Abstract. Let \( p_k(n) \) be given by the \( k \)-th power of the Euler Product \( \prod_{n=1}^{\infty} (1 - q^n)^{k} = \sum_{n=0}^{\infty} p_k(n) q^n. \) By investigating the properties of the modular equations of the second and the third order under the Atkin \( U \)-operator, we determine the generating functions of \( p_{8k}(2^2 \alpha n + \frac{k(2^{2\alpha} - 1)}{2}) \) (1 \( \leq \) \( k \) \( \leq \) 3) and \( p_{8k}(3^{2\beta} n + \frac{k(3^{2\beta} - 1)}{2}) \) (1 \( \leq \) \( k \) \( \leq \) 8) in terms of some linear recurring sequences. Combining with a result of Engstrom about the periodicity of linear recurring sequences modulo \( m \), we obtain infinite families of congruences for \( p_k(n) \) modulo any \( m \geq 2 \), where 1 \( \leq \) \( k \) \( \leq \) 24 and 3|\( k \) or 8|\( k \). Based on these congruences for \( p_k(n) \), infinite families of congruences for many partition functions such as the overpartition function, \( t \)-core partition functions and \( \ell \)-regular partition functions are easily obtained.

Keywords: the Euler Product, congruences, modular equations, partition functions.

AMS Classification: 11P83, 05A15, 05A17, 05A30.

1 Introduction

This paper is devoted to the congruence properties of \( p_k(n) \) modulo arbitrary integer \( m \geq 2 \), where \( p_k(n) \) is defined by the \( k \)-th power of the Euler Product

\[
f_k^n := \prod_{n=1}^{\infty} (1 - q^n)^k = \sum_{n=0}^{\infty} p_k(n) q^n.
\]

When \( k = 1 \), \( f_1 \) is called the Euler Product, and by Euler’s pentagonal number theorem [12], [1, pp. 11], we see that

\[
p_1(n) = \begin{cases} (-1)^{m}, & \text{if } n = m(3m \pm 1)/2; \\ 0, & \text{otherwise}. \end{cases}
\]

When \( k = -1 \), \( p_{-1}(n) \) is the ordinary partition function \( p(n) \), which counts the number of ways of writing \( n \) as a sum of a non-increasing sequence of positive integers.

Recently, the congruences for \( p_k(n) \) have drawn much more attention since many partition functions possess the same congruence properties as \( p_k(n) \), such as \( \ell \)-regular partition functions, the overpartition function, \( t \)-core partition functions. Recall that for any \( \ell \geq 2 \), a
A partition is called an $\ell$-regular partition if none of its parts is divisible by $\ell$. Let $b_\ell(n)$ denote the number of $\ell$-regular partitions of $n$, and the generating function of $b_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \prod_{n=1}^{\infty} \left(1 - \frac{q^{\ell n}}{1 - q^n}\right)$$

with the convention that $b_\ell(0) = 1$. For any prime $\ell$ and $j \geq 0$, it is easy to see that

$$\sum_{n=0}^{\infty} b_{\ell^j}(n)q^n \equiv f_{\ell}^{\ell^j-1} \pmod{\ell}.$$

Combining with (1.1), we have for any $n \geq 0$,

$$b_{\ell^j}(n) \equiv p_{\ell^j-1}(n) \pmod{\ell}.$$

In [10], Cui, Gu and Huang obtained some infinite families of congruences for $p_k(n)$ ($1 \leq k \leq 24$) modulo 2 or 3 by using the modular equations of the fifth and the seventh order. To be specific, they established the following congruences

$$p_k \left( \ell^{a_n} + \frac{(24i + \ell k) \cdot \ell^{a_n} - k}{24} \right) \equiv 0 \pmod{2 \text{ or } 3},$$

where $1 \leq k \leq 24$, $a \geq 1$ and $\ell = 5$ or 7, the positive integer $t$ is determined by $k$, $\ell$ and the modulus 2 or 3. Moreover, they also obtained many infinite families of congruences for $\ell$-regular partition functions and generalized Frobenius partition functions. For example, they proved that for any $a \geq 1$, $n \geq 0$ and $1 \leq i \leq 4$,

$$b_{16} \left( 5^{4n} + \frac{3 \cdot 5^{4a+1} - 15}{24} \right) \equiv 0 \pmod{2}.$$ 

Later, Xia [31] used the modular equations of the fifth and the seventh, as well as the thirteenth order to establish the generating functions of

$$p_k \left( \ell^{2a+1} n + \frac{k(\ell^{2a+2} - 1)}{24} - \ell^{2a+1} t(\ell, k) \right),$$

where $1 \leq k \leq 24$, $a \geq 0$ and $\ell = 5, 7$ or 13 and $t(5, k) = \lfloor \frac{k}{5} \rfloor$, $t(7, k) = \lfloor \frac{2k}{7} \rfloor$ and $t(13, k) = \lfloor \frac{7k}{13} \rfloor$, respectively. Based on the above generating functions, he derived that for a fixed prime $\ell \in \{5, 7, 13\}$, $1 \leq k \leq 24$ and any $m \geq 2$, if there exists $r(k, m)$ satisfying some restricted conditions, then for any $a \geq 1$, $n \geq 0$ and $1 \leq i \leq \ell - 1$,

$$p_k \left( \ell^{2r(k,m)a_n} + \frac{(24i + \ell k) \cdot \ell^{2r(k,m)a_n} - k}{24} \right) \equiv 0 \pmod{m}.$$

Moreover, he obtained many infinite families of congruences for $\ell$-regular partition functions, partition functions related to mock theta functions and generalized Frobenius partition functions. For example, he proved that for any $a \geq 1$, $n \geq 0$ and $1 \leq i \leq 12$,

$$b_{23} \left( 13^{22a_n} + \frac{(12i + 143) \cdot 13^{22a-1} - 11}{12} \right) \equiv 0 \pmod{23}.$$
In this paper, we determine the generating functions of $p_{8k}(2^{2\alpha}n + \frac{k(2^{2\alpha} - 1)}{3})$ ($1 \leq k \leq 3$) and $p_{3k}(3^{2\beta}n + \frac{k(3^{2\beta} - 1)}{8})$ ($1 \leq k \leq 8$) based on the modular equations of the second and the third order. Furthermore, combining with a result of Engstrom [11] about the periodicity of linear recurring sequences modulo $m$, we obtain infinite families congruences for $p_{8k}(n)$ and $p_{3k}(n)$ with respect to arbitrary modulus $m \geq 2$.

To state our results more precisely, we adopt the notion

$$f_t := \prod_{n=1}^{\infty} (1 - q^{nt})$$

for any integer $t \geq 1$. Note that the case of $t = 1$ is coincident with the Euler Product. By means of the modular equations of the second and the third order, we derive the following generating functions.

**Theorem 1.1.** For any $\alpha \geq 0$ and $1 \leq k \leq 3$, we have

$$\sum_{n=0}^{\infty} p_{8k}(2^{2\alpha}n + \frac{k(2^{2\alpha} - 1)}{3})q^n = A_k(2\alpha)f_1^{8k} + B_k(2\alpha)q^{\frac{k}{2}}f_1^{8k-24}f_2^{24}$$  \hspace{1cm} (1.2)

and

$$\sum_{n=-\lfloor k/2 \rfloor}^{\infty} p_{8k}(2^{2\alpha+1}n + \frac{k(2^{2\alpha+2} - 1)}{3})q^n = A_k(2\alpha + 1)f_2^{8k} + B_k(2\alpha + 1)q^{-\frac{k}{2}}f_1^{24}f_2^{8k-24},$$  \hspace{1cm} (1.3)

where the coefficients $A_k(\alpha)$ and $B_k(\alpha)$ satisfy the same recurrence relation

$$A_k(\alpha + 4) = f(k)A_k(\alpha + 2) + g(k)A_k(\alpha).$$  \hspace{1cm} (1.4)

The values of $f(k)$ and $g(k)$ ($1 \leq k \leq 3$) and the initial values of $A_k$ and $B_k$ ($1 \leq k \leq 3$) are listed in Table 3.

**Theorem 1.2.** For any $\beta \geq 0$ and $1 \leq k \leq 8$, we have

$$\sum_{n=0}^{\infty} p_{3k}(3^{2\beta}n + \frac{k(3^{2\beta} - 1)}{8})q^n = C_k(2\beta)f_1^{3k} + \sum_{i=1}^{\lfloor k/3 \rfloor} D_{k,i}(2\beta)q^{i}f_1^{3k-12i}f_3^{12i}$$  \hspace{1cm} (1.5)

and

$$\sum_{n=-\lfloor k/3 \rfloor}^{\infty} p_{3k}(3^{2\beta+1}n + \frac{k(3^{2\beta+2} - 1)}{8})q^n = C_k(2\beta + 1)f_3^{3k} + \sum_{i=1}^{\lfloor k/3 \rfloor} D_{k,i}(2\beta + 1)q^{-i}f_1^{12i}f_3^{3k-12i},$$  \hspace{1cm} (1.6)

where the coefficients $C_k(\beta)$ and $D_{k,i}(\beta)$ satisfy the same recurrence relation

$$C_k(\beta + 4) = h(k)C_k(\beta + 2) + r(k)C_k(\beta).$$  \hspace{1cm} (1.7)

The values of $h(k)$ and $r(k)$ ($1 \leq k \leq 8$) and the initial values of $C_k$ and $D_{k,i}$ ($1 \leq k \leq 8$) are listed in Table 4.
Utilizing a result of Engstrom [11], we prove that for any \( m \geq 2 \), there always exist positive integers \( \alpha \) and \( \beta \) satisfying \( B_k(2\alpha - 1) \equiv 0 \) (mod \( m \)) and \( D_{k,i}(2\beta - 1) \equiv 0 \) (mod \( m \)) for \( 1 \leq i \leq \left\lfloor \frac{k}{3} \right\rfloor \) in Theorem 3.2. Therefore we may define \( \mu_m(k) \) and \( \nu_m(k) \) by

\[
\mu_m(k) = \min \{ \alpha \geq 1 \mid B_k(2\alpha - 1) \equiv 0 \pmod{m} \},
\]

and

\[
\nu_m(k) = \min \{ \beta \geq 1 \mid D_{k,i}(2\beta - 1) \equiv 0 \pmod{m} \text{ for } 1 \leq i \leq \left\lfloor \frac{k}{3} \right\rfloor \}. \tag{1.8}
\]

In addition, let \( 0 \leq c_1 \leq m - 1 \) be the integer such that \( c_1 \equiv A_k(2\mu_m(k) - 1) \) (mod \( m \)), and \( 0 \leq c_2 \leq m - 1 \) be the integer such that \( c_2 \equiv C_k(2\nu_m(k) - 1) \) (mod \( m \)).

We can establish the following infinite families of congruences for \( p_{sk}(n) \) and \( p_{3k}(n) \) modulo \( m \) with the aid of generating functions (1.2) and (1.5).

**Theorem 1.3.** For any \( m \geq 2 \) and \( 1 \leq k \leq 3 \), the following statements hold:

(i) If \( c_1^\alpha \equiv 0 \pmod{m} \) for any \( \alpha \geq 1 \), then for any \( n \geq 0 \),

\[
p_{sk} \left( 2^{2\mu_m(k)\alpha}n + \frac{(2k + 3) \cdot 2^{2\mu_m(k)\alpha - 1} - k}{3} \right) \equiv 0 \pmod{m}.
\]

(ii) If there exists a positive integer \( \alpha \) such that \( c_1^\alpha \equiv 0 \pmod{m} \), then for any \( n \geq 0 \),

\[
p_{sk} \left( 2^{2\mu_m(k)\alpha - 1}n + \frac{k(2^{2\mu_m(k)\alpha} - 1)}{3} \right) \equiv 0 \pmod{m}.
\]

**Theorem 1.4.** For any \( m \geq 2 \) and \( 1 \leq k \leq 8 \), the following statements hold:

(i) If \( c_2^\beta \equiv 0 \pmod{m} \) for any \( \beta \geq 1 \), then for any \( n \geq 0 \) and \( i = 1 \) or \( 2 \),

\[
p_{3k} \left( 3^{2\nu_m(k)\beta}n + \frac{(3k + 8i) \cdot 3^{2\nu_m(k)\beta - 1} - k}{8} \right) \equiv 0 \pmod{m}.
\]

(ii) If there exists a positive integer \( \beta \) such that \( c_2^\beta \equiv 0 \pmod{m} \), then for any \( n \geq 0 \),

\[
p_{3k} \left( 3^{2\nu_m(k)\beta - 1}n + \frac{k(3^{2\nu_m(k)\beta} - 1)}{8} \right) \equiv 0 \pmod{m}.
\]

It is worth mentioning that there are another two approaches to obtain some cases of Theorem 1.3 and Theorem 1.4. For the first one, the case of \( k = 1 \) in Theorem 1.3 and the case of \( k = 2 \) in Theorem 1.4 can also be obtained from Newman’s Theorem [25, Theorem 1]. For the second one, the cases of \( (k, m) = (1, \geq 2) \) in Theorem 1.3 and the cases of \( (k, m) = (2, \geq 2), (k, m) = (3, 12), (k, m) = (4, 12), (k, m) = (5, 1836), (k, m) = (7, 53028) \) in Theorem 1.4 can be deduced with the aid of eigenforms. For more details, please refer to [10] and [26].
As applications of Theorem 1.3 and Theorem 1.4, we obtain many infinite families of congruences for different kinds of partition functions. For example, we derive some congruences for \( \ell \)-regular partition functions modulo 2, 3, 5, 17 and 19, some of which are stated as follows:

for any \( \alpha \geq 1 \), \( \beta \geq 0 \) and \( i = 1 \) or \( 2 \),

\[
b_{25} \left( 2^{4\beta}3^{4\alpha} \cdot n + (i + 3) \cdot 2^{4\beta}3^{4\alpha-1} - 1 \right) \equiv 0 \pmod{5}
\]

and

\[
b_{25} \left( 2^{4\alpha}3^{4\beta} \cdot n + 2^{4\alpha-1}3^{4\beta+1}-1 \right) \equiv 0 \pmod{5}.
\]

We also obtain some infinite families of congruences for the overpartition function and \( t \)-core partition functions by applying Theorem 1.3 and Theorem 1.4. Recall that an overpartition of \( n \) is a partition of \( n \) where the first occurrence of each distinct part may be overlaid \([9]\). Denote by \( \overline{p}(n) \) the number of overpartitions of \( n \). The congruences for \( \overline{p}(n) \) have been extensively studied, see, for example, \([7,13,21,24,29]\). We obtain some new infinite families of congruences for the overpartition function modulo the powers of 2,

\[
\overline{p}(4 \cdot 3^{2\alpha}n + (4i + 3) \cdot 3^{2\alpha-1}) \equiv 0 \pmod{2^2},
\]

and

\[
\overline{p}(4 \cdot 3^{2\alpha}n + (4i + 6) \cdot 3^{2\alpha-1}) \equiv 0 \pmod{2^3},
\]

where \( \alpha \geq 1 \), \( n \geq 0 \) and \( i = 1 \) or \( 2 \).

For \( t \geq 1 \), a partition is called a \( t \)-core partition if none of its hook lengths is divisible by \( t \). Let \( a_t(n) \) denote the number of \( t \)-core partitions of \( n \). For congruences for \( t \)-core partition functions, see, for example, \([15,16,18,19,27]\). We obtain some infinite families of congruences for 2-core and 4-core partition functions, for example, for any \( \alpha \geq 1 \), \( \beta \geq 0 \), \( n \geq 0 \) and \( i = 1 \) or \( 2 \), we have

\[
a_2 \left( 3^{2\alpha}n + 3^{2\alpha-1}i + \frac{3^{2\alpha} - 1}{8} \right) \equiv 0 \pmod{2},
\]

\[
a_2 \left( \frac{3^{2\beta} - 1}{8} \right) \equiv 1 \pmod{2},
\]

and

\[
a_4 \left( 3^{2\alpha}n + \frac{(8i + 15) \cdot 3^{2\alpha-1} - 5}{8} \right) \equiv 0 \pmod{2}.
\]

Note that this yields a special case of Hirschhorn and Seller’s conjecture \([20]\) when \( \alpha = 1 \). This conjecture has been proved by Chen \([6]\).

The rest of this paper is organized as follows. In Section 2, we give the proofs of Theorem 1.1 and Theorem 1.2 by investigating the properties of the modular equations of the second and the third order under the Atkin \( U \)-operator. In Section 3, we first prove the existence of \( \mu_m(k) \) and \( \nu_m(k) \) via linear recurrence relations (1.4) and (1.7). Combining this with the generating functions in Theorem 1.1 and Theorem 1.2, we give the proofs of Theorem 1.3 and Theorem 1.4. Section 4 is devoted to infinite families of congruences for the overpartition function, \( t \)-core partition functions and \( \ell \)-regular partition functions with the aid of Theorem 1.3 and Theorem 1.4.
2 Generating functions and modular equations

To derive the generating functions of $p_8^k(2^{2\alpha}n + \frac{k(2\alpha - 1)}{3})$ (1 ≤ $k$ ≤ 3) and $p_3^k(3^{2\beta}n + \frac{k(3\beta - 1)}{8})$ (1 ≤ $k$ ≤ 8), we first introduce the following three operators acting on $\sum_{n=-\infty}^{\infty} a(n)q^n$. Let $p = 2$ or 3 and define

$$U_p \left\{ \sum_{n=-\infty}^{\infty} a(n)q^n \right\} = \sum_{n=-\infty}^{\infty} a(pm)q^n,$$

$$g_p(k) \left\{ \sum_{n=-\infty}^{\infty} a(n)q^n \right\} = \sum_{n=-\infty}^{\infty} a(pm + k)q^n,$$

$$G_p(k) \left\{ \sum_{n=-\infty}^{\infty} a(n)q^n \right\} = \sum_{n=-\infty}^{\infty} a(p^2 n + k)q^n.$$

Clearly, we have $U_p g_p(k) = G_p(k)$. Note that $U_p$ is the Atkin $U$-operator [4], and Cui, Gu and Huang [10] introduced the notation of $g_p$ and $G_p$ for $p = 5$ or 7.

In the notation of these three operators, the left hand sides of (1.2), (1.3), (1.5) and (1.6) can be written as

$$G^\alpha_2(k)\{f_8^k\} = \sum_{n=0}^{\infty} p_8^k \left( 2^{2\alpha}n + \frac{k(2\alpha - 1)}{3} \right) q^n,$$

$$g_2 G^\alpha_2(k)\{f_8^k\} = \sum_{n=\lfloor k/2 \rfloor}^{\infty} p_8^k \left( 2^{2\alpha+1}n + \frac{k(2\alpha + 2 - 1)}{3} \right) q^n,$$

$$G^\beta_3(k)\{f_3^k\} = \sum_{n=0}^{\infty} p_3^k \left( 3^{2\beta}n + \frac{k(3\beta - 1)}{8} \right) q^n,$$

and

$$g_3 G^\beta_3(k)\{f_3^k\} = \sum_{n=\lfloor k/3 \rfloor}^{\infty} p_3^k \left( 3^{2\beta+1}n + \frac{k(3\beta + 2 - 1)}{8} \right) q^n.$$

Thus, the proof of Theorem 1.1 (resp. Theorem 1.2) is equivalent to compute $G^\alpha_2(k)\{f_8^k\}$ and $g_2 G^\alpha_2(k)\{f_8^k\}$ (resp. $G^\beta_3(k)\{f_3^k\}$ and $g_3 G^\beta_3(k)\{f_3^k\}$).

In order to obtain $G^\alpha_2(k)\{f_8^k\}$ and $g_2 G^\alpha_2(k)\{f_8^k\}$, we need the following modular equation of the second order (2.1).

Set

$$S = S(q) = \frac{f_8^{24}}{q f_2^{24}} \quad \text{and} \quad T = T(q) = \frac{f_8^{8}}{q f_2^{8}}.$$

Then the modular equation of the second order is

$$S(q^2) = T^2 + 16T. \quad (2.1)$$
This celebrated identity was discovered by Jacobi [30, p. 470], which also appears in [28, p. 833] and [17, p. 394].

The next lemma shows that \( U_2\{T^n\} \) is a polynomial of \( S \) or \( S^{-1} \).

**Lemma 2.1.** For any \( n \geq 0 \), \( U_2\{T^n\} \) is an integer polynomial in \( S \) with degree at most \( \lfloor \frac{n}{2} \rfloor \). Similarly, for \( n < 0 \), \( U_2\{T^n\} \) is an integer polynomial of \( S^{-1} \) with degree at most \( -n \).

**Proof.** First, we prove the case of \( n \geq 0 \) by induction on \( n \). For \( n = 0 \), we have done since \( U_2\{T^0\} = 1 \). By the following 2-dissection of \( f_1^4 \) [5, p. 40, Entry 25]:

\[
f_1^4 = \frac{f_4^{10}}{f_2^8 f_8^2} - 4q^2 \frac{f_2^4 f_8^4}{f_4^2},
\]

we obtain

\[
T = \frac{f_2^4}{q f_8^4} = \frac{1}{q f_4^4} \left( \frac{f_4^{10}}{f_2^8 f_8^2} - 8q f_4^8 + 16q^2 \frac{f_2^4 f_8^4}{f_4^2} \right) = -8 + \frac{f_4^{12}}{q f_2^4 f_8^4} + 16q^2 \frac{f_2^4 f_8^4}{f_1^2}.
\]

It implies that \( U_2\{T\} = -8 \). Therefore, the lemma holds for \( n = 0 \) and 1. Suppose that the lemma holds for the case of \( 0 \leq n \leq \lambda \) (\( \lambda \geq 1 \)). By means of the modular equation of the second order (2.1), we get that

\[
U_2\{T^{\lambda+1}\} = U_2\{S(q^2)T^{\lambda-1}\} - 16U_2\{T^\lambda\} = S \cdot U_2\{T^{\lambda-1}\} - 16U_2\{T^\lambda\},
\]

which means that \( U_2\{T^{\lambda+1}\} \) is also an integer polynomial of \( S \). Because \( \lfloor \frac{\lambda+1}{2} \rfloor + 1 = \lfloor \frac{\lambda+1}{2} \rfloor \geq \lfloor \frac{\lambda}{2} \rfloor \), the degree of \( U_2\{T^{\lambda+1}\} \) is at most \( \lfloor \frac{\lambda+1}{2} \rfloor \). This proves the case of \( n \geq 0 \).

From (2.1), we see that

\[
U_2\{T^{-\lambda-1}\} = S^{-1}U_2\{T^{-\lambda+1}\} + 16S^{-1}U_2\{T^{-\lambda}\}.
\]

(2.2)

For \( n < 0 \), it can be shown that

\[
U_2\{T^{-1}\} = 8S^{-1} \quad \text{and} \quad U_2\{T^{-2}\} = 128S^{-2} + S^{-1},
\]

which imply that the lemma holds for \( n = -1 \) and \( -2 \). Suppose that the statement holds for \( -\lambda \leq n \leq -1 \) (\( \lambda \geq 2 \)). It follows from (2.2) that \( U_2\{T^{-\lambda-1}\} \) is an integer polynomial of \( S^{-1} \), and its degree is at most \( \lambda + 1 \).

For ease of calculation, we list \( U_2\{T^n\} \) for \( -2 \leq n \leq 3 \) in Table 1.

| \( n \)  | \(-2\) | \(-1\) | \(0\) | \(1\) | \(2\) | \(3\) |
|--------|-------|-------|-------|-------|-------|-------|
| \( U_2\{T^n\} \) | \(128S^{-2} + S^{-1}\) | \(8S^{-1}\) | \(1\) | \(-8\) | \(S + 128\) | \(-24S - 2048\) |

Table 1: \( U_2\{T^n\} \) for \(-2 \leq n \leq 3 \).

For the sake of brevity, we abbreviate \( G_2(k)\{f_1^{sk}P(S)\} \) (resp. \( g_2(k)\{f_1^{sk}P(S)\} \)) as \( G_2\{f_1^{sk}P(S)\} \) (resp. \( g_2\{f_1^{sk}P(S)\} \)) if no confusions arise, where \( P(S) \) is a power series of \( S \).
Lemma 2.2. For any $\alpha \geq 0$ and $1 \leq k \leq 3$, $f_2^{-8k} g_2^{\alpha} f_1^{8k}$ is an integer polynomial of $S$ with degree $\leq \lfloor \frac{k}{2} \rfloor$, and $f_1^{-8k} G_2^{\alpha} f_1^{8k}$ is an integer polynomial of $S^{-1}$ with degree $\leq \lfloor \frac{k}{2} \rfloor$.

Proof. We prove this lemma by induction on $\alpha$. For $\alpha = 0$, we see that
\[ g_2 f_1^{8k} = g_2 \{ q^k f_1^{8k} T^k \} = f_2^{8k} U_2 \{ T^k \}. \]
This, together with Lemma 2.1, yields that $f_2^{-8k} g_2 f_1^{8k}$ is a polynomial of $S$ with integral coefficients, and its degree is at most $\lfloor \frac{k}{2} \rfloor$. Suppose that this lemma holds for $\alpha$, then we set
\[ g_2 G_2^{\alpha} f_1^{8k} = f_2^{8k} \sum_{i=0}^{\lfloor k/2 \rfloor} a_i S^i, \tag{2.3} \]
where $a_i \ (0 \leq i \leq \lfloor \frac{k}{2} \rfloor)$ are integers. Applying $U_2$ to both sides of (2.3), we obtain
\[ G_2^{\alpha+1} f_1^{8k} = U_2 g_2 G_2^{\alpha} f_1^{8k} = U_2 \left\{ f_2^{8k} \sum_{i=0}^{\lfloor k/2 \rfloor} a_i S^i \right\}. \tag{2.4} \]
By the definition of $U_2$, we have for any integer $i$,
\[ U_2 f_2^{8k} S^i = U_2 \{ q^{2i} f_2^{8k-24i} f_1^{24i} T^{3i} \} = q^i f_1^{8k-24i} f_2^{24i} U_2 \{ T^{3i} \} = f_1^{8k} S^{-i} U_2 \{ T^{3i} \}. \tag{2.5} \]
Combining (2.5) with (2.4), we arrive at
\[ G_2^{\alpha+1} f_1^{8k} = f_1^{8k} \sum_{i=0}^{\lfloor k/2 \rfloor} a_i S^{-i} U_2 \{ T^{3i} \}. \]
For $1 \leq k \leq 3$, it is immediate from Lemma 2.1 that $S^{-i} U_2 \{ T^{3i} \}$ is an integer polynomial of $S^{-1}$ with degree at most $\lfloor \frac{k}{2} \rfloor$ if $0 \leq i \leq \lfloor \frac{k}{2} \rfloor$. So we see that $f_1^{-8k} G_2^{\alpha+1} f_1^{8k}$ is an integer polynomial of $S^{-1}$ with degree at most $\lfloor \frac{k}{2} \rfloor$. Since $f_1^{-8k} G_2^{0} f_1^{8k} = 1$, the second statement of this lemma holds. Thus, we assume that
\[ G_2^{\alpha+1} f_1^{8k} = f_1^{8k} \sum_{i=0}^{\lfloor k/2 \rfloor} b_i S^{-i}, \tag{2.6} \]
where $b_i \ (0 \leq i \leq \lfloor \frac{k}{2} \rfloor)$ are integers. Since
\[ g_2 \{ f_1^{8k} S^i \} = g_2 \{ q^{-i} f_1^{8k+24i} f_2^{24i} U_2 \{ T^{k+3i} \} \} = f_1^{8k} S^{-i} U_2 \{ T^{k+3i} \} \tag{2.7} \]
holds for any integer $i$, it follows from (2.6) that
\[ g_2 G_2^{\alpha+1} f_1^{8k} = f_2^{8k} \sum_{i=0}^{\lfloor k/2 \rfloor} b_i S^i U_2 \{ T^{k-3i} \}. \]
Note that $k - 3i < 0$ may hold only if $i = \lfloor \frac{k}{3} \rfloor$. In this case, from Lemma 2.1, it can be seen that the degree of $U_2 \{ T^{k-3i} \}$, as an integer polynomial of $S^{-1}$, is at most $3i - k = 3 \lfloor \frac{k}{3} \rfloor - k \leq \lfloor \frac{k}{2} \rfloor = i$. So $S^i U_2 \{ T^{k-3i} \}$ is an integer polynomial of $S$ with degree $\leq \lfloor \frac{k}{2} \rfloor$. If $k - 3i \geq 0$, then the power of $S$ in $S^i U_2 \{ T^{k-3i} \}$ is at most $i + \lfloor \frac{k-3i}{2} \rfloor = \lfloor \frac{k-3i}{2} \rfloor \leq \frac{k}{2}$. Therefore, $f_2^{-8k} g_2 G_2^{\alpha+1} f_1^{8k} = \sum_{i=0}^{\lfloor k/2 \rfloor} b_i S^i U_2 \{ T^{k-3i} \}$ is an integer polynomial of $S$, and its degree is at most $\lfloor \frac{k}{2} \rfloor$. So the first statement is true. This completes the proof. \[\blacksquare\]
We are ready to prove Theorem 1.1 based on Lemma 2.2.

Proof of Theorem 1.1. We illustrate our method of proving Theorem 1.1 with an example of the case $k = 2$. By Lemma 2.2, we assume that for $\alpha \geq 0$

$$f_1^{-16}G_2^\alpha \{ f_1^{16} \} = A'_2(2\alpha) + B'_2(2\alpha)S^{-1},$$  \hspace{1cm} (2.8)
and

$$f_2^{-16}g_2G_2^\alpha \{ f_1^{16} \} = A'_2(2\alpha + 1) + B'_2(2\alpha + 1)S.$$

(2.9)

It follows from (2.8) that

$$g_2G_2^\alpha \{ f_1^{16} \} = A'_2(2\alpha)g_2\{ f_1^{16} \} + B'_2(2\alpha)g_2\{ f_1^{16}S^{-1} \}.$$ By (2.7), we obtain

$$g_2G_2^\alpha \{ f_1^{16} \} = A'_2(2\alpha)f_2^{16}U_2\{ T^2 \} + B'_2(2\alpha)f_2^{16}S \cdot U_2\{ T^{-1} \}.$$ Combining this with the expressions of $U_2\{ T^2 \}$ and $U_2\{ T^{-1} \}$ in Table 1, we have

$$f_2^{-16}g_2G_2^\alpha \{ f_1^{16} \} = (128A'_2(2\alpha) + 8B'_2(2\alpha)) + A'_2(2\alpha)S.$$ Comparing (2.10) with (2.9) implies the recurrence relations

$$\begin{cases} A'_2(2\alpha + 1) = 128A'_2(2\alpha) + 8B'_2(2\alpha), \quad (2.11) \\
B'_2(2\alpha + 1) = A'_2(2\alpha). \end{cases}$$

On the other hand, multiplying $f_2^{16}$ on both sides of (2.9) and then applying $U_2$, we obtain

$$G_2^{\alpha+1} \{ f_1^{16} \} = A'_2(2\alpha + 1)U_2\{ f_1^{16} \} + B'_2(2\alpha + 1)U_2\{ f_2^{16}S \}.$$ In light of (2.5), the equality (2.12) yields

$$G_2^{\alpha+1} \{ f_1^{16} \} = A'_2(2\alpha + 1)f_1^{16} + B'_2(2\alpha + 1)f_1^{16}S^{-1}U_2\{ T^3 \}.$$ (2.13)

Substituting the expression of $U_2\{ T^3 \}$ (see Table 1) into (2.13), we get

$$f_1^{-16}G_2^{\alpha+1} \{ f_1^{16} \} = (A'_2(2\alpha + 1) - 24B'_2(2\alpha + 1)) - 2048B'_2(2\alpha + 1)S^{-1}. \quad (2.14)$$ Comparing (2.14) with (2.8), we see that

$$\begin{cases} A'_2(2\alpha + 2) = A'_2(2\alpha + 1) - 24B'_2(2\alpha + 1), \quad (2.15) \\
B'_2(2\alpha + 2) = -2048B'_2(2\alpha + 1). \end{cases}$$

It is obvious that $A'_2(0) = 1$ and $B'_2(0) = 0$. Hence, $A'_2(\alpha)$ and $B'_2(\alpha)$ are determined by the recurrence relations (2.11) and (2.15) and the initial values $A'_2(0) = 1$ and $B'_2(0) = 0$. It is easy to check that $A'_2(\alpha)$ (resp. $B'_2(\alpha)$) and $A_2(\alpha)$ (resp. $B_2(\alpha)$) satisfy the same recurrence relation and share the same initial values. Thus, $A_2(\alpha) = A'_2(\alpha)$ and $B_2(\alpha) = B'_2(\alpha)$. This completes the proof.
To study $G_3^\beta(k\{f_1^{3k}\})$ and $g_3G_3^\beta(k\{f_1^{3k}\})$ \((1 \leq k \leq 8)\), we require the following modular equation of the third order \((2.16)\).

Define
\[
X = X(q) = \frac{f_1^{12}}{qf_3^{12}} \quad \text{and} \quad Y = Y(q) = \frac{f_3^{13}}{qf_9^{13}}.
\]

Then the modular equation of the third order is of the form
\[
X(q^3) = Y^3 + 9Y^2 + 27Y. \tag{2.16}
\]
For more details, please refer to [8, Theorem 9.1] and [17, p. 397].

Now we proceed to derive some properties related to the modular equation of the third order.

**Lemma 2.3.** For any \(n \geq 0\), \(U_3\{Y^n\}\) is an integer polynomial in \(X\) with degree at most \(\lfloor \frac{n}{3} \rfloor\). Similarly, for \(n < 0\), \(U_3\{Y^n\}\) is an integer polynomial of \(X^{-1}\) with degree at most \(-n\).

**Proof.** We first show the statements of the case of \(n \geq 0\) by induction on \(n\). Due to Hirschhorn [17, p. 397, (43.3.14)], it is easy to deduce that
\[
U_3\{Y^0\} = 1, \quad U_3\{Y\} = -3 \quad \text{and} \quad U_3\{Y^2\} = 9. \tag{2.17}
\]

Therefore, the lemma holds for \(n = 0, 1\) and 2. Suppose that the lemma holds for \(0 \leq n \leq \lambda\) \((\lambda \geq 2)\). By means of the modular equation of the third order \((2.16)\), we get that
\[
U_3\{Y^{\lambda+1}\} = X \cdot U_3\{Y^{\lambda-2}\} - 27U_3\{Y^{\lambda-1}\} - 9U_3\{Y^\lambda\},
\]
which means that \(U_3\{Y^{\lambda+1}\}\) is also an integer polynomial of \(X\). Because \(\lfloor \frac{\lambda - 2}{3} \rfloor + 1 = \lfloor \frac{\lambda + 1}{3} \rfloor \geq \lfloor \frac{\lambda - 1}{3} \rfloor\), the degree of \(U_3\{Y^{\lambda+1}\}\) is at most \(\lfloor \frac{\lambda + 1}{3} \rfloor\). This proves the case of \(n \geq 0\).

From \((2.16)\), we derive that
\[
U_3\{Y^{-\lambda-1}\} = X^{-1}U_3\{Y^{-\lambda+2}\} + 9X^{-1}U_3\{Y^{-\lambda+1}\} + 27X^{-1}U_3\{Y^{-\lambda}\}. \tag{2.18}
\]

For \(n < 0\), combining \((2.18)\) with \((2.17)\), we have
\[
U_3\{Y^{-1}\} = 9X^{-1},
\]
\[
U_3\{Y^{-2}\} = 6X^{-1} + 243X^{-2},
\]
\[
U_3\{Y^{-3}\} = X^{-1} + 234X^{-2} + 6561X^{-3},
\]
which show that the lemma holds for \(n = -1, -2\) and \(-3\). Suppose that the statement holds for \(-\lambda \leq n \leq -1\) \((\lambda \geq 3)\). From \((2.18)\), we derive that \(U_3\{Y^{-\lambda-1}\}\) is also an integer polynomial of \(X^{-1}\), and its degree is at most \(\lambda + 1\). This completes the proof for \(n < 0\).
For ease of calculation, we list $U_3\{Y^n\}$ for $-3 \leq n \leq 8$ in Table 2.

For the sake of convenience, we abbreviate $G_3(k)\{f_3^{3k}P(X)\}$ (resp. $g_3(k)\{f_3^{3k}P(X)\}$) as $G_3\{f_3^{3k}P(X)\}$ (resp. $g_3\{f_3^{3k}P(X)\}$) if no confusions arise, where $P(X)$ is a power series of $X$.

We also obtain the following equalities similar to (2.5) and (2.7), for any integer $i$,

$$U_3\{f_3^{3k}X^i\} = U_3\{q^{-i}f_1^{12i}f_3^{3k-12i}\} = q^i f_1^{3k-12i} f_3^{12i} U_3\{Y^{4i}\} = f_1^{3k} X^{-i} U_3\{Y^{4i}\}, \quad (2.19)$$

and

$$g_3\{f_3^{3k}X^i\} = g_3\{q^{-i}f_1^{3k+12i}f_3^{-12i}\} = q^i f_1^{-12i} f_3^{3k+12i} U_3\{Y^{k+4i}\} = f_3^{3k} X^{-i} U_3\{Y^{k+4i}\}. \quad (2.20)$$

Based on the above two equalities, we can derive the following statements parallel to Lemma 2.2.

**Lemma 2.4.** For any $\beta \geq 0$ and $1 \leq k \leq 8$, $f_3^{-3k}g_3 G_3^\beta \{f_3^{3k}\}$ is an integer polynomial of $X$ with degree $\leq \left\lfloor \frac{4\beta}{3} \right\rfloor$, and $f_1^{-3k}G_3^\beta \{f_1^{3k}\}$ is an integer polynomial of $X^{-1}$ with degree $\leq \left\lfloor \frac{4\beta}{3} \right\rfloor$.

**Proof.** The proof is analogous to the proof of Lemma 2.2, and hence is omitted.

We prove Theorem 1.2 with the aid of Lemma 2.4.

**Proof of Theorem 1.2.** Since there are eight cases should be considered for $1 \leq k \leq 8$, we only prove this theorem for $k = 7$ without loss of generality. Other cases can be derived similarly. By Lemma 2.4, we assume that for any $\beta \geq 0$,

$$f_1^{-21} G_3^\beta \{f_1^{21}\} = C_7^\prime(2\beta) + D_{7,1}^\prime(2\beta) X^{-1} + D_{7,2}^\prime(2\beta) X^{-2}, \quad (2.21)$$

and

$$f_3^{-21} g_3 G_3^\beta \{f_1^{21}\} = C_7^\prime(2\beta + 1) + D_{7,1}^\prime(2\beta + 1) X + D_{7,2}^\prime(2\beta + 1) X^2. \quad (2.22)$$
Multiplying \( f_1^{21} \) on both sides of (2.21) and applying \( g_3 \), we have

\[
g_3 G_3^\beta \{ f_1^{21} \} = C_7'(2\beta)g_3 \{ f_1^{21} \} + D_{7,1}'(2\beta)g_3 \{ f_1^{21} X^{-1} \} + D_{7,2}'(2\beta)g_3 \{ f_1^{21} X^{-2} \}.
\]

Combining this with (2.20), we obtain that

\[
g_3 G_3^\beta \{ f_1^{21} \} = C_7'(2\beta) f_3^{21} U_3 \{ Y^7 \} + D_{7,1}'(2\beta) f_3^{21} X U_3 \{ Y^3 \} + D_{7,2}'(2\beta) f_3^{21} X^2 U_3 \{ Y^{-1} \}. \tag{2.23}
\]

Substituting the expressions of \( U_3 \{ Y^{-1} \} \), \( U_3 \{ Y^3 \} \) and \( U_3 \{ Y^7 \} \) (see Table 2) into (2.23), we get

\[
f_3^{-21} g_3 G_3^\beta \{ f_1^{21} \} = 59049 C_7'(2\beta) + (1701 C_7'(2\beta) + 9 D_{7,2}'(2\beta)) X + (-21 C_7'(2\beta) + D_{7,1}'(2\beta)) X^2. \tag{2.24}
\]

Matching the coefficients of like powers of \( X \) in (2.24) and (2.22), we obtain

\[
\begin{align*}
C_7'(2\beta + 1) &= 59049 C_7'(2\beta), \\
D_{7,1}'(2\beta + 1) &= 1701 C_7'(2\beta) + 9 D_{7,2}'(2\beta), \\
D_{7,2}'(2\beta + 1) &= -21 C_7'(2\beta) + D_{7,1}'(2\beta). \tag{2.25}
\end{align*}
\]

Meanwhile, the equality (2.22) implies that

\[
G_3^{\beta+1} \{ f_1^{21} \} = C_7'(2\beta + 1) U_3 \{ f_3^{21} \} + D_{7,1}'(2\beta + 1) U_3 \{ f_3^{21} X \} + D_{7,2}'(2\beta + 1) U_3 \{ f_3^{21} X^2 \}.
\]

By (2.19), we have

\[
G_3^{\beta+1} \{ f_1^{21} \} = C_7'(2\beta + 1) f_1^{21} + D_{7,1}'(2\beta + 1) f_1^{21} X^{-1} U_3 \{ Y^4 \} + D_{7,2}'(2\beta + 1) f_1^{21} X^{-2} U_3 \{ Y^8 \}.
\]

Due to the expressions of \( U_3 \{ Y^4 \} \) and \( U_3 \{ Y^8 \} \) (see Table 2), we arrive at

\[
G_3^{\beta+1} \{ f_1^{21} \} = (C_7'(2\beta + 1) - 12 D_{7,1}'(2\beta + 1) + 252 D_{7,2}'(2\beta + 1)) f_1^{21} - 243 D_{7,1}'(2\beta + 1) f_1^{21} X^{-1} - 177147 D_{7,2}'(2\beta + 1) f_1^{21} X^{-2}.
\]

Combining this with (2.21), we see that

\[
\begin{align*}
C_7'(2\beta + 2) &= C_7'(2\beta + 1) - 12 D_{7,1}'(2\beta + 1) + 252 D_{7,2}'(2\beta + 1), \\
D_{7,1}'(2\beta + 2) &= -243 D_{7,1}'(2\beta + 1), \\
D_{7,2}'(2\beta + 2) &= -177147 D_{7,2}'(2\beta + 1). \tag{2.26}
\end{align*}
\]

Clearly, \( C_7'(0) = 1 \), \( D_{7,1}'(0) = D_{7,2}'(0) = 0 \). Hence, \( C_7'(\beta) \), \( D_{7,1}'(\beta) \) and \( D_{7,2}'(\beta) \) are uniquely determined by (2.25), (2.26) and the initial values \( C_7'(0) = 1 \), \( D_{7,1}'(0) = D_{7,2}'(0) = 0 \). By direct calculations, we know that \( C_7'(\beta) \), \( D_{7,1}'(\beta) \) and \( D_{7,2}'(\beta) \) enjoy the same recurrence relation and share the same initial values as \( C_7(\beta) \), \( D_{7,1}(\beta) \) and \( D_{7,2}(\beta) \), respectively. Thus, for any \( \beta \geq 0 \), we have \( C_7'(\beta) = C_7(\beta) \), \( D_{7,1}'(\beta) = D_{7,1}(\beta) \) and \( D_{7,2}'(\beta) = D_{7,2}(\beta) \). This completes the proof. \[\blacksquare\]
3 Congruences for $p_{8k}(n)$ and $p_{3k}(n)$

In this section, we first prove that for any $m \geq 2$, $\mu_m(k)$ ($1 \leq k \leq 3$) and $\nu_m(k)$ ($1 \leq k \leq 8$) always exist. Then we establish infinite families of congruences modulo $m$ for $p_{8k}(n)$ ($1 \leq k \leq 3$) and $p_{3k}(n)$ ($1 \leq k \leq 8$) based on the generating functions in Theorem 1.1 and Theorem 1.2.

Recall that

$$\mu_m(k) = \min\{\alpha \geq 1 \mid B_k(2\alpha - 1) \equiv 0 \pmod{m}\},$$

and

$$\nu_m(k) = \min\left\{\beta \geq 1 \mid D_{k,i}(2\beta - 1) \equiv 0 \pmod{m} \text{ for } 1 \leq i \leq \left\lfloor \frac{k}{3} \right\rfloor\right\},$$

where $B_k(\alpha)$ and $D_{k,i}(\beta)$ are obtained in Theorem 1.1 and Theorem 1.2, respectively.

Now we point out that for any $m \geq 2$, $\mu_m(k)$ ($1 \leq k \leq 3$) and $\nu_m(k)$ ($1 \leq k \leq 8$) must exist. In order to prove the existence, we need the definition of period of recurring sequences. For details, refer to [23, Chapter 6].

Let $\{s_n\}_{n=0}^{\infty}$ be an integer sequence. If there exist positive integers $r$ and $n_0$ such that $s_{n+r} \equiv s_n \pmod{m}$ for all $n \geq n_0$, then the sequence is called ultimately periodic for the modulus $m$ and $r$ is called a period of the sequence. The smallest number among all the possible periods of an ultimately periodic sequence for the modulus $m$ is called the least period of the sequence. It is easy to prove that every period of an ultimately periodic sequence for the modulus $m$ is divisible by the least period. An ultimately periodic sequence $\{s_n\}_{n=0}^{\infty}$ for the modulus $m$ with least period $r$ is called periodic if $s_{n+r} \equiv s_n \pmod{m}$ holds for all $n \geq 0$.

Let $d$ be a positive integer and $a_0, \ldots, a_{d-1}$ be given integers with $a_0 \neq 0$. An integer sequence $s_0, s_1, \ldots$ is called a $d$th-order linear recurring sequence if it satisfies the relation

$$s_{n+d} = a_{d-1}s_{n+d-1} + a_{d-2}s_{n+d-2} + \cdots + a_0s_n, \quad \text{for } n \geq 0$$

with the initial values $s_0, s_1, \ldots, s_{d-1}$.

Engstrom [11] studied the periodic property of linear recurring sequences for modulus $m$.

**Lemma 3.1.** [11, p. 210] Let $s_0, s_1, \ldots$ be a $d$th-order linear recurring sequence satisfying

$$s_{n+d} = a_{d-1}s_{n+d-1} + a_{d-2}s_{n+d-2} + \cdots + a_0s_n, \quad \text{for } n \geq 0$$

with $a_0 \neq 0$ and the initial values $s_0, s_1, \ldots, s_{d-1}$. If $m$ is prime to $a_0$, then the $d$th-order linear recurring sequence $s_0, s_1, \ldots$ is periodic; otherwise, it is ultimately periodic for the modulus $m$.

We have proved that $\{B_k(2\alpha)\}_{\alpha=0}^{\infty}$ (resp. $\{D_{k,i}(2\beta)\}_{\beta=0}^{\infty}$) is a linear recurring sequence in Theorem 1.1 (resp. Theorem 1.2). Using the above result of Engstrom, we prove that for arbitrary $m \geq 2$, $\mu_m(k)$ (resp. $\nu_m(k)$) defined by (1.8) (resp. (1.9)) always exists.
Theorem 3.2. For the sequence \( \{B_k(2\alpha - 1)\}_{\alpha=1}^\infty \) (resp. \( \{D_{k,i}(2\beta - 1)\}_{\beta=1}^\infty \)) obtained in Theorem 1.1 (resp. 1.2), the corresponding \( \mu_m(k) \) (resp. \( \nu_m(k) \)) always exists for any \( m \geq 2 \).

Proof. We only prove the case of \( \nu_m(k) \), and the case of \( \mu_m(k) \) is completely similar. It is sufficient to show there exists a positive integer \( \beta_0 \) such that \( D_{k,i}(2\beta_0 - 1) \equiv 0 \pmod{m} \) for all \( 1 \leq i \leq \left\lfloor \frac{k}{3} \right\rfloor \). According to Theorem 1.2 and Table 4, we observe that the corresponding \( a_0 \) of these linear recurring sequences \( \{D_{k,i}(2\beta)\}_{\beta=0}^\infty \) are of the form \( \pm 3^t \) for some integer \( t \). Therefore, by Lemma 3.1, if \( m \) is coprime to 3, then the integer sequences \( \{D_{k,i}(2\beta)\}_{\beta=0}^\infty \) are periodic for the modulus \( m \). This implies that there exist positive integers \( \beta_0 \) such that \( D_{k,i}(2\beta_0) \equiv 0 \pmod{m} \) since \( D_{k,i}(0) = 0 \). Observe that if \( 1 \leq i \leq \left\lfloor \frac{k}{3} \right\rfloor \) then \( -3^w D_{k,i}(2\beta - 1) = D_{k,i}(2\beta) \) for some positive integer \( w \). So we deduce that \( D_{k,i}(2\beta_0 - 1) \equiv 0 \pmod{m} \) for all \( 1 \leq i \leq \left\lfloor \frac{k}{3} \right\rfloor \).

It remains to show that \( \nu_m(k) \) also exists when \( m \) is not prime to 3. Assume that \( m = 3^j \cdot \ell \) with \( 3 \nmid \ell \) and \( j \geq 1 \). We discuss the property of \( D_{k,i}(2\beta - 1) \) modulo \( 3^j \) and \( \ell \), respectively. First, it should be noticed that for any \( 1 \leq i \leq \left\lfloor \frac{k}{3} \right\rfloor \), there always exists a positive integer \( N_i \) such that for any \( \beta \geq N_i \),

\[
D_{k,i}(2\beta - 1) \equiv 0 \pmod{3^j}.
\]

This is because all the coefficients \( h(k) \) and \( r(k) \) in the recurrence relation (1.7) are divisible by 3. Second, by Lemma 3.1 again, we see that the integer sequences \( \{D_{k,i}(2\beta)\}_{\beta=0}^\infty \) are periodic for the modulus \( \ell \). Let \( r_i \) be the least period of the \( \{D_{k,i}(2\beta)\}_{\beta=0}^\infty \) modulo \( \ell \). Then the least common multiple \( r \) of all \( r_i \) \((1 \leq i \leq \left\lfloor \frac{k}{3} \right\rfloor) \) is a period of all sequences \( \{D_{k,i}(2\beta)\}_{\beta=0}^\infty \), which implies

\[
0 \equiv D_{k,i}(0) \equiv D_{k,i}(2r) \equiv D_{k,i}(4r) \equiv D_{k,i}(6r) \equiv \cdots \pmod{\ell}, \quad \text{for } 1 \leq i \leq \left\lfloor \frac{k}{3} \right\rfloor.
\]

Combining this with the fact that if \( 1 \leq i \leq \left\lfloor \frac{k}{3} \right\rfloor \) then \( -3^w D_{k,i}(2\beta - 1) = D_{k,i}(2\beta) \) for some positive integer \( w \), we have

\[
0 \equiv D_{k,i}(2r - 1) \equiv D_{k,i}(4r - 1) \equiv D_{k,i}(6r - 1) \equiv \cdots \pmod{\ell}, \quad \text{for } 1 \leq i \leq \left\lfloor \frac{k}{3} \right\rfloor.
\]

Let \( n_0 \) be an integer such that \( n_0 r \geq N = \max_{1 \leq i \leq \left\lfloor \frac{k}{3} \right\rfloor} N_i \) and set \( \beta_0 = n_0 r \). Combining (3.1) with (3.2), we have \( D_{k,i}(2\beta_0 - 1) \equiv 0 \pmod{m} \). Therefore, we conclude that \( \nu_m(k) \) must exist. Similarly, we can show that \( \mu_m(k) \) always exists for any \( m \geq 2 \). This completes the proof. \( \square \)

Recall that \( c_1 \) is the integer such that \( 0 \leq c_1 \leq m - 1 \) and \( c_1 \equiv A_k(2\mu_m(k) - 1) \pmod{m} \) and \( c_2 \) is the integer such that \( 0 \leq c_2 \leq m - 1 \) and \( c_2 \equiv C_k(2\nu_m(k) - 1) \pmod{m} \). According to Theorem 3.2, it follows from (1.2) and (1.5) that for any \( m \geq 2 \),

\[
g_2 G_2^{\mu_m(k)-1} \{f_1^{8k}\} \equiv c_1 \cdot f_1^{8k} \pmod{m},
\]

and

\[
g_3 G_3^{\nu_m(k)-1} \{f_1^{3k}\} \equiv c_2 \cdot f_3^{3k} \pmod{m}.
\]

14
Thus we can establish infinite families of congruences for $p_{8k}(n)$ ($1 \leq k \leq 3$) and $p_{3k}(n)$ ($1 \leq k \leq 8$).

**Proof of Theorem 1.3.** For any $m \geq 2$, based on Theorem 3.2, we have
\[ g_2G_{2^{\mu_m(k)-1}}^{\mu_m(k)}\{f_1^{8k}\} \equiv c_1 \cdot f_2^{8k} \mod m, \]
where $0 \leq c_1 \leq m - 1$ and $c_1 \equiv A_k(2^{\mu_m(k)} - 1) \mod m$. Acting the operator $U_2$, we obtain that
\[ G_{2^{\mu_m(k)}}^{\mu_m(k)}\{f_1^{8k}\} \equiv c_1 \cdot f_1^{8k} \mod m. \]

Then for $\alpha \geq 1$, the following congruence holds
\[ G_{2^{\mu_m(k)(\alpha-1)}}^{\mu_m(k)(\alpha-1)}\{f_1^{8k}\} \equiv c_1^{\alpha-1} \cdot f_1^{8k} \mod m. \]

Acting the operator $g_2G_2^{\mu_m(k)-1}$, we have
\[ g_2G_{2^{\mu_m(k)}}^{\mu_m(k)}\{f_1^{8k}\} \equiv c_1^{\alpha} \cdot f_2^{8k} \mod m. \]

Equivalently,
\[ \sum_{n=0}^{\infty} p_{8k} \left( 2^{2^{\mu_m(k)(\alpha-1)}}n + \frac{k(2^{2^{\mu_m(k)(\alpha-1)}} - 1)}{3} \right) q^n \equiv c_1^{\alpha} f_2^{8k} \mod m. \]

(3.3)

As a consequence, if $c_1^{\alpha} \not\equiv 0 \mod m$ for any $\alpha \geq 1$, then we have
\[ p_{8k} \left( 2^{2^{\mu_m(k)(\alpha-1)}}(2n + 1) + \frac{k(2^{2^{\mu_m(k)(\alpha-1)}} - 1)}{3} \right) \equiv 0 \mod m. \]

Otherwise, there exists a positive integer $\alpha$ such that $c_1^{\alpha} \equiv 0 \mod m$. Then for such positive integer $\alpha$, it follows from (3.3) that
\[ p_{8k} \left( 2^{2^{\mu_m(k)(\alpha-1)}}n + \frac{k(2^{2^{\mu_m(k)(\alpha-1)}} - 1)}{3} \right) \equiv 0 \mod m. \]

This completes the proof.

Comparing the constant terms on both sides of (3.3) yields Corollary 3.3.

**Corollary 3.3.** Let $m$ and $k$ be integers with $m \geq 2$ and $1 \leq k \leq 3$. Then for any $\alpha \geq 1$,
\[ p_{8k} \left( \frac{k \cdot 2^{2^{\mu_m(k)(\alpha-1)}} - k}{3} \right) \equiv c_1^{\alpha} \mod m. \]
We are now ready to complete the proof of Theorem 1.4.

**Proof of Theorem 1.4.** For any \( m \geq 2 \), based on Theorem 3.2,

\[
g_3 G_3^{\nu_m(k)-1} \{ f_1^{3k} \} \equiv c_2 \cdot f_3^{3k} \pmod{m},
\]

where \( 0 \leq c_2 \leq m - 1 \) and \( c_2 \equiv C_k(2\nu_m(k) - 1) \pmod{m} \). Acting the operator \( U_3 \), we obtain that

\[
G_3^{\nu_m(k)} \{ f_1^{3k} \} \equiv c_2 \cdot f_3^{3k} \pmod{m}.
\]

Then for \( \beta \geq 1 \), we get

\[
G_3^{\nu_m(k)(\beta-1)} \{ f_1^{3k} \} \equiv c_2^{\beta-1} \cdot f_3^{3k} \pmod{m}.
\]

Acting the operator \( g_3 G_3^{\nu_m(k)-1} \), we have

\[
g_3 G_3^{\nu_m(k)(\beta-1)} \{ f_1^{3k} \} \equiv c_2^\beta \cdot f_3^{3k} \pmod{m},
\]

that is,

\[
\sum_{n=0}^{\infty} p_{3k} \left( 3^{2\nu_m(k)\beta-1} n + \frac{k(3^{2\nu_m(k)\beta} - 1)}{8} \right) q^n \equiv c_2^\beta \cdot f_3^{3k} \pmod{m}. \tag{3.4}
\]

Consequently, if \( c_2^\beta \not\equiv 0 \pmod{m} \) for any \( \beta \geq 1 \), then we have

\[
p_{3k} \left( 3^{2\nu_m(k)\beta-1} (3n + i) + \frac{k(3^{2\nu_m(k)\beta} - 1)}{8} \right) = p_{3k} \left( 3^{2\nu_m(k)\beta} n + \frac{(3k + 8i)3^{2\nu_m(k)\beta-1} - k}{8} \right) \equiv 0 \pmod{m},
\]

where \( i = 1 \) or 2. Otherwise, let \( \beta \) be an integer satisfying \( c_2^\beta \equiv 0 \pmod{m} \), then we have

\[
p_{3k} \left( 3^{2\nu_m(k)\beta-1} n + \frac{k(3^{2\nu_m(k)\beta} - 1)}{8} \right) \equiv 0 \pmod{m}
\]

and hence the proof is complete. \( \blacksquare \)

Equating the constants on both sides of (3.4) directly implies Corollary 3.4.

**Corollary 3.4.** Let \( m \) and \( k \) be integers with \( m \geq 2 \) and \( 1 \leq k \leq 8 \). Then for any \( \beta \geq 1 \),

\[
p_{3k} \left( \frac{k \cdot 3^{2\nu_m(k)\beta} - k}{8} \right) \equiv c_2^\beta \pmod{m}.
\]
4 Congruences for certain partition functions

In this section, we derive infinite families of congruences for certain partition functions, such as the overpartition function, $t$-core partition functions, $\ell$-regular partition functions in light of Theorems 1.1, 1.2, 1.3 and 1.4. The values of $\mu_m(k)$ and $\nu_m(k)$ play an important role in deducing congruences, and one can refer to Table 5 for the values about some small primes $m$.

4.1 Congruences for the overpartition function

Recall that an overpartition of $n$ is a partition of $n$, where the first occurrence of each distinct part may be overlined. We denote the number of overpartitions of $n$ by $p(n)$. The generating function of $p(n)$ is

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} = \frac{f_2}{f_1^2}. $$

In order to obtain the congruences of $p(n)$, we employ a 4-dissection of $f_2/f_1^2$. This identity was proved by Andrews, Passary, Sellers and Yee [3]. Here we give a new proof.

**Lemma 4.1.** We have

$$\frac{f_2}{f_1^2} = \frac{f_8^{19}}{f_4^{14}f_{16}^6} + 2q\frac{f_8^{13}}{f_4^{12}f_{16}^2} + 4q^2\frac{f_8^7f_{16}^2}{f_{16}^6} + 8q^3\frac{f_8^6f_{16}^6}{f_8^2}. $$

**Proof.** Employing the 2-dissection [5, p. 40, Entry 25] of $1/f_1^2$

$$\frac{1}{f_1^2} = \frac{f_5^5}{f_8^5} + 2q\frac{f_2^2f_{16}^2}{f_8^2f_8},$$

we deduce that

$$\frac{f_2}{f_1^2} = f_2 \left( \frac{f_5^5}{f_8^5} + 2q\frac{f_2^2f_{16}^2}{f_8^2f_8} \right) = \frac{1}{f_4^2} \left( \frac{f_5^5}{f_{16}^6} + 2q\frac{f_4^2f_{16}^2}{f_8} \right).$$

Thanks to the 2-dissection [5, p. 40, Entry 25] of $1/f_1^4$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_2^2f_8^4}{f_2^4},$$

we derive that

$$\frac{f_2}{f_1^2} = \left( \frac{f_4^{14}}{f_2^{14}f_{16}} + 4q^2\frac{f_8^2f_{16}^4}{f_4^{10}} \right) \left( \frac{f_5^5}{f_{16}^6} + 2q\frac{f_4^2f_{16}^2}{f_8} \right)$$

$$= \frac{f_8^{19}}{f_4^{14}f_{16}^6} + 2q\frac{f_8^{13}}{f_4^{12}f_{16}^2} + 4q^2\frac{f_8^7f_{16}^2}{f_{16}^6} + 8q^3\frac{f_8^6f_{16}^6}{f_8^2}. $$

This completes the proof. \(\blacksquare\)
The generating functions of $p(4n+k)$ ($k = 0, 1, 2$ and $3$) immediately follow from Lemma 4.1. In fact, Fortin, Jacob and Mathieu [13], and Hirschhorn and Sellers [21] established the 2-, 3-, and 4-dissections of the generating function for $p(n)$.

**Corollary 4.2.** We have

\[
\sum_{n=0}^{\infty} \frac{p(4n)q^n}{2} = \frac{f_2^{19}}{f_1^{14}f_4},
\]

\[
\sum_{n=0}^{\infty} \frac{p(4n+1)q^n}{4} = 2\frac{f_2^{13}}{f_1^{12}f_4},
\]

\[
\sum_{n=0}^{\infty} \frac{p(4n+2)q^n}{4} = 4\frac{f_2^{7}f_4}{f_1^{10}},
\]

and

\[
\sum_{n=0}^{\infty} \frac{p(4n+3)q^n}{8} = 8\frac{f_2f_6^4}{f_1^8}.
\]

So we deduce that

\[
\sum_{n=0}^{\infty} \frac{p(4n+1)q^n}{2} \equiv f_1^6 \pmod{2},
\] (4.1)

\[
\sum_{n=0}^{\infty} \frac{p(4n+2)q^n}{4} \equiv f_1^{12} \pmod{2},
\] (4.2)

and

\[
\sum_{n=0}^{\infty} \frac{p(4n+3)q^n}{8} \equiv f_1^{18} \pmod{2}.\] (4.3)

Applying Theorem 1.4 to the above identities, we obtain some congruences for $p(n)$.

**Theorem 4.3.** For any $\alpha \geq 1$, $n \geq 0$ and $i = 1$ or 2, we have

1. $p(4 \cdot 3^{2\alpha}n + (4i + 3) \cdot 3^{2\alpha-1}) \equiv 0 \pmod{2^2};$
2. $p(4 \cdot 3^{2\alpha}n + (4i + 6) \cdot 3^{2\alpha-1}) \equiv 0 \pmod{2^3};$
3. $p(4 \cdot 3^{4\alpha}n + (4i + 9) \cdot 3^{4\alpha-1}) \equiv 0 \pmod{2^4}.$

**Proof.** (1) For any $n \geq 0$, it follows from (4.1) that

\[
\frac{p(4n+1)}{2} \equiv p_6(n) \pmod{2}.\] (4.4)

Due to Theorem 1.2, we deduce that $\nu_2(2) = 1$ and $g_3\{f_1^6\} \equiv f_1^6 \pmod{2}$. Combining this with Theorem 1.4, we see that for any $\alpha \geq 1$ and $n \geq 0$, the congruences

\[
p_6 \left(3^{2\alpha}n + 3^{2\alpha-1}i + \frac{3^{2\alpha}-1}{4} \right) \equiv 0 \pmod{2}
\]

hold for $i = 1$ and 2. Then by (4.4) we arrive at

\[
p(4 \cdot 3^{2\alpha}n + (4i + 3) \cdot 3^{2\alpha-1}) \equiv 0 \pmod{2^2}.
\]
(2) By (4.2), we have for any $n \geq 0$,
\[
\frac{p(4n + 2)}{4} \equiv p_{12}(n) \pmod{2}. \tag{4.5}
\]
Due to Theorem 1.2, we deduce that $\nu_2(4) = 1$ and $g_3(f_{12}^{12}) \equiv f_3^{12} \pmod{2}$. Combining this with Theorem 1.4, we derive that for any $\alpha \geq 1$, $n \geq 0$ and $i = 1$ or 2,
\[
p_{12}\left(3^{2\alpha}n + 3^{2\alpha-1}i + \frac{3^{2\alpha} - 1}{2}\right) \equiv 0 \pmod{2}.
\]
Substituting this into (4.5) yields
\[
\frac{p(4 \cdot 3^{2\alpha}n + 4i \cdot 3^{2\alpha-1})}{2} \equiv 0 \pmod{2}.
\]
(3) In terms of (4.3), we have for any $n \geq 0$,
\[
\frac{p(4n + 3)}{8} \equiv p_{18}(n) \pmod{2}. \tag{4.6}
\]
Due to Theorem 1.2, we deduce that $\nu_2(6) = 2$ and $g_3G_3(f_{18}^{18}) \equiv f_3^{18} \pmod{2}$. In light of Theorem 1.4, we obtain that for any $\alpha \geq 1$, $n \geq 0$ and $i = 1$ or 2,
\[
p_{18}\left(3^{4\alpha}n + 3^{4\alpha-1}i + \frac{3^{4\alpha+1} - 3}{4}\right) \equiv 0 \pmod{2}.
\]
Substituting this into (4.6) yields
\[
\frac{p(4 \cdot 3^{4\alpha}n + 4i + 9 \cdot 3^{4\alpha-1})}{2} \equiv 0 \pmod{2}.
\]
This completes the proof.

Remark 4.4. Yang, Cui and Lin [32] proved some infinite families of congruences modulo powers of 2 for $p(n)$. For example, for $\alpha \geq 0$, $n \geq 0$ and $i = 1$ or 2, they showed that
\[
p(8 \cdot 3^{4\alpha+4}n + (4i + 9) \cdot 3^{4\alpha+3}) \equiv 0 \pmod{2^8},
\]
which contains the case of (3) in Theorem 4.3 when $n$ is even.

4.2 Congruences for $t$-core partition functions

Next we will use Theorem 1.4 to prove some congruences for $t$-core partition functions. The following generating function of the number of $t$-core partitions of $n$ is given by Garvan, Kim and Stanton [16],
\[
\sum_{n=0}^{\infty} a_t(n)q^n = \frac{f_t^t}{f_t^t}.
\]
Theorem 4.5. For any $\alpha \geq 1, \beta \geq 0, n \geq 0$ and $i = 1$ or $2$,

$$a_2 \left( 3^{2\alpha} n + 3^{2\alpha - 1} i + \frac{3^{2\alpha} - 1}{8} \right) \equiv 0 \pmod{2} \quad (4.7)$$

and

$$a_2 \left( \frac{3^{2\beta} - 1}{8} \right) \equiv 1 \pmod{2}.$$

Proof. Since $\sum_{n=0}^{\infty} a_2(n) q^n = \frac{f_2^2}{f_1} \equiv f_3^3 \pmod{2}$, it follows that for any $n \geq 0,$

$$a_2(n) \equiv p_3(n) \pmod{2}.$$

By Theorem 1.2, we have $\nu_2(1) = 1$ and $g_3(f_1) \equiv f_3^3 \pmod{2}$. Then Theorem 1.4 implies that for any $\alpha \geq 1$ and $n \geq 0,$

$$p_3 \left( 3^{2\alpha} n + 3^{2\alpha - 1} i + \frac{3^{2\alpha} - 1}{8} \right) \equiv 0 \pmod{2},$$

where $i = 1$ or $2$. Then the congruence (4.7) holds as claimed. By Corollary 3.4, we have for any $\beta \geq 0$,

$$a_2 \left( \frac{3^{2\beta} - 1}{8} \right) \equiv p_3 \left( \frac{3^{2\beta} - 1}{8} \right) \equiv 1 \pmod{2}.$$

This completes the proof.

Hirschhorn and Sellers [20] conjectured that for any $t \geq 2$ and $k = 0, 2$,

$$a_{2^t} \left( \frac{3^{2^{t-1}-1}(24n + 8k + 7) - 4^{t-1}i}{8} \right) \equiv 0 \pmod{2}$$

holds for all $n \geq 0$, which has been proved by Chen [6]. We can also deduce congruences for the 4-core partition function. More specifically, we have for any $\alpha \geq 1, \beta \geq 0, n \geq 0$ and $i = 1$ or $2$

$$a_4 \left( 3^{2\alpha} n + \frac{(8i + 15) \cdot 3^{2\alpha - 1} - 5}{8} \right) \equiv 0 \pmod{2}, \quad (4.8)$$

and

$$a_4 \left( \frac{5 \cdot 3^{2\beta} - 5}{8} \right) \equiv 1 \pmod{2}.$$

It is obvious that setting $\alpha = 1$ in (4.8) yields Hirschhorn and Seller’s conjecture for $t = 2$. Moreover, Gao, Cui and Guo [14, Theorem 1.2] contains the case of $i = 1$. 

20
4.3 Congruences for ℓ-regular partition functions

Now we study arithmetic properties for ℓ-regular partition functions. Recall that the generating function of \( b_\ell(n) \) is given by

\[
\sum_{n=0}^{\infty} b_\ell(n) q^n = \frac{f_\ell}{f_1}. \]

**Theorem 4.6.** For any \( \alpha \geq 1, \beta \geq 0, n \geq 0 \) and \( i = 1 \) or 2,

\[
b_{25}(2^{4\alpha}n + 3 \cdot 2^{4\alpha - 1} - 1) \equiv 0 \pmod{5}, \tag{4.9}
\]

\[
b_{25}(3^{4\alpha}n + (i + 3) \cdot 3^{4\alpha - 1} - 1) \equiv 0 \pmod{5}, \tag{4.10}
\]

and

\[
b_{25}(3^{4\beta} - 1) \equiv 1 \pmod{5}. \tag{4.11}
\]

**Proof.** It is easy to see that

\[
\sum_{n=0}^{\infty} b_{25}(n) q^n \equiv f_{24}^2 \pmod{5}. \tag{4.12}
\]

By Theorem 1.1 we find that \( \mu_5(3) = 2 \) and

\[ g_2 G_2 \{f_1^{24}\} \equiv f_2^{24} \pmod{5}. \tag{4.13} \]

By Theorem 1.2 we obtain that \( \nu_5(8) = 2 \) and

\[ g_3 G_3 \{f_1^{24}\} \equiv f_3^{24} \pmod{5}. \tag{4.14} \]

Then Theorem 1.3 and Theorem 1.4 imply that (4.9) and (4.10) hold, respectively.

Combining (4.13) with Corollary 3.4, we find that for any \( \beta \geq 0, \)

\[
b_{25}(3^{4\beta} - 1) \equiv 1 \pmod{5}. \tag{4.15}
\]

This completes the proof.

By considering Theorem 1.3 and Theorem 1.4 together, we obtain a generalized form of Theorem 4.6.

**Corollary 4.7.** For any \( \alpha \geq 1, \beta \geq 0 \) and \( i = 1 \) or 2, we have

\[
b_{25}(2^{4\beta}3^{4\alpha} \cdot n + (i + 3) \cdot 2^{4\beta}3^{4\alpha - 1} - 1) \equiv 0 \pmod{5}, \tag{4.16}
\]

and

\[
b_{25}(2^{4\alpha}3^{4\beta} \cdot n + 2^{4\alpha - 1}3^{4\beta + 1} - 1) \equiv 0 \pmod{5}. \tag{4.17}
\]
Proof. By (4.12) and (4.13), we find that for any $\beta \geq 0$,

$$G_2^\beta \{f_1^{24}\} \equiv f_1^{24} \pmod{5} \quad \text{and} \quad G_3^\beta \{f_1^{24}\} \equiv f_1^{24} \pmod{5}.$$  

It follows from (4.11) that

$$b_{25} \left(2^{4\beta} n + 2^{4\beta} - 1\right) \equiv b_{25}(n) \pmod{5} \quad (4.14)$$

and

$$b_{25} \left(3^{4\beta} n + 3^{4\beta} - 1\right) \equiv b_{25}(n) \pmod{5}. \quad (4.15)$$

Substitute (4.9) and (4.10) into (4.15) and (4.14), respectively and the proof follows.  

We also obtain some congruences for other $\ell$-regular partition functions. Andrews et al. [2, Theorem 3.5] showed that for $\alpha \geq 0$ and $n \geq 0$,

$$b_4 \left(3^{2\alpha+2} n + \frac{11 \cdot 3^{2\alpha+1} - 1}{8}\right) \equiv 0 \pmod{2}, \quad (4.16)$$

$$b_4 \left(3^{2\alpha+2} n + \frac{19 \cdot 3^{2\alpha+1} - 1}{8}\right) \equiv 0 \pmod{2}. \quad (4.17)$$

The congruences (4.16) and (4.17) can be easily derived by Theorem 1.4. Since the proofs are similar to the proof of Theorem 4.6, we list the congruences for 9-, 17-, 19-regular partition functions without proof. For any $\alpha \geq 1$, $\beta \geq 0$, $n \geq 0$ and $i = 1$ or $2$, we have

$$b_9 \left(2^{2\alpha} n + \frac{5 \cdot 2^{2\alpha-1} - 1}{3}\right) \equiv 0 \pmod{3}, \quad (4.18)$$

$$b_9 \left(\frac{2^{2\beta} - 1}{3}\right) \equiv 1 \pmod{3},$$

$$b_{17} \left(2^{18\alpha} n + \frac{7 \cdot 2^{18\alpha-1} - 2}{3}\right) \equiv 0 \pmod{17},$$

and

$$b_{19} \left(3^{10\alpha} n + \frac{(4i + 9) \cdot 3^{10\alpha-1} - 3}{4}\right) \equiv 0 \pmod{19}.$$

Note that the congruence (4.18) is coincident with Keith [22, Theorem 3].

Acknowledgements. This work was supported by the 973 Project and the National Science Foundation of China.
## Appendix

| $k$ | $f(k)$ | $g(k)$ | Initial values |
|-----|--------|--------|----------------|
| 1   | $-2^3$ | 0      | $A_1(0) = 1; A_1(1) = -8.$ |
| 2   | $2^3 \cdot 13$ | $-2^{14}$ | $A_2(0) = 1, A_2(2) = 104; A_2(1) = 128, A_2(3) = -3072;\ A_2(0) = 0, B_2(2) = -2048; B_2(1) = 1, B_2(3) = 104.$ |
| 3   | $-2^6 \cdot 5 \cdot 11$ | $-2^{22}$ | $A_3(0) = 1, A_3(2) = -1472; A_3(1) = -2048, A_3(3) = 3014656;\ B_3(0) = 0, B_3(2) = 49152; B_3(1) = -24, B_3(3) = 84480.$ |

Table 3: The values of $f(k)$, $g(k)$ and the initial values of $A_k(\alpha)$ and $B_k(\alpha)$.

| $k$ | $h(k)$ | $r(k)$ | Initial values |
|-----|--------|--------|----------------|
| 1   | $-3$   | 0      | $C_1(0) = 1, C_1(1) = -3.$ |
| 2   | $3^2$  | 0      | $C_2(0) = 1, C_2(1) = 9.$ |
| 3   | $-2^2 \cdot 3$ | $-3^7$ | $C_3(0) = 1, C_3(2) = -12; C_3(1) = 0, C_3(3) = -2187;\ C_3(2) = 0, D_3,1(2) = -243; D_3,1(1) = 1, D_3,1(3) = -12.$ |
| 4   | $-2 \cdot 3^2 \cdot 19$ | $-3^{10}$ | $C_4(0) = 1, C_4(2) = -99; C_4(1) = -243, C_4(3) = 24057;\ C_4(2) = 0, D_4,1(2) = 2916; D_4,1(1) = -12, D_4,1(3) = 4104.$ |
| 5   | $2^2 \cdot 3^3 \cdot 17$ | $-3^{13}$ | $C_5(0) = 1, C_5(2) = 1107; C_5(1) = 2187, C_5(3) = 2421009;\ C_5(0) = 0, D_5,1(2) = -21870; D_5,1(1) = 90, D_5,1(3) = 165240.$ |
| 6   | $-2 \cdot 3^2 \cdot 17 \cdot 23$ | $-3^{16}$ | $C_6(0) = 1, C_6(2) = -7038; C_6(1) = -13122, C_6(3) = 49305915;\ C_6(2) = 0, D_6,1(2) = 118098; D_6,1(1) = -486, D_6,1(3) = 3420468;\ D_6,2(0) = 0, D_6,2(2) = -177147; D_6,2(1) = 1, D_6,2(3) = -7038.$ |
| 7   | $2^2 \cdot 3^3 \cdot 491$ | $-3^{19}$ | $C_7(0) = 1, C_7(2) = 33345; C_7(1) = 59049, C_7(3) = 1968988905;\ C_7(2) = 0, D_7,1(2) = 413343; D_7,1(1) = 1701, D_7,1(3) = 90200628;\ D_7,2(0) = 0, D_7,2(2) = 3720087; D_7,2(1) = -21, D_7,2(3) = -1131588.$ |
| 8   | $-2 \cdot 3^4 \cdot 5 \cdot 359$ | $-3^{22}$ | $C_8(0) = 1, C_8(2) = -113643; C_8(1) = -177147, C_8(3) = 20131516521;\ C_8(2) = 0, D_8,2(2) = -44641044; D_8,2(1) = 252, D_8,2(3) = -73279080.$ |

Table 4: The values of $h(k)$, $r(k)$ and the initial values of $C_k(\beta)$ and $D_{k,i}(\beta)$. 
Table 5: Values for $\mu_m(k)$ and $\nu_m(k)$.

References

[1] G.E. Andrews, The Theory of Partitions, Addison-Wesley Publishing Co., 1976.

[2] G.E. Andrews, M.D. Hirschhorn and J.A. Sellers, Arithmetic properties of partitions with even parts distinct, Ramanujan J. 23 (2010) 169-181.

[3] G.E. Andrews, D. Passary, J. Sellers and A.J. Yee, Congruences related to the Ramanujan/Watson mock theta functions $\omega(q)$ and $\nu(q)$, Ramanujan J. 43 (2016) 347-357.

[4] A.O.L. Atkin and J. Lehner, Hecke Operators on $\Gamma_0(M)$, Math. Ann. 185 (1970) 134-160.

[5] B.C. Berndt, Ramanujan’s Notebooks, Part III, Springer, New York, 1991.

[6] S.-C. Chen, Congruences for $t$-core partition functions, J. Number Theory 133 (2013) 4036-4046.

[7] W.Y.C. Chen and E.X.W. Xia, Proof of a conjecture of Hirschhorn and Sellers on overpartitions, Acta Arith. 163(1) (2014) 59-69.

[8] S. Cooper, Ramanujan’s Theta Functions, Springer, 2017.

[9] S. Corteel and J. Lovejoy, Overpartitions, Trans. Amer. Math. Soc. 356 (2004) 1623-1635.

[10] S.-P. Cui, N.S.S. Gu and A.X. Huang, Congruence properties for a certain kind of partition functions, Adv. Math. 290 (2016) 739-772.

[11] H.T. Engstrom, On sequences of integers defined by recurrence relations, Trans. Amer. Math. Soc. 33 (1931) 210-218.
L. Euler, The expansion of the infinite product \((1 - x)(1 - xx)(1 - x^3)(1 - x^4)(1 - x^5)(1 - x^6)\) etc. into a single series, English translation from the Latin by Jordan Bell on arXiv:math.HO/0411454.

J.-F. Fortin, P. Jacob and P. Mathieu, Jagged partitions, Ramanujan J. 10 (2005) 215-235.

Y.Y. Gao, S.P. Cui and L.T. Guo, New Ramanujan-type congruences for 4-core partitions, Colloquium Mathematicum, 148 (2017) 157-164.

F. Garvan, Some congruence properties for partitions that are \(t\)-cores, Proc. London Math. Soc. 66 (1993).

F. Garvan, D. Kim and D. Stanton, Cranks and \(t\)-cores, Invent. Math. 101 (1990) 1-17.

M.D. Hirschhorn, The Power of \(q\), Developments in Mathematics, vol. 49, Springer, 2017.

M.D. Hirschhorn and J.A. Sellers, Some amazing facts about 4-cores, J. Number Theory 60 (1996) 51-69.

M.D. Hirschhorn and J.A. Sellers, Two congruences involving 4-cores, Electron. J. Combin. 3 (1996) #R10.

M.D. Hirschhorn and J.A. Sellers, Some parity results for 16-cores, Ramanujan J. 3 (1999) 281-296.

M.D. Hirschhorn and J.A. Sellers, Arithmetic relations for overpartitions, J. Combin. Math. Combin. Comput. 53 (2005) 65-73.

W.J. Keith, Congruences for 9-regular partitions modulo 3, Ramanujan J. 35 (2014) 157-164.

R. Lidl and H. Niederreiter, Introduction to Finite Fields and Their Applications, Cambridge University Press, 1986.

K. Mahlburg, The overpartition function modulo small powers of 2, Discret. Math. 286 (2004) 263-267.

M. Newman, An identity for the coefficients of certain modular forms, Canad. J. Math. 10 (1958) 577–586.

K. Ono, The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and \(q\)-Series, CBMS, vol. 102, Amer. Math. Soc., Providence, RI, 2004.

K. Ono and L. Sze, 4-core partitions and class numbers, Acta Arith. 65 (1997) 249-272.

P. Paule and C.-S. Radu, The Andrews-Sellers family of partition congruences, Adv. Math. 230 (2012) 819-838.
[29] S. Treneer, Congruences for the coefficients of weakly holomorphic modular forms, Proc. Lond. Math. Soc. 93 (2006) 304-324.

[30] E.T. Whittaker and G.N. Watson, A Course of Modern Analysis, Cambridge University Press, 1965.

[31] E.X.W. Xia, The powers of Euler’s product and congruences for certain partition functions, in preparation.

[32] X. Yang, S.P. Cui and B.L.S. Lin, Overpartition function modulo powers of 2, Ramanujan J. 44(2017) 89-104.