Vector fields on certain quotients of the complex Stiefel manifolds.

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Let $M$ be a closed connected manifold of dimension $n$. 

(i) If $n \equiv 0 \pmod{2}$, then $\text{span}(M) = 0$ or $\text{stable span}(M)$.

(ii) If $n \equiv 1 \pmod{4}$ and $w_2^1(M) = 0$, then $\text{span}(M) = 1$ or $\text{stable span}(M)$.

(iii) If $n \equiv 3 \pmod{8}$ and $w_2^1(M) = w_2(M) = 0$, then $\text{span}(M) = 3$ or $\text{stable span}(M)$.
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Definition ($m$- projective Stiefel manifold)

Let $m \in \mathbb{N}$. 

$\Gamma_m \subset S^1$, $\Gamma_m$ subgroup generated by a primitive $m$-th root of unity.

$\Gamma_m$ acts on $W^n_k$, by $z \cdot (v_1, \ldots, v_k) = (zv_1, \ldots, zv_k)$ where $z \in \Gamma_m$.

We denote the orbit space by $W^n_k; m$ and it will be referred to as the $m$-projective Stiefel manifolds.

Notice that when $k = 1$, $W^n_1; m = L^n_m$ ($m$) lens space.

One can see that $W^n_k; m = U(n) / (Z_m \times U(n-k))$.

Also, $Z_m \subset U(1) = Z(U(n))$ and $U(n) / (Z_m \times U(n-k)) = Z_m \setminus U(n) / U(n-k)$. 

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Notice that when $k = 1$, $W_{n,1;m} = L^n(m)$ lens space.

One can see that $W_{n,k;m} = U(n)/(\mathbb{Z}_m \times U(n-k))$. Also,  
$\mathbb{Z}_m \subset U(1) = \mathbb{Z}(U(n))$ and $U(n)/(\mathbb{Z}_m \times U(n-k)) = \mathbb{Z}_m \backslash U(n)/U(n-k)$. 

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Let $\beta_{n,k}$ denote complex $(n-k)$-plane bundle over $PW_{n,k}$ whose fibre over $[v_1, \ldots, v_k]_0$ is the vector space $\{v_1, \ldots, v_k\}^\perp \subset \mathbb{C}^n$ where the orthogonal complement is taken with respect to standard Hermitian product on $\mathbb{C}^n$. 

\[ \tau_{W_{n,k};m} \sim R^k \xi_{n,k;m} \oplus \beta_{n,k;m} \]

where $\beta_{n,k;m} := \pi^* m(\beta_{n,k})$ and $\xi_{n,k;m}$ denote the complex line bundle associated to the principal $U(1)$-bundle obtained by extension of structure group via the character $\Gamma_m \subset U(1)$ of the $\Gamma_m$-bundle $W_{n,k;m} \rightarrow W_{n,k; m}$. 

Let $\pi_m: W_{n,k;m} \longrightarrow PW_{n,k}$ denote canonical quotient map (which is a principle $U(1)$-bundle).

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$$\tau W_{n,k;m} \cong_{\mathbb{R}} k\xi_{n,k;m} \oplus \beta_{n,k;m} \oplus k^2 \varepsilon_{\mathbb{R}}.$$ 

where $\beta_{n,k;m} := \pi_m^*(\beta_{n,k})$ and $\xi_{n,k;m}$ denote the complex line bundle associated to the principal $U(1)$-bundle obtained by extension of structure group via the character $\Gamma_m \subset U(1)$ of the $\Gamma_m$-bundle $W_{n,k} \longrightarrow W_{n,k;m}$. 
Let $M$ be a smooth manifold of dimension $n$. If the tangent bundle $\tau M$ of $M$ admits $n - p$ everywhere linearly independent vector fields, (where $0 \leq p \leq n$) then $w_{p+1}(\tau M) = \cdots = w_n(\tau M) = 0$. Similarly understanding of Chern class and Pontrjagin class can be of some help while solving vector field problem.
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Condition on $n, k$ and $m$ which would imply $\text{span}(W_{n,k};m) = \text{stable span}(W_{n,k};m)$

$$\tau W_{n,k;m} \oplus k^2 \mathbb{C}_\mathbb{R} \cong nk \xi_{n,k;m}$$
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Calculating $w_1(W_{n,k;m})$ and $w_2(W_{n,k;m})$ by elementary methods and using criteria mentioned above we get,
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**Proposition**

Let $2 \leq k < n$ and let $m \geq 2$. 

Condition on \( n, k \) and \( m \) which would imply \( \text{span}(W_{n,k;m}) = \text{stable span}(W_{n,k;m}) \)

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**Proposition**

Let \( 2 \leq k < n \) and let \( m \geq 2 \). One has

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Proposition

Let $2 \leq k < n$ and let $m \geq 2$. One has

$$\text{span}(W_{n,k;m}) = \text{stable span}(W_{n,k;m})$$

in each of the following cases: (i) $k$ is even, (ii) $n$ is odd, and (iii) $n \equiv 2 \pmod{4}$. 
Cohomology of $W_{n,k;m}$

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\begin{array}{c}
S^1 / \Gamma_m \\
\downarrow \\
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\end{array}
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Cohomology of $W_{n,k;m}$
Notations:

(i) Let $N := 2N'$ where $N'$ is defined as:

$N' = \min \{ n - k + 1 \leq j \leq n \mid (n^j) \not\equiv 0 \pmod{p} \}$.

(Note that the value of $N'$ depends on $n$, $k$, and $p$.)

(ii) We shall label (homogeneous) generators of a graded algebra by their degrees. Thus $|x_j| = j$ when $x_j \in H^*(X; \mathbb{R})$.

(iii) $\Lambda_{\mathbb{Z}_p}(x_1, \ldots, x_r)$ denotes any graded commutative algebra $A$ over $\mathbb{Z}_p$ in which square-free monomials in $x_1, \ldots, x_r$ form a basis.
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Theorem

Suppose that $2 \leq k < n$ and $m \geq 2$. 

(i) If $p$ is any prime not dividing $m$, then $H^*_{W_n, k; m; Z_p} \cong H^*_{W_n, k; Z_p}$. 

(ii) If $p$ is an odd prime that divides $m$, then $H^*_{W_n, k; m; Z_p} \cong Z_p[y]/\langle y^N \rangle \otimes \Lambda_{Z_p}(y_1, y_2, \ldots, y_{2^{n-2k+1}}, \ldots, y_{2^{n-1}})$ where $N, N'$ are as defined above (As usual, $\hat{\cdot}$ stands for omission of the variable). Also $y_1 = c^1_{\xi_{2^{n-2k+1}}}$ mod $p$.

(iii) Suppose $m \equiv 2 \pmod{4}$. Then $H^*_{W_n, k; m; Z_2} \cong Z_2[y]/\langle y^N \rangle \otimes \Lambda_{Z_2}(y_2, y_3, \ldots, y_{2^{n-2k+1}}, \ldots, y_{2^{n-1}})$.

(iv) Suppose that $m \equiv 0 \pmod{4}$. Then $H^*_{W_n, k; m; Z_2} \cong Z_2[y]/\langle y^N' \rangle \otimes \Lambda_{Z_2}(y_1, y_2, \ldots, y_{2^{n-2k+1}}, \ldots, y_{2^{n-1}})$, where $y_2 = 0$. 


Theorem

Suppose that $2 \leq k < n$ and $m \geq 2$.

(i) If $p$ is any prime not dividing $m$, then

$$p^*_m : H^*(W_{n,k}; \mathbb{Z}_p) \cong H^*(W_{n,k}; \mathbb{Z}_p) = \Lambda_{\mathbb{Z}_p}(v_{2n-2k+1}, \ldots, v_{2n-1}).$$

is an isomorphism of algebras.

(ii) If $p$ is an odd prime that divides $m$, then

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where $N$, $N'$ are as defined above (As usual, $\hat{y}$ stands for omission of the variable). Also $y_1 = 0$.

(iii) Suppose that $m \equiv 2 \pmod{4}$. Then

$$H^*(W_{n,k}; m; \mathbb{Z}_2) \cong \mathbb{Z}_2[y]/\langle y_N \rangle \otimes \Lambda_{\mathbb{Z}_2}(y_{2n-2k+1}, \ldots, y_{2n-1}).$$

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where $y_2 \equiv c(\xi, k, m) \pmod{p}$.
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(ii) If $p$ is an odd prime that divides $m$, then $H^*(W_{n,k;m};\mathbb{Z}_p) \cong \mathbb{Z}_p[y_2]/\langle y_{2N}' \rangle \otimes \Lambda_{\mathbb{Z}_p}(y_1, y_{2n-2k+1}, y_{2n-2k+3}, \ldots, \hat{y}_{N-1}, \ldots, y_{2n-1})$ where $N, N'$ are as defined above (As usual, $\hat{\text{stands for omission of the variable})$. Also $y_2 = c_1(\xi_{n,k;m}) \mod p$.
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Suppose that $2 \leq k < n$ and $m \geq 2$.

(i) If $p$ is any prime not dividing $m$, then

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(ii) If $p$ is an odd prime that divides $m$, then

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(iii) 1. Suppose $m \equiv 2 \pmod{4}$. Then $H^*(W_{n,k;m}; \mathbb{Z}_2) \cong \mathbb{Z}_2[y_1]/\langle y_1^{N} \rangle \otimes \Lambda_{\mathbb{Z}_2}(y_{2n-2k+1}, y_{2n-2k+3}, \ldots, \hat{y}_{N-1}, \ldots, y_{2n-1})$.

2. Suppose that $m \equiv 0 \pmod{4}$. Then $H^*(W_{n,k;m}; \mathbb{Z}_2) \cong \mathbb{Z}_2[y_2]/\langle y_2^{N'} \rangle \otimes \Lambda_{\mathbb{Z}_2}(y_1, y_{2n-2k+1}, y_{2n-2k+3}, \ldots, \hat{y}_{N-1}, \ldots, y_{2n-1})$, where $y_1^2 = 0$. 
We give a partial description of the integral cohomology of $W_{n,k;m}$. 

Theorem
We will assume that, 
1 < k ≤ n − 2 and m ≥ 2. Let $m_r := m$ if $r ≤ n − k$ and $m_r := \gcd\{m, (n_j)\}_{n − k < j ≤ r}$ if $n − k < r ≤ n$. Then:

(i) The (additive) order of $y_r^2 ∈ H_2^r(W_{n,k,m};Z)$ is $m_r$ for $1 ≤ r ≤ n$. In particular the height of $y_2^2 ∈ H_2^2(W_{n,k,m};Z)$ is the largest integer $h$, $n − k < h ≤ n$, such that $m_h > 1$.

(ii) One has the total Pontrjagin class $p(W_{n,k,m}) = (1 + y_2)^{nk}$ for all $r ≥ 1$. 

The total Stiefel-Whitney class $w(W_{n,k,m}) = (1 + y_1)^{nk}$, where it is understood that $y_1 = 0$ when $m$ is odd.
We give a partial description of the integral cohomology of $W_{n,k;m}$. We calculate the (additive) order of $y_2^r \in H^{2r}(W_{n,k;m}; \mathbb{Z})$ where $y_2 = c_1(\xi_{n,k;m})$ and we give explicit formulae for the total Stiefel-Whitney class and the total Pontrjagin class of $W_{n,k;m}$.
We give a partial description of the integral cohomology of \( W_{n,k;m} \). We calculate the (additive) order of \( y_2^r \in H^{2r}(W_{n,k;m}; \mathbb{Z}) \) where \( y_2 = c_1(\xi_{n,k;m}) \) and we give explicit formulae for the total Stiefel-Whitney class and the total Pontrjagin class of \( W_{n,k;m} \).

**Theorem**

We will assume that, \( 1 < k \leq n - 2 \) and \( m \geq 2 \). Let \( m_r := m \) if \( r \leq n - k \) and \( m_r := \gcd\{m, \binom{n}{j}; n - k < j \leq r\} \) if \( n - k < r \leq n \). Then:
We give a partial description of the integral cohomology of $W_{n,k;m}$. We calculate the (additive) order of $y_2^r \in H^{2r}(W_{n,k;m}; \mathbb{Z})$ where $y_2 = c_1(\xi_{n,k;m})$ and we give explicit formulae for the total Stiefel-Whitney class and the total Pontrjagin class of $W_{n,k;m}$.

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(i) The (additive) order of $y_2^r \in H^{2r}(W_{n,k;m}; \mathbb{Z})$ is $m_r$ for $1 \leq r \leq n$. In particular the height of $y_2 \in H^2(W_{n,k;m}; \mathbb{Z}) \cong \mathbb{Z}_m$ is the largest integer $h$, $n - k < h \leq n$, such that $m_h > 1$. 
We give a partial description of the integral cohomology of $W_{n,k;m}$. We calculate the (additive) order of $y_2^r \in H^{2r}(W_{n,k;m}; \mathbb{Z})$ where $y_2 = c_1(\xi_{n,k;m})$ and we give explicit formulae for the total Stiefel-Whitney class and the total Pontrjagin class of $W_{n,k;m}$.

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(ii) One has the total Pontrjagin class $p(W_{n,k;m}) = (1 + y_2^2)^{nk}$ for all $r \geq 1$. The total Stiefel-Whitney class $w(W_{n,k;m}) = (1 + y_1^2)^{nk}$, where it is understood that $y_1 = 0$ when $m$ is odd.
Theorem

If there exists an \( r \geq 1 \) such that \( \binom{n^k}{r} \) is not divisible by \( m_{2r} \), then \( W_{n,k;m} \) is not stably parallelizable. In particular, if \( W_{n,k;m} \) is stably parallelizable, then \( m \) divides \( nk \).
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\(X\) - a paracompact, Hausdorff topological space.

\(\omega\) - a complex vector bundle over \(X\).
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We continue to denote the class of \(\omega\) in \(K(X)\) by \([\omega]\). Note that \([\omega] = [\omega']\) if and only if \(\omega \oplus \eta \cong_{\mathbb{C}} \omega' \oplus \eta\) for some complex vector bundle \(\eta\) over \(X\).
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$G$- compact connected Lie group, $\text{Tor}(\pi_1(G)) = 0.$

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Restriction map $\text{Res}: R(G) \longrightarrow R(H)$ makes $R(H)$, an $R(G)$-module and $\mathbb{Z}$ is $R(G)$-module via augmentation map $\epsilon_G: R(G) \longrightarrow R\{1\} = \mathbb{Z}$. 

Theorem

$K$-ring of $W_{n,k;2}$ is given by

$$K^\ast(W_{n,k;2}) = \Lambda_\mathbb{Z}^\ast(w_{n-k+1}, \ldots, w_{n-2}, \tilde{L}, \tilde{\vartheta}) \otimes_{\mathbb{Z}} \mathbb{Z}[z]/\mathcal{I},$$

were $\Lambda_\mathbb{Z}^\ast$ denotes the exterior algebra in the indicated generators. The generators $w_{n-k+1}, \ldots, w_{n-2}, \tilde{L}, \tilde{\vartheta}$ are of degree 1 and $z$ is of degree zero and the ideal $\mathcal{I} = \langle z^2 + 2z, 2^\alpha z, \tilde{L}z \rangle$.

Here $\alpha = \min\{n - 1, i - 1 + \nu_2(^n_i) \mid n - k < i < n\}$ where $\nu_2(n)$ denotes the largest number such that $2^{\nu_2(n)}$ divides $n$. 
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Currently we are working on $K(W_{n,1;m})$ i.e. $K$ ring of lens spaces.
Thank you.
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(i) Let \( \Gamma = \mathbb{Z}[\alpha_1, \ldots, \alpha_r] \) for some \( 1 \leq r \leq n \) such that \( R(H) \) is a free \( \Gamma \)-module and \( \text{Res}(\alpha_i) = \beta_i \) for \( i = 1, \ldots, r \) then

\[
\text{Tor}^*_R(G)(R(H); \mathbb{Z}) = \text{Tor}^*_A(B; \mathbb{Z}),
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where \( A = R(G)/\langle \alpha_1, \ldots, \alpha_r \rangle \) and \( B = R(H)/\langle \beta_1, \ldots, \beta_r \rangle \).
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(ii) Let \( a_1, \ldots, a_s \in A \) such that \( \text{Res}(a_i) = 0 \in B \) for \( 1 \leq i \leq s \) then

\[
\text{Tor}_A^*(B; \mathbb{Z}) = \Lambda_\mathbb{Z}^*[t_1, \ldots, t_s] \otimes \text{Tor}_{A'}^*(B; \mathbb{Z})
\]

where \( A' = A/\langle a_1, \ldots, a_s \rangle \) and degree of \( t_i \) is 1 for all \( i = 1, \ldots, s \).
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Let $\Lambda_k := \sum_{1 \leq i_1 < \ldots < i_k \leq n} (x_{i_1} - 1) \cdots (x_{i_k} - 1)$ for $1 \leq k \leq n$, for all $1 \leq k \leq n$ and $R(G) = \mathbb{Z}[\Lambda_1, \ldots, \Lambda_{n-1}, \sigma_n]$ where $\sigma_n = x_1 \cdot x_2 \cdots x_n$ is one dimensional.

Similarly we can write $R(H) = \mathbb{Z}[z, \lambda_1, \ldots, \lambda_{n-1} - k - 1, \mu_{n-k}] / J$ where $\epsilon_H(z) = \epsilon_H(\lambda_i) = 0$ and $\epsilon_H(\mu_{n-k}) = 1$ and $J = \langle z^m + \sum_{1 \leq j \leq m-1} (m^j)z^j \rangle$. 


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**Lemma**

\( Tor^*_{RG}(RH, \mathbb{Z}) \cong Tor^*_A(B; \mathbb{Z}). \)

We have the following recurring relations in \( B \) for \( n - k + 1 \leq i \leq n - 1 \):

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\overline{\text{Res}}(\bar{\Lambda}_i) + \sum_{J=n-k}^{i-1} \binom{n-J}{i-J} 2^{i-J} \overline{\text{Res}}(\bar{\Lambda}_J) = \binom{n}{i} 2^{i-1} z.
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\]

where \( A' = \mathbb{Z}[\Lambda_{n-k}, V] \) and \( B = \mathbb{Z}[z]/\langle z^2 + 2z \rangle \).