SPECTRAL PROPERTIES OF HARMONIC TOEPLITZ OPERATORS AND APPLICATIONS TO THE PERTURBED KREIN LAPLACIAN

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Abstract. We consider harmonic Toeplitz operators $T_V = PV : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$ where $P : L^2(\Omega) \to \mathcal{H}(\Omega)$ is the orthogonal projection onto $\mathcal{H}(\Omega) = \{u \in L^2(\Omega) | \Delta u = 0 \text{ in } \Omega\}$, $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded domain with boundary $\partial \Omega \in C^\infty$, and $V : \Omega \to \mathbb{C}$ is an appropriate multiplier. First, we complement the known criteria which guarantee that $T_V$ is in the $p$th Schatten-von Neumann class $S_p$, by simple sufficient conditions which imply $T_V \in S_p,w$, the weak counterpart of $S_p$. Next, we assume that $\Omega$ is the unit ball in $\mathbb{R}^d$, and $V = \nabla^2$ possesses a partial radial symmetry, and investigate the eigenvalue asymptotics of the compact operator $T_V$ in the case where $V$ admits a power-like decay at $\partial \Omega$ or $V$ is compactly supported in $\Omega$. Further, we consider general $\Omega$ and $V \geq 0$ which is regular in $\Omega$, and admits a power-like decay of rate $\gamma > 0$ at $\partial \Omega$, and we show that in this case $T_V$ is unitarily equivalent to a pseudo-differential operator of order $-\gamma$, self-adjoint in $L^2(\partial \Omega)$. Utilizing this unitary equivalence, we obtain the main asymptotic term of the eigenvalue counting function for the operator $T_V$, and establish a sharp remainder estimate. Finally, we introduce the Krein Laplacian $K$, self-adjoint in $L^2(\Omega)$; it is known that $K \geq 0$, $\text{Ker } K = \mathcal{H}(\Omega)$, and the zero eigenvalue of $K$ is isolated. We perturb $K$ by a multiplier $V \in C(\overline{\Omega}; \mathbb{R})$, and show that $\sigma_{\text{ess}}(K + V) = V(\partial \Omega)$. Assuming that $V \geq 0$ and $V|_{\partial \Omega} = 0$, we study the asymptotic distribution of the discrete spectrum of $K + V$ near the origin, and find that the effective Hamiltonian which governs this distribution is the harmonic Toeplitz operator $T_V$.

Keywords: Harmonic Toeplitz operators, Krein Laplacian, essential spectrum, eigenvalue asymptotics, effective Hamiltonian

2010 AMS Mathematics Subject Classification: 47B35, 35J25, 35P15, 35P20

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain, i.e. a bounded open, connected, non-empty set. Suppose that $\partial \Omega \in C^\infty$.

Let $\mathcal{H}(\Omega)$ be the subspace of $L^2(\Omega)$ consisting of functions harmonic in $\Omega$, i.e.

\[(1.1) \quad \mathcal{H}(\Omega) := \{u \in L^2(\Omega) | \Delta u = 0 \text{ in } \Omega\}.\]

It is well known that $\mathcal{H}(\Omega)$ is a closed subspace of $L^2(\Omega)$ (see e.g. [14]). Let $P : L^2(\Omega) \to L^2(\Omega)$ be the orthogonal projection onto $\mathcal{H}(\Omega)$. Assume that $V : \Omega \to \mathbb{C}$ is locally integrable in $\Omega$, and satisfies certain regularity conditions near $\partial \Omega$. Then it can happen that the operator $T_V := PV : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$ called harmonic Toeplitz operator with symbol $V$, is bounded or even compact.
The article is devoted mostly to the study of the spectral properties of compact $T_V$.

First, in Section 2 we recall some known criteria for the boundedness of $T_V$, its compactness, and its membership to the Schatten-von Neumann classes $S_p$. Moreover, in Section 2 we establish simple sufficient conditions which guarantee $T_V \in S_{p,w}$, the weak Schatten-von Neumann class.

In Section 3 we consider the special case where $\Omega$ is the unit ball in $\mathbb{R}^d$, while $V$ is radially symmetric and vanishes on $\partial \Omega$. Then the eigenvalues and the eigenfunctions of the compact operator $T_V$ could be written explicitly. Using these explicit calculations, we obtain the main asymptotic term of the eigenvalue counting function for $T_V$ in the case where $V$ possesses a partial radial symmetry, and admits a power-like decay at $\partial \Omega$ (see Proposition 3.1), or is compactly supported in $\Omega$ (see Proposition 3.2).

Further, in Section 4 we assume that $V$ admits a prescribed power-like decay at $\partial \Omega$, and establish in Proposition 4.3 a unitary equivalence between $T_V$ and a certain pseudo-differential operator acting in $L^2(\partial \Omega)$. We apply these results in order to investigate in Theorem 4.1 the asymptotic distribution of the discrete spectrum of $T_V$.

Finally, in Section 5 we introduce the Krein Laplacian $K$, self-adjoint in $L^2(\Omega)$. We have $K \geq 0$, Ker $K = \mathcal{H}(\Omega)$, and the zero eigenvalue of $K$ is isolated (see [24, 18, 4]). We perturb $K$ by the real-valued multiplier $V \in C(\overline{\Omega})$ and show that $\sigma_{\text{ess}}(K + V) = V(\partial \Omega)$. If $V \geq 0$ and $V|_{\partial \Omega} = 0$, we show that, generically, there exists a sequence of negative (resp., positive) discrete eigenvalues of the operator $K - V$ (resp., $K + V$), which accumulate to the origin from below (resp., from above). We show that the effective Hamiltonian governing the asymptotics of these sequences is the harmonic Toeplitz operator $T_V$. Using the results of the previous sections we obtain the main asymptotic terms of the corresponding eigenvalue counting functions.

2. Compactness and membership to Schatten-von Neumann Classes of harmonic Toeplitz operators $T_V$

In this section we recall some known criteria for the boundedness, compactness and membership to the Schatten-von Neumann classes $S_p$, $p \in [1, \infty)$, of the harmonic Toeplitz operator $T_V$, which we borrow mainly from [14]. Moreover, we establish simple sufficient conditions which guarantee $T_V \in S_{p,w}$, $p \in (1, \infty)$, where $S_{p,w}$ is the $p$th weak Schatten-von Neumann class.

First, we introduce the notations we need. Let $X$ and $Y$ be separable Hilbert spaces. We denote by $\mathcal{L}(X,Y)$ (resp., $S_\infty(X,Y)$) the class of linear bounded (resp., compact) operators $T : X \rightarrow Y$. Let $T \in S_\infty(X,Y)$. Then $\{s_j(T)\}_{j=1}^{\text{rank } T}$ is the set of the non-zero singular values of $T$, enumerated in non-increasing order. Next, $S_p(X,Y)$, $p \in (0, \infty)$, is $p$th Schatten-von Neumann class, i.e. the class of compact operators $T : X \rightarrow Y$ for which the functional

$$\|T\|_p := \left(\sum_{j=1}^{\text{rank } T} s_j(T)^p\right)^{1/p}$$
Let $V \in C(\Omega)$. Assume at first that $V \in C(\Omega)$; then, evidently, $T_V$ is bounded. Our first proposition deals with the location of $\sigma_{\text{ess}}(T_V)$, and contains a criterion for the compactness of $T_V$.

**Proposition 2.1.** Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with boundary $\partial\Omega \in C^\infty$. Let $V \in C(\Omega)$.

(i) [14, Theorem 4.5] We have $\sigma_{\text{ess}}(T_V) = V(\partial\Omega)$.

(ii) [14, Corollary 4.7] The operator $T_V$ is compact in $\mathcal{H}(\Omega)$ if and only if $V = 0$ on $\partial\Omega$. 

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is finite. Similarly, $S_{p,w}(X,Y)$, $p \in (0,\infty)$, is the $p$th weak Schatten-von Neumann class, i.e. the class of operators $T \in S_\infty(X,Y)$ for which the functional

$$
\|T\|_{p,w} := \sup_{j \geq 1} j^{1/p} s_j(T)
$$

is finite. If $X = Y$, we write $\mathcal{L}(X)$, $S_p(X)$, and $S_{p,w}(X)$, instead of $\mathcal{L}(X,X)$, $S_p(X,X)$, and $S_{p,w}(X,X)$, respectively. Moreover, whenever appropriate, we omit $X$ and $Y$ in the notations $\mathcal{L}$, $S_p$, and $S_{p,w}$.

If $p \geq 1$, then $\| \cdot \|_p$ is a norm, and $S_p$ is a Banach space. If $p > 1$, then there exists a norm in $S_{p,w}$ which is equivalent to the functional $\| \cdot \|_{p,w}$, and $S_{p,w}$, equipped with this norm, is again a Banach space. Moreover, evidently, if $0 < p_1 \leq p_2 < p_3$, then $S_{p_1} \subset S_{p_2,w} \subset S_{p_3}$, and all inclusions are strict.

For further references, we introduce here the eigenvalue counting functions for compact operators. Let $T = T^* \in S_\infty$. For $s > 0$ set

$$
(2.1) \quad n_{\pm}(s; T) := \text{Tr} \mathbb{1}_{(s,\infty)}(\pm T).
$$

Here and in the sequel $\mathbb{1}_S$ denotes the characteristic function of the set $S$; thus $\mathbb{1}_T(T)$ is the spectral projection of $T$ corresponding to the interval $I \subset \mathbb{R}$, and $n_+(s; T)$ (resp., $n_{-}(s; T)$) is just the number of the eigenvalues of the operator $T$ larger than $s$ (resp., smaller than $-s$), counted with their multiplicities. In other words, if $\{ \varepsilon_j^+(T) \}_{j=1}^{\text{rank} T_+}$ (resp., $\{ \varepsilon_j^-(T) \}_{j=1}^{\text{rank} T_-}$) is the set of the positive (resp., negative) eigenvalues of $T$, enumerated in non-increasing (resp., non-decreasing) order, then

$$
n_{\pm}(s; T) = \# \{ j \in \mathbb{N} | \pm \varepsilon_j^+(T) > s \}, \quad s > 0.
$$

If $T_j = T^*_j \in S_\infty(X)$, $j = 1, 2$, then the Weyl inequalities

$$
(2.2) \quad n_{\pm}(s_1 + s_2; T_1 + T_2) \leq n_{\pm}(s_1; T_1) + n_{\pm}(s_2; T_2)
$$

hold for $s_j > 0$, $j = 1, 2$, (see e.g. [9, Theorem 9, Section 9.2]).

Let $T \in S_\infty(X,Y)$. For $s > 0$ set

$$
(2.3) \quad n_+(s; T) := n_+(s^2; T^* T).
$$

Thus, $n_+(s; T)$ is the number of the singular values of the operator $T$, larger than $s$, and counted with their multiplicities.
Further, it is well known that the projection $P$ onto $\mathcal{H}(\Omega)$ (see (1.1)) admits an integral kernel $R \in C^\infty(\Omega \times \Omega)$, called the reproducing kernel of $P$ (see e.g. [23, 14]). Thus

$$(Pu)(x) = \int_\Omega R(x,y)u(y)dy, \quad x \in \Omega, \quad u \in L^2(\Omega).$$

Let $\{\varphi_j\}_{j \in \mathbb{N}}$ be an orthogonal basis in $\mathcal{H}(\Omega)$. Then we have

$$(2.4) \quad R(x,y) = \sum_{j \in \mathbb{N}} \varphi_j(x)\varphi_j(y), \quad x,y \in \Omega,$$

the series being locally uniformly convergent in $\Omega \times \Omega$. Evidently, $R(x,y)$ is independent of the choice of the basis $\{\varphi_j\}_{j \in \mathbb{N}}$. Moreover, the kernel $R$ is real-valued and symmetric. For $x \in \Omega$ put

$$\rho(x) := R(x,x).$$

Then, (2.4) implies that

$$|R(x,y)| \leq \rho(x)^{1/2} \rho(y)^{1/2}, \quad x,y \in \Omega.$$

For $x, y \in \Omega$, set

$$(2.5) \quad r(x) := \text{dist}(x,\partial \Omega), \quad d(x,y) := |x-y| + r(x) + r(y).$$

**Lemma 2.1.** [23 Theorem 1.1] For any multiindices $\alpha, \beta \in \mathbb{Z}_+^n$ there exists a constant $C_{\alpha,\beta} \in (0, \infty)$ such that

$$|D_x^\alpha D_y^\beta R(x,y)| \leq \frac{C_{\alpha,\beta}}{d(x,y)^{d+|\alpha|+|\beta|}}, \quad x,y \in \Omega.$$

Moreover, there exists a constant $C \in (0, \infty)$ such that

$$(2.7) \quad \rho(x) \geq Cr(x)^{-d}, \quad x \in \Omega.$$

For a Borel set $A \subset \Omega$ set $\rho(A) := \int_A \rho(x)dx$. By (2.6) with $\alpha = \beta = 0$, and (2.7), $\rho$ is an infinite $\sigma$-finite measure on $\Omega$ which is absolutely continuous with respect to the Lebesgue measure.

The following proposition contains criteria for the boundedness, compactness and membership to $S_p$, $p \in [1, \infty)$, of $T_V$ in the case where $0 \leq V \in L^1(\Omega)$. In fact, following [14], we will formulate these results in a more general setting, considering harmonic Toeplitz operators $T_\mu$ associated with finite Borel measures $\mu \geq 0$ on $\Omega$. In this case, $T_\mu$ is defined by

$$(T_\mu u)(x) := \int_\Omega R(x,y)u(y)d\mu(y), \quad u \in \mathcal{H}(\Omega), \quad x \in \Omega.$$

If $d\mu(x) = V(x)dx$ with $0 \leq V \in L^1(\Omega)$, then, of course, $T_\mu = T_V$. Define the Berezin transform $\tilde{\mu}$ of the measure $\mu$ by

$$\tilde{\mu}(x) := \rho(x)^{-1} \int_\Omega R(x,y)^2d\mu(y), \quad x \in \Omega.$$
In what follows we write $A \asymp B$ if there exist constants $0 < c_1 \leq c_2 < \infty$ such that $c_1 A \leq B \leq c_2 A$.

**Proposition 2.2.** Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with boundary $\partial \Omega \in C^\infty$. Let $\mu \geq 0$ be a finite Borel measure on $\Omega$, and let $\tilde{\mu}$ be its Berezin transform.  

(i) [14, Theorem 3.5, Theorem 3.9] We have $T_\mu \in \mathcal{L}(\mathcal{H}(\Omega))$ if and only if $\tilde{\mu}$ is bounded on $\Omega$. Moreover,

\begin{equation}
(2.8) \quad \|T_\mu\| \asymp \sup_{x \in \Omega} \tilde{\mu}(x).
\end{equation}

(ii) [14, Theorem 3.11, Theorem 3.12] We have $T_\mu \in S_\infty(\mathcal{H}(\Omega))$ if and only if \[ \lim_{x \to \partial \Omega} \tilde{\mu}(x) = 0. \]

(iii) [14, Theorem 3.13] Let $p \in [1, \infty)$. We have $T_\mu \in S_p(\mathcal{H}(\Omega))$ if and only if $\tilde{\mu} \in L^p(\Omega; \omega)$. Moreover,

\begin{equation}
(2.9) \quad \|T_\mu\|_p \asymp \|	ilde{\mu}\|_{L^p(\Omega; \omega)}.
\end{equation}

Our next goal is to establish conditions which guarantee $T_V \in S_{p,w}(\mathcal{H}(\Omega))$, $p \in (1, \infty)$. As a by-product we obtain also simple-looking sufficient conditions which imply $T_V \in S_p(\mathcal{H}(\Omega))$, $p \in [1, \infty)$.

For $p \in (0, \infty)$ define $L^p_w(\Omega; \omega)$ as the class of $\omega$-measurable functions $u : \Omega \to \mathbb{C}$ for which the quasinorm

\[ \|u\|_{L^p_w(\Omega; \omega)} := \sup_{t > 0} t \rho(\{x \in \Omega \mid |u(x)| > t\})^{1/p} \]

is finite. If $p > 1$, then there exists a norm in $L^p_w(\Omega; \omega)$ which is equivalent to the functional $\| \cdot \|_{L^p_w(\Omega; \omega)}$, and $L^p_w(\Omega; \omega)$, equipped with this norm, is a Banach space.

**Proposition 2.3.** Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$ be a bounded domain with boundary $\partial \Omega \in C^\infty$. 

(i) Assume $V \in L^p(\Omega; \omega)$, $p \in [1, \infty)$. Then $T_V \in S_p(\mathcal{H}(\Omega))$ and

\begin{equation}
(2.10) \quad \|T_V\|_p \leq \|V\|_{L^p(\Omega; \omega)}.
\end{equation}

(ii) Assume $V \in L^p_w(\Omega; \omega)$, $p \in (1, \infty)$. Then $T_V \in S_{p,w}(\mathcal{H}(\Omega))$ and

\begin{equation}
(2.11) \quad \|T_V\|_{p,w} \leq \|V\|_{L^p_w(\Omega; \omega)}.
\end{equation}

**Proof.** Let us consider the operator $PVP$ as defined on $L^2(\Omega)$. Evidently,

\begin{equation}
(2.12) \quad \|T_V\|_p = \|PVP\|_p, \quad \|T_V\|_{p,w} = \|PVP\|_{p,w}, \quad p \in (0, \infty).
\end{equation}

We have $PVP = F^* e^{i \arg V} F$ where $F : L^2(\Omega) \to L^2(\Omega)$ is the operator with integral kernel

\[ |V(x)|^{1/2} R(x, y), \quad x, y \in \Omega. \]

Assume $V \in L^1(\Omega; \omega)$. Then

\begin{equation}
(2.13) \quad \|PVP\|_1 \leq \|F^*\|_2 \|e^{i \arg V}\|_2 \|F\|_2 = \|F\|_2^2 = \|V\|_{L^1(\Omega; \omega)}.
\end{equation}
Assume now \( V \in L^\infty(\Omega; d\rho) \). Since \( \|P\| = 1 \) and \( d\rho \) is absolutely continuous with respect to the Lebesgue measure,

\[
\|PVP\| \leq \|V\|_{L^\infty(\Omega)} = \|V\|_{L^\infty(\Omega; d\rho)}.
\]  

Interpolating between (2.13) and (2.14), and applying [8, Theorem 3.1], we find that

\[
\|PVP\|_p \leq \|V\|_{L^p(\Omega; d\rho)}, \quad p \in [1, \infty),
\]

\[
\|PVP\|_{p,w} \leq \|V\|_{L^{p,w}(\Omega; d\rho)}, \quad p \in (1, \infty),
\]

which combined with (2.12), implies (2.10) and (2.11).

\[\Box\]

Remarks: (i) We consider that the main part of Proposition 2.3 is the second one, while the first part is just a by-product of the interpolation method applied, and is obviously less sharp than Proposition 2.2 (iii). Let us still point out some of the aspects of estimates (2.10) which we consider valuable:

- The estimating constant in (2.10) is just equal to one while the constants in (2.9) are not explicit and depend on \( \Omega \).
- The boundedness of \( \Omega \) in Proposition 2.2 is essential, while estimates (2.10) remain valid for generic unbounded domains.
- Estimates (2.10) are given in terms of \( V \) itself, while estimates (2.9) are given in terms of its Berezin transform.

(ii) The Berezin-Toeplitz operators related to the Fock-Segal-Bargmann holomorphic subspace of \( L^2(\mathbb{R}^2) \), and their generalizations corresponding to higher Landau levels, are known to play an important role in the spectral and scattering theory of quantum Hamiltonians in constant magnetic fields (see e.g. [28, 29, 17, 10, 27, 11, 26]). Proposition 3.6 of [26] is an analogue of our Proposition 2.3 for such operators (see also [28, Lemma 5.1] and [17, Lemma 3.1] where however no weak Schatten-von Neumann classes were considered). A further development of these results is contained in [27, Theorem 1.6] and [26, Corollary 3.2].

Finally, we establish a result which shows that if the symbol \( V \) is compactly supported in \( \Omega \), then \( T_V \in S_p \) for any \( p \in (0, \infty) \), i.e. the singular numbers of \( T_V \) decay very rapidly, even if the behaviour of \( V \) is quite irregular. In fact, we will replace in this case \( V \) by \( \phi \in \mathcal{E}'(\Omega) \), the class of distributions over \( \mathcal{E}(\Omega) := C^\infty(\Omega) \). We recall that \( \phi \in \mathcal{D}'(\Omega) \), the class of distributions over \( \mathcal{D}(\Omega) := C_0^\infty(\Omega) \), is in \( \mathcal{E}'(\Omega) \), if and only if \( \text{supp } \phi \) is compact in \( \Omega \). If \( \phi \in \mathcal{E}'(\Omega) \), we define \( T_\phi : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega) \) as the operator with integral kernel

\[ K_\phi(x, y) := (\phi, \mathcal{R}(x, \cdot)\mathcal{R}(\cdot, y)), \quad x, y \in \Omega, \]

where \( \langle \cdot, \cdot \rangle \) denotes the pairing between \( \mathcal{E}'(\Omega) \) and \( \mathcal{E}(\Omega) \). Of course, if \( \phi = \mu \) and \( \mu \geq 0 \) is a finite Borel measure such that \( \text{supp } \mu \) is compact in \( \Omega \), then \( T_\phi = T_\mu \).

Since \( \text{supp } \phi \) is compact in \( \Omega \), we have \( K_\phi \in C^\infty(\overline{\Omega} \times \overline{\Omega}) \). Therefore,

\[ s_j(T_\phi) = O(j^{-m}), \quad \forall m \in (0, \infty), \]

(see e.g. [8, Proposition 2.1]). Thus, we arrive at
Proposition 2.4. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with boundary $\Omega \in C^\infty$. Assume that $\phi \in \mathcal{E}'(\Omega)$. Then we have $T_\phi \in S_p(\mathcal{H}(\Omega))$ for any $p \in (0, \infty)$, and, hence,

$$n_*(\lambda; T_\phi) = O(\lambda^{-\alpha}), \quad \lambda \downarrow 0,$$

for any $\alpha \in (0, \infty)$.

Remarks: (i) In Section 3 we will show that if $\Omega$ is the unit ball in $\mathbb{R}^d$, and $V \geq 0$ is compactly supported and possesses a partial radial symmetry, then the eigenvalues of $T_V$ decay exponentially fast. Hopefully, in a future work we will extend these results to more general domains, and more general compactly supported $V$.

(ii) Harmonic Toeplitz operators $T_\phi$ with $\phi \in \mathcal{E}'(\Omega)$ were considered in [3] where, in particular, it was proved that rank $T_\phi < \infty$, if and only if supp $\phi$ is finite.

3. Spectral properties of radially symmetric $T_V$

In this section we assume that $\Omega = B_1$ where

$$B_R := \{ x \in \mathbb{R}^d \mid |x| < R \}, \quad d \geq 2, \quad R \in (0, \infty).$$

Thus, $\partial \Omega = S^{d-1} := \{ x \in \mathbb{R}^d \mid |x| = 1 \}$. The space $\mathcal{H}(B_1)$ admits an explicit orthonormal eigenbasis which we are now going to describe. Recall that $k(k+d-2)$, $k \in \mathbb{Z}_+$, are the eigenvalues of the Beltrami-Laplace operator $-\Delta_{S^{d-1}},$ self-adjoint in $L^2(S^{d-1})$ (see e.g. [31, Section 22]). Moreover,

$$\dim \ker (-\Delta_{S^{d-1}} - k(k+d-2)I) =: m_k = \binom{d+k-1}{d-1} - \binom{d+k-3}{d-1}$$

where $\binom{m}{n} = \frac{m!}{(m-n)!n!}$ if $m \geq n$, and $\binom{m}{n} = 0$ if $m < n$ (see e.g. [31, Theorem 22.1]). Set $M_k := \sum_{j=0}^k m_j$, $k \in \mathbb{Z}_+$. By induction, we easily find that

$$M_k = \binom{d+k-1}{d-1} + \binom{d+k-2}{d-1}, \quad k \in \mathbb{Z}_+.$$

In particular,

$$M_k = \frac{2k^{d-1}}{(d-1)!} \left( 1 + O \left( k^{-1} \right) \right), \quad k \to \infty,$$

(see e.g. [1, Eq. 6.1.47]). Let $\psi_{k,\ell}$, $\ell = 1, \ldots, m_k$, be an orthonormal basis in $\ker (-\Delta_{S^{d-1}} - k(k+d-2)I)$, $k \in \mathbb{Z}_+$. It is well known that $\psi_{k,\ell}$ are restrictions on $S^{d-1}$ of homogeneous polynomials of order $k$, harmonic in $\mathbb{R}^d$ (see e.g. [31, Section 22]).

Then the functions $\phi_{k,\ell}(x) := \sqrt{2k+d} |x|^k \psi_{k,\ell}(x/|x|)$, $x \in B_1$, $\ell = 1, \ldots, m_k$, $k \in \mathbb{Z}_+$, form an orthonormal basis in $\mathcal{H}(B_1)$. Let $\mathcal{H}_k(B_1)$, $k \in \mathbb{Z}_+$, be the subspace of $\mathcal{H}(B_1)$ generated by $\phi_{k,\ell}$, $\ell = 1, \ldots, m_k$.

Further, let $V(x) = v(|x|)$, $x \in B_1$, and let $v : [0, 1) \to \mathbb{R}$ satisfy $\lim_{r \uparrow 1} v(r) = 0$, $v \in L^1((0,1); r^{d-1}dr)$. Then the operator $T_V$ is self-adjoint and compact in $\mathcal{H}(B_1)$, and

$$T_V u = \mu_k u, \quad u \in \mathcal{H}_k(B_1),$$

where $\mu_k$ are the eigenvalues of $T_V$. In particular, $\mu_k \to +\infty$ as $k \to \infty$. Moreover, $\mu_k$ and the corresponding eigenfunctions $\phi_{k,\ell}$ are bounded in $\mathcal{H}(B_1)$, but unbounded in $L^2(B_1)$. Finally, the eigenfunctions of $T_V$ satisfy $\phi_{k,\ell} = \phi_{k,\ell}^\prime$, $\ell = 1, \ldots, m_k$, where $\phi_{k,\ell}^\prime$ are the first derivatives of $\phi_{k,\ell}$ with respect to $x$. Consequently, $T_V$ is a self-adjoint operator in $\mathcal{H}(B_1)$.
where
\begin{equation}
\mu_k(v) := (2k + d) \int_0^1 v(r)r^{2k+d-1}dr, \quad k \in \mathbb{Z}_+.
\end{equation}

Set
\[\nu_\pm(s; v) = \# \{k \in \mathbb{Z}_+ \mid \mu_k(\pm v) > s\}, \quad s > 0.\]

Let us calculate the eigenvalues of $T_V$ in two simple model situations where, in particular, $v \geq 0$ so that $T_V \geq 0$.

(i) Let $v_1(r) = a(1 - r)^\gamma, \quad r \in [0, 1)$, with $a > 0$ and $\gamma > 0$. Then, according to (3.4), we have
\begin{equation}
\mu_k(v_1) = a(2k + d)B(\gamma + 1, 2k + d) = a\Gamma(\gamma + 1)\frac{\Gamma(2k + d + 1)}{\Gamma(2k + d + 1 + \gamma)}, \quad k \in \mathbb{Z}_+,
\end{equation}
where $B$ and $\Gamma$ are the Euler beta and gamma functions respectively. Let us show that the sequence $\{\mu_k(v_1)\}_{k \in \mathbb{Z}_+}$ is decreasing. To this end, it suffices to check that
\begin{equation}
f_\gamma'(x) < 0, \quad x > 0,
\end{equation}
where $f_\gamma(x) := \frac{\Gamma(x)}{\Gamma(x + \gamma)}$. We have
\begin{equation}
f_\gamma'(x) = f_\gamma(x)(\Psi(x) - \Psi(x + \gamma))
\end{equation}
where $\Psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}$, $x > 0$, is the digamma function. Since
\[\Psi'(x) = \int_0^\infty \frac{te^{-xt}}{1 - e^{-t}}dt > 0, \quad x \in (0, \infty),\]
(see e.g. [1, Eq. (6.1.47)]), and $\gamma > 0$, we obtain $\Psi(x) - \Psi(x + \gamma) < 0$ so that (3.7) implies (3.6). Setting $V_1(x) := v_1(|x|), \quad x \in \mathbb{R}^d$, we get
\begin{equation}
n_+(\lambda; T_{V_1}) = M_{\nu_+(\lambda; v_1)}, \quad \lambda > 0,
\end{equation}
the counting function $n_+$ being defined in (2.1). Let us discuss briefly the asymptotics of $n_+(\lambda; T_{V_1})$ as $\lambda \downarrow 0$. By (3.5) and [1, Eq. 6.1.47], we have
\[\mu_k(v_1) = a\Gamma(\gamma + 1)(2k + d)^{-\gamma}(1 + O(k^{-1})), \quad k \to \infty.
\]
Therefore,
\begin{equation}
\nu_+(\lambda; v_1) = \frac{1}{2}(a\Gamma(\gamma + 1))^{1/\gamma}\lambda^{-1/\gamma}(1 + o(1)), \quad \lambda \downarrow 0.
\end{equation}
Applying (3.8), (3.2), and (3.9), we find that
\begin{equation}
n_+(\lambda; T_{V_1}) = \frac{2^{-d+2}}{(d - 1)!}(a\Gamma(\gamma + 1))^{(d-1)/\gamma}\lambda^{-(d-1)/\gamma}(1 + o(1)), \quad \lambda \downarrow 0,
\end{equation}
which is equivalent to
\[\xi_j^+(T_{V_1}) = \left(\frac{2^{-d+2}}{(d - 1)!}\right)^{\gamma/(d-1)}a\Gamma(\gamma + 1)j^{-\gamma/(d-1)}(1 + o(1)), \quad j \to \infty.
\]
(ii) Let $v_2(r) = b \mathbf{1}_{[0,c]}(r)$, $r \in [0,1)$, with $b > 0$, and $c \in (0,1)$. Then (3.4) implies (3.11)
\[
\mu_k(v_2) = b c^{2k+d}, \quad k \in \mathbb{Z}_+.
\]
Evidently, the sequence $\{\mu_k(v_2)\}_{k \in \mathbb{Z}_+}$ is decreasing. Setting $V_2(x) := v_2(|x|)$, $x \in \mathbb{R}^d$, similarly to (3.8), we get (3.12)
\[
n_+(\lambda; T_{V_2}) = M_{\nu_+(\lambda; v_2)}, \quad \lambda > 0.
\]
Let us discuss the asymptotics of $n_+(\lambda; T_{V_2})$ as $\lambda \downarrow 0$. By (3.11), (3.13)
\[
\nu_+(\lambda; v_2) = \frac{1}{2} |\ln \lambda| + O(1), \quad \lambda \downarrow 0.
\]
By (3.12), (3.2), and (3.13), we get (3.14)
\[
n_+(\lambda; T_{V_2}) = 2^{-(d+2)} \frac{2^{-(d+2)}}{(d-1)!} |\ln \lambda|^{d-1} + O \left( |\ln \lambda|^{-d+2} \right), \quad \lambda \downarrow 0,
\]
which implies (3.15)
\[
\ln \kappa_j^+(T_{V_2}) = 2 \ln c \left( (d-1)!/2 \right)^{1/(d-1)} j^{1/(d-1)} (1 + o(1)), \quad j \to \infty.
\]

**Remark:** The fact that the basis $\{\phi_{k,\ell}\}$ diagonalizes the operator $T_V$ with radially symmetric symbol $V$, acting in $\mathcal{H}(B_1)$, was noted in [30, Part 2.3.2], and was used there, in particular, to obtain asymptotic relations of type (3.14) and (3.15). The fact that the Toeplitz operators with radially symmetric symbols, acting in the holomorphic Fock-Segal-Bargmann space, are diagonalized in a certain canonic basis, was utilized already in [29, 21]. A similar result concerning Toeplitz operators with radially symmetric symbols, acting in the holomorphic Bergman space, can be found in [20].

Next, we use (3.10) and (3.14) to study the spectral asymptotics for Toeplitz operators with symbols $V$ which possess partial radial symmetry.

**Proposition 3.1.** Let $\Omega = B_1$. Assume that $V \in L^1(B_1; \mathbb{R})$. Suppose moreover, that there exist $\gamma > 0$ and $a \in \mathbb{R}$ such that $\lim_{|x| \to 1} (1 - |x|)^{-\gamma} V(x) = a$, uniformly with respect to $x/|x| \in \mathbb{S}^{d-1}$.

(i) Let $a > 0$. Then we have (3.16)
\[
\lim_{\lambda \downarrow 0} \lambda^{(d-1)/\gamma} n_+(\lambda; T_V) = 2^{-d+2} (d-1)! a \Gamma(\gamma+1)^{(d-1)/\gamma}.
\]

(ii) Let $a < 0$. Then we have (3.17)
\[
n_+(\lambda; T_V) = O(\lambda^{-\alpha}), \quad \lambda \downarrow 0,
\]
for any $\alpha > 0$.

**Proof.** (i) Pick $\varepsilon \in (0,1)$, and $\delta \in (0,1)$ such that
\[
V_-(x) + \tilde{V}_-(x) \leq V(x) \leq V_+(x) + \tilde{V}_+(x), \quad x \in B_1,
\]
where
\[
V_\pm(x) := (1 \pm \varepsilon)a(1 - |x|)^{\gamma},
\]
\[ \tilde{V}_-(x) := (V(x) - (1 - \varepsilon)(1 - |x|)\gamma) \mathbb{1}_{B_1-\delta}(x), \quad \tilde{V}_+(x) := V(x) \mathbb{1}_{B_1-\delta}(x), \quad x \in B_1. \]

Then the mini-max principle and the Weyl inequalities \text{(2.2)} imply

\[ n_+((1 + \varepsilon)\lambda; T\tilde{V}_-) - n_- (\varepsilon\lambda; T\tilde{V}_-) \leq n_+(\lambda; TV) \leq n_+((1 - \varepsilon)\lambda; T\tilde{V}_+) + n_+ (\varepsilon\lambda; T\tilde{V}_+), \quad \lambda > 0. \]

By \text{(3.10)} and \text{(2.15)}, we find that it follows from \text{(3.18)} that

\[ \lim_{\lambda \downarrow 0} \lambda^{(d-1)/\gamma} n_+ (\lambda; TV) \leq \limsup_{\lambda \downarrow 0} \lambda^{(d-1)/\gamma} n_+ (\lambda; TV) \leq \liminf_{\lambda \downarrow 0} \lambda^{(d-1)/\gamma} n_+ (\lambda; TV) \leq \lim_{\lambda \downarrow 0} \lambda^{(d-1)/\gamma} n_+ (\lambda; TV). \]

Letting \( \varepsilon \downarrow 0 \), we obtain \text{(3.16)}.

(ii) Since \( a < 0 \) there exists \( c \in (0, 1) \) such that

\[ V(x) \leq |V(x)| \mathbb{1}_{B_c}(x), \quad x \in B_1. \]

Now \text{(3.17)} follows from \text{(3.19)}, the mini-max principle, and Proposition \text{2.1}.

**Proposition 3.2.** Let \( \Omega = B_1 \). Assume that \( V : B_1 \to [0, \infty) \) satisfies \( V \in L^\infty(B_1) \) and \( \operatorname{supp} V = \overline{B_c} \) for some \( c \in (0, 1) \). Suppose moreover that for any \( \delta \in (0, c) \) we have \( \operatorname{ess inf}_{x \in B_\delta} V(x) > 0 \). Then

\[ \lim_{\lambda \downarrow 0} |\ln \lambda|^{-d+1} n_+ (\lambda; TV) = \frac{2^{-d+2}}{(d-1)! |\ln c|^{d-1}}. \]

**Proof.** Pick \( \delta \in (0, c) \). Then for almost every \( x \in B_1 \) we have

\[ b_- \mathbb{1}_{B_\delta}(x) \leq V(x) \leq b_+ \mathbb{1}_{B_\delta}(x), \]

where

\[ b_- := \operatorname{ess inf}_{x \in B_\delta} V(x), \quad b_+ := \operatorname{ess sup}_{x \in B_\delta} V(x). \]

Then the mini-max principle and \text{(3.14)} imply

\[ \liminf_{\lambda \downarrow 0} |\ln \lambda|^{-d+1} n_+ (\lambda; TV) \leq \limsup_{\lambda \downarrow 0} |\ln \lambda|^{-d+1} n_+ (\lambda; TV) \leq \frac{2^{-d+2}}{(d-1)! |\ln c|^{d-1}}. \]

Letting \( \delta \uparrow c \), we obtain \text{(3.20)}.

\[ \square \]
4. Spectral asymptotics of \( T_V \) for general \( V \) admitting power-like decay at the boundary

4.1. Statement of the main results. In this section we assume that \( V : \overline{\Omega} \to [0, \infty) \) is sufficiently regular near \( \partial \Omega \) and admits a power-like decay at \( \partial \Omega \), and investigate the asymptotic behaviour of the discrete spectrum of \( T_V \) near the origin. We obtain the main asymptotic term of \( n_+(\lambda; T_V) \) as \( \lambda \downarrow 0 \), and give a sharp estimate of the remainder (see Theorem 4.1 below).

For the statement of Theorem 4.1 we need the following notations. We consider \( \partial \Omega \) as a compact \((d-1)\)-dimensional Riemannian manifold with metric tensor \( g(y) := \{g_{jk}(y)\}_{j,k=1}^{d-1}, \ y \in \partial \Omega \), generated by the Euclidean metrics in \( \mathbb{R}^d \). For \( y \in \partial \Omega \) and \( \eta \in T_y^* \partial \Omega = \mathbb{R}^{d-1} \) we set

\[
|\eta| = |\eta|_y := \left( \sum_{j,k=1}^{d-1} g_{jk}(y) \eta_j \eta_k \right)^{1/2},
\]

where \( \{g_{jk}(y)\}_{j,k=1}^{d-1} \) is the matrix inverse to \( g(y) \). Let \( dS(y) \) be the measure induced by \( g \) on \( \partial \Omega \). As usually, we denote by \( L^2(\partial \Omega) \) the Hilbert space \( L^2(\partial \Omega; dS(y)) \).

Let \( a, \tau \in C^\infty(\overline{\Omega}) \) satisfy \( a > 0 \) on \( \overline{\Omega} \), \( \tau > 0 \) on \( \Omega \), and \( \tau = \gamma \cdot r := \text{dist}(\cdot, \partial \Omega) \) (see (2.5)) in a vicinity of \( \partial \Omega \). Assume that

\[
(4.1) \quad V(x) = \tau(x) \gamma a(x), \quad \gamma \geq 0, \quad x \in \Omega.
\]

Set \( a_0 := a|_{\partial \Omega} \).

**Theorem 4.1.** Assume that \( V \) satisfies (4.1) with \( \gamma > 0 \). Then we have

\[
(4.2) \quad n_+(\lambda; T_V) = C \lambda^{-\frac{d-1}{d-\gamma}} \left( 1 + O(\lambda^{1/\gamma}) \right), \quad \lambda \downarrow 0,
\]

where

\[
(4.3) \quad C := \omega_{d-1} \left( \frac{\Gamma(\gamma + 1)^{1/\gamma}}{4\pi} \right)^{d-1} \int_{\partial \Omega} a_0(y)^{\frac{d-1}{d}} dS(y),
\]

and \( \omega_n = \pi^{n/2}/\Gamma(1+n/2) \) is the Lebesgue measure of the unit ball \( B_1 \subset \mathbb{R}^n \), \( n \geq 1 \).

The proof of Theorem 4.1 is contained in the following subsection.

*Remark:* Let \( V \) satisfy the assumptions of Theorem 4.1 and \( 0 \leq \phi \in \mathcal{E}'(\Omega) \). Set \( T_{V+\phi} := T_V + T_\phi \). Then Theorem 4.1 combined with the Weyl inequalities (2.2) and estimate (2.15) easily implies

\[
\lim_{\lambda \downarrow 0} \lambda^{\frac{d-1}{d-\gamma}} n_+(\lambda; T_{V+\phi}) = C.
\]

A more precise analysis shows that we can modify the proof of Theorem 4.1 so that (4.2) remains valid if we replace \( T_V \) by \( T_{V+\phi} \). We omit the proof of this fact in order to avoid unnecessary purely technical complications.
4.2. **Proof of Theorem 4.1.** For $s \in \mathbb{R}$ denote by $H^s(\Omega)$ and $H^s(\partial\Omega)$ the Sobolev spaces on $\Omega$ and $\partial\Omega$ respectively. Assume that $f \in H^s(\partial\Omega)$, $s \in \mathbb{R}$. Then the boundary-value problem

\[
\begin{aligned}
\Delta u = 0 & \quad \text{in} \quad \Omega, \\
 u = f & \quad \text{on} \quad \partial\Omega,
\end{aligned}
\]

admits a unique solution $u \in H^{s+1/2}(\Omega)$, there exists a constant $c$ such that

\[
\|u\|_{H^{s+1/2}(\Omega)} \leq c\|f\|_{H^s(\partial\Omega)},
\]

and the mapping $f \mapsto u$ defines an isomorphism between $H^s(\partial\Omega)$ and $H^{s+1/2}(\Omega)$ (see \[25\] Sections 5, 6, 7, Chapter 2]).

If $s = 0$, we set

\[
u \mapsto u
\]

By (4.5) with $s = 0$, and the compactness of the embedding of $H^{1/2}(\Omega)$ into $L^2(\Omega)$, we find that the operator $G : L^2(\partial\Omega) \to L^2(\Omega)$ is compact. By \[16\] Theorem 12, Section 2.2, we have

\[
u \mapsto u
\]

where

\[
u \mapsto u
\]

\[
u \mapsto u
\]

\[
u \mapsto u
\]

$\mathcal{G}$ is the Dirichlet Green function associated with $\Omega$, and $\nu$ is the unit outer normal vector at $\partial\Omega$. Note that

\[
u \mapsto u
\]

**Lemma 4.1.** We have

\[
u \mapsto u
\]

\[
u \mapsto u
\]

\[
u \mapsto u
\]

**Proof.** Relation (4.10) follows from the uniqueness of the solution of (4.4) with $s = 0$. Let us check (4.11). Pick $u \in \mathcal{H}(\Omega)$. Then, by (4.4) with $s = -1/2$, we have $f := u|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$. Let $f_n \in L^2(\partial\Omega)$, $n \in \mathbb{N}$, and

\[
u \mapsto u
\]

set $u_n := G f_n$. Then $u_n \in \text{Ran} G$, $n \in \mathbb{N}$, and by (4.5) with $s = -1/2$, and (4.12), we have $\lim_{n \to \infty} \|u_n - u\|_{L^2(\Omega)} = 0$ which implies (4.11). $\square
Set $J := G^* G$. Then the operator $J = J^* \geq 0$ is compact in $L^2(\partial \Omega)$. Due to (4.10), we have $\text{Ker} J = \{0\}$. Let $\{\lambda_j\}_{j \in \mathbb{N}}$ be the non-increasing sequence of the eigenvalues $\lambda_j > 0$ of $J$, and let $\{\phi_j\}_{j \in \mathbb{N}}$ be the corresponding orthonormal eigenbasis in $L^2(\partial \Omega)$ with $J \phi_j = \lambda_j \phi_j$, $j \in \mathbb{N}$. Define the operator $J^{-1}$, self-adjoint in $L^2(\partial \Omega)$, by

$$J^{-1} u := \sum_{j \in \mathbb{N}} \lambda_j^{-1} \langle u, \phi_j \rangle \phi_j, \quad \text{Dom } J^{-1} := \left\{ u \in L^2(\partial \Omega) \mid \sum_{j \in \mathbb{N}} \lambda_j^{-2} |\langle u, \phi_j \rangle|^2 < \infty \right\},$$

$\langle \cdot, \cdot \rangle$ being the scalar product in $L^2(\partial \Omega)$. Evidently, $JJ^{-1} = J^{-1} J = I$.

Further, write the polar decomposition of the operator $G = U |G| = U J^{1/2}$ where $U : L^2(\partial \Omega) \to L^2(\Omega)$ is an isometric operator. By Lemma 4.1, we have $\text{Ker } U = \{0\}$ and $\text{Ran } U = \mathcal{H}(\Omega)$. Thus, we obtain the following

**Proposition 4.1.** The orthogonal projection $P$ onto $\mathcal{H}(\Omega)$ satisfies

$$P = G J^{-1} G^* = U U^*.$$  

Assume that $V$ satisfies (4.1) with $\gamma \geq 0$, and set $J_V := G^* VG$; from this point of view, we have $J = J_1$.

**Proposition 4.2.** Let $V$ satisfy (4.1) with $\gamma \geq 0$. Then the operator $T_V$ is unitarily equivalent to (the closure of) the operator $J^{-1/2} J_V J^{-1/2}$.

**Proof.** By (4.14), we have

$$P V P = U J^{-1/2} G^* V G J^{-1/2} U^* = U J^{-1/2} J_V J^{-1/2} U^*,$$

and the operator $U$ maps unitarily $L^2(\partial \Omega)$ onto $\mathcal{H}(\Omega)$. \qed

**Proposition 4.3.** Under the assumptions of Proposition 4.2 the operator $J^{-1/2} J_V J^{-1/2}$ is a $\Psi$DO with principal symbol

$$2^{-\gamma} \Gamma(\gamma + 1) |\eta|^{-\gamma} a_0(y), \quad (y, \eta) \in T^* \partial \Omega.$$  

**Proof.** Using the pseudo-differential calculus due to L. Boutet de Monvel (see [12, 13], M. Engliš showed recently in [15] Sections 6, 7] that if $V$ satisfies (4.1) with $\gamma \geq 0$, then the operator $J_V$ is a $\Psi$DO with principal symbol

$$2^{-\gamma - 1} \Gamma(\gamma + 1) |\eta|^{-\gamma - 1} a_0(y), \quad (y, \eta) \in T^* \partial \Omega.$$  

In particular, $J = J_1$ is a $\Psi$DO with principal symbol $2^{-1} |\eta|^{-1}$. Then the pseudo-differential calculus (see e.g. [31] Chapters I, II]) easily implies that $J^{-1/2}$ is a $\Psi$DO with principal symbol $2^{1/2} |\eta|^{1/2}$, and $J^{-1/2} J_V J^{-1/2}$ is a $\Psi$DO with principal symbol defined in (4.15). \qed

Now we are in position to prove Theorem 4.1. It is easy to see that under its assumptions we have $\text{Ker } J^{-1/2} J_V J^{-1/2} = \{0\}$. Using the spectral theorem, define the operator

$$A := (J^{-1/2} J_V J^{-1/2})^{-1/\gamma}$$
(cf. (4.13)). Then, by the pseudo-differential calculus, $A$ is a ΨDO with principal symbol

$$2\Gamma(\gamma + 1)^{-1/\gamma}|\eta|a_0(y)^{-1/\gamma}, \quad (y, \eta) \in T^*\partial\Omega.$$ 

By Proposition 4.2 and the spectral theorem, we have

$$n_+(\lambda; T_V) = n_+(\lambda; J^{-1/2}J_VJ^{-1/2}) = \text{Tr} \mathbf{1}_{(\infty, \lambda^{-1/\gamma})}(A), \quad \lambda > 0.$$ 

A classical result of L. Hörmander [22] easily implies that

$$\text{Tr} \mathbf{1}_{(-\infty, E)}(A) = CE^{d-1}(1 + O(E^{-1})), \quad E \to \infty,$$

the constant $C$ being defined in (4.3). Combining (4.16) and (4.17), we arrive at (4.2).

5. Applications to the spectral theory of the perturbed Krein Laplacian

In this section we introduce the Krein Laplacian $K$, perturb it by a multiplier $V \in C(\Omega; \mathbb{R})$, and investigate the spectral properties of the perturbed operator $K + V$.

For $s \in \mathbb{R}$, we denote, as usual, by $H^s_0(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the topology of the Sobolev space $H^s(\Omega)$. Set also $H^2_D(\Omega) := H^2(\Omega) \cap H^1_0(\Omega)$. Define the minimal Laplacian

$$\Delta_{\text{min}} := \Delta, \quad \text{Dom } \Delta_{\text{min}} = H^2_0(\Omega).$$

As is well known, $\Delta_{\text{min}}$ is symmetric but not self-adjoint in $L^2(\Omega)$, since we have

$$\Delta_{\text{min}}^* =: \Delta_{\text{max}} = \Delta, \quad \text{Dom } \Delta_{\text{max}} = \{ u \in L^2(\Omega) \mid \Delta u \in L^2(\Omega) \}.$$ 

$
\Delta u$ being the distributional Laplacian of $u \in L^2(\Omega)$. Note that we have

$$\text{Ker } \Delta_{\text{max}} = \mathcal{H}(\Omega).$$

Lemma 5.1. The domain $\text{Dom } \Delta_{\text{max}}$ admits the direct-sum decomposition

$$\text{Dom } \Delta_{\text{max}} = \mathcal{H}(\Omega) + H^2_D(\Omega).$$

Proof. Let us first show that the sum at the r.h.s. of (5.2) is direct. Assume that $u_1 \in \mathcal{H}(\Omega)$, $u_2 \in H^2_D(\Omega)$, and $u_1 + u_2 = 0$. Then $u_2 \in H^2(\Omega)$ satisfies the homogeneous boundary-value problem

$$\begin{cases} 
\Delta u_2 = 0 \quad \text{in } \Omega, \\
 u_2 = 0 \quad \text{on } \partial\Omega.
\end{cases}$$

Hence, $u_2 = 0$, and $u_1 = 0$. Evidently, if $u_1 \in \mathcal{H}(\Omega)$, $u_2 \in H^2_D(\Omega)$, then $u_1 + u_2 \in \text{Dom } \Delta_{\text{max}}$. Pick now $u \in \text{Dom } \Delta_{\text{max}}$, and let us check the existence of $u_1$ and $u_2$ such that

$$u_1 \in \mathcal{H}(\Omega), \quad u_2 \in H^2_D(\Omega), \quad u = u_1 + u_2.$$ 

Define the Dirichlet Laplacian

$$\Delta_D := \Delta, \quad \text{Dom } \Delta_D := H^2_D(\Omega).$$

Set

$$u_2 := \Delta^{-1}_D u, \quad u_1 := u - u_2.$$ 

Evidently, $u_1$ and $u_2$ satisfy (5.3). □
Introduce the Krein Laplacian
\[ K := -\Delta, \quad \text{Dom } K = \mathcal{H}(\Omega) + H^2_0(\Omega). \]

The operator \( K \geq 0 \), self-adjoint in \( L^2(\Omega) \), is the von Neumann - Krein “soft” extension of \( -\Delta_{\min} \), remarkable for the fact that any other self-adjoint extension \( S \geq 0 \) of \( -\Delta_{\min} \) satisfies \( (S + I)^{-1} \leq (K + I)^{-1} \) (see [32, 24]). Evidently, \( \text{Ker } K = \mathcal{H}(\Omega) \). The domain \( \text{Dom } K \) admits a more explicit description in the terms of the Dirichlet-to-Neumann operator \( D \). For \( f \in C^\infty(\partial\Omega) \), \( Df \) is defined by
\[
Df = \partial u / \partial \nu_{|\partial\Omega},
\]
where \( u \) is the solution of the boundary-value problem
\[
\begin{align*}
\Delta u &= 0 \quad \text{in } \Omega, \\
u &= f \quad \text{on } \partial\Omega.
\end{align*}
\]
The operator \( D \) is a first-order elliptic operator; by the elliptic regularity, it extends to a bounded operator form \( H^s(\partial\Omega) \) into \( H^{s-1}(\partial\Omega) \), \( s \in \mathbb{R} \). Then we have
\[
\text{Dom } K = \left\{ u \in \text{Dom } \Delta_{\max} \mid \left. \frac{\partial u}{\partial \nu_{|\partial\Omega}} = D(u_{|\partial\Omega}) \right. \right\}
\]
(see [18 Theorem III.1.2]). The Krein Laplacian \( K \) arises naturally in the so called abstract buckling problem (see e.g. [19, 5]).

Denote by \( L \) the restriction of \( K \) onto \( \text{Dom } K \cap \mathcal{H}(\Omega)^\perp \) where \( \mathcal{H}(\Omega)^\perp := L^2(\Omega) \ominus \mathcal{H}(\Omega) \). Then, \( L \) is self-adjoint in the Hilbert space \( \mathcal{H}(\Omega)^\perp \).

**Proposition 5.1.** [24, 4 Theorem 5.1] The spectrum of \( L \) is purely discrete and positive, and, hence, \( L^{-1} \in S_\infty(\mathcal{H}(\Omega)^\perp) \). As a consequence, \( \sigma_{\text{ess}}(K) = \{0\} \), and the zero is an isolated eigenvalue of \( K \) of infinite multiplicity.

Let \( V \in C(\overline{\Omega}; \mathbb{R}) \). Then the operator \( K + V \) with domain \( \text{Dom } K \) is self-adjoint in \( L^2(\Omega) \). In the sequel, we will investigate the spectral properties of \( K + V \).

**Remarks:** (i) In many aspects, the assumption \( V \in C(\overline{\Omega}) \) is too restrictive, the operator \( K + V \) could also be self-adjoint on \( \text{Dom } K \) for less regular potentials \( V \). Moreover, the sum \( K + V \) could be defined in the sense of quadratic forms. However, the description of an optimal class of singular \( V \) for which the sum \( K + V \) is well defined in the operator or form sense requires additional technical work which is left for a possible future article. (ii) It should be underlined here that the perturbations \( K_V \) of the Krein Laplacian \( K \) discussed in [6] are of different nature than the perturbations \( K + V \) considered here. Namely, the authors of [6] assume that \( V \geq 0 \), define the maximal operator \( K_{V,\text{max}} \) as
\[
K_{V,\text{max}} := -\Delta + V, \quad \text{Dom } K_{V,\text{max}} := \text{Dom } \Delta_{\max},
\]
and set
\[
K_V := -\Delta + V, \quad \text{Dom } K_V := \text{Ker } K_{V,\text{max}} + H^2_0(\Omega).
\]
Thus, if \( V \neq 0 \), then the operators \( K_V \) and \( K_0 = K \) are self-adjoint on different domains, while the operators \( K + V \) introduced here are self-adjoint on the same domain \( \text{Dom } K \).

It is shown in [6] that for any \( 0 \leq V \in L^\infty(\Omega) \) we have \( K_V \geq 0 \), \( \sigma_{\text{ess}}(K_V) = \{0\} \), and the zero is an isolated eigenvalue of \( K_V \) of infinite multiplicity. As we will see in what follows, the spectral properties of \( K + V \) could be quite different.

**Theorem 5.1.** Let \( V \in C(\overline{\Omega}; \mathbb{R}) \). Then we have

\[
\sigma_{\text{ess}}(K + V) = V(\partial \Omega).
\]

In particular, \( \sigma_{\text{ess}}(K + V) = \{0\} \) if and only if \( V|_{\partial \Omega} = 0 \).

**Proof.** First, we will show that

\[
(K + V - i)^{-1} - (K + PVP - i)^{-1} = \]

\[
-(K + V - i)(K - i)^{-1}(QVQ + PVQ + QVP)(K - i)^{-1}(K - i)(K + PVP - i)^{-1}.
\]

Evidently,

\[
(K + V - i)^{-1}(K - i), \quad (K - i)(K + PVP - i)^{-1}, \quad P, V \in \mathcal{L}(L^2(\Omega)).
\]

Moreover, using the orthogonal decomposition \( L^2(\Omega) = \mathcal{H}(\Omega) \oplus \mathcal{H}(\Omega)^{\perp} \), and bearing in mind Proposition 5.1, we find that

\[
Q(K - i)^{-1}, \quad (K - i)^{-1}Q \in S_\infty(L^2(\Omega)).
\]

Now (5.5) follows from (5.6)-(5.8). Therefore,

\[
\sigma_{\text{ess}}(K + V) = \sigma_{\text{ess}}(K + PVP).
\]

Further, we have \( K + PVP = T_V \oplus L \) in \( L^2(\Omega) = \mathcal{H}(\Omega) \oplus \mathcal{H}(\Omega)^{\perp} \), and, hence,

\[
\sigma_{\text{ess}}(K + PVP) = \sigma_{\text{ess}}(T_V) \cup \sigma_{\text{ess}}(L).
\]

By Proposition 5.1 (i), we have \( \sigma_{\text{ess}}(T_V) = V(\partial \Omega) \), and by Proposition 5.1 \( \sigma_{\text{ess}}(L) = \emptyset \).

Thus, (5.9) and (5.10) imply (5.4). \( \square \)

In the rest of the section we assume that \( 0 \leq V \in C(\overline{\Omega}) \) with \( V|_{\partial \Omega} = 0 \), and investigate the asymptotic distribution of the discrete spectrum of the operators \( K \pm V \), adjoining the origin.

For \( \lambda > 0 \) set

\[
\mathcal{N}_-(\lambda) := \text{Tr} \mathbf{1}_{(-\infty,-\lambda)}(K - V).
\]

Set \( \lambda_0 := \inf \sigma(L) \). By Proposition 5.1 we have \( \lambda_0 > 0 \). For \( \lambda \in (0, \lambda_0) \) set

\[
\mathcal{N}_+(\lambda) := \text{Tr} \mathbf{1}_{(\lambda, \lambda_0)}(K + V).
\]
**Theorem 5.2.** Assume that $0 \leq V \in C(\overline{\Omega})$ and $V_{\partial \Omega} = 0$. Then for any $\varepsilon \in (0, 1)$ we have

\begin{equation}
(5.11) \quad n_+(\lambda; T_V) \leq \mathcal{N}_-(\lambda) \leq n_+((1 - \varepsilon)\lambda; T_V) + O(1),
\end{equation}

\begin{equation}
(5.12) \quad n_+((1 + \varepsilon)\lambda; T_V) + O(1) \leq \mathcal{N}_+(\lambda) \leq n_+((1 - \varepsilon)\lambda; T_V) + O(1),
\end{equation}

as $\lambda \downarrow 0$.

**Proof.** By the Birman-Schwinger principle [7, Lemma 1.1], we have

\begin{equation}
(5.13) \quad \mathcal{N}_-(\lambda) = n_+(1; (K + \lambda)^{-1/2}V(K + \lambda)^{-1/2}) = n_+(1; V^{1/2}(K + \lambda)^{-1/2}), \ \lambda > 0.
\end{equation}

It follows from the mini-max principle that

\begin{equation}
(5.14) \quad n_+(1; (K + \lambda)^{-1/2}V(K + \lambda)^{-1/2}) \geq n_+(1; P(K + \lambda)^{-1/2}V(K + \lambda)^{-1/2}P) = n_+(\lambda; PV\lambda) = n_+(\lambda; T_V),
\end{equation}

which, combined with the first equality in (5.13), implies the lower bound in (5.11). Further, by the Weyl inequalities (2.2), we have

\begin{equation}
(5.15) \quad n_+(1; V^{1/2}(K + \lambda)^{-1}V^{1/2}) \leq n_+(1; (1 - \varepsilon)\lambda; V^{1/2}PV^{1/2}) + n_+(\varepsilon; V^{1/2}Q(K + \lambda)^{-1}V^{1/2}), \ \lambda > 0.
\end{equation}

Evidently,

\begin{equation}
(5.16) \quad n_+(s; V^{1/2}PV^{1/2}) = n_+(s; PV\lambda) = n_+(s; T_V), \quad s > 0,
\end{equation}

while Proposition 5.1 easily implies that for any $\varepsilon > 0$ we have

\begin{equation}
(5.17) \quad n_+(\varepsilon; V^{1/2}Q(K + \lambda)^{-1}V^{1/2}) = O(1), \quad \lambda \to 0.
\end{equation}

Putting together (5.13) and (5.14) – (5.16), we obtain the upper bound in (5.11).

In order to prove (5.12), we recall that the generalized Birman-Schwinger principle (see e.g. [2, Theorem 1.3]), easily entails

\begin{equation}
(5.18) \quad n_+(1; V^{1/2}PV^{1/2}) + n_-(\varepsilon; V^{1/2}Q(K - \lambda)^{-1}V^{1/2})
\end{equation}

hold true for every $\varepsilon \in (0, 1)$. Now (5.17), (5.18), (5.15), and (5.16), imply (5.12). \qed

Combining Theorem 5.2 and the results of Section 2, 3, and 4, we could obtain rich information concerning the spectrum of the operator $K \pm V$, adjoining the origin. In particular, estimates (5.11) – (5.12) on one hand, and Proposition 3.1, Proposition 3.2, or Theorem 4.1 on the other, imply the following three corollaries.
Corollary 5.1. Let \( \Omega = B_1 \subset \mathbb{R}^d, \ d \geq 2, \ 0 \leq V \in C(\overline{B}_1) \). Assume that there exist \( \gamma > 0 \) and \( a > 0 \) such that
\[
\lim_{x \to 1} (1 - |x|)^{-\gamma} V(x) = a.
\]
Then we have
\[
\lim_{\lambda \downarrow 0} \lambda^{(d-1)/\gamma} N_\pm(\lambda) = \frac{2^{-d+2}}{(d-1)!} (a \Gamma(\gamma + 1))^{(d-1)/\gamma}.
\]

Corollary 5.2. Let \( \Omega = B_1 \subset \mathbb{R}^d, \ d \geq 2, \ 0 \leq V \in C(\overline{B}_1) \). Assume that \( \operatorname{supp} V = \overline{B}_c \) for some \( c \in (0, 1) \), and that for any \( \delta \in (0, c) \) we have \( \inf_{x \in B_\delta} V(x) > 0 \). Then
\[
\lim_{\lambda \downarrow 0} |\ln \lambda|^{-(d+1)} N_\pm(\lambda) = \frac{2^{-d+2}}{(d-1)! |\ln c|^{d-1}}.
\]

Corollary 5.3. Assume that \( V \) satisfies (4.1) with \( \gamma > 0 \). Then we have
\[
\lim_{\lambda \downarrow 0} \lambda^{(d-1)/\gamma} N_\pm(\lambda) = \frac{1}{\Gamma(1 + (d - 1)/2)} \left( \frac{\Gamma(\gamma + 1)^{1/\gamma}}{4 \pi^{1/2}} \right)^{d-1} \int_{\partial \Omega} a_0(y)^{d-1} dS(y).
\]

Acknowledgements. The work on this article was initiated during the second author’s visit to the University of Bordeaux, France, in January 2014. He thanks this institution for hospitality.

Both authors gratefully acknowledge the partial support of the French Research Project ANR-2011-BS01019-01, of the Chilean Scientific Foundation Fondecyt under Grant 1130591, and of Núcleo Milenio de Física Matemática RC120002.

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