Event-Triggered Consensus of Matrix-Weighted Networks Subject to Actuator Saturation

Lulu Pan, Member, IEEE, Haibin Shao, Member, IEEE, Yuanlong Li, Dewei Li, and Yugeng Xi, Senior Member, IEEE

Abstract—This paper examines the event-triggered global consensus of matrix-weighted networks subject to actuator saturation. A distributed protocol design is proposed for this category of networks to guarantee its global consensus subject to both event-triggered communication and actuator saturation. It is shown that the largest singular value of matrix-valued edge weights plays a crucial role in both protocol design and network performance, which renders the proposed framework more general than existing results that are only applicable to scalar-weighted networks. Conditions under which the global consensus can be guaranteed for leaderless matrix-weighted multi-agent networks are derived. However, the average consensus on the initial agents’ states cannot be achieved due to the nonlinearities introduced in the closed-loop dynamics by the actuator saturation constraint. We further examine the scenario of leader-follower consensus for matrix-weighted multi-agent networks under the constraints of both event-triggered communication and actuator saturation. The applicability of the proposed protocol design framework to time-varying matrix-weighted networks is also examined. It is shown that the Zeno phenomenon can be excluded under the proposed interaction protocols. Simulation results in the context of the hearing-only cooperative formation of multi-vehicle systems are provided to demonstrate the effectiveness of theoretical results.

Index Terms—Actuator saturation, bearing-only formation, event-triggered mechanism, global consensus, largest singular value, matrix-weighted networks.

I. INTRODUCTION

Consensus problem on matrix-weighted networks is becoming a recent concern since, as an immediate generalization of scalar-weighted networks, matrix-weighted networks naturally capture interdependency complexity among vector-valued states of neighboring agents in a multi-agent network [1], [2], [3], [4], [5], [6], [7], [8], [9], [10]. In this view, scalar-weighted networks are only a special case of matrix-weighted networks, where the technical treatment of the latter is more intricate than the former. Actually, matrix-weighted networks arise in scenarios such as effective resistance based distributed control and estimation [11], [12], logical interdependency of multiple topics in opinion evolution [13], bearing-based formation control [14], array of coupled LC oscillators [15] as well as consensus and synchronization on matrix-weighted networks [3], [4].

In contrast to scalar-weighted networks, properties of matrix-valued edge weights now play an important role in not only the characterization of consensus (e.g., connectivity alone does not translate to achieving consensus on matrix-weighted networks), but also in the design of interaction protocols subject to physical constraints [8], [9]. In the literature, positive/negative definiteness of weight matrices has been employed to provide consensus conditions [2], [3], [4], [16]. In the meantime, it is worth noting that the matrix-weighted network is a generalized category of multi-agent networks, recent trends in this line of research involve the constraints of physical systems encountered in real-world applications. For instance, beyond the first-order local dynamics, consensus conditions for second-order multi-agent systems on matrix-weighted networks are provided [17], [18]. However, a comprehensive investigation of matrix-weighted networks subject to physical constraints is still lacking. Typically, these constraints can arise from input, output, and communication, which may bring nonlinearities in the closed-loop dynamics [19], [20], [21], [22], [23], [24], [25].

For distributed control of practical multi-agent systems, the control input is often subject to saturation constraints due to physical limitations. For instance, the bearing-based formation control of a drone swarm can be naturally formulated by using matrix-weighted Laplacian, where the actuator saturation problem arises from the amplitude of thruster on each drone [14]; cooperative formation of networked robotic manipulators is another notable example captured by matrix-weighted networks, where the torque provided by the motor at each joint is bounded [26]. In the existing literature, insightful efforts have been devoted to cooperative control of multi-agent systems subject to input saturation via continuous-time information exchange under the framework of traditional scalar-weighted networks. For instance, the global consensus problems of single-integrator and double-integrator multi-agent systems with input saturation were examined in [27], [28]. Moreover, it was shown in [28], [29] that a global leader-following consensus of neutraly stable linear multi-
agent systems with input saturation can be achieved using linear local feedback laws. By using the low-gain feedback design technique, semi-global consensus can be achieved for linear multi-agent systems with input saturation whose open-loop poles are all located in the closed left-half plane [30], [31].

However, in the aforementioned investigations, simultaneous information exchange and transmission between neighboring agents are needed, which is expensive from the perspective of both communication and computation. The event-triggered mechanism turns out to be efficient in handling this issue, where the control actuation or the information transmission was determined by the designed event [32], [33]. Decentralized event-triggered control for single-integrator multi-agent systems was initially proposed in [34] where the event-triggered function for the agent depends on the continuous information monitoring of its neighbors. In order to overcome this limitation, the distributed event-triggered functions proposed in [35] where only the state of neighboring agents at the last event-triggered time was employed to avoid the continuous information exchange between neighboring agents. However, this method was not satisfactory in the respect of avoiding Zeno behaviors. In [36], distributed event-triggered consensus control of single-integrator multi-agent systems was examined and it was shown that dynamic parameters ensure fewer triggering instants and played essential roles in avoiding Zeno behaviors. For more details about the event-triggered problems of multi-agent systems, one can refer to the recent survey papers [32], [33].

In contrast to the numerous results on multi-agent systems with saturated control or event-triggered control, few works address the consensus problem of multi-agent systems under the constraints of both input saturation and event-triggered communication. In this line of work, the influence of actuator saturation on event-triggered control for single systems is examined in [37]. In [38], a distributed event-triggered control strategy is proposed to achieve consensus for multi-agent systems subject to input saturation through output feedback, however, the Zeno behavior therein cannot be avoided. In [39], LMI techniques are employed to design leader-following consensus protocol for multi-agent systems subject to input saturation, but the design depends on the global information of graph Laplacian. Recently, the event-triggered global consensus problem for leaderless multi-agent systems with input saturation constraints using a triggering function whose threshold depends on time rather than the state has been studied [40].

Although the event-triggered consensus problem with input saturation constraint for scalar-weighted networks has been investigated, it turns out that the existing methods are only applicable to scalar-weighted networks. For the case of matrix-weighted networks, it becomes more challenging since specific properties of weight matrices have to be involved in the design of the interaction protocol for multi-agent networks under the constraints of input saturation and event-triggered communication, which makes this work non-trivial. To the best of our knowledge, this paper is the first attempt to examine the interaction protocol design problem for matrix-weighted networks subject to both actuator saturation and event-triggered communication.

**Contributions.** In this paper, a novel distributed event-triggered coordination protocol design framework is introduced for the multi-agent systems on matrix-weighted networks subject to actuator saturation. The dynamic parameters are employed to avoid Zeno behavior, whose update is determined by the local measurement error, the saturated state difference between each agent and its neighbors at triggering instants, and notably the largest singular value of matrix-valued edge weights associated with neighboring agents. Conditions under which the global consensus can be guaranteed for leaderless matrix-weighted multi-agent networks are derived. Note that the average consensus on the initial agents’ states cannot be achieved since nonlinearities are introduced in the closed-loop dynamics by the actuator saturation constraint. We further examine the scenario of leader-follower consensus for matrix-weighted multi-agent networks under the constraints of both event-triggered communication and actuator saturation. The applicability of the proposed design framework to time-varying matrix-weighted networks is further examined. The proposed design framework is more general than existing results that are only applicable to scalar-weighted networks [40]. Simulation results in the context of the bearing-only cooperative formation of multi-vehicle systems are provided to demonstrate the effectiveness of theoretical results.

The remainder of this paper is organized as follows. The preliminaries of matrix analysis and graph theory are introduced in Section II as well as fundamental facts of matrix-weighted networks. Then, the problem formulation is provided in Section III and the main results on the design of event-triggered bipartite consensus protocol for leaderless matrix-weighted networks and leader-follower matrix-weighted networks are provided in Section IV and Section V, respectively. The applicability of the proposed protocol design framework to time-varying matrix-weighted networks is further examined in Section VI, which is followed by the simulation results in Section VII. The concluding remarks are finally given in Section VIII.

**II. PRELIMINARIES**

In this section, we provide notations and background knowledge of matrix-weighted networks.

**A. Notations**

Let $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}_+$ be the set of real numbers, natural numbers, and positive integers, respectively. Denote $n = \{1, 2, \ldots, n\}$ for a $n \in \mathbb{Z}_+$. For a symmetric matrix $M$, if $M$ is positive definite (resp., negative definite), we write $M > 0$ (resp., $M < 0$); if $M$ is positive (resp., negative) semi-definite, we write $M \geq 0$ (resp., $M \leq 0$). The absolute value of a symmetric matrix $M \in \mathbb{R}^{n \times n}$ is denoted by $|M|$ such that $|M| = M$ if $M > 0$ or $M \geq 0$ and $|M| = -M$ if $M < 0$ or $M \leq 0$. The absolute value of a vector $z = (z_1, z_2, \ldots, z_n)^T \in \mathbb{R}^n$ is denoted by $|z|$ such that $|z| = (|z_1|, |z_2|, \ldots, |z_n|)^T$. We
shall write $z > 0$ if $z = |z|$ and $z \neq 0$. The null space of a matrix $M \in \mathbb{R}^{n \times n}$ is denoted by $\text{null}(M)$. Let $\lambda_i(M)$ and $\sigma_i(M)$ denote the $i$-th smallest eigenvalue and singular value of a symmetric matrix $M \in \mathbb{R}^{n \times n}$. Let $1_n \in \mathbb{R}^n$ and $0_{n \times n} \in \mathbb{R}^{n \times n}$ designate the vector whose components are all $1$'s and the matrix whose components are all $0$'s, respectively. Let $I_d \in \mathbb{R}^{d \times d}$ denote the identity matrix. The sign function $\text{sgn}(\cdot) : \mathbb{R}^{n \times n} \rightarrow \{0, -1, 1\}$ satisfies $\text{sgn}(M) = 1$ if $M \geq 0$ or $M > 0$, $\text{sgn}(M) = -1$ if $M \leq 0$ or $M < 0$, and $\text{sgn}(M) = 0$ if $M = 0_{n \times n}$.

B. Matrix-Weighted Networks

Let $G = (V, E, A)$ be a matrix-weighted network where the node set and the edge set of $G$ are denoted by $V = \{1, 2, \ldots, n\}$ and $E \subseteq V \times V$, respectively. The weight matrix for edges in $G$ is a symmetric matrix $A_{ij} \in \mathbb{R}^{d \times d}$ such that $|A_{ij}| \geq 0$ or $A_{ij} > 0$ if $(i, j) \in E$ and $A_{ij} = 0_{d \times d}$ otherwise for all $i, j \in V$. Thereby, the matrix-valued adjacency matrix $A = [A_{ij}] \in \mathbb{R}^{n \times d \times d}$ is a block matrix such that the block located in the $i$-th row (block) and the $j$-th column (block) is $A_{ij}$. We shall assume that $A_{ij} = A_{ji}$ for all $i \neq j \in V$ and $A_{ii} = 0_{d \times d}$ for all $i \in V$, which are analogous to the assumptions of the undirected and simple graphs in a normal sense. The neighbor set of an agent $i \in V$ is denoted by $N_i = \{j \in V | (i, j) \in E\}$. Denote $\mathcal{D} = \text{diag}(D_1, D_2, \ldots, D_n) \in \mathbb{R}^{d \times d \times n}$ as the matrix-weighted degree matrix of a graph where $D_i = \sum_{j \in N_i} |A_{ij}| \in \mathbb{R}^{d \times d}$. The matrix-valued Laplacian matrix of a matrix-weighted graph is defined as $L(G) = \mathcal{D} - A$.

In signed networks, the concept of structural balance (can be traced back to the seminal work [41]) turns out to be an important graph-theoretic object. The concept of structural balance for matrix-weighted networks is introduced in [3].

Definition 2.1: [3] A matrix-weighted network $G = (V, E, A)$ is structurally balanced if there exists a bipartition of the node set $V$, say $V_1$ and $V_2$ ($V_1 = V \cup V_2$ and $V_1 \cap V_2 = \emptyset$), such that the matrix weights on the edges within each subset is positive definite or positive semi-definite, but negative definite or negative semi-definite for the edges between the two subsets. A matrix-weighted network is structurally imbalanced if it is not structurally balanced.

Let $G = (V, E, A)$ be a matrix-weighted network with a node bipartition $V_1$ and $V_2$ and $d \in \mathbb{N}$ represent the dimension of matrix-valued edge weight. The gauge transformation for this node bipartition $V_1$ and $V_2$ is performed by a diagonal matrix $D' = \text{diag}(\eta_1, \eta_2, \ldots, \eta_n)$ where $\eta_i = I_d$ if $i \in V_1$ and $\eta_i = -I_d$ if $i \in V_2$.

Lemma 2.2 ([3]): If a matrix-weighted network $G = (V, E, A)$ is structurally balanced, then there exists a gauge transformation $D'$ such that $D'AD' = [[A_{ij}]] \in \mathbb{R}^{d \times n \times d}$.

The following result characterizes the structure of the null space of matrix-valued Laplacian of a matrix-weighted network, which is different from the Laplacian matrix of scalar-weighted networks whose null space is always $\text{span}(1_n)$.

Lemma 2.3 ([3]): Let $G = (V, E, A)$ be a structurally balanced matrix-weighted network. Then the associated matrix-valued Laplacian matrix $L$ of $G$ is positive semi-definite and the structure of its null space can be characterized by $\text{null}(L) = \text{span}(\mathcal{R}, \mathcal{H})$, where

$$\mathcal{R} = \text{range}\{(D^*1_n \otimes I_d)\}$$

and

$$\mathcal{H} = \{v = (v_1^T, v_2^T, \ldots, v_n^T)^T \in \mathbb{R}^{dn} | (v_i - \text{sgn}(A_{ij})v_j) \in \text{null}(A_{ij}), (i, j) \in E\}.$$
bipartite consensus under saturated interaction protocol. Consider the following interaction protocol,

$$u_i(t) = -\sum_{j \in \mathcal{N}_i} |A_{ij}|(\mathbf{x}_i(t) - \text{sgn}(A_{ij})\mathbf{x}_j(t)), \; i \in \mathcal{V},$$  \hspace{1cm} (7)$$

the overall dynamics of (3) can be characterized by

$$\dot{\mathbf{z}}(t) = \text{sat}_\Delta(-L\mathbf{z}(t)), $$ \hspace{1cm} (8)$$

where \(\mathbf{z}(t) = (\mathbf{z}_1^T(t), \mathbf{z}_2^T(t), \ldots, \mathbf{z}_n^T(t))^T \in \mathbb{R}^{dn}.$$

**Remark 4.1:** It is worth noting that a variety of scenarios can be modeled by (8) with further consideration of saturation constraint. For instance, array of LC oscillators [15], coupled multiple-link pendulums [42], cooperative manipulation of robotic arms [26], multi-topic opinion dynamics [43] and so forth.

**Lemma 4.2:** Let Assumption 1 hold. Then, the matrix-weighted multi-agent network (3) achieves global bipartite consensus under the interaction protocol (7) for arbitrary saturation level \(\Delta > 0.$$

**Proof:** Consider the following Lyapunov function candidate,

$$V(t) = \mathbf{z}^T(t)L\mathbf{z}(t),$$ \hspace{1cm} (9)$$

computing the time derivative of \(V(t)\) along with (8) yields,

$$\dot{V}(t) = 2\mathbf{z}^T(t)L\dot{\mathbf{z}}(t) = \mathbf{z}^T(t)L\text{sat}_\Delta(-L\mathbf{z}(t)) \leq 0.$$ \hspace{1cm} (10)$$

One can see that \(\dot{V}(t) = 0\) if and only if \(L\mathbf{z}(t) = 0\), i.e., \(\mathbf{z}(t) = \text{sgn}(A_{ij})\mathbf{z}_j(t), \forall i, j \in \mathcal{V} \). Thus according to LaSalle’s invariance principle [44],

$$\lim_{t \to \infty} \mathbf{z}(t) = \text{sgn}(A_{ij})\mathbf{z}_j(t) = 0, \forall i, j \in \mathcal{V}.$$ \hspace{1cm} (11)$$

That is, the matrix-weighted multi-agent network (3) achieves global bipartite consensus under the interaction protocol (7) for arbitrary saturation level \(\Delta > 0.$$

**Remark 4.3:** If the saturation level \(\Delta\) is large enough, the effect of the saturation function on system (8) vanishes. Then the result in Lemma IV.2 is in accordance with the result shown in [3], [4]. Actually, for matrix-weighted multi-agent network (3), due to the fact that \(\dot{V}(t) \leq 0\), one can see that the saturation constraint is no longer effective after a finite time which depends on the initial values of agents, the saturation function, matrix-valued edge weights, and the network topology.

From the above analysis, one can see that, to implement consensus protocol (8) with saturation, continuous states from neighbors are needed. However, continuous communication is impractical in physical applications. To avoid continuously sending information among agents and updating controls in the following, we shall equip the consensus protocol (8) with an event-triggered communication strategy, in this setting, the control signal is only updated when the triggering condition is satisfied.

### B. Event-Triggered Mechanism Design

Denote by \(\mathbf{z}^e(t)\) as the last broadcast state of agent \(i \in \mathcal{V}\) at any given time \(t \geq 0\), consider the following protocol for the leaderless multi-agent system with input saturation and event-triggered constraint,

$$\dot{\mathbf{x}}_i(t) = \text{sat}_\Delta(\mathbf{u}_i(t)), \; i \in \mathcal{V},$$ \hspace{1cm} (12)$$

and

$$\dot{\mathbf{u}}_i(t) = -\sum_{j \in \mathcal{N}_i} |A_{ij}|(\mathbf{z}_i(t) - \text{sgn}(A_{ij})\mathbf{z}_j(t)), $$ \hspace{1cm} (13)$$

let \(\mathbf{z}(t) = [\mathbf{z}_1^T(t), \mathbf{z}_2^T(t), \ldots, \mathbf{z}_n^T(t)]^T \in \mathbb{R}^{dn}, \) then the system (12) under the interaction protocol (13) can be written in a compact form as

$$\dot{\mathbf{z}}(t) = \text{sat}_\Delta(-L\mathbf{z}(t)).$$ \hspace{1cm} (14)$$

Define the state-based measurement error between the last broadcast state of agent \(i \in \mathcal{V}\) and its current state at time \(t \geq 0\) as

$$e_i(t) = \mathbf{z}_i(t) - \mathbf{z}(t),$$ \hspace{1cm} (15)$$

then the system-wise measurement error is denoted by \(e(t) = [e_1^T(t), e_2^T(t), \ldots, e_n^T(t)]^T. \) For each agent \(i \in \mathcal{V}\), the triggering time sequence is initiated from \(t_1^i = 0\) and subsequently determined by,

$$t_{k+1}^i = \max_{r \geq t_k^i} \{r | \theta_i(\varpi_i \| e_i(t) \|^2 - \rho \| \mathbf{u}_i(t) \| \text{sat}_\Delta(\mathbf{u}_i(t))) \leq \psi_i(t), \forall t \in [t_k^i, r]\},$$ \hspace{1cm} (16)$$

where \(k \in \mathbb{Z}_+, \rho_i \in [0, 1], \theta_i, \varpi_i \) are the design parameters and \(\psi_i(t)\) is an auxiliary system for each agent \(i \in \mathcal{V}\) such that,

$$\psi_i(t) = -\beta_i \psi_i(t) + \delta_i(\rho_i \| \mathbf{u}_i(t) \| \text{sat}_\Delta(\mathbf{u}_i(t)) - \varpi_i \| e_i(t) \| ^2),$$ \hspace{1cm} (17)$$

with \(\psi_i(0) > 0, \beta_i > 0\) and \(\delta_i \in [0, 1]. \)

**Remark 4.4:** For the problem considered in this work, the reason to employ an auxiliary system is that the Zeno behavior can be explicitly excluded.

We present our main result in this part as follows.

**Theorem 4.5:** Consider the multi-agent system (14) under the matrix-weighted network \(\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)\) satisfying Assumption 1. Let \(\theta_i, \varpi_i\) be such that \(\theta_i > \frac{1}{\beta_i k}\) and

$$\varpi_i = n \left( \sum_{j \in \mathcal{N}_i} \sigma_j(A_{ij})^2 + \sum_{j \in \mathcal{N}_i} \sigma^2_j(A_{ij}) \right),$$ \hspace{1cm} (18)$$

for all \(i \in \mathcal{V}\), respectively, the triggering time sequence is determined by (16) for agent \(i\) with \(\psi_i(t)\) defined in (17). Then the multi-agent system (14) admits global bipartite consensus.

**Proof:** Consider the Lyapunov function candidate as follows,

$$V(t) = V_1(t) + V_2(t),$$ \hspace{1cm} (19)$$
where $V_1(t) = x^T(t)Lx(t)$, and $V_2(t) = \sum_{i=1}^n \psi_i(t)$. For any $t \geq 0$, from the equations in (16) and (17), one has,
\[ \psi_i(t) \geq -\beta_i \psi_i(t) - \delta_i \psi_i(t), \]
and
\[ \psi_i(t) \geq \psi_i(0)e^{-(\beta_i+\delta_i)t} > 0, \]
therefore, one can get that $V(t) \geq 0$.

Computing the time derivative of $V_1(t)$ along with (14) yields,
\[
\dot{V}_1(t) = \dot{x}^T(t)Lx(t) + x^T(t)L\dot{x}(t)
= 2x^T(t)Lsat_x(u(t))
= -2u^T(t) sat_x(u(t)).
\]
Let $\phi(t) = [\phi_1^T(t), \phi_2^T(t), \ldots, \phi_n^T(t)]^T \in \mathbb{R}^{dn}$ and $\dot{\phi}(t) = \dot{u}(t) - \dot{u}(t)$, then one has,
\[
\dot{V}_1(t) = -2u^T(t) sat_x(u(t)) + 2\phi^T(t) sat_x(u(t))
= -\sum_{i=1}^n 2u_i^T(t) sat_x(u_i(t)) + \sum_{i=1}^n \phi_i^T(t) sat_x(u_i(t))
\leq -\sum_{i=1}^n 2u_i^T(t) sat_x(u_i(t)) + \sum_{i=1}^n \phi_i^T(t) \phi_i(t)
+ \sum_{i=1}^n sat_x(u_i(t))^T sat_x(u_i(t)).
\]
According to Lemma III.1, one has,
\[
sat_x(u_i(t))^T sat_x(u_i(t)) \leq \overline{u}_i^T(t) sat_x(u_i(t)),
\]
therefore,
\[
\dot{V}_1(t) \leq -\sum_{i=1}^n \overline{u}_i^T(t) sat_x(u_i(t)) + \sum_{i=1}^n \phi_i^T(t) \phi_i(t).
\]
Since,
\[
\phi_i(t) = \sum_{j \in N_i} |A_{ij}| (sgn(A_{ij})x_j(t) - x_i(t))
- \sum_{j \in N_i} |A_{ij}| (sgn(A_{ij})x_j(t) - x_i(t))
= |A_{ij}| (sgn(A_{ij})x_j(t) - x_i(t)),
\]
then one has,
\[
\phi_i(t)^T \phi_i(t)
\leq \left( \sum_{j \in N_i} |A_{ij}| \right) \| e_i(t) \| + \sum_{j \in N_i} |A_{ij}| \| e_i(t) \|
\leq (|N_i| + 1) \left( \sum_{j \in N_i} |A_{ij}| \right) \| e_i(t) \|^2
+ (|N_i| + 1) \sum_{j \in N_i} |A_{ij}| \| e_i(t) \|^2.
\]
Hence,
\[
\sum_{i=1}^n \phi_i(t)^T \phi_i(t) \leq \sum_{i=1}^n \sum_{j \in N_i} \sigma_d(A_{ij}) \| e_i(t) \|^2
+ \sum_{i=1}^n \sum_{j \in N_i} \sigma_d^2(A_{ij}) \| e_i(t) \|^2
= \sum_{i=1}^n \omega_i \| e_i(t) \|^2.
\]
where,
\[
\omega_i = n \left( \sum_{j \in N_i} \sigma_d(A_{ij}) \right)^2 + n \sum_{j \in N_i} \sigma_d^2(A_{ij}).
\]
Thus,
\[
\dot{V}_1(t) \leq -\sum_{i=1}^n \omega_i \| e_i(t) \|^2 - \sum_{i=1}^n \overline{u}_i^T(t) sat_x(u_i(t)).
\]

Now, we are in the position to consider the Lyapunov function candidate $V(t)$ in (19), one has,
\[
\dot{V}(t) = V_1(t) + \sum_{i=1}^n \psi_i(t)
\leq \sum_{i=1}^n \omega_i \| e_i(t) \|^2 - \sum_{i=1}^n \overline{u}_i^T(t) sat_x(u_i(t))
+ \sum_{i=1}^n \left( \delta_i \rho_i \overline{u}_i^T(t) sat_x(u_i(t)) - \omega_i \| e_i(t) \|^2 \right)
+ \sum_{i=1}^n (-\beta_i \psi_i(t))
= -\sum_{i=1}^n (1 - \delta_i \rho_i) \overline{u}_i^T(t) sat_x(u_i(t))
+ \sum_{i=1}^n (1 - \delta_i) \omega_i \| e_i(t) \|^2 - \sum_{i=1}^n \beta_i \psi_i(t)
= -\sum_{i=1}^n \beta_i \psi_i(t) + \sum_{i=1}^n (1 - \delta_i) \omega_i \| e_i(t) \|^2
- \sum_{i=1}^n \overline{u}_i^T(t) sat_x(u_i(t)) + \sum_{i=1}^n \rho_i \overline{u}_i^T(t) sat_x(u_i(t))
- \sum_{i=1}^n (1 - \delta_i) \rho_i \overline{u}_i^T(t) sat_x(u_i(t))
\leq -\sum_{i=1}^n \left( \beta_i - \frac{1 - \delta_i}{\theta_i} \right) \psi_i(t)
- \sum_{i=1}^n (1 - \rho_i) \overline{u}_i^T(t) sat_x(u_i(t))
- \sum_{i=1}^n (1 - \delta_i) \rho_i \overline{u}_i^T(t) sat_x(u_i(t))
\leq -\sum_{i=1}^n \left( \beta_i - \frac{1 - \delta_i}{\theta_i} \right) \psi_i(t)
- (1 - \max_{i \in \mathbb{N}} \{ \rho_i \}) \sum_{i=1}^n \overline{u}_i^T(t) sat_x(u_i(t)).
\]
Therefore, $\dot{V}(t) \leq 0$. Due to $V(t) \geq 0$ and $\dot{V}(t) \leq 0$, which implies that $\lim_{t \to \infty} \dot{V}(t) = 0$. Thus, one has $\lim_{t \to \infty} \psi_i(t) = 0$ and
\[
\lim_{t \to \infty} \hat{u}_i(t) = 0. \]
Due to the fact that
\[
0 \leq \| \mathbf{e}_i(t) \|^2 \leq \frac{\rho_i}{\omega_i} \hat{u}_i^T(t) \mathbf{sat}_\Delta(\hat{u}_i(t)) + \frac{1}{\omega_i} \psi_i(t),
\]
therefore, \( \lim_{t \to \infty} \mathbf{e}_i(t) = 0 \). Then, one has,
\[
\hat{V}_1(t) = \hat{\mathbf{x}}^T(t) L \mathbf{z}(t) + \mathbf{z}(t)^T L \mathbf{z}(t) = -2 \mathbf{z}(t)^T L \mathbf{sat}_\Delta(L(\mathbf{z}(t) + \mathbf{e}(t))),
\]
thus, \( \lim_{t \to \infty} L \mathbf{z}(t) = 0 \) and
\[
\lim_{t \to \infty} (\mathbf{z}(t) - \operatorname{sgn}(A_{ij}) \mathbf{z}(t)) = 0, \quad \forall i, j \in \mathbb{N}.
\]
That is, the multi-agent system (14) achieves global bipartite consensus.

Remark 4.6: Notably, owing to the nonlinearity induced by actuator saturation, the multi-agent system does not always achieve average bipartite consensus. The final consensus value of the network is eventually influenced by the saturation level \( \Delta \). Specifically, the consensus value is the average (after a proper gauge transformation) of the agents’ states at the last time instance \( T_{sd} \) that exist saturated control inputs in the multi-agent system, that is, \( D'(1_n \otimes (\frac{1}{2} (1_n^T \otimes I_d) D' \mathbf{z}(T_{sd}))) \). After \( T_{sd} \), the saturation constraint on the multi-agent system is eliminated until the achievement of the final bipartite consensus.

Remark 4.7: The proposed design framework is not only applicable to the matrix-weighted networks but also to the scalar-weighted networks. Note that (3) degenerates into the scalar-weighted case when setting \( A_{ij} = a_{ij} I_d \) where \( a_{ij} \in \mathbb{R} \). In this case, one can choose \( \sigma_d(A_{ij}) = |a_{ij}| \) in the design procedure. Therefore, the triggering function (16) is also suitable for scalar-weighted networks.

Remark 4.8: We notice that the proposed distributed protocol design can also be applied to multi-agent systems without satisfying Assumption 1. Specifically, from (33) in proof of Theorem 4.5, one has
\[
\lim_{t \to \infty} \sum_{(i,j) \in \mathcal{E}} (\mathbf{z}(t) - \operatorname{sgn}(A_{ij}) \mathbf{z}(t))^T |
\]
\[
A_{ij} | (\mathbf{z}(t) - \operatorname{sgn}(A_{ij}) \mathbf{z}(t)) = 0,
\]
therefore, the steady-state of multi-agent system (14) admit,
\[
\lim_{t \to \infty} (\mathbf{z}(t) - \operatorname{sgn}(A_{ij}) \mathbf{z}(t)) \in \mathbf{null}(\| \mathbf{A}_{ij} \|).
\]
In the following discussion, we shall prove that Zeno behavior can be avoided using the aforementioned event-triggered strategy. We have the following result.

Theorem 4.9: Under the global bipartite consensus condition in Theorem 4.5, the Zeno behavior of multi-agent system (14) under the matrix-weighted network \( \mathcal{G} = (V, E, A) \) can be avoided, i.e., there are no infinite triggering instants in a finite time.

Proof: By contradiction. Suppose that there exists Zeno behavior. Then, there at least exists one agent \( i \) such that
\[
\lim_{k \to \infty} t_k^i = T_0 \quad \text{where} \quad T_0 > 0.
\]
From the above analysis, we know that there exists a positive constant \( M_0 > 0 \) satisfying \( \| \mathbf{z}_i(t) \| \leq M_0 \) for all \( t \geq 0 \) and \( i \in \mathbb{N} \). Then one has
\[
\| \mathbf{u}_i(t) \| \leq 2 M_0 \sum_{j \in \mathbb{N}} \sigma_d(A_{ij}),
\]
for any \( t \geq 0 \). Choose
\[
\varepsilon_0 = \left( 4 M_0 \sum_{j \in \mathbb{N}} \sigma_d(A_{ij}) \right)^{-1} \sqrt{\frac{\psi_i(0)}{\theta_i \omega_i}} e^{-\frac{1}{2} (\rho_i + \frac{1}{\gamma}) T_0},
\]
according to the definition of limits, there exists a positive integer \( N(\varepsilon_0) \) such that for any \( k \geq N(\varepsilon_0) \),
\[
t_k^i \in [T_0 - \varepsilon_0, T_0].
\]
Then one sufficient condition to guarantee the inequality in (16) is
\[
\| \mathbf{e}_i(t) \| \leq \sqrt{\frac{\psi_i(0)}{\theta_i \omega_i}} e^{-\frac{1}{2} (\rho_i + \frac{1}{\gamma}) t_k^i}.
\]
In addition,
\[
\| \mathbf{e}_i(t) \| = \| \hat{\mathbf{z}}_i(t_k^i) - \mathbf{z}_i(t) \| = \| \mathbf{z}_i(t_k^i) - \mathbf{z}_i(t) \| \leq \int_{t_k^i}^t \| \mathbf{z}_i(t) \| \ d(t) \leq (t - t_k^i) \left( 2 M_0 \sum_{j \in \mathbb{N}} \sigma_d(A_{ij}) \right),
\]
then another sufficient condition to guarantee that the inequality in (16) holds if
\[
(t - t_k^i) \left( 2 M_0 \sum_{j \in \mathbb{N}} \sigma_d(A_{ij}) \right) \leq \sqrt{\frac{\psi_i(0)}{\theta_i \omega_i}} e^{-\frac{1}{2} (\rho_i + \frac{1}{\gamma}) t_k^i},
\]
(41) Let \( t_{N(\varepsilon_0)+1} \) and \( t_{N(\varepsilon_0)+1}^i \) denote the next triggering time determined by the inequalities in (16) and (41), respectively. Then,
\[
\begin{align*}
& t_{N(\varepsilon_0)+1}^i \geq \bar{t}_{N(\varepsilon_0)+1} - t_{N(\varepsilon_0)}^i \\
& = \left( 2 M_0 \sum_{j \in \mathbb{N}} \sigma_d(A_{ij}) \right)^{-1} \sqrt{\frac{\psi_i(0)}{\theta_i \omega_i}} e^{-\frac{1}{2} (\rho_i + \frac{1}{\gamma}) t_{N(\varepsilon_0)+1}} \\
& \geq \left( 2 M_0 \sum_{j \in \mathbb{N}} \sigma_d(A_{ij}) \right)^{-1} \sqrt{\frac{\psi_i(0)}{\theta_i \omega_i}} e^{-\frac{1}{2} (\rho_i + \frac{1}{\gamma}) T_0} = 2 \varepsilon_0,
\end{align*}
\]
(42) which contradicts with the equation in (38). Therefore, Zeno behavior is excluded.
V. LEADER-FOLLOWER MATRIX-WEIGHTED NETWORKS

Note that reaching an average consensus (an important building block of distributed algorithms) cannot be guaranteed in leaderless matrix-weighted networks since nonlinearities are introduced in the closed-loop dynamics by the actuator saturation constraint. We further examine the leader-follower consensus for matrix-weighted multi-agent networks, where a subset of agents are selected as leaders or informed agents, denoted by $\mathcal{V}_{\text{leader}} \subset \mathcal{V}$, to steer the network to the desired state. The remaining agents are referred to as followers, denoted by $\mathcal{V}_{\text{follower}} = \mathcal{V} \setminus \mathcal{V}_{\text{leader}}$. The set of external input signals is denoted by $\mathcal{W} = \{w_1, \ldots, w_m\}$ where $w_i \in \mathbb{R}^d$, $i \in m$ and $m \in \mathbb{Z}_{+}$. In the following discussion, we shall assume that the input signal is homogeneous, i.e., $w_i = w_j = w_0$ for all $1, \ldots, m$. Denote by the edge set between external input signals and the leaders as $\mathcal{E}'$, and a corresponding set of matrix weights as $B = [B_{il}] \in \mathbb{R}^{nd \times nd}$ where $|B_{il}| \geq 0$ or $|B_{il}| > 0$ if agent $i$ is influenced by the input $w_i$ and $B_{il} = 0_{rd \times rd}$ otherwise. The graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \tilde{A})$ is directed with $\mathcal{V} = \mathcal{V} \cup \mathcal{W}, \mathcal{E} = \mathcal{E}' \cup \mathcal{E}, \tilde{A} = A \cup B$.\[\text{\textbf{A. Actuator Saturation}}\]

Similar to the leaderless case, we now first consider the following leader-follower control protocol without the event-triggered communication constraint,

$$\dot{x}_i(t) = \text{sat}_\Delta(q_i(t)), i \in \mathcal{V},$$

(43)

where

$$q_i(t) = - \sum_{j \in N_i} |A_{ij}|(x_i(t) - \text{sgn}(A_{ij})x_j(t)) - \sum_{i=1}^m |B_{il}|(x_i(t) - \text{sgn}(B_{il})w_i), i \in \mathcal{V}.$$ (44)

The collective dynamics of (43) can subsequently be characterized by

$$\dot{x} = \text{sat}_\Delta(-L_B\mathcal{G})x + Bw,$$ (45)

where $x = (x_1^T(t), x_2^T(t), \ldots, x_n^T(t))^T \in \mathbb{R}^{nd}$, $w = (w_1^T, w_2^T, \ldots, w_m^T)^T \in \mathbb{R}^{nd}$ and

$$L_B\mathcal{G} = L_B\tilde{G} + \text{blkdiag}\left(\sum_{l=1}^m |B_{il}|\right).$$ (46)

Definition 5.1: For $i \in \mathcal{V}$ and an arbitrary $x_0(t) \in \mathbb{R}^d$, the multi-agent system (45) is said to admit global bipartite leader-follower consensus if

$$\lim_{t \to \infty} |x_i(t)| = |w_0|.$$

Assumption 2: The matrix-weighted network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \tilde{A})$ is structurally balanced and $\sum_{i=1}^n \sum_{l=1}^m |B_{il}|$ is positive definite.

Remark 5.2: Assumption 1 and Assumption 2 together guarantee that the leader-follower multi-agent network (45) without saturation admits a bipartite leader-follower consensus [3, 45].

In the following, we shall analyze the convergence situation of the leader-follower multi-agent system (45) on matrix-weighted networks.

Lemma 5.3: Let Assumptions 1 and 2 hold. Then, the multi-agent system (45) achieves global bipartite leader-follower consensus.

Proof: Let

$$\xi(t) = x(t) - D^* (1_n \otimes w_0),$$ (47)

where $D^*$ is the gauge transformation corresponding to the matrix-weighted network. Then one has,

$$\dot{\xi}(t) = \text{sat}_\Delta(-L_B\xi(t)).$$ (48)

Consider the Lyapunov function candidate as follows,

$$V(t) = \frac{1}{2} \xi^T(t)L_B\xi(t),$$ (49)

computing the time derivative of $V(t)$ along with (48) yields

$$\dot{V}(t) = \xi^T(t)L_B\xi(t) = \xi^T(t)L_B\text{sat}_\Delta(-L_B\xi(t)) \leq 0.$$ (50)

It is obvious that $\dot{V}(t) = 0$ if and only if $L_B\xi(t) = 0$, i.e., $\xi(t) = 0$. Thus according to LaSalle’s invariance principle,

$$\lim_{t \to \infty} |x_i(t)| = |w_0| = 0, \forall i \in \mathcal{V}.$$ (51)

That is, the multi-agent system (45) achieves global bipartite leader-follower consensus. ■

B. Event-Triggered Mechanism Design

In order to avoid continuous information exchange amongst agents and updating actuators, we proceed to equip the protocol (45) with an event-triggered communication mechanism. Consider the following protocol for leader-follower multi-agent system with input saturation and event-triggered constraint,

$$\dot{x}_i(t) = \text{sat}_\Delta(q_i(t)), i \in \mathcal{V},$$ (52)

where

$$q_i(t) = - \sum_{j \in N_i} |A_{ij}|(|x_i(t) - \text{sgn}(A_{ij})x_j(t)|$$

$$- \sum_{i=1}^m |B_{il}|(|x_i(t) - \text{sgn}(B_{il})w_i|), i \in \mathcal{V}.$$ (53)

The collective dynamics of (52) can subsequently be characterized by

$$\dot{x}(t) = \text{sat}_\Delta(-L_B\tilde{x}(t) + Bw),$$ (54)

where $\tilde{x}(t) = (x_1^T(t), x_2^T(t), \ldots, x_n^T(t))^T \in \mathbb{R}^{dn}$. Define the state-based measurement error between the last broadcast state of agent $i \in \mathcal{V}$ and its current state at time $t \geq 0$ as
\[ e_i(t) = \hat{x}_i(t) - x_i(t), \] (55)

then the system-wise measurement error is denoted by \( e(t) = [e_1^T(t), e_2^T(t), \ldots, e_N^T(t)]^T \). For agent \( i \in \mathcal{V} \), the triggering time sequence is initiated from \( t^*_i = 0 \) and subsequently determined by,

\[
t_{k+1}^i = \max\{r \geq t_i^* \mid \theta_i(\|e_i(t)\|^2 - \rho_i \Phi_i(t) \sigma_{\delta_3}(\tilde{q}_i(t))) \leq \psi_i(t), \forall t \in [t_i^*, r] \},
\] (56)

where \( k \in \mathbb{Z}_+ \), \( \rho_i \in (0,1) \), \( \theta_i \), and \( \psi_i \) are the design parameters and \( \psi_i(t) \) is an auxiliary system for each agent \( i \in \mathcal{V} \) such that

\[
\dot{\psi}_i(t) = -\beta_i \psi_i(t) + \delta_i(\rho_i \Phi_i(t) \sigma_{\delta_3}(\tilde{q}_i(t))) - \omega_i \|e_i(t)\|^2,
\] (57)

with \( \psi_i(0) > 0 \), \( \beta_i > 0 \), and \( \delta_i \in [0,1] \).

**Theorem 4.5**: Consider the multi-agent system (54) under the matrix-weighted network \( G = (\mathcal{V}, \mathcal{E}, \Lambda) \) satisfying Assumptions 1 and 2. Let \( \theta_i \) and \( \psi_i \) be such that \( \theta_i > \frac{1-\rho_i}{\rho_i} \) and

\[
\omega_i = n \left( \sum_{j \in N_i} \sigma_d(\Lambda_{ij}) + \sum_{i=1}^m \sigma_d(\Lambda_{ij}) \right)^2 + n \sum_{j \in N_i} \sigma_d^2(\Lambda_{ij}).
\] (58)

for all \( i \in \mathcal{V} \), the triggering time sequence is determined by (56) for agent \( i \) with \( \psi_i(t) \) defined in (57). Then the multi-agent system (54) admits a global bipartite leader-follower consensus. Moreover, there is no Zeno behavior.

**Proof**: Let \( \xi(t) = x(t) - D^*(I_n \otimes w_0) \), where \( D^* \) is the gauge transformation corresponding to the matrix-weighted network \( G = (\mathcal{V}, \mathcal{E}, \Lambda) \). Then one has,

\[
\dot{\xi}(t) = -L_{D}\xi(t).
\] (59)

Consider the Lyapunov function candidate as \( V(t) = V_1(t) + V_2(t) \), where \( V_1(t) = \xi^T(t)L_D^2 \xi(t) \), and \( V_2(t) = \sum_{i=1}^n \psi_i(t) \).

Different from the leaderless case, in the leader-follower situation, denote by \( \Phi(t) = \tilde{q}(t) - q(t) \) and \( \phi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_n(t))^T \), where \( q(t) \) and \( \tilde{q}(t) \) are defined in (44) and (53), respectively. Then one has,

\[
\|\phi(t)\|^2 \leq \omega_i \|e_i(t)\|^2, \quad \text{where}
\]

\[
\omega_i = n \left( \sum_{j \in N_i} \sigma_d(\Lambda_{ij}) + \sum_{i=1}^m \sigma_d(\Lambda_{ij}) \right)^2 + n \sum_{j \in N_i} \sigma_d^2(\Lambda_{ij}).
\] (60)

Then, similar to the proof of Theorem 4.5 and Theorem 4.9, one can get the conclusion.

**Remark 5.5**: Similar to the leaderless case, the event-triggered strategy proposed for the matrix-weighted leader-following system with saturation can be applied to the scalar-weighted leader-follower case directly.

**VI. MATRIX-WEIGHTED SWITCHING NETWORKS**

In this section, we shall further extend the proposed event-triggered coordination strategy to switching matrix-weighted networks, where the matrix-weighted network \( G \) switches among a set of possible networks with the same finite node sets, say \( \{G_1, G_2, \ldots, G_M\} \) where \( M \in \mathbb{Z}_+ \). A matrix-weighted switching network \( G(t) \) is said to be simultaneously structurally balanced if there exists a time-invariant bipartition of the node set \( \mathcal{V} \), say \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \), such that the matrix weights on the edges within each subset is positive definite or positive semi-definite, but negative definite or negative semi-definite for the edges between the two subsets. A matrix-weighted switching network is simultaneously structurally imbalanced if it is not simultaneously structurally balanced. In order to deal with switching matrix-weighted networks, we need the following assumptions.

**Assumption 3**: There exists a sequence \( \{h_k \mid k \in \mathbb{N} \} \) such that \( \lim_{k \to \infty} h_k = \infty \) and the dwell time satisfies \( \Delta h_k = h_{k+1} - h_k \geq \alpha \) for all \( k \in \mathbb{N} \), where \( \alpha > 0 \), \( h_0 = 0 \), and \( G(t) \) is time-invariant for \( t \in [h_k, h_{k+1}) \) for all \( k \in \mathbb{N} \). Moreover, \( G(t) \) is simultaneously structurally balanced.

Since the matrix-weighted network \( G \) switches among a finite number of networks, then there exist uniform upper bounds on the largest singular value of weight matrices, denoted by \( \gamma_{ij} \), such that \( \sigma_d(A_{ij}(t)) \leq \gamma_{ij} \) for all \( t \geq 0 \) and \( i,j \in \mathcal{V} \).

**Assumption 4**: There exists a subsequence \( \{h_k \mid k \in \mathbb{N} \} \), denoted by \( \{h_{k_1}, h_{k_2}, \ldots\} \), such that \( h_{k_{i+1}} - h_{k_i} < \infty, i \in \mathbb{N} \), and a scalar \( q \in (0,1) \), such that \( \lambda_{d+1}(\Phi(h_{k_{i+1}}), h_{k_i}) \leq q \) for all \( l \in \mathbb{N} \), where \( \Phi(h_{k_{i+1}}), h_{k_i}) \) is the state transition matrix over the time span \( [h_{k_i}, h_{k_{i+1}}) \).
In the case of switching networks, the trigger time instances can be determined by
\[ t_{k+1}^i = \min \left\{ t_{k_a}^i, t_{k_b}^i \right\}, \quad i \in \mathcal{V}, \tag{64} \]
where
\[ t_{k_a}^i = \max \{ r \mid \theta_i(e_i(t) \| e_i(t)) \leq \psi_i(t) \forall t \in [t_k^i, r) \}, \tag{65} \]
and \( t_{k_b}^i = \min \{ t \mid A_{ij}(t) \neq A_{ij}(t), t \geq t_k^i \} \), where \( k \in \mathbb{Z}^+ \), \( \rho_i \in [0, 1] \), \( \theta_i \) and \( \omega^i \) are the design parameters and \( \psi_i(t) \) is an auxiliary system for each agent \( i \in \mathcal{V} \) such that,
\[ \psi_i(t) = -\rho_i \psi_i(t) + \delta_i (\rho_i \tilde{u}_i^t(t) \text{ sat}_A(\tilde{u}_i(t))) - \omega^i \| e_i(t) \|^2, \tag{66} \]
with \( \psi_i(0) > 0, \beta_i > 0 \) and \( \delta_i \in [0, 1] \).

**Theorem 6.2:** Consider the multi-agent system (62) under the matrix-weighted switching network \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, A) \) satisfying Assumptions 3 and Assumptions 4. Let \( \theta_i \) and \( \omega^i \) be such that \( \theta_i > \frac{\beta_i}{\rho_i} \) and
\[ \omega^i = n \left( \sum_{j \in N_i} \gamma_{ij} \right)^2 + n \sum_{j \in N_i} \gamma_{ij}^2, \tag{67} \]
for all \( i \in \mathcal{V} \), respectively, the triggering time sequence is determined by (64) for agent \( i \) with \( \psi_i(t) \) defined in (66). Then the multi-agent system (62) admits global bipartite consensus.

**Proof:** Consider the Lyapunov function candidate
\[ V(t) = V_1(t) + V_2(t), \quad \text{where} \quad V_1(t) = x^T(t) L(t) x(t), \quad \text{and} \quad V_2(t) = \sum_{i=1}^n \psi_i(t). \]
Different from the time-invariant case,
\[ \sum_{i=1}^n \phi_i(t) \phi_i(t) \leq \sum_{i=1}^n \left( \sum_{j \in N_i} \sigma(A_i(t)) \right) \| e_i(t) \|^2 + \sum_{i=1}^n \sum_{j \in N_i} \sigma^2(A_i(t)) \| e_j(t) \|^2 + \omega^i \| e_i(t) \|^2. \tag{68} \]
where,
\[ \omega^i = n \left( \sum_{j \in N_i} \gamma_{ij} \right)^2 + n \sum_{j \in N_i} \gamma_{ij}^2. \tag{69} \]
Then, similar to the proof of Theorem 4.5, one can get the conclusion.

**Remark 6.3:** Compare with the matrix-weighted time-invariant network, note that the events related to agent \( i \) include all of the instants when that agent establishes or loses a connection with another agent.

**Theorem 6.4:** Under the global bipartite consensus condition in the Theorem 6.2, the Zeno behavior of multi-agent system (62) under the matrix-weighted switching network \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, A) \) can be avoided, i.e., there are no infinite triggering instants in a finite time.

**Proof:** By contradiction, suppose that there exists Zeno behavior. Let
\[ \varepsilon_0 = \frac{4M_o \sum_{j \in N_i} \sigma_d(A_{ij})}{\theta_i \omega^i} \left( \psi_i(t) - \varepsilon \rho_i \tilde{u}_i^t(t) \text{ sat}_A(\tilde{u}_i(t)) \right)^{-1} \frac{1}{\theta_i \omega^i} e_i^T e_i - \varepsilon \rho_i \tilde{u}_i^t(t) \text{ sat}_A(\tilde{u}_i(t)) \tag{70} \]
and choose \( \varepsilon = \min \{ \varepsilon_0, \varepsilon \} \), then according to the proof of Theorem 4.9, there exists a positive integer \( N(\varepsilon) \) such that for any \( k \geq N(\varepsilon) \),
\[ t_{k+1}^i \in [T_0 - \varepsilon, T_0]. \tag{71} \]
Let \( t_{N(\varepsilon)+1}^i \) denote the next triggering time determined by the inequality
\[ \theta_i(e_i(t) \| e_i(t)) = \varepsilon \rho_i \tilde{u}_i^t(t) \text{ sat}_A(\tilde{u}_i(t)) \leq \psi_i(t), \tag{72} \]
one has \( t_{N(\varepsilon)+1}^i - t_{N(\varepsilon)}^i \geq 2\varepsilon_0 \). Let \( t_{N(\varepsilon)+1}^i \) denote the next triggering time determined by the inequality in (64), then one has \( t_{N(\varepsilon)+1}^i - t_{N(\varepsilon)}^i \geq 2\varepsilon, \) which contradicts the equation in (71). Therefore, Zeno behavior can be excluded.

**VIII. BEARING-ONLY COOPERATIVE FORMATION CONTROL**

Notice that the bearing-only cooperative formation control of multi-robot systems is a natural application scenario for matrix-weighted networks [14, 46, 47, 48, 49]. In this section, we proceed to provide simulation results in the context of the bearing-only cooperative formation of multi-vehicle systems to demonstrate the effectiveness of the proposed protocol design. We first provide the notations and setup of bearing-only cooperative formation control.

Consider a team of \( n \) vehicles with the underlying interaction network \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, A) \) and denote by \( \mathbf{p} \in \mathbb{R}^d \) the position of vehicle \( i \) in a \( d \)-dimensional Euclidean space. In practical applications, the vehicles are modeled in either 2-dimensional or 3-dimensional spaces, in the subsequent discussion, one can think of \( d \) as either equal to 2 or 3. The spatial configuration of all the vehicles will be denoted by \( \mathbf{p} = (\mathbf{p}_1^T, \ldots, \mathbf{p}_n^T)^T \in \mathbb{R}^{dn} \).

For each edge \( (i, j) \in \mathcal{E} \), define the displacement vector \( \mathbf{e}_{ij} = \mathbf{p}_i - \mathbf{p}_j \) and the bearing vector is a unit vector from \( \mathbf{p}_i \) to \( \mathbf{p}_j \) (as shown in Fig. 2), namely,
\[ g_{ij} = \frac{\mathbf{p}_i - \mathbf{p}_j}{\| \mathbf{p}_i - \mathbf{p}_j \|} = \frac{\mathbf{e}_{ij}}{\| \mathbf{e}_{ij} \|}, \quad (i, j) \in \mathcal{E}. \]

In our simulation, we consider the following first-order dynamics of \( \mathbf{p}_i(t) \),
\[ \dot{\mathbf{p}}_i(t) = \mathbf{v}_i(t), \tag{73} \]
where the velocity \( \mathbf{v}_i(t) \) acts as the cooperative control protocol that reads
Consider a square-shape spatial configuration $\mathbf{p}$ with square shape (right).

![Diagram of a matrix-weighted network $\mathcal{G}$ (left) and a desired formation configuration $\mathbf{p}$ with square shape (right).](image)

Fig. 1. A matrix-weighted network $\mathcal{G}$ (left) and a desired formation configuration $\mathbf{p}$ with square shape (right).

$$\mathbf{p}_i(t) = - \sum_{j \in N_i} P_{g_{ij}} (\mathbf{p}_j(t) - \mathbf{p}_i(t)), \quad (74)$$

the projection matrix $P_{g_{ij}}$ associated with $(i, j) \in \mathcal{E}$ is

$$P_{g_{ij}} = I_d - \frac{\mathbf{e}_{ij} \mathbf{e}_{ij}^T}{\| \mathbf{e}_{ij} \|} \in \mathbb{R}^{d \times d}. \quad (75)$$

Note that the matrix-valued edge weight $P_{g_{ij}}$ naturally appears in the bearing-only cooperative formation control protocol (74), which is a special case of protocol (7). Moreover, the saturation on velocity $\mathbf{v}_i(t)$ (control input) is natural since each vehicle has only limited thrust; the inter-vehicle communication can be expensive for a multi-vehicle system to achieve a collective task cooperatively, therefore the event-triggered communication is necessary for this scenario.

Notice that $P_{g_{ij}}$ is positive semi-definite and $\text{null}(P_{g_{ij}}) = \text{span}\{\mathbf{e}_{ij}\}$, according to Remark IV.8, one has that $\lim_{t \to \infty} (\mathbf{p}_j(t) - \mathbf{p}_i(t)) \in \text{null}(P_{g_{ij}})$. Therefore, the distributed protocol design in this work can also guarantee that a multi-vehicle system adopting bearing-only formation control protocol (74) achieves the desired spatial configuration subject to both event-triggered communication and actuator saturation constraints.

### A. Bearing-Only Shape Formation

We provide two formation shapes in this part, namely, square and triangle.

1) **Square Shape**: Consider a square-shape spatial configuration $\mathbf{p}$ in a plane shown in Fig. 1 (right), where $\mathbf{p}_1 = (0, 0)^T \in \mathbb{R}^2$, $\mathbf{p}_2 = (0, 1)^T \in \mathbb{R}^2$, $\mathbf{p}_3 = (1, 1)^T \in \mathbb{R}^2$, and $\mathbf{p}_4 = (1, 0)^T \in \mathbb{R}^2$. Then $\mathbf{p} = (\mathbf{p}_1^T, \ldots, \mathbf{p}_4^T)^T \in \mathbb{R}^8$. The projection matrices (namely, the matrix-valued weight in (74)) are

\[
P_{g_{12}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_{g_{13}} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad P_{g_{14}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_{g_{23}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_{g_{24}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_{g_{34}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Let the saturation level be $\Delta = 0.05$ for each dimension of control input $\mathbf{v}_i(t)$. Choose $\rho_i = 0.9$, $\delta_i = 1$, $\beta_i = 1$, and $\psi_i(0) = 0.5$. By computing the singular value of the weight matrices, one has $\varpi_i$ in (18) as $\varpi_1 = 48$, $\varpi_2 = 24$, $\varpi_3 = 48$, and $\varpi_4 = 24$. According to Theorem 4.5, one can choose $\beta_i = 0.5$ which satisfies $\beta_i > \frac{1-\rho_i}{\varpi_i}$. Each dimension of the initial value corresponding to each agent is randomly chosen from the interval $[0, 1]$. Using the above parameters, the desired formation $\mathbf{p}$ can be achieved, as shown in Fig. 3 from a 2-dimensional view and in Fig. 4 from an entry-wise view. The dimensions of the control protocol for each vehicle are illustrated in Fig. 5 and the triggered time instances in Fig. 6.

2) **Triangle Shape**: Consider a triangle-shape spatial configuration $\mathbf{p}'$ in Fig. 7, where $\mathbf{p}'_1 = (0, 0)^T \in \mathbb{R}^2$, $\mathbf{p}'_2 = (0, 1)^T \in \mathbb{R}^2$, $\mathbf{p}'_3 = (1, \frac{1}{2})^T \in \mathbb{R}^2$, and $\mathbf{p}'_4 = (1, 0)^T \in \mathbb{R}^2$. By straightforward computation, one has $\mathbf{p}'_{g_{12}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,
Let the saturation level be $\Delta = 0.05$ for each dimension of control input $u_i(t)$. Choose $\rho_i = 0.9$, $\delta_i = 1$, $\beta_i = 1$, and $\psi_i(0) = 0.5$. By computing the singular value of the weight matrices, one has $\varpi_i$ in (18) as $\varpi_1 = 48$, $\varpi_2 = 24$, $\varpi_3 = 48$, and $\varpi_4 = 24$. According to Theorem 4.5, one can choose $\theta_i = 0.5$ which satisfies $\theta_i > \frac{1-\rho_i}{\rho_i}$. Each dimension of the initial value corresponding to each agent is the same as that in the previous simulation example. Using the above parameters, the desired formation $p'$ can be achieved, as shown in Fig. 8 from a 2-dimensional view and in Fig. 9 from an entry-wise view. The dimensions of the control protocol for each vehicle are illustrated in Fig. 10 and the triggered time instances in Fig. 11.

**B. Formation Switching**

We further consider a scenario of formation switching for multi-vehicle systems from square to triangle subject to both event-triggered communication and actuator saturation. In
the bearing-only formation setup, we choose agent 3 to act as a leader agent who has been indicated the formation switching signal, namely, from \( p_3 = (1, 1)^T \) to \( p_3 = (\frac{1}{2}, \frac{1}{2})^T \). Then, agent 3 needs to negotiate with all its neighbors to update their respective matrix-valued projection matrices, namely, from \( P_{13}, P_{23}, P_{34} \) to \( P_{01}, P_{02}, P_{03} \). Let the saturation level be \( \Delta = 0.05 \) for each dimension of control input \( v_i(t) \). By computing the singular value of the weight matrices, one has \( \omega_1 \) in (18) as \( \omega_1 = 48 \), \( \omega_2 = 24 \), \( \omega_3 = 48 \), and \( \omega_4 = 24 \). According to Theorem 4.5, one can choose \( \theta_i = 0.5 \) which satisfies \( \theta_i > \frac{1}{\omega_i} \). Using the above parameters, the desired formation \( \mathbf{p} \) can be achieved, as shown in Fig. 12 from a 2-dimensional view and in Fig. 13 from an entry-wise view. The dimensions of the control protocol for each agent are illustrated in Fig. 14 and the triggered time instances in Fig. 15.
VIII. Conclusion

In this paper, we examined the event-triggered global bipartite consensus problem for multi-agent systems on matrix-weighted networks subject to input saturation constraints. Dynamic event-triggered distributed protocols for both leaderless and leader-follower cases were provided, where each agent only needs to broadcast its own state at its triggering time instances and listen to incoming information from its neighbors at their triggering times, which reduces the limited communication resource and avoids the continuous communication among agents. Then, criteria were further derived to guarantee the leaderless and leader-follower global bipartite consensus of the matrix-weighted multi-agent networks. Also, the proposed triggering laws were shown to be free of the Zeno phenomenon by proving that the triggering time sequence of each agent is divergent. The proposed event-triggered protocol design approach has also been extended to the case when the matrix-weighted network is switching over time. Simulation results in the context of the bearing-only cooperative formation of multi-vehicle systems have been provided to demonstrate the effectiveness of the proposed design approach.

Along this line of research, future work can be the investigation of event-triggered consensus problems for the matrix-weighted multi-agent networks subject to actuator saturation where the behavior of agents is characterized by general linear systems or Euler–Lagrangian systems, as well as their applications in real-world multi-agent networks.

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