On construction of projection operators

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The problem of construction of projection operators on eigen-subspaces of symmetry operators is considered. This problem arises in many approximate methods for solving time-independent and time-dependent quantum problems, and its solution ensures proper physical symmetries in development of approximate methods. The projector form is sought as a function of symmetry operators and their eigenvalues characterizing the eigen-subspace of interest. This form is obtained in two steps: 1) identification of algebraic structures within a set of symmetry operators (e.g. groups and Lie algebras), and 2) construction of the projection operators for individual symmetry operators. The first step is crucial for efficient projection operator construction because it allows for using information on irreducible representations of the present algebraic structure. The discussed approaches promise to stimulate further developments of variational approaches for electronic structure of strongly correlated systems and in quantum computing.

I. INTRODUCTION

Projection operators on eigen-subspaces of operators commuting with the Hamiltonian are beneficial for development of various approximate methods of solving both time-independent (TI) and time-dependent (TD) Schrödinger equations (SEs). The operators commuting with the system Hamiltonian are also known as symmetries, and their expectation values are conserved throughout the dynamics. In TI-SE, being able to project the Hamiltonian on one of the symmetries irreducible eigen-subspace ensures that any trial wavefunction that is non-orthogonal to the selected eigen-subspace will have a proper symmetry after the variational procedure and projection. In dynamical problems, approximate Hamiltonians (or Liouwillians) that do not commute with symmetries due to introduced approximations can be symmetrically restored by applying the corresponding projectors, which enforces the approximate dynamics to respect symmetries of the original exact Hamiltonian.

The projection becomes crucial if various variable mappings are employed for devising approximations, for example fermionic to qubit Jordan-Wigner or Bravyi-Kitaev transformations, in quantum computing, or the mapping between discrete quantum states and continuous variables in quantum-classical dynamics. Very frequently, introducing approximations in the mapped problem violates proper physical symmetries in the original formulation and introducing the projection operators is a straightforward way to restore the physical symmetries.

Recently, projection techniques were also extensively used in developing efficient (e.g. polynomial computational cost) methods for treatment of strongly correlated systems. In strongly correlated systems, mean-field approaches very frequently undergo spontaneous symmetry breaking. Projections techniques can be used either after variational procedures accounting for electron correlation (projection-after-variation) or within them (variation-after-projection). In both cases, projectors preserve correct physical symmetries in the final wavefunction, moreover, in variation-after-projection, they allow the variational procedure to use the variational flexibility more efficiently.

A naive view on the construction of a projector for a symmetry operator \( \hat{O} \) can be using the form \( \hat{P}_k = |\phi_k\rangle \langle \phi_k| \), where \( |\phi_k\rangle \) is a corresponding eigenfunction \( \hat{O} |\phi_k\rangle = \omega_k |\phi_k\rangle \). Yet, this approach is not acceptable because of at least two difficulties: 1) the \( |\phi_k\rangle \langle \phi_k| \) form may not be compactly presentable in the Hilbert space of the problem (e.g. in the coordinate space this operator is non-local integral operator), 2) there can be a degenerate subspace corresponding to a single eigenvalue \( \omega_k \) containing potentially infinite number of different terms like \( |\phi_k\rangle \langle \phi_k| \). Thus, it is more prudent to search for the projection operator in the form \( \hat{P}_k = F(\hat{O}, \omega_k) \), where \( F \) is a function of two arguments. The basic principles of construction such functions will be discussed in this work.

Another complication in construction of projectors is that, usually, there is not a single operator but a whole set of operators \( \{\hat{O}_i\} \) commuting with the Hamiltonian, \( [\hat{H}, \hat{O}_i] = 0 \). However, in general, these operators do not commute with each other \( [\hat{O}_i, \hat{O}_j] \neq 0 \), and thus do not have a common set of eigenfunctions. Therefore, the projection using all known symmetries cannot be simply a projection on an eigen-subspace of a particular operator. The most natural algebraic structure for this set of operators is the Lie algebra. This is quite intuitive if we consider that any commutator of two operators, \( [\hat{O}_i, \hat{O}_j] \), is again an operator commuting with the Hamiltonian. Thus, if an initial set of operators commuting with \( \hat{H} \) is known, this set can be extended by forming all possible commutators

\[
[\hat{O}_i, \hat{O}_j] = \sum_k c_{ij}^{(k)} \hat{O}_k,
\]
here $c_{ij}^{(k)}$ are some constants. An extended set of operators $\{\hat{O}_k\}$ that is closed with respect to the commutation operation in Eq. (1) forms a Lie algebra. This Lie algebra will consists of all operators commuting with the Hamiltonian.  

To impose symmetries, the Lie algebraic structure suggests to construct projectors on irreducible representations of the Lie algebra. This process is well described in mathematical literature on classification of Lie algebras. Here, I will provide only a basic summary and examples while referring the interested reader to more specialized literature. One of the essential differences from other works in the literature on symmetry operators will be considering not only irreducible subspaces of Lie algebras but rather construction of projection operators on these irreducible subspaces as functions of the symmetry operators.

In what follows I present a theory that allows one to account for algebraic properties of a set of operators commuting with the Hamiltonian and thus to use all available symmetry information to construct projectors on irreducible subspaces of symmetry operators. Even though I will consider a set of operators commuting with the Hamiltonian, the methods discussed here can be applied for a set of operators not necessarily related to symmetries of the Hamiltonian, for example this can be operators commuting with the Liouvillian.

\section{II. THEORY}

First, treatment of algebraic properties of the symmetry operator set will be considered. Chemists, of course, are more familiar with the situation when $\{\hat{O}_i\}$ form a finite group, $\hat{O}_i \hat{O}_j = \hat{O}_k$, as in the case of point group symmetries. However, I would like to argue that this group structure is only a useful addition to the more general underling Lie algebraic structure. Nevertheless, it is convenient to separate two cases based on whether $\{\hat{O}_i\}$ form a multiplicative group or not. It will be shown that in the latter (more general case) accounting for the Lie algebraic structure leads to the problem of construction projection operators for some subset of symmetry operators (e.g. $S_z$, and $S^2$ for the algebra of spin). Thus, possible ways for such constructions will be considered next.

\subsection{A. Projectors for groups}

Here, I consider the case when operators $\{\hat{O}_k\}$ belong to multiplicative group $G$. Existence of the group structure allows one to generate projectors on the group irreducible representations following the standard procedure

\begin{equation}
\hat{P}_\Gamma = \frac{d_\Gamma}{|G|} \sum_{k=1}^{|\Gamma|} \chi_\Gamma^*(\hat{O}_k)\hat{O}_k,
\end{equation}

where $\Gamma$ is the irreducible representation of interest, $d_\Gamma$ is the dimension of $\Gamma$, $|G|$ is the number of the group elements, and $\chi_\Gamma(\hat{O}_k)$ are characters for the group elements. In this case, one does not need to deal with Lie algebras and projection operators are simply expressed as a linear function of all operators forming the group.

\subsection{B. Projectors for Lie algebras}

In the case when $\{\hat{O}_i\}$’s only form the Lie algebra, one can obtain a continuous Lie group using the exponentiation of the algebra elements (see appendix A for general exposition). Then the same standard machinery as in the group case can be used for projection construction. This approach can be illustrated on a simple example of a single symmetry Hermitian operator $\hat{O}$. Exponentiation of $\hat{O}$, $g(\hat{O}, \phi) = \exp[\phi \hat{O}]$, where $\phi \in [0, 2\pi)$, allows one to create a continuous compact cyclic group $G$ with elements $g(\hat{O}, \phi)$. Then the continuous analogue of Eq. (2) can be written as

\begin{equation}
\hat{P}_j = \frac{1}{2\pi} \int_0^{2\pi} e^{i\phi(\hat{O} - o_j)} d\phi,
\end{equation}

where $o_j$ is a particular eigenvalue of $\hat{O}$. All cyclic groups are abelian and have one-dimensional irreducible representations; each irreducible representation is characterized by the eigenvalue $o_j$, hence, the characters of the irreducible representations are $\exp[i\phi o_j]$.

However, switching from the algebra to the group is not necessary to obtain the projectors on the irreducible representations of the algebra. Moreover, historically, Lie algebras were introduced to simplify analysis of the irreducible representations of Lie groups. In any case, understanding irreducible representations of the symmetry Lie algebra is a necessary step for the group pathway (see appendix A) and for a simpler method avoiding the group construction, which is described below.

For all simple or semi-simple Lie algebras (e.g. $su(2)$, the electron spin) the standard procedure to construct irreducible representations is to select maximal commuting sub-algebra (i.e. the Cartan sub-algebra), this sub-algebra will form the maximal set of all mutually commuting operators with the Hamiltonian and the corresponding eigen-values represent good quantum numbers, while the eigen-functions form the basis for the irreducible representations. For the well-known $su(2)$-case, the usual choice of the Cartan sub-algebra is the $\hat{S}_2$ operator. To further characterize the irreducible representations one can use Casimir operators, which commute with all elements of the algebra. By Schur’s lemma this commutativity makes any Casimir operator to be equivalent to the
identity multiplied by a constant for any irreducible representation. These constants are eigen-values of Casimir operators on irreducible representations and along with the full set of quantum numbers fully characterize the basis of irreducible representations. In the $su(2)$-case, $\hat{S}^2$ is the Casimir operator and its eigenvalue $S(S+1)$ along with that for $\hat{S}_z$, $M = -S,...,S$, fully characterize the basis of all irreducible representations. Thus, to construct projectors on the basis states of irreducible representations it is enough to construct projectors on eigenstates of all operators of the Cartan sub-algebra and Casimir operators, I will refer to these operators as the fully commuting set.

For each operator $\hat{O}_i$ in the fully commuting set, individual projectors for eigen-subspaces $\hat{P}_j^{(i)}$ will be build as a function that depends on $\hat{O}_i$ and its eigenvalue $\phi_j^{(i)}$ determining the eigen-subspace, $\hat{P}_j^{(i)} = F(\hat{O}_i, \phi_j^{(i)})$. A total projector on a particular irreducible representation of the Lie algebra can be written as $\hat{P} = \prod_i \hat{P}_i(\hat{O}_i, \phi_j^{(i)})$, where the eigenvalues $\phi_j^{(i)}$ should be chosen so that the projectors in the product are not orthogonal to each other (e.g. in the $su(2)$-case, $M \in \{-S,...,S\}$). The order of the $\hat{P}_i$ functions does not matter because if $\hat{O}_i$ operators commute their eigen-subspace projectors also commute (see appendix B).

C. Role of non-commuting elements

Both algebras and groups, generally contain elements that do not commute. It is important to point out a significance of this non-commutativity. It always indicates degeneracy of some eigenstates in the Hamiltonian. In other words, if we characterize some basis functions for irreducible representations using only commuting sub-algebras or sub-groups, non-commuting parts contain information that some of these basis functions form multi-dimensional irreducible representations corresponding to the degenerate eigenstates of the Hamiltonian. If it is not for the non-commuting elements, any Hamiltonian degeneracy would be perceived as completely accidental. Therefore, non-commuting elements providing more symmetry related information on the Hamiltonian spectrum, they are accounted in Eq. (2) for groups and translated through the Casimir operators in algebras.

D. Projector as an indicator function of the operator

For a single projector $\hat{P}_j^{(i)}$, $F(\hat{O}_i, \phi_j^{(i)})$ can be constructed as some differentiable representation of the Kronecker-delta function. The Kronecker-delta function naturally appears in the spectral decomposition of the projector (here, we consider the non-degenerate case, while the degenerate case can be treated similarly with more cumbersome notation)

$$\hat{P}_j^{(i)} = \sum_n \phi_n^{(j)} \langle \phi_n^{(i)} | \delta_{nj} \rangle$$

$$= \sum_n \phi_n^{(j)} \langle \phi_n^{(i)} | F(x, \phi_j^{(i)}) \rangle |_{x = \phi_n^{(i)}} \quad (4)$$

where we substituted the Kronecker-delta function (also known as an indicator function) with the differentiable function

$$F(x, \phi_j^{(i)}) = \begin{cases} 1, & x = \phi_j^{(i)}, \\ 0, & x = \phi_n^{(i)}, n \neq j, \\ \xi(x) \in [0,1], & x \neq \phi_n^{(i)}, \forall n, \end{cases} \quad (5)$$

where $\xi(x)$ can be any smooth function for intermediate values of $x$. Due to its differentiability we can expand $F$ in the Taylor series, and this expansion can define $F(\hat{O}_i, \phi_j^{(i)})$. Then using the Taylor expansion of $F$ and the projector property of $|\phi_n^{(i)}\rangle \langle \phi_n^{(i)}|^2 = |\phi_n^{(i)}\rangle \langle \phi_n^{(i)}|$ one can obtain

$$\hat{P}_j^{(i)} = F(\hat{O}_i, \phi_j^{(i)}). \quad (6)$$

There are multiple ways to define differentiable representation of the Kronecker-delta function $F(x, \phi_j^{(i)})$, here we list several forms:

1) Integration over a unit circle in real space:

$$F(x, \phi_j^{(i)}) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\phi(x-\phi_j^{(i))})} d\phi \quad (7)$$

2) Integration over a unit circle in the complex plane:

$$F(x, \phi_j^{(i)}) = \frac{1}{2\pi i} \oint_{|z|=1} z^{(x-\phi_j^{(i))}-1} dz \quad (8)$$

This function exploits the same idea as the previous one but using the complex plane.

3) The Lagrange interpolation product:

$$F(x, \phi_j^{(i)}) = \prod_{n \neq j} \frac{x - \phi_n^{(i)}}{\phi_j^{(i)} - \phi_n^{(i)}}, \quad (9)$$

which is less restrictive since for $x$-values in between the eigenvalues the functional value is not fixed to zero or one. This polynomial function is used in the Lagrange interpolation method.\textsuperscript{19} The substitution, $x \rightarrow \hat{O}_i$ in Eq. (10) leads to the Löwdin projector operator used for the spin projection.\textsuperscript{20,21}

4) Integration over an arbitrary contour $C(\phi_j^{(i)})$ encircling only $z = \phi_j^{(i)}$ in the complex plane:

$$F(x, \phi_j^{(i)}) = \frac{1}{2\pi i} \oint_{C(\phi_j^{(i)})} \frac{dz}{x-z} \quad (10)$$
This function with $x \to \hat{H}$ is the resolvent used in investigation of the perturbation series.\textsuperscript{22,23} 5) The difference between the limits of logistic functions:

$$F(x, o_j^{(i)}) = \lim_{k \to \infty} \left(1 + e^{-k(x-o_j^{(i)}+\epsilon)}\right)^{-1} - \lim_{k \to \infty} \left(1 + e^{-k(x-o_j^{(i)}-\epsilon)}\right)^{-1}, \quad (12)$$

where $\epsilon = \min |o_j^{(i)} - o_{j+1}^{(i)}|/2$. These limits of logistic functions correspond to the Heaviside functions.

6) “Bump” function (or mollifier):

$$F(x, o_j^{(i)}) = \begin{cases} \exp \left[\frac{1}{(x-o_j^{(i)})^2-\epsilon}\right], & x \in (o_j^{(i)} - \epsilon, o_j^{(i)} + \epsilon) \\ 0, & x \not\in (o_j^{(i)} - \epsilon, o_j^{(i)} + \epsilon) \end{cases}. \quad (13)$$

Unfortunately, this last example cannot be extended to $x \to \hat{O}_i$ because of the branch choice based on $x$-value in its definition.

Equation (10) is especially useful to build projectors for operators with a finite number of eigenvalues because then the product contains a finite number of terms. Interestingly, for such operators, projectors built based on Eqs. (8) and (10) are the same. This is a consequence of only a finite number of linear independent powers for an operator with a finite spectrum. Using the Cayley-Hamilton theorem\textsuperscript{24} one can show that any function of such an operator is equivalent to $N-1$ polynomial, where $N$ is the number of eigenvalues.

Another interesting connection can be found between projectors based on Eq. (8) and generalization of the group projector in Eq. (2) to an infinite continuous one-parametric cyclic group in Eq. (3).

In Eq. (4), the spectrum of the symmetry operator is assumed to be discrete, if it is not the case, the Kronecker-delta function needs to be substituted by the Dirac-delta function and its numerous representations as limits of continuous functions.

E. Construction of the indicator function using orthogonality

Another way to present the Kronecker-delta function in Eq. (4) is to build the $F$ function as an expansion $F(x, y) = \sum_n f_n(x)c_n(y)$ that satisfies the following relations

$$\hat{P}_j^{(i)}|\phi_k^{(i)}\rangle = F(\hat{O}_i, o_j^{(i)})|\phi_k^{(i)}\rangle = \sum_n c_n(o_j^{(i)})f_n(\hat{O}_i)|\phi_k^{(i)}\rangle$$

$$\quad \quad \quad = \sum_n c_n(o_j^{(i)})f_n(\hat{O}_i)|\phi_k^{(i)}\rangle = \sum_n c_n(o_j^{(i)})f_n(\hat{O}_i)|\phi_k^{(i)}\rangle$$

$$\sum_n c_n(o_j^{(i)})f_n(\hat{O}_i)|\phi_k^{(i)}\rangle = \langle c(o_j^{(i)}), f(\hat{O}_i)\rangle = \delta_{kj}. \quad (16)$$

Here vectors $|f(o_j^{(i)})\rangle$ and $|c(o_j^{(i)})\rangle$ are defined as

$$|f(o_k^{(i)})\rangle = \{f_1(o_k^{(i)}), f_2(o_k^{(i)}), ..., f_M(o_k^{(i)})\},$$

$$|c(o_j^{(i)})\rangle = \{c_1(o_j^{(i)}), c_2(o_j^{(i)}), ..., c_M(o_j^{(i)})\}$$

and are orthonormal for all eigenvalues. The natural question is how to choose $f_n(x)$ and $c_n(y)$? One of the choices closely related to the group theory construction of the projectors is to take $f_k(x) = c_k(x) = \exp(ikx)$ and to switch to a continuous version of $k$ with substituting summation by integration

$$F(\hat{O}_i, o_j^{(i)}) = \frac{1}{2\pi} \int_{0}^{2\pi} c_k^*(o_j^{(i)})f_k(\hat{O}_i)dk, \quad (17)$$

we arrived to the projector already introduced in Eq. (3). Applying further restrictions on operator $\hat{O}_i$, one can generate finite sum expansions for the projector on its eigenspaces. Such restrictions are: equidistant separation between neighboring eigenvalues and finite number of eigenvalues. The later condition is less crucial because its violation only leads to projectors that can separate eigen-states within a finite subset. The basic idea of this finite construction is on the following representation of the Kronecker-delta

$$\delta_{nm} = \frac{1}{M} \sum_{k=1}^{M} e^{2\pi i k(n-m)/M}. \quad (18)$$

If $\hat{O}_i$ has a finite and equidistant spectrum with the distance between its eigenstates $d$ then the projector can be written as

$$F(\hat{O}_i, o_j^{(i)}) = \frac{1}{M} \sum_{k=1}^{M} e^{2\pi i k(\hat{O}_i-o_j^{(i)})/(dM)}$$

$$= \frac{1}{M} \sum_{k=1}^{M} c_k^*(o_j^{(i)})f_k(\hat{O}_i), \quad (19)$$

where $c_k(x) = f_k(x) = \exp(2\pi i kx/(dM))$. This approach can be used for the electron spin projection $\hat{S}_z$ and the number of electrons $N$ operators.

III. CONCLUSIONS

I reviewed various approaches to construct projectors on irreducible eigen-subspaces of symmetry operators. There are two aspects of this problem: 1) accounting for available algebraic structures of the set of symmetry operators, and 2) construction of individual projection operators as functions of symmetry operators and their eigenvalues. Two algebraic structures, groups and Lie algebras, were discussed. For both structures standard methods of construction of irreducible representations were developed in mathematical literature. Knowledge of irreducible representations helps to construct functions of symmetry operators into the projection operator for a
particular irreducible representation. For Lie algebras, various approaches to construct individual projection operators for each symmetry operator were considered. The origin of various projection functions was found in variety of representations for the Kronecker-delta function.

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APPENDIX A: CONSTRUCTING PROJECTORS FOR INFINITE GROUPS

Symmetry operators $\{\hat{O}_k\}$’s commuting with $\hat{H}$ form Lie algebra $\mathcal{L}$:

$$[\hat{O}_1, \hat{O}_2] = \sum_k c_{12}^{(k)} \hat{O}_k, \hat{O}_k \in \mathcal{L}$$  \hspace{1cm} (21)

$$a\hat{O}_1 + b\hat{O}_2 \in \mathcal{L}, a, b \in \mathbb{C}$$  \hspace{1cm} (22)

Let us assume that we have $K$ linear independent elements of $\mathcal{L}$. There is a simple way to organize a compact Lie group $\mathcal{G}$ (which is an infinite continuous group):

$$g(\phi_1, \ldots, \phi_K) = \prod_{k=1}^{K} e^{i\phi_k \hat{O}_k} \in \mathcal{G}, \phi_k \in [0, 2\pi).$$  \hspace{1cm} (23)

It is possible to verify the group axioms using the closure relation for the algebra $\mathcal{L}$ (Eq. (21)). Of course, due to non-commutativity there are many ways to parametrize $\mathcal{G}$, two simple alternatives can be

$$g'(\phi_1, \ldots, \phi_K) = \exp \left[ \sum_{k=1}^{K} i\phi_k \hat{O}_k \right]$$  \hspace{1cm} (24)

$$g''(\phi_1, \ldots, \phi_K) = e^{i\phi_K \hat{O}_{K-1} \prod_{k=1}^{K-1} e^{i\phi_k \hat{O}_k}},$$  \hspace{1cm} (25)

where in the last equation the parametrization effectively introduces $\hat{O}_K$ via commutations between all other $\hat{O}_k$’s. There is a standard expression for projectors on irreducible representations of a compact group

$$P_\Gamma = \int d\tilde{\phi} \chi_\Gamma(\tilde{\phi})^* g(\tilde{\phi}),$$  \hspace{1cm} (26)

where $\tilde{\phi} = (\phi_1, \ldots, \phi_K)$, $\chi_\Gamma(\tilde{\phi})$ is the character of the $\Gamma$ irreducible representation, and $d\tilde{\phi}$ is the Haar measure defined over the group domain.
To illustrate Eq. (26) in action let us consider the SO(3) group where the group is parametrized using the Euler angles (this parametrization is similar to that in Eq. (25))

\[ g(\alpha, \beta, \gamma) = e^{i \alpha \hat{J}_z} e^{i \beta \hat{J}_y} e^{i \gamma \hat{J}_z}, \]

\[ \alpha, \gamma \in [0, 2\pi), \beta \in [0, \pi] \]

and the projector to a particular \( \hat{J}^2 \) and \( \hat{J}_z \) eigenstate is

\[ \hat{P}_{jm} = \frac{2j + 1}{8\pi^2} \int d\Omega \langle jm| g(\Omega) |jm \rangle \int d\Omega \]

\[ \hat{P}_{jm} = \left( j + \frac{1}{2} \right) \int_0^\pi d\beta \sin(\beta) \langle jm| e^{i \beta \hat{J}_y} |jm \rangle e^{i \beta \hat{J}_y} \]

The eigenstates \( |jm \rangle \) can be found independently as the basis of irreducible representations for the Lie algebra using the highest weight theorem and the corresponding method.\(^16\) Thus, knowledge of irreducible representations for the corresponding Lie algebra is essential element of constructing projectors via the Lie group exponential map.

**APPENDIX B: COMMUTATION OF PROJECTORS FOR COMMUTING OPERATORS**

Commutation of projectors on eigen-subspaces for commuting operators is straightforward to show if we consider all such operators in the common eigenstate basis

\[ \hat{O}_i = \sum_j o_j^{(i)} \sum_k |\phi_{j,k}\rangle \langle \phi_{j,k}| \]

\[ = \sum_j o_j^{(i)} \hat{P}^{(i)}_j, \quad i = 1, 2 \]

Here index \( k \) enumerates eigen-basis states corresponding to the same eigenvalue. Then, the commutator of the projector operators is

\[ \left[ \hat{P}^{(1)}_1, \hat{P}^{(2)}_2 \right] = \sum_{k,k'} |\phi_{j,k}\rangle \langle \phi_{j,k'}| \langle \phi_{l,k'}| \hat{P}^{(i)}_{j} \hat{P}^{(i)}_{l} |\phi_{j,k}\rangle \langle \phi_{j,k}| = 0. \]

The last equality is a consequence of the equalities for inner products \( \langle \phi_{l,k'}| \phi_{j,k} \rangle = \delta_{kk'} \delta_{jl} \) and \( \langle \phi_{j,k}| \phi_{l,k'} \rangle = \delta_{kk'} \delta_{jl} \). In the case when these inner products are 0, \( \hat{P}^{(1)}_1 \hat{P}^{(2)}_2 \equiv 0 \) thus pairing such projectors will not give rise to non-trivial operators. For example, if the product of two projectors is \( \hat{P}^{(1)}_1 \hat{P}^{(2)}_2 S^2_0 S_z = 1 \), it clearly is \( \equiv 0 \) because this combination violates the usual conditions on the ranges of eigenvalues of \( S^2 \) and \( S_z \) operators. Thus, knowledge of irreducible representations of the corresponding Lie algebra is crucial to avoid pairings of projectors that produce trivial operators.