De Rham Cohomology Of Rigid Spaces
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Abstract

We define de Rham cohomology groups for rigid spaces over non-archimedean fields of characteristic zero, based on the notion of dagger space introduced in [12]. We establish some functorial properties and a finiteness result, and discuss the relation to the rigid cohomology as defined by P. Berthelot [2].

Introduction

Let $k$ be a field, complete with respect to a non-archimedean valuation. V. Berkovich, R. Huber and others have worked out satisfying foundations for the étale cohomology theory of $k$-rigid spaces. In this paper we assume $\text{char}(k) = 0$ and propose a definition of de Rham cohomology groups $H^*_{dR}(X)$ for $k$-rigid analytic spaces $X$. In [20] it is shown that for smooth $X$ the naive definition $H^*(X, \Omega^\bullet_X)$ meets certain minimal axiomatic requirements. However, if $X$ presents “boundaries” there are also serious pathologies entailed by this definition. For example, if $X = \text{Sp}(T_1)$, the closed unit disk, then $H^1(X, \Omega^\bullet_X)$ is an infinite dimensional $k$-vector space; formal integration preserves the radius of convergence, but does not preserve convergence on the boundary. The classical idea from the paper [19] of Monsky and Washnitzer to remedy this is to use only overconvergent power series for the definition of $H^*_{dR}(X)$. In [12] we introduced a category of ‘rigid spaces with overconvergent structure sheaf’ which we called $k$-dagger spaces, and in [13] we proved that $H^*(Y, \Omega^\bullet_Y)$ for many smooth $k$-dagger spaces $Y$ is indeed finite dimensional. Moreover, in [12] we associated to a $k$-dagger space functorially a $k$-rigid space with the same underlying $G$-topological space. Therefore, ideally we should define $H^*_{dR}(X)$ for a smooth $k$-rigid analytic space $X$ as $H^*(Y, \Omega^\bullet_Y)$, where $Y$ is a smooth $k$-dagger space whose associated rigid space is $X$. (For affinoid $k$-rigid spaces $X$, for which such a $Y$ exists, this is the approach of M. van der Put [22]). Our results here imply that this definition is indeed independent, up to canonical isomorphism, of the choice of $Y$ — if such a $Y$ exists. But if $X$ is not smooth, or if $X$ is not associated with a dagger space $Y$, another definition is needed. We propose: Let $W$ be a smooth $k$-dagger space and let $\phi : X \to W'$ be a closed embedding into its associated $k$-rigid space $W'$. Let $w : W' \to W$ be the natural morphism of ringed spaces (actually an isomorphism of underlying Grothendieck topologies). Then we set

$$H^*_{dR}(X) = H^*(X, (w \circ \phi)^{-1} \Omega^\bullet_{W/k}).$$

The content of the paper is organized as follows. In section 0 we recall some facts on dagger spaces, and in section 1 we show that $H^*_{dR}(X)$ is well-defined. In section 2 we
establish some functorial properties, similar to [14]. In section 3 we specialize to the case where \( k = \text{Frac}(R) \) with a complete discrete valuation ring \( R \) of mixed characteristic, with residue field \( \bar{k} \). From [13] we derive the finiteness of \( H^*_\text{dR}(X) \) for a big class of \( k \)-rigid spaces \( X \). We explain how the rigid cohomology, defined by Berthelot in [2], of a finite type \( \bar{k} \)-scheme \( Y \) can be expressed (or: be redefined) as the de Rham cohomology of the tube of \( Y \) in a smooth formal \( R \)-scheme with an embedding of \( Y \). A Gysin sequence for the de Rham cohomology of tubes in a semi-stable formal \( R \)-scheme is presented, generalizing the Gysin sequences for rigid cohomology from [3], [18]. In section 4 we briefly discuss for a given admissible proper formal \( R \)-scheme the relation between the rigid cohomology of its special fibre and the de Rham cohomology of its generic fibre.

0 Dagger spaces

Let \( k \) be a field complete with respect to a non-trivial non-archimedean valuation \( | \cdot | \), and of characteristic zero. We denote by \( k_a \) its algebraic closure with value group \( \Gamma^* = |k_a^*| = |k^*| \otimes \mathbb{Q} \).

We gather some facts from [12]. For \( \rho \in \Gamma^* \) the \( k \)-affinoid algebra \( T_n(\rho) \) consists of all series \( \sum a_\nu X^\nu \in k[[X_1, \ldots, X_n]] \) such that \( |a_\nu| \rho^{|\nu|} \) tends to zero if \( |\nu| \to \infty \). The algebra \( W_n \) is defined to be \( W_n = \bigcup_{\rho > 1} T_n(\rho) \). A \( k \)-dagger algebra \( A \) is a quotient of some \( W_n \); a surjection \( W_n \to A \) endows it with a norm which is the quotient of the Gauss norm on \( W_n \). All \( k \)-algebra morphisms between \( k \)-dagger algebras are continuous with respect to these norms, and the completion of a \( k \)-dagger algebra \( A \) is a \( k \)-affinoid algebra \( A' \) in the sense of [6]. There is a tensor product \( \otimes^1_k \) in the category of \( k \)-dagger algebras. As for \( k \)-affinoid algebras, one has for the set \( \text{Sp}(A) \) of maximal ideals of \( A \) the notions of rational and affinoid subdomains, and for these the analogue of Tate’s acyclicity theorem ([6],8.2.1) holds. The natural map \( \text{Sp}(A') \to \text{Sp}(A) \) of sets is bijective, and via this map the affinoid subdomains of \( \text{Sp}(A) \) form a basis for the strong \( G \)-topology on \( \text{Sp}(A') \) from [6]. Imposing this \( G \)-topology on \( \text{Sp}(A) \) one gets a locally \( G \)-ringed space, an affinoid \( k \)-dagger space. (Global) \( k \)-dagger spaces are built from affinoid ones precisely as in [6].

The fundamental concepts and properties from [6] translate to \( k \)-dagger spaces.

There is a faithful functor from the category of \( k \)-dagger spaces to the category of \( k \)-rigid spaces, assigning to a \( k \)-dagger space \( X \) a \( k \)-rigid space \( X' \) (to which we will refer as the associated rigid space). There is a natural morphism of ringed spaces \( x : X' \to X \) which induces isomorphisms between the underlying \( G \)-topological spaces and between the stalks of the structure sheaves. \( X \) is smooth if and only if \( X' \) is smooth. A smooth \( k \)-rigid space \( Y \) admits an admissible open affinoid covering \( Y = \bigcup V_i \) such that \( V_i = U'_i \) for uniquely determined (up to non canonical isomorphisms) affinoid \( k \)-dagger spaces \( U_i \).
A separated rigid space $X$ is called partially proper, if there are admissible open affinoid coverings $X = \bigcup_{j \in J} X_j = \bigcup_{j \in J} \tilde{X}_j$ with $X_j \subset \subset \tilde{X}_j$ for every $j \in J$ (where $\subset \subset$ is defined as in [6]).

For a $k$-dagger algebra (resp. $k$-affinoid algebra) $A$, one has a universal $k$-derivation of $A$ into finite $A$-modules, $d : A \to \Omega^1_A$. In the usual way it gives rise to de Rham complexes $\Omega^\bullet_X = \Omega^\bullet_{X/k}$ on $k$-dagger (resp. $k$-rigid) spaces $X$.

By a dagger space not specified otherwise, we will mean a $k$-dagger space, and similarly for dagger algebras, rigid spaces etc..

In the sequel, all dagger spaces and rigid spaces are assumed to be quasi-separated. We denote by $D = \{x \in k; |x| \leq 1\}$ (resp. $D^0 = \{x \in k; |x| < 1\}$) the unit disk with (resp. without) boundary, with its canonical structure of $k$-dagger or $k$-rigid space, depending on the context. For $\epsilon \in \Gamma^*$, the ring of global functions on the polydisk $\{x \in k^n; \text{all } |x_i| \leq \epsilon\}$, endowed with its canonical structure of $k$-dagger space, will be denoted by $k < \epsilon^{-1}.X_1,\ldots,\epsilon^{-1}.X_n >^\dagger$. The dimension $\dim(X)$ of a dagger space $X$ is the maximum of all $\dim(\mathcal{O}_{X,x})$ for $x \in X$. We say $X$ is pure dimensional if $\dim(X) = \dim(\mathcal{O}_{X,x})$ for all $x \in X$.

All rigid spaces and all dagger spaces are to be understood as spaces over $k$, unless otherwise specified; they are all assumed to be quasi-separated. For a smooth *dagger* space $X$ we set

$$R\Gamma_{dR}(X/k) = R\Gamma_{dR}(X) = R\Gamma(X, \Omega^\bullet_{X/k}).$$

1 The Definition

1.1 Let $T$ be a rigid space, $S$ a dagger space with associated rigid space $S'$, and let $\phi : T \to S'$ be a closed immersion. We denote by $\Psi(\phi, S)$ the set of admissible open subsets $U$ of $S$ for which $\phi$ factors as

$$T \to U' \to S'$$

where $U' \to S'$ is the embedding of rigid spaces associated with the embedding of dagger spaces $U \to S$. Usually we denote the open immersion $U \to S$ by $j_U$. For an abelian sheaf $\mathcal{F}$ on $S$, we define

$$\mathcal{F}^\phi = \lim_{U \in \Psi(\phi, S)} j_{U*}\mathcal{F}|_U.$$

Lemma 1.2. In 1.1, the natural morphism $\mathcal{F} \to \mathcal{F}^\phi$ is an epimorphism, and the functor $\mathcal{F} \mapsto \mathcal{F}^\phi$ is exact.
Proof: Let $S$ be affinoid and $\sigma \in \Gamma(S, \mathcal{F}^\phi)$. Then $\sigma$ comes from a section $\tau \in \mathcal{F}(U_1)$ for some $U_1 \in \Psi(\phi, S)$. Choose another $U_2 \in \Psi(\phi, S)$ such that $S = (S - U_2) \cup U_1$ is an admissible covering. Obviously $\sigma|_{(S - U_2)} = 0 \in \Gamma(S - U_2, \mathcal{F}^\phi)$ and the first claim follows. It implies the right exactness of (?)$^\phi$; its left exactness is clear.

Proposition 1.3. Let $Z$ be a rigid space, let $Y_1, Y_2$ be smooth dagger spaces, let $\phi_1 : Z \to Y_1'$ be a closed immersion of $Z$ into the rigid space associated with $Y_1$, and let $\psi : Y_1 \to Y_2$ be a smooth morphism of dagger spaces such that the induced morphism of rigid spaces $\psi^* \circ \phi_1 = \phi_2 : Z \to Y_2'$ is also a closed immersion. Then the natural map

$$R\Gamma(Y_2, (\Omega^*_2)^{\phi_2}) \to R\Gamma(Y_1, (\Omega^*_1)^{\phi_1})$$

is an isomorphism.

Proof: Set $Y_2' = Y_1' \times_{Y_2'} Z$ and let $J \subset O_{Y_2'}$ be the ideal defining $Z$ in $Y_2'$ (embedded diagonally). Then $J/J^2$ is locally free over $O_Z$ since $J$ defines a section of the smooth morphism $Y_2' \to Z$ (use [10] 0,19.5.4). Passing to admissible coverings (the claim is local) we may suppose $Y_2'$ is affinoid and $J/J^2$ is free. Choose a basis $t_1, \ldots, t_m$ of $J/J^2$, lift it to sections $t_1, \ldots, t_m \in O_{Y_2'}$ and let $L$ be the quotient of the relative differential module $\Omega^1_{Y_1'/Y_2'}$ divided by the submodule generated by $dt_1, \ldots, dt_m$. This is a coherent $O_{Y_1'}$-module and therefore has a Zariski closed support ([6] 9.5.2/4). It follows from Nakayama’s lemma that $L_x$ vanishes for all $x \in Y_2'$. Thus, passing to an admissible open subset of $Y_1'$, we may suppose that $\Omega^1_{Y_1'/Y_2'}$ has $dt_1, \ldots, dt_m$ as a basis. Let $W$ be the Zariski closed subspace of $Y_1'$ defined by the ideal $(t_1, \ldots, t_m) \subset O_{Y_1'}$. As in [10] IV 17.4.1, 17.6.1, 17.12.1, we see that $W \to Y_2'$ is étale. Covering admissibly and shrinking we may, in view of [14] below, even suppose that $W \to Y_2'$ is an isomorphism, that is, we may suppose $\psi^*$ has a section $\sigma : Y_2' \to Y_1'$, compatible with $\phi_1$ and $\phi_2$. We may suppose $Y_2$ and $Y_1'$ are affinoid, and by [17], 1.18 we may suppose that there is an isomorphism

$$\alpha : Y_1' \cong Y_2' \times \text{Sp}(k \leq \delta^{-1}.T_1, \ldots, \delta^{-1}.T_m >)$$

for some $\delta \in \Gamma^*$, where the section $\sigma$ on the left hand side corresponds to the zero section on the right hand side. For affinoid $U \in \Psi(\phi_2, Y_2)$ and $0 < \epsilon \leq \delta$ let $Y_1(\epsilon, U) \subset Y_1$ be the open dagger subspace of $Y_1$ defined by the open rigid subspace

$$Y_1'(\epsilon, U) = \alpha^{-1}(U' \times \text{Sp}(k \leq \epsilon^{-1}.T_1, \ldots, \epsilon^{-1}.T_m >))$$

of $Y_1'$. Given such a $U$ we have $\psi(\alpha(\epsilon, U)) \subset U$ for sufficiently small $\epsilon$. If $U$ is a Weierstrass domain in $Y_2$, then $Y_1'(\epsilon, U)$ is a Weierstrass domain in $Y_1'$, so $Y_1(\epsilon, U)$ is a Weierstrass
domain in $Y_1$ (if necessary, modify the defining functions slightly to get overconvergent ones). In particular, $Y_1(\epsilon, U)$ is affinoid, hence

$$Rj_{Y_1(\epsilon, U)}\ast \Omega_{Y_1(\epsilon, U)}^\bullet = j_{Y_1(\epsilon, U)}\ast \Omega_{Y_1(\epsilon, U)}^\bullet.$$  

The set of all such $Y_1(\epsilon, U)$ with $0 < \epsilon \leq \delta$ and Weierstrass domains $U \in \Psi(\phi_2, Y_2)$ is cofinal in the system $\Psi(\phi_1, Y_1)$. Since $Y_1$ and $Y_2$ are quasi-compact, cohomology commutes with the direct limit, so it is now enough to show that $\psi|_{Y_1(\epsilon, U)}$ induces an isomorphism

$$R\Gamma_{dR}(U) \to R\Gamma_{dR}(Y_1(\epsilon, U))$$

for all such $U, \epsilon$. By [5] we can find a map of dagger spaces $\eta : U \to Y_1(\epsilon/2, U)$ whose completion (on the level of algebras) is close to the map of rigid spaces $\sigma|_{U'} : U' \to Y_1'(\epsilon/2, U)$, namely so close that $(\psi|_{Y_1(\epsilon, U)}) \circ \eta$ is close to the identity. In particular $(\psi|_{Y_1(\epsilon, U)}) \circ \eta$ is an automorphism of $U$, thus induces an automorphism of $R\Gamma_{dR}(U)$. Therefore it is enough to see that $\eta$ induces an isomorphism in de Rham cohomology. Let $\delta : Y_1'(\epsilon, U) \to U'$ be the composition (of the respective restrictions) of $\sigma$ with the projection onto the first factor in the target of $\sigma$. By [5] we can approximate $\delta$ by a map of dagger spaces

$$\gamma : Y_1(\epsilon, U) \to U$$

such that $\gamma' \circ \sigma : U' \to U'$ is close to the identity, hence $\gamma \circ \eta : U \to U$ is close to the identity. In particular $\gamma \circ \eta$ is an automorphism and induces an automorphism of $R\Gamma_{dR}(U)$. So we only need to show that $\gamma$ induces an isomorphism in de Rham cohomology. We can find $S_i \in \mathcal{O}_{Y_1(\epsilon, U)}(Y_1'(\epsilon, U))$ close to $\alpha^*(T_i) \in \mathcal{O}_{Y_1'(\epsilon, U)}(Y_1'(\epsilon, U))$ such that the rule $\epsilon^{-1}.T_i \mapsto \epsilon^{-1}.S_i$ defines an extension

$$\tilde{\gamma} : Y_1(\epsilon, U) \to U \times \text{Sp}(k < \epsilon^{-1}.T_1, \ldots, \epsilon^{-1}.T_m > \dagger)$$

of $\gamma$ and such that the completion $\tilde{\gamma}'$ of $\tilde{\gamma}$ is close to the map induced by $\alpha$. In particular $\tilde{\gamma}'$ and hence $\tilde{\gamma}$ is an isomorphism. But then $\gamma$ must induce a automorphism isomorphism in de Rham cohomology.

**Lemma 1.4.** Let $Z \xrightarrow{j} X \xrightarrow{f} Y$ be morphisms of affinoid rigid spaces such that $f$ is étale and $i$ and $j = f \circ i$ are closed immersions. Then there is an admissible covering $Y = \bigcup_{i \in I} Y_i$ and for all $i \in I$ there are open neighbourhoods $U_i \subset Y_i$ of $Y_i \cap j(Z)$ and $V_i \subset X_i = f^{-1}(Y_i)$ of $X_i \cap i(Z)$ such that $f$ induces isomorphisms $V_i \cong U_i$.

**Proof:** According to [8] 3.1.4 we may assume, after passing to an admissible covering of $Y$, that there is a finite étale morphism $\bar{f} : \bar{X} \to Y$ and an open immersion $l : X \to \bar{X}$
for some rigid space \( \tilde{X} \) such that \( f = \tilde{f} \circ l \). By \([1] \) 2.5 we may assume \( Z \) is connected. Since \( \tilde{f} \) is étale, there is a decomposition \( \tilde{f}^{-1}(j(Z)) = i(Z) \bigcup Q \). Again by \([1] \) 2.5 we find an admissible open \( T \subset \tilde{X} \) which decomposes as \( T = T_1 \bigcup T_2 \) with \( i(Z) \subset T_1 \) and \( Q \subset T_2 \). By \([1] \) 2.4 there is an open connected \( S \subset Y \) with \( j(Z) \subset S \) and \( \tilde{f}^{-1}(S) \subset T \). Let \( W \) be the connected component of \( \tilde{f}^{-1}(S) \cap T_1 \) which contains \( i(Z) \). Then \( \mathcal{O}_W \) is a locally free finite rank \( \mathcal{O}_S \)-module — its rank is one since this is so modulo the ideal defining \( Z \) in \( S \). Therefore \( W \to S \) is an isomorphism, and \( V = W \cap X \subset X \) and \( U = f(V) \subset Y \) do the job.

**1.5** Let \( T, S, \phi \) be as in 1.1 and suppose \( S \) is smooth. Let \( s : S' \to S \) be the natural morphism of ringed spaces. Define the de Rham cohomology of \( T \) by

\[
R\Gamma_dR(T/k) = R\Gamma_dR(T) = R\Gamma(T, (s \circ \phi)^{-1}\Omega^{\bullet}_{S/k})
\]

and let \( H^q_dR(T) = H^q(R\Gamma_dR(T)) \). Clearly

\[
R\Gamma_dR(T) = R\Gamma(S, (\Omega^{\bullet}_{S/k})^\phi)
\]

and it is this formulation to which we refer in our proof of the welldefinedness of \( R\Gamma_dR(T/k) \).

**Proposition 1.6.** \( R\Gamma_dR(T) \) is independent of \( S \) and \( \psi \); it depends only on the reduced structure of \( T \). The de Rham cohomology is a contravariant functor in \( T \), with the following property: Given \( T_1, S_1, \phi_1 \) and \( T_2, S_2, \phi_2 \) as above and morphisms \( \beta : S_1 \to S_2 \) and \( \gamma : T_1 \to T_2 \) such that for the map \( \beta' : S'_1 \to S'_2 \) associated with \( \beta \) we have \( \beta' \circ \phi_1 = \phi_2 \circ \gamma \), then the map of functoriality \( R\Gamma_dR(T_2) \to R\Gamma_dR(T_1) \) induced by \( \gamma \) is the natural map

\[
R\Gamma(S_2, (\Omega^{\bullet}_{S_2})^{\phi_2}) \to R\Gamma(S_1, (\Omega^{\bullet}_{S_1})^{\phi_1})
\]

induced by \( \beta \).

**Proof:** Let \( S_1, S_2 \) be smooth dagger spaces with associated rigid spaces \( S'_1, S'_2 \), and let \( S'_1 \xleftarrow{\phi_1} T \xrightarrow{\phi_2} S'_2 \) be closed immersions. We compare with the diagonal embedding \( \phi_{1,2} = (\phi_1, \phi_2) : T \to S'_1 \times S'_2 \): By \([1] \) 2.3 the projections \( S_{12} = S_1 \times S_2 \to S_i \), which are smooth, induce isomorphisms

\[
R\Gamma(S_i, (\Omega^{\bullet}_{S_i})^{\phi_i}) \cong R\Gamma(S_{12}, (\Omega^{\bullet}_{S_{12}})^{\phi_{12}})
\]

for \( i = 1, 2 \). Composing we get the wanted isomorphism

\[
R\Gamma(S_2, (\Omega^{\bullet}_{S_2})^{\phi_2}) \cong R\Gamma(S_1, (\Omega^{\bullet}_{S_1})^{\phi_1}),
\]

compatible with those for a third choice \( S_3, \phi_3 \). Now let \( T_1, S_1, \phi_1 \) and \( T_2, S_2, \phi_2 \) be as above, and let \( \gamma : T_1 \to T_2 \) be a morphism of rigid spaces. We have

\[
R\Gamma_dR(T_1) = R\Gamma(S_{12}, (\Omega^{\bullet}_{S_{12}})^{(\phi_1, \phi_2 \circ \gamma)})
\]
\[ R\Gamma_{dR}(T_2) = R\Gamma(S_2, (\Omega^\bullet_{S_2})^{\phi_2}) \]

and the map of functoriality \( R\Gamma_{dR}(T_2) \to R\Gamma_{dR}(T_1) \) induced by \( \gamma \) is by definition the one induced from the natural projection \( S_{12} = S_1 \times S_2 \to S_2 \). Again one shows that it is independent of the \( S_i, \phi_i \).

Now let in addition \( \beta : S_1 \to S_2 \) with \( \beta' \circ \phi_1 = \phi_2 \circ \gamma \) be given. Then \( \beta \) defines a morphism \( \sigma : S_1 \to S_{12} \) such that the induced morphism \( S'_1 \to S'_{12} \) is compatible with the \( T_1 \)-embeddings. We have to show that the morphism
\[ R\Gamma(S_{12}, (\Omega^\bullet_{S_{12}})^{(\phi_1, \phi_2 \circ \gamma)}) \to R\Gamma(S_1, (\Omega^\bullet_{S_1})^{\phi_1}) \]

induced by \( \sigma \) coincides with the isomorphism which underlies the well-definedness of \( R\Gamma_{dR}(T_1) \) as described above. But this follows immediately from the fact that \( \sigma \) defines a section for the canonical projection \( S_1 \times S_{12} \to S_1 \) onto the first factor.

1.7 If the rigid space \( T \) admits no immersion into a rigid space which is associated with a smooth dagger space, then to define \( R\Gamma_{dR}(T) \) one can use embeddings of open pieces of \( T \), as in [14], p.28f.

1.8 Examples. (a) \( H^i_{dR}(\text{Sp}(T_n)) = 0 \) if \( i > 0 \) and \( H^0_{dR}(\text{Sp}(T_n)) = k \), because this is the de Rham cohomology of the dagger space \( \text{Sp}(W_n) \).
(b) If the rigid space \( X \) is partially proper and smooth, then we recover the naive definition, i.e.
\[ H^*_{dR}(X) = H^*(X, \Omega^\bullet_X). \]

To see this note that \( X \) is the rigid space associated to a uniquely determined smooth dagger space \( \tilde{X} \) (by [12] 2.27), and the canonical map
\[ H^*(\tilde{X}, \Omega^\bullet_{\tilde{X}}) \to H^*(X, \Omega^\bullet_X) \]

is an isomorphism. Indeed, by [12] 3.2, the maps
\[ H^*(\tilde{X}, \Omega^i_{\tilde{X}}) \to H^*(X, \Omega^i_X) \]

are isomorphisms for any \( i \) since \( X \) is partially proper. Apply this to the morphism between the respective Hodge-de Rham spectral sequences.
(c) For some computations of the de Rham cohomology of smooth affinoid curves and of hypersurfaces, see [22]. They show that, in these cases, the numbers \( \text{dim}_k(H^i_{dR}(\text{?)}) \) are finite and are, in fact, the "correct" Betti numbers.
2 Functorial properties and some exact sequences

**Proposition 2.1.** Let $T$ be a rigid space, let $T_1, T_2$ be Zariski closed subspaces of $T$ such that $T = T_1 \cup T_2$. Suppose there exists a pair $(\phi : T \to S', S)$ as before. Then there is a long exact sequence

$$\ldots \to H^q_{dR}(T) \to H^q_{dR}(T_1) \oplus H^q_{dR}(T_2) \to H^q_{dR}(T_1 \cap T_2) \to H^{q+1}_{dR}(T) \to \ldots.$$  

**Proof:** Let $\phi_i : T_i \to S'$ for $i = 1, 2$ and $\phi_{12} : T_{12} = T_1 \cap T_2 \to S'$ be induced by $\phi$. It is enough to show that for every $q \geq 0$ the sequence

$$L = [0 \to (\Omega^q_S)^{\phi} \to (\Omega^q_S)^{\phi_1} \oplus (\Omega^q_S)^{\phi_2} \to (\Omega^q_S)^{\phi_{12}} \to 0]$$

is exact. The claim is local on $S$. If $S$ is affinoid, the set of all $U_1 \cup U_2$, resp. of all $U_i$, resp. of all $U_1 \cap U_2$, where the $U_i$ run through the affinoid $U_i \in \Psi(\phi_i, S)$ for $i = 1, 2$, is cofinal in $\Psi(\phi, S)$, resp. in $\Psi(\phi_i, S)$, resp. in $\Psi(\phi_{12}, S)$. By the sheaf property and the commutation of direct limits with $\Gamma(V,?)$ on quasi-compact spaces $V$, we see that $\Gamma(V, L)$ is exact on the left and in the middle for all admissible open affinoid $V \subset S$, hence $L$ is exact on the left and in the middle. Since $\Omega^q_S \to (\Omega^q_S)^{\phi_{12}}$ is an epimorphism [1,2], $L$ is also exact on the right. This implies what we want.

**Proposition 2.2.** Let $f : T_1 \to T_2$ be a quasi-compact morphism of rigid spaces. Let $Z_2$ be a Zariski closed subspace of $T_2$ and let $Z_1 = Z_2 \times_{T_2} T_1$. Assume that $T_1 - Z_1$ maps isomorphically to $T_2 - Z_2$. Assume furthermore that there exist smooth dagger spaces $S_1, S_2$, closed immersions $\phi_1 : T_1 \to S'_1$ and $\phi_2 : T_2 \to S'_2$ into their associated rigid spaces, and a quasi-compact morphism $g : S_1 \to S_2$ such that the induced morphism $g' : S'_1 \to S'_2$ satisfies $g' \circ \psi_1 = \psi_2 \circ f$ and maps $S'_1 - g'^{-1}(Z_2)$ isomorphically to $S'_2 - Z_2$. Then there is a long exact sequence

$$\ldots \to H^q_{dR}(T_2) \to H^q_{dR}(T_1) \oplus H^q_{dR}(Z_2) \to H^q_{dR}(Z_1) \to H^{q+1}_{dR}(T_2) \to \ldots.$$  

**Proof:** First suppose $T_i = S'_i$ for $i = 1, 2$. Note that we have a natural transformation

$$(g_* (\cdot))^{\phi_2} \to g_* (\cdot)^{\phi_1}$$

of functors of abelian sheaves on $S_1$. We claim that

$$(Rg_* \mathcal{F})^{\phi_2} \to Rg_* (\mathcal{F}^{\phi_1})$$

for an abelian sheaf $\mathcal{F}$ on $S_1$ is an isomorphism. Indeed, one has

$$(R^i g_* \mathcal{F})^{\phi_2} = \lim_{U \in \Psi(\phi_2, S_2)} j_{U,*} (R^i g_* \mathcal{F}|_U) = \lim_{U \in \Psi(\phi_2, S_2)} R^i g_* (j_{g^{-1}(U),*} \mathcal{F}|_{g^{-1}(U)})$$
$$R^ig_*(\lim_{U \in \Psi(\phi_2, S_2)} j_{g^{-1}(U) \ast} \mathcal{F}_{|_{g^{-1}(U)}}) \cong R^ig_*(\lim_{U \in \Psi(\phi_1, S_1)} j_{U \ast} \mathcal{F}_{|U}) = R^ig_*(\mathcal{F}^{\phi_1}).$$

Here (1) holds since $g$ is quasi-compact, and (2) holds since the set of all $g^{-1}(U)$ for $U \in \Psi(\phi_2, S_2)$ is cofinal within $\Psi(\phi_1, S_1)$, also because $g$ is quasi-compact ([11] 2.4). We obtain

$$(Rg_! \Omega^\bullet_{S_1})^{\phi_2} \cong Rg_! ((\Omega^\bullet_{S_2})^{\phi_2}).$$

Now let $Q$ be the mapping cone of the natural map $\Omega^\bullet_{S_2} \to Rg_! \Omega^\bullet_{S_1}$. The exactness of $(?)^{\phi_2}$ and the isomorphism just seen tell us that $Q^{\phi_2}$ is the mapping cone of $(\Omega^\bullet_{S_2})^{\phi_2} \to Rg_!((\Omega^\bullet_{S_1})^{\phi_1})$. On the other hand, our assumptions imply that $Q \to Q^{\phi_2}$ is an isomorphism, i.e. $\Omega^\bullet_{S_2} \to Rg_! \Omega^\bullet_{S_1}$ and $(\Omega^\bullet_{S_2})^{\phi_2} \to Rg_!((\Omega^\bullet_{S_1})^{\phi_1})$ have an isomorphic mapping cone. By a diagram chase according to the pattern of [14] p.44, we conclude in this case.

The general case follows formally from this and 2.1, precisely as in [14] 4.4.

**Theorem 2.3.** Let $X$ be a $k$-scheme of finite type, let $H^*_{dR}(X)$ be its algebraic de Rham cohomology as defined in [14] and let $X^{an}$ be the rigid analytification of $X$. Then there is a canonical isomorphism

$$H^*_{dR}(X) = H^*_{dR}(X^{an}).$$

**Proof:** We assume for simplicity that there is a closed embedding $X \to Y$ into a smooth $k$-scheme $Y$. By definition,

$$H^*_{dR}(X) = H^*(Y, \hat{\Omega}^\bullet_Y)$$

where $\hat{\Omega}^\bullet_Y$ denotes the formal completion of the de Rham complex $\Omega^\bullet_Y$ on $Y$ along $X$. Similarly we may define an auxiliary de Rham cohomology theory $\hat{H}^*_{dR}$ for rigid spaces as follows: Given a rigid space $W$, choose a closed embedding $W \to Z$ into a smooth rigid space $Z$ and let $\hat{H}^*_{dR}(W) = H^*(Z, \hat{\Omega}^\bullet_Z)$ where $\hat{\Omega}^\bullet_Z$ denotes the formal completion of the de Rham complex $\Omega^\bullet_Z$ on $Z$ along $W$. Based on Kiehl’s extension [15] of the theorem of formal functions to the rigid analytic context one shows just as in [14] that this definition is independent on the choice of embedding $W \to Z$. Moreover for the resulting de Rham cohomology theory $\hat{H}^*_{dR}$ we have just as in [14] Propositions 4.1 and 4.4 long exact sequences for blowing up and for decomposition into Zariski closed subspaces, i.e. the analogs of Propositions 2.1 and 2.2 above. Now from the definitions we get natural maps

$$H^*_{dR}(X) \to \hat{H}^*_{dR}(X^{an}),$$

$$H^*_{dR}(X^{an}) \to \hat{H}^*_{dR}(X^{an}).$$

We claim that these maps are isomorphisms. The proof, which is the same for both maps in question, is by induction on the dimension of $X$. First we reduce to the case where $X$ is
irreducible, using Proposition 2.1 resp. [14] Proposition 4.1 and its analog for \( \hat{H}_{dR}^* \). Then we perform a resolution of singularities [4]: using Proposition 2.2 resp. [14] Proposition 4.4 and its analog for \( \hat{H}_{dR}^* \) this reduces the claim to the case where \( X \) is smooth. But then it follows for \( H_{dR}^*(X) \to \hat{H}_{dR}^*(X^{an}) \) from [17] and for \( H_{dR}^*(X^{an}) \to \hat{H}_{dR}^*(X^{an}) \) from 1.8 (b) (observing that \( X^{an} \) is partially proper).

**Proposition 2.4.** Let \( T \) be a rigid space, let \( k \subset k_1 \) be a finite field extension, let \( T_1 = T \times_{Sp(k)} Sp(k_1) \). There is a canonical isomorphism

\[ R\Gamma_{dR}(T/k) \otimes_k k_1 \cong R\Gamma_{dR}(T_1/k_1). \]

**Proof:** Choose \( S \) and \( \phi : T \to S' \) as in 1.5. Then \( S_1 = S \times_{Sp(k)} Sp(k_1) \) is a smooth \( k_1 \)-dagger space, and \( \phi_1 = (\phi \times_{Sp(k)} Sp(k_1)) : T_1 \to S' \times_{Sp(k)} Sp(k_1) \) is a closed immersion into its associated \( k_1 \)-rigid space. We may regard \( S_1 \) also as a \( k \)-rigid space, and we have a map \( q : S_1 \to S \) of \( k \)-rigid spaces. It induces the wanted map

\[ R\Gamma(S, (\Omega^\bullet_{S/k})^\phi) \otimes_k k_1 \to R\Gamma(S_1, (\Omega^\bullet_{S_1/k_1})^{\phi_1}). \]

Since the isomorphy claim is local, we may assume \( S \) and \( S_1 \) quasi-compact. Then \( R\Gamma \) commutes with the direct limits. Since \( \{U \times_{Sp(k)} Sp(k_1) ; U \in \Psi(\phi, S)\} \) is a fundamental system in \( \Psi(\phi_1, S_1) \) (use [11] 2.4), we therefore only need to check that

\[ R\Gamma_{dR}(U/k) \otimes_k k_1 \to R\Gamma_{dR}(U \times_{Sp(k)} Sp(k_1)/k_1) \]

is an isomorphism for all \( U \in \Psi(\phi, S) \). One can assume that \( U \) is affinoid, and then it follows immediately from the cohomological acyclicity of \( U \) resp. of \( U \times_{Sp(k)} Sp(k_1) \) for coherent \( O_{U-} \)- resp. \( O_{U \times_{Sp(k)} Sp(k_1)} \)-modules ( [12] 3.1).

**Proposition 2.5.** Let \( f : Z \to X \) be a closed immersion of smooth pure dimensional rigid spaces, associated with a closed immersion \( Z^{\dagger} \to X^{\dagger} \) of dagger spaces. With \( c = \text{codim}(f) \), there is a long exact Gysin sequence

\[ \ldots H_{dR}^{i-2c}(Z) \to H_{dR}^i(X) \to H_{dR}^i(X - Z) \to H_{dR}^{i-2c+1}(Z) \to \ldots. \]

**Proof:** This results from the corresponding Gysin sequence for \( Z^{\dagger} \to X^{\dagger} \), established in [13] 1.16.

### 3 Finiteness and Formal Models

Now we assume \( k = \text{Frac}(R) \) for a complete discrete valuation ring \( R \) of mixed characteristic, and we denote by \( \bar{k} \) its residue field. For an admissible formal \( \text{Spf}(R) \)-scheme \( \mathcal{X} \) with
generic fibre (as rigid space) \( X_k \) and specialization map \( s_p : X_k \to X \), and a subscheme \( Z \subset X_k \) of its special fibre, we denote by \( ]Z[\chi = sp^{-1}(Z) \) the tube of \( Z \) in \( X \), an admissible open subset of \( X_k \).

For \( \epsilon \in \Gamma^* \) we denote by \( D(\epsilon) \) (resp. \( D^0(\epsilon) \)) the closed (resp. open) disk of radius \( \epsilon \), as rigid spaces; in particular we let \( D^0 = D^0(1) \).

A rigid space \( L \) is called quasi-dagger, if \( L \) admits an admissible covering \( L = \bigcup_{i \in I} L_i \) such that each \( L_i \) is the rigid space associated with a dagger space. A closed immersion \( N \to L \) of rigid spaces is called quasi-dagger if \( L \) admits an admissible covering \( L = \bigcup_{i \in I} L_i \) such that each \( N \times_L L_i \to L_i \) is associated with a closed immersion of dagger spaces.

Smooth rigid spaces are quasi-dagger; open subspaces of quasi-dagger spaces are quasi-dagger; analytifications of algebraic \( k \)-schemes are quasi-dagger.

**Theorem 3.1.** Let \( L, M \) and \( N \) be quasi-compact rigid spaces. Suppose we are given a quasi-dagger closed immersion \( N \to L \) and an open immersion \( M \to L \). If we let \( T = L - (M \cup N) \), then \( H^q_{dR}(T) \) is finite dimensional for all \( q \in \mathbb{Z} \).

**Proof:** Induction on \( \dim(L) \). First we use 2.1 and the induction hypothesis to reduce to the case where \( L \) is irreducible, and clearly we can also assume \( L \) is reduced. Since \( N \to L \) is quasi-dagger, we may assume, after passing to a finite covering, that \( N \to L \) is associated with a closed immersion \( N^0 \to L^0 \) into a reduced and irreducible affinoid dagger space \( L^0 \). As explained in [13] 0.1, the results of [4] 1.10, [21] imply resolution of singularities for affinoid dagger spaces. Performing a resolution of singularities in our situation, we may in view of 2.2 and the induction hypothesis assume that \( L^0 \) is smooth. But in this case we can apply [13] 3.5, 3.6 to conclude.

**Corollary 3.2.** Let \( X \) be a smooth rigid Stein space, or a smooth affinoid dagger space. All differentials \( d^i_X : \Omega^i_X(X) \to \Omega^{i+1}_X(X) \) have a closed image.

**Proof:** See [16] for the definition of a Stein space. A Stein space \( X \) admits (in particular) an admissible covering \( X = \bigcup_{j \in \mathbb{N}} U_j \) by affinoid rigid spaces \( U_j \) with \( U_j \subset U_{j+1} \). Each \( \Omega^i_{U_j}(U_j) \) is a Banach space, and we endow \( \Omega^i_X(X) = \lim \Omega^i_{U_j}(U_j) \) with the inverse limit topology, hence get a Fréchet space. The differentials \( d^i_X \) are continuous. Note that since \( X \) is partially proper, and acyclic for coherent \( O_X \)-modules by [16], we have

\[
H^1_{dR}(X) = \text{Ker}(d^1_X)/\text{Im}(d^0_X) \]

by 1.8. Now first consider the special case of smooth Stein spaces of the type

\[
X = \cup_{\rho < \rho'} \text{Sp}(T_n(\rho')/T_n(\rho'))
\]


for some fixed \( \rho \in \Gamma^* \) and \( I < T_n(\rho) \) such that \( \text{Sp}(T_n(\rho)/I) \) is smooth. In this case, \( H_{dR}^j(X) \) is finite dimensional by [31], i.e. the image of the continuous map of Fréchet spaces \( \iota_{\delta_1} : \Omega_{X,\delta_1}^{-1}(X) \rightarrow \text{Ker}(d_{X,\delta_1}^{-1}) \) is of finite codimension, hence is closed. A general Stein space \( X \) admits an admissible covering \( X = \bigcup_{j \in \mathbb{N}} V_j \) by open Stein subspaces \( V_j \) of the type just considered, and such that \( V_j \subset V_{j+1} \). We claim

\[
H_{dR}^j(X) = \lim_{\overset{\rightarrow}{j}} H_{dR}^j(V_j) = \lim_{\overset{\rightarrow}{j}} \frac{\text{Ker}(d_{V_j}^i)}{\text{Im}(d_{V_j}^{i-1})} = \frac{\lim_{\overset{\rightarrow}{j}} \text{Ker}(d_{V_j}^i)}{\lim_{\overset{\rightarrow}{j}} \text{Im}(d_{V_j}^{i-1})} = \text{Ker}(d_X^i) \quad \text{for } i = 1, 2.
\]

Indeed, (1) follows from [20] ch.2, Cor.5 since all \( H_{dR}^j(V_j) \) are finite dimensional. Moreover, one has

\[ R^1 \lim_{\overset{\rightarrow}{j}} \Omega_{V_j}^{-1}(V_j) = 0 \]

by [16], and \( R^2 \lim_{\overset{\rightarrow}{j}} \text{Ker}(d_{V_j}^{i-1}) = 0 \) because of \( \text{coh.dim}(\mathbb{N}) = 1 \). Together this implies \( R^1 \lim_{\overset{\rightarrow}{j}} \text{Im}(d_{V_j}^{i-1}) = 0 \) and thus equality in (2). In other words, we have \( \text{Im}(d_X^{i-1}) = \lim_{\overset{\rightarrow}{j}} \text{Im}(d_{V_j}^{i-1}) \). Since \( \Omega_X^i(X) = \lim_{\overset{\rightarrow}{j}} \Omega_{V_j}^i(V_j) \) is a topological isomorphism, we see that \( \text{Im}(d_X^{i-1}) \) is closed in \( \Omega_X^i(X) \) since each \( \text{Im}(d_{V_j}^{i-1}) \) is closed in \( \Omega_{V_j}^i(V_j) \).

Now let \( X \) be a smooth affinoid dagger space. Then we can find a \( \delta \in \Gamma^* \) and an ideal \( I \subset T_n(\delta) \) such that, if for \( \delta' \) with \( 1 < \delta' < \delta \) we set \( X_{\delta'} = \text{Sp}(T_n(\delta')/I.T_n(\delta')) \), each \( X_{\delta'} \) is a smooth affinoid rigid space and such that

\[ \Omega_X^i(X) = \lim_{\overset{\rightarrow}{1 < \delta'}} \Omega_{X_{\delta'}}^i(X_{\delta'}). \]

Via this isomorphism, we define the topology on \( \Omega_X^i(X) \) as the direct limit topology in the category of locally \( k \)-convex topological vector spaces (compare [12] 4.2; it must not be confused with the norm topology, which is coarser). Of course, if we define the Stein spaces \( X_{\delta'}^0 = \bigcup_{\delta' < \delta} X_{\delta'} \), then it is also the direct limit topology for the isomorphism

\[ \Omega_X^i(X) = \lim_{\overset{\rightarrow}{1 < \delta'}} \Omega_{X_{\delta'}}^i(X_{\delta'}^0), \]

and in view of \( \text{Im}(d_X^{i-1}) = \lim_{\overset{\rightarrow}{j}} \text{Im}(d_{X_{\delta'}}^{i-1}) \) and \( \text{Ker}(d_X^i) = \lim_{\overset{\rightarrow}{j}} \text{Ker}(d_{X_{\delta'}}^i) \), our claim follows from that for Stein spaces.

It follows that the de Rham cohomology groups of smooth \( k \)-rigid Stein spaces are topologically separated for their canonical topology, hence are Fréchet spaces. In the particular case of Drinfeld’s \( p \)-adic symmetric spaces this has been proved directly by Schneider and Teitelbaum.
Proposition 3.3. Let $T_1, T_2$ be smooth rigid spaces. There exists a canonical isomorphism

$$R\Gamma_{dR}(T_1) \otimes_k R\Gamma_{dR}(T_2) \xrightarrow{\cong} R\Gamma_{dR}(T_1 \times T_2).$$

Proof: One easily constructs such a map and sees that the isomorphism claim is local. Therefore we can assume $T_1$ and $T_2$ are associated with smooth dagger spaces. We then conclude by [12] 4.12 (which is deduced from [12] 4.7, i.e. our 3.2).

3.4 Due to 3.2, one can also derive some duality formulas from [12] section 4.

Lemma 3.5. Let $f : X \to Y$ be an immersion of smooth formal $\text{Spf}(R)$-schemes, let $Z \to X$ be a closed immersion into its special fibre. The canonical map

$$R\Gamma_{dR}(]Z[\gamma) \to R\Gamma_{dR}(]Z[\chi)$$

is an isomorphism.

Proof: The claim is local, so by [1] 4.3 we may suppose

$$]Z[\gamma \cong ]Z[\chi \times (D^0)^r$$

such that the zero section on the right hand side corresponds to the embedding $\phi : ]Z[\chi \to ]Z[\gamma$ induced by $f$. We may also suppose that $Y_k$ has a smooth underlying dagger space $Y_0^\psi$; let $]Z[\gamma^\psi$ be its open subspace defined by $]Z[\gamma \subset Y_k$. Then our task is to show that

$$R\Gamma_{dR}(]Z[\gamma^\psi) \to R\Gamma(]Z[\gamma^\psi, (\Omega^\bullet_{]Z[\gamma^\psi/k})^\phi)$$

is an isomorphism. For an affinoid open subspace $V \subset ]Z[\chi$ and $\epsilon \in \Gamma^* \cap ]0,1[$ define $\tilde{V} \subset ]Z[\gamma^\psi$ resp. $V_\epsilon \subset ]Z[\gamma^\psi$ as the open dagger subspace corresponding to the rigid subspace $V \times (D^0)^r \subset ]Z[\gamma$ resp. to $V \times (D(\epsilon))^r \subset ]Z[\gamma$. It is enough to show that

$$R\Gamma_{dR}(\tilde{V}) \to R\Gamma(\tilde{V}, (\Omega^\bullet_{]Z[\gamma^\psi/k})^\phi)$$

is an isomorphism for all such $V$. Now $\tilde{V}$ is quasi-compact, so

$$R\Gamma(\tilde{V}, (\Omega^\bullet_{]Z[\gamma^\psi/k})^\phi) = \lim_{U \in \Psi(\phi) ]Z[\gamma^\psi} R\Gamma(\tilde{V}, j_{U,!*} \Omega^\bullet_{U/k}) = \lim_{U \in \Psi(\phi[V, \tilde{V})} R\Gamma_{dR}(U);$$

since the $V_\epsilon$ are cofinal in $\Psi(\phi[V, \tilde{V})$, we only need to show that each

$$R\Gamma_{dR}(\tilde{V}) \to R\Gamma_{dR}(V_\epsilon)$$
is an isomorphism. Switching back to associated rigid spaces, it is enough to show that each
\[ R\Gamma_{dR}(\tilde{V}') \to R\Gamma_{dR}(V'_e) \]
is an isomorphism; but this is a consequence of \[3.3\] and the triviality of \[R\Gamma_{dR}((D^0)')\] and \[R\Gamma_{dR}((D(\epsilon))').\]

**Proposition 3.6.** Let \( \mathcal{X} \) be a smooth formal \( \text{Spf}(R) \)-scheme and let \( Z \to \mathcal{X}_k \) be a closed immersion. There is for all \( q \in \mathbb{Z} \) a canonical isomorphism
\[ \alpha : H^q_{\text{rig}}(Z) \cong H^q_{dR}(\mathbb{Z}[\mathcal{X}]). \]
If \( \mathcal{X}_i \) for \( i = 1, 2 \) are two smooth formal \( \text{Spf}(R) \)-schemes with closed immersions \( Z_i \to (\mathcal{X}_i)_k \), and if there is a morphism \( \mathcal{X}_1 \to \mathcal{X}_2 \) covering a morphism \( \rho : Z_1 \to Z_2 \), in particular inducing a morphism \( \sigma : [Z_1|_{\mathcal{X}_1}] \to [Z_2|_{\mathcal{X}_2}] \), then \( \alpha_1 \circ H^q_{\text{rig}}(\rho) = H^q_{dR}(\sigma) \circ \alpha_2 \).

**Proof:** First assume there is an embedding \( \mathcal{X} \to \mathcal{Y} \) into a smooth proper \( \text{Spf}(R) \)-scheme \( \mathcal{Y} \). By \[12\] 2.27 the proper rigid space \( \mathcal{Y}_k \) has a unique underlying dagger space \( \mathcal{Y}_k^\dagger \); denote by \( [Z|_{\mathcal{Y}^\dagger}] \) its admissible open subset corresponding to \( [Z|_{\mathcal{Y}}] \subset \mathcal{Y}_k^\dagger \). Then \( H^q_{dR}(\mathbb{Z}[\mathcal{X}]) = H^q_{dR}(\mathbb{Z}[\mathcal{Y}]) \) by \[3.5\] and \( H^q_{dR}(\mathbb{Z}[\mathcal{Y}]) = H^q_{dR}(\mathbb{Z}[\mathcal{Y}^\dagger]) \) by definition of \( H^q_{dR}(\mathbb{Z}[\mathcal{Y}]) \). Thus our wanted isomorphism is obtained from the isomorphism
\[ H^q_{\text{rig}}(Z) \cong H^q_{dR}(\mathbb{Z}[\mathcal{Y}^\dagger]) \]
we constructed in \[12\] 5.1. In general, if there is no global embedding \( \mathcal{X} \to \mathcal{Y} \) as above, one covers \( \mathcal{X} \) by open affine formal subschemes and works with simplicial formal schemes and analytic spaces as usual. The functoriality assertion follows similarly as in \[1.6\].

From \[3.6\] it follows in particular that the rigid cohomology groups of algebraic \( \bar{k} \)-schemes depend functorially on their rigid analytic tubes in arbitrary smooth formal \( \text{Spf}(R) \)-schemes, i.e. morphisms between these tubes are enough to define morphism in rigid cohomology; no extension requirements are needed.

**Proposition 3.7.** Let \( X = \text{Sp}(A) \) be a smooth affinoid rigid space, let \( f_1, \ldots, f_m \) be elements of \( A \), and let \( \epsilon \in \Gamma^* \). Set \( Z = \mathcal{V}(f_1, \ldots, f_m) \) and \( U = \text{Sp}(A < (\epsilon^{-1} f_i)^{-1}) \), and then \( U = \bigcup_{i=1}^m U_i \). Suppose there is an isomorphism
\[ X - U \cong (D^0(\epsilon))^m \times Z \]
where the functions $f_1, \ldots, f_m$ on the left hand side correspond to the standard coordinates on the right hand side. Then the natural homomorphism

$$R\Gamma_{dR}(X - Z) \to R\Gamma_{dR}(U)$$

is an isomorphism.

**Proof:** For $\delta \in \Gamma^*, \delta \leq \epsilon$, and $1 \leq i \leq m$ let

$$V_\delta = \text{Sp}(A < \delta^{-1}.f_1, \ldots, \delta^{-1}.f_m >)$$

$$U_{\delta,i} = \text{Sp}(A < (\delta^{-1}.f_i)^{-1} >)$$

and let $U_\delta = \bigcup_{i=1}^m U_{\delta,i}$. Since for each $\delta$, the covering $X = U_\delta \cup V_\delta$ is admissible, one easily sees that it suffices to prove that the natural maps

(1) \[ \lim_{\delta < \epsilon} R\Gamma_{dR}(U_\delta) \to R\Gamma_{dR}(U) \]

(2) \[ R\Gamma_{dR}(V_\delta - Z) \to R\Gamma_{dR}(V_\delta \cap U_\delta) \quad (\delta < \epsilon) \]

are isomorphisms. Since $X$ is smooth, it is quasi-algebraic ([12] 2.18), hence we may assume it is associated with a smooth affinoid dagger space $X^\dagger$. For $\delta \leq \epsilon$ let $U^{\dagger}_{\delta,i}$, resp. $U^{\dagger}_{\delta}$ be the open subspace of $X^\dagger$ corresponding to $U_{\delta,i}$, resp. $U_\delta$. Then

$$R\Gamma_{dR}(U) = R\Gamma_{dR}(U^{\dagger}_{\delta}) \quad \text{and} \quad R\Gamma_{dR}(U_\delta) = R\Gamma_{dR}(U^{\dagger}_{\delta}),$$

and we have admissible coverings $U^{\dagger}_{\delta} = \bigcup_{i=1}^m U^{\dagger}_{\delta,i}$ for all $\delta \leq \epsilon$. By the exactness of direct limits, that (1) is an isomorphism will follow once we know that for all $I \subset \{1, \ldots, m\}$, the map

$$\lim_{\delta < \epsilon} R\Gamma_{dR}(\cap_{i \in I} U^{\dagger}_{\delta,i}) \to R\Gamma_{dR}(\cap_{i \in I} U^{\dagger}_{\epsilon,i})$$

is an isomorphism. But since all $\cap_{i \in I} U^{\dagger}_{\delta,i}$ for $\delta \leq \epsilon$ are affinoid, this follows from the very definition of a dagger algebra. Now fix $\delta < \epsilon$. Observe that $Z$ is a smooth rigid space since its fibre product with $(D^0(\epsilon))^m$ is smooth, therefore we find as above an affinoid dagger space $Z^\dagger$ such that $Z$ is the associated rigid space of $Z^\dagger$. Thus we get an isomorphism of the rigid space $(D^0(\epsilon))^m \times Z$ and hence of the rigid space $X - U$ with the associated rigid space of the dagger space $(D^0(\epsilon)^\dagger)^m \times Z^\dagger$ (by $D^0(\epsilon)^\dagger$ we mean the open disk of radius $\epsilon$ with its structure of dagger space). Under this isomorphism, the open immersion $(V_\delta \cap U_\delta) \to (V_\delta - Z)$ is the one associated with the open immersion of dagger spaces $W^{\dagger}_{\delta} \to Y^{\dagger}_{\delta}$, where we defined

$$Y^{\dagger}_{\delta} = (\text{Sp}(k < \delta^{-1}.T_1, \ldots, \delta^{-1}.T_m >^\dagger) - \{0\}) \times Z^\dagger,$$
$$W_{\delta,i}^\dagger = (\text{Sp}(k < \delta^{-1}T_1, \ldots, \delta^{-1} . T_m, (\delta^{-1} . T_i)^{-1} >)^\dagger) \times Z^\dagger$$

and $$W_{\delta}^\dagger = \cup_{i=1}^m W_{\delta,i}^\dagger$$, regarded as open subspaces of $$(D^0(e)^\dagger)^m \times Z^\dagger$$. Therefore, to prove the isomorphy of (2), we only need to prove the isomorphy of

$$R\Gamma_{dR}(Y_{\delta}^\dagger) \rightarrow R\Gamma_{dR}(W_{\delta}^\dagger).$$

There is a natural morphism from the Cech spectral sequence computing $$R\Gamma_{dR}(Y_{\delta}^\dagger)$$ by means of the admissible covering $$Y_{\delta}^\dagger = \cup_{i=1}^m (Y_{\delta}^\dagger - V(T_i))$$, to the Cech spectral sequence computing $$R\Gamma_{dR}(W_{\delta}^\dagger)$$ by means of the admissible covering $$W_{\delta}^\dagger = \cup_{i=1}^m W_{\delta,i}^\dagger$$. It shows that we only need to prove that

$$R\Gamma_{dR}(\cap_{i\in I} Y_{\delta}^\dagger - V(T_i)) \rightarrow R\Gamma_{dR}(\cap_{i\in I} W_{\delta,i}^\dagger)$$

is an isomorphism for all $$I \subset \{1, \ldots, m\}$$. By the Künneth formula (3.3), this is reduced to proving that

$$R\Gamma_{dR}(\text{Sp}(k < \delta^{-1}T >)^\dagger - \{0\}) \rightarrow R\Gamma_{dR}(\text{Sp}(k < \delta^{-1}T, (\delta^{-1}T)^{-1} >)^\dagger))$$

($$T$$ a single variable) is an isomorphism, which results from a simple computation of both sides.

**Corollary 3.8.** (a) Let $$\pi$$ be a uniformizer of $$R$$ and let $$\mathcal{Z} \rightarrow \mathcal{X}$$ be a closed immersion of formal $$R$$-schemes. Assume that locally on $$\mathcal{X}$$ there exist étale morphisms of formal $$R$$-schemes

$$q : \mathcal{X} \rightarrow \text{Spf}(R < X_1, \ldots, X_n > / (X_1 \ldots X_r - \pi))$$

such that $$\mathcal{Z} = \cap_{j=r+1}^{m} V(q^*X_j)$$ for some $$1 \leq r \leq m \leq n$$. Then the natural map

$$R\Gamma_{dR}(\mathcal{X}_k - \mathcal{Z}_k) \rightarrow R\Gamma_{dR}(\mid \mathcal{X}_k - \mathcal{Z}_k[\mathcal{X}])$$

is an isomorphism.

(b) In (a), the canonical map $$R\Gamma_{dR}(\mid \mathcal{Z}_k[\mathcal{X}] \rightarrow R\Gamma_{dR}(\mathcal{Z}_k)$$ is an isomorphism.

(c) In the description of the setting in (a), replace $$\mathcal{Z} = \cap_{j=r+1}^{m} V(q^*X_j)$$ by $$\mathcal{Z} = \cup_{j=r+1}^{m} V(q^*X_j).$$ Then the same statement as in (a) holds.

**Proof:** (a) Since the assertion is local on $$\mathcal{X}$$, we may assume there exists globally a map $$q$$ as described. By [13] 2.6, there exists an isomorphism

$$(\ast) \quad \mid \mathcal{Z}_k[\mathcal{X}] \cong \mathcal{Z}_k \times (D^0)^{m-r}$$

where the $$q^*X_j$$ for $$r+1 \leq j \leq m$$ correspond to the standard coordinates. Thus for (a) we can cite [3.7]. Also (b) follows from $$(\ast)$$. Assertion (c) follows formally from (a) by Cech complex arguments.
Corollary 3.9. In 3.8 (a), assume that the closed embedding $Z_k \to X_k$ of rigid spaces is associated with a closed embedding of dagger spaces. Then there is a long exact Gysin sequence

$$\ldots \to H^{i-2(m-r)}_{dR}(\bar{Z}_k) \to H^{i}_{dR}(X_k) \to H^{i}_{dR}(X_k - Z_k) \to H^{i-2(m-r)+1}_{dR}(\bar{Z}_k) \to \ldots$$

**Proof:** Combine 3.8 with 2.5.

**Remark:** (1) In 3.9 the assumption that $Z_k \to X_k$ is associated with a closed embedding of dagger spaces, is fulfilled in particular if $Z \to X$ is obtained by $\pi$-adic completion from a closed immersion of $R$-schemes of finite type, or if $X$ is proper.

(2) Suppose $r = 1$ in 3.9. Then $X$ and $Z$ are smooth, and we recover the long exact Gysin sequence

$$\ldots \to H^{i-2(m-r)}(\bar{Z}_k) \to H^{i}_{rig}(X_k) \to H^{i}_{rig}(X_k - Z_k) \to H^{i-2(m-r)+1}_{rig}(\bar{Z}_k) \to \ldots$$

for rigid cohomology, constructed in [3] and [18]. For general $r$, our sequence might be thought of as its version for logarithmic rigid cohomology, where the base $\text{Spf}(R)$ is endowed with its canonical log structure.

(3) From 2.5 and 3.7 one can derive more general Gysin sequences, without knowledge of reductions. Such occurred for example in [9] p. 186, p. 190.

4 Vanishing cycles

Let $\mathcal{X}$ be an admissible proper formal $\text{Spf}(R)$-scheme. Attached to it are two de Rham type cohomology theories, finite dimensional over $k$: The rigid cohomology $R\Gamma_{rig}(\mathcal{X}_k/k)$ of its special fibre $\mathcal{X}_k$, and the de Rham cohomology $R\Gamma_{dR}(\mathcal{X}_k)$ of its generic fibre (as rigid space) $\mathcal{X}_k$. We want to compare these two.

For simplicity we assume that there exists a smooth admissible formal $\text{Spf}(R)$-scheme $\mathcal{Y}$ and an embedding $\mathcal{X} \to \mathcal{Y}$. It induces an embedding $\mathcal{X}_k \to \mathcal{Y}_k$ into the generic fibre (as rigid space) $\mathcal{Y}_k$ of $\mathcal{Y}$. Since $\mathcal{X}_k$ is proper we do not need dagger spaces to define $R\Gamma_{dR}(\mathcal{X}_k)$:

We have

$$R\Gamma_{dR}(\mathcal{X}_k) = R\Gamma(\mathcal{X}_k / \mathcal{Y}, \lim_U j_{U*} \Omega_U^\bullet)$$

where $j_U : U \to \mathcal{X}_k / \mathcal{Y}$ runs through the inclusions of admissible open subsets of $\mathcal{X}_k / \mathcal{Y}$ containing $\mathcal{X}_k$. On the other hand we have

$$R\Gamma_{rig}(\mathcal{X}_k/k) = R\Gamma(\mathcal{X}_k / \mathcal{Y}, \Omega_{\mathcal{X}_k / \mathcal{Z}}^\bullet).$$
Denote by $s : \mathcal{X}_k[Y] \to \mathcal{X}_k$ the specialization map. Note that

$$s_* \lim_{\mathcal{U}} j_{U,*} \Omega^*_U = Rs_* \lim_{\mathcal{U}} j_{U,*} \Omega^*_U$$

$$s_* \Omega^*|_{\mathcal{X}_k[Y]} = Rs_* \Omega^*|_{\mathcal{X}_k[Y]}.$$

This follows from the acyclicity of coherent modules on quasi-Stein spaces \[16\]. Thus

$$R\Gamma_{dR}(\mathcal{X}_k) = R\Gamma(\mathcal{X}_k, s_* \lim_{\mathcal{U}} j_{U,*} \Omega^*_U)$$

$$R\Gamma_{rig}(\mathcal{X}_k/k) = R\Gamma(\mathcal{X}_k, s_* \Omega^*|_{\mathcal{X}_k[Y]}).$$

The canonical restriction map of sheaf complexes on $\mathcal{X}_k$

$$\tau : s_* \Omega^*|_{\mathcal{X}_k[Y]} \to s_* \lim_{\mathcal{U}} j_{U,*} \Omega^*_U$$

is injective. In other words, we can filter the sheaf complex $s_* \lim_{\mathcal{U}} j_{U,*} \Omega^*_U$ on $\mathcal{X}_k$ which computes $R\Gamma_{dR}(\mathcal{X}_k)$ by a subcomplex $\text{Im}(\tau)$ which computes $R\Gamma_{rig}(\mathcal{X}_k/k)$.

The situation is similar to that in Bruno Chiarellotto’s paper \[7\]: There, for a proper semistable $\bar{k}$-log scheme $Y$ he shows that the Hyodo-Steenbrink complex $W \mathcal{A}^*$ which computes the Hyodo-Kato cohomology of $Y$ — which is isogenous to the de Rham cohomology of a semistable lift of $Y$, if such a lift exists — can be filtered by a subcomplex which computes the rigid cohomology of $Y$, provided the $p$-adic monodromy-weight conjecture holds for $Y$.

Returning to our situation, define the sheaf complex $V^*_{\mathcal{X}, Y}$ on $\mathcal{X}_k$ by the exact sequence

$$0 \to s_* \Omega^*|_{\mathcal{X}_k[Y]} \to s_* \lim_{\mathcal{U}} j_{U,*} \Omega^*_U \to V^*_{\mathcal{X}, Y} \to 0.$$

One might call it the complex of vanishing cycles for $\mathcal{X}$ with respect to $\mathcal{Y}$: It measures the difference between $R\Gamma_{dR}(\mathcal{X}_k)$ and $R\Gamma_{rig}(\mathcal{X}_k/k)$; we have $R\Gamma_{dR}(\mathcal{X}_k) = R\Gamma_{rig}(\mathcal{X}_k/k)$ if $H^q(\mathcal{X}_k, V^*_{\mathcal{X}, Y}) = 0$ for all $q \geq 0$ (somewhat different to the $l$-adic vanishing cycles). For example, if $\mathcal{X}_k$ is smooth, or more specifically if $\mathcal{X}$ is the formal completion of a proper $R$-scheme of finite type with smooth generic fibre, this might be a starting point for a rigid analytic investigation of the monodromy on $H^*_{dR}(\mathcal{X}_k)$ due to bad reduction (note that even if $\mathcal{X}_k$ is smooth there is no obvious subcomplex of the sheaf complex $\Omega^*|_{\mathcal{X}_k}$ on $\mathcal{X}_k$ which computes $R\Gamma_{rig}(\mathcal{X}_k/k)$; so even if $\mathcal{X}_k$ is smooth the construction of this paper has its assets).

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