Fractional electric charge of a magnetic vortex at nonzero temperature

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Abstract

An ideal gas of twodimensional Dirac fermions in the background of a pointlike magnetic vortex with arbitrary flux is considered. We find that this system acquires fractional electric charge at finite temperatures and determine the functional dependence of the thermal average and quadratic fluctuation of the charge on the temperature, the vortex flux, and the continuous parameter of the boundary condition at the location of the vortex.

I Introduction

Spontaneous breakdown of continuous symmetries can give rise to topological defects (texture solitons) with rather interesting properties. A topological defect in threedimensional space, which is characterized by the nontrivial second homotopy group, is known as a magnetic monopole [1, 2], see also genuine Ref. [3]. Vacuum fluctuations of quantized Dirac fields result in the monopole becoming a CP symmetry violating dyon, i.e. acquiring nonzero (and fractional) electric charge [4, 5, 6]. More recently the effect of thermal fluctuations of quantized Dirac fields in the presence of the monopole has been considered, yielding the temperature dependence of the induced charge [7, 8, 9].

A topological defect in twodimensional space, which is characterized by the nontrivial first homotopy group, is a cross section of the Abrikosov-Nielsen-Olesen magnetic vortex [10, 11]. The vortex defect is described in terms of a spin-0 field which condenses and a

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spin-1 field corresponding to the spontaneously broken gauge group; the former is coupled
to the latter in the minimal way with constant $e_{\text{cond}}$. Single-valuedness of the condensate
field and finiteness of the vortex energy imply that the vortex flux is related to $e_{\text{cond}}$:

$$\Phi = \frac{1}{2\pi} \oint d\mathbf{x} \mathbf{V}(\mathbf{x}) = \frac{1}{e_{\text{cond}}},$$

where $\mathbf{V}(\mathbf{x})$ is the vector potential of the spin-1 field, and the integral is over a path enclosing
once the vortex tube. The quantized fermion field is coupled minimally to the spin-1 field
with constant $e$ - the elementary charge; thus, quantum effects depend on the value of $e\Phi$. The case of $e_{\text{cond}} = 2e$ ($e\Phi = 1/2$) is realized in ordinary Bardeen-Cooper-Schrieffer
superconductors where the Cooper pair field condenses and, in addition, there are normal
electron (pair-breaking) excitations. It remains still to be elucidated, whether other values
of $e\Phi$ are realized in nature, although there are claims that vortices with fractional $e\Phi \neq 1/2$
exist in chiral superfluids and chiral and two-gap superconductors [12, 13].

The aim of the present paper is to consider the effect of thermal fluctuations of quantized
Dirac fields \(^1\) in the presence of the vortex defect with arbitrary value of $e\Phi$, which results
in the vortex acquiring fractional electric charge; the zero-temperature effect was considered
earlier [17, 18, 19]. Since continuous symmetry is not spontaneously broken at the core of
the defect, it seems reasonable to exclude the region of the defect and to impose a boundary
condition for quantized fields at the edge of this region. Thus, quantum effects depend both
on $e\Phi$ and real continuous quantity $\Theta$ which parameterizes the most general varieties of
boundary conditions (for more details see next Section). This setup should not be confused
with the setup when fermions are quantized in the presence of an extensive magnetic field
with finite flux and the region of the nonvanishing field strength is not excluded. The
induced charge in the latter case was considered in Refs. [20, 21, 22](zero temperature) and
Refs. [23, 24, 25](nonzero temperature), and we shall compare the results of both setups in
Section V.

The operator of the second-quantized fermion field in a static background can be pre-
sented in the form

$$\Psi(\mathbf{x}, t) = \sum_{E>0} \int e^{-iEt} \langle \mathbf{x}|E, \lambda \rangle a_{E\lambda} + \sum_{E<0} \int e^{-iEt} \langle \mathbf{x}|E, \lambda \rangle b_{E\lambda}^+, \quad (1.2)$$

where $a_{E\lambda}$ and $a_{E\lambda}^+$ ($b_{E\lambda}$ and $b_{E\lambda}^+$) are the fermion (antifermion) creation and destruction
operators satisfying anticommutation relations,

$$[a_{E\lambda}, a_{E'\lambda}^+]_+ = [b_{E\lambda}, b_{E'\lambda}^+]_+ = \langle E, \lambda|E', \lambda' \rangle, \quad (1.3)$$

and $\langle \mathbf{x}|E, \lambda \rangle$ is the solution to the stationary Dirac equation,

$$H \langle \mathbf{x}|E, \lambda \rangle = E \langle \mathbf{x}|E, \lambda \rangle, \quad (1.4)$$

$H$ is the Dirac Hamiltonian, $E$ is the energy and $\lambda$ is the set of other parameters (quan-
tum numbers) specifying a state; symbol $\sum_{\lambda}$ means the summation over discrete and the

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\(^1\)This may be relevant for various particle physics models with applications ranging from early Universe
cosmology to hot nuclear matter phenomenology, and even for condensed matter models, because effectively
quasirelativistic fermions arise, in particular, in $d$-wave type II superconductors (see, e.g., Refs. [14, 15, 16].
integration (with a certain measure) over continuous values of all quantum numbers. Conventionally, the operators of dynamical invariants are defined as bilinears of the fermion field operators, and, thus, comprizing:

the energy operator (temporal component of the energy-momentum vector),

\[ \hat{P}_0 = \frac{i}{4} \int d^d x \left( [\Psi^+, \partial_t \Psi] - [\partial_t \Psi^+, \Psi] \right), \tag{1.5} \]

and the fermion number operator,

\[ \hat{N} = \frac{1}{2} \int d^d x \left[ \Psi^+, \Psi \right], \tag{1.6} \]

where \( d \) is the space dimension. Operators (1.5) and (1.6) commute and are thus diagonal in the fermion and antifermion creation and destruction operators.

The thermal average of the fermion number operator over the canonical ensemble is defined as (see, e.g., Ref.[26])

\[ \langle \hat{N} \rangle = \frac{Sp \hat{N} \exp(-\beta \hat{P}_0)}{Sp \exp(-\beta \hat{P}_0)} = (k_B T)^{-1}, \tag{1.7} \]

where \( T \) is the equilibrium temperature, \( k_B \) is the Boltzmann constant, and \( Sp \) is the trace or the sum over the expectation values in the Fock state basis created by operators in Eq.(1.3). Appropriately, the electric charge of the quantum fermionic system in thermal equilibrium is given by expression

\[ Q(T) \equiv e \langle \hat{N} \rangle = -\frac{e}{2} \int_{-\infty}^{\infty} dE \tau(E) \tanh \left( \frac{1}{2} \beta E \right), \tag{1.8} \]

where the last equality is obtained by transforming the right hand side of Eq.(1.7) into an integral over the spectrum of the Dirac Hamiltonian (see, e.g., Ref.[23]), and the spectral density of the Dirac Hamiltonian (or density of states) is

\[ \tau(E) = \frac{1}{\pi} \text{Im} \text{Tr} \frac{1}{H - E - i0}, \tag{1.9} \]

where Tr is the trace of an integro-differential operator in functional space: \( \text{Tr} U = \int d^d x \text{tr}(\mathbf{x} | U | \mathbf{x}) \); tr denotes the trace over spinor indices only; note that the functional trace should be regularized and renormalized by subtraction, if necessary.

Similarly, one gets expression for the quadratic fluctuation of the electric charge:

\[ \Delta^2_{Q(T)} \equiv e^2 \left[ \langle \hat{N}^2 \rangle - \left( \langle \hat{N} \rangle \right)^2 \right] = \frac{e^2}{4} \int_{-\infty}^{\infty} dE \frac{\tau(E)}{\cosh^2 \left( \frac{1}{2} \beta E \right)}. \tag{1.10} \]

Evidently, if the quadratic fluctuation becomes nonvanishing, then the corresponding dynamical invariant ceases to be a sharp quantum observable.

In the present paper we shall find electric charge (1.8) and its fluctuation (1.10) in the \( d = 2 \) quantum fermionic system in the background of a single static topological defect which is a twodimensional cross section of the magnetic vortex.

3
II Self-adjointness of the Dirac Hamiltonian in the background of the pointlike vortex defect

The Dirac Hamiltonian in external magnetic field takes form

\[ H = -i\gamma^0 \gamma [\partial - ieV(x)] + \gamma^0 m . \] (2.1)

In 2 + 1-dimensional space-time \((x, t) = (x^1, x^2, t)\), the Clifford algebra has two inequivalent irreducible representations which can be differed in the following way:

\[ i\gamma^0 \gamma^1 \gamma^2 = s, \quad s = \pm 1 . \] (2.2)

Choosing the \(\gamma^0\) matrix in the diagonal form,

\[ \gamma^0 = \sigma_3 , \] (2.3)

one gets

\[ \gamma^1 = e^{\frac{i}{2} \sigma_3 \chi_s} i\sigma_1 e^{-\frac{i}{2} \sigma_3 \chi_s}, \quad \gamma^2 = e^{\frac{i}{2} \sigma_3 \chi_s} i\sigma_2 e^{-\frac{i}{2} \sigma_3 \chi_s}, \] (2.4)

where \(\sigma_1, \sigma_2,\) and \(\sigma_3\) are the Pauli matrices, and \(\chi_1\) and \(\chi_{-1}\) are the parameters varying in interval \(0 < \chi_s < 2\pi\) to go over to the equivalent representation. Note also that in odd-dimensional space-time the \(m\) parameter in Eq.(2.1) can take both positive and negative values; a change of sign of \(m\) corresponds to going over to the inequivalent representation.

A solution to stationary Dirac equation (1.4) with Hamiltonian (2.1) can be presented as

\[ \langle x|E, n \rangle = \begin{pmatrix} f_n(r, E) e^{in\varphi + i\chi_s} \\ g_n(r, E) e^{i(n+s)\varphi} \end{pmatrix}, \quad n \in \mathbb{Z} , \] (2.5)

where polar coordinates \(r = \sqrt{(x^1)^2 + (x^2)^2}\) and \(\varphi = \arctan(x^2/x^1)\) are introduced, and \(\mathbb{Z}\) is the set of integer numbers. The magnetic field strength and its flux in the units of \(2\pi\) are given by expressions (compare with Eq.(1.1)):

\[ B(x) = \partial \times V(x) , \quad \Phi = \frac{1}{2\pi} \int d^2x B(x) . \] (2.6)

Since a single defect is considered, a support of the magnetic field strength is localized in a certain, let it be central, region of two-dimensional space. It is evident that different functions \(B(x)\) can give the same value of \(\Phi\). In general, a solution to the Dirac equation in an external magnetic field depends on the field configuration in two ways: there is a direct, or local, impact of \(B(x)\) on the solution at the same point \(x\) (similar to the action of the classical Lorentz force), and there is an indirect (through vector potential) influence of the field strength on the behaviour of the solution in regions out of the field strength’s support (similar to the quantum-mechanical Bohm-Aharonov effect [27]). Namely the latter effects are those which interest us, since, as it has been already noted in Introduction, the central region (i.e. the region of the field strength’s support) is excluded, and at its edge a boundary condition is imposed on the solution to the Dirac equation.

It might be anticipated that the solution in the outer region depends on the flux rather than the local features of the field strength in the central region. However, a closer look at
the situation when the central region is not excluded suggests that the local features of the field strength may influence the behaviour of the solution out of the central region. Really, changes of a profile of the field strength influence strongly the behaviour of the solution in the central region, and this, due to the continuity and smoothness properties of the solution as a solution to a differential equation, entails changes in its behaviour in the outer region. Thus, when the central region is excluded, our purpose is not to stick to a limited set of boundary conditions but, instead, to extend this set maximally in order to cover all possible types of the behaviour of the solution near the boundary and, perhaps, all plausible profiles of the field strength in the excluded region.

How to achieve this purpose in general, remains to be a question. However, a recipe is available under a simplifying assumption that finite size of the excluded region is neglected: then the most general conditions are those ensuring self-adjointness of the Dirac Hamiltonian, and they are labelled by self-adjoint extension parameter $\Theta$ (see, e.g., Ref.[28]). Although $\Theta$ is physically interpreted as the CP violating vacuum angle in the $d = 3$ case of the monopole defect [4,5,6], the direct physical interpretation of $\Theta$ in the $d = 2$ case of the vortex defect is yet lacking. Both in the monopole and vortex cases parameter $\Theta$ is involved into a condition for just one of the modes of the solution to the Dirac equation.

When the transverse size of the vortex defect is shrinked to zero, the magnetic field strength takes form

$$B(x) = 2\pi \Phi \delta(x),$$

and the vector potential can be chosen as

$$V^1(x) = -\Phi r^{-1} \sin \phi, \quad V^2(x) = \Phi r^{-1} \cos \phi.$$ (2.8)

Then Dirac Hamiltonian (2.1) takes form

$$H = -i\gamma^0 \gamma^r \partial_r - i r^{-1} \gamma^0 \gamma^\phi (\partial_\phi - ie\Phi) + \gamma^0 m,$$ (2.9)

where

$$\gamma^r = \gamma^1 \cos \phi + \gamma^2 \sin \phi, \quad \gamma^\phi = -\gamma^1 \sin \phi + \gamma^2 \cos \phi.$$ (2.10)

Using explicit form of $\gamma$ matrices (2.3)-(2.4), one finds that the Dirac equation in the back-
ground of a pointlike defect is reduced to following set of equations for the modes of Eq.(2.5):

$$\begin{pmatrix}
  m \\
  -\partial_r + s(n - e\Phi + s)r^{-1}
\end{pmatrix}
\begin{pmatrix}
  f_n \\
  g_n
\end{pmatrix} = E
\begin{pmatrix}
  f_n \\
  g_n
\end{pmatrix}.$$ (2.11)

Partial Hamiltonians are essentially self-adjoint for all $n$, with the exception of $n = n_0$, where

$$n_0 = \lfloor e\Phi \rfloor + \frac{1}{2} - \frac{1}{2}s,$$ (2.12)

$\lfloor u \rfloor$ is the integer part of quantity $u$ (i.e., the largest integer which is less than or equal to $u$). Correspondingly, the modes with $n \neq n_0$ are regular at $r = 0$ (i.e., at the location of the defect). The partial Hamiltonian at $n = n_0$ requires a self-adjoint extension according to the Weyl-von Neumann theory of self-adjoint operators (see, e.g., Ref.[29]), which upon implementation yields following condition for the corresponding mode [30,18,19]:

$$\cos \left( s \frac{\Theta}{2} + \frac{\pi}{4} \right) \lim_{r \to 0} (|m|r)^F f_{n_0} = -\text{sgn}(m) \sin \left( s \frac{\Theta}{2} + \frac{\pi}{4} \right) \lim_{r \to 0} (|m|r)^{1-F} g_{n_0},$$ (2.13)
where
\[
\text{sgn}(u) = \begin{cases} 
1, & u > 0 \\
-1, & u < 0 
\end{cases},
\]
\(\Theta\) is the self-adjoint extension parameter, and
\[
F = s[e^\Phi] + \frac{1}{2} - \frac{1}{2}s,
\tag{2.14}
\]
\([u] = u - [u]\) is the fractional part of quantity \(u\), \(0 \leq [u] < 1\); note here that Eq.(2.13) implies that \(0 < F < 1\), since in the case of \(F = \frac{1}{2} - \frac{1}{2}s\) both \(f_n\) and \(g_n\) obey the condition of regularity at \(r \to 0\). Note also that Eq.(2.13) is periodic in \(\Theta\) with period \(2\pi\).

So far solutions corresponding to the continuous spectrum, \(|E| > |m|\), are concerned, that obey the "orthonormality" condition
\[
\int d^2x \langle E, n|x|n'\rangle \langle x|E, n'\rangle = \delta(E - E') \delta_{nn'}.
\tag{2.15}
\]
Owing to Eq.(2.13), an additional solution corresponding to the bound state with energy \(E = E_{BS}\), \(|E_{BS}| < |m|\), appears at \(\cos \Theta < 0\), that obeys usual normalization condition
\[
\int d^2x \langle E_{BS}, n_0|x|n_0\rangle = 1.
\tag{2.16}
\]
Its energy is determined as a real root of algebraic equation
\[
\frac{(1 + m^{-1}E_{BS})^{1-F}}{(1 - m^{-1}E_{BS})^F} = -A,
\tag{2.17}
\]
where
\[
A = 2^{1-2F} \frac{\Gamma(1-F)}{\Gamma(F)} \tan \left( \frac{s \Theta}{2} + \frac{\pi}{4} \right),
\tag{2.18}
\]
\(\Gamma(u)\) is the Euler gamma function. The bound state energy is zero, \(E_{BS} = 0\), at \(A = -1\); otherwise, we get
\[
\text{sgn}(E_{BS}) = \frac{1}{2} \text{sgn}(m) \left[ \text{sgn}(1 + A^{-1}) - \text{sgn}(1 + A) \right].
\tag{2.19}
\]
At \(\cos \Theta > 0\) \((A > 0)\) the right hand side of Eq.(2.19) turns to zero, which corresponds to the absence of bound state in this case.

### III Resolvent and spectral density

The kernel of the resolvent (the Green’s function) of the Dirac Hamiltonian in the coordinate representation is defined as
\[
G^\omega(r, \varphi; r', \varphi') = \langle r, \varphi| (H - \omega)^{-1} |r', \varphi'\rangle,
\tag{3.1}
\]
where $\omega$ is a complex parameter with dimension of energy. The expansion of Eq. (3.1) in modes takes form

$$G^\omega(r, \varphi; r', \varphi') = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{in(\varphi-\varphi')} \begin{pmatrix} a_n(r; r') & d_n(r; r')e^{-is(\varphi'-\chi_s)} \\ b_n(r; r') & c_n(r; r')e^{is(\varphi-\varphi')} \end{pmatrix}. \quad (3.2)$$

In the case of $H$ given by Eq. (2.9) radial components of $G^\omega(r, \varphi; r', \varphi')$ (3.2) satisfy equations (compare with Eq. (2.11)):

$$\begin{pmatrix} -\omega - m & \partial_r + s(n-e\Phi + s)r^{-1} \\ -\partial_r + s(n-e\Phi)r^{-1} & -\omega - m \end{pmatrix} \begin{pmatrix} a_n(r; r') & d_n(r; r') \\ b_n(r; r') & c_n(r; r') \end{pmatrix} = \begin{pmatrix} a_n(r; r') & b_n(r; r') \\ d_n(r; r') & c_n(r; r') \end{pmatrix} = \frac{\delta(r - r')}{\sqrt{rr'}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.3)$$

Off-diagonal radial components are expressed through the diagonal ones:

$$b_n(r; r') = (\omega + m)^{-1} \left[ -\partial_r + s(n-e\Phi)r^{-1} \right] a_n(r; r') = (\omega - m)^{-1} \left[ \partial_r + s(n-e\Phi + s)r^{-1} \right] c_n(r; r'), \quad (3.4)$$

$$d_n(r; r') = (\omega - m)^{-1} \left[ \partial_r + s(n-e\Phi + s)r^{-1} \right] c_n(r; r') = (\omega + m)^{-1} \left[ -\partial_r + s(n-e\Phi)r^{-1} \right] a_n(r; r'). \quad (3.5)$$

In Appendix A we determine the diagonal radial components which can be presented in the following way,

- type 1 ($l = s(n-n_0) > 0$):
  $$a_n(r; r') = \frac{i\pi}{2}(\omega + m) \left[ \theta(r' - r)H_{l+1-F}^{(1)}(kr)J_{l-F}(kr') + \theta(r' - r)J_{l-F}(kr)H_{l+1-F}^{(1)}(kr') \right], \quad (3.6)$$

  $$c_n(r; r') = \frac{i\pi}{2}(\omega - m) \left[ \theta(r - r')H_{l+1-F}^{(1)}(kr)J_{l+1-F}(kr') + \theta(r' - r)J_{l+1-F}(kr)H_{l+1-F}^{(1)}(kr') \right]; \quad (3.7)$$

- type 2 ($l' = -s(n-n_0) > 0$):
  $$a_n(r; r') = \frac{i\pi}{2}(\omega + m) \left[ \theta(r' - r)H_{l'+1-F}^{(1)}(kr)J_{l'-F}(kr') + \theta(r' - r)J_{l'+F}(kr)H_{l'+1-F}^{(1)}(kr') \right], \quad (3.8)$$

  $$c_n(r; r') = \frac{i\pi}{2}(\omega - m) \left[ \theta(r - r')H_{l'+1-F}^{(1)}(kr)J_{l'-1+F}(kr') + \theta(r' - r)J_{l'+1-F}(kr)H_{l'+1-F}^{(1)}(kr') \right]; \quad (3.9)$$
Here $r$ is the first-kind Hankel function of order $\lambda$, $J_{\lambda}(u)$ is the Bessel function of order $\lambda$, and $H^{(1)}_{\lambda}(u)$ is the first-kind Hankel function of order $\lambda$, and

$$\tan \nu_\omega = \frac{k^{2F}}{\omega + m} \sgn(m)|m|^{1-2F}A,$$  \hspace{1cm} (3.12)

where $A$ is given by Eq. (2.13). Note that the type 1 and type 2 components are regular at $r = 0$ (or $r' = 0$), whereas the type 3 components are irregular at $r = 0$ (or $r' = 0$), satisfying conditions (compare with Eq. (2.13)):

$$\cos \left( s \frac{\Theta}{2} + \frac{\pi}{4} \right) \lim_{r \to 0} (|m|r)^F a_{n_0}(r; r') = -\sgn(m) \sin \left( s \frac{\Theta}{2} + \frac{\pi}{4} \right) \lim_{r \to 0} (|m|r)^{1-F}b_{n_0}(r; r'),$$  \hspace{1cm} (3.13)

$$\cos \left( s \frac{\Theta}{2} + \frac{\pi}{4} \right) \lim_{r \to 0} (|m|r)^F d_{n_0}(r; r') = -\sgn(m) \sin \left( s \frac{\Theta}{2} + \frac{\pi}{4} \right) \lim_{r \to 0} (|m|r)^{1-F}c_{n_0}(r; r'),$$  \hspace{1cm} (3.14)

and the ones at $r' \to 0$, which are obtained from Eqs. (3.13) and (3.14) by interchange $b_{n_0} \leftrightarrow d_{n_0}$.

Taking $r' > r$ for definiteness, we get relations

$$\int_0^{2\pi} d\varphi \ tr \ G^{\omega}(r, \varphi; r', \varphi) = \sum_{n \in \mathbb{Z}} [a_n(r; r') + c_n(r; r')] =$$

$$= \sum_{l \in \mathbb{Z}} \sum_{l' \geq 1} [(\omega + m) I_{l-F}(-ikr) K_{l+F}(-ikr') + (\omega - m) I_{l+1-F}(-ikr) K_{l+1+F}(-ikr')] +$$

$$+ \sum_{l' \in \mathbb{Z}} \sum_{l' \geq 1} [(\omega + m) I_{l+F}(-ikr) K_{l+F}(-ikr') + (\omega - m) I_{l+1+F}(-ikr) K_{l+1+F}(-ikr')] +$$

$$+ (\omega + m) I_{F}(-ikr) K_{F}(-ikr') + (\omega - m) I_{1-F}(-ikr) K_{1-F}(-ikr') + \frac{2 \sin(F\pi)}{\pi (\tan \nu_\omega + e^{iF\pi})} \times$$

$$\times [(\omega + m) \tan \nu_\omega K_{F}(-ikr) K_{F}(-ikr') + (\omega - m) e^{iF\pi} K_{1-F}(-ikr) K_{1-F}(-ikr')], \hspace{1cm} (3.15)$$
and
\[ \int_0^{2\pi} d\varphi \, tr \bigg| G^\omega(r, \varphi; r', \varphi) \bigg|_{e\Phi=0} = \sum_{n \in \mathbb{Z}} [a_n(r; r') + c_n(r; r')] \bigg|_{e\Phi=0} = 2\omega \sum_{n \in \mathbb{Z}} I_n(-ikr)K_n(-ikr), \] (3.16)

where \( I_\lambda(u) \) is the modified Bessel function of order \( \lambda \), and
\[ K_\lambda(u) = \frac{\pi}{2\sin(\lambda\pi)} [I_{-\lambda}(u) - I_\lambda(u)] \]
is the Macdonald function of order \( \lambda \); note that the last equalities in Eqs. (3.15) and (3.16) are obtained under condition \( \text{Im} \, k > 0 \). Using relations (see, e.g., Ref. [31])
\[ I_\lambda(kr)K_\lambda(kr') = \frac{1}{2} \int_0^\infty \frac{dy}{y} \exp \left( -\frac{\kappa^2 r r'}{2y} - \frac{r^2 + r'^2}{2r r'} y \right) I_\lambda(y), \quad \text{Re} \, \kappa^2 > 0, \]

we perform summation in Eqs. (3.15) and (3.16) and get in the case of \( \text{Im} \, k > |\text{Re} \, k| \):
\[ \int_0^{2\pi} d\varphi \, tr \left[ G^\omega(r, \varphi; r', \varphi) - G^\omega(r, \varphi; r', \varphi) \right]_{e\Phi=0} = \frac{2 \sin(\pi) \nu_\omega}{\pi (\tan \nu_\omega + e^{i\pi})} \times \]
\[ \sum_{n \in \mathbb{Z}} I_{t+\lambda}(y) = -\frac{1}{2\lambda} \left\{ e^y \int_0^y du \, e^{-u} I_\lambda(u) - y [I_\lambda(y) + I_{\lambda+1}(y)] \right\}, \quad \text{Re} \, \lambda > -1, \]

where \( \kappa = -ik \). Taking the limit \( r' \to r \) in Eq. (3.17) and integrating it over the radial variable, we get the renormalized (finite) trace of the resolvent operator
\[ Tr (H - \omega)^{-1} \equiv \int_0^{2\pi} dr \int_0^{2\pi} d\varphi \, tr \left[ G^\omega(r, \varphi; r, \varphi) - G^\omega(r, \varphi; r, \varphi) \right]_{e\Phi=0} = \]
\[ = \frac{1}{\omega^2 - m^2} \left[ \omega \left( \frac{2F - 1}{e^{-i\pi} \tan \nu_\omega + 1} - F^2 \right) + m \left( \frac{1}{e^{-i\pi} \tan \nu_\omega + 1} - F \right) \right]; \] (3.18)
note that the last result can be continued analytically to the whole complex $\omega$-plane. Note also that Eq. (3.18) can be rewritten in an equivalent form:

$$
Tr (H - \omega)^{-1} = \frac{1}{\omega^2 - m^2} \left\{ \omega \left[ \frac{2F - 1}{e^{-i(1-F)\pi} \cot \nu - 1} - (1 - F)^2 \right] + m \left[ \frac{1}{e^{-i(1-F)\pi} \cot \nu - 1} + 1 - F \right] \right\}.
$$

(3.19)

Taking the imaginary part of Eq. (3.18) or Eq. (3.19) at $\omega = E + i0$, we get spectral density $\tau(E)$, see Eq. (1.9).

IV Thermal average and fluctuation of the charge

Taking into account Eq. (1.9), one can get the following contour integral representation for induced charge (1.8) and its quadratic fluctuation (1.10):

$$
Q(T) = -\frac{e}{2} \int_C \frac{d\omega}{2\pi i} \tan \left( \frac{\beta \omega}{2} \right) Tr(H - \omega)^{-1},
$$

(4.1)

and

$$
\Delta^2 Q(T) = \frac{e^2}{4} \int_C \frac{d\omega}{2\pi i} \sech^2 \left( \frac{\beta \omega}{2} \right) Tr(H - \omega)^{-1},
$$

(4.2)

where $C$ is the contour $(-\infty + i0, +\infty + i0)$ and $(+\infty - i0, -\infty - i0)$ in the complex $\omega$-plane. Substituting Eq. (3.18) for $Re \omega > 0$ and Eq. (3.19) for $Re \omega < 0$ into Eqs. (4.1) and (4.2), we obtain

$$
Q(T) = -\frac{e}{2} \text{sgn}(m) \left\{ \frac{1}{2} \left[ \text{sgn}(1 + A^{-1}) - \text{sgn}(1 + A) \right] \tan \left( \frac{\beta |E_{BS}|}{2} \right) + 
\frac{2 \sin(F\pi)}{\pi} \int_0^{\infty} \frac{du}{u\sqrt{u+1}} \tan \left( \frac{\beta |m| \sqrt{u+1}}{2} \right) \times
\frac{FAu - (1 - F)A^{-1}u^{-1} - u \cos (F\pi) + (F - \frac{1}{2}) u (Au + A^{-1}u^{-1} - F)}{[AuF - A^{-1}u^{-1} + 2 \cos (F\pi)]^2 + 4(u+1) \sin^2 (F\pi)} \right\},
$$

(4.3)

and

$$
\Delta^2 Q(T) = \frac{e^2}{4} \left\{ \frac{1}{2} \left[ 1 - \text{sgn}(A) \right] \sech^2 \left( \frac{\beta |E_{BS}|}{2} \right) - F(1 - F) \sech^2 \left( \frac{\beta |m|}{2} \right) + 
\frac{2 \sin(F\pi)}{\pi} \int_0^{\infty} \frac{du}{u} \sech^2 \left( \frac{\beta |m| \sqrt{u+1}}{2} \right) \times
\frac{FAuF + (1 - F)A^{-1}u^{-1} - (2F - 1)u \cos (F\pi)}{[AuF - A^{-1}u^{-1} + 2 \cos (F\pi)]^2 + 4(u+1) \sin^2 (F\pi)} \right\},
$$

(4.4)
where $A$ is given by Eq. (2.18).

In the cases of $A = 0$ and $A^{-1} = 0$ expressions for the charge and its fluctuation simplify:

$$Q(T) = -\frac{e}{2} \left( F - \frac{1}{2} \pm \frac{1}{2} \right) \tanh \left( \frac{1}{2} \beta m \right), \quad \Theta = \pm \frac{s \pi}{2} \text{ (mod } 2\pi),$$

(4.5)

and

$$\Delta^2_{Q(T)} = \frac{e^2}{4} \left( F - \frac{1}{2} \pm \frac{1}{2} \right)^2 \text{sech}^2 \left( \frac{1}{2} \beta |m| \right), \quad \Theta = \pm \frac{s \pi}{2} \text{ (mod } 2\pi);$$

(4.6)

note that Eq. (1.35) at $\Theta = \frac{s \pi}{2} \text{ (mod } 2\pi)$ was obtained in Ref. [7].

In the limit $T \to 0$ ($\beta \to \infty$) the charge tends to finite value (see Ref. [19]):

$$Q(0) = \begin{cases} \frac{e}{2} \text{sgn}(m)(1 - F), & -1 < A < \infty \\ -\frac{e}{2} \text{sgn}(m)F, & A^{-1} = -1, \ A^{-1} = 0 \\ -\frac{e}{2} \text{sgn}(m)(1 + F), & -\infty < A < -1 \\ -\frac{e}{2} \text{sgn}(m)F, & -1 < A^{-1} < \infty \\ \frac{e}{2} \text{sgn}(m)(1 - F), & A = -1, \ A = 0 \\ \frac{e}{2} \text{sgn}(m)(2 - F), & -\infty < A^{-1} < -1 \end{cases}, \quad 0 < F < \frac{1}{2}$$

(4.7)

$$Q(0) = \begin{cases} -\frac{e}{\pi} \text{sgn}(m) \arctan \left( \tan \frac{\Theta}{2} \right), & \Theta \neq \pi \text{ (mod } 2\pi) \\ 0, & \Theta = \pi \text{ (mod } 2\pi) \end{cases} \quad F = \frac{1}{2},$$

(4.8)

whereas the fluctuation tends exponentially to zero for almost all values of $\Theta$ with the exception of one corresponding to the zero bound state energy, $E_{BS} = 0$ ($A = -1$):

$$\Delta^2_{Q(0)} = \begin{cases} 0, & A \neq -1 \\ \frac{e^2}{4}, & A = -1 \end{cases}$$

(4.9)

In the high-temperature limit the charge tends to zero:

$$Q(T \to \infty) = \begin{cases} \frac{e}{2} \text{sgn}(m) \frac{\sin(F \pi)}{\pi} \frac{\Gamma(1 - F)}{\Gamma(1 + F)} \tan \left( \frac{\Theta}{2} + \frac{\pi}{4} \right) \left( \frac{|m|}{k_B T} \right)^{1 - 2F}, & 0 < F < \frac{1}{2} \\ -\frac{e}{8k_B T} \text{sm} \sin \Theta, & F = \frac{1}{2} \\ -\frac{e}{2} \text{sgn}(m) \frac{\sin(F \pi)}{\pi} \frac{\Gamma(F)}{\Gamma(2 - F)} \cot \left( \frac{\Theta}{2} + \frac{\pi}{4} \right) \left( \frac{|m|}{k_B T} \right)^{2F - 1}, & \frac{1}{2} < F < 1 \end{cases}$$

(4.10)
whereas the fluctuation tends to finite value, see Appendix B:

\[
\lim_{T \to \infty} \Delta^2_{Q(T)} = \begin{cases} 
\frac{e^2}{4}(1-F)^2, & \text{if } \Theta \neq s\frac{\pi}{2} \mod 2\pi \\
\frac{e^2}{4}F^2, & \text{if } \Theta = s\frac{\pi}{2} \mod 2\pi \\
\frac{e^2}{4}(1-F)^2, & \text{if } \Theta \neq -s\frac{\pi}{2} \mod 2\pi \\
\frac{e^2}{4}F^2, & \text{if } \Theta = -s\frac{\pi}{2} \mod 2\pi
\end{cases}
\]

(4.11)

\[
\frac{e^2}{4}(1-F)^2, \quad \Theta \neq s\frac{\pi}{2} \mod 2\pi \\
\frac{e^2}{4}F^2, \quad \Theta = s\frac{\pi}{2} \mod 2\pi
\]

(4.12)

At half-integer values of \(e\Phi\) one has

\[
A\big|_{\Phi = \frac{1}{2}} = \tan \left( \frac{\Theta}{2} + \frac{\pi}{4} \right),
\]

and the charge and its fluctuation take form

\[
Q(T)\big|_{\Phi = \frac{1}{2}} = -\frac{e}{4} \left\{ [1 - \text{sgn}(\cos \Theta)] \tanh \left( \frac{1}{2} \beta m \sin \Theta \right) + \\
\frac{\sin 2\Theta}{2\pi} \int_1^{\infty} \frac{dv}{\sqrt{v(v-1)}} \frac{\tanh \left( \frac{1}{2} \beta m \sqrt{v} \right)}{v - \sin^2 \Theta} \right\},
\]

(4.13)

and

\[
\Delta^2_{Q(T)}\big|_{\Phi = \frac{1}{2}} = \frac{e^2}{8} \left\{ [1 - \text{sgn}(\cos \Theta)] \text{sech}^2 \left( \frac{1}{2} \beta |m \sin \Theta| \right) - \frac{1}{2} \text{sech}^2 \left( \frac{1}{2} \beta |m| \right) + \\
\cos \Theta \int_1^{\infty} \frac{dv}{\sqrt{v(v-1)}} \text{sech}^2 \left( \frac{1}{2} \beta |m| \sqrt{v} \right) \right\}.
\]

(4.14)

An alternative representation for the charge and its fluctuation is obtained by deforming contour \(C\) to encircle poles of the \(\tanh \left( \frac{1}{2} \beta \omega \right)\) and \(\text{sech}^2 \left( \frac{1}{2} \beta \omega \right)\) functions, which occur along the imaginary axis at the Matsubara modes \(\omega_n = (2n + 1)\frac{i\pi}{\beta}\), see Appendix C:

\[
Q(T) = -e \text{sgn}(m) \left\{ \frac{1}{2} \left( F - \frac{1}{2} \right) \tanh \left( \frac{\pi}{2\xi} \right) + \\
\frac{\xi}{\pi} \sum_{n \in \mathbb{Z}} 2(2F-1)(2n+1)^2\xi^2 + A[1 + (2n + 1)^2\xi^2]^F - A^{-1}[1 + (2n + 1)^2\xi^2]^{-F} \right\},
\]

(4.15)
and

\[
\Delta^2_{Q(T)} = \frac{e^2}{8} \left[ 1 - 2F(1 - F) \right] \text{sech}^2 \left( \frac{\pi}{2\xi} \right) + \\
+ e^2 \xi^2 \sum_{n \in \mathbb{Z}} \frac{1}{\left[ 1 + (2n + 1)^2 \xi^2 \right]^2} \left\{ A[1 + (2n + 1)^2 \xi^2]^F + 2 + A^{-1}[1 + (2n + 1)^2 \xi^2]^{-1-F} \right\} \times \\
\times \left\{ (2F - 1)[(2n + 1)^2 \xi^2 - 1] \left\{ A[1 + (2n + 1)^2 \xi^2]^F - A^{-1}[1 + (2n + 1)^2 \xi^2]^{-1-F} \right\} + \\
+ 2\left\{ 1 - [3 - 4F(1 - F)](2n + 1)^2 \xi^2 \right\} - 4(2n + 1)^2 \xi^2 \times \\
\right. \\
\left. \times \frac{(2F - 1) \left\{ A[1 + (2n + 1)^2 \xi^2]^F - A^{-1}[1 + (2n + 1)^2 \xi^2]^{-1-F} \right\} - 1 + (2F - 1)^2(2n + 1)^2 \xi^2}{A[1 + (2n + 1)^2 \xi^2]^F + 2 + A^{-1}[1 + (2n + 1)^2 \xi^2]^{-1-F}} \right\} ,
\]

(4.16)

where \( \xi = \pi/|\beta|m| \).

V Discussion

In the present paper we consider an ideal gas of two-dimensional relativistic massive electrons in the background of a static pointlike magnetic vortex. This system at thermal equilibrium is found to acquire electric charge: its average \( Q(T) \) is given by Eq.(4.3), and its quadratic fluctuation \( \Delta^2_{Q(T)} \) is given by Eq.(4.4). The most general boundary conditions (parametrized by the self-adjoint extension parameter \( \Theta \)) at the location of the vortex are employed, and arbitrary values of the vortex flux \( \Phi \) are permitted; our results are periodic in \( \Theta \) with period \( 2\pi \) at fixed \( \Phi \) and periodic in \( \Phi \) with period \( e^{-1} \) at fixed \( \Theta \) (e is the electron charge).

Note that Eqs.(4.3) and (4.4) can be regarded as the Sommerfeld-Watson transforms of the infinite sum representation, Eqs.(4.15) and (4.16). Note also that the charge is odd and its fluctuation is even under transition to the inequivalent representation of the Clifford algebra (\( s \rightarrow -s \) or \( m \rightarrow -m \)).

Eq.(4.3) can rewritten in the form

\[
Q(T) = Q(0) + \tilde{Q}(T),
\]

(5.1)

where \( Q(0) \) is given by Eqs.(4.7)-(4.8), and

\[
\tilde{Q}(T) = \frac{e}{2} \text{sgn}(m) \left\{ \frac{\text{sgn}(1 + A^{-1}) - \text{sgn}(1 + A)}{\exp(\beta|m|) + 1} + \frac{2F - 1}{\exp(\beta|m|) + 1} + \\
+ \frac{\beta|m|}{2\pi} \int dw \text{sech}^2 \left( \frac{1}{2} \beta|m|w \right) \arctan \left[ \frac{A(w^2 - 1)^F - A^{-1}(w^2 - 1)^{-1-F} + 2\cos(F\pi)}{2w \sin(F\pi)} \right] \right\};
\]

(5.2)

recall that \( F \) is related to the fractional part of \( e\Phi \) by Eq.(2.14), \( A \) is related to \( \Theta \) by Eq.(2.18), and bound state energy \( E_{BS} \) is determined implicitly by Eq.(2.17).
Our result should be compared with the result of Refs. [23, 24, 25]

\[ Q(T) = -\frac{e^2}{2}s\Phi \tanh \left( \frac{1}{2} \beta m \right), \]  

(5.3)

where \( \Phi \) is the flux of a magnetic field with an extensive support, and it is implied that the region of the support is not excluded. Thus, result (5.3) describes the direct effect of the field strength, whereas our result describes the indirect, through the vector potential, effect of the field strength from the excluded region. In contrast to Eq. (5.3), our expressions for \( Q(T) \) and \( \Delta^2_{Q(T)} \) are periodic in the value of the flux, vanishing at integer values of \( e\Phi \), and this can be regarded as a manifestation of the Bohm-Aharonov effect [27] in quantum field theory at nonzero temperature.

The nonvanishing of the charge quadratic fluctuation signifies that the charge of the system is not a sharp quantum observable and has to be understood as a thermal expectation value only. In the high-temperature limit the average charge tends to zero (4.10) and the fluctuation tends to finite value (4.11). In the zero-temperature limit quantities \( \Delta^2_{Q(T)} \) and \( \tilde{Q}(T) \) tend exponentially to zero and the charge becomes a sharp quantum observable with finite value \( Q(0) \) (4.7)-(4.8). However, the last statement is true for almost all values of \( \Theta \) with the exception of one corresponding to the zero bound state energy, \( E_{BS} = 0 \) (\( A = -1 \)), since in this case the zero-temperature fluctuation is nonzero, see Eq. (4.9).

At half-integer values of \( e\Phi \) the average charge takes form of Eq. (4.13) which coincides (after substituting \( s \) for \( 2eg \), where \( g \) is the magnetic monopole charge, \( 2eg = n \) is the Dirac quantization condition) with the expression for the thermally induced charge in the monopole background in threedimensional space [7, 8, 9]. It should be emphasized that at non-half-integer values of \( e\Phi \) the behaviour of the charge as a function of \( \Theta \) differs drastically from the one at half-integer \( e\Phi \).

To see this explicitly, we plot the charge and its fluctuation as functions of \( \Theta \) for several values of the vortex flux and temperature on Figs. 1-5: \( F = 0.1, 0.3, 0.5, 0.7, 0.9 \). Here values \( (k_B T/|m|) = 5^{-1}, 1, 5 \) correspond to two dashed (with longer and shorter dashes) and one dotted lines, and values \( T = 0 \) and \( T = \infty \) correspond to solid lines; the latter cannot lead to confusion, since, as it has been already noted, the charge at \( T = \infty \) equals to zero everywhere, while the fluctuation at \( T = 0 \) equals to zero almost everywhere with the exception of one point (\( A = -1 \)). Two dashed lines coincide practically in the utmost left parts of Figs.1a, 2a and in the utmost right parts of Figs.4a, 5a. The qualitative difference between the \( F = 1/2 \) and \( F \neq 1/2 \) cases is most evident at zero temperature and is persisting with the increase of temperature, notwithstanding the dying of the charge on the whole at high temperature.

In the \( F \neq \frac{1}{2} \) case the charge at zero temperature is given by a step function with two jumps. As temperature increases, the jump corresponding to the zero bound state energy (\( A = -1 \)) is smoothed out, while another jump is persisting. The charge at \( A = -1 \) is not a sharp quantum observable even at zero temperature, which is explicated by the nonvanishing of the fluctuation in this case. As temperature departs from zero, the fluctuation develops a maximum at \( A = -1 \) and a minimum close to the position of the persisting jump of the charge, but out of the region where bound state exists. With the increase of temperature the maximum is widening and disappearing, while the minimum is narrowing with its position
approaching the position of the charge jump and its width tending to zero in the high-temperature limit.

In the $F = \frac{1}{2}$ case the charge at zero temperature is linear in $\Theta$ with one jump at $A = -1$ ($\Theta = s\pi \ (\text{mod} \ 2\pi)$) where the charge is not a sharp quantum observable. As temperature increases, this jump is smoothed out. Appropriately, the fluctuation is symmetric with respect to the position of this jump, and a maximum of the fluctuation is smoothed out with the increase of temperature.

In conclusion we note that the system considered can acquire, in addition to the charge, also other quantum numbers. In the case of zero temperature this issue is comprehensively elucidated in Refs. [19, 32, 33], and an appropriate generalization to the case of nonzero temperature will be studied elsewhere.

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**Appendix A**

The diagonal elements of $G^\omega(r, \varphi; r', \varphi')$ (3.2) satisfy second-order equations:

$$[-r^{-1}\partial_r r \partial_r + r^{-2}(n - e\Phi)^2 - \omega^2 + m^2] a_n(r; r') = (\omega + m) \frac{\delta(r - r')}{\sqrt{rr'}}, \quad (A.1)$$

$$[-r^{-1}\partial_r r \partial_r + r^{-2}(n - e\Phi + s)^2 - \omega^2 + m^2] c_n(r; r') = (\omega - m) \frac{\delta(r - r')}{\sqrt{rr'}}. \quad (A.2)$$

The general solution to, say, Eq.(A.1) has the form

$$a_n(r; r') = \frac{i\pi}{4}(\omega + m) \left\{ \theta(r - r') \left[ H_{s(n-e\Phi)}^{(1)}(kr)H_{s(n-e\Phi)}^{(2)}(kr') - H_{s(n-e\Phi)}^{(2)}(kr)H_{s(n-e\Phi)}^{(1)}(kr') \right] + H_{s(n-e\Phi)}^{(1)}(kr)\rho_n^{(a)}(r') + H_{s(n-e\Phi)}^{(2)}(kr)\tilde{\rho}_n^{(a)}(r') \right\}, \quad (A.3)$$

where

$$H^{(1)}_\lambda(u) = \frac{i}{\sin(\lambda\pi)} \left[ e^{-i\lambda\pi} J_\lambda(u) - J_{-\lambda}(u) \right] \quad \text{and} \quad H^{(2)}_\lambda(u) = \frac{i}{\sin(\lambda\pi)} \left[ J_{-\lambda}(u) - e^{i\lambda\pi} J_\lambda(u) \right]$$

are the first- and second-kind Hankel functions of order $\lambda$. Without a loss of generality one can choose a physical sheet for square root $k = \sqrt{\omega^2 - m^2}$ as $0 < \text{Arg} \ k < \pi \ (Im \ k > 0)$. Then we impose the condition that solution (A.3) behaves asymptotically (at $r \to \infty$) as an
outgoing wave \( \left( \frac{\exp(i kr)}{2\pi \sqrt{r}} \right) \), and this yields: \( \tilde{\rho}^{(a)}(r') = H^{(1)}_{s(n-\epsilon \Phi)}(kr') \). Thus we get

\[
a_n(r; r') = \frac{i\pi}{4} (\omega + m) \left\{ \left[ \theta(r - r') H^{(1)}_{s(n-\epsilon \Phi)}(kr) H^{(2)}_{s(n-\epsilon \Phi)}(kr') + + \theta(r' - r) H^{(2)}_{s(n-\epsilon \Phi)}(kr) H^{(1)}_{s(n-\epsilon \Phi)}(kr') \right] + H^{(1)}_{s(n-\epsilon \Phi)}(kr) \rho^{(a)}_n(r') \right\}. \tag{A.4}
\]

In a similar way we get for the solution to Eq. (A.2)

\[
c_n(r; r') = \frac{i\pi}{4} (\omega - m) \left\{ \left[ \theta(r - r') H^{(1)}_{s(n-\epsilon \Phi)}(kr) J^{(1)}_{n-\epsilon \Phi}(kr') + + \theta(r' - r) J^{(1)}_{n-\epsilon \Phi}(kr) H^{(1)}_{s(n-\epsilon \Phi)}(kr') \right] + H^{(1)}_{s(n-\epsilon \Phi)}(kr) \rho^{(c)}_n(r') \right\} \tag{A.5}
\]

Quantities \( \rho^{(a)}_n(r') \) and \( \rho^{(c)}_n(r') \) are determined by the condition at \( r \to 0 \). As it has been discussed in Section II, the condition of regularity at \( r \to 0 \) is imposed in the case of \( n \neq n_0 \), and this yields: \( \rho^{(a)}_n(r') = H^{(1)}_{s(n-\epsilon \Phi)}(kr') \) and \( \rho^{(c)}_n(r') = H^{(1)}_{s(n-\epsilon \Phi)}(kr') \) \( n \neq n_0 \). Thus we get

\[
a_n(r; r') = \frac{i\pi}{2} (\omega + m) \left[ \theta(r - r') H^{(1)}_{n-\epsilon \Phi}(kr) J^{(1)}_{n-\epsilon \Phi}(kr') + + \theta(r' - r) J^{(1)}_{n-\epsilon \Phi}(kr) H^{(1)}_{n-\epsilon \Phi}(kr') \right], \ n \neq n_0, \tag{A.6}
\]

and

\[
c_n(r; r') = \frac{i\pi}{2} (\omega - m) \left[ \theta(r - r') H^{(1)}_{n-\epsilon \Phi+\sigma}(kr) J^{(1)}_{n-\epsilon \Phi+\sigma}(kr') + + \theta(r' - r) J^{(1)}_{n-\epsilon \Phi+\sigma}(kr) H^{(1)}_{n-\epsilon \Phi+\sigma}(kr') \right], \ n \neq n_0, \tag{A.7}
\]

which gives the type 1 and the type 2 solutions (3.6)-(3.9).

In the case of \( n = n_0 \) the solutions to Eqs. (3.11) and (3.2) are not regular at \( r \to 0 \), but their irregular behaviour has to be matched with the one of the \( b_{n_0} \) and \( d_{n_0} \) components correspondingly, owing to conditions (3.13)-(3.14). Using Eqs. (3.4) and (3.5), we get

\[
b_{n_0}(r; r') = \frac{i\pi}{4} k \left\{ \left[ \theta(r - r') H^{(1)}_{1-F}(kr) H^{(2)}_{1-F}(kr') + + \theta(r' - r) H^{(2)}_{1-F}(kr) H^{(1)}_{1-F}(kr') \right] + H^{(1)}_{1-F}(kr) \rho^{(a)}_{n_0}(r') \right\}. \tag{A.8}
\]

\[
d_{n_0}(r; r') = \frac{i\pi}{4} k \left\{ \left[ \theta(r - r') H^{(1)}_{1-F}(kr) H^{(2)}_{1-F}(kr') + + \theta(r' - r) H^{(2)}_{1-F}(kr) H^{(1)}_{1-F}(kr') \right] + H^{(1)}_{1-F}(kr) \rho^{(c)}_{n_0}(r') \right\}. \tag{A.9}
\]

Substituting the pair of Eq. (A.4) at \( n = n_0 \) and Eq. (A.8) into Eq. (3.13) and the pair of Eq. (A.3) at \( n = n_0 \) and Eq. (A.9) into Eqs. (3.11), we determine \( \rho^{(a)}_{n_0}(r') \) and \( \rho^{(c)}_{n_0}(r') \), and obtain the type 3 solution (3.10)-(3.11).
In the absence of the vortex defect radial components for all \( n \) are regular at \( r \to 0 \):

\[
a_n(r; r') \big|_{\Phi = 0} = \frac{i\pi}{2} (\omega + m) \left[ \theta(r - r') H_n^{(1)}(kr) J_n(kr') + \right.
\]
\[
+ \theta(r' - r) J_n(kr) H_n^{(1)}(kr') \bigg], \quad (A.10)
\]

\[
c_n(r; r') \big|_{\Phi = 0} = \frac{i\pi}{2} (\omega - m) \left[ \theta(r - r') H_n^{(1)}(kr) J_n(kr') + \right.
\]
\[
+ \theta(r' - r) J_n(kr) H_n^{(1)}(kr') \bigg]. \quad (A.11)
\]

**Appendix B**

In the high-temperature limit Eq. (4.4) takes form

\[
\lim_{T \to \infty} \Delta^2 \frac{Q(T)}{\pi} = \frac{e^2}{4} \left\{ \frac{1}{2} \left[ 1 - \text{sgn}(A) \right] - F(1 - F) + \right.
\]
\[
\left. + 2 \sin(F\pi) \int_0^\infty \frac{du}{u} \frac{FAu^F + (1 - F)A^{-1} u^{1-F} - (2F - 1)u \cos(F\pi)}{[Au^F - A^{-1}u^{1-F} + 2\cos(F\pi)]^2 + 4(u + 1)\sin^2(F\pi)} \right\}. \quad (B.1)
\]

Using relation

\[
\frac{d}{du} \arctan \frac{(Au^F + A^{-1}u^{1-F}) \sin(F\pi)}{(Au^F - A^{-1}u^{1-F}) \cos(F\pi) + 2} = \frac{2\sin(F\pi)}{u} \frac{FAu^F + (1 - F)A^{-1} u^{1-F} - (2F - 1)u \cos(F\pi)}{[Au^F - A^{-1}u^{1-F} + 2\cos(F\pi)]^2 + 4(u + 1)\sin^2(F\pi)}, \quad (B.2)
\]

we get in the case of \(-\infty < A < 0\):

\[
\frac{2\sin(F\pi)}{\pi} \int_0^\infty \frac{du}{u} \frac{FAu^F + (1 - F)A^{-1} u^{1-F} - (2F - 1)u \cos(F\pi)}{[Au^F - A^{-1}u^{1-F} + 2\cos(F\pi)]^2 + 4(u + 1)\sin^2(F\pi)} = \quad
\]
\[
= \begin{cases} 
\frac{1}{\pi} \arctan \left( \frac{(Au^F + A^{-1}u^{1-F}) \sin(F\pi)}{(Au^F - A^{-1}u^{1-F}) \cos(F\pi) + 2} \right) \big|_{u=0}^{u=\infty} = -F, & 0 < F < \frac{1}{2} \\
\frac{1}{\pi} \arctan \left( \frac{(Au^F + A^{-1}u^{1-F}) \sin[(1 - F)\pi]}{(-Au^F + A^{-1}u^{1-F}) \cos((1 - F)\pi) + 2} \right) \big|_{u=0}^{u=\infty} = -1 + F, & \frac{1}{2} < F < 1 
\end{cases}
\]

(B.3)

In the case of \(0 < A < \infty\) one should note that the integration extends over both the principal and neighboring sheets of the Arctan function (here Arctan \( z = \arctan z + n\pi \), and
\[ \tan(\arctan z) = z; \]
\[
\frac{2 \sin(F \pi)}{\pi} \int_0^\infty \frac{du}{u} \frac{F A u^F + (1 - F) A^{-1} u^{-1 - F} - (2F - 1)u \cos(F \pi)}{[A u^F - A^{-1} u^{-1 - F} + 2 \cos(F \pi)]^2 + 4(u + 1) \sin^2(F \pi)} = \\
= \begin{cases} 
1 + \frac{1}{\pi} \arctan \left( \frac{A u^F + A^{-1} u^{-1 - F}) \sin(F \pi)}{A u^F - A^{-1} u^{-1 - F}) \cos(F \pi) + 2} \right) & \text{if } F = 0, \ 0 < F < \frac{1}{2} \\
1 + \frac{1}{\pi} \arctan \left( \frac{(A u^F + A^{-1} u^{-1 - F}) \sin((1 - F) \pi)}{(A u^F - A^{-1} u^{-1 - F}) \cos((1 - F) \pi) + 2} \right) & \text{if } \frac{1}{2} < F < 1
\end{cases}
\]

(B.4)

In the cases of \( A = 0 \) and \( A^{-1} = 0 \) we use Eq.(4.6), and in the case of \( F = \frac{1}{2} \) we use Eq.(4.14). Thus, we get Eq.(4.11) as the high-temperature limit of the fluctuation.

**Appendix C**

If a function of complex variable \( \omega \) has a pole of \( l \)-th order at \( \omega = \omega_n \), then the integral over a contour encircling this pole is given by expression

\[
\oint d\omega f(\omega) = 2\pi i \lim_{\omega \to \omega_n} \frac{1}{(l - 1)!} \frac{d^{l-1}}{d\omega^{l-1}} [f(\omega)(\omega - \omega_n)^l].
\]

By deforming contour \( C \) in Eqs.(4.1) and (4.2) to encircle poles of first and second orders, we get

\[
Q(T) = -\frac{e}{\beta} \sum_{n \in \mathbb{Z}} Tr (H - \omega_n)^{-1},
\]

and

\[
\Delta_Q(T) = -\frac{e^2}{\beta^2} \sum_{n \in \mathbb{Z}} Tr (H - \omega_n)^{-2},
\]

where \( \omega_n = (2n + 1)\frac{i\pi}{\beta} \). Using Eq.(3.18), we get

\[
Tr (H - \omega)^{-2} = \frac{1}{(\omega^2 - m^2)^2} \left\{ F(F \omega^2 + 2\omega m + F m^2) - 2F \frac{(2F - 1)\omega^2 + 2\omega m + m^2}{e^{-iF \pi} \tan \nu + 1} + \frac{[(2F - 1)\omega + m]^2}{[e^{-iF \pi} \tan \nu + 1]^2} \right\}.
\]

(C.3)

Consequently, we obtain

\[
Tr (H - \omega_n)^{-1} = \frac{1}{m[1 + (2n + 1)\xi^2]} \left\{ F + F^2 i(2n + 1)\xi \text{sgn}(m) - \frac{1 + (2F - 1)i(2n + 1)\xi \text{sgn}(m)}{1 + A[1 + (2n + 1)^2\xi^2]F[1 + i(2n + 1)\xi \text{sgn}(m)]^{-1}} \right\}.
\]

(C.4)
\[
\begin{align*}
\text{Tr} (H - \omega_n)^{-2} = & \frac{1}{m^2[1 + (2n + 1)^2 \xi^2]^2} \left\{ F^2[1 - (2n + 1)^2 \xi^2] + 2Fi(2n + 1)\xi \text{sgn}(m) - \\
& - 2F - \frac{1 - (2F - 1)(2n + 1)^2 \xi^2 + 2i(2n + 1)\xi \text{sgn}(m)}{1 + A[1 + (2n + 1)^2 \xi^2]^2[1 + i(2n + 1)\xi \text{sgn}(m)]^{-1}} + \\
& + \frac{1 + A[1 + (2n + 1)^2 \xi^2]^2[1 + i(2n + 1)\xi \text{sgn}(m)]^{-1}}{(1 + A[1 + (2n + 1)^2 \xi^2]^2[1 + i(2n + 1)\xi \text{sgn}(m)]^{-1})^2} \right\}, \quad (C.5)
\end{align*}
\]

where \( \xi = \pi/(\beta|m|) \). Summing over \( n \), we get Eqs. (4.15) and (4.16).

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Figure 1: \( F = 0.1 \)
Figure 2: $F = 0.3$
Figure 3: $F = 0.5$
Figure 4: $F = 0.7$
$e^{-1} Q(T)$

Figure 5: $F = 0.9$