An alternative to the Simon tensor

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The Simon tensor gives rise to a local characterization of the Kerr-NUT family in the stationary class of vacuum spacetimes. We find that a symmetric and traceless tensor in the quotient space of the stationary Killing trajectory offers a useful alternative to the Simon tensor. Our tensor is distinct from the spatial dual of the Simon tensor and illustrates the geometric property of the three dimensional quotient space more manifest. The reconstruction procedure of the metric for which the generalized Simon tensor vanishes is spelled out in detail. We give a four dimensional description of this tensor in terms of the Coulomb part of the imaginary selfdual Weyl tensor, which corresponds to the generalization of the three-index tensor defined by Mars. This allows us to establish a new and simple criterion for the Kerr-NUT family: the gradient of the Ernst potential becomes the non-null eigenvector of the Coulomb part of the imaginary selfdual Weyl tensor. We also discuss the SU(1,2) covariant extension of the obstruction tensor into the Einstein-Maxwell system as an intrinsic characterization of the Kerr-Newman-NUT family.

I. INTRODUCTION

During the last couple of years, we have witnessed a considerable advance in our ability to access black holes in our universe. With the advent of gravitational wave astronomy [1] and the direct detection of black hole shadow with unprecedented precision [2], the Kerr solution [3]—describing a stationary and axisymmetric rotating black hole in vacuum spacetimes—has been a rekindled subject of intensive research. First and foremost, central to the analysis of black holes in our universe is the uniqueness theorem [4–6], which allows us to center exclusively on the Kerr solution, provided that the spacetime eventually settles down to an equilibrium final state. On top of this astrophysical significance, the Kerr solution provides an interesting and valuable arena for the (pseudo-)Riemannian geometry. This topic is considerably vast and has been tackled by numerous authors.

The first noticeable earmark of the Kerr metric is that there exist two distinct shear-free null geodesic congruences. Due to the theorem provided by Goldberg and Sachs [7], these optical aspects are closely tied to the algebraic property of the Weyl tensor. Working in the Newman-Penrose formalism [8], one finds a preferred frame for which Einstein’s equations are simplified, allowing one to obtain a number of algebraically special solutions of physical significance in a closed form [9]. This property is the key for the original discovery of the Kerr solution [3].

Another distinctive feature of the Kerr solution is that it admits a nondegenerate Killing-Yano tensor [10]. We refer the readers to [11] for a recent comprehensive review. The Killing-Yano tensor is an anti-symmetric generalization of the Killing vector and is viewed as a “square-root” of the Killing tensor, the latter of which enables one to get the constant of geodesic motion [12, 13]. By dint of the integrability conditions, the existence of the Killing-Yano tensor demands the algebraically special nature of curvature. Additional intriguing properties emerge when Wick-rotated to the Riemannian signature, for which the Euclidean Kerr solution admits an integrable complex structure [14] and a conformal Kähler structure [15]. These properties are combined with the Killing-Yano tensor to generate the enhanced Euclidean supersymmetry [15, 16] when the Weyl tensor is self-dual.

The present paper specializes to yet another characterization of the Kerr solution provided by Simon [17]. The three-index Simon tensor $S_{abc}$ is defined on the manifold of trajectory for the stationary Killing field and vanishes for the Kerr(-NUT) solution. Its construction is based on the fact that it reduces in the static case to the Cotton tensor, which measures the obstruction to the conformal flatness in three space and thereby describes the derivation from the Schwarzschild solution in the vacuum case [18, 21]. In the paper [17], it has been advocated that the local portion of asymptotically flat vacuum spacetime admitting the vanishing Simon tensor is isometrically embedded into the Kerr spacetime. Subsequently, Mars has given the elegant spacetime description of the Simon tensor in terms of the imaginary self-dual complexified Weyl tensor and its eigen-twoform [22, 23]. Some pertinent topics regarding

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these tensors have been studied in \[24, 27\].

In this paper, we would like to revisit the issue of local characterization of the Kerr solution from the perspective of Simon \[17\]. As a matter of first priority, the geometric interpretation of the Simon tensor has remained puzzling. To the best of our knowledge, the utmost available expertise about the Simon tensor is merely a complex generalization of the Cotton tensor. It therefore appears imperative to appreciate at a deeper level the geometric origin of the Simon tensor. Our second impetus emerges from the static case. In our recent paper \[28\], we have found a novel tensor field \(H_{ab}\), which deserves a global characterization of the Schwarzschild metric (see (2.28) for definition). Accordingly, this tensor \(H_{ab}\) supersedes the Cotton tensor in the static vacuum spacetime. The tensor \(H_{ab}\) is more user-friendly than the three-index Cotton tensor and has been exploited to derive a noteworthy equation of divergence type, which is of essential help in proving the uniqueness theorem of static black holes. It is therefore natural to seek a similar simplification in the stationary case. This would yield insights into the geometric structure of the stationary quotient space and would be likely to alleviate a lot of anguish in deriving nontrivial equations of divergence type.

This paper is intended to provide a useful alternative to the Simon tensor. In common with the static counterpart, our proposed tensorial field \(N_{ab}\) is symmetric and tracefree, but is complex (see (3.3) for definition). It turns out that the Simon tensor \(S_{abc}\) is completely represented in terms of our tensor \(N_{ab}\). The vanishing of the tensor \(N_{ab}\) enforces a tight restriction to the geometry which is compatible with equations of motion. As demonstrated by Perjes \[29\] (see also \[30\]), some degenerate cases of \(S_{abc} = 0\) give rise to solutions whose explicit metric forms are indeterminate. To the contrary, the metrics with \(N_{ab} = 0\) are constrained more severely including the degenerate cases. Likewise, one must fix the integration constant arising from \(S_{abc} = 0\) by boundary conditions to exclude the “topological version” of the Kerr-NUT solution, whereas this extra task is unnecessary in our case. It therefore seems fair to say that our tensor field \(N_{ab}\) deserves an appropriate local characterization of the Kerr-NUT family. Another applicability of our tensor is that it can be spacetime covariant, \textit{mutatis mutandis}, in a simple way. We uncover that for the Kerr-NUT solution the derivative of the Ernst potential becomes a non-null eigenvector of the Coulomb part of the imaginary selfdual Weyl tensor. Furthermore, the extension of \(N_{ab}\) into the electrovacuum case is fairly straightforward without any drastic jump in logic, while the electrovacuum Simon tensor in \[25\] has been deduced from the Mars tensor. See also \[31\] for the detailed supplementary conditions to be imposed for the spacetime characterization of the Kerr-Newman solution.

The remainder of the current paper is constructed as follows. In the ensuing section, we review the argument of Simon for the stationary vacuum spacetimes and discuss a static prototype for generalizing the Simon tensor. Our proposed alternative to the Simon tensor is given in section III. A systematic classification for the local metrics admitting \(N_{ab} = 0\) is implemented. We also give a spacetime description of this tensor. Section IV discusses the extension to the Einstein-Maxwell system. Our conclusion is summed up in section V.

We use Greek letters \(\mu, \nu, \ldots\) for spacetime indices and Latin letters \(a, b, \ldots\) for indices on the quotient space of the stationary Killing field. We adopt \(c = G = 1\) units throughout the paper.

II. PRELIMINARY

A. Vacuum spacetime with a stationary Killing vector

Let us consider the vacuum solution to Einstein’s field equations \(R_{\mu\nu} = 0\) endowed with a stationary Killing vector field \(\xi^\mu\). In an adapted coordinate system \(x^\mu = (t, x^a)\) with \(\xi = \partial/\partial t\), the metric can be written locally as

\[
ds^2 = -f(dt + \chi_a dx^a)^2 + f^{-1} h_{ab} dx^a dx^b,\]

where \(f = -g_{\mu\nu}\xi^\mu\xi^\nu\) and \(\chi_a\) describe respectively the norm and the rotation of the Killing vector. \(h_{ab}\) stands for the metric on the quotient space of the stationary Killing field—which we shall refer to as the base space \(\mathcal{B}\)—and the factor \(f^{-1}\) has been inserted for convenience. Here and in what follows, we raise and lower the Latin indices \(a, b, \ldots\) by \(h_{ab}\) and its inverse \(h^{ab}\). Every metric component in (2.1) is independent of \(t\). In this paper, we are primarily concerned with a strictly stationary region \(f > 0\) unless otherwise stated.

Since \(\chi_a\) describes a massless vector on the three-dimensional base space in the spirit of Kaluza-Klein reduction, it can be transformed into a three-dimensional scalar \(\psi\) via

\[
\omega_\mu \equiv \epsilon_{\mu\nu\rho\sigma} \xi^\nu \nabla_\rho \xi^\sigma = \nabla_\mu \psi,
\]

by virtue of the identity (see e.g., \[32\])

\[
\nabla_\mu \omega_\nu = -\epsilon_{\mu\nu\rho\sigma} \xi^\rho R^\tau_{\sigma} \xi^\tau.
\]
The system is now reduced to the three dimensional gravity coupled to the nonlinear sigma model as

\[ \mathcal{L}_3 = \sqrt{h} \left( R - h^{ab} G_{AB}(X) \partial_a X^A \partial_b X^B \right), \tag{2.4} \]

where \( R \) is a scalar curvature built out of \( h_{ab} \) and \( X^A = (f, \psi) \). \( G_{AB} \) is a metric of the target space \( \text{SU}(1,1)/\text{U}(1) \) represented by \( \mathcal{E} \equiv f - i\psi \) denotes the Ernst potential and the bar stands for the complex conjugation. In terms of the stereographic coordinate

\[ w \equiv \frac{1 - \mathcal{E}}{1 + \mathcal{E}}, \tag{2.6} \]

the target space \( \mathcal{M} \) boils down to

\[ ds_T^2 = \frac{2dw d\bar{w}}{\Theta^2}, \tag{2.7} \]

where

\[ \Theta \equiv 1 - w\bar{w}. \tag{2.8} \]

The field equations are cast into

\[ E_{ab} \equiv R_{ab} - 2\Theta^{-2} D(a w D_b) \bar{w} = 0, \tag{2.9a} \]
\[ E^{(S)} \equiv D^a D_a w + 2\Theta^{-1} \bar{w} D^a w D_a w = 0, \tag{2.9b} \]

where \( D_a \) is the derivative operator compatible with \( h_{ab} \). One can bring some field equations into the divergence form

\[ D^a (\Theta^{-1} D_a w - 2i\Theta^{-2} A_a w) = 0, \tag{2.10a} \]
\[ D^a (\Theta^{-2} A_a) = 0, \tag{2.10b} \]

where

\[ A_a \equiv \text{Im}(w D_a \bar{w}). \tag{2.11} \]

The system \( \mathcal{M} \) is invariant under the Möbius transformation

\[ w \rightarrow w' = \frac{\alpha w + \beta}{\beta w + \alpha}, \tag{2.12} \]

where \( \alpha \) and \( \beta \) are complex constants subjected to

\[ |\alpha|^2 - |\beta|^2 = 1. \tag{2.13} \]

This \( \text{SU}(1,1) \simeq \text{SL}(2,\mathbb{R}) \) isometry group of the target space \( \mathcal{M} \) is exploited to generate a new solution from an old one.

Since the variable \( w \) transforms nonlinearly under \( \text{SU}(1,1) \), it is more propitious to work with the homogeneous coordinates \( Z^A (A = 0, 1) \) defined by

\[ w = \frac{Z^1}{Z^0}. \tag{2.14} \]

The coordinate \( w \) is regarded on the space of equivalence classes \( Z^A \sim cZ^A \ (c \neq 0) \in \mathbb{C} \) for the nonzero complex vectors \( Z^A \). This means that one can define a trivial fibre bundle \( \pi : E = B \times \mathbb{C}P^2 \rightarrow B \) with a structure group \( \text{SU}(1,1) \).
It follows that the global SU(1, 1) transformation acts linearly on $Z^A$ as
\[
Z^A \rightarrow \Lambda^A_B Z^B, \quad \Lambda = \left( \begin{array}{cc} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{array} \right),
\]  
(2.15)
where constants $\alpha$ and $\beta$ obey (2.18) for $\Lambda^A_B$ to be an element of SU(1, 1) as
\[
\Lambda^i \eta_A = \eta, \quad \eta_{AB} = \text{diag}(1, -1).
\]  
(2.16)
Since one of $Z^A$ is redundant, one can always choose $Z^A$ to satisfy
\[
\langle Z, \bar{Z} \rangle D^a D_a Z^A = 2 Z_A D^a Z^B D_a Z^A.
\]  
(2.17)
where $\bar{Z}_A = \eta_{AB} \bar{Z}^B$ and $\langle Z, \bar{Z} \rangle = \eta_{AB} Z^A \bar{Z}^B$ is an SU(1, 1) invariant norm of the complex vector $Z^A$. The three dimensional covariant derivative acts on $Z^A$ trivially, i.e., $D_a Z^A = \partial_a Z^A$.

### B. Simon tensor

As a complex generalization of the Cotton tensor which describes the departure from the Schwarzschild metric in the static case, Simon has introduced the tensor field $S_{abc}$ as an obstruction to the Kerr solution. In terms of $w$, the Simon tensor reads \[17\]
\[
S_{abc} = 4 \Theta^{-2} (D_a D_b D_c w - h_{a[b} u_{c]}) ,
\]  
(2.18)
where
\[
u_a = D^b D_b D_a w ,
\]  
(2.19)
satisfying $S_{abc} = S_{a[bc]}$ and $S^a_{ab} = 0$. Simon has demonstrated that the Simon tensor vanishes in the asymptotically flat vacuum spacetime if and only if the spacetime is locally isometric to the Kerr solution. It is a simple exercise to verify 'if' part of this statement. In the Boyer-Lindquist coordinates, the Kerr metric reads
\[
ds^2 = - \frac{\Delta(r)}{\Sigma^2} (dt - a \sin^2 \theta d\phi)^2 + \frac{\sin^2 \theta}{\Sigma^2} \left( (r^2 + a^2) d\theta - ad\psi \right)^2 + \frac{\Sigma^2}{\Delta(r)} dr^2 + \Sigma^2 d\theta^2 ,
\]  
(2.20)
where
\[
\Delta(r) = r^2 - 2mr + a^2, \quad \Sigma^2 = r^2 + a^2 \cos^2 \theta .
\]  
(2.21)
The dimensional reduction along the time coordinate gives rise to the sigma model variables
\[
f = \frac{\Delta(r) - a^2 \sin^2 \theta}{\Sigma^2} , \quad \psi = - \frac{2m \cos \theta}{\Sigma^2} ,
\]  
(2.22)
i.e., $w = m/(r - m + i a \cos \theta)$ together with a base space
\[
h_{ab} dx^a dx^b = \left( (\Delta(r) - a^2 \sin^2 \theta) \left( \frac{dr^2}{\Delta(r)} + d\theta^2 \right) + \Delta(r) \sin^2 \theta d\phi^2 \right).
\]  
(2.23)
One can confirm that the Simon tensor \[2.18\] vanishes for the Kerr solution.

The Cotton tensor $C_{abc} \equiv 2 D_a \left[ R_{b[c} - \frac{4}{3} R h_{b[c}] \right]$ is computed by equations of motion \[2.19\] and expressed in terms of the Simon tensor as
\[
C_{[abc]} = \frac{1}{2} \left( S_{c[a|b]} + \bar{S}_{c[a\bar{b}]} \right) + \frac{4}{\Theta^3} D_{[a} w D_{b]} \bar{w} (\bar{w} D_{c} w - w D_{c} \bar{w})
\]  
\[- \frac{2}{\Theta^3} h_{c[a} (\bar{w} D_{b]} w D^d w - w D_{b]} \bar{w} D^d \bar{w}) D_d (w - \bar{w})
\]  
\[- \frac{1}{\Theta^2} (2 D_{c} D_{[a} (w - \bar{w}) D_{b]} (w - \bar{w}) + h_{c[a} D_{b]} D_{d} (w - \bar{w}) D^d (w - \bar{w}) \right) .
\]  
(2.24)
This makes it clear that the Simon tensor $S_{abc}$ reduces to the Cotton tensor $C_{cba}$ when $w$ is real.
C. Static exemplar

The original motivation by Simon to incorporate the tensor $S_{abc}$ (2.18) is to derive the divergence identities for the global characterization of the Kerr solution. Obviously, this hope is not attainable since the stationary metric (2.1) is singular on the ergosurface $f = 0$ and thus fails to cover the whole domain of outer communications. Nevertheless, it is illustrative here to observe where this idea is sprung from. What has inspired Simon is the structure of field equations in the static spacetime

$$\text{d}s^2 = -V^2 \text{d}t^2 + V^{-2} h_{ab} \text{d}x^a \text{d}x^b,$$  \hfill (2.25)

where (square of) $V$ is the norm of the static Killing vector $\partial/\partial t$. The vacuum Einstein’s equations are boiled down to

$$D^a D_a \log V = 0, \quad R_{ab} = \frac{2}{V^2} D_a V D_b V. \hfill (2.26)$$

The crux of the program trying to classify the static black holes in [18–21] is the Cotton tensor $C_{abc}$ for the three dimensional metric $h_{ab}$. With the aid of the field equations (2.26), the Cotton tensor takes the form

$$C_{abc} = \frac{2}{V^2} [2D_c D_{[a} V D_{b]} V + h_{c[|a} D_{b]} D^d V (D_d V) - V^{-1}(DV)^2 h_{c[a} D_{b]} V]. \hfill (2.27)$$

For the Schwarzschild solution, the Cotton tensor vanishes.

In our recent paper [28], we have demonstrated that there exists a meritorious tensor field which can be qualified as a global characterization of the Schwarzschild black hole among the static solution (2.25). The desired tensor is of the following form

$$H_{ab} \equiv D_a D_b V + \frac{2(1 + V^2)}{V(1 - V^2)} D_a V D_b V - \frac{1 + V^2}{V(1 - V^2)} (DV)^2 h_{ab}. \hfill (2.28)$$

The traceless property $H^a_a = 0$ follows from the equations of motion (2.26). The previous article [28] has given a new uniqueness proof of the Schwarzschild black hole among the static and asymptotically flat solutions to the vacuum Einstein’s equations. The first step of the proof consists of showing that the vanishing of $H_{ab}$ is tantamount to the Schwarzschild solution. This can be easily seen if we decompose the quantities on three dimensional base space by the level set $\Sigma_V = \{V = \text{const.}\}$. The induced metric on $\Sigma_V$ is given by $\gamma_{ab} = h_{ab} - n_a n_b$, where $n_a = \rho D_a V$ is the outward-pointing unit normal to $\Sigma_V$ and $\rho$ is the lapse function such that $\rho^2 = (DV)^2$. The extrinsic curvature of $\Sigma_V$ in $t = \{\text{const.}\}$ surface is $K_{ab} = \gamma_c^a D_c h_{ab}$, which is further decomposed into $K_{ab} = \sigma_{ab} + (1/2) K \gamma_{ab}$ where $\sigma_{ab} \gamma^{ab} = 0$ and $K = \gamma^{ac} K_{ac}$. In terms of these ingredients, we have

$$H_{ab} = \rho^{-2} \sigma_{ab} - 2 \rho^{-2} n_{(a} D_{b)} \rho + \frac{1}{2 \rho} \left( K - \frac{2(1 + V^2)}{\rho V(1 - V^2)} \right) (\gamma_{ab} - 2 n_a n_b), \hfill (2.29)$$

where $D_a$ is a derivative operator built out of $\gamma_{ab}$. To obtain the above, we have used a relation $\partial (\log \rho)/\partial V = \rho K - 1/V$ derivable from the first equation of (2.26). Since each term on the right hand side of (2.29) is tensorially independent, $H_{ab} = 0$ enforces three independent conditions

$$\sigma_{ab} = 0, \quad D_a \rho = 0, \quad K = \frac{2(1 + V^2)}{\rho V(1 - V^2)}. \hfill (2.30)$$

Upon integration, we immediately get the Schwarzschild solution. The deviation from the Schwarzschild metric is therefore encoded in $H_{ab}$.

The uniqueness proofs in [18–21] made essential use of the Cotton tensor as an obstruction to the Schwarzschild solution. This is consistent with the aforementioned assertion, since the Cotton tensor (2.27) is rephrased as

$$C_{abc} = \frac{2}{V^2} (2 h_{c[a} D_{b]} V + h_{c[|a} H_{b]} d) D^d V). \hfill (2.31)$$

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1 Note that $h_{ab}$ is conformally transformed by a factor $V^{-2}$, compared with notational conventions in [18–20, 28].

2 Note that the tensor $\tilde{C}_{ab} \equiv \frac{1}{2} e_{acd} C^{cd_b}$ dual to the Cotton tensor is also symmetric and tracefree $\tilde{C}_{ab} = C_{(ab)}$ and $\tilde{C}^a_a = 0$, but is distinct from $H_{ab}$. 
Evidently, \( H_{ab} = 0 \) implies \( C_{abc} = 0 \). The utility of \( H_{ab} \) lies in the easy handling compared to the three-index tensor \( C_{abc} \). This becomes prominent when one tries to derive the global divergence identities. The three parameter family of divergence identities constructed based upon \( H_{ab} \) in (2.31) surpasses the two parameter identities in [28], inasmuch as the proof based on \( H_{ab} \) can be readily generalized into nonvacuum case and into higher dimensions.

Last but by no means least, \( C_{abc} = 0 \) is not equivalent to \( H_{ab} = 0 \). A typical example is the topological extension of the Schwarzschild solution \( ds^2 = -f(r)dt^2 + dr^2/f(r) + r^2d\Omega_2 \), where \( f = k - 2m/r \) and \( d\Sigma_2 \) being the metric of Einstein space with curvature \( k \). The condition \( C_{abc} = 0 \) is satisfied for the constant time slice of this solution, but \( H_{ab} = 0 \) is not. Since the vanishing of \( H_{ab} \) requires \( k = 1 \) with \( d\Sigma_2^2 \) being a metric of a unit sphere, \( H_{ab} = 0 \) puts a stronger restriction on the three dimensional geometry. On the other hand, the condition \( C_{abc} = 0 \) puts only the first two restrictions in (2.30), which can be recognized by

\[
C_{abc}C^{abc} = \frac{8}{V^4} \left( \sigma_{ab}\sigma^{ab} + \frac{(D\rho)^2}{2\rho^2} \right). 
\tag{2.32}
\]

In the conventional approach pertaining to the Cotton tensor, the last condition \( K = 2(1 + V^2)/[\rho V(1 - V^2)] \) must be additionally imposed using the rest of Einstein’s equations and the resulting integration constant must be fixed by prescribing boundary conditions. By comparison to the argument in [13 [21], the uniqueness proof of [28] based upon \( H_{ab} \) thereby turns out “one in go.”

**III. GENERALIZATION OF THE SIMON TENSOR**

**A. Alternative to the Simon tensor**

In the last section, we have seen that the introduction of symmetric traceless tensor \( H_{ab} \) defined by (2.28) significantly streamlines the analysis of static system, in lieu of the three-index Cotton tensor \( C_{abc} \). A natural question to be asked is if a similar simplification occurs for the three-index complex Simon tensor (2.18). We are now going to discuss that the answer is affirmative. We put forward the following tracefree complex tensors as an adequate replacement of the Simon tensor:

\[
N^A_{ab} = D_a D_b Z^A - \frac{1}{(Z^A)(1 - (Z^A))} \bar{Z}_b D^c Z^B D_c Z^A h_{ab} + \frac{1 + 2\langle Z, \bar{Z} \rangle}{(Z^A)(1 - (Z^A))} \bar{Z}_b D(a Z^B D_b Z^A). \tag{3.1}
\]

By construction, these tensors satisfy \( N^A_{ab} = N^A_{(ab)} \) and \( N^A_{ab} h^{ab} = 0 \), and linearly transform under \( SU(1, 1) \) as a doublet. To be specific, we take a parametrization as

\[
Z^A = \begin{pmatrix} 1 \\ w \end{pmatrix}, \tag{3.2}
\]

for which \( N^0_{ab} \) vanishes trivially, whereas \( N^1_{ab} (= N_{ab}) \) is simplified to

\[
N_{ab} = D_a D_b w + \frac{(Dw)^2}{w(1 - |w|^2)} h_{ab} - \frac{3 - 2|w|^2}{w(1 - |w|^2)} D_a w D_b w. \tag{3.3}
\]

A simple calculation shows that \( N_{ab} = 0 \) is obeyed by the Kerr solution. One can also verify that \( N_{ab} = -4V(1 + V^2)^{-2} H_{ab} \) when \( w = \bar{w} \), i.e., \( N_{ab} \) recovers the obstruction for the Schwarzschild metric in the static limit. It is noteworthy that the Simon tensor (2.18) can be expressed in terms of this tensor as

\[
S_{abc} = \frac{2}{\Theta^2} \left( 2 N_{a[b} D_{c]} w + h_{a[b} N_{c]d} D^d w \right). \tag{3.4}
\]

This expression enjoys the same structure as the one in the static case (2.31). It turns out that the vanishing of \( N_{ab} \) leads to \( S_{abc} = 0 \). The Cotton tensor (2.21) is also expressible by \( N_{ab} \) more conveniently as

\[
C_{abc} = \frac{1}{\Theta^2} \left[ 2(N_{c[a} D_b \bar{w} + \bar{N}_{c[a} D_b w) + h_{c[a}(N_{b]d} D^d \bar{w} + \bar{N}_{b]d} D^d w) \right] + \frac{3}{|w|^2 \Theta^3} (\bar{w} D^d w - w D^d \bar{w}) h_{c[a}(D_{b]} \bar{w} D_d w - D_{b]} w D_d \bar{w}) \right] + \frac{6}{|w|^2 \Theta^3} D_{a[w} D_{b]} \bar{w} (\bar{w} D_c w - w D_c \bar{w}) . \tag{3.5}
\]
Several remarks are in order. Since $N_{ab} = 0$ gives rise to nontrivial second-order differential equations for $w$, the existence of the solution to these equations is far from obvious beforehand. To evince that neither inconsistency nor overdetermination arises, let us prospect the integrability conditions

\[
D_{[a}N_{b]c} = \frac{3 - 2|w|^2}{w(1 - |w|^2)}D_{[a}wN_{b]c} + \frac{2}{w(1 - |w|^2)}D^bwN_{d[a}h_{b]c} - D_{[a}wE_{b]c} + \frac{1}{2}E_{d[a}d_{b]c} - E_{[a}d_{b]c}D_dw,
\]

and

\[
D^bN_{ab} = -\frac{1 + 2|w|^2}{w(1 - |w|^2)}N_{ab}D_aD_bw + E_{ab}D^bw + D_aE^{(S)} - \frac{3 - 2|w|^2}{w(1 - |w|^2)}E^{(S)}D_aw,
\]

where $(E_{ab}, E^{(S)})$ have been given in (3.9). It follows that the condition $N_{ab} = 0$ does not conflict with Einstein’s equations of motion $E_{ab} = E^{(S)} = 0$.

By the reasoning identical to the static case, the conditions imposed by $S_{abc} = 0$ are more relaxed than $N_{ab} = 0$. The Kerr metric satisfies both, but its topological generalization fails to fulfill $N_{ab} = 0$. In this light, the condition $N_{ab} = 0$ gives rise to an unequivocal characterization of the Kerr solution.

It is also worthwhile to stress that we have fixed the gauge as $Z^A = (1, w)$ to derive the explicit expression of $N_{ab}$ in (3.3). We caution the reader that we are not allowed to perform a further SU(1, 1) transformation by merely replacing $w$ in $N_{ab}$ as (2.12), since $N_{ab}^2$ transforms as a fundamental representation of SU(1, 1). As a matter of fact, the explicit form of $N_{ab}$ nontrivially changes and $N_{ab}^2$ becomes no longer identically trivial under (2.12), for which both of the components $N_{ab}$ play the role of obstruction for the Kerr-NUT family. The only allowed transformation that leaves the form of (3.3) invariant is $Z^A = (1, w) \rightarrow (e^{i\theta}w, e^{-i\theta}w)$ with $\theta \in \mathbb{R}$, which corresponds to the maximal subgroup U(1) of SU(1, 1). On the other hand, the Simon tensor $S_{abc}$ is inert under SU(1, 1), since equation (3.3) can be recast into an SU(1, 1) singlet as

\[
S_{abc} = \frac{2}{(Z, Z)^2}\epsilon_{AB\epsilon CD}Z^A Z^C \left(2N_{[a[B}Z_{D]} + h_{[aB}N_{c]d}D^DZ^D\right),
\]

where $\epsilon_{AB}$ is an alternating tensor.

Bearing this gauge fixing in mind, we are now going to derive a number of equations of the divergence type by making use of $N_{ab}$. To this aim, we notice the following identities

\[
D^2\rho = -\rho^3(D_aD_bw)^2 + \frac{3}{\rho}(D\rho)^2 - \rho^3D^aw[D_a(D^bw) + R_{ab}D^bw]
\]

\[
= -\rho^3(D_aD_bw)^2 + \frac{3}{\rho}(D\rho)^2 + \frac{2\bar{w}^2}{\rho\Theta^2} - \frac{4\bar{w}}{\rho\Theta}D_a\rho D^aw,
\]

and

\[
(D_aD_bw)^2 = N_{ab}N^{ab} + \frac{2(3 - 2|w|^2)}{w\Theta}N_{ab}D^aD^bw + \frac{1}{\rho^4w^2\Theta^2}(6 - 8|w|^2 + 4|w|^4),
\]

where we have supposed

\[
\rho \equiv (D^aD_bw)^{-1/2}
\]

is well-defined. Squaring (3.4) yields

\[
S_{abc}S^{abc} = \frac{8}{\rho^4\Theta^4} \left(N_{ab}N^{ab} - \frac{3}{2\rho^2}N_aN^a\right),
\]

where

\[
N_a \equiv \frac{D_a\rho + 2D_aw}{w} = -\rho^2N_{ab}D^bw.
\]
Capitalizing on these relations, one obtains the following divergence equations\(^3\)

\[
D^a[\Theta^{-1}D_a k - 2i\Theta^{-2}k A_a] = \frac{1}{16} k^{-7} \Theta^3 S_{abc} S^{abc},
\]

\[
D^a[\Theta^{-1} (w D_a k - k D_a w)] = \frac{1}{16} w k^{-7} \Theta^3 S_{abc} S^{abc} + \frac{k w}{2 \Theta^2} N_a (\bar{w} D^a w - w D^a \bar{w}).
\]

where \(A_a\) has been defined by (2.11) and

\[
k \equiv (D^a w D_a w)^{1/4} = \rho^{-1/2},
\]

has been introduced to conform to the notation in [17]. In addition to those constructed in [17], we obtain the following more manifest argument of the metric reconstruction for \(S\), where \(A\) and \(c\) where

\[
\text{Case (I-i).}
\]

In this case, the variable \(f = 0\). The stationary metric form is not well adapted to the global boundary value problems. Another adversity is that the right hand side of these equations (3.14)–(3.17) remain complex. To the contrary, the corresponding expressions in the static case can be made positive-definite, so that these divergence equations are utilized for proving the uniqueness for global boundary value problems. It therefore seems no direct applicability of these equations in Lorentzian signature. Nevertheless, these equations would be of great help for the uniqueness proof of gravitational instanton solutions in Euclidean signature [36].

B. Taxonomy of solutions with \(N_{ab} = 0\)

This section provides the local form of the four dimensional stationary metric admitting \(N_{ab} = 0\). The original argument of the metric reconstruction for \(S_{abc} = 0\) is given by Perjes [29]. Since the condition \(N_{ab} = 0\) is stronger than \(S_{abc} = 0\), our proof allows one to obtain the explicit metric form and discloses the underlying geometric structure more manifest.

The classification divides into subcases according to (I) \(D_a w\) is not null and (II) \(D_a w\) is null. Case (I) is further categorized into (I-i) \(dw \wedge d\bar{w} \neq 0\) and (I-ii) \(dw \wedge d\bar{w} = 0\).

Case (I-i). In this case, the variable \(w\) and its complex conjugation can be exploited for the coordinates in the base space. One can then define the following real vector field

\[
U^a = \frac{i(1 - |w|^2)}{(w \bar{w})^3} c^{abc} D_b w D_c \bar{w},
\]

which satisfies \(U^a D_a w = 0\), i.e., \(U^a\) is orthogonal to \(D_a w\) and \(D_a \bar{w}\). From \(N_{ab} = 0\), it can be straightforwardly checked that

\[
D_{(a} U_{b)} = 0, \quad U_{[a} D_{b} U_{c]} = 0.
\]

Namely, \(U^a\) defines a hypersurface-orthogonal Killing vector of the base space. It is also advantageous to adopt the following symmetric tensor

\[
K_{ab} \equiv \frac{1 - w \bar{w}}{w^3 \bar{w}^3} \left( (D^c w)(D_c \bar{w}) h_{ab} - D_{(a} w D_{b)} \bar{w} \right).
\]

When \(N_{ab} = 0\), this tensor satisfies

\[
D_{(a} K_{bc)} = 0, \quad \mathcal{L}_U K_{ab} = 0.
\]

\(^3\) It should be noted that the second term on the right hand side of (3.15) is missing in eq. (22) of [17].
It therefore turns out that $K_{ab}$ defines the Killing tensor of the base space, which is invariant under the action of the Killing vector $U^a$. The Killing tensor (3.20) is irreducible in the sense that it is neither proportional to the metric nor expressed as a symmetric product of Killing vectors. This class of space is considerably restrained and its canonical metric falls into the Benenti-Francaviglia form [37]. Without adopting $N_{ab}$, it would have been impossible to identify this unexpected hidden symmetry. This represents one of the advantages of using $N_{ab}$ relative to $S_{abc}$.

Let us move on to introduce the local coordinate system admitting $N_{ab} = 0$. Since $U^a$ is a Killing vector, it is opportune to employ the coordinate $(w, \bar{w}, \varphi)$ in such a way that $U^a = \partial / \partial \varphi^a$. Specifically, the inverse metric $h^{ab}$ can be put into

$$h^{ab} \partial_a \partial_b = U^{-2} \partial_\varphi^2 + \rho^{-2} \partial_w^2 + \bar{\rho}^{-2} \partial_{\bar{w}}^2 + \frac{2}{\rho \bar{\rho}} \partial_w \partial_{\bar{w}},$$  \hspace{1cm} (3.22)

where $(U, \Omega)$ are undetermined real functions of $(w, \bar{w})$ and $h_{ww} = (Dw)^2 = \rho^{-2}$ comes from the definition (3.11). An inspection of (3.13) immediately yields

$$\rho = C_w^2,$$  \hspace{1cm} (3.23)

where $C$ is a complex constant. Computing the norm of (3.18), one finds

$$U(w, \bar{w}) = 1 - \frac{|w|^2}{|C|^2 |w|^2} \sqrt{\Omega^2 - 1}.$$  \hspace{1cm} (3.24)

By $N_{ab}D^b \bar{w} = 0$ and $N_{ab}U^b = 0$, $\Omega$ is determined to be

$$\Omega(w, \bar{w}) = \frac{1}{2|w|^2(1 - |w|^2)} \left( \frac{\bar{C}}{C} w^2 + \frac{C}{\bar{C}} \bar{w}^2 + 2C_0 w^2 \bar{w}^2 \right),$$  \hspace{1cm} (3.25)

where $C_0$ is a real constant.

We have exhausted all the constraints coming from $N_{ab} = 0$. One verifies that all the Einstein’s equations of motion (2.9) are satisfied. To cast the metric into a more recognizable form, let us change $(w, \bar{w})$ to real coordinates $(q, p)$ by $w = C/(q + ip)$. Denoting

$$C = m - in, \quad (m, n) \in \mathbb{R},$$  \hspace{1cm} (3.26)

let us further shift $q \rightarrow q - m$ and $p \rightarrow p + n$, i.e.,

$$w = \frac{m - in}{(q - m) + i(p + n)}.$$  \hspace{1cm} (3.27)

Introducing a new real constant $a$ by

$$C_0 = -1 + \frac{2a^2}{m^2 + n^2},$$  \hspace{1cm} (3.28)

we find

$$h_{ab}dx^a dx^b = (Q(q) - P(p)) \left( \frac{dq^2}{Q(q)} + \frac{dp^2}{P(p)} \right) + Q(q)P(p)d\sigma^2,$$  \hspace{1cm} (3.29)

where

$$Q(q) \equiv q^2 - 2mq + a^2 - n^2, \quad P(p) \equiv a^2 - (n + p)^2,$$  \hspace{1cm} (3.30)

and

$$\varphi = \frac{1}{2}(m^2 + n^2)^2 \sigma.$$  \hspace{1cm} (3.31)

The sigma model variables are

$$f = \frac{Q(q) - P(p)}{q^2 + p^2}, \quad \psi = -\frac{2(mp + nq)}{q^2 + p^2}.$$  \hspace{1cm} (3.32)
By means of these coordinates, the Killing tensor \( K_{ab}dx^a dx^b \) is
\[
K_{ab}dx^a dx^b = \frac{1}{(m^2 + n^2)^2} \left[ (Q(q) - P(p)) \left( \frac{Q(q)}{P(q)} dq^2 + \frac{P(p)}{Q(p)} dp^2 \right) + (Q(q) + P(p))Q(q)P(p)d\sigma^2 \right].
\]  
(3.33)

This Killing tensor is responsible for the separability of the geodesic motion \( k^b D_b k^a = 0 \) for the three dimensional space \( \mathbb{S}^2 \).

Letting \( \tau \) be a stationary coordinate, the four dimensional metric then reads
\[
ds^2 = -\frac{Q(q)}{q^2 + p^2}(d\tau - p^2 d\sigma)^2 + \frac{P(p)}{q^2 + p^2}(d\tau + q^2 d\sigma)^2 + (q^2 + p^2) \left( \frac{dq^2}{Q(q)} + \frac{dp^2}{P(p)} \right).
\]  
(3.34)

This is nothing but the Carter form \([12]\) of the Kerr-NUT solution, where \( m, a, n \) denote respectively the mass, the specific angular momentum and the NUT charge. The physical interpretation of the NUT charge has been given from various perspectives in \([35, 41]\). The Kerr metric \([2.20]\) is recovered by
\[
q = r, \quad p = a \cos \theta, \quad n = 0, \quad \tau = t - a \phi, \quad \sigma = -\frac{\phi}{a}.
\]  
(3.35)

It appears that the three dimensional Killing tensor \([3.33]\) presented above is not related to the eminent four dimensional Killing tensor \([12, 13]\).

**Case (I-ii).** Letting \( w = w_R + i w_I \) \((w_R, w_I \in \mathbb{R})\), the condition \( dw \wedge d\bar{w} = 0 \) implies that \( w_I \) is a function of \( w_R \). Equivalently, \( f = f(w) \) expressed in terms of the original variable \( w = [1 - (f - i \dot{w})]/[1 + (f - i \dot{w})] \). This class of solutions is referred to as a Papapetrou’s class \([42]\) (see also section 20.3 in \([43]\)). Since \( \psi = 0 \) is exhausted by the Schwarzschild solution, we postulate \( d\psi \neq 0 \) in the sequel. Inserting \( f = f(w) \) into equations of motion \([2.9b]\) of \( w \), we thus get
\[
\psi = C_2 + \sqrt{C_1 - f^2}, \quad D^a D_a \left( \text{arctanh} \frac{\sqrt{C_1 - f^2}}{\sqrt{C_1}} \right) = 0,
\]  
(3.36)

where \( C_1 \) and \( C_2 \) are real constants. Equation \([3.35]\) implies that the base space \( h_{ab} \) is conformally flat. Note that this conclusion is not drawn immediately from \([2.24]\). The conformal flatness permits one to foliate the base space by the level set \( f = \text{const.} \) as \([19, 21]\)
\[
h_{ab}dx^a dx^b = F_1^2(f) df^2 + F_2^2(f) \gamma_{ij} (y^k) dy^i dy^j,
\]  
(3.37)

where \( \gamma_{ij} \) is a two-dimensional metric on the \( f = \text{const.} \) surface and can be taken to be a conformally flat form \( \gamma_{ij} (y^k) dy^i dy^j = e^{2\Phi} dz d\bar{z} \). The second equation in \([3.36]\) is integrated to give
\[
F_2(f) = \left( C_3 f (C_1 - f^2)^{1/2} F_1(f) \right)^{1/2},
\]  
(3.38)

where \( C_3 \) is a constant. Substitution of this into Einstein’s equations \( E_{ab} D^a f D^b f = 0 \) yields
\[
F_1(f) = C_5 \frac{f + (\sqrt{C_1 - f} C_1 C_2^2 + \sqrt{C_1}(1 + 2 C_1 \sqrt{C_1 - f^2})}{\sqrt{C_1 - f^2} (\sqrt{C_1 + f} - (\sqrt{C_1 - f}) C_2 C_1^2)^2}.
\]  
(3.39)

where \( C_4 \) and \( C_5 \) are integration constants. Plugging this into \( N_{ab} = 0 \), one finds
\[
C_2 = -\sqrt{C_1 - 1}, \quad C_4 = \sqrt{C_1 + 1} \frac{1}{\sqrt{C_1} (C_1 - 1)}.
\]  
(3.40)

The rest of Einstein’s equations is satisfied, provided that the conformal factor in \( \gamma_{ij} (y^k) dy^i dy^j = e^{2\Phi} dz d\bar{z} \) satisfies the Liouville equation
\[
\partial_z \partial_{\bar{z}} \Phi = -\frac{1}{4} k e^{2\Phi}, \quad k = \frac{C_1 C_3}{(\sqrt{C_1 - 1}) C_5}.
\]  
(3.41)
The local solution to this equation can be chosen to be
\[ e^\Phi = \frac{1}{1 + kz\bar{z}/4}, \] (3.42)
for which \(d\Sigma_k^2 \equiv e^{2\Phi}dz\bar{z}\) is a space of constant curvature \(k\), which can be normalized to be \(k = \pm 1, 0\). Define a new variable \(r\) by \(r = \int F_1(f)df\), i.e.,
\[ f = \frac{r^2 - n^2 - 2mr}{r^2 + n^2}, \quad \psi = \frac{2n(r - m)}{r^2 + n^2}, \] (3.43)
where we have renamed the constants \(C_k = 2kn\) and \(C_1 = (m^2 + n^2)/n^2\). In the region \(r^2 - 2mr - n^2 > 0\), \(F_2\) becomes real for \(k = 1\). Changing to \(z = 2\tan(\theta/2)e^{i\phi}\), the base space therefore admits \(SO(3) \simeq SU(2)\) symmetry
\[ h_{ab}dx^a dx^b = dr^2 + (r^2 + n^2)f(r)(d\theta^2 + \sin^2 \theta d\phi^2). \] (3.44)
The rotation of the Killing vector reads \(\chi = 2n\cos \theta d\phi\). It follows that the four dimensional metric gives rise to the Taub-NUT solution [44, 45].

**Case (II).** Also in this case, the vector \(U^a\) defined in (3.18) is well-defined. For \(D^a w D_a w = 0\), this vector satisfies \(D_a U_b = 0\), i.e., \(U_a\) is a Killing vector that is covariantly constant. One can then cast the metric into
\[ h_{ab}dx^a dx^b = 2f_1(w, \bar{w})dw\bar{d}w + dv^2, \] (3.45)
where \(U^a = (\partial/\partial v)^a\) and \(f_1\) is a real function. The condition \(N_{ab} = 0\) is immediately integrated as
\[ f_1 = \frac{D_1^2(1 - |w|^2)}{w^3 w^\dagger}, \] (3.46)
where \(D_1\) is a real constant. Equations of motion (3.40) demand no more restrictions. Introducing new coordinates \((r, \theta)\) by \(w^{-1} = re^{i\theta}\), the base space simplifies to
\[ h_{ab}dx^a dx^b = 2D_1^2(r^2 - 1)(dr^2 + r^2 d\theta^2) + dv^2. \] (3.47)
The twist one-form \(\chi = \chi_a dx^a\) of the Killing vector can be chosen to
\[ \chi = \frac{2(1 + r \cos \theta)}{r^2 - 1} dv. \] (3.48)
The vacuum four dimensional metric (2.1) takes a more suggestive form by \(t = u + v\) as
\[
\begin{align*}
    ds^2 &= -\frac{r^2 - 1}{r^2 + 1 + 2r \cos \theta} du^2 - 2dvdu + 2D_1^2(r^2 + 1 + 2r \cos \theta)(dr^2 + r^2 d\theta^2) \\
    &= -\left(1 - \sqrt{D_1}\frac{\sqrt{D_1}}{\sqrt{2\xi}} - \sqrt{\sqrt{2\xi}}\right) du^2 - 2dvdu + 2d\zeta d\bar{\zeta},
\end{align*}
\] (3.49)
where \(\zeta = D_1(1 + re^{i\theta})^2/2\). When promoted into four dimensions, the vector \(\partial/\partial v\) is null and covariantly constant. This class of solutions is referred to as a pp-wave [43].

**C. Four dimensional description**

During the course of our analyses so far, we have limited the discussion for the Simon tensor and its generalization within the framework of the quotient space associated with the stationary Killing field. This formulation inevitably encounters some obstacles as described at the end of section III A. To partially overcome this shortcoming, Mars has provided a four dimensional counterpart of the Simon tensor (2.18) in [22].

The Mars tensor \(M_{\mu\nu\rho}\) is defined as
\[ M_{\mu\nu\rho} = 4E^+_{\mu[\nu} \sigma_{\rho]} - \frac{1}{2} h_{\mu[\nu} \xi^C \epsilon_{\rho]} (d\xi^+)^{\sigma\tau}, \] (3.50)
where

\[ E^+_{\mu
u} = C^+_{\mu\rho\sigma} \xi^\rho \xi^\sigma, \quad \sigma_\mu \equiv \xi^\nu (d\xi^+)_{\nu\mu}, \]  

(3.51)

and

\[ C^+_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + \frac{i}{2} \epsilon_{\mu\nu\tau} C^\tau_{\rho\sigma}, \]  

(3.52a)

\[ (d\xi^+)_{\mu\nu} = 2\nabla_{[\mu} \xi_{\nu]} + i \epsilon_{\mu\nu\rho\sigma} \nabla^\rho \xi^\sigma, \]  

(3.52b)

\[ h_{\mu\nu} = f g_{\mu\nu} + \xi_\mu \xi_\nu. \]  

(3.52c)

Imposing the vacuum Einstein’s equations, the vector \( \sigma_\mu \) can be represented by the Ernst potential as \( \sigma_\mu = \nabla_\mu (f - i \psi) = \nabla_\mu \xi. \) Owing to \( h_{\mu\nu} \xi^{\nu} = 0, \) \( h_{\mu\nu} \) can be viewed as a projection operator orthogonal to \( \xi^\mu \) and \( h_{\mu\nu} \) is identified as an induced metric on the base space. The Mars tensor satisfies \( M_{\mu\rho\nu} = M_{\nu[\mu\rho]}, \) \( M^\mu_{\mu\nu} = 0 \) and \( M_{\mu\nu\rho\sigma} = 0 = M_{\mu\nu\rho} \xi^\rho, \) so that the only nonvanishing components are obtained by projecting onto the base space. When \( R_{\mu\nu} = 0, \) the relation \( 23 \) implies

\[ \nabla_\mu (d\xi^+)_{\nu\rho} = 2C^+_{\mu\rho\sigma} \xi^\sigma. \]  

(3.53)

Together with \( (d\xi^+)_{\mu\nu}(d\xi^+)_{\mu\nu} = -(4/f)\sigma_\mu \sigma^\mu, \) the Mars tensor reduces to

\[ M_{\mu\rho\nu} = 2 \left( \nabla_\mu \xi_{[\nu} - \nabla_\nu \xi^\sigma (d\xi^+)_{\sigma[\nu]} \sigma^\rho \right) + \frac{1}{2} h_{\mu\nu} \nabla_{[\rho}(f^{-1} \sigma_{\sigma^\rho}). \]  

(3.54)

To simplify the second term in the above equation, one resorts to

\[ \nabla_\mu \xi_{[\nu} = -\frac{1}{2} f^{-1} \epsilon_{\mu\rho\sigma\omega} \xi^\rho \omega^\sigma_\nu - f^{-1} \xi_{[\mu} \nabla_\nu f, \quad (d\xi^+)_{\mu\nu} = -\frac{1}{f} (2 \xi_{[\mu} \sigma_{\nu]} + i \epsilon_{\mu\nu\rho\sigma} \xi^\rho \sigma^\sigma_\nu), \]  

(3.55)

where \( \omega_\mu \) is defined by \( 22, \) leading to

\[ \nabla_\mu \xi^\sigma (d\xi^+)_{\sigma\nu} = \frac{1}{2} f \nabla_\mu f \sigma_\nu - \frac{1}{2 f^2} \epsilon_{\mu\rho\sigma\omega} \xi^\rho \omega^\sigma_\nu \nabla_\nu f + \frac{i}{2 f} (\omega_{\nu} \sigma_\rho - \sigma_\rho f \nabla_\mu \omega_{\nu} \sigma^\rho - \frac{1}{2} f \xi_{(\mu} \sigma_{\rho\nu)} \nabla^\rho f \xi^\omega \omega^\sigma_\nu. \]  

(3.56)

By projecting onto the base space guided by the method in \( 33, \) some calculations show that

\[ M_{abc} = 2 D_a \sigma_b \sigma_c + \sigma^d h_{ab} \left( D_c \sigma_d - \frac{1}{f} \sigma^d \sigma^c \right) = \frac{2(1 - |w|^2)^2}{(1 + w)^2} S_{abc}. \]  

(3.57)

This demonstrates the identification of these tensors up to the multiplicative factor. As contrasted to the fact that the Simon tensor \( 2.1b \) is defined only outside the ergosphere, the Mars tensor \( 3.50 \) does not suffer from this difficulty. Furthermore, the Mars tensor is defined irrespective of the existence of Ernst potentials.

Inspecting the form of the Mars tensor \( 3.50, \) it is tempting to inquire the spacetime picture of the tensor \( 4.30. \) For this purpose, the following tensor is a well-suited candidate of this kind

\[ N_{\mu\nu} = E^+_{\mu\nu} + \frac{1}{2 \sigma^2} \xi^{\rho\sigma} (f^{-1} h_{\mu\nu} - \frac{3 \sigma_\rho \sigma^\sigma}{\sigma^2}), \]  

(3.58)

where \( \sigma^2 \equiv \sigma_\mu \sigma^{\mu} \neq 0 \) has been assumed. By a straightforward computation, it can be verified that \( N_{\mu\nu} = 0 \) is satisfied by the Kerr-NUT metric \( 3.34. \) This statement can be slightly strengthened in such a way that \( N_{\mu\nu} = 0 \) is indeed satisfied for the metric \( 3.34 \) with arbitrary structure functions \( (Q(q), P(p)), \) i.e., insensitive to the satisfaction of Einstein’s equations. The above tensor obeys \( N_{\mu\nu} = N_{(\mu\nu)}, \) \( N^\mu_{\mu\nu} = 0 \) and \( N_{\mu\nu} \xi^{\nu} = 0, \) so that \( N_{ab} \) is identified as a tensor on the base space.

To demonstrate that the on-shell value of \( N_{\mu\nu} \) recovers \( 4.30 \) when projected onto the base space, it is useful to record

\[ E^+_{ab} = \frac{1}{2} D_a D_b \xi + \frac{1}{4 f} D_a D_b \xi - \frac{1}{4 f} D^{\rho} \xi D_{\rho} h_{ab}. \]  

(3.59)

We then find

\[ N_{ab} = -\frac{1}{4} (1 + \xi^{\rho})^2 \left[ N_{ab} + \frac{1}{2 (D^2)^2} N_{cd} D^{\rho} \xi D^{\rho} D_{cd} \xi \left( h_{ab} - \frac{3 D_a D_b \xi}{(D^2)^2} \right) \right]. \]  

(3.60)
Accordingly, the on-shell $N_{\mu\nu}$ stores the geometric data identical to $N_{ab}$, as we wanted to show.

By noting $\xi^\lambda C_{\lambda\mu\nu}\sigma^\rho, (d\xi^\tau)^{\sigma\tau} = -(A/f) E^+_{\mu\nu}\sigma^\rho$, one can rewrite the Mars tensor (3.50) into

$$M_{\mu\nu\rho} = 2 \left( 2N_{\mu[\sigma}\sigma_{\rho]} + f^{-1} h_{\mu[\rho}N_{\nu]\sigma}\sigma^\sigma \right).$$  \hfill (3.61)

This equation retains a striking pattern reminiscent of (3.34) and (3.35).

It follows from these parallel structures that equation (3.60) defines the spacetime characterization of (3.3) that admits a description simpler than the Mars tensor. However, this should not be taken too literally. The expression in (3.58) incorporates the inverse of $f = -g_{\mu\nu}\xi^\mu\xi^\nu$ and also $\sigma^2 = f h^{ab}D_aE D_bE$, both of which are not well-defined at the ergosurface. Unfortunately, the global property that has been a major meliority of the Mars tensor is now lost. This is a fundamental limitation of our tensor (3.58).

Nevertheless, the use of $N_{\mu\nu}$ rather than $M_{\mu\nu\rho}$ unveils a new geometric condition for the Kerr-NUT family in the following way. By construction, we have

$$N_{\mu\nu}\sigma^\nu = E^+_{\mu\nu}\sigma^\nu - \frac{E^+_{\rho\sigma}}{\sigma^2}\sigma^\mu = 0.$$  \hfill (3.62)

This is an eigenvalue problem which is well-defined insensitive to the sign of $f$, provided $\sigma^2 \neq 0$.\footnote{For the \textit{pp}-wave metric (3.39), we have $\sigma^2 = 0$ and $E^+_{\mu\nu}\sigma^\nu = 0.$} It follows that the configuration $N_{\mu\nu} = 0$ implies that $\sigma^\nu$ is an eigenvector of $E^+_{\mu\nu}$. This is the four dimensional covariant condition free from restriction $f > 0$ and symbolizes the (off-shell) Kerr-NUT family (3.34). This simple criterion is appealing and has been unnoticed in the literature. Note that the diagonalizability of $E^+_{\mu\nu}$ with doubly degenerate eigenvalues amounts to the Petrov D condition. As a consistency check, the condition (3.62) is not satisfied by the most general Petrov-D vacuum metric constructed by Plebański and Demiański [40].

### IV. ELECTROVACUUM

Let us now extend the discussion of the vacuum case in the previous section into the electrovacuum. The field equations to the Einstein-Maxwell system read

$$R_{\mu\nu} = 2 \left( F_{\mu\rho}F_{\nu\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \right), \quad \text{d}F = \text{d}\ast F = 0.$$  \hfill (4.1)

Assuming that the Maxwell field is also invariant under the flow of the stationary Killing vector $Z F = 0$, the Maxwell equation and the Bianchi identity imply the existence of local scalar functions $(E, B)$ such that

$$\nabla_\mu E = F_{\mu\nu}\xi^\nu, \quad \nabla_\mu B = -\ast F_{\mu\nu}\xi^\nu.$$  \hfill (4.2)

The twist $\omega_\mu = \epsilon_{\mu\rho\sigma\nu}\xi^\rho\nabla^\sigma\xi^\nu$ of the Killing vector satisfies (2.3), leading to

$$\nabla_{[\mu}\omega_{\nu]} = 4\nabla_{[\mu}E\nabla_{\nu]}B.$$  \hfill (4.3)

This implies the local existence of a twist scalar $\psi$ such that

$$\omega_\mu = \nabla_\mu \psi + 2(E\nabla_\mu B - B\nabla_\mu E).$$  \hfill (4.4)

Then, the system is reduced to the gravity-coupled sigma model (2.4) with the target space $SU(1,2)/S(U(1) \times U(1,1))$ [47, 48]

$$\text{d}s^2_T = 2f^2 \left[ |d\psi|^2 + (d\psi + 2E \text{d}B - 2B \text{d}E)^2 \right] - \frac{2}{f} (dE^2 + dB^2) = \frac{1}{2(\text{Re}E^2 + |\Phi|^2)^2} (d\text{E}^2 + 2\Phi \text{d}\Phi)(d\text{E} + 2\Phi \text{d}\Phi) - \frac{2}{\text{Re}E^2 + |\Phi|^2} \text{d}\Phi \text{d}\Phi,$$  \hfill (4.5)

where

$$E \equiv f - i\psi - (E^2 + B^2), \quad \Phi \equiv -E + iB.$$  \hfill (4.6)
For the spacelike reduction $f < 0$, this is a negative curvature complex projective space $\mathbb{CP}^{1,1}$, which admits the simultaneous Kähler and quaternionic structures. In the stationary reduction $f > 0$, the target space $\mathbb{CP}^{1,1}$ has the $(+,+,−,−)$ signature.

By the following holomorphic transformation to the new complex variables $w^i$ ($i = 1, 2$)

$$w^1 = \frac{1 - \mathcal{E}}{1 + \mathcal{E}}, \quad w^2 = \frac{2\Theta}{1 + \mathcal{E}},$$

the field equations reduce to

$$R_{ab} = 2\Theta^{-1}D_{(a}w^iD_{b)}\bar{w}_i + 2\Theta^{-2}w_i\bar{w}_jD_{(a}\bar{w}^iD_{b)}w^j,$$  \hspace{1cm} (4.8a)

$$D^aD_aw^i = -2\Theta^{-1}\bar{w}_jD^aw^iD_aw^j,$$  \hspace{1cm} (4.8b)

where

$$w_i = \eta_{ij}w^j, \quad \eta_{ij} = \text{diag}(1, -1),$$

and

$$\Theta = 1 - w \cdot \bar{w}, \quad v_1 \cdot \bar{v}_2 = \eta_{ij}v^i_1\bar{v}^j_2.$$  \hspace{1cm} (4.10)

One can introduce the complex vectors $Z^A$ ($A = 0, 1, 2$) by

$$w^1 = \frac{Z^1}{Z^0}, \quad w^2 = \frac{Z^2}{Z^0}.$$  \hspace{1cm} (4.11)

Apart from the distinction that the indices $A, B$ now run from 0 to 2 with

$$\eta_{AB} = \text{diag}(1, -1, 1),$$

these quantities $Z^A$ can be chosen to satisfy (2.17), where $\langle Z, \bar{Z} \rangle = \eta_{AB}Z^A\bar{Z}^B$ and $\bar{Z}_A = \eta_{AB}\bar{Z}^B$ as before.

The generalization of the complex tensor $N_{ab}^A$ into the Einstein-Maxwell system is straightforward and is given by the same form as (3.1):

$$N_{ab}^A = D_aD_bZ^A = \frac{1}{(Z, \bar{Z})(1 - \langle Z, \bar{Z} \rangle)}\bar{Z}_BD^cZ^BD_cZ^Ah_{ab} + \frac{1 + 2\langle Z, \bar{Z} \rangle}{(Z, \bar{Z})(1 - \langle Z, \bar{Z} \rangle)}\bar{Z}_BD(a)Z^BD_bZ^A,$$  \hspace{1cm} (4.13)

where $Z^A$ is now regarded as a triplet of $\text{SU}(1, 2)$. Besides this, the obstruction must supplemented by

$$\mathcal{N}_{ab} = \epsilon_{ABC}Z^AD_aZ^BD_bZ^C.$$  \hspace{1cm} (4.14)

Here $\epsilon_{ABC}$ is an alternate tensor of $\text{SU}(1, 2)$. We affirm that both of $N_{ab}^A$ and $\mathcal{N}_{ab}$ fulfill a role as an obstruction to the Kerr-Newman-NUT family. Because of the formal similarity, the most of the discussion in the vacuum case carries over to the electrovacuum case.

Choosing the homogeneous coordinates as $Z^A = (1, w^i)$, these tensors are boiled down to

$$N_{ab}^i = D_aD_bw^i + \frac{1}{w \cdot \bar{w}\Theta}w_kD^cw^kD_aw^i\bar{h}_b - \frac{3 - 2w \cdot \bar{w}}{w \cdot \bar{w}\Theta}w_kD_{(a}w^kD_{b)}w^i,$$  \hspace{1cm} (4.15)

and

$$\mathcal{N}_{ab} = 2D_{[a}w^1D_{b]}w^2.$$  \hspace{1cm} (4.16)

This symmetric tensor satisfies $N_{ab}^a = 0$ in view of the Einstein-Maxwell field equations (4.8). When the electromagnetic field is switched off $w^2 = 0$, one recovers $N_{ab}^1 \to N_{ab}$ in (3.3). We stress that the form of the tensors (4.13) and (4.16) remains unchanged for

$$Z^A = \begin{pmatrix} 1 & w^1 \\ w^2 & 0 \end{pmatrix} \mapsto A^A_{B}Z^B, \quad A = \begin{pmatrix} e^{2\theta} & 0 & 0 \\ 0 & e^{-\theta} & e^{-\theta} \alpha \\ 0 & e^{-\theta} \beta & e^{-\theta} \alpha \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1,$$  \hspace{1cm} (4.17)

corresponding to the stability subgroup $\text{SU}(1 \times \text{U}(1))$ of $\text{SU}(1, 2)$. 
Let us now construct solutions admitting \( N_{ab} = \mathcal{N}_{ab} = 0 \). The condition (4.16) implies that \( w^2 \) is a function only of \( w^1 \), viz, \( w^2 = h(w^1) \). Inserting this into the equations of motion (4.85), one arrives at \((Dw^1)^2 \partial h/\partial (w^1)^2 = 0 \). For concreteness, we shall assume that that \( D_\alpha w^1 \) is not null or zero. Then, this leads to

\[
w^2 = \gamma w^1 + \delta, \tag{4.18}
\]

where \( \gamma \) and \( \delta \) are complex constants. We denote \( w^1 = w \) and \( w^2 = \gamma w + \delta \) in what follows.

From \( N^1_{ab} D^b w = 0 \), one finds a relation \( D_\alpha (\log \rho) \propto D_\alpha w \) for \( \rho^{-2} = (Dw)^2 \). The integrability \( D_\nu D_\mu \log \rho = 0 \) of this equation is assured only for (i) \( \rho \) is real or (ii) \( \delta = 0 \). For real \( \rho \) with \( \delta \neq 0 \), one can foliate \( h_{ab} dx^a dx^b = \rho^2(w) dw^2 + \Xi(w)^2 e^{2\Phi(z)} dz d\bar{z} \), for which one finds no solutions compatible with Einstein’s equations. It follows that the only allowed possibility is (ii) \( \delta = 0 \).

The rest of the proof is fairly straightforward. For the sake of clarity, we focus on the case \( dw \wedge d\bar{w} \neq 0 \). The vanishing of \( N^1_{ab} \) implies that the following vector is a hypersurface-orthogonal Killing vector

\[
U^a = \frac{i[1 - (1 - |\gamma|^2)|w|^2]}{(w\bar{w})^3} \epsilon^{abc} D_b w D_c \bar{w}, \tag{4.19}
\]

By \( N^a_{\mu \nu} D^\nu w = 0 \), we have \( \rho = C w^{-2} \) with \( C \) being a complex constant. For the metric ansatz (3.22), we thus obtain

\[
U = \frac{1 - (1 - |\gamma|^2)|w|^2}{|C|^2|w|^2} \sqrt{\Omega^2 - 1}, \quad \Omega = \frac{1}{2|w|^2[1 - (1 - |\gamma|^2)|w|^2]} \left( \frac{\bar{C}}{C} w^2 + \frac{C}{\bar{C}} \bar{w}^2 + 2C \bar{w}^2 w^2 \right). \tag{4.20}
\]

Choosing

\[
C = m - in, \quad C_0 = -1 + \frac{2a^2 + Q^2_a + Q^2_m}{m^2 + n^2}, \quad \gamma = \frac{Q_e + iQ_m}{m - in}, \tag{4.21}
\]

and transforming to \( w = (m - in)/[q - m + i(p + n)] \) with \( \phi = \frac{1}{2}(m^2 + n^2)^2 \sigma \), we get the base space (3.20) with structure functions

\[
Q(q) = q^2 - 2mq + a^2 - n^2 + Q^2_a + Q^2_m, \quad P(p) = a^2 - (n + p)^2. \tag{4.22}
\]

This recovers the Kerr-Newman-NUT family \((51)\) with the electric charge \( Q_e \) and the magnetic charge \( Q_m \).

V. SUMMARY

This paper has endeavored to give a coherent description for the local characterization of the Kerr-NUT family \((17)\). We have proposed a symmetric traceless complex tensor (3.1) or (3.3) defined on the orbit space of the stationary Killing vector. This tensor takes the place of the Simon tensor in several respects. Our tensor (3.1) is an SU(1,1) vector-valued symmetric tensor and is simplified in the gauge (3.22) to the form (3.23), by means of which the Simon tensor is entirely described as (3.4). Most notably, the practical benefit of our tensor (3.3) is its simple tensorial structure, which allows us to elucidate the hidden symmetry (4.18) on top of the Killing symmetry (4.17). Apart from the degenerate cases which correspond to Taub-NUT and pp-waves, our proposed tensor is entitled as a new criterion for the local identification of the Kerr-NUT family. Moreover, the obstruction tensor is readily generalized into the electrovacuum case, while leaving the SU(1,2) invariance unbroken. This is the main importance of the present result.

We have further undertaken to covariantize the tensor (3.3) in the language of four dimensions, following the philosophy of Mars (22). Since our expression (3.3) is not globally defined, it does not seem to have a significant advantage over the Mars tensor. Despite this unsatisfactory feature, we have found a new criterion (3.62) as a four dimensional covariant obstruction to the Kerr-NUT metric, which is much more manageable and fully amendable to analytic study.

Several applications of our results are conceivable. One can consider various relatives of the Kerr-NUT metric. The Wahlquist class of solutions (51) describing a rigidly rotating perfect fluid in stationary and axisymmetric family is one of the nonvacuum extensions of the Carter solution, since it admits a Killing-Yano tensor with a three-form torsion (52). One can verify that the Wahlquist metric satisfies \( \mathcal{N}_{\mu \nu} = 0 \) (and \( M_{\mu \nu \rho} = 0 \)) only on-shell. This traces back to the fact that the Wahlquist metric belongs to type I for off-shell, whereas to type D for on-shell (53). It would be an interesting future direction to explore the relation to the (torsionful) Killing-Yano tensor and the tensor \( \mathcal{N}_{\mu \nu} \) and to the quotient space interpretation.

Throughout the paper, we have addressed the local characterization of the Kerr-NUT solution. For the application of the global boundary value problems for the stationary and axisymmetric system, we need to perform the first
dimensional reduction along the Killing vector that generates $U(1)$ instead of the stationary Killing vector. To date, no adequate counterpart of the Simon tensor has been constructed for the spacelike reduction. It would be definitely worthwhile to pursue this direction for the new uniqueness proof of rotating black holes. Work along this direction is in progress.

An intrinsic labeling of the Kerr-NUT metric has been also given in [55], where no stationary assumption is made. Alternatively, additional conditions should be presumed therein. They gave two criteria: (i) the eigen-twoform of the complex selfdual Weyl tensor in Petrov-D space should satisfy a certain differential equation, (ii) the gradients of the Weyl scalar invariants for Petrov-D space should satisfy a certain algebraic equation. The analysis of [55] follows in part the spirit of the Cartan-Karlhede program, according to which the equivalence problem and the isometry group of a given metric can be addressed by the Riemann tensor and its covariant derivatives. Recently, this Cartan-Karlhede program has been streamlined substantially into a practical form by [56–58] in three dimensions. It is then a promising route to examine generalizations of these algorithms into $3 + 1$ dimensions, for the labeling the Kerr-NUT solution under more relaxed conditions.

The method developed here seems to be operative only in four spacetime dimensions. The Myers-Perry metric possibly with a cosmological constant and a NUT charge can be written into the Carter form in arbitrary dimensions [60]. It would be interesting to explore the analogue of the Simon tensor for the dimensionally reduced space of these higher dimensional metrics.

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