LOGARITHMIC MEAN OSCILLATION ON THE POLYDISC, 
ENDPOINT RESULTS FOR MULTI-PARAMETER 
PARAPRODUCTS, AND COMMUTATORS ON BMO

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Abstract. We study boundedness properties of a class of multiparameter para-
products on the dual space of the dyadic Hardy space $H^d_T(T^N)$, the dyadic product 
BMO space $BMO^d(T^N)$. For this, we introduce a notion of logarithmic mean os-
cillation on the polydisc. We also obtain a result on the boundedness of iterated 
commutators on $BMO([0, 1]^N)$.

1 Introduction and notation

In recent years, multi-parameter paraproducts have generated much interest [1, 6, 9, 11], both in their own right, and as building blocks for other operators such as commutators and Hankel operators. In this paper, we characterise bound-
edness of dyadic paraproducts on the endpoint spaces $BMO^d(T^N)$ and $H^1_T(T^N)$. Here, the spaces $H^1(T^N)$ and $BMO(T^N)$ and their dyadic counterparts $BMO^d(T^N)$ and $H^1_d(T^N)$ on the polydisc are the product spaces in the sense of Chang and Fefferman [4]. Our main interest is in the paraproduct denoted below by $\Pi$ on the space $BMO^d(T^N)$. We prove a characterisation of boundedness in terms of a natural notion of logarithmic mean oscillation in the polydisc. We then apply the results on paraproducts to obtain a result on the boundedness of iterated commu-
tators with the Hilbert transforms on compactly supported functions in $BMO(\mathbb{R}^2)$. This is motivated by the classical one-parameter results in [8] on Hankel operators or, equivalently, commutators with the Hilbert transform on $BMO(\mathbb{T})$ and by the more recent results of Ferguson, Lacey and Terwilleger on iterated commutators with the Hilbert transforms on $L^2(\mathbb{R}^N)$; see [6, 10].

The notion of logarithmic mean oscillation was originally introduced in the one-parameter setting for the characterisation of multipliers of $BMO$ and Toeplitz

∗supported by a Heisenberg fellowship of the German Research Foundation (DFG)
†received funding from the “Irish Research Council for Science, Engineering and Technology”
operators on $H^1$ [17, 19]. The corresponding multiparameter results, which rely on our results here, are the subject of a forthcoming paper [14]. The adjoint paraproduct denoted by $\Pi^{(1,1)}$ below and continuous analogues have been considered on $\text{BMO}^d(\mathbb{T}^N)$ and $\text{BMO}(\mathbb{T}^N)$ before; see [1, 9]. We restrict most of our presentation to the two-dimensional case. As the general case follows in the same way, we just give the corresponding results.

The paper is organised as follows. In Section 2, we prove the main technical results on the paraproduct $\Pi^1$. In Section 3, we give conditions on the boundedness of the other paraproducts. The general $N$-parameter case is treated in Section 4. In Section 5, we first consider paraproducts of functions on $\mathbb{R}^N$ rather than $\mathbb{T}^N$. For this, local versions of the results of Sections 2 and 3 are required. These results are then used to prove boundedness estimates for commutators with the so-called dyadic shift on product $\text{BMO}^d$. These in turn lead to a result on the boundedness of iterated commutators with the Hilbert transforms on a suitable product $\text{BMO}$ space, by means of the decomposition of the Hilbert transform into dyadic shifts and new results on the relation between dyadic and continuous product $\text{BMO}$ spaces.

**Notation.** Let $\mathbb{T}$ denote the unit circle. We identify $\mathbb{T}$ with the interval $[0, 1)$ in the usual way and write $\mathcal{D}$ for the set of all dyadic subintervals. We denote by $\mathcal{R}$ the set of all dyadic rectangles $R = I \times J$, where $I, J \in \mathcal{D}$. Let $h_I$ denote the Haar wavelet adapted to the dyadic interval $I$, $h_I = |I|^{-1/2}(\chi_I - \chi_{I^c})$, where $I^c$ and $I^c$ are the right and left halves of $I$, respectively.

For any rectangle $R \in \mathcal{R}$, the product Haar wavelet adapted to $R = I \times J = h_I \otimes h_J$ is defined by $h_R(s, t) = h_I(s)h_J(t)$. These wavelets form an orthonormal basis of $L^2(\mathbb{T}^2) = \left\{ f \in L^2(\mathbb{T}^2) : \int_{\mathbb{T}} f(s, t)dt = 0, \int_{\mathbb{T}} f(s, t)ds = 0 \text{ for a.e. } s, t \in \mathbb{T} \right\}$, with

$$f = \sum_{R \in \mathcal{R}} \langle f, h_R \rangle h_R = \sum_{R \in \mathcal{R}} f_R h_R \quad (f \in L^2(\mathbb{T}^2)).$$

We write $m_R f$ for the mean of $f \in L^2(\mathbb{T}^2)$ over the dyadic rectangle $R = I \times J$ and $f_R = f_{IJ}$ for the Haar coefficient $\langle f, h_R \rangle = \langle f, h_I \otimes h_J \rangle$.

The space of functions of dyadic bounded mean oscillations in $\mathbb{T}^2$, $\text{BMO}^d(\mathbb{T}^2)$, is the space of all functions $b \in L^2(\mathbb{T}^2)$ such that

$$||b||_{\text{BMO}^d}^2 := \sup_{\Omega \subset \mathbb{T}^2} \frac{1}{|\Omega|} \sum_{R \in \Omega} |b_R|^2 = \sup_{\Omega \subset \mathbb{T}^2} \frac{1}{|\Omega|} ||P_\Omega b||_2^2 < \infty,$$
where the supremum is taken over all open sets $\Omega \subset \mathbb{T}^2$ and $P_\Omega$ is the orthogonal projection on the subspace spanned by Haar functions $h_R, R \in \mathbb{R}$ and $R \subset \Omega$.

It is well known (see, e.g., [3], [1]) that $\text{BMO}^d(\mathbb{T}^2)$ is the dual space of the dyadic product Hardy space $H^1_d(\mathbb{T}^2)$ defined in terms of the dyadic square functions $S$. This means that $H^1_d(\mathbb{T}^2) = \{ f \in L^1(\mathbb{T}^2) : S[f] \in L^1(\mathbb{T}^2) \}$, where

$$S[f] = \left( \sum_{R \in \mathbb{R}} \frac{\chi_R}{|R|} |f_R|^2 \right)^{1/2}.$$

For $I$ a dyadic interval and $\varepsilon \in \{0, 1\}$, we define $h_\varepsilon^I$ by

$$h_\varepsilon^I = \begin{cases} h_I & \text{if } \varepsilon = 0, \\ |I|^{-1/2}|h_I| & \text{if } \varepsilon = 1. \end{cases}$$

For $R = I \times J \in \mathbb{R}$ and $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2)$, with $\varepsilon_j \in \{0, 1\}$, we write $h_\vec{\varepsilon}^R = h_{\varepsilon_1}^I \otimes h_{\varepsilon_2}^J$.

We consider operators of the following general form:

$$B_{\vec{\varepsilon}, \vec{\delta}, \vec{\beta}}(\phi, f) := \sum_{R \in \mathbb{R}} \langle \phi, h_\vec{\varepsilon}^R \rangle \langle f, h_\vec{\delta}^R \rangle h_\vec{\beta}^R.$$  

(3)

They appear naturally in the study of many other operators in complex analysis and harmonic analysis. In this paper, we consider the paraproducts appearing as pieces of the usual product in the Haar expansion and corresponding to non-diagonal terms in this expansion. Some of the other operators of the form given in (3) on endpoint spaces appear in [14]. In other words, we consider here paraproducts $B_{\vec{\varepsilon}, \vec{\delta}, \vec{\beta}}(\phi, \cdot)$ with symbol $\phi$, corresponding to triples $(\vec{\varepsilon}, \vec{\delta}, \vec{\beta})$ with $\vec{\varepsilon} = (0, 0)$ and

$$\delta_j = \begin{cases} 1 & \text{if } \beta_j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, for simplicity, we can just denote the corresponding paraproducts by $\Pi^{\vec{\beta}}$. One easily sees that there are exactly four in dimension $N = 2$. We occasionally write $\vec{1} = (1, 1), \vec{0} = (0, 0)$.

As usual, for $\vec{j} = (j_1, j_2) \in \mathbb{N}_0 \times \mathbb{N}_0$ we define the $j_1$th generation of dyadic intervals and the $j_2$th generation of dyadic rectangles

$$\mathcal{D}_{j_1} = \{ I \in \mathcal{D} : |I| = 2^{-j_1} \},$$

$$\mathcal{R}_{j} = \mathcal{D}_{j_1} \times \mathcal{D}_{j_2} = \{ I \times J \in \mathcal{R} : |I| = 2^{-j_1}, |J| = 2^{-j_2} \},$$

the product Haar martingale difference $\Delta \vec{j}f = \sum_{R \in \mathcal{R}_{j}} \langle f, h_R \rangle h_R$, the expectations $E_{\vec{j}}f = \sum_{\vec{k} \in \mathbb{N}_0 \times \mathbb{N}_0, \vec{k} < \vec{j}} \Delta \vec{k}f$ (where $(k_1, k_2) = \vec{k}, \vec{j} = (j_1, j_2)$ for $k_1 < j_1, k_2 < j_2$.
and correspondingly \((k_1, k_2) = \bar{k} \leq \bar{j} = (j_1, j_2)\) for \(k_1 \leq j_1, k_2 \leq j_2\), \(E_i^{(1)}f = \sum_{k \in \mathbb{N}_0 \times \mathbb{N}_0, k_1 < i} \Delta_k f\), and \(E_j^{(2)}f = \sum_{k \in \mathbb{N}_0 \times \mathbb{N}_0, k < j} \Delta_k f\) for \(f \in L^2(\mathbb{T})\), \(\bar{j} \in \mathbb{N}_0 \times \mathbb{N}_0\).

We also require the operators on \(L^2(\mathbb{T}^2)\) given by

\[
Q_j f = \sum_{k \geq j} \Delta_k f
\]

\(Q_i^{(1)}f = \sum_{k \in \mathbb{N}_0 \times \mathbb{N}_0, k_i \geq i} \Delta_k f\), and \(Q_j^{(2)}f = \sum_{k \in \mathbb{N}_0 \times \mathbb{N}_0, k \geq j} \Delta_k f\). Note that contrary to the one-parameter situation, \(Q_{\bar{k}}\) is not the orthogonal complement of the expectation \(E_{\bar{k}}\). In fact, we have the relation

\[
f = E_{\bar{k}}f + E_{k_1}^{(1)}Q_{k_2}^{(2)}f + Q_{k_1}^{(1)}E_{k_2}^{(2)}f + Q_{\bar{k}}f \quad \text{for} \quad \bar{k} \in \mathbb{N}_0 \times \mathbb{N}_0, \quad f \in L^2(\mathbb{T}^2).
\]

Let \(\phi \in L^2(\mathbb{T}^2)\). The (main) paraproduct \(\Pi_\phi\) is defined by

\[
\Pi_\phi f = \Pi(\phi, f) := \sum_{j \in \mathbb{N}_0 \times \mathbb{N}_0} (\Delta_j f)(E_j f) = \sum_{R \in \mathcal{R}} h_R \phi_R m_R f
\]

on functions with finite Haar expansion. This is just the paraproduct \(\Pi^{(0,0)}\) introduced above.

We now define the space of functions of dyadic logarithmic mean oscillation on the bidisc, \(\text{LMO}^d(\mathbb{T}^2)\).

**Definition 1.1.** Let \(\phi \in L^2(\mathbb{T}^2)\). We say that \(\phi \in \text{LMO}^d(\mathbb{T}^2)\), if there exists \(C > 0\) with \(\|Q_j \phi\|_{\text{BMO}^d(\mathbb{T}^2)} \leq C/((j_1 + 1)(j_2 + 1))\) for all \(j = (j_1, j_2) \in \mathbb{N}_0 \times \mathbb{N}_0\).

The infimum of such constants is denoted by \(\|\phi\|_{\text{LMO}^d}\).

An alternative characterisation, which is closer in spirit to the one-parameter case, is the following.

**Proposition 1.2.** Let \(\phi \in L^2(\mathbb{T}^2)\). Then \(\phi \in \text{LMO}^d(\mathbb{T}^2)\) if and only if there exists \(C > 0\) such that for each dyadic rectangle \(R = I \times J\) and each open set \(\Omega \subseteq R\),

\[
(\log 4/|I|)^2(\log 4/|J|)^2 \sum_{Q \in \mathcal{R}, Q \subseteq \Omega} |\phi_Q|^2 \leq C.
\]

**Proof.** Let \(\phi \in \text{LMO}^d(\mathbb{T}^2)\) in the sense of Definition 1.1, \(R = I \times J\) be a dyadic rectangle with \(|I| = 2^{-j_1}, |J| = 2^{-j_2}\), and \(\Omega \subseteq R\) be open. Let \(j = (j_1, j_2)\). Then

\[
\sum_{Q \in \mathcal{R}, Q \subseteq \Omega} |\phi_Q|^2 = \|P_\Omega \phi\|_2^2 = \|P_\Omega Q_j \phi\|_2^2 \leq |\Omega| \|Q_j \phi\|_{\text{BMO}^d}^2 \leq |\Omega| \frac{1}{(j_1 + 1)^2(j_2 + 1)^2} \|\phi\|_{\text{LMO}^d}^2 \lesssim \left(\log \frac{4}{|I|}\right)^{-2} \left(\log \frac{4}{|J|}\right)^{-2} |\Omega| \|\phi\|_{\text{LMO}^d}^2.
\]
Conversely, suppose that $\phi \in L^2(T^2)$ and that (6) holds. Let $\vec{j} = (j_1, j_2) \in \mathbb{N}_0^2$, and let $\Omega \subseteq T^2$ open. Then

$$
\|P_{\Omega} Q_{\vec{j}} \phi\|_2^2 = \sum_{R \in \mathcal{R}_{\vec{j}}} \|P_{R \cap \Omega} Q_{\vec{j}} \phi\|_2^2 \lesssim C \frac{1}{(j_1 + 1)^2(j_2 + 1)^2} \sum_{R \in \mathcal{R}_{\vec{j}}} |R \cap \Omega| = C \frac{1}{(j_1 + 1)^2(j_2 + 1)^2} |\Omega|.
$$

This holds for all $\Omega \subseteq T^2$ open; hence $\|Q_{\vec{j}} \phi\|_{BMO^d} \lesssim 1((j_1 + 1)(j_2 + 1))$. □

\section{The main paraproduct}

The main result of this section is the following.

**Theorem 2.1.** Let $\phi \in L^2(T^2)$. Then $\phi \in \text{LMO}^d(T^2)$ if and only if $\Pi_{\phi} : \text{BMO}^d(T^2) \to \text{BMO}^d(T^2)$ is bounded. Moreover, $\|\Pi_{\phi}\|_{\text{BMO} \to \text{BMO}} \approx \|\phi\|_{\text{LMO}^d}$.

Before proving Theorem 2.1, we introduce some more notation. Given an integrable function $f$ on $T^2$ and intervals $I$ and $J$ in $T$, we write $m_I f = \int_I f(s, t) ds/|I|$, $m_J f = \int_J f(s, t) dt/|J|$, and $m_R f = \int_R f(s, t) dsdt/|R|$, $R = I \times J$. We remark that $m_I f$ is in fact a function of the second variable while $m_J f$ is a function in the first variable. We require the following lemma on the growth of averages and restrictions of functions in BMO.

**Lemma 2.2.** Let $b \in \text{BMO}^d(T^2)$, $\vec{k} = (k_1, k_2) \in \mathbb{N}_0 \times \mathbb{N}_0$. Then

$$
|m_R b| \lesssim (k_1 + 1)(k_2 + 1)\|b\|_{\text{BMO}^d(T^2)} \quad (R \in \mathcal{R}_{\vec{k}}),
$$

$$
\|m_I b\|_{\text{BMO}^d(T^2)} \lesssim (k_1 + 1)\|b\|_{\text{BMO}^d(T^2)} \quad (I \in \mathcal{D}_{\vec{k}}),
$$

$$
\|\chi_R b\|_2^2 \lesssim (k_1 + 1)^2(k_2 + 1)^2|R|\|b\|_{\text{BMO}^d(T^2)}^2 \quad (R \in \mathcal{R}_{\vec{k}}),
$$

$$
\|\chi_I P_J b\|_2^2 \lesssim (k_1 + 1)^2|I||J|\|b\|_{\text{BMO}^d(T^2)}^2 \quad (R \in \mathcal{R}_{\vec{k}});
$$

and this is sharp.
Proof. Let $R = I \times J$, $|I| = 2^{-k_1}$, $J = 2^{-k_2}$, $\vec{k} = (k_1, k_2)$, $j \in \mathbb{N}_0$. For the first inequality, observe that

$$\sup_{b \in \text{BMO}^d, \|b\|_{\text{BMO}^d} = 1} |m_R b| = \sup_{b \in \text{BMO}^d, \|b\|_{\text{BMO}^d} = 1} \left| \langle b, \frac{\chi_R}{|R|} \rangle \right| \leq \left\| \frac{\chi_R}{|R|} \right\|_{L^2(I^2)} = \left\| \frac{\chi_I}{|I|} \right\|_{L^2(I^2)} \left\| \frac{\chi_J}{|J|} \right\|_{L^2(J^2)} \lesssim \log \frac{4}{|I|} \log \frac{4}{|J|} \approx (k_1 + 1)(k_2 + 1),$$

where we have used $H^1_d(\mathbb{T}^2)$-BMO$^d(\mathbb{T}^2)$ duality in the first line and a known one-variable result in the last line.

For the second inequality, note that

$$\|m_I b\|_{\text{BMO}^d(\mathbb{T})} \approx \sup_{f \in H^2_d(\mathbb{T}), \|f\|_{H^4_d} \leq 1} \left| \int_T \int_T \left| \frac{1}{|I|} \chi_I(s) f(t) b(s,t) dsdt \right| \right| \leq \|b\|_{\text{BMO}^d(\mathbb{T}^2)} \sup_{f \in H^2_d(\mathbb{T}), \|f\|_{H^4_d} \leq 1} \left| \int_T \chi_I(s) f(t) \right|_{H^2_d(\mathbb{T}^2)} = \|b\|_{\text{BMO}^d(\mathbb{T}^2)} \left| \chi_I \right|_{H^2_d(\mathbb{T}^2)} \lesssim (k_1 + 1) \|b\|_{\text{BMO}^d(\mathbb{T}^2)}.$$

For the third inequality, write

$$\chi_R b(s,t) = P_R b(s,t) + \chi_R(s,t)m_I b(t) + \chi_R(s,t)m_J b(s) - \chi_R(s,t)m_R b.$$ Clearly $\|P_R b\|_{\text{BMO}^d} \leq |R| \|b\|_{\text{BMO}^d}$, and

$$\|\chi_R m_R b\|^2_{\text{BMO}^d} = |m_R b|^2 |R| \lesssim (k_1 + 1)^2 (k_2 + 1)^2 |R| \|b\|_{\text{BMO}^d}^2$$

by the first inequality in Lemma 2.2. The results for the remaining terms follow from the one-dimensional John-Nirenberg inequality, since

$$\|\chi_R(s,t)m_I b(t)\|_2 = |I|^{1/2} \|\chi_I(t)m_I b(t)\|_2 \lesssim |I|^{1/2} \|J|^{1/2} (k_2 + 1) \|m_I b(t)\|_{\text{BMO}^d(\mathbb{T})} \lesssim |I|^{1/2} \|J|^{1/2} (k_1 + 1)(k_2 + 1) \|b\|_{\text{BMO}^d(\mathbb{T}^2)},$$

by the second inequality.
For the last inequality, note that

$$\| \chi_I(s)m_I(P_Jb)\|_2^2 = |I|\|m_I(P_Jb)\|_2^2$$

$$\lesssim |I|J\|m_Ib(o)\|_{BMO(T)}$$

$$\lesssim |I|J(k_1 + 1)^2\|b\|_{BMO(T)}^2,$$

by the second inequality.

Hence

$$\| \chi_I(s)P_Jb\|_2 \leq \| \chi_I(s)m_I(P_Jb)\|_2 + \| P_{I \times 1}b\|_2$$

$$\leq |I|^{1/2}|J|^{1/2}(k_1 + 1)\|b\|_{BMO(T)} + |I|^{1/2}|J|^{1/2}\|b\|_{BMO(T)}$$

$$\lesssim |I|^{1/2}|J|^{1/2}(k_1 + 1)\|b\|_{BMO(T)},$$

by the second inequality.

Sharpness follows easily from the one-dimensional case, by forming an appropriate product of $BMO^d(T)$ functions in the two different variables.  

Next, for $k \in \mathbb{N}_0 \times \mathbb{N}_0$ and $b \in L^2(T^2)$, we consider the operator $\Pi_bE_k = \Pi(b, E_k \cdot)$ on $L^2(T^2)$, given by $\Pi_bE_kf = \Pi(b, E_kf)$, $f \in L^2(T^2)$.

**Lemma 2.3.** Let $b \in L^2(T^2)$ and let $k = (k_1, k_2) \in \mathbb{N}_0 \times \mathbb{N}_0$. Then

$$\| \Pi_bE_k\|_{L^2 \to L^2} = \| \Pi(\sigma_kb, \cdot)\|_{L^2 \to L^2},$$

where $\sigma_kb$ is given by

$$(\sigma_kb)_{I,J} = \begin{cases} b_{I,J} & \text{if } |I| > 2^{-k_1}, |J| > 2^{-k_2}, \\ \left( \sum_{J \subseteq I} |b_{I,J}|^2 \right)^{1/2} & \text{if } |I| > 2^{-k_1}, |J| = 2^{-k_2}, \\ \left( \sum_{I \subseteq J} |b_{I,J}|^2 \right)^{1/2} & \text{if } |I| = 2^{-k_1}, |J| > 2^{-k_2}, \\ \left( \sum_{I \subseteq J, J^c \subseteq I} |b_{I,J}|^2 \right)^{1/2} & \text{if } |I| = 2^{-k_1}, |J| = 2^{-k_2}, \\ 0 & \text{otherwise}. \end{cases}$$

In particular, $S^2[\sigma_kb] = E_kS^2[b]$, where $S^2[b]$ denotes the square of the dyadic square function in (2), and $\|\sigma_kb\|_2 = \|b\|_2$. 

Proof. Let $f \in L^2(\mathbb{T}^2)$. Then

$$\| \Pi_b E_{\bar{k}} f \|^2 = \left\| \sum_j (\Delta_j b) E_j E_{\bar{k}} f \right\|^2 = \sum_{j \geq \bar{k}} \| (\Delta_j b) E_{\bar{k}} f \|^2 + \sum_{j_1 \geq k_1, j_2 < k_2} \| (\Delta_j b) E_{(j_1, j_2)} f \|^2$$

$$+ \sum_{j_1 < k_1, j_2 \geq k_2} \| (\Delta_j b) E_{(j_1, k_2)} f \|^2 + \sum_{j \geq \bar{k}} \left\| \left( \sum_{j_1 \geq k_1} |\Delta_j b|^2 \right)^{1/2} E_{\bar{k}} f \right\|^2$$

$$+ \sum_{j_1 < k_1} \left\| \left( \sum_{j_2 \geq k_2} |\Delta_j b|^2 \right)^{1/2} E_{(j_1, k_2)} f \right\|^2 + \sum_{j < \bar{k}} \| (\Delta_j b) E_j f \|^2$$

$$= \left\| \sum_{j \leq \bar{k}} \Delta_j (\sigma_{\bar{k}} b) E_j f \right\|^2 = \| \Pi(\sigma_{\bar{k}} b, f) \|^2.$$

The remaining identities for $\sigma_{\bar{k}} b$ follow directly from the definition. \qed

Here is our main technical lemma.

Lemma 2.4. Let $\phi, b \in \text{BMO}^d(\mathbb{T}^2)$ and $\bar{k} = (k_1, k_2) \in \mathbb{N}_0 \times \mathbb{N}_0$. Then

$$\| \Pi \left( \Pi(\phi, b), E_{\bar{k}} \cdot \right) \|_{L^2 \to L^2} \lesssim (k_1 + 1)(k_2 + 1) \| \phi \|_{\text{BMO}^d} \| b \|_{\text{BMO}^d}.$$

Proof. By Lemma 2.3, we have to estimate the $\text{BMO}^d$ norm of $\sigma_{\bar{k}}(\Pi \phi b) = \sigma_{\bar{k}}(\Pi(\phi, b))$. Clearly,

$$\sigma_{\bar{k}}(\Pi \phi b) = \sigma_{\bar{k}}(E_{\bar{k}} \Pi \phi b) + \sigma_{\bar{k}}(E_{k_1}^{(1)} Q_{k_2}^{(2)} \pi \phi b) + \sigma_{\bar{k}}(E_{k_2}^{(2)} Q_{k_1}^{(1)} \Pi \phi b) + \sigma_{\bar{k}}(Q_{\bar{k}} \Pi \phi b)$$

$$= E_{\bar{k}} \Pi \phi b + \sigma_{\bar{k}}(E_{k_1}^{(1)} Q_{k_2}^{(2)} \Pi \phi b) + \sigma_{\bar{k}}(E_{k_2}^{(2)} Q_{k_1}^{(1)} \Pi \phi b) + \sigma_{\bar{k}}(Q_{\bar{k}} \Pi \phi b)$$

$$= I + II + III + IV.$$

(Compare this decomposition to the one in the definition of $\sigma_{\bar{k}}$ in Lemma 2.3).

We start with term I. For any open set $\Omega \subseteq \mathbb{T}^2$,

$$\frac{1}{|\Omega|} \left\| P_{\Omega} E_{\bar{k}} \Pi \phi b \right\|^2_{L^2} = \frac{1}{|\Omega|} \sum_{R \cap \Omega \neq \emptyset} \left| \phi_R \right|^2 |m_R b|^2$$

$$\lesssim \frac{(k_1 + 1)^2(k_2 + 1)^2}{|\Omega|} \sum_{R \cap \Omega \neq \emptyset} \left| \phi_R \right|^2 \| b \|^2_{\text{BMO}^d}$$

$$\lesssim (k_1 + 1)^2(k_2 + 1)^2 \| \phi \|^2_{\text{BMO}^d} \| b \|^2_{\text{BMO}^d}.$$
by Lemma 2.2.

For term II, note that since $\sigma_k(E_{k_1}^{(1)} Q_{k_2}^{(2)} \Pi \phi b)$ has only nontrivial Haar coefficients for those $R = I \times J$ with $|J| = 2^{-k_2}$ and $|I| > 2^{-k_1}$ (this corresponds to the second term in the definition of $\sigma_k$ in Lemma 2.3), it is sufficient to check the BMO norm on rectangles $R = I \times J$ with $|J| = 2^{-k_2}$ and $|I| > 2^{-k_1}$. Then

$$
\frac{1}{|R|} \left\| P_R \sigma_k(E_{k_1}^{(1)} Q_{k_2}^{(2)} \Pi \phi b) \right\|_2^2 = \frac{1}{|R|} \sum_{I' \subseteq I} \left| \left( \sigma_k(E_{k_1}^{(1)} Q_{k_2}^{(2)} \Pi \phi b) \right)_{I', J} \right|^2
$$

$$
= \frac{1}{|R|} \sum_{I' \subseteq I, J' \subseteq J, \substack{I' \times J' \in R \\substack{I' \times J' \in R}}} |\phi_{I' \times J'}|^2 |m_{I' \times J'} b|^2
$$

$$
\leq \frac{1}{|R|} \| \Pi \phi \chi_R b \|_2^2 \lesssim \| \phi \|_{BMO}^2 \frac{1}{|R|} \| \chi_R b \|_2^2
$$

$$
\lesssim (k_1 + 1)^2 (k_2 + 1)^2 \| \phi \|_{BMO}^2 \| b \|_{BMO}^2.
$$

Term III is dealt with analogously. For term IV, note that since $\sigma_k(Q_{k-}^j \Pi \phi b)$ has only nontrivial Haar coefficient for $R \in R_k$, it suffices to check the BMO norm on rectangles of this type. We obtain, for $R = I \times J \in R_k$,

$$
\frac{1}{|R|} \int_R |P_R \sigma_k(Q_{k-}^j \Pi \phi b)|^2 dsdt \leq \frac{1}{|I||J|} \sum_{I' \subseteq I, J' \subseteq J} |\phi_{I' \times J'}|^2 |m_{I' \times J'} b|^2
$$

$$
= \frac{1}{|R|} \| \Pi \phi \chi_R b \|_2^2
$$

$$
\lesssim \frac{1}{|R|} \| \phi \|_{BMO}^2 \| \chi_R b \|_2^2
$$

$$
\lesssim (k_1 + 1)^2 (k_2 + 1)^2 \| \phi \|_{BMO}^2 \| b \|_{BMO}^2
$$

by Lemma 2.2. □

In particular, we have the following.

**Lemma 2.5.** Let $\phi \in \text{LMO}^d(\mathbb{T}^2)$, $b \in \text{BMO}^d(\mathbb{T}^2)$ and $\vec{k}, \vec{j} \in \mathbb{N}_0 \times \mathbb{N}_0$. Then

$$
\| \Pi(\Pi(Q_{\vec{j}}^j \phi, b), E_{\vec{k}^-}) \|_{L^2 \rightarrow L^2} \lesssim \frac{|\vec{k} + \vec{j}|}{|\vec{j} + \vec{1}|} \| \phi \|_{\text{LMO}^d} \| b \|_{\text{BMO}^d},
$$

where $|\vec{k} + \vec{j}| = (k_1 + 1)(k_2 + 1)$ and $|\vec{j} + \vec{1}| = (j_1 + 1)(j_2 + 1)$.

**Proof.** Definition 1.1 and Lemma 2.4. □

**Proof of Theorem 2.1.** We begin by proving necessity. Suppose that the mapping $\Pi_{\phi} : \text{BMO}^d(\mathbb{T}^2) \rightarrow \text{BMO}^d(\mathbb{T}^2)$ is bounded. Let $R = I \times J$ be a dyadic rectangle, with $|I| = 2^{-k}$ and $|J| = 2^{-l}$, and let $\Omega \subseteq R$ be open. It is easy to
see that there exists a function $b \in \text{BMO}^d(\mathbb{T}^2)$ with $b|_R \equiv (k + 1)(l + 1)$ and $\|b\|_{\text{BMO}} \leq C$, where $C$ is a constant independent of $R$. Such a function can, for example, be found by forming the product $b_1 \otimes b_2$ of two one-variable functions $b_1, b_2$, which have the corresponding properties for the intervals $I$ and $J$, respectively. For details on the construction in the one-dimensional case, see [16]. Then

$$\frac{(\log 4/|I|)^2(\log 4/|J|)^2}{|\Omega|} \sum_{Q \in \mathcal{Q}, Q \subseteq \Omega} |\phi_Q|^2 \approx \frac{(k + 1)^2(l + 1)^2}{|\Omega|} \sum_{Q \in \mathcal{Q}, Q \subseteq \Omega} |\phi_Q|^2$$

$$= \frac{1}{|\Omega|} \sum_{Q \in \mathcal{Q}, Q \subseteq \Omega} |\phi_Q|^2 |m_Q|^2 \leq \|\Pi \phi b\|_{\text{BMO}}^2 \leq C^2 \|\Pi \phi\|_{\text{BMO}^d \to \text{BMO}^d}^2.$$ 

Thus $\phi \in \text{LMO}^d(\mathbb{T}^2)$ by Proposition 1.2, with the appropriate norm estimate.

To prove sufficiency of the $\text{LMO}^d$ condition for boundedness of the paraproduct on $\text{BMO}^d$, let $\phi \in \text{LMO}^d(\mathbb{T}^2)$ and $b \in \text{BMO}^d(\mathbb{T}^2)$. We estimate $\|\Pi \phi b\|_{\text{BMO}^d} \approx \|\Pi \Pi(\phi, b)\|_{L^2 \to L^2} = \|\Pi(\phi, b, \cdot)\|_{L^2 \to L^2}$ by means of Cotlar’s lemma and use Lemma 2.5 to control off-diagonal decay.

For $N, K \in \mathbb{N}_0$, let

$$P_{N,K} = \sum_{j_1=2^{N-1}}^{2^{N+1}-2} \sum_{j_2=2^K-1}^{2^{K+1}-2} \Delta j, \quad P^{N,K} = \sum_{j_1=2^{N-1}}^{\infty} \sum_{j_2=2^K-1}^{\infty} \Delta j,$$

and $T_{N,K} = \Pi \Pi(\phi, b)P_{N,K} = \Pi(\Pi(\phi, b), P_{N,K} \cdot )$. This means that we wish to estimate the $L^2 - L^2$ operator norm of $\Pi(\Pi(\phi, b), \cdot) = \sum_{N, K=0}^{\infty} T_{N,K}$. Clearly, $T_{N,K}T_{N',K'}^* = 0$ for $N \neq N'$ or $K \neq K'$. Therefore, we only have to estimate the norm of $T_{N,K}^*$ for $N, N', K, K' \in \mathbb{N}$. Letting $\overline{N} = \max\{N, N'\}$, $\underline{N} = \min\{N, N'\}$, $\overline{K} = \max\{K, K'\}$, $\underline{K} = \min\{K, K'\}$, and using the elementary identities

$$T_{N,K} = \sum_{j_1=2^{N-1}}^{\overline{N}} \sum_{j_2=2^{K-1}}^{\overline{K}} \Delta j,$$

and $T_{N,K}^* = \sum_{j_1=\underline{N}}^{2^{N+1}-2} \sum_{j_2=2^{K'}-1}^{2^{K'+1}-2} \Delta j$.
\[ \Pi(\Pi(\phi, b), \cdot)P_{N,K} = \Pi(\Pi(P^{N,K} \phi, b), \cdot)P_{N,K} = \Pi(P^{N,K} \Pi(\phi, b), \cdot)P_{N,K} = P^{N,K} \Pi(\Pi(\phi, b), \cdot)P_{N,K} \]

and \( P^{N,K} \Pi(\Pi(\phi, b), \cdot) = \Pi(\Pi(P^{N,K} \phi, b), \cdot) \), we obtain

\[
\|T_{N,K}^* T_{N',K'}\| = \|P_{N,K} (\Pi(\Pi(\phi, b), \cdot))^* \Pi(\Pi(\phi, b), P_{N',K'} \cdot)) \| \\
= |P_{N,K} \Pi(\Pi(\phi, b), \cdot)^* P^{N',K'} \Pi(\Pi(\phi, b), \cdot)P_{N',K'}\| \\
= |P_{N,K} \Pi(\Pi(\phi, b), \cdot)^* P^{N',K} \Pi(\Pi(\phi, b), \cdot)P_{N',K'}\| \\
\leq |P_{N,K} \Pi(\Pi(P^{N,K} \phi, b), \cdot)^* \Pi(\Pi(P^{N,K} \phi, b), \cdot)P_{N',K'}\| \\
\leq |\Pi(\Pi(P^{N,K} \phi, b), P_{N,K})\Pi(\Pi(P^{N,K} \phi, b), P_{N',K'})\| \\
\leq \frac{2^{N+1} \frac{2K+1}{2} \frac{2^{N'+1} 2K+1}{2^{N/2}} \|\phi\|_{LMO'}^2 \|b\|_{BMO'}^2}{2^{N-K+1} 2^{N' - K+1}} \\
\leq 2^{-|N-N'|} 2^{-|K-K'|} \|\phi\|_{LMO'}^2 \|b\|_{BMO'}^2 \\
\leq 2^{-|N-N'|} 2^{-|K-K'|} \|\phi\|_{LMO'}^2 \|b\|_{BMO'}^2. 
\]

by Lemma 2.5. In particular, the \( T_{N,K} \) are uniformly bounded in norm, and there exists a positive sequence \((a(i, j))_{i,j \geq 0}\) with \(\sum_{i,j=1}^{\infty} a(i, j)^{1/2} < \infty\) such that

\[
\|T_{N,K}^* T_{N',K'}\| \leq a(|N - N'|, |K - K'|) \|\phi\|_{LMO'}^2 \|b\|_{BMO'}^2.
\]

Thus, by Cotlar’s lemma, \( T = \Pi(\Pi(\phi, b), \cdot) \) is bounded, and there exists an absolute constant \( C > 0 \) with \( \|\Pi(\Pi(\phi, b), \cdot)\| \leq C \|\phi\|_{LMO'} \|b\|_{BMO'} \). Hence \( \|\Pi(\phi, b)\|_{BMO'} \leq C \|\phi\|_{LMO'} \|b\|_{BMO'} \). \( \square \)

In the previous theorem, sharp estimates of \( L^2 \) norms of restrictions of BMO functions to rectangles were required. We do not know such sharp estimates for restrictions of BMO functions to general open sets.

**Question 2.6.** By duality, \( |m_\Omega b| \lesssim \|\chi_\Omega/|\Omega|\|_{L^p_{1,1}(T^2)} |b|_L^{BMO'} \), \( b \in BMO^d(T^2) \) for all open sets \( \Omega \), and this is sharp for each individual set \( \Omega \). Is it true that “estimates for \( p \)-norms are no worse than the estimate for the average \( m_\Omega b \)”, i.e., \( \|\chi_\Omega b\|_p \lesssim \|\chi_\Omega/|\Omega|\|_{L^p_{1,1}(T^2)} |\Omega| \) for \( 1 < p < \infty \)?

The John-Nirenberg Theorem [3] gives \( \|\chi_\Omega b\|_p \lesssim (\log(4/|\Omega|))^{2p} |\Omega| \), which is easily seen not to be sharp for certain sets \( \Omega \) (for example, by considering “long thin” rectangles and using Lemma 2.2).
3 The other paraproductions

There are four dyadic paraproductions in two variables, namely the paraproduct \( \Pi^{(1,1)}(\phi, f) = \Delta \phi f = \Delta(\phi, f) = \sum_{R \in \mathbb{R}} \frac{\chi_R}{|R|} \phi_R f_R, \)
and the mixed paraproducts given by
\[
\Pi^{(1,0)}(\phi, f) = \sum_{I \times J \in \mathbb{R}} \frac{\chi_I(s)}{|I|} h_J(t) \phi_{I \times J} m_J f_I,
\]
\[
\Pi^{(0,1)}(\phi, f) = \sum_{I \times J \in \mathbb{R}} h_I(s) \frac{\chi_J(t)}{|J|} \phi_{I \times J} m_I f_J;
\]
see [1]. Here and in the following, the variables are sometimes included to explain dependency on the different variables rather than to indicate pointwise equality.

Interestingly, all four paraproductions have different boundedness behaviour on \( \text{BMO}^d(\mathbb{T}^2). \)

**Definition 3.1.** Let \( \phi \in L^2(\mathbb{T}^2). \) We say that \( \phi \in \text{LMO}^d(\mathbb{T}^2) \) if there exists \( C > 0 \) with \( \|Q_i^{(1)} \phi\|_{\text{BMO}^d} \leq C/(i+1) \) for all \( i \in \mathbb{N}_0. \) We say that \( \phi \in \text{LMO}^d(\mathbb{T}^2) \) if there exists \( C > 0 \) with \( \|Q_j^{(2)} \phi\|_{\text{BMO}^d} \leq C/(j+1) \) for all \( j \in \mathbb{N}_0. \) The infimum of such constants is denoted by \( \|\phi\|_{\text{LMO}^d}, \|\phi\|_{\text{LMO}^d}, \) respectively.

**Theorem 3.2.** Let \( \phi \in L^2(\mathbb{T}^2). \) Then the following hold.

1. \( \Delta \phi : \text{BMO}^d(\mathbb{T}^2) \to \text{BMO}^d(\mathbb{T}^2) \) is bounded if and only if \( \phi \in \text{BMO}^d. \)
   Moreover, \( \|\Delta \phi\|_{\text{BMO}^d} \to \text{BMO}^d \approx \|\phi\|_{\text{BMO}^d}. \)

2. \( \Pi^{(1,0)}(\phi, \cdot) : \text{BMO}^d(\mathbb{T}^2) \to \text{BMO}^d(\mathbb{T}^2) \) is bounded if \( \phi \in \text{LMO}^d(\mathbb{T}^2). \)
   Moreover, \( \Pi^{(1,0)}(\phi, \cdot) \|_{\text{BMO}^d(\mathbb{T}^2) \to \text{BMO}^d(\mathbb{T}^2)} \lesssim \|\phi\|_{\text{LMO}^d(\mathbb{T}^2)}. \)

3. \( \Pi^{(0,1)}(\phi, \cdot) : \text{BMO}^d(\mathbb{T}^2) \to \text{BMO}^d(\mathbb{T}^2) \) is bounded if \( \phi \in \text{LMO}^d(\mathbb{T}^2). \)
   Moreover, \( \Pi^{(0,1)}(\phi, \cdot) \|_{\text{BMO}^d(\mathbb{T}^2) \to \text{BMO}^d(\mathbb{T}^2)} \lesssim \|\phi\|_{\text{LMO}^d(\mathbb{T}^2)}. \)

**Proof.** Assertion (1) was proved in [1]. To show (2), we follow a simplified version of the ideas of the proof of Theorem 2.1.

**Lemma 3.3.** Let \( b \in L^2(\mathbb{T}^2) \) and let \( k \in \mathbb{N}. \) Then
\[
\|\Pi_b E_k^{(1)}\|_{L^2 \to L^2} = \|\Pi_{\sigma_k^{(1)} b}\|_{L^2 \to L^2},
\]
where
\[
(\sigma_k^{(1)} b)_{I, J} = \begin{cases} 
  b_{I, J} & \text{if } |I| > 2^{-k} \\
  \left( \sum_{I' \subseteq I} |b_{I', J}|^2 \right)^{1/2} & \text{if } |I| = 2^{-k} \\
  0 & \text{otherwise.}
\end{cases}
\]
Proof. As in Lemma 2.3. □

It remains to prove the following.

**Lemma 3.4.** Let $\phi, b \in \text{BMO}^d(T^2)$, $k \in \mathbb{N}$. Then

$$\left\| \Pi \left( \Pi^{(0,1)}(\phi, b), E^{(1)}_k \right) \right\|_{L^2 \to L^2} \lesssim (k + 1) \| \phi \|_{\text{BMO}^d} \| b \|_{\text{BMO}^d}.$$

**Proof.** We write $E$ for $E^{(1)}$, and $\sigma$ for $\sigma^{(1)}$. Following the results in Lemma 3.3, we estimate

$$\| \sigma_k(\Pi^{(0,1)}(\phi, b)) \|_{\text{BMO}^d} \leq \| \sigma_k(\Pi^{(0,1)}(E_k \phi, b)) \|_{\text{BMO}^d} + \| \sigma_k(\Pi^{(0,1)}(Q_k \phi, b)) \|_{\text{BMO}^d}.$$

We start with the second term. Since $\Pi^{(0,1)}(Q_k \phi, b)$ has no nontrivial Haar terms in the first variable for intervals $I$ with $|I| > 2^{-k}$,

$$\sigma_k(\Pi^{(0,1)}(Q_k \phi, b)) = \sum_{J \in \mathcal{D}} \sum_{|I| = 2^{-k}} h_I(s) \left( \sum_{I' \subseteq I} \left| \phi_{I'J} \right|^2 \| m_{I'} b_J \|_{L^2}^2 \right)^{1/2} \chi_J(t),$$

and this has only nontrivial Haar terms in the first variable for intervals $I$ with $|I| = 2^{-k}$. As before, the BMO$^d$ condition now only has to be considered on rectangles of the form $R = I \times J$, $|I| = 2^{-k}$. Thus

$$\| P_R \sigma_k(\Pi^{(0,1)}(Q_k \phi, b)) \|_2^2 = \| P_R \sigma_k(\Pi^{(0,1)}(P_R Q_k \phi, b)) \|_2^2 \leq \| \sigma_k(\Pi^{(0,1)}(P_R Q_k \phi, b)) \|_2^2 \leq \| \Pi^{(0,1)}(P_R Q_k \phi, b) \|_2^2 = \Pi^{(0,1)}(P_R \phi, \chi_I(s) P_J b) \|_2^2 \lesssim \| P_R \phi \|_{\text{BMO}^d} \| \chi_I P_J b \|_2^2 \lesssim (k + 1)^2 |R| \| \phi \|_{\text{BMO}^d}^2 \| b \|_{\text{BMO}^d}^2.$$
Now we have to deal with the first term $\| \sigma_k(\Pi^{(1,0)}(E_k \phi, b)) \|_{\text{BMO}}^d$. Let $\Omega \subseteq \mathbb{T}^2$ be open and write $\partial I = \bigcup_{J \in \mathcal{D}, I \times J \subseteq \Omega} J$ for $I \in \mathcal{D}$. Then

$$
\| P_{\Omega}(\sigma_k(\Pi^{(0,1)}(\phi, b))) \|_2^2 = \| P_{\Omega} \sigma_k(\Pi^{(0,1)}(P_{\Omega} E_k \phi, b)) \|_2^2 \\
\leq \| \sigma_k(\Pi^{(0,1)}(P_{\Omega} E_k \phi, b)) \|_2^2 = \| \Pi^{(0,1)}(P_{\Omega} E_k \phi, b) \|_2^2 \\
= \left( \sum_{I \in \mathcal{D}, |I| > 2^{-k}} \sum_{J \subseteq \mathcal{D}_I} h_I(s) \frac{1}{|I|}(I) \phi_{IJ} m_I b_J \right)^2 \\
= \sum_{I \in \mathcal{D}, |I| > 2^{-k}} \left( \sum_{J \subseteq \mathcal{D}_I} \frac{1}{|J|}(I) \phi_{IJ} m_I b_J \right)^2 \\
\leq \sum_{I \in \mathcal{D}, |I| > 2^{-k}} \| \Delta_{m_I b} P_{\partial I} \phi_I \|_2^2 \\
\lesssim \sum_{I \in \mathcal{D}, |I| > 2^{-k}} \| m_I b \|_{\text{BMO}}^d \| P_{\partial I} \phi_I \|_2^2 \\
\lesssim (k + 1)^2 \| b \|_{\text{BMO}}^d \sum_{I \in \mathcal{D}} \| P_{\partial I} \phi_I \|_2^2 \\
\lesssim (k + 1)^2 \| b \|_{\text{BMO}}^d \| P_{\Omega} \phi \|_2^2 \\
\lesssim (k + 1)^2 \| b \|_{\text{BMO}}^d \| \phi \|_{\text{BMO}}^d |\Omega|,
$$

by Lemma 2.2.

As in the last section, we immediately deduce

$$\| \Pi(\Pi^{(0,1)}(Q_j^{(1)} \phi, b), E_k^{(1)} \cdot) \|_{L^2 \rightarrow L^2} \lesssim \frac{k + 1}{j + 1} \| \phi \|_{\text{LMO}} \| b \|_{\text{BMO}}^d. \tag{8}$$

The remainder of the proof of (2) is now exactly analogous to the proof of Theorem 2.1, defining $T_N = \Pi(\Pi^{(0,1)}(\phi, \cdot), P_N \cdot)$, where $P_N = \sum_{i=2^{N+1}-1}^{2^{N+1} - 2} \Delta_i^{(1)}$, and using Cotlar’s lemma in one parameter. Finally, (3) follows by simply switching variables. \qed

4 Generalisation to more than two variables

The results of Sections 2 and 3 generalise easily to more than two variables. We only state the results here; the proofs are very similar to those in the previous sections.

Let $N \in \mathbb{N}$, $\mathcal{R} = \{ R = R_1 \times \cdots \times R_N \in \mathbb{T}^N : R_j \in \mathcal{D} \}$ the $N$-fold Cartesian product of the set of dyadic intervals $\mathcal{D}$, and let $(h_R)_{R \in \mathcal{R}}$ denote the corresponding product Haar basis of $L_0^2(\mathbb{T}^N)$. Recall that a function $b \in L^2(\mathbb{T}^N)$ is in the dyadic product BMO space $\text{BMO}^d(\mathbb{T}^N)$ if $\| b \|_{\text{BMO}^d}^2 = \sup_{\Omega \in \mathbb{T}^N \text{open}} \sum_{R \subseteq \Omega} |b_R|^2 / |\Omega| < \infty$. 
Definition 4.1. Suppose \( \phi \in L^2(\mathbb{T}^N) \), \( \vec{\delta} = (\delta_1, \ldots, \delta_N) \), \( \delta_j \in \{0, 1\} \). Then \( \phi \in \text{LMO}^d(\mathbb{T}^N) \) if and only if there exists \( C > 0 \) such that for each dyadic rectangle \( R = R_1 \times R_2 \times \cdots \times R_N \in \mathcal{D}^N \) and each open set \( \Omega \subseteq R \),

\[
\frac{(\log(4/|R_{\delta_1}|))^2 \cdots (\log(4/|R_{\delta_N}|))^2}{|\Omega|} \sum_{Q \in \mathcal{R}, Q \subseteq \Omega} |\phi_Q|^2 \leq C,
\]

where

\[
R_{\delta_j} = \begin{cases} R_j & \text{if } \delta_j = 0, \\ \mathbb{T} & \text{otherwise}. \end{cases}
\]

When \( \vec{\delta} = \vec{0} = (0, \ldots, 0) \), \( \text{LMO}^d(\mathbb{T}^N) \) corresponds to the generalisation of \( \text{LMO}^d(\mathbb{T}^2) \) and is denoted by \( \text{LMO}^d(\mathbb{T}^N) \). It is not difficult to see that for \( \vec{\delta} = (1, \ldots, 1) = \vec{1} \), the corresponding space is just the space \( \text{BMO}^d(\mathbb{T}^N) \).

As before, we consider paraproducts as defined in (3) for triples \( (\vec{\epsilon}, \vec{\delta}, \vec{\beta}) \) with \( \vec{\epsilon} = (0, \ldots, 0) \), and

\[
\delta_j = \begin{cases} 1 & \text{if } \beta_j = 0, \\ 0 & \text{otherwise}. \end{cases}
\]

For simplicity, we write

\[
(9) \quad \Pi_{\vec{\phi}}^{\vec{\beta}} f = B_{\vec{\epsilon}, \vec{\delta}, \vec{\beta}}(\phi, f) := \sum_{R \in \mathcal{R}} \phi_R \langle f, h_R^{\vec{\beta}} \rangle h_R^{\vec{\beta}}.
\]

As in the case \( N = 2 \), \( \Pi \) and \( \Delta \) are given by \( \Pi_{\vec{\beta}}^{\vec{\phi}} \) for \( \vec{\beta} = (0, \cdots, 0) \) and \( \vec{\beta} = (1, \cdots, 1) \), respectively.

Here is the result on the boundedness of \( \Pi_{\vec{\phi}}^{\vec{\beta}} \) on \( \text{BMO}^d(\mathbb{T}^N) \).

**Theorem 4.2.** Let \( \phi \in L^2_0(\mathbb{T}^N) \), \( \vec{\beta} = (\beta_1, \cdots, \beta_n) \), \( \beta_j \in \{0, 1\} \). Then

1. \( \Pi_{\vec{\phi}}^{(0, \ldots, 0)} = \Pi_{\vec{\phi}} : \text{BMO}^d(\mathbb{T}^N) \to \text{BMO}^d(\mathbb{T}^N) \) is bounded if and only if \( \phi \in \text{LMO}^d(\mathbb{T}^N) \). Moreover, \( \|\Pi_{\vec{\phi}}\|_{\text{BMO}^d \to \text{BMO}^d} \approx \|\phi\|_{\text{LMO}^d} \).
2. \( \Pi_{\vec{\phi}}^{(1, \ldots, 1)} = \Delta_{\vec{\phi}} : \text{BMO}^d(\mathbb{T}^N) \to \text{BMO}^d(\mathbb{T}^N) \) is bounded if and only if \( \phi \in \text{BMO}^d(\mathbb{T}^N) \). Moreover, \( \|\Delta_{\vec{\phi}}\|_{\text{BMO}^d \to \text{BMO}^d} \approx \|\phi\|_{\text{BMO}^d} \).
3. For \( \vec{\beta} \neq (0, \cdots, 0), (1, \cdots, 1) \), \( \Pi_{\vec{\phi}}^{\vec{\beta}} : \text{BMO}^d(\mathbb{T}^N) \to \text{BMO}^d(\mathbb{T}^N) \) is bounded if \( \phi \in \text{LMO}^d(\mathbb{T}^N) \). Moreover, \( \|\Pi_{\vec{\phi}}^{\vec{\beta}}\|_{\text{BMO}^d(\mathbb{T}^N) \to \text{BMO}^d(\mathbb{T}^N)} \lesssim \|\phi\|_{\text{LMO}^d} \).

5 Hankel operators, commutators and dyadic shifts

In this section, we are interested in applying the previous results to boundedness of iterated commutators with Hilbert transform on \( H^1(\mathbb{R}^N) \), i.e., the pre-dual of
BMO($\mathbb{R}^N$) as defined by A. Chang and R. Fefferman [3]. Again, for simplicity, we restrict our presentation to the two dimensional case, as the general case follows the same way.

We write $H_1$ and $H_2$ for the Hilbert transform in the first and second variable, respectively. Let us first recall the following result.

**Theorem 5.1** ([3, 5, 6]). Let $b \in \text{BMO}(\mathbb{R}^2)$. Then

$$[H_1, [H_2, \phi]] : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$$

is bounded, and $\| [H_1, [H_2, \phi]] \|_{L^2 \to L^2} \approx \| \phi \|_{\text{BMO}}$.

We would like to characterise boundedness of commutators on the endpoint spaces $H^1(\mathbb{R}^2)$ and BMO$(\mathbb{R}^2)$ in an analogous fashion in terms of a suitable notion of LMO$(\mathbb{R}^2)$. However, this is not possible even in one parameter; see, e.g., [8, Remark 4.1]. The reason is the slow decay of the kernel $1/x$ of the Hilbert transform. This means that for $\phi$ with compact support and an atom $f \in H^1(\mathbb{R})$, $\phi Hf$ is integrable, but $\phi f$ does not in general have average zero. Thus, $H(\phi f)$ behaves like $1/x$ at $\infty$ and is therefore not integrable at $\infty$. On the dual side, this amounts to saying that one should consider only functions with compact support in BMO$(\mathbb{R}^2)$.

In terms of our main result on paraproducts, Theorem 2.1, one sees easily that there is no good estimate for averages of BMO$^d(\mathbb{R})$ functions, and the theorem does not hold for BMO$^d(\mathbb{R}^2)$. However, we prove a local estimates for paraproducts and commutators. Our tools are adapted for the case of $\mathbb{R}^N$. It would be interesting to see the result in the case of the polydisc.

Let

\begin{equation}
\text{BMO}([0, 1]^2) := \{ f \in \text{BMO}(\mathbb{R}^2) : \text{supp} f \subseteq [0, 1]^2 \}.
\end{equation}

and BMO$^d([0, 1]^2) = \{ f \in \text{BMO}^d(\mathbb{R}^2) : \text{supp} f \subseteq [0, 1]^2 \}$. We say that $f$ is in LMO$^d([0, 1]^2)$ if $f \in \text{BMO}^d([0, 1]^2)$ and there exists $C > 0$ with

$$\| Q_{\vec{k}} f \|_{\text{BMO}^d(\mathbb{R}^2)} \leq C \frac{1}{(k_1 + 1)(k_2 + 1)} \quad \text{for} \; \vec{k} = (k_1, k_2) \in \mathbb{N}_0 \times \mathbb{N}_0.$$

The spaces LMO$^d_{\vec{\beta}}([0, 1]^2)$ are defined correspondingly. Here are our local estimates on paraproducts.

**Theorem 5.2.** Let $\phi \in L^2_0([0, 1]^2)$, $\vec{\beta} = (\beta_1, \beta_2), \beta_j \in \{0, 1\}$. Then

\begin{enumerate}
\item $\Pi^{(0,0)}_{\phi} = \Pi_{\phi} : \text{BMO}^d([0, 1]^2) \to \text{BMO}^d(\mathbb{R}^2)$ is bounded if and only if $\phi \in \text{LMO}^d([0, 1]^2)$. Moreover, $\| \Pi_{\phi} \|_{\text{BMO}^d([0, 1]^2) \to \text{BMO}^d(\mathbb{R}^2)} \approx \| \phi \|_{\text{LMO}^d([0, 1]^2)}$.
\end{enumerate}
(2) $\Pi^{(1,1)}_\phi = \Delta_\phi : \text{BMO}^d([0, 1]^2) \to \text{BMO}^d(\mathbb{R}^2)$ is bounded if and only if $\phi \in \text{BMO}^d([0, 1]^2)$. Moreover, $\|\Delta_\phi\|_{\text{BMO}^d([0, 1]^2) \to \text{BMO}^d(\mathbb{R}^2)} \approx \|\phi\|_{\text{BMO}^d([0, 1]^2)}$.

(3) For $\vec{\beta} \neq (0, 0), (1, 1)$, $\Pi^{\vec{\beta}}_\phi : \text{BMO}^d([0, 1]^2) \to \text{BMO}^d(\mathbb{R}^2)$ is bounded if $\phi \in \text{LOMO}^d([0, 1]^2)$. Moreover, $\|\Pi^{\vec{\beta}}_\phi\|_{\text{BMO}^d([0, 1]^2) \to \text{BMO}^d(\mathbb{R}^2)} \lesssim \|\phi\|_{\text{LOMO}^d([0, 1]^2)}^2$.

The theorem relies only on the appropriate version of Lemma 2.2 and a slight change of the decomposition of the identity in the proofs of Theorem 2.1 and 3.2.

**Lemma 5.3.** For a bounded (not necessarily dyadic) interval $I \subset \mathbb{R}$, let

$$s(I) = \begin{cases} \log |I|^{-1} + 1 & \text{for } |I| \leq 1, \\ 1 & \text{for } |I| > 1. \end{cases}$$

For a bounded (not necessarily dyadic) axis-parallel rectangle $R = I \times J \subset \mathbb{R}^2$, let $s(R) = s(I)s(J)$.

Then for each $b \in \text{BMO}([0, 1]^2)$ and each rectangle $R = I \times J \subset \mathbb{R}^2$,

1. $|m_{Rb}| \lesssim s(R)\|b\|_{\text{BMO}(\mathbb{R}^2)}$;
2. $\|m_{Tb}\|_{\text{BMO}(\mathbb{R})} \lesssim s(I)\|b\|_{\text{BMO}(\mathbb{R}^2)}$;
3. $\|\chi_R b\|_{L^2}^2 \lesssim s(R)^2\|b\|^2_{\text{BMO}(\mathbb{R}^2)}$;
4. $\|\chi_I P_J b\|_{L^2}^2 \lesssim s(I)^2 |I| |J| \|b\|^2_{\text{BMO}(\mathbb{R}^2)}$;
5. $\|\chi_I m_J b\|_{L^2}^2 \lesssim s(J)^2 |I| |J| \|b\|^2_{\text{BMO}(\mathbb{R}^2)}$;

and this is sharp. Here, $P_J b(s, t) = \chi_J(t)(b(s, t) - m_J b(s))$.

Before proving this lemma, let us first turn to the relation of $\text{BMO}([0, 1]^2)$, $\text{BMO}(\mathbb{R}^2)$, $\text{BMO}(\mathbb{T}^2)$, and $\text{BMO}^d(\mathbb{R}^2)$.

First let us consider the relation between $\text{BMO}(\mathbb{R}^2)$ and $\text{BMO}^d(\mathbb{R}^2)$. This was clarified only quite recently in [13, 18]. Given $\alpha = (\alpha_j)_{j \in \mathbb{Z}} \in \{0, 1\}^\mathbb{Z}$ and $r \in [1, 2)$, we denote by $\mathcal{D}^{\alpha, r} = r\mathcal{D}^{\alpha}$ the dilated and translated standard dyadic grid $\mathcal{D}$ of $\mathbb{R}$ in the sense of [7]. For $\vec{\alpha} = (\alpha^1, \alpha^2) \in \{0, 1\}^\mathbb{Z} \times \{0, 1\}^\mathbb{Z}$ and $\vec{r} = (r_1, r_2) \in [1, 2)^2$, we define $\mathcal{D}^{\vec{\alpha}, \vec{r}}$ to be the dilated and translated product dyadic grid in $\mathbb{R}^2$, i.e., $Q = Q_1 \times Q_2 \in \mathcal{D}^{\vec{\alpha}, \vec{r}}$ if $Q_1 \in r_1^{-1}\mathcal{D}^{\alpha^1}$ and $Q_2 \in r_2^{-1}\mathcal{D}^{\alpha^2}$.

The work in [13, 18] implies in particular that

$$\text{BMO}(\mathbb{R}^2) = \bigcap_{\vec{\alpha} \in \{0, 1\}^\mathbb{Z} \times \{0, 1\}^\mathbb{Z}, \vec{r} \in [1, 2)^2} \text{BMO}^{d, \vec{\alpha}, \vec{r}}(\mathbb{R}^2) = \bigcap_{\vec{\alpha} \in \{0, 1\}^\mathbb{Z} \times \{0, 1\}^\mathbb{Z}} \text{BMO}^{d, \vec{\alpha}, \vec{r}_0}(\mathbb{R}^2) \text{ for any } \vec{r}_0 \in [0, 1)^2,$$

where $\text{BMO}^{d, \vec{\alpha}, \vec{r}}(\mathbb{R}^2)$ is the dyadic $\text{BMO}(\mathbb{R}^2)$ defined with respect to the product dyadic grid $\mathcal{D}^{\vec{\alpha}, \vec{r}}$. One also has

$$\text{BMO}([0, 1]^2) = \bigcap_{\vec{\alpha} \in \{0, 1\}^\mathbb{Z} \times \{0, 1\}^\mathbb{Z}, \vec{r} \in [1, 2)^2} \text{BMO}^{d, \vec{\alpha}, \vec{r}}([0, 1]^2).$$
Now let us consider the relationship between $\text{BMO}([0, 1]^2)$ and $\text{BMO}(\mathbb{T}^2)$. It is easy to see that under the usual identification of $[0, 1)$ and $\mathbb{T}$, $\text{BMO}([0, 1]^2) \neq \text{BMO}(\mathbb{T}^2)$. A simple example is the function

$$b(s, t) = \log(\min(s, 1 - s)) \cdot \log(\min(t, 1 - t)) \quad (s, t) \in [0, 1]^2,$$

which is in $\text{BMO}(\mathbb{T}^2)$, but not in $\text{BMO}([0, 1]^2)$. On the other hand, for each $a < 0$, $b > 1$, we can extend $b \in \text{BMO}([0, 1]^2)$ to a doubly $(b - a)$-periodic function in $\text{BMO}(\mathbb{R}^2)$ by first considering it as a function on $[a, b]^2$ and then extending this function doubly periodically. Of course, the subspace of doubly 1-periodic functions in $\text{BMO}(\mathbb{R}^2)$ can be identified with $\text{BMO}(\mathbb{T}^2)$, and for any $b > a$, the subspace of doubly $b - a$ periodic functions in $\text{BMO}(\mathbb{R}^2)$ can be identified with $\text{BMO}(\mathbb{T}^2)$ by means of an appropriate dilation.

**Proof of Lemma 5.3.** The proof follows mostly from Lemma 2.2, but we have to attend to a few technicalities. Let $R = I \times J$ be a rectangle. We need only consider the case $I \times J \cap [0, 1]^2 \neq \emptyset$, that is, $I \cap [0, 1] \neq \emptyset$ and $J \cap [0, 1] \neq \emptyset$. If $|I|, |J| \leq 1$, then $R \cup [0, 1]^2 \subset [-1, 2]^2$. By first considering the functions in $\text{BMO}([0, 1]^2)$ as functions on $[-1, 2)^2$, then extending doubly periodically with period 3 and identifying the space of doubly periodic functions with period 3 in $\text{BMO}(\mathbb{R}^2)$ with $\text{BMO}(\mathbb{T}^2)$, we can apply Lemma 2.2 to obtain the desired estimates (1) - (5). Note that (5), which didn’t appear explicitly in Lemma 2.2, is a simple consequence of (2) and (3), applied in the one-dimensional case.

Then (1) is obtained in general by writing $I' = I \cap [0, 1]$, $J' = J \cap [0, 1]$, and observing that $|I'|s(I'), |J'|s(J') < 2$, which yields

$$|m_{I \times J}b| = \frac{|I'|}{|I|} \frac{|J'|}{|J|} m_{I' \times J'}b \leq \frac{|I'|}{|I|} \frac{|J'|}{|J|} s(I')s(J')\|b\|_{\text{BMO}}$$

$$\lesssim s(I)s(J)\|b\|_{\text{BMO}} \quad \text{for } |I|, |J| > 1$$

and

$$|m_{I \times J}b| = \frac{|I'|}{|I|} m_{I' \times J}b \leq \frac{|I'|}{|I|} s(I)s(J)\|b\|_{\text{BMO}}$$

$$\lesssim s(I)s(J)\|b\|_{\text{BMO}} \quad \text{for } |I| > 1 \text{ and } |J| \leq 1.$$

To obtain estimate (2) in case $|I| \leq 1$, we need to check boundedness of $\|P_J m_I b\|_2/|J|^{1/2}$ for arbitrary intervals $J$ with $J \cap [0, 1] \neq \emptyset$. For $|J| \leq 1$, we get the desired estimate as above. For $|J| > 1$, write $J' = J \cap [0, 1]$, $J'' = J \cap [0, 1]^c$
and obtain

\[ P_J m_I b(t) = \chi_J(t)(m_I b(t) - m_{I \times J} b) \]

\[ = \chi_J(t) \left( m_I b(t) - \frac{|J'|}{|J|} m_{I \times J} b \right) - \chi_{J''}(t) \frac{|J''|}{|J|} m_{I \times J} b \]

\[ = \chi_J(t)(m_I b(t) - m_{I \times J} b) + \chi_J(t) \frac{|J''|}{|J|} m_{I \times J} b \]

\[ - \chi_{J'}(t) \frac{|J'|}{|J|} m_{I \times J} b. \]

Thus

\[ \| P_J m_I b(t) \|_2 \leq \| \chi_J(t)(m_I b(t) - m_{I \times J} b) \|_2 \]

\[ + \| \chi_J(t) \frac{|J''|}{|J|} m_{I \times J} b \|_2 + \| \chi_{J'}(t) \frac{|J'|}{|J|} m_{I \times J} b \|_2. \]

The first summand is estimated by the previous argument for the case \( |J| \leq 1 \). For the second and third summand, we observe that by (1), \( |m_{I \times J} b| \lesssim s(I)s(J') \| b \|_{BMO} \); consequently, as \( |J'| \leq 1 \),

\[ \| \chi_J(t) \frac{|J''|}{|J|} m_{I \times J} b \|_2 \lesssim \frac{|J'|^{1/2}|J''|}{|J|} s(I)s(J') \| b \|_{BMO} \lesssim s(I) \| b \|_{BMO} \]

and

\[ \| \chi_{J'}(t) \frac{|J'|}{|J|} m_{I \times J} b \|_2 \lesssim \frac{|J''|^{1/2}|J'|}{|J|} s(I)s(J') \| b \|_{BMO} \lesssim s(I) \| b \|_{BMO}. \]

Now consider the case \( |I| > 1, I \cap [0, 1] \neq \emptyset \). Similarly to the above, let \( I' = I \cap [0, 1] \). Writing \( m_I b = ([I']/|I|) m_I b \), we obtain the same result.

It is useful for further estimates to prove (5) at this point. It is clear for \( |I|, |J| \leq 1 \); otherwise,

\[ \| \chi_I m_J b \|_2^2 = \frac{|J'|}{|J|^2} \| \chi_I m_J b \|_2^2 \lesssim \frac{|J'|}{|J|^2} s(J')^2 |I'| s(I) \| b \|_{BMO}^2 \]

\[ \lesssim s(J')^2 s(I) \| b \|_{BMO}^2. \]

For (3), write for \( R = I \times J \)

\[ \chi_R b = P_R b + \chi_{R m_I} b + \chi_{R m_J} b - \chi_{R m_I} b \]

and use (1) and (5).

For (4), write \( \chi_I(s) P_J b(s, t) = \chi_I(s) \chi_J(t) b(s, t) - \chi_I(s) m_J b(s) \) and use (3) and (5).

**Proof of Theorem 5.2.** We only prove assertion (1). The proof for the other paraproducts uses the same ideas combined with those in the proof of Theorem 3.2.
We want to prove that given $\phi \in \text{LMO}^d([0,1]^2)$, $b \in \text{BMO}^d([0,1]^2)$ and $f \in L^2(\mathbb{R}^2)$, the function $\Pi(\Pi(\phi, b), f)$ belongs to $L^2(\mathbb{R}^2)$, with the appropriate norm estimate.

We now work with the standard system $\mathcal{D}(\mathbb{R})$ of dyadic intervals in $\mathbb{R}$, the Haar basis $(h_I \otimes h_J)_{I,J \in \mathcal{D}(\mathbb{R})} = (h_R)_{R \in \mathcal{D}(\mathbb{R}) \times \mathcal{D}(\mathbb{R})}$ of $L^2(\mathbb{R}^2)$, and the decomposition

$$f = \sum_{j_1=-\infty}^{\infty} \sum_{j_2=-\infty}^{\infty} \Delta_j f,$$

where

$$\Delta_j f = \sum_{|I|=2^{-j_1}, |J|=2^{-j_2}} h_I(s) h_J(t) \langle f, h_I \otimes h_J \rangle = \sum_{R \in \mathcal{D}_{j_1}(\mathbb{R}) \times \mathcal{D}_{j_2}(\mathbb{R})} h_R \langle f, h_R \rangle$$

for $j_1, j_2 \in \mathbb{Z}$.

Then

$$T := \Pi(\Pi(\phi, b), \cdot) = P_{(0,1)^2} T + P_{(0,1) \times (0,1)^y} T + P_{(0,1)^x \times (0,1) T} + P_{(0,1)^x \times (0,1)^y} T,$$

where

$$P_{(0,1) \times (0,1)} = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \Delta_j,$$

$$P_{(0,1)^x \times (0,1)} = \sum_{j_1=-\infty}^{-1} \sum_{j_2=0}^{\infty} \Delta_j,$$

$$P_{(0,1)^x \times (0,1)^y} = \sum_{j_1=-\infty}^{-1} \sum_{j_2=-\infty}^{-1} \Delta_j,$$

$$P_{(0,1)^x \times (0,1)^y} = \sum_{j_1=0}^{\infty} \sum_{j_2=-\infty}^{-1} \Delta_j.$$

Hence we only need to check the $L^2$-boundedness of each of the four terms in the right hand side of the above identity. The estimate for $P_{(0,1)^2} \Pi(\Pi(\phi, b), f) = \Pi(\Pi(P_{(0,1)^2} \phi, b), f)$ for $f \in L^2(\mathbb{R}^2)$ is obtained exactly as in the proof of Theorem 2.1, with the help of the growth estimate in Lemma 5.3.

For the fourth term, we observe that with

$$P_{(0,1)^x \times (0,1)^y} \Pi(\Pi(\phi, b), \cdot) = \Pi \left( \Pi(P_{(0,1)^x \times (0,1)^y} \phi, b), \cdot \right),$$

we only have to check that for given $\phi \in \text{LMO}^d([0,1]^2)$ and $b \in \text{BMO}^d([0,1]^2)$, $P_{(0,1)^x \times (0,1)^y} \Pi(\phi, b)$ belongs to $\text{BMO}^d(\mathbb{R}^2)$. Using the fact that for $R = I \times J \in \mathcal{R}$ with $|I|, |J| \geq 1$, $|m_R b| \lesssim \|b\|_{\text{BMO}}$, one obtains directly that for any open set $\Omega \subset \mathbb{R}^2$, $\|P_\Omega P_{(0,1)^x \times (0,1)^y} \Pi(\phi, b)\|_2 \leq \|P_\Omega \phi\|_2 \|b\|_{\text{BMO}}^2$, which proves that this term is bounded on $L^2(\mathbb{R}^2)$.

As the second and third terms are symmetric, we only prove the boundedness of the second one. For this, we need to go back to the proof of Theorem 3.2. Again, we use the equality $P_{(0,1) \times (0,1)^y} \Pi(\Pi(\phi, b), \cdot) = \Pi \left( \Pi(P_{(0,1) \times (0,1)^y} \phi, b), \cdot \right)$.
Lemma 5.4. Let $\phi, b \in \text{BMO}^d(\mathbb{T}^2)$, $k \in \mathbb{N}_0$. Then

$$\|\Pi(\Pi(P_{(0,1)\times(0,1)^r}\phi, b), E_k^{(1)})\|_{L^2 \to L^2} \lesssim (k + 1)\|\phi\|_{\text{BMO}}\|b\|_{\text{BMO}}.$$

Proof. We follow the proof of Lemma 3.4. Again, we write $E$ for $E^{(1)}$ and $\sigma$ for $\sigma^{(1)}$. As in Lemma 3.3, we need to estimate

$$\|\sigma_k(\Pi(P_{(0,1)\times(0,1)^r}\phi, b))\|_{\text{BMO}} \leq \|\sigma_k(\Pi(P_{(0,1)\times(0,1)^r}E_k\phi, b))\|_{\text{BMO}}$$

$$+ \|\sigma_k(\Pi(P_{(0,1)\times(0,1)^r}Q_k\phi, b))\|_{\text{BMO}}$$

$$= \|\Pi(P_{(0,1)\times(0,1)^r}E_k\phi, b))\|_{\text{BMO}}$$

$$+ \|\sigma_k(\Pi(P_{(0,1)\times(0,1)^r}Q_k\phi, b))\|_{\text{BMO}}.$$

Starting with the first term, we obtain for any open set $\Omega \subset \mathbb{R}^2$,

$$\frac{1}{|\Omega|}\|P_\Omega \left(\Pi(P_{(0,1)\times(0,1)^r}E_k\phi, b)\right)\|_2^2 = \frac{1}{|\Omega|} \sum_{R = I \times J, |I| > 2^{-k}, |J| > 1, R \subset \Omega} |\phi_R|^2 |m_R b|^2$$

$$\lesssim \frac{(k + 1)^2}{|\Omega|} \sum_{R = I \times J, |I| > 2^{-k}, |J| > 1, R \subset \Omega} |\phi_R|^2$$

$$\lesssim (k + 1)^2 \|\phi\|_{\text{BMO}}^2 \|b\|_{\text{BMO}}^2,$$

where we have used Lemma 5.3.

For the second term, we observe that $\sigma_k \left(\Pi(P_{(0,1)\times(0,1)^r}Q_k\phi, b)\right)$ has only non-trivial coefficients for those rectangles $R = I \times J$ with $|I| = 2^{-k}$ and $|J| > 1$. Hence, it is enough to check the BMO-norm on rectangles $R = I \times J$ with $|I| = 2^{-k}$ and $|J| > 1$. We obtain

$$\frac{1}{|R|} \|P_R \sigma_k \left(\Pi(P_{(0,1)\times(0,1)^r}Q_k\phi, b)\right)\|_2^2 = \frac{1}{|R|} \sum_{J \subseteq J} \|\sigma_k \left(\Pi(P_{(0,1)\times(0,1)^r}Q_k\phi, b)\right)\|_2^2$$

$$= \frac{1}{|R|} \sum_{I \subseteq I} \sum_{J \subseteq J} |\phi_{I,J}|^2 |m_{I,J} b|^2$$

$$= \frac{1}{|R|} \|\Pi(P_R\phi, \chi_R b)\|_2^2$$

$$\lesssim \frac{1}{|R|} \|P_R\phi\|_{\text{BMO}} \|\chi_R b\|_2^2$$

$$\lesssim (k + 1)^2 \|\phi\|_{\text{BMO}}^2 \|b\|_{\text{BMO}}^2.$$

The remainder of the proof of boundedness of $P_{(0,1)\times(0,1)^r} \Pi(\Pi(\phi, b), \cdot)$ follows now with Cotlar’s lemma exactly as in the proof of Theorem 3.2. \qed
Before stating the main result of this section, we introduce the space \( \text{LMO}(0, 1]^2 \).

**Definition 5.5.** Let \( f \in L^2(\mathbb{R}^2) \). We say that \( f \in \text{LMO}([0, 1]^2) \) whenever \( \text{supp} f \subseteq [0, 1]^2 \) and there exists a constant \( C > 0 \) such that for any \( \vec{a} \in \mathbb{R} \times \mathbb{R}, \vec{r} \in [1, 2)^2 \) and \( \vec{j} = (j_1, j_2) \in \mathbb{N}_0 \times \mathbb{N}_0 \), \( \|Q_j^{\vec{a}, \vec{r}} f\|_{\text{BMO}^{\vec{a}, \vec{r}}([0, 1]^2)} \leq C/(j_1 + 1)(j_2 + 1) \).

Here, \( Q_j^{\vec{a}, \vec{r}} \) denotes the projection as in (4), but relative to the dyadic grid \( D_j^{\vec{a}, \vec{r}} \). More precisely,

\[
Q_j^{\vec{a}, \vec{r}} f(s, t) = \sum_{r_1 |I| \leq 2^{-j_1}, r_2 |J| \leq 2^{-j_2}} \langle f, h_I^{a_1, r_1} h_J^{a_2, r_2}\rangle h_I^{a_1, r_1}(s) h_J^{a_2, r_2}(t),
\]

where \( h_I^{a_1, r_1} \) is the Haar wavelet adapted to \( I \in r_I D_{a_I}, I = 1, 2 \).

Clearly, \( \text{LMO}([0, 1]^2) \) continuously embeds into \( \text{BMO}([0, 1]^2) \). Moreover, if we denote by \( \text{LMO}^{d, \vec{a}, \vec{r}}([0, 1]^2) \) the subset of \( \text{BMO}^{d, \vec{a}, \vec{r}}([0, 1]^2) \) of functions \( f \) such that there exists \( C > 0 \) with

\[
\|Q_j^{\vec{a}, \vec{r}} f\|_{\text{BMO}^{d, \vec{a}, \vec{r}}([0, 1]^2)} \leq C/(j_1 + 1)(j_2 + 1)
\]

for any \( \vec{j} \in \mathbb{N}_0 \times \mathbb{N}_0 \),

then of course

\[
\text{LMO}([0, 1]^2) = \bigcap_{\vec{a} \in \{0, 1\}^2 \times [0, 1]^2, \vec{r} \in [1, 2)^2} \text{LMO}^{d, \vec{a}, \vec{r}}([0, 1]^2).
\]

The spaces \( \text{LMO}_1([0, 1]^2) \) and \( \text{LMO}_2([0, 1]^2) \) along with their dyadic counterparts are defined analogously.

Here is the main result of this section.

**Theorem 5.6.** Let \( \phi \in \text{LMO}([0, 1]^2) \). Then

\[
[H_1, [H_2, \phi]] : \text{BMO}([0, 1]^2) \to \text{BMO}(\mathbb{R}^2),
\]

is bounded, and \( \|[H_1, [H_2, \phi]]\|_{\text{BMO}([0, 1]^2) \to \text{BMO}(\mathbb{R}^2)} \lesssim \|\phi\|_{\text{LMO}([0, 1]^2)} \).

To prove Theorem 5.6, we use the representation of the Hilbert transform as averages of dyadic shifts from \([12, 7]\). Let \( S : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) be the bounded linear operator defined by \( \text{Sh}_I = h_I - h_{I'}, I \in D \). Define \( S^{(1)} = S \otimes 1, S^{(2)} = 1 \otimes S \), as operators on \( L^2(\mathbb{R}^2) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \). For the averaging technique, we need to investigate the iterated commutator \( [S^{(1)}, [S^{(2)}, \phi]] \). We first prove the following dyadic analogue of the commutator theorem.

**Theorem 5.7.** Let \( \phi \in \text{LMO}^d([0, 1]^2) \). Then

\[
[S^{(1)}, [S^{(2)}, \phi]] : \text{BMO}^d([0, 1]^2) \to \text{BMO}^d(\mathbb{R}^2)
\]

is bounded, and \( \|[S^{(1)}, [S^{(2)}, \phi]]\|_{\text{BMO}^d([0, 1]^2) \to \text{BMO}^d(\mathbb{R}^2)} \lesssim \|\phi\|_{\text{LMO}^d([0, 1]^2)}. \)
Proof. We formally decompose the multiplication operator with $\phi$ into nine parts: $\Pi_\phi$, $\Delta_\phi$, $\Pi^{(1)}$, $\Pi^{(2)}$, $R_{\Delta_\phi}$, $R_{\Pi_\phi}$, $\Delta_{R_\phi}$, $R_{R_\phi}$, corresponding to the matrix elements $\langle M_\phi h_I(t), h_I'(t) \rangle$ for $I' \subset I$, $I' = I$, $I' \subset I$, $I' \supset I$, $J' \subset J$, $J' = J$, $J' \supset J$. Notice that the operator $R$ denotes the Haar-diagonal part of the multiplication operator.

It is easy to see that $S^{(1)}$ and $S^{(2)}$ are bounded on $BMO^d(\mathbb{R}^2)$. Thus, after considering Theorems 2.1, and 3.2 and symmetry of variables, we are left to consider $[S^{(1)}, [S^{(2)}, R_{R_\phi}]]$, $[S^{(1)}, [S^{(2)}, \Pi_\phi]]$, $[S^{(1)}, [S^{(2)}, \Pi_{R_\phi}]]$, $[S^{(1)}, [S^{(2)}, \Delta_{R_\phi}]]$. We recall that

$$R_{R_\phi} b(s, t) = \sum_{l, j} b_{l, j} m_{l, j}(\phi) h_I(s) h_J(t),$$

$$\Pi_{R_\phi} b(s, t) = \sum_{l, j} m_{l, j}(\phi) m_{l, j}(b_J) h_I(s) h_J(t),$$

$$\Delta_{R_\phi} b(s, t) = \sum_{l, j} m_{l, j}(\phi) b_{l, j} h_I(s) h_J(t).$$

We start with $[S^{(1)}, [S^{(2)}, R_{R_\phi}]]$. One verifies that

$$[S^{(1)}, [S^{(2)}, R_{R_\phi}]] h_{I, J} = (m_{l, j} \phi - m_{l, j} \phi - m_{l, j} \phi + m_{l, j} \phi) h_{I, J} + (m_{l, j} \phi - m_{l, j} \phi - m_{l, j} \phi + m_{l, j} \phi) h_{I, J} + (m_{l, j} \phi - m_{l, j} \phi - m_{l, j} \phi + m_{l, j} \phi) h_{I, J} + (m_{l, j} \phi - m_{l, j} \phi - m_{l, j} \phi + m_{l, j} \phi) h_{I, J}.$$

Thus $[S^{(1)}, [S^{(2)}, R_{R_\phi}]]$ preserves the orthogonality of the Haar system $(h_{I, J})_{I, J \in \mathcal{D}}$. Letting $\Phi = E_{k+1, l+1} \phi$ for $|I| = 2^{-k}$, $|J| = 2^{-l}$, we find that

$$\|[S^{(1)}, [S^{(2)}, R_{R_\phi}]] h_{I, J} \|^2 = \frac{1}{|I||J|} \|\Phi(s, t) - m_{l, j} \Phi(t) - m_{l, j} \Phi(s) + m_{l, j} \Phi \|^2 < \frac{1}{|I||J|} \|P_{l \times J} \Phi \|^2 \leq \|\Phi\|^2_{BMO^d_{\text{rect}}}.$$

Hence for $b = \sum_{I, J \in \mathcal{D}} h_{I, J} b_{I, J} \in BMO^d(\mathbb{R}^2)$, $\Omega \subset \mathbb{R}^2$ open,

$$\|P_\Omega [S^{(1)}, [S^{(2)}, R_{R_\phi}]] b \|^2 \leq \|[S^{(1)}, [S^{(2)}, R_{R_\phi}]] P_\Omega b \|^2 \leq \sum_{l \times J \in \Omega} \|\Phi\|^2_{BMO^d_{\text{rect}}} \|b_{I, J}\|^2 \leq \|\Phi\|^2_{BMO^d_{\text{rect}}} \|b\|^2_{BMO^d(\mathbb{R}^2)} \Omega | \leq 4 \|\Phi\|^2_{BMO^d_{\text{rect}}} \|b\|^2_{BMO^d(\mathbb{R}^2)} \Omega |,$$

where $\Omega = \bigcup_{I, J \in \mathcal{D}, l \times J \in \Omega} I \times J$ and $I$, $J$ are the parents of $I$, $J$. Thus

$$\|[S^{(1)}, [S^{(2)}, R_{R_\phi}]]\|_{BMO^d(\mathbb{R}^2) \to BMO^d(\mathbb{R}^2)} \leq 2 \|\Phi\|^2_{BMO^d_{\text{rect}}(\mathbb{R}^2)^{0, 1^2}}.$$
Here, $\text{BMO}^d_{\text{rect}}(\mathbb{R}^2)$ is the dyadic rectangular BMO space which continuously contains $\text{BMO}^d(\mathbb{R}^2)$ and consists of function $f \in L^2_0(\mathbb{R}^2)$ such that

$$\sup_{I \times J \in \mathcal{D}(\mathbb{R}^2)} \|P_{I \times J} f\| = \sup_{I \times J \in \mathcal{D}(\mathbb{R}^2)} \frac{1}{|I||J|} \|f(s, t) - m_{I, J} f(t) - m_{J, I} f(s) + m_{I \times J} f\|^2 < \infty.$$ 

To establish the boundedness of $[S^{(1)}, [S^{(2)}, \Pi_{R\phi}]]$ and $[S^{(1)}, [S^{(2)}, \Delta_{R\phi}]]$ from $\text{BMO}^d([0, 1]^2)$ to $\text{BMO}^d(\mathbb{R}^2)$, we remark that since $S^{(1)}$ is bounded on $\text{BMO}^d(\mathbb{R}^2)$, we only need to establish that $[S^{(2)}, \Delta_{R\phi}]$ and $[S^{(2)}, \Pi_{R\phi}]$ are bounded from $\text{BMO}^d([0, 1]^2)$ to $\text{BMO}^d(\mathbb{R}^2)$.

Straightforward computations give, for $b \in \text{BMO}^d([0, 1]^2)$,

$$[S^{(2)}, \Delta_{R\phi}](b)(s, t) = \sum_{I \times J} b_{IJ} \phi_{IJ} h^2_I(s) \frac{h_{J^+}(t) - h_{J^-}(t)}{|J|^{1/2}} = \frac{1}{2\sqrt{2}} \sum_{I \times J} b_{IJ} \phi_{IJ} h^2_I(s) \left( \frac{\chi_{J^+}(t)}{|J^+|} - \frac{\chi_{J-}(t)}{|J-|} - \frac{\chi_{J^+}(t)}{|J^+|} + \frac{\chi_{J^+}(t)}{|J^+|} \right)$$

where

$$\tilde{b}(s, t) = \sum_{I \times J} b_{IJ} h_I(s) (h_{J^+}(t) - h_{J^+}(t) - h_{J^+}(t) + h_{J^-}(t)),$$

$$\tilde{\phi}(s, t) = \sum_{I \times J} \phi_{IJ} h_I(s) (h_{J^+}(t) - h_{J^-}(t) - h_{J^+}(t) + h_{J^+}(t)).$$

In the same way, we obtain

$$[S^{(2)}, \Pi_{R\phi}](b)(s, t) = \sum_{I \times J} \phi_{IJ} m_I(b_J) h_I(s) \frac{h_{J^+}(t) - h_{J^-}(t)}{|J|^{1/2}} = \frac{1}{2\sqrt{2}} \sum_{I \times J} \phi_{IJ} m_I(b_J) h_I(s) \sum_{I \times J} \left( \frac{\chi_{J^+}(t)}{|J^+|} - \frac{\chi_{J^+}(t)}{|J^+|} - \frac{\chi_{J^-}(t)}{|J^+|} + \frac{\chi_{J^+}(t)}{|J^+|} \right)$$

Since $\|\tilde{b}\|_{\text{BMO}^d([0, 1]^2)} \lesssim \|b\|_{\text{BMO}^d([0, 1]^2)}$ and $\|\tilde{\phi}\|_{\text{LMO}^d([0, 1]^2)} \lesssim \|\phi\|_{\text{LMO}^d([0, 1]^2)}$, we obtain

$$\|[[S^{(1)}, [S^{(2)}, \Delta_{R\phi}]]\|_{\text{BMO}^d([0, 1]^2)} \lesssim \|\phi\|_{\text{BMO}^d([0, 1]^2)},$$

$$\|[[S^{(1)}, [S^{(2)}, \Pi_{R\phi}]]\|_{\text{BMO}^d([0, 1]^2)} \lesssim \|\phi\|_{\text{LMO}^d([0, 1]^2)}.$$
by Theorem 3.2. Swapping variables yields

\[ \|S(\phi), [S, R]_{\alpha, \phi} \|_{\text{BMO}^d([0,1]^2) \to \text{BMO}^d(\mathbb{R}^2)} \lesssim \|\phi\|_{\text{BMO}^d([0,1]^2)}, \]

(15) \[ \|S(\phi), [S, R]_{\alpha, \phi} \|_{\text{BMO}^d([0,1]^2) \to \text{BMO}^d(\mathbb{R}^2)} \lesssim \|\phi\|_{\text{LMO}^d([0,1]^2)} . \]

This finishes the proof of Theorem 5.7. \[
\]

To finish the proof of Theorem 5.6, we need to consider the relation between BMO(\mathbb{R}^N) and BMO^d(\mathbb{R}^N) established in [13, 18].

Let us momentarily return to the one-variable setting and recall the following result which simplifies the one in [12].

**Theorem 5.8** (Theorem 1.1 of [7]). For \( r \in [1, 2) \) and \( \beta \in \{0, 1\}^Z \), let \( \beta_{\text{dyadic}} \) be the dyadic shift associated to the dyadic system \( r^\beta_{\text{dyadic}} \). Let \( \mu \) stand for the canonical probability measure on \( \{0, 1\}^Z \) which makes the coordinate functions independent with \( \mu(\beta_j = 0) = \mu(\beta_j = 1) = 1/2 \). Then for all \( p \in (1, \infty) \) and \( f \in L^p(\mathbb{R}) \),

(17) \[ Hf(x) = -\frac{8}{\pi} \int_1^2 \int_{\{0,1\}^Z} S_{\beta, r} f(x) d\mu(\beta) \frac{dr}{r}, \]

where the integral converges both pointwise for a.e. \( x \in \mathbb{R} \) and also in the sense of an \( L^p(\mathbb{R}) \)-valued Bochner integral.

**Proof of Theorem 5.6.** Returning to the two-variable setting, we obtain for \( b \in \text{BMO}([0,1]^2), \phi \in \text{LMO}([0,1]^2) \),

\[ [H_1, [H_2, \phi]]b = \frac{64}{\pi^2} \int_1^2 \int_{\{0,1\}^Z} \int_{\{0,1\}^Z} [S^{\alpha_1, r_1}, [S^{\alpha_2, r_2}, \phi]] b d\mu(\alpha_1) \frac{dr_1}{r_1} d\mu(\alpha_2) \frac{dr_2}{r_2} . \]

Now since \( b \in \text{BMO}([0,1]^2), \phi \in \text{LMO}([0,1]^2) \), we have that \( b \in \text{BMO}^{d, \tilde{a}, \tilde{r}}(\mathbb{R}^2) \), and \( \phi \in \text{LMO}^{d, \tilde{a}, \tilde{r}}(\mathbb{R}^2) \) for each \( \tilde{a} = (\alpha_1, \alpha_2) \in \{0,1\}^2 \times \{0,1\}^Z \) and \( \tilde{r} = (r_1, r_2) \in [1, 2)^2 \) with uniformly bounded norm (see, e.g., [6]). Thus, there exists a constant \( C > 0 \) such that

\[ \|[S^{\alpha_1, r_1}, [S^{\alpha_2, r_2}, \phi]]b\|_{\text{BMO}^{d, \tilde{a}, \tilde{r}}} \leq C \|b\|_{\text{BMO}} \|\phi\|_{\text{LMO}} \text{ for all } (\alpha_1, \alpha_2, r_1, r_2) \]

by Theorem 5.7. By [18, Remark 0.5], (see also [13]), it follows that

\[ \frac{64}{\pi^2} \int_1^2 \int_{\{0,1\}^Z} \int_{\{0,1\}^Z} [S^{\alpha_1, r_1}, [S^{\alpha_2, r_2}, \phi]] b d\mu(\alpha_1) \frac{dr_1}{r_1} d\mu(\alpha_2) \frac{dr_2}{r_2} \in \text{BMO}(\mathbb{R}^2) \]

with norm controlled by \( \|b\|_{\text{BMO}} \|\phi\|_{\text{LMO}} \). The proof is complete. \[
\]

**Remark.** At this point, it is natural to ask whether the converse of Theorem 5.6 holds, that is, whether we have an analogue of Theorem 5.1 in this setting. In an appropriate sense, this is the case, but the proof is involved and will be the subject of a forthcoming paper.
5.1 Acknowledgements. It is a pleasure to thank Aline Bonami for many very useful discussions and comments. Professor Bonami’s visit at the University of Glasgow was made possible through a grant of the Edinburgh Mathematical Society. The authors also gratefully acknowledge support of the Fields Institute of Mathematical Sciences and the “Thematic Program on New Trends in Harmonic Analysis”, where part of this work was completed.

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(Received September 4, 2008 and in revised form December 20, 2011)