A note on the Petrov-like boundary condition at finite cutoff surface in Gravity/Fluid duality

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Abstract

Previously it has been shown that imposing a Petrov-like boundary condition on a hypersurface may reduce the Einstein equation to the incompressible Navier-Stokes equation, but all these correspondences are established in the near horizon limit. In this note, we remark that this strategy can be extended to an arbitrary finite cutoff surface which is spatially flat, and the Navier-Stokes equation is obtained by employing a non-relativistic long-wavelength limit.
I. INTRODUCTION

It has been known that the excitations of a black hole horizon dissipate very much like those of a fluid since the 70’s of last century [1–4]. From then on, the gravity/fluid duality has been heavily investigated and lots of important progress have been made [5–40]. Remarkably, recent progress in AdS/CFT correspondence has shed more insightful light on this duality. The method of hydrodynamical expansion of the metric was initially proposed to study the dual fluid living on the boundary of spacetime, in which the regularity condition is imposed on the horizon and a long-wavelength expansion is needed [24]. Later, an alternative way was proposed to reduce the Einstein equation to the Navier-stokes equation by imposing a Petrov-like boundary condition on the cutoff surface [28]. The key idea of this strategy is to consider the perturbations of the extrinsic curvature of the cutoff surface directly, rather than those of the metric. As a result, the Brown-York stress tensor is treated as the fundamental variable which due to the holographic dictionary can be identified with the stress-energy tensor of a fluid living on the cutoff surface [24, 40]. In another word, we may extract the hydrodynamical behavior of gravity directly, needless to solve the perturbation equation for the explicit form of the perturbed metric. In literature the advantages of this strategy have been continuously disclosed. It has been successfully applied to a spacetime with a spatially curved cutoff surface [31], or a spacetime with a cosmological constant as well as matter fields [32, 36]. In particular, our recent investigation indicates that it can be applicable for a very general spacetime which is only required to contain a weakly isolated horizon without rotation [38]. Nevertheless, comparing with the conventional hydrodynamical expansion method, the method of imposing Petrov-like boundary condition contains an obvious weak point, which sticks to the near horizon limit. Namely, in this approach we always take the non-relativistical limit with the near horizon limit simultaneously. While recently our understandings on the gravity/fluid duality have been significantly pushed forward by investigating the hydrodynamical behavior of gravity at finite cutoff surface based on the Wilsonian approach or the renormalization group point of view [14, 20, 21]. One key observation in this approach is that any interacting quantum field theory at finite temperature should be described by hydrodynamics when viewed at sufficiently long length scales. Since the radial coordinate $r$ of the AdS bulk corresponds to the energy scale of the boundary field theory, the near horizon limit only captures the low-frequency limit of linear response.
of the boundary theory fluid. Therefore, if one intends to move away from the low frequency limit, he need consider fluid membrane at a hypersurface with constant-radius and finite distance from the horizon. A flow equation for the radius-dependent response function has been derived, for instance in [14], which can be viewed as a renormalization group flow to link the gravity/fluid duality near horizon and that at infinity. In this note we intend to establish such duality at finite cutoff surface with the Petrov-like boundary condition method.

We will demonstrate through explicit models that once the near horizon limit is replaced by the long-wavelength limit, the incompressible Navier-Stokes equation can still be derived by directly imposing Petrov-like boundary condition on the finite cutoff surface such that the gravity/fluid duality can be established. Of course in this extension we will only focus on the cutoff surface which is spatially flat since the long-wavelength limit is introduced.

To keep this note in a concise version, we will just present our main results in the main body, but leave all the detailed calculation in the appendix.

II. PETROV-LIKE BOUNDARY CONDITION ON THE FINITE CUTOFF SURFACE FOR RINDLER SPACETIME

The framework of imposing Petrov-like boundary condition on the cutoff surface has been introduced in previous literature, and we refer to Refs. [28, 31] for details. Here we just repeat its basic definition and setting. The Petrov-like boundary condition on a hypersurface $\Sigma_c$ is defined as

$$C(\ell)_{ij}(\ell)_{j} \equiv \ell^\mu m_i^\nu \ell^\alpha m_j^\beta C_{\mu \nu \alpha \beta} = 0,$$

We stress that, although in original Ref. [24] it has been pointed out that the perturbed metric in bulk obtained by hydrodynamical expansion may be subject to the Petrov-type condition at finite cutoff surface (also see the similar check in [37]), conversely the explicit demonstration that any perturbation constrained by the Petrov-like boundary condition at finite cutoff surface would lead to Navier-Stokes equation is still missing.

The need of a long-wavelength limit in finite cutoff case can be understood from the viewpoint of holography. The radius of the cutoff corresponds to the energy scale of a dual field theory on the boundary. In near horizon case, the energy scale of dual theory approaches to zero, which means any perturbation of the dual theory are low energy modes which can be described by hydrodynamics. However, in finite cutoff case, the energy scale of dual theory is not low enough such that not all perturbations have contributions to the hydrodynamical degrees of freedom. In this sense we need the long-wavelength limit to pick out those low energy perturbation which corresponds to the degree of freedom of hydrodynamics.
where \( C \) is the Weyl tensor and the Newman-Penrose-like vector fields satisfy the relations

\[
\ell^2 = k^2 = 0, \quad (k, \ell) = 1, \quad (k, m_i) = (\ell, m_i) = 0, \quad (m^i, m_j) = \delta^i_j. \tag{2}
\]

As the simplest example we firstly demonstrate how to derive the Navier-Stokes equation at finite cutoff surface in Rindler spacetime. A \( p + 2 \)-dimensional metric is

\[
ds_{p+2}^2 = -r dt^2 + 2 dt dr + \delta_{ij} dx^i dx^j, \quad i, j = 1, \ldots, p. \tag{3}
\]

Setting \( r = r_c \), then we obtain an embedded hypersurface \( \Sigma_c \) and the induced metric on \( \Sigma_c \) reads as

\[
ds_{p+1}^2 = -r_c dt^2 + \delta_{ij} dx^i dx^j \equiv -(dx^0)^2 + \delta_{ij} dx^i dx^j. \tag{4}
\]

In coordinate system \((t, x^I)\), one can easily check that the non-vanishing component of the extrinsic curvature \( K_{ab} \) is \( K_{tt} = -\sqrt{r_c}/2 \). In order to extract the dynamical behavior of the geometry in the long-wavelength limit as well as the non-relativistic limit simultaneously, we introduce a parameter \( \lambda \) by rescaling the time coordinate with \( x^0 = \frac{1}{\lambda} \tau \) and the space coordinates with \( x^i = \frac{1}{\sqrt{\lambda}} x^I \) such that

\[
ds_{p+1}^2 = -\frac{1}{\lambda^2} d\tau^2 + \frac{1}{\lambda} \delta_{IJ} dx^I dx^J. \tag{5}
\]

Obviously the non-relativistic limit and long-wavelength limit can be implemented by taking \( \lambda \to 0 \).

Next we consider the perturbations of gravity. As we have adopted before in \[31\], \[32\] and \[36\], we keep the intrinsic metric of the hypersurface fixed, and then take the Brown-York stress tensor as the fundamental variable, which is defined as \( t_{ab} \equiv Kh_{ab} - K_{ab} \). In coordinate system \((\tau, x^I)\), we expand the components of Brown-York tensor in powers of \( \lambda \) as

\[
\begin{align*}
t^\tau_\tau &= 0 + \lambda t^\tau_\tau^{(1)} + \ldots \\
t^\tau_I &= 0 + \lambda t^\tau_I^{(1)} + \ldots \\
t^I_J &= \frac{1}{2\sqrt{r_c}} \delta^I_J + \lambda t^I_J^{(1)} + \ldots \\
t &= \frac{p}{2\sqrt{r_c}} + \lambda t^{(1)} + \ldots,
\end{align*}
\tag{6}
\]
where $t$ is the trace of Brown-York tensor. In terms of the Brown-York tensor, the Hamiltonian constraint can be written as

$$
(t^\tau_\tau)^2 - \frac{2}{\lambda^2} h^{IJ} t^I_\tau t^J_\tau + t^I_J t^J_\tau - \frac{t^2}{p} = 0.
$$

(7)

Directly taking the perturbation expansion, we find the leading order is trivially satisfied by the background while the sub-leading order with $\lambda^1$ reads as

$$
t^\tau_\tau^{(1)} = -2\sqrt{r_c} \delta^{IJ} t^I_\tau^{(1)} t^J_\tau^{(1)}.
$$

(8)

Now we turn to the Petrov-like boundary condition. In terms of the Brown-York tensor in $(\tau, x^I)$, this condition becomes

$$
\lambda t^\tau_\tau t^I_J + \frac{2}{\lambda} h^{IK} t^I_K t^\tau_\tau - 2 \lambda^2 t^I_J t^\tau_\tau - \lambda t^I_K t^J_K - 2 h^{IK} t^\tau_\tau (K,J) + \lambda \delta^I_J \left[ \frac{t^I}{p} - \frac{t^\tau_\tau}{p} \right] + 2 \lambda \partial^I \frac{t^\tau_\tau}{p} = 0.
$$

(9)

Similarly, one finds the leading order of the expansion is automatically satisfied by the background, while the sub-leading order with $\lambda^2$ gives

$$
t^I_J^{(1)} = 2\sqrt{r_c} \delta^{IK} t^\tau_\tau^{(1)} t^K_J + \sqrt{r_c} \delta^{IK} \partial_J t^K_\tau^{(1)} - \sqrt{r_c} \delta^{IK} \partial_K t^\tau_\tau^{(1)} + \delta^I_J \frac{t^{(1)}}{p}.
$$

(10)

Until now we have obtained the sub-leading order of the Hamiltonian constraint and the Petrov-like boundary condition. The next step is plugging these results into the momentum constraint. Its time component and space component will be identified as the incompressible condition and the Navier-Stokes equation respectively. Such technical steps have been used in Refs. [28, 31, 32] and [36]. Hence, substituting all these results into the momentum constraint

$$
\partial_a t^a_b = 0,
$$

(11)

and identifying

$$
t^\tau_\tau^{(1)} = \frac{1}{2\sqrt{r_c}} \tilde{v}_I, \quad t^{(1)} = \frac{p}{2\sqrt{r_c}} \tilde{P},
$$

(12)

as the velocity and the pressure fields of the dual fluid, we obtain the incompressible condition and the Navier-Stokes equation on a finite cutoff surface as

$$
\partial_I v^I = 0,
$$

(13)

$$
\partial_\tau v_I + \delta^{JK} v_K \partial_J v_I - \sqrt{r_c} \delta^{JK} \partial_J v_K + \partial_I \tilde{P} = 0.
$$

(14)

\footnote{The original definitions about the Hamiltonian constraint and the momentum constraint on the cutoff surface can be found in [28]. Moreover, their specific forms for the models in our current paper have previously been presented in [28, 32] and [36] respectively.}
We find that the kinematic viscosity $\nu_c$ is cutoff dependent with $\nu_c = \sqrt{r_c}$. First of all, we remark that we have obtained the same results as those obtained by hydrodynamical expansion of the metric in $[24]^4$. Secondly, we find that the previous results obtained for the cutoff surface in near horizon limit in $[28]$ can be treated as a special case of our current work. As a matter of fact, transforming the coordinate system $(\tau, x^I)$ into $(\tau, x^i)$ which is applied in $[28]$ we find

$$\partial_\tau v_i + v^j \partial_j v_i - \partial^j \partial_j v_i + \partial_i P = 0,$$

(15)

where $v_i$ is defined as $dx^i/d\tau$ correspondingly. Therefore, in near horizon limit one obtains the standard incompressible Navier-Stokes equation with unit shear viscosity.

III. PETROV-LIKE BOUNDARY CONDITION ON THE FINITE CUT-OFF SURFACE FOR A BLACK BRANE BACKGROUND

Next we will treat the Petrov-like boundary condition on the finite cutoff surface for different backgrounds in a parallel way. The general framework for Petrov-like boundary condition in this context is presented in $[32]$. Firstly we consider a black brane background with a metric as

$$ds^2_{p+2} = -f(r)dt^2 + 2dt dr + r^2 \delta_{ij} d\bar{x}^i d\bar{x}^j, \quad i, j = 1, \ldots p,$$

$$f(r) = r^2 (1 - r_h^{p+1}/r^{p+1}), \quad \Lambda = -\frac{p(p+1)}{2}.$$  

(16)

Where $r_h$ is the position of the horizon. Setting $r = r_c$, we have the embedded hypersurface $\Sigma_c$ and its metric reads as

$$ds^2_{p+1} = -f(r_c)dt^2 + r_c^2 \tilde{\delta}_{ij} d\bar{x}^i d\bar{x}^j \equiv -(dx^0)^2 + \delta_{ij} dx^i dx^j.$$  

(17)

It is obvious that this is a intrinsically flat embedding, so that

$$^{p+1}\tilde{R}_{ij} = \tilde{p} \tilde{R}_{ij} = 0.$$  

(18)

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$^4$ Transforming the coordinate system from $(\tau, x^I)$ to $(t, x^i)$, we easily find the kinematic viscosity in $(t, x^i)$ is $r_c$ as derived in $[24]$. Moreover, we point out that the Navier-Stokes equation has the same form (or has the same kinematic viscosity) in $(\tau, x^I)$ and $(x^0, x^i)$ coordinate systems, which can be easily proved by dimension analysis and can be viewed as an alternative representation of the scaling symmetry of NS equation presented in $[24]$. Thus we will always identify our derived equations in $(\tau, x^I)$ system to those in $(x^0, x^i)$ system as well.
Similarly, the non-relativistic and long-wavelength limit is complemented by rescaling the coordinates as
\[ ds_{p+1}^2 = -\frac{1}{\lambda^2}d\tau^2 + \frac{1}{\lambda}\delta_{IJ}dx^I dx^J. \]  

Straightforwardly, we take the perturbation expansion for Brown-York stress tensor and substitute it into the Petrov-like boundary condition and constraint equations step by step. We find in this case the sub-leading order of the Hamiltonian constraint becomes
\[ t_\tau^{(1)} = \frac{2\sqrt{f}r_c}{-r_c\partial_r f + (p-2)f} \delta^{MN}t_M^{(1)}t_N^{(1)} + \frac{2f}{-r_c\partial_r f + 2f}t^{(1)}. \]  

While from the Petrov-like boundary condition we have
\[ t_I^{(1)} = \frac{2\sqrt{f}r_c}{r_c\partial_r f + (p-2)f} \delta^{JK} t_K^{(1)} t_J^{(1)} - \frac{2\sqrt{f}r_c}{r_c\partial_r f + (p-2)f} \delta^{JK} t_K^{(1)} t_J^{(1)} - \frac{f}{r_c\partial_r f} \delta_{IJ} t^{(1)} + \frac{f}{r_c\partial_r f + (p-2)f} \delta_{IJ} t^{(1)}. \]  

If we identify
\[ t_I^{(1)} = \frac{r_c\partial_r f + (p-2)f}{2\sqrt{f}r_c} v_I, \]  
\[ \tilde{P} = \frac{f}{r_c\partial_r f - 2f} \delta^{MN} v_M v_N + \frac{2\sqrt{f}r_c^2\partial_r f}{p[r_c\partial_r f + (p-2)f](r_c\partial_r f - 2f)} t^{(1)}, \]  

then the momentum constraint leads to the incompressible condition and the Navier-Stokes equation as
\[ \partial_I v^I = 0, \]  
\[ \partial_r v_I + \delta^{JK} v_K \partial_J v_I - \nu_c \delta^{JK} \partial_J v_K + \partial_I \tilde{P} = 0. \]

Where \( \nu_c = \frac{\sqrt{f}r_c}{r_c\partial_r f + (p-2)f} \) is the viscosity of the dual fluid. As argued in previous section, the form of Navier-Stokes equation will not change when one transforms the coordinate system to \((x^0, x^i)\)
\[ \partial_0 v^i = 0, \]  
\[ \partial_0 v_i + v^j \partial_j v_i - \nu_c \partial_i v^j + \partial_i P = 0, \]

with the same viscosity \( \nu_c = \frac{\sqrt{f}r_c}{r_c\partial_r f + (p-2)f} \).

Specially, when

\[ r_c \to r_h, \quad \nu_c = 0; \]
\[ r_c \to \infty, \quad \nu_c = \frac{1}{p}. \]
First of all, after transforming the coordinate system \((x^0, x^i)\) into \((\tau, x^i)\), we find that the previous results obtained for the cutoff surface in near horizon limit in [32] can be treated as a special case of our current work. Secondly, comparing our results with the previous results presented in [27], the viscosity of both results are equal to zero when the hypersurface moves to horizon. However, when the hypersurface moves to infinity, our viscosity approaches to \(\frac{1}{p}\), in contrast to the results in [27] in which the viscosity tends to divergence. Such a difference implies that the dual field theory that we obtained from the Petrov-like boundary condition may be different from that obtained through the hydrodynamical expansion of the metric.

IV. PETROV-LIKE BOUNDARY CONDITION ON THE FINITE CUTOFF SURFACE FOR A BACKGROUND WITH MATTER

The last model is on the gravity/fluid duality in spacetime with matter fields. The general framework for Petrov-like boundary condition in this context is presented in [36]. Here we consider a 4-dimensional magnetic black brane, which is a solution to the Einstein equation coupled to the electromagnetic field with a metric as

\[
\begin{align*}
    ds_4^2 &= -f(r)dt^2 + 2dtdr + r^2\delta_{ij}dx^i dx^j, \quad i, j = 1, 2, \\
    f(r) &= r^2 - \frac{2\mu}{r} + \frac{Q_m^2}{r^2}, \quad \Lambda = -3.
\end{align*}
\]  

Here \(\mu\) is the mass parameter and \(Q_m\) is the magnetic charge. The electromagnetic field strength is given by

\[
F = \sqrt{2}Q_m dx^1 \wedge dx^2.
\]  

After a straightforward but tedious calculation (the relevant detailed calculation is presented in the appendix A, B and C), one obtains the sub-leading order of the Hamiltonian constraint from the expansion as

\[
t^{\tau\tau}_{(1)} = \frac{2\sqrt{T}r_c}{-r_c \partial_r f + 2f} \delta^{MN} t^{\tau M}_{(1)} t^{\tau N}_{(1)} + \frac{2f}{-r_c \partial_r f + 2f} t^{I(1)}.
\]  

While from the Petrov-like boundary condition, the sub-leading order of the expansion reads as

\[
t^I J^{(1)} = \frac{2\sqrt{T}}{\partial_r f} \delta^{IK} t^{\tau K (1)} t^\tau J^{(1)} - \frac{2\sqrt{T}}{\partial_r f} \delta^{IK} t^{(K,J)} + \frac{f}{r_c \partial_r f} \delta^I J^{(1)} + \frac{r_c \partial_r f + 2f}{2r_c \partial_r f} \delta^I J^{(1)}.
\]
In the presence of matter fields, we note the momentum constraint becomes
\[ \partial_a v^a = T_{\mu b} n^\mu, \quad (32) \]
where \( T_{\mu b} \) is energy-momentum tensor of the matter field. Similarly, if we identify
\[ t^I = \frac{\partial_{c f}}{2\sqrt{f}} v_I, \quad (33) \]
\[ \tilde{P} = \frac{f}{r_c \partial_{c f} f - 2f} \delta^{MN} u_M v_N + \frac{\sqrt{f}}{r_c \partial_{c f} f - 2f} t^{(1)}_I, \quad (34) \]
\[ \tilde{J}_I = -\frac{2\sqrt{f}}{\partial_{c f} f} F_{n I} (1), \quad (35) \]
then from the momentum constraint we obtain the incompressible condition and the standard incompressible magnetofluid equation as
\[ \partial_I v^I = 0, \quad (36) \]
\[ \partial_\tau v_I + \delta^{JK} v_K \partial_J v_I - \nu_c \delta^{JK} \partial_J \partial_K v_I + \partial_I \tilde{P} = f_I, \quad (37) \]
Where \( \nu_c = \frac{\sqrt{f}}{\partial_{c f} f} \) is the viscosity of the dual fluid and \( f_I = \tilde{J}_J F^{j I} \) as an external force term appears on the right hand side of the equation due to the coupling of the background and the perturbations of the electromagnetic field.

As argued in previous section, the Navier-Stokes equation has the same form in coordinate systems \((\tau, x^I)\) and \((x^0, x^i)\). Thus, we have the incompressible condition and the standard incompressible magnetofluid equation in \((x^0, x^i)\) as
\[ \partial_i v^i = 0, \quad (38) \]
\[ \partial_0 v_i + v^j \partial_j v_i - \nu_c \partial^i \partial_j v_i + \partial_i \tilde{P} = f_i, \quad (39) \]
with the same viscosity \( \nu_c = \frac{\sqrt{f}}{\partial_{c f} f} \). Specially, its asymptotic behavior is
\[ r_c \rightarrow r_h, \quad \nu_c = 0; \]
\[ r_c \rightarrow \infty, \quad \nu_c = \frac{1}{2}. \]

V. SUMMARY AND DISCUSSIONS

By explicit construction we have extended the Petrov-like boundary condition to the finite cutoff surface and derived the incompressible Navier-Stokes equation in the long-wavelength
limit. In general the kinematic viscosity is cutoff dependent and such a dependence asks for further understanding from the side of holographic renormalization group flow. In special case when the cutoff surface approaches to the horizon, our results go back to the previous ones without employing a long-wavelength limit, implying a deep analogy between the near horizon limit and the long-wavelength limit.

This work, as well as previous works imposing the Petrov-like boundary condition in the near horizon limit, only involves the electromagnetic field as the most simple matter field in the bulk (see, however, [39] for the perfect fluid case as a step further). More general matter fields may lead to further problems, such as the anisotropy caused by the axion field [29], which is rather interesting. It is also challenging to extend this framework to a finite cutoff surface which may be spatially curved. When the spatial part of the hypersurface is compact, the long-wavelength limit seems not applicable. We leave these issues for further investigation in future.

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Appendix:

A. The Petrov-like boundary condition in the last model

From now on, we will present the detailed calculation of the last model with respect to the gravity/fluid duality in spacetime with matter fields. We have the embedded hypersurface \( \Sigma_c \) and its metric reads as

\[
ds^2_{p+1} = -f(r_c)dt^2 + r_c^2 \bar{\delta}_{ij} d\bar{x}^i d\bar{x}^j \\
\equiv -(dx^0)^2 + \delta_{ij} dx^i dx^j \\
= -\frac{1}{\lambda^2} d\tau^2 + \frac{1}{\lambda} \delta_{IJ} dx^I dx^J.
\]
Similarly as we fix the induced metric $h_{ab}$ on the cutoff surface, we also fix $F_{ab}\big|_{\Sigma_c}$, which could be regarded as the Dirichlet-like boundary condition. Then we have

$$F_{\tau I}\big|_{r_c} = 0.$$ 

$F_n^b$, $F_a^b$ and $F^{ab}$ could be written in terms of $F_{\mu\nu}$ on $\Sigma_c$ as

$$F_n^\tau\big|_{r_c} = F_{n,\tau} h^{\tau \tau}, \quad F_n^I\big|_{r_c} = F_{n,I} h^{I J},$$
$$F_\tau^I\big|_{r_c} = F_{\tau,I} h^{I J} = 0, \quad F^{I J}\big|_{r_c} = F_{K L} h^{K I} h^{L J}.$$ 

Then, the perturbation of electromagnetic field should take the following form

$$F_{n\tau} = 0 + \lambda F_{n\tau}^{(1)},$$
$$F_{nI} = 0 + \lambda F_{nI}^{(1)}.$$ 

Now we will give the detailed calculation from the Petrov-like boundary condition to equation (31). In terms of Brown-York stress tensor, the Petrov-like boundary condition can be written as

$$\lambda t^\tau_\tau t^I_J + \frac{2}{\lambda} h^{IK} t^K_\tau t^\tau_J - 2\lambda^2 t^I_J t^\tau_\tau - \lambda t^I_K t^K_J - 2 h^{IK} t^\tau_{(K,J)} + \lambda \delta^I_J \left[ \frac{t}{p - t^\tau_\tau} \right] + 2\lambda \partial_\tau \frac{t}{p}$$
$$+ \lambda \left( T_{\delta \beta} n^\beta n^\delta + 2\Lambda + T^2 t^\tau_\tau - 2\lambda T t^\tau_\tau - 2\lambda T \delta^I_J \delta^I_J - \lambda T^2 t^\tau_\tau \right) = 0.$$ 

The energy-momentum tensor of electromagnetic field takes the form

$$T_{\mu\nu} = \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} - F_{\mu\rho} F^{\rho \nu}.$$ 

We have

$$T_{\delta \beta} n^\beta n^\delta = T_{nn} = \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} - F_{n\rho} F^{\rho n},$$
$$T = \frac{p - 2}{4} F_{\rho\sigma} F^{\rho\sigma},$$
$$\lambda^2 T_{\tau\tau} = -\frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} - \lambda^2 F_{\tau\rho} F^{\rho \tau},$$
$$-2\lambda T_{\delta \tau} n^\delta = -2\lambda T_{n\tau} = 2\lambda F_{n\rho} F^{\rho \tau},$$
$$-T^I_J = -\frac{1}{4} \delta^I_J F_{\rho\sigma} F^{\rho\sigma} + F^{I \rho} F_{J \rho}.$$ 

\footnote{Our calculation is applicable for a general spacetime with matter fields, thus we keep $p$ as general until we get back to the last model with a magnetic black brane, in which $p$ is set to 2.}
The Petrov-like boundary condition further becomes

\[
\lambda t^{\tau}t_{\tau}^{I} + \frac{2}{\lambda} h^{IK} t^{I}_{K} t^{\tau}_{J} - 2\lambda^{2} t^{I}_{J,\tau} - \lambda t^{I}_{K} t^{K}_{J} - 2 h^{IK} t^{\tau}_{(K,J)} + \lambda \delta^{I}_{J} \left[ \frac{t^{\tau}}{\rho} (1 - t^{\tau}) + 2 \lambda \delta^{I}_{J} \right] + \frac{1}{p} \left( -\frac{1}{2} F_{\rho \sigma} F^{\rho \sigma} - F_{n \rho} F_{n}^{\rho} - \lambda^{2} F_{\tau \rho} F_{\tau}^{\rho} + 2 \lambda F_{n \rho} F_{n}^{\rho} + 2 \lambda \right) \delta^{I}_{J} + \lambda F^{I \rho} F_{J \rho} = 0.
\]

Moreover

\[
-\frac{1}{2} F_{\rho \sigma} F^{\rho \sigma} = -F_{n \tau} F_{n \tau} t^{\tau} t^{J} - F_{n I} F_{n J} t^{I J} - \frac{1}{2} F_{I J} F_{K L} h^{K I} h^{L J},
\]

\[
-\lambda^{2} F_{\tau \rho} F_{\tau}^{\rho} = -\lambda^{2} F_{n \tau} F_{n \tau},
\]

\[
2 \lambda F_{n \rho} F_{n}^{\rho} = 2 \lambda (F_{n \tau} F_{n \tau} + F_{n I} F_{n J}) = 0,
\]

\[
F^{I \rho} F_{J \rho} = F_{n J} F_{n L} h^{I L} + F_{J K} F_{L M} h^{L I} h^{M K}.
\]

So, the Petrov-like boundary condition reads as

\[
\lambda t^{\tau} t_{\tau}^{I} + \frac{2}{\lambda} h^{IK} t^{I}_{K} t^{\tau}_{J} - 2\lambda^{2} t^{I}_{J,\tau} - \lambda t^{I}_{K} t^{K}_{J} - 2 h^{IK} t^{\tau}_{(K,J)} + \lambda \delta^{I}_{J} \left[ \frac{t^{\tau}}{\rho} (1 - t^{\tau}) + 2 \lambda \delta^{I}_{J} \right] + \frac{1}{p} \left( -\frac{1}{2} F_{\rho \sigma} F^{\rho \sigma} - F_{n \rho} F_{n}^{\rho} - \lambda^{2} F_{\tau \rho} F_{\tau}^{\rho} + 2 \lambda F_{n \rho} F_{n}^{\rho} + 2 \lambda \right) \delta^{I}_{J} + \lambda F^{I \rho} F_{J \rho} = 0.
\]

After plugging \( p = 2 \) into above equation, we get the Petrov-like boundary condition

\[
\lambda t^{\tau} t_{\tau}^{I} + \frac{2}{\lambda} h^{IK} t^{I}_{K} t^{\tau}_{J} - 2\lambda^{2} t^{I}_{J,\tau} - \lambda t^{I}_{K} t^{K}_{J} - 2 h^{IK} t^{\tau}_{(K,J)} + \lambda \delta^{I}_{J} \left[ \frac{t^{\tau}}{2} (1 - t^{\tau}) + \lambda \delta^{I}_{J} \right] + \lambda F^{I \rho} F_{J \rho} = 0.
\]

Taking the perturbation expansion for Brown-York stress tensor and electromagnetic field, we find the leading order of the expansion is automatically satisfied by the background while the sub-leading order with \( \lambda^{2} \) reads as

\[
t^{I}_{J} = \frac{2\sqrt{T}}{\partial^{\tau}_{r c} f} \delta^{I K} t^{K}_{J} t^{\tau}_{J} - \frac{2\sqrt{T}}{\partial^{\tau}_{r c} f} \delta^{I K} t^{\tau}_{(K,J)} + \frac{f}{r c \partial^{\tau}_{r c} f} \delta^{I \tau}_{J} t^{\tau}_{J} + \frac{1}{r c \partial^{\tau}_{r c} f} \delta^{I \tau}_{J} t^{\tau}_{J}.
\]
B. The Hamiltonian constraint in the last model

Here we give the detailed calculation from the Hamiltonian constraint to equation (30). The Hamiltonian constraint is

\[ p + K_{ab} K_{ab} - K^2 = 2\Lambda + 2T_{\mu\nu} n^\mu n^\nu, \quad a, b = 0, \ldots, p, \quad \mu, \nu = 0, \ldots, p + 1. \]

In terms of \( t_{ab} = K h_{ab} - K_{ab} \) in coordinate system \((\tau, x^I)\), we get

\[ (t_{\tau}^\tau)^2 - \frac{2}{\lambda^2} h^{IJ} t_{\tau}^I t_{\tau}^J + t_{J}^I t_{J}^I - \frac{t^2}{p} - 2\Lambda - 2T_{\mu\nu} n^\mu n^\nu = 0. \]

Considering the last term on the left-hand side of the above equation

\[ -2T_{\mu\nu} n^\mu n^\nu = -2T_{n\tau} = F_{n\tau} F_{n\tau} h^{\tau\tau} + F_{nI} F_{nJ} h^{IJ} - \frac{1}{2} F_{IJ} F_{K\ell} h^{K\ell} h^{IJ}, \]

then the Hamiltonian constraint becomes

\[ (t_{\tau}^\tau)^2 - \frac{2}{\lambda^2} h^{IJ} t_{\tau}^I t_{\tau}^J + t_{J}^I t_{J}^I - \frac{t^2}{p} - 2\Lambda + F_{n\tau} F_{n\tau} h^{\tau\tau} + F_{nI} F_{nJ} h^{IJ} - \frac{1}{2} F_{IJ} F_{K\ell} h^{K\ell} h^{IJ} = 0. \]

Now, considering the perturbation of the electromagnetic field and meanwhile taking the perturbation expansion for Brown-York stress tensor, we find the leading order of the expansion is automatically satisfied by the background while the sub-leading order with \( \lambda^1 \) reads as

\[ t_{\tau}^{(1)} = \frac{2\sqrt{f}}{r_c} \frac{r_c}{f} \delta^{MN} t_{M}^{(1)} t_{N}^{(1)} + \frac{2f}{r_c \partial_r f + 2f} t^{(1)}. \]

C. The momentum constraint in the last model

Following discussion is about the momentum constraint

\[ \partial_a t^a_b = T_{\mu\nu} n^\mu. \]

The time component of the equation is

\[ \partial_a t^a_\tau = T_{\mu\nu} n^\mu. \]

Because

\[ \partial_a t^a_\tau = \partial_\tau t^\tau_\tau + \partial_t t^I_\tau = \lambda \partial_\tau t^\tau_\tau (1) - \frac{1}{\lambda} \partial^I t^I_\tau (1) + \ldots, \]

\[ T_{\mu\tau} n^\mu = T_{n\tau} = 0, \]
then at leading order it gives rise to
\[ \partial^\mu t^\mu_I^{(1)} = 0. \]

The space component of the equation is
\[ \partial_a t^a_I = T_{\mu I} n^\mu. \]

Similarly, since
\[ \partial_a t^a_I = \partial_\tau t^\tau_I + \partial_J t^J_I \]
\[ = \lambda \partial_\tau t^\tau_I^{(1)} + \lambda \partial_J t^J_I^{(1)}, \]
\[ T_{\mu I} n^\mu = T_{n I} \]
\[ = -(0 + \lambda F_{n J}^{(1)}) F_I^J \]
\[ = -\lambda F_{n J}^{(1)} F_I^J, \]

then at leading order we have
\[ \partial_\tau t^\tau_I^{(1)} + \partial_J t^J_I^{(1)} = -F_{n J}^{(1)} F_I^J. \]

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