Perfect colouring of the graph with its kinds

A A Bhangé* and H R Bhapkar

MIT ADT University, India

*Corresponding author e-mail: archana.bhangé@manetpune.edu.in

Abstract. The Perfect colouring ($χ^p$) of a graph is an assignment of colours to all elements (vertices, edges, and regions) of the graph such that no two adjacent elements receive the same colour. In this paper, we determined the tight bounds of perfect colouring as $χ''(G) \leq χ^p(G) \leq χ''(G) + 4$, where $χ''(G)$ is total colouring of the graph $G$. Depending on these bounds, the perfect colouring is divided into five different kinds, and the results of these for some standard graphs are presented in the paper.

1. Introduction

All graphs considered in this paper are planar. Graph colouring is having numerous applications in various fields for aircraft scheduling, frequency assignment, exam timetabling, etc. [1]. Graph colouring concept came into the form when Appel and Haken [2] gave the proof of four colour theorem. Four colour theorem states that every planar graph can be coloured with four or fewer colours. Bhapkar [3] gave the proof of four colour theorem using the pivot region number of the graph. Graph colouring is basically deals with vertex colouring, edge colouring, and region colouring [4]. Vertex colouring is nothing but colouring the vertices of the graph so that no two neighbouring vertices should receive the same colour. The least number of colours required to colour the graph using vertex colouring is the chromatic number of that graph, and it is denoted by $χ$. In a similar way edge colouring is assignment of colours to edges so that no two adjacent edges should receive the same colour and minimum of such number is called chromatic index ($χ'$). Also, face colouring is colouring the regions of the graphs so that no two adjacent region should receive same colour. Bezhad [5] proposed the total colouring concept, and Vizing V G [6] proposed the bounds for total colouring of graph i.e., the total colouring of graphs holds the inequality $Δ(H) + 1 \leq χ''(H) \leq Δ(H) + 1$, that for any graph $H$. Further this total colouring conjecture was proved for a graph with degree 3 [7]. It’s proven that every cubic graph is totally colorable with five colors [8].

Formanowicz and Tana [9] surveyed different types of vertex, edge, and total coloring. It is stated that vertex coloring can be categorized into equitable vertex colouring, circular vertex colouring, acyclic vertex colouring, and star vertex colouring. Edge coloring is divided into circular edge coloring, acyclic edge coloring, Berge-Fulkerson colouring and Fan-Raspaud colouring, and further defined list colouring, path colouring, and total colouring with examples. Kostochka A V [10] proved that every multigraph with maximal degree four has a total chromatic number six. Murthy T S [11] proved the total colouring conjecture using the algebraic method over a finite field $Z_p$. Sudha S and Manikandan K [12] discussed the total colouring of the center of graphs like path graph, star graph, cycle graph, etc. S.K. Vaidya and Isaac [13] investigated the total colouring number for the middle graph, total graph, shadow graph of cycle, and one point union of cycle graphs. Zatesko [14] determined the least total colouring.
of the graph, which is a join graph or cobipartite graph. Also, presented the algorithm for solution for the problem, which gives the upper bound of the total chromatic number of these graphs. Vignesh [15] proved the total colouring theorem for certain classes of graphs of deleted lexicographic product, path graph, and double graph.

This paper is organized as section 1 is an introduction, giving details about different types of colouring and the research development in graph colouring. Section 2 focuses on the concept of perfect colouring and its kinds. Section 3 is having results for a few standard graphs, and section 4 is stating the conclusion of the paper.

2. Methods
In this paper we have consider a graph G as demarcate and directionless with vertex set and edge set as \{g_1, g_2, g_3, \ldots g_n\} and \{(g_1, g_2), (g_1, g_3), (g_2, g_3) \ldots \} respectively. Also C\{g_i\} denotes the colour of vertex g_i and C\{g_i, g_j\} denotes edge (g_i, g_j).

2.1. Perfect colouring of graph
The minimum colours will be required to colour planar graph such that adjacent vertices have different colours, incident edges have different colours, adjacent regions have different colours, a region, boundary edges, and boundary vertices of that region have different colours, this type of colouring is known as Perfect Colouring of graph. The minimum number of colours required to colour any graph using perfect colouring is called the perfect chromatic number of H or PC number of H, denoted as \(\chi^c(H)\).

2.2. P kind graphs
The graphs or family of graphs which gives a particular relation between total colouring \(\chi''(H)\) and perfect colouring \(\chi^c(H)\) are called P kind graphs. If H is any graph then the five kinds of P kind graphs are given as.

2.2.1. \(P_0\) kind graphs. It is a kind of graph in which perfect colouring is equal to total colouring of the graph, i.e., \(P_0(H) = \chi''(H)\).

2.2.2. \(P_1\) kind graphs. The graphs in which perfect colour \(\chi^c(H)\) is one more than that of total colour \(\chi''(H)\) are called \(P_1\) kind graphs, i.e., \(P_1(H) = \chi''(H) + 1\).

2.2.3. \(P_2\) kind graphs. The graphs in which perfect colour \(\chi^c(H)\) is two more than that of total colour \(\chi''(H)\) are called \(P_2\) kind graphs, i.e., \(P_2(H) = \chi''(H) + 2\).

2.2.4. \(P_3\) Kind graphs. It is a kind of graph in which total colouring \(\chi''(H)\) differs with perfect colouring \(\chi^c(H)\) by 3 colours, i.e., \(P_3(H) = \chi''(H) + 3\).

2.2.5. \(P_4\) Kind graphs. It is a kind of graph in which total colouring \(\chi''(H)\) differs with perfect colouring \(\chi^c(H)\) by 4 colours, i.e., \(P_4(H) = \chi''(H) + 4\).

3. Result and discussion
Theorem 1. There is no graph having \(\chi^c = \chi'' + 5\).

Proof. Consider there exist a graph H with \(\chi^c(H) = \chi''(H) + 5\). Hence five colours are required to colour graph H. Therefore, H is five colourable which is a contradiction to four colour theorem. This is a contradiction to our assumption. Hence there is no graph having \(\chi^c = \chi'' + 5\).

Theorem 2. A diamond graph \(D_n\), \(n \geq 6\), is \(P_0\) kind graph.

Proof. Consider a diamond graph \(D_n\) graph as shown in figure 1, consist of \(n\) regions and \(n + 2\) vertices.
Assign colours to edges as
\[ C\{u,x_i\} = i \quad \text{and} \quad C\{u,y_j\} = \frac{n}{2} + j; \quad \forall i, j = 1, 2, ..., \frac{n}{2} \quad \text{and} \quad C\{u,v\} = n + 1 \]

Also,
\[ C\{v,x_i\} = \frac{n}{2} + i \quad \text{and} \quad C\{u,y_j\} = j; \quad \forall i, j = 1, 2, ..., \frac{n}{2} \]

Colour the vertices as
\[ C\{x_k\} = k + 1, \quad C\{y_l\} = \frac{n}{2} + l; \quad \forall k = 1, 2, ..., \frac{n}{2}, \quad l = 1, 2, ..., \frac{n-2}{2} \quad \text{and} \quad C\{y_{\frac{n}{2}}\} = 1 \]

And finally colour vertices u and v as
\[ C\{u\} = n + 2, \quad C\{v\} = n + 3 \]

Which gives \( \chi''(D_n) = n + 3 \). To calculate perfect colouring, colour the regions as
\[ C\{R_m\} = m + 3; \quad \forall m = 1, 2, ..., n - 3 \]
\[ C\{R_{n-2}\} = 3, \quad C\{R_{n-1}\} = 2, \quad C\{R_n\} = 1 \]

Also colour the unbounded region \( C\{R_0\} = n + 1 \). Hence, \( \chi^P(D_n) = n + 3 \).

Hence a diamond graph \( D_n \) is \( P_0 \) kind graph.

**Theorem 3.** Null graphs are \( P_1 \) kind graphs.

**Proof.** Consider a null graph N as shown in figure 2. Colour the vertices as
Figure 2. Null graph N.

As there are no edges,

\[ \chi''(N) = 1 \] and \( C(R_0) = 2 \)

\[ \therefore \chi^P(N) = 2 \]

Which implies \( \chi^P(N) = \chi''(N) + 1 \). Hence Null graphs are \( P_1 \) kind graphs.

**Theorem 4.** Trees are \( P_1 \) kind graphs.

**Proof.** Consider the tree graph T. Colour all the vertices and edges. Let the total colouring \( \chi'' = n \) As trees are not closed graphs there is no bounded region so colour only the unbounded region by colour apart from previous \( n \) colours which gives \( \chi^P(T) = n + 1 \) which implies \( \chi^P(T) = \chi''(T) + 1 \). Hence tree graph T is \( P_1 \) kind graph.

**Theorem 5.** Friendship graph \( F_n \), \( n \geq 2 \) are \( P_1 \) kind graphs.

**Proof.** Consider a Friendship graph \( F_n \), as shown in figure 3 and is defined as

\[ V(F_n) = \bigcup_{i=1}^{2n} (f_i \cup O) \]

\[ E(F_n) = \bigcup_{i=1}^{2n} (O, f_i) \cup \bigcup_{i=1}^{n} (f_{2i-1}, f_{2i}) \]

Colour the vertices as

\[ C\{f_{2i-1}\} = 1, \ C\{f_{2i}\} = 3; \ \forall i = 1, 2, ..., n \] and \( C\{f_{2n+1}\} = 2 \)

Also colour the edges as

\[ C\{f_{2i-1}, f_{2i}\} = 2; \ \forall i = 1, 2, ..., n; \ \{f_{2n+2}, f_{2i+1}\} = 2; \ \forall i = 1, 2, ..., n - 1; \ j = 1, 2, ..., n \]

\[ C\{f_{2n+1}, f_{2i}\} = 2j + 1; \ \forall i, j = 1, 2, ..., n \]

\[ \therefore \chi''(F_n) = 2n + 1 \]

To calculate the perfect colouring, colour the regions as

\[ C\{R_i\} = C\{f_{2n+1}, f_{2i+1}\}, \ C\{R_{n}\} = C\{f_{2n+1}, f_{2i}\}; \ \forall i, j = 1, 2, ..., n - 1 \]

And colour the open region \( C\{R_o\} = 2n + 2 \). Hence \( \chi^P(F_n) = 2n + 2 \)
Figure 3. Friendship graph $F_n$.

**Theorem 6.** Ladder Rung graphs are $P_1$ kind graphs.

**Proof.** Consider the $n$-ladder rung graph, as shown in figure 4, consist of $n$ isolated path graphs.

Figure 4. Ladder rung graph.

Colour the vertices of the graph as

$$C\{d_i\} = 1; \quad C\{f_i\} = 3; \quad \forall i = 1, 2, \ldots, n$$

And colour the edges as

$$C\{d_i, f_i\} = 2; \quad \forall i = 1, 2, \ldots, n$$

Which gives $\chi'' = 3$. Also, colour the open unbounded region as $C\{R_o\} = 4$. Hence $\chi^p = 4$. Which implies $\chi^p = \chi'' + 1$. This shows that Ladder Rung graphs are $P_1$ kind graphs.

**Theorem 7.** Prism graphs or circular ladder graphs $Y_{3n,1}$ are $P_2$ kind graphs for all $n \geq 1$. 
Proof. Consider prism graph $Y_{3n,1}$ as shown in figure 5 and is defined as

$$V(Y_{n,1}) = \bigcup_{i=1}^{n} [y_i \cup P_i]$$

$$E(Y_{n,1}) = \bigcup_{i=1}^{n} [(p_i, p_{i+1}) \cup (y_i, y_{i+1}) \cup (p_i, y_i) \cup (p_n, p_1) \cup (y_n, y_1)]$$

For all $i = 1, 2, \ldots, n$

Assign colours to vertices as

$$C\{P_{3i-2}\} = 1$$

$$C\{P_{3i-1}\} = C\{Y_{3i-2}\} = 3$$

$$C\{P_{3i}\} = C\{Y_{3i-1}\} = 2$$

And Colour the edges as

$$C\{P_{3i-1}, P_{3i}\} = C\{Y_{3i-2}, Y_{3i-1}\} = 1,$$

$$C\{P_{3i-2}, P_{3i-1}\} = C\{Y_{3i-1}, Y_{3i+1}\} = 2,$$

$$C\{P_{3i}, P_{3i+1}\} = C\{Y_{3i-1}, Y_{3i}\} = 3,$$

and

$$C\{P_{3i}, Y_{3i}\} = 4; \; \forall i = 1, 2, \ldots, 3n.$$

Hence $\chi'' (Y_{3n,1}) = 4$. To find perfect colour of $\chi'p$, colour the regions as

$$C\{R_{2i-1}\} = 5, \; C\{R_{2i}\} = 6; \; \forall i = 1, 2, \ldots, n$$

Colour the inner region and open unbounded region as

$$C\{R_o\} = C\{R_o\} = 4$$
Hence $\chi^p(Y_{3n,1}) = 6$. Which implies $\chi^p(Y_{3n,1}) = \chi''(Y_{3n,1}) + 2$
Hence it is $P_2$ kind graph.

**Theorem 8.** Ladder graph $L_n$ are $P_3$ kind graphs for all $n = 3m + 1$

**Proof.** Consider a ladder graph $L_n$ as shown in figure 6. And is defined as

$$V(L_n) = \bigcup_{i=1}^{n} [S_i \cup t_i]$$

$$E(L_n) = \bigcup_{i=1}^{n-1} [(S_i, S_{i+1}) \cup (t_i, t_{i+1})] \cup \bigcup_{i=1}^{n} (S_i, t_i)$$

Assign colours to vertices for all $i = 1, 2, \ldots, n$

$$C\{T_{3i-2}\} = C\{S_{3i}\} = 1,$$

$$C\{T_{3i-1}\} = C\{S_{3i-2}\} = 3,$$ and

$$C\{T_{3i}\} = C\{S_{3i-1}\} = 2$$

Colour the edges for all $i = 1, 2, \ldots, n$

$$\{ \{ T_{3i-2}, T_{3i} \} = C\{S_{3i-2}, S_{3i-1}\} = 1 \}
\{ \{ T_{3i-2}, T_{3i-1} \} = C\{S_{3i}, S_{3i+1}\} = 2 \},$$

$$\{ \{ T_{3i}, T_{3i+1} \} = C\{S_{3i-1}, S_{3i}\} = 3 \},$$ and

$$\{ \{ T_{3i}, T_{3i} \} = 4; \ \forall i = 1, 2, \ldots, 3n + 1 \}$$

Hence $\chi''(L_n) = 4$. To find the perfect colour $\chi^p$, colour the regions as

**Figure 6.** Ladder graph $L_n$.
\[ C\{R_{2i-1}\} = 5, \quad C\{R_{2i}\} = 6; \quad \forall i = 1, 2, ..., n \]

And the unbounded region as \( C(R_o) = 7 \). Hence \( \chi_T(L_o) = 7 \). Which implies \( \chi_T(L_o) = \chi''(L_o) + 3 \). Hence it is \( P_3 \) kind graph.

**Theorem 9.** Helm graph \( H_3 \) is \( P_4 \) kind graph.

**Proof.** Consider a helm graph \( H_3 \) as shown in figure 7 and is defined as

\[
V(H_n) = \bigcup_{i=1}^{n} (h_i \cup u_i \cup O)
\]

\[
E(H_n) = \bigcup_{i=1}^{n-1} [(h_i, u_i) \cup (O, h_i)] \bigcup_{i=1}^{n} (h_i, h_{i+1})
\]

![Figure 7. Helm graph \( H_3 \).](image)

Colour vertices as

\[ C\{h_1\} = C\{u_3\} = 3, \]
\[ C\{h_2\} = 4, C\{h_3\} = 2, \quad \text{and} \]
\[ C\{O\} = C\{u_i\} = C\{u_2\} = 1 \]

Colour the edges as

\[ C\{u_1, h_1\} = C\{O, h_2\} = 2 \]
\[ C\{u_2, h_2\} = C\{O, h_3\} = 3, \]
\[ C\{u_3, h_3\} = C\{h_1, h_2\} = 1 \]
\[ C\{u_2, h_2\} = C\{O, h_3\} = 3 \text{ and } \]
\[ C\{h_1, h_3\} = 4 \]

Hence \( \chi''(H_3) = 5 \)

Also, Colour the region as
\[ C\{R_1\} = 6, \; C\{R_2\} = 8, \; C\{R_3\} = 7 \]

And the open region \( C\{Ro\} = 9 \). Which gives \( \chi^P(H_3) = 9 \). Hence \( \chi^P(H_3) = \chi''(H_3) + 4 \). Which shows helm graph \( H_3 \) is \( P_4 \) kind graph.

4. Conclusion

The paper defines the perfect colouring of the graph and its kinds depending upon its relationship with the total colouring of the graph. Also investigated tight bounds of perfect colouring as \( \chi''(G) \leq \chi^P(G) \leq \chi''(G) + 4 \). We have proved the results for diamond graphs, friendship graphs, ladder graphs, prism graphs, etc.

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