Poisson type relativistic spheres

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Static spherically symmetric exact solutions of the Einstein’s field equations in isotropic coordinates representing distributions of matter made of a fluid both perfect and anisotropic from Newtonian potential-density pairs are investigated. For perfect fluid spheres, the approach is illustrated with three simple examples based in the seed potential-density pairs corresponding to a harmonic oscillator (homogeneous sphere), the well-known Plummer model and a massive spherical dark matter halo model with a logarithm potential. For anisotropic spheres, the models are built assuming also the circular speed profile and, as a simple application, an analytical family of anisotropic spheres is considered. Moreover, the geodesic motion of test particles in stable circular orbits around such structures is studied. The models constructed satisfy all the energy conditions.

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I. INTRODUCTION

Matter distributions with spherical symmetry have played an important role in astrophysics as models of dwarf spheroidal galaxies [1], bulges of disc galaxies [2, 3], galactic nuclei [4, 5], globular clusters [6], clusters of galaxies and dark matter haloes [7, 8]. In relativistic astrophysics, such matter configurations have been used as models of neutron stars, highly dense stars, gravastars, dark energy stars, quark stars, galactic nuclei and certain star clusters where relativistic effects are expected to be important. In relation to static spherically symmetric fields, several exact solutions have been obtained over the years [9, 10]. In particular, anisotropic relativistic models of spherically symmetric matter distributions from various Newtonian potential-density pairs in isotropic coordinates were investigated in Refs. [11, 12] for Schwarzschild type space-times, also used in [13] for same space-time, and in Ref. [14] for Majumdar-Papapetrou type fields. This same approach and spacetime was also used in [15] in the construction of three-dimensional axisymmetric sources.

In this work, we constructed anisotropic and perfect fluid sources for static spherically symmetric fields from a Newtonian potential-density pair using isotropic coordinates. The paper is structured as follows. In Sect. II we present the method to build different compact sources for static spherically symmetric fields from given solutions of the Poisson’s equation using isotropic coordinates. We also analyse the geodesic circular motion of test particles around such structures and the stability of the orbits against radial perturbations.

In Sects. III-IV we present some simple examples of matter configurations built using such approach. For perfect fluid sources, the method is illustrated with three examples based in the seed potential-density pairs of a homogeneous sphere, the well-known Plummer model and a logarithm potential. These potential-density pairs have been used by some authors as models of dark matter haloes [1, 3, 25] and although such structures are essentially Newtonian, its anisotropic description can be interesting from the theoretical point of view and in the analysis of their interaction with large scale cosmic evolution and relativistic effects such as gravitational lenses or gravitational waves [16]. For anisotropic matter distributions, a analytical family of relativistic spheres is considered for a circular speed profile imposed. Finally, in Sect. V we summarize and discuss the results obtained.

II. POISSON TYPE RELATIVISTIC SPHERES AND MOTION OF PARTICLES

The line element for a static spherically symmetric spacetime in isotropic coordinates is given by [10]

$$ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}(dr^2 + r^2 d\Omega^2),$$

(1)

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$, and $\nu$ and $\lambda$ are functions of only $r$.

The Einstein’s gravitational field equations $G_{ab} = 8\pi GT_{ab}$ yield the following non-zero components of the energy-momentum tensor

$$T^t_\nu = \frac{1}{4\pi G} e^{-2\lambda} \left[ \lambda'' + \frac{2\lambda'}{r} + \frac{(\lambda')^2}{2} \right],$$

(2a)

$$T^r_r = \frac{1}{8\pi G} e^{-2\lambda} \left[ (\lambda')^2 + 2\lambda' \nu' + \frac{2}{r} (\lambda' + \nu') \right],$$

(2b)

$$T^\theta_\theta = T^\phi_\phi = \frac{1}{8\pi G} e^{-2\lambda} \left[ \lambda'' + \nu'' + (\nu')^2 + \frac{1}{r} (\lambda' + \nu') \right],$$

(2c)

where primes indicate differentiation with respect to $r$.

With respect to the orthonormal tetrad or comoving observer $e_{(a)}^b = \{U^b, X^b, Y^b, Z^b\}$, where

$$U^a = \frac{1}{\sqrt{-g_{00}}} \delta^a_0 = e^{-\nu} \delta^a_0,
X^a = \frac{1}{\sqrt{|g_{11}|}} \delta^a_1 = e^{-\lambda} \delta^a_1,$$

(3a)

$$Y^a = \frac{1}{\sqrt{g_{22}}} \delta^a_2 = \frac{1}{r} e^{-\lambda} \delta^a_2,
Z^a = \frac{1}{\sqrt{|g_{33}|}} \delta^a_3 = \frac{1}{r \sin \theta} e^{-\lambda} \delta^a_3,$$

(3b)

the energy density is $\rho = -T^0_0$, and the stresses (pressure or tensions) are $p_i = T^i_i$.

By setting

$$e^{2\lambda} = \left(1 - \frac{\phi}{2}\right)^4$$

(4)
we get the following Poisson type non-linear equation for the energy density

$$\rho = \frac{\nabla^2 \phi}{4\pi G \left(1 - \frac{\phi}{2}\right)^5}. \quad (5)$$

Hence, the metric potential $\phi$ can be chosen of the fact that in Newtonian limit when $\phi \ll 1$, the relativistic energy density must reduce to Poisson’s equation

$$\nabla^2 \Phi = 4\pi G \rho_N. \quad (6)$$

This is possible by choosing $\phi = \Phi$. For example, for the point-mass potential $\phi = -\frac{MG}{r}$ obtains the external Schwarzschild solution in isotropic coordinates. With this choice of the metric potential $\phi$, the energy density takes the form

$$\rho = \frac{\rho_N}{\left(1 - \frac{\Phi}{2}\right)^5}. \quad (7)$$

To obtain the other metric function $\nu$ an additional assumption must be imposed. In addition, in order to have physically meaningful sources the parameters of the solutions must be chosen so that the energy conditions are satisfied. For a diagonal energy-momentum tensor, the weak energy condition reads $\rho \geq 0$, whereas the dominant energy condition states that $|\rho| \geq |p_i|$. The strong energy condition requires that $\rho_{eff} = \rho + p_r + p_\theta + p_\phi \geq 0$, where $\rho_{eff}$ is the “effective Newtonian density”.

Thus, this approach allows to construct different static spherically symmetric matter configurations in isotropic coordinates from given solutions of the Poisson’s equation.

### A. Stable circular orbits

An important quantity related to the circular motion of test particles around matter distributions is the circular speed (rotation profile) $v_c$. With respect to the comoving frame of reference $(3a)$ - $(3b)$, the 4-velocity of the particles $u$ has components

$$u^{(a)} = e^{(a)}_b u^b, \quad (8)$$

and the 3-velocity

$$v^{(i)} = \frac{u^{(i)}}{u^{(0)}} = \frac{e^{(i)}_a u^a}{e^{(0)}_a u^a}. \quad (9)$$

For circular orbits $\dot{r} = 0$ and $u = u^0(1,0,\omega)$, where $\omega = \frac{d\Omega}{dt}$ is the angular speed of the test particles. In this case the only nonvanishing velocity component is

$$[u^{(\Omega)}]^2 = v_c^2 = -\frac{g_{00} \omega^2}{g_{tt}}, \quad (10)$$

where $v_c$ represents the circular speed (rotation curves) of the particles measured by an inertial observer far from the source. The angular speed $\omega$ can be calculated considering the geodesic motion of the particles. The Lagrangian density for a massive test particle is defined as

$$2\mathcal{L} = g_{ab} \dot{x}^a \dot{x}^b = -e^{2\nu} \dot{r}^2 + e^{2\lambda} (\dot{r}^2 + \dot{r}^2 \dot{\Omega}^2), \quad (11)$$

where the overdot denotes derivative with respect to the proper time $\tau$. Since the Lagrangian does not explicit depend on $t$ and $\Omega$, the Lagrange’s equations

$$\frac{d}{ds} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}^a} \right) - \frac{\partial \mathcal{L}}{\partial x^a} = 0 \quad (12)$$

yield the following constants of motion (generalized momenta)

$$E = -p_t/m = -\frac{\partial \mathcal{L}}{\partial t} = e^{2\nu} \dot{r}, \quad (13a)$$

$$L = p_\Omega/m = \frac{\partial \mathcal{L}}{\partial \Omega} = e^{2\lambda} \dot{r} \Omega, \quad (13b)$$
where \( E \) represents the relativistic specific energy and \( L \) the total specific angular momentum. Therefore, the Lagrangian can be written as
\[
2\mathcal{L} = e^{2\lambda}r^2 + e^{-2\lambda} \frac{L^2}{r^2} - e^{-2\nu}E^2.
\]  
(14)

Normalizing \( u^a \), that is requiring \( g_{ab}u^au^b = -1 \), we obtain
\[
e^{2\nu+2\lambda}r^2 + e^{2\nu} \left(1 + e^{-2\lambda} \frac{L^2}{r^2}\right) = E^2.
\]  
(15)

This expression allows to define an effective potential \( V_{\text{eff}} \) as
\[
V_{\text{eff}} = e^{2\nu} \left(1 + e^{-2\lambda} \frac{L^2}{r^2}\right).
\]  
(16)

For a circular motion have that
\[
E^2 = e^{2\nu} \left(1 + e^{-2\lambda} \frac{L^2}{r^2}\right).
\]  
(17)

Assuming the Lagrangian as \( \tilde{\mathcal{L}} = 2\mathcal{L} + 1 \) and using the condition of normalizing (15), the radial motion equation for circular orbits (extreme motion) reads
\[
\frac{dV_{\text{eff}}}{dr} = 0.
\]  
(18)

This condition implies
\[
\frac{L^2}{r^2} = \frac{r\nu_r e^{2\nu-2\lambda}E^2}{1 + r\lambda_r}.
\]  
(19)

From Eqs. (17) and (19) we find
\[
E^2 = \frac{e^{2\nu}}{1 + r\lambda_r},
\]  
(20a)
\[
L^2 = \frac{r^3\nu_r e^{2\lambda}}{1 + r(\lambda_r - \nu_r)}.
\]  
(20b)

By using the conserved quantities (13b) and the expression (19) we obtain
\[
\omega^2 = \left(\frac{d\Omega}{dt}\right)^2 = \frac{\dot{\Omega}^2}{L^2} = e^{4(\nu - \lambda)} \frac{L^2}{r^4} \frac{E^2}{r^4} = -\frac{g_{tt,r}}{g_{\Omega\Omega,r}}.
\]  
(21)

Thus, for motion of particles in stable circular orbit the circular speed is given by
\[
v_c^2 = \frac{r\nu_r}{1 + r\lambda_r},
\]  
(22)

and the angular momentum can be cast as
\[
L^2 = \frac{r^2 e^{2\lambda}v_c^2}{1 - v_c^2}.
\]  
(23)

In addition, in order to have stable circular orbits against radial perturbations the following condition must be satisfied
\[
\left.\frac{d^2V_{\text{eff}}}{dr^2}\right|_{\text{extr}} > 0,
\]  
(24)

or explicitly
\[
\nu_{,rr} + \nu_r \left(2\lambda_r - 2\nu_r + \frac{3}{r} - \frac{\lambda_r + r\lambda_{,rr}}{1 + r\lambda_r}\right) > 0.
\]  
(25)

In terms of the angular momentum the stability condition (25) reads
\[
\frac{dL^2}{dr} > 0,
\]  
(26)

which is an extension of the Rayleigh criteria of stability of a fluid in rest in a gravitational field [18, 19].
III. RELATIVISTIC PERFECT FLUID SPHERES

To a perfect fluid source,
\[ T^{ab} = (\rho + p)U^a U^b + pg^{ab}, \]
the assumption to determine the metric function \( \nu \) is the condition of pressure isotropy which in isotropic coordinates reads
\[ \lambda'' + \nu'' + (\nu')^2 - (\lambda')^2 - 2\lambda' \nu' - \frac{1}{r} (\lambda' + \nu') = 0 \]
This expression is a Riccati equation in either \( \lambda' \) or \( \nu' \). It can also be cast as
\[ L G_{xx} = 2 G L_{xx}, \quad L \equiv e^{-\lambda}, \quad G \equiv Le^{\nu}, \quad x \equiv r^2. \]

A. Harmonic oscillator type spheres

In Newtonian gravity the gravitational potential of a sphere of radius \( a \) and constant mass density \( \rho_N \) is
\[ \Phi = \begin{cases} -2\pi G \rho_N (a^2 - \frac{1}{3} r^2) & (r < a), \\ -\frac{4\pi G \rho_N a^3}{3r} & (r > a). \end{cases} \]
For \( r < a \) the potential corresponds to a harmonic oscillator potential and has been used to model extended dark matter haloes \[1\]. The circular speed is
\[ v_{cN} = \sqrt{\frac{4\pi G \rho_N}{3}} r. \]

Solving the condition of pressure isotropy \[29\], the interior solution is
\[ e^\nu = \left[ C_1 \left( \frac{r^2}{a^2} - \frac{3(b+2)}{b} \right) \right]^4 + C_2 \left( \frac{r^2}{a^2} - \frac{3(b+2)}{b} \right)^{-3} \left[ 1 + \frac{b}{2} \left( 1 - \frac{1}{3} \frac{r^2}{a^2} \right) \right]^2, \]
\[ e^\lambda = \left[ 1 + \frac{b}{2} \left( 1 - \frac{1}{3} \frac{r^2}{a^2} \right) \right]^2, \]
where \( b = 2\pi G a^2 \rho_N \) is the parameter that measures the strength of the gravitational field, and \( C_1 \) and \( C_2 \) are constants of integration which obtain by demanding the continuity of the metric functions and of its first derivatives (Darmois Conditions) at the boundary \( r = a \) \[20, 21\] between the above interior space-time and the exterior Schwarzschild metric. Then it follows
\[ C_1 = \frac{-9b^4(4b^2 + 12b + 27)}{112(3 + 2b)^2(b + 3)b^6}, \]
\[ C_2 = \frac{432(4b^3 + 10b^2 - 21b - 45)}{7b^4(3 + 2b)^2}. \]

The relativistic energy density is
\[ \rho = \frac{\rho_N}{\left[ 1 + \frac{b}{2} \left( 1 - \frac{1}{3} \frac{r^2}{a^2} \right) \right]^5}. \]

Since \( r^2/a^2 \leq 1 \) this quantity is always positive in accordance with the weak energy condition. Expressions for other physical quantities are very cumbersome and accordingly the analysis is better done graphically. In figure \[1\] we plot the energy density \( \tilde{\rho} = 2\pi G a^2 \rho \), the isotropic pressure \( \tilde{p} = 8\pi G p \), the relativistic and Newtonian circular speed \( v^2_c \) and \( v^2_{cN} \), and the specific angular momentum \( \tilde{h}^2 = h^2/a^2 \) for the relativistic analogue of a homogeneous sphere with gravitational parameter \( b = 0.2, 0.3, 0.4 \), as functions of \( \tilde{r} = r/a \). We see that for these values of parameters the energy conditions are all satisfied, the stresses are positive (pressure) and the circular orbits of test particles are stables against radial perturbations. We also observer that the relativistic effects increase everywhere the speed of particles and they become more important as we move away from the central region.
B. Plummer type spheres

A simple Newtonian potential-density pair is the Plummer’s model \[6\]
\[
\Phi = -\frac{GM}{\sqrt{r^2 + a^2}},
\]
\[
\rho_N = \left(\frac{3M}{4\pi a^3}\right) \left(1 + \frac{r^2}{a^2}\right)^{-\frac{5}{2}},
\]
where \(a\) is a non-zero constant with the dimension of length. The Plummer’s sphere is used to model star clusters, the central spherical nucleus of spiral galaxies [22, 23] and also as models of dark matter haloes [3]. The circular speed is
\[
v_{cN}^2 = \frac{GMr^2}{(r^2 + a^2)^{3/2}}.
\]

The relativistic version of this model is a particular case of the Buchdahl’s solutions [24]. Therefore, the metric functions are
\[
e^{2\nu} = \left(\frac{1 - \frac{GM}{2a\sqrt{1 + \tilde{r}^2}}}{1 + \frac{GM}{2a\sqrt{1 + \tilde{r}^2}}}\right)^2,
\]
\[
e^{2\lambda} = \left(1 + \frac{GM}{2a\sqrt{1 + \tilde{r}^2}}\right)^4.
\]

The main relativistic physical quantities associated with these structures are
\[
\tilde{\rho} = \frac{3b}{2\pi (\sqrt{1 + \tilde{r}^2} + b)^5},
\]
\[
\tilde{\rho} = \frac{b}{4\pi (\sqrt{1 + \tilde{r}^2} - b) (\sqrt{1 + \tilde{r}^2} + b)^5},
\]
\[
v_{c}^2 = \frac{2b\tilde{r}^2 \sqrt{1 + \tilde{r}^2}}{\left(\sqrt{1 + \tilde{r}^2} - b\right) \left[(1 + \tilde{r}^2)^{3/2} + b(1 - \tilde{r}^2)\right]},
\]
\[
h^2 = \frac{2b\tilde{r}^4 \left(\sqrt{1 + \tilde{r}^2} + b\right)^4}{(1 + \tilde{r}^2)^{3/2} \left[-4b\tilde{r}^2 \sqrt{1 + \tilde{r}^2} + b^2(\tilde{r}^2 - 1) + (1 + \tilde{r}^2)^2\right]}.
\]

where \(\tilde{\rho} = Ga^2\rho, \tilde{\rho} = (a^3/M)p, \tilde{r} = r/a\) and \(b = GM/(2a)\). The energy density always is positive in agreement with the weak energy condition and for \(b < 1\) we have pressure.

In figure 2 we graph, as functions of \(\tilde{r}\), the relativistic and Newtonian density profiles \(\tilde{\rho}\) and \(\tilde{\rho}_N = Ga^2\rho_N\), and the isotropic pressure \(\tilde{p}\) for Plummer type spheres with gravitational parameter \(b = 0.2, 0.4, 0.52\). We find that the energy density presents a maximum at \(r = 0\) and then decreases rapidly with \(r\). We also find that when the gravitational field is increased the relativistic density profile increases at the central region of the matter distribution and then decreases. In turn, the relativistic corrections increase everywhere the energy density.

In figure 3 we plot the relativistic and Newtonian rotation curves \(v_{c}^2\) and \(v_{cN}^2\), and the specific angular momentum \(h^2\), also as functions of \(\tilde{r}\). We see that both the gravitational field and relativistic effects increase the circular speed of particles and also that such corrections are more important in the regions around its maximum value and, as expected, for velocities comparable to the speed of light. Moreover, we observer that the increase in the gravitational field can make unstable the orbits of particles. Thus, the orbits with parameter \(b = 0.52\) are non-physical. For this value of the parameter the dominant energy condition is also satisfied.
C. Logarithmic potential type spheres

Dark matter haloes can be modeled in Newtonian theory with a logarithmic potential of the form \(25\)

\[
\Phi = \frac{1}{2} v_0^2 \ln\left(\frac{r^2 + a^2}{b}\right),
\]

(39)

where \(a\) and \(b\) are constants, and \(v_0\) is circular speed at large radii, also a constant. When \(a = 0\) this potential is often referred to as the singular isothermal sphere. The mass density distribution is

\[
\rho_N = \frac{v_0^2 (r^2 + 3a^2)}{4\pi G (r^2 + a^2)^2},
\]

(40)

and the circular speed in radius \(r\) is

\[
v_{cN} = \frac{v_0}{\sqrt{r^2 + a^2}}.
\]

(41)

This potential yields an asymptotically flat rotation curve.

Solving (29), a solution is

\[
e^\nu = C \left(r^2 + a^2\right) \left[1 - \frac{1}{4} \frac{v_0^2}{c^2} \ln \left(\frac{r^2 + a^2}{b}\right)\right]^6,
\]

(42a)

\[
e^\lambda = \left[1 - \frac{1}{4} \frac{v_0^2}{c^2} \ln \left(\frac{r^2 + a^2}{b}\right)\right]^2,
\]

(42b)

where \(C\) is an integration constant. The main relativistic physical quantities associated with the distribution of matter are

\[
\dot{\rho} = \frac{v_0^2 \left(r^2 + 3a^2\right)}{(r^2 + a^2)^2} \left[1 - \frac{1}{4} \frac{v_0^2}{c^2} \ln \left(\frac{r^2 + a^2}{b}\right)\right]^{-5},
\]

(43a)

\[
\dot{\rho} = \frac{2048 \left(71v_0^2 - 4\right)r^2 + \left(5v_0^2 - 4\right)a^2\right) \ln \left(\frac{r^2 + a^2}{b}\right) + \left(28672 v_0^4 - 49152 v_0^2 + 16384\right) r^2 - 32768 a^2 \left(v_0^2 - \frac{1}{2}\right)}{4096 \left(r^2 + a^2\right)^2 \left[1 - \frac{1}{4} \frac{v_0^2}{c^2} \ln \left(\frac{r^2 + a^2}{b}\right)\right]^6},
\]

(43b)

\[
v_c^2 = \frac{2r^2 \left[v_0^2 \ln \left(\frac{r^2 + a^2}{b}\right) + 6 \frac{v_0^2}{c^2} - 4\right]}{v_0^2 \left(r^2 + a^2\right) \ln \left(\frac{r^2 + a^2}{b}\right) + 4\left(v_0^2 - 1\right) r^2 - 4a^2},
\]

(43c)

\[
h^2 = \frac{r^4 \left[v_0^2 \ln \left(\frac{r^2 + a^2}{b}\right) - 4\right]^4 \left[v_0^2 \ln \left(\frac{r^2 + a^2}{b}\right) + 6 \frac{v_0^2}{c^2} - 4\right]}{128 \left(\frac{r^2 - a^2}{v_0^2}\right) \ln \left(\frac{r^2 + a^2}{b}\right) + \frac{r^2 + 2a^2}{2} + 4r^2 \left(1 - 2\frac{v_0^2}{c^2} - 4a^2\right)},
\]

(43d)

where \(\dot{\rho} = 4\pi G \rho\) and \(\dot{\rho} = 8\pi G p\). In order for the weak energy condition \(\rho \geq 0\) to be satisfied a cut-off radius \(r_c\) must be imposed such that \(r_c^2 + a^2 = b\). Again, the other relationships will be analyzed using a graphic method. In figure \(4(a)\) we graph, as functions of \(r\), the relativistic energy density \(\dot{\rho}\), the Newtonian density profile \(\rho_N\) and the isotropic pressure \(\rho\) for dark matter haloes constructed from a logarithmic seed potential with parameters \(v_0^2 = 0.458, a = 3\) and \(r_c = 10\). We observer that the energy density presents a maximum at the center of the distribution of matter, and then decreases rapidly with the radial distance \(r\) which permits define a cut-off radius \(r_c\) and so, in principle, consider the structure as a compact object. We also see that the relativistic effects decrease the density profile everywhere of the dark halo and they become more important in the central region of the distribution. At large \(r\) both density profiles (relativistic and Newtonian) coincide. For these values of the parameters we have pressure.

In figure \(4(b)\) we show the relativistic and Newtonian rotation curves \(v_c^2\) and \(v_{cN}^2\), and the specific angular momentum \(h^2 \times 10^{-4}\) for the same values of parameters, also as function of \(r\). In contrast to the energy density, we find that the relativistic effects increase everywhere the circular speed of the particles and they become more important as we move away from the central region. We also see that relativistic rotation curve is flattered after certain value of \(r\) as observational data indicate. The speed of the particles always is less than light speed (dominant energy condition). We find that for these values of parameters the motion of particles is stable against radial perturbations.
Another possibility to calculate the metric function $\nu$ is to assume the tangential speed of the particles $v_c$. In this case $\nu$ obtains by integrating (22) which can be cast as

$$\nu_r = \frac{v_r^2 (1 - \frac{\Phi}{2} - v_N^2)}{(1 - \frac{\Phi}{2})}. \quad (44)$$

A physically reasonable way to choose $v_c$ is by requiring that in the Newtonian limit when $\phi \ll 1$ it reduces to its observational value $v_N^2 = r \Phi, r$ and that it also satisfies everywhere the dominant energy condition. A simple expression for $v_c$ that satisfies such conditions and for which $\nu$ can be obtained analytically is

$$v_c^2 = \frac{v_N^2 (1 - \frac{\Phi}{2})^{\gamma-1}}{1 - \frac{\Phi}{2} - v_N^2}, \quad (45)$$

where $\gamma$ is a constant. Integrating (44), we obtain

$$\nu = \begin{cases} -2 \ln(1 - \frac{\Phi}{2}), \quad \gamma = 1, \\ -\frac{2(1 - \frac{\Phi}{2})^{\gamma-1}}{\gamma-1} + C, \quad \text{otherwise,} \end{cases} \quad (46)$$

where $C$ is an integration constant which chosen so that the solution is asymptotically flat. The case $\gamma = 1$ was studied in Ref. [14] and describes a Majumdar-Papapetrou type spacetime. These fields satisfy the energy conditions but have negative radial stresses. However, the average stresses are positive. Thus, the above solutions can be called generalized Majumdar-Papapetrou type fields.

As a simple application, we consider a Newtonian density profiles with a two-power law [10]

$$\rho_N = \frac{\rho_0}{(r/a)^\alpha (1 + r/a)^{\beta - \alpha}}, \quad (47)$$

where $\rho_0$, $a$ and $\alpha$ are constants. $\beta = 4$ corresponds to the Dehnen models [4] which provide reasonable models of the centers of elliptical galaxies, and have the models Hernquist $\alpha = 1$ [7] and Jaffe $\alpha = 2$ [26] as special cases. These potential-density pairs have been used as successful analytic models for elliptical galaxies and bulges of disk galaxies. Meanwhile, dark matter haloes are describe with $(\alpha, \beta) = (1, 3)$ and corresponds to the NFW model [8].

For the important cases of the Hernquist, Jaffe, and NFW models the gravitational potentials are

$$\Phi_{Her} = -v_0^2 \frac{1}{2(1 + r/a)}, \quad (48a)$$
$$\Phi_J = -v_0^2 \ln(1 + a/r), \quad (48b)$$
$$\Phi_{NFW} = -v_0^2 \ln(1 + r/a) \frac{1}{r/a}, \quad (48c)$$

$v_0^2 = 4\pi G\rho_0 a^2$ being the parameter that measures the strength of the gravitational field. These potentials satisfy our requirement that at large $r$ the metric function $\phi \ll 1$. In turns, the tangential velocities are

$$v_{N-Her} = \frac{1}{2} v_0^2 \left( \frac{r/a}{(1 + r/a)^2} \right), \quad (49a)$$
$$v_{N-J} = \frac{v_0^2}{1 + r/a}, \quad (49b)$$
$$v_{N-NFW}^2 = v_0^2 \frac{\ln(1 + r/a)}{r/a} - \frac{1}{1 + r/a}. \quad (49c)$$

We consider the case with $\gamma = 2$. It follows that $C = 2$ and in consequence $\nu = \Phi$. The main physical quantities
associated to the systems are: For Hernquist type fields

\[ \rho = \frac{256v_0^2(1 + \tilde{r})^2}{\pi G\tilde{r}(4\tilde{r} + v_0^2 + 4)^5}, \]  
\[ p_r = \frac{32v_0^2(1 + \tilde{r})\left[4 - v_0^2\right] + v_0^2 + 4}{\pi G\tilde{r}(4\tilde{r} + v_0^2 + 4)^6}, \]  
\[ p_\varphi = \frac{8v_0^2\left[8\tilde{r}^3 + (8v_0^2 + 32)v_0^2 + (v_0^2 + 12v_0^2 + 40)v_0^2 + 4v_0^2 + 16\right]}{\pi G\tilde{r}(4\tilde{r} + v_0^2 + 4)^6}, \]  
\[ v_c^2 = \frac{v_0^2\tilde{r}(4\tilde{r} + v_0^2 + 4)}{8\tilde{r}^3 + 2(12 - v_0^2)v_0^2 + 24\tilde{r} + 2v_0^2 + 8}, \]  
\[ L^2 = \frac{v_0^2(4\tilde{r} + v_0^2 + 4)\tilde{r}^3}{[8\tilde{r}^3 + (24 - 6v_0^2)v_0^2 + (24 - v_0^2 - 4v_0^2)v_0^2 + 8]\left(1 + \tilde{r}\right)^4}, \]  

where \( \tilde{r} = r/a \), for Jaffe type fields

\[ \rho = \frac{8v_0^2}{\pi G\tilde{r}^2(\tilde{r} + 1)^2[v_0^2\ln(1 + \frac{1}{\tilde{r}}) + 2]], \]  
\[ p_r = \frac{8v_0^2\left[\ln(1 + \frac{1}{\tilde{r}})\left(v_0^2\tilde{r}(\tilde{r} + 1)\ln(1 + \frac{1}{\tilde{r}}) + \tilde{r} + 1 - v_0^2\right)\right]}{\pi G\tilde{r}^2(\tilde{r} + 1)^2[v_0^2\ln(1 + \frac{1}{\tilde{r}}) + 2]^6}, \]  
\[ p_\varphi = \frac{4v_0^2\left[\ln(1 + \frac{1}{\tilde{r}})\left(v_0^2\tilde{r}(\tilde{r} + 1)\ln(1 + \frac{1}{\tilde{r}}) + \tilde{r} + 1 + v_0^2\right)\right]}{\pi G\tilde{r}^2(\tilde{r} + 1)^2[v_0^2\ln(1 + \frac{1}{\tilde{r}}) + 2]^6}, \]  
\[ v_c^2 = \frac{v_0^2\left[\ln(1 + \frac{1}{\tilde{r}}) + 2\right]^5}{v_0^2(\tilde{r} + 1)\ln(1 + \frac{1}{\tilde{r}}) + 2\tilde{r} + 2 - 2v_0^2}, \]  
\[ L^2 = \frac{v_0^2\tilde{r}^2\left[\ln(1 + \frac{1}{\tilde{r}}) + 2\right]}{16\left[v_0^2\ln(1 + \frac{1}{\tilde{r}}) + 2\tilde{r} + 2 - 4v_0^2\right]^6}, \]  

and for NFW type fields

\[ \rho = \frac{8v_0^2}{\pi G\tilde{r}^2(1 + \tilde{r})^2\left[v_0^2\ln(1 + \frac{1}{\tilde{r}}) + 2\right]^5}, \]  
\[ p_r = \frac{8v_0^2\tilde{r}^3\left[\ln\left(\frac{1 + \tilde{r}}{\tilde{r}}\right)\left(v_0^2 - \frac{1}{2}v_0^2(1 + \tilde{r})\ln\left(\frac{1 + \tilde{r}}{\tilde{r}}\right)\right) + 1\right]\left(1 + \tilde{r}\right)^{\ln(1 + \frac{1}{\tilde{r}})} - 1}{\pi G\tilde{r}^2(1 + \tilde{r})^2\left[v_0^2\ln(1 + \frac{1}{\tilde{r}}) + 2\right]^6}, \]  
\[ p_\varphi = \frac{4v_0^2\left[\frac{1}{2}v_0^2(1 + \tilde{r})^2\left(\frac{\ln(1 + \frac{1}{\tilde{r}})}{\tilde{r}}\right)^4 + 2\left(1 + \tilde{r}\right)\left(1 + \tilde{r} - \frac{2}{3}v_0^2\right)\left(\frac{\ln(1 + \frac{1}{\tilde{r}})}{\tilde{r}}\right)^3 + ((\tilde{r}^2 + 1)^2 + \frac{1}{2}v_0^4 - \frac{7}{2}v_0^2\tilde{r} - 4v_0^2)\left(\ln(1 + \frac{1}{\tilde{r}})\right)^2\right]}{\pi G\tilde{r}^2(1 + \tilde{r})^2\left[v_0^2\ln(1 + \frac{1}{\tilde{r}}) + 2\right]^6}, \]  
\[ v_c^2 = \frac{v_0^2\left[\left(1 + \tilde{r}\right)\ln(1 + \frac{1}{\tilde{r}}) - 1\right]\left(v_0^2\ln(1 + \frac{1}{\tilde{r}}) + 2\right)}{2\left(\tilde{r} + 1 + v_0^2 - \frac{1}{2}v_0^2(1 + \tilde{r})\ln(1 + \frac{1}{\tilde{r}})\right)^5}, \]  
\[ L^2 = \frac{v_0^2\tilde{r}^2\left[\left(1 + \tilde{r}\right)\ln(1 + \frac{1}{\tilde{r}}) - 1\right]\left(v_0^2\ln(1 + \frac{1}{\tilde{r}}) + 2\right)^5}{32\left(\tilde{r} + 1 + 2v_0^2 - \frac{1}{2}v_0^2(1 + \tilde{r})\left(\ln(1 + \frac{1}{\tilde{r}})\right)^2 - \frac{1}{2}v_0^2\left(\tilde{r} + 1 - \frac{1}{2}v_0^2\right)\ln(1 + \frac{1}{\tilde{r}})\right).} \]  

In figures 5 - 2 we have plotted, as functions of \( \tilde{r} \), the relativistic energy density \( \rho \), the radial pressure \( p_r \), the azimuthal pressure \( p_\varphi \), the rotation curves \( v_c^2 \), and the specific angular momentum \( L^2 \) for Hernquist, Jaffe and NFW type spheres with \( G = 1 \) and parameters \( v_0^2 = 1, 1.5, 2, v_0^2 = 0.4, 0.7, 0.82 \) and \( v_0^2 = 1, 1.3, 1.6 \), respectively. In all models, the energy density always is a positive quantity and, as in the Newtonian case, presents a central cusp and then it decreases rapidly with the radius. The main stresses are also positive, that is we have pressure. The speed of
the particles always is less than light speed in according to the dominant energy condition. We also observer in all models that when the gravitational field is increased the rotation curves also increase. In all cases, we find stables circular orbits against radial perturbations of test particles moving around such structures. However, the increasing the gravitational field can make the orbits unstable. For other values of \( \gamma > 2 \) we find the same behavior. However, the solutions with \( \gamma < 1 \) satisfy also the energy conditions but yield negative stresses (tension). Therefore, the physically significant solutions are when \( \gamma \geq 1 \).

V. CONCLUSIONS

In this work, we have construct anisotropic and perfect fluid sources for static spherically symmetric fields in isotropic coordinates from given solutions of the Poisson’s equation. For perfect fluid spheres, the method was illustrated with three simple examples based in the potential-density pairs corresponding to a harmonic oscillator, the well-known Plummer model and a massive spherical dark matter halo model with a logarithm potential. For anisotropic spheres, the tangential speed profiles was also assumed by requiring that in the Newtonian limit it reduces to its observational value and that it also satisfies everywhere the dominant energy condition. As an application, an analytical family of anisotropic spheres was considered and illustrated with Hernquist, Jaffe and NFW type sphere models in the case \( \gamma = 2 \). These fields can be called generalized Majumdar-Papapetrou type solutions.

In all models considered, the energy conditions are satisfied and also found that the main stresses are positive, that is we have pressure. Furthermore, in all the cases were found stable circular orbits, but for Plummer type perfect fluid fields and the models of anisotropic spheres was observed that the increase in the gravitational field can make unstable against radial perturbations the motion of the particles.

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FIG. 1. (a) The relativistic energy density $\tilde{\rho}$, (b) the isotropic pressure $\tilde{p}$, (c) the relativistic and Newtonian circular speed $v^2_c$ (top curves) and $v^2_{cN}$ (bottom curves), and (d) the specific angular momentum $\tilde{h}^2$ for the relativistic analogue of a homogeneous sphere with parameters $b = 0.2$ (solid curves), $0.3$ (dashed curves), $0.4$ (dotted curves), as functions of $\tilde{r}$. 
FIG. 2. The relativistic and Newtonian density profiles (a) $\tilde{\rho}$, (b) $\tilde{\rho}_N$ and (c) the isotropic pressure $\tilde{p}$ for Plumer type spheres with parameter $b = 0.2$ (solid curves), $0.4$ (dashed curves), $0.52$ (dotted curves), as functions of $\tilde{r}$. 
FIG. 3. (a) The relativistic and Newtonian rotation curves $v_\text{c}^2$ (solid curves) and $v_\text{cN}^2$ (dashed curves) for Plumer type spheres with parameter $b = 0.2$ (bottom curves), 0.4, 0.52 (top curves), as functions of $\tilde{r}$. (b) the specific angular momentum $h^2$ with parameter $b = 0.2$ (bottom curve), 0.4, 0.52 (top curve), also as functions of $\tilde{r}$.

FIG. 4. (a) The relativistic energy density $\tilde{\rho}$ (solid curve), Newtonian density distribution $\tilde{\rho}_N$ (dashed curve), the isotropic pressure $\tilde{p}$ (dotted curve), (b) the relativistic circular speed $v_\text{c}^2$ (solid curve), Newtonian rotation curve $v_\text{cN}^2$ (dashed curve) and the specific angular momentum $h^2 \times 10^{-4}$ (dotted curve), for logarithmic potential type dark matter haloes with parameters $v_0^2 = 0.458$, $a = 3$ and $r_c = 10$, as functions of $r$. 
FIG. 5. (a) The relativistic energy density $\rho$, (b) the radial pressure $p_r$, (c) the azimuthal pressure $p_\varphi$, (d) the rotation curves $v_\varphi^2$, and (d) the specific angular momentum $L^2$ for Hernquist type fields with parameter $\gamma = 2$, $v_0^2 = 1$ (dot curves), 1.5 (dashed curves), and 2 (solid curves), as functions of $\tilde{r}$. 
FIG. 6. (a) The relativistic energy density $\rho$, (b) the radial pressure $p_r$, (c) the azimuthal pressure $p_\varphi$, (d) the rotation curves $v_0^2$, and (d) the specific angular momentum $L^2$ for Jaffe type fields with parameter $\gamma = 2$, $v_0^2 = 0.4$ (dot curves), 0.7 (dashed curves), and 0.82 (solid curves), as functions of $\tilde{r}$. 
FIG. 7. (a) The relativistic energy density $\rho$, (b) the radial pressure $p_r$, (c) the azimuthal pressure $p_\phi$, (d) the rotation curves $v_c^2$, and (d) the specific angular momentum $L^2$ for NFW type fields with parameter $\gamma = 2$, $v_0^2 = 1$ (dot curves), 1.3 (dashed curves), and 1.6 (solid curves), as functions of $\tilde{r}$. 