STABILIZATION RESULTS OF A LORENZ PIEZOELECTRIC BEAM WITH PARTIAL VISCOS DAMPINGS

MOHAMMAD AKIL\textsuperscript{1} , ABDELAZIZ SOUFYANE\textsuperscript{2} AND YOUSSEF BELHAMADIA\textsuperscript{3}

Abstract. In this paper, we investigate the stabilization of a one-dimensional Lorenz piezoelectric (Stretching system) with partial viscous dampings. First, by using Lorenz gauge conditions, we reformulate our system to achieve the existence and uniqueness of the solution. Next, by using General criteria of Arendt-Batty, we prove the strong stability in different cases. Finally, we prove that it is sufficient to control the stretching of the center-line of the beam in $x$–direction to achieve the exponential stability. Numerical results are also presented to validate our theoretical result.


c{tableofcontents}

1. Introduction
2. Reformulation and Wellposedness
3. Strong Stability
4. The stretching of the centreline of the beam in $x$–direction and electrical field component in $(x$ and $z$)–direction are damped "$(a, b, c) \neq (0, 0, 0)$"
5. The electrical field component in $(x$ and $z$)–direction are damped "$a = 0$ and $(b, c) \neq (0, 0)$"
6. The stretching of the centreline of the beam in $x$–direction and electrical field component in $z$–direction are damped "$b = 0$ and $(a, c) \neq (0, 0)$"
7. The stretching of the centreline of the beam in $x$–direction and electrical field component in $x$–direction are damped "$c = 0$ and $(a, b) \neq (0, 0)$"
8. The stretching of the centreline of the beam in $x$–direction only is damped and "$a \neq 0$ and $(b, c) = (0, 0)$"
9. Numerical Results
10. Conclusion

References

Keywords. Lorenz Gauge - Piezoelectric beams - Stabilization - Electromagnetic potentials-Exponential Stability.

1. Introduction

Piezoelectric materials have become more promising in aeronautic, civil and space structures. It is known, since the 19th century that materials such as quartz, Rochelle salt and barium titanate under pressure produces electric charge/voltage, this phenomenon is called the direct piezoelectric effect and was discovered by brothers...

\textsuperscript{1}\textit{Univ. Polytechnique Hauts-de-France, INSA Hauts-de-France, CERAMATHS-Laboratoire de Matériaux Céra-miques et de Mathématiques, F-59313 Valenciennes, France,} \textsuperscript{2} \textit{Department of Mathematics, College of Science, University of Sharjah, P.O.Box 27272, Sharjah, UAE.,} \textsuperscript{3} \textit{Department of Mathematics and Statistics, American University of Sharjah, Sharjah, UAE., mohammad.akil@uphf.fr, asoufyane@sharjah.ac.ae , ybelhamadia@aus.edu}
Pierre and Jacques Curie in 1880. This same materials, when subjected to an electric field, produce proportional geometric tension. Such a phenomenon is known as the converse piezoelectric effect and was discovered by Gabriel Lippmann in 1881 [21, 26, 28]. In many studies related to piezoelectric structures, the magnetic effect is neglected and only the mechanical the mechanical effects are considered. In general the mechanical effects are modelled by using Kirchhoff, Euler-Bernoulli or Midlin-Timoshenko assumptions for small displacements [5, 7, 23, 28], and electrical and magnetic effects are added to the system generally using electrostatic, quasi-static and fully dynamic approaches ([6, 9, 10, 27]). Morris and Özer in [12, 13], proposed a variational approach, a piezoelectric beam model with a magnetic effect, based on the Euler-Bernoulli and Rayleigh beam theory for small displacement. They considered an elastic beam covered by a piezoelectric material on its upper and lower surfaces, isolated at the edges and connected to an external electrical circuit to feed charge to the electrodes. It is worth mentioning that it is well known that piezoelectric beams without the magnetic effect, in which they exists a few results on piezoelectric material with different kind of dampings [1, 24, 19, 15, 17, 2, 3].

Recently, in [16], a nouvel infinite-dimensional models, by a through variational approach, are introduced to describe vibrations on a piezoelectric beam. Electro-Magnetic effects due to Maxwell’s equations factor in the models via the electric and magnetic potentials. This system is described by a wave equation [13], are exactly observable [11] and exponentially stable [25]. Also, there are modelled by using Kirchhoff, Euler-Bernoulli or Midlin-Timoshenko assumptions for small displacements [13, 26, 28]. In many studies related to piezoelectric structures, the magnetic effect geometric tension. Such a phenomenon is known as the converse piezoelectric effect and was discovered by Gabriel Lippmann in 1881 [21, 26, 28]. In many studies related to piezoelectric structures, the magnetic effect was discovered by Pierre and Jacques Curie in 1880. This same materials, when subjected to an electric field, produce proportional geometric tension. Such a phenomenon is known as the converse piezoelectric effect. The natural physical constants $\rho$, $\alpha$, $\gamma$, $\varepsilon_1$, $\varepsilon_3$, $\mu$ denotes the mass density per unit volume, elastic stifness, piezoelectric coupling coefficient, permittivity in $x$ and $z$ directions, and magnetic permeability respectively. The conditions (1.1)$_5$-(1.1)$_7$, represents respectively beam clamped on the left, Lateral force. First charge moment, Current. The applied current $i_s(t)$ at the electrodes effects only the stretching motion and the surface electrical continuity is satisfied

$$\frac{di_s(x,t)}{dx} = 0.$$  

The author proved that this model fail to be asymptotically stable if the material parameters satisfy certain conditions. To achieve at least asymptotic stability the author proposed an additional controller. In this paper, we study system (1.1) without current acting on the electrode and with different partial viscous damping acting on the stretching of the centreline of the beam in $x$–direction, electrical field component in $x$–direction, electrical field component in $z$–direction and magnetic field component in $y$–direction. This system in described by

$$\begin{aligned}
\rho v_{tt} - \alpha v_{xx} - \gamma (\phi + \eta)_{x} &= 0, & (x, t) \in (0, L) \times (0, \infty), \\
-\xi (\phi + \theta)_{x} + (\eta + \phi) - \frac{\gamma}{\varepsilon_3} v_{x} &= 0, & (x, t) \in (0, L) \times (0, \infty), \\
(\theta_t + \phi_x) - \frac{\mu}{\varepsilon_3} (\eta_x - \theta_x) &= \frac{i_s(t)}{\varepsilon_3 h}, & (x, t) \in (0, L) \times (0, \infty), \\
(\eta_t + \phi_x) - \frac{\mu}{\varepsilon_3} (\eta_x - \eta_x) - \frac{\gamma}{\varepsilon_3} v_{tx} &= 0, & (x, t) \in (0, L) \times (0, \infty), \\
v(0, t) = \alpha v_x(L, t) + \gamma (\phi(L, t) + \gamma \eta(L, t) &= 0, & t \in (0, \infty), \\
\xi \varepsilon_3 (\theta_t + \phi_x)(0, t) = \xi \varepsilon_3 (\theta_t + \phi_x)(L, t) &= 0, & t \in (0, \infty), \\
\mu (\theta - \eta_x)(0, t) = \mu (\theta - \eta_x)(L, t) &= 0 \\
\end{aligned}$$

where $v, \theta_t + \phi_x, \eta_t + \phi$ and $\theta - \eta_x$ represents respectively, the stretching of the centreline of the beam in $x$–direction, electrical field component in $x$–direction, electrical field component in $z$–direction and magnetic field component in $y$–direction and $\xi = \frac{\alpha^2 L^2}{12 \varepsilon_3}$. The natural physical constants $\rho$, $\alpha$, $\gamma$, $\varepsilon_1$, $\varepsilon_3$, $\mu$ denotes the mass density per unit volume, elastic stifness, piezoelectric coupling coefficient, permittivity in $x$ and $z$ directions, and magnetic permeability respectively. The conditions (1.1)$_5$-(1.1)$_7$, represents respectively beam clamped on the left, Lateral force. First charge moment, Current. The applied current $i_s(t)$ at the electrodes effects only the stretching motion and the surface electrical continuity is satisfied
and \( \theta \) and (2.1) respectively reduce to (Stretching) of motion (Stretching)

\[
\begin{align*}
\rho v_{tt} - \alpha v_{xx} - \gamma (\phi + \eta_t)_x + av_t &= 0, \\
-\xi (\phi_x + \theta_t)_x + (\eta_t + \phi) - \frac{\gamma}{\varepsilon_3} v_x &= 0, \\
(\theta_t + \phi_e)_t - \frac{\mu}{\varepsilon_3} (\eta_x - \theta) + b(\theta_t + \phi_x) &= 0, \\
(\eta_t + \phi_x)_t - \frac{\mu}{\varepsilon_3} (\eta_x - \theta)_x - \frac{\gamma}{\varepsilon_3} v_{tx} + c(\eta_t + \phi) &= 0,
\end{align*}
\]

\( (x, t) \in (0, L) \times (0, \infty) \),

where \( a, b, c > 0 \). In the first section we reformulate and we prove the well-posedness of our system. In the second part we prove the strong stability of system (Stretching). Next, we prove the exponential stability under partial viscous damping on the centreline of the beam in \( x \)-direction and/or electrical field component in \( x \) and \( z \)-direction. Finally, we numerically illustrate the exponential stability decay of the natural energy \( E(t) \) of (Lorenz) system.

2. Reformulation and Wellposedness

System (Stretching) does not yield a unique solution since:

- The magnetic potential vector component \( \theta, \eta \) and the electrical potential \( \phi \) are not uniquely defined (see Equation (1) in [16] and [17]).
- The Lagrangian is invariant under certain transformations [17].

To obtain a unique solution, particular gauge conditions are presented in electro-magnetic theory to completely decouple the electromagnetic equations in (Stretching). One of the most widely used gauges is Lorenz Gauges [13, 16, 17, 14, 22]). For the piezoelectric beam model, the Lorenz Gauge condition is given by

\[
(LGC) \quad -\xi \theta_x + \eta = \frac{\xi \varepsilon_3}{\mu} \phi_t,
\]

with the boundary conditions

\[
(2.1) \quad \theta(0, t) = \theta(L, t) = 0.
\]

In the case of (LGC), the term \(-\xi \theta_x + \eta \) in (Stretching)_2 is transformed into \( \frac{\xi \varepsilon_3}{\mu} \phi_t \). As well, the terms \( \phi_{tx} - \frac{\mu}{\varepsilon_3} (\eta_x - \theta) \) and \( \phi_t - \frac{\mu}{\varepsilon_3} (\eta_x - \theta)_x \) in (Stretching)_1 and (Stretching)_4 are transformed into \( -\frac{\mu}{\varepsilon_3} (\xi \theta_x - \theta) \) and \( -\frac{\mu}{\varepsilon_3} (\xi \eta_x - \eta) \), respectively. This transformation not only the \( \phi \)-equation to a wave equation but also the \( \theta \) and \( \eta \) equations. Therefore, both electric and magnetic equations are wave equations. Then, the equations of motion (Stretching)-(2.1) respectively reduce to

\[
(Lorenz) \quad \begin{align*}
\rho v_{tt} - \alpha v_{xx} - \gamma (\phi + \eta_t)_x + av_t &= 0, \\
\phi_{tt} - \frac{\mu}{\varepsilon_3} \phi_{xx} + \frac{\mu}{\xi \varepsilon_3} \phi - \frac{\gamma}{\varepsilon_3} v_x &= 0, \\
\theta_{tt} - \frac{\mu}{\varepsilon_3} \theta_{xx} + \frac{\mu}{\xi \varepsilon_3} \theta + b(\theta_t + \phi_x) &= 0, \\
\eta_{tt} - \frac{\mu}{\varepsilon_3} \eta_{xx} + \frac{\mu}{\xi \varepsilon_3} \eta - \frac{\gamma}{\varepsilon_3} v_{tx} + c(\eta_t + \phi) &= 0,
\end{align*}
\]

\( (x, t) \in (0, L) \times (0, \infty) \),

Lemma 2.1. The natural energy \( E(t) \) associated to (Lorenz) system is the sum of kinetic, potential, magnetic and electrical energies, i.e.,

\[
(2.2) \quad E(t) = E_k(t) + E_p(t) + E_B(t) + E_{elec}(t),
\]
where
\[
\begin{align*}
E_k(t) &= \frac{\rho}{2} \int_0^L |v_t|^2 dx, \\
E_p(t) &= \frac{\alpha}{2} \int_0^L |v_x|^2 dx, \\
E_B(t) &= \frac{\mu}{2} \int_0^L |\theta - \eta|^2 dx,
\end{align*}
\]
and
\[
E_{elc}(t) = \frac{1}{2} \int_0^L \left[ \xi \varepsilon_3 (\theta_t + \phi_x)^2 + \varepsilon_5 |\eta_t + \phi|^2 \right] dx
\]
(2.3)

Differentiating (2.3), we get
\[
\frac{d}{dt} E(t) = -a \int_0^L |v_t|^2 dx - b \xi \varepsilon_3 \int_0^L |\theta_t + \phi_x|^2 dx - c \varepsilon_3 \int_0^L |\eta_t + \phi|^2 dx.
\]
(2.4)

**Proof:** Multiplying (Lorenz) by $\pi_t$, integrating by parts over $(0, L)$ and taking the real part, we get
\[
\frac{d}{dt} E_k(t) + \frac{d}{dt} E_p(t) + \Re \left( \gamma \int_0^L (\phi + \eta) \bar{\omega}_x dx \right) + a \int_0^L |v_t|^2 dx = 0.
\]

Multiplying (Lorenz) by $\xi \varepsilon_3 (\theta_t + \phi_x)$, integrating over $(0, L)$, we get
\[
\frac{\xi \varepsilon_3}{2} \frac{d}{dt} \int_0^L |\theta_t|^2 dx + \frac{\mu}{2} \frac{d}{dt} \int_0^L |\theta|^2 dx + \Re \left( \xi \varepsilon_3 \int_0^L \theta_t \bar{\omega}_x dx \right) + \Re \left( \mu \int_0^L \theta_x (\theta_{tx} + \phi_{xx}) dx \right)
\]
(2.5)

Using (LGC) in the fourth integral in (2.5), we obtain
\[
\Re \left( \mu \int_0^L \theta_t \bar{\omega}_x dx \right) + \Re \left( \xi \varepsilon_3 \int_0^L \phi_x \bar{\theta}_x dx \right)
\]
(2.6)

Inserting (2.6) in (2.5), we get
\[
\frac{\xi \varepsilon_3}{2} \frac{d}{dt} \int_0^L |\theta_t + \phi_x|^2 dx + \frac{\mu}{2} \frac{d}{dt} \int_0^L |\theta|^2 dx - \Re \left( \mu \int_0^L \eta_x \bar{\theta}_x dx \right) - \Re \left( \mu \int_0^L \eta_x \bar{\phi}_x dx \right)
\]
(2.7)

Multiplying (Lorenz) by $\varepsilon_3 (\eta_t + \phi)$, integrating by parts over $(0, L)$, we get
\[
\frac{\varepsilon_3}{2} \frac{d}{dt} \int_0^L |\eta_t|^2 dx + \frac{\mu}{2} \frac{d}{dt} \int_0^L |\eta_x|^2 dx + \Re \left( \varepsilon_3 \int_0^L \eta_t \bar{\phi}_x dx \right) + \Re \left( \mu \int_0^L \eta_x \bar{\phi}_x dx \right)
\]
(2.8)

Using (LGC) in the fifth integral over (2.8) and integrating by parts over $(0, L)$, we obtain
\[
\Re \left( \frac{\mu}{\xi} \int_0^L \eta_t \bar{\phi}_x dx \right) = \Re \left( \frac{\varepsilon_3}{2} \frac{d}{dt} \int_0^L |\phi|^2 dx + \Re \left( \varepsilon_3 \int_0^L \phi_t \bar{\eta}_x dx \right) \right)
\]
(2.9)

Inserting (2.9) in (2.8), we get
\[
\frac{\varepsilon_3}{2} \frac{d}{dt} \int_0^L |\eta_t + \phi|^2 dx + \frac{\mu}{2} \frac{d}{dt} \int_0^L |\eta_x|^2 dx - \Re \left( \mu \int_0^L \eta_t \bar{\phi}_x dx \right) + \Re \left( \mu \int_0^L \eta_x \bar{\phi}_x dx \right)
\]
(2.10)

\[
- \Re \left( \mu \int_0^L \theta \bar{\phi}_x dx \right) - \Re \left( \gamma \int_0^L v_{xt} (\phi + \eta) dx \right) + \varepsilon_3 \int_0^L |\eta_t + \phi|^2 dx = 0.
\]
Adding (2.4), (2.7) and (2.10), we get
\[
\frac{d}{dt}(E_k(t) + E_p(t) + E_B(t) + E_{\text{elec}}(t)) = -a \int_0^L |v_1|^2 - b \xi \varepsilon_3 \int_0^L |\theta_t + \phi_x|^2 \, dx - c \varepsilon_3 \int_0^L |\eta_t + \phi|^2 \, dx.
\]
Thus, we obtain (2.3). The proof has been completed. \(\square\)

**Lemma 2.2.** If \(E(t) = 0\) then \(v = \phi = \eta = \theta = 0\).

**Proof.** By (2.2), \(E(t) = 0\) implies that
\[
v_t + v_x(t) = \theta + \eta_x(t) \quad \text{and} \quad \theta_t + \phi_x + \eta(t) + \phi(t) = 0.
\]
Using the fact that \(v(0, t) = 0\) in (2.11), we get
\[
v(x, t) = 0 \quad \text{in} \quad (0, L) \times (0, \infty).
\]
Using (2.12) and (Lorenz), we get the following system
\[
\begin{align*}
\phi_x(t, x) - \frac{\phi}{\xi_x} \phi(t, x) + \frac{\phi}{\xi_3} \phi(t, x) = 0 & \quad \text{in} \quad (0, L) \times (0, \infty), \\
\phi(x, 0) = \phi(L, t) = 0 & \quad \text{in} \quad (0, \infty), \\
\phi(x, 0) = \phi_t(x, 0) = 0 & \quad \text{in} \quad (0, L).
\end{align*}
\]
Applying Fourier Transforms with respect to the variable \(x\) on (2.13) and using the zeros initial conditions, we get
\[
\phi(x, t) = 0.
\]
Using (2.14) and (2.11), we get \(\theta_t = 0\). Using the fact that \(\theta_t = 0\) and (Lorenz), we get
\[
\begin{align*}
\xi \theta_x(x, t) - \theta(x, t) = 0 & \quad \text{in} \quad (0, L) \times (0, \infty), \\
\theta(0, t) = \theta(L, t) = 0 & \quad \text{in} \quad (0, \infty).
\end{align*}
\]
The solution of (2.15) is \(\theta(x, t) = 0\). Finally, using (LGC) and the fact that \(\theta(x, t) = 0\), we get \(\eta(x, t) = 0\). The proof has been completed. \(\square\)

Now, we define the following state
\[
U = (v, z, u^1, u^2, u^3)
\]
such that \(z = u_t, u^1 = \theta - \eta_x, u^2 = \theta_t + \phi_x \) and \(u^3 = \eta_t + \phi\). with the following initial condition
\[
U(\cdot, 0) = U_0 = (v(\cdot, 0), z(\cdot, 0), \theta(\cdot, 0) - \eta_x(\cdot, 0), \theta_t(\cdot, 0), \eta_t(\cdot, 0)).
\]
By the choices of the states, (Lorenz) and (LGC), we obtain the following compatibility condition:
\[
\xi u_x^2 - u^3 + \frac{\gamma}{\varepsilon_3} v_x = 0.
\]
We define the linear space
\[
\mathcal{H} = \left\{ U \in \left( H^1_0(0, L) \times L^2(0, L) \right)^4 : u_x^2 \in L^2(0, L), u^2(0) = u^2(L) = 0, \xi u_x^2 - u^3 + \frac{\gamma}{\varepsilon_3} v_x = 0 \right\}
\]
and the bilinear form on \( \mathcal{H} \times \mathcal{H} \)
\[
b(U, \bar{U}) = \int_0^L \left( \alpha u_x \bar{v}_x + \rho \xi u_x \bar{v}_x + \mu u^1 \bar{u}^1 + \xi \varepsilon_3 u^2 \bar{u}^2 + \varepsilon_3 u^3 \bar{u}^3 \right) \, dx.
\]

**Remark 2.3.** Using (2.16), the bilinear form \( b \) can be written as
\[
b(U, \bar{U}) = \int_0^L \left( \alpha u_x \bar{v}_x + \rho \xi u_x \bar{v}_x + \mu u^1 \bar{u}^1 + \xi \varepsilon_3 u^2 \bar{u}^2 + \varepsilon_3 u^3 \bar{u}^3 \right) \left( \xi \frac{u_x}{\varepsilon_3} \bar{v}_x + \frac{\gamma}{\varepsilon_3} \bar{v}_x \right) \, dx.
\]

**Lemma 2.4.** (See [16]) The bilinear form \( b \) is symmetric, continuous and coercive on \( \mathcal{H} \times \mathcal{H} \).

**Proof.** The bilinear form (2.18) is symmetric and by using the Poincaré’s inequality on \( u^2 \) terms, we can check easily the continuity. For the coercivity, using (2.19) and the generalized Young’s inequality, we get
\[
b(U, U) \geq \int_0^L \left( |z|^2 + \mu |u|^2 \right) + \xi \varepsilon_3 |u^2|^2 + \left( \alpha + \frac{\gamma^2}{\varepsilon_3} - \frac{2\xi}{k} \right) |v_x|^2 + (\xi \Delta \xi^2 - \gamma k) |u^2|^2 \, dx.
\]
By choosing
\[ \frac{\gamma \xi}{\alpha + \frac{\gamma}{\xi_3}} < k < \frac{\xi_3}{\gamma} \]
then the coefficients of \(|v_x|^2\) and \(|u_x^2|^2\) are positive. Therefore,
\[ b(U, U) \geq C \|U\|_H, \]
where \(C = \min \left( \rho, \mu, \xi_3, \left( \alpha + \frac{\gamma^2}{\xi_3} - \frac{\gamma \xi}{k} \right), (\xi_3 \xi_2^2 - \gamma \xi k) \right)\). The proof has been completed. \(\square\)

**Lemma 2.5.** \(H\) is a Hilbert space equipped by the inner product \(b(U, \tilde{U})\).

We define the unbounded linear operator \(A : D(A) \rightarrow H\), by
\[
AU = \begin{pmatrix} z \\ \frac{\alpha}{\rho} v_{xx} + \frac{\gamma}{\rho} u_x^3 \\ u^2 - u_x^3 \\ -\frac{\mu}{\xi_3} u_1^1 \\ -\frac{\mu}{\xi_3} \frac{u_1^1}{\xi_3} + \frac{\gamma}{\xi_3} z_x \end{pmatrix}
\]
and
\[
D(A) = \left\{ U = (v, z, u_1, u_2, u_3) \in H; \ v \in H^2(0, L), v \in H^2(0, L) \cap H^1_0(0, L), \ u_1, u_2 \in H^1_0(0, L), \ u_3 \in H^1(0, L) \quad \text{and} \quad \alpha v (L) + \gamma u_3 (L) = 0 \right\}
\]

**Proposition 2.6.** (See [16]) We have:

(1) \(0 \in \rho(A)\).

(2) The operator \(A\) satisfies \(A^* = -A\) on \(H\), and \(A\) is a generator of a unitary semigroup \((e^{tA})_{t\geq0}\).

The system (Lorenz) can be written as
\[ (2.20) \quad U_t = (A - B) U, \quad U(0) = U_0. \]
where
\[ BU := \begin{pmatrix} 0, \frac{a}{\rho}, 0, bu^2, cu^3 \end{pmatrix}^T. \]

It is easy to see that the operator \(B\) is a bounded operator. Let us denote \(A_{a,b,c} = A - B\). The operator \(A_{a,b,c}\) defined by (2.20) with domain \(D(A_{a,b,c}) = D(A)\) is densely defined in \(H\). Moreover, \(A_{a,b,c}\) is the infinitesimal generator of \(C_0\)-semigroup of contractions. Therefore, by Lumer-Phillips theorem if \(U_0 \in D(A)\) solution of (2.20) then \(U \in C([0, T]; D(A)) \cap C^1([0, T]; H)\).

### 3. Strong Stability

The aim of this section is to analyse the strong stability of system (2.20). The main result of this section is the following theorems.

**Theorem 3.1.** The \(C_0\)-semigroup of contractions \((e^{tA_{a,b,c}})_{t\geq0}\) is strongly stable in \(H\) is the sense that
\[
\lim_{t \rightarrow +\infty} \|e^{tA_{a,b,c}}U_0\|_H = 0, \text{ in the following cases:}
\]

**Case 1:** \((a, b, c) \neq (0, 0, 0)\).

**Case 2:** \(a = 0\) and \((b, c) \neq (0, 0)\).

**Case 3:** \(b = 0\) and \((a, c) \neq (0, 0)\).

**Case 4:** \(c = 0\) and \((a, b) \neq (0, 0)\).

**Case 5:** \(a \neq 0\) and \((b, c) = (0, 0)\).

**Case 6:** \(b \neq 0\) and \((a, c) = (0, 0)\).
Proof. Since the resolvent of $A_{a,b,c}$ is compact in $H$, then according to Arendt-Batty theorem see (Page 837 in [4]), system (Lorenz) is strongly stable if and only if $A$ doesn’t have pure imaginary eigenvalues, that is, $\sigma(A) \cap i\mathbb{R} = \emptyset$. We have already shown that $0 \in \rho(A_{a,b,c})$, and still need to show that $\sigma(A_{a,b,c}) \cap i\mathbb{R}^* = \emptyset$. For this aim, suppose by contradiction that there exists $\lambda \in \mathbb{R}^*$ and $U \in D(A_{a,b,c}) \setminus \{0\}$ such that

$$A_{a,b,c}U = i\lambda U.$$  

Equivalently, we have:

$$z = i\lambda v,$$  

$$\lambda^2 \rho v + \alpha v_{xx} + \gamma u_x^3 - az = 0,$$  

$$u^2 - u_x^3 = i\lambda u^1,$$  

$$-\frac{\mu}{\xi_3} u_1^1 - bu^2 = i\lambda u^2,$$  

$$-\frac{\mu}{\xi_3} u_1^1 + \frac{\gamma}{\xi_3} i\lambda v_x - cu^3 = i\lambda u^3.$$  

A straightforward calculation gives:

$$0 = \Re \langle i\lambda U, U \rangle_H = \Re \langle A_{a,b,c}U, U \rangle_H = -a \int_0^L |z|^2 dx - b\xi_3 \int_0^L |u^2|^2 dx - c\xi_3 \int_0^L |u^3|^2 dx.$$  

Consequently, we deduce that:

$$az = bu^2 - cu^3 = 0.$$  

Case 1: From (3.7), we get $z = u^2 = u^3 = 0$. Using the fact that $\lambda \neq 0$, (3.2) and (3.5), we get $v = 0$ and $u^1$. Thus, $U = 0$ and consequently $A$ has no pure imaginary eigenvalues.

Case 2: From (3.7), we get $u^2 = u^3 = 0$. Then, from (3.5) we obtain

$$u^1 = 0.$$  

Using (3.7), (3.8) and the fact that $\lambda \neq 0$ in (3.6), we get $v_x = 0$. Using the boundary condition $v(0) = 0$ and the fact that $\lambda \neq 0$, we obtain $v = 0$. Using the fact that that $v = 0$ and $\lambda \neq 0$ in (3.2), we get $z = 0$. Thus, $U = 0$ and consequently $A$ has no pure imaginary eigenvalues.

Case 3: From (3.7), we get $z = u^3 = 0$. Then, from (3.2) and the fact that $\lambda \neq 0$, we get $v = 0$. Using the fact that $v = u^2 = 0$ and $u^1(0) = 0$ in (3.6), we get $u^1 = 0$. Using the fact that $u^3 = u^1 = 0$ in (3.4), we get $u^3 = 0$. Thus, $U = 0$ and consequently $A$ has no pure imaginary eigenvalues.

Case 4: From (3.7), we get $z = u^2 = 0$. Then, from (3.2), (3.5) and the fact that $\lambda \neq 0$, we get $v = u^1 = 0$. Using $v = u^1 = 0$ and the fact that $\lambda \neq 0$ in (3.6), we get $u^3 = 0$. Thus, $U = 0$ and consequently $A$ has no pure imaginary eigenvalues.

Case 5: From (3.7), we get $z = 0$. Using the facts that $z = 0$ and $\lambda \neq 0$ in (3.2), we get $v = 0$. Using $z = v = 0$ in (3.3), we obtain $u_x^3 = 0$, it follows that

$$u^3 = k.$$  

Using the fact that $v = 0$ and inserting (3.9) in (3.6), we get

$$-\frac{\mu}{\xi_3} u^1 = i\lambda kx + k_1.$$  

Using the fact that $u^1(0) = u^1(L) = 0$ in (3.10), we get $k = k_1 = 0$, it follows that

$$u^1 = u^3 = 0.$$  

Inserting (3.11) in (3.5) and using the fact that $\lambda \neq 0$, we get $u^2 = 0$. Thus, $U = 0$ and consequently $A$ has no pure imaginary eigenvalues.

Case 6: From (3.7), we get $u^2 = 0$. Then, from (3.5), we get $u^1 = 0$. Using the fact that $u^2 = u^1 = 0$ in (3.4), we get $u_x^3 = 0$. Using the fact that $u^1 = u_x^3 = 0$ in (3.6), we get

$$\frac{\gamma}{\xi_3} v_x = u^3.$$  

7
Deriving the above equation and using the fact that \( u_3^3 = 0 \), we obtain

\[(3.13) \quad v_{xx} = 0.\]

Inserting (3.13) in (3.3) and using the fact that \( \lambda \neq 0 \), we get \( v = 0 \). Then, from (3.12), we obtain \( u^3 = 0 \). Thus, \( U = 0 \) and consequently \( A \) has no pure imaginary eigenvalues. \( \square \)

**Theorem 3.2.** Assume that \( c \neq 0 \) and \( (a, b) = (0, 0) \). Then, the \( C_0 \)-semigroup of contractions \( (e^{tA_0,c}) \) is strongly stable on \( H \) in the sense that \( \lim_{t \to +\infty} \|e^{tA_0,c}\| = 0 \) for all \( U_0 \in H \) if and only if

\[(SC) \quad \frac{\mu \rho}{\xi \varepsilon^3 \alpha} \neq \frac{(2n + 1)^2 \pi^2}{4L^2}.\]

**Proof.** We suppose by contradiction that there exists \( \lambda \in \mathbb{R}^* \) and \( U \in D(A_{0,0,c}) \setminus \{0\} \) such that

\[(3.14) \quad A_{0,0,c}U = i\lambda U.\]

A straightforward computation gives:

\[0 = \Re \langle i\lambda U, U \rangle_H = \Re \langle A_{0,0,c}U, U \rangle_H = -c\varepsilon_3 \int_0^L |u^3|^2 dx.\]

Consequently, we deduce that

\[(3.15) \quad u^3 = 0.\]

Detailing (3.14) and using (3.15)

\[(3.16) \quad z = i\lambda v,\]

\[(3.17) \quad \lambda^2 \rho v + \alpha v_{xx} = 0,\]

\[(3.18) \quad u^2 = i\lambda u_1,\]

\[(3.19) \quad -\frac{\mu}{\xi \varepsilon^3} u_1 = i\lambda u^2,\]

\[(3.20) \quad -\frac{\mu}{\xi \varepsilon^3} u_x^1 + \frac{\gamma}{\varepsilon^3} \lambda v_x = 0.\]

Inserting (3.18) in (3.19), we get

\[(3.21) \quad \left( \lambda^2 - \frac{\mu}{\xi \varepsilon^3} \right) u^1 = 0.\]

We distinguish two cases:

**Case 1:** If \( \lambda^2 \neq \frac{\mu}{\xi \varepsilon^3} \), it follows that \( u^1 = 0 \). Using the fact that \( u^1 = 0 \) in (3.18), we obtain \( u^2 = 0 \). Using the fact that \( \lambda \neq 0 \) and \( u^1 = 0 \) and \( v(0) = 0 \) in (3.20), we get \( v = 0 \) then \( z = 0 \). Thus, \( U = 0 \) and consequently \( A \) has no pure imaginary eigenvalues.

**Case 2:** If \( \lambda^2 = \frac{\mu}{\xi \varepsilon^3} \). From, (3.20) and the fact that \( v(0) = u^1(0) = 0 \), we get

\[(3.22) \quad u^1 = i\lambda \frac{\gamma}{\mu} v.\]

Using (3.15), the compatibility condition (2.16) and the facts that \( v(0) = u^2(0) = 0 \), we get

\[(3.23) \quad u^2 = -\frac{\gamma}{\varepsilon^3 \xi} v.\]

The general solution of (3.17) with \( v(0) = 0 \), is given by

\[(3.24) \quad v(x) = B \sin \left( \lambda \sqrt{\frac{\rho}{\alpha}} x \right).\]

Using the fact that \( u^3 = 0 \) and the boundary condition \( \alpha v_x(L) + \gamma u^3(L) = 0 \), we get \( v_x(L) = 0 \). Using \( v_x(L) = 0 \) in (3.24), we obtain \( B_o \lambda \sqrt{\frac{\rho}{\alpha}} \cos (\lambda \sqrt{\frac{\rho}{\alpha}} L) = 0 \). If,

\[(3.25) \quad \cos \left( \lambda \sqrt{\frac{\rho}{\alpha}} L \right) = 0,\]
then,

\[(3.26)\]

\[\lambda = \frac{(2n+1)\pi}{2L} \sqrt{\frac{\alpha}{\rho}}.\]

Using the fact that \(\lambda^2 = \frac{\mu}{\xi_3^3}\) in (3.26), we get

\[(3.27)\]

\[\frac{\mu\rho}{\xi_3^3} = \frac{(2n+1)^2\pi^2}{4L^2}.\]

This contradicts (SC), consequently hypothesis (3.25) is not true, and so \(v = 0\), then from (3.16), (3.22) and (3.23), we get \(u^1 = u^2 = z = 0\), which yields to \(U = 0\). Consequently, if (SC) holds, then \(i\lambda\) is not an eigenvalue of \(A\). Thus

\[\ker (i\lambda I - A_{0,1}) = \{0\}.\]

On the other hand, if condition (SC) is not true (i.e., if (3.27) holds), then \(i\lambda\) (where \(\lambda\) is given in (3.26)) is an eigenvalue of \(A_{0,1}\) with the corresponding eigenvector

\[U = \left( v, i\lambda v, i\lambda \frac{\gamma}{\mu} v, -\frac{\gamma}{\varepsilon_3} v, 0 \right),\]

such that \(v\) is given in (3.24). The proof is thus complete. □

4. The stretching of the centreline of the beam in \(x\)-direction and electrical field component in \((x\) and \(z\))-direction are damped \(\"(a,b,c) \neq (0,0,0)\"

The aim of this part is to prove the exponential stability of Lorenz system (Lorenz). The stretching of the centreline of the beam in \(x\)-direction and electrical field component in \((x\) and \(z\))-direction are damped (i.e. \((a; b, c) \neq (0,0,0))\). The main result of this part is the following theorem.

**Theorem 4.1.** The \(C_0\)-semigroup of contractions \((e^{tA_{a,b,c}})_{t \geq 0}\) is exponentially stable; i.e., there exist constants \(M \geq 1\) and \(\epsilon > 0\) independent of \(U_0\) such that

\[\|e^{tA_{a,b,c}}U_0\|_{\mathcal{H}} \leq Me^{-\epsilon t}\|U_0\|_{\mathcal{H}}.\]

According to Huang and Prüss [8, 18], we have to check if the following conditions hold:

(H1) \(i\mathbb{R} \subset \rho(A_{a,b,c})\)

and

(H2) \(\sup_{\lambda \in \mathbb{R}} \| (i\lambda I - A_{a,b,c})^{-1} \|_{\mathcal{L}(\mathcal{H})} = O(1).\)

Condition (H1) is already proved in Theorem 3.1. The next proposition is a technical result to be used in the proof of (H2) given below.

**Proposition 4.2.** Let \(\lambda, U := (v, z, u^1, u^2, u^3) \in \mathbb{R}^5 \times D(A_{a,b,c})\), with \(|\lambda| \geq 1\), such that

\[(4.1)\]

\[(i\lambda I - A_{a,b,c}) U = f := (f^1, f^2, f^3, f^4, f^5) \in \mathcal{H},\]

detailed as

\[(4.2)\]

\[i\lambda v - z = f^1,\]

\[(4.3)\]

\[i\lambda z - \frac{\alpha}{\rho} v_{xx} - \frac{\gamma}{\rho} u^3_x + \frac{a}{\rho} z = f^2,\]

\[(4.4)\]

\[i\lambda u^1 - u^2_x + u^3_x = f^3,\]

\[(4.5)\]

\[i\lambda u^2 + \frac{\mu}{\xi_3} u^1_x + bu^2 = f^4,\]

\[(4.6)\]

\[i\lambda u^3 + \frac{\mu}{\xi_3} u^1_x - \frac{\gamma}{\varepsilon_3} z_x + cu^3 = f^5.\]

Then, we have the following inequality

\[(4.7)\]

\[\|U\|_{\mathcal{H}} \leq K\|F\|_{\mathcal{H}}.\]

Here and below we denote by \(K_j\) a positive constant number independent of \(\lambda\). For the proof of Proposition 4.2, we need the following lemmas.
Lemma 4.3. The solution \((v, z, u^1, u^2, w^3) \in D(A_{a,b,c})\) of equation (4.1) satisfies the following estimates:

\[(4.8)\]
\[\int_0^L |z|^2 dx \leq K_1 \|U\|_\mathcal{H} \|F\|_\mathcal{H} \quad \text{where} \quad K_1 = \frac{1}{a},\]

\[(4.9)\]
\[\int_0^L |u^2|^2 dx \leq K_2 \|U\|_\mathcal{H} \|F\|_\mathcal{H} \quad \text{where} \quad K_2 = \frac{1}{b \epsilon},\]

\[(4.10)\]
\[\int_0^L |u^3|^2 dx \leq K_3 \|U\|_\mathcal{H} \|F\|_\mathcal{H} \quad \text{where} \quad K_3 = \frac{1}{c \epsilon}.\]

Proof. First, taking the inner product of (4.1) with \(U \in \mathcal{H}\), we obtain

\[(4.11)\]
\[a \int_0^L |z|^2 + b \epsilon z \int_0^L |u^2|^2 dx + c \epsilon z \int_0^L |u^3|^2 dx = \mathbb{R} (A_{a,b,c} U, U) \leq \|U\|_\mathcal{H} \|F\|_\mathcal{H}.\]

Then, we obtain (4.8)-(4.10). The proof has been completed. □

Lemma 4.4. The solution \((v, z, u^1, u^2, w^3) \in D(A_{a,b,c})\) of equation (4.1) satisfies the following estimation:

\[(4.12)\]
\[\alpha \int_0^L |v_x|^2 dx \leq K_4 \|U\|_\mathcal{H} \|F\|_\mathcal{H} \quad \text{where} \quad K_4 = 2 \left( \rho K_1 + 2 \sqrt{\frac{b}{\alpha}} + \frac{a}{\alpha} + \frac{\gamma^2}{K_3} \right).\]

Proof. Multiplying (4.3) by \(\rho \pi\), integrating by parts over \((0, L)\), we get

\[(4.13)\]
\[i \lambda \rho \int_0^L z \pi dx + \alpha \int_0^L |v_x|^2 dx + \gamma \int_0^L u^3 \pi dx + a \int_0^L z \pi dx = \rho \int_0^L f^2 \pi dx.\]

From (4.2), we get

\[\alpha \int_0^L |v_x|^2 dx = \rho \int_0^L |z|^2 dx + \rho \int_0^L z \pi dx - \gamma \int_0^L u^3 \pi dx - a \int_0^L z \pi dx + \rho \int_0^L f^2 \pi dx,\]

consequently, we obtain

\[(4.14)\]
\[\alpha \int_0^L |v_x|^2 dx \leq \rho \int_0^L |z|^2 dx + \rho \int_0^L |z| |f|^2 dx + \rho \int_0^L |f|^2 |v_x|^2 dx + \gamma \int_0^L u^3 |v_x|^2 dx + a \int_0^L |z|^2 |v|^2 dx.\]

Using the fact that \(\sqrt{\rho} \leq \|U\|_\mathcal{H}, \sqrt{\alpha} \leq \|F\|_\mathcal{H}, \sqrt{\alpha} |f_2| \leq \|F\|_\mathcal{H}\) and Poincaré inequality, we get

\[(4.15)\]
\[\left\{ \begin{array}{l}
\rho \int_0^L |z| |f|^2 dx \leq \rho c_p \|U\|_\mathcal{H} \|F\|_\mathcal{H}, \\
\rho \int_0^L |f|^2 |v_x|^2 dx \leq \rho c_p \|U\|_\mathcal{H} \|F\|_\mathcal{H}.
\end{array} \right.\]

Applying Young inequality, Poincaré inequality and using (4.8) and (4.10), we get

\[(4.16)\]
\[a \int_0^L |z||v|^2 dx \leq \frac{a}{2r_1} \int_0^L |z|^2 dx + \frac{ar_1 c_p^2}{2} \int_0^L |v_x|^2 dx \leq \frac{1}{2r_1} \|U\|_\mathcal{H} \|F\|_\mathcal{H} + \frac{ar_1 c_p^2}{2} \int_0^L |v_x|^2 dx\]

and

\[(4.17)\]
\[\gamma \int_0^L |u^3| |v_x|^2 dx \leq \frac{\gamma^2}{2r_1} \int_0^L |u^3|^2 dx + \frac{r_2}{2} \int_0^L |v_x|^2 dx \leq \frac{\gamma^2}{2r_1} K_3 \|U\|_\mathcal{H} \|F\|_\mathcal{H} + \frac{r_2}{2} \int_0^L |v_x|^2 dx.\]

Inserting (4.15)-(4.17) in (4.14) and using (4.8), we get

\[(4.18)\]
\[\left( \alpha - \frac{ar_1 c_p^2}{2} - \frac{r_2}{2} \right) \int_0^L |v_x|^2 dx \leq \left( \rho K_1 + 2 \frac{b}{\alpha} c_p + \frac{a}{\alpha} + \frac{\gamma^2}{K_3} \right) \|U\|_\mathcal{H} \|F\|_\mathcal{H}.\]

Taking \(r_1 = a r_1 c_p^2\) and \(r_2 = \frac{\alpha}{2}\) in (4.18), we get

\[\frac{\alpha}{2} \int_0^L |v_x|^2 dx \leq \left( \rho K_1 + 2 \frac{b}{\alpha} c_p + \frac{a}{\alpha} + \frac{\gamma^2}{K_3} \right) \|U\|_\mathcal{H} \|F\|_\mathcal{H}.\]

Thus, we obtain (4.12). The proof has been completed. □
Lemma 4.5. The solution \((v, z, u^1, u^2, u^3) \in D(A_{a,b,c})\) of equation (4.1) satisfies the following estimation:

\[
\mu \int_0^L |u|^2 dx \leq K_5 \|U\|_H \|F\|_H,
\]

where \(K_5 = 2\xi_3 \left(1 + \frac{b^2\xi_3}{2\mu} \right) K_2 + \left(1 + \frac{\gamma^2}{2\xi_3^2} \right) K_3 + \frac{1}{2} K_4 + \frac{2}{\sqrt{\xi_3\mu}} \).

**Proof.** Multiplying (4.5) by \(u^2\) integrating over \((0, L)\), we get

\[
i\lambda \int_0^L u^2 u^2 dx + \frac{\mu}{\xi_3} \int_0^L |u|^2 dx + b \int_0^L u^2 u^1 dx = \int_0^L f^1 u^1 dx.
\]

Multiplying (4.4) by \(u^2\) integrating by parts over \((0, L)\), we get

\[
i\lambda \int_0^L u^1 u^2 dx - \int_0^L |u|^2 dx - \int_0^L u^3 u^2 dx = \int_0^L f^3 u^2 dx.
\]

Adding (4.20)-(4.21) and taking the real part, we get

\[
\Re \left( \int_0^L f^1 u^1 dx \right) + \Re \left( \int_0^L f^3 u^2 dx \right) \leq \frac{1}{\sqrt{\xi_3\mu}} \|U\|_H \|F\|_H.
\] (4.22)

Using the fact that \(\sqrt{\xi_3} \|F^i\| \leq \|F\|_H\), we get

\[
\left\{ \begin{array}{l}
\Re \left( \int_0^L f^1 u^1 dx \right) \leq \frac{1}{\sqrt{\xi_3\mu}} \|U\|_H \|F\|_H, \\
\Re \left( \int_0^L f^3 u^2 dx \right) \leq \frac{1}{\sqrt{\xi_3\mu}} \|U\|_H \|F\|_H.
\end{array} \right.
\] (4.23)

Using Young inequality and (4.9), we get

\[
\left\| \int_0^L u^3 u^2 dx \right\| \leq \frac{b^2}{2r_3} \int_0^L |u|^2 dx + \frac{r_3}{2} \int_0^L |u|^1 dx \leq \frac{b^2}{2r_3} K_2 \|U\|_H \|F\|_H + \frac{r_3}{2} \int_0^L |u|^1 dx.
\] (4.24)

Now, we give an estimation on \(\Re \left( \int_0^L u^3 u^2 dx \right)\). Using compatibility condition (2.16), we get

\[
\Re \left( \int_0^L u^3 u^2 dx \right) = \frac{1}{\xi_3} \int_0^L |u^3|^2 dx - \frac{\gamma}{\xi_3} \int_0^L u^3 \xi_3 dx.
\]

Applying Young inequality in the above estimation and using (4.10) and (4.12), we get

\[
\left\| \int_0^L u^3 u^2 dx \right\| \leq \left(1 + \frac{\gamma^2}{2\xi_3} \right) \int_0^L |u^3|^2 dx + \frac{1}{2} \int_0^L |u|^1 dx \leq \left(1 + \frac{\gamma^2}{2\xi_3} \right) \|U\|_H \|F\|_H.
\] (4.25)

Inserting (4.23), (4.24), (4.25) and using (4.9) in (4.22), we get

\[
\left(\frac{\mu}{\xi_3} - \frac{r_3}{2}\right) \int_0^L |u|^1 dx \leq \left( K_2 + \frac{b^2}{2\xi_3} K_2 + \left(1 + \frac{\gamma^2}{2\xi_3^2} \right) K_3 + \frac{1}{2} K_4 + \frac{2}{\sqrt{\xi_3\mu}} \right) \|U\|_H \|F\|_H.
\]

Taking \(r_3 = \frac{\mu}{\xi_3}\) in the above estimation, we get

\[
\mu \int_0^L |u|^1 dx \leq 2\xi_3 \left(1 + \frac{b^2\xi_3}{2\mu} \right) K_2 + \left(1 + \frac{\gamma^2}{2\xi_3^2} \right) K_3 + \frac{1}{2} K_4 + \frac{2}{\sqrt{\xi_3\mu}} \|U\|_H \|F\|_H.
\]

The proof has been completed. \(\square\)
Proof of Proposition 4.2. Adding (4.8), (4.9), (4.10), (4.12) and (4.19), we get
\[ \|U\|_H^2 = \alpha \|v_x\|^2 + \rho \|z\|^2 + \mu \|u^1\|^2 + \varepsilon_3 \|u^2\|^2 + \varepsilon_3 \|u^3\|^2 \leq K \|U\|_H \|F\|_H. \]
Thus, we obtain (4.7) where \( K = \rho K_1 + \xi \varepsilon_3 K_2 + \varepsilon_3 K_3 + K_4 + K_5 \).

Proof of Theorem 4.1 For all \( U \in D(A) \) according to Proposition (4.2), we get
\[ \|U\|_H \leq K \|(i\lambda I - A_{a,b,c})U\|_H. \]
Thus, we have
\[ \|(i\lambda I - A_{a,b,c})^{-1}V\|_H \leq K \|V\|_H, \quad \forall V \in H. \]
Therefore, from the above equation, we get (H2) holds. Thus, we get the conclusion by applying Huang and Prüss Theorem.

5. The electrical field component in \((x \text{ and } z)\)-direction are damped "\(a = 0 \text{ and } (b,c) \neq (0,0)\)"
The aim of this part is to prove the exponential stability of Lorenz system (Lorenz) with the damping acting on the electrical field component in \((x - z)\)-direction i.e. \((a = 0 \text{ and } (b,c) \neq (0,0))\). The main result of this part is the following theorem.

Theorem 5.1. The \(C_0\)-semigroup of contractions \(e^{tA_{0,b,c}}\) \((t \geq 0)\) is exponentially stable; i.e., there exist constants \(M \geq 1 \text{ and } \epsilon > 0\) independent of \(U_0\) such that
\[ \|e^{tA_{0,b,c}}U_0\|_H \leq Me^{-\epsilon t}\|U_0\|_H. \]
From Theorem 3.1, we have seen that \(i\mathbb{R} \subset \rho(A_{0,b,c})\); then for the proof of Theorem 5.1, we still to prove that (H3)
\[ \sup_{\lambda \in \mathbb{R}} \|(i\lambda I - A_{0,b,c})^{-1}\|_{L(H)} = O(1). \]
The next proposition is a technical result to be used in the proof of Theorem 5.1 given below.

Proposition 5.2. Let \((\lambda, U) := (v, z, u^1, u^2, u^3) \in \mathbb{R}^* \times D(A_{0,b,c}), \) with \(|\lambda| \geq 1\), such that
\[ (i\lambda I - A_{0,b,c})U = F := (f^1, f^2, f^3, f^4, f^5) \in H, \]
detailed as
\[ (i\lambda v - z) = f^1, \]
\[ (i\lambda z - \frac{\alpha}{\rho} v_{xx} - \frac{\gamma}{\rho} u^3) = f^2, \]
\[ (i\mu u^1 - u^2 + u_x^3) = f^3, \]
\[ (i\mu u^2 + \frac{\mu}{\varepsilon_3} u^1 + bu^2) = f^4, \]
\[ (i\mu u^3 + \frac{\mu}{\varepsilon_3} u^1 + \frac{\gamma}{\varepsilon_3} x_x + cu^3) = f^5. \]
Then, we have the following inequality
\[ \|U\|_H \leq M\|F\|_H. \]
For the proof of Proposition 5.2, we need the following lemmas.

Lemma 5.3. The solution \((v, z, u^1, u^2, u^3) \in D(A_{0,b,c})\) of equation (5.1) satisfies the following estimations:
\[ \int_0^L |u^2|^2 dx \leq M_1 \|U\|_H \|F\|_H \quad \text{where} \quad M_1 = \frac{1}{L\varepsilon_3}, \]
\[ \int_0^L |u^3|^2 dx \leq M_2 \|U\|_H \|F\|_H \quad \text{where} \quad M_2 = \frac{1}{\varepsilon_3}, \]
Proof. By using the argument in Lemma 4.3, we get (5.8)-(5.9). The proof has been completed. \(\square\)
Lemma 5.4. The solution \((v, z, u^1, u^2, u^3) \in D(A_{b,c})\) of equation (5.1) satisfies the following estimation:

\[
\alpha \int_0^L |v_x|^2 dx \leq \mathcal{M}_3\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}},
\]

where \(\mathcal{M}_3 = \frac{2\varepsilon_3^2}{\beta} \left( \frac{(b-c)^2\varepsilon_3}{\gamma} M_2 \right)\).

Proof. First, inserting (5.2) in (5.6), we get

\[
i\lambda u^3 + \frac{\mu}{\varepsilon_3} u^3_x - \frac{\gamma}{\varepsilon_3} i\lambda v_x + \frac{\gamma}{\varepsilon_3} f^1_x + cu^3 = f^5.
\]

Multiplying (5.11) by \(i\lambda^{-1}v_x\), integrating over \((0, L)\), we get

\[
-\int_0^L u^2 v_x dx + i\frac{\mu}{\varepsilon_3} \lambda^{-1} \int_0^L u^1 v_x dx + \frac{\gamma}{\varepsilon_3} \int_0^L |v_x|^2 dx + i\lambda^{-1} \int_0^L f^1 v_x dx = 0.
\]

Using the compatibility condition (2.16) in the first integral in (5.12), we get

\[
-\xi \int_0^L u^2 v_x dx + i\frac{\mu}{\varepsilon_3} \lambda^{-1} \int_0^L u^1 v_x dx + i\lambda^{-1} \int_0^L f^1 v_x dx = 0.
\]

Deriving (5.5) with respect to \(x\) and multiplying the result by \(-i\xi\lambda^{-1}v_x\), we get

\[
\xi \int_0^L u^2 v_x dx - i\frac{\mu}{\varepsilon_3} \lambda^{-1} \int_0^L u^1 v_x dx - i\lambda^{-1} \int_0^L f^1 v_x dx = 0.
\]

Adding (5.12) and (5.13), we get

\[
-b\xi \int_0^L u^2 v_x dx + \frac{\gamma}{\varepsilon_3} \int_0^L f^1 v_x dx + \frac{\gamma}{\varepsilon_3} \int_0^L f^2 v_x dx = 0.
\]

Again, using the compatibility condition (2.16) in the above equation, we get

\[
\frac{\gamma}{\varepsilon_3} \int_0^L |v_x|^2 dx = (b - c) \int_0^L u^3 v_x dx - b\xi \int_0^L f^1 v_x dx = 0.
\]

Since \(F \in \mathcal{H}\), then \((f^1, f^2, f^3, f^4, f^5)\) satisfies the compatibility condition

\[
\xi f^4_x - f^5_x + \frac{\gamma}{\varepsilon_3} f^1_x = 0.
\]

Combining (5.15) and (5.14), we get

\[
\frac{\gamma}{\varepsilon_3} b \int_0^L |v_x|^2 dx = (b - c) \int_0^L u^3 v_x dx.
\]

It follows that

\[
\frac{\gamma}{\varepsilon_3} b \int_0^L |v_x|^2 dx \leq |b - c| \int_0^L |u^3||v_x|dx + \frac{2\gamma}{\varepsilon_3} \int_0^L |f^1_x||v_x|dx + 2 \int_0^L |f^5||v_x|dx.
\]

Applying Young Inequality

\[
|b - c| \int_0^L |u^3||v_x|dx \leq \frac{(b - c)^2\varepsilon_3}{2\gamma} \int_0^L |u^3|^2 dx + \frac{b\gamma}{2\varepsilon_3} \int_0^L |v_x|^2 dx,
\]

\[
\leq \frac{(b - c)^2\varepsilon_3}{2\gamma} M_2\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{b\gamma}{2\varepsilon_3} \int_0^L |v_x|^2 dx.
\]

Inserting (5.17) in (5.16), we get

\[
\frac{\gamma}{2\varepsilon_3} b \int_0^L |v_x|^2 dx \leq \left( \frac{(b - c)^2\varepsilon_3}{2\gamma} M_2 \right) \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.
\]

Thus, we obtain (5.10). The proof has been completed. \(\square\)
Lemma 5.5. The solution \((v, z, u^1, u^2, u^3) \in D(A_{A_0,b,c})\) of equation \((5.1)\) satisfies the following estimation:

\[
(5.18) \quad \int_0^L |z|^2 dx \leq M_4 ||u||_{\mathcal{H}} ||F||_{\mathcal{H}},
\]

where \(M_4 = 2 + \frac{\gamma}{\sqrt{\varepsilon_3}} + \frac{\gamma}{M_2} + \left(1 + \frac{\gamma}{2} \right) M\).

**Proof.** Multiplying \((5.3)\) by \(-i\lambda^{-1} \rho\sigma\), integrating by parts over \((0, L)\), we get

\[
(5.19) \quad \rho \int_0^L |z|^2 dx - i\lambda^{-1} \alpha \int_0^L v_x z_x dx - i\gamma \lambda^{-1} \int_0^L u^3 z_x dx = -i\lambda^{-1} \rho \int_0^L f^2 \varpi dx.
\]

From \((5.2)\), we have

\[
(5.20) \quad -i\lambda^{-1} \varpi_x = -\varpi + i\lambda^{-1} \varpi_x
\]

Inserting \((5.20)\) in \((5.19)\), we get

\[
\rho \int_0^L |z|^2 dx = \alpha \int_0^L |v_x|^2 dx - i\lambda^{-1} \alpha \int_0^L v_x z_x dx + \gamma \int_0^L u^3 \varpi_x dx - i\gamma \lambda^{-1} \int_0^L u^3 \varpi_x dx - i\lambda^{-1} \rho \int_0^L f^2 \varpi dx.
\]

Consequently, we get

\[
(5.21) \quad \rho \int_0^L |z|^2 dx \leq \alpha \int_0^L |v_x|^2 dx + |\lambda|^{-1} \alpha \int_0^L |v_x||f_1|^2 dx
\]

\[
+ \gamma \int_0^L |u^3||v_x| dx + \gamma |\lambda|^{-1} \int_0^L |u^3||f_1|^2 dx + \rho |\lambda|^{-1} \int_0^L |f^2||x| dx.
\]

Using the fact that \(\sqrt{\alpha} |v_x| \leq ||U||_{\mathcal{H}}\), \(\sqrt{\rho} |x| \leq ||F||_{\mathcal{H}}\), \(\sqrt{\varepsilon_3} |u^3| \leq ||U||_{\mathcal{H}}\), \(\sqrt{\alpha} |f_1|^2 \leq ||F||_{\mathcal{H}}\) and \(|\lambda| \geq 1\), we get

\[
(5.22) \quad \begin{cases} 
|\lambda|^{-1} \alpha \int_0^L |v_x||f_1|^2 dx \leq ||U||_{\mathcal{H}} ||F||_{\mathcal{H}}, \\
\gamma |\lambda|^{-1} \int_0^L |u^3||f_1|^2 dx \leq \frac{\gamma}{\sqrt{\varepsilon_3}} ||U||_{\mathcal{H}} ||F||_{\mathcal{H}}, \\
\rho |\lambda|^{-1} \int_0^L |f^2||x| dx \leq ||U||_{\mathcal{H}} ||F||_{\mathcal{H}}.
\end{cases}
\]

Applying Young inequality and using \((5.9)\) and \((5.10)\), we get

\[
(5.23) \quad \gamma \int_0^L |u^3| \varpi_x dx \leq \frac{\gamma}{2} \int_0^L |u^3|^2 dx + \frac{\gamma}{2} \int_0^L |v_x|^2 dx \leq \frac{\gamma}{2} \left(M_2 + \frac{1}{\alpha} M_3 \right) ||U||_{\mathcal{H}} ||F||_{\mathcal{H}}.
\]

Inserting \((5.22)\) and \((5.23)\) in \((5.21)\) and using \((5.10)\), we get

\[
\rho \int_0^L |z|^2 dx \leq 2 + \frac{\gamma}{\sqrt{\varepsilon_3}} \left(M_2 + \frac{1}{\alpha} M_3 \right) ||U||_{\mathcal{H}} ||F||_{\mathcal{H}}.
\]

The proof has been completed. \(\Box\)

Lemma 5.6. The solution \((v, z, u^1, u^2, u^3) \in D(A_{A_0,b,c})\) of equation \((5.1)\) satisfies the following estimation:

\[
(5.24) \quad \mu \int_0^L |u^1|^2 dx \leq M_5 ||u||_{\mathcal{H}} ||F||_{\mathcal{H}},
\]

where \(M_5 = \xi_3 \left(1 + \frac{\sqrt{\varepsilon_3}}{2\mu} \right) M_1 + \left(1 + \frac{\sqrt{\varepsilon_3}}{2\mu} \right) M_2 + \frac{1}{2} M_3 + \frac{\gamma}{\sqrt{\varepsilon_3}} \mu \).

**Proof.** By proceeding the same techniques used in Lemma 4.5, we get \((5.24)\). The proof has been completed. \(\Box\)

**Proof of Proposition 5.2.** Adding \((5.8)\), \((5.9)\), \((5.10)\), \((5.18)\) and \((5.24)\), we get

\[
||U||_{\mathcal{H}}^2 = \alpha ||v_x||^2 + \rho ||z||^2 + \mu ||u^1||^2 + \xi_3 ||u^2||^2 + \varepsilon_3 ||u^3||^2 \leq M ||U||_{\mathcal{H}} ||F||_{\mathcal{H}},
\]

where \(M = M_3 + M_1 + M_5 + \xi_3 M_1 + \varepsilon M_2\). Then, \(||U||_{\mathcal{H}} \leq M ||F||_{\mathcal{H}}\). The proof has been completed.

**Proof of Theorem 5.1.** For all \(U \in D(A_{A_0,b,c})\) according to Proposition \((5.2)\), we get

\[
||U||_{\mathcal{H}} \leq M ||(i\lambda I - A_{A_0,b,c}) U||_{\mathcal{H}}.
\]
Thus, we have
\[ \|(i\lambda I - A_{0,b,c})^{-1}V\|_{\mathcal{H}} \leq \mathcal{M}\|V\|_{\mathcal{H}}, \quad \forall V \in \mathcal{H}. \]
Therefore, from the above equation, we get (H3) holds. Thus, we get the conclusion by applying Huang and Prüss Theorem.

6. THE STRETCHING OF THE CENTRELINE OF THE BEAM IN \( x \)-DIRECTION AND ELECTRICAL FIELD COMPONENT IN \( z \)-DIRECTION ARE DAMPED "\( b = 0 \quad \text{AND} \quad (a,c) \neq (0,0)"\)

The aim of this part is to prove the exponential stability of Lorenz system (Lorenz) with a damping acting on the stretching of the centerline of the beam in \( x \)-direction and electrical field component in \( x \)-direction. (i.e. \( b = 0 \quad \text{AND} \quad (a,c) \neq (0,0)\)). The main result of this pat is the following theorem.

**Theorem 6.1.** The \( C_0 \)-semigroup of contractions \( (e^{tA_{a,0,c}})_{t \geq 0} \) is exponentially stable; i.e., there exist constants \( M \geq 1 \) and \( \epsilon > 0 \) independent of \( U_0 \) such that
\[ \|e^{tA_{a,0,c}}U_0\|_{\mathcal{H}} \leq Me^{-\epsilon t}\|U_0\|_{\mathcal{H}}. \]

According to Huang and Prüss, we have to check if the following conditions hold:

(H1) \( i\mathbb{R} \subset \rho(A_{a,0,c}) \)

and

(H5) \( \sup_{\lambda \in \mathbb{R}} \|((i\lambda I - A_{a,0,c})^{-1}\|_{\mathcal{L}(\mathcal{H})} = O(1). \)

Condition (H1) is already proved in Theorem 3.1. The next proposition is a technical result to be used in the proof of (H5) given below.

**Proposition 6.2.** Let \( (\lambda, U := (v, z, u^1, u^2, u^3)) \in \mathbb{R}^* \times D(A_{a,0,c}), \) with \( |\lambda| \geq 1, \) such that
\[ (i\lambda I - A_{a,0,c})U = f := (f^1, f^2, f^3, f^4, f^5) \in \mathcal{H}, \]
detailed as
\[ i\lambda^2 - z = f^1, \]
\[ i\lambda z - \frac{\alpha}{\rho} v_{xx} + \frac{\gamma}{\rho} u^2 + \frac{\alpha}{\rho} z = f^2, \]
\[ i\lambda u^1 - u^2 + u^3 = f^3, \]
\[ i\lambda u^2 + \frac{\mu}{\varepsilon_3} u^1 = f^4, \]
\[ i\lambda u^3 + \frac{\mu}{\varepsilon_3} u^1 - \frac{\gamma}{\varepsilon_3} z_x + cu^3 = f^5. \]

Then, we have the following inequality
\[ \|U\|_{\mathcal{H}} \leq \mathcal{N}\|F\|_{\mathcal{H}}. \]
For the proof of Proposition 6.2, we need the following lemmas.

**Lemma 6.3.** The solution \( (v, z, u^1, u^2, u^3) \in D(A_{a,0,c}) \) of equation (6.1) satisfies the following estimates:
\[ \int_0^L |z|^2dx \leq \mathcal{N}_1\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} \quad \text{where} \quad \mathcal{N}_1 = \frac{1}{a}, \]
\[ \int_0^L |u^3|^2dx \leq \mathcal{N}_2\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} \quad \text{where} \quad \mathcal{N}_2 = \frac{1}{\varepsilon_3}. \]

**Proof.** By using the same argument used in Lemma 4.3, we get (6.8)-(6.9). The proof has been completed. \( \square \)

**Lemma 6.4.** The solution \( (v, z, u^1, u^2, u^3) \in D(A_{a,0,c}) \) of equation (6.1) satisfies the following estimation:
\[ \alpha \int_0^L |v_x|^2dx \leq \mathcal{N}_3\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} \quad \text{where} \quad \mathcal{N}_3 = 2\left(\frac{\rho}{\alpha} + \frac{2\rho c}{\alpha} + \frac{\gamma^2}{\alpha} + 2\frac{1}{\varepsilon_3}\right). \]
Proof. Using the same arguments in Lemma 4.4, we get (6.10). The proof has been completed.

Lemma 6.5. The solution \((v, z, u^1, u^2, u^3) \in D(A_{a,0,c})\) of equation (6.1) satisfies the following estimation:

\[
\xi_3 \int_0^L |u^2|^2 dx \leq N_4 \|U\|_H \|F\|_H, \quad \text{where} \quad N_4 = 2 \left( \frac{\xi_3 N_2 + \gamma^2}{\xi_3 \xi_\alpha} \right).
\]

Proof. Multiplying the compatibility condition (2.16) by \(\xi_3 u^2_x\), integrating over \((0, L)\), we get

\[
\xi_3 \int_0^L |u^2_x|^2 dx = \xi_3 \int_0^L u^2 v u^2_x dx - \gamma \int_0^L v_x u^2_x dx,
\]

it yields that

\[
\xi_3 \int_0^L |u^2|^2 dx \leq \xi_3 \int_0^L |u|^3 |u^2| dx + \gamma \int_0^L |v_x| |u^2_x| dx.
\]

Applying Young inequality in (6.12), we get

\[
\xi_3 \int_0^L |u^2|^2 dx \leq \frac{\xi_3}{\xi r} \int_0^L |u|^3 |u^2| dx + \frac{\gamma^2}{\xi r} \int_0^L |v_x|^2 dx + \frac{\gamma}{2} \xi_3 \int_0^L |u^2|^2 dx
\]

By taking \(r = 1\) in the above estimation and using (6.9) and (6.10), we get

\[
\frac{\xi_3}{2} \int_0^L |u^2|^2 dx \leq \frac{\xi_3}{\xi} \int_0^L |u|^3 |u^2| dx + \frac{\gamma^2}{\xi^2} \int_0^L |v_x|^2 dx \leq \left( \frac{\xi_3}{\xi} N_2 + \frac{\gamma^2}{\xi^2} \right) \|U\|_H \|F\|_H.
\]

Thus, we obtain (6.10). The proof is thus completed.

Lemma 6.6. The solution \((v, z, u^1, u^2, u^3) \in D(A_{a,0,c})\) of equation (6.1) satisfies the following estimation:

\[
\mu \int_0^L |u^1|^2 dx \leq N_5 \|U\|_H \|F\|_H,
\]

where \(N_5 = 2 \xi_3 \left( 1 + \frac{\gamma^2}{2 \xi^2} \right) N_2 + \frac{1}{2} N_3 + \frac{1}{2} \sqrt{\xi \xi_\alpha} \).

Proof. By proceeding the same technics used in Lemma 4.5, we get (6.14). The proof has been completed.

Proof of Proposition 6.2. Adding (6.8), (6.9), (6.10), (6.11) and (6.14), we get

\[
\|U\|_H^2 = \alpha \|v_x\|^2 + \rho \|z\|^2 + \mu \|u^1\|^2 + \xi_3 \|u^2\|^2 + \xi_\alpha \|u^3\|^2 \leq N \|U\|_H \|F\|_H,
\]

where \(N = N_2 + \rho N_1 + N_5 + N_4 + \xi_3 N_2\). Then, \(\|U\|_H \leq N \|F\|_H\). The proof has been completed.

Proof of Theorem 6.1 For all \(U \in D(A_{a,0,c})\) according to Proposition (4.2), we get

\[
\|U\|_H \leq K \|(i\lambda I - A_{a,0,c})U\|_H.
\]

Thus, we have

\[
\|(i\lambda I - A_{a,0,c})^{-1} V\|_H \leq K \|V\|_H, \quad \forall V \in H.
\]

Therefore, from the above equation, we get (H5) holds. Thus, we get the conclusion by applying Huang and Prüss Theorem.

7. The stretching of the centerline of the beam in x-direction and electrical field component in x-direction are damped "c = 0 and \((a, b) \neq (0, 0)\)"

The aim of this part is to prove the exponential stability of Lorenz system (Lorenz) with a damping acting on the stretching of the centerline of the beam in x-direction and electrical field component in z-direction. (i.e. \(c = 0\) and \((a, b) \neq (0, 0)\)). The main result of this pat is the following theorem.

Theorem 7.1. The \(C_0\)-semigroup of contractions \((e^{tA_{a,b,0}})_{t \geq 0}\) is exponentially stable; i.e., there exist constants \(M \geq 1\) and \(\epsilon > 0\) independent of \(U_0\) such that

\[
\|e^{tA_{a,b,0}} U_0\|_H \leq M e^{-\epsilon t} \|U_0\|_H.
\]
According to Huang and Prüss, we have to check if the following conditions hold:

(H1) \[ i\mathbb{R} \subset \rho(A_{a,b,0}) \]
and

(H6) \[ \sup_{\lambda \in \mathbb{R}} \| (i\lambda I - A_{a,b,0})^{-1} \|_{\mathcal{L}(\mathcal{H})} = O(1). \]

Condition (H1) is already proved in Theorem 3.1. The next proposition is a technical result to be used in the proof of (H6) given below.

**Proposition 7.2.** Let \( (\lambda, U := (v, z, u^1, u^2, u^3)) \in \mathbb{R}^5 \times D(A_{a,b,0}), \) with \( |\lambda| \geq 1, \) such that

\[ (7.1) \quad (i\lambda I - A_{a,b,0}) U = f := (f^1, f^2, f^3, f^4, f^5) \in \mathcal{H}, \]
detailed as

\[ (7.2) \quad i\lambda v - z = f^1, \]
\[ (7.3) \quad i\lambda z - \frac{\alpha}{\rho} \varepsilon_{xx} \gamma v^3 + \frac{a}{\rho} = f^2, \]
\[ (7.4) \quad i\lambda u^1 - u^2 + u^3 = f^3, \]
\[ (7.5) \quad i\lambda u^2 + \frac{\mu}{\xi_3} u^1 + b u^2 = f^4, \]
\[ (7.6) \quad i\lambda u^3 + \frac{\mu}{\xi_3} u^1 - \frac{\gamma}{\varepsilon_3} z = f^5. \]

Then, we have the following inequality

\[ (7.7) \quad \| U \|_{\mathcal{H}} \leq S \| F \|_{\mathcal{H}}. \]

For the proof of Proposition 7.2, we need the following lemmas.

**Lemma 7.3.** The solution \((v, z, u^1, u^2, u^3) \in D(A_{a,b,0})\) of equation (7.1) satisfies the following estimates:

\[ (7.8) \quad \int_0^L |z|^2 dx \leq S_1 \| U \|_{\mathcal{H}} \| F \|_{\mathcal{H}} \quad \text{where} \quad S_1 = \frac{1}{a}, \]
\[ (7.9) \quad \int_0^L |u^2|^2 dx \leq S_2 \| U \|_{\mathcal{H}} \| F \|_{\mathcal{H}} \quad \text{where} \quad S_2 = \frac{1}{b \xi_3}, \]

**Proof.** By using the same arguments used in Lemma 4.3, we get (7.8)-(7.9). The proof has been completed. \(\square\)

**Lemma 7.4.** The solution \((v, z, u^1, u^2, u^3) \in D(A_{a,b,0})\) of equation (7.1) satisfies the following estimates:

\[ (7.10) \quad \frac{a}{2} \left( \int_0^L |v_x|^2 dx + \int_0^L |u^1|^2 dx \right) \leq S_3 \| U \|_{\mathcal{H}} \| F \|_{\mathcal{H}}, \]

where \( S_3 = 2 \left( \sqrt{\frac{\alpha}{a}} c_p + b^{-1} + \frac{\alpha b^{-1}}{\sqrt{a \varepsilon_3}} + \frac{a c^2}{4a} \right) + S_1. \)

**Proof.** The proof of this Lemma, is divided into several Steps.

**Step 1.** The aim of this step is to prove the following equation

\[ (7.11) \quad \alpha \int_0^L |v_x|^2 dx + \gamma \int_0^L u^3 v_x dx = \rho \int_0^L |z|^2 dx + \rho \int_0^L z f dx + \alpha \int_0^L \int_0^L z v \rho dx + \rho \int_0^L f^2 v dx. \]

For this aim, multiplying (7.3) by \( \rho \tau \), integrating by parts over \((0, L)\), we get

\[ i\lambda \rho \int_0^L z \tau dx + \alpha \int_0^L |v_x|^2 dx + \gamma \int_0^L u^3 v_x dx + \alpha \int_0^L u^3 v x dx + \alpha \int_0^L z \varepsilon_3 \rho dx = \rho \int_0^L f^2 v dx. \]

Using (7.2) in the above equation, we get (7.11).

**Step 2.** The aim of this step is to prove the following equation

\[ (7.12) \quad \varepsilon_3 \int_0^L |u^3|^2 dx - \gamma \int_0^L v_x \overline{w^3} dx = b^{-1} \varepsilon_3 \int_0^L f^2_4 \overline{w^3} dx - b^{-1} \varepsilon_3 \int_0^L f^2_5 \overline{w^3} dx + \gamma b^{-1} \int_0^L f^2_5 \overline{w^3} dx. \]
For this aim, inserting (7.2) in (7.6), we get
\[ i\lambda u^3 + \frac{\mu}{\xi_3} u_z^1 - \frac{\gamma}{\xi_3} i\lambda v_x + \frac{\gamma}{\xi_3} f_z^1 = f^5. \]
Multiplying the above equation by \(-i\lambda^{-1}\xi_3\), integrating over \((0, L)\), we get
\[ \varepsilon_3 \int_0^L |u^3|^2 dx - i\lambda^{-1}\mu \int_0^L u_z^1 \overline{u^3} dx - \gamma \int_0^L v_x \overline{u^3} dx = -i\lambda^{-1}\xi_3 \int_0^L f^5 \overline{u^3} dx. \]
Differentiating (7.5) with respect to \(x\), we obtain
\[ i\lambda u_z^2 + \frac{\mu}{\xi_3} u_z^1 + bu_z^2 = f_z^4. \]
Multiplying the above equation by \(i\lambda^{-1}\xi_3\), integrating over \((0, L)\), we get
\[ -\varepsilon_3 \int_0^L u_z^2 \overline{u^3} dx + i\lambda^{-1}\mu \int_0^L u_z^1 \overline{u^3} dx + \mu \lambda^{-1}\xi_3 \int_0^L u_z^2 \overline{u^3} dx = i\lambda^{-1}\xi_3 \int_0^L f_z^4 \overline{u^3} dx. \]
Using the compatibility condition (2.16) in the first term of (7.14), we get
\[ -\varepsilon_3 \int_0^L |u^3|^2 dx + \gamma \int_0^L v_x \overline{u^3} dx = \int_0^L u_z^2 \overline{u^3} dx. \]
We get
\[ b\varepsilon_3 \int_0^L u_z^2 \overline{u^3} dx = \varepsilon_3 \int_0^L f_z^4 \overline{u^3} dx - \varepsilon_3 \int_0^L f^5 \overline{u^3} dx + \gamma \int_0^L f_z^4 \overline{u^3} dx. \]
Again, using the compatibility condition (2.16) in (7.16), we get (7.12).

**Step 3.** The aim of this step is to prove (7.10). For this aim adding (7.11) and (7.12) and taking the real part, we get
\[ \alpha \int_0^L |v_x|^2 dx + \varepsilon_3 \int_0^L |u^3|^2 dx = \rho \int_0^L |z|^2 dx + \Re \left( \rho \int_0^L z\overline{\sigma} dx \right) - \Re \left( \rho \int_0^L f^2 \overline{\sigma} dx \right) + \Re \left( b^{-1}\varepsilon_3 \int_0^L f_z^4 \overline{u^3} dx \right) - \Re \left( b^{-1}\varepsilon_3 \int_0^L f^5 \overline{u^3} dx \right) + \Re \left( b^{-1}\varepsilon_3 \int_0^L f_z^4 \overline{u^3} dx \right).
\]
It follows that
\[ \alpha \int_0^L |v_x|^2 dx + \varepsilon_3 \int_0^L |u^3|^2 dx \leq \rho \int_0^L |z|^2 dx + \rho \int_0^L |z| |\sigma| |dx| + \rho \int_0^L |f^2| |v| |dx| + \rho \int_0^L |f_z^4| |u^3| |dx| + \rho \int_0^L |f^5| |u^3| |dx| + \rho \int_0^L |f_z^4| |u^3| |dx|.
\]
Using the facts that, \(\sqrt{\rho} |z| \leq ||U||_{H^1}, \sqrt{\varepsilon_3} |u^3| \leq ||U||_{H^1}, \sqrt{\rho} |f_z^4| \leq ||F||_{H^1}, \sqrt{\varepsilon_3} |f^5| \leq ||F||_{H^1} \) and Poincaré inequality, we get
\[ \begin{aligned}
&\rho \int_0^L |z| |f^2| |dx| \leq \sqrt{\frac{\rho}{\alpha}} c_p ||U||_{H^1} ||F||_{H^1}, \\
&b^{-1}\varepsilon_3 \int_0^L |f_z^4| |u^3| |dx| \leq b^{-1} ||U||_{H^1} ||F||_{H^1}, \\
&\gamma b^{-1} \int_0^L |f_z^4| |u^3| |dx| \leq \frac{\gamma b^{-1}}{\sqrt{\alpha\varepsilon_3}} ||U||_{H^1} ||F||_{H^1}, \\
&\rho \int_0^L |f^2| |v| |dx| \leq \sqrt{\frac{\rho}{\alpha}} c_p ||U||_{H^1} ||F||_{H^1}.
\end{aligned}
\]
Applying Cauchy-Schwarz and Young inequality, and using (7.8), we get
\[ a \int_0^L |z| |v| |dx| \leq \frac{a}{2r_1} \int_0^L |z|^2 dx + \frac{ar_1}{2} \int_0^L |v_x|^2 dx \leq \frac{1}{2r_1} ||U||_{H^1} ||F||_{H^1} + \frac{ar_1}{2} \int_0^L |v_x|^2 dx.
\]
Since $F$ in $\mathcal{H}$, then the components of $F$ satisfies the compatibility condition (2.16), then we get
\begin{equation}
(7.20) \\
\xi f^4_x - f^5 + \frac{\gamma}{\varepsilon^3} f^1_x = 0.
\end{equation}
Using (7.18), (7.20), $\sqrt{\varepsilon^3}\|u^3\| \leq \|U\|_{\mathcal{H}}$, $\sqrt{\varepsilon^3}\|f^5\| \leq \|F\|_{\mathcal{H}}$ and $\sqrt{\gamma}\|f^1_x\| \leq \|F\|_{\mathcal{H}}$, we get
\begin{equation}
(7.21) \\
b^{-1}\xi\varepsilon^3 \int_0^L |f_x^4|^2 \, dx \leq b^{-1}\varepsilon^3 \left( \|f^5\| + \frac{\gamma}{\varepsilon^3}\|f_x^1\| \right) \|u^3\| \leq b^{-1} \left( 1 + \frac{\gamma}{\varepsilon^3\alpha} \right) \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.
\end{equation}
Inserting (7.18), (7.19) and (7.21), in (7.17)
\[ \left( \alpha - \frac{ar_1^2\varepsilon^3}{2} \right) \mu \int_0^L |u_x|^2 \, dx + \varepsilon_3 \left( \int_0^L |u|^2 \, dx \right) \leq 2 \left( \sqrt{\frac{\rho}{4\alpha}} + b^{-1} \sqrt{\frac{\gamma}{\varepsilon^3\alpha}} + \frac{1}{4r_1^2} \right) S_1 \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \]
Taking $r_1 = \frac{\alpha}{ar_0^2}$ in the above inequality, we get (7.10). The proof has been completed.

Lemma 7.5. The solution $(v, z, u^1, u^2, u^3) \in D(A_{a,b,0})$ of equation (7.1) satisfies the following estimation:
\begin{equation}
(7.22) \\
\mu \int_0^L |u_x|^2 \, dx \leq S_4 \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}},
\end{equation}
where $S_4 = \xi\varepsilon^3 \left( 1 + \frac{b^2\varepsilon^3}{2\mu} \right) S_2 + \left( \frac{1}{2} + \frac{\gamma^2}{4\mu\varepsilon^3} \right) \varepsilon_3^{-1} + \frac{2}{\sqrt{\varepsilon^3\mu}} S_3 + \frac{2}{\sqrt{\varepsilon^3\mu}}$.

\textbf{Proof.} By proceeding the same technics used in Lemma 4.5, we get (7.22). The proof has been completed.

\textbf{Proof of Proposition 7.2.} From (7.8), (7.9), (7.10) and (7.22), we get
\[ \|U\|^2_{\mathcal{H}} = \alpha \|v_x\|^2 + \rho \|z\|^2 + \mu \|u_1\|^2 + \xi\varepsilon^3 \|u^2\|^2 + \varepsilon_3 \|u^3\|^2 \leq S_4 \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}, \]
where $S = \rho S_1 + \xi\varepsilon^3 S_2 + 3S_3 + S_4$. The proof has been completed.

\textbf{Proof of Theorem 7.1} For all $U \in D(A_{a,b,0})$ according to Proposition (7.2), we get
\[ \|U\|_{\mathcal{H}} \leq S \|(i\lambda I - A_{a,0,c}) U\|_{\mathcal{H}}. \]
Thus, we have
\[ \|(i\lambda I - A_{a,b,0})^{-1} V\|_{\mathcal{H}} \leq S \|V\|_{\mathcal{H}}, \quad \forall V \in \mathcal{H}. \]
Therefore, from the above equation, we get (H6) holds. Thus, we get the conclusion by applying Huang and Prüss Theorem.

8. **The stretching of the centreline of the beam in x-direction only is damped and \(a \neq 0\) and \((b, c) = (0, 0)\)**

In this section, we prove that the Lorenz system with only one damping acting on the stretching of the centreline still be exponentially stable. The main result of this section is the following theorem:

\textbf{Theorem 8.1.} Assume that $a \neq 0$ and $(b, c) = (0, 0)$. Then, the $C_0-$semigroup of contraction $e^{tA}$ is exponentially stable; i.e. there exists constants $M \geq 1$ and $\epsilon > 0$ independent of $U_0$ such that
\[ \|e^{tA_{a,b,0}} U_0\|_{\mathcal{H}} \leq Me^{-\epsilon t}\|U_0\|_{\mathcal{H}}. \]

According to Huang and Prüss, we have to check if the following conditions hold:
\begin{enumerate}
\item[(H1)] $i\mathbb{R} \subset \rho(A_{a,0,0})$
\item[(H7)] $\sup_{\lambda \in \mathbb{R}} \|(i\lambda I - A_{a,0,0})^{-1}\|_{\mathcal{L}(\mathcal{H})} = O(1)$.
\end{enumerate}

Since $i\mathbb{R} \subset \rho(A_{a,0,0})$, then condition (H1) is satisfied. We will prove condition (H7) by a contradiction argument. For this purpose, suppose that (H7) is false, then there exists \(\{\lambda^n, U^n\}_{n \geq 1} \subset \mathbb{R}^* \times D(A)\) with
\begin{enumerate}
\item[(8.1)] $|\lambda^n| \to \infty$ and $\|U^n\|_{\mathcal{H}} = \|(v_n, z_n, u^1_n, u^2_n, u^3_n)^T\|_{\mathcal{H}} = 1$,
\item[(8.2)] $(i\lambda^n I - A_{a,0,0}) U^n = F^n := (f^n_1, f^n_2, f^n_3, f^n_4, f^n_5) \to 0$ in $\mathcal{H}$.
\end{enumerate}
For simplicity, we drop the index \( n \). Equivalently, from (8.2), we have \( \lambda p = f_1 \), \( \lambda z - \frac{\alpha}{\rho} u_x - \frac{\gamma}{\rho} u_x^3 + \frac{\alpha}{\rho} z = f_2 \), \( \lambda u^1 - u^2 + u^3 = f_3 \), \( \lambda u^2 + \frac{\mu}{\xi} u^1 = f_4 \), \( \lambda u^3 + \frac{\mu}{\xi} u_{x}^1 - \frac{\gamma}{\xi} u_{x} = f_5 \).

Lemma 8.2. The solution \((v, z, u^1, u^2, u^3) \in D(A_{a,0,0})\) of equation (8.2) satisfies the following estimates:

\[ \int_0^L |v|^2 dx = o(1) \quad \text{and} \quad \int_0^L |\lambda v|^2 dx = o(1). \]

Proof. First, taking the inner product of (8.2) with \( U \) in \( \mathcal{H} \), we obtain

\[ a \int_0^L |v|^2 dx = -\Re(A_{a,0,0}U, F)_{\mathcal{H}} = \Re(F, U)_{\mathcal{H}} \leq ||U||_{\mathcal{H}} ||F||_{\mathcal{H}}. \]

Thus, from the above estimation and the fact that \( ||F||_{\mathcal{H}} = o(1) \) and \( ||U||_{\mathcal{H}} = 1 \), we obtain the first estimation in (8.8). From (8.3), we deduce that

\[ \int_0^L |\lambda v|^2 dx \leq \int_0^L |v|^2 dx + \int_0^L |f_1|^2 dx. \]

Finally, from (8.10), the first estimation in (8.8), we get the second estimation in (8.8). The proof is thus complete.

Lemma 8.3. The solution \((v, z, u^1, u^2, u^3) \in D(A_{a,0,0})\) of equation (8.2) satisfies the following estimates:

\[ \int_0^L |v|^2 dx = o(1). \]

Proof. Multiplying (8.4) by \( \rho v \) integrating by parts over \((0, L)\), we get

\[ i\lambda \rho \int_0^L z v dx + \alpha \int_0^L |v|^2 dx + \gamma \int_0^L u^3 \bar{v} dx + a \int_0^L z v dx = \rho \int_0^L f^2 \bar{v} dx. \]

Using Lemma 8.2 and the fact that \( ||f^2||_{L^2(0,L)} = o(1) \), we get

\[ i\lambda \rho \int_0^L z v dx = o(1), \quad a \int_0^L z v dx = o(|\lambda|^{-1}) \quad \text{and} \quad \rho \int_0^L f^2 \bar{v} dx = o(|\lambda|^{-1}). \]

Inserting (8.13) in (8.12), we get

\[ \alpha \int_0^L |v|^2 dx + \gamma \int_0^L u^3 \bar{v} dx = o(1). \]

Now, multiplying (8.5) by \( \gamma \bar{v} \) integrating by parts over \((0, L)\), we get

\[ i\lambda \gamma \int_0^L u^1 \bar{v} dx - \gamma \int_0^L u^2 \bar{v} dx - \gamma \int_0^L u^3 \bar{v} dx + \gamma u^3(L) \bar{v}(L) = \gamma \int_0^L f^3 \bar{v} dx. \]

Using the facts that \( u^1, u^2 \) are uniformly bounded in \( L^2(0,L) \), equation (8.8) and the fact that \( ||f^3||_{\mathcal{H}^1(0,L)} = o(1) \), we get

\[ i\lambda \gamma \int_0^L u^1 \bar{v} dx = o(1), \quad \gamma \int_0^L u^2 \bar{v} dx = o(|\lambda|^{-1}) \quad \text{and} \quad \gamma \int_0^L f^3 \bar{v} dx = o(|\lambda|^{-1}). \]
From (8.5), it is easy to see that \( |u^3_{|L^2(0, L)} = O(|\lambda|) \). Using Galgianxo-Niremb inequality, \( \|u^3\| \) and \( \|v_x\| \) are uniformly bounded in \( L^2(0, L) \) and (8.8), we get

\[
\begin{align*}
|u^3(L)| \lesssim \|u_x^2\|^2 |u^3|^{\frac{1}{2}} + \|u^3\| & \lesssim O(|\lambda|^2) , \\
|v(L)| \lesssim \|v_x\|^2 |v|^{\frac{1}{2}} + \|v\| & \lesssim O(|\lambda|^2) .
\end{align*}
\]

Inserting (8.16) and (8.17) in (8.15), we get

\[
\begin{align*}
\left| \int_0^L u^3v_x dx \right| = o(1).
\end{align*}
\]

Inserting the above estimation in (8.14), we get (8.11). The proof has thus been completed. \( \square \)

**Lemma 8.4.** The solution \((v, z, u^1, u^2, u^3) \in D(A_{\alpha,0,0}) \) of equation (8.2) satisfies the following estimates:

\[
\begin{align*}
\int_0^L |u^2|^2 dx = o(1) \quad \text{and} \quad \int_0^L |u|^2 dx = o(1) .
\end{align*}
\]

**Proof.** Multiplying (8.4) by \( \bar{u}^2 \), integrating by parts over \((0, L) \), we get

\[
\begin{align*}
i\lambda \int_0^L \bar{u}^2 dx + \alpha \int_0^L v_x \bar{u}^2 dx + \frac{\gamma}{\rho} \int_0^L u^3 \bar{u}^2 dx + \frac{\alpha}{\rho} \int_0^L \bar{u}^2 dx = \int_0^L \bar{f} dx .
\end{align*}
\]

From the compatibibility condition, we have

\[
u^3 = \xi u^2_x + \gamma \xi v_x .
\]

Inserting the above equation in (8.19), we get

\[
\begin{align*}
i\lambda \int_0^L \bar{u}^2 dx + \left( \alpha \rho + \gamma \xi \bar{\xi} \right) \int_0^L v_x \bar{u}^2 dx + \frac{\gamma}{\rho} \int_0^L |u^3|^2 dx + \frac{\alpha}{\rho} \int_0^L \bar{u}^2 dx = \int_0^L \bar{f} dx .
\end{align*}
\]

Using the facts that \( u^2_x \) and \( u^2 \) are uniformly bounded in \( L^2(0, L) \), (8.11) and \( \|\bar{f}\|_{L^2(0, L)} = o(1) \), we get

\[
\begin{align*}
\left| \int_0^L v_x \bar{u}^2 dx \right| = o(1) , \quad \left| \int_0^L \bar{u}^2 dx \right| = o(1) \quad \text{and} \quad \left| \int_0^L \bar{f} dx \right| = o(1) .
\end{align*}
\]

Inserting the above estimations in (8.20), we get

\[
i\lambda \int_0^L \bar{u}^2 dx + \frac{\gamma}{\rho} \int_0^L |u^3|^2 dx = o(1) .
\]

Multiplying (8.6) by \( \bar{u} \), integrating over \((0, L) \), using the fact that \( u^1 \) is uniformly bounded in \( L^2(0, L) \), \( \|\bar{f}\|_{L^2(0, L)} = o(1) \) and (8.8), we get we get

\[
i\lambda \int_0^L u^2 dx + \frac{\mu}{\xi_3} \int_0^L u^4 dx = \left[ \int_0^L \bar{f} dx \right]_{o(1)} .
\]

Inserting the above estimation in (8.21), we get the first estimation in (8.18). Using the first estimation in (8.18) and Poincaré inequality, we get the second estimation in (8.18). The proof has been completed. \( \square \)

**Lemma 8.5.** The solution \((v, z, u^1, u^2, u^3) \in D(A_{\alpha,0,0}) \) of equation (8.2) satisfies the following estimates:

\[
\begin{align*}
\int_0^L |u|^2 dx = o(1) \quad \text{and} \quad \int_0^L |u^1|^2 dx = o(1) .
\end{align*}
\]

**Proof.** First, we prove the first estimation in (8.22). For this aim, Using the compatibility condition (2.16), (8.18) and (8.11), we get

\[
\begin{align*}
\int_0^L |u|^2 dx \leq 2\xi^2 \int_0^L |u^2|^2 dx + 2\frac{\alpha}{\rho} \int_0^L |v_x|^2 dx \equiv o(1) .
\end{align*}
\]
Now, we prove the second estimation in (8.22). For this aim, multiplying (8.6) by $u^1$, integrating over $(0, L)$ and using the facts that $u^1$ is uniformly bounded in $L^2(0,L)$ and $\|f^4\|_{L^2(0,L)} = o(1)$, we get

\begin{equation}
(8.24) \quad i\lambda \int_0^L u^2 u^1 dx + \mu \xi \int_0^L |u^1|^2 dx = o(1).
\end{equation}

Now, multiplying (8.5) by $u^2$, integrating by parts over $(0, L)$ and using the fact that $\|f^3\|_{L^2(0,L)} = o(1)$, (8.18) and the first estimation in (8.22), we get

\begin{equation}
(8.25) \quad i\lambda \int_0^L u^1 u^2 dx - \int_0^L u^1 u^2 dx = o(1).
\end{equation}

Inserting (8.21) in (8.20), we get the second estimation in (8.18). The proof has been completed.

**Proof of Theorem 8.1.** From Lemmas 8.2-(8.5), we obtain $\|U\|_{H} = o(1)$, which contradicts (8.1). This implies that

$$
\sup_{\lambda \in \mathbb{R}} \| (i\lambda I - A_{a,0,0})^{-1} \|_{H} < \infty.
$$

Finally, according to Huang-Pruss theorem, we obtain the desired result. The proof is thus complete.

### 9. Numerical Results

In this section, we will numerically illustrate the exponential decay of the natural energy $E(t)$ associated to (Lorenz) system. To carry out the numerical simulations, we first re-write the second-order Lorenz system in a first-order form in time and then we discretize the resulted system using a second-order centered finite difference approximation for space and the second-order implicit backward differentiation formula for time.

The computational domain considered is $[0, 1]$ and the time interval is $[0, 100]$. For simplicity all the parameter in Lorenz system are set to one. The following initial conditions are used:

$$
(v, \phi, \theta, \eta)(\cdot, 0) = (10^{-2} \sin 3\pi x, \cos \pi x, \sin \pi x, \pi \cos \pi x)
$$

$$
(v_t, \phi_t, \theta_t, \eta_t)(\cdot, 0) = (10^2 \sin 3\pi x, 0, 0, 0).
$$

Our results are presented in Figures 1, 2 and 3. First, in figure 1 we show $v_t, v_x, \theta + \phi_x, \eta_t + \phi, \theta - \eta_x$ as well as the natural energy $E(t)$ in the case where $a = 0, b = 0$ and $c = 0$. The conservation of the natural energy is clearly shown in this case. Then, we consider the following six cases:

**Case1:** $(a, b, c) = (1, 1, 1)$.

**Case2:** $a = 0$ and $(b, c) = (1, 1)$.

**Case3:** $b = 0$ and $(a, c) = (1, 1)$.

**Case4:** $c = 0$ and $(a, b) = (1, 1)$.

**Case5:** $a = 1$ and $(b, c) = (0, 0)$.

**Case6:** $c = 1$ and $(a, b) = (0, 0)$.

The results for case 5, where $a = 1, b = 0$ and $c = 0$, is presented in figure 2. As can be seen, we obtained an exponential decay of the numerical solutions $v_t, v_x, \theta + \phi_x, \eta_t + \phi, \theta - \eta_x$ as well as the natural energy $E(t)$. This is consistent with our theoretical results. For all the above mentioned cases, we obtained numerical results similar to figure 2 showing an exponential decay of the solutions as expected by our theoretical results. The figures are not presented here to avoid repetition. However, we present the natural energy for all case in figure 3.
Figure 1. Space and time numerical solutions when $a = 0$, $b = 0$ and $c = 0$.

Figure 2. Space and time numerical solutions when $a = 1$, $b = 0$ and $c = 0$ (Case 5).
Figure 3. The natural energy $E(t)$ for all cases.

10. Conclusion

In this paper, we investigate the exponential stability of a Lorenz Piezoelectric beam with partial viscous damping. Different cases have been studied. We remark that it sufficient to control the stretching of the centreline of the beam in $x$–direction to achieve the exponential stability. The case where $b \neq 0$ and $(a, c) = (0, 0)$ is still an open problem. However, based on our numerical results, we remark that we do not obtain the exponential stability in the case where $b = 1$ and $(a, c) = (0, 0)$ (See Figure 4).

Figure 4. Space and time numerical solutions when $a = 0$, $b = 1$ and $c = 0$ (open problem).
[1] M. Afilal, A. Soufyane, and M. de Lima Santos. Piezoelectric beams with magnetic effect and localized damping. *Mathematical Control & Related Fields, 0:* , 2021.
[2] M. Akil. Stability of piezoelectric beam with magnetic effect under (coleman or pipkin)-gurtin thermal law, 2022.
[3] Y. An, W. Liu, and A. Kong. Stability of piezoelectric beams with magnetic effects of fractional derivative type and with/without thermal effects. 2021.
[4] W. Arendt and C. J. K. Batty. Tauberian theorems and stability of one-parameter semigroups. *Trans. Amer. Math. Soc.,* 306(2):837–852, 1988.
[5] H. T. Banks, R. C. Smith, and Y. Wang. Smart material structures: Modeling, estimation, and control. 1996.
[6] P. Destuynder, I. Legrain, L. Castel, and N. Richard. Theoretical, numerical and experimental discussion on the use of piezoelectric devices for control-structure interaction. *European Journal of Mechanics A-solids,* 11:181–213, 1992.
[7] S. Hansen. Analysis of a plate with a localized piezoelectric patch. In *Proceedings of the 37th IEEE Conference on Decision and Control (Cat. No.98CH36171)*, volume 3, pages 2952–2957 vol.3, 1998.
[8] F. L. Huang. Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces. *Ann. Differential Equations,* 1(1):43–56, 1985.
[9] B. Kapitonov, B. Miara, and G. P. Menzala. Boundary observation and exact control of a quasi-electrostatic piezoelectric system in multilayered media. *SIAM Journal on Control and Optimization,* 46(3):1080–1097, 2007.
[10] I. Lasiecka and B. Miara. Exact controllability of a 3d piezoelectric body. *Comptes Rendus Mathematique,* 347(3):167–172, 2009.
[11] J. L. Lions. Exact controllability, stabilization and perturbations for distributed systems. *SIAM Review,* 30(1):1–68, 1988.
[12] K. Morris and A. Özer. Strong stabilization of piezoelectric beams with magnetic effects. pages 3014–3019, 2013.
[13] K. A. Morris and A. Ö. Özer. Modeling and stabilizability of voltage-activated piezoelectric beams with magnetic effects. *SIAM J. Control. Optim.,* 52:2371–2398, 2014.
[14] T. D. Oh, S.J. Local well-posedness of the (4 + 1)-dimensional maxwell–klein–gordon equation at energy regularity. *Ann. PDE* 2, 2, 2016.
[15] A. Ö. Özer. Potential formulation for charge or current-controlled piezoelectric smart composites and stabilization results: Electrostatic versus quasi-static versus fully-dynamic approaches. *IEEE Transactions on Automatic Control,* 64:989–1002, 2019.
[16] Özer, Ahmet Özkan. Stabilization results for well-posed potential formulations of a current-controlled piezoelectric beam and their approximations. *Applied Mathematics & Optimization,* 84:877–914, 2021.
[17] Özer, Ahmet Özkan and Morris, Kirsten A. Modeling and stabilization of current-controlled piezo-electric beams with dynamic electromagnetic field. *ESAIM: COCV,* 26:8, 2020.
[18] J. Prüss. On the spectrum of C0-semigroups. *Trans. Amer. Math. Soc.,* 284(2):847–857, 1984.
[19] A. J. A. Ramos, M. M. Freitas, D. S. Almeida, S. S. Jesus, and T. R. S. Moura. Equivalence between exponential stabilization and boundary observability for piezoelectric beams with magnetic effect. *Zeitschrift für angwandte Mathematic und Physik,* 70(2):60, Mar 2019.
[20] Ramos, Anderson J.A., Gonçalves, Cledson S.L., and Corrêa Neto, Silvério S. Exponential stability and numerical treatment for piezoelectric beams with magnetic effect. *ESAIM: M2AN,* 52(1):255–274, 2018.
[21] N. Rogacheva. *The Theory of Piezoelectric Shells and Plates.* CRC Press. Cambridge University Press, (1st ed.) edition, 1994.
[22] S. Selberg and A. Tesfahun. Finite-energy global well-posedness of the maxwell–klein–gordon system in lorentz gauge. *Communications in Partial Differential Equations,* 35(6):1029–1057, 2010.
[23] R. Smith. *Smart Material Systems.* Frontiers in Applied Mathematics. Society for Industrial and Applied Mathematics, 1 edition, 2005.
[24] A. Soufyane, M. Afilal, and M. L. Santos. Energy decay for a weakly nonlinear damped piezoelectric beams with magnetic effects and a nonlinear delay term. *Zeitschrift für angwandte Mathematic und Physik,* 72(4), Aug. 2021.
[25] L. T. Tebou and E. Zuazua. Uniform boundary stabilization of the finite difference space discretization of the 1–d wave equation. *Advances in Computational Mathematics,* 26(1):337, Dec 2006.
[26] H. Tiersten. *Linear Piezoelectric Plate Vibrations.* CRC Press. Springer New York, NY, 1 edition, 1969.
[27] H. Tzou. *Piezoelectric Shells: Sensing, Energy Harvesting, and Distributed Control.* Solid Mechanics and its Applications. Springer Dordrecht, second edition edition, 2019.
[28] J. Yang. *An Introduction to the Theory of Piezoelectricity.* Advances in Mechanics and MAthematics. Springer New York, NY, 1 edition, 2005.