Given a principal bundle on an orientable closed surface with compact connected structure group, we endow the space of based gauge equivalence classes of smooth connections relative to smooth based gauge transformations with the structure of a Fréchet manifold. Using Wilson loop holonomies and a certain characteristic class determined by the topology of the bundle, we then impose suitable constraints on that Fréchet manifold that single out the based gauge equivalence classes of central Yang-Mills connections but do not directly involve the Yang-Mills equation. We also explain how our theory yields the based and unbased gauge equivalence classes of all Yang-Mills connections and deduce the stratified symplectic structure on the space of unbased gauge equivalence classes of central Yang-Mills connections. The crucial new technical tool is a slice analysis in the Fréchet setting.

**Keywords:** Yang-Mills connection on a closed Riemann surface, moduli space of smooth Yang-Mills connections on a closed Riemann surface in the Fréchet setting, stratified symplectic structure on the moduli space of Yang-Mills connections on a closed Riemann surface, moduli space of holomorphic vector bundles on a projective curve, Fréchet slice analysis

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1 Introduction

We rework and extend, in the framework of Fréchet manifolds, the approach of Atiyah and Bott [AB82] to the moduli spaces of Yang-Mills connections on a closed Riemann surface.

Let $G$ be a compact connected Lie group and $M$ a Riemannian manifold, and let $\xi: P \to M$ be a principal $G$-bundle. Yang-Mills theory proceeds by constructing moduli spaces of solutions of the relevant partial differential equations modulo gauge transformations. In the case of a manifold $M$ of arbitrary finite dimension, smooth solutions need not exist and, even if they exist, constructing solutions is a rather delicate endeavor. For example, Uhlenbeck established the local solvability in the Coulomb gauge [Uhl82]. The main difficulty in Yang-Mills theory resides, perhaps, in the fact that the space of connections is too big, with too much unwanted gauge freedom or gauge ambiguity.

We will here concentrate on the case where the base manifold $M$ is an orientable closed (compact) Riemann surface, and we write this surface as $\Sigma$ and denote by $\text{vol}_\Sigma$ a (suitably normalized) volume form on $\Sigma$. The situation now greatly simplifies. Indeed, the central Yang-Mills connections $A$ on $\xi$ (the Yang-Mills connections having central curvature) are characterized by the equation

$$\text{curv}_A = -X_\xi \cdot \text{vol}_\Sigma,$$

(1.1)

for a certain characteristic class $X_\xi$ in the center $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of $G$ determined by the topology of the bundle $\xi$. Concerning the sign, see Section 2 below. In a fundamental paper [AB82], Atiyah and Bott showed that, in the case at hand, the Yang-Mills solution space decomposes into a disjoint union of solution spaces of central Yang-Mills connections for reductions of the structure group. Furthermore, they described the structure of such a central Yang-Mills solution space using infinite dimensional techniques including symplectic reduction phrased in terms of suitable Sobolev spaces. They emphatically raised the issue of understanding the singularities of this kind of space. In the presence of singularities, the concept of a stratified symplectic space due to [SL91] satisfactorily isolates the singular structure of a space that arises by symplectic reduction in finite dimensions. This approach has so far not been available for a space which results from an infinite-dimensional construction, such as the solution space of central Yang-Mills connections. In particular, a rigorous theory of Poisson brackets in infinite dimensions is lacking and functional analytic issues obscure the underlying geometry.

We circumvent these technical issues and reduce the analysis of the singularities of the moduli space of central Yang–Mills connections to a finite-dimensional problem. As a new approach to the issue of gauge freedom, we perform reduction by stages, first relative to based gauge transformations and thereafter relative to the residual action of the structure group. We deliberately write “reduction by stages” rather than “symplectic reduction by stages”, since there is no obvious momentum mapping for the group of based gauge transformations. We therefore take the entire orbit space
relative to the action, on the space of smooth connections, of the Fréchet Lie group of based gauge transformations. We show that this orbit space is a Fréchet manifold. We then show that the Wilson loop mapping developed by the second-named author in [Hue98] yields a smooth map from the Fréchet manifold of based gauge equivalence classes of connections to the product manifold \( G^{2g} \) (technically the space of \( G \)-valued homomorphisms defined on the free group \( F \) on a family of \( 2\ell \) chosen generators of the fundamental group of \( \Sigma \)) where \( \ell \) denotes the genus of \( \Sigma \). We then impose, in terms of the holonomies around all contractible loops bounded by a disk, suitable constraints on that Fréchet manifold which single out the based gauge equivalence classes of smooth central Yang-Mills connections. These constraints recover the central curvature condition in \[AB82\] and thereby yield a replacement for the Yang-Mills equation. We show that the Wilson loop mapping (more precisely, the map (7.4) below) yields a homeomorphism between the space of based gauge equivalence classes of smooth connections satisfying the appropriate constraint (equivalently: space of based gauge equivalence classes of smooth central Yang–Mills connections) endowed with its Fréchet topology and a suitable space of homomorphisms into the structure group \( G \), realized within the product manifold \( G^{2g} \). We also show that, relative to the \( G \)-orbit stratifications, the homeomorphism under discussion is an isomorphism of stratified spaces which, on each stratum, restricts to a diffeomorphism. See Theorem 7.1 below for details.

A crucial step in the proof of Theorem 7.1 consists in establishing the fact that the Wilson loop mapping from that space of based gauge equivalence classes to that space of homomorphisms into the structure group \( G \) is a local homeomorphism. The continuity of that Wilson loop mapping is a consequence of the smoothness of the Wilson loop mapping defined on the ambient space. A construction in \[AB82, Section 6\] yields the inverse of the map under discussion, see Remark 7.3 below. While, on the one hand, the continuity of that inverse map is a consequence of Uhlenbeck compactness \[Uhl82\], that compactness theorem relies on Sobolev space techniques. We avoid these techniques and develop a proof that merely involves Fréchet space techniques. The proof we give includes the statement of the compactness theorem for our Fréchet space situation over a surface. See Remark 7.4 and Lemma 7.5 below.

Our basic tool is a slice analysis technique for groups of smooth gauge transformations in the Fréchet setting. This technique was developed in \[ACM89\] and \[Sub86\] and has thereafter been extended by the first-named author in \[Die13\]. It is phrased within the differential calculus due to Michal and Bastiani, see \[Nee06\] for further remarks concerning the differential calculus on locally convex spaces, and crucially involves the version of the Nash–Moser inverse function theorem given in \[Ham82\] in the tame Fréchet setting. Below we reproduce that result as Theorem 11.5. This differential calculus, in turn, has since been applied to global analytic problems in \[Ham82\], \[Mil84\], \[Nee06\] and elsewhere. The slice analysis technique enables us to reconstruct, in our framework, the stratified symplectic structure on such a Yang–Mills solution space developed previously \[Hue95b\], \[HJ94\]. The upshot of the present paper is that the theory developed in \[Hue95a\], \[Hue96\], and \[Hue98\] – see \[Hue01\] for a leisurely introduction – can be
built in terms of the appropriate spaces of smooth connections. Pushing a bit further, we endow the Fréchet manifold of based gauge equivalence classes of connections with a quasi-hamiltonian $G$-space structure relative to the structure group $G$, and the Yang-Mills moduli space we are interested in then arises by reduction relative to $G$. See Remark 10.2 below for details. This observation, together with the idea of imposing suitable constraints on the space of based gauge equivalence classes of smooth connections, justifies, perhaps, the term Fréchet reduction in the title. The analysis of the topology of the Yang-Mills solution spaces in [AB82, Section 6] enables us to reconstruct, entirely in terms of Fréchet topologies on spaces of smooth connections, the moduli space of all Yang-Mills connections on the fixed bundle $\xi$ and, furthermore, that of all Yang-Mills connections relative to $G$ over the surface $\Sigma$. See Section 8 for details. Thus, we can understand the Yang-Mills solution spaces over a closed surface in terms of spaces of ordinary smooth connections. This raises the issue whether we can understand the Yang-Mills solution spaces over a general compact Riemannian manifold in terms of spaces of smooth connections.

For a general gauge theory situation, an alternate model of the space of based holonomies was developed by the second-named author in [Hue99]. That model yields a rigorous approach to lattice gauge theory. For the special case explored in the present paper, the alternate model comes down to the extended moduli space construction developed in [HJ94], [Hue95b], [Jef97], see Section 10 below. The construction in the present paper is somewhat a special case of that in [Hue99], but now in the framework of Fréchet manifolds.

In Section 2, following [AB82], we spell out the universal example in terms of the (necessarily unique) Schur cover of the fundamental group of $\Sigma$, tailored to our purposes and, in Section 3, we recall the requisite topological classification of principal bundles over $\Sigma$. In Section 4 we introduce the Fréchet manifold structure on the space of based equivalence classes of connections. In Section 5 we explore the smoothness of the Wilson loop mapping and in Section 6 we establish a technical result. In Section 7 we then proceed to describe the constraints and to spell out and prove the main result, Theorem 7.1, and we complete the proof in Section 9. In Section 8 we explain how we can recover, entirely in terms of Fréchet topologies on spaces of smooth connections, the moduli space of all Yang-Mills connections relative to the structure group $G$. In Section 10 we briefly recall the resulting singular symplectic geometry, phrased in the language of stratified symplectic spaces, and the final section, Section 11, is devoted to the requisite Fréchet space technology.

2 The universal example

We maintain the notation $G$ for a compact connected Lie group. Our base manifold is an orientable (real) closed (compact) connected surface $\Sigma$ of genus $\geq 1$. Our notation for the differential forms on a smooth manifold $M$ is $\mathcal{A}(M, \cdot)$. Ordinary Yang-Mills theory necessitates a choice of Riemannian metric on the structure group and on the base manifold. Our approach does not involve such metrics...
except when we link it to ordinary Yang-Mills theory, and a choice of complex structure of \( \Sigma \) will play no role either, except possibly for the orientation it determines. As a side remark we note that the Yang-Mills equation \( d_A \ast \text{curv}_A = 0 \) makes sense for a principal bundle having as structure group a general Lie group when we interpret the operator \( \ast \) as having its values in the forms with values in the dual to the adjoint bundle.

Let \( \ell \geq 1 \) denote the genus of \( \Sigma \), let \( Q \) be a point of \( \Sigma \), fixed throughout and taken henceforth as base point, and let \( \pi_1 = \pi_1(\Sigma, Q) \) denote the fundamental group of \( \Sigma \) at the point \( Q \). Consider the standard presentation

\[
\langle x_1, y_1, \ldots, x_\ell, y_\ell; r \rangle, \quad r = [x_1, y_1] \cdots [x_\ell, y_\ell] \tag{2.1}
\]

of \( \pi_1 \). Let \( u_1, v_1, \ldots, u_\ell, v_\ell \) be a canonical system (in the sense of \([ZVC80]\)) of (closed) curves in \( \Sigma \) that have \( Q \) as starting point; that is to say: Cutting \( \Sigma \) successively along these curves yields a planar disk \( e^2 \); see, e.g., \([ZVC80, \S3.3.2]\) for details. We suppose things arranged in such a way that these curves represent the respective generators \( x_1, y_1, \ldots, x_\ell, y_\ell \) of the fundamental group \( \pi_1 = \pi_1(\Sigma, Q) \) of \( \Sigma \) and that the boundary path of the disk \( e^2 \) yields the relator \( r \). Thus the standard cell decomposition of \( \Sigma \) with a single 2-cell \( e^2 \) corresponding to \( r \) results.

We denote by \( F \) the free group on \( x_1, y_1, \ldots, x_\ell, y_\ell \) and by \( N \) the normal closure of \( r \) in \( F \). Consider the quotient group \( \Gamma = F/[F, N] \). The image \([r] \in \Gamma \) of \( r \) generates the central subgroup \( \mathbb{Z}\langle [r] \rangle = N/[F, N] \) of \( \Gamma \), and the resulting extension

\[
0 \longrightarrow \mathbb{Z}\langle [r] \rangle \longrightarrow \Gamma \longrightarrow \pi_1 \longrightarrow 1 \tag{2.2}
\]

is central. Since the transgression homomorphism \( H_2(\pi_1) \rightarrow \mathbb{Z}\langle [r] \rangle \) is an isomorphism, the extension (2.2) is a maximal stem extension (Schur cover) and since, furthermore, the abelianization of \( \pi_1 \) is a free abelian group, that maximal stem extension is unique up to within isomorphism \([Gru70, \S9.9 \text{Theorem 5 p. 214}]\). Atiyah and Bott use the terminology “universal central extension” to refer to this situation \([AB82, \S6]\).

Relative to the embedding \( \mathbb{Z}\langle [r] \rangle \rightarrow \mathbb{R} \) of abelian groups given by the assignment to \([r] \) of \( 2\pi \in \mathbb{R} \), similarly as in \([AB82, \S6 \text{p. 560}]\), let \( \Gamma_\mathbb{R} \) denote the 1-dimensional Lie group characterized by the requirement that

\[
0 \longrightarrow \mathbb{Z}\langle [r] \rangle \longrightarrow \Gamma \longrightarrow \pi_1 \longrightarrow 1 \tag{2.3}
\]

be a commutative diagram of central extensions. Consider a principal \( U(1) \)-bundle \( \xi_{M_\Sigma} : M_\Sigma \rightarrow \Sigma \) on \( \Sigma \) having Chern class 1, cf., e.g., \([Hir95, I.4 \text{p. 58 ff}]\) for the precise meaning of having Chern class 1. Similarly as in \([AB82, \S6]\), we realize (2.2) in terms of \( \xi_{M_\Sigma} \) as follows (in \([AB82, \S6]\), the compact 3-manifold \( M_\Sigma \) is written as \( Q \)): Choose a smooth connection form

\[
\omega_{M_\Sigma} : TM_\Sigma \longrightarrow \mathfrak{d} \mathbb{R} = \text{Lie}(U(1)) \subset \text{Lie}(\text{GL}(1, \mathbb{C})) = \mathbb{C} \tag{2.4}
\]
on $\xi_M$ having non-degenerate curvature form and write the curvature form as $-2\pi i \text{vol}_\Sigma \in \mathcal{A}^2(\Sigma, i\mathbb{R})$. This defines the volume form $\text{vol}_\Sigma$ on $\Sigma$ and fixes an orientation of $\Sigma$ and, when we orient $\epsilon^2$ consistently with $\Sigma$, fixes an orientation of the closed path $[u_1, v_1] \cdot \cdots \cdot [u_l, v_l]$, the “boundary path” of $\epsilon^2$, as well. Unlike the approach in [AB82], in our setting, there is no need to choose a Riemannian metric on $\Sigma$ and to arrange for $\text{vol}_\Sigma$ to be the volume form associated to that Riemannian metric. We can, of course, choose a (positive) complex structure on $\Sigma$ that is compatible with the volume form $\text{vol}_\Sigma$ (as a symplectic structure) and work with the resulting Kähler metric on $\Sigma$. Then the 2-form $\text{vol}_\Sigma$ is the corresponding normalized Riemannian volume form on $\Sigma$, and the orientation of $\Sigma$ coincides with the orientation arising from the complex structure. This reconciles our approach with that in [AB82] and shows that the sign in the above expression for the curvature form is consistent with standard Chern-Weil theory, cf. [Che95].

The group $\Gamma$ is isomorphic to the fundamental group of $M_\Sigma$. To make the identification of $\Gamma$ with the fundamental group of $M_\Sigma$ explicit, we choose a pre-image $Q_{M_\Sigma} \in M_\Sigma$ of the chosen base point $Q$ of $\Sigma$ and, thereafter, closed lifts $u_{1, M_\Sigma}, v_{1, M_\Sigma}, \ldots, u_{l, M_\Sigma}, v_{l, M_\Sigma}$, to $M_\Sigma$, of the canonical curves having $Q_{M_\Sigma} \in M_\Sigma$ as starting point. Though this is not strictly necessary, we note that, since, up to gauge transformations, the curvature determines the connection only up to a member of $H^1(\Sigma, U(1))$, we can rechoose the smooth principal $U(1)$-connection form $\omega_{M_\Sigma}$ in such a way that its holonomies with respect to $Q_{M_\Sigma}$ and the canonical curves in $\Sigma$ are trivial. See also Remark 7.6 below. Such a connection is unique up to gauge transformations. The respective horizontal lifts $u_{1, M_\Sigma}, v_{1, M_\Sigma}, \ldots, u_{l, M_\Sigma}, v_{l, M_\Sigma}$ in $M_\Sigma$ having $Q_{M_\Sigma}$ as starting point are then closed, and we can take their based homotopy classes as generators of $\pi_1(M_\Sigma, Q_{M_\Sigma}) \cong \Gamma$.

We take the universal covering projection $R \rightarrow U(1)$ to be given by the association $t \mapsto e^{it}$ ($t \in \mathbb{R}$). The $U(1)$-bundle projection $\xi_M : M_\Sigma \rightarrow \Sigma$ lifts to a principal $R$-bundle projection $\tilde{\xi}_M : \tilde{M}_\Sigma \rightarrow \tilde{\Sigma}$ defined on the universal covering manifold $\tilde{M}_\Sigma$ of $M_\Sigma$ having as base the universal covering manifold $\tilde{\Sigma}$ of $\Sigma$, and the $R$-action and $\Gamma$-action on $\tilde{M}_\Sigma$ combine to a $\Gamma_R$-action on $\tilde{M}_\Sigma$ turning the resulting projection $\tilde{\xi}_M : \tilde{M}_\Sigma \rightarrow \Sigma$ into a principal bundle having $\Gamma_R$ as its structure group. The connection form $\omega_{M_\Sigma}$ on $\xi_M$ determines a connection form $\tilde{\omega}_{\tilde{M}_\Sigma} : T\tilde{M}_\Sigma \rightarrow \mathbb{R}$ on $\tilde{\xi}_M : \tilde{M}_\Sigma \rightarrow \tilde{\Sigma}$ and hence a connection having curvature form $-2\pi i \text{vol}_\Sigma \in \mathcal{A}^2(\Sigma, \mathbb{R})$, and this connection is necessarily central. Here and henceforth we refer to a connection as being central when the values of its curvature form lie in the center of the structure group. The universal covering projection $R \rightarrow U(1)$ induces the map $\text{Lie}(\Gamma_R) = R \rightarrow iR = \text{Lie}(U(1))$ given by $t \mapsto it$ ($t \in \mathbb{R}$) and, by construction, the connection and curvature forms on $\xi_M$ and $\tilde{\xi}_M$ correspond via that induced map.

3 Topology of principal bundles over a closed oriented surface

To unveil, in the Fréchet setting, the structure of the moduli space of Yang-Mills connections on a fixed principal $G$-bundle over $\Sigma$ we exploit a certain topological characteristic class that lies in the center of the Lie algebra of $G$, cf. [AB82, §6 p. 560].
For intelligibility and for later reference, we briefly recall how this class arises and how it relates to the topological classification of principal $G$-bundles on $\Sigma$.

By structure theory, the compact connected Lie group $G$ is generated by its maximal semi-simple subgroup $S = [G, G]$ and the connected component $H$ of the identity of the center of $G$ and, accordingly, we write $G$ as $G = H \times_D S$, for the finite discrete central subgroup $D = H \cap S$ of $G$. The injection $H \to G$ induces an isomorphism $\overline{\Pi} = H/D \to G/S$ onto the abelianized group $G_{\text{Ab}} = G/S$, and the group $\overline{\Pi}$ is a torus group.

Let $\mathfrak{h}$ denote the Lie algebra of $\overline{\Pi}$ (and of $H$). The assignment to $\varphi \in \text{Hom}(U(1), \overline{\Pi})$ of the induced principal $\overline{\Pi}$-bundle $\xi_{\varphi} \times_{\varphi} \overline{\Pi} : M_{\overline{\Pi}} \to \Sigma$ topologically classifies principal $\overline{\Pi}$-bundles on $\Sigma$. When we assign to a member $\varphi$ of $\text{Hom}(U(1), \overline{\Pi})$, viewed as a 1-parameter subgroup of $\overline{\Pi}$, its tangent vector $X_\varphi \in \mathfrak{h}$ at the origin (so that $\varphi(e^{it}) = \exp(tX_\varphi)$), we obtain an isomorphism

$$\text{Hom}(U(1), \overline{\Pi}) \to \ker(\exp : \mathfrak{h} \to \overline{\Pi}) = \pi_1(\overline{\Pi})$$

of discrete abelian groups. Observing that Poincaré duality with respect to the orientation of $\Sigma$ chosen in Section 2 above identifies $H^2(\pi_1, \pi_1(\overline{\Pi}))$ canonically with $\pi_1(\overline{\Pi})$ reconciles this description with the ordinary topological classification of principal $\overline{\Pi}$-bundles on $\Sigma$. In this language, our reference bundle $\xi_{\xi_{\overline{\Pi}}} : M_{\overline{\Pi}} \to \Sigma$ introduced in Section 2 above has characteristic class $X_{\xi_{\overline{\Pi}}} = 2\pi i \in \mathbb{R} = \text{Lie}(U(1))$.

Consider a principal $G$-bundle $\xi : P \to \Sigma$ on $\Sigma$. The bundle projection factors through the induced map from the orbit manifold $P/S$ to $\Sigma$, necessarily a smooth principal $\overline{\Pi}$-bundle, and we denote this bundle by $\xi_{\overline{\Pi}} : P/S \to \Sigma$ and by $X_\xi \in \mathfrak{h}$ the single topological characteristic class of $\xi_{\overline{\Pi}}$. This yields the requisite topological characteristic class for the bundle $\xi$, cf. [AB82, §6 p. 560].

Consider the relator map

$$r : \text{Hom}(F, G) \to G, \; r(\chi) = [\chi(x_1), \chi(y_1)] \cdots [\chi(x_i), \chi(y_i)] \in G.$$  

The injection $\text{Hom}(\Gamma, G) \subset \text{Hom}(F, G)$ induced by the surjection $F \to \Gamma$ identifies a certain subspace of $\text{Hom}(\Gamma, G)$ with the subspace $r^{-1}(\exp(X_\xi))$ of $\text{Hom}(F, G)$, and we denote that subspace of $\text{Hom}(\Gamma, G)$ by $\text{Hom}_\chi(\Gamma, G)$. Viewed as a subspace of $\text{Hom}(F, G)$, the space $\text{Hom}_\chi(\Gamma, G)$ is necessarily compact.

Let $\text{Hom}_\chi(\Gamma \cap \Gamma, G)$ denote the space of homomorphisms $\chi$ from $\Gamma \cap \Gamma$ to $G$ that have the property that $\chi(t[r]) = \exp(tX_\xi)$ ($t \in \mathbb{R}$). Given $\chi \in \text{Hom}_\chi(\Gamma, G)$, with a slight abuse of the notation $\chi$, setting $\chi(t[r]) = \exp(tX_\xi)$ ($t \in \mathbb{R}$), we obtain an extension of $\chi$ to a homomorphism $\chi : \Gamma \cap \Gamma \to G$, uniquely determined by $\chi$ and $X_\xi$, that is, the restriction $\text{Hom}_\chi(\Gamma \cap \Gamma, G) \to \text{Hom}_\chi(\Gamma, G)$ is a bijection. Hence, given $\chi \in \text{Hom}_\chi(\Gamma, G)$, the associated principal $G$-bundle $\xi_\chi : M_{\overline{\Pi}} \times_\chi G \to \Sigma$ on $\Sigma$ is defined.

**Proposition 3.1:** The assignment to $\chi \in \text{Hom}_\chi(\Gamma, G)$ of $\xi_\chi$ yields a bijection between the connected components of $\text{Hom}_\chi(\Gamma, G)$ and the topological types of principal $G$-bundles on $\Sigma$ having $X_\xi \in \mathfrak{h}$ as its corresponding characteristic class, the number of components being given by the order $|\pi_1(S)|$ of $\pi_1(S)$. 

The reasoning in [AB82, Section 6] yields a proof of this proposition. Suffice it to note the following: Let \( S = G/H = S/D \). The bundle \( \xi \) is topologically classified by its characteristic class in \( H^2(\Sigma, \pi_1(G)) \) and, likewise, the principal \( \mathcal{H} \)-bundle \( \xi_{\mathcal{H}} \) is topologically classified by its characteristic class in \( H^2(\Sigma, \pi_1(\mathcal{H})) \). Under the induced homomorphism \( H^2(\Sigma, \pi_1(G)) \to H^2(\Sigma, \pi_1(\mathcal{H})) \), the characteristic class of \( \xi \) goes to the characteristic class of \( \xi_{\mathcal{H}} \). Poincaré duality relative to the fundamental class \( [\Sigma] \) identifies that homomorphism with the induced homomorphism \( \pi_1(G) \to \pi_1(\mathcal{H}) \), and the induced homomorphism \( \pi_1(S) \to \pi_1(G) \) identifies that kernel with \( \pi_1(S) \). We leave the details to the reader.

We denote by \( \text{Hom}_X(\Gamma, G) \) the connected component that corresponds to the principal \( G \)-bundle \( \xi \).

### 4 Fréchet manifold structure on the space of based gauge equivalence classes of connections

Consider a principal \( G \)-bundle \( \xi : P \to M \) on a smooth compact connected manifold \( M \). Let \( g \) denote the Lie algebra of \( G \). We use the standard formalism with structure group \( G \) acting on \( P \) from the right etc. We denote by \( \text{ad}_\xi : P \times_G g \to M \) the (infinitesimal) adjoint bundle associated to \( \xi \) and write the graded object of differential forms on \( M \) with values in \( \text{ad}_\xi \) as \( \mathcal{A}^*(M, \text{ad}_\xi) \).

Consider the space \( \mathcal{A}_\xi \) of smooth \( G \)-connections on \( \xi : P \to M \). The Atiyah sequence [Ati57] associated with \( \xi \), spelled out here for the total spaces, has the form

\[
0 \to P \times_G g \to (TP)/G \to TM \to 0,
\]

and we realize the space \( \mathcal{A}_\xi \) as that of vector bundle sections \( TM \to (TP)/G \) for (4.1). Thus the points of \( \mathcal{A}_\xi \) are in bijective correspondence with the sections of an affine bundle over \( M \) and in this way carry a natural tame Fréchet manifold structure modeled on the vector space \( \mathcal{A}_1^*(M, \text{ad}_\xi) \) [ACMM86].

Let \( G_{\xi} \) denote the group of smooth gauge transformations of \( \xi \). We realize this group as the group \( \Gamma^\infty(\text{Ad}_\xi) \) of smooth sections of the associated adjoint bundle \( \text{Ad}_\xi : P \times_G G \to M \), endowed with the pointwise group structure. In this manner, \( G_{\xi} \) acquires the structure of a tame Fréchet Lie group [ACMM86].

As usual, we identify the Lie algebra of \( G_{\xi} \) with the Lie algebra \( \text{gau}_{\xi} = \Gamma^\infty(\text{ad}_\xi) \) of smooth sections of the adjoint bundle \( \text{ad}_\xi \) on \( M \), endowed with the pointwise Lie bracket. This Lie algebra acquires the structure of a tame Fréchet Lie algebra and, relative to the Fréchet structures, the exponential map \( \exp : \text{gau}_{\xi} \to G_{\xi} \) is a local diffeomorphism at \( 0 \), see [CM85, Section IV]. In these topologies, the standard left \( G_{\xi} \)-action on \( \mathcal{A}_\xi \) is tame smooth, proper and admits slices at every point [ACM89, Die13]. In particular the orbit space \( G_{\xi} \backslash \mathcal{A}_\xi \) is stratified by tame Fréchet manifolds.

Choose a base point \( Q_P \) for the total space \( P \) of \( \xi : P \to M \) and let \( Q = \xi(Q_P) \in M \).
The evaluation map $\ev_{Q_p} : \mathcal{G}_\xi \to G$ characterized by the identity

$$\phi(Q) = [Q_p, \ev_{Q_p}(\phi)] \in P \times_G G,$$

as $\phi$ ranges over $\mathcal{G}_\xi$ (the group of sections of $\Ad_\xi$), is a morphism of Lie groups. The group of smooth based gauge transformations associated to the data is the kernel of $\ev_{Q_p}$. This group is a normal, locally exponential Lie subgroup of $\mathcal{G}_\xi$ [Nee06, Prop. IV.3.4]. While $\ev_{Q_p}$ depends on the choice of $Q_p$, the kernel of $\ev_{Q_p}$ is independent of the particular choice of $Q_p$ in $\xi^{-1}(Q)$ and depends only on $Q$; we therefore denote the group of based gauge transformations by $\mathcal{G}_{\xi,Q}$.

Relative to $\mathcal{G}_\xi$, the group $\mathcal{G}_{\xi,Q}$ has finite codimension equal to the dimension of $G$. Theorem 11.4 below entails that the obvious surjection $\mathcal{G}_\xi \to \mathcal{G}_\xi/\mathcal{G}_{\xi,Q}$ is a smooth right principal $\mathcal{G}_{\xi,Q}$-bundle; we refer to this kind of situation by the phrase “$\mathcal{G}_{\xi,Q}$ is a principal Lie subgroup of $\mathcal{G}_\xi$.”

Proposition 11.9 below will say that slices for the $\mathcal{G}_\xi$-action on $\mathcal{A}_\xi$ yield slices for the $\mathcal{G}_{\xi,Q}$-action on $\mathcal{A}_\xi$ and thus implies the following:

**Proposition 4.1:** Relative to the Fréchet manifold structures, the $\mathcal{G}_{\xi,Q}$-action on $\mathcal{A}_\xi$ is smooth and free; furthermore, it admits slices at every point. Hence the orbit space $B_{\xi,Q} = \mathcal{G}_{\xi,Q}/\mathcal{A}_\xi$ acquires a smooth Fréchet manifold structure, and the canonical projection $\mathcal{A}_\xi \to B_{\xi,Q}$ is a smooth principal $\mathcal{G}_{\xi,Q}$-bundle.

## 5 Wilson loops

Consider a principal $G$-bundle $\xi : P \to M$ on a smooth connected manifold $M$, not necessarily compact. Choose a base point $Q_p$ of $P$ and consider a smooth closed path $u : [0, 1] \to M$ in $M$ starting at the point $\xi(Q_p)$ of $M$. With respect to $Q_p$ and a smooth connection $A$ on $\xi : P \to M$, we denote by $\Hol_{u,Q_p}(A) \in G$ the holonomy of $A$ along $u$.

For later reference, we recall a description, cf. [Bry95, Appendix to Lecture 3], [KN63, Proof of Proposition II.3.1 p. 69].

Let $u_\omega : [0, 1] \to P$ be a lift of $u$ having $Q_p$ as its starting point, and let $\omega_A$ denote the connection form associated to $A$. The solution $u_G : [0, 1] \to G$ of the differential equation

$$\omega_A((u_F u_G)') = 0, \ u_G(0) = e,$$

then yield the desired horizontal lift $u_A : [0, 1] \to P$ of $u$ as $u_A = u_F u_G$.

Now $(u_F u_G)' = u_\omega u_G + u_F \dot{u}_G$, the sum being evaluated, for given $t \in [0, 1]$, in the vector space $T_{u_F(u_G)(t)}P$, and

$$\omega_A(u_F u_G + u_\omega u_G) = \omega_A(u_F u_G) + \omega_A(u_\omega u_G) = \Ad_{u_G^{-1}} \omega_A(u_\omega) + u_G^{-1} \dot{u}_G,$$

whence (5.1) is equivalent to

$$\dot{u}_G u_G^{-1} = -\omega_A(u_\omega), \ u_G(0) = e.$$
Then the identity
\[ Q_p \operatorname{Hol}_{u,Q_p}(A) = u_A(1) = u_p(1)u_G(1) \] (5.4)
characterizes \( \operatorname{Hol}_{u,Q_p}(A) \in G \).

The following observation is a Fréchet version of [Hue98, Theorem 2.1].

**Lemma 5.1:** Given a smooth loop \( u : [0, 1] \to M \) in \( M \) having starting point \( \xi(Q_p) \), the horizontal lift map
\[ \operatorname{Hor}_{u,Q_p} : A_{\xi} \to \Gamma^\infty(u^*\xi), \quad A \mapsto u_A, \] (5.5)
with respect to \( Q_p \) is smooth and, at a smooth connection \( A \), the derivative is given by
\[ T_A\operatorname{Hor}_{u,Q_p} : \mathcal{A}^1(M, \text{ad}) \to C^\infty([0, 1], \mathfrak{g}), \quad \vartheta \mapsto f_\vartheta, \] (5.6)
where
\[ f_\vartheta(t) = \int_0^t \vartheta_{u_A(\tau)}(\dot{u}_A(\tau))d\tau, \quad t \in [0, 1]. \]

To guide the reader through (5.6), we note that, for any \( \tau \in [0, 1] \), the notation \( u_A(\tau) \) refers to a point of the total space \( P \) of \( \xi \) and \( \dot{u}_A(\tau) \) to a tangent direction at \( u_A(\tau) \).

The idea that underlies the proof we are about to give is the same as that for the proof of [Hue98, Theorem 2.1]. However, we here consider the horizontal lift as a map \( A_{\xi} \to \Gamma^\infty(u^*\xi) \) into the space of smooth sections of the induced bundle \( u^*\xi \) rather than as a map of the kind \( A_{\xi} \times [0, 1] \to \mathbb{P} \). Moreover, we exploit a very natural and well-established notion of smoothness instead of the ad hoc concept of a map that is “smooth on every finite-dimensional submanifold” used in [Hue98].

**Proof.** Let \( u : [0, 1] \to M \) be a smooth loop in \( M \) having starting point \( \xi(Q_p) \), let \( A \) be a smooth connection on \( \xi \), and let \( u_A \) denote the horizontal lift of \( u \) with respect to \( A \) and starting at \( Q_p \).

The induced bundle \( u^*\xi \) trivializes. We take the connection \( A \) as reference connection, use the horizontal lift \( u_A \) of \( u \) relative to \( A \) to trivialize the bundle \( u^*\xi \) and argue henceforth in terms of the total space \([0, 1] \times G\). The trivialization identifies the space \( \Gamma^\infty(u^*\xi) \) of smooth sections of \( u^*\xi \) with the space \( C^\infty([0, 1], G) \) of smooth \( G \)-valued maps on \([0, 1] \). By construction, in the chosen trivialization, the original path \( u \) amounts to the identity path \([0, 1] \to [0, 1] \). Accordingly, given \( \vartheta \in \mathcal{A}^1(M, \text{ad}) \), the component \( u_{A+\vartheta} : [0, 1] \to G \) into \( G \) of the horizontal lift \([0, 1] \to [0, 1] \times G \) of the identity path of \([0, 1] \) determines the horizontal lift of \( u \) relative to \( A + \vartheta \). Thus the horizontal lift map takes the form
\[ \operatorname{Hor}_{u,Q_p} : \mathcal{A}^1(M, \text{ad}) \to C^\infty([0, 1], G), \quad \vartheta \mapsto u_{A+\vartheta}. \] (5.7)
In particular, \( u_A \) is the trivial path at the neutral element \( e \) of \( G \).

We use the variable \( t \) as coordinate on \([0, 1] \). Let \( \vartheta \in \mathcal{A}^1(M, \text{ad}) \). This 1-form, restricted to the bundle \( u^*\xi \), now takes, along the factor \([0, 1] \) of the decomposition \([0, 1] \times G \) of the total space, the form \( h_{\vartheta}dt \) for some smooth \( g \)-valued function \( h_{\vartheta} \) on \([0, 1] \) uniquely determined by \( \vartheta \), and the path \( u_{A+\vartheta} \) is the solution of the differential
equation
\[ u_{A+\theta}^{-1} u_{A+\theta}^{-1} = h_\theta. \] (5.8)

The sign here is consistent with (5.3). Indeed, in our realization of connections as vector bundle sections for (4.1), the connection \( A+\theta \) on \( \xi \) has connection form \( \omega_A - \theta : TP \to \mathfrak{g} \), where we identify \( \text{ad}_t \)-valued 1-forms on \( M \) with \( G \)-equivariant 1-forms \( TP \to \mathfrak{g} \) as usual. The right-hand side of equation (5.8) depends continuously on the parameter \( \theta \in A^1(M, \text{ad}_t) \). The continuous dependence of solutions of differential equations on parameters [Die69, 10.7.1] now implies that the horizontal lift map (5.7) is a continuous map.

Let \( s \) denote a real variable. Differentiating the identity
\[ \dot{u}_{A+\theta} u_{A+\theta}^{-1} = sh_\theta \] (5.9)

with respect to \( s \) yields the identity
\[ \frac{d}{ds}(u_{A+\theta}) u_{A+\theta}^{-1} + u_{A+\theta} \frac{d}{ds}(u_{A+\theta}^{-1}) = h_\theta. \]

However, the path \( u_A \) is constant whence, for \( s = 0 \), the differential equation reduces to
\[ \frac{d}{dt} \bigg|_{s=0} (u_{A+\theta}) = h_\theta. \]

Interchanging the order of differentiation, we write this equation as
\[ \frac{d}{dt} \frac{d}{ds} \bigg|_{s=0} (u_{A+\theta}) = h_\theta. \]

Consequently
\[ \frac{d}{ds} \bigg|_{s=0} (u_{A+\theta})(t) = \int_0^t h_\theta(\tau)d\tau = \int_0^t \dot{\theta}(\tau) \dot{u}_A(\tau) d\tau. \]

The associated Magnus series provides more insight: Since (5.7) is a continuous map, the horizontal lift with respect to a connection near \( A \) lies in an appropriate tubular neighborhood of \( u_A \). Hence, when \( \theta \) is sufficiently close to zero, we can write the path \( u_{A+\theta} \) as
\[ u_{A+\theta} = \exp(W_{A+\theta}) \]

for some path \( W_{A+\theta} : [0, 1] \to \mathfrak{g} \). Using the familiar identity
\[ \frac{d}{dt} \exp(Y(t)) \exp(Y(t))^{-1} = \frac{\exp[\text{ad}(Y(t))] - 1}{\text{ad}(Y(t))} \dot{Y}(t), \]
letting \( Y = W_{A_0} \), we rewrite equation (5.8) as
\[
\frac{e^{\text{ad}(Y)} - 1}{\text{ad}(Y)} \dot{Y} = h_0. 
\] (5.10)

The first two terms of a Magnus expansion \( Y(t) = \sum_{j=1} Y^{(j)}(t) \) of the solution \( Y \) of (5.10) read
\[
Y^{(1)}(t) = \int_0^t d\tau \ h_0(\tau), 
\] (5.11)
\[
Y^{(2)}(t) = -\frac{1}{2} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 [h_0(\tau_1), h_0(\tau_2)], 
\] (5.12)
and the higher order terms involve higher order commutators of \( h_0 \) evaluated at different times. The expansion yields explicit expressions for the derivatives of the horizontal lift map (5.7), since every order in the series equates to the order of differentiation. In particular, this argument confirms that (5.7) is smooth and that the expression (5.6) yields the derivative.

We return to our surface \( \Sigma \) and maintain the choice of base point \( q \) and of a canonical system \( u_1, v_1, \ldots, u_\ell, v_\ell \) of closed curves in \( \Sigma \) whose homotopy classes generate \( \pi_1(\Sigma, Q) \), cf. Section 2. Consider a principal G-bundle \( \xi : P \rightarrow \Sigma \) on \( \Sigma \) and choose a base point \( Q_0 \) of \( P \) over the base point \( Q \) of \( \Sigma \). Henceforth the base point \( Q_0 \) of \( P \) remains fixed. Given a smooth connection \( A \) on \( \xi \), for \( 1 \leq j \leq \ell \), let \( u_{A_j} \) and \( v_{A_j} \) denote the horizontal lifts, in the total space \( P \) of \( \xi \), of \( u_j \) and \( v_j \), respectively, with reference to \( A \) and \( Q_0 \). The Wilson loop mapping
\[
\rho : \mathcal{A}_\xi \rightarrow G^{2\ell} \quad (5.13)
\]
relative to \( Q_0 \) and \( u_1, v_1, \ldots, u_\ell, v_\ell \) assigns to a smooth connection \( A \) on \( \xi \) the point
\[
\rho(A) = (\text{Hol}_{u_1, Q_0}(A), \text{Hol}_{v_1, Q_0}(A), \ldots, \text{Hol}_{u_\ell, Q_0}(A), \text{Hol}_{v_\ell, Q_0}(A)) \quad (5.14)
\]
of \( G^{2\ell} \) [Hue98, (2.6)]. This map depends on the choices made to carry out its construction. In Section 9 below we explore the dependence on these choices.

For \( q \in G \), we denote the induced operation of left translation by
\[
L_q : g \mapsto T_q G \rightarrow T_q G.
\]

The subsequent result is the Fréchet version of [Hue98, Theorem 2.7], spelled out for the special case where the base manifold is a closed surface.

**Theorem 5.2:** In the Fréchet topologies, the Wilson loop mapping \( \rho : \mathcal{A}_\xi \rightarrow G^{2\ell} \), cf. (5.13) above, is smooth. At a smooth connection \( A \), the value (5.14) under \( \rho \) being written as
\[
\rho(A) = (a_1, b_1, \ldots, a_\ell, b_\ell) \in G^{2\ell},
\]
the tangent map

\[ T_A \rho : T_A A_\xi \longrightarrow T_{\rho(A)} G^{2d} = T_{a_1} G \times T_{b_1} G \times \cdots \times T_{a_i} G \times T_{b_i} G \]  

(5.15)

of \( \rho \) sends \( \vartheta \in A^1(\Sigma, \text{ad}_\xi) = T_A A_\xi \) to

\[
\begin{bmatrix}
L_{a_1} \int_{a_{a_{1}}} \vartheta, & L_{b_1} \int_{a_{b_{1}}} \vartheta, & \cdots, & L_{a_i} \int_{a_{a_{i}}} \vartheta, & L_{b_i} \int_{a_{b_{i}}} \vartheta
\end{bmatrix},
\]

(5.16)

by construction a vector in \( T_{a_1} G \times T_{b_1} G \times \cdots \times T_{a_i} G \times T_{b_i} G \).

Proof. This is a consequence of Lemma 5.1 since, given a smooth closed path \( u : [0, 1] \rightarrow \Sigma \) starting at the base point \( Q \), evaluation at 1 is a smooth map \( \Gamma^{u}(u' \xi) \rightarrow P_G = G \), and the derivative thereof also comes down to evaluation at 1.

\[ \square \]

6 Preparing for the Description of the Fréchet Constraints

To explain the idea as to how the constraints arise, consider a 2-form \( \beta \) on \( \Sigma \) such that, with respect to the volume form \( \text{vol}_\Sigma \) on \( \Sigma \) chosen in Section 2 above, \( \int_{a} \beta = \int_{a} \text{vol}_\Sigma \) for any disk \( \Delta \) in \( \Sigma \). Then \( \int_{a}(\beta - \text{vol}_\Sigma) = 0 \), for any disk \( \Delta \) in \( \Sigma \) whence

\[ \beta = \text{vol}_\Sigma + d\alpha \]

for some 1-form \( \alpha \) on \( \Sigma \). Now

\[ \int_{a_{\Delta}} \alpha = \int_{\Delta} d\alpha = \int_{\Delta} (\beta - \text{vol}_\Sigma) = 0, \]

for any disk \( \Delta \) in \( \Sigma \). However, a 1-form whose integral over any closed contractible path vanishes is an exact form. Hence \( da = 0 \) and so \( \beta = \text{vol}_\Sigma \). In other words, the functionals given by integration over arbitrary disks separate points of \( A^2(\Sigma, R) \).

We return to our principal \( G \)-bundle \( \xi : P \rightarrow \Sigma \) on \( \Sigma \) and maintain the choice of base point \( Q_p \) of \( P \) over the base point \( Q \) of \( \Sigma \). For convenience, we slightly vary the classical construction that reduces a principal bundle with connection to the holonomy bundle at a chosen base point of the total space, cf., e. g., the reduction theorem [KN63, II.7.1]. Recall the choice of a principal U(1)-connection form \( \omega_{\tilde{M}_\Sigma} : T_{\tilde{M}_\Sigma} \rightarrow iR \) on \( \tilde{\xi}_{\tilde{M}_\Sigma} : \tilde{M}_\Sigma \rightarrow \Sigma \) and its lift \( \tilde{\omega}_{\tilde{M}_\Sigma} : T\tilde{M}_\Sigma \rightarrow R \) to a principal \( \Gamma_R \)-connection form on the principal \( \Gamma_R \)-bundle \( \xi_{\tilde{M}_\Sigma} : \tilde{M}_\Sigma \rightarrow \Sigma \) and, furthermore, the choice of a base point \( Q_{\tilde{M}_\Sigma} \) of \( \tilde{M} \) over \( Q \). Let \( A \) be a smooth connection on \( \xi \), and let \( P_{\text{hor}} \) denote the space of paths \( u : [0, 1] \rightarrow \tilde{M}_\Sigma \) in \( \tilde{M}_\Sigma \) that start at \( Q_{\tilde{M}_\Sigma} \) and are horizontal relative to \( A \), and let \( ev : P_{\text{hor}} \rightarrow \tilde{M}_\Sigma \) denote the evaluation map which sends a path in \( P_{\text{hor}} \) to its end point. This map is surjective. Indeed, since the principal \( \Gamma_R \)-bundle \( \xi_{\tilde{M}_\Sigma} : \tilde{M}_\Sigma \rightarrow \Sigma \) coincides with its holonomy bundle at \( Q_{\tilde{M}_\Sigma} \) with respect to the connection form \( \omega_{\tilde{M}_\Sigma} \), any point \( T \)
of $\widetilde{M}_E$ is the end point of a smooth horizontal path joining $Q_{\widetilde{M}_E}$ to $T$.

We define the pre-reduction map $\hat{\lambda}_{Q_{\widetilde{M}_E},Q_P, A} : P_{\text{hor}} \rightarrow P$ of $A$ relative to $Q_{\widetilde{M}_E}$ and $Q_P$ as follows: Given a member $\bar{u} : [0, 1] \rightarrow \widetilde{M}_E$ of $P_{\text{hor}}$, that is, a path in $\widetilde{M}_E$ that is horizontal relative to $\omega_{\widetilde{M}_E}$ and starts at the point $Q_{\widetilde{M}_E}$ of $\widetilde{M}_E$, let $u$ be the path in $\Sigma$ obtained by projecting $\bar{u}$ into $\Sigma$, and let $u_P : [0, 1] \rightarrow P$ be the unique lift of $u$ that is horizontal for $A$ and has starting point $Q_P$; define the value $\hat{\lambda}_{Q_{\widetilde{M}_E},Q_P, A}(\bar{u})$ to be the end point of $u_P$. We say that the pre-reduction map is defined on $\widetilde{M}_E$ when it factors through the evaluation map $\text{ev} : P_{\text{hor}} \rightarrow \widetilde{M}_E$; we then write the resulting map as $\hat{\lambda}_{Q_{\widetilde{M}_E},Q_P, A} : \widetilde{M}_E \rightarrow P$.

Given a disk $\Delta$ in $\Sigma$, let $a_\Delta = \int_{\Delta} \text{vol}_\Sigma$ denote the area of $\Delta$ with respect to $\text{vol}_\Sigma$. Recall the topological characteristic class $X_\xi \in \mathfrak{h}$ of $\xi$, see Section 3.

**Lemma 6.1:** Consider a smooth connection $A$ on $\xi : P \rightarrow \Sigma$. The following are equivalent:

(i) The values of the curvature form $\text{curv}_A$ of $A$ lie in the center $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of $G$ in such a way that $\text{curv}_A = -X_\xi \cdot \text{vol}_\Sigma$.

(ii) The values of the curvature form $\text{curv}_A$ of $A$ lie in the center $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of $G$, and the curvature form $\text{curv}_A$ of $A$, being accordingly viewed as an ordinary $\mathfrak{h}$-valued 2-form on $\Sigma$, satisfies the identity

$$\int_\Delta \text{curv}_A = -a_\Delta X_\xi \left( -\int_\Delta \text{vol}_\Sigma X_\xi \right) \in \mathfrak{h} \quad (6.1)$$

whenever $\Delta \subset \Sigma$ is a disk in $\Sigma$.

(iii) For every smooth contractible loop $u$ in $\Sigma$ starting at some point $q$ of $\Sigma$ and bounded by a disk $\Delta$ in $\Sigma$,

$$\text{Hol}_{u,q_p}(A) = \exp(a_\Delta X_\xi) \in H, \quad (6.2)$$

for any choice of pre-image $q_p \in P$ of $q \in \Sigma$.

(iv) The pre-reduction map of $A$ relative to $Q_{\widetilde{M}_E}$ and $Q_P$ is defined on $\widetilde{M}_E$ and yields a morphism

$$\begin{array}{cccc}
\Gamma_R & \longrightarrow & \widetilde{M}_E & \xrightarrow{\xi_{\Omega_{\mathfrak{h}}}} & \Sigma \\
\downarrow \chi_A & & \downarrow \lambda_{Q_{\widetilde{M}_E},Q_P, A} & & \\
G & \longrightarrow & P & \xrightarrow{\xi} & \Sigma
\end{array} \quad (6.3)$$

of principal bundles with connection in such a way that the following hold:

(a) The homomorphism $\rho(A) : F \rightarrow G$ which arises as the value of $A$ under the Wilson loop mapping (5.13) coincides with the composite

$$F \longrightarrow \Gamma \longrightarrow \Gamma_R \xrightarrow{\chi_A} G,$$

the unlabeled arrows being the obvious maps.
(b) The homomorphism $\chi_A$ satisfies the identity $\chi_A(t[r]) = \exp(tX_\xi)$, for $t \in \mathbb{R}$.

**Remark 6.2.** Since the genus $t$ of $\Sigma$ is (supposed to be) positive, given a contractible loop $u$ in $\Sigma$ bounded by a disk, that disk is uniquely determined by $u$ (up to reparametrization). In this sense, the right-hand side of (6.2) depends only on $u$.

**Proof of Lemma 6.1.** In view of the observation spelled out at the beginning of the present section, the equivalence of (i) and (ii) is immediate. It is also immediate that (ii) implies (iii). We now show that (iii) implies (i).

When $G$ is semisimple, the topological characteristic class $X_\xi$ of the bundle $\xi$ is zero, and the constraints (6.2) simplify to

$$\text{Hol}_{u,q_\Omega}(A) = \{e\} \text{ whenever } u \text{ is a contractible loop in } \Sigma, \quad (6.4)$$

for a choice $q \in \Sigma$ of starting point of $u$ and of a pre-image $q_\Omega \in P$ of $q$.

Consider now the case where the center of $G$ has dimension $\geq 1$ and, as before, let $\mathfrak{h}$ denote the center of the Lie algebra $\mathfrak{g}$ of $G$. The group $G/H = \mathcal{S} \cong \mathcal{S}/D$ is semisimple, and the bundle projection $\xi : P \to \Sigma$ factors through the induced map from the orbit manifold $P/H$ to $\Sigma$, necessarily a principal $\mathcal{S}$-bundle, and we denote this bundle by $\xi_\mathcal{S} : P/H \to \Sigma$.

Consider a smooth $G$-connection $A$ on $\xi$ that satisfies the constraints (6.2), and let $A_\mathcal{S}$ denote the $\mathcal{S}$-connection it induces on $\xi_\mathcal{S}$. Relative to $\xi_\mathcal{S}$, the corresponding constraints (6.2) say that, given a smooth contractible loop $u$ in $\Sigma$, for a choice $q \in \Sigma$ of starting point of $u$ and of a pre-image $q_\Omega \in P/H$ of $q \in \Sigma$,

$$\text{Hol}_{u,q_\Omega}(A_\mathcal{S}) = e \in \mathcal{S}.$$ 

Hence the connection $A_\mathcal{S}$ on $\xi_\mathcal{S}$ is flat. Consequently the curvature form $\text{curv}_A \in \mathcal{A}^2(\Sigma, \text{ad}_\xi)$ of $A$ is **central** in the sense that the values of $\text{curv}_A$ lie in the center $\mathfrak{h}$ of $\mathfrak{g}$.

Relative to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ of the Lie algebra $\mathfrak{g}$ into its center $\mathfrak{h}$ and its semisimple constituent $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$, the connection form $\omega_A$ of $A$ decomposes as $\omega_A = \omega_\mathfrak{h} + \omega_\mathfrak{s}$ and, accordingly, the curvature form $\text{curv}_A$ of $A$ decomposes as

$$\text{curv}_A = d\omega_\mathfrak{h} + d\omega_\mathfrak{s} + \frac{1}{2}[\omega_\mathfrak{s}, \omega_\mathfrak{s}]. \quad (6.5)$$

Since the values of the curvature form $\text{curv}_A$ lie in the center $\mathfrak{h}$ of $\mathfrak{g}$, the constituent $d\omega_\mathfrak{s} + \frac{1}{2}[\omega_\mathfrak{s}, \omega_\mathfrak{s}]$ is zero.

Consider a contractible loop $u : [0,1] \to \Sigma$ in $\Sigma$ and a disk $\Delta$ in $\Sigma$ that bounds $u$. Choose a starting point $q$ of $u$ in $\Sigma$ with $w(0) = q$ and a pre-image $q_\Omega \in P$ of $q$. Since the loop $u$ is contractible, the summand $\omega_\mathfrak{s}$ of the decomposition $\omega_A = \omega_\mathfrak{h} + \omega_\mathfrak{s}$ does not contribute to the $G$-holonomy of $u$ relative to $A$ and $q_\Omega$.

To justify this claim, consider the restriction of $\xi$ to the disk $\Delta$. This restriction trivializes, and we fix a smooth $G$-equivariant embedding $\Delta \times G \to P$ that covers the embedding $\Delta \to \Sigma$ in such a way that the embedding sends the point $(q, e)$ of $\Delta \times G$ to the point $q_\Omega$ of $P$. Consider the restrictions $\eta_\mathfrak{h} \in \mathcal{A}^1(\Delta, \mathfrak{h})$ and $\eta_\mathfrak{s} \in \mathcal{A}^1(\Delta, \mathfrak{s})$, to $\Delta$, of
the respective 1-forms $\omega_h$ and $\eta_s$. The restriction, to $\Delta$, of the connection form $\omega_A$ of $A$ now coincides with the sum $\eta_A = \eta_h + \eta_s \in \mathcal{A}^1(\Delta, \mathfrak{g})$. This sum yields a connection form relative to the covering group $H \times S$ of $G = H \times_D S$.

The closed path $u$ now amounts to the boundary path $u : [0, 1] \to \Delta$ of $\Delta$ having starting point $q \in \Delta$, and the holonomy with respect to $A$ and $(q_p, e)$, in $H \times S$, of $u$ is the end point of the path $(u_H, u_S) : [0, 1] \to H \times S$ that solves the two differential equations

\[
\begin{align*}
\dot{u}_H u_H^{-1} &= -\eta_h(u), \quad u_H(0) = e \in H, \\
\dot{u}_S u_S^{-1} &= -\eta_s(u), \quad u_S(0) = e \in S.
\end{align*}
\]

Since $\eta_h$ arises from a flat connection and since the path $u$ is contractible, the path $u_S$ is closed. Consequently that holonomy, in $H \times S$, of $u$ is the end point of the path $(u_H, e) : [0, 1] \to H \times S$.

As in Section 5, let $u_P : [0, 1] \to P$ denote a closed lift of $u$ having the point $q_P$ as its starting point. We can take the composite of the boundary path $u : [0, 1] \to \Delta$ with the embedding $\Delta \to \Delta \times G \to P$ but this is not strictly necessary. Since the summand $\omega_s$ of $\omega_A = \omega_h + \omega_s$ does not contribute to the $G$-holonomy of $u$ relative to $A$ and $q_P$, in view of (5.3),

$$\text{Hol}_{u, q_P}(A) = \exp \left( - \int_{u_P} \omega_h \right).$$

(6.6)

Since

$$\exp \left( - \int_{u_P} \omega_h \right) = \exp \left( - \int_{\Delta} d\omega_h \right) = \exp \left( - \int_{\Delta} \text{curv}_A \right),$$

(6.7)

the constraint (6.2) implies that

$$\exp \left( - \int_{\Delta} \text{curv}_A \right) = \exp(a_\Delta X_\xi) \in H \text{ whenever } \Delta \text{ is a disk in } \Sigma.$$  

(6.8)

Consider the $\mathfrak{h}$-valued 2-form $\sigma = \text{curv}_A + X_\xi \cdot \text{vol}_S$ on $\Sigma$. This 2-form has the property that $\exp \left( \int_\Delta \sigma \right) = e \in H$, for any disk $\Delta$ in $\Sigma$. Consequently $\int_\Delta \sigma \in \pi_1(H) \subseteq \mathfrak{h}$, for any disk $\Delta$ in $\Sigma$. Now fix a disk $\Delta$. Within $\Delta$, consider a descending sequence $\{\Delta_\epsilon\}_\epsilon$ of disks whose intersection is a single point. In a suitable coordinate patch, we realize the disks $\Delta_\epsilon$ as ordinary disks having radius $\epsilon \in \mathbb{R}$. Now, on the one hand, the map $f : I \to \pi_1(H)$ given by $f(\epsilon) = \int_{\Delta_\epsilon} \sigma$ is continuous and hence constant, and on the other hand, $\lim_{\epsilon \to 0} f(\epsilon) = \lim_{\epsilon \to 0} \int_{\Delta_\epsilon} \sigma = 0$, whence $\int_{\Delta} \sigma = 0$. Since $\Delta$ is an arbitrary disk, we conclude $\sigma = 0$, that is, $\text{curv}_A = -X_\xi \cdot \text{vol}_S$ as asserted.

We now show that (i) implies (iv). Consider first the special case where the group $G$ is abelian, i.e., in our notation, coincides with the compact connected abelian subgroup $H$ of $G$. Thus $\xi : P \to \Sigma$ is momentarily a principal $H$-bundle with an $H$-connection $A$
having curvature \( \text{curv}_A = -X^\xi \cdot \text{vol}_\Sigma \).

Recall from Section 3 above that the characteristic class \( X^\xi \) determines the corresponding homomorphism \( \chi^\xi : U(1) \to H \) via the identity \( \chi^\xi(t) = \exp(tX^\xi) \) \( (t \in [0, 2\pi]) \).

Let \( \xi_A : P_A \to \Sigma \) denote the principal \( H \)-bundle associated to our reference \( U(1) \)-bundle \( \xi_M : M_\Sigma \to \Sigma \) via \( \chi^\xi \), and denote the resulting smooth map from \( M_\Sigma \) to \( P_A \) by \( \lambda \). The commutative diagram

\[
\begin{array}{ccc}
U(1) & \longrightarrow & M_\Sigma \\
\times_i & \downarrow & \downarrow \lambda \\
H & \longrightarrow & P_A \\
\end{array}
\]

(6.9)

displays the resulting bundle map. Our reference connection form \( \omega_{M_\Sigma} : TM_\Sigma \to \mathfrak{t} \mathbb{R} \) on \( \xi_M : M_\Sigma \to \Sigma \) induces an \( H \)-connection form on \( \xi_A \) having curvature form \( -X^\xi \cdot \text{vol}_\Sigma \), and we denote this connection form by \( (\chi^\xi)_* \omega_{M_\Sigma} : TP_A \to \mathfrak{t} \mathbb{R} \). The bundles \( \xi \) and \( \xi_A \) are topologically equivalent. Hence there is an \( H \)-equivariant diffeomorphism \( \Phi : P \to P_A \) covering the identity of \( \Sigma \) that induces an isomorphism \( \xi_A \to \xi \) of principal \( H \)-bundles. The \( H \)-connection form \( (\chi^\xi)_* \omega_{M_\Sigma} \) induces, via \( \Phi \), an \( H \)-connection form \( \omega_{\xi, \Phi} : TP \to \mathfrak{t} \mathbb{R} \) on \( \xi \) having, likewise, curvature form \( -X^\xi \cdot \text{vol}_\Sigma \). Since this is the curvature form of \( A \) as well, for a suitable bundle automorphism (gauge transformation) \( \beta : P \to P \), pulling back the connection form \( \omega_A \) of \( A \) via \( \beta \) yields \( \omega_{\xi, \Phi} \) as \( \omega_{\xi, \Phi} = \beta^* \omega_A \). By construction, \( \lambda \) has the form of a pre-reduction map, now defined on \( M_\Sigma \) rather than on \( \tilde{M}_\Sigma \), and \( \Phi \) and \( \beta \) send horizontal paths to horizontal paths. The composite \( \beta^* \Phi \circ \lambda \) yields the morphism

\[
\begin{array}{ccc}
U(1) & \longrightarrow & M_\Sigma \\
\times_i & \downarrow & \downarrow \beta^* \Phi \circ \lambda \\
H & \longrightarrow & P \\
\end{array}
\]

(6.10)

of principal bundles with connection. Claim (iv) now boils down to the observation that the composite of \( \beta^* \Phi \circ \lambda \) with the universal covering projection \( M_\Sigma \to M_\Sigma \) admits the description spelled out as pre-reduction map.

Now we consider the case of a general principal \( G \)-bundle \( \xi : P \to \Sigma \) with a \( G \)-connection \( A \) whose curvature form reads \( \text{curv}_A = -X^\xi \cdot \text{vol}_\Sigma \). Maintain the notation \( \bar{\Pi} = G/S \cong H/D \) and \( \xi_{\bar{\Pi}} : P/S \to \Sigma \) introduced in Section 3 above and let \( \bar{\Lambda} \) denote the induced \( \bar{\Pi} \)-connection on the principal \( \bar{\Pi} \)-bundle \( \xi_{\bar{\Pi}} \). Let \( Q_{P/S} \) denote the image of \( Q_{P} \) in \( P/S \). The above argument shows that, with a slight abuse of the notation \( \lambda_{Q_{M_\Sigma} Q_{P/S}} \), the \( U(1) \)-bundle \( \xi_{M_\Sigma} : M_\Sigma \to \Sigma \) being endowed with the reference connection form \( \omega_{M_\Sigma} : TM_\Sigma \to \mathfrak{t} \mathbb{R} \), the expression for the pre-reduction map of \( \bar{\Lambda} \) relative to \( Q_{M_\Sigma} \) and \( Q_{P/S} \) yields a morphism

\[
\begin{array}{ccc}
U(1) & \longrightarrow & M_\Sigma \\
\times_{\bar{\Pi}} & \downarrow & \downarrow \lambda_{Q_{M_\Sigma} Q_{P/S}} \\
\bar{\Pi} & \longrightarrow & P/S \\
\end{array}
\]

(6.11)
of principal bundles with connection. Since the summand $\omega_b$ in the decomposition $\omega_A = \omega_b + \omega_b$ of the connection form $\omega_A$ of $A$ is the connection form of a flat $S$-connection $A_s$ on the principal $S$-bundle $P \to P/S$, the expression for the pre-reduction map of $A$ relative to $Q_{\overline{M}_S}$ and $Q_P$ yields the unique lift $\tilde{M}_S \to P$ of $\lambda_{Q_{\overline{M}_S},Q_P,\pi} : \tilde{M}_S \to P/S$ with respect to the flat connection $A_s$ and the choice of base points $Q_{\overline{M}_S}$ of $\tilde{M}_S$ and $Q_P$ of $P$.

The commutative diagram

$$
\begin{array}{c}
\tilde{M}_S \\
\downarrow \\
M_S \\
\downarrow \\
P/S
\end{array}
\xrightarrow{\lambda_{Q_{\overline{M}_S},Q_P,\pi}}
\begin{array}{c}
P \\
\downarrow \\
P/S
\end{array}
\tag{6.12}
$$

displays the situation.

Let $u_r$ denote the “boundary path” $\prod u_i v_j u_j^{-1} v_j^{-1}$ of the 2-cell $e^2$ of $\Sigma$; its horizontal lift $\tilde{u}_r$ in $\tilde{M}_S$ joins $Q_{\overline{M}_S}$ to $Q_{\overline{M}_S}[r] \in \tilde{M}_S$. Let $u_{p,r}$ denote the horizontal lift in $P$ of $u_r$ relative to $A$ having starting point $Q_P$. Then $Q_P \circ \text{Hol}_{u_r,Q_P}(A)$ coincides with the end point $u_{p,r}(1)$ of $u_{p,r}$. By construction,

$$Q_P X_A([r]) = \lambda_{Q_{\overline{M}_S},Q_P,\pi}(Q_{\overline{M}_S}[r]) = u_{p,r}(1) = Q_P \circ \text{Hol}_{u_r,Q_P}(A). \tag{6.13}$$

By (iii), $\text{Hol}_{u_r,Q_P}(A) = \exp(X_t)$ whence $X_A([r]) = \exp(X_t)$. Since the homomorphism $X_A$ and the surjective homomorphism $\Gamma_{\mathbb{R}} \to U(1)$ associated with the universal covering projection $\tilde{M}_S \to M_S$ render the diagram

$$
\begin{array}{c}
\Gamma_{\mathbb{R}} \\
\downarrow \\
U(1)
\end{array}
\xrightarrow{X_A}
\begin{array}{c}
G \\
\downarrow \\
\overline{H}
\end{array}
$$

commutative, the homomorphism $X_A$ satisfies the identity $X_A(t[r]) = \exp(tX_t)$, for $t \in \mathbb{R}$.

It is immediate that (iv) implies (i). This completes the proof. \hfill \square

7 \hspace{1em} \textsc{The constraints to be imposed on the based gauge orbit manifold}

In the Atiyah–Bott approach, the moduli space arises by symplectic reduction with respect to the group of gauge transformations, the vector space $A^2(\Sigma, \text{ad}_\xi)$ being viewed as the dual of the Lie algebra of infinitesimal gauge transformations, and the map which assigns to a connection its curvature being interpreted as a momentum mapping. In the Fréchet setting, the dual of an infinite-dimensional vector space is a delicate notion. Below we substitute for that momentum mapping a suitable infinite family of smooth $G$-equivariant $G$-valued maps on the orbit manifold $B_{\xi,0} = G_{\xi,0}/A_{\xi}$, and we refer to these maps as \textit{constraints}. The Wilson loop mapping $\rho$, cf. (5.13), is invariant under the
action of the group of based gauge transformations and hence induces a smooth map

$$\rho_\ast : B_{\xi,Q} \rightarrow G^{2t} \quad (7.1)$$

between Fréchet manifolds. With a slight abuse of terminology, we refer to $\rho_\ast$ as Wilson loop mapping as well. Via evaluation, the choice of generators for the presentation (2.1) of $\pi_1(\Sigma, Q)$ induces an obvious diffeomorphism from $\text{Hom}(F, G)$ onto $G^{2t}$, and we identify the two manifolds. The constraints will single out a subspace such that the restriction of the Wilson loop mapping $\rho_\ast$ to that subspace is a homeomorphism onto the subspace $\text{Hom}_{\xi,}(\Gamma, G)_\xi$ of $\text{Hom}(F, G) \equiv G^{2t}$ introduced at the end of Section 3 above. The subspace of $B_{\xi,Q}$ thus cut out by the constraints will recover the space of based gauge equivalence classes of smooth central Yang-Mills connections on $\xi$.

For a smooth connection $A$ on $\xi : P \rightarrow \Sigma$, we denote by $[A]$ its class in the orbit manifold $B_{\xi,Q} = G_{\xi,Q}\backslash A_\xi$. Consider the maps

$$c_{u,a,\xi,Q} : B_{\xi,Q} \rightarrow G, \ [A] \mapsto \text{Hol}_{u,Q}(a) \exp(a_{\Lambda} X_\xi), \quad (7.2)$$

as $u$ ranges over contractible loops in $\Sigma$ having the base point $Q$ of $\Sigma$ as starting point and bounded by disks $\Delta$ in $\Sigma$. Lemma 5.1 entails that these maps are smooth. Let

$$\mathcal{CYM}_{\xi,Q} \subseteq B_{\xi,Q}$$

be the subspace of $B_{\xi,Q}$ cut out by the constraints

$$c_{u,a,\xi,Q}([A]) = e \in G. \quad (7.3)$$

Lemma 6.1 entails that $\mathcal{CYM}_{\xi,Q}$ recovers the space of smooth based gauge equivalence classes of smooth central Yang–Mills connections on $\xi$ and that the values of the restriction of the Wilson loop mapping (7.1) to $\mathcal{CYM}_{\xi,Q}$ lie in $\text{Hom}_{\xi,}(\Gamma, G)_\xi$. We endow $\mathcal{CYM}_{\xi,Q}$ with the topology induced from the Fréchet topology of $B_{\xi,Q}$ and refer to this topology as the Fréchet topology on $\mathcal{CYM}_{\xi,Q}$.

**Theorem 7.1:** The space $\mathcal{CYM}_{\xi,Q}$ is non-empty, the values of the restriction of the Wilson loop mapping $\rho_\ast$, cf. (7.1), to the subspace $\mathcal{CYM}_{\xi,Q}$ of $B_{\xi,Q}$ lie in $\text{Hom}_{\xi,}(\Gamma, G)_\xi$, and the resulting map

$$\rho' : \mathcal{CYM}_{\xi,Q} \rightarrow \text{Hom}_{\xi,}(\Gamma, G)_\xi, \quad (7.4)$$

defined on $\mathcal{CYM}_{\xi,Q}$ endowed with its Fréchet topology, is a $G$-equivariant homeomorphism. Furthermore, abstractly, the space $\mathcal{CYM}_{\xi,Q}$ depends only on the bundle $\xi$ but not on the choices made to carry out its construction. Finally, with respect to the $G$-orbit stratifications on both sides, $\rho'$ is an isomorphism of stratified spaces which, on each stratum, restricts to a diffeomorphism onto the corresponding stratum of the target space.

We split the proof of Theorem 7.1 into Lemmata 7.2 and 7.5. In Section 9 below, we establish the “Furthermore” statement claiming the independence of the choices.

**Lemma 7.2:** The map $\rho' : \mathcal{CYM}_{\xi,Q} \rightarrow \text{Hom}_{\xi,}(\Gamma, G)_\xi$ given as (7.4) above is a bijection. More precisely: For every member $\chi$ of $\text{Hom}_{\xi,}(\Gamma, G)_\xi$, there is a connection $A$ on $\xi$ repre-
resenting a point of \( \mathcal{C}Y \mathcal{M}_{\xi,Q} \), unique up to based gauge transformations, such that the composite \( \Gamma \to \Gamma_R \xrightarrow{\chi} G \), with the injection \( \Gamma \to \Gamma_R \), of the constituent \( \chi_A : \Gamma_R \to G \) in the associated pre-reduction map (6.3) of \( A \) relative to \( Q_{\tilde{M}_\xi} \) and \( Q_P \) in Lemma 6.1 (iv) coincides with \( \chi : \Gamma \to G \).

Proof. Given \( \chi \in \text{Hom}_{\xi}(\Gamma, G) \), via the extended homomorphism \( \chi : \Gamma_R \to G \) given by \( \chi([t\{r\}]) = \exp(tX_\xi) \), for \( t \in \mathbb{R} \), the \( \Gamma_R \)-connection form \( \omega_{\tilde{M}_\xi} : T\tilde{M}_\xi \to \mathbb{R} \) on the principal reference \( \Gamma_R \)-bundle \( \tilde{M}_\xi \to \Sigma \) and the Maurer-Cartan form of \( G \) yield a \( G \)-connection form \( \omega_P : TP_\xi \to \mathfrak{g} \) on the associated principal \( G \)-bundle \( \xi_P : P : = \tilde{M}_\xi \times_G G \to \Sigma \). Below we sometimes write the resulting smooth principal \( G \)-connection on \( \xi_P \) as \( A_\xi \). By construction, the derivative \( d\chi : \mathbb{R} \to \mathfrak{g} \) of \( \chi : \Gamma_R \to G \) is given by \( d\chi(2\pi) = X_\xi \in \mathfrak{g} \). Since \( \text{curv}_{\omega_{\tilde{M}_\xi}} = -2\pi \text{vol}_\xi \in \mathcal{A}_2(\Sigma, \mathbb{R}) \), we conclude that \( \text{curv}_{\omega_P} = -X_\xi \cdot \text{vol}_\xi \in \mathcal{A}_2(\Sigma, \mathbb{R}) \).

In view of the definition of the connected component \( \text{Hom}_{\xi}(\Gamma, G) \) of \( \text{Hom}_{\xi}(\Gamma, G) \), cf. Section 3 above, the bundles \( \xi \) and \( \xi_P \) have the same topological type, that is, are isomorphic. Since the principal \( G \)-bundles \( \xi \) and \( \xi_P \) are isomorphic, there exists a (vertical) diffeomorphism \( \Phi : P \to P_\xi \) that induces an isomorphism \( \xi \to \xi_P \) of principal \( G \)-bundles. Let \( Q_{P_\xi} = [Q_{\tilde{M}_\xi}, e] \in P_\xi \). A suitable gauge transformation of \( \xi_P \) carries \( \Phi(Q_P) \) to \( Q_{P_\xi} \). Hence we can arrange for the diffeomorphism \( \Phi \) to be based in the sense that \( \Phi(Q_P) = Q_{P_\xi} \).

Pulling back the connection from \( \omega_P \) on \( \xi_P \) along \( \Phi \) yields a \( G \)-connection \( \omega_{\xi_P} \Phi \) on \( \xi \). The curvature of \( \omega_{\xi_P} \Phi \) still equals \( -X_\xi \cdot \text{vol}_\xi \). Another choice \( \tilde{\Phi} : P \to P_\xi \) of based diffeomorphism inducing an isomorphism \( \xi \to \xi_P \) of principal \( G \)-bundles yields a \( G \)-connection form \( \omega_{\xi,\tilde{\Phi}} \) on \( \xi \) in such a way that \( \omega_{\xi,\Phi} \) and \( \omega_{\xi,\tilde{\Phi}} \) are based gauge equivalent. Lemma 6.1 now implies the claim.

Remark 7.3. The proof of Lemma 7.2 essentially involves the argument for \([AB82, Proposition 6.16]\), with the connection \( \omega_{\tilde{M}_\xi} \) on \( \xi_{\tilde{M}_\xi} : \tilde{M}_\xi \to \Sigma \) substituted for the harmonic Yang-Mills connection in \([AB82, \S 6]\). As noted in the introduction, the novelty of our approach resides in the characterization of the space \( \mathcal{C}Y \mathcal{M}_{\xi,Q} \) merely in terms of constraints imposed on the Fréchet manifold of smooth based gauge equivalence classes of all smooth \( G \)-connections on \( \xi \).

Lemma 7.2 entails in particular that the subspace \( \mathcal{C}Y \mathcal{M}_{\xi,Q} \) of \( B_{\xi,Q} \) cut out by the constraints (7.3) is non-empty.

Remark 7.4. By Theorem 5.2, the Wilson loop mapping (5.13) is smooth, and hence the Wilson loop mapping (7.1) is smooth. Consequently \( \rho : \mathcal{C}Y \mathcal{M}_{\xi,Q} \to \text{Hom}_{\xi}(\Gamma, G) \), cf. (7.4), is a continuous bijection onto the compact subspace \( \text{Hom}_{\xi}(\Gamma, G) \) of the smooth compact manifold \( \text{Hom}(F, G) \). By the strong Uhlenbeck compactness theorem \([Uhl82]\), for any sequence \( \{A_j\} \) of smooth Yang-Mills connections with constant curvature (in fact, the hypothesis that the curvature be uniformly \( L^2 \)-bounded suffices), there exists a subsequence \( \{A_{j'}\} \) of smooth Yang-Mills connections and a sequence \( \{\beta_j\} \) of smooth gauge transformations such that the sequence \( \{\beta_j \cdot A_{j'}\} \) converges uniformly in the \( C^{\infty} \)-topology with all derivatives, that is, in the Fréchet topology, to a smooth connection \( A \).
Hence, in the Fréchet topology, the space \( \mathcal{CYM}_{\xi,Q} \) is as well compact. Consequently the map \( \rho^\xi \) is a homeomorphism as asserted. However, the Fréchet slice analysis in the proof of Lemma 7.5 below avoids that theorem. On the other hand, that slice analysis recovers the statement of the Uhlenbeck compactness theorem in the case at hand.

**Lemma 7.5:** The map \( \rho^\xi : \mathcal{CYM}_{\xi,Q} \to \text{Hom}_{\mathcal{X}}(\Gamma, G)_\xi \) given as (7.4) above is a \( G \)-equivariant homeomorphism that is compatible with the \( G \)-orbit stratifications on both sides.

**Proof:** First we show that the bijection \( \rho^\xi \) is a local homeomorphism.

Consider a smooth connection \( A \) on \( \xi \) that represents a point of \( \mathcal{CYM}_{\xi,Q} \). Since the connection \( A \) is central, the operator \( d_A \) of covariant derivative turns \( A^*(\Sigma, \text{ad}_\xi) \) into a cochain complex. We denote the resulting cohomology by \( \mathcal{H}_\xi \) and we use the notation

\[
Z^0_A = Z^0_A(\Sigma, \text{ad}_\xi) = \ker(d_A : \mathcal{A}^0(\Sigma, \text{ad}_\xi) \to \mathcal{A}^1(\Sigma, \text{ad}_\xi)) \\
Z^1_A = Z^1_A(\Sigma, \text{ad}_\xi) = \ker(d_A : \mathcal{A}^1(\Sigma, \text{ad}_\xi) \to \mathcal{A}^2(\Sigma, \text{ad}_\xi)) \\
B^1_A = B^1_A(\Sigma, \text{ad}_\xi) = \text{im}(d_A : \mathcal{A}^0(\Sigma, \text{ad}_\xi) \to \mathcal{A}^1(\Sigma, \text{ad}_\xi)) \\
B^2_A = B^2_A(\Sigma, \text{ad}_\xi) = \text{im}(d_A : \mathcal{A}^1(\Sigma, \text{ad}_\xi) \to \mathcal{A}^2(\Sigma, \text{ad}_\xi)).
\]

In particular, the cokernel of \( d_A : \mathcal{A}^1(\Sigma, \text{ad}_\xi) \to \mathcal{A}^2(\Sigma, \text{ad}_\xi) \) yields the associated second cohomology group \( \mathcal{H}^2_A \). We choose a complement \( \mathcal{H}^2_A \subseteq \mathcal{A}^2(\Sigma, \text{ad}_\xi) \) for \( B^2_A \subseteq \mathcal{A}^2(\Sigma, \text{ad}_\xi) \) such that the direct sum decomposition \( \mathcal{A}^2(\Sigma, \text{ad}_\xi) = B^2_A \oplus \mathcal{H}^2_A \) is a homeomorphism in the Fréchet topology. Then the projection \( \mathcal{A}^2(\Sigma, \text{ad}_\xi) \to \mathcal{H}^2_A \), restricted to \( \mathcal{H}^2_A \), is an isomorphism of finite-dimensional vector spaces.

Let \( S_A \subseteq A_\xi \) be a Fréchet slice for the action of the group \( G_\xi \) of gauge transformations on \( A_\xi \). Then

\[
T_A A_\xi = B^2_A \oplus T_A(S_A). \tag{7.9}
\]

Consider the smooth map

\[
h : S_A \to B^2_A, \quad A' \mapsto \text{curv}_{A'} \mod \mathcal{H}^2_A, \quad A' \in S_A. \tag{7.10}
\]

The derivative \( T_A h : T_A S_A \to B^2_A \) at the point \( A' \) of \( S_A \) is the composite

\[
T_A : S_A \xrightarrow{d_A} A^2 \xrightarrow{\text{proj}} B^2_A. \tag{7.11}
\]

By construction, the derivative \( T_A h = d_A \) at the point \( A \) is surjective. Our aim is to conclude that, for an open neighborhood \( U \) of \( A \) in \( S_A \), the pre-image \( h^{-1}(0) \cap U \) is a smooth manifold, necessarily finite-dimensional, having as tangent space at \( A \) the finite-dimensional vector space \( Z^1_A \cap T_A S_A = H^1_A \). To this end, we use the Nash–Moser theorem to show that \( h \) is a submersion at \( A \). Hence we need to verify that the derivative \( T_A h \) is surjective in an open neighborhood of \( A \) in \( S_A \) and that the family of right inverses is tame smooth.

Let \( A + \alpha \) be a point of \( A_\xi \), where \( \alpha \in \mathcal{A}^1(\Sigma, \text{ad}_\xi) \), and suppose that \( A + \alpha \in S_A \).
Consider the operators

\[ L_\alpha : \mathcal{A}^1(\Sigma, \text{ad}_\beta) \rightarrow \mathcal{A}^2(\Sigma, \text{ad}_\beta), \quad \beta \mapsto d_{\alpha} \beta = d_\alpha \beta + [\alpha \wedge \beta], \]

\[ T_\alpha = L_\alpha \circ L_0^* = d_{\alpha} \partial_\alpha : \mathcal{A}^2(\Sigma, \text{ad}_\beta) \rightarrow \mathcal{A}^2(\Sigma, \text{ad}_\beta). \]

The operator \( T_\alpha \) satisfies the identity

\[ T_\alpha(\eta) = \partial_\alpha(\eta) + [\alpha \wedge d_\alpha(\eta)] \quad (\eta \in \mathcal{A}^2(\Sigma, \text{ad}_\beta)). \]

In particular, \( T_0 \) is the covariant Laplacian on \( \mathcal{A}^2(\Sigma, \text{ad}_\beta) \) relative to \( A \) and so is an elliptic operator of index zero. Since the symbol map is continuous, for small \( \alpha \), the deformed operator \( T_\alpha \) is still elliptic and has index zero. We claim that the induced operator

\[ B^2_A \xrightarrow{\text{inj}} \mathcal{A}^2(\Sigma, \text{ad}_\beta) \rightarrow \mathcal{A}^2(\Sigma, \text{ad}_\beta) \xrightarrow{B^2_A} \]

(7.12) is invertible. In view of the Fredholm alternative, since \( T_\alpha \) has index zero, it is enough to show that \( \ker T_\alpha = H^2_\alpha \). The identity \((d_\alpha d_\alpha \eta, \eta)_{L_2} = [d_\alpha \eta, d_\alpha \eta]_{L_2}\) shows that \( H^2_\alpha = \ker T_0 = \ker d_\alpha d_\alpha \) coincides with the kernel of \( d_\alpha^* \). Hence \( \ker T_0 \subset \ker T_\alpha \). The converse inclusion follows from the upper semi-continuity of the dimension of the kernel of an elliptic operator [H07, Corollary 19.1.6], that is, \( \dim \ker T_\alpha \leq \dim \ker T_0 \) for small \( \alpha \). Hence the induced operator (7.12) is invertible and, with a slight abuse of notation, we denote this operator on \( B^2_A \) by \( T_\alpha \) as well. The inverse operator \( T_\alpha^{-1} \) arises from the Green’s operator and, by [Ham82, II.3.3.3 Theorem p. 158], is therefore tame.

Since \( L_\alpha \circ (L_0^* T_\alpha^{-1}) = \text{Id}_{B^2_A} \) for small \( \alpha \), we obtain the same family \( \{ L_0^* T_\alpha^{-1} \}_\alpha \) of right inverses for the family \( \{ T_\alpha, h \}_\alpha \). Hence the Nash–Moser inverse function theorem implies that \( h \) is a submersion near \( A \). In other words, there exists an open neighborhood \( U \) of \( A \) in \( S_A \) such that the pre-image \( h^{-1}(0) \cap U \) is a smooth manifold \( M_A \) having as tangent space at \( A \) the finite-dimensional vector space \( Z_A \cap T_A S_A \cong H^1_\alpha \). Since the canonical map \( H^1_\alpha \rightarrow H^1_\alpha \) is a linear isomorphism, the manifold \( M_A \) contains connections \( A' \in S_A \) that are \( C^\infty \)-close to \( A \) and have curvature map \( \text{curv}_{A'} : P \rightarrow g \) constant on the holonomy bundle of \( A \).

The action of the group \( G_\xi \) of gauge transformations on \( A_\xi \) restricts to a smooth action of the stabilizer subgroup \( Z_A \) of \( A \) on \( M_A \). Moreover, in a \( Z_A \)-neighborhood of \( A \) in \( M_A \), the projection to \( B_\xi, q \) is a diffeomorphism onto a smooth finite-dimensional \( Z_A \)-submanifold \( M_{A, Q} \) of \( B_\xi, q \). The restriction to \( M_{A, Q} \) of the Wilson loop mapping \( \rho : B_\xi, Q \rightarrow G^{2t} \) given as (7.1) above yields a smooth map

\[ \rho : M_{A, Q} \rightarrow \text{Hom}(F, G) \cong G^{2t} \]

(7.13)

having injective tangent map

\[ T_{[A]} \rho : T_{[A]} M_{A, Q} \rightarrow T_{\rho(A)} \text{Hom}(F, G) \]

at the point \([A]\) and, up to left translation by \( \rho(A) \in \text{Hom}(F, G) \cong G^{2t} \), that tangent
map takes the form

\[ T_{[A]} \rho : T_{[A]} M_{A,Q} \to \mathfrak{g}^{2't}. \]  

We shall shortly justify the injectivity claim. Passing, if need be, to a smaller open submanifold of \( M_{A,Q} \) containing the point \([A]\), we can arrange for the image \( M_{\rho(A),Q} := \rho(M_{A,Q}) \subset \text{Hom}(F,G) \) to be an ordinary (embedded) submanifold of \( \text{Hom}(F,G) \).

Since the connection \( A \) is central, the homomorphism \( \chi_A = \rho(A) : F \to G \) descends to a homomorphism \( \pi_1 \to G/H \equiv \mathbb{S} \) and hence induces, via the adjoint action of \( G \), a \( \pi_1 \)-module structure on \( \mathfrak{g} \). We write the resulting \( \pi_1 \)-module as \( \mathfrak{g}_A \) or \( \mathfrak{g}_{[A]} \). Then the cellular cochain complex of \( \Sigma \) with local coefficients is defined and takes the form

\[ \mathfrak{g}_{[A]} \xrightarrow{d_{[A]}} \mathfrak{g}_{[A]}^{2t} \xrightarrow{d_{[A]}} \mathfrak{g}_{[A]}. \]  

Twisted integration yields a morphism of cochain complexes

\[ A^0(\Sigma, \text{ad}_\ell) \xrightarrow{d_A} A^1(\Sigma, \text{ad}_\ell) \xrightarrow{d_A} A^2(\Sigma, \text{ad}_\ell) \]

inducing an isomorphism on cohomology, in particular an isomorphism \( H^1_A(\Sigma, \text{ad}_\ell) \to H^1(\Sigma, \mathfrak{g}_{[A]}). \) Given \( \chi : \Gamma \to G \) such that the value \( \chi([r]) \) lies in \( H \) (the connected component of the center of \( G \)), we use the notation

\[ Z^0_{\chi} = Z^0_{\chi}(\Sigma, \mathfrak{g}) = \ker(d_{\chi} : C^0(\Sigma, \mathfrak{g}) \to C^1(\Sigma, \mathfrak{g})), \]

\[ Z^1_{\chi} = Z^1_{\chi}(\Sigma, \mathfrak{g}) = \ker(d_{\chi} : C^1(\Sigma, \mathfrak{g}) \to C^2(\Sigma, \mathfrak{g})), \]

\[ B^1_{\chi} = B^1_{\chi}(\Sigma, \mathfrak{g}) = \text{im}(d_{\chi} : C^0(\Sigma, \mathfrak{g}) \to C^1(\Sigma, \mathfrak{g})), \]

\[ B^2_{\chi} = B^2_{\chi}(\Sigma, \mathfrak{g}) = \text{im}(d_{\chi} : C^1(\Sigma, \mathfrak{g}) \to C^2(\Sigma, \mathfrak{g})). \]

By Theorem 5.2, up to left translation by the value \( \rho(A) \in \text{Hom}(F,G) \equiv G^{2't} \), the twisted integration mapping \( f : A^1(\Sigma, \text{ad}_\ell) \to \mathfrak{g}_{[A]}^{2t} \) is the derivative (5.15) of the Wilson loop mapping \( \rho \) at the point \( A \) of \( A_\Sigma \). Hence the image of \( H^1_{\chi} = T_{[A]} M_{A,Q} \) under \( T_{[A]} \rho : T_{[A]} M_{A,Q} \to \mathfrak{g}_{[A]}^{2t} \) is a linear subspace of \( \mathfrak{g}_{[A]}^{2t} \) which lies in the subspace \( Z^1_{\chi} \subset \mathfrak{g}_{[A]}^{2t} \) of 1-cocycles relative to \( d_{[A]} : \mathfrak{g}_{[A]}^{2t} \to \mathfrak{g}_{[A]}^{2t} \). The projection from \( Z^1_{\chi} \) to the cohomology group \( H^1(\Sigma, \mathfrak{g}_{[A]}) \), isomorphically onto that cohomology group. Consequently, \( T_{[A]} \rho \) is injective and thus \( \rho \) is a diffeomorphism \( M_{A,Q} \to M_{\rho(A),Q} \) onto its image \( M_{\rho(A),Q} \), at least in a neighborhood of \([A]\) in \( M_{A,Q} \).

The evaluation map \( ev_{\mathcal{Q}_r} : G \to G \) identifies the stabilizer \( Z_A \) of \( A \) with the stabilizer \( Z_{\chi_A} \subset G \) of the value \( \chi_A = \rho(A) \in M_{\rho(A),Q} \). Thus the diffeomorphism \( \rho : M_{A,Q} \to M_{\rho(A),Q} \) extends to a diffeomorphism

\[ \rho : G \times Z_A M_{A,Q} \to G \times Z_{\chi_A} M_{\rho(A),Q} \subset \text{Hom}(F,G) \equiv G^{2't}. \]
Consequently the restriction
\[ (G \times_{\mathbb{A}} M_{\mathcal{A}Q}) \cap \mathcal{C}\mathcal{Y}\mathcal{M}_{\xi Q} \to (G \times_{\mathbb{A}} M_{\mathcal{A}(A)Q}) \cap \text{Hom}_{X_\xi}(\Gamma, G) \] (7.22)
is a homeomorphism. By construction, this homeomorphism is the restriction, to the intersection \((G \times_{\mathbb{A}} M_{\mathcal{A}Q}) \cap \mathcal{C}\mathcal{Y}\mathcal{M}_{\xi Q}\), of the map \(\rho^\ast\) given as (7.4) above.

Since our reasoning works for any point \([A]\) of \(\mathcal{C}\mathcal{Y}\mathcal{M}_{\xi Q}\), we conclude that \(\rho^\ast\) is a homeomorphism. Moreover, the above Fréchet slice analysis shows that \(\rho^\ast\) is compatible with the orbit type stratifications on both sides and that, on each stratum, \(\rho^\ast\) restricts to a diffeomorphism onto the corresponding stratum of the target space. \(\square\)

**Remark 7.6.** Let \(G = \text{U}(1)\) and endow our reference \(\text{U}(1)\)-bundle \(\xi_{M_\Sigma} : M_\Sigma \to \Sigma\) introduced in Section 3 above with a connection \(A\) whose curvature form \(\text{curv}_A\) equals \(-2\pi i\text{vol}_2\), that is, represents a point of \(\mathcal{C}\mathcal{Y}\mathcal{M}_{\xi_{M_\Sigma}Q}\). Under these circumstances, the pre-reduction map of \(A\) relative to \(Q_{M_\Sigma}\) and \(Q_{M_\Sigma}\) induces a gauge transformation of \(\xi_{M_\Sigma}\) that identifies the reference connection chosen in Section 3 with \(A\). In particular, the space \(\mathcal{C}\mathcal{Y}\mathcal{M}_{\xi_{M_\Sigma}Q}\) reduces to a point. This recovers the fact that a complex line bundle on \(\Sigma\) has a harmonic connection that is unique up to gauge transformations.

## 8 Fréchet-reconstruction of the space of gauge equivalence classes of all Yang-Mills connections

In this section we reconstruct the space of gauge equivalence classes of Yang-Mills connections on \(\Sigma\) within our Fréchet setting. Our procedure parallels the corresponding classical description of the topological decomposition of the space of Yang-Mills connections given in [AB82, Section 6].

The obvious surjection can : \(F \to \Gamma\) defined on the free group \(F\) on the chosen generators \(x_1, y_1, \ldots, x_r, y_r\) of the fundamental group \(\pi_1(\Sigma, Q)\) extends to the surjective homomorphism
\[ \mathbb{R} \times F \longrightarrow \Gamma_{\mathbb{R}}, \quad (t, w) \longmapsto t\lfloor r\rfloor \text{can}(w), \quad t \in \mathbb{R}, \quad w \in F. \] (8.1)

This surjective homomorphism, in turn, induces an injection of \(\text{Hom}(\Gamma_{\mathbb{R}}, G)\) into \(\text{Hom}\left(\mathbb{R} \times F, G\right)\). The restriction to the central copy of \(\mathbb{R}\) in \(\Gamma_{\mathbb{R}}\) generated by \([r]\) of a homomorphism \(\phi : \mathbb{R} \times F \to G\) is a 1-parameter subgroup of \(G\) and therefore of the kind \(t \mapsto \exp(tX_\phi)\) \((t \in \mathbb{R})\) for some uniquely determined member \(X_\phi\) of \(\mathfrak{g}\). Hence the assignment to \(\phi \in \text{Hom}(\mathbb{R} \times F, G)\) of \((X_\phi, \phi(x_1), \phi(y_1), \ldots, \phi(x_r), \phi(y_r))\) injects \(\text{Hom}(\mathbb{R} \times F, G)\) into \(\mathfrak{g} \times G^{2r}\) and hence yields an injection
\[ \text{Hom}(\Gamma_{\mathbb{R}}, G) \longrightarrow \mathfrak{g} \times G^{2r}. \] (8.2)

We endow \(\text{Hom}(\Gamma_{\mathbb{R}}, G)\) with the induced topology.

Let \(X \in \mathfrak{g}\). We denote by \(G_X \leq G\) the stabilizer of \(X\) under the adjoint action, necessarily a closed connected subgroup of \(G\), cf. [GS84, §32 Theorem 32.16 p. 259], and
we denote by $\text{Hom}_X(\Gamma_R, G)$ the space, if non-empty, of homomorphisms $\chi$ from $\Gamma_R$ to $G$ having the property that $\chi(t[r]) = \exp(tX)$. Since the copy of $R$ in $\Gamma_R$ generated by $[r]$ is central, the values of any $\varphi \in \text{Hom}_X(\Gamma_R, G)$ lie in $G_X$, that is, the canonical injection map from $\text{Hom}_X(\Gamma_R, G_X)$ to $\text{Hom}_X(\Gamma_R, G)$ is a homeomorphism. Given an adjoint orbit $\mathcal{O} \subset g$, we denote by $\text{Hom}_0(\Gamma_R, G)$ the space of homomorphisms $\chi$ from $\Gamma_R$ to $G$ having the property that $\chi(t[r]) = \exp(tY)$, for some $Y \in \mathcal{O}$. The space $\text{Hom}_0(\Gamma_R, G)$ is the total space of a fiber bundle over $\mathcal{O}$ having compact fiber, the fiber over $Y \in \mathcal{O}$ being the compact space $\text{Hom}_Y(\Gamma_R, G_Y)$, and so $\text{Hom}_0(\Gamma_R, G)$ is therefore itself compact.

We return to our principal $G$-bundle $\xi : P \to \Sigma$. Given an adjoint orbit $\mathcal{O}$, we denote by $\text{Hom}_0(\Gamma_R, G)^{\mathcal{O}}$ the subspace, if non-empty, of $\text{Hom}_0(\Gamma_R, G)$ that consists of the homomorphisms $\varphi : \Gamma_R \to G$ in $\text{Hom}_0(\Gamma_R, G)$ which have the property that the associated principal $G$-bundle $\xi_{\mathcal{O}} \times_\varphi G$ is topologically equivalent to $\xi$. When $\mathcal{O}$ consists of a single point $X$, we use the notation $\text{Hom}_X(\Gamma_R, G)^{\mathcal{O}}$. We then denote by $\text{Hom}(\Gamma_R, G)^{\mathcal{O}}$ the disjoint union of the spaces of the kind $\text{Hom}_0(\Gamma_R, G)^{\mathcal{O}}$, where $\mathcal{O}$ ranges over the adjoint orbits such that $\text{Hom}_0(\Gamma_R, G)^{\mathcal{O}}$ is non-empty. In terms of the notation thus established, the restriction mapping

$$\text{Hom}_X(\Gamma_R, G) \to \text{Hom}_X(\Gamma, G)$$

yields a homeomorphism from $\text{Hom}_X(\Gamma_R, G)^{\mathcal{O}}$ onto the space $\text{Hom}_X(\Gamma, G)^{\mathcal{O}}$ used in Theorem 7.1 above.

Recall from the proof of [AB82, Theorem 6.7] that Yang–Mills connections on $\xi$ arise from central Yang–Mills connections on certain subbundles. Indeed, consider a connection $A$ on $\xi$ that satisfies the Yang–Mills equation. Then the $G$-equivariant map $\ast \text{curv} A : P \to g$ is constant along horizontal paths. Let $X = \ast \text{curv} A(Q_p)$ and $\mathcal{P}_A = \{ p \in P : \ast \text{curv} A(p) = X \}$. The bundle projection $\xi$ restricts to a principal $G_X$-bundle projection $\xi_X : \mathcal{P}_A \to \Sigma$, and the connection $A$ restricts to a connection $A_X$ on $\xi_X$ having the property that $\text{curv}_{A_X} = -X \cdot \text{vol}_X$.

For $X \in g$, denote by $\mathcal{Y} \mathcal{M}_{\xi, Q_X}$ the subspace, if non-empty, of $B_{\xi, Q}$ that consists of the points $[A]$ characterized as follows: There exists a reduction of structure group $\xi_X : P_X \to \Sigma$ of $\xi$ to $G_X$ that is based in the sense that, relative to the associated injection of $P_X$ into the total space $P$ of $\xi$, the base point $Q_p$ of $P$ lies in $P_X$, such that some connection $A$ representing the class $[A]$ restricts to a connection $A_X$ on $\xi_X$ having the property that $\text{curv}_{A_X} = -X \cdot \text{vol}_X$. The residual $G$-action then corresponds to the adjoint action on $g$, i.e., $x \in G$ maps $\mathcal{Y} \mathcal{M}_{\xi, Q_X}$ to $\mathcal{Y} \mathcal{M}_{\xi, Q_{AdX}}$. This leads us to define, for a given adjoint orbit $\mathcal{O}$, the $G$-invariant subspace $\mathcal{Y} \mathcal{M}_{\xi, Q, \mathcal{O}}$ of $B_{\xi, Q}$ to be the union of the spaces $\mathcal{Y} \mathcal{M}_{\xi, Q_X}$ for $X \in \mathcal{O}$. The above argument shows that the space $\mathcal{Y} \mathcal{M}_{\xi, Q} \subset B_{\xi, Q}$ of based gauge equivalence classes of Yang–Mills connections on $\xi$ is the disjoint union of the spaces of the kind $\mathcal{Y} \mathcal{M}_{\xi, Q, \mathcal{O}}$, where $\mathcal{O}$ ranges over those adjoint orbits such that $\mathcal{Y} \mathcal{M}_{\xi, Q, \mathcal{O}}$ is non-empty. The space $\mathcal{Y} \mathcal{M}_{\xi, Q}$ is our Fréchet version of the space of based gauge equivalence classes of Yang–Mills connections on $\xi$. Likewise we take the Fréchet version $\mathcal{Y} \mathcal{M}_{\xi, Q, \mathcal{O}}$ of the space of based gauge equivalence classes of all Yang–Mills connections over $\Sigma$ relative to $G$ to be the disjoint union of the spaces of the kind $\mathcal{Y} \mathcal{M}_{\xi, Q}$, where $\xi$ ranges over all topological types of $G$-bundles on
We denote by \( \text{Hom} \) the space of smooth connections on \( \Sigma \). By construction, each connected component of \( \mathcal{YM}_{\xi, Q} \) is characterized by Fréchet constraints, and the same is true of \( \mathcal{YM}_{G \Sigma, Q} \). In this sense, the spaces \( \mathcal{YM}_{\xi, Q} \) and \( \mathcal{YM}_{G \Sigma, Q} \) are entirely characterized by Fréchet constraints. We say that a space of the kind \( \mathcal{YM}_{\xi, Q, \varnothing} \) and, likewise, one the kind \( \text{Hom}_\varnothing(\Gamma_R, G)_\xi \) is defined when its defining condition is non-vacuously satisfied.

**Theorem 8.1:** Given an adjoint orbit \( \mathcal{O} \) in \( \mathfrak{g} \), the space \( \mathcal{YM}_{\xi, Q, \varnothing} \) is defined if and only if the space \( \text{Hom}_\varnothing(\Gamma_R, G)_\xi \) is defined and, when this happens to be the case, the Wilson loop mapping \( \rho : B_{\xi, Q} \to G^{2\pi} \), cf. (7.1), induces a \( G \)-equivariant homeomorphism

\[
\mathcal{YM}_{\xi, Q, \varnothing} \to \text{Hom}_\varnothing(\Gamma_R, G)_\xi.
\]

Hence the Wilson loop mapping \( \rho \) induces a \( G \)-equivariant homeomorphism

\[
\mathcal{YM}_{\xi, Q} \to \text{Hom}(\Gamma_R, G)_\xi.
\]

Furthermore, as \( \xi \) ranges over all topological types of principal \( G \)-bundles on \( \Sigma \), the Wilson loop mapping \( \rho \) induces a homeomorphism

\[
\mathcal{YM}_{G \Sigma, Q} \to \text{Hom}(\Gamma_R, G).
\]

Moreover, abstractly the spaces \( C \mathcal{YM}_{\xi, Q} \) and \( C \mathcal{YM}_{G \Sigma, Q} \) do not depend on the choices made to carry out their construction. Finally, each of the above homeomorphisms is compatible with the \( G \)-orbit type stratifications and is, hence, an isomorphism of stratified spaces.

**Proof.** Consider \( X \in \mathfrak{g} \) such that \( \text{Hom}_X(\Gamma_R, G) \) is non-empty and choose \( \chi \in \text{Hom}_X(\Gamma_R, G) \). We denote by \( \text{Hom}_X(\Gamma_R, G)_\chi \) the connected component of \( \text{Hom}_X(\Gamma_R, G) \) that contains \( \chi \). Since \( [r] \) is central in \( \Gamma_R \), the values of any \( \varphi \in \text{Hom}_X(\Gamma_R, G)_\chi \) lie in \( G_X \) whence the canonical injection from \( \text{Hom}_X(\Gamma_R, G)_\chi \) to \( \text{Hom}_X(\Gamma_R, G)_\chi \) is a homeomorphism. The associated bundle

\[
\xi_X = \xi_{M} \times_X G_X : P_X = \tilde{M} \times_X G_X \to \Sigma
\]

is a principal \( G_X \)-bundle, and the canonical restriction map

\[
\text{Hom}_X(\Gamma_R, G)_\chi \to \text{Hom}_X(\Gamma, G_{\xi_X})
\]

is a homeomorphism. We take the point \( Q_X = [(Q_{M}, e)] \) as the base point of \( P_X \). By Theorem 7.1,

\[
\rho^* : C \mathcal{YM}_{\xi_X, Q} \to \text{Hom}_X(\Gamma, G_{\xi_X})
\]

is a \( G \)-equivariant homeomorphism.

Extension of the structure group \( G_X \) to \( G \) yields the principal \( G \)-bundle

\[
\xi_{G, G} : P_X \times_{G_X} G \to \Sigma
\]

on \( \Sigma \). The action \( G_{\xi_X, G} \times A_{\xi_X, G} \to A_{\xi_X, G} \) of the group \( G_{\xi_X, G} \) of smooth gauge transformations of \( \xi_{X, G} \) on the space \( A_{\xi_X, G} \) of smooth connections on \( \xi_{X, G} \) induces a \( G \)-equivariant
homeomorphism
\[ G \times G \xrightarrow{C} \mathcal{YM}_{\xi, Q} \to \mathcal{YM}_{\xi, G, \varnothing, Q}. \]

Likewise, the conjugation action
\[ G \times \text{Hom}(\Gamma, G) \to \text{Hom}(\Gamma, G), \quad (x, \chi) \mapsto x\chi(y)x^{-1}, \quad x \in G, \ y \in \Gamma, \ \chi \in \text{Hom}(\Gamma, G), \]
of \( G \) on \( \text{Hom}(\Gamma, G) \) induces a \( G \)-equivariant homeomorphism
\[ G \times G \xrightarrow{C} \text{Hom}_\phi(\Gamma, G)_{\xi, G}. \]

Consequently, the Wilson loop mapping induces a \( G \)-equivariant homeomorphism
\[ \mathcal{YM}_{\xi, G, \varnothing, Q} \to \text{Hom}_\phi(\Gamma, G)_{\xi, G} \]
of the kind (8.4).

The rest of the proof is straightforward. We leave the details to the reader. \( \square \)

**Remark 8.2.** Theorem 8.1 is a based version of [AB82, Theorem 6.7], but phrased in our Fréchet setting. We can paraphrase Theorem 8.1 by saying that our Fréchet construction precisely recovers the Atiyah–Bott spaces of based gauge equivalence classes of Yang-Mills connections.

To spell out what Theorems 7.1 and 8.1 entail for the ordinary moduli spaces of Yang-Mills connections, we use the notation
\[ \mathcal{YM}_{\xi, Q} = G \backslash \mathcal{YM}_{\xi, Q} \]
\[ \mathcal{YM}_{\xi} = G \backslash \mathcal{YM}_{\xi, Q} \]
\[ \mathcal{YM}_{G, \Sigma} = G \backslash \mathcal{YM}_{G, \Sigma, Q} \]

and
\[ \text{Rep}_{\chi}(\Gamma_\xi, G)_\xi = G \backslash \text{Hom}_{\chi}(\Gamma_\xi, G)_\xi (\equiv G \backslash \text{Hom}_{\chi}(\Gamma, G)_\xi) \]
\[ \text{Rep}(\Gamma_\xi, G)_\xi = G \backslash \text{Hom}(\Gamma_\xi, G)_\xi \]
\[ \text{Rep}(\Gamma_\xi, G) = G \backslash \text{Hom}(\Gamma_\xi, G). \]

Plainly, the homeomorphism \( \rho^* \), cf. (7.4), induces homeomorphisms
\[ \mathcal{YM}_{\xi} \to \text{Rep}_{\chi}(\Gamma_\xi, G)_\xi, \]
\[ \mathcal{YM}_{\xi} \to \text{Rep}(\Gamma_\xi, G)_\xi, \]
\[ \mathcal{YM}_{G, \Sigma} \to \text{Rep}(\Gamma_\xi, G). \]

and these homeomorphism preserve the stratifications on both sides. In view of Theorems 7.1 and 8.1, the spaces \( \mathcal{YM}_{\xi}, \mathcal{YM}_{\xi}, \) and \( \mathcal{YM}_{G, \Sigma} \) do not depend on the choices made to construct the Wilson loop mapping. The orbit spaces \( \mathcal{YM}_{\xi}, \mathcal{YM}_{\xi}, \) and
\( \mathcal{YM}_{G, \Sigma} \) recover, respectively, the moduli space of gauge equivalence classes of central Yang-Mills connections on \( \xi \), that of Yang-Mills connections on \( \xi \), and that of all Yang-Mills connections on \( \Sigma \) relative to \( G \).

9 Dependence on the Choices

The Wilson loop mapping \( \rho \), cf. (7.1), and the resulting map \( \rho^* \), cf. (7.4), depend on the choices of the point \( Q^p \) of the total space \( P \) of \( \xi \) and the canonical system \( u_1, v_1, \ldots, u_t, v_t \) of curves in \( \Sigma \). Let \( \overline{u}_1, \overline{v}_1, \ldots, \overline{u}_t, \overline{v}_t \) be another canonical system of curves in \( \Sigma \) that start at \( Q \). Their based homotopy classes \( \overline{x}_1, \overline{y}_1, \ldots, \overline{x}_t, \overline{y}_t \) constitute a system of generators of the fundamental group \( \pi_1 = \pi_1(\Sigma, Q) \) of \( \Sigma \) at \( Q \). Define the relator by \( \overline{r} = [\overline{x}_1, \overline{y}_1] \cdots [\overline{x}_t, \overline{y}_t] \). The association

\[
(u_1, v_1, \ldots, u_t, v_t) \mapsto (\overline{u}_1, \overline{v}_1, \ldots, \overline{u}_t, \overline{v}_t)
\]

induces an automorphism of \( \Gamma \) and one of \( \pi_1 \) that make the diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{Z}\langle [r] \rangle \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{Z}\langle [r] \rangle \\
\end{array}
\]

\[
\begin{array}{ccc}
\Gamma & \longrightarrow & \pi_1 \\
\downarrow & & \downarrow \\
\Gamma & \longrightarrow & \pi_1 \\
\end{array}
\]

(9.1)

commutative. Since the closed paths \([u_1, v_1] \cdots [u_t, v_t] \) and \([\overline{u}_1, \overline{v}_1] \cdots [\overline{u}_t, \overline{v}_t] \) acquire the same orientation, the two members \([r] \) and \([\overline{r}] \) of \( \Gamma \) coincide. This is a consequence of the fact that, up to within isomorphism, the group \( \Gamma \) is the uniquely determined Schur cover of \( \pi_1 = \pi_1(\Sigma, Q) \). Hence the identity symbol in (9.1) makes sense. That automorphism, in turn, determines a member of the mapping class group of \( \Sigma \). These observations yield a proof of the following.

Proposition 9.1: In the situation of Theorem 7.1 above, the base point \( Q^p \) of the total space \( P \) of \( \xi \) being fixed, two isomorphisms \( C\mathcal{YM}_{\xi, Q} \to \text{Hom}_{X_1}(\Gamma, G)_{\xi} \) of stratified spaces that arise from two Wilson loop mappings of the kind (7.4) associated to distinct choices of canonical systems of curves in \( \Sigma \) differ by a homeomorphism \( \text{Hom}_{X_1}(\Gamma, G)_{\xi} \to \text{Hom}_{X_1}(\Gamma, G)_{\xi} \), necessarily an automorphism of stratified spaces, induced by an automorphism of \( \Gamma \) that represents a member of the mapping class group of \( \Sigma \).

Proposition 9.2: In the situation of Theorem 7.1 above, the base point \( Q \in \Sigma \) being fixed, two isomorphisms of stratified spaces of the kind (7.4) that arise from the Wilson loop mappings associated to the same choice of canonical system of curves in \( \Sigma \) but one relative to \( Q^p \) and the other one relative to \( Q' \) for some \( x \in G \) differ by the homeomorphism \( \text{Hom}_{X_1}(\Gamma, G)_{\xi} \to \text{Hom}_{X_1}(\Gamma, G)_{\xi} \) induced by conjugation with \( x \in G \), necessarily an automorphism of stratified spaces.

Proposition 9.3: In the situation of Theorem 7.1 above, with a new choice of base points \( Q' \) of \( \Sigma \) and \( Q^{m_2} \) of \( M_\Sigma \), a choice of a homotopy class of paths in \( M_\Sigma \) from \( Q^{m_2} \) to \( Q^{m_2} \) induces an isomorphism of stratified spaces from \( \text{Hom}_{X_1}(\pi_1(M_\Sigma, Q^{m_2}), G)_{\xi} \) onto
\( \text{Hom}(\pi_{1}(M_{\xi}, Q_{\xi}), G)_{\xi} \).

The three propositions imply the independence statement in Theorem 7.1 above as well as that in Theorem 8.1 above.

10 Stratified Symplectic Structure

Recall that, by definition, the algebra \( C^{\infty}(\text{Hom}(\Gamma_{R}, G)_{\xi}) \) of Whitney smooth functions on \( \text{Hom}(\Gamma_{R}, G)_{\xi} \) (relative to the embedding into \( \text{Hom}(F, G) \)) is the algebra of continuous functions on \( \text{Hom}(\Gamma_{R}, G)_{\xi} \) that are restrictions of smooth functions on the ambient space \( \text{Hom}(F, G) \equiv G^{\infty} \); cf. [Whi34], [Whi57]. Let \( C^{\infty}(\text{Rep}(\Gamma_{R}, G)_{\xi}) = C^{\infty}(\text{Hom}(\Gamma_{R}, G)_{\xi})^{G} \)
be the algebra of \( G \)-invariant functions in \( C^{\infty}(\text{Hom}(\Gamma_{R}, G)_{\xi}) \), viewed as an algebra of continuous functions on the orbit space \( \text{Rep}(\Gamma_{R}, G)_{\xi} \) in an obvious manner. The algebras \( C^{\infty}(\text{Hom}(\Gamma_{R}, G)_{\xi}) \) and \( C^{\infty}(\text{Rep}(\Gamma_{R}, G)_{\xi}) \) are what has come to be known as a smooth structure on \( \text{Hom}(\Gamma_{R}, G)_{\xi} \) and \( \text{Rep}(\Gamma_{R}, G)_{\xi} \), respectively.

Recall that a differential space is a topological space \( T \) together with a subalgebra \( C \) of the algebra of continuous functions on \( T \) subject to certain conditions [Sik72]. A smooth structure on a topological space \( T \) is an algebra of functions that turns \( T \) into a differential space, subject to suitable additional conditions depending on context.

As \( B_{\xi, \varnothing} \) is a Fréchet manifold, it acquires a natural algebra \( C^{\infty}(B_{\xi, \varnothing}) \) of smooth functions on \( B_{\xi, \varnothing} \). Furthermore, since the Wilson loop mapping \( \rho : B_{\xi, \varnothing} \rightarrow \text{Hom}(F, G) \), cf. (7.1), is smooth, the image \( \rho^{*}(C^{\infty}(\text{Hom}(F, G))) \subseteq C^{\infty}(B_{\xi, \varnothing}) \) yields a smooth structure on \( B_{\xi, \varnothing} \) as well, but not every smooth function on \( B_{\xi, \varnothing} \) arises as the pull back, under the Wilson loop mapping, of a smooth function on \( \text{Hom}(F, G) \). When we consider the space \( \mathcal{Y}M_{\xi, \varnothing} \), the situation changes:

**Theorem 10.1:** The smooth structure on \( \mathcal{Y}M_{\xi, \varnothing} \) that arises via restriction of functions in \( C^{\infty}(B_{\xi, \varnothing}) \) coincides with the smooth structure that arises via restriction of functions in \( \rho^{*}(C^{\infty}(\text{Hom}(F, G))) \). In other words, the Wilson loop mapping \( \rho^{*} : \mathcal{Y}M_{\xi, \varnothing} \rightarrow \text{Hom}_{\mathcal{X}}(\Gamma, G)_{\xi} \), cf. (7.4), is an isomorphism of smooth spaces.

**Proof.** This is a consequence of the Fréchet slice analysis in the proof of Lemma 7.5. □

Accordingly, we take the smooth structure \( C^{\infty}(\mathcal{Y}M_{\xi, \varnothing}) \) on \( \mathcal{Y}M_{\xi, \varnothing} \) to be the algebra of continuous functions on \( \mathcal{Y}M_{\xi, \varnothing} \) that arise from ordinary smooth functions on \( \text{Hom}(F, G) \) via pull back under the Wilson loop mapping. Moreover, we take the smooth structure \( C^{\infty}(\mathcal{Y}M_{\xi}) \) on \( \mathcal{Y}M_{\xi} \) to be the algebra
\[ C^{\infty}(\mathcal{Y}M_{\xi}) = C^{\infty}(\mathcal{Y}M_{\xi, \varnothing})^{G} \]
of \( G \)-invariant functions, viewed as an algebra of continuous functions on \( \mathcal{Y}M_{\xi} = G \backslash \mathcal{Y}M_{\xi, \varnothing} \) in the obvious way. By construction, the homeomorphism \( \rho_{r} \), cf. (7.4), induces isomorphisms
\[
C^{\infty}(\mathcal{Y}M_{\xi, \varnothing}) \rightarrow C^{\infty}(\text{Hom}(\Gamma_{R}, G)_{\xi}) \tag{10.1}
\]
\[
C^{\infty}(\mathcal{Y}M_{\xi}) \rightarrow C^{\infty}(\text{Rep}(\Gamma_{R}, G)_{\xi}) \tag{10.2}
\]
of real algebras.

The extended moduli space construction developed in [Hue95b], [Jef97], [HJ94], see [Hue01] for a leisurely introduction and more references, yields the stratified symplectic structure

\[(C^\infty(\text{Rep}(\Gamma, G)_\ell), \{\cdot, \cdot\})\]

on \(\text{Rep}(\Gamma, G)_\ell\) determined by the data. On each stratum of \(\text{Rep}_{\ell}(\Gamma, G)\), that stratified symplectic structure comes down to a symplectic structure. On the other hand, the symplectic structure that was constructed in [AB82] from the familiar expression

\[
\omega(\alpha, \beta) = \int_\Sigma \alpha \wedge \beta, \quad \alpha, \beta \in \mathcal{A}_1(\Sigma, \text{ad}_\ell) = \mathcal{T}_A(A_\ell), \quad A \in \mathcal{A}_\ell,
\]

by infinite dimensional symplectic reduction involving Sobolev space techniques induces a symplectic structure on the top stratum of \(\mathcal{YM}_\ell\); here the exterior product \(\alpha \wedge \beta\) of the two \(\text{ad}_\ell\)-valued 1-forms \(\alpha\) and \(\beta\) on \(\Sigma\) is evaluated in terms of the inner product on \(\mathfrak{g}\). With respect to these symplectic structures, the Wilson loop mapping induces a symplectomorphism from the top stratum of \(\text{Rep}_{\ell}(\Gamma, G)\) onto the top stratum of \(\mathcal{YM}_\ell\).

Suitably interpreted, the Wilson loop mapping also induces a symplectomorphism from each stratum of \(\text{Rep}_{\ell}(\Gamma, G)\) onto the corresponding stratum of \(\mathcal{YM}_\ell\).

Remark 10.2. We can even push a bit further: In [AMM98], the authors reworked the extended moduli space construction and introduced the terminology quasi-hamiltonian \(G\)-space. Via the Wilson loop mapping (7.1), the 2-form on \(\text{Hom}(F, G)\) constructed in [HJ94], [Hue95b], [Jef97], pulls back to a 2-form on \(B_{\xi, Q}\) but, beware, this 2-form is not closed. We can then view the composite

\[r \circ \rho : B_{\xi, Q} \longrightarrow G\]

as a Lie group valued momentum mapping with respect to the pulled back 2-form on \(B_{\xi, Q}\). Then \(B_{\xi, Q}\) together with the \(G\)-action on \(B_{\xi, Q}\) and the map \(r \circ \rho\) is a quasi-hamiltonian \(G\)-space except that on \(B_{\xi, Q}\) (as opposed to \(\text{Hom}(F, G)\)) the non-degeneracy condition is to be interpreted appropriately. Reduction with respect to the structure group \(G\) then yields the moduli space of central Yang-Mills connections as a stratified symplectic space. Suitably interpreted, we can view \(B_{\xi, Q}\) as the reduced space relative to the group of based gauge transformations. This is, perhaps, not strictly correct – we did not make precise the momentum mapping for the group of based gauge transformations – and the purported space that arises by reduction relative to the group of based gauge transformations would have to be a proper subspace of \(B_{\xi, Q}\). However, to use some physics terminology, we are interested only in what happens on shell, that is, on the subspace \(\mathcal{YM}_{\xi, Q}\), and we can interpret Theorem 7.1 by saying that, on shell, the Fréchet manifold \(B_{\xi, Q}\) has the appropriate behavior as if it did arise by reduction relative to the group of based gauge transformations.
11 Fréchet Slices

Our references for terminology and notation in the framework of infinite-dimensional differential geometry are [Ham82] for the Fréchet case and [Nee06] for the locally convex setting. Thus we freely use the terms smooth manifold, Lie group, Fréchet structure etc. without further explanation.

11.1 Principal Lie subgroups

In finite dimensions, for any Lie subgroup \( H \subseteq G \), the space of left cosets \( G/H \) carries the structure of a smooth manifold that turns the natural projection \( \pi : G \to G/H \) into a right principal \( H \)-bundle. Recall that the local diffeomorphism

\[
\mu : \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h} \to G, \quad (X, Y) \mapsto \exp(X) \exp(Y), \quad X \in \mathfrak{k}, \ Y \in \mathfrak{h},
\]

plays an important role in the construction of charts on \( G/H \). This concept needs some refinement for general locally convex manifolds since in general \( \mu \) fails to be a local diffeomorphism (and the exponential map need not exist in the first place).

**Proposition 11.1** ([GN, Proposition 7.1.21]): Let \( G \) be a Lie group and \( H \subseteq G \) a subgroup. Given a Lie group structure on \( H \), the statements (1) and (2) below are equivalent:

1. The group \( H \) is a splitting Lie subgroup, and there exists a unique smooth structure on the space of left cosets \( G/H \) such that the canonical projection \( \pi : G \to G/H \) defines a smooth principal \( H \)-bundle structure. In this case, a map \( f : G/H \to M \) to some manifold \( M \) is smooth if and only if the induced map \( \hat{f} = f \circ \pi : G \to M \) is smooth.

2. The inclusion \( i : H \to G \) is a morphism of Lie groups, and there exist an open subset \( V \) containing 0 in some locally convex space \( \mathfrak{k} \) and a smooth map \( \sigma : V \to G \) with \( \sigma(0) = e \) such that

\[
\mu : V \times H \to G, \quad (X, y) \mapsto \sigma(X)y, \quad X \in V, \ y \in H,
\]

is a diffeomorphism onto an open subset of \( G \). In this case, \( \mu(V \times H) \) is a tube around \( H \) in \( G \).

Direct inspection of the proof of that proposition in [GN] shows the following.

**Complement 11.2:** The equivalence of (1) and (2) in Proposition 11.1 is in particular valid in the category of tame smooth Fréchet manifolds and tame smooth maps.

Given a Lie group \( G \) and a subgroup \( H \), we refer to \( H \) as a principal Lie subgroup when one (and hence both) of the conditions in Proposition 11.1 are met. In [GN] such a subgroup \( H \) is termed a split Lie subgroup but, as the requirement of being a splitting submanifold is not enough to ensure the properties in the above proposition we prefer the terminology principal Lie subgroup. When the Lie group \( G \) admits an exponential map, a local product structure around the identity already suffices since the local product structure can then be extended to the whole subgroup.

The subsequent proposition is a consequence of [GN, Proposition 7.1.24]. Since the
final version of this book is not yet available, for ease of exposition, we spell out a complete statement.

**Proposition 11.3:** Let $G$ be a Lie group with smooth exponential map and $H \subset G$ a splitting Lie subgroup. Denote a complement of $\mathfrak{h}$ in $\mathfrak{g}$ by $\mathfrak{k}$, and let $V \subset \mathfrak{k}$ be an open neighborhood of 0 in $\mathfrak{k}$. If the map

$$
\mu : V \times H \to G, \quad (X, y) \mapsto \exp(X)y, \quad X \in V, \ y \in H,
$$

(11.3)
is a local diffeomorphism at $(0, e)$, then $H$ is a principal Lie subgroup.

**Proof.** Since $\mu$ is a local diffeomorphism at $(0, e)$ there exists a neighborhood $U_H \subset H$ of $e$ in $H$ such that the restriction $\mu|_{V \times U_H}$ is a diffeomorphism onto an open neighborhood $U_G \subset G$ of $e$ in $G$ (after shrinking $V$ if need be). Because of the identity

$$
\mu(X, qy) = \mu(X, q)y, \quad X \in V, \ q, y \in H,
$$

the map $\mu$ is a local diffeomorphism at every point of $V \times H$.

Thus, to apply Proposition 11.1, it suffices to show that $\mu$ is injective. Since $U_H$ is open in $H$, there exists an open neighborhood $W_G$ of $e$ in $G$ such that $U_H = H \cap W_G$. By shrinking $V$ further, we can arrange for the set $\exp(-V) \exp(V)$ to lie completely in $W_G$. Now let $(X, a)$ and $(Y, b)$ be two points of $V \times H$ having the same image under $\mu$. Then

$$
\exp(-Y) \exp(X) = ba^{-1}.
$$

(11.4)
The left-hand side lies in $W_G$ by assumption and the right-hand side is an element of $H$ whence both expressions are contained in $U_H$. On the other hand, $\mu$ is bijective on $V \times U_H$, and hence the calculation

$$
\mu(Y, ba^{-1}) = \mu(Y, b)a^{-1} = \mu(X, a)a^{-1} = \mu(X, e)
$$

establishes the desired result $X = Y$ and $a = b$. \hfill $\square$

Since the derivative $\mu'_{0,e} : \mathfrak{v} \times \mathfrak{h} \to \mathfrak{g}$ of $\mu$ at the point $(0, e)$ of $V \times H$ is just the direct sum isomorphism $\mathfrak{v} \oplus \mathfrak{h} \to \mathfrak{g}$, the inverse function theorem yields the necessary local diffeomorphism so that Proposition 11.3 applies to Banach Lie groups; thus all splitting Lie subgroups of a Banach Lie group are principal.

Exploiting the Nash–Moser theorem, we obtain the following important consequence for tame Fréchet Lie groups.

**Theorem 11.4:** Let $G$ be a tame Fréchet Lie group with tame smooth exponential map which is a local diffeomorphism close to 0. Every splitting Lie subgroup $H \subset G$ of finite dimension or finite codimension is a principal Lie subgroup.

**Proof.** Denote the complement of $\mathfrak{h}$ in $\mathfrak{g}$ by $\mathfrak{k}$, and let $V \subset \mathfrak{k}$ be an open neighborhood of 0. In view of Proposition 11.3 above, we have to show that

$$
\mu : V \times H \to G, \quad (X, y) \mapsto \exp(X)y, \quad X \in V, \ y \in H,
$$

(11.5)
is a local diffeomorphism at \((0, e)\).

The derivative of \(\mu\) at the point \((X, e)\) of \(V \times H\) is given by:

\[
\mu'_{(X,e)} : \mathfrak{g} \times \mathfrak{h} \rightarrow T_{\exp(X)} G, \quad (Y, Z) \mapsto \exp'_X Y + (L_{\exp(X)})'_e Z, \ Y \in \mathfrak{g}, \ Z \in \mathfrak{h}.
\] (11.6)

In the case of a finite-dimensional subgroup \(H\), apply Proposition 11.6 below to

\[
(\exp^{-1} \circ \mu)'_{(X,e)} : \mathfrak{g} \times \mathfrak{h} \rightarrow \mathfrak{g}, \quad (Y, Z) \mapsto Y + (\exp^{-1} \circ L_{\exp(X)})'_e Z, \ Y \in \mathfrak{g}, \ Z \in \mathfrak{h},
\] (11.7)

and conclude that \(\mu'_{(Y,0)}\) is invertible with tame inverse for every \(Y \in V\) near \(X\). By \(H\)-translation the same statement holds true at every point \((Y, h)\) near \((X, e)\). By the version of the Nash–Moser inverse function given in [Die13, Theorem 3.4.2], we conclude that \(\mu\) is a local diffeomorphism at \((0, e)\).

A similar reasoning, applied to the map

\[
(L_{\exp(-X)^e} \circ \mu)'_{(X,e)} : \mathfrak{g} \times \mathfrak{h} \rightarrow \mathfrak{g}, \quad (Y, Z) \mapsto (L_{\exp(-X)^e} \circ \exp)'_X Y + Z, \ Y \in \mathfrak{g}, \ Z \in \mathfrak{h},
\] (11.8)

establishes the case of finite codimension.

For intelligibility, we now recall the tame Fréchet setting and the Nash–Moser inverse function theorem, cf. [Ham82, Die13]. A Fréchet space is called graded if it carries a distinguished increasing fundamental system of seminorms \(\|\cdot\|_k\). If the growth of the sequence \(k \mapsto \|\cdot\|_k\) is controlled by the exponential map, then the graded Fréchet space is called tame (see [Ham82, Definition II.1.3.2] for the exact statement). A smooth map \(f\) between graded Fréchet spaces is \(r\)-tame smooth if \(f\) and all its derivatives satisfy a local estimate of the form

\[
\|f(x)\|_k \lesssim 1 + |x|^k r.
\] (11.9)

Roughly speaking, this means that \(f\) has a 'maximal loss of \(r\) derivatives'.

**Theorem 11.5 ([HAM82, SECTION III.1]):** Let \(X\) and \(Y\) be tame Fréchet spaces, let \(U\) be an open subset of \(X\), and let \(f : U \rightarrow Y\) be tame smooth. Suppose that the derivative \(f'\) has a tame smooth family \(\Psi^f\) of inverses, that is, \(\Psi^f : U \times Y \rightarrow X\) is a tame smooth map, and the family \(\Psi^f_x : X \rightarrow Y\) is inverse to \(f'_x\) for every \(x \in U\). Then the map \(f\) is locally bijective and the inverse is a tame smooth map.

**Proposition 11.6 ([SUB85, PP. 47FF]):** Let \(A\) and \(X\) be tame Fréchet spaces, and let \(E \subset X\) and \(F \subset X\) be closed subspaces. Suppose that \(E\) is finite-dimensional. Moreover, let \(\Phi : A \times (E \times F) \rightarrow X\) be a tame smooth family of linear maps which decomposes into

\[
\Phi_a(u, v) = \varphi_a(u) + v, \quad a \in A, \ u \in E, \ v \in F,
\] (11.10)

where \(\varphi : A \times E \rightarrow X\) is a tame smooth family of injective, linear maps. If, for some \(a_0 \in A\), the partial map \(\Phi_{a_0} : E \times F \rightarrow X\) is a linear and topological isomorphism \(E \oplus F \rightarrow X\) then there is an open neighborhood \(U \subset A\) of \(a_0\) in \(A\) such that \(\Phi_a\) is a bijection for every \(a \in U\) in such a way that the inverses constitute a tame smooth map \(U \times X \rightarrow E \times F\).
11.2 Definition of a slice

In finite dimensions, a smooth retraction and a tubular neighborhood are equivalent to the existence of a slice, see [OR04, Theorem 2.3.26]. However, the proof relies on the inverse function theorem; indeed, this theorem entails that an infinitesimal splitting generates a local product decomposition. This argument does not carry over to arbitrary infinite-dimensional manifolds. For this reason and for later reference, we now make precise the notion of slice we subsequently use; this is the strongest concept of a slice we could possibly think of.

Definition 11.7. Let $\Upsilon: G \times M \to M$ be a smooth action of a Lie group $G$ on a manifold $M$. Let $q$ be a point of $M$ and suppose that the stabilizer $G_q \subseteq G$ at $q \in M$ is a principal Lie subgroup. A slice at $q \in M$ is a submanifold $S$ of $M$ that contains $q$ in such a way that the following conditions are met:

(S1) The $G$-closure $G \cdot S$ of $S$ is an open neighborhood of the orbit $G \cdot q$ in $M$ and $S$ is closed in $G \cdot S$.

(S2) The submanifold $S$ is closed under the induced action of $G_q$ in the sense that $G_q \cdot S \subseteq S$.

(S3) Any $x \in G$ such that $(x \cdot S) \cap S \neq \emptyset$ necessarily lies in $G_q$.

(S4) The smooth submersion $G \to G/G_q$ admits a local section $\chi: U \to G$ defined on an open neighborhood $U \subseteq G/G_q$ of the identity coset in such a way that the map

$$\chi^\delta: U \times S \to M, \quad (x, y) \mapsto \chi(x) \cdot y, \; x \in U, \; y \in S,$$

is a diffeomorphism onto an open neighborhood $V$ of $q$ in $M$.

A Lie group action $\Upsilon: G \times M \to M$ is said to admit a slice at $q \in M$ if the stabilizer $G_q$ of $q$ in $G$ is a principal Lie subgroup of $G$ and there is a slice at $q \in M$.

Proposition 11.8: Let $\Upsilon: G \times M \to M$ be a smooth Lie group action admitting a slice $S$ at the point $q$ of $M$. The following hold:

1. Given $y \in S$, the stabilizer $G_y$ of $y$ is a subgroup of the stabilizer $G_q$ of $q$. That is, the point $q$ has the maximal stabilizer of the whole slice.

2. The point $q$ has a neighborhood $V$ in $M$ such that any point of $V$ has a stabilizer that is conjugate to a subgroup of $G_q$.

Proof. The first claim follows directly from property (S3).

Let $V$ be a neighborhood of the kind characterized in (S4). Since every $v \in V$ is obtained by translation of the slice $S$ along the orbit, for some $x \in G$, the point $x \cdot v$ lies in $S$. Applying the first statement together with the equivariance of stabilizer subgroups we conclude that $x G_v x^{-1} \subseteq G_q$ for some $x \in G$. 

$\square$
11.3 Slice for the action of a subgroup

Let \((M, G)\) be a \(G\)-manifold, and let \(q\) be a point of \(M\) having the property that the action admits a slice at \(q\). In this section we discuss how one can construct a slice at \(q\) for the induced action of a subgroup \(H\) of \(G\).

**Proposition 11.9:** Let \(M\) be a manifold with a smooth \(G\)-action \(\mathcal{Y}\) admitting a slice at \(q \in M\). Let \(H \subset G\) be a normal Lie subgroup. If the induced action of \(H\) on \(M\) is free and the product \(G_q H\) is a principal Lie subgroup of \(G\), then there exists a slice at \(q\) for the \(H\)-action as well.

In the gauge theory context in Section 4 above, we apply this proposition with \(G\) being the group \(G_{\xi}\) of all gauge transformations and \(H\) the subgroup \(G_{\xi, Q}\) of based gauge transformations with respect to the chosen base point \(Q\) of the manifold written in Section 4 as \(M\) (but presently the notation \(M\) refers to a \(G\)-manifold). It is a standard fact that \(G_{\xi, Q}\) acts freely on the space of connections and that it is a normal subgroup of \(G_{\xi}\). By Theorem 11.4, \(G_{\xi, Q}\) is a principal Lie subgroup of \(G_{\xi}\). Furthermore, given a smooth connection \(A\) on \(\xi\), the stabilizer \(G_{\xi, A}\) of \(A\) in \(G_{\xi}\) is finite-dimensional, and the group \(G_{\xi, A} G_{\xi, Q}\) has finite codimension in \(G_{\xi}\) (indeed, \(\dim(G) - \dim(G_{\xi, A})\)) and hence, by Theorem 11.4, the group \(G_{\xi, A} G_{\xi, Q}\) is a principal Lie subgroup of \(G_{\xi}\).

**Proof.** Let \(S\) be a slice at \(q \in M\) for the \(G\)-action. The group \(G_q H\) is a principal Lie subgroup of \(G\) by assumption. Hence there is a closed subspace \(t_q\) of \(g\), an open neighborhood \(V\) of \(0\) in \(t_q\), and a smooth map \(\sigma : V \to G\) with \(\sigma(0) = e\) such that the map

\[
\mu : V \times G_q H \to G, \quad (X, xy) \mapsto \sigma(X) xy, \quad x \in G_q, \; y \in H,
\]

is a diffeomorphism onto an open subset of \(G\). Since \(H\) is a normal subgroup of \(G\),

\[
H \sigma(V) = \sigma(V)H = \mu(V, H).
\]

We claim that \(\hat{S} = \sigma(V) \cdot S\) is a slice for the \(H\)-action. Indeed, \(\hat{S}\) is a submanifold of \(M\) containing \(q\) since, in view of Proposition 11.8 (1), for any point \(y\) of \(S\), the stabilizer \(G_y\) of \(y\) is a subgroup of \(G_q\). Consequently the restriction of the action \(\mathcal{Y}\) to \(\sigma(V) \times S\) is a diffeomorphism onto \(\hat{S}\); we note that \(\sigma(X)\) lies in \(G_q\) only for the trivial element \(X = 0\). Furthermore, all defining properties of a slice are met:

(\(\hat{S}1\)) The \(H\)-closure \(H \cdot \hat{S} = H \sigma(V) \cdot S = \mu(V, H) \cdot S\) is open in \(M\) since \(G \cdot S \subset M\) is open and since \(V\) is an open subset of \(t_q\). Furthermore, since \(V \times \{e\}\) is closed in \(V \times H\), the space \(\hat{S} = \mu(V, e) \cdot S\) is closed in \(H \cdot \hat{S} = \mu(V, H) \cdot S\).

(\(\hat{S}2\)) The manifold \(\hat{S}\) is clearly invariant under the trivial stabilizer \(H_q = \{e\}\).

(\(\hat{S}3\)) Let \(x \in H\) and \(y \in \hat{S}\) such that \(x \cdot y\) lies in \(\hat{S}\) again. We have to show \(x = e\). By the construction of \(\hat{S}\) there are \(X, X' \in V\) and \(s, s' \in S\) with \(x \sigma(X) \cdot s = \sigma(X') \cdot s'\). Now property (\(\hat{S}3\)) for the \(G\)-slice \(S\) implies that, for some \(w \in G_q\),

\[
x \sigma(X) = \sigma(X) \sigma(X)^{-1} h \sigma(X) = \sigma(X') w.
\]
Since $H$ is a normal subgroup of $G$, the element $\tilde{x} := \sigma(X)^{-1}x\sigma(X)$ lies in $H$ again and hence

$$\mu(X, e) = \sigma(X) = \sigma(X') w y^{-1} = \mu(X', w y^{-1}).$$

The map $\mu : V \times G_q H \to G$ is injective and thus yields $X = X'$ and $w = e$, $\tilde{x} = e$. Consequently $x = e$.

(\S 4) Since $S$ is a slice, there is a smooth map $\chi : \Upsilon \to G$ defined on an open neighborhood $U$ of $0$ in $t \times h$ such that the maps

$$\chi^\tilde{S} : U \times S \to M, \quad (X, y) \mapsto \chi(X) \cdot y, \quad X \in U, \; y \in S,$$

$$v : U \times G_q \to G, \quad (X, y) \mapsto \chi(X)y, \quad X \in U, \; y \in G_q,$$

are diffeomorphisms onto open neighborhoods $W_M$ of $q$ in $M$ and $W_G$ of $e$ in $G$, respectively. Shrink $V$ (and hence the slice $\tilde{S}$) and choose an open and conjugation invariant subset $V_H$ of $H$ such that $\mu(V, V_H) \subseteq \nu(U, e)$. Then the map $\lambda : V \times V_H \leftrightarrow U$ defined by $\lambda = v^{-1} \mu$ is a diffeomorphism onto an open subset of $U$.

Finally, the map

$$\eta : V_H \times \tilde{S} \to M, \quad (v, \sigma(X) \cdot y) \mapsto v \sigma(X) \cdot y, \quad v \in V_H, \; X \in V, \; y \in S,$$

is a diffeomorphism onto an open neighborhood of $q$ in $M$. Indeed, this map can be written as the composite

$$V_H \times \tilde{S} \xrightarrow{Id \times \mu^{-1}} V_H \times V \times S \xrightarrow{\lambda \times 1} U \times S \xrightarrow{\chi^\tilde{S}} W_M$$

made explicit by the association

$$(v, \sigma(X) \cdot y) \quad \mapsto \quad (v, X, y) \quad \mapsto \quad (v^{-1} \mu(v, X), y) \quad \mapsto \quad \mu(v, X) \cdot y,$$

where $v \in V_H, X \in V, y \in S$. 

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