Weighted Well-Covered Graphs without Cycles of Length 4, 6 and 7

Vadim E. Levit and David Tankus
Department of Computer Science and Mathematics
Ariel University Center of Samaria, ISRAEL
{levitv, davidta}@ariel.ac.il

Abstract

A graph is well-covered if every maximal independent set has the same cardinality. The recognition problem of well-covered graphs is known to be co-NP-complete. Let \( w \) be a weight function defined on the vertices of \( G \). Then \( G \) is \( w \)-well-covered if all maximal independent sets of \( G \) are of the same weight. The set of weight functions \( w \) for which a graph is \( w \)-well-covered is a vector space. We prove that finding the vector space of weight functions under which an input graph is \( w \)-well-covered can be done in polynomial time, if the input graph does not contain cycles of length 4, 6 and 7.

1 Introduction

Throughout this paper \( G = (V, E) \) is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set \( V = V(G) \) and edge set \( E = E(G) \).

Let \( S \subseteq V \) be a set of vertices, and let \( i \in \mathbb{N} \). Then

\[
N_i(S) = \{w \in V \mid \min_{s \in S} d(w, s) = i\},
\]

where \( d(x, y) \) is the minimal number of edges required to construct a path between \( x \) and \( y \). If \( i \neq j \) then, obviously, \( N_i(S) \cap N_j(S) = \phi \). If \( S = \{v\} \) for some \( v \in V \), then \( N_i(\{v\}) \) is abbreviated to \( N_i(v) \).

A set of vertices \( S \subseteq V \) is independent if for every \( x, y \in S, x \) and \( y \) are not adjacent. It is clear that an empty set is independent. The independence number of a graph \( G \), denoted \( \alpha(G) \), is the size of a maximum cardinality independent set in \( G \). A graph is well-covered if every maximal independent set has the same cardinality, \( \alpha(G) \). Finding the independence number of an input graph is generally an NP-complete problem. However, if the input is restricted to well-covered graphs then the problem can be solved polynomially by applying the greedy algorithm.

A well-covered graph \( G \) is 1-well-covered if and only if for every vertex \( v \in G \), the graph \( G - \{v\} \) is well-covered and \( \alpha(G) = \alpha(G - \{v\}) \).

Let \( T \subseteq V \). Then \( S \) dominates \( T \) if \( S \cup N_1(S) \supseteq T \). If \( S \) and \( T \) are both empty, then \( N_1(S) = \phi \), and therefore \( S \) dominates \( T \). If \( S \) is a maximal independent set of \( G \), then it dominates the whole graph.
Two adjacent vertices, $x$ and $y$, in $G$ are said to be related if there is an independent set $S$, containing neither $x$ nor $y$, such that $S \cup \{x\}$ and $S \cup \{y\}$ are both maximal independent sets in the graph. If $x$ and $y$ are related, then $xy$ is a relating edge. It is proved in [1] that deciding whether an edge in an input graph is relating is an NP-complete problem.

**Theorem 1.1** [1] The following problem is NP-complete:

*Input:* A graph $G = (V, E)$ and an edge $xy \in E$.

*Question:* Is $xy$ a relating edge?

However, if the input graph contains neither $C_4$ nor $C_6$ then the problem is polynomial.

**Theorem 1.2** [7] The following problem is polynomially solvable:

*Input:* A graph $G = (V, E)$, which does not contain simple cycles of length 4 and 6, and an edge $xy \in E$.

*Question:* Is $xy$ a relating edge?

The recognition of well-covered graphs is known to be co-NP-complete. The problem remains co-NP-complete even when the input is restricted to $K_{1,4}$-free graphs [3]. However, the problem is polynomially solvable for $K_{1,3}$-free graphs [9][10], for graphs with girth at least 5 [4], for graphs that contain neither 4- nor 5-cycles [5], for graphs with a bounded maximal degree [2], or for chordal graphs [8]. Recognizing 1-well-covered graphs with no 4-cycles can be implemented in polynomial time [9].

Brown, Nowakowski and Zverovich investigated well-covered graphs with no cycles of length 4, and presented the following open problem.

**Problem 1.3** [1] What is the complexity of determining whether an input graph with no cycles of length 4 is well-covered?

Levit and Tankus proposed the following.

**Conjecture 1.4** [7] The following problem can be solved in polynomial time:

*Input:* A graph $G = (V, E)$ which does not contain cycles of length 4 and 6.

*Question:* Is $G$ well-covered?

### 2 Generating Subgraphs

Let $G = (V, E)$ be a graph, and let $w : V \rightarrow R$ be a weight function defined on its vertices. The weight of a set $S \subseteq V$ is defined by: $w(S) = \sum_{s \in S} w(s)$. The graph $G$ is $w$-well-covered if all maximal independent sets of $G$ are of the same weight [2].

Let $B$ be a complete bipartite induced subgraph of $G$, and denote its sides by $B_X$ and $B_Y$. Then $B$ is a generating subgraph of $G$ if there exists an independent set $S$ of $G$ such that $S \cup B_X$ and $S \cup B_Y$ are both maximal independent sets of $G$. In this case $B$ produces the constraint that $B_X$ and $B_Y$ are of the same weight. $B \approx K_{1,1}$ is a generating subgraph if and only if its two vertices are related. Hence the notion of related vertices introduced in [1], is an instance of a generating subgraph, for the case that this subgraph is isomorphic to $K_{1,1}$. The following theorem is a generalization of Theorem [1][2] for the case that the input graph does not admit a $C_7$. 

2
**Theorem 2.1** The following problem can be solved in polynomial time:

**Input:** A graph $G = (V, E)$ which does not contain cycles of length 4, 6 and 7, and a complete bipartite induced subgraph $B$ of $G$.

**Question:** Is $B$ a generating subgraph of $G$?

**Proof.** Let us recall that the sides of $B$ are denoted by $B_X$ and $B_Y$. Assume, without loss of generality, that $|B_X| \leq |B_Y|$. Notice that since the graph $G$ does not contain cycles of length 4, the set $B_X$ contains just one element, i.e., $|B_X| = 1$.

Let $B_X = \{x\}$ and $B_Y = \{y_1, ..., y_k\}$. The absence of cycles of length 4, 6 and 7 from the graph implies that:

- $C_4$: $\forall 1 \leq i < j \leq k$ \( N_1(y_i) \cap N_1(y_j) = \{x\} \).
- $C_4$, $C_6$: $\forall 1 \leq i < j \leq k \ N_2(y_i) \cap N_2(y_j) \cap N_3(x) = \phi$.
- $C_4$, $C_7$: For every $1 \leq i < j \leq k$ there are no edges between $N_2(y_i) \cap N_3(x)$ and $N_2(y_j) \cap N_3(x)$.
- $C_4$: For every $1 \leq i \leq k$, every connectivity component of $N_1(y_i) \cap N_2(x)$ contains at most one edge.
- $C_4$, $C_6$, $C_7$: Every connectivity component of $N_3(x)$ contains at most one edge.
- $C_4$, $C_6$: Every vertex of $N_3(x)$ is adjacent to exactly one vertex of $N_2(x)$.

For every $P \in \{B_X, B_Y\}$, let $Q = B - P$, and define

$$M_1(P) = N_1(P) \cap N_2(Q), M_2(P) = N_1(M_1(P)) - B.$$ 

The subgraph $B$ is generating if and only if there exists an independent subset of the set $M_2(B_X) \cup M_2(B_Y)$, which dominates $M_1(B_X) \cup M_1(B_Y)$.

The fact that the graph $G$ does not contain cycles of length 6 implies the following:

- There are no edges connecting vertices of $M_2(B_X)$ with vertices of $M_2(B_Y)$.
- The set $M_2(B_X) \cap M_2(B_Y)$ is independent.
- There are no edges between the vertices belonging to $M_2(B_X) \cap M_2(B_Y)$ and the other vertices of $M_2(B_X) \cup M_2(B_Y)$.

Consequently, if $S_x \subseteq M_2(B_X)$ and $S_y \subseteq M_2(B_Y)$ are independent, then $S_x \cup S_y$ is independent as well. Therefore, it is easy to prove that one can decide in polynomial time whether there exists an independent subset of the set $M_2(P)$ dominating $M_1(P)$, where $P \in \{B_X, B_Y\}$.

Let us note that:

- Every vertex of $M_2(P)$ is adjacent to exactly one vertex of $M_1(P)$, or otherwise the graph $G$ contains a $C_4$.
- Every connectivity component of $M_2(P)$ contains at most 2 vertices, or otherwise the graph $G$ contains either a $C_4$ or a $C_6$ or a $C_7$.
Let $A_1, ..., A_k$ be the connectivity components of $M_2(P)$. Define a flow network

$$F_P = \{ G_F = (V_F, E_F), s \in V_F, t \in V_F, w: E_F \rightarrow R \}$$

as follows.

Let

$$V_F = M_1(P) \cup M_2(P) \cup \{ a_1, ..., a_k, s, t \},$$

where $a_1, ..., a_k, s, t$ are new vertices, $s$ and $t$ are the source and sink of the network, respectively.

The directed edges $E_F$ are:

- the directed edges from $s$ to each vertex of $M_1(P)$;
- all directed edges $v_1v_2$ s.t. $v_1 \in M_1(P)$, $v_2 \in M_2(P)$ and $v_1v_2 \in E$;
- the directed edges $va_i$, for each $1 \leq i \leq k$ and for each $v \in A_i$;
- the directed edges $a_it$, for each $1 \leq i \leq k$.

Let $w \equiv 1$. Invoke any polynomial time algorithm for finding a maximum flow in the network, for example, Ford and Fulkerson’s algorithm. Let $S_P$ be the set of vertices in $M_2(P)$ in which there is a positive flow. Clearly, $S_P$ is independent. The maximality of $S_P$ implies that $|M_1(P) \cap N_1(S_P)| \geq |M_1(P) \cap N_1(S'_P)|$, for any independent set $S'_P$ of $M_2(P)$.

Let us conclude the proof with the recognition algorithm for generating subgraphs.

For each $P \in \{ B_X, B_Y \}$, build a flow network $F_P$, and find a maximum flow. Let $S_P$ be the set of vertices in $M_2(P)$ in which there is a positive flow. If $S_P$ does not dominate $M_1(P)$ the algorithm terminates announcing that $B$ is not generating. Otherwise, let $S$ be any maximal independent set of $G - B$ which contains $S_{B_X} \cup S_{B_Y}$. Each of $S \cup B_X$ and $S \cup B_Y$ is a maximal independent set of $G$, and $B$ is generating.

This algorithm can be implemented in polynomial time: One iteration of Ford and Fulkerson’s algorithm includes:

- Updating the flow function. (In the first iteration the flow equals 0.)
- Constructing the residual graph.
- Finding an augmenting path, if exists. The residual capacity of every augmenting path is equal to 1.

Each of the above can be implemented in $O(|V| + |E|)$ time. In each iteration the number of vertices in $M_2(P)$ with a positive flow increases by 1. Therefore, the number of iterations can not exceed $|V|$, and Ford and Fulkerson’s algorithm terminates in $O(|V| (|V| + |E|))$ time. Our algorithm invokes Ford and Fulkerson’s algorithm twice, and terminates in $O(|V| (|V| + |E|))$ time. ■
3 Hereditary Systems

A hereditary system is a pair $H = (S, F)$, where $S$ is a finite set and $F$ is a family of subsets of $S$, where $f \in F$ and $f' \subseteq f$ implies $f' \in F$. The members of $F$ are called the feasible sets of the system.

A weighted hereditary system is a pair $(H, w)$, where $H = (S, F)$ is a hereditary system, and $w : S \rightarrow R$ is a weight function on $S$. The weight of a set $S' \subseteq S$ is defined by:

$$w(S') = \sum_{s' \in S'} w(s').$$

A greedy weighted hereditary system is a weighted hereditary system $(H, w)$ for which all maximal feasible sets are of the same weight.

Theorem 3.1 \[11\] Let 

$$(H = (S, F), w : S \rightarrow R)$$

be a weighted hereditary system. Then $(H, w)$ is not greedy if and only if there exist two maximal feasible sets, $A$ and $B$, of $F$ with different weights, $w(A) \neq w(B)$, such that for each $a \in A \setminus B$ and for each $b \in B \setminus A$, the set $(A \cap B) \cup \{a, b\}$ is not feasible.

Let $(G, w) = (V, E, w)$ be a weighted graph with the weighted function $w : V \rightarrow R$. Then the weighted graph $(G, w)$ with the family of all its independent sets clearly forms a weighted hereditary system. This system is greedy if and only if $G$ is $w$-well-covered.

Theorem 3.2 Let $(G, w)$ be a weighted graph. Then $G$ is $w$-well-covered if and only if it obeys all the constrains produced by generating subgraphs of $G$.

Proof. Clearly, if $G$ is $w$-well-covered and $B$ is a generating subgraph of $G$, then the sides of $B$ must have equal weights.

Assume that $G$ is not $w$-well-covered. By Theorem 3.1 there exist two maximal independent sets, $A$ and $B$, of $G$ such that $w(A) \neq w(B)$ and the subgraph induced by $A \Delta B$ is complete bipartite. Let $H$ be the complete bipartite subgraph of $G$ induced by $A \Delta B$. The union of $A \cap B$ with either side of $H$ is a maximal independent set of the graph. Therefore, $H$ is generating.

4 The Vector Space

Let $G = (V, E)$ be a graph. The set of all weight functions $w : V \rightarrow R$ for which $G$ is $w$-well-covered is a vector space \[2\]. Assume that $G$ does not contain cycles of length 4, 6 and 7. In Section 1 we proved that for every complete bipartite subgraph $B$ of $G$ it is possible to decide in polynomial time whether $B$ is generating. In Section 3 it was proved that the union of constrains produced by all generating subgraphs of $G$ is the vector space of weight functions under which $G$ is $w$-well-covered. However, the number of generating subgraphs of $G$ is not necessarily polynomial. In this section we supply an algorithm to find the requested vector space in polynomial time.

For every $v \in V$, define $L_v$ to be the vector space of weight functions of $G$ obeying the union of all constrains produced by subgraphs $B$ of $G$ with $B_X = \{v\}$. Suppose $w$ is a weight function defined on $V$. Then $G$ is $w$-well-covered if and only if $w \in \bigcap_{v \in V} L_v$. 

5
**Theorem 4.1** Let $G = (V, E)$ be a graph that does not contain cycles of length 4, 6 and 7. For every $v \in V$ it is possible to find $L_v$ in polynomial time.

**Proof.** Let $v \in V$. For every vertex $y \in N_1(v)$, define

$$M_1(y) = N_1(y) \cap N_2(v) \text{ and } M_2(y) = N_2(y) \cap N_3(v).$$

Let $D(v)$ be the set of all vertices $y$ of $N_1(v)$ such that there exists an independent set of $M_2(y)$ which dominates $M_1(y)$. Note that $y \in D(v)$ if and only if $v$ and $y$ are related in the subgraph of $G$ induced by $\{v, y\} \cup M_1(y) \cup M_2(y)$. Hence it is possible to find $D(v)$ by invoking the algorithm presented in the proof of Theorem 2.1 for each $y \in N_1(v)$.

Since $G$ does not contain $C_4$, every connected component of $D(v)$ contains at most 2 vertices. For every $y \in D(v)$, let $S_y$ be an independent set of $M_2(y)$ which dominates $M_1(y)$. Clearly, $\bigcup_{y \in D(v)} S_y$ is independent. Construct a bipartite graph $B^*$ as follows: one side of $B^*$ is $\{v\}$, and the other side contains exactly one vertex from every connected component of $D(v)$.

Define $F_v$ to be the family of the following bipartite subgraphs of $G$:

- $B^* \in F_v.$
- If $B^* \neq K_{1,1}$ then $B^* - \{y\} \in F_v$, for every $y \in D(v)$.
- $B^* \Delta C \in F_v$, for every connectivity component $C$ of $D(v)$ with 2 vertices.

For each member of $F_v$ decide whether it is a generating subgraph of $G$, using the algorithm of Theorem 2.1. Define $F_v^*$ to be the set of all generating members of $F_v$. Let $B$ be a generating subgraph of $G$ with $B_X = \{v\}$. Then every weight function of $G$ obeying all constraints produced by $F_v^*$, obeys the constraint produced by $B$. The algorithm of finding a base of $L_v$ is completed.

The complexity of the algorithm: For each $y \in N_1(v)$ the decision whether $y \in D(v)$ takes $O(|V| (|V| + |E|))$ time. Hence, $D(v)$ can be found in $O \left( |V|^2 \left( |V| + |E| \right) \right)$ time. The size of $F_v$ is bounded by $|V|$, and for each member of $F_v$ is it possible to decide whether it is generating in $O \left( |V| \left( |V| + |E| \right) \right)$ time. The total time required to find $L_v$ is $O \left( |V|^2 \left( |V| + |E| \right) \right)$. \[\blacksquare\]

**Theorem 4.2** Let $G = (V, E)$ be a graph that does not contain cycles of length 4, 6 and 7. Then it is possible to find in polynomial time the vector space of weight functions $w$ under which the graph $G$ is $w$-well-covered.

**Proof.** According to Theorem 3.2, the vector space of weight functions of $G$ under which the graph is $w$-well-covered is the maximum linear subspace satisfying all the constraints produced by generating subgraphs of $G$. Since $G$ does not contain cycles of length 4, one of the sides of every generating subgraph comprises only one vertex. Hence, the required vector space is $\bigcap_{v \in V} L_v$.

By Theorem 4.1 for every $v \in V$ it is possible to find $L_v$ in $O \left( |V|^2 \left( |V| + |E| \right) \right)$ time. Consequently, $\{L_v\}_{1 \leq i \leq |V|}$ can be found in $O \left( |V|^3 \left( |V| + |E| \right) \right)$ time. In order to find the intersection $\bigcap_{v \in V} L_v$, which is the vector space of weight functions under which the graph is $w$-well-covered, one has to apply the Gaussian elimination procedure to a matrix of size
\[ \left( \sum_{i=1}^{|V|} g_i \right) \cdot |V|, \] where \( g_i \) is the number of generating subgraphs of \( G \) belonging to \( F_{v_i} \). Since \( \sum_{i=1}^{|V|} g_i \leq |V|^2 \), the time complexity of the Gaussian elimination procedure for this matrix is bounded by \( O\left(|V|^4\right) \). Finally, \( \bigcap_{v \in V} L_v \) may be constructed in \( O\left(|V|^3(|V| + |E|)\right) \). \[ \blacksquare \]

5 Open Problem

Our main conjecture reads as follows.

**Conjecture 5.1** The following problem can be solved in polynomial time:

*Input:* A graph \( G \) which does not contain cycles of length 4 or 6.

*Question:* Find the vector space of weight functions \( w \) under which the graph \( G \) is \( w \)-well-covered.

References

[1] J. I. Brown, R. J. Nowakowski, I. E. Zverovich, *The structure of well-covered graphs with no cycles of length 4*, Discrete Mathematics 307 (2007) 2235 – 2245.

[2] Y. Caro, N. Ellingham, G. F. Ramey, *Local structure when all maximal independent sets have equal weight*, SIAM Journal on Discrete Mathematics 11 (1998) 644-654.

[3] Y. Caro, A. Sebő, M. Tarsi, *Recognizing greedy structures*, Journal of Algorithms 20 (1996) 137-156.

[4] A. Finbow, B. Hartnell, R. Nowakowski, *A characterization of well-covered graphs of girth 5 or greater*, Journal of Combinatorial Theory Ser. B. 57 (1993) 44-68.

[5] A. Finbow, B. Hartnell, R. Nowakowski *A characterization of well-covered graphs that contain neither 4- nor 5-cycles*, Journal of Graph Theory 18 (1994) 713-721.

[6] B. Hartnell, *A characterization of the 1-well-covered graphs with no 4 cycles*, Graph Theory Trends in Mathematics (2006) 219-224.

[7] V. Levit, D. Tankus, *On relating edges in well-covered graphs without cycles of length 4 and 6*, Graph Theory, Computational Intelligence and Thought: Essays Dedicated to Martin Charles Golumbic on the Occasion of His 60th Birthday, Lecture Notes in Computer Science 5420 (2009) 144-147.

[8] E. Prisner, J. Topp and P. D. Vestergaard, *Well-covered simplicial, chordal and circular arc graphs*, Journal of Graph Theory 21 (1996), 113–119.

[9] D. Tankus, M. Tarsi, *Well-covered claw-free graphs*, Journal of Combinatorial Theory Ser. B. 66 (1996) 293-302.

[10] D. Tankus, M. Tarsi, *The structure of well-covered graphs and the complexity of their recognition problems*, Journal of Combinatorial Theory Ser. B. 69 (1997) 230-233.

[11] D. Tankus, M. Tarsi, *Greedily constructing Hamiltonian paths, Hamiltonian cycles and maximum linear forests*, Discrete Mathematics 307 (2007) 1833-1843.