Justification of Diffusion limit for the Boltzmann Equation with a non-trivial Profile

Feimin Huang\textsuperscript{a,b}, Yi Wang\textsuperscript{a,b}, Yong Wang\textsuperscript{a}, and Tong Yang\textsuperscript{c}

\textsuperscript{a}Institute of Applied Mathematics, AMSS, CAS, Beijing 100190, China
\textsuperscript{b}Beijing Center of Mathematics and Information Sciences, Beijing 100048, P.R.China
and
\textsuperscript{c}Department of Mathematics, City University of Hong Kong, Hong Kong

Abstract

Under the diffusion scaling and a scaling assumption on the microscopic component, a non-classical fluid dynamic system was derived in \cite{3} that is related to the system of ghost effect derived in \cite{41} in different settings. This paper aims to justify this limit system for a non-trivial background profile with slab symmetry. The result reveals not only the diffusion phenomena in the temperature and density, but also the flow of higher order in Knudsen number due to the gradient of the temperature. Precisely, we show that the solution to the Boltzmann equation converges to a diffusion wave with decay rates in both Knudsen number and time.

Keywords: Boltzmann equation, Knudsen number, diffusive scaling, diffusion wave

AMS: 35Q35, 35B65, 76N10

Contents

1 Introduction 2

2 Construction of Profile and the Main Result 4
  2.1 Construction of profile . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7
  2.2 Main result . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 13

3 Stability Analysis 14
  3.1 Reformulated system . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 14
  3.2 Lower order estimate . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 18
  3.3 Derivative estimate . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 27

4 The Proof of Main Result 36

*Corresponding author.
Email addresses: fhuang@amt.ac.cn(Feimin Huang), wangyi@amss.ac.cn(Yi Wang), yongwang@amss.ac.cn(Yong Wang), matyang@cityu.edu.hk(Tong Yang)
1 Introduction

Consider the Boltzmann equation with slab symmetry under the diffusive scaling

$$\varepsilon \partial_t f^\varepsilon + \xi_1 f^\varepsilon_t = \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon), \quad (t, x, \xi) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^3. \quad (1.1)$$

Here $f^\varepsilon(t, x, \xi) \geq 0$ is the distribution density of particles at $(t, x)$ with velocity $\xi$, $Q(f, f)$ is the collision operator which is a non-local bilinear operator in the velocity variable with a kernel determined by the physics of particle interaction. For monatomic gas, the rotational invariance of the particle leads to the collision operator $Q(f, f)$ as a bilinear collision operator in the form of, cf. [5]:

$$Q(f, g)(\xi) \equiv \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \left( f(\xi') g(\xi'_*) + f(\xi'_*) g(\xi') - f(\xi) g(\xi) - f(\xi_*) g(\xi_*') \right) B(|\xi - \xi_*|, \theta) d\xi_* d\Omega,$$

with $\theta$ being the angle between the relative velocity and the unit vector $\Omega$. Here $\mathbb{S}^2_+ = \{ \Omega \in \mathbb{S}^2 : (\xi - \xi_*) \cdot \Omega \geq 0 \}$. The conservation of momentum and energy gives the following relation between velocities before and after collision:

$$\begin{cases}
\xi' = \xi - [(\xi - \xi_*) \cdot \Omega] \Omega, \\
\xi_*' = \xi_* + [(\xi - \xi_*) \cdot \Omega] \Omega.
\end{cases}$$

In this paper, we will consider the two basic models, i.e., the hard sphere model and the hard potential with angular cut-off, for which the collision kernel $B(|\xi - \xi_*|, \theta)$ takes the form of

$$B(|\xi - \xi_*|, \theta) = |\xi - \xi_*|^p b(\theta), \quad b(\theta) \in L^1([0, \pi]), \quad p \geq 5,$$

respectively. Here, $p$ is the index in potential of the inverse power law that is proportional to $r^{1-p}$ with $r$ being the distance between two particles.

Motivated by [41], the following macroscopic and microscopic decomposition with scalings was introduced in [31]:

$$f^\varepsilon = M(\rho^\varepsilon, e^\varepsilon, \psi_0, \psi_1, \psi_2, \psi_3) + \varepsilon G^\varepsilon. \quad (1.2)$$

Here $M(\rho^\varepsilon, e^\varepsilon, \psi_0, \psi_1, \psi_2, \psi_3)$ is the local Maxwellian and $G^\varepsilon$ is the microscopic component. Moreover, the local Maxwellian $M(\rho^\varepsilon, e^\varepsilon, \psi_0, \psi_1, \psi_2, \psi_3)$ is defined by the five conserved quantities, that is, the mass density $\rho^\varepsilon(t, x)$, momentum density $m^\varepsilon(t, x) = \varepsilon \rho^\varepsilon(t, x) u^\varepsilon(t, x)$ and energy density $e^\varepsilon(t, x) = \frac{1}{2} |e^\varepsilon(t, x)|^2$ given by

$$\begin{cases}
\rho^\varepsilon(t, x) \equiv \int_{\mathbb{R}^3} f^\varepsilon(t, x, \xi) d\xi, \\
m^\varepsilon_i(t, x) \equiv \int_{\mathbb{R}^3} \psi_i(\xi) f^\varepsilon(t, x, \xi) d\xi \quad \text{for } i = 1, 2, 3, \\
\left[ \rho^\varepsilon \left( e^\varepsilon + \frac{\varepsilon^2}{2} |u^\varepsilon|^2 \right) \right](t, x) \equiv \int_{\mathbb{R}^3} \psi_4(\xi) f^\varepsilon(t, x, \xi) d\xi,
\end{cases} \quad (1.3)$$

as

$$M(t, x, \xi) \equiv \frac{\rho^\varepsilon(t, x)}{\sqrt{(2\pi R^2)^3}} \exp \left( -\frac{|\xi - \varepsilon u^\varepsilon(t, x)|^2}{2 \varepsilon R^2(t, x)} \right). \quad (1.4)$$

Here $\psi_i(\xi)$ are the collision invariants:

$$\begin{cases}
\psi_0(\xi) \equiv 1, \\
\psi_i(\xi) \equiv \xi_i \quad \text{for } i = 1, 2, 3, \\
\psi_4(\xi) \equiv \frac{1}{2} |\xi|^2.
\end{cases}$$
satisfying
\[ \int_{\mathbb{R}^3} \psi_j(\xi)Q(h, g) d\xi = 0, \quad \text{for} \quad j = 0, 1, 2, 3, 4. \]

Here, \( \theta^\varepsilon \) is the temperature related to the internal energy \( e^\varepsilon \) by \( e^\varepsilon = \frac{2}{3} R \theta^\varepsilon \) with \( R \) being the gas constant, and \( u^\varepsilon \) is the bulk velocity. Note that even though \( u^\varepsilon \) is of higher order, it is the scaled velocity that appears in the equations for the macroscopic variables \( \rho^\varepsilon \) and \( \theta^\varepsilon \).

The Boltzmann equation is a fundamental equation in statistical physics for rarefied gas which describes the time evolution of particle distribution. There has been tremendous progress on the mathematical theories for the Boltzmann equation with \( \varepsilon \) being a fixed constant, such as the global existence of weak (renormalized) solution for large data in \[11\] and classical solutions as small perturbations of equilibrium states (Maxwellian) in \[20, 34, 42\] and the references therein, etc.

On the other hand, the study on the hydrodynamic limit of Boltzmann equation is important and challenging. For this, it is well known that the classical works of Hilbert, Chapman-Enskog reveal the relation of the Boltzmann equation to the classical systems of fluid dynamics through asymptotic expansions with respect to the Knudsen number. For the hydrodynamic limit of Boltzmann equation to the compressible Euler system, we refer \[2, 3\] for the formal derivation.

If the Euler system is assumed to have smooth solution, this hydrodynamic limit is proved rigorously in \[13, 0\] with and without initial layer respectively.

However, it is well known that the compressible Euler system develops singularity in finite time even for sufficiently smooth initial data. The Riemann problem is the basic problem to the compressible Euler system, and its solution turns out to be fundamental in the theory of hyperbolic conservation laws because it not only captures the local and global behavior of solutions but also reveals the effect of nonlinearity in the structure of the solutions. There are three basic wave patterns for the Euler system, that is, shock wave, rarefaction wave, and contact discontinuity. For the hydrodynamic limit of the Boltzmann equation in the setting of Riemann solutions, we refer \[23, 24, 25, 44, 45\].

Under the diffusive scaling, usually, the density function \( f^\varepsilon(t, x, \xi) \) is set as a perturbation of a global Maxwellian \( M_{[1,0,1]} \), i.e.
\[
 f^\varepsilon(t, x, \xi) = M_{[1,0,1]} + M_{[1,0,1]} \left( \varepsilon f_1(t, x, \xi) + \cdots + \varepsilon^n f_n(t, x, \xi) \right). 
\]

There has been extensive study on the hydrodynamic limit \( \varepsilon \to 0 \) of the Boltzmann equation to the incompressible Navier-Stokes-Fourier system, for example, to justify the DiPerna-Lions’ renormalized solution in \[11\] of the Boltzmann equation to the Leray-Hopf weak solutions of the incompressible Navier-Stokes-Fourier system. For this, Bardos-Golse-Levermore \[2\] first studied this problem under certain a priori assumption. Recently, a breakthrough was achieved by Golse-Raymond in \[17\] which established a proof of such limit for certain class of collision kernels. After that, some progress was made for more general collision kernels, cf. \[30\]. In fact, there are also a lot of important contributions on this problem over the years, see \[15, 18, 31, 32, 38, 39, 40\] and the references therein.

In the framework of classical solutions to the incompressible Navier-Stokes-Fourier system, it was proved in \[41\] that one can find a Boltzmann solution \( f^\varepsilon(t, x, \xi) \) such that \( f_2^\varepsilon \) is of order \( \varepsilon^2 \), but it is not clear about the amplitude \( f_2^\varepsilon \) at the initial time. Later, the Navier-Stokes-Fourier limit was proved for \( f^\varepsilon(0, x, \xi) \) with small data in \[41\]. Recently, Guo in \[21\] justified the diffusive expansion \[1.5\] when \( f_1(0, x, \xi) \) has small amplitude while \( f_i^\varepsilon(0, x, \xi) \) can have arbitrarily large amplitude for \( i \geq 2 \) in a torus. This work was later generalized to some other settings, cf. \[33, 28, 29\]. Moreover, based on the \( L^2-L^\infty \) estimate, Esposito-Guo-Kim-Marra \[13\] proved the hydrodynamic limit of the rescaled Boltzmann equation to the incompressible Navier-Stokes-Fourier system in a bounded domain if the initial data is small.
Notice that all the results under the diffusive scaling mentioned above are either about large perturbation of vacuum or small perturbation of a global Maxwellian. A natural question to ask is how about the perturbation of a non-trivial profile. The purpose of this paper is to study this problem in the setting of \( (1.2) \).

In fact, under the assumption \( (1.2) \), when \( \varepsilon \to 0 \), formally we have
\[
f^\varepsilon = M_{[\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon]} + \varepsilon G^\varepsilon \to M_{[\rho, 0, \theta]},
\]
which shows that in the macroscopic level, only the unknown limit functions \( \rho, \theta \) survive because the macroscopic velocity is zero. However, as shown in [3] and will be recalled in the next section, the equations of \( \rho \) and \( \theta \) are actually closely related to the scaled velocity \( u \). Indeed, this diffusive scaling induces diffusion phenomenon for both the temperature \( \theta \) and density \( \rho \), and the non-zero gradient of temperature induces a non-trivial flow in the higher order along the same direction. In this paper, we will construct such diffusion wave and study the hydrodynamic limit of the rescaled Boltzmann equation to such a diffusion wave global in time.

The rest of the paper will be organized as follows. The construction of the diffusion wave and the main theorem will be given in the next section. We will reformulate the problem and derive some a priori estimates in Section 3. Based on the a priori estimates, the main theorem will be proved in Section 4.

**Notations:** Throughout this paper, the positive generic constants that are independent of \( \varepsilon \) are denoted by \( c, C, C_i (i = 1, 2, 3, \cdots) \). And we will use \( \| \cdot \| \) to denote the standard \( L_2(\mathbb{R}; dz) \) norm, and \( \| \cdot \|_{H^i} (i = 1, 2, 3, \cdots) \) to denote the standard Sobolev \( H^i(\mathbb{R}; dz) \) norm with \( z = x \) or \( y \). Sometimes, we also use \( O(1) \) to denote a uniform bounded constant independent of \( \varepsilon \).

2 Construction of Profile and the Main Result

We will drop the superscript \( \varepsilon \) in the case of no confusion for simple notation. The inner product of \( h, g \) in \( L_2(\mathbb{R}^3) \) with respect to a given Maxwellian \( \tilde{M} \) is defined by:
\[
\langle h, g \rangle_{\tilde{M}} \equiv \int_{\mathbb{R}^3} \frac{1}{M} h(\xi)g(\xi)d\xi,
\]
when the integral is well defined. If \( \tilde{M} \) is the local Maxwellian \( M \), with respect to this inner product, the macroscopic space is spanned by the following five pairwise orthogonal functions
\[
\left\{
\begin{aligned}
\chi_0(\xi) &\equiv \frac{1}{\sqrt{\rho}} M, \\
\chi_i(\xi) &\equiv \frac{\xi_i - \varepsilon u_i}{\sqrt{R\theta \rho}} M \quad \text{for } i = 1, 2, 3, \\
\chi_4(\xi) &\equiv \frac{1}{\sqrt{6\rho}} \left( \frac{|\xi - \varepsilon u|^2}{R\theta} - 3 \right) M, \\
\langle \chi_i, \chi_j \rangle &\equiv \delta_{ij}, \quad i, j = 0, 1, 2, 3, 4.
\end{aligned}
\right.
\]
Using these functions, we define the macroscopic projection \( P_0 \) and microscopic projection \( P_1 \) as follows:
\[
\left\{
\begin{aligned}
P_0 h &\equiv \sum_{j=0}^4 \langle h, \chi_j \rangle \chi_j, \\
P_1 h &\equiv h - P_0 h.
\end{aligned}
\right.
\]
The projections \( P_0 \) and \( P_1 \) are orthogonal:
\[
P_0 P_0 = P_0, \quad P_1 P_1 = P_1, \quad P_0 P_1 = P_1 P_0 = 0.
\]
A function \( h(\xi) \) is called microscopic or non-fluid if
\[
\int h(\xi)\psi_j(\xi)d\xi = 0, \quad j = 0, 1, 2, 3, 4.
\]
Under this decomposition, the solution \( f(t,x,\xi) \) of the Boltzmann equations satisfies
\[
P_0f = M, \quad P_1f = \varepsilon G,
\]
and the Boltzmann equation becomes
\[
(\varepsilon M + \varepsilon^2 G)_t + \xi_1(M + \varepsilon G)_x = 2Q(M, G) + \varepsilon Q(G, G),
\]
which is equivalent to the following fluid-type system for the fluid components (see [34] and [36] for details):
\[
\begin{cases}
    \varepsilon \rho_t + (\varepsilon \rho u_1)_x = 0, \\
    \varepsilon (\varepsilon \rho u_1)_t + (\varepsilon^2 \rho u_1^2 + p)_x = -\varepsilon \int \xi_1^2 G_x d\xi, \\
    \varepsilon (\varepsilon \rho u_i)_t + (\varepsilon^2 \rho u_i u_i)_x = -\varepsilon \int \xi_1 \xi_i G_x d\xi, \quad i = 2, 3, \\
    \varepsilon [\rho(e + \frac{|u|^2}{2})]_t + [\varepsilon \rho u_1(e + \frac{|u|^2}{2}) + \varepsilon pu_1]_x = -\varepsilon \int \frac{1}{2} \xi_1 |\xi|^2 G_x d\xi,
\end{cases}
\]
or more precisely,
\[
\begin{cases}
    \varepsilon \rho_t + (\varepsilon \rho u_1)_x = 0, \\
    \varepsilon (\varepsilon \rho u_1)_t + (\varepsilon^2 \rho u_1^2 + p)_x = \frac{4}{3} \varepsilon (\mu(\theta)\varepsilon u_{1x})_x - \varepsilon \int \xi_1^2 \Theta_x d\xi, \\
    \varepsilon (\varepsilon \rho u_i)_t + (\varepsilon^2 \rho u_i u_i)_x = \varepsilon (\mu(\theta)\varepsilon u_{ix})_x - \varepsilon \int \xi_1 \xi_i \Theta_x d\xi, \quad i = 2, 3, \\
    \varepsilon [\rho(e + \frac{|u|^2}{2})]_t + [\varepsilon \rho u_1(e + \frac{|u|^2}{2}) + \varepsilon pu_1]_x = \varepsilon (\kappa(\theta)\theta)_x \\
    + \frac{4}{3} \varepsilon (\varepsilon^2 \mu(\theta)u_{1x})_x + \sum_{i=2}^{3} \varepsilon (\varepsilon^2 \mu(\theta)u_{ix})_x - \varepsilon \int \frac{1}{2} \xi_1 |\xi|^2 \Theta_x d\xi,
\end{cases}
\]

\begin{equation}
\varepsilon^2 G_t + P_1(\xi_1 M_x) + \varepsilon P_1(\xi_1 G_x) = L_M G + \varepsilon Q(G, G),
\end{equation}

where
\[
G = L_M^{-1}(P_1(\xi_1 M_x)) + \Theta,
\]
and
\[
\Theta = L_M^{-1}(\varepsilon^2 G_t + \varepsilon P_1(\xi_1 G_x) - \varepsilon Q(G, G)).
\]
Here \( L_M \) is the linearized operator of the collision operator with respect to the local Maxwellian \( M \):
\[
L_M h = Q(M, h) + Q(h, M),
\]
and the null space \( N \) of \( L_M \) is spanned by the macroscopic variables:
\[
\chi_j, \quad j = 0, 1, 2, 3, 4.
\]
Furthermore, there exists a positive constant \( \sigma_0(\rho, u, \theta) > 0 \) such that for any function \( h(\xi) \in N^\perp \), see [19],
\[
(\langle h, L_M h \rangle \leq -\sigma_0(\nu(|\xi|)h, h),
\]

where
\[
\nu(|\xi|) = |\xi|^2, \quad |\xi|^2 = \sum_{i=1}^{d} \xi_i^2.
\]
where $\nu(|\xi|)$ is the collision frequency. For the hard sphere and the hard potential with angular cut-off, the collision frequency $\nu(|\xi|)$ has the following property

$$0 < \nu_0 < \nu(|\xi|) < c(1 + |\xi|)^\beta,$$

for some positive constants $\nu_0, c$ and $0 < \beta \leq 1$.

In the above presentation, we normalize the gas constant $R$ to be $\frac{3}{2}$ for simplicity so that $e = \frac{3}{2}R\theta = \theta$ and $p = R\theta = \frac{3}{2}\rho\theta$. Notice also that the viscosity coefficient $\mu(\theta) > 0$ and the heat conductivity coefficient $\kappa(\theta) > 0$ are smooth functions of the temperature $\theta$. And the following relation holds between these two functions, \[8, 19\],

$$\kappa(\theta) = \frac{15}{4}R\mu(\theta) = \frac{5}{2}\mu(\theta),$$  \[(2.4)\]

after taking $R = \frac{3}{2}$. It should be pointed out that (2.4) is crucially used in the following analysis.

In fact, in our analysis, it is required that

$$\inf_{\theta} \kappa(\theta) > \frac{5}{4} \sup_{\theta} \mu(\theta)$$

for all $\theta$ under consideration. By (2.4), it is known that the above condition holds provided that the variation of the temperature is suitably small.

Now we are in a position to derive the limit equations for $(\rho, u, \theta)$ in the diffusive limit (1.6) formally. As [3], we assume that

$$p^\varepsilon = \text{const} + O(1)\varepsilon^2,$$  \[(2.5)\]

then, as $\varepsilon \to 0$, (2.2), (2.2) and (2.2) yields formally that

$$\left\{ \begin{array}{l}
p = \text{const}, \\
\rho_t + (\rho u_1)_x = 0, \\
(\rho u_1)_t + (\rho u_1^2)_x + P^*_x = \frac{4}{3}(\mu(\theta)u_1)_x, \\
(\rho\theta)_t + (\rho u_1\theta + pu_1)_x = (\kappa(\theta)\theta_x)_x,
\end{array} \right.$$  \[(2.6)\]

where $P^*$ is unknown function. The equation (2.6) reveals how the zero order function $\rho, \theta$ depend on the scaled velocity even though the macroscopic velocity tends to zero.

With slab symmetry, in the macroscopic level, it is more convenient to rewrite the system by using the Lagrangian coordinates as in the study of conservation laws. That is, consider the coordinate transformation:

$$(x, t) \to \left( \int_{(0, 0)}^{(x,t)} \rho(y, s)dy - (\rho u_1)(y, s)ds, s \right),$$

which is still denoted as $(x, t)$ without confusion. Denote that $v = \frac{1}{\rho}$, the system (1.1) and (2.1) in the Lagrangian coordinates become

$$\varepsilon f_t - \varepsilon u_1 \frac{v}{f_x} + \frac{\xi_1}{\varepsilon} f_x = \frac{1}{\varepsilon} Q(f, f),$$  \[(2.7)\]

and

$$\left\{ \begin{array}{l}
\varepsilon v_t - \varepsilon u_{1x} = 0, \\
\varepsilon^2 u_{1t} + p_x = -\varepsilon \int \xi_1^2 G_x d\xi, \\
\varepsilon^2 u_{ix} = -\varepsilon \int \xi_1 \xi_i G_x d\xi, \ i = 2, 3, \\
\varepsilon(\varepsilon + \frac{|\xi|^2}{2})_t + (\varepsilon pu_1)_x = -\varepsilon \int \frac{1}{2} \xi_1 |\xi|^2 G_x d\xi,
\end{array} \right.$$  \[(2.8)\]
respectively. Moreover, (2.2) and (2.3) take the form
\[
\begin{align*}
\varepsilon v_t - \varepsilon u_{1x} &= 0, \\
\varepsilon^2 u_{1t} + p_x &= \frac{4}{3} \varepsilon^2 \left( \frac{\mu(\theta)}{v} u_{1x} \right)_x - \varepsilon \int \xi_1^2 \Theta_{1x} d\xi, \\
\varepsilon^2 u_{it} &= \varepsilon^2 \left( \frac{\mu(\theta)}{v} u_{ix} \right)_x - \varepsilon \int \xi_1 \xi_i \Theta_{1x} d\xi, \quad i = 2, 3, \\
\varepsilon (e + \frac{1}{2} |u|^2)_t + (\varepsilon pu_{1x})_x &= \varepsilon \frac{\kappa(\theta)}{v} \theta_x + \frac{4}{3} \varepsilon^3 \left( \frac{\mu(\theta)}{v} u_1 u_{1x} \right)_x \\
&\quad + \sum_{i=2} \varepsilon^3 \left( \frac{\mu(\theta)}{v} u_i u_{ix} \right)_x - \varepsilon \int \frac{1}{2} \xi_1^2 |\xi|^2 \Theta_{1x} d\xi,
\end{align*}
\]
and
\[
\varepsilon^2 G_t - \frac{\varepsilon^2 u_1}{v} G_x + P_1 \left( \frac{\xi_1}{v} M_x \right) + \varepsilon P_1 \left( \frac{\xi_1}{v} G_x \right) = L_M G + \varepsilon Q(G, G),
\]
with
\[
G = L_M^{-1} (P_1 \left( \frac{\xi_1}{v} M_x \right) + \Theta_1,
\]
and
\[
\Theta_1 = L_M^{-1} \left( \varepsilon^2 G_t - \frac{\varepsilon^2 u_1}{v} G_x + \frac{\varepsilon}{v} P_1 (\xi_1 G_x) - \varepsilon Q(G, G) \right).
\]
The limiting equation (2.6) becomes
\[
\begin{align*}
p &= \text{const}, \\
v_t - u_{1x} &= 0, \\
u_{1t} + P^*_x &= \frac{4}{3} \left( \frac{\mu(\theta)}{\theta} u_{1x} \right)_x, \\
\theta_t + p u_{1x} &= \left( \frac{\kappa(\theta)}{\theta} \theta_x \right)_x.
\end{align*}
\]

2.1 Construction of profile

We will construct a background solution to (2.12) in this subsection. Without loss of generality, set
\[
p = \frac{2\theta}{3v} = \frac{2}{3},
\]
that is
\[
v = \theta.
\]
Assume the boundary conditions at the far fields given by
\[
\lim_{x \to \pm\infty} (v, \theta)(x, t) = (v_\pm, \theta_\pm), \quad \text{and} \quad \frac{\theta_+}{v_+} = \frac{\theta_-}{v_-} = 1, \quad \text{with} \ \theta_- \neq \theta_+.
\]
Note that if \( \theta_- = \theta_+ \), then \( v = \theta = 1, u_1 = 0 \) is a trivial solution to (2.12), and the diffusive limit of the rescaled Boltzmann equation to the incompressible Navier-Stokes-Fourier system is well studied as mentioned in the introduction.

Noting (2.14), the equation (2.12) is rewritten as
\[
\theta_t + \frac{2}{3} u_{1x} = \left( \frac{\kappa(\theta)}{\theta} \theta_x \right)_x.
\]
Substituting \((2.12)\) into \((2.16)\) and noting \((2.14)\), we have the following scalar nonlinear diffusion equation

$$
\theta_t = (a(\theta)\theta_x)_x, \quad a(\theta) = \frac{3\kappa(\theta)}{5\theta}, \quad \text{with} \quad \lim_{x \to \pm \infty} \theta(x,t) = \theta_{\pm}.
$$

(2.17)

From [1] and [10], it is known that the nonlinear diffusion equation \((2.17)\) admits a self-similar solution \(\hat{\theta}(\eta)\) with \(\eta = \frac{x}{\sqrt{1+t}}\) satisfying the boundary conditions \(\hat{\theta}(\pm \infty, t) = \theta_{\pm}\). Furthermore, \(\hat{\theta}(\eta)\) is a monotonic function. Let \(\delta = |\theta_+ - \theta_-|\), then \(\hat{\theta}(t,x)\) has the property that

$$
\hat{\theta}_x(t,x) = O(1) \delta \frac{e^{-4\kappa(\theta)(t+1)}}{1+t}, \quad \text{as} \quad x \to \pm \infty.
$$

(2.18)

Define

$$
(\tilde{v}, \tilde{u}_1, \tilde{\theta}) = (\hat{\theta}, a(\hat{\theta})\hat{\theta}_x, \hat{\theta})(x,t),
$$

(2.19)

then it is easy to check that \((\tilde{v}, \tilde{u}_1, \tilde{\theta})\) satisfying \((2.12)\) as

\[
\begin{align*}
\tilde{p} &= 2\hat{\theta} \frac{2}{3\tilde{v}} = \frac{2}{3}, \\
\tilde{v}_t - \tilde{u}_1x &= 0, \\
\tilde{u}_1 + P^* &= \frac{4}{3}(\frac{\mu(\hat{\theta})}{\tilde{v}}\tilde{u}_1x), \\
\tilde{\theta}_t + \tilde{p}_1x &= (\frac{\kappa(\hat{\theta})}{\tilde{v}}\tilde{\theta}_x)_x,
\end{align*}
\]

(2.20)

where \(P^* = -a(\hat{\theta})\hat{\theta}_t + \frac{4\mu(\hat{\theta})}{\tilde{v}}(a(\hat{\theta})\hat{\theta}_x)_x\).

**Remark 2.1** By \((2.19)\) and \((2.20)\), we actually construct a diffusion wave to the limit system. On the other hand, if \(\theta_- < \theta_+\), then \(\tilde{u}_1 = a(\hat{\theta})\hat{\theta}_x > 0\), that is, the variation of temperature along the \(x\)-axis induces a nontrivial scaled flow along the same direction, see Figure 1. The case \(\theta_- > \theta_+\) is similar, see Figure 2.

**Remark 2.2** The construction of the profile \((\tilde{v}, \tilde{u}_1, \tilde{\theta})\) is motivated by the viscous contact wave of compressible Navier-Stokes equations, see [22], [26] and [27]. The viscous contact wave is used to approximate the contact discontinuity for compressible Euler equation and its pressure keeps constant.

In order to justify the hydrodynamic limit of the rescaled Boltzmann equation to the limit system \((2.20)\), if we use the profile \((\tilde{v}, \tilde{u}_1, \tilde{\theta})\), then some non-integrable error terms with respect to time coming from the non-fluid component for the system about perturbation. Therefore, one needs to construct another profile \((\bar{v}, \bar{u}, \bar{\theta})\) for the rescaled Boltzmann equation, based on \((\tilde{v}, \tilde{u}_1, \tilde{\theta})\). For this, we require that the approximate pressure \(p\) satisfies

$$
p = \frac{2\tilde{\theta}}{3\tilde{v}} = \frac{2}{3} + O(1)\varepsilon^2 = p_+ + O(1)\varepsilon^2.
$$

(2.21)
Motivating by \[27\], we first notice that the main part of the non-fluid component in the solution $G$ and part of $\Theta_1$ defined in \[(2.11),\] are given by

$$w = \frac{1}{v}L_M^{-1}\{P_1(\xi_1 M_x)\} = \frac{1}{Rv\theta}L_M^{-1}\{P_1[\xi_1(\frac{|\xi - \varepsilon u|^2}{2\theta} \theta_x + \xi \cdot \varepsilon u_x)M]\},$$

and

$$\dot{\Theta}_1 = L_M^{-1}\{(\varepsilon P_1(\xi_1 w_x) - \varepsilon Q(w, w))\},$$

respectively. To distinguish the leading term coming from the non-fluid component, we rewrite the Boltzmann equation \[(2.9)\] as

$$\begin{cases}
\varepsilon v_t - \varepsilon u_{1x} = 0, \\
\varepsilon^2 u_{it} + p_x = \frac{4}{3}\varepsilon^2(\frac{\mu(\theta)}{v} u_{1x})_x - \sum_{j=1}^2 \varepsilon \int \xi_1^2 \Theta_{1x}^j d\xi, \\
\varepsilon^2 u_{it} = \varepsilon^2(\frac{\mu(\theta)}{v} u_{1x})_x - \sum_{j=1}^2 \varepsilon \int \xi_1 \xi_1 \Theta_{1x}^j d\xi, \quad i = 2, 3, \\
\varepsilon (\varepsilon + \frac{|\varepsilon u|^2}{2})_t + (\varepsilon pu_{1x})_x = \varepsilon(\frac{\kappa(\theta)}{v} \theta_x)_x - \sum_{j=1}^2 \varepsilon \int \frac{1}{2} \xi_1 |\varepsilon|^2 \Theta_{1x}^j d\xi + H_x,
\end{cases} \tag{2.22}$$

with

$$\varepsilon^2 \tilde{G}_t - L_M \tilde{G} = -\frac{1}{Rv\theta}P_1[\xi_1(\frac{|\xi - \varepsilon u|^2}{2\theta} \theta_x + \xi \cdot (\varepsilon u - \varepsilon \bar{u})_x)M]$$

$$+ \frac{\varepsilon^2 u_1}{v} + \varepsilon G_x \varepsilon P_1(\xi_1 G_x) + \varepsilon Q(G, G) - \varepsilon^2 \tilde{G}_t, \tag{2.23}$$

where

$$\begin{align*}
\tilde{G} &= \frac{1}{Rv\theta}L_M^{-1}\{P_1[\xi_1(\frac{|\xi - \varepsilon u|^2}{2\theta} \theta_x + \xi \cdot (\varepsilon u - \varepsilon \bar{u})_x)M]\}, \quad \tilde{G} = G - \tilde{G}, \\
H &= \frac{4}{3}\varepsilon^2(\frac{\mu(\theta)}{v} u_{1x})_x + \sum_{i=2}^3 \frac{3}{2}\varepsilon^3(\frac{\mu(\theta)}{v} u_{1x})_x, \\
\Theta_1^1 &= L_M^{-1}\{(\varepsilon P_1(\xi_1 G_x) - \varepsilon Q(G, \tilde{G})\}, \tag{2.24} \\
\Theta_1^2 &= L_M^{-1}\{(\varepsilon^2 G_t - \frac{\varepsilon^2 u_1}{v} + \varepsilon G_x) + \varepsilon P_1(\xi_1 G_x) - \varepsilon Q(G, \tilde{G}) - 2\varepsilon Q(G, \tilde{G})\},
\end{align*}$$

satisfying

$$\sum_{j=1}^2 \Theta_{1x}^j = \Theta_1 = L_M^{-1}(\varepsilon^2 G_t - \frac{\varepsilon^2 u_1}{v} + \varepsilon G_x) + \varepsilon P_1(\xi_1 G_x) - \varepsilon Q(G, G)).$$

Here, the function $(\bar{v}, \varepsilon \bar{u}, \bar{\theta})(x, t)$ is the profile to be constructed.

Since the velocity $\varepsilon u$ decays faster than $(v, \theta)$ in time, the leading terms in the energy equation \[(2.22)\] are

$$\varepsilon \theta_t + \varepsilon pu_{1x} = \varepsilon(\frac{\kappa(\theta)}{v} \theta_x)_x - \varepsilon \int \frac{1}{2} \xi_1 |\varepsilon|^2 \Theta_{1x}^1 d\xi. \tag{2.25}$$

By the definition of $\Theta_1^1$, it holds that
where the coefficients $f_{ij}, j = 1, 2, 3, 4$ are smooth functions of $(v, \varepsilon u, \theta)$. By (2.21), it is expected that the profile $(\tilde{v}, \varepsilon \tilde{u}, \tilde{\theta})$ for the Boltzmann equation satisfies $\tilde{\theta} \approx \tilde{v}$. Thus, by choosing only the leading term in (2.25), one obtains that

$$\varepsilon \theta_t = \varepsilon (a(\theta) \theta_x)_x + \frac{3\varepsilon^2}{5} N_{1x},$$

(2.27)

where $a(\theta)$ is given in (2.17). Thus the leading part of (2.27) is the nonlinear diffusion equation (2.17) and an explicit solution $\hat{\theta}(\frac{x}{\sqrt{1+t}})$ is given with the boundary conditions $\hat{\theta}(\pm \infty, t) = \theta_\pm$.

To include more microscopic effect, let the profile $\theta = \hat{\theta}(\frac{x}{\sqrt{1+t}}) + \varepsilon \theta^{nf}(x, t)$, where $\theta^{nf}(x, t)$ represents the part of the nonlinear diffusion wave coming from the non-fluid component. Moreover, the term $\theta^{nf}(x, t)$ in the form of $\frac{1}{(\sqrt{1+t})^D} \frac{d}{dx} (\frac{x}{\sqrt{1+t}})$ is from $N_1$ in (2.27). Note that $\theta^{nf}(x, t)$ decays faster than $\hat{\theta}(x, t)$ so that it can be viewed as a perturbation around profile $\hat{\theta}(x, t)$. To construct $\theta^{nf}(x, t)$, we linearize the equation (2.27) around $\hat{\theta}(x, t)$ and keep only the linear terms. This leads to a linear equation for $\theta^{nf}(x, t)$ from (2.27)

$$\theta^{nf}_t = (a(\hat{\theta}) \theta^{nf}_x)_x + (a'(\hat{\theta}) \theta_x \theta^{nf})_x + \frac{3}{5} \dot{N}_{1x},$$

(2.28)

where $\dot{N}_1 = (\tilde{f}_{11} + \tilde{f}_{12} + \tilde{f}_{13}) (\hat{\theta}_x)^2 + \hat{f}_{14} \hat{\theta}_{xx}$ with $\tilde{f}_{ij} = f_{1j}(\tilde{v}, 0, \hat{\theta}), j = 1, 2, 3, 4$. Let

$$g_1(x, t) = \int_{-\infty}^{x} \theta^{nf}(x, t) dx,$$

then integrating (2.28) with respect to $x$ yields that

$$g_{1t} = a(\hat{\theta}) g_{1xx} + a'(\hat{\theta}) \hat{\theta}_x g_{1x} + \frac{3}{5} \dot{N}_1.$$

(2.29)

Note that $\dot{N}_1$ takes the form of $\frac{1}{1+t} D_2(\frac{x}{\sqrt{1+t}})$ and satisfies

$$|\dot{N}_1| = O(1) \delta (1 + t)^{-1} e^{-\frac{4a(\hat{\theta})^2}{4a(\hat{\theta})^2}}$$

as $x \to \pm \infty$.

We can check that there exists a self-similar solution $g_1(\eta), \eta = x/\sqrt{1+t}$ for (2.29) with the boundary condition $g_1(-\infty, t) = 0, g_1(+\infty, t) = \delta_1$. Here $\delta_1$ satisfies $0 < \delta_1 < \delta$. Note that even though the function $g_1(x, t)$ depends on the constant $\delta_1, \theta^{nf}(x, t) = g_{1x}(x, t) \to 0$ as $x \to \pm \infty$. That is, the choice of the constant $\delta_1$ has no influence on the ansatz as long as $|\delta_1| < \delta$. From now on, we fix $\delta_1$ so that the function $g_1(x, t)$ is uniquely determined and its derivative $g_{1x} = \theta^{nf}$ has the property

$$|\theta^{nf}| = |g_{1x}| = O(\delta)(1 + t)^{-\frac{1}{2}} e^{-\frac{4a(\hat{\theta})^2}{4a(\hat{\theta})^2}}$$

as $x \to \pm \infty$.

Now we follow the same procedure to construct the second and third components of the velocity profile denoted by $\varepsilon \tilde{u}_i, i = 2, 3$. That is, the leading part of the equation for $\varepsilon u_i$ coming from (2.22) is

$$\varepsilon^2 u_{it} = \varepsilon^2 (\frac{\mu(\theta)}{\theta} u_{ix})_x - \varepsilon \int \xi_1 \xi_i \Theta_1 d\xi,$$

(2.30)
For \(i = 2, 3\), one gets
\[
\begin{cases}
- \varepsilon \int \xi \xi_1 \Theta_1 d\xi = \varepsilon^2 N_i + \varepsilon^3 F_i, \\
N_i = f_{i1}\theta_1 \theta_x + f_{i2}v \theta_x + f_{i3} \theta^2 + f_{i4} \theta_{xx}, \\
|F_i| = O(1)(|v_x| + |\theta_x| + \varepsilon |u_x| + \varepsilon |\bar{u}_x|)|\bar{u}_x| + |u_x| |\theta_x| + |\bar{u}_{xx}|),
\end{cases}
\]
(2.31)
with smooth functions \(f_{ij}, i = 2, 3, j = 1, 2, 3, 4\). Notice that the symbols \(N_i\) and \(F_i, i = 2, 3\), used here are for the convenience of notations.

From (2.30) and (2.31), we expect that the profile \(\bar{u}_i(x, t)\) takes the form of \(\frac{1}{\sqrt{1+\varepsilon t}} h_i(\frac{x}{\sqrt{1+\varepsilon t}})\) and satisfies the following linear equation
\[
\varepsilon^2 \bar{u}_{it} = \varepsilon^2 \left( \frac{\mu(\hat{\theta})}{\hat{\theta}} \bar{u}_{ix} \right)_x + \varepsilon^2 \hat{N}_i, \quad i = 2, 3,
\]
(2.32)
where \(\hat{N}_i = (\hat{f}_{i1} + \hat{f}_{i2} + \hat{f}_{i3})(\hat{\theta}_x)^2 + \hat{f}_{i4} \hat{\theta}_{xx}, \hat{f}_{ij} = f_{ij}(\hat{v}, 0, \hat{\theta}), i = 2, 3, j = 1, 2, 3, 4\).

Denote
\[
g_i(x, t) = \int_{-\infty}^{x} \bar{u}_i(x, t) dx,
\]
then integrating (2.32) with respect to \(x\), one has
\[
g_{it} = \frac{\mu(\hat{\theta})}{\hat{\theta}} g_{ixx} + \hat{N}_i. \quad (2.33)
\]

For given \(\hat{\theta}\), we can check that there exists a self-similar solution \(g_i(\eta)\) with \(\eta = \frac{x}{\sqrt{1+\varepsilon t}}\) with the boundary conditions \(g_i(-\infty, t) = 0, g_i(+\infty, t) = \delta_i\), where \(\delta_i\) satisfies \(0 < \delta_i < \delta\). As we explained before, the choice of the constant \(\delta_i\) is not essential. From (2.18), we fix \(\delta_i\) so that the function \(g_i(x, t)\) is uniquely determined and the derivative \(g_{ix} = \bar{u}_i (i = 2, 3)\) has the following property
\[
|\varepsilon \bar{u}_i| = |\varepsilon g_{ix}| = O(1) \delta \varepsilon (1 + t) - \frac{\varepsilon}{4(\delta + 1)} \bar{u}_i(1 + t), \quad \text{as } x \to \pm \infty,
\]
where \(b(\theta) = \max\left\{a(\theta), \frac{\mu(\theta)}{\theta}\right\}\).

In summary, one can define the profile \((\bar{v}, \varepsilon \bar{u}, \hat{\theta})\) for the Boltzmann equation as follows. To satisfy the conservation of mass, one needs
\[
\varepsilon \bar{v}_t - \varepsilon \bar{u}_{tx} = 0.
\]
By letting \(\bar{v} = \hat{\theta} + \varepsilon \theta^n f\), one gets
\[
\varepsilon \bar{u}_1 = \varepsilon[a(\hat{\theta}) \hat{\theta}_x + \varepsilon a(\hat{\theta}) \theta^n f + \varepsilon a'(\hat{\theta}) \hat{\theta}_x \theta^n f] + \frac{3\varepsilon^2}{5} \hat{N}_i. \quad (2.34)
\]
However, by plugging (2.34) into the momentum equation of (2.22), we have a non-conservative term containing \(\varepsilon^2 \hat{N}_{1t}\). To avoid this, one defines
\[
\varepsilon \bar{u}_1 = \varepsilon[a(\hat{\theta}) \hat{\theta}_x + \varepsilon a(\hat{\theta}) \theta^n f + \varepsilon a'(\hat{\theta}) \hat{\theta}_x \theta^n f].
\]
Similarly, to avoid the non-conservative term \(|\bar{u}|^2\_t\) in the energy equation, set
\[
\hat{\theta} = \theta^n s + \varepsilon \theta^n f - \frac{1}{2} |\varepsilon \bar{u}|^2.
\]
Therefore, the profile \((\bar{v}, \varepsilon \bar{u}, \bar{\theta})\) is finally defined as:

\[
\begin{align*}
\bar{v} &= \hat{\theta} + \varepsilon \theta^{nf}, \\
\varepsilon \bar{u}_1 &= \varepsilon [a(\hat{\theta}) \hat{\theta}_x + \varepsilon a(\hat{\theta}) \theta^{nf} + \varepsilon a'(\hat{\theta}) \hat{\theta}_x \theta^{nf}], \\
\varepsilon \bar{u}_i &= \varepsilon g_i x, \quad i = 2, 3, \\
\bar{\theta} &= \hat{\theta} + \varepsilon \theta^{nf} - \frac{1}{2} \varepsilon |\bar{u}|^2,
\end{align*}
\]  

(2.35)

where \(\hat{\theta}\) is given by \((2.17)\), \(\theta^{nf}\) by \((2.28)\) and \(g_i, i = 2, 3\) by \((2.33)\). Then a direct but tedious computation shows that

\[
\begin{align*}
\varepsilon \bar{v}_t - \varepsilon \bar{u}_{1x} &= \frac{3 \varepsilon^2}{\bar{v}} \hat{N}_{1x}, \\
\varepsilon^2 \bar{u}_{1t} + \bar{p}_x &= \frac{4 \varepsilon^2}{3} \left(\frac{\mu(\hat{\theta})}{\bar{v}} \bar{u}_{1x}\right)_x + \bar{R}_{1x}, \\
\varepsilon^2 \bar{u}_{it} &= \varepsilon^2 \left(\frac{\mu(\hat{\theta})}{\bar{v}} \bar{u}_{ix}\right)_x + \varepsilon^2 \hat{N}_{ix} + \bar{R}_{ix}, \quad i = 2, 3, \\
\varepsilon \left(\bar{e} + \frac{|\bar{u}|^2}{2}\right)_x + (\varepsilon \bar{p})_1 &= \varepsilon \left(\frac{\kappa(\hat{\theta})}{\bar{v}} \hat{\theta}_x\right)_x + \bar{H}_x + \varepsilon^2 \hat{N}_{1x} - \frac{2 \varepsilon^2}{3} \hat{N}_{ix} + \bar{R}_{4x},
\end{align*}
\]  

(2.36)

where

\[
\begin{align*}
\bar{R}_1 &= \varepsilon^2 [a(\hat{\theta}) \hat{\theta}_t + (a(\hat{\theta}) \theta^{nf})_t] + \bar{p} - p + \frac{4 \varepsilon^2}{3} \left(\frac{\mu(\hat{\theta})}{\bar{v}} \varepsilon \bar{u}_{1x}\right) \\
&= O(1)\delta \varepsilon^2 (1 + t)^{-1} e^{-\frac{1}{2}(\varepsilon^2 e^{2 \hat{u}})|\bar{v}|^2 (1 + t)}, \quad \text{as} \ x \to \pm \infty,
\end{align*}
\]  

(2.37)

\[
\begin{align*}
\bar{R}_i &= \varepsilon^2 \left[\frac{\mu(\hat{\theta})}{\bar{v}} \varepsilon \bar{u}_{ix}\right] + \varepsilon^2 (\hat{N}_i - \hat{N}_i) \\
&= O(1)\delta \varepsilon \varepsilon^3 (1 + t)^{-3/2} e^{-\frac{1}{2}(\varepsilon^2 e^{2 \bar{u}})|\bar{v}|^2 (1 + t)}, \quad \text{as} \ x \to \pm \infty, \quad i = 2, 3,
\end{align*}
\]  

(2.38)

\[
\begin{align*}
\bar{R}_i &= \left[\frac{5}{3} \varepsilon (a(\hat{\theta}) \hat{\theta}_x + a(\hat{\theta}) \theta^{nf}_x + a'(\hat{\theta}) \hat{\theta}_x \theta^{nf}) - \varepsilon \frac{\kappa(\hat{\theta})}{\bar{v}} \hat{\theta}_x\right] \\
&\quad + (\bar{p} - p + \varepsilon \bar{u}_x) + \varepsilon^2 (\hat{N}_i - \hat{N}_1) - \bar{H} \\
&= O(\delta \varepsilon^3 (1 + t)^{-3/2} e^{-\frac{1}{2}(\varepsilon^2 e^{2 \bar{u}})|\bar{v}|^2 (1 + t)}), \quad \text{as} \ x \to \pm \infty,
\end{align*}
\]  

(2.39)

\[
\begin{align*}
\hat{N}_i &= O(1)\delta (1 + t)^{-1} e^{-\frac{1}{2}(\varepsilon^2 e^{2 \bar{u}})|\bar{v}|^2 (1 + t)}), \quad \text{as} \ x \to \pm \infty, \quad i = 1, 2, 3,
\end{align*}
\]  

(2.40)

with \(c(\theta_x) = \max\{a(\theta_x), \frac{1}{2}b(\theta_x)\}\), \(\hat{N}_i, i = 1, 2, 3\), and \(\bar{H}\) are the corresponding functions defined in \((2.24), (2.26)\) and \((2.31)\) by substituting the variable \((v, \varepsilon u, \theta)\) by the profile \((\bar{v}, \varepsilon \bar{u}, \bar{\theta})\). Note that the decay rate of \(R_i, i = 2, 3, 4\) is of order \(\varepsilon^3(1 + t)^{-3/2}\). Furthermore, even though the decay rate of \(R_1\) is still \(\varepsilon^2(1 + t)^{-1}\), it is sufficient to obtain the desired a priori estimates through some subtle analysis coming from the intrinsic dissipation mechanism in the momentum equations as shown in the following.

Define

\[
\tilde{M} = \frac{\bar{v}^{-1}}{\sqrt{2\pi R^3}} \exp \left(-\frac{1}{2} \frac{\xi - \varepsilon \bar{u}_x}{\bar{v}} \right), \quad \tilde{G}_0 = L_{\tilde{M}} \left( \frac{1}{\bar{v}} \tilde{P}_1 (\xi_1 \tilde{M}_x) \right),
\]

and

\[
\tilde{f} = \tilde{M} + \varepsilon \tilde{G}_0.
\]

Then it follows from \((2.36)\) that

\[
\varepsilon \tilde{f}_t - \frac{\varepsilon \bar{u}_1}{\bar{v}} \tilde{f}_x + \frac{1}{\bar{v}} \varepsilon \xi_1 \tilde{f}_x = L_{\tilde{M}} \tilde{G}_0 + \varepsilon Q(\tilde{G}_0, \tilde{G}_0) + \tilde{R}_f,
\]  

(2.41)
Corollary 2.5
Under the conditions of Theorem 2.4, from
that

\[ v \to \text{the diffusion wave solution} \ (\tilde{C}, \tilde{\theta}) \ \text{implies that} \]

\[ |(v - \tilde{v}, u_1 - \tilde{u}_1, \theta - \tilde{\theta})(x, t)| = O(1) \delta (1 + t)^{-\frac{3}{2}} e^{-\frac{\delta}{4\varepsilon^2(1+\varepsilon^2)}|\xi|^{\frac{3}{2}}} \text{, as } x \to \pm \infty. \]

Remark 2.3
From the definition of \( \delta \) in (2.19) and the definition of \((\tilde{v}, \tilde{u}_1, \tilde{\theta})\) in (2.35), it holds that

\[ |(\tilde{v} - \bar{v}, \bar{u}_1 - \tilde{u}_1, \bar{\theta} - \tilde{\theta})(x, t)| = O(1) \delta \varepsilon (1 + t)^{-\frac{3}{2}} e^{-\frac{\delta}{4\varepsilon^2(1+\varepsilon^2)}|\xi|^{\frac{3}{2}}}, \]

that implies that the ansatz \((\tilde{v}, \tilde{u}_1, \tilde{\theta})\) well approximates \((\bar{v}, \bar{u}_1, \bar{\theta})\) when \( \varepsilon \) is small.

2.2 Main result

Now we consider the system (2.9)-(2.10) with the initial data

\[ (v, u, \theta)|_{t=0} = (\bar{v}, \bar{u}, \bar{\theta})(x, 0), \quad G(x, t)|_{t=0} = \tilde{G}(x, 0). \quad (2.43) \]

Then the main result in this paper can be stated as follows.

Theorem 2.4
Let \((\tilde{v}, \bar{u}, \bar{\theta})(x, t)\) be the profile defined in (2.35) with strength \( \delta = |\theta_+ - \theta_-| \).
Then there exist small positive constants \( \delta_0 \) and \( \varepsilon_0 \) and a global Maxwellian \( M_\star = M_{[v, u, \theta, \bar{\theta}]} \), such that when \( \delta \leq \delta_0 \) and \( \varepsilon \leq \varepsilon_0 \), the Cauchy problem (2.9)-(2.10) with the initial data (2.43) has a unique global solution \((v, u, \theta, G)\) satisfying, for any sufficiently small but fixed positive constant \( \vartheta > 0 \),

\[
\begin{align*}
\| (v - \tilde{v}, \varepsilon u - \varepsilon \bar{u}, \theta - \bar{\theta})(t) \|_{L^2_{x,t}} & \leq C \sqrt{\delta} \varepsilon (1 + t)^{-1 + \vartheta + C_0 \sqrt{\delta}}, \\
\| (v - \tilde{v}, \varepsilon u - \varepsilon \bar{u}, \theta - \bar{\theta})_x(t) \|_{L^2_{x,t}} & \leq C \sqrt{\delta} \varepsilon^2 (1 + t)^{-\frac{3}{2} + \vartheta + C_0 \sqrt{\delta}}, \\
\| f_{xx}(t) \|_{L^2_{x,t}(\frac{\varepsilon^2}{1+\varepsilon^4})} & + \| (v - \tilde{v}, \varepsilon u - \varepsilon \bar{u}, \theta - \bar{\theta})_{xx}(t) \|_{L^2_{x,t}} \leq C \sqrt{\delta} (1 + t)^{-\frac{3}{2} + \vartheta + C_0 \sqrt{\delta}}, \\
\| (G - \tilde{G})(t) \|_{L^2_{x,t}(\frac{1}{\varepsilon^2})} & \leq C \sqrt{\delta} (1 + t)^{-\frac{1}{2}}, \\
\| (G - \tilde{G})_x(t) \|_{L^2_{x,t}(\frac{1}{\varepsilon^2})} & \leq C \sqrt{\delta} (1 + t)^{-\frac{3}{2} + \vartheta + C_0 \sqrt{\delta}},
\end{align*}
\]

that implies that

\[
\begin{align*}
\| (v - \tilde{v}, \varepsilon u - \varepsilon \bar{u}, \theta - \bar{\theta})(t) \|_{L^\infty_{x,t}} & \leq C \delta^{\frac{1}{4}} \varepsilon^{\frac{1}{2}} (1 + t)^{-\frac{3}{2} + \vartheta + \frac{1}{2}}, \\
\| (v - \tilde{v}, \varepsilon u - \varepsilon \bar{u}, \theta - \bar{\theta})_x(t) \|_{L^\infty_{x,t}} & \leq C \delta^{\frac{1}{4}} \varepsilon^{\frac{1}{2}} (1 + t)^{-\frac{3}{2} + \vartheta + \frac{1}{2}},
\end{align*}
\]

where \( C \) and \( C_0 \) are positive constants independent of \( \varepsilon \) and \( \delta \).

The following result justifies the hydrodynamic limit of the rescaled Boltzmann equation (1.1) to the diffusion wave \((\tilde{v}, \tilde{u}_1, \tilde{\theta})\) global in time.

Corollary 2.5
Under the conditions of Theorem 2.4 from (2.42) and (2.45), it holds that

\[
\begin{align*}
\| (v - \tilde{v}, \theta - \tilde{\theta})(x, t) \| & \leq C \varepsilon (1 + t)^{-\frac{3}{2}} \to 0, \\
\| (u_1 - \tilde{u}_1)(x, t) \| & \leq C \varepsilon^{\frac{1}{2}} (1 + t)^{-\frac{3}{2}} \to 0, \quad \text{as } \varepsilon \to 0,
\end{align*}
\]

that is, the fluid part \((v, u_1, \theta)\) of the solution of the rescaled Boltzmann equation (1.1) converges to the diffusion wave solution \((\tilde{v}, \tilde{u}_1, \tilde{\theta})\) of (2.20) in the sense of (2.46) as \( \varepsilon \to 0 \), which reveals that \( v \) and \( \theta \) are diffusive.
Remark 2.6 The above Corollary shows that if the zero order function in (1.5) is not a global Maxwellian, then one has to consider the effect of diffusive wave in the diffusive limit of rescaled Boltzmann equation (1.1).

Since the scaled velocity \( \tilde{u}_1 \) is actually induced by the variation of temperature \( \tilde{\theta} \), i.e., \( \tilde{u}_1 = a(\tilde{\theta})\tilde{\theta}_x \). The following result shows that the scaled velocity \( u_1 \) is also induced by the variation of temperature \( \theta \) in some sense when \( \varepsilon \) is small. From the definition of \( \hat{\theta}(\eta) \) with \( \eta = \frac{x}{\sqrt{1+t}} \) in (2.17) and (2.18), it can be seen that \( \hat{\theta} \) is monotonic. To be definite and without loss of generality, let us assume that \( \theta_- < \theta_+ \), that is, \( \hat{\theta} \) is monotonically increasing. Then there exists a positive constant \( \eta_0 > 0 \) such that

\[
\hat{\theta}'(\eta) > c_{\eta_0}\delta, \quad \text{for } |\eta| \leq \eta_0, \tag{2.47}
\]

where \( c_{\eta_0} \) depends on \( \eta_0 \) and \( c_{\eta_0} \to 0 \) as \( \eta_0 \to +\infty \).

Corollary 2.7 Under the conditions of Theorem 2.4 and \( \theta_- < \theta_+ \), for any fixed \( \eta_0 > 0 \), there exists a small positive constant \( \varepsilon_1 = \varepsilon_1(\eta_0) \leq \varepsilon_0 \), such that if \( \varepsilon \leq \varepsilon_1 \), then it follows from (2.47) and (2.45) that

\[
\begin{align*}
0 < \frac{c_{\varepsilon_0}\delta}{C_1\sqrt{1+t}} &< \frac{1}{C_1}\hat{\theta}_x \leq u_1(x,t) \leq C_1\hat{\theta}_x, \\
0 < \frac{1}{2}\delta_x &< \theta_x(x,t) \leq \frac{3}{2}\delta_x,
\end{align*}
\]

for \( |x| \leq \eta_0(1+t)^{\frac{1}{2}}, \ t \geq 0 \), \( \tag{2.48} \)

that is

\[
\frac{2}{3C_1}\theta_x(x,t) \leq u_1(x,t) \leq 2C_1\theta_x(x,t), \ \text{for } |x| \leq \eta_0(1+t)^{\frac{1}{2}}, \ t \geq 0,
\]

where \( C_1 \) is a suitably large positive constant depending only on \( \theta_+ \). In particular, (2.49) implies that variation of the temperature induces a non-trivial flow of higher order in the following parabolic region

\[
\left\{ (x,t) : |x| \leq \eta_0(1+t)^{\frac{1}{2}}, \ t \geq 0 \right\}.
\]

3 Stability Analysis

In this section, we will investigate the stability of the profile constructed in (2.36) for the Boltzmann equation (1.1). This section is organized as follows: in Section 3.1, the fluid type system (2.2) is reformulated in terms of the integrated variables; Section 3.2 is devoted to the lower order estimate, while Section 3.3 is for the derivative estimate.

3.1 Reformulated system

We now reformulate the system by introducing a scaling for the independent variables. Set

\[
y = \frac{x}{\varepsilon}, \quad \tau = \frac{t}{\varepsilon^2}.
\]

In the following, we will also use the notations \( (v, u, \theta)(\tau, y) \) and \( (\tilde{v}, \tilde{u}, \tilde{\theta})(\tau, y) \), etc., in the scaled independent variables. Set the perturbation around the profile \( (\tilde{v}, \tilde{u}, \tilde{\theta})(\tau, y) \) by

\[
\phi = v - \tilde{v}, \psi = \varepsilon u - \varepsilon\tilde{u}, \ z = \theta - \tilde{\theta},
\]
and
\[
(\Phi, \Psi, \bar{W})(y, \tau) = \int_{-\infty}^{y} \left( \phi, \psi, (\theta + \frac{|\epsilon u|^2}{2}) - (\bar{\theta} + \frac{|\epsilon \bar{u}|^2}{2}) \right)(z, \tau) dz.
\]

Then we have \((\phi, \psi) = (\Phi, \Psi)_y\) and \(\zeta + \frac{1}{2} |\Psi_y|^2 + \sum_{i=1}^{3} \epsilon \bar{u}_i \Psi_{iy} = \bar{W}_y\). Subtracting \([2.36]\) from the equation \((2.22)\) and integrating the reduced system yield
\[
\begin{aligned}
\begin{cases}
\Phi_\tau - \Psi_{1y} &= -\frac{2}{5p_+} \epsilon^2 \bar{N}_1, \\
\Psi_{1\tau} + p - \bar{p} &= \frac{4\epsilon}{3} \left( \frac{\mu(\bar{\theta})}{v} u_{1y} - \frac{4}{3} \frac{\mu(\bar{\theta})}{v} \bar{u}_{1y} \right) - \epsilon \sum_{j=1}^2 \int \xi_j^2 \Theta_j^2 d\xi - \bar{R}_1, \\
\Psi_{1\tau} &= \epsilon \left( \frac{\mu(\bar{\theta})}{v} u_{1y} - \frac{4}{3} \frac{\mu(\bar{\theta})}{v} \bar{u}_{1y} \right) + \epsilon^2 (N_i - \bar{N}_i) + \epsilon^3 F_i - \epsilon \int \xi_j \Theta_j^2 d\xi - \bar{R}_i, \quad i = 2, 3, \quad (3.2) \\
W_\tau + \epsilon p_{1y} - \epsilon \bar{p}_1 &= \left( \frac{\kappa(\bar{\theta})}{v} \theta - \frac{\kappa(\bar{\theta})}{v} \bar{\theta} \right) + (H - \bar{H}) + \epsilon^2 (N_i - \bar{N}_i) + \epsilon^3 F_i \\
&- \epsilon \int \frac{1}{2} \xi_1 |\xi|^2 \Theta_1^2 d\xi - \bar{R}_4 + \frac{2}{5} \epsilon^2 \bar{N}_4.
\end{cases}
\end{aligned}
\]

Since the variable \(\bar{W}\) is the anti-derivative of the total energy, not the temperature, it is more convenient to introduce another variable
\[
W = \bar{W} - \epsilon \bar{u}_1 \Psi_1.
\]

It follows that
\[
\zeta = W_y - \bar{y}, \quad \text{with} \quad \bar{y} = \frac{1}{2} |\Psi_y|^2 - \epsilon \bar{u}_{1y} \Psi_1 + \epsilon \bar{u}_2 \Psi_{2y} + \epsilon \bar{u}_3 \Psi_{3y}.
\]

Using the new variable \(W\) and linearizing the left hand side of the system \((3.2)\) by using the formula of \(H\) in \((2.24)\) give that
\[
\begin{aligned}
\begin{cases}
\Phi_\tau - \Psi_{1y} &= -\frac{3}{5} \epsilon^2 \bar{N}_1, \\
\Psi_{1\tau} - \bar{p}_+ \frac{\mu(\bar{\theta})}{v} \Phi_y + \frac{2}{3} \rho \bar{W}_y &= \frac{4}{3} \frac{\mu(\bar{\theta})}{v} \Psi_{1yy} + \frac{4}{3} \frac{\mu(\bar{\theta})}{v} \bar{u}_{1yy} - \epsilon \sum_{j=1}^2 \int \xi_j^2 \Theta_j^2 d\xi + J_1 + \frac{2}{3} \rho \bar{W}_y - \bar{R}_1 = \frac{4}{3} \frac{\mu(\bar{\theta})}{v} \Psi_{1yy} + Q_1, \\
\Psi_{1\tau} &= \epsilon \left( \frac{\mu(\bar{\theta})}{v} u_{1yy} - \frac{4}{3} \frac{\mu(\bar{\theta})}{v} \bar{u}_{1yy} \right) + \epsilon^2 (N_i - \bar{N}_i) + \epsilon^3 F_i \\
&- \epsilon \int \xi_j \Theta_j^2 d\xi - \bar{R}_1 = \frac{4}{3} \frac{\mu(\bar{\theta})}{v} \Psi_{1yy} + Q_i, \quad i = 2, 3, \quad (3.3) \\
W_\tau + \rho \Psi_{1y} &= \frac{\kappa(\bar{\theta})}{v} W_{yy} + \left( \frac{\kappa(\bar{\theta})}{v} - \frac{\kappa(\bar{\theta})}{v} \right) \bar{\theta}_y + \epsilon^2 (N_i - \bar{N}_i) + \epsilon^3 F_i + \frac{4\epsilon}{3} \frac{\mu(\bar{\theta})}{v} \bar{u}_{1yy} \\
&+ \epsilon^3 \sum_{i=2}^3 \frac{\mu(\bar{\theta})}{v} u_{1iyy} - \epsilon \frac{\mu(\bar{\theta})}{v} \bar{u}_{1iyy} - \epsilon \bar{u}_{1y} \Psi_1 + J_2 = \epsilon \int \frac{1}{2} \xi_1 |\xi|^2 \Theta_1^2 d\xi \\
&+ \epsilon^2 \bar{u}_1 \sum_{j=1}^2 \int \xi_j^2 \Theta_j^2 d\xi - \frac{\kappa(\bar{\theta})}{v} \bar{Y}_y + \frac{2}{5} \epsilon^2 \bar{N}_4 + \epsilon \bar{u}_1 \bar{R}_1 - \bar{R}_4 \\
&= \frac{\kappa(\bar{\theta})}{v} W_{yy} + \frac{2}{5} \epsilon^2 \bar{N}_4 + Q_4.
\end{cases}
\end{aligned}
\]
where

\[
J_1 = \frac{\tilde{p} - p}{\tilde{v}} \Phi_y - [p - \tilde{p} + \frac{\tilde{p}}{\tilde{v}} \Phi_y - \frac{2}{3\tilde{v}}(\theta - \tilde{\theta})] = O(1)(\Phi_y^2 + (\theta - \tilde{\theta})^2 + |\varepsilon \tilde{u}|^4),
\]

\[
J_2 = (p_+ - p)\Psi_{1y} = O(1)(\Phi_y^2 + \Psi_{1y}^2 + (\theta - \tilde{\theta})^2 + |\varepsilon \tilde{u}|^4),
\]

\[
Q_1 = \frac{4\varepsilon}{3}(\frac{\mu(\theta)}{v} - \frac{\mu(\tilde{\theta})}{v})u_{1y} - \varepsilon \sum_{j=1}^{2} \int \xi_j^2 \Theta_j^2 d\xi + J_1 + \frac{2}{3\tilde{v}}Y - \tilde{R}_1,
\]

\[
Q_i = \varepsilon \left(\frac{\mu(\tilde{\theta})}{v} - \frac{\mu(\theta)}{v}\right)u_{iy} + \varepsilon^2(N_i - N_i) + \varepsilon^3 F_i - \varepsilon \int \xi_i \xi_j \Theta_j^2 d\xi - \tilde{R}_i, \quad i = 2, 3,
\]

\[
Q_4 = \frac{3}{3}(\frac{\kappa(\theta)}{v} - \frac{\kappa(\tilde{\theta})}{v})\theta_y + \varepsilon^2(N_1 - N_1) + \varepsilon^3 F_1 + \frac{4\varepsilon}{3}(\frac{\mu(\tilde{\theta})}{v})u_{1y} \Psi_{1y}
\]

\[
+ \varepsilon^3 \sum_{i=2}^{2} \frac{(\mu(\tilde{\theta})}{v} u_{iy} - \frac{(\mu(\theta)}{v} u_{iy} - \varepsilon u_1 \Psi_{1y} + J_2 - \varepsilon \int \frac{1}{2} \xi_1 |\xi|^2 \Theta_1^2 d\xi
\]

\[+ \varepsilon^2 u_1 \sum_{j=1}^{2} \int \xi_j^2 \Theta_j^2 d\xi - \frac{\kappa(\tilde{\theta})}{v}Y_y + \varepsilon \tilde{u}_1 \tilde{R}_1 - \tilde{R}_4.\]

The equation of microscopic component \(\tilde{G}\) given in (2.23) in the coordinate \((y, \tau)\) becomes

\[
v \tilde{G}_\tau - vL_M \tilde{G} = \frac{1}{R^\theta}P_1[\xi_1(\frac{|\xi - u|}{2\theta} \epsilon \frac{1}{\epsilon \tilde{y}_y} + \xi \cdot \frac{1}{\epsilon \tilde{y}_y} M] + \varepsilon u_1 G_y - vP_1(\xi_1 G_y) + \varepsilon vQ(G, G) - vG_\tau.
\]

In the scaling of (3.1), the equation (2.7) reads

\[
f_\tau - \frac{u_1}{v} f_y + \frac{\xi_1}{v} f_y = \varepsilon L_M G + \varepsilon^2 Q(G, G).
\]

Set

\[
\tilde{f} = f - f,
\]

then from (3.6) and (2.4), we have

\[
v \tilde{f}_\tau - \varepsilon u_1 \tilde{f}_y + \xi_1 \tilde{f}_y = \varepsilon vL_M \tilde{G} + \varepsilon [vL_M \tilde{G} - \tilde{v}L_M \tilde{G_0}] + \varepsilon^2 [vQ(G, G) - \tilde{v}Q(G_0, G_0)]
\]

\[- \phi \tilde{f}_\tau + \psi \tilde{f}_y - \varepsilon G_\tau.
\]

Note that to prove the main theorem in this paper, it is sufficient to prove the following \textit{a priori} estimate in the scaled independent variables based on the construction of the approximate profile.

\textbf{Theorem 3.1 (A priori estimate)} For any sufficiently small and fixed positive constant \(\vartheta > 0\), there exist small positive constants \(\delta_2 > 0, \varepsilon_2 > 0\) and a global Maxwellian \(M_0 = M_{[p_-, u_-, \theta_0]}\) such that if \(\delta \leq \delta_2\) and \(\varepsilon \leq \varepsilon_2\), then the Cauchy problem (3.3), (3.5) and (3.8) admits a unique smooth solution satisfying

\[
\left\{\begin{array}{l}
||\Phi, \Psi, W||_{L^\infty} \leq C\sqrt{\delta} \varepsilon, \\
||\phi, \psi, \zeta||_{L^2} \leq C\sqrt{\delta} \varepsilon^2 (1 + \varepsilon^2 \tau)^{-1 + C_0 \sqrt{\delta}}, \\
||\phi, \psi, \zeta||_{L^4} + ||\phi, \psi, \zeta||_{yy} \leq C\varepsilon (1 + \varepsilon^2 \tau)^{-1 + C_0 \sqrt{\delta}}, \\
\varepsilon \int_R \int_R \int_R \tilde{G}^2 M_0 d\xi dy \leq C\varepsilon (1 + \varepsilon^2 \tau)^{-1 + C_0 \sqrt{\delta}}.
\end{array}\right.
\]

where \(C, C_0\) are positive constants independent of \(\delta\) and \(\varepsilon\).
In the next subsection, we will work on the reformulated system (3.3) and (3.8). Since the local existence of the solution can be proved similarly as the discussion in [20] and [12], we will omit it here for brevity. To prove the global existence, it is sufficient to close the following a priori estimate:

\[
N(\tau) = \sup_{0 \leq s \leq \tau} \left\{ \epsilon^{-1} \| (\Phi, \Psi, W) \|_{L^\infty}^2 + \epsilon^{-2} \| (\phi, \psi, \zeta) \|_{L^2}^2 + \epsilon^{-3} \| (\phi_y, \psi_y, \zeta_y) \|_{L^2}^2 + \left\| \int_{\mathbb{R}^3} \frac{\tilde{G}^2}{M^*} d\xi \right\|_{L^\infty} + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \sum_{|\alpha|=1} \epsilon^{-1} \frac{\partial\alpha \tilde{G}^2}{M^*} + \sum_{|\alpha|=2} \epsilon^{-3} \frac{\partial\alpha \tilde{f}^2}{M^*} \right) d\xi dy \right\} \leq \lambda_0^2, \tag{3.10}
\]

where \(\lambda_0\) is a positive small constant depending on the initial data and \(M^*\) is a global Maxwellian to be chosen later.

Before proving the a priori estimate (3.10), we list some lemmas based on the celebrated H-theorem for later use. The first one is from [16].

**Lemma 3.2** There exists a positive constant \(C > 0\) such that

\[
\int_{\mathbb{R}^3} \frac{\nu(|\xi|)^{-Q}(f, g)^2}{M} d\xi \leq C \left\{ \int_{\mathbb{R}^3} \frac{\nu(|\xi|) f^2}{M} d\xi \cdot \int_{\mathbb{R}^3} \frac{\bar{g}^2}{M} d\xi + \int_{\mathbb{R}^3} \frac{f^2}{M} d\xi \cdot \int_{\mathbb{R}^3} \frac{\nu(|\xi|) g^2}{M} d\xi \right\},
\]

where \(M\) can be any Maxwellian so that the above integrals are well defined.

Based on Lemma 3.2, the following three lemmas are from [35].

**Lemma 3.3** If \(\theta/2 < \theta_* < \theta\), then there exist two positive constants \(\bar{\sigma} = \bar{\sigma}(\rho, u, \theta; \rho_*, u_*, \theta_*) > 0\) and \(\eta_0 = \eta_0(\rho, u, \theta; \rho_*, u_*, \theta_*) > 0\) such that if \(|\rho - \rho_*| + |\theta - \theta_*| < \eta_0\), we have for \(h(\xi) \in N^\perp\),

\[
- \int_{\mathbb{R}^3} \frac{h L_M h}{M^*} d\xi \geq \bar{\sigma} \int_{\mathbb{R}^3} \frac{\nu(|\xi|) h^2}{M^*} d\xi,
\]

where \(M^* = M_{[\rho, u, \theta, \theta_*]}\) and the definition of \(M = M_{[\rho, u, \theta]}\) can be found in (1.4).

**Lemma 3.4** Under the assumptions in Lemma 3.3, we have

\[
\left\{ \begin{aligned}
\int_{\mathbb{R}^3} \frac{\nu(|\xi|)}{M} |L_M h|^2 d\xi & \leq \bar{\sigma}^{-2} \int_{\mathbb{R}^3} \frac{\nu(|\xi|)^{-1} h^2}{M} d\xi, \\
\int_{\mathbb{R}^3} \frac{\nu(|\xi|)}{M^*} |L_M h|^2 d\xi & \leq \bar{\sigma}^{-2} \int_{\mathbb{R}^3} \frac{\nu(|\xi|)^{-1} h^2}{M^*} d\xi,
\end{aligned} \right.
\]

for each \(h(\xi) \in N^\perp\).

**Lemma 3.5** Under the conditions in Lemma 3.3, there exists a constant \(C > 0\) such that for positive constants \(k\) and \(\lambda\), we have

\[
\int_{\mathbb{R}^3} g_1 P_1 (|\xi|^k g_2) d\xi - \int_{\mathbb{R}^3} g_1 |\xi|^k g_2 d\xi \leq C \int_{\mathbb{R}^3} \lambda |g_1|^2 + \lambda^{-1} |g_2|^2 d\xi.
\]

Note that (3.10) also gives the a priori estimates on \(\| (\phi, \psi, \zeta, \phi_y, \psi_y, \zeta_y) \|, \| \partial^\alpha (\phi, \psi, \zeta) \|\) and \(\int \int \frac{\partial^\alpha \tilde{G}^2}{M^*} d\xi dx\) (|\alpha| = 2). In fact, from (3.1), (2.8) and (3.10), one has

\[
\| (\phi, \psi, \zeta, \phi_y, \psi_y, \zeta_y) \|^2 \leq C \| (\psi, \epsilon \psi_y, \theta, \psi_y, \theta_y) \|^2 + C \delta \epsilon^3 (1 + \epsilon^2 \tau)^{-\frac{3}{2}}
\]

\[
\leq C \left( \| (p_y - \bar{p}_y, \epsilon \psi u_{1y} - \epsilon \bar{p}_y u_{1y}, \psi_y, \bar{p}_y) \|^2 + \| (\bar{p}_y, \epsilon \bar{p}_y u_{1y}) \|^2 + \epsilon^2 \int \frac{\tilde{G}_y^2}{M^*} d\xi dy \right) + C \delta \epsilon^3 (1 + \epsilon^2 \tau)^{-\frac{3}{2}}
\]

\[
\leq C \left( \| (\phi, \psi_y, \zeta_y) \|^2 + \delta^2 \epsilon^2 \| (\phi, \psi, \zeta) \|^2 + \epsilon^2 \int \frac{\tilde{G}_y^2}{M^*} d\xi dy \right) + C \delta \epsilon^3 (1 + \epsilon^2 \tau)^{-\frac{3}{2}}
\]

\[
\leq C (\delta + \lambda_0^2) \epsilon^3,
\]  

(3.11)
where we have used the fact that
\[ \int \left( \int \xi_1^2 G_y \, dy \right) \, dx \leq C \int \int \frac{G^2_y}{M_\ast} \, dx \, dy. \]
To derive the a priori assumption on \( \| \partial^\alpha (\phi, \psi, \zeta) \| \), \(|\alpha| = 2\), we use the definition of \( \rho, \, m = \varepsilon \rho u \) and \( \rho(\theta + \frac{1}{2} |\varepsilon u|^2) \). Let \(|\alpha| = 2\), by (3.3), one can obtain
\[ \| \partial^\alpha (\mu, m, \rho \theta) \| \leq C \int \int \frac{\partial^\alpha f^2}{M_\ast} \, dx \, dy \]
\[ \leq C \int \int \left( \frac{\partial^\alpha f^2}{M_\ast} \right) \, dx \, dy \leq C \int \int \frac{\partial^\alpha (M - M)}{M_\ast} \, dx \, dy \]
This yields that
\[ \sum_{|\alpha| = 2} \| \partial^\alpha (\phi, \psi, \zeta) \|^2 \leq C \int \int \frac{\partial^\alpha f^2}{M_\ast} \, dx \, dy + \sum_{|\beta| = 1} \| \partial^\beta (\phi, \psi, \zeta) \|^2 + C \varepsilon^3 (1 + t)^{-\frac{3}{2}} \leq C (\lambda_0^2 + \delta) \varepsilon^3. \]
Finally, one has
\[ \varepsilon^2 \int \int \frac{\partial^\alpha \tilde{G}^2}{M_\ast} \, dx \, dy \leq C \int \int \frac{\partial^\alpha \tilde{f}^2}{M_\ast} \, dx \, dy + C \int \int \frac{\partial^\alpha (M - M)}{M_\ast} \, dx \, dy \]
\[ \leq C \int \int \frac{\partial^\alpha \tilde{f}^2}{M_\ast} \, dx \, dy + C \sum_{|\alpha| = 1, 2} \| \partial^\alpha (\phi, \psi, \zeta) \|^2 \leq C (\varepsilon_0 + \delta)^2 \varepsilon^3, \quad |\alpha| = 2. \]

### 3.2 Lower order estimate

We are now ready to derive the lower order estimate. Multiplying (3.3) by \( p, \phi \), (3.3) by \( \Psi_i \), (3.3) by \( \Psi \), (3.3) by \( \frac{1}{\varepsilon} W \) with \( p_\ast = \frac{1}{2} \) respectively and adding all the equations, one can obtain
\[ \left( \frac{p_\ast}{2} \phi^2 + \frac{1}{3p_\ast} W^2 + \frac{\beta}{2} \Psi_1^2 + \frac{1}{2} \sum_{i=2}^3 \Psi_i^2 \right)_\tau + \frac{4\mu(\theta)}{3} \Psi_1^2 + \sum_{i=2}^3 \frac{\mu(\theta)}{\bar{v}} \Psi_i^2 + \frac{2\kappa(\theta)}{3p_\ast \bar{v}} W^2 \]
\[ = -\frac{5}{2} \varepsilon^2 \tilde{N}_1 (\phi + \frac{2}{3p_\ast} W) + \frac{1}{2} \bar{v}_1 \Psi_1^2 + \bar{v} Q_1 \Psi_1 + \sum_{i=2}^3 Q_i \Psi_i + \frac{2}{3p_\ast} W Q_4 \]
\[ - \frac{2 \mu(\theta)}{3} \Psi_1^2 - \sum_{i=2}^3 \frac{\mu(\theta)}{\bar{v}} \Psi_i \Psi_i = \frac{2 \lambda(\theta)}{3p_\ast \bar{v}} W W_y + (\cdots)_y. \]

Here and in the sequel the notation \((\cdots)_y\) represents the term in the conservative form so that it vanishes after integration. Since it has no effect on the energy estimates, we do not write them out in detail.

Note that the term \( Q_1 \Psi_1 \) contains \((1 + t)^{-1} \Psi_1 \) which can not be controlled by the dissipation from the viscosity and heat conductivity. So is the term \( \tilde{N}_1 (\phi + \frac{2}{3p_\ast} W) \). As we will see later, an intrinsic dissipation associated with the profile is derived by the diagonal method and weighted energy estimate to control the above two terms. Let us consider the equations for the conservation of the mass, the first component of velocity and energy by defining
\[ V = (\Phi, \Psi_1, W)_t, \]
where \((\cdot, \cdot, \cdot)_t\) means the transpose of the vector \((\cdot, \cdot, \cdot)\). Then from (3.3), we have
\[ V_r + A_1 V_y = A_2 V_{yy} + A_3, \]
Let a large positive integer $n$ such that

$$
\sqrt{\frac{n}{5p_+}}.
$$

Direct computation shows that the eigenvalues of the matrix $A_1$ are $\lambda_1, 0, \lambda_3$. Here $\lambda_3 = -\lambda_1 = \sqrt{\frac{5p_+}{3p_+}}$. The corresponding normalized left and right eigenvectors can be chosen as

$$
l_1 = \sqrt{\frac{3}{10}}(-1, -\frac{5}{3\lambda_3}, \frac{2}{3p_+}), \quad l_2 = \sqrt{2/5}(1, 0, \frac{1}{p_+}), \quad l_3 = \sqrt{3/10}(-1, \frac{5}{3\lambda_3}, \frac{2}{3p_+}),
$$

$$
r_1 = \sqrt{3/10}(-1, -\lambda_3, p_+)^t, \quad r_2 = \sqrt{2/5}(1, 0, \frac{3}{2}p_+)^t, \quad r_3 = \sqrt{3/10}(-1, \lambda_3, p_+)^t,
$$

such that

$$
l_i r_j = \delta_{ij}, \quad i, j = 1, 2, 3, \quad LA_1 R = \Lambda = \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix},
$$

with

$$
L = (l_1, l_2, l_3)^t, \quad R = (r_1, r_2, r_3).
$$

Let

$$
B = LV = (b_1, b_2, b_3),
$$

then multiplying the equations \((3.16)\) by the matrix $L$ yields that

$$
B_r + \Lambda B_y = LA_2 RB_{yy} + 2LA_2 R_y B_y + [(L_r + AL_y)R + LA_2 R_{yy}] B + LA_2.
$$

A direct computation shows that $LA_2 R = A_1$ is a non-negative matrix. From \((3.17)\), we will apply weighted energy method to derive an intrinsic dissipation. Since we have assumed that $\tilde{\theta}_y > 0$. Let $v_1 = \frac{\tilde{\theta}}{\tilde{\theta}_y}$, then $|v_1 - 1| \leq C\delta$. Multiplying \((3.17)\) by $\tilde{B} = (v_1^n b_1, b_2, v_1^{-n} b_3)$ with a large positive integer $n$ which will be chosen later, we have

$$
\begin{align*}
\left(\frac{1}{2}v_1^n b_1^2 + \frac{1}{2}b_2^2 + \frac{1}{2}v_1^{-n} b_3^2\right) - \frac{v_1^n}{2} b_1^2 - \frac{v_1^{-n}}{2} b_3^2 + \tilde{B}_y A_1 B_y + \tilde{B} A_1 y \tilde{B}_y \\
- \frac{1}{2}v_1^{-n} (n \lambda_1 v_1 y + v_1 \lambda_3 y) b_3^2 + \frac{1}{2}v_1^{-n} (n \lambda_3 v_1 y - v_1 \lambda_3 y) b_3^2
\end{align*}
$$

(3.18)

$$
= 2BLA_2 R_y B_y + \tilde{B}[L_r + LA_2 R_{yy}] B + B \Lambda L_x R B + \tilde{B} LA_A + (\cdots)_x.
$$

Let

$$
E_1 = \int \left(\frac{p_+}{2} \Psi^2 + \frac{1}{3p_+} W^2 + \frac{v}{2} \Psi_1^2 + \frac{1}{2} \sum_{i=2}^3 \Psi_i^2\right) dy + \int \left(\frac{v_1^n}{2} b_1^2 + \frac{1}{2} b_2^2 + \frac{v_1^{-n}}{2} b_3^2\right) dy,
$$

$$
K_1 = \int \left(\frac{4\mu(\tilde{\theta})}{3} \Psi_1^2 + \sum_{i=2}^3 \frac{\mu(\tilde{\theta})}{v} \Psi_i^2 \right) + \frac{2\lambda(\tilde{\theta})}{3p_+ v} W_y^2 + B_y A_1 B_y) dy.
$$

Note that

$$
\left| \int (\tilde{B} - B) y A_1 B_y dy \right| \leq C\delta \int |B_y|^2 dy + C\delta^{-1} \int |\tilde{\theta}_y|^2 |B|^2 dy
\leq C\varepsilon^2 \delta (1 + t)^{-1} E_1 + C\delta K_1 + C\delta \int |\Phi_y|^2 dy.
$$

(3.19)
Thus the terms in the last second line of (3.18), $B\Lambda yRB$, $B\Lambda_{2}R yB_{y}$ and $\tilde{B}[L_{r}R + L\Lambda_{2}R_{y}]B$ satisfy the same estimate. For $B\Lambda yRB$ and $B\Lambda_{3}$, we need to use the explicit presentation. By the choice of the characteristic matrix $L$ and $R$, we have

\[
AL_{y}R = \frac{1}{2}\lambda_{3y} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad LA_{3} = \begin{pmatrix} \sqrt{\frac{2}{15}p_{+}}(\varepsilon^{2}\hat{N}_{1} + Q_{4}) - \sqrt{\frac{2}{5}Q_{4}} \\ \sqrt{\frac{2}{5}Q_{4}} \\ \sqrt{\frac{2}{15}p_{+}}(\varepsilon^{2}\hat{N}_{1} + Q_{4}) + \sqrt{\frac{2}{5}Q_{4}} \end{pmatrix}.
\]

Thus

\[
\tilde{B}AL_{y}RB = \frac{1}{2}\lambda_{3y}(v_{1}^{n}b_{1}^{2} + v_{1}^{-n}b_{1}b_{3} - v_{1}^{-n}b_{3}^{-2}),
\]

\[
\tilde{B}LA_{3} = \sqrt{\frac{2}{15}p_{+}}\varepsilon^{2}\hat{N}_{1}(v_{1}^{n}b_{1} + v_{1}^{-n}b_{3}) + q_{1}v_{1}^{n}b_{1} + q_{2}b_{2} + q_{3}v_{1}^{-n}b_{3},
\]

where

\[
q_{1} = \sqrt{\frac{2}{15}p_{+}}Q_{4} - \sqrt{\frac{5}{6}Q_{4}}, \quad q_{2} = \sqrt{\frac{2}{5}Q_{4}}, \quad q_{3} = \sqrt{\frac{2}{15}p_{+}}Q_{4} + \sqrt{\frac{5}{6}Q_{4}}.
\]

Combine (3.15), (3.18), (3.19)-(3.20), we have by choosing $n$ sufficiently large,

\[
E_{1r} + \frac{1}{2}K_{1} + 2\int |\hat{\theta}_{y}|(b_{1}^{2} + b_{3}^{2})|dy| \leq C\varepsilon^{2}\delta(1 + t)^{-1}(E_{1} + 1) + C\delta \int \Phi_{y}^{2}dy + I_{nf},
\]

where

\[
I_{nf} = \int vQ_{1}\Psi_{1}dy + \sum_{i=2}^{3}Q_{i}\Psi_{i}dy + \int \frac{2}{3p_{+}}WQ_{4}dy + \int (q_{1}v_{1}^{n}b_{1} + q_{2}b_{2} + q_{3}v_{1}^{-n}b_{3})dy.
\]

Here we have used the fact that

\[
-\Phi + \frac{2}{3p_{+}}W = \sqrt{5/6}(b_{1} + b_{3}),
\]

and

\[
\varepsilon^{2}\int |\hat{N}_{1}|(|b_{1}| + |b_{3}|)|dy| \leq C\delta \int |\hat{\theta}_{y}|(b_{1}^{2} + b_{3}^{2})|dy| + C\varepsilon^{2}\delta(1 + t)^{-1},
\]

and for $n$ large enough,

\[
-\frac{1}{2}v_{1}^{n-1}(n\lambda_{1}v_{1y} + 2v_{1}\lambda_{1y})b_{1}^{2} + \frac{1}{2}v_{1}^{-n-1}(n\lambda_{3}v_{1y} - 2v_{1}\lambda_{3y})b_{3}^{2} - \tilde{B}AL_{y}RB \geq 3|\hat{\theta}_{y}|(b_{1}^{2} + b_{3}^{2}).
\]

Even though $Q_{1}$ contains the term $R_{1}$ with the decay rate $\frac{\varepsilon^{2}}{177}$, the terms in (3.22) involving $Q_{1}$ have factor $b_{1}$ or $b_{3}$ because

\[
\Psi_{1} = \sqrt{3/10}\lambda_{3}(b_{3} - b_{1}).
\]

Thus the terms $vQ_{1}\Psi_{1}$, $q_{1}v_{1}^{n}b_{1}$ and $q_{3}v_{1}^{-n}b_{3}$ can be controlled by the intrinsic dissipation on $b_{1}$ and $b_{3}$ as shown later. The estimates on the other terms involving $Q_{i}$ ($i = 2, 3, 4$) are straightforward because from (2.34)-(2.40) and (3.4), they decay at least in the order of $\varepsilon^{3}(1 + t)^{-3/2}$. For brevity, we only estimate $\int vQ_{1}\Psi_{1}dy$ and $\int q_{2}b_{2}dy$ as follows for illustration.

**Estimate on** $\int vQ_{1}\Psi_{1}dy$: 
From (3.24), we have

\[
\int \bar{v}Q_1 \Psi_1 dy = \sqrt{\frac{3}{10}} \int \bar{v}Q_1 \lambda_3 (b_3 - b_1) dy. \quad (3.25)
\]

Here we only consider the integral

\[
I_1 = \int \bar{v}Q_1 \lambda_3 b_1 dy,
\]

and the other term in (3.25) can be estimated similarly. By the definition of \(Q_1\) in (3.4), we have

\[
I_1 = \int \bar{v} \lambda_3 b_1 \left[ \frac{A v}{3} \left( \frac{\mu(\theta)}{v} - \frac{\mu(\bar{\theta})}{\bar{v}} \right) u_{1y} + J_1 + \frac{2}{3v} Y \right] dy - \int \bar{v} \lambda_3 b_1 \bar{R}_1 dy - \varepsilon \int \bar{v} \lambda_3 b_1 \sum_{j=1}^{2} \int \xi_j^2 \Theta^j_1 d\xi dy
\]

\[
= I_1^1 + I_1^2 + I_1^3.
\]

Since

\[
\int \left| \frac{A v}{3} \left( \frac{\mu(\theta)}{v} - \frac{\mu(\bar{\theta})}{\bar{v}} \right) u_{1y} \right| |b_1| dy
\]

\[
\leq C(\delta + \lambda_0)(K_1 + \|\Phi_y\|^2) + C\delta \varepsilon^2(1 + t)^{-1} E_1 + C(\delta + \lambda_0)\|\psi_{1y}\|^2 + C\delta \varepsilon^5(1 + t)^{-2},
\]

and

\[
\int \left| (J_1 + \frac{Y}{v}) \right| |b_1| dy \leq C(\delta + \lambda_0)(K_1 + \|\Phi_y\|^2) + C\delta \varepsilon^2(1 + t)^{-1} E_1 + C\delta \varepsilon^5(1 + t)^{-\frac{5}{2}},
\]

we obtain

\[
I_1^1 \leq C(\delta + \lambda_0)(K_1 + \|\Phi_y\|^2) + C(\delta + \lambda_0)\|\psi_{1y}\|^2 L_2 + C\delta \varepsilon^2(1 + t)^{-1} E_1 + C\delta \varepsilon^5(1 + t)^{-\frac{5}{2}}. \quad (3.26)
\]

On the other hand, from (2.37), we have

\[
\bar{R}_1 = O(1) \delta \varepsilon^2(1 + t)^{-1} e^{-\frac{x^2}{4(\lambda_0)(1 + t)}}, \quad \text{as } x \to \pm \infty.
\]

From (2.18), \(\hat{\theta}_y\) satisfies

\[
|\hat{\theta}_y| = O(\delta) \varepsilon(1 + t)^{-\frac{1}{2}} e^{-\frac{x^2}{4(\lambda_0)(1 + t)}}, \quad \text{as } x \to \pm \infty.
\]

Thus, by (2.4) and the assumption on the profile, we have

\[
\kappa(\theta_\pm) = \frac{5}{2} \mu(\theta_\pm) > \frac{5}{4} \mu(\theta_\pm). \quad (3.27)
\]

Since \(a(\theta_\pm) = \frac{3\kappa(\theta_\pm)}{3\kappa(\theta_\pm)}\), \(b(\theta_\pm) = \max\{a(\theta_\pm), \frac{\mu(\theta_\pm)}{\lambda_\pm}\}\) and \(c(\theta_\pm) = \max\{a(\theta_\pm), \frac{1}{2} b(\theta_\pm)\}\), it follows from (3.27) that \(a(\theta_\pm) > \frac{3}{2} c(\theta_\pm)\), which leads to

\[
|I_1^2| \leq \frac{1}{16} \int |\hat{\theta}_y| b_1^2 dy + C\delta \varepsilon^2(1 + t)^{-1}. \quad (3.28)
\]
We now estimate the integral $I_1^3$. Let $M_s$ be a global Maxwellian with the state $(\rho_*, u_*, \theta_*)$ satisfying $\frac{1}{2} \theta < \theta_* < \theta$ and $|\rho - \rho_*| + |\rho u - u_*| + |\theta - \theta_*| \leq \eta_0$ so that Lemma 3.3 holds. Note that,

$$I_1^3 = -\varepsilon \int \bar{v} \lambda_3 b_1 \int \xi_1^2 \Theta_1^2 d\xi dy - \varepsilon \int \bar{v} \lambda_3 b_1 \int \xi_1^2 \Theta_1^2 d\xi dy =: I_1^{31} + I_1^{32}.$$  

(3.29)

The estimation on $I_1^{31}$ is straightforward by using the intrinsic dissipation on $b_1$ and (2.24).

$$|I_1^{31}| = | - \varepsilon \int \bar{v} \lambda_3 b_1 \int \xi_1^2 L_M^{-1} \frac{1}{\nu} P_1 (\xi_1 \bar{G}_y) - \varepsilon Q(\bar{G}, \bar{G}) | d\xi dy |
\leq C \int |b_1| \left( |(\varepsilon \bar{u}_y, \bar{\theta}_y)|^2 + |(\varepsilon \bar{u}_y, \bar{\theta}_y)| \right) dy 
\leq C \frac{\delta}{\nu} \int |b_1|^2 dy + C \delta \varepsilon^2 (1 + t)^{-1} + C \delta \| (\phi, \psi, y) \|^2.  
(3.30)

The estimation on $I_1^{32}$ is more complicated and it will be divided into five parts as follows. From (2.24), it holds that

$$I_1^{32} = \varepsilon \int \bar{v} \lambda_3 b_1 \int \xi_1^2 L_M^{-1} (G_r) d\xi dy + \varepsilon \int \bar{v} \lambda_3 b_1 \int \xi_1^2 \frac{\nu_1}{\nu} L_M^{-1} (G_g) d\xi dy 
- \varepsilon \int \bar{v} \lambda_3 b_1 \frac{1}{\nu} \int \xi_1^2 L_M^{-1} (P_1 (\xi_1 \bar{G}_y)) d\xi dy + \varepsilon \int \bar{v} \lambda_3 b_1 \int \xi_1^2 \frac{1}{\nu} L_M^{-1} (Q(\bar{G}, \bar{G})) d\xi dy 
+ 2 \varepsilon \int \bar{v} \lambda_3 b_1 \int \xi_1^2 L_M^{-1} (Q(\bar{G}, \bar{G})) d\xi dy =: \sum_{i=1}^{5} I_1^{32i}.  
(3.31)

For the integral $I_1^{321}$, one has

$$I_1^{321} = -\varepsilon \int \bar{v} \lambda_3 b_1 \int \xi_1^2 L_M^{-1} (\bar{G}_r) d\xi dy - \varepsilon \int \bar{v} \lambda_3 b_1 \int \xi_1^2 L_M^{-1} (\bar{G}_r) d\xi dy =: I_1^{3211} + I_1^{3212}.  
(3.32)

Note that the linearized operator $L_M^{-1}$ satisfies that, for any $h \in N^\perp$,

$$(L_M^{-1} h)_r = L_M^{-1} (h_r) - 2 L_M^{-1} \{Q(L_M^{-1} h, M_r)\}, 
(L_M^{-1} h)_y = L_M^{-1} (h_y) - 2 L_M^{-1} \{Q(L_M^{-1} h, M_y)\}.  
(3.33)

Then it follows that

$$I_1^{3211} = -\varepsilon \int \bar{v} \lambda_3 b_1 \int \xi_1^2 (L_M^{-1} \bar{G}) d\xi dy - 2 \varepsilon \int \bar{v} \lambda_3 b_1 \int \xi_1^2 L_M^{-1} (Q(L_M^{-1} \bar{G}, M_r)) d\xi dy 
= -\varepsilon \left( \int \bar{v} \lambda_3 b_1 \int \xi_1^2 L_M^{-1} (\bar{G}) d\xi dy 
+ \varepsilon \int (\bar{v} \lambda_3 b_1)_r \int \xi_1^2 \bar{G} d\xi dy \right)  
- 2 \varepsilon \int \bar{v} \lambda_3 b_1 \int \xi_1^2 L_M^{-1} (Q(L_M^{-1} \bar{G}, M_r)) d\xi dy.  
(3.34)

The Hölder inequality and Lemma 3.3 yield that

$$\int \xi_1^2 L_M^{-1} (\bar{G}) d\xi \leq C \int \xi_1^4 \nu (|\xi|)^{-1} M_r d\xi \cdot \int \frac{\nu (|\xi|)}{M_r} |L_M^{-1} \bar{G}|^2 d\xi \leq C \int \frac{\nu (|\xi|)}{M_r} |\bar{G}|^2 d\xi.  

Moreover, from Lemmas 3.2, 3.4 one has

$$\int \xi_1^2 L_M^{-1} (Q(L_M^{-1} \bar{G}, M_r)) d\xi \leq C \int \frac{\nu (|\xi|)}{M_r} |L_M^{-1} (Q(L_M^{-1} \bar{G}, M_r))|^2 d\xi 
\leq C \int \frac{\nu (|\xi|)}{M_r} |Q(L_M^{-1} \bar{G}, M_r)|^2 d\xi \leq C \int \frac{\nu (|\xi|)}{M_r} |L_M^{-1} \bar{G}|^2 d\xi \cdot \int \frac{\nu (|\xi|)}{M_r} |M_r|^2 d\xi 
\leq C (v_1^2 + \varepsilon u_1 + \theta_1^2) \int \frac{\nu (|\xi|)}{M_r} |\bar{G}|^2 d\xi.  
(3.35)$$
Combining (3.34)-(3.35) gives that

\[ I_1^{3211} \leq -\left( \varepsilon \int \tilde{v}_\lambda b_1 \int \xi_1^2 L_M^{-1} \tilde{G} d\xi dy \right) + C \beta \| (\Phi_\tau, \Psi_\tau, W_\tau) \|^2 + C \delta \varepsilon^2 (1 + t)^{-1} E_1 \\
+ C_\beta \varepsilon^2 \int \int \frac{\nu(|\xi|)}{M_a} |\tilde{G}|^2 d\xi dy + C \lambda_0^2 |(\phi, \psi, \zeta)|^2, \]  

(3.36)

where and in the sequel \( \beta \) is a small positive constant to be chosen later and \( C_\beta \) is a positive constant depending on \( \beta \). By the definition of \( \tilde{G} \) in (2.24), similar to the estimate in (3.30), one has

\[ |I_1^{3212}| = |\varepsilon \int \tilde{v}_\lambda b_1 \int \xi_1^2 L_M^{-1} (\tilde{G}_x) d\xi dy| \]
\[ \leq C \delta \varepsilon^2 (1 + t)^{-1} E_1 + C \delta \varepsilon^3 (1 + t)^{-\frac{3}{2}} + C \delta \| (\phi, \psi, \zeta) \|^2. \]  

(3.37)

Substituting (3.36) and (3.37) into (3.32) implies that

\[ I_1^{321} \leq -\left( \varepsilon \int \tilde{v}_\lambda b_1 \int \xi_1^2 L_M^{-1} \tilde{G} d\xi dy \right) + C \beta \| (\Phi_\tau, \Psi_\tau, W_\tau) \|^2 + C \delta \varepsilon^2 (1 + t)^{-1} E_1 \\
+ C_\beta \varepsilon^2 \int \int \frac{\nu(|\xi|)}{M_a} |\tilde{G}|^2 d\xi dy + C(\delta + \lambda_0) \| (\phi, \psi, \zeta) \|^2 + C \delta \varepsilon^3 (1 + t)^{-\frac{3}{2}}. \]  

(3.38)

The estimation on \( I_1^{32i} \) \((i = 2, 4, 5)\) is straightforward by using the Cauchy inequality and Lemmas 3.2-3.4. First, it holds that

\[ |I_1^{322}| \leq C \delta \varepsilon^2 (1 + t)^{-1} E_1 + C \lambda_0 K_1 + C \varepsilon^2 \int \int \frac{\nu(|\xi|)}{M_a} |\tilde{G}_y|^2 d\xi dy \\
+ C \delta \varepsilon^3 (1 + t)^{-\frac{3}{2}} + C \delta \varepsilon^2 |(\phi, \psi, \zeta)_y|^2. \]  

(3.39)

Since

\[ |\int \xi_1^2 L_M^{-1} \{Q(\tilde{G}, \tilde{G})\} d\xi|^2 \leq C \int \frac{\nu(|\xi|)}{M_a} |L_M^{-1} \{Q(\tilde{G}, \tilde{G})\}|^2 d\xi \]
\[ \leq C \int \frac{\nu(|\xi|)}{M_a} |Q(\tilde{G}, \tilde{G})|^2 d\xi \leq C \int \frac{\nu(|\xi|)}{M_a} |L_M^{-1} \tilde{G}|^2 d\xi \int \frac{\nu(|\xi|)}{M_a} |\tilde{G}|^2 d\xi \]
\[ \leq C(\varepsilon \tilde{u}_x, \tilde{\theta}_x)^2 \int \frac{\nu(|\xi|)}{M_a} |\tilde{G}|^2 d\xi, \]

and

\[ |\int \xi_1^2 L_M^{-1} \{Q(\tilde{G}, \tilde{G})\} d\xi| \leq C(\int \frac{\nu(|\xi|)}{M_a} |L_M^{-1} \{Q(\tilde{G}, \tilde{G})\}|^2 d\xi)^\frac{1}{2} \]
\[ \leq C(\int \frac{\nu(|\xi|)}{M_a} |Q(\tilde{G}, \tilde{G})|^2 d\xi)^\frac{1}{2} \leq C \int \frac{\nu(|\xi|)}{M_a} |\tilde{G}|^2 d\xi, \]

it follows that

\[ |I_1^{324}| + |I_1^{325}| \leq C(\delta + \lambda_0) \varepsilon^2 \int \int \frac{\nu(|\xi|)}{M_a} |\tilde{G}|^2 d\xi dy + C \delta \varepsilon^2 (1 + t)^{-1} E_1. \]  

(3.40)

The estimate on \( I_1^{323} \) is similar to the one for \( I_1^{321} \). First, notice that

\[ P_1(\xi_1 \tilde{G}) = \{P_1(\xi_1 \tilde{G})\}_y + \sum_{j=0}^4 (\xi_1 \tilde{G}, \chi_j)P_1(\chi_{jy}). \]
From (3.33) and Lemmas 2.3-4.4, we have

\[ I_1^{33} = \varepsilon \int \left( \bar{v} \lambda_3 b_1 \right) y \int \xi^2_4 L_M^{1-1} \{ P_1(\xi \bar{G}) \} d\xi dy - \varepsilon \int \bar{v} \lambda_3 b_1 \int \xi^2_4 L_M^{1-1} \{ \sum_{j=0}^4 \langle \xi \bar{G}, \chi_j \rangle P_1(\chi_j y) \} d\xi dy \]

\[-2\varepsilon \int \bar{v} \lambda_3 b_1 \int \xi^2_4 L_M^{1-1} \{ Q(\xi^2_4 L_M^{1-1} \{ P_1(\xi \bar{G}) \}, M_y) \} d\xi dy \]

\[ \leq C_\beta \varepsilon^2 \int \int \frac{\nu(|\xi|)}{M_y} |\bar{G}|^2 d\xi dy + C_\delta \varepsilon^2 (1 + t)^{-1} E_1 + C(\lambda_0 + \beta)(K_1 + \| \Phi_y \|^2) + C\lambda_0 \|(\phi_y, \psi_y, \zeta_y)\|^2, \]

(3.41)

where we have used the fact that

\[ |\langle \xi \bar{G}, \chi_j \rangle|^2 \leq C \int \frac{\nu(|\xi|)}{M_y} \bar{G}^2 d\xi. \]

Substituting (3.38), (3.39), (3.40) and (3.41) into (3.31) gives that

\[ I_1^{32} \leq -\varepsilon \int \bar{v} \lambda_3 b_1 \int \xi^2_4 L_M^{1-1} \bar{G} d\xi dy + C_\delta \varepsilon^2 (1 + t)^{-1} E_1 + C(\lambda_0 + \beta)(K_1 + \| \Phi_y \|^2) + C\beta \|(\Phi, \Psi, W)\|_\tau^2 + C_\beta \varepsilon^2 \int \int \frac{\nu(|\xi|)}{M_y} \bar{G}^2 d\xi dy + C\varepsilon^2 \int \int \frac{\nu(|\xi|)}{M_y} \bar{G}_y^2 d\xi dy \]

\[ + C(\delta + \lambda_0) \sum_{|\alpha|=1} \| \partial^\alpha (\phi, \psi, \zeta) \|^2 + C\delta \varepsilon^3 (1 + t)^{-\frac{3}{2}}, \]

(3.42)

which implies by (3.29) and (3.30) that

\[ I_1 \leq -\varepsilon \int \bar{v} \lambda_3 b_1 \int \xi^2_4 L_M^{1-1} \bar{G} d\xi dy + C_\delta \varepsilon^2 (1 + t)^{-1} E_1 + C(\lambda_0 + \beta)(K_1 + \| \Phi_y \|^2) + C\beta \|(\Phi, \Psi, W)\|_\tau^2 + C_\beta \varepsilon^2 \int \int \frac{\nu(|\xi|)}{M_y} \bar{G}^2 d\xi dy + C\varepsilon^2 \int \int \frac{\nu(|\xi|)}{M_y} \bar{G}_y^2 d\xi dy \]

(3.43)

\[ + C(\delta + \lambda_0) \sum_{|\alpha|=1} \| \partial^\alpha (\phi, \psi, \zeta) \|^2 + C\delta \varepsilon^3 (1 + t)^{-1} + \frac{1}{16} \int |\bar{b}|^2 dy, \]

which completes the estimate on the term \( \int \bar{v} Q_1 \Psi_1 dx. \)

**Estimate on \( \int q_2 b_2 dy. \)**

Notice that the profile has no intrinsic dissipation on \( b_2. \) Fortunately, it holds that \( q_2 = \sqrt{\frac{\varrho_2}{\varrho}} \) and \( Q_4 \) has the decay rate as \( \varepsilon^3 (1 + t)^{-\frac{3}{2}}. \) Thus the estimation on \( \int q_2 b_2 dy \) can be directly obtained even though there is no intrinsic dissipation on \( b_2. \) For example,

\[ |\int \varepsilon \bar{v}_1 R_1 b_2 dy| \leq C_\delta \varepsilon^2 (1 + t)^{-1} E_1 + C_\delta \varepsilon^3 (1 + t)^{-\frac{3}{2}}, \]

\[ |\int \varepsilon^2 \bar{v}_1 b_2^2 \Theta_1 d\xi dy| \leq C_\delta \varepsilon^2 (1 + t)^{-1} E_1 + C(\delta + \lambda_0) \sum_{|\alpha|=1} \| \partial^\alpha (\phi, \psi, \zeta) \|^2 \]

\[ + C\delta \varepsilon^2 \int \frac{\nu(|\xi|)}{M_y} \bar{G}_x^2 d\xi dy + C \int \frac{\nu(|\xi|)}{M_y} (\bar{G}_x^2 + \bar{G}_y^2) d\xi dy + C\delta \varepsilon^3 (1 + t)^{-\frac{3}{2}}. \]
And the term \( \varepsilon \int \int \xi_1 |\xi|^2 \Theta_1^2 b_2 d\xi dy \) can be estimated similarly as for \( I_1^2 \) where the intrinsic dissipation on \( b_1, b_3 \) is not needed. Notice also that all the other terms in \( q_2 \) are of higher order. Therefore, one has

\[
I_2 = \int q_2 b_2 d\xi dy \leq (\varepsilon \int \int \hat{A}(\xi, b_2) L_M^{-1} \tilde{G} d\xi dy) + C \delta \varepsilon^2 (1 + t)^{-1} E_1 + C(\lambda_0 + \beta)(K_1 + \|\Phi_y\|^2)
+ C\beta \|\Phi, \Psi, W\|_r^2 + C \varepsilon^2 \int \int \frac{\nu(|\xi|)}{M_s}|\tilde{G}|^2 d\xi dy + C\varepsilon^2 \int \int \frac{\nu(|\xi|)}{M_s}|\tilde{G}_y|^2 d\xi dy
+ C(\delta + \lambda_0) \sum_{|\alpha|=1} \|\partial^\alpha (\phi, \psi, \zeta)\|^2 + C \delta \varepsilon^2 (1 + t)^{-1},
\]

(3.44)

where \( \hat{A}(\xi, b_2) \) is a linear function of \( b_2 \) and a polynomial function of \( \xi \). Using (3.33), (3.44) and (3.21), we get

\[
E_{1r} + \left( \int \int \varepsilon \hat{A}_1(\xi, B) L_M^{-1} \tilde{G} d\xi dy \right) + \frac{1}{4} K_1 + \int |\partial_y|(b_1^2 + b_3^2)dy \leq C \delta \varepsilon^2 (1 + t)^{-1} E_1 + C(\lambda_0 + \beta)(K_1 + \|\Phi_y\|^2) + C \beta \|\Phi, \Psi, W\|_r^2
+ C \varepsilon^2 \int \int \frac{\nu(|\xi|)}{M_s}|\tilde{G}|^2 d\xi dy + C\varepsilon^2 \int \int \frac{\nu(|\xi|)}{M_s}|\tilde{G}_y|^2 d\xi dy
+ C(\delta + \lambda_0) \sum_{|\alpha|=1} \|\partial^\alpha (\phi, \psi, \zeta)\|^2 + C \delta \varepsilon^2 (1 + t)^{-1},
\]

(3.45)

where we have used the smallness of \( \delta \) and \( \varepsilon_0 \). Here \( \hat{A}_1 \) is a linear function of \( B = (b_1, b_2, b_3)^t \) and a polynomial function of \( \xi \).

Note that \( K_1 \) does not contain the norm \( \|\Phi_y\|^2 \). To complete the lower order inequality, we have to estimate \( \Phi_y \). From (3.32), we have

\[
\frac{4\mu(\theta)}{3b} \Phi_y - \Psi_1 - \frac{p_+}{v} \Phi_y = \frac{2}{3b} W_y - \frac{8\mu(\theta)}{15p_+v} \varepsilon^2 \hat{N}_1y - Q_1.
\]

(3.46)

Multiplying (3.46) by \( \Phi_y \) yields

\[
(\frac{2\mu(\theta)}{3b} \Phi_y^2)_r - (\frac{2\mu(\theta)}{3b})_r \Phi_y^2 - \Phi_y \Psi_1_r + \frac{p_+}{v} \Phi_y = \left( \frac{2}{3b} W_y - \frac{8\mu(\theta)}{15p_+v} \varepsilon^2 \hat{N}_1y - Q_1 \right) \Phi_y.
\]

Since

\[
\Phi_y \Psi_1 = (\Phi_y \Psi_1)_r - (\Phi_r \Psi_1)_y + \Psi_1^2 - \frac{2}{5p_+} \varepsilon^2 \hat{N}_1 \Psi_1 y,
\]

we can obtain

\[
\left( \int \frac{2\mu(\theta)}{3b} \Phi_y^2 dy \right) + \int \frac{p_+}{2v} \Phi_y^2 dy \leq C \|\Psi_1, W_y\|^2 + C \delta \varepsilon^3 (1 + t)^{-3/2} + \int Q_1^2 dy.
\]

(3.47)

The formula [3.4] for \( Q_1 \) and the Cauchy inequality directly yield

\[
\int Q_1^2 dy \leq C(\delta + \lambda_0)(K_1 + \|\Phi_y\|^2) + C\lambda_0 \sum_{|\alpha|=1} \|\partial^\alpha (\phi, \psi, \zeta)\|^2
+ C \delta \varepsilon^3 (1 + t)^{-3/2} + C \int \int \xi_1^2 \Theta_1 d\xi|^2 dy.
\]

(3.48)
And using Lemmas 3.2 and 3.4 implies
\[
\int | \xi^2 \Theta_1 d\xi^2 d\xi dy \leq C\epsilon^2 \int \int \frac{\nu(|\xi|)}{M_*} (|\hat{G}_y|^2 + |\hat{G}_\tau|^2) d\xi dy + C(\delta + \lambda_0)\epsilon^4 \int \int \frac{\nu(|\xi|) |\hat{G}|^2}{M_*} d\xi dy \\
+ C(\delta + \lambda_0) \sum_{|\alpha|=1} \| \partial^\alpha (\phi, \psi, \zeta) \|^2 + C\delta \epsilon^3 (1 + t)^{-3/2}.
\]
(3.49)

Substituting (3.48) and (3.49) into (3.47) yields
\[
\left( \int \frac{2\mu(\theta)}{3\theta} \Phi_y^2 - \Phi_y \Phi_1 dy \right)_\tau + \int \frac{p_+}{4\theta} \Phi_y^2 dy \\
\leq C_2 K_1 + C_2 \epsilon^2 \int \int \frac{\nu(|\xi|)}{M_*} (|\hat{G}_y|^2 + |\hat{G}_\tau|^2) d\xi dy + C_2(\delta + \lambda_0)\epsilon^4 \int \int \frac{\nu(|\xi|) |\hat{G}|^2}{M_*} d\xi dy \\
+ C_2(\delta + \lambda_0) \sum_{|\alpha|=1} \| \partial^\alpha (\phi, \psi, \zeta) \|^2 + C_2 \delta \epsilon^3 (1 + t)^{-3/2}.
\]
(3.50)

Multiplying (3.5) by \( \epsilon^2 \frac{\hat{G}}{M_*} \), one can obtain
\[
\left( \epsilon^2 \frac{\nu \hat{G}^2}{2M_*} \right)_\tau - \epsilon^2 \frac{\nu \hat{G}}{M_*} L_M \hat{G} = \left\{ - \frac{1}{R\theta} P_1 [\xi_1 (|\xi - \epsilon u|^2 1 - \epsilon y + \xi \cdot \epsilon y) M] \\
+ \epsilon u_1 G_y - P_1 (\xi_1 G_y) + \epsilon v Q(G, G) - v G_\tau \right\} \cdot \epsilon \frac{\hat{G}}{M_*}.
\]
(3.51)

Integrating (3.51) with respect to \( \xi \) and \( y \) and using the Cauchy inequality and Lemmas 3.2 and 3.4, one has
\[
\left( \epsilon^2 \int \int \frac{1}{2M_*} |\hat{G}|^2 d\xi dy \right)_\tau + 3\epsilon^2 \frac{\hat{G}}{4} \int \int \frac{\nu(|\xi|) |\hat{G}|^2}{M_*} d\xi dy \\
\leq C_3 \delta \epsilon^3 (1 + t)^{-3/2} + C_3 \sum_{|\alpha|=1} \| \partial^\alpha (\phi, \psi, \zeta) \|^2 + C_3 \epsilon^2 \int \int \frac{\nu(|\xi|) |\hat{G}|^2}{M_*} d\xi dy.
\]
(3.52)

On the other hand, since \( (\Phi, \Psi, W)_\tau \) can be represented by \( (\Phi, \Psi, W)_y \) and \( (\Phi, \Psi, W)_{yy} \) from the equation (3.3), we can get an estimate for \( (\Phi, \Psi, W)_\tau \) as follows.
\[
\| (\Phi, \Psi, W)_\tau \|^2 \leq C_4 (K_1 + \| \Phi_y \|^2) + C_4 \sum_{|\alpha|=1} \| \partial^\alpha (\phi, \psi, \zeta) \|^2 + C_4 \delta \epsilon^3 (1 + t)^{-\frac{3}{2}} \\
+ C_4 \epsilon^2 \int \int \frac{\nu(|\xi|) (|\hat{G}_y|^2 + |\hat{G}_\tau|^2) d\xi dy + C_4(\delta + \lambda_0)\epsilon^4 \int \int \frac{\nu(|\xi|) |\hat{G}|^2}{M_*} d\xi dy}.
\]
(3.53)

Now we can complete the lower order estimate. Since \( \hat{A}_1 \) is a linear function of the vector \( B \) and a polynomial of \( \xi \), we get
\[
|\epsilon | \int \int \hat{A}_1 (\xi, B) L^{-1}_M \hat{G} d\xi dy | \leq \frac{1}{4} E_1 + C \epsilon^2 \int \int \frac{|\hat{G}|^2}{M_*} d\xi dy.
\]

We choose large constants \( C_1 > 1, C_2 > 1, C_3 > 1 \) and small constant \( \beta \) such that
\[
C_1 E_1 + C_1 \epsilon \int \int \hat{A}_1 L^{-1}_M \hat{G} d\xi dy + C_2 \int \frac{2\mu(\theta)}{3\theta} \Phi_y^2 - \Phi_y \Phi_1) dy + C_3 \epsilon^2 \int \int \frac{|\hat{G}|^2}{2M_*} d\xi dy \\
\geq \frac{1}{2} C_1 E_1 + C_2 \int \frac{\mu(\theta)}{3\theta} \Phi_y^2 dy + \frac{C_4}{4} \epsilon^2 \int \int \frac{G^2}{M_*} d\xi dy,
\]
Hence, by multiplying \( (3.45) \) by \( \bar{\Theta} \),

\[
\int C_1 C_3 \delta + \lambda_0 - C_3 \bar{C}_1 - C_2 \varepsilon^2 (\delta + \lambda_0) \geq \frac{\sigma}{4} C_3.
\]

Hence, by multiplying \((3.45)\) by \( \bar{C}_1\), \((3.50)\) by \( \bar{C}_2\), \((3.52)\) by \( \bar{C}_3\), \((3.53)\) by \( C_1(\delta + \varepsilon_0 + \varepsilon) \bar{C}_1 \) and adding all these inequalities together, we have

\[
E_{2r} + K_2 + \int |\hat{\theta}_y| (b^2 + b_2^2) dy \leq C_5 \delta \varepsilon^2 (1 + t)^{-1} (E_2 + 1)
\]

\[
+ C_5 \varepsilon^2 \int \int \frac{\nu(|\xi|)}{M_2} (|\bar{G}_y|^2 + |\bar{G}_x|^2) d\xi dy + C_5 \sum_{|\alpha| = 1} \| \partial^\alpha (\phi, \psi, \zeta) \|^2,
\]

where

\[
E_2 = \bar{C}_1 E_1 + \bar{C}_1 \int \varepsilon A_1 L_M^{-1} \bar{G} d\xi dy + \bar{C}_2 \int (\frac{2\mu(\bar{\theta})}{3v} \Phi_y - \Phi_y) dy + \bar{C}_3 \varepsilon^2 \int \int \frac{|\bar{G}|^2}{2M_\varepsilon} d\xi dy
\]

\[
K_2 = \frac{\bar{C}_1}{16} K_1 + \bar{C}_2 \int \frac{p^2}{\bar{M}} \Phi_y dy + \|(\Phi, \Psi, W)_\tau\|^2 + \frac{\sigma}{8} C_3 \varepsilon^2 \int \int \frac{\nu(|\xi|)|\bar{G}|^2}{M_\varepsilon} d\xi dy.
\]
and $N_i, F_i \ (i = 1, 2, 3, 4)$ are defined in \([2.26], [2.31]\) and \([3.55]\) respectively and $\tilde{N}_i \ (i = 1, 2, 3, 4)$ is the corresponding function of $N_i \ (i = 1, 2, 3, 4)$ by substituting the variable $(v, u, \theta)$ by the profile $(\bar{v}, \bar{u}, \bar{\theta})$.

We will use the convex entropy for the fluid system to obtain the first-order derivative estimates of $(\Phi_y, \Psi_y, W_y)$. Multiplying \([3.56]_2\) by $\psi_1$ and \([3.56]_3\) by $\psi_i$, one has

$$
\left(\frac{1}{2} \sum_{i=1}^{3} \psi_i^2 \right)_\tau - (p - \bar{p})\psi_{1y} + \frac{4\varepsilon}{3} \left( \frac{\mu(\theta)}{v} u_{1y} - \frac{\mu(\bar{\theta})}{\bar{v}} \bar{u}_{1y} \right) \psi_{1y} + \varepsilon \left( \frac{\mu(\theta)}{v} u_{iy} - \frac{\mu(\bar{\theta})}{\bar{v}} \bar{u}_{iy} \right) \psi_{iy} = \sum_{i=1}^{3} Q_{4+i}\psi_i + (\cdots)_y.
$$

Since $p - \bar{p} = \frac{2}{3} \bar{\theta}(-1 + \frac{1}{v}) + \frac{2\zeta}{3\bar{v}}$, we obtain

$$
\left(\frac{1}{2} \sum_{i=1}^{3} \psi_i^2 \right)_\tau - \frac{2}{3} \bar{\theta}(-1 + \frac{1}{v}) \phi_r + \frac{2}{3\bar{v}} \zeta \psi_{1y} + \frac{4}{3} \frac{\mu(\theta)}{v} \psi_{1y}^2 + \sum_{i=2}^{3} \frac{\mu(\theta)}{v} \psi_{iy} + \frac{4\varepsilon}{3} \left( \frac{\mu(\theta)}{v} - \frac{\mu(\bar{\theta})}{\bar{v}} \right) \bar{u}_{1y} \psi_{1y} + \varepsilon \left( \frac{\mu(\theta)}{v} - \frac{\mu(\bar{\theta})}{\bar{v}} \right) \bar{u}_{iy} \psi_{iy} = \sum_{i=1}^{3} \psi_i Q_{4+i} + \frac{2\zeta}{3} \bar{\theta}(-1 + \frac{1}{v}) \bar{N}_{1y} + (\cdots)_x.
$$

Let

$$
\hat{\Phi}(s) = s - 1 - \ln s,
$$

then it holds that

$$
\left\{ \frac{2}{3} \bar{\theta} \hat{\Phi}(\frac{v}{\bar{v}}) \right\}_\tau = \frac{2}{3} \bar{\theta} \hat{\Phi}(\frac{v}{\bar{v}}) + \frac{2}{3} \bar{\theta}(-1 + \frac{1}{v}) \bar{\phi}_r + \frac{2}{3} \bar{\theta}(-\frac{v}{\bar{v}^2} + \frac{1}{\bar{v}}) \bar{\psi}_r + \frac{2}{3} \bar{\theta}(-1 + \frac{1}{v}) \bar{\psi}_r
$$

$$
= \frac{2}{3} \bar{\theta}(-1 + \frac{1}{v}) \bar{\phi}_r - \bar{\psi}_r \hat{\Phi}(\frac{v}{\bar{v}}) \bar{\phi}_r + \bar{\psi}_r \hat{\Phi}(\frac{v}{\bar{v}}),
$$

(3.58)

where

$$
\hat{\Psi}(s) = s^{-1} - 1 + \ln s.
$$

It is easy to check that $\hat{\Phi}(1) = \hat{\Psi}(1) = \hat{\Phi}(1) = \hat{\Psi}(1) = 0$ and $\hat{\Phi}(s), \hat{\Psi}(s)$ are strictly convex around $s = 1$. Substituting (3.58) into (3.57) yields that

$$
\left(\frac{1}{2} \sum_{i=1}^{3} \psi_i^2 + \frac{2}{3} \bar{\theta} \hat{\Phi}(\frac{v}{\bar{v}}) \right)_\tau = \frac{2}{3\bar{v}} \zeta \psi_{1y} + \frac{4}{3} \frac{\mu(\theta)}{v} \psi_{1y}^2 + \sum_{i=2}^{3} \frac{\mu(\theta)}{v} \psi_{iy} + \frac{4\varepsilon}{3} \left( \frac{\mu(\theta)}{v} - \frac{\mu(\bar{\theta})}{\bar{v}} \right) \bar{u}_{1y} \psi_{1y} + \varepsilon \left( \frac{\mu(\theta)}{v} - \frac{\mu(\bar{\theta})}{\bar{v}} \right) \bar{u}_{iy} \psi_{iy} = \sum_{i=1}^{3} \psi_i Q_{4+i} + \frac{2\zeta}{3} \bar{\theta}(-1 + \frac{1}{v}) \bar{N}_{1y} - \bar{\psi}_r \hat{\Phi}(\frac{v}{\bar{v}}) \bar{\phi}_r + \bar{\psi}_r \hat{\Phi}(\frac{v}{\bar{v}}),
$$

(3.59)

On the other hand, multiplying \([3.56]_4\) by $\zeta / \theta$, it holds that

$$
\frac{\zeta}{\theta} \zeta_r + \varepsilon (pu_{1y} - \bar{p}u_{1y}) \zeta \theta \bar{\theta} = \left( \frac{\kappa(\theta)}{v} \theta_y - \frac{\kappa(\bar{\theta})}{\bar{v}} \bar{\theta}_y \right) \zeta \theta \bar{\theta} + Q_8 \zeta \theta.
$$

(3.60)

One can compute that

$$
\frac{\zeta}{\theta} \zeta_r = \left( \bar{\theta} \hat{\Phi}(\frac{\theta}{\bar{\theta}}) \right)_\tau + \bar{\phi}_r \hat{\Psi}(\frac{\theta}{\bar{\theta}}) = \left( \bar{\theta} \hat{\Phi}(\frac{\theta}{\bar{\theta}}) \right)_\tau + O(1)\delta \varepsilon^2 (1 + t)^{-1} |\zeta|^2,
$$

(3.61)

$$
\varepsilon (pu_{1y} - \bar{p}u_{1y}) \zeta \theta \bar{\theta} = \frac{2\zeta}{3v} \psi_{1y} + \varepsilon (p - \bar{p})u_{1y} \zeta \theta = \frac{2\zeta}{3v} \psi_{1y} + O(1)\delta \varepsilon^2 (1 + t)^{-1} |(\psi, \zeta)|^2,
$$

(3.62)
and
\[
\left( \frac{\kappa(\theta)}{v} \frac{\partial_y}{v} - \frac{\kappa(\theta)}{v} \right) \zeta_y = \left( \cdots \right)_y - \frac{\partial \kappa(\theta)}{v \theta^2} \zeta_y - \frac{\kappa(\theta) \partial_y \zeta_y}{v \theta^2} - \frac{\partial \theta_y \zeta_y - |\partial_y|^2 \zeta}{\theta^2} \left( \frac{\kappa(\theta)}{v} - \frac{\kappa(\theta)}{v} \right) \tag{3.63}
\]
\[
\leq \left( \cdots \right)_y - \frac{3 \partial \kappa(\theta)}{4 v \theta^2} \zeta_y^2 + C \delta \varepsilon^2 (1 + t)^{-1} |(\phi, \zeta)|^2.
\]
Substituting (3.61), (3.62) and (3.63) into (3.60) yields that
\[
\left( \frac{\partial \Phi(\theta)}{\theta} \right)_\tau + \frac{2 \zeta}{3 \theta} \psi_1 + \frac{3 \partial \kappa(\theta)}{4 v \theta^2} \zeta_y^2 \leq \left( \cdots \right)_y + C \delta \varepsilon^2 (1 + t)^{-1} |(\phi, \zeta)|^2 + |Q_h \zeta| \tag{3.64}
\]
Combining (3.64) and (3.59) and using Cauchy inequality, one has
\[
E_{4r} + \frac{3}{4} K_3 \leq C \delta \varepsilon^2 (1 + t)^{-1} E_3 + C \delta \varepsilon^4 (1 + t)^{-2} + C \varepsilon^2 (1 + t)^{-1} \int |\partial_y| (b_1^2 + b_3^2) dy + \int \frac{2}{5} \varepsilon^2 N_{1y} \left( - \frac{\partial}{v \theta} \phi + \frac{\zeta}{\theta} \right) dy + \sum_{i=1}^{4} |I_{i+2}|, \tag{3.65}
\]
where
\[
\begin{align*}
E_3 &= \int \left( \frac{1}{3} \sum_{i=1}^{3} \psi_i^2 + R \frac{\partial \Phi(\theta)}{\theta} + \frac{\partial \Phi(\theta)}{\theta} \right) dy, \\
K_3 &= \int \left( \frac{4}{3} \frac{\mu(\theta)}{v} \psi_i^2 + \sum_{i=2}^{3} \frac{\mu(\theta)}{v} \psi_i^2 + \frac{3 \partial \kappa(\theta)}{4 v \theta^2} \zeta^2 \right) dy,
\end{align*}
\]
and
\[
\begin{align*}
I_3 &= \int \varepsilon^2 (N_{4y} - \bar{N}_{4y}) \psi_1 dy + \int \varepsilon^2 F_{4y} \psi_1 dy - \varepsilon \int \xi_1^2 \Theta_{1y} \psi_1 d\xi dy, \\
I_{2+i} &= \int \varepsilon^2 (N_{iy} - \bar{N}_{iy}) \psi_i dy + \int \varepsilon^2 F_{iy} \psi_1 dy - \varepsilon \int \xi_1 \xi_i \Theta_{iy} \psi_i d\xi dy, \quad i = 2, 3, \\
I_6 &= \int \varepsilon^2 (N_{1y} - \bar{N}_{1y}) \frac{\zeta}{\theta} dy + \int \varepsilon^2 F_{1y} \frac{\zeta}{\theta} dy - \frac{1}{2} \varepsilon \int \xi_1 |\xi|^2 \Theta_{1y} \frac{\zeta}{\theta} d\xi dy + \sum_{i=1}^{3} \varepsilon^2 u_i \int \xi_i \xi_1 \Theta_{iy} \frac{\zeta}{\theta} d\xi dy.
\end{align*}
\]
In the estimate of (3.65), we have used the estimate like
\[
\begin{align*}
| \int \varepsilon^2 \bar{N}_{iy} - \bar{R}_{iy} \psi_i dy | &= | \int \varepsilon^2 \bar{N}_{iy} - \bar{R}_{iy} \psi_i dy | \leq C \int | \varepsilon^2 \bar{N}_{iy} - \bar{R}_{iy} | | b_1 | + | b_3 | dy, \\
&\leq C \delta \varepsilon^4 (1 + t)^{-2} + C \varepsilon^2 (1 + t)^{-1} \int |\partial_y| (b_1^2 + b_3^2) dy.
\end{align*}
\]
Now, we calculate the terms on the right hand side of (3.65). Firstly, a direct calculation yields
\[
- \frac{\partial}{v \theta} \phi + \frac{\zeta}{\theta} = \frac{1}{2} \frac{2}{3 \theta} \psi_1 W_1 - \Phi_y + O(1) \left[ |(\phi, \psi, \zeta)|^2 + |\varepsilon \bar{u}_1 \psi_1| + \delta \varepsilon (1 + t)^{-1} \right],
\]
and thus by combining (3.23) and (3.24), it holds that
\[
\begin{align*}
| \int \frac{2}{5} \varepsilon^2 N_{1y} \left( - \frac{\partial}{v \theta} \phi + \frac{\zeta}{\theta} \right) dy | \\
\leq C \delta \varepsilon^2 (1 + t)^{-1} E_3 + C \delta \varepsilon^4 (1 + t)^{-2} + C \int \varepsilon^2 |(\bar{N}_{1y} \frac{1}{\theta})_y| + |\bar{N}_{1y} \varepsilon \bar{u}_1| \cdot |(b_1, b_3)| dy \\
\leq C \delta \varepsilon^2 (1 + t)^{-1} E_3 + C \delta \varepsilon^4 (1 + t)^{-2} + C \varepsilon^2 (1 + t)^{-1} \int |\partial_y| (b_1^2 + b_3^2) dy. \tag{3.67}
\end{align*}
\]
By using the definition of $N_4$ and $F_4$ in (3.55), one can obtain

$$
\left| \int \varepsilon^2 (N_{4y} - \bar{N}_{4y}) \psi_1 dy + \int \varepsilon^3 F_{4y} \psi_1 dy \right| \leq \frac{1}{352} K_3 + \int \varepsilon^4 (N_{4} - \bar{N}_{4})^2 dy + \varepsilon^6 F_{4}^2 dy
$$

$$
\leq \frac{1}{32} K_3 + C\delta \| \phi_y \|^2 + C\delta \varepsilon^2 (1 + t)^{-1} E_3 + C\delta \varepsilon^4 (1 + t)^{-2}.
$$

(3.68)

And by using Lemma 3.2, 3.3, one has that

$$
|\varepsilon| \int \int \xi^2 \Theta \psi_1 d\xi dy \leq \frac{1}{32} K_3 + C\varepsilon^2 \int \int \xi^2 \Theta^2 d\xi^2 dy
$$

$$
\leq \frac{1}{32} K_3 + C\varepsilon^2 \int \int \left( \frac{\nu(|\xi|)}{M_\ast} |(\bar{G}_y, \bar{G}_r)|^2 d\xi dy + C\delta \sum_{|\alpha| = 1} \| \partial^\alpha (\phi, \psi, \zeta) \|^2 + C\delta \varepsilon^2 (1 + t)^{-1} E_3
$$

$$
+ C\delta \varepsilon^4 (1 + t)^{-2} + C\varepsilon^4 \left[ \delta (1 + t)^{-1} + \| \int \frac{|\bar{G}|^2}{M_\ast} d\xi \|_{L^\infty} \right] \int \int \frac{\nu(|\xi|) |\bar{G}|^2}{M_\ast} d\xi dy.
$$

(3.69)

Combining (3.68) and (3.69) yields that

$$
I_3 \leq \frac{1}{32} K_3 + C\varepsilon^2 \int \int \left( \frac{\nu(|\xi|)}{M_\ast} |(\bar{G}_y, \bar{G}_r)|^2 d\xi dy + C\delta \sum_{|\alpha| = 1} \| \partial^\alpha (\phi, \psi, \zeta) \|^2 + C\delta \varepsilon^2 (1 + t)^{-1} E_3
$$

$$
+ C\delta \varepsilon^4 (1 + t)^{-2} + C\varepsilon^4 \left[ \delta (1 + t)^{-1} + \| \int \frac{|\bar{G}|^2}{M_\ast} d\xi \|_{L^\infty} \right] \int \int \frac{\nu(|\xi|) |\bar{G}|^2}{M_\ast} d\xi dy.
$$

(3.70)

Similarly, $I_4, I_5, I_6$ can be controlled by the right hand side of (3.70). Substituting (3.67) and (3.70) into (3.65) gives that

$$
E_{3r} + \frac{1}{2} K_3 \leq C_6 \varepsilon^2 \int \int \left( \frac{\nu(|\xi|)}{M_\ast} |(\bar{G}_y, \bar{G}_r)|^2 d\xi dy + C_6 \delta \sum_{|\alpha| = 1} \| \partial^\alpha (\phi, \psi, \zeta) \|^2
$$

$$
+ C_6 \delta \varepsilon^2 (1 + t)^{-1} E_3 + C_6 \delta \varepsilon^4 (1 + t)^{-2} + C_6 \varepsilon^2 (1 + t)^{-1} \int \int |\partial_y (b_1^2 + b_2^2) d\xi dy
$$

$$
+ C_6 \varepsilon^4 \left[ \delta (1 + t)^{-1} + \| \int \frac{|\bar{G}|^2}{M_\ast} d\xi \|_{L^\infty} \right] \int \int \frac{\nu(|\xi|) |\bar{G}|^2}{M_\ast} d\xi dy.
$$

(3.71)

Note that the norm $\| \phi_y \|$ is not included in $K_3$ (see (3.66)). To complete the first-order derivative estimate, we follow the same way as to estimate $\Phi_y$ in the previous section. By using the (3.56), we can rewrite the equation (3.56)_{12} as

$$
\frac{4}{3} \frac{\mu(\bar{\theta})}{\bar{v}} \psi_{yy} - \psi_{1ry} - (p - \bar{p})_y = -\frac{4}{3} \frac{\mu(\bar{\theta})}{\bar{v}} \bar{N}_{1y} + \frac{4}{3} \left( \frac{\mu(\bar{\theta})}{\bar{v}} \right)_{yy} \psi_{1y} + \varepsilon \int \xi^2 \Theta_{1y} d\xi + R_{1y}.
$$

(3.72)

Multiplying the equation (3.72) by $\phi_y$, one has

$$
\frac{2}{3} \left( \frac{\mu(\bar{\theta})}{\bar{v}} \phi_y^2 \right)_{y} + \frac{2}{3} \left( \frac{\mu(\bar{\theta})}{\bar{v}} \right)_{yy} \phi_y^2 - \psi_{1y} \phi_y - (p - \bar{p})_y \phi_y
$$

$$
= \left\{ -\frac{4}{5} \frac{\mu(\bar{\theta})}{\bar{v}} \bar{N}_{1yy} - \frac{4}{3} \left( \frac{\mu(\bar{\theta})}{\bar{v}} \right)_{yy} \psi_{1yy} + \frac{4}{3} \left( \frac{\mu(\bar{\theta})}{\bar{v}} - \frac{\mu(\bar{\theta})}{\bar{v}} u_{1yy} \right) + \varepsilon \int \xi^2 \Theta_{1y} d\xi + R_{1y} \right\} \phi_y.
$$

(3.73)

Since

$$
-(p - \bar{p})_y = \bar{p} \phi_y - \frac{2}{3e} \xi_y + \left( \frac{p}{v} - \frac{\bar{p}}{v} \right) v_y - \frac{2}{3} \left( \frac{1}{v} - \frac{1}{\bar{v}} \right) \theta_y,
$$

(3.74)
and
\[ \phi_y \psi_{1\tau} = (\phi_y \psi_1)_\tau - (\phi_{1\tau} \psi_1)_y + \psi_{1y}^2 - \frac{3 \varepsilon^2}{5} \bar{N}_{1y} \psi_{1y}, \]
integrating (3.73) with respect to \( y \) and using the Cauchy inequality yield
\[
\left( \int \frac{2 \mu(\theta)}{3 \bar{v}} \phi_y^2 - \phi_y \psi_{1y} dy \right) + \int \frac{\bar{p}}{3 \bar{v}} \phi_y^2 dy \leq C_7 K_3 + C_7 \delta \varepsilon^2 (1 + t)^{-1} E_3 + C_7 \delta \varepsilon^5 (1 + t)^{-\frac{3}{2}}
+ C_7 (\delta + \lambda_0) \varepsilon \sum_{|\alpha| = 2} \| \partial^\alpha (\phi, \psi, \zeta) \|^2 + C_7 \delta \| \partial_{\tau} (\phi, \psi, \zeta) \|^2 + C \varepsilon^2 \int | \int \xi_1^2 \Theta_{1y} d\xi|^2 dy, \tag{3.74}
\]
where we have used the fact that
\[
\int \left( \frac{\bar{p}}{v} - \frac{\bar{p}}{v} \right) \psi_y - \frac{2}{3} \left( 1 - \frac{1}{v} \right) \theta_y \| \phi_y \| dy \leq \frac{1}{8} \| \phi_y \|^2 + C \delta \varepsilon^2 (1 + t)^{-1} E_3 + C K_3.
\]
It follows from (2.24) and Lemmas 3.2-3.3 that
\[ \varepsilon^2 \int | \int \xi_1^2 \Theta_{1y} d\xi|^2 dy \leq C \delta \varepsilon^5 (1 + t)^{-\frac{3}{2}} + C \delta \varepsilon^4 (1 + t)^{-1} \sum_{|\alpha| = 1} \| \partial^\alpha (\phi, \psi, \zeta) \|^2
+ C \delta \varepsilon^2 \sum_{|\alpha| = 2} \| \partial^\alpha (\phi, \psi, \zeta) \|^2 + C J_3, \tag{3.75}
\]
where
\[
J_3 = \left[ \varepsilon^2 \sum_{|\alpha| = 2} \int \left( \frac{\nu(|\xi|)}{M_s} \right) \| \partial^\alpha \tilde{G} \|^2 d\xi dy + \varepsilon^4 (\delta + \lambda_0) \sum_{|\alpha| = 1} \int \left( \frac{\nu(|\xi|)}{M_s} \right) \| \partial^\alpha \tilde{G} \|^2 d\xi dy
+ \varepsilon^2 (\delta \varepsilon (1 + t)^{-1} + \int \int \frac{|G_{xy}|^2}{M_s} d\xi dy + \int \int \frac{|G_{yy}|^2}{M_s} d\xi dy \right) \int \int \left( \frac{\nu(|\xi|)}{M_s} \right) \| \tilde{G} \|^2 d\xi dy \right]. \tag{3.76}
\]
To estimate \( \| \partial_{\tau} (\phi, \psi, \zeta) \|^2 \), we use (3.56) to obtain
\[
\| \partial_{\tau} (\phi, \psi, \zeta) \|^2 \leq C_6 (K_3 + \| \phi_y \|^2) + C_8 \delta \varepsilon^2 (1 + t)^{-1} E_3 + C_8 \delta \varepsilon^5 (1 + t)^{-\frac{3}{2}}
+ C_8 \sum_{|\alpha| = 2} \| \partial^\alpha (\phi, \psi, \zeta) \|^2 + C_8 J_3. \tag{3.77}
\]
Thus we choose large constants \( \bar{C}_4 \) and \( \bar{C}_5 \) so that
\[
\bar{C}_4 E_3 + \bar{C}_5 \int \frac{2 \mu(\theta)}{3 \bar{v}} \phi_y^2 - \phi_y \psi_1 dy \geq \frac{\bar{C}_4}{2} E_3 + \bar{C}_5 \int \frac{\mu(\theta)}{3 \bar{v}} \phi_y^2 dy,
\]
and
\[
\frac{1}{2} \bar{C}_4 - \bar{C}_5 C_7 - C_8 \geq \frac{1}{8} \bar{C}_4, \quad \bar{C}_5 \int \frac{\bar{p}}{2 \bar{v}} \phi_y^2 dy - C_8 \| \phi_y \|^2 \geq \frac{\bar{C}_5}{4} \int \frac{\bar{p}}{\bar{v}} \phi_y^2 dy.
\]
Let
\[
E_4 = \bar{C}_4 \varepsilon^{-2} E_3 + \bar{C}_5 \varepsilon^{-2} \int \frac{2 \mu(\theta)}{3 \bar{v}} \phi_y^2 - \phi_y \psi_1 dy,
\]
and
\[
K_4 = \frac{1}{8} \bar{C}_4 \varepsilon^{-2} K_3 + \frac{\bar{C}_5}{4} \varepsilon^{-2} \int \frac{\bar{p}}{\bar{v}} \phi_y^2 dy + \varepsilon^{-2} \| (\phi_{\tau}, \psi_{\tau}, \zeta_{\tau}) \|^2.
\]
Then from (3.71), (3.74), (3.75) and (3.77), we have the following estimate on the $(\phi, \psi, \zeta)$

$$E_4 + K_4 \leq C_9 \delta \varepsilon^2 (1 + t)^{-1} E_4 + C_9 \delta \varepsilon^2 (1 + t)^{-2} + C_9 \varepsilon^{-2} \sum_{|\alpha|=2} \|\partial^\alpha (\phi, \psi, \zeta)\|^2$$

$$+ C_9 (1 + t)^{-1} \int |\hat{\partial}_y| (b_1^2 + b_3^2) dy + C_9 \varepsilon^{-2} J_3, \quad (3.78)$$

where $J_3$ is defined in (3.76).

Define

$$E_5 = E_4 + \varepsilon \int \int \frac{\tilde{G}}{2M_s} d\xi dy, \quad K_5 = K_4 + \frac{\bar{\theta}}{4} \varepsilon \int \frac{\nu(|\xi|)}{M_s} \tilde{G}^2 d\xi dy. \quad (3.79)$$

Then from (3.78) and (3.52), one has

$$E_{5r} + K_5 \leq C_{10} \delta \varepsilon^2 (1 + t)^{-1} E_5 + C_{10} \delta \varepsilon^2 (1 + t)^{-3} + C_{10} \varepsilon^{-2} \sum_{|\alpha|=2} \|\partial^\alpha (\phi, \psi, \zeta)\|^2$$

$$+ C_{10} (\delta + \lambda_0) \varepsilon^2 \sum_{|\alpha|=1} \int \int \frac{\nu(|\xi|)}{M_s} |\partial^\alpha \tilde{G}|^2 d\xi dy + C_{10} \sum_{|\alpha|=2} \int \int \frac{\nu(|\xi|)}{M_s} |\partial^\alpha \tilde{G}|^2 d\xi dy$$

$$+ C_{10} (1 + t)^{-1} \int |\hat{\partial}_y| (b_1^2 + b_3^2) dy. \quad (3.80)$$

Next we derive the higher order derivative estimate. Applying $\hat{\partial}_y$ to (3.56) yields that

$$\begin{aligned}
\phi_{yy} - \psi_{yy} &= -\frac{3}{5} \varepsilon^2 \hat{N}_{1yy}, \\
\psi_{yy} + p_+ \psi_y - p_+ \phi_{yy} &= \frac{4 \varepsilon}{3} \left( \frac{\mu(\theta)}{v} u_{1yy} - \frac{\mu(\bar{\theta})}{\bar{v}} \bar{u}_{1yy} \right) + Q_y, \\
\psi_{yy} &= \varepsilon \left( \frac{\mu(\theta)}{v} u_{yy} - \frac{\mu(\bar{\theta})}{\bar{v}} \bar{u}_{yy} \right) + Q_{8+i}, \quad i = 2, 3, \\
\xi_{yy} + p_+ \psi_{yy} &= \left( \frac{\kappa(\theta)}{v} \theta_y - \frac{\kappa(\bar{\theta})}{\bar{v}} \bar{\theta}_y \right) + \frac{\varepsilon}{5} \hat{N}_{1yy} + Q_{12},
\end{aligned}$$

where

$$Q_y = \frac{p - \bar{p}}{v} \phi_y + \left( \frac{p}{v} - \frac{\bar{p}}{\bar{v}} \right) \phi_{yy} + O(1)(|\bar{\psi}_{yy}| \cdot |(\phi, \zeta)| + |\phi y_{yy}|)$$

$$- \frac{4}{3} \left( \frac{\theta}{v^2} \bar{v} \bar{\theta}_y - \frac{\theta}{\bar{v}^2} \bar{v} \bar{\theta}_y \right) + \frac{4}{3} \left( \frac{\bar{\theta}}{\bar{v}^2} \bar{v} \bar{\theta}_y - \frac{\bar{\theta}}{v^2} \bar{v} \bar{\theta}_y \right) - \varepsilon \int \xi_1^2 \Theta_{1yy} d\xi - \bar{R}_{1yy},$$

$$Q_{i+8} = -\varepsilon \int \xi_1 \xi_i \Theta_{1yy} d\xi - \bar{R}_{iyy}, \quad i = 2, 3,$$

$$Q_{12} = -\varepsilon \bar{u}_{1yy}(p - \bar{p}) - \varepsilon (p_y u_y - \bar{p_y} \bar{u}_y) + (p_+ - \bar{p}) \psi_{1yy} + Q_{13y}$$

$$- \frac{1}{2} \varepsilon \int \xi_1 \xi_i \Theta_{1yy} d\xi + \sum_{i=1}^3 \varepsilon^2 (u_i \int \xi_1 \xi_i \Theta_{1yy} d\xi)$$

$$Q_{13} = \frac{4 \mu(\theta)}{v} \varepsilon^2 u_{1yy}^2 + \varepsilon^2 \sum_{i=2}^3 \frac{\mu(\theta)}{v} u_{1yy}^2 - \bar{H}_{1yy} - \bar{R}_{4yy} + \frac{1}{2} (|\varepsilon \bar{u}|^2)_{tr} + \varepsilon \bar{p}_y \bar{u}_1.$$
Multiplying (3.80) by $p_+\phi_y$, (3.80) by $\bar{v}_1\psi_y$, (3.80) by $\psi_y$, (3.80) by $\zeta_y$, we have

\[
\left[ \int \left( \frac{p_+}{2} \phi_y^2 + \frac{\bar{v}}{2} \psi_y^2 + \sum_{i=2}^3 \psi_i^2 + \frac{1}{2} \xi^2 \right) dy \right] + \frac{3}{4} \int \left[ \frac{4\mu(\theta)}{3v} \psi_{1yy}^2 + \sum_{i=2}^3 \frac{\mu(\theta)}{v} \psi_{iyy}^2 + \frac{\kappa(\theta)}{v} \xi^2 \right] dy 
\leq C\delta \varepsilon (1 + t)^{-\frac{1}{2}} \|(\phi_y, \psi_y, \zeta_y)^2 + C\delta \varepsilon^3 (1 + t)^{-\frac{3}{2}} \|(\phi, \psi)^2 + C\|(\phi, \psi)\|\|(\phi, \psi, \zeta)^2
\]
\[+ C\|(\phi_y, \psi_y, \zeta_y)^2 + C\delta \varepsilon^5 (1 + t)^{-\frac{3}{2}} + C\delta \varepsilon^2 \sum_{|\alpha|=2} \|\partial^\alpha (\phi, \psi, \zeta)^2 + C J_3, \quad (3.81)\]

where we have used (3.75) in the last inequality.

Let

\[E_6 = \int \left[ \frac{p_+}{2} \phi_y^2 + \frac{\bar{v}}{2} \psi_y^2 + \sum_{i=2}^3 \psi_i^2 + \frac{1}{2} \xi^2 \right] dy, \quad K_6 = \int \left[ \frac{4\mu(\theta)}{3v} \psi_{1yy}^2 + \sum_{i=2}^3 \frac{\mu(\theta)}{v} \psi_{iyy}^2 + \frac{\kappa(\theta)}{v} \xi^2 \right] dy, \]

then (3.81) implies

\[E_{6tr} + \frac{1}{2} K_6 \leq C_{11} \delta \varepsilon (1 + t)^{-\frac{1}{2}} \sum_{|\alpha|=1} \|\partial^\alpha (\phi, \psi, \zeta)^2 + C_{11} \delta \varepsilon^3 (1 + t)^{-\frac{3}{2}} \|(\phi, \psi, \zeta)^2 + C_{11} \delta \varepsilon^5 (1 + t)^{-\frac{3}{2}}
\]
\[+ C_{11}\|(\phi, \psi)\|\|(\phi_y, \psi_y, \zeta_y)^2 + C_{11}\|(\phi, \psi, \zeta)^2 \| + C_{11} \delta \varepsilon^2 \sum_{|\alpha|=2} \|\partial^\alpha (\phi, \psi, \zeta)^2 + C_{11} J_3. \quad (3.82)\]

To get the estimate on $\phi_{yy}$, we use the momentum equation (2.8). Applying $\partial_y$ on (2.8), it holds that

\[\psi_{1yr} + (p - \bar{p})_{yy} + \varepsilon \bar{u}_{1yr} + \bar{p}_{yy} = -\varepsilon \int \xi_1^2 G_{yy} d\xi. \quad (3.83)\]

Note that

\[(p - \bar{p})_{yy} = -\frac{p}{v} \phi_{yy} + \frac{2}{3v} \zeta_{yy} - \frac{1}{v} (p - \bar{p}) \psi_{yy} - \frac{\phi}{v} \bar{p}_{yy} - \frac{2v}{v} (p - \bar{p})_y - \frac{2\bar{p}}{v} \phi_y, \]

then multiplying (3.83) by $-\phi_{yy}$ and integrating the reduced equation with respect to $y$ give that

\[\left( - \int \psi_{1y} \phi_{yy} dy \right)_r + \int \frac{p}{2v} \phi_{yy}^2 dy \leq C_{12} K_6 + C_{12} \delta \varepsilon (1 + t)^{-\frac{1}{2}} \|(\phi, \psi, \zeta_y)^2 + C_{12} \delta \varepsilon^3 (1 + t)^{-\frac{3}{2}} \|(\phi, \psi, \zeta)^2 + C_{12} \delta \varepsilon^5 (1 + t)^{-\frac{3}{2}}
\]
\[+ C_{12} \|(\phi_y, \psi_y, \zeta_y)^2 + C_{12} \delta \varepsilon^2 \int \frac{\mu(\xi)}{M_\ast} |\tilde{G}_{yy}|^2 d\xi dy. \quad (3.84)\]

To estimate $(\phi, \psi, \zeta)_{yrt}$ and $(\phi, \psi, \zeta)_{rr}$, we also use the original fluid-type equation (2.8). Here we only consider the term $\int \psi_{1yr}^2 dy$ because the other terms can be estimated similarly. It follows from (2.8) that

\[\psi_{1yr} = -(p - \bar{p})_{yy} - \varepsilon \bar{u}_{1yr} - \bar{p}_{yy} - \varepsilon \int \xi_1^2 G_{yy} d\xi. \quad (3.85)\]
By (3.85) and using the Cauchy inequality, it holds that

\[
\|\psi_{1\tau}\|^2 \leq C_{13}(K_6 + \|\phi_{yy}\|^2) + C_{13}\delta\varepsilon(1 + t)^{-\frac{3}{2}}\|\phi, \psi, \zeta\|^2 + C_{13}\delta\varepsilon^3(1 + t)^{-\frac{3}{2}}\|\phi, \psi, \zeta\|^2 + C_{13}\delta\varepsilon^5(1 + t)^{-\frac{5}{2}} + C_{13}\|\phi_y, \psi_y, \zeta_y\|_{L^1}^\frac{10}{3} + C_{13}\varepsilon^2\int \frac{\nu(|\xi|)}{M_*}\hat{G}_{yy}^2d\xi dy. \tag{3.86}
\]

Let $C_6$ and $C_7$ be suitably large constants, then it follows from (3.82), (3.84) and (3.86) that

\[
C_7\left(C_6E_6 - \int \psi_{1y}\phi_{yy}d\tau\right) + \sum_{|\alpha|=2}\|\partial^\alpha(\phi, \psi, \zeta)\|^2 \\
\leq C_{14}\delta\varepsilon(1 + t)^{-\frac{3}{2}}\sum_{|\alpha|=2}\|\partial^\alpha(\phi, \psi, \zeta)\|^2 + C_{14}\delta\varepsilon^3(1 + t)^{-\frac{3}{2}}\|\phi, \psi, \zeta\|^2 + C_{14}\delta\varepsilon^5(1 + t)^{-\frac{5}{2}} + C_{14}\|\phi_y, \psi_y, \zeta_y\|_{L^3}^3 + C_{14}\|\phi_y, \psi_y, \zeta_y\|_{L^3}^2 + C_{14}J_3, \tag{3.87}
\]

where $J_3$ is defined in (3.76).

To close the a priori argument, we need to estimate the non-fluid component $\partial^\alpha\hat{G}, |\alpha| = 1, 2$. Applying $\partial_y$ on (3.5), we have

\[
v\hat{G}_{yy} - vL_M\hat{G}_y = -v_y\hat{G}_y + v_yL_M\hat{G}_y + 2Q(M_y, \hat{G}) - \left\{\frac{1}{R\theta}P_1[\xi_1(\frac{\xi - \varepsilon u}{2\theta} + \frac{1}{\varepsilon}\psi_y + \xi \cdot \frac{1}{\varepsilon}\psi_y)M]\right\}_y \\
+ \left\{\varepsilon u_1\hat{G}_y - P_1(\xi_1\hat{G}_y) + \varepsilon vQ(G, G) - v\hat{G}_y\right\}_y. \tag{3.88}
\]

Multiplying (3.88) by $\frac{\hat{G}_y}{M_*}$, then integrating the reduced equation with respect to $\xi$ and $y$ and using the Cauchy inequality and Lemmas 3.2-3.4, we have

\[
\left(\int \int \frac{v|\hat{G}_y|^2}{2M_*}d\xi dy\right)_\tau + \frac{3\sigma}{4} \int \int \frac{\nu(|\xi|)}{M_*}|\hat{G}_y|^2d\xi dy \\
\leq C \sum_{|\alpha|=2} \int \int \frac{\nu(|\xi|)}{M_*}|\partial^\alpha\hat{G}|^2d\xi dy + C_3\delta\varepsilon^3(1 + t)^{-5/2} + C_3\varepsilon^2 \sum_{|\alpha|=2}\|\partial^\alpha(\phi, \psi, \zeta)\|^2 \\
+ C_3\delta(1 + t)^{-1} \sum_{|\alpha|=1}\|\partial^\alpha(\phi, \psi, \zeta)\|^2 + C_3\varepsilon^2 \sum_{|\alpha|=1}\|\partial^\alpha(\phi, \psi, \zeta)\|^6 \\
+ C_3 \sum_{|\alpha|=1}\|\partial^\alpha(v, u, \theta)\|_{L^\infty}^2 \int \int \frac{\nu(|\xi|)}{M_*}|\hat{G}|^2d\xi dy. 
\]

Similarly, we can obtain the estimate for $\hat{G}_\tau$. Hence, one obtains that

\[
\left(\sum_{|\alpha|=1} \int \int \frac{v|\partial^\alpha\hat{G}|^2}{2M_*}d\xi dy\right)_\tau + \frac{3\sigma}{4} \sum_{|\alpha|=1} \int \int \frac{\nu(|\xi|)}{M_*}|\partial^\alpha\hat{G}|^2d\xi dy \\
\leq C \sum_{|\alpha|=2} \int \int \frac{\nu(|\xi|)}{M_*}|\partial^\alpha\hat{G}|^2d\xi dy + C_3\delta\varepsilon^3(1 + t)^{-5/2} + C_3\varepsilon^2 \sum_{|\alpha|=2}\|\partial^\alpha(\phi, \psi, \zeta)\|^2 \\
+ C_3\delta(1 + t)^{-1} \sum_{|\alpha|=1}\|\partial^\alpha(\phi, \psi, \zeta)\|^2 + C_3\varepsilon^2 \sum_{|\alpha|=1}\|\partial^\alpha(\phi, \psi, \zeta)\|^6 \\
+ C_3 \sum_{|\alpha|=1}\|\partial^\alpha(v, u, \theta)\|_{L^\infty}^2 \int \int \frac{\nu(|\xi|)}{M_*}|\hat{G}|^2d\xi dy. \tag{3.89}
\]
Finally, we need the highest order estimate to control \( \sum_{|\alpha|=2} \int \int \frac{\nu(|\xi|)}{M_*} |\partial^{\alpha}\tilde{G}|^2 d\xi dy \) and \( \int \psi_1 \phi_{yy}dy \) in (3.87). To estimate \( \int \psi_1 \phi_{yy}dy \), it is sufficient to study the a priori estimate for \( \sum_{|\alpha|=2} \int \int \frac{v|\partial^{\alpha}\tilde{f}|^2}{2M_*} d\xi dy \) due to (3.12) and (3.13). Applying \( \partial^{\alpha} \), \( |\alpha| = 2 \) to (3.8), one obtains that

\[
v\partial^{\alpha}\tilde{f}_r - \varepsilon vL_M \partial^{\alpha} \tilde{G} - \varepsilon u_1 \partial^{\alpha} \tilde{f}_y + \xi_1 \partial^{\alpha} \tilde{f}_y = -\partial^{\alpha} v\tilde{f}_r + \varepsilon \partial^{\alpha} u_1 \tilde{f}_y - \sum_{|\beta|=1} [\partial^{\alpha-\beta} v\partial^{\beta} \tilde{f}_r - \varepsilon \partial^{\alpha-\beta} u_1 \partial^{\beta} \tilde{f}_y] + \varepsilon \partial^{\alpha} [vL_M \tilde{G} - vL_M G_0] + \varepsilon^2 \partial^{\alpha} [vQ(G, G) - vQ(G_0, G_0)] + \partial^{\alpha} \left[ -\phi \tilde{f}_r + \psi \tilde{f}_y - \varepsilon v\tilde{R}_f \right].
\]

(3.90)

Multiplying (3.90) by \( \frac{\partial^{\alpha}\tilde{f}}{M_*} \), integrating the reduced equation with respect to \( \xi \) and \( y \) and using the Cauchy inequality and Lemmas 3.2.3.5, similar to the argument used in [24], one gets that

\[
\left( \sum_{|\alpha|=2} \int \int \frac{v|\partial^{\alpha}\tilde{f}|^2}{2M_*} d\xi dy \right) + \frac{3\sigma}{4} \sum_{|\alpha|=2} \varepsilon^2 \int \int \frac{\nu(|\xi|)}{M_*} |\partial^{\alpha}\tilde{G}|^2 d\xi dy \leq C_3 \delta \varepsilon^5 (1 + t)^{-5/2} + C_3 (\delta + \eta_0 + \frac{1}{\lambda_0}) \sum_{|\alpha|=2} \|\partial^{\alpha}\phi, \psi, \zeta\|^2 + C \sum_{|\alpha|=2} \|\partial^{\alpha}\phi, \psi, \zeta\||^{10} \frac{\eta}{\sigma} + C_3 \delta \varepsilon^3 (1 + t)^{-\frac{3}{2}} \|\phi, \psi, \zeta\|^2
\]

\[
+ \frac{C_3}{\lambda_0} \left( \delta \varepsilon^4 (1 + t)^{-2} + \sum_{|\alpha|=1} \|\partial^{\alpha}\phi, \psi, \zeta\| \right) \int \int \frac{\nu(|\xi|)}{M_*} |\tilde{G}|^2 d\xi dy
\]

\[
+ C_3 (\delta + \eta_0 + \frac{1}{\lambda_0}) \varepsilon^2 \int \int \frac{\nu(|\xi|)}{M_*} |\tilde{G}|^2 d\xi dy.
\]

(3.91)

Choose large constants \( \tilde{C}_6 > 1 \) and \( \tilde{C}_9 > 1 \) such that

\[
E_7 = \frac{\tilde{C}_6}{\varepsilon} \left( \tilde{C}_9 E_6 - \int \psi_1 \phi_{yy}dy \right) + \frac{1}{\varepsilon} \sum_{|\alpha|=1} \int \int \frac{v|\partial^{\alpha}\tilde{f}|^2}{2M_*} d\xi dy + \frac{C_9}{\varepsilon^3} \sum_{|\alpha|=2} \int \int \frac{|\partial^{\alpha}\tilde{f}|^2}{2M_*} d\xi dy
\]

\[
\geq \frac{c_1}{\varepsilon^2} \left( \|\phi, \psi, \zeta\|_y^2 + \sum_{|\alpha|=2} \|\partial^{\alpha}\phi, \psi, \zeta\|^2 \right) + \frac{1}{\varepsilon^2} \sum_{|\alpha|=2} \int \int \frac{|\partial^{\alpha}\tilde{f}|^2}{2M_*} d\xi dy
\]

\[
+ \frac{c_1}{\varepsilon} \sum_{|\alpha|=1} \int \int \frac{|\partial^{\alpha}\tilde{G}|^2}{2M_*} d\xi dy - C\delta (1 + t)^{-\frac{3}{2}}.
\]

Let

\[
K_7 = \frac{\tilde{C}_8}{4\varepsilon^3} \sum_{|\alpha|=2} \|\partial^{\alpha}\phi, \psi, \zeta\|^2 + \frac{\sigma}{4\varepsilon} \sum_{1 \leq |\alpha| \leq 2} \int \int \frac{\nu(|\xi|)}{M_*} |\partial^{\alpha}\tilde{G}|^2 d\xi dy.
\]

Then from (3.87), (3.89) and (3.91), one obtains that

\[
E_{7r} + K_7 \leq C \delta \varepsilon^2 (1 + t)^{-5/2} + C \frac{1}{\varepsilon^3} \sum_{|\alpha|=1} \|\partial^{\alpha}\phi, \psi, \zeta\|^{10} \frac{\eta}{\sigma} + C \delta (1 + t)^{-\frac{3}{2}} \|\phi, \psi, \zeta\|^2
\]

\[
+ C \left[ \delta (1 + t)^{-\frac{1}{2}} + \frac{1}{\varepsilon} \left( \|\phi, \psi\| \cdot \|\phi_y, \psi_y, \zeta_y\| \right) \right] \sum_{|\alpha|=1} \frac{1}{\varepsilon^2} \|\partial^{\alpha}\phi, \psi, \zeta\|^2 + \frac{2}{\varepsilon^2} \sum_{|\alpha|=2} \|\partial^{\alpha}\phi, \psi, \zeta\|^2 + \varepsilon \int \int \frac{|\partial^{\alpha}\tilde{f}|^2}{2M_*} d\xi dy \int \int \frac{\nu(|\xi|)}{M_*} |\tilde{G}|^2 d\xi dy.
\]

(3.92)
From (3.78), (3.79) and (3.92) and using the smallness of $\delta, \lambda_0$ and $\varepsilon$, we have
\[
(E_4 + E_7)_\tau + \frac{1}{2}(K_4 + K_7) \leq C \delta \varepsilon^2 (1 + t)^{-1} E_4 + C \delta \varepsilon^2 (1 + t)^{-2} + C(1 + t)^{-1} \int |\hat{\theta}_y| (b_1^2 + b_2^2) dy
\]
\[
+ C \left[ \delta \varepsilon (1 + t)^{-1} + \sum_{|\alpha| = 1} \left( \frac{1}{\varepsilon} \|\partial^\alpha (\phi, \psi, \zeta)\|_2^2 + \varepsilon \int \frac{|\partial^\alpha \hat{G}|^2}{M_s} d\xi dy \right) \right] \int \frac{\nu(|\xi|)}{M_s} |\hat{G}|^2 d\xi dy.
\]
and
\[
(E_5 + E_7)_\tau + \frac{1}{2}(K_5 + K_7)
\]
\[
\leq C \delta \varepsilon^2 (1 + t)^{-1} E_5 + C \delta \varepsilon^2 (1 + t)^{-\frac{3}{2}} + C(1 + t)^{-1} \int |\hat{\theta}_y| (b_1^2 + b_2^2) dy.
\]

4 The Proof of Main Result

For a suitable large constant $\hat{C}_9$, by combining (3.54) and (3.94) and using the smallness of $\delta, \lambda_0$ and $\varepsilon$, we have
\[
E_{8\tau} + K_8 \leq C \delta \varepsilon^2 (1 + \varepsilon^2 \tau)^{-1} E_8 + C \delta \varepsilon^2 (1 + \varepsilon^2 \tau)^{-1},
\]
where
\[
E_8 = C_0 E_2 + E_5 + E_7, \quad K_8 = \frac{1}{4}(K_2 + K_5 + K_7) + \int |\hat{\theta}_y| (b_1^2 + b_2^2) dy.
\]
Note that
\[
E_8 \geq \|\Phi, \Psi, W\|_2^2 + \left\{ \frac{c_0}{\varepsilon^2} \|\phi, \psi, \zeta\|_2^2 + \varepsilon \int \frac{\nu |\hat{G}|^2}{M_s} d\xi dy \right\}
\]
\[
+ \left\{ \frac{c_1}{\varepsilon^2} \|\phi, \psi, \zeta\|_2^2 + \sum_{|\alpha| = 0} \|\partial^\alpha (\phi, \psi, \zeta)\|_2^2 + \sum_{|\alpha| = 0} \int \frac{|\partial^\alpha |f|^2}{M_s} d\xi dy \right\}
\]
\[
+ \frac{1}{\varepsilon} \sum_{|\alpha| = 1} \int \frac{\nu |\partial^\alpha \hat{G}|^2}{2M_s} d\xi dy - C \delta (1 + \varepsilon^2 \tau)^{-\frac{3}{2}} \right\},
\]
\[
K_8 \geq \sum_{|\beta| = 1} \|\partial^\beta (\Phi, \Psi, W)\|_2^2 + c_2 \left\{ \frac{1}{\varepsilon} \left\{ \sum_{|\alpha| = 0} \|\partial^\alpha (\phi, \psi, \zeta)\|_2^2 + \varepsilon \int \frac{\nu |\xi|}{2M_s} |\hat{G}|^2 d\xi dy \right\}
\]
\[
+ \left\{ \frac{c_1}{\varepsilon^2} \sum_{|\alpha| = 0} \|\partial^\alpha (\phi, \psi, \zeta)\|_2^2 + \frac{1}{\varepsilon} \sum_{|\alpha| = 1, 2} \int \frac{\nu |\xi|}{M_s} |\partial^\alpha \hat{G}|^2 d\xi dy \right\} \right\} + \frac{1}{2} \int |\hat{\theta}_y| (b_1^2 + b_2^2) dy,
\]
and
\[
\varepsilon^2 E_7 \leq C \delta \varepsilon^2 (1 + \varepsilon^2 \tau)^{-\frac{3}{2}} + C(K_4 + K_7), \quad \text{and} \quad \varepsilon^2 (E_5 + E_7) \leq C \delta \varepsilon^2 (1 + \varepsilon^2 \tau)^{-\frac{3}{2}} + C K_8.
\]

Then the Gronwall inequality yields that
\[
E_8 \leq C \sqrt{\delta} (1 + \varepsilon^2 \tau)^{C_0 \sqrt{\delta}}, \quad \int_0^\tau K_8 ds \leq C \sqrt{\delta} (1 + \varepsilon^2 \tau)^{C_0 \sqrt{\delta}}.
\]
Hence, it holds that
\[
\|\Phi, \Psi, W\|_2^2 \leq C \sqrt{\delta} (1 + \varepsilon^2 \tau)^{C_0 \sqrt{\delta}}.
\]
Multiplying (3.94) by \((1 + \varepsilon^2 \tau)\) gives
\[
[(1 + \varepsilon^2 \tau)(E_5 + E_7)]_\tau + \frac{1}{2}(1 + \varepsilon^2 \tau)(K_5 + K_7) \\
\leq C\varepsilon^2 (E_5 + E_7) + C\delta \varepsilon^2 (1 + \varepsilon^2 \tau)^{-\frac{1}{2}} + C_9 \int |\tilde{\theta}_y|(b_1^2 + b_2^2)dy.
\] (4.4)

Integrating (4.4) with respect to \(\tau\) and using (4.2) and (4.1), one has that
\[
(1 + \varepsilon^2 \tau)(E_5 + E_7) + \int_0^\tau \frac{1}{2}(1 + \varepsilon^2 s)(K_5 + K_7)ds \\
\leq C\varepsilon^2 \int_0^\tau (E_5 + E_7)ds + C\sqrt{\delta}(1 + \varepsilon^2 \tau)^{\frac{1}{2}} \\
\leq C\sqrt{\delta}(1 + \varepsilon^2 \tau)^{\frac{1}{2}} + C \int_0^\tau K_8ds \leq C\sqrt{\delta}(1 + \varepsilon^2 \tau)^{\frac{1}{2}},
\]
which yields
\[
(E_5 + E_7) \leq C\sqrt{\delta}(1 + \varepsilon^2 \tau)^{-\frac{1}{2}}.
\] (4.5)

In particular, one has
\[
\varepsilon \int \int |\tilde{G}|^2 M^* d\xi dy \leq C\sqrt{\delta}(1 + \varepsilon^2 \tau)^{-\frac{1}{2}}.
\] (4.6)

On the other hand, multiplying (3.94) by \((1 + \varepsilon^2 \tau)\), it holds
\[
\int_0^\tau (1 + \varepsilon^2 s)^{\frac{1}{2}}(K_5 + K_7)ds \leq C\sqrt{\delta}(1 + \varepsilon^2 \tau)^{C_0\sqrt{\delta}}.
\] (4.7)

Multiplying (3.93) by \((1 + \varepsilon^2 \tau)\) and using (4.5) and (4.1), one can obtain
\[
[(1 + \varepsilon^2 \tau)(E_4 + E_7)]_\tau + \frac{1}{2}(1 + \varepsilon^2 \tau)(K_4 + K_7) \leq C\varepsilon^2 (E_4 + E_7) + C\delta \varepsilon^2 (1 + \varepsilon^2 \tau)^{-1} \\
+ C_9 \int |\tilde{\theta}_y|(b_1^2 + b_2^2)dy + C\varepsilon(1 + \varepsilon^2 \tau)^{\frac{1}{2}} \int \frac{\nu(|\xi|)}{M^*} |\tilde{G}|^2 d\xi dy \\
\leq C\delta \varepsilon^2 (1 + \varepsilon^2 \tau)^{-1} + CK_8 + C(1 + \varepsilon^2 \tau)^{\frac{1}{2}} K_5.
\] (4.8)

Integrating (4.8) with respect to \(\tau\) and using (4.2) and (4.7), one has
\[
(E_4 + E_7) \leq C\sqrt{\delta}(1 + \varepsilon^2 \tau)^{-1+C_0\sqrt{\delta}}, \quad \int_0^\tau (1 + \varepsilon^2 s)(K_4 + K_7)ds \leq C\sqrt{\delta}(1 + \varepsilon^2 \tau)^{C_0\sqrt{\delta}}.
\] (4.9)

Therefore, it holds
\[
\|\left(\phi, \psi, \zeta \right)(\tau)\|^2 \leq C\sqrt{\delta}(1 + \varepsilon^2 \tau)^{-1+C_0\sqrt{\delta}}.
\] (4.10)

Multiplying (3.92) by \((1 + \varepsilon^2 \tau)^{\frac{1}{2}}\) with \(\vartheta > 0\) in Theorem 3.1 and using (3.14), (4.1), (4.7),
and the smallness of \( \delta \), one has

\[
[(1 + \varepsilon^{2} \tau)^{\frac{2}{3} - \vartheta} E_{\tau}]_{s} = (\frac{3}{2} - \vartheta)(1 + \varepsilon^{2} \tau)^{\frac{1}{2} - \vartheta} E_{\tau} + (1 + \varepsilon^{2} \tau)^{\frac{2}{3} - \vartheta} E_{\tau},
\]

\[
\leq C \delta K_{9} + C(1 + \varepsilon^{2} \tau)^{\frac{1}{2} - \vartheta} (K_{4} + K_{7}) + C \delta(1 + \varepsilon^{2} \tau) \sum_{|\alpha|=1} \frac{1}{\varepsilon^{2}} \| \partial^{\alpha} (\phi, \psi, \zeta) \|^2
\]

\[
+ \frac{C}{\varepsilon^{3}}(1 + \varepsilon^{2} \tau)^{\frac{1}{2} - \vartheta} \sum_{|\alpha|=1} \| \partial^{\alpha} (\phi, \psi, \zeta) \|^{2} + \frac{C}{\varepsilon^{3}}(1 + \varepsilon^{2} \tau)^{\frac{1}{2} - \vartheta} \| (\phi, \psi, \zeta) \| \sum_{|\alpha|=1} \| \partial^{\alpha} (\phi, \psi, \zeta) \|^3
\]

\[
+ C \varepsilon(1 + \varepsilon^{2} \tau)^{\frac{1}{2} - \vartheta + C_{0} \sqrt{\delta}} \int \int \frac{\nu([\theta]\right)}{M_{*}} \| \tilde{G} \|^2 d\xi dy + C \delta \varepsilon^{2}(1 + \varepsilon^{2} \tau)^{\frac{1}{2} - \vartheta}
\]

\[
\leq C \delta K_{9} + C \delta(1 + \varepsilon^{2} \tau)(K_{4} + K_{7}) + C(1 + \varepsilon^{2} \tau)^{\frac{1}{2} - \vartheta + C_{0} \sqrt{\delta}} K_{5} + C \delta \varepsilon^{2}(1 + \varepsilon^{2} \tau)^{\frac{1}{2} - \vartheta} \quad(4.11)
\]

\[
+ \frac{C}{\varepsilon^{3}}(1 + \varepsilon^{2} \tau)^{\frac{1}{2} - \vartheta} \sum_{|\alpha|=1} \| \partial^{\alpha} (\phi, \psi, \zeta) \|^{2} \leq \frac{\delta}{\varepsilon} \|
\]

\[
\leq C \varepsilon \int \int (1 + \varepsilon^{2} \tau)^{\frac{1}{2} - \vartheta} K_{4} ds \leq C \sqrt{\delta} \varepsilon(1 + \varepsilon^{2} \tau)^{C_{0} \sqrt{\delta}}, \quad(4.12)
\]

provided that \( C_{0} \sqrt{\delta} \leq \vartheta \). Thus integrating (4.11) over \([0, \tau]\) and using (4.2), (4.7), (4.9) and (4.12) yield that

\[
E_{\tau} \leq C \sqrt{\delta}(1 + \varepsilon^{2} \tau)^{-\frac{3}{2} + \vartheta + C_{0} \sqrt{\delta}},
\]

which immediately implies

\[
\frac{1}{\varepsilon^{3}} \left( \| (\phi, \psi, \zeta) \|^{2} + \sum_{|\alpha|=2} \| \partial^{\alpha} (\phi, \psi, \zeta) \|^{2} + \sum_{|\alpha|=2} \int \int \frac{| \partial^{\alpha} \tilde{f} |^{2}}{M_{*}} d\xi dy \right)
\]

\[
+ \frac{1}{\varepsilon} \sum_{|\alpha|=1} \int \int \frac{| \partial^{\alpha} \tilde{G} |^{2}}{2M_{*}} d\xi dy \leq C \sqrt{\delta}(1 + \varepsilon^{2} \tau)^{-\frac{3}{2} + \vartheta + C_{0} \sqrt{\delta}}. \quad(4.13)
\]

**Proof of Theorem 3.1**

Combining (4.3), (4.6), (4.13) and (4.10) and using the Sobolev inequality, it holds that

\[
\| (\Phi, \Psi, W) \|_{L^{\infty}}^{2} \leq C \| (\Phi, \Psi, W) \| \left( \| (\phi, \psi, \zeta) \| + \delta \varepsilon(1 + \varepsilon^{2} \tau)^{-\frac{1}{2}} \right) \leq C \sqrt{\delta} \varepsilon, \quad(4.14)
\]

and

\[
\int \int \frac{| \tilde{G} |^{2}}{M_{*}} d\xi L^{\infty} \leq \left( \int \int \frac{| \tilde{G} |^{2}}{M_{*}} d\xi \right)^{\frac{1}{2}} \left( \int \int \frac{| \tilde{G}_{y} |^{2}}{M_{*}} d\xi dy \right)^{\frac{1}{2}} \leq C \sqrt{\delta}(1 + \varepsilon^{2} \tau)^{-\frac{1}{2}}. \quad(4.15)
\]

Therefore, (4.6), (4.10), (4.13), (4.14) and (4.15) verify the a priori assumption (3.10) if we choose \( \lambda_{0} = \delta \frac{\pi}{2} \). Hence, the proof of Theorem 3.1 is completed.
Proof of Theorem 2.4: The proof of (2.44) can be obtained directly from (3.9) by using the transformation (3.1) of the scaled variables \((y, \tau)\) and the original variables \((x, t)\). By combining (2.44) and Sobolev inequality, (2.45) can be derived immediately. Thus the proof of Theorem 2.4 is completed. □

Acknowledgments. The authors would like to thank Renjun Duan for insightful discussion during the third author’s visit in the Chinese University of Hong Kong. Feimin Huang is partially supported by National Basic Research Program of China (973 Program) under Grant No. 2011CB808002 and by National Center for Mathematics and Interdisciplinary Sciences, AMSS, CAS and the CAS Program for Cross & Cooperative Team of the Science & Technology Innovation. Yi Wang is supported by National Natural Sciences Foundation of China No. 10801128 and No. 11171326. Yong Wang is partially supported by National Natural Sciences Foundation of China No. 11371064 and 11401565. Tong Yang is supported by the General Research Fund of Hong Kong, CityU 103412.

References

[1] F. V. Atkinson and L. A. Peletier, Similarity solutions of the nonlinear diffusion equation, Arch. Rat. Mech. Anal., 54 (1974), 373–392.

[2] C. Bardos, F. Golse and D. Levermore, Fluid dynamic limits of kinetic equations, I. Formal derivations, J. Statis. Phys., 63, (1991), 323-344; II. Convergence proofs for the Boltzmann equation, Comm. Pure Appl. Math., 46, (1993), 667-753.

[3] C. Bardos, C. Levermore, S. Ukai, and T. Yang, Kinetic equations: fluid dynamical limits and viscous heating, Bull. Inst. Math. Acad. Sin. (N.S.) 3 (2008), 1-49.

[4] C. Bardos and S. Ukai, The classical incompressible Navier-Stokes limit of the Boltzmann equation, Math. Models Methods Appl. Sci., 1 (1991), 235-257.

[5] L. Boltzmann, (translated by Stephen G. Brush), “Lectures on Gas Theory,” Dover Publications, Inc. New York, 1964.

[6] R. E. Caflisch, The fluid dynamical limit of the nonlinear Boltzmann equation, Comm. Pure Appl. Math., 33 (1980), 491-508.

[7] C. Cercignani, R. Illner and M. Pulvirenti, “The Mathematical Theory of Dilute Gases,” Springer-Verlag, Berlin, 1994.

[8] S. Chapman and T. G. Cowling, “The Mathematical Theory of Non-Uniform Gases,” 3rd edition, Cambridge University Press, 1990.

[9] A. De Masi, R. Esposito and J. L. Lebowitz, Incompressible Navier-Stokes and Euler limits of the Boltzmann equation, Comm. Pure Appl. Math. 42 (1989), no. 8, 1189-1214.

[10] C. T. Duyn and L. A. Peletier, A class of similarity solution of the nonlinear diffusion equation, Nonlinear Analysis, T.M.A., 1 (1977), 223-233.

[11] R. J. DiPerna and P. L. Lions, On the Cauchy Problem for Boltzmann Equations: Global Existence and Weak Stability, Annals of Mathematics, 130 (1989), no. 2, 321-366.
[12] R. Esposito and M. Pulvirenti, *From particle to fluids*, in “Handbook of Mathematical Fluid Dynamics,” Vol. **III**, North-Holland, Amsterdam, (2004), 1-82.

[13] R. Esposito, Y. Guo, C. Kim and R. Marra, *Stationary solutions to the Boltzmann equation in the Hydrodynamic limit*. Preprint, [arXiv:1502.05324](http://arxiv.org/abs/1502.05324).

[14] F. Golse, *The Boltzmann equation and its hydrodynamic limits*, Evolutionary equations. Vol. II, 159-301, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2005.

[15] F. Golse and D. Levermore, *Stokes-Fourier and acoustic limits for the Boltzmann equation: convergence proofs*, Comm. Pure Appl. Math. **55** (2002), no. 3, 336-393.

[16] F. Golse, B. Perthame and C. Sulem, *On a boundary layer problem for the nonlinear Boltzmann equation*, Arch. Ration. Mech. Anal., **103** (1986), 81-96.

[17] F. Golse and L. Saint-Raymond, *The Navier-Stokes limit of the Boltzmann equation for bounded collision kernels*, Invent. Math. **155** (2004), no. 1, 81-161.

[18] F. Golse and L. Saint-Raymond, *The incompressible Navier-Stokes limit of the Boltzmann equation for hard cutoff potentials*, J. Math. Pures Appl., **91** (2009), 508-552.

[19] H. Grad, “Asymptotic Theory of the Boltzmann Equation II,” in “Rarefied Gas Dynamics” (J. A. Laurmann, ed.), Vol. **I**, Academic Press, New York, (1963), 26–59.

[20] Y. Guo, *The Boltzmann equation in the whole space*, Indiana Univ. Math. J., **53** (2004), 1081-1094.

[21] Y. Guo, *Boltzmann diffusive limit beyond the Navier-Stokes approximation*. Comm. Pure Appl. Math. **59** (2006), no. 5, 626-687.

[22] F. M. Huang, A. Matsumura and Z. P. Xin, *Stability of Contact Discontinuities for the 1-D Compressible Navier-Stokes Equations*, Arch. Rat. Mech. Anal., **179**(2005), 55-77.

[23] F. M. Huang, Y. Wang, Y. Wang and T. Yang, *The Limit of the Boltzmann Equation to the Euler Equations for Riemann Problems*, SIAM J. Math. Anal., **45** (2013), no. 3, 1741-1811.

[24] F. M. Huang, Y. Wang and T. Yang, *Hydrodynamic limit of the Boltzmann equation with contact discontinuities*, Comm. Math. Phy., **295** (2010), 293-326.

[25] F. M. Huang, Y. Wang and T. Yang, *Fluid dynamic limit to the Riemann solutions of Euler equations: I. Superposition of rarefaction waves and contact discontinuity*, Kinet. Relat. Models, **3** (2010), 685-728.

[26] F. M. Huang, Z. P. Xin and T. Yang, *Contact discontinuity with general perturbations for gas motions*, Adv. Math., **219** (4) (2008), 1246-1297.

[27] F. M. Huang and T. Yang, *Stability of contact discontinuity for the Boltzmann equation*, J. Differential Equations, **229** (2006), 698-742.

[28] J. Jang, *Vlasov-Maxwell-Boltzmann diffusive limit*, Arch. Ration. Mech. Anal. **194** (2009), no. 2, 531-584.

[29] N. Jiang and L. J. Xiong, *Diffusive limit of the Boltzmann equation with fluid initial layer in the periodic domain*, to appear in SIAM, 2015.
[30] D. Levermore and N. Masmoudi, *From the Boltzmann equation to an incompressible Navier-Stokes-Fourier system*, Arch. Ration. Mech. Anal. **196** (2010), no. 3, 753-809.

[31] P.L. Lions and N. Masmoudi, *From the Boltzmann equations to the equations of incompressible fluid mechanics. I*, Arch. Ration. Mech. Anal. **158** (2001), no. 3, 173-193.

[32] P.L. Lions and N. Masmoudi, *From the Boltzmann equations to the equations of incompressible fluid mechanics. II*, Arch. Ration. Mech. Anal. **158** (2001), no. 3, 195-211.

[33] S. Q. Liu and H. J. Zhao, *Diffusive expansion for solutions of the Boltzmann equation in the whole space*, J. Differential Equations, **250** (2011), no. 2, 623-674.

[34] T. Liu, T. Yang, and S. H. Yu, *Energy method for the Boltzmann equation*, Physica D, **188** (2004), 178–192.

[35] T. Liu, T. Yang, S. H. Yu and H. J. Zhao, *Nonlinear stability of rarefaction waves for the Boltzmann equation*, Arch. Rat. Mech. Anal., **181** (2006), 333–371.

[36] T. Liu and S. H. Yu, *Boltzmann equation: Micro-macro decompositions and positivity of shock profiles*, Commun. Math. Phys., **246** (2004), 133-179.

[37] J. C. Maxwell, *On the dynamical theory of gases*, Phil. Trans. Roy. Soc. London, 157(1866), 49-88.

[38] N. Masmoudi, *Examples of singular limits in hydrodynamics*, Handbook of differential equations: evolutionary equations. Vol. III, 195-275, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2007.

[39] N. Masmoudi and L. Saint-Raymond, *From the Boltzmann equation to the Stokes-Fourier system in a bounded domain*, Comm. Pure Appl. Math. **56** (2003), no. 9, 1263-1293.

[40] L. Saint-Raymond, *From the BGK model to the Navier-Stokes equations*, Ann. Sci. cole Norm. Sup. **36** (2003), no. 2, 271-317.

[41] Y. Sone, *Molecular Gas Dynamics, Theory, Techniques, and Applications*, Birkhäuser, Boston, 2006.

[42] S. Ukai, *Solutions of the Boltzmann equation, Pattern and Waves - Qualitative Analysis of Nonlinear Differential Equations* (eds. M.Mimura and T.Nishida), Studies of Mathematics and Its Applications 18, 37-96, Kinokuniya-North-Holland, Tokyo, 1986.

[43] S. Ukai and K. Asano, *The Euler limit and initial layer of the Boltzmann equation*, Hokkaido Math. J, **12**(1983), 311-332.

[44] Z. P. Xin and H. H. Zeng, *Convergence to the rarefaction waves for the nonlinear Boltzmann equation and compressible Navier-Stokes equations*, J. Diff. Eqs., **249** (2010), 827–871.

[45] S. H. Yu, *Hydrodynamic limits with shock waves of the Boltzmann equations*, Commun. Pure Appl. Math, **58** (2005), 409–443.