Gordian unknots

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Abstract

Numerical simulations indicate that there exist conformations of the unknot, tied on a finite piece of rope, entangled in such a manner, that they cannot be disentangled to the torus conformation without cutting the rope. The simplest example of such a gordian unknot is presented.

Knots are closed, self-avoiding curves in the 3-dimensional space. The shape and size of a knot, i.e. its conformation, can be changed in a very broad range without changing the knot type. The necessary condition to keep the knot type intact is that during all transformations applied to the knot the curve must remain self-avoiding. From the topological point of view, all conformations of a knot are equivalent but if the knot is considered as a physical object, it may be not so. Let us give a simple example. Take a concrete, knotted space curve $K$. Imagine, that $K$ is inflated into a tube of diameter $D$. If $K$ is scaled down without scaling down $D$, then there is obviously a minimum size below which one cannot go without changing the shape of $K$. Diminishing, in a thought or computer experiment, the size of a knot one arrives to the limit below which in some places of the knot the impenetrability of the tube on which it has been tied would be violated.

Consider a knot tied on a piece of a rope. If the knot is tied in a loose manner, one can easily change its shape. However, the range of transformations available in such a process is much more narrow than in the case of knots tied on an infinitely thin rope. Limitations imposed on the transformations used to change the knot shape by the fixed thickness and length of the rope may make some conformations of the knot inaccessible from each other. The limitations can be in an elegant manner represented by the single condition that the global curvature of the knot cannot be larger than $2/D$ \textsuperscript{[1]}. That it is the case we shall try to demonstrate in the most simple case of the unknot. The knot is a particular one since we know for it the shape of the ideal, least rope consuming conformation \textsuperscript{[2]}. The simplest shape of the unknot is obviously circular. If the knot is tied on the rope of diameter $D$ the shortest piece of rope one must use to
form it has the length $L_{min} = \pi D$. If one starts from the circular conformation of the unknot tied on a longer piece of rope, the length of the rope can be subsequently reduced without changing the circular shape until the $L_{min}$ value is reached.

Consider now a different, entangled conformation of the unknot tied on a piece of rope having the length $L > L_{min}$. Can it be disentangled to the canonical circular shape? Are there such conformations of the unknot, which cannot be disentangled to a circle without elongating the rope? For obvious reasons we propose to call such conformations gordian. In what follows we shall report results of numerical experiments suggesting existence of the gordian conformations of the unknot.

Imagine that the entangled conformation of the unknot is tied on piece the ideal rope of diameter $D$ and length $L > L_{min}$. The ideal rope is perfectly flexible but at the same time perfectly hard. Its perpendicular cross-sections remain always circular. The diameters of all the cross-sections are equal $D$. None of the circular cross-sections overlap. The surface of the rope is perfectly slippery. In such conditions one may try to force the knot to disentangle itself just by shortening the rope length. Such a process, in which the knot is tightened, can be easily simulated with a computer. The details of SONO (Shrink-On-No-Overlaps), the simulation algorithm we developed, are described elsewhere[3]. As shown in[3], SONO disentangles some simple conformations of the unknot. See Fig.1. It manages to cope also with the more complex conformation proposed by Freedman[4] disentangled previously by the Kusner and Sullivan algorithm minimizing the Möbius energy[5].

The steps of the construction of the Freedman conformation, are as follows
Figure 2: Evolution of the length of the rope in a process in which SONO disentangles the Freedman’s $F(3_1, 3_1)$ conformation of the unknot. Initially, the loose $F(3_1, 3_1)$ conformation is rapidly tightened. Then, the evolution slows down. At the end of the slow stage one of the end knots becomes untied. Subsequently, the other of the end knots becomes untied. Eventually the conformation becomes disentangled and the unknot reaches its ideal, circular shape. The lower curve shows the evolution of the rope length in the much faster process in which the unknot shown in Fig.1 becomes disentangled.

3:
1. Take a circular unknot and splash it into a flat double rope band.
2. Tie overhand knots on both ends of the band and tighten them. (From the point of view of the knot theory, the overhand knots are open trefoil knots.)
3. Open and slip the end loops over the bodies of the overhand knots, so that they meet in the central part of the band.
4. Move the rope through both overhand knots so that the loops become smaller.

In what follows we shall refer to the conformation as $F(3_1, 3_1)$. To disentangle $F(3_1, 3_1)$, one must slip the loops back all around the bodies of the overhand knots, which is difficult, since the move needs first making the loops bigger.

How the SONO algorithm copes with this task is shown in Fig.2, where con-
secutive stages of the disentangling process are shown. Tightening the $F(3_1, 3_1)$ conformation SONO algorithm brings it to the very compact state, which seems at the first sight to be impossible to disentangle. The end loops are very tight and they seem to be too small to slip back over the bodies of the overhand knots. However, as the computer simulations prove, there exists a path in the configurational space of the knot along which the loops slowly become bigger and one of them slips over the body of the overhand knot. Then, the disentangling process proceeds without any problems. Results of the computer experiments we performed suggest strongly, that the $F(3_1, 3_1)$ conformation is not gordian.

The construction of original Freedman entanglement may be modified making it more difficult to disentangle. The simplest way of doing this is to change the end trefoil knots to some more complex knots. For the sake of brevity we will use $F(K^{(1)}, K^{(2)})$ symbols to indicate with what kind of the Freedman conformation of the unknot we are dealing with. Results of computer simulations we performed prove that the $F(4_1, 4_1)$ conformation is also disentangled in the knot tightening process. However, the $F(5_1, 5_1)$ conformation proves to be resistant to SONO algorithm. Fig.3 shows consecutive stages of the tightening process. The initial conformation, loose, it becomes tight soon. Then the evolution process slows down and eventually stops. The final conformation proves to be stable. The gordian conformation has been reached.

Eperimenting with knots tied on real, macroscopic ropes or tubes is by no means easy. First of all, the surface of any real rope is never smooth and strong friction often stops the walk within the configurational space of a knot tied on such a rope. The role of friction was exposed by Kauffman. Friction can be significantly reduced, however, when a knot is tied on a smooth nanoscopic filament, e.g. a nanotube, or on a thermally fluctuating polymer.

Figure 3: SONO tightens the $F(5_1, 5_1)$ conformation of the unknot, but does not manage to disentangle it.
molecule. There exists another, less obvious, factor which makes laboratory experiments on knots difficult: the Berry’s phase, to be more precise, its classical counterpart - the Hannay’s angle. Modern ropes are often constructed in the following manner: a parallel bundle of smooth filaments is kept together by a tube-like, plaited cover. As easy to check, such ropes are much easier to bend than to twist. Forming a knot on a rope, one has to deform it. In view of what was said above, the deformation applied is rather bending than twisting. Avoiding the twist deformations one follows the procedure known as the parallel transport. As a result, when at the final stage of the knot tying procedure the ends of the rope meet, they are in general rotated in relation to each other: the misfit angle is the Hannay’s angle. As shown in and , the Hannay’s angle stays in a simple relation,

\[ 1 + W_r = \left( \frac{A}{2\pi} \right) \mod 2 \]

with the writhe of the knot into which the rope has been formed. Splicing the ends of the rope one fixes the misfit angle . Consequently, the writhe value becomes fixed as well. As a result, any further changes of the conformation of the knot become very difficult and are basically restricted to the manifold of constant writhe. (The specific construction of the Freedman conformations makes them achiral. Their writhe is equal zero.)

The natural question arises, if the impossibility of disentangling the gordian conformation does not stem from the described above friction and writhe factors. We feel emphasize, that it is not the case. The rope simulated by the SONO algorithm is perfect: it is frictionless and utterly flexible. It has no internal, parallel bundle structure and it accepts any twist. Problems with disentangling the gordian conformations are purely steric. Tightening the Freedman conformation SONO brings it into a cul-de-sac of what mathematicians call thickness energy. To get out of it, one needs elongate the rope. By how much? We do not know yet the answer to this question.

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