On Relations of Hyperelliptic Weierstrass al Functions

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Abstract

We study relations of the Weierstrass’s hyperelliptic al-functions over a non-degenerated hyperelliptic curve $y^2 = f(x)$ of arbitrary genus $g$ as solutions of sine-Gordon equation using Weierstrass’s local parameters, which are characterized by two ramified points. Though the hyperelliptic solutions of the sine-Gordon equation had already obtained, our derivations of them is simple; they need only residual computations over the curve and primitive matrix computations.

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§1. Introduction

The sine-Gordon equation is a famous nonlinear integrable differential equations. For a hyperelliptic curve $C_g$ ($y^2 = f(x) = (x - b_1) \cdots (x - b_{2g+1})$) of genus $g$, the hyperelliptic solutions of the sine-Gordon equation were formulated in [Mu 3.241] in terms of Riemann theta functions. In [Mu], $(U, V, W)$ representation of symmetric product space of the $g$ curves $\text{Symm}^g C_g$ is defined; especially, $U$ is defined by $U(z) := (x_1 - z) \cdots (x_g - z)$ a for a point $((x_1, y_1), \cdots, (x_g, y_g))$ in $\text{Symm}^g C_g$. (In this article, we will denote $U$ by $F(z)$ on later following the conventions in [Ba1, 2, 3, Ma].) Using the relation between $U$ and the Riemann theta functions in [Mu 3.113], the solutions [Mu 3.241] can be rewritten as,

$$\frac{\partial}{\partial t_P} \frac{\partial}{\partial t_Q} \log([2P - 2Q]) = A([2P - 2Q] - [2Q - 2P]),$$ (1-3)

where $P$ and $Q$ are ramified points of $C_g$, $A$ is a constant number, $[D]$ is a meromorphic function over $\text{Sym}^g(C_g)$ with a divisor $D$ for each $C_g$ and $t_P$ is a coordinate in the Jacobi
variety such that it is identified with a local parameter at a branch point \( P' \) up to constant. In other words, for a finite branch point \( (b_i, 0) \) \( U(b_i) \) is identified with \( [2(b_i, 0) - 2\infty] \) up to constant factor.

In the formulations in [Mu], local parameters \( t_P' \) were not concretely treated. In this article, we will give more explicit representations of (1-3) using concrete local parameters \( t_{Ba2, W2, 3} \) and present simpler derivations of (1-3) without using any \( \theta \)-function. This article is an application of a scheme developed in [Ma] to the sine-Gordon equation, which is based upon [Ba3].

In [W1, W2], Weierstrass defined al function by \( \gamma_r \sqrt{U(b_r)} \) using a constant factor \( \gamma_r \). In Theorem 3.1, we will give

\[
\frac{\partial^2}{\partial u_1^{(r)} \partial u_g^{(r)}} \log \text{al}_{r}(u^{(r)}) = \frac{1}{2} \left( \frac{\text{al}_{r}^2(u^{(r)})}{\gamma_r^2} - \frac{f'(b_r) \gamma_r^2}{\text{al}_{r}^2(u^{(r)})} \right),
\]

(1-4)
in terms of a coordinate system \( u^{(r)} \)'s defined in (2-5). \text{al}_{r}(u) \) has the single order zero at \( (b_r, 0) \) and a singularity of the single order at \( \infty \) as a function of \( x_i \in C_g \).

Further we give another representation in Theorem 4.1 in terms of \( v \)'s defined in (2-6) \( (a_1 := b_r, a_2 := b_s) \) [W2, 3],

\[
\frac{\partial^2}{\partial v_1 \partial v_2} \log \frac{\text{al}_{r}(v)}{\text{al}_{s}(v)} = \frac{1}{2(b_r - b_s)} \left( f'(b_s) \frac{\gamma_s^2 \text{al}_{r}(v)^2}{\gamma_r^2 \text{al}_{s}(v)^2} + f'(b_r) \frac{\gamma_r^2 \text{al}_{s}(v)^2}{\gamma_s^2 \text{al}_{r}(v)^2} \right).
\]

(1-5)

The function \( \text{al}_{s}(v)/\text{al}_{r}(v) \) vanishes with order one when \( x_i \) is at \( (b_s, 0) \) whereas it diverges with order one if \( x_i \) approaches to \( (b_r, 0) \). As they were discovered by Weierstrass [W2, 3] and they play the essential roles in the investigation in [W2, 3] and in \( \S 4 \). Thus we have called them \textit{Weierstrass parameters}.

In these proofs, we will use only residual computations using the data of curve itself without any \( \theta \) functions as the derivation of hypereilliptic solutions of the modified Korteweg-de Vries equations in [Ma]. The curve is sometimes given by an affine equation with special coefficients. Then it might be important to study the relation between the properties of line-bundle over the curve and these coefficients. As (1-4) and (1-5) can be explicitly expressed by data of curve \( C_g \), the author believes that they have some advantage as relations of special functions.
§2. Differentials of a Hyperelliptic Curve

In this section, we will give the conventions and notations of the hyperelliptic functions in this article. We denote the set of complex numbers by \( \mathbb{C} \) and the set of integers by \( \mathbb{Z} \).

2.1 Hyperelliptic Curve. We deal with a hyperelliptic curve \( C_g \) of genus \( g \) \((g > 0)\) given by the affine equation,

\[
y^2 = f(x) = \lambda_{2g+1}x^{2g+1} + \lambda_{2g}x^{2g} + \cdots + \lambda_2x^2 + \lambda_1x + \lambda_0 = (x - b_r)h_r(x), \tag{2-1}
\]

where \( \lambda_{2g+1} \equiv 1 \) and \( \lambda_j \)'s are complex numbers. We use the expressions,

\[
f(x) := (x - b_1)(x - b_2)\cdots(x - b_{2g})(x - b_{2g+1}) = P(x)Q(x),
\]

\[
P(x) := (x - a_1)(x - a_2)\cdots(x - a_g),
\]

\[
Q(x) := (x - c_1)(x - c_2)\cdots(x - c_g)(x - c), \tag{2-2}
\]

where \( b_j \)'s \((b_i = a_i, b_{g+i} = c_i)\) are complex numbers.

It is noted that the permutation group acts on these \( \{b_r\} \) and \( \{a_r\} \).

2.2 Definition [Ba1, Ba2, W2, 3].

1. For a point \((x_i, y_i) \in C_g\), the unnormalized differentials of the first kind are defined by,

\[
du^{(r,i)}_1 := \frac{dx_i}{2y_i}, \quad du^{(r,i)}_2 := \frac{(x_i - b_r)dx_i}{2y_i}, \quad \cdots, \quad du^{(r,i)}_g := \frac{(x_i - b_r)^{g-1}dx_i}{2y_i}. \tag{2-3}
\]

\[
dv^{(i)}_1 := \frac{P(x_i)dx_i}{2P'(a_1)(x_i - a_1)y_i}, \quad dv^{(i)}_2 := \frac{P(x_i)dx_i}{2P'(a_2)(x_i - a_2)y_i}, \quad \cdots, \quad dv^{(i)}_g := \frac{P(x_i)dx_i}{2P'(a_g)(x_i - a_g)y_i}. \tag{2-4}
\]

2. Let us define the Abel maps for \( g \)-th symmetric product of the curve \( C_g \),

\[
u^{(r)} := (u^{(r)}_1, \cdots, u^{(r)}_g) : \text{Sym}^g(C_g) \rightarrow \mathbb{C}^g,
\]

3
\( u_k^{(r)} ((x_1, y_1), \cdots, (x_g, y_g)) := \sum_{i=1}^{g} \int_{\infty}^{(x_i, y_i)} d u_k^{(r,i)} \), \hspace{1cm} (2-5)

\( v := (v_1, \cdots, v_g) : \text{Sym}^g(C_g) \rightarrow \mathbb{C}^g \),

\( v_k((x_1, y_1), \cdots, (x_g, y_g)) := \sum_{i=1}^{g} \int_{\infty}^{(x_i, y_i)} d v_k^{(i)} \). \hspace{1cm} (2-6)

These coordinates are universal covering of the related Jacobian \( J \). The definition (2-6) [Ba2 p.382] is due to Weierstrass [W2, 3] and we call (2-6) *Weierstrass parameter*, though we choose different constant factor from the original one [W2, 3]. This parameterization is a key of the second solutions mentioned in §4.

### 2.3 Definition.

1. Hyperelliptic \( \alpha \) function is defined by [Ba2 p.340, W2, 3],

\( \alpha_r(u) := \gamma_r \sqrt{F(b_r)}, \) \hspace{1cm} (2-7)

where \( \gamma_r := \sqrt{-1/P'(b_r)} \) and

\[ F(x) := (x - x_1) \cdots (x - x_g) \]

\[ = (x - b_r - x_1 + b_r) \cdots (x - b_r - x_g + b_r). \] \hspace{1cm} (2-8)

On the choice of \( \gamma_r \), we will employ the convention of Baker [Ba2] instead of original one [W2, 3]. We note that \( \alpha_r \)'s have mutually algebraic relations.

For later convenience, a polynomial associated with \( F(x) \) is introduced by

\[ \pi_i^{(r)}(x) := \frac{F(x)}{x - x_i} = \chi_i^{(r)}(x - b_r)^{g-1} + \chi_{i,g-2}^{(r)}(x - b_r)^{g-2} + \cdots + \chi_{i,1}^{(r)}(x - b_r) + \chi_{i,0}^{(r)}. \]

Then we have \( \chi_{i,g-1}^{(r)} \equiv 1 \) and \( \chi_{i,0}^{(r)} = F(b_r)/(x_i - b_r) \). Further we introduce \( g \times g \)-matrices,

\[ \mathcal{W}^{(r)} := \begin{pmatrix} \chi_{1,0}^{(r)} & \chi_{1,1}^{(r)} & \cdots & \chi_{1,g-1}^{(r)} \\ \chi_{2,0}^{(r)} & \chi_{2,1}^{(r)} & \cdots & \chi_{2,g-1}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ \chi_{g,0}^{(r)} & \chi_{g,1}^{(r)} & \cdots & \chi_{g,g-1}^{(r)} \end{pmatrix}, \quad \mathcal{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_g \end{pmatrix}, \]
\[
\mathcal{M} := \begin{pmatrix}
1 & 1 & \cdots & 1 \\
x_1 - a_1 & x_2 - a_1 & \cdots & x_g - a_1 \\
x_1 - a_2 & x_2 - a_2 & \cdots & x_g - a_2 \\
\vdots & \vdots & \ddots & \vdots \\
x_1 - a_g & x_2 - a_g & \cdots & x_g - a_g \\
\end{pmatrix},
\]
\[
\mathcal{P} = \begin{pmatrix}
\sqrt{P(x_1)/Q(x_1)} \\
\sqrt{P(x_2)/Q(x_2)} \\
\vdots \\
\sqrt{P(x_g)/Q(x_g)} \\
\end{pmatrix},
\]
\[
\mathcal{A} = \begin{pmatrix}
P'(a_1) \\
P'(a_2) \\
\vdots \\
P'(a_g) \\
\end{pmatrix}, \quad \mathcal{F}' = \begin{pmatrix}
F'(x_1) \\
F'(x_2) \\
\vdots \\
F'(x_g) \\
\end{pmatrix},
\]
where \( F'(x) := dF(x)/dx \).

### 2.3 Lemma

For these matrices, the following relations hold:

1. The inverse matrix of \( \mathcal{W}^{(r)} \) is given by \( \mathcal{W}^{(r)-1} = \mathcal{F}'^{(r)-1} \mathcal{V}^{(r)} \), where \( \mathcal{V}^{(r)} \) is the Vandermonde matrix,

\[
\mathcal{V}^{(r)} = \begin{pmatrix}
(x_1 - b_r) & (x_1 - b_r)^2 & \cdots & (x_1 - b_r)^{g-1} \\
(x_2 - b_r) & (x_2 - b_r)^2 & \cdots & (x_2 - b_r)^{g-1} \\
\vdots & \vdots & \ddots & \vdots \\
(x_g - b_r) & (x_g - b_r)^2 & \cdots & (x_g - b_r)^{g-1} \\
\end{pmatrix}.
\]

2. The determinant of \( \mathcal{M} \) is given by

\[
\det \mathcal{M} = \frac{(-1)^{g(g-1)/2} P(x_1, \ldots, x_g) P(a_1, \ldots, a_g)}{\prod_{k,l} (x_k - a_l)},
\]
where

\[
P(z_1, \ldots, z_g) := \prod_{i<j} (z_i - z_j).
\]

3. \( (\mathcal{M} \mathcal{P})^{-1} \mathcal{A} = \left[ \begin{array}{c}
\frac{2y_j F(a_j)}{F'(x_i)(a_j - x_i)} \\
\vdots \\
\frac{2y_j F(a_j)}{F'(x_i)(a_j - x_i)} \\
\end{array} \right] \).
Proof. (1) is obtained by direct computations. (2) is a well-known result [T]. Since the zero and singularity in the left hand side give the right hand side as

\[ CP(x_1, \cdots, x_g)P(a_1, \cdots, a_g)/\prod_{k,l}(x_k - a_l), \]

for a certain constant \( C \). In order to determine \( C \), we multiply \( \prod_{k,l}(x_k - a_l) \) both sides and let \( x_1 = a_1, x_2 = a_2, \cdots, \) and \( x_g = a_g \). Then \( C \) is determined as above. (3) is obtained by the Laplace formula using the minor determinant for the inverse matrix.

Then we have following corollary.

2.5 Corollary. Let \( \partial_{u_i}^{(r)} := \partial/\partial u_i^{(r)}, \) \( \partial_{v_i}^{(r)} := \partial/\partial v_i, \) and \( \partial_{x_i} := \partial/\partial x_i. \)

\[
\begin{pmatrix}
\partial_{u_1}^{(r)} \\
\partial_{u_2}^{(r)} \\
\vdots \\
\partial_{u_g}^{(r)}
\end{pmatrix}
= 2\mathcal{F}^{-1} \cdot t\mathcal{W}^{(r)}
\begin{pmatrix}
\partial_{x_1} \\
\partial_{x_2} \\
\vdots \\
\partial_{x_g}
\end{pmatrix},
\begin{pmatrix}
\partial_{v_1} \\
\partial_{v_2} \\
\vdots \\
\partial_{v_g}
\end{pmatrix}
= 2(\mathcal{M}\mathcal{P})^{-1} A
\begin{pmatrix}
\partial_{x_1} \\
\partial_{x_2} \\
\vdots \\
\partial_{x_g}
\end{pmatrix}.
\] (2-10)

§3. Relations between Hyperelliptic al Functions \((b_r, \infty)-\text{type}\)

In this section, we will give the first relation of hyperelliptic al function using the parameters \( u_1^{(r)} \) and \( u_g^{(r)} \) in (2-5).

3.1 Theorem.

\[
\frac{\partial}{\partial u_1^{(r)}} \frac{\partial}{\partial u_g^{(r)}} \log a_{r} = \frac{1}{2} \left( \frac{a_{r}^2}{\gamma_{r}^2} - \frac{f'(b_r)\gamma_{r}^2}{a_{r}^2} \right).
\] (3-1)

Here we will give a comment on Theorem 3.1. Let us fix the parameters \( x_2, \cdots, x_g \) and regard al\( r \) as a function of a parameter related to \( x_1 \) over \( C_g \). Then its divisor is \( (a_{r}) = (b_r, 0) - \infty \). Further by letting \( t^2 = (x_i - b_r) \) around \( (b_r, 0) \), the definition (2-3) shows,

\[
du_1^{(r, i)}|_{(b_r, 0)} = \frac{2}{\sqrt{f'(b_r)}} dt.
\]
while for $s^2 = 1/x$ around $\infty$,

$$du_g^{(r,i)}|_{(\infty)} = -2ds.$$ 

Hence (3-1) can be regarded as an explicit representation of (1-3).

**Proof.** Instead of (3-1), we will prove following formula (3-2) in remainder in this section.

$$\frac{\partial}{\partial u_1^{(r)}} \frac{\partial}{\partial u_g^{(r)}} \log F(b_r) = F(b_r) - \frac{f'(b_r)}{F(b_r)}.$$  \hspace{1cm} (3-2)

The strategy is essentially the same as [Ba3, Ma]. First we translate the words of the Jacobian into those of the curves; we rewrite the differentials $u^{(r)}$’s in terms of the differentials over curves as in (3-3). We count the residue of an integration and use a combinatorial trick. Then we will obtain (3-2).

From (2-10), we will express $u^{(r)}$’s by the affine coordinates $x_i$’s,

$$\frac{\partial}{\partial u_g^{(r)}} = \sum_{i=1}^g \frac{2y_i}{F'(x_i)} \frac{\partial}{\partial x_i},$$

$$\frac{\partial}{\partial u_1^{(r)}} = \sum_{i=1}^g 2y_i x_i^{(r)} \frac{\partial}{\partial x_i} = F(b_r) \sum_{i=1}^g \frac{2y_i}{(x_i - b_r)F'(x_i)} \frac{\partial}{\partial x_i}.$$  \hspace{1cm} (3-3)

Hence the right hand side of (3-2) becomes

$$-\frac{\partial^2}{\partial u_1^{(r)} \partial u_g^{(r)}} \log F(b_r) = F(b_r) \sum_{j=1, i=1}^g \frac{2y_j}{(x_i - b_r)^2 F'(x_j)} \frac{\partial}{\partial x_j} \frac{2y_i}{F'(x_i)(x_i - b_r)^2}. \hspace{1cm} (3-4)$$

Here we will note the derivative of $F(x)$, which is shown by direct computations.

$$\frac{\partial}{\partial x_k} \left( \frac{\partial}{\partial x} F(x) \right)_{x=x_k} = \frac{1}{2} \left[ \frac{\partial^2}{\partial x^2} F(x) \right]_{x=x_k}.$$ 

Then (3-4) can be written as,

$$-\frac{\partial}{\partial u_1^{(r)}} \frac{\partial}{\partial u_g^{(r)}} \log F(b_r) = F(b_r) \sum_{i=1}^g \frac{1}{F'(x_i)} \left[ \frac{\partial}{\partial x} \left( \frac{f(x)}{(x - b_r)F'(x)} \right) \right]_{x=x_i}$$

$$-F(b_r) \sum_{k, l, k \neq l} \frac{4y_k y_l}{(b_r - x_k)(b_r - x_l)(x_k - x_l)F'(x_k)F'(x_l)}.$$

The proof of Theorem 3.1 finishes due to the following lemma. 

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3.2 Lemma. Following relations hold:

\[
\sum_{k=1}^{g} \frac{1}{F'(x_k)} \left[ \frac{\partial}{\partial x} \left( \frac{f(x)}{(x-b_r)^2 F'(x)} \right) \right]_{x=x_k} = 1 - \frac{f'(b_r)}{F(b_r)^2}. \tag{3-5}
\]

\[
\sum_{k,l,k \neq l} \frac{2y_k y_l}{(b_r-x_k) (b_r-x_l) (x_k-x_l) F'(x_k) F'(x_l)} = 0. \tag{3-6}
\]

Proof.: (3-5) will be proved by the following residual computations: Let \( \partial C_g^o \) be the boundary of a polygon representation \( C_g^o \) of \( C_g \),

\[
\oint_{\partial C_g^o} \frac{f(x)}{(x-b_r)^2 F(x)^2} \, dx = 0. \tag{3-7}
\]

The divisor of the integrand of (3-7) is given by,

\[
\left( \frac{f(x)}{(x-b_r)^2 F(x)^2} \right) dx = \frac{f(x)}{(x-b_r)^2 F(x)^2} dx = 2 \sum_{i=1}^{g} (b_i,0) - (b_r,0) - 2 \sum_{i=1}^{g} (x_i,y_i) - 2 \sum_{i=1}^{g} (x_i,-y_i) - \infty.
\]

We check these poles: First we consider the contribution around \( \infty \) point. Noting that the local parameter \( t \) at \( \infty \) is \( x = 1/t^2 \),

\[
\text{res}_{\infty} \frac{f(x)}{(x-b_r)^2 F(x)^2} \, dx = -2.
\]

Since the local parameter \( t \) at \((x_k, \pm y_k)\) is \( t = x - x_k \), we have

\[
\text{res}_{(x_k, \pm y_k)} \frac{f(x)}{(x-b_r)^2 F(x)^2} \, dx = \frac{1}{F'(x_k)} \left[ \frac{\partial}{\partial x} \left( \frac{f(x)}{(x-b_r)^2 F'(x)} \right) \right]_{x=x_k}. 
\]

For each branch point \((b_r,0)\), the local parameter \( t \) is \( t^2 = x - b_r \) and thus

\[
\text{res}_{(b_r,0)} \frac{f(x)}{(x-b_r)^2 F(x)^2} \, dx = 2 \frac{f'(b_r)}{F(b_r)^2}.
\]

By arranging them, we obtain (3-5).

On the other hand, (3-6) can be proved by using a trick: for \( i \neq j \),

\[
\frac{1}{(b_r-x_k)(b_r-x_l)(x_k-x_l)} = \frac{1}{(x_k-x_l)^2} \left( \frac{1}{(b_r-x_k)} - \frac{1}{(b_r-x_l)} \right).
\]

\[
\frac{1}{(b_r-x_k)(b_r-x_l)(x_k-x_l)} = \frac{1}{(x_k-x_l)^2} \left( \frac{1}{(b_r-x_k)} - \frac{1}{(b_r-x_l)} \right).
\]

\[
\frac{1}{(b_r-x_k)(b_r-x_l)(x_k-x_l)} = \frac{1}{(x_k-x_l)^2} \left( \frac{1}{(b_r-x_k)} - \frac{1}{(b_r-x_l)} \right).
\]
§4. Relations between Hyperelliptic al Functions: \((a_1, a_2)\)-type

In the previous section, we have a solution with a duality between a finite ramified point and \(\infty\)-point. In this section, we will give a relation between hyperelliptic al functions using the Weierstrass parameter (2-6). The relation has a duality between finite ramified points \((a_r, 0)\) and \((a_s, 0)\).

4.1 Theorem. For \(r \neq s\), we obtain

\[
\frac{\partial}{\partial v_r} \frac{\partial}{\partial v_s} \log \frac{a_r}{a_s} = \frac{1}{2(a_r - a_s)} \left( f'(a_r) \frac{\gamma_r^2 a_r^2}{\gamma_s^2 a_s^2} + f'(a_s) \frac{\gamma_s^2 a_s^2}{\gamma_r^2 a_r^2} \right). \tag{4-1}
\]

Before we prove it, we will give some comments: Let us fix the parameter \(x_2, \cdots, x_g\) and regard \(a_r/a_s(\propto \sqrt{F(a_r)/F(a_s)})\) as a function of \(x_1\) over \(C_g\). Then its divisor is \((a_r/a_s) = (a_r, 0) - (a_s, 0)\). By letting \(t_r^2 = (x_1 - a_r)\) around \((a_r, 0)\), infinitesimal value of Weierstrass parameter (2-4) is given,

\[
dv_r^{(i)} \big|_{(a_r, 0)} = \frac{1}{\sqrt{f'(a_r)}} dt_r.
\]

Thus (4-1) is also a concrete expression of (1-3).

Proof. Similar to the proof of Theorem 3.1, let us prove the theorem. Without loss of generality, we will prove the following relation instead of (4-1):

\[
\frac{\partial}{\partial v_1} \frac{\partial}{\partial v_2} \log \frac{F(a_1)}{F(a_2)} = \frac{F(a_1)F(a_2)}{(a_1 - a_2)} \left( \frac{f'(a_1)}{F(a_1)^2} + \frac{f'(a_2)}{F(a_2)^2} \right). \tag{4-2}
\]

From (2-9) and (2-10), the derivative \(v\)'s are expressed by the affine coordinate \(x_i\)'s,

\[
\frac{\partial}{\partial v_r} = F(a_r) \sum_{j=1}^{g} \frac{2y_j}{F'(x_j)(x_j - a_r)} \frac{\partial}{\partial x_j}.
\]

The right hand side of (4-2) becomes,

\[
\frac{\partial^2}{\partial v_1 \partial v_2} \log \frac{F(a_1)}{F(a_2)} = F(a_1) \sum_{j=1}^{g} \frac{2y_j}{(x_i - a_1)F'(x_j)} \frac{\partial}{\partial x_j} F'(x_i) \left( \frac{2y_i F(a_2)}{F'(x_i)(x_i - a_2)} \frac{(a_1 - a_2)}{(x_i - a_1)(x_i - a_2)} \right). \tag{4-3}
\]
The right hand side of (4-3) is

\[
F(a_1)F(a_2) \sum_{i=1}^{g} \frac{1}{F'(x_i)} \left[ \frac{\partial}{\partial x} \left( \frac{f(x)(a_2 - a_1)}{(x - a_1)^2(x - a_2)^2F'(x)} \right) \right]_{x=x_i}
\]

\[
-F(a_1)F(a_2) \sum_{k,l,k\neq l} \frac{2y_ky_l(a_2 - a_1)}{F'(x_k)F'(x_l)(x_l - a_1)(x_k - a_2)(x_k - a_1)(x_l - a_2)(x_l - x_k)}.
\]

Then the proof of Theorem 4.1 is completely done due to the following lemma.

4.2 Lemma. Following relations hold:

\[
\sum_{i=1}^{g} \frac{1}{F'(x_i)} \left[ \frac{\partial}{\partial x} \left( \frac{f(x)}{(x - a_1)^2(x - a_2)^2F'(x)} \right) \right]_{x=x_i} = \frac{1}{(a_1 - a_2)^2} \left( \frac{f'(a_1)}{F(a_1)^2} - \frac{f'(a_1)}{F(a_1)^2} \right).
\]

\[
\sum_{k,l,k\neq l} \frac{2y_ky_l(a_2 - a_1)}{F'(x_k)F'(x_l)(x_l - a_1)(x_k - a_2)(x_k - a_1)(x_l - a_2)(x_l - x_k)} = 0.
\]

Proof. : Similar to Lemma 3-2, we consider an integral,

\[
\oint_{\partial C_g} \frac{f(x)}{(x - a_1)^2(x - a_2)^2F(x)^2}dx = 0.
\]

As the divisor of the integrand of (4-6) is

\[
\left( \frac{f(x)}{(x - a_1)^2(x - a_2)^2F(x)^2} \right)
\]

\[
= 3 \sum_{i=1}^{2g+1} (b_i,0) - (a_1,0) - (a_2,0) - 2 \sum_{i=1}^{g} (x_i,y_i) - 2 \sum_{i=1}^{g} (x_i,-y_i) + 3\infty,
\]

we count residual contributions from each terms as in the proof of Lemma 3-2 and obtain (4-4). Considering the symmetry, (4-5) is easily obtained.

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