$[x, y] = -mx^2$$

(20)

There are also similar commutation relations between the elements of $T$, $f$ and $\Theta$, as well as cross-relations between $X$ and $\Theta$.

4.2 Coalgebra Relations and Antipode

The coalgebraic structure of the Hopf algebra is the usual one:

$$\Delta(T) = T \otimes T \quad \epsilon(T) = I_3$$

(21)

with antipode

$$S(T) = \left( \begin{array}{cc} e & -e\Theta T^{-1} \\ -T^{-1}Xe & T^{-1}Xe\Theta T^{-1} + T^{-1} \end{array} \right)$$

(22)

where we append to the algebra, the element $e = (f - \Theta T^{-1}X)^{-1}$. In terms of these elements, the quantum determinant of the $T$-matrix $T$ is

$$D = \det(T) = e^{-1}\delta$$

(23)

and so, in the usual way, we can add $\xi = D^{-1}$ to the algebra to obtain the full Hopf algebra.

5 The Inhomogeneous Multiparameter Jordanian Quantum Group

$IGL_J(2)$

We define $H$ to be the space of all monomials containing at least one element of $\Theta$. It is straightforward to prove the following:

1. $H$ is a two–sided ideal in $GL_J(3)$.
2. $H$ is a co–ideal i.e. $\Delta(H) \subseteq H \otimes GL_J(3) + GL_J(3) \otimes H$ and $\epsilon(H) = 0$.
3. $S(H) \subseteq H$.

Thus $H$ is a two–sided Hopf ideal and so we can define a canonical projection from $GL_J(3)$ to the quotient space $GL_J(3)/H$ which respects the Hopf–algebraic structure (i.e. the $RTT$–relations). Consequently the quotient is a Hopf algebra which we denote $IGL_J(2)$.

The algebra sector for this quantum group has commutation relations formally obtained from $GL_J(3)$ by setting the generator set $\Theta = 0$ and this gives rise to the commutation relations explicitly detailed in the previous section. The $T$–matrix for the coalgebra is given by

$$T = \left( \begin{array}{cc} f & 0 \\ X & T \end{array} \right)$$

(24)
which gives the coproduct

\[ \Delta(T) = T \otimes T = \left( \begin{array}{cc} f \otimes f & 0 \\ T \otimes X + X \otimes f & T \otimes T \end{array} \right) \]  

(25)

counit \( \epsilon(T) = I_3 \) and antipode

\[ S(T) = \left( \begin{array}{cc} f^{-1} & 0 \\ -T^{-1}Xf^{-1} & T^{-1} \end{array} \right) \]  

(26)

The quantum determinant \( D = f \delta \) is group–like but, since \( f \) is not central, it cannot be made simultaneously central with \( \delta \) unless the whole algebraic structure collapses to a trivial extension of the single–parameter Jordanian deformation of \( GL(2) \). This is analogous to the situation in the \( q \)–deformed case.

This procedure also shows that it is possible to view the Jordanian quantum plane, \( C_J(2) \), as the quantum homogeneous space \( IGL_J(2)/GL_J(2)^* \) where \( GL_J(2)^* \) is the Hopf algebra formed by appropriately appending the “dilatation element” \( f \) to \( GL_J(2) \). The comultiplication in \( IGL_J(2) \) can then be viewed as a coaction of the quantum group \( GL_J(2)^* \) on the quantum plane \( C_J(2) \) generated by the elements \( X \). However, unlike the usual case, there is a non–trivial braiding between the elements of the quantum group and quantum plane.

6 Conclusion

We have shown that it is possible to construct the inhomogeneous Jordanian deformation \( IGL_J(2) \) as a quotient group by factoring out a Hopf ideal from \( GL_J(3) \). It would be of interest to construct the differential calculus on the Jordanian quantum plane by a reduction of the bicovariant differential calculus on \( GL_J(3) \). This would allow the investigation of physical models with \( GL_J(N) \) symmetry similar to that of Cho et al \[5\] and Madore and Steinacker \[6\]. Work on this problem is underway.

References

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