ASSOCIATOR EQUATIONS AND DEPTH FILTERATIONS - I

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Abstract. This is the first part of a work whose goal is study associator equations in a way which is adapted to the framework of crystalline pro-unipotent fundamental groupoids. Our general goal is to reformulate (some natural consequences of) the associator equations as an explicit comparison between the respective modules of coefficients of an associator and its image under a certain automorphism, this comparison being compatible with their respective depth filtrations and defined over a ring of rational coefficients whose denominators have their $p$-adic norms bounded in a certain specific way.

In this first paper, we achieve this goal for a certain part of the associator equations. We deduce from our result new proofs to known properties of "depth reduction" for multiple zeta values, i.e. the vanishing of certain depth-graded multiple zeta values; this shows that associator equations can be adapted to the study of depth-graded multiple zeta values.

One of these depth reductions has a specific application to $p$-adic multiple zeta values, which involves finite multiple zeta values, and the notion of adjoint multiple zeta values which we introduced in [J2] and which we used for studying $p$-adic multiple zeta values. We also interpret others of our results and our approach in terms of the crystalline pro-unipotent fundamental groupoid of $\mathbb{P}^1 - \{0, 1, \infty\}$.

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1. Introduction

1.1. The notion of associator has been defined by Drinfeld in [Dr]. Its abstract definition comes from the axioms of a quasi-triangular quasi-Hopf algebra. A concrete example of associator from [Dr], §2, which is significative for our purposes, is the generating series of multiple zeta values, which are the following real numbers: for $d \in \mathbb{N}^*$, $s_d, \ldots, s_1 \in \mathbb{N}^*$ such that $s_d \geq 2$, and $(\epsilon_{s_d + \ldots + s_1}, \ldots, \epsilon_1) = (0, \ldots, 0, 1, \ldots, 0, 0, 1)$,

\begin{equation}
\zeta(s_d, \ldots, s_1) = \sum_{0 < n_1 < \ldots < n_d} \frac{1}{n_1^{s_1} \cdots n_d^{s_d}} = (-1)^d \int_{0 < t_1 < \ldots < t_{sd + \ldots + s_1} < 1} \lambda_{i=1}^{s_d + \ldots + s_1} \frac{dt_i}{t_i - \epsilon_i} \in \mathbb{R}
\end{equation}

One says that the word $(s_d, \ldots, s_1)$ is of weight $s_d + \ldots + s_1$ and of depth $d$.

In [Dr], §2, the generating series of multiple zeta values is viewed implicitly as arising from the Betti-De Rham comparison of the pro-unipotent fundamental groupoid $(\pi_1^{\mathrm{un}})$, in the sense of [Dr], of $\mathcal{M}_{0,4} = \mathbb{P}^1 - \{0, 1, \infty\}$ and $\mathcal{M}_{0,5}$ where, for $n \in \mathbb{N}^*$ such that $n \geq 4$, $\mathcal{M}_{0,n} = \{(x_1, \ldots, x_n) \in (\mathbb{P}^1)^n \mid \text{for all } i,j, x_i \neq x_j\}$
$x_j$ / $\text{PGL}_2$ is the moduli space of curves of genus zero with $n$ marked points. In other words, multiple zeta values are periods of $\pi_1^{\text{un}}(\mathbb{P}^1 - \{0, 1, \infty\})$, and this follows from their integral representation in [1]. Multiple zeta values being periods, one is led to study the polynomial equations over $\mathbb{Q}$ existing in the $\mathbb{Q}$-algebra generated by multiple zeta values using algebraic geometry. The fact that the generating series of multiple zeta values is an associator actually amounts to families of such polynomial equations; examples in low weight are $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(2, 1) = \zeta(3)$. It is conjectured that the associator equations imply all polynomial equations over $\mathbb{Q}$ satisfied by multiple zeta values. There exist natural generalizations of multiple zeta values associated with the pro-unipotent fundamental groupoid of $\mathbb{P}^1 - \{0, \mu_N, \infty\}$, $N \in \mathbb{N}^*$; they are called cyclotomic multiple zeta values.

1.2. This paper is originated in our study ([Jn], $n = 1, \ldots, 12$) of the crystalline pro-unipotent fundamental groupoid, in the sense of in [B] §11, [CL], [S1], [S2], of $(\mathbb{P}^1 - \{0, \mu_N, \infty\})/\mathbb{F}_q$ with $N$ prime to $q$. That study is in particular a study of the $p$-adic analogues of cyclotomic multiple zeta values, which are defined via the Frobenius of $\pi_1^{\text{un,crys}}((\mathbb{P}^1 - \{0, \mu_N, \infty\})/\mathbb{F}_q)$.

The preliminary step of that study, [L1], is to solve the differential equation satisfied by the Frobenius, in a way which is compatible with the depth filtration and, especially, implies a certain type of bounds of valuations of overconvergent $p$-adic hyperlogarithms, which depend in a specific way on $p$, and on the weight and on the depth of the indices. Those bounds of valuations are the key technical lemma for proving the main result of [L2], on which all our theory relies.

Our proof of those bounds of valuations requires to use that the Frobenius satisfies a certain algebraic relation; when $N = 1$, that algebraic relation is an associator equation, and we guess that for any $N$ it is a consequence of cyclotomic associator equations in the sense of Enriquez [E]. It was natural to consider intrinsically the analogue of the generating series of $p$-adic multiple zeta values defined by iterated integration from 0 to $\infty$, instead of from 0 to 1, and to view that algebraic relation as a comparison between the coefficients of those two generating series. It was essential to use that this algebraic relation is compatible with the depth filtrations and, moreover, that it has coefficients in $\mathbb{Z}$ and thus not too big $p$-adically.

This led us to the present work: its first goal is to understand more generally, and intrinsically, the compatibilities between associator equations, or some of their natural consequences, with the depth filtration, and the $p$-adic norms of the coefficients of those equations, as well as the possibility to view associator associations as comparisons of the respective modules of coefficients of the associator itself and its image under an automorphism of $\pi_1^{\text{un,DR}}(\mathcal{M}_{0,4})$ or $\pi_1^{\text{un,DR}}(\mathcal{M}_{0,5})$ induced by an automorphism of $\mathcal{M}_{0,4}$ or $\mathcal{M}_{0,5}$.

1.3. The goal described just above has the following $p$-adic motivation. Let us consider, more generally, the crystalline pro-unipotent fundamental groupoid of a smooth algebraic variety $X_0$ over a perfect field of characteristic $p > 0$. Its Frobenius is still compatible with its groupoid structure, its morphisms of local monodromy at tangential base-points, it is still functorial, and it still acts by the multiplication by $\frac{1}{p}$ at tangential base-points. When we take $X_0$ to be the moduli spaces $\mathcal{M}_{0,n}$, these properties imply, by formal arguments, that the generating series of $p$-adic multiple zeta values is a degenerated associator, as one can see by the proof of that result by Ünver in [L] which relies on $\pi_1^{\text{un,crys}}(\mathcal{M}_{0,4})$ and $\pi_1^{\text{un,crys}}(\mathcal{M}_{0,5})$.

In the case of a more general variety, under certain natural assumptions on $X_0$, these properties imply, by the same formal arguments, that the $p$-adic periods associated with $\pi_1^{\text{un,crys}}(X_0)$ satisfy certain algebraic relations.

These algebraic relations could be crucial for conducting explicit computations in $\pi_1^{\text{un,crys}}(X_0)$ as they were in the case of $\mathbb{P}^1 - \{0, \mu_N, \infty\}$. We will study them in another paper.

1.4. The second motivation for this paper is the study of the role played by the depth filtration regarding the algebraic relations among multiple zeta values; and, in particular, the study of depth-graded multiple zeta values.
The depth filtration on the $\mathbb{Q}$-vector space $Z$ of multiple zeta values is the data of, for each $d \in \mathbb{N}^*$, the $\mathbb{Q}$-vector space $Z_{\leq d}$ generated by multiple zeta values of depth $\leq d$. Conjecturally, the polynomial relations over $\mathbb{Q}$ among multiple zeta values are homogeneous for the weight, but this is not true for the depth; for example, we have $\zeta(3) = \zeta(2, 1)$. One calls depth-graded multiple zeta values the images of multiple zeta values in the graded space associated to the depth filtration, i.e. for each $d \in \mathbb{N}^*$, the images of the $\zeta(s_1, \ldots, s_d)$’s in $Z_{\leq d}/Z_{\leq d-1}$. Dual to the notion of depth-graded multiple zeta values is the notion of depth reduction: a $\mathbb{Q}$-linear combination of multiple zeta values of depth $d \in \mathbb{N}^*$ is said to admit a depth reduction if it can be written as a polynomial over $\mathbb{Q}$ of multiple zeta values of depth $\leq d - 1$ and $2\pi i$ i.e. if the corresponding depth-graded multiple zeta value is zero modulo $2\pi i$ - we add the condition modulo $2\pi i$ here because it will be convenient in the next paragraphs.

The depth is significant in the study of algebraic relations in the following ways.

The depth filtration "is motivic", which means that it is compatible with the motivic Galois action on $\pi_{1,\text{un},\text{DR}}(\mathbb{P}^1 - \{0, 1, \infty\}, -\Gamma_1, \Gamma_0)$, and thus descends to a filtration on motivic multiple zeta values in the sense of [BI]. Thus, one has a notion of motivic depth-graded multiple zeta values, and it makes sense to view depth-graded multiple zeta values as periods.

Another reason to be interested in the algebraic relations among depth-graded multiple zeta values is that they are known to be connected to algebraic relations between periods of the $\pi_{1,\text{un}}$ of the moduli spaces $M_{1,n}$, in particular, certain iterated integrals of modular forms. This was suggested by Broadhurst-Kreimer’s conjecture reviewed in [IKZ], and this is studied in [B2].

1.5. Aside from associator relations, the other standard family of polynomial relations among multiple zeta values is the regularized double shuffle relations (see [IKZ] for a detailed review). They are immediate consequences of equation (1) and yet, it is conjectured, as is for associator relations, that they imply all polynomial equations over $\mathbb{Q}$ satisfied by multiple zeta values.

An interesting aspect of the regularized double shuffle relations is that they are naturally adapted to the depth filtration. This means that their depth-graded version gives instantly a good conjectural description of algebraic relations between depth graded multiple zeta values. A significative part of [IKZ] is a description of the "depth-graded double shuffle relations" and of some of their consequences.

By contrast, the associator equations are not naturally adapted to the depth filtration. For each $n \in \mathbb{N}^*$, the highest depth appearing in the term of weight $n$ of an associator equation is essentially equal to $n$, and the corresponding term is not very significative. Yet, it is conjectured that double shuffle relations and associator relations are equivalent to each other.

What we develop in this paper is a partial analogue for the associator relations of the description of depth-graded double shuffle relations of [IKZ]. We are going to rewrite part of the associator equations, in a way for which the highest depth terms are significative, in a certain sense which we will make precise. By a non-trivial result of Furusho [F3], associators satisfy the double shuffle relations, and we will also retrieve directly certain properties of associators known indirectly via double shuffle relations.

Whereas the associator equations depend on an associator $\Phi$, our reformulation will be made in terms of a couple $(\Phi, \Phi_\infty)$ where $\Phi_\infty$ is the image of $\Phi$ by a certain automorphism of the ring of formal power series containing $\Phi$; and we will consider the depth filtration for $\Phi$ and for $\Phi_\infty$ at the same time. In our main example of multiple zeta values, $\Phi = \Phi_{KZ}$ will be the generating series of multiple zeta values and $\Phi_\infty = \Phi_{KZ,\infty}$ will be its image by the automorphism $(z \mapsto \frac{\sqrt{z}}{\pi})$ of $\pi_{1,\text{un}}(\mathbb{P}^1 - \{0, 1, \infty\})$.

The couple $(\Phi, \Phi_\infty)$ is the natural parameter for encoding a certain automorphism of the ring of formal power series containing $\Phi$ which, in the example of multiple zeta values, is defined by the monodromy of the connection on $\pi_{1,\text{un},\text{DR}}(\mathbb{P}^1 - \{0, 1, \infty\})$; such a type of automorphism is also the natural parameter of Kashiwara-Vergne equations, which are satisfied if $\Phi$ is an associator, by [AT], [AET]; in the $p$-adic context, the Frobenius of $\pi_{1,\text{un},\text{cryst}}(\mathbb{P}^1 - \{0, 1, \infty\} / \mathbb{F}_p)$, subjacent to $p$-adic multiple zeta values, is such...
an automorphism. The equations which we will write are partially related, but not equivalent to the usual formulation of a part of Kashiwara-Vergne equations.

1.6. In [J2] we have defined a notion of adjoint multiple zeta values, which is motivated by our study of \( \pi_{1,\text{un}}^{\text{crys}}((\mathbb{F}_q^{1} - \{0,\mu_{N},\infty\})/\mathbb{F}_q) \), and which appears in most of the papers [Jn], \( n \geq 3 \). Adjoint multiple zeta values are the coefficients of \( \text{Ad}_{\Phi_{KZ}}(e_1) \), where \( \text{Ad} \) is the adjoint action on \( \pi_{1,\text{un}}^{\text{DR}}(\mathcal{M}_{0,4}, -\bar{\mathcal{I}}, \bar{\Gamma}_0) \) viewed canonically as the algebraic group \( \pi_{1,\text{un}}^{\text{DR}}(\mathcal{M}_{0,4}, -\bar{\mathcal{I}}) \), and \( e_1 \) is one of the generators of the Lie algebra of \( \pi_{1,\text{un}}^{\text{DR}}(\mathcal{M}_{0,4}, -\bar{\mathcal{I}}) \). One can express easily adjoint multiple zeta values and multiple zeta values in terms of each other, by polynomial expressions over \( \mathbb{Q} \). The \( p \)-adic cyclotomic adjoint multiple zeta values appeared naturally in [J2] and [J3]. We have studied their algebraic properties in [J4] and [J5]. This paper is also an occasion to continue a little further the study of certain algebraic aspects of adjoint multiple zeta values, as we explain in §5.4.

1.7. The main result of this paper is the following; we omit the technical details in this introduction.

**Theorem 1.** (see Theorem 3.15 for a precise statement)
The 'one-dimensional' associator equations, in the sense of Definition 3.1 satisfied by \( \Phi \) imply (and, when \( \mu = 0 \), are equivalent to) an explicit equality between modules of coefficients of \( \Phi \) and of its variant \( \Phi_{\infty} \), over a filtered ring of rational coefficients with explicitly bounded denominators, and which is compatible with the depth filtration in a certain sense.

In order to see the meaning of this result from the point of view of algebraic relations among the coefficients of \( \Phi \), let us state and discuss here two of its corollaries. These are two examples of depth reduction, which had been proved by other methods, in particular, via the double shuffle equations, and which have been singled out in the literature, because of a special meaning or a special application. By our theorem, we find here different and sometimes simpler proofs.

**Corollary 1.** (see Corollary 4.2 for a more precise result; Tsumura, [Ts], §1, Theorem ; Ihara-Kaneko-Zagier, [IKZ], §8, corollary 8, Panzer, [P])
Let \( d \in \mathbb{N}^* \) and \( s_d, \ldots, s_1 \in \mathbb{N}^* \) such that \( s_d + \ldots + s_1 - d \) is odd. Then \( \zeta(s_d, \ldots, s_1) \) admits a depth reduction.

This result goes back to Euler in depth one and two. The depth one part is nothing else than the equality

\[
\zeta(2n) = \frac{|B_{2n}|}{2(2n)!} \pi^{2n} \text{ for all } n \in \mathbb{N}^*
\]

The depth two part is that multiple zeta values of depth two and odd weight admit a depth reduction; first example :

\[
\zeta(3) = \zeta(2, 1)
\]

The second corollary which we want to point out here has an application to \( p \)-adic and finite multiple zeta values, as we explain in §5.2.2, and is at the same time a property of adjoint multiple zeta values, as we explain in §5.4.

**Corollary 2.** (see Corollary 4.4 for a more precise result ; Yasuda, [Y], corollary 3.3 ; announced by Zagier)
Let \( d \in \mathbb{N}^* \) and \( s_d, \ldots, s_1 \in \mathbb{N}^* \). Then \( \zeta(s_d, \ldots, s_1) + (-1)^{s_d+\ldots+s_1} \zeta(s_1, \ldots, s_d) \) admits a depth reduction. Precisely, there is a natural way to write in depth \( \leq d - 1 \) the number

\[
\sum_{d'=0}^{d} (-1)^{s_{d'+1}+\ldots+s_d} \zeta(s_{d'+1}, \ldots, s_d) \zeta(s_{d'}, \ldots, s_1)
\]
1.8. Our method, using associators and the parameter \((\Phi, \Phi_\infty)\), has the following particularities:

i) for certain of these results (in particular, Corollary 2), it gives proofs which are simpler than using the double shuffle relations (as one can see by comparing our proof with the proof of [Y], corollary 3.3). This is primarily because in the framework of associators, many natural operations such as the inversion, adjoint action, algebraic automorphisms, Ihara product on \(\pi^{\text{un,DR}}_1(M_{0,4}, -\bar{I}_1, \bar{I}_0)\) can be used directly, and one can work with generating series until the last steps of the computation.

ii) it teaches us that the depth reductions of Corollary 1 and Corollary 2 have closely related origins: they follow from a same "part" of associator equations.

iii) it also provides natural counterparts for \(\Phi_\infty\) to all our results concerning \(\Phi\).

iv) it has a natural proximity with the \(p\)-adic world, as suggested by §1.2, §1.3: we will explain in §5 why, unlike for multiple zeta values, for \(p\)-adic multiple zeta values, \(\Phi_\infty\) is as natural as \(\Phi\) and unavoidable in the study of \(p\)-adic multiple zeta values, as well as why the depth filtration for \(\Phi_\infty\) is as natural as is the depth filtration for \(\Phi\).

v) it has a natural proximity with the study of adjoint multiple zeta values (this notion being itself naturally adapted for studying \(p\)-adic multiple zeta values).

In a second paper, we will prove a result similar to Theorem 1, but which will use the whole of associator equations and in a different way. Knowing that, by Furusho [F3], the pentagon equation implies the whole of associator equations, it should reimplies non-trivially the present results.

Outline. The definition of associators is recalled in §2, together with certain notations. A reformulation of certain "one-dimensional" associator equations and its depth-graded version are obtained in §3 and the depth reductions are deduced in §4. Applications and interpretations related to complex, \(p\)-adic and finite multiple zeta values, as well as to (complex or \(p\)-adic) adjoint multiple zeta values, are explained in §5; the main emphasis is on the \(p\)-adic applications.

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2. Definitions and notations

We review the definitions and some basic properties of associators. Since our applications and motivations concern the pro-unipotent fundamental groupoid, we will use notations which refer to \(\pi^\text{un}(M_{0,4}) = \mathbb{P}^1 - \{0, 1, \infty\}\) and \(\pi^\text{un}(M_{0,5})\).

2.1. The De Rham pro-unipotent fundamental groupoid of \(M_{0,4}\) and \(M_{0,5}\) at the canonical base-point. For \(X\) equal to \(M_{0,4}\) or \(M_{0,5}\) over \(Q\), let \(\omega_{\text{DR}} = \omega_{\text{DR}}(X)\) be the canonical base-point of the De Rham pro-unipotent fundamental groupoid of \(X\), in the sense of [D], §12.

Notation 2.1. Let the two pro-unipotent group schemes \(\Pi = \Pi^\text{un}(M_{0,4}) = \pi^\text{un,DR}_1(M_{0,4}, \omega_{\text{DR}})\) and \(\Pi^\text{un}(M_{0,5}) = \pi^\text{un,DR}_1(M_{0,5}, \omega_{\text{DR}})\).

The following presentations of \(\Pi^\text{un}_1(M_{0,4})\) and \(\Pi^\text{un}_1(M_{0,5})\) follow from [D], §12.

Proposition 2.2. i) The group scheme \(\Pi^\text{un}_1(M_{0,4})\) is the exponential of the pro-nilpotent Lie algebra over \(Q\) defined by the generators \(e_0, e_1, e_\infty\) and the relation \(e_0 + e_1 + e_\infty = 0\) (i.e. freely generated by \(e_0, e_1\)).

ii) The group scheme \(\Pi^\text{un}_1(M_{0,5})\) is the exponential of the pro-nilpotent Lie algebra over \(Q\) defined by the generators \(e_{ij}, 1 \leq i, j \leq 4\), and the relations \(e_{ii} = 0, e_{ij} = e_{ji}, [e_{jk} + e_{jl}, e_{kl}] = 0\) if \(j, k, l\) are pairwise distinct, and \([e_{ij}, e_{kl}] = 0\) if \(i, j, k, l\) are pairwise distinct.
We now focus on $\Pi = \Pi^m(M_{0,4})$ since we will work mostly with it. The next definitions and characterizations concerning $\Pi^m(M_{0,4})$ admit analogues for $\Pi^m(M_{0,5})$.

**Definition 2.3.** Let $\mathcal{O}^m$ be the shuffle Hopf algebra over $\mathbb{Z}$ associated with the alphabet $\{e_0, e_1\}$. As a $\mathbb{Z}$-module, it is freely generated by the words on $\{e_0, e_1\}$ (we take the convention to read them from the right to the left), including the empty word. Its product $\mathfrak{m}$ is the shuffle product, defined by the formula
\[
e_i \cdots e_{i_1} \mathfrak{m} e_{i_{r+2}} \cdots e_{i_{r+1}} = \sum_{\text{\scriptsize$\sigma$, permutation of $\{1, \ldots, r+1\}$}} \sigma(1) \cdots \sigma(r+1) e_{i_{\sigma(1)-1}} \cdots e_{i_{\sigma(r+1)-1}}\]
; its coproduct is the deconcatenation of words ; its antipode is defined by $S: e_i \mapsto (-1)^r e_1 \cdots e_i$ ; its counity $\epsilon$ is the augmentation map.

The number of letters of the words on $\{e_0, e_1\}$ is called their weight, and the weight defines a grading of the Hopf algebra $\mathcal{O}^m$.

We fix $K$ a field of characteristic zero.

**Notation 2.4.** We denote by $K\langle\langle e_0, e_1 \rangle\rangle$ the non-commutative ring of polynomials resp. formal power series over the non-commutative variables $e_0, e_1$.

We denote an element of $K\langle\langle e_0, e_1 \rangle\rangle$ in the following way : $f = \sum_w f[w]w$ where the sum is over the words on the alphabet $\{e_0, e_1\}$ (including the empty word) and $f[w] \in K$ for all $w$.

The notation $f[w]$ extends by linearity to $w$ equal to any element of $\mathcal{O}^m$.

Moreover, $K\langle\langle e_0, e_1 \rangle\rangle$ is the completion of the universal enveloping algebra of the pro-nilpotent Lie algebra freely generated by $e_0, e_1$, and is canonically endowed with a Hopf algebra structure.

**Proposition-Definition 2.5.** i) We have $\Pi = \text{Spec}(\mathcal{O}^m)$, and we have
\[
\Pi(K) = \{f \in K\langle\langle e_0, e_1 \rangle\rangle \mid \text{for all words } w, w', f[w \mathfrak{m} w'] = f[w]f[w'] \text{, and } f[\emptyset] = 1\}
\]
The equation appearing in (2) i) 'for all words $w, w'$, $f[w \mathfrak{m} w'] = f[w]f[w']^*$', is called the shuffle equation.

ii) The Hopf algebra structure on $K\langle\langle e_0, e_1 \rangle\rangle$ is dual to $\mathcal{O}^m$. Its coproduct $\Delta_m$ is such that the shuffle equation amounts to $\Delta_m(f) = f \otimes f$ i.e. it characterizes the elements which are primitive for $\Delta_m$.

Its Lie algebra consists the elements which satisfy shuffle equation modulo products, i.e. for all non-empty words $w, w'$, $f[w \mathfrak{m} w'] = 0$ ; this characterizes the elements which are primitive for $\Delta_m$, i.e. the elements $f$ satisfying $\Delta_m = f \otimes 1 + 1 \otimes f$.

The words over the alphabet $\{e_0, e_1\}$ are usually denoted by regrouping the consecutive letters $e_0$, in the following way : $e_0^{s_0-1} e_1 \cdots e_0^{s_1-1} e_1 e_0^{s_0-1}$, with $s_i \in \mathbb{N}^*$. In order to make certain formulas more readable, let us add to it an abbreviation :

**Notation 2.6.** We denote $e_0^{s_0-1} e_1 \cdots e_0^{s_1-1} e_1 e_0^{s_0-1}$ by $e_0^{s_0-1} \cdots e_0^{s_1-1} e_1$, and when $s_0 = 1$, we also denote it by $e_0^{s_0-1} \cdots e_0^{s_1-1} e_1$.

**Notation 2.7.** Let respectively $\mathcal{O}_D^{\leq d}$, $\mathcal{O}_D^{= d}$, $\mathcal{O}_D^{= d}$, $\mathcal{O}_D^{\geq d}$, be the sub-$\mathbb{Z}$-module of $\mathcal{O}^m$ spanned respectively by the shuffle polynomials of words whose depth is $\leq d$, $\leq d$ and $\geq 1, \geq d$.

We denote their subspans of weight $s \in \mathbb{N}^*$ by adding $\text{Wt} = s$ in the subscript : $\mathcal{O}_{\text{Wt}}^{= d}$, etc.

**Notation 2.8.** Let $\bar{\Pi}$ i.e. $\Pi^m(M_{0,4})$, resp. $\bar{\Pi}^m(M_{0,5})$ be the subscheme of $\Pi$ i.e. $\Pi^m(M_{0,4})$ resp. $\Pi^m(M_{0,5})$ defined by the equations $f[w] = 0$ for all $w$ of weight one".

**Fact 2.9.** i) Let $f$ be an element of $K\langle\langle e_0, e_1 \rangle\rangle$ satisfying for all words $w$, $f[w e_0 e_0] = 0$. For all $s_0, \ldots, s_l \in \mathbb{N}^*$, we have : $f[e_0^{s_0-1} e_1 \cdots e_0^{s_1-1} e_1] = \sum_i e_0^{s_i-1} e_1 \cdots e_0^{s_{l-1}-1} e_1$ $f[e_0^{s_0-1} e_1 \cdots e_0^{s_1-1} e_1] = \sum_i e_0^{s_i-1} e_1 \cdots e_0^{s_{l-1}-1} e_1$ $f[e_0^{s_0-1} e_1 \cdots e_0^{s_1-1} e_1] = \sum_i e_0^{s_i-1} e_1 \cdots e_0^{s_{l-1}-1} e_1$ ii) Let $f$ be an element of $K\langle\langle e_0, e_1 \rangle\rangle$ satisfying for all words $w$, $f[w e_0 e_0] = 0$. For all words $w$ and

\footnote{and to read the groupoid multiplication in $\Pi^m_{\text{DR}}(M_{0,4})$ from the right to the left : the reason for this choice is the correspondence between words and sequences of differential forms under iterated integrals, knowing that iterated integration corresponds to a composition of operators, and the composition of functions is usually denoted from the right to the left.}
n ∈ N*, we have: n.f[e^n_1e_0w] + f[e^n_1e_0(e1+m w)] = 0.

iii) In particular, if f ∈ /K) or if f is of the form f^−1e_2f, with z ∈ {0, 1, ∞} and f ∈ /K), then the Z-module f(Σ[K d.,w)] is generated by the coefficients of the form f(e^n s−1...s−1e_1) with s_d ≥ 2.

(Indeed, elements of the form f^−1e_2f with f ∈ /K) are primitive for ∆_m thus satisfy i) and ii), since the e_s’ are primitive and ∆_m is multiplicative.)

**Remark**: Let f ∈ /K) be the linear map removing the furthest to the left, resp. furthest to the right letter of words, i.e. the dual of f ∈ /K⟨⟨e_0,e_1⟩⟩ → (e_0 + e_1)f ∈ /K⟨⟨e_0,e_1⟩⟩, resp. the dual of f ∈ /K⟨⟨e_0,e_1⟩⟩ → f(e_0 + e_1) ∈ /K⟨⟨e_0,e_1⟩⟩.

We have ∂ o ∂ = ∂ o ∂.

**2.2. Associators.** Let K be a field of characteristic 0 and μ ∈ K.

**Definition 2.11.** (Drinfeld, [D]) The scheme of associators with coupling constant μ is the subscheme M_μ of /K(0,5) defined by the following equations on Φ(e_{12}, e_{23}) :

(3) Φ(e_{12}, e_{23} + e_{24})Φ(e_{13} + e_{23} + e_{34}) = Φ(e_{23} + e_{34})Φ(e_{12} + e_{13} + e_{24} + e_{34})Φ(e_{12}, e_{23})

(4) e^{μe_1} e + e^{μe_1}Φ(e_{13} + e_{12}) e^{μe_1}Φ(e_{31} + e_{23}) = e^{μe_1}Φ(e_{23}, e_{31}) = e^{μe_1}Φ(e_{12}, e_{23})^{-1}

(5) Φ(f_{23}, f_{12}) = Φ(e_{12}, e_{23})^{-1}

where equations [3], [4] and [5] are called, respectively, pentagon, hexagon and duality equations.

**Definition 2.12.** (Drinfeld, [D]) The Grothendieck-Teichmüller group GRT_1 is the subscheme of /K(0,5) defined by the equations :

(6) Φ(e_0, e_1)Φ(e_1, e_0) = 1

(7) Φ(e_∞, e_0)Φ(e_1, e_∞)Φ(e_0, e_1) = 1

(8) Φ(e_{12}, e_{23} + e_{24})Φ(e_{13} + e_{23} + e_{34}) = Φ(e_{23} + e_{34})Φ(e_{12} + e_{13} + e_{24} + e_{34})Φ(e_{12}, e_{23})

(9) e_0 + Φ^−1(e_0, e_1)c_1Φ(e_0, e_1) + Φ(e_0, e_∞)^−1Φ(e_0, e_∞) = 0

where equations [6], [7] and [8] are called, respectively 2-cycle or duality, 3-cycle, and 5-cycle of pentagon.

**Proposition-Definition 2.13.** (Drinfeld)

i) ([D], Proposition 5.9) The conjunction of the equations [3], [7] and [8] implies [6], and GRT_1 = M_0.

ii) ([D], equation (5.16)) GRT_1 is a group scheme with the Ihara product defined by (g_2 ◦ g_1)(e_0, e_1) = g_2(e_0, e_1)g_1(e_0, g_2^−1(e_0, e_2).

iii) ([D], Proposition 5.5) The left multiplication by the Ihara product defines an action of GRT_1 on M_μ which makes M_μ into a GRT_1-torsor.

**Proposition 2.14.** (Alekseev-Enriquez-Torossian, [AET], equation (3) in the introduction, proof in §5.2) For each μ ∈ K, an associator with coupling constant μ satisfies the following equation called the "equation of special automorphisms"

(10) − Φ^−1(e_0, e_1)e^{μe_0}Φ(e_0, e_1)e^{−μe_0} = e^{μe_0}Φ(e_0, e_∞)^−1e^{μe_∞}Φ(e_0, e_∞)e^{−μe_0}

**Remark 2.15.** The equation [10] is the coefficient of degree 1 with respect μ of equation [10]. In the next paragraphs, we could unify the statements for μ ≠ 0 and for μ = 0 as part of a same statement. Nevertheless, for readability, and also because we will take μ = 2iπ for the application to multiple zeta values and μ = 0 for the application to p-adic multiple zeta values, we will distinguish the two cases in most of the statements and proofs.

**Convention 2.16.** It will sometimes be convenient to say that μ has depth 0 and weight 1.
3. A quotient of associator equations as a comparison between $\Phi$ and $\Phi_\infty$

We define a certain quotient of the associator equations (§3.1) ; we define notations enabling to keep track of $p$-adic information on the rational coefficients of the algebraic equations that we will write (§3.2) ; we formulate differently the equations of §3.1 (§3.3) ; we write their depth-graded version of our reformulation (§3.4) ; we conclude by the main result (§3.5).

In the rest of the paper, $K$ is any field of characteristic 0.

3.1. Definition of a quotient of associator equations. Since $M_{0,4}$ is of dimension 1 and $M_{0,5}$ is of dimension 2, we will call "one-dimensional" equations some consequences of associator equations which can be expressed in terms of $\Pi_1^m(M_{0,4})$.

The ideal of $\text{Lie}(\Pi_1^m(M_{0,5}))$ generated by $e_{12} + e_{23} + e_{31}$ is a Lie ideal, and defines a quotient map $\text{Lie}(\Pi_1^m(M_{0,5})) \to \text{Lie}(\Pi_1^m(M_{0,4}))$. This map sends the hexagon equation (11) to the following equation
\[ \epsilon^2 e_0 \Phi(e_\infty, e_0) e^{\vec{e}_1} \Phi(e_1, e_\infty) e^{\vec{e}_1} \Phi(e_0, e_1) = 1 \]

**Definition 3.1.** We call the following equations "associator equations of dimension one" with coupling constant $\mu$ : the equations (5), (11) and (11) if $\mu \neq 0$, and the equations (6), (7) and (9) if $\mu = 0$.

3.2. A filtered ring of coefficients. For the rest of the paper, we must replace $O^m$ by a filtered module over a certain filtered ring ; this is a way to keep a track of the denominators of the rational coefficients of the algebraic equations that we are going to write.

**Definition 3.2.** i) Assume that $\mu \neq 0$. Let $C_\mu$ be the filtered ring $(\mathbb{Q}, \text{Fil})$ where $\text{Fil} = (\text{Fil}_s)_{s \in \mathbb{N}}$ is the filtration defined by : $\text{Fil}_s = \sum_{s_1 + s_2 \geq s} Z(q) Z(q^2)$.

ii) Let $C_\mu = Z$, with the trivial filtration ($\text{Fil}_s = Z$ for all $s \in \mathbb{N}$).

**Definition 3.3.** Let $\mathcal{O}^m \otimes C_\mu$, be the filtered module over $C_\mu$ defined as $\oplus_{s \in \mathbb{N}} (\mathcal{O}^m_{\text{Fil}_s} \otimes \text{Fil}_s(C_\mu))$.

3.3. A reformulation of the of equations of §3.1. We rewrite the equations of §3.1 in a way which will be more convenient for our purposes.

3.3.1. Preliminary : injectivity of the adjoint action on $e_i$. Let $i \in \{0, 1, \infty\}$. Consider the maps $K(\langle e_0, e_1 \rangle) \to K(\langle e_0, e_1 \rangle)$ defined as $\text{Ad}(e_i) : u \mapsto u^{-1} e_i u$ and $\text{Ad}(e^{\mu e_i}) : u \mapsto u^{-1} e^{\mu e_i} u$. (This notation is opposed to the usual one where $\text{Ad}(x)$ means $g \mapsto gxg^{-1}$ ; our notation is compatible with our convention to read words and the groupoid multiplication in $\pi_1^{\text{in,DR}}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ from the right to the left, explained in Definition 2.3.)

**Fact 3.4.** Then, for $f, g \in \Pi(K)$, we have
\[ \text{Ad}_f(e_i) = \text{Ad}_{e_i}(g) \Leftrightarrow \text{Ad}_f(e^{\mu e_i}) = \text{Ad}_g(e^{\mu e_i}) \Leftrightarrow f^{-1} g = \exp(\lambda e_i) \text{ for a } \lambda \in K \]

In particular, the restrictions of $\text{Ad}(e_i)$ and $\text{Ad}(e^{\mu e_i})$ to the $\tilde{\Pi}(K)$ are injective.

**Proof.** It is sufficient to restrict to the case where, for example, $i = 1$. Let thus $\partial_{e_0}, \partial_{e_1} : \mathcal{O}^m \to \mathcal{O}^m$ be the linear functions defined by, for all words $w$, $\partial_{e_0}(e_1 w) = w$, $\partial_{e_0}(e_0 w) = 0$, $\partial_{e_1}(we_1) = w$, $\partial_{e_1}(we_0) = 0$ and $\partial_{e_0}(\emptyset) = \partial_{e_1}(\emptyset) = 0$.

1) Let $u \in K(\langle e_0, e_1 \rangle)$ such that $u$ commutes to $e_1$. Let $w$ a word which is not of the form $e_1^n$, $n \in \mathbb{N}^*$. It can be written in a unique way in the form $e_1^a(w) e_0 z$, with $a(w) \in \mathbb{N}$ and $z$ a word. We have $f[w] = (we_1)[we_1] = (e_1 w)[we_1] = u(\partial_{e_1}(w)e_1)$. By induction on $a(w)$, this shows that $f[w] = 0$ for all words $w$ containing at least one letter $e_0$. Thus $u \in K(\langle e_1 \rangle)$ (see 3.1, §5 for a similar proof).

2) Let $u \in K(\langle e_0, e_1 \rangle)$ such that $u$ commutes to $e^{\mu e_1}$. Let a word $w$ which contains at least one letter $e_0$. It can be written in a unique way in the form $e_1^{a(w)} z e_1^{b(w)}$ with $(a(w), b(w)) \in (\mathbb{N})^2$ and $z$ a word such that $\partial_{e_1}(z) = \partial_{e_0}(z) = 0$ (i.e. $z = e_0$ or $z$ is of the form $e_0 \ldots e_0$). We have $(e^{\mu e_1} - 1)u =
Using a) and b), this proves that $f$ satisfies the duality equation and the hexagon equations. By induction on $(a(w), b(w))$ for the lexicographical order on $\mathbb{N}^2$, this shows that $f[w] = 0$ for all words $w$ containing at least one letter $e_0$, and the end of the proof is similar to 1). Thus $u \in K\langle\langle e_1 \rangle\rangle$.

3) It is trivial to check that, among all elements of $K\langle\langle e_1 \rangle\rangle$, those who satisfy the shuffle equation are those of the form $\exp(\lambda e_1)$ with $\lambda \in K$.

### 3.3.2. Reformulation of the equations of §3.1.

**Proposition 3.5.** Let $f$ a point of $\Pi$ and $\mu \in K$. We have an equivalence between:

1. $f$ satisfies the equations of Definition 3.1.
2. $f$ satisfies the duality and the hexagon equations $\Leftrightarrow f$ satisfies the duality equation and

$$e^{\frac{\mu e_1}{\mu e_0}} f(e_0, e_\infty) = f(e_\infty, e_1)^{-1} e^{\frac{\mu e_1}{\mu e_0}} f(e_0, e_1) e^{\frac{\mu e_0}{\mu e_1}}$$

3. $f$ satisfies the equations of Proposition 3.5.

**Proof.** Essentialy, there is something to prove only for $\mu = 0$, where we have eliminated an equation.

a) Rewriting of the hexagon equation: in both cases $\mu \neq 0$ and $\mu = 0$, we have the following equivalence, for $f$ a point of $\Pi$: $f$ satisfies the duality and the hexagon equations $\Leftrightarrow f$ satisfies the duality equation and

$$e^{\frac{\mu e_1}{\mu e_0}} f(e_0, e_\infty) = f(e_\infty, e_1)^{-1} e^{\frac{\mu e_1}{\mu e_0}} f(e_0, e_1) e^{\frac{\mu e_0}{\mu e_1}}$$

b) Replacing the duality by its conjugated version: because of Fact 3.4, we have the equivalence: $f$ satisfies the duality equation $\Leftrightarrow (f^{-1} e_f)(e_1, e_0) = f e_0 f^{-1} \Leftrightarrow (f^{-1} e_{\mu e_1 f})(e_1, e_0) = f e_{\mu e_0 f} f^{-1}$, where the last equivalence holds if $\mu \neq 0$.

c) End of the proof when $\mu = 0$. Let $f$ satisfying the equation of the special automorphism with $\mu = 0$ (9); let us apply to the special automorphism equation, on the one hand, the conjugation by $f^{-1}$, and, on the other hand, the change of variables $(e_0, e_1) \rightarrow (e_1, e_0)$. We obtain

$$f(e_0, e_1) f(e_0, e_1)^{-1} + e_1 + f(e_0, e_1) f(e_0, e_\infty)^{-1} e_\infty f(e_0, e_\infty) f(e_0, e_1)^{-1} = 0$$

$$e_1 + f(e_1, e_0) f(e_1, e_0)^{-1} + f(e_1, e_\infty)^{-1} e_\infty f(e_1, e_\infty) = 0$$

Using a) and b), this proves that $f$ satisfies the duality equation (9) if and only if $f$ satisfies the 3-cycle equation (7), whence the elimination of the duality equation.

**Remark 3.6.** The analog of c) in the proof above for $\mu \neq 0$ gives only the following equations, where $X = f(e_1, e_0)^{-1} e_{\mu e_1} f(e_1, e_0)$ and $Y = e_{\mu e_1}^2$:

$$X^{-1} Y^{-1} = e^{\frac{\mu e_1}{\mu e_0}} f(e_1, e_\infty)^{-1} e_{\mu e_0} f(e_1, e_\infty) e^{-\frac{\mu e_1}{\mu e_0}}$$

$$Y^{-1} X^{-1} = e_{\mu e_0} f(e_0, e_1)^{-1} e^{\frac{\mu e_1}{\mu e_0}} f(e_0, e_\infty) e^{-\frac{\mu e_1}{\mu e_0}} f(e_0, e_1)^{-1}$$

This does not yield the same elimination result.

### 3.3.3. Definition of the maps $l_\mu$, $l_\infty^\mu$, $\tilde{l}_\mu$, $\tilde{l}_\infty^\mu$. The Proposition 3.5 leads us to the following definitions.
Definitio{n 3.7.} We define \( I_\mu, I_\mu^\infty : \mathcal{O}^m \otimes_\mathbb{Z} \mathcal{O}^m \otimes \frac{\mathbb{Z}[\mu]}{\text{weight}(\mu)} \rightarrow \mathcal{O}^m \otimes_\mathbb{Z} \mathbb{Z}[\mu] \) as duals of some automorphisms of \( \Pi \times_\mathbb{Z} \mathbb{Z}[\mu] \):

\[
I_\mu = \left( f(e_0, e_1) \mapsto f(e_\infty, e_1)^{-1}e^{e_0}f(e_0, e_1)e^{e_0} \right)^{\vee}
I_\mu^\infty = \left( g(e_0, e_1) \mapsto e^{e_\infty}g(e_0, e_1) \right)^{\vee}
\]

If \( \mu \neq 0 \), we define similarly \( I_\mu \) and \( I_\mu^\infty \):

\[
I_\mu = \left( f(e_0, e_1) \mapsto f^{-1}(e_0, e_1)e^{-e_0}f(e_0, e_1)e^{e_0} \right)^{\vee}
I_\mu^\infty = \left( g(e_0, e_1) \mapsto e^{e_\infty}g(e_0, e_1)^{-1}e^{e_0}g(e_0, e_1)e^{-e_0} \right)^{\vee}
\]

For \( \mu = 0 \), we define \( I_0, I_0^\infty : \mathcal{O}^m \otimes_\mathbb{Z} \mathbb{Z}[\mu] \rightarrow \text{Lie}(\Pi) \otimes_\mathbb{Z} \mathbb{Z}[\mu] \) as duals of maps \( \Pi \rightarrow \text{Lie}(\Pi)^{\vee} \otimes_\mathbb{Z} \mathbb{Z}[\mu] \):

\[
I_0 = \lim_{\mu \to 0} \frac{d}{d\mu} I_\mu = \left( f(e_0, e_1) \mapsto -f(e_0, e_1)^{-1}e_1 f(e_0, e_1) - e_0 \right)^{\vee}
I_0^\infty = \lim_{\mu \to 0} \frac{d}{d\mu} I_\mu^\infty = \left( g(e_0, e_1) \mapsto g(e_0, e_1)^{-1}e_\infty g(e_0, e_1) \right)^{\vee}
\]

With these definitions, equation (13) \((\mu \neq 0)\) and equation (15) \((\mu = 0)\) are respectively equivalent to (17)

\[
f \circ I_\mu = f_\infty \circ \mu_\mu
\]

Equation (13) \((\mu \neq 0)\) and equation (16) \((\mu = 0)\) are respectively equivalent to (18)

\[
f \circ I_\mu = f_\infty \circ \mu_\mu
\]

3.4. Depth-graded versions of the maps \( l_\mu, I_\mu^\infty, I_\mu, I_\mu^\infty \). Here we observe that \( l_\mu, I_\mu^\infty, I_\mu, I_\mu^\infty \) respect the depth filtration and we write explicit formulas for their 'depth-graded’ versions in the sense of the following definition.

3.4.1. Definition.

Definitio{n 3.8.} Let \( F : \mathcal{O}^m \otimes C_\mu \rightarrow \mathcal{O}^m \otimes C_\mu \) be a linear map which preserves the depth filtration, i.e. for any word of depth \( d \in \mathbb{N}^* \), \( F(w) \) is a linear combination of words of depth \( \leq d \). We call depth-graded version of \( F \), and denote \( \text{gr}_D(F) \) the map which associates to a word of depth \( d \in \mathbb{N}^* \) the terms of depth \( d \) in \( F(w) \), where the shuffle polynomials of words of depth \( < d \) are viewed as having depth \( < d \).

In this \( \S 3.3 \), we write the depth-graded version of several standard operations on \( \pi_1^{un,DR}(\mathbb{P}^1 - \{0, 1, \infty\}) \), in view of writing in \( \S 3.4 \) the depth-graded version of the equations of \( \S 3.2 \).

3.4.2. Preliminaries : depth-graded versions of some other operations. The group \( \text{Aut}(\mathbb{P}^1 - \{0, 1, \infty\}) \) is isomorphic to \( S_3 \) represented as the group of permutations of \( \{0, 1, \infty\} \). Each of its elements induces by functoriality an automorphism of \( \pi_1^{un,DR}(\mathbb{P}^1 - \{0, 1, \infty\}) \), and thus an automorphism of \( \Pi \). For example, the automorphism \( z \mapsto \frac{1}{z} \) of \( \mathbb{P}^1 - \{0, 1, \infty\} \) induces the automorphism \( f(e_0, e_1) \mapsto f(e_\infty, e_1) \) of \( \Pi \).

Lemma 3.9. The map \( (z \mapsto \frac{1}{z})' \) preserves the depth filtration and we have :

\[
\text{gr}_D \left( z \mapsto \frac{1}{z} \right) : w \mapsto (-1)^{\text{weight}(w) - \text{depth}(w)}w
\]

Proof. The dual of \( f(e_0, e_1) \in K(\langle e_0, e_1 \rangle) \mapsto f(e_\infty, e_1) \in K(\langle e_0, e_1 \rangle) \) is the map \( w(e_0, e_1) \in \mathcal{O}^m \mapsto w(-e_0, -e_0 + 1) \in \mathcal{O}^m \).

Lemma 3.10. The dual of the map \( f \mapsto f^{-1} \) of \( \tilde{\Pi}(K) \), viewed as a linear function \( \mathcal{O}^m \rightarrow \mathcal{O}^m \), preserves the depth filtration and we have :

\[
\text{gr}_D(f \mapsto f^{-1}) = -w
\]
Proof. We view $\tilde{\Pi}(K)$ inside $K\langle \epsilon_0, \epsilon_1 \rangle$, where the inversion is the one of power series, and the invertible elements are those of the form $c(1 + \epsilon)$, where $c \in K - \{0\}$ and $\epsilon$ is in the augmentation ideal of $K\langle \epsilon_0, \epsilon_1 \rangle$. Thus the inverse of $f$ such that $f[0] = 1$ is $f^{-1} = f[\sum_{a \geq 1} (-1)^a (1 - f)^a]$; and, of course, $(1 - f)[w] = -f[w]$ for all non-empty words $w$. Since we have started with $\tilde{\Pi}$, we deal with elements $f$ such that $f[0] = 0$; thus, any contribution to $(f \mapsto f^{-1})^\infty(w)$ arising from $(1 - f)^a[w]$ with $a \geq 2$ is a shuffle polynomial of words of depth strictly lower than the depth of $w$, and does not appear in $gr_D(f \mapsto f^{-1})^\infty(w)$.

3.4.3. Depth-graded maps $l_\mu$, $l_\infty^\mu$, $\tilde{l}_\mu$, $\tilde{l}_\infty^\mu$. We now write formulas for the depth-graded versions of the maps of Definition def of maps $l$. In this paragraph, $w$ is a word of depth $d \in \mathbb{N}^*$, of the form $\epsilon_0^{s_d-1} \cdots \epsilon_1 \epsilon_0^{s_0-1}$, with $s_d, \ldots, s_0 \in \mathbb{N}^*$.

Lemma 3.11. $l_\mu$ preserves the depth filtration and we have

$$gr_D l_\mu(w) = \sum_{s' = 0}^{s_0 - 1} (1 - (-1)^{\text{weight}(w) - \text{depth}(w) - s'}) \left( \frac{1}{s'!} \right) (\frac{\mu}{2})^{s'} \partial^{s'} w$$

In particular, if $s_d \geq 2$ and $s_0 = 1$, we have:

$$gr_D l_\mu(w) = (1 - (-1)^{\text{depth}(w) - \text{weight}(w)}) w$$

Proof. Clear by Lemma [3.10] and Lemma [3.11]

Lemma 3.12. $l_\infty^\mu$ preserves the depth filtration and we have

$$gr_D l_\infty^\mu(w) = \sum_{s' = 0}^{s_0 - 1} \frac{1}{s'!} (\frac{\mu}{2})^{s'} \partial^{s'} w$$

Proof. Clear.

Lemma 3.13. $\tilde{l}_\mu$ preserves the depth filtration and we have

$$gr_D \tilde{l}_\mu(w) = 0$$

Let us assume that $s_d \geq 2$, $s_0 = s_1 = 1$ (i.e. $w = \epsilon_0^{s_d-1} \cdots \epsilon_0^{s_2-1} e_1$ and $s_d - 1 \geq 1$); then, more precisely, we have, if $\mu \neq 0$:

$$\tilde{l}_\mu(w) \equiv -\mu \partial w \mod O_{D \leq d - 2}^\infty \otimes C_\mu$$

and, for $\mu = 0$:

$$\tilde{l}_0(w) \equiv -\partial w + O_{D \leq d - 2}^\infty \otimes C_\mu$$

Proof. Clear by Lemma [3.10]

Lemma 3.14. $\tilde{l}_\infty^\mu$ preserves the depth filtration and we have:

i) If $\mu \neq 0$:

$$gr_D \tilde{l}_\infty^\mu(w) = \sum_{0 \leq s_d \leq s_0 - 1} (-1)^{s_d} \left( \frac{\mu}{2} \right)^{s_d + s_0} \left( \sum_{0 \leq l \leq s_d - 1, l' \leq s_0} \frac{l'}{l!} \partial^{l+s'_d} \partial^{s'_0} w - \sum_{0 \leq l \leq s_0 - 1, l' \leq s_d} \frac{l'}{l!} \partial^{l+s'_d} \partial^{s'_0+l} w \right)$$

In particular, if $s_d \geq 2$ and $s_0 = 1$, (i.e. $w = \epsilon_0^{s_d-1} \cdots \epsilon_0^{s_2-1} \epsilon_1$ with $s_d \geq 1$), we have:

$$gr_D \tilde{l}_\infty^\mu(w) = \sum_{0 \leq s_d \leq s_0 - 1} \frac{1}{s'_d!} (\frac{\mu}{2})^{s'_d} \left( \sum_{0 \leq l \leq s_d - 1, l' \leq s_0} \frac{l'}{l!} \partial^{l+s'_d} \partial^{s'_0+l} w \right)$$

ii) For $\mu = 0$, we have:

$$gr_D \tilde{l}_0^\infty(w) = -\mathbb{I}_{s_d \geq 2} \partial w + \mathbb{I}_{s_0 \geq 2} \partial w$$
In particular, if \( s_d \geq 2 \) and \( s_0 = 1 \) (i.e. \( w = e_0^{s_d-1}e_1 \ldots e_0^{s_1-1}e_1 \) with \( s_d \geq 1 \)), we have:

\[
\text{gr}_D \tilde{\Gamma}_0^w(w) = -\partial w
\]

**Proof.** Clear by Lemma 3.10. \( \square \)

### 3.5. Conclusion : main result.

**Theorem 3.15.** (comparison between the modules of coefficients of \( \Phi \) and \( \Phi_\infty \), compatible with the depth filtration, and on a ring of rational coefficients having prescribed bounds of the denominators)

Let \( (\Phi, \mu) \in \tilde{\Pi}(K) \times K \) satisfying the equations of Definition 3.7. Then we have, for all \( d \in \mathbb{N}^* \):

\[
\Phi(O_{1 \leq D \leq d-1}^m \otimes C_\mu) = \Phi_\infty(O_{1 \leq D \leq d}^m \otimes C_\mu)
\]

And, more precisely, for all \( s_d, \ldots, s_1 \in \mathbb{N}^* \) such that \( s_d \geq 2 \), resp. \( s_d+1, \ldots, s_1 \in \mathbb{N}^* \) such that \( s_{d+1} \geq 2 \), one has the following congruences:

If \( \mu \neq 0 \):

\[
-\Phi[e_0^{s_d-1}e_1 \ldots e_0^{s_1-1}e_1] \equiv \sum_{0 \leq s_d' \leq s_d-1} \frac{1}{s_d'} \left( \frac{\mu}{2} \right)^{s_d'} \sum_{1 \leq s_d-1-s_d'} \frac{\mu^l}{l!} \Phi_\infty[e_0^{s_d-1-l-s_d'}e_1 \ldots e_0^{s_1-1}e_1] \mod (O_{1 \leq D \leq d-1}^m)
\]

\[
\sum_{0 \leq s_d' \leq s_d+1-1} \frac{1}{s_d'+1} \left( \frac{\mu}{2} \right)^{s_d'+1} \sum_{1 \leq l \leq s_d+1-s_d'} \frac{\mu^l}{l!} \Phi_\infty[e_0^{s_d+1-1-l-s_d'+1}e_1 \ldots e_0^{s_1-1}e_1] \equiv -\Phi[e_0^{s_d+1-1}e_1 \ldots e_1 e_0^{s_1-1}] \mod (O_{1 \leq D \leq d-1}^m)
\]

If \( \mu = 0 \):

\[
\Phi[e_0^{s_d-1}e_1 \ldots e_0^{s_1-1}e_1] \equiv \Phi_\infty[e_0^{s_d-2}e_1 \ldots e_0^{s_1-2}e_1] \mod (O_{1 \leq D \leq d-1}^m)
\]

\[
\Phi_\infty[e_0^{s_d+1-2}e_1 \ldots e_0^{s_1-1}e_1] \equiv -\Phi[e_0^{s_d+1-1}e_1 \ldots e_1 e_0^{s_1-1}] \mod (O_{1 \leq D \leq d-1}^m)
\]

And furthermore, for any \( \mu \):

\[
(1 - (-1)^{\text{weight}(w) - \text{depth}(w)})\Phi[e_0^{s_d-1}e_1 \ldots e_0^{s_1-1}e_1] \equiv \sum_{s' = 0}^{s_d-1} \frac{1}{s'!} \left( \frac{\mu}{2} \right)^{s'} \Phi_\infty[e_0^{s_d-1-s'}e_1 \ldots e_0^{s_1-1}e_1] \mod (O_{1 \leq D \leq d-2}^m \otimes C_\mu)
\]

Moreover, when \( \mu = 0 \), if one replaces the congruences above by the full equalities which arise from the previous lemmas, the converse implication is true.

**Proof.** 1) We prove as follows the equality of modules \( 27 \) by induction on \( d \), using Proposition 3.13 and the Lemmas 3.11 and Lemmas 3.14.

By Lemma 3.13, we have, for all \( s_d, \ldots, s_2 \in \mathbb{N}^* \) with \( s_d \geq 2 \),

\[
\Phi[e_0^{s_d-1}e_1 \ldots e_0^{s_2-1}e_1] \in \Phi_\infty(O_{1 \leq D \leq d}^m \otimes C_\mu) + \Phi(O_{1 \leq D \leq d-2}^m \otimes C_\mu)
\]

the induction hypothesis implies \( \Phi(O_{1 \leq D \leq d-2}^m \otimes C_\mu) \subset \Phi_\infty(O_{1 \leq D \leq d-1}^m \otimes C_\mu) \); by definition \( \Phi_\infty(O_{1 \leq D \leq d-1}^m \otimes C_\mu) \subset \Phi_\infty(O_{1 \leq D \leq d}^m \otimes C_\mu) \), whence \( \Phi[e_0^{s_d-1}e_1 \ldots e_0^{s_2-1}e_1] \in \Phi_\infty(O_{1 \leq D \leq d}^m \otimes C_\mu) \); by Fact 2.9 iii), this last equality implies the inclusion

\[
\Phi(O_{1 \leq D \leq d-1}^m \otimes C_\mu) \subset \Phi_\infty(O_{1 \leq D \leq d}^m \otimes C_\mu)
\]
On the other hand, by Lemma 3.14 we have, for all \( s_d, \ldots, s_1 \in \mathbb{N}^+ \) with \( s_d \geq 2 \),
\[
\Phi_\infty \left[ \sum_{s_d, t_2 \geq 0 \atop 0 \leq s_d + t_2 \leq s_d - 1} \frac{\mu^{s_d+t_2}}{s_d!t_2!^2} e_0^{s_d-1-s_d-t_2} e_0 e_1^{s_d-1-1} e_1 \ldots e_1^{s_d-1-1} e_1 \right] \in \Phi_\infty (\mathcal{O}_{D, \leq d-1}^{\mathbb{Z}} \otimes C_\mu) + \Phi (\mathcal{O}_{D, \leq d-1}^{\mathbb{Z}} \otimes C_\mu)
\]
the induction hypothesis implies \( \Phi_\infty (\mathcal{O}_{D, \leq d-1}^{\mathbb{Z}} \otimes C_\mu) \subset \Phi (\mathcal{O}_{D, \leq d-2}^{\mathbb{Z}} \otimes C_\mu) \); by definition, \( \Phi (\mathcal{O}_{D, \leq d-2}^{\mathbb{Z}} \otimes C_\mu) \subset \Phi (\mathcal{O}_{D, \leq d-1}^{\mathbb{Z}} \otimes C_\mu) \), whence \( \Phi_\infty \sum_{s_d, t_2 \geq 0 \atop 0 \leq s_d + t_2 \leq s_d - 1} \frac{\mu^{s_d+t_2}}{s_d!t_2!^2} e_0^{s_d-1-s_d-t_2} e_0 e_1^{s_d-1-1} e_1 \ldots e_1^{s_d-1-1} e_1 \in \Phi (\mathcal{O}_{D, \leq d-1}^{\mathbb{Z}} \otimes C_\mu) \); by Fact 2.14 iii), using an induction on \( s_d \), this implies the converse inclusion
\[
\Phi_\infty (\mathcal{O}_{D, \leq d}^{\mathbb{Z}} \otimes C_\mu) \subset \Phi (\mathcal{O}_{D, \leq d-1}^{\mathbb{Z}} \otimes C_\mu)
\]
2) The congruences (28), (29), (30), (31) express depth-graded versions of equation (18), using the formulas of Lemma 3.13 and Lemma 3.14 specialized to words of the form \( e_0^{s_d-1} e_1 \ldots e_1^{s_d-1} e_1 \) and \( e_0^{s_d+1} e_1 \ldots e_1^{s_d-1} e_1 \), with \( s_d \geq 2 \), resp. \( s_d+1 \geq 2 \); they use respectively : equation (21) of Lemma 3.13 ; equation (23) of Lemma 3.14 ; equation (22) of Lemma 3.13 ; equation (26) of Lemma 3.14 ; and knowing the equality of modules equation (27). 3) The congruence (32) is the depth-graded version of equation (17), using the formulas of Lemma 3.11 Lemma 3.12 and knowing the equality of modules (27).

Remark 3.16. There are other possible ways to write Theorem 3.15: the special automorphism equation combined with the duality equation relative to a couple \((\mu, f)\) implies :
\[
f (e_0, e_1) e^{-\mu e_0} f (e_0, e_1) \cdot e^{-\mu e_1} = e^{\Phi (\mu) f (e_1, e_\infty)} e^{-\Phi (e_1)}
\]

Remark 3.17. In the case where \( \mu = 0 \), although we have \( C_\mu = \mathbb{Z} \) (with the trivial ring filtration on \( \mathbb{Z} \)), this machinery does not produce only algebraic equations with coefficients in \( \mathbb{Z} \), as we are going to see in §4.3.

4. Depth reductions

We now show that the main result of §3 implies certain properties of depth reductions. They follow, in brief, from the redundancies or cancellations in the formulas for the comparison between \( \Phi \) and \( \Phi_\infty \) obtained in §3.

In this paragraph, \( K \) is again a field of characteristic zero, and \((\Phi, \mu) \in \tilde{\Phi} (K) \times K \) satisfying the equations of Definition 3.1; for example, \( \Phi \) an associator with coupling constant \( \mu \). We emphasize equally the depth reductions for \( \Phi \) and \( \Phi_\infty \). The reason for giving such a role to \( \Phi_\infty \) is its interpretation in the case of \( p \)-adic multiple zeta values, which is explained in the case of Definition 5.1.

4.1. Definition

Definition 4.1. If \((f, \mu) \in \tilde{\Phi} (K) \times K \) and \( d, d' \in \mathbb{N}^+ \) with \( d > d' \), let \( \text{DRed}_{d \rightarrow d'} (f, C_\mu) \) be the filtered sub-\( C_\mu \)-module of \( \mathcal{O}_{D, \leq d}^{\mathbb{Z}} \otimes C_\mu \) generated by words \( w \) such that \( f[w] \) is a polynomial over \( \mathbb{Q} \) of coefficients of \( f \) or depth \( \leq d - r \) and of \( \mu \) (viewed according to Convention 2.16).

4.2. The "parity" depth reduction, for \( \Phi \) and \( \Phi_\infty \). Let \((s_d, \ldots, s_1)\) an index (corresponding to the word \( \epsilon_0^{s_d-1} e_1 \ldots e_1^{s_d-1} e_1 \)).

Corollary 4.2. If \( s_d + \ldots + s_1 - d \) is odd, then \((s_d, \ldots, s_1) \in \text{DRed}_{d \rightarrow d-1} (\Phi, C_\mu) \).

Proof. This is equation (32) combined to equation (27).

The counterpart of this corollary for \( \Phi_\infty \) is :

Corollary 4.3. Assume that \( s_d + \ldots + s_1 - d \) is odd. Then we have
\[
\sum_{\sigma = 0}^{s_d-1} \left( \frac{\mu}{2} \right)^{s_d - \sigma - s_d-1} \ldots (s_d - \sigma) \in \text{DRed}_{d \rightarrow d-1} (\Phi_\infty, C_{mu})
\]
4.3. The "adjoint" depth reduction, for Φ and Φ∞. In this paragraph, for simplicity, and because the application which we have in mind is specifically p-adic, we restrict ourselves to μ = 0, and we leave to the reader the analogous facts for μ ≠ 0.

**Corollary 4.4.** We have \((s_d, \ldots, s_1) + (-1)^{s_d+\ldots+s_1} (s_1, \ldots, s_d) \in \text{DR}_{d \to d-1}(\Phi, C_\mu)\)

If μ = 0, we have more precisely, with z = \(e_0^{s_d-1} \cdots s_1 e_1\):

\[
\Phi[e_0^{s_d-1} \cdots s_1 e_1] + (-1)^{s_d+\ldots+s_1} \Phi[e_0^{s_1-1} s_d e_1]
\]

\[
\equiv (-1)^{s_1} \sum_{s_1} \sum_{l_1+\ldots+l_d-1=s_d} \prod_{i=1}^d \left(\frac{-s_i}{l_i}\right) \Phi[e_0^{s_1+l_1-1} \cdots s_d e_1] 
\]

\[
+ \sum_{l_2+\ldots+l_d=s_1} \prod_{i=2}^d \left(\frac{-s_i}{l_i}\right) \Phi[e_0^{s_d-1} \cdots s_2 e_1] \mod \Phi(O_{DRed}^{\mu})
\]

**Proof.** We specialize the equation (18) to the coefficients of the form \(e_1 e_0^{s_d-1} \cdots s_1 e_1\). We obtain a formula for \((\Phi^{-1} e_1 \Phi)[e_1 e_0^{s_d-1} \cdots s_1 e_1]\) obtain that they are in \(\Phi_\infty(O_{DRed}^{\mu})\). Applying the expression of \(\Phi_\infty\) in depth \(d\) in terms of \(\Phi\) in depth \(\leq d - 1\) given by \(\Phi \circ \tilde{L}_\mu = \Phi_\infty \circ \tilde{L}_\mu\) gives the result. □

We call this result the 'adjoint' depth reduction; the justification for this terminology is explained by §5.4.3.

**Example 4.5.** The corollary 4.4 with μ = 0, gives in depth one and two: for all \(s \in \mathbb{N}^*\), and \((s_1, s_2) \in (\mathbb{N}^*)^2\),

\[
(\Phi^{-1} e_1 \Phi)[e_1 e_0^{s_1-1} e_1] = 0
\]

\[
(\Phi^{-1} e_1 \Phi)[e_1 e_0^{s_2-1} s_1 e_1] \equiv (-1)^{s_1} \left(\frac{s_1 + s_2}{s_1}\right) \Phi[e_0^{s_1} s_2 e_1]
\]

The Corollary 4.0 below is one way among others to write the counterpart of Corollary 4.4 for \(\Phi_\infty\); it is the one which we found the most significative. A reason for our choice of formulation is the intepretation of Corollary 4.0 in the particular case of \(p\)-adic multiple zeta values, which we will explain in §5.3.2.

Another reason is the following. The fact that \(\Phi_\infty\) in depth \(d\) can be written in terms of \(\psi\) in depth \(\leq d - 1\), implied by the equation (27), is a restrictive condition - it implies for example that \(\Phi_\infty\) vanishes in depth one, which is not true in general. Intuitively, this condition means, intrinsically on \(\Phi_\infty\), that \(\Phi_\infty\) depends on "one too much variable" as a function of indices \(s_d, \ldots, s_1\). The Corollary 4.0 can be viewed as a formalization of this idea.

**Corollary 4.6.** If μ = 0, then, for all \(d \in \mathbb{N}^*, s_d, \ldots, s_0 \in (\mathbb{N}^*)^d\), \(r_d, r_0 \in \mathbb{N}\):

\[
\left(s_0 + s_d + r_0 + r_d - 1 \right) \Phi_\infty e_0^{s_d e_0} \equiv (-1)^{s_1} \left(\frac{s_1 + s_2}{s_1}\right) \Phi[e_0^{s_1} s_2 e_1] 
\]

\[
- \sum_{u_2 + \cdots + u_0 = r_d + r_0} \prod_{i=1}^{d-1} \left(\frac{-s_i}{u_i}\right) \Phi[e_0^{s_d-1} s_{d-1} e_1] \in \text{DR}_{d \to d-1}(\Phi_\infty, C_\mu)
\]

When we take \(s_d = s_0 = 1\), we obtain that all coefficients of \(\Phi_\infty\) can be written, modulo lesser depth, as coefficients on words of the form \(\Phi_\infty[e_1 \cdots e_1]\).
Proof. For convenience, we start directly with the special automorphism equation with $\mu = 0$ instead of applying Theorem 3.15. \[ e_0 \Phi_\infty - \Phi_\infty e_0 = -(e_1 \Phi_\infty - \Phi_\infty e_1) + \Phi_\infty \widetilde{Ad}_p(e_1) \] (38)

We note that, since $\Phi[e_0] = 0$, $\widetilde{Ad}_p(e_1)$ vanishes in depth 0 and 1. This and the equation (27) imply that we have

\[ \Phi_\infty \widetilde{Ad}_p(e_1)(\Omega_d^{\mu} D_{D_{\leq d-2}}) \subset \Phi_\infty (\Omega_d^{\mu}) \]

Let us take $d \in \mathbb{N}^*$ and $t_1, \ldots, t_0 \in \mathbb{N}$. By considering the coefficient of $e_0^{t_0+1} t_d^{t_d-1} \cdots t_1^{t_1+1}$ in (38), we obtain

\[ \Phi_\infty \Phi_\infty(e_0^{t_0+1} t_d^{t_d-1} \cdots t_1^{t_1+1}) = (\Phi_\infty \widetilde{Ad}_p(e_1)(e_0^{t_0+1} t_d^{t_d-1} \cdots t_1^{t_1+1})) \]

On the other hand, the shuffle relation $\Phi_\infty[e_0 \in (e^{t_d} t_d-1 \cdots t_1^{t_1+1})] = 0$ gives

\[ (t_d + 1) \Phi_\infty(e_0^{t_0+1} t_d^{t_d-1} \cdots t_1^{t_1+1}) + (t_0 + 1) \Phi_\infty[e_0^{t_0+1} e_0^{t_1+1} \cdots e_0^{t_d+1}] = - \sum_{k=1}^{d-1} (t_k + 1) \Phi_\infty(e_0^{t_0+1} e_0^{t_1+1} \cdots e_0^{t_d+1}) \]

The combination of (40) and (41) is a linear system which is inverted into the following system, after a change of variable replacing $t_0$ by $t_0 - 1$, resp. $t_d$ by $t_d - 1$:

\[ \Phi_\infty(e_0^{t_0+1} e_0^{t_1+1} \cdots e_0^{t_d+1}) = \sum_{k=1}^{d-1} \frac{t_k + 1}{t_d + t_0 + 1} \Phi_\infty(e_0^{t_0+1} e_0^{t_1+1} \cdots e_0^{t_d+1}) = \frac{t_0 + 1}{t_d + t_0 + 1} \]

\[ \Phi_\infty(e_0^{t_0+1} e_0^{t_1+1} \cdots e_0^{t_d+1}) = \sum_{k=1}^{d-1} \frac{t_k + 1}{t_d + t_0 + 1} \Phi_\infty(e_0^{t_0+1} e_0^{t_1+1} \cdots e_0^{t_d+1}) = \frac{t_0 + 1}{t_d + t_0 + 1} \]

Let $S = (a_d, a_0, 0) \leq e_r + r_0$ be any sequence of elements of $\{0, \ldots, r_d\} \times \{0, \ldots, r_0\}$, satisfying:

\[
\begin{cases}
(a_d, a_0, 0) = (r_d, r_0) & \text{and} \quad (a_d, r_d + r_0, a_0, r_d + r_0) = (0, 0) \\
\text{For all } i \in \{0, \ldots, r_d + r_0 - 1\}, \quad (a_{d,i+1}, a_0, i+1) \in \{(a_{d,i} - 1, a_0, i), (a_{d,i}, a_0, i)\}
\end{cases}
\]

The proof of the lemma follows by induction on $(r_d, r_0)$ for the lexicographical order, using the linear system formed by (42), and equation (69) which enables to eliminate the $\Phi_\infty \widetilde{Ad}_p(e_1)$ terms: we apply (42), resp. (42), inductively to $(t_d, \ldots, t_0) = (s_d + 1 + a_d, s_{d-1} + 1, \ldots, s_1 - 1, s_0 - 1 + a_d, i)$ if $a_{0,i+1} = a_0, i - 1$, resp. $a_{d,i+1} = a_d, i - 1$.

4. Consequences of both depth reductions combined.

Corollary 4.7. (Ihara-Kaneko-Zagier [1], Yasuda [3]) Assume that $(s_1 + \ldots + s_d) - d$ is even. Then $(s_d + \ldots, s_1) = (-1)^{s_d + \ldots + s_1} (s_1 + \ldots + s_d) \in D_{Red, d + d - 2}(\Phi, C_{\mu})$.

Proof. Follows from Corollary 4.2 and Corollary 4.1 combined. 

5. Applications and interpretations related to (p-adic) multiple zeta values

We review the fact that complex resp. p-adic multiple zeta values provide an example of associators with $\mu = 2\pi i$ resp. $\mu = 0$, as well as the notion of finite multiple zeta values and its relation with p-adic multiple zeta values (§5.1). We explain some specific applications of our results to complex and p-adic multiple zeta values (§5.2). We obtain further interpretations of our results related to p-adic multiple
zeta values (§5.3) and to our notion of adjoint multiple zeta values which is itself a way to study $p$-adic multiple zeta values (§5.4). As explained in the introduction, the matters related, directly or indirectly, to $p$-adic multiple zeta values are our main concern.

5.1. Complex, $p$-adic, finite multiple zeta values and associators.

5.1.1. Multiple zeta values and associators.

**Proposition-Definition 5.1.** (Drinfeld, [DR], §2) Let $\Phi_{KZ} \in \Pi^{un}(M_{0,4})(\mathbb{R})$ be the regularized holonomy on the straight path from 0 to 1 ($\gamma : [0,1] \to [0,1]$) of the KZ equation on the trivial bundle of fiber $\mathcal{C}((e_0,e_1))$ on $\mathbb{P}^1 - \{0,1,\infty\}$: $\nabla_{KZ} : f \mapsto df - (e_0 \frac{dz}{z} + e_1 \frac{dz}{1-z})$. $\Phi_{KZ}$ is an associator with $\mu = 2i\pi$, called the Knizhnik-Zamolodchikov (KZ) associator.

The connection between $\Phi_{KZ}$ and multiple zeta values, which has been observed later, is the following : for all $d \in \mathbb{N}^*$, $s_d, \ldots, s_1 \in \mathbb{N}^*$, $s_d \geq 2$,

\[ \zeta(s_d, \ldots, s_1) = (-1)^d \Phi_{KZ}[e_{0}^{s_{d-1}}e_{1} \cdots e_{0}^{s_{1}}] \]

By equation (44), by $\Phi_{KZ} \in \Pi^{un}(M_{0,4})(\mathbb{R})$, because we have moreover $\Phi_{KZ}[e_0] = \Phi_{KZ}[e_1] = 0$, and by Fact 2.9 one can write all the coefficients of $\Phi_{KZ}$ as $Q$-linear combinations of multiple zeta values.

In terms of the notion of pro-unipotent fundamental groupoid : $\nabla_{KZ}$ is the canonical connection on $\pi_1^{un,DR}(\mathbb{P}^1 - \{0,1,\infty\})$ and we have $\Phi_{KZ} \in \pi_1^{un,DR}(\mathbb{P}^1 - \{0,1,\infty\}, -\Gamma_1, \Gamma_0)(\mathbb{R})$.

5.1.2. $p$-adic multiple zeta values and associators. Two different but closely related notions of $p$-adic multiple zeta values have been defined respectively by Furusho ([F1], [F2]) and by Deligne and Goncharov ([DG], §5.28) independently. They are coefficients of elements of $\Pi(Q_p)$ which we denote respectively by $\Phi_{p,KZ}$ and $\Phi_p$.

We denote also by $\Phi_p = \Phi_{p,1}$ and more generally, following [IL], for $\alpha \in \mathbb{N}^*$, let $\Phi_{p,\alpha}$ the natural generalization of $\Phi_{p,1}$ associated with the Frobenius of $\pi_1^{un,crys}(\mathbb{P}^1 - \{0,1,\infty\})$ iterated $\alpha$ times ; let $\Phi_{p,\infty} = \Phi_{p,KZ}$ (the justification for this notation comes from [J3]) ; finally, for all $\alpha \in \mathbb{N} \cup \{\infty\}$, let $\Phi_{p,-\alpha}$ be the inverse of $\Phi_{p,\alpha}$ for the Ihara product in the sense of Proposition-Definition 2.13. The definition of $p$-adic multiple zeta values uses an analogy with equation (44) : for all words $w$, and all $\alpha \in \mathbb{N}^* \cup -\mathbb{N}^* \cup \{\pm \infty\}$ :

\[ \zeta_{p,\alpha}(w) = (-1)^{\text{depth}(w)} \Phi_{p,\alpha}[w] \]

Any notion of $p$-adic multiple zeta values as above (corresponding to a value of $\alpha$) can expressed in terms of any other one (associated to a different value $\alpha'$) through the following formula : for all $\alpha \in \mathbb{N}^*$, denoting by $c^{\text{DR}}$ the Ihara product :

\[ \Phi_{p,\alpha}(c_0, e_1) \circ^{\text{DR}} \Phi_{p,\infty}(p^n c_0, p^\alpha e_1) = \Phi_{p,\infty}(c_0, e_1) \]

Conjecturally, all the notions of $p$-adic multiple zeta values are equivalent to each other arithmetically, and the algebraic relations within any family of $p$-adic multiple zeta values are those of multiple zeta values modulo the ideal generated by $\zeta(2) = \pi^2/6$. We have indeed $\zeta_{p,\alpha}(2) = 0$ for all $\alpha$.

**Theorem 5.2.** (Ünver, [I]) We have $\Phi_{p,KZ} \in \text{GRT}_1(Q_p)$

Note that there are several ways to write the equations defining $\text{GRT}_1$ : the one used by Ünver requires to prove additionally that it is a commutator (see [F2]). Since $\text{GRT}_1$ is a group for the Ihara product (Proposition-Definition 2.13), by equation (45) and by the definitions of $\Phi_{p,\alpha}$ for $\alpha \in -\mathbb{N}^* \cup \{-\infty\}$, one obtains :

**Corollary 5.3.** (for $\alpha \in \mathbb{N}^*$: Furusho, [F2], proof of Proposition 3.1) We have, for every $\alpha \in \mathbb{N}^* \cup -\mathbb{N}^* \cup \{\pm \infty\}$, $\Phi_{p,\alpha} \in \text{GRT}_1(Q_p)$, i.e. $\Phi_{p,\alpha}$ is a degenerated associator, i.e. an associator with $\mu = 0$. 

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Remark 5.4. According to unpublished work of Yamashita, one has functors of crystalline realization of categories of mixed Tate motives, and a lift of $p$-adic multiple zeta values to elements of a Fontaine ring (which surjects onto $C_p$) in which the analogue of $2i\pi$ is non-zero. Thus, we see that the $\mu \neq 0$ part of our results should also have ultimately applications to the $p$-adic setting, even if this is far from our current purposes.

Remark 5.5. Moreover, when $p \neq 2$ and $\mu \neq 0$, the rational coefficients appearing in our filtered ring of coefficients $C_\mu$ have denominators at most in $\frac{1}{\text{weight}}\mathbb{Z}_p$ (and in $\frac{1}{\text{weight}}\mathbb{Z}_2[\frac{1}{2}]$ when $p = 2$); these are as well the denominators appearing in the integral structure on the crystalline pro-unipotent fundamental groupoid of $\mathbb{P}^1 - \{0, 1, \infty\}$, arising from the cohomological interpretation of the pro-unipotent fundamental groupoid (and the divided powers on log-crystalline cohomology).

Thus, the algebraic relations that we have written in this paper preserve the integral structure on $\pi_1^{\text{un,crys}}(\mathbb{P}^1 - \{0, 1, \infty\})$.

5.1.3. **Finite multiple zeta values.** Let the $\mathbb{Q}$-algebra of $^\ast$integers modulo infinitely large primes$^\ast$

$$\mathcal{A} = \left( \prod_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z} \right) / \left( \oplus_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z} \right)$$

and let multiple harmonic sums be the following rational numbers: for $n \in \mathbb{N}^*$ and $(s_d, \ldots, s_1) \in \Pi_{d \in \mathbb{R}}((\mathbb{N}^*)^d)$:

$$h_n(s_d, \ldots, s_1) = \sum_{0<n_1<\cdots<n_d<n} \frac{1}{n_1 \cdots n_d} \in \mathbb{Q}$$

**Definition 5.6.** (Zagier) Let finite multiple zeta values be the following numbers, for all $(s_d, \ldots, s_1) \in \Pi_{d \in \mathbb{R}}((\mathbb{N}^*)^d)$:

$$\zeta_A(s_d, \ldots, s_1) = \left( h_p(s_d, \ldots, s_1) \mod p \right)_{p \text{ prime}} \in \mathcal{A}$$

One also has earlier definitions of notions related to this one by Hoffman [H] and Zhao [Z].

**Conjecture 5.7.** (Kaneko-Zagier) The following formula defines an isomorphism between the $\mathbb{Q}$-algebra of finite multiple zeta values and the $\mathbb{Q}$-algebra of multiple zeta values moded out by the ideal $(\zeta(2))$:

$$(46) \quad \zeta_A(s_d, \ldots, s_1) \mapsto \sum_{d' = 0}^d (-1)^{s_{d'+1} + \cdots + s_d} \zeta(s_{d'+1}, \ldots, s_d) \zeta(s_d, \ldots, s_1)$$

where, in the right right hand-side, $\zeta(w)$ means $\Phi_{KZ}(w)$ when the sum of series defining $\zeta(w)$ is divergent.

In our $p$-adic point of view, the main interest of finite multiple zeta values was that it suggested a relation between the reduction modulo large primes of $p$-adic multiple zeta values and some simple and explicit sequences of multiple harmonic sums, and justified to look for readable explicit formulas for $p$-adic multiple zeta values.

5.1.4. **Finite multiple zeta values and the reduction modulo $p$ of $p$-adic multiple zeta values for $p$ large.** We have proved in [J2] that, for all $d \in \mathbb{N}^*$, $s_d, \ldots, s_1 \in \mathbb{N}^*$, $\alpha \in \mathbb{N}^*$, we have:

$$(47) \quad (p^\alpha)^{s_{d'+1} + \cdots + s_d} h_{p^\alpha}(s_d, \ldots, s_1)
= (-1)^d \sum_{l_{d'+1}, \ldots, l_d \geq 0} \prod_{i=d'-1}^{d} (-1)^{s_i} \left( \frac{-s_1}{l_i} \right) \zeta_{p,\alpha}(s_{d'+1} + l_{d'+1}, \ldots, s_d + l_d) \zeta_{p,\alpha}(s_d, \ldots, s_1) \in \mathbb{Q}_p$$

This is a corollary of the main theorem of [J2]. For $\alpha = 1$, this had been conjectured by Akagi, Hirose and Yasuda, who have furthermore proved [AHY] the following property of integrality of $p$-adic multiple zeta values (as a formal consequence of the logarithmic generalization [Q] of Mazur’s theorem of comparison between the slopes of the Frobenius and the Hodge filtration on crystalline cohomology [M]): for all

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words $w$, 
\[ \zeta_{p,KZ}(w) \in \sum_{s > \text{weight}(w)} p^s \mathbb{Z}_p \]

They have deduced immediately from (48) and (47), that, for $d \in \mathbb{N}^*$, $s_d, \ldots, s_1 \in \mathbb{N}^*$ and $p$ a prime number, we have, if $p > s_d + \ldots + s_1$:
\[ h_p(s_d, \ldots, s_1) \equiv p^{-\sum (s_d + \ldots + s_1)} \sum_{d' = 0}^d (-1)^{s_{d'+1} + \ldots + s_d} \zeta_p(s_{d'+1}, \ldots, s_d) \zeta_p(s_{d'}, \ldots, s_1) \mod p \]

where the right-hand side in (49) is in $\mathbb{Z}_p$ by (48) and (45). By (49), there is an expression of finite multiple zeta values in $\mathcal{A}$ which is entirely in terms of \p-adic multiple zeta values which coincides with the formula in the conjecture of Kaneko and Zagier (46) and thus sheds light on that conjecture: it reduces it to the usual conjectures of periods for complex and \p-adic multiple zeta values plus a property of injectivity of the reduction of \p-adic multiple zeta values modulo large primes.

5.2. Applications of the main result to complex, \p-adic, finite multiple zeta values.

5.2.1. Applications to complex multiple zeta values. By Proposition-Definition 5.14 and its consequences to $\Phi_{KZ}$ with $\mu = 2i\pi$. Since multiple zeta values are in $\mathbb{R}$, one obtains equations in $\mathbb{R}[2i\pi]^\ast \mathbb{Q}[\text{multiple zeta values}] = \mathbb{Q}[\text{multiple zeta values}] \oplus 2i\pi \mathbb{Q}[\text{multiple zeta values}]$, where one expresses the even powers of $2i\pi$ in terms of $\zeta(2) = \pi^2/6$ or, more generally, all the even zeta values $\zeta(2n) = \frac{B_{2n}}{2(2n)!} \pi^{2n}$, $n \in \mathbb{N}^*$. Thus:

**Fact 5.8.** Any depth-graded equation among multiple zeta values arising from §3 or §4 is a congruence modulo a $R[2i\pi]$-module for a filtered ring $R \subset \mathbb{Q}$ and is equivalent to two congruences modulo $R$-modules corresponding respectively to its real and imaginary parts.

**Remark 5.9.** In [Y], Corollary 3.3, Yasuda proves the adjoint depth reduction (Corollary 4.4) of the present paper by using regularized double shuffle relations. His proof, which works for any solution to the regularized double shuffle equations, is highly influenced by the representation of multiple zeta values as sums of series (11): see Remark 5.16. His formula depends on the parity of weight($w$) − depth($w$) for each word $w$ under consideration; it coincides with our formula (53) only when $s_d + \ldots + s_1 - d$ is odd. In particular, it is interesting to note that the exact formulas beyond the congruences expressing depth reduction can always be written, but that the ones obtain via double shuffle equations and via associator equations are not the same a priori.

5.2.2. Applications to \p-adic multiple zeta values: the adjoint depth reduction and the reduction modulo large primes of \p-adic multiple zeta values. By Corollary 5.3, one can apply the Theorem 3.15 and its consequences to $\Phi_{p,\alpha}$ for any $\alpha \in \pm \mathbb{N} \cup \{\pm \infty\}$, with $\mu = 0$. Let us consider in particular the adjoint depth reduction (Corollary 4.4): we get an explicit depth reduction for the numbers
\[ \sum_{d' = 0}^d (-1)^{s_{d'+1} + \ldots + s_d} \zeta_{p,\alpha}(s_{d'+1}, \ldots, s_d) \zeta_{p,\alpha}(s_{d'}, \ldots, s_1) \]

and that property of depth reduction gives an expression of finite multiple zeta values of depth $d$ in terms of \p-adic multiple zeta values of depth $\leq d - 1$, for all $d \in \mathbb{N}^*$.

**Application 5.10.** For every $d$, one can write algorithms, relying on all known algebraic relations among multiple zeta values, checking that the (\p-adic) multiple zeta values of depth $\leq d - 1$ appearing in the right-hand side of equation (44) in Corollary 4.4 generate the spaces of (\p-adic) multiple zeta values of depth $\leq d - 1$.

Taking the reduction modulo $p$ of that fact and applying 44 for $\alpha = 1$ (or, more generally, for any
\[ \alpha \in \mathbb{N}^* \), this gives an algorithm for writing the reduction modulo large primes of all \( p \)-adic multiple zeta values in terms of finite multiple zeta values.

In low depth, one can compare easily the formulas of Example 15 for the adjoint depth reduction with the following ones, obtained by elementary computations on multiple harmonic sums; the two can be related through equation (49):

**Facts 5.11.** i) (classical) For all \( s \in \mathbb{N}^* \), \( \zeta_A(s) = 0 \). More precisely, \( b_p(s) \) is congruent to \(-1\) mod \( p \) if \( p \) divides \( s \), and to \( 0 \) mod \( p \) otherwise.

ii) (follows from [1], theorem 6.1) For all \( s_2, s_1 \in \mathbb{N}^* \), \( \zeta_A(s_2, s_1) \) depends only on the weight \( s_2 + s_1 \) up to a rational coefficient, and vanishes if the weight is even. More precisely, we have, if \( p > s_1 + s_2 \), \( b_p(s_2, s_1) \equiv (-1)^{s_2-1} \binom{s_1+s_2}{s_1} \frac{B_{s_2-1}}{s_1+s_2} \mod p \equiv \frac{1}{s_1+s_2} \binom{s_1+s_2}{s_1} b_p(s_1 + s_2 - 1, 1) \mod p \).

The ii) above corresponds to the fact that, for each value of the weight, there is, up to multiplication by a rational number, one zeta value of depth one and of the given weight, and that we have \( \zeta_{p, \alpha}(2s) = 0 \) for all \( s \in \mathbb{N}^* \). One deduces that, for all \( s \in \mathbb{N}^* \), the analogue of \( \zeta(2s + 1) \) in the algebra of finite multiple zeta values is the image of \( (-\frac{B_{2s+1}}{2s+1}) \mod p \) \( p > 2s+1 \) in \( A \).

5.3. Further \( p \)-adic interpretations.

5.3.1. Interpretation of the \( \Phi_\infty \) and the depth filtration for \( \Phi_\infty \) in terms of \( \pi_{1, \text{crys}}^\text{un}(\mathbb{P}^1 - \{0, 1, \infty\})/\mathbb{F}_p \).

For all \( d \in \mathbb{N}^* \), multiple zeta values of depth \( d \) are expressed in terms of sums of series iterated \( d \) times: this is equation (1). In the \( p \)-adic setting, we have actually proved a similar result: for all \( d \in \mathbb{N}^* \), our formulas for \( p \)-adic multiple zeta values \( \zeta_{p, \pm \alpha} \) of depth \( \leq d \), involve convergent infinite sums of \( \mathbb{Q} \)-linear combinations, with coefficients independent of \( p \), of \( \zeta \)-values of depth \( \leq d \) only ([1], [2], [3]).

Essentially, these two results have the same origin; yet, in the \( p \)-adic setting, there is an additional subtlety behind the result, and the fact that equation (4) is compatible with the depth filtration, which is part of Theorem 5.13 is hidden behind it, in the following way.

The computation of \( p \)-adic multiple zeta values is made by considering the differential equation satisfied by the Frobenius of \( \pi_{1, \text{crys}}^\text{un}(\mathbb{P}^1 - \{0, 1, \infty\})/\mathbb{F}_p \): it is a differential equation on a trivial bundle on the rigid analytic space \( \mathbb{P}^1,\text{an}/\mathbb{Q}_p \). One has to choose an open affine covering of \( \mathbb{P}^1,\text{an}/\mathbb{Q}_p \), write the equation in coordinates, and solve it, on each open affine of the covering.

The Frobenius \( z \mapsto z^p \) of \( \mathbb{P}^1(\mathbb{F}_p) \) has the natural \( p \)-adic lift \( F_{0, \infty} : z \mapsto z^p \), which is a good lift at 0 and \( \infty \), but not at 1: the preimages of 0 and \( \infty \) by \( F_{0, \infty} \) are 0 and \( \infty \) with multiplicity \( p \) whereas 1 has as preimages the \( p \)-th roots of unity.

Thus, it is convenient to consider the open affine subspace \( X_{0, \infty} = \mathbb{P}^1,\text{an} - \{z \mid |z|_p < 1\} \). Then, \( \Phi_\infty \) appears as the evaluation at \( \infty \) of the solution to the differential equation of the Frobenius on \( X_{0, \infty} \), and is involved in the resolution. This already suggests why \( \Phi_\infty \) is a natural object in the example of \( p \)-adic multiple zeta values.

Using equation (4), which relates \( \Phi \) and \( \Phi_\infty \) in the case of \( p \)-adic multiple zeta values, it is sufficient to consider only \( X_{0, \infty} \), and it is not necessary to consider another open affine subspace of \( \mathbb{P}^1,\text{an} \). This framework involving \( X_{0, \infty} \) appears in Deligne’s paper [1], §19.6 in depth one, Ünver’s paper [11] in depth one and two, and appears in our paper [1].
of depth \( \leq d+1 \), where \( \alpha \in \pm \mathbb{N}^* \) and the depth of a word over \( \{ \omega_1, \omega_1, \omega_p^{(n)} \} \) is the sum of the numbers of letters \( \omega_1 \) and of letters \( \omega_p^{(n)} \); then, that the regularizations of those iterated integrals of depth \( \leq d+1 \), viewed as overconvergent functions on \( X_{0\infty} \), have their "coefficients" (in a certain sense made precise in [1]) equal to sums of series involving multiple harmonic sums of depth \( \leq d \).

If one uses instead the open affine subset \( \mathbb{A}^n_p = \mathbb{P}^1 \mathbb{A} - \{ z \mid |z - \infty|_p < 1 \} \), the good lift of Frobenius on this space is the conjugation of \( z \mapsto z^p \) by \( z \mapsto \frac{1}{1+z} \). Dealing directly with power series \( \sum a_n z^n \) twisted by \( F_01 \), instead of referring to \( F_{0\infty} : z \mapsto z^p, \Phi_{\infty} \) and equation (30) may seem different at first sight; actually, it can be decomposed into steps which amount to use implicitly the differential equation on \( X_{0\infty} \).

As a conclusion, making a reference, at least implicitly, to \( \Phi_{\infty} \) seems impossible to avoid.

**Observation 5.12.** Conclusion: in the framework of \( \pi_1^{un,crys}(\mathbb{P}^1 - \{ 0, 1, \infty \}/\mathbb{F}_p) \) and \( p \)-adic multiple zeta values:

i) \( \Phi_{\infty} \) and \( \Phi \) are equally natural

ii) the depth of \( \Phi_{\infty} \) is has a \( p \)-adic interpretation as the natural intermediate relating the depth of \( p \)-adic multiple zeta values and the depth of the explicit sums of series which express them. Alternatively, we can also say that the depth of the explicit sums of series expressing \( p \)-adic multiple zeta values is canonically attached to the depths of \( \Phi \) and \( \Phi_{\infty} \) at the same time rather than to the depth of \( \Phi \) only.

5.3.2. Interpretation of the polar factor at 1 of the Kubota-Leopoldt zeta function in terms of the depth reduction for \( \Phi_{\infty} \). Let \( L_p \) be the \( p \)-adic zeta function of Kubota and Leopoldt. It is defined by \( p \)-adic interpolations of the values of the Riemann zeta function at negative integers, which are rational numbers. Let \( \omega \) be Teichmüller’s character. By a result of Coleman ([Co], Theorem 1), we have, for all \( s \in \mathbb{N}^* \):

\[
\zeta_p(s) = L_p(s, \omega^{1-s})
\]

The Kubota-Leopoldt zeta function has a simple pole at \( s = 1 \). We are going to show that there exists an algebraic counterpart to this fact, which is valid for all associators in all depths, and the two coincide in the case of \( p \)-adic multiple zeta values and depth one.

**Corollary 5.13.** For all points \( \Phi \) of \( GRT_1 \), we have, for all \( s_d, \ldots, s_1 \in \mathbb{N}^* \) such that \( s_d \geq 2 \):

\[
\Phi_{\infty} \left[ e_0^{s_d, s_d-1, \ldots, s_1-1} e_1 \right] + \frac{1}{s_d - 1} \sum_{u_d + \ldots + u_1 = s_d - 2} \prod_{d' = 1}^{d-1} \left( \frac{-s_{d'}}{u_{d'}} \right) \Phi_{\infty} \left[ e_0^{s_d-2, s_d-1, \ldots, s_1-1} \right] \in \text{DR}^{d-d-1}(\Phi)
\]

**Proof.** The Corollary 4.6 applied to \( s_d = 0 = 1, r_0 = 0 \) implies, after renaming \( r_d \) into \( s_d - 1 \):

\[
\Phi_{\infty} \left[ e_0^{s_d-2, s_d-1, \ldots, s_1-1} e_1 \right] - \frac{1}{s_d - 1} \sum_{u_d + \ldots + u_1 = s_d - 2} \prod_{d' = 1}^{d-1} \left( \frac{-s_{d'}}{u_{d'}} \right) \Phi_{\infty} \left[ e_0^{s_d-2, s_d-1, \ldots, s_1-1} \right] \in \text{DR}^{d-d-1}(\Phi_{\infty})
\]

On the other hand, by equation (30) and equation (27), we have

\[
\Phi \left[ e_0^{s_d-1} e_1 \ldots e_0^{s_1-1} e_1 \right] + \Phi \left( \mathcal{O}^\mathbb{M}_{d \leq d-1} \right) = -\Phi_{\infty} \left[ e_0^{s_d-2} e_1 \ldots e_0^{s_1-2} e_1 \right] + \Phi_{\infty} \left( \mathcal{O}^\mathbb{M}_{D \leq d} \right)
\]

Whence the result. 

By applying again the Corollary 4.6 to the second term in (50), we obtain an expression of the values of the form \( \Phi \left[ e_0^{s_d-1} e_1 \right] \) with \( s_d \geq 2 \) involving the coefficients of the form \( \Phi_{\infty} \left[ e_1 \ldots e_1 \right] \) and some rational coefficients having "poles" attained for certain tuples satisfying \( s_d = 1 \). In [1], we study \( p \)-adic multiple zeta values \( \zeta_{p, \omega}(s_d, \ldots, s_1) \) (or rather their adjoint variants in the sense of §5.4.1) as functions
of \((s_d, \ldots, s_1)\) viewed as a tuple of p-adic integers. We prove some properties of partial continuity and interpolation. The presence of polar factors is a key element of description of these functions.

**Example 5.14.** In low depth, we re-obtain the following equations written by Ünver in [U1] in the particular case of \(\Phi_{p, -1}\) : for all \(s_1 \in \mathbb{N}^*\) such that \(s_1 \geq 2\) we have (U1, §5.11):

\[
\Phi[e_0^{s_1-1}e_1] = \frac{(-1)^{s_1-1}}{s_1 - 1}\Phi_\infty[e_1e_0^{s_1-1}e_1]
\]

for all \(s_1, s_2 \in \mathbb{N}^*\) such that \(s_2 \geq 2\) and \(s_2 + s_1\) is odd, we have (U1, §5.14):

\[
\Phi[e_0^{s_2-1}e_1e_0^{s_1-1}e_1] = \frac{(-1)^{s_2-1}}{s_2 - 1} \sum_{l=0}^{s_2-2} \left( s - 1 + l \right) \Phi_\infty[e_1e_0^{s_1-1+l}e_1e_0^{s_2-2+l}] + \Phi_\infty[e_0^{s_2-1}e_1e_0^{s_1-1}e_1]
\]

5.4. *"Adjoint" interpretations.*

5.4.1. **Adjoint multiple zeta values.** We have defined in [J2], and studied algebraically in [J1], [J5], [J6], the following notion.

**Definition 5.15.** Let adjoint multiple zeta values be the following real numbers : for \(b \in \mathbb{N}\), and \(s_d, \ldots, s_1 \in \mathbb{N}^*\),

\[
\zeta^{\text{Ad}}(b; s_d, \ldots, s_1) = (\Phi^{-1}e_1\Phi)[e_0^b e_1 e_0^{s_d-1} e_1 \ldots e_0^{s_1-1} e_1]
\]

and let

\[
\zeta^{\text{Ad}}(s_d, \ldots, s_1) = \zeta^{\text{Ad}}(0; s_d, \ldots, s_1)\left( = \sum_{d'=0}^{d} (-1)^{s_{d'+1} + \ldots + s_d} \zeta(s_{d'+1}, \ldots, s_d) \zeta(s_{d'}, \ldots, s_1) \right)
\]

The same definition holds if one replaces complex multiple zeta values by their p-adic or motivic analogues, and the primary interest for this notion is p-adic. Indeed, because of the formulas for the Frobenius of \(\pi_{1}^\text{un,crys}(\mathbb{P}^1 - \{0, 1, \infty\}/\mathbb{F}_p)\), when computing p-adic multiple zeta values, it is natural to actually compute adjoint p-adic multiple zeta values and use afterwards the relation between usual and adjoint p-adic multiple zeta values. The same phenomenon holds when one wants to study the algebraic properties of p-adic multiple zeta values through explicit formulas.

The notion also has a more general interest, not only p-adic : as we explain in [J4], it sheds light on certain aspects of the algebraic relations between multiple zeta values. In [J4], we ultimately conclude that adjoint multiple zeta values are just another way to view multiple zeta values, which is more practical for certain purposes, among them p-adic purposes.

**Remark 5.16.** An interpretation of the notion of adjoint multiple zeta values with \(b = 0\) in terms of sums of series can be found in Yasuda’s paper [Y], where it is used to prove that the adjoint depth reduction (Corollary 4.3) holds for solutions to the regularized double shuffle relations. It is the following : let for all \((s_d, \ldots, s_1) \in \Pi_{d\in \mathbb{N}^*}^{\mathbb{N}^*})^d\), (the series regularization of)

\[
\zeta^\mathcal{F}(s_d, \ldots, s_1) = \lim_{N \to \infty} \sum_{\substack{n_1, \ldots, n_d \in \mathbb{Z} \\text{ such that} \ 0 < |n_1| < \ldots < |n_d| < N \ \text{and}}} \frac{1}{n_1^{s_1} \ldots n_d^{s_d}}
\]

Following [Y], §6.1, beginning of the proof of Theorem 6.1, we have, for all indices :

\[
\zeta^\mathcal{F}(s_d, \ldots, s_1) = \sum_{d'=0}^{d} (-1)^{s_{d'+1} + \ldots + s_d} \zeta(s_{d'+1}, \ldots, s_d) \zeta(s_{d'}, \ldots, s_1)
\]
where the subscript * means "regularization of iterated sums" ; and it follows from the comparison of regularizations ([C], equation (162)) that, in the \( \mathbb{Q} \)-algebra of multiple zeta values, one has

\[
\sum_{d=0}^{d} (-1)^{s_{1}+\ldots+s_{d}} \zeta_{*}(s_{d+1}, \ldots, s_{d}) \zeta_{*}(s_{d}, \ldots, s_{1}) \\
\equiv \sum_{d=0}^{d} (-1)^{s_{1}+\ldots+s_{d}} \zeta_{*}(s_{d+1}, \ldots, s_{d}) \zeta_{*}(s_{d}, \ldots, s_{1}) \pmod{\zeta(2)}
\]

where the subscript \( \ast \) means "regularization of iterated sums" and where we have, because of Definition 5.1.1, \( \zeta_{*}(w) = \Phi_{KZ}[w] \) for all words \( w \).

5.4.2. The adjoint variant of the couple \( (\Phi, \Phi_{\infty}) \). In the framework of adjoint multiple zeta values, the role of \( \Phi \) is played by \( \text{Ad}_{\Phi}(\epsilon) \) and the role of \( \Phi_{\infty} \) is played by \( \text{Ad}_{\Phi}(\epsilon) - \epsilon_{0} \).

Thus, aside from the \( p \)-adic context, we have a second one where \( \Phi_{\infty} \) is a natural object : the framework of adjoint multiple zeta values (which we use a lot to study \( p \)-adic multiple zeta values) simply suppresses the differences between \( \Phi \) and \( \Phi_{\infty} \).

5.4.3. The adjoint depth reduction in terms of adjoint multiple zeta values. Our terminology "adjoint depth reduction" for Corollary 4.4 comes from that this result finds its most natural statement in terms of adjoint multiple zeta values : in the particular case of \( \Phi = \Phi_{p,\alpha} \) we have

\[
\zeta_{\Phi,\alpha}^{\text{Ad}}(s_{d}, \ldots, s_{1}) \equiv (-1)^{\sum_{i=1}^{d} s_{i}} \sum_{l_{1}, \ldots, l_{d-1} \in \mathbb{N}} \prod_{i=1}^{d} \left( -\frac{s_{i}}{l_{i}} \right) \zeta_{p,\alpha}(s_{1} + l_{1}, \ldots, s_{d-1} + l_{d-1}) \\
+ \sum_{l_{2}, \ldots, l_{d} \in \mathbb{N}} \prod_{i=2}^{d} \left( -\frac{s_{i}}{l_{i}} \right) \zeta_{p,\alpha}(s_{d} + l_{d}, \ldots, s_{2} + l_{2}) \pmod{\Phi_{p,\alpha}(C_{D}^{\infty})}
\]

This leaves to imagine that one can use this equation to study further certain algebraic aspects of adjoint multiple zeta values.

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