The Picard group of vertex affinoids in the first Drinfeld covering

BY JAMES TAYLOR

Mathematical Institute, Andrew Wiles Building, University of Oxford, Radcliffe Observatory Quarter, Woodstock Road, Oxford, OX2 6GG.

e-mail: james.taylor@maths.ox.ac.uk

(Received 23 March 2022; revised 20 March 2023; accepted 15 March 2023)

Abstract

Let $F$ be a finite extension of $\mathbb{Q}_p$. Let $\Omega$ be the Drinfeld upper half plane, and $\Sigma^1$ the first Drinfeld covering of $\Omega$. We study the affinoid open subset $\Sigma^1_v$ of $\Sigma^1$ above a vertex of the Bruhat–Tits tree for $\text{GL}_2(F)$. Our main result is that $\text{Pic}(\Sigma^1_v)[p]=0$, which we establish by showing that $\text{Pic}(Y)[p]=0$ for $Y$ the Deligne–Lusztig variety of $\text{SL}_2(\bar{\mathbb{F}}_q)$. One formal consequence is a description of the representation $H^1_{\text{ét}}(\Sigma^1_v, \mathbb{Z}_p(1))$ of $\text{GL}_2(\mathcal{O}_F)$ as the $p$-adic completion of $\mathcal{O}(\Sigma^1_v)^\times$.

2020 Mathematics Subject Classification: 11S37, 14G22 (Primary); 14C22 (Secondary)

1. Introduction

Let $p$ be a prime, $F$ a finite extension of $\mathbb{Q}_p$, and $K$ the completion of the maximal unramified extension of $F$. Let $\mathcal{M}_0$ be the disjoint union of $\mathbb{Z}$ copies of $\Omega$, where $\Omega$ is the Drinfeld upper half plane: the rigid analytic space over $K$ defined by removing all $F$-rational points from $\mathbb{P}^1_{\mathbb{K}}^{\text{an}}$. The work of Drinfeld [14] implies the existence of a tower of finite étale coverings $(\mathcal{M}_n)_{n\geq 0}$ of $\mathcal{M}_0$ equipped with compatible actions of $\text{GL}_2(F)$, which has been shown to realise both the local Langlands and Jacquet–Langlands correspondence in its étale cohomology [4, 5, 18, 19]. On the other hand, there is at present no formulated $p$-adic local Langlands correspondence for $\text{GL}_2(F)$ for general finite extensions $F$. The Drinfeld tower is expected to be of importance in yielding natural representations of $\text{GL}_2(F)$ that should appear in any such correspondence. For example, the geometric $p$-adic étale cohomology of the Drinfeld tower has been shown to encode the $p$-adic local Langlands correspondence for $F=\mathbb{Q}_p$ [7].

The preimage of the index zero piece $\Omega \hookrightarrow \mathcal{M}_0$ in the tower $(\mathcal{M}_n)_{n\geq 0}$ defines a tower $(\Sigma^n)_{n\geq 0}$ of finite étale coverings of $\Sigma^0 = \Omega$. The transition morphisms are equivariant for the action of the stabilising subgroup $\text{GL}_2(F)^+=\{g \in \text{GL}_2(F)\mid \det(g) \in \mathcal{O}^\times\}$. Let $\mathcal{T}$ be the Bruhat–Tits tree for $\text{GL}_2(F)$, $\nu$ the central vertex of $\mathcal{T}$, and $r: \Sigma^1 \rightarrow \Omega \rightarrow \mathcal{T}$ the retraction map. In this paper we study the open affinoid subset $\Sigma^1_v := r^{-1}(\nu)$ of $\Sigma^1$. This is stable under the action of $\text{GL}_2(\mathcal{O}_F)$ and after a finite extension of $K$, $\Sigma^1_v$ splits up into $g-1$ geometrically connected components, each isomorphic to $\text{Sp}(B)$, where

$$B = A[z]/\left(z^{q+1} - (x^q - x)\right), \text{ for } A = K\left[x, \frac{1}{x^q - x}\right].$$
The group $GL_2(F)^+$ acts with two orbits on the set of vertices of $\mathcal{T}$, and one can show that for any vertex $w$ adjacent to $v$, $\Sigma^1_w \cong \Sigma^1_v$. As any such $w$ will be in the other orbit from $v$, $\Sigma^1_w \cong \Sigma^1_v$ for all vertices $w \in \mathcal{T}$, and consequently this open subset often determines global properties of $\Sigma^1$. For example, the first de-Rham cohomology $H^1_{dR}(\Sigma^1)$ as a representation of $GL_2(F)$ is determined by $H^1_{dR}(\Sigma^1) [21$, theorem 6.1].

Our main result is that $\text{Pic}(\Sigma^1_v) [p] = 0$ (Theorem 3.2). The $p$-adic étale cohomology groups of Drinfeld spaces are of considerable interest [3, 6–9, 24], and one immediate consequence of Theorem 3.2 is a description of the $GL_2(O_F)$-representation $H^1_{\acute{e}t}(\Sigma^1_v, \mathbb{Z}_p(1))$, as the $p$-adic completion of $O(\Sigma^1_v)^\times$ (Theorem 3.4). This description is very explicit, as the unit group $O(\Sigma^1_v)^\times$ has been described by Junger [22, theorem 5.1].

Our main interest in Theorem 3.2 is the following. A precise statement of the $p$-adic local Langlands correspondence is formulated when $F = \mathbb{Q}_p$ [10], and Dospinescu and Le Bras [13] have used this to show that for $F = \mathbb{Q}_p$, and all $n \geq 1$, the representation $O(\Sigma^n)$ is naturally a coadmissible module over $D(G, K)$, the distribution algebra of $G$.

In an effort to remove the restriction on $F$, Ardakov and Wadsley show in their forthcoming work [1] using $p$-adic $D$-modules that the representation $O(\Sigma^1)$ splits up naturally into a direct sum of coadmissible $D(G, K)$-modules. This decomposition contains $O(\Omega)$, and all other components are shown to be topologically irreducible $D(G, K)$-modules. The benefits of this approach over that of [13], are that it holds for general field extensions $F$, is purely local, and establishes topological irreducibility. The obvious disadvantage is that it describes $O(\Sigma^n)$ only for $n = 1$. One would like to establish similar results for $O(\Sigma^n)$ for $n \geq 2$, where the situation is significantly more complicated. This is partially due to the fact that $\Sigma^n \to \Sigma^{n-1}$ has degree a power of $p$, whereas the degree of $\Sigma^1 \to \Omega$ is coprime to $p$.

The methods of [1] use the standard result that $\text{Pic}(\Omega) = 0$, and in attempting to transfer these methods to $O(\Sigma^2)$, one considers the group $\text{Pic}(\Sigma^1) [p]$ instead. Almost nothing is known about $\text{Pic}(\Sigma^1) [p]$, which is strongly expected to be non-zero. Our result that $\text{Pic}(\Sigma^1) [p] = 0$ is therefore slightly surprising. It also provides the first steps towards computing $\text{Pic}(\Sigma^1) [p]$ (by choosing an appropriate Čech cover), and allows one the possibility of using similar methods to [1] locally.

In order to prove Theorem 3.2, we consider the affine curve $Y$ defined by,

$$x^qy^q - yx^q = 1,$$

over the residue field of $K$, where $\mathbb{F}_q$ is the residue field of $F$. This curve was first considered by Drinfeld, who showed that all the discrete series representations of $SL_2(\mathbb{F}_q)$ can be realised in the cohomology of $Y$ [2, preface]. Inspired by this, these ideas were generalised to all reductive groups $\mathbb{G}$ by Deligne and Lusztig in their landmark paper [12]. They introduce what are now called Deligne–Lusztig varieties, which assign to $\mathbb{G}(\mathbb{F}_q)$ and $w \in W$, the Weyl group, a base space $X(w)$ and a finite covering $Y(w)$, and it is in the étale cohomology of $Y(w)$ that the cuspidal representations are realised. These are spaces of considerable interest, and the Picard groups of the base spaces $X(w)$ have been considered in [17]. Here we consider $Y = Y(w)$ in the special case of $\mathbb{G} = SL_2$, and $w \neq 1$. It would be interesting to study the Picard groups of $Y(w)$ more generally.
2. Deligne–Lusztig curves

Throughout this section, let $\mathbb{F}$ be an algebraic field extension of $\mathbb{F}_q$. We consider the affine curve,

$$
Y = \text{Spec} \left( \frac{\mathbb{F}[x, y]}{xy^q - yx^q = 1} \right),
$$

and its projective closure,

$$
Z = \text{Proj} \left( \frac{\mathbb{F}[X, Y, Z]}{XY^q - YX^q = Z^{q+1}} \right).
$$

We also consider the projective curve,

$$
W = \text{Proj} \left( \frac{\mathbb{F}[U, V, W]}{UV^q + TV^q = W^{q+1}} \right).
$$

We would first like to show that $\text{Pic}(Z)[p] = 0$.

**Lemma 2.1.** $Z$ is a smooth integral projective curve over $\mathbb{F}$. Furthermore, if $\mathbb{F}_q^4 \subset \mathbb{F}$, then $W \cong Z$.

**Proof.** The polynomial $P(X, Y, Z) = Z^{q+1} - (XY^q - YX^q) \in \mathbb{F}[X, Y, Z]$ is prime, which follows from Eisenstein’s criterion for $P \in \mathbb{F}[X, Y][Z]$, at the prime ideal $(X)$. Therefore $Z$ is integral. Furthermore, $Z$ is smooth, because the system $\partial_X P = \partial_Y P = \partial_Z P = 0$ has no solutions over $Z(\mathbb{F})$. For the isomorphism, let $\lambda \in \mathbb{F}_q^2$ with $\lambda^q = -1$, and let $\mu \in \mathbb{F}$ with $\mu^{q+1} = \lambda^q$. The element $\mu$ lies in $\mathbb{F}_q^4$, as,

$$
\mu^q = (\lambda^q)^{q-1} \mu = -\mu,
$$

so,

$$
\mu^q = (-\mu)^q = -(-\mu) = \mu.
$$

Then the claimed isomorphism is given by,

$$
U = X, \quad V = \lambda Y, \quad W = \mu Z.
$$

Indeed,

$$
X(\lambda Y)^q + (\lambda Y)X^q = \lambda^q (XY^q - YX^q),
$$

$$
= \lambda^q Z^{q+1} = (\mu Z)^{q+1},
$$

and similarly $U(\lambda^{-1} V)^q - (\lambda^{-1} V)U^q = (\mu^{-1} W)^{q+1}$.

**Proposition 2.2.** $\text{Pic}(Z)[p] = 0$.

**Proof.** By Lemma 2.1, $Z_{\overline{\mathbb{F}}} \cong W_{\overline{\mathbb{F}}}$, and thus the group $\text{Pic}(Z_{\overline{\mathbb{F}}})[p] \cong \text{Pic}(W_{\overline{\mathbb{F}}})[p] \cong J(\overline{\mathbb{F}})[p]$, where $J$ is the Jacobian of $W$. $W$ is known as the Hermitian curve, defined by affine equation $w^{q+1} = v^q + v$, and is maximal over $\mathbb{F}_q^2$ [26, lemma 6.4.4], hence $J(\overline{\mathbb{F}})[p] = 0$ by [15, corollary 2.5]. Then, because pullback induces an exact sequence $0 \to \text{Pic}(Z) \to \text{Pic}(Z_{\overline{\mathbb{F}}})$ [25, Tag 0CC5], and $p$-torsion is left exact, $\text{Pic}(Z)[p] = 0$. 

https://doi.org/10.1017/S0305004123000221 Published online by Cambridge University Press
Our next goal is to establish that $\text{Pic}(Y)[p] = 0$.

**Lemma 2.3.** $\mathbf{Z}(\overline{F}) \setminus \mathbf{Y}(\overline{F})$ consists of the $q + 1$ points,

$$\mathcal{P} := \{(a : b : 0) | (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}.$$  

Furthermore, $\mathcal{P} = \mathbf{Z}(\mathbb{F}_q)$.

**Proof.** If $(a : b : c) \in \mathbf{Z}(\overline{F})$ with $c = 0$, then $b^q a - a^q b = 0$, so $b^q a = a^q b$. If $a \neq 0$, then $(b/a)^q = b/a$, so $b/a \in \mathbb{F}_q$, and $(a : b) \in \mathbb{P}^1(\mathbb{F}_q)$. Similarly, if $b \neq 0$, $(a : b : c) \in \mathbf{Z}(\overline{F})$ with $c = 1$, because if so then $1 = ab^q - ba^q = ab - ba = 0$, as $a, b \in \mathbb{F}_q$.

Therefore the closed points of $\mathbf{Z} \setminus \mathbf{Y}$ are $\mathcal{P}$ [16, proposition 5-4], which we enumerate by $\mathcal{P} = \{P_0, \ldots, P_q\}$. From [27, exercise 5-12 (a)] we have an exact sequence,

$$\mathbb{Z}^{q+1} \longrightarrow \text{Cl}(\mathbf{Z}) \longrightarrow \text{Cl}(\mathbf{Y}) \longrightarrow 0,$$

where the first map sends,

$$(m_0, \ldots, m_q) \longmapsto \sum_{i=0}^q m_i[P_i],$$

and the second sends, for $I$ a finite set of closed points of $\mathbf{Z}$,

$$\sum_{P \in I} n_P[P] \longmapsto \sum_{P \in I \setminus \mathcal{P}} n_P[P].$$

Let $\Gamma = \langle [P_0], \ldots, [P_q] \rangle \subset \text{Cl}(\mathbf{Z})$ be the image of $\mathbb{Z}^{q+1}$ in $\text{Cl}(\mathbf{Z})$. The resulting exact sequence,

$$0 \longrightarrow \Gamma \longrightarrow \text{Cl}(\mathbf{Z}) \longrightarrow \text{Cl}(\mathbf{Y}) \longrightarrow 0,$$

yields the long exact sequence,

$$0 \longrightarrow \Gamma[p] \longrightarrow \text{Cl}(\mathbf{Z})[p] \longrightarrow \text{Cl}(\mathbf{Y})[p] \longrightarrow \Gamma/p\Gamma \longrightarrow \text{Cl}(\mathbf{Z})/p\text{Cl}(\mathbf{Z}) \longrightarrow \text{Cl}(\mathbf{Y})/p\text{Cl}(\mathbf{Y}) \longrightarrow 0,$$

from the right derived functors of $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, -)$. Then from Proposition 2.2 and the above discussion we have the following.

**Proposition 2.4.** There is an exact sequence

$$0 \longrightarrow \text{Cl}(\mathbf{Y})[p] \longrightarrow \Gamma/p\Gamma \longrightarrow \text{Cl}(\mathbf{Z})/p\text{Cl}(\mathbf{Z}),$$

where the map $\Gamma/p\Gamma \rightarrow \text{Cl}(\mathbf{Z})/p\text{Cl}(\mathbf{Z})$ is that induced by the inclusion $\Gamma \hookrightarrow \text{Cl}(\mathbf{Z})$.

**Remark.** We note that if $\mathbf{Z} \setminus \mathbf{Y}$ contained exactly one degree 1 closed point $Q$, then we could establish that $\text{Pic}(\mathbf{Y})[p] = 0$ almost immediately in the following way. In the exact sequence,

$$\mathbb{Z} \longrightarrow \text{Cl}(\mathbf{Z}) \longrightarrow \text{Cl}(\mathbf{Y}) \longrightarrow 0,$$
The Picard group of vertex affinoids in the first Drinfeld covering

the map \( Z \to \text{Cl}(Z) \) is actually injective and split by the degree homomorphism, hence \( \text{Cl}(Z) \cong \mathbb{Z} \times \text{Cl}(Y) \) so,

\[
0 = \text{Cl}(Z)[p] \cong \mathbb{Z}[p] \times \text{Cl}(Y)[p] = \text{Cl}(Y)[p].
\]

In particular, this can be applied to show that the class groups of affine dehomogenisations of \( Z \) with respect to both \( X \) and \( Y \) both have no \( p \)-torsion.

We want to show that \( \text{Cl}(Y)[p] = 0 \), and so in light of Proposition 2.4, we want to show that,

\[
\frac{\Gamma}{p\Gamma} \rightarrow \text{Cl}(Z)/p\text{Cl}(Z),
\]

is injective. In order to do so, we now examine the structure of \( \Gamma \). First we compute the principal divisors of some rational functions on \( Z \).

**Definition 2.5.** For \((a : b) \in \mathbb{P}^1(F_q)\), we let \( P_{(a : b)} \) be the closed point of \( Z \) defined by \((a : b : 0) \in \mathbb{P}^1(F_q)\).

**Lemma 2.6.** Let \((a : b), (c : d) \in \mathbb{P}^1(F_q)\) with \((a : b) \neq (c : d)\). Then the rational function,

\[
f := \frac{bX - aY}{dX - cY},
\]

has associated principal divisor,

\[
(f) = (q + 1) [P_{(a : b)}] - (q + 1) [P_{(c : d)}].
\]

**Proof.** Consider the morphism \( \zeta : Z \rightarrow \mathbb{P}^1 \) corresponding to the extension of function fields \( F(\mathbb{P}^1) \rightarrow F(Z) \), which sends,

\[
S \quad \mapsto \quad bX - aY,
\]

\[
T \quad \mapsto \quad dX - cY,
\]

where \( \mathbb{P}^1 = \text{Proj}(F[S, T]) \), and \( F(\mathbb{P}^1) = F(S/T) \). On \( \overline{F} \)-points, \( \zeta : Z \rightarrow \mathbb{P}^1 \) is given by,

\[
\zeta(x : y : z) = (bx - ay : dx - cy).
\]

This extension \( F(\mathbb{P}^1) \rightarrow F(Z) \) has degree \( q + 1 \) because it differs by an automorphism of \( \mathbb{P}^1 \) from the extension \( F(\mathbb{P}^1) \rightarrow F(Z) \), defined by,

\[
S \quad \mapsto \quad X,
\]

\[
T \quad \mapsto \quad Y,
\]

which clearly has degree \( q + 1 \). Let \( Q_0, Q_\infty \) be the closed points of \( \mathbb{P}^1 \) defined by \((0 : 1), (1 : 0) \in \mathbb{P}^1(\overline{F})\) respectively. By [23, corollary 3.9], we have that,

\[
(f) = \xi^*((S/T)) = \xi^*([Q_0]) - \xi^*([Q_\infty]),
\]

and \( \deg(\xi^*([Q_0])) = \deg(\xi^*([Q_\infty])) = [F(\mathbb{P}^1) : F(Z)] = q + 1 \). But \( \xi^*([Q_0]) \) is some integer multiple of \([P_{(a : b)}]\) and \( \xi^*([Q_\infty]) \) some integer multiple of \([P_{(c : d)}]\), hence,

\[
(f) = (q + 1) [P_{(a : b)}] - (q + 1) [P_{(c : d)}].
\]
Let $\Gamma^0 \subset \Gamma$ be the degree 0 subgroup of $\Gamma$, and $\text{Cl}^0(\mathbb{Z}) \subset \text{Cl}(\mathbb{Z})$ the degree 0 subgroup of $\text{Cl}(\mathbb{Z})$.

**Lemma 2.7.** The function $\phi : \mathbb{Z} \times (\mathbb{Z}/(q + 1)\mathbb{Z})^q \to \Gamma$,

$$\phi : (n_0, \ldots, n_q) \mapsto n_0[P_0] + n_1([P_1] - [P_0]) + \ldots + n_q([P_q] - [P_0]),$$

is a surjective homomorphism. In particular, $\Gamma^0$ is a quotient of $(\mathbb{Z}/(q + 1)\mathbb{Z})^q$.

**Proof.** For each $P_k \in \mathcal{P}$, we can write $P_k = P(a_k : b_k)$ for some $a_k, b_k \in \mathbb{F}_q$. For each $0 \leq i \neq j \leq q$, consider the rational function,

$$f = \frac{b_j X - a_i Y}{b_j X - a_j Y}.$$

Taking the divisor of $f$,

$$0 = (f) = (q + 1)[P_i] - (q + 1)[P_j],$$

in $\Gamma$, by Lemma 2.6. Therefore, $\phi$ is a well-defined homomorphism, which is surjective because $\{[P_0], \ldots, [P_q]\}$ generate $\Gamma$. Finally, as $\Gamma^0 = \langle [P_1] - [P_0], \ldots, [P_q] - [P_0]\rangle$, then $\Gamma^0$ is a quotient of $(\mathbb{Z}/(q + 1)\mathbb{Z})^q$.

We are finally in a position to prove the main result of this section.

**Theorem 2.8.** $	ext{Pic}(\mathbb{Y})[p] = 0$.

**Proof.** We can split the degree homomorphism with $[P_0]$, as $[P_0]$ has degree 1 and $\langle [P_0]\rangle$ is free [27, exercise 5.12 (b)]. Then,

$$\psi : \text{Cl}(\mathbb{Z}) \to \text{Cl}^0(\mathbb{Z}) \times \mathbb{Z},$$

$$Q \mapsto (Q - \deg(Q)[P_0], \deg(Q)),$$

is an isomorphism, which restricts to,

$$\Gamma \cong \Gamma^0 \times \mathbb{Z}.$$

We then obtain the following commutative diagram,

$$\begin{array}{ccc}
\text{Cl}(\mathbb{Z}) & \xrightarrow{\sim} & \text{Cl}^0(\mathbb{Z}) \times \mathbb{Z} \\
\text{Cl}(\mathbb{Z}) / p\text{Cl}(\mathbb{Z}) & \xrightarrow{\sim} & \text{Cl}^0(\mathbb{Z}) / p\text{Cl}(\mathbb{Z}) \times \mathbb{Z} / p\mathbb{Z}.
\end{array}$$

Here, the vertical maps are induced from the inclusions of $\Gamma$ into $\text{Cl}(\mathbb{Z})$ and of $\Gamma^0$ into $\text{Cl}^0(\mathbb{Z})$, the left horizontal maps are induced by $\psi$, and the right horizontal maps are the standard identifications.

Now, by Lemma 2.7, $\Gamma^0$ is a quotient of $(\mathbb{Z}/(q + 1)\mathbb{Z})^q$, thus $\Gamma^0 / p\Gamma^0 = 0$. Consequently, $\Gamma / p\Gamma \to \text{Cl}(\mathbb{Z}) / p\text{Cl}(\mathbb{Z})$ is an injection. Therefore, $\text{Cl}(\mathbb{Y})[p] = 0$, by the exact sequence of Proposition 2.4.
3. Rigid curves

Let $F$ be a finite extension of $\mathbb{Q}_p$ with uniformiser $\pi$ and residue field $\mathbb{F}_q$. Let $K$ be a complete field extension of $F$ with residue field $\mathbb{F}$, such that $\mathbb{F}$ is an algebraic extension of $\mathbb{F}_q$. Let $R$ be the ring of integers of $K$ and $\varpi \in K$ an element with $0 < |\varpi| < 1$.

Let $A$ be the affinoid algebra,

$$A = K\left(x, \frac{1}{x^q - x}\right),$$

for which the associated rigid space $\text{Sp}(A)$ has admissible formal model $\text{Spf}(A_0)$, where,

$$A_0 = R\left(x, \frac{1}{x^q - x}\right).$$

Let $u := x^q - x \in A_0^\times \subset A^\times$, and let $B$ be the affinoid algebra,

$$B := A[z]/\left(z^{q+1} - u\right).$$

Consider the ring extension,

$$B_0 := A_0[z]/\left(z^{q+1} - u\right).$$

$B_0$ is $\varpi$-torsion free, and the natural map,

$$B_0 = A_0[z]/\left(z^{q+1} - u\right) \longrightarrow R\left(x, \frac{1}{x^q - x}, z\right)/\left(z^{q+1} - u\right),$$

is an isomorphism (because $u \in A_0^\times$ is a unit), hence $B_0$ is an admissible $R$-algebra. The special fibre of $\text{Spf}(B_0)$ is,

$$\text{Spec}(B_0 \otimes_R \mathbb{F}) = \text{Spec}\left(\mathbb{F}[y, 1/v, t]/(t^{q+1} - v)\right),$$

where $v = y^q - y$, and the generic fibre of $\text{Spf}(B_0)$ is $\text{Sp}(B_0 \otimes_R K) = \text{Sp}(B)$.

**Lemma 3.1.** \(\text{Pic}(\text{Spf}(B)) \cong \text{Pic}(\text{Spec}(B))\).

*Proof.* First note that there is an isomorphism of $\mathbb{F}$-algebras,

$$\mathbb{F}[r, s]/(rs^q - sr^q - 1) \xrightarrow{\sim} \mathbb{F}[y, 1/v, t]/(t^{q+1} - v),$$

given by $r \mapsto 1/t$, $s \mapsto y/t$, with inverse $y \mapsto s/r$, $t \mapsto 1/r$. Thus $\text{Spec}(B_0 \otimes_R \mathbb{F}) \cong \text{Y}$, and $\text{Spf}(B_0)$ is a smooth admissible formal model of $\text{Sp}(B)$. Therefore by [20, lemma 3.6], the natural maps,

$$\text{Pic}(\text{Spf}(B)) \xleftarrow{\sim} \text{Pic}(\text{Spf}(B_0)) \xrightarrow{\sim} \text{Pic}(\text{Spec}(B_0 \otimes_R \mathbb{F})),$$

are isomorphisms and we’re done.

We can now state our main results. If $K$ contains $\hat{F}$ the completion of the maximal unramified extension of $F$, then we can consider the rigid analytic space $\Sigma^1$ defined over any such $K$. For an overview of the construction and properties of $\Sigma^1$ see [22, section 2]. If $v \in \mathcal{T}$ is

https://doi.org/10.1017/503050041230000221 Published online by Cambridge University Press
the central vertex of the Bruhat-Tits tree, then the open affinoid subset $\Sigma_v^1 := r^{-1}(v) \subset \Sigma^1$ has coordinate ring isomorphic to,

$$O\left( \Sigma_v^1 \right) \cong A[\bar{z}] / \left( \bar{z}^{q^2-1} - (\pi u^{q-1}) \right),$$

by [22, theorem 2.7].

Let $\omega$ be a primitive $(q^2 - 1)$st root of $\pi$ in $\bar{F}$. From now on we strengthen our assumption on the complete field extension $K$ of $F$ and assume that,

\[ K \text{ contains } \bar{F}(\omega) \text{ and } F \text{ is an algebraic extension of } \mathbb{F}_q. \]

We note that this forces $\mathbb{F}$ to be an algebraic closure of $\mathbb{F}_q$, and that this assumption holds for any complete field extension $K$ of $\bar{F}(\omega)$ which is contained in $\mathbb{C}_p$.

**Theorem 3.2.** $\text{Pic}(\Sigma_v^1) [p] = 0$.

**Proof.** Because $K$ contains $\omega$,

$$O\left( \Sigma_v^1 \right) \cong B^{q-1},$$

and therefore,

$$\text{Pic}(\Sigma_v^1) \cong \text{Pic}(\text{Sp}(B^{q-1})) = \text{Pic}(\text{Sp}(B))^{q-1} \cong \text{Pic}(Y)^{q-1},$$

by Lemma 3.1. But then $\text{Pic}(\Sigma_v^1) [p] \cong \text{Pic}(Y)[p]^{q-1}$, which is zero by Theorem 2.8.

Recall that $\Sigma_v^1 = r^{-1}(v)$ is the pre-image of $v$, the central vertex of the Bruhat–Tits tree. The vertex $v$ is fixed by $\text{GL}_2(O_F)$, and because $r$ is equivariant, $\text{GL}_2(O_F)$ acts on $\Sigma_v^1$.

**Corollary 3.3.** The natural map,

$$O\left( \Sigma_v^1 \right)^\times / O\left( \Sigma_v^1 \right)^\times p^n \longrightarrow H^1_{\text{et}}(\Sigma_v^1, \mu_{p^n}),$$

arising from the Kummer exact sequence is an isomorphism of $\text{GL}_2(O_F)$-modules.

**Proof.** Because $K$ has characteristic 0, we can consider the Kummer exact sequence for rigid analytic spaces [11, section 3.2]. Then the result follows from Theorem 3.2 after taking the long exact sequence in étale cohomology, using that $\text{Pic}(\Sigma_v^1) \cong H^1_{\text{et}}(\Sigma_v^1, \mathbb{G}_m)$ [11, proposition 3.2-4].

As a consequence, we may now compute $H^1_{\text{et}}(\Sigma_v^1, \mathbb{Z}_p(1))$ as the $p$-adic completion of $O(\Sigma_v^1)^\times$. This is completely explicit, as the group $O(\Sigma_v^1)^\times$ has been computed by Junger [22, theorem 5.1].

**Theorem 3.4.** There is an isomorphism of $\mathbb{Z}_p$-linear representations of $\text{GL}_2(O_F)$,

$$H^1_{\text{et}}\left( \Sigma_v^1, \mathbb{Z}_p(1) \right) \cong \lim_{\longrightarrow} \left( O\left( \Sigma_v^1 \right)^\times / O\left( \Sigma_v^1 \right)^\times p^n \right).$$
Proof. For all $n \geq 1$ the diagram,
\[
\begin{array}{ccc}
\mathcal{O}(\Sigma^1_v) \times / \mathcal{O}(\Sigma^1_v) \times \mathbb{P}^{n+1} & \longrightarrow & H^1_{\text{ét}}(\Sigma^1_v, \mu_{p^{n+1}}) \\
\downarrow & & \downarrow \\
\mathcal{O}(\Sigma^1_v) \times / \mathcal{O}(\Sigma^1_v) \times \mathbb{P}^n & \longrightarrow & H^1_{\text{ét}}(\Sigma^1_v, \mu_{p^n})
\end{array}
\]
commutes. Then by the definition of $H^1_{\text{ét}}(\Sigma^1_v, \mathbb{Z}_p(1))$ and Corollary 3.3,
\[
H^1_{\text{ét}}(\Sigma^1_v, \mathbb{Z}_p(1)) = \lim_{\leftarrow} H^1_{\text{ét}}(\Sigma^1_v, \mu_{p^n}) \sim \lim_{\leftarrow} \mathcal{O}(\Sigma^1_v) \times / \mathcal{O}(\Sigma^1_v) \times \mathbb{P}^n.
\]

Acknowledgements. The author would like to thank Konstantin Ardakov, Damien Junger and the referee for their comments on this paper. This research was financially supported by the EPSRC.

REFERENCES

[1] K. ARDAKOV and S. WADSLEY. Irreducibility of global sections of Drinfeld line bundles. In preparation.
[2] C. BONNAFÉ. Representations of $\text{SL}_2(\mathbb{F}_q)$. Algebra Appl., vol. 13 (Springer-Verlag London, ltd., London, 2011).
[3] G. BOSCO. On the $p$-adic pro-étale cohomology of Drinfeld symmetric spaces, arXiv:2110.10683 (2021).
[4] P. BOYER. Mauvaise réduction des variétés de Drinfeld et correspondance de Langlands locale. Invent. Math. 138(3) (1999), 573–629.
[5] H. CARAYOL. Nonabelian Lubin–Tate theor. Automorphic forms, Shimura varieties and $L$-functions, vol. II (Ann Arbor, MI, 1988), pp. 15–39.
[6] P. COLMEZ, G. DOSPINESCU, J. HAUSEUX and W. NIZIOL. $p$-adic étale cohomology of period domains. Math. Ann. 381(1-2) (2021), 105–180.
[7] P. COLMEZ, G. DOSPINESCU and W. NIZIOL. Cohomologie $p$-adique de la tour de Drinfeld: le cas de la dimension 1. J. Amer. Math. Soc. 33(2) (2020), 311–362.
[8] P. COLMEZ, G. DOSPINESCU and W. NIZIOL. Cohomology of $p$-adic Stein spaces. Invent. Math. 219(3) (2020), 873–985.
[9] P. COLMEZ, G. DOSPINESCU and W. NIZIOL. Integral $p$-adic étale cohomology of Drinfeld symmetric spaces. Duke Math. J. 170(3) (2021), 575–613.
[10] P. COLMEZ, G. DOSPINESCU and V. PAŠKŪNAS. The $p$-adic local Langlands correspondence for $\text{GL}_2(\mathbb{Q}_p)$. Camb. J. Math. 2(1) (2014), 1–47.
[11] J. DE JONG and M. VAN DER PUT. Étale cohomology of rigid analytic spaces. Doc. Math. 1(1) (1996), 1–56.
[12] P. DELIGNE and G. LUSZTIG. Representations of reductive groups over finite fields. Ann. of Math. (2) 103(1) (1976), 103–161.
[13] G. DOSPINESCU and A.-C. LE BRAS. Revêtements du demi-plan de Drinfeld et correspondance de Langlands p-adique. Ann. of Math. (2) 186(2) (2017), 321–411.
[14] V. G. DRINFELD. Coverings of $p$-adic symmetric domains. Funkcional. Anal. i Priložen. 10(2) (1976), 29–40.
[15] A. GARCIA and S. TAFAZOLIAN. Certain maximal curves and Cartier operators. Acta Arith. 135(3) (2008), 199–218.
[16] U. GÖRTZ and T. WEDHORN. Algebraic Geometry I. Schemes, second edition, (Springer Spektrum, Wiesbaden, 2020).
[17] S. H. HANSEN. Picard groups of Deligne–Lusztig varieties—with a view toward higher codimensions. Beiträge Algebra Geom. 43(1) (2002), 9–26.
[18] M. Harris. Supercuspidal representations in the cohomology of Drinfeld upper half spaces; elaboration of Carayol’s program. *Invent. Math.* **129**(1) (1997), 75–119.

[19] M. Harris and R. Taylor. *The geometry and cohomology of some simple Shimura varieties*. *Ann. of Math. Stud.* vol. 151 (Princeton University Press, Princeton, NJ, 2001). With an appendix by Vladimir G. Berkovich.

[20] B. Heuer. Line bundles on perfectoid covers: case of good reduction. arXiv:2105.05230 (2021).

[21] D. Junger. Cohomologie de de rham du revtement modr de l’espace de Drinfeld. arXiv:2204.06363 (2022).

[22] D. Junger. Équations pour le premier revêtement de l’espace symétrique de Drinfeld. arXiv:2202.01018 (2022).

[23] Q. Liu. *Algebraic geometry and arithmetic curves*. Oxf. Grad. Texts Math. vol. 6 (Oxford University Press, Oxford, 2002). Translated from the French by Reinie Erné, Oxford Science Publications.

[24] S. Orlik. The pro-étale cohomology of Drinfeld’s upper half space. *Doc Math.* **26** (2021), 1395–1421.

[25] The Stacks Project Authors. *The Stacks Project* 9 2021.

[26] H. Stichtenoth. *Algebraic function fields and codes*, Second edition. Grad. Texts in Math. vol. 254 (Springer-Verlag, Berlin, 2009).

[27] C. A. Weibel. *The K-book*. Grad. Stud. Math. vol. 145. (Amer. Math. Soc. Providence, RI, 2013). An introduction to algebraic $K$-theory.