The van der Pol oscillator under hysteretic control: regular and chaotic dynamics

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Abstract. In this work we provide the novel hysteretic approach to control the chaotic dynamics of the van der Pol oscillator under periodic excitation. Using the phenomenological Bouc-Wen model, as well as in the frame of the small parameter approach we investigate the influence of the hysteretic control to the dynamical characteristics of this system. Based on the analysis of numerical results in the form of bifurcation diagrams and Lyapunov exponents, the efficiency of the stabilizing role of the hysteretic control element is established. In addition, we investigate the synchronization of oscillations in the system of coupled van der Pol oscillators (with “simple” linear coupling, as well as under hysteretic control). The stabilizing role of the hysteretic control in this case is also demonstrated.

1. Introduction
The van der Pol equation is one of the most important equations in the theory of nonlinear oscillations. By means of this equation the universal mechanism of appearance of self-oscillation through the Andronov-Hopf bifurcation is described. Also, using the van der Pol equation the appearance of both quasi-harmonic and relaxation oscillations can be described in the natural way.

The number of examples of self-oscillatory systems whose behavior is modeled using the van der Pol equation is quite large. Here we would like to note various systems in radio engineering, such as a triode generator and a generator on a tunnel diode, modeling the dynamics of the heart rate [1], as well as various applications in modern robotics, for example the model of the turning robot [2]. The systems of coupled van der Pol oscillators were quite successfully applied to the analysis of complex systems, including systems with distributed parameters such as tropical cyclones, where the property of oscillators was used to generate self-oscillatory modes. The same property was used in the investigation of ionization waves [3], as well as in a wide range of tasks related to models of processes occurring in the human body. In this case, the method
based on the shortened (or reduced) equations (often called the Landau-Stewart equations) was used as the main method for studying the resulting system. Such an approach is possible when the control parameter, coupling magnitude, and frequency detuning of oscillators are relatively small. In this case the method of slowly varying amplitudes is suggested as a main approach.

It is known that the chaotic regimes of motion can be realized in the system of van der Pol oscillators [4]. In this case the chaos observed in this system may have a hyperbolic nature. This means that such a chaotic motion is stable with respect to small perturbations of the system parameters. One of the classical problems connected to the chaotic dynamics is the control problem. Here we would like to note the most well-known methods of control of chaotic dynamics [5, 6, 7], which use various modifications of the feedback principle in combination with the methods of the theory of delayed differential equations. The algorithmic approach to control of the chaotic dynamics is based on the phenomenon of forced synchronization, when the system adjusts to an external periodic excitation. One of the promising methods of control of chaotic dynamics is based on the hysteretic principles when the hysteretic element (or hysteretic converter) is placed in a control loop, which is formed by the feedback technique. It is known that hysteresis operators at each cycle of oscillations absorb energy proportional to the loop area. In this sense the hysteretic control of chaotic systems seems promising and significant in the frame of modern nonlinear dynamics.

Hysteretic elements (or hysteretic converters) in control elements were considered in a number of works [8, 9]. As components of oscillatory systems they considered in [10, 11, 12]. Let us note also that two approaches are traditionally used describing hysteresis models: the first one is based on an operator interpretation (or constructive approach) in terms of converters with the corresponding state space and input-output relations [13]; the second one is based on a phenomenological approach where the hysteresis loop is described by means of differential equations and algebraic relations (the most popular in this sense is the well-known Bouc-Wen model [14]).

2. Van der Pol oscillator under periodic excitation

The van der Pol oscillator under periodic external excitation with a period close to the period of autonomous oscillations is described by an equation of the form:

$$\begin{cases} \ddot{x} - (\lambda - x^2)\dot{x} + \omega_0^2 x = A \cos(\omega t), \\ x(0) = x_0, \dot{x}(0) = x_1. \end{cases}$$

(1)

where $\lambda$ is a control parameter, $\omega_0$ is a frequency of self-oscillations, $\omega$ and $A$ are frequency and amplitude of an external excitation, respectively.

The dynamics of the system (1) is characterized by the following features: in a certain frequency range of an external force, the oscillations of the system are synchronized with the frequency of the external force, and this frequency range (synchronization band) becomes wider while the intensity of external excitation growth. This effect (the so-called synchronization by external force) is observed in systems of the different nature: in radio-engineering and electronic devices, in lasers, in mechanical systems, in oscillating chemical reactions, in biological objects, etc. In addition to synchronization at the frequency of external force, synchronization at harmonics and sub-harmonics can also be realized, when the frequencies of excitation and response are multiples of each other or, in the most general case, are in some rational relation.

Analyzing the dynamics of the system (1), we can note the following: at weak external force there are three periodic solutions, but only one of them is stable. As the amplitude $A$ increases, there will be a single solution. For a fixed quantity of $A$, while the frequency detuning $(\omega_0 - \omega)$ increases, one can observe the emergence of other types of motion – quasi-periodic and chaotic motions.
Further we consider the following model:
\[
\begin{aligned}
\ddot{x} - (\lambda - x^2)\dot{x} + \omega_0^2 x &= A \cos(\omega t) + \Phi_{BW}(x, t), \\
\Phi_{BW}(x, t) &= \alpha x(t) + (1 - \alpha) Dz(t), \\
\dot{z}(t) &= D^{-1} \left[ A_1 \dot{x}(t) - \beta |\dot{x}(t)||z(t)|^{n-1}z(t) - \gamma \dot{x}(t)|z(t)|^n \right].
\end{aligned}
\]

(2)

Here we use the standard notations as in [14]. The system (2) describes the van der Pol oscillator under periodic excitation, as well as under hysteretic control formalized by means of the Bouc-Wen model. The initial and boundary conditions are defined in the same manner as to the system (1).

Let us analyze the effect of the hysteresis element on the dynamics of the system (2). Assuming the smallness of the nonlinear terms, the corresponding external periodic excitation and hysteretic control, we can write the Eq. (2) in the following form (here $\varepsilon$ is a small parameter):
\[
\ddot{x} + \omega_0^2 x = \varepsilon \left( (\lambda - x^2)\dot{x} + A \cos(\omega t) + \Phi_{BW}(x, t) \right)
\]

(3)

The solution to this equation can be presented in the form:
\[
x = B \cos(\psi) + \varepsilon u_1(B, \psi) + \ldots,
\]

(4)

where $\psi = \omega t + \varphi(t)$ and $u_1(B, \psi)$ are unknown functions without resonant terms. Then, $B$ and $\psi$ are amplitude and phase of oscillations satisfying the following equations:
\[
\dot{B} = \varepsilon f_1(B, \psi) + \ldots, \quad \dot{\psi} = -\Delta + \varepsilon F_1(B, \psi) + \ldots,
\]

(5)

and $\Delta = \omega - \omega_0$ is a frequency detuning; $F_1$ and $f_1$ are unknown functions that should be determined from the conditions for the absence of resonant terms in the function $u_1$. Following the algorithm described in details in [15], in the first approximation by $\varepsilon$, taking into account the condition $u_1(B, \psi) = 0$, we get:
\[
\dot{B} = \frac{B}{2} \left( \lambda - \frac{7B^2}{4} \right), \quad \dot{\psi} = -\Delta + \frac{\Phi_{BW}}{2\Delta} - \frac{A \cos[\varphi]}{2B\Delta},
\]

(6)

Obviously, due to the presence of nonlinear terms of the type $\Phi_{BW}$, the system of equations (6) does not allow an explicit analytic solution, so the solution can be obtained using only numerical methods with the corresponding standard software packages.

2.1. Numerical results

An analysis of various dynamic regimes of system (1) showed that, along with the periodic and quasi-periodic regimes, at a certain value of parameters the chaotic regime may occur. In order to make sure that some sets of parameters correspond to the chaotic regime, using the standard approach based on the Wolf algorithm (the Gram-Schmidt orthonormalization), the Lyapunov exponent spectra were calculated. Taking into account the following values of the parameters describing the van der Pol oscillator under periodic perturbation: $\lambda = 3.4$, $\omega_0^2 = \pi$, $A = 2\pi$, $\omega = 0.7$, the values of Lyapunov exponents are $[0.109414, -2.41502, 0.0]$. These values of Lyapunov exponents confirm the chaotic behavior of the system.

The following Figures 1 and 2 show the solution to the systems (1) and (2) and their phase portraits, respectively.

The results of numerical simulation for the systems (1) and (2) show significant differences in the dynamics of these systems. This can be explained by the fact that the presence of a hysteretic part leads to an energy dissipation and, as a result, to a change in the dynamic characteristics.
Figure 1. Solutions to the system (1) (left panel) and (2) (right panel) at the following values of parameters: \( \lambda = 3.4, \omega_0^2 = \pi, A = 2\pi, \omega = 0.7 \).

Figure 2. Phase portraits of (1) (left panel) and (2) (right panel) at the following values of parameters: \( \lambda = 3.4, \omega_0^2 = \pi, A = 2\pi, \omega = 0.7 \).

of the considered system. As it can be seen from right panels of Figures 1 and 2, periodic oscillations are established in the system (2). This fact is also confirmed by the corresponding Lyapunov exponents: for the system (2) with the parameters \( \lambda = 3.4, \omega_0^2 = \pi, A = 2\pi, \omega = 0.7 \) the Lyapunov exponents are \([0.000177937, -0.935628, -11.1694]\).

Below we plot the bifurcation diagrams for the system (1) and (2) depending on the amplitude of the external excitation \( A \) (Figure 3).

It is known that the qualitative features of the dynamics of nonlinear systems are reflected in the spectral characteristics (Fourier spectra). Below we present the Fourier spectra of solutions to the system (1) and (2) (Figure 4).

As it can be seen from the left panel of Figure 4, the Fourier spectrum of the solution to the system (1) is continuous. As is well known, this continuous spectrum corresponds to chaotic behavior of the system. In contrast, from the right panel it is clear that the system (2) exhibits multi-frequency motion.

3. Coupled van der Pol oscillators

We now turn to the coupled van der Pol oscillators system. The first equation of this system coincides with the equation of the system (1), while the right hand side of the second equation contains the term connecting it with the oscillator \( x \).

\[
\begin{align*}
\dot{x} - (\lambda - x^2)\dot{x} + \omega_0^2 x &= A\cos[\omega t], \\
\dot{y} - (\lambda - y^2)\dot{y} + \omega_0^2 y &= \mu x.
\end{align*}
\]
Figure 3. Bifurcation diagrams for the system (1) (left panel) and (2) (right panel) depending on the amplitude of the external excitation at the following values of parameters: $\lambda = 3.4$, $\omega_0^2 = \pi$, $\omega = 0.7$.

Figure 4. Fourier spectra of the system (1) (left panel) and (2) (right panel).

System parameters are defined in a similar way with the systems (1) and (2), namely: $\lambda = 3.4$, $\omega_0^2 = \pi$, $A = 2\pi$, $\omega = 0.7$. In the Figure 5 we show the bifurcation diagram corresponding to behavior of the oscillator $y$ in the system (7) depending on the parameter $\mu$.

Figure 5. Bifurcation diagram of the oscillator $y$ in the system (7) depending on the parameter $\mu$.

From the analysis of the numerical results for bifurcation diagram, we can make the following
conclusion: as soon as the value of the parameter $\mu$ becomes big enough, the second oscillator receives enough energy to realize the chaotic regime. It is also should be noted that with a further increase in the parameter $\mu$, the synchronization (over the period of oscillations) of oscillators $x$ and $y$ takes place.

Below we present the results of numerical simulation for the dynamics of the oscillator $y$ (Figure 6), its phase portraits (Figure 7), as well as the corresponding Fourier spectra (Figure 8) at different values of the parameter $\mu$. Since the oscillator $y$ does not affect the behavior of the oscillator $x$, the results of numerical simulation will coincide with the corresponding results obtained earlier for the single oscillator (see the previous section).

![Figure 6](image1.png)

**Figure 6.** The dynamics of the oscillator $y$ in the system (7) at different values of the parameter $\mu$: left panel – $\mu = 0.1$; middle panel – $\mu = 2$; right panel – $\mu = 6$.

![Figure 7](image2.png)

**Figure 7.** Phase portraits of the oscillator $y$ in the system (7) at different values of the parameter $\mu$: left panel – $\mu = 0.1$; middle panel – $\mu = 2$; right panel – $\mu = 6$.

![Figure 8](image3.png)

**Figure 8.** Fourier spectra of the oscillator $y$ in the system (7) at different values of the parameter $\mu$: left panel – $\mu = 0.1$; middle panel – $\mu = 2$; right panel – $\mu = 6$.

Further, we investigate the regularizing role of the hysteretic term in the similar manner to the model (2). To do this, we will add a hysteretic control formalized by the phenomenological BoucWen model to the right hand side of the system (7) describing the $x$ oscillator. Then, the
resulting system can be written in the following form:

\[
\begin{align*}
\ddot{x} - (\lambda - x^2) \dot{x} + \omega_0^2 x &= A \cos(\omega t) + b \Phi_{BW}(x, t), \\
\ddot{y} - (\lambda - y^2) \dot{y} + \omega_0^2 y &= \mu x, \\
\Phi_{BW}(x, t) &= \alpha x(t) + (1 - \alpha) Dz(t), \\
\dot{z}(t) &= D^{-1} \left[ A_1 \dot{x}(t) - \beta |\dot{x}(t)| z(t)^{n-1} z(t) - \gamma \dot{x}(t) |z(t)|^n \right].
\end{align*}
\]

In the Figure 9 we show the bifurcation diagram for the oscillator \( y \) in the obtained system (8) depending on the parameter \( b \) at fixed value of \( \mu = 2 \).

![Bifurcation Diagram](image)

**Figure 9.** Bifurcation diagram for the oscillator \( y \) in the system (8) depending on the parameter \( b \).

As soon as the influence of the hysteretic control becomes significant, regular dynamics is established in the system, both for the \( x \) oscillator and for the \( y \) oscillator. When this happens, the synchronization between oscillators (by amplitude and frequency) occurs in the system. In the Figure 10 we present the numerical results for solutions to the system (8).

![Dynamics of System](image)

**Figure 10.** Dynamics of the system (8) at \( b = 1.7 \): left panel – the \( x \) oscillator; right panel – the \( y \) oscillator.

To confirm the obtained numerical results, we calculate the values of the Lyapunov exponents for the system (7) and (8) at the following values of the parameters: \( \lambda = 3.4, \omega_0^2 = \pi, A = 2\pi, \omega = 0.7, \mu = 2 \), the parameter \( b \) in the system (8) is \( b = 1.7 \). Then, the Lyapunov exponents for the system (7) are: \([0.110292, -0.156914, -2.41723, -4.93154]\); the Lyapunov exponents for the system (8) are: \([0.000189798, -0.322927, -0.843493, -6.63479, -11.7175]\). Using these
numerical values for Lyapunov exponents we can conclude that the periodic regime for both $x$ and $y$ oscillators establishes. Moreover, the synchronization of oscillations takes place. This fact also indicates the stabilizing role of the hysteretic element.

4. Conclusions
In this paper we provided the novel hysteretic approach to control the chaotic dynamics of the van der Pol oscillator under periodic excitation. Based on the analysis of the numerical results in the form of bifurcation diagrams and the corresponding dynamics of the Lyapunov exponents, the efficiency of the stabilizing role of the hysteretic control element is established.

The results were extended to the system of coupled van der Pol oscillators. It was possible to show that, as soon as the influence of the hysteretic element becomes significant, periodic regimes in the system of coupled oscillators are established. Furthermore, at some values of the external parameters (such as an intensity of hysteretic control) oscillations in this system are synchronized.

5. References
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