I. INTRODUCTION

There is a revival of interest in the light hadron spectroscopy. Several experiments are currently carrying out searches for new and exotic resonances and more facilities are planned for the near future. This is because the current knowledge of the spectrum is still very poor; in particular in the meson sector the Particle Data Group lists approximately a dozen or so of well-established resonances and a plethora of states with poorly determined characteristics whose existence, in some cases, is even questionable. Establishing resonance parameters requires partial wave analysis and in particular knowledge of amplitudes in an unphysical region of the energy. This necessitates use of methods, such as dispersion relations, that explore analytical properties of the physical amplitudes. Since the unitarity condition relates the measured cross sections to the discontinuities of the amplitude, dispersion relations can be solved to obtain amplitudes for any complex (unphysical) value of the energy. Unfortunately, even in the simplest cases, no complete information about discontinuities of the amplitude is available and approximations must be devised to solve the dispersion relations. Nevertheless, application of dispersion relations can lead to precision studies of resonance parameters, as was recently shown to be the case of the σ resonance in S-wave ππ scattering \( \pi \pi \) scattering [2][4]. Not only that but, exploring the relation between chiral effective field theory and QCD parameters, it was possible to shed new light onto the resonance structure [9][12]. Further insight into the microscopic nature of a resonance can be obtained by studying its shape in different production processes [13][15]. For example, a spatially extended source that produces a pair of mesons is not expected to couple strongly to a resonance that is dominated by valence quarks, i.e. whose wave function is spatially compact. Thus a compact source such as charmonium, when decaying into light mesons, is expected to couple to short-range components of intermediate states and resonances.

It thus follows that in partial wave analysis it is important to use amplitudes with proper analytical behavior across both the physical (right) and unphysical (left) cut. The discontinuity of the amplitude across the right hand cut (rhc) is constrained by unitarity and relates partial wave amplitudes in production and scattering reactions. The latter "contains" resonances and these will also appear in the production process albeit with modified characteristics, that depend on the nature of production dynamics. The left hand cut (lhc) determines the production dynamics and may enhance or suppress a particular resonance. For example, the QCD nature of the \( a_1 \) as a member of the quark multiplet was established by application of analytical methods to the \( 3\pi \) production in \( \pi p \) collisions [16]. More recently, an analysis of the \( \pi \pi \) production in \( \gamma \gamma \) fusion based on dispersion relations showed that the \( \sigma \) meson is expected to contain a substantial short range component, besides the dominant \( \pi \pi \) component [17]. The long range component originates from long range interactions between pions and, only in the limit of asymptotically large number of colors, it is expected to be suppressed compared to the quark component.

Unfortunately it is a common practice to ignore the above mentioned intricacies of amplitude analysis and rely on simple parameterizations i.e. in terms of a superposition of Breit-Wigner (BW) resonances. More sophisticated analyses would use pole parameterizations for the \( K \)-matrix and implement unitarity restrictions across the right hand cut, but the role of the left hand cut is often overlooked. The main motivation of this paper is illustrate its role using as an example the \( \pi \pi P \)-wave amplitude which is dominated by the well-known \( \rho(770) \) resonance. We will show how the general analytical properties of amplitudes discussed above emerge in this particular case and how the behavior of the amplitude on the lhc can modify the resonance production.

The paper is organized as follows. In the next section we summarize the analytical properties of partial wave amplitudes, both in the case of scattering and production. In Section II we solve the dispersion relation for...
the $P$-wave amplitude and discuss a simple parameterizations showing how the shape of the $\rho$-meson changes depending on the characteristics of the production process. Summary and outlook are given in Section IV.

II. ANALYTICAL PROPERTIES OF SCATTERING AND PRODUCTION PARTIAL WAVE AMPITUDENS

In this work we focus on partial wave amplitudes which are a function of a single energy variable. This applies, for example to photoproduction of pion pairs on the nucleon, $\gamma p \to \pi^+ \pi^- p$. At fixed photon energy $E_\gamma$, momentum transfer, $t$ between the target and recoil nucleon and photon and nucleon helicities $\lambda_i$, the di-pion production amplitude, $A$ is a function of the di-pion invariant mass squared $s$, and the spherical angle $\Omega = (\theta, \phi)$ which describes the direction of motion of the $\pi^+$ in the di-pion rest frame,

$$ A = A(s, \Omega, E_\gamma, t, \lambda_i). \quad (1) $$

The angular dependence can be expanded in a series of partial waves labeled by the spin of the di-pion pair, $l$ and its projection on, e.g. the photon direction (t-channel helicity frame)

$$ A = \sum_{lm} A_{lm}(s; E_\gamma, t) Y_{lm}(\Omega). \quad (2) $$

Finally the charged di-pion state can be decomposed into states of total isospin, $I = 0, 1, 2$

$$ A_{lm} = \sum_{I, I_3} A_{lm, I I_3}(11, 1 - 1|I I_3). \quad (3) $$

The partial wave analysis can now be performed in bins of $E_\gamma$ and $t$ for a set of partial wave amplitudes $A_{lm, I I_3}$ of a single variable $s$ -- the di-pion mass squared. The singularities of these amplitudes in the complex-$s$ plane will in general depend on $t$. This parametrization was recently used in the analyses of the two pion photoproduction data from CLAS at JLab [18, 19] and is currently under way for the hadronic decays of light charmed hadrons using CLEO and BES data [20].

In the following we will concentrate on production of pion pairs. The case of heavier mesons and/or baryons can be formulated analogously, albeit with complications arising from presence of sub-threshold cuts and spin. As a function of the di-pion invariant mass squared, $s = 4(q^2 + m^2)$, with $q$ being the relative momentum, and at fixed values of other kinematical variables, the $t$-th partial wave production amplitude is factorized into the angular momentum barrier factor, $F$ and the amplitude $F(s)$ that is free from kinematical singularities

$$ A(s) = q^3 F(s). \quad (4) $$

The amplitude $F(s)$ is a real analytical function with a right hand cut starting at threshold, $s_{th} = 4m^2$ and with each open inelastic channel, $s_i > s_{th}$ contributing to the discontinuity across the cut. The value of this discontinuity is constrained by unitarity. For $s$ on the positive real axis, defining $F^\pm(s) \equiv F(s \pm i\epsilon)$ as the boundary value of the function $F(s)$ on the upper (lower) lip of the cut, unitarity implies

$$ \text{Im} F^+(s) = t^-(s)\rho(s)F^+(s)\theta(s-s_{th}) + \sigma(s)\theta(s-s_i). \quad (5) $$

Here $t^-(s) = t(s - i\epsilon)$ is the boundary value on the lower lip on the right hand cut of the $\pi\pi$ l-wave scattering amplitude $t(s)$ considered as an analytical function of $s$. The two body phase space is given by $\rho = (1 - s_{th}/s)^{1/2}$ and $\sigma(s)$ represents the contribution from production of inelastic channels above $s_i$. We are using the following normalization of the $\pi\pi$ amplitude,

$$ \eta(s)e^{2i\delta(s)} = 1 + 2i\rho(s)t^+(s), \quad (6) $$

where $\delta, \eta$ is the phase shift and inelasticity, respectively. In addition to the right hand unitarity cut, the production amplitude is discontinuous for negative $s$. The location of the left hand cut $s < s_0 \leq 0$ depends on the underlying production dynamics. It is related to thresholds for particle production in the crossed channels and plays the role of the driving term in the integral equation for the amplitude that follows from a dispersion relation. It has a similar role to that of the potential in the non-relativistic Lippmann-Schwinger equation of the Schrödinger theory. Thus for $s$ real and $s < s_0$,

$$ \text{Im} F^+(s) = \text{Im} F_L(s + i\epsilon)\theta(s_0 - s), \quad (7) $$

with the ”potential” $F_L(s)$ defined as a real analytic function in the complex $s$-plane with the lhc discontinuity given by Eq. (7),

$$ F_L(s) = \frac{1}{\pi} \int_{-\infty}^{s_0} ds_L \frac{\text{Im} F^+(s_L)}{s_L - s}. \quad (8) $$

From Eqs. (5), (7) it follows that $F(s)$ satisfies an integral equation,

$$ F(s) = F_L(s) + \frac{1}{\pi} \int_{s_{th}}^{s_0} ds_R \frac{t^-(s_R)F^+(s_R)}{s_R - s} + \frac{1}{\pi} \int_{s_i}^{\infty} ds_R \frac{\sigma(s_R)}{s_R - s}. \quad (9) $$

where we assumed that the integrands vanish as $|s| \to \infty$. If $F(s)$ is bound by a polynomial in $s$, the dispersive integral can be made convergent by subtractions. These introduce additional parameters, that are related to the asymptotic behavior of the amplitude. In general little is known about the left hand cut discontinuity, i.e. $F_L(s)$. For example in bootstrap calculations it is approximated by particle, or more generally Regge exchanges with parameters adjusted so that the solution of the dispersion relation reproduces the known resonances. More recently
chiral effective field theory has been used to construct approximation to $F_L$ at low energies by expanding it in powers of $s/\Lambda^2$, where $\Lambda = 4\pi f_\pi$ is the chiral scale [4,5,21,23]. The dispersion relation in Eq.(9) is an integral equation for the production amplitude $F(s)$ which as input takes i) the scattering amplitude $t$, ii) the amplitude representing contributions from production of inelastic channels $\sigma$ and iii) the "potential", $F_L$. The analytical solution of Eq.(9) is known, and involves the function $D(s)$ from the $N/D$ decomposition of the scattering amplitude, $t(s)$ [24,20]. Even though the $N/D$ method for solving partial wave dispersion relations has been extensively studied in the past, due to its important role in partial wave analysis we summarize its main features in the following paragraphs.

A. N/D representation of the scattering amplitude

The dispersion relation for the partial wave scattering amplitude $t(s)$ is given by

$$t(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} ds_L \frac{Imt^+(s_L)}{s_L - s} + \frac{1}{\pi} \int_{s_{th}}^{\infty} ds_R \frac{Imt^+(s_R)}{s_R - s}. \tag{10}$$

It becomes an integral equation for $t(s)$, once unitarity is implemented. It relates the $rhc$ discontinuity in $t$ to the amplitude itself via,

$$Imt^+(s_R) = \rho(s_R) |t(s_R)|^2 \theta(s_R - s_{th})$$

$$+ \frac{1 - \eta^2(s_R)}{4} \theta(s_R - s_i). \tag{11}$$

The solution of Eqs.(10), (11) is expressed in the form,

$$t(s) = (s - s_{th})^l \frac{N(s)}{D(s)}, \tag{12}$$

where the angular momentum barrier factor has been explicitly factored out and $N(s)$, and $D(s)$ are defined so to have only left and right hand cut, respectively. The definition of $N$ and $D$ is unique up to an overall constant, which we fix by normalizing $D$ at the elastic threshold, $D(s_{th}) = 1$. For $l \geq 1$ the uniqueness follows from the threshold behavior, $t(s_{th}) = 0$ and the asymptotic behavior at large $s$, $t((\infty + i\epsilon)) < O(1)$. For $S$-waves, one subtraction may be needed to make the integrals in Eq.(10) convergent and consequently the dispersion relation for $t$ (or $N$ or $D$) contains one undetermined constant (i.e. scattering length).

From now on we focus on the $P$-wave, $l = 1$ amplitude. From the dispersion relation for $t$ it follows that,

$$N(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} ds_L \frac{Imt(s_L)D(s_L)}{(s_L - s_{th})(s_L - s)}$$

$$D(s) = 1 - \frac{s - s_{th}}{s_{th}} \int_{s_{th}}^{\infty} ds_R \frac{\rho(s_R)N(s_R)R(s_R)}{s_R - s}, \tag{13}$$

where $R \equiv Imt^+/|t|^2$, differs from unity for $s_R > s_i$ because of inelastic channels opening. Substitution of the equation for $N$ into the equation for $D$ leads to an integral equation,

$$D(s) = 1 - \int_{-\infty}^{s_{th}} ds_L K(s,s_L)D(s_L) \tag{14}$$

where the kernel

$$K(s,s_L) = \frac{s - s_{th}}{\pi^2} \int_{s_{th}}^{\infty} ds_R \frac{\rho(s_R)R(s_R)Imt(s_L)}{(s_L - s_{th})(s_L - s_R)(s_R - s)}, \tag{15}$$

is given in terms of the "potential", i.e. discontinuity of $t$ across the left hand cut, and the inelastic contribution proportional to $R - 1$. Since $|t((\infty + i\epsilon))| < O(1)$, it follows from Eq.(13) that asymptotically $|N(s)| < O(1/s)$ and $|D(s)| < O(1)$ and no subtractions are needed.

The dispersion relation for $D$ in Eq.(14) assumes $t \neq 0$, which is always the case in the inelastic region when $\eta \neq 1$. Otherwise zeros of $t$ correspond to poles of $D$, the so called CDD poles that are not accounted for by Eq.(13) [27]. Therefore in presence of zeros of the scattering amplitude, the CDD poles have to be added "by hand" and result in,

$$D(s) = 1 - \frac{s - s_{th}}{s_{th}} \int_{s_{th}}^{\infty} ds_R \frac{\rho(s_R)N(s_R)R(s_R)}{s_R - s}$$

$$-(s - s_{th})\Pi_{p=1}^{N_p} \frac{\gamma_p}{(s - s_p)(s - s_{th})}, \tag{16}$$

where $\gamma_p$ and $s_p$ are the (real) residue and position of the $p$-th CDD pole, respectively. At every CDD pole the phase of the amplitude passes through $180^0$ and Levinson theorem relates the number of CDD poles to the phase shift at infinity, $\delta(\infty)/\pi = N_p$ [28]. If the residue of a CDD pole is small, then $D(s)$ will develop a zero on the second sheet near the position of the pole, i.e. will produce a resonance. Thus, in the past it has been proposed to identify CDD poles with the elementary quark bound states that turn into physical resonances when coupled to the continuum channels. Indeed it has been shown that in potential models describing, for example scattering off a static source with internal structure, CDD poles in the dispersion relation for a scattering amplitude correspond to excitations of the target. There is, however no proof of such correspondence in QCD, and as we will show in the following section, a correspondence between CDD poles and quark model states may be more complicated than in potential theory.

It follows that, even though unitarity and the left hand cut discontinuity do not yield a unique solution of the dispersion relation for the scattering amplitude $t$, the CDD ambiguity disappears if the phase shift and inelasticity are known. In this case, the solution of Eq.(16) is given by,

$$D(s) = \Pi_{p=1}^{N_p} \left[ \frac{s_{th} - s_p}{s - s_p} \right] \Omega_\phi(s), \tag{17}$$
where the first factor comes from the CDD poles and the Omnes-Muskelishvili function $\Omega$ \cite{29,30}.

$$\Omega_\phi = \exp \left( -\frac{s - s_{th}}{\pi} \int_{s_{th}}^{\infty} ds_R \frac{\phi(s_R)}{(s_R - s)(s_R - s_{th})} \right)$$

is given in terms of the phase of $t$, $t = |t| \exp(i\phi)$, which is identical to the phase shift $\delta$ in the elastic region. Outside the elastic region the phase $\phi$ is determined from phase shift and inelasticity and is bound between 0 and $\pi$.

B. N/D representation of the production amplitude

The solution of the dispersion relation, Eq.\cite{19} for the production amplitude can be represented in a number of equivalent ways \cite{31}. From the point of view of data parametrization, a particularly useful representation is given by

$$F(s) = \frac{G(s)}{D(s)} = \frac{F(s_{th}) + G_L(s) - G_i(s)}{D(s)},$$

where $D(s)$ is given by Eq.\cite{17} (or \cite{16}), the numerator function $G_L$ contains only the left hand cut,

$$G_L(s) = \frac{s - s_{th}}{\pi} \int_{s_{th}}^{s_0} ds_L \frac{ImF_L(s_L)D(s_L)}{(s_L - s)(s_L - s_{th})}$$

while the function $G_i(s)$ describes inelastic contribution to production,

$$G_i(s) = \frac{s - s_{th}}{\pi} \int_{s_i}^{\infty} ds_R \frac{Im\sigma(s_R)D(s_R)}{t^{-}(s_R)\rho(s_R)(s_R - s)(s_R - s_{th})}.$$  \hspace{1cm} (21)

We have assumed that at most one subtraction is needed, which is the case if $D$ has a CDD pole at infinity that as we will discuss later is the likely case for the $P$-wave. Expression for the production amplitude in Eq.\cite{19} can easily be adopted to fits of experimental data. Assuming that $D(s)$ is a known function (we will discuss special cases in the next Section), the numerator in Eq.\cite{19} depends on the inelastic contribution ($\sigma$) and the production dynamics ($F_L$). These are the quantities one should extract from the data. The form in Eq.\cite{19} suggests that, instead of parametrizing $F_L$ and $\sigma$, it would be more efficient to parametrize the entire expression appearing in the numerator. It represents a sum of two analytical functions one with the left an the other with a right hand cut that begins at the first inelastic threshold. Thus one can conveniently use a series expansion,

$$G_L(s) = \sum_{i=0}^{\infty} c_i L z^i_L(s), \quad G_i(s) = \sum_{i=0}^{\infty} c_{i,R} z^i_R(s),$$

where $s \to z_{L,R}(s)$ represents a conformal mapping of a cut plane on a unit circle \cite{32}.

$$z_L(s) = \frac{1 - \sqrt{1 - s/s_L}}{1 + \sqrt{1 - s/s_L}}$$

$$z_R(s) = \frac{1 - \sqrt{1 - s/s_i}}{1 + \sqrt{1 - s/s_i}}.$$  \hspace{1cm} (23)

One would then determine the coefficients $c_{i,L}, c_{i,R}$ from fitting $F(s)$ to the data.

Another way of representing the solution of Eq.\cite{19}, which in particular amplifies the spacial characteristics of the production process, is obtained by representing $D(s_L)$ in $G_L(s)$ through its dispersion relation (c.f. Eq.\cite{16}), resulting in

$$F(s) = F_L(s) + \frac{F(s_{th}) - F_L(s_{th})}{D(s)}$$

$$-\frac{s - s_{th}}{\pi D(s)} \int_{s_{th}}^{\infty} ds_R \frac{ImD(s_R)F_L(s_R)}{(s_R - s)(s_R - s_{th})}$$

$$-\frac{s - s_{th}}{\pi D(s)} \int_{s_i}^{\infty} ds_R \frac{Im\sigma(s_R)D(s_R)}{t^{-}(s_R)\rho(s_R)(s_R - s)(s_R - s_{th})}.$$  \hspace{1cm} (24)

Both representation, Eq.\cite{19} and Eq.\cite{24}, below the inelastic threshold $s < s_i$, satisfy the final state interaction theorem, $\arg F(s) = -\arg D(s) = \arg t(s)$. In Eq.\cite{24}, the production amplitude is expressed in terms of the production amplitude $F_L$ evaluated in the physical $(s > s_{th})$ kinematics. Parametrizing $F_L$ via the conformal map (c.f. Eq.\cite{23}),

$$F_L(s) = \sum_i c_i z^i_L(s),$$

and replacing the inelastic contribution with the fit function $G_i$ from Eq.\cite{25} leads to

$$F(s) = \frac{F(s_{th})}{D(s)} + \sum_i \left[ z^i_L(s) - \frac{z^i(s_{th})}{D(s)} + I_i(s) \right] - \frac{G_i(s)}{D(s)}$$

with the functions $I_i(s)$ given by

$$I_i(s) = -\frac{s - s_{th}}{\pi} \int_{s_{th}}^{\infty} ds_R \frac{ImD(s_R)z^i_L(s_R)}{(s_R - s)(s_R - s_{th})}.$$  \hspace{1cm} (27)

The asymptotic behavior is encoded in the rate of convergence of $\sum_i c_i z^i_L$ at the circle $|z_L| = 1$. The applicability of Eq.\cite{27}, in experimental data fit is however based on the assumption that only a few orders in the expansion in powers of $z_L$ are needed to describe a particular data set, i.e. one wants the fits to be insensitive to the poorly known asymptotic behavior.

In the elastic region $s_{th} < s < s_i$ from Eq.\cite{24} one finds

$$F(s) = e^{i\delta(s)} \left[ F_L(s) \cos \delta(s) - B(s) \frac{\sin \delta(s)}{\rho(s)} \right],$$

where $s \to z_{L,R}(s)$ represents a conformal mapping of a cut plane on a unit circle \cite{32}.
where $\delta$ is the elastic phase shift and $B$ is a real function given by

$$B(s) = \frac{F_L(s_{th}) - F(s_{th})}{t(s_{th})D(s)} + \frac{s - s_{th}}{\pi t(s_{th})D(s)} \int_{s_{th}}^{\infty} ds_R \frac{I_mD(s_R)F_L(s_R)}{(s_R - s)(s_R - s_{th})} + \frac{s - s_{th}}{\pi t(s)D(s)} \int_{s}^{\infty} ds_R \frac{I_m(s_R)D(s_R)}{(s_R - s)(s_R - s_{th})},$$

with $P.V.$ standing for the principal value. In particular, if at some $s = s_r$ the phase shift passes through a resonance, i.e $90^\circ$, the first term in Eq. (28) will produce a zero in the amplitude, while the second term will produce a peak there. The zero is due to a destructive interference between the direct production of two-particles and their final state interaction. In the case where production is short-ranged, as discussed earlier, the start of the left hand cut in $F_L(s)$, $s_0$ is far away from the physical region, $s_0 << s_{th}$, the principal value integral extends over a large interval in $s_R$, $s_R < |s_0|$ and the contribution from the $B$ term in Eq. (28) is significant. On the other hand, if the production source is diffuse and inelasticity is small the resonance peak may be significantly distorted by the zero from the $\cos \delta$ term. Thus analyzing the shape of a resonance can shed light on its production characteristics and thus its nature. We also note that the first term in the rhs. of Eq. (29) contributes if the dispersion relation for $G(s)$ requires subtraction. It is needed if $D(s \to \infty) \sim O(s)$ i.e $D(s)$ has a CDD pole at infinity.

III. P-WAVE $\pi\pi$ SCATTERING AND PRODUCTION

The solution of the dispersion relation for the scattering amplitude requires knowledge of the inelasticity ($R - 1$) and the left hand discontinuity. In the case of the $P$ wave the former is known up to $\sqrt{s} = 1.9$ GeV and the latter has been evaluated in Ref. [35]. In the case of $\pi\pi$ scattering, crossing symmetry gives an additional constraint between the “potential” and the right hand discontinuity. In Ref. [33] a particular parametrization, for the left hand discontinuity

$$Imt(s) = \frac{a + bs}{s^2} + c[1 - \cos(2\pi x)]$$

with $a = 0.48$ GeV$^4$, $b = 1.21$ GeV$^2$, $c = 0.601$ and $x = (-0.320/(0.283 + s/\text{GeV}^2))^{0.36}$, for $s \leq -32m_\rho^2$ has been shown to faithfully represent this constraint. For $-32m_\rho^2 < s < s_0 = 0$, crossing symmetry leads to

$$Imt(s) = \frac{2m_\rho^2}{s - s_{th}} \int_{s_{th}}^{s} ds_R P_1 \left(1 + \frac{2s_R}{s - s_{th}}\right) \times \sum_{l=0}^{2} C_{l1}\sum_{l=0}^{\infty}(2l + 1)Imt_l(s_R)P_{l1} \left(1 + \frac{2s}{s_R - s_{th}}\right).$$

(31)

Since the integration range is limited between threshold and $36m_\rho^2 \sim 0.675$ GeV$^2$, the sum can be truncated to include only a few low partial waves and the corresponding amplitudes $Imt_{l1}$ ($t = t_{l1}$) expressed in terms of known phase shifts and inelasticities. For $R$ in Eq.(13) we use the data from [34–36] and assume $\eta = 1$ for $\sqrt{s} > 2$ GeV. The kernel in Eq.(15) is then inverted numerically and the $D$ function is obtained from computing

$$D(s) = \int_{-\infty}^{s_0} ds_L [1 + K]^{-1}(s, s_L)P(s_L)$$

(32)

where $P(s_L)$ if no CDD poles are assumed, or $P(s_L) = 1 + \gamma^2(s_L - s_{th})/(s_p - s_{th})/(s_p - s)$ for the case of one CDD pole. Since the $P$-wave phase shift stays below 180$^\circ$ all the way up to the first relevant inelastic threshold associated with $K\bar{K}$ production, no CDD poles are expected. The exception is a pole at infinity. In our case, since we have assumed that inelasticity is negligible at large $s$, the CDD pole at infinity could in fact appear at a finite $s_p > (2 \text{GeV})^2$. Once $D(s)$ for $s < 0$ is obtained from Eq.(32), both $N(s)$ and $D(s)$ in the physical region can be computed from Eq.(13) and compared with the measured phase shift and inelasticity. The results are shown in Fig. 1, respectively with the dashed-dotted line showing the result computed without the CDD pole and the solid with one CDD pole, whose residue and pole position have been fitted to the scattering data [34–36]. It is clear that the resonance behavior corresponding to the $\rho(770)$ is due to the presence of a CDD pole. The exact location of the pole cannot be established due to the unknown behavior of the phase shift at high energies. In the current fit we find the CDD pole with $\gamma = 50.4$ GeV$^2$ and $s_p = 6.9$ GeV$^2$ which for all practical purposes as far as the $\rho$ meson is concerned might be taken to be at infinity. We also note that in [31] it was erroneously stated that the $P$-wave has a CDD pole at threshold. Vanishing of the amplitude at threshold is due to the angular momentum barrier and not to the presence of a CDD pole, the latter being related to the dynamics of the scattering process.

Next explore the connection between the CDD pole description of the $\rho$ resonance with that of the quark model. From the quark model perspective, the $\rho$ meson is considered as a quark-antiquark bound state which becomes a resonance when coupling to the continuum $\pi\pi$ channel is allowed. Generically, with such dynamical assumptions, the $P$-wave $\pi\pi$ scattering amplitude can be expressed as a solution of a separable integral equation shown in Fig.[2]
FIG. 1: $P$-wave phase shift (upper panel) and inelasticity (lower panel). Data from [34–36], dashed-dotted (solid) line solution of dispersion relation without (with) a CDD pole. Dashed line is the fit of the quark model from Eq. (33).

FIG. 2: Quark model representation of the $\rho$ resonance. The bare state with unrenormalised mass $m_B$ couples to a single, two-particle, open channel.

which leads to

$$t(s) = \frac{(s - s_{th}) f^2(s)}{m_B^2 - s - I_{ff}(s)}$$

(33)

Here $f(s)$ is a vertex function which represents the coupling of the bare, quark model $\rho$ state to the $\pi\pi$ channel, i.e. is given in terms of the overlap of the quark model $\rho$ meson and the pion wave functions. In general $f(s)$ is analytical in the $s$-plane, except for "potential"-like singularities for negative $s$. In the pole approximation it is then given by ($s_f > 0$),

$$f(s) = \frac{\lambda_f s_f}{s + s_f}$$

(34)

The contribution from the open $\pi\pi$ channel results in the modification of the bare $\rho$ propagator $(m_B^2 - s)^{-1}$ by loop a integral given by

$$I_{ff}(s) = \frac{1}{\pi} \int_{s_{th}}^{\infty} ds_R \frac{\rho(s_R) f^2(s_R)}{s_R - s}.$$  

(35)

In terms of the $N/D$ representation (c.f. Eq. (12)) the amplitude of Eq. (33) is becomes,

$$D(s) = 1 - \frac{s - s_{th}}{\pi} \int_{s_{th}}^{\infty} ds_R \frac{\rho(s_R) N(s_R)}{s_R^2 - s^2} - (s - s_{th}) C,$$

$$N(s) = C f^2(s)$$

(36)

where the (positive) constant $C$ is defined by

$$C^{-1} = m_B^2 - s_{th} - \frac{1}{\pi} \int_{s_{th}}^{\infty} ds_R \rho(s_R) f^2(s_R).$$

(37)

Comparing with Eq. (16) shows that the quark model representation of the $\rho$ meson, in the absence of higher excited ($\rho'$) states corresponds to a single CDD pole at infinity i.e. with $s_{\rho'}, \gamma_{\rho'} \to \infty$, $\gamma_{\rho'}/s_{\rho'}^2 = $ fixed. We fit the three constants $s_f, \lambda_f$ to the $P$-wave phase shift below $\sqrt{s} = 1.2$ GeV, i.e. below the energy at which inelasticity becomes sizable (since $t$ in Eq. (33) describes a single channel). The result of the fit is shown in Fig. 1 by the dashed line. Below $\sqrt{s} = 1.5$ GeV the solution of the dispersion relation and the simple quark model parametrization are essentially indistinguishable. The comparison between $N$, $D$ and $t$ from dispersion relation calculation and the quark model is shown in Fig. 3 and 4 respectively. For $s > 0$ and in the $\rho$ region, $t$ computed from dispersion relations and the model should agree since both reproduce the phase shift and inelasticity. The $D$ function is sensitive to the phase of the amplitude over the whole energy range and since the phases of $t$
in both cases approach 180° there is also little difference below the inelastic region. The only noticeable difference is in the “potential”, i.e. the numerator function $N$. In the case of the model in Eq. (33) there is a double pole at $s = -s_f = -3.1 \text{ GeV}^2$ imposed by the simple parametrization of Eq. (34), while the actual solution of the dispersion relation shows a structure at $s \sim -1 \text{ GeV}^2$ and $s \sim 2.5 \text{ GeV}^2$. It is clear that knowledge of the amplitude in the physical region alone does not allow one to reconstruct the potential. This is because the $\rho$ is dominated by the CDD pole and the “potential” interaction, i.e. rescattering of the pions, has little effect on the $\rho$. This can also be seen in Fig. 4 where the comparison between the full solution and the no-CDD solution is shown.

A. $\rho$ meson production

We finally illustrate the role of the left hand cut on the $\rho$ shape in a specific production process. In a quark model description the production amplitude $A(s) = qF(s) = (s - s_{th})^{1/2}F(s)$, (c.f. Eq. (4)) becomes

$$F(s) = g(s) + \frac{f(s)I_{gf}(s)}{m_B^2 - s - I_{ff}(s)}, \quad (38)$$

where

$$I_{gf}(s) = \frac{1}{\pi} \int_{s_{th}}^{\infty} ds_R \frac{(s_R - s_{th})\rho(s_R)g(s_R)f(s_R)}{s_R - s}, \quad (39)$$

and is depicted in Fig. 6. The first term in Eq. (38) corresponds to direct production of the two-pions, and is given by a “potential” term, $g(s)$, which we parametrize...
as

\[ g(s) = \frac{\lambda_g s_g}{s + s_g} \]  

with \( s_g > 0 \) characterizing the range of interaction or the size (\( \sim 1/\sqrt{s_g} \)) of the source. A compact, or point-like source corresponds to \( g(s) \to \text{const.} \), i.e. \( s_g \to \infty \) while a spatially extended source has \( s_g \to 0 \) (with \( \lambda_g s_g \) fixed). The production amplitude in Eq. (38) can be cast in the form of the general representation of Eq. (19). In this case \( G_i(s) = 0 \) since the model contains no inelastic channels, and

\[ G(s) = C \left[ g(s)(m_B^2 - s) - (g(s)I_{ff}(s) - f(s)I_{gf}(s)) \right]. \]  

(41)

Indeed, the right hand cut singularities of \( I_{ff} \) and \( I_{gf} \) cancel out and \( G(s) \) has only the left hand cut singularities of \( g(s) \) and \( f(s) \). Similarly Eq. (38) can be written in the form of Eq. (28) with

\[ \text{Im}F_L(s) = -\pi\delta(s+s_g)\lambda_g s_g F_g - \pi\delta(s+s_f)\lambda_f s_f F_f, \]  

(42)

with

\[ F_g = C m_B^2 + s_g - I_{ff}(-s_g) \]  

\[ D(-s_g) \]

\[ F_f = C I_{gf}(-s_f) D(-s_f) \]  

(43)

and so

\[ F_L(s) = F_g g(s) + F_f f(s). \]  

(44)

In the case of a diffuse source the interaction range is small, and the principal value integral in Eq. (29) is suppressed. In this case the production amplitude is suppressed at the resonance mass, since the other term is proportional to \( \cos \delta \) which vanishes. In the quark model model case, however, we see that regardless of the range of \( \pi\pi \) production, determined by \( g(s) \), the production potential also contains a term responsible for direct production of the quark state, i.e. the term proportional to \( f(s) \) in Eq. (44). Thus the resonance associated with a quark state or CDD pole is expected to show up as a bump in a production process. As expected, the \( g \) term in Eq. (44) becomes more relevant as for diffuse sources, i.e. when \( s_g \ll s_f \) and \( P.V. I_{gf} < P.V. I_{ff} \). This effect is illustrated in Fig. 7.

IV. SUMMARY AND OUTLOOK

Development of analytical parameterizations of hadron production amplitudes is necessary for a successful partial wave analysis. Here we presented the analysis of the \( \pi\pi \) P-wave amplitude. The dominant feature in the
elastic region is the \( \rho(770) \) resonance which cannot be accounted for by “potential” interactions \( i.e. \) the discontinuity of the amplitude across the left hand cut. Instead it seems to originate from internal structure of the scatterers \( i.e. \) the CDD pole. The precise location of the pole cannot be determined without better knowledge of the phase shift and inelasticity in the high \( (\sqrt{s} \gtrsim 2 \text{ GeV}) \) energy range, however for practical applications, it may be assumed that the CDD pole is at infinite energy, \( i.e. \) \( \eta(\infty) \rightarrow 1, \delta(\infty) \rightarrow \pi \). While this is consistent with the expectation for the asymptotic behavior of the pion form factor \[ F(s) \] it contradicts the expectation that inelastic open channels dominate at high energies \( (\eta(\infty) \rightarrow 0) \) and requires further studies. The CDD pole in a partial wave amplitude is equivalent to presence of a “bare” state \( e.g. \) quark bound state, which couples to the \( \pi \pi \) continuum. It is thus not surprising that a generic quark model description of \( P \)-wave \( \pi \pi \) scattering with a single bound state faithfully describes the \( P \)-wave amplitude up to \( \sqrt{s} \sim 1.6 \text{ GeV} \) where inelasticity becomes significant indicating a large \( K\bar{K} \) component of the \( \rho'(1600) \). While the \( \rho \) resonance shape can be distorted if the production amplitude varies strongly near threshold, to our best knowledge such features has not been observed in the experimental data. Even though the left hand cut behavior of the \( P \)-wave amplitude in scattering and production does not seem to drive the \( \rho \) resonance, this does not need to be the case for other resonances \( e.g. \) the sigma meson leading to the conclusion that the left hand cut properties of amplitudes require attention in partial wave analyses.

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