Research Article

Option Pricing for Path-Dependent Options with Assets Exposed to Multiple Defaults Risk

Taoshun He

1Numerical Simulation Key Laboratory of Sichuan Province, College of Mathematics and Information Science, Neijiang Normal University, Neijiang 641110, China
2Data Recovery Key Laboratory of Sichuan Province, College of Mathematics and Information Science, Neijiang Normal University, Neijiang 641100, China

Correspondence should be addressed to Taoshun He; hstled@163.com

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1. Introduction

Barrier and lookback options are among the most popular path-dependent derivatives traded in exchanges and over-the-counter markets worldwide. The barrier option is a financial derivative contract that is activated or deactivated when the price of the underlying asset crosses a certain level (called barrier) from above or below. And the standard floating lookback call (put) option confers the holder the right to buy (sell) an asset at its lowest (highest) price during the life of the contract. For a complete description of these and other related contracts, refer to Hull [1]. There has been extensive research in option pricing for path-dependent option. For example, Merton [2] and Goldman et al. [3] derive closed-form solutions for barrier and lookback options in the standard Black-Scholes model. Davydov and Linetsky [4] derive analytical formulas for the prices of path-dependent options, such as barrier and lookback options, with the asset price process under Constant Elasticity of Variance (CEV) diffusion model. Kou et al. [5] present analytical solutions for two-dimensional Laplace transforms of barrier option prices under a double exponential jump diffusion model.

Multiple defaults risks include exogenous counterparty default risk and endogenous default risk, where the exogenous counterparty default risk may not be unique. However, for simplicity, we assume that the exogenous counterparty default risk is sole in the following parts. In the financial market, an exogenous counterparty default usually has important influences in various contexts. In terms of credit spreads, one observes, in general, a positive jump of the default intensity which is called the contagious jump (see [6]). According to asset values for a firm, the default of a counterparty will in general induce a drop of its value process (refer to [7]). Jiao et al. analyze the impact of the single exogenous counterparty risk and the multiple exogenous counterparty risk on the optimal investment problem, we can refer to [8, 9] for more detail. In this paper, we study the impact of the multiple defaults risk on option pricing problem. In particular, we focus on the pricing of path-dependent option with the underlying asset subject to multiple defaults risk such that the instantaneous loss of the
asset at the exogenous counterparty default time and the asset price instantaneously become zero at endogenous default time.

Ma et al., in [10], obtain that the explicit valuation of European options with the asset exposed to exogenous counterparty default risk. Yan derives analytical formulas for lookback and barrier options on underlying assets that are subject to an exogenous counterparty risk in [11]. The explicit pricing formulas for European option with asset exposed to multiple defaults risk is given by He [12]. However, to the best of our knowledge, the derivation of the analytic formula for pricing barrier and lookback options under the multiple defaults risk model has not been performed in the previous literature. The main difficulty lies in that the distribution of the first passage time is difficult to derive owing to the multiple defaults and the continuous trading of the underlying asset after the exogenous counterparty default time. We use the conditional density approach of default, which is particularly suitable to study what goes on after the default and was adopted by Jiao and Pham [8] for the optimal investment problem, to derive the explicit distribution of the first passage time and then obtain the analytic formulas for valuation of the barrier and lookback options. We also compare the pricing results of the multiple defaults risk model with Merton’s [2] default-free option model and Yan’s [11] exogenous counterparty default option model.

The organization of the rest of this paper is as follows. First, a financial model is introduced in Section 2. Next, an analytical formula for barrier option on underlying asset that can be exposed to multiple defaults risk is derived in Section 3. Then, in Section 4 we derive the formula for pricing lookback options under this model. Finally, we conclude the paper in final Section.

2. Financial Model

In this section, we consider a financial market model with a risk asset (stock) subject to multiple defaults risks. We denote the stock by \((S_t)_{t \in [0,T]}\) the dynamic of the stock is affected by not only the possibility of the exogenous counterparty default but also the possibility of the endogenous default. However, this stock still exists and can be traded after the exogenous counterparty default.

Assume \((\Omega, \mathcal{F}, \mathbb{P})\) is a complete probability space satisfying the usual conditions. Let \((W_t)_{t \in [0,T]}\) be a Brownian motion with horizon \(T < \infty\) on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and denote by \(F = (\mathcal{F}_t)_{t \in [0,T]}\) the natural filtration of \(W\). Let \(t_1\) and \(t_2\) be both, almost surely, nonrandom real variables on \((\Omega, \mathcal{F}, \mathbb{P})\), representing the stock of the exogenous counterparty default time and the endogenous default time, respectively. Then, \((\mathcal{F}_t)_{t \in [0,T]}\) is defined by \(\mathcal{F}_t = \sigma(H^1_u, u \leq t)\), where \(H^1_t = 1_{\{t_1 \leq t\}}\), which equals 0 if \(t_1 > t\) and 1 otherwise. Similar, \((\mathcal{H}^2_t)_{t \in [0,T]}\) is defined by \(\mathcal{H}^2_t = \sigma(H^2_u, u \leq t)\), where \(H^2_t = 1_{\{t_2 \leq t\}}\). Denote \(\mathcal{H}^1 = (\mathcal{H}^1_t)_{t \in [0,T]}\) and \(\mathcal{H}^2 = (\mathcal{H}^2_t)_{t \in [0,T]}\), then the progressively enlarged filtration \(G = (\mathcal{F}_t)_{t \in [0,T]} = F \vee \mathcal{H}^1 \vee \mathcal{H}^2\), representing the structure of information available for the investors over \([0, T]\). The market model is given by the following stochastic differential equation (SDE):

\[
\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t - \gamma_t dH^1_t - dH^2_t, \quad 0 \leq t \leq T, \tag{1}
\]

where \(\mu_t, \sigma_t, \) and \(\gamma_t\) are \(G\)-predictable processes, \(\mu_t\) and \(\sigma_t\) are the drift rate and volatility rate of the stock \(S\), respectively, and \(\gamma_t\) is the (percentage) loss on the stock price induced by the defaults of the counterparty. At default time \(t_1\), the stock price \(S\) falls to zero at default time \(t_2\).

Let us define the following (mutually exclusive and exhaustive) events ordering the default times:

\[
A = T < t_1 \leq t_2, \\
B = t_1 < t_2 \leq t_1, \\
C = t_2 \leq t_1 < T, \\
D = t_2 < T < t_1, \\
E = t_1 < T < t_2, \\
F = t_1 \leq t_2 < T. 
\tag{2}
\]

Then, according to Pham [13], the dynamic of stock price process (1) can be decomposed to the following four situations under physical measure:

**Situation 1:** if the stock is in absence of any default in the life of the option, i.e., the default times satisfy \(A \cup B\), then we have

\[
dS^F_t = S_t^F(\mu_t^F dt + \sigma_t^F dW_t^F), \quad 0 \leq t \leq T. \tag{3}
\]

**Situation 2:** if the default times satisfy \(C \cup D\), then we have

\[
dS^F_t = S_t^F(\mu_t^F dt + \sigma_t^F dW_t^F), \quad 0 \leq t < t_2, \\
S_{t_2}^F(t_2) = 0, \quad t_2 \leq t \leq T. \tag{4}
\]

**Situation 3:** if the stock has only exogenous counterparty default in the life of the option, i.e., the default times satisfy \(E\), then we obtain

\[
dS^F_t = S_t^F(\mu_t^F dt + \sigma_t^F dW_t^F), \quad 0 \leq t < t_1, \\
dS_{t_1}^F(t_1) = S_{t_1}^F(t_1)(\mu_t^F(t_1) dt + \sigma_t^F(t_1) dW_t^F), \quad t_1 < t \leq T, \\
S_{t_1}^F(t_1) = S_{t_1}^F(t_1)^{1 - \gamma_t^F}. \tag{5}
\]

**Situation 4:** if the stock has both endogenous default and exogenous counterparty default in the life of the option and the exogenous default time is early than the endogenous default time, i.e., the default times satisfy \(F\), then we obtain

\[
dS^F_t = S_t^F(\mu_t^F dt + \sigma_t^F dW_t^F), \quad 0 \leq t < t_1, \\
dS_{t_1}^F(t_1) = S_{t_1}^F(t_1)(\mu_t^F(t_1) dt + \sigma_t^F(t_1) dW_t^F), \quad t_1 < t < t_2, \\
S_{t_1}^F(t_2) = 0, \quad t_2 \leq t \leq T, \\
S_{t_1}^F(t_1) = S_{t_1}^F(t_1)^{1 - \gamma_t^F}. \tag{6}
\]
where $\mu^F_t, \sigma^F_t, \gamma^F_t$, and $S^F_t$ are $\mathbb{F}$-adapted process and $\mu^d_t(\tau_1), \sigma^d_t(\tau_1)$, and $S^d_t(\tau_1)$ are $\mathcal{F}_t \otimes \mathcal{B}([0, t))$-measurable functions for all $t \in [0, T]$. When the counterparty default, the drift, and volatility coefficient ($\mu, \sigma$) of the stock price switch from $(\mu^F_t, \sigma^F_t)$ to $(\mu^d_t(\tau_1), \sigma^d_t(\tau_1))$, the after default coefficients may depend on the default time $\tau_1$. However, when the stock itself default, the drift and diffusion coefficient ($\mu, \sigma$) of the stock price switch from $(\mu^d_t, \sigma^d_t)$ to $(0, 0)$ due to the stock price identically vanishing. Here, for simplicity we assume that

$$
\begin{align*}
\mu^F_t &= \mu_1, \\
\sigma^F_t &= \sigma_1, \\
\mu^d_t(\tau_1) &= \mu_2, \\
\sigma^d_t(\tau_1) &= \sigma_2,
\end{align*}
$$

with $\mu_1, \sigma_1, \mu_2,$ and $\sigma_2$ are nonnegative constants and the distribution of $\gamma$ ($\gamma < 1$) fixed. Moreover, $\gamma, \tau_1, \tau_2,$ and $W_t$ are independent and $\tau_1$ and $\tau_2$ are all the exponential variables with parameter $\lambda_1$ and $\lambda_2$, respectively. For more details refer to Jiao and Pham [8].

Assume that $r$ is a risk-free interest rate and denote $n = E(\gamma)$. Let us define the $\mathbb{G}$-adapted process:

$$
\theta_t = \frac{\mu_t - r - \lambda_1 n 1_{[t \leq \tau_1]} - \lambda_2 1_{[t \leq \tau_2]}}{\sigma_t},
$$

By assuming $E[\int_0^T (1/2) |\theta_t|^2 dt] < \infty$, we define a probability measure $\mathbb{Q}$ which is equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F})$ with Radon–Nikodym density:

$$
\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{Z}^G_T = \exp\left\{-\int_0^T \theta_t dW_t - \frac{1}{2} \int_0^T \sigma_t^2 dt\right\},
$$

under which, by Girsanov’s theorem, $\tilde{W}_t = W_t + \int_0^t \theta_s ds$ is a $(\mathbb{Q}, \mathcal{F})$-Brownian motion. And thus we can rewrite (1) as follows:

$$
\begin{align*}
\frac{dS_t}{S_t} &= \left[\left(r + \lambda_1 n 1_{[t \leq \tau_1]} + \lambda_2 1_{[t \leq \tau_2]}\right) dt + \sigma_t d\tilde{W}_t \\
&\quad - \gamma_t dH_t^1 - dH_t^2\right], \\
0 \leq t &\leq T,
\end{align*}
$$

that is, by changing measure, the four situations to decompose of the stock price $S_t$ under the physical measure $\mathbb{P}$ can be transformed into the corresponding following four forms under the equivalent martingale measure $\mathbb{Q}$:

**Situation I:** if the stock is in absence of any default in the life of the option, i.e., the default times satisfy $A \cup B$, then we have

$$
\frac{dS^F_t}{S_t} = \left[\left(r + \lambda_1 n + \lambda_2\right) dt + \sigma_t d\tilde{W}_t\right], \\
0 \leq t &\leq T.
$$

**Situation II:** if the default times satisfy $C \cup D$, then we obtain

$$
\frac{dS^F_t}{S_t} = \left[\left(r + \lambda_1 n + \lambda_2\right) dt + \sigma_t d\tilde{W}_t\right], \\
0 \leq t &\leq T.
$$

**Situation III:** if the stock has only exogenous counterparty default in the life of the option, i.e., the default times satisfy $E$, then we have

$$
\begin{align*}
\frac{dS^F_t}{S_t} &= \left[\left(r + \lambda_1 n + \lambda_2\right) dt + \sigma_t d\tilde{W}_t\right], \\
0 \leq t &\leq \tau_1, \\
\frac{dS^F_t}{S_t} &= \left[\left(r + \lambda_2\right) dt + \sigma_2 d\tilde{W}_t\right], \\
\tau_1 &< t \leq T.
\end{align*}
$$

**Situation IV:** if the stock has both endogenous default and exogenous counterparty default in the life of the option and the exogenous default time is early than the endogenous default time, i.e., the default times satisfy $F$, then we obtain

$$
\begin{align*}
\frac{dS^F_t}{S_t} &= \left[\left(r + \lambda_1 n + \lambda_2\right) dt + \sigma_1 d\tilde{W}_t\right], \\
0 \leq t &\leq \tau_1, \\
\frac{dS^F_t}{S_t} &= \left[\left(r + \lambda_2\right) dt + \sigma_2 d\tilde{W}_t\right], \\
\tau_1 &< t \leq \tau_2, \\
\frac{dS^F_t}{S_t} &= \left(1 - \gamma\right), \\
\tau_2 &< t \leq T.
\end{align*}
$$

In practice, we may assume $\gamma$ is a discrete random variable to simplify the computation; in what follows, we further assume that $\gamma$ takes value $\gamma_i$ with probability $p_i$ for $i = 1, 2, 3$, where $0 < \gamma_1 < 1$ (loss), $\gamma_2 = 0$ (no change), and $\gamma_3 < 0$ (gain).

### 3. Analytic Formula for Pricing Barrier Options

In this section we derive an analytic formula for pricing barrier options under the model (1). The barrier options include up-and-out, up-and-in, down-and-out, and down-and-in puts and calls. Since the approaches for deriving the formulas for pricing these kinds of barrier options are similar, we only study the up-and-out barrier call in this section.

Consider an up-and-out barrier call option expiring at time $T$, with strike price $K$ and barrier level $B$. We assume that $K < B$ and denote the maximum of the stock price up to time $T$ by

$$
Y_T = \max_{0 \leq t \leq T} S_t.
$$

Then, the option knocks out (i.e., payoff equals to 0) if and only if $Y_T > B$, on the other hand, the option pay off is $(S_T - K)^+$ when $Y_T \leq B$. In other words, the payoff of the option is

$$
V_T^{\text{uo}}(S_T) = (S_T - K)^+ 1_{\{Y_T \leq B\}} = (S_T - K)^+ 1_{\{S_T \leq K, Y_T \leq B\}}.
$$
Thus, the risk-neutral price of the up-and-out barrier call option at initial time is
\[
V_0^{uo} = \mathbb{E}_0 \left[ e^{-rT} (S_T - K) 1_{\{S_T \leq B\}} \right].
\] (17)

According to compute (17), we obtain the risk-neutral price of the up-and-out barrier call option at time 0 under multiple defaults risks model as follows.

\[
V_0^{uo} = e^{-(\lambda_1 + \lambda_2)T} V_0^{uo} + e^{-(\lambda_1 + \lambda_2)T} \sum_{i=1}^{\infty} P_i \int_0^T e^{-(\lambda_1 + \lambda_2)\tau} \int_{-\infty}^{(1/\sigma)\ln (B/S_0)} (1/\sigma)\ln (B/S_0) u^* \\
\cdot \left\{ \phi(y_i, u) \cdot e^{\lambda_1 T} \left[ \Phi\left( \delta^+_{\phi}\left( \frac{\phi(y_i, u)}{K} \right) \right) - \Phi\left( \delta^+_{\phi}\left( \frac{\phi(y_i, u)}{B} \right) \right) \right] \\
- e^{-r(T-T_i)} \left[ \Phi\left( \delta^-_{\phi}\left( \frac{\phi(y_i, u)}{K} \right) \right) - \Phi\left( \delta^-_{\phi}\left( \frac{\phi(y_i, u)}{B} \right) \right) \right] \\
- e^{\lambda_1 T} B \left( \frac{\phi(y_i, u)}{B} \right)^{-(2(\lambda_1 + \lambda_2)/\sigma^2)} \left[ \Phi\left( \delta^-_{\phi}\left( \frac{B^2}{K\phi(y_i, u)} \right) \right) - \Phi\left( \delta^-_{\phi}\left( \frac{B}{\phi(y_i, u)} \right) \right) \right] \\
+ e^{-r(T-T_i)} \left[ \phi(y_i, u) \right]^{-(2(\lambda_1 + \lambda_2)/\sigma^2)} \left[ \Phi\left( \delta^-_{\phi}\left( \frac{B^2}{K\phi(y_i, u)} \right) \right) - \Phi\left( \delta^-_{\phi}\left( \frac{B}{\phi(y_i, u)} \right) \right) \right] \\
\right\} 
\]

\[
= \frac{2(2m - w)}{t} \int_{\sigma^2/2}^\infty e^{\sigma^2/2} dm \int_{-\infty}^{\infty} \Phi\left( \frac{\phi(y_i, u)}{B} \right) \phi(y_i, u) \mathrm{d}u
\]

where \( \Phi \) is the standard normal distribution function and

\[
V_0^{uo} = S_0 e^{(\lambda_1 + \lambda_2)T} \left[ \Phi\left( \delta^+_{\phi}\left( \frac{S_0}{K} \right) \right) - \Phi\left( \delta^+_{\phi}\left( \frac{S_0}{B} \right) \right) \right] \\
- e^{(\lambda_1 + \lambda_2)T} B \left( \frac{S_0}{B} \right)^{-(2(\lambda_1 + \lambda_2)/\sigma^2)} \left[ \Phi\left( \delta^-_{\phi}\left( \frac{B^2}{K\phi(y_i, u)} \right) \right) - \Phi\left( \delta^-_{\phi}\left( \frac{B}{\phi(y_i, u)} \right) \right) \right] \\
+ e^{-rT} K \left( \frac{S_0}{B} \right)^{-((2(\lambda_1 + \lambda_2))/\sigma^2)} \left[ \Phi\left( \delta^-_{\phi}\left( \frac{B^2}{K\phi(y_i, u)} \right) \right) - \Phi\left( \delta^-_{\phi}\left( \frac{B}{\phi(y_i, u)} \right) \right) \right].
\] (19)

**Theorem 1.** If \( 0 < S_0 \leq B \), then the risk-neutral price of an up-and-out barrier call option at time 0 under model (1) is given by
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with

\[ \tilde{\delta}_1^+(s) = \frac{1}{\sigma_1 \sqrt{T}} \left[ \ln(s) + \left( r + \lambda_1 n + \lambda_2 \pm \frac{\sigma_1^2}{2} \right) T \right], \]

\[ \tilde{\delta}_2^+(s) = \frac{1}{\sigma_2 \sqrt{T} - t} \left[ \ln(s) + \left( r + \lambda_2 \pm \frac{\sigma_2^2}{2} \right) (T - t) \right], \]

\[ \alpha_1 = \frac{1}{\sigma_1} \left( r + \lambda_1 n + \lambda_2 - \frac{\sigma_1^2}{2} \right), \]

\[ \varphi(y_i, w) = S_0 (1 - y_i)e^{\alpha_i w}. \]

(20)

Proof. See Appendix A.

Remark 1

(1) If \( \lambda_2 = 0 \), i.e., there is no exist endogenous default risk in model (1), then the risk-neutral price of the up-and-out barrier call option at time 0 under this model becomes

\[ V_0^{uo} = e^{-\lambda_1 T} V_0^{uo} + \sum_{i=1}^{3} p_i \int_0^T \int_{-\infty}^{(1/\alpha_i)\ln(B/S_0)} \left[ \frac{\varphi(y_i, w)}{B} \right] d\varphi dw dt, \]

where

\[ V_0^{uo} = S_0 e^{\lambda_1 nT} \left[ \Phi \left( \frac{S_u}{K} \right) - \Phi \left( \frac{S_u}{B} \right) \right] - e^{-\lambda_1 T} K \left[ \Phi \left( \frac{S_t}{K} \right) - \Phi \left( \frac{S_t}{B} \right) \right] - \Phi \left( \frac{S_t}{B} \right) + \frac{2 (2m - w)}{\sqrt{2\pi} T} e^{\alpha_i w - (1/\alpha_i)\ln(B/S_0)}, \]

(21)

with

\[ \tilde{\delta}_1^+(s) = \frac{1}{\sigma_1 \sqrt{T}} \left[ \ln(s) + \left( r + \lambda_1 n + \lambda_2 \pm \frac{\sigma_1^2}{2} \right) T \right], \]

\[ \tilde{\delta}_2^+(s) = \frac{1}{\sigma_2 \sqrt{T} - t} \left[ \ln(s) + \left( r + \lambda_2 \pm \frac{\sigma_2^2}{2} \right) (T - t) \right], \]

\[ \alpha_1 = \frac{1}{\sigma_1} \left( r + \lambda_1 n + \lambda_2 - \frac{\sigma_1^2}{2} \right), \]

\[ \varphi(y_i, w) = S_0 (1 - y_i)e^{\alpha_i w}. \]

It is obvious that the value of up-and-out barrier call option at time 0 under model (1) is the same as the one at time 0 with the stock exposed to counterparty risk (see Yan [11]).

(2) If \( \lambda_1 = 0 \) and \( \lambda_2 = 0 \), i.e., there is no any default in model (1), then the risk-neutral price of up-and-out

barrier call option at time 0 under this model becomes

\[ V_0^{uo} = S_0 \left[ \Phi \left( \frac{S_u}{K} \right) - \Phi \left( \frac{S_u}{B} \right) \right] - e^{-\lambda_1 T} K \left[ \Phi \left( \frac{S_t}{K} \right) - \Phi \left( \frac{S_t}{B} \right) \right] - \Phi \left( \frac{S_t}{B} \right) + \frac{2 (2m - w)}{\sqrt{2\pi} T} e^{\alpha_i w - (1/\alpha_i)\ln(B/S_0)} \]

(24)

with \( \delta_k(s) = (1/\sigma_k \sqrt{T}) [\ln(s) + (s + (\sigma_k^2)/2) T] \). It is obvious that (24) is standard Black–Scholes formula for up-and-out call option.
4. Analytic Formula for Pricing Lookback Options

In this section, we price a floating strike lookback option, whose payoff is the difference between the maximum asset price over the life time and the asset price at expiration.

By formula (15), the risk-neutral price of the lookback option at initial time can be written as follows:

\[
V_0 = S_0e^{-(r+\lambda_1)T}V_0 + S_0e^{(\lambda_1n-\lambda_2)T} \left[ 1 + \frac{\sigma_1^2}{2(r + \lambda_1n + \lambda_2)} \right] \Phi((\alpha_1 + \sigma_1)\sqrt{T}) \\
+ S_0e^{-(r+\lambda_1)T} \left[ 1 - \frac{\sigma_1^2}{2(r + \lambda_1n + \lambda_2)} \right] \left[ 1 - \Phi(\alpha_1\sqrt{T}) \right] + S_0e^{-rT}V_0 + \lambda_2S_0e^{-rT} \\
\cdot \int_0^T \left[ e^{(r+\lambda_1n-\lambda_2)u} \left[ 1 + \frac{\sigma_1^2}{2(r + \lambda_1n + \lambda_2)} \right] \Phi((\alpha_1 + \sigma_1)\sqrt{u}) + e^{-(\lambda_1+\lambda_2)u} \left[ 1 - \frac{\sigma_1^2}{2(r + \lambda_1n + \lambda_2)} \right] \left[ 1 - \Phi(\alpha_1\sqrt{u}) \right] \right] du - S_0,
\]

where

\[
\begin{align*}
V_0 &= \sum_{i=1}^3 \int_0^T \lambda_i e^{-\lambda_i t} \left[ \int_0^{\infty} \int_{\infty}^{\infty} \int_0^{\infty} \int_{\infty}^{\infty} e^{\alpha_1m} \Phi(\delta_1^i) + (1 - \gamma_i) \left( 1 + \frac{\sigma_1^2}{2(r + \lambda_2)} \right) \right] e^{\sigma_1u + r(T-t)\Phi(\delta_2^i)} \\
&\quad - \frac{\sigma_1^2}{2(r + \lambda_2)} (1 - \gamma_i) \left[ (2r + \lambda_2) (m - \omega) \right] e^{\sigma_1u + r(T-t)\Phi(\delta_2^i)} \\
&\quad \cdot \int_0^T \int_0^{\infty} \int_0^{\infty} e^{\sigma_1u + r(T-t)\Phi(\delta_2^i)} dt, \\
\end{align*}
\]

with

\[
\begin{align*}
\alpha_1 &= 1 \left( r + \lambda_1n + \lambda_2 - \frac{\sigma_1^2}{2} \right), \\
\beta_2 &= r + \lambda_2 - \frac{\sigma_1^2}{2}, \\
\delta_1^i &= \frac{\sigma_1 (m - \omega) - \ln(1 - \gamma_i) - \beta_2 (T-t)}{\sigma_1 \sqrt{T-t}}, \\
\delta_2^i &= \frac{-\sigma_1 (m - \omega) + \ln(1 - \gamma_i) + (\beta_2 + \sigma_1^2) (T-t)}{\sigma_1 \sqrt{T-t}}.
\end{align*}
\]

Remark 2

(1) If \( \lambda_2 = 0 \), i.e., there is no exist endogenous default risk in model (1), then the risk-neutral price of the lookback option at time 0 under this model becomes

\[
V_0 = \mathbb{E}[e^{-rT}(Y_T - S_T)] = \mathbb{E}[e^{-rTY_T}] - \mathbb{E}[e^{-rTS_T}].
\]

According to the calculation of (25), we obtain the following theorem.

**Theorem 2.** The risk-neutral price of the lookback option at initial time under model (1) is given by

\[
V_0 = \mathbb{E}[e^{-rT}(Y_T - S_T)] = \mathbb{E}[e^{-rTY_T}] - \mathbb{E}[e^{-rTS_T}].
\]
\[
\begin{align*}
\bar{V}_0 &= \sum_{i=1}^{3} p_i \int_0^T \lambda_i e^{-\lambda_i t} \left[ \int_0^{\infty} \int_0^r e^{\sigma^2 m \Phi (\bar{\delta}_i^+)} (1 - \gamma_i) \left( 1 + \frac{\sigma^2}{2r} \right) e^{\sigma m (T-t) \Phi (\bar{\delta}_i^-)} \right] \frac{2(2m - \omega)}{t \sqrt{2 \pi t}} e^{\sigma (m - (1/2) \sigma^2)} \, dm \, dw \, dt,
\end{align*}
\]

with
\[
\begin{align*}
\alpha_1 &= \frac{1}{\sigma_1} \left( r + \lambda_1 n - \frac{\sigma_1^2}{2} \right), \\
\beta_2 &= r - \frac{\sigma_2^2}{2}, \\
\bar{\delta}_1^+ &= \pm \frac{\sigma_1 (m - w)}{\sqrt{\sigma_1^2 (T-t)}} - \frac{\beta_2 (T-t)}{\sigma_1 \sqrt{T-t}}, \\
\bar{\delta}_2 &= -\frac{\sigma_1 (m - w) + \ln (1 + \gamma_i) + (\beta_2 + \sigma_2^2) (T-t)}{\sigma_2 \sqrt{T-t}}.
\end{align*}
\]

It is obvious that the value of lookback option at time 0 under model (1) is the same as the one at time 0 with the stock exposed to counterparty risk (see Yan [11]).

(2) If \( \lambda_1 = 0 \) and \( \lambda_2 = 0 \), i.e., there is no any default in model (1), then the risk-neutral price of lookback option at time 0 under this model becomes

\[
V_0 = S_0 \left( 1 + \frac{\sigma_1^2}{2r} \right) \Phi \left( \frac{\alpha_1 + \sigma_1 \sqrt{T}}{\sigma_1} \right) + S_0 e^{-rt} \left( 1 - \frac{\sigma_1^2}{2r} \right) \left[ 1 - \Phi \left( \frac{\alpha_1 \sqrt{T}}{\sigma_1} \right) \right] - S_0,
\]

with \( \alpha_1 = (1/\sigma_1) (r - (\sigma_1^2/2)) \). It is obvious that (32) is standard Black–Scholes formula for lookback option.

5. Conclusions

The explicit analytical formulas for European call and put options with asset exposed to multiple defaults risks have been derived. However, it is still very challenging to obtain the explicit analytical formulas for path-dependent options under this model. This is because the multiple default risks cause the difficulty in deriving the density of the first passage time for the maximum asset price. In this paper, the conditional density approach, which is developed by Jiao and Pham [8] for optimal investment, is utilized to overcome the difficulty and derive the formulas for lookback and barrier options when the underlying asset is subject to multiple defaults risks. Future research lies in deriving analytic formulas for the path-dependent options with two underlying assets exposed to loop contagion risks.

Appendix

A. Proof of Theorem 1

We can rewrite (17) as follows:

\[
V_0^{uo} = \mathbb{E} \left[ e^{-rT} (S_T - K) 1_{[S_T \geq K, Y_T \leq \bar{B}]} \right]_{\{AUB, CDEC, DEUF\}}
\]

and (A.1)

\[
+ \mathbb{E} \left[ e^{-rT} (S_T - K) 1_{[S_T \geq K, Y_T > \bar{B}]} \right]_{\{D, E \cup F\}}
\]

If the default times satisfy situation I, then the dynamic of stock price process takes the form as (11). By Ito’s lemma and (11), we can obtain

\[
S_T^F = S_0 e^{(r + \lambda_1 n + \lambda_2 - (\sigma_2^2/2)) t + \sigma_1 \bar{W}_i} = S_0 e^{\sigma_1 \bar{W}_i^D},
\]

where \( \bar{W}_i^D = a_1 t + \bar{W}_i \) with \( a_1 = (1/\sigma_1) (r + \lambda_1 n + \lambda_2 - (\sigma_2^2/2)) \).

We define \( M_T^F = \max_{0 \leq t \leq T} \bar{W}_i^D \), so by (15) we derive that

\[
Y_T^F = S_0 e^{\sigma_1 M_T^F}.
\]

The first term on the right-hand side of (A.1) can be calculated as

\[
\mathbb{E} \left[ e^{-rT} (S_T - K) 1_{[S_T \geq K, Y_T \leq \bar{B}]} \right]_{\{A, B\}}
\]

\[
= \mathbb{E} \left[ e^{-rT} (S^F_U - K) 1_{[S^F_U \geq K, Y_T \leq \bar{B}]} \right]_{\{U\}}
\]

\[
= \mathbb{E} \left[ e^{-rT} (S^F_T - K) 1_{[S^F_T \geq K, \bar{W}_i \leq \bar{D}]} \right]_{\{D\}}
\]

Notice that \( V_0^{uo} \) corresponds to the case when there is no any default. Then, we use the following identity:
\[
\frac{1}{\sqrt{2\pi T}} \int_{k}^{b} e^{\beta t+\omega-(1/2T)dw} = \frac{1}{\sqrt{2\pi T}} \int_{k}^{b} e^{-(1/2T)(\omega-\eta)^2-(1/2)\eta^2T+\beta dw} \\
= e^{-(1/2\eta^2T+\beta)} \frac{1}{\sqrt{2\pi}} \int_{k}^{b} e^{-(\eta^2(1/k-1))dy,}
\]

(A.5)

\[
\Phi[\bar{\delta}_i(S)] - \Phi[\bar{\delta}_i(S_0)] - e^{-\eta T} K \left[ \Phi[\bar{\delta}_i(S_0)] - \Phi[\bar{\delta}_i(S_0)] \right] \\
+ e^{-\eta T} K \left[ \Phi[\bar{\delta}_i(S_0)] - \Phi[\bar{\delta}_i(S_0)] \right] \\
\]

(A.6)

where \( \bar{\delta}_i(s) \) is defined in Theorem 1.

If the stock has endogenous default in the life of the option, i.e., the default time satisfy situation II and situation IV, then the price of the stock at expiration \( T \) is zero, and thus we obtain \( 1_{[S_0=K,Y_0<\infty]} 1_{[\omega,\eta]|a,b]} = 0 \). Therefore, the second term on the right-hand side of (A.1) is equal to 0.

If the default times satisfy situation III, then the dynamic of stock price process such that (13). Using Ito’s lemma, the solution to SDE (13) for the stock price is

\[
S_t^d = S_0 e^{\bar{\delta}_i t Y_{d-}}, 0 \leq t < \tau_1, \\
S_{\tau_1}^d = S_{\tau_1} e^{\bar{\delta}_i \left[ \bar{W}^{d-} - \bar{W}_{\tau_1} \right]}, \quad \tau_1 < t \leq T, \\
S_{\tau_1}^d = S_{\tau_1} (1 - \gamma),
\]

(A.7)

where \( \bar{W}^{d-}_t = (1/\sigma_1) (r + \lambda_1 n + \lambda_2 - (\sigma_2^2/2)) 1 + \bar{W}_t = a_t + \bar{W}_t \) and \( \bar{W}^{d-}_{\tau_1} = (1/\sigma_1) (r + \lambda_1 n + \lambda_2 - (\sigma_2^2/2)) 1 + \bar{W}_t = a_t + \bar{W}_t \). Denote \( \bar{\omega}^{d,\tau_1} = \max_{0 \leq s \leq \tau_1} \bar{W}^{d-}_s \) and \( \bar{Y}^{d,\tau_1} = \max_{0 \leq s \leq \tau_1} S_t^d \), then we have

\[
S_{\tau_1}^d (t) = S_0 (1 - \gamma) e^{\bar{\delta}_i \bar{W}^{d-}_t}, \\
S_{\tau_1}^d (t) = e^{\bar{\delta}_i \bar{W}^{d-}_t - \bar{W}^{d-}_t}, \quad \tau_1 < t \leq T, \\
S_{\tau_1}^d (t) = S_{\tau_1} (1 - \gamma),
\]

(A.8)

It is obvious that \( S_{\tau_1}^d (t)/S_{\tau_1}^d (t) \) and \( Y_{\tau_1}^d/S_{\tau_1}^d (t) \) are independent on \( \mathcal{F}_t \). Let \( x = S_{\tau_1}^d (t) \), we calculate

\[
\mathbb{E} \left[ e^{-\eta T} (S_{\tau_1}^d (t) - K) 1_{[S_{\tau_1}^d (t) \geq K, Y_{\tau_1}^d \leq \infty]} \right] = \mathbb{E} \left[ e^{-\eta T} (x e^{\delta_i (\bar{W}^{d-}_t - \bar{W}_{\tau_1})} - K) 1_{[x e^{\delta_i (\bar{W}^{d-}_t - \bar{W}_{\tau_1})} \geq K, x e^{\max_{0 \leq s \leq \tau_1} (\bar{W}^{d-}_s - \bar{W}_{\tau_1})} \leq \infty]} \right] y = y_1
\]

(A.9)

where \( k = (1/\sigma_1) \ln (K/x) \) and \( b = (1/\sigma_2) \ln (B/x) \). Notice that the expectation

\[
\mathbb{E} \left[ e^{-\eta T} (x e^{\delta_i (\bar{W}^{d-}_t - \bar{W}_{\tau_1})} - K) 1_{[x e^{\delta_i (\bar{W}^{d-}_t - \bar{W}_{\tau_1})} \geq K, x e^{\max_{0 \leq s \leq \tau_1} (\bar{W}^{d-}_s - \bar{W}_{\tau_1})} \leq \infty]} \right] = y_1
\]

(A.10)
corresponds to the case when there is no default. Therefore using the techniques as in calculate (A.6), we have

\[
\mathbb{E}\left[e^{-r(T-t)}\left(xe^{\sigma\bar{W}^I_{T-t}} - K\right)I_{\left\{\bar{W}^I_{T-t} > Y_d\right\}} \mid y = y\right] = xe^{\lambda_T} \left[\Phi\left(\bar{\sigma}_2\left(\frac{x}{K}\right)\right) - \Phi\left(\bar{\sigma}_2\left(\frac{x}{B}\right)\right)\right] \\
- e^{-r(T-t)} K \left[\Phi\left(\bar{\sigma}_1\left(\frac{x}{K}\right)\right) - \Phi\left(\bar{\sigma}_1\left(\frac{x}{B}\right)\right)\right] \\
- e^{\lambda_T} B \left(\frac{x}{B}\right) \left(\left(2(r+\lambda_2)\sigma_2^2\right)\right) \\
\cdot \left[\Phi\left(\bar{\sigma}_1\left(\frac{B^2}{Kx}\right)\right) - \Phi\left(\bar{\sigma}_1\left(\frac{B}{x}\right)\right)\right],
\]

(A.11)

where \(\bar{\sigma}_2\) is defined in Theorem 1.

Then, the third term on the right-hand side of (A.1) can be calculated as follows:

\[
\mathbb{E}\left[e^{-r(T-t)}(S_T - K)1_{\left\{S_T \leq K, Y \leq B\right\}} \mid y = y\right] = \mathbb{E}\left[e^{-r(T-t)}(S_T - K)1_{\left\{S_T \leq K, Y \leq B\right\}} \mid \tau_1 = t, \tau_2 = u\right] \\
= \int_0^T \lambda_t e^{-\lambda_t u} \mathbb{E}\left[e^{-r(u)}(S_u - K)1_{\left\{S_u \leq K, Y_u \leq B\right\}} \mid \mathcal{F}_u\right] du \\
= e^{-\lambda_T} \int_0^T \lambda_t e^{-\lambda_t u} \mathbb{E}\left[e^{-r(u)}(S_u - K)1_{\left\{S_u \leq K, Y_u \leq B\right\}} \mid \mathcal{F}_u\right] du.
\]

(A.12)

According to Shreve [14], the joint density function under \(\bar{\mathbb{P}}\) of \((\bar{W}^I_t, \bar{M}^I_t)\) involved with \(Y^I_t\) and \(S^{I,P}_t(t)\) is

\[
\tilde{f}_{\bar{W}^I_t, \bar{M}^I_t} (w, m) = \frac{2(2m - w)}{t \sqrt{\pi}} e^{-\frac{w - (1/2)m^2}{t - (1/2)(2m - w)^2}}, \quad w \leq m, m \geq 0.
\]

(A.13)

Substituting (A.11) and \(x = S^{I,P}_t(t)\) into (A.9) and using (A.13), we can continue to calculate (A.12) and obtain

\[
\mathbb{E}\left[e^{-r(T-t)}(S_T - K)1_{\left\{S_T \leq K, Y \leq B\right\}} \mid y = y\right]
\]

where \(\varphi(y, w)\) is defined in Theorem 1. Combining (A.6) and (A.14), we obtain formula (18). Thus, the proof of Theorem 1 is complete.

**B. Proof of Theorem 2**

The risk-neutral price of the lookback option (25) can be rewritten as

\[
\mathbb{E}\left[e^{-r(T-t)}(S_T - K)1_{\left\{S_T \leq K, Y \leq B\right\}} \mid y = y\right] = \int_0^T \lambda_t e^{-\lambda_t u} \mathbb{E}\left[e^{-r(u)}(S_u - K)1_{\left\{S_u \leq K, Y_u \leq B\right\}} \mid \mathcal{F}_u\right] du.
\]

(A.14)
\( V_0 = \mathbb{E}[e^{-rT}Y_{T1_{[AUB,T]}D_{[J,B,F]}]}] - \mathbb{E}[e^{-rT}S_T]. \) (B.1)

\[
F(S) = e^{-(\lambda_1 + \lambda_2)T} \Phi \left( \frac{\ln(S/S_0) - a(T)}{b(T)} \right) + (1 - e^{-\lambda_1 T}) + e^{-\lambda_1 T} \sum_{i=1}^{3} p_i \int_{0}^{T} \lambda_1 e^{-\lambda_1 t} \Phi \left( \frac{1}{b(t)} \left( \ln \left( \frac{S}{S_0 (1 - Y_i)} \right) - a(t) \right) \right) dt, \tag{B.2}
\]

with \( a(t) = \left( r + \lambda_1 n + \lambda_2 - \left( a(t)^2/2 \right) \right) t + \left( r + \lambda_2 - \left( a(t)^2/2 \right) \right) (T - t) \) and \( b(t) = \sqrt{\sigma_1^2 t + \sigma_2^2 (T - t)} \). Combining the distribution function \( F \) in (B.2) and the following identity

\[
\int_{0}^{\infty} S \Phi \left( \frac{1}{b} \left( \ln \left( \frac{S}{C} \right) - A \right) \right) = Ce^{\lambda r(B/2)}, \tag{B.3}
\]

the second term on the right-hand side of (B.1) can be calculated as follows:

\[
\mathbb{E}[e^{-rT}S_T] = e^{-rT} \int_{0}^{\infty} S dF(S) = e^{-(\lambda_1 + \lambda_2)T} T \int_{0}^{\infty} S d \Phi \left( \frac{1}{b(T)} \left( \ln \left( \frac{S}{S_0 (1 - Y_i)} \right) - a(T) \right) \right) + e^{-(\lambda_1 + \lambda_2)T} \sum_{i=1}^{3} p_i \int_{0}^{T} \lambda_1 e^{-\lambda_1 t} \Phi \left( \frac{1}{b(t)} \left( \ln \left( \frac{S}{S_0 (1 - Y_i)} \right) - a(t) \right) \right) dt \]

According to [12], the distribution function of the stock price at expire time \( T \) is given by

\[
\text{where we use } \sum_{i=1}^{3} p_i = 1 \text{ and } \sum_{i=1}^{3} p_i Y_i = n \text{ to obtain the last equality in formula (B.4).}
\]

Next we aim to calculate the first term on the right-hand side of (B.1):

\[
\mathbb{E}[e^{-rT}Y_{T1_{[AUB,T]}D_{[J,B,F]}]}] = \mathbb{E}[e^{-rTY_{T1_{[AUB,T]}D_{[J,B,F]}]}]} \]

If the default times satisfy situation 1, then by (11) and Ito’s lemma, we can obtain (A.2) and (A.3). Thus, the first term of the right-hand side of (B.5) can be calculated as

\[
\mathbb{E}[e^{-rTY_{T1_{[AUB,T]}D_{[J,B,F]}]}]} = \mathbb{E}[e^{-rTY_{T1_{[t_1,t_2,T]}D_{[J,B,F]}]}]} \]

\[
\text{According to Shreve [14], the density function of } \tilde{M}_T^P \text{ under } \mathbb{P} \text{ is given by}
\]

\[
\tilde{f}_{M_N^T}(m) = \frac{2}{\sqrt{2\pi T}} e^{-(1/2T)(m - a(T))} - 2 \alpha_1 e^{2\alpha_1 m} \Phi \left( \frac{m - a(T)}{\sqrt{T}} \right), \quad m \geq 0,
\]

so we have

\[
\mathbb{E}[Y_{T1_{[AUB,T]}D_{[J,B,F]}]}] = \int_{0}^{\infty} S_0 e^{\sigma m} \left[ \frac{2}{\sqrt{2\pi T}} e^{-(1/2T)(m - a(T))} - 2 \alpha_1 e^{2\alpha_1 m} \Phi \left( \frac{m - a(T)}{\sqrt{T}} \right) \right] dm \]

\[
= 2S_0 e^{(\lambda_1 + \lambda_2)T} \Phi \left( (\alpha_1 + \sigma_1) \sqrt{T} \right) - 2S_0 \alpha_1 \int_{0}^{\infty} e^{(2\alpha_1 + \sigma_1)m} \left[ \frac{1}{\sqrt{2\pi}} e^{-(1/2T)(m - \alpha_1 T)} \right] dm \]

\[
= 2S_0 e^{(\lambda_1 + \lambda_2)T} \Phi \left( (\alpha_1 + \sigma_1) \sqrt{T} \right) - 2S_0 \alpha_1 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{(2\alpha_1 + \sigma_1)m - (T/2)} dm \]

\[
= S_0 e^{(\lambda_1 + \lambda_2)T} \left[ 1 + \frac{\sigma_1^2}{2(r + \lambda_1 n + \lambda_2)} \right] \Phi \left( (\alpha_1 + \sigma_1) \sqrt{T} \right) + S_0 \left[ 1 - \frac{\sigma_1^2}{2(r + \lambda_1 n + \lambda_2)} \right] \Phi \left( (\alpha_1 + \sigma_1) \sqrt{T} \right), \quad m \geq 0.
\]

(B.8)
and thus the value of the expectation in (B.6) can be calculated as follows:

\[
\mathbb{E}\left[ e^{-rT} Y_T^{1_{[A,B]}} \right] = S_0 e^{\left( \lambda_t n - \lambda_t \right) T} \left[ 1 + \frac{\sigma_1^2}{2(r + \lambda_t n + \lambda_t)} \right] \Phi\left( (\alpha_1 + \sigma_1) \sqrt{T} \right) + S_0 e^{-\left( \lambda_t n + \lambda_t \right) T} \left[ 1 - \frac{\sigma_1^2}{2(r + \lambda_t n + \lambda_t)} \right] \left[ 1 - \Phi\left( \alpha_1 \sqrt{T} \right) \right].
\]

(B.9)

Similar, the second term of the right-hand side of (B.5) can be calculated as

\[
\mathbb{E}\left[ e^{-rT} Y_T^{1_{(C,D)}} \right] = \mathbb{E}\left[ \mathbb{E}\left[ e^{-rT} Y_T^{1_{(C,D)}} \bigg| \tau_1, \tau_2 \right] \right] = e^{-rT} \int_0^T \lambda_t e^{-\lambda_t u} \int_u^{\infty} \lambda_t e^{-\lambda_t \tau_u} \mathbb{E}\left[ Y_T^{1_{(C,D)}} \bigg| \tau_1 = t, \tau_2 = u \right] \, dt \, du
\]

\[
= e^{-rT} \int_0^T \lambda_t e^{-\lambda_t u} \int_u^{\infty} \lambda_t e^{-\lambda_t \tau_u} \mathbb{E}\left[ Y_T^F \right] \, dt \, du = e^{-rT} \int_0^T \lambda_t e^{-\left( \lambda_t n + \lambda_t \right) u} \mathbb{E}\left[ Y_T^F \right] \, du
\]

\[
= \lambda_t S_0 e^{-rT} \int_0^T e^{\left( \lambda_t n + \lambda_t \right) u} \left[ 1 + \frac{\sigma_1^2}{2(r + \lambda_t n + \lambda_t)} \right] \Phi\left( (\alpha_1 + \sigma_1) \sqrt{u} \right) + e^{-\left( \lambda_t n + \lambda_t \right) u} \left[ 1 - \frac{\sigma_1^2}{2(r + \lambda_t n + \lambda_t)} \right] \left[ 1 - \Phi\left( \alpha_1 \sqrt{u} \right) \right] \, du.
\]

(B.10)

We shall calculate the third term of the right-hand side of (B.5). To this end, we define

\[
G(t, x) = \mathbb{E}\left[ \max\left( x, \frac{Y_t^{d_1}}{S_t^{d_1}}(t) \right) \bigg| \gamma = y \right],
\]

(B.11)

\[
R(T - t, m) = \mathbb{P}\left( M^{d_1}_{T-t} \leq m \right).
\]

By definition of the \( G(t, x) \), we have

\[
G(t, x) = \mathbb{E}\left[ \max\left( x, \frac{Y_t^{d_1}}{S_t^{d_1}}(t) \right) \bigg| \gamma = y \right] = \mathbb{E}\left[ \exp\left( \max\left( \ln x, \max_{t \leq T} \ln \sigma_2 \left( \tilde{W}_t^{d_1} - \tilde{W}_t^{d_1} \right) \right) \right) \bigg| \gamma = y \right]
\]

\[
= \mathbb{E}\left[ \exp\left( \max\left( \ln x, \sigma_2 \tilde{M}_{T-t}^{d_1} \right) \right) \bigg| \gamma = y \right] = \mathbb{E}\left[ x \mathbb{1}_{\{\ln x \leq \sigma_2 \tilde{M}_{T-t}\}} \bigg| \gamma = y \right] + \mathbb{E}\left[ \sigma_2 \tilde{M}_{T-t}^{d_1} \mathbb{1}_{\{\ln x > \sigma_2 \tilde{M}_{T-t}\}} \bigg| \gamma = y \right] = I_1 + I_2.
\]

(B.12)

According to Shreve [14], we obtain

\[
\mathbb{P}\left( \tilde{M}^{d_1}_{T-t} \leq m \right) = \Phi\left( \frac{m - \alpha_2 (T - t)}{\sigma_2 \sqrt{T - t}} \right)
\]

\[
- e^{2\alpha_2 m} \phi\left( \frac{m - \alpha_2 (T - t)}{\sigma_2 \sqrt{T - t}} \right), \quad m \geq 0.
\]

(B.13)

Then, combining (B.11) and (B.13), the first term of the right-hand side of (B.12) can be calculated as

\[
I_1 = x R\left( T - t, \ln x, \sigma_2 \left( \tilde{W}_t^{d_1} - \tilde{W}_t^{d_1} \right) \right) = x \left[ \Phi\left( \frac{\ln x - (r + \lambda_t - (\sigma_2^2 / 2)(T - t))}{\sigma_2 \sqrt{T - t}} \right) \right]
\]

\[
- x \left( \frac{\ln x - (r + \lambda_t - (\sigma_2^2 / 2)(T - t))}{\sigma_2 \sqrt{T - t}} \right) \frac{\phi\left( \frac{-\ln x - (r + \lambda_t - (\sigma_2^2 / 2)(T - t))}{\sigma_2 \sqrt{T - t}} \right)}{\sigma_2 \sqrt{T - t}}
\]

\[
= x \left[ \Phi\left( \delta_1^* \right) - x^{2\sigma_2^2 / \sigma_2^2} \Phi\left( \delta_1^* \right) \right].
\]

(B.14)

and we use some technique in calculated integral to obtain that the second term of the right-hand side of (B.12) is
\[ I_2 = \int_{\ln x/\sigma}^{\infty} e^{\epsilon_s m} R_m \left( T - t, m \right) dm \]
\[ = \int_{\ln x/\sigma}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} e^{\left( (2\alpha_2 + \sigma_2)m \right)} e^{-\left( (m - \alpha_2(T-t))^2/2(T-t) \right)} \, dm \]
\[ + \int_{\ln x/\sigma}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} e^{\epsilon_s m} e^{-\left( (m - \alpha_2(T-t))^2/2(T-t) \right)} \, dm \]
\[ - \int_{\ln x/\sigma}^{\infty} 2\alpha_2 e^{(2\alpha_2 + \sigma_2)m} \Phi \left( \frac{-m - \alpha_2(T-t)}{\sqrt{T-t}} \right) \, dm \]
\[ = \left( 1 + \frac{\sigma_2^2}{2(r + \lambda_2)} \right) e^{\epsilon(T-t)} \Phi \left( \delta_2 \right) - \left( 1 - \frac{\sigma_2^2}{2(r + \lambda_2)} \right) \Phi \left( \delta_2^\prime \right), \]
\[ \text{(B.15)} \]

where
\[ \beta_2 \equiv r + \lambda_2 - \frac{\sigma_2^2}{2}, \]
\[ \delta_2^\prime = \frac{\pm \ln x - \beta_2 (T-t)}{\sigma_2 \sqrt{T-t}} \]
\[ \delta_{2} = -\frac{\ln x + (\beta_2 + \sigma_2^2) (T-t)}{\sigma_2 \sqrt{T-t}} \]
\[ \text{(B.16)} \]

Substituting (B.14) and (B.15) into (B.12), we have
\[ G(t, x) = x \left[ \Phi \left( \delta_2^\prime \right) - x^{2\beta_2/\sigma_2^2} \Phi \left( \delta_2 \right) \right] + \left( 1 + \frac{\sigma_2^2}{2(r + \lambda_2)} \right) \Phi \left( \delta_2 \right) \]
\[ - \left( 1 - \frac{\sigma_2^2}{2(r + \lambda_2)} \right) \Phi \left( \delta_2^\prime \right) \]
\[ = x \Phi \left( \delta_2^\prime \right) + \left( 1 + \frac{\sigma_2^2}{2(r + \lambda_2)} \right) e^{\epsilon(T-t)} \Phi \left( \delta_2 \right) \]
\[ - \frac{\sigma_2^2}{2(r + \lambda_2)} x^{2(r+\lambda_2)/\sigma_2^2} \Phi \left( \delta_2^\prime \right). \]
\[ \text{(B.17)} \]

By the independence lemma (refer to [14]), we have
\[ \mathbb{E} \left[ \max \left( Y_T^f, Y_{\tau_1}^d \right) \right] = G \left( t, Y_T^f, t, Y_{\tau_1}^d \right). \]
\[ \text{(B.18)} \]

If the default times satisfy situation III, then by (13) and Ito’s lemma, we can obtain (A.7) and (A.8). Thus, the third term of the right-hand side of (B.5) can be calculated as
\[ \mathbb{E} \left[ e^{-r \tau_1} Y_{\tau_1}^d \right] = \mathbb{E} \left[ e^{-r \tau_1} Y_{\tau_1}^d \right] \mid \tau_1, \tau_2 \]
\[ = e^{-r \tau_1} \int_0^{\tau_1} \lambda_1 e^{-\lambda_1 t} t \left[ \max \left( Y_T^f, Y_{\tau_1}^d \right) \right] dt \]
\[ = e^{-r \tau_1} \int_0^{\tau_1} \lambda_1 e^{-\lambda_1 t} t \left[ \max \left( Y_T^f, Y_{\tau_1}^d \right) \right] dt \]
\[ = e^{-r \tau_1} \int_0^{\tau_1} \lambda_1 e^{-\lambda_1 t} \left[ \max \left( Y_T^f, Y_{\tau_1}^d \right) \right] dt \]
\[ = e^{-r \tau_1} \int_0^{\tau_1} \lambda_1 e^{-\lambda_1 t} \left[ \max \left( Y_T^f, Y_{\tau_1}^d \right) \right] dt \]
\[ \text{(B.19)} \]

Substituting (B.17) and (B.18) into (B.19) and using (A.13), we obtain that
\[ \mathbb{E} \left[ e^{-r \tau_1} Y_{\tau_1}^d \right] = \mathbb{E} \left[ e^{-r \tau_1} \right] \sum_{i=1}^{3} p_i \mathbb{E} \left[ Y_T^f \mid \tau_1, \tau_2 \right] \]
\[ + \frac{\sigma_2^2}{2(r + \lambda_2)} \int_0^{\tau_1} \lambda_1 e^{-\lambda_1 t} t \left[ \max \left( Y_T^f, Y_{\tau_1}^d \right) \right] dt \]
\[ = S_0 e^{-r \tau_1} \sum_{i=1}^{3} p_i \left[ \int_0^{\tau_1} \lambda_1 e^{-\lambda_1 t} t \left[ \max \left( Y_T^f, Y_{\tau_1}^d \right) \right] dt \right] \]
\[ = S_0 e^{-r \tau_1} \sum_{i=1}^{3} p_i \left[ \int_0^{\tau_1} \lambda_1 e^{-\lambda_1 t} t \left[ \max \left( Y_T^f, Y_{\tau_1}^d \right) \right] dt \right] \]
\[ \text{(B.20)} \]
Similar, the last term of the right-hand side of (B.5) can be calculated as

\[ \mathbb{E}\left[e^{-rT} Y_{ \tau_1, \tau_2} \right] = \mathbb{E}\left[ \mathbb{E}\left[e^{-rT} Y_{ \tau_1, \tau_2} \big| \tau_1, \tau_2 \right] | \tau_1, \tau_2 \right] \]

\[ = e^{-rT} \int_0^T \int_0^u \mathbb{E}\left[ \max\left( Y_{\tau_1, \tau_2} \right) | \tau_1 = t, \tau_2 = u \right] dt \, du \]

\[ = e^{-rT} \int_0^T \int_0^u \mathbb{E}\left[ \sum_{i=1}^k p_i S_i(t) \left( \max\left( \frac{Y_{\tau_1, \tau_2}}{S_i(t)}, \frac{Y_{\tau_1, \tau_2}}{S_i(t)} \right) \right) | \tau_1, \tau_2 \right] dt \, du \]

\[ = S_0 e^{-rT} \int_0^T \int_0^u \mathbb{E}\left[ \sum_{i=1}^k p_i S_i(t) \left( \max\left( \frac{Y_{\tau_1, \tau_2}}{S_i(t)}, \frac{Y_{\tau_1, \tau_2}}{S_i(t)} \right) \right) | \tau_1, \tau_2 \right] dt \, du \]

\[ = S_0 e^{-rT} \int_0^T \int_0^u \mathbb{E}\left[ \sum_{i=1}^k p_i S_i(t) \left( \max\left( \frac{Y_{\tau_1, \tau_2}}{S_i(t)}, \frac{Y_{\tau_1, \tau_2}}{S_i(t)} \right) \right) | \tau_1, \tau_2 \right] dt \, du \]

\[ = S_0 e^{-rT} \int_0^T \int_0^u \mathbb{E}\left[ \sum_{i=1}^k p_i S_i(t) \left( \max\left( \frac{Y_{\tau_1, \tau_2}}{S_i(t)}, \frac{Y_{\tau_1, \tau_2}}{S_i(t)} \right) \right) | \tau_1, \tau_2 \right] dt \, du \]

\[ \approx S_0 e^{-rT} \int_0^T \int_0^u \mathbb{E}\left[ \sum_{i=1}^k p_i S_i(t) \left( \max\left( \frac{Y_{\tau_1, \tau_2}}{S_i(t)}, \frac{Y_{\tau_1, \tau_2}}{S_i(t)} \right) \right) | \tau_1, \tau_2 \right] dt \, du \]

where \( \overline{\delta}_i \) and \( \tilde{\delta}_i \) are defined in Theorem 2. Combining (B.9), (B.10), (B.20), and (B.21) gives the value of \( \mathbb{E}\left[e^{-rT} Y_{ \tau_1, \tau_2} \right] \), and then using (B.4) complete the proof of this theorem.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The author declares that there are no conflicts of interest regarding the publication of this paper.

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