TROPICALIZING THE SPACE OF ADMISSIBLE COVERS

RENZO CAVALIERI, HANNAH MARKWIG, AND DHARUV RANGANATHAN

ABSTRACT. We study the relationship between tropical and classical Hurwitz moduli spaces. Following recent work of Abramovich, Caporaso and Payne, we outline a tropicalization for the moduli space of generalized Hurwitz covers of an arbitrary genus curve. Our approach is to appeal to the geometry of admissible covers, which compactify the Hurwitz scheme. We define and construct a moduli space of tropical admissible covers, and study its relationship with the skeleton of the Berkovich analytification of the classical space of admissible covers. We use techniques from non-archimedean geometry to show that the tropical and classical tautological maps are compatible via tropicalization, and that the degree of the classical branch map can be recovered from the tropical side. As a consequence, we obtain a proof, at the level of moduli spaces, of the equality of classical and tropical Hurwitz numbers.

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1. INTRODUCTION

The primary objective of this paper is to establish a geometric and functorial relationship between the moduli space of Hurwitz covers of an algebraic curve, and a moduli space of tropical Hurwitz covers, which we introduce here. Such a relationship is given by a morphism of cone complexes from the skeleton of the Berkovich analytification of the space of admissible covers to the moduli space of tropical Hurwitz covers. We show that tropicalization commutes with the natural tautological source and branch maps on these moduli spaces. Consequently, we recover the equality between tropical and classical Hurwitz numbers, originally proved via combinatorial and topological methods in [7, 11].

1.1. Results. Fix a vector of partitions \( \vec{\mu} = (\mu^1, \ldots, \mu^r) \) of an integer \( d > 0 \). Denote by \( \mathcal{H}_{g \to h, d}(\vec{\mu}) \) the space of degree \( d \) admissible covers \([D \to C]\) of genus \( h \) curves by genus \( g \) curves with ramification \( \mu^i \) over smooth marked points \( p_i \) of \( C \), and simple ramification over smooth marked points \( q_1, \ldots, q_s \). We denote by \( \mathcal{H}_{\text{an}}^{g \to h, d}(\vec{\mu}) \) (resp. \( \mathcal{M}_{\text{an}}^{g,n} \)) the Berkovich analytification of the space of admissible covers (resp. the moduli space of stable curves). We use \( \mathcal{H}_{\text{trop}}^{g \to h, d}(\vec{\mu}) \) to denote the space of tropical admissible covers of genus \( h \) tropical curves by genus \( g \) tropical curves, with expansion factors along infinite edges prescribed by \( \vec{\mu} \). See Sections 2, 3 for definitions and background.

Our first result studies the relationship between the “naive” set theoretic tropicalization map from \( \mathcal{H}_{\text{an}}^{g \to h, d}(\vec{\mu}) \) to \( \mathcal{H}_{\text{trop}}^{g \to h, d}(\vec{\mu}) \) (see Definition 24) and the canonical projection to the skeleton from the analytification to its skeleton \( \Sigma(\mathcal{H}_{\text{an}}^{g \to h, d}(\vec{\mu})) \).

Theorem 1. The set theoretic tropicalization map \( \text{trop} : \mathcal{H}_{\text{an}}^{g \to h, d}(\vec{\mu}) \to \mathcal{H}_{\text{trop}}^{g \to h, d}(\vec{\mu}) \) factors through the canonical projection from the analytification to its skeleton \( \Sigma(\mathcal{H}_{\text{an}}^{g \to h, d}(\vec{\mu})) \).

\[
\begin{array}{ccc}
\mathcal{H}_{\text{an}}^{g \to h, d}(\vec{\mu}) & \xrightarrow{\text{trop}} & \mathcal{H}_{\text{trop}}^{g \to h, d}(\vec{\mu}) \\
\downarrow{\scriptstyle \text{trop}} & & \downarrow{\scriptstyle \text{trop}} \\
\Sigma(\mathcal{H}_{\text{an}}^{g \to h, d}(\vec{\mu})) & \xrightarrow{\text{trop}_\Sigma} & \mathcal{H}_{\text{trop}}^{g \to h, d}(\vec{\mu})
\end{array}
\]

Furthermore the map \( \text{trop}_\Sigma \) is a face morphism of cone complexes, i.e. the restriction of \( \text{trop}_\Sigma \) to any cone of \( \Sigma(\mathcal{H}_{\text{an}}^{g \to h, d}(\vec{\mu})) \) is an isomorphism onto a cone of the tropical moduli space \( \mathcal{H}_{\text{trop}}^{g \to h, d}(\vec{\mu}) \). The map \( \text{trop}_\Sigma \) extends naturally and uniquely to the extended complexes \( \Sigma(\mathcal{H}_{\text{an}}^{g \to h, d}(\vec{\mu})) \to \mathcal{H}_{\text{trop}}^{g \to h, d}(\vec{\mu}) \).

The Hurwitz moduli spaces and their compactifications are natural vehicles for the enumerative geometry of target curves. The (classical) Hurwitz number \( h_{g \to h, d}(\vec{\mu}) \) counts the number of covers with the above invariants and a fixed branch divisor. Our next result shows that the tropical moduli space (and skeleton) contain enough information to recover Hurwitz numbers.

Theorem 2. Let \( \sigma_\Gamma \) be any fixed top dimensional cone of the tropical moduli space \( \mathcal{M}_{h, r+s}^{\text{trop}} \). Denote by \( \sigma_\Theta \to \sigma_\Gamma \) a cone in the moduli space \( \mathcal{H}_{\text{trop}}^{g \to h, d}(\vec{\mu}) \) of combinatorial type \( \Theta \) such that the base graph of \( \Theta \) is equal to \( \Gamma \). The restriction of the tropical branch map is a surjective morphism of cones with integral structure and consequently has a dilation factor which we denote \( d_{\Theta}(\text{br}) \).
The Hurwitz number is equal to:

\[
\text{h}_{g\rightarrow h,d}(\vec{\mu}) = \sum_{\sigma_{\Theta}^{\rightarrow} \sigma_r} \omega(\Theta) \cdot d_{\Theta}(\text{br}^{\text{trop}}),
\]

where \(\omega(\Theta)\) is the weight associated to the combinatorial type \(\Theta\) as in Definition 22.

The right hand side of (2) coincides with the definition of tropical Hurwitz numbers in [7]. Theorem 2 then provides a geometric proof for the following correspondence theorem.

**Theorem 3** (Bertrand–Brugallé–Mikhalkin [7]). Classical and tropical Hurwitz numbers coincide, i.e. we have \(\text{h}_{g\rightarrow h,d}(\vec{\mu}) = \text{h}^{\text{trop}}_{g\rightarrow h,d}(\vec{\mu})\).

For us Theorem 2 is a consequence of the functoriality of the tropicalization map. More precisely, the Hurwitz number arises geometrically as the degree of a tautological map called the *branch map*, which takes a cover to its base curve, marked at its branch points. We also study the source map, taking a cover to its source curve, marked at the entire inverse image of the branch locus. We have the diagram:

\[
\begin{array}{c}
\overline{\mathcal{H}}_{g\rightarrow h,d}(\vec{\mu}) \xrightarrow{\text{src}} \overline{\mathcal{M}}_{g,n} \\
\downarrow \text{br} \\
\overline{\mathcal{M}}_{h,r+s}
\end{array}
\]

where \(s\) is the number of simple branch points. There are analogous tautological morphisms on spaces of admissible covers of tropical curves. Tropicalization is compatible with these two tautological morphisms to the moduli space of curves in the following sense.

**Theorem 4.** Let \(\text{br}\) denote the branch map \(\overline{\mathcal{H}}_{g\rightarrow h,d}(\vec{\mu}) \rightarrow \overline{\mathcal{M}}_{h,r+s}\), and \(\text{src}\) denote the source map \(\overline{\mathcal{H}}_{g\rightarrow h,d}(\vec{\mu}) \rightarrow \overline{\mathcal{M}}_{g,n}\), where \(n\) is the number of smooth points in the inverse image of the branch locus. Then the following diagram is commutative:

\[
\begin{array}{c}
\overline{\mathcal{H}}_{g\rightarrow h,d}(\vec{\mu}) \xrightarrow{\text{src}^{\text{an}}} \overline{\mathcal{M}}_{g,n} \\
\downarrow \text{br}^{\text{an}} \\
\overline{\mathcal{M}}_{h,r+s} \xrightarrow{\text{trop}} \overline{\mathcal{M}}^{\text{trop}}_{h,r+s}
\end{array}
\]

The induced map on skeleta of the branch (resp. source) morphism factors as a composition of the map \(\text{trop}_\Sigma\) to \(\Sigma(\overline{\mathcal{H}}_{g\rightarrow h,d}(\vec{\mu}))\), followed by the tropical branch (resp. source) map, so \(\text{br}^{\text{trop}} = \text{trop}_\Sigma \circ \text{br}^{\text{rat}}\) (resp. \(\text{src}^{\text{trop}} = \text{trop}_\Sigma \circ \text{src}^{\text{rat}}\)).

It is proved in [1] that the skeleton \(\Sigma(\overline{\mathcal{M}}_{g,n})\) is identified with \(\overline{\mathcal{M}}^{\text{trop}}_{g,n}\). In other words, the analogue of the map \(\text{trop}_\Sigma\) for \(\overline{\mathcal{M}}_{g,n}\) is an isomorphism. A crucial aspect of the present work is analyzing precisely how the skeleton relates to \(\overline{\mathcal{H}}_{g\rightarrow h,d}(\vec{\mu})\). The map \(\text{trop}_\Sigma\) records the combinatorial data of an admissible cover. In particular, the failure of \(\text{trop}_\Sigma\) to be an isomorphism is due to two phenomena.
Given a weighted dual graph $\Gamma$, there is a unique irreducible stratum of the moduli space $\overline{M}_{g,n}$ corresponding to it. In our setting, for one combinatorial type $\Theta = [\Gamma_{\text{src}} \to \Gamma_{\text{tgt}}]$ for a cover, there are multiple irreducible components in the stratum in $\overline{\mathcal{M}}_{g}\to h,d([\mu])$ having dual graph $\Theta$. In particular, there are multiple zero strata corresponding to the same combinatorial data.

The stack of admissible covers arises as the normalization of the stack of generalized Harris–Mumford covers $\overline{\mathcal{M}}_{g}\to h,d([\mu])$. There are multiple cones of the skeleton $\Sigma(\overline{\mathcal{M}}_{g}\to h,d([\mu]))$ (corresponding to the multiple analytic branches at a point of the moduli space $\overline{\mathcal{M}}_{g}\to h,d([\mu])$) which all map isomorphically to the same cone of the tropical moduli space. We discuss this in detail in Section 4.2.3.

All the results share two common ingredients. The first is the technology developed in [1, 34] that allows to study skeletons of toroidal compactifications of Deligne–Mumford stacks over trivially valued fields. The second is a careful study of the boundary stratification and deformation theory of the stack of admissible covers from [2, 22, 28].

1.2. Context and Motivation. Classical Hurwitz theory studies ramified maps between algebraic curves. Hurwitz numbers count the number of covers of a genus $h$ curve by a genus $g$ curve, with prescribed degree, ramification data, and branch points. As often is the case in enumerative geometry, there is a tight dictionary between the enumerative data of Hurwitz numbers and the intersection theory on the moduli spaces parameterizing Hurwitz covers.

Moduli spaces of smooth covers (Hurwitz spaces) are not proper: limits of smooth covers may be singular. For many applications, including enumerative geometry, it is desirable to compactify the Hurwitz space. There are multiple approaches to compactifying this space, each with its pros and cons. In this work we focus on the compactification by admissible covers.

The notion of admissible covers was first introduced by Harris and Mumford in [22]. The fundamental idea is that source and target curves must degenerate together; branch points are not allowed to come together; as branch points approach, a new component of the base curve “bubbles off”, and simultaneously the source curve splits into a nodal curve.

The admissible covers that Harris and Mumford consider are covers of genus 0 curves, having only simple ramification — namely, such that all ramification profiles are given by $(2,1,1,\ldots,1)$. Their work is generalized by Mochizuki in his thesis [28]. Mochizuki uses logarithmic geometry to understand the geometry of the admissible cover space for covers of arbitrary genus and arbitrary ramification profiles. Abramovich, Corti, and Vistoli, in [2], reinterpret admissible covers using the theory of twisted stable maps to classifying stacks $\mathcal{B}S_d$. A map from a curve $C$ to the stack $\mathcal{B}S_d$ produces, by definition, a principal $S_d$-bundle on $C$. Given such a principal $S_d$-bundle $P \to C$, one can associate a finite étale cover of degree $d$, $D \to C$, where $D = P/S_{d-1}$. In fact, this gives an equivalence of categories between principal $S_d$-bundles and finite étale covers of $C$ of degree $d$. By allowing orbifold structure at points and nodes of $C$ one introduces ramification over such points, and a map from the orbicurve to $\mathcal{B}S_d$ corresponds to an admissible cover $D \to C$ where $D = P/S_{d-1}$. The orbifold structure at the nodes of $C$ is required to be balanced, which is precisely the condition allowing the nodes to be smoothly deformed.

The Abramovich–Corti–Vistoli stack of twisted stable maps is the normalization of the Harris–Mumford admissible covers. It is common abuse of terminology, which we adopt as well, to call the normalized stack the stack of admissible covers.
The Hurwitz enumeration problem is one that moves freely between disciplines. It provides ubiquitous and deep connections between the representation theory of the symmetric group, enumerative geometry, intersection theory on moduli spaces, and combinatorics. For example, see [18, 19]. In fact, Hurwitz theory is essentially equivalent to Gromov–Witten theory for target curves [30]. Tropical Hurwitz theory was first introduced in [11], where the case of double Hurwitz numbers for genus 0 targets is investigated. Further steps in the theory of tropical Hurwitz covers have since been made by Bertrand, Brugallé, and Mikhalkin [7], and by Buchholz and the second author [9].

At the base of any successful application of tropical methods to enumerative geometry are so-called correspondence theorems, which establish equality between classical and tropical enumerative invariants. The first instance of such a result was demonstrated by Mikhalkin [27], in his study of the Gromov–Witten invariants of toric surfaces. His correspondence result follows from a direct bijection between the set of algebraic curves satisfying fixed incidence conditions and the (weighted) set of corresponding tropical curves. Subsequent breakthroughs in tropical enumerative geometry have shared this feature of establishing direct set-theoretic bijections between the tropical and classical objects. However, enumerative invariants often represent degree zero Chow cycles on an natural moduli space, and traditional correspondence theorems do not link the classical and tropical problems at the level of moduli spaces. Tropical moduli spaces and their intersection theory have been studied in order to express tropical enumerative invariants, analogously to the algebraic setting, as intersection products on a suitable moduli space [17]. In light of this, it is natural to seek an understanding of the equality of classical and tropical enumerative invariants at the level of moduli spaces.

In this paper, we present the first instance of such a result, by equating classical Hurwitz numbers with tropical ones. We do so by appealing to machinery of Berkovich analytic spaces. The zero cycles representing Hurwitz numbers are obtained as the degree of a naturally defined branch morphism on an appropriate compactification of the Hurwitz space, namely the stack of admissible covers. This recovers previously known results on double Hurwitz numbers [11]. The present framework also applies equally well to more general settings, such as higher genus targets, and arbitrary ramification profiles. In particular, we can also reprove the general correspondence theorem for tropical Hurwitz numbers from [7] at the level of moduli spaces.

Our method for comparing classical and tropical moduli spaces of admissible covers follows closely the work of Abramovich, Caporaso and Payne [1] on the moduli space of curves. The moduli space of genus g tropical curves, roughly speaking, parametrizes vertex weighted metric graphs of a given genus [8, 10, 13]. This moduli space bears some striking similarities to the moduli space of genus g Deligne–Mumford stable curves. The spaces share the same dimension, and have similar stratifications. These analogies are put on firm ground by realizing the space $\overline{M}_{g,n}^{\text{trop}}$ as a skeleton of the Berkovich analytification of the stack $\overline{M}_{g,n}$. The relationship between $\overline{M}_{g,n}$ and $\overline{M}_{g,n}^{\text{trop}}$ is not unlike the relationship between a toric variety and its fan. This intuition is made precise by using a natural toroidal structure of $\overline{M}_{g,n}$ to obtain a skeleton of $\overline{M}_{g,n}^{\text{trop}}$, Abramovich, Caporaso, and Payne developed techniques, building on work of Thuillier [34], to construct the skeleton of the
analytification of any toroidal Deligne–Mumford stack, and relate it to the cone complex naturally associated to the toroidal embedding [24]. As stacks of admissible covers are smooth with a toroidal structure induced by the (not necessarily simple) normal crossing boundary, these techniques apply directly to our setting.

The study of Hurwitz numbers of \( \mathbb{P}^1 \) has also interacted fruitfully with the theory of stable maps and Gromov–Witten invariants of \( \mathbb{P}^1 \). In fact, the moduli space of Hurwitz covers sits inside the moduli space of degree \( d \) relative stable maps to \( \mathbb{P}^1 \), relative to the (special) branch divisor. As a result, the techniques of Gromov–Witten theory have been applied to obtain powerful results connecting Hurwitz numbers to the tautological intersection theory on the moduli space of curves: for example the ELSV formula [15, 20] has had some remarkable applications, including a famous re-proof of Kontsevich’s Theorem/Witten’s Conjecture, see [31].

In [11], the first and second authors, together with Paul Johnson, constructed and studied a tropical analogue of the Gromov–Witten moduli stack of stable maps to \( \mathbb{P}^1 \), relative to a two-point special branch divisor. The corresponding classical moduli space is a singular, non-equidimensional stack, and does not afford a direct application of the techniques of Abramovich, Caporaso, and Payne.

However, the space of relative stable maps can be seen as a “hybrid” theory, between admissible covers and (absolute) stable maps. More precisely, admissible covers is a theory of relative stable maps when the entire branch divisor on the target curves is made relative (and it is allowed to “move”). The admissible cover compactification of the Hurwitz scheme admits a rational map to the relative stable map space which is dominant on the main (expected dimensional) component. As a result, we see the study of admissible covers as a natural first step towards a functorial and geometric understanding of tropical relative Gromov–Witten theory.

The remainder of our paper is organized as follows. In Section 2 we collect the relevant combinatorial constructions needed in the present work. In Section 3 we recall the basic theory of admissible covers and the admissible cover compactification of Hurwitz space. We then define tropical admissible covers and construct a moduli space of tropical admissible covers. In Section 4 we describe the tropicalization map on the moduli space of admissible covers. We then present a proof of Theorem 1. In Section 5 we study the tropical and classical tautological morphisms on spaces of admissible covers, namely the source and branch map. Finally in Section 6 we study the relationship between the present work and previous work on tropical Hurwitz numbers. We recover the known correspondence results for Hurwitz numbers from [7, 11].

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2. COMBINATORIAL CONSTRUCTIONS

A unifying feature of the numerous instances of tropicalization is that they associate combinatorial and polyhedral structures to algebraic varieties. Examples of these are finite and metric graphs, cones, fans, and cone complexes. Often, these polyhedral structures are finite approximations to an appropriate Berkovich analytic space. In this section we briefly recall the concepts relevant to this work.
2.1. Dual graphs. If \( C \) is a nodal curve, one may associate a \emph{vertex weighted dual graph} or simply \emph{dual graph} \( \Gamma_C \) as follows:

(i) the vertices of \( \Gamma_C \) are the irreducible components of \( C \);
(ii) the edges of \( \Gamma_C \) are the nodes of \( C \); an edge \( e \) is incident to a vertex \( v \) if the node associated to \( e \) is contained in the component corresponding to \( v \);
(iii) a vertex \( v \) is given a weight \( g(v) \), equal to the geometric genus of the component corresponding to \( v \).

If \( C \) is a pointed curve, with marked points \( p_1, \ldots, p_n \), then we add a \emph{half-edge} or \emph{infinite edge} for each \( p_i \), incident to the vertex whose component contains \( p_i \). We say that a dual graph is \emph{totally degenerate} if all vertices carry genus 0.

We call a dual graph \emph{stable} if a nodal curve having that dual graph is stable. In other words, all genus 0 vertices must be at least trivalent (counting infinite edges), and all genus 1 vertices must be incident to an edge.

2.2. Tropical curves and morphisms. For our purposes, an \( n \)-pointed tropical curve is essentially a metrization of the dual graph of an \( n \)-pointed nodal curve, where half-edges are metrized as \( \mathbb{R}_{>0} \cup \{\infty\} \), and the metric must be singular at the ends of half-edges. We sometimes refer to edges as \emph{interior edges} to distinguish them from half-edges.

\[ \Gamma := \lim_{u \to 0} \pi_0(U \setminus p), \]

where the limit is taken over neighborhoods of \( p \) homeomorphic to a star with \( r \) branches. A tangent direction may be thought of as a germ of an edge. The size of this set equals the valence of the point.

A tropical curve \( \Gamma \) is a connected metric graph with a weighting \( g: \Gamma \to \mathbb{Z}_{\geq 0} \), which is zero outside finitely many points of \( \Gamma \). This weight \( g(p) \) should be thought of as the genus of a virtual algebraic curve lying above \( p \). (This intuition can be made concrete in terms of metrized complexes of curves and Berkovich skeleta, in the sense of [4].) The genus of a graph \( \Gamma \) is given by

\[ g(\Gamma) = h^1(\Gamma) + \sum_{p \in \Gamma} g(p). \]

Throughout the text, we consider \( n \)-pointed tropical curves, i.e. tropical curves with \( n \) marked infinite edges. An \( (n\text{-pointed}) \) tropical curve is \emph{stable} if every genus 0 vertex is at least trivalent.

One can associate to any (pointed) tropical curve \( \Gamma \) a finite graph, which we refer to as a \emph{combinatorial type}, by taking its minimal finite graph model. That is, we take the vertices to be those points of valence different from 2, or whose genus is nonzero. The edges are
formed in the obvious way. Conversely, given a combinatorial type, a metrization is an assignment of lengths to the edges. Half-edges are metrized as \([0, \infty]\).

2.2.1. Harmonic morphisms of tropical curves. Let \(\varphi : \Gamma' \to \Gamma\) be a morphism of metric graphs which is piecewise linear with integer slopes. For an edge \(e'\) of \(\Gamma'\), let \(d_{e'}(\varphi)\) be the slope of this affine map along the edge \(e'\), we refer to this number as the \textit{expansion factor along} \(e'\). A morphism of tropical curves is a harmonic morphism between the underlying metric graphs with integer affine slopes, in the sense of [4, Section 1]. Recall that a piecewise affine morphism of graphs with integer slopes is \textit{harmonic at} \(p'\) if for each tangent direction \(e \in T_{\varphi(p')}\Gamma\), the number

\[
d_{p'}(\varphi) := \sum_{e' \in T_{p'}\Gamma'} d_{e'}(\varphi),
\]

is independent of \(e\). In other words, the sum of outgoing slopes at \(p'\) along tangent directions mapping to a chosen tangent direction \(e\) is independent of \(e\). The morphism \(\varphi\) is harmonic if it is surjective and harmonic at all points of \(\Gamma'\).

Example 5. Consider the non-harmonic morphism depicted in Figure 2. Observe that on one side of the central vertex in the target, the sum of degrees mapping to it is 5, while on the other side, it is 4. In particular, there is no well defined notion of degree for such a map.

![Figure 2. An example of a nonharmonic morphism of graphs.](image)

2.3. Cones and cone complexes. A \textit{polyhedral cone with integral structure} is a pair \((\sigma, M)\) consisting of a topological space \(\sigma\), together with a finitely generated abelian group \(M\) of continuous real valued functions on \(\sigma\), such that the natural map \(\sigma \to \text{Hom}(M, \mathbb{R})\) is a homeomorphism onto a (strictly convex) polyhedral cone. We will only consider \textit{rational} cones, i.e. cones whose image is rational with respect to the dual lattice \(\text{Hom}(M, \mathbb{Z})\). A good example to have in mind is the cone defining an affine toric variety, where \(M\) is the character lattice of the dense torus.

Let \(\sigma\) be a cone. The \textit{dual monoid} \(S_\sigma\) of \(\sigma\) consists of those functions \(f \in M\) that are nonnegative on \(\sigma\). We can recover the original cone from its dual monoid as

\[\sigma = \text{Hom}(S_\sigma, \mathbb{R}_{\geq 0}),\]

where \(\mathbb{R}_{\geq 0}\) is taken with its usual additive monoid structure. The associated \textit{extended cone} is

\[\sigma = \text{Hom}(S_\sigma, \mathbb{R}_{\geq 0} \cup \{\infty\}).\]

A \textit{(rational polyhedral) cone complex} is a finite disjoint union of polyhedral cones with integral structures, obtained by gluing cones along isomorphic faces such that each cone maps homeomorphically onto its image. Fans are of course examples of cone complexes. We remark however that, in a cone complex, the intersection of faces of cones is allowed to be a union of faces, rather than a single face of each. Additionally, such cone complexes make no reference to an abstract vector space in which it is embedded. See [24] for further details. \textit{Extended cone complexes} are obtained analogously from extended cones.
A useful tool for us is the process of barycentric subdivision. The barycenter of $\sigma$ is the ray in its interior spanned by the sum of the primitive generators of the one-dimensional faces of $\sigma$. The barycentric subdivision of a cone complex $\Sigma$ is the iterated stellar subdivision of cones in $\Sigma$, in decreasing order of dimension. A more elegant, if less concrete, definition of the barycentric subdivision is that it is obtained as the poset of chains in the face poset of $\Sigma$, ordered by inclusion. The following proposition is often useful.

**Proposition 6 (3).** Let $\Sigma$ be any cone complex. Then the barycentric subdivision of $\Sigma$ is isomorphic to a simplicial fan.

A face morphism is a morphism between cone complexes such that every cone maps isomorphically onto a cone in the image. A generalized cone complex is an arbitrary finite colimit of cones $\sigma_i$ and face morphisms $\psi_i$:

$$\Sigma = \lim_{\to} (\sigma_i, \psi_i).$$

These objects are built to allow two principal operations: gluing cones along isomorphic faces, and taking quotients of cones by a group of automorphisms.

**Warning 7.** The relationship between generalized extended cone complexes and various barycentric subdivisions is subtle, for instance the extended complex of the subdivision is not the subdivision of the extended complex. However, we do not use these facts explicitly in this paper. A detailed discussion may be found in [1, Section 2].

3. Classical and tropical admissible covers

3.1. Classical admissible covers and their moduli. Let $(C, p_1, \ldots, p_n)$ be a genus $g$ $n$-pointed stable nodal curve.

**Definition 8.** An admissible cover $\pi : D \to C$ of degree $d$ is a finite morphism of stable pointed curves such that:

(i) The map $\pi$ restricted to the complement of the inverse image of the marked points, $D \setminus (\bigcup \pi^{-1}(p_i))$, is étale of constant degree $d$.

(ii) The set of nodes of $D$ is precisely the preimage under $\pi$ of the set of nodes of $C$.

(iii) The set of smooth branch points of $\pi$ are the marked points of $C$. All inverse images of marked points of $C$ are marked in $D$.

(iv) Over a node, étale locally, $D$, $C$ and $\pi$ are described by

$$\begin{align*}
D : & \quad y_1 y_2 = a \\
C : & \quad x_1 x_2 = a^\ell \\
\pi : & \quad x_1 = y_1^{\frac{1}{\ell}}, \; x_2 = y_2^{\frac{1}{\ell}}
\end{align*}$$

for $\ell \leq d$.

**Remark 9.** Intuitively, the condition (i), (ii) and (iii) are forcing the source and target to have nodes “only when they have to”. Condition (iv) is crucial to ensure that the admissible cover is “controlled” by the base curve. In particular it implies a natural kissing condition: if $\pi$ is lifted to $\tilde{\pi}$ on the normalizations of $D$ and $C$, then for each node of $D$, the ramification indices of $\tilde{\pi}$ at the two points lying above the node coincide.

There are several slight variations of moduli spaces of admissible covers. For instance, the base curve can be fixed, or allowed to vary in moduli. Fixing both the base curve and the branch locus, the moduli space becomes zero dimensional; the degree of the fundamental class is called a Hurwitz number.

Fix a vector of partitions $\vec{\mu} = (\mu^1, \ldots, \mu^r)$ of an integer $d > 0$. 
Definition 10. Fix \((r + s)\) points \(p_1, \ldots, p_r, q_1, \ldots, q_s\) on a smooth genus \(h\) curve \(C\). The Hurwitz number, denoted \(h_{g \rightarrow h, d}(\vec{\mu})\), is the weighted number of degree \(d\) covers \([\pi : D \rightarrow C]\) such that \(\pi\) is unramified over the complement of \(\{p_i, q_j\}_{i,j}\), with ramification profile \(\vec{\mu}\) over \(p_i\) and simple ramification over \(q_j\). Each cover is weighted by \(1/|\text{Aut}(\pi)|\).

Example 11. Here are some examples of Hurwitz numbers.

\[
\begin{align*}
    h_{0 \rightarrow 0, d}(d, d) &= \frac{1}{d} \\
    h_{1 \rightarrow 0, 2} &= \frac{1}{2} \quad \text{(all ramification is simple)} \\
    h_{1 \rightarrow 0, 4}((2, 2), (4)) &= 14 \cdot 2 \cdot 2^3.
\end{align*}
\]

The first number is an elementary fact. The second number can be computed by explicitly counting factorizations of the identity in the symmetric group. For an example computation of the third number via monodromy graphs, see [11] Example 4.5. We have a factor of 2 because we mark the two points giving the profile \((2, 2)\), contrary to [11], and a factor of \(2^3\) because the unramified inverse images above each branch point are also marked.

Notice that nodal covers can have arbitrary ramification profiles above the nodes, not specified directly by the discrete data \((d, g, h, \vec{\mu})\). This observation is important in understanding the tropical picture. The space of admissible covers is in general a non-normal stack, however the normalization is always smooth. A modular interpretation of the normalization was given by Abramovich, Corti, and Vistoli in [2].

Definition 12. We denote by \(\mathcal{H}_{g \rightarrow h, d}(\vec{\mu})\), the Abramovich–Corti–Vistoli stack of admissible covers of degree \(d\) of a genus \(h\) curve with \((r + s)\) marked branch points \((p_1, \ldots, p_r, q_1, \ldots, q_s)\), by curves of genus \(g\), having ramification profiles \(\vec{\mu}\) over \(p_i\) and simple ramification over \(q_j\) (and no further ramification). Denote by \(\mathcal{HM}_{g \rightarrow h, d}(\vec{\mu})\), the open substack parametrizing covers whose source and target are smooth curves.

Remark 13. We adopt a common abuse terminology and refer to the normalized stack as the stack of admissible covers. When we need to specifically point our attention to the Harris–Mumford spaces of admissible covers, we will explicitly say so and denote this stack by \(\mathcal{HM}_{g \rightarrow h, d}(\vec{\mu})\).

Remark 14. The open parts of these moduli spaces coincide, that is, \(\mathcal{HM}_{g \rightarrow h, d}(\vec{\mu}) \cong \mathcal{H}_{g \rightarrow h, d}(\vec{\mu})\). In other words, the non-normality manifests in how boundary strata intersect. We will witness this non-normality (as well as why the normalization is smooth) when we study the local rings of these moduli spaces in Section 3.1.1.42.1.

Convention 15. The number \(s\) always denotes the number of simple branch points, the number \(h\) the genus of the base curve. To avoid burdensome notation, we suppress \(h\) and \(s\) and use \(\mathcal{H}_{g, d}(\vec{\mu})\) to denote our moduli space, with the understanding that \(h\) may be arbitrarily chosen but is fixed, and \(s\) is determined by the Riemann–Hurwitz formula.

3.1.1. Toroidal embeddings. A toroidal scheme is a pair \(U \rightarrow X\) which “locally analytically” looks like the inclusion of the dense torus into a toric variety. That is, at every point \(p \in X\), there is an étale (or formal) neighborhood \(\varphi : V \rightarrow X\), which admits an étale map \(\psi : V \rightarrow V_\sigma\) to an affine toric variety, such that \(\psi^{-1}T = \varphi^{-1}U\), where \(T\) is the dense open torus. (Toroidal embeddings may be defined using either formal or étale neighborhoods, and the resulting theories are equivalent. See [14] [24].) Let \(\mathcal{E}\) be
a Deligne–Mumford stack over a field, with coarse space \( X \). Let \( \mathcal{U} \subset X \) be an open substack. For any morphism from a scheme \( V \to X \), denote \( \mathcal{U}_V \subset V \) the fiber product \( \mathcal{U} \times_X V \).

**Definition 16.** The inclusion \( \mathcal{U} \hookrightarrow X \) is a toroidal embedding of Deligne–Mumford stacks if, for every morphism from a scheme \( V \to X \), the inclusion \( \mathcal{U}_V \hookrightarrow V \) is a toroidal embedding of schemes.

Toric varieties are obvious examples of toroidal embeddings. Another relevant example is the inclusion of the complement of a normal crossings divisor, \((X - D) \hookrightarrow X\). In fact, in this case, all local toric models can be taken to be affine spaces. The moduli space of stable pointed curves, \( \overline{M}_{g,n} \) is hence an example of a toroidal Deligne–Mumford stack, since the boundary \( \overline{M}_{g,n} \setminus M_{g,n} \) is a divisor with normal crossings. If in addition \( g = 0 \), the boundary divisor has strict normal crossings (i.e. the irreducible components have no self intersections). Similarly, the stack \( \overline{H}_{g,d}(\vec{\mu}) \) is a smooth Deligne–Mumford stack, and the boundary \( \overline{H}_{g,d}(\vec{\mu}) \setminus H_{g,d}(\vec{\mu}) \) is a divisor with normal crossings, allowing us to apply the techniques developed by Abramovich, Caporaso, and Payne in this setting. For admissible covers as well, if \( h = 0 \), the boundary divisor has strict normal crossings.

### 3.2. Dual graphs of covers.

Just as one can associate a dual graph to any nodal pointed curve, given an admissible cover, we can associate to it the dual graphs of source and target, and a map between them. This map is a well defined morphism of graphs by the axioms placed on admissible covers. That is, irreducible components map to irreducible components, so vertices map to vertices. The inverse image of the set of nodes of the target is precisely the set of nodes of the source, so adjacent edges map to adjacent edges. Furthermore, nodes map to nodes, so no edges are contracted in this map. Thus, given an algebraic admissible cover, we obtain a map of graphs as follows.

**Source and target curves.** Take the dual graph of the source and target curves in the above sense. Call these graphs \( \Gamma_{\text{src}} \) and \( \Gamma_{\text{tgt}} \) respectively. Recall here that all branch and ramification points are marked.

**The map.** For an admissible cover, a component of the source maps onto precisely one component of the target, yielding a map of vertices. Since nodes map to nodes, edges map to edges.

**Ramification.** We mark edges of \( \Gamma_{\text{src}} \) with integers recording the ramification at the corresponding node or marked point of the source curve. That is, if an edge \( \tilde{e} \) of \( \Gamma_{\text{src}} \) maps to an edge \( e \) of \( \Gamma_{\text{tgt}} \), this corresponds to a special point \( \tilde{p} \) of the source curve mapping to a special point \( p \) of the target. We decorate \( \tilde{e} \) with the ramification at \( \tilde{p} \).

### 3.2.1. Tropical admissible covers.

We wish to study covers of genus \( g \) tropical curves, with prescribed ramification data over \( r \) points and simple ramification over the remaining \( s \) points. We say that a map of tropical curves satisfies the **local Riemann–Hurwitz condition** if, when \( v' \mapsto v \) with local degree \( d \), then

\[
2 - 2g(v') = d(2 - 2g(v)) - \sum (m_e - 1),
\]

where \( e \) ranges over edges incident to \( v \), and \( m_e \) is the expansion factor of the morphism along \( e \). In the present work, a **Hurwitz cover** of a tropical curve will be a harmonic map of tropical curves that satisfies the local Riemann–Hurwitz equation at every point.

**Remark 17.** We require that covers satisfy local Riemann–Hurwitz with equality rather than inequality. This latter, weaker condition is sometimes referred to as **effectiveness**, and
allows unmarked ramification inside the residue curve that is not visible at the level of metric graphs. By marking all ramification points, we do not allow this, c.f. the generically étale condition in [4].

We note that our definition of tropical Hurwitz covers essentially coincides with the definition given in [7], aside from our choice to mark all branch and ramification points. We now define and construct a moduli space of tropical Hurwitz covers, and study its relationship to the classical space of admissible covers. Continue to fix a vector of partitions \( \vec{\mu} = (\mu^1, \ldots, \mu^r) \) of an integer \( d > 0 \).

**Definition 18.** The tropical Hurwitz space \( \mathcal{H}_{trop}^{g \rightarrow h, d}(\vec{\mu}) \) is the moduli space parametrizing Hurwitz covers of a tropical genus \( h \), \((r + s)\)-pointed curve, by a genus \( g \), \( n \)-pointed stable tropical curve (where \( n = \sum \ell(\mu^i) + s(d - 1) \)), having ramification profiles over the \( i \)th marked infinite edge given by the partition \( \mu^i \).

![Figure 3](image3.png)

**Figure 3.** A combinatorial type in the space \( \mathcal{H}_{trop}^{1 \rightarrow 0, 4}((4), (2, 2)) \) representing a top dimensional stratum.

A compactification \( \overline{\mathcal{M}}_{g,n}^{trop} \) is obtained from \( \mathcal{M}_{g,n} \) by allowing edge lengths of interior edges to become infinity. This idea has a nice interpretation in terms of analytifications of curves. An infinite edge is thought of as two infinite leaves whose 1-valent vertices are identified. Thus, a graph with infinite edge length corresponds to a skeleton of a nodal curve whose set of punctures contains a node, see [6, Section 1.4]. This compactification carries the structure of a generalized extended cone complex.

![Figure 4](image4.png)

**Figure 4.** An interior infinite edge, topologized as two infinite edges with the points at infinity identified.

In identical fashion, we obtain a compactification of the tropical Hurwitz space.

**Definition 19.** The space of tropical admissible covers \( \mathcal{H}_{trop}^{g \rightarrow h, d}(\vec{\mu}) \) is the compactification of the tropical Hurwitz space \( \mathcal{H}_{trop}^{g \rightarrow h, d}(\vec{\mu}) \) obtained by allowing edge lengths to become infinite.

3.2.2. Constructing the tropical moduli space: fixed combinatorial type. The tropical Hurwitz space \( \mathcal{H}_{trop}^{g \rightarrow h, d}(\vec{\mu}) \) is constructed as a topological colimit of cones. This construction is a slight variation to the procedure used in [1], which we briefly outline.

A combinatorial type \( \Theta \) of a tropical admissible cover is the data of a tropical admissible cover without the metric. Note that this data includes the expansion factors of the cover along edges. We stress that a metrization of the base graph fully determines the tropical Hurwitz cover.
Definition 20. Let $\Theta = [\theta : \Gamma_{src} \to \Gamma_{tgt}]$ be a combinatorial type for an admissible cover. Then an automorphism of $\Theta$ is the data of a commuting square of automorphisms of base and target

$$
\begin{array}{ccc}
\Gamma_{src} & \xrightarrow{\varphi_{src}} & \Gamma_{src} \\
\downarrow \quad \theta & & \downarrow \quad \theta \\
\Gamma_{tgt} & \xrightarrow{\varphi_{tgt}} & \Gamma_{tgt},
\end{array}
$$

where $\varphi_{src}$ and $\varphi_{tgt}$ are automorphisms, and $\varphi_{src}$ is required to preserve expansion factors on edges.

We use $\text{Aut}(\Theta)$ to denote the (finite) group of automorphisms of $\Theta$. We denote by $\text{Aut}_0(\Theta)$ the subgroup of $\text{Aut}(\Theta)$ where $\varphi_{tgt}$ is the identity, that is, automorphisms of $\Gamma_{src}$, preserving expansion factors, which cover the identity map.

Let $\Theta = [\theta : \Gamma_{src} \to \Gamma_{tgt}]$ be a combinatorial type for an admissible cover. Associated to a vertex $\tilde{v}$ of $\Gamma_{src}$, are local Hurwitz numbers. That is, for every $\tilde{v} \in \theta^{-1}(v)$, there is a Hurwitz number $h_{g,h,d}(\vec{\mu})$. Here, $g$ is the genus of $\tilde{v}$, $h$ is the genus of $v$, $d$ is the local degree of $\theta$ at $v$ and $\vec{\mu}$ is given by the expansion factors along tangent directions at $\tilde{v}$. It will sometimes be convenient to study all Hurwitz numbers above a given vertex simultaneously. We denote by $H(v)$ the product of all local Hurwitz numbers over vertices lying above $v$. When all ramification is marked, and the base graph is trivalent and totally degenerate, these local Hurwitz numbers encode the number of ways in which a morphism of metric graphs may be promoted to a morphism of nodal curves (or metrized complexes).

Let $B$ be the number of interior edges of the base curve of $\Theta$. A cover in this combinatorial type is determined by a choice of lengths in $\mathbb{R}_{\geq 0}$, since $\Theta$ contains data about expansion factors. Denote by $\sigma_{\Theta}^0 = (\mathbb{R}_{\geq 0})^B$. The moduli space $\mathcal{H}_{\Theta}^{trop}$, parametrizing covers with combinatorial type $\Theta$ is defined by $B \sigma_{\Theta}^0 / \text{Aut}(\Theta)$. See for instance Figure 5. The extended cone $\mathcal{H}_{\Theta}^{trop}$ is the quotient of its closure by $\text{Aut}(\Theta)$. We only consider cones such that $H(v)$ is nonzero for all vertices $v$ in $\Gamma_{tgt}$.

3.2.3. Constructing the tropical moduli space: graph contractions, gluing, and weights. A weighted graph contraction of $\Gamma$ is a composition of edge contractions of the underlying graph $\alpha : \Gamma \to \Gamma'$, endowed with a canonical genus function, $g_{\Gamma'}(v) = g_{\Gamma}(\alpha^{-1}v)$.

Given a graph contraction of $\Gamma_{tgt}$, there is a naturally induced contraction of $\Gamma_{src}$. These are the only allowable contractions for covers. Observe that as the length of an edge in the base curve tends to zero, the lengths of its preimages also tend to zero. (Loosely speaking, admissible covers compactify the Hurwitz space by allowing source and target to degenerate simultaneously — letting the source acquire nodes only when the target does and vice versa).

The moduli space $\mathcal{H}_{g,d}^{trop}(\vec{\mu})$ is constructed as the topological colimit

$$
\lim_{\to} \{\sigma_{\Theta}^0, j_\omega\},
$$

where $j_\omega$ is a contraction of tropical covers. Here, we think of automorphisms as graph contractions where no edge is actually contracted. This space carries a natural integral cone complex structure by barycentrically subdividing. The extended cones are glued similarly to obtain $\mathcal{H}_{g,d}^{trop}(\vec{\mu})$, which naturally carries an extended cone complex structure.

Automorphisms and weights on combinatorial types. A stable tropical curve with no loop edges will generically have no automorphisms, since generic edge lengths will be distinct, and
automorphisms are required to be isometries. However, an admissible cover may have automorphisms generically. Take for instance the cover depicted in Figure 6. These automorphisms will become relevant when we extract enumerative information from the degree of the branch map.

Remark 21. Recall that such automorphisms are familiar in the classical setting. The hyperelliptic locus of $\mathcal{M}_g$ can be understood as the space $\mathcal{H}_{g \rightarrow (2), \ldots, (2)}$, which is the stack quotient of $\mathcal{M}_{0,2g+2}$ by the trivial $\mu_2$ action. Here, the $\mu_2$ is naturally seen as acting on the covering curve, c.f. Figure 5.

The construction of Abramovich, Caporaso and Payne involves taking a colimit of cone complexes in the category of topological spaces and not in the category of topological stacks. As a consequence, we need to explicitly remember the data of stabilizers in our enumerative calculations.

Definition 22. We give top dimensional cells of combinatorial type $\Theta$ a weight $\omega(\Theta)$ defined as the product of:

(W1) A factor of $\frac{1}{|\text{Aut}_0(\Theta)|}$ for automorphisms.
(W2) A factor of local Hurwitz numbers $\prod_{v \in \Gamma_{\text{tgt}}} \mathbb{H}(v)$.
(W3) A factor of $M = \prod_{e \in \Gamma_{\text{tgt}}} M_e$, where $M_e$ is the product of the ramification indices above the $i$th edge, divided by their LCM.

Weight (W1) accounts for automorphisms.
Term (W2) encodes the fact that there may be many zero strata in $\T_{g,d}(\vec{\mu})$ which have the same dual graph. The map

$$\text{trop}_\Sigma : \Sigma(\T_{g,h,d}(\vec{\mu})) \to \T_{g,h,d}(\vec{\mu})$$

defined in Theorem 1, identifies the distinct cones of $\Sigma(\T_{g,h,d}(\vec{\mu}))$ with a given dual graph to a single cone in $\T_{g,d}(\vec{\mu})$. See Section 4.2.3.

Finally (W3) can be thought either as “ghost automorphisms”, or as arising from the normalization of the Harris-Mumford admissible cover space. We discuss this further in Section 4.2.3 after discussing the deformation theory of $\T_{g,d}(\vec{\mu})$. (W3) is the generalization of the index of the matrix of “length constraint equations” studied in [9, 11].

### 3.3. Skeleta of toroidal embeddings

Associated to any toroidal embedding $U \hookrightarrow X$ is a cone complex with integral structure which we refer to as the skeleton $\Sigma(X)$. The idea is that locally analytically near a point $x$, $X$ looks like an affine toric variety $V_\sigma$. Thus, we can build a cone complex from these cones $\sigma$. The key difference between fans and abstract cone complexes is that abstract cone complexes do not come with a natural embedding into a vector space. In the case that the toroidal embedding is a toric variety, this cone complex is precisely the fan. It is worth observing though that unlike a toric variety, where the fan determines the variety, the cone complex $\Sigma(X)$ is far from determining $X$.

For our purposes, the most important example of a toroidal embedding is the inclusion of the complement of a divisor with normal crossings, and we now explore this. Let $X$ be a normal scheme of dimension $n$. If $U \hookrightarrow X$ is given by the complement of a divisor with (not necessarily strict) normal crossings, then there is a natural stratification on $X$. That is, the 0-strata are the $n$-fold intersections of divisors $D_i$, the 1-strata are the $(n-1)$-fold intersections and so on. The top dimensional stratum is $U$. Consider a zero stratum $x \in X$. A formal neighborhood of $x$ looks like $n$ hyperplanes meeting at $x$. Locally near $x$, up to scaling, we obtain defining equations of these hyperplanes, say $f_1, \ldots, f_n$. These equations yield a system of formal local monomial coordinates near $x$. The completion of the local ring at $x$ is the coordinate ring of a formal affine space. The cone associated to this point is the standard cone for the toric variety $\mathbb{A}^n$. Call this cone $\sigma$. For a 1-stratum $W$, we get $(n-1)$ defining equations, giving a formal system of coordinates for an $(n-1)$ dimensional affine space. The cone associated to $W$ is the standard cone for $\mathbb{A}^{n-1}$. Call this cone $\tau$. Moreover, if $x \in W$, then the associated cone complex naturally identifies $\tau$ as a face of $\sigma$. This construction generalizes in the natural way, and the cones assemble into a cone complex $\Sigma(X)$. This yields an order reversing bijection between strata of the toroidal scheme $X$ and cones of the cone complex $\Sigma(X)$.

In [34], Thuillier shows that this cone complex lives naturally inside the Berkovich analytification of $X$. More precisely, given a toroidal embedding $U \hookrightarrow X$ with $X$ proper, he constructs a continuous (non-analytic) self-map of the Berkovich analytification of $X$,

$$p_X : X^{an} \to X^{an}$$

**Definition 23.** The image of $p_X$ is the skeleton and is denoted $\Sigma(X)$. The map $p_X$ is referred to as the retraction to the skeleton.

The crucial fact for our purposes is the existence of such a map and its properties. For an explicit realization in coordinates, see [1, Section 5.2]. Abramovich, Caporaso, and Payne extend this construction to toroidal compactifications of Deligne–Mumford stacks, and produce a generalized (extended) cone complex and a retraction from the Berkovich
analytification. We briefly describe the construction of the cone complex here. A detailed discussion of this retraction map in the setting of log structures may also be found in [36].

3.3.1. Local toric models. A toroidal scheme \( U \hookrightarrow X \) is described in a formal neighborhood of every point \( x \in X \) by a toric chart \( V_\sigma \). The cone \( \sigma \) is described as follows. Let \( M \) be the group of Cartier divisors supported on the complement of \( U \). Let \( M^+ \) be the submonoid of effective Cartier divisors. Then the cone \( \sigma \) is identified with the space of homomorphisms to the (additively written) monoid \( \mathbb{R}_{\geq 0} \),

\[
\text{Hom}(M^+, \mathbb{R}_{\geq 0}),
\]

equipped with the natural structure of a rational polyhedral cone with integral structure. In the language of logarithmic geometry, sheafifying \( M^+ \) produces the characteristic monoid sheaf, and \( M \) produces the characteristic abelian sheaf. The connections between tropical geometry and log geometry are being explored by Gross and Siebert, and Ulirsch, see [21, 36].

4. TROPICALIZATION OF THE MODULI SPACE OF ADMISSIBLE COVERS

4.1. Abstract tropicalization for admissible covers. In this section we describe an abstract tropicalization for admissible covers. The following tropicalization map — which we denote \( \text{trop} \), is obtained in direct analogy with the moduli space of curves, discussed in [1].

Let \( \mathcal{H}^{\text{an}}_{g,d}(\vec{\mu}) \) denote the Berkovich analytification of \( \mathcal{H}_{g,d}(\vec{\mu}) \). A point \( [D \to C] \) of \( \mathcal{H}^{\text{an}}_{g,d}(\vec{\mu}) \) is represented by an admissible cover of curves over \( \text{Spec}(K) \) where \( K \) is a valued field extension of \( C \). By properness of the stack \( \mathcal{H}_{g,d}(\vec{\mu}) \), this extends to a map of curves over \( \text{Spec}(R) \), where \( R \) is a rank 1 valuation ring with valuation \( \text{val}(\cdot) \). Let \( [\Gamma_D \to \Gamma_C] \) be the associated morphism of dual graphs of the special fibers. This morphism is well defined by the axioms placed on admissible covers. The ramification data of the admissible cover determines the expansion factors on all edges, therefore, we obtain a tropical admissible cover by metrizing these dual graphs. Let \( e \) be an edge of \( \Gamma_C \) corresponding to a node \( q \). Choose an étale neighborhood of the node at \( q \). The local equation is given by \( x_1 x_2 = f \). We metrize the edge \( e \) as \([0, \text{val}(f)]\).

**Definition 24.** Let \( [D \to C] \) be a point of \( \mathcal{H}^{\text{an}}_{g,d}(\vec{\mu}) \). With the notation above, we define the map

\[
\text{trop} : \mathcal{H}^{\text{an}}_{g,d}(\vec{\mu}) \to \mathcal{H}^{\text{trop}}_{g,d}(\vec{\mu})
\]

\[
[D \to C] \mapsto [\Gamma_D \to \Gamma_C].
\]

**Warning 25.** Note that in general it is not true that the analytification of a map of positive genus curves restricts to a finite harmonic map on minimal skeletons. Admissible covers are sufficiently well behaved that this is true, cf. [4, Section 4].

**Remark 26.** The result would be unchanged if the dual graph of the source is metrized as a nodal curve in its own right, as the defining equations of its nodes are completely controlled by the ramification. Nonetheless, we choose to think of a metrization of a combinatorial cover as a metrization of the base being lifted to one on the cover.

4.2. Functorial tropicalization for the stack of admissible covers. In [1], it is shown that there is a generalized extended cone complex that is functorially associated to any toroidal compactification of a Deligne–Mumford stack, which lives as a retract of the Berkovich analytification. To describe the construction in our specific case, we first recall some facts about the deformation theory of admissible covers. Along the way we gather facts which will be useful in studying the tautological maps on \( \mathcal{T}_{g \to h, d}(\vec{\mu}) \).
4.2.1. Deformation spaces. We analyze the completed local rings of points of the moduli space $\mathcal{M}_{g \to h,d}(\vec{\mu})$ by explicitly normalizing the local rings of the Harris–Mumford stack $\mathcal{M}_{g \to h,d}(\vec{\mu})$.

We begin with a toy example that illustrates the key features of the general case. Let $[D \to C]$ be an admissible cover in $\mathcal{M}_{g \to h,d}(\vec{\mu})$. Assume that $[D \to C]$ has no automorphisms. Let $z$ be a node of $C$ and assume that there are two nodes $\tilde{z}_1, \tilde{z}_2$ above $z$, with ramification 2 and 3 respectively. Let $\xi$ be the deformation parameter of the node $z$, and let $\tilde{\xi}_1$ and $\tilde{\xi}_2$ be the deformation parameters of the nodes $\tilde{z}_1, \tilde{z}_2$. The situation is depicted in terms of dual graphs in Figure 7.

As we deform $z$, we need to deform $\tilde{z}_1$ and $\tilde{z}_2$ in accordance with the ramification profiles. Thus, the coordinate ring of the versal deformation space is

$$C[\xi, \tilde{\xi}_1, \tilde{\xi}_2]/(\xi - \tilde{\xi}_1, \xi - \tilde{\xi}_2) \cong C[\tilde{\xi}_1, \tilde{\xi}_2]/(\tilde{\xi}_1^2 - \tilde{\xi}_2^3).$$

This is the completed local ring of a cuspidal cubic, at the cusp, and it is not integrally closed. Its integral closure is given by $C[\zeta]$. The normalization map is given by $\tilde{\xi}_1 \mapsto \zeta^3$ and $\tilde{\xi}_2 \mapsto \zeta^2$. Thus, there is a unique point in the normalization $\mathcal{M}_{g \to h,d}(\vec{\mu})$ lying above $[D \to C]$. In particular, notice that $\xi \mapsto \zeta^6$.

If we replace the ramification index 3 above with 4, then the completed local ring is the completed local ring at $(0,0)$ of two parabolas meeting at the origin in $\mathbb{A}^2$. That is, the completed local ring at $(0,0)$ of the affine place curve $V(\tilde{\xi}_1^4 - \tilde{\xi}_2^3)$. In particular, notice that these completed local rings not only fail to be integrally closed, but even fail to be an integral domain. In this case the normalization is $C[\zeta] \times C[\zeta]$, and $\xi \mapsto (\zeta^4, \zeta^4)$.

We now tackle the general case, which is essentially the same as the example above, with certain clerical difficulties.

Consider a point $[\pi : D \to C] \in \mathcal{M}_{g \to h,d}(\vec{\mu})$. Let $z_1, \ldots, z_k$ be the nodes of $C$, and let $\pi^{-1}(z_i) = \{z_{i,1}, \ldots, z_{i,r(i)}\}$ be the preimages of the $i$th node. Assume that the ramification
at $z_{i,j}$ is given by $p(i,j)$. Again assume that $[D \to C]$ has no automorphisms. As argued in [22], the completed local ring of $\mathcal{H}_g \to h, d(\vec{\mu})$ is given by

$$\mathcal{O}_{[D \to C]} = C[\xi_1, \ldots, \xi_{3h-3}, \xi_{i,j}] / (\xi_{i,j}^{p(i,j)} - \xi_1).$$

Here, the first $k$ $\xi_i$ correspond to deformation parameters of the nodes of $C$, and the remaining are the parameters for deformations of complex structure on $C$. In general, if $[D \to C]$ has automorphisms, we take the ring of invariants of this ring under the appropriate automorphism group.

The completed local ring $\mathcal{O}_{[D \to C]}$ is in general not integrally closed, indeed it is not even an integral domain. However, its normalization is a semi-local ring all of whose local rings are regular. Note that the only relations that appear in the ideal above are between $\xi_i$ and $\xi_{i,j}$ for varying $j$. Thus, it suffices to treat the case of a single node, and understand the normalization of rings of the form $C[\xi_i, \xi_{i,j}] / (\xi_{i,j}^{p(i,j)} - \xi_i)$ for each $i$. A straightforward simplification shows that the normalization is

$$M \prod_{i=1}^{M} C[\xi_1, \ldots, \xi_{3h-3}],$$

where $M = \prod_{\ell \in \Gamma_{gt}} M_{\ell}$ and $M_{\ell}$ is the product of the ramification indices above the $\ell$th node, divided by their LCM. Moreover, we have a map $\xi_i \mapsto \xi_i^{N_i}$ (on each factor of the product) where $N_i$ is the LCM of the ramification indices over the $i$th node. This gives us an explicit formula for how the deformation parameters for the nodes of $C$ pull back to the deformation parameters of the corresponding node in $[D \to C]$.

A main technical result of [1] is that the skeleton of a toroidal Deligne–Mumford stack decomposes as a disjoint union of extended open cones corresponding to each stratum of the stack, modulo the appropriate monodromy.

Recall from Definition 23 that a point of the skeleton $\Sigma(\mathcal{H}^\text{an}_{g, d}(\vec{\mu}))$ is a point in the image of the retraction map $p_{tt} : \mathcal{H}^\text{an}_{g, d}(\vec{\mu}) \to \mathcal{H}^\text{an}_{g, d}(\vec{\mu})$. Thus, given a point of the skeleton, represented by an admissible cover $[D \to C]$ over a valued extension field of $C$, we obtain, by taking dual graphs of the special fiber, a point $[\Gamma_D \to \Gamma_C]$ of the space $\mathcal{H}^\text{trop}_{g, d}(\vec{\mu})$.

**Definition 27.** The map $\text{trop}_\Sigma : \Sigma(\mathcal{H}^\text{an}_{g, d}(\vec{\mu})) \to \mathcal{H}^\text{trop}_{g, d}(\vec{\mu})$ is obtained as above, by sending an admissible cover $[D \to C]$ to the tropical cover $[\Gamma_D \to \Gamma_C]$.

### 4.2.2. Proof of Theorem 1

The proof naturally breaks up into three steps. First, we analyze, stratum-by-stratum, the structure of the skeleton, and compare it with the structure of the tropical moduli space. Next, we analyze how the strata of the skeleton glue together to form the extended cone complex. Finally, we study the projection map to the skeleton, and the map $\text{trop}_\Sigma$.

**The cones of the skeleton.** Consider a combinatorial type of admissible cover $\Theta = [\Gamma_{src} \to \Gamma_{gt}]$, and suppose for the moment that $\Gamma_{gt}$ is trivalent. Any zero stratum of $\mathcal{H}_g \to h, d(\vec{\mu})$ corresponds to one such admissible cover, though in general there are multiple zero strata corresponding to a given $\Theta$. Let $\{w_i = [D^1 \to C^1]\}_{i=1,...,k}$ be the zero strata of $\mathcal{H}_g \to h, d(\vec{\mu})$ with type $\Theta$.

The deformation parameters for the cover corresponding to the nodes of $C^1$ yield local coordinates for a formal affine space near $w_i$. The open cone corresponding to $w_i$ is a copy of the standard (affine space) cone of dimension $3h - 3 + n$, quotiented out by the automorphisms of $\Theta$. The general case for higher dimensional strata is similar. The deformation
There are multiple lifts of a (trivalent, totally degenerate) combinatorial type of the image of \( \tau \) in the skeleton is compatible with the gluing of the extended cones in \( \mathcal{T}^{\text{trop}}_{g,d}(\bar{\mu}) \) and \( \mathcal{T}^{\text{trop}}_{g,d}(\bar{\mu}) \) is a contraction of the cone in \( \mathcal{T}^{\text{trop}}_{g,d}(\bar{\mu}) \) of the corresponding combinatorial type.

**Gluing.** We now study how the cones of the skeleton are assembled to form the cone complex. First, note that in an étale neighborhood \( V \) of a point \( [D \to C] \) in \( \mathcal{T}^{\text{an}}_{g,d}(\bar{\mu}) \), the locus parametrizing covers where the \( i \)th node of \( C \) persists is a divisor. We may assume that the locus in \( V \) parametrizing covers of singular curves is a union of divisors. In fact, the combinatorial type of any cover parametrized by a point in \( V \) is a contraction of the type \( [\Gamma_D \to \Gamma_C] \). As a result, we see that if a stratum \( W_0 \) is contained in the closure of \( W_1 \), then there exists a graph contraction taking the combinatorial type associated to \( W_0 \) to that of \( W_1 \). Consequently, observe that if a cone \( \tau \) is a face of a cone \( \sigma \) in \( \Sigma(\mathcal{T}^{\text{an}}_{g,d}(\bar{\mu})) \), then the image of \( \tau \) under \( \text{trop}_\Sigma \) is a face of the image of \( \sigma \) under \( \text{trop}_\Sigma \). We conclude that the gluing of the extended cones of \( \Sigma(\mathcal{T}^{\text{an}}_{g,d}(\bar{\mu})) \) is compatible with the gluing of the extended cones in \( \mathcal{T}^{\text{trop}}_{g,d}(\bar{\mu}) \).

**Projecting to the skeleton.** It remains only to check that the set theoretic tropicalization map agrees with the projection to the skeleton followed by the map \( \text{trop}_\Sigma \). That is,

\[
\text{trop} = \text{trop}_\Sigma \circ p_H.
\]

Let \( p = [D \to C] \) be a point of \( \mathcal{T}^{\text{an}}_{g,d}(\bar{\mu}) \). Such a point is represented by an admissible cover over a valued field \( K \). By properness of the moduli space, we obtain a family of admissible covers over the valuation ring \( R \), as previously discussed. The map \( \text{trop} \) takes \( p \) to the metrized dual graph \( [\Gamma_D \to \Gamma_C] \) as discussed in Section 4.1. Recall that the length of the edge \( e_i \) of the base \( \Gamma_C \) is \( \text{val}(f_i) \) where \( f_i \) is the defining equation of the corresponding node in the special fiber. As previously discussed, the \( f_i \) form a basis for the monoid of effective divisors \( M^+ \). It now easily follows from arguments in \( \Pi \) Section 5.2] that \( \text{trop} = p_H \circ \text{trop}_\Sigma \), as desired. Finally, notice that for any given stratum \( W \) of the moduli space \( \mathcal{T}_{g-h,d}(\bar{\mu}) \), formally locally, the picture is no different to that of \( \mathcal{M}_{g,n} \), which is pursued in \( \Pi \) Section 7. The cone \( \sigma_W \) associated to \( W \) in the skeleton is easily seen to be isomorphic to the cone in \( \mathcal{T}^{\text{trop}}_{g-h,d}(\bar{\mu}) \) via the map \( \text{trop}_\Sigma \), and hence \( \text{trop}_\Sigma \) is a face morphism.

\[\square\]

4.2.3. *The map from the skeleton to the tropical moduli space.* For the moduli of curves, the natural map \( \Sigma(\mathcal{M}^{\text{an}}_{g,n}) \to \mathcal{M}^{\text{trop}}_{g,n} \) is an isomorphism. This is not the case for admissible covers. The map

\[
\text{trop}_\Sigma : \Sigma(\mathcal{T}^{\text{an}}_{g,d}(\bar{\mu})) \to \mathcal{T}^{\text{trop}}_{g,d}(\bar{\mu}),
\]

does not distinguish between strata which are “combinatorially indistinguishable”. Said more precisely, there are multiple zero strata in \( \mathcal{T}^{\text{an}}_{g,d}(\bar{\mu}) \) corresponding to the same combinatorial type \( \Theta = [\Gamma_{src} \to \Gamma_{tgt}] \). However, this loss of information can be completely characterized. Distinct cones in the skeleton become identified in \( \mathcal{T}^{\text{trop}}_{g,d}(\bar{\mu}) \) for the following reasons.

(A) There are multiple lifts of a (trivalent, totally degenerate) combinatorial type \( \Theta \) to an admissible cover of nodal curves \( [C \to D] \),

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(B) A nodal admissible cover \([D \to C]\) is really an element of the Harris–Mumford moduli space \(\overline{\mathcal{M}}_{g,d}(\vec{\mu})\). As discussed in Section 4.2.1, there are multiple points in the normalization \(\overline{\mathcal{M}}_{g,d}(\vec{\mu})\) that lie above \([D \to C]\). As we have seen, there are precisely \(M\) such points, where \(M = \prod_{e \in \Gamma_{\text{tgt}}} M_i\), and \(M_i\) is the product of the ramification indices above the \(i\)th node, divided by their LCM.

These explain the weights on the tropical moduli space described in Section 3.2.3. The product of the local Hurwitz numbers from (A) above gives \((W_2)\). The weight \((W_3)\) on \(\mathcal{H}_{g \to h,d}(\vec{\mu})\) is an artifact of (B) above.

5. Classical and Tropical Tautological Maps

5.1. The branch maps. The classical Hurwitz space \(\overline{\mathcal{M}}_{g,d}(\vec{\mu})\) admits a branch map, recording the base curve, marked at its branch points:

\[
br : \overline{\mathcal{M}}_{g,d}(\vec{\mu}) \to \mathcal{M}_{h,r+s}
\]

\([D \to C] \mapsto [(C,p_1,\ldots,p_r,q_1,\ldots,q_s)]\).

The degree of this map is the corresponding Hurwitz number. Intuitively, fixing a point \(p\) in \(\mathcal{M}_{h,r+s}\), the cardinality of the fiber over \(p\) is the number of covers having given discrete data, \((d,g,h,\vec{\mu})\) over the specific pointed stable curve \(C_p\).

The tropical branch map is defined similarly, taking values in the space of pointed tropical \((r+s)\)-pointed curves.

\[
br^{\text{trop}} : \overline{\mathcal{H}}_{g,d}(\vec{\mu}) \to \mathcal{M}_{h,r+s}^{\text{trop}}
\]

\([\Gamma_{\text{src}} \to \Gamma_{\text{tgt}}] \mapsto [\Gamma_{\text{tgt}}]\).

5.2. Tropicalizing the branch map. We want to compare the classical and tropical branch maps in a functorial manner. The process of taking skeletons for toroidal Deligne–Mumford stacks is compatible with toroidal and subtoroidal morphisms. Recall that a toroidal morphism \(\phi : \mathcal{X} \to \mathcal{Y}\) is a morphism such that for every point \(x \in \mathcal{X}\), there exist compatible étale toric charts around \(x\) and \(\phi(x)\), on which the morphism is given by a dominant equivariant map of toric varieties. We remark that although this is not pursued in [1], we can impose a weaker condition on morphisms between toroidal embedding for which there is an induced map on skeleta.

Definition 28. A morphism \(\varphi : \mathcal{X} \to \mathcal{Y}\) of toroidal embeddings is said to be locally analytically toric if for every point \(x \in \mathcal{X}\), there exist étale toric charts around \(x\) and \(\phi(x)\) such \(\phi\) is given by an equivariant toric morphism. That is, we have the following diagram

\[
\begin{array}{ccc}
\mathcal{V}_\sigma & \longrightarrow & \mathcal{V}_x \\
\downarrow & & \downarrow \\
\mathcal{V}_\tau & \longrightarrow & \mathcal{V}_{\phi(x)} \\
\end{array}
\]

where the top row is a local toric chart around \(x\) in \(\mathcal{X}\), and the bottom row is a local toric chart around \(\varphi(x)\) in \(\mathcal{Y}\). The leftmost vertical arrow is torus equivariant.
It is easy to see that any morphism that is locally analytically toric induces a map on skeleta, and the statements for toroidal and subtoroidal morphisms in [1] go through without any substantial changes. The crucial ingredient is that monomial coordinates on the target pull back to monomial coordinates on the source.

**Lemma 29.** The branch map \( br: \overline{H}_{g,d}(\vec{\mu}) \to \overline{M}_{h,r+s} \) is locally analytically toric, where \( \overline{M}_{h,r+s} \) is given the toroidal structure from the inclusion \( \overline{M}_{h,r+s} \hookrightarrow \overline{M}_{h,r+s} \).

**Proof.** To prove this, we need to find compatible local toric models for the points \( x \in \overline{H}_{g,d}(\vec{\mu}) \) and \( br(x) \in \overline{M}_{h,r+s} \), such that \( br \) is given by an equivariant locally analytic toric morphism. It is sufficient to check that locally analytically \( br \) pulls back monomials to monomials. However, this is clear from the discussion of deformation theory in Section 4.2.1: deformations of an admissible cover being controlled by deformations of the target curve amounts precisely to saying that the local affine models around a boundary point are identified via the branch map. \( \square \)

Identifying the skeleton of \( \overline{M}_{h,r+s}^{an} \) with the tropical moduli space \( \overline{M}_{h,r+s}^{trop} \), we immediately have the following consequence.

**Corollary 30.** The analytified branch map \( br^{an}: \overline{H}_{g,d}^{an}(\vec{\mu}) \to \overline{M}_{h,r+s}^{an} \) induces a map on skeleta, \( \Sigma(\overline{H}_{g,d}^{an}(\vec{\mu})) \to \overline{M}_{h,r+s}^{trop} \).

**Proof.** Locally analytically, monomials are pulled back to monomials. Thus, it follows that there is an induced map on each cone of \( \Sigma(\overline{H}_{g,d}^{an}(\vec{\mu})) \) to the skeleton \( \overline{M}_{h,r+s}^{trop} \). The fact that these maps glue to give a global map is straightforward. \( \square \)

5.2.1. **Proof of Theorem 4, part one: branch map.** It is clear from the description of the abstract tropicalization map for covers and for curves, that the tropical and classical branch maps fit together in a commutative diagram:

\[
\begin{array}{ccc}
\overline{H}_{g,d}^{an}(\vec{\mu}) & \xrightarrow{\text{trop}} & \overline{M}_{h,r+s}^{trop} \\
\downarrow{br^{an}} & & \downarrow{br^{trop}} \\
\overline{M}_{h,r+s}^{an} & \xrightarrow{\text{trop}} & \overline{M}_{h,r+s}^{trop}
\end{array}
\]

We recall from [1, Section 6], taking skeletons is functorial for toroidal morphisms. We thus also obtain the following more detailed commutative diagram. The commutativity of the left square follows by functoriality of taking skeletons. We use [1, Theorem 1.2.1] to identify the skeleton of \( \overline{M}_{h,r+s}^{an} \) with the tropical moduli space \( \overline{M}_{h,r+s}^{trop} \). Theorem 1 asserts that \( \text{trop}_\Sigma \) is an isomorphism on each cone. Thus we get an extension of left square to the full diagram below.

---

1We have been informed by the authors of [1] that these morphisms will be incorporated into a revised version of their paper.
The tropical branch map \( \text{br}^{\text{trop}} \) fills in the far right vertical arrow, making the entire diagram commute, since \( \psi \) is an isomorphism, and \( \text{trop}_\Sigma \) is an isomorphism when restricted to any cone. It follows that the tropical branch map is indeed the tropicalization of the classical branch map, as desired. \( \square \)

5.3. The source maps. The “classical” source map takes a cover \( [D \to C] \) to its source curve \( [D \to C] \in \overline{M}_{g,n} \) where \( n \) is the number of smooth ramification points, which is equal to the sum of the lengths of the partitions \( \mu^i \). Similarly, there is a tropical source map taking \( [\Gamma_{\text{src}} \to \Gamma_{\text{tgt}}] \) to the source graph \( \Gamma_{\text{src}} \). We wish to show that the tropical source map is naturally identified with the tropicalization of the analytified source map.

5.3.1. Proof of Theorem 4, part two: source map. We want to show that in an étale neighborhood of a point \( x \in \overline{M}_{g,d}(\vec{\mu}) \), the map \( \text{src} \) is given by a toric morphism. Let \( x = [D \to C] \), and let \( [\Gamma_{\text{src}} \to \Gamma_{\text{tgt}}] \) be its (unmetrized) dual graph. In \( \overline{M}_{g,d}(\vec{\mu}) \), the local monomial coordinates are given by the deformation parameters of the nodes of \( C \). Let us focus on a single node \( p \in C \) and on \( \tilde{\tau}_1, \ldots, \tilde{\tau}_m \) the nodes of \( D \) mapping to \( \tilde{\tau}_i \). The nodes \( \tilde{\tau}_i \) can be independently deformed, and their deformation parameters \( \xi_1, \ldots, \xi_m \) are the local monomial coordinates on \( \overline{M}_{g,n} \). The deformations of \( [D] \) which come from deformations of the map \( [D \to C] \) satisfy the relations

\[
\xi_1^{w_1} = \xi_2^{w_2} = \cdots = \xi_m^{w_m},
\]

where \( w_i \) is the ramification on the node of \( [D] \) corresponding to \( \tilde{\tau}_i \). Thus, locally analytically, \( \text{src} \) maps \( \overline{M}_{g,d}(\vec{\mu}) \) via a toric morphism. This map induces a map on skeleta.

To metrize \( [\Gamma_{\text{src}} \to \Gamma_{\text{tgt}}] \) the above equations yield length conditions on edges \( \tilde{e}_i \) of \( \Gamma_{\text{src}} \) mapping to a fixed edge \( e \) of \( \Gamma_{\text{tgt}} \):

\[
w_1 \ell(\tilde{e}_1) = w_2 \ell(\tilde{e}_2) = \cdots = w_m \ell(\tilde{e}_m),
\]

where \( w_i \) is the expansion factors along \( \tilde{e}_i \). This collection of linear conditions cuts out a subcone of \( \overline{M}_{g,n}^{\text{trop}} \). The result now follows from similar arguments to Section 5.2.1. \( \square \)

5.4. The degree of the branch map. In this section we prove Theorem 2 by extracting the degree of the branch morphism (the Hurwitz number) from the associated map on skeleta. We can compute this degree over any point of the moduli space \( \overline{M}_{h,r+s} \). By choosing a zero-stratum, i.e. the most degenerate base curve, the degree can be computed in the local analytic toric coordinates, as we now explain.

Choose a zero stratum \( P \in \overline{M}_{h,n} \). The versal deformation parameters give local analytic coordinates in a formal neighborhood \( V_P \) of this point. In fact, \( V_P \) is a formal affine space with the chosen coordinates. Let the points \( \tilde{P}_1, \ldots, \tilde{P}_h \) be the preimages of \( P \) in \( \overline{M}_{g,d}(\vec{\mu}) \).
Then the deformation parameters at for the nodes of the cover corresponding to \( \tilde{p}_i \) give local analytic coordinates in a formal neighborhood \( U_\tilde{p}_i \). As we argued previously, the branch map \( \text{br} \) yields a torus equivariant map \( U_\tilde{p}_i \to V_p \). We compute the degree of \( \text{br} \) by computing the local degree at each \( \tilde{p}_i \) and adding the contributions. The local contribution at \( \tilde{p}_i \) can then be computed by considering the induced map on the cone associated to the toric neighborhoods \( U_\tilde{p}_i \) and \( V_p \).

![Figure 9. Cones in \( \Sigma(\mathcal{T}_{g,d}(\bar{\mu})) \) lying above a top dimension stratum in \( \mathcal{M}_{h,r+s}^{\text{trop}} \).](image)

5.4.1. Proof of Theorem 2. The proof proceeds in two steps. We first show that the degree of the branch map can be recovered from the maps on skeleta. We then proceed to show that with the weighting introduced in Section 3.2.3, the degree of the tropical branch map is equal to the degree of the map on skeleta.

Cones of \( \mathcal{M}_{h,r+s}^{\text{trop}} \) and \( \Sigma(\mathcal{T}_{g,d}(\bar{\mu})) \) may carry generic stabilizers. To avoid working directly with a stacky structure on the cone complex we adopt a standard procedure and rigidify the problem. Let \( \Theta = [\Gamma_{\text{src}} \to \Gamma_{\text{tgt}}] \) be a combinatorial type and let \( \sigma_\Theta \) be a cone of \( \Sigma(\mathcal{T}_{g,d}(\bar{\mu})) \) having \( \Theta \) as its combinatorial type. Under the branch map \( \text{br}_\Sigma \), \( \sigma_\Theta \) maps to the cone \( \sigma_{\Gamma_{\text{tgt}}} \) in \( \mathcal{M}_{h,r+s}^{\text{trop}} \). We pass to covers \( \tilde{\sigma}_\Theta \) and \( \tilde{\sigma}_{\Gamma_{\text{tgt}}} \) of source and target cones, such that the quotient by the respective automorphism groups of the combinatorial types yield \( \sigma_{\Gamma_{\text{tgt}}} \) and \( \sigma_\Theta \). If \( \Gamma_{\text{tgt}} \) contains loop edges, we mark an additional point, adding an infinite edge attached to each loop edge. We do the same for the loops of \( \Gamma_{\text{src}} \) mapping to this loop edge, allowing each infinite edge to map with expansion factor \( 1 \). This eliminates the automorphisms, and lifting the branch maps to these covers, we obtain the following diagram.

\[
\begin{array}{ccc}
\tilde{\sigma}_\Theta & \longrightarrow & \sigma_\Theta \\
\downarrow \text{br}_\Sigma & & \downarrow \text{br}_\Sigma \\
\tilde{\sigma}_{\Gamma_{\text{tgt}}} & \longrightarrow & \sigma_{\Gamma_{\text{tgt}}}
\end{array}
\]

After barycentrically subdividing and lifting to covers, the map \( \text{br}_\Sigma \) is a morphism of standard affine space cones.

The map \( \text{br} \) analytifies to a map \( \text{br}_{\text{an}} : \mathcal{T}_{g,d}(\bar{\mu}) \to \mathcal{M}_{h,r+s}^{\text{an}} \). Since \( \text{br} \) is flat, proper and quasi-finite, we may compute its degree by computing the degree of \( \text{br}_{\text{an}} \) on an analytic domain in \( \mathcal{T}_{g,d}(\bar{\mu}) \). A standard way to produce domains on which the map can be easily understood is to pull back polyhedral regions in the skeleton, via the canonical projection map.

Consider the two projection maps \( p_H : \mathcal{T}_{g,d}(\bar{\mu}) \to \Sigma(\mathcal{T}_{g,d}(\bar{\mu})) \) and \( p_M : \mathcal{M}_{h,r+s}^{\text{an}} \to \mathcal{M}_{h,r+s}^{\text{trop}} \). Pulling back the cones \( \tilde{\sigma}_\Theta \) and \( \tilde{\sigma}_{\Gamma_{\text{tgt}}} \) via these projection maps produces polyhedral domains in analytic tori. Let \( \mathcal{U}_\Theta = p_H^{-1}(\sigma_\Theta) \) and \( \mathcal{U}_M = p_M^{-1}(\sigma_{\Gamma_{\text{tgt}}}) \). The analytified branch map \( \text{br}_{\text{an}} \), induces a map

\[\varphi : \mathcal{U}_H \to \mathcal{U}_M,\]
between polyhedral domains in analytic tori (see [33, Section 6]).

Furthermore, we have coordinates on these analytic tori, given by the deformation parameters, as previously discussed. Let \( \xi_i \) be coordinates on \( U_M \) and \( \tilde{\xi}_i \) the coordinates on \( U_H \). To compute the degree of this map, we need to understand \( \phi^*(\xi_i) \). It follows from the discussion in Section 4.2.1 that \( \phi^*(\xi_i) = \tilde{\xi}_N^i \) where \( N \) is the LCM of the ramification indices at the nodes lying over the \( i \)th node. This is precisely equal to the dilation factor in the \( i \)th coordinate for this map of covers of cones. Hence, locally analytically, the degree of this map is equal to the product of the LCM’s of the ramifications over each node. Accounting for the stacky structure on each cone on source and target, we recover the degree of the branch map by adding up these dilation factors for each once lying over \( M_\Gamma \).

We now need to pass from the skeleton \( \Sigma(\text{H}_{\text{an}}(\vec{\mu})) \) to the tropical space \( \text{H}_{\text{trop}}(\vec{\mu}) \). As we have discussed, several cones of the skeleton are identified via \( \text{trop}\Sigma \). However, it is clear from the discussion in Section 4.2.1 that the contributions to the degree from the branch map of each cone in the skeleton of the same combinatorial type (i.e. whose image under \( \text{trop}\Sigma \) coincide) are equal.

We continue to fix the base graph \( \Gamma \), and the combinatorial type \( \Theta \) of the admissible cover. We need to understand how many points of \( \text{H}_{g,d}(\vec{\mu}) \) have dual graph \( \Gamma \). Notice that \( \Gamma \) is trivalent with totally degenerate genus function since we are working with a top dimensional stratum of the moduli space. As a result, we can uniquely build a curve \( C \) whose dual graph is \( \Gamma \) by placing the nodes at 0, 1 and \( \infty \) on each \( \mathbb{P}^1 \) component. The number of Harris–Mumford algebraic admissible covers in a zero stratum of \( \text{H}_{\text{CM}}(\vec{\mu}) \) is by definition the product of the local Hurwitz numbers of \( \Theta \). Furthermore, notice that for each such algebraic admissible cover \( [D \to C] \) in \( \text{H}_{\text{CM}}(\vec{\mu}) \), the number of preimages of \( [D \to C] \) in \( \text{H}_{g,d}(\vec{\mu}) \) is given by the weight (W3) in Section 3.2.3, namely, the product \( \prod_{e \in \Gamma_{\text{tgt}}} M_i \), where \( M_i \) is the product of the ramification indices above the \( i \)th node, divided by their LCM.

These precisely are the weights on the tropical admissible cover space, and thus, we see that the weighted degree of the map \( \text{br}_{\text{trop}} \) is equal to the degree of the map \( \text{br}_{\Sigma} \). The result follows.

\[ \square \]

6. Applications to Previous Work

In this section, we recover known correspondence theorems for tropical Hurwitz numbers at the level of moduli spaces. In Section 6.1 we return to the motivating case of double Hurwitz numbers. The first correspondence theorem for double Hurwitz numbers was proved in [11]. In that work, a tropical analogue of the relevant relative stable map space is constructed, in order to carry out the relevant intersection theory computations. We pay special attention to the relation of our tropical admissible cover spaces to the ones used there. The equality of tropical and classical double Hurwitz numbers can also be deduced from the general correspondence theorem of [7], which we recover at the level of moduli spaces in Section 6.2.

6.1. Monodromy graphs and tropical double Hurwitz numbers. We now frame the “monodromy graphs” computation for the double Hurwitz numbers, introduced by the first two authors and Paul Johnson, in a geometric context. First, we briefly recall the relevant aspects of [11].

We fix two partitions \( \mu^1 = (\mu^1_1, \ldots, \mu^1_k) \) and \( \mu^2 = (\mu^2_1, \ldots, \mu^2_\ell) \) of degree \( d \), and denote \( s = 2g - 2 + \ell + k \), the number of simple branch points, determined by Riemann–Hurwitz.
Definition 31. Monodromy graphs project to the segment \([0, s + 1]\) and are constructed as follows:

(i) Start with \(k\) small segments over \(0\), with weights \(\mu_1, \ldots, \mu_k\).
(ii) Over the point \(1\), create a trivalent vertex by either joining two strands or splitting two strands. When joining two strands, label the outgoing edge with the sum of the incoming weights. In case of a cut, label the two new strands in all possible positive ways of adding the weight of the split edge. Each choice of split produces a distinct monodromy graph.
(iii) Repeat this process for integers up to \(s\).
(iv) Retain all connected graphs that terminate with \(\ell\) points of weights \(\mu_1^2, \ldots, \mu_\ell^2\) over \(s + 1\).

It is proved in [11] that such graphs produce a formula for the double Hurwitz number.

Theorem 32 (C–Johnson–M). The double Hurwitz number is equal to
\[ h_{g \to 0,d}(\vec{\mu}) = \sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} \prod w(e), \]
where we take the sum over isomorphism classes of monodromy graphs, and the product of interior edge weights of each graph (i.e. edges not over \(0\) or \(s + 1\)).

Example 33. Consider for instance the monodromy graph depicted in Figure 10. This graph has two automorphisms, coming from the double edge (“wiener”). We ignore the automorphisms coming from the “fork” since the ends are now marked. The product of the weights of interior edges is 16, so this graph contributes 8 to the sum in the preceding theorem.

![Figure 10. A genus 1 monodromy graph for degree 4 covers of \(\mathbb{P}^1\) with ramification profiles \((4)\) over \(0\) and \((2, 2)\) over \(\infty\).](image)

We now study the monodromy graphs in terms of admissible covers, and recover the above formula. Moreover, we reinterpret the expected dimensional cells in the tropical moduli space of parametrized covers as a collection of cones in \(\mathcal{M}_{g,2+s}\), lying over a chosen cell in \(\mathcal{M}_{0,2+s}\). From this vantage point, we see the factors appearing in the above sum from the geometric perspective developed in previous sections.

6.1.1. The CJM covers of \(\mathbb{P}^1_{\text{trop}}\). In order to reinterpret the CJM formula in our framework, we need to build tropical admissible covers from the tropical relative stable maps considered in [11]. Not all relative stable maps produce admissible covers, but those combinatorial types of expected dimension do produce admissible covers. Denote by \(\mathbb{P}^1_{\text{trop}}\), the “two pointed” tropical \(\mathbb{P}^1\), \(\mathbb{R} \cup \{\pm \infty\}\). We first recall the definition of tropical covers of \(\mathbb{P}^1_{\text{trop}}\) as stated in [11].

Definition 34. Let \(\mu^1, \mu^2\) be partitions of \(d\). Let \(\Gamma\) be a genus \(g\) \(\ell((\mu^1) + \ell(\mu^2))\)-pointed tropical curve. A parametrized tropical curve of genus \(g\) and degree \((\mu^1, \mu^2)\) in \(\mathbb{P}^1\) is an integral harmonic morphism \(\theta : \Gamma \to \mathbb{P}^1_{\text{trop}}\), where \(\Gamma\) has genus \(g\), such that
(i) The image of $\Gamma$ without its infinite edges is inside $\mathbb{R}$.
(ii) The multiset of expansion factors over the $+\infty$ segment is given by $\mu^1$, and the multiset of expansion factors over the $-\infty$ segment is given by $\mu^2$.

A combinatorial type of a parametrized tropical curve in $\mathbb{P}^1$ is the data obtained from dropping the edge length data and remembering only the source curve together with its expansion factors. For a combinatorial type $[\alpha]$ of covers, we build an unbounded, open unbounded convex polyhedron, formed by varying edge lengths. These cells glue together to form a moduli space $M_g(\mathbb{P}^1_{trop}, \mu^1, \mu^2)$. We refer to [11] for details on the construction.

The key difference between parametrized tropical curves in $\mathbb{P}^1_{trop}$ and admissible covers is that parametrized curves in $\mathbb{P}^1_{trop}$ may contract subgraphs. However, if there is a combinatorial type with a contracted component, it will not be of expected dimension, and will not contribute to the degree. In fact, the moduli space of tropical covers constructed in [11] does not consider cells where the associated combinatorial type is not of expected dimension.

It is shown in [11] that the moduli space admits a natural branch map to $((\mathbb{P}^1_{trop})^s$ where $s$ is the number of simple branch points. Moreover, by weighting this moduli space appropriately, the degree of this branch map essentially recovers the formula above. In particular, the factor $\prod w(e)$ arises as a product of the determinant of the branch map, times a certain weight on each cone of $M_g(\mathbb{P}^1_{trop}, \mu^1, \mu^2)$. We remark that in the construction of $M_g(\mathbb{P}^1_{trop}, \mu^1, \mu^2)$, it is necessary to disregard cells of unexpected dimension, in order to obtain a well defined degree.

Given a parametrized tropical cover whose combinatorial type is of expected dimension, we obtain an admissible cover by first giving the base $\mathbb{P}^1_{trop}$ the natural structure of a 2-pointed tropical curve as follows. We mark the images of all branch points of $\Gamma$. Since the combinatorial type is of expected dimension, there are precisely $s$ points which are marked, where $s$ is the number of simple branch points.

Additionally, we subdivide $\Gamma$ such that vertices map to vertices. This amounts to making each point in the preimage of a branch point into a vertex, see Figure 11. Finally, we add an infinite edge to each of the $s$ marked points on the base obtaining what we call the path graph on $s$ vertices. The ramification over these infinite edges is simple. By the local Riemann–Hurwitz condition, there is a unique way to add $(d-1)$ infinite edges mapping to each new infinite edge added on the base graph, such that the ramification over each infinite edge is simple. We record the following observation.

**Proposition 35.** Let $\Gamma$ be the path graph on $s$ vertices, and let $\mathcal{H}_\Gamma$ be subcomplex of $\mathcal{H}^{trop}_{g,d}(\mu^1, \mu^2)$ such that $\Gamma_{tgt} = \Gamma$. Then, there is an identification of cone complexes

$$\mathcal{H}_\Gamma \cong M_g(\mathbb{P}^1_{trop}, \mu^1, \mu^2).$$

The proof follows immediately from the preceding discussion.

When covers have no contracted components, and the cell of the tropical moduli space of the expected dimension, we essentially recover a cone of tropical admissible covers. The only change is that here, we drop the data of the root vertex, which is irrelevant, since the combinatorics does not change if we translate the $s$ branch points by a fixed real number. It is easy to see that the branch map defined in [11] transforms naturally into the map $\text{br}^{trop}$ defined here. With this translation, we recover the previously defined tropical double Hurwitz numbers, as we now demonstrate.
6.1.2. The CJM formula for the double Hurwitz number. We now turn our attention to the expression of double Hurwitz numbers in terms of monodromy graphs, studied in [11]. The following useful proposition is due to Lando and Zvonkin [25].

Proposition 36. Let $\mu^1 = (d)$ and $\mu^2$ be arbitrary with $t$ parts. Then we have the formula

$$h_{0 \to 0, d}(\mu^1, \mu^2) = (t - 1)! d^{t-2}.$$ 

Observe that if $\mu^2$ is a two-part partition, i.e. $t = 2$, then $h_{0 \to 0, d}(\mu^1, \mu^2) = 1$.

Warning 37. Given a parametrized tropical curve, when we subdivide, we create new vertices and consequently new interior edges. There is a unique expansion factor on his new interior edges by the harmonicity condition. Let $\nu$ be a new vertex created in such a manner by subdivision. The local Hurwitz number at $\nu$ is given by $h_{0 \to 0, d}((d), (d))$, which we know to be $1/d$. It will be crucial in the forthcoming discussion that the quantity $h_{0 \to 0, d}((d), (d))$, times the weight on the new bounded edge is 1. See Figure 11. For a related issue, see the discussion in [9, Lemma 3.5].

Proof of the CJM formula. We work with the tropical admissible cover space $\mathcal{M}_{g \to 0, d}^{\text{trop}}(\mu^1, \mu^2)$. Since the degree of the tropical branch map is constant, we may choose to compute it over a fixed top dimensional cell in $\mathcal{M}_{0, 2+s}^{\text{trop}}$, where $s$ is the number of simple branch points. We choose the locus of curves whose combinatorial type is a path graph, augmented with one infinite edge at every bivalent vertex, as shown in Figure 12. We denote this combinatorial type by $[\Gamma]$, and by $\mathcal{M}_\Gamma$ the corresponding cell of the tropical moduli space.

It was observed in [11] Remark 5.2 that the combinatorial types lying over $\mathcal{M}_\Gamma$ have a totally degenerate genus function on the source curve. Furthermore, after contracting the infinite edges corresponding to the $s$ simple branch points and their preimages, every edge of $\Gamma_{\text{src}}$ is trivalent. Consequently we see that the profiles for the local Hurwitz numbers of $\Gamma_{\text{src}}$ are given by $(d)$ (total ramification) and a two-part ramification profile. Thus, all local Hurwitz numbers are 1 by Proposition 36 except those introduced by subdivision, which were discussed in Warning 37.

![Figure 11. A parametrized tropical curve of genus 1 on the left, and the corresponding admissible cover on the right. The infinite edges carry simple ramification $(2,1,\ldots,1)$. The expansion factors of 2 occur on trivalent vertices of the parametrized tropical curve.](image1)

![Figure 12. The chosen cell $\mathcal{M}_\Gamma$ in $\mathcal{M}_{0, 2+s}^{\text{trop}}$.](image2)
The cones mapping onto $\overline{M}_g$ via the branch map are precisely those cones $\mathcal{H}_\Theta$ where $\Theta = \Gamma_{\text{src}} \rightarrow \Gamma$. Here the $\Gamma_{\text{src}}$ precisely correspond to the monodromy graphs of $[11]$. To compute the degree of the branch map over this cell $M_g$, we need only compute the degrees of maps from the individual top dimensional cells lying over $M_g$, and add the resulting contributions.

Consider an admissible cover $[D \rightarrow C]$ in $\overline{\mathcal{M}}_{g,h,d}(\vec{\mu})$ lying over the stable nodal genus $0$ curve $[C]$ in $\overline{M}_{0,2+s}$. We denote by $\xi_i$ the deformation parameter of the $i$th node of $[C]$, and by $\tilde{\xi}_i$ the deformation parameter of the $i$th node of the base of the admissible cover.

With the above discussion in mind, fix a top dimensional cell $\mathcal{H}_\Theta$ in the tropical Hurwitz space. We need to understand the dilation factor that this map induces on integral structures. Recall that the coordinates on the cone $M_{\Gamma_{\text{src}}}$ are given by $\text{val}(\xi_i)$, the valuation of the deformation parameters. Recall from our deformation theory computations in Section 4.2.1

$$\text{br}^* (\xi_i) = \tilde{\xi}_i^{N_i},$$

where $N_i$ is the LCM of the ramification above the nodes of $D$ lying above the $i$th node of $C$. It follows that

$$\text{val}(\text{br}^* (\xi_i)) = N_i \cdot \text{val}(\xi_i).$$

However, as we discussed in Section 4.2.1 there are $M = \prod_{e \in \Gamma_{\text{tgt}}} M_i$ zero strata in the space $\overline{\mathcal{M}}_{g,h,d}(\vec{\mu})$ for each chosen nodal cover, where $M_i$ is the product of the ramification indices above the $i$th node, divided by their LCM. Clearly the total contribution $M_i N_i$ is the product of the ramification indices above a node. Ranging over all nodes, we see that the total degree is the product of the edge weights. Keeping in mind Warning 37, we see that for each combinatorial type, we recover the weight in the CJM formulae. Appropriately taking into account automorphisms, we recover the desired formula

$$h_{g \rightarrow 0,d}(\vec{\mu}) = \sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} \prod w(e).$$

\[\square\]

Remark 38. The induced map $\text{br}^\Sigma$ on skeleta is an isomorphism on each cone if we forget about the integral structure. The same is true for $\text{br}^{\text{trop}}$. The enumerative information relies crucially on the integral structure of these cones and the weights, which in turn, relies heavily on the deformation theory.

Remark 39. Although tropicalization is a relatively new to the study of Hurwitz numbers, the spirit of these results is quite classical – namely, using degeneration techniques to study (enumerative) geometry. In the above computations, we choose a suitable degeneration of the base curve to a rational curve with desirable properties. For instance, the caterpillar curve above (Figure 12) has the property that it allows us to easily compute local Hurwitz numbers. This strategy was actualized for double Hurwitz numbers in [11].

6.2. The general correspondence theorem for Hurwitz numbers. Totally analogously, we can reprove the general correspondence theorem for Hurwitz numbers from [7] at the level of moduli spaces using our newly developed techniques.
Recall from [7] that for a tropical admissible cover of combinatorial type $\Theta = [\theta : \Gamma_{src} \to \Gamma_{tgt}]$ as in Section 3.2.1, the multiplicity (depending only on the combinatorial type) is defined to be

$$\frac{1}{|\text{Aut}_0(\Theta)|} \cdot \prod_{v \in \Gamma_{tgt}} H(v) \cdot \prod_{e \in \Gamma_{src}} d_e(\theta),$$

where the second product goes over all interior edges $e$ of $\Gamma_{src}$ and $d_e(\theta)$ denotes their expansion factors. We decide to mark preimages of branch points, resulting in a simplification in our expression of the local Hurwitz numbers compared to [7].

For a fixed trivalent target tropical curve of genus $h$ with totally degenerate genus function $\Gamma_{tgt}$, in [7] the tropical Hurwitz number $h_{g \to h,d}(\vec{\mu})$ is defined to be the weighted number of admissible covers of $\Gamma_{tgt}$, satisfying the prescribed genus and ramification conditions, counted with the multiplicity defined in (3). This number does not depend on the choice of $\Gamma_{tgt}$.

**Proof of Theorem 3** From Theorem 2 we know already that $h_{g \to h,d}(\vec{\mu})$, which equals the degree of the branch map, also equals the degree of the tropical branch map. All that remains to be seen is that the multiplicity of an admissible cover defined above equals the dilation factors times the weight of the cone of the corresponding combinatorial type. This follows analogously to the proof of the monodromy graph formula as in Section 6.1.2.

The monodromy graph formula is implied by Theorem 3 using the method of attaching infinite edges in the manner described in Section 6.1.

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