Vector measures of bounded $\gamma$-variation and stochastic integrals

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Abstract. We introduce the class of vector measures of bounded $\gamma$-variation and study its relationship with vector-valued stochastic integrals with respect to Brownian motions.

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1. Introduction

It is well known that stochastic integrals can be interpreted as vector measures, the identification being given by the identity

$$F(A) = \int_A \phi \, dB.$$ 

Here, the driving process $B$ is a (semi)martingale (for instance, a Brownian motion), and $\phi$ is a stochastic process satisfying suitable measurability and integrability conditions. This observation has been used by various authors as the starting point of a theory of stochastic integration for vector-valued processes.

Let $X$ be a Banach space. In [5] we characterized the class of functions $\phi : (0,1) \to X$ which are stochastically integrable with respect to a Brownian motion $(W_t)_{t \in [0,1]}$ as being the class of functions for which the operator $T_\phi : L^2(0,1) \to X$,

$$T_\phi f := \int_0^1 f(t)\phi(t) \, dt,$$

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belongs to the operator ideal $\gamma(L^2(0, 1), X)$ of all $\gamma$-radonifying operators. Indeed, we established the Itô isomorphism

$$E \left\| \int_0^1 f \, dW \right\|^2 = \|Tf\|_{\gamma(L^2(0, 1), X)}^2.$$ 

The linear subspace of all operators in $\gamma(L^2(0, 1), X)$ of the form $T = Tf$ for some function $f : (0, 1) \to X$ is dense, but unless $X$ has cotype 2 it is strictly smaller than $\gamma(L^2(0, 1), X)$. This means that in general there are operators $T \in \gamma(L^2(0, 1), X)$ which are not representable by an $X$-valued function. Since the space of test functions $\mathcal{D}(0, 1)$ embeds in $L^2(0, 1)$, by restriction one could still think of such operators as $X$-valued distributions. It may be more intuitive, however, to think of $T$ as an $X$-valued vector measure. We shall prove (see Theorem 2.3 and the subsequent remark) that if $X$ does not contain a closed subspace isomorphic to $c_0$, then the space $\gamma(L^2(0, 1), X)$ is isometrically isomorphic in a natural way to the space of $X$-valued vector measures on $(0, 1)$ which are of bounded $\gamma$-variation. This gives a ‘measure theoretic’ description of the class of admissible integrands for stochastic integrals with respect to Brownian motions. The condition $c_0 \not\subseteq X$ can be removed if we replace the space of $\gamma$-radonifying operators by the larger space of all $\gamma$-summing operators (which contains the space of all $\gamma$-radonifying operators isometrically as a closed subspace).

Vector measures of bounded $\gamma$-variation behave quite differently from vector measures of bounded variation. For instance, the question whether an $X$-valued vector measure of bounded $\gamma$-variation can be represented by an $X$-valued function is not linked to the Radon-Nikodým property, but rather to the type 2 and cotype 2 properties of $X$ (see Corollaries 2.5 and 2.6).

In section 3 we consider yet another class of vector measures whose variation is given by certain random sums, and we show that a function $\phi : (0, 1) \to X$ is stochastically integrable with respect to a Brownian motion $(W_t)_{t \in [0, 1]}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ if and only if the formula $F(A) := \int_A \phi \, dW$ defines an $L^2(\Omega; X)$-valued vector measure $F$ in this class.

2. Vector measures of bounded $\gamma$-variation

Let $(S, \Sigma)$ be a measurable space, $X$ a Banach space, and $(\gamma_n)_{n \geq 1}$ a sequence of independent standard Gaussian random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

**Definition 2.1.** We say that a countably additive vector measure $F$ has bounded $\gamma$-variation with respect to a probability measure $\mu$ on $(S, \Sigma)$ if $\|F\|_{\gamma(V, \mu; X)} < \infty$, where

$$\|F\|_{\gamma(V, \mu; X)} := \sup \left( E \left\| \sum_{n=1}^N \gamma_n \frac{F(A_n)}{\sqrt{\mu(A_n)}} \right\|^2 \right)^{\frac{1}{2}} ,$$

the supremum being taken over all finite collections of disjoint sets $A_1, \ldots, A_N \in \Sigma$ such that $\mu(A_n) > 0$ for all $n = 1, \ldots, N$. 
It is routine to check (e.g. by an argument similar to [4, Proposition 5.2]) that the space $V_\gamma(\mu; X)$ of all countably additive vector measures $F : \Sigma \rightarrow X$ which have bounded $\gamma$-variation with respect to $\mu$ is a Banach space with respect to the norm $\| \cdot \|_{V_\gamma(\mu; X)}$. Furthermore, every vector measure which is of bounded $\gamma$-variation is of bounded 2-semivariation.

In order to give a necessary and sufficient condition for a vector measure to have bounded $\gamma$-variation we need to introduce the following terminology. A bounded operator $T : H \rightarrow X$, where $H$ is a Hilbert space, is said to be $\gamma$-summing if there exists a constant $C$ such that for all finite orthonormal systems $\{h_1, \ldots, h_N\}$ in $H$ one has

$$E \left\| \sum_{n=1}^N \gamma_n Th_n \right\|^2 \leq C^2.$$  

The least constant $C$ for which this holds is called the $\gamma$-summing norm of $T$, notation $\|T\|_{\gamma_n(H, X)}$. With respect to this norm, the space $\gamma_\infty(H, X)$ of all $\gamma$-summing operators from $H$ to $X$ is a Banach space which contains all finite rank operators from $H$ to $X$. In what follows we shall make free use of the elementary properties of $\gamma$-summing operators. For a systematic exposition of these we refer to [2, Chapter 12] and the lecture notes [4].

**Theorem 2.2.** Let $\mathcal{A}$ be an algebra of subsets of $S$ which generates the $\sigma$-algebra $\Sigma$, and let $F : \mathcal{A} \rightarrow X$ be a finitely additive mapping. If, for some $1 \leq p < \infty$, $T : L^p(\mu) \rightarrow X$ is a bounded operator such that

$$F(A) = T1_A, \quad A \in \mathcal{A},$$

then $F$ has a unique extension to a countably additive vector measure on $\Sigma$ which is absolutely continuous with respect to $\mu$. If $T : L^2(\mu) \rightarrow X$ is $\gamma$-summing, then the extension of $F$ has bounded $\gamma$-variation with respect to $\mu$ and we have

$$\|F\|_{V_\gamma(\mu; X)} \leq \|T\|_{\gamma_\infty(L^2(\mu), X)}.$$  

**Proof.** We define the extension $F : \Sigma \rightarrow X$ by $F(A) := T1_A$, $A \in \Sigma$. To see that $F$ is countably additive, consider a disjoint union $A = \bigcup_{n=1}^N A_n$ with $A_n, A \in \Sigma$. Then $\lim_{N \rightarrow \infty} \mathbf{1}_{\bigcup_{n=1}^N A_n} = 1_A$ in $L^p(\mu)$ and therefore

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N F(A_n) = \lim_{N \rightarrow \infty} T \sum_{n=1}^N 1_{A_n} = T1_A = F(A).$$

The absolute continuity of $F$ is clear. To prove uniqueness, suppose $\tilde{F} : \Sigma \rightarrow X$ is another countably additive vector measure extending $F$. For each $x^* \in X^*$, $\langle \tilde{F}, x^* \rangle$ and $\langle F, x^* \rangle$ are finite measures on $\Sigma$ which agree on $\mathcal{A}$, and therefore by Dynkin’s lemma they agree on all of $\Sigma$. This being true for all $x^* \in X^*$, it follows that $\tilde{F} = F$ by the Hahn-Banach theorem.

Suppose next that $T : L^2(\mu) \rightarrow X$ is $\gamma$-summing, and consider a finite collection of disjoint sets $A_1, \ldots, A_N$ in $\Sigma$ such that $\mu(A_n) > 0$ for all $n = 1, \ldots, N$. 


The functions \(f_n = 1_{A_n}/\sqrt{\mu(A_n)}\) are orthonormal in \(L^2(\mu)\) and therefore
\[
\mathbb{E}\left\| \sum_{n=1}^{N} \gamma_n \frac{F(A_n)}{\sqrt{\mu(A_n)}} \right\|^2 = \mathbb{E}\left\| \sum_{n=1}^{N} \gamma_n T f_n \right\|^2 \leq \|T\|_{\gamma_\infty(L^2(\mu),X)}^2.
\]
It follows that \(F\) has bounded \(\gamma\)-variation with respect to \(\mu\) and that \(\|F\|_{\gamma_\infty(X)} \leq \|T\|_{\gamma_\infty(L^2(\mu),X)}\).

**Theorem 2.3.** For a countably additive vector measure \(F : \Sigma \to X\) the following assertions are equivalent:

1. \(F\) has bounded \(\gamma\)-variation with respect to \(\mu\);
2. There exists a \(\gamma\)-summing operator \(T : L^2(\mu) \to X\) such that
   \[
   F(A) = T 1_{A}, \quad A \in \Sigma.
   \]

In this situation we have
\[
\|F\|_{\gamma(X)} = \|T\|_{\gamma_\infty(L^2(\mu),X)}.
\]

**Proof.** (1)\(\Rightarrow\)(2): Suppose that \(F\) has bounded \(\gamma\)-variation with respect to \(\mu\). For a simple function \(f = \sum_{n=1}^{N} c_n 1_{A_n}\), where the sets \(A_n \in \Sigma\) are disjoint and of positive \(\mu\)-measure, define
\[
T f := \sum_{n=1}^{N} c_n F(A_n).
\]
By the Cauchy-Schwarz inequality, for all \(x^* \in X^*\) we have
\[
|\langle Tf, x^* \rangle| = \left| \mathbb{E} \sum_{n=1}^{N} \gamma_n c_n \sqrt{\mu(A_m)} \cdot \sum_{n=1}^{N} \gamma_n \frac{\langle F(A_n), x^* \rangle}{\sqrt{\mu(A_n)}} \right| \leq \left( \mathbb{E} \sum_{n=1}^{N} \gamma_n c_n \sqrt{\mu(A_n)} \right)^{\frac{1}{2}} \left( \mathbb{E} \sum_{n=1}^{N} \gamma_n \frac{\|F(A_n), x^*\|}{\sqrt{\mu(A_n)}} \right)^{\frac{1}{2}} \leq \left( \sum_{n=1}^{N} \|c_n\|^2 \mu(A_n) \right)^{\frac{1}{2}} \|F\|_{\gamma_\infty(X)} \|x^*\| \\|T\|_{L^2(\mu)} \|F\|_{\gamma_\infty(X)} \|x^*\|.
\]
It follows that \(T\) is bounded and \(\|T\|_{\mathcal{L}(L^2(\mu),X)} \leq \|F\|_{\gamma_\infty(X)}\). To prove that \(T\) is \(\gamma\)-summing we shall first make the simplifying assumption that the \(\sigma\)-algebra \(\Sigma\) is countably generated. Under this assumption there exists an increasing sequence of finite \(\sigma\)-algebras \((\Sigma_n)_{n \geq 1}\) such that \(\Sigma = \bigvee_{n \geq 1} \Sigma_n\). Let \(P_n\) be the orthogonal projection in \(L^2(\mu)\) onto \(L^2(\Sigma_n, \mu)\) and put \(T_n := T \circ P_n\). These operators are of finite rank and we have \(\lim_{n \to \infty} T_n \to T\) in the strong operator topology of \(\mathcal{L}(L^2(\mu),X)\).

Fix an index \(n \geq 1\) for the moment. Since \(\Sigma_n\) is finitely generated there exists a partition \(S = \bigcup_{j=1}^{N} A_j\), where the disjoint sets \(A_1, \ldots, A_N\) generate \(\Sigma_n\). Assuming that \(\mu(A_j) > 0\) for all \(j = 1, \ldots, M\) and \(\mu(A_j) = 0\) for \(j = M+1, \ldots, N\),
the functions \( g_j = 1_{A_j}/\sqrt{\mu(A_j)} \), \( j = 1, \ldots, M \), form an orthonormal basis for \( L^2(\Sigma_n, \mu) \) and
\[
\|T_n\|_{\gamma_\infty(L^2(\mu), X)}^2 = \|T_n\|_{\gamma_\infty(L^2(\Sigma_n, \mu), X)}^2 = \mathbb{E}\left[ \sum_{j=1}^M \gamma_j Tg_j \right]^2 = \mathbb{E}\left[ \sum_{j=1}^M \gamma_j Tg_j \right] = \mathbb{E}\left[ \sum_{j=1}^M \gamma_j \frac{F(A_j)}{\sqrt{\mu(A_n)}} \right]^2 \leq \|F\|_{V, (\mu; X)}^2,
\]
the first identity being a consequence of [4, Corollary 5.5] and the second of [4, Lemma 5.7]. It follows that the sequence \( (T_n)_{n \geq 1} \) is bounded in \( \gamma_\infty(L^2(\mu), X) \). By the Fatou lemma, if \( \{f_1, \ldots, f_k\} \) is any orthonormal family in \( L^2(\mu) \), then
\[
\mathbb{E}\left[ \sum_{j=1}^k \gamma_j T_{f_j} \right]^2 \leq \liminf_{n \to \infty} \mathbb{E}\left[ \sum_{j=1}^k \gamma_j T_{f_j} \right]^2 \leq \|T_n\|_{\gamma_\infty(L^2(\mu), X)}^2 \leq \|F\|_{V, (\mu; X)}^2.
\]
This proves that \( T \) is \( \gamma \)-summing and \( \|T\|_{\gamma_\infty(L^2(\mu), X)} \leq \|F\|_{V, (\mu; X)} \).

It remains to remove the assumption that \( \Sigma \) is countably generated. The preceding argument shows that if we define \( T \) in the above way, then its restriction to \( L^2(\Sigma', \mu) \) is \( \gamma \)-summing for every countably generated \( \sigma \)-algebra \( \Sigma' \subseteq \Sigma \), with a uniform bound
\[
\|T\|_{\gamma_\infty(L^2(\Sigma', \mu), X)} \leq \|F\|_{V, (\mu; X)}.
\]
Since every finite orthonormal family \( \{f_1, \ldots, f_k\} \) in \( L^2(\mu) \) is contained in \( L^2(\Sigma', \mu) \) for some countably generated \( \sigma \)-algebra \( \Sigma' \subseteq \Sigma \), we see that
\[
\mathbb{E}\left[ \sum_{j=1}^k \gamma_j T_{f_j} \right]^2 \leq \|T\|_{\gamma_\infty(L^2(\Sigma', \mu), X)}^2 \leq \|F\|_{V, (\mu; X)}^2.
\]
It follows that \( T \) is \( \gamma \)-summing and \( \|T\|_{\gamma_\infty(L^2(\mu), X)} \leq \|F\|_{V, (\mu; X)} \).

(2) \( \Rightarrow \) (1): This implication is contained in Theorem 2.2. \( \square \)

By a theorem of Hoffmann-Jørgensen and Kwapien [3, Theorem 9.29], if \( X \) is a Banach space not containing an isomorphic copy of \( c_0 \), then for any Hilbert space \( H \) one has
\[
\gamma_\infty(H, X) = \gamma(H, X),
\]
where by definition \( \gamma(H, X) \) denotes the closure in \( \gamma_\infty(H, X) \) of the finite rank operators from \( H \) to \( X \). Since any operator in this closure is compact we obtain:

**Corollary 2.4.** If \( X \) does not contain an isomorphic copy of \( c_0 \) and \( F : \Sigma \to X \) has bounded \( \gamma \)-variation with respect to \( \mu \), then \( F \) has relatively compact range.

Using the terminology of [5], a theorem of Rosiński and Suchanecki [6] asserts that if \( X \) has type 2 we have a continuous inclusion \( L^2(\mu; X) \to \gamma(L^2(\mu), X) \) and that if \( X \) has cotype 2 we have a continuous inclusion \( \gamma_\infty(L^2(\mu), X) \to L^2(\mu; X) \). In both cases the embedding is contractive, and the relation between the operator \( T \) and the representing function \( \phi \) is given by
\[
Tf = \int_{\Sigma} f \phi \, d\mu, \quad f \in L^2(\mu).
\]
If \( \dim L^2(\mu) = \infty \), then in the converse direction the existence of a continuous embedding \( L^2(\mu; X) \hookrightarrow \gamma_\infty(L^2(\mu), X) \) (respectively \( \gamma(L^2(\mu), X) \hookrightarrow L^2(\mu; X) \)) actually implies the type 2 property (respectively the cotype 2 property) of \( X \).

**Corollary 2.5.** Let \( X \) have type 2. For all \( \phi \in L^2(\mu; X) \) the formula

\[
F(A) := \int_A \phi \, d\mu, \quad A \in \Sigma,
\]

defines a countably additive vector measure \( F : \Sigma \to X \) which has bounded \( \gamma \)-variation with respect to \( \mu \). Moreover,

\[
\| F \|_{V_{\gamma}(\mu; X)} \leq \| \phi \|_{L^2(\mu; X)}. \]

If \( \dim L^2(\mu) = \infty \), this property characterises the type 2 property of \( X \).

**Proof.** By the theorem of Rosiński and Suchanecki, \( \phi \) represents an operator \( T \in \gamma(L^2(\mu), X) \) such that \( T1_A = \int_A \phi \, d\mu = F(A) \) for all \( A \in \Sigma \). The result now follows from Theorem 2.2. The converse direction follows from Theorem 2.3 and the preceding remarks. \( \square \)

**Corollary 2.6.** Let \( X \) have cotype 2. If \( F : \Sigma \to X \) has bounded \( \gamma \)-variation with respect to \( \mu \), there exists a function \( \phi \in L^2(\mu; X) \) such that

\[
F(A) = \int_A \phi \, d\mu, \quad A \in \Sigma.
\]

Moreover,

\[
\| \phi \|_{L^2(\mu; X)} \leq \| F \|_{V_{\gamma}(\mu; X)}. \]

If \( \dim L^2(\mu) = \infty \), this property characterises the cotype 2 property of \( X \).

**Proof.** By Theorem 2.3 there exists an operator \( T \in \gamma_\infty(L^2(\mu), X) \) such that \( F(A) = T1_A \) for all \( A \in \Sigma \). Since \( X \) has cotype 2, \( X \) does not contain an isomorphic copy of \( c_0 \) and therefore the theorem of Hoffmann-Jørgensen and Kwapień implies that \( T \in \gamma(L^2(\mu), X) \). Now the theorem of Rosiński and Suchanecki shows that \( T \) is represented by a function \( \phi \in L^2(\mu; X) \). The converse direction follows from Theorem 2.2 and the preceding remarks. \( \square \)

### 3. Vector measures of bounded randomised variation

Let \((S, \Sigma)\) be a measurable space and \((r_n)_{n \geq 1}\) a Rademacher sequence, i.e., a sequence of independent random variables with \( \Pr(r_n = \pm 1) = \frac{1}{2} \).

**Definition 3.1.** A countably additive vector measure \( F : \Sigma \to X \) is of **bounded randomised variation** if \( \| F \|_{V_{r}(\mu; X)} < \infty \), where

\[
\| F \|_{V_{r}(\mu; X)} = \sup \left( \mathbb{E} \left( \left( \sum_{n=1}^{N} r_n F(A_n) \right)^2 \right)^{\frac{1}{2}} \right),
\]

the supremum being taken over all finite collections of disjoint sets \( A_1, \ldots, A_N \in \Sigma \).
Clearly, if $F$ is of bounded variation, then $F$ is of bounded randomised variation. The converse fails; see Example 1. If $X$ has finite cotype, standard comparison results for Banach space-valued random sums \([2, 3]\) imply that an equivalent norm is obtained when the Rademacher variables are replaced by Gaussian variables.

It is routine to check that the space $V^\gamma(\mu; X)$ of all countably additive vector measures $F : \Sigma \to X$ of bounded randomised variation is a Banach space with respect to the norm $\| \cdot \|_{V^\gamma(\mu; X)}$.

In Theorem 3.2 below we establish a connection between measures of bounded randomised variation and the theory of stochastic integration. For this purpose we need the following terminology. A Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ indexed by another probability space $(S, \Sigma, \mu)$ is a mapping $W : \Sigma \to L^2(\Omega)$ such that:

(i) For all $A \in \Sigma$ the random variable $W(A)$ is centred Gaussian with variance $\mathbb{E}(W(A))^2 = \mu(A)$;

(ii) For all disjoint $A, B \in \Sigma$ the random variables $W(A)$ and $W(B)$ are independent.

A strongly $\mu$-measurable function $\phi : S \to X$ is stochastically integrable with respect to $W$ if for all $x^* \in X^*$ we have $\langle \phi, x^* \rangle \in L^2(\mu)$ (i.e., $f$ belongs to $L^2(\mu)$ scalarly) and for all $A \in \Sigma$ there exists a strongly measurable random variable $Y_A : \Omega \to X$ such that for all $x^* \in X^*$ we have

$$\langle Y_A, x^* \rangle = \int_A \langle \phi, x^* \rangle d\mu$$

almost surely. Note that each $Y_A$ is centred Gaussian and therefore belongs to $L^2(\Omega; X)$ by Fernique’s theorem; the above equality then holds in the sense of $L^2(\Omega)$. We define the stochastic integral of $\phi$ over $A$ by $\int_A \phi dW := Y_A$. For more details and various equivalent definitions we refer to [5].

**Theorem 3.2.** Let $W : \Sigma \to L^2(\Omega)$ be a Brownian motion. For a strongly $\mu$-measurable function $\phi : S \to X$ the following assertions are equivalent:

1. $\phi$ is stochastically integrable with respect to $W$;
2. $\phi$ belongs to $L^2(\mu)$ scalarly and there exists a countably additive vector measure $F : \Sigma \to X$, of bounded $\gamma$-variation with respect to $\mu$, such that for all $x^* \in X^*$ we have

$$\langle F(A), x^* \rangle = \int_A \langle \phi, x^* \rangle d\mu, \quad A \in \Sigma;$$

3. $\phi$ belongs to $L^2(\mu)$ scalarly and there exists a countably additive vector measure $G : \Sigma \to L^2(\Omega; X)$ of bounded randomised variation such that for all $x^* \in X^*$ we have

$$\langle G(A), x^* \rangle = \int_A \langle \phi, x^* \rangle dW, \quad A \in \Sigma.$$. 
In this situation we have
\[ \|F\|_{V,\gamma(\mu;X)} = \|G\|_{V^\gamma(\mu;L^2(\Omega;X))} = \left( \mathbb{E} \left\| \int_S \phi \, dW \right\|^2 \right)^{\frac{1}{2}}. \]

Proof. (1)⇔(2): This equivalence is immediate from Theorem 2.3 and the fact, proven in [5], that \( \phi \) is stochastically integrable with respect to \( W \) if and only there exists an operator \( T \in \gamma(L^2(\mu),X) \) such that
\[ Tf = \int_S f \phi \, d\mu, \quad f \in L^2(\mu). \]
In this case we also have
\[ \|T\|_{\gamma(L^2(\mu),X)} = \left( \mathbb{E} \left\| \int_S \phi \, dW \right\|^2 \right)^{\frac{1}{2}}. \]

In view of Theorem 2.3, this proves the identity
\[ \|F\|_{V,\gamma(\mu;X)} = \left( \mathbb{E} \left\| \int_S \phi \, dW \right\|^2 \right)^{\frac{1}{2}}. \]

(1)⇒(3): Define \( G : \Sigma \to L^2(\Omega;X) \) by
\[ G(A) := \int_A \phi \, dW, \quad A \in \Sigma. \]
By the \( \gamma \)-dominated convergence theorem [5], \( G \) is countably additive. To prove that \( G \) is of bounded randomised variation we consider disjoint sets \( A_1, \ldots, A_N \in \Sigma \). If \( (\tilde{r}_n)_{n \geq 1} \) is a Rademacher sequence on a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\), then by randomisation we have
\[
\mathbb{E} \left\| \sum_{n=1}^N \tilde{r}_n G(A_n) \right\|_{L^2(\Omega;X)}^2
= \mathbb{E} \left\| \sum_{n=1}^N \int_{A_n} \phi \, dW \right\|^2
= \mathbb{E} \left\| \sum_{n=1}^N \int_{A_n} \phi \, dW \right\|^2 \leq \mathbb{E} \int_S \phi \, dW^2.
\]
with equality if \( \bigcup_{n=1}^N A_n = S \). In the second identity we used that the \( X \)-valued random variables \( \int_{A_n} \phi \, dW \) are independent and symmetric. The final inequality follows by, e.g., covariance domination [5] or an application of the contraction principle. It follows that \( G \) is a countably additive vector measure of bounded randomised variation and
\[ \|G\|_{V^\gamma(\mu;X)} = \left( \mathbb{E} \left\| \int_S \phi \, dW \right\|^2 \right)^{\frac{1}{2}}. \]

(3)⇒(1): This is immediate from the definition of stochastic integrability. \( \square \)

Example 1. If \( W \) is a standard Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P})\) indexed by the Borel interval \((0,1], \mathcal{B}, m)\), then \( W \) is a countably additive vector measure with values in \( L^2(\Omega) \) which is of bounded randomised variation, but of unbounded variation. The first claim follows from Theorem 3.2 since \( W(A) = \int_A 1 \, dW \) for all
Borel sets $A$. To see that $W$ is of unbounded variation, note that for any partition
$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = 1$ we have
$$\sum_{n=1}^{N} \|W((t_{n-1}, t_n))\|_{L^2(\Omega)} = \sum_{n=1}^{N} \sqrt{t_n - t_{n-1}}.$$  
The supremum over all possible partitions of $[0,1]$ is unbounded.

References

[1] J. Diestel and J.J. Uhl, Vector measures. Mathematical Surveys, Vol. 15, Amer. Math. Soc., Providence (1977).
[2] J. Diestel, H. Jarchow and A. Tonge, Absolutely summing operators. Cambridge Studies in Adv. Math., Vol. 34, Cambridge, 1995.
[3] M. Ledoux and M. Talagrand, Probability in Banach spaces. Ergebnisse d. Math. u. ihre Grenzgebiete, Vol. 23, Springer-Verlag, 1991.
[4] J.M.A.M. van Neerven, Stochastic evolution equations. Lecture notes of the 11th International Internet Seminar, TU Delft, downloadable at [http://fa.its.tudelft.nl/~isemwiki](http://fa.its.tudelft.nl/~isemwiki).
[5] J.M.A.M. van Neerven and L. Weis, Stochastic integration of functions with values in a Banach space. Studia Math. 166 (2005), 131–170.
[6] J. Rosiński and Z. Suchanecki, On the space of vector-valued functions integrable with respect to the white noise. Colloq. Math. 43 (1980), 183–201.

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