On A New Form of Darboux-Bäcklund Transformation for DNLS Equation-Mixed and Rational Type Solutions.

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Abstract

A new form of Darboux-Bäcklund transformation and its higher order form is derived for Derivative Nonlinear Schrödinger Equation (DNLS). The new form arises due to the different form of Lax pair. It is observed that by a special choice of the eigenvalue of DB transformation one can generate a mixed form of solution (containing both algebraic and exponential dependence on \((x, t)\) can be generated. On the other hand by adopting a new methodology due to Neugebauer et. al. it is found that purely rational solution can be constructed. The two different approach yields different class of solution and are compared.

PACS Number(s): 05.45.Pq, 05.45.Ac, 05.45.-a.
Keywords: Darboux-Bäcklund, Neugebauer, Multisoliton state, Rational Solution.

1 Introduction

Darboux-Bäcklund transformation (DBT) is known to be a very useful tool to construct new solution of various non-linear equations which are associated with a Lax pair\(^1\). Of course there exist other approaches for the generation of multisolitary states. Most versatile of them is the Inverse Scattering transform\(^2\). But that requires a detailed analyticity study of the spectral problem in the complex eigenvalue plain\(^3\) for establishing the Gelfand-Levitan-Marchenko integro-differential equation\(^4\). Such an analysis is quite elaborate and time consuming. That is why Darboux-Bäcklund transformation is now a days so popular for the construction of a bigger class of solitary solutions\(^5\). Also, in this approach one has the freedom of choice for the seed solutions. Here, we have adopted both the traditional approach\(^6\), which usually uses two eigenfunction at a time and the Neugebauer method\(^7\) for which it is sufficient to fix one eigenvalue and corresponding eigenfunction\(^8\). These two methods lead to two different class of solutions, one is a mixture of rational and exponential function and the other is purely rational solution.

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2 Formulation

There exist various forms of Lax pair for the Derivative Nonlinear Schrodinger Equation (DNLS), from which we consider the following one given in reference (9). The equations are

\[ \begin{align*}
q_t + iq_{xx} + (rq^2)_x &= 0 \\
r_t - ir_{xx} + (r^2q)_x &= 0 \quad (1)
\end{align*} \]

and the Lax pair is

\[ \begin{align*}
\Psi_x &= M \Psi \\
\Psi_t &= N \Psi \quad (2)
\end{align*} \]

with

\[ \begin{align*}
M &= J\lambda^2 - R\lambda \\
N &= 2J\lambda^4 - 2R\lambda^3 + qrJ\lambda^2 + U\lambda \quad (3)
\end{align*} \]

where

\[ \begin{align*}
J &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad (4) \\
R &= \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \quad (5) \\
U &= \begin{pmatrix} 0 & -iq_x - rq^2 \\ ir_x - r^2q & 0 \end{pmatrix} \quad (6)
\end{align*} \]

The eigenfunction \( \Psi \) is written as

\[ \Psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} \quad (7) \]

If the seed solution is chosen as \( q = q_0 \exp[i(kx - \omega t)] \) and \( r = r_0 \exp[-i(kx - \omega t)] \) with \( q_0 \) and \( r_0 \) being constants and satisfying equation (1) then the Lax eigenfunctions turn out to be

\[ \begin{align*}
\psi_{11}(x, t) &= \frac{i\lambda q_0}{\alpha_+ - \lambda^2} \exp[i(\alpha_+ x - \beta_+ t)] \\
\psi_{21}(x, t) &= -\frac{i(P(\lambda) + \beta_+)}{Q(\lambda)} \exp[i((\alpha_+ - k)x - (\beta_+ - \omega)t)] \\
\psi_{12}(x, t) &= \frac{i\lambda q_0}{\alpha_- - \lambda^2} \exp[i(\alpha_- x - \beta_- t)] \\
\psi_{22}(x, t) &= -\frac{i(P(\lambda) + \beta_-)}{Q(\lambda)} \exp[i((\alpha_- - k)x - (\beta_- - \omega)t)] \quad (8)
\end{align*} \]

with

\[ \begin{align*}
P(\lambda) &= 2\lambda^4 + \lambda^2 q_0 r_0 \\
Q(\lambda) &= -\lambda r_0 q_0^2 + \lambda k q_0 - 2\lambda^3 q_0 \\
R(\lambda) &= -2\lambda^3 r_0 + \lambda k r_0 - \lambda r_0^2 q_0 \quad (9)
\end{align*} \]
\[ \alpha_{\pm} = \frac{1}{2} [k \pm \{k^2 + 4\lambda^2(\lambda^2 - k - q_0r_0)\}^{1/2}] \]
\[ \beta_{\pm} = \frac{1}{2} [\omega \pm \{\omega^2 + 4(P^2 + \omega P - RQ)\}^{1/2}] \] (10)

Next we choose the Darboux-Bäcklund transformation
\[ \Psi' = D\Psi \] (11)
where, \( \Psi' \) satisfies the similar Lax equations as (2) with new nonlinear fields \((q', r')\). In our case \( D(\lambda) \)
\[ D(\lambda) = \lambda^2 \begin{pmatrix} a_2 & 0 \\ 0 & d_2 \end{pmatrix} + \lambda \begin{pmatrix} 0 & b_1 \\ c_1 & 0 \end{pmatrix} + \begin{pmatrix} a_0 & 0 \\ 0 & d_0 \end{pmatrix} \] (12)

The equation satisfied by \( D(\lambda) \) is
\[ D_x = M'D - DM \\
D_t = N'D - DN \] (13)

It may be added at this point that the usual choice of \( D \)
\[ D(\lambda) = \lambda \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \] (14)

does not hold in our case because the equation satisfied by \( D(\text{equation(13)}) \) leads to a trivial solution \( D = \text{null matrix} \). This is happening due to the form of the Lax pair(equation(2)) which does not contain any term free from \( \lambda \).

From the condition that \( \det D(\lambda) = 0 \) at \( \lambda = \lambda_1 \) and \( \lambda_2 \), we can determine
\[ a_2 = \frac{a_0(\lambda_1 \psi_2 \phi_2 - \lambda_2 \psi_1 \phi_1)}{\lambda_1 \lambda_2 (\lambda_1 \psi_1 \phi_2 - \lambda_2 \psi_2 \phi_1)} \]
\[ d_2 = \frac{d_0(\lambda_2 \psi_1 \phi_2 - \lambda_1 \psi_1 \phi_1)}{\lambda_1 \lambda_2 (\lambda_2 \psi_1 \phi_2 - \lambda_1 \psi_2 \phi_1)} \]
\[ b_1 = \frac{a_0(\lambda_2^2 - \lambda_1^2) \psi_1 \phi_1}{\lambda_1 \lambda_2 (\lambda_1 \psi_1 \phi_2 - \lambda_2 \psi_2 \phi_1)} \]
\[ c_1 = -\frac{d_0(\lambda_2^2 - \lambda_1^2) \psi_2 \phi_2}{\lambda_1 \lambda_2 (\lambda_2 \psi_1 \phi_2 - \lambda_1 \psi_2 \phi_1)} \] (15)

along with the relations
\[ q' = a_2^2 q + 2ib_1 a_2 \]
\[ r' = d_2^2 r - 2ic_1 d_2 \] (16)

If one now uses eigenfunctions(8) in these one can construct a pair of nontrivial solutions, by starting with a constant solution. These can be finally written as
\[ q' = \frac{\Lambda^2(\lambda_2, \lambda_1)}{\Lambda^2(\lambda_1, \lambda_2)} \left( q + 2i \frac{\lambda_2^2 - \lambda_1^2}{\Lambda(\lambda_2, \lambda_1)} \psi_1 \phi_1 \right) \]
\[ r' = \frac{\Lambda^2(\lambda_1, \lambda_2)}{\Lambda^2(\lambda_2, \lambda_1)} \left( r + 2i \frac{\lambda_1^2 - \lambda_2^2}{\Lambda(\lambda_1, \lambda_2)} \psi_2 \phi_2 \right) \] (17)
where

\[
\Lambda(\lambda_2, \lambda_1) = \lambda_2 \psi_1 \phi_2 - \lambda_1 \psi_2 \phi_1 \\
= \lambda_1 \lambda_2 \phi_0 \Theta
\]  

(18)

where

\[
\Theta = \left( \frac{P(\lambda_2) + \sigma_2}{(\nu_1 - \lambda_1^2)Q(\lambda_2)} - \frac{P(\lambda_1) + \sigma_1}{(\nu_2 - \lambda_2^2)Q(\lambda_1)} \right) \exp(i \int (\mu_1, \sigma_1, \delta_1, k, \omega, x, t)) \\
+ k_2 \left( \frac{P(\lambda_2) + \delta_2}{(\mu_1 - \lambda_1^2)Q(\lambda_2)} - \frac{P(\lambda_1) + \delta_1}{(\mu_2 - \lambda_2^2)Q(\lambda_1)} \right) \exp(i \int (\mu_1, \sigma_1, \delta_1, k, \omega, x, t)) \\
+ k_1 \left( \frac{P(\lambda_2) + \sigma_2}{(\nu_1 - \lambda_1^2)Q(\lambda_2)} - \frac{P(\lambda_1) + \delta_1}{(\nu_2 - \lambda_2^2)Q(\lambda_1)} \right) \exp(i \int (\nu_1, \sigma_1, \delta_1, k, \omega, x, t)) \\
+ k_1 k_2 \left( \frac{P(\lambda_2) + \sigma_2}{(\nu_1 - \lambda_1^2)Q(\lambda_2)} - \frac{P(\lambda_1) + \delta_1}{(\nu_2 - \lambda_2^2)Q(\lambda_1)} \right) \exp(i \int (\nu_1, \sigma_1, \delta_1, k, \omega, x, t))
\]  

(19)

Here, the function \( \int \) is given by

\[
\int (a, b, c, d, k, \omega, x, t) = (a + b - k)x - i(c + d - \omega)t
\]  

(20)

with a similar expressions for other \( \Lambda \)'s.

### 3 Second Order Transformation

One of the most important properties of Darboux-Bäcklund transformation it can be repeated for finite number of times to generate higher solutions. Let us consider that the second time transformed eigenfunction is \( \Psi''; \Psi'' = D_1 \Psi' = D_1 D \Psi \). Here, \( D_1 \) stands for the next DB operator written as

\[
D_1(\lambda) = \lambda^2 \left( \begin{array}{cc} A_2 & 0 \\ 0 & D_2 \end{array} \right) + \lambda \left( \begin{array}{cc} 0 & B_1 \\ C_1 & 0 \end{array} \right) + \left( \begin{array}{cc} A_0 & 0 \\ 0 & D_0 \end{array} \right)
\]

(21)

As per previous formulae

\[
A_2 = \frac{A_0(\lambda_1 \phi_1' \psi_2' - \lambda_2 \psi_2' \phi_1')}{\lambda_1 \lambda_2(\lambda_1 \phi_1' \phi_2' - \lambda_2 \psi_2' \phi_1')} \\
D_2 = \frac{D_0(\lambda_2 \phi_2' \psi_1' - \lambda_1 \phi_1' \phi_2')}{\lambda_1 \lambda_2(\lambda_2 \phi_2' \phi_1' - \lambda_1 \psi_1' \phi_2')} \\
B_1 = \frac{A_0(\lambda_2^2 - \lambda_1^2) \psi_1' \phi_1'}{\lambda_1 \lambda_2(\lambda_2 \phi_1' \phi_1' - \lambda_1 \psi_1' \phi_2')} \\
C_1 = -\frac{D_0(\lambda_2^2 - \lambda_1^2) \psi_2' \phi_2'}{\lambda_1 \lambda_2(\lambda_2 \phi_2' \phi_1' - \lambda_1 \psi_2' \phi_1')}
\]

(22)

where, \( \Psi' = D \Psi \). After a lot of simplification we get

\[
q'' = \frac{\Lambda^2(\lambda_2, \lambda_1)}{\Lambda^2(\lambda_1, \lambda_2)} \left( q' + 2i \frac{\lambda_1^2 - \lambda_2^2}{\Lambda' \Lambda''} \psi_1' \phi_1' \right)
\]

\[
r'' = \frac{\Lambda^2(\lambda_1, \lambda_2)}{\Lambda^2(\lambda_2, \lambda_1)} \left( r' + 2i \frac{\lambda_1^2 - \lambda_2^2}{\Lambda' \Lambda''} \psi_2' \phi_2' \right)
\]

(23)
4 Mixed Rational Solution

4.1 Case 1

Now let us start from a set of constant seed solutions i.e.; \( q_0 = \text{const} \) and \( r_0 = \text{const} \) and an eigenvalue \( \lambda = \sqrt{q_0 r_0} \). Then it is easy to ascertain that the Lax eigenfunctions are

\[
\psi_1(\lambda_1) = \psi_{11}(\lambda_1) + k_1 \psi_{12}(\lambda_1) \\
= (A + k_1 A')x \\
\psi_2(\lambda_1) = \psi_{21}(\lambda_1) + k_1 \psi_{22}(\lambda_1) \\
= (C + k_1 C')x + (D + k_1 D')t
\]

(24)

Let us choose \( \lambda_2 = \lambda_0 \) any fixed constant eigenvalue and then the second set of eigenfunction turns out to be

\[
\psi_{11}(\lambda_0) = \exp[\zeta_+ x + \delta_+ t] \\
\psi_{21}(\lambda_0) = \exp[\zeta_- x + \delta_- t]
\]

(25)

whence;

\[
\phi_1(\lambda_2) = \phi_1(\lambda_0) \\
= \psi_{11}(\lambda_0) + k_2 \psi_{12}(\lambda_0) \\
= (1 + k_2) \exp[\zeta_+ x + \delta_+ t] \\
\phi_1(\lambda_2) = (1 + k_2) \exp[\zeta_- x + \delta_- t]
\]

(26)

where, \( \delta_\pm \) are solutions of

\[
\delta^2 + (2\lambda_0^4 + \lambda_0^2 q_0 r_0)^2 - (2q_0 \lambda_0^3 + r_0 q_0^2)(2\lambda_0^3 r_0 - r_0 q_0) = 0
\]

(27)

Now if these two sets of eigenfunctions are used in formula\( \text{e}\) (17) we get

\[
q' = q_0 \left[ \mu_1 x t \exp(\zeta_- x + \delta_- t) - (\mu_2 x + \mu_3 x) \exp(\zeta_+ x + \delta_+ t) \right] \\
+ 2i \left[ \frac{\mu_4 x + \mu_5 t}{(\mu_4 x + \mu_5 t) \exp(\zeta_+ x + \delta_+ t) - \mu_6 x t \exp(\zeta_- x + \delta_- t)} \right] \\
\times (\lambda_0^2 - q_0^2) \mu_10 x t \exp(\zeta_+ x + \delta_+ t)
\]

(28)

Where, \( \mu_i \) are constants being combination of those occurring in the expressions of \( \phi_i \), \( \psi_i \)'s. The second formula\( \text{e}\) of (17) can be used similarly to construct an expression for \( r' \) also. It is interesting to see that the above expression is a mixture of algebraic expressions in \( (x, t) \) and also exponential functions. \( \mu_1 = -\mu_6 = \mu_10 = \mu_7 = q_0 = r_0 = 1, \mu_2 = \mu_5 = \mu_8 = 2i \) and \( \mu_3 = \mu_4 = \mu_9 = 3i \) with \( \zeta_+ = \zeta_- = \delta_- = -\delta_+ = 1 \) and \( \lambda_0 = \sqrt{2} \). Fig(\( \text{e}\)) represents \( U = | q' |^2 \) for \( \mu_1 = -\mu_6 = \mu_10 = \mu_7 = q_0 = r_0 = 1, \mu_2 = \mu_5 = \mu_8 = 2i \) and \( \mu_3 = \mu_4 = \mu_9 = 3i \) with \( \zeta_+ = \zeta_- = \delta_- = -\delta_+ = 1 \) and \( \lambda_0 = \sqrt{2} \). Fig(\( \text{e}\)) represents the \( V = | q' |^2 \) choosing \( \delta_+ = 1 \) retaining all the other values of the constants unchanged.

4.2 Case 2

On the other hand a pair of nonconstant seed solutions of the following form

\[
q = q_0 \exp(-iq_0^2 x) \\
r = -q_0 \exp(iq_0^2 x)
\]

(29)
Figure 1: $U$ vs space\;time

Figure 2: $a$
Figure 3: a
leads us to mixed-rational solutions for eigenvalues $\lambda = q_0 \frac{i+1}{2}$. Following the same procedure as described above we obtain the 1st order transformed solution of Darboux-Bäcklund type

$$q' = q_0 \exp(-iq_0^2 x) \left[ \frac{1 + 2iq_0^2 x - 2q_0^4(1 - q_0^4)t - q_0^4 x^2 - 2iq_0^6 xt}{1 - 2iq_0^2 x + 2q_0^4(1 - q_0^4)t + q_0^4 x^2 + 2iq_0^6 xt} \right]^2 \times \left[ 1 - 2iq_0^2 x - 2q_0^4(1 - q_0^4)t - q_0^4 x^2 - 2iq_0^6 xt \right]$$

and

$$r' = -q_0 \exp(iq_0^2 x) \left[ \frac{1 - 2iq_0^2 x + 2q_0^4(1 - q_0^4)t + q_0^4 x^2 + 2iq_0^6 xt}{1 + 2iq_0^2 x - 2q_0^4(1 - q_0^4)t - q_0^4 x^2 - 2iq_0^6 xt} \right]^2 \times \left[ 1 + 2iq_0^2 x + 2q_0^4(1 - q_0^4)t + q_0^4 x^2 + 2iq_0^6 xt \right]$$

Fig() and fig() represent $Q = |q'|^2$ and $R = |r'|^2$ with $q_0 = 1$. Fig() and fig() represent $Q$ vs $x$ keeping $t$ fixed and $Q$ vs $t$ keeping $x$ fixed while fig() and fig() represent the corresponding diagrams for $R$.

5 Comments and Discussion

In our above analysis we have shown that by starting with two constant solutions (case-1) or non-constant seed solutions (case-2) it is possible to generate a new form of mixed rational solutions (for the DNLS equation) which contain both polynomial in $(x,t)$ as well as exponential functions of the same. This was accomplished with help of Darboux-Bäcklund transformation and a special choice of $(\lambda_1, \lambda_2)$. On the other hand a different approach to DBT was proposed by Neugebauer et. al. which does not require two such eigenfunctions and eigenvalues. In this approach we write our Lax operator as

$$L = \chi^2 L_2 - \chi L_1$$

and the starting one as $L_0$, then DBT matrix say $P$ should satisfy

$$LP = P_x + PL_0$$
Figure 5: Space vs Q at time

Figure 6: Space vs R at time

Figure 7: two dimensional plot
where $P$ is written as

$$ P(x, t, \lambda) = \sum_{j=1}^{N-1} \lambda^j P_j + \lambda^N \mathbf{1} $$

with

$$ P_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} $$

whence equation (33) written in full can be seen to be equal to

$$ \begin{bmatrix} i\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \end{bmatrix} \left[ \sum_{j=1}^{N-1} \lambda^j \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} + \lambda^N \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] $$

$$ = \sum_{j=1}^{N-1} \lambda^j \begin{pmatrix} A_{jx} & B_{jx} \\ C_{jx} & D_{jx} \end{pmatrix} $$

$$ + \left[ \sum_{j=1}^{N-1} \lambda^j \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} + \lambda^N \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \left[ i\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 0 & q_0 \\ r_0 & 0 \end{pmatrix} \right] $$

Equating the coefficients of $\lambda^{N+1}$ we get

$$ q = q_0 + 2iB_{N-1} $$

$$ r = r_0 - 2iC_{N-1} $$

where as the coefficients $B_N$, $C_N$ etc. are obtained as solutions of

$$ \sum_{j=1}^{N-1} (A_j + \beta_i B_j) \lambda_i^j = -\lambda_i^N $$

$$ \sum_{j=1}^{N-1} (D_j + \beta_i^{-1} C_j) \lambda_i^j = -\lambda_i^N $$

(39)
with \((\lambda_i \mid i = 1, 2, \cdots 2N)\) being \(2N\) solutions of \(\det P\). Solving equation(40) by Cramer’s rule leads to

\[
B_{N-1} = \frac{\Delta^{(B)}}{\Delta_1}
\]

\[
C_{N-1} = \frac{\Delta^{(C)}}{\Delta_2}
\]

(40)

where \(\Delta^s\)’s are determinants written below

\[
\Delta_1 = \begin{vmatrix} 
1 & \beta_1 & \lambda_1 & \beta_1 \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{N-1} & \lambda_1^{N-1} \beta_1 \\
1 & \beta_2 & \lambda_2 & \beta_2 \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{N-1} & \lambda_2^{N-1} \beta_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \beta_{2N} & \lambda_{2N} & \beta_{2N} \lambda_{2N} & \lambda_{2N}^2 & \ldots & \lambda_{2N}^{N-1} & \lambda_{2N}^{N-1} \beta_{2N} 
\end{vmatrix}
\]

(41)

\[
\Delta^{(B)} = \begin{vmatrix} 
1 & \beta_1 & \lambda_1 & \beta_1 \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{N-1} & \lambda_1^{N} \\
1 & \beta_2 & \lambda_2 & \beta_2 \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{N-1} & \lambda_2^{N} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \beta_{2N} & \lambda_{2N} & \beta_{2N} \lambda_{2N} & \lambda_{2N}^2 & \ldots & \lambda_{2N}^{N-1} & \lambda_{2N}^{N} 
\end{vmatrix}
\]

(42)

\[
\Delta_2 = \begin{vmatrix} 
1 & \beta_1^{-1} & \lambda_1 & \beta_1^{-1} \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{N-1} & \lambda_1^{N-1} \beta_1^{-1} \\
1 & \beta_2^{-1} & \lambda_2 & \beta_2^{-1} \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{N-1} & \lambda_2^{N-1} \beta_2^{-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \beta_{2N}^{-1} & \lambda_{2N} & \beta_{2N}^{-1} \lambda_{2N} & \lambda_{2N}^2 & \ldots & \lambda_{2N}^{N-1} & \lambda_{2N}^{N-1} \beta_{2N}^{-1} 
\end{vmatrix}
\]

(43)

\[
\Delta^{(C)} = \begin{vmatrix} 
1 & \beta_1^{-1} & \lambda_1 & \beta_1^{-1} \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{N-1} & \lambda_1^{N} \\
1 & \beta_2^{-1} & \lambda_2 & \beta_2^{-1} \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{N-1} & \lambda_2^{N} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \beta_{2N}^{-1} & \lambda_{2N} & \beta_{2N}^{-1} \lambda_{2N} & \lambda_{2N}^2 & \ldots & \lambda_{2N}^{N-1} & \lambda_{2N}^{N} 
\end{vmatrix}
\]

(44)

where \(\beta_i\) stands for

\[
\beta_i = \frac{\psi_{21}^0 - b_i \psi_{22}^0}{\psi_{11}^0 - b_i \psi_{12}^0}
\]

(45)

Suppose \(\lambda = \sqrt{\theta_0 r_0}\) and \(P(\lambda)\) is so chosen that it has a root at \(\lambda_1\), then

\[
\beta_i = \beta_i(\lambda_1) = \frac{\psi_{21}^0(\lambda_1) - b_i(\lambda_1) \psi_{22}^0(\lambda_1)}{\psi_{11}^0(\lambda_1) - b_i(\lambda) \psi_{12}^0(\lambda_1)}
\]

(46)

Choose as before \(\psi_{21}^0 = Axt, \psi_{22}^0 = A'xt, \psi_{11}^0 = Cx + Dt\) and \(\psi_{12}^0 = C'x + D't\), whence

\[
\beta_1 = \frac{(A - b_1 A')xt}{(C - b_1 C')x + (D - b_1 D')t}
\]

(47)

the whole form of the solution equation(38) is composed of \(\beta_1\) and its powers. So it is now purely rational one. In a similar vein for nonconstant seed solutions as taken in equation(29)
one can construct $\beta_1$ corresponding to both $\lambda_\pm = q_0\frac{i+1}{2}$ with respective seed eigenfunctions given below.

For $\lambda = \lambda_+$

\[
\psi_{11}^0(\lambda_+) = A_0[1 - q_0^2(x + q_0^2(i - 1)t)] \exp(-iq_0^2x/2) \\
\psi_{21}^0(\lambda_+) = B_0[1 + q_0^2(x + q_0^2(i - 1)t)] \exp(iq_0^2x/2)
\]

(48)

For $\lambda = \lambda_-$

\[
\psi_{11}^0(\lambda_-) = A'_0[1 + q_0^2(x + q_0^2(i - 1)t)] \exp(-iq_0^2x/2) \\
\psi_{21}^0(\lambda_-) = B'_0[1 - q_0^2(x + q_0^2(i - 1)t)] \exp(iq_0^2x/2)
\]

(49)

Assuming similar expressions for $\psi_{12}$ and $\psi_{22}$ corresponding $\beta_1$ can be constructed involving both rational and exponential part. this when put in the expressions of $q'$ and $r'$ gives mixed-rational solutions. So, with the help of Nuegebauer approach we have been able to construct both purely rational and mixed solutions of DNLS equation. The basic difference between these two methodologies is that in the Neugebauer approach we need one eigenfunction and one eigenvalue in contrast to the traditional one. So it can be summarized to DBT can be effectively utilized to generate new class of solutions of integrable nonlinear systems.

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