A Finsler Geometrical Programming for the Nonlinear Complementarity Problem of Traffic Equilibrium

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Abstract

This work is a geometrical approach to the optimization problem motivated by transportation system management. First, an attempt has been made to furnish a comprehensive account of geometric programming based on the elementary Finsler geometry in $\mathbb{R}^n$. Then, a Finslerian dynamical model for the nonlinear complementarity problem of traffic equilibrium is presented that may be applied for different equilibrium problems.

1 Introduction

Finsler metrics, as a natural extension of Riemannian and Euclidean metrics, becomes more present in mathematical programming. Particularly, Finsler metrics originated in the study of calculus of variation by P. Finsler in the first quarter of twentieth century, is more reliable in optimization theory than other two metrics, see for instance [3, 2, 4, 19, 21]. In this paper, we restrict our attention to the Finsler metrics defined on $\mathbb{R}^n$ in order to formulate the problems of nonlinear complementarity problem (NCP) and traffic equilibrium with geometric models.

In a general sense, an optimisation problem or a mathematical programming mainly refers to choosing the best (optimal) elements from a set of available alternatives. If the set includes different paths between two fixed points, we may be able to consider a metric on the set and define the optimised paths as the shortest paths or geodesics of this metric. Definition and axioms of the geometry of a space basically depend on the definition of the shortest path in that space. For a well-known metric (e.g. Euclidean, Riemannian, or Finslerian metric), this happens by establishing the relevant geometry to have a better understanding of its geometric behaviour. This phenomenon may be called the geometrisation of mathematical programming. Here, we present some examples of geometrisation of complementarity problems and traffic equilibriums. This work may serve as the first stage of applying Finsler metrics in these types of mathematical programming.

The need for measures to reduce congestion and manage traffic demand in the metropolitan areas is becoming more serious as the population grows. A functioning society depends on the mobility provided by the transportation network to enable its members to participate in essential activities such as production, consumption, communication, and recreation. However, it is necessary for society also to introduce congestion-relief measures for the quality of life, the environment, and the safety of the citizens not to deteriorate.

Any well-founded traffic model recognises the individual network user’s right to decide about when, where, and how to travel [25]. However, there is always a conflict of interest in the transportation system: A typical traveller expected to choose an optimal route, the route that minimises the combined travel time and cost, given the network conditions during the travel time (the Wardrop’s principle). But, society’s goal often is to reduce the average travel times and the environment’s damage. The aggregate effect of these decisions is a network flow which does not minimise the total

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system costs. The Wardrop conditions are frequently used in urban traffic planning to predict and analyse traffic flows. The equilibrium conditions are based on the assumption of rational route choice behaviour by individual users and define a situation where the travellers’ routes are the shortest paths between the given set of origins and destinations.

In the design and management of urban transportation systems, there is a need for efficient planning tools for analysing and predicting future scenarios. Such tools rely on the mathematical model of the transportation system, the users of the system and their aggregate behaviour under different traffic conditions [6]. From the mathematical point of view, it is assumed that the area under study is strongly connected and there is at least one path between any origin-destination pair. Furthermore, the transportation system is represented by a network that describes transportation possibilities (i.e., roads, transit lines, etc.) between origin-destination pairs.

The nonlinear complementarity problem (NCP) is an important problem in mathematical programming that has many applications in different fields, see for instance [14]. One of the well-known NCP functions is the Fischer-Burmister function [11] that is used extensively in the solution of nonlinear complementarity and variational inequality problems [8]. In 1999, Qi in [27] showed that the and its several variants, such as the Tseng-Luo NCP function ([24]) and the Kanzow-Kleinmichel NCP function ([17]) are smooth in the areas away from the origin and are strongly semi-smooth at the origin. The Fischer-Burmister function and its variants are also irrational. Moreover, Qi proposed a class of piecewise rational NCP functions with the same strongly semi-smooth property, see [27].

Here, we study the traffic equilibrium problem from the Finsler geometrical point of view and show that the shortest paths of traffic problem are solutions of a system of nonlinear equations known as geodesics of a Randers metric. Also, we show that for a dynamical transition network, optimised routes are minimal of an integration of the Fischer-Burmister function. This leads to a Finsler geometrical model for the dynamical Wardrop’s user equilibrium problem.

The remainder of this paper is structured as follows. In Section 2, we briefly define a Finsler metric on \( \mathbb{R}^n \), the NCP and the traffic equilibrium problem. In Section 3, we present a Finsler geometrical model for the traffic problem and show that the optimal routes (shortest paths) of traffic problem can be presented as geodesics of a Randers metric. In Section 4, first we indicate that the Fischer-Burmister function can be considered as a special Randers metric. Then, we present a dynamical model for solving NCP. Finally, we propose a mathematical model for solving the dynamical Wardrop’s user equilibrium problem. These results can apply for different equilibrium problems.

## 2 Preliminaries

In this section, we briefly introduce Finsler metrics on \( \mathbb{R}^n \), the nonlinear complementarity problem (NCP) and the traffic equilibrium problem. We assume that the reader has no background knowledge of this topic.

### 2.1 Finsler metrics on \( \mathbb{R}^n \)

Let \( M = \mathbb{R}^n \) be the real n-dimensional space. The set of all tangent vectors at the point \( x \in M \) is called the tangent space of \( M \) and denoted by \( T_x M \). The set of all tangent spaces at \( x \in M \) is called the tangent bundle of \( M \) and denoted by \( TM := \bigcup_{x \in M} T_x M \). Each element of \( TM \) has the form \((x,y)\) consisting of the point \( x \in M \) and the tangent vector \( y \in T_x M \) at the point \( x \). Let the natural projection \( \pi : TM \to M \) be defined as \( \pi(x,y) := x \). The pair \((M,F)\) is said to be a Finsler space where \( F : TM \to [0,\infty) \) is a function with the following properties:

1. \( F(x,y) \) is differentiable on \( TM \setminus \{0\} \).
2. \( F \) is positively homogeneous of degree one in \( y \), i.e. \( F(x,\lambda y) = \lambda F(x,y), \forall \lambda > 0 \).
3. The Hessian matrix of $F^2$,

$$g = (g_{ij}) := \left(\frac{1}{2} \left[ \frac{\partial^2}{\partial y^i \partial y^j} F^2 \right] \right),$$

is positive-definite on $TM \setminus \{0\}$ where $i, j \in 1, \ldots, n$.

The function $F$ is called a Finsler structure and $g$ is called its associated Finsler metric on manifold $M$. Here, we use the notation $\overline{F}$ for the Finsler structure, rather than the usual notation $F$, to avoid any confusion with the function $F$ in the nonlinear complementarity problem.

We can define a norm function on the vector space $T_xM$ based on any Finsler metric $g$ on $M$. Let $\{U, (x^i)\}$ or simply $(x^1, \ldots, x^n) = (x^i) : U \rightarrow \mathbb{R}^n$ be a local coordinate system on an open subset $U \subset M$ around the point $x \in M$. The coordinate system $\{U, (x^i)\}$ induces a coordinate basis $\frac{\partial}{\partial x^i}$ on $T_xM$. Hence, we can write each tangent vector in the form $y = y^i \frac{\partial}{\partial x^i}$, where we apply the Einstein summation convention or Einstein notation, that is, whenever an index variable appears twice in a single term, once in an upper (superscript) and once in a lower (subscript) position, it implies that we are summing over all of its possible values.

Let $x$ be a point of $M$ with the local coordinate system $(x^i)$, then it generates a local coordinate system $(x^i, y^j)$ on $TM$. The pair $(x^i, y^j)$ consisting of the position $x \in M$ and the direction $y \in T_xM$, is known as the line element of $(M, \overline{F})$. The Finsler structure $\overline{F}(x,y)$ is said to be Riemannian if it is independent of the direction $y$ and by homogeneity of $\overline{F}$ it is equivalent to say that $\frac{\partial \overline{F}}{\partial y^j} = 0$ for $i \in 1, \ldots, n$. The Finsler structure $\overline{F}(x,y)$ is said to be Euclidean if $g_{ij}(x,y) = \delta_{ij}$, where $g_{ij}$ is defined by (1) and $\delta_{ij}$ is the Kronecker delta.

We denote by $\Gamma(p,q)$ the collection of all piecewise $C^\infty$ curves $\sigma : [a, b] \subset \mathbb{R} \rightarrow M$ on $(M, \overline{F})$ with $\sigma(a) = p$ and $\sigma(b) = q$. Then the length of $\sigma$ is defined by

$$J(\sigma) := \int_a^b \overline{F}(\sigma, \frac{d\sigma}{dt}) dt,$$

where $\frac{d\sigma}{dt} = \frac{d\sigma^i}{dt} \frac{\partial}{\partial x^i} \in T_{\sigma(i)}M$. Define the map

$$d : M \times M \rightarrow [0, \infty)$$

$$d(p,q) := \inf_{\Gamma(p,q)} J(\sigma).$$

It can be shown that $(M,d)$ satisfies the axioms of a metric space, except the symmetry property, see [2] and [28]. In the original sense, a geodesic is a generalization of the notion of a "straight line" in a Euclidean space. However, when the metric is not Euclidean, geodesics are not necessarily straight lines and define as follows. A piecewise $C^\infty$ curve $\sigma : [a, b] \rightarrow M$ with $\sigma(a) = p$ and $\sigma(b) = q$ on the space $M$ with the Finsler structure $\overline{F}$ is said to be a geodesic if it is a minimal curve in $\Gamma(p,q)$, with a constant velocity, i.e. $J(\sigma) = d(p,q)$. Hence, geodesics are defined to be (locally) the shortest path between points in the space.

For any Finsler metric $g$ on $M$, we can define an inner product $g(\ldots) := g\langle \ldots \rangle$ on the tangent space $T_xM$ (here, for $M = \mathbb{R}^n$ we have , $T_xM \cong \mathbb{R}^n$). In the local coordinate system $\{U, (x^i)\}$, for all $y, z \in T_xM$ we have $g(y, z) = g_{ij}y^i z^j$. Each inner product defines a norm $||y||_g := g\langle y, y \rangle^{1/2}$. A vector $y \in T_xM$ with respect to $g$. Hence, a vector $y$ on the tangent space $T_xM$ can have different norms according to different Finsler metrics defined on $M$. For a Riemannian metric $g_{ij} \beta^i = g_{ij}y^i z^j$, where $\beta = g(\beta)\beta^i$ is a 1-form on $M$ with $||\beta||_g < 1$, is a Finsler structure on $M$ and its associated Finsler metric is called a Randers metric, see [28] for more details.
2.2 Nonlinear Complementarity Problem

The nonlinear complementarity problems (NCPs) arise from optimisation theory, engineering and economic applications. The restricted NCP functions which can be used to reformulate NCPs as constrained optimisation problems are first introduced by Yamashita [30]. Furthermore, the discretisation of infinite-dimensional variational inequalities leads to linear and nonlinear complementarity problems with many degrees of freedom. Luo and Tseng [24] proposed a class of merit functions for the NCPs and showed that they satisfy several interesting properties under some assumptions. Kanzow, Yamashita and Fukushima have used the similar idea and proposed new merit functions for the NCP in [18].

For a given smooth mapping \( F : \mathbb{R}^n \to \mathbb{R}^n \), the NCP consists of finding a vector \( x \in \mathbb{R}^n \) such that

\[
\begin{align*}
x & \geq 0, \\
F(x) & \geq 0, \\
F(x)^T . x & = 0,
\end{align*}
\]

where \( F(x)^T \) is the transpose of \( F(x) \). Complementarity problems can be reformulated as systems of nonlinear equations in several ways. A large number of methods has been developed based on Newton method and its generalisations. In order to overcome some disadvantages of the class of non-smooth Newton methods and the class of the so-called smoothing methods, a third class known as Jacobian smoothing methods developed in [16]. A Jacobian smoothing method is a mixture of Newton and non-smooth methods and is based on solving the mixed Newton equation: a combination of the original semi-smooth function and the Jacobian of the smooth operator of the Clarke generalised Jacobian [7]. There are also several related algorithms entitled Newtonian algorithms, see for instance [9, 15, 16, 26, 30]. Further, a solution of NCP with neural networks algorithm was presented in [23]. Among these algorithms, the modified Newton algorithm is preferred.

The modified Newton algorithm is a reformulation of system of differential equations (2) that converts it to a non-modal optimisation problem which is convergent under some conditions, see Section 8.7 of [5]. This reformulation is given by the system of equations:

\[
\phi(x) = \begin{pmatrix}
\phi_1(x) \\
\vdots \\
\phi_n(x)
\end{pmatrix} = \begin{pmatrix}
\phi(x_1, F_1(x)) \\
\vdots \\
\phi(x_n, F_n(x))
\end{pmatrix} = 0,
\]

where the NCP-function \( \phi \) is the Fisher-Burmeister NCP-function \([10, 11]\) defined as follows,

\[
\phi : \mathbb{R}^2 \to \mathbb{R}, \quad \phi(a, b) = \sqrt{a^2 + b^2} - (a + b),
\]

where \( \phi(a, b) = 0 \iff a \geq 0, b \geq 0, ab = 0 \). The resulting system of nonlinear equations (2) is semi-smooth and the NCP has an solution \( x \) if and only if \( x \) is a solution for (3). Note that the function \( \phi \) is locally Lipschitz and continuous at every point. Thus, the Clarke generalised Jacobian at every point is defined. In [12], the NCP is recast as an unconstrained minimisation problem and the natural merit function \( G : \mathbb{R}^n \to \mathbb{R} \) given by

\[
G(x) = \frac{1}{2} \sum_{i=1}^{n} \phi(x_i, F_i(x))^2 = \frac{1}{2} \| \phi(x) \|^2,
\]

is considered. Also, it is proved that any stationary point of the above function is a solution of the NCP if the mapping \( F \) involved in Eq. (2) is continuously differentiable and monotone. Hence, the solution of the NCP are given by solving

\[
\min_{x \in \mathbb{R}^n} G(x).
\]

The NCP mainly applied in solving the traffic equilibrium problem which is studied here, and the Karush-Kuhn-Tucker (KKT) optimality conditions [13].
2.3 Traffic equilibrium problem

Network equilibrium models have a variety of applications in urban transportation, energy distribution, game theory, spatially separated economic markets, electrical networks, and water resource planning, [1]. Although the idea of traffic equilibrium originated as early as 1924 with Frank Knight [20], the mathematical description of equilibrium was provided for the first time in 1952 by Wardrop. The Wardrop’s equilibrium principals [29] are related to the idea of Nash equilibrium [22] in game theory which is developed separately. However, analysing the transportation networks with many players involved is more difficult than analysing games with a few number of players. The Wardrop’s principals states as follows:

**Wardrop’s first principle**  The travel times in all routes actually used are equal and less than those which would be experienced by a single vehicle on any unused route. Each user non-cooperatively seeks to minimise their transportation cost/time. A traffic flow that satisfies this principle is usually named **user equilibrium (UE) flow**.

**Wardrop’s second principle**  At equilibrium the average journey time is minimum. This implies that each user behaves cooperatively in choosing their route to ensure the minimum cost of the whole system. A traffic flow satisfying Wardrop’s second principle is generally referred to as **system optimal (SO) flow**.

Wardrop’s principles can be presented mathematically as follows. Assume that $R_{ab}$ denotes the set of simple (loop-free) routes for the origin-destination pair $(a, b)$, $h_r$ denotes the flow on the route $r \in R_{ab}$, and $c_r$ denotes the travel time on the route $r \in R_{ab}$ as experienced by an individual user. If the flow does not depend on time, i.e. in a static model (against the dynamic model), the Wardrop’s principles are written as

$$h_r^T (c_r(h_r) - \pi_{ab}) = 0, \quad c_r(h_r) - \pi_{ab} \geq 0, \quad h_r = d_r(\pi_{ab}), \quad h_r \geq 0, \quad \pi_{ab} \geq 0, \quad (7)$$

where $\pi_{ab}$ is the minimal rout time and $d_r$ is the demand function of the route $r \in R_{ab}$.

In [11], an equilibrium model for predicting traffic flow on a congested transportation network is proposed based on using the Wardrop’s first principle and write it as an NCP (Eq. (2)) in the following form:

$$F(x)^T \cdot x = (c_r(h_r) - \pi_{ab}) \cdot (h_r - d_r(\pi_{ab})) = 0, \quad x = \left( \frac{h_r}{\pi_{ab}} \right) \geq 0, \quad F(x) = \left( \frac{c_r(h_r) - \pi_{ab}}{h_r - d_r(\pi_{ab})} \right) \geq 0, \quad (8)$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. For reducing this NCP to an unconstrained global minimisation problem, gap function $G$ which is smooth, convex and bounded is applied, see [9]:

$$G(x) = \sum_{i=1}^{n} \phi(x_i, F(x_i)), \quad (9)$$

where $\phi = \frac{1}{4} \phi^2$ and $\phi$ is the Fischer-Burmeister function given by (4). So, solving the NCP (8) is equivalent to finding a general unconstrained solution for Eq. (6) with $G$ given in (9).

3 A Finsler geometric model of traffic problem

In this section, we apply geometric methods to provide a mathematical model for the transportation network. Unless stated otherwise we will assume further that

1 All origin-destination pairs are distinct.
II The network is strongly connected, that is, at least one route joins each origin-destination pair.

III The transport demand is either a constant number or a function of travel cost/time.

IV All travellers have perfect information about their travel. Therefore, we have a deterministic model, unlike a stochastic model where travellers choose their paths randomly.

For any given pair of the origin-destination points in the vehicle network, we usually have different route options. These routes are made by a string of streets and cross-sections and for each route we have an estimated travel time. This time depends on some factors such as the capacity of streets, the number of stops behind cross-sections and the travel demand of that specific route.

Now assume that we have a Riemannian metric \( g \) on \( \mathbb{R}^2 \) and its corresponding norm-squared for tangent vectors \( y \in T_x \mathbb{R}^2 \) is given by \( ||y||^2_g := g(y,y) \). In traffic modelling, we can interpret it as the time it takes, using a car with fixed power, to travel from the base point of the vector \( y \) to its tip.

Let \( u \in T_x \mathbb{R}^2 \) be a unit vector and denote the traffic congestion or any external factor that increases the traffic congestion by a vector \( \omega \). After taking the effect of the vector \( \omega \) into account, we have \( v = u - \omega \) instead. This is because traffic congestion is a vector in the opposite direction of flow. The measure of this new vector is not equal to 1 (\( ||v||_g \neq 1 \)). This argument for using vector’s length is the key idea of a technique known as Okubo’s technique [2]. In fact, the geometry of movement in the former case (without considering the external factor) is the Riemannian geometry, rather than the Finslerian geometry in the latter case. Here, we consider the effect of traffic congestion (the vector field \( \omega \)) and introduce a new metric \( \mathcal{F} \) such that \( v \) be a unit vector with respect to the new norm corresponding to \( \mathcal{F} \).

**Theorem 3.1.** Let \( g \) be a Riemannian metric and \( \omega \) a vector field on \( \mathbb{R}^2 \) such that \( ||\omega||_g < 1 \). If \( \omega \) indicates the traffic congestion, then the travel time for a car with a fixed power to pass through a vector field \( y \in T_x \mathbb{R}^2 \) is measured by the Randers metric \( \mathcal{F}(x,y) = \frac{||y||_g}{1-||\omega||_g} \).

**Proof.** Let \( u \in T_x \mathbb{R}^2 \) be a unit vector. After taking the effect of the vector \( \omega \) into account, we have \( v = u - \omega \), where \( ||v||_g = 1 - ||\omega||_g \). Now, a Finsler structure \( \mathcal{F} \) that satisfies \( \mathcal{F}(v) = 1 \) would be \( \mathcal{F}(v) = \frac{||v||_g}{1 - ||\omega||_g} \). For an arbitrary vector field \( y \in T_x \mathbb{R}^2 \), the corresponding Finsler metric \( \mathcal{F} \) is

\[
\mathcal{F}(x,y) = \frac{||y||_g}{1 - ||\omega||_g}.
\]

If we put \( \lambda = 1 - ||\omega||_g^2 \) then \( \mathcal{F}(x,y) \) can be written as

\[
\mathcal{F}(x,y) = \frac{||y||_g}{\lambda} + \frac{||\omega||_g||y||_g}{\lambda} = \frac{\sqrt{g_{ij}y^i y^j}}{\lambda} + \frac{g_{ij}\omega^i y^j}{\lambda},
\]

(10)

which is obviously a Randers metric.

Therefore, the optimal/shortest route between any arbitrary origin-destination pair \( (p,q) \) is a geodesic of the corresponding Randers metric passing through these points. Equivalently, we need to find minimums of the following integral

\[
J_{\mathcal{F}}(C) = \int_0^1 \mathcal{F}(x(t),y(t))dt, \quad y(t) = \frac{dx}{dt},
\]

(11)

where \( \mathcal{F} \) is given by (10) and \( C : [0,1] \to \mathbb{R} \) defined by \( t \to x(t) \) such that \( x(0) = p \) and \( x(1) = q \). It can be shown that by assuming \( L = \frac{\mathcal{F}^2}{2} \), any minimal point \( x(t) \) of the length integral (11) is a solution of the following system of differential equations, see Chapter 3 of [28]:

\[
L_{ij} \frac{d^2 x^i}{dt^2} + \frac{\partial L_{ij}}{\partial x^j} \frac{dx^i}{dt} - \frac{\partial L}{\partial x^i} = 0,
\]

(12)
where \( L_i = \frac{\partial L}{\partial y^i} \), \( L_{ij} = \frac{\partial^2 L}{\partial y^i \partial y^j} \). Therefore,

**Corollary 3.2.** If we consider the traffic congestion as an external factor that effects on vehicle transition network, then the differential equation of the shortest route between any origin-destination pair is given by Eq. (12).

4 **A Finslerian model for Wardrop’s user equilibrium problem**

In this section we show that the Fischer-Burmeister function is a special Randers metric, and therefore, the equations of its geodesics are equations of the minimal paths. Then, we present a Finsler geometrical model for solving the dynamical Wardrop’s user equilibrium problem that may applied in solving different equilibrium problems.

4.1 **Finsler geometry and traffic equilibrium**

Here, we illustrate that the Fischer-Burmeister function is a special Randers metric.

**Proposition 4.1.** The Fischer-Burmeister function is a Finsler metric of Randers type and its geodesics are the optimised paths in the NCP.

**Proof.** Let \( g \) be a Euclidean metric on \( \mathbb{R}^2 \) given by \( g(a, b) = \sqrt{a^1 b^1 + a^2 b^2} \), where \( a = (a^1, a^2) \) and \( b = (b^1, b^2) \) are two arbitrary points on \( \mathbb{R}^2 \). The local coordinate system \( (x^1, x^2) \) on \( \mathbb{R}^2 \) induces the local fields of frames \( \left( \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \right) \) and \( (dx^1, dx^2) \) on the tangent space \( T_x \mathbb{R}^2 \) and its dual \( T^*_x \mathbb{R}^2 \). Therefore, we have \( g \left( \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \right) = \sqrt{(dx^1)^2 + (dx^2)^2} \) and from Eq. (4), the Fischer-Burmeister function on \( \mathbb{R}^2 \) is given by:

\[
\phi(dx^1, dx^2) = \sqrt{(dx^1)^2 + (dx^2)^2} - (dx^1 + dx^2). \tag{13}
\]

The first term in the right hand side of (13) can be written as \( \sqrt{\delta_{ij} dx^i dx^j} \), where \( \delta_{ij} \) is the Kronecker symbol and \( i, j \) run over the range 1,2. So, it is a Euclidean (or Riemannian) metric on \( \mathbb{R}^2 \). The second term in (13) is a differential 1-form \( \beta = \omega_i dx^i \), where \( \forall i : \omega_i = -1 \). Therefore, the Fischer-Burmeister function is a Randers metric.

By virtue of the above proposition, solutions of the NCP, i.e. the optimised paths of the Fischer-Burmeister function, are geodesics of a Randers metric.

4.2 **The dynamical transition network**

The traffic congestion and transport demand in some hours of a day (peak hours) are extremely increasing. A typical solution to address this issue, instead of building additional infrastructure, is introduction of dynamic elements to the road traffic management. This includes for instance using sensors to measure traffic flows and automatic interconnected guidance systems (for example traffic signs which open a lane in different directions depending on the time of day) to manage traffic in peak hours. Here, we generalise the method of solving the NCP (in the previous subsection) to dynamic systems.

**Theorem 4.2.** If the traffic flow and the travel time are functions of time \( t \), then the solutions of traffic equilibrium problem are minimal of the following integration

\[
G(t) = \int_{t_0}^{t_1} \phi(h(t), c(t)) h(t) dt, \tag{14}
\]

where \( \phi = \frac{1}{2} \phi \) and \( \phi \) is the Fischer-Burmeister function, \( h(t) \) is the traffic flow, \( c(t) \) is the travel time, \( t_0 \) is the start of travel time, and \( t_1 \) is the end of travel time.
Proof. Let the traffic flow \( h(t) \) and the travel time \( c(t) \) (which is in general a function of \( h \)) be functions of time and assume that the minimal rout time \( \pi \) in (7) is zero. Then, using the Fischer-Burmeister function \( \phi \) on \( \mathbb{R}^2 \) (Eq. (13)), the Eq. (5) can be written as

\[
G(h) = \int_{h_0}^{h_1} \phi(h, c(h)) dh,
\]

where \( \phi = \frac{1}{2}\phi \). Since \( dh = h'(t) dt \), this equation is equivalent to Eq. (14). Now by virtue of a well-known fact in the calculus of variation the minimal of integration (14) are solutions of traffic equilibrium problem or present the shortest route (the route which needs the shortest time duration to passing through it).

4.3 The Finsler geometrical model

Now, we can present a Finsler geometrical model for solving the dynamical Wardrop’s user equilibrium problem:

Lemma 4.3. Any Finsler structure \( \mathcal{F}(x, y) \) on \( \mathbb{R}^2 \) can be considered as a gap function in solving the NCP.

Proof. Assume that the transit network has all conditions I-IV in Section 3 and also traffic flow and the other ingredients in the transit network are functions of time. Further assume that \( \mathcal{F}(x, y) : T\mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^+ \) is a Finsler metric on \( \mathbb{R}^2 \) and a car with a fixed power travels from point \( a \) to point \( b \) trough a smooth curve \( r \) on \( \mathbb{R}^2 \). From the calculus of variation the minimal of the following integration provide the shortest curve between points \( a \) and \( b \):

\[
L_{\mathcal{F}}(r) = \int_a^b \mathcal{F}(x(t), y(t)) dt,
\]

(15)

where \( y(t) = \frac{dx}{dt} \).

Since \( \mathcal{F} \) is a Finsler structure, the following conditions are satisfied (see for instance [2, 28]):

1. The length of the curve \( r \) is independent of the parameter \( t \).
2. The length of the curve \( r \) is always a positive number.
3. If we put \( (g_{ij}) := \left( \frac{1}{2} \left[ \frac{\partial^2}{\partial y^i \partial y^j} \mathcal{F}^2 \right] \right) \), then we have \( \text{det}(g_{ij}) \neq 0 \).
4. \( \mathcal{F}(x, y) \) is a differentiable function.

Therefore, \( \mathcal{F}(x, y) \) meets all the conditions of a gap function in solving the NCP, see [5]. □

Theorem 4.4. The optimised routes for the traffic equilibrium problem in a dynamical transition network are geodesics of a Finsler structure \( \mathcal{F}(x, y) \) of Randers type, where \( x(t) \) denotes the traffic flow at time \( t \) and \( y(t) = \dot{x}(t) \) is the velocity of a car with fixed power at time \( t \).

Proof. Let \( x(t) \) be the traffic flow and \( y(t) = \dot{x}(t) \) the velocity of a car with fixed power at time \( t \). Then, the Wardrop’s user equilibrium principle can be written as

\[
\mathcal{F}(x, y) = 0 \Leftrightarrow x \geq 0, \quad y \geq 0, \quad x y = 0.
\]

Here, \( \mathcal{F}(x, y) = 0 \) implies that the traffic congestion is extremely increased and the car velocity is zero \( (y = 0) \) or the car stops, so we have \( x, y = 0 \). In Section 3, we show that the geometrical method of solving the traffic problem leads to finding the geodesics of a Randers metric. On the other hand, Proposition 4.1 implies that the common applied mathematical method, i.e. finding the minimums of the Fischer-Burmeister function, can be seen as a special case of this geometrical method. Thus, if the given Finsler metric in Lemma 4.3 be a Randers metric, it is a solution of the NCP. Therefore, from Theorem 4.2, the equations of optimised routes for the traffic equilibrium problem in a dynamical transition network are given by the geodesic equations of a Finsler metric of Randers type. □
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