General non-asymptotic and asymptotic formulas in channel resolvability and identification capacity and their application to wire-tap channel

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Abstract—Several non-asymptotic formulas are established in channel resolvability and identification capacity, and they are applied to wire-tap channel. By using these formulas, the \( \epsilon \) capacities of the above three problems are considered in the most general setting, where no structural assumptions such as the stationary memoryless property are made on a channel. As a result, we solve an open problem proposed in Han & Verdú[2] and Han [3]. Moreover, we obtain lower bounds of the exponents of error probability and the wire-tapper’s information in wire-tap channel.

Index Terms—identification code, channel resolvability, information spectrum, wire-tap channel, non-asymptotic setting

I. INTRODUCTION

In 1989, Ahlswede & Dueck [1] proposed the identification code as a new framework for communication system using noisy channels. However, the upper bound of the rate of the reliable identification codes was not solved in their paper. In 1993, for analysis of the converse part of this problem, Han & Verdú[2] proposed the channel resolvability problem, in which we approximate the output distribution to a desired output distribution by using a uniform input distribution with smaller support. In particular, the capacity of this problem is defined as the rate of the maximal number of the size of support for every desired output distribution. In order to discuss the channel resolvability problem, they introduced the concepts of ‘general sequence of channels’ and the ‘information spectrum method’. They gave the relation between identification code and channel resolvability, and succeeded in proving the converse part of the capacity of identification code for the discrete memoryless channel. In this method it is essential that the performances of several problems be characterized by using the probability distribution of the random variable with a form of ‘likelihood’ function in this method. This insight is very useful for obtaining the overview of information theory[3]. In particular, it gives a useful insight into quantum information theory [12], [11], [10]. Therefore, Han & Verdú’s paper[2] is undoubtedly the landmark of information spectrum.

However, while Han & Verdú’s paper gives the capacity of channel resolvability for general sequence of channels[2], their proof of the converse part contains mistakes as is recognized in section 6.3. of Han[3]. They proved the achievability of channel resolvability with the asymptotic zero error setting for a general sequence of channels. Concerning the converse part, their proof is valid for the asymptotic \( \epsilon \) error setting when the general sequence of channels has a strong converse property. However, their proof is not valid in the general channel even in the asymptotic zero error setting.

In this paper, we give several useful non-asymptotic formulas for identification code and channel resolvability, which are divided into two parts. One is the direct part of the identification code. The existence of a good identification code is proved in Theorem 1. This construction is much improved from Ahlswede & Dueck’s construction. The other is the direct part of channel resolvability. The existence of a good approximation regarding the output statistics is proved in the two criteria, variational distance and K-L divergence as in Theorem 2. In this discussion, we derived upper bounds of the average of the variational distance and K-L divergence between the output distribution of a given distribution \( p \) and the output distribution of the input uniform distribution on \( M \) elements of the input signal space, when the \( M \) elements are randomly chosen with the distribution \( p \) (Lemma 2). Combining Han & Verdú’s relation between identification code and channel resolvability, we derived the capacity of the channel resolvability for general sequence of channels with the asymptotic zero error setting, which was conjectured by Han & Verdú[2] (26) and (27) of Theorem 3. This discussion is valid even though the strong converse property does not hold.

As another application, we give an upper bound of the capacity of the channel resolvability for a general sequence of channels with the asymptotic \( \epsilon \) error. As a byproduct, we show that there exists a sequence of codes whose second error probability goes to 0 in any general sequence of channels, and only the first error probability is asymptotically related to the probability distribution of the random variable with the form of ‘likelihood’ (24) and (25) of Theorem 4. We also derived several lower bounds of exponent of channel resolvability in the stationary memoryless setting with respective error criteria (Theorem 5).

Moreover, we apply our non-asymptotic formulas for channel resolvability to wire-tap channel, in which there are two receivers i.e., the eavesdropper and the normal receiver. Wyner[4] introduced this wire-tap channel, and proved that its capacity is greater than the difference between the normal receiver’s information and the eavesdropper’s information. Csiszár & Narayan [16] showed that the capacity does not depend on the following two conditions for eavesdropper’s in-
formation: i) The eavesdropper’s information must be less than $ne$ for given $\epsilon > 0$, where $n$ is the number of transmissions. ii) The eavesdropper’s information must go to 0 exponentially. However, there are no results giving an explicit lower bound of the optimal exponents of wire-tapper’s information.

Indeed, this problem is closely related to the channel resolvability as follows. In Wyner’s proof [4], in the asymptotic i.d. setting with a large enough number $M$, he essentially showed that when $M$ elements of the input signal space are randomly chosen with a given distribution $p$, the output distribution of the distribution $p$ can be approximated with a high probability by the output distribution of the input uniform distribution on the above $M$ elements of the input signal space. This idea is also applied in Devetak[5] and Winter, Nascimento & Imai [6]. Using the same idea in the non-asymptotic setting, we can apply our formulas of channel resolvability to wire-tap channel, and derive a good non-asymptotic formula for wire-tap channel (Theorem 5). As consequence we obtain the capacity of general sequence of wire-tap channel (Theorem 5), and lower bounds of the exponents of error probability and the wire-tapper’s information in the stationary memoryless setting (Theorem 7). We can expect that these results will be applied to evaluations of the security of channels.

Finally, we should remark that our non-asymptotic resolvability formula regarding variational distance can be regarded as essentially the same results as Oohama[9]’s formula, where he treated the partial resolvability. Furthermore, he also derived a lower bound of exponent of channel resolvability by type method[8].

II. IDENTIFICATION CODE IN NON-ASYMPTOTIC SETTING

Let $W : x \rightarrow W_x$ be an arbitrary channel with the input alphabet $\mathcal{X}$ and the output alphabet $\mathcal{Y}$. The identification channel code for the channel $W$ is defined in the following way. First, let $\mathcal{N} = \{1, \ldots, N\}$ be a set of messages to be transmitted, and denote by $\mathcal{P}(\mathcal{X})$ the set of all probability distribution over $\mathcal{X}$. A transmitter prepares $N$ probability distributions $Q_1, \ldots, Q_N \in \mathcal{P}\left(\mathcal{X}\right)$. If the transmitter wants to send a message $i \in \mathcal{N}$, an encoder generates an input sequence $x_i \in \mathcal{X}$ randomly subject to the probability distribution $Q_i$. In this case, the output signal $y$ obeys the distribution $W_{Q_i}$, where the output distribution $W_p$ of a given input distribution $p$ is defined as

$$W_p(y) \overset{\text{def}}{=} \sum_x p(x)W_x(y).$$

On the other hand, at the decoder side an $N$-tuple of decoders is prepared. For every $i = 1, \ldots, N$, the $i$-th decoder judges that $i \in \mathcal{N}$ is transmitted if a channel output $y$ belongs to $D_i$, where $\{D_1, \ldots, D_N\}$ are $N$ subsets of $\mathcal{Y}$ in advance. The $i$-th decoder judges that a message different from $i \in \mathcal{N}$ if $y \not\in D_i$. Here, $D_i$ is called the decoding region, of the message $i$. It is not required that $D_1, \ldots, D_N$ be disjoint. In the identification coding problem, the $i$-th decoder is only interested in transmission of the corresponding message $i$. Thus, we call the tuple of $\Phi \overset{\text{def}}{=} (\mathcal{N}, \{Q_1, \ldots, Q_N\}, \{D_1, \ldots, D_N\})$ an identification code of channel $W$. The performance of this code can be characterized by the following three quantities. One is the size $N$ of the message sent and is denoted by $|\Phi|$, and the others are the maximum values of the two-type error probabilities given as:

$$\mu(\Phi) \overset{\text{def}}{=} \max_i W_{Q_i}(D_i^c), \quad \lambda(\Phi) \overset{\text{def}}{=} \max_{i \neq j} W_{Q_i}(D_i),$$

where $D_i^c$ is the complement set of $D_i$. Concerning this problem, as discussed in the following theorem, the ‘likelihood’ function $\frac{W_{Q_i}(y)}{W_p(y)}$ suitably characterizes the performance of good identification codes.

Theorem 1: Assume that real numbers $\alpha, \alpha', \beta, \beta', \tau, \kappa > 0$ satisfy

$$\kappa \log\left(\frac{1}{\tau} + 1\right) > \log 2 + 1, \frac{1}{3} > \tau > 0, \ 1 > \kappa > 0, \ (1)$$

$$1 > \frac{1}{\alpha} + \frac{1}{\alpha'} > \frac{1}{\beta} + \frac{1}{\beta'} > 0. \quad (2)$$

Then, for any integer $M > 0$, any real number $C > 0$, any channel $W$, and any probability distribution $p \in \mathcal{P}(\mathcal{X})$, there exists an identification code $\Phi$ such that

$$\mu(\Phi) \leq \alpha\beta E_{p,x} W_x \left\{ y : \frac{W_{Q_i}(y)}{W_p(y)} \leq C \right\}$$

$$\lambda(\Phi) \leq \kappa + \alpha' \beta' \frac{1}{C} \left\lfloor \frac{M}{\gamma} \right\rfloor, \quad |\Phi| = \left\lfloor \frac{e^\tau M}{Me} \right\rfloor$$

if

$$\beta E_{p,x} W_x \left\{ y : \frac{W_{Q_i}(y)}{W_p(y)} \leq C \right\} + \alpha' \beta' \frac{1}{C} \left\lfloor \frac{M}{\gamma} \right\rfloor < 1, \quad (3)$$

where $E_{p,x}$ denotes the expectation concerning random variable $x$ obeying the probability distribution $p$.

In the following, we omit $x$ or $p$ in the notation $E_{p,x}$, and abbreviate the set $\left\{ y : \frac{W_{Q_i}(y)}{W_p(y)} \leq C \right\}$ as $\left\{ \frac{W_{Q_i}(y)}{W_p(y)} \leq C \right\}$. We also denote the probability that the random variable $X$ belongs to the set $D$, by $P_X(D)$ or $P_X D$. If we do not need to take note of the random variable $X$, we simplify it to $P(D)$ or $PD$. This theorem is proven by using the following lemma.

Lemma 1: (Ahlswede and Dueck[1]) Let $M$ be an arbitrary finite set of the size $M = |\mathcal{M}|$. Choose constants $\tau$ and $\kappa$ satisfying the condition (1). Then there exist $\mathcal{N}(\overset{\text{def}}{=} \left\lfloor e^\tau M / Me \right\rfloor)$ subsets $A_1, \ldots, A_N \subseteq \mathcal{M}$ satisfying

$$|A_i| = \lceil \tau M \rceil, \quad |A_i \cap A_j| < \kappa \lceil \tau M \rceil (i \neq j). \quad (4)$$

Proof of Theorem 1: In this proof, the subset $U_x \overset{\text{def}}{=} \left\{ \frac{W_{Q_i}(y)}{W_p(y)} > C \right\}$ plays an important role. First, we assume the existence of $M$ distinct elements $x_1, \ldots, x_M$ of $\mathcal{X}$ satisfying

$$W_{x_i}(U_{x_j}) \leq \alpha\beta E_{p}W_x \left\{ \frac{W_{Q_i}(y)}{W_p(y)} \leq C \right\}, \quad (5)$$

$$W_{x_i} \left( \bigcup_{j \neq i} U_{x_j} \right) \leq \alpha' \beta' \frac{1}{C} \left\lfloor \frac{M}{\gamma} \right\rfloor. \quad (6)$$

From Lemma 1 we can choose $N = \left\lfloor e^\tau M / Me \right\rfloor$ subsets $A_i, \ldots, A_N$ of the set $\{x_1, \ldots, x_M\}$ satisfying (4). Let $Q_i$ be
the uniform distribution on the subset $A_i$ whose cardinality is $\lceil \tau M \rceil$, that is, $Q_i$ is defined as

$$Q_i(x) \overset{\text{def}}{=} \frac{1}{|A_i|} \sum_{x' \in A_i} 1_{x'}(x), \quad (7)$$

where $1_{x'} = 1_{x'}(x)$ is an indicator function taking value 1 if $x = x'$ and 0 otherwise. Defining the subset $D_i$ as $D_i \overset{\text{def}}{=} \cup_{x \in A_i} MU_x$, we evaluate

$$W_{Q_i}(D_i) = \sum_{x \in A_i} \frac{1}{|A_i|} W_x(D_i) \geq \sum_{x \in A_i} \frac{1}{|A_i|} W_x(U_x)$$

$$\geq 1 - \alpha \beta \sum_{x \in A_i} \frac{1}{|A_i|} W_x(y) \leq C \} \right.$$ 

$$W_{Q_i}(D_j) = \sum_{x \in A_j} \frac{1}{|A_j|} W_x(D_j)$$

$$= \sum_{x \in A_j} \frac{1}{|A_j|} W_x(D_j)$$

$$\leq \frac{|A_j \cap A_j|}{|A_j|} + \sum_{x \in A_j \cap A_j} \frac{1}{|A_j|} W_x \left( \bigcup_{x' \neq x} U_{x'} \right)$$

$$\leq \kappa + \alpha' \beta' \frac{1}{C} \frac{M}{\gamma}.$$ 

Therefore, we obtain the desired argument.

Next, we prove the existence of $M$ elements and $M$ subsets satisfying (5) and (6) by a random coding method. Let $M'$ be $\lceil \frac{M}{\gamma} \rceil$, and $X = (X_1, \ldots, X_{M'})$ be $M$ independent and identical random variables subject to the probability distribution $p \in \mathcal{P}(\mathcal{X})$, then we have

$$W_p(U_x) \leq \frac{1}{C} W_x(U_x) \leq \frac{1}{C}.$$ 

Using this inequality, we obtain

$$E_X \sum_{i=1}^{M'} W_{X_i} \left( \bigcup_{j \neq i} U_{X_j} \right) \leq E_X \sum_{i=1}^{M'} \sum_{j \neq i} W_{X_i}(U_{X_j})$$

$$= \sum_{j=1}^{M'} E_X W_j \left( U_{X_j} \right) \leq \sum_{j=1}^{M'} E X \left( \frac{M' - 1}{M' C} \right) \leq \frac{M' - 1}{C},$$

Further,

$$E_X \sum_{i=1}^{M'} W_{X_i}(U_{X_j}) = \sum_{i=1}^{M'} E_X W_{X_i}(U_{X_j}) = W_{p}(U_x).$$

Using the Markov inequality $P_X \{ X > \alpha EX \} < \frac{1}{\alpha}$, i.e.,

$$P_X \{ X \leq \alpha EX \} > 1 - \frac{1}{\alpha},$$

we can show that

$$P_X \left( \frac{1}{M'} \sum_{i=1}^{M'} W_{X_i} \left( \bigcup_{j \neq i} U_{X_j} \right) \leq \frac{\alpha' M' - 1}{C} \right) > 1 - \frac{1}{\alpha'}$$

$$P_X \left( \frac{1}{M'} \sum_{i=1}^{M'} W_{X_i}(U_{X_j}) \leq \alpha E_p W_x(U_x) \right) > 1 - \frac{1}{\alpha}.$$ 

Since $(1 - \frac{1}{\alpha'}) + (1 - \frac{1}{\alpha}) > 1$, there exist $M$ elements $x_1, \ldots, x_{M'}$ such that

$$\frac{1}{M'} \sum_{i=1}^{M'} W_{X_i} \left( \bigcup_{j \neq i} U_{X_j} \right) \leq \frac{\alpha' M' - 1}{C}$$

$$\frac{1}{M'} \sum_{i=1}^{M'} W_{X_i}(U_{X_j}) \leq \alpha E_p W_x(U_x).$$

In the following, the above $M$ elements $x_1, \ldots, x_{M'}$ are fixed, and we only focus on the random variable $i$ subject to the uniform distribution on the set $\{1, \ldots, M'\}$. Combining Markov inequality and the preceding inequalities, we have

$$P_i \left( \frac{1}{M'} \sum_{i=1}^{M'} W_{X_i} \left( \bigcup_{j \neq i} U_{X_j} \right) \leq \frac{\alpha' M' - 1}{C} \right) > 1 - \frac{1}{\beta'}$$

$$P_i \left( \frac{1}{M'} \sum_{i=1}^{M'} W_{X_i}(U_{X_j}) \leq \alpha E_p W_x(U_x) \right) > 1 - \frac{1}{\beta}.$$ 

Hence, we obtain

$$P_i \left( W_{X_i} \left( \bigcup_{j \neq i} U_{X_j} \right) \leq \frac{\beta' \alpha' M' - 1}{C} \right)$$

$$> (1 - \frac{1}{\beta'}) + (1 - \frac{1}{\beta}) = 1,$$

which yields

$$\left\{ i \right\} \left( W_{X_i} \left( \bigcup_{j \neq i} U_{X_j} \right) \leq \frac{\beta' \alpha M' - 1}{C} \right) \right.$$ 

$$\times (\bigcup_{j \neq i} U_{X_j}) \leq \alpha E_p W_x(U_x) \right) \right.$$ 

$$> \gamma M'.$$

Since $\gamma M' = [\gamma \frac{M}{\gamma}] \geq M$, there exist $M$ elements of $\mathcal{X}$ satisfying (5) and (6). Here, one may think that these $M$ elements may not be distinct. However, if $x_i = x_{i'} (i \neq i')$, the relation $W_{X_i}(U_{X_j}) + W_{X_{i'}}(U_{X_j}) \geq 1$ holds. From condition (3), this contradicts (5) and (6). Hence, we obtain the desired bound.

### III. CHANNEL RESOLVABILITY IN NON-ASYMPTOTIC SETTING

In the channel resolvability, we choose $M$ elements $x_1, \ldots, x_M$ in the input set $\mathcal{X}$ for every probability distribution $p \in \mathcal{P}(\mathcal{X})$, such that the output distribution of the input distribution

$$\sum_{i=1}^{M'} \frac{1}{M'} x_i,$$

close enough to the output distribution of $p$ through the channel $W$. In particular, we call the distribution with the preceding form an $M$-type. In this setting, our purpose is to disenable the receiver of the given channel $W$ to distinguish whether the sender generates the input signal based on ‘the given distribution $p$’ or ‘the $M$-type $\sum_{i=1}^{M'} \frac{1}{M'} x_i$ with a smaller number $M'$. This kind of indistinguishability cannot be applied to any realistic model, directly, but it can be technically related to wire-tap channel. In particular, we prove Lemma 2 in this section as the technically essential part, but this lemma is also the technically essential part for the direct part of wire-tap channel.
In the following, we call the pair of the integer $M$ and the $M$ elements $x_1, \ldots, x_M$ of $\mathcal{X}$, a resolvability code $\Psi$ with the size $|\Psi| \overset{\text{def}}{=} M$. The performance of a resolvability code $\Psi$ is characterized by its size $|\Psi|$ and the variational distance
\[
\epsilon_\Psi(W_p) \overset{\text{def}}{=} d \left( \sum_{i=1}^{M} \frac{1}{M} W_{x_i}, W_p \right),
\]
where the variational distance $d(p, q)$ defined by
\[
d(p, q) = \sum_y |p(y) - q(y)|,
\]
which equals the $l_1$ norm $\|p - q\|_1$. Another characterization of its performance is given by K-L divergence
\[
D_\Psi(W_p, W_q) \overset{\text{def}}{=} D \left( \sum_{i=1}^{M} \frac{1}{M} W_{x_i} \parallel W_p \right),
\]
where $D(p||q) \overset{\text{def}}{=} \sum_y p(y) \log \frac{p(y)}{q(y)}$.

**Theorem 2:** For any integer $M > 0$, any real number $C > 0$ and any probability distribution $p \in \mathcal{P}(\mathcal{X})$, there exists a resolvability code $\Psi$ such that $|\Psi| = M$ and
\[
e_\Psi(W_p, W_q) \leq 2\delta_{p,W,C} + \sqrt{\frac{\delta_{p,W,C}}{M}},
\]
where
\[
\delta_{p,W,C} \overset{\text{def}}{=} E_p W_x \left\{ \frac{W_x(y)}{W_p(y)} > C \right\},
\]
and
\[
\delta_{p,W,C}^\prime \overset{\text{def}}{=} E_p W_x \left\{ \frac{W_x(y)}{W_p(y)} \leq C \right\}.
\]
If the cardinality $|\mathcal{Y}|$ is finite, for any $0 > t \geq -\frac{1}{2}$, there exists a resolvability code $\Psi$ such that $|\Psi| = M$ and either of
\[
D(\Psi', W_p) \leq \frac{\log(1 + M^t e^{\phi(t|W,p)})}{-t},
\]
holds, where
\[
\eta(x) \overset{\text{def}}{=} -x \log x \quad \text{and} \quad \phi(t|W,p) \overset{\text{def}}{=} \log \sum_y (E_p W_x)^t \left( \frac{W_x(y)}{W_p(y)} \right)^{1-t}.
\]

**Remark 1:** The partial resolvability version of inequality (9) has been obtained by Oohama(9). Inequality (9) can be regarded as the essentially same result as Oohama’s inequality.

**Proof:** In the following, the indicator functions $I_x$ and $I_x^c$ on the sets $\mathcal{U}_x = \left\{ \frac{W_x(y)}{W_p(y)} > C \right\}$ and their compliment sets $\mathcal{U}_x^c$ play important roles. In our proof of Theorem 2, we use the random coding method, i.e., we consider the $M$ independent and identical random variables $X = (X_1, \ldots, X_M)$ subject to $p$. Using the notations:
\[
W_{\alpha}^\gamma (y) \overset{\text{def}}{=} W_x(y) I_{\gamma}^\alpha(y), \quad W_{\beta}^\gamma (y) \overset{\text{def}}{=} W_x(y) I_{\beta}^\gamma(y),
\]
we have the following lemma.

**Lemma 2:** The $M$ random variables $X = (X_1, \ldots, X_M)$ satisfy the following inequality
\[
E_X \left\| W_{X}^M - W_p \right\|_1 \leq 2\delta_{p,W,C} + \sqrt{\frac{\delta_{p,W,C}}{M}}
\]
for $0 > t \geq -\frac{1}{2}$. If the cardinality $|\mathcal{Y}|$ is finite, the inequality
\[
E_X D(W_{X}^M || W_p) \leq \eta(\delta_{p,W,C}) + \delta_{p,W,C} \log |\mathcal{Y}| + \frac{\delta_{p,W,C}^\prime}{M}
\]
holds.

Since there exists a resolvability code $\Psi$ with the size $M$ such that
\[
e_\Psi(W_p, W_q) \leq E_X \left\| W_{X}^M - W_p \right\|_1,
\]
the inequality (12) guarantees the existence of a resolvability code $\Psi$ satisfying (9). On the other hand, the relation $\frac{W_x(y)}{W_p(y)} \leq C$ holds. Thus,
\[
\delta_{p,W,C}^\prime = \sum_y W_x(y) \frac{W_x(y)}{W_p(y)} \leq C.
\]
Similarly, since there exists a resolvability code $\Psi$ with the size $M$ such that
\[
D(\Psi', W_p) \leq E_X D(W_{X}^M || W_p),
\]
the inequalities (13) and (14) guarantees the existence of a resolvability code $\Psi$ satisfying (10) and (11). □

**Proof of Lemma 2**

First, we show (12). Since
\[
\delta_{p,W,C} = E_{p,x} I_x(U_x) = E_{p} \left\| W_{\alpha}^\gamma \right\| = \left\| W_{\beta}^\gamma \right\|
\]
we can evaluate
\[
E_X \left\| W_{X}^M - W_p \right\|_1 = E_{p,x} I_x(U_x) = \sum_{i=1}^{M} \frac{1}{M} E_X \left\| W_{\alpha}^\gamma \right\| + \left\| W_{\beta}^\gamma \right\|
\]
Next, we focus on the Schwartz inequality regarding the random variable $l_x(y) \overset{\text{def}}{=} \frac{W_x(y)}{W_p(y)}$ and the sign function $\tilde{l}_x(y) \overset{\text{def}}{=} \frac{1}{l_x(y)}$. (we can check that $\tilde{l}_x = 1$), then we obtain
\[
\left( \left\| W_{\alpha}^\gamma \right\| \right)^2 = (E_{p,x} l_x(y))^2 = (E_{p,x} l_x(y) \tilde{l}_x(y))^2 \leq E_{p,x} \tilde{l}_x(x) = E_{p,x} l_x(y).
\]
Thus, the Jensen inequality yields that
\[
E_X \left\| W_{X}^M - W_p \right\|_1 \leq E_{p,x} \left\| W_{\alpha}^\gamma \right\| \leq E_X \left\| W_{X}^M - W_p \right\|_1.
\]
Since $E_x \frac{W^p_{\nu}(y)}{W^p_p(y)} = \frac{W^p_{\nu}(y)}{W^p_p(y)}$, we have
\[
E_x E_w l^2_X = E_w E_x l_{w x}^2(y)
\]
\[
= E_w p_x \frac{1}{M^2} \sum_{i=1}^{M} \left( \frac{W^p_{\nu}(y)}{W^p_p(y)} - \frac{W^p_{\nu}(y)}{W^p_p(y)} \right)^2
\]
\[
= E_w p_x \frac{1}{M} \left( \frac{W^p_{\nu}(y)}{W^p_p(y)} - \frac{W^p_{\nu}(y)}{W^p_p(y)} \right)^2
\]
\[
\leq E_w \frac{1}{M} E_w \left( \frac{W^p_{\nu}(y)}{W^p_p(y)} \right)^2 = \frac{\delta_{p,w,c}}{M}.
\]
Therefore, we obtain
\[
E_x \left\| \sum_{i=1}^{M} \frac{1}{M} W_{X_i} - W_p \right\|_1 \leq 2\delta_{p,w,c} + \sqrt{\frac{\delta_{p,w,c}}{M}}.
\]
Hence, we obtain \([12]\).

Next, we show \([13]\). Since $\frac{W^p_{\nu}(y)}{W^p_p(y)} \leq \frac{1}{W^p_p(y)}$, by using the inequality $\log x \leq x - 1$, we can evaluate
\[
E_x D(W^M_{X} || W_p) = E_x \sum_{X} W^M_{X}(y) \log \frac{W^M_{X}(y)}{W^p_p(y)} + \sum_{Y} W^\beta_{Y}(y) \log \frac{W^M_{X}(y)}{W^p_p(y)}
\]
\[
\leq E_x \sum_{X} \left( W^M_{X}(y) \left( \frac{W^M_{X}(y)}{W^p_p(y)} - 1 \right) + W^\beta_{Y}(y) \log \frac{W^M_{X}(y)}{W^p_p(y)} \right)
\]
\[
= \sum_{X} E_x W^M_{X}(y) \left( \frac{W^M_{X}(y)}{W^p_p(y)} - 1 \right) + \sum_{Y} W^\beta_{Y}(y) \log \frac{1}{W^p_p(y)}.
\]
Regarding the first term, we can calculate
\[
\sum_{Y} E_x W^M_{X}(y) \left( \frac{W^M_{X}(y)}{W^p_p(y)} - 1 \right)
\]
\[
= \sum_{X} \frac{1}{M^2} \sum_{i,j} W^M_{X_i}(y) \left( \frac{W^M_{X_i}(y)}{W^p_p(y)} - 1 \right)
\]
\[
= \sum_{X} \frac{1}{M} E_{y^{x}} W^\alpha_{y}(y) \left( \frac{W^M_{X}(y)}{W^p_p(y)} - 1 \right)
\]
\[
\leq \sum_{X} \frac{1}{M} E_{y^{x}} W^\alpha_{y}(y) \frac{\delta_{p,w,c}}{M},
\]
where we use the relation $E_x W^\alpha_{y}(y) \left( \frac{W^M_{X_i}(y)}{W^p_p(y)} - 1 \right) = 0$ for $i \neq j$. Concerning the second term, letting $K \overset{\text{def}}{=} \sum_{Y} W^\beta_{Y}(y)$, we have
\[
\sum_{Y} W^\beta_{Y}(y) \log \frac{1}{W^p_p(y)}
\]
\[
= -K \log K - K \sum_{Y} \frac{W^\beta_{Y}(y)}{K} \log \frac{W^\beta_{Y}(y)}{K}
\]
\[
\leq \eta \left( \sum_{Y} W^\beta_{Y}(y) \right) \sum_{Y} \frac{W^\beta_{Y}(y)}{K} \log |Y|
\]
because $\log |Y|$ is the maximal entropy of the distribution on the probability space $Y$. Since $\sum_{Y} W^\beta_{Y}(y) = \delta_{p,w,c}$, we obtain \([12]\).

Finally, we prove \([13]\) by a different method. The quantity $E_x D(W^M_{X} || W_p)$ can be regarded as the mutual information of channel $X \rightarrow W^M_{X}$ with the input probability $p^M_{X}(x)$ which equals the $M$-fold i.i.d. of $p$. We can check that the function $t \rightarrow \phi(t|W^M_{X}, p^M)$ satisfies the following property:
\[
\phi(0|W^M_{X}, p^M) = 0
\]
\[
\frac{d\phi(t|W^M_{X}, p^M)}{dt} \bigg|_{t=0} = -E_x D(W^M_{X} || W_p),
\]
\[
\frac{d^2 \phi(t|W^M_{X}, p^M)}{dt^2} \geq 0.
\]
Hence, its convexity guarantees the inequality
\[
-t E_x D(W^M_{X} || W_p) \leq \phi(t|W^M_{X}, p^M),
\]
which implies the inequality
\[
E_x D(W^M_{X} || W_p) \leq \frac{\phi(t|W^M_{X}, p^M)}{-t} (16)
\]
for $0 > t \geq -\frac{1}{2}$.

Let $1 + s = \frac{1}{t}$, then $1 \geq s > 0$ and $t = \frac{1}{1+s}$. Since $x \rightarrow x^s$ is concave,
\[
E_x \left( \sum_{X} W_{X_i}(y)^s \right) \leq \left[ E_x \sum_{X} W_{X_i}(y) \right]^s = (M-1)^s W^s_p(y).
\]
(16)

Using \([16]\) and the relation $(x+y)^s \leq x^s + y^s$ for two positive real numbers $x, y$, we obtain
\[
e^{-\phi(t|W^M_{X}, p^M)} = \sum_{y} \left( E_x (W^M_{X})^{1+s}(y) \right)^{1+s}
\]
\[
= \frac{1}{M} \sum_{y} \left( E_x \sum_{i=1}^{M} W_{X_i}(y) \left( W_{X_i}(y) + \sum_{j \neq i} W_{X_j}(y) \right)^s \right)^{1+s}
\]
\[
\leq \frac{1}{M} \sum_{y} \left( E_x \sum_{i=1}^{M} W_{X_i}(y) \left( W_{X_i}(y) + \sum_{j \neq i} W_{X_j}(y) \right)^s \right)^{1+s}
\]
\[
\leq \frac{1}{M} \sum_{y} \left( \sum_{i=1}^{M} E_x W_{X_i}^{1+s}(y) \right)^{1+s}
\]
\[
+ \sum_{i=1}^{M} E_x W_{X_i}(y) \left( \sum_{j \neq i} W_{X_j}(y) \right)^{1+s}
\]
\[
\leq \frac{1}{M} \sum_{y} \left( \sum_{i=1}^{M} E_x W_{X_i}^{1+s}(y) \right)^{1+s}
\]
\[
\leq \frac{1}{M} \sum_{y} \left( \sum_{i=1}^{M} E_x W_{X_i}^{1+s}(y) + \sum_{i=1}^{M} (M-1)^s W_{X_i}^{1+s}(y) \right)^{1+s}
\]
\[
\leq \frac{1}{M} \sum_{y} \left( ME_x W_{X_i}^{1+s}(y) + M(M-1)^s W_{X_i}^{1+s}(y) \right)^{1+s}
\]
\[
= \sum_{y} \left( E_x W_{X_i}^{1+s}(y) \right)^{1+s} + (M-1)^s W_{X_i}^{1+s}(y)
\]
\[
\leq 1 + \frac{1}{M^{1+s}} \sum_{y} \left( E_x W_{X_i}^{1+s}(y) \right)^{1+s} = 1 + M^s e^{\phi(t|W^M_{X}, p^M)}.
\]
Since $-t$ is positive, the desired inequality \([13]\) follows from \([15]\) and the above inequality.
Next, we proceed to the relation with identification codes. In order to discuss this relation, we focus on channel resolvability of the worst input case, and define the following values:

\[\epsilon(M, W) \overset{\text{def}}{=} \max_{p \in P(X)} \min_{\Psi : |\Psi| \leq M} \epsilon(\Psi, W_p),\]

\[D(M, W) \overset{\text{def}}{=} \max_{p \in P(X)} \min_{\Psi : |\Psi| \leq M} D(\Psi, W_p),\]

which satisfies

\[\epsilon(M, W) \leq 2 \max_{p} E_p W_x \left\{ \frac{W^L}{W^p}(y) > C \right\} + \sqrt{\frac{C}{M}},\] (17)

for any real number \(C > 0\).

**Lemma 3:** (Han & Verdú[2]) If the cardinality \(|X|\) is finite, and if an identification code \(\Phi\) and an integer \(M\) satisfy

\[1 - \mu(\Phi) - \lambda(\Phi) > \epsilon(M, W),\]

then

\[|X|^M \geq |\Phi|.\] (18)

**Proof:** Let the identification code \(\Phi\) be a triplet \((N, \{Q_1, \ldots, Q_N\}, \{D_1, \ldots, D_N\})\), then there exist \(N\) \(M\)-types \(Q'_1, \ldots, Q'_N\) such that

\[d(Q_i, Q'_j) \leq \epsilon(M, W).\]

Since the inequalities

\[2\epsilon(M, W) + d(Q_i, Q'_j) \geq d(Q_i, Q'_j) + d(Q_i, Q'_j) + d(Q'_i, Q'_j) \geq 2(1 - \mu(\Phi) - \lambda(\Phi)),\]

hold for any \(i \neq j\), we can show

\[d(Q_i, Q'_j) > 0,\]

which implies that \(Q'_i\) is different from \(Q'_j\). However, the total number of \(M\)-types is less than \(|X|^M\). Therefore, we obtain (18).}

**IV. WIRE-TAP CHANNEL IN NON-ASYMPTOTIC SETTING**

Next, we discuss the message transmission with the wire-tapper who has less information than the main receiver. This problem is formulated as follows. Let \(\mathcal{U}\) be the probability space of the main receiver, and \(\mathcal{Z}\) be the space of the wire-tapper, then the main channel from the transmitter to the main receiver is described by \(W^B : x \mapsto W^B_x\), and the wire-tapper channel from the transmitter to the wire-tapper is described by \(W^E : x \mapsto W^E_x\). In this setting, the transmitter choose \(M\) distributions \(Q_1, \ldots, Q_M\) on \(X\), and he generates \(x \in X\) subject to \(Q_i\) when he wants to send the message \(i \in \{1, \ldots, M\}\). The normal receiver prepares \(M\) disjoint subsets \(D_1, \ldots, D_M\) of \(\mathcal{U}\) and judges that a message is \(i\) if \(y\) belongs to \(D_i\). Therefore, the triplet \((M, \{Q_1, \ldots, Q_M\}, \{D_1, \ldots, D_M\})\) is called a code, and is described by \(\Phi\). Its performance is given by the following quantities. One is the size \(M\), which is denoted by \(|\Phi|\). The second one is the average error probability \(\epsilon_B(\Phi)\):

\[\epsilon_B(\Phi) \overset{\text{def}}{=} \frac{1}{M} \sum_{i=1}^{M} W^B_{Q_i}(D'_i),\]

and the third one is the wire-tapper’s information regarding the transmitted message \(I_E(\Phi)\):

\[I_E(\Phi) \overset{\text{def}}{=} \sum_{i} \frac{1}{M} D(W^E_{Q_i} \parallel W^E_F), \quad W^E_F \overset{\text{def}}{=} \sum_{i} \frac{1}{M} W^E_{Q_i}.\]

A different measure of the wire-tapper’s information is given by the average variational distance \(d_E(\Phi)\):

\[d_E(\Phi) \overset{\text{def}}{=} \frac{1}{M(M-1)} \sum_{i \neq j} d(W^E_{Q_i}, W^E_{Q_j}).\]

**Theorem 3:** There exists a code \(\Phi\) for any integers \(L, M\), any real numbers \(C, C' > 0\), and any probability distribution \(p\) on \(X\) such that

\[|\Phi| = M,\]

\[\epsilon_B(\Phi) \leq 3 \min_{0 \leq s \leq 1} (ML)^s \sum_{y} \left( E_p(W^B_x(y))^{1/(1+s)} \right)^{1+s},\] (19)

\[\epsilon_B(\Phi) \leq 3 \left( E_p W_x \left\{ \frac{W^B}{W^p}(y) \leq C' \right\} + \frac{ML}{C'} \right),\] (20)

\[I_E(\Phi) \leq 3 \left( \eta(\delta_{p,W^E,C}) + \delta_{p,W^E,C} \log |Z| + \frac{\delta_{p,W^E,C}}{L} \right),\] (21)

\[d_E(\Phi) \leq 6 \left( 2\epsilon_B \log(1 + L \epsilon^B(t)) \right),\] (23)

**Proof:** We prove Theorem 3 by a random coding method. Let \(X = (X_{l,m})\) be \(LM\) independent and identical random variables subject to the distribution \(p\) on \(X\) for integers \(l = 1, \ldots, L\) and \(m = 1, \ldots, M\), and \(D'_{l,m}(X)\) be the maximum likelihood decoder of the code \(X_{l,m}\), then we can evaluate as follows by Gallager upper bound[7].

\[E_X \frac{1}{ML} \sum_{l,m} W^B_{X_{l,m}}(D'_{l,m}(X)^c) \leq \min_{0 \leq s \leq 1} (ML)^s \sum_{y} \left( E_p(W^B_x(y))^{1/(1+s)} \right)^{1+s}.\]

Since the maximum likelihood decoder is better than the code \(\mathcal{D}_{l,m}'(X) = \left\{ \frac{W^B}{W^p}(y) > C' \right\} \setminus \cup_{(l',m') \neq (l,m)} \left\{ \frac{W^B}{W^p}(y) > C' \right\}\), we have another evaluation
as

\[
E_X \frac{1}{ML} \sum_{l,m} W_{X,m,l}^B (D'_{l,m}(X)^c)
\]

\[
\leq E_X \frac{1}{ML} \sum_{l,m} W_{X,m,l}^B (D''_{l,m}(X)^c)
\]

\[
\leq E_X \frac{1}{ML} \sum_{l,m} W_{X,m,l}^B \left\{ \frac{W_{X,m,l}^B(y) \leq C'}{W_P^B(y) \leq C'} \right\}
\]

\[
+ E_X \frac{1}{ML} \sum_{l,m} W_{X,m,l}^B \sum_{(l',m') \neq (l,m)} \left\{ \frac{W_{X,m',l'}^B(y) \leq C'}{W_P^B(y) \leq C'} \right\}
\]

\[
\leq E_{p,x} W_x^B \left\{ \frac{W_x^B(y) \leq C'}{W_P^B(y) \leq C'} \right\}
\]

\[
+ W_P^B (ML - 1) E_{p,x} \left\{ \frac{W_x^B(y) \leq C'}{W_P^B(y) \leq C'} \right\}
\]

\[
\leq E_{p,x} W_x^B \left\{ \frac{W_x^B(y) \leq C'}{W_P^B(y) \leq C'} \right\} + ML C'.
\]

Using Markov inequality, we obtain

\[
P_X \{ \epsilon_B(\Phi(X)) \leq 3E\epsilon_B(\Phi(X)) \}^c < \frac{1}{3^3},
\]

\[
P_X \{ I_E(\Phi(X)) \leq 3EI_E(\Phi(X)) \}^c < \frac{1}{3^3},
\]

\[
P_X \{ d_E(\Phi(X)) \leq 3Ed_E(\Phi(X)) \}^c < \frac{1}{3}.
\]

Therefore, there exists a code \( \Phi \) satisfying desired conditions.

V. General asymptotic setting

A. Identification code and channel resolvability

Next, we focus on an arbitrary sequence of channels \( \mathbf{W} = \{ W_n \}_{n=1}^\infty \), in which \( W^n \) is an arbitrary channel from \( X^n \) to \( Y^n \). In this setting, two-types of \((\mu, \lambda)\)-identification capacities are defined by

\[
D(\mu, \lambda|\mathbf{W}) \quad \text{def} \quad \sup_{\{ \Phi_n \}} \left\{ \lim_{n} \frac{1}{n} \log \log |\Phi_n| \left| \lim_{n} \mu(\Phi_n) < \mu, \lim_{n} \lambda(\Phi_n) \leq \lambda \right\}
\]

\[
D^T(\mu, \lambda|\mathbf{W}) \quad \text{def} \quad \sup_{\{ \Phi_n \}} \left\{ \lim_{n} \frac{1}{n} \log \log |\Phi_n| \left| \lim_{n} \mu(\Phi_n) < \mu, \lim_{n} \lambda(\Phi_n) \leq \lambda \right\}.
\]

However, in the case of \( \mu = 0 \), we replace \( \lim_{n} \mu(\Phi_n) < \mu \) by \( \mu \), \( \lim_{n} \lambda(\Phi_n) = 0 \), \( \lim_{n} \lambda(\Phi_n) = 0 \). In this case, the above two definitions. On the other hand, two-types \( \epsilon \)-resolvability capacities are defined by

\[
S(\epsilon|\mathbf{W}) \quad \text{def} \quad \sup \left\{ R | \lim_{n} \epsilon(e^{\epsilon R}, Y_n) \leq \epsilon \right\}
\]

\[
S^T(\epsilon|\mathbf{W}) \quad \text{def} \quad \sup \left\{ R | \lim_{n} \epsilon(e^{\epsilon R}, Y_n) \leq \epsilon \right\},
\]

where the case of \( \epsilon = 2 \), we replace \( \leq \epsilon \) by \( < \epsilon \) at the above two definitions.

In the information spectrum method, the following quantities are defined for arbitrary sequence \( p = \{ p^n \}_{n=1}^\infty \) of input probability distributions:

\[
\mathbf{7}(\epsilon|p, \mathbf{W}) \quad \text{def} \quad \inf \left\{ a \left| \lim_{n} E_{p^n} W_x^n \left\{ \frac{1}{n} \log \frac{W^n}{W^n_{p^n}}(y) > a \right\} \leq \epsilon \right\}
\]

\[
\mathbf{I}(\epsilon|p, \mathbf{W}) \quad \text{def} \quad \inf \left\{ a \left| \lim_{n} E_{p^n} W_x^n \left\{ \frac{1}{n} \log \frac{W^n}{W^n_{p^n}}(y) > a \right\} \leq \epsilon \right\},
\]

where the case of \( \epsilon = 1 \), we replace \( \leq \) by \( < \) at the above definitions. These quantities have another expression as

\[
\mathbf{7}(\epsilon|p, \mathbf{W}) = \sup \left\{ a \left| \lim_{n} E_{p^n} W_x^n \left\{ \frac{1}{n} \log \frac{W^n}{W^n_{p^n}}(y) \leq a \right\} < 1 - \epsilon \right\},
\]

\[
\mathbf{I}(\epsilon|p, \mathbf{W}) = \sup \left\{ a \left| \lim_{n} E_{p^n} W_x^n \left\{ \frac{1}{n} \log \frac{W^n}{W^n_{p^n}}(y) \leq a \right\} < 1 - \epsilon \right\},
\]
Theorem 4: Assume that $|X^n| = d^n$, then the above quantities satisfy the following relations.

$$
\sup_p I(\epsilon|p, W) \leq D(1 - \epsilon, 0|W) \leq S(\epsilon|W)
$$

$$
\sup_p I(\epsilon|p, W) \leq \frac{D(1 - \epsilon, 0|W)}{2}
$$

$$
\sup_p I(\epsilon|p, W) \leq \frac{D(1 - \epsilon, 0|W)}{2}
$$

$$
\sup_p I(\epsilon|p, W) \leq \frac{D(1 - \epsilon, 0|W)}{2}
$$

for any real number $0 \leq \epsilon < 1$. However, the first inequalities in (24) and (25) hold for $0 \leq \epsilon \leq 1$, and the third ones hold for $0 \leq \epsilon \leq 2$. In particular, we obtain

$$
\sup_p I(0|p, W) = D(1, 0|0W) = S(0|W)
$$

which is desired in Han & Verdú[2] and Han[3].

This theorem indicates the existence of a code satisfying the following: The second error probability $\lambda$ is asymptotically independent for the behavior of the distribution of the random variable of likelihood and always goes to 0, and only the second error probability $\mu$ asymptotically depends on it.

Remark 2: Steinberg[14] claims the inequalities

$$
\sup_p I(\epsilon|p, W) \geq D(\lambda_1, \lambda_2|W),
$$

$$
\sup_p I(\epsilon|p, W) \geq D(\lambda_1, \lambda_2|W)
$$

for $\lambda_1 + \lambda_2 < 1 - \epsilon$. If they are proved, by combining the above inequalities and Theorem 4 we can prove the equalities of the above inequalities in the continuous case. However, it seems that his paper has a gap in counting the maximum number of different pairs of a partial response and an $M'$-type measure at the proof of Lemma 2, which is essential for these inequalities. That is, he estimated the total number of positive functions on $X \times Y$ with the form

$$
f(x, y) = \frac{1}{M'} \sum_{i=1}^{M'} 1_{x_i}(x) \sum_{(x', y') \in F} 1_{(x', y')}(x, y),
$$

where $F$ is an arbitrary subset of $X \times Y$. The total measure of $f$, i.e., $\sum_{(x, y) \in X \times Y} f(x, y)$, is not necessarily less than 1, while he indicated that it is less than 1. Hence, this total number cannot be bounded by $|X|^M$.

Proof: In order to prove the first inequalities, we choose an arbitrary real number $R \leq \sup_p I(\epsilon|p, W)$ and a sequence of input probability distributions $p$ such that $R < R' \overset{def}{=} I(1 - \mu|p, W)$. Substitute $M = e^{nR'}, C = e^{nR}, \alpha = \beta = 1 + \frac{1}{n+2}, \gamma = \frac{\log 2 + 1}{\log n}$ in Theorem 6 then the conditions (1) and (2) are satisfied and $\gamma = \frac{1}{n+2}$.

Thus, there exists an identification code $\Phi_n$ such that

$$
|\Phi_n| = \left\lfloor \frac{e^{nR}}{e^{1+nR}} \right\rfloor
$$

$$
\mu(\Phi_n) \leq \left\{ 1 + \frac{2}{n} \right\}^{2e^{nR}} W_n \left\{ \frac{1}{n} \log \frac{W_n}{W_{p^n}}(y) \leq R' \right\}
$$

$$
\lambda(\Phi_n) \leq \log 2 + 1 + (n + 2)^2 \frac{1}{e^{nR}} \left( (n + 2) e^{nR} \right)
$$

$$
\leq \log 2 + 1 + (n + 2)^2 e^{-n(R'-R)}.
$$

Therefore, we obtain

$$
\lim \frac{1}{n} \log \log |\Phi_n| = R,
$$

$$
\lim \mu(\Phi_n) \leq \lim \sup_p W_n \left\{ \frac{1}{n} \log \frac{W_n}{W_{p^n}}(y) \leq R' \right\} < \mu
$$

$$
\lim \lambda(\Phi_n) = 0,
$$

which implies that $D(\mu, 0|W) \geq R'$. Thus, we obtain the first inequality in (24) for $0 \leq \epsilon < 1$. In the case of $\epsilon = 1$, we need to replace $\epsilon$ by $\mu$ at (28). By replacing $\lim$ by $\lim$ at (28), we can similarly prove $D(1 - \epsilon, 0|W) \geq \sup_p I(1 - \mu|p, W)$.

Next, we proceed to the second inequalities. Let $R$ be an arbitrary real number such that $R > D(1 - \epsilon, 0|W)$. Then, there exists a sequence $\{\Phi_n\}$ of identification codes such that

$$
R = \lim \frac{1}{n} \log |\Phi_n|, \lim \mu(\Phi_n) < 1 - \epsilon, \lim \lambda(\Phi_n) = 0.
$$

Therefore, we can choose an integer $N$ large enough, such that $1 - \mu(\Phi_n) - \lambda(\Phi_n) \geq 1 - \lim \mu(\Phi_n) > \epsilon$. Moreover, we choose a strictly increasing sequence $\{a_n\}$ of integers such that $a_1 \geq N$ and $1 - \lim \mu(\Phi_n) > \epsilon e^{-nR'}, W_{p^n}$, where $R' = S(1|\epsilon, W)$.

Thus, Lemma 4 yields that $(d^n)^{e^R R'} \geq |\Phi_n|$, which implies that $R' \geq R$. We obtain the second inequalities in (24). We can prove the second inequalities in (25) by choosing a strictly increasing sequence $\{a_n\}$ of integers such that $1 - \mu(\Phi_n) - \lambda(\Phi_n) \geq 1 - \lim \mu(\Phi_n) > \epsilon$.

Finally, we prove the third inequalities by using another expression of $\sup_p I(\epsilon|p, W)$:

$$
\sup_p I(\epsilon|p, W) = \inf \left\{ a \left| \lim \sup_p W_n \left\{ \frac{1}{n} \log \frac{W_n}{W_{p^n}}(y) > a \right\} \leq \epsilon \right\}
$$

Let $R$ and $R'$ be arbitrary real numbers such that $R > \sup_p I(\epsilon/2|p, W)$ and $R > R' > \sup_p I(\epsilon|p, W)$, then the inequality (17) yields that

$$
\epsilon(e^{nR/2} W_n) \leq \lim \inf \sup_p W_n \left\{ \frac{1}{n} \log \frac{W_n}{W_{p^n}}(y) > R' \right\} + e^{-n(R'-R)/2}.
$$

Taking the limit $\lim$, we obtain

$$
\lim \epsilon(e^{nR} W_n) \leq \epsilon,
$$

(29)
which implies $S^I(\mathcal{E}, \mathbf{W}) \leq R$. Thus, we obtain the third inequality in (24) for $0 \leq \epsilon < 1$. In the case of $\epsilon = 2$, we need to replace $\leq \epsilon$ by $< 1$ at (29). By replacing $\lim \text{by} \lim$ in the above, we can prove the third one in (25).

B. Wire-tap channel

Next, we focus on a general sequence $(\mathbf{W}^B = \{W^{B,n}\}, \mathbf{W}^E = \{W^{E,n}\})$ of wire-tap channels, and define the following two kinds of capacities by

$$C_d(\mathbf{W}^B, \mathbf{W}^E) \overset{\text{def}}{=} \sup_{\{\Phi_n\}} \left\{ \lim \frac{1}{n} \log |\Phi_n| \mid \lim \epsilon_B(\Phi_n) = \lim d_E(\Phi_n) = 0 \right\}$$

$$C_t(\mathbf{W}^B, \mathbf{W}^E) \overset{\text{def}}{=} \sup_{\{\Phi_n\}} \left\{ \lim \frac{1}{n} \log |\Phi_n| \mid \lim \epsilon_B(\Phi_n) = \lim \frac{I_E(\Phi_n)}{n} = 0 \right\}.$$  

**Lemma 4:** The inequality

$$C_d(\mathbf{W}^B, \mathbf{W}^E) \geq I(1|p, \mathbf{W}^B) - T(0|p, \mathbf{W}^E)$$  

(30)

holds for any sequence of input distributions $p = \{p^n\}$. Furthermore, if $|Z^n| = d^n$,

$$C_d(\mathbf{W}^B, \mathbf{W}^E) \geq I(1|p, \mathbf{W}^B) - T(0|p, \mathbf{W}^E).$$  

(31)

This theorem is an information spectrum version of Wyner’s result [4], that will be mentioned in the next section.

**Proof:** Let $R' > T(0|p, \mathbf{W}^E)$, $R < I(1|p, \mathbf{W}^B) - R'$ and choose a real number $a$ such that $0 < a < \min \{I(1|p, \mathbf{W}^B) - (R + R'), R' - T(0|p, \mathbf{W}^E)\}$. Substituting $M = e^{nR}, L = e^{nR'}, C = e^{nR-a}, C' = e^{n(R + R' + a)}$, we can show that the right hand side of (20) goes to 0, and that

$$\delta_{p^n, W^{E,n}, e^{n(R-a)}} \to 0, \quad \frac{\delta_{p^n, W^{E,n}, e^{n(R-a)}}}{e^{nR'}} \to 0.$$

Hence, the right hand side of (23) go to 0. Concerning (21), the relations

$$\frac{1}{n} \left( \frac{\eta(\delta_{p^n, W^{E,n}, e^{n(R-a)}})}{e^{nR'}} + \delta_{p^n, W^{E,n}, e^{n(R-a)}} \log |Z^n| \right) + \frac{\delta_{p^n, W^{E,n}, e^{n(R-a)}}}{e^{nR'}} \rightarrow 0$$

$$\frac{1}{n} \eta(\delta_{p^n, W^{E,n}, e^{n(R-a)}}) + \delta_{p^n, W^{E,n}, e^{n(R-a)}} \log d + \frac{\delta_{p^n, W^{E,n}, e^{n(R-a)}}}{e^{nR'}}$$

$$\rightarrow 0$$

hold. Therefore, we obtain (30) and (31).

Conversely, we obtain the following lemma.

**Lemma 5:** Let $\mathcal{Q} = \{Q^n\}$ be a sequence of channels from arbitrary set $\tilde{X}^n$ to the set $X^n$ and $p = \{p^n\}$ be a sequence of distributions on $X^n$. Then, the inequalities

$$C_d(\mathbf{W}^B, \mathbf{W}^E) \sup_{p\in \mathcal{P}} \left\{ I(1|p, \mathbf{W}^B) - T(0|p, \mathbf{W}^E) \right\}$$  

(32)

$$C_t(\mathbf{W}^B, \mathbf{W}^E) \sup_{p\in \mathcal{P}} \left\{ I(1|p, \mathbf{W}^B) - T(0|p, \mathbf{W}^E) \right\}$$  

(33)

hold, where $\mathbf{W}^Q = \{W^nQ^n\}$ denotes the sequence of channels from $X^n$ to $Y^n$:

$$(W^nQ^n)_{\tilde{z}}(y) \overset{\text{def}}{=} \sum_{x \in X^n} W^n_x(y)Q^n_{\tilde{z}}(x)$$

for a sequence of channels $\mathbf{W} = \{W^n\}$ from $X^n$ to $Y^n$.

Hence, applying Lemma 3 to the sequence of the channels $\mathbf{W}^Q, \mathbf{W}^E, \mathbf{W}^Q, \mathbf{W}^E$, we obtain the following theorem.

**Theorem 5:**

$$C_d(\mathbf{W}^B, \mathbf{W}^E) = C_t(\mathbf{W}^B, \mathbf{W}^E)$$

$$= \sup_{p \in \mathcal{P}} \left\{ I(1|p, \mathbf{W}^B) - T(0|p, \mathbf{W}^E) \right\}.$$  

**Proof of Lemma 5** Let $\{\Phi_n = (M_n, \{Q^n_1, \ldots, Q^n_{M_n}\}, \{D^n_1, \ldots, D^n_{M_n}\})\}$ be a sequence of codes of wire-tap channel such that

$$R = \lim \frac{1}{n} \log |\Phi_n|, \quad \lim \epsilon_B(\Phi_n) = 0, \quad \lim I_E(\Phi_n) = 0.$$  

Hence, Verdú-Han’s result [19] yields that the transmission capacity of the sequence of channel $\mathbf{W}^Q$ is less than $I(1|p, \mathbf{W}^B)$, which implies

$$R \leq I(1|p, \mathbf{W}^B).$$

Furthermore, the property $\lim I_E(\Phi_n) = 0$ implies that $S(0|\mathbf{W}^EQ) = 0$. Hence, we have

$$T(0|p, \mathbf{W}^E) = 0.$$  

(34)

Thus, we obtain

$$R = I(1|p, \mathbf{W}^B) - T(0|p, \mathbf{W}^E),$$

which implies (32).

Next, we assume that a sequence of codes of wire-tap channel $\{\Phi_n = (M_n, \{Q^n_1, \ldots, Q^n_{M_n}\}, \{D^n_1, \ldots, D^n_{M_n}\})\}$ satisfies that

$$R = \lim \frac{1}{n} \log |\Phi_n|, \quad \lim \epsilon_B(\Phi_n) = 0, \quad \lim I_E(\Phi_n) = 0.$$  

Since the mutual information

$$I_E(\Phi_n) = \frac{1}{M_n} \sum_{i=1}^{M_n} (W^{E,n}Q^n_i) \log \frac{(W^{E,n}Q^n_i)}{\sum_{i=1}^{M_n}(W^{E,n}Q^n_i)}$$

can be regarded as KL-divergence, Lemma 6 yields that

$$\frac{1}{M_n} \sum_{i=1}^{M_n} (W^{E,n}Q^n_i) \left\{ \frac{1}{n} \log \frac{(W^{E,n}Q^n_i)}{\sum_{i=1}^{M_n}(W^{E,n}Q^n_i)} \geq a \right\} \leq \frac{I_E(\Phi_n) + \frac{1}{n} \log \left| \frac{a}{na} \right|}{M_n} \rightarrow 0$$

for any $a > 0$. Thus, we obtain (32). Therefore, similarly to (32), we obtain (33).

**Lemma 6:** Assume that $p$ and $q$ are two probability distributions on $\Omega$. Then, we have

$$D(p||q) + 1 \geq \alpha \cdot p \left\{ \log \frac{p(\omega)}{q(\omega)} \geq \alpha \right\}. \tag{35}$$

**Proof:** We focus on the two probability distributions on $\Omega_0 \overset{\text{def}}{=} \left\{ \omega \mid \frac{p(\omega)}{q(\omega)} < a \right\}$.

$$p_0(\omega) \overset{\text{def}}{=} \frac{p(\omega)}{p(\Omega_0)}, \quad q_0(\omega) \overset{\text{def}}{=} \frac{q(\omega)}{q(\Omega_0)}.$$
Hence,
\[ D(p||q) = \sum_{\omega \in \Omega} p(\omega) \log \frac{p(\omega)}{q(\omega)} + \sum_{\omega \in \Omega} p(\omega) \log \frac{p(\omega)}{q(\omega)} \geq \alpha p\{\Omega_0\} + \sum_{\omega \in \Omega} p_0(\omega) \left( \log \frac{p_0(\Omega_0)}{q(\Omega_0)} + \log \frac{p_0(\omega)}{q_0(\omega)} \right) \]
\[ = \alpha p\{\Omega_0\} + \alpha p\{\Omega_0\} \log \frac{p_0(\Omega_0)}{q(\Omega_0)} + D(p_0||q_0) \geq \alpha p\{\Omega_0\} + \alpha p\{\Omega_0\} \log \frac{p_0(\Omega_0)}{q(\Omega_0)} = \alpha p\{\Omega_0\} + p\{\Omega_0\} \log p\{\Omega_0\}. \]

Finally, the convexity of the map \( x \mapsto x \log x \) guarantees that \( p\{\Omega_0\} \log p\{\Omega_0\} \geq -\frac{1}{e} \). We obtain (35).

\section{VI. Exponents in Stationary Memoryless Channel}

\subsection{A. Channel Resolvability}

Next, we proceed to the stationary memoryless channel of a given channel \( W \) as a special case.

First, we treat channel resolvability. As was shown by Han & Verdù [2] and Han [3], the information spectrum quantities of discrete memoryless channel of \( W \) is calculated as
\[ \sup_p T(\epsilon|p, W) = \sup_p I(\epsilon|p, W) = \max_p I(p; W) \]
for \( 1 \geq \epsilon \geq 0 \), where
\[ I(p; W) \overset{\text{def}}{=} \text{E}_p D(W_x||W_p). \]

Hence, Theorem \[ \textbf{4} \] yields
\[ S(\epsilon|W) = S'(\epsilon|W) = \max_p I(p; W), \]
which has been obtained by Han & Verdù [2]. Furthermore, using Theorem \[ \textbf{4} \] we can discuss these problems in more details by treating the following optimal exponents:
\[ e_\epsilon(R|W, p) \]
\[ \overset{\text{def}}{=} \sup_{\{\psi_n\}} \left\{ \lim_{n} \frac{1}{n} \log \epsilon(\psi_n, W^n_x) \right\} \]
\[ = \sup_{\{\psi_n\}} \left\{ \lim_{n} \frac{1}{n} \log D(\psi_n, W^n_x) \right\}, \]
and
\[ e_\epsilon(R|W) \overset{\text{def}}{=} \lim_{n} \frac{1}{n} \log e^{nR}, W^n, \]
\[ e_D(R|W) \overset{\text{def}}{=} \lim_{n} \frac{1}{n} \log D(e^{nR}, W^n), \]
where \( p^n \) is the \( n \)-fold identical independent distribution of \( p \). As is discussed by Oohama [8], by using Lemma \[ \textbf{7} \] the exponent \( e_\epsilon(R, W) \) gives a lower bound of strong converse exponent of identification code.

\begin{theorem}
Assume that the cardinality \( |\mathcal{Y}| \) is finite, then
\[ e_\epsilon(R|W, p) \geq \max_{0 \leq t \leq -1/2} \left\{ -\psi(W,s) + sR \right\} \]
\[ e_D(R|W, p) \geq \max_{0 \leq t \leq -1/2} \left\{ -\phi(t|W,p) - tR \right\} \]
\[ e_\epsilon(R|W) \geq \max_{0 \leq t \leq -1/2} \left\{ -max \phi(t|W,p) - tR \right\}, \]
\end{theorem}
where \( \psi(s|W, p) \overset{\text{def}}{=} \log \text{E}_p \sum_y W^{1+s} (y) W^{-s} (y) \) and \( \phi(t|W, p) \overset{\text{def}}{=} \log \max_p \sum_y \left( E_p W^{1+t+s}(y) \right)^{-1-s} \).

Using Pinsker’s inequality \( D(p||q) \geq ||p-q||^2 \), we obtain two inequalities \( \frac{1}{2} e_D(R|W, p) \leq e_\epsilon(R|W, p) \) and \( \frac{1}{2} e_D(R|W) \leq e_\epsilon(R|W) \), which implies different lower bounds of exponents:
\[ e_\epsilon(R|W, p) \geq \frac{1}{2} \max_{0 \leq t \leq -1/2} \left\{ -\epsilon(t|W, p) - tR \right\} \]
\[ e_\epsilon(R|W) \geq \frac{1}{2} \max_{0 \leq t \leq -1/2} \left\{ -\max \epsilon(t|W, p) - tR \right\}. \]

We can derive different lower bounds of \( e_D(R|W, p) \) and \( e_D(R|W) \) from the inequality (31). However, these bounds are smaller than the bound presented here.

\begin{remark}
Arimoto’s strong converse exponent [15] of channel coding of transmission code equals
\[ \max_{0 \leq t \leq -1} \left\{ -\max \epsilon(t|W, p) - tR \right\}, \]
which is a bit greater than the RHS of (37) when \( R \) is sufficiently large.

\begin{remark}
By using inequality (9) and type method, Oohama [8] has obtained a lower bound of \( e_\epsilon(R|W) \):
\[ \frac{1}{2} \max_{0 \leq t \leq -1} \left\{ -\max \phi(t|W, p) - tR \right\}, \]
which is a bit better than (41) when \( R \) is sufficiently large. It is interesting that his approach is in contrast to our approach to (41), which is based on (10) not on (9).
\end{remark}

\begin{remark}
It is difficult to treat the exponent of the sum of two error probabilities in identification code based on Theorem \[ \textbf{1} \]. For this purpose, we need a modified version of Theorem \[ \textbf{1} \].
\end{remark}

The following lemma is a preparation of our proof of Theorem \[ \textbf{6} \].

\begin{lemma}
For any \( s \geq 0 \) and \( 0 \geq t > -1 \), the equalities
\[ \max_{p \in \mathcal{P}(X^n)} \sum_{y \in \mathcal{Y}^n} \left( E_p W_x^n (y^n)^{1+s} y^n \right)^{1-s} \]
\[ = \left( \max_{p \in \mathcal{P}(X^n)} \sum_{y \in \mathcal{Y}^n} \left( E_p W_x^n (y^n)^{1+s} y^n \right)^{1-s} \right)^n \]
\[ = \left( \max_{p \in \mathcal{P}(X^n)} \sum_{y \in \mathcal{Y}^n} \left( E_p W_x^n (y^n)^{1+s} y^n \right)^{1-s} \right)^n \]
hold.
\end{lemma}
Proof: Since \[ p \mapsto \max_{p \in P(X)} \sum_y (E_p W^{1+s}(y))^{-s} \] is continuous and convex function, if and only if \( f(p^*) = \max_p f(p) \), there exists a constant \( \lambda \) such that
\[
\sum_y p(x) \lambda = \sum_y p(x) \sum_x W^{1+s}(y) \left( \sum_x p^*(x) W^{1+s}(y) \right)^{-s} = \left( \sum_x p^*(x) W^{1+s}(y) \right)^{-s}.
\]
Indeed, \( \lambda \) is calculated as
\[
\sum_x p(x) \lambda = \sum_x p(x) \sum_y W^{1+s}(y) \left( \sum_x p^*(x) W^{1+s}(y) \right)^{-s} = \left( \sum_x p^*(x) W^{1+s}(y) \right)^{-s}.
\]
Thus, if and only if \( f(p^*) = \max_p f(p) \),
\[
\sum_y W^{1+s}(y) \left( \sum_x p^*(x) W^{1+s}(y) \right)^{-s} = \left( \sum_x p^*(x) W^{1+s}(y) \right)^{-s}
\]
which is a necessary and sufficient condition for \( p^* \) gives the maximum. Hence, if \( p^* \) satisfies the above condition, \( (p^*)^n \) also satisfies the following condition:
\[
\sum_{y^n} (W^n_x)^{1+s}(y^n) \left( \sum_{x^n} (p^*)^n(x^n) (W^n_x)^{1+s}(y^n) \right)^{-s} = \left( \sum_{x^n} (p^*)^n(x^n) (W^n_x)^{1+s}(y^n) \right)^{-s},
\]
which is a necessary and sufficient condition for
\[
\sum_{y^n} (W^n_x)^{1+s}(y^n) \left( \sum_{x^n} (p^*)^n(x^n) (W^n_x)^{1+s}(y^n) \right)^{-s} = \max_{p \in P(Y^n)} \sum_{y^n} \left( E_p (W^n_x(y^n))^{1+s} \right)^{-s}.
\]
It implies the equation \( 42 \).

Proof of Theorem 6: By inequality \( 10 \) of Theorem 2, we have
\[
D(e^{nR}, W^n_p) \leq \log(1 + (e^{nR}) e^{tW^n_p}) - t \leq \frac{(e^{nR}) e^{tW^n_p}}{1 + e^{R}} = \frac{e^{(t+R)^n}}{1 + e^{nR}}.
\]
for \( 0 > t \geq -1/2 \), where the second inequality follows from \( \log(1 + x) \leq x \). From \( 43 \), we obtain
\[
\epsilon_D(R|W,p) \geq -\phi(t|W,p) - tR.
\]
We proceed to the proof of (53). By inequalities (29) and (50), we obtain
\[
\epsilon(e^{nR}, W^n) \leq \max_{p \in P(X^n)} 3e^{\psi(s|W^n, p)} + e^{3e^{sR}}
\] (53)

Now, we estimate an upper bound of
\[
e^{\psi(s|W^n, p)} = E_p \sum_y W_1^{1+s}(y) W_p^{-s}(y).
\] (54)

Since the map \(x \mapsto x^{1+s}\) is convex, we have
\[W_p(y) = E_p W_x(y) \geq E_p W_1^{1+s}(y),\]
which imply that
\[W_p^{-s}(y) \leq (E_p(W_x(y))^{1+s})^{-s}.
\] Hence, the relations
\[
E_p \sum_y W_1^{1+s}(y) W_p^{-s}(y) = \sum_y E_p W_1^{1+s}(y) W_p^{-s}(y)
\] \leq \sum_y (E_p W_1^{1+s}(y))^{1-s}
\] (55)

hold. Using (55) and Lemma 7, we can evaluate
\[
\max_{p \in P(X^n)} E_p \sum_{y \in Y^n} W_1^n(y) W_p^{-s} W_1^n(y) \leq \max_{p \in P(X^n)} \sum_y (E_p(W_1^n(y))^{1+s})^{1-s}
\] \[= \left( \max_{p \in P(X^n)} \sum_y (E_p W_1^{1+s}(y))^{1-s} \right)^n = e^{n(\psi(s|W))}.
\] (56)

Combining (55) and (56), we have
\[
\epsilon(e^{nR}, W^n) \leq e^{\psi(\epsilon^{s|W^n, p}) + sR}.
\] (57)

for any \(1 \geq s \geq 0\). In a manner similar to the derivation of (56) from (51), we can derive (58) from (57).

\section{B. Wire-tap channel}

Next, we proceed to discrete memoryless wire-tap channel. Applying Theorem 4 to this case with the input identical and independent distribution, we obtain
\[
C(W^B, W^E) = \max_{\{\Phi_n\}} \left\{ \lim_{n} 1/n \log |\Phi_n| \right\} \lim_{n} \epsilon_B(\Phi_n) = \lim_{n} d_E(\Phi_n) = 0
\] \[= \max_{p} \left\{ I(p; W^B) - I(p; W^E) \right\},
\] (58)

which has been obtained by Wyner [4]. Hence, Theorem 4 can be regarded as a general extension of Wyner’s result. Moreover, using Lemma 5 we derived several explicit lower bounds of exponents.

\section{7. Comparison of lower bounds of exponents}

Finally, we compare the lower bounds (38), (39), (40), and (41) of error exponents of channel resolvability.

\textbf{Theorem 7:} Assume that the cardinality \(|Z|\) is finite, then there exists a sequence \(\{\Phi_n\}\) of codes for any real numbers \(R, R'\) and any probability distribution \(p\) such that
\[
\lim_{n} \frac{1}{n} \log |\Phi_n| = R
\] \[\lim_{n} \frac{1}{n} \log \epsilon_B(\Phi_n) \geq \max_{1 \geq s \geq 0} \left\{ -\phi(s|W^B, p) - s(R + R') \right\}
\] \[\lim_{n} \frac{1}{n} \log I_E(\Phi_n) \geq \max_{0 \geq t \geq 1/2} \left\{ -\phi(t|W^B, p) - tR' \right\}
\] \[\lim_{n} \frac{1}{n} \log d_E(\Phi_n) \geq \max_{0 \geq t \geq 1/2} \left\{ -\phi(t|W^B, p) - tR' \right\}.
\] (59)

Indeed, these exponents are very useful for evaluating error and wire-tapper’s information for a finite \(n\).

\textbf{Proof:} The inequality (38) immediately follows from (19). By using an evaluation similar to (37), we can show (39) from (21). Furthermore, by using an evaluation similar to (40), we can show (40) from (22).

\section{VII. Comparison of lower bounds of exponents}

Finally, we compare the lower bounds (38), (39), (40), and (41) of error exponents of channel resolvability.

\textbf{Theorem 8:} Assume that \(\Delta \equiv R - I(p; W)\) is sufficiently small. Then, RSHs of (40) and (41) (which are lower bounds of exponent of the variational distance) are approximately calculated as
\[
\text{RHS of (40)} \frac{1}{2} \max_{0 \geq \tau \geq 1/2} \left\{ -\phi(\tau|W, p) - tR \right\} \approx \frac{\Delta^2}{8J(p; W)}
\] where
\[
J(p; W) \overset{\text{def}}{=} \frac{1}{2} (E_{p,W} \log W_x(y) - \log W_p(y))^2 - I^2(p; W).
\]

Moreover, RHSs of (38) and (41) (which are lower bounds of exponent of the worst variational distance) are approximately calculated as
\[
\text{RHS of (38)} \max_{s \geq 0} \left\{ -\psi(s|W) + sR \right\} \overset{\Delta^2}{=} 4J(p; W) + E_{p, H}(W_x),
\] \[\text{RHS of (41)} \frac{1}{2} \max_{0 \geq \tau \geq 1/2} \left\{ -\phi(\tau|W, p) - tR \right\} \overset{\Delta^2}{=} 8J(p; W),
\] where \(p_0 \equiv \arg \max_p I(p; W).\) Thus, when \(R\) is sufficiently close to \(\max_p I(p; W), (38)\) gives a better lower bound than (41). Of course, this comparison can be applied to exponents of eavesdropper’s information in wire-tap channel, \(i.e.,\) the comparison of RHSs of (40) and (41).
Thus, \( E_{p_0} H(W_x) \leq \frac{1}{2} \left( E_{p,y} E_{W_{xy}} (\log W_x(y) - \log W_p(y))^2 - I^2(p;W) \right) \).

Therefore, although \( R - \max_{s} I(p;W) \) is small enough, the relation between bounds \([38]\) and \([41]\) is not clear.

**Proof:** By using a Taylor expansion, we obtain the approximations:

\[
\psi(s|W, p) \approx I(p;W)s + J(p;W)s^2 \\
\phi(t|W, p) \approx -I(p;W)t + J(p;W)t^2 \\
\psi(s|W) \approx I(p_0;W)s + (J(p_0;W) + E_{p_0} H(W_x))s^2,
\]

Thus,

\[
\begin{aligned}
\max_{1 \geq s \geq 0} & \left\{ -\frac{\psi(s|W, p) + sR}{1 + s} \right\} \\
\approx & \max_{1 \geq s \geq 0} \left\{ -I(p;W)s - J(p;W)s^2 + (I(p;W) + \Delta s) \right\} \\
\max_{0 \geq t \geq -1/2} & \left\{ -\phi(t|W, p) - tR \right\} \\
\approx & \max_{0 \geq t \geq -1/2} \left\{ I(p;W)t - J(p;W)t^2 - (I(p;W) + \Delta)t \right\} \\
= & \max_{0 \geq t \geq -1/2} \left\{ -J(p;W)t^2 - \Delta t \right\} = \frac{\Delta^2}{4J(p;W)} \\
\max_{s \geq 0} & \left\{ -\frac{\psi(s|W) + sR}{1 + s} \right\} \\
\approx & \max_{s \geq 0} \left\{ -I(p_0;W)s - (J(p_0;W) + E_{p_0} H(W_x))s^2 \right\} \\
& \quad + s(I(p_0;W) + \Delta) \frac{1}{1 + s} \\
\approx & \max_{s \geq 0} \left\{ -(J(p_0;W) + E_{p_0} H(W_x))s^2 + \Delta s \right\} \\
= & \frac{\Delta^2}{4(J(p_0;W) + E_{p_0} H(W_x))}.
\end{aligned}
\]

**VIII. Conclusion**

We give several non-asymptotic formulas in identification code, channel resolvability, and wire-tap channel. Using these formulas, we give the achievable rate channel resolvability for the general channel, which had been an open problem. Also, we derived several asymptotic relations among divergence rates, capacities of identification code, and \( \epsilon \) capacities of channel resolvability.

From these non-asymptotic formulas, we obtained lower bounds of error exponents of channel resolvability in the stationary memoryless setting. Moreover, we derived lower bounds of error probability and wire-tapper’s information in the stationary memoryless setting in wire-tap channel.

Concerning the quantum setting, wire-tap channel has been discussed in the discrete memoryless channel case by Devetak [5], Winter et. al.[6] and Cai & Yeung[17], and identification codes has been discussed by Ahlswede & Winter[18]. Hence, several quantum extensions of the results presented here can be expected. Some has been obtained by the author. And some of them have appeared in the author’s textbook[13]. Those not already presented will appear in a forthcoming paper.

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