THE DOUBLE OF REPRESENTATIONS OF COHOMOLOGICAL HALL ALGEBRA FOR $A_1$-QUIVER

XINLI XIAO

Abstract. We compute two representations of COHA for $A_1$-quiver, and show that they can be combined into a representation of $D_{n+1}$ Lie algebra.

1. Introduction

The aim of this paper is to define and discuss two representations of the Cohomological Hall algebras, and combine them into a single representation of the algebra which is called “full” (or “double”) COHA in [9].

Cohomological Hall algebra (COHA for short) was introduced in [5]. The definition is similar to the definition of conventional Hall algebra (see e.g. [8]) or its motivic version (see e.g. [4]). Instead of spaces of constructible functions on the stack of objects of an abelian category, one considers cohomology groups of the stacks. The product is defined through the pullback-pushforward construction. Details can be found in [5].

By analogy with conventional Hall algebra of a quiver, which gives the “positive” part of a quantization of the corresponding Lie algebra, one may want to define the “double” COHA, for which the one defined in [5] would be a “positive part”. Following the discussion in [9], we study the double of representations of COHA, and hope to find the double of COHA through its representations.

This paper focuses on $A_1$-quiver. Stable framed representations of the quiver are used to produce two representations of COHA. Since the moduli spaces of stable framed representations of $A_1$-quiver are Grassmannians, we actually define two representations on the cohomology of Grassmannians. We show that the operators from these two representations form $D_{n+1}$-Lie algebra. This confirms the conjecture from [9] that the double of $A_1$-COHA is the infinite Clifford algebra.

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2. Two geometric representations of $A_1$-COHA

2.1. COHA. Let $Q$ be a quiver with $N$ vertices. Given a dimension vector $\gamma = (\gamma_i)_{i=1}^N$, $M_\gamma$ is the space of complex representations with fixed underlying vector space $\bigoplus_{i=1}^N \mathbb{C}^{\gamma_i}$ of dimension vector $\gamma$, and $G_\gamma = \prod_{i=1}^N GL_{\gamma_i}(\mathbb{C})$ is the associated gauge group. $[M_\gamma/G_\gamma]$ is the stack of representations of $Q$ with fixed dimension vector $\gamma$. As a vector space, COHA of $Q$ is defined to be $\mathcal{H} := \bigoplus_{\gamma} \mathcal{H}_\gamma := \bigoplus_{\gamma} H^*([M_\gamma/G_\gamma]) := \bigoplus_{\gamma} H^*_{G_\gamma}(M_\gamma)$. Here by equivariant cohomology of a complex algebraic variety $M_\gamma$ acted by a complex algebraic group $G_\gamma$ we mean the usual (Betti) cohomology with coefficients in $\mathbb{Q}$ of the bundle $EG_\gamma \times_{G_\gamma} M_\gamma$ associated to the universal $G_\gamma$-bundle $EG_\gamma \to BG_\gamma$ over the classifying space of $G_\gamma$. The product $*: \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ is defined by means of the pullback-pushforward construction in [5].

2.2. $A_1$-COHA. Let $Q$ be $A_1$. $N = 1$. Since there is only one representation with fixed underlying vector space $\mathbb{C}^d$ of dimension $d$, $M_d$ is a point and $G_d = GL_d(\mathbb{C})$. Therefore $\mathcal{H}_d = H^*_{GL_d(\mathbb{C})}(M_d) = \mathbb{Q}[x_{1,d}, \ldots, x_{d,d}]$ is the algebra of symmetric polynomials in variables $x_{1,d}, \ldots, x_{d,d}$. It is possible to talk about the geometric interpretation of these variables. They can be treated as the first Chern classes of the tautological bundles over the classifying space of $G_d$. For details see e.g. [10].

The COHA $\mathcal{H}$ for quiver $A_1$ is described in [5]. It is the infinite exterior algebra generated by odd elements $\phi_0, \phi_1, \phi_2, \ldots$ with wedge $\wedge$ as its product. Generators $(\phi_i)_{i \geq 0}$ correspond to the additive generators $(x_{1,i})_{i \geq 0}$ of $\mathcal{H}_1 = \mathbb{Q}[x_{1,1}]$. A monomial in the exterior algebra

$$\phi_{k_1} \wedge \ldots \wedge \phi_{k_d} \in \mathcal{H}_d, \quad 0 \leq k_1 < \ldots < k_d$$

corresponds to the Schur symmetric polynomial $s_\lambda(x_{1,d}, \ldots, x_{d,d})$, where $\lambda = (\lambda_d, \ldots, \lambda_1) = (k_d - d + 1, k_{d-1} - d + 2, \ldots, k_1)$ is a partition.

Let $\Phi_k = \phi_{k_1} \wedge \ldots \wedge \phi_{k_d}$ with index $k = (k_1, \ldots, k_d)$, $0 \leq k_1 < \ldots < k_d$. Denote by $k(\lambda)$ the index related to the partition $\lambda$ and by $\lambda(k)$ the partition related to the basis index $k$. Then we have $\Phi_{k(\lambda)} = s_{\lambda(k)}$.

2.3. Stable framed representations. Fix a dimension vector $n = (n_i)_{i=1}^N$. A framed representation of $Q$ of dimension vector $\gamma$ is a pair $(V, f)$, where $V$ is an ordinary representation of $Q$ of dimension $\gamma$ and $f = (f_i)_{i=1}^N$ is a collection of linear maps from $\mathbb{C}^{n_i}$ to $V_i$. The set of framed representations of dimension vector $\gamma$ with framed structure dimension vector $n$ is denoted by $\hat{M}_{\gamma,n}$. It carries a natural gauge group $G_\gamma$-action. See e.g. [7].

For the notion of stable framed representation of a quiver, see e.g. [6] (more general framework of triangulated categories can be found in [9]). We focus on the trivial
stability condition. In this case, a framed representation is called \textit{stable} if there is no proper (ordinary) subrepresentation of $V$ which contains the image of $f$. The set of stable framed representations of dimension vector $\gamma$ with framed structure dimension vector $n$ is denoted by $\hat{M}_{\gamma,n}^{st}$. The gauge group $G_{\gamma}$ of $M_{\gamma,n}$ induces a $G_{\gamma}$-action on $\hat{M}_{\gamma,n}^{st}$. The stack of stable framed representations $[\hat{M}_{\gamma,n}^{st}/G_{\gamma}]$ is in fact a smooth projective scheme. We denote it by $M_{\gamma,n}^{st}$ and call it the \textit{smooth model} of quiver $Q$ with dimension $\gamma$ and framed structure $n$.

The pullback-pushforward construction is applied to the cohomology of the scheme of stable framed representations. This construction leads to two representations of COHA for the quiver $Q$ which we describe below.

Fix two dimension vectors $\gamma_1$ and $\gamma_2$. Set $\gamma = \gamma_1 + \gamma_2$. Consider the scheme consisting of diagrams

$$M_{\gamma_2,\gamma,n}^{st} := \{ 0 \to E_1 \to E \to E_2 \to 0 \}, \quad (2.1)$$

where $E_1 \in M_{\gamma_1}$, $(E, f) \in M_{\gamma,n}^{st}$, $(E_2, f_2) \in M_{\gamma_2,n}^{st}$, $f : \mathbb{C}^n \to E$ and $f_2 : \mathbb{C}^n \to E_2$ are the framed structures attached to $E$ and $E_2$ respectively. The subgroup of the automorphism group of $E$ which preserves the embedding of $E_1$ is denoted by $P_{\gamma_1,\gamma,n}$. It plays the role of the automorphism group of $M_{\gamma_2,\gamma,n}^{st}$. The natural projections from the diagram to its components give the following diagram:

$$M_{\gamma,n}^{st} \quad (2.2)$$

where $\phi_1 \in H_{\gamma_1}$ and $\phi_2 \in H^{*}(M_{\gamma_2,n}^{st})$. This morphism induces a representation of $\mathcal{H} = \bigoplus_{\gamma} \mathcal{H}_{\gamma}$ on $\bigoplus_{\gamma} H^{*}(M_{\gamma,n}^{st})$. It is called the \textit{increasing representation} of COHA for the quiver $Q$, and denoted by $R_{n}^{+}$. Similarly, the map $(p_2^*)(\phi_1) \cup p_2^*(\varphi)$ for $\phi_1 \in \mathcal{H}_{\gamma_1}$ and $\varphi \in H^{*}(M_{\gamma_2,n}^{st})$ gives the \textit{decreasing representation} $R_{n}^{-}$ on the cohomology of the smooth model. In order to have well-defined representations one needs to show that $p$ and $p_2$ are proper morphisms. For $A_1$-case the properness is obvious (see Section 2.5 below).
2.4. A₁-case. Let \( n \) be the framed structure dimension. A framed representation \((\mathbb{C}^d, f)\) of A₁-quiver is stable if and only if \( f : \mathbb{C}^n \to \mathbb{C}^d \) is surjective. Thus the stable framed moduli space \( M_{d,n}^{st} \) is the Grassmannian (of quotient spaces) \( Gr(d, n) \) for \( 0 \leq d \leq n \), and empty for \( d > n \).

It is well known (see e.g. [3], p.161) that the cohomology of full flag variety \( Fl(n) \) is isomorphic to \( R(n) = \mathbb{Q}[x_1, \ldots, x_n]/(e_1(x_1, \ldots, x_n), \ldots, e_n(x_1, \ldots, x_n)) \), where \( e_i(x_1, \ldots, x_n) \) represents the \( i \)-th elementary symmetric polynomial. The cohomology of Grassmannian \( Gr(d, n) \) is a subalgebra of \( R(n) \) which is generated by Schur polynomials in variables \( x_1, \ldots, x_d \). Thus we can use \( s_i(x_1, \ldots, x_d) \) to represent classes in \( H^*(Gr(d, n)) \). There is a natural projection \( \pi : Fl(n) \to Gr(d, n) \). By abuse of notations, we use the same symbol \( x_i \) to denote the classes in \( Gr(d, n) \) whose pullback \( \pi^*(x_i) \) is \( x_i \in H^*(Fl(n)) \).

Classes in \( H^*(Gr(d, n)) \) have an alternative presentation.

**Lemma 2.1.** In \( H^*(Gr(d, n)) \), \( s_d(x_1, \ldots, x_d) = (-1)^{|\lambda|} s_{\lambda'}(x_{d+1}, \ldots, x_n) \), where \( \lambda' \) is the transpose partition of \( \lambda \).

**Proof:** The above identity can be easily deduced from the identity \( \prod_{i=1}^d \frac{1}{1-x_it} = \prod_{i=d+1}^n (1-x_it) \) (see e.g. [3], p.163) in the ring \( R(n)[t] \). \( \square \)

In the following we will use this “transpose” presentation to do some computations.

2.5. Two representations of A₁-COHA. The scheme \([M_{d_2,d,n}^{st}/P_{d_2,d,n}]\) in A₁-quiver case is isomorphic as a scheme to the two-step flag variety \( F_{d_2,d,n} \), which is variety of the flags \( \{\mathbb{C}^n \to \mathbb{C}^d \to \mathbb{C}^{d_2}\} \). Let \( \phi_i \) be a generator of \( \mathcal{H}_1 \), and \( s_\lambda \) be the Schur polynomial considered as an element of the cohomology of the Grassmannian \( H^*(Gr(d_2, n)) \) whose partition is \( \lambda \). In this case, \( p \) is the obvious projection from \( F_{d_2,d,n} \) to \( Gr(d, n) \) and \( p_2 \) is the obvious projection from \( F_{d_2,d,n} \) to \( Gr(d_2, n) \). Therefore both \( p \) and \( p_2 \) are proper morphisms of stacks (which are in fact schemes), and the increasing and decreasing representations introduced in Section 2.3 are well defined.

Now we want to compute the increasing representation by the formula \( p_*(p_1^*(\phi_i) \cup p_2^*(s_\lambda)) \). Note that in this case, \( d_1 = 1 \). Recall that \( \phi_i \) represents the polynomial \( \phi_i(x_{1,1}) = x_{1,1}^i \). Using the geometric interpretation, \( x_{1,1}^i \) is treated as the first Chern class of the tautological line bundle \( \mathcal{O}(-i) \) over the classifying space of \( G_1 \). \( \mathcal{O}(-i) \) will be pulled back through \( p_1 \) to the line bundle over \( F_{d_2,d,n} \) associated to the corresponding character of \( G_{d_1} \) when treating \( G_{d_1} \) as a subquotient of \( P_{d_2,d,n} \). Hence \( p_1^*(\phi_i) \) will be the first Chern class of the line bundle described above, which is \( \phi_i(x_{d_2+1}) = x_{d_2+1}^i \).

As homogenous spaces, \( Gr(d, n) \approx GL_n(\mathbb{C})/P_{d,n} \), \( Gr(d_2, n) \approx GL_n(\mathbb{C})/P_{d_2,n} \) and \( F(d_2, d, n) \approx GL_n(\mathbb{C})/P_{d_2,d,n} \). We use the formula in [11] to compute the pushforward.
**Theorem 2.2.** [1] Let $G$ be a connected reductive algebraic group over $\mathbb{C}$ and $B$ a Borel subgroup. Choose a maximal torus $T \subset B$ with Weyl group $W$. The set of all positive roots of the root system of $(G, T)$ is denoted by $\Delta^+$. Let $P \supset B$ be a parabolic subgroup of $G$, with the set of positive roots $\Delta^+(P)$ and Weyl group $W_P$. Let $L_{\alpha}$ be the complex line bundle over $G/B$ which is associated to the root $\alpha$. The Gysin homomorphism $f_* : H^*(G/B) \to H^*(G/P)$ is given by

$$f_*(p) = \sum_{w \in W/W_P} w \cdot \frac{p}{\prod_{\alpha \in \Delta^+(P)} c_1(L_\alpha)}.$$  

(2.3)

Applying Thm[2.2] for $s_i \in H^*(Gr(d_2, n))$,

$$(\phi_i^+ \cdot s_i)(x_1, \ldots, x_{d_2+1}) = \sum_{i_1 < \ldots < i_{d_2}} \frac{s_j(x_{i_1}, \ldots, x_{i_{d_2}}, x_{i_{d_2+1}})\phi_i(x_{i_{d_2}})}{\prod_{j=1}^{d_2}(x_{i_j} - x_{i_{d_2+1}})}.$$  

(2.4)

Similarly, the formula of the decreasing actions is

$$(\phi_i^- \cdot s_i)(x_1, \ldots, x_{d_2-1}) = \sum_{i_1 < \ldots < i_{d_2}} \frac{s_j(x_{i_1}, \ldots, x_{i_{d_2}})\phi_i(x_{i_{d_2}})}{\prod_{j=d_2+1}^{n}(x_{i_d} - x_j)}.$$  

(2.5)

**Remark 2.3.** In Formula (2.5), variables $x_i$ for $i > d_2 - 1$ appear on the right side, which do not belong to the variables on the left side. This is not a contradiction because of the formula $s_d(x_1, \ldots, x_d) = (-1)^d s_d(x_{d+1}, \ldots, x_n)$ by Lemma[2.1]. More details will be discussed in the following section.

3. Increasing and Decreasing Operators

3.1. Increasing operators. The key result of this subsection is adapted from [2].

**Proposition 3.1.** [2] The increasing representation structure is induced by the open embedding $j : \tilde{M}_{d,n}^a \to \tilde{M}_{d,n}$. The induced map $j^* : \mathcal{H} \to R_{n}^+$ is $\mathcal{H}$-linear and surjective. The kernel of $j^*$ equals $\bigoplus_{\rho \geq 0, \rho > 0} \mathcal{H}_\rho \wedge (e_\rho \cup \mathcal{H}_\rho)$, where $e_\rho = \prod_{i=1}^{d} x_i$.

**Proof:** In [2], the similar result for $n = 1$ is proved. It can be easily generalized to $n > 1$ case for $A_1$-quiver. \[\square\]

The next lemma follows immediately from the definition of Schur polynomials.

**Lemma 3.2.** $s_{(d_2+1, d_2+1, \ldots, d_2+1)} = e_d s_d$ for $s_d \in \mathbb{Q}[x_1, \ldots, x_d]^{S_d}$ and $e_d = \prod_{i=1}^{d} x_i$. Thus $e_\rho \cup \Phi_k = \Phi_{k+n}$ for $\Phi_k \in \mathcal{H}_\rho$, and $n = (n, n, \ldots, n)$.

Finally, we come to the result, whose proof is straightforward.
Lemma 3.4. if \( \phi \) is an ordinary presentation. Thus \( R^+_n \) is a quotient of \( \mathcal{H} = \bigwedge^\ast(\mathcal{H}_l) \) whose kernel is the subalgebra generated by \( \{\phi_i\}_{i \geq 0} \). Thus \( R^+_n \) is isomorphic to \( \bigwedge^\ast(V(n)) \) where \( V(n) \) is the linear space spanned by \( \phi_0, \ldots, \phi_{n-1} \) and the action is given by wedge product from left. Then \( \{\phi_{k_1} \land \ldots \land \phi_{k_d}, k_1 < \ldots < k_d, 0 \leq d \leq n-1 \} \) form a basis of \( R^+_n \).

3.1.1. Two presentations of classes in the cohomology of Grassmannian. Proposition 3.3 implies that we can use the notations introduced in section 2.2 to represent cohomology classes of Grassmannians, as well as those in COHA, since they share the same product structure. Thus in \( H^\ast(Gr(d, n)) \), \( \Phi_k \) is a basis of the Schur polynomial \( s_{\lambda(k)}(x_1, \ldots, x_d, x_{d+1}, \ldots, x_n) \), where \( 0 \leq k_1 < \ldots < k_d \leq n-1 \) and \( \lambda = (\lambda_1, \ldots, \lambda_k) = (k_d - d + 1, k_{d-1} - d + 2, \ldots, k_1) \) is a partition of length \( \leq n \).

Let \( \lambda' \) be the transpose partition of \( \lambda \), and \( k' = k(\lambda') \). By Lemma 2.1 \( \Phi_k(x_1, \ldots, x_d) = (-1)^{|k|} \Phi_{k'}(x_{d+1}, \ldots, x_n) \). \( \Phi_k \) is called the ordinary presentation of the correspondent class \( s_\lambda \), and \( (-1)^{|k|} \Phi_{k'} \) is called the transpose presentation.

3.2. Decreasing operators. Our goal is to understand the decreasing representation using the basis \( \{\Phi_k\}_k \) of \( R^+_n \). From Section 3.1.1, the equation (2.5) can be rewritten as

\[
(\phi_i^+ \cdot \Phi_k)(x_1, \ldots, x_{d_i-1}) = \sum_{i_1 < \ldots < i_{d_i}} \frac{\Phi_k(x_{i_1}, \ldots, x_{i_{d_i}}) \phi_i(x_{i_{d_i}})}{\prod_{j=1}^{d_i+1}(x_{i_{d_i}} - x_j)} = (-1)^{|k|} \sum_{i_{d_1+1} < \ldots < i_{d_2}} \frac{\Phi_{k'}(x_{i_{d_2}}, \ldots, x_{i_{d_1}}) \phi_i(x_{i_{d_1}})}{\prod_{j=1}^{n}(x_{i_{d_2}} - x_j)} = (-1)^{|k|+d_i-1} (\phi_i^+ \cdot \Phi_{k'}) (x_{d_i}, \ldots, x_n).
\]

This formula suggests an algorithm. Start from an ordinary presentation of a class \( \phi_k = \phi_{k_1} \land \ldots \land \phi_{k_d} \) in \( H^\ast(Gr(d, n)) \), where \( k = (k_1, \ldots, k_d) \), and \( 0 \leq k_1 < \ldots < k_d \leq n-1 \). First we change \( \Phi_k(x_1, \ldots, x_d) \) to \( (-1)^{|k|} \Phi_{k'}(x_{d+1}, \ldots, x_n) \) by Lemma 2.1. Then apply \( \phi_i^+ \) to \( \Phi_{k'} \) using formula (3.1) and Proposition 3.3. Finally change the result back to the ordinary presentation.

We need the following lemma to help us to do these transformations.

Lemma 3.4. If \( \phi \) appears in \( \Phi_{k(\lambda)} \), \( \phi_{n-r-1} \) will not appear in \( \Phi_{k(\lambda')} \). On the other hand, if \( \phi \) doesn’t appear in \( \Phi_{k(\lambda)} \), \( \phi_{n-r-1} \) will appear in \( \Phi_{k(\lambda')} \).

Proof: From Section 3.3, \( \lambda = (\lambda_d, \ldots, \lambda_1) = (k_d - d + 1, k_{d-1} - d + 2, \ldots, k_1) \) is a partition of length \( \leq n \). The transpose partition is defined by \( \lambda'_j = \#(\lambda_i \geq n - d + 1 - j) \) for
\[ 1 \leq j \leq n - d. \text{ Thus we have } \]

\[ \lambda_{d-i+1} = \begin{cases} 
  n - d & \text{if } 1 \leq i \leq \lambda'_1, \\
  n - d - j & \text{if } \lambda'_j + 1 \leq i \leq \lambda'_{j+1} \text{ for } 1 \leq j \leq n - d - 1, \\
  0 & \text{if } \lambda'_{n-d} + 1 \leq i \leq d. 
\end{cases} \tag{3.2} \]

From \( \lambda = (\lambda_d, \ldots, \lambda_1) = (k_d - d + 1, k_{d-1} - d + 2, \ldots, k_1) \), it immediately implies

\[ k_{d-i+1} = \begin{cases} 
  n - i & \text{if } 1 \leq i \leq \lambda'_1, \\
  n - i - j & \text{if } \lambda'_j + 1 \leq i \leq \lambda'_{j+1} \text{ for } 1 \leq j \leq n - d - 1, \\
  d - i & \text{if } \lambda'_{n-d} + 1 \leq i \leq d. 
\end{cases} \tag{3.3} \]

Then \( n - k'_{j+1} = n - j - \lambda'_{j+1} \leq k_{d-i+1} = n - i - j \leq n - j - \lambda'_j - 1 = n - k'_j - 2 \) if \( \lambda'_j + 1 \leq i \leq \lambda'_{j+1} \) for \( 1 \leq j \leq n - d - 1 \), or \( 0 = d - d \leq k_{d-i+1} = d - i \leq d - \lambda'_{n-d} - 1 = n - k'_{n-d} - 2 \) if \( \lambda'_{n-d} + 1 \leq i \leq d \), or \( n - k'_1 = n - \lambda'_1 \leq k_d = n - i \leq n - 1 \). Therefore \( k_{d-i+1} \) would run over all integers between \( n - k'_{j+1} \) and \( n - k'_j - 2 \), or between 0 and \( n - k'_{n-d} - 2 \), or between \( n - k'_1 \) and \( n - 1 \). If \( \varphi_r \) doesn’t appear in \( \Phi_{k'(s)} \), there are three cases. If \( k'_s < r < k'_{s+1} \) for \( 1 \leq s \leq n - d - 1, n - k'_{s+1} \leq n - r - 1 \leq n - k'_s - 2 \). If \( k'_{n-d} < r \leq d, 0 \leq n - r - 1 \leq n - k'_{n-d} - 2 \). If \( 0 \leq r < k'_1, n - k'_1 \leq n - r - 1 \leq n - 1 \). This means that there exists some \( 1 \leq i \leq d \) such that \( k_{d-i+1} = n - r - 1 \).

If \( \varphi_r \) appear in \( \Phi_{k'(i)} = \phi_{k'_1} \wedge \ldots \wedge \phi_{k'_{n-d}} \), let \( r = k'_s \). Then \( k_{d-i+1} \) can never be \( n - k'_s - 1 = n - r - 1 \) for \( 1 \leq i \leq d \).

\[ \square \]

**Definition 3.5.** We introduce the right partial derivative operator \( \partial_i^R : \wedge^s(V(n)) \to \wedge^s(V(n)) \) to state the following proposition. For \( \Phi_k = \phi_{k_1} \wedge \ldots \wedge \phi_{k_d} \), if \( \varphi_i \) appears in \( \Phi_k, \partial_i^R(\Phi_k) = (-1)^{d-i} \phi_{k_i} \wedge \ldots \wedge \phi_{k_d} \). If \( \varphi_i \) does not appear in \( \Phi_k, \partial_i^R(\Phi_k) = 0. \)

**Proposition 3.6.** The decreasing operators are the right partial derivative operators on \( \wedge^*(V(n)) \): \( \phi_r^- \cdot \Phi_k = \partial^R_{n-r-1}(\Phi_k) \).

**Proof:** What we want is to compute \( \phi_r^- \cdot \Phi_k \). Based on formula (3.1), we have

\[ (\phi_r^- \cdot \Phi_k)(x_1, \ldots, x_{d-1}) = (-1)^{|\lambda|+n-d}(\phi_r^+ \cdot \Phi_k)(x_d, \ldots, x_n) \]

\[ = (-1)^{|\lambda|+n-d}(\varphi_r \wedge \phi_{k'_1} \wedge \ldots \wedge \phi_{k'_{n-d}})(x_d, \ldots, x_n). \tag{3.4} \]

If \( \varphi_{n-r-1} \) is not in the \( \Phi_k, \varphi_r \) will appear in \( \Phi_k' \). Thus \( \phi_r^- \cdot \Phi_k(x_1, \ldots, x_{d-1}) = (\varphi_r \wedge \phi_{k'_1} \wedge \ldots \wedge \phi_r \wedge \ldots \wedge \phi_{k'_{n-d}})(x_d, \ldots, x_n) = 0. \)
If \( \phi_{n-r-1} \) appears in \( \Phi_k = \phi_{k_1} \land \ldots \land \phi_{k_d} \), \( \phi_r \) won’t be in \( \Phi_{k'} = \phi_{k'_1} \land \ldots \land \phi_{k'_{n-d}} \). Assume \( k'_s < r < k'_{s+1} \). We have

\[
\phi_r \land \phi_{k'_1} \land \ldots \land \phi_{k'_{n-d}} = (-1)^s \phi_{k'_{s}} \land \ldots \land \phi_{k'_{s+1}} \land \phi_r \land \phi_{k'_{s+1}} \ldots \land \phi_{k'_{n-d}}. \tag{3.5}
\]

We have to change this back to the ordinary presentation. First, let’s find the partition associated to this polynomial. The index \( \lambda' = (l'_1, \ldots, l'_{n-d+1}) \) is given by

\[
l'_i = \begin{cases} 
k'_{i-1} & s + 2 \leq i \leq n - d + 1, \\
r & i = s + 1, \\
k'_i & 1 \leq i \leq s. 
\end{cases}
\tag{3.6}
\]

Then the new partition \( \mu' = (\mu'_{n-d+1}, \ldots, \mu'_1) \) is given by

\[
\mu'_i = \begin{cases} 
\lambda'_{i-1} - 1 & s + 2 \leq i \leq n - d + 1, \\
\lambda'_i - s & i = s + 1, \\
\lambda'_i & 1 \leq i \leq s.
\end{cases}
\tag{3.7}
\]

Next step is to recover the partition \( \mu \) from its transpose \( \mu' \). From the definition of transpose partition, \( \mu'_j = \#\{\mu_i \geq n - d + 2 - j\} \) for \( 1 \leq j \leq n - d - 1 \). Then

\[
\mu_{d-i} = \begin{cases} 
n - d - j & \text{if } \lambda'_j \leq i \leq \lambda'_{j+1} - 1 \text{ and } s + 1 \leq j \leq n - d - 1, \\
n - d - s & \text{if } r - s + 1 \leq i \leq \lambda'_{s+1} - 1, \\
n - d + 1 - s & \text{if } \lambda'_s + 1 \leq i \leq r - s, \\
n - d + 1 - j & \text{if } \lambda'_j + 1 \leq i \leq \lambda'_{j+1} \text{ and } 2 \leq j \leq s - 1, \\
n - d + 1 & \text{if } 1 \leq i \leq \lambda'_1.
\end{cases}
\tag{3.8}
\]

By comparing it with

\[
\lambda_{d-i+1} = \begin{cases} 
in - d - j & \text{if } \lambda'_j + 1 \leq i \leq \lambda'_{j+1} \text{ and } 2 \leq j \leq n - d - 1, \\
n - d & \text{if } 1 \leq i \leq \lambda'_1,
\end{cases}
\tag{3.9}
\]

we notice that \( \mu_i = \lambda_{i+1} + 1 \) for \( d - r + s \leq i \leq d - 1 \) and \( \mu_i = \lambda_i \) for \( 1 \leq i \leq d - r + s - 1 \).

Therefore, since \( l_i = \mu_i + i - 1 \) for \( 1 \leq i \leq d - 1 \) and \( k_j = \lambda_j + j - 1 \) for \( 1 \leq j \leq d \), it is easy to see that \( l_i = k_{i+1} \) for \( d - r + s \leq i \leq d - 1 \) and \( l_i = k_i \) for \( 1 \leq i \leq d - r + s - 1 \). Thus the resulted presentation is \((-1)^{n-d+s+|d|+|d|} \phi_{k_1} \land \ldots \land \phi_{n-r-1} \land \ldots \land \phi_{k_d} = (-1)^{r+s} \phi_{k_1} \land \ldots \land \phi_{n-r-1} \land \ldots \land \phi_{k_d} = \hat{\partial}_{n-r-1}^R(\Phi_k) \), which is \( \Phi_k \) applied by the right partial derivative of \( \phi_{n-r-1} \). If \( r < k'_1 \) or \( r > k'_{n-d} \), the similar process will lead to the same result. \( \square \)
4. The double of representations

Let $V(n)$ be the $n$-dimensional vector space spanned by $\{\phi_i\}_{i=0}^{n-1}$. The increasing and decreasing representations can be realized as creation operators $\{\alpha_i^+\}_{i=0}^{n-1}$ and annihilation operators $\{\alpha_i^-\}_{i=0}^{n-1}$ on $\bigwedge^\perp(V(n))$. Here $\alpha_i^+ = \phi_i^+$ is the left wedge product, and $\alpha_i^- = \phi_{n-i-1}$ is the right partial derivative $\partial_i^R$.

Define $H = [\alpha_0^+, \alpha_0^-]$ and the following operators for $0 \leq i \leq n - 1$:

$$\begin{align*}
T_i &= \frac{\alpha_i^+ + [H, \alpha_i^+]}{2}, \\
S_i &= \frac{\alpha_i^- - [H, \alpha_i^-]}{2}.
\end{align*}$$

Then define the following operators

$$\begin{align*}
E_0 &= -\frac{\alpha_0^+ + [H, \alpha_0^+]}{2}, \\
F_0 &= \frac{\alpha_0^- - [H, \alpha_0^-]}{2}, \\
E_1 &= S_0, \\
F_1 &= T_0, \\
E_i &= [T_{i-2}, S_{i-1}], \\
F_i &= [T_{i-1}, S_{i-2}], \quad \text{for } 2 \leq i \leq n, \\
H_i &= [E_i, F_i], \quad \text{for } 0 \leq i \leq n.
\end{align*}$$

In the following, let $P_k$ be an arbitrary degree $k$ monomial in $\bigwedge^\perp(V(n))$. Denote by $R_i^j$ the operator which change the factor $\phi_i$ in $P_k$ to $\phi_j$.

**Lemma 4.1.** For $2 \leq i \leq n$,

1. $H(P_k) = (-1)^{k-1} P_k$.
2. $E_0(P_k) = -\partial_0^R(P_k)$ if $k$ is even, and $\phi_0$ is included in $P_k$. Otherwise it’s 0.
3. $F_0(P_k) = \phi_0 \wedge P_k$ if $k$ is odd, and $\phi_0$ is NOT included in $P_k$. Otherwise it’s 0.
4. $E_1(P_k) = \partial_0^R(P_k)$ if $k$ is odd, and $\phi_0$ is included in $P_k$. Otherwise it’s 0.
5. $F_1(P_k) = \phi_0 \wedge P_k$ if $k$ is even, and $\phi_0$ is NOT included in $P_k$. Otherwise it’s 0.
6. $S_{i-1}(P_k) = \partial_i^R(P_k)$ if $k$ is odd, and $\phi_{i-1}$ is included in $P_k$. Otherwise it’s 0.
7. $T_{i-1}(P_k) = \phi_{i-1} \wedge P_k$ if $k$ is even, and $\phi_{i-1}$ is NOT included in $P_k$. Otherwise it’s 0.
8. $E_i(P_k) = R_{i-2}^i(P_k)$ if $\phi_{i-1}$ is included in $P_k$ and $\phi_{i-2}$ is NOT. Otherwise it’s 0.
9. $F_i(P_k) = R_{i-2}^i(P_k)$ if $\phi_{i-2}$ is included in $P_k$ and $\phi_{i-1}$ is NOT. Otherwise it’s 0.
10. $H_0(P_k) = \begin{cases} -P_k & \text{k is even and } \phi_0 \text{ is included in } P_k \\ P_k & \text{k is odd and } \phi_0 \text{ is NOT included in } P_k \\ 0 & \text{otherwise} \end{cases}$
11. $H_1(P_k) = \begin{cases} -P_k & \text{k is odd and } \phi_0 \text{ is included in } P_k \\ P_k & \text{k is even and } \phi_0 \text{ is NOT included in } P_k \\ 0 & \text{otherwise} \end{cases}$
Proof: The proof of the lemma is straightforward. 

The main theorem below implies that the combination of two representations $R^+_n$ and $R^-_n$ of $A_1$-COHA forms an $D_{n+1}$-Lie algebra.

**Theorem 4.2.** The above operators satisfy the Serre relations for $0 \leq i, j \leq n$:

1. $[H_i, H_j] = 0$,
2. $[E_i, F_j] = \delta_{ij} H_i,$
3. $[H_i, E_j] = a_{ij} E_j, \quad [H_i, F_j] = -a_{ij} F_j,$
4. $(adE_i)^{-a_{ij}+1}(E_j) = 0$, if $i \neq j$.
5. $(adF_i)^{-a_{ij}+1}(F_j) = 0$, if $i \neq j$.

where $(a_{ij})$ is the Cartan matrix for $D_{n+1}$-Lie algebras.

Proof: The first statement holds since each $H_i$ is diagonal by Lemma 4.1. The second is due to the definition of $H_i$ for $\delta_{ij} = 1$. For the other relations, we need to check the following relations, which can be easily solved by Lemma 4.1:

1. $a_{ii} = 2$, for $0 \leq i \leq n$,
2. $a_{21} = a_{20} = a_{12} = a_{02} = a_{i-1,i} = a_{i-1,i} = -1$ for $3 \leq i \leq n$,
3. $a_{10} = a_{01} = a_{0,i} = a_{i,0} = a_{i,i} = a_{i,i} = a_{i,j} = a_{j,i} = 0$, for $3 \leq i \leq n$, $2 \leq j \leq n$ and $|i - j| > 1$.
4. $[E_0, F_1] = [E_1, F_0] = [E_0, F_j] = [E_1, F_j] = [E_i, F_0] = [E_i, F_1] = [E_i, F_j] = 0$, for $2 \leq i \neq j \leq n$.
5. $[E_2, [E_2, E_0]] = [E_0, [E_0, E_2]] = [F_2, [F_2, F_0]] = [F_0, [F_0, F_2]] = 0$,
6. $[E_{i-1}, [E_{i-1}, E_i]] = [E_i, [E_i, E_{i-1}]] = [F_{i-1}, [F_{i-1}, F_i]] = [F_i, [F_i, F_{i-1}]] = 0$, for $2 \leq i \leq n$.
7. $[E_0, E_i] = [E_0, E_i] = [E_i, E_j] = 0$ for $3 \leq i \leq n$, $2 \leq j \leq n$ and $|i - j| > 1$.
8. $[F_0, F_i] = [F_0, F_i] = [F_i, F_i] = [F_i, F_j] = 0$, for $3 \leq i \leq n$, $2 \leq j \leq n$ and $|i - j| > 1$.

□
5. Further discussions

For fixed $n$, the double of $R_n^+$ and $R_n^-$ forms $D_{n+1}$-Lie algebra. This leads to the following conjecture stated in [9].

**Conjecture 5.1.** [9] Full COHA for the quiver $A_1$ is isomorphic to the infinite Clifford algebra $Cl_c$ with generators $\phi_n^\pm$, $n \in \mathbb{Z}$ and the central element $c$, subject to the standard anticommuting relations between $\phi_n^+$ (resp. $\phi_n^-$) as well as the relation $\phi_n^+ \phi_m^- + \phi_m^- \phi_n^+ = \delta_{n,m}^c$.

**Remark 5.2.** As stated in [9], in the case of finite-dimensional representations we have $c \mapsto 0$ and we see two representations of the infinite Grassmann algebra, which are combined in the representations of the orthogonal Lie algebra.

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Mathematical Department, Kansas State University, Cardwell Hall 128, Manhattan, Kansas, 66502

E-mail address: xiaoxl@math.ksu.edu