Abstract

We give a new proof of the fact that Milnor-Witt K-theory has geometric transfers. The proof yields to a simplification of Morel’s conjecture about transfers on contracted homotopy sheaves.

Keywords — Cycle modules, Milnor-Witt K-theory, Chow-Witt groups, A1-homotopy

MSC — 14C17, 14C35, 11E81

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1 Introduction

1.1 Current work

In [Mor12 Chapter 3], Morel introduced the Milnor-Witt K-theory of a field. Following ideas of Bass and Tate [BT73], one can define geometric transfer maps

\[ \text{Tr}_{x_1, \ldots, x_r/E} = \text{Tr}_{x_r/E(x_1, \ldots, x_{r-1})} \circ \cdots \circ \text{Tr}_{x_1/E} : K^\text{MW}_{x}(E(x_1, \ldots, x_r), \omega_{E(x_1, \ldots, x_r)/E}) \to K^\text{MW}_{x}(E) \]

on \( K^\text{MW} \) for finite extensions \( E(x_1, \ldots, x_r)/E \). Morel proved in [Mor12 Chapter 4] that any homotopy sheaves of the form \( M_{-1} \) admit such transfers and that they are functorial. In particular, this applies to Milnor-Witt K-theory.

In this article, we give an alternative proof of this result:

**Theorem 1** (Theorem 2.2.5). The transfers maps

\[ \text{Tr}_{x_1, \ldots, x_r/E} : K^\text{MW}_{x}(E(x_1, \ldots, x_r), \omega_{E(x_1, \ldots, x_r)/E}) \to K^\text{MW}_{x}(E) \]

do not depend on the choice of the generating system \((x_1, \ldots, x_r)\).

The idea is to reduce to the case of \( p \)-primary fields (see Definition 2.1.3) then study the transfers manually, as Kato originally did for Milnor K-theory (see [GS17] for a modern exposition). More elementary, the proof does not apply in full generality to the case of contracted homotopy sheaves \( M_{-1} \). However, we obtain as a corollary a reduction of Morel’s conjecture [Fel20 Conjecture 4.1.13].

**Theorem 2** (Theorem 2.2.17). In order to prove that a contracted homotopy sheaf \( M_{-1} \) has functorial transfers, it suffices to consider the case of \( p \)-primary fields (where \( p \) is a prime number).

1.2 Outline of the chapter

In Subsection 2.1, we recall the basic properties of some fields that we call \( p \)-primary fields. For \( p \) a prime number, a \( p \)-primary field has no nontrivial finite extension prime to \( p \) (see Definition 2.1.3).

In Subsection 2.2, we prove that Milnor-Witt K-theory admit transfer maps that are functorial. The proof is similar to the original proof of Kato for Milnor K-theory (see [GS17]): we reduce to the case of \( p \)-primary fields then study the transfers manually.

Notation

Throughout the paper, we fix a (commutative) field \( k \) and we assume moreover that \( k \) is perfect (of arbitrary characteristic).

By a field \( E \) over \( k \), we mean a finitely generated extension of fields \( E/k \).

Let \( f : X \to Y \) be a morphism of schemes. Denote by \( L_f \) (or \( L_{X/Y} \)) the virtual vector bundle over \( Y \) associated with the cotangent complex of \( f \), and by \( \omega_f \) (or \( \omega_{X/Y} \)) its determinant. Recall that if \( p : X \to Y \) is a smooth morphism, then \( L_p \) is (isomorphic to) \( T_p = \Omega_{X/Y} \) the space of (Kähler) differentials. If \( i : Z \to X \) is a regular closed immersion, then \( L_i \) is the normal cone \( -N_ZX \). If \( f \) is the composite \( Y \xrightarrow{i} \mathbb{P}^n_X \xrightarrow{p} X \) with \( p \) and \( i \) as previously (in other words, if \( f \) is lci projective), then \( L_f \) is isomorphic to the virtual tangent bundle \( i^*T_{\mathbb{P}^n/X} - N_Y(\mathbb{P}^n_X) \).

Let \( X \) be a scheme and \( x \in X \) a point, we denote by \( L_x = (m_x/m_x^2)^\vee \) and \( \omega_x \) its determinant. Similarly, let \( v \) a discrete valuation on a field, we denote by \( \omega_v \) the line bundle \( (m_v/m_v^2)^\vee \).

Let \( E \) be a field (over \( k \)) and \( v \) a valuation on \( E \). We will always assume that \( v \) is discrete. We denote by \( \mathcal{O}_v \) its valuation ring, by \( m_v \) its maximal ideal and by \( \kappa(v) \) its residue class field. We consider only valuations of geometric type, that is we assume: \( k \subset \mathcal{O}_v \), the residue field \( \kappa(v) \) is finitely generated over \( k \) and satisfies \( \text{tr.deg}_{k}(\kappa(v)) + 1 = \text{tr.deg}_{k}(E) \).
Let $E$ be a field. We denote by $\text{GW}(E)$ the Grothendieck-Witt ring of symmetric bilinear forms on $E$. For any $a \in E^*$, we denote by $\langle a \rangle$ the class of the symmetric bilinear form on $E$ defined by $(X,Y) \mapsto aXY$ and, for any natural number $n$, we put $n_ε = \sum_{i=1}^{n} (-1)^{i-1}$.

To any natural number $n$, we can associate an element in $\text{GW}(E)$ denoted by $n_ε = \sum_{i=1}^{n} (-1)^{i-1}$. Recall that, if $n$ and $m$ are two natural numbers, then $(nm)_ε = n_εm_ε$.

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2 Transfers on Milnor-Witt K-theory

2.1 On $p$-primary fields

We recall some facts about fields (See [Sha82, §1] and [BT73, Section 5]). Let $E$ be a field and $p$ a prime number. Fix a separable closure $E_s$ of $E$ and consider the set of all sub-extensions of $E_s$ that contain $E$ and that can be realized as a union of finite prime-to-$p$ extensions of $E$. Zorn’s lemma implies that this set contains a maximal element $E_{(p)}$ for the inclusion.

**Proposition 2.1.1.** If $F$ is a finite extension of $E$ contained in $E_{(p)}$, then its degree $[F : E]$ is prime to $p$.

**Proof.** Write $F = E(x_1, \ldots, x_r)$ with $x_i \in F$. Each $x_i$ is contained in a prime-to-$p$ extension of $E$ hence a degree prime to $p$.

**Proposition 2.1.2.** If $F$ is a finite extension of $E_{(p)}$, then its degree $[F : E_{(p)}]$ is equal to $p^n$ for some natural number $n$.

**Proof.** Let $x$ be any element in $F$ and denote by $P_x$ its irreducible polynomial over $E_{(p)}$. We prove that its degree is a power of $p$. All the coefficients lie in a finite prime-to-$p$ extension of $E$. If the degree of $x$ over $E_{(p)}$ is prime to $p$, then $E_{(p)}(x)$, which is a nontrivial extension of $E_{(p)}$, contradicts the maximality of $E_{(p)}$. Write $p^nm$ the degree of $x$ over $E_{(p)}$ with $n,m \geq 1$ and $(n,p) = 1$. Let $F_N$ be the normal closure of $F$ in $E_s$; it is a Galois extension of $E_{(p)}$ whose degree over $E_{(p)}$ is divisible by $p^nm$. If $n \neq 1$, then a Sylow $p$-subgroup $S(p)$ of $\text{Gal}(F_N/E_{(p)})$ is a nontrivial proper subgroup and the fixed field $F_N^{S(p)}$ is a nontrivial prime-to-$p$ extension of $E_{(p)}$, which is absurd. Thus $n = 1$ and the result follows.

The previous result leads to the following definition.

**Definition 2.1.3.** A field that has no nontrivial finite extension prime to $p$ is called $p$-primary.

**Proposition 2.1.4.** Let $F$ be a nontrivial finite extension of $E_{(p)}$ contained in $E_s$ and let $p^n$ be the degree $[F : E_{(p)}]$. Then there is a tower of fields

$$E_{(p)} = F_1 \subset F_2 \subset \cdots \subset F_n = F$$

such that $[F_i : F_{i-1}] = p$.

**Proof.** We prove the result by induction on $n$. We need to find a subfield $K$ of $F$ whose degree over $E_{(p)}$ is $p^{n-1}$. The group $G = \text{Gal}(E_s/E_{(p)})$ is a pro-$p$-group since all finite extensions of $E_{(p)}$ contained in $E_s$ are $p$-power extensions. Galois theory implies that $E$ is the fixed subfield of a subgroup $H$ of $G$ with $[G : H] = p^n$. We will find a subgroup $H_1$, such that $H \subset H_1 \subset G$ and $[G : H_1] = p^{n-1}$. Letting $K = E_s^{H_1}$, we will get the desired subfield $K$. 

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The group $H$ is subgroup of $G$ of finite index hence is open. By the class equation, it also follows that $H$ has only a finite number of conjugates in $G$. Let $H' = \cap_{x \in G} x^{-1} H x$, then $H'$ is an open normal subgroup of $G$ containing $H$. The group $G/H'$ is a finite $p$-group containing $H/H'$. By the Sylow theorems, we can find $H_1$, normal in $G$, with $H \subset H_1 \subset G$ and $[G : H_1] = p^{n-1}$. This ends to proof. □

Similarly, we obtain the following result.

Lemma 2.1.5. Let $p$ be a prime number and $E$ a $p$-primary field. Let $F/E$ be a finite extension.

1. The field $F$ inherits the property of having no nontrivial finite extension of degree prime to $p$.

2. If $F \neq E$, then there exists a subfield $E' \subset F'$ such that $F'/E$ is a normal extension of degree $p$.

2.2 Transfers on Milnor-Witt K-theory

We refer to [Mor12 §3] or [Fel18 §1] for the definitions and basic properties regarding Milnor-Witt K-theory. Recall the definition of transfers on Milnor-Witt K-theory, this follows from the original definition of Bass-Tate (see [BT73], see also [GS17]).

Theorem 2.2.1 (Homotopy invariance). Let $F$ be a field and $F(t)$ the field of rational fractions with coefficients in $F$ in one variable $t$. We have a split short exact sequence

$$0 \to \mathbb{K}^{MW}_*(F) \xrightarrow{i_*} \mathbb{K}^{MW}_*(F(t)) \xrightarrow{d_*} \bigoplus_{x \in (\mathbb{A}^1_k)^{\text{t}}(1)} \mathbb{K}^{MW}_*(\kappa(x), \omega_x) \to 0$$

where $d = \bigoplus_{x \in (\mathbb{A}^1_k)^{\text{t}}(1)} \partial_x$ is the usual differential.

Proof. See [Mor12 Theorem 5.38]. □

2.2.2. Let $\varphi : E \to F$ be a monogenic finite field extension and choose $x \in F$ such that $F = E(x)$. The homotopy exact sequence implies that for any $\beta \in \mathbb{K}^{MW}_*(E, \omega_{E/k})$ there exists $\gamma \in \mathbb{K}^{MW}_*(E(t), \omega_{E(t)/k})$ with the property that $d(\gamma) = \beta$. Now the valuation at $\infty$ yields a morphism

$$\partial_{\infty} : \mathbb{K}^{MW}_{n+1}((E(t), \omega_{E(t)/k}) \to \mathbb{K}^{MW}_*(E, \omega_{E/k})$$

which vanishes on the image of $i_*$. We denote by $\varphi^*(\beta)$ or by $\text{Tr}_{x/E}(\beta)$ the element $-\partial_{\infty}(\gamma)$; it does not depend on the choice of $\gamma$. This defines a group morphism

$$\text{Tr}_{x/E} : \mathbb{K}^{MW}_*(E(x), \omega_{F/k}) \to \mathbb{K}^{MW}_*(E, \omega_{E/k})$$

called the transfer map and also denoted by $\text{Tr}_{x/E}$. The following result completely characterizes the transfer maps.

Lemma 2.2.3. Keeping the previous notations, let

$$d = (\bigoplus_x d_x) \oplus d_{\infty} : \mathbb{K}^{MW}_{n+1}(E(t), \omega_{F(t)/k}) \to (\bigoplus_x \mathbb{K}^{MW}_*(E(x), \omega_{E(x)/k}) \oplus \mathbb{K}^{MW}_*(E, \omega_{E/k})$$

be the total twisted residue morphism (where $x$ runs through the set of monic irreducible polynomials in $E(t)$). Then, the transfer maps $\text{Tr}_{x/E}$ are the unique morphisms such that $\sum_x \langle \text{Tr}_{x/E} \circ d_x \rangle + d_{\infty} = 0$.

Proof. Straightforward (see [Mor12 §4.2]). □

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1In fact, Morel does not use twisted sheaves but chooses a canonical generator for each $\omega_x$ instead, which is equivalent.
**Definition 2.2.4.** Let $F = E(x_1, x_2, \ldots, x_r)$ be a finite extension of a field $E$ and consider the chain of subfields

$$E \subset E(x_1) \subset E(x_1, x_2) \subset \cdots \subset E(x_1, \ldots, x_r) = F.$$ 

Define by induction:

$$\text{Tr}_{x_1, \ldots, x_r/E} = \text{Tr}_{x_r/E(x_1, \ldots, x_{r-1})} \circ \cdots \circ \text{Tr}_{x_2/E(x_1)} \circ \text{Tr}_{x_1/E}$$

We give an elementary proof of the fact that the definition does not depend on the choice of the factorization (see [Mor12, Theorem 4.27] for the original proof):

**Theorem 2.2.5.** The maps $\text{Tr}_{x_1, \ldots, x_r/E} : \mathbf{K}^*_MW(F) \rightarrow \mathbf{K}^*_MW(E)$ do not depend on the choice of the generating system $(x_1, \ldots, x_r)$. 

We begin with a series of lemmas aimed at reducing the theorem to the case of $p$-primary fields.

**Lemma 2.2.6.** Let $F = E(x)/E$ be a simple extension of degree $n$ of characteristic zero fields and consider the transfer map $\text{Tr}_{F/E} : \text{GW}(F) \rightarrow \text{GW}(E)$. If $n$ is odd, then

$$\text{Tr}_{F/E}(1) = n_x.$$ 

If $n$ is even, then

$$\text{Tr}_{F/E}(1) = (n-1)x + \langle -N_{F/E}(x) \rangle.$$ 

where $N_{F/E}(x)$ is the classical norm of $x \in F^\times$.

**Proof.** See [Lam05, VII.2.2].

**Lemma 2.2.7.** Let $F/E$ be a finite extension of degree $n$ of characteristic zero fields and consider the transfer map $\text{Tr}_{F/E} : \text{GW}(F) \rightarrow \text{GW}(E)$. If $n$ is odd, then

$$\text{Tr}_{F/E}(1) = n_x.$$ 

If $n$ is even, then there exist $a_1, \ldots, a_n \in E^\times$ such that

$$\text{Tr}_{F/E}(1) = \sum_i (a_i).$$

**Proof.** When the extension is simple, this is Lemma 2.2.6. We conclude by induction on the number of generators.

**Lemma 2.2.8.** Let $E$ be a field of characteristic $p > 0$. Let $\alpha \in \text{GW}(E)$ be an element in the kernel of the rank morphism $\text{GW}(E) \rightarrow \mathbb{Z}$. Then $\alpha$ is nilpotent in $\text{GW}(E)$.

**Proof.** (See [MILZ16, Lemma B.4]) As the set of nilpotent elements in the commutative ring $\text{GW}(E)$ is an ideal, we may assume $\alpha = \langle t \rangle = 1$ where $t \in E^\times$. We have $(1 + \alpha)^2 = \langle t^2 \rangle = 1$, so that $\alpha^2 = -2\alpha$. By induction, we get $\alpha^n = (-2)^{n-1}\alpha$ for $n \geq 1$: we have to show that $\alpha$ is annihilated by a power of two. If $p = 2$, $2\alpha = 0$ holds (see [Mor12, Lemma 3.9]), i.e. $\alpha^2 = 0$. Now we assume $p \geq 3$ so that there is no danger thinking in terms of usual quadratic forms. We first consider $\mu := \langle -1 \rangle + 1 \in \text{GW}(F_p)$. The quadratic form $-x^2 - y^2$ over $F_p$ represents $1$ (see [Ser77, Proposition 4.4IV.1.7]) so that $\langle -1 \rangle + 1 = (1 + 1) \in \text{GW}(F_p)$, i.e. $2\mu = 0 \in \text{GW}(F_p)$, which gives $\mu^2 = 0$. Let $t \in E^\times$ be any nonzero element in an extension $E$ of $F_p$. The quadratic form $q(x, y) := x^2 - y^2 = (x + y)(x - y)$ represents $t$ (this is $q((1 + t)/2, (1 - t)/2)$), which easily implies that $\langle 1 \rangle + \langle -1 \rangle = \langle t \rangle + \langle -t \rangle$. This is equivalent to saying $(2 + \mu)\alpha = 0 \in \text{GW}(E)$. It follows that $4\alpha = (2 - \mu)(2 + \mu)\alpha = 0$, and then $\alpha^3 = 0$.

**Lemma 2.2.9.** Consider two finite extensions $F/E$ and $L/E$ of coprime degrees $n$ and $m$, respectively. Let $x \in \mathbf{K}^*_MW(E)$ such that $\text{res}_{F/E}(x) = 0 = \text{res}_{L/E}(x)$. Then $x = 0$. 


Proof. Applying the transfer map to \( \text{res}_{F/E}(x) \) and \( \text{res}_{L/E}(x) \), we see that \( x \) is killed by \( \text{Tr}_{F/E}(1) \) and \( \text{Tr}_{L/E}(1) \). In characteristic zero, up to swapping \( n \) and \( m \), we may assume that \( n \) is odd, hence \( \text{Tr}_{F/E}(1) = n \) and \( \text{Tr}_{L/E}(1) = \sum_i (a_i) \) for some \( a_1, \ldots, a_m \in E^\times \). Write \( n = 2^r + 1 \). There exist \( a, b \in \mathbb{Z} \) such that \( an + bm = r \) since \( n \) and \( m \) are coprime. Recall that the hyperbolic form \( h = 1 + (-1) \) satisfies \( (a_i)h = h \) for any \( i \) (see [Mor12, Lemma 3.7]). Hence \( rh = (an + b \sum_i (a_i))h \) and \( 1 = n - rh = (1 - ah) \text{Tr}_{F/E}(1) - bh \text{Tr}_{L/E}(1) \) kills \( x \).

In characteristic \( p > 0 \), there exist two nilpotent \( \alpha \) and \( \alpha' \) in \( \text{GW}(E) \) such that \( \text{Tr}_{F/E}(1) = \alpha + \alpha' \) and \( \text{Tr}_{L/E}(1) = m + \alpha' \), according to Lemma 2.2.8. Hence for a natural number \( s \) large enough, the element \( x \) is killed by the coprime numbers \( \alpha^s \) and \( \alpha'^s \) so that \( x = 0 \).

**Lemma 2.2.10.** Let \( E \) be a field of characteristic \( p > 0 \). Let \( F_1, \ldots, F_n \) be finite extensions of coprime degrees \( d_1, \ldots, d_n \). Let \( \delta \in K^\text{MW}_*(E) \) be an element such that \( \text{res}_{F_i/E}(\delta) = 0 \) for any \( i \). Then, \( \delta \) is zero.

Proof. In zero characteristic, this follows as in Lemma 2.2.9. Assume the characteristic of \( E \) to be nonzero. Let \( 1 \leq i \leq n \), the projection formula proves that \( \delta \) is killed by \( \text{Tr}_{F_i/E}(1) \). Thus, according to Lemma 2.2.8, there exist a nilpotent element \( a_i \) in \( \text{GW}(E) \) such that \( d_i + a_i \) kills \( \delta \). Since the degrees \( d_i \) are coprime, a Bezout combination yields a nilpotent element \( \alpha' \) in \( \text{GW}(E) \) such that \( 1 + \alpha' \) kills \( \delta \). Finally, we can find a natural number \( n \) large enough such that \( \delta = 1 + (\alpha')^n \cdot \delta = (1 + \alpha')^n \cdot \delta = 0 \).

**Lemma 2.2.11.** Let \( F/E \) be a field extension and \( w \) be a valuation on \( F \) which restricts to a non trivial valuation \( v \) on \( E \) with ramification \( e \). We have a commutative square

\[
\begin{array}{ccc}
K^\text{MW}_*(E) & \xrightarrow{\partial_v} & K^\text{MW}_{*-1}(\kappa(v), \omega_v) \\
\text{res}_{F/E} & \downarrow & \text{res}_{F/E} \\
K^\text{MW}_*(F) & \xrightarrow{\partial_w} & K^\text{MW}_{*-1}(\kappa(w), \omega_w)
\end{array}
\]

where \( e_v = \sum_{i=1}^{\epsilon} (-1)^{i-1} \).

Proof. See [Mor12, Lemma 3.19].

**Lemma 2.2.12.** Let \( F/E \) be a field extension and \( x \in (\mathbb{A}^1_E)_{(1)} \) a closed point. Then the following diagram

\[
\begin{array}{ccc}
K^\text{MW}_*(E(x), \omega_{E(x)/k}) & \xrightarrow{\text{Tr}_{E/E}} & K^\text{MW}_*(E, \omega_{E/k}) \\
\oplus_v \text{res}_{F(y)/E(x)} & \downarrow & \text{res}_{F/E} \\
\bigoplus_y \text{res}_{F(y)/E(x)} K^\text{MW}_*(F(y), \omega_{F(y)/k}) & \xrightarrow{\sum_v e_y \cdot \text{Tr}_{E/F}} & K^\text{MW}_*(F, \omega_{E/F})
\end{array}
\]

is commutative, where the notation \( y \mapsto x \) stands for the closed points of \( \mathbb{A}^1_E \) lying above \( x \), and \( e_{y,x} = \sum_{i=1}^{\epsilon} (-1)^{i-1} \) is the quadratic form associated to the ramification index of the valuation \( v_y \) extending \( v_x \) to \( F(t) \).

Proof. According to Lemma 2.2.11, the following diagram

\[
\begin{array}{ccc}
K^\text{MW}_*(E(t)) & \xrightarrow{\partial_e} & K^\text{MW}_{*-1}(E(x), \omega_x) \\
\oplus_v \text{res}_{F(y)/E(x)} & \downarrow & \bigoplus_y e_y \cdot \text{res}_{F(y)/E(x)} \\
K^\text{MW}_*(F(t)) & \xrightarrow{\oplus_v \partial_y} & \bigoplus_y \text{res}_{F(y)/E(x)} K^\text{MW}_{*-1}(F(y), \omega_y)
\end{array}
\]

is commutative hence so does the diagram.
where \( \rho_x \) is the canonical splitting (see Theorem 2.2.1). Then, we conclude according to the definition of the Bass-Tate transfer maps 2.2.2.

**Remark 2.2.13.** The multiplicities \( e_y \) appearing in the previous lemma are equal to

\[ [E(x) : E]_i/[F(y) : F]_i \]

where \([E(x) : E]_i\) is the inseparable degree.

**Theorem 2.2.14 (Strong R1c).** Let \( E \) be a field, \( F/E \) a finite field extension and \( L/E \) an arbitrary field extension. Write \( F = E(x_1, \ldots, x_r) \) with \( x_i \in F \), \( R = F \otimes E L \) and \( \psi_p : R \to R/p \) the natural projection defined for any \( p \in \Spec(R) \). Then the diagram

\[
\begin{array}{ccc}
\bigoplus_{p \in \Spec(R)} K^\MW_i(F/p, \omega_{F/k}) & \xrightarrow{\text{Tr}_{x_1,\ldots,x_r/E}} & K^\MW_i(E, \omega_{E/k}) \\
\otimes p \text{res}(R/p)/F & & \text{res}_L/E \\
\end{array}
\]

is commutative where \( m_p \) the length of the localized ring \( R_p \).

**Proof.** We prove the theorem by induction. For \( r = 1 \), this is Lemma 2.2.12. Write \( E(x_1) \otimes F L = \prod_i R_{ij} \) for some Artin local \( L \)-algebras \( R_{ij} \), and decompose the finite dimensional \( L \)-algebra \( F \otimes_{E(x_1)} R_{ij} \) as \( F \otimes_{E(x_1)} R_{ij} = \prod_i R_{ij} \) for some local \( L \)-algebras \( R_{ij} \). We have \( F \otimes_{E(x_1)} L \simeq \prod_i R_{ij} \). Denote by \( L_{ij} \) (resp. \( L_{ij} \)) the residue fields of the Artin local \( L \)-algebras \( R_{ij} \) (resp. \( R_{ij} \)), and \( m_j \) (resp. \( m_{ij} \)) for their geometric multiplicity. We can conclude as the following diagram commutes

\[
\begin{array}{cccc}
M_{-1}(F, \omega_{F/k}) & \xrightarrow{\text{Tr}_{x_1,\ldots,x_r/E}} & M_{-1}(E(x_1), \omega_{E(x_1)/k}) & \xrightarrow{\text{Tr}_{x_1/E}} & M_{-1}(E, \omega_{E/k}) \\
\otimes_{i,j} \text{res}_{L_{ij}/E} & & \otimes_{i,j} \text{res}_{L_{ij}/E} & & \text{res}_{L_{ij}/E} \\
\bigoplus_{i,j} M_{-1}(L_{ij}, \omega_{L_{ij}/k}) & \xrightarrow{\sum_{i,j}(m_{ij}) \text{Tr}_{\omega_{ij}(x_1),\ldots,\omega_{ij}(x_r)/L}} & \bigoplus_{i,j} M_{-1}(L_{ij}, \omega_{L_{ij}/k}) & \xrightarrow{\sum_{i,j}(m_{ij}) \text{Tr}_{\omega_{ij}(x_1)/L}} & M_{-1}(L, \omega_{L/k}) \\
\end{array}
\]

since both squares are commutative by the inductive hypothesis and the multiplicity formula \( (mn)_e = m_en_e \) for any natural numbers \( m, n \).
of $F$. Let $\alpha \in M_{-1}(F)$ and denote by $\delta$ the element $\text{Tr}_{x_1 \ldots x_r/E}(\alpha) - \text{Tr}_{y_1 \ldots y_r/E}(\alpha)$. Fix $p$ a prime number and let $L$ be a maximal prime to $p$ extension of $E$ ($L$ has no nontrivial finite extension of degree prime to $p$). With the notation of Theorem \[2.2.14\] the map $\sum_{m_p} (m_p) \cdot \text{Tr}_{y_{p(x_1)} \ldots y_{p(x_r)}/L}$ does not depend on the choice of $x_i$ according to the assumption. Hence $\text{res}_{L/E}(\delta) = 0$ and we can find a finite extension $L_p/E$ of degree prime to $p$ such that $\text{res}_{L_p/E}(\delta) = 0$. Since this is true for all prime number $p$, we see that the assumption of Lemma \[2.2.11\] are satisfied. Thus $\delta = 0$ and the theorem is proved.

**Remark 2.2.16.** More generally, one may replace $K^{\text{MW}}_*$ by any contracted homotopy sheaf $M_{-1}$ and apply the proof verbatim. In particular, we have the following simplification of Morel’s conjecture [Fe20, Conjecture 4.1.13].

**Theorem 2.2.17.** In order to prove that a contracted homotopy sheaf $M_{-1}$ has functorial transfers, it suffices to consider the case of $p$-primary fields (where $p$ is a prime number).

**Proposition 2.2.18 (Bass-Tate-Morel Lemma).** Let $F(x)$ be a monogenous extension of $F$. Then $K^{\text{MW}}_*(F(x))$ is generated as a left $K^{\text{MW}}_*(F)$-module by elements of the form

$$\eta^m \cdot [p_1(x), p_2(x), \ldots, p_n(x)]$$

where the $p_i$ are monic irreducible polynomials of $F[t]$ satisfying

$$\deg(p_1) < \deg(p_2) < \cdots < \deg(p_n) \leq d - 1$$

where $d$ is the degree of the extension $F(x)/F$.

**Proof.** Straightforward computations (see also [Mor12 Lemma 3.25.1]).

**Corollary 2.2.19.** Let $F/E$ be a finite field extension and assume one of the following conditions holds:

- $F/E$ is a quadratic extension,
- $F/E$ is a prime degree $p$ extension and $E$ has no nontrivial extension of degree prime to $p$.

Then $K^{\text{MW}}_*(F)$ is generated as a left $K^{\text{MW}}_*(E)$-module by $F^\times$.

**Proof.** In both cases, the extension $F/E$ is simple and the only monic irreducible polynomial in $E[t]$ of degree strictly smaller than $[F : E]$ are the polynomials of degree 1. We conclude by Proposition \[2.2.18\] and the fact that we fix a prime number $p$ and $E$ a $p$-primary field.

**Proposition 2.2.20.** Let $F = E(x)$ be a monogenous extension of $E$ of degree $p$. Then the transfers $\text{Tr}_{x/E} : K^{\text{MW}}_*(E(x), \omega_{E(x)/k}) \to K^{\text{MW}}_*(E, \omega_{E/k})$ do not depend on the choice of $x$.

**Proof.** According to Lemma \[2.2.19\] the group $K^{\text{MW}}_*(F, \omega_{F/k})$ is generated by products of the form $\text{res}_{F/E}(\alpha) \cdot [\beta]$ with $\alpha \in K^{\text{MW}}_*(E, \omega_{E/k})$ and $\beta \in F^\times$. The projection formula yields

$$\text{Tr}_{x/E}(\text{res}_{F/E}(\alpha) \cdot [\beta]) = \alpha \cdot \text{Tr}_{F/E}([\beta])$$

which does not depend on a $x$ (see [Fas19 §1]).

2.2.21. We may now use the notation $\text{Tr}_{F/E} : K^{\text{MW}}_*(F, \omega_{F/k}) \to K^{\text{MW}}_*(E, \omega_{E/k})$ for extensions of prime degree $p$.

**Proposition 2.2.22.** Let $F$ be a field complete with respect to a discrete valuation $v$, and $F'/F$ a normal extension of degree $p$. Denote by $v'$ the unique extension of $v$ to $F'$. Then the diagram
Proof. See [Mor12, Remark 5.20].

Corollary 2.2.23. Let $F/E$ be a normal extension of degree $p$ and let $x \in (\mathbb{A}^1_E)$. Then the diagram

\[
\begin{array}{ccc}
\mathbf{K}_*^{MW}(F', \omega_{F'/k}) & \xrightarrow{\delta_y} & \mathbf{K}_*^{MW}(\kappa(v'), \omega_{\kappa(v')}) \\
\bigoplus_{y \rightarrow x} & & \\
\mathbf{K}_*^{MW}(F) & \xrightarrow{\partial_x} & \mathbf{K}_*^{MW}(\kappa(v))
\end{array}
\]

is commutative.

Proof. Denote by $\hat{E}_x$ (resp. $\hat{F}_y$) the completions of $E(t)$ (resp. $F(t)$) with respect to the valuations defined by $x$ (resp. $y$). Consider the following diagram

\[
\begin{array}{ccc}
\mathbf{K}_*^{MW}(F(t), \omega_{F(t)/k}) & \xrightarrow{\delta_y} & \bigoplus_{y \rightarrow x} \mathbf{K}_*^{MW}(\hat{E}_x, \omega_{\hat{E}_x/k}) \\
\bigoplus_{y \rightarrow x} & & \\
\mathbf{K}_*^{MW}(E(t), \omega_{E(t)/k}) & \xrightarrow{\partial_x} & \mathbf{K}_*^{MW}(\hat{F}_y, \omega_{\hat{F}_y/k})
\end{array}
\]

The left-hand square is commutative according to Theorem 2.2.14. The right-hand square commute according to Proposition 2.2.22. Hence the corollary.

Lemma 2.2.24. Let $L/E$ be a normal extension of degree $p$, and let $E(a)/E$ be a monogenous finite extension. Assume that $L$ and $E(a)$ are both subfields of some algebraic extension of $E$, and denote by $L(a)$ their composite. Then the following diagram

\[
\begin{array}{ccc}
\mathbf{K}_*^{MW}(L(a), \omega_{L(a)/k}) & \xrightarrow{\delta_y} & \mathbf{K}_*^{MW}(L, \omega_{L/k}) \\
\bigoplus_{y \rightarrow x} & & \\
\mathbf{K}_*^{MW}(E(a), \omega_{E(a)/k}) & \xrightarrow{\partial_x} & \mathbf{K}_*^{MW}(E, \omega_{E/k})
\end{array}
\]

is commutative.

Proof. Let $x$ (resp. $y_0$) be the closed point of $\mathbb{A}^1_E$ (resp. $\mathbb{A}^1_L$) defined by the minimal polynomial of $a$ over $E$ (resp. $L$). Given $a \in \mathbf{K}_*^{MW}(L(a), \omega_{L(a)/k})$, we have $\text{Tr}_{a/L}(\alpha) = -\partial_{\alpha}(\beta)$ for some $\beta \in \mathbf{K}_*^{MW}(L(t), \omega_{L(t)/k})$ satisfying $\partial_\alpha(\beta) = \alpha$ and $\partial_\beta = 0$ for $y \neq y_0$. By Corollary 2.2.23

\[
\partial_x(\text{Tr}_{L(t)/E(t)}(\beta)) = \sum_{y \rightarrow x} \text{Tr}_{\kappa(y)/\kappa(x)}(\partial_\beta(\beta)) = \text{Tr}_{\kappa(y)/\kappa(x)}(\alpha),
\]

and, similarly, $\partial_{x'}(\text{Tr}_{L(t)/E(t)}(\beta)) = 0$ for $x \neq x'$. Hence by definition of the transfer map $\text{Tr}_{a/E}$ we have

\[
\text{Tr}_{a/E}(\text{Tr}_{L(a)/E(a)}(\alpha)) = -\partial_{\alpha}(\text{Tr}_{L(t)/E(t)}(\beta)).
\]
Moreover, since the only point of $\mathbb{P}^1$ above $\infty$ is $\infty$, another application of Corollary 2.2.23 gives
\[
\partial_\infty(\text{Tr}_{L(t)/E(t)}(\beta)) = \text{Tr}_{L/E}(\partial_\infty(\beta)).
\]
Hence the result.
\[
\text{Tr}_{a/E}(\text{Tr}_{L(a)/E(a)}(\alpha)) = -\text{Tr}_{L/E}(\partial_\infty(\beta)) = \text{Tr}_{L/E}(\text{Tr}_{a/E}(\alpha)).
\]

**Proof of Theorem 2.2.5.** We keep the previous notations. We already know that it suffices to treat the case when $E$ has no nontrivial extension of degree prime to $p$. Let $p^m$ be the degree of the extension $F/E$. We prove the result by induction on $m$. The case $m = 1$ follows from Proposition 2.2.21

Consider two decompositions
\[
E \subset E(x_1) \subset E(x_1, x_2) \subset \cdots \subset E(x_1, \ldots, x_r) = F.
\]
and
\[
E \subset E(y_1) \subset E(y_1, y_2) \subset \cdots \subset E(y_1, \ldots, y_s) = F.
\]
of $F$. By Lemma 2.2.4, the extension $E(x_1)/E$ contains a normal subfield $E(x_1')$ of degree $p$ over $E$. Applying Lemma 2.2.21 with $a = x_1$ and $L = E(x_1')$ yields $\text{Tr}_{x_1/E} = \text{Tr}_{x_1/E} \circ \text{Tr}_{x_1/E(x_1')}$. Hence, without loss of generality, we may assume that $x_1 = x_1'$ and, similarly, $[E(y_1) : E] = p$. Write $F_0$ for the composite of the fields $E(x_1)$ and $E(y_1)$ in $F$ and write $F = F_0(z_1, \ldots, z_i)$ with $z_i \in F$. The fields $E(x_1)$ and $E(y_1)$ have no nontrivial prime to $p$ extension, thus we may conclude by the induction hypothesis that the triangles
\[
\begin{array}{ccc}
K^\ast_{\text{MW}}(F, \omega_{F/k}) & \xrightarrow{\text{Tr}_{x_2, \ldots, x_r/E(x_1)}} & K^\ast_{\text{MW}}(E(x_1), \omega_{E(x_1)/k}) \\
\text{Tr}_{x_1, \ldots, x_r/F_0} & & \text{Tr}_{F_0/E(x_1)} \\
K^\ast_{\text{MW}}(F_0, \omega_{F_0/k})
\end{array}
\]
and
\[
\begin{array}{ccc}
K^\ast_{\text{MW}}(F, \omega_{F/k}) & \xrightarrow{\text{Tr}_{y_2, \ldots, y_s/E(y_1)}} & K^\ast_{\text{MW}}(E(y_1), \omega_{E(y_1)/k}) \\
\text{Tr}_{y_1, \ldots, y_s/F_0} & & \text{Tr}_{F_0/E(y_1)} \\
K^\ast_{\text{MW}}(F_0, \omega_{F_0/k})
\end{array}
\]
are commutative.

Moreover, Lemma 2.2.24 for $a = x_1$ and $L = E(y_1)$ implies that the following diagram
\[
\begin{array}{ccc}
K^\ast_{\text{MW}}(F_0, \omega_{F_0/k}) & \xrightarrow{\text{Tr}_{F_0/E(y_1)}} & K^\ast_{\text{MW}}(E(x_1), \omega_{E(x_1)/k}) \\
\text{Tr}_{F_0/E(x_1)} & & \text{Tr}_{x_1/E} \\
K^\ast_{\text{MW}}(E(y_1), \omega_{E(y_1)/k}) & \xrightarrow{\text{Tr}_{y_1/E}} & K^\ast_{\text{MW}}(E, \omega_{E/k})
\end{array}
\]
is commutative. Putting everything together, we conclude that $\text{Tr}_{x_1, \ldots, x_r/E} = \text{Tr}_{y_1, \ldots, y_s/E}$.\hfill $\square$
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