On limit laws of multi-dimensional stochastic synchronization models

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Abstract. Lévy stochastic processes and related fine analytic properties of probability distributions such as infinite divisibility play an important role in construction of stochastic models of various distributed networks (e.g., local clock synchronization), of some physical systems (e.g., anomalous diffusions, quantum probability models), of finance etc. Nevertheless, little is known about limit probability laws resulted from the long time behavior of such stochastic systems. In this paper we will focus on the impact of interaction graph topologies on limit laws of multicomponent synchronization models.

Models with synchronizing interaction. We recall a notion stochastic synchronization model (SMM) in a short descriptive manner referring readers to [1] for detailed definitions. There are $N$ components and $x_j \in \mathbb{R}^d$ denotes a state of the component $j \in \{1, \ldots, N\}$. The system $x(t) = (x_1(t), \ldots, x_N(t))$ evolves in continuous time $t \in \mathbb{R}_+$ and the evolution is composed of two parts: a free dynamics and a synchronizing pairwise interaction. The interaction is defined by the set of nonnegative parameters $\alpha_{jk}, j \neq k$. It is assumed that the component $j$ sends messages to the component $k$ with the rate $\alpha_{jk}$. Messages contain information on the sender’s current state $x_j$ and immediately reach their destinations. We may imagine that there is a network with nodes $1, \ldots, N$ and each node shares information with the other nodes by using messages. If $\alpha_{j'k} = 0$ for some $j' \neq k'$ then it means that there are no messages from $j'$ to $k'$. For any $j \neq k$ the message flow $T_{j\rightarrow k}$ is a Poissonian point process. It is assumed that stochastic flows $T_{j\rightarrow k}$ are independent for different pairs $(j, k)$. Thus points of $T = \bigcup_{j \neq k} T_{j\rightarrow k}$ form a sequence $T = \{0 < T_1 < T_2 < \cdots\}$ of random epochs when nodes of the network exchange messages and interact in the following way. Imagine, for example, that at time $T_q$ the node $j_1^{(q)}$ sends a message to some another node $j_2^{(q)}$. For brevity notation let us write $j_1 = j_1^{(q)}$ and $j_2 = j_2^{(q)}$. After receiving the message from $j_1$ the node $j_2$ adjusts its state to the value $x_{j_1}$: $x_{j_2}(T_q + 0) = x_{j_1}(T_q)$. This is the only jump in the system at the time $T_q$: $x_j(T_q + 0) = x_j(T_q)$ for all $j \neq j_2$. Between successive epochs of interaction all $x_k(t)$ evolve independently. Namely there is a family of mutually independent stochastic processes $x_k^\pi(t), k = 1, \ldots, N$, called the free...
dynamics of components, such that
\[ x_k(t) - x_k(T_{n-1} + 0) = x_k^0(t) - x_k^0(T_{n-1} + 0), \quad T_{n-1} < t \leq T_n, \quad n \geq 1, \quad k \in \{1, \ldots, N\}. \]

It is very natural to assume that any \( x_k^0(t) \) is an \( \mathbb{R}^d \)-valued Lévy process, i.e., a process having independent increments, starting from \( x_k^0(0) = 0 \) and being stochastically continuous [2]. A remarkable feature of Lévy processes is the following property of their increments [2].

\[ \mathbb{E} \exp \left( i \lambda_j \cdot (x_j^0(t) - x_j^0(s)) \right) = \exp \left( -(t-s) \eta_j(\lambda_j) \right), \quad \lambda_j \in \mathbb{R}^d, \quad 0 \leq s < t. \] (1)

Here the function \( \eta_j : \mathbb{R}^d \to \mathbb{C} \) is called the Lévy exponent of \( x_k^0(t) \). The LHS of (1) is the characteristic function \( \psi_{x_j^0(t) - x_j^0(s)}(\lambda_j) \), i.e., the Fourier transform of the probability distribution of \( x_j^0(t) - x_j^0(s) \). The RHS of the identity (1) provides a very convenient tool for analytical study of Lévy-driven dynamics. It should be noted that the class of Lévy processes is wide enough to cover many important examples such as homogeneous random walks with light-tailed or heavy-tailed jumps, jump-diffusion processes etc. In particular, if \( x_j^0(t) \) is a classical Brownian motion with diffusion matrix \( \Sigma \) and drift vector \( v_j \in \mathbb{R}^d \) then \( \eta_j(\lambda_j) = -i v_j \cdot \lambda_j + \frac{1}{2} (\Sigma \Sigma^\top \lambda_j) \cdot \lambda_j \).

To summarize, the SSMs are a special class of continuous time \( (\mathbb{R}^d)^N \)-valued Markov processes determined by two sets of parameters: the matrix \((\alpha_{jk})\) and the Lévy exponents \{\( \eta_j = \eta_j(\lambda_j) \)\}.

Motivations of SSMs. The synchronizing interaction is motivated by many computer science applications with message passing mechanism (parallel computations [3–5], wireless sensor networks [6–8] etc.). SSMs can be also considered as agreement algorithms [9, 10] of a special kind. The idea to use Lévy processes instead of classical random walks is rather popular in physics [11–14], especially in subdomains of social and biological physics.

Known results on the long time behavior. The rate matrix \((\alpha_{jk})\) determines a message routing graph (MRG) \( \mathcal{G} = \mathcal{G}(N, (\alpha_{jk})) \) having vertices 1, \ldots, \( N \) and directed edges \((j, k)\) with weights \( \alpha_{jk} > 0 \). The presence of the edge \((j, k)\) in \( \mathcal{G} \) indicates a possibility of directly sending messages from \( j \) to \( k \). We will always assume that the digraph \( \mathcal{G} \) is strongly connected. Then the SSM demonstrates a long time synchronization in the following (stochastic) sense [1]: all differences \( x_j(t) - x_k(t) \) have limits in distribution as \( t \to \infty \),

\[ r_{jk}(x(t)) = x_j(t) - x_k(t) \xrightarrow{d} r_{jk}^\infty \] (2)

where \( r_{jk}(x) = x_k - x_j \). Moreover, in regular situations the state vector \( x(t) = (x_1(t), \ldots, x_N(t)) \) has no limit in distribution [1, Th. 1]. Denote by \( \ell_{jk}^\infty \) the probability law of the random vector \( r_{jk}^\infty \in \mathbb{R}^d \). In [15] some general results for probability laws \( \ell_{jk}^\infty \) were obtained. It was discovered that under particular cases we have the famous Linnik distributions among them.

Infinite divisibility in probability and applications. The probability law of a random vector \( \xi \in \mathbb{R}^d \) is called infinitely divisible (ID) iff for any \( n \) there exist independent identically distributed random vectors \( \xi_{n,1}, \ldots, \xi_{n,n} \) such that \( \xi_{n,1} + \cdots + \xi_{n,n} \) has the same distribution as \( \xi \). It is a classical result [16] that the ID laws are appearing as limits of sums independent uniformly asymptotically negligible random variables. Since the Central Limit Theorem and the Poisson Limit Theorem are special cases of such summations it is not surprising that the Gaussian and Poisson laws which we often meet in physics and other application domains are ID. At the same time the uniform distribution in the segment \([0, 1]\) is not ID.

The ID property is very desirable when choosing probability laws for constructing new models in statistics [17, 18], physics [19, 20], biology [21, p. 451], actuarial science [18] etc.
Denoting by $\psi_\xi(\nu) := E \exp(i\nu \cdot \xi)$, $\nu \in \mathbb{R}^d$, the characteristic function (CF) of $\xi$, we can reformulate the definition of ID as follows: $\xi$ is ID iff for any $n \in \mathbb{N}$ there exists a CF $\psi_\nu(n)(\nu)$ such that $\psi_\xi(\nu) = (\psi_\nu(n)(\nu))^n$. Of course, there is no warranty the the $n$-th root of a CF is again a CF of some other distribution. In general, proofs that concrete probability laws are ID (or are not ID) can be extremely complicated. It is well known [2], however, and it is readily seen from (1) that any Lévy process $L(t)$ is ID for any $t > 0$.

**Infinite divisibility in synchronization models.** Since the free dynamics of SSM is ID for any $t$ it is natural to ask whenever this property is inherited by limit laws of $\ell_{jk}^\infty$. This is the main problem discussed in the present paper. The problem is complicated and as we will show soon the answer depends on topology of the graph $\mathcal{G}$. We start from a useful representation for CFs of $\ell_{jk}^\infty$. Introduce $b_2(\nu) = (b_{jk}, j, k = 1, N, j \neq k)$, a set of variables $b_{jk} \in \mathbb{C}$. Consider CFs $\psi_{rjk(x(t))}(\nu) = E e^{i\nu(x_k(t) - x_j(t))}$. Theorems 2 and 3 from [15] can be rephrased as follows.

**Theorem 1.** Let the graph $\mathcal{G}$ be strongly connected. Then

(i) for all $j \neq k$ and for all $\nu \in \mathbb{R}^d$ limits $\varphi_{(jk)}(\nu) := \lim_{t \to \infty} \psi_{rjk(x(t))}(\nu)$ exist;

(ii) the functions $\varphi_{(jk)}(\nu)$ are CFs of some probability laws in $\mathbb{R}^d$ and can be represented as composition $\varphi_{(j1k)}(\nu) = (h_{j1k1} \circ b_{(2)}(\nu))$ where

- $b_2(\nu) : \mathbb{R}^d \to \mathbb{C}^{N(N-1)}$ such that $b_{jk} = \eta_j(-\nu) + \eta_k(\nu)$,
- functions $h_{j1k1} = h_{j1k1}(b_{(2)})$ are solutions of the following system of $(N-1)N$ equations

$$h_{j1k1}Z_{j1k1} = \alpha_{j1k1} + \alpha_{k1j1} + \sum_{j \notin \{j1, k1\}} \alpha_{jj1}h_{j1k1} + \sum_{k \notin \{j1, k1\}} \alpha_{kk1}h_{j1k1}, \quad j1, k1 = 1, N, \; j_1 \neq k_1,$$

and $Z_{j1k1} := b_{j1k1} + \sum k; k \neq k_1 \alpha_{kk1} + \sum j; j \neq j_1 \alpha_{jj1}$.

Item (i) means that $x_j(t) - x_k(t)$ have limits in distribution as $t \to \infty$, i.e., that (2) holds. Hereafter, we will consider the case of identical components

$$\eta_j(\lambda) = \eta(\lambda) \quad \forall j \quad (5)$$

i.e., we assume the probability law of free dynamics $x_j^\infty(t)$ of any component $j$ is the same. Recall that by assumption the processes $\left(x_j^\infty(t), \; t \in \mathbb{R_+}\right)$ are independent. It follows from (5) that the substitution $b_{jk} = \eta_j(-\nu) + \eta_k(\nu) = \eta(-\nu) + \eta(\nu) = 2 \Re \eta(\nu)$ does not depend on $(j, k)$ so $\varphi_{(j1k)}(\nu) = h_{j1k1}(1)$ where $1$ is the vector of all ones.

**New probabilistic interpretation for limit laws.** Coalescing MCs and first passage distributions. Let $X_t, t \in \mathbb{R}_+$, be a particle walking on the set $\{1, \ldots, N\}$ with jump rates $j_1 \overset{\alpha_{j1j2}}{\rightarrow} j_2$. Consider a continuous time Markov chain (MC) $Y_t = (X_t^{(1)}, X_t^{(2)})$ with state space $S = \{1, \ldots, N\}^2$ defined as follows. $X_t^{(1)}$ and $X_t^{(2)}$ are coordinates of two particles started from different points $k_0^{(1)}$ and $k_0^{(2)}$ and moving according to the rules of $X_t$. These two particles are walking independently until they meet. Afterwards they are moving together. So we have coalescing MCs. Let $\sigma$ be the coalescing time $\sigma := \inf \left\{ s > 0 : X_s^{(1)} = X_s^{(2)} \right\}$. In terms of the two-dimensional MC $Y_t$ the
random variable $\sigma$ is the first hitting time of $Y_t$ to the subset $\{(m, m) : m = 1, N\} \subset S$. Let $G_{(jk)}(c) := E \left( \exp(-ct) \left| (X_0^{(1)}, X_0^{(2)}) = (j, k) \right. \right)$, $c \geq 0$, denote the Laplace transform (LT) of the probability distribution of $\sigma$. It is clear from this definition that $G_{(jk)}(c) = G_{(kj)}(c)$. Consider the SSM $x(t)$ with identical components. The next result states that $h_{jk}(c1) = G_{(jk)}(c)$.

**Theorem 2.** Let the graph $G$ be strongly connected and assumption (5) holds. Then

$$\psi_{(jk)}(\nu) = G_{(jk)}(c) \bigg|_{c=2\Re\nu} = G_{(jk)}(2\Re\eta(\nu)) \quad \forall j \neq k.$$ 

Since $G_{(jk)}(c) = G_{(kj)}(c)$ it follows that limit laws $\ell_{jk}^\infty$ are symmetric.

Using basic properties of ID laws it is easy to get the following conclusion.

**Corollary 3.** If $G_{(jk)}(c)$ is the Laplace transform of an ID law then $\ell_{jk}^\infty$ is ID.

We see that the question about ID of $\ell_{jk}^\infty$ is related to the problem of ID of first passage times (FPTs) in very special MCs.

**Example 4.** The two-component SSM provides the simplest example with $N = 2$ and two rate parameters $\alpha_{12} > 0$ and $\alpha_{21} > 0$. It is easy to see that for the starting point $Y_0 = (X_0^{(1)}, X_0^{(2)}) = (1, 2)$ the coalescing time $\sigma$ is exponential with rate $\gamma = \alpha_{12} + \alpha_{21}$, so it has probability density $p(t) = \gamma e^{-\gamma t}$, $t \geq 0$, and LT $G_{(12)}(c) = \gamma / (\gamma + c)$. It is well known that the exponential distribution is ID.

**Example 5.** Let $N \geq 2$ be arbitrary and $G$ be a complete graph with $\sigma_{jk} = \alpha > 0$, $j \neq k$. This case corresponds to the totally symmetric networks discussed in [15, Sect. 3.2]. Under assumption (5) it follows from (4) that $G_{(jk)}(c) = G_{(12)}(c)$. So (4) reduces to a single equation and we get $G_{(jk)}(c) = 2\alpha / (2\alpha + c)$. These algebraic considerations may be replaced by purely probabilistic arguments. Indeed, if $\sigma_{jk}$ is distributed as the coalescing time in $Y_t$ with the starting point $Y_0 = (j, k)$ then by symmetry $\sigma_{jk} \sim \sigma_{12}$ (i.e., they all have the same distribution). Applying the one step analysis [22] to $Y_t$ it is not hard to derive that distribution of $\sigma_{12}$ coincides with the mixture of an exponential and the distribution of $\sigma_{12}$ itself,

$$p_{\sigma_{12}}(t) dt = \frac{1}{N-1} 2\alpha \exp(-2\alpha t) dt + \frac{N-2}{N-1} p_{\sigma_{12}}(t) dt, \quad t > 0,$$

and, evidently, $p_{\sigma_{12}}(t) = 2\alpha \exp(-2\alpha t)$. So $\sigma$ is again exponential and thus it is ID.

The first passage problem is of significant interest for many applied domains such as queueing systems [23], financial models or physics [14, 24]. Despite of the existing progress in modelling many interesting phenomena and in the related asymptotic analysis, results on the ID property of FPTs are far to be exhaustive. There is a huge probabilistic literature on this topic [22, 25–29] but there is no answer which would be applicable to the analysis of a general $N$-component SSM. Nevertheless, as we will show below, some of these results are useful for particular cases of SSMs.

Computational aspects related to inversion of the LTs of ID laws are also rather complicated [23].

**General results about the LT of $\sigma$ for synchronization models.** Analysis of the equations (4) carried out in [15] results in the following conclusions.

**Theorem 6.** Let the graph $G$ be strongly connected. Then

- any $G_{(jk)}(c)$, $j < k$, the LT of the first passage time $\sigma$, is a rational function $P_{(jk)}(c)/Q_{(jk)}(c)$ with real coefficients.
all poles of \(G_{(jk)}(c)\) being considered as a function of the complex variable \(c \in \mathbb{C}\) belong to 
\(\{c \in \mathbb{C} : \text{Re} \, c < 0\} \);

- \(\deg P_{(jk)} < \deg Q_{(jk)}, \ \deg P_{(jk)} \leq \frac{1}{2} (N^2 - N) - 1, \ \deg Q_{(jk)} \leq \frac{1}{2} (N^2 - N)\).

There are no known results about ID precisely applicable to this situation. Paper [28] contains a study of rational LTs having negative poles and zeros. The above theorem 6 covers more general LTs with complex (non real) poles and zeros. For many reasons (including computational aspects) it is important to describe digraphs \(G\) for which degrees of \(G_{(jk)}\) are less than for the general case. As we will see below this can be done for specific topologies of the MRG \(G\).

Symmetric digraphs. In the case of identical components we assume that \(N\) is arbitrary and the digraph \(G\) is symmetric: \(\alpha_{jk} = \alpha_{kj}\) for all \(j \neq k\). Example 5 is a very particular subcase of this situation. For symmetric digraphs we can slightly improve the general result of Theorem 6. It appears that the rational function \(P_{(jk)}(Q_{(jk)})\) has only real negative poles 
\(-\vartheta_M < \cdots < -\vartheta_1<0\) and the following representation holds

\[
G_{(jk)}(c) = \sum_{m=1}^{M} \frac{d_{jk,m}}{c + \vartheta_m} \quad (6)
\]

with some coefficients \(d_{jk,m} \in \mathbb{R}, \ d_{jk,1} > 0, \sum_m d_{jk,m} = 1\) and \(M \leq \frac{1}{2} (N^2 - N)\). The key idea is to connect the system of equations (4) with spectral properties of some symmetric matrix. The proof is rather straightforward and so we omit details.

Many researches were interested in sufficient conditions on parameters \(d_{jk,m}\) and \(\vartheta_m\) for ensuring that the RHS of (6) is a LT of an ID probability law.

If all \(d_{jk,m} > 0\) then we have a mixture of exponential laws which is ID (see [25, Sect. XIII.7]). For \(d_{jk,m}\) of different signs and ordered \(\vartheta_m (0 < \vartheta_1 < \vartheta_2 < \cdots )\) Steutel [29] provided a particular answer by proving that finite mixtures with one change of signs are ID.

Unfortunately, neither of these analytical results is applicable to general stochastic synchronization models. In general, coefficients \(d_{jk,m}\) in (6) can be positive or negative with two or more changes of signs. It is not clear how to control them in terms of parameters \(\alpha_{jk}\).

Another possible way is to use the probabilistic interpretation of \(G_{(jk)}(c)\) as the LT of a first passage time in a special Markov chain.

On ID property of first passage times in continuous time Markov chains. Consider a finite MC with no symmetry assumption. Let \(n\) be a number of its states. Miller [27] proved that if \(n \leq 4\) then the FPT from any state to any other state is ID. By providing a corresponding example he also proved that for \(n \geq 5\) the statement is not true.

Ring topology of \(G\). Let \(N\) be arbitrary. We say that a digraph \(G\) is a ring iff \(\alpha_{jk} = \gamma_{k-j} \pmod{N}\). This message routing topology depends on \(N - 1\) parameters \(\gamma_1, \gamma_2, \ldots, \gamma_{N-1}\). In general, \(\gamma_m \neq \gamma_{-m}\) hence the digraph is not symmetric. It is easy to see that the LT \(G_{(jk)}(c) = H_{k-j}(c)\), i.e., it depends only on \(k-j \pmod{N}\). Such model was studied in [30]. It was proved that \(G_{(jk)}(c)\) is a fraction whose nominator and denominator are rational functions of a special form. Using that representation it is easy to see that \(G_{(jk)}(c) = P_{(jk)}(c)/Q_{(jk)}(c)\) with \(\deg P_{(jk)}(c) \leq \lfloor N/2 \rfloor - 1, \ \deg Q_{(jk)}(c) \leq \lfloor N/2 \rfloor\). Evidently, the degrees here are much smaller than those in the general case of Theorem 6. Developing the approach of [30] we get a result similar to, but better than, (6).

**Theorem 7.** Let a (nonsymmetric) graph be a ring. Then all poles \(G_{(jk)}(c)\) are strictly negative and the representation (6) holds with some \(M \leq \lfloor N/2 \rfloor\).
It was also proved in [30] that for $N \leq 5$ all first passage times $LT$ are ID. It is worth noting that even for $N = 6$ there is an example of $G$ such that for some pair $(j,k)$ the first passage time $LT_{G}(jk)(c)$ has the form (6) with $M = 3$, $d_{jk,1} > 0$, $d_{jk,2} < 0$, $d_{jk,3} > 0$. This is the case of two changes of signs and the Steutel’s theorem [29] does not work here. Nevertheless, applying the state-augmentation method and the Miller’s result we can extend results of [30].

**Theorem 8.** Let a stochastic synchronization model consist of $N \leq 7$ identical components and its digraph $G$ is a ring. Then all FPTs $\sigma_{jk}$ are ID and hence all limit laws $\ell_{\infty}^{jk}$ are ID.

**NNE-ring topology.** The so called birth-death processes and skip-free random walks on the one-dimensional lattice are the only classes of continuous time MCs with arbitrary number of states for which many fine probabilistic properties of the first passage times were successfully studied [22, 25, 26, 31]. In these classes, in going from one state to another, the chain must pass all intermediate states at least once. It was proved that the first passage times in such Markov chains are ID. To make use of there result we define the following subclass of ring topologies.

We say that the ring topology satisfies a NNE (nearest neighbors edges) condition iff $\gamma_{m} = 0$ for $|m| \neq 1 \pmod{N}$.

**Theorem 9.** For arbitrary $N$ in stochastic synchronization models with the NNE-ring topology all Laplace transforms $G_{jk}(c)$ are ID.

The idea of the proof is to construct an auxiliary birth-death process (BDP) having the same FPT distributions as the MC $Y_{t} = (X_{1}^{(1)}_{t}, X_{1}^{(2)}_{t})$. As in [30] we consider $U_{t} = X_{t}^{(1)} - X_{t}^{(2)}$ as a random walk on $\mathbb{Z}_{N}$ and reduce the problem to the first passage times to the state $0 \in \mathbb{Z}_{N}$. Identifying states $j$ and $N - j$ for all $1 \leq j < N/2$ we get a BDP with the finite state space $\{0, 1, \ldots, [N/2]\}$. Now the ID property follows from known results.

**Conclusions.** We considered a class of stochastic models with synchronizing interaction and studied the question if their limit probability laws inherit the infinite divisibility property from individual dynamics of components. It was shown that this problem is heavily dependent on topology of the message routing graph. An important byproduct of this study is the demonstration of the fact that a complexity of numerical procedures for finding these limit laws may also depend on a chosen network topology. Since Lévy-driven dynamics are popular tools for constructing many physical models we hope that our results will be useful not only in the context of stochastic synchronization models.

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