Reinforcement Learning in Linear Quadratic Deep Structured Teams: Global Convergence of Policy Gradient Methods

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Abstract—In this paper, we study the global convergence of model-based and model-free policy gradient descent and natural policy gradient descent algorithms for linear quadratic deep structured teams. In such systems, agents are partitioned into a few sub-populations wherein in each sub-population are coupled in the dynamics and cost function through a set of linear regressions of the states and actions of all agents. Every agent observes its local state and the linear regressions of states, called deep states. For a sufficiently small risk factor and/or sufficiently large population, we prove that model-based policy gradient methods globally converge to the optimal solution. Given an arbitrary number of agents, we develop model-free policy gradient and natural policy gradient algorithms for the special case of risk-neutral cost function. The proposed algorithms are scalable with respect to the number of agents due to the fact that the dimension of their policy space is independent of the number of agents in each sub-population. Simulations are provided to verify the theoretical results.

I. INTRODUCTION

In today’s world, networked control systems are ubiquitous, ranging from smart grids and economics to communication networks and epidemics. Such systems often consist of many decision makers (nodes) with complex interactions. In general, finding an optimal (or even sub-optimal) solution in networked control systems is difficult. This difficulty exacerbates when practical restrictions are taken into account such as limited number of computation and communication resources and incomplete knowledge of the model.

It is well known that the number of computational elements (such as memory and time) increases exponentially with the number of decision makers in stochastic dynamic control systems. In addition, the lack of centralized communication among the decision makers can lead to different perspectives at the agent level, where solving a simple linear quadratic problem is challenging; see [1] for a counterexample in which the resultant optimization problem is non-convex. Furthermore, the above challenges are worsened when the underlying model is not completely known. Therefore, it is of special interest in control theory to find a class of models in which the above challenges can be addressed to some extent.

Motivated from recent developments in artificial intelligence, deep structured teams and games have been introduced in [2], [3], [4], [5], [6], [7], which may be viewed as the generalization of mean-field teams proposed in [8] and showcased in [9], [10], [11], [12], [13], [14]. In such systems, the interaction between the decision makers is modelled by a set of linear regressions (weighted averages) of the states and actions, where the weights represent the dominant features of the model. We call such models deep structured because the interaction between the decision makers is similar to that between the neurons of a deep feed-forward neural network.

In this paper, we study the global convergence of the model-based and model-free policy gradient descent and natural policy gradient descent algorithms for linear quadratic deep structured teams. In such systems, agents are partitioned into S ∈ N disjoint sub-populations (sub-systems) with

II. PROBLEM FORMULATION

In this article, 1(·) is the indicator function, \( \rho(\cdot) \) is the spectral radius of a matrix, diag(·) is a block diagonal matrix, \( \| \cdot \|_F \) is the Frobenius norm of a matrix, and \( \text{var}(\cdot) \) is the variance of a random variable. For any \( n \in \mathbb{N}, x_{1:n} \) is the vector \( (x_1, \ldots, x_n) \) and \( \mathbb{N}_n \) is the finite set \( \{1, \ldots, n\} \). For any vectors \( x, y \) and \( z \), vec(\( x, y, z \)) = \( [x^\top, y^\top, z^\top]^\top \) and for any matrices \( A, B \) and \( C \) with the same number of columns, row(\( A, B, C \)) = \( [A^\top, B^\top, C^\top]^\top \). For any square matrix \( A \), \( A \geq 0 \) and \( A > 0 \) mean that matrix \( A \) is positive semi-definite and positive definite, respectively. Also, \( I \) refers to the identity matrix and \( 0 \) to a matrix with zero arrays.

A. Model

Consider a decentralized stochastic control system with \( n \in \mathbb{N} \) decision makers (agents). The agents are partitioned into \( S \in \mathbb{N} \) disjoint sub-populations (sub-systems) with
For any feature \( j \in \mathbb{N}_{f(s)} \) of sub-population \( s \in \mathbb{N}_S \), define the following linear regressions:

\[
\hat{x}_t^j(s) = \frac{1}{n(s)} \sum_{i=1}^{n(s)} \alpha_{i,j}(s)x_t^i, \quad \bar{u}_t^j(s) = \frac{1}{n(s)} \sum_{i=1}^{n(s)} \alpha_{i,j}(s)u_t^i. 
\]

From [2], [3], [4], [5], [6], we refer to the above linear regressions as deep states and deep actions. At any time \( t \in \mathbb{N} \), define \( \bar{x}_t := \text{vec}(\hat{x}_t^j(s))_{j=1}^{f(s)} \) and \( \bar{u}_t := \text{vec}(\bar{u}_t^j(s))_{j=1}^{f(s)} \). Let the initial states of the agents of each sub-population \( s \in \mathbb{N}_S \) be independent and identically distributed (i.i.d.) Gaussian random vectors with positive covariance matrix \( \Sigma_s(s) \). The per-step cost of agent \( i \in \mathbb{N}_{n(s)} \) in sub-population \( s \in \mathbb{N}_S \) is described by:

\[
x_{t+1} = A(s)x_t^i + B(s)u_t^i + \sum_{j=1}^{f(s)} \alpha_{i,j}(s)(\bar{A}_j(s)\bar{x}_t + \bar{B}_j(s)\bar{u}_t) + w_t^i,
\]

where \( \{w_t^i\}_{t=1}^{\infty} \) is an i.i.d. zero-mean Gaussian random vector with positive covariance matrix \( \Sigma_w(s) \). The social welfare cost function at time \( t \in \mathbb{N} \) is given by:

\[
\bar{c}_t = \sum_{s=1}^{S} \frac{\mu(s)}{n(s)} \sum_{i=1}^{n(s)} c_t^i,
\]

where \( \mu(s) > 0 \) determines the importance of the cost of agents of sub-population \( s \in \mathbb{N}_S \) with respect to other sub-populations. It is assumed that the primitive random vectors \( \{(x_t^i)_{i=1}^{n(s)}\}_{s=1}^{S}, \{(w_t^i)_{i=1}^{n(s)}\}_{s=1}^{S}, \{(u_t^i)_{i=1}^{n(s)}\}_{s=1}^{S}, \ldots \) are defined on a common probability space, and are mutually independent across time and space.

**Definition 1** (Weakly coupled agents [4]). The agents are said to be weakly coupled in the dynamics if the coupling term in (2) can be expressed as: \( \sum_{j=1}^{f(s)} \alpha_{i,j}(s)(\bar{A}_j(s)\bar{x}_t + \bar{B}_j(s)\bar{u}_t) \). Similarly, the agents are said to be weakly coupled in the cost function if the coupling term in (3) can be expressed as: \( \sum_{j=1}^{f(s)} \alpha_{i,j}(s)(\bar{x}_t^j(s))\Sigma_j(s)\bar{x}_t^j(s) + (\bar{u}_t^j(s))\Sigma_j(s)\bar{u}_t^j(s) \). Weakly coupling often arises in natural systems with equivariant structure.

The information structure considered here is called **deep-state sharing** (DSS), where each agent \( i \in \mathbb{N}_{n(s)} \) of sub-population \( s \in \mathbb{N}_S \) observes its local state as well as the deep states, i.e., \( u_t^i = g_t^i(x_t^i, \bar{x}_t) \), where \( g_t^i \) is the control law at time \( t \in \mathbb{N} \). Notice that DSS is a non-classical information structure wherein each agent has a different information set.

**B. Problem statement**

Given any risk factor \( \lambda > 0 \), define the following objective function: \( J_{n,\lambda} := \lim_{T \to \infty} \frac{1}{T} \log E[e^{\lambda \sum_{i=1}^{T} e_t^i}] \). Note that for a small risk factor \( \lambda \), one has:

\[
J_{n,\lambda} \approx E[\lim_{T \to \infty} \frac{1}{T} \sum_{i=1}^{T} e_t^i] + \frac{\lambda}{2} \text{var}(\lim_{T \to \infty} \frac{1}{T} \sum_{i=1}^{T} e_t^i).
\]

From (5), it implies that risk-factor \( \lambda \) balances the trade off between optimality (where \( \lambda \to 0 \)) and robustness (where robustness is defined in terms of minimum variance). To have a well-posed problem, it is assumed that all matrices defined above are uniformly bounded in time and space, and that the set of admissible actions are square integrable. Let \( g := \{(g_t^i)_{n(s)}\}_{s=1}^{S} \) denote the strategy of all agents.

**Problem 1.** Develop model-based gradient descent and natural policy gradient descent algorithms to compute the optimal risk-sensitive strategy \( g^* \) such that for any strategy \( g \), the following inequality holds: \( J_{n,\lambda}(g^*) \leq J_{n,\lambda}(g) \).

**Problem 2.** Develop model-free gradient descent and natural policy gradient descent algorithms to learn the optimal risk-neutral strategies \( g^* \), i.e., when \( \lambda \to 0 \).

**Remark 1.** For the special case of single sub-population (i.e. \( S = 1 \)), single agent (i.e. \( n(s) = 1, s \in \mathbb{N}_S \)) and single feature (i.e. \( f(s) = 1, s \in \mathbb{N}_S \)), deep structured teams reduce to the classical single-agent control problems [18], [19], [20].

**III. MAIN RESULTS FOR PROBLEM 1**

In this section, we first present the solution of Problem 1 in terms of Riccati equations. Then, we establish the global convergence of model-based policy gradient algorithms. From [4], we define a gauge transformation for any agent \( i \in \mathbb{N}_{n(s)} \) in sub-population \( s \in \mathbb{N}_S \) at time \( t \in \mathbb{N} \):

\[
\begin{align*}
\Delta x_t^i &:= x_t^i - \sum_{j=1}^{f(s)} \alpha_{i,j}(s)\bar{x}_t^j(s), \\
\Delta u_t^i &:= u_t^i - \sum_{j=1}^{f(s)} \alpha_{i,j}(s)\bar{u}_t^j(s), \\
\Delta w_t^i &:= w_t^i - \sum_{j=1}^{f(s)} \alpha_{i,j}(s)\bar{w}_t^j(s),
\end{align*}
\]

where \( \bar{w}_t^j(s) := \frac{1}{\mu(s)} \sum_{i=1}^{n(s)} \alpha_{i,j}(s)w_t^i, \forall j \in \mathbb{N}_{f(s)} \). The gauge transformation induces the following linear dependences:

\[
\begin{align*}
\sum_{i=1}^{n(s)} \sum_{j=1}^{f(s)} \alpha_{i,j}(s) \Delta x_t^j &:= 0, \\
\sum_{i=1}^{n(s)} \sum_{j=1}^{f(s)} \alpha_{i,j}(s) \Delta u_t^j &:= 0, \\
\sum_{i=1}^{n(s)} \sum_{j=1}^{f(s)} \alpha_{i,j}(s) \Delta w_t^j &:= 0.
\end{align*}
\]

Subsequently, the dynamics of the \( j \)-th deep state of sub-population \( s \in \mathbb{N}_S \) can be represented as follows:

\[
\bar{x}_{t+1}^j(s) = A(s)\bar{x}_t^j(s) + B(s)\bar{u}_t^j(s) + \bar{A}_j(s)\bar{x}_t + \bar{B}_j(s)\bar{u}_t + \bar{w}_t^j(s).
\]

\[\hat{\bar{u}}_t^j(s) = \frac{1}{\mu(s)} \sum_{i=1}^{n(s)} \alpha_{i,j}(s)\bar{u}_t^i.\]
For any $s \in \mathbb{N}_S$ and $t \in \mathbb{N}$, define the following matrices:

\[
\begin{align*}
\mathbf{A}_t(s) &= \text{diag}(A_t(s))_{f(s)}, \\
\mathbf{B}_t(s) &= \text{diag}(B_t(s))_{f(s)}, \\
\tilde{\mathbf{A}}_t(s) &= \begin{bmatrix} 0 \| f(s) \|_2 & f(1) \|_2^2 & \cdots & 0 \| f(s) \|_2 & f(S) \|_2^2 \end{bmatrix}, \\
\tilde{\mathbf{B}}_t(s) &= \begin{bmatrix} 0 \| f(s) \|_2 & f(1) \|_2^2 & \cdots & 0 \| f(s) \|_2 & f(S) \|_2^2 \end{bmatrix}, \\
\hat{\mathbf{A}}_t &= \text{row}(\tilde{\mathbf{A}}_1(1), \ldots, \tilde{\mathbf{A}}_t(S)), \\
\hat{\mathbf{B}}_t &= \text{row}(\tilde{\mathbf{B}}_1(1), \ldots, \tilde{\mathbf{B}}_t(S)).
\end{align*}
\]

One can then write: $\mathbf{x}_{t+1} = \hat{\mathbf{A}}_t \hat{\mathbf{x}}_t + \hat{\mathbf{B}}_t \hat{\mathbf{u}}_t + \hat{\mathbf{w}}_t$, where $\hat{\mathbf{w}}_t := \text{vec}((\tilde{w}_0(s))_{f(s)})_{j=1}^S$. From (2) and (5), it follows that for any $i \in \mathbb{N}_n(s)$ and $s \in \mathbb{N}_S$: $\Delta x_{t+1} = A(s)\Delta x_t + B(s)\Delta u_t + \Delta w_t$. In addition, one has the following orthogonal relations:

\[
\begin{align*}
\sum_{i=1}^n \sum_{j=1}^f \alpha_{ij}^s(\Delta x_t^s)\tilde{Q}(s)\tilde{x}_t^s = 0, \\
\sum_{i=1}^n \sum_{j=1}^f \alpha_{ij}^s(\Delta u_t^s)\tilde{R}(s)\tilde{u}_t^s = 0.
\end{align*}
\]

Define the following matrices $\mathbf{Q}_t(s) := \text{diag}(Q(s))_{f(s)}$, $\mathbf{R}_t(s) := \text{diag}(R(s))_{f(s)}$, $s \in \mathbb{N}_S$, $t \in \mathbb{N}$, and

\[
\begin{align*}
\mathbf{Q} := \text{diag}(\mu(s)\text{diag}(Q(s), \ldots, Q(s)))_{f(s)}^S + \sum_{s=1}^S \mu(s)\mathbf{Q}(s), \\
\tilde{\mathbf{R}} := \text{diag}(\mu(s)\text{diag}(R(s), \ldots, R(s)))_{f(s)}^S + \sum_{s=1}^S \mu(s)\tilde{\mathbf{R}}(s).
\end{align*}
\]

The proof follows directly from (1) and (6).

Theorem 1 (Model-known solution [4]). Let Assumption 1 hold. There exists a unique stationary optimal linear strategy $\nu_t^*(s)$, $s \in \mathbb{N}_S$, where the certainty equivalence theorem holds.

Lemma 1. For any $s \in \mathbb{N}_S$, the followings hold for any $i \neq j \in \mathbb{N}_n(s)$, $j \neq m \in \mathbb{N}_n(s)$ and $t \in \mathbb{N}$:

\[
\begin{align*}
\mathbb{E}[\Delta w_t^i(\Delta u_t^j)^\top] &= 0, \\
\mathbb{E}[\Delta w_t^i(\Delta u_t^i)^\top] &= \mathbb{E}[\Delta u_t^i(\Delta w_t^i)^\top] = 0.
\end{align*}
\]

Proof. The proof follows directly from (1) and (5).

Remark 2. For the weakly-coupled case in Definition 1, the Riccati equation decomposes further into $S + \sum_{s=1}^S f(s)$ smaller Riccati equations, where (10) can be expressed for any $j \in \mathbb{N}_f(s)$ and any $s \in S$ as follows:

\[
\begin{align*}
\tilde{P}_t^j(s) &= Q(s) + A_t^j(s)\tilde{P}_t^j(s)A_t^j(s) - A_t^j(s)\tilde{P}_t^j(s)B_t^j(s) \times (R_t^j(s)+B_t^j(s)\tilde{P}_t^j(s)B_t^j(s))^{-1}B_t^j(s)\tilde{P}_t^j(s)A_t^j(s), \\
\tilde{P}_n^j(s) &= P_n(s)(I_{d_x^j \times d_x^j} - 2\Delta w(s)\Sigma u(s)P_n(s))^{-1}, \\
\Sigma u(s) &= \text{diag}(\frac{1}{n} \sum_{s=1}^S \text{diag}(\sum w(s), \ldots, \sum w(s)))_{f(s)}^S.
\end{align*}
\]

The dimensions of the above Riccati equations are independent of the number of agents in each sub-population $s \in \mathbb{N}_S$.

Remark 3. Part (I) of Assumption 7 is a standard convexity condition and Part (II) is required to ensure that the system is stabilizable. Part (III) is a standard condition in risk-sensitive LQ problems that guarantees the deep Riccati equation, presented in (9) and (10), admit a unique positive definite solution. Suppose matrices in the dynamics (2) and cost functions (3) are independent of the size of sub-populations $n(s)$, $s \in \mathbb{N}_S$; then, if the risk-factor $\lambda$ decreases and/or the number of agents (i.e. $n(s)$, $s \in \mathbb{N}_S$) increases, the positiveness condition in Part (III) gets more relaxed such that it automatically holds if $\lambda = 0$ and/or $n(s) = \infty$.

Since the certainty equivalence theorem does not hold in the risk-sensitive case, which is in contrast to the risk-neutral model, we present a few key covariance properties.

Theorem 1 (Model-known solution [4]). Let Assumption 7 hold. There exists a unique stationary optimal linear strategy such that for any $i \in \mathbb{N}_n(s)$ and $s \in \mathbb{N}_S$ at time $t \in \mathbb{N}$:

\[
u_t^*(s)x_t^i = \theta_n^*(s)x_t^i - \sum_{j=1}^f \alpha_{ij}^*(s)\theta_n^*(s)x_t^j + \sum_{j=1}^f \alpha_{ij}^*(s)\theta_n^*(s)x_t^j.
\]

Proof. The proof follows from the linearly dependent equations (7), orthogonal relations (8) and covariance properties in Lemma 1 leading to a low-dimensional representation of the solution. For more details, see [4, Theorem 1].
For the special case of weakly coupled agents, the common Riccati equation decomposes into $\sum_{s=1}^{S} f(s) d_{s}^{2}$. For the special case of risk-neutral agents, the common Riccati equation decomposes into $\sum_{s=1}^{S} f(s)$ smaller Riccati equations. In this case, every agent needs to solve only $f(s) + 1$ Riccati equations in (9) and (11) with the dimensions $d_{s}^{2} \times d_{s}^{2}$. During the control process, each agent computes its action according to (12) based on the above Riccati solutions, its local (private) state and influence factors as well as common (public) deep states.

A. Model-based approach

From Theorem 1, the optimization problem in action space is strictly convex and there is no loss of optimality in restricting attention to stationary linear strategies of the form (5)\textsuperscript{(12)}. However, the convexity in action space does not lead to the convexity in policy space; see a simple counterexample in [15]. In what follows, we provide an analytical proof showing that policy gradient methods converge to the globally optimal solution (12) based on the above Riccati solutions, its local (private) state and influence factors as well as common (public) deep states.

The solution of Riccati equation in the risk-sensitive model decomposes into $\sum_{s=1}^{S} f(s)^{-1}$ for the case of weakly coupled agents, the common Riccati equation converges to that of the risk-neutral one. This observation enables us to establish an asymptotic global convergence result for the risk-sensitive cost function.

Theorem 2. Let Assumptions 1\textsuperscript{[7]} and 2\textsuperscript{[2]} hold. For a sufficiently small risk factor $\lambda$ and/or sufficiently large population $n(s)$, $\forall s \in S$, the policy gradient algorithms in (15) and (16) converge to the globally optimal solution $\theta^{*}$ for an adequately small step size $\eta$.

Proof. Let $\theta$ have a finite cost. Following from [15, Lemma 11], we obtain an upper bound on the distance between the cost function and its optimal value in terms of $\theta(s)$ and $\Sigma(s)$, $s \in S$, as well as $E_{\theta}$ and $\Sigma_{\theta}$ (which represent the gradients in (13)). Therefore, there exists a positive constant $L_{1}(\theta^{*})$ such that $|J(\theta) - J(\theta^{*})| \leq L_{1}(\theta^{*}) \|
abla_{\theta} J(\theta^{*})\|_{F}^{2}$. This inequality is known as gradient domination (or PL inequality [22]). Furthermore, we proceed according to [23, Lemmas 15 and 16] to show that the cost and gradient are locally Lipschitz functions in the neighbourhood of $\theta$, where the Lipschitz constants depend on $\varepsilon(\theta)$. In particular, given any $\theta'$ satisfying the inequality $|\theta' - \theta|_{F} < \varepsilon(\theta)$, there exist positive constants $L_{2}(\theta)$ and $L_{3}(\theta)$ such that $|J(\theta') - J(\theta)| \leq L_{2}(\theta)\|\theta' - \theta\|_{F}$ and $\|
abla_{\theta} J(\theta') - \nabla_{\theta} J(\theta)\|_{F} \leq L_{3}(\theta)\|\theta' - \theta\|_{F}$. Following the proof technique proposed in [15, Theorem 7] and [7, Theorem 2], we select a sufficiently small step size $\eta$ such that the value of the cost decreases at each iteration. In particular, for the natural policy gradient descent and a sufficiently large number of iterations $K$, one has: $J(\theta_{K+1}) - J(\theta^{*}) \leq 0$. The above recursion is contractive for a sufficiently small step size $\eta \leq \eta_{\text{max}}$.

IV. MAIN RESULTS FOR PROBLEM 2

In this subsection, we propose model-free policy gradient descent and natural policy gradient algorithms for the special case of risk-neutral $\lambda \rightarrow 0$ and/or $n(s) \rightarrow \infty$, one arrives at

\begin{align}
E_{\theta}(s) & := (R(s) + B^{T}(s) P_{\theta}(s) B(s))\theta(s) - B^{T} P_{\theta}(s) A(s), \\
\Sigma_{\theta}(s) & := \sum_{t=1}^{\infty} (I - 2\lambda \Sigma_{\alpha}(s))^{t} \Delta x_{i}(\Delta x_{i})^{T}, \\
E_{\theta} & := (R + B^{T} P_{\theta} B)\theta - B^{T} P_{\theta} A, \\
\Sigma_{\theta} & := \sum_{t=1}^{\infty} \Delta x_{i}(\Delta x_{i})^{T}. \quad (14)
\end{align}

For simplicity, it is assumed in [21] that the initial states have zero mean.

We propose two gradient methods described below, where $k \in \mathbb{N}$ denotes the iteration.

- **Policy gradient descent:**

\[
\begin{aligned}
\theta_{k+1}(s) &= \theta_{k}(s) - \eta \nabla_{\theta}(s) J(\theta), \quad s \in S, \\
\theta_{k+1} &= \theta_{k} - \eta \nabla_{\theta} J(\theta).
\end{aligned}
\]

- **Natural policy gradient descent:**

\[
\begin{aligned}
\theta_{k+1}(s) &= \theta_{k}(s) - \eta \nabla_{\theta}(s) J(\theta) \Sigma_{\theta}(s)^{-1}, \quad s \in S, \\
\theta_{k+1} &= \theta_{k} - \eta \nabla_{\theta} J(\theta) \Sigma_{\theta}^{-1}.
\end{aligned}
\]

We now make an assumption that the initial policy is stable, which is a standard assumption.

Assumption 2. For the initial policy, $\rho(A(s) - B(s)\theta_{0}(s)) < 1$, $s \in S$, and $\rho(A - B\theta_{0}) < 1$. In addition, part III of Assumption 7 holds for $P_{\theta}(s) > 0$, $s \in S$, and $P_{\theta_{i}} > 0$.
case of risk neutral problem (where \( \lambda \to 0 \)) such that \( J(\theta) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}[\sum_{t=1}^{T} c_t] \).

**Lemma 2** (Finite-horizon approximation). Let \( \tilde{J}_T(\theta) := \mathbb{E}[\sum_{t=1}^{T} c_t] \) for any \( \theta \) with a finite cost function. Let also \( \varepsilon(T) \) be a positive function such that \( \lim_{T \to \infty} \varepsilon(T) = 0 \). Then, there exists a sufficiently large horizon \( T \) for which \( |\tilde{J}_T(\theta) - J(\theta)| \leq \varepsilon(T) \).

**Proof.** The proof follows from [15, Lemma 26].

Denote by \( S_r \), a set of uniformly distributed points with norm \( r > 0 \) and by \( B_r \) the set of all uniformly distributed points whose norms are at most \( r \). Thus, \( J(\theta) = \mathbb{E}_{\hat{\theta} \sim B_r}[J(\theta + \hat{\theta})] \). Let \( \tilde{\theta} := \{\tilde{\theta}(1), \ldots, \tilde{\theta}(S), \hat{\theta}\} \) be a set of independent random matrices whose Frobenius norm is \( r \).

**Lemma 3** (Zeroth-order optimization). For a smoothing factor \( r > 0 \), \( \nabla_{\hat{\theta}(s)} J(\theta) = \frac{d}{dt} \mathbb{E}_{\hat{\theta} \sim S_r}[J(\theta + \hat{\theta}(s))] \), \( s \in S_r \), and \( \nabla_{\theta} J(\theta) = \sum_{s=1}^{S} f(s) d_s^* \mathbb{E}_{\hat{\theta} \sim S_r}[J(\theta + \hat{\theta}(s))] \).

**Proof.** The proof follows directly from the zeroth-order optimization approach [24, Lemma 1] and the fact that the cost function gets decoupled into \( S + 1 \) additive terms.

**Lemma 4.** Given any \( s \in S_r \), let \( \hat{\theta}_1(s), \ldots, \hat{\theta}_L(s), L \in \mathbb{N} \), be i.i.d. samples drawn uniformly from \( S_r \). For any \( \varepsilon(L) > 0 \), the following average converges to an \( \varepsilon(L) \)-neighbourhood of the gradient \( \nabla J_\theta(s)(\theta) \) in the Frobenius norm with a probability greater than \( 1 - \left( \frac{d_s^*}{e(L)} \right)^{-2} \).

\[
\nabla_{\theta} J(\theta) := \frac{1}{L} \sum_{l=1}^{L} \nabla_{\theta} J(\theta) = \frac{1}{L} \sum_{l=1}^{L} \left( \sum_{s=1}^{S} f(s) d_s^* \right) \mathbb{E}_{\hat{\theta} \sim S_r}[J(\theta + \hat{\theta}(s))] \hat{\theta}.
\]

**Proof.** Following the steps proposed in [15, Lemma 30], the proof follows from Lemma 3 and Bernstein’s inequality.

We now compute an empirical gradient for a sufficiently large number of samples \( L \) and rollsouts \( T \):

\[
\begin{cases}
\nabla_{\theta(s)} J(\theta) := \frac{1}{L} \sum_{l=1}^{L} d_s^* \nabla_{\theta} J(\theta(s)), \\
\nabla_{\theta} J(\theta) := \frac{1}{L} \sum_{l=1}^{L} \left( \sum_{s=1}^{S} f(s) d_s^* \right) \mathbb{E}_{\hat{\theta} \sim S_r}[J(\theta + \hat{\theta}(s))],
\end{cases}
\]

\[
(17)
\]

**Theorem 3.** Let Assumptions 1 and 2 hold. There exists a sufficiently small step size \( \eta \) such that the following inequality holds with a probability converging to one as the number of samples \( L \) and rollsouts \( T \) tend to infinity, \( J_{L,T}(\theta) - J(\theta^*) \leq \varepsilon(L,T) \), where \( \varepsilon(L,T) = \text{poly}(1/L, 1/T) \).

**Proof.** It results from [15, Theorem 31] and [7, Theorem 3] that the following inequality at iteration \( K \in \mathbb{N} \) holds for a sufficiently small step size \( \eta \leq \eta_{\max}: J(\theta_{K+1}) - J(\theta^*) \leq (1 - \eta_{\max} J(\theta_{K+1}) - J(\theta^*)) \). At iteration \( K \), let \( \nabla_{\theta} J(\theta) \) denote the empirical gradient in (17) and \( \theta_{K+1} = \theta_{K} - \eta \nabla_{\theta} J(\theta) \). Denote the iterate with the empirical gradient. Due to the locally Lipschitz continuity, Lemmas 3 and Bernstein inequality, the approximate \( \theta_{K+1} \) converges to its exact value \( \theta_{K+1} \) as the number of samples and rollsouts tend to infinity with a probability greater than \( 1 - \left( \frac{z}{e(L,T)} \right)^{-2} \), where \( z := \left( \sum_{s=1}^{S} f(s) + 1 \right)^2 \). Subsequently, one gets \( J(\theta_{K+1}) - J(\theta^*) \leq (1 - \frac{1}{2} \eta_{\max}) (J(\theta_{K}) - J(\theta^*)) \), when \( J(\theta_{K}) - J(\theta^*) \leq \varepsilon(L,T) \). This recursion is contractive, which is similar to the proof of Theorem 2.

**Remark 4.** From [25], it can be shown that the policy gradient algorithms can be extended to actor-critic ones.

**V. NUMERICAL EXAMPLES**

In this section, we provide two numerical examples.

**Example 1.** Consider a risk-sensitive LQ deep structured team with the following parameters: \( n = 10, A = 0.9, B = 0.4, Q = 2, R = 1, \eta = 5, T = 10, L = 100, \sigma_{\lambda} = 10^{-6} = \sqrt{0.5}, \alpha^7 = \sqrt{1.5}, \alpha^8 = 1, \alpha^9 = \sqrt{2}, \alpha^{10} = \sqrt{2.5}, \nu_t \sim \text{norm}(0, 0.1), \) and \( x_t \sim \text{norm}(0, 0.1) \). It is observed in Figure 1 that the model-based policy gradient descent algorithm converges to the global optimal solution when the risk factor is \( \lambda = 0.1 \).

**Example 2.** Consider a risk-neutral LQ deep structured teams with the following parameters: \( n = 10, A = B = Q = 1, \tilde{Q} = 2, R = 2, \tilde{R} = 1, \eta = 0.2, T = 10, L = 100, \alpha^{10} = 10^{-10} = \sqrt{0.1}, \nu_t \sim \text{norm}(0, 0.02), \) and \( x_t \sim \text{norm}(0, 0.1) \). The learning trajectory of the model-free policy gradient descent algorithm is depicted in Figure 2.
10 random seeds. The simulation was run on a 2.7 GHz Intel Core i5 processor and took roughly 2 hours.

VI. IMPLEMENTATION

In practice, agents can use different methods to implement the RL algorithms. Below, we mention three types.

- **Team (common) learner:** All agents have access to a common exploration noise, meaning that the empirical gradient is identical for all agents. This way, all agents run the same learning algorithm with an identical solution, given that the step sizes are identical for all agents.

- **Single learner and multiple imitators:** This is when a single agent learns the optimal strategy while other agents act as imitators and are passive during the learning process. In particular, one agent explores the system from its point of view and others employ the updated (learned) strategy at each iteration to decide their next actions. It is also possible to select the learner randomly at each iteration in order to have a fair implementation. This type of implementation is similar to the notion of person-by-person optimality, which is different from the global optimality, in general. Its advantage over the above team implementation is that the single player may use an individualized observation. For example, in the natural policy gradient [16], the centralized information \( \Sigma \dot{\theta}(s) \) in [14] may be replaced by the individualized information \( \sum_{i=1}^{\infty} \Delta x^i (\Delta x^i)^\top \), because \( \Delta x^i \) has an identical distribution for all players in sub-population \( s \).

- **Many independent learners:** To avoid discrepancy between the agents during the learning process, the above implementations allow for only one common rule of learning. However, if the number of agents is very large, all agents can independently learn the strategy, because their explorations are decoupled from one another. In such a case, the trajectory of the deep state (the coupling term) is independent of i.i.d. exploration noises.

VII. CONCLUSIONS

In this paper, we investigated the convergence of model-based and model-free gradient descent and natural policy gradient descent algorithms in linear quadratic deep structured teams. The size of the parameter space of the proposed algorithms is independent of the number agents in each sub-population, making the algorithms applicable to large-scale problems. By using the notions of gradient domination and locally Lipschitz continuity, we presented an analytical proof for the global convergence of the above algorithms. The theoretical findings were verified by some simulations. The obtained results naturally extend to other variants of reinforcement learning methods such as actor-critic.

REFERENCES

[1] H. Witsenhausen, “A counterexample in stochastic optimum control,” SIAM Journal on Control and Optimization, vol. 6, pp. 131–147, 1968.

[2] J. Arabneydi and A. G. Aghdam, “Deep teams: Decentralized decision making with finite and infinite number of agents,” IEEE Transactions on Automatic Control, DOI: 10.1109/TAC.2020.2966035, 2020.

[3] J. Arabneydi, A. G. Aghdam, and R. P. Malhamé, “Explicit sequential equilibria in LQ deep structured games and weighted mean-field games,” conditionally accepted in Automatica, 2020.

[4] J. Arabneydi and A. G. Aghdam, “Deep structured teams with linear quadratic model: Partial equivariance and gauge transformation,” [Online]. Available at https://arxiv.org/abs/1912.03951, 2019.

[5] ——, “Deep structured teams and games with Markov-chain model: Finite and infinite number of players,” Submitted, 2019.

[6] J. Arabneydi, M. Roudneshin, and A. G. Aghdam, “Reinforcement learning in deep structured teams: Initial results with finite and infinite valued features,” in Proceedings of IEEE Conference on Control Technology and Applications, 2020.

[7] M. Roudneshin, J. Arabneydi, and A. G. Aghdam, “Reinforcement learning in nonzero-sum Linear Quadratic deep structured games: Global convergence of policy optimization,” in Proceedings of the 59th IEEE Conference on Decision and Control, 2020.

[8] J. Arabneydi, “New concepts in team theory: Mean field teams and reinforcement learning.” Ph.D. dissertation, Dep. of Electrical and Computer Engineering, McGill University, Montreal, Canada, 2016.

[9] J. Arabneydi and A. Mahajan, “Linear quadratic mean field teams: Optimal and approximately optimal decentralized solutions,” Available at https://arxiv.org/abs/1609.00056, 2016.

[10] ——, “Team-optimal solution of finite number of mean-field coupled LQG subsystems,” in Proceedings of the 54th IEEE Conference on Decision and Control, 2015, pp. 5308 – 5313.

[11] M. Roudneshin, J. Arabneydi, and A. G. Aghdam, “Near-optimal control strategy in leader-follower networks: A case study for linear quadratic mean-field teams,” in Proceedings of the 57th IEEE Conference on Decision and Control, 2018, pp. 3288–3293.

[12] ——, “Minmax mean-field team approach for a leader-follower network: A saddle-point strategy,” IEEE Control Systems Letters, vol. 4, no. 1, pp. 121–126, 2019.

[13] J. Arabneydi, M. Baharloo, and A. G. Aghdam, “Optimal distributed control for leader-follower networks: A scalable design,” in Proceedings of the 31st IEEE Canadian Conference on Electrical and Computer Engineering, 2018, pp. 1–4.

[14] J. Arabneydi and A. G. Aghdam, “Optimal dynamic pricing for binary demands in smart grids: A fair and privacy-preserving strategy,” in Proceedings of American Control Conference, 2018, pp. 5368–5373.

[15] M. Fazel, R. Ge, S. M. Kakade, and M. Mesbahi, “Global convergence of policy gradient methods for the linear quadratic regulator,” arXiv preprint arXiv:1801.05039, 2018.

[16] Y. Luo, Z. Yang, Z. Wang, and M. Kolar, “Natural actor-critic converges globally for hierarchical linear quadratic regulator,” arXiv at https://arxiv.org/pdf/1912.06875.pdf, 2019.

[17] L. Lewis, Z. Yang, L. Yuchen, and Z. Wang, “Decentralized policy gradient method for mean-field linear quadratic regulator with global convergence,” ICML, 2020.

[18] D. Jacobson, “Optimal stochastic linear systems with exponential performance criteria and their relation to deterministic differential games,” IEEE Transactions on Automatic control, vol. 18, no. 2, pp. 124–131, 1973.

[19] P. Whittle, “Risk-sensitive linear/quadratic/Gaussian control,” Advances in Applied Probability, vol. 13, no. 4, pp. 764–777, 1981.

[20] T. Başar and P. Bernhard, H-infinity optimal control and related minmax design problems: A dynamic game approach. Birkhäuser Basel, 2008.

[21] K. Zhang, B. Hu, and T. Basar, “Policy optimization for H2 linear control with H∞ robustness guarantee: Implicit regularization and global convergence,” arXiv preprint arXiv:1910.09496, 2019.

[22] B. T. Polyak, “Gradient methods for the minimisation of functionals,” USSR Computational Mathematics and Mathematical Physics, vol. 3, no. 4, pp. 864–878, 1963.

[23] D. Malik, A. Pananjady, K. Bhatia, K. Khamaru, P. L. Bartlett, and M. J. Wainwright, “Derivative-free methods for policy optimization: Guarantees for linear quadratic systems,” Journal of Machine Learning Research, vol. 21, no. 21, pp. 1–51, 2020.

[24] A. D. Flaxman, A. T. Kalai, and H. B. McMahan, “Online convex optimization in the bandit setting: Gradient descent without a gradient,” in Proceedings of ACM-SIAM Symposium on Discrete Algorithms. Society for Industrial and Applied Mathematics, 2005, p. 385–394.

[25] Z. Yang, Y. Chen, M. Hong, and Z. Wang, “Provably global convergence of actor-critic: A case for linear quadratic regulator with ergodic cost,” in Advances in Neural Information Processing Systems, 2019, pp. 8353–8365.