Measures of noncompactness in some new lacunary difference sequence spaces

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Abstract: In this research article, we establish some identities and estimates for the operator norms and the Hausdorff measures of noncompactness of certain operators on some lacunary difference sequence spaces defined by Orlicz function. Moreover, we apply our results to characterize some classes of compact operators on those spaces by using the Hausdorff measure of noncompactness.

Key Words: BK-space; Matrix transformation; Compact operator; Hausdorff measure of noncompactness; lacunary operator, Orlicz function; Difference operator.

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1 Introduction

Measures of noncompactness were first introduced and later on applied in fixed point theory by Kuratowski [11] and Darbo [8]. Hausdorff measure of noncompactness was introduced by Goldenstein et al. and later on it was studied in broad sense by Eberhard Malkowsky et al. [5], Feyzi Basar et al. [7], Eberhard Malkowsky and Ekrem Savas [6], Mohammed Mursaleen et al. [12,13] and many others. Some identities or estimations for the operator norms and the Hausdorff measures of noncompactness of certain matrix operators on some sequence spaces

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were studied and established.

An important application of the Hausdorff measure of noncompactness of bounded linear operators between Banach spaces is the characterization of compact matrix transformations between BK spaces. W.L.C. Sargent proved that the characterizations of compact matrix operators between the classical sequence spaces in almost all cases.

Let $S$ and $M$ be subsets of a metric space $(X,d)$ and if for $\varepsilon > 0$ and for every $x \in M$ there exists $s \in S$ such that $d(x,s) < \varepsilon$, then $S$ is called an $\varepsilon$-net of $M$ in $X$.

Let $\mathcal{M}_X$ be a collection of all bounded subsets of a metric space $(X,d)$. The Hausdorff measure of noncompactness of the set $Q$, denoted by $\chi(Q)$, is defined by, $\chi(Q) = \inf \{ \varepsilon > 0 : Q$ has a finite $\varepsilon$ – net in $X \}$, where $Q \in \mathcal{M}_X$. The function $\chi : \mathcal{M}_X \to [0, \infty)$ is called the Hausdorff measure of noncompactness.

If $Q, Q_1$ and $Q_2$ are bounded subsets of a metric space $(X,d)$, then [see Malkowsky [5]),

$\chi(Q) = 0$ if and only if $Q$ is totally bounded ,

$Q_1 \subset Q_2$ implies $\chi(Q_1) \leq \chi(Q_2)$.

Further, the function $\chi$ has some additional properties connected with the linear structure, e.g.

$\chi(Q_1 + Q_2) \leq \chi(Q_1) + \chi(Q_2),$

$\chi(\alpha Q) = |\alpha|\chi(Q)$, for all $\alpha \in C$.

Let $X$ and $Y$ be Banach spaces and $L \in B(X,Y)$. Then, the Hausdorff measure of noncompactness of $L$, denoted by $\|L\|_\chi$, can be defined by,

$$\|L\|_\chi = \chi(L(S_X)) = \chi(L(B_X))$$ (1)

and we have,

$L$ is compact if and only if $\|L\|_\chi = 0$. (2)
2 Some preliminary concepts

By a lacunary sequence, we mean an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$, where the intervals determined by $\theta$ will be denoted by $J_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ is defined by $\phi_r$.

For any lacunary sequence $\theta = (k_r)$, the space $N_\theta$ is defined as, (Freedman et al. [2])

$$N_\theta = \{(x_k) : \lim_{r \to \infty} h_r^{-1} \sum_{k \in J_r} |x_k - L| = 0, \text{ for some } L\}.$$ 

The space $N_\theta$ is a BK space with the norm,

$$\|(x_k)\|_\theta = \sup_r h_r^{-1} \sum_{k \in J_r} |x_k|.$$ 

An Orlicz function is defined as a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

The idea of Orlicz function was used to construct the sequence space, [see Lindenstrauss and Tzafriri [10]]

$$\ell_M = \left\{(x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\}$$

which is a Banach space with the norm, called as Orlicz sequence space,

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}.$$ 

The difference sequence spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ of crisp sets are defined as $Z(\Delta) = \{x = (x_k) : (\Delta x_k) \in Z\}$, for $Z = \ell_\infty, c$ and $c_0$, where $\Delta x_k = (\Delta x_k) = (x_k - x_{k+1})$, for all $k \in \mathbb{N}$, which can be a Banach space with $\|x\|_\Delta = |x_1| + \sup_{k} |\Delta x_k|$.

The generalized difference sequence spaces are defined as, for $m \geq 1$ and $n \geq 1$,

$$Z(\Delta_m^n) = \{x = (x_k) : (\Delta_m^n x_k) \in Z\}, \text{ for } Z = \ell_\infty, c \text{ and } c_0.$$
Let $X$ is any subset of $w$, then a matrix domain of an infinite matrix $A$ in $X$ is defined by, $X_A = \{x \in w : Ax \in X\}$. If $x \supset \phi$ is a BK-space and $a = (a_k) \in w$, then we define,

$$
\|a\|_X = \sup_{x \in X} \left| \sum_{k=0}^{\infty} a_k x_k \right| .
$$  \hspace{0.5cm} (3)

### 3 The difference sequence spaces $c^\lambda_0(M, \Delta, s, \theta)$, $c^\lambda(M, \Delta, s, \theta)$ and $\ell^\lambda_{\infty}(M, \Delta, s, \theta)$

Consider $\lambda = (\lambda_k)_{k=0}^\infty$ to be a strictly increasing sequence of positive reals such that $\lambda_k \to \infty$ as $k \to \infty$. We define the infinite matrix $\Lambda = (\lambda_{nk})_{n,k=0}^\infty$ by,

$$
\Lambda_{nk} = \begin{cases} 
\frac{(\lambda_{k-1} - \lambda_k) - (\lambda_{k+1} - \lambda_k)}{\lambda_k}; & (k < n), \\
\frac{\lambda_k - \lambda_{k-1}}{\lambda_k}; & (k = n), \\
0; & (k > n), 
\end{cases}
$$  \hspace{0.5cm} (4)

where, we shall use the convention that any term with a negative subscript is equal to zero. Mursaleen and Noman [13] introduced the difference sequence spaces $c^\lambda_0(\Delta)$ and $\ell^\lambda_{\infty}(\Delta)$ as the matrix domains of the triangle $\Lambda$ in the spaces $c_0$ and $\ell_{\infty}$ respectively.

In this paper, we study the sequence spaces $c^\lambda_0(M, \Delta, s, \theta)$, $c^\lambda(M, \Delta, s, \theta)$ and $\ell^\lambda_{\infty}(M, \Delta, s, \theta)$ and try to estimate for the operator norms and the Hausdorff measures of noncompactness of certain operators on these spaces. The spaces $c^\lambda_0(M, \Delta, s, \theta)$, $c^\lambda(M, \Delta, s, \theta)$ and $\ell^\lambda_{\infty}(M, \Delta, s, \theta)$ are BK-spaces with the norm given by,

$$
\|x\|_{c^\lambda_0(M, \Delta, s, \theta)} = \|\Lambda(x)\|_{c^\lambda_0(M, \Delta, s, \theta)} = \inf \left\{ \rho > 0 : \lim_{r \to 0} \frac{1}{r} \sum_{k=1}^{\infty} \left( M \left( \frac{|\Lambda_k(x)|}{\rho} \right) \right)^{\theta_k} \leq 1 \right\} . \hspace{0.5cm} (5)
$$

The $\beta$–duals of a subset $X$ of $w$ are respectively defined by,

$$
X^\beta = \{a = (a_k) \in w : ax = (a_k x_k) \in cs, \text{ for all } x = (x_k) \in X\}
$$

**Lemma 3.1.** Let $X$ denote any of the spaces $c^\lambda_0(M, \Delta, s, \theta)$ or $\ell^\lambda_{\infty}(M, \Delta, s, \theta)$. Then, we have,
\[ \|a\|_X^* = \|\overline{a}\|_\ell_1 = \sum_{k=0}^{\infty} |\overline{a}_k| < \infty \]  

(6)

for all \( a = (a_k) \in X^\beta \), where,

\[ \overline{a}_k = \lambda_k \left[ \frac{a_k}{\lambda_k - \lambda_{k-1}} + \left( \frac{1}{\lambda_k - \lambda_{k-1}} - \frac{1}{\lambda_{k+1} - \lambda_k} \right) \sum_{j=k+1}^{\infty} a_j \right]; (k \in N) \]  

(7)

**Proof:** Let \( Y \) be the respective one of the spaces \( c_0 \) or \( \ell_\infty \).

Assume \( a = (a_k) \in X^\beta \) and \( y = \overline{A}(x) \) be the associated sequence defined by,

\[ y_k = \sum_{j=0}^{k} \left( \frac{\lambda_j - \lambda_{j-1}}{\lambda_k} \right) (x_j - x_{j-1}); (k \in N). \]  

(8)

Taking \( y = \overline{A}(x) \) as the associated sequence, we have \( \overline{a} = (\overline{a}_k) \in \ell_1 \) such that for every \( x = (x_k) \in X \),

\[ \sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} \overline{a}_k y_k, \]  

(9)

Since \( x \in S_X \) if and only if \( y \in S_Y \), (followed by (7)), we can derive that, (by (1) and (9))

\[ \|a\|_X^* = \inf \left\{ \rho > 0 : \lim_{r \to 1} \frac{1}{h_r} \sum_{k=0}^{\infty} \left( M \left( \frac{|a_k x_k|}{\rho} \right) \right)^{\gamma_k} \leq 1, x \in S_X \right\} \]

\[ = \inf \left\{ \rho > 0 : \lim_{r \to 1} \frac{1}{h_r} \sum_{k=0}^{\infty} \left( M \left( \frac{|\overline{a}_k y_k|}{\rho} \right) \right)^{\gamma_k} \leq 1, y \in S_Y \right\} \]

\[ = \|\overline{a}\|_Y^* \]

It is known that \( \|\cdot\|_X^\beta = \|\cdot\|_{X^\beta} \) on \( X^\beta \), where \( \|\cdot\|_{X^\beta} \) denotes the natural norm on the dual space \( X^\beta \) and \( X = c_0, c, \ell_\infty \) or \( \ell_p (1 \leq p < \infty) \).

So if \( \overline{a} \in \ell_1 \), we obtain that \( \|a\|_X^* = \|\overline{a}\|_Y^* = \|\overline{a}\|_{\ell_1} < \infty \), which concludes the proof.
Let $A = (a_{nk})$ be an infinite matrix and $\overline{A} = (\overline{a}_{nk})$ is the associated matrix defined by,

$$\overline{a}_{nk} = \lambda_k \left[ \frac{a_{nk}}{\lambda_k - \lambda_{k-1}} + \left( \frac{1}{\lambda_k - \lambda_{k-1}} - \frac{1}{\lambda_{k+1} - \lambda_k} \right) \sum_{j=k+1}^{\infty} a_{nj} \right] ; (n, k \in N) \quad (10)$$

**Lemma 3.2.** Let $X$ be any of the spaces $c_0^\lambda (M, \Delta, s, \theta)$ or $\ell_1^\lambda (M, \Delta)$ and $Z$ be a sequence space. If $A \in (X, Z)$, then $\overline{A} \in (Y, Z)$ such that $Ax = \overline{A}y$ for all sequences $x \in X$ and $y \in Y$, here $Y$ is the respective one of the spaces $c_0$ or $\ell_1$.

**Proof.** Suppose that $A \in (X, Z)$.

For any sequence $x = (x_k) \in w$ and the associated sequence $y = \overline{\lambda}(x)$ defined in (8), we have,

$$x_k = \sum_{j=0}^{k} \left( \frac{\lambda_j y_j - \lambda_{j-1} y_{j-1}}{\lambda_j - \lambda_{j-1}} \right) ; (k \in N). \quad (11)$$

Then, $A_n \in X^\beta$ for all $n \in N$. Also, $\overline{A}_n \in \ell_1 \beta \in Y^\beta$ for all $n \in N$ and the equality $Ax = \overline{A}y$, followed by equations (8), (9) and (10). Hence, $\overline{A}y \in Z$. Further, by (11), we get that every $y \in Y$ is the associated sequence of some $x \in X$. Thus, it can be deduced that $\overline{A} \in (Y, Z)$, which completes the proof.

**Lemma 3.3.** Let $X$ be any of the spaces $c_0^\lambda (M, \Delta, s, \theta)$ or $\ell_1^\lambda (M, \Delta, s, \theta)$, $A = (a_{nk})$ an infinite matrix and $\overline{A} = (\overline{a}_{nk})$ the associated matrix. If $A$ is in any of the classes $(X, c_0), (X, c)$ or $(X, \ell_1)$, then,

$$||L_A|| = ||A||_{(X, c_0)} = \sup_n \left\{ \inf \left( \rho > 0 : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k=0}^{\infty} M \left( \frac{||\overline{a}_{nk}||}{\rho} \right)^{s_k} \leq 1 \right) \right\} < \infty$$
4 Compact operators on the spaces $c^\lambda_0(M, \Delta, s, \theta)$ and $\ell^\lambda_\infty(M, \Delta, s, \theta)$

In this section, we are trying to establish some identities or estimates for the Hausdorff measures of noncompactness of certain matrix operators on the spaces $c^\lambda_0(M, \Delta, s, \theta)$ and $\ell^\lambda_\infty(M, \Delta, s, \theta)$. Also, the results obtained by examining these sequence spaces are applied to characterize some classes of compact operators on those spaces.

**Remark:** 4.1. Let $X$ denote any of the spaces $c_0$ or $\ell_\infty$. If $A \in (X, c)$, then we have,

- $\alpha_k = \lim_{n \to \infty} a_{nk}$ exists for every $k \in \mathbb{N}$,
- $\alpha = (\alpha_k) \in \ell_1$,
- $\sup_n \left( \sum_{k=0}^\infty |a_{nk} - \alpha_k| \right) < \infty$,
- $\lim_{n \to \infty} A_n(x) = \sum_{k=0}^\infty \alpha_k x_k$ for all $x = (x_k) \in X$.

**Theorem 4.2.** Assume $A = (a_{nk})$ be an infinite matrix and $\overline{A} = (\overline{a}_{nk})$ the associated matrix defined by (10). Then, we have the following results on the Hausdorff measures of noncompactness on the sequence spaces $X = c^\lambda_0(M, \Delta, s, \theta)$, $c^\lambda(M, \Delta, s, \theta)$ or $\ell^\lambda_\infty(M, \Delta, s, \theta)$.

(a) If $A \in (X, c_0)$, then,

$$
\|L_A\|_X = \lim_{n \to \infty} \sup \left\{ \rho > 0 : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k=0}^\infty \left( M \left( \frac{\overline{a}_{nk}}{\rho} \right) \right)^{s_k} \leq 1 \right\}.
$$

(b) If $A \in (X, c)$, then,

$$
\frac{1}{2} \lim_{n \to \infty} \sup \left\{ \rho > 0 : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k=0}^\infty \left( M \left( \frac{\overline{a}_{nk} - \alpha_k}{\rho} \right) \right)^{s_k} \leq 1 \right\}
$$
\[ \leq \|L_A\|_\chi \leq \lim_{n \to \infty} \sup \left\{ \rho > 0 : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k=0}^{\infty} \left( M \left( \frac{a_{nk} - \alpha_k}{\rho} \right) \right)^{\delta_k} \leq 1 \} \right\}, \tag{13} \]

where \( \alpha_k = \lim_{n \to \infty} a_{nk} \) for all \( k \in \mathbb{N} \).

(c) If \( A \in (X, \ell_\infty) \), then,

\[ 0 \leq \|L_A\|_\chi \leq \lim_{n \to \infty} \sup \left\{ \rho > 0 : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k=0}^{\infty} \left( M \left( \frac{a_{nk}}{\rho} \right) \right)^{\delta_k} \leq 1 \} \right\}. \tag{14} \]

**Proof:** Following Lemma 3.3, it can be easily proved that the expressions in (12) and (13) exist.

Similarly, following Remark 4.1. and Lemma 3.2, we can deduce that the expression in (15) also exists.

Let \( X \supset \phi \) and \( Y \) be BK-spaces. Then, we have, \((X, Y) \subset B(X, Y)\), that is, every matrix \( A \in (X, Y) \) defines an operator \( L_A \in B(X, Y) \) by \( L_A(x) = Ax \) for all \( x \in X \).

So,

\[ \|L_A\|_\chi = \chi(AS), \text{ where } S = S_X, \text{ by (1)} \tag{15} \]

Let \( P_r : c_0 \to c_0 \) \((r \in \mathbb{N})\) be the operator defined by \( P_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots) \) for all \( x = (x_n) \in c_0 \). Then, we have, for \( Q \in \mathcal{M}_{c_0} \),

\[ \chi(Q) = \lim_{r \to \infty} \sup_{x \in Q} \| (I - P_r)(x) \|_{\ell_\infty}. \]

where \( I \) is the identity operator on \( c_0 \).

Further, every \( z = (z_n) \in c \) has a unique representation as \( z = \overline{z}e + \sum_{n=0}^{\infty} (z_n - \overline{z})e^{(n)} \),

where \( \overline{z} = \lim_{n \to \infty} z_n \). The projectors \( P_r : c \to c \) \((r \in \mathbb{N})\) are obtained by,

\[ P_r(z) = \overline{z}e + \sum_{n=0}^{r} (z_n - \overline{z})e^{(n)}; \text{ (r \in \mathbb{N})} \tag{16} \]

for all \( z = (z_n) \in c \) with \( \overline{z} = \lim_{n \to \infty} z_n \).
Let $AS \in \mathcal{M}_{c_0}$. Then,

$$
\chi(AS) = \lim_{r \to \infty} \sup_{x \in S} \|(I - P_r)(A)\|_{\ell_{\infty}},
$$

(17)

where $P_r : c_0 \to c_0(r \in N)$.

This implies that,

$$
\|(I - P_r)(Ax)\|_{\ell_{\infty}} = \sup_{n > r} |A_n(x)|, \text{ for all } x \in X \text{ and every } r \in N.
$$

(18)

For an infinite matrix $A = (a_{nk})_{n,k=0}^{\infty}$ we have the $A$-transform of $x$ as the sequence $Ax = (A_n(x))_{n=0}^{\infty}$ where $A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k$, for $x \in w$ and $n \in N$.

Thus, we get, (by (3) and Lemma 3.1)

$$
\sup_{x \in S} \|(I - P_r)(Ax)\|_{\ell_{\infty}} = \sup_{n > r} \|A_n\|^{*}_{(\ell_{\infty}(M \Delta X, \theta))} = \sup_{n > r} \|A_n\|_{\ell_1}, \text{ for every } r \in N.
$$

Which implies that, (using above with(17))

$$
\chi(AS) = \lim_{r \to \infty} (\sup_{n > r} \|A_n\|_{\ell_1}) = \lim_{r \to \infty} \sup_{n > r} \|A_n\|_{\ell_1}.
$$

This concludes the proof of (a).

To prove (b), let us take $Q \in \mathcal{M}_c$ and $P_r : c \to c(r \in N)$ be the projector onto the linear span of $\{e, e^{(0)}, e^{(1)}, \ldots, e^{(0)}\}$. Then, we have

$$
\frac{1}{2} \lim_{r \to \infty} \left( \sup_{x \in Q} \|(I - P_r)(x)\|_{\ell_{\infty}} \right) \leq \chi(Q) \leq \lim_{r \to \infty} \left( \sup_{x \in Q} \|(I - P_r)(x)\|_{\ell_{\infty}} \right),
$$

where $I$ is the identity operator on $c$.

Since we have $AS \in \mathcal{M}_c$. We can get an estimate for the value of $\chi(AS)$ in (15). For this, let $P_r : c \to c(r \in N)$ be the projectors defined by (16).

Then, we have for every $r \in N$ that,
\[(I - P_r)(z) = \sum_{n=r+1}^{\infty} (z_n - \bar{z}) e_n\]

and hence,

\[\| (I - P_r)(z) \|_{\ell_\infty} = \sup_{n > r} |z_n - \bar{z}| \quad (19)\]

for all \( z = (z_n) \in c \) and every \( r \in N \), where \( \bar{z} = \lim_{n \to \infty} z_n \) and \( I \) is the identity operator on \( c \).

By using (15), we obtain,

\[\frac{1}{2} \lim_{r \to \infty} \sup_{x \in S} \| (I - P_r)(Ax) \|_{\ell_\infty} \leq \| L_A \|_X \leq \lim_{r \to \infty} \sup_{x \in S} \| (I - P_r)(Ax) \|_{\ell_\infty} \quad (20)\]

On the other hand, it is given that \( X = c_0^\lambda (M, \Delta, s, \theta) \) or \( X = \ell_\infty^\lambda (M, \Delta, s, \theta) \), and let \( Y \) be the respective one of the spaces \( c_0 \) or \( \ell_\infty \). Also, let \( y \in Y \) be the associated sequence defined by (8). Since \( A \in (X, c) \), we have from Lemma 3.2 that \( \bar{A} \in (Y, c) \) and \( Ax = \bar{A} y \). Further, we have the limits \( \bar{a}_k = \lim_{n \to \infty} a_{nk} \) exist for all \( k, \bar{a} = (\bar{a}_k) \in \ell_1 = Y^\beta \) and \( \lim_{n \to \infty} A_n(y) = \sum_{k=0}^{\infty} \bar{a}_k y_k \). (Remark 4.1)

Consequently,

\[\| (I - P_r)(Ax) \|_{\ell_\infty} = \| (I - P_r)(\bar{A} y) \|_{\ell_\infty}\]

\[= \sup_{n > r} |\bar{A}_n(y) - \sum_{k=0}^{\infty} \bar{a}_k y_k|\]

\[= \sup_{n > r} \sum_{k=0}^{\infty} (\bar{a}_{nk} - \bar{a}_k) y_k, \text{ for all } r \in N \text{ (by (21))}.\]

Moreover, since \( x \in S = S_X \) if and only if \( y \in S_Y \), we get,

\[\sup_{x \in S} \| (I - P_r)(Ax) \|_{\ell_\infty} = \sup_{n > r} \left( \sup_{y \in S_Y} \sum_{k=0}^{\infty} (\bar{a}_{nk} - \bar{a}_k) y_k \right)\]

\[= \sup_{n > r} |\bar{A}_n \bar{a}|_Y\]

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\[
= \sup_{n>r} \| \overline{A_n a} \|_{\ell_1}
\]
for all \( r \in N \).

This concludes the proof.

Finally, to prove (c), let us define the operators \( P_r : \ell_\infty \to \ell_\infty (r \in N) \) as in the proof of part (a) for all \( x = (x_k) \in \ell_\infty \). Then, we have,

\[
AS \subset P_r(AS) + (I - P_r)(AS); (r \in N).
\]

Thus, following the elementary properties of the function \( \chi \), we have,

\[
0 \leq \chi(AS) \leq \chi(P_r(AS)) + \chi((I - P_r)(AS))
= \chi((I - P_r)(AS))
\leq \sup_{x \in S} \| (I - P_r)(Ax) \|_{\ell_\infty}
= \sup_{n>r} \| A_n \|_{\ell_1}, \text{ for all } r \in N.
\]

Hence,

\[
0 \leq \chi(AS) \leq \lim_{r \to \infty} (\sup_{n>r} \| A_n \|_{\ell_1})
= \lim_{n \to \infty} \sup_{n>r} \| A_n \|_{\ell_1}.
\]

Combining this together with (15), imply (14), which completes the proof.

**Corollary 4.3.** Let \( X \) denote any of the spaces \( c_0^\lambda (M, \Delta, s, \theta) \) or \( \ell_\infty^\lambda (M, \Delta, s, \theta) \). Then, we have,

(a) If \( A \in (X, c_0) \), then,

\[
L_A \text{ is compact iff } \lim_{n \to \infty} \left( \inf \left\{ \rho > 0 : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k=0}^{\infty} \frac{M\left( \overline{a_{nk}} \right)^{s_k}}{\rho} \leq 1 \right\} \right) = 0.
\]

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(b) If $A \in (X, c)$, then,

$$L_A \text{ is compact iff } \lim_{n \to \infty} \left( \inf \left\{ \rho > 0 : \lim_{h \to 0} \frac{1}{h} \sum_{k=0}^{\infty} M \left( \left| \frac{\overline{a}_{nk} - \overline{a}_k}{\rho} \right| \right)^{s_k} \right\} \leq 1 \right) = 0 \text{ where } \overline{a}_k = \lim_{n \to \infty} \overline{a}_{nk} \text{ for all } k \in N.$$

(c) If $A \in (X, \ell_\infty)$, then,

$$L_A \text{ is compact if } \lim_{n \to \infty} \left( \inf \left\{ \rho > 0 : \lim_{h \to 0} \frac{1}{h} \sum_{k=0}^{\infty} M \left( \left| \frac{\overline{a}_{nk}}{\rho} \right| \right)^{s_k} \right\} \leq 1 \right) = 0.$$

### 5 Some applications

By applying the previous results, in this section, we are trying to establish some identities or estimates for the operator norms and the Hausdorff measure of non-compactness of certain matrix operators that map any of the spaces $c_0^\lambda(M, \Delta, s, \theta), c^\lambda(M, \Delta, s, \theta)$ and $\ell_\infty^\lambda(M, \Delta, s, \theta)$ into the matrix domains of triangles in the spaces $c_0, c$ and $\ell_\infty$. Further, we deduce the necessary and sufficient conditions for such operators to be compact.

**Lemma 5.1.** Let $T$ be a triangle. Then, we have,

- For arbitrary subsets $X$ and $Y$ of $w$, $A \in (X, Y_T)$ if and only if $B = TA \in (X, Y)$.
- Further, if $X$ and $Y$ are BK spaces and $A \in (X, Y_T)$, then $\|L_A\| = \|L_B\|$.

Throughout, we assume that $A = (a_{nk})$ is an infinite matrix and $T = (t_{nk})$ is a triangle, and we define the matrix $B = (b_{nk})$ by $b_{nk} = \sum_{m=0}^{n} t_{nm}a_{mk}; (n, k \in N)$, that is $B = TA$ and hence,

$$B_n = \sum_{m=0}^{n} t_{nm}A_m = \left( \sum_{m=0}^{n} t_{nm}a_{mk} \right)_{k=0}^{\infty}; (n \in N).$$

Consider $\overline{A} = (\overline{a}_{nk})$ and $\overline{B} = (\overline{b}_{nk})$ be the associated matrices of $A$ and $B$, respectively. Then it can easily be seen that,
\[ \overline{b}_{nk} = \sum_{m=0}^{n} t_{nm} \overline{a}_{mk}; (n, k \in N). \]

Hence, \[ \overline{B}_n = \sum_{m=0}^{n} t_{nm} \overline{A}_m = \left( \sum_{m=0}^{n} t_{nm} \overline{a}_{mk} \right)_{k=0}^{\infty}; (n \in N). \]

Moreover, we define the sequence \( \overline{a} = (\overline{a}_k)_{k=0}^{\infty} \) by,

\[ \overline{a}_k = \lim_{n \to \infty} \left( \sum_{m=0}^{n} t_{nm} \overline{a}_{mk} \right); (k \in N) \]

provided the above limits exist for all \( k \in N \) which is the case whenever \( A \in (c^1_0(M, \Delta, \theta, c_T)) \) or \( A \in (\ell^1_\infty(M, \Delta, \theta, c_T)) \) by lemmas 5.1, 3.2 and Remark 4.1.

Now using the above results, we have the following results:

**Theorem 5.2.** Let \( X \) be any of the spaces \( c^1_0(M, \Delta, \theta) \) or \( \ell_\infty^1(M, \Delta, \theta) \), \( T \) a trangle and \( A \) an infinite matrix. If \( A \) is in any of the classes \( (X, (c_0)_T), (X, c_T) \) or \( (X, (\ell_\infty)_T) \), then

\[ \|L_A\| = \|A\|_{(X, (\ell_\infty)_T)} = \sup_n \left\{ \inf \left\{ \rho > 0 : \lim_{t \to \infty} \frac{1}{h_t} \sum_{k=0}^{\infty} \sum_{m=0}^{n} \left( M \left( \frac{|t_{nm} \overline{a}_{mk}|}{\rho} \right) \right)^{s_k} \leq 1 \right\} \right\} < \infty. \]

**Theorem 5.3.** Let \( T \) be a triangle. If either \( A \in (\ell_\infty^1(M, \Delta, \theta, (c_0)_T)) \) or \( A \in (\ell_\infty^1(M, \Delta, \theta, (c_T)) \) then \( L_A \) is compact.

**Theorem 5.4.** Let \( T \) be a triangle. Then, we have,

1. If \( A \in (c^1_0(M, \Delta, \theta), (c_0)_T) \), then,

\[ \|L_A\|_X = \limsup_{n \to \infty} \left\{ \inf \left\{ \rho > 0 : \lim_{t \to \infty} \frac{1}{h_t} \sum_{k=0}^{\infty} \sum_{m=0}^{n} \left( M \left( \frac{|t_{nm} \overline{a}_{mk}|}{\rho} \right) \right)^{s_k} \leq 1 \right\} \right\}. \]

and \( L_A \) is compact if and only if

\[ \lim_{n \to \infty} \left\{ \inf \left\{ \rho > 0 : \lim_{t \to \infty} \frac{1}{h_t} \sum_{k=0}^{\infty} \sum_{m=0}^{n} \left( M \left( \frac{|t_{nm} \overline{a}_{mk}|}{\rho} \right) \right)^{s_k} \leq 1 \right\} \right\} = 0. \]
2. If \( A \in (c_0^\lambda (M, \Delta, s, \theta), c_T) \), then

\[
\frac{1}{2} \limsup_{n \to \infty} \left\{ \inf \left( \rho > 0 : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k=0}^{\infty} \sum_{m=0}^{n} \left( M \left( \frac{|t_{nm} \overline{a}_{mk} - \overline{a}_{k}|}{\rho} \right) \right)^{\delta_k} \leq 1 \right) \right\} 
\]

\[
\leq \|L_A\|_\infty \leq \limsup_{n \to \infty} \left\{ \inf \left( \rho > 0 : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k=0}^{\infty} \sum_{m=0}^{n} \left( M \left( \frac{|t_{nm} \overline{a}_{mk} - \overline{a}_{k}|}{\rho} \right) \right)^{\delta_k} \leq 1 \right) \right\}
\]

and \( L_A \) is compact if and only if

\[
\lim_{n \to \infty} \left\{ \inf \left( \rho > 0 : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k=0}^{\infty} \sum_{m=0}^{n} \left( M \left( \frac{|t_{nm} \overline{a}_{mk} - \overline{a}_{k}|}{\rho} \right) \right)^{\delta_k} \leq 1 \right) \right\} = 0.
\]

3. If either \( A \in (c_0^\lambda (M, \Delta, s, \theta), (\ell_\infty)^T) \) or \( A \in (\ell_\infty^\lambda (M, \Delta, s, \theta), (\ell_\infty)^T) \), then

\[
0 \leq \|L_A\|_\infty \leq \limsup_{n \to \infty} \left\{ \inf \left( \rho > 0 : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k=0}^{\infty} \sum_{m=0}^{n} \left( M \left( \frac{|t_{nm} \overline{a}_{mk} - \overline{a}_{k}|}{\rho} \right) \right)^{\delta_k} \leq 1 \right) \right\}
\]

and \( L_A \) is compact if

\[
\lim_{n \to \infty} \left\{ \inf \left( \rho > 0 : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k=0}^{\infty} \sum_{m=0}^{n} \left( M \left( \frac{|t_{nm} \overline{a}_{mk} - \overline{a}_{k}|}{\rho} \right) \right)^{\delta_k} \leq 1 \right) \right\} = 0.
\]

Particular cases: Let \( \lambda' = (\lambda'_k)^\infty_{k=0} \) be a strictly increasing sequence of positive reals tending to infinity and \( \Lambda' = (\lambda'_k) \) be the triangle defined by (4), with the sequence \( \lambda' \) instead of \( \lambda \). Also, let \( c_0^{\lambda'} (M, \Delta, s, \theta) \), \( c^{\lambda'} (M, \Delta, s, \theta) \) and \( \ell_\infty^{\lambda'} (M, \Delta, s, \theta) \) be the matrix domains of the triangle \( \Lambda' \) in the spaces \( c_0, c \) and \( \ell_\infty \) respectively.

**Particular Case 5.5.** Let \( X \) be any of the spaces \( c_0^{\lambda'} (M, \Delta, s, \theta) \) or \( \ell_\infty^{\lambda'} (M, \Delta, s, \theta) \) and \( A \) an infinite matrix. If \( A \) is in any of the classes \( (X, c_0^{\lambda'} (M, \Delta, s, \theta)), (X, c^{\lambda'} (M, \Delta, s, \theta)) \) or \( (X, \ell_\infty^{\lambda'} (M, \Delta, s, \theta)) \), then

\[
\|L_A\|_\infty = \|A\|_{(X, c_0^{\lambda'} (M, \Delta, s, \theta))} = \sup_n \left\{ \inf \left( \rho > 0 : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k=0}^{\infty} \sum_{m=0}^{n} \left( M \left( \frac{|\lambda'_k \overline{a}_{mk} - \overline{a}_{k}|}{\rho} \right) \right)^{\delta_k} \leq 1 \right) \right\}.
\]

**Particular Case 5.6.** If either \( A \in (\ell_\infty^\lambda (M, \Delta, s, \theta), c_0^{\lambda'} (M, \Delta, s, \theta)) \) or \( A \in (\ell_\infty^\lambda (M, \Delta, s, \theta), c^{\lambda'} (M, \Delta, s, \theta)) \), then \( L_A \) is compact.
Similarly, we get some identities or estimates for the Hausdorff measures of noncompactness of operators given by matrices in the classes \((c_0^r(M, \Delta, s, \theta), c_0^\infty(M, \Delta, s, \theta))\), \((c_0^r(M, \Delta, s, \theta), c_0^\infty(M, \Delta, s, \theta))\), \((\ell^r_\infty(M, \Delta, s, \theta), \ell^\infty_\infty(M, \Delta, s, \theta))\), and deduce the necessary and sufficient (or only sufficient) conditions for such operators to be compact.

Let \(bs, cs\) and \(cs_0\) be the spaces of all sequences associated with bounded, convergent and null series, respectively. Then, we have the following results associated with these sequence spaces,

**Corollary 5.7.** Let \(X\) be any of the spaces \(c_0^r(M, \Delta, s, \theta)\) or \(\ell^r_\infty(M, \Delta, s, \theta)\) and \(A\) an infinite matrix. If \(A\) is in any of the classes \((X, cs_0), (X, cs)\) or \((X, bs)\), then,

\[
\|L_A\| = \|A\|_{(X, bs)} = \sup_n \left\{ \inf \left\{ \rho > 0 : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k=0}^{\infty} \sum_{m=0}^{n} \left( M \left( \frac{|a_{mk}|}{\rho} \right) \right)^{s_k} \right\} \leq 1 \right\} < \infty.
\]

**Corollary 5.8.** If either \(A \in (\ell^r_\infty(M, \Delta, s, \theta), cs_0)\) or \(A \in (\ell^r_\infty(M, \Delta, s, \theta), cs)\), then \(L_A\) is compact.

**Corollary 5.9.** We have

1. If \(A \in (c_0^r(M, \Delta, s, \theta), cs_0)\), then

\[
\|L_A\|_X = \limsup_{n \to \infty} \left\{ \inf \left\{ \rho > 0 : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k=0}^{\infty} \sum_{m=0}^{n} \left( M \left( \frac{|a_{mk}|}{\rho} \right) \right)^{s_k} \right\} \leq 1 \right\}
\]

and \(L_A\) is compact if and only if

\[
\lim_{n \to \infty} \left\{ \inf \left\{ \rho > 0 : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k=0}^{\infty} \sum_{m=0}^{n} \left( M \left( \frac{|a_{mk}|}{\rho} \right) \right)^{s_k} \right\} \leq 1 \right\} = 0.
\]

2. If \(A \in (c_0^r(M, \Delta, s, \theta), cs)\), then

\[
\frac{1}{2} \limsup_{n \to \infty} \left\{ \inf \left\{ \rho > 0 : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k=0}^{\infty} \sum_{m=0}^{n} \left( M \left( \frac{|a_{mk} - c_{mk}|}{\rho} \right) \right)^{s_k} \right\} \leq 1 \right\}
\]
≤ \|L_A\|_X \leq \limsup_{n \to \infty} \left\{ \inf \left( \rho > 0 : \lim r \frac{1}{h_r} \sum_{k=0}^{\infty} \sum_{m=0}^{n} \left| M \left( \frac{|a_{mk} - \bar{a}_k|}{\rho} \right) \right|^s \right) \leq 1 \right\}

and $L_A$ is compact if and only if

$$
\lim_{n \to \infty} \left\{ \inf \left( \rho > 0 : \lim r \frac{1}{h_r} \sum_{k=0}^{\infty} \sum_{m=0}^{n} \left| M \left( \frac{|a_{mk} - \bar{a}_k|}{\rho} \right) \right|^s \right) \leq 1 \right\} = 0,
$$

where $\bar{a}_k = \lim_{n \to \infty} \left( \sum_{m=0}^{n} a_{mk} \right)$ for all $k \in \mathbb{N}$.

3. If either $A \in (c_0^1(M, \Delta, s, \theta), bs)$ or $A \in (\ell_{\infty}^1(M, \Delta, s, \theta), bs)$, then

$$
0 \leq \|L_A\|_X \leq \limsup_{n \to \infty} \left\{ \inf \left( \rho > 0 : \lim r \frac{1}{h_r} \sum_{k=0}^{\infty} \sum_{m=0}^{n} \left| M \left( \frac{|a_{mk} - \bar{a}_k|}{\rho} \right) \right|^s \right) \leq 1 \right\}
$$

and $L_A$ is compact if

$$
\lim_{n \to \infty} \left\{ \inf \left( \rho > 0 : \lim r \frac{1}{h_r} \sum_{k=0}^{\infty} \sum_{m=0}^{n} \left| M \left( \frac{|a_{mk} - \bar{a}_k|}{\rho} \right) \right|^s \right) \leq 1 \right\} = 0.
$$

References

[1] A. Alotaibi, B. Hazarika and S. A. Mohiuddine (2014). On lacunary statistical convergence of double sequences in locally solid Riesz spaces. J. Comput. Anal. Appl. 17 (1) : 156165.

[2] A.R. Freedman, J.J. Sember and M. Raphael (1978). Some Cesàro type summability space. Proceedings of the London Mathematical Society. 37(3):508520.

[3] A. Gökhan, M. Et and M. Mursaleen (2009). Almost lacunary statistical and strongly almost lacunary convergence of sequences of fuzzy numbers. Mathematical and Computer Modelling.49:548-555.
[4] B. Hazarika, S.A. Mohiuddine and M. Mursaleen (2014). Some inclusion results for lacunary statistical convergence in locally solid Riesz spaces. Iran. J. Sci. Technol. Trans. A Sci. 38(1):6168.

[5] E. Malkowsky (2010). Compact matrix operators between some BK spaces, in: M. Mursaleen (Ed.). Modern Methods of Analysis and Its Applications, Anamaya Publ., New Delhi. 86-120.

[6] E. Malkowsky and E. Savas (2004). Matrix transformations between sequence spaces of generalized weighted means. Applied Mathematics and Computation. 147:333-345.

[7] F. Basar and E. Malkowsky (2011). The characterization of compact operators on spaces of strongly summable and bounded sequences. Applied Mathematics and Computation. 217:5199-5207.

[8] G. Darbo (1955). Punti uniti in transformazioni a condominio non compatto. Rend. Sem. Math. Univ. Padova. 24:84-92.

[9] H. Kizmaz (1981). On certain sequence spaces. Canadian Mathematical Bulletin. 24(2):169-176.

[10] J. Lindenstrauss and L. Tzafriri (1971). On Orlicz sequence spaces. Israel Journal of Mathematics. 10:379-390.

[11] K. Kuratowski (1930). Sur les espaces complets. Fund. Math. 15:301-309.

[12] M. Mursaleen and A.K. Noman (2010). Compactness by the Hausdorff measure of noncompactness. Nonlinear Analysis. 73:2541-2557.

[13] M. Mursaleen and A.K. Noman (2010). On some new difference sequence spaces of non-absolute type. Math. Comput. Modelling. 52(3-4): 603-617.