Covariance and objectivity in mechanics and turbulence

A revisiting of definitions and applications

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Abstract

Form-invariance (covariance) and frame-indifference (objectivity) are two notions in classical continuum mechanics which have attracted much attention and controversy over the past decades. Particularly in turbulence modelling it seems that there still is a need for clarification. The aim and purpose of this study is fourfold: (i) To achieve consensus in general on definitions and principles when trying to establish an invariant theory for modelling constitutive structures and dynamic processes in mechanics, where special focus is put on the principle of Material Frame-Indifference (MFI), a general principle to model the response of a material in solid and fluid mechanics, stating that only frame-indifferent (objective) terms should enter the constitutive equations. (ii) To show that in constitutive modelling MFI can only be regarded as an approximation that needs to be reduced to a weaker statement when trying to advance it to an axiom of nature. (iii) To convince that in dynamical modelling, as in turbulence, MFI may not be utilized as a modelling guideline, not even in an approximate sense. Instead, its reduced form has to be supplemented by a second, independent axiom that includes the microscopic (fluctuating) description of the dynamical processes. Concerning Navier-Stokes turbulence, the axiom of Turbulent Frame-Indifference (TFI) is stated in which turbulence has to be modelled consistently with the invariant properties of the deterministic Navier-Stokes equations, and finally (iv) to propose a novel invariant modelling ansatz both for constitutive and dynamical modelling that allows to include the (mean) velocity field as an own independent modelling variable, however, not in an absolute but only in the relative sense as a velocity difference; a result that would systematically improve current modelling procedures in extended thermodynamics and turbulence theory to be more consistent with physical observations.

Keywords: Tensors, Form-invariance, Covariance, Frame-indifference, Objectivity, Symmetry, Inertial and Non-inertial systems, Galilean and Euclidean transformations, Mach’s Principle, Constitutive equations, Acceleration-sensitive materials, Extended Thermodynamics, Turbulence

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1. Introduction

The still ongoing confusion between form-invariance and frame-indifference is the root for many wrong conclusions when trying to model constitutive structures and dynamical processes in continuum mechanics. The aim of this study is to clarify the essential difference between these two concepts as precise as possible. As a result a universal axiom for invariant modelling is formulated that clearly separates the statements on form-invariance and frame-indifference, allowing thus to see that the velocity field itself expressed as a velocity difference is indeed a valid modelling variable, both for constitutive and dynamical modelling, and, hence, that it would be inappropriate to exclude it from the modelling process as standardly practiced in the community of classical continuum mechanics. For turbulence modelling, however, an additional, independent axiom is needed to ensure full consistency with the known invariant properties of the deterministic Navier-Stokes equations. Worthwhile to mention here is that the following mathematical development could entirely be formulated in a 3D context, without changing the geometrical description to a 4D framework, as it had been done in Frewer (2009a) — nevertheless an adequate 4D-embedding in classical mechanics is still the most natural framework to discuss form-invariance and frame-indifference and its consequences for constitutive modelling in general (Havas, 1964; Matolcsi & Ván, 2006, 2007; Rouhaud et al., 2013; Panicaud & Rouhaud, 2014; Panicaud et al., 2016; Ván, 2015) and turbulence modelling in particular (Frewer, 2009b,c).

Before the difference between form-invariance and frame-indifference can be made clear, it is necessary to have a clear picture of what a ‘tensor’ is. First of all, it is not correct to call every object that carries indices automatically a tensor. A tensor is more than just an array or an ordered collection of numbers or components. It has to satisfy a well-defined transformation law under a certain coordinate transformation, namely that if any ordered collection of numbers or components make up a tensor relative to a specified coordinate transformation then this collection has to transform homogeneously and multilinearly (locally in each space-time point). Exactly this behavior defines the tensor-concept as it was originally introduced by Voigt, Ricci and Levi-Civita in the end of the 19th century, and not otherwise. For example, a particular feature of a tensor is that if it is zero in one coordinate frame, then it is zero in all frames relative to the coordinate transformations considered.†

Now, for an array of mathematical objects which transforms as a tensor, one says that this array transforms form-invariantly. But form-invariance, i.e., the tensor property of this array does not imply frame-indifference for this array. In other words, although each observer will ‘see’ a tensor, or a tensor relation, in the same structural form, its numerical evaluation, however, will be different for each observer due to their frame used. Hence frame-indifference under a specified coordinate transformation is extraordinarily much more than just form-invariance; it is a much stronger statement (see also e.g. Sadiki & Hutter (1996)). To be more precise: Form-invariance applies if a mathematical object transforms homogeneously and multilinearly under a given coordinate transformation, while frame-indifference applies if this coordinate transformation additionally also acts as a symmetry transformation on that object. Therefore frame-indifference must be proven independently and cannot be read off from any form-invariant expression, or, stated differently: Frame-indifference implies form-invariance under pure space-time coordinate transformations, but not vice versa.

Before continuing the discussion on the physical relevance of these two invariant formulations, some remarks ought to be made about the terminology standardly used in the literature. In the physics community the concept of ‘form-invariance’ is also termed as ‘covariance’, stemming from the notion of working with relativistic theories. Unfortunately in the engineering community, however, it is frequently observed that ‘form-invariance’ is misleadingly termed as

†Since the tensor concept is defined on coordinate transformations that have to be invertible (non-zero Jacobian), this particular feature of a tensor can also be stated oppositely: If a tensor is non-zero in one coordinate frame, then it is non-zero in all frames relative to the coordinate transformations considered. Of course, in general this feature is only to be understood in a local sense around each point in the manifold separately and not globally for all points simultaneously (Schrödinger, 1950).
‘frame-indifference’ (see e.g. Truesdell & Noll (1965); Speziale (1998); Dafalias (2011); Kirwan (2016)), thus leading to great confusion in the literature between the form-invariance of an equation and its frame-independence. In this regard it is important to note that the word ‘objectivity’ will be used herein in the following as a synonym for ‘frame-indifference’, unlike some authors in classical continuum mechanics (see e.g. Hutter & Jöhnk (2004); Ariki (2015); Liu & Lee (2016)) who take it as synonym for ‘form-invariance’ due to its relation to geometric invariance (see Appendix A), which, in my opinion, does not properly match as a synonym. The word ‘objective’ should be used as it is defined in the general linguistic sense, as something opposing a subjective impression or a measurement. Frame-indifference is of that category, while form-invariance is not.

For example, transforming between inertial coordinate systems within a physically closed environment is a frame-indifferent (objective) process, experimentally first realized by Galilei that one inertial system can not be distinguished from any other one; no matter which physical experiment is conducted, they will always give the same exact (numerical) results in each and every chosen inertial system — also known as the equivalence principle of inertial reference frames, which constitutes a fundamental principle of nature for all physics, without any exceptions. In clear contrast when transforming between inertial and non-inertial or between different non-inertial coordinate systems, which always, at least in 4D, is only a form-invariant process, but definitely not a frame-indifferent (objective) one, since for each observer it is always possible to distinguish one non-inertial system from any other one by performing an appropriate physical experiment (e.g. the Foucault pendulum to demonstrate the rotation of the earth within a closed room without any sensuous contact to the outside world). Hence an equivalence principle for non-inertial reference frames does not exist as it does for inertial frames.

2. The mathematical description of form-invariance vs. frame-indifference

For the defining coordinate transformation, let’s consider for simplicity in the following only a change in spatial coordinates given by an orientated uniform (time-dependent) rotation

\[ \dot{x} = Qx, \text{ with } QQ^T = 1, \text{ det}(Q) = 1, \text{ and constant spin } \Omega := QQ^T, \Omega = 0, \]

(2.1)

which can be interpreted either as a passive or as an active rotation (Frewer, 2009a). Hence the bold-face notation used in this study is contextual, depending on whether (2.1) is interpreted passively (change of frame) or actively (change of state): For a passive interpretation the bold-face notation of the coordinate \( x \) has to be seen only as a compact short-hand notation to symbolize the collection of all components \( x^I = (x^1, x^2, x^3) \), thus being equivalent to the index notation; it does not symbolize the geometrical, rotationally invariant coordinate vector \( x \) itself.

\[ \Omega = \left( \begin{array}{ccc} 0 & \Omega_{12} & \Omega_{13} \\ -\Omega_{12} & 0 & \Omega_{23} \\ -\Omega_{13} & -\Omega_{23} & 0 \end{array} \right), \]

(2.2)

\[ \Omega = \left( \begin{array}{ccc} 0 & \Omega_{12} & \Omega_{13} \\ -\Omega_{12} & 0 & \Omega_{23} \\ -\Omega_{13} & -\Omega_{23} & 0 \end{array} \right), \]

(2.3)

Within a classical 3D formulation, the general form-invariance of physics between arbitrary space-time coordinate transformations cannot be naturally seen; for that the theory has to be geometrically reformulated into a true 4D framework, which can always achieved without changing the physical content of theory (see also Kretschmann’s objection in Section 4.1). For example, in Frewer (2009a,b,c) it has been demonstrated how the theory of the classical Navier-Stokes equations can be put into a general form-invariant (covariant) form for arbitrary space-time transformations without changing the physical content of these equations, by just reformulating them onto a true 4D space-time manifold (either flat or curved) within Newtonian physics.

Important to note here is that Einstein’s theory of general relativity is not a theory on general frame-independency. In general relativity it is still possible for an observer to distinguish between inertial and non-inertial systems. For example, a free falling observer as an astronaut will notice the effect of his acceleration induced by gravity if he inspects larger space-time regions than his immediate surrounding (e.g. for him the planetary orbits are no straight lines). In this sense the name ‘general relativity theory’ can be criticized, since this theory presents no true generalization of the relativity or equivalence principle for inertial reference frames towards a relativity or equivalence principle for non-inertial systems. Essentially, general relativity is ‘only’ a theory on connecting inertial frames locally, in contrast to special relativity which is a global theory on inertial frames.
which, as an element of vector space with a specifically chosen basis \( \mathbf{g}_i \), would be represented as \( \mathbf{x} = q^i \mathbf{g}_i \), being invariant under a rotated change of frame (passive coordinate transformation): \( \mathbf{x} = q^i \mathbf{g}_i = \tilde{q}^i \mathbf{g}_i \). Hence, under a passive transformation, \( \tilde{\mathbf{x}} \) and \( \mathbf{x} \) in (2.1) do not represent invariant geometrical objects but only the (frame-dependent) set of components of the point vector in each frame, i.e., under a passive transformation, \( \tilde{\mathbf{x}} = Q \mathbf{x} \) equivalently stands for

\[
\mathbf{x} \xrightarrow{\text{passive transf.}} \tilde{x}^i = (Q)_j^i x^j. \tag{2.2}
\]

However, for an active interpretation of transformation (2.1), the bold-face notation of \( \mathbf{x} \) has to be seen as an element of vector space \( \mathbf{x} = x^i \mathbf{g}_i \) being mapped by a linear mapping \( Q \) into itself, \( \mathbf{x} \rightarrow \tilde{\mathbf{x}} = Q \mathbf{x} \), where the components of the new (transformed) vector \( \tilde{\mathbf{x}} \) in the old basis \( \mathbf{g}_i \) are given as (Saxl & Urbantke, 2001): \( \tilde{x}^j = Q(x^i \mathbf{g}_i) = x^i Q(\mathbf{g}_i) = x^i (Q)_j^i \mathbf{g}_j = \tilde{x}^j \mathbf{g}_j \), with \( \tilde{x}^j = (Q^T)_j^i x^i \), i.e., under an active transformation, \( \tilde{\mathbf{x}} = Q \mathbf{x} \) equivalently stands for

\[
\mathbf{x} \xrightarrow{\text{active transf.}} \tilde{x}^i = (Q^T)_j^i x^j. \tag{2.3}
\]

Hence, the actively transformed components (2.3) relative to one frame (one observer, two states) transform contragrediently to the passively transformed ones (2.2) between two frames (two observers, one state). Of course, with respect to the new basis \( \mathbf{g}_i \), the actively rotated vector \( \tilde{\mathbf{x}} \) (2.1) has the same components as \( \mathbf{x} \) with respect to the original one \( \mathbf{g}_i \).

As already said, since the defining transformation \( \tilde{\mathbf{x}} = Q \mathbf{x} \) in (2.1) can be interpreted either passively or actively in the straightforward way as shown above, it is not necessary to always explicitly distinguish between these two cases. Hence, as long as no ambiguity arises, I will thus consider \( \tilde{\mathbf{x}} = Q \mathbf{x} \) (2.1) in the following as a generic transformation between two systems \( \tilde{\mathbf{x}} \) and \( \mathbf{x} \), without explicitly specifying each time whether these two systems correspond to two different observers looking at one state or to only one observer looking at two different states.

### 2.1. The tensor field

By considering first an arbitrary single-valued function \( \phi(\mathbf{x}) \), it will transform under (2.1) as

\[
\phi(\mathbf{x}) = \phi(Q^T \tilde{\mathbf{x}}) =: \tilde{\phi}(\tilde{\mathbf{x}}), \tag{2.4}
\]

where for notational abbreviation an additional symbol \( \tilde{\phi} \) has been introduced to show that the functional dependence on the new coordinates will in general be different than the functional dependence on the old coordinates. This relation \( \phi(\mathbf{x}) = \tilde{\phi}(\tilde{\mathbf{x}}) \), however, only tells us that the single-valued function \( \phi \) transforms as a scalar, or as a tensor of rank 0, i.e. form-invariantly or covariantly. It does not tell us that \( \phi \) transforms frame-indifferently or objectively. In order to transform frame-indifferently, the function itself must show in addition to this scalar relation an invariant structure, namely such that if the new coordinates are inserted into \( \phi \), the functional dependence will stay unchanged

\[
\tilde{\phi}(\tilde{\mathbf{x}}) = \phi(\mathbf{x}) = \phi(Q^T \tilde{\mathbf{x}}) = \phi(\tilde{\mathbf{x}}), \quad \text{i.e.} \quad \tilde{\phi}(\cdot) = \phi(\cdot). \tag{2.5}
\]

Such a coordinate transformation (2.1) is then called a symmetry transformation for \( \phi(\mathbf{x}) \), while \( \phi(\mathbf{x}) \) in this specific case is called an isotropic (rotationally frame-indifferent) function, an invariant function of the restricted form

\[
\phi(\mathbf{x}) = \phi(\|\mathbf{x}\|), \tag{2.6}
\]

where \( \|\mathbf{x}\| = \sqrt{x^T \mathbf{x}} \) is the Euclidean norm (magnitude) of \( \mathbf{x} \). Indeed, restriction (2.6) globally satisfies the frame-indifference condition (2.5) for the specific coordinate transformation (2.1).

A tensor in the next higher rank can be constructed by taking the gradient of the scalar field \( \mathbf{G}(\mathbf{x}) := \nabla \phi(\mathbf{x}) \). Using the above scalar transformation law for \( \phi \) (2.4) and the chain rule for the derivative, the gradient of a scalar field transforms as

\[
\mathbf{G}(\mathbf{x}) = \nabla \phi(\mathbf{x}) = \nabla \tilde{\phi}(\tilde{\mathbf{x}}) = \nabla \tilde{\phi}(\tilde{\mathbf{x}}) Q =: \tilde{\mathbf{G}}(\tilde{\mathbf{x}}) Q. \tag{2.7}
\]
By inverting this relation we get the transformation law for the gradient of an arbitrary scalar field under a uniform rotation:
\[ \hat{\mathbf{G}}(\hat{x}) = \mathbf{G}(x)Q^T. \] (2.8)

As before, this relation only tells us that the gradient of a scalar field transforms as a tensor of rank 1, or more precisely as a covariant tensor of rank 1 due to the appearance of the transpose (inverse) transformation matrix. But this relation does not imply that \( \mathbf{G} \) transforms frame-indifferently or objectively. Only if \( \mathbf{G} \) has the following invariant property \( \mathbf{G}(\cdot) = \mathbf{G}(\cdot) \) when inserting the new coordinates, then \( \mathbf{G} \) is said to transform frame-indifferently. But this property can only be achieved if \( \mathbf{G} \) behaves as follows
\[ \mathbf{G}(x) = \mathbf{G}(Q^T x) = \mathbf{G}(\hat{x})Q, \quad \text{i.e.} \quad \mathbf{G}(\hat{x}) = \mathbf{G}(x)Q^T, \] (2.9)

because only then the frame-indifferent transformation rule is obtained:
\[ \hat{\mathbf{G}}(\hat{x}) = \mathbf{G}(x)Q^T = \mathbf{G}(Q^T x)Q^T = \mathbf{G}(\hat{x})QQ^T = \mathbf{G}(\hat{x}), \quad \text{i.e.} \quad \hat{\mathbf{G}}(\cdot) = \mathbf{G}(\cdot). \] (2.10)

A typical example for an isotropic covariant tensor of rank 1 is obviously
\[ \mathbf{G}(x) = \nabla \phi(||x||) =: \frac{\partial}{\partial x} \phi(||x||), \] (2.11)

where \( \phi \) is again some arbitrary scalar function of the Euclidean norm \( ||x|| = \sqrt{x^T \cdot x} \). Indeed, structure (2.11) behaves as (2.9):
\[ \mathbf{G}(\hat{x}) = \frac{\partial}{\partial \hat{x}} \phi(||\hat{x}||) = \frac{\partial}{\partial x} \frac{\partial x}{\partial \hat{x}} \phi(||x||) = \frac{\partial}{\partial x} \phi(||x||)Q^T = \mathbf{G}(x)Q^T. \] (2.12)

Again, it is important to observe the small but decisive difference between the condition of form-invariance given in (2.8) and the condition of frame-independence given in (2.9): In relation (2.8) we are dealing with two different tensor-functions \( \hat{\mathbf{G}} \) and \( \mathbf{G} \), while relation (2.9) refers to only one tensor-function \( \mathbf{G} \).

The above construction procedure can now be straightforwardly extended to arbitrary order in the rank and type of a tensor. For example, for a tensor field \( \mathbf{C} = \mathbf{C}(x) \) of rank 2 we can have the following form-invariant transformation relations
\[ \hat{\mathbf{C}}(\hat{x}) = Q \mathbf{C}(x) Q^T, \quad \hat{\mathbf{C}}(\hat{x}) = (Q^{-1})^T \mathbf{C}(x) Q^{-1}, \quad \hat{\mathbf{C}}(\hat{x}) = Q \mathbf{C}(x) Q^{-1}, \] (2.13)

depending on whether the tensor is contravariant \( (\mathbf{C})^{ij} \), covariant \( (\mathbf{C})_{ij} \), or of mixed form \( (\mathbf{C})^{ij} \), respectively. However, since the considered transformation matrix \( Q \) is orthogonal, i.e. since \( Q^{-1} = Q^T \), the above contravariant and covariant tensors all transform in the same way for the specifically considered case of uniform rotations, namely as
\[ \hat{\mathbf{C}}(\hat{x}) = Q \mathbf{C}(x) Q^T, \] (2.14)

but careful, this does not mean that the contravariant and covariant tensors themselves coincide. The universal relation (2.14) only states that the components of contravariant and covariant tensors
\[ \mathbf{C} = \mathbf{C}(x) \]

in the rank and type of a tensor. For example, for a tensor field \( \mathbf{C} = \mathbf{C}(x) \) of rank 2 we can have the following form-invariant transformation relations
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\[ \hat{\mathbf{C}}(\hat{x}) = Q \mathbf{C}(x) Q^T, \] (2.14)

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\[ \hat{\mathbf{C}}(\hat{x}) = Q \mathbf{C}(x) Q^T, \] (2.13)

depending on whether the tensor is contravariant \( (\mathbf{C})^{ij} \), covariant \( (\mathbf{C})_{ij} \), or of mixed form \( (\mathbf{C})^{ij} \), respectively. However, since the considered transformation matrix \( Q \) is orthogonal, i.e. since \( Q^{-1} = Q^T \), the above contravariant and covariant tensors all transform in the same way for the specifically considered case of uniform rotations, namely as
\[ \hat{\mathbf{C}}(\hat{x}) = Q \mathbf{C}(x) Q^T, \] (2.14)
tensors transform in the same way under orthogonal transformations, and not that they coincide, since in general \((C)^{ij} \neq (C)_{ij} \neq (C)^{ij}\). These components will coincide only if we would introduce an orthonormal basis of the vector space for the coordinate system considered.¹

That \(C\) transforms form-invariantly as a tensor \((2.14)\) is of course not a guarantee that it also transforms frame-indifferently or objectively. For that it additionally needs to transform as

\[ C(\tilde{x}) = QC(x)Q^T, \]  

(2.15)

because only then the frame-indifferent transformation rule is again obtained:

\[ \tilde{C}(\tilde{x}) = QC(x)Q^T = QQ^T C(\tilde{x})QQ^T = C(\tilde{x}), \]  

(2.16)

A typical example for a frame-indifferent (objective) second rank tensor under uniform rotations is the isotropic tensor

\[ C(x) = \phi(\|x\|) x \otimes x + \psi(\|x\|) 1, \]  

(2.17)

where \(\phi\) and \(\psi\) are two arbitrary scalar functions of the spatial Euclidean norm of coordinate \(x\).

### 2.2. The composite tensor field

The tensor relations given above refer directly to the coordinates to be transformed. The question now is: How will these relations change if we consider nested tensor functions, i.e., composite tensor functions? Imagine we would have the following scalar function \(\phi(\overline{f}(x))\), where for simplicity we will assume that the scalar function \(\phi\) is composed of a contravariant tensor function \(f\) of rank 1, i.e., that it transforms as \(\tilde{f}(\tilde{x}) = Qf(x)\). This composed scalar function will then transform as

\[ \phi(f(x)) = \phi(Q^T \tilde{f}(\tilde{x})) := \tilde{\phi}(\tilde{f}(\tilde{x})), \]  

(2.18)

displaying a relation, which formally, if \(f(x)\) is known or specified, can be written also as

\[ \phi^*(x) = \tilde{\phi}^*(\tilde{x}), \]  

(2.19)

where \(\phi^*\) denotes the function constructed from evaluating the composite function \(\phi(\overline{f}(x))\) relative to the coordinates \(x\), and likewise the notation \(\tilde{\phi}^*\) for evaluating the transformed composite function, i.e., \(\phi(f(x)) \equiv \phi^*(x)\) and \(\phi(\overline{f}(\overline{x})) \equiv \tilde{\phi}^*(\overline{x})\). Obviously, as already assumed, such a construction is only possible if the inner function \(f\) is explicitly known or specified. Nevertheless, the latter representation \((2.19)\) refers directly to the transformed coordinates, whereas the former representation \((2.18)\) only refers indirectly via the inner tensor function \(f\) to the transformed coordinates. Hence, frame-indifference for this scalar field \(\phi\) can occur in two variants:

- Explicit frame-indifference, if \(\phi(\cdot) = \tilde{\phi}(\cdot), \)  
  \(\text{or,} \quad \text{Full frame-indifference, if } \phi^*(\cdot) = \tilde{\phi}^*(\cdot). \)

(2.20, 2.21)

It is intuitively clear that full frame-indifference \((2.21)\) is a stronger restriction on \(\phi\) than explicit frame-indifference \((2.20)\). Full frame-indifference of \(\phi(f(x))\) can be seen as a superposition of frame-indifference originating from the outer dependence \(f\) (explicit frame-indifference) and the inner nested dependence \(x\) (implicit frame-indifference). To illustrate this difference of restriction explicitly, let’s consider, for example, the following simple specification \(^1\)

\[ \phi(f(x)) = \|f(x)\|, \quad \text{with } f(x)^T = (x^1, 0, 0), \]  

(2.22)

\(^1\)Note that under orientated orthogonal transformations (rotations) any chosen basis of a vector space, irrespective of whether being a contravariant, covariant or orthonormal basis, remains class invariant.

\(^{1}\)Note that the explicit structure of transformed function \(\tilde{f}(\tilde{x})\) is determined from the untransformed function \(f(x)\) \((2.22)\) via its defined tensor transformation rule as \(\tilde{f}(\tilde{x}) \equiv Qf(Q^T \tilde{x})\), which results to: \(\tilde{f}(\tilde{x}) = (Q)^1, (Q^T \tilde{x})^1\).
which, as a result, induces the functional relation directly on the coordinates as

$$\phi(\mathbf{f}(\mathbf{x})) = \|\mathbf{f}(\mathbf{x})\| = |x^1|, \text{ i.e. } \phi^*(\mathbf{x}) = |x^1|. \quad (2.23)$$

Obviously, the scalar function $\phi(\mathbf{f}(\mathbf{x}))$, as specified in (2.22), only shows the weaker explicit frame-indifference (2.20), since

$$\tilde{\phi}(\tilde{\mathbf{f}}(\tilde{\mathbf{x}})) = \phi(\mathbf{f}(\mathbf{x})) = \phi(Q^T\mathbf{f}(\mathbf{x})) = \|Q^T\tilde{\mathbf{f}}(\tilde{\mathbf{x}})\| = \|\tilde{\mathbf{f}}(\tilde{\mathbf{x}})\| = \phi(\tilde{\mathbf{f}}(\tilde{\mathbf{x}})), \text{ i.e. } \phi(\cdot) = \tilde{\phi}(\cdot), \quad (2.24)$$

and not the stronger full frame-indifference (2.21), since

$$\tilde{\phi}^*(\tilde{\mathbf{x}}) = \tilde{\phi}(\tilde{\mathbf{f}}(\tilde{\mathbf{x}})) = \phi(\mathbf{f}(\mathbf{x})) = \phi^*(\mathbf{x}) = \phi^*(Q^T\mathbf{x}) = |(Q^T)^j_i\tilde{x}^j| \neq |\tilde{x}^1| = \phi^*(\tilde{\mathbf{x}}), \text{ i.e. } \phi^*(\cdot) \neq \tilde{\phi}^*(\cdot).$$

However, as already said before, the representation (2.19) and its condition for frame-indifference (2.21) is only then of a concern if the inner function $\mathbf{f}$ is known. Usually this is not the case, since normally such functions $\mathbf{f}(\mathbf{x})$ represent field variables satisfying certain differential or integral equations which first have to be solved in order to get hold of the explicit functional structure of $\mathbf{f}(\mathbf{x})$. Henceforth I will not consider the case of full frame-indifference (2.21) any further, that is, throughout this study the notion ‘frame-indifference’ for composite tensor fields will only be understood as ‘explicit frame-indifference’ as formulated in (2.20).

Surely, the specific dependence $\phi = \phi(\mathbf{f}(\mathbf{x}))$ considered so far is not the most general case for a composite scalar function. The general case would be

$$\phi = \phi(T_1(\mathbf{x}), \ldots, T_n(\mathbf{x}); \mathbf{x}), \quad (2.25)$$

depending explicitly on the coordinates $\mathbf{x}$ and implicitly on $n$ different field functions $T_n(\mathbf{x})$ of any type, not necessarily tensors. In general these functions are unknown system variables satisfying certain balance equations, but, of course, with a known transformation behavior $T_n(\mathbf{x}) \to T_n(\tilde{\mathbf{x}}), \forall n$, otherwise an a priori invariance analysis on (2.25) would not be possible. Form-invariance of (2.25) is then given if $\phi$ transforms as a true scalar, namely as

$$\phi(T_1(\mathbf{x}), \ldots, T_n(\mathbf{x}); \mathbf{x}) = \tilde{\phi}(T_1(\tilde{\mathbf{x}}), \ldots, T_n(\tilde{\mathbf{x}}); \tilde{\mathbf{x}}), \quad (2.26)$$

and frame-indifference in the sense as (2.20), if $\phi$ additionally behaves as

$$\phi(T_1(\mathbf{x}), \ldots, T_n(\mathbf{x}); \mathbf{x}) = \tilde{\phi}(T_1(\tilde{\mathbf{x}}), \ldots, T_n(\tilde{\mathbf{x}}); \tilde{\mathbf{x}}). \quad (2.27)$$

The concept of measuring or constructing explicit frame-indifference as formulated in (2.20) is conferrable to any composite tensor of any rank. For example, when looking at the transformation law for the tensor composition $G(T_1(\mathbf{x}), \ldots, T_n(\mathbf{x}); \mathbf{x})$ of rank 1, where $G$ is the gradient of the scalar field (2.25)

$$G(T_1(\mathbf{x}), \ldots, T_n(\mathbf{x}); \mathbf{x}) = \nabla \phi(T_1(\mathbf{x}), \ldots, T_n(\mathbf{x}); \mathbf{x}), \quad (2.28)$$

which transforms form-invariantly

$$G(T_1(\tilde{\mathbf{x}}), \ldots, T_n(\tilde{\mathbf{x}}); \tilde{\mathbf{x}}) = G(T_1(\mathbf{x}), \ldots, T_n(\mathbf{x}); \mathbf{x}) Q^T, \quad (2.29)$$

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1Full frame-indifference can only be obtained if we would specify the contravariant vector field $\mathbf{f}(\mathbf{x})$ frame-indifferently, e.g., as the isotropic function $\mathbf{f}(\mathbf{x}) = \psi(\|\mathbf{x}\|)\mathbf{x}$, where $\psi$ is some arbitrary scalar function, and where the direct coordinate dependence of $\phi$ would then read as $\phi^*(\mathbf{x}) = \psi(\|\mathbf{x}\|)\|\mathbf{x}\|$, which now, obviously, is fully frame-indifferent, since $\phi^*(\cdot) \neq \tilde{\phi}^*(\cdot)$. 

since, obviously, according to (2.26), $G$ explicitly transforms as

$$G(T_1(x), \ldots, T_n(x) ; x) = \nabla \phi(T_1(x), \ldots, T_n(x) ; x)$$

$$= \nabla \tilde{\phi}(\tilde{T}_1(\tilde{x}), \ldots, \tilde{T}_n(\tilde{x}) ; \tilde{x}) = \nabla \frac{\partial \tilde{x}}{\partial x} \tilde{\phi}(\tilde{T}_1(\tilde{x}), \ldots, \tilde{T}_n(\tilde{x}) ; \tilde{x})$$

$$= \tilde{G}((\tilde{T}_1(\tilde{x}), \ldots, \tilde{T}_n(\tilde{x}) ; \tilde{x}) Q,$$  \hspace{1cm} (2.30)

the property frame-indifference for $G$ in the sense (2.20), i.e. $G(\cdot) = \tilde{G}(\cdot)$, is then given, if, next to the transformation law (2.30), $G$ additionally transforms as

$$G(T_1(x), \ldots, T_n(x) ; x) = G(T_1(x), \ldots, T_n(x) ; x) Q^T,$$  \hspace{1cm} (2.31)

a rule analogous to (2.9), because only then the condition $G(\cdot) = \tilde{G}(\cdot)$ is guaranteed:

$$\tilde{G}(T_1(\tilde{x}), \ldots, T_n(\tilde{x}) ; \tilde{x}) = G(T_1(x), \ldots, T_n(x) ; x) Q^T$$

$$= G(T_1(x), \ldots, T_n(x) ; x), \text{ i.e. } \tilde{G}(\cdot) = G(\cdot).$$  \hspace{1cm} (2.32)

However, in view of the redefinition problems arising in the specific examples to be introduced and discussed next, it is already important to note that to measure or construct explicit frame-indifference for any tensor $C$ as $\tilde{C}(\cdot) = C(\cdot)$, only works for absolute tensors, but not for (redefined) relative tensors. The notion of absolute and relative tensors is defined in Appendix B, while the criterion to measure frame-indifference (objectivity) for a relative tensor is defined in Appendix C, where it is made clear that it is necessary to distinguish between relative and absolute objectivity.

2.3. The velocity field — example for a relative, non-objective tensor

A key quantity in continuum mechanics is the velocity field $u = u(x, t)$, which formally gets its transformation rule by taking the time derivative of the transformed coordinates (2.1)

$$\dot{x} = Q\dot{x} + Qx$$

$$= Q\dot{x} + QQ^T \dot{x} = Q\dot{x} + \Omega^T \dot{x} = Q\dot{x} - \Omega \ddot{x} \quad \implies \quad \ddot{u} + \Omega \ddot{x} = Qu,$$  \hspace{1cm} (2.33)

which then, when seen as the transition going from the Lagrangian to the Eulerian description, yields the well-known transformation rule for the velocity field\(^1\)

$$\dot{u}(x) + \Omega \ddot{x} = Qu(x).$$  \hspace{1cm} (2.34)

It is clear that the inhomogeneous term proportional to the spin $\Omega$ destroys the tensor property of the velocity field in 3D. Nevertheless, as shown and discussed in Appendix B, it is possible within the 3D framework to reformulate (2.34) into a form-invariant, but still frame-dependent (non-objective) tensor relation

$$\dot{u}_Q(x) = Qu_Q(x),$$  \hspace{1cm} (2.35)

with $u_Q(x) := u(x) + \Omega x$ being the new redefined velocity field relative to the spin term $\Omega$, which itself, however, transforms as a non-tensor

$$\Omega = Q\Omega Q^T + Q\dot{Q}^T.$$  \hspace{1cm} (2.36)

\(^1\)To simplify notation in the following, the explicit time dependence for the velocity field $u$ will be suppressed, which generally shows the full coordinate dependence $u = u(x, t)$.  

As introduced in Appendix B, the redefined velocity field \( \mathbf{u}_\Omega(x) \) cannot be classified as a true (absolute) tensor, but only as a relative one, since obviously \( \mathbf{u}(x) \neq \mathbf{u}_\Omega(x) \), for \( \Omega \neq 0 \). While a true tensor can either be frame-indifferent (objective) or frame-dependent (non-objective), a relative tensor can never be objective in an absolute sense, since it always, per se, depends on the reference frame used, for example, as already seen at the relative velocity \( \mathbf{u}_\Omega(x) = \mathbf{u}(x) + \Omega x \), which explicitly depends on the spin value \( \Omega \) of the chosen frame. The upcoming examples, along with Appendix C, will clarify this issue even further.

2.4. The strain rate — example for an absolute, objective composite tensor

The next example is the strain rate \( \mathbf{S} \), defined as the following symmetric composite function of the velocity gradient \( \mathbf{L}(x) := \nabla \otimes \mathbf{u}(x) \):

\[
\mathbf{S}(\mathbf{L}(x)) = \frac{1}{2} \left( \mathbf{L}(x) + \mathbf{L}(x)^T \right),
\]

which, as is well known, transforms form-invariantly as a tensor of rank 2 under a uniform rotation:

\[
\tilde{\mathbf{S}}(\tilde{\mathbf{L}}(\tilde{x})) = \mathbf{Q} \mathbf{S}(\mathbf{L}(x)) \mathbf{Q}^T.
\]

In contrast to the velocity gradient \( \mathbf{L}(x) \) which itself does not transform as tensor, due to the appearance of an inhomogeneous term arising from the non-tensor transformation property of the velocity field \( \mathbf{u}(x) \) (2.34):

\[
\mathbf{L}(x) = \nabla \otimes \mathbf{u}(x) = \nabla \frac{\partial \mathbf{x}}{\partial x} \otimes \left( \mathbf{Q}^T \mathbf{u}(x) + \mathbf{Q}^T \mathbf{x} \right) = \nabla \frac{\partial \mathbf{x}}{\partial x} \otimes \left( \mathbf{Q}^T \mathbf{u}(x) + \mathbf{Q}^T \Omega \mathbf{x} \right) = \mathbf{Q}^T \left( \nabla \otimes \mathbf{u}(x) + \nabla \otimes \Omega \mathbf{x} \right) \frac{\partial \mathbf{x}}{\partial x} = \mathbf{Q}^T \mathbf{L}(x) \mathbf{Q} + \mathbf{Q}^T \Omega \mathbf{Q},
\]

that means, we obtain the non-tensor relation

\[
\tilde{\mathbf{L}}(\tilde{x}) = \mathbf{QL}(x)\mathbf{Q}^T - \Omega \neq \mathbf{QL}(x)\mathbf{Q}^T.
\]

Hence, since the spin matrix \( \Omega \) is antisymmetric, i.e. \( \Omega^T = -\Omega \), we can see the reason why the strain rate \( \mathbf{S} \) (2.37) transforms as a tensor formulated by (2.38), while its argument, the velocity gradient \( \mathbf{L} \), not.\(^1\) Due to adding \( \mathbf{L} \) and \( \mathbf{L}^T \), the inhomogeneous spin term cancels \( \Omega + \Omega^T = 0 \).

Again, as stated several times before for other quantities, relation (2.38) only tells us that the strain rate \( \mathbf{S} \) is transforming as a tensor of rank 2; it does not automatically tell us that \( \mathbf{S} \) is also transforming frame-indifferently under uniform rotations. Such a property has to be analyzed separately. A closer inspection, however, reveals that the composite structure (2.37) is indeed frame-indifferent in an absolute sense by showing the transformation behavior \( \tilde{\mathbf{S}}(\cdot) = \mathbf{S}(\cdot) \), since

\[
\tilde{\mathbf{S}}(\tilde{\mathbf{L}}(\tilde{x})) = \mathbf{Q} \mathbf{S}(\mathbf{L}(x)) \mathbf{Q}^T = \mathbf{Q} \left( \frac{1}{2} \left( \mathbf{L}(x) + \mathbf{L}(x)^T \right) \right) \mathbf{Q}^T
\]

\[
= \mathbf{Q} \left( \frac{1}{2} \left( \mathbf{Q}^T \tilde{\mathbf{L}}(\tilde{x}) \mathbf{Q} + \mathbf{Q}^T \Omega \mathbf{Q} + \mathbf{Q}^T \tilde{\mathbf{L}}(\tilde{x})^T \mathbf{Q} + \mathbf{Q}^T \Omega^T \mathbf{Q} \right) \right) \mathbf{Q}^T = \frac{1}{2} \left( \mathbf{L}(x) + \tilde{\mathbf{L}}(\tilde{x})^T \right)
\]

\[
= \mathbf{S}(\tilde{\mathbf{L}}(\tilde{x})).
\]

\(^1\)In the formulation of Appendix B, note that the strain rate \( \mathbf{S} \) obviously transforms as an absolute tensor, since it need not to be redefined in order to obtain a tensor relation for its transformed values, i.e., since for \( \Omega \neq 0 \) it obviously has the property \( \mathbf{S} = \mathbf{S}_\Omega \). In contrast of course to its argument, the velocity gradient \( \mathbf{L} \) which need to be redefined as \( \mathbf{L} \rightarrow \mathbf{L}_\Omega = \mathbf{L} + \Omega \) if one wants to turn (2.39) into a form-invariant tensor relation as presented in (B.17). Hence, since \( \mathbf{L} \neq \mathbf{L}_\Omega \) for \( \Omega \neq 0 \), the velocity gradient thus only transforms as a relative tensor.
2.5. Vorticity — example for a relative, non-objective composite tensor

A different example is the vorticity matrix, formulated as the antisymmetric composite function of the velocity gradient $\mathbf{L}$

$$
\mathbf{W}(\mathbf{L}(x)) = \frac{1}{2} \left( \mathbf{L}(x) - \mathbf{L}(x)^T \right),
$$

(2.41)

which, obviously, does not transform as a tensor, since

$$
\mathbf{W}(\mathbf{L}(x)) = \frac{1}{2} \left( \mathbf{L}(x) - \mathbf{L}(x)^T \right) = \frac{1}{2} \left( Q^T \mathbf{L}(\tilde{x}) Q + Q^T \Omega Q - Q^T \mathbf{L}(\tilde{x})^T Q - Q^T \Omega^T Q \right)
$$

$$
= Q^T \mathbf{W}(\mathbf{L}(\tilde{x})) Q + Q^T \Omega Q,
$$

(2.42)

that means, we obtain the non-tensor relation

$$
\mathbf{W}(\mathbf{L}(\tilde{x})) = Q \mathbf{W}(\mathbf{L}(x)) Q^T - \Omega \neq Q \mathbf{W}(\mathbf{L}(x)) Q^T,
$$

(2.43)

being identical to the transformation rule of its argument, the velocity gradient $\mathbf{L}$ (2.39). Now, since the vorticity matrix $\mathbf{W}$ (2.41) does not transform as a tensor under uniform rotations, it is clear that it then also cannot show any frame-indifference: While frame-indifference (the symmetry property) always implies form-invariance (the tensor property) under pure space-time coordinate transformations, the opposite obviously cannot be stated. Said differently, if some quantity does not transform as a tensor then it also cannot be frame-indifferent. This can be readily seen by the simple argument that if the considered quantity is zero in one frame then it is, due to its non-tensor property, necessarily not zero and thus different in another (transformed) frame, exhibiting thus a clear explicit frame-dependence (non-objectivity).

Up to now for the flow quantities mentioned, it is only the strain rate $\mathbf{S}$ (2.37) which transforms as an absolute tensor being additionally also frame-indifferent under uniform rotations, while the other three quantities, the velocity field $\mathbf{u}$, the velocity gradient $\mathbf{L}$ and the vorticity $\mathbf{W}$ only transform as relative tensors, since in each case they need to be redefined in order to obtain for their transformed values (within a 3D spatial framework) a form-invariant, but still frame-dependent (non-objective) relation (see Appendix B):

$$
\begin{align*}
\tilde{\mathbf{u}}(\tilde{x}) + \Omega \tilde{x} &= Q \mathbf{u}(x) \rightarrow \tilde{\mathbf{u}}_\Omega(\tilde{x}) = Q \mathbf{u}_\Omega(x), \\
\tilde{\mathbf{L}}(\tilde{x}) + \Omega &= Q \mathbf{L}(x) Q^T \rightarrow \tilde{\mathbf{L}}_\Omega(\tilde{x}) = Q \mathbf{L}_\Omega(x) Q^T, \\
\tilde{\mathbf{W}}(\tilde{\mathbf{L}}(\tilde{x})) + \Omega &= Q \mathbf{W}(\mathbf{L}(x)) Q^T \rightarrow \tilde{\mathbf{W}}(\tilde{\mathbf{L}}_\Omega(\tilde{x})) = Q \mathbf{W}(\mathbf{L}_\Omega(x)) Q^T,
\end{align*}
$$

(2.44)

satisfying then in each redefined case the multilinear and homogeneous transformation property of a (3D spatial) tensor. Moreover, as shown in Appendix C, although the relative vorticity $\mathbf{W}(\mathbf{L}_\Omega(x))$ is frame-dependent (non-objective) in the absolute sense, it nevertheless exhibits a certain relative frame-indifference (relative objectivity) between different uniform rotating (non-inertial) reference frames.

2.6. Example for an absolute, non-objective composite tensor

For the last example, we want to construct a flow quantity $\mathbf{Z}$ that transforms as an absolute tensor but that does not show frame-indifference. Such quantities are easy to construct and become relevant when opting for explicit (non-inertial) frame-dependence in modelling relations. For example, let us consider $\mathbf{Z}$ as a composite contravariant tensor field of rank 2 of the form $\mathbf{Z} = \mathbf{Z}(\mathbf{L}(x))$ with the following specification for its components

$$
Z^{ij}(\mathbf{L}(x)) = \delta^i_j \Delta L^1_1,
$$

(2.45)

where $L^1_1 = \partial_1 u^1$ is the $(1,1)$-component of the velocity gradient matrix $\mathbf{L}(x) = \nabla \otimes \mathbf{u}(x)$, and $\Delta = \nabla^T \nabla$ the Laplacian, being, of course as a scalar, invariant under arbitrary uniform rotations: $\tilde{\Delta} = \Delta$. That the collection of all components (2.45) of $\mathbf{Z}$ transforms as a (contravariant,
second rank) tensor is easily shown, since
\[ Z^{ij}(L(x)) = \delta_i^j \Delta L^1 = (Q^T Q)_1^1 \Delta (Q^T L Q + Q^T \Omega Q)_1^1 \]
\[ = (Q^T)_i^k (Q)_k^j (Q^T \Delta \tilde{L} Q)_1^1 (Q^T)_1^l (Q)_l^j \]
\[ =: (Q^T)_i^j \tilde{Z}^{kl}(\tilde{L}(\tilde{x}))(Q)_l^j, \] (2.46)
that means, we get the tensor relation
\[ Z(L(x)) = Q^T \tilde{Z}(\tilde{L}(\tilde{x}))Q, \]
where the components of \( Z \) in the transformed domain are collectively then given as
\[ \tilde{Z}^{ij}(\tilde{L}(\tilde{x})) = (Q)_1^i (Q^T \Delta \tilde{L} Q)_1^1 (Q^T)_1^j. \] (2.48)

Now, although \( Z \) (2.45) transforms as a tensor in the absolute sense, it does not transform frame-indifferently, i.e., \( \tilde{Z}(\cdot) \neq Z(\cdot) \), or, when formulated in its components, \( \tilde{Z}^{ij}(\cdot) \neq Z^{ij}(\cdot) \), since obviously
\[ \tilde{Z}^{ij}(\tilde{L}(\tilde{x})) = (Q)_1^i (Q^T \Delta \tilde{L} Q)_1^1 (Q^T)_1^j \]
\[ = (Q)_1^i (Q^T)_1^k \Delta \tilde{L}^k_l (Q)_l^j (Q^T)_1^j \neq \delta_i^j \delta_l^l \Delta \tilde{L}^l_l = Z^{ij}(\tilde{L}(\tilde{x})). \] (2.49)

3. Galilean form-invariance vs. frame-indifference in a physical application

To see the difference and consequences of form-invariance (covariance) and frame-indifference (objectivity) within a physical application as clearly as possible, I will only consider in this section the theory of classical mechanics for a single particle, however, which in each step can be easily transferred to the notions of continuum mechanics.

The equation of motion for a single particle in an inertial system in 3D is given by Newton’s second law
\[ m\ddot{x} = F, \] (3.1)
where \( m \) is the mass and \( F \) the given force to cause the acceleration \( \ddot{x} \) of the particle. The coordinate transformations connecting all inertial systems in classical Newtonian physics are the Galilei transformations
\[ x' = R x + v t + c, \quad t' = t + \tau, \] (3.2)
forming mathematically a proper, orthochronous Lie-group with 10 constant parameters: the eigenvector (axis of rotation) of the 3D rotation matrix \( R \) (having the property \( RR^T = 1 \), \( \det(R) = 1 \)), the velocity boost \( v \), the spatial shift \( c \) and the temporal off-set \( \tau \). Now, transforming Newton’s law of motion (3.1) according to (3.2), one obtains the defining result that it is form-invariant in all inertial systems, since, after transformation, the same form of (3.1) is again obtained
\[ m\ddot{x}' = F', \] (3.3)
where \( F' \) is the same force \( F \) only expressed in the new ‘primed’ coordinates related to the components of \( F \) by
\[ F' = R F. \] (3.4)
Hence all observers experience the same law of motion (3.1) only expressed in their coordinates (3.3), that is, \( F \) and \( F' \) is the same physical force only in different component representations belonging to different observers. Obviously this form-invariance is caused by the fact that the acceleration \( \ddot{x} \) in (3.1) transforms as a tensor under (3.2)
\[ \ddot{x} = \frac{d}{dt} \frac{dx}{dt} = \frac{d}{dt} \frac{\partial x}{\partial x'} \frac{dx'}{dt} + \frac{\partial x}{\partial t'} dt' = \frac{d}{dt} \left( R^T \frac{dx'}{dt} - R^T v dt' \right) = \frac{d}{dt} \left( R^T R F - R^T v \right) = R^T F', \]
explicitly showing its tensor property \( \mathbf{x}' = \mathbf{R} \mathbf{x} \) in transforming linearly and homogeneously.\(^\dagger\) And since Newton’s law (3.1) forms a mathematical equation, the proven tensor property of \( \mathbf{x} \) on the left-hand side of equation (3.1) then defines its right-side, the force \( \mathbf{F} \), to be a tensor of the same type, too. Hence the form-invariant tensor relation (3.4), stating that if the physical force \( \mathbf{F} \) is observed as zero in one inertial system, it is observed as zero in all inertial systems.

However, up to now we have only shown the weaker form-invariance of Newton’s law of motion (3.1) under the Galilei transformations (3.2), and not the stronger frame-indifference. According to the notions and laws of mechanics, space and time are such that no point and no direction in (empty) space is distinguished, that time can be chosen arbitrarily, and that an absolute velocity (except for the speed of light in vacuum) cannot be defined. In classical Newtonian mechanics all these facts are expressed by the Galilei transformations (3.2), where experiments, first conducted by Galilei, show that within a physically closed environment\(^\ddagger\) no inertial system is distinguished from any other one in the sense that, for any physical experiment conducted, all inertial systems give the same exact (numerical) results. This experimental fact forms the basis for the equivalence principle of inertial reference frames, stating that the laws of motion for closed systems are not only form-invariant (covariant), but also frame-indifferent (objective). In other words, not only do the laws have the same form in all inertial systems, but also the processes (induced by these laws) run in the same way in all of these systems.

Hence, according to this principle the tensorial equation (3.1) also has to be frame-indifferent under the Galilei transformations (3.2). As a consequence, the force \( \mathbf{F} \) can no longer be treated as a fully arbitrary quantity anymore, it has to be specified in terms of the basic mechanical quantities in order to obtain a particular system of differential equations such that the Galilei transformations (3.2) are admitted as symmetries. In other words, the force \( \mathbf{F} \) has to be identified and formulated as a constitutive relation which has to obey the principle of Galilean frame-indifference — similar to the process of modelling constitutive equations in continuum mechanics according to the principle of MFI, or in its correctly stated form, according to r-MFI (see definition and discussion in Section 4.2 & 4.3). In general the force \( \mathbf{F} \) in (3.1) may explicitly depend on time, space, direction and velocity, as well as on the position and velocity of the particle:

\[
\mathbf{F} = \mathbf{F}(\mathbf{x}, \mathbf{\dot{x}}, t; \mathbf{x}_0^0, \mathbf{v}_0^0, t_0^0),
\]

(3.5)

where \( \mathbf{x}_0^0, \mathbf{v}_0^0 \) and \( t_0^0 \) are some arbitrary but fixed reference values of the system, subject to be transformed in the same way as the position \( \mathbf{x} \), the velocity \( \mathbf{x} \), and the evolution coordinate \( t \) of the particle for some constant value, respectively. Since \( \mathbf{F} \) (3.4) transforms as a (contravariant) tensor field of rank 1, the condition for frame-indifference \( \mathbf{F}'(\cdot) = \mathbf{F}(\cdot) \), as it was correspondingly derived in (2.9) for a covariant tensor field, leads then to the following restricting relation for the force

\[
\mathbf{F}(\mathbf{x}', \mathbf{\dot{x}}', t'; \mathbf{x}_0^0, \mathbf{v}_0^0, t_0^0) = \mathbf{R} \mathbf{F}(\mathbf{x}, \mathbf{\dot{x}}, t; \mathbf{x}_0^0, \mathbf{v}_0^0, t_0^0).
\]

(3.6)

To solve such a relation in the most general way is part of a Lie-group classification problem (see e.g. Bihlo et al. (2012); Ibragimov et al. (2016); Kontogiorgis & Sophocleous (2016)).

---

\(^\dagger\)Note that for \( \mathbf{v} \neq \mathbf{0} \) and \( \mathbf{c} \neq \mathbf{0} \) the spatial coordinates \( \mathbf{x}' \) (3.2) themselves do not transform form-invariantly as a tensor; see also Appendix A.

\(^\ddagger\)Under a ‘closed system’ we only understand a system in which the sources of all forces are part of the system, i.e., a system where no external forces occur (Stephani, 2004); see also second footnote on p. 35. Of course, a closed system is only an approximation, because, as correctly stated by Stephani (2004), how far do we have to go to get a really closed system? Is our Galaxy sufficient, or do we have to take the whole universe?

\(^\ddagger\)To note here is that the reference values \( \mathbf{x}_0^0, \mathbf{v}_0^0 \) and \( t_0^0 \) do not represent the initial position \( \mathbf{x}_0 := \mathbf{x}(t_0) \) and velocity \( \mathbf{x}_0 := \mathbf{\dot{x}}(t_0) \) of the particle at the initial time \( t_0 \). In other words these reference values are not characteristics of the particle, but of the system in which this particle evolves. For example, the absolute point \( \mathbf{x}_0^0 \) can be some fixed reference point in the laboratory, \( \mathbf{v}_0^0 \) the absolute velocity with which the experiment may move or be superposed with in that laboratory, and \( t_0^0 \) the absolute time point in the day or in the week at which the experiment is conducted.
most obvious solution of (3.6) is the particular solution
\[
F(x, \dot{x}, t; x_0^r, v_0^r, t_0^r) = f_1 \cdot (x - x_0^r) + f_2 \cdot (\dot{x} - v_0^r),
\]
where the two \(f_i\)'s are arbitrary scalar functions of the general form
\[
f_i = f_i(\|x - x_0^r\|, \|\dot{x} - v_0^r\|, (x - x_0^r)^T \cdot (\dot{x} - v_0^r), t - t_0^r),
\]
which ultimately states that for closed systems the laws of nature do not permit an experimental verification, or a sensible definition, of an absolute location in space and time, or of an absolute direction in space, or, particularly in classical Newtonian mechanics, of an absolute velocity. Only relative distances, time differences, directional differences and relative speeds enter the constitutive relation (3.5).

Specifying the Galilean frame-indifferent constitutive relation (3.7) leads to a particular physical problem. For example, choosing \(f_1 = -\kappa\) as a constant and \(f_2 = 0\), leads in 1D to Hooke’s law, turning Newton’s law of motion (3.1) into the problem of the harmonic oscillator
\[
m\ddot{x} = -\kappa (x - x_0^r),
\]
where the absolute reference point \(x_0^r\) can be for example the suspension point of the spring. As correctly noted by Horzela et al. (1991), it is important to stress here the fact that the individual terms on the right-hand side in (3.9) do not have the meaning of a force.† Only the difference on the right-hand of (3.9) has a physical meaning of a force and it is erroneously to think that the right-hand side is a difference of two forces.

Another example would be to choose \(f_1 = -mg/\|x - x_0^r\|\), where \(g = G \cdot M_E/r_E^2\) is the Earth’s gravitational acceleration on its surface with \(G\) as the universal gravitational constant and \(M_E\) and \(r_E\) as the Earth’s mass and radius, and \(f_2 = a\) as a constant proportional to the drag coefficient of air, then Newton’s law of motion (3.1) turns into the problem of gravitational motion (close to the Earth’s surface, i.e. where \(\|x - x_0^r\| \sim r_E\)) under the influence of air friction
\[
m\ddot{x} = -a(\dot{x} - v_0^r) - mg \frac{x - x_0^r}{\|x - x_0^r\|},
\]
where \(x_0^r\) is the absolute position of the Earth’s center and \(v_0^r\) the absolute air speed (wind) experienced in the system when the particle would be at rest (Stephani, 2004). By construction, the law of motion (3.10) is simultaneously form-invariant (covariant) and frame-indifferent (objective) under Galilei transformations (3.2), even if, e.g., the wind speed would be experienced in one inertial system as zero, i.e. as \(v_0^r = 0\), the law (3.10) remains indifferent since in another inertial system the wind speed may be experienced as non-zero, in particular as \(v_0^r = v\), according to (3.2), such that the velocity difference \(\Delta \dot{x} := \dot{x} - v_0^r\) always remains covariant: \(\Delta \dot{x}^r = R \Delta \dot{x}\).

Obviously, when transforming to a non-inertial system, say by the following Euclidean (time-dependent Galilei) transformation
\[
x^r = R(t) x + c(t), \quad t^r = t + \tau,
\]
the form of (3.10) changes and thus is not frame-indifferent anymore.‡ This change in form then reads
\[
m\ddot{x}^r = -a(\dot{x}^r - v_0^{r^*}) - mg \frac{x^r - x_0^{r^*}}{\|x^r - x_0^{r^*}\|} + F_{\text{Inertial}},
\]

†This misleading interpretation stems from the fact that one is always used to solve the harmonic oscillator in its non-covariant form, namely as \(m\ddot{x} = -\kappa x\), being however only valid in one particular reference frame in which the center (equilibrium point) of the oscillator is at rest (Horzela et al., 1991; Horzela & Kapuščik, 1992).
‡As was already noted before, under pure coordinate transformations frame-indifference (objectivity) implies form-invariance (covariance), but not vice versa. In other words, if a relation is form-invariant then it has not to be also frame-indifferent, or, stated oppositely, if a relation is not form-invariant then it cannot be frame-indifferent.
with the fictitious or inertial force given as

\[ F^*_{\text{Inertial}} = m\ddot{c} - mR\dot{R}^T(x^* - c) - 2mR\dot{R}^T(x^* - \dot{c}) - aR\dot{R}^T(x^* - \dot{c}), \]  

(3.13)

where here, in the non-inertial case, it is important to know that, while the velocity of the particle transforms as

\[ \dot{x}^* = R\dot{x} + \dot{R}\dot{x} + \dot{c}, \]  

(3.14)

the wind speed \( v_0^* \) only transforms as it is known in the inertial case

\[ v_0^{r*} = R v_0^* + \dot{c}, \]  

(3.15)

simply because the wind speed in an inertial system \( v_0^* \) is a constant vector, where its components in any new frame are then just rotated by \( R \) and translated by \( \dot{c} \), in contrast to the particle’s velocity \( \dot{x} \), which is a differential quantity and thus needs to be treated differently.

Due to this different non-inertial transformation behavior between (3.14) and (3.15) we thus obtain the following important result that should be kept in mind for the discussion in the next sections: When transforming to a non-inertial system, it is not only the dynamic part \( \ddot{x} \) in Newton’s second law (3.1) that gives rise to fictitious or inertial forces, but also the constitutive relation for the force \( F \) itself may give frame-dependent contributions in a very natural way. In the particular case (3.10), it is the velocity difference \( a(\dot{x} - v_0^*) \) of the specified constitutive relation (3.7) that gives rise to an own contribution \( aR\dot{R}^T(x^* - c) \) in the resulting inertial force (3.13). In other words, although the constitutive relation (3.7) itself is manifestly frame-indifferent for inertial systems by construction, it nevertheless picks up frame dependent terms when changing to a non-inertial system.

4. A comprehensive overview on covariance and objectivity in physics

A view into the literature reveals that when posing form-invariance (covariance) or frame-indifference (objectivity) as a principle of nature they both have been praised and criticized. Let me give a detailed overview on this controversial situation in each case.

4.1. The criticism on form-invariance as a principle of nature

When Einstein formulated his general theory of relativity in 1915, he was proud to present a theory that was generally form-invariant, or as how he first called it, a theory that was generally covariant: Its equations retained their form under arbitrary transformations of the space-time coordinate system. Einstein had the following argument for general covariance: the physical content of a theory is exhausted by a catalog of events, which must be preserved under arbitrary coordinate transformations; all we do in coordinate transformations is just relabelling the space-time coordinates assigned to each event. Therefore a physical theory should be generally covariant. Einstein thus claimed that his general theory of relativity rests on three physical pillars: on the principle of general covariance, on the principle of constancy in the speed of light for all local inertial reference frames, and on the principle of equivalence between inertial and gravitational mass; only when taking all three principles together, the theory of general relativity arrives at a new description of gravitation in terms of a curved four-dimensional space-time.

Shortly after, Kretschmann (1917) pointed out and concurred in by Einstein (1918a) that the principle of general covariance is fully devoid of any physical content. For, Kretschmann urged, it is unessential to declare general covariance as a principle of nature, since any space-time theory whatever can be formulated in a generally covariant form as long as one is prepared to put sufficient energy into the task of reformulating it; thus the theory of general relativity rests on

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†This section is an excerpt from Sec. 8.1.2 in Frewer (2009a).
only two pillars, on the principle of a constant speed of light for all observers in their immediate
neighbourhood and on the mass equivalence principle. In arriving at general relativity, Einstein
had used the tensor calculus of Ricci and Levi-Civita, where Kretschmann pointed to this cal-
culus as a mathematical tool that made the task of finding generally covariant formulations of
theories tractable. In his objection, Kretschmann agreed that the physical content of space-time
theories is exhausted by a catalog of events and that they should be preserved under any co-
ordinate transformation, but this, he argued, is no peculiarity of the new gravitational theory
presented by Einstein. For this very reason all space-time theories can be given generally covari-
ant formulations. For a further discussion of Kretschmann’s objection, Einstein’s response and
of the still active debate that follows, see the articles of Norton (1993, 1995) and Dieks (2006).

Kretschmann’s objection that general covariance is physically vacuous, in that it does not
limit or restrict the range of acceptable theories, is really a non-trivial objection when it comes
down to constructing and formulating ‘new’ physical theories, as it was at that time in 1917 for
the theory of general relativity. However, for already existing physical theories his objection is
more or less obvious from a pure theoretical point of view, since at the end, only a new mathe-
matical representation is given for the theory. But this does not mean that it is always an easy
task to put any given theory into a generally covariant form; sometimes it’s a challenge to the
mathematician’s or physicist’s ingenuity.

Even to the classical laws of fluid motion and to the physics of turbulence modelling,
Kretschmann’s objection does seem sustainable. In Frewer (2009a) it is shown how it is possible
to rewrite the Navier-Stokes equations as well as the Reynolds averaged equations in a general
manifest form-invariant way without changing the physical content of the theory. For that,
however, one has to change to a four-dimensional space-time formulation.†

Hence, fully in accord with Kretschmann’s objection, we see that no new physics is implied
when reformulating the existing laws of classical fluid motion into their generally form-invariant
form. This insight inevitably brings about the question of what the necessity or even the
advantage is when rewriting existing laws into their general form-invariant representation? The
answer surely depends on what one intends to do. If for example the equations are to be solved
numerically or even analytically such reformulations do not bring any advantages at all, but if for
example the equations are not closed and need to be modelled, e.g., as the constitutive equations
for materials or the equations of turbulence, such form-invariant reformulations automatically
bring along consistent and structured modelling arguments in the most natural way when they
are based on invariant principles (Frewer, 2009b,c; Ariki, 2015; Ván, 2015; Panicaud et al., 2016).

4.2. The criticism on frame-indifference as a principle of nature

The idea to pose frame-indifference as a principle of nature has its formal roots in rational
continuum mechanics as introduced by Truesdell & Noll (1965), in particular from the question
how to systematically model constitutive equations for materials in solid and fluid mechanics.‡
When put as a general principle to model the response of a material, it states that only frame-
indifferent (objective) terms should enter the constitutive equation, independent of whether an
inertial or non-inertial system is considered. Hence in the community of continuum mechanics
this principle is also coined as the principle of material frame-indifference (MFI). Whether or not
MFI deserves the status of a principle in mechanics has been subject of heated debates (Frewer,
2009a), and there still is a need to clarify this issue (Dafalias, 2011; Muschik, 2012a,b,c, 2013;
Romano & Barretta, 2013; Liu & Sampaio, 2014; Kirwan, 2016; Yang et al., 2016).

†For the specific case of Euclidean (time-dependent Galilei) coordinate transformations, Sadiki & Hutter (1996)
have already shown earlier how it is possible within a 3D framework to equivalently reformulate the (unclosed)
turbulent balance equations of classical continuum mechanics into a manifest form-invariant set of relations.
‡Historically, Hooke in 17th and Poisson and Cauchy in the 19th century were the first to be inspired by their
experiments that the response of materials can show frame-indifferent behavior. For more details on this account,
see e.g. Sec. 2 in Frewer (2009a) and the references therein.
For the discussion on MFI it is important to keep in mind that if the constitutive equations for the considered material are modelled explicitly as frame-indifferent, then this does not imply that the dynamics of this material in space and time evolves frame-indifferently too, since in general the dynamical equations themselves are not free from frame-dependent (non-inertial) terms.

Constitutive relations are of a different physical nature than the field equations of motion into which they are inserted to obtain a dynamically closed system. The former are modeling statements to describe the response of a material to its intended environment, while the latter constitute specific dynamical rules (driven by conservation or balance laws) that determine the motion of this material in this environment. For that difference, it can well be that a material’s response is frame-indifferent (objective), while at the same time its dynamics evolves frame-dependently (non-objectively). Hence, from the outset, it is not unreasonable to assume MFI as a systematic modelling restriction for constitutive relations.

The first controversy on MFI, however, was raised by Müller (1972). By taking into account the microscopic description of the material based on the kinetic theory of Boltzmann, Müller argued that this theory does not support MFI, according to which constitutive functions must be generally frame-independent in all reference systems, including the non-inertial systems. In particular, Müller concluded that MFI cannot be regarded as an absolute principle. It rather has to be seen only as an approximative principle (Müller, 1972, 1976): Depending on the numerical ratio of the microscopic time scale $t_\mu$ (defined proportional to the mean free path length of the internal colliding constituents) of the material and the macroscopic time scale $t_M$ of the non-inertial frame environment, one can systematically determine and decide whether MFI gives a good or a poor approximation for the material within the environment considered. If there is a clear-cut separation of time scales the decision is easy: a sufficiently small ratio $t_\mu/t_M \ll 1$ certifies a very good approximation, while a very large ratio $t_\mu/t_M \gg 1$ a poor approximation. For solids and ordinary dense fluids the MFI-principle is a valid principle to model their constitutive relations (of course only as long as the macroscopic time scale $t_M$ of the non-inertial frame environment does not approach a relativistic regime). This, however, is no longer the case for materials if the characteristic size of their microstructure gets bigger as in suspensions or polymer solutions, or if the mean free path length in the microprocesses gets longer as in gases and rarefied gases. In all these cases MFI may not be used as a guiding modelling principle anymore.

However, with the subsequent development of extended thermodynamics as a new systematic procedure to model constitutive equations, new insight is provided into the principle of MFI. According to Müller, the frame sensitivity of the stress and heat flux discussed by him in Müller (1972, 1976) can now ultimately be explained on a pure macroscopic level without the need to delve into a microscopic description of the problem (Müller & Liu, 1983): It is only the natural frame dependence of the basic balance equations for stress and heat flux which solely give rise to the observed frame-dependence, not the constitutive theory itself, which, according to him, still is frame-indifferent. In other words, the microscopic analysis performed in Müller (1972, 1976), particularly on the time scale ratio $t_\mu/t_M$, only measures the effect and the relative strength of frame-dependence in its entirety, but does not make any statements from where this frame-dependence has its origin.

So what happened? On the one side Müller rejects MFI (Müller, 1972, 1976), while on the other side he later agrees on it (Müller & Liu, 1983). The answer depends on how a constitutive equation or relation is defined. In classical or ordinary thermodynamics, when based on the (unclosed) hierarchy of moments of kinetic theory, the constitutive relations are identified as fluxes in an iterative Maxwell scheme† to close a fixed set of balance equations, as was done in

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†As noted in Müller (1972), this naming goes back to Ikenberry & Truesdell (1956) who invented this iterative scheme, which they named after Maxwell, since Maxwell had sketched the beginning of it: In an iterative manner this scheme determines the constitutive relation between the fluxes and the densities (the field variables) of the balance equations within some microscopic interaction model (mostly Maxwellian molecules) for the underlying probability density function obeying the Boltzmann equation.
Müller (1972, 1976), while in extended thermodynamics the constitutive relations are identified in a first step as densities satisfying their own balance equations, extending thus the fixed set of balance equations to higher order, and then, in a second step, again as Maxwell iterates in order to establish back a relationship to the constitutive relations of ordinary thermodynamics by approximating the higher-order balance equations near the equilibrium state of the fields. Hence, according to Müller, the procedure of extended thermodynamics allows to “see this violation of material frame-indifference in a new light” (Müller & Liu, 1983, p. 330). As defined and briefly reviewed in Appendix D, classical thermodynamics operates (after a second order Maxwell iteration) with constitutive relations of the form

$$F = F^{(c)}(\phi, \nabla \phi, \nabla^2 \phi, \Gamma),$$

(4.1)

where $\phi$ are the densities (mostly mass, momentum and internal energy) and $F$ the fluxes (mostly stress and heat) of the considered system. The parameters $\Gamma$ collectively denote all possible frame-dependent quantities which arise as soon as the considered system is transformed into a new one via a coordinate transformation (see Appendix D). Hence, the parameters $\Gamma$ appearing in the material law (4.1) ultimately characterize the state of the material if observed from a new frame of reference or if transformed into a new frame of state. Extended thermodynamics, however, operates with higher-order constitutive relations of the form

$$J = J^{(c)}(\phi, F), \quad \Xi = \Xi^{(c)}(\phi, F),$$

(4.2)

where the higher-order or so-called extended fluxes $J$ then depend on the densities $\phi$ and the fluxes $F$ of classical thermodynamics, but at the price of gaining new unclosed production terms $\Xi$ in the balance laws for $J$. In contrast now to classical thermodynamics, the modelling procedure in extended thermodynamics only allows for local and frame-indifferent constitutive relations as shown in (4.2); gradients and frame-dependent parameters do not enter the higher-order constitutive relations (4.2) as in (4.1) for classical thermodynamics.

The standard argument for this difference in frame dependency for the two types of constitutive relations is that the classical non-local ones (4.1) are not true constitutive equations but only approximations of balance laws, and hence do not need to satisfy MFI, while the local ones (4.2) are true constitutive relations that have to obey MFI. In the words of Ruggeri & Sugiyama (2015), p. 360: “Extended thermodynamics seems to indicate in clear manner that nonlocal relations are not constitutive equations but approximations of balance laws. The true constitutive equations are in local form and they obey the material frame difference”. However, this argument of extended thermodynamics, favored by Müller & Ruggeri (1998), Liu & Sampaio (2014) and Ruggeri & Sugiyama (2015), namely to push MFI in its general statement as an axiom of nature for local constitutive equations, is not convincing when considering the following three aspects:

(i) As explained and discussed in detail by Muschik (2012a, b, 2013), any constitutive law, does not matter if local or non-local, will pick up frame dependent terms as soon as the balance equations are formulated or transformed to non-inertial systems. That means for any non-inertial system all constitutive relations in nature inherently show explicit frame dependence through a collection of parameters characterizing this particular non-inertial system, which we denoted as $\Gamma$ in (4.1) and which Muschik (2012a, b, 2013) calls the “second entry” in constitutive mappings to correctly describe the motion of the material within any non-inertial system. In particular, this “second entry” emerges independently of whether the material is passively observed from a non-inertial system or whether the material itself is actively transformed into a non-inertial system.

However, as also further correctly addressed by Muschik (2012a, b, 2013), in order to ensure physical consistency of this non-inertial frame dependency $\Gamma$, all balance laws and all constitutive relations must be formulated form-invariantly as true geometrical tensor relations, in fact for all physically realizable frames, independent of whether an inertial or non-inertial one is considered.
Only this will ensure that the motion of the material is described by the same laws for each observer. In this respect, it is important to note that the notions “objective”, “tensor”, “observer-independence” and “observer-invariance” are all used as synonyms in Muschik (2012a, b, 2013) to declare the general notion of form-invariance, implying that all observers will see the same laws of physics, i.e., where no observer is distinguished, not even the inertial one.

(ii) In order to satisfy MFI under Euclidean (time-dependent Galilei) transformations, the local constitutive relations (4.2) of extended thermodynamics are standardly restricted to be isotropic functions of their variables and not to depend on the velocity field (Müller & Ruggeri, 1998, p. 58). The latter restriction, however, is too strong as it would unnecessarily exclude any velocity field dependence per se. As frame-indifference restrictions under Galilei transformations for inertial systems already show, it is not necessary to exclude the velocity entirely, but that, instead, it is fully sufficient to only consider velocity differences relative to some fixed reference velocity of the system (see Section 3, in particular Eq. (3.7)). When transforming to a non-inertial system, then it is this velocity difference in the constitutive relations for example which will give raise to frame-dependent terms as shown in the last contribution of Eq. (3.13) and as explained above in (i), independent of whether a non-local or local constitutive relation is considered, since in continuum mechanics the velocity field is defined as a local (non-gradient) quantity and thus may enter both constitutive relations, but, as said, as a velocity difference only. Hence, to state that “in extended thermodynamics with its local and instantaneous constitutive functions material frame indifference under [non-inertial] Euclidean transformations provides no more restrictive conditions than material frame indifference under [inertial] Galilean transformations does” (Müller & Ruggeri, 1998, p. 73), is not correct.

(iii) In all of this discussion on MFI, one always has to keep in mind that when regarding frame-indifference, i.e., when regarding true objectivity and not the weaker form-invariance, there only exists an equivalence principle for inertial frames in nature, but not for non-inertial ones. In other words, frame-indifference for physical laws can only be demanded for inertial systems, a physical principle which applies without exception. In this sense the inertial frame is distinguished from all other frames: While the weaker principle of form-invariance has to apply for all physically realizable frames, i.e., for inertial as well as non-inertial ones, the stronger principle of frame-indifference (true objectivity), however, may only be applied to inertial frames and not to non-inertial ones. How inertial frame indifference expresses itself or how it should be enforced in physical models will be discussed in the next section.

It is still a mystery of nature why the inertial frame is particularly distinguished from all other frames. For example, the distinction between an inertial and an accelerated system is not yet fully understood. The question remains of relative to what an inertial system is not accelerated. Only to say that it moves with a constant velocity relative to a sky of fixed stars is certainly not precise enough. This discussion was first introduced by Mach (1883) and is still an open discussion today (Friedman, 1983; Barbour & Pfister, 1995; Barbour, 2004; Penrose, 2005; Pfister & King, 2015). Mach’s ideas cumulated to Mach’s Principle, a name given by Einstein which served him as a guiding criterion to develop his general theory of relativity. Today we understand Mach’s principle as a hypothesis that all mass of the universe determines the structure and behavior of an inertial system. The question, however, in how far this principle is incorporated or reflected by Einstein’s theory is still open.

† In reality the inertial frame is only an approximation, whereby the error, of course, can be very small. For example, on Earth a true inertial system cannot be established or generated due to its own rotation, in planetary range outside Earth also not since our Solar System rotates around the center of the Milky Way Galaxy, and so on. Nevertheless, one cannot exclude that our universe exhibits some local inertial regions in deep space that are close to perfection.
4.3. Formulating a universal axiom for invariant modelling

Accounting for all three aspects mentioned above, we can state for general continuum mechanics the following **reduced axiom of Material Frame-Indifference (r-MFI)**:†

(I) **Regarding form-invariance (covariance)**, all balance equations and all constitutive relations have to be formulated as tensor relations for all physically realizable frames in order to satisfy the general principle of form-invariance so that no observer is distinguished, not even the inertial one. The most natural framework to achieve this is to formulate all physical laws in a 4D space-time setting. How to stay in the realm of classical Newtonian physics, two approaches exist: Either by formulating directly a 4D Newtonian manifold endowed with a time-like and a space-like metric (Havas, 1964; Matolcsi & Ván, 2007; Frewer, 2009a,b,c; Ván, 2015), or indirectly by using Einstein’s space-time theory in the classical limit of small velocities with respect to the speed of light (Rouhaud et al., 2013; Panicaud et al., 2016).

(II) **Regarding frame-indifference (objectivity)**, the inertial frame is distinguished from all other frames in physics since only for these a general equivalence principle exists: All inertial frames are indistinguishable. The connecting coordinate transformations are the Galilei transformations in classical and the Lorentz transformations in relativistic continuum mechanics. For balance equations and constitutive relations this equivalence principle expresses itself differently:

(a) Balance equations are to be formulated frame-indifferently under Galilei or Lorentz transformations only for closed systems, i.e., if no external body forces act on the considered thermodynamic system, since such forces may break the space-time symmetries necessary for Galilean or Lorentzian invariance. Practically this is achieved in three steps, first by deactivating all external forces in the balance laws by putting them formally to zero, then, in the second step, restricting these resulting balance laws to Galilean frame-indifference, and then, in the last step, to re-activate all external forces back to their original form again.

(b) The constitutive relations will automatically be restricted to Galilean frame-indifference as soon as the above process (a) is initiated. As a consequence, the constitutive equations will not depend on any absolute locations in space and time, nor on any absolute directions in space nor on any absolute velocities. If modelled as such, only relative distances, times differences, directional differences and relative velocities enter the constitutive relations (see Section 3), independent of whether the modelling concept of classical or extended thermodynamics is employed. Of course, the latter concept is superior to the former one in that it not only adopts more independent variables by incorporating non-equilibrium variables such as viscous stress and heat flux into the theory, but also by modelling the constitutive relations locally and instantaneously without gradients, having the twofold effect then that on the one side the theory is governed by hyperbolic field equations allowing only for finite speeds of propagations, and, on the other side, having simple access to the principle of frame-indifference for closed inertial systems.

Hence a thermodynamical system which is formulated according to the above r-MFI principle will involve constitutive equations that contain information about the motion of the material (Muschik, 2012b, 2013), irrespective of whether this motion is induced by observing the material passively from a new frame of reference or by actively transforming the material into a new frame of state. However, this frame-dependence or frame-sensitivity of the material will only emerge when changing to a non-inertial system, since for all inertial systems the material behaves frame-

†The add-on ‘reduced’ refers here to the fact that MFI, regarding its ingredient ‘frame-indifference’, is only considered as a general principle for inertial Galilean transformations, and not for the non-inertial Euclidean transformations as-standardly and incorrectly assumed in the community of continuum mechanics. Of course, assuming MFI for Euclidean transformations can be a good approximation for some practical problems in engineering mechanics, since mostly in these cases the influence of the material’s motion (either induced through an active or passive coordinate transformation) on the constitutive equation can be neglected (Muschik, 2012a).
indifferently by construction. In other words, all constitutive equations, irrespective of whether local or non-local, will pick up frame-dependent terms as soon as one changes to a non-inertial system, reflecting essentially the material’s sensitivity upon acceleration. In Muschik (2012b) this acceleration-sensitivity of materials is regulated by the MMD-axiom, the axiom of ‘Material Motion Dependence’.

To close this section, it is worthwhile to mention the note of Muschik (2012a,b), that the existence of acceleration-sensitive materials is well known in physics. A famous example is the Barnett-effect (Barnett, 1915), where upon rotation a magnetization of an uncharged body is induced;† irrespective of whether the body is passively observed from a rotating frame or if actively put into rotation, the magnetization is always proportional to the angular velocity and thus parallel to the angular velocity of the body (Matsuo et al., 2015). Hence, it may well be that the Barnett-effect gives a relevant contribution towards the Earth’s magnetic field due to being in a state of constant rotation.

4.4. On form-invariance and frame-indifference in turbulence modelling

Using the concepts of form-invariance and frame-indifference in turbulence modelling has its motivation from MFI when interpreting the unclosed and modelled turbulent flow quantities as constitutive relations which are necessary to close the underlying dynamical equations of motion. Despite this similarity, it is clear that physically one has to distinguish between modelling a turbulent flow and modelling constitutive equations of a continuous material: The closure problem in turbulence modelling is a pure dynamical problem which originates from the lack of knowledge in quantifying the complex flow behavior of a fluid material, and not from the lack of knowledge concerning the structure and behavior of the material itself. Yet, there is a common link between these two entities, as correctly stated by Dafalias (2011): Both must be valid for any motion and in regard to any frame, if they are to be something more than just a convenient fitting for the solution of particular problems associated with particular motions and particular frames. It is exactly this quest of validity for “any motion and any frame” that makes it possible to impose objectivity requirements and, as a result, draw conclusions for the appropriate analytical representation of either constitutive or dynamical relations.

However, for the same reasons as stated before in the previous section, it is clear that not MFI, but only r-MFI can apply to turbulence modelling. But for turbulence modelling this limitation is a stringent condition, in contrast to material modelling where MFI can still be used as a reasonable approximation if the influence of the material’s motion on its constitutive equation can be neglected (Muschik, 2012a). An indicator for frame sensitivity in general is the time-scale ratio $t_\mu/t_M$ introduced in Section 4.2, to compare the microscopic time scale of the material’s constituents with the macroscopic time scale of its frame environment. Although this ratio measures frame sensitivity only as a whole, not distinguishing between the frame-dependence originating from the balance laws or from the constitutive relations, it nevertheless is a global indicator of whether we face a weak ($t_\mu/t_M \ll 1$) or a strong ($t_\mu/t_M \gg 1$) dependence of the material on the frame.

Now, when applying this indicator to turbulent motion of a fluid, Lumley (1970, 1983) correctly pointed out that a general principle of frame-indifference, as stated by MFI, cannot be utilized for turbulence modelling, not even in an approximative sense as it can be done in rational continuum mechanics when modelling the response of a material. The reason is simply the huge spectrum of (microscopic) time-scales within a turbulent flow where there always exists a band of scales which are in the (macroscopic) time-scale range of the underlying non-inertial frame of reference. In other words, in a turbulent flow there is no clear-cut separation of microscopic and macroscopic time-scales, as it would be the case, for example, in material modelling when considering a dense material within a certain slow changing non-inertial frame environment,

†Obviously the Barnett-effect can only be observed for magnetizable bodies, e.g. made of iron or nickel.
where obviously, due to $t_\mu/t_M \ll 1$, frame-indifference as stated by MFI can be used as a reasonable guiding principle to model such materials in such environments.

Hence, since MFI may not be used as a guiding principle to model turbulent flows, not even in an approximative sense, makes the role of r-MFI in turbulent flows even more explicit. But as defined and formulated in the previous section, is r-MFI really a sufficient principle to account for all invariant aspects of a turbulent flow? Particularly in view of the fact that in contrast to material modelling, turbulence is based on an underlying deterministic description where the direct equations of motion are known exactly — in classical fluid mechanics, these are the Navier-Stokes equations, where, in the following, we will only consider the incompressible approximation.

It is clear that r-MFI forms the basis as a principle, providing a rule of procedure that strictly applies to any modelling ansatz in physics, without exception. But, particularly for turbulence modelling, where there is a direct link down to the underlying and exactly known microscopic (fine-grained) equations of motion, the central question will be, if r-MFI can be extended with regard to its second part (II), the frame-indifference (objectivity) principle, to include more symmetries than just the Galilean (or Lorentzian) symmetry? The answer is yes and no, which will be discussed next.

5. Supplementing r-MFI to model incompressible turbulent flow

Although the study of Speziale (1998) shows the serious drawback that it can be misunderstood to due a systematic confusion of the concepts ‘form-invariance’ and ‘frame-indifference’,\ † it still marks the key study in how turbulence should be modelled when based on invariant principles. Basically this study addresses two points: Firstly, MFI under 3D Euclidean coordinate transformations, as generally stated in rational continuum mechanics, does not apply to turbulence, and secondly, in order to model turbulence according to the principles of invariance correctly, it is necessary to also include the fine-grained (microscopic or fluctuating) description of turbulence, saying that it is not expedient to focus solely on the coarse-grained (macroscopic or mean) description of turbulence.

Speziale is thus in accordance with the criticism of Lumley (1970, 1983) as it was discussed in the previous section,‡ yet he goes beyond this criticism in asking, if not MFI, what then are the correct invariance principles for turbulence? The solution to this problem is to look at the symmetry properties of the transport equations for the fluctuating variables, since the unclosed terms are just an averaged multiplicative combination of these, or in his words: “Since closure relations are built up from solutions of the fluctuation dynamics, we must look at its invariance groups” (Speziale, 1998, p. 498).

\[ \tau = Q \tau Q^T \] (Speziale, 1998, Eqs. (70)-(74)). However, this just shows form-invariance (covariance) of the Reynolds-stress $\tau$, and not its frame-indifference (objectivity). Hence Speziale’s confusing statement has to be correctly read as: “The fact that the Reynolds-stress is a tensor, i.e. form-invariant, does not establish the validity of Material Frame-Indifference”; a statement which now turns out to be a trivial statement from the viewpoint of the notions defined herein in this study (see e.g. second footnote on p. 12).

\[ \text{For example, Speziale leaves the reader with the following confusing statement: “The fact that the Reynolds stress tensor is a frame-indifferent tensor does not establish the validity of Material Frame-Indifference” (Speziale, 1998, p. 498); a statement which unfortunately generated misguidance as for example in study of Dafalias (2011). The problem is that Speziale understands ‘form-invariance’ as ‘frame-indifference’, by showing that under arbitrary 3D Euclidean (non-inertial) transformations the Reynolds-stress transforms as a tensor of rank 2, namely as } \tau = Q \tau Q^T \] (Speziale, 1998, Eqs. (70)-(74)). However, this just shows form-invariance (covariance) of the Reynolds-stress $\tau$, and not its frame-indifference (objectivity). Hence Speziale’s confusing statement has to be correctly read as: “The fact that the Reynolds-stress is a tensor, i.e. form-invariant, does not establish the validity of Material Frame-Indifference”; a statement which now turns out to be a trivial statement from the viewpoint of the notions defined herein in this study (see e.g. second footnote on p. 12).

\[ \text{Speziale’s work is to be separated into two classes, those studies concerning material modelling where he favors MFI as a valid modelling principle in its full 3D original formulation (Speziale, 1984, 1987, 1998), and those studies concerning turbulence where he denies it (Speziale, 1989, 1990, 1998). It is not true that Speziale favored full MFI in turbulence modelling before writing his last review (Speziale, 1998) and comment (Spalart & Speziale, 1999), except for his very first study on this issue (Speziale, 1979), where took a contrary position. He always stated that MFI in turbulence cannot be valid in 3D, but only in 2D under certain restrictions. However, when excluding 3D rotation, he constantly supplemented his statement by the argument that, nevertheless, in the specific case of linear accelerations a 3D-MFI for turbulence can be formulated according to an extended set of Galilei transformations.} \]
However, in this step caution has to be exercised, because in order to avoid any wrong interpretations and conclusions from the outset, it proves to be useful not to analyze the pure fluctuating but rather the full instantaneous equations for symmetries, which include both the mean and fluctuating dynamics of the flow. The simple reason is that if the symmetry analysis is not carried out careful enough by providing and revealing all information originating from the pure fluctuating dynamics, the governing equations will turn into an unclosed system due to the appearance of an infinite hierarchy of statistical equations triggered by the mean velocity field. A condition which is obtrusive to any symmetry analysis, because, as explicitly shown e.g. in Frewer (2015a,b); Khujadze & Frewer (2016); Frewer & Khujadze (2016c), to determine the symmetries for such an unclosed and infinite equational system is basically ill-defined: If the equations are unclosed, so are their symmetries.\footnote{Note that to use the word ‘symmetry’ in the context of unclosed systems is mathematically incorrect: Any invariant transformation admitted by an unclosed system of equations can only be identified as an equivalence transformation and not as a true symmetry transformation; see e.g. Frewer et al. (2014a) and the references therein.}

Performing a symmetry analysis on such unclosed and infinite systems has the negative consequence that one is not only faced with complete arbitrariness in generating symmetries, but that also unphysical symmetries in this process can be generated which violate the classical principle of cause and effect (Frewer et al., 2014a, 2015a, 2016a,b; Frewer & Khujadze, 2016a).\footnote{To the peer-reviewed publications (Frewer et al., 2014a, 2015a, 2016b; Frewer & Khujadze, 2016a), please also see the reactions (Frewer et al., 2014b, 2015b, 2016c; Frewer & Khujadze, 2016b), respectively.}

Now, in order to extend the second part (II) of the axiom r-MFI, one has to determine all invertible coordinate transformations (within Newtonian physics)

\begin{equation}
\hat{t} = t + c_0, \quad \hat{x} = \hat{x}(x, t),
\end{equation}

that leave the incompressible Navier-Stokes equations

\begin{equation}
\nabla \cdot \mathbf{u} = 0, \quad \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u},
\end{equation}

frame-indifferent. Important to note here is that for pure spatial (time-independent) transformations \(\hat{x} = \hat{x}(x)\), equations (5.2) are 3D-form-invariant if all spatial operators are identified as 3D covariant derivatives.\footnote{See Appendix B for a general definition of the covariant derivative (B.6), where the Greek 4D indices have to be replaced by the Latin 3D indices in order to obtain a 3D spatial covariant derivative. For example, in (5.2) the 3D-form-invariant continuity equation then has the explicit componental form

\begin{equation}
0 = \nabla \cdot \mathbf{u} = \nabla_i u^i = \partial_i u^i + \Gamma^i_{jk} u^k = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^j} \left( \sqrt{|g|} u^i \right),
\end{equation}

where \(\Gamma^i_{jk}\) is the affine connection and \(g = \det(g_{ij})\) the determinant of the metric tensor \(g_{ij}\) of the considered 3D spatial manifold.} However, a consistent 3D tensor relation of \((\nabla p)^i = \nabla^i p = g^{ij} \nabla_j p\), where \(g^{ij}\) are the contravariant coefficients of the metric tensor with the standard defining property \(g^{ik} g_{kj} = \delta^i_j\), and, of course, the convective term as \(\mathbf{u} \cdot \nabla \mathbf{u} \equiv (\mathbf{u}^T \cdot \nabla^T) \mathbf{u}\).

Now, in order to extend the second part (II) of the axiom r-MFI, one has to determine all invertible coordinate transformations (within Newtonian physics)
if these two fields are known in the initial frame, then (5.4) represents the explicit construction rule for these fields in the transformed frame

\[
\begin{align*}
\hat{\mathbf{u}}(\hat{x}, \hat{t}) & \equiv \frac{\partial \hat{x}}{\partial x} \bigg|_{x = x(\hat{x}, \hat{t}); \ t = \hat{t} - c_0} \cdot \mathbf{u}(\hat{x}, \hat{t}, \hat{t} - c_0) + \frac{\partial \hat{x}}{\partial t} \bigg|_{x = x(\hat{x}, \hat{t}); \ t = \hat{t} - c_0}, \\
\hat{p}(\hat{x}, \hat{t}) & \equiv p(x(\hat{x}, \hat{t}), \hat{t} - c_0).
\end{align*}
\]

(5.5)

and

\[p(\hat{x}, \hat{t}) \equiv p(x(\hat{x}, \hat{t}), \hat{t} - c_0).
\]

(5.6)

The reason why we only look at pure coordinate transformations (5.1) when trying to extend axiom r-MFI, is that it’s a principle that by construction only relates to coordinate transformations, in particular since r-MFI is based on the concept of tensors, which is a well-defined concept only for coordinate transformations concerning the change of a considered reference frame.

For simplicity, we will formulate the symmetry results for the Navier-Stokes equations (5.2) only in 3D Cartesian (rectilinear) coordinates, i.e., in the following only for \(g_{ij} = \delta_{ij}\). It is clear that for all other spatial (curvilinear) coordinate transformations the symmetry property of the Navier-Stokes equations remains unchanged, since a pure spatial coordinate transformation in 3D only results into a pure relabelling action of the coordinates, not changing the physics of the Navier-Stokes equations (5.2), due to being formulated as a 3D-form-invariant tensor relation. Hence, each symmetry transformation that will be listed below will represent an independent symmetry of the Navier-Stokes system modulo (up to) all its relabelling symmetries.

Performing a systematic Lie-group symmetry analysis on the incompressible Navier-Stokes equations (5.2), as first done by Danilov (1967), Bytev (1972) and Pukhnachev (1972), reveals that the only set of continuous coordinate transformations admitted as a symmetry are the Galilei transformations

\[\mathbf{G} : \ \tilde{t} = t + c_0, \quad \tilde{x} = \mathbf{A}x + c_1 t + c_2 \ \Rightarrow \ \tilde{\mathbf{u}} = \mathbf{A} \mathbf{u} + c_1, \quad \tilde{p} = p, \]

(5.7)

where \(\mathbf{A}\) is a constant rotation matrix and \(c_0, c_1, c_2\) arbitrary constants. The two remaining continuous symmetries of the 3D Navier-Stokes equations

\[\begin{align*}
\mathbf{S}_1 : \ \tilde{t} & = e^{2\varepsilon} t, \quad \tilde{x} = e^\varepsilon x, \quad \tilde{\mathbf{u}} = e^{-\varepsilon} \mathbf{u}, \quad \tilde{p} = e^{-2\varepsilon} p, \\
\mathbf{S}_2 : \ \tilde{t} & = t, \quad \tilde{x} = x + \mathbf{f}(t), \quad \tilde{\mathbf{u}} = \mathbf{u} + \dot{\mathbf{f}}(t), \quad \tilde{p} = p - x^T \cdot \dot{\mathbf{f}}(t) + g(t), \quad \text{for } \dot{\mathbf{f}}(t) \neq 0.
\end{align*}\]

(5.8)

are no coordinate transformations, since the fields do not transform as (5.4). In particular for the so-called ‘extended Galilei transformation’ \(\mathbf{S}_2\), the pressure is not transforming as a scalar function, i.e., not invariantly and thus generating a frame-dependence in the solution of the equations: If the pressure is (locally) zero in one frame \(p = 0\), then it is not zero in another frame \(\tilde{p} \neq 0\), and vice versa, thus distinguishing (locally at a certain point) that frame where the pressure is zero from all other frames where it is not zero.\(^1\) It is thus misleading to call \(\mathbf{S}_2\) an extension of the Galilei transformation \(\mathbf{G}\) (5.7), as standardly done in the community of fluid mechanics (see e.g. Speziale (1998)), because \(\mathbf{S}_2\) is physically of a different nature than \(\mathbf{G}\), being a coordinate transformation while \(\mathbf{S}_2\) not.

It is exactly in this sense that r-MFI cannot be extended to include more symmetries than the Galilei transformation (G), since all other continuous symmetries of the Navier-Stokes equations (\(\mathbf{S}_1, \mathbf{S}_2\)) are no coordinate transformations for which r-MFI applies. But the symmetries (5.8) cannot be ignored, since they reflect a characteristic physical behavior of the Navier-Stokes equations: \(\mathbf{S}_1\) ensures the correct scaling of all physical dimensions involved in the flow, while \(\mathbf{S}_2\) is a re-gauging of the pressure field when changing to a linearly accelerated frame of reference. Hence, when modelling turbulent flow according to the incompressible Navier-Stokes equations, the axiom r-MFI needs to be supplemented by a second, independent axiom, to be called:

\(^1\)Note that, in contrast to the pressure field \(p\), the velocity field \(\mathbf{u}\) in \(\mathbf{S}_2\) transforms form-invariantly as (5.4); its form-invariant tensor structure, however, can only be explicitly seen in a 4D formulation (Freuer, 2009a,b,c).
Turbulent Frame-Indifference (TFI): All turbulence models have to be consistent with the invariant properties of the Navier-Stokes equations (modulo relabelling). In particular, besides the principle of r-MFI, all turbulence models have to respect the two continuous symmetries (5.8), along with the two further discrete symmetries† of the 3D Navier-Stokes equations (5.2)

\[\begin{align*}
S_3: & \quad \tilde{t} = t, \quad \tilde{x}^i = -x^i, \quad \tilde{v}^i = -v^i, \quad \tilde{\rho} = \rho, \quad \text{with } j \neq i,
S_4: & \quad \tilde{t} = -t, \quad \tilde{x} = x, \quad \tilde{u} = -u, \quad \tilde{\rho} = \rho, \quad \tilde{\nu} = -\nu.
\end{align*}\]

Additional symmetries may also be utilized for turbulence modelling, but only in an approximative sense. For example, if in the turbulent flow domain there is a (localized) region where the influence of viscosity on the flow is negligibly small, then, independent from \(S_1\), a second, but approximative scaling symmetry can be considered as a modelling restriction, namely the additional 3D scaling symmetry admitted exactly only by the inviscid \((\nu = 0)\) Euler equations

\[\begin{align*}
S_3^{\text{(approx.)}}: & \quad \tilde{t} = t, \quad \tilde{x} = e^a x, \quad \tilde{u} = e^a u, \quad \tilde{\rho} = e^{2a} \rho, \tag{5.10}
\end{align*}\]

or, if some acting body force in the system drives the 3D turbulent flow into a 2D state, as for example the Coriolis force in a rapidly rotating turbulent flow in the long time limit (Gallet, 2015; Machicoane et al., 2016), then an approximative uniform rotation symmetry can be considered, admitted exactly only by the 2D Navier-Stokes equations

\[\begin{align*}
S_6^{\text{(approx.)}}: & \quad \tilde{t} = t, \quad \tilde{x}^i = Q^j_i x^i, \quad \tilde{u}^i = Q^j_i u^j + \hat{Q}^i_j x^j, \quad \text{for } i, j = 1, 2,
\text{and } \tilde{\rho} = \rho + \frac{1}{2} \omega \delta_{ij} x^i x^j + 2 \omega z \psi, \quad \text{with } \psi = - \int_C \left( u^2 dx^1 - u^1 dx^2 \right), \tag{5.11}
\end{align*}\]

where \(Q^j_i\) represents a two-dimensional uniform rotation matrix with \(\omega\) being the constant angular velocity and \(\psi\) the corresponding stream function defined as a planar curve integral over the two-componental velocity field \(u^i\). This exact symmetry of the 2D Navier-Stokes equations was first derived independently by Pukhnachev (1960); Cantwell (1978); Speziale (1981), and it was Speziale who misleadingly coined this invariance as ‘2D-MFI’, which is misleading in so far as the symmetry \(S_6^{\text{(approx.)}}\) does not form a coordinate but only a re-gauging transformation in the same way as discussed before for the non-coordinate transformation \(S_2\) (5.8).

To see how the two invariance principles r-MFI and TFI act as modelling restrictions for turbulence, let us for simplicity first consider the option to model the unclosed Reynolds stresses algebraically, i.e., directly as a constitutive equation in the analogous way as it is done in classical or ordinary thermodynamics (see Appendix D), by relating the Reynolds stresses to the fields and their gradients of the underlying balance equations, here to the mean incompressible continuity and momentum equations of (5.2), which again, for reasons of simplicity and clarity, will only be demonstrated explicitly for Cartesian (rectilinear) coordinates†

\[\begin{align*}
\partial_i \langle u^i \rangle = 0, \quad \partial_i \langle u^i \rangle + \langle u^j \rangle \partial_j \langle u^i \rangle = -\delta^{ij} \partial_j \langle p \rangle + \nu \delta^{ij} \partial_k^2 \langle u^i \rangle - \partial_j \tau^{ij}, \tag{5.12}
\end{align*}\]

†Mathematically correct, the invariance \(S_4\) in (5.9) is not a symmetry, but only an equivalence transformation (see e.g. Sec. 2 in Frewer et al. (2014a) and the references therein). Although this transformation is not physically realizable, since it is impossible to construct a fluid having negative molecular viscosity, it nevertheless represents a mathematical property which is characteristic for the Navier-Stokes equations. In particular this equivalence transformation will serve as guideline when developing, e.g., a molecular viscosity expansion within a certain turbulence model (Frewer, 2008c). In this regard, it should be noted that the discrete equivalence \(S_4\) is not the only equivalence transformation which the Navier-Stokes equations admit when trying to change the value of the molecular viscosity: because from the viewpoint of a continuous transformation, an infinite Lie-algebra of such equivalence transformations exist (Ünal, 1994, 1995).

‡In the following, only the concept of ‘frame-indifference’ is considered, i.e., only the concept as how to incorporate the symmetries of the Navier-Stokes equations into the turbulence modelling process, and not so much the concept of ‘form-invariance’, which has been done and discussed elsewhere; see e.g. Sadiki & Hutter (1996); Frewer (2009a,b,c); Ariki (2015), where the form-invariant (covariant) Reynolds-averaged Navier-Stokes are formulated and discussed both in 3D and 4D.
where

$$\tau^{ij} = \langle u^i u^j \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} u^{i(n)} u^{j(n)},$$

are the Reynolds stresses defined as an ensemble average over \( n \) flow realizations of the velocity product \( u^i u^j \) for the fluctuating field \( u^i = u^i - \langle u^i \rangle \). Each realization of the fluctuating velocity field satisfies the same continuity and momentum equations obtained by subtracting the mean Navier-Stokes equations (5.12) from the Reynolds decomposed instantaneous ones (5.2):

$$\begin{align*}
\partial_t u^{i(n)} + u^{j(n)} \partial_j u^{i(n)} + u^{j(n)} \partial_j \langle u^i \rangle & = -\delta^{ij} \partial_j p^{(n)} + \nu \delta^{jk} \partial_{j,k} u^{i(n)} + \partial_j \tau^{ij}, \\
\partial_t u^{j(n)} + u^{i(n)} \partial_i u^{j(n)} + u^{i(n)} \partial_i \langle u^j \rangle & = -\delta^{ij} \partial_j p^{(n)} + \nu \delta^{ij} \partial_{i,j} u^{j(n)} + \partial_i \tau^{ij}.
\end{align*}$$

(5.14)

Formally the system (5.12)-(5.14) constitutes a closed system of equations\(^1\) which admits the same set of symmetry transformations (5.7)-(5.11) as the instantaneous Navier-Stokes equations, however, now in the specific Reynolds decomposed form\(^2\)

\[
\begin{align*}
G: & \quad \tilde{t} = t + c_0, \quad \tilde{x}^i = A^i_j x^j + c^i_1 t + c^i_2, \quad \langle \tilde{u}^i \rangle = A^i_j \langle u^j \rangle + c^i_1, \quad \langle \tilde{p} \rangle = \langle p \rangle, \\
& \quad \tilde{u}^{i(n)} = A^i_j u^{j(n)}, \quad \tilde{p}^{(n)} = p^{(n)}, \\
S_1: & \quad \tilde{t} = e^{2\xi} t, \quad \tilde{x}^i = e^{\xi} x^i, \quad \langle \tilde{u}^i \rangle = e^{-\xi} \langle u^i \rangle, \quad \langle \tilde{p} \rangle = e^{-2\xi} \langle p \rangle, \\
& \quad \tilde{u}^{i(n)} = e^{-\xi} u^{i(n)}, \quad \tilde{p}^{(n)} = e^{-2\xi} p^{(n)}, \\
S_2: & \quad \tilde{t} = t, \quad \tilde{x}^i = x^i + f^i(t), \quad \langle \tilde{u}^i \rangle = \langle u^i \rangle + \tilde{f}^i(t), \quad \langle \tilde{p} \rangle = \langle p \rangle - \delta_{ij} x^i \tilde{f}^j(t) + \langle g(t) \rangle, \\
& \quad \tilde{u}^{i(n)} = u^{i(n)}, \quad \tilde{p}^{(n)} = p^{(n)} + g(t), \quad \text{for} \quad \tilde{f}^i(t) \neq 0, \\
S_3: & \quad \tilde{t} = t, \quad \tilde{x}^i = -x^i, \quad \tilde{x}^j = x^j, \quad \langle \tilde{u}^i \rangle = -\langle u^i \rangle, \quad \langle \tilde{u}^j \rangle = \langle u^j \rangle, \quad \langle \tilde{p} \rangle = \langle p \rangle, \\
& \quad \tilde{u}^{i(n)} = -u^{i(n)}, \quad \tilde{u}^{j(n)} = u^{j(n)}, \quad \tilde{p}^{(n)} = p^{(n)}, \quad \text{with} \quad j \neq i, \\
S_4: & \quad \tilde{t} = -t, \quad \tilde{x}^i = x^i, \quad \langle \tilde{u}^i \rangle = -\langle u^i \rangle, \quad \tilde{p} = p, \quad \tilde{v} = -v, \\
& \quad \tilde{u}^{i(n)} = -u^{i(n)}, \quad \tilde{p}^{(n)} = p^{(n)},
\end{align*}
\]

(5.15)

(5.16)

(5.17)

(5.18)

(5.19)

along with the two approximate symmetries, valid in a flow regime where, respectively, the influence of viscosity is negligibly small

$$S_5^{\text{approx}}: \quad \tilde{t} = t, \quad \tilde{x}^i = e^{\alpha} x^i, \quad \langle \tilde{u}^i \rangle = e^{\alpha} \langle u^i \rangle, \quad \langle \tilde{p} \rangle = e^{2\alpha} \langle p \rangle, \\
& \quad \tilde{u}^{i(n)} = e^{\alpha} u^{i(n)}, \quad \tilde{p}^{(n)} = e^{2\alpha} p^{(n)}.
$$

(5.20)

\(^1\)The fact that system (5.12)-(5.14) can be considered as formally closed, is due to taking along the definition of the ensemble average (5.13), which forms the defining relation between the fine-grained (microscopic) and the coarse-grained (macroscopic) description of turbulence. If this information (5.13) is not provided to an analysis, e.g., on symmetries, the mean equations (5.12) obviously turn into an unclosed set of equations, for which, as was already discussed before, a true symmetry analysis is ill-defined due to the effect of having to deal with an infinite hierarchy of statistical moment equations triggered by the lowest order equation (5.12).

\(^2\)Indeed, the transformations (5.15)-(5.19) are admitted as symmetries, and (5.20)-(5.21) as approximate symmetries of the combined system (5.12)-(5.14). A systematic Lie-group symmetry analysis in fact even shows that the instantaneous Navier-Stokes symmetries can only be decomposed into its mean and fluctuating part as given in (5.15)-(5.21). Any other decomposition would not be admitted as a symmetry of this system (5.12)-(5.14).
and/or the 3D turbulent flow is driven into a 2D state:†
\[
S_5^{(\text{approx.)}} : \tilde{t} = t, \quad \tilde{x}^i = Q^i_j x^j, \quad \langle \tilde{u}^i \rangle = Q^i_j \langle u^j \rangle + \dot{Q}^i_j x^j, \quad \tilde{u}^{ij}_{(n)} = Q^i_j u^{ij}_{(n)}, \quad \text{for } i, j = 1, 2.
\]
and \(\langle \tilde{p} \rangle = \langle p \rangle + \frac{1}{2} \tilde{\omega}_2 \delta_{ij} x^i x^j + 2 \omega_2 \langle \psi \rangle,\) with \(\langle \psi \rangle = - \int_C \left( (u^2) dx^1 - \langle u^1 \rangle dx^2 \right),\)
\[
\tilde{p}^{ij}_{(n)} = p^{ij}_{(n)} + 2 \omega_2 \psi^{ij}_{(n)}, \quad \text{with } \psi^{ij}_{(n)} = - \int_C \left( u^{i2}_{(n)} dx^1 - u^{i1}_{(n)} dx^2 \right). \tag{5.21}
\]

Now, the usual algebraic modelling ansatz for the Reynolds stresses, as standardly favored in the turbulence community, is to collectively choose them as an autonomous function of the mean velocity field and its gradient only, not to include the mean pressure field and the coordinates. Although there is no reason to exclude the mean pressure and its gradient as modelling variables (Frewer, 2009,b,c), we will, for the sake of simplicity, only proceed with the reduced, but still general ansatz
\[
\tau^{ij} = \tau^{ij} \left( t, x^i, \nu; \langle u^i \rangle, \partial_t \langle u^i \rangle \right). \tag{5.22}
\]
The aim is to model the Reynolds stresses such that they are consistent with the Navier-Stokes equations, in particular to show the same frame-indifference. Based on the results obtained in Section 3, a self-evident proposal would be, for example,
\[
(\nabla \cdot \tau)^T = \phi_1 \cdot (x - x_0^i) + \phi_2 \cdot (\langle u \rangle - u_0^i) + \phi_3 \cdot \left( \nabla \otimes \langle u \rangle + (\nabla \otimes \langle u \rangle)^T \right) \cdot (x - x_0^i)
\]
\[
+ \phi_4 \cdot \left( (\langle u \rangle - u_0^i)^T \cdot \nabla^T \right) \langle u \rangle + \phi_5 \cdot \Delta \langle u \rangle, \tag{5.23}
\]
already expressed for the gradient of the Reynolds stresses in a 3D covariant form. The \(\phi_i\) are arbitrary scalar functions with Euclidean-invariant arguments in all possible combinations
\[
\phi_i = \phi_i (\nu, t - t_0^i, \|x - x_0^i\|, \|\langle u \rangle - u_0^i\|, \ldots, (x - x_0^i)^T \cdot (\langle u \rangle - u_0^i), (x - x_0^i)^T \cdot \Delta \langle u \rangle, \ldots),
\]
but restricted to transform homogeneously in the following way
\[
\begin{align*}
S_1 : & \quad \tilde{\phi}_1 = e^{-4\epsilon} \phi_1, \quad \tilde{\phi}_2 = e^{-2\epsilon} \phi_2, \quad \tilde{\phi}_3 = e^{-3\epsilon} \phi_3, \quad \tilde{\phi}_4 = \phi_4, \quad \tilde{\phi}_5 = \phi_5, \\
S_3 : & \quad \tilde{\phi}_1 = \phi_1, \quad \tilde{\phi}_2 = \phi_2, \quad \tilde{\phi}_3 = \phi_3, \quad \tilde{\phi}_4 = \phi_4, \quad \tilde{\phi}_5 = \phi_5, \\
S_4 : & \quad \tilde{\phi}_1 = \phi_1, \quad \tilde{\phi}_2 = -\phi_2, \quad \tilde{\phi}_3 = -\phi_3, \quad \tilde{\phi}_4 = \phi_4, \quad \tilde{\phi}_5 = -\phi_5,
\end{align*}
\tag{5.24}
\]
in order to guarantee frame-indifference under the transformations (5.15)-(5.19). Note again that only relative quantities enter the constitutive relation (5.23), where \(t_0^i, x_0^i\) and \(u_0^i\) are some fixed absolute reference values characterizing the environment of the system in which the fluid flows. As explained in Section 3, these absolute system values transform in the same way as the respective variables characterizing the flow itself for some constant value, that is, in the same way as if the time coordinate \(t\), the spatial coordinate \(x \) and the mean velocity \(\langle u \rangle\) were constant, independent of whether \(t_0^i, x_0^i\) and \(u_0^i\) are specified in one particular frame as zero or not.

Moreover, when also including the remaining limit of the two approximative symmetries (5.20) and (5.21), the scalar functions in the constitutive relation (5.23) are restricted further in that they are to be modelled such that also the following limiting behavior emerges
\[
\begin{align*}
S_5^{(\text{approx.)}} (\nu \to 0) : & \quad \tilde{\phi}_1 = \phi_1, \quad \tilde{\phi}_2 = \phi_2, \quad \tilde{\phi}_3 = \phi_3, \quad \tilde{\phi}_4 = \phi_4, \quad \tilde{\phi}_5 = e^{2\alpha} \phi_5, \\
S_6^{(\text{approx.)}} (3D \to 2D) : & \quad \phi_2 \to 0, \quad \phi_4 \to 0.
\end{align*}
\tag{5.25}
\]

Hence, in contrast to traditional (algebraic) turbulence models for the Reynolds stresses, the present model (5.23) exhibits an explicit dependence on the mean velocity field \(\langle u \rangle\), however, not

† This confirms the result of Speziale (1998) that the Reynolds stresses (5.13) are transforming form-invariantly under all symmetries of the Navier-Stokes equations. Note that although the Reynolds stresses (5.13) do not transform as a tensor defined on a change of coordinates for the two scaling symmetries \(S_1\) (5.16) and \(S_5^{(\text{approx.)}}\) (5.20), they nevertheless transform form-invariantly in the sense of changing linearly and homogeneously.

‡ The dependence on the velocity difference \(\langle u \rangle - u_0^i\) has to vanish also in the contributing scalars \(\phi_1, \phi_3\) and \(\phi_5\).
absolutely, but only in a relative manner, and particularly such that in the instantaneous or fluctuating 2D limit (5.25) these velocity contributions via \( \phi_2 \) and \( \phi_4 \) will vanish. This is a novel modelling feature which would be worth to investigate further in a future study.

To close this section and this study, it is to mention that as one prolongs from classical or ordinary thermodynamics to extended thermodynamics by identifying the unclosed flux as a density satisfying its own balance equation (see Appendix D), one can analogously prolong from the mean momentum equations (5.12) to the Reynolds stress transport equations by identifying the Reynolds stress fluxes \( \tau \) as a set of densities, to then obtain the extended system (in 3D covariant form)

\[
\nabla \cdot \langle u \rangle = 0, \\
\partial_t \langle u \rangle + \langle u \rangle \cdot \nabla \langle u \rangle + \nabla \cdot \tau = -\nabla \langle p \rangle + \nu \Delta \langle u \rangle, \\
\partial_t \tau + \langle u \rangle \cdot \nabla \tau + \nabla \cdot T = \nu \Delta \tau + S, 
\]

but, of course, at the price of gaining new unclosed terms: The extended flux \( T \) known as the turbulent transport driven by velocity and pressure fluctuations, and the turbulent source term \( S \) containing both positive (productive) and negative (dissipative) contributions. In the spirit of extended thermodynamics one can now assume for the unclosed terms solely a local dependence on the densities \( \langle u \rangle \) and \( \tau \), including a possible dependence on the space-time coordinates, but excluding gradients and the pressure field \( \langle p \rangle \), which in the incompressible case cannot be identified as an own independent density evolving in time. A general ansatz for the two constitutive relations would thus have the form

\[
T = T(\nu, t, x, \langle u \rangle, \tau), \quad S = S(\nu, t, x, \langle u \rangle, \tau). 
\] (5.27)

The aim again is to model these relations such that they are consistent with the Navier-Stokes equations, in particular to show the same frame-indifference, that is, to be in accordance with the Navier-Stokes symmetries (5.15)-(5.21). Although a detailed analysis of this problem would be beyond the scope of this section, one can nevertheless expect again an explicit dependence on the mean velocity field, however, as before in (5.23), only as a velocity difference; and the same for the coordinates if such a dependence is necessary to take into account:

\[
T = T(\nu, t - t_0^\nu, x - x_0^\nu, \langle u \rangle - u_0^\nu, \tau), \quad S = S(\nu, t - t_0^\nu, x - x_0^\nu, \langle u \rangle - u_0^\nu, \tau). 
\] (5.28)

Hence, similar as the algebraic relation (5.23), the extended constitutive relations (5.28) will be frame-indifferent only for those transformations (5.15)-(5.21) which are admitted as symmetries or approximative symmetries by the instantaneous Navier-Stokes equations. For all other transformations, like e.g. a 3D uniform (time-dependent) rotation, the constitutive relations (5.23) and (5.28) will no longer be frame-indifferent relations anymore, as they will pick up frame dependent terms \( \Gamma \) in the sense as explained in Section 4.2. In other words, when generally formulating or transforming the balance equations (5.26) to non-inertial systems, the corresponding constitutive relations (5.28) will inherently show an explicit frame-dependence of the form

\[
T = T(\nu, t - t_0^\nu, x - x_0^\nu, \langle u \rangle - u_0^\nu, \tau; \Gamma), \quad S = S(\nu, t - t_0^\nu, x - x_0^\nu, \langle u \rangle - u_0^\nu, \tau; \Gamma). 
\] (5.29)

Hence, modelling the constitutive relations (5.29) generally frame-indifferent for all non-inertial systems would therefore be incorrect; a problem faced, for example, in the work of Dafalias & Younis (2007, 2009); Dafalias (2011), where in particular the pressure-strain-rate correlations in \( S \) are modelled explicitly frame-indifferent for all Euclidean transformations, including the 3D time-dependent rotations.

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\(^1\)In how far this local ansatz inspired from extended thermodynamics is sufficient to model turbulence is a different problem and has to be investigated in a separate study. Maybe this ansatz is not sufficient and the field gradients together with mean pressure field have to be taken along as additional modelling variables.

\(^2\)The dependence on the Reynolds stresses \( \tau \) need not to be considered as a difference, since \( \tau \) itself is already transforming form-invariantly under all symmetries (5.15)-(5.21) of the Navier-Stokes equations.
Appendix A. On the notion of geometrical invariance

The aim of this section is to show that not every geometric object is automatically geometrically invariant when observed from several different frames of reference: A geometric object which stays invariant when observed from two different frames, may not necessarily stay invariant when observed from another, a third different frame. For example, let us consider as a geometric object first a single point \( x \) of physical 3D space, which we observe from a fixed but arbitrarily chosen frame \( F \) spanned by a basis \((g_1, g_2, g_3)\).\(^1\) The geometric point \( x \) in \( F \) then has the unique representation

\[
x = x^i g_i, \tag{A.1}
\]

where \( x^i \) are the three components of the point. It is clear that both the components \( x^i \) as well as the basis \( g_i \) are observer dependent and thus relative quantities which are assigned only to the specific frame used, here currently to \( F \). Now, let us observe this geometric point \( x \) from a second, different frame \( \hat{F} \) spanned by a basis \((\hat{g}_1, \hat{g}_2, \hat{g}_3)\), and let the components in each frame be related by a constant homogeneous linear transformation (with an invertible matrix \( A \))

\[
\hat{x}^i = (A)^j_i x^j, \tag{A.2}
\]

then it is clear that the geometric point \( x \) itself is unaffected from the process of which frame it is being observed, since with (A.2) we only perform a relabelling of that point.\(^1\) Hence, under the change of frame \( F \rightarrow \hat{F} \), according to (A.2), the point \( x \) (A.1) stays geometrically invariant

\[
x = x^i g_i = \hat{x}^i \hat{g}_i. \tag{A.3}
\]

This invariance then defines the transformation rule for the basis as

\[
\hat{g}_i = (A^{-T})^j_i g_j, \tag{A.4}
\]

where \( A^{-T} := (A^{-1})^T \) is the short-hand notation for the transpose of the inverse matrix of \( A \). Now, let us consider relative to the initial frame \( F \) a third frame \( F' \), which is spanned by the basis \((g'_1, g'_2, g'_3)\) and where the components of the geometric point \( x \) are now just related by a constant non-zero shift

\[
x'^i = x^i + b^i. \tag{A.5}
\]

Since this transformation \( F \rightarrow F' \) (A.5) only resembles a parallel frame shift, it is clear that the new basis \( g'_i \) is equivalent to the initial basis \( g_i \), i.e., \( g'_i = g_i \), \( \forall i \). Thus, in contrast to the first (homogeneous) linear transformation (A.2), the second (inhomogeneous) linear transformation (A.5) does not leave the considered geometric point \( x \) invariant as in (A.3), since

\[
x = x^i g_i = (x'^i - b^i) g'_i = x'^i g'_i - b^i g'_i \neq x'^i g'_i. \tag{A.6}
\]

Instead of absolute points, as \( x \), only relative connections as \( r := x_2 - x_1 \) between two absolute points \( x_1 \) and \( x_2 \) can serve as geometric invariants under (A.5), since only then obviously the invariance-breaking shift term cancels

\[
r = (x^i_2 - x^i_1) g_i = (x'^i_2 - x'^i_1) g'_i. \tag{A.7}
\]

Also under the first linear transformation \( F \rightarrow \hat{F} \) (A.2), the relative connection \( r \) remains to be a geometrical invariant

\[
r = (x^i_2 - x^i_1) g_i = (\hat{x}^i_2 - \hat{x}^i_1) \hat{g}_i, \tag{A.8}
\]

\(^1\)For reasons of simplicity we will only consider frame representations relative to a covariant basis. To distinguish in this section in each frame between a covariant basis \( g_i \) and a contravariant basis \( g^i \) would only unnecessarily complicate the analysis, without gaining new insight.

\(^2\)A transformation as (A.2), connecting two different frames observing the same physical object, i.e., two observers one object, is also known as a passive transformation; opposed to an active transformation where the observed object itself is transformed.
as can be readily verified, by applying (A.4) and (A.2) for each absolute point \( x_1 \) and \( x_2 \) separately. Hence, while the relative connection \( r \) behaves geometrically invariant under all three frames \( \mathcal{F} \), \( \hat{\mathcal{F}} \) and \( \mathcal{F}' \) considered so far, the absolute geometric point \( x \) is only invariant when observed from \( \mathcal{F} \) and \( \hat{\mathcal{F}} \), but not when observed from \( \mathcal{F}' \). Although this example already suffices to show that a geometric object need not to be necessarily invariant when observed from different frames of reference, it is expedient to continue this geometric process of invariance-breaking to gain further insights. For example, the frame to be considered next will break the geometrical invariance of the finite difference vector \( r \).

Let us now consider, relative to the initial frame \( \mathcal{F} \), a fourth frame \( \mathcal{F}^* \), spanned by a basis \( (g_1^*, g_2^*, g_3^*) \) which is local in space, i.e., \( g_i^* = g_i^*(x) \), \( \forall i \), and where the components of any geometric point \( x \) in each frame are related by an arbitrary and locally invertible spatial coordinate transformation

\[
x^{*i} = x^{i}(x^j).
\]  

Since the new frame \( \mathcal{F}^* \) can be identified as a linear space locally in each point \( x \), it is clear that the new basis \( (g_1^*, g_2^*, g_3^*) \) can always be expressed as a linear combination of the initial basis \( (g_1, g_2, g_3) \) locally in each point of the initial linear space \( \mathcal{F} \) — similar to (A.4), however only locally in each (geometric) point

\[
g_i^*(x) = (J^{-T}(x))^j_i g_j.
\]  

Obviously the finite difference vector \( r \) cannot be observed as a geometrical invariant from the fourth and local frame \( \mathcal{F}^* \), since for each point one faces the non-invariant and meaningless relation

\[
r = (x_2 - x_1)g_i = x_2^i (J^T(x_2))^j_i g_j(x_2) |_{x_2^j = x_2^j(x_2^i)} - x_1^i (J^T(x_1))^j_i g_j(x_1) |_{x_1^j = x_1^j(x_1^i)}.
\]  

A meaningful transformation relation can only be obtained in the infinitesimal limit \( x_2 \to x_1 \), i.e., only for an infinitesimal difference vector \( dr \), which in the local frame \( \mathcal{F}^* \) now even turns out to be geometrically invariant, since according to (A.9) and (A.10) we obtain the reduction

\[
dr = dx^i g_i = \frac{\partial x^i(x^k)}{\partial x^j} dx^j g_i(x) = dx^j g_j(x) |_{x^j = x^j(x^i)} =: dx^i g_i^*,
\]  

if the local linear mapping \( J \) (A.10) is identified as the Jacobi matrix

\[
(J(x))^i_j = \frac{\partial x^i(x^k)}{\partial x^j}.
\]  

Since the two linear transformation (A.2) and (A.5) are both contained as special transformations of (A.9), we can conclude that the geometrical object \( dr \) is invariant under all four frames \( \mathcal{F} \), \( \hat{\mathcal{F}} \), \( \mathcal{F}' \) and \( \mathcal{F}^* \) considered so far, while the invariance of \( r \) between the three former frames is broken for the later frame \( \mathcal{F}^* \).

As the last example, we now will consider a frame which also breaks the geometrical invariance of \( dr \). Relative again to the initial frame \( \mathcal{F} \), let us consider a fifth frame \( \hat{\mathcal{F}} \) spanned by a basis \( (\hat{g}_1, \hat{g}_2, \hat{g}_3) \) where the components of a geometric point \( x \) in each frame are again related by the homogenous linear transformation (A.2), but now through a time-dependent and thus non-constant matrix \( A = A(t) \). This dependence on a fourth coordinate breaks the invariant property of the infinitesimal spatial 3D-vector \( dr \), since the total derivative of the underlying transformation for the coordinates connecting the frames \( \mathcal{F} \) and \( \hat{\mathcal{F}} \)

\[
\hat{x}^i = \hat{x}^i(x^j, t) = (A(t))^i_j x^j, \quad \text{with its local inverse:} \quad x^i = x^i(\hat{x}^j, t) = (A^{-1}(t))^i_j \hat{x}^j,
\]  

is now given as

\[
d\hat{x}^i = \frac{\partial \hat{x}^i}{\partial x^j} dx^j + \frac{\partial \hat{x}^i}{\partial t} dt = (A(t))^i_j dx^j + (\dot{A}(t))^i_j x^j dt,
\]  

(A.15)
which thus, due to respecting the variation also along the time coordinate $t$, leads to the non-invariant relation

\[ dr = dx^i g_i = \left( (A^{-1}(t))^i_j \ dx^j + \left( \frac{\partial A^{-1}(t)}{\partial t} \right)^i_j \ dx^j \ dt \right) (A^T(t))_k^i \ g_k \]

\[ = d\tilde{x}^i \tilde{g}_i + (A(t) \frac{\partial A^{-1}(t)}{\partial t})^i_j \tilde{x}^j \ dt \tilde{g}_k \neq d\tilde{x}^i \tilde{g}_i, \]  

(A.16)

where we used the obvious fact that the basis still transforms as formulated in (A.4), however, in the current case, locally now for each time step.

A geometrical invariance for $dr$ under time-dependent transformations can only achieved when changing the framework from 3D space to a 4D space-time. If the aim is to remain within the realm of classical Newtonian physics then it is necessary to only consider space-time transformations in which the time coordinate transforms absolutely, i.e. invariantly up to some constant time shift. For that, let us reformulate the two frames $F$ and $\tilde{F}$ currently considered as Newtonian frames in a true 4D space-time setting, $F^{4D}$ and $\tilde{F}^{4D}$, spanned now by a 4D basis $(g_0, g_1, g_2, g_3)$ and $(\tilde{g}_0, \tilde{g}_1, \tilde{g}_2, \tilde{g}_3)$, respectively. The components of a geometric space-time point $x^{4D}$ in each frame $F^{4D}$ and $\tilde{F}^{4D}$ are again related by (A.14), however, now reformulated into a 4D Newtonian framework:

\[ \tilde{x}^\alpha = \tilde{x}^\alpha(x^\beta), \text{ where } x^0 = x^0, \text{ and } \tilde{x}^i = (A(x^0))_i^j \ x^j. \]  

(A.17)

The infinitesimal 4D-vector $dr^{4D}$ transforms geometrically invariant

\[ dr^{4D} = dx^\alpha g_\alpha = \frac{\partial x^\alpha}{\partial x^\beta} d\tilde{x}^\beta \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \tilde{g}_\gamma = \delta^\gamma_\beta d\tilde{x}^\beta \tilde{g}_\gamma = d\tilde{x}^\alpha \tilde{g}_\alpha, \]  

(A.18)

since the basis $g_0$ for the time coordinate compensates the invariance-breaking term (A.16) in the pure 3D case. In particular, the basis $(g_\alpha) = (g_0, g_i)$ transforms as

\[ \begin{aligned}
    \tilde{g}_0 &= \frac{\partial x^\alpha}{\partial x^0} g_\alpha = g_0 + \frac{\partial x^i}{\partial x^0} g_i = g_0 + \left( \frac{\partial A^{-1}(t)}{\partial t} \right)_j^i \tilde{x}^j \tilde{g}_i, \\
    \tilde{g}_i &= \frac{\partial x^\alpha}{\partial x^i} g_\alpha = \frac{\partial x^i}{\partial x^j} g_j = (A^{-T}(t))_i^j g_j.
\end{aligned} \]  

(A.19)

This completes the investigation on geometrical invariance, to show that it depends on the frame used whether a geometrical object is observed invariantly or not. Worthwhile to note in this respect is that the notion of geometrical invariance is intimately linked to form-invariance and the definition of a tensor (see Sections 1 and 2): For example, in the frame relation $F \leftrightarrow \tilde{F}$ the absolute space point $x$ is observed as a geometric invariant because the coordinates (A.2) transform as a (contravariant) tensor of rank 1, while in the frame relation $F \leftrightarrow F'$ this point is not observed as a invariant, because in this case the coordinates (A.5) do not transform as a tensor. Or, for example, $dr$ which is observed as a geometrical invariant in $F \leftrightarrow F'$, but not in $F \leftrightarrow \tilde{F}$, simply because in this case again the coordinates (A.15) do not transform as a tensor, but only in a 3D formulation, while in true 4D formulation they do.

**Appendix B. The non-tensors for uniform rotations in a 3D formulation**

The non-tensors considered in this study are the velocity field $u(x)$, its gradient $L(x) = \nabla \otimes u(x)$, and the vorticity $W(L(x)) = \frac{1}{2} (L(x) - L(x)^T)$. Within a true 4D framework, however, all three quantities naturally appear as tensors (Frewer, 2009a), but not in a 3D framework as formulated

\[ \text{The Greek indices run from 0 to 3, the Latin indices from 1 to 3, and } x^0 = t \text{ is defined as the time coordinate. In Frewer (2009a,b,c) it is demonstrated that the metric of the underlying 4D manifold allowing for transformations of the kind (A.17) degenerates into a singular twofold metric, namely into a time-like and into a space-like metric.} \]
herein. Nevertheless, it is indeed possible to turn these three quantities into tensors without changing to a 4D framework, namely by just making use of a similar procedure from differential geometry when defining an affine connection to overcome the general non-tensor property of the second and higher-order derivatives; ultimately thus supporting once more the objection of Kretschmann (see discussion in Section 4.1). To demonstrate this, consider first the central non-tensor in this list, the velocity field $\mathbf{u}(\mathbf{x})$, which transforms as (2.34)

$$\tilde{\mathbf{u}}(\tilde{\mathbf{x}}) + \Omega \tilde{\mathbf{x}} = \mathbf{Q}\mathbf{u}(\mathbf{x}),$$

(B.1)

where again it is clear that the inhomogeneous term proportional to the spin $\Omega$ destroys the tensor property of the velocity field in 3D — if everything, however, would be reformulated into a true 4D framework (within Newtonian physics), this ‘non-tensor’-problem would not exist in the first place (Frewer, 2009a, b), since the 4D velocity field $u^\alpha(x^\gamma) = (1, \mathbf{u}(x^0, \mathbf{x}))$ would then naturally always transform as a tensor $^1$

$$\tilde{\mathbf{u}}(\tilde{\mathbf{x}}) = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta} u^\beta(x^\gamma), \quad \text{where} \quad \frac{\partial \tilde{x}^i}{\partial x^j} = (Q)^i_j, \quad \frac{\partial \tilde{x}^0}{\partial x^j} = (Q)^i_j x^j, \quad \text{and} \quad \frac{\partial \tilde{x}^0}{\partial x^0} = 0, \quad \frac{\partial \tilde{x}^0}{\partial x^0} = 1. \quad \text{(B.2)}$$

But our aim in this study is to reside within 3D and not to change the framework to 4D. Hence a different argument has to be used when trying to reformulate (B.1) into a 3D tensor relation. For that a short excursion into differential geometry is necessary, by recalling the fact that also in a true 4D framework one naturally runs into non-tensors. The most natural one is the derivative of any covariant vector field $A_\alpha = A_\alpha(x^\gamma)$, which for arbitrary space-time coordinate transformations changes as (Frewer, 2009a)

$$\frac{\partial A_\alpha}{\partial \tilde{x}^\beta} = \frac{\partial}{\partial \tilde{x}^\beta} \left( \frac{\partial x^\rho}{\partial \tilde{x}^\alpha} A_\rho \right) = \frac{\partial x^\rho}{\partial \tilde{x}^\alpha} \frac{\partial}{\partial x^\beta} A_\rho + \frac{\partial^2 x^\rho}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta} A_\rho$$

$$= \frac{\partial x^\rho}{\partial \tilde{x}^\alpha} \frac{\partial x^\sigma}{\partial \tilde{x}^\beta} \frac{\partial A_\rho}{\partial x^\sigma} + \frac{\partial^2 x^\rho}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta} A_\rho \neq \frac{\partial x^\rho}{\partial \tilde{x}^\alpha} \frac{\partial x^\sigma}{\partial \tilde{x}^\beta} \frac{\partial A_\rho}{\partial x^\sigma}, \quad \text{(B.3)}$$

where for a better readability the explicit functional coordinate dependence was dropped. As a result, we see that the second derivative term as an inhomogeneous term destroys the tensor property, similar as the spin term $\Omega$ in (B.1). However, by redefining expressions, it is possible to turn (B.3) in a tensor relation. The usual procedure, as taken from Schrödinger (1950), is first to introduce the short-hand notation

$$\tilde{\Gamma}^\mu_{\alpha\beta} := \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial^2 x^\rho}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta} \neq 0, \quad \text{(B.4)}$$

and then to rewrite relation (B.3) equivalently as

$$\frac{\partial A_\alpha}{\partial \tilde{x}^\beta} - \tilde{\Gamma}^\mu_{\alpha\beta} A_\mu = \frac{\partial x^\rho}{\partial \tilde{x}^\alpha} \frac{\partial x^\sigma}{\partial \tilde{x}^\beta} \left( \frac{\partial A_\rho}{\partial x^\sigma} - \Gamma^\mu_{\rho\sigma} A_\mu \right), \quad \text{where} \quad \Gamma^\mu_{\rho\sigma} = 0, \quad \text{(B.5)}$$

which allows us to see that the formal quantity $^4$

$$\nabla_\sigma A_\rho := \frac{\partial A_\rho}{\partial x^\sigma} - \Gamma^\mu_{\rho\sigma} A_\mu, \quad \text{(B.6)}$$

$^1$Note that the Newtonian 4D velocity field $u^\alpha(x^\gamma) = (1, \mathbf{u}(x^0, \mathbf{x}))$ is a tensor that never gets zero, i.e. it is non-zero in all reference frames and thus does not distinguish any particular frame, a feature inherent to the definition of a tensor (see footnote on p.1). Finally note here again that all Greek indices run from 0 to 3, the Latin indices from 1 to 3, and $x^0 = t$ is defined as the time coordinate, which itself stays invariant under the spatial coordinate transformation (2.1), i.e. $\tilde{x}^0 = x^0$.

$^4$In the literature the expression $\nabla_\sigma A_\rho$ is sometimes also denoted as $A_{\rho,\sigma}$, opposed to the usual partial derivative denoted as $\partial_\sigma A_\rho = A_{\rho,\sigma}$. 


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transforms as a covariant tensor of rank 2
\[ \tilde{\nabla}_m \tilde{A}_\alpha = \frac{\partial x^\rho}{\partial x^m} \frac{\partial x^\sigma}{\partial \tilde{x}^\beta} \nabla_\sigma A_\rho, \]  
(B.7)

where \( \nabla_\alpha \) is known as the covariant derivative satisfying all linear rules of ordinary differentiation as \( \partial_\alpha \). Although we managed with (B.7) to define a form-invariant differentiation process from the non-tensor relation (B.3), the explicit frame dependence could not be removed and still is to be found in the transformed quantity \( \nabla_\beta \tilde{A}_\alpha = \partial_\beta \tilde{A}_\alpha - \Gamma^\mu_{\alpha\beta} \tilde{A}_\mu \), being, due to \( \Gamma^\mu_{\alpha\beta} \), a frame-dependent (non-objective) quantity, except in the special case when considering linear space-time coordinate transformations connecting inertial frames of reference, which, within classical Newtonian physics, are the Galilei transformations. Otherwise, for all other (non-inertial) frames of references, the transformed covariant derivative \( \nabla_\beta \tilde{A}_\alpha \) shows a regular dependence on the used frame through its driving and defining constituent \( \Gamma^\mu_{\alpha\beta} \), which itself transforms as a non-tensor again. To determine this transformation it is necessary to identify (B.5) as an equation for \( \tilde{\Gamma}^\mu_{\alpha\beta} \) and then to solve it in terms of \( \Gamma^\mu_{\rho\sigma} \), which can be achieved by rewriting (B.5) equivalently as
\[ -\tilde{\Gamma}^\mu_{\alpha\beta} \tilde{A}_\mu = -\frac{\partial \tilde{A}_\alpha}{\partial \tilde{x}^\beta} + \frac{\partial x^\rho}{\partial x^\alpha} \frac{\partial x^\sigma}{\partial \tilde{x}^\beta} \left( \frac{\partial A_\rho}{\partial x^\sigma} - \Gamma^\mu_{\rho\sigma} A_\mu \right) \]
\[ = -\left( \frac{\partial x^\rho}{\partial x^\alpha} \frac{\partial x^\sigma}{\partial \tilde{x}^\beta} A_\rho + \frac{\partial^2 x^\rho}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta} A_\rho \right) + \frac{\partial x^\rho}{\partial x^\alpha} \frac{\partial x^\sigma}{\partial \tilde{x}^\beta} \left( \frac{\partial A_\rho}{\partial x^\sigma} - \Gamma^\mu_{\rho\sigma} A_\mu \right) \]
\[ = -\frac{\partial^2 x^\rho}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta} A_\rho \frac{\partial x^\rho}{\partial x^\alpha} \frac{\partial x^\sigma}{\partial \tilde{x}^\beta} \Gamma^\mu_{\rho\sigma} A_\nu = -\left( \frac{\partial^2 x^\rho}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta} \frac{\partial \tilde{x}^\mu}{\partial x^\sigma} + \frac{\partial x^\rho}{\partial x^\alpha} \frac{\partial x^\sigma}{\partial \tilde{x}^\beta} \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \Gamma^\nu_{\rho\sigma} \right) \tilde{A}_\mu, \]  
(B.8)

from which then the solution can be read off as (since this relation must hold for all \( \tilde{A}_\mu \))
\[ \tilde{\Gamma}^\mu_{\alpha\beta} = \frac{\partial \tilde{x}^\mu}{\partial \tilde{x}^\alpha} \frac{\partial \tilde{x}^\rho}{\partial \tilde{x}^\beta} \Gamma^\nu_{\rho\sigma} + \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial x^\sigma}{\partial \tilde{x}^\beta} \frac{\partial \tilde{x}^\nu}{\partial x^\rho} \Gamma^\rho_{\nu\sigma}. \]  
(B.9)

Thus \( \Gamma^\mu_{\alpha\beta} \) transforms as a non-tensor, that is, if it vanishes in one coordinate system \( \Gamma^\mu_{\alpha\beta} = 0 \) (initial coordinate system as used in (B.5)), then it does not necessarily vanish in any other coordinate system \( \Gamma^\mu_{\alpha\beta} \neq 0 \). Hence \( \Gamma^\mu_{\alpha\beta} \) is a non-trivial quantity known as the affine connection which can be arbitrarily assigned in one coordinate system determining then in that considered system the meaning of parallel displacement in space-time by connecting nearby tangent spaces. To conclude, we thus managed to turn the non-tensor relationship (B.3) into the form-invariant, but still non-objective tensor relationship (B.7).

Obviously, the idea of the above procedure in defining an affine connection can now also be applied to the 3D non-tensor relation (B.1). By equivalently rewriting this relation in the way as it was done in (B.5)
\[ \dot{\tilde{u}}(\mathbf{x}) + \dot{\tilde{\Omega}} \dot{\mathbf{x}} = Q \left( \tilde{u}(\mathbf{x}) + \tilde{\Omega} \mathbf{x} \right), \]  
where \( \tilde{\Omega} := Q \dot{Q}^T \neq 0 \), and \( \Omega = 0 \),
(B.10)

and then, similar as in (B.7), redefining this expression into the tensor relation
\[ \dot{\tilde{u}}_Q(\tilde{\mathbf{x}}) = Q \tilde{u}_Q(\mathbf{x}), \]  
(B.11)

we finally yield a 3D form-invariant representation\(^\dagger\) of (B.1), with \( \tilde{u}_Q(\mathbf{x}) := u(\mathbf{x}) + \tilde{\Omega} \mathbf{x} \) being the new redefined velocity field relative to the spin term \( \tilde{\Omega} \). Although being manifestly form-invariant, the explicit frame-dependence originating from (B.1), however, could not be removed.

\(^\dagger\)Note that the specifically designed 3D-form-invariance (B.11) is not equivalent to the form-invariance that would be achieved in a true 4D formulation (Frewer, 2009a,b,c). Form-invariance of the velocity field in 4D is of a different nature than (B.11). The former concept is absolute, while the latter one is relative; see the definitions at the end of this section.
in (B.11): The transformed redefined velocity $\tilde{u}_\Omega(\tilde{x})$ still shows a regular dependency on the uniform rotating frame through its spin $\tilde{\Omega}$, which itself transforms as a non-tensor. Similar as in the procedure outlined above for the affine connection $\Gamma^\alpha_{\beta\gamma}$, the transformation of $\Omega$ is determined from (B.10) by equivalently rewriting it as

$$\dot{\tilde{\Omega}} \tilde{x} = -\tilde{u}(\tilde{x}) + Q\left( u(x) + \Omega x \right) = -\left( Qu(x) + Qx \right) + Q\left( u(x) + \Omega x \right) = -\tilde{Q}Q^T \tilde{x} + \tilde{Q}\Omega Q^T \tilde{x},$$

and then to solve it for $\dot{\tilde{\Omega}}$ (valid for all $\tilde{x}$) to obtain the (non-objective) transformation rule

$$\tilde{\Omega} = Q\Omega Q^T + \tilde{Q}Q^T,$$

which can be formally compared to the situation in (B.9): The spin $\Omega$ transforms as a non-tensor, that is, if it vanishes in one coordinate system $\Omega = 0$ (initial coordinate system as used in (B.10)), then it does not necessarily vanish in any other coordinate system $\tilde{\Omega} \neq 0$. Hence, the spin $\Omega$ can be identified as an own independent quantity which can be arbitrarily assigned to feature one uniform rotating system which then, via (B.13), can be connected to another uniform rotating system $\tilde{\Omega}$. The two different (uniform rotating) systems, tilde and non-tilde, are then connected by a relative coordinate transformation

$$\tilde{x} = Qx, \quad \text{with} \quad QQ^T = 1, \quad \det(Q) = 1, \quad \text{and} \quad \Omega_R := QQ^T, \quad \dot{\Omega}_R = 0,$$

having again, of course, the same property of an orientated uniform rotation as initially defined in (2.1), but, with a relative spin $\Omega_R$ which in general is different now to the initially chosen (untransformed) spin $\Omega$, which itself can be assigned arbitrarily. In other words, from a more constructive point of view, we thus face an initial uniform rotating system, characterized by an arbitrarily assigned $\Omega$, that gets transformed by (B.14) to then obtain a new uniform rotating frame characterized by $\tilde{\Omega}$, or, from a preset point of view, we already face a configuration of two different uniform rotating frames $\Omega$ and $\tilde{\Omega}$ which then are both connected by the relative transformation (B.14) with spin $\Omega_R$. In both cases, if the first spin $\Omega$ is chosen arbitrarily, then the set of values for the second spin $\tilde{\Omega}$ are uniquely fixed by relation (B.13) as

$$\tilde{\Omega} = Q\Omega Q^T + \Omega_R, \quad \text{for any given} \quad \Omega_R = Q\dot{\Omega}^T.$$  

To conclude, we thus managed to turn the non-tensor relationship (B.1) into the form-invariant, but still non-objective tensor relationship (B.11). This relation allows us now to also turn the non-tensor relationships of the velocity gradient and the vorticity

$$\begin{align*}
\tilde{L}(\tilde{x}) + \Omega &= QL(x)Q^T, \\
\tilde{W}(\tilde{L}(\tilde{x})) + \Omega &= QW(L(x))Q^T,
\end{align*}$$

into their 3D-form-invariant, but still frame-dependent (non-objective) tensor relations

$$\begin{align*}
\tilde{L}_\Omega(\tilde{x}) &= QL_\Omega(x)Q^T, \\
\tilde{W}(\tilde{L}_\Omega(\tilde{x})) &= QW(L_\Omega(x))Q^T,
\end{align*}$$

with $L_\Omega(x) := \nabla \otimes u_\Omega(x) = L(x) + \Omega$ and $W(L_\Omega(x)) := \frac{1}{2}(L_\Omega(x) - L_\Omega(x)^T)$ being, relative to the spin term $\Omega$, the new redefined velocity gradient and vorticity, respectively.

Despite the resemblance of redefining the generally transformed derivative of a covariant vector field in 4D, on the one hand, and the uniform rotated velocity field in 3D, on the other,
B.7

2.3

B.13

2

C.2

B.11

2.6

B.17

B.15

-

B.7

2.3

B.13

2

C.2

B.11

2.6

B.17

B.15

-

as the form-invariant tensor relation \((B.7)\) and \((B.11)\), respectively, there is nevertheless a decisive difference between these two approaches: While the former construction procedure can be straightforwardly generalized to derivatives of tensors of any kind, namely as \((\text{Schrödinger, 1950})\)

\[
\partial_\nu A^{\kappa\lambda...}_{\rho\sigma...} \rightarrow \nabla_\nu A^{\kappa\lambda...}_{\rho\sigma...} = \partial_\nu A^{\kappa\lambda...}_{\rho\sigma...} + \Gamma^\kappa_{\alpha\nu} A^{\lambda...}_{\rho\sigma...} + \cdots = \frac{\partial A^{\kappa\lambda...}_{\rho\sigma...}}{\partial x^\nu} - \Gamma^\kappa_{\nu\sigma} A^{\lambda...}_{\rho\sigma...} - \cdots, 
\]

without knowing the explicit structure of the general tensor \(A^{\kappa\lambda...}_{\rho\sigma...}\) itself, one fails to do so for the latter construction process if one would generalize to some arbitrary non-tensor

\[
N(x) \rightarrow N_\Omega(x) = N(x) + F(\Omega, N, \{X_n\}), \quad (B.18)
\]

since, without knowing the explicit structure of \(N\), it is not clear beforehand how this quantity transforms under a uniform rotation induced by \(\Omega\), i.e., the redefining function \(F\) is not known beforehand unless the functional structure of \(N\) is explicitly known — the set of extra variables \(X_n\) in \(F\) denote all quantities that may arise when explicitly transforming \(N\) into a uniform rotating frame.\(^\dagger\) In other words, the introduced process in 3D of redefining non-tensors to tensors is context-related; in clear contrast to the similar process considered in 4D when redefining the partial to a covariant derivative. Hence, if a redefinition for \(\Omega \neq 0\) is possible, we thus still have to distinguish between those tensors which have the property \(N = N_\Omega\), i.e., where \(F = 0\), which appropriately can be classified as absolute tensors or simply as tensors, and those where \(F \neq 0\) as relative tensors. This classification is used in Sections 2.3-2.6.

**Appendix C. The explicit frame-dependence of the relative vorticity**

As introduced in the previous Appendix B, the relative vorticity \(W(L_\Omega(x))\) transforms form-invariantly as \((B.17)\)

\[
W(L_\tilde{\Omega}(\tilde{x})) = QW(L_\Omega(x))Q^T, \quad (C.1)
\]

where \(\tilde{\Omega}\) and \(\Omega\) are two unequal spins featuring two different uniform rotating (non-inertial) reference frames which are connected through a coordinate transformation \(\tilde{x} = Qx\) \((B.14)\) via the relation \((B.15)\)

\[
\tilde{\Omega} = Q\Omega Q^T + \Omega_R, \quad \text{with} \quad \Omega_R = Q\tilde{Q}^T. \quad (C.2)
\]

That the relative vorticity \(W(L_\Omega(x))\) constitutes a frame-dependent (non-objective) quantity is manifestly clear, since its evaluation \(W(L_\Omega(x)) = W(L(x)) + \Omega\) explicitly depends on the spin value \(\Omega\) of the chosen frame — the same is of course also true for the relative vorticity in the transformed domain \(W(L_{\tilde{\Omega}}(\tilde{x})) = W(L(\tilde{x})) + \tilde{\Omega}\), which explicitly depends, via \((C.2)\), on the spin value \(\tilde{\Omega}\) of the second frame. Hence the correct approach to measure explicit frame-dependence for relative tensors, as in the case here for the relative vorticity, is to directly compare the two vorticity expressions in each frame separately:

For a relative tensor we face explicit frame-dependence, if \(W(L_\Omega(x)) \neq W(L(x))\),

\[
\text{or equivalently, if } W(L_\tilde{\Omega}(\tilde{x})) \neq W(L(\tilde{x})). \quad (C.3)
\]

The standard approach to measure frame-dependency as introduced in Section 2, namely to verify whether \(\tilde{W}(\cdot) \neq W(\cdot)\) holds or not, only works for absolute tensors, but not for relative tensors where this approach degenerates to an action which only measures the relative dependency between two different (non-inertial) frames \(\tilde{\Omega}\) and \(\Omega\), and not the wanted frame-dependency within one (non-inertial) frame \(\Omega\) or \(\tilde{\Omega}\). Hence, obviously, the relative vorticity only shows a relative frame-indifference (relative objectivity), since when evaluating the relative

\(^\dagger\)Moreover, it is to note that a redefinition \(N \rightarrow N_\Omega = N + F\) is only valid if the redefining function \(F\) is also compatible with the transformation rule \((B.13)\) for \(\Omega\).
tensor relation (C.1) by using the definition of the vorticity $\mathbf{W}$ (2.41) and the transforation rule for the relative velocity gradient $\mathbf{L}_\Omega$ (B.17), one obtains the invariance

$$\tilde{\mathbf{W}}(\tilde{\mathbf{L}}_\Omega(\tilde{x})) = QW(\mathbf{L}_\Omega(x))Q^T = Q\left(\frac{1}{2}(\mathbf{L}_\Omega(x) - \mathbf{L}_\Omega(x)^T)\right)Q^T$$

$$= Q\left(\frac{1}{2}(Q^T\mathbf{L}_\Omega(\tilde{x})Q - Q^T\tilde{\mathbf{L}}_\Omega(\tilde{x})^TQ)\right)Q^T = \frac{1}{2}(\tilde{\mathbf{L}}_\Omega(\tilde{x}) - \tilde{\mathbf{L}}_\Omega(\tilde{x})^T)$$

$$= \mathbf{W}(\mathbf{L}_\Omega(x)),$$

i.e. $\tilde{\mathbf{W}}(\cdot) = \mathbf{W}(\cdot)$, \(\text{(C.4)}\)

which explicitly shows that the relative vorticity tensor is an objective quantity only in the relative sense, since $\tilde{\mathbf{W}}(\cdot) = \mathbf{W}(\cdot)$, but, as discussed before, not in the absolute sense, due to its explicit frame dependence $\mathbf{W}(\mathbf{L}_\Omega(x)) = \mathbf{W}(\mathbf{L}(x)) + \Omega \neq \mathbf{W}(\mathbf{L}(x))$.

Appendix D. Classical vs. extended thermodynamics from the view of MFI

In a nutshell, the difference in the modelling procedure of classical and extended thermodynamics, described in detail in Müller & Ruggeri (1998); Jou et al. (2010); Siginer (2014); Ruggeri & Sugiyama (2015), boils down to the following: Extended thermodynamics extends the number of fields beyond the ones of ordinary or classical thermodynamics which are mostly the densities of mass, momentum and energy. Typical extensions are the fluxes of momentum and energy.

A simple way to obtain the evolution equations for these fluxes from a macroscopic basis is to generalize the classical thermodynamic procedure, however, always strictly in accord with the two incontrovertible principles of entropy and relativity, namely to satisfy the entropy balance law with a non-negative entropy production, and to ensure that all laws have the same form and all processes induced by these laws run in the same way in all inertial systems, either Galilean or Lorentzian, depending on whether the theory is non-relativistic or relativistic.

Hence, in extended thermodynamics the fluxes are no longer considered as mere control parameters but as own independent variables, thus enlarging the range of applicability of non-equilibrium thermodynamics to a vast domain of phenomena where memory, non-local, and non-linear effects are relevant. In particular for high-frequency phenomena the independent character of the fluxes is made evident by the fact that the fluxes are fast variables that decay to their local-equilibrium values after a short relaxation time, thus allowing to describe phenomena at (high) frequencies comparable to the inverse of the (low) relaxation times of the fluxes. Also the aspect that the field equations of ordinary thermodynamics are parabolic while extended thermodynamics is governed by hyperbolic systems allowing only finite speeds of propagation, shows that the procedure of extended thermodynamics is superior over the ordinary or classical approach of thermodynamics for irreversible processes.

Not summarized so far is the difference between these two approaches concerning the principle of material frame-indifference (MFI). For that, however, a closer look is necessary, which I briefly want to illustrate in terms of a generic example without specifying physical details, in order to bring out the essence of the modelling procedure in each case from the perspective of MFI:

In classical thermodynamics (for irreversible processes) the aim is to determine the space-time evolution of some fields, say $\phi = (\phi_1, \ldots, \phi_n, \ldots, \phi_N)$, also called densities, satisfying a set of unclosed balance equations

$$\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{F} = \Pi,$$ \(\text{(D.1)}\)

where the quantities $\mathbf{F}$ are called the fluxes and $\Pi$ the productions; in classical thermodynamics the latter are mostly predetermined and thus known quantities of the system which can be expressed in terms of $\phi$ and $\mathbf{F}$ and their gradients. If the balance law (D.1) for some component $\phi_n$ represents a conservation law, then the corresponding production $\Pi_n$ vanishes globally, if

\[1\] If the theory is not consistent with the second law of thermodynamics, dissipative phenomena cannot be modelled correctly.
not, then \( \Pi_n \) only vanishes locally in all states of equilibrium. In order to solve for the densities \( \phi \), the above unclosed equations of balance must be supplemented by constitutive relations which relate the unknown fluxes \( F \) to the densities \( \phi \) in a manner characteristic for the material considered. In classical thermodynamics, when employing a microscopic description based on the Boltzmann equation for some microscopic interaction model (mostly Maxwellian molecules), the constitutive relations for the fluxes will have the form

\[
F = F^{(e)}(\phi, \nabla \phi, \nabla^2 \phi, \Gamma), \tag{D.2}
\]

when applying the iterative Maxwell scheme \(^1\) up to second order (which mostly suffices as a good approximation to the exact values) — for example, for the fluxes of momentum and energy the first order Maxwell iterates represent the classical constitutive equations of Navier-Stokes and Fourier. The result (D.2) thus depends at each point on the values of the fields at that point and on their values in the immediate neighbourhood, symbolically denoted as the gradients \( \nabla \phi \) and \( \nabla^2 \phi \), but also on all parameters collectively denoted as \( \Gamma \) characterizing the state of the material, either if passively observed within a non-inertial frame of reference or if actively transformed into a new non-inertial state. \(^2\) Hence, it is in this sense that MFI is violated as expressed in Müller (1972, 1976). In other words, to demand frame-indifference for the constitutive relations (D.2) within the procedure of ordinary thermodynamics would obviously be wrong.

In extended thermodynamics, however, the number of fields is extended by identifying the constitutive flux \( F \) in (D.1) as a density satisfying its own balance equation, thus augmenting the set of equations (D.1) of classical thermodynamics with a new set of equations of higher order

\[
\begin{align*}
\frac{\partial \phi}{\partial t} + \nabla \cdot F &= \Pi, \\
\frac{\partial F}{\partial t} + \nabla \cdot J &= \Xi,
\end{align*} \tag{D.3}
\]

where \( J \) and \( \Xi \) are, respectively, the new (unclosed) fluxes and productions of the combined system \((\phi, F)\), i.e., in contrast to the productions \( \Pi \) of classical thermodynamics, the productions \( \Xi \) of extended thermodynamics are unknowns of the system which need to be modelled, too. Thus in order to solve for these densities, the balance equations (D.3) must be supplemented again by constitutive relations, which now, in the extended version, have to relate \( J \) and \( \Xi \) to the densities \((\phi, F)\). In the framework of extended thermodynamics, all constitutive relations of the augmented system (D.3) are modelled locally and instantaneously in space-time as

\[
J = J^{(e)}(\phi, F), \quad \Xi = \Xi^{(e)}(\phi, F), \tag{D.4}
\]

so that the fluxes \( J \) and the productions \( \Xi \) at a point and a time depend only on the densities \((\phi, F)\) at that point and time, in contrast to the form of the constitutive relations (D.2) in ordinary thermodynamics, which standardly depend also on the gradient of the densities. As

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\(^1\)See footnote on p. 15.

\(^2\)The explicit frame-dependency \( \Gamma \) in the iterative result (D.2) is essentially picked up through the balance equations (D.1) themselves when formulated for, or transformed to any non-inertial system. For inertial systems or transformations connecting inertial systems, however, all frame-dependency parameters \( \Gamma \) will vanish if the set of all balance equations (D.1) show frame-indifference under the inertial Galilei transformations (or Lorentz transformations in relativistic continuum mechanics), which for closed systems is a strict principle of nature and must always be satisfied — where here it is to note that the requirement ‘closed system’ need not to be closed in the thermodynamic sense, which obviously would be too restrictive. For example, the classical Navier-Stokes equations without external body forces are frame-indifferent and thus fully symmetric under Galilei transformations, although they are not thermodynamically closed due to their dissipative nature. Hence, under a ‘closed system’ regarding frame-indifference, we will henceforth only understand the much weaker requirement of no external body forces acting on the thermodynamic system, as such forces may break the space-time symmetries necessary for Galilean or Lorentzian invariance; for example, a spatially and temporally constant external force breaks the isotropy of space.
explained and discussed in detail in Müller & Ruggeri (1998) and Ruggeri & Sugiyama (2015), the local and instantaneous modelling ansatz (D.4) is superior to any other ansatz as this naturally leads to a thermodynamic system which is governed by hyperbolic field equations allowing only finite speeds of propagation, in contrast to the field equations of ordinary thermodynamics which, mainly due to the non-instantaneous structure of the fluxes (D.2), are in general of parabolic type allowing for unphysical propagations of infinite speed.

Now, with the aim to see how extended thermodynamics relates to classical thermodynamics, it is necessary to construct from the augmented balance equation (D.3) a constitutive relation for the fluxes \( \mathbf{F} \) of the classical type (D.2). This relation is provided again by a formal iterative scheme akin to the Maxwell scheme of classical thermodynamics: The first iterates of the fluxes \( \mathbf{F} \) are obtained from the right-hand sides of the extended balance laws (D.3) by putting the equilibrium values into the left-hand sides, while the second iterates are again obtained from the right-hand sides but now by putting the first iterates into the left-hand sides, and so on. Ultimately this iteration can be considered as a power expansion of the thermodynamic relaxation times of the fluxes \( \mathbf{F} \), where with each iterate the thermodynamic state is described further and further away from equilibrium.\footnote{Note that in a far-from-equilibrium case it is still a major open problem whether the solutions via Maxwellian iterations converge or not. A rigorous proof for convergence is still missing (Ruggeri & Sugiyama, 2015).} For small relaxation times, however, i.e. close or not far from equilibrium, the iteration can already be terminated at second order to yield a sufficiently good approximation for the fluxes, giving then the comparative result to (D.2) of classical thermodynamics

\[
\mathbf{F} = \mathbf{F}^{(c)}(\phi, \nabla \phi, \nabla^2 \phi, \Gamma) \sim \mathbf{F}^{(c)}(\phi, \nabla \phi, \nabla^2 \phi, \Gamma),
\]

showing that classical thermodynamics is only a valid theory close to equilibrium for small relaxation times. In other words, extended thermodynamics not only specifically shows that classical constitutive relations as (D.2), which (due to the appearance of gradients) are non-local in space, are approximations of some higher-order balance laws involving only certain spatially local constitutive relations (D.4), but also more generally that parabolic systems of classical theories in principle appear as approximations of corresponding higher-order hyperbolic systems when some relaxation times are negligible.

When regarding the problem of MFI, however, the above procedure of extended thermodynamics shows the striking feature that while the Maxwell iterate (D.5) for the fluxes \( \mathbf{F} \) picks up again the frame-dependency \( \Gamma \) when the balance laws (D.3) are formulated or transformed to a non-inertial system, the constitutive equations of the extended fluxes and productions (D.4), however, remain frame-indifferent. Not because of the fact that they do not transform, but because of the fact that they are modelled as such. Any frame-dependency \( \Gamma \) is manually eliminated from (D.4) by only allowing for frame-indifferent operators and fields, thus constituting a strong restriction on the possible functional structures of (D.4). For example, for non-inertial (time-dependent) rotations, the structure of (D.4) is restricted to isotropic functions.

However, as explained and discussed in Section 4.2, to restrict the extended constitutive relations (D.4) according to MFI and to proclaim it as an axiom of nature, is not convincing. For an appropriate axiom in physics to model form-invariantly (covariantly) and frame-indifferently (objectively) in accordance with all physical observations existing, the general principle of MFI, which is based on the non-inertial Euclidean transformations, has to be replaced by the reduced principle r-MFI (as formulated in Section 4.3) restricting constitutive modelling only according to the inertial Galilean transformations. For particular modelling cases, however, as turbulence modelling, where the microscopic description is exactly known, the axiom r-MFI can be naturally supplemented by including additional modelling restrictions as worked out in Section 5.
References

ARIKI, T. 2015 Covariance of fluid-turbulence theory. Phys. Rev. E 91 (5), 053001.

BARBOUR, J. B. 2004 Absolute or Relative Motion: The Deep Structure of General Relativity. Oxford University Press.

BARBOUR, J. B. & PFISTER, H. 1995 Mach’s Principle: From Newton’s Bucket to Quantum Gravity. Birkhäuser Verlag.

BARNETT, S. J. 1915 Magnetization by rotation. Physical Review 6 (4), 239–270.

BIHLO, A., CARDOSO-BIHLO, E. D. S. & POPOVYCH, R. O. 2012 Complete group classification of a class of nonlinear wave equations. J. Math. Phys. 53 (12), 123515.

BYTEV, V. O. 1972 Group properties of the Navier-Stokes equations. Chislennye metody mehaniki sploshnoy sredy (Numerical methods of continuum mechanics) 3, 13–17.

CANTWELL, B. J. 1978 Similarity transformations for the two-dimensional, unsteady, stream-function equation. J. Fluid Mech. 85 (2), 257–271.

DAFALIAS, Y. F. 2011 Objectivity in turbulence under change of reference frame and superposed rigid body motion. Journal of Engineering Mechanics 137 (10), 699–707.

DAFALIAS, Y. F. & YOUNIS, B. A. 2007 Objective tensorial representation of the pressure-strain correlations of turbulence. Mechanics Research Communications 34, 319–324.

DAFALIAS, Y. F. & YOUNIS, B. A. 2009 Objective model for the fluctuating pressure-strain-rate correlations. J. Eng. Mech. 135 (9), 1006–1014.

DANILOV, Y. A. 1967 Group properties of the Maxwell and Navier-Stokes equations. Acad. Sci. USSR. Preprint, Khurchatov Inst. Nucl. Energy, Moscow.

DIEKS, D. 2006 Another look at general covariance and the equivalence of reference frames. Studies in History and Philosophy of Modern Physics 37, 174–191.

EINSTEIN, A. 1918a Dialog über Einwände gegen die Relativitätstheorie. Naturwissenschaften 6, 697–702.

FREWER, M. 2009a More clarity on the concept of material frame-indifference in classical continuum mechanics. Acta Mechanica 202, 213–246.

FREWER, M. 2009b Proper invariant turbulence modelling within one-point statistics. J. Fluid Mech. 639, 37–64.

FREWER, M. 2009c Invariant algebraic wall-bounded turbulence modeling. CTR Annual Research Briefs 2009, 233–246.

FREWER, M. 2015a An example elucidating the mathematical situation in the statistical non-uniqueness problem of turbulence. arXiv:1508.06962.

FREWER, M. 2015b Application of Lie-group symmetry analysis to an infinite hierarchy of differential equations at the example of first order ODEs. arXiv:1511.00002.

FREWER, M. & KHUJADZE, G. 2016a Comments on Janocha et al. Lie symmetry analysis of the Hopf functional-differential equation”. Symmetry 8 (4), 23.
Frewer, M. & Khujadze, G. 2016b An example of how a methodological mistake aggravates erroneous results when only correcting the results and not the method itself. *ResearchGate*, 1–10.

Frewer, M. & Khujadze, G. 2016c On the use of applying Lie-group symmetry analysis to turbulent channel flow with streamwise rotation. *arXiv:1609.08155*.

Frewer, M., Khujadze, G. & Foysi, H. 2014a On the physical inconsistency of a new statistical scaling symmetry in incompressible Navier-Stokes turbulence. *arXiv:1412.3061*.

Frewer, M., Khujadze, G. & Foysi, H. 2014b Referee Reports - arxiv:1412.3061; Supplementary material to support the discussion on open peer review. *ResearchGate*.

Frewer, M., Khujadze, G. & Foysi, H. 2015a Comment on “Statistical symmetries of the Lundgren-Monin-Novikov hierarchy”. *Phys. Rev. E* 92, 067001.

Frewer, M., Khujadze, G. & Foysi, H. 2015b Objections to a Reply of Oberlack et al. *ResearchGate*, 1–9.

Frewer, M., Khujadze, G. & Foysi, H. 2016a A note on the notion “statistical symmetry”. *arXiv:1602.08039*.

Frewer, M., Khujadze, G. & Foysi, H. 2016b Comment on “Application of the extended Lie group analysis to the Hopf functional formulation of the Burgers equation”. *J. Math. Phys.* 57, 034102.

Frewer, M., Khujadze, G. & Foysi, H. 2016c On a new technical error in a further Reply by Oberlack et al. and its far-reaching effect on their original study. *ResearchGate*, 1–6.

Friedman, M. 1983 *Foundations of Space-Time Theories: Relativistic Physics and Philosophy of Science*. Princeton University Press.

Gallet, B. 2015 Exact two-dimensionalization of rapidly rotating large-Reynolds-number flows. *J. Fluid Mech.* 783, 412–447.

Havas, P. 1964 Four-dimensional formulations of Newtonian mechanics and their relation to the special and the general theory of relativity. *Rev. Mod. Phys.* 36, 938–965.

Horzela, A. & Kapuścik, E. 1992 Galilean covariant harmonic oscillator. In *Workshop on Harmonic Oscillators* (ed. D. Han, Y. S. Kim & W. W. Zachary), pp. 295–305. NASA Conference Publication.

Horzela, A., Kapuścik, E. & Kempczyński, J. 1991 On the Galilean covariance of classical mechanics. *Tech. Rep.* INP–1556/PH. Institute of Nuclear Physics, Kraków (Poland).

Hutter, K. & Jöhnk, K. 2004 *Continuum Methods of Physical Modeling*. Springer Verlag.

Ibragimov, N. H., Gandarias, M. L., Galiakberova, L. R., Bruzon, M. S. & Avdonina, E. D. 2016 Group classification and conservation laws of anisotropic wave equations with a source. *J. Math. Phys.* 57 (8), 083504.

Ikenberry, E. & Truesdell, C. 1956 On the pressures and the flux of energy in a gas according to Maxwell’s kinetic theory. *J. Rat. Mech. Anal.* 5, 1–54.

Jou, D., Casas-Vázquez, J. & Lebon, G. 2010 *Extended Irreversible Thermodynamics*, 4th edn. Springer Verlag.
Khujadze, G. & Frewer, M. 2016 Revisiting the Lie-group symmetry method for turbulent channel flow with wall transpiration. arXiv:1606.08396.

Kirwan, A. D. 2016 On objectivity, irreversibility and non-Newtonian fluids. Fluids 1 (1), 3.

Kontogiorgis, S. & Sophocleous, C. 2016 Group classification of systems of diffusion equations. Mathematical Methods in the Applied Sciences, doi:10.1002/mma.4094.

Kretschmann, E. 1917 Über den physikalischen Sinn der Relativitätspostulate, A. Einsteins neue und seine ursprüngliche Relativitätstheorie. Ann. Physik 53, 575–614.

Liu, I.-S. & Lee, J. D. 2016 On material objectivity of intermolecular force in molecular dynamics. Acta Mechanica pp. 1–8.

Liu, I.-S. & Sampaio, R. 2014 Remarks on material frame-indifference controversy. Acta Mechanica 225 (2), 331–348.

Lumley, J. L. 1970 Toward a turbulent constitutive relation. J. Fluid Mech. 41, 413–434.

Lumley, J. L. 1983 Turbulence modeling. J. Appl. Mech. 50, 1097–1103.

Mach, E. 1883 Die Mechanik in ihrer Entwicklung, 9th edn. Wissenschaftliche Buchgesellschaft Darmstadt, Reprografischer Nachdruck 1991.

Machicoane, N., Moisy, F. & Cortet, P.-P. 2016 Two-dimensionalization of the flow driven by a slowly rotating impeller in a rapidly rotating fluid. arXiv:1611.00905.

Matolcsi, T. & Ván, P. 2006 Can material time derivative be objective? Physics Letters A 353 (2), 109–112.

Matolcsi, T. & Ván, P. 2007 Absolute time derivatives. J. Math. Phys. 48 (5), 053507.

Matsuo, M., Ieda, J. & Maekawa, S. 2015 Mechanical generation of spin current. Frontiers in Physics 3 (54), 1–10.

Müller, I. 1972 On the frame dependence of stress and heat flux. Arch. Rational Mech. Anal. 45, 241–250.

Müller, I. 1976 On the frame dependence of electric current and heat flux in a metal. Acta Mechanica 24, 117–128.

Müller, I. & Liu, I.-S. 1983 Extended thermodynamics of classical and degenerate ideal gases. Arch. Rat. Mech. Anal. 83, 285–332.

Müller, I. & Ruggeri, T. 1998 Rational Extended Thermodynamics, 2nd edn. Springer Verlag.

Muschik, W. 2012a Is the heat flux density really non-objective? A glance back, 40 years later. Continuum Mechanics and Thermodynamics 24 (4), 333–337.

Muschik, W. 2012b Objectivity and frame indifference of acceleration-sensitive materials. Journal of Theoretical and Applied Mechanics 50 (3), 807–817.

Muschik, W. 2012c Comment on: I-Shih Liu: Constitutive theory of anisotropic rigid heat conductors. Continuum Mechanics and Thermodynamics 24 (2), 175–180.

Muschik, W. 2013 Brief remark on material motion dependence. J. Non-Equilib. Thermodyn. 38 (4), 391–398.
Ünal, G. 1994 Application of equivalence transformations to inertial subrange of turbulence. *Lie Groups Appl.* **1** (1), 232–240.

Ünal, G. 1995 Equivalence transformations of the Navier-Stokes equations and the inertial range of turbulence. In *Continuum Models and Discrete Systems* (ed. K. Z. Markov), pp. 480–487. World Scientific.

Norton, J. D. 1993 General covariance and the foundations of general relativity: eight decades of dispute. *Rep. Prog. Phys.* **56**, 791–858.

Norton, J. D. 1995 Did Einstein stumble? The debate over general covariance. *Erkenntnis, Kluwer Academic Publishers* **42**, 223–245.

Panicaud, B. & Rouhaud, E. 2014 A frame-indifferent model for a thermo-elastic material beyond the three-dimensional Eulerian and Lagrangian descriptions. *Continuum Mechanics and Thermodynamics* **26** (1), 79–93.

Panicaud, B., Rouhaud, E., Altmeyer, G., Wang, M., Kerner, R., Roos, A. & Ameline, O. 2016 Consistent hypo-elastic behavior using the four-dimensional formalism of differential geometry. *Acta Mechanica* **227** (3), 651–675.

Penrose, R. 2005 *The Road to Reality: A Complete Guide to the Laws of the Universe*. A. Knopf Publishing.

Pfister, H. & King, M. 2015 *Inertia and Gravitation: The Fundamental Nature and Structure of Space-time*. Springer Verlag.

Pukhnachev, V. V. 1960 Group properties of the equations of Navier-Stokes in the plane. *Journal of Appl. Mech. and Tech. Phys.* **1**, 83–90.

Pukhnachev, V. V. 1972 Invariant solutions of the Navier-Stokes equations describing motion with a free boundary. *Dokl. Acad. Sci. USSR* **202** (2), 302–305.

Romano, G. & Barretta, R. 2013 Geometric constitutive theory and frame invariance. *International Journal of Non-Linear Mechanics* **51**, 75–86.

Rouhaud, E., Panicaud, B. & Kerner, R. 2013 Canonical frame-indifferent transport operators with the four-dimensional formalism of differential geometry. *Computational Materials Science* **77**, 120–130.

Ruggeri, T. & Sugiyama, M. 2015 *Rational Extended Thermodynamics beyond the Monatomic Gas*. Springer Verlag.

Sadiki, A. & Hutter, K. 1996 On the frame dependence and form invariance of the transport equations for the Reynolds stress tensor and the turbulent heat flux vector: its consequences on closure models in turbulence modelling. *Cont. Mech. Thermodyn.* **8**, 341–349.

Schrödinger, E. 1950 *Space-Time Structure*, 4th edn. Cambridge University Press, 1985.

Sexl, R. U. & Urbantke, H. K. 2001 *Relativity, Groups, Particles: Special Relativity and Relativistic Symmetry in Field and Particle Physics*, 4th edn. Springer Verlag.

Signer, D. A. 2014 *Stability of Non-Linear Constitutive Formulations for Viscoelastic Fluids*. Springer Verlag.

Spalart, P. R. & Speziale, C. G. 1999 A note on constraints in turbulence modelling. *J. Fluid Mech.* **391**, 373–376.
Speziale, C. G. 1979 Invariance of turbulent closure models. *Phys. Fluids* **22** (6), 1033–1037.

Speziale, C. G. 1981 Some interesting properties of two-dimensional turbulence. *Phys. Fluids* **24** (8), 1425–1427.

Speziale, C. G. 1984 On the invariance of constitutive equations according to the kinetic theory of gases. *J. Stat. Phys.* **35** (3), 457–470.

Speziale, C. G. 1987 Comments on the “material frame-indifference” controversy. *Phys. Rev. A* **36** (9), 4522–4525.

Speziale, C. G. 1989 Turbulence modeling in noninertial frames of reference. *Theoretical and Computational Fluid Dynamics* **1**, 3–19.

Speziale, C. G. 1991 Analytical methods for the development of Reynolds-stress closures in turbulence. *Annu. Rev. Fluid Mech.* **23**, 107–157.

Speziale, C. G. 1998 A review of material frame-indifference in mechanics. *Appl. Mech. Rev.* **51**, 489–504.

Stephani, H. 2004 *Relativity: An Introduction to Special and General Relativity*, 3rd edn. Cambridge University Press.

Truesdell, C. & Noll, W. 1965 *The Non-Linear Field Theories of Mechanics*, 3rd edn. Springer Verlag, 2004.

Ván, P. 2015 Galilean relativistic fluid mechanics. *arXiv:1508.00121*.

Yang, Z., Lee, J. D., Liu, I.-S. & Eskandarian, A. 2016 On non-equilibrium molecular dynamics with Euclidean objectivity. *Acta Mechanica*, doi:10.1007/s00707-016-1735-x.