Universal dynamics of a soliton after an interaction quench

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Received 16 April 2015, revised 4 June 2015
Accepted for publication 5 June 2015
Published 24 June 2015

Abstract

We propose a new type of experimentally feasible quantum quench protocol in which a quantum system is prepared in a coherent, localized excited state of a Hamiltonian. During the evolution of this solitonic excitation, the microscopic interaction is suddenly changed. We study the dynamics of solitons after this interaction quench for a wide class of systems using a hydrodynamic approach. We find that the post-quench dynamics is universal at short times, i.e. it does not depend on the microscopic details of the physical system. Numerical support for these results is presented using generalized nonlinear Schrödinger equation, relevant for the implementation of the proposed protocol with ultracold bosons, as well as for the integrable Calogero model in harmonic potential. Finally, it is shown that the effects of integrability breaking by a parabolic potential and by a power-law nonlinearity do not change the universality of the short-time dynamics.

Online supplementary data available from stacks.iop.org/jpa/48/28FT01/mmedia

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Recent years have witnessed a rising interest in non-equilibrium dynamics, which has been largely motivated by extraordinary progresses in the manipulation of many-body quantum systems [1–12]. This interplay between experimental and theoretical research is allowing the investigation of longstanding questions, such as under which conditions systems driven out of equilibrium can show universal behaviors and whether and how a system reaches equilibrium and possibly thermalizes [13–23], in particular in cold atom systems [24–28].

A particularly interesting protocol used to identify paradigms of out-of-equilibrium dynamics is provided by the quantum quenches, in which, typically, a system described by a Hamiltonian $H$ is prepared in its ground state and then, at a given moment of time, let evolved using a different Hamiltonian $H'$ [29]. Although a unitary evolution cannot relax the system to a true equilibrium, mounting evidences support a picture for which, in most cases, the expectation value of most local operators tending to a stationary value in the long time limit can be calculated through an effective density matrix describing the system [15–17, 30–36]: that is, an arbitrary out-of-equilibrium many-body state in general evolves toward a state that effectively cannot be distinguished from a mixed state.

In this work, inspired by these progresses and motivated by experimental considerations, we propose a new protocol based on a quench of the interaction on an inhomogeneous initial state localized in space: this protocol has the merit of showing universal dynamics already at short times after the quench, while to date most of the studies on universal effects in quantum quenches focus on long times. We focus in particular on one-dimensional (1D) systems, where localized solitonic states are present and stable in a variety of models, even though we expect that our results are valid in any dimension provided that localized excitations exist and are stable and that the hydrodynamic description of the dynamics is valid. Specifically, we propose to prepare a 1D system in a particular localized, coherent excited state, one that in the hydrodynamic limit can be characterized as a soliton, that is, an excitation whose density and velocity profiles evolve in time without (almost) changing their shapes. At some moment during its evolution, a control parameter of the microscopic Hamiltonian is changed suddenly: this modification of the interaction strength reflects itself, for instance, in the change of the scattering length and of the speed of sound and can be triggered experimentally in ultracold atom setups through the trapping or with an external magnetic field [1]. After this quench, the system is let evolved according to the new Hamiltonian. We find that, immediately after the quench, the initial profile splits into two ’bumps’ (over the background density): one moving in the same direction as the initial excitation, the other in the opposite. We predict the velocities and shape of these chiral profiles for short time after the quench and we find them to be expressible in a universal way. These findings for a global quench on a localized excitation have to be contrasted with the usual quantum quench protocol discussed earlier, in which universality typically emerges at very large times.

The hydrodynamic description

To study the effect of an interaction quench on a localized excitation, one needs to prepare the quantum system in an excited state that would propagate for a certain time without dispersing
(or dissipating). From a hydrodynamic point of view, such configuration is called a soliton. However, to date, the construction of a quantum state with the degree of correlation necessary to evolve like a soliton has been challenging. In fact, different attempts to superimpose excited states to generate stable localized excitations have generated unstable configurations which in time lose coherence and fall apart [37, 38]. This failure of theoretical attempts indicates that some clever insight is needed to generate a quantum state that propagates without changing its macroscopic properties, while such configurations are easily produced in the laboratories as a results of certain non-equilibrium dynamics. Thus, for the moment we abstain from discussing the quantum nature of a solitonic excitation and we proceed from the empirical observation that such excitations are commonly observed in experiments and can nowadays be manipulated with remarkable precision [39–41]. From a mathematical point of view, solitons (and in particular multi-soliton solutions) are a characteristic feature of integrable differential equations [42]. Nonetheless, solitonic waves are commonly observed in nature, both in classical physics [43], as well as a collective behavior of a quantum many-body systems [39], proving that, in certain regimes, one can neglect all other contributions and describe the system accurately by an integrable equation, thus justifying the existence of very long-lived excitations.

We will focus on systems whose quantum dynamics can be described by a hydrodynamic approach: this approach is successfully used for a wide class of systems [44] including in particular trapped ultracold atoms [45]. The hydrodynamics of a quantum system may be derived, for instance, from an effective single-particle mean-field equation, provided a functional energy is guessed or derived (this point of view is typical of density functional theory approaches [46]): a major example is given by weakly interacting Bose gases where the nonlinear Schrödinger equation (NLSE) provides equilibrium [47] and dynamical [48] properties of the corresponding quantum system.

For quantum systems for (and the regimes in) which the hydrodynamic approach is valid, one can write as usual [44] the continuity and Euler equations for the density and velocity fields

\[\dot{\rho} + \partial (\rho v) = 0; \quad \dot{v} + \partial A = 0, \tag{1}\]

\[A \equiv \frac{v^2}{2} + \omega (\rho) - A (\rho) \left( \partial \sqrt{\rho} \right)^2 - A (\rho) \frac{\partial \sqrt{\rho}}{\sqrt{\rho}}, \tag{2}\]

where \(\omega (\rho)\) is the specific enthalpy and \(A (\rho)\) is related to the quantum pressure: both \(A\) and \(\omega\) are model-dependent.

Linearization of the hydrodynamic equation around a constant background leads to sound waves (the Bogolioubov modes). As derived in [49, 50], in 1D for configurations with small, smooth perturbations over the background \(\rho_0\) it is possible to retain the leading nonlinear and dispersive terms, which, under the condition of locality of the interactions, have the universal form of the KdV equation:

\[\dot{u}_x \equiv \partial \left[ cu_x + \frac{\zeta}{2} u_x^2 - a \partial^2 u_x \right] = 0, \tag{3}\]

where \(c \equiv \sqrt{\rho_0 \omega_0'}\) is the sound velocity \((\omega_0' \equiv \partial, \omega |_{\rho_0})\) and the nonlinear and dispersive coefficients are given by

\[\zeta \equiv \frac{c}{\rho_0} + \frac{\partial c}{\partial \rho_0}, \quad a \equiv \frac{A (\rho_0)}{4c}. \tag{4}\]
Here, a formal (small) expansion parameter $\epsilon$ was introduced, so that the fields could be expanded around the static background as

$$\rho(x, t) = \rho_0 + \epsilon \rho^{(1)}(x, t) + \epsilon^2 \rho^{(2)}(x, t) + \ldots, \tag{5}$$

$$v(x, t) = \epsilon v^{(1)}(x, t) + \epsilon^2 v^{(2)}(x, t) + \ldots. \tag{6}$$

The first order terms evolve according to the KdV:

$$u_\pm = \rho^{(1)} = \frac{c}{\partial_t} v^{(1)}. \tag{7}$$

Since KdV is a chiral equation, the $\pm$ refers to the two chiral components of the fields. We neglected the interaction between the left and right moving sectors, since locality implies that this effect is relevant only when the two are overlapping, but this happens for short times, as they pass through each other with a relative velocity of approximately $2c$ [50].

**The interaction quench**

Let us denote with $g$ some microscopical parameter capturing the strength of the interaction in the quantum Hamiltonian. Suddenly, during the evolution of the soliton, we change the interparticle coupling to $g'$, which reflects itself in the change of the effective parameters $\omega$ and $A$ in (1), (2) and hence of $c$, $\zeta$, and $\alpha$ in (3). We denote with a prime all the post-quench parameters, calculated using $g'$. As a consequence of this global quantum quench, the initial profile is not anymore a soliton of the new system and hence will not be stable under the new evolution.

The profile experiences the quench as an external perturbation, to which it reacts by splitting into a transmitted and reflected component, exactly as it would happen to a linear wave, if the sound velocity was suddenly changed. A typical nonlinear quench dynamics of this type for the dark soliton of the NLSE is shown in figure 1: one profile moves forward in
the same direction as the initial soliton (the transmitted one), while the other moves in the opposite direction (the reflected one).

Hence, we make the ansatz that immediately after the quench we have

\[ u(x, t) = u'(x - V_t t) + u'(x - V_t t), \tag{8} \]

assuming that for short times after the quench the two chiral profiles evolve without changing their shapes significantly. If the initial soliton has \( V > 0 \), the reflected velocity \( V_r < 0 \) and the transmitted one is \( V_t > 0 \). Imposing continuity of the solution and the conservation of momentum at \( t = 0 \) constraint the post-quench configuration to be

\[ u(x, t) = R \ s(x - V_t t) + T \ s(x - V_t t), \tag{9} \]

that is, we find that the two chiral profiles maintain the same functional shape as the original soliton, but with their heights reduced by the reflection and transmission coefficients, which are found to be

\[ R(V, V_r, V_t) = \frac{V_t - V}{V_t - V_r}, \quad T(V, V_r, V_t) = \frac{V - V_r}{V_t - V_r}. \tag{10} \]

We remark that, so far, we only applied kinematic considerations (we made no use of the KdV), which alone are not sufficient to determine the velocities \( V_r, V_t \).

In the corresponding linear problem, all waves would move at the speed of sound, that is \( V = c \) and \( V_r = -V_t = c' \). Hence, sound waves would be scattered by an interaction quench with

\[ R_{\text{linear}} = \frac{1}{2} \left[ 1 - \frac{c}{c'} \right], \quad T_{\text{linear}} = \frac{1}{2} \left[ 1 + \frac{c}{c'} \right]. \tag{11} \]

Naturally, for a small quench \((c' \approx c)\) the reflected wave has vanishing amplitude, while for large quenches \((c' \gg c)\) both waves are comparable.

The nonlinear terms in (3) make the different profiles move at different velocities. The stability of the initial soliton is given by a balancing of opposite effects, coming from the nonlinear and dispersive terms in (3). It can be shown that the average velocity of the solitonic solution is that of its barycenter, where the dispersion is perfectly balanced and \( V \) is determined only by the nonlinear contribution. Using this observation, we can estimate the velocities of the two chiral profiles to be\(^8\)

\[ V_r = -[c - \eta R(c - V)] \frac{c'}{c}, \tag{12} \]

\[ V_t = [c - \eta T(c - V)] \frac{c'}{c}, \tag{13} \]

where we introduced the universal parameter

\[ \eta \equiv \frac{c}{c'} \frac{c'}{c} = \frac{1 + \frac{\rho_0}{c} \frac{\partial c'}{c' \partial \rho_0}}{1 + \frac{\rho_0}{c} \frac{\partial c}{c \partial \rho_0}}. \tag{14} \]

\(^8\) For the explicit calculation, see the supplementary material of universal dynamics of a soliton after a quantum quench. (stacks.iop.org/jpa/48/28FT01/mmedia)
and where the $T$ and $R$ are found consistently to be

\[
R = \frac{1}{2}\left[ 1 - \frac{c}{c'} \eta \frac{V}{V + (1 - \eta) c} \right],
\]

\[
T = \frac{1}{2}\left[ 1 + \frac{c}{c'} \eta \frac{V}{V + (1 - \eta) c} \right]
\]

We note that these expressions are completely universal and depend on the microscopics of the system only through the parameter $\eta$, which in turn depends on $A$ and $\omega$ ($\eta \approx 1$ for the 1D NLSE). In particular, everything can be made dimensionless by measuring velocities in unit of the sound velocity (for instance, before the quench). Experimentally, the velocities and heights of the two profiles can be measured and checked against our predictions through a time-of-flight experiment, once $\eta$ is measured using (14).

If we assume that $\omega(\rho)$ in (2) is a simple monomial of the density ($\omega(\rho) \propto \rho^{\gamma-1}$), then $\eta = 1$ and the formulae simplify to

\[
R = \frac{1}{2}\left[ 1 - \frac{c}{c'} \right], \quad T = \frac{1}{2}\left[ 1 + \frac{c}{c'} \right],
\]

\[
V_r = -(T c + R V)\frac{c'}{c}, \quad V_t = (R c + T V)\frac{c'}{c}.
\]

We note that these reflection and transmission coefficients are the same as of the linear process (11), although accompanied by non trivial velocities for the profiles.

**1D ultracold bosons**

Localized excitations are easy to prepare in a cold atom experiments: using a phase mask it is possible to create long-lived excitations that can travel through the system without significantly changing their shape. The initial momentum can be given by suitably varying the trap potential, as it has been done to induce and study the dynamics of dark solitons in ultracold bosonic [40] or fermionic [41] condensates. Moreover, the interaction between bosons can be varied by changing the scattering length [45]: our interaction quench can be accomplished by a rapid change of the magnetic field, easily feasible in present-day experiments. Moreover, the splitting of the solitons can be straightforwardly detected by interference patterns measurements.

We write the 1D NLSE in the generalized form

\[
i\hbar \partial_t \psi = \left\{ -\frac{\hbar^2}{2m} \partial_{xx} + f(\rho) \right\} \psi.
\]

where $\rho(x, t) = |\psi(x, t)|^2$. The choice $f(\rho) = g\rho$ corresponds to the usual NLSE describing bosons with contact interaction in the weakly interacting limit. The generalized NLSE (GNLSE) reduces to (1), (2) with the ansatz $\psi = \sqrt{\rho} e^{it\int_{-\infty}^x v(x') dx'}$, with $\omega(\rho) = \frac{\int_0^\rho}{m}$ and $A = \frac{\hbar^2}{2m}$ (of course $\omega(\rho) = \frac{\rho}{m}$ for the usual NLSE). Several 1D classical mean-field equations have been proposed and used to reproduce the quantum dynamics of the 1D Bose gas for not so small interactions [51–54] and agreement between NLSE results, dynamics of the quantum model and experimental findings is found, and expected to be more robust at short times, that is, the regime we are considering. In particular short times have to be considered the ones much smaller than $\hbar/\Delta E$, where $\Delta E$ is the energy difference between the expectation values.
of the Hamiltonian post- and pre-quench, even though numerical simulations show that the results are stable also for longer times.

The validity of the hydrodynamic approach for the 1D Bose gas has been shown in [54, 55], while the hydrodynamic equations (1), (2) have been derived for 1D ultracold systems in [49]. We observe that in our approach the NLSE and the GNLSE are treated on equal footing since the analytical results of the GNLSE (19) are obtained using the correct $\omega(\rho)$ in the hydrodynamic equations.
Therefore, without loss of generality, in the following we present our results for the dark soliton of the NLSE [50] subjected to the interaction quench. In figures 2–3 we plot the results of our numerical simulations versus the analytical predictions discussed so far, finding a remarkable agreement. As the reflected and transmitted profiles are created superimposed at the time of the quench, they need to evolve for a certain time before they can be distinguished. In this time, their shapes evolve (and they also interact with one-another) and this introduces additional noise in the simulations. Moreover, in this way we measure the velocity of the peaks of the profile, which could differ from the center-of-mass velocity due to dispersion, since these profiles are not solitons. Nonetheless, a remarkable good agreement is found, which gets better for larger interaction quenches, since the dispersion effects we neglect are expected to become smaller. Furthermore, as expected for darker grey solitons (i.e., for larger solitons) the agreement becomes worse.

In figure 2 we plot the measured ratio between the heights of the two chiral profiles versus the ratio $R/T$ from (17) for different quenches and with different velocities of the initial soliton, while in figure 3 we compare the measured peak velocities with (18). In all the numerical points the values are computed at the mid-time between the instant in which there are two bumps and the one in which a third bump emerges (see footnote 7). In addition to the NLSE, to test the universality of our results, we also computed the same quantities using a parabolic confinement (i.e., adding $V\psi$ to the NLSE with $V = (1/2)\omega^2 x^2$) or a power-law NLSE (i.e., having a nonlinear term of the form $|\psi|^\alpha \psi$): both are non-integrable deformations of the NLSE and support the universality of our results.

**Calogero model**

As a further check on our ansatz (8), we compare our predictions against a microscopical evolution. The rationale for such check is that, even if we know that for a (classical or quantum) system the hydrodynamic description is valid, our ansatz (8) may be wrong in presence of general non-local interactions. For this experiment, we use the classical Calogero model, given by the Hamiltonian

$$H = \frac{1}{2m} \sum_{j=1}^{N} p_j^2 + \frac{\hbar^2}{2m} \sum_{j<k} \frac{\lambda^2}{(x_j - x_k)^2} + \frac{\omega^2}{2} \sum_{j=1}^{N} x_j^2,$$

(20)

with a dimensionless coupling constant $\lambda$. This is a classical model which is integrable even in the presence of an external parabolic potential. Its hydrodynamic description is thus not the KdV, but a different integrable equation, in the family of the Benjamin–Ono equation [56]. It differs from the KdV by its dispersive term and by the fact that its solitons have longer (power-law) tails. Another difference is that this system supports supersonic bright solitons, instead of the subsonic dark ones of local models [57]. Nonetheless, it has $\eta = 1$ and we find that the center-of-mass velocities and transmission/reflection coefficients have the same universal expression (17), (18). In our numerical simulation, we simulated the Newtonian evolution of $N = 301$ particles which initially evolve as a soliton of the pre-quench system and we follow their dynamics after the quench. We notice the correct splitting of the density

9 It is possible to include this effect by accounting for dispersion and the estimate for the peak velocity are also expressible in universal terms.

10 The data for the power-law NLSE are generated starting from the integrable NLSE soliton. This approximation and the effect of dispersion are probably accountable for some deviation from the expected behavior in figure 3.

11 We also quenched the trapping frequency $\omega$ of the same amount, in order to freeze the background (ground-state) density, that otherwise would oscillate, as discussed in [58].
field into the two chiral profile and we see from figures 2–3 that this microscopic dynamics agrees remarkably well with our analytical predictions. We stress that, although in this case the numerical experiment simulates a classical, Newtonian particle dynamics, it still shows a clear chiral splitting of the profiles due to the quench.

Conclusions

We have proposed a novel quantum quench protocol, where we initially prepare the system in a localized excitation and evolve it after changing the interaction strength and we studied this interaction quench of quantum systems using a hydrodynamic approach. We found that the dynamics immediately after the quench is universal, i.e. it does not depend on the details of interaction, but only on the strength of the quench, measured from macroscopic parameters, such as the speed of sound, see (12), (13), (15) and (16). We checked our analytical predictions against the numerical simulation of the 1D NLSE also in presence of a parabolic potential and with power-law nonlinearity) and through the Newtonian evolution of the Calogero model in harmonic potential finding a remarkable agreement. We finally remark that the universal nature of our results implies that this protocol can also have ubiquitous applications in quantum optics, nonlinear waveguides and in other nonlinear systems.

Note added

During the final stages of this work, we were made aware of a parallel development by Gamayun et al [59], concerning the long time behavior of the NLSE after the quench protocol we discussed, where the integrability of the hydrodynamics leads to remarkable effects. After the submission of our paper on arXiv, a paper by Chiocchetta et al appeared [60] discussing the universality of the scaling at short times of a quantum system after a quench.

Acknowledgments

We acknowledge discussions with S Sotiriadis, D Schneble and A Polychronakos. FF was supported by a Marie Curie International Outgoing Fellowship within the 7th European Community Framework Programme (FP7/2007-2013) under the grant PIOF-PHY-276093. MK gratefully acknowledges support from the Professional Staff Congress City University of New York award # 68193-00 46.

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