L-Visibility Drawings of IC-planar Graphs

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Abstract

An IC-plane graph is a topological graph where every edge is crossed at most once and no two crossed edges share a vertex. We show that every IC-plane graph has a visibility drawing where every vertex is of the form \{L, \mid, \cap, \top\}, and every edge is either a horizontal or vertical segment. As a byproduct of our drawing technique, we prove that an IC-plane graph has a RAC drawing in quadratic area with at most two bends per edge.

1 Introduction

A visibility drawing \(\Gamma\) of a planar graph \(G\) maps the vertices of \(G\) into non-overlapping horizontal segments (bars), and the edges of \(G\) into vertical segments (visibilities), each connecting the two bars corresponding to its two end-vertices. Visibilities intersect bars only at their extreme points. \(\Gamma\) is a strong visibility drawing if there exists a visibility between two bars if and only if there exists an edge in \(G\) between the corresponding vertices. Every biconnected planar graph admits a strong visibility drawing (see, e.g., [13]). Conversely, if a visibility may not correspond to an edge of the graph, then \(\Gamma\) is a weak visibility drawing. Since every planar graph can be augmented to a biconnected planar graph, every planar graph admits a weak visibility drawing.

The problem of extending visibility drawings to non-planar graphs has been first addressed by Dean et al. [3]. They introduce bar \(k\)-visibility drawings, which are visibility drawings where each bar can see through at most \(k\) distinct bars. In other words, each visibility segment can intersect at most \(k\) bars, while each bar can be intersected by arbitrary many visibility segments. The graphs that admit a bar 1-visibility drawing are called 1-visible. Brandenburg et al. and independently Evans et al. prove that 1-planar graphs, i.e., those graphs that can be drawn with at most one crossings per edge, are 1-visible [1, 6]. They focus on a weak model, where there is a visibility through at most \(k\) bars if there is an edge, while the converse may not be true. In fact, having a strong model would be too restrictive in the case of bar \(k\)-visibility drawings. For example, it is easy to see that a cycle of length at least four does not admit a strong bar 1-visibility drawing [1]. In terms of readability, a clear benefit of bar \(k\)-visibility drawings is that the crossings form right angles. Right-angle crossing (RAC) drawings and their advantages in terms of readability have been extensively studied in the graph drawing literature (see, e.g., [5, 9]). However, in a bar \(k\)-visibility drawing crossings involve bars and visibilities, i.e., vertices and edges. These crossings are arguably less intuitive than crossings between edges.

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Evans et al. introduce a new model of visibility drawings, called *L-visibility drawings* [1]. Their aim is to simultaneously represent two plane *st*-graphs $G_r$ and $G_b$ (whose union might be non-planar). They assume a *strong* model, where each vertex is represented by a horizontal bar and a vertical bar that share an extreme point, i.e., it is an *L-shape* in the set \{L, J, T, R\}. Two L-shapes are connected by a vertical (horizontal) visibility segment if and only if there exists an edge in $G_r$ ($G_b$) between the corresponding vertices, no two L-shapes cross one another, and visibilities intersect bars only at their extreme points. A clear advantage of this kind of drawing is that the only possible crossings are between vertical and horizontal visibilities, i.e., between edges of the graph. Furthermore, similarly as bar $k$-visibilities, these crossings form right angles.

In this paper we initiate the study of *weak* L-visibility drawings of non-planar graphs. We focus on the class of graphs that admit a drawing where each edge is crossed at most once, and no two crossed edges share an end-vertex. These graphs are called *IC-planar graphs* (see Fig. 1(a) for an example). Their chromatic number is at most five [10], and they have at most $13n/4 - 6$ edges, which is a tight bound [15]. Recognizing IC-planar graphs is NP-hard [2].

Our main contribution is summarized by the following theorem, proved in Section 3. See Fig. 1(b) for an example of a L-visibility drawing computed by using Theorem 1.

**Theorem 1.** Every $n$-vertex IC-plane graph $G$ admits a L-visibility drawing in $O(n^2)$ area, which can be computed in $O(n)$ time.

We remark that Theorem 1 contributes to the rapidly growing literature devoted to the problem of drawing graphs that are “nearly planar” in some sense, i.e., graphs where only some types of edge crossings are allowed (for example, an edge can be crossed at most a constant number of times); see e.g., [11] for references. In particular, Brandenburg et al. have recently described a cubic-time algorithm that computes IC-planar drawings with right-angle crossings and straight-line edges. However these drawings may require exponential area, which is proved to be worst-case optimal [2]. Brandenburg et al. leave as an open problem to study techniques that compute IC-planar drawings in polynomial area and with good crossing resolution [2]. As a byproduct of Theorem 1 we obtain the following corollary (see Fig. 1(c) for an example).

**Corollary 1.** Every $n$-vertex IC-plane graph $G$ admits a RAC drawing with at most two bends per edge in $O(n^2)$ area, which can be computed in $O(n)$ time.
2 Preliminaries

We assume familiarity with basic graph drawing concepts and recall in this section only those definitions and facts that will be used in the rest of the paper, see also [4]. In particular, we shall recall some definitions also given in [2].

**Planarity and connectivity.** A graph $G = (V, E)$ is simple, if it contains neither loops nor multiple edges. We consider simple graphs, if not otherwise specified. A drawing $\Gamma$ of $G$ maps each vertex of $V$ to a point of the plane and each edge of $E$ to a Jordan arc between its two end-points. We only consider simple drawings, i.e., drawings such that the arcs representing two edges have at most one point in common, which is either a common end-vertex or a common interior point where the two arcs properly cross. A drawing is planar if no two arcs representing two edges cross. A planar drawing divides the plane into topologically connected regions, called faces. The unbounded region is called the outer face. A planar embedding of a graph is an equivalence class of planar drawings that define the same set of faces. A graph with a given planar embedding is a plane graph. For a non-planar drawing, we can still talk about embedding considering that the boundary of a face may consist of edge segments between vertices and/or crossing points of edges. The unbounded region is still called the outer face.

A graph is biconnected if it remains connected after removing any one vertex. A directed graph (a digraph for short) is biconnected if its underlying undirected graph is biconnected. A topological numbering of a digraph is an assignment, $X$, of numbers to its vertices such that $X(u) < X(v)$ for every edge $(u, v)$. A graph admits a topological numbering if and only if it is acyclic. An acyclic digraph with a single source $s$ and a single sink $t$ is called an $st$-graph. A plane $st$-graph is an $st$-graph that is planar and embedded such that $s$ and $t$ are on the boundary of the outer face. In any $st$-graph, the presence of the edge $(s, t)$ guarantees that the graph is biconnected. In the following we consider $st$-graphs that contain the edge $(s, t)$, as otherwise it can be added without violating planarity. Let $G$ be a plane $st$-graph, then for each vertex $v$ of $G$ the incoming edges appear consecutively around $v$, and so do the outgoing edges. Vertex $s$ only has outgoing edges, while vertex $t$ only has incoming edges. This particular transversal structure is known as a bipolar orientation [12, 13]. Each face $f$ of $G$ is bounded by two directed paths with a common origin and destination, called the left path and right path of $f$.

**IC-planar graphs.** A drawing is IC-planar if each edge is crossed at most once, and any two crossed edges do not share an end-vertex. See Fig. 1(a) for an illustration. An IC-planar embedding is an embedding derived from an IC-planar drawing. A graph with a given IC-planar embedding is an IC-plane graph. Thomassen [14] characterized the possible crossing

\[ \text{(a)} \]
\[ \text{(b)} \]

Figure 2: (a) An X-configuration and (b) a B-configuration.
configurations that occur in a 1-planar drawing, i.e., a drawing where each edge is crossed at most once. This characterization applied to IC-planar drawings gives rise to the following property, where an X-crossing is of the type described in Fig. 2(a) and a B-crossing is of the type described in Fig. 2(b) (the solid edges only).

Property 1. Every crossing of an IC-planar drawing is either an X- or a B-crossing.

A kite $K$ is a graph isomorphic to $K_4$ together with an embedding such that all the vertices are on the boundary of the outer face. This implies that two edges of $K$ cross each other, while the other four edges are not crossed and belong to the boundary of the outer face; see Fig. 2(a).

Consider a pair of crossing edges of an IC-plane graph $G$, such that their four end-vertices induce a kite $K$. The kite $K$ is empty, if in $G$ there is no other vertex inside the internal faces of $K$.

Visibility model. In a L-visibility drawing $\Gamma$ of a graph $G$, every vertex is represented by a horizontal and a vertical segment sharing an end-point, i.e., by an L-shape in the set \{L, \top, \bot, \Gamma\}. Each edge of $G$ is drawn in $\Gamma$ as either a horizontal or a vertical visibility segment joining the two L-shapes corresponding to its two end-vertices. Clearly, horizontal visibilities only cross vertical visibilities at right angles. Moreover, no two L-shapes intersect. If $G$ is an IC-plane graph, then each visibility is crossed at most once and no two crossed visibilities are incident to the same L-shape. In Fig. 1(b) a L-visibility representation $\Gamma$ of an IC-plane graph $G$ is shown. Finally, we adopt a weak model, where a visibility may not imply the existence of the corresponding edge in the graph. For example, in Fig. 1(b) the L-shapes of vertices 8 and 10 can be joined by a horizontal visibility, although the edge $(8, 10)$ does not exist in $G$.

3 Proof of Theorem 1

The proof of Theorem 1 is constructive and is based on a drawing algorithm that takes as input an IC-plane graph $G$ and returns a L-visibility drawing $\Gamma$ of $G$. We begin by removing from $G$ all B-configurations and by ensuring that each X-configuration induces an empty kite (see Section 2). Next lemma is a consequence of the more general Lemma 1 in [2]. For completeness, we provide a more direct proof of our statement.

Lemma 1. Let $G = (V, E)$ be an $n$-vertex IC-plane graph. It is possible to augment $G$ to a biconnected IC-plane graph $G^+ = (V, E^+)$, where $E \subseteq E^+$, such that the end-vertices of each pair of crossing edges of $G^+$ induce an empty kite. This operation can be done in $O(n)$ time.

Proof. We augment $G$ by adding edges such that, for each pair of crossing edges $(a, c)$ and $(b, d)$, the subgraph induced by vertices $\{a, b, c, d\}$ is $K_4$; see the dashed edges in Figs. 2(a) and 2(b). We then convert each B-configuration into an X-configuration. Since $G$ is IC-plane, it has no two B-configurations sharing an edge $(a, b)$. Thus, we remove a B-configuration with vertices $\{a, b, c, d\}$ by rerouting the edge $(a, b)$ to follow the edge $(a, c)$ from vertex $a$ until the crossing point, then edge $(b, d)$ until vertex $b$, as shown by the dotted edge in Fig. 2(b). This is always possible, because edges $(a, c)$ and $(b, d)$ only cross each other, hence, following their curves, we do not introduce any new crossing. At this point, each pair of crossing edges induces a kite $K$. If $K$ is not empty, then it suffice to reroute its four external edges in such a way that they follow the curves of the two crossed edges. Finally, if the computed graph is not biconnected,
we can remove one edge for each pair of crossing edges so to make it planar, augment it to a biconnected planar graph, and then reinsert the crossing edges. The resulting IC-plane graph \( G^+ \) is computed from \( G \) in \( O(n) \) time, as every step can be clearly executed in linear time.

Let \( G^+ \) be the biconnected IC-plane graph obtained from \( G \) applying Lemma 1. We now remove from \( G^+ \) all pairs of crossing edges and orient the resulting graph \( P \) to an \( st \)-graph. The computed orientation is such that, when reinserting a pair of crossing edges in the corresponding planar face of \( P \), one of them is always incident to the origin and the destination of the face. In other words, each face of \( P \) that corresponds to an empty kite of \( G^+ \), is oriented so that its left and right paths contain exactly one vertex each. To this aim, we first need to introduce additional notation. Let \( f \) be a face of a plane graph \( G \). Let \( v_1, \ldots, v_h \) be the \( h \geq 3 \) vertices that belong to the boundary of \( f \), and let \( N(v) \) be the set of neighbors of a vertex \( v \) of \( G \). The \textit{contraction} of \( f \) is the operation defined as follows. Add to \( G \) a vertex \( v_f \) and connect \( v_f \) to the vertices in \( N(v_1) \cup \ldots \cup N(v_h) \). Then remove \( v_1, \ldots, v_h \) from \( G \). The resulting (multi)graph is still planar. Moreover, the contraction operation can be performed so to preserve the planar embedding of \( G \). Namely, the circular order of the edges incident to \( v_f \) is the same circular order encountered walking along the boundary of \( f \). See also Fig. 3(a) for an illustration. Notice that, since \( G^+ \) is an IC-plane graph, no two faces of \( P \) share a vertex, and thus each vertex \( v_K \) of \( P \) corresponds to exactly one face of \( P \). Hence, we can contract the faces following an arbitrary order. The resulting graph \( P_C \) is a plane (multi)graph.

\begin{lemma}
 Let \( C \) be the set of crossing edges in \( G^+ \), and consider the plane graph \( P = G^+ \setminus C \). \( P \) can be oriented to an \( st \)-graph such that each pair of crossing edges of \( C \) has been removed from a face of \( P \) whose left and right paths contain exactly one vertex each. This operation can be done in \( O(n) \) time.
\end{lemma}

\begin{proof}
 By Lemma 1 each pair of crossing edges of \( G^+ \) induces an empty kite \( K \), and thus corresponds to a single face \( f_K \) in \( P \) having exactly four vertices on its boundary. As a first step, we contract each face \( f_K \) of \( P \) (corresponding to a kite \( K \) in \( G^+ \)). See also Fig. 3(a) for an illustration. Notice that, since \( G^+ \) is an IC-plane graph, no two faces of \( P \) share a vertex, and thus each vertex \( v_K \) of \( P_C \) corresponds to exactly one face of \( P \). Hence, we can contract the faces following an arbitrary order. The resulting graph \( P_C \) is a plane (multi)graph.

As a second step, we orient \( P_C \) to an \( st \)-graph. In the third step, we expand one by one all the vertices \( v_K \) corresponding to a contracted face \( f_K \). After expanding a vertex \( v_K \), we orient the four reinserted edges of the face \( f_K \) maintaining the following invariants:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Illustration for the proof of Lemma 2.}
\end{figure}
I1. The resulting graph has a single source and a single sink;
I2. The left and right paths of $f_K$ contain exactly one vertex each.

Invariants I1 and I2 imply that the resulting graph after expanding $v_K$ is still an $st$-graph, as it has a single source and a single sink by I1 and it acyclic by I2.

Let $\{a, b, c, d\}$ be the four vertices belonging to the boundary of the face $f_K$, encountered in this order clockwise around the boundary of the face. To ensure I2 we then need to orient the edges of $f_K$ such that the origin and the destination of $f_K$ are two non-adjacent vertices, i.e., either $a$ and $c$ or $b$ and $d$. Moreover, in order to ensure I1, recall that the incoming edges of $v_K$ appear consecutive around it and so do the outgoing edges, unless $v_K$ is the source or the sink of the graph. Thus, if $v_K$ is neither the source nor the sink of the graph, then that at most two of the reinserted vertices will have both incoming and outgoing edges, whereas at most three will have only incoming or only outgoing edges. We distinguish between the following three cases.

Case 1. No vertex of $f_K$ has both incoming and outgoing edges. See Fig. 3(b) for an illustration. Then we consider the following subcases.

Case 1a. There are three vertices having only outgoing edges, say $a$, $b$ and $c$, which implies that all the edges of $d$ are incoming. Then we orient the edges of $f_K$ so that $b$ is the destination and $d$ is the origin of the face, which ensures I2. All vertices has now both incoming and outgoing edges, and thus I1 is also maintained.

Case 1b. There are three vertices having only incoming edges, say $a$, $b$ and $c$, which implies that all the edges of $d$ are outgoing. Then the orientation that ensures I1 and I2 is the one where $d$ is the destination and $b$ is the origin of the face.

Case 1c. Two (consecutive) vertices only have incoming edges, say $c$ and $d$, and two (consecutive) vertices only have outgoing edges, $a$ and $b$. Then to maintain I1 and I2 we let $a$ to be the destination and $c$ the origin of the face. This particular case is shown in Fig. 3(b).

Case 2. Only one vertex of $f_K$, say $a$, has both incoming and outgoing edges. Moreover, assume that the incoming edges of $a$ are between the edge $(a, d)$ and the outgoing edges of $a$, as in Fig. 3(c) since the other case is symmetric. We have two subcases.

Case 2a. The other three vertices only have incoming (resp., outgoing) edges. Then we orient the edges of $f_K$ so that $c$ (resp., $a$) is the origin and $a$ (resp., $c$) is the destination. This choice ensures both I1 and I2.

Case 2b. Two vertices only have incoming edges and the other vertex only have outgoing edges. Due to the bipolar orientation of $v_K$, the two vertices having only incoming edges are $c$ and $d$. Then we choose $d$ as origin of the face and $b$ as destination, which ensures both I1 and I2. This particular case is shown in Fig. 3(c). The same orientation works also if two vertices only have outgoing edges ($b$ and $c$) and the other vertex only have incoming edges ($d$).

Case 3. Two vertices of $f_K$ have both incoming and outgoing edges. We have two subcases.

Case 3a. Suppose first that these two vertices, say $a$ and $b$, are adjacent in $P$. Moreover, assume that the incoming edges of $a$ are between the edge $(a, d)$ and the outgoing edges of $a$, as in Fig. 3(d) since the other case is symmetric. This implies that the incoming edges of $b$ are between the outgoing edges of $b$ and the edge $(b, c)$. Moreover, $c$ and $d$ only have incoming edges. Then, if we let $a$ to be the destination of the face and $c$ the origin, I1 and I2 are maintained.
Case 3b. Suppose now that the two vertices, say $a$ and $c$, are not adjacent in $P$. Moreover, assume that the incoming edges of $a$ are between the edge $(a, d)$ and the outgoing edges of $a$, as in Fig. 3(d) since the other case is symmetric. This implies that the incoming edges of $c$ are between the outgoing edges of $c$ and the edge $(c, d)$. Moreover, $b$ and $d$ only have outgoing and incoming edges, respectively. Then, if we orient the edges of $f_K$ so that $b$ is the destination of the face and $d$ the origin, then $I_1$ and $I_2$ are ensured.

Finally, suppose that $v_K$ is the source (resp., sink) of the graph. Then all the vertices of $f_K$ have either no incident oriented edges, or all outgoing (resp., incoming) edges. Also, at least one of them has at least one outgoing (resp., incoming) edge, say $a$. We orient the edges of $f_K$ so that $a$ is the destination (resp., origin) and $c$ the origin (resp., destination) of $f_K$. This choice ensures both $I_1$ and $I_2$.

The described algorithm works in $O(n)$ time. Namely, the graph $P_C$ can be constructed and oriented to an $st$-graph in $O(n)$ time (see, e.g., [8]). Furthermore, orienting the edges of an expanded face requires first to analyze the orientation of the edges incident to each vertex of the face, and then to orient the edges of the face. Since the faces are vertex-disjoint, this costs at most $\sum_{v \in P} \deg(v) = 2m_P$, where $m_P$ is the number of edges of $P$, which is $O(n)$.

Let $f_K$ be a face of the plane $st$-graph $P$ which corresponds to an (empty) kite $K$ in $G^+$. By Lemma 2 $f_K$ is such that its left and right path have both length one. This implies that one of the two crossing edges of $K$ is incident to the origin and to the destination of $f_K$, whereas the other one is incident to the two vertices belonging one to the left and one to the right path of $f_K$. Let $P^+$ be the biconnected plane graph obtained from $P$ by reinserting, for each pair of crossing edges of $G^+$, the edge incident to the origin and to the destination of the corresponding face in $P$. Furthermore, let $\Gamma'$ be a strong visibility drawing of $P^+$ in $O(n^2)$ area. Recall that, $\Gamma'$ can be computed in $O(n)$ time by applying the algorithm by Tamassia and Tollis [4] [13].

Lemma 3. $\Gamma'$ can be transformed into a L-visibility drawing $\Gamma$ of $G$ that requires $O(n^2)$ area. This operation can be done in $O(n)$ time.

Proof. Consider a face $f_K$ of $P$ corresponding to an empty kite $K$ in $G^+$. Let $\{a, b, c, d\}$ be the four vertices of $f_K$, encountered in this order clockwise around the boundary of the face. See also Fig. 4(a). Without loss of generality, let $a$ be the destination of the face. Then Lemma 2
imply the vertical segments of \( \ell \) have non-zero length, then \( r \) is the point where they touch. Otherwise, \( r \) is the midpoint of the segment of \( \ell \) having non-zero length. Replace each vertical visibility segment with a polyline as follows. Let \( s \) be a visibility segment connecting the two L-shapes \( \ell_1 \) and \( \ell_2 \). Let \( r_1 \) and \( r_2 \) be the representative points of \( \ell_1 \) and \( \ell_2 \), respectively. Also, let \( p_1 \) and \( p_2 \) be the points that \( s \) shares with \( \ell_1 \) and \( \ell_2 \), respectively. Suppose that \( r_1 \) is below \( r_2 \) (and thus \( p_1 \) is below \( p_2 \)). Replace \( s \) with the polyline starting at \( r_1 \), bending 0.25 grid units above \( p_1 \), bending again 0.25 units below \( p_2 \), and ending at \( r_2 \). With a symmetric operation we can also replace each horizontal visibility segment. Finally, replace each L-shape with its representative point. The resulting drawing is an IC-plane drawing of \( G \) where edges are polylines with (at most) two bends that cross at right-angles. Finally, multiplying by a factor 4 the grid of the drawing we restore integer coordinates. Fig. \ref{fig:4(c)} shows a RAC drawing computed from the L-visibility drawing in Fig. \ref{fig:4(b)}.

### 4 Conclusions and Open Problems

We have proved that every IC-plane graph \( G \) has a L-visibility drawing which can be computed in linear time. As a corollary, our result implies that \( G \) has a RAC drawing in quadratic area and at most two bends per edge which can also be computed in linear time. We conclude the paper.
with two open problems:  
(i) Does every 1-planar graph admit a visibility drawing where the shape associated with each vertex is either an L-shape, a T-shape, or a + -shape? 
(ii) Does every IC-plane graph admit a RAC drawing with at most one bend per edge in polynomial area?

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