BACKWARD CONTROLLABILITY OF PULLBACK TRAJECTORY ATTRACTIONS WITH APPLICATIONS TO MULTI-VALUED JEFFREYS-OLDROYD EQUATIONS

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(Communicated by Irena Lasiecka)

Abstract. This paper analyzes the time-dependence and backward controllability of pullback attractors for the trajectory space generated by a non-autonomous evolution equation without uniqueness. A pullback trajectory attractor is called *backward controllable* if the norm of its union over the past is controlled by a continuous function, and *backward compact* if it is backward controllable and pre-compact in the past on the underlying space. We then establish two existence theorems for such a backward compact trajectory attractor, which leads to the existence of a pullback attractor with the backward compactness and backward boundedness in two original phase spaces respectively. An essential criterion is the existence of an increasing, compact and absorbing brochette. Applying to the non-autonomous Jeffreys-Oldroyd equations with a backward controllable force, we obtain a backward compact trajectory attractor, and also a pullback attractor with backward compactness in the negative-exponent Sobolev space and backward boundedness in the Lebesgue space.

1. Introduction. This paper is concerned with backward controllability of trajectory attractors for a non-autonomous evolution equation without uniqueness:

\[ u'(t) = S(t, u(t)), \quad t \geq \tau, u(\tau) = u_0, \] (1)

where \( S : \mathbb{R} \times E_S \rightarrow E, E_S \subset E \) and \( E \) is a reflexive Banach space.

We concern whether a pullback attractor (which is a family of time-indexed sets) can be controlled by a bounded set or a compact set, uniformly in the past. The notion of backward controllability may not be the one used in Control Theory.

Let us explain the idea of a pullback trajectory attractor introduced by Vorotnikov[22]. Take another larger space \( E_0 \) such that \( E \hookrightarrow E_0 \) and let

\[ \mathcal{T} = C(\mathbb{R}_+; E_0) \cap L^{\text{loc}}_\infty(\mathbb{R}_+; E) \] (2)

be the underlying space. Let a brochette \( \mathcal{H}^+ \) over \( \mathcal{T} \) be the trajectory space generated by some admissible weak solutions of Eq.(1) under the translation sense. Then

2000 Mathematics Subject Classification. Primary: 35B40, 35B41; Secondary: 37B55.

Key words and phrases. Trajectory attractor, pullback attractor, backward controllability, backward compactness, nonautonomous Jeffreys-Oldroyd equations.

Y. Li and R. Wang are supported by National Natural Science Foundation of China grant 11571283 and L. She is supported by the Natural Science Foundation of Guizhou Province:KY[2016]103.

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a set-valued mapping \( t \mapsto \mathcal{P}_t \subset T \) is said to be a **pullback trajectory attractor** if it is \( T \)-compact, invariant under the translation operator and it pullback attracts all trajectories in \( \mathcal{H}^+ \) under the topology of \( C(\mathbb{R}_+; E_0) \). Some existence conclusions of trajectory attractors were established in the nonautonomous case, see [22, 24], in the autonomous case, see [2, 3, 20, 21].

The main distinction between nonautonomous and autonomous attractor is that one is time-indexed and another is time-independent.

In this paper, we just reveal time-dependence of a pullback trajectory attractor. In particular, we wonder how controllability of a pullback attractor in the past?

In Sec.2, we establish a theoretical framework on **backward compact trajectory attractors** (BCTA) for the trajectory space \( \mathcal{H}^+ \) generated from Eq.(1). We define a BCTA by a minimal trajectory attractor with the backward controllability in \( L^\infty_{loc}(\mathbb{R}_+; E) \) and backward compactness in \( C(\mathbb{R}_+; E_0) \) (see Defs.2.1, 2.5). These backward properties describe that the trajectory attractor is relatively smaller and uniformly controllable in the past.

We then establish two theorems (see Theorems 2.10, 2.11) to show the existence of a BCTA if an absorbing brochette is increasingly \( T \)-compact or backward \( T \)-compact respectively. Moreover, the union of a BCTA over the past is \( T \)-precompact and attracts all trajectories at the past time.

In Sec.3, we discuss the backward topological property of a pullback attractor in the original spaces \( E \) and \( E_0 \). The concept of a pullback attractor we use here depends on the trajectory space and the translation operator, which is slightly different from the stand one defined by an evolution process (see [1, 4, 8, 9, 12, 13, 25]), or a multi-valued process (see [5, 6, 10, 11, 26, 30]), but they have some connections.

A minimal pullback attractor over \( E \) (thus over \( E_0 \)) is called a **backward-compact attractor** (BCA) if it is backward-compactness in \( E_0 \) and backward-bounded in \( E \). The existence theorem of a BCA is established from the existence of a BCTA (see Theorem 3.2). This concept give some further topological properties such as backward controllability, backward compactness, backward boundedness and so on. They lead to the existence of a backward compact brochette, which pullback attracts all trajectories over the past.

In Sec.4, we apply the above theoretical results on the following nonautonomous 3D Jeffreys-Oldroyd equations:

\[
\frac{\partial u}{\partial t} + \sum_{i=1}^3 u_i \frac{\partial u}{\partial x_i} + \text{grad } p = \text{Div } \sigma + f(x, t), \quad t \geq \tau, \tag{3}
\]

\[
\sigma + \lambda_1 \left( \frac{\partial \sigma}{\partial t} + \sum_{i=1}^3 u_i \frac{\partial \sigma}{\partial x_i} \right) = 2\eta E + 2\eta \lambda_2 \left( \frac{\partial E}{\partial t} + \sum_{i=1}^3 u_i \frac{\partial E}{\partial x_i} \right), \tag{4}
\]

where the parameters and boundary conditions will be special in Sec. 4.

This model is often used in physics, material science, geology and other subjects. For instance, the Navier-Stokes type equation (3) is used properly to describe the motion of the incompressible medium with constant density in the form of Cauchy. Eq.(4) is the Jeffreys constitutive law with corresponding linearizations, which makes the problem has no such regulation. The coupled equations (3)-(4) describe materials like solutions of polymers, bitumens, concrete and the earth’s crust.
Although the existence of weak solutions for Jeffreys-Oldroyd equations had been proved by using an approximating topological method (see [23, 24, 31, 32]), the weak solution may not be unique. This phenomenon is similar to the 3D Navier-Stokes equation, the uniqueness of weak solutions remains open (cf. [11, 19, 21, 22]). However, those weak solutions will generate a trajectory space, which consists of possibly multi-valued trajectories.

Zvyagin and Kondratyev [31] proved that the problem (3)-(4) possesses a minimal trajectory attractor and a pullback attractor for a special trajectory space if the body force \( f \) is assumed to have a tempered condition.

In the present paper, we need to further assume that the force \( f \) satisfies a backward tempered condition, which is equivalent to the backward translation boundness. In this case, by using the approximating topological method, we can prove the existence of weak solutions in the admissible sense. These admissible weak solutions construct the required trajectory space, which may be slightly different from that given in [31].

For the above trajectory space, we then prove the existence of an increasing, compact and absorbing brochette if we take \( E \) (resp. \( E_0 \)) by the Lebesgue space (resp. the negative-exponent Sobolev space). Then the abstract result can be applied to prove the existence of a BCTA in the underlying space and also obtain a backward compact pullback attractor with backward compactness in the negative-exponent Sobolev space and backward boundedness in the Lebesgue space.

### 2. Theoretical results on backward compact trajectory attractors.

#### 2.1. Backward compact brochettes in the underlying space.

Throughout this paper, we take two phase spaces \( E \) and \( E_0 \) such that they are Banach spaces, \( E \hookrightarrow E_0 \) and \( E \) is reflexive. The underlying space is given by \( \mathcal{T} = C(\mathbb{R}_+; E_0) \cap L^\infty(\mathbb{R}_+; E) \), where \( C(\mathbb{R}_+; E_0) \) is the Fréchet space defined by the semi-norm

\[
||u||_{C(\mathbb{R}_+; E_0)} = \sum_{i=1}^{\infty} 2^{-i} \frac{\|u\|_{C([0,i]; E_0)}}{1 + \|u\|_{C([0,i]; E_0)}},
\]

(5)

Since \( E \) is reflexive, it follows from the Lions-Magenes lemma given in [33] that \( \mathcal{T} \subset C_u(\mathbb{R}_+; E) \), which implies that \( u(s) \in E \) for all \( s \geq 0 \) if \( u \in \mathcal{T} \).

A brochette \( \mathcal{A} \) over a metric space \( X \) means a time-dependent set-valued mapping \( t \mapsto \mathcal{A}_t \) from \( \mathbb{R} \) to the collection of all subsets in \( X \). The inclusion and intersection are defined by \( \mathcal{A}_t \subset \mathcal{B}_t \) and \( (\mathcal{A} \cap \mathcal{B})_t = \mathcal{A}_t \cap \mathcal{B}_t \) for each \( t \in \mathbb{R} \). A brochette \( \mathcal{A} \) is called **increasing** (resp. **decreasing**) if \( \mathcal{A}_s \subset \mathcal{A}_t \) (resp. \( \mathcal{A}_s \supset \mathcal{A}_t \)) for all \( s \leq t \).

We are concerned with some special brochettes over \( \mathcal{T} \) as follows.

**Definition 2.1.** A brochette \( \mathcal{P} = \{\mathcal{P}_t\}_{t \in \mathbb{R}} \) over \( \mathcal{T} \) is said to be

(i) **controllable** if there is a function \( \phi_1 : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \phi_1(t, \cdot) \) is continuous and \( ||u(s)||_E \leq \phi_1(t, s) \) for each \( t \in \mathbb{R} \), \( s \geq 0 \) and \( u \in \mathcal{P}_t \),

(ii) **backward controllable** if there exists a function \( \phi_2 : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \phi_2(t, \cdot) \) is continuous and \( ||u(s)||_E \leq \phi_2(t, s) \) for each \( t \in \mathbb{R} \), \( s \geq 0 \) and \( u \in \cup_{r \leq t} \mathcal{P}_r \),

(iii) **\( \mathcal{T} \)-compact** if it is controllable and \( \mathcal{P}_t \) is compact in \( C(\mathbb{R}_+; E_0) \) for each \( t \in \mathbb{R} \),

(iv) **backward \( \mathcal{T} \)-compact** if it is backward controllable, \( \mathcal{T} \)-compact and \( \cup_{r \leq t} \mathcal{P}_r \) is pre-compact in \( C(\mathbb{R}_+; E_0) \) for each \( t \in \mathbb{R} \).

We give some easy-to-verify criteria for the backward controllability as follows.
Lemma 2.2. (a) A brochette $\mathcal{P}$ is backward controllable if there exists a function $\phi_3 : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\sup_{r \leq t} \phi_3(t, \cdot)$ is finite and continuous for fixed $t \in \mathbb{R}$, and $\|u(s)\|_E \leq \phi_3(t, s)$ for $s \geq 0$ and $u \in \mathcal{P}_t$.

(b) $\mathcal{P}$ is backward controllable if there exists a function $\phi_4 : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\phi_4(t, \cdot)$ is continuous, $\phi_4(\cdot, s)$ is non-decreasing, and $\|u(s)\|_E \leq \phi_4(t, s)$ for each $s \geq 0$ and $u \in \mathcal{P}_t$ with $t \in \mathbb{R}$.

(c) $\mathcal{P}$ is backward controllable if and only if there exist a function $\phi_5 : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\phi_5(t, \cdot)$ is continuous and $\|u\|_{L_\infty(s,s+1;E)} \leq \phi_5(t, s)$ for all $s \geq 0$, $t \in \mathbb{R}$ and $u \in \cup_{t \leq t} \mathcal{P}_r$.

(d) If $\mathcal{P}$ is backward controllable, then, for each $t \in \mathbb{R}$, $\cup_{r \leq t} \mathcal{P}_r$ is bounded in $L_\infty(\mathbb{R}_+; E)$, which means the backward-boundedness.

Proof. (a) We take the controlled function by $\phi_0(t, s) := \sup_{r \leq t} \phi_3(r, s)$ for $s \geq 0$ and $t \in \mathbb{R}$. Then $\phi_0(t, \cdot)$ is a continuous function for fixed $t \in \mathbb{R}$. On the other hand, for each $u \in \cup_{r \leq t} \mathcal{P}_r$, we have $u \in \mathcal{P}_r$ for some $r \leq t$, and so $\|u(s)\|_E \leq \phi_3(t, s)$ for all $s \geq 0$, $t \in \mathbb{R}$ and $u \in \cup_{r \leq t} \mathcal{P}_r$.

(b) For each $u \in \cup_{r \leq t} \mathcal{P}_r$ with $t \in \mathbb{R}$, we have $u \in \mathcal{P}_r$ for some $r \leq t$. Since $\phi_4(\cdot, s)$ is non-decreasing for fixed $s \geq 0$, we have $\|u(s)\|_E \leq \phi_4(r, s) \leq \phi_4(t, s)$, which means $\phi_0$ is a backward controlled function.

(c) If $\mathcal{P}$ is backward controllable with the controlled function $\phi_3$, then the continuity of $\phi_2$ implies that $\phi_5(t, s) := \sup_{r \in [s,s+1]} \phi_2(t, r)$ is finite and $\|u\|_{L_\infty(s,s+1;E)} \leq \sup_{r \in [s,s+1]} \phi_2(t, r) = \phi_5(t, s)$ for all $s \geq 0$, $t \in \mathbb{R}$ and $u \in \cup_{r \leq t} \mathcal{P}_r$. It suffices to prove the continuity of $\phi_5(t, \cdot)$. In fact, we can show a general conclusion: if $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, then $\tilde{\phi}(s) = \sup_{r \in [s,s+1]} \phi(r)$ is also a continuous function on $\mathbb{R}_+$.

Indeed, by the continuity of $\phi$, for each $\varepsilon > 0$, there is a $\delta > 0$ such that, for all $s \in [s_0, s_0 + \delta]$, we have $\sup_{r \in [s_0+1,s+1]} \phi(r) - \phi(s_0 + 1) < \varepsilon$, which implies that

$$\tilde{\phi}(s) - \tilde{\phi}(s_0) = \sup_{r \in [s_0+1,s+1]} \phi(r) - \sup_{r \in [s_0,s_0+1]} \phi(r) \leq \sup_{r \in [s_0+1,s+1]} \phi(r) - \sup_{r \in [s_0,s_0+1]} \phi(r) \leq \sup_{r \in [s_0+1,s+1]} 0 < \varepsilon.$$ 

By the continuity of $\phi$, again, for each $s \in [s_0, s_0 + \delta]$, we know $\phi(s) \geq \phi(s_0) - \varepsilon$ and $\sup_{r \in [s_0,s]} \phi(r) - \phi(s_0) < \varepsilon$, which implies that

$$\tilde{\phi}(s_0) - \tilde{\phi}(s) = \sup_{r \in [s_0+1,s+1]} \phi(r) - \sup_{r \in [s_0,s_0+1]} \phi(r) \leq \sup_{r \in [s_0+1,s+1]} \phi(r) - \sup_{r \in [s_0,s_0+1]} \phi(r) \leq \sup_{r \in [s_0+1,s+1]} 0 < \varepsilon.$$ 

Hence, $\lim_{s \to s_0^+} \tilde{\phi}(s) = \tilde{\phi}(s_0)$, which proves the right continuity. It is similar to prove the left continuity.

(d) By the continuity of $\phi_2(t, \cdot)$, we have $\|u\|_{L_\infty(0,T;E)} \leq \sup_{s \in [0,T]} \phi_2(t, s) < \infty$ for each $u \in \cup_{r \leq t} \mathcal{P}_r$ and $T > 0$, which obviously implies $\mathcal{P}$ is backward-bounded in $L_\infty((0, \infty); E)$.

In general, backward controllability implies controllability, while backward $\mathcal{T}$-compactness implies $\mathcal{T}$-compactness. For the converse assertion, we have
Lemma 2.3. (I) Let $\mathcal{P}$ be an increasing brochette over $\mathcal{T}$, then it is backward $\mathcal{T}$-compact (resp. backward controllable) if and only if it is $\mathcal{T}$-compact (resp. controllable).

(II) Conversely, if a brochette $\mathcal{P}$ is $\mathcal{T}$-compact (resp. controllable), then there exists another increasing brochette $\mathcal{Q}$ such that $\mathcal{Q} \subset \mathcal{P}$ and so $\mathcal{Q}$ is backward $\mathcal{T}$-compact (resp. backward controllable).

(III) The intersection of a family of backward $\mathcal{T}$-compact brochettes is backward $\mathcal{T}$-compact.

Proof. (I) Since $\mathcal{P}$ is increasing, it is easy to prove $\cup_{s \leq t} \mathcal{P}_s = \mathcal{P}_t$ for each $t \in \mathbb{R}$. Let $\mathcal{P}$ be controllable with the controlled function $\phi_1$, then $\phi_1$ controls all elements in $\mathcal{P}_t = \cup_{s \leq t} \mathcal{P}_s$ and so $\mathcal{P}$ is backward controllable. Other assertions also follow from this equality of sets immediately.

(II) If $\mathcal{P}$ is $\mathcal{T}$-compact, we define another brochette $\mathcal{Q}$ by

$$\mathcal{Q}_t := \bigcap_{r \geq t} \mathcal{P}_r, \quad \forall t \in \mathbb{R},$$

with the closure in $C(\mathbb{R}^+; E_0)$. (6)

Obviously, $\mathcal{Q}$ is increasing such that $\mathcal{Q}_t \subset \overline{\mathcal{P}_t} = \mathcal{P}_t$. Since $\mathcal{P}$ is controllable and $\cup_{s \leq t} \mathcal{Q}_s = \mathcal{Q}_t \subset \mathcal{P}_t$, it follows that $\mathcal{Q}$ is backward controllable. Since $\mathcal{Q}_t$ is closed and $\mathcal{P}_t$ is compact, $\mathcal{Q}$ is $\mathcal{T}$-compact. Therefore, the backward $\mathcal{T}$-compactness of $\mathcal{Q}$ follows from (I) and the increasing property of $\mathcal{Q}$.

(III) Let $\mathcal{P}^i (i \in I)$ be a family of backward $\mathcal{T}$-compact brochettes. It is easy to show that $\cap_{i \in I} \mathcal{P}^i$ is backward controllable. As an intersection of compact sets, $(\cap_{i \in I} \mathcal{P}^i)_t = \cap_{i \in I} \mathcal{P}^i_t$ is compact in $C(\mathbb{R}^+; E_0)$ for each $t \in \mathbb{R}$. Finally, since

$$\bigcup_{r \leq t} \bigcap_{i \in I} \mathcal{P}^i_r = \bigcup_{r \leq t} \bigcap_{i \in I} \mathcal{P}^i_r \subset \bigcap_{i \in I} \mathcal{P}^i_t$$

and the last set is pre-compact, it follows that the first set is pre-compact in $C(\mathbb{R}^+; E_0)$. Therefore, $\cap_{i \in I} \mathcal{P}^i$ is backward $\mathcal{T}$-compact.

We now consider a translation operator $T(h)$ on $\mathcal{T}$ if $h \geq 0$ or on $C(\mathbb{R}, E_0) \cap L^\infty_{\text{loc}}(\mathbb{R}, E)$ if $h \in \mathbb{R}$:

$$(T(h)u)(s) = u(s + h), \quad \text{for } s \geq 0 \text{ or } s \in \mathbb{R}.$$ 

Given a brochette $\mathcal{P}$ over $\mathcal{T}$, the translation brochette $T(h)\mathcal{P}$ is defined by

$$(T(h)\mathcal{P})_t = T(h)\mathcal{P}_{t-h} \quad \text{for all } t \in \mathbb{R} \text{ and } h \geq 0.$$ 

Lemma 2.4. Let $\mathcal{P}$ be backward $\mathcal{T}$-compact (resp. $\mathcal{T}$-compact, controllable or backward controllable), then $T(h)\mathcal{P}$ has the corresponding property for each $h \geq 0$.

Proof. If $\mathcal{P}$ is backward controllable, then there exists a function $\phi : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t, \cdot)$ is continuous and $\|u(s)\|_E \leq \phi(t, s)$ for each $s \geq 0$ and $u \in \cup_{r \leq t} \mathcal{P}_r$ with $t \in \mathbb{R}$. Let $h \geq 0$ and define $\phi_h(t, s) = \phi(t-h, s+h)$ for $t \in \mathbb{R}$ and $s \geq 0$. Then $\phi_h(t, \cdot)$ is continuous since so is $\phi$. On the other hand, if $u \in \cup_{r \leq t} (T(h)\mathcal{P})_r$, then there exist $r_0 \leq t$ such that $u \in (T(h)\mathcal{P})_{r_0} = T(h)\mathcal{P}_{r_0-t-h}$. So there exist $v \in \mathcal{P}_{r_0-t-h} \subset \cup_{r \leq t} \mathcal{P}_r$ such that $u(s) = v(s+h)$ for all $s \geq 0$ and thus

$$\|u(s)\|_E = \|v(s+h)\|_E \leq \phi(t-h, s+h) = \phi_h(t, s), \quad \forall t \in \mathbb{R}, \quad s \geq 0,$$

which proves that $T(h)\mathcal{P}$ is backward controllable with the controlled function $\phi_h$. It is similar (but relatively easy) to prove that $T(h)\mathcal{P}$ is controllable.

On the other hand, by using the semi-norm given by (5), it is easy to prove that $T(h)$ is a continuous operator on $C(\mathbb{R}^+; E_0)$, which implies that $T(h)C$ is
compact in \( C(\mathbb{R}_+; E_0) \) if \( C \) is a compact set in \( C(\mathbb{R}_+; E_0) \). From this fact, it follows that \( (T(h)P)_t := T(h)P_{t-h} \) is compact in \( C(\mathbb{R}_+; E_0) \) if \( P \) is \( T \)-compact, and \( \cup_{r \leq t}(T(h)P)_r = T(h)\cup_{r \leq t-h}P_r \) is pre-compact in \( C(\mathbb{R}_+; E_0) \) if \( P \) is backward \( T \)-compact.

2.2. Backward compact trajectory attractors. We need to introduce the concept of trajectory space induced by the abstract non-autonomous equation (1).

For each \( u_0 \in E \) and \( \tau \in \mathbb{R} \), we assume that Eq.(1) has at least one weak solution \( u := u(t) \ (t \geq \tau) \) such that \( v \in T = C(\mathbb{R}_+; E_0) \cap L^\infty_{\text{loc}}(\mathbb{R}_+; E) \), where \( v = T(\tau)u \), that is, \( v(s) = u(s + \tau) \ (s \geq 0) \). We denote the trajectory by

\[
\mathcal{H}^+_\tau(u_0) = \{ v \in T : u(t) = v(t - \tau) \ (t \geq \tau) \text{ is a solution of Eq.(1) with } v(0) = u(\tau) = u_0 \},
\]

which defines a brochette \( \mathcal{H}^+ \) over \( T \), called a trajectory space. Note that \( \mathcal{H}^+_\tau(u_0) \) may not be a single point since we have not assumed the uniqueness of solutions. In application, we may give other restricted conditions for the trajectory space, for example, we require the weak solution is admissible in some senses, see Sec.4.

We use \( \mathfrak{D} \) to denote a universe of some non-empty brochettes \( \mathcal{D} = \{ \mathcal{D}_\tau \}_{\tau \in \mathbb{R}} \) over \( E \) and then define a brochette \( \mathcal{H}(\mathcal{D}) = \{ \mathcal{H}_\tau(\mathcal{D}) \}_{\tau \in \mathbb{R}} \) over \( T \) by

\[
\mathcal{H}_\tau(\mathcal{D}) = \{ u \in \mathcal{H}_\tau^+(u_0) : u(0) = u_0 \in \mathcal{D}_\tau \}.
\]

Definition 2.5. A brochette \( \mathcal{P} = \{ \mathcal{P}_t \}_{t \in \mathbb{R}} \) over \( T \) is called a backward compact trajectory attractor (BCTA) for the trajectory space \( \mathcal{H}^+ \) of Eq.(1) if it is

(i) \textbf{backward} \( T \)-
compact, i.e. \( T(h)\mathcal{P} = \mathcal{P} \) for any \( h \geq 0 \);

(ii) \textbf{invariant}, i.e. \( \mathcal{P} \cap \mathcal{H}_\tau(\mathcal{D}) = 0 \), where the Hausdorff semi-distance is \( \text{dist}(A, B) = \sup_{a \in A} \text{inf}_{b \in B} d(a, b) \);

(iii) \textbf{pullback} attracting, i.e. for each brochette \( \mathcal{D} \in \mathfrak{D} \) and \( t \in \mathbb{R} \),

\[
\lim_{\tau \to -\infty} \text{dist}_{C(\mathbb{R}_+; E_0)}(T(t-\tau)\mathcal{H}_\tau(\mathcal{D}), \mathcal{P}_t) = 0,
\]

where the minimality indicates a BCTA must be unique if it exists. The attraction always means in the pullback sense, but we often omit the word ‘pullback’.

The concept of a BCTA is used to consider some further topological properties of the usual trajectory attractor (TA), where a TA means a \( T \)-compact, invariant and attracting brochette (cf. [22, 31]). A \( T \)-compact and attracting brochette \( \mathcal{P} \) is called a trajectory semi-attractor (TSA) if it is

(v) \textbf{positively invariant}, i.e. \( T(h)\mathcal{P} \subset \mathcal{P} \) for any \( h \geq 0 \).

In general, the intersection of two attracting brochettes may not be attracting. However, the intersection of two \( T \)-compact attracting brochettes must be attracting. The following lemma can be found in [22, Lemma 3.15-3.16].

Lemma 2.6. (a) Let \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) be two \( T \)-compact and attracting brochettes. Then \( \mathcal{P}_1 \cap \mathcal{P}_2 \) is an attracting brochette.

(b) If \( \mathcal{P}_i \) (\( i \in I \)) is a family of positively invariant (resp. invariant) brochettes, then \( \cap_{i \in I} \mathcal{P}_i \) is also positively invariant (resp. invariant).

(c) If \( \mathcal{P} \) is attracting (resp. invariant or positively invariant), then \( T(h)\mathcal{P} \) has the corresponding property for each \( h \geq 0 \).

Definition 2.7. A brochette \( \mathcal{P} \) over \( T \) is called \( \mathfrak{D} \)-absorbing for \( \mathcal{H}^+ \) if for each \( \mathcal{D} \in \mathfrak{D} \) and \( t \in \mathbb{R} \), there exist \( \tau_0 = \tau_0(\mathcal{D}, t) \leq t \) such that \( \inf_{r \geq t} \tau_0(\mathcal{D}, r) > -\infty \) and

\[
T(t-\tau)\mathcal{H}_\tau(\mathcal{D}) \subset \mathcal{P}_t, \quad \forall \tau \leq \tau_0.
\]
We show the existence of a backward Proof. We divide into 3 steps to finish the proof.

Suppose there exists an increasing, Theorem 2.10.

subsection, we establish the abstract existence result of a \(\text{BCTA}.\)

2.3. Existence theorem of backward compact trajectory attractors. In this subsection, we establish the abstract existence result of a \(\text{BCTA}.\)

**Lemma 2.8.** A brochette \(\mathcal{P}\) is \(\mathcal{D}\)-absorbing in the sense of Def.2.7 if and only if, for each \(\mathcal{D} \in \mathcal{D}\) and \(t \in \mathbb{R}\), there exist \(\tau_1 = \tau_1(\mathcal{D}, t) \leq t\) such that \(\tau_1(\mathcal{D}, \cdot)\) is non-decreasing and (8) holds true for all \(\tau \leq \tau_1\).

**Proof.** Let \(\mathcal{P}\) be an absorbing brochette and let \(\tau_0\) be given in Def.2.7. Setting \(\tau_1(\mathcal{D}, t) = \inf_{r \geq t} \tau_0(\mathcal{D}, r)\), we know that \(\tau_1(\mathcal{D}, t)\) is finite, \(\tau_1(\mathcal{D}, t) \leq \tau_0(\mathcal{D}, t) \leq t\) and \(\tau_1(\mathcal{D}, \cdot)\) is non-decreasing for each fixed \(\mathcal{D}\). Furthermore, (8) holds true for all \(\tau \leq \tau_1(\mathcal{D}, t)\) since \(\tau_1(\mathcal{D}, t) \leq \tau_0(\mathcal{D}, t)\).

Conversely, since \(\tau_1(\mathcal{D}, \cdot)\) is non-decreasing, it follows that \(\inf_{r \geq t} \tau_1(\mathcal{D}, r) = \tau_1(\mathcal{D}, t) \neq -\infty\).

The following useful lemma about a net of compact sets can be founded in [33, Lemma 4.2.6] or [22, Lemma 3.20].

**Lemma 2.9.** Let \(\{K_\alpha : \alpha \in \Lambda\}\) be a net of nonempty compact sets in a metric space \(X\) such that the intersection of arbitrary two elements from the net belongs to this net. Then the intersection \(K_0 = \cap_{\alpha \in \Lambda} K_\alpha\) is nonempty compact, and for each \(\varepsilon > 0\) there is \(\alpha_\varepsilon \in \Lambda\) such that \(\text{dist}_X(K_\alpha, K_0) < \varepsilon\).

**2.3. Existence theorem of backward compact trajectory attractors.** In this subsection, we establish the abstract existence result of a \(\text{BCTA}.\)

**Theorem 2.10.** Suppose there exists an increasing, \(\mathcal{T}\)-compact and \(\mathcal{D}\)-absorbing brochette, then there is a \(\text{BCTA}\) for the trajectory space \(\mathcal{H}^+\).

**Proof.** We divide into 3 steps to finish the proof.

**Step 1.** We show the existence of a backward \(\mathcal{T}\)-compact TSA.

Indeed, let \(\mathcal{P}\) be an increasing, \(\mathcal{T}\)-compact and \(\mathcal{D}\)-absorbing brochette. Then, by Lemma 2.8, for every \(\mathcal{D} \in \mathcal{D}\) and \(t \in \mathbb{R}\), there exist \(\tau_1 = \tau_1(\mathcal{D}, t) \leq t\) such that \(\tau_1(\mathcal{D}, \cdot)\) is non-decreasing and

\[
T(t - \tau)\mathcal{H}_+ (\mathcal{D}) \subset \mathcal{P}_t, \quad \text{for all } \tau \leq \tau_1(\mathcal{D}, t).
\]

(9)

We construct the needed brochette \(\mathcal{Q} = \{\mathcal{Q}_t\}_{t \in \mathbb{R}}\) by

\[
\mathcal{Q}_t = \bigcup_{\mathcal{D} \in \mathcal{D}} \bigcup_{\tau \leq \tau_1(\mathcal{D}, t)} T(t - \tau)\mathcal{H}_+ (\mathcal{D}) \quad \text{for each } t \in \mathbb{R},
\]

(10)

where the overline means the closure in \(C(\mathbb{R}_+; E_0)\).

We need to show that \(\mathcal{Q}\) is backward \(\mathcal{T}\)-compact. Indeed, by (9)-(10) and the closedness of \(\mathcal{P}\) in \(C(\mathbb{R}_+; E_0)\), we have

\[
\mathcal{Q}_t \subset \overline{\mathcal{P}_t} = \mathcal{P}_t \quad \text{for each } t \in \mathbb{R},
\]

which implies \(\mathcal{Q}_t\) is pre-compact and thus compact in \(C(\mathbb{R}_+; E_0)\) since \(\mathcal{P}_t\) is compact. Note that \(\mathcal{P}\) is assumed to be increasing, we have

\[
\bigcup_{s \leq t} \mathcal{Q}_s \subset \bigcup_{s \leq t} \mathcal{P}_s = \mathcal{P}_t \quad \text{for each } t \in \mathbb{R},
\]

(11)

which implies that \(\bigcup_{s \leq t} \mathcal{Q}_s\) is pre-compact in \(C(\mathbb{R}_+; E_0)\). On the other hand, since \(\mathcal{P}\) is controllable, we take a controlled function \(\phi\) such that it controls all elements
in $\mathcal{P}_t$. Hence, by (11), the function $\phi$ controls all elements in $\cup_{s \leq t} Q_s$. Therefore, $Q$ is backward controllable and so $Q$ is backward $T$-compact.

We then prove the positive invariance of $Q$. Since $\tau_1(\mathcal{D}, \cdot)$ is non-decreasing, it follows that

$$T(h) \bigcup_{\mathcal{D} \in \mathcal{D}} \bigcup_{\tau \leq \tau_1(\mathcal{D}, t-h)} T(t-h - \tau) \mathcal{H}_T(\mathcal{D}) = \bigcup_{\mathcal{D} \in \mathcal{D}} \bigcup_{\tau \leq \tau_1(\mathcal{D}, t-h)} T(t-\tau) \mathcal{H}_T(\mathcal{D}) \subset \bigcup_{\mathcal{D} \in \mathcal{D}} \bigcup_{\tau \leq \tau_1(\mathcal{D}, t)} T(t-\tau) \mathcal{H}_T(\mathcal{D}).$$

(12)

It is easy to prove that the shift operator $T(h)$ is continuous on $C(\mathbb{R}_+; E_0)$. Then, by taking the closure of two sides in the inclusion (12) under the topology of $C(\mathbb{R}_+; E_0)$, and by (10), we have $T(h) Q_{t-h} \subset Q_t$, i.e. $(T(h) Q)_t \subset Q_t$ and so $T(h) Q \subset Q$. Therefore, $Q$ is positively invariant.

By (10), it is obvious that $T(t-\tau) \mathcal{H}_T(\mathcal{D}) \subset Q_t$ for all $\mathcal{D} \in \mathcal{D}$ and $\tau \leq \tau_1(\mathcal{D}, t)$, which immediately implies that $Q$ is an attracting brochette. So far, we have proved that $Q$ is indeed a backward $T$-compact TSA.

**Step 2.** We show that, if there is a backward $T$-compact TSA $Q$, then there exist $\mathfrak{A} \subset Q$ such that $\mathfrak{A}$ is just the minimal backward $T$-compact TSA.

Indeed, let $\{K^\alpha_t : \alpha \in \Lambda\}$ be the family of all backward $T$-compact TSAs and set $\mathfrak{A} = \cap_{\alpha \in \Lambda} K^\alpha_t$. By Lemma 2.3(III), $\mathfrak{A}$ is backward $T$-compact. By Lemma 2.6(b), $\mathfrak{A}$ is positively invariant. It suffices to prove the attraction of $\mathfrak{A}$.

Fix $t \in \mathbb{R}$ and $\mathcal{D} \in \mathcal{D}$. By Lemma 2.6(a), the intersection of any two sets from the net $\{K^\alpha_t : \alpha \in \Lambda\}$ belongs to this net. By Lemma 2.9, we see that $\mathfrak{A}_t = \cap_{\alpha \in \Lambda} K^\alpha_t$ is nonempty compact in $C(\mathbb{R}_+; E_0)$ and, for each $\varepsilon > 0$, there is a $\alpha_\varepsilon \in \Lambda$ such that

$$\text{dist}_{C(\mathbb{R}_+; E_0)}(K^\alpha_t, \mathfrak{A}_t) < \frac{\varepsilon}{2}. \quad (13)$$

By the attraction of $K^\alpha_t$, there is a $\tau_\varepsilon$ such that, for all $\tau \leq \tau_\varepsilon$,

$$\text{dist}_{C(\mathbb{R}_+; E_0)}(T(t-\tau) \mathcal{H}_T(\mathcal{D}), K^\alpha_t) < \frac{\varepsilon}{2}. \quad (14)$$

Both (13) and (14) imply that, for all $\tau \leq \tau_\varepsilon$,

$$\text{dist}_{C(\mathbb{R}_+; E_0)}(T(t-\tau) \mathcal{H}_T(\mathcal{D}), \mathfrak{A}_t) < \varepsilon,$$

which yields the attraction of $\mathfrak{A}$. So $\mathfrak{A}$ is indeed a backward $T$-compact TSA. Obviously, it is the minimal one and $\mathfrak{A} \subset Q$.

**Step 3.** We show that the brochette $\mathfrak{A}$ given in step 2 is actually a BCTA.

In fact, by step 2, $\mathfrak{A}$ is backward $T$-compact, attracting and positively invariant.

We then prove the negatively invariance of $\mathfrak{A}$. Indeed, by Lemma 2.4, for each $h \geq 0$, $T(h) \mathfrak{A}$ is backward $T$-compact. By Lemma 2.6(c), $T(h) \mathfrak{A}$ is attracting and positively invariant (i.e. $T(h) \mathfrak{A} \subset \mathfrak{A}$). So $T(h) \mathfrak{A}$ is yet a backward $T$-compact TSA. By step 2, $\mathfrak{A}$ is the minimal one in such brochettes and so $\mathfrak{A} \subset T(h) \mathfrak{A}$. Therefore $\mathfrak{A} = T(h) \mathfrak{A}$ for each $h \geq 0$, which proves the invariance.

Now, let $K$ be another backward $T$-compact, invariant and attracting brochette, then $K$ is certainly a backward $T$-compact TSA. By the minimality in step 2, $\mathfrak{A} \subset K$, which proves the minimality of $\mathfrak{A}$. So far, we have prove $\mathfrak{A}$ is indeed a BCTA. $\Box$

By the proof of Theorem 2.10, we have some useful conclusions as follows.

**Proposition 1.** (i) If there is a brochette $Q$ such that $Q$ is a backward $T$-compact TSA, then there exists a BCTA $\mathfrak{A}$ such that $\mathfrak{A} \subset Q$. 


(ii) Conversely, a BCTA must be the minimal backward $\mathcal{T}$-compact TSA.

Proof. The assertion (i) follows immediately from step 2 and 3 in the proof of Theorem 2.10. We then use (i) to prove (ii). Let $\mathfrak{A}$ be a BCTA and $Q$ a backward $\mathcal{T}$-compact TSA. By Def.2.5, a BCTA is the minimal and thus unique, it follows from (i) that $\mathfrak{A} \subset Q$, which prove the minimality.

Note that, by Lemma 2.3, backward $\mathcal{T}$-compactness is weaker than the increasing $\mathcal{T}$-compactness. However, we can replace the assumption in Theorem 2.10 by this weaker condition. Then we obtain the following stronger result.

**Theorem 2.11.** Suppose there exists a backward $\mathcal{T}$-compact and $\mathcal{D}$-absorbing brochette for $\mathcal{H}^+$, then the abstract equation (1) has a BCTA.

Proof. We need only to reverse step 1 in the proof of Theorem 2.10. Let now $\mathcal{P}$ be a backward $\mathcal{T}$-compact and $\mathcal{D}$-absorbing brochette and let $\mathcal{D} \in \mathcal{D}$, $t \in \mathbb{R}$ and $\tau_1 = \tau_1(\mathcal{D}, t)$ satisfying (9). In this case, we define

$$K_t = \bigcup_{D \in \mathcal{D}} \bigcup_{r \leq t} \bigcup_{\tau \leq \tau_1(D, r)} T(r - \tau)\mathcal{H}_r(D)$$

for each $t \in \mathbb{R}$ \hspace{1cm} (15)

and then prove $K$ is a backward $\mathcal{T}$-compact TSA. Indeed, by the absorption of $\mathcal{P}$, we have $K_t \subset \bigcup_{r \leq t} \overline{P_r}$. By (15), it is easy to see that $K$ is an increasing brochette. Hence,

$$\bigcup_{r \leq t} K_r = K_t \subset \bigcup_{r \leq t} P_r$$

for each $t \in \mathbb{R}$.

Then $K$ is backward controllable since so is $\mathcal{P}$, and both $K$ and $\bigcup_{r \leq t} K_r$ is compact in $C(\mathbb{R}_+; E_0)$ since so is $\bigcup_{r \leq t} \overline{P_r}$. Therefore, $K$ is backward $\mathcal{T}$-compact.

We then prove the positive invariance of $K$. Since $\tau_1(\mathcal{D}, \cdot)$ is non-decreasing, it follows that

$$T(h) \bigcup_{D \in \mathcal{D}} \bigcup_{r \leq t-h} \bigcup_{\tau \leq \tau_1(D, r)} T(r - \tau)\mathcal{H}_r(D) = \bigcup_{D \in \mathcal{D}} \bigcup_{r \leq t-h} \bigcup_{\tau \leq \tau_1(D, r)} T(r + h - \tau)\mathcal{H}_r(D) = \bigcup_{D \in \mathcal{D}} \bigcup_{s \leq t} \bigcup_{\tau \leq \tau_1(D, s-h)} T(s - \tau)\mathcal{H}_r(D) \subset \bigcup_{D \in \mathcal{D}} \bigcup_{s \leq t} \bigcup_{\tau \leq \tau_1(D, s)} T(s - \tau)\mathcal{H}_r(D).$$

Taking the closure on the two-side of the above inclusion, by the continuity of the shift operator $T(h)$ on $C(\mathbb{R}_+; E_0)$, we have $T(h)K_{t-h} \subset K_t$ and so $(T(h)K)_{t} \subset K_t$. Therefore, $K$ is positively invariant.

Obviously, $K \supset Q$, where $Q$ is defined by (10). Since $Q$ is an attracting brochette, the larger $K$ is still an attracting brochette. Therefore, $K$ is indeed a backward $\mathcal{T}$-compact TSA.

The following result indicates the existence of a $\mathcal{T}$-compact brochette which pullback attracts all trajectories over the past.

**Proposition 2.** Under the same assumptions as given in Theorems 2.10 or 2.11, there is a $\mathcal{T}$-precompact brochette $K$ such that $K$ pullback attracts all trajectories in the past, that is, for each $t \in \mathbb{R}$ and $\mathcal{D} \in \mathcal{D}$,

$$\lim_{\tau \to -\infty} \text{dist}_{C(\mathbb{R}_+; E_0)}(T(s - \tau)\mathcal{H}_r(D), K_t) = 0, \forall s \leq t.$$ \hspace{1cm} (17)
Proof. By Theorems 2.10 or 2.11, there is a BCTA denoted by $\mathfrak{A}$. Let $\mathcal{K}_t = \bigcup_{s \leq t} \mathfrak{A}_s$ for $t \in \mathbb{R}$, then $\mathcal{K}$ is $T$-precompact. For each $D \in \mathcal{D}$ and $s \leq t$, we have

$$\text{dist}_{C(\mathbb{R}^+;\mathcal{E}_0)}(T(s - \tau)\mathcal{H}_r(D), \mathcal{K}_t) \leq \text{dist}_{C(\mathbb{R}^+;\mathcal{E}_0)}(T(s - \tau)\mathcal{H}_r(D), \mathfrak{A}_s) \to 0$$

as $\tau \to -\infty$. So $\mathcal{K}$ pullback attracts all trajectories in the past.\hfill\Box

Remark 1. Although we have established an increasing attracting brochette in Theorem 2.10 or an increasing TSA in Theorem 2.11, this is not enough to ensure that the attractor is increasing. We give a simple example to illustrate this fact.

Let an evolution process $S$ on $\mathbb{R}$ be defined by

$$S(t, t - \tau)x_0 = e^{-\tau}x_0 + \int_{-\tau}^{0} e^{\tau}f(r + t)dr, \quad \forall t \in \mathbb{R}, \quad \tau \geq 0, x_0 \in \mathbb{R}.$$ 

If $f(s) = e^s$ for $s \in \mathbb{R}$, then it is easy to prove that the process has a pullback attractor $\mathcal{A}_t = \{a_t\}$ for each $t \in \mathbb{R}$, where $a_t = e^t/2$. Hence $\bigcup_{s \leq t} \mathcal{A}(s) = \bigcup_{s \leq t} \{e^s/2\} = (0, e^t/2]$, which is bounded and thus pre-compact in $\mathbb{R}$. So $\mathcal{A}$ is a backward compact pullback attractor. It is obvious that $\mathcal{A}$ is not an increasing brochette. However, the family of sets $[0, e^t/2]$ is an increasing, compact and attracting brochette.

3. Existence of backward-compact pullback attractors. After all preparations are done, we now go back to study the existence of a pullback attractor over $E$ in the backward compact sense.

In general, we say a brochette $\mathcal{A}$ over a metric space $X$ is backward-compact if both $\mathcal{A}_t$ and $\bigcup_{s \leq t} \mathcal{A}_s$ are compact and backward-bounded if $\bigcup_{s \leq t} \mathcal{A}_s$ is bounded in $X$ for each $t \in \mathbb{R}$.

Definition 3.1. A brochette $\mathcal{A}$ over $E$ is called a backward-compact attractor (BCA) for $\mathcal{H}^+$ if $\mathcal{A}$ is (i) backward-compact in $E_0$ and backward-bounded in $E$,

(ii) pullback attracting in $E_0$, that is, for all $D \in \mathcal{D}$ and $t \in \mathbb{R}$,

$$\lim_{\tau \to -\infty} \sup_{u \in \mathcal{H}_r(D)} \inf_{v \in \mathcal{A}_t} \|u(t - \tau) - v\|_{E_0} = 0,$$

(iii) minimal, that is, if $\mathcal{B}$ satisfies (i) and (ii), then $\mathcal{A} \subset \mathcal{B}$.

We now generalize the result of [22, Theorem 3.14] in the backward compact sense.

Theorem 3.2. If Eq.(1) has a BCTA denoted by $\mathfrak{A}$, then there exists a backward-compact attractor $\mathcal{A}$ over $E$ constructed by $\mathcal{A} = \mathfrak{A}(0)$.

Proof. We will verify $\mathcal{A}$ is indeed a BCA, where

$$\mathcal{A} = \mathfrak{A}(0) = \{\mathfrak{A}_t(0)\}_{t \in \mathbb{R}}, \quad \text{and} \quad \mathfrak{A}_t(0) = \{u(0) \in E : u \in \mathfrak{A}_t\}. \quad (18)$$

Firstly, $\mathcal{A}$ is backward-bounded in $E$. Indeed, since $\mathfrak{A}$ is backward controllable in $\mathcal{T}$, there is a non-negative function $\phi(\cdot, \cdot)$ such that $\|u(s)\|_E \leq \phi(t, s)$ for all $s \geq 0$, $u \in \bigcup_{t \leq s} \mathfrak{A}_t$, with $t \in \mathbb{R}$. In particular, we take $s = 0$ to obtain that $\|x\|_E \leq \phi(t, 0)$ for each $x \in \bigcup_{t \leq s} \mathfrak{A}_s = \bigcup_{r \leq t} \mathfrak{A}_r(0)$ with $t \in \mathbb{R}$, which means $\bigcup_{s \leq t} \mathfrak{A}_s$ is bounded in $E$ for each $t \in \mathbb{R}$, and thus $\mathcal{A}$ is backward-bounded in $E$.

Secondly, $\mathcal{A}$ is backward-compact in $E_0$. Indeed, let $t \in \mathbb{R}$ be fixed and take a sequence $\{x_n\}$ from $\bigcup_{r \leq t} \mathfrak{A}_r$, then, by (18), $x_n = u_n(0)$ for a sequence $u_n \in \bigcup_{r \leq t} \mathfrak{A}_r$. Since $\mathfrak{A}$ is backward $\mathcal{T}$-compact, it follows that $u_n$ has a convergent subsequence.
(not relabeled) in \(C(\mathbb{R}^+; E_0)\). Let \(u\) be the limit, then by the semi-norm (5), we have

\[
\lim_{n \to \infty} \|u_n - u\|_{C([0,1]; E_0)} = \lim_{n \to \infty} \sup_{s \in [0,1]} \|u_n(s) - u(s)\|_{E_0} = 0,
\]

which means that \(u_n(s) \to u(s)\) uniformly in \(s \in [0,1]\). In particular, \(x_n = u_n(0) \to u(0)\) in \(E_0\) and so \(\cup_{r\leq t} A_r\) is pre-compact in \(E_0\). It is similar to prove the compactness of \(A\) from the \(T\)-compactness of \(A\).

Thirdly, \(A\) is pullback attracting in \(E_0\). Indeed, by the attraction of \(A\) in \(C(\mathbb{R}_+; E_0)\), it is easy to prove the attraction of \(\mathfrak{A}(0)\) in \(E_0\).

Finally, it is relatively difficult to show the minimality. Let a brochette \(K\) (over \(E\)) be pullback attracting, backward-compact in \(E_0\) and backward-bounded in \(E\). We need to prove \(\mathfrak{A} \subset K\). For this end, we define a brochette \(K = \{K_t\}_{t \in \mathbb{R}}\) over \(T\) by

\[
K_t = \{u \in \mathfrak{A}_t : u(h) \in K_{t+h} \text{ for any } h \geq 0\}, \quad \forall t \in \mathbb{R}. \tag{19}
\]

Our next goal is to prove that \(K\) is a backward \(T\)-compact TSA. We claim that \(K\) is backward \(T\)-compact. Indeed, by (19), \(K \subset \mathfrak{A}\), but \(\mathfrak{A}\) is backward controllable, hence \(K\) is backward controllable still. Similarly, for each \(t \in \mathbb{R}\), both \(K_t\) and \(\cup_{r\leq t} K_r\) are pre-compact in \(C(\mathbb{R}_+; E_0)\) since so is \(\mathfrak{A}\). To prove the closeness of \(K_t\) in \(C(\mathbb{R}_+; E_0)\), we take \(\{u_n\} \subset K_t\) such that \(u_n \to u\) with \(u \in C(\mathbb{R}_+; E_0)\). In particular, we have

\[
\lim_{n \to \infty} \|u_n(h) - u(h)\|_{E_0} = 0, \quad \text{for any } h \geq 0. \tag{20}
\]

For each \(h \geq 0\), we have \(u_n(h) \in K_{t+h}\), which is closed in \(E_0\). Then by (20)

\[
u(h) \in K_{t+h} \text{ for any } h \geq 0. \tag{21}
\]

Since \(\{u_n\} \subset K_t \subset \mathfrak{A}_t\), it follows from the \(T\)-compactness of \(\mathfrak{A}\) that the limit \(u \in \mathfrak{A}_t\). Then (21) implies \(u \in K_t\) and proves the closedness of \(K_t\) in \(C(\mathbb{R}_+; E_0)\).

We prove that \(K\) is positively invariant, i.e. \(T(s)K \subset K\), which is equivalent to prove that \(T(s)K_{t-s} \subset K_t\) for each \(s \geq 0\) and \(t \in \mathbb{R}\). Let \(u = T(s)v\) with \(v \in K_{t-s} \subset \mathfrak{A}_{t-s}\). The invariance of \(\mathfrak{A}\) yields \(u \in \mathfrak{A}_t\), and by (19), for all \(h \geq 0\),

\[
u(h) = (T(s)v)(h) = v(s + h) \in K_{t-s} + (s + h) = K_{t+h}. \tag{22}
\]

By (19) again, \(u \in K_t\) as required.

The proof for the attraction of \(K\) follows from the idea in the proof of [22, Theorem 3.14]. But we still write down the proof for completeness. If \(K\) is not attracting, then there exist \(\delta > 0\), \(t \in \mathbb{R}\), \(\tau_k \to -\infty\) and \(u_k \in \mathcal{H}_{\tau_k}(D)\) with \(D \in \mathcal{D}\) such that

\[
d_{C([0,\infty); E_0)}(T(t - \tau_k)u_k, K_t) > \delta, \quad \forall k \in \mathbb{N}. \tag{23}
\]

Since \(\mathfrak{A}_t\) is attracting and compact in \(C(\mathbb{R}_+; E_0)\), there is a subsequence of \(\{k\}\) (not relabeled) and \(v \in \mathfrak{A}_t\) such that

\[
\|T(t - \tau_k)u_k - v\|_{C(\mathbb{R}_+; E_0)} \to 0 \quad \text{as } k \to \infty, \tag{24}
\]

which together with (23) implies \(v \notin K_t\), that is, there is an \(h \geq 0\) such that \(v(h) \notin K_{t+h}\).

On the other hand, by the assumption, \(K_{t+h}\) is attracting and compact in \(E_0\), then there is a subsequence of \(\{k\}\) (not relabeled) and \(y \in K_{t+h}\) such that \(u_k(t + h - \tau_k) \to y\) in \(E_0\), that is, \((T(t - \tau_k)u_k)(h) \to y\) in \(E_0\). However, we see from
(24) that \((T(t - \tau_k)u_k)(h) \rightarrow v(h)\) in \(E_0\). Therefore, \(v(h) = y \in K_{t+h}\), which is a contradiction.

So far, we have proved that \(\mathcal{K}\) is indeed a backward \(\mathcal{T}\)-compact TSA. Then, by Proposition 1(ii), the BCTA \(\mathfrak{A}\) is the minimal backward \(\mathcal{T}\)-compact TSA, so \(\mathfrak{A} \subset \mathcal{K}\). Hence, it follows from (18) and (19) that \(\mathcal{A} = \mathfrak{A}(0) \subset K(0) \subset K\), which completes the proof of the minimality. Therefore, \(\mathcal{A}\) is indeed a BCA. \(\Box\)

**Remark 2.** A pullback attractor given in Def.3.1 involves two phases spaces, in this respect, the concept coincides with bi-spatial attractors (see e.g. \([7, 14, 15, 16, 18, 28, 29]\)). However, a bi-spatial attractor is sufficiently regular, which means it is compact and attracting in the terminate space \(E\) (rather than \(E_0\) only). It is possible to develop the regularity of both trajectory attractor and pullback attractor.

4. **Application to the Jeffreys-Oldroyd equations.** In this section, we consider the application on the following nonautonomous Jeffreys-Oldroyd equations defined on a bounded domain \(\Omega \subset \mathbb{R}^3\) with smooth boundary \(\partial \Omega\):

\[
\frac{\partial u}{\partial t} + \sum_{i=1}^{3} u_i \frac{\partial u}{\partial x_i} + \text{grad } p = \text{Div } \sigma + f(x,t), \quad t \geq \tau, \quad (25)
\]

\[
\sigma + \lambda_1 \left( \frac{\partial \sigma}{\partial t} + \sum_{i=1}^{3} u_i \frac{\partial \sigma}{\partial x_i} \right) = 2\eta \mathcal{E} + 2\eta \lambda_2 \left( \frac{\partial \mathcal{E}}{\partial t} + \sum_{i=1}^{3} u_i \frac{\partial \mathcal{E}}{\partial x_i} \right), \quad (26)
\]

\[
\text{div } u = 0, \quad u|_{\partial \Omega} = 0, \quad u(\tau) = u_0, \quad \sigma(\tau) = \sigma_0. \quad (27)
\]

The parameters have some physical meanings as follow.

- \(\lambda_1\) is the relaxation time, \(\lambda_2\) is the retardation time, and \(\lambda_1 > \lambda_2 > 0\).
- \(\eta > 0\) is the viscosity of the medium.
- \(f := f(x,t)\) is the body force vector.

The unknown functions are given as follows.

- \(u := u(x,t) = (u_1, u_2, u_3)\) is the velocity vector, \(p := p(x,t)\) is the pressure function.
- \(\sigma = (\sigma_{ij}(x,t))\) is the unknown function of stress deviator as a symmetric matrix with

\[
\text{Div } \sigma = \left( \sum_{i=1}^{3} \frac{\partial \sigma_{1i}}{\partial x_i}, \sum_{i=1}^{3} \frac{\partial \sigma_{2i}}{\partial x_i}, \sum_{i=1}^{3} \frac{\partial \sigma_{3i}}{\partial x_i} \right).
\]

- \(\mathcal{E} = (\mathcal{E}_{ij}(x,t))\) is the strain velocity tensor as a symmetric matrix of order 3 given by

\[
\mathcal{E}_{ij} = \mathcal{E}_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i, j = 1, 2, 3. \quad (28)
\]

4.1. **Transform of variables and space of functions.** As usual (cf.\([23, 24]\)), we put \(\mu_1 = \eta \lambda_2 / \lambda_1\), \(\mu_2 = (\eta - \mu_1) / \lambda_1\) and make a proper change of variable:

\[
\varsigma = \sigma - 2\mu_1 \mathcal{E}(u).
\]

Then the problem (25)-(27) is equivalent to the followings.

\[
\frac{\partial u}{\partial t} - \mu_1 \Delta u + \sum_{i=1}^{3} u_i \frac{\partial u}{\partial x_i} - \text{Div } \varsigma + \text{grad } p = f, \quad (29)
\]
where we replace $\varsigma$ and $\beta$ by $H$ spaces respectively. Let $K$ such that $V \subset (\Omega, M_3)$, where $M_3 \geq 3$. The closed subspaces of $V$ in this paper are taken by $\mathcal{H}$ following chain:

A weak solution ($\psi$, $v$, $\varsigma$) of problem (32)-(34) is called admissible in the following sense: $v$ such that $v$ is an admissible weak solution for the Jeffreys-Oldroyd equations was referred to [31, Def. 2.8]. Under a basic assumption that $f \in L^2(\mathbb{R}; V^*)$, the existence conclusion of a weak solution can be founded in [31, Theorem 2.4]. However, we require the weak solution to be admissible in the following sense:

**Definition 4.1.** A weak solution $(v, \varsigma)$ of problem (32)-(34) is called admissible if there are constants $c_1, c_2 > 0$ such that, for all $h \geq 0$,

$$
\|v\|_{L^\infty(\Omega, M_3; H)}^2 + \|\varsigma\|_{L^\infty(\Omega, M_3; L^2)}^2 + \|v\|_{L^2(\Omega, M_3; V)}^2 \\
\leq c_1 e^{-\gamma h} (\|v(0)\|_{L^2}^2 + \|\varsigma(0)\|_{L^2}^2) + c_2 \int_0^{h+1} e^{(r-h)} \|f(r + \tau)\|_{V^*}^2 dr,
$$

where $\tau \in \mathbb{R}$. In this case, the pair $(u(t), \varsigma(t)) = (v(t - \tau), \varsigma(t - \tau)), t \geq \tau$, is called an admissible weak solution of problem (29)-(31).

We take some ideas from [31] to prove a priori estimate for the admissible weak solution of problem (32)-(34).
Proposition 3. Suppose \( f \in L^2_{\text{loc}}(\mathbb{R}, V^*) \) and let \( F = T(\tau) f \) with any \( \tau \in \mathbb{R} \). Then, for any \((v_0, s_0) \in H \times L^2_2\), problem (32)-(34) has an admissible weak solution \((v, \varsigma)\) satisfying: for all \( h \geq 0 \),

\[
\|v\|_{L^\infty(h, h+1; H)}^2 + \frac{1}{2\mu_2} \|\varsigma\|^2_{L^\infty(h, h+1; L^2_2)} + \mu_1 \|v\|^2_{L^2_2(h, h+1; V)} \\
\leq 2e^{-\gamma h}(\|v_0\|^2_H + \|s_0\|^2_{L^2_2}) + 16 \mu_1 \int_0^{h+1} e^{-\gamma(r-h)} \|F(r)\|_{V'}^2 dr,
\]

where \( \gamma := \min\left\{ \frac{\mu_1 K_0}{2}, \frac{2}{\lambda_1} \right\} \). Furthermore, the derivatives of the weak solution satisfy

\[
\|v\|_{L^2_4(h, h+1; V^*)} + \|\varsigma\|_{L^2_2(h, h+1; H^{-2})} \\
\leq K(\|v\|_{L^\infty(h, h+1; H)}, \|v\|_{L^2_2(h, h+1; V)}, \|\varsigma\|_{L^\infty(h, h+1; L^2_2)}, \|F\|_{L^2_2(h, h+1; V^*)}),
\]

where, the function \( K \) continuously depends on its argument.

Proof. By [31, Theorem 2.4], the problem (32)-(34) has a weak solution \((v_m, s_m)\) such that it is the weak limit of the approximated solution \((v, s)\) in (39) and \( \Phi \in \frac{2}{\lambda_1} v \) in (40), then sum the results to obtain

\[
\frac{1}{2} \frac{d}{dt} \|v\|^2_H + \frac{1}{4\mu_2} \frac{d}{dt} \|\varsigma\|^2_{L^2_2} + \mu_1 \|v\|^2_V \\
+ \frac{1}{2\lambda_1 \mu_2} \|\varsigma\|^2_{L^2_2} + \frac{\varepsilon}{2\lambda_1 \mu_2} \|\nabla \varsigma\|^2_{L^2_2} - I_1 + I_2 = \langle F, v \rangle,
\]

where, by a similar calculation as given in [33, Subsection 6.1.2], we have

\[
I_1 := \sum_{i=1}^n \left( \frac{v_i v_i}{1 + \delta(\frac{|\varsigma|^2}{\mu_2} + |v|^2)} \frac{\partial v}{\partial x_i} + \frac{v_i \varsigma_i}{1 + \delta(\frac{|\varsigma|^2}{\mu_2} + |v|^2)} \frac{\partial \varsigma}{\partial x_i} \right) = 0,
\]

\[
I_2 := (v, \text{Div} \varsigma) + (\varsigma, \nabla v) = 0.
\]

On the other hand, by the Young inequality and Poincaré inequality,

\[
|\langle F, v \rangle| \leq \frac{4}{\mu_1} \|F\|_{V^*}^2 + \frac{\mu_1}{4} \|v\|_{V'}^2, \quad \text{and}
\]

\[
\mu_1 \|v\|_{V'}^2 \geq \frac{\mu_1}{4} \|v\|_V^2 + \frac{\mu_1 K_0}{4} \|v\|_H^2 + \frac{\mu_1}{2} \|v\|_{V'}^2.
\]

Substituting all above estimates into (41) and setting \( \gamma := \min\left\{ \frac{\mu_1 K_0}{2}, \frac{2}{\lambda_1} \right\} \), we find

\[
\frac{d}{dt}(\|v\|^2_H + \frac{1}{2\mu_2} \|\varsigma\|^2_{L^2_2}) + \gamma(\|v\|^2_H + \frac{1}{2\mu_2} \|\varsigma\|^2_{L^2_2}) + \mu_1 \|v\|^2_V \leq \frac{8}{\mu_1} \|F\|_{V'}^2.
\]
Hence, it follows from the Gronwall lemma on (42) that
\[
\|v(t)\|^2_H + \frac{1}{2\mu_2} \|\varsigma(t)\|^2_{L^2} + \mu_1 \int_0^t e^{-\gamma(t-r)} \|v(r)\|^2_V \, dr \
\leq e^{-\gamma t} (\|v_0\|^2_H + \|\varsigma_0\|^2_{L^2}) + \frac{8}{\mu_1} \int_0^t e^{-\gamma(t-r)} \|F(r)\|^2_V \, dr.
\] (43)

By taking \( t = h + 1 \) in (43), we know the approximated solution \( v_m \) satisfies
\[
\mu_1 \|v_m\|^2_{L^2([h,h+1;V])} = \mu_1 \int_h^{h+1} \|v_m(r)\|^2_V \, dr \
\leq e^{-\gamma h} \mu_1 \int_h^{h+1} e^{-\gamma(h+1-r)} \|v_m(r)\|^2_V \, dr \
\leq e^{-\gamma h} \mu_1 \int_h^{h+1} e^{-\gamma(h+1-r)} \|v_m(r)\|^2_V \, dr \
\leq e^{-\gamma h} (\|v_0\|^2_H + \|\varsigma_0\|^2_{L^2}) + \frac{8}{\mu_1} \int_0^{h+1} e^{-\gamma(h-r)} \|F(r)\|^2_V \, dr.
\] (44)

By taking \( t \in [h, h+1) \) in (43), it is easy to see that
\[
\|v_m\|^2_{L^\infty([h,h+1;H])} + \frac{1}{2\mu_2} \|\varsigma_m\|^2_{L^\infty([h,h+1;L^2])} \
\leq e^{-\gamma h} (\|v_0\|^2_H + \|\varsigma_0\|^2_{L^2}) + \frac{8}{\mu_1} \int_0^{h+1} e^{-\gamma(h-r)} \|F(r)\|^2_V \, dr.
\] (45)

Since \((v_m, \varsigma_m) \to (v, \varsigma)\), by the lower semi-continuity of the corresponding norm, it follows from (44)-(45) that the weak solution \((v, \varsigma)\) satisfies (37) and it also satisfies (38) from [33].

Motivating from Proposition 3, we can define the required trajectory space as follows. For each \( \tau \in \mathbb{R} \), the trajectory space \( \mathcal{H}^\tau_+ \) is defined by the set of all admissible weak solutions \((v, \varsigma)\) such that they satisfy (38) and for all \( h \geq 0 \),
\[
\|v\|^2_{L^\infty([h,h+1;H])} + \|\varsigma\|^2_{L^\infty([h,h+1;L^2])} + \|v\|^2_{L^2([h,h+1;V])} \
\leq c_3 e^{-\gamma h} (\|v(0)\|^2_H + \|\varsigma(0)\|^2_{L^2}) + \frac{8c_3}{\mu_1} \int_0^{h+1} e^{-\gamma(h-r)} \|f(r)\|^2_V \, dr,
\] (46)

where \( c_3 = 2/\min\{1,1/(2\mu_2), \mu_1\} \). By Proposition 3, the trajectory space \( \mathcal{H}^\tau_+ \) (starting at any element in \( E \)) is nonempty.

4.3. Backward compact dynamics for Jeffreys-Oldroyd equations. We need some assumptions on the force.

**Hypothesis F.** Suppose \( f \in L^2_{loc}(\mathbb{R}; V^*) \) and it is backward translation bounded, i.e.
\[
\sup_{s \leq t} \int_{s-a}^s \|f(r)\|^2_V \, dr < \infty, \text{ for some } t \in \mathbb{R} \text{ and some } a > 0.
\] (47)

In the appendix, we will prove the following equivalence.

**Lemma 4.2.** Let \( f \in L^2_{loc}(\mathbb{R}; V^*) \). Then the following statements are equivalent.
(A) (47) holds true for some \( t \in \mathbb{R} \) and some \( a > 0 \);
(B) (47) holds true for all \( t \in \mathbb{R} \) and all \( a > 0 \).
(C) $f$ is backward tempered, given $\gamma = \min\{\frac{\mu_kK_0}{2}, \frac{2}{\gamma}\}$,
\[
\sup_{s \leq t} \int_{-\infty}^{s} e^{\gamma(r-s)} \|f(r)\|_V^2 dr < \infty, \quad \text{for all } t \in \mathbb{R}.
\] (48)

**Examples.** In general, an increasing function (e.g. $f(t) = e^t$) or a periodic function (e.g. $f(t) = \sin t$) satisfy the hypothesis $F$. On the other hand, the function $f(t) = e^{-\gamma t/4}$ does not satisfy the hypothesis $F$. So, it is not backward tempered in the sense of (48). However, it is tempered at present:
\[
\int_{-\infty}^{t} e^{\gamma(r-t)} |f(r)|^2 dr = \int_{-\infty}^{t} e^{\gamma(r-t)} e^{-\frac{2}{\gamma} r} dr = \frac{2}{\gamma} e^{-\frac{2}{\gamma} t} < \infty.
\]

To specify a universe $D$ over $E$, we use $R$ to denote the set of all continuous functions $r(\cdot) : \mathbb{R} \to \mathbb{R}_+$ such that the function $\tau \mapsto e^{\gamma \tau} r(\tau)$ is non-decreasing and $\lim_{\tau \to -\infty} e^{\gamma \tau} r(\tau) = 0$, where $\gamma$ is the constant given in Proposition 3.

We then define the universe $D$ to be the family of all brochettes $D$ over $E$ such that, for some $r_D \in R$,
\[
\|v_0\|^2_{\mathcal{L}_2} + \|s_0\|^2_{\mathcal{L}_2} \leq r_D(\tau), \quad \text{for } (v_0, s_0) \in D_\tau, \quad \tau \in \mathbb{R}.
\] (49)

The main result of this section is stated and proved as follows.

**Theorem 4.3.** Suppose $f$ satisfies the hypothesis $F$. Then, the Jeffreys-Oldroyd equations (29)-(31) have a backward compact trajectory attractor $\mathcal{A}$ over $\mathcal{T}$ and a backward-compact attractor $A$ over $E$ such that $A = \mathcal{A}(0)$.

**Proof.** By Theorem 2.10 and Theorem 3.2, it suffices to construct an increasing, $\mathcal{T}$-compact and $D$-absorbing brochette for $H^+$. In fact, for each $t \in \mathbb{R}$, we take $P_t$ to be the closure in $C(\mathbb{R}^+; E_0)$ of the set $\tilde{P}_t$, which is consisted of all pairs $(v, \varsigma) \in \mathcal{T}$ satisfying
\[
\|v\|^2_{L_\infty(h,h+1;H)} + \|\varsigma\|^2_{L_\infty(h,h+1;\mathcal{L}_2)} + \|v\|^2_{L_2(h,h+1;V)} \\
\leq c_3 + \frac{8c_3}{\mu_1} \sup_{s \leq t} \int_{-\infty}^{h+1} e^{\gamma(r-h)} \|f(r+s)\|_V^2 dr, \quad (50)
\]
\[
\|v\|_{L_2(h,h+1;V^*)} + \|\varsigma\|_{L_2(h,h+1;\mathcal{L}_2^*)} \leq K(\|v\|_{L_\infty(h,h+1;H)}), \quad (51)
\]

for any $h \geq 0$, where $K$ is the function given in (38).

It is obvious that the brochette $P = \{P_t : t \in \mathbb{R}\}$ is increasing since the function
\[
\phi(t, h) := \sup_{s \leq t} \int_{-\infty}^{h+1} e^{\gamma(r-h)} \|f(r+s)\|_V^2 dr
\] (52)

is non-decreasing with respect to $t \in \mathbb{R}$ for each fixed $h \geq 0$.

To prove $P$ is controllable, we need to prove that $\phi(t, h)$ is finite for $t \in \mathbb{R}$ and $h \geq 0$. Indeed,
\[
\phi(t, h) = e^{-\gamma h} \sup_{s \leq t} \int_{-\infty}^{s+h+1} e^{\gamma(r-s)} \|f(r)\|_V^2 dr \\
\leq e^{-\gamma h} \left( \sup_{s \leq t} \int_{-\infty}^{s} e^{\gamma(r-s)} \|f(r)\|_V^2 dr + \sup_{s \leq t} \int_{s}^{s+h+1} e^{\gamma(r-s)} \|f(r)\|_V^2 dr \right) \\
\leq \sup_{s \leq t} \int_{-\infty}^{s} e^{\gamma(r-s)} \|f(r)\|_V^2 dr + e^{\gamma} \sup_{s \leq t \leq h+1} \int_{s}^{s+h+1} \|f(r)\|_V^2 dr.
\] (53)
By Lemma 4.2(B), the last term in (53) is finite, and by Lemma 4.2(C) the first term in (53) is finite. So \( \phi(t, h) \) is finite. On the other hand, by the absolute continuity of the Lebesgue integral, we know that the function \( h \to \int_{-\infty}^{s} e^{s-r} \| f(r + s) \|^2 dr \) is continuous for each \( s \leq t \). Then, by the same method as given in the proof of Lemma 2.2(c), one can prove the continuity of \( \phi(t, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+ \) for each fixed \( t \in \mathbb{R} \). So \( \phi(t, h) = c_3 + \frac{8\kappa}{\mu^2} \phi(t, h) \) satisfies the controllable condition in Lemma 2.2(c). Therefore, \( \mathcal{P} \) is controllable and actually backward controllable since \( \mathcal{P} \) is increasing.

Let \( t \in \mathbb{R} \) and \( h \geq 0 \) be fixed, by (50), \( \tilde{P}_t \) is bounded in \( L_{\infty}(h, h + 1; H \times L_2) \), in fact, the norms of all elements in \( \tilde{P}_t \) are less than \( \phi(t, h) \). On the other hand, by (51), the set \( \{ (v', \xi') : (v, \xi) \in \tilde{P}_t \} \) is bounded in \( L_{4/3}(h, h + 1; V^* \times \mathcal{H}^{-2}) \). Since \( E = H \times L_2 \subset E_0 = V_0^* \times H^{-\alpha} \subset V^* \times \mathcal{H}^{-2} \) and \( E \) is embedding compactly into \( E_0 \), it follows from the Aubin-Lions Lemma that \( \tilde{P}_t \) is relatively compact in \( C([h, h + 1]; E_0) \) for any \( h \geq 0 \), which easily implies \( \bar{P}_t \) is relatively compact in \( C([0, h]; E_0) \) for any \( k \in \mathbb{N} \). Then, by a diagonal technique, it easily follows from (5) that \( \bar{P}_t \) is relatively compact in \( C(\mathbb{R}_+; E_0) \). So far, we have proved that \( \mathcal{P} \) is increasing and \( \mathcal{T} \)-compact. By Lemma 2.3(I), it actually is backward \( \mathcal{T} \)-compact.

Next, we show the pullback \( \mathcal{D} \)-absorption of \( \mathcal{P} \). It suffices to prove that, for fixed \( \mathcal{D} \in \mathcal{D} \) and \( t \in \mathbb{R} \), one can choose \( \tau_0 = \tau_0(D, t) \leq t \) such that \( \inf_{r \geq t} \tau_0(D, r) > -\infty \) and

\[
T(t - \tau) \mathcal{H}^+_t(D) \subset \tilde{P}_t \subset \mathcal{P}_t, \text{ whenever } \tau \leq \tau_0. \tag{54}
\]

Since \( \mathcal{D} \in \mathcal{D} \), we can take a function \( \tau \in \mathcal{R} \) such that

\[
\| v_0 \|^2_H + \| \xi_0 \|^2_{L_2} \leq \rho(\tau), \text{ for all } (v_0, \xi_0) \in \mathcal{D}_\tau, \tau \in \mathbb{R}, \tag{55}
\]

where \( \rho \in \mathcal{R} \) means the function \( \rho(\tau) := e^{\tau r} \rho(\tau) \) is non-negative, continuous and non-decreasing in \( \tau \in \mathbb{R} \) and \( \rho(\tau) \to 0 \text{ as } \tau \to -\infty \). The monotonicity of \( \rho \) implies that it has an inverse function \( \rho^{-1} \). By the continuity of \( \rho \) and \( \rho(\tau) \to 0 \text{ as } \tau \to -\infty \), we know that the definition domain of \( \rho^{-1} \) satisfies

\[
(0, \sup_{r \in \mathbb{R}} \psi(r)) \subset \text{Dom } \psi^{-1} = \text{Range } \psi \subset [0, \sup_{r \in \mathbb{R}} \psi(r)]. \tag{56}
\]

We remark here that the first inclusion in (56) may not hold if we do not assume the continuity of \( r(\cdot) \). We now define the entry time \( \tau_0 \) by

\[
\tau_0(D, t) = \begin{cases} 
\min \{ \psi^{-1}(e^{\tau}), t \}, & \text{if } e^{\tau t} \geq \sup_{r \in \mathbb{R}} \psi(r), \\
\min \{ \psi^{-1}(e^{\tau}) \}, & \text{if } e^{\tau t} < \sup_{r \in \mathbb{R}} \psi(r). 
\end{cases} \tag{57}
\]

By (56), we know \( e^{\tau t} \in \text{Dom } \psi^{-1} \) in the second case of (57), then \( \tau_0(D, t) \) is well-defined. It is obvious that \( \tau_0(D, t) \leq t \). On the other hand, since \( \psi^{-1} \) is still non-decreasing, it follows that

\[
\inf_{r \geq t} \tau_0(D, r) \geq \min \{ \psi^{-1}(e^{\tau}), t \} \neq -\infty,
\]

which means that \( \tau_0(D, t) \) can be regarded really as an entry time in Def. 2.7. We then claim

\[
e^{-\gamma(t-\tau)} \rho(\tau) \leq 1, \text{ for all } \tau \leq \tau_0(D, t). \tag{58}
\]

Indeed, if \( e^{\tau t} \geq \sup_{r \in \mathbb{R}} \psi(r) \), we know \( \tau_0(D, t) = t \), then, for all \( \tau \leq \tau_0(D, t) \),

\[e^{-\gamma(t-\tau)} \rho(\tau) = e^{-\gamma t} \rho(\tau) \leq e^{-\gamma t} e^{\gamma t} = 1.\]
If \( e^{\gamma t} < \sup_{r \in \mathbb{R}} \psi(r) \), by (57), \( \tau_0(D,t) \leq \psi^{-1}(e^{\gamma t}) \), then, for each \( \tau \leq \tau_0 \leq \psi^{-1}(e^{\gamma t}) \), it follows from the monotonicity of \( \psi \) that \( \psi(\tau) \leq \psi^{-1}(e^{\gamma t}) = e^{\gamma t} \), which also implies \( e^{-\gamma(t-\tau)} \tau(\tau) = e^{-\gamma t} \psi(\tau) \leq 1 \). Hence (58) holds true, which together with (55) implies

\[
e^{-\gamma(t-\tau)}(\|v_0\|_{L^2}^2 + \|s_0\|_{L^2}^2) \leq 1, \text{ for all } (v_0, s_0) \in D_\tau, \tau \leq \tau_0(D, t). \tag{59}\]

It remains to show (54) for \( \tau \leq \tau_0(D, t) \). We take \((\bar{v}, \bar{\zeta}) \in T(t-\tau)\mathcal{H}^+_e(D) \) and write \((\bar{v}, \bar{\zeta}) = T(t-\tau)(v, \zeta) \) with \((v, \zeta) \in \mathcal{H}^+_e(D) \), then \((v(0), \zeta(0)) \in D_\tau \). By (59), we have, for all \( h \geq 0 \),

\[
\|\bar{v}\|_{L^\infty(h,h+1;H)}^2 + \|\bar{\zeta}\|_{L^\infty(h,h+1;L^2)}^2 + \|\bar{v}\|_{L^2(h,h+1;V)}^2 \leq c_3 e^{\gamma(t-\tau)}(\|v(0)\|_{L^2}^2 + \|s(0)\|_{L^2}^2) + \frac{8c_3}{\mu_1} \int_{-\infty}^{h+1} e^{-\gamma(r-(h+1-\tau))} \|f(r+\tau)\|_{V^*}^2 dr \\
\leq c_3 e^{\gamma(t-\tau)}(\|v(0)\|_{L^2}^2 + \|s(0)\|_{L^2}^2) + \frac{8c_3}{\mu_1} \int_{-\infty}^{h+1} e^{-\gamma(r-h)} \|f(r+t)\|_{V^*}^2 dr \\
\leq c_3 + \frac{8c_3}{\mu_1} \sup_{s \leq h} \int_{-\infty}^{h} e^{-\gamma(r-h)} \|f(r+s)\|_{V^*}^2 dr.
\]

The above estimate shows that the pair \((\bar{v}, \bar{\zeta}) \) satisfies (50). Since the trajectory \((v, \zeta) \) satisfies (38), it is easy to prove that \((\bar{v}, \bar{\zeta}) \) satisfies (51). So \((\bar{v}, \bar{\zeta}) \in \bar{P}_t \subset P_t \), which proves the absorption as required. 

**Remark 3.** Comparing with [31], we have an extra assumption for the attracted universe that \( r(\cdot) \) is **continuous** and so \( \tau \to \psi(\tau) = e^{\gamma \tau} r(\tau) \) is **continuous**, which ensures the entry time \( \tau_0 \) in the second case of (57) is well-defined in view of (56). However, in [31], the entry time \( \tau_0 \) may not be well-defined, since it is possible that \( e^{\gamma t} \notin \text{Dom } \psi^{-1} \) without the continuity assumption. Also, in the proof of [22, Theorem 4.3] for the 3D Navier-Stokes equation, the entry time may be undefined without the continuity assumption.

5. **Appendix.** In this appendix, we prove Lemma 4.2 in the general case.

**Lemma 5.1.** Suppose \((X, \| \cdot \|)\) is a Banach space and \( f \in L^\infty_2(\mathbb{R}; X) \). Then the following statements are equivalent to each other.

(A) \( f \) is backward translation bound, i.e. there exist some \( t_0 \in \mathbb{R} \) and \( h_0 > 0 \) such that

\[
I(t_0, h_0) := \sup_{s \leq t_0} \int_{s-h_0}^s \|f(r)\|^2 dr < \infty. \tag{60}\]

(B) \( I(t, h) < \infty \) for all \( t \in \mathbb{R} \) and all \( h > 0 \).

(C) \( f \) is backward tempered, that is, there exist some \( \beta_0 > 0 \) and \( t_0 \in \mathbb{R} \) such that

\[
J(t_0, \beta_0) := \sup_{s \leq t_0} \int_{-\infty}^{s} e^{\beta_0(r-s)} \|f(r)\|^2 dr < \infty. \tag{61}\]

(D) \( J(t, \beta) < \infty \) for all \( \beta > 0 \) and all \( t \in \mathbb{R} \).
Proof. \((A) \Rightarrow (B).\) Suppose \(I(t_0, h_0) < \infty\) for some \(t_0 \in \mathbb{R}\) and \(h_0 > 0.\) If \(t \in \mathbb{R}\) and \(h > 0\) are arbitrary, we can write \(h = kh_0 + \delta,\) where \(k \in \mathbb{N} \cup \{0\}\) and \(\delta \in [0, h_0).\) Then

\[
I(t_0, h) = \sup_{s \leq t_0} \int_{s-h}^{s} \|f(r)\|^2 dr \leq \sup_{s \leq t_0} \sum_{i=0}^{k} \int_{s-(i+1)h_0}^{s-ih_0} \|f(r)\|^2 dr
\]

\[
\leq \sum_{i=0}^{k} \sup_{s \leq t_0} \int_{s-(i+1)h_0}^{s-ih_0} \|f(r)\|^2 dr = \sum_{i=0}^{k} \sup_{s \leq t_0} \int_{\xi-iho}^{\xi} \|f(r)\|^2 dr
\]

\[
\leq \sum_{i=0}^{k} \sup_{s \leq t_0} \int_{\xi-iho}^{\xi} \|f(r)\|^2 dr = (k + 1)I(t_0, h_0) < \infty. \tag{62}
\]

By (62), if \(t \leq t_0\) then obviously \(I(t, h) \leq I(t_0, h) < \infty.\) If \(t > t_0,\) by the assumption \(f \in L^2_{\infty}(\mathbb{R}; X),\) we also have

\[
I(t, h) = \sup_{s \leq t} \int_{s-h}^{s} \|f(r)\|^2 dr
\]

\[
\leq \sup_{s \leq t_0} \int_{s-h}^{s} \|f(r)\|^2 dr, \sup_{t_0 \leq s \leq t} \int_{s-h}^{s} \|f(r)\|^2 dr
\]

\[
\leq \sum_{i=0}^{k} \sup_{s \leq t_0} \int_{s-(i+1)h_0}^{s-ih_0} \|f(r)\|^2 dr
\]

\[
\leq \sup \{I(t_0, h), \int_{t_0-h}^{t} \|f(r)\|^2 dr\} < \infty. \tag{63}
\]

\((B) \Rightarrow (C).\) Let \(\beta_0 > 0\) and \(t_0 \in \mathbb{R}.\) Given any \(h > 0,\) we have

\[
J(t_0, \beta_0) = \sum_{s \leq t_0} \sum_{m=0}^{\infty} \int_{s-(m+1)h}^{s-mh} e^{\beta_0(r-s)} \|f(r)\|^2 dr
\]

\[
\leq \sum_{s \leq t_0} \sum_{m=0}^{\infty} e^{-m\beta_0 h} \int_{s-(m+1)h}^{s-mh} \|f(r)\|^2 dr
\]

\[
\leq \sum_{m=0}^{\infty} e^{-m\beta_0 h} \sup_{\rho \leq t_0} \int_{\rho-h}^{\rho} \|f(r)\|^2 dr
\]

\[
= \frac{1}{1 - e^{-\beta_0 h}} I(t_0, h) < \infty. \tag{64}
\]

\((C) \Rightarrow (D).\) Suppose \(J(t_0, \beta_0) < \infty\) for some \(\beta_0 > 0\) and \(t_0 \in \mathbb{R}.\) Then we have

\[
I(t_0, 1) = \sup_{s \leq t_0} \int_{s-1}^{s} \|f(r)\|^2 dr \leq e^{\beta_0} \sup_{s \leq t_0} \int_{s-1}^{s} e^{\beta_0(r-s)} \|f(r)\|^2 dr
\]

\[
\leq e^{\beta_0} \sup_{s \leq t_0} \int_{s-1}^{s} e^{\beta_0(r-s)} \|f(r)\|^2 dr = e^{\beta_0} J(t_0, \beta_0). \tag{65}
\]

Let \(t \in \mathbb{R}\) be arbitrary and assume \(t > t_0\) without lose of generality. Then it follows from (63) and (65) that

\[
I(t, 1) \leq \sup \{I(t_0, 1), \int_{t_0-1}^{t} \|f(r)\|^2 dr\}
\]

\[
\leq \sup \{e^{\beta_0} J(t_0, \beta_0), \int_{t_0-1}^{t} \|f(r)\|^2 dr\} < \infty.
\]
Let now $\beta > 0$ be arbitrary. Then, by taking $h = 1$ in (64), we have
\[ J(t, \beta) \leq \frac{1}{1 - e^{-\beta}} I(t, 1) < \infty. \]

(D)$⇒(A). In fact, this assertion has been proved by (65).

\textbf{Remark 4.} Lemma 5.1 generalizes the corresponding results given in [5, 17, 27].

Note that the assertion (C)$\Rightarrow$(D) may not be true if we omit the supremum. For example, let $\beta < \beta_0$ and $\|f(r)\|^2 = e^{-(\beta + \delta)r}$ with $\delta \in (0, \beta_0 - \beta)$, then we have
\[
\int_{-\infty}^{0} e^{\beta r} \|f(r)\|^2 dr = \int_{-\infty}^{0} e^{\beta r} e^{-(\beta + \delta)r} dr = \frac{1}{\beta_0 - \beta - \delta} < \infty,
\]
\[
\int_{-\infty}^{0} e^{\beta r} \|f(r)\|^2 dr = \int_{-\infty}^{0} e^{\delta r} e^{-(\beta + \delta)r} dr = \int_{-\infty}^{0} e^{-\delta r} dr = \infty.
\]

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Received August 2017; revised May 2018.

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