Mixed state entanglement
by efficient separation of quantum from classical correlations

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Entanglement is the key resource for quantum technologies and is at the root of exciting many-body phenomena. Nevertheless, quantifying the entanglement in a real-world quantum system is challenging when it interacts with its environment, as the latter mixes classical correlations with entanglement. Here, we devise an entanglement measure for such realistic open systems by filtering the entanglement spectrum of the mixed state. We first establish which spectral values encode entanglement and then develop a filtering algorithm that is efficiently realizable using a tensor network representation of the system’s density matrix. We showcase our scheme for spinless particles moving on a chain in presence of dephasing. Crucially, our approach distinguishes classical from quantum correlations for a broad range of systems and motivates efficient experimental entanglement quantification.

In quantum mechanics, particles can become far more correlated than classically possible. Such correlations, dubbed entanglement, are a key resource in the present-day quantum revolution. For example, entanglement is harvested in quantum information processing devices [1–4], error-correction schemes [5–9], quantum detectors that break sensitivity limits [10–12], or secure quantum communication protocols [13–15]. Entanglement can also lead to unique effects such as teleportation [16–19], the formation of strong correlations in many-body systems [20–40], and the high efficiency of light-harvesting processes [41–43].

The premise of quantum mechanics relies on having a wavefunction description for particles moving in a closed system. The wavefunction entails probability amplitudes for the state to be in different locations in the Hilbert space of the system. Commonly, the Hilbert space is very large, and entanglement has become a modern tool for compressing the required information needed to properly describe a quantum state [44, 45]. For example, in tensor network representations of quantum states, the Hilbert space is truncated such that only entangled regions are kept [46–48]. As such, measures for quantifying entanglement (e.g., entanglement entropy [1, 49]) were developed to assess the potential usefulness of quantum resources, as well as to compress their representation.

In reality, however, all quantum systems are open, i.e., they are coupled to an environment and become correlated with it. The direct result of such coupling is that the state of the system can become mixed, i.e., lose its entanglement. This is best described by considering the system’s density matrix, which is kin to a covariance matrix of the state’s probability amplitudes. As the density matrix describes both classical and quantum correlations of mixed states, it is challenging to distill the amount of entanglement in the system. Many mixed-state entanglement measures have been proposed, e.g., (Rényi) negativity [50–54], squashed entanglement [55], reflected entropy [56], number entanglement [57], or the entropy of the operator space entanglement spectrum (OSES) [58–61]. However, so far none of them fulfill all of the following criteria for acting as a useful entanglement measure: (i) entanglement monotone – not increase under local non-entangling operations; (ii) pure-state limit – map to a known measure when the system is not mixed with the environment; and (iii) efficient computation.

In this work, we identify which OSES values encode entanglement and establish that their sum is an entanglement measure. By construction, our measure fulfills criterion (i). By mapping it to the negativity in the pure-state limit, we validate criterion (ii). Furthermore, we present an efficient tensor network algorithm for the computation of our measure, thus fulfilling criterion (iii). Our insights explain how classical correlations corrupt the ability of the operator space entanglement entropy (OSEE) [58, 59] to work as an entanglement measure for mixed states. We showcase our measure by quantifying the entanglement in an open quantum system during Lindblad evolution, thus motivating its broad applicability.

The entanglement spectrum (ES) of a quantum state $|\psi\rangle$ is defined relative to a bipartition (cut) of the system into two parts $A$ and $B$ [see Figs. 1(a) and (b)] as the spectrum of the reduced state $\rho_A = \text{Tr}_B\{|\psi\rangle\langle\psi|\}$. Concurrently, the Schmidt decomposition of the state relative to this cut is

$$|\psi\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i} |i,\mu_i\rangle,$$

where $r \geq 1$ is the Schmidt rank, $\sqrt{\lambda_i} \geq 0$ are real-valued Schmidt values, and $|i,\mu_i\rangle = |i\rangle_A \otimes |\mu_i\rangle_B$ with suitable orthonormal sets of states for systems $A$ and $B$. The ES of state (1) is given by the squares $\lambda_i$ of its Schmidt values [62]. The corresponding von Neumann entropy, $S_{\mathcal{N}} \equiv -\text{Tr}(\rho_A \log \rho_A) = -\sum_i \lambda_i \log \lambda_i$, serves as an entanglement measure for pure states.

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Similarly to the pure case, we can define the OSES of a density matrix \( \rho \) relative to a bipartition of the system into two parts \( A \) and \( B \). The density matrix can be written relative to this bipartition as

\[
\rho = \sum_{i,j \in A} \sum_{\mu, \nu \in B} \rho_{i,\mu; j,\nu} |i, \mu \rangle \langle j, \nu |,
\]

where \( |i, \mu \rangle = |i \rangle_A \otimes |\mu \rangle_B \), and \( |i \rangle_A \) and \( |\mu \rangle_B \) are basis states of the two parts of the system. The density matrix is Hermitian, and hence the prefactors in Eq. (2) follow the relation \( \rho_{i,\mu; j,\nu} = \rho_{j,\nu; i,\mu}^* \). We define the vectorized density matrix as

\[
|\rho \rangle \equiv \sum_{i,j \in A} \sum_{\mu, \nu \in B} \rho_{i,\mu; j,\nu} |i, \mu; j, \nu \rangle,
\]

which is obtained by stacking the columns of the density matrix (2) into a column vector. The inner product over such column vectors in terms of their respective density operators is defined as \( \langle \langle \rho | \sigma \rangle \rangle \equiv \text{Tr} \rho^\dagger \sigma \). The OSES of \( \rho \) consists of the eigenvalues of the matrix [44]

\[
\mathcal{C} = \text{Tr}_B \{ |\rho \rangle \langle \rho | \} = \sum_{i,j,k,l \in A} \sum_{\mu,\nu \in B} \rho_{i,\mu; j,\nu} \rho_{k,\nu; l,\mu}^* \langle \langle i, \mu | j, \nu \rangle | k, \nu \rangle | l, \mu \rangle,
\]

where \( \text{Tr}_B \mathcal{O} = \sum_{\mu, \nu \in B} \langle \langle \mu, \nu | \mathcal{O} | \mu, \nu \rangle \rangle \) is the partial trace over subsystem \( B \). The matrix (4) is a positive operator that involves correlations up to fourth order in the state’s probability amplitudes, and we dub it the kurtosis matrix. As we show below, the OSES contains values encoding both classical correlations as well as entanglement, raising doubts concerning its naming convention. Interestingly, the sum over both classical and quantum OSES values equals to the purity of the system, i.e., \( \text{Tr} \mathcal{C} = \text{Tr} \{ |\rho \rangle \langle \rho | \} = \text{Tr} \rho^2 \equiv \mathcal{P}(\rho) \).

Ostensibly, we would like to employ the OSEE, \( S \equiv -\text{Tr} \{ \mathcal{C} \log C \} \), as an entanglement measure. Yet, it appears to be sensitive to both classical and quantum correlations [59, 63]. The latter is evident from our construction (4) using a counterexample: consider a pure state with a single excitation residing solely within subsystem \( A \), e.g., subsystem \( A \) is composed of states |1⟩ and |2⟩, whereas subsystem \( B \) of state |3⟩, see Fig. 1(b). We take the quantum state to be in an equal superposition, \( |\psi \rangle = (|1⟩ + |2⟩)/\sqrt{2} \). The corresponding OSES has a single nonvanishing value \( \Lambda_A \) [cf. Eq. (8) and discussion below]. Hence, for our pure system \( \Lambda_A = \text{Tr} \{ \mathcal{C} \} = \mathcal{P}(\rho) = 1 \). Correspondingly, \( S \equiv -\Lambda_A \log \Lambda_A = 0 \) as expected for a pure product state. We now locally couple subsystem \( A \) to a dephasing environment, i.e., no particles leak out, but the system decoheres into a mixed state \( \rho' \) after some time. As the particle remains in subsystem \( A \), we still have a single eigenvalue \( \Lambda'_A \) that corresponds to a reduced purity \( \mathcal{P}(\rho') < 1 \) of the system. We, thus, obtain that the entanglement entropy increases to \( S' = -\Lambda'_A \log \Lambda'_A > 0 \) even though the local operation on subsystem \( A \) cannot generate entanglement between subsystems \( A \) and \( B \). This is a first important observation of this work.

In Fig. 1(c), we show the outcome of our counterexample with increasing dephasing. The latter is obtained by mixing the pure state with the classical mixture of the particle being either in state |1⟩ or |2⟩, namely

\[
\rho_p = (1 - p) |\psi \rangle \langle \psi | + p \sigma ,
\]

with \( \sigma = (|1⟩ \langle 1 | + |2⟩ \langle 2 |)/2 \), see Fig. 1(b). The entanglement entropy increases with increasing weight \( p \) of the separable classical mixture, confirming the deficiency of the OSEE as an entanglement measure for mixed states. We identify that part of the OSES bears no entanglement information, e.g., \( \Lambda_A \) in our example. Hence, any entanglement measure should filter out such values, which may be challenging for a many-body system on a large Hilbert space.

For pure states |ψ⟩, however, the filtering is relatively straightforward: we can write the \( C \)-matrix (4) of a density matrix |ψ⟩⟨ψ| using the state’s Schmidt basis [cf. Eq. (1)]

\[
C = \sum_i \Lambda_i^2 |i, i⟩ \langle i, i | + \sum_{j \neq i} \lambda_i \lambda_j |i, j⟩ \langle j, i |.
\]

In this basis, the \( C \)-matrix is diagonal and its spectrum consists of \( r \) eigenvalues of type \( \lambda_i^2 \) and \( r(r - 1)/2 \) two-fold degenerate eigenvalues of type \( \lambda_i \lambda_j \). Thus, in this pure limit, the OSES of the density matrix is equivalent to the outer product of the ES of the state [44, 64]. We
FIG. 2. (a) Examples of configurations of two particles on four sites with respect to a bipartition in the middle. Yellow shadings mark entanglement. (b) and (c) OSES corresponding to the configurations in (a) along a mixing interpolation [cf. Eq. (5)]. (1) marks an avoided crossing between spectral values of a classically correlated and an entangled $AB$ configuration. (d) MPO representation of a density matrix on 4 sites. (e) Negativity $N$, purity $\mathcal{P}$, and entanglement measures $\mathcal{M}_1$ and $\mathcal{M}_2$ for the spectrum in (b) and (c).

can also verify using Eq. (6) that only a single nonvanishing $\lambda_i^2$ value appears in the pure limit of the example of Fig. 1(b).

Now, recall that a pure state (1) is entangled if and only if its Schmidt rank is $r > 1$. As the second sum in Eq. (6) vanishes for $r = 1$ and is finite and positive for $r > 1$, we propose the sum over these eigenvalues as our entanglement measure,

$$\mathcal{M} := \sum_{j \neq i} \lambda_i \lambda_j . \quad (7)$$

In other words, we define our entanglement measure for pure states as the sum over inherently degenerate eigenvalues of the matrix $\mathcal{C}$ and filter out the $\lambda_i^2$ values.

Crucially, our measure (7) is closely related to the negativity of the state, which is defined as the absolute value of the sum over all negative eigenvalues of the partial transpose $\rho^{T_B}$ of the density matrix [50]. Using the Schmidt basis (1), the negativity reads $\mathcal{N} = \frac{1}{2} \sum_{j \neq i} \sqrt{\lambda_j / \lambda_i}$. Thus, our Eq. (7) inherits the desirable properties of the negativity as an entanglement measure for pure states and simultaneously defines a new way to calculate it. This is a second important observation of this work. Furthermore, the definition of the measure (7) via the matrix $\mathcal{C}$ lends a natural extension to open systems. The remaining challenge involves the selection of $\mathcal{C}$-matrix eigenvalues that should be included in the entanglement measure for mixed states.

It is illuminating to start with a particle-number conserving system, e.g., with $N$ spinless particles. We write the density matrix (2) using basis states $|i_n, \mu_n\rangle$ with $0 \leq n \leq N$ particles in subsystem $A$ and $N - n$ particles in subsystem $B$. The $\mathcal{C}$-matrix in this basis is block diagonal,

$$\mathcal{C} = \bigoplus_{n,n'} \mathcal{C}_{n,n'} , \quad (8)$$

where $\mathcal{C}_{n,n'} = \sum_{i_n,j_n;k_n,l_n} c_{i_n,j_n;k_n,l_n} |i_n,j_n\rangle \langle k_n,l_n|$, with coefficients $c_{i_n,j_n;k_n,l_n} = \sum_{\mu_n,\nu_n} \rho_{i_n,\mu_n;j_n,\nu_n} \rho_{k_n,\mu_n;l_n,\nu_n}$. A graphical derivation of this block diagonal form is shown in Fig. 1(d).

Importantly, we can interpret the blocks in terms of configurations of the $N$ particles with respect to the bipartition: the block $\mathcal{C}_{n,n'}$ contains all information on $\min(n,n')$ particles that are fully in subsystem $A$, $N - \max(n,n')$ particles that are fully in subsystem $B$, and $\max(n,n') - \min(n,n')$ particles that are coherently distributed across the cut.

In the following, we compare the eigenvalues of the blocks $\mathcal{C}_{n,n'}$ with the pure case limit (6) to define an entanglement measure for mixed states. We accompany our discussion with an example of a chain with $N = 2$ spinless particles residing on 4 sites, see Fig. 2. We begin with the blocks $\mathcal{C}_{0,0}$ and $\mathcal{C}_{N,N}$, which are rank 1 and have eigenvalues $\Lambda_{0,0} = \sum_{\mu,\nu} |\rho_{0,\mu,0,\nu}|^2$ and $\Lambda_{N,N} = \sum_{i,j} |\rho_{i,0,j,0}|^2$, corresponding respectively to the scenario where all $N$ particles reside solely in subsystem $B$ or $A$, see Fig. 2(a).

Therefore, these eigenvalues do not contain any information about cross-boundary coherence and should not contribute to our entanglement measure. Indeed, these values are generally non-degenerate, and we identify that they reduce to eigenvalues of type $\lambda_i^2$ in the pure case limit, cf. Eq. (6). Furthermore, such values do not vanish for a fully mixed state [see Fig. 2(b)], justifying our choice to not include them in our entanglement measure (7).

The blocks $\mathcal{C}_{n,n'}$ for $n \neq n'$ describe a scenario where at least one of the particles is in a coherent cross-boundary state and clearly encode entanglement. The blocks $\mathcal{C}_{n,n'}$ and $\mathcal{C}_{n',n}$ generate the same eigenvalues as they are related via a unitary transformation, $\mathcal{C}_{n,n'} = \mathcal{U} \mathcal{C}_{n',n} \mathcal{U}^{-1}$, with the permutation $U[i,j] = |j',i\rangle$. Hence, such coherent eigenvalues of $\mathcal{C}$ are inherently degenerate, and must map to the eigenvalues of type $\lambda_i \lambda_j$ in the pure case limit (6). Therefore, we include these eigenvalues in our extension of the entanglement measure to the mixed case, i.e., by summing over them [cf. Eq. (7)]

$$\text{Tr} \{\mathcal{C}_{n \neq n'}\} = \sum_{n \neq n'} \sum_{\mu_n,\nu_n} |\rho_{i_n,\mu_n,j_n,\nu_n}|^2 . \quad (9)$$

Crucially, the sum (9) contains only off-diagonal elements of the density matrix, which vanish in the fully decohered case, in conjunction with the fact that the fully decohered state contains no quantum correlations, see Figs. 2(a) and (b).
The remaining blocks with \( n = n' \notin [0, n] \) describe both classical correlations and entanglement. This is evident from their pure limit, where they exhibit both non-degenerate and degenerate eigenvalues, of which only the latter contribute to the measure (7), see Fig. 2(c). Note that when the state is mixed, the degeneracy may be lifted via avoided crossings between classical-correlation and entanglement values. Conversely, the entanglement values must vanish in the fully mixed limit. This behaviour allows us to identify the values that we include into our measure, namely, those that start degenerate in the pure limit and would cross spectral gaps to reach zero in the fully mixed limit. Identifying the spectral values of the \( C \)-matrix that contribute to an entanglement measure of mixed states with a proper pure-state limit is the first main result of this work.

The remaining challenge involves the efficient spectral filtering of a density matrix \( \rho \) describing mixed states of realistic open quantum systems. We propose the following algorithm that can be efficiently realized using a matrix product density operator (MPDO) decomposition of \( \rho \) [65] [cf. Fig. 2(d)]: (a) obtain the OSES of \( \rho \) using the MPDO; (b) interpolate the given distribution to a pure-state limit [66] [cf. Eq. (5)], and sum over the values that smoothly end up degenerate under this homotopy to obtain \( \mathcal{M}_1 \); and (c) similarly interpolate [also using Eq. (5)] to the fully mixed limit, and sum over the values that vanish under this second homotopy to obtain \( \mathcal{M}_2 \). Importantly, the interpolation is efficient in the MPDO representation because the bond dimension of a sum of MPDOs is bounded by the sum of the individual bond dimensions. Note, however, that due to the avoided crossings in the eigenvalues of blocks with \( n = n' \notin [0, n] \) [cf. Fig. 2(e)], the two homotopies do not necessarily identify the same values, \( \mathcal{M}_1 \neq \mathcal{M}_2 \), see Fig. 2(e): Using \( \mathcal{M}_1 \) we find the correct entanglement measure value in the pure limit, but overestimate the entanglement in the fully mixed limit, whereas using \( \mathcal{M}_2 \) we correctly identify zero entanglement in the fully mixed limit, but underestimate the entanglement of the pure state. This discrepancy can be fixed by weighting the filtered OSES values by the overlaps between the eigenstates of \( C \) before and after avoided crossings. Importantly, whereas our analytical separation of the OSES values above relies on a fixed particle number, our filtering algorithm does not suffer from this restriction. The algorithm is the second main result of this work.

We turn now to showcase our algorithm by measuring the entanglement in a realistic open system scenario. We consider a system of \( N = 2 \) spinless particles moving on a 1D chain with 12 sites in the presence of dephasing. For this system, we can readily obtain an exact MPDO representation [67]. For larger systems requiring a truncated MPDO description, our algorithm will yield a tight lower bound on the entanglement. The time evolution follows a Lindblad master equation

\[
\partial_t \rho = -i[H, \rho] + \gamma \sum_i (2n_i \rho n_i - \{n_i, \rho\}) ,
\]  

with a hopping Hamiltonian \( H = J \sum_i c_i^\dagger c_{i+1} + h.c. \), and local density operators \( n_i = c_i^\dagger c_i \). The parameters \( J \) and \( \gamma \) are the hopping amplitude and the dephasing coupling rate to local baths, respectively.

We initialise the system in a product state of one particle on site 1 and the other on site 11, and evolve Eq. (10) using time evolving block decimation (TEBD) with the JULIA ITensors package [68]. In Fig. 3(a), we present the resulting particle density of the two particles. The OSES associated with a half-chain bipartition is directly obtained from the MPDO representation throughout the time evolution [48], see Fig. 3(b). As expected for a product state, the OSES at \( t = 0 \) consists of a single value only, describing one particle in each of the subsystems to the left and right of the cut. Along the time evolution, the particles delocalize across the cut, leading to entanglement, whereas the Lindblad terms decrease the purity of their distribution, see Fig. 3(c). Using our filtering algorithm at different time steps [see inset of Fig. 3(b)], we can extract our entanglement measure from the OSES, see Fig. 3(c). Moreover, we also obtain the classical correlations in the system by subtracting our measure from the purity. Note that by repeated entanglement analysis during the time evolution, we avoid large discrepancies.
between $M_1$ and $M_2$ [see minor jumps in the entanglement measure and the classical correlations in Fig. 3], and for simplicity do not include the weighted sum with cross-gap overlap tracking.

Our algorithm can be applied to any system with a suitable low-rank MPDO representation, such as infinite size dissipative quantum chains [69], open many-body localized systems [70–72], strongly thermalizing systems [73], exciton dynamics [74], the quantum Heisenberg magnet [75], and temporal entanglement in many-body Floquet dynamics [76, 77]. Moreover, for fixed particle number, our analysis of the $C$-matrix blocks allows calculation of the MPDO rank and explains which correlations dominate it [67]. Experimentally, our measure can be obtained by estimating the purity of the mixed state [78, 79] and subtracting the values encoding classical correlations; the latter are constructed out of local density measurements. Our results facilitate the study of entanglement in contemporary noisy intermediate-scale quantum era systems [80, 81] and motivate further OSES-based measures and complexity estimates.

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See Supplementary Material for more details.

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Supplemental Material for
Mixed state entanglement
by efficient separation of quantum from classical correlations

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1. RANK OF THE $\mathcal{C}$-MATRIX

In the main text, we introduce the $\mathcal{C}$-matrix whose eigenvalues define the operator-space entanglement spectrum (OSES). The matrix has a block diagonal structure for the spinless particle-number conserving case. Here, we discuss the rank of these blocks in more detail.

In a matrix product density operator (MPDO) representation of the mixed state density matrix \cite{Verstraete2004}, the rank of the $\mathcal{C}$-matrix defines the necessary bond dimension $\chi_{\text{exact}}$ of the MPDO to represent the state exactly. We can calculate the rank by summing over the ranks of the individual blocks of the $\mathcal{C}$-matrix.

The blocks are given by

$$
\mathcal{C}_{n,n'} = \sum_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}} c_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}} |i_n, j_{n'}\rangle \langle k_n, l_{n'}|,
$$

(S1)

with coefficients

$$
c_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}} = \sum_{\mu_n, \nu_{n'}} \rho_{\mathbf{i},\mu_n; \mathbf{j},\nu_{n'}} \rho_{\mathbf{k},\mu_n; \mathbf{l},\nu_{n'}}.
$$

(S2)

Reordering the summation, we can write the block \(S1\) as

$$
\mathcal{C}_{n,n'} = \sum_{\mu_n, \nu_{n'}} v_{\mu_n, \nu_{n'}} v_{\mu_n, \nu_{n'}}^\dagger,
$$

(S3)

with vectors

$$
v_{\mu_n, \nu_{n'}} = \sum_{\mathbf{i}, \mathbf{n}} \rho_{\mathbf{i}, \mu_n; \mathbf{j}, \nu_{n'}} |i_n, j_{n'}\rangle.
$$

(S4)

The size of the vector \(S4\) and thus the size of the block \(S1\) is determined by the number of possible supervectors \(|i_n, j_{n'}\rangle\). If we denote the size of subsystem $A$ by $L_A$, then there are \(L_A^n\) possible states \(|i_n\rangle\) of $n$ particles in subsystem $A$. It follows that the size of the block \(S1\) is given by \(L_A^{2n}\). At the same time, the form \(S3\) reveals that the block \(\mathcal{C}_{n,n'}\) is a sum over rank-1 matrices \(v_{\mu_n, \nu_{n'}} v_{\mu_n, \nu_{n'}}^\dagger\). Similarly, there are \(L_B^{N-n}\) of these matrices, with $L_B$ the size of subsystem $B$. As the rank of a matrix sum is bounded by the sum of the ranks of the summands, we find that the block \(\mathcal{C}_{n,n'}\) has a maximal rank

$$
\text{rank} \mathcal{C}_{n,n'} = \min \left( \left( \begin{array}{c} L_A \\ n \end{array} \right), \left( \begin{array}{c} L_A \\ n' \end{array} \right), \left( \begin{array}{c} L_B \\ N-n \end{array} \right), \left( \begin{array}{c} L_B \\ N-n' \end{array} \right) \right),
$$

(S5)

with $L_A$ ($L_B$) the size of subsystem $A$ ($B$). It follows that the maximal bond dimension for $N$ particles is given by

$$
\chi_{\text{max}} = \sum_{n,n'=0}^N \min \left( \left( \begin{array}{c} L_A \\ n \end{array} \right), \left( \begin{array}{c} L_A \\ n' \end{array} \right), \left( \begin{array}{c} L_B \\ N-n \end{array} \right), \left( \begin{array}{c} L_B \\ N-n' \end{array} \right) \right).
$$

(S6)

The maximal rank \(S6\) for $N$ particles in a system of size $L$ scales as $\chi_{\text{max}} \propto L^N$ (stemming from the blocks with $n + n' = N$). Thus, if the number of particles is fixed, the maximal rank has a power-law scaling with system size, as opposed to an exponential scaling as for the general setting. This permits simulation of very large system sizes when the particle number is fixed. Table S1 shows the maximal ranks per block for the example of $N = 2$ particles. As expected, the maximal bond dimension scales as $L^2$.

| block \((n, n')\) | degeneracy | maximal rank |
|------------------|-------------|--------------|
| AA \((2, 2)\)    | 1           | 1            |
| AC \((2, 1)\)    | 2           | $\frac{L}{2}$|
| AB \((1, 1)\)    | 1           | $\left(\frac{L}{2}\right)^2$|
| BC \((0, 1)\)    | 2           | $\frac{L}{2}$|
| BB \((0, 0)\)    | 1           | 1            |
| CC \((2, 0)\)    | 2           | $\frac{L}{2} \left(\frac{L}{2} - 1\right)$|

Table S1. Maximal ranks of the blocks of the $\mathcal{C}$-matrix for $N = 2$ particles on a chain of length $L$ and bipartition in the middle. The maximal bond dimension is obtained as the sum over the maximal ranks times the corresponding degeneracies.

Several factors can reduce the maximal bond dimension \(S6\). In particular, under the assumption of local decoherence, cf. Ref. \cite{vanNieuwenburg2018}, the blocks $\mathcal{C}_{N,0}$ and $\mathcal{C}_{0,N}$ become rank 1.

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