Application of Method of Special Series for Representation of Solutions Describing Stationary Gas Flow

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Abstract. A method of special series with recurrently computed coefficients is considered for solutions of nonlinear partial differential equations. To represent solutions of equation of velocity potential for stationary gas flows the series by the powers of special functions, which provide the recurrent coefficients determining from a sequence of linear differential equations. This makes possible to investigate the convergence of the series, and the functional arbitrariness allows to solve problem of internal flows in a nozzles.

1. Introduction

At present, a prospective direction to obtain approximate solutions of nonlinear partial differential equations is a combination of numerical and analytical methods. This paper considers one of the areas of creation of effective methods for constructing solutions of nonlinear equations in the form of special series with recurrently computed coefficients. This method is developed in the papers [1, 2, 3, 4, 5]. The method is an expansion of the solution of the equation in powers of specially choosed functions that allow to find the coefficients recurrently.

Definition 1. If we will find a solution of an initial problem for nonlinear equation in the form of series by the powers of a special function, which allows the series coefficients to be calculated recurrently, then this function we shall call a basic function (BF).

The systematic application of the method of special series investigation of the basic functions were started with [2] in consideration of the following Cauchy problem:

\[ u_t = F(t, u, \frac{\partial u}{\partial x}, \cdots, \frac{\partial^m u}{\partial x^m}), \quad u(0, x) = u^0(x), \]

where \( F \) is a polynomial of the unknown function \( u(t, x) \) and its derivatives with respect to the space variable.

The proposed series constructions used only the mentioned property of the function \( F \) and may contain a functional arbitrariness that was used to prove the solvability of the initial boundary value problem for the generalized Korteweg-de Vries equation [6, 7].

In some cases it is possible to prove global convergence of the constructed series, including in unlimited domains, where the use of numerical methods meets essential difficulties [5]. In some cases it is possible to exactly satisfy zero boundary conditions by the choice of the basic
functions (for example, for description of nonlinear vibrating string with fixed end points [1], or membrane with fixed edges [8]).

The solutions in the form of special convergent series are practically important for nonlinear evolution equations such as nonlinear equation of filtration [9, 10], nonlinear wave equations [3, 4] and Lin-Reissner-Tsien equation which describes unsteady transonic flows under the transonic approximation [11, 12, 13]. Convergence of such series is proved, as a rule, in an unbounded domain.

Constructed series converge fast, that allows to use it for testing of numerical methods and for creation new numerical-analytical methods. For example, by special series a nonstationary transonic flow around a wedge was calculated, which describes the transition from non-stationary gas flow to the steady, and it has been shown analytically that this transition is exponential [11]. From a formal point of view, a characteristic feature of the special series, is independency of its form from the type and construction of the equations to be solved (usually, the type of the equation is closely related to study of convergence of the series). Convergence of special series is proved for a broad class of non-linear equations and systems of continuum mechanics. Another area in the theory of the method of special series are constructions of series, taking into account the specificity of the considered equations [14]. Accounting for the specifics of the considered nonlinear equations leads to expansion of classes of basic functions that allow to solve initial boundary value problems and investigate equations with peculiarities.

In some cases special series are transformed into finite sums and then generate exact solutions [15]. For real problems, in which satisfaction of various boundary conditions is required it is difficult to manage using of special series, or exact solutions, for example, for long-term problems of forecasting of degradation of permafrost [16, 17, 18]. However, the presence of an exact solution allows sometimes to simplify original mathematical models and to choose numerical methods, as it was done for a geothermal problems [19, 18].

In this paper new results related with with construction and study of special series to the representation of solutions of nonlinear partial differential equations, which have peculiarities.

2. General scheme of construction of series with recurrently calculated coefficients in powers of consistent functions

Let find a solution of equation (1) in the form of the series

$$u(t, x) = \sum_{n=0}^{\infty} u_n(t) P^n(t, x)$$

(2)

by the powers of basic function $P(t, x)$ satisfying the overdetermined system

$$P_x = A(t, P), \quad P_t = B(t, P)$$

(3)

with functions $A(t, P)$ and $B(t, P)$ being analytic respect to $P$ and such that $A(t, 0) \equiv 0$ and $B(t, 0) \equiv 0$. It was shown that if the initial conditions are written in the form

$$u^0(x) = \sum_{n=0}^{\infty} u_n^0 P^n(0, x),$$

(4)

then substituting series (2) into equation (1), collecting similar terms, and taking into account relations (3), we obtain the sequence of first-order ordinary differential equations for the coefficients $u_n(t)$

$$u'_n = F_n(t, u_n, \ldots, u_0), \quad u_n(0) = u_n^0, \quad n = 1, 2, \ldots$$

(5)
where the right-hand sides $F_n$ include only $u_j$ with $j \leq n$ and the coefficients $u_n$ may be linearly contained only. Thus, the coefficients of (2) are calculated recurrently. Examples of basic functions (3) are presented, for example, in [2].

**Definition 2.** Let the functions satisfying relations (3) be called universal basic functions. Let consider nonlinear equation

$$L_t u = F\left(t, x, \frac{\partial u}{\partial x}, \cdots, \frac{\partial^m u}{\partial x^m}\right),$$

(6)

where

$$L_t = d_1(t) \frac{\partial}{\partial t} + \cdots + d_l(t) \frac{\partial^l}{\partial^l t}, \quad d_k(t) \in C[0, \infty), \quad k = 1, \cdots, l, \quad d_l(t) \neq 0, \quad t \geq 0,$$

and $F_1$ is a nonlinear function, which form will be defined in the following. Note, that function $F_1$ may depend on the spatial variable. The solution of equation (6) will be find in the form of

$$u(x, t) = \sum_{n=0}^{\infty} u_n(t) R^n(x, t).$$

(7)

Let function $R(x, t)$ satisfy the differential relations

$$\left(\frac{\partial R}{\partial x}\right)^2 = \sum_{k=k_0}^{\infty} a_k(t) R^k,$$

$$\frac{\partial R}{\partial t} = \sum_{k=1}^{\infty} b_k(t) R^k.$$  

(8)

Here $k_0 \geq 2$ and functions $a_k(t), b_k(t) \in C^1[0, \infty)$ such that series in the right-hand sides of relations (8) are assumed to be absolutely convergent for $|R| \leq R_0, \ R_0 > 0 \ and \ t \geq 0$. It is easy to show that the system (8) is consistent, if the functions $a_k(t)$ and $b_k(t)$ are finid from differential equations

$$a_k' = \sum_{l=0}^{k} a_l b_n(2n - l).$$

**Definition 3.** Functions, which satisfy (8), we will call consistent basic functions, and the special series by the powers of these functions will call consistent series.

If a consistent basic function (8) depends from $x$ only, then instead of relation (8) the following differential equation should be valid

$$\left(\frac{dR}{dx}\right)^2 = \sum_{k=k_0}^{\infty} c_k R^k, \quad c_k = \text{const.}$$

(9)

Examples of consistent basic functions (9) are

$$R_1(x) = \frac{1}{x^2 + A}, \quad (R_1')^2 = 4R_3^3 - 4AR_4^4 \quad A = \text{const};$$

$$R_2(x) = a_2^2 a_3^{-2} \left(tg^2 0.5a_2 x + 1\right), \quad (R_2')^2 = -a_2^2 R_2^3 + a_3^2 R_2^3, \quad a_2, a_3 = \text{const};$$

$$R_3(x) = \frac{1}{\cos^2 x + A}, \quad (R_3')^2 = -4R_3^3 + (4 + 8A)R_3^3 + 4A(A - 1)R_3^4;$$

$$R_4(x) = \frac{4}{\left(e^x + e^{-x}\right)^2}, \quad (R_4')^2 = 8R_4^2 - R_4^3.$$  

Examples of consistent basic functions with functional arbitrariness are $R_1(x, t) = (x^2 + f(t))^{-1}, \ R_3(x, t) = (\cos^2 x + f(t))^{-1}$, for which differential relations (8) are valid.
We describe a class of equations that allow the construction of solutions in a special series in powers of both universal and consistent basic functions. Let consider a class of nonlinear equations

\[ L_t u = \sum_{j=0}^{\infty} A_{p_j}(t) u^{k_{ij}} \left( \frac{\partial u}{\partial x} \right)^{l_{ij}} \left( \frac{\partial^2 u}{\partial x^2} \right)^{r_{ij}} \cdots \left( \frac{\partial^{2m-1} u}{\partial x^{2m-1}} \right)^{k_{mj}} \left( \frac{\partial^{2m} u}{\partial x^{2m}} \right)^{l_{mj}}, \quad (10) \]

where \( A_{p_j}(t) \in C[0, \infty), \) \( p_j = (k_{ij}, k_{ij}, l_{ij}, \ldots, m_{ij}, l_{mj}), \) \( k_{ij} + \sum_{l=1}^{m} (k_{ij} + l_{ij}) = j, \) the right-hand side of equation (10) is assumed to be absolutely convergent in a some domain. It is obvious that the special series by the powers of consistent basic function is a formal solution for class of equations (10).

We check that the series by the powers of consistent basic function is a formal solution for class of equations (10).

The following theorem is valid:

**Theorem 1.** Consistent series (7), (8) is a formal solution of equation (10), if for the powers \( k_{ij}, \) for any \( j \) the conditions \( \sum_{i=1}^{m} k_{ij} = 2N_j, \) are satisfied, where \( N_j \) are integers.

**Proof** is carried out by substitution of series (7), (8) into equation (10). Consider the even derivatives of \( u \) by \( x \) and its powers in the right hand side of equation (10). It is easy to verify that its may be represented by consistent series (7), (8). Indeed, we calculate, for example, \( u_{xx} \)

\[ u_{xx} = \sum_{n \geq 2} 2n(n-1)u_{n}(t)R^{n-2}R_x^2 + \sum_{n \geq 1} nu_{n}(t)R^{n-1}R_{xx}. \]

From the first relation of (8) it follows that \( R_{xx} \) is also expanded into a series in powers of consistent basic function \( R \)

\[ R_{xx} = \sum_{k=k_0-1}^{\infty} \frac{(k+1)a_{k+1}(t)}{2} R^k, \]

therefore

\[ u_{xx} = \sum_{m=2}^{\infty} \sum_{k=k_0}^{m} 2m(m-1)u_{m}(t)a_k(t)R^{m-2+k} \]
\[ + \sum_{m=1}^{\infty} \sum_{k=k_0-1}^{m} nu_m(t)\frac{(k+1)a_{k+1}(t)}{2} R^{m-1+k} \]
\[ + \sum_{n=2}^{\infty} \left[ \sum_{m+k+n+2, k \geq k_0} 2m(m-1)u_{m}(t)a_k(t) + \sum_{m+k+n+1, k \geq k_0-1} nu_m(t)\frac{(k+1)a_{k+1}(t)}{2} \right] R^n \]

Therefore, the derivative of \( u_{xx} \) and its powers are represented in a power series by \( R. \) Similarly it is shown, that the even derivatives of \( u \) by \( x \) can also be written in a series by the powers of \( R. \) Consequently,

\[ \frac{\partial^{2m} u}{\partial x^{2m}} = \sum_{k=0}^{\infty} g_{mk}(t)R^k. \quad (11) \]

Let us now consider the odd derivatives of \( u \) by \( x, \) included in the right hand side of equation (10). Formula (11) implies that the product of two odd derivatives can be written as a series in the powers of consistent basic function \( R. \) Indeed,

\[ \frac{\partial^{2m_1+1} u}{\partial x^{2m_1+1}} \frac{\partial^{2m_2+1} u}{\partial x^{2m_2+1}} = \sum_{k_1=1}^{\infty} g_{m_1k_1}(t)R^{k_1-1}R_x \sum_{k_2=1}^{\infty} g_{m_2k_2}(t)R^{k_2-1}R_x \]
Thus, to here make it possible to write the right hand side of equation (10), after substituting in the series (7), (8), also in the form of a series in the powers (8), it is sufficient to require that the terms, which include the odd derivatives should consist of an even number of odd derivatives, including their powers. That is, if the condition \( \sum_{i=1}^{m} k_{ij} = 2N_j \) is valid, the right hand side of equation (10) can be represented as a series by the powers of consistent basic function \( R \). The even derivatives and its powers possess this property, if its are in the right hand side of original equation.

In order to ensure that series (7) is a formal solution of equation (10), we need to check that the coefficients of the consistent series \( u_i(t) \) may be found recurrently. Note that the recurrence of finding the coefficients \( u_i(t) \) achieved by the specificity of the first equation (8), namely, the fact that the terms in the right hand side of the first equation starts no less than the second power of \( R \) \(( k \geq k_0 \geq 2) \). Theorem 1 is proved.

Thus, as function \( F_1 \) is possible to take the right hand side of equation (10) with the restrictions of Theorem 1.

The consistent series are also useful for constructing solution of nonlinear equations, for which the special series in powers of universal basic functions are not applicable.

3. Investigation of convergence of consistent series

Consider the equation of the velocity potential, describing the stationary gas flow [20],

\[
\Phi_z^2 \Phi_{zz} + \Phi_r^2 \Phi_{rr} + 2 \Phi_r \Phi_z \Phi_{rz} - \Theta (\Phi_{rr} + \Phi_{zz} + N \Phi_r/r) = 0. \tag{12}
\]

Here \( \Theta = (\gamma - 1)[K - 0.5(\Phi_z^2 + \Phi_r^2)], \gamma = \text{const} \) is the gas adiabatic index, \( K = \text{const}, \Phi_r(r, z), \Phi_z(r, z) \) are the gas velocity in \( r \) and \( z \), respectively, \( N = 0 \) for a plane case, \( N = 1 \) for axisymmetric case. Consider the following problem for equation (12):

\[
\Phi(0, z) = \Phi_0(z); \quad \Phi_r(0, z) = 0, \quad -\infty < z < \infty, \tag{13}
\]

t.i. in the axis \( r = 0 \) velocity potential distribution \( \Phi_0(z) \) is given, for which we will find the current lines (nozzle walls), which create the distribution. Review of numerical and analytical methods for solving the inverse problem (12), (13) is presented in [20].

Solution of equation (12) we will find in the form of consistent series

\[
u_i(z) R_i(r, z), \quad u_0 = \text{const},
\]

\[
R_i(r, z) = \frac{1}{r^2 + \varphi(z)}, \quad \varphi(z) \in C^2(-\infty, \infty).
\tag{14}
\]

Let series (12) substitute into equation (12) and we get an equation for \( u_i(z) \)

\[
u_i''(M_\infty^2 - 1) = (1 - i)[4i - 2\gamma - 1 + M_\infty^2]\varphi' u_i - (i - 1)(i - 2) 
\cdot [(1 + M_\infty^2)(\varphi')^2 - 4\gamma]u_{i-2} + 2M_\infty^2(i - 1)u_{i-1}\varphi' + R_i(z), \quad i \geq 1;
\tag{15}
\]

here

\[
R_i(z) = (\gamma - 3)M_\infty \sum_{m+n=i} G_1(m)G_2(n) - M_\infty^2[8 + 2N(\gamma - 1)] \sum_{m+n=i} u_{n-1}(n-1)G_1(m)
\]
\[ +\frac{\gamma - 3}{2} \sum_{m+n+k=i} G_1(m)G_1(n)G_2(k) - 8 \sum_{n+l=i} G_3(n)[(l-1)(2l-1)u_{l-1} - 2\varphi(l-2)(l-1)u_{l-2} - [8 + N(\gamma - 1)] \sum_{m+n+k=i} u_{m-1}(m-1)G_1(n)G_1(k) \]
\[ + 8\varphi \sum_{m+n+k=i} (m-1)u_{m-1}G_1(n)(k-1)G_1(k-1) \]
\[ + 8\varphi \sum_{m+k=i} u_{m-1}(k-1)G_1(k-1) + 2(\gamma - 1) \sum_{m+n=i} G_3(n)[G_2(l) - 2N(l-1)u_{l-1}], \]

\[ G_1(n) = u'_n - (n-1)\varphi' u_{n-1}; \quad G_2(n) = \frac{dG_1(n)}{dz}, \]
\[ G_3(n) = \sum_{m+k=n} [m(n-1)u_m u_{n-1} - \varphi(m-1)(n-1)n_{m-1}u_{n-1}]. \]

Constants, which describe gas parameters, are for \( u_0 = M_\infty \) (\( M_\infty \) is the Mach number of the incident flow). So, the solutions of equation (15) are representable in the form

\[
\frac{1}{M_\infty^2 - 1} \int \int \{(1-i)[4i - 2N - (1 + M_\infty^2)\varphi''(\tau)]u_{i-1}(\tau) - (i-1)(i-2)(1 + M_\infty^2)\varphi''(\tau) - [4\varphi]u_{i-2}(\tau) + 2M_\infty^2(i-1)u'_i(\tau)\varphi(\tau) + R_i(\tau)\}d\tau dz + C_1z + D_i,
\]

where \( C_i, D_i = \text{const}, i \geq 1. \)

The theorem is valid.

**Theorem 2.** Let the following conditions be fulfilled:

1. an arbitrary function \( \varphi(z) \in C^2(-\infty, \infty) \), which

\[
1 < \varphi_0 \leq \varphi(z) \leq \varphi_1, \quad |\varphi'|, |\varphi''| \leq M \exp(b|z|), \quad 0 < M \leq M_0 \quad 0 < b_0 \leq b;
\]

2. for the constant \( C_n, D_n \) the inequalities

\[
|C_n|, |D_n| \leq \frac{M}{3n^4}, \quad n \geq 1, \quad \varphi_0, \varphi_1, M, b, M_0, b_0 = \text{const}, \quad \text{are valid.}
\]

Then, \( L > 0 \) exists, that for \( M_\infty \neq 1 \) series (14) uniformly converges to a solution of equation (12) for all \( 0 \leq r < \infty, |z| \leq L. \)

The proof of Theorem 2 is obtained after estimations for coefficients \( u_n(z) \) with \( |z| \leq L \)

\[
|u_n(z)| \leq \frac{M}{n^4} \exp(bn|z|), \quad n \geq 1,
\]
\[
|u'_n| \leq \frac{M_1}{n^3} \exp(bn|z|), \quad |u''_n| \leq \frac{M_2}{n^2} \exp(bn|z|), \quad M_1, M_2 = \text{const},
\]

which are established by induction for direct estimates of expressions (16).

**Remark.** The second condition (13) is fulfilled out automatically due to the specifics of the basic function \( R(r, z) \). In addition, the arbitrariness in the basic function \( \varphi(z) \) generates a velocity distribution on the axis \( r = 0 \). Therefore, if we set \( r = h(z) \), then on this line the impermeability condition \( h'\Phi_z(h(z), z) = \Phi_r(h(z), z) \) has to be fulfilled, from which we can try to find the function \( \varphi(z) \) approximately.

The uniform convergence of the consistent series, proved in Theorem 2, allows to solve the problems of the internal flow in nozzles [20] due to the specific of the series (13).
4. Conclusion
A method of special series is suggested for representation of solutions of nonlinear partial differential equations. The method consists in the expansion of the solution of the equation in the form of series with recurrently computed coefficients, which may be found from a sequence of linear ordinary differential equations. The recurrence of finding the coefficients is achieved by a special choice of the basic functions, which powers makes the series to be a solution. The basic functions can be functions of two types: universal basic functions and consistent basic functions, and can contain a functional arbitrariness. Universal basic functions are useful for representation of solutions for a wide class of nonlinear partial differential equations. Consistent basic functions take into account the specificity of the original non-linear equations and applicable to a wider class of nonlinear equations than universal basic functions.

Application of consistent basic functions is shown for the equation that describes an axisymmetric stationary flows of gas. We obtain a sequence of linear ordinary differential equations, which are equations for the series coefficients. The conditions under which this series converges to the solution of the equation are formulated. It turns out that a choice the consistent basic function allows to exactly satisfy the condition on the axis of symmetry, and by using the arbitrary function, and we can approximate and solve the problem of the internal gas flow in a nozzle.

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