Connecting monomiality questions with the structure of rational group algebras*†

Gurmeet K. Bakshi ‡
Centre for Advanced Study in Mathematics,
Panjab University, Chandigarh 160014, India
email: gkbakshi@pu.ac.in

and

Gurleen Kaur §
Department of Mathematics,
Sri Guru Gobind Singh College, Chandigarh 160019, India
email: gurleenkaur992gk@gmail.com

Abstract

In recent times, there has been a lot of active research on monomial groups in two different directions. While group theorists are interested in the study of their normal subgroups and Hall subgroups, the interest of group ring theorists lie in the structure of their rational group algebras due to varied applications. The purpose of this paper is to bind the two threads together. Revisiting Dade’s celebrated embedding theorem which states that a finite solvable group can be embedded inside some monomial group, it is proved here that the embedding is indeed done inside some generalized strongly monomial group. The so called generalized strongly monomial groups arose in a recent work of authors while understanding the algebraic structure of rational group algebras. Still unresolved monomiality questions have been correlated by proving that all the classes of monomial groups where they have been answered are generalized strongly monomial. The study also raises some intriguing questions weaker than those asked by Dornhoff and Isaacs in their investigations.

* 2010 Mathematics Subject Classification: Primary 16S34, 20C15, Secondary 20C05.
† Keywords and phrases: Monomial group, supermonomial group, generalized strongly monomial group, linear limit, Sylow tower.
‡Research supported by Science and Engineering Research Board (SERB), DST, Govt. of India under the scheme Mathematical Research Impact Centric Support (sanction order no MTR/2019/001342) is gratefully acknowledged.
§Corresponding author
1 Introduction

Throughout this paper, we assume that all groups are finite and all characters are complex characters. A group is said to be a monomial group if each of its irreducible complex character is monomial, i.e., it is induced from a linear character of some subgroup. Dade proved that every finite solvable group can be embedded inside a monomial group (see [11], Theorem 9.7). A direct consequence of Dade’s result is that an arbitrary subgroup of a monomial group is not necessary monomial. In 1967, Dornhoff [7] proved that normal Hall subgroups of monomial groups are monomial and asked that whether the Hall subgroups and normal subgroups of monomial groups are again monomial. For the past half century, the focus of group theorists has been to establish connections of monomial groups with their Hall subgroups and normal subgroups. In 2005, Fukushima [8] settled the question regarding Hall subgroups of monomial groups in negative by constructing a family of monomial groups where it fails. Regarding normal subgroups of monomial groups, in 1973, Dade [6] gave an example of a monomial group of even order having a non monomial normal subgroup. In his example, the prime 2 played a fundamental role and so the monomiality question regarding normal subgroups of odd order monomial groups was left open. Since then several researchers have been trying hard to answer this question. Parks [20] showed that every normal subgroup of a nilpotent-by-supersolvable monomial group of odd order is monomial. In [9], Gunter proved that every normal subgroup of a monomial group with a Sylow tower is monomial. In [18], Loukaki proved that every normal subgroup of a monomial group of order $p^aq^b$ is monomial, where $p$ and $q$ are odd primes. Later, generalizing the ideas contained in Parks’s and Loukaki’s work, the theory of linear limits was introduced by Dade and Loukaki [5]. Using this theory, Chang, Zheng and Jin [4] generalized the work of Gunter [9] for a class of solvable groups with technical conditions involved in it. Extending the work of Parks [20], the monomiality of normal subgroups and Hall subgroups of a class of solvable groups with some restrictions on its supersolvable residual was given by Zheng and Jin [23]. In [12], Isaacs has pointed that the problem being faced while working with monomial character is that every character inducing it need not necessary be monomial (also see Chapter 9 of [11]), and so he defined supermonomial characters. Isaacs, in [12], called an irreducible complex character $\chi$ of a group $G$ to be supermonomial if every character inducing $\chi$ is monomial and conjectured that every monomial group of odd order is a supermonomial group, i.e., each of its irreducible complex character is supermonomial. Recently Lewis, in [16], proved that if this conjecture
is true, then the question about the monomiality of normal subgroups of odd order monomial groups can be settled in affirmative.

We now draw attention to the other thread on monomial groups which is connected with the study of their rational group algebras. One of the fundamental problems in group rings is to understand the precise Wedderburn decomposition of the rational group algebra of a finite group. In this connection, Olivieri, del Río and Simón [19] defined strongly monomial groups (including abelian-by-supersolvable groups) and gave precise description of the simple components of their rational group algebras. Later, in [2], we generalized this concept and accordingly defined generalized strongly monomial groups. We had seen, in [2], that many interesting classes of monomial groups turn out to be generalized strongly monomial covering the class $C$ of finite groups, introduced by Huppert (see [10], Chapter 24), that consists of all finite groups whose each subquotient is either abelian or has a non central abelian normal subgroup. A significant result proved in [2] is the explicit description of the primitive central idempotents and the corresponding simple components of the rational group algebra of a generalized strongly monomial group from its subgroup structure. For an extensive survey on this topic, we refer to [1] and a quick recollection is done in section 2.

The objective of this paper is to provide a coherent unification of these two threads. In section 3, we have revisited Dade’s embedding theorem and proved the following in Theorem 1:

Every finite solvable group is isomorphic to a subgroup of some generalized strongly monomial group.

This motivates us to investigate the closeness of monomial groups to generalized strongly monomial groups. In sections 4-7, we examined all the groups where monomiality questions have been answered and surprisingly found that all of them are indeed generalized strongly monomial. To be precise, we have shown in series of theorems (Theorems 2-10) that all the classes of groups discussed in the work of Parks [20], Gunter [9], Loukaki [18], Lewis [16], Chang, Zheng and Jin [1] and Zheng and Jin [23] turn out to be generalized strongly monomial. Below is the list of some well known classes of groups which have been proved to be generalized strongly monomial:

- supermonomial groups (Theorem 2);
- monomial groups of order $p^aq^b$ ($p$ and $q$ odd primes) (Theorem 6);
- monomial groups of odd order which are nilpotent-by-supersolvable (Theorem 7);
• monomial groups of odd order which are nilpotent-by-nilpotent-by-Sylow abelian (Theorem 8);
• monomial groups with Sylow towers (Theorem 9, Corollary 5).

All of these results correlate generalized strongly monomial groups with the monomiality questions mentioned in the beginning. Also they accordingly arouse interest in investigating the counter example of Fukushima where monomiality is not preserved by Hall subgroups, and that given by Dade where monomiality is not inherited by normal subgroups. We have found that both of these examples are also generalized strongly monomial (Theorem 4 and Remark 1).

The results proved in this paper give an evidence to expect that under fairly general circumstances a given monomial group is generalized strongly monomial. The results obtained raise some questions weaker than those asked by leading group theorists which will be stated in section 8. Any attempt in answering to the questions raised in section 8, in addition to contribution in group theory, will also contribute to understanding of semisimple group algebras which is a problem of core interest in group rings.

2 Background on the algebraic structure of rational group algebras and generalized strongly monomial groups

Let $\mathbb{Q}G$ be the rational group algebra of a finite group $G$. By Maschke’s theorem (see [21], Theorem 3.4.7), $\mathbb{Q}G$ is a semisimple ring, which means that each left ideal of $\mathbb{Q}G$ is its direct summand. Consequently, the structure theorem due to Wedderburn and Artin (see [21], Theorem 2.6.18) implies that $\mathbb{Q}G$ is uniquely written as $\bigoplus_{1 \leq i \leq k} M_{n_i}(D_i)$, a direct sum of matrix algebras over finite dimensional division algebras $D_i$ over $\mathbb{Q}$. This is called the Wedderburn decomposition of $\mathbb{Q}G$. The summands $M_{n_i}(D_i)$ in this decomposition are called the simple components of $\mathbb{Q}G$. Furthermore, each simple component $M_{n_i}(D_i)$ is a two sided ideal of $\mathbb{Q}G$ generated by a central idempotent $e_i$ which is primitive, i.e., $e_i$ has the property that it can’t further be written as a sum of two mutually orthogonal central idempotents of $\mathbb{Q}G$; such central idempotents of $\mathbb{Q}G$ are called primitive central idempotents. The set $\{e_i \mid 1 \leq i \leq k\}$ of all the primitive central idempotents is uniquely determined by the rational group algebra $\mathbb{Q}G$ and is called the complete and irredundant set of primitive central idempotents of $\mathbb{Q}G$. In practice, it is quite a hard problem to explicitly determine the complete and irredundant set of primitive central idem-
potents and the Wedderburn decomposition of a given semisimple group algebra \( \mathbb{Q}G \). Furthermore, an understanding of this problem is a tool to deal with several problems concerning group algebras.

Given a group \( G \), a natural approach to find the precise and explicit Wedderburn decomposition of \( \mathbb{Q}G \) including the primitive central idempotents is using the character theory of \( G \). The classical method to find primitive central idempotents of \( \mathbb{Q}G \) begins with the computation of the primitive central idempotent \( e(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g)g^{-1} \) of the complex group algebra \( \mathbb{C}G \), followed by summing up all the primitive central idempotents of the form \( e(\sigma \circ \chi) \) with \( \sigma \in \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q}) \) for \( \chi \in \text{Irr}(G) \), where \( \text{Irr}(G) \) is the set of all the irreducible complex characters of \( G \), \( \mathbb{Q}(\chi) \) is the field obtained by adjoining all the values \( \chi(g) \), for \( g \in G \), to \( \mathbb{Q} \) and \( \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q}) \) is the Galois group of \( \mathbb{Q}(\chi) \) over \( \mathbb{Q} \) (see [22] for details). The primitive central idempotent so obtained, i.e., \( \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} e(\sigma \circ \chi) \) of \( \mathbb{Q}G \), is commonly denoted by \( e_\mathbb{Q}(\chi) \) and is called the primitive central idempotent of \( \mathbb{Q}G \) realized by \( \chi \). This approach has computational difficulty because it is difficult to determine \( \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q}) \) for a given \( \chi \) and moreover the complete information about the character table of \( G \) may not be known.

For a monomial character \( \chi \), Olivieri, del Río and Simón [19] gave the expression of \( e_\mathbb{Q}(\chi) \) that avoids the knowledge of \( \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q}) \). They found that if \( \chi \) is an irreducible complex character of \( G \) which is induced from a linear character \( \lambda \) of some subgroup \( H \) of \( G \) with \( K \) as its kernel, i.e., \( K = \ker \lambda \), then the expression of \( e_\mathbb{Q}(\chi) \) can be written in terms of \( G, H \) and \( K \). In [19], a pair \((H,K)\) (with \( H/K \) cyclic) of subgroups of \( G \) with the property that linear character of \( H \) with kernel \( K \) induces irreducibly to \( G \) is termed as a Shoda pair of \( G \). The term Shoda pair is in honor of Shoda who gave the criterion for the irreducibility of induced monomial characters (see [15], Corollary 3.2.3). More precisely, a Shoda pair is defined as follows:

**Definition 1.** Shoda Pair ([19], Definition 1.4) A Shoda pair \((H,K)\) of \( G \) is a pair of subgroups of \( G \) satisfying the following:

(i) \( K \triangleleft H, H/K \) is cyclic;

(ii) if \( g \in G \) and \( [H,g] \cap H \subseteq K \), then \( g \in H \).

For \( K \triangleleft H \leq G \), define:

\[
\hat{H} := \frac{1}{|H|} \sum_{h \in H} h,
\]

\[
\varepsilon(H,K) := \begin{cases} \hat{K}, & H = K; \\ \prod(\hat{K} - \hat{L}), & \text{otherwise}, \end{cases}
\]
where $L$ runs over all the minimal normal subgroups of $H$ containing $K$ properly, and
\[
\varepsilon(G, H, K) := \text{the sum of all the distinct } G\text{-conjugates of } \varepsilon(H, K).
\]

If $( H, K )$ is a Shoda pair of $G$ and $\lambda$ is a linear character of $H$ with kernel $K$, then Theorem 2.1 of [19] tells that $e_Q(\lambda^G)$ is a rational multiple (unique) of $e(G, H, K)$. It may be pointed out that this knowledge was not enough to accomplish the task of determining the structure of the corresponding simple component $\mathbb{Q}Ge_Q(\lambda^G)$. However, this task was achieved by Olivieri, del Río and Simón [19] by imposing certain constraints on the Shoda pair $(H, K)$. The Shoda pairs with these added constraints are termed as strong Shoda pairs and are defined as follow:

**Definition 2.** Strong Shoda Pair ([19], Definition 3.1, Proposition 3.3) A Shoda pair $(H, K)$ of $G$ is called a strong Shoda pair if
\begin{enumerate}[(i)]  \item $H \trianglelefteq N_G(K)$;  \item $\varepsilon(H, K)\varepsilon(H, K)^g = 0$ for all $g \in G \setminus N_G(K)$, where $\varepsilon(H, K)^g = g^{-1}\varepsilon(H, K)g$. \end{enumerate}

**Definition 3.** Strongly monomial character (see [15], p.104) An irreducible complex character $\chi$ of a group $G$ is said to be a strongly monomial character of $G$ if $\chi = \lambda^G$ for a linear character $\lambda$ of a subgroup $H$ with kernel $K$ such that $(H, K)$ is a strong Shoda pair of $G$.

**Definition 4.** Strongly monomial group ([15], p.104) A group $G$ is said to be a strongly monomial group if each of its irreducible complex character is strongly monomial.

It is proved in Theorem 4.4 of [19] that every abelian-by-supersolvable group is strongly monomial.

Recently, in [2], we have given a generalization of the concept of strong Shada pairs, and correspondingly strongly monomial groups, and called them generalized strong Shada pairs and generalized strongly monomial groups, respectively.

**Definition 5.** Generalized strong Shada pair ([2], p.422): If $(H, K)$ is a Shada pair of $G$ and $\lambda$ is a linear character of $H$ with kernel $K$, then we say that the pair $(H, K)$ is a generalized strong Shada pair of $G$ if there is a chain $H = H_0 \leq H_1 \leq \cdots \leq H_n = G$ (called strong inductive chain from $H$ to $G$) of subgroups of $G$ such that the following conditions hold for all $0 \leq i \leq n - 1$:
\begin{enumerate}[(i)]  \item $H_i \trianglelefteq \text{Cen}_{H_{i+1}}(e_Q(\lambda^{H_i}))$; \end{enumerate}
(ii) the distinct $H_{i+1}$-conjugates of $e_Q(\lambda^{H_i})$ are mutually orthogonal.

It is easy to see that every strong Shoda pair $(H, K)$ of $G$ is a generalized strong Shoda pair (with strong inductive chain $H \leq G$) because, by ([10], Lemma 1.2, Proposition 3.3), $e_Q(\lambda) = \varepsilon(H, K)$ and $Cen_G(\varepsilon(H, K)) = N_G(K)$.

**Definition 6.** Generalized strongly monomial character: An irreducible complex character $\chi$ of a group $G$ is said to be a generalized strongly monomial character of $G$ if $\chi = \lambda^G$ for a linear character $\lambda$ of some subgroup $H$ with kernel $K$ such that $(H, K)$ is a generalized strong Shoda pair of $G$.

**Definition 7.** Generalized strongly monomial group ([2], p.423): A group $G$ is said to be a generalized strongly monomial group if each of its irreducible complex character is generalized strongly monomial.

Beside strongly monomial groups, a list of important families of groups which are generalized strongly monomial is produced in Theorem 1 of [2]. The beauty of this class of generalized strongly monomial groups is due to the explicit description of the primitive central idempotents and the corresponding simple components of their rational group algebras. This work appears in [2].

3 Embedding solvable group inside generalized strongly monomial group

One of the significant property of monomial groups proved by Dade (see [11], Theorem 9.7) is that every solvable group can be embedded inside some monomial group. Given a finite solvable group $G$, consider a subnormal series \{e\} = G_0 \unlhd G_1 \unlhd \cdots \unlhd G_n = G$ with factor groups $G_i/G_{i-1}$, $1 \leq i \leq n$, of prime order. Denote $G_i/G_{i-1}$ by $C_i$. The crucial steps towards Dade’s proof are the following:

1. $G$ is isomorphic to a subgroup of $(((C_1 \wr C_2) \wr C_3) \wr \cdots \wr C_n)$, where $A \wr B$ denotes the wreath product of $A$ by $B$. This is a direct consequence of Lemma 9.6 of [11];

2. the wreath product of a monomial group by a cyclic group of prime order is monomial (see Lemma 9.5 of [11]).

In this section, we will show in Proposition [1] that the wreath product of a generalized strongly monomial group by a cyclic group of prime order is generalized strongly monomial. Consequently, prime order groups being generalized strongly monomial, it follows that $(((C_1 \wr C_2) \wr C_3) \wr \cdots) \wr C_n$ is generalized strongly monomial and hence we obtain the following:
**Theorem 1.** Every finite solvable group is isomorphic to a subgroup of some generalized strongly monomial group.

As said earlier, all we require is the following:

**Proposition 1.** If \( W = A \triangleright C \) is the wreath product of a generalized strongly monomial group \( A \) by a cyclic group \( C \) of prime order, then \( W \) is generalized strongly monomial.

To prove Proposition 1, a bit of preparation is needed and the following lemma plays a key role.

**Lemma 1.** Let \( W = G \rtimes C \) be the semidirect product of a finite group \( G \) by a cyclic group \( C \) of prime order \( p \). Let \( \psi \in \text{Irr}(G) \) be a generalized strongly monomial character which is \( C \)-invariant so that it extends to \( W \). Assume \( \psi = \chi^G \) for some linear character \( \chi \) of a subgroup \( H \) of \( G \) and that there is a strong inductive chain \( H = H_0 \leq H_1 \leq \cdots \leq H_n = G \) from \( H \) to \( G \) such that \( C \) normalizes \( H_i \) and stabilizes \( \chi^{H_i} \) for all \( 0 \leq i \leq n \). Then every extension of \( \psi \) to \( W \) is generalized strongly monomial.

**Proof.** Let \( \chi^G = \psi \) has a strong inductive chain \( H = H_0 \leq H_1 \leq \cdots \leq H_n = G \) from \( H \) to \( G \) so that \( C \) normalizes \( H_i \) and stabilizes \( \chi^{H_i} \) for all \( 0 \leq i \leq n \). Observe that for any \( i, \ 0 \leq i \leq n \), \( H_i C \) is a subgroup of \( W \) and \( \chi^{H_i} \) extends to \( H_i C \) and that also in \( p \)-ways. In particular, for \( i = 0 \), it gives that \( \chi \) extends to \( H C \). Let \( \varphi_1, \varphi_2, \ldots, \varphi_p \) be all the extensions of \( \chi \) to \( H C \). Now, for any \( j \), \( (\varphi_j^{H_i C})_{H_i} = ((\varphi_j)_H)^{H_i} = \chi^{H_i} \) and so \( \varphi_j^{H_i C} \) is an extension of \( \chi^{H_i} \). Furthermore, if \( \chi \) is any extension of \( \chi^{H_i} \) to \( H_i C \), then by Gallaghar’s theorem, \( \chi = \beta_{\varphi_1}^{H_i C} \) for some \( \beta \in \text{Irr}(H_i C/H_i) \). Since \( \beta_{\varphi_1}^{H_i C} = (\beta_{HC} \varphi_1)^{H_i C} \) and the restriction of \( \beta_{HC} \varphi_1 \) to \( H \) is \( \lambda \), \( \beta_{HC} \varphi_1 \) equals \( \varphi_j \) for some \( j \). Therefore, \( \chi = \varphi_j^{H_i C} \) and hence \( \varphi_j^{H_i C} \) for \( 1 \leq j \leq p \) are precisely all the extensions of \( \chi^{H_i} \) to \( H_i C \). For \( i = n \), it gives \( \varphi_j^W \) for \( 1 \leq j \leq p \) are precisely all the extensions of \( \chi^G \) to \( W \). Thus, to prove the lemma, we need to show that \( \varphi_j^W \), for all \( 1 \leq j \leq p \), are generalized strongly monomial.

Let \( \varphi \) be any of the \( \varphi_j \), \( 1 \leq j \leq p \). We will show that for the character \( \varphi^W \), \( HC = H_0 C \leq H_1 C \leq \cdots \leq H_n C = GC = W \) is a strong inductive chain from \( HC \) to \( W \), i.e., the following hold for all \( 0 \leq i \leq n - 1 \):

(i) \( H_i C \leq \text{Cen}_{H_{i+1} C}(e_{\varphi}(\varphi^{H_i C})) \);

(ii) distinct \( H_{i+1} C \)-conjugates of \( e_{\varphi}(\varphi^{H_i C}) \) are mutually orthogonal.

This will be proved in steps:
Step 1 \( H_i C \trianglelefteq Cen_{H_{i+1}}(e_\varphi(\lambda^{H_i}))C \).

Note that \( C \) normalizes \( Cen_{H_{i+1}}(e_\varphi(\lambda^{H_i})) \). For if \( x \in Cen_{H_{i+1}}(e_\varphi(\lambda^{H_i})) \) and \( y \in C \), then \( (\lambda^{H_i})^{-1}xy = (\lambda^{H_i})^{xy} = (\sigma \circ \lambda^{H_i})^y \) for some \( \sigma \in \text{Gal}(Q(\lambda^{H_i})/Q) \) which is further equal to \( \sigma \circ \lambda^{H_i} \), since \( C \) stabilizes \( \lambda^{H_i} \). Thus 
\[
e_\varphi(\lambda^{H_i})^{-1}xy = e_\varphi((\lambda^{H_i})^{-1}xy) = e_\varphi(\sigma \circ \lambda^{H_i}) = e_\varphi(\lambda^{H_i}),
\]
implying that \( y^{-1}xy \in Cen_{H_{i+1}}(e_\varphi(\lambda^{H_i})) \).

Since \( C \) and \( Cen_{H_{i+1}}(e_\varphi(\lambda^{H_i})) \) both normalizes \( H_i \), all we need to show is that
if \( a \in C \) and \( x \in Cen_{H_{i+1}}(e_\varphi(\lambda^{H_i})) \), then \([a, x] \in H_i C \). We will indeed show that \([a, x] \in H \). As \( C \) normalizes \( Cen_{H_{i+1}}(e_\varphi(\lambda^{H_i})) \), \([a, x] \in Cen_{H_{i+1}}(e_\varphi(\lambda^{H_i})) \). In view of \([2], \text{Theorem 2}\), \([a, x] \) will belong to \( H \) if it stabilizes \( \lambda^{H_i} \). Let us see this now.

Since \( H_i \trianglelefteq Cen_{H_{i+1}}(e_\varphi(\lambda^{H_i})) \) and \( C \) stabilizes \( \lambda^{H_i} \), the character \( (\lambda^{H_i})^{axa^{-1}x^{-1}} \) equals to \( (\lambda^{H_i})^{x^{a^{-1}x^{-1}}} \). Further, \( (\lambda^{H_i})^{x} = \sigma \circ \lambda^{H_i} \) for some \( \sigma \) belonging to \( \text{Gal}(Q(\lambda^{H_i})/Q) \), as \( x \in Cen_{H_{i+1}}(e_\varphi(\lambda^{H_i})) \). Therefore, \( (\lambda^{H_i})^{axa^{-1}x^{-1}} = (\lambda^{H_i})^{x^{a^{-1}x^{-1}}} = (\sigma \circ \lambda^{H_i})^{-1} = (\sigma \circ \lambda^{H_i}) = \lambda^{H_i} \). This proves step 1.

Step 2 If \( x \in H_{i+1} \setminus Cen_{H_{i+1}}(e_\varphi(\lambda^{H_i}))C \), then 
\( e_\varphi(\varphi^{H_i} \varphi)^x = 0 \).

Consider \( x = yz \), where \( y \in H_{i+1} \setminus Cen_{H_{i+1}}(e_\varphi(\lambda^{H_i})) \) and \( z \in C \). Then
\[
e_\varphi(\varphi^{H_i} \varphi)^y e_\varphi(\varphi^{H_i} \varphi)^{yz} = e_\varphi(\varphi^{H_i} \varphi)^{y^{z^{-1}yz}} = e_\varphi(\varphi^{H_i} \varphi)^{z^{-1}yz}.
\]
Since \( C \) normalizes both \( H_{i+1} \) and \( Cen_{H_{i+1}}(e_\varphi(\lambda^{H_i})) \), we have \( z^{-1}yz \) belongs to \( H_{i+1} \) but is not in \( Cen_{H_{i+1}}(e_\varphi(\lambda^{H_i})) \) and so
\[
e_\varphi(\lambda^{H_i}) e_\varphi(\lambda^{H_i})^{z^{-1}yz} = 0. \quad (1)
\]
Since \( \lambda^{H_i} \) extends to \( \varphi^{H_i} \), we have by \([2], \text{Lemma 1}\) that
\[
e_\varphi(\lambda^{H_i}) e_\varphi(\varphi^{H_i}) = e_\varphi(\varphi^{H_i}) e_\varphi(\lambda^{H_i}). \quad (2)
\]
From eqns \( 1 \) and \( 2 \) it turns out that \( e_\varphi(\varphi^{H_i}) e_\varphi(\varphi^{H_i})^{z^{-1}yz} = 0 \), as desired.

Step 3 \( Cen_{H_{i+1}}(e_\varphi(\varphi^{H_i})) \) = \( Cen_{H_{i+1}}(e_\varphi(\lambda^{H_i}))C \).

This follows from step 2.

Step 4 \( H_i C \trianglelefteq Cen_{H_{i+1}}(e_\varphi(\varphi^{H_i})) \).

This is an immediate consequence of steps 1 and 3.

Step 5 \( e_\varphi(\varphi^{H_i}) e_\varphi(\varphi^{H_i})^x = 0 \) for all \( x \in Cen_{H_{i+1}}(e_\varphi(\lambda^{H_i}))C \setminus Cen_{H_{i+1}}(e_\varphi(\varphi^{H_i})) \).

Let \( x \in Cen_{H_{i+1}}(e_\varphi(\lambda^{H_i}))C \setminus Cen_{H_{i+1}}(e_\varphi(\varphi^{H_i})) \). By step 1, \( H_i C \) is a normal subgroup of \( Cen_{H_{i+1}}(e_\varphi(\lambda^{H_i}))C \). Thus both \( e_\varphi(\varphi^{H_i}) \) and \( e_\varphi(\varphi^{H_i})^x \) are primitive central idempotents of \( H_i C \) which must be either same or mutually orthogonal. Since \( x \notin Cen_{H_{i+1}}(e_\varphi(\varphi^{H_i})) \), they can’t be same and hence mutually orthogonal. This proves step 5 and completes the proof of the lemma. □
Lemma 2. A finite direct product of generalized strongly monomial groups is generalized strongly monomial.

Proof. It is enough to show that if $\chi_1$ and $\chi_2$ are generalized strongly monomial characters of arbitrary groups $G_1$ and $G_2$ respectively, then $\chi_1 \times \chi_2$ is a generalized strongly monomial character of their direct product $G_1 \times G_2$. Denote $G_1 \times G_2$ by $G$ and $\chi_1 \times \chi_2$ by $\chi$. Since $\chi_1$ is a generalized strongly monomial character of $G_1$, there is a strong inductive chain $H = H_0 \leq H_1 \leq \cdots \leq H_t = G_1$ from $H$ to $G_1$ and a linear character $\lambda$ of $H$ with $\lambda^{G_1} = \chi_1$, $H_i \leq \text{Cen}_{H_{i+1}}(e_G(\lambda^{H_i}))$ and distinct $H_{i+1}$-conjugates of $e_G(\lambda^{H_i})$ are mutually orthogonal for all $0 \leq i \leq t-1$. Similarly for $\chi_2$, there is a strong inductive chain $L = L_0 \leq L_1 \leq \cdots \leq L_k = G_2$ from $L$ to $G_2$ and a linear character $\vartheta$ of $L$ with $\vartheta^{G_2} = \chi_2$, $L_i \leq \text{Cen}_{L_{i+1}}(e_G(\vartheta^{L_i}))$ and distinct $L_{i+1}$-conjugates of $e_G(\vartheta^{L_i})$ are mutually orthogonal for all $0 \leq i \leq k-1$. We can assume that $t \leq k$. We will show that for $\chi$, $H \times L = H_0 \times L_0 \leq H_1 \times L_1 \leq \cdots \leq H_t \times L_t \leq \cdots \leq H_k \times L_k = G_1 \times G_2$, where $H_i = G_1$ for all $t \leq i \leq k$ is a strong inductive chain from $H \times L$ to $G_1 \times G_2$.

Observe that $H_i \times L_i \leq \text{Cen}_{H_{i+1}}(e_G(\lambda^{H_i})) \times \text{Cen}_{L_{i+1}}(e_G(\vartheta^{L_i}))$ for all $0 \leq i \leq k-1$, as $H_i \leq \text{Cen}_{H_{i+1}}(e_G(\lambda^{H_i}))$ and $L_i \leq \text{Cen}_{L_{i+1}}(e_G(\vartheta^{L_i}))$.

Firstly, we are going to show that if $(x, y) \in H_{i+1} \times L_{i+1}$ and doesn’t belong to $\text{Cen}_{H_{i+1}}(e_G(\lambda^{H_i})) \times \text{Cen}_{L_{i+1}}(e_G(\vartheta^{L_i}))$, then $e_G(\lambda^{H_i} \times \vartheta^{L_i})$ and $e_G(\lambda^{H_i} \times \vartheta^{L_i})$ are mutually orthogonal. Let $(x, y)$ belong to $H_{i+1} \times L_{i+1}$ and doesn’t belong to $\text{Cen}_{H_{i+1}}(e_G(\lambda^{H_i})) \times \text{Cen}_{L_{i+1}}(e_G(\vartheta^{L_i}))$. Then either $x \notin \text{Cen}_{H_{i+1}}(e_G(\lambda^{H_i}))$ or $y \notin \text{Cen}_{L_{i+1}}(e_G(\vartheta^{L_i}))$. W.l.o.g. assume that $x \notin \text{Cen}_{H_{i+1}}(e_G(\lambda^{H_i}))$. Then

$$e_G(\lambda^{H_i})^x e_G(\lambda^{H_i}) = 0. \quad (3)$$

Consider the restriction of $\lambda^{H_i} \times \vartheta^{L_i}$ on $H_i \times \{e\}$ and observe that the restriction is homogeneous and $\lambda^{H_i} \times 1_{\{e\}}$ is the irreducible component, where $1_{\{e\}}$ is the principal character of the trivial subgroup $\{e\}$. We now use Lemma 1 of [2] and obtain that

$$e_G(\lambda^{H_i} \times \vartheta^{L_i}) e_G(\lambda^{H_i} \times 1_{\{e\}}) = e_G(\lambda^{H_i} \times \vartheta^{L_i}) = e_G(\lambda^{H_i} \times \vartheta^{L_i}) = e_G(\lambda^{H_i} \times 1_{\{e\}}) e_G(\lambda^{H_i} \times \vartheta^{L_i}) = e_G(\lambda^{H_i} \times \vartheta^{L_i}). \quad (4)$$

View the rational group algebra of $H_i \times L_i$, as the tensor product $\mathbb{Q} H_i \otimes_{\mathbb{Q}} \mathbb{Q} L_i$ and note that $e_G(\lambda^{H_i} \times 1_{\{e\}}) = e_G(\lambda^{H_i}) \otimes e$. Also $e_G(\lambda^{H_i} \times 1_{\{e\}})^{(x, y)} = e_G(\lambda^{H_i})^x \otimes e$. Using eqn (3) we see that

$$e_G(\lambda^{H_i} \times 1_{\{e\}})^{(x, y)} e_G(\lambda^{H_i} \times 1_{\{e\}}) = e_G(\lambda^{H_i})^x e_G(\lambda^{H_i}) \otimes e = 0 \otimes e = 0. \quad (5)$$

Now, in view of eqn (4) $e_G(\lambda^{H_i} \times \vartheta^{L_i})^{(x, y)} e_G(\lambda^{H_i} \times \vartheta^{L_i})$ is equal to

$$e_G(\lambda^{H_i} \times \vartheta^{L_i})^{(x, y)} e_G(\lambda^{H_i} \times 1_{\{e\}})^{(x, y)} e_G(\lambda^{H_i} \times 1_{\{e\}}) e_G(\lambda^{H_i} \times \vartheta^{L_i}) = e_G(\lambda^{H_i} \times \vartheta^{L_i}),$$
which is zero, using eqn 5 as desired.

As a consequence of what has been shown in the above paragraph, we have that $\text{Cent}_{H_{i+1} \times L_{i+1}}(e_Q(\lambda B_1 \times \vartheta L_i))$ is subgroup of $\text{Cent}_{H_{i+1}}(e_Q(\lambda H_i)) \times \text{Cent}_{L_{i+1}}(e_Q(\vartheta L_i))$ and hence $H_i \times L_i$ is normal in $\text{Cent}_{H_{i+1} \times L_{i+1}}(e_Q(\lambda H_i \times \vartheta L_i))$.

It now only remains to show that if $(x, y) \in \text{Cent}_{H_{i+1}}(e_Q(\lambda H_i)) \times \text{Cent}_{L_{i+1}}(e_Q(\vartheta L_i))$ but $(x, y) \notin \text{Cent}_{H_{i+1} \times L_{i+1}}(e_Q(\lambda H_i \times \vartheta L_i))$, then $e_Q(\lambda H_i \times \vartheta L_i)^{(x,y)}$ and $e_Q(\lambda H_i \times \vartheta L_i)$ are mutually orthogonal. Since $H_i \times L_i \leq \text{Cent}_{H_{i+1}}(e_Q(\lambda H_i)) \times \text{Cent}_{L_{i+1}}(e_Q(\vartheta L_i))$, $e_Q(\lambda H_i \times \vartheta L_i)$ and $e_Q(\lambda H_i \times \vartheta L_i)^{(x, y)}$ are primitive central idempotents of the rational group algebra of $H_i \times L_i$, and are also distinct, as $(x, y)$ doesn’t belong to $\text{Cent}_{H_{i+1} \times L_{i+1}}(e_Q(\lambda H_i \times \vartheta L_i))$. Therefore, they must be mutually orthogonal. This finishes the proof. \hfill \Box

**Lemma 3.** If $W = A \wr C$ is the wreath product of a generalized strongly monomial group $A$ by a cyclic group $C$ of order $n$ and if $\chi \in \text{Irr}(W)$ restricts to an irreducible character $\psi$ of the base group $A \times \cdots \times A$ ($n$ copies) of $W$ then $\psi$ is generalized strongly monomial and has a strong inductive chain satisfying the conditions stated in Lemma 2.

**Proof.** Since $A$ is a generalized strongly monomial group, by Lemma 2 we have that the base group $B$, i.e., $A \times A \times \cdots \times A$ ($n$ copies) is generalized strongly monomial. Therefore, $\psi$ is a generalized strongly monomial character of $B$. We can write $\psi$ as $\psi_1 \times \psi_2 \times \cdots \times \psi_n$, where $\psi_i \in \text{Irr}(A)$ for all $1 \leq i \leq n$. The way elements of $C$ act on $B$ and $\psi$ being $C$-invariant, it turns out that all $\psi_i$ are equal, say equal to $\psi'$. Since $\psi'$ is a generalized strongly monomial character of $A$, we have that $\psi' = \lambda A$ and $H = H_0 \leq H_1 \leq \cdots \leq H_r = A$ is a strong inductive chain from $H$ to $A$. Now $\psi' = \lambda A$ implies that $\psi = (\lambda \times \cdots \times \lambda)^B$. Furthermore, $H \times \cdots \times H \leq H_1 \times \cdots \times H_1 \leq \cdots \leq H_r \times \cdots \times H_r$ ($n$ copies) is a strong inductive chain of $\psi$ as shown in the proof of Lemma 2. From the action of $C$ on $B$, it follows that $C$ normalizes $H_i \times \cdots \times H_i$ and stabilizes $(\lambda \times \cdots \times \lambda)^{H_i \times \cdots \times H_i}$, as desired. \hfill \Box

**Proof of Proposition 1.** If $\chi \in \text{Irr}(W)$, then there is an irreducible character $\psi$ of the base group $B = A \times \cdots \times A$ ($p$-copies) of $W$ such that either $\chi_B = \psi$ or $\chi = \psi^G$.

If $\chi_B = \psi$, then by Lemma 3 $\psi$ is generalized strongly monomial and has a strong inductive chain satisfying conditions stated in Lemma 1. Consequently Lemma 1 yields that $\chi$ is generalized strongly monomial.

Suppose $\chi = \psi^G$. In this case, it is easy to see that for the character $\psi$ if $H = H_0 \leq H_1 \leq \cdots \leq H_n = B$ is a strong inductive chain from $H$ to $B$, then $\chi$...
has a strong inductive chain $H = H_0 \leq H_1 \leq \cdots \leq H_n = B \leq W$ from $H$ to $W$ and hence the result is proved. 

4 Supermonomial groups

Let $\chi \in \text{Irr}(G)$. Let $N \trianglelefteq G$ and let $\psi \in \text{Irr}(N|\chi_N)$, i.e., $\psi$ is an irreducible constituent of $\chi_N$. Let $T = I_G(\psi)$, i.e., the inertia group of $\psi$ in $G$. By Clifford’s correspondence theorem ([14], Theorem 6.11), there is a unique Clifford correspondent $\theta \in \text{Irr}(T)$ of $\chi$ w.r.t. $\psi$ such that $\chi = \theta^G$ and $\theta_N = e\psi$, where $e = \langle \theta_N, \psi \rangle$. Now repeat this process with $\chi$ replaced by $\theta$ and a normal subgroup $N$ of $G$ replaced by a normal subgroup $S$ of $T$ and consider the Clifford correspondent of $\theta$ w.r.t. an irreducible constituent of $\theta_S$. This iterative process can be continued. A character $\theta$ obtained through any number of such iterations is called a compound Clifford correspondent of $\chi$. Those compound Clifford correspondents of $\chi$ which themselves have no proper Clifford correspondents (in other words which are quasi-primitive) are called the stabilizer limits of $\chi$. This terminology was introduced by Isaacs in [13].

**Proposition 2.** Let $G$ be a solvable group, $\chi \in \text{Irr}(G)$, and $\theta$ a compound Clifford correspondent of $\chi$. If $\theta$ is generalized strongly monomial, then so is $\chi$.

The above proposition is a key for the following:

**Theorem 2.** Supermonomial groups are generalized strongly monomial.

**Theorem 3.** If $\chi \in \text{Irr}(G)$ is a supermonomial character, then it is generalized strongly monomial.

Theorem 2 is a direct consequence of Theorem 3 which we will prove in this section. We begin by proving Proposition 2 which is also crucial for the results in the forthcoming sections.

**Proof of Proposition 2.** Observe that it is enough to prove the result when $\theta$ is a Clifford correspondent of $\chi$. Suppose $N \trianglelefteq G$, $\psi \in \text{Irr}(N|\chi_N)$, and $\theta \in \text{Irr}(T)$, where $T = I_G(\psi)$, is the Clifford correspondent of $\chi$ w.r.t $\psi$. We will show that if $\theta$ is generalized strongly monomial, then so is $\chi$. For this purpose, it is sufficient to prove the following:

(i) $T \trianglelefteq \text{Cen}_G(e_Q(\theta))$;

(ii) the distinct $G$-conjugates of $e_Q(\theta)$ are mutually orthogonal.
For if \( \theta \) is generalized strongly monomial, then \( \theta = \lambda^T \) for some linear character \( \lambda \) of a subgroup \( H \) of \( T \) and a strong inductive chain \( H = H_0 \leq H_1 \leq \cdots \leq H_m = T \) from \( H \) to \( T \), i.e., \( H_i \leq \text{Cen}_{H_{i+1}}(e_Q(\lambda^{H_i})) \) and the distinct \( H_{i+1} \)-conjugates of \( e_Q(\lambda^{H_i}) \) are mutually orthogonal, for all \( 0 \leq i \leq m-1 \). Now, if (i) and (ii) as stated above hold then we obtain that \( H = H_0 \leq H_1 \leq \cdots \leq H_m = T \leq G \) is a strong inductive chain from \( H \) to \( G \), showing that \( \lambda^G = \chi \) is generalized strongly monomial.

We now move to prove (i) and (ii). In the first step, we show that \( T \trianglelefteq \text{Cen}_G(e_Q(\psi)) \). Let \( x \in \text{Cen}_G(e_Q(\psi)) \). Then \( e_Q(\psi)^{x^{-1}} = e_Q(\psi) \) which gives that \( \psi^{x^{-1}} = \sigma \circ \psi \), where \( \sigma \in \text{Gal}(\mathbb{Q}(\psi)/\mathbb{Q}) \). Let \( t \in T \). Now \( x^{-1}tx \in T \) if, and only if, \( \psi^{tx^{-1}t^{-1}}(n) = \psi(n) \) for all \( n \in N \). Note that \( \psi(x^{-1}tx^{-1}t^{-1}x) = \psi^{tx^{-1}}(txnx^{-1}t^{-1}) = \sigma \circ \psi(txnx^{-1}t^{-1}) = \frac{1}{e}(\sigma \circ \theta(txnx^{-1})) = \sigma \circ \psi(txnx^{-1}) = \psi(n) \), where \( \theta_N = e\psi \) with \( e = \langle \theta_N, \psi \rangle \). Therefore, it follows that \( x^{-1}tx \in T \). This proves that \( T \trianglelefteq \text{Cen}_G(e_Q(\psi)) \).

Our next step is to prove that \( \text{Cen}_G(e_Q(\theta)) \) is a subgroup of \( \text{Cen}_G(e_Q(\psi)) \), which will prove (i) in view of the above step. This, however, follows from the following stronger statement which we will prove:

\[
e_Q(\theta)e_Q(\theta)^g = 0 \quad \forall \ g \in G \setminus \text{Cen}_G(e_Q(\psi)). \tag{6}\]

Due to Lemma 1 of [2], we have

\[
e_Q(\theta)e_Q(\psi) = e_Q(\theta) = e_Q(\psi)e_Q(\theta).
\]

Consequently, for any \( g \in G \),

\[
e_Q(\theta)e_Q(\theta)^g = e_Q(\theta)e_Q(\psi)e_Q(\psi)^g e_Q(\theta)^g. \tag{7}\]

As \( N \trianglelefteq G \), \( e_Q(\psi) \) and \( e_Q(\psi)^g \) are both primitive central idempotents of \( \mathbb{Q}N \). Since \( g \not\in \text{Cen}_G(e_Q(\psi)) \), \( e_Q(\psi) \) and \( e_Q(\psi)^g \) can’t be same and hence they are mutually orthogonal. Therefore, it follows from eqn (7) that \( e_Q(\theta) \) and \( e_Q(\theta)^g \) are mutually orthogonal. Hence eqn (7) is proved.

Note that eqn (7) partially proves (ii) also. To finish the proof of (ii), it only remains to show that \( e_Q(\theta)e_Q(\theta)^g = 0 \) if \( g \in \text{Cen}_G(e_Q(\psi)) \setminus \text{Cen}_G(e_Q(\theta)) \). Consider \( g \in \text{Cen}_G(e_Q(\psi)) \). In view of the first step, i.e., \( T \trianglelefteq \text{Cen}_G(e_Q(\psi)) \), we have that both \( e_Q(\theta) \) and \( e_Q(\theta)^g \) are primitive central idempotents of \( QT \). Hence either they are same or mutually orthogonal. But if \( g \not\in \text{Cen}_G(e_Q(\theta)) \), they can’t be same and therefore are mutually orthogonal. This finishes the proof of (ii) and hence completes the proof of the proposition. \( \square \)

**Proof of Theorem 3.** Let \( \chi \in \text{Irr}(G) \) be a supermonomial character. Consider
Due to Theorem 2, many interesting classes of groups turn out to be generalized strongly monomial:

**Corollary 1.** The following solvable groups being supermonomial are generalized strongly monomial:

(i) groups whose all primitive characters are linear and every proper subgroup is monomial, in particular, subgroup closed monomial groups;

(ii) supersolvable-by-Sylow abelian groups and Sylow abelian-by-supersolvable groups, where Sylow abelian denotes the class of solvable groups with all its Sylow subgroups abelian;

(iii) monomial groups of odd order whose irreducible characters are of prime power degree;

(iv) groups whose order is a product of distinct odd primes;

(v) groups of odd order with every irreducible character being a \( \{p\} \)-lift for possibly different primes \( p \).

**Proof.** (i) Lewis, in Lemma 2.3 of [16], proved that such groups are supermonomial.

(ii) Since supersolvable-by-Sylow abelian groups and Sylow abelian-by-supersolvable groups are subgroup closed monomial, we are done, using (i).

(iii) From ([11], Theorem 9.14), such groups are supermonomial.

(iv) Let \( \chi \) be an arbitrary irreducible character of a subgroup, say \( H \), of \( G \). Consider a stabilizer limit, say \( \theta \in \text{Irr}(S) \) where \( S \leq H \), of \( \chi \). Then \( \chi = \theta^H \) with \( \theta \) being a primitive character. Due to Theorem 2.18 of [11], \( \theta(1)^2 \) divides \( |S : Z(S)| \), where \( Z(S) \) denotes the center of \( S \). However, the order of \( G \) is a product of distinct primes. Therefore \( \theta \) must be linear. This proves that every irreducible character of \( H \) is monomial. Since \( H \) is an arbitrary subgroup of \( G \), we have that \( G \) is a subgroup closed monomial group. This proves (iv), in view of (i).
Theorem 4. Dade’s example of a monomial group where monomiality is not inherited by normal subgroups is generalized strongly monomial.

Proof. The example given by Dade in [6] is that of a group $G$ (of even order) which has a central subgroup $Z$ such that $G/Z$ is abelian-by-supersolvable. So any irreducible character of $G$ with $Z$ in its kernel is supermonomial and hence generalized strongly monomial. If an irreducible character $\chi$ of $G$ does not contain $Z$ in its kernel, then it’s shown while proving the monomiality of such character in [6] that $\chi$ is induced from a supermonomial character, of a normal subgroup of $G$ (also see [10], Example 24.11). It can be readily verified that if $\varphi = \vartheta^G$, where $\vartheta$ is an irreducible character of a normal subgroup, say $H$, of $G$ and if $\vartheta$ is generalized strongly monomial, i.e., $\vartheta = \lambda^H$, where $\lambda$ is a linear character of $T$, having a strong inductive chain $T = H_0 \leq H_1 \leq \cdots \leq H_n = H$ from $T$ to $H$, then $\varphi$ is also generalized strongly monomial having a strong inductive chain $T = H_0 \leq H_1 \leq \cdots \leq H_n = H \leq G$ from $T$ to $G$. Consequently, $\chi$ is also generalized strongly monomial. This proves that any arbitrary irreducible character of $G$ is generalized strongly monomial.

In [13], Isaacs proved that if $\chi$ is an irreducible complex character of an arbitrary finite solvable group $G$ such that its restriction to the fitting subgroup of $G$ is irreducible, then all the primitive characters that induce $\chi$ must have same degree. So if, in addition, $\chi$ is monomial then it turns out to be supermonomial and thus generalized strongly monomial as well. This gives the following:

Corollary 2. If $\chi$ is a monomial character of a solvable group $G$ such that its restriction to the fitting subgroup of $G$ is irreducible, then it is generalized strongly monomial.

Finally, we prove a slight generalization of Theorem 2.

Theorem 5. Any central extension of a cyclic group by a supermonomial group of coprime order is generalized strongly monomial.
Proof. Let $G$ be a group with a cyclic central normal subgroup $N$ such that $G/N$ is supermonomial and $\gcd(|N|, |G/N|) = 1$. Consider $\chi \in \text{Irr}(G)$. In view of Proposition 2, it is enough to show that there is a compound Clifford correspondent of $\chi$ which is linear. Let $\psi$ be an irreducible constituent of $\chi_N$. As $N$ is central in $G$, $\psi$ is $G$-invariant. Since $\gcd(|N|, |G/N|) = 1$, we have that $\psi$ extends to $G$ (see Corollary 8.16 of [14]). Let $\delta$ be one such extension. By Gallagher’s theorem ([14], Corollary 6.17), $\chi = \delta\theta$, where $\theta \in \text{Irr}(G)$ has $N$ in its kernel. Let $\overline{\theta}$ be $\theta$ modulo $N$. Since $G/N$ is supermonomial, every stabilizer limit of $\overline{\theta}$ being quasi-primitive is primitive and hence linear. Consequently, there is a linear stabilizer limit of $\theta$. We now show that if $\theta \in \text{Irr}(G)$ has a linear stabilizer limit, then so does $\delta\theta$.

Let $M \trianglelefteq G$, $\zeta \in \text{Irr}(M|\theta_M)$, and $\phi \in \text{Irr}(I_G(\zeta))$ be the Clifford correspondent of $\theta$ w.r.t $\zeta$. Let $\zeta^* = \delta_M \zeta$. Clearly $\zeta^*$ is an irreducible constituent of $(\delta\theta)_M$. As $\delta$ is $G$-invariant, $I_G(\zeta^*) = I_G(\zeta)$. Set $\phi^* = \delta_{I_G(\zeta)} \phi$. One can check that $\phi^* \in \text{Irr}(I_G(\zeta^*))$ is the Clifford correspondent of $\delta\theta$ w.r.t $\zeta^*$. By the repeated application of this step, it follows that if $H$ is a subgroup of $G$ and $\lambda \in \text{Irr}(H)$ is a linear stabilizer limit of $\theta$, then $\lambda^* = \delta_H \lambda$ is a compound Clifford correspondent of $\delta\theta$. Since $\lambda^*$ is linear, it has to be a stabilizer limit of $\delta\theta$ and therefore the result follows. \hfill \Box

5 Groups of order $p^aq^b$

We recall some of the terminologies given by Dade and Loukaki in [5]. Let $G$ be a finite group. Consider a triple $\tau = (G, N, \psi)$ with $N \trianglelefteq G$ and $\psi \in \text{Irr}(N)$. The center $Z(\tau)$ of $\tau$ is defined to be the center $Z(\psi^G)$ of the induced character $\psi^G$.

A triple $\tau_1 = (G_1, N_1, \psi_1)$ is called a linear reduction of the triple $\tau = (G, N, \psi)$ whenever there exists $L \trianglelefteq G$ contained in $N$, and a linear character $\lambda \in \text{Irr}(L)$ lying below $\psi$, such that $G_1 = I_G(\lambda)$, $N_1 = I_N(\lambda)$ and $\psi_1$ is the unique Clifford correspondent of $\psi$ w.r.t $\lambda$. Furthermore, a triple $\tau'$ is said to be a multilinear reduction of $\tau$ if there is a series of triples $\tau = \tau_0, \tau_1, \ldots, \tau_n = \tau'$ such that each $\tau_i$ is a linear reduction of $\tau_{i-1}$ for $1 \leq i \leq n$, and $\tau'$ is called a linear limit if it is a multilinear reduction of $\tau$ in a way that the only possible linear reduction of $\tau'$ is $\tau'$ itself.

Also, we recollect some notions introduced in [4] which are related to the work in [5 17 18]. For a triple $\tau = (G, N, \psi)$ with $N \trianglelefteq G$ and $\psi \in \text{Irr}(N)$, the section of $\tau$, denoted $\text{Sec}\tau$, is defined to be the quotient group $N/Z(\psi^G)$. It is to be noted that an arbitrary triple may have different linear limits but their sections are isomorphic to each other, in view of the main theorem of [5]. A triple $\tau$ is said to have a trivial linear limit $\tau'$ if the section $\text{Sec}\tau'$ is trivial and is said to have a
nilpotent linear limit if its section is nilpotent.

In the next theorem, we will show that the class of groups studied by Loukaki in [18] turns out to be generalized strongly monomial.

**Theorem 6.** Monomial groups of order \( p^aq^b \), where \( p \) and \( q \) are odd primes, are generalized strongly monomial. Furthermore, their normal subgroups are also generalized strongly monomial.

**Lemma 4.** Let \( G \) be a finite group and let \( \chi \in \text{Irr}(G) \). If the triple \((G,G,\chi)\) has a nilpotent linear limit, then \( \chi \) is generalized strongly monomial.

**Proof.** Suppose \( \chi \in \text{Irr}(G) \) is such that \( \tau = (G,G,\chi) \) has a nilpotent linear limit, say \( \tau' = (G',G',\chi') \). Since \( \chi' \) is a compound Clifford correspondent of \( \chi \), in view of Proposition 2 it is enough to show that \( \chi' \) is generalized strongly monomial. Clearly, \( \mathcal{Z}(\chi')/\ker \chi' \) is a central subgroup of \( G'/\ker \chi' \) (see Lemma 2.27 of [14]). The triple \( \tau' \) being a nilpotent linear limit of \( \tau \) yields that the factor group \( G'/\mathcal{Z}(\chi') \) is nilpotent. Consequently, \( G'/\ker \chi' \) is nilpotent. Now the generalized strong monomiality of \( \chi' \) follows by going modulo \( \ker \chi' \) and using the fact that nilpotent groups being subgroup closed monomial are also generalized strongly monomial.

**Proof of Theorem 6.** In [18], it is proved that if \( G \) is a monomial group of order \( p^aq^b \), where \( p \) and \( q \) are odd primes, then for every normal subgroup \( N \) of \( G \) and \( \psi \in \text{Irr}(N) \), the triple \((G,N,\psi)\) has nilpotent linear limit. In particular, \((G,G,\chi)\) has a nilpotent linear limit for all \( \chi \in \text{Irr}(G) \). Hence, by Lemma 4, \( G \) is generalized strongly monomial. Finally all normal subgroups, being monomial by [18], are also generalized strongly monomial.

6 Groups with solvability length atmost 3 and fitting length atmost 2

In this section, Parks [20] notion of successful search of the triple \((G,N,\psi)\) where \( N \) is a nilpotent normal subgroup of \( G \) and \( \psi \in \text{Irr}(N) \) plays a key role. This notion is related to that of trivial linear limits. In [20], Parks proved that if \( G \) is a finite monomial group and \( N \trianglelefteq G \) with \( N \) nilpotent and \( G/N \) supersolvable of odd order then for every \( \psi \in \text{Irr}(N) \) there is a successful \((G,N,\psi)\) search. Observe that the existence of a successful search of \((G,N,\psi)\), in Theorem 7.4 of [20], implies that it has a trivial linear limit. Following the notation of Chang, Zheng and Jin [4], we write \( N \propto G \) to denote those normal subgroups \( N \) of \( G \) for which the triple
(G, N, ψ) has a trivial linear limit for each ψ ∈ Irr(N). Hence Parks’s result [20] can be rephrased as follows: if N is a nilpotent normal subgroup of a monomial group G such that the factor group G/N is supersolvable of odd order, then N ∼ G.

Consider a monomial group G having a normal subgroup N of G with N nilpotent such that the factor group G/N is supersolvable of odd order. In this section we will show that such groups are generalized strongly monomial.

**Proposition 3.** Assume that G is a finite monomial group and N is a normal subgroup of G with G/N supersolvable of odd order. If N ∼ G, then G is a generalized strongly monomial group.

**Proof.** Let χ ∈ Irr(G) and ψ ∈ Irr(N|χN). Let (G′, N′, ψ′) be a trivial linear limit of (G, N, ψ). The construction of linear limits tells that there is a compound Clifford correspondent χ′ ∈ Irr(G′) which lies above ψ′ and moreover ψ′ is a homogeneous restriction of χ′. By Proposition 2 to show that χ is generalized strongly monomial, it is enough to show that χ′ is so. As χ′|N′ is integral multiple of ψ′, ker χ′ ∩ N′ = ker ψ′ and so ker ψ′ ⊆ ker χ′. By going modulo ker ψ′, it is enough to see that χ′ is a generalized strongly monomial character of G′/ker ψ′. The triple (G′, N′, ψ′) being a trivial linear limit of (G, N, ψ) yields that its section N′/Z(ψ′G′) is trivial, i.e., N′ = Z(ψ′G′). Hence ψ′ is linear and N′/ker ψ′ is abelian, using Lemma 2.27 of [14]. Thus, G′/ker ψ′ is abelian-by-supersolvable, due to Lemma 2.2 of [4]. Since abelian-by-supersolvable groups are generalized strongly monomial, we have that χ′ is generalized strongly monomial. Consequently the same is true for χ and subsequently for χ. This finishes the proof of the theorem. □

As an immediate consequence, we obtain the following:

**Theorem 7.** Monomial groups of odd order which are nilpotent-by-supersolvable are generalized strongly monomial. In particular, monomial groups of odd order with fitting length atmost 2 are generalized strongly monomial.

Next, we will see that monomial groups of odd order which are nilpotent-by-nilpotent-by-Sylow abelian are also generalized strongly monomial.

**Theorem 8.** Monomial groups of odd order which are nilpotent-by-nilpotent-by-Sylow abelian are generalized strongly monomial. In particular, monomial groups with solvability length atmost 3 are generalized strongly monomial.

**Proof.** Let G be a monomial group of odd order with a nilpotent-by-nilpotent normal subgroup N such that the factor group G/N is Sylow abelian. Let χ ∈ Irr(G) and let ψ be an irreducible constituent of χN. Due to Theorem C of [17], (G, N, ψ)
has a nilpotent linear limit, say \((G', N', \psi')\). This gives a compound Clifford correspondent \(\chi' \in \text{Irr}(G')\) of \(\chi\) which lies above the \(G'\)-invariant character \(\psi' \in \text{Irr}(N')\). In view of Proposition 2, it is enough to show that \(\chi'\) is generalized strongly monomial. Observe that \(G'/N'\) is Sylow abelian. Next \(N'/\ker \psi'\) is nilpotent as \(N'/Z(\psi^{G'})\) is nilpotent and \(\psi'\) is \(G'\)-invariant. Therefore, \(G'/\ker \psi'\) is nilpotent-by-Sylow abelian and hence generalized strongly monomial (see Corollary 1). Since \(\ker \psi' \leq \ker \chi'\), by going modulo \(\ker \psi'\), it turns out that \(\overline{\chi'}\) and hence \(\chi'\) is generalized strongly monomial. This completes the proof.

7 Towers of groups with pairwise coprime order

We recall some notions introduced by Chang, Zheng and Jin [4]. A solvable group \(N\) is called an \(L\)-group, if whenever \(G\) is a monomial group and \(N \leq G\) then the triple \((G, N, \psi)\) has a nilpotent linear limit for every \(\psi \in \text{Irr}(N)\). Furthermore, \(N\) is said to be an \(\tilde{L}\)-group if each subquotient of \(N\) is an \(L\)-group.

Theorem 9. Let \(G\) be a monomial group with a series of normal subgroups

\[ 1 = G_0 \leq G_1 \leq \cdots \leq G_{n-1} \leq G_n = G, \]

such that

(a) \(|G_i/G_{i-1}|\) are pairwise coprime for \(1 \leq i \leq n\);
(b) \(G_i/G_{i-1}\) is an \(\tilde{L}\)-group for each \(1 \leq i \leq n - 1\);
(c) all the subquotients of \(G/G_{n-1}\) are also monomial.

Then \(G\) is generalized strongly monomial and moreover all its normal subgroups are also generalized strongly monomial.

Proposition 4. Assume that \(G\) is a monomial group and \(N \leq G\) with \(G/N\) a subgroup closed monomial group. If \(N \cong G\), and \(\gcd(|N|, |G/N|) = 1\), then \(G\) is generalized strongly monomial.

Proof. Consider any \(\chi \in \text{Irr}(G)\). We have to show that \(\chi\) is a generalized strongly monomial character.

If the restriction of \(\chi\) to \(N\) is the principal character, then \(N\) is a subgroup of \(\ker \chi\). So by going modulo \(N\), and using the fact that subgroup closed monomial groups are generalized strongly monomial (by Corollary 1), it follows that \(\overline{\chi}\) and hence \(\chi\) is generalized strongly monomial.
Assume that the restriction of $\chi$ to $N$ is not the principal character. Let $\psi$ be an irreducible constituent of $\chi_N$. Since $N \propto G$, the triple $(G, N, \psi)$ has a trivial linear limit, say $(G', N', \psi')$. Suppose $\chi' \in \text{Irr}(G')$ which lies above $\psi'$ is a compound Clifford correspondent of $\chi$. To prove that $\chi$ is generalized strongly monomial, it is enough to show that $\chi'$ is generalized strongly monomial. Observe that $\ker \psi' \leq \ker \chi'$. As $N' = Z(\psi')$ and $\psi'$ is $G'$-invariant, we see that $G'/\ker \psi'$ has a cyclic central subgroup $N'/\ker \psi'$, using Lemma 2.27 of [14]. Also, its quotient is subgroup closed monomial. Furthermore, the order of $N'/\ker \psi'$ is coprime to its index in $G'/\ker \psi'$. Hence Theorem 5 is applicable and we get that $\chi'$ modulo $\ker \psi'$ is generalized strongly monomial and consequently so is $\chi'$. This finishes the proof.

**Corollary 3.** Any extension of an abelian group by a subgroup closed monomial group of coprime order is generalized strongly monomial.

**Proof of Theorem 9.** By Theorem A of [4], $G_{n-1} \propto G$ and so Proposition 4 gives that $G$ is generalized strongly monomial. Furthermore, by Theorem B of [4], every normal subgroup of $G$ is monomial and thus the hypothesis of the theorem is preserved by its normal subgroups and hence they are generalized strongly monomial as well.

**Corollary 4.** Let $G$ be a monomial group with a series of normal subgroups

$$\{e\} = G_0 \leq G_1 \leq \cdots \leq G_{n-1} \leq G_n = G,$$

such that

(a) $|G_i/G_{i-1}|$ are pairwise coprime for $1 \leq i \leq n$;

(b) $G_i/G_{i-1}$ is either nilpotent(odd order)-by-nilpotent or Sylow abelian-by-nilpotent for $1 \leq i \leq n-1$;

(c) all the subquotients of $G/G_{n-1}$ are also monomial.

Then all normal subgroups of $G$ are generalized strongly monomial.

**Proof.** Lemma 3.4 of [1] yields that nilpotent(odd order)-by-nilpotent or Sylow abelian-by-nilpotent are $\tilde{L}$ groups. Hence the result immediately follows from the above theorem.

Another immediate consequence of Theorem 9 is the following:
Corollary 5. If $G$ is a monomial group with a Sylow tower, then all its subnormal subgroups are generalized strongly monomial.

Proof. Clearly a monomial group $G$ with a Sylow tower is of the type discussed in Theorem [9] and hence it is generalized strongly monomial. Also, by the work of Gunter [9], every subnormal subgroup of $G$ is monomial and thus being with a Sylow tower it is also generalized strongly monomial. \[\square\]

Remark 1. In [8], Fukushima gave an example of a monomial group which has a Hall subgroup that is not monomial. Since the example of Fukushima is that of a monomial group with a Sylow tower, it turns out to be generalized strongly monomial as well.

Theorem 10. Let $G$ be a monomial group with a series of normal subgroups $1 \leq L \leq K \leq G$ such that:

(a) the factor group $G/K$ is supersolvable, $K/L$ is nilpotent and $L$ has a normal series $1 = L_0 \leq L_1 \leq \cdots \leq L_{n-1} \leq L = L_n$ where all factors $L_i/L_{i-1}$ are abelian with pairwise coprime orders, for $i = 1, \cdots, n$;

(b) either $|G/K|$ or $|K/L|$ is odd;

(c) $|K/L|$ and $|L|$ are coprime.

Then all normal subgroups of $G$ are generalized strongly monomial. Furthermore, if $L$ is abelian then all Hall subgroups of $G$ are also generalized strongly monomial.

Proof. Consider a group $G$ which have a series of normal subgroups as mentioned in the statement and $\chi \in \text{Irr}(G)$. If $K \leq \text{ker} \chi$, then by going modulo $K$ and using the fact that supersolvable groups are strongly monomial (and hence generalized strongly monomial) it follows that $\chi$ is so. If $K \nsubseteq \text{ker} \chi$, consider the triple $(G, K, \psi)$, where $\psi$ is an irreducible constituent of $\chi_K$. By Theorem A of [23], $(G, K, \psi)$ has a linear limit, say $(G', N', \psi')$, where $\psi'$ is linear. There now exists $\chi' \in \text{Irr}(G'|\psi')$ which induced to $G$ is $\chi$ and is its compound Clifford correspondent. The generalized strong monomiality of $\chi'$ is seen by going modulo $\text{ker} \psi'$ and using the fact that cyclic-by-supersolvable are again supersolvable and hence generalized strongly monomial. Finally, the generalized strong monomiality of $\chi$ follows from that of $\chi'$.

Next, by Theorem B of [23], all normal subgroups of $G$ are monomial and consequently being of the same type are also generalized strongly monomial.

Finally, if $L$ is abelian then, by Theorem C of [23], all Hall subgroups of $G$ are monomial and thus the result follows. \[\square\]
8 Questions

Dornhoff’s question regarding whether arbitrary normal subgroups of monomial groups are monomial was answered in negative by Dade [6] by constructing an example of a monomial group having a non monomial normal subgroup of even order and having even index in the group. The prime 2 played a key role in Dade’s example and thus the question regarding monomiality of normal subgroups of odd order monomial groups was left open. This question has been studied extensively in literature. For various classes of monomial groups, an affirmative answer to this question has been provided. In the earlier sections, we have seen that all such monomial groups where monomiality is inherited by normal subgroups turn out to be generalized strongly monomial. This leads one to ask the following:

Question 1: Is a normal subgroup of a generalized strongly monomial group of odd order itself monomial?

As mentioned earlier, Isaacs conjectured that every monomial group of odd order is a supermonomial group. In light of Theorem 2, where we have shown that supermonomial groups are generalized strongly monomial, a weak form of Isaacs’s conjecture is raised:

Question 2: Whether every generalized strongly monomial group of odd order is supermonomial?

Another question which is of interest is the following:

Question 3: Is an arbitrary monomial group of odd order generalized strongly monomial?

An affirmative answer to both Questions 2 and 3 is equivalent to proving Isaacs conjecture and it also answers the open question regarding monomiality of normal subgroups of odd order monomial groups in affirmative, in view of Lewis’s work in [16].

References

[1] Gurmeet K. Bakshi, Can we explicitly determine the structure of rational group algebras?, The Mathematics Student 89 (2020), no. 1-2, 01–27.

[2] Gurmeet K. Bakshi and Gurleen Kaur, A generalization of strongly monomial groups, J. Algebra 520 (2019), 419–439. MR 3883245
[3] T. R. Berger, *Primitive solvable groups*, J. Algebra **33** (1975), 9–21. MR 360806

[4] Xuewu Chang, Huijuan Zheng, and Ping Jin, *On M-groups with Sylow towers*, Arch. Math. (Basel) **105** (2015), no. 6, 519–528. MR 3422856

[5] Everett Dade and Maria Loukaki, *Linear limits of irreducible characters*, arXiv preprint math/0412385 (2004).

[6] Everett C. Dade, *Normal subgroups of M-groups need not be M-groups*, Math. Z. **133** (1973), 313–317. MR 325748

[7] Larry Dornhoff, *M-groups and 2-groups*, Math. Z. **100** (1967), 226–256. MR 217174

[8] Hiroshi Fukushima, *Hall subgroups of M-groups need not be M-groups*, Proc. Amer. Math. Soc. **133** (2005), no. 3, 671–675. MR 2113913

[9] Elsa L. Gunter, *M-groups with Sylow towers*, Proc. Amer. Math. Soc. **105** (1989), no. 3, 555–563. MR 955459

[10] Bertram Huppert, *Character theory of finite groups*, De Gruyter Expositions in Mathematics, vol. 25, Walter de Gruyter & Co., Berlin, 1998. MR 164 5304

[11] I. M. Isaacs, *Characters of solvable groups*, The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979), Proc. Sympos. Pure Math., vol. 37, Amer. Math. Soc., Providence, R.I., 1980, pp. 377–384. MR 604607

[12] ______, *Characters and sets of primes for solvable groups*, Finite and locally finite groups (Istanbul, 1994), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 471, Kluwer Acad. Publ., Dordrecht, 1995, pp. 347–376. MR 1362816

[13] I. Martin Isaacs, *Character stabilizer limits relative to a normal nilpotent subgroup*, J. Algebra **102** (1986), no. 2, 367–375. MR 853249

[14] ______, *Character theory of finite groups*, AMS Chelsea Publishing, Providence, RI, 2006, Corrected reprint of the 1976 original [Academic Press, New York; MR0460423]. MR 2270898

[15] Eric Jespers and Ángel del Río, *Group ring groups. Vol. 1. Orders and generic constructions of units*, De Gruyter Graduate, De Gruyter, Berlin, 2016. MR 3618092
[16] Mark L. Lewis, *Groups where all the irreducible characters are super-monomial*, Proc. Amer. Math. Soc. 138 (2010), no. 1, 9–16. MR 2550165

[17] Maria Loukaki, *Extendible characters and monomial groups of odd order*, J. Algebra 299 (2006), no. 2, 778–819. MR 2228340

[18] Maria I. Loukaki, *Normal subgroups of odd order monomial P(A)Q(B)-groups*, ProQuest LLC, Ann Arbor, MI, 2001, Thesis (Ph.D.)–University of Illinois at Urbana-Champaign. MR 2702427

[19] Aurora Olivieri, Ángel del Río, and Juan Jacobo Simón, *On monomial characters and central idempotents of rational group algebras*, Comm. Algebra 32 (2004), no. 4, 1531–1550. MR 2100373

[20] Alan E. Parks, *Nilpotent by supersolvable M-groups*, Canad. J. Math. 37 (1985), no. 5, 934–962. MR 806649

[21] César Polcino Milies and Sudarshan K. Sehgal, *An introduction to group rings*, Algebra and Applications, vol. 1, Kluwer Academic Publishers, Dordrecht, 2002. MR 1896125

[22] Toshihiko Yamada, *The Schur subgroup of the Brauer group*, Lecture Notes in Mathematics, Vol. 397, Springer-Verlag, Berlin-New York, 1974. MR 0347957

[23] Huijuan Zheng and Ping Jin, *On the supersolvable residual of an M-group*, J. Algebra Appl. 17 (2018), no. 7, 1850138, 9. MR 3813711