Estimate for concentration level of the Adams functional and extremals for Adams-type inequality*

José Francisco Alves de Oliveira
Departamento de Matemática
Universidade Federal do Piauí
64049-550 Teresina, PI, Brazil
jfoliveira@ufpi.edu.br

Abiel Costa Macedo
Instituto de Matemática e Estatística
Universidade Federal de Goiás
74001-970 Goiânia, GO, Brazil
abielcosta@ufg.br

Abstract

This paper is mainly concerned with the existence of extrema ls for the Adams inequality. We first establish an upper bound for the classical Adams functional along of all concentrated sequences in the higher order Sobolev space with homogeneous Navier boundary conditions $W_{m,N}^{m,n} (\Omega)$, which in particular includes the classical Sobolev space $W_0^{m,n} (\Omega)$, where $\Omega$ is a smooth bounded domain in Euclidean $n$-space. Secondly, based on the Concentration-compactness alternative due to Do Ó and Macedo, we prove the existence of extremals for the Adams inequality under Navier boundary conditions for second order derivatives at least for higher dimensions when $\Omega$ is an Euclidean ball.

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1 Introduction

Let $\Omega$ be a smooth domain in $\mathbb{R}^n$, $n \geq 2$, with $n$-measure $|\Omega| < \infty$, and $W_0^{m,n} (\Omega)$ be the completion of $C_0^\infty (\Omega)$ in $W^{m,n} (\Omega)$, for positive integer $m < n$. Given $u \in C_0^\infty (\Omega)$ we will denote

$$
\nabla^m u = \begin{cases} 
\Delta^{m/2} u, & \text{if } m \text{ is even} \\
\nabla \Delta^{(m-1)/2} u, & \text{if } m \text{ is odd}
\end{cases}
$$

Adams in [1] proved that

$$
\sup_{u \in W_0^{m,n} (\Omega), \| \nabla^m u \| \leq 1} \int_{\Omega} e^{\beta |u|^{n-m}} \, dx < \infty, \quad \text{if and only if } \beta \leq \beta_0,
$$

where

$$
\beta_0 = \beta_0 (m,n) = \begin{cases} 
\frac{n}{\omega_{n-1}} \left[ \frac{\pi^{n/2} \Gamma (\frac{n-m}{2})}{\Gamma (\frac{n}{2})} \right]^{n/(n-m)}, & \text{if } m \text{ is odd} \\
\frac{n}{\omega_{n-1}} \left[ \frac{\pi^{n/2} \Gamma (\frac{n-m}{2})}{\Gamma (\frac{n}{2})} \right]^{n/(n-m)}, & \text{if } m \text{ is even}
\end{cases}
$$

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in which $\Gamma(x) = \int_0^1 (-\ln t)^{x-1} dt$, $x > 0$ is the gamma Euler function and $\omega_{n-1}$ is the area of the surface of the unit $n$-ball. Inequality (1.1) is the extension for higher order derivatives of that classical one due to Moser [30], which improved the earlier results due to Trudinger [42], Pohozaev [33] and Yudovich [24] and it is currently known as Adams inequality or Adams-Moser-Trudinger inequality.

Adams inequality has a broad range of applications in partial differential equations and geometric analysis, see for instance [14, 15, 37, 38, 5, 6, 18], and there are a lot of extensions and generalizations, among which we point out the works [3, 2] for Lorentz spaces, [8, 41] for Zygmund spaces, [21] for extensions to Riesz subcritical potentials, [26, 35, 36, 9] for the entire space $\Omega = \mathbb{R}^n$, and [17, 31, 44, 45] for Riemannian manifolds. For more related results, see [10, 20, 25, 16, 46, 13] and references quoted therein.

Tarsi [41] extends (1.1) to functions with homogeneous Navier boundary conditions. More precisely, it was proved that

$$\sup_{u \in W^m, n_0 N(\Omega)} \int_{\Omega} e^{\beta |u|^\frac{n}{n-m}} dx < \infty, \text{ if and only if } \beta \leq \beta_0,$$

(1.3)

where

$$W^m, n_0 N(\Omega) := \{ u \in W^m, n N(\Omega) : u_{|\partial\Omega} = \Delta^j u_{|\partial\Omega} = 0 \text{ in the sense of trace}, 1 \leq j < m/2 \}.$$

We are interested in finding extremal function for the Adams Inequality. In this direction we provide the following estimate for Adams functional along of all concentrated sequences:

**Theorem 1.1.** Let $m, n$ be positive integers, $n \geq 2$ and $n > m$, and $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$. Let $(u_i) \subset W^m, n_0 N(\Omega)$, with $\|\nabla^m u_i\|_{\frac{n}{n-m}} = 1$ be a sequence concentrating at $x_0 \in \Omega$, i.e.,

$$\lim_{i \to \infty} \int_{\Omega \setminus B_r(x_0)} \|\nabla^m u_i\|_{\frac{n}{n-m}} dx = 0, \text{ for any } r > 0.$$

Then

$$\lim_{i} \sup \int_{\Omega} e^{\beta_0 |u|_m} dx \leq |\Omega| \left( 1 + e^{\psi(\frac{n}{n-m}) + \gamma} \right),$$

where $\gamma = \lim_{n \to \infty} \left( \sum_{j=1}^{n} (1/j) - \ln n \right)$ is the Euler-Mascheroni constant and $\psi(x) = \frac{d}{dx} (\ln \Gamma(x))$ is the classical Psi-function.

Since $W^m, n_0 N(\Omega)$ is a subspace of $W^m, n N(\Omega)$, as a direct consequence of the Theorem 1.1, we can highlight the following:

**Corollary 1.2.** For $\gamma$ and $\psi$ as in Theorem 1.1, we have

$$\lim_{i} \sup \int_{\Omega} e^{\beta_0 |u|_m} dx \leq |\Omega| \left( 1 + e^{\psi(\frac{n}{n-m}) + \gamma} \right),$$

(1.4)

for any $(u_i) \subset W^m, n_0 N(\Omega)$ under the same hypotheses of Theorem 1.1.

Although Corollary 1.2 is an easy consequence of Theorem 1.1 it is new and has merit itself. Indeed, from the Concentration-Compactness alternative [14, Theorem 1] (Theorem 2.3 below), in order to ensure the existence of extremal functions for the classical Adams inequality (1.1), it is now sufficient to show that there are test functions $u \in W^m, n_0 N(\Omega)$ such that

$$\|\nabla^m u\|_{\frac{n}{n-m}} = 1 \text{ and } \int_{\Omega} e^{\beta_0 |u|_m} dx > |\Omega| \left( 1 + e^{\psi(\frac{n}{n-m}) + \gamma} \right).$$
Hence, Corollary 1.2 sheds some new light on the existence of extremal functions for the inequality (1.1) in the critical case $\beta = \beta_0$. Actually, for $m = 1$ the existence of extremals it was proved in the series of papers [7, 19, 27, 39]; however concerning the higher order case $m > 1$, as far as we know, there are not so many results and we can only mention Lu and Yang [29], which proved the existence of extremals in the case $H^m_0(\Omega)$ with $\Omega \subset \mathbb{R}^{2m}$ is proved. We believe that the Corollary 1.2 is a significant contribution to solve completely this question.

With this approach, using the Concentration-Compactness alternative [14, Theorem 1] and Theorem 1.1, we will state the existence of extremals for Adams inequality under homogeneous Navier boundary conditions (1.3) for second order derivatives, at least when $\Omega$ is an Euclidean ball and $n$ is large enough.

**Theorem 1.3.** Let $B_R$ be the unit ball with radius $R > 0$ centered at $0 \in \mathbb{R}^n$. Then, there exists $u_0 \in W^{2, \frac{n}{n-2}}_{\lambda}(B_R)$, with $\|\Delta u_0\|_{\frac{n}{n-2}} \leq 1$ such that

$$
C_{\beta_0}(B_R) = \sup_{u \in W^{2, \frac{n}{n-2}}_{\lambda}(B_R) : \|\Delta u\|_{\frac{n}{n-2}} \leq 1} \int_{B_R} e^{\beta_0|u|^{\frac{n}{n-2}}} \, dx = \int_{B_R} e^{\beta_0|u_0|^{\frac{n}{n-2}}} \, dx
$$

provided that $n \geq 2T_0$, where $T_0$ is the smallest positive integer such that

$$
T_0 \geq 1 + \frac{1 + 36\sigma}{17 - 24\gamma} + \left[ 1 + \left( \frac{1 + 36\sigma}{17 - 24\gamma} \right)^2 + \frac{72\sigma}{17 - 24\gamma} \right]^\frac{1}{2} \approx 51.9233
$$

where $\sigma = 1 + 2/\sqrt{3}$ and $\gamma$ is the Euler-Mascheroni constant.

Theorem 1.3 is the first result on the existence of extremal function for Adams inequality under homogeneous Navier boundary conditions [41].

This paper is arranged as follows. In Section 2, we present some notations and results which will be used in the next sections. Section 3 is devoted to proof both the general estimate of Carleson and Chang type (cf. [7]) and the Theorem 1.1 for $m = 2$. In Section 4 we perform some test function computations to show that the supremum (1.5) surpass the estimation given in Theorem 1.1, which proves the Theorem 1.3. Finally, in Section 5 we prove Theorem 1.1 in the general case $m \geq 2$.

## 2 Preliminaries

In this section we present both the comparison theorem of Talenti [40, Theorem 1] (and some generalization) and the Concentration-Compactness alternative [14, Theorem 1]. We also present some estimates for the best constant of Hardy type inequalities due to Opic and Kufner [32].

### 2.1 Comparison theorem and concentration-compactness alternative

Let $\Omega^* = B_R$ be the ball of radio $R > 0$ centered at $0$ in $\mathbb{R}^n$ such that $|\Omega^*| = |\Omega|$. Let $u : \Omega \to \mathbb{R}$ be a measurable function. We denote by

$$
\#(s) := \inf \{ t \geq 0 : \{|x \in \Omega : |u(x)| > t| < s \}, \quad \forall s \in [0,|\Omega|],
$$

the decreasing rearrangement of $u$ and by

$$
*(x) := \#(\omega_n|x|^n), \quad \forall x \in \Omega^*,
$$
the spherically symmetric decreasing rearrangement of \( u \), where \( \omega_n \) is the volume of the unit ball on \( \mathbb{R}^n \).

In [40, Theorem 1] Talenti presented the result known as Talenti comparison principle, which in particular implies in the following result

**Theorem 2.1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain and \( f \in C_0^\infty(\Omega) \). If \( u \) is a solution of

\[
\begin{cases}
-\Delta u = f & \text{in } \Omega \\
u = 0 & \text{in } \partial\Omega,
\end{cases}
\]

and \( v \) is a solution of

\[
\begin{cases}
-\Delta v = f^* & \text{in } \Omega^* \\
v = 0 & \text{in } \partial\Omega^*.
\end{cases}
\]

Then \( v \geq u^* \) a.e. on \( \Omega^* \).

We note that the result is also true for a more general domain and function \( f \), see [40, Theorem 1]. Now by iterating the Theorem 2.1 together with the Maximum Principle we can obtain some comparison principle to the polyharmonic equation with Navier boundary condition which can be found in [22, Proposition 3], that is

**Proposition 2.2.** Let \( \Omega \subset \mathbb{R}^n, n \geq 2 \) be a smooth bounded domain and let \( q \geq 2n/(n+2) \). Let \( f \in C_0^\infty(\Omega) \) and let \( u \in W^{2k,q}_{0}(\Omega) \) be the unique strong solution of

\[
\begin{cases}
(\Delta)ku = f & \text{in } \Omega \\
\Delta^j u = u = 0 & \text{in } \partial\Omega, \quad j = 1, 2, \ldots, k-1.
\end{cases}
\]

Let \( v \in W^{2k,q}_{0}(\Omega^*) \) be the unique strong solution of

\[
\begin{cases}
(\Delta)kv = f^* & \text{in } \Omega^* \\
\Delta^j v = v = 0 & \text{in } \partial\Omega^*, \quad j = 1, 2, \ldots, k-1.
\end{cases}
\]

Then, \( v \geq u^* \) a.e on \( \Omega^* \).

In [14, Theorem 1], more precisely on [14, Remark 3], is established the concentration-compactness alternative that will be used in this work to prove Theorem 1.3.

**Theorem 2.3.** Let \( m \) be a positive integer with \( m < n \). Let \( u_i, u \in W^{m,p}_N(\Omega) \) and \( \mu \) be a Radon measure on \( \overline{\Omega} \) such that \( \|\nabla^m u_i\|_p = 1, \ u_i \rightharpoonup u \) in \( W^{m,p}_N(\Omega) \) and \( |\nabla^m u_i|^p \rightharpoonup \mu \) in \( \mathcal{M}(\Omega) \).

(i) If \( u \equiv 0 \) and \( \mu = \delta_{x_0} \), the Dirac mass concentrated at some \( x_0 \in \Omega \), then, up to a subsequence,

\[
e^{\beta_0 |u_i|^{p/(p-1)}} \rightharpoonup e\delta_{x_0} + \mathcal{L}_n \quad \text{in } \mathcal{M}(\overline{\Omega}), \quad \text{for some } c \geq 0,
\]

where \( \mathcal{L}_n \) is the Lebesgue measure in \( \mathbb{R}^n \).

(ii) If \( u \equiv 0 \) and \( \mu \) is not a Dirac mass concentrated at one point, then there are \( \gamma > 1 \) and \( C = C(\gamma, \Omega) > 0 \) such that

\[
\limsup_i \int_{\Omega} e^{\beta_0 \gamma |u_i|^{p/(p-1)}} \, dx \leq C.
\]
(iii) If \( u \neq 0 \), then, for \( \gamma \in [1, \eta) \), there is a constant \( C = C(\gamma, \Omega) > 0 \) such that

\[
\limsup_i \int_{\Omega} e^{\beta_0 \gamma |u|^p/(p-1)} \, dx \leq C,
\]

where

\[
\eta = \eta_{m,n}(u) := \begin{cases} 
(1 - \|\nabla (\Delta^k u)^*\|_p^{-1/(p-1)})^{-1} & \text{if } m = 2k + 1, \\
(1 - \|\nabla^m u\|_p^{-1/(p-1)})^{-1} & \text{if } m = 2k.
\end{cases}
\]

The above result represents an extension of Lions concentration-compactness alternative [28] (see also [43]) for higher order derivatives.

### 2.2 Hardy type inequalities

Let \( AC_{loc}(a,b) \) be the set of all locally absolutely continuous functions on interval \((a,b)\). Denote by \( W(a,b) \) the set of all functions measurable, positive and finite almost everywhere on \((a,b)\). Further, denote by \( AC_L(a,b) \) and \( AC_R(a,b) \) the set of all functions \( u \in AC_{loc}(a,b) \) such that

\[
\lim_{r \to a^+} u(r) = 0
\]

and

\[
\lim_{r \to b^-} u(r) = 0,
\]

respectively. In [32], Opic and Kufner have investigated the conditions on \( p, q \in (1, \infty) \) and \( v, w \in W(a,b) \) for which there is a constant \( C > 0 \) such that

\[
\left( \int_a^b |u(r)|^q w(r) \, dr \right)^{\frac{1}{q}} \leq C \left( \int_a^b |u'(r)|^p v(r) \, dr \right)^{\frac{1}{p}},
\]

holds for every \( u \in AC_L(a,b) \) (or \( u \in AC_R(a,b) \)). In this direction, Opic and Kufner were able to prove the following (see [32, Theorem 1.14 and Theorem 6.2]):

**Theorem 2.4.** Let \( 1 < p \leq q < \infty \) and \( v, w \in W(a,b) \). Set

\[
k(q,p) = \left( 1 + \frac{q(p-1)}{p} \right)^{\frac{1}{p}} \left( 1 + \frac{p}{q(p-1)} \right)^{\frac{p-1}{p}}.
\]

(i) Inequality 2.5 holds for every \( u \in AC_L(a,b) \) if and only if

\[
B_L = \sup_{x \in (a,b)} \left( \int_x^b w(r) \, dr \right)^{\frac{1}{q}} \left( \int_a^x v^{\frac{1}{1-p}}(r) \, dr \right)^{\frac{p-1}{p}} < +\infty.
\]

Also, if \( C_L \) is the best possible constant in (2.5) we must have

\[
B_L \leq C_L \leq k(q,p) B_L.
\]

(ii) Inequality 2.5 holds for every \( u \in AC_R(a,b) \) if and only if

\[
B_R = \sup_{x \in (a,b)} \left( \int_a^x w(r) \, dr \right)^{\frac{1}{q}} \left( \int_x^b v^{\frac{1}{1-p}}(r) \, dr \right)^{\frac{p-1}{p}} < +\infty.
\]

Also, if \( C_R \) is the best possible constant in (2.5) we must have

\[
B_R \leq C_R \leq k(q,p) B_R.
\]
The following consequence of Theorem 2.4 will be used in the next sections.

**Corollary 2.5.** Let \( p, q \in (1, \infty) \), \( 0 < R < \infty \) and \( \alpha, \theta \in \mathbb{R} \). Then, the inequality

\[
\left( \int_0^R |u(r)|^{q r^\theta} dr \right)^{\frac{1}{q}} \leq C \left( \int_0^R |u'(r)|^p r^\alpha dr \right)^{\frac{1}{p}}
\]

holds for some constant \( C = C(p, q, \theta, \alpha, R) \) under the following conditions:

(i) for \( u \in AC_L(0, R) \) if the conditions \( \alpha - p + 1 < 0 \) and \( q(\alpha - p + 1) \leq p(\theta + 1) \) are fulfilled. Further, if we also suppose \( p = q = \alpha - \theta \) the best possible constant \( C_L \) must satisfy

\[
(p - 1) \frac{\frac{p-1}{p-1-\alpha}}{p-1-\alpha} \leq C_L \leq \frac{p}{p-1-\alpha}
\]

(ii) for \( u \in AC_R(0, R) \) if the conditions \( \alpha - p + 1 > 0 \) and \( q(\alpha - p + 1) \leq p(\theta + 1) \) are fulfilled. Further, if we also suppose \( p = q = \alpha - \theta \) the best possible constant \( C_R \) must satisfy

\[
(p - 1) \frac{\frac{p-1}{\alpha-p-1}}{\alpha-p-1} \leq C_R \leq \frac{p}{\alpha-p-1}
\]

**Proof.** We will apply Theorem 2.4 with the special weight functions \( w(r) = r^\theta \) and \( v(r) = r^\alpha \), for \( \alpha, \theta \in \mathbb{R} \) on the interval \( (a, b) = (0, R) \). Indeed, a direct calculus shows

\[
\int_x^R w(r) dr = \begin{cases} O \left( \frac{1}{x} \right)_{x \to 0^+}, & \text{if } \theta > -1 \\ O \left( \ln x \right)_{x \to 0^+}, & \text{if } \theta = -1 \\ O \left( x^{\theta+1} \right)_{x \to 0^+}, & \text{if } \theta < -1 \end{cases}
\]

and

\[
\int_0^x v^{-\frac{1}{p}}(r) dr = O \left( x^{\frac{\alpha+1}{p}} \right)_{x \to 0^+}, \quad \text{if } \alpha - p + 1 < 0.
\]

If \( \alpha - p + 1 < 0 \) it is clear that \( B_L \) is finite for \( \theta \geq -1 \). In addition, in the case \( \theta < -1 \), by using the assumption \((\theta + 1)p \geq (\alpha - p + 1)q\) we can see that \( B_L < +\infty \). Finally, if \( \alpha - p + 1 < 0 \) and \( p = q = \alpha - \theta \) we must have \( \theta < -1 \), and thus

\[
B_L = (p - 1) \frac{\frac{p-1}{p-1-\alpha}}{p-1-\alpha} \quad \text{and} \quad k(p,p) = p(p - 1) \frac{\frac{p-1}{\alpha-p-1}}{\alpha-p-1}
\]

which implies that (i) holds. Now, the both conditions \( \alpha - p + 1 > 0 \) and \( q(\alpha - p + 1) \leq p(\theta + 1) \) imply \( \theta > -1 \) and since we have

\[
\int_0^x w(r) dr = O \left( x^{\theta+1} \right)_{x \to 0^+}, \quad \text{if } \theta > -1
\]

and

\[
\int_x^R v^{-\frac{1}{p}}(r) dr = O \left( x^{-\frac{\alpha+1}{p}} \right)_{x \to 0^+}, \quad \text{if } \alpha - p + 1 > 0
\]

it is clear that \( B_R < \infty \). Also, for \( p = q = \alpha - \theta \), it follows that

\[
B_R = (p - 1) \frac{\frac{p-1}{\alpha-p-1}}{\alpha-p-1}
\]

which proves (ii). \( \square \)
3 Adams functional along of concentrated sequences

In this section we will prove Theorem 1.1 for $m = 2$. Indeed, we provide an upper bound for Adam’s functional along of all concentrated sequences $(u_i) \subset W^{2,p}_N(\Omega)$.

3.1 General estimate of Carleson-Chang type

Let $\Gamma$ be the gamma Euler function and $\psi(x) = \frac{d}{dx}(\ln \Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)}$ the classical psi-function. We recall the following properties of the psi-function which can be found in [4, Theorem 1.2.5]

\begin{equation}
\psi(1) = -\gamma, \quad \psi(x) - \psi(1) = \sum_{k=0}^{\infty} \left( \frac{1}{k+1} - \frac{1}{x+k} \right), \quad \text{and} \quad \psi'(x) = \sum_{k=0}^{\infty} \frac{1}{(x+k)^2}
\end{equation}

where

\[ \gamma = \lim_{n \to \infty} \left( \sum_{j=1}^{n} \frac{1}{j} - \ln n \right) \]

is the Euler-Mascheroni constant.

For $2 \leq p \in \mathbb{N}$ integer number, the next result was proved by Carleson and Chang [7], and for general $2 \leq p \in \mathbb{R}$ it was proved by Hudson and Leckband [23] (see also [11]).

**Lemma 3.1.** Let $p \geq 2$, $a > 0$ and $\delta > 0$ be real numbers. For each $w \in C[0, \infty)$, nonnegative piecewise differentiable function satisfying $\int_{a}^{\infty} |w'(t)|^p dt \leq \delta$ we have

\[ \int_{a}^{\infty} e^{w(t)-t} dt \leq e^{w(a)-a} \frac{1}{1-\delta^{1/(p-1)}} \exp \left\{ \left( \frac{p-1}{p} \right)^{p-1} \frac{c^{p-1}}{p} + \psi(p) + \gamma \right\}, \]

where $\gamma_p = \delta(1 - \delta^{1/(p-1)})^{1-p}$, $c = qw^{a-1}(a)$ and $q = p/(p-1)$.

Now, inspired by Carleson and Chang [7], we will apply the above result to obtain the following one-dimensional estimate:

**Theorem 3.2.** Let $p \geq 2$ be a real number. Let $(g_i)$ be a sequence of nonnegative continuous piecewise differentiable functions on $[0, \infty)$. Suppose

\[ g_i(0) = 0, \quad \int_{0}^{\infty} |g^{\prime}_i|^p dt \leq 1 \quad \text{and} \quad \lim_{i \to \infty} \int_{A}^{\infty} |g^{\prime}_i|^p dt \to 0, \quad \forall \ A > 0. \]

Then

\[ \limsup_{i} \int_{0}^{\infty} e^{g_i(t)-t} dt \leq 1 + e^{\psi(p)+\gamma}, \]

where $q = p/(p-1)$ and $\psi, \gamma$ are given by (3.1).

**Proof.** Firstly, since $\psi(p) + \gamma > 0$ (cf. (3.1)), we can assume that

\[ \limsup_{i} \int_{0}^{\infty} e^{g_i(t)-t} dt > 2. \]

By Hölder inequality

\[ g_i(t) = \int_{0}^{t} g_i'(s) ds \leq \left( \int_{0}^{\infty} |g_i'|^p dt \right)^{\frac{1}{p}} t^{\frac{1}{q}} \leq t^{\frac{1}{q}}, \quad t > 0 \]
and consequently
\[ g_i^q(t) \leq t, \quad \forall t \geq 0. \] (3.4)
For each i large enough, we claim that there exists \( a_i \) smallest number in \([1, \infty)\) satisfying
\[ g_i^q(a_i) = a_i - 2 \ln a_i \quad \text{and} \quad \lim_{i \to \infty} a_i = \infty. \] (3.5)
Indeed, the inequality (3.4) yields \( g_i^q(t) \leq t - 2 \ln^+ t \), for all \( t \in [0, 1] \). Also, if we assume that \( g_i^q(t) < t - 2 \ln^+ t \), for all \( t \in [1, \infty) \) it follows that
\[ \int_0^{\infty} e^{g_i^q(t) - t} dt \leq 1 + \int_1^{\infty} e^{-2 \ln^+ t} dt = 2, \]
which contradicts (3.3). Thus, we have \( Z_i = \{ t \in [1, \infty) : g_i^q(t) = t - 2 \ln^+ t \} \neq \emptyset \) and we can choose \( a_i = \inf Z_i \). Of course we have the \( a_i \in [1, \infty) \) and \( g_i^q(a_i) = a_i - 2 \ln a_i \). In addition, for each \( A > 0 \) the assumption (3.2) ensures \( i_0 = i_0(A) \) such that
\[ \left( \int_0^A |g_i^q(s)|^p ds \right)^{1/(p-1)} < \eta, \quad \forall i \geq i_0 \]
where \( \eta > 0 \) is chosen small enough such that \( \eta t \leq t - 2 \ln^+ t \), for all \( t \in [0, \infty) \). Hence, by Hölder inequality
\[ g_i^q(t) \leq \left( \int_0^A |g_i^q(s)|^p ds \right)^{1/(p-1)} t < \eta t \leq t - 2 \ln^+ t, \]
for all \( 0 \leq t \leq A \) and \( i \geq i_0 \) which forces \( a_i \geq A \), for all \( i \geq i_0 \). Thus, we get \( a_i \to \infty \), as \( i \to \infty \) and (3.5) holds.

Now, we are using the Lemma 3.1 to complete our proof. Indeed, choosing \( w = g_i, a = a_i, \delta = \delta_i = \int_{a_i}^\infty |g_i'|^p dt, \) we obtain
\[ \int_{a_i}^{\infty} e^{g_i^q(t) - t} dt \leq \frac{1}{1 - \delta_i^{1/(p-1)}} e^{K_i + \psi(p) + \gamma} \] (3.6)
where
\[ K_i = g_i^q(a_i) \left[ 1 + \frac{\delta_i}{(p - 1)(1 - \delta_i^{1/(p-1)})^{p-1}} \right] - a_i. \]
We are going to show that
\[ \delta_i \to 0 \quad \text{and} \quad K_i \to 0, \quad \text{as} \quad i \to \infty. \]
We have
\[ g_i^q(a_i) \leq \left( \int_0^{a_i} |g_i'|^p dt \right)^{\frac{q}{p}} a_i \leq (1 - \delta_i)^{1/(p-1)} a_i, \]
which combined with (3.5) imply
\[ \delta_i \leq 1 - \left( 1 - \frac{2 \ln^+ a_i}{a_i} \right)^{p-1} \leq (p - 1) \frac{2 \ln^+ a_i}{a_i} \to 0, \quad \text{as} \quad i \to \infty, \]
because \( 1 - t^d \leq d(1 - t), \quad t \geq 0 \) for any \( d \geq 1 \). We also have, for all \( i \geq i_0, \)
\[ K_i \leq g_i^q(a_i) \left[ 1 + \frac{\delta_i}{p - 1} + \frac{2 \rho \delta_i^q}{p - 1} \right] - a_i \leq 2(p - 1)^q a_i \left( \frac{2 \ln^+ a_i}{a_i} \right)^q, \]
which implies $K_i \to 0$ as $i \to \infty$. Letting $i \to \infty$ in (3.6), we obtain

$$\limsup_{i \to \infty} \int_{a_i}^{\infty} e^{\eta(t)-t} \, dt \leq e^{\psi(p)+\gamma}. \tag{3.7}$$

It follows from (3.2) that $g_i \to 0$ uniformly on compact sets. Thus, given $\epsilon > 0$ and $A > 0$, we have $g_i(t)^q \leq \epsilon$ for all $0 \leq t \leq A$ and $i$ sufficiently large. Since $a_i$ is the smallest number such that $g_i(t)^q \geq t - 2 \ln^+ t$, we obtain

$$\int_0^{a_i} e^{\eta(t)-t} \, dt = \int_0^A e^{\eta(t)-t} \, dt + \int_{a_i}^A e^{\eta(t)-t} \, dt \leq e^\epsilon \left(1 - \frac{1}{e^A}\right) + \left(\frac{1}{A} - \frac{1}{a_i}\right) \leq 1,$$

for $\epsilon$ small enough and $A$ sufficiently large. On the other hand,

$$\int_0^{a_i} e^{\eta(t)-t} \, dt \geq \int_0^{a_i} e^{-t} \, dt = 1 - e^{-a_i} \to 1, \quad \text{as} \quad i \to \infty.$$

Combining the above limits, we have

$$\lim_{i \to \infty} \int_0^{a_i} e^{\eta(t)-t} \, dt = 1$$

which together with (3.7) complete the proof. \hfill \Box

### 3.2 Proof of the Theorem 1.1: case $m = 2$

In order to prove Theorem 1.1 we will use Theorem 2.1 to replace the concentration sequence $(u_i)$ by another one $(v_i) \subset W_0^{1,n}(B_R) \cap W^{2,2}_c(B_R)$ which is also concentrated and satisfies $u_i^* \leq v_i$. Then, we use $(v_i)$ to define a new sequence $(g_i)$ proper to apply the Carleson-Chang type estimate Theorem 3.2. Throughout this section, without loss of generality, we will assume that the functions $u_i$ belong to $C^\infty(\Omega)$.

**Lemma 3.3.** Let $(u_i) \subset W^{2,2}_c(\Omega)$ be a concentrated sequence such that $\|\Delta u_i\|^\frac{n}{2} = 1$. Then there exists $(v_i) \subset W_0^{1,n}(B_R) \cap W^{2,2}_c(B_R)$ such that $u_i^* \leq v_i$ a.e. in $B_R$, $\|\Delta v_i\|^\frac{n}{2} = 1$ and

$$\lim_{i \to \infty} \int_{B_R \setminus B_r} |\Delta v_i|^\frac{n}{2} \, dx = 0, \quad \text{for all} \quad 0 < r < R$$

where $B_R$ is the ball centered at the origin such that $|B_R| = |\Omega|$.

**Proof.** Let $x_0 \in \Omega$ such that

$$\lim_{i \to \infty} \int_{\Omega \setminus B_r(x_0)} |\Delta u_i|^\frac{n}{2} \, dx = 0, \quad \text{for any} \quad r > 0.$$

For each $i$, Theorem 2.1 ensures that $v_i : B_R \to \mathbb{R}$ given by

$$v_i(x) = \frac{1}{n^2 \omega_n^\frac{n}{2}} \int_{B_r(x_0)^c} s^{\frac{n}{2} - 2} \int_0^s \gamma^\#(t) \, dt \, ds, \tag{3.8}$$

satisfies $u_i^* \leq v_i$. Note that

$$\int_{B_R} |\Delta v_i|^\frac{n}{2} \, dx = 1. \tag{3.9}$$

In addition,

$$\int_{B_r(0)} |\Delta v_i|^\frac{n}{2} \, dx = \int_{B_r(0)} |(\Delta u_i)^*|^\frac{n}{2} \, dx \geq \int_{B_r(x_0) \cap \Omega} |\Delta u_i|^\frac{n}{2} \, dx \to 1, \tag{3.10}$$

with
In addition, we have
\[ \lim_{i \to \infty} \int_{B_R \setminus B_r} |\Delta v_i|^\frac{n}{n-2} \, dx = 1 - \lim_{i \to \infty} \int_{B_r} |\Delta v_i|^\frac{n}{n-2} \, dx = 0, \]  \tag{3.11}
for any \( r \in (0, R) \).

At this point, it is convenient to compare the Adams functional along of both sequences \((u_i)\) and \((v_i)\) which were described in Lemma 3.3. Let us first denote
\[ v_i(x) = w_i(r), \quad r = |x|. \]  \tag{3.12}
Thus,
\[ \int e^{\beta |u_i|^\frac{n}{n-2}} \, dx = \int_{B_R} e^{\beta_0(u_i')^\frac{n}{n-2}} \, dx \leq \int_{B_R} e^{\beta_0 v_i^\frac{n}{n-2}} \, dx = \omega_{n-1} \int_{B_R} e^{\beta_0 v_i^\frac{n}{n-2}} r^{n-1} \, dr. \]  \tag{3.13}
In addition, we have \( r^{n-1}w_i'(r) \in AC_L(0, R) \). Further, setting
\[ p = q = \frac{n}{2}, \quad \alpha = \frac{n}{2} \frac{n^2}{2} + n - 1 \quad \text{and} \quad \theta = \frac{n}{2} - \frac{n^2}{2} + \frac{n - 1}{2} \]  \tag{3.14}
we have \( \alpha - p + 1 = n(1 - n/2) < 0, \ n > 2 \) and \( q = p(\theta + 1)/(\alpha - p - 1) \). Hence, the Corollary 2.5, item (i) yields
\[ \int_0^R |w_i'|^{\frac{n}{n-2}} r^{\frac{n}{n-2} - 1} \, dr \leq C \int_0^R |r^{1-n} (r^{n-1} w_i')'|^{\frac{n}{n-2}} r^{n-1} \, dr = \frac{C}{w_{n-1}} \int_{B_R} |\Delta v_i|^\frac{n}{n-2} \, dx = \frac{C}{w_{n-1}}. \]  \tag{3.15}

In order to give a sharp estimate for the Adams functional along of the sequence \((w_i)\), we are going to estimate the constant \( C \) in (3.15).

**Lemma 3.4.** For any \( u \in AC_L(0, R) \), we have
\[ \int_0^R |u|^{\frac{n}{n-2}} r^{n(1-\frac{n}{n-2}) - 1} \, dr \leq \frac{1}{(n-2)^{\frac{n}{n-2}}} \int_0^R |u'|^{\frac{n}{n-2}} r^{n(1-\frac{n}{n-2}) + \frac{n}{n-2} - 1} \, dr. \]
In particular, for \( w_i \) defined as in (3.12)
\[ (n-2)^\frac{n}{n-1} \int_0^R |w_i'|^{\frac{n}{n-2}} r^{\frac{n}{n-2} - 1} \, dr \leq \omega_{n-1} \int_0^R |r^{1-n} (r^{n-1} w_i')'|^{\frac{n}{n-2}} r^{n-1} \, dr = \int_{B_R} |\Delta v_i|^\frac{n}{n-2} \, dx = 1 \]  \tag{3.16}
holds.

**Proof.** It is a direct consequence of Corollary 2.5 with the choice (3.14). In fact, that choice also satisfy \( p = \alpha - \theta \) and \( p/(p - 1 - \alpha) = 1/(n - 2) \), then the best positive possible constant \( C_L \) must satisfy
\[ (p - 1)^{1-\frac{1}{\alpha}} \frac{1}{p - 1 - \alpha} \leq C_L \leq \frac{p}{p - 1 - \alpha} = \frac{1}{n - 2} \]
which completes the proof. \( \square \)

Next, we prove that the sequence \((w_i)\) defined in (3.12) is concentrated at the origin.

**Lemma 3.5.** Let \((w_i)\) be the sequence defined in (3.12). Then, for any \( r \in (0, R) \)
\[ \int_r^R |w_i'|^{\frac{n}{n-2}} r^{\frac{n}{n-2} - 1} \, dt \to 0, \quad \text{as} \ i \to \infty. \]
Proof. Note that
\[ w_1(r) = \frac{1}{n^2 \omega_n} \int_{\omega_n}^{R_n} s^{\frac{n}{2}} \int_0^s (\Delta u_i)^\#(t)dt ds, \]
and so
\[ w_1'(s) = -\frac{1}{n \omega_n} s^{1-n} \int_0^{R_n s^n} (\Delta u_i)^\#(t)dt = -s^{1-n} \int_0^s (\Delta u_i)^\#(\omega_{n-1} t^n) t^{n-1} dt, \quad \text{for all } s \in (0, R). \]
Hence, the Hölder inequality and (3.9) yield
\[ |s^{n-1} w_1'(s)| \leq \left( \frac{R^n}{n} \right)^{\frac{n}{n-2}} \left( \int_{B_R} |\Delta v_i|^{\frac{n}{2}} dx \right)^{\frac{2}{n}} \leq \left( \frac{R^n}{n} \right)^{\frac{n}{n-2}}, \quad \forall \ i \in \mathbb{N}. \] (3.17)
In addition, for 0 < r < R and s, t ∈ [r, R], with t < s, we obtain
\[ |s^{n-1} w_1'(s) - t^{n-1} w_1'(t)| \leq \int_t^s (\Delta u_i)^\#(\omega_{n-1} t^n) t^{n-1} dt \]
\[ \leq \left( \frac{s^{n-1} - t^{n-1}}{n} \right)^{\frac{n}{n-2}} \left( \int_{B_{r\setminus B_t}(0)} |\Delta v_i|^{\frac{n}{2}} dx \right)^{\frac{2}{n}}, \] (3.18)
for any \( i \in \mathbb{N} \). It follows from (3.17) and (3.18) that \( f_i : [r, R] \to \mathbb{R} \) such that \( f_i(s) = s^{n-1} w_1'(s) \) becomes uniformly bounded and equicontinuous sequence. Thus, up to a subsequence, we have \( f_i \to g \) uniformly on \([r, R] \). From Lemma 3.3, by setting \( i \to \infty \) in (3.18) we conclude that \( g \) must be a constant \( c_w \).

Now we claim that \( c_w = 0 \). Suppose \( c_w > 0 \), then is possible to choose \( i(r) \) large enough such that for all \( i \geq i(r) \)
\[ s^{n-1} w_1'(s) \geq c_w - \epsilon > 0, \quad \forall s \in (r, R), \]
for \( \epsilon > 0 \) sufficiently small. Thus,
\[ \int_r^R |w_1'|^{\frac{n}{2}} s^{\frac{n}{2}-1} ds \geq \int_r^R \left( \frac{c_w - \epsilon}{s^{n-1}} \right)^{\frac{n}{2}} s^{\frac{n}{2}-1} ds \]
\[ = (c_w - \epsilon)^{\frac{n}{2}} \int_r^R s^{n-1 - \frac{n}{2}} ds = O \left( r^{n-\frac{n}{2}} \right)_{r \to 0}, \]
which contradicts (3.15). Analogously, the assumption \( c_w < 0 \) leads a contradiction and our claim is proved.

Therefore, \( w_1' \) converge uniformly to 0 on \([r, R] \), with \( r \in (0, R) \) which completes the proof.

Next we will complete the proof of Theorem 1.1. Consider the change of variable
\[ r = Re^{-\frac{t}{n}} \quad \text{and} \quad g_i(t) = \omega_n^{\frac{2}{n-1}} n^{\frac{n-2}{n}} (n-2) w_i(r). \]
From (3.16), we have
\[ \int_0^\infty |g_i'|^{\frac{n}{2}} dt = (n-2)^{\frac{n}{2}} \omega_n^{\frac{2}{n-1}} \int_0^R |w_i'(r)|^{\frac{n}{2}} r^{\frac{n}{2}-1} dr \leq 1. \] (3.19)
Since \( \omega_n = 2\pi^{\frac{n}{2}} / \Gamma \left( \frac{n}{2} \right) \) and \( \Gamma(x) = \Gamma(x + 1) / x \), for \( x > 0 \) we have
\[ \beta_0 = \beta_0(2, n) = \frac{n}{\omega_n} \left[ \frac{4\pi^{\frac{n}{2}}}{\Gamma \left( \frac{n}{2} - 1 \right)} \right]^{n/(n-2)} \]
\[ = \frac{n}{\omega_n} \left[ \frac{2(n-2)\pi^{\frac{n}{2}}}{\Gamma \left( \frac{n}{2} \right)} \right]^{n/(n-2)} \]
\[ = \frac{n}{\omega_n} \left[ \frac{2(n-2)\omega_n}{(n-2)^{n/(n-2)}} \right] = \left[ \frac{2}{\omega_n} \right]^{n/(n-2)} (n-2)^{n/(n-2)}. \] (3.20)
Hence, we can write (cf. (3.13))
\[ \int_{\Omega} e^{\beta |u_i|^2} \, dx \leq \omega_{n-1} \int_0^R e^{\beta |u_i|^2} r^{n-1} \, dr = |B_R| \int_0^\infty e^{\beta n(t)^{\frac{n-2}{n}}} \, dt. \] (3.21)
Taking into account (3.21) and Lemma 3.5, Theorem 3.2 yields
\[ \limsup_i \int_{\Omega} e^{\beta |u_i|^2} \, dx \leq |B_R| \limsup_i \int_0^\infty e^{\beta n(t)^{\frac{n-2}{n}}} \, dt \leq |\Omega| \left( 1 + e^{\psi(\frac{n}{2}) + \gamma} \right). \] (3.22)

4 Proof of Theorem 1.3

According to Theorem 2.3 we only need to find some test function such that the integral in (1.1) surpass the upper bound for the concentration level given in Theorem 1.1. For \( t \in [0, \infty) \), we set
\[ u(t) = \begin{cases} 
\frac{n - 2}{n} \left( \frac{n - 2}{2} \right)^{-\frac{2}{n}} t, & \text{if } 0 \leq t \leq \frac{n}{2} \\
(t - 1)^{\frac{n-2}{n}}, & \text{if } \frac{n}{2} < t \leq \lambda \\
\frac{n - 2}{3} (\lambda - 1)^{-\frac{2}{n}} \left( 1 - e^{\frac{n}{2}(\lambda - t)} \right)^{\frac{n-2}{n}} + (\lambda - 1)^{\frac{n-2}{n}}, & \text{if } t \geq \lambda
\end{cases} \] (4.1)
where \( \lambda > 0 \) will be chosen later. Then, we set
\[ u(x) = \omega_{n-1}^{-\frac{2}{n}} u - \frac{n-2}{n} (n - 2)^{-1} w \left( n \ln \frac{R}{|x|} \right), \quad 0 < |x| \leq R. \] (4.2)
Hence, we obtain \( u \in W^2_\infty N^\frac{2}{n}(B_R) \), where \( B_R \) is the ball with radius \( R > 0 \) centered at the origin. Since \( u \) is a radially symmetric function, we can write
\[ \Delta u = \omega_{n-1}^{-\frac{2}{n}} \left( \frac{n}{n - 2} w'' \left( n \ln \frac{R}{r} \right) - w' \left( n \ln \frac{R}{r} \right) \right) \frac{1}{r^2}, \quad 0 < r = |x| \leq R. \] (4.3)
Thus,
\[ \|\Delta u\|_{\frac{2}{n}} = \left( \int_0^\infty |L(w'', w')|^{\frac{2}{n}} \, dt \right)^{\frac{n}{2}}. \] (4.4)
where
\[ L(w'', w')(t) = \frac{n}{n - 2} w''(t) - w'(t), \quad t \geq 0. \] (4.5)

Lemma 4.1. For \( n \geq 15 \), there exists \( \lambda > n/2 \) such that \( \|\Delta u\|_{\frac{2}{n}} \leq 1 \). In fact, we can choose
\[ \lambda = 1 + \frac{n - 2}{2} e^{b-s}. \] (4.6)
where \( 0 < s < b \), with \( s \) and \( b \) depending on \( n \).

Proof. Note that
\[ L(w'', w')(t) = \begin{cases} 
\frac{n - 2}{n} \left( \frac{n - 2}{2} \right)^{-\frac{2}{n}}, & \text{if } 0 \leq t \leq \frac{n}{2} \\
- \frac{1}{n} \left( (n - 2)(t - 1)^{-\frac{2}{n}} + 2(t - 1)^{-\frac{n-2}{n}} \right) & \text{if } \frac{n}{2} < t \leq \lambda \\
\left( \frac{n+1}{n} \right) (\lambda - 1)^{-\frac{2}{n}} e^{\frac{n}{2} \lambda} e^{-\frac{2}{n} t} & \text{if } t \geq \lambda
\end{cases} \] (4.7)
and
\[
\int_0^\infty |L(w'', w')|^{\frac{2}{n}} \ dt \\
= \int_0^\infty \left| \chi(0, \Phi)L(w'', w') - \chi(\Phi, \lambda) \frac{1}{n} (n-2)(t-1)^{-\frac{2}{n}} + 2(t-1)^{-\frac{n+2}{n}} \right| + \chi(\lambda, \infty)L(w'', w') \left|^{\frac{2}{n}} \ dt.
\]

Hence, the Minkowski inequality yields
\[
\left( \int_0^\infty |L(w'', w')|^{\frac{2}{n}} \ dt \right)^{\frac{n}{2}} \leq \frac{2}{n} \left( \int_0^\lambda (t-1)^{-\frac{n+2}{n}} \ dt \right)^{\frac{n}{2}}
\]
\[+ \left( \int_0^\infty \left| \chi(0, \Phi)L(w'', w') - \chi(\Phi, \lambda) \frac{1}{n} (n-2)(t-1)^{-\frac{2}{n}} + 2(t-1)^{-\frac{n+2}{n}} \right| \ dt \right)^{\frac{n}{2}} \]
\[= \frac{2}{n} \left( \frac{2}{n-2} \frac{2}{n} - \left( \frac{1}{\lambda-1} \right) \right)^{\frac{n}{2}} (4.12)
\]
\[+ \left( \int_0^\infty |L(w'', w')|^{\frac{2}{n}} \ dt + (n-2) \int_0^\lambda \frac{1}{t-1} \ dt + \int_0^\infty |L(w'', w')|^{\frac{2}{n}} \ dt \right)^{\frac{n}{2}}
\]
\[= \frac{2}{n} \left( \frac{2}{n-2} \frac{2}{n} - \left( \frac{1}{\lambda-1} \right) \right)^{\frac{n}{2}} (4.13)
\]
\[+ \left( \frac{n-2}{n} \left( \frac{n-2}{n} \right)^{\frac{n}{2}} \right) + \left( \frac{n-2}{n} \right)^{\frac{n}{2}} \ln \left( \frac{2(\lambda-1)}{n-2} \right) + \frac{2}{3} \left( \frac{n+1}{n} \right)^{\frac{n}{2}} \frac{1}{\lambda-1} \right)^{\frac{n}{2}}.
\]

From (4.4), we have
\[
\| \Delta u \|^{\frac{n}{2}} \leq \frac{2}{n} \left( \frac{2}{n-2} \frac{2}{n} - \left( \frac{1}{\lambda-1} \right) \right)^{\frac{n}{2}} (4.8)
\]
\[+ \left( \frac{n-2}{n} \left( \frac{n-2}{n} \right)^{\frac{n}{2}} \right) + \left( \frac{n-2}{n} \right)^{\frac{n}{2}} \ln \left( \frac{2(\lambda-1)}{n-2} \right) + \frac{2}{3} \left( \frac{n+1}{n} \right)^{\frac{n}{2}} \frac{1}{\lambda-1} \right)^{\frac{n}{2}}. (4.9)
\]

Setting
\[
\lambda_s := 1 + \frac{n-2}{2} e^{b-s}, \quad \text{with} \quad 0 < s < b
\]
we can write
\[
\| \Delta u \|^{\frac{n}{2}} \leq \frac{4}{n(n-2)} \left( 1 - \left( \frac{1}{e^{b-s}} \right) \right)^{\frac{n}{2}} (4.10)
\]
\[+ \left( \frac{n-2}{n} \left( \frac{n-2}{n} \right)^{\frac{n}{2}} \right) (b-s) + \frac{2}{3} \left( \frac{n+1}{n} \right)^{\frac{n}{2}} \frac{2}{n-2} e^{b-s} \right)^{\frac{n}{2}}. (4.11)
\]

Now, setting
\[
b := \left( \frac{n}{n-2} \right)^{\frac{n}{2}} - \frac{n}{n-2} (4.12)
\]
we have

\[ \| \Delta u \|_{L^2} \leq \frac{4}{n(n-2)} + \left( 1 - s \left( \frac{n-2}{n} \right)^{\frac{n}{2}} + \frac{4}{3} \left( \frac{n+1}{n} \right)^{\frac{n}{2}} - \frac{1}{n-2} \right)^{\frac{n}{2}}, \] for any \( 0 < s < b. \) \hspace{1cm} (4.13)

Finally, we set

\[ s := \left( \frac{n}{n-2} \right)^{\frac{n}{2}} \left[ 1 + 4 \left( \frac{n+1}{n} \right)^{\frac{n}{2}} - \frac{1}{n-2} \right] \] \hspace{1cm} (4.14)

From (4.13), the above choice for \( s \) clearly ensures \( \| \Delta u \|_{L^2} \leq 1 \) provided that \( 0 < s < b. \) Bernoulli’s inequality yields

\[ \left( 1 - \frac{4}{n(n-2)} \right)^{\frac{n}{2}} \geq 1 - \frac{2}{n-2}, \text{ for } n \geq 4 \]

and since \( 2 \leq (1 + \frac{1}{k})^k < 3, k \geq 2, \) we can write

\[
\frac{s}{b} = \left( \frac{1 + \frac{1}{n-2}}{n-2} \right)^{\frac{n-2}{2}} \left[ 1 + \frac{4}{3} \left( \frac{n+1}{n} \right)^{\frac{n}{2}} - \frac{1}{n-2} \right] - \left( \frac{1 - \frac{4}{n-2}}{n-2} \right)^{\frac{n}{2}} - 1
\]

\[
\leq 4 \left( \frac{1 + \frac{1}{n}}{n-2} \right)^{\frac{n}{2}} - \frac{1}{n-2} + \frac{6}{n-2}
\]

\[
= \frac{6}{n-2} \left[ 2 \left( \frac{1 + \frac{1}{n}}{n-2} \right)^{\frac{n}{2}} + 1 \right] < \frac{6}{n-2} \left( \frac{2\sqrt{3}}{3} + 1 \right),
\]

where the right side is decreasing on \( n \) and for \( n = 15 \)

\[
\frac{6}{13} \left( \frac{2\sqrt{3}}{3} + 1 \right) < 1.
\]

Hence, we have \( s < b \) for \( n \geq 15. \) \hfill \( \square \)

**Remark 4.2.** Note that the Bernoulli’s inequality and \( (1 + \frac{1}{n})^{n/2} < \sqrt{3} \) allow us to obtain the estimate

\[
0 < s \leq \left( \frac{n}{n-2} \right)^{\frac{n}{2}} \left[ \frac{4}{3} \left( 1 + \frac{1}{n} \right)^{\frac{n}{2}} - \frac{1}{n-2} + \frac{2}{n-2} \right] \leq \left( \frac{n}{n-2} \right)^{\frac{n}{2}} \left[ \left( \frac{2\sqrt{3}}{3} + 1 \right) \frac{2}{n-2} \right], \text{ for } n \geq 4. \] \hspace{1cm} (4.15)

**Lemma 4.3.** Let \( u \) given in (4.2) with \( \lambda = 1 + \frac{n-2}{2} e^{b-s}, \) where \( b \) and \( s \) are given in (4.12) and (4.14), respectively. Then,

\[
\int_{B_R} e^{\frac{|u|^2}{n-2}} \, dx = |B_R| \int_{0}^{\infty} e^{\frac{w^{n-2}}{n-2} \gamma(t)} \, dt > |B_R| \left( 1 + e^{w(\frac{2}{3})+\gamma} \right), \] \hspace{1cm} (4.16)

for \( n \geq 2T_0, \) where \( T_0 \) is the smallest positive integer such that

\[
T_0 \geq 1 + \frac{1 + 36\sigma}{17 - 24\gamma} + \left[ 1 + \left( \frac{1 + 36\sigma}{17 - 24\gamma} \right)^2 + \frac{72\sigma}{17 - 24\gamma} \right]^{\frac{1}{2}} \approx 51.9233.
\]
Proof. Firstly, we have
\[
\int_0^n e^{w_n \frac{n-1}{n} (t-\tau)} dt = \frac{n}{2} \int_0^1 e^{\phi(\tau)} d\tau,
\]
where
\[
g(\tau) = \left(\frac{n}{2} - 1\right) \tau - \frac{n}{2} \tau, \quad \tau \in [0,1].
\]
It is easy to see that \( g \) is a decreasing function with \( g(1) = -1 \) and \( g(\tau) \geq -\frac{n}{2} \tau \), for all \( 0 < \tau \leq 2 \). Thus
\[
\int_0^1 e^{\phi(\tau)} d\tau \geq \int_0^\frac{n}{2} e^{-\frac{n}{2} \tau} d\tau + \frac{1}{e} \left(1 - \frac{2}{n}\right) = \frac{2}{n} \left(1 - \frac{1}{e}\right) + \frac{1}{e} \left(1 - \frac{2}{n}\right).
\]
Hence,
\[
\int_0^n e^{w_n \frac{n-1}{n} (t-\tau)} dt \geq \frac{n}{2} e^{-\frac{n}{2} \tau} \int_0^1 e^{\phi(\tau)} d\tau + \frac{1}{e} \left(1 - \frac{2}{n}\right) + \frac{1}{e} \left(1 - \frac{2}{n}\right).
\]
(4.17)

In addition,
\[
\int_\lambda^n e^{\frac{n}{n-1} \lambda - t} dt = \frac{1}{e} \left(\lambda - \frac{n}{2}\right) = -\frac{1}{e} \left(\frac{n}{2} - 1\right) + \left(\frac{n}{2} - 1\right) e^{b-s-1}.
\]
(4.18)

Finally,
\[
\int_\lambda^\infty e^{\frac{n}{n-1} \lambda - t} dt \geq e^{\lambda-1} \int_\lambda^\infty e^{-t} dt = \frac{1}{e}.
\]
(4.19)

It follows from (4.17),(4.18) and (4.19) that
\[
\int_0^\infty e^{w_n \frac{n-1}{n} (t-\tau)} dt \geq 1 + \left(\frac{n}{2} - 1\right) e^{b-s-1}.
\]
(4.20)

Hence, it remains to prove that
\[
\left(\frac{n}{2} - 1\right) e^{b-s-1} > e^{\psi(\frac{n}{2})+\gamma},
\]
or equivalently
\[
\psi\left(\frac{n}{2}\right) + \gamma + s - b + 1 - \ln\left(\frac{n}{2} - 1\right) < 0,
\]
for \( n \geq 2T_0 \). We will use the auxiliary function
\[
\eta(t) = \psi(t) + \gamma + \left(\frac{t}{t-1}\right) t \left[\left(\frac{2\sqrt{3}}{3} + 1\right) \frac{1}{t-1} - 1\right] + \frac{t}{t-1} + 1 - \ln (t-1), \quad t \geq 2.
\]
Thus, taking into account (4.12) and (4.15), we obtain
\[
\psi\left(\frac{n}{2}\right) + \gamma + s - b + 1 - \ln\left(\frac{n}{2} - 1\right) < \eta\left(\frac{n}{2}\right), \quad n \geq 4.
\]
Hence, it is now enough to prove the following:

Assertion 4.4. Let \( \sigma = 1 + \frac{2\sqrt{3}}{3} \). Then, the function
\[
\eta(t) = \psi(t) + \gamma + \left(\frac{t}{t-1}\right) t \left[\frac{\sigma}{t-1} - 1\right] + \frac{t}{t-1} + 1 - \ln (t-1), \quad t \geq 2.
\]
is strictly decreasing on \([2, \infty)\) and \(\eta(T_0) < 0\).
Firstly, we shall prove that $\eta'(t) < 0$, for all $t \geq 2$. From (3.1), we can write

$$\psi'(t) = \sum_{k=0}^{+\infty} \frac{1}{(k+t)^2} \leq \sum_{k=0}^{+\infty} \frac{1}{(k+t-\frac{1}{2})} \left( \frac{t}{t-\frac{1}{2}} \right)^2 = \frac{1}{t-\frac{1}{2}}$$

and consequently

$$\eta'(t) = \psi'(t) - \left( \frac{t}{t-1} \right)^2 \left\{ \frac{t-(\sigma+1)}{t-1} \left[ \ln \left( \frac{t}{t-1} \right) - \frac{1}{t-1} \right] + \frac{\sigma}{(t-1)^2} \right\}$$

$$- \frac{1}{(t-1)^2} - \frac{1}{t}$$

$$\leq \left( \frac{1}{t-\frac{3}{2}} \right) - \frac{1}{t-1} - \frac{1}{(t-1)^2}$$

$$- \left( \frac{1}{t-1} \right)^2 \left\{ \frac{t-(\sigma+1)}{t-1} \left[ \ln \left( \frac{t}{t-1} \right) - \frac{1}{t-1} \right] + \frac{\sigma}{(t-1)^2} \right\}.$$

It follows that

$$\eta'(t) < - \left( \frac{t}{t-1} \right)^2 \left\{ \frac{t-(\sigma+1)}{t-1} \left[ \ln \left( \frac{t}{t-1} \right) - \frac{1}{t-1} \right] + \frac{\sigma}{(t-1)^2} \right\}$$

and it is sufficient to show

$$\frac{t-(\sigma+1)}{t-1} \left[ \ln \left( \frac{t}{t-1} \right) - \frac{1}{t-1} \right] + \frac{\sigma}{(t-1)^2} \geq 0, \quad t \geq 2.$$ 

Setting $x = t/(t-1)$ the above inequality is equivalent to show

$$h(x) := |x - (x-1)(\sigma + 1)|[\ln x - x + 1] + \sigma(x-1)^2 \geq 0, \quad 1 < x \leq 2.$$ 

We have $h(x) \to 0$, as $x \to 1^+$. We claim that $h$ is an increasing function on $(1, 2]$. Indeed, we have

$$h'(x) = 4\sigma x + \frac{\sigma + 1}{x} - 5\sigma - \sigma \ln x - 1.$$ 

In addition,

$$x^2 h''(x) = 4\sigma x^2 - \sigma x - (\sigma + 1).$$

Since $y(x) = 4\sigma x^2 - \sigma x - (\sigma + 1), x \in \mathbb{R}$ is a convex parable with minimal point at $x = \frac{1}{2}$ and $y'(1) = 2\sigma - 1 > 0$, we obtain $x^2 h''(x) > 0$ for $x \in (1, 2]$. Hence, $h'>0$ on $(1, 2]$ since it is an increasing function with $h'(x) \to 0$ as $x \to 1^+$. 

Next, we will prove $\eta(T_0) < 0$. Firstly, since $\Gamma(x+1) = x\Gamma(x), x > 0$ we have $\ln \Gamma(x+1) = \ln \Gamma(x) + \ln x$ and thus

$$\psi(x+1) = \psi(x) + \frac{1}{x}.$$ 

Thus, since $\psi(1) + \gamma = 0$ (cf. (3.1)) we have

$$\psi(k+1) + \gamma = H_k := \sum_{j=1}^{k} \frac{1}{k}, \text{ for any } k \in \mathbb{N}.$$ 

In particular,

$$\eta(k+1) = 2 + \gamma k + \left( 1 + \frac{1}{k} \right)^{k+1} \left( \frac{\sigma}{k} - 1 \right) + \frac{1}{k},$$

$$\left( 1 + \frac{1}{k} \right)^{k+1} \left( \frac{\sigma}{k} - 1 \right) + \frac{1}{k}.$$
where $\gamma_k = H_k - \ln k$. In addition, from [34, Corollary 2.13] the estimate
\[
\gamma_k < \gamma + \frac{1}{2k} - \frac{\beta}{k^2}, \quad \text{with } \beta = \gamma - \frac{1}{2}.
\]
holds. Therefore, we can write
\[
\eta(k+1) < 2 + \gamma + \frac{1}{k} + \frac{1}{2k^2} + \left(1 + \frac{1}{k}\right)^{k+1} \left(\frac{\sigma}{k} - 1\right) - \frac{\beta}{k^2}.
\]
For $k \geq 4$, it is easy to see that
\[
\left(1 + \frac{1}{k}\right)^{k+1} \geq \left(\frac{k+1}{k}\right)^{k+1} + \frac{1}{k} + \frac{1}{2k} - \frac{\beta}{k^2} - \left(1 + \frac{1}{k}\right)^{k+1} \frac{\sigma}{k} \left(1 + \frac{1}{k}\right) < \left(\gamma - \frac{17}{24}\right) + \left(\frac{1}{12} + 3\sigma\right) \frac{1}{k} + \left(\frac{17}{24} - \gamma + 3\sigma\right) \frac{1}{k^2}.
\]
It follows that
\[
k^2\eta(k+1) < \left(\gamma - \frac{17}{24}\right) \left[k^2 - \frac{1 + 36\sigma}{12} \frac{24}{17 - 24\gamma} - \frac{72\sigma}{17 - 24\gamma} - 1\right]
\]
\[
= \left(\gamma - \frac{17}{24}\right) \left[k^2 - \frac{2 + 72\sigma}{17 - 24\gamma} - \frac{72\sigma}{17 - 24\gamma} - 1\right]
\]
\[
= \left(\gamma - \frac{17}{24}\right) \left[k - \frac{1 + 36\sigma}{17 - 24\gamma}\right]^2 - \left(\frac{1 + 36\sigma}{17 - 24\gamma}\right)^2 - \frac{72\sigma}{17 - 24\gamma} - 1\right].
\]
Thus, since $\gamma < 17/24$ we have $\eta(k+1) < 0$ provided that
\[
k \in \mathbb{N} \quad \text{and} \quad k \geq \frac{1 + 36\sigma}{17 - 24\gamma} + \left[1 + \left(\frac{1 + 36\sigma}{17 - 24\gamma}\right) + \frac{72\sigma}{17 - 24\gamma}\right].
\]

\section{Proof of the Theorem 1.1: case $m \geq 2$}

First, as well as in Section 3, we only need to consider $u_i \in C^m(\Omega) \cap W^m_{N,2}(\Omega)$. Second, with the help of Proposition 2.2 and Pólya-Szegö inequality we iterate the same argument of the Lemma 3.3 in order to obtain our result for $m \geq 2$. 

\begin{flushright}
\Box
\end{flushright}
Lemma 5.1. Let $\Omega \subset \mathbb{R}^n$ be a smooth domain and $B_R \subset \mathbb{R}^n$ is the ball centered at the origin such that $|B_R| = |\Omega|$. Let $(u_i) \subset W^{m,\frac{m}{n}}_N(\Omega)$ be a concentrated sequence such that $\|\nabla^m u_i\|_{\frac{m}{n}} = 1$. Then there exists $(v_i) \subset W^{m,\frac{m}{n}}_N(B_R)$ such that $u^*_i \leq v_i$. In addition, one has

\[
\begin{aligned}
&\|\nabla^m v_i\|_{\frac{m}{n}} = \|\nabla^m u_i\|_{\frac{m}{n}} = 1 \quad \text{and} \quad \lim_{i \to \infty} \int_{B_R \setminus B_r} |\Delta^\frac{m}{2} v_i|^{\frac{m}{n}} \, dx = 0, \quad \text{if } m \text{ is even} \\
&\|\nabla^m v_i\|_{\frac{m}{n}} \leq \|\nabla^m u_i\|_{\frac{m}{n}} = 1 \quad \text{and} \quad \lim_{i \to \infty} \int_{B_R \setminus B_r} |\Delta^\frac{m-1}{2} v_i|^{\frac{m}{n}} \, dx = 0, \quad \text{if } m \text{ is odd}.
\end{aligned}
\]

Proof. Initially, for $m$ odd, we will prove that up to a subsequence

\[
\lim_{i \to \infty} \int_{\Omega \setminus B_r(x_0)} |\Delta^\frac{m-1}{2} u_i|^{\frac{m}{n}} \, dx = 0, \quad \text{for any } r > 0,
\]

where $x_0 \in \overline{\Omega}$ is the concentration point of the sequence $u_i \in W^{m,\frac{m}{n}}_N(\Omega)$. Poincaré inequality yields

\[
\int_U |\Delta^\frac{m-1}{2} u_i - (\Delta^\frac{m-1}{2} u_i)_U|^{\frac{m}{n}} \, dx \leq C' \int_U |\nabla \Delta^\frac{m-1}{2} u_i|^{\frac{m}{n}} \, dx,
\]

for $U = \Omega \setminus B_r(x_0)$ and $(\Delta^\frac{m-1}{2} u_i)_U = |U|^{-1} \int_U \Delta^\frac{m-1}{2} u_i \, dx$.

Since $(\Delta^\frac{m-1}{2} u_i) \subset W^{1,\frac{m}{n}}_0(\Omega)$ is a bounded sequence, we have $\Delta^\frac{m-1}{2} u_i \rightharpoonup h$ in $W^{1,\frac{m}{n}}_0(\Omega)$ and the compact embedding gives $\Delta^\frac{m-1}{2} u_i \to h$ in $L^{\frac{m}{n}}(\Omega)$. Thus,

\[
\int_U |\Delta^\frac{m-1}{2} u_i|^{\frac{m}{n}} \, dx \to \int_U |h|^{\frac{m}{n}} \, dx \quad \text{and} \quad (\Delta^\frac{m-1}{2} u_i)_U \to (h)_U.
\]

Therefore, since $(u_i)$ is a concentrated sequence and (5.2) holds we obtain $h = (h)_U$ a.e. in $U$. Finally, $(\Delta^\frac{m-1}{2} u_i)_{|\partial \Omega} = 0$ and $\Delta^\frac{m-1}{2} u_i \to h$ a.e. in $U$ imply $h \equiv 0$ and thus (5.1).

In view of (5.1) for both cases $m$ odd and $m$ even, we can apply Proposition 2.2, the same argument used in the proof of Lemma 3.3 and the Pólya-Szegö inequality to finish the proof. \hfill \Box

To apply Proposition 2.2 in the proof of Lemma 5.1, for either $m = 2k$ or $m = 2k + 1$, with $k \geq 2$ and for $f_i = \Delta^k u_i$ we can rewrite the problems (2.1) and (2.2) as the following systems

\[
\begin{aligned}
&\begin{cases}
-\Delta u_i = f_i & \text{in } \Omega \\
u_i = 0 & \text{in } \partial \Omega
\end{cases} \quad \begin{cases}
-\Delta v_i = f_i & \text{in } \Omega^* \\
v_i = 0 & \text{in } \partial \Omega^*
\end{cases} \\
&\begin{cases}
-\Delta u_i = u_i^{j-1} & \text{in } \Omega \\
u_i = 0 & \text{in } \partial \Omega
\end{cases} \quad \begin{cases}
-\Delta v_i = v_i^{j-1} & \text{in } \Omega^* \\
v_i = 0 & \text{in } \partial \Omega^*,
\end{cases}
\end{aligned}
\]

for $j = 2, \ldots, k$. Note that $u_i^k = u_i$, $v_i^k = v_i$. As well as in (3.12), we denote

\[
w_i(|x|) = v_i(x).
\]

In (5.3), setting $h_i^{j-1}(\omega_{n-1}|x|^n) = v_i^{j-1}(x)$ we have

\[
v_i^j(x) = \frac{1}{n^2 \omega_n^{\frac{m}{n}}} \int_{\omega_n|x|^n} s^{\frac{m}{n}-2} \int_0^s h_i^{j-1}(t) \, dt \, ds.
\]

(5.5)

With this notation, we are able to prove the following.

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Lemma 5.2. Let \((w_i)\) be the sequence in (5.4). Then, for any \(r \in (0, R)\) we have
\[
\int_0^R |w'_i|^\frac{np}{n} t^\frac{n}{np} dt \to 0, \quad i \to \infty.
\]

Proof. First note that \(w_i(|x|) = v^i_k(x)\), which is given iteratively by (5.3). Thus, we only need to iterate the argument used in Lemma 3.5, Hölder inequality and (5.5) to get the result.

Now, as well as in the case \(m = 2\), we consider the change of variable
\[
r = R e^{-\frac{t}{R}} \quad \text{and} \quad g_i(t) = \beta_0(m, n)\frac{nq}{nq - np} w_i(r).
\]

In order to complete the proof of Theorem 1.1, it is sufficient to show that the sequence \((g_i)\) is under the hypotheses of Theorem 3.2. To accomplish this task we will first use the result proved in [16, Proposition 3.1]. For completeness, we will prove the necessary version of this result.

Proposition 5.3. Let \(p, q > 1\) and \(0 < R < \infty\). Consider \(n\) satisfying \(n - 2q > 0\). Then, for any \(u \in AC_{\text{loc}}(0, R)\) such that \(\lim_{r \to R} u(r) = 0\) and \(\lim_{r \to R} r^{1-n}(r^{n-1}u) = 0\) we have
\[
\left(\int_0^R |u|^p t^{\frac{np}{n} - 1} dt\right)^{\frac{1}{p}} \leq \frac{q^2}{(q-1)n(n-2q)} \left(\int_0^R |r^{1-n}(r^{n-1}u)'|^p t^{\frac{np}{n} - 1} dt\right)^{\frac{1}{p}},
\]
where \(q^* = \frac{np}{n-2q}\).

Proof. Consider the following change of variable
\[
w(t) = u(Rt^{\frac{1}{n-2q}}), \quad t \geq 1.
\]

Note that
\[
\int_0^R |u|^p t^{\frac{np}{n} - 1} dt = \frac{R^{\frac{np}{n-2q}}}{n-2} \int_1^\infty |w|^p t^{\frac{np}{n} - 1} dt = \frac{R^{\frac{np}{n-2q}}}{n-2} \int_1^\infty |w|^p t^{\frac{(n-2q)p}{n(n-2q)} - 1} dt \quad (5.8)
\]
and
\[
\int_0^R |r^{1-n}(r^{n-1}u)'|^p t^{\frac{np}{n} - 1} dt = (n-2)^2 \frac{R^{\frac{np}{n-2q}}}{n-2} \int_1^\infty |w''(t)|^p t^{\frac{2p(n-1)-p}{n(n-2q)} - 1} dt. \quad (5.9)
\]

By choosing
\[
a = \frac{p-1}{p} 2q(n-1) - n \quad \text{and} \quad \frac{p}{q(n-2)}
\]
we have
\[
w(t) = \left(\int_1^\infty w'(z) dz\right)^p = \left(\int_1^t \int_z^\infty -w''(s)dsdz\right)^p = \left(\int_1^t \int_z^\infty -w''(s)s^a dsdz\right)^p \leq \left(\int_1^t \int_z^\infty |w''(s)|^{p}\frac{s^{ap}}{s^a} dsdz\right)^{1/p} \left(\int_1^t \int_z^\infty s^{-ap'} dsdz\right)^{1/p'}^p \leq \left[\int_1^t \int_z^\infty |w''(s)|^{p}\frac{s^{ap}}{s^a} dsdz\right] \frac{q^2(n-2)^2}{n(q-1)(n-2q)} \left(t^{\frac{n-2q}{n(n-2q)} - 1}\right)^{p/p'}. \quad (5.10)
\]
Then, we have
\[
\int_1^\infty |w|^p t^{-\frac{(n-2)p}{q(n-2)}} dt \leq \int_1^\infty \left[ \int_z^t \int_z^\infty |w''(s)|^p s^{-\rho} ds dz \right] dt^{-\frac{(n-2)p}{q(n-2)-\rho}} dt
\]
\[
= \left( \frac{q^2(n-2)^2}{(n-1)(n-2)} \right)^{\frac{p-1}{p}} \int_1^\infty \int_1^s |w''(s)|^p s^{\rho} \int_0^\infty t^{-\frac{n-2q}{q(n-2)-\rho}} dt dz ds
\]
\[
\leq \frac{q^2(n-2)^2}{(n-1)(n-2)} \int_1^\infty |w''(s)|^p s^{\rho} \int_0^\infty t^{-\frac{n-2q}{q(n-2)-\rho}} dt ds
\]
Combining this last inequality with (5.8) and (5.9) we get (5.7).

The next result represents the version of Lemma 3.4 for \( m \geq 2 \).

**Lemma 5.4.** For any \( u \in AC_L(0, R) \), we have
\[
\int_0^R |u|^{\frac{m}{n}} r^{\frac{m}{n}(1-n)+\frac{m}{n}-1} dr \leq \frac{1}{(n-2)^{\frac{m}{n}}} \int_0^R |u'|^{\frac{m}{n}} r^{\frac{m}{n}(1-n)+\frac{m}{n}-1} dr.
\]
In particular, for \( w_i \) defined as in (5.4)
\[
(n-2)\omega_{n-1} \int_0^R |w_i|^{\frac{m}{n}} r^{\frac{m}{n}-1} dr \leq \omega_{n-1} \int_0^R |r^{1-n} (r^{n-1} w_i') |^{\frac{m}{n}} r^{\frac{m}{n}-1} dr \tag{5.10}
\]
holds.

**Proof.** It is a consequence of the Corollary 2.5, item (i) with the choice
\[
p = q = \frac{n}{m}, \quad \alpha = \frac{n}{m} (1-n) + \frac{2n}{m} - 1 \quad \text{and} \quad \theta = \frac{n}{m} (1-n) + \frac{n}{m} - 1.
\]
Indeed, we can check that this choice satisfies \( \alpha - p + 1 = \frac{m}{n} (2-n) < 0 \) and \( p = \alpha - \theta \). Hence, we must have
\[
(p-1)^{1-\frac{m}{n}} \frac{1}{p-1-\alpha} \leq C_L \leq \frac{p}{p-1-\alpha} = \frac{1}{n-2}
\]
which completes the proof.

**Remark 5.5.** Analogous to the identity (3.20), since
\[
\omega_{n-1} = \frac{2\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)} \quad \text{and} \quad \Gamma(x+p) = \Gamma(x) \prod_{j=0}^{p-1} (x+j), \quad p \in \mathbb{N}
\]
we can write the following expressions for the critical exponent \( \beta_0(m,n) \): If \( m = 2k \) is even
\[
\frac{n}{\omega_{n-1}} \left[ \frac{\pi^{\frac{m}{2n}} 2^n \Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m-n}{2}\right)} \right]^{\frac{n}{n-m}} = \left[ \omega_{n-1}^{\frac{n}{n-m}} (n-2) \prod_{j=0}^{k-2} (n-m+2j)(m-2j-2) \right]^{\frac{n}{n-m}}
\]
and, for \( m = 2k + 1 \) odd, we have
\[
\frac{n}{\omega_{n-1}} \left[ \frac{\pi^{\frac{n-m}{2}} \Gamma \left( \frac{m+1}{2} \right)}{\Gamma \left( \frac{n-m+1}{2} \right)} \right]^{n/(n-m)} = \left[ \frac{n^{\frac{n-m}{2}} \omega_{n-1} \pi^{\frac{k-1}{2}} \prod_{j=0}^{k-1} (n-m+2j)(m-2j-3)}{\prod_{j=0}^{k-1} (n-m+2j)(m-2j-3)} \right]^{n/(n-m)}.
\]

Finally, we will prove that the sequence \((g_i)\) in (5.6) satisfies the conditions of Carleson-Chang type estimate Theorem 3.2.

**Proposition 5.6.** Let \((g_i)\) be the sequence given in (5.6). Then
\[
\lim_{i \to \infty} \int_0^A |g_i(t)| dt = 0 \quad \text{and} \quad \int_0^+ |g_i(t)| dt \leq 1
\]
for any \( A > 0 \).

**Proof.** Initially, directly from Lemma 5.2 we get Proposition 5.6.

In order to prove the inequality we will divide the estimation in two cases:

**Even case:** \( m = 2k \). Proposition 5.3 with the choice \( p = \frac{n}{m} = \frac{n}{2k} \), \( q_j = q_j^* = \frac{nq_j}{n-2q_j} \) with \( q = p = q_0 = \frac{n}{2k} \) and \( u = \Delta^j w_i \) on the interval \((0, R)\), yields
\[
\left( \int_0^R |\Delta^{k-j-1} w_i|^{p \frac{n}{n-1}} \right)^{\frac{1}{p}} \leq \frac{q_j^2}{(q_j - 1)n(n-2q_j)} \left( \int_0^R |\Delta^j w_i|^{p \frac{n}{n-1}} \right)^{\frac{1}{p}}. \tag{5.12}
\]
Here we are denoting \( \Delta u = r^{1-n} \left(r^{n-1}u'\right)' \) and \( \Delta^j u = \Delta \Delta^{j-1} u \). By iterating (5.12) we get
\[
\left( \int_0^R |\Delta w_i|^{p \frac{n}{n-1}} \right)^{\frac{1}{p}} \leq \left( \int_0^R |\Delta w_i|^{p \frac{n}{n-1}} \right)^{\frac{1}{p}} \leq \prod_{j=0}^{k-2} \left( \frac{q_j^2}{(q_j - 1)n(n-2q_j)} \right)^{\frac{1}{p}} \left( \int_0^R |\Delta^j w_i|^{p \frac{n}{n-1}} \right)^{\frac{1}{p}}.
\]
Since
\[
q_{j+1} = \frac{np}{n-2jp}
\]
we can also write
\[
\left( \int_0^R |\Delta w_i|^{p \frac{n}{n-1}} \right)^{\frac{m}{m}} \leq \prod_{j=0}^{k-2} \left( \frac{1}{(n-m+2j)(m-2j-2)} \right)^{\frac{m}{m}} \left( \int_0^R |\Delta^j w_i|^{p \frac{n}{n-1}} \right)^{\frac{m}{m}}. \tag{5.13}
\]
In view of Remark 5.5, we can write
\[
r = Re^{-\frac{m}{n}} \quad \text{and} \quad g_i(t) = \left( \frac{n-m}{n-m} \omega_{n-1} \prod_{j=0}^{k-2} (n-m+2j)(m-2j-2) \right) w_i(r). \tag{5.14}
\]

Then (5.13), (5.10) and (5.14) yield
\[
\int_0^\infty |g_i(t)| dt = 0 \quad \left( (n-m) \prod_{j=0}^{k-2} (n-m+2j)(m-2j-2) \right)^{\frac{m}{m}} \leq 1.
\]
Odd case: $m = 2k + 1$. First we will prove that

$$
\left( \int_0^R |\Delta^k w_i| \frac{m}{m-1} r^{m-1} dr \right)^{\frac{m}{m-1}} \leq \frac{1}{m-1} \left( \int_0^R \left| (\Delta^k w_i)' \right|^{\frac{m}{m-1}} r^{m-1} dr \right)^{\frac{m}{m-1}}.
$$

(5.15) 

We will apply the Corollary 2.5, item (ii). Firstly, note that $u = \Delta^k w_i \in AC_R(0, R)$. Further, the choice

$$p = q = \frac{n}{m} \quad \alpha = n - 1 \quad \text{and} \quad \theta = \frac{n}{m} - 1$$

implies $\alpha - p + 1 = n (1 - 1/m) > 0$ and $p = \alpha - \theta$. Then, the best possible constant $C_R$ must satisfy

$$
(p - 1) \frac{m}{\alpha - p - 1} \leq C_R \leq \frac{p}{\alpha - p - 1} = 1 
$$

This proves (5.15). Now, similarly to the even case, by iterating the Proposition 5.3 with $p = \frac{n}{m} = \frac{2k}{m}$, $q_{j+1} = q_j = \frac{nq_j}{m - 2q_j}$ with $q = q_0 = \frac{n}{m - 1} = \frac{n}{2k}$, we obtain

$$
\left( \int_0^R \left| \Delta^k w_i \right| \frac{m}{m-1} r^{m-1} dr \right)^{\frac{m}{m-1}} \leq \prod_{j=0}^{k-2} \frac{1}{(n-m+2j+1)(m-2j-3)} \left( \int_0^R \left| \Delta^k w_i \right| \frac{m}{m-1} r^{m-1} dr \right)^{\frac{m}{m-1}}.
$$

(5.16) 

Then, by Lemma 5.4, inequality (5.15) and (5.16), if we take (cf. Remark 5.5)

$$g_i(t) = \left( n^{\frac{m}{m-1}} \omega_{n-1} \prod_{j=0}^{k-1} (n-m+2j+1)(m-2j-3) \right) w_i(r),$$

we have

$$
\int_0^\infty |g_i|^\frac{m}{m-1} dt = \omega_{n-1} \left( \prod_{j=0}^{k-1} (n-m+2j+1)(m-2j-3) \right) \int_0^R \left| w_i(r) \right|^\frac{m}{m-1} r^{m-1} dr \leq 1.
$$
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