Abstract

The BPS D3 brane has a non-supersymmetric cousin, called the non-susy D3 brane, which is also a solution of type IIB string theory. The corresponding counterpart of black D3 brane is the ‘black’ non-susy D3 brane and like the BPS D3 brane, it also has a decoupling limit, where the decoupled geometry (in the case we are interested, this is asymptotically AdS$_5 \times S^5$) is the holographic dual of a non-conformal, non-supersymmetric QFT in (3 + 1)-dimensions. In this QFT we compute the entanglement entropy (EE), the complexity and the Fisher information metric holographically using the above mentioned geometry for spherical subsystems. The fidelity and the Fisher information metric have been calculated from the regularized extremal volume of the codimension one time slice of the bulk geometry using two different proposals in the literature. Although for AdS black hole both the proposals give identical results, the results differ for the non-supersymmetric background.
1 Introduction

In recent years, significant amount of work has been done to understand the gravity duals of certain measures of quantum information \[1\], namely, the entanglement entropy (EE) \[2–10\], the fidelity susceptibility or Fisher information metric \[11–17\], the Bures metric \[18\] and so on. The AdS/CFT correspondence \[19, 20\] appears to be the most useful tool for this purpose. The primary motivation in this came from the seminal work by Ryu and Takayanagi \[21, 22\], where they gave a proposal to quantify the EE holographically, particularly, in case of spacetimes with constant negative curvature. Results obtained using their holographic proposal matched exactly with those of the corresponding CFT duals in low dimensions. As it is quite well-known by now, the EE is a good way to measure the amount of quantum information of a bipartite system. One way to quantify this information is to calculate the von Neumann entropy of a bipartite system where the system is divided into two parts named, \(A\) and \(B\). The von Neumann entropy of part \(A\) is defined as

\[
S_A = -\text{Tr}(\rho_A \log \rho_A),
\]

where \(\rho_A = \text{Tr}_B(\rho_{\text{tot}})\) is the reduced density matrix on \(A\) obtained by tracing out on \(B\), the complement of \(A\), of the density matrix of the total system \(\rho_{\text{tot}}\). Holographically it can be computed from the Ryu-Takayanagi formula (as proposed by them) \[21, 22\]

\[
S_E = \frac{\text{Area}(\gamma_{\text{min}}^A)}{4G_N} \tag{1}
\]

where \(\gamma_{\text{min}}^A\) is the \(d\)-dimensional minimal area (time-sliced) surface in AdS\(_{d+2}\) space whose boundary matches with the boundary of the subsystem \(A\), i.e., \(\partial\gamma_{\text{min}}^A = \partial A\) and \(G_N\) is the \((d + 2)\)-dimensional Newton’s constant. As mentioned earlier, for lower spacetime dimensions (AdS\(_3\)/CFT\(_2\)) the corresponding results matched. Since then, this dual description has been checked for several cases and it’s regime of application has been extended to cases of higher dimensional and asymptotically AdS spacetimes \[23–25\]. For asymptotic AdS cases, one finds extra finite contributions to the EE other than that of the pure AdS spacetimes which has also been studied in details in several works. These terms are found to follow certain relations analogous to the thermodynamical relations called, the entanglement thermodynamics \[23, 24, 26, 27\].

Complexity is another measure of entanglement between quantum systems and a holographic definition in the context of eternal AdS black hole \[28\] was originally proposed by Susskind et. al. \[29–32\] through two different notions, one from the volume of the Einstein-Rosen bridge (ERB) connecting the two boundaries of the black hole and the other is by the action of its Wheeler-DeWitt patch as given below,

\[
C_V = \left(\frac{V(\gamma)}{R G_N}\right), \quad C_A = \frac{I_{WDW}}{\pi \hbar} \tag{2}
\]

where \(R\) is the AdS radius, \(V(\gamma)\) is the maximum volume of the co-dimension one bulk surface (time-sliced) bounded by the two boundaries of the black hole and \(I_{WDW}\) is the action of the Wheeler-DeWitt patch.

Motivated by Susskind et. al., another definition of holographic complexity has been proposed by Alishahiha \[33\] by the codimension one volume of the time-slice of the bulk geometry enclosed by the extremal codimension two Ryu-Takayanagi (RT) hypersurface used for the computation of holographic EE. This is usually referred to as the subregion complexity \[34, 35\] and the relation between these two notions has been clarified in some recent works given in \[36, 37\]. The subregion complexity, which we calculate in this paper, is defined in a very similar way as,

\[
C_V = \frac{V_{RT}(\gamma)}{8\pi R G_N}, \quad \text{where } V_{RT}\text{ denotes the}
\]
volume enclosed by the RT surface. This is closely related to quantum complexity, a concept borrowed from quantum information theory. Formally, complexity of a CFT state (corresponding to a particular quantum circuit) is defined as the minimum number of simple unitaries (or gates) required to prepare the state from a fixed reference state. So, if $| \psi(R) \rangle$ denotes the reference state and $| \psi(T) \rangle$ denotes the final target state, then complexity of $| \psi(T) \rangle$ with respect to the reference state would be the minimum number of quantum gates to form a single unitary $U(R,T)$ satisfying the equation,

$$| \psi(T) \rangle = U(R,T)| \psi(R) \rangle$$

(3)

with some tolerance or limit of how close one can reach to the target state. However, a clear field theory description of holographic complexity is not yet known (see, however, [40–42]). Using the geometric approach of Nielsen to the quantum circuit model [43,44], the complexity of only some free field theory states have been found to resemble the holographic complexity in [45–48]. But, for interacting field theory this is far from clear. Many different descriptions and proposals have been given in the literature to relate holographic complexity to the fidelity susceptibility or the quantum Fisher information metric, the Bures metric and so on [15–17]. In the same way as EE, complexity can be calculated perturbatively in case of asymptotically AdS solutions. One of the recent proposals relates the holographic complexity as the dual to quantum Fisher information metric [17]. It relates the regularized second order change of RT volume with respect to a perturbation parameter to the fidelity $F \sim (V^m_{RT} - V_{AdS})$, where $m$ is a perturbation parameter. The corresponding Fisher information metric has been proposed to have a form

$$G_{F,mm} = \partial^2_m F$$

(4)

In this paper, we work with the decoupled geometry of ‘black’ non-susy D3 brane solution of type IIB string theory [25,49–51]. This geometry is the gravity dual of a non-supersymmetric, non-conformal QFT in (3+1) dimensions at finite temperature which is also confining and has running coupling constant very much like QCD. As we will see the geometry is asymptotically AdS$_5$ which means that it can be thought of as some non-supersymmetric and non-conformal deformation of the CFT which is $\mathcal{N} = 4$, $D = 4$ SU($N$) super Yang-Mills theory at large $N$. We compute the EE, complexity and Fisher information metric holographically in this background for spherical subsystems. The goal of this study would be to gain better understanding of the various phases of QCD-like theories and the transitions among them since it is believed that the EE and the complexity are possibly related to some universal properties like order paramater or some renormalization group flow [45]. However, this will be more clear once we have a clearer picture of the holographic complexity in the (strongly coupled) interacting field theories.

The non-susy D3-brane solution is given by the following field configurations consisting of a Einstein-frame metric, a dilaton and a self-dual 5-form field strength,

$$ds^2 = F_1(\rho)^{-\frac{1}{2}} G(\rho)^{-\frac{1}{2}} \left[ -G(\rho)^{\frac{1}{2}} dt^2 + \sum_{i=1}^{3} (dx^i)^2 \right] + F_1(\rho)^{\frac{1}{2}} G(\rho)^{\frac{1}{2}} \left[ \frac{d\rho^2}{G(\rho)} + \rho^2 d\Omega_5^2 \right]$$

$$e^{2\phi} = G(\rho)^{-\frac{2\phi}{2\phi_0}} + \frac{2\phi_0}{\phi}, \quad F_5 = \frac{1}{\sqrt{2}} (1 + *) Q \text{Vol}(\Omega_5).$$

(5)
where the functions $G(\rho)$ and $F(\rho)$ are defined as,

$$G(\rho) = 1 + \frac{\rho_0^4}{\rho^4}, \quad F_1(\rho) = G(\rho) \frac{2}{\rho^2} \cosh^2 \theta - G(\rho) \frac{2}{\rho^2} \sinh^2 \theta$$

(6)

Here $\delta_1, \delta_2, \alpha_1, \beta_1, \theta, \rho_0, Q$ are the parameters characterizing the solution. The parameters satisfy $\alpha_1 = \beta_1 = (1/2) \sqrt{10 - (21/2)\delta_2^2 - (49/4)\delta_1^2 + 21\delta_1 \delta_2}$ and $Q = 2\alpha_1 \rho_0^3 \sinh 2\theta$. For simplicity, we put $\alpha_1 + \beta_1 = 2$ (in this case $S^5$ has a constant radius), implying $\alpha_1 = \beta_1 = 1$ and $\delta_1, \delta_2$ satisfy $42\delta_2^2 + 49\delta_1^2 - 84\delta_1 \delta_2 = 24$.

The decoupled geometry can be obtained by zooming into the region $\rho \sim \rho_0 \ll \rho_0 \cosh^2 \theta$ and the metric in (5) then reduces to the form (without the product $S^5$ part),

$$ds^2 = \frac{\rho^2}{R_1^2} G(\rho)^{1/4 - 2\delta_2} \left[ -G(\rho)^{\delta_2} \, dt^2 + \sum_{i=1}^3 (dx_i^2) \right] + \frac{R_1^2}{\rho^2} \frac{d\rho^2}{G(\rho)}$$

(7)

This geometry is asymptotically ($\rho \rightarrow \infty$) AdS$_5$ with radius $R_1 = \rho_0 \cosh \frac{\theta}{2}$ and so, it can be regarded as an excited state over AdS$_5$. The solution reduces to AdS$_5$ black hole once we take $\delta_2 = -2$ and $\delta_1 = -12/7$. In general, the non-susy solution can be shown [52] to have a temperature $T_{\text{nonsusy}} = (-\frac{\delta_2}{2})^{1/4} \frac{1}{\rho_0 \cosh \theta}$ (although there is a singularity at $\rho = 0$) which gives standard AdS$_5$ black hole temperature for $\delta_2 = -2$. In our earlier work [25] we computed the change in EE for this asymptotically AdS$_5$ state and obtained the entanglement thermodynamics up to first order (in perturbation parameter $m$), considering only a small strip type subsystem. In this paper we compute holographically the change in EE as well as the change in complexity in both the first and the second order for the small spherical subsystem. Here $m \equiv \frac{1}{z_0^2} \equiv \frac{\delta_2}{R_1^2}$ will be treated as the perturbation parameter we mentioned before. Also as noted earlier, from the second order change in complexity which is related to the second order change in regularized RT volume, we obtain the form of fidelity and Fisher information metric of the boundary QFT.

The rest of the paper is organized as follows. The holographic EE and the complexity computation of AdS$_5$ black hole for spherical subsystem have been reviewed in section 2. In section 3, we give the holographic EE and the holographic complexity for decoupled ‘black’ nonsusy D3 brane for spherical subsystem. Finally we conclude in section 4.

2 EE and complexity of AdS$_5$ black hole for spherical subsystem

The AdS$_5$ black hole geometry can be obtained from the decoupling limit of non-extremal D3-brane solution of type IIB string theory [53] and the metric has the form

$$ds^2 = \frac{R_1^2}{z^2} \left[ -\left(1 - \frac{z^4}{z_0^4}\right) \, dt^2 + \sum_{i=1}^3 (dx_i^2) + \frac{dz^2}{(1 - z^4)} \right]$$

(8)

where $z = z_0$ is the location of the horizon. Note that this metric reduces to AdS$_5$ metric for $z \rightarrow 0$ and therefore it is asymptotically AdS. Now replacing $\frac{1}{z_0}$ by $m$, the metric takes the form

$$ds^2 = \frac{R_1^2}{z^2} \left[ -(1 - mz^4) \, dt^2 + \sum_{i=1}^3 (dx_i^2) + \frac{dz^2}{(1 - mz^4)} \right]$$

(9)
In the following we give the metric by rewriting the three dimensional Euclidean part in spherical polar coordinates as,

\[ ds^2 = \frac{R_1^2}{z^2} \left[ -(1 - mz^4) \, dt^2 + \frac{dz^2}{(1 - mz^4)} + dr^2 + r^2 \, d\Omega_2^2 \right] \] (10)

This is very similar to the metric studied earlier in [23] and [33]. Here, we briefly review their calculation of holographic EE and complexity for the purpose of the calculation of the same in more general 'black' non-susy D3 brane background to be discussed in section 3. We also compute the second order change of both holographic EE and holographic subregion complexity. The subsystem in this case is given by a round ball \( r^2 = \sum_{i=1}^{3} x_i^2 \leq \ell^2 \) on the boundary. The embedding of the surface in the bulk is specified by \( r = r(z) \). The area of the embedded surface can be written as (we assume, \( m\ell^4 \ll 1 \), where \( m \) is related to the black hole horizon \( z_h \) by the relation \( m z_h^4 = 1 \)),

\[ A_{BH} = 4\pi R_1^3 \int_{z = \epsilon}^{L} \frac{dz}{r(z)} \sqrt{\frac{1}{1 - mz^4} + r'(z)^2} \] (11)

Here \( L \) is a parameter (the turning point of the RT surface) which is closely related to the radius of the sphere \( \ell \) (the turning point in case of pure AdS$_5$). The entanglement entropy would be calculated from the area (11) upto second order in \( m \). We try to find the functional form of \( r(z) \) by solving the Euler-Lagrange equation after expanding it upto second order in \( m \). For this we work with the ansatz

\[ r(z) = \sqrt{L^2 - z^2} + mr_1(z) + m^2 r_2(z) \] (12)

and solve the differential equation (obtained from the Euler-Lagrange equation for \( r(z) \)) of \( r_1(z) \) and \( r_2(z) \) perturbatively. The boundary condition we use is: \( \text{Lim}_{z \to L} r_{1,2}(z) = 0 \). We, therefore, first solve \( r_1(z) \) which has the following form,

\[ r_1(z) = \frac{2L^6 - z^4 (z^2 + L^2)}{10 \sqrt{(L^2 - z^2)}} \] (13)

Now using this, we can solve for \( r_2(z) \) which is of the form

\[ r_2(z) = \frac{\sqrt{L^2 - z^2} \left( 164L^8 + 380L^6 z^2 + 228L^4 z^4 + 328L^2 z^6 + 175 z^8 \right)}{4200} \] (14)

Using this form, we first get a relationship between \( \ell \) and \( L \), by taking \( r(z = 0) = \ell \) as,

\[ \ell = r(0) = L + \frac{L^5 m}{5} + \frac{58L^9 m^2}{525} \] (15)

and the inverse relation thus looks like

\[ L = \ell - \frac{m\ell^5}{5} + \frac{47\ell^9 m^2}{525} \] (16)

Then using (12) with derived versions of \( r_1(z) \) and \( r_2(z) \) in the area integral (11) and expanding it
upto the second order in $m$ we get the minimal area of RT surface as

\begin{align*}
A_{0(BH)} &= 4\pi R_1^4 \int_\epsilon^L dz \frac{\sqrt{L^2 - z^2}}{z^3} \\
A_{1(BH)} &= 4\pi R_1^3 m \int_\epsilon^L dz \frac{\sqrt{L^2 - z^2} \left(4L^4 + 2L^2 z^2 + 9z^4\right)}{10z^3} \\
A_{2(BH)} &= 4\pi R_1^3 m^2 \int_\epsilon^L dz \frac{\sqrt{L^2 - z^2}}{4200z^3} \left[1096L^8 + 548L^6 z^2 + 1608L^4 z^4 + 1256L^2 z^6 + 3367z^8\right]
\end{align*}

Note that here $A_{0,1,2}$ are not the true zero, first and second order forms of the area due to the appearance of $L$ which also has an expansion in $m$ due to the inverse relation given in [10]. So, to get the correct forms of area at different orders we evaluate the sum of the three integrals $A_{0(BH)} + A_{1(BH)} + A_{2(BH)}$ and then use the expansion of $L$. Finally, we put $\epsilon \to 0$. This way the zeroth order term will be the pure AdS result which is divergent, but the first and second order terms are finite and give the first and second order change in area. Using these the first and second order change in holographic entropy can be written in the form

\begin{equation}
\Delta S_{EE(BH)}^{(1)} = \frac{\pi R_1^4}{10G_5} m \ell^4
\end{equation}

and

\begin{equation}
\Delta S_{EE(BH)}^{(2)} = -\frac{\pi R_1^3}{525G_5} m^2 \ell^8.
\end{equation}

We note that (20) is precisely the relation obtained in [23] for the first order change in EE, on the other hand, (21) matches with the second order results obtained in [54].

Now we will extend this calculation to compute the holographic complexity for AdS$_5$ black hole. The volume integral here takes the form

\begin{equation}
V_{BH} = \frac{4\pi R_1^4}{3} \int_\epsilon^L dz \frac{r(z)^3}{z^4 \sqrt{1 - mz^2}}
\end{equation}

As before we replace $r(z)$ by the functional form given in (12) after using $r_1$ and $r_2$. Then we expand the integral upto the second order in $m$. The zeroth, the first and the second order terms of the integral are given respectively as,

\begin{align*}
V_{0(BH)} &= 4\pi R_1^4 \int_\epsilon^L dz \frac{(L^2 - z^2)^{3/2}}{z^4} = V_{\text{AdS}} \\
V_{1(BH)} &= \left(\frac{4\pi R_1^4}{15}\right) m \int_\epsilon^L dz \frac{\sqrt{L^2 - z^2}}{z^4} \left(3L^6 + L^2 z^4 - 4z^6\right) \\
V_{2(BH)} &= \left(\frac{4\pi R_1^4}{1050}\right) m^2 \int_\epsilon^L dz \frac{\sqrt{(L^2 - z^2)^{1/2}}}{z^4} \left(158L^{10} + 21L^8 z^2 + 67L^6 z^4 - 17L^4 z^6 + 9L^2 z^8 - 238z^{10}\right)
\end{align*}

Here again, the integrals are taken care of in the way mentioned previously in case of area integrals. In case of non-supersymmetric solution, we would not repeat this statement anymore. But the calculations

\footnote{Note that for the computation of the second order change in holographic EE, it is actually enough to work with only the first order change in embedding, i.e., to take $r(z)$ upto $r_1(z)$, but this is not the case for the computation of the second order change in subregion complexity. Here we necessarily have to take the second order change in embedding, i.e., we have to take $r(z)$ upto $r_2(z)$.}
are done in the same way. After all these steps one find that the first order change in volume in the
\( \epsilon \to 0 \) limit is zero. The second order change in complexity/volume can be obtained from the second
order change of regularized Ryu-Takayanagi volume which we write as,

\[
\Delta V_{(BH)}^{(2)} = \frac{4\pi R_1^4}{3} \left( \frac{3\pi}{1280} \right) (m\ell^4)^2
\]

and so the second order change in complexity for AdS\(_5\) black hole is given by,

\[
\Delta C_{V(BH)}^{(2)} = \frac{4\pi R_1^3}{24\pi G_5} \left( \frac{3\pi}{1280} \right) (m\ell^4)^2 = \frac{\pi R_1^3}{2560 G_5} (m\ell^4)^2
\]

Similar results have been obtained in [33] only for AdS\(_3\) and AdS\(_4\) black hole with spherical subsystem.
Now for spherical subsystem the change of energy is given as [24, 55, 56],

\[
\Delta E = \frac{4\pi}{3} \ell^3 \langle T_{tt} \rangle
\]

Using \( \langle T_{tt} \rangle = \frac{3R_1^3 m}{16\pi G_5} \), in (28) we get,

\[
\Delta E = \frac{R_1^3 m\ell^3}{4G_5}
\]

Thus the entanglement temperature computed from (21) and (29) comes out to be

\[
T_{ent(BH)} = \frac{5}{2\pi \ell}
\]

consistent with the results of [23]. The change in entanglement entropy can be written as \( \Delta S_{EE(BH)}^{(1)} = \Delta E/T_{ent(BH)} \), similarly, one can write \( \Delta C_{V(BH)}^{(2)} \sim (\Delta E/T_{ent(BH)})^2 \) as,

\[
\Delta C_{V(BH)}^{(2)} = \frac{5G_5}{128\pi R_1^3} \left( \frac{\Delta E}{T_{ent(BH)}} \right)^2
\]

Once we have the change in Ryu-Takayanagi volume (regularized) for an excited state, we can easily
compute the fidelity and from there obtain the Fisher information metric following the proposal of [17].

In general \( d + 1 \) dimensions the change in Ryu-Takayanagi volume and the corresponding fidelity are
given as,

\[
\Delta V^{(2)} = \frac{R_1^d \Omega_{d-2}}{d-1} \mathcal{A}_d (m\ell^d)^2
\]

\[
\mathcal{F} = \frac{\pi \frac{\pi}{2} (d-1)^2 \Gamma (d+1) \Gamma \left( d + \frac{3}{2} \right) R_1 \mathcal{A}_d}{G_{(d+1)} 2^{d+6} (d+1) \Gamma(d+3/2)} \Delta V^{(2)}
\]

where \( \mathcal{A}_d \) is a \( d \)-dependent constant. Now comparing the expression (32) for \( d = 4 \) with (26) we easily
identify \( \mathcal{A}_4 = \frac{4\pi}{1280} \). Then the fidelity for the AdS\(_5\) black hole can be calculated from (33) for \( d = 4 \) and (26) as,

\[
\mathcal{F}_{BH} = \frac{\pi}{525 G_5} R_1^4 m^2 \ell^8
\]

The corresponding Fisher information metric for the AdS\(_5\) black hole w.r.t \( \lambda = m\ell^4 \) therefore takes the
Here, we would like to remark that the definition of fidelity in [17] contains a $d$-dependent constant $\mathcal{A}_d$ in the denominator (see eq.(33)). This is deliberately chosen such that it precisely cancels the same constant in $\Delta V^{(2)}$ and the result coincides with that obtained from the relative entropy calculation given in [14,54] for AdS black hole. However, we can directly calculate the Fisher information from the second order change in EE using the definition given in [54] as,

$$G_{FBH,\lambda} = \frac{d^2}{d\lambda^2} (\Delta H - \Delta S_{EE(BH)}) = -\frac{2}{\lambda^2} \Delta S_{EE(BH)}^{(2)}.$$

In the above, $\lambda (= m\ell^4)$ is the perturbation parameter and $\Delta H$ is the change of modular Hamiltonian, which is defined in information theory as $H = - \log \rho$, where $\rho$ is the density matrix. One can show that in case of spherical subregion, $\Delta H$ is only first order in $\lambda$ (all higher order changes are zero) and equal to $\Delta E$. Using this fact and the above identity, we find a relation between Fisher Information and the change in volume in general dimension (given in our recent paper [57]) as,

$$G_{FBH,d,\lambda} = \frac{(d-1)^2 \Gamma(d+1) \Gamma\left(\frac{d}{2} - 1\right)}{8G_{d+1}\lambda^2 R_1 \Gamma\left(d + \frac{3}{2}\right) \Gamma\left(\frac{d+1}{2}\right)} \Delta V^{(2)}_{(BH),d}.$$

For $d = 4$, this relation reduces to,

$$G_{FBH,\lambda} = \frac{128}{105\pi R_1^2 G_5} \Delta V^{(2)}_{(BH)} = \frac{2\pi}{525G_5} R_1^3,$$

where we have used [26] for $\Delta V^{(2)}$. This precisely matches with the expression we obtained in (35) according to the definition used in [17]. We will however find that these two ways of calculating the Fisher information metric yield different results for the decoupled geometry of ‘black’ non-susy D3 brane to which we turn in the next section.

### 3 EE and complexity for (decoupled) ‘black’ non-susy D3 brane in case of spherical subsystem

The decoupled geometry of ‘black’ non-susy D3 brane is given in eq.(7). To compare our results with those of the previous section we will not directly use this geometry, but instead try to recast the solution in a form very similar to the AdS$_5$ black hole geometry. For this purpose we first make a coordinate transformation $\tilde{\rho}_4 = \rho_4 + \rho_0^4$. Then we make another coordinate transformation by taking $\tilde{\rho} = \frac{\rho_4}{z^2}$. With these transformations the decoupled geometry of non-susy ‘black’ D3 brane (7) takes the form

$$ds^2 = \frac{R_1^2}{z^2}\left[-(1 - mz^4)^{\frac{1}{2}} - \frac{2mz^2}{x^2} dt^2 + (1 - mz^4)^{\frac{1}{2}} + \frac{2mz^2}{x^2} \sum_{i=1}^{3} (dx_i)^2 + \frac{dz^2}{1 - mz^4}\right].$$

where $m = \frac{1}{z_0}$ and $z_0 = \frac{R_1^2}{m}$. To compute the entanglement entropy, the complexity and the associated Fisher information metric for the decoupled geometry of ‘black’ non-susy D3 brane we will use the metric given in (39) with choice of a spherical subsystem. We remark that as the Ryu-Takayanagi area or the
volume formula uses the Einstein-frame metric, we also use the Einstein frame metric for the decoupled ‘black’ non-susy D3 brane geometry. This, in turn, takes into account that we have a non-trivial dilaton in the background. The area integral, after taking the embedding $r = r(z)$, in this case takes the form,

$$A_{nsD3} = 4\pi R_3^3 \int dz \frac{r(z)^2(1 - mz^4)\frac{dr}{dz} + \frac{1}{2}}{z^3} \left[ 1 + (1 - mz^4)\frac{dr}{dz} + \frac{1}{2} \right]^{\frac{1}{2}} \quad (40)$$

Again, as before, we are assuming the small subsystem and consider up to the second order change in the metric. To minimize this area, we use Euler-Lagrange equation of motion once we consider the area as action integral. The equation of motion is a bit long and so we do not write it explicitly here. We just give its solution. As mentioned earlier, we know that by taking $m = 0$, we can get back the pure AdS case. Thus we again take our solution as a perturbation over pure AdS and as before work with the ansatz

$$r(z) = \sqrt{L^2 - z^2} + m r_1(z) + m^2 r_2(z). \quad (41)$$

Now solving the equation of motion with this ansatz, and with proper boundary conditions and regularity conditions (similar to the AdS black hole case discussed in the previous section), we get $r_1(z)$ and $r_2(z)$ to be of the form

$$r_1(z) = \frac{1}{80} \sqrt{L^2 - z^2} \left[ (10 - 3\delta_2)L^4 + (10 - 3\delta_2)L^2 z^2 + (\delta_2 + 10)z^4 \right],$$

$$r_2(z) = \frac{1}{806400} \sqrt{L^2 - z^2} \left[ (\delta_2(5207\delta_2 - 18900) + 30460)L^8 + 8(\delta_2(58\delta_2 - 189) + 302)L^6 z^2 + 3(\delta_2(683\delta_2 - 2100) + 7660)L^4 z^4 + 8(\delta_2(263\delta_2 - 1260) + 4300)L^2 z^6 + 175(\delta_2 + 10)(\delta_2 + 26)z^8 \right] \quad (42)$$

As before, we can now use this form of $r(z)$ to get the relation between $\ell$ and $L$, but, what we really need is the inverse of that. This comes out as,

$$L = \ell + \frac{1}{80} m \ell^5 (3\delta_2 - 10) + m^2 \ell^9 \frac{(463\delta_2^2 - 18900\delta_2 + 32540)}{806400} \quad (43)$$

Using the form of $r(z)$ along with (42) in the area integral (40) and then expanding the integral in the second order in $m$ in the way we mentioned before, we perform the integrals up to order $m^2$.

After performing the integral (as done before) and replacing $L$ by (43), we get the first and second order change of EE with respect to $m$ as,

$$\Delta S^{(1)}_{EE(nsD3)} = -\frac{\pi \delta_2 R_1^3}{20 G_5} m \ell^4 \quad (44)$$

$$\Delta S^{(2)}_{EE(nsD3)} = \frac{(\delta_2^2 - 10) \pi R_1^3}{3150 G_5} m^2 \ell^8 \quad (45)$$

Note that both of these match precisely with the change in EE we obtained for AdS black hole in (20) and (21) once we put $\delta_2 = -2$ and provides a consistency check of our result (44) and (45).

Now to compute the complexity we have to find the RT volume from the geometry given in (39). The
volume integral has the form,

\[ V_{nsD3} = \frac{4\pi R_1^4}{3} \int_{\epsilon}^{L} \frac{dz}{z^4} T(z)^3 \left( 1 - \frac{\delta_2}{8} mz^4 \right) \left( \frac{\pi^2}{6} - \frac{\pi^2}{9} \right) \]  

(46)

Putting the functional form of \( r(z) \) and expanding up to second order in \( m \), we get the integrals upto second order.

Evaluating these integrals and taking \( \epsilon \to 0 \) limit, we find that the change of complexity upto first order in \( m \) is zero similar to the case of AdS\(_5\) black hole. On the other hand, the change of complexity in the second order in \( m \) is

\[ \Delta C^{(2)}_{V_{nsD3}} = \frac{4\pi R_1^2}{24\pi G_5} \left( \frac{\pi (60 - 9\delta_2^2)}{10240} \right) (m\ell^4)^2 = \frac{\Delta V_{V_{nsD3}}^{(2)}}{8\pi R_1 G_5} \]  

(47)

This can be seen to match with the change in AdS\(_5\) black hole complexity given in (27) once we take \( \delta_2 = -2 \). Now using the \( \langle T_{tt} \rangle \) calculated in [25], the change in energy for the non-susy geometry can be obtained as

\[ \langle T_{tt} \rangle = -\frac{3R_3^2}{32\pi G_5}, \quad \Delta E = \frac{4\pi \ell^3}{3} \langle T_{tt} \rangle = -\frac{\delta_2 R_3^2 m\ell^3}{8G_5}. \]  

(48)

Thus we see that again we can write the change in EE in the form \( \Delta S^{(1)}_{EE_{nsD3}} = \frac{\Delta E}{T_{ent_{nsD3}}} \), where \( T_{ent_{nsD3}} = \frac{\delta_2}{m\ell^4} \). Note that the entanglement temperature remains the same as for the AdS\(_5\) black hole. Similarly, we can express the change in complexity in (47) as,

\[ \Delta C^{(2)}_{V_{nsD3}} = \frac{5}{256\pi \delta_2^2 R_1^4} \left( \frac{\Delta E}{T_{ent_{nsD3}}} \right)^2 \]  

(49)

We also compute the fidelity and Fisher information metric for the non-susy geometry. Comparing this with the general expression of change of volume [32], we identify the \( d \)-dependent constant \( A_4 \) and fidelity in this case as,

\[ A_4 = \frac{\pi (60 - 9\delta_2^2)}{10240}, \quad F_{nsD3} = \frac{\pi}{525G_5} R_1^3 m^2 \ell^8 \]  

(50)

The corresponding Fisher information metric has the form

\[ G_{F_{nsD3},\lambda} = \partial^2_{\lambda} F_{nsD3} = \frac{2\pi}{525G_5} R_1^3 \]  

(51)

Interestingly, here we observe that both the fidelity and the Fisher information metric do not depend on the non-susy parameter \( \delta_2 \) and by comparison we see that they have exactly the same value as those of the AdS\(_5\) black hole. But this is due to the choice of the \( d \)-dependent constant \( A_d \) in the denominator of the fidelity used in [17]. Next, we consider the direct way of calculating the Fisher information metric as discussed in the previous section. Using the definition given in (36) and also the relation for \( \Delta V_{V_{nsD3}}^{(2)} \) in (47) we get,

\[ G_{F_{nsD3},\lambda} = -\frac{2}{\lambda^2} \Delta S^{(2)}_{EE_{nsD3}} = \frac{512}{105\pi (60 - 9\delta_2^2)} \pi R_1 \lambda^2 G_5 \Delta V_{V_{nsD3}}^{(2)} \]  

(52)
Here we find that the Fisher information metric indeed depends on the non-supersymmetric parameter $\delta_2$ which at $\delta_2 = -2$ gives back the AdS black hole result $G_{FBH,\lambda}$. It is, therefore, clear that the definition of fidelity, used in [17] which contains the $d$-dependent constant $A_d$ needed to get the correct AdS black hole result for Fisher information metric, does not produce the correct result for the non-supersymmetric background. This calculation gives the Fisher information metric which is independent of the non-supersymmetric parameter $\delta_2$ and has precisely the same value as that of the AdS$_5$ black hole. However, a direct way of calculating the Fisher information metric given in [54], yields a different result and in this case it depends on the non-supersymmetric parameter $\delta_2$ as expected and for $\delta_2 = -2$, it gives the correct AdS$_5$ black hole result. We observe from (52) that the decoupled ‘black’ non-susy D3 brane geometry stores more quantum Fisher information than its AdS$_5$ black hole counterpart. For $\delta_2 = 0$ which corresponds to the zero temperature non-susy solution, in fact, stores the most quantum Fisher information, whereas for $\delta_2 = -2$, which corresponds to the AdS$_5$ black hole stores the least.

4 Conclusion

To conclude, in this paper we have holographically computed the EE and the complexity of the QFT whose gravity dual is given by the decoupled geometry of ‘black’ non-susy D3 brane of type IIB string theory for spherical subsystems. We start with a brief review of the computation of holographic EE and the complexity of the AdS$_5$ black hole for spherical subregion done before (complexity was done explicitly for AdS$_3$ and AdS$_4$ in [33]) and then compute the entanglement entropy and subregion complexity for the decoupled ‘black’ non-susy D3 brane geometry up to the second order in perturbation parameter using the prescription of Ryu and Takayanagi. We have extended our calculation of complexity to compute the fidelity and the Fisher information metric using the definition given earlier [17] for both the AdS$_5$ black hole and the decoupled ‘black’ non-susy D3 brane geometry for spherical subsystem. Since the decoupled geometry of ‘black’ non-susy D3 brane reduces to the standard AdS$_5$ black hole when its parameter $\delta_2$ takes value $-2$, we have observed that both the EE and the complexity for the former geometry indeed reduce to those of the AdS$_5$ black hole when we put $\delta_2 = -2$, giving a consistency check of our results.

We have also checked the entanglement thermodynamics to be consistent for the spherical subsystem and gives the same entanglement temperature as the AdS$_5$ black hole. We have further observed that although the fidelity and the Fisher information metric of the QFT dual to decoupled ‘black’ non-susy D3 brane geometry remain the same as those of the AdS$_5$ black hole when one uses proposal of [17], using a more exact relation [54] without any arbitrary constant gives us a different value of Fisher information in case of the non-supersymmetric solution, which is parameter dependent. Putting the right parameter value gives back the AdS$_5$ black hole result, indicating (52) is a more general relation which includes the AdS$_5$ black hole relation (37) as well.

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