Solution of Einstein's Field Equations for the Static Fluid Sphere

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Abstract. In this paper we deals about some exact static spherical solution of Einstein’s field equations with \( \Lambda = 0 \) (cosmological constant) and \( p = \rho \) (taking suitable choice of \( g_{11} \) and \( g_{44} \). We have \( e^\psi = km^{5/4} \) and \( e^{-\chi} = 1 \), which help to investigate the value of \( e^\chi \). Here some previously known solutions are contained as a particular case. The explicit expressions for rotation, shear scalar of expansion and fluid velocity have also investigated. We get some previously known solution for distinct values of \( n \). Here \( \Lambda = 0 \), this implies that Einstein element would degenerate into a line element of special relativity for flat space time. It also helpful to investigates solution for the perfect fluid core.

Keywords. Cosmology, Static fluid sphere, Shear tensor, Scalar of expansion, Metric space, Homogeneous density

1. Introduction

This paper discussed the cosmological constant and the exact solutions of the Einstein’s field equations for the perfect fluid. The paper focused for a spatially homogeneous and anisotropic cosmological model. The \( \Lambda \) (cosmological constants) and the \( G \) (gravitational constant) are the two parameters for the Einstein’s field equation. In present time Cosmological models with a cosmological constant have important role to describe the dynamics and evolution of the Universe [Pradhan (2013), Saha (2012), Singh (2008)]. The rise of interest in the theory of General Relativity as a tool for studying the evolution and behaviour of various cosmological models has been rapid expensive. Since the early 1920's to the present, the Einstein's theory of relativity has been used extensively as a tool in the prediction and modelling of the cosmos. One reason for the prominence of modern relativity is its success in predicting the behaviour of large scale phenomena where gravitation plays a dominant role [Dicke (1962), Feinstein (1987)]. Various researcher in theory of relativity have focused their mind to the study of solution of Einstein’s field equation with \( \Lambda = 0 \) (cosmological constant) and \( p = \rho \) (equation of state). Solution of Einstein’s field equation of state \( p = \rho \) have been obtained by various authors e.g., Latelier [1975], Letelier and Tabensky [1975], Tabensky, R., et.al. [1973] and Yadav [2007]. Singh and Yadav [1981] have also discussed the static fluid sphere with the equation of the state \( p = \rho \). Further study in the line has been done by Yadav and Saini [1991], which is more general than one due to Singh and Yadav [1981]. Narlikar [1968] studied about relative mass \( m \) of a particle in the gravitational field related to its proper mass \( m_0 \). The perfect fluid spheres with isotropic pressure and homogeneous density in general relativity for the solutions of relativistic field equations obtained by Schwarzschild [1916]. A mathematical method for solving the model of static fluid spheres applied in the Einstein's field equations developed by Tolman [1939].
Letelier and Tabensky [1975], Letelier [1939] and Singh and Yadav [1981]) obtained solutions for the Einstein’s equations for an irrotational perfect fluid have \( p = \rho \) (equations of state). Solutions to Einstein’s equations with a various equations of state have been found in some distinct cases, e.g. (Buchdahl [1964]) have obtained solution for \( p = 1 + a \sqrt{\rho} + a \); (Buchdahl and Land [1968] and Allnutt [1980]) obtained solution for \( p = \rho + \text{constant} \); (Whittaker [1968]) obtained solution for \( p + 3\rho = \text{constant} \) and (Klein [1947]) obtained solution for \( p = 3 \rho \) respectively. [Tooper (1964), Klein (1953), Buchdahl (1967)] have obtained solution for polytrophic fluid sphere \( \rho = a \rho^{1/n} \). The static fluid sphere with equation of state for stiff matter i.e \( p = \rho \) studied by Yadav and Saini [1991]. Tolman [1939], Yadav and Purushottam [2001], Thomas E Kiess [2009], Karmer [1988], Singh. et.al. [1973], Raychaudhari [1979], Walecka [1974], Kandalkar [2009], Yadav, et.al. [2012], Yadav and Singh [2003], Singh [2019], Maurya [2017], Fuloria [2018] are some of the authors who have important research in different dimensions of interacting fields in the framework of Einstein’s field equations for the perfect fluid with specified equation of state and general relativity.

In this paper we deals about some exact static spherical solution of Einstein’s field equations with \( \Lambda = 0 \) (cosmological constant) and \( p = \rho \) (equation of state). Here we discussed about two different cases \( e^\psi = km^{5/4} \) and \( e^{-\chi} = l \), (where \( k \) and \( l \) are constants). We get some previously known solution for distinct values of \( n \). This paper has eight sections. First section is introduction. Second and third section deals about field equations and its solutions with different cases. Fourth section discussed about solution for the perfect fluid core. Finally in fifth, sixth, seventh and eight section deals about discussions, applications, future prospects and limitation of finding results respectively.

### 2. The Field Equations

Let us consider the static spherically symmetric metric given by

\[
\text{ds}^2 = e^\psi \text{dt}^2 - e^\chi \text{dm}^2 - m^2 \text{d}^2 \theta - m^2 \sin^2 \theta \text{d}^2 \phi
\]

where \( \chi \) and \( \psi \) are functions of \( m \) only.

Taking cosmological constant \( \Lambda \) into account, we obtain the field equations

\[
(2.2a) \quad R^i_j - \frac{1}{2} R \delta^i_j + \Lambda \delta^i_j = -8\pi T^i_j
\]

For \( \Lambda = 0 \), (2.2a) gives

\[
(2.2b) \quad R^i_j - \frac{1}{2} R \delta^i_j = -8\pi T^i_j
\]

For the metric (2.1) are (Tolman [1939])

\[
(2.3) \quad -8\pi T^1_1 = e^{-\chi} \left( \psi' + \frac{1}{m^2} \right) - \frac{1}{m^2}
\]

\[
(2.4) \quad -8\pi T^2_2 = -8\pi T^3_3
\]

\[
= e^{-\chi} \left( \frac{\psi'}{2} - \frac{\psi' \psi}{4} + \frac{\psi'^2}{4} + \frac{\psi^2}{2m} \right)
\]
\[ (2.5) \quad -8\pi T_4^4 = e^{-\chi} \left( \frac{\psi'}{m} - \frac{1}{m^2} \right) + \frac{1}{m^2} \]

Here ('\prime') prime shows that differentiation with respect to \( m \).

The energy momentum tensor
\[ (2.6) \quad T_i^j = (\rho + p)u^iu_j - \delta_i^j p \]

Specified by the equation of state
\[ (2.7) \quad p = \varsigma \rho \]
we use co-moving co-ordinates so that
\[ u^1 = u^2 = u^3 = 0 \quad \text{and} \quad u^4 = e^{-\frac{\psi}{2}} \]

The non-vanishing components of the energy momentum tensor are
\[ T_1^1 = T_2^2 = T_3^3 = -p \quad \text{and} \quad T_4^4 = \rho \]

Now we write the field equations as below
\[ (2.8) \quad 8\pi p = e^{-\chi} \left( \frac{\psi'}{m} + \frac{1}{m^2} \right) - \frac{1}{m^2} \]
\[ (2.9) \quad 8\pi p = e^{-\chi} \left( \frac{\psi''}{2} - \frac{\psi' \psi'}{4} + \frac{\psi' + \psi'}{2m} \right) \]
\[ (2.10) \quad 8\pi \rho = e^{-\chi} \left( \frac{\psi'}{m} - \frac{1}{m^2} \right) + \frac{1}{m^2} \]

3. Solution of the Field Equations

Using equations (2.7), (2.8) & (2.10) with \( \varsigma = 1 \), we have
\[ (3.1) \quad e^{-\chi} \left( \frac{\psi'}{m} + \frac{1}{m^2} \right) - \frac{1}{m^2} = e^{-\chi} \left( \frac{\psi'}{m} - \frac{1}{m^2} \right) + \frac{1}{m^2} \]

From [3.1] we see that if \( \psi \) is known, \( \chi \) can be obtained, so we choose

Case. I
\[ (3.2) \quad e^\psi = km^{5/4}, \text{ (where k is constant)} \]

Using (3.2) in equation (3.1) goes to form
\[ (3.3) \quad \frac{de^{-\chi}}{dm} + \frac{13}{4m} e^{-\chi} = \frac{2}{m} \]

Substituting \( \tau = e^{-\chi} \), the equation (3.3) is reduced to,
\[ (3.4) \quad \frac{d\tau}{dm} + \frac{13}{4m} \tau = \frac{2}{m} \]

The solution of linear differential equation is given by
\[ (3.5) \quad \tau = \frac{8}{13} + \frac{l}{m^{13/4}} \]

or
\( e^{-\chi} = \frac{8}{13} + \frac{L}{m^{13/4}} \)

where \( L \) is integration constant.

Hence the metric (2.1) yields

\[
\text{(3.7)} \quad ds^2 = km^{5/4}dt^2 - \left(\frac{8}{13} + \frac{L}{m^{13/4}}\right)^{-1} dm^2 - m^2(d\theta^2 + \sin^2\theta.d\phi^2)
\]

Now we absorbing the constant \( k \) in \( dt \) and put \( L = 0 \) the metric (3.7) goes to the form

\[
\text{(3.8)} \quad ds^2 = m^{5/4}dt^2 - \frac{13}{8} dm^2 - m^2(d\theta^2 + \sin^2\theta.d\phi^2)
\]

The non-zero component of Riemann-christoffel curvature tensor \( R_{hijk} \) for the metric (3.8) is

\[
\text{(3.9)} \quad \sin^2\theta R_{2424} = R_{3434} = \frac{13}{8} m^{13/4}\sin^2\theta = R_{2323}
\]

The fluid velocity \( v' \) (for the metric [3.8]) is given by

\[
\text{(3.10)} \quad v^1 = v^2 = v^3 = 0, \quad v^4 = m^{-5/8} = \frac{1}{m^{5/8}}
\]

In the usual notation, we have the tensor of rotation \( \omega_{ij} = v_{ij} - v_{ji} \) and shear tensor \( \sigma_{ij} = \frac{1}{2}(v_{ij} + v_{ji}) - \frac{1}{3}H_{ij} \) gives results for metric (3.8) as

\[
\text{(3.11)} \quad \Theta = 0, \quad \omega_{14} = -\omega_{41} = -\frac{5}{8} m^{-3/8} = -\frac{5}{8m^{3/8}},
\]

and

\[
\text{(3.12)} \quad \sigma_{14} = \sigma_{41} = \frac{5}{8} m^{-3/8} = \frac{5}{8m^{3/8}}.
\]

\textbf{Case, II}

\[
\text{(3.13)} \quad e^{-\chi} = l, \quad \text{(where} l \text{ is constant)}
\]

Using (3.13), equation (3.1) goes to the –

\[
\text{(3.14)} \quad \psi' - \chi' + \frac{2}{m} \left[ 1 - \frac{1}{l} \right] = 0
\]

Since \( e^{-\chi} = l \), then \( \dot{\chi} = 0 \) and hence (3.14) reduces to

\[
\text{(3.15)} \quad \psi' + \frac{2}{m} \left[ 1 - \frac{1}{l} \right] = 0
\]

Now (3.15) integrate w.r.t \( m \) we get

\[
\text{(3.16)} \quad e^\Psi = Am^{2(1-\frac{2}{l})}
\]

where \( A \) is a integration constant.

If we consider \( l = 3 \), then we get

\[
\text{(3.17)} \quad e^\Psi = Am^{4/3}
\]

Hence, using the equation (3.17) the metric (3.1) yields

\[
\text{(3.18)} \quad ds^2 = Am^{4/3}dt^2 - \frac{1}{3}(dm^2) - m^2(d\theta^2 + \sin^2\theta.d\phi^2)
\]

Now we absorbing the constant \( k \) in \( dt \) and put \( L = 0 \) the metric (3.18) goes to the Form
\( ds^2 = m^{4/3} dt^2 - 1/3(dm^2) - m^2(d\theta^2 + \sin^2\theta. d\varphi^2) \)

The non-zero component of Riemann-christoffel curvature tensor \( R_{hijk} \) for the metric (3.19) are

\[
\sin^2\theta R_{2424} = R_{3434} = -\frac{1}{2} m^2 \sin^2\theta = R_{2323}
\]

We observe that \( R_{hijk} \to 0 \) as \( m \to \infty \). Therefore, it follows that the space time is asymptotically Homaloidal.

The fluid velocity \( v' \) for the metric (3.19) is given by

\[
v_1 = v_2 = v_3 = 0 \quad \text{and} \quad v_4 = \frac{1}{m} = m^{-1}
\]

\( \Theta = 0 \) i.e., the scalar of expansion \( \Theta = v_i \) is symmetrically zero and we have the tensor of rotation \( \omega_{ij} = v_{ij} - v_ji \) we get

\[
\omega_{14} = -\omega_{41} m = m^0 = 1
\]

The shear tensor \( \sigma_{ij} \) have component defined by \( \sigma_{ij} = \frac{1}{2} (v_{ij} + v_{ji}) - \frac{1}{3} H_{ij} \), with the projection tensor \( H_{ij} = g_{ij} - v_i v_j \) are

\[
\sigma_{14} = \sigma_{41} = \frac{1}{2} m^0 = \frac{1}{2}
\]

4. Solution for the Perfect Fluid Core

From Eqn. (3.7-3.8), Pressure and density for the metric are

\[
8\pi p = 8\pi \rho = \frac{9}{4m^2} \left[ \frac{8}{13} + \frac{L}{m^2} \right] - \frac{1}{m^2}
\]

If we consider \( L = 0 \), then equation (4.1) reduces to

\[
8\pi p = 8\pi \rho = \frac{9}{4m^2} \left[ \frac{8}{13} \right] - \frac{1}{m^2}
\]

\[
8\pi p = 8\pi \rho = \frac{5}{13m^2}
\]

Eqn. (4.1 - 4.3), It shows that the \( \rho \to \infty \) (density of the distribution tends to infinity) as \( m \to 0 \). We get singularity at \( m = 0 \) for the density and we obtained that the distribution has a core of radius \( r_o \) and constant \( \rho_o \) respectively. Here the field inside the core is given by Schwarzschild internal solution.

\[
e^{-\chi} = 1 - \frac{m^2}{R^2}
\]

\[
e^{\psi} = \left[ L - M \left( 1 - \frac{m^2}{R^2} \right) \right]^{2}
\]

\[
8\pi p = \frac{1}{R^2} \left[ \frac{3M \left( 1 - \frac{m^2}{R^2} \right) - L}{L - M \left( 1 - \frac{m^2}{R^2} \right)^2} \right]
\]

where \( L, M \) are constants and \( R^2 = \frac{3}{8\pi \rho} \).
The continuity condition for the metric (3.7-3.8) and (4.4a - 4b - 4c) at the boundary gives

\[ R^2 = \frac{r_0^2}{\left(\frac{5}{13} - \frac{L}{r_0^2}\right)} \]

\[ L = r_0^{5/9} + \frac{5R^2}{3r_0^{11/9}} \left(1 - \frac{r_0^2}{R^2}\right) \]

\[ M = \frac{5R^2}{3r_0^{17/9}} \left(1 - \frac{r_0^2}{R^2}\right)^{1/2} \]

\[ C = r_0^{13/4} \left(\frac{5}{13} - \frac{r_0^2}{R^2}\right) \]

and the density of the core

\[ \rho_o = \frac{3}{8m^2} \left(\frac{5}{13} - \frac{L}{r_0^2}\right) \]

It satisfies the solution for the perfect fluid core of radius \( r_o \) surrounded by considered fluid. (Here we use the energy condition \( T_{ij}u^iu_j > 0 \) and the Hawking and Penrose condition)

\[ (T_{ij} - \frac{1}{2}g_{ij}T)u^iu_j > 0, \]

Both reduces to \( \rho > 0 \), which is obviously satisfied.

5. Discussion

In this paper we have discussed about exact static spherical solution of Einstein’s field equation with \( \Lambda = 0 \) and \( p = \rho \). We have shown that when cosmological constant \( \Lambda = 0 \), then if pressure and density are equal there is no electromagnetic field. Our assumption is \( e^\psi = km^{5/4} \) and \( e^{-\chi} = 1 \), which investigate the value of \( e^\chi \) and \( e^\psi \) respectively. It describe several important cases, e.g.- relativistic model, fluid velocity, rotation, shear tensor, scalar of expansion. It also helpful to investigates solution for the perfect fluid core.

6. Applications

[a.] In this paper we get the value of \( e^\chi \) and metric. Now we can easily obtained the metric for any given value of \( n \), where \( n \) is the power of \( m \).

[b.] It helpful to study of staler body, radiation and fluid core (especially for the perfect fluid core).

[c.] To study of ideal gas (for casual limit) has also for \( \rho = p \) and relativistic model, fluid velocity, rotation, shear tensor, scalar of expansion.

7. Future Prospects

The investigation on this topic can be further taken up in different directions:

- It is important in a natural way to make a search for exact solutions of theories of gravitation for different types of distributions of matter and for different type of symmetries of space time.
- This topic has been a proliferation of works on higher dimensional space times both in localized and cosmological domains.
- This also helpful to provide the idea about to study of early stages of the formation of the cosmos.
8. Limitation
If we consider $l = 0$ for equations (3.16), then the value of $e^{\psi}$ becomes undefined but for $l = 1$ it gives a constant value.

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