Oscillations and convective motion in stars with URCA shells

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Abstract

It is shown that in presence of URCA shell pulsational energy losses due to neutrino emission and nonequilibrium beta heating are much less than energy losses by excitation of short-wavelength acoustic waves. Convective motion in presence of URCA shell is considered, and equations generalizing the mean free path model of convection are derived.

2 Introduction

It was suggested in [15] that in the convective region cooling of matter may be enhanced in presence URCA shells appearing when matter contains an isotope with a threshold Fermi energy for an electron capture, corresponding to a density less then the central one. Presence of such isotope leads to existence of a jump in the composition at a density, corresponding to a threshold energy. During convective motion the matter in eddies around this density crosses periodically the boundary. That implies continuous beta capture and beta decay in the matter of these eddies.

Because of heating of a degenerate matter due to nonequilibrium beta processes [4], with account of convective URCA shell different conclusions had been done with respect to stabilizing or destabilizing the carbon burning in the convective degenerate core [16, 14, 13, 11, 12, 2, 14, 17]. Here we calculate damping of stellar oscillations in presence of URCA shell (see [1]), analyze physical processes in the convective URCA shells and formulate approximate quantitative approach to the solution of this problem.

3 Linear oscillations of a slab with a phase transition

Consider the plane-parallel layer in the constant gravitational field with an acceleration $g$, with a phase transition at the pressure $P_*$, and polytropic equation of state $P = K \rho^\gamma$, with $K = K_1$ at $P < P_*$ and $K = K_2$ at $P > P_*$, $K_1 > K_2$. In static equilibrium $P = gM \left(1 - \frac{m}{M}\right)$, $\rho = \left(\frac{gM}{K_2}\right)^{1/\gamma} \left(1 - \frac{m}{M}\right)^{1/\gamma}$. Here the pressure is continuous, but the density $\rho$ has a jump at $P_*$ due to the jump of the constant $K$, $M$ is the mass of one cm$^2$ of the slab, $m$ is the mass of one cm$^2$ of the slab under the layer with a Lagrangian coordinate $x$, which is related to the density as $\rho = \left(\frac{gM}{K_2}\right)^{1/\gamma} \left(C - gx\right)^{1/\gamma}$. In presence of a phase transition we have $\rho_0 = \left(\frac{gM}{K_2}\right)^{1/\gamma}$, $P_0 = gM$, $gx_0 = C_2 - \frac{x_0}{\gamma - 1} P_0$, $x_0 = C_1^0$, $C_2 = \frac{x_0}{\gamma - 1} (gM)^{\gamma - 1} K_2^{1/\gamma}$, $C_1 = C_2 + \frac{x_0}{\gamma - 1} P_0 (\lambda - 1)$. Here $\rho_0$ and $P_0$ are the density and the pressure at the bottom of the slab, $x_0$ and $x_*$ are
Here tend to infinity at the total thickness and thickness of the inner denser phase layer of the slab, $\lambda = \rho_2/\rho_1$. The phase transition in the slab happens only if its specific mass $M > P_*/g = M_*$. Linear oscillations of the slab are reduced to Bessel equation for perturbations $\tilde{P}, \tilde{v} = \tilde{P}_a, \tilde{v}_a \exp(-i\omega t)$ with solutions

$$\tilde{P}_a = A\sqrt{\gamma} J_{1/\gamma} (\eta) + B\sqrt{\gamma} Y_{1/\gamma} (\eta), \quad (1)$$

$$\tilde{v}_a = \frac{i}{\sqrt{\gamma P_0 \rho_0}} \eta^{1/2} \left[ AJ_{1/\gamma} (\eta) + BY_{1/\gamma} (\eta) \right], \quad (2)$$

where for two phases $\eta_2 = \frac{2\gamma}{\gamma - 1} \frac{M_1 \rho_0}{\sqrt{\gamma^2 - 1}} \gamma_1, \eta_1 = \eta_2 \sqrt{\lambda}, \ z_* = 1 - \frac{m_*}{M_*}$. The frequency of oscillations $\omega$ and relations between constants $A_1, B_1, A_2, B_2$ are obtained from boundary conditions and from relations on the phase jump [3, 5, 10]. The dispersion equation are obtained analytically for a frozen

$$J_{1/\gamma} (\Omega \sqrt{\gamma} z_*) J_{1/\gamma} (\Omega \gamma) - J_{1/\gamma} (\Omega \gamma) J_{1/\gamma} (\Omega \sqrt{\gamma} z_*) = 0, \quad (3)$$

$$-\sqrt{\lambda} J_{1/\gamma} \left( \Omega \sqrt{\gamma} z_*^2 \right) \left[ J_{1/\gamma} (\Omega \gamma_1) Y_{1/\gamma} (\Omega \gamma_1) - J_{1/\gamma} (\Omega \gamma_1) Y_{1/\gamma} (\Omega \gamma_1) \right]$$

$$- \left[ \sqrt{\lambda} J_{1/\gamma} (\Omega \gamma_1) \left( \Omega \sqrt{\gamma} z_*^2 \right) - (\lambda - 1) \frac{\gamma - 1}{2} \Omega z_*^2 J_{1/\gamma} (\Omega \sqrt{\gamma} z_*^2) \right]$$

$$\times \left[ J_{1/\gamma} (\Omega \gamma_1) Y_{1/\gamma} (\Omega \gamma_1) - J_{1/\gamma} (\Omega \gamma_1) Y_{1/\gamma} (\Omega \gamma_1) \right] = 0, \quad (4)$$

and equilibrium phase transition

$$J_{1/\gamma} (\Omega \sqrt{\gamma} z_*^2) \left[ J_{1/\gamma} (\Omega \gamma_1) Y_{1/\gamma} (\Omega \gamma_1) - J_{1/\gamma} (\Omega \gamma_1) Y_{1/\gamma} (\Omega \gamma_1) \right]$$

$$- \left[ \sqrt{\lambda} J_{1/\gamma} (\Omega \sqrt{\gamma} z_*^2) - (\lambda - 1) \frac{\gamma - 1}{2} \Omega z_*^2 J_{1/\gamma} (\Omega \sqrt{\gamma} z_*^2) \right]$$

$$\times \left[ J_{1/\gamma} (\Omega \gamma_1) Y_{1/\gamma} (\Omega \gamma_1) - J_{1/\gamma} (\Omega \gamma_1) Y_{1/\gamma} (\Omega \gamma_1) \right] = 0. \quad (5)$$

In the limiting case $m_* = 0$, when the boundary between phases is on the inner boundary of the layer, and $z_* = 1$ the dispersion equation is reduced to $J_{1/\gamma} (\Omega \sqrt{\lambda}) = 0$ for a frozen, and

$$\sqrt{\lambda} J_{1/\gamma} (\Omega \sqrt{\lambda}) - (\lambda - 1) \frac{\gamma - 1}{2} \Omega J_{1/\gamma} (\Omega \sqrt{\lambda}) = 0 \quad (6)$$

for an equilibrium phase transitions. At $m_* \to M, z_* \to 0$ when the level between phases is moving to the outer boundary, we have the dispersion equation $J_{1/\gamma} (\Omega) = 0$ in both cases. Here

$$\Omega = \frac{2\gamma}{\gamma - 1} \frac{M \omega}{\sqrt{\gamma P_0 \rho_0}}. \quad (6)$$

To investigate the dependence of oscillation modes with an ideal phase transition on $\gamma$ it is convenient to introduce $\tilde{\Omega} = \frac{\gamma - 1}{\gamma} \Omega$, with the first root $\tilde{\Omega}^2 \to \frac{1}{\lambda}$ at $\gamma \to \infty$. All other roots tend to infinity at $\gamma \to \infty$. First three roots of equations [3, 10] are presented in Figs.1,2,3.

**Caption to Figures 1,2,3**
Frequencies $\Omega$ of the slab oscillations as functions of $z_*$ of the basic mode (Fig.1), modes with one (Fig.2) and two (Fig.3) nodes. Upper and lower curves correspond to frozen and equilibrium cases, relatively. Frequencies of these two cases coincide, when one of the node coincides with the phase transition.

\(^1\text{Numerical solution of dispersion equations had been done by O.V.Shorokhov.}\)
4 Damping of oscillations due to URCA shell in a highly degenerate matter

Consider ultrarelativistic degenerate electron gas a good approximation in most URCA shells. It corresponds to the polytrope with \( \gamma = 4/3 \), and constant \( K = \frac{\rho}{12\pi^2} \left( \frac{3\pi^2}{\mu_0 m_p} \right)^{4/3} \), where \( \mu_0 = \left( xZ/Z + x_{Z-1}Z_{-1} \right)^{-1} \) is the average number of nucleons on one electron. Here a two component mixture is considered consisting of elements with an atomic weight \( A \) and atomic numbers \( Z \) and \( Z-1 \), with a beta transitions between them, \( xZ \) and \( x_{Z-1} \) are mass concentrations of these elements, \( xZ + x_{Z-1} = 1 \).

Let \( u \) and \( \delta \) be Fermi energy plus rest mass energy of the electrons, and threshold energy for a beta capture, in units of \( m_e c^2 \); \( g \) and \( g_{Z-1} \) be statistical weights of the elements \( (A, Z) \) and \( (A, Z-1) \); \( Ft_{1/2} \) be a nondimensional value measured in the beta-decay experiments, or estimated theoretically. For small difference \( |\delta - u| < 1 \) we have simple expressions \footnote{Eq. (3) for an entropy increase during beta decay and capture in a fully degenerate matter} for the pressure may be represented by an expansion

\[
\rho T \frac{\partial S}{\partial t} = \Phi(\delta - u)^{4}n_{Z-1} - \rho T \frac{\partial S}{\partial t} = \Phi(u - \delta)^{4}gZ^{-1}n_{Z}, \quad \Phi = m_e^2 c^2 \ln \frac{2}{12(Ft_{1/2})^{Z-1}}.
\]

Here the rate of the entropy increase is equal to \( 1/3 \) of the energy loss rate by neutrino emission. During linear oscillations the beta reactions take place only in a thin layer of matter, crossing in its motion the boundary \( x = x_*, m = m_*, P = P_*. \) In this layer the perturbation \( \tilde{P} \) be a perturbation of the pressure determined by the beta reactions \( P = g(m - m_*) + \tilde{P} + \tilde{p} \), with \( \tilde{p} = g_{Z-1}P_{Z-1}^{3}\cos 4\omega t \), \( P_{eq} < P \), \( P > P_* \); \( \tilde{P} = g_{Z-1}P_{Z-1}^{3}\cos 4\omega t \), \( P_{eq} < P \), \( P > P_*. \)

Equations describing change of concentrations during oscillations averaged over the layer \( \Delta m = \tilde{P}a/g \), are written as

\[
\frac{d\tilde{x}_Z}{dt} = -R \frac{P^{-9/4}}{256} \tilde{P}a^3 \cos 4\omega t gZ^{-1} gZ, \quad P_{eq} < P, \quad P > P_*; \quad \frac{d\tilde{x}_{Z-1}}{dt} = -R \frac{P^{-9/4}}{256} \tilde{P}a^3 \cos 4\omega t, \quad P_{eq} > P, \quad P < P_*.
\]

Here \( R = \ln \left( \frac{2(\delta^2 - 1)^{1/2}}{3(Ft_{1/2})^{Z-1}} \right) \left( \frac{12x^2 h^3}{m_e^2 c^2} \right)^{3/4} \). To derive an equation describing decrease of the pulsatinal amplitude, we should take into account the change of pressure due to change of the electron concentration. Let \( \tilde{\rho} \) be a perturbation of the pressure determined by the beta reactions \( P = g(m - m_*) + \tilde{P} + \tilde{p} \), with \( \tilde{p} = \frac{4}{3}P_{Z-1}^{3}\tilde{x}_Z^2 \), \( P_{eq} < P, \quad P > P_*; \quad \tilde{P} = \frac{4}{3}P_{Z-1}^{3}\tilde{x}_Z^{-1}, \quad P_{eq} > P, \quad P < P_* \).

In presence of damping we represent the velocity perturbation in a form \( \tilde{v} = \tilde{v}_a(t) \sin \omega t \) and define \( V(t) = \tilde{v}_a,m_*(t) \). so that \( \tilde{v} = V \sin \omega t \). The amplitude of the perturbed outside pressure my be written as a function of \( V \) as \( \tilde{P}_a = \frac{7}{4} \left( \frac{\pi^2}{\sqrt{A}} \right) \sqrt{\frac{J_A(m_*)}{J_A(m_*)}} V \), \( \gamma = \frac{4}{3} \). The equation of motion gives the relation describing damping of oscillations in the layer \( \Delta m(t) = \tilde{P}_a \cos \omega t \), after subtraction of the proper oscillations of the slab at the frozen composition and averaging over the motion of the whole slab.
\[
\sin \omega t \frac{dV}{dt} = -\frac{\Delta \bar{p}}{\Delta m(t)} \frac{\Delta m}{M} \approx \frac{g \tilde{p}(m_\ast)}{P_0 \cos \omega t} \frac{\Delta m}{M},
\]
where \( \Delta m = \frac{\tilde{p}}{g} \). Taking into account, \( M = gP_0 \), we get
\[
\sin \omega t \frac{dV}{dt} = \frac{g \tilde{p}(m_\ast)}{P_0 \cos \omega t}.
\]
Averaging over the oscillation period we obtain equation for decreasing of the amplitude of oscillations in the form
\[
\frac{dV}{dt} = -DV^3, \quad D = \frac{RgP_0^{-5/4}}{768\omega P_0 Z} \left( \frac{gZ-1}{gZ} \right) \left( \gamma \lambda P_0 \rho_0 \right)^{3/2} \frac{2^4 J_3^3(\eta_\ast)}{J_4^3(\eta_\ast)}.
\]

5 Energy balance and damping of oscillations

Averaging equations (7) over the time and space we get an expression for heating rate of the oscillating slab due to nonequilibrium beta processes in the layer around the URCA shell
\[
\dot{Q}_\nu = R_1 \frac{P_*^{-3}}{g 32 \times 75\pi} \tilde{p}_a^5 \left( 1 + \frac{gZ-1}{gZ} \right), \quad R_1 = \ln \frac{12(\delta^2 - 1)^{1/2}\delta}{12(F\eta_{1/2})Z \cdot \Lambda m_p} \frac{12\pi^2 h^3}{m_e^2 c^3}.
\]

The rate of the neutrino energy losses \( L_\nu \) (ergs/s/cm\(^2\)) is obtained by averaging over a time of losses in the whole oscillating layer. We get similar to (13)
\[
L_\nu = 3R_1 \frac{P_*^{-3}}{g 32 \times 75\pi} \tilde{p}_a^5 \left( 1 + \frac{gZ-1}{gZ} \right).
\]

Similar dependence of neutrino energy losses during oscillations because of URCA shell had been obtained by Tsuruta and Cameron (1970), who also took (14) for the rate of loss of kinetic energy of oscillations. In the approximation of strong degeneracy the matter will be heated during oscillations with the rate (13), and neutrino luminosity is determined by (14). The source of both kind of energy fluxes is the pulsation energy of the slab, giving the rate of pulsation energy losses directly connected with beta reactions as
\[
\dot{E}_{\text{pul}}^{(\beta)} = -(\dot{Q}_\nu + L_\nu) = -\frac{R_1 P_*^{-3}}{600\pi g} \tilde{p}_a^5 \left( 1 + \frac{gZ-1}{gZ} \right).
\]
Defining \( E_{\text{pul}} \approx \frac{1}{2} MV^2 \), and using approximately \( V = \tilde{p}_a \sqrt{\gamma P_0 \rho_0} \), for URCA shell in the middle of the slab, we get from (12) the equation for decreasing of pulsational energy, connected with hydrodynamic processes, in the form
\[
\dot{E}_{\text{pul}}^{(\text{dyn})} = -\frac{RP_*^{-5/4}}{768\omega Z^\gamma \sqrt{\gamma P_0 \rho_0}} \tilde{p}_a^4 \left( 1 + \frac{gZ-1}{gZ} \right).
\]
A ratio of damping rates of oscillation on \( i \)-th mode is \( \frac{\dot{E}_{\text{pul}}^{(\beta)}}{\dot{E}_{\text{pul}}^{(\text{dyn})}} \approx \tilde{p}_a \frac{\omega_1}{\tilde{p}_a \omega_1} \). It follows that the main source of damping of oscillations in presence of URCA shell is connected not with the neutrino emission / nonequilibrium heating, but with a dynamical action of the nonequilibrium layer.
of the slab, where beta reactions take place. This action leads to an exitation of short-wavelength acoustic waves with length $l \sim \Delta m/\rho_0 \ll x_0$. When the wavelength of the excited eigen-mode approaches the thickness of the nonequilibrium layer $\Delta x$, formed by oscillations, both mechanisms of damping become comparable. For $l_i \sim \Delta x = \frac{P_0}{g\rho_0}$, with account of relations $\omega_1 \approx \frac{1}{x_0} \sqrt{\frac{P_0}{\rho_0}}$, $x_0 \sim \frac{P_0}{g\rho_0}$, $\omega_i \approx \frac{1}{l_i} \sqrt{\frac{P_0}{\rho_0}}$, and $\rho_0 \sim \rho_0$, we get $\dot{E}_{\text{pul}}^{(\beta)}/\dot{E}_{\text{pul}}^{(\text{dyn})} \sim 1$. Importance of the dynamical damping of stellar oscillations in presence of URCA shell is connected with non-linearity of weak interaction rates, and deviations from eigen-oscillations under an action of weak interactions in this layer, leading to excitation of acoustic waves. The dissipation of pressure oscillations, connected with excitation of sound waves is inherent to any kind of dissipation, when the main term is nonlinear. The linear mechanism, connected with a conventional bulk viscosity, does not change the form of the eigenfunction of oscillations, and so no additional waves are excited.

6 URCA shell in a convective motion

It was concluded in [2] that nonequilibrium heating is balanced by the change in convective flow, leading to the net cooling due to convective URCA shell. Nine years later same authors [17] changed their mind, concluding that "convective URCA process can reduce the rate of heating by nuclear reactions but cannot result in a net decrease in entropy, and hence in temperature, for a constant or increasing density." This conclusion, as well as opposite one, made by using thermodynamic relations only, seems to be not convincing. Following this line, let us present two plausible scenarios, leading to two opposite conclusions.

A. Due to action of a nonlinear bulk viscosity, the convection is damping in the vicinity of the URCA shell, decreasing the convective heat flux from the central part of the star. In the general heat balance of the star it means that cooling become less effective, and nuclear reactions become thermally unstable and lead to a nuclear explosion earlier, than without a presence of the URCA shell. Nonequilibrium heating give additional heating, supporting the earlier nuclear explosion.

B. Due to action of a nonlinear bulk viscosity, the convection is damping in the vicinity of the URCA shell, decreasing the convective heat flux from the central part of the star. Due to local decrease of the heat flux from the core the average temperature gradient increases, leading finally to increase of the convective flux soon after entering an URCA shell into a convective zone. If the increase of the convective flux prevails the nonequilibrium heating in the URCA shell, the general heat balance would be shifted to a larger temperature with more effective cooling, and the boundary of the thermal explosion would be postponed in time, if not eliminated.

I cannot choose between these two scenario without construction of the numerical model taking into account all processes mentioned above. In such a highly nonlinear system, as a star with nuclear reactions, neutrino losses, degeneracy, convection, and many feedback influences it seems to be impossible to make a conclusion about the direction of process under the action of additional URCA shell, basing only on thermodynamical ground.

Convective modes belong to $g$-mode family, in which the local pressure perturbations are small, and could be neglected when the convective velocity is much less than the velocity of the sound. In this situation the sound wave dissipation of the convective modes imposed by URCA shell is negligible. The equations of stellar evolution in presence of URCA shell should take into account the following physical processes.
1. Loss of energy due to neutrino emission in the URCA shell
2. Heating of the matter in the convective region around URCA shell due to nonequilibrium beta processes.
3. Decrease of the convective velocity in the layer around the URCA shell due to energy dissipation connected with the nonequilibrium beta processes. Kinetic energy of the convection is the source of energy for neutrino losses and nonequilibrium heating of the matter. In the condition of static equilibrium only energy and heat transfer equations should be modified.

In the energy equation
\[
\frac{dS}{dt} = \frac{dE}{dt} - \frac{P}{\rho^2} \frac{d\rho}{dt} = \epsilon_n - \epsilon_\nu + \epsilon_\nu^{CU} - \frac{1}{4\pi \rho^2} \frac{dL_\nu}{dr}, \tag{17}
\]
in addition to other neutrino cooling processes \(\epsilon_\nu\), the new term \(\epsilon_\nu^{CU}\) is connected with heating due to nonequilibrium beta processes around the URCA shell. Having in mind strong degeneracy of electrons in this region, Neutrino emission in nonequilibrium URCA processes is accompanied by heating at high degeneracy, because the positive term \(\sum \mu_i d\mu_i\) exceeds the energy carried away by neutrino (Bisnovatyi-Kogan and Seidov, 1970). Convective motion, consisting of convective vortexes around an URCA shell is a source of additional neutrino energy losses, and of heating of the matter. This dissipation of convective energy may be described in the same way, as corresponding dissipation and heating during stellar pulsations. Therefore we use for description of these processes the formulae from the previous sections. If we accept that the pressure difference in the convective vortex is about one half of the local pressure, we use for the amplitude of pressure pulsations in (13) and (16) \(\tilde{P}_a = \alpha_\rho \tilde{P}_g\). Taking also approximately \(P_0 = P_\star\), \(\rho_0 = \rho_\star\), and \(u_{fe} = \delta\) we get
\[
\hat{Q}_\nu = \frac{\hat{R} m_\nu c^2}{A m_p} g \frac{\delta^2 \alpha_\rho^5}{2^{10} \times 75\pi} \sum \mu_i d\mu_i, \quad \hat{R} = \ln 2 \frac{(\delta^2 - 1)^{1/2} \delta}{12(F t_{1/2})_{Z-1}} \frac{m_\nu^4 c^5}{12 \pi^2 \hbar^3} \left(1 + \frac{g_{Z-1}}{g_Z}\right). \tag{18}
\]
Equation (18) is related to energy losses averaged over the whole slab. In the convective motion the losses are localized in the layer around the URCA shell radius \(r_\star\)
\[
r_\star + l_{conv} < r < r_\star - l_{conv}, \quad l_{conv} = \alpha_\rho \frac{P}{\nabla P}, \tag{19}
\]
here \(l_{conv}\) is taken from the mean free path model. The local rate is obtained from (18), if we take into account that the whole heating is concentrated inside the layer (19). We get than
\[
\epsilon_\nu^{CU} = \frac{\hat{Q}_\nu}{2 l_{conv}}. \tag{20}
\]
Convective velocity suffers from additional damping due to URCA shell, because kinetic energy of the convection is a source of both nonequilibrium heating and of additional neutrino losses. The only relation of the convetional mixing length model of the convetion (see e.g. [3]) should be modified, determining the convective velocity with an additional damping
\[
\frac{1}{2} \rho v_{conv}^2 = \frac{1}{8} (\Delta \nabla T)^2 \left(\frac{\partial \rho}{\partial T}\right) |_P - \frac{i \hat{E}_\nu^{(\beta)}}{v_{conv}}, \tag{21}
\]
where \(\hat{E}_\nu^{(\beta)} = 4 \hat{Q}_\nu\) is found from (18). These relations may be applied for description of URCA shell convection only for sufficiently strong convective motion, when the first term in (21) exceeds considerably the second one. The equation (21) has roots only when
\[
\dot{E}_{\text{conv}}^{(\beta)} < \frac{1}{8 \sqrt{\rho}} \left[ \frac{1}{3} (\Delta \nabla T)^2 \left( \frac{\partial \rho}{\partial T} \right)_P \right]^{3/2} g.
\] (22)

Violation of this inequality may result in an abrupt termination of the convection in the layer around the URCA shell. When analysing the URCA shell convection in star, it would be premature to predict the results of evolutionary calculations with account of convective URCA shell before such calculations are done. Two possibilities may be expected. One is connected with obtaining of a definite result which has a little sensitivity to the input parameters of the problem, such as \(\alpha_p, F_{l1/2}\), accepted rates of nuclear reactions, neutrino losses etc. Another possibility could be a great sensitivity of the result to the same input parameters. If the second possibility would be realized we could still remain in situation of ambiguity, because the set of the input parameters for presupernovae model cannot be established with a sufficient precision.

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