On Goldfeld’s Proof of Siegel’s Theorem

Zihao Liu

*International Department, The Affiliated High School of SCNU, Email: mailto:travor_lzh@163.com

ABSTRACT

In this paper, we give a detailed account of Goldfeld’s proof of Siegel’s theorem. Particularly, we present complete proofs of the nontrivial assumptions made in his paper.

Keywords: Analytic number theory, Dirichlet L-function, Primes in arithmetic progressions, Siegel’s theorem, Siegel-Walfisz theorem

I. INTRODUCTION

As stated in Theorem 1, Siegel’s theorem is a result in multiplicative number theory concerning the lower bound of Dirichlet L-functions associated with quadratic primitive characters.

Theorem 1 (Siegel). For all $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that for any primitive quadratic character $\chi$ modulo $q > 1$,

$$L(1, \chi) > C(\varepsilon)q^{-\varepsilon}$$

(1)

By applying the mean value theorem for differentiable real-valued functions*, Theorem 1 allows us to establish a zero-free region† for $L(s, \chi)$:

Theorem 2 (Siegel). For all $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that for any primitive quadratic character $\chi$ modulo $q > 1$, its associated $L(s, \chi)$ is free of zeros in

$$s > 1 - C(\varepsilon)q^{-\varepsilon}$$

(2)

Siegel’s theorem in the form of Theorem 2 is significant because it provides the prime number theorem in arithmetic progressions with an error term independent of the choice of modulus:

Theorem 3 (Siegel-Walfisz†). Let $A$ be any fixed positive number and $\pi(x; q, a)$ denote the number of primes that are $\equiv a \pmod q$ and $\leq x$. Then for all $(q, a) = 1$ we have

$$\pi(x; q, a) = \frac{1}{\varphi(q)} \int_{\frac{x}{q}}^{x} \frac{du}{\log u} + O_A \left( \frac{x}{\log^A x} \right)$$

(3)

*Details can be found in §11.2 of [4] and §21 of [1]
†In fact, every lower bound of $L(1, \chi)$ associated with quadratic $\chi$ can be converted to a certain zero-free region using this argument.
‡See §22 of [1]
From a historical perspective, Theorem 1 is an outcome of Siegel’s [5] investigation on the algebraic properties of the Dedekind zeta function associated with the quartic number field \( K = \mathbb{Q}(\sqrt{q_1}, \sqrt{q_2}) \):

\[
f(s) = \zeta(s) L(s, \chi_1) L(s, \chi_2) L(s, \chi_1\chi_2) \tag{4}
\]

where \( \chi_1, \chi_2 \) denote primitive quadratic characters modulo \( q_1, q_2 \) respectively.

Siegel’s original proof of Theorem 1 uses algebraic number theory, but purely analytic proofs have been developed by Estermann[2] and Goldfeld[3]. Although Goldfeld’s method can lead to a proof of Theorem 1, he did not justify all the steps he took in the derivation. Notably, Goldfeld states without justification that

\[
1 \ll \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} f(s + \beta) \frac{x^s}{s(s + 1)(s + 2)(s + 3)(s + 4)} ds \tag{5}
\]

holds for all \( 0 < \beta < 1 \). The right hand side of (5) is undeniable positive, but if the \( \ll \) constant relies on the choice of \( q_2 \) then the subsequent steps in his paper will not lead to a valid proof of Theorem 1, so in this paper, we present a more complete version of Goldfeld’s proof that addresses these unproven assumptions.

II. PLAN FOR THE PROOF

Following the convention, we let \( \lambda \) denote the residue of \( f(s) \) at \( s = 1 \).

\[
\lambda = L(1, \chi_1) L(1, \chi_2) L(1, \chi_1\chi_2) \tag{6}
\]

Basically, we attack the problem by giving lower estimate for \( \lambda \). When its lower bound is combined with the upper bounds of various \( L(1, \chi) \), Theorem 1 comes out.

To obtain a positive underestimate of \( \lambda \), Goldfeld applied a variant of Perron’s formula to \( f(s) \), so the lower bound emerges from an application of residue theorem.

Since Perron’s formula connects partial sums with Dirichlet series, it would be necessary to investigate both of them before proving Theorem 1. In section III, we perform an extensive study of the partial sum associated with \( f(s) \) to deduce

**Theorem 4.** Let \( a_n \) be the Dirichlet series coefficient of \( f(s) \) and \( A_0(x, w) \) denote the partial sum of \( f(s) \):

\[
A_0(x, w) = \sum_{n \leq x} \frac{a_n}{n^w} \tag{7}
\]

and

\[
A_k(x, w) = \int_0^x A_{k-1}(y, w) dy \tag{8}
\]

Then for every nonnegative integer \( k \), there exists an absolute constant \( c_k \) and \( x_k \) such that \( A_k(x, w) \geq c_k x^k \) whenever \( x \geq x_k \).

Then, in section IV, we study the analytic properties of \( f(s) \) to derive a sharp upper bound for \( f(s) \) on the right half plane:

**Theorem 5.** For every \( \varepsilon > 0 \) and \( \sigma \geq -\varepsilon \). As \( |t| \to \infty \), there is

\[
f(s) \ll_\varepsilon (q_1 q_2)^{1+\varepsilon} |t|^{2+\varepsilon} \tag{9}
\]

Finally, in section V, we combine Theorem 4 and Theorem 5 via Perron’s formula to deduce Theorem 1.
III. ARITHMETICAL PROPERTIES OF \( f(s) \)

Before investigating the partial sum, we first focus on the properties of \( a_n \).

**Lemma 1.** \( a_1 = 1 \) and \( a_n \geq 0 \) for all \( n \in \mathbb{Z}^+ \).

**Proof.** It follows from the properties of Dirichlet series and Dirichlet convolution that \( a_n \) is multiplicative, meaning \( a_1 = 1 \). Taking logarithms on both sides, we see

\[
\log f(s) = \sum_p \sum_{m \geq 1} \frac{1}{mp^m} [1 + \chi_1(p^m)][1 + \chi_2(p^m)]
\]

(10)

This indicates that the Dirichlet series coefficients for \( \log f(s) \) is nonnegative, so are the those of \( f(s) \). Q.E.D.

**Corollary 1.** \( A_0(x, w) \geq 1 \) whenever \( x \geq 1 \).

Performing an induction on Corollary 1, we can prove Theorem 4.

**Proof of Theorem 4.** Suppose Theorem 4 is true for \( k = m - 1 \geq 0 \), then by definition

\[
A_m(x, w) \geq c_{m-1} \int_{x_{m-1}}^{x} y^{m-1} dy = \frac{c_{m-1}}{m} (x^m - x_{m-1})
\]

(11)

The rightmost quantity is \( \gg m x^m \), so we conclude there exists admissible \( c_m > 0 \) and \( x_m > 0 \) such that \( A_m(x, w) \geq c_m x^m \) whenever \( x \geq x_m \). Q.E.D.

**Remark.** Since constants appearing on the right hand side of (11) only depends on \( m \), Theorem 4 virtually provides positive lower bounds for \( A_k(x, w) \) that are independent of \( \chi_1 \) and \( \chi_2 \).

IV. ANALYTIC PROPERTIES OF \( f(s) \)

To obtain an upper estimate for \( f(s) \), we quote a classical result from literature:

**Lemma 2** (Corollary 10.10 of [4]). Let \( \chi \) be a primitive character modulo \( q > 1 \), and suppose \( \sigma \) lies in a fixed interval and \( |t| \to \infty \), then

\[
|L(s, \chi)| \asymp (q|t|)^{1/2 - \sigma} |L(1 - s, \overline{\chi})|^{\delta}
\]

(12)

This allows us to conclude that

**Lemma 3.** Let \( \chi \) be a primitive character modulo \( q > 1 \). For every \( \varepsilon > 0 \) and \( \sigma \geq -\varepsilon \), as \( |t| \to \infty \) we have

\[
L(s, \chi) \ll_{\varepsilon} (q|t|)^{1/2 + \varepsilon}
\]

(13)

**Proof.** By definition, we see that when \( \sigma = -\varepsilon < 0 \), the Dirichlet series expansion for \( L(1 - s, \chi) \) converges absolutely. Combining this fact with Lemma 2, we see that (13) holds for \( \sigma = -\varepsilon \). Now, it follows from Phragmén-Lindelöf theorem\(^8\) that (13) is valid throughout \( \sigma \geq -\varepsilon \) uniformly. Q.E.D.

**Remark.** We can develop an argument analogous to the proof of Lemma 3 to deduce for all \( \sigma \geq 0 \) and \( \varepsilon > 0 \),

\[
\zeta(s) \ll_{\varepsilon} |t|^{1/2 + \varepsilon}
\]

(14)

Plugging Lemma 3 and (14) into (4), we obtain Theorem 4.

\(^8\)We use the convention \( s = \sigma + it \) (where \( \sigma, t \in \mathbb{R} \)) throughout the entire paper.

\(^\dagger\)See §5.65 of [6].
V. Proof of Theorem 1

Applying Perron’s formula[7], we see that for all \( x \geq 1 \),
\[
A_1(x, w) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} f(s + w) \frac{x^{s+1}}{s(s+1)} \, ds \tag{15}
\]
Integrating on both side of (15) for three times and applying Theorem 4, we have the following result:

**Lemma 4** (Justification of (5)). There exists absolute constants \( X, M > 0 \) such that the following inequality
\[
M \leq \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} f(s + w) \frac{x^s}{s(s+1)(s+2)(s+3)(s+4)} \, ds := J(x, w) \tag{16}
\]
holds uniformly for \( x \geq X \).

To study the integral \( J(x, w) \), we set \( \lambda = L(1, \chi_1)L(1, \chi_2)L(1, \chi_1\chi_2) \) so that shifting the line of integration to \( 0 > \sigma = -w > -1 \) gives
\[
J(x, w) = \frac{\lambda x^{1-w}}{(1-w)w(w+1)(w+2)(w+3)} + f(w) \frac{x^w}{4!} \tag{17}
\]
\[
+ \int_{-w-i\infty}^{-w+i\infty} f(s + w) \frac{x^s}{s(s+1)(s+2)(s+3)(s+4)} \, ds \tag{18}
\]
The upper bound for the remaining integral can be deduced using Theorem 5:
\[
\int_{-w-i\infty}^{-w+i\infty} \ll \frac{(q_1q_2)^{1+\varepsilon}x^{-w}}{w(1-w)} \tag{20}
\]
Choosing \( w \) subtly allows us to omit the contribution of (18) when estimating \( J(x, w) \):

**Lemma 5.** For every \( \varepsilon > 0 \) there exists a primitive quadratic \( \chi_1 \) modulo \( q_1 \) and \( 1 - \varepsilon < \beta < 1 \) such that \( f(\beta) \leq 0 \) for all quadratic primitive character \( \chi_2 \).

**Proof.** If no quadratic primitive \( \chi \) can be found such that \( L(s, \chi) \) has a real zero in \((1 - \varepsilon, 1)\). Then we can use the fact that \( \lambda > 0 \)\(^1\) and the fact that \( \zeta(\sigma) < 0 \) for \( 0 < \sigma < 1 \)** to conclude \( f(\beta) < 0 \) for any \( 1 - \varepsilon < \beta < 1 \).

If such quadratic primitive \( \chi \) does exist, then we let \( \chi_1 = \chi \) and \( \beta \) be the real zero of \( L(s, \chi_1) \) in \((1 - \varepsilon, 1)\) so that \( f(\beta) = 0 \) independent of what \( \chi_2 \) is. Q.E.D.

**Remark.** Lemma 5 explains why the implied constant in Theorem 1 is not effectively computable because \( \chi_1 \) and \( \beta \) cannot be determined within finitely many steps.

Since \( \beta \) and \( q_1 \) are only associated with \( \varepsilon > 0 \), we can simplify \( J(x, w) \) significantly using Lemma 5 and (20) when \( w = \beta \):
\[
J(x, \beta) \leq \frac{\lambda x^{1-\beta}}{(1-\beta)(1-\varepsilon)} + O_{\varepsilon} \left( q_2^{1+\varepsilon}x^{-\beta} \right) \tag{21}
\]
Finally, we can start proving Siegel’s theorem. During the proof, \( b_1(\varepsilon), b_2(\varepsilon), \ldots \) always denote positive constants that only depend on \( \varepsilon > 0 \).

\(^1\)See Theorem 4.9 of [4]
**See Corollary 1.14 of [4]
Proof of Theorem 1. Plugging (21) into Lemma 4, we know that
\[ b_1(\varepsilon) < \lambda x^{1-\beta} + q_2^{1+\varepsilon} x^{-\beta} \] (22)

Now we choose \( x \) large enough so that the latter term is less than \( b_1(\varepsilon) \). This means that we can choose \( x \) large enough so that \( b_2(\varepsilon) - q_2^{1+\varepsilon} x^{-\beta} > 0 \), meaning that we can pick
\[ x^{\beta} = b_3(\varepsilon) q_2^{1+\varepsilon} \] (23)

where \( b_3(\varepsilon) > 0 \) is a large constant depending on \( \varepsilon \). Without loss of generality, we assume \( \varepsilon < 1/3 \), so that plugging (23) into (22) gives
\[ \lambda > b_4(\varepsilon) x^{-(1-\beta)} = b_4(\varepsilon) q_2^{-(1+\varepsilon)(1-\beta)/\beta} \] (24)
\[ > b_4(\varepsilon) q_2^{-(1+\varepsilon)/(1-\varepsilon)} > b_4(\varepsilon) q_2^{-2\varepsilon} \] (25)

To transfer the lower bound for \( \lambda \) to \( L(1, \chi_2) \), it suffices to note that for every nontrivial character \( \chi \) modulo \( q \)
\[ L(1, \chi) = \sum_{n \leq T} \frac{\chi(n)}{n} + O\left( \frac{q}{T} \right) \] (26)
as it indicates \( L(1, \chi) \ll \log q \) after setting \( T = q \). Consequently, we have
\[ L(1, \chi_2) > b_5(\varepsilon) q_2^{-2\varepsilon} (\log q_1)^{-1} (\log q_1 q_2)^{-1} \] (27)
\[ > b_6(\varepsilon) q_2^{-2\varepsilon} (\log q_1 q_2)^{-1} \] (28)

If we make \( q_2 \) be sufficiently large, then (28) gets simplified into
\[ L(1, \chi_2) > b_7(\varepsilon) q_2^{-2\varepsilon} (\log q_2)^{-1} > b_7(\varepsilon) q_2^{-3\varepsilon} \] (29)

Finally, we make \( b_7(\varepsilon) \) very small to ensure that (29) hold for small values of \( q_2 \), so the proof of Theorem 1 is complete. Q.E.D.

References

[1] Harold Davenport. *Multiplicative Number Theory*, volume 74 of *Graduate Texts in Mathematics*. Springer New York, New York, NY, 1980.

[2] T. Estermann. On Dirichlet’s \( L \) functions. *J. Lond. Math. Soc.*, 23:275–279, 1949.

[3] D. M. Goldfeld. A Simple Proof of Siegel’s Theorem. *Proceedings of the National Academy of Sciences*, 71(4):1055–1055, April 1974.

[4] Hugh L. Montgomery and Robert C. Vaughan. *Multiplicative number theory I: classical theory*. Number 97 in Cambridge studies in advanced mathematics. Cambridge University Press, Cambridge, UK ; New York, 2007. OCLC: ocm61757122.

[5] Carl Siegel. Über die classenzahl quadratischer zahlkörper. *Acta Arithmetica*, 1(1):83–86, 1935.

[6] E. C. Titchmarsh. *The theory of functions*. Oxford science publications. Oxford Univ. Press, Oxford, 2. ed., reprinted edition, 2002. OCLC: 249703508.

[7] E. C. Titchmarsh and D. R. Heath-Brown. *The theory of the Riemann zeta-function*. Oxford science publications. Oxford University Press, New York, 2nd ed edition, 1986.