COHERENT STATES OF PARAMETRIC OSCILLATORS
IN THE PROBABILITY REPRESENTATION
OF QUANTUM MECHANICS†

Vladimir N. Chernega1 and Olga V. Man’ko1,2∗

1Lebedev Physical Institute, Russian Academy of Sciences
Leninskii Prospect 53, Moscow 119991, Russia
2Bauman Moscow State Technical University
The 2nd Baumanskaya Str. 5, Moscow 105005, Russia
∗Corresponding author e-mail: mankoov@lebedev.ru

Abstract

Glauber coherent states of quantum systems are reviewed. We construct the tomographic probability distributions of the oscillator states. The possibility to describe quantum states by tomographic probability distributions (tomograms) is presented on an example of coherent states of parametric oscillator. The integrals of motion linear in the position and momentum are used to explicitly obtain the tomogram evolution expressed in terms of trajectories of classical parametric oscillator.

Keywords: dichotomic random variables, probability representation of quantum states, linear invariants, parametric oscillator, coherent states, Wigner function.

1. Introduction

In quantum mechanics, the states of a particle, e.g., of the harmonic oscillator, are identified with the wave functions ψ(x,t) satisfying the Schrödinger evolution equation [1], where x is the oscillator position and t is time. The energy levels and stationary states of the oscillator and other systems are obtained by solving the stationary Schrödinger equation ĤψE(x) = EψE(x), where Ĥ is the quantum system Hamiltonian. Among all solutions of the evolution Schrödinger equations, there are specific Gaussian-packet solutions for which the probability distribution P(x,t) = |ψ(x,t)|2 of the oscillator position at a given time moment is described by the normal probability distribution of the position x, with a given mean value 〈x(t)〉 and the dispersion σ(t) = 〈x²(t)〉 − (〈x(t)〉)². Such packets were studied by Schrödinger [1], and these oscillator states are similar to classical oscillator states with fluctuating position and momentum. In 1963, while studying the coherence properties of photons, Roy Glauber [2] introduced the notion and terminology of the field coherent states; see also [3–9].

†We dedicate this paper to the memory of Roy Jay Glauber, the great scientist and Nobel Prize Winner, on his first death anniversary, December 26, 2019. Ad Memoriam of Roy Glauber and George Sudarshan is published in [10,11] and is also available on link.springer.com/article/10.1007/s10946-019-09805-4 and www.mdpi.com/2624-960X/1/2/13.
For a single mode, the field is modeled by the quantum harmonic oscillator, and the wave function $\psi_\alpha(x,t)$ of the harmonic oscillator is the Gaussian packet satisfying the Schrödinger evolution equation. Generic Gaussian states and entropic inequalities for these states for multimode photon states were studied in [12].

The function $\psi_\alpha(x, t = 0)$ is the eigenfunction of the photon annihilation operator $\hat{a}\psi_\alpha(x) = \alpha \psi_\alpha(x)$, where $\alpha$ is the complex number $\alpha = \alpha_1 + i\alpha_2$; see also [13,14].

The photon annihilation $\hat{a}$ and creation $\hat{a}^\dagger$ operators satisfy the bosonic commutation relations $[\hat{a}, \hat{a}^\dagger] = 1$. In the coherent states of the harmonic oscillator, the Heisenberg uncertainty relations [15] for dimensionless position and momentum have the property $(\bar{x}^2 - \bar{x}^2)(\bar{p}^2 - \bar{p}^2) = 1/4$ and $\bar{x}^2 - \bar{x}^2 = \bar{p}^2 - \bar{p}^2 = 1/2$. The coherent state properties of the oscillator were studied in [2–4]. The properties of the coherent states of photons were also considered by Sudarshan [16].

The aim of our work is to discuss the coherent states of the parametric oscillator, i.e., of the oscillator with time-dependent frequency $\omega(t)$. The Schrödinger evolution equation for such oscillator was solved in [17]. There is no energy levels of the parametric oscillator, and the energy is not the integral of motion. For a classical parametric oscillator, the integral of motion, being quadratic in the position and momentum, was found by Ermakov [18]. The quantum operator quadratic in the position and momentum, being the integral of motion, contains an explicit dependence on time in the Schrödinger representation, as was found in [19]. This quantum integral of motion is an analog of the classical Ermakov invariant, and it was used to find different solutions to the Schrödinger equation in [19].

It was shown in [20] that the parametric oscillator has the linear (in the position and momentum) integrals of motion $\hat{A}(t)$ and $\hat{A}^\dagger(t)$, which have the commutation properties of bosonic annihilation and creation operators, i.e., $[\hat{A}(t), \hat{A}^\dagger(t)] = 1$. In view of what we said above, one can extend the construction of Glauber coherent states to the case of the parametric oscillator; see, e.g., [21]). In view of developing the technique of homodyne tomography of photon states [22] based on the relation between the Radon transform [23] of the Wigner function [24] of the quantum system state with optical tomogram, which is a fair probability distribution of the photon quadrature found in [25,26], the suggestion to identify the quantum state with the probability distribution as a primary object was done in [27]; see also the review [28].

The kinetic equation for the tomographic probability distribution, which is the optical tomogram of the quantum state, with the wave function obeying the Schrödinger evolution equation, was obtained in [29,30]. This equation is compatible with the kinetic equation for the symplectic tomogram of quantum states introduced and studied in [27,31]. Such tomogram exists and obeys the kinetic equation for the fair probability distributions also in the case of a spin-1/2 particle, with the wave function satisfying the Pauli equation [32]. Thus, in addition to the review of Glauber’s coherent states for the wave function of the parametric oscillator, we consider the oscillator coherent states in the probability representation of quantum mechanics.

We present the evolution for the tomographic probability distributions determining the oscillator states and construct the probability distributions of the oscillator position in the form of normal distribution with time-dependent parameters. The tomographic probability distributions identified with the coherent states satisfy the kinetic equations equivalent to the Schrödinger equation for the wave function and the von-Neumann equation for the density matrix of the parametric oscillator. As an application of the formalism, we discuss the stimulated Raman scattering process in the probability representation of quantum mechanics in [33–35]. The problem of parametric oscillator was studied using different methods in [36–43].
This paper is organized as follows.

In Sec. 2, we present the method of linear integrals of motion to find coherent states of a parametric oscillator. In Sec. 3, we give a review of the conditional probability representation of quantum states of the parametric oscillator. In Sec. 4, we construct the joint probability distribution of three random variables for the parametric oscillator in coherent states. In Sec. 5, we consider the evolution of the parametric oscillator in the probability representation of quantum mechanics. Our conclusions and prospectives are given in Sec. 6.

2. Integrals of Motion of Parametric Oscillator and Coherent States

The parametric oscillator has the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{\omega^2(t)\hat{q}^2}{2}. \quad (1)$$

We assume the Planck constant $\hbar = 1$, the oscillator mass $m = 1$, and frequency at time $t = 0$, i.e., $\omega(0) = 1$. The Schrödinger equation for the oscillator wave function $\psi(x, t)$ in the position representation

$$i \frac{\partial \psi(x, t)}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi(x, t)}{\partial t^2} + \frac{\omega^2(t)x^2}{2} \psi(x, t) \quad (2)$$

was solved in [17], and various methods to study this equation and its solutions were suggested in [20]. The method based on finding the system’s integrals of motion, which are operators quadratic in the position and momentum, was used in [19].

The Ermakov integral of motion for a classical parametric oscillator was found in [18]. The quantum version of the classical Ermakov invariant depends on the solution of the classical nonlinear equation [36–38,40,41]. Invariants, which are linear in the position and momentum operators, were found in [20].

The time-dependent operators $\hat{A}(t)$ and $\hat{A}^\dagger(t)$ of the form

$$\hat{A}(t) = i \frac{1}{\sqrt{2}} (\epsilon(t)\hat{p} - \dot{\epsilon}(t)\hat{q}), \quad \hat{A}^\dagger(t) = -i \frac{1}{\sqrt{2}} (\epsilon^*(t)\hat{p} - \dot{\epsilon}^*(t)\hat{q}) \quad (3)$$

are the linear integrals of motion satisfying the conditions

$$\langle \psi(t)|\hat{A}(t)|\psi(t)\rangle = \langle \psi(0)|\hat{A}(0)|\psi(0)\rangle, \quad \langle \psi(t)|\hat{A}^\dagger(t)|\psi(t)\rangle = \langle \psi(0)|\hat{A}^\dagger(0)|\psi(0)\rangle$$

for function $\epsilon(t)$ satisfying the equation of motion for the classical parametric oscillator $\ddot{\epsilon}(t) + \omega^2(t)\epsilon(t) = 0$.

For initial conditions of the function $\epsilon(t)$ of the form $\epsilon(0) = 1$, $\dot{\epsilon}(t) = i$, the integrals of motion (2) and (3) satisfy the commutation relation

$$\left[\hat{A}(t), \hat{A}^\dagger(t)\right] = 1, \quad (4)$$

and these operators coincide for $t = 0$ with the annihilation $\hat{a}$ and creation $\hat{a}^\dagger$ operators of the harmonic oscillator, i.e.,

$$\hat{A}(0) = \hat{a} = \frac{1}{\sqrt{2}} (\hat{q} + i\hat{p}), \quad \hat{A}^\dagger(0) = \hat{a}^\dagger = \frac{1}{\sqrt{2}} (\hat{q} - i\hat{p}).$$

The coherent states of the parametric oscillator $|\alpha, t\rangle$ satisfying the equation $\hat{A}(t)|\alpha, t\rangle = \alpha|\alpha, t\rangle$, where eigenvalues of the integral of motion $\hat{A}(t)$ do not depend on time, are obtained from an analog of the ground state of harmonic oscillator $|0, t\rangle$ satisfying the Schrödinger equation and the condition $\hat{A}(t)|0, t\rangle = 0$ by means of the Weyl operator, which is the integral of motion

$$\hat{D}(\alpha) = \exp \left[ \alpha \hat{A}^\dagger(t) - \alpha^* \hat{A}(t) \right].$$

The coherent state reads

$$|\alpha, t\rangle = \exp \left[ \alpha \hat{A}^\dagger(t) - \alpha^* \hat{A}(t) \right]|0, t\rangle. \quad (6)$$

One can check that the function

$$\psi_0(x, t) = \langle x|0, t\rangle = \pi^{-1/4} \epsilon(t)^{-1/4} \exp \left( -\epsilon(t)^{-1/2} x^2 \right),$$

is the normalized solution to the Schrödinger equation (2); for $t = 0$, it is equal to the wave function $\psi_0(x) = \pi^{-1/4} \exp(-x^2/2)$ of the oscillator ground state satisfying the condition $\hat{a}\psi_0(x) = 0$.

The Fock states of the parametric oscillator $|n, t\rangle$ satisfying the Schrödinger equation and the condition $\hat{A}^\dagger(t)\hat{A}(t)|n, t\rangle = n|n, t\rangle$, where $n = 0, 1, 2, \ldots$, are given by the formula

$$|n, t\rangle = \frac{(\hat{A}^\dagger)^n(t)}{\sqrt{n!}}|0, t\rangle. \quad (8)$$

The coherent states of the parametric oscillator (6) are expressed in terms of Fock states (8),

$$|\alpha, t\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n, t\rangle. \quad (9)$$

Since the coherent state $|\alpha, t\rangle$ is given by Eq. (6), which provides the relation

$$|\alpha, t\rangle = e^{-|\alpha|^2/2} \exp \left( \alpha \hat{A}^\dagger(t) \right)|0, t\rangle,$$

one has an explicit expression for the wave function of the coherent state in the position representation; it reads

$$\psi_\alpha(x, t) = \langle x|\alpha, t\rangle = \psi_0(x, t) \exp \left( -\frac{|\alpha|^2}{2} + \frac{\sqrt{2} \alpha x}{\epsilon(t)} - \frac{\alpha^2 \epsilon^*(t)}{2 \epsilon(t)} \right). \quad (10)$$

The wave function $\psi_n(x, t) = \langle x|n, t\rangle$ can be obtained using the generating function for Hermite polynomials

$$e^{-t^2 + 2tx} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x),$$

and formula (9), where the parameter $\alpha$ is used to get the coefficient in the series determining the vector $|n, t\rangle$ and consequently the wave function $\psi_n(x, t)$ in the decomposition of the coherent-state wave function (10). We obtain the wave function $\psi_n(x, t)$ in an explicit form as follows:

$$\psi_n(x, t) = \left( \frac{\epsilon(t)}{\epsilon(t)} \right)^{n/2} \psi_0(x, t) \frac{1}{\sqrt{2^n n!}} H_n \left( \frac{x}{|\epsilon(t)|} \right). \quad (12)$$
For $\omega(t) = 1$ and $\epsilon(t) = e^{it}$, the coherent-state wave function becomes

$$\psi_\alpha(x,t) = \frac{-(|\alpha|^2/2) - it/2}{\pi^{1/4}} \exp \left( -\frac{x^2}{2} + \sqrt{2}ae^{-it}x - \frac{e^{-2it}\alpha^2}{2} \right).$$  \hspace{1cm} (13)

The wave function (12) has the standard form

$$\psi_n(x,t) = e^{-it(n+1/2)} \frac{e^{-x^2/2}}{\pi^{1/4} \sqrt{2^n n!}} H_n(x).$$  \hspace{1cm} (14)

### 3. Tomographic Probability Representation of the Parametric Oscillator States

The density matrix $\rho_\alpha(x,x',t)$ of coherent states (6) of the parametric oscillator has the Gaussian form,

$$\rho_\alpha(x,x',t) = \psi_\alpha(x,t)\psi_\alpha^*(x',t) = \langle x|\hat{\rho}_\alpha(t)|x'\rangle.$$  \hspace{1cm} (15)

In [27], the construction of the symplectic tomographic probability representation of the system states with continuous variables, like the oscillator, was proposed using the invertible map of the state density operators at time $t = 0$ onto fair conditional probability distributions $w_{\rho}(X|\mu,\nu)$ of a random variable (oscillator position) $-\infty \leq X \leq \infty$. It depends also on the parameters $-\infty < \mu, \nu < \infty$ characterizing the reference frame in the phase space $(q,p)$, where this position is measured. The map is given by the relation

$$w_{\rho_j}(X|\mu,\nu) = T r \hat{\delta}(X\hat{1} - \mu\hat{q} - \nu\hat{p}).$$  \hspace{1cm} (16)

The function is called the symplectic tomogram of the oscillator state. The given formula can be used to express the density operator in terms of the tomogram (probability distribution) $w_{\rho_j}(X|\mu,\nu)$, i.e.,

$$\hat{\rho} = \frac{1}{2\pi} \int dX d\mu d\nu w_{\rho_j}(X|\mu,\nu) \exp i(X\hat{1} - \mu\hat{q} - \nu\hat{p}).$$  \hspace{1cm} (17)

In (16) and (17), operators $\hat{q}$ and $\hat{p}$ are the position and momentum operators, respectively; also we assume the Planck constant $\hbar = 1$ as well as the oscillator mass $m = 1$.

For pure states $|\psi\rangle$, the expressions for the symplectic tomogram can be given in terms of the fractional Fourier transform of the wave function [44],

$$w_{\psi_j}(X|\mu,\nu) = \frac{1}{2\pi |\nu|} \left| \int \psi(y) \exp \left( \frac{i\mu y^2}{2\nu} - \frac{iyX}{\nu} \right) dy \right|^2.$$  \hspace{1cm} (18)

Tomograms of pure and mixed states are nonnegative and satisfy the normalization condition for arbitrary values of parameters $\mu$ and $\nu$, i.e.,

$$\int w_{\rho_j}(X|\mu,\nu) \, dX = 1.$$

The tomogram is related to the state’s Wigner function $W(q,p)$ [24] given by the Fourier transform of the density matrix $\rho(x,x')$,

$$W(q,p) = \int \rho \left( q + \frac{u}{2}, q - \frac{u}{2} \right) e^{-imu} \, du.$$  \hspace{1cm} (19)
The relation is given by the Radon transform \[23\]
\[
w(X|\mu,\nu) = \int W(q,p)\delta(X - \mu q - \nu p)\frac{dq dp}{2\pi}.
\] (20)

The Wigner function can be reconstructed if the tomogram is known,
\[
W(q,p) = \frac{1}{2\pi} \int w_\rho(X,\mu,\nu) \exp(i(X - \mu q - \nu p)) dX d\mu d\nu.
\] (21)

For experimental study of photon states, optical tomograms \(w^{(\text{opt})}(X|\theta)\) measured by homodyne detectors, where \(X\) is the photon quadrature and \(\theta\) is a local oscillator phase, are used to reconstruct the Wigner function \[22\].

The symplectic tomogram determines the optical tomogram \(w_\rho^{(\text{opt})}(X|\theta)\) given by the relation
\[
w_\rho^{(\text{opt})}(X|\theta) = \text{Tr} \left( \hat{\rho} \delta(X \hat{1} - \hat{q} \cos \theta - \hat{p} \sin \theta) \right),
\] (22)
which can be rewritten in terms of the Wigner function using the Radon transform
\[
w_\rho^{(\text{opt})}(X|\theta) = \int W_\rho(q,p)\delta(X - q \cos \theta - p \sin \theta)\frac{dq dp}{2\pi}.
\] (23)

This means that
\[
w_\rho^{(\text{opt})}(X|\theta) = w_\rho(X|\mu = \cos \theta, \nu = \sin \theta)
\] (24)
and, in view of the Dirac delta-function property \(\delta(\lambda x) = |\lambda|^{-1}\delta(x)\), the optical tomogram determines the symplectic tomographic probability distribution
\[
w_\rho(X|\mu,\nu) = \frac{1}{\sqrt{\mu^2 + \nu^2}} w_\rho^{(\text{opt})}\left( \frac{X}{\sqrt{\mu^2 + \nu^2}}|\theta = \arctan \frac{\nu}{\mu} \right).
\] (25)

For the parametric oscillator state with the wave function (7), the symplectic tomographic probability distribution is the normal distribution of a random variable \(X\); it has the form
\[
w_0(X|\mu,\nu,t) = \frac{1}{\sqrt{2\pi\sigma(\mu,\nu,t)}} \exp\left( -\frac{X^2}{2\sigma(\mu,\nu,t)} \right).
\] (26)

Here, the dispersion parameter reads
\[
\sigma(\mu,\nu,t) = \mu^2\frac{|\kappa(t)|^2}{2} + \nu^2\frac{|\tilde{\kappa}(t)|^2}{2} + 2\mu\nu\sigma_{qp}(t).
\] (27)

This expression follows from the relation determined by the contribution of the Dirac delta-function term in the density operator (17)
\[
X \hat{1} = \mu\hat{q} + \nu\hat{p},
\] (28)
which provides the equality
\[
\langle X^2 \hat{1} \rangle = \mu^2\langle \hat{q}^2 \rangle_0 + \nu^2\langle \hat{p}^2 \rangle_0 + 2\mu\nu\left( \frac{\hat{q}\hat{p} + \hat{p}\hat{q}}{2} \right)_0.
\] (29)
For the parametric oscillator state with the wave function (7), one has
\[
\langle \hat{q} \rangle = 0, \quad \langle \hat{p} \rangle = 0, \quad \langle \hat{q}^2 \rangle_0 = \frac{|\epsilon(t)|^2}{2}, \quad \langle \dot{p}^2 \rangle_0 = \frac{1}{2},
\]
and the covariance term satisfies the relation depending on the correlation coefficient
\[
r^2 = \frac{\sigma_{qp}^2}{\langle \hat{q}^2 \rangle_0 \langle \dot{p}^2 \rangle_0}, \quad \sigma_{qp} = \frac{\langle \hat{q} \dot{p} + \dot{q} \hat{p} \rangle}{2} = \frac{1}{4} \cdot \frac{1}{1 - r^2}.
\]
This equality means that the state (7) provides the bound in the Schrödinger–Robertson [45,46] uncertainty relation.

Thus, we have the following property of the quantum parametric oscillator state (7). The variances and covariances of these oscillator state are determined by the solution \( \epsilon(t) \) and \( \dot{\epsilon}(t) \) for the classical parametric oscillator motion. The correlation coefficient \( r(t) \), being dependent on time \( t \), is expressed also in terms of the trajectories \( \epsilon(t) \) and \( \dot{\epsilon}(t) \) of the classical parametric oscillator [47],
\[
r^2(t) = 1 - |\epsilon(t)\dot{\epsilon}(t)|^{-2}.
\]

For the coherent state of the parametric oscillator with the wave function (10), the tomographic probability distribution \( w_\alpha(X|\mu,\nu,t) \) has the form of the normal probability distribution
\[
w_\alpha(X|\mu,\nu,t) = \frac{1}{\sqrt{2\pi\sigma_\alpha(\mu,\nu,t)}} \exp\left( -\frac{(X - \tilde{X}_\alpha(\mu,\nu,t))^2}{2\sigma_\alpha(\mu,\nu,t)} \right),
\]
where
\[
\tilde{X}_\alpha(\mu,\nu,t) = \mu \langle \hat{q} \rangle_\alpha + \nu \langle \hat{p} \rangle_\alpha
\]
and
\[
\langle \hat{q} \rangle_\alpha = \sqrt{2} \Re(\alpha \epsilon^*(t)), \quad \langle \hat{p} \rangle_\alpha = \sqrt{2} \Im(\alpha \dot{\epsilon}(t)).
\]
The parameter \( \sigma_\alpha(\mu,\nu,t) = \sigma(\mu,\nu,t) \) is given by (27).

Thus, the fair probability distribution (32) describes coherent states of the parametric oscillator, and this probability distribution contains complete information on the state.

The optical tomographic probability distribution \( w_\alpha^{(\text{opt})}(X|\theta) \) of the coherent state, which can be measured by the homodyne detector, has the form (32) with the parameters \( \mu = \cos \theta \) and \( \nu = \sin \theta \). This means that, for the parametric oscillator, the optical tomogram of the coherent state reads
\[
w_\alpha(X|\theta) = \frac{1}{\sqrt{2\pi\sigma_\alpha(\theta,t)}} \exp\left[ -\frac{(X - \tilde{X}(\theta,t))^2}{2\sigma_\alpha(\theta,t)} \right],
\]
where the mean photon quadrature is
\[
\tilde{X}(\theta,t) = \cos \theta \sqrt{2} \Re(\alpha \epsilon^*(t)) + \sin \theta \sqrt{2} \Im(\alpha \dot{\epsilon}(t))
\]
and
\[
\sigma_\alpha(\theta,t) = \cos^2 \theta \frac{|\epsilon(t)|^2}{2} + \sin^2 \theta \frac{|\dot{\epsilon}(t)|^2}{2} + \sin 2\theta (|\epsilon(t)\dot{\epsilon}(t)|^2 - 1)^{1/2}.
\]

The physical properties of coherent states of the parametric oscillator depend on frequency \( \omega(t) \). Varying the frequency, one can create both the squeezing phenomenon, i.e., \( |\epsilon(t)|^2 < 1 \) or \( |\dot{\epsilon}(t)|^2 < 1 \), as well as the correlation phenomenon, when \( |\epsilon(t)\dot{\epsilon}(t)|^2 > 1 \). The uncertainty relation [45,46] guarantees that, for the classical complex trajectory \( \epsilon(t) \), one has the inequality \( |\epsilon(t)\dot{\epsilon}(t)|^2 \geq 1 \). One can conjecture that, in the case of correlated coherent states discussed in [47], the squeezing phenomenon of Gaussian states can also take place.
4. Conditional and Joint Probability Distributions Determining the Oscillator’s Coherent States

The symplectic tomographic probability distribution \( w_\psi(X|\mu, \nu) \) of the parametric oscillator state with the wave function \( \psi(x) \) is determined in terms of the fractional Fourier transform of the wave function (18) [44], where \( X \) is the oscillator position measured in the reference frame of the oscillator phase space determined by real parameters \( \mu \) and \( \nu; -\infty < \mu, \nu < \infty \).

In the case of the classical parametric oscillator, one has the relation \( X = \mu q + \nu p \); for \( \mu = s \cos \theta \) and \( \nu = s^{-1} \sin \theta \), the reference frame parameters \( s \) and \( \theta \) provide the scale changes of the form \( q \rightarrow q' = sq \) and \( p \rightarrow p' = s^{-1}p \), along with the rotation of the axes \( q' \rightarrow X = \cos \theta q' + \sin \theta p' \) and \( p' \rightarrow \mathcal{P} = \sin \theta q' + \cos \theta p' \). The tomogram does not depend on the variable \( \mathcal{P} \).

The dependence on the parameters \( \mu \) and \( \nu \) of the symplectic tomogram \( w_\psi(X|\mu, \nu) \) provides the interpretation of the tomographic conditional probability distribution of the system position in the given reference frame. Using the Bayes’ formula, the joint probability distribution \( w_\psi(X, \mu, \nu) \) with the corresponding conditional probability distribution (18) can be introduced [48]

\[
w_\psi(X, \mu, \nu) = w_\psi(X|\mu, \nu) \mathcal{P}(\mu, \nu),
\]

where \( \mathcal{P}(\mu, \nu) \) is an arbitrary normalized marginal probability distribution \( 0 \leq \mathcal{P}(\mu, \nu) \leq 1 \), i.e.,

\[
\int \mathcal{P}(\mu, \nu) \, d\mu \, d\nu = 1.
\]

For example, the function \( \mathcal{P}(\mu, \nu) \) can be chosen as the normal distribution \( \mathcal{P}(\mu, \nu) = \frac{1}{\pi} \exp(-\mu^2 - \nu^2) \).

For a harmonic oscillator with frequency \( \omega = 1 \) and \( m = 1 \) described by the wave function of the coherent state (13), the tomogram \( w_\alpha(X, \mu, \nu, t = 0) \) is given as the normal joint probability distribution of three random variables

\[
w_\alpha(X, \mu, \nu, t = 0) = \frac{1}{\sqrt{2\pi \sigma}} \exp \left[ -\frac{(X - \bar{X})^2}{2\sigma} \right] \mathcal{P}(\mu, \nu),
\]

where \( \bar{X} = \sqrt{2}(\mu \text{Re} \alpha + \nu \text{Im} \alpha) \) and \( \sigma = (\mu^2 + \nu^2)/2 \).

For coherent state, the parametric oscillator tomogram \( w_\alpha(X|\mu, \nu, t) \) determines the joint probability distribution

\[
w_\alpha(X, \mu, \nu, t) = \frac{\exp(-\mu^2 - \nu^2)}{\pi \sqrt{2\pi \sigma_p(t)}} \exp \left[ -\frac{(X - \bar{X}(t))^2}{2\sigma_p(t)} \right],
\]

where \( \bar{X} = \mu \bar{q}(t) + \nu \bar{p}(t), \quad \bar{q}(t) = \sqrt{2} \text{Re} [\alpha \dot{\epsilon}(t)], \) and \( \bar{p}(t) = \sqrt{2} \text{Re} [\alpha \epsilon^*(t)] \).

The joint probability distribution (39) of three random variables \( X, \mu, \) and \( \nu \) provides the possibility to reconstruct tomogram (16) using the Bayes’ formula

\[
w_\rho(X|\mu, \nu) = w_\alpha(X, \mu, \nu) \mathcal{P}^{-1}(\mu, \nu).
\]
5. The Parametric State Evolution in the Probability Representation

For a given Hamiltonian of the parametric oscillator, the unitary evolution of the state vector $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$ provides the evolution of the density operators $\hat{\rho}_\psi(t) = \hat{U}(t)|\psi(0)\rangle\langle\psi(0)|\hat{U}^\dagger(t)$ and the evolution of the tomographic probability distribution of the form

$$w(X|\mu, \nu, t) = \int G(X, \mu, \nu, X', \mu', \nu', t)w(X'|\mu', \nu', t = 0) dX' d\mu' d\nu'. \quad (41)$$

In this section, we demonstrate that the evolution is given, using a specific change of the variables $X, \mu, \nu \rightarrow X(t), \mu(t), \nu(t)$ determined by the classical trajectories $\epsilon(t)$ and $\dot{\epsilon}(t)$.

In view of the properties of the Dirac delta-function $\delta(\lambda y) = |\lambda|^{-1}\delta(y)$, the tomogram $w(X|\mu, \nu, t)$ for arbitrary time $t$ has the property

$$w(\lambda X|\lambda \mu, \lambda \nu, t) = |\lambda|^{-1}w(X|\mu, \nu, t). \quad (42)$$

The density operator $\hat{\rho}(t)$ of an arbitrary state of the parametric oscillator evolves according to the following form of the solution

$$\dot{\hat{\rho}} = \hat{U}(t)\hat{\rho}(0)\hat{U}^\dagger(t) \quad (43)$$

of the von Neumann equation

$$\dot{\hat{\rho}}(t) + i[\hat{H}(t), \hat{\rho}(t)] = 0. \quad (44)$$

Here, the unitary operator $\hat{U}(t)$ is the solution of the Schrödinger equation

$$i\frac{\partial \hat{U}(t)}{\partial t} = \hat{H}(t)\hat{u}(t), \quad \hat{U}(0) = 1. \quad (45)$$

Calculating the tomographic probability distribution $w_\rho(X|\mu, \nu, t)$, in view of (16), we arrive at

$$w_\rho(X|\mu, \nu, t) = \text{Tr}(\hat{U}(t)\hat{\rho}(0)\hat{U}^\dagger(t)\delta(X\hat{1} - \mu\hat{q} - \nu\hat{p})) \quad (46)$$

Using the relation

$$\hat{U}^\dagger(t)\delta(X\hat{1} - \mu\hat{q} - \nu\hat{p})\hat{U}(t) = \delta(X\hat{1} - \mu\hat{q}_H(t) - \nu\hat{p}_H(t)), \quad (47)$$

where operators $\hat{q}_H(t)$ and $\hat{p}_H(t)$ are the position and momentum operators of the parametric oscillator in the Heisenberg representation, we obtain the tomogram $w_\rho(X|\mu, \nu, t)$ as follows:

$$w_\rho(X|\mu, \nu, t) = w_\rho(X|\mu_H(t), \nu_H(t), t = 0). \quad (48)$$

Parameters $\mu_H(t)$ and $\nu_H(t)$ are linear combinations of the parameters $\mu$ and $\nu$ with coefficients depending on the functions $\epsilon(t)$ and $\dot{\epsilon}(t)$. The integrals of motion $\hat{A}(t)$ and $\hat{A}^\dagger(t)$ (3), satisfying comutation relations (4) and the conditions

$$\frac{d\hat{A}(t)}{dt} + i[\hat{H}(t), \hat{A}(t)] = 0, \quad \frac{d\hat{A}^\dagger(t)}{dt} + i[\hat{H}(t), \hat{A}^\dagger(t)] = 0 \quad (49)$$

provide the possibilities to obtain the Heisenberg position operator $\hat{q}_H(t)$ and momentum operator $\hat{p}_H(t)$ satisfying the equations

$$\frac{\partial \hat{q}_H(t)}{\partial t} - i[\hat{H}(t), \hat{q}_H(t)] = 0, \quad \frac{\partial \hat{p}_H(t)}{\partial t} - i[\hat{H}(t), \hat{p}_H(t)] = 0, \quad (50)$$

19
as the linear combination of operators $\hat{q}$ and $\hat{p}$. This means that we obtain the following transform of the Dirac delta-function:

$$\delta(X\hat{1} - \mu\hat{q}_H(t) - \nu\hat{p}_H(t)) = \delta(X\hat{1} - \mu_H(t)\hat{q} - \nu_H(t)\hat{p}).$$  \hspace{1cm} (51)

Finally, we have explicit expressions for operators $\hat{q}_H(t)$ and $\hat{p}_H(t)$ in terms of complex functions $\epsilon(t)$ and $\dot{\epsilon}(t)$; they read

$$\hat{q}_H(t) = \frac{1}{2}\{\dot{\epsilon}(t)(\hat{q} + i\hat{p}) + \epsilon(\hat{q} - i\hat{p})\},$$

$$\hat{p}_H(t) = \frac{1}{2}\{\dot{\epsilon}(t)(\hat{q} + i\hat{p}) + \epsilon(\hat{q} - i\hat{p})\}. \hspace{1cm} (52)$$

In view of these explicit expressions, we arrive at

$$\mu_H(t) = \frac{1}{2}\{\mu[(\epsilon^*(t) + \epsilon(t)] + \nu[\dot{\epsilon}^*(t) + \epsilon(t)]\},$$

$$\nu_H(t) = \frac{1}{2}\{\mu[(\epsilon^*(t) - \epsilon(t)] + \nu[\dot{\epsilon}^*(t) - \epsilon(t)]\}. \hspace{1cm} (53)$$

Thus, for an arbitrary state of the parametric oscillator $\hat{\rho}(0)$, the initial tomogram $w_{\rho(0)}(X|\mu, \nu)$ becomes the tomographic probability distribution with the time dependence given by formula (48), where the parameters $\mu_H(t)$ and $\nu_H(t)$ are given by (53). Such kind of tomographic probability evolution takes place for arbitrary systems with Hamiltonians quadratic in the position and momentum.

### 6. Conclusions

To conclude, we point out the main results of our work.

We reviewed the known solution to the Schrödinger equation for parametric oscillator. We constructed the probability distributions, which can be identified with coherent states of a parametric oscillator. The dynamics of symplectic and optical tomographic probability distributions for the states of a quantum parametric oscillator is expressed in terms of classical trajectories of the classical parametric oscillator. Coherent states of a quantum parametric oscillator, which describe the phenomenon of squeezing and correlation of the oscillator’s position and momentum, are considered in the probability representation of quantum mechanics, and the optical and symplectic tomograms of the oscillator are obtained explicitly.

Different aspects of the tomographic approach to studying photon states, oscillator states, and qubit states were considered in [49–51].

The tomographic probability distributions can also describe classical oscillator states identified with the probability densities in the phase space. The classical oscillator states with Gaussian probability density in the phase space have symplectic and optical tomograms, which are normal probability distributions $w_{cl}(X|\mu, \nu)$ as in the case of quantum parametric oscillator considered in this work. But the set of such states for a classical parametric oscillator contains tomograms violating the Schrödinger–Robertson uncertainty relation. If one reconstructs the formal density operator, using such tomographic probability distribution of the classical parametric oscillator state with Gaussian tomogram and Gaussian probability density in the phase space, the formal density operator will have negative eigenvalues. The relation of tomograms of classical and quantum oscillators, as well as the case of multimode parametric oscillator and its coherent states, will be discussed in future publications.
References

1. E. Schrödinger, Naturwissenschaften, 14, 664 (1926).
2. R. J. Glauber, Phys. Rev. Lett., 10, 84 (1963).
3. C. E. Cahill and R. J. Glauber, Phys. Rev., 177, 1882 (1969).
4. R. J. Glauber, Phys. Rev., 131, 2766 (1963).
5. R. Glauber and V. I. Man’ko, “Damping and fluctuations in the systems of two entangled quantum oscillators,” in: A. A. Komar (Ed.), Group Theory, Gravitation, and Physics of Elementary Particles, Proceedings of the Lebedev Physical Institute, Nauka, Moscow (1986), Vol. 167 [English translation by Nova Science, Commack, New York (1987), Vol. 167].
6. R. Glauber and V. I. Man’ko, Zh. Éksp. Teor. Fiz., 87, 790 (1984) [Sov. Phys. JETP, 60, 450 (1984)].
7. R. Glauber and V. I. Man’ko, “Damping and fluctuations in the systems of two entangled quantum oscillators,” in: A. A. Komar (Ed.), Group Theory, Gravitation, and Physics of Elementary Particles, Proceedings of the Lebedev Physical Institute, Nauka, Moscow (1986), Vol. 167 [English translation by Nova Science, Commack, New York (1987), Vol. 167].
8. R. Glauber and V. I. Man’ko, “Damping and fluctuations in the systems of two entangled quantum oscillators,” in: A. A. Komar (Ed.), Group Theory, Gravitation, and Physics of Elementary Particles, Proceedings of the Lebedev Physical Institute, Nauka, Moscow (1986), Vol. 167 [English translation by Nova Science, Commack, New York (1987), Vol. 167].
9. R. Glauber and V. I. Man’ko, “Damping and fluctuations in the systems of two entangled quantum oscillators,” in: A. A. Komar (Ed.), Group Theory, Gravitation, and Physics of Elementary Particles, Proceedings of the Lebedev Physical Institute, Nauka, Moscow (1986), Vol. 167 [English translation by Nova Science, Commack, New York (1987), Vol. 167].
10. I. Ya. Doskoch and M. A. Man’ko, J. Russ. Laser Res., 40, 1 (2019).
11. I. Ya. Doskoch and M. A. Man’ko, Quantum Rep., 1(2), 130 (2019).
12. M. A. Man’ko, Phys. Scr., 87, 038013 (2013).
13. M. Scully and M. S. Zubairy, Quantum Optics, Cambridge University Press (1997).
14. W. Schleich, Quantum Optics in Phase Space, Wiley-VCH (2001).
15. W. Heisenberg, Z. Phys., 43, 172 (1927).
16. E. G. Sudarshan, Phys. Rev. Lett., 10, 277 (1863).
17. K. Husini, Proc. Phys. Math. Soc. Jpn., 22, 264 (1940).
18. V. Ermakov, “Second-order differential equation. Conditions of complete integrability,” Kiev University Izvestia, Series III, 9, 1 (1980) [English translation: A. O. Harin, Appl. Anal. Discrete Math., 2, 123 (2008)].
19. H. R. Lewis and W. B. Reisenfeld, J. Math. Phys., 10, 1458 (1969).
20. I. Ya. Doskoch and M. A. Man’ko, J. Russ. Laser Res., 40, 1 (2019).
21. I. Ya. Doskoch and M. A. Man’ko, Quantum Rep., 1(2), 130 (2019).
22. M. A. Man’ko, Phys. Scr., 87, 038013 (2013).
23. M. Scully and M. S. Zubairy, Quantum Optics, Cambridge University Press (1997).
24. W. Schleich, Quantum Optics in Phase Space, Wiley-VCH (2001).
25. W. Heisenberg, Z. Phys., 43, 172 (1927).
26. E. G. Sudarshan, Phys. Rev. Lett., 10, 277 (1863).
27. K. Husini, Proc. Phys. Math. Soc. Jpn., 22, 264 (1940).
28. V. Ermakov, “Second-order differential equation. Conditions of complete integrability,” Kiev University Izvestia, Series III, 9, 1 (1980) [English translation: A. O. Harin, Appl. Anal. Discrete Math., 2, 123 (2008)].
19. H. R. Lewis and W. B. Reisenfeld, J. Math. Phys., 10, 1458 (1969).
20. I. Ya. Doskoch and M. A. Man’ko, J. Russ. Laser Res., 40, 1 (2019).
21. I. Ya. Doskoch and M. A. Man’ko, Quantum Rep., 1(2), 130 (2019).
22. M. A. Man’ko, Phys. Scr., 87, 038013 (2013).
23. M. Scully and M. S. Zubairy, Quantum Optics, Cambridge University Press (1997).
24. W. Schleich, Quantum Optics in Phase Space, Wiley-VCH (2001).
25. W. Heisenberg, Z. Phys., 43, 172 (1927).
26. E. G. Sudarshan, Phys. Rev. Lett., 10, 277 (1863).
27. K. Husini, Proc. Phys. Math. Soc. Jpn., 22, 264 (1940).
28. V. Ermakov, “Second-order differential equation. Conditions of complete integrability,” Kiev University Izvestia, Series III, 9, 1 (1980) [English translation: A. O. Harin, Appl. Anal. Discrete Math., 2, 123 (2008)].
19. H. R. Lewis and W. B. Reisenfeld, J. Math. Phys., 10, 1458 (1969).
20. I. Ya. Doskoch and M. A. Man’ko, J. Russ. Laser Res., 40, 1 (2019).
21. I. Ya. Doskoch and M. A. Man’ko, Quantum Rep., 1(2), 130 (2019).
22. M. A. Man’ko, Phys. Scr., 87, 038013 (2013).
23. M. Scully and M. S. Zubairy, Quantum Optics, Cambridge University Press (1997).
24. W. Schleich, Quantum Optics in Phase Space, Wiley-VCH (2001).
25. W. Heisenberg, Z. Phys., 43, 172 (1927).
26. E. G. Sudarshan, Phys. Rev. Lett., 10, 277 (1863).
27. K. Husini, Proc. Phys. Math. Soc. Jpn., 22, 264 (1940).
28. V. Ermakov, “Second-order differential equation. Conditions of complete integrability,” Kiev University Izvestia, Series III, 9, 1 (1980) [English translation: A. O. Harin, Appl. Anal. Discrete Math., 2, 123 (2008)].
19. H. R. Lewis and W. B. Reisenfeld, J. Math. Phys., 10, 1458 (1969).
20. I. Ya. Doskoch and M. A. Man’ko, J. Russ. Laser Res., 40, 1 (2019).
21. I. Ya. Doskoch and M. A. Man’ko, Quantum Rep., 1(2), 130 (2019).
22. M. A. Man’ko, Phys. Scr., 87, 038013 (2013).
23. M. Scully and M. S. Zubairy, Quantum Optics, Cambridge University Press (1997).
24. W. Schleich, Quantum Optics in Phase Space, Wiley-VCH (2001).
25. W. Heisenberg, Z. Phys., 43, 172 (1927).
26. E. G. Sudarshan, Phys. Rev. Lett., 10, 277 (1863).
27. K. Husini, Proc. Phys. Math. Soc. Jpn., 22, 264 (1940).
28. V. Ermakov, “Second-order differential equation. Conditions of complete integrability,” Kiev University Izvestia, Series III, 9, 1 (1980) [English translation: A. O. Harin, Appl. Anal. Discrete Math., 2, 123 (2008)].
42. Z. Blanco-Garcia, O. Rosas-Ortiz, and K. Zelaya, “Interplay between Riccati, Ermakov and Schrödinger equations to produce complex-valued potentials with real energy spectrum,” *Math. Methods Appl. Sci.* (2018), p. 1.
43. D. J. Fernández and V. Hussin, *J. Phys. A: Math. Gen.*, **32**, 3603 (1999).
44. V. I. Man’ko and R. V. Mendes, *Phys. Lett. A*, **263**, 53 (1999).
45. E. Schrödinger, *Ber. Kgl. Akad. Wiss. Berlin*, 296 (1930).
46. H. P. Robertson, *Phys. Rev. A*, **35**, 667 (1930).
47. V. V. Dodonov, E. V. Kurmushev, and V. I. Man’ko, *Phys. Lett. A*, **79**, 150, (1980).
48. M. A. Man’ko and V. I. Man’ko, *Entropy*, **20**, 692:1 (2018).
49. O. V. Man’ko and V. I. Man’ko, *Laser Phys.*, **19**, 1804 (2009).
50. O. V. Man’ko and V. I. Man’ko, *Fortschritte Phys.*, **57**, 1054 (2009).
51. O. V. Man’ko and V. I. Man’ko, *Phys. Scr.*, **T140**, 014028 (21010).