DISCOVERING OF BOUNDEDNESS AND CONTINUITY
OF RANDOM FIELDS BY MEANS OF
PARTITION ENTROPIC SCHEME.

Eugene Ostrovsky, Leonid Sirota.
Department of Mathematics, Bar-Ilan University,
Ramat Gan, 52900, Israel,
e-mails: eugostrovsky@list.ru;
sirota3@bezeqint.net.

Abstract.
We construct a new sufficient conditions for boundedness or continuity of arbitrary random fields relying on the so-called partition scheme, alike in the classical majorizing measure method.
We deduce also the used in the practice (statistics, method Monte-Carlo etc.) exact exponential estimates for tail of distribution of maximum for random field satisfying formulated in this report conditions.

Key words and phrases: Tails of distributions, separable random process (r.p.) and random fields (r.f.), distance and semi-distance, metric entropy and metric entropy conditions, Young-Orlicz function, boundedness and continuity of random fields almost everywhere, probabilistic module of continuity, partition scheme, sub-gaussian variables, diameter, ordinary and Grand Lebesgue norm and spaces, Orlicz norm and spaces, disjoint sets and functions, majorizing measure method.

Mathematics Subject Classification (2000): primary 60G17; secondary 60E07; 60G70.

1 Introduction. Notations. Statement of problem.

Let \((T, d)\) be arbitrary compact metric space with a set \(T\) equipped with a distance (or, more generally, semi-distance) function \(d = d(t, s), \ t, s \in T\). Let also \(\xi = \xi(t), \ t \in T\) be numerical valued separable stochastic continuous random process or separable random field. Denote by the \(P_e(\xi)\) a probability that almost every path of \(\xi = \xi(t)\) is continuous:

\[
P_e(\xi) \overset{def}{=} P(\xi(\cdot) \in C(T, d)), \quad (1.1_c)
\]
as well as denote by the $P_b(\xi)$ a probability that almost every path of $\xi = \xi(t)$ is bounded:

$$P_b(\xi) = P(\sup_{t \in T} |\xi(t)| < \infty). \quad (1.1_b)$$

It may be interest in many applications: method Monte-Carlo, statistics etc. the concrete estimate of tail for maximum distribution

$$P_T(u) = P_{T,\xi}(u) = P(\sup_{t \in T} |\xi(t)| > u), \quad u \geq C > 0,$$

as well as the tail estimate for module of continuity of the r.f. $\xi(\cdot)$, see [9], [10], [19], [20] etc.

We will talk about the sufficient conditions imposed on the finite-dimensional distributions of the r.f. $\xi(t)$ under which $P_c(\xi) = 1$ or $P_b(\xi) = 1$.

This problem may be named as classical. There are at least two approaches for his solving: the so-called entropy approach, where the metric entropy is calculated relative the distance, more precisely, semi-distance function generated by the r.f. $\xi(t)$ itself, see e.g. [13], [14], [31], [1], [3], [4], [5], [20], [23], [28]; the second one is named as majorizing measure approach, see [3], [32] - [35], [19] etc.

S E V E R A L  N O T A T I O N S ,  D E F I N I T I O N S  A N D  F A C T S .

A. A triplet $(\Omega, \mathcal{B}, \mathbb{P})$, where $\Omega = \{\omega\}$ or $\Omega = \{x\}$ is arbitrary set, $\mathcal{B}$ is non-trivial sigma-algebra of subsets $\Omega$ and $\mathbb{P}$ is non-zero non-negative completely additive normed: $\mathbb{P}(\Omega) = 1$ measure defined on the $\mathcal{B}$ is called as ordinary a probabilistic space.

We denote as usually for the random variable $\xi$ (r.v.) (i.e. measurable function $\Omega \to R$) its classical Lebesgue-Riesz norm by

$$|\xi|_p = [\mathbb{E}|\xi|^p]^{1/p} = \left[\int_{\Omega} |\xi(\omega)|^p \mathbb{P}(d\omega)\right]^{1/p}, \quad p \geq 1; \quad (1.2)$$

$$L_p = \{\xi, \ |\xi|_p < \infty\}.$$

B. The so-called Grand Lebesgue Space $G\psi$ equipped with a norm $\|\cdot\|_{G\psi}$ consists by definition on all the numerical valued random variables defined on the our probability space and having a following finite norm

$$G\psi = \{\xi, \ |\xi|_{G\psi} < \infty\}, \quad \|\xi\|_{G\psi} \overset{def}{=} \sup_{p \geq 1} \left[\frac{|\xi|_p}{\psi(p)}\right]. \quad (1.3)$$

Here $\psi = \psi(p)$ is some continuous strictly positive function such that there exists $\lim_{p \to \infty} \psi(p) = \infty$.

These spaces are Banach rearrangement invariant complete spaces. The detail investigation of this spaces (and more general ones) see in [17], [24]. See also [6], [7], [11], [12], [14] etc.
An important for us fact about considered here spaces is proved in [14], [20], [22]: they coincide with some exponential Orlicz’s spaces $Or(\Phi_\psi)$. For instance, if $\psi(p) = \psi_{1/2}(p) := \sqrt{p}$, $p \geq 1$, then the space $G\psi_{1/2}$ consists on all the subgaussian (non-centered, in general case) r.v. $Or(\Phi_{\psi_{1/2}})$ for which by definition $\Phi_{\psi_{1/2}}(u) = \exp(u^2/2) - 1$.

The Gaussian distributed r.v. $\eta$ belongs to this space. Another example: let $\Omega = (0, 1)$ with usually Lebesgue measure and $f_{1/2}(\omega) = \sqrt{|\log \omega|}$, $\omega > 0$; $f_{1/2}(0) = 0$.

It is easy to calculate using Stirling’s formula for the Gamma function:

$$|f_{1/2}|_p \asymp \sqrt{p}, \quad p \in (1, \infty).$$

The correspondent tail behavior:

$$P(f_{1/2} > u) = \exp(-u^2).$$

More detail, let the function $\psi(\cdot) \in G\Psi = G\Psi_\infty$ be such that the new generated by $\psi$ function

$$\nu(p) = \nu_\psi(p) := p \ln \psi(p), \quad p \in [1, \infty)$$

is convex. The r.v. $\eta$ belongs to the space $G\psi$ if and only if it belongs to the Orlicz’s space $L(N_\psi)$ with the correspondent exponential continuous Young-Orlicz function

$$N_\psi(u) := \exp\left(-\nu_\psi^*(\ln |u|)\right), \quad |u| \geq e,$$

$$N_\psi(u) := C \ u^2, \quad |u| < e, \quad C \ e^2 = \exp\left(-\nu_\psi^*(1)\right),$$

and herewith of course both the Banach space norms: $|| \cdot ||_{L(N_\psi)}$ and $|| \cdot ||_{G\psi}$ are equivalent.

One can also complete characterize (under formulated here conditions) the belonging of the non-zero r.v. $\xi$ to the space $G\psi$ by means of its tail behavior:

$$\xi \in G\psi \iff \exists K = \text{const} \in (0, \infty), \quad \max(P(\xi > u), P(\xi < -u)) \leq \exp\left(-\nu_\psi^*(\ln |u|/K)\right), \quad u \geq Ke,$$

see [14], [20], p. 33-35.

The case when in (1.3) the supremum is calculated over finite interval is investigated in [17], [24], [25]:

$$G_{b\psi} = \{\xi, \ ||\xi||_{G_{b\psi}} < \infty\}, \ ||\xi||_{G_{b\psi}} \overset{def}{=} \sup_{1 \leq p < b} \left[\frac{|||\xi||_p}{\psi(p)}\right], \quad b = \text{const} > 1,$$  \hfill (1.4)

but in (1.4) $\psi = \psi(p)$ is continuous function in the semi-open interval $1 \leq p < b$ such that $\lim_{p \uparrow b} \psi(p) = \infty$; the case when $\psi(b - 0) < \infty$ is trivial.
We define formally in the case when \( b < \infty \) \( \psi(p) := +\infty \) for all the values \( p > b \).

An used further example:

\[
\psi(\beta,b)(p) \overset{\text{def}}{=} (b - p)^{-\beta}, \quad 1 \leq p < b, \beta = \text{const} > 0; \ G_{\beta,b}(p) := G_b\psi(\beta,b)(p).
\]

C. Recall that sets \( A_1, A_2, A_i \in B \) are named disjoint, if \( A_1 \cap A_2 = \emptyset \). The sequence of functions \( \{h_n\}, n = 1,2,3\ldots \) is said to be disjoint, or more exactly pairwise disjoint, if

\[
\forall i,j : \ i \neq j \Rightarrow h_i \cdot h_j \overset{\text{a.e.}}{=} 0. \quad (1.5)
\]

If the sequence of (measurable) functions \( \{h_n\} \) is pairwise disjoint, then

\[
\left| \sum_n h_n \right|^p = \sum_n |h_n|^p, \quad \sup_n |h_n(x)|^p = \sum_n |h_n(x)|^p, \quad p = \text{const} > 0. \quad (1.6)
\]

D. We denote as ordinary for any measurable set \( A, A \in B \) its indicator function by \( I_A(\omega) \).

E. Let \( \xi = \xi(t), t \in T \) be again separable random field (process) such that

\[
\exists b = \text{const} \in (1, \infty], \forall p \in [1, b) \Rightarrow \sup_{t \in T} |\xi(t)|_p < \infty.
\]

Then the r.f. \( \xi(\cdot) \) generated the so-called natural \( G\psi \) - function by the formula

\[
\psi(p) = \psi(\xi)(p) \overset{\text{def}}{=} \sup_{t \in T} |\xi(t)|_p, \quad 1 \leq p < b. \quad (1.7)
\]

Evidently,

\[
\forall t \in T \Rightarrow \xi(t) \in G\psi(\xi) \quad \text{and moreover} \quad \sup_{t \in T} ||\xi(t)||G\psi(\xi) = 1. \quad (1.8)
\]

F. Let \( \psi = \psi(p) \) be some function from the class \( G\psi_b, \ b = \text{const} \in (1, \infty] \), such that all the values \( \xi(t), t \in T \) belongs uniformly to the space \( G\psi \). One can suppose without loss of generality

\[
\sup_{t \in T} ||\xi(t)||G\psi = 1. \quad (1.9)
\]

For instance, as the capacity of the function \( \psi(\cdot) \) may be picked the natural function for the r.f. \( \xi : \psi(p) := \psi(\xi)(p) \), if of course there exists.

Define by means of the function \( \psi(\cdot) \) the so - called natural (bounded) distance (more precisely, semi-distance) \( d_\psi(t,s), \ t,s \in T \) on the set \( T \):

\[
d_\psi(t,s) := ||\xi(t) - \xi(s)||G\psi; \quad (1.10)
\]

so that \( d_\psi(t,s) \leq 2 \).

Denote also by \( D = D_\psi \) the diameter of the set \( T \) relative the distance \( d_\psi : \)
\[ D = D(T, d_\psi) = \text{diam}(T, d_\psi) \overset{\text{def}}{=} \sup_{t, s \in T} d_\psi(t, s), \] (1.11)

and by \( H(T, d_\psi, \epsilon) \) the metric entropy of the set \( T \) relative the distance \( d_\psi \) at the point \( \epsilon, \ 0 < \epsilon < D. \)

The so-called entropy integral has by definition a form

\[ \Theta(T, d_\psi, \delta) \overset{\text{def}}{=} 9 \int_0^\delta \exp \left[ v_*(\ln 2 + H(T, d_\psi, \epsilon)) \right] \ d\epsilon, \ 0 < \delta \leq D, \] (1.12)

so that

\[ \Theta(T, d_\psi, D) \overset{\text{def}}{=} 9 \int_0^D \exp \left[ v_*(\ln 2 + H(T, d_\psi, \epsilon)) \right] \ d\epsilon, \] (1.12a)

where

\[ v(y) = \ln \psi(1/y), \ y \in (1/b, 1); \quad v_*(x) \overset{\text{def}}{=} \inf_{y \in (1/b, 1)} (xy + v(y)). \] (1.13)

The transform \( v \to v_* \) is named as co-transform of Young-Fenchel, or Legendre, in contradiction to the classical Young-Fenchel transform

\[ f^*(x) \overset{\text{def}}{=} \sup_{y \in \text{dom } f} (xy - f(y)). \] (1.14)

It is proved in particular in the monograph [20], page 172, theorem 3.17.1, that if for the considered r.f. \( \Theta(T, d_\psi, D) < \infty \), then \( \xi(t) \) is \( d_\psi(\cdot, \cdot) \) continuous almost surely and herewith

\[ \| \sup_{t \in T} |\xi(t)| \|G_\psi \leq \Theta(T, d_\psi, D), \] (1.15)

and following

\[ P\left( \sup_{t \in T} |\xi(t)| > u \right) \leq \exp \left\{ -v_*^* \left[ \ln (u/\Theta(T, d_\psi, D)) \right] \right\}, \ u \geq 0. \] (1.16)

Moreover,

\[ \| \sup_{t, s : d_\psi(t, s) \leq \delta} |\xi(t) - \xi(s)| \|G_\psi \leq 9 \int_0^\delta \exp \left[ v_*(\ln 2 + H(T, d_\psi, \epsilon)) \right] \ d\epsilon = \Theta(T, d_\psi, \delta). \] (1.17)

The value

\[ \omega_{\xi, G_\psi, d_\psi}(\delta) \overset{\text{def}}{=} \| \sup_{t, s : d_\psi \leq \delta} |\xi(t) - \xi(s)| \|G_\psi \]

is named as ordinary probabilistic module of continuity for the (uniform continuous) r.f. \( \xi(\cdot) \) relative the distance function \( d_\psi \). Obviously,
\[
\lim_{\delta \downarrow 0} \omega_{\xi,G,\psi,d}(\delta) = 0 \Rightarrow P_c(\xi) = 1.
\]

Analogous estimates holds true is the r.f. \( \xi = \xi(t) \) satisfies the so-called majorizing measure condition, in particular, if \( \sup_{t \in T} |\xi(t)||G\psi| < \infty \), see \cite{16}, \cite{32}-\cite{35}.

2 An interest example.

The following hypothesis has been formulated in the article \cite{27}, 2008 year:

"Let \( \theta = \theta(t), t \in T \) be arbitrary separable random field, centered: \( E\theta(t) = 0 \), bounded with probability one: \( \sup_{t \in T} |\theta(t)| < \infty \) (mod \( P \)), moreover, may be continuous, if the set \( T \) is compact metric space relative some distance.

Assume in addition that for some Young (or Young-Orlicz) function \( \Phi(\cdot) \) and correspondent Orlicz norm \( || \cdot ||_{Or}(\Phi) \)

\[
\sup_{t \in T} ||\theta(t)||_{Or}(\Phi) < \infty. \tag{2.1}
\]

Recall that the Luxemburg norm \( ||\xi||_{Or}(\Phi) \) of a r.v. (measurable function) \( \xi \) is defined as follows:

\[
||\xi||_{Or}(\Phi) = \inf_{k,k>0} \left\{ \int_{\Omega} \Phi(|\xi(\omega)|/k) \, dP(\omega) \leq 1 \right\}.
\]

The Young function \( \Phi(\cdot) \) is by definition arbitrary even convex continuous strictly increasing on the non-negative right-hand semi-axis such that

\[
\Phi(0) = 0, \lim_{u \to \infty} \Phi(u) = \infty.
\]

Let also \( \Psi(\cdot) \) be arbitrary another Young function such that \( \lim_{u \to \infty} \Psi(u) = \infty, \quad \Psi << \Phi \), which denotes by definition

\[
\forall \lambda > 0 \Rightarrow \lim_{u \to \infty} \frac{\Psi(\lambda u)}{\Phi(u)} = 0, \tag{2.2}
\]

see \cite{29}, p.16.

Recall that \( \Psi << \Phi \) implies in particular that the (unit) ball in the space \( Or(\Psi) \) is precompact set in the space \( Or(\Phi) \).

Open question: there holds (or not)"

\[
|| \sup_{t \in T} |\theta(t)| ||_{Or}(\Psi) < \infty. \tag{2.3}
\]

A more slight question: there holds (or not)

\[
|| \sup_{t \in T} |\theta(t)| ||_{Or}(\Phi) < \infty. \tag{2.3a}
\]
One can formulate the analogous question replacing the Orlicz spaces by Grand Lebesgue ones, see [27].

The detail investigation of the theory of Orlicz’s spaces including the case of unbounded source measure $\mathbf{P}$ may be found in the monographs [29], [30].

This conclusions are true for the centered (separable) Gaussian fields, [3], if the field $\theta(\cdot)$ satisfies the so-called entropy or generic chaining condition [18], [20], [19], [16], [32], [33], [34]; in the case when $\theta(\cdot)$ belongs to the domain of attraction of Law of Iterated Logarithm [21] etc.

Notice that if the field $\theta(t)$ is continuous (mod $\mathbf{P}$) and satisfies the condition (2.1), then there exists an Young function $\Psi(\cdot)$, $\Psi(\cdot) \ll \Phi(\cdot)$ for which the inequality (2.3) there holds, see [18].

The negative answers on these questions are obtained: in the article [26], the case of Orlicz spaces; in [27], more general and more strictly case of Grand Lebesgue Spaces. We recall briefly using further the correspondent example from the last report [27].

**Example 2.1.**

1. We choose in the sequel in this pilcrow as the capacity of compact metric space $(T, d)$ the set of positive integer numbers with infinite associated point which we denote by $\infty$:

$$T = \{1, 2, 3, \ldots, \infty\}. \quad (2.4)$$

The distance $d$ is defined as follows:

$$d(i, j) = \left| \frac{1}{i} - \frac{1}{j} \right|, \quad i, j < \infty; \quad d(i, \infty) = d(\infty, i) = \frac{1}{i}, \quad i < \infty; \quad (2.5)$$

and obviously $d(\infty, \infty) = 0$.

The pair $(T, d)$ is compact (closed) metric space and the set $T$ has an unique limit point $t_0 = \infty$. For instance, $\lim_{n \to \infty} d(n, \infty) = 0$.

2. Let $\Omega = (0, 1)$ with ordinary Lebesgue measure. Let also $f = f(x), \ x \in \Omega = (0, 1)$ be non-zero non-negative integrable function belonging to the space $L_6$. Define a following $\psi$ – function:

$$\nu(p) = |f|^p = \left[ \int_0^1 |f(x)|^p \, dx \right]^{1/p}, \quad 1 \leq p \leq 4. \quad (2.6)$$

On the other words, $\nu(\cdot)$ is a natural function for the function $f$. Evidently, $\nu(\cdot) \in G_{\psi_4} = G_{\psi_{(0,4)}}$.

Introduce also the following numerical sequences

$$c_n := n^\beta, \quad \beta = \text{const} > 0, \ n = 2, 3, \ldots; \quad (2.7)$$

$$\Delta_n := C(\beta) \cdot n^{-4\beta - 1}, \quad C(\beta) : \sum_{n=1}^\infty \Delta_n = 1; \quad a_n = a(n) := \sum_{m=n}^\infty \Delta_n; \quad (2.8)$$
and define also sequence of functions and likewise the following positive random process \( \theta(t) = g_n, \ n = t, \ t, n \in T, \ \Omega = \{x\}, \)

\[
g_n(x) = c(n) \left( \frac{x - a(n)}{\Delta(n)} \right) I_{(a(n+1),a(n))}(x), \ x \in \Omega, \ g_{\infty}(x) = 0; \tag{2.9}
\]

\[
g(x) = \sum_{n=1}^{\infty} g_n(x) = \sum_{n=1}^{\infty} c_n \left( \frac{x - a(n)}{\Delta(n)} \right) I_{(a(n+1),a(n))}(x). \tag{2.10}
\]

Note that the sequence of r.v. \( \{g_n(x)\} \) consists on the non-negative and disjoint functions, therefore

\[
\sup_n g_n(x) = \sum_n g_n(x) = g(x), \quad \left| \sup_n g_n \right|^p = \sum_n \left| g_n \right|^p. \tag{2.11}
\]

Note also that the functions \( g_n \) are disjoint and following \( \sup_n |g_n(x)| < \infty \) almost surely.

We calculate using the relations (2.7)-(2.11):

\[
|g_n|^p = c^p(n) \Delta_n \nu^p(p) = C(\beta) n^{p\beta - 4\beta - 1} \nu^p(p), \ 1 \leq p \leq 4, \tag{2.12}
\]

therefore

\[
\sup_n \sup_{p \in [1,4]} \sup_n |g_n|^p \leq C(\beta) \nu^4(4) < \infty \tag{2.13a}
\]

or equivalently

\[
\sup_n |g_n(\cdot)|_4 < \infty. \tag{2.13b}
\]

Moreover, \( g_n \to 0 \) almost everywhere. Indeed, let \( \epsilon \) be arbitrary positive number. We get applying the estimate (2.12) at the value \( p = 1 \) and Tchebychev-Markov inequality

\[
\sum_n P(|g_n| > \epsilon) \leq C(\beta) \sum_n \frac{n^{-3\beta - 1}}{\epsilon} < \infty.
\]

Our conclusion follows immediately from the lemma of Borel-Cantelli.

So, the random process \( \theta(t) = g_n, \ n = t, \ t, n \in T, \ \Omega = \{x\}, \)

Let us now find the exact up to multiplicative constant expression for the natural function of the r.v. \( \sup_n |g_n(x)| \) as \( p \to 4 - 0 \). We have:

\[
| \sup_n |g_n| |^p = \sum_n |g_n|_p^p = \sum_n c^p(n) \Delta_n \nu^p(p) = \]

\[
= C(\beta) \nu^p(p) \sum_n n^{p\beta - 4\beta - 1} \sim \frac{C_1(\beta)}{4 - p}, \tag{2.14}
\]

\[
| \sup_n |g_n| |_p \sim C_2(\beta)(4 - p)^{-1/4}. \tag{2.15}
\]
Thus, we can choose for our proposition as capacity the $\Psi$-function $\psi(p)$ the function $\psi(4)(p)$, which is in turn equivalent to the following $\Psi$-function

$$\psi(4)(p) := 1, \quad 1 \leq p < 4,$$

and correspondingly to take

$$\phi_0(p) := (4 - p)^{-1/8 \ defined as} \psi^{(1/8,4)}(p), \quad 1 \leq p < 4. \quad (2.16)$$

Obviously,

$$\phi_0(\cdot) \ll \psi^{(0,4)}(\cdot) \quad (2.17)$$

and

$$\| \sup_n |g_n| \| G\phi_0 = \infty, \quad (2.18)$$

**Remark 2.0.** In order to obtain the centered needed process $\theta(t)$ with at the same properties, we consider the sequence $\tilde{g}_n(x) = \epsilon(n) \cdot g_n(x)$, where $\{\epsilon(n)\}$ is a Rademacher sequence independent on the $\{g_n\}$ defined perhaps on some sufficiently rich probability space:

$$P(\epsilon(n) = 1) = P(\epsilon(n) = -1) = 1/2; \quad (2.19)$$

then

$$|\tilde{g}_n(x)| = |g_n(x)|, \quad |\tilde{g}_n|_p = |g_n|_p \quad (2.20)$$

and the sequence $\{\tilde{g}_n\}$ is also pairwise disjoint (Rademacher's symmetrization).

This completes the grounding of using for us properties of our (counter - - ) example.

**Remark 2.1.** The constructed process $\theta(t)$ give us a new example of centered continuous random process with relatively light tails of finite-dimensional distribution, but for which the so-called entropy and generic chains series divergent.

**Remark 2.2.** The properties of our example remains true if we use instead the space of continuous function $C(T,d)$ arbitrary separable Banach space.

### 3 Partition scheme. Main result-boundedness.

**Definition 3.1.** The representation of the form

$$T = \cup_{m=1}^\infty T_m, \quad (3.1)$$

write $T \sim \{T_m\}$, on the (measurable) subsets $T_m$, not necessary to be disjoint, is said to be a *partition*, or equally *covering* of the set $T$.

We have for any partition
\[ P_T(u) \leq \sum_{m=1}^{\infty} P_{T_m}(u), \quad u > 0. \] (3.2)

In order to estimate each summand in (3.2), we need to introduce an "individual" natural parameters for the r.f. \( \xi(\cdot) \) on the arbitrary subset \( T_m \). In detail, denote:

\[ \psi_m(p) := \sup_{t \in T_m} \| \xi(t) \|_p, \] (3.3)

and assume

\[ \exists b_m \in (1, \infty], \forall p < b_m \Rightarrow \psi_m(p) < \infty. \] (3.4)

Define as before formally in the case \( b_m < \infty \forall p > b_m \Rightarrow \psi(p) = +\infty. \)

Further, put

\[ v^{(m)}(y) := \ln \psi_m(1/y), \quad y \in (0, 1); \quad D_m := \text{diam}(T_m, d_{\psi_m}), \] (3.5)

\[ Z(m) := \Theta_m(T_m, d(\psi_m), D_m), \] (3.6)

\[ Y(\{T_m\}, u) \overset{def}{=} \sum_{m=1}^{\infty} \exp \left\{ -\nu_*^{\psi_m} \left[ \ln(u/Z(m)) \right] \right\}. \] (3.7)

**Theorem 3.1.** Suppose that for some partition \( T \sim \{T_m\} \)

\[ \lim_{u \to \infty} Y(\{T_m\}, u) = 0. \] (3.8)

Then the r.f. \( \xi(t) \) is bounded with probability one and moreover

\[ P(\sup_{t \in T} |\xi(t)| > u) \leq Y(\{T_m\}, u). \] (3.9)

**Proof.** We have using (1.16)

\[ P\left( \sup_{t \in T_m} |\xi(t)| > u \right) \leq \exp \left\{ -\nu_*^{\psi_m} \left[ \ln(u/Z(m)) \right] \right\}, \quad u \geq 0. \] (3.10)

The proposition (3.9) follows immediately from the estimate (3.2).

**Corollary 3.1.** We conclude under conditions of theorem (3.1)

\[ P(\sup_{t \in T} |\xi(t)| > u) \leq \inf_{\{T_m\}} Y(\{T_m\}, u) \overset{def}{=} R(u), \quad u \geq 0. \] (3.11)

where "inf" in (3.11) is calculated over all the partitions \( T \sim \{T_m\} \) of the set \( T \).

**Remark 3.1.** The estimates (3.10) and (3.11) are exponential non-improvable still in the previous entropy approach, i.e. without partition, see [20], chapters 3,4.

**An example.** It is easy to see that the our example in the second section satisfies all the conditions of theorem 3.1. The partition for the considered therein r.f. is trivial.
4 Partition scheme. Main result-continuity.

The condition (3.8) guarantee us only the boundedness of almost all the paths of the r.f. \( \xi(t) \). We will discuss in this section the sufficient conditions relative the appropriate distance for the continuity of \( \xi(\cdot) \) based on the offered in this article partition scheme.

We will follow the article of V.A.Dmitrovsky [2].

Some new notations. Introduce a new \( \psi \) - function \( \tau = \tau(p) \) as follows:

\[
\tau(p) := \left[ p \int_0^\infty u^{p-1} R(u) \, du \right]^{1/p},
\]

so that

\[
\left[ \mathbb{E} \left\{ \sup_{t \in T} |\xi(t)| \right\} \right]^{1/p} \leq \tau(p),
\]

and suppose its finiteness at last for one value \( p > 1 \); define \( b = \sup\{p, \, p > 1, \tau(p) < \infty\} \); then \( \sup_t ||\xi(t)||_{G\tau} \leq 1 \).

The correspondent bounded natural distance \( \rho = \rho(t, s) \) may be defined as follows:

\[
\rho(t, s) := ||\xi(t) - \xi(s)||_{G\tau}.
\]

We find as before using (1.17)

\[
|| \sup_{t, s: \rho(t, s) \leq \delta} |\xi(t) - \xi(s)| ||_{G\tau} \leq \Theta(T, \rho, \delta),
\]

or equally

\[
\omega_{\xi, G\tau, \rho}(\delta) \leq \Theta(T, \rho, \delta), \quad \delta \in (0, \text{diam}(T, \rho)).
\]

Thus, we proved in fact the following statement.

**Proposition 4.1.** We conclude under formulated above in this section notations and conditions: the r.f. \( \xi = \xi(t) \) is \( \rho(\cdot, \cdot) \) is uniform continuous with probability one.

**Example 4.1 - 2.1.** Let us return to the example 2.1 considered in the second section. We deduce after some calculations taking into account proposition 4.1

\[
P \left( \sup_n |g_n| > u \right) \leq \frac{C_1(\beta)}{u^4} \frac{\ln u}{u^4}, \quad u > e.
\]

The continuity a.e. of the r.p. \( g_n \) is proved in the second section; we intend to prove further the probabilistic continuity. It is sufficient to consider the unique limit point \( n = \infty \).

Define the next Young function

\[
\Phi(u) = e^2 u^2, \quad |u| \leq e, \quad \Phi(u) = \frac{u^4}{\ln |u|}, \quad |u| > e.
\]

The distance function \( \rho(n, m) \) is here equivalent to the Orlicz’s natural distance.
\[ r(n, m) := \| g_n - g_m \| L(\Phi) \]

and is equivalent in turn to the source distance \( d(m, n), m, n = 1, 2, \dots, \infty \).

Our statement:

\[
\lim_{n \to \infty} \| \sup_{k \geq n} |g_k| \| L(\Phi) = 0,
\]

by virtue of proposition 4.1.

References

[1] Dudley R.M. *The sizes of compact of Hilbert space and continuity of Gaussian processes.* J. Functional Analysis, (1967), B. 1 pp. 290-330.

[2] Dmitrovsky V.A. (1981). *On the distributions of maximum and local properties of realizations of the pre-gaussian fields.* Theory of Probab. and Math. Stat.; (in Russian). Kiev, KSU, 25, 154-164.

[3] Fernique X. (1975). *Regularite des trajectoires des function aleatoires gaussiennes.* Ecole de Probabilite de Saint-Flour, IV, Lecture Notes in Mathematics; 480, 1, 96, Springer Verlag, Berlin.

[4] Fernique X, *Caracterisation de processus de trajectoires majores ou continues.* Seminaire de Probabilites XII. Lecture Notes in Math., 649, (1978), 691706, Springer, Berlin.

[5] Fernique X. *Regularite de fonctions aleatoires non gaussiennes.* Ecolee de Ete de Probabilit’es de Saint-Flour XI, Lecture Notes in Math., 976, (1983), 174, Springer, Berlin.

[6] Capone C., Fiorenza A., Krbec M. *On the Extrapolation Blowups in the \( L_p \) Scale.* Collectanea Mathematica, 48, 2, (1998), 71-88.

[7] Fiorenza A. *Duality and reflexivity in grand Lebesque spaces.* Collectanea Mathematica (electronic version), 51, 2, (2000), 131-148.

[8] Fiorenza A., and Karadzhov G.E. *Grand and small Lebesque spaces and their analogs.* Consiglio Nationale Delle Ricerche, Instituto per le Applicazioni del Calcoto Mauro Picine, Sezione di Napoli, Rapporto tecnico n. 272/03, (2005).

[9] Frolov A.S., Tchentzov N.N. *On the calculation by the Monte-Carlo method definite integrals depending on the parameters.* Journal of Computational Mathematics and Mathematical Physics, (1962), V. 2, Issue 4, p. 714-718 (in Russian).
[10] Grigorjeva M.L., Ostrovsky E.I. Calculation of Integrals on discontinuous Functions by means of depending trials method. Journal of Computational Mathematics and Mathematical Physics, (1996), V. 36, Issue 12, p. 28-39 (in Russian).

[11] Iwaniec T., and Sbordone C. On the integrability of the Jacobian under minimal hypotheses. Arch. Rat. Mech. Anal., 119, (1992), 129143.

[12] Iwaniec T., P. Koskela P., and Onninen J. Mapping of finite distortion: Monotonicity and Continuity. Invent. Math. 144 (2001), 507-531.

[13] Kolmogorov A.N. About analitical methods in the probability theory. Surveys of Soviet Math., (1938), B.17, issue 5, 5-41.

[14] Kozachenko Yu. V., Ostrovsky E.I. (1985). The Banach Spaces of random Variables of subgaussian type. Theory of Probab. and Math. Stat.; (in Russian). Kiev, KSU, 32, 43-57.

[15] Kurbanmuradov O., Sabelfeld K. (2007). Exponential bounds for the probability deviation of sums of random fields. Preprint. Weierstraß - Institut für Angewandte Analysis und Stochastik (WIAS), ISSN 0946-8633, p. 1-16.

[16] Ledoux M., Talagrand M. (1991) Probability in Banach Spaces. Springer, Berlin, MR 1102015.

[17] Liflyand E., Ostrovsky E., SirotA L. Structural Properties of Bilateral Grand Lebesgue Spaces. Turk. Journal of Math., 34, (2010), 207-219.

[18] Ostrovsky E. Support of Borelian Measures in separable Banach Spaces. arXiv:0808.03248v1 [math.FA] 24 Aug 2008.

[19] Ostrovsky E., Rogover E. Exact exponential Bounds for the random field Maximum Distribution via the majorizing Measures (Generic Chaining). arXiv:0802.0349v1 [math.PR] 4 Feb 2008.

[20] Ostrovsky E.I. (1999). Exponential estimations for random Fields and its applications, (in Russian). Moscow-Obninsk, OINPE.

[21] Ostrovsky E.I. (1994.) Exponential Bounds in the Law of Iterated Logarithm in Banach Spaces. Math. Notes, 56, 5, p. 98-107.

[22] Ostrovsky E., SirotA L. Fourier Transforms in exponential rearrangement invariant Spaces. arXiv:040639v1 [math.FA], 20 Jun 2004.

[23] Ostrovsky E., Rogover E. Maximal Inequalities in bilateral Grand Lebesgue Spaces over unbounded Measure. arXiv:0808v1 [math.FA], 24 Aug 2008.

[24] Ostrovsky E. and SirotA L. Moment Banach spaces: theory and applications. HIAT Journal of Science and Engineering, C, Volume 4, Issues 1-2, pp. 233-262, (2007).
[25] Ostrovsky E. and Sirota L. Moment and Tail Inequalities for polynomial Martingales. The case of heavy tails.
arXiv: 1112.2768v1 [math.PR] 13 Dez 2011.

[26] Ostrovsky E. and Sirota L. A counterexample to a hypothesis of light tail of maximum distribution for continuous random processes with light finite-dimensional tails.
arXiv:1208.6281v1 [math.PR] 30 Aug 2012

[27] Ostrovsky E. and Sirota L. Can the tail for maximum of continuous random field be significantly more heavy than maximum of tails?
arXiv:1508.05646v1 [math.PR] 23 Aug 2015

[28] Pizier G. Condition d’entropic assupant la continuite de certains processus et applications a l’analyse harmonique. Seminaire d’analyse fonctionnelle. (1980), Exp.13, p. 23-34.

[29] Rao M.M., Ren Z.D. Theory of Orlicz Spaces. Marcel Dekker Inc., 1991. New York, Basel, Hong Kong.

[30] Rao M.M., Ren Z.D. Applications of Orlicz Spaces. Marcel Dekker Inc., 2002. New York, Basel, Hong Kong.

[31] Slutsky E.E. Some propositions about the theory of random functions. Proceedings of Middle-Asia University, Tashkent, Ser. Math., (5), 31, (1949), 3-15. (in Russian).

[32] Talagrand M. (1996). Majorizing measure: The generic chaining. Ann. Probab. 24 1049-1103. MR1825156

[33] Talagrand M. (2001). Majorizing Measures without Measures. Ann. Probab. 29, 411-417. MR1825156

[34] Talagrand M. (2005). The Generic Chaining. Upper and Lower Bounds of Stochastic Processes. Springer, Berlin. MR2133757.

[35] Talagrand M.(1990). Sample boundedness of stochastic processes under increment conditions. Ann. Probab., 18, 1-49.