MACROSCOPIC LIMIT OF THE KINETIC BLOCH EQUATION

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Abstract. This work concerns the existence of solution of the kinetic spinor Boltzmann equation as well as the asymptotic behavior of such solution when $\varepsilon \to 0$, that is when the time relaxation of the spin-flip collisions is very small in comparison to the time relaxation parameter of the collisions with no spin reversal. Due to the lack of regularity of the weak solution, the switching term $H_\varepsilon \times M_\varepsilon$ is not stable under the weak convergences. Hence we establish new estimates of the solutions in a weighted Sobolev space of order 3.

1. Introduction. The kinetic equation called the spinor Boltzmann equation, describes the change in the distribution function of noninteracting electrons in a metal, including the interaction of their spins with the magnetic field. In this model, the distribution $F = F(t, x, v)$ is a $2 \times 2$ hermitian matrix which obeys to the following PDE (see [5, 9, 10] for example for physical details)

$$
\frac{\partial}{\partial t} F + (v \cdot \nabla_x) F - \frac{e}{c\hbar} \left((E + v \times H) \cdot \nabla_v\right) F - \frac{i\mu}{\hbar} \{H; \sigma; F\} + (\frac{\partial F}{\partial t})_{\text{coll}} = 0,
$$

(1)

for $(t, x, v) \in (0, T) \times \mathbb{R}^3 \times \mathbb{R}^3$ where $T > 0$, $t$ is the time, $x$ is the position and $v$ is the velocity. In this equation, the point $\cdot$ between two vectors stands for the scalar product in $\mathbb{R}^3$ and the cross $\times$ for the vectorial product, the physical constants $\mu$, $e$, $c$ and $\hbar$ are respectively the magnetic moment, the electron charge, the velocity of light and the Planck constant. $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the triple of Pauli matrices given by

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
$$

where $i^2 = -1$. The product $v \cdot \sigma$ for a vector $v \in \mathbb{C}^3$ is defined as the matrix $v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3$, whereas $[A; B] = AB - BA$ is the Poisson bracket between matrices $A, B$.

In this paper, the electric field $E$ is neglected and the magnetic field $H$ satisfies

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the magnetostatic equations \( \text{div} \, B = -\text{div} \, H_e \) \( \text{curl} \, H = 0 \) where \( B = H + M \) is the magnetic induction and \( H_e \) is the given applied magnetic field. The magnetization field \( M \) is defined by

\[
M_i(t, x) = \frac{\mu}{8\pi^3} \int_{\mathbb{R}^3} \text{Tr} \left( \sigma_i F \right) dv, \quad i = 1, 2, 3,
\]

where \( \text{Tr} (A) \) is the trace of the matrix \( A \). The properties of the Pauli matrices allow to write \( F \) in the form

\[
F = fI + f \cdot \sigma,
\]

where \( I \) is the unit matrix, \( f = \frac{1}{2} \text{Tr} F \) and \( f = (f_1, f_2, f_3) \), \( f_i = \frac{1}{2} \text{Tr} (\sigma_i F) \), \( i = 1, 2, 3 \). The scalar function \( f \) represents the charge distribution and the vector valued function \( f \) represents the spin-vector part of the distribution \( F \). With this decomposition of \( F \), \( M \) can be rewritten as

\[
M(t, x) = \gamma \int_{\mathbb{R}^3} f(t, x, v) \, dv,
\]

with \( \gamma = \frac{\mu}{4\pi} \) and using the relation

\[
[v \cdot \sigma, w \cdot \sigma] = 2i (v \times w) \cdot \sigma, \quad v, w \in \mathbb{C}^3,
\]

we see that the scalar function \( f \) satisfies the following kinetic equation

\[
\partial_t f + v \cdot \nabla_x f - \alpha (v \times H) \cdot \nabla_v f + (\partial f/\partial t)_{\text{coll}} = 0,
\]

and the vector valued function \( f \) satisfies the following one

\[
\partial_t f + (v \cdot \nabla_x) f - \alpha ((v \times H) \cdot \nabla_v) f + \beta H \times f + (\partial f/\partial t)_{\text{coll}} = 0,
\]

with \( \alpha = \frac{e \hbar}{2m} \) and \( \beta = \frac{2\mu}{\hbar} \).

In [1, 7] for example, the authors established existence of solutions to kinetic equations for spin polarized Fermi system, in the matrix framework. Since the aim of this paper is to determine the equation satisfied by the magnetization \( M \) when some parameters tend to 0, we restrict ourselves to the study of the equation (4) satisfied by the vectorial function \( f \). The collision operator in this equation takes the form

\[
(\partial f/\partial t)_{\text{coll}} = (\partial f/\partial t)_{\tau, f} + (\partial f/\partial t)_{\tau, sf},
\]

where \( (\partial f/\partial t)_{\tau, f} \) represents the change in \( f \) due to the momentum-changing collisions, but with no spin reversal and \( (\partial f/\partial t)_{\tau, sf} \) represents only spin-flip collisions, \( \tau > 0 \) and \( \tau_{sf} > 0 \) being the corresponding relaxation times. We assume that \( \tau_{sf} < \tau \) and introduce the small parameter \( \varepsilon = \tau_{sf}/\tau \), the collision operators are then defined by (see [5])

\[
(\partial f/\partial t)_{\tau} = \frac{\gamma f - \chi_0 \psi H}{\tau}, \quad (\partial f/\partial t)_{\tau, f} = \frac{\gamma f - \psi M}{\tau \varepsilon},
\]

where \( \chi_0 > 0 \) is a constant and \( \psi \) is the normalized Maxwellian

\[
\psi(v) = (2\pi)^{-3/2} e^{-|v|^2}, \quad v \in \mathbb{R}^3.
\]
Finally, we shall discuss in this paper the system labeled problem \((P_ε)\) coupling the kinetic equation
\[
\partial_t f_ε + (v \cdot \nabla_x) f_ε - \alpha ((v \times H_ε) \cdot \nabla_v) f_ε + \beta H_ε \times f_ε + \frac{1}{\tau} (\gamma f_ε - \chi_0 \psi H_ε) + \frac{1}{\tau ε} (\gamma f_ε - \psi M_ε) = 0, \quad (t, x, v) \in (0, T) \times \mathbb{R}^3 \times \mathbb{R}^3, \tag{6}
\]
and the magnetostatic equations
\[
\operatorname{div} (H_ε + M_ε) = -\operatorname{div} H_ε, \quad \operatorname{curl} H_ε = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^3, \tag{7}
\]
where
\[
M_ε(t, x) = \gamma \int_{\mathbb{R}^3} f_ε(t, x, v) \, dv, \tag{8}
\]
and the initial data \(f_0\) and the applied magnetic field \(H_ε\) are independent of the parameter \(ε\).

The macroscopic equation satisfied by the magnetization \(M_ε\) writes as
\[
\partial_t M_ε + \nabla_x \cdot J_ε + \beta H_ε \times M_ε + \frac{\gamma}{\tau} (M_ε - \chi_0 H_ε) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^3, \tag{9}
\]
\[
J_ε(0, x) = M_0(x) := \gamma \int_{\mathbb{R}^3} f_0(x, v) \, dv, \quad x \in \mathbb{R}^3,
\]
\(J_ε\) being the associated current matrix defined by
\[
J_ε(t, x) = \gamma \int_{\mathbb{R}^3} v \otimes f_ε(t, x, v) \, dv, \tag{10}
\]
where the symbol \(\otimes\) stands for the tensor product of vectors in \(\mathbb{R}^3\) and the derivative \(\nabla_x \cdot J_ε\) is the vector of components \(\nabla_x \cdot J_{εi}\) where \(J_{εi} = \gamma \int v_i f_ε \, dv, i = 1, 2, 3\).

Equations (7) and (9) form the problem labeled \((B_ε)\) and the aim of this work is to determine the limiting problem when \(ε \to 0\). In section 3, a first response is given, see Proposition 3, based only on the \(L^2\)-estimates of the solutions \((f_ε, H_ε)\) of problem \((P_ε)\). Unfortunately, the limit of the cross product \(H_ε \times M_ε\) reveals a concentration effect that we cannot characterize. Therefore in the rest of the paper, we will concentrate on regular solutions to problem \((P_ε)\) which are local in time and for technical reasons related to Sobolev embeddings, the \(H^3\)-regularity of the fields is required to avoid successfully the concentration effect.

Before stating our main results, let us precise some notations. For \(1 \leq p \leq \infty\), let \(L^p\) denote the Lebesgue spaces \(L^p(\mathbb{R}^3)\) or \(L^p(\mathbb{R}^3 \times \mathbb{R}^3)\) (of scalar functions or vectorial fields) with norms denoted \(\| \cdot \|_p\) or simply \(\| \cdot \|\) if \(p = 2\) while \(H^s\) stands for the usual Sobolev spaces of order \(s \in \mathbb{R}\) (of scalar functions or vectorial fields). Sometimes, we will precise the variable under consideration by writing \(L^p_{\nu} \) or \(H^s_{\nu}\).

For a general Banach space \(V\), the symbol \(\| \cdot \|_V\) stands for the norm and the bracket notation \(\langle \cdot , \cdot \rangle_{V' \times V}\) (or simply \(\langle \cdot , \cdot \rangle\) if no confusion arises) will be reserved for pairings between \(V\) and its dual \(V'\). We will also make use of the well known spaces \(L^p(0, T; V), H^s(0, T; V)\) and \(C([0, T]; V)\).

Then we define some weighted Hilbert spaces related to the velocity-dependent function \(ψ = (2π)^{-3/2} e^{-\frac{x^2}{2}}\). First we set
\[
L^2_ψ = \left\{ g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \text{ measurable} : \int \frac{|g(x, v)|^2}{ψ} \, dx dv < \infty \right\}, \tag{11}
\]
equipped with the scalar product and the norm
\[(u, g)_{L^2_\psi} = \int \frac{u \cdot g}{\psi} \, dx \, dv, \quad \|g\|_{L^2_\psi} = \left( \int \frac{|g(x, v)|^2}{\psi} \, dx \, dv \right)^{1/2}. \tag{12}\]

Here and from now on, an integral without a precise domain is taken over the whole space which can be either \(\mathbb{R}^3 \times \mathbb{R}^3\) or \(\mathbb{R}^3\).

To avoid any confusion, we denote for \(\delta \in \mathbb{N}^3\) by \(\partial^\delta\) the derivative \(\partial^\delta\) with respect to the \(x\)-variable and by \(D^\delta\) the derivative \(\partial^\delta\) with respect to the \(v\)-variable. Then for \(m \in \mathbb{N}^*\), we define the weighted Hilbert space
\[\mathbb{H}^m_\psi = \left\{ g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \text{ measurable; } \partial^\delta D^\psi g \in L^2_\psi, \text{ for } |\delta| + |\psi| \leq m \right\}, \tag{13}\]
endowed with the norm
\[\|g\|_{\mathbb{H}^m_\psi} = \left( \sum_{|\delta| + |\psi| \leq m} \|\partial^\delta D^\psi g\|_{L^2_\psi}^2 \right)^{1/2}, \tag{14}\]
and we set
\[\mathbb{H}^0_\psi = L^2_\psi, \quad \mathbb{H}^{-m}_\psi = (\mathbb{H}^m_\psi)' \tag{15}\]

In the sequel, \(C > 0\) denotes various constants which depend on the parameters appearing in the equations. Sometimes we denote by \(C_\nu\) or \(C_\varepsilon\) positive constants depending on the terms indicated as subscripts.

Our main results are the following. Theorem 1.1 gives the existence result of a regular but local in time solution of problem \((P_\varepsilon)\) for \(\varepsilon > 0\) fixed and Theorem 1.2 the convergence when \(\varepsilon\) goes to 0 in problem \((B_\varepsilon)\).

**Theorem 1.1.** Assume \(f_0 \in \mathbb{H}^3_\psi\) and \(H_\varepsilon \in L^2(0,T^*;H^3_\varepsilon)\). Then there exists \(T^* \in [0,T]\) such that for all \(\varepsilon > 0\), problem \((P_\varepsilon)\) admits a local solution \((f_\varepsilon, H_\varepsilon)\) defined on \([0,T^*]\) such that
\[f_\varepsilon \in L^\infty(0,T^*;H^3_\psi), \quad H_\varepsilon \in L^2(0,T^*;H^3), \tag{16}\]
\[M_\varepsilon = \gamma \int f_\varepsilon \, dv \in L^\infty(0,T^*;H^3) \cap H^1(0,T^*;H^2), \quad J_\varepsilon = \gamma \int v \otimes f_\varepsilon \, dv \in L^\infty(0,T^*;H^2).\]

Moreover the following estimates are satisfied
\[\|f_\varepsilon\|_{L^\infty(0,T^*;H^3)} + \|M_\varepsilon\|_{L^\infty(0,T^*;H^3)} + \|H_\varepsilon\|_{L^2(0,T^*;H^3)} \leq C^*, \tag{17}\]
\[\|H_\varepsilon \times M_\varepsilon\|_{L^2(0,T^*;H^2)} + \|J_\varepsilon\|_{L^\infty(0,T^*;H^2)} + \|\partial_t M_\varepsilon\|_{L^2(0,T^*;H^2)} \leq C^*, \tag{18}\]
where \(C^* > 0\) depends upon \(f_0\) and \(H_\varepsilon\) but not of \(\varepsilon\).

**Theorem 1.2.** Under assumptions of Theorem 1.1, there exists a subsequence still labeled \((M_\varepsilon, H_\varepsilon)\) and
\[(M, H) \in (L^\infty(0, T^*; H^3) \cap H^1(0, T^*; H^2)) \times L^2(0, T^*; H^3),\]
such that when $\varepsilon \to 0$, we get the limits
\[
\begin{align*}
M_\varepsilon & \to M \quad \text{weakly} - \ast \text{ in } L^\infty(0,T^*;H^3), \\
\partial_t M_\varepsilon & \to \partial_t M \quad \text{weakly in } L^2(0,T^*;H^2), \\
M_\varepsilon & \to M \quad \text{strongly in } L^2(0,T^*;H^2_{loc}), \\
H_\varepsilon & \to H \quad \text{weakly in } L^2(0,T^*;H^3).
\end{align*}
\]  
Moreover
\[
\begin{align*}
M_\varepsilon \times H_\varepsilon & \to M \times H \quad \text{weakly in } L^2(0,T^*;H^2), \\
J_\varepsilon & \to 0 \quad \text{weakly} - \ast \text{ in } L^\infty(0,T^*;H^3),
\end{align*}
\]  
and $(M, H)$ satisfies the coupling of Bloch equation and magnetostatic equations
\[
\begin{align*}
\partial_t M + \beta H \times M + \frac{\gamma}{2}(M - \chi_0 H) &= 0 \quad \text{in } (0,T^*) \times \mathbb{R}^3, \\
M(0) &= M_0 := \gamma \int f_0 \, dv \quad \text{in } \mathbb{R}^3, \\
\text{div}(H + M) &= -\text{div} H_\varepsilon, \quad \text{curl} H = 0 \quad \text{in } (0,T^*) \times \mathbb{R}^3.
\end{align*}
\]  

The rest of the paper is organized as follows. Section 2 is devoted to some preliminary results, this preparation will simplify the rest of the presentation. In Section 3, we establish some formal $L^2$-estimates satisfied by any weak solution of problem $(\mathcal{P}_\varepsilon)$ and perform the limit when $\varepsilon \to 0$ in problem $(\mathcal{B}_\varepsilon)$. When dealing with the convergence of the nonlinear term $M_\varepsilon \times H_\varepsilon$, the lack of uniform regularity on the fields may induce a concentration effect that we cannot precise, see Proposition 3. To overcome this difficulty, we will consider strong solutions of the problem to obtain the desired Bloch equation of (22). In section 4, we provide for $\varepsilon > 0$ fixed, a result of existence and uniqueness of a global-in-time solution to problem $(\mathcal{P}_\varepsilon)$, stated in Theorem 4.1. The proof is based on a regularization method: we define a regularized problem of $(\mathcal{P}_\varepsilon)$ labeled $(\mathcal{Q}_\nu')$ depending on a small parameter $\nu > 0$, that we solve by using an iterative method, then we pass to the limit as $n \to \infty$ to obtain a solution $(f_\nu', H_\nu')$ of problem $(\mathcal{Q}_\nu')$. Section 5 is devoted to prove a regularity result on the solution $(f_\nu', H_\nu')$: assuming the data $f_0$ and $H_\varepsilon$ to verify hypotheses of Theorem 1.1, we prove that $(f_\nu', H_\nu')$ has the regularity $H^3_{\psi} \times H^3$ and satisfies uniform estimates with respect to $\nu$ in this space, locally-in-time. This allows to prove Theorem 1.1 in Section 6 by letting $\nu \to 0$. Finally in Section 7, we perform the limit when $\varepsilon \to 0$ and achieve the proof of Theorem 1.2.

2. Preliminary results. We collect some preliminary results to prepare the framework. We begin by some properties on the Hilbert spaces $H^m_{\psi}$, then we analyze the magnetostatic equations and finally, we prove entropy inequalities that will be used to establish estimates satisfied by the solutions $(f_\varepsilon, H_\varepsilon)$ of problem $(\mathcal{P}_\varepsilon)$. In this section, for simplicity we momentarily ignore the dependance of the functions upon the time-variable $t$, the extension to time-dependent functions can be easily deduced.

2.1. The weighted spaces. We consider the Hilbert weighted spaces $L^1_{\psi}$ and $H^m_{\psi}$ $(m \geq 1)$ defined by (11) and (13) respectively. We recall that $\psi(v) = (2\pi)^{-3/2} e^{-\frac{|v|^2}{2}}$ and we denote by $D(\mathbb{R}^3 \times \mathbb{R}^3)$, the space of infinitely differentiable functions with compact support in $\mathbb{R}^3 \times \mathbb{R}^3$. Let us verify the following properties.
Lemma 2.1. $\mathbb{H}^m_\psi \subset L^2_\psi \subset L^2$ continuously and
\[ \|g\| \leq \|g\|_{L^2_\psi}, \quad \forall g \in L^2_\psi. \] (23)
Moreover $\mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^3)$ is dense into $L^2_\psi$ and $\mathbb{H}^m_\psi$. Therefore identifying $L^2_\psi$ with its dual, we get in particular $\mathbb{H}^m_\psi \subset L^2_\psi \subset \mathbb{H}^m_\psi$ continuously and densely.

Proof. The first property is trivial, let us prove the following results.

1. The density of $\mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^3)$ in $L^2_\psi$: let $g \in L^2_\psi$ then $h = \frac{g}{\sqrt{\psi}} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ and since $\mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^3)$ is dense in $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, there exists a sequence $(h_n)_n \subset \mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^3)$ such that $h_n \to h$ strongly in $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$. Let $g_n = \sqrt{\psi} h_n$, since $\sqrt{\psi} \in C^\infty(\mathbb{R}^3)$ then $(g_n)_n \subset \mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^3)$ and $g_n \to g$ strongly in $L^2_\psi$.

2. The density of $\mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^3)$ in $\mathbb{H}^m_\psi$: first, every function $g \in \mathbb{H}^m_\psi$ can be approximated by a sequence of functions $(g_n)_n \subset \mathbb{H}^m_\psi$ with compact support. Indeed we can take $g_n = c_n g$ where $(c_n)_n$ is a truncation sequence with respect to $x$ and $v$ and we easily verify that $g_n \to g$ strongly in $\mathbb{H}^m_\psi$. Now if $h \in \mathbb{H}^m_\psi$ is compactly supported, then $h$ can be approximated by a sequence of functions $(h_n)_n \subset \mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^3)$ for the convergence in $\mathbb{H}^m_\psi$. Indeed $k = \frac{h}{\sqrt{\psi}} \in H^m(\mathbb{R}^3 \times \mathbb{R}^3)$ so there exists a sequence $(k_n)_n \subset \mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^3)$ such that $k_n \to k$ in $H^m(\mathbb{R}^3 \times \mathbb{R}^3)$. One can take $k_n = \rho_n * k$ where $\rho_n = \rho_n(x,v)$ is a regularizing sequence, so that all the supports of the functions $k_n$ are contained in the same compact independent of $n$. Setting $h_n = \sqrt{\psi} k_n$, we see that $h_n \to h$ in $\mathbb{H}^m_\psi$.

$\square$

2.2. The perturbed magnetostatic equations. Let $M, H_e \in L^2(\mathbb{R}^3)$, we consider the magnetostatic equations (7) that we rewrite as
\[ H = \nabla \varphi, \quad -\Delta \varphi = \text{div} (M + H_e). \]

In order to solve the elliptic equation of $\varphi$ in $H^1(\mathbb{R}^3)$ by Lax-Milgram theorem, we introduce a perturbation using a small parameter $\nu > 0$ and we consider the perturbed magnetostatic equations
\[ H = \nabla \varphi, \quad -\Delta \varphi + \nu \varphi = \text{div} (M + H_e). \] (24)

Therefore there exists a unique solution $\varphi \in H^1(\mathbb{R}^3)$ obtained by solving the weak formulation
\[ \int \nabla \varphi \cdot \nabla \xi \, dx + \nu \int \varphi \xi \, dx = -\int (M + H_e) \cdot \nabla \xi \, dx, \quad \forall \xi \in H^1(\mathbb{R}^3). \] (25)

Taking $\xi = \varphi$ in (25), we get
\[ \|\nabla \varphi\|^2 + \nu \|\varphi\|^2 = -\int M \cdot \nabla \varphi \, dx - \int H_e \cdot \nabla \varphi \, dx, \] (26)
leading to
\[ \int H \cdot M \, dx = -\|H\|^2 - \nu \|\varphi\|^2 - \int H_e \cdot H \, dx, \]
\[ \|H\| \leq \|M\| + \|H_e\|, \quad \sqrt{\nu} \|\varphi\| \leq \|M\| + \|H_e\|. \] (27)
It is useful to introduce the following notation for the solution of problem (25)
\[ \varphi = \mathcal{H}^\nu(H_e, M). \] (28)

Using elliptic regularity results, we obtain that if \( M \) and \( H_e \) are regular, since for \( \delta \in \mathbb{N}^3 \), \( \partial^\delta \mathcal{H}^\nu(H_e, M) = \mathcal{H}^\nu(\partial^\delta H_e, \partial^\delta M) \), we deduce the following properties.

**Lemma 2.2.** Let \((H_e, M) \in H^m \times H^m, m \in \mathbb{N}\). Then \( \varphi \in H^{m+1} \) and it holds for \(|\delta| \leq m\) that
\[
\int \partial^\delta H \cdot \partial^\delta M \, dx = -\|\partial^\delta H\|^2 - \nu\|\partial^\delta \varphi\|^2 - \int \partial^\delta H_e \cdot \partial^\delta H \, dx,
\]
\[
\|\partial^\delta H\| \leq \|\partial^\delta M\| + \|\partial^\delta H_e\|,
\]
\[
\sqrt{\nu}\|\partial^\delta \varphi\| \leq \|\partial^\delta M\| + \|\partial^\delta H_e\|.
\]
Therefore operator \( \mathcal{H}^\nu : H^m \times H^m \to H^{m+1} \) is continuous, for all \( m \in \mathbb{N} \).

2.3. **The entropy inequalities.** To simplify the presentation, we introduce the following operators. Let \( \delta, \theta \in \mathbb{N}^3 \), for functions \( f = f(x, v), M = M(x), H = H(x) \), we set
\[
A^\delta_0(f, M) = \gamma \frac{\partial^\delta D^\theta f - D^\theta \psi \partial^\delta M}{\tau \varepsilon} \quad \text{and} \quad A = A^0_0,
\]
\[
B^\delta_0(f, H) = \gamma \frac{\partial^\delta D^\theta f - \chi_0 D^\theta \psi \partial^\delta H}{\tau} \quad \text{and} \quad B = B^0_0,
\]
and
\[
\mathcal{M}(f) = \gamma \int f \, dv, \quad f \in L^2_\psi.
\]

In this paragraph, we will prove some entropy inequalities involving \( A^\delta_0(f, M) \) and \( B^\delta_0(f, H) \) for a vector field \( f \), the magnetization \( M = \mathcal{M}(f) \) and the corresponding magnetic field \( H = \nabla \varphi \) where \( \varphi = \mathcal{H}^\nu(H_e, M) \) (see (28)). These inequalities will play an important role in the sequel. Let us prove first that:

**Proposition 1.** Let \( f \in H^m_\psi, m \in \mathbb{N} \), then \( M = \mathcal{M}(f) \in H^m \) and it holds
\[
\|\partial^\delta M\| \leq \gamma \|\partial^\delta f\| \|\sqrt{\psi}\|_{L^2_\psi}, \quad |\delta| \leq m,
\] (32)
\[
\left( A^\delta_0(f, M), \partial^\theta D^\delta f \right)_{L^2_\psi} \geq 0, \quad |\delta| + |\theta| \leq m \leq 3.
\] (33)

In particular, the linear operator \( \mathcal{M} \) is continuous from \( H^m_\psi \) to \( H^m \).

**Proof.** We easily verify that for \(|\delta| \leq m\), the (distributional ) derivative \( \partial^\delta M \) of \( M \) is given by
\[
\partial^\delta M = \mathcal{M}(\partial^\delta f) = \gamma \int \partial^\delta f \, dv = \gamma \int \sqrt{\psi} \frac{\partial^\delta f}{\sqrt{\psi}} \, dv,
\]
so using Cauchy-Schwarz inequality, we get (32) which implies that
\[
\|\partial^\delta M\|^2 = \gamma \int \partial^\theta M \cdot \left( \int \partial^\delta f \, dv \right) \, dx \leq \gamma \|\partial^\delta f\|^2 \|\sqrt{\psi}\|_{L^2_\psi}.
\]
This inequality is nothing else that (33) for \( \theta = 0 \). Next we remark that
\[
\left( A^\delta_0(f, M), \partial^\delta D^\delta f \right)_{L^2_\psi} = \left( A^\delta_0(\partial^\delta f, \partial^\delta M), D^\delta(\partial^\delta f) \right)_{L^2_\psi}.
\]
so to achieve the proof of (33), it is enough to consider the case where \( \delta = 0 \) and \( \theta \neq 0 \) and show the inequality
\[
\left( A_0^0(f, M), D^\theta f \right)_{L^2_\psi} = \frac{1}{\tau \varepsilon} \left( \gamma \|D^\theta f\|^2_{L^2_\psi} - \int M \cdot \left( \int \frac{D^\theta \psi}{\psi} D^\theta f \, dv \right) \, dx \right) \geq 0. \tag{34}
\]

The proof of (34) needs to compute the integrals
\[
\int D^\theta \psi \psi D^\theta f \, dv
\]
for \( 1 \leq |\theta| \leq 3 \).

Recall that
\[
D^\theta = \partial_{\theta_1} \partial_{\theta_2} \partial_{\theta_3} \quad \text{for} \quad \theta = \sum_{i=1}^3 \theta_i e_i, \quad \theta_i \in \mathbb{N}, \quad \{e_1, e_2, e_3\} \text{ being the canonical basis of } \mathbb{R}^3
\]
and the derivatives \( \partial^n_{\theta_i} \psi \) are given by
\[
\partial^n_{\theta_i} \psi = \begin{cases} 
-v_i & \text{if } \theta = e_i, \quad i = 1, 2, 3, \\
v_i v_j & \text{if } \theta = e_i + e_j, \quad i, j = 1, 2, 3, \quad i \neq j, \\
-1 + v_i^2 & \text{if } \theta = 2e_i, \quad i = 1, 2, 3, \\
3v_i - v_i^3 & \text{if } \theta = 3e_i, \quad i = 1, 2, 3, \\
-v_j(-1 + v_i^2) & \text{if } \theta = 2e_i + e_j, \quad i, j = 1, 2, 3, \quad i \neq j, \\
-v_1 v_2 v_3 & \text{if } \theta = e_1 + e_2 + e_3,
\end{cases}
\]
so for \( 1 \leq |\theta| \leq 3 \), \( \frac{D^\theta \psi}{\psi} \) is the polynomial of degree \(|\theta|\) given by
\[
\mathcal{D}^\theta \left( \frac{D^\theta \psi}{\psi} \right) = (-1)^{|\theta|} \theta!.
\]

Now we will establish the following formula
\[
\int \frac{D^\theta \psi}{\psi} D^\theta f \, dv = \frac{\theta!}{\gamma} M. \tag{35}
\]

Using a density argument, see Lemma 2.1, it is enough to prove formula (35) for a function \( f \in \mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^3) \). In this case, integrations by parts lead to
\[
\int \frac{D^\theta \psi}{\psi} D^\theta f \, dv = (-1)^{|\theta|} \int \mathcal{D}^\theta \left( \frac{D^\theta \psi}{\psi} \right) f \, dv = \theta! \int f \, dv = \frac{\theta!}{\gamma} M.
\]

Hence relation (35) and Cauchy-Schwarz inequality lead to
\[
\int M \cdot \left( \int \frac{D^\theta \psi}{\psi} D^\theta f \, dv \right) \, dx = \frac{\gamma}{\theta!} \left( \int \int \frac{D^\theta \psi}{\psi} D^\theta f \, dv \right)^2 \, dx \leq \frac{\gamma}{\theta!} \left( \int \left| \frac{D^\theta \psi}{\psi} \right|^2 \, dv \right) \left( \int \left| D^\theta f \right|^2 \, dv \right) \, dx \, dv.
\]

Therefore we obtain inequality (34), once we verify that
\[
\int \left| \frac{D^\theta \psi}{\psi} \right|^2 \, dv = \theta!, \quad 1 \leq |\theta| \leq 3. \tag{36}
\]
For this purpose, we check that for \( i, j = 1, 2, 3 \)
\[
\int v_i \psi \, dv = I_1, \quad \int (-1 + v_i^2)^2 \psi \, dv = I_2, \quad \int (3v_i - v_i^3)^2 \psi \, dv = I_3,
\]
\[
\int v_i^2 \psi \, dv = I_1^2, \quad \int v_i^2 (-1 + v_i^2)^2 \psi \, dv = I_1 I_2 = 2, \quad \text{for } i \neq j,
\]
\[
\int v_i^2 v_j^2 \psi \, dv = I_1^3,
\]
where
\[
I_1 = \frac{1}{\sqrt{2\pi}} \int s^2 e^{-s^2/2} \, ds = 1,
\]
\[
I_2 = \frac{1}{\sqrt{2\pi}} \int (-1 + s^2)^2 e^{-s^2/2} \, ds = 2,
\]
\[
I_3 = \frac{1}{\sqrt{2\pi}} \int (3s - s^3)^2 e^{-s^2/2} \, ds = 6.
\]
This ends the proof of Proposition 1.

Proposition 2. Let \( H_\epsilon \in H^m, \, f \in \mathbb{H}^m, \, m \in \mathbb{N} \) and let \( M = \mathcal{M}(f) \). Then \( \varphi = \mathcal{H}^\nu(H_\epsilon, M) \in H^{m+1} \) and setting \( H = \nabla \varphi \), the following inequality holds for \( |\delta| + |\theta| \leq m \leq 3 \)
\[
\begin{align*}
\left( B^\delta_\theta (f, H), \partial^\delta H \psi \right)_{L^2_\psi} & \geq \frac{\gamma}{2} \| \partial^\delta H \psi \|_{L^2_\psi}^2 + \frac{\theta\gamma \lambda_0}{2\tau} \left( \| \partial^\delta H \|_{L^2_{\psi}}^2 + 2\nu \| \partial^\delta \varphi \|_{L^2_{\psi}}^2 \right). \\
\end{align*}
\]  
(37)

Proof. By using formula (35) (also valuable for \( \theta = 0 \)), we write
\[
\begin{align*}
\left( B^\delta_\theta (f, H), \partial^\delta H \psi \right)_{L^2_\psi} & = \frac{\gamma}{2} \| \partial^\delta H \psi \|_{L^2_\psi}^2 - \frac{\lambda_0}{\tau} \int \partial^\delta H \cdot \left( \partial^\delta \int \frac{D^\theta \psi}{\psi} f \, dv \right) \, dx \\
& = \frac{\gamma}{2} \| \partial^\delta H \psi \|_{L^2_\psi}^2 - \frac{\theta\gamma \lambda_0}{2\tau} \int \partial^\delta H \cdot \partial^\delta M \, dx.
\end{align*}
\]
Therefore we get the desired result thanks to (29).

In the next section, we discuss the stability of the system (6)-(7) by using the uniform bounds given in (41) and the weak convergences (48) stated below. The switching term \( H_\epsilon \times M_\epsilon \) is uniformly bounded in \( L^\infty(0, T; \mathbb{R}^3) \) then concentration effect represented by a Radon measure \( \mu_\epsilon(dx) \) may appear in the weak convergence. The characterization of this Radon measure is not yet established.

3. Formal estimates and limit when \( \epsilon \to 0 \). We assume that the data satisfy the hypotheses
\[
f_0 \in L^2_\psi \quad \text{and} \quad H_\epsilon \in L^2((0, T) \times \mathbb{R}^3),
\]  
(38)
and set
\[
\mathcal{E}_0 = \| f_0 \|_{L^2_\psi}^2 + \frac{\chi_0}{\gamma T} \int_0^T \| H_\epsilon(t) \|_{L^2_{\psi}}^2 \, dt.
\]  
(39)
Let \( \epsilon > 0 \) be fixed, we consider problem \( (P_\epsilon) \) defined by equations (6)-(7) and we assume that it admits a weak solution
\[
(f_\epsilon, H_\epsilon) \in L^\infty(0, T; L^\infty_{\psi} \times L^2).
\]  
(40)
Let \( M_\epsilon \) and \( J_\epsilon \) be respectively the magnetization field and the current matrix defined in (8) and (10). In this section, first we derive formally some uniform estimates...
satisfied by \((f_\varepsilon, H_\varepsilon, M_\varepsilon, J_\varepsilon)\), then we perform the limit when \(\varepsilon \to 0\) in problem \((\mathcal{B}_\varepsilon)\) (see equations (7)-(9)) verified by \((M_\varepsilon, H_\varepsilon)\).

### 3.1. Formal estimates.

**Lemma 3.1.** Under hypotheses (38) and (40), we have \(M_\varepsilon, J_\varepsilon \in L^\infty(0, T; L^2)\) and for a.e. \(t \in (0, T)\) it holds

\[
\|f_\varepsilon(t)\|_{L^2_\psi}^2 + \frac{2\gamma}{T} \int_0^t \|f_\varepsilon(s)\|_{L^2_\psi}^2 ds + \frac{\chi_0}{\gamma T} \int_0^t \|H_\varepsilon(s)\|_{L^2}^2 ds \leq \mathcal{E}_0, \tag{41}
\]

\[
\|M_\varepsilon(t)\| \leq \gamma^2 \mathcal{E}_0, \quad \|J_\varepsilon(t)\| \leq 3\gamma \mathcal{E}_0,
\]

where \(\mathcal{E}_0\) is given in (39).

**Proof.** We will use the result of Proposition 1. We easily see that \(M_\varepsilon \in L^\infty(0, T; L^2)\) and satisfies for a.e. \(t \in (0, T)\) the following inequality

\[
\|M_\varepsilon(t)\|_{L^2} \leq \gamma \|f_\varepsilon(t)\|_{L^2_\psi}. \tag{42}
\]

To prove the first estimate of the lemma, we multiply (6) by \(\frac{f_\varepsilon}{\psi}\) and integrate with respect to \((x, v)\) to get

\[
\frac{1}{2} \frac{d}{dt} \|f_\varepsilon\|_{L^2_\psi}^2 + \frac{1}{\gamma} \int \gamma f_\varepsilon - \chi_0 \psi H_\varepsilon \cdot \frac{f_\varepsilon}{\psi} \, dx \, dv + \frac{1}{\tau \varepsilon} \int \gamma f_\varepsilon - \psi M_\varepsilon \cdot \frac{f_\varepsilon}{\psi} \, dx \, dv = 0.
\]

First, we see that the entropy inequality (33) leads to

\[
\frac{1}{\tau \varepsilon} \int \gamma f_\varepsilon - \psi M_\varepsilon \cdot \frac{f_\varepsilon}{\psi} \, dx \, dv = \frac{1}{\tau \varepsilon} \left( \gamma \|f_\varepsilon\|_{L^2_\psi}^2 - \int M_\varepsilon \cdot \left( \int f_\varepsilon \, dv \right) \, dx \right) \geq 0. \tag{43}
\]

Next we have

\[
\int H_\varepsilon \cdot \left( \int f_\varepsilon \, dv \right) \, dx = \frac{1}{\gamma} \int H_\varepsilon \cdot M_\varepsilon \, dx = -\frac{1}{\gamma} \|H_\varepsilon\|^2 - \frac{1}{\gamma} \int H_\varepsilon \cdot H_\varepsilon \, dx,
\]

the last equality follows by writing \(H_\varepsilon = \nabla \varphi_\varepsilon\) and multiplying the magnetostatic equation (7) by \(\varphi_\varepsilon\) and integrating by parts. Therefore, using Cauchy-Schwarz inequality, we deduce that

\[
\frac{1}{2} \frac{d}{dt} \|f_\varepsilon\|_{L^2_\psi}^2 + \frac{\gamma}{\tau} \|f_\varepsilon\|_{L^2_\psi}^2 + \frac{\chi_0}{2\gamma T} \|H_\varepsilon\|^2 \leq \frac{\chi_0}{2\gamma T} \|H_\varepsilon\|^2. \tag{44}
\]

Integrating (44) between 0 and \(t\), we derive the first estimate of the lemma, then the second one using (42). Finally, writing \(J_\varepsilon = \gamma \int \sqrt{\psi v} \otimes \frac{f_\varepsilon}{\sqrt{\psi}} \, dv\) and using Cauchy-Schwarz inequality, we see that \(J_\varepsilon \in L^\infty(0, T; L^2)\) and satisfies the last bound of the lemma.

Moreover we have:

**Lemma 3.2.** Under hypotheses of Lemma 3.1, there exists a constant \(C > 0\) which depends on \(\mathcal{E}_0\) such that for all \(\varepsilon > 0\), it holds

\[
\|\gamma f_\varepsilon - \psi M_\varepsilon\|_{H^{-\frac{s}{2}}((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)} \leq \varepsilon C, \quad \text{for } s > 5/2,
\]

\[
\|J_\varepsilon\|_{H^{-\frac{s}{2}}((0, T) \times \mathbb{R}^3)} \leq \varepsilon C, \quad \text{for } s > 3/2. \tag{45}
\]
Proof. We test equation (6) with functions $\Phi \in \mathcal{D}((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ to get
\[
\frac{1}{\varepsilon^2} \int_0^T \int \left( \gamma f_\varepsilon - \psi M_\varepsilon \right) \cdot \Phi \, dx \, dv \, dt = \int_0^T \int \left( \partial_t \Phi + (v \cdot \nabla_x) \Phi \right) \, dx \, dv \, dt
\]
\[-\alpha \int_0^T \int ((v \times H_\varepsilon) \cdot \nabla_v) \Phi \cdot f_\varepsilon \, dx \, dv \, dt - \beta \int_0^T \int H_\varepsilon \times f_\varepsilon \cdot \Phi \, dx \, dv \, dt
\]
\[-\frac{1}{\tau} \int_0^T \int (\gamma f_\varepsilon - \chi_0 \psi H_\varepsilon) \cdot \Phi \, dx \, dv \, dt.
\]
We estimate each integral in the right hand side of (46), using the bounds of Lemma 3.1. The first term is bounded as follows
\[
\left| \int_0^T \int \frac{f_\varepsilon}{\sqrt{\psi}} \cdot (\partial_t \Phi + (v \cdot \nabla_x) \Phi) \sqrt{\psi} \, dx \, dv \, dt \right| \leq C \|f_\varepsilon\|_{L^2(0, T; L^2_\varepsilon)} \|\nabla (t, x) \Phi\|_{L^2(0, T; L^2)}
\]
\[\leq C \sqrt{\mathcal{E}_0} \|\Phi\|_{H^1((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)}.
\]
Then, using the Sobolev embedding $H^s(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$ for $s > 3/2$, we get for the second term
\[
\left| \int_0^T \int ((\sqrt{\psi} v \times H_\varepsilon) \cdot \nabla_v) \Phi \cdot f_\varepsilon \, dx \, dv \, dt \right| \leq C \|f_\varepsilon\|_{L^\infty(0, T; L^3_\varepsilon)} \|H_\varepsilon\|_{L^2(0, T; L^2)} \|\nabla_v \Phi\|_{L^2(0, T) \times \mathbb{R}^3 \times L^\infty}
\]
\[\leq C \mathcal{E}_0 \|\nabla_v \Phi\|_{L^2((0, T) \times \mathbb{R}^3; H^3_\varepsilon)} \leq C \mathcal{E}_0 \|\Phi\|_{H^{s+1}(0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)}.
\]
Similarly for $s > 3/2$, we have
\[
\left| \int_0^T \int H_\varepsilon \times f_\varepsilon \cdot \Phi \, dx \, dv \, dt \right| \leq C \|f_\varepsilon\|_{L^\infty(0, T; L^3_\varepsilon)} \|H_\varepsilon\|_{L^2(0, T; L^2)} \|\Phi\|_{L^2((0, T) \times \mathbb{R}^3 \times L^\infty)}
\]
\[\leq C \mathcal{E}_0 \|\Phi\|_{H^s((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)},
\]
and the last term is bounded as follows
\[
\left| \int_0^T \int (\gamma f_\varepsilon - \chi_0 \psi H_\varepsilon) \cdot \Phi \, dx \, dv \, dt \right| \leq C (\|f_\varepsilon\|_{L^2(0, T; L^3_\varepsilon)} + \|H_\varepsilon\|_{L^2(0, T; L^2)}) \|\Phi\|_{L^2((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)}
\]
\[\leq C \sqrt{\mathcal{E}_0} \|\Phi\|_{L^2((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)}.
\]
Adding these estimates, we see that for $s > 5/2$
\[
\left| \int_0^T \int (\gamma f_\varepsilon - \psi M_\varepsilon) \cdot \Phi \, dx \, dv \, dt \right| \leq \varepsilon C \|\Phi\|_{H^s((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)},
\]
so the first bound of Lemma 3.2 is proved. Next, since $\int v_i \psi \, dv = 0$ we can write for $i = 1, 2, 3$
\[J_{\varepsilon i} = \int (\gamma f_\varepsilon - \psi M_\varepsilon)v_i \, dv,
\]
so for $\phi \in \mathcal{D}((0, T) \times \mathbb{R}^3)$, setting $\Psi = \phi(t, x)v_i$, we get as for (46)

$$
\frac{1}{\varepsilon T} \int_0^T \int J_{\varepsilon i} \cdot \phi \, dx dt = \frac{1}{\varepsilon T} \int_0^T \int (\gamma f_{\varepsilon} - \psi M_{\varepsilon}) \cdot \Psi \, dx dv dt =
$$

$$
\int_0^T \int f_{\varepsilon} \cdot (\partial_t \Psi + (v \cdot \nabla_x) \Psi) \, dx dv dt - \alpha \int_0^T \int ((v \times H_{\varepsilon}) \cdot \nabla v) \Psi \cdot f_{\varepsilon} \, dx dv dt
$$

$$
- \beta \int_0^T \int H_{\varepsilon} \times f_{\varepsilon} \cdot \Psi \, dx dv dt - \frac{1}{T} \int_0^T \int (\gamma f_{\varepsilon} - \chi_0 \psi H_{\varepsilon}) \cdot \Psi \, dx dv dt,
$$

and we estimate each integral of the right-hand side to show that

$$
\left| \int_0^T J_{\varepsilon i} \cdot \phi \, dx dt \right| \leq \varepsilon \|\phi\|_{H^s((0, T) \times \mathbb{R}^3)} \quad \text{for } s > 3/2. \quad (47)
$$

The proof is very similar to the previous one and omitting the details, we get the bounds

$$
\left| \int_0^T \int \frac{f_{\varepsilon}}{\sqrt{\psi}} \cdot (\partial_t \phi + (v \cdot \nabla_x) \phi) v_i \sqrt{\psi} \, dx dv dt \right| \leq C \sqrt{E_0} \|\phi\|_{H^1((0, T) \times \mathbb{R}^3)},
$$

$$
\left| \int_0^T \int ((v \times H_{\varepsilon}) \cdot \nabla v) (v_i \phi) \cdot f_{\varepsilon} \, dx dv dt \right| \leq C E_0 \|\phi\|_{L^2(0, T; H^s)},
$$

$$
\left| \int_0^T \int H_{\varepsilon} \times f_{\varepsilon} \cdot \phi v_i \, dx dv dt \right| \leq C E_0 \|\phi\|_{L^2(0, T; H^s)},
$$

$$
\left| \int_0^T \int (\gamma f_{\varepsilon} - \chi_0 \psi H_{\varepsilon}) \cdot \phi v_i \, dx dv dt \right| \leq C \sqrt{E_0} \|\phi\|_{L^2((0, T) \times \mathbb{R}^3)}.
$$

Combining these estimates, we obtain the bound (47), which completes the proof of Lemma 3.2. \hfill \Box

3.2. **Passing to the limit as $\varepsilon \to 0$.** The uniform bounds given in Lemmas 3.1 and 3.2 allow to deduce that

**Lemma 3.3.** There exists a subsequence still labeled $(f_{\varepsilon}, H_{\varepsilon}, M_{\varepsilon}, J_{\varepsilon})$ and $(f, H, M, J)$ such that when $\varepsilon \to 0$, we have

$$
f_{\varepsilon} \rightharpoonup f \quad \text{weakly-\ast in } L^\infty(0, T; L^2_{\nu^2}),
$$

$$
H_{\varepsilon} \rightharpoonup H \quad \text{weakly in } L^2(0, T; L^2),
$$

$$
M_{\varepsilon} \rightharpoonup M \quad \text{weakly-\ast in } L^\infty(0, T; L^2),
$$

$$
J_{\varepsilon} \rightharpoonup J \quad \text{weakly-\ast in } L^\infty(0, T; L^2). \quad (48)
$$

Moreover

$$
M = \gamma \int f \, dv, \quad J = 0 \quad \text{and} \quad f = \frac{1}{\gamma} \psi M. \quad (49)
$$

**Proof.** We have only to prove (49). For all functions $\phi \in L^1(0, T; L^2(\mathbb{R}^3))$, since $\phi \psi \in L^1(0, T; L^2_{\nu^2})$ then

$$
\lim_{\varepsilon \to 0} \int_0^T \int M_{\varepsilon} \cdot \phi \, dx dt = \gamma \lim_{\varepsilon \to 0} \int_0^T \int f_{\varepsilon} \cdot \phi \psi \frac{1}{\psi} \, dx dv dt =
$$

$$
\gamma \int_0^T \int f \cdot \phi \psi \frac{1}{\psi} \, dx dv dt = \gamma \int_0^T \int (\int f \, dv) \cdot \phi \, dx dt, \quad (50)
$$

\begin{align*}
\int_0^T \int M_{\varepsilon} \cdot \phi \, dx dt & = \gamma \lim_{\varepsilon \to 0} \int_0^T \int f_{\varepsilon} \cdot \phi \psi \frac{1}{\psi} \, dx dv dt \\
\gamma \int_0^T \int f \cdot \phi \psi \frac{1}{\psi} \, dx dv dt & = \gamma \int_0^T \int (\int f \, dv) \cdot \phi \, dx dt,
\end{align*}

\text{for } s > 3/2.
The sense of measures given field, independent of $\varepsilon$. In that case, we have clearly that $\mu$ is a Radon measure. Thus passing to the limit when $\varepsilon \to 0$ in $(B_\varepsilon)$ we get

\[ H_\varepsilon \times M_\varepsilon \to H \times M + \mu_t(dx). \] (51)

Thus passing to the limit when $\varepsilon \to 0$ in $(B_\varepsilon)$ we get

**Proposition 3.** Under hypotheses (38) and (40), there exists a parameterized Radon measure $\mu_t(dx)$ such that when $\varepsilon \to 0$, the weak limit $(M, H)$ of $(M_\varepsilon, H_\varepsilon)$ in $L^\infty(0, T; L^2) \times L^2(0, T; L^2)$ satisfies the Bloch equation coupled to the magnetostatic equations below

\[
\begin{align*}
\partial_t M + \beta H \times M + \frac{\gamma}{T} (M - \chi_0 H) &= -\beta \mu_t(dx) \quad \text{in } (0, T) \times \mathbb{R}^3, \\
M(0) &= M_0 \quad \text{in } \mathbb{R}^3, \\
\text{div } (H + M) &= -\text{div } H_\varepsilon, \quad \text{curl } H = 0 \quad \text{in } (0, T) \times \mathbb{R}^3.
\end{align*}
\] (52)

**Proof.** We have only to verify that the trace $M(0)$ makes sense and that $M(0) = M_0$. From equation (52) we get $\partial_t M \in L^2(0, T; H^{-s})$ for some $s > 0$ large enough and then $M \in C([0, T], H^{-r})$ for $0 < r < s$. Hence $M(0)$ makes sense in $H^{-r}$ and from the weak formulation of problem $(B_\varepsilon)$ we deduce that $M(0) = M_0$. \hfill $\Box$

The characterization of the measure $\mu_t(dx)$ in equation (52) remains an open problem. In a recent paper [6], the authors discussed the propagation of oscillations in a conservative system and in particular for the simplified Bloch equation $\partial_t m_\varepsilon - h_\varepsilon \times m_\varepsilon = g$ with $m_\varepsilon(0) = m_0 \in L^\infty(\mathbb{R}^3)$ and $h_\varepsilon \to h$ in $L^\infty((0, T) \times \mathbb{R}^3)$ weakly-*. They proved that the interactions of oscillations of $h_\varepsilon$ and $m_\varepsilon$ induce a nonlocal-in-time effect. The case where the fields $m_\varepsilon$ and $h_\varepsilon$ are bounded in $L^\infty(0, T; L^2)$ and $L^2(0, T; L^2)$ respectively is, in our knowledge, an open problem.

In [9, 5] for example, it is assumed that the magnetic field $H \in L^2(0, T; L^2)$ is a given field, independent of $\varepsilon$. In that case, we have clearly that $\mu_t(dx) = 0$, if we prove uniform estimates for $f_\varepsilon$ and $M_\varepsilon$. Indeed we have

**Proposition 4.** Assume $f_0 \in L^2_\varepsilon$ and $H \in L^2(0, T; L^2)$, and let $f_\varepsilon \in L^\infty(0, T; L^2_\varepsilon)$ a solution of the kinetic equation (6) for fixed $\varepsilon > 0$ (with $H_\varepsilon = H$) and let $M_\varepsilon = \gamma \int f \, dv$. Similarly $v_1 \phi \psi \in L^1(0, T; L^2_\varepsilon)$ so

\[
\lim_{\varepsilon \to 0} \int_0^T \int J_{\varepsilon 1} \cdot \phi \, dx \, dt = \gamma \lim_{\varepsilon \to 0} \int_0^T \int v_1 f_\varepsilon \cdot \phi \psi \frac{1}{\psi} \, dx \, dv \, dt = \\
\gamma \int_0^T \int v_1 f \cdot \phi \psi \frac{1}{\psi} \, dx \, dv \, dt.
\]

Hence $J_{\varepsilon 1} \to J_1 = \gamma \int v_1 f \, dv$ weakly-* in $L^\infty(0, T; L^2)$ but by Lemma 3.2, $J_\varepsilon$ converges strongly to 0 in $H^{-s}((0, T) \times \mathbb{R}^3)$, $s > 3/2$ then we conclude that $J = 0$. Finally, the first estimate of Lemma 3.2 implies that $f = \frac{1}{\gamma} \psi M$ and then we retrieve that $J = 0$. \hfill $\Box$
\( \gamma \int f_\varepsilon \, dv \). Then there exists a subsequence still labeled \( f_\varepsilon \) such that when \( \varepsilon \to 0 \)
\begin{align*}
f_\varepsilon &\rightharpoonup f \quad \text{weakly} - * \quad \text{in} \ L^\infty(0,T;L^2\nu), \\
M_\varepsilon &\rightharpoonup M \quad \text{weakly} - * \quad \text{in} \ L^\infty(0,T;L^2),
\end{align*}
where \( M = \gamma \int f \, dv \) verifies the Bloch equation
\[ \partial_t M + \beta H \times M + \frac{\gamma}{T} (M - \chi_0 H) = 0 \quad \text{in} \ (0,T) \times \mathbb{R}^3, \]
\begin{equation}
M(0) = M_0 \quad \text{in} \ \mathbb{R}^3.
\end{equation}

**Proof.** We verify that \( f_\varepsilon \) remains uniformly bounded in \( L^\infty(0,T;L^2\nu) \). Following the proof of Lemma 3.1, we have only to estimate the integral \( \int H \cdot f_\varepsilon \, dv \). Using Cauchy-Schwarz and Young inequalities, we write
\[ \frac{\chi_0}{\tau} \int H \cdot f_\varepsilon \, dv = \frac{\chi_0}{\tau} \int \sqrt{\psi} H \cdot \frac{f_\varepsilon}{\sqrt{\psi}} \, dv \leq \frac{\gamma}{2\tau} \| f_\varepsilon \|^2_{L^2\nu} + \frac{\lambda_0^2}{2\gamma \tau} \| H \|^2, \]
so we deduce the following estimate
\[ \| f_\varepsilon(t) \|^2_{L^2} + \frac{\gamma}{\tau} \int_0^t \| f_\varepsilon(s) \|^2_{L^2} \, ds \leq \| f_0 \|^2_{L^2} + \frac{\lambda_0^2}{\gamma \tau} \int_0^t \| H(s) \|^2 \, ds. \]
From here, we see that the bounds for \( M_\varepsilon \) and \( J_\varepsilon \) given in Lemmas 3.1 and 3.2 are valid, by replacing \( \mathcal{E}_0 \) by \( \mathcal{E}_0 = \| f_0 \|^2_{L^2} + \frac{\lambda_0^2}{\gamma \tau} \int_0^t \| H(s) \|^2 \, ds \). This ends the proof of the proposition. \( \square \)

4. **The regularized problem** (\( \mathcal{Q}^\nu \)). Let \( \varepsilon > 0 \) be fixed and let \( \nu > 0 \) be a small parameter. We introduce a smoothing sequence \( \rho_\nu = \rho_\nu(x) \) of indefinitely differentiable real valued functions on \( \mathbb{R}^3 \) with support in the ball of center 0 and radius \( \nu \), such that \( \int \rho_\nu(x) \, dx = 1 \) and we set \( \rho_\nu * K(t,x) = \int \rho_\nu(y) K(t,x-y) \, dy \) for a function \( K = K(t,x) \). We modify the kinetic equation (6) as follows
\begin{align*}
\partial_t f - \nu (\Delta_x f + \Delta_v f + (v \cdot \nabla_v) f) + (\psi^\nu v \cdot \nabla_x) f \\
- \alpha [\psi^\nu v \times (\rho_\nu * H)] \cdot \nabla_v f + \beta (\rho_\nu * H) \times f \\
+ \frac{1}{\tau} (\gamma f - \chi_0 \psi M) + \frac{1}{\tau \varepsilon} (\gamma f - \psi M) = 0 \quad \text{in} \ (0,T) \times \mathbb{R}^3 \times \mathbb{R}^3,
\end{align*}
\begin{equation}
f(0, x, v) = f_0(x, v) \quad \text{in} \ \mathbb{R}^3 \times \mathbb{R}^3,
\end{equation}
and recall that \( M(t,x) = \gamma \int f(t,x,v) \, dv \) and \( \psi \) is the Maxwellian given in (5). We have introduced the function \( \psi^\nu(v) = (\psi(v))^\nu \) in order to have \( \psi^\nu v \in L^\infty(\mathbb{R}^3) \) and we see that as \( \nu \to 0 \), \( \psi^\nu(v) \to 1 \) for all \( v \in \mathbb{R}^3 \).

Equation (54) coupled to the perturbed magnetostatic equations (24) will be labeled problem (\( \mathcal{Q}^\nu \)).

It is useful to rewrite equation (54) in a condensate form, more adapted to our purpose. For this, we use operators \( \mathcal{A} \) and \( \mathcal{M} \) defined in (30) and (31) and introduce further the operators defined hereafter
\begin{align*}
\mathcal{T}^\nu(f, H) &= (\psi^\nu v \cdot \nabla_x) f - \alpha [\psi^\nu v \times (\rho_\nu * H)] \cdot \nabla_v f + \beta (\rho_\nu * H) \times f, \\
\mathcal{Q}^\nu(f) &= -\nu (\Delta_x f + \Delta_v f + (v \cdot \nabla_v) f).
\end{align*}

\( \mathcal{Q}^\nu \)
With these notations, equation (54) takes the form
\[ \partial_t f + D^\nu(f) + \mathcal{T}^\nu(f, H) + A(f, M) + \frac{\gamma}{\tau} f = \frac{\chi_0}{\tau} \psi H \quad \text{in } (0, T), \]
\[ f(0) = f_0. \]
To solve this problem, we use the Hilbert space \( H^1_\psi \) defined by (13) and we prove the following result.

**Theorem 4.1.** Assume that \( f_0 \) and \( H_e \) satisfy hypotheses (38). Then for \( \nu > 0 \), there exists a unique solution
\[ (f^{\nu}, \varphi^{\nu}) \in \left( C([0, T]; \mathbb{L}^2_{\psi}) \cap L^2(0, T; H^1_\psi) \right) \times L^2(0, T; H^1) \]
of the regularized problem \( (\mathcal{Q}^{\nu}) \) which satisfies for \( t \in [0, T] \) the following estimate
\[ \| f^{\nu}(t) \|^2_{L^2_{\psi}} + \frac{\gamma}{2\tau} \int_0^T \| f^{\nu}(s) \|^2_{L^2_{\psi}} ds + 2\nu \int_0^T \| \nabla_x \varphi^{\nu}(s) \|^2_{L^2_{\psi}} ds \leq C e^{C t}, \]
where \( C > 0 \) is a constant independent of \( \nu \). In addition, if \( H_e \in L^2(0, T; H^1) \) then \( \varphi^{\nu} \in L^2(0, T; H^1) \).

The proof is based on an iterative method described below.

4.1. **Iterative scheme.** Let us present the iterative scheme used to solve problem \( (\mathcal{Q}^{\nu}) \).

First we set \( f^{(0)} = 0 \), then assuming that \( f^{(n)} \) is known at order \( n \in \mathbb{N} \), we set
\[ M^{(n)} = M(f^{(n)}), \quad \varphi^{(n)} = \mathcal{H}^{\nu}(H_e, M^{(n)}), \quad H^{(n)} = \nabla \varphi^{(n)}, \quad n \in \mathbb{N}. \]
Next, we define for all \( n \in \mathbb{N} \), \( f^{(n+1)} \) as the solution of the following linearized problem
\[ \partial_t f^{(n+1)} + D^\nu(f^{(n+1)}) + \mathcal{T}^\nu(f^{(n+1)}, H^{(n)}) + A(f^{(n+1)}, M(f^{(n+1)})) + \frac{\gamma}{\tau} f^{(n+1)} = \frac{\chi_0}{\tau} \psi H^{(n)} \quad \text{in } (0, T), \]
\[ f^{(n+1)}(0) = f_0. \]
To prove that the iterative scheme is well defined, we will use the following result.

**Lemma 4.2.** Let \( K = K(t, x) \in L^2(0, T; L^2) \), for all \( u_0 \in L^2_{\psi} \) and \( U \in L^2(0, T; H^{-1}_\psi) \), the equation
\[ \partial_t u + D^\nu(u) + \mathcal{T}^\nu(u, K(t)) + A(u, M(u)) + \frac{\gamma}{\tau} u = U \quad \text{in } (0, T), \]
\[ u(0) = u_0, \]
admits a unique weak solution \( u \in C([0, T]; \mathbb{L}^2_{\psi}) \cap L^2(0, T; H^1_\psi) \).

**Proof.** We use a variational method for parabolic equations. We easily see that equation (59) is equivalent to the following weak problem
\[ \frac{d}{dt} (u, g)_{L^2_{\psi}} + \Theta^\nu(t; u, g) = (U, g)_{H^{-1}_\psi, H^1_\psi}, \quad \forall g \in H^1_\psi, \]
\[ u(0) = u_0, \]
where
\[ \Theta^\nu(t, u, g) = a^\nu(u, g) + b^\nu(t; u, g) + c(u, g), \quad (u, g) \in H^1_\psi \times H^1_\psi, \]
and for \((u, g) \in \mathbb{H}^{1}_{\psi} \times \mathbb{H}^{1}_{\psi}\)
\[
a^\nu(u, g) = \frac{\gamma}{\tau} (u, g)_{\mathbb{L}^{2}} + \nu(\nabla u, \nabla g)_{\mathbb{L}^{2}} + \nu(\nabla u, \nabla g)_{\mathbb{L}^{2}},
\]
\[
b^\nu(t; u, g) = (T^\nu(u, K(t)), g)_{\mathbb{L}^{2}},
\]
\[
c(u, g) = (A(u, M(u)), g)_{\mathbb{L}^{2}}.
\]

We will verify that the bilinear form \(\Theta^\nu(t; \cdot, \cdot)\) is measurable with respect to time, continuous on \(\mathbb{H}^{1}_{\psi} \times \mathbb{H}^{1}_{\psi}\) and satisfies
\[
\Theta^\nu(t; u, u) \geq \nu \left( \|u\|_{\mathbb{H}^{1}_{\psi}}^{2} - \|u\|_{\mathbb{L}^{2}}^{2} \right) + \frac{\gamma}{\tau} \|u\|_{\mathbb{L}^{2}}^{2}, \quad \forall u \in \mathbb{H}^{1}_{\psi}.
\]

Indeed,
1. The bilinear form \(a^\nu\) is continuous on \(\mathbb{H}^{1}_{\psi} \times \mathbb{H}^{1}_{\psi}\) and
\[
a^\nu(u, u) = \frac{\gamma}{\tau} \|u\|_{\mathbb{L}^{2}}^{2} + \nu \left( \|u\|_{\mathbb{H}^{1}_{\psi}}^{2} - \|u\|_{\mathbb{L}^{2}}^{2} \right), \quad \forall u \in \mathbb{H}^{1}_{\psi}.
\]
2. Since \(\psi^\nu \in L^{\infty}\) and \(\rho_{\nu} * K \in L^{2}(0; T; L^{\infty})\), then the bilinear form \(b^\nu(t; \cdot, \cdot)\) is well defined, is measurable with respect to time and continuous on \(\mathbb{H}^{1}_{\psi} \times \mathbb{H}^{1}_{\psi}\) and we have
\[
b^\nu(t; u, u) = 0, \quad \forall u \in \mathbb{H}^{1}_{\psi}.
\]
3. The bilinear form \(c\) is continuous on \(\mathbb{H}^{1}_{\psi} \times \mathbb{H}^{1}_{\psi}\) and verifies (thanks to the entropy inequality (33) for \(\delta = \theta = 0\))
\[
c(u, u) \geq 0, \quad \forall u \in \mathbb{H}^{1}_{\psi}.
\]

Hence, applying a theorem of J.L. Lions (see [2] p. 218), we infer that problem (60) has a unique solution \(u\) in \(C([0, T]; \mathbb{L}^{2}_{\psi}) \cap L^{2}(0, T; \mathbb{H}^{1}_{\psi})\). Therefore we deduce in a classical way that \(u\) is the unique solution satisfying equation (59) in \(L^{2}(0, T; \mathbb{H}^{1}_{\psi})\) and thus in the sense of distributions, and the initial condition in \(\mathbb{L}^{2}_{\psi}\).

Now consider the iterative scheme (58) defined above. Using Lemma 4.2, we will prove the following result.

**Proposition 5.** Under hypotheses (38), the sequence \((f^{(n)}, \varphi^{(n)})\) is well defined, verifies
\[
f^{(n)} \in C([0, T]; \mathbb{L}^{2}_{\psi}) \cap L^{2}(0, T; \mathbb{H}^{1}_{\psi}), \quad \varphi^{(n)} \in L^{2}(0, T; H^{1}),
\]
\[
M^{(n)} = M(f^{(n)}) \in C([0, T], L^{2}) \cap L^{2}(0, T; H^{1}),
\]
and for \(t \in (0, T)\) it holds
\[
\|M^{(n)}(t)\| \leq \gamma \|f^{(n)}(t)\|_{\mathbb{L}^{2}} + \gamma \|\nabla f^{(n)}(t)\|_{\mathbb{L}^{2}},
\]
\[
\|M^{(n)}(t)\| \leq \gamma \|f^{(n)}(t)\|_{\mathbb{L}^{2}} + \|H_{e}(t)\|,
\]
\[
\|H^{(n)}(t)\| \leq \gamma \|f^{(n)}(t)\|_{\mathbb{L}^{2}} + \|H_{e}(t)\|,
\]
where \(H^{(n)} = \nabla \varphi^{(n)}\). Moreover if \(H_{e} \in L^{2}(0, T, H^{1})\), then \(\varphi^{(n)} \in L^{2}(0, T; H^{2})\) and we have for \(t \in (0, T)\)
\[
\|\nabla H^{(n)}(t)\| \leq \gamma \|\nabla f^{(n)}(t)\|_{\mathbb{L}^{2}} + \|\nabla H_{e}(t)\|.
\]
Proof. We proceed by induction and use the results of Section 2. Assume that \( f^{(n)} \in C([0, T]; L^2_\varphi) \cap L^2(0, T; L^2_{\varphi} \cap L^2_\psi) \), since \( \nabla_x M(f^{(n)}) = M(\nabla_x f^{(n)}) \) then using (32) leads to \( M^{(n)} \in C([0, T]; L^2) \cap L^2(0, T; H^1) \) and satisfies (64). From Lemma 2.2, we see that \( \varphi^{(n)} \in L^2(0, T; H^1) \) and so \( H^{(n)} \in L^2(0, T; L^2) \). Using (27) and (64) we get (66) and if \( H_\epsilon \in L^2(0, T; H^1) \), then \( \varphi^{(n)} \in L^2(0, T; H^2) \) and from (29) and (64) we get (67). Now we solve problem (58) by applying Lemma 4.2 with the following data

\[
K = H^{(n)}, \quad u_0 = f_0, \quad U = \frac{\chi_0}{T} \psi H^{(n)} \in L^2(0, T; L^2_\varphi).
\]

Hence problem (58) admits a unique weak solution \( f^{(n+1)} \in C([0, T]; L^2_\varphi) \cap L^2(0, T; L^2_\psi \cap L^2_\varphi) \). Moreover \( f^{(n+1)} \) satisfies equation (58) in the sense of distributions and the initial condition in \( L^2_\varphi \). Proposition 5 is then proved.

4.2. Uniform estimates and weak convergence. Now, let us prove this uniform estimate with respect to \( n \).

Lemma 4.3. Under hypotheses (38), let for \( \nu > 0 \) fixed, \( (f^{(n)}, \varphi^{(n)})_n \) be the sequence provided by Proposition 5. Then it holds for all \( n \in \mathbb{N} \) and \( t \in [0, T] \) the following inequality

\[
\|f^{(n+1)}(t)\|_{L^2_\varphi}^2 + 2\nu \int_0^t \|\nabla_{x,v}f^{(n+1)}(s)\|_{L^2_\varphi}^2 \, ds + \frac{\gamma}{2\tau} \int_0^t \|f^{(n+1)}(s)\|_{L^2_\varphi}^2 \, ds \leq C e^{Ct},
\]

where \( C > 0 \) is a constant independent of \( n \) and \( \nu \). Therefore the sequences \( (f^{(n)}, M^{(n)})_n \) and \( (\varphi^{(n)})_n \) are uniformly bounded (with respect to \( n \)) in \( C([0, T]; L^2_\psi \cap L^2(0, T; H^1)) \) and \( L^2(0, T; H^1) \) respectively. Moreover, if \( H_\epsilon \in L^2(0, T, H^1) \) then \( \varphi^{(n)} \) is uniformly bounded in \( L^2(0, T; H^2) \).

Proof. We recall inequality (63) that we rewrite in the form

\[
\Theta^\nu(t; u, u) \geq \nu \left( \|\nabla_x u\|_{L^2_\varphi}^2 + \|\nabla_v u\|_{L^2_\varphi}^2 \right) + \frac{\gamma}{\tau} \|u\|_{L^2_\varphi}^2, \quad \forall u \in H^1_\varphi.
\]

Therefore the equality

\[
\frac{1}{2} \frac{d}{dt} \|f^{(n+1)}\|_{L^2_\varphi}^2 + \Theta^\nu(t; f^{(n+1)}, f^{(n+1)}) = \frac{\chi_0}{T} \psi H^{(n)}, f^{(n+1)}),
\]

yields to

\[
\frac{1}{2} \frac{d}{dt} \|f^{(n+1)}\|_{L^2_\varphi}^2 + \nu \|\nabla_{x,v}f^{(n+1)}\|_{L^2_\varphi}^2 + \frac{\gamma}{\tau} \|f^{(n+1)}\|_{L^2_\varphi}^2 \leq \frac{\chi_0}{T} \int H^{(n)} \cdot f^{(n+1)} \, dx dv.
\]

Using Cauchy-Schwarz inequality and (66) to estimate the right hand side, we get

\[
\frac{\chi_0}{T} \left| \int H^{(n)} \cdot f^{(n+1)} \, dx dv \right| \leq \frac{\chi_0}{T} \|H^{(n)}\| \|f^{(n+1)}\|_{L^2_\varphi} \leq \frac{\chi_0}{T} \left( \gamma \|f^{(n)}\|_{L^2_\varphi} + \|H_\epsilon\| \right) \|f^{(n+1)}\|_{L^2_\varphi} \leq \frac{\gamma}{2\tau} \|f^{(n+1)}\|_{L^2_\varphi}^2 + \frac{\chi_0^2}{T\gamma} \left( \gamma^2 \|f^{(n)}\|_{L^2_\varphi}^2 + \|H_\epsilon\|^2 \right).
\]
This leads to
\[
\frac{d}{dt} \|f^{(n+1)}\|_{L^2_0}^2 + 2\nu \|\nabla (x,v) f^{(n+1)}\|_{L^2_0}^2 + \frac{\gamma}{T} \|f^{(n+1)}\|_{L^2_0}^2 \leq 2\chi^2 \gamma \tau \|f^{(n)}\|_{L^2_0}^2 + 2\chi^2 \gamma \|H_c(t)\|^2. \tag{70}
\]

Therefore we deduce that for \(n \in \mathbb{N}\) and \(t \in [0, T]\), it holds
\[
\|f^{(n+1)}(t)\|_{L^2_0}^2 \leq \|f_0\|_{L^2_0}^2 + 2\chi^2 \gamma \tau \|H_c\|_{L^2(0,T;\mathbb{R}^3)}^2 + 2\chi^2 \gamma \int_0^t \|f^{(n)}(s)\|_{L^2_0}^2 \, ds \leq C \left(1 + \int_0^t \|f^{(n)}(s)\|_{L^2_0}^2 \, ds\right),
\]
where \(C = \max \left(\|f_0\|_{L^2_0}^2 + 2\chi^2 \gamma \tau \|H_c\|_{L^2(0,T;\mathbb{R}^3)}^2, \frac{2\chi^2 \gamma}{\tau}\right)\). Setting \(a_n(t) = \|f^{(n)}(t)\|_{L^2_0}^2\), this inequality reduces to
\[
a_{n+1}(t) \leq C \left(1 + \int_0^t a_n(s) \, ds\right),
\]
which implies that
\[
a_n(t) \leq C \sum_{k=0}^{n-1} \frac{(Ct)^k}{k!}, \quad \forall n \in \mathbb{N}^*, \quad t \in [0, T].
\]

Therefore we conclude that
\[
\|f^{(n)}(t)\|_{L^2_0}^2 \leq C e^{Ct}, \quad \forall n \in \mathbb{N}, \quad t \in [0, T]. \tag{71}
\]

Coming back to (70), we achieve the proof of (68) and the proof of the lemma by using the bounds (64), (66) and (67).

These results allow to deduce directly the following weak convergence results.

**Corollary 1.** Under assumptions of Lemma 4.3, for any \(\nu > 0\) there exist functions \(f^{\nu}, M^{\nu}, \varphi^{\nu}\) such that up to a subsequence labeled again \((f^{(n)}, M^{(n)}, \varphi^{(n)})_n\) we get as \(n \to \infty\)
\[
(f^{(n)}, M^{(n)}) \to (f^{\nu}, M^{\nu}) \quad \text{weakly} \to \text{ in } L^\infty(0,T;L^1_0 \times L^2),
\]
\[
(f^{(n)}, M^{(n)}) \to (f^{\nu}, M^{\nu}) \quad \text{weakly in } L^2(0,T;H^1_0 \times H^1), \tag{72}
\]
\[
\varphi^{(n)} \to \varphi^{\nu} \quad \text{weakly in } L^2(0,T;H^1).
\]

Moreover \(M^{\nu} = M(f^{\nu})\) and if \(H_c \in L^2(0,T;H^1)\), then
\[
\varphi^{(n)} \to \varphi^{\nu} \quad \text{weakly in } L^2(0,T;H^2).
\]

Unfortunately, these weak convergences are not sufficient to perform the limit in problem (58) as \(n \to +\infty\). In the next paragraph, we aim to prove the strong convergence of the iterative scheme.
The Kinetic Bloch Equation

4.3. Strong convergence. Let us prove that:

**Lemma 4.4.** Under assumptions of Lemma 4.3, the sequence \((f^{(n)}, M^{(n)}, \varphi^{(n)})_n\) converges strongly towards \((f^\nu, M^\nu, \varphi^\nu)\) in 
\[C([0, T]; L^2_\psi) \times C([0, T]; L^2) \times L^2(0, T; H^1).\]

Moreover \(M^\nu = \mathcal{M}(f^\nu)\) and \(\varphi^\nu = \mathcal{H}(H_\nu, M^\nu)\) solve the perturbed magnetostatic equations (24).

**Proof.** We have only to prove the strong convergence. Setting for \(n \geq 1\)
\[g^{(n)} = f^{(n)} - f^{(n-1)}, \quad \phi^{(n)} = \varphi^{(n)} - \varphi^{(n-1)}, \quad h^{(n)} = H^{(n)} - H^{(n-1)},\]
then \(g^{(n+1)}\) satisfies the following equation
\[
\partial_t g^{(n+1)} + \mathcal{D}^\nu(g^{(n+1)}) + \mathcal{T}^\nu(g^{(n+1)}, H^{(n)}) + \mathcal{A}(g^{(n+1)}, \mathcal{M}(g^{(n+1)}))
+ \frac{\gamma}{\tau} g^{(n+1)} = G_1^{(n)} + G_2^{(n)} \quad \text{in} \quad (0, T),
\]
(73)
\[g^{(n+1)}(0) = 0,\]
and \(\phi^{(n)}\) satisfies the equations
\[
\nabla \phi^{(n)} = h^{(n)}, \quad \text{div} \ (h^{(n)} + m^{(n)}) - \nu \phi^{(n)} = 0,
\]
with \(m^{(n)} = \mathcal{M}(g^{(n)})\) and
\[G_1^{(n)} = \alpha \left( (\psi^\nu v \times (\rho_\nu \ast h^{(n)}) \cdot \nabla) \, f^{(n)} \right) \in L^2(0, T; L^2_\psi^2),\]
\[G_2^{(n)} = \frac{\chi_0}{\tau} \psi \, h^{(n)} - \beta (\rho_\nu \ast h^{(n)}) \times f^{(n)} \in L^2(0, T; L^2_\psi^2).\]

Hence \(\phi^{(n)} = \mathcal{H}^\nu(0, m^{(n)})\) and using (29) and (32), we see that for \(t \in (0, T)\) it holds
\[
\|h^{(n)}(t)\| \leq \|m^{(n)}(t)\| \leq \gamma \|g^{(n)}(t)\|_{L^2_\psi}, \quad \sqrt{\nu} \|\phi^{(n)}(t)\| \leq \gamma \|g^{(n)}(t)\|_{L^2_\psi},
\]
(74)
Next, from the relation
\[
\frac{1}{2} \frac{d}{dt} \|g^{(n+1)}\|_{L^2_\psi}^2 + \nu \|\nabla(x, v) g^{(n+1)}\|_{L^2_\psi}^2 + \frac{\gamma}{\tau} \|g^{(n+1)}\|_{L^2_\psi}^2 \leq (G_1^{(n)} + G_2^{(n)}, g^{(n+1)})_{L^2_\psi},
\]
we get as in the proof of Lemma 4.3
\[
\frac{1}{2} \frac{d}{dt} \|g^{(n+1)}\|_{L^2_\psi}^2 + \nu \|\nabla(x, v) g^{(n+1)}\|_{L^2_\psi}^2 + \frac{\gamma}{\tau} \|g^{(n+1)}\|_{L^2_\psi}^2 \leq (G_1^{(n)} + G_2^{(n)}, g^{(n+1)})_{L^2_\psi}.
\]
Let us estimate the right hand side of this inequality. First using (68) and (74) we obtain
\[
\|G_2^{(n)}\|_{L^2_\psi} \leq \frac{\chi_0}{\tau} \|h^{(n)}\| + \beta \|\rho_\nu\| \|h^{(n)}\| \|f^{(n)}\|_{L^2_\psi}
\]
\[
\leq \gamma \|g^{(n)}\|_{L^2_\psi} \left( \frac{\chi_0}{\tau} + \beta C_\nu C e^{C t} \right) \leq C_\nu \|g^{(n)}\|_{L^2_\psi}.
\]
where $C_\nu > 0$ denotes different constants which are independent of $n$. To deal with the term $(G^{(n)}_1, g^{(n+1)})_{L^2}$, we use an integration by parts to get

\[
(G^{(n)}_1, g^{(n+1)})_{L^2} = \alpha \sum_i \int \partial_{\nu_i} \left( (\psi^{(n)} v \times (\rho_\nu \ast h^{(n)})), f^{(n)} \right) \frac{g^{(n+1)}}{\psi} \, dxdv
\]

\[
= -\alpha \sum_i \int (\psi^{(n)} v \times (\rho_\nu \ast h^{(n)}), f^{(n)}) \left( \frac{\partial_{\nu_i} g^{(n+1)}}{\psi} + \frac{v_i g^{(n+1)}}{\psi} \right) \, dxdv.
\]

\[
= -\alpha \int \left( (\psi^{(n)} v \times (\rho_\nu \ast h^{(n)})) \cdot \nabla_v \right) g^{(n+1)} \cdot \frac{f^{(n)}}{\psi} \, dxdv.
\]

Hence, we have

\[
\left| (G^{(n)}_1, g^{(n+1)})_{L^2} \right| \leq C_{\nu} \| f^{(n)} \|_{L^2} \| h^{(n)} \| \| \nabla_v g^{(n+1)} \|_{L^2},
\]

and using (68) and (74), we obtain

\[
\left| (G^{(n)}_1, g^{(n+1)})_{L^2} \right| \leq C_{\nu} \| h^{(n)} \| \| \nabla_v g^{(n+1)} \|_{L^2} \leq \frac{\nu}{2} \| \nabla_v g^{(n+1)} \|_{L^2} + C_{\nu} \| g^{(n)} \|_{L^2},
\]

which leads to

\[
\frac{1}{2} \frac{d}{dt} \| g^{(n+1)} \|_{L^2}^2 + \frac{\nu}{2} \| \nabla (x,v) g^{(n+1)} \|_{L^2}^2 + \frac{\gamma}{2r} \| g^{(n+1)} \|_{L^2}^2 \leq C_{\nu} \| g^{(n)} \|_{L^2}^2.
\]

Thus, we deduce that for all $n \in \mathbb{N}^*$ and $t \in [0, T]$

\[
\| g^{(n+1)}(t) \|_{L^2}^2 \leq C_{\nu} \| g^{(1)} \|_{L^2}^2 L^{\infty}([0,T];L^2) \frac{t^{n-1}}{(n-1)!},
\]

which means that $(f^{(n)})_n$ is a Cauchy sequence in $C([0,T];L^2)$. Going back to (74) we conclude that $(M^{(n)})_n$ and $(\varphi^{(n)})_n$ are Cauchy sequences respectively in $C([0,T];L^2)$ and $L^2(0,T;H^1)$. \hfill $\Box$

### 4.4. End of proof of Theorem 4.1

The results of Lemma 4.4 allow to perform the limit when $n \to \infty$ in problem (58) and we get that $(f^{(n)}, \varphi^{(n)}) \in (C([0,T];L^2) \cap L^2(0,T;H^1)) \times L^2(0,T;H^1)$, satisfies estimates (57), the equations of the regularized problem $(Q^{(n)})$ in the sense of distributions and the initial condition almost everywhere in $\mathbb{R}^3 \times \mathbb{R}^3$.

Moreover if $H_e \in L^2(0,T;H^1)$ then $\varphi^{(n)} \in L^2(0,T;H^2)$ and satisfies the perturbed magnetostatic equations almost everywhere in $(0,T) \times \mathbb{R}^3$.

To achieve the proof of Theorem 4.1, let us establish the uniqueness of the solution: let $(f_i, \varphi_i) \in (C([0,T];L^2) \cap L^2(0,T;H^1)) \times L^2(0,T;H^1)$ be two solutions of problem $(Q^{(n)})$ and let $M_i = M(f_i) \in C([0,T];L^2) \cap L^2(0,T;H^1)$ and $H_i = \nabla \varphi_i \in L^2((0,T) \times \mathbb{R}^3)$ for $i = 1, 2$. We set

\[
g = f_1 - f_2, \quad \phi = \varphi_1 - \varphi_2, \quad m = M_1 - M_2, \quad h = H_1 - H_2.
\]

Then $(g, \phi)$ satisfies the following problem

\[
\begin{align*}
\partial_t g + D^{(n)}(g) + T^{(n)}(g,H_1) + A(g,M(g)) + \gamma g &= \frac{\chi_0}{\tau} \psi h + G \quad \text{in} \ (0,T), \\
g(0) &= 0,
\end{align*}
\]

where

\[
G = \alpha \left( (\psi^{(n)} v \times (\rho_\nu \ast h)) \cdot \nabla_v \right) f_2 - \beta (\rho_\nu \ast h) \times f_2,
\]
coupled to the magnetostatic equations
\[
\text{div } (h + m) - \nu \phi = 0, \ h = \nabla \phi. \quad (78)
\]

Therefore, it holds that
\[
\frac{1}{2} \frac{d}{dt} \|g\|_{L^2}^2 + \nu \|\nabla_{(x,v)} g\|_{L^2}^2 + \frac{\gamma}{\tau} \|g\|_{L^2}^2 \leq \frac{\chi_0}{\tau} \int h \cdot g \, dx \, dv + (G,g)_{L^2}.
\]

Since \( \int h \cdot g \, dx \, dv = \frac{1}{2} \int h \cdot m \, dx \) and \( \phi = H^\nu(0, m) \), using (27) we obtain
\[
\int h \cdot g \, dx \, dv = - \frac{1}{\gamma} \|h\|^2 - \frac{\nu}{\gamma} \|\phi\|^2,
\]
and
\[
\|h\| \leq \|m\| \leq \gamma \|g\|_{L^2}^2. \quad (79)
\]

Next, using an integration by parts, (57) and (79) (see also (75)), we deduce that
\[
(G,g)_{L^2} \leq C_\nu \|f_2\|_{L^2} \|h\| (\|\nabla_{x,v} g\|_{L^2} + \|g\|_{L^2})
\]
\[
\leq C_\nu \|g\|_{L^2} (\|\nabla_{x,v} g\|_{L^2} + \|g\|_{L^2}) \leq \frac{\nu}{2} \|\nabla_{x,v} g\|_{L^2}^2 + C_\nu \|g\|_{L^2}^2,
\]
so we arrive at the inequality
\[
\frac{1}{2} \frac{d}{dt} \|g\|_{L^2}^2 + \frac{\gamma}{\tau} \|g\|_{L^2}^2 + \frac{\chi_0}{\gamma \tau} (\|h\|^2 + \nu \|\phi\|^2) \leq C_\nu \|g\|_{L^2}^2. \quad (80)
\]

In particular
\[
\frac{1}{2} \frac{d}{dt} \|g\|_{L^2}^2 \leq C_\nu \|g\|_{L^2}^2,
\]
and since \( g(0) = 0 \) we get \( \|g(t)\|_{L^2} = 0 \) for all \( t \in [0,T] \) which leads to \( \phi(t) = 0 \) for a.e. \( t \in (0,T) \). This ends the proof of Theorem 4.1.

5. Regularity of the solution for problem \((Q^\nu)\). In this section, we prove a regularity result for the solution of \((Q^\nu)\) provided by Theorem 4.1. We establish also new uniform estimates with respect to \( \nu \) which are local in time.

**Theorem 5.1.** Let \( f_0 \in H^1_\psi \), \( H \in L^2(0,T;H^2) \) and let for \( \nu > 0 \) fixed, \((f^\nu,\varphi^\nu)\) the solution of the regularized problem \((Q^\nu)\) defined on \([0,T]\) provided by Theorem 4.1. Then we have
\[
f^\nu \in C([0,T];\mathbb{H}^3) \cap L^2(0,T;\mathbb{H}^4) \cap H^1(0,T;H^2), \quad \varphi^\nu \in L^2(0,T;H^4). \quad (81)
\]

Moreover there exists a time \( T^* \in ]0,T[ \) independent of \( \nu \) and \( \varepsilon \) such that \((f^\nu,\varphi^\nu)\) satisfies the following uniform bounds
\[
\|f^\nu\|_{C([0,T^*];H^3_2)} + \|M^\nu\|_{C([0,T^*];H^3)} + \|H^\nu\|_{L^2(0,T^*;H^3)} \leq C^*,
\]
\[
\sqrt{\nu} (\|\nabla^2 f^\nu\|_{L^2(0,T^*;H^3)} + \|\varphi^\nu\|_{L^2(0,T^*;H^3)}) \leq C^*.
\]
\[
\|\partial_t f^\nu\|_{L^2(0,T^*;L^2)} \leq \frac{C^*}{\varepsilon},
\]
where \( M^\nu = M(f^\nu) \) and \( H^\nu = \nabla \varphi^\nu \). The constant \( C^* > 0 \) depends on \( T^* \), \( \|f_0\|_{H^3_2} \) and \( \|H\|_{L^2(0,T^*;H^3)} \) but not of \( \nu \) and \( \varepsilon \).
Proof. We will consider the equation satisfied by each derivative of \( f \), up to order 3. For simplicity, we denote by \( \partial^d \) (resp \( D^d \)) any derivative \( \partial^d \) (resp \( \partial^d \)) of order \( |\delta| = d \). Accordingly, Leibnitz formula will be simply written as follows for operator \( \partial^d \)

\[
\partial^d(\varphi \xi) = \sum_{0 \leq d_1 \leq d} \binom{d}{d_1} \partial^{d-d_1} \varphi \partial^{d_1} \xi,
\]

the same being valuable for \( D^d \). The proof is divided into several steps.

Step 1. We will prove the regularity result. To this purpose, we will only verify the result stated in Lemma 5.2 below, therefore using a recursive reasoning, we will get the result stated in (81) for \( f^\nu \). Finally Proposition 1 implies that \( M^\nu = \mathcal{M}(f^\nu) \in L^2(0,T; H^3) \) and using Lemma 2.2, we get that \( \varphi^\nu \in L^2(0,T; H^4) \).

Let us prove that:

**Lemma 5.2.** If \( f_0 \in \mathbb{H}^1_\psi \) and \( H_\nu \in L^2(0,T; H^1) \) then \( \varphi^\nu \in L^2(0,T; H^2) \) and

\[
f^\nu \in C([0,T] ; \mathbb{H}^1_\psi) \cap L^2(0,T; \mathbb{H}^2_\psi) \cap H^1(0,T; L^2), \tag{84}
\]

and if in addition \( H_\nu \in L^2(0,T; H^2) \) then \( \varphi^\nu \in L^2(0,T; H^3) \).

**Proof.** From Theorem 4.1, we know that \( \varphi^\nu \in L^2(0,T; H^2) \) and \( f^\nu \in C([0,T]; \mathbb{L}^2_\psi) \cap L^2(0,T; \mathbb{H}^1_\psi) \) satisfies the linear parabolic equation

\[
\partial_t f^\nu - \nu \Delta x f^\nu - \nu \Delta_v f^\nu = F^\nu \quad \text{in} \quad (0,T), \quad f^\nu(0) = f_0,
\]

where

\[
F^\nu = \nu (v \cdot \nabla v) f^\nu - \nu f^\nu H^\nu - \mathcal{A}(f^\nu, \mathcal{M}(f^\nu)) - \frac{\gamma}{\tau} f^\nu + \frac{\lambda}{\tau} \psi H^\nu \in L^2(0,T; L^2),
\]

with \( H^\nu = \nabla \varphi^\nu \) and \( f_0 \in H^1 \). Therefore \( f^\nu \in C([0,T]; H^1) \cap L^2(0,T; H^2) \) and the derivative \( \partial^1 f^\nu \) satisfies the equation

\[
\partial_t u^\nu - \nu \Delta_x u^\nu - \nu \Delta_v u^\nu = \partial^1 F^\nu \quad \text{in} \quad (0,T), \quad u^\nu(0) = \partial^1 f_0. \tag{85}
\]

Henceforth \( \partial^1 f^\nu \in C([0,T]; L^2) \cap L^2(0,T; H^1) \) and \( \partial^1 f^\nu \) is the unique solution of the equation in this space. We rewrite equation (85) in the form

\[
\partial_t u^\nu + \mathcal{D}^{\nu}(u^\nu) + \mathcal{T}^{\nu}(u^\nu, H^\nu) + \mathcal{A}(u^\nu, \mathcal{M}(u^\nu)) + \frac{\gamma}{\tau} u^\nu = U^\nu \quad \text{in} \quad (0,T), \tag{86}
\]

\[
u(0) = \partial^1 f_0,
\]

where

\[
U^\nu = \alpha \left( (\psi^\nu v \times (\rho_\nu \ast \partial^1 H^\nu)) \cdot \nabla v \right) f^\nu - \beta \left( \rho_\nu \ast \partial^1 H^\nu \right) \cdot f^\nu + \frac{\lambda}{\tau} \psi H^\nu \in L^2(0,T; \mathbb{L}^2_\psi).
\]

By Lemma 4.2, equation (86) has a unique solution in \( C([0,T]; \mathbb{L}^2_\psi) \cap L^2(0,T; \mathbb{H}^1_\psi) \subset C([0,T]; L^2) \cap L^2(0,T; H^1) \) so we conclude that

\[
u(0) = \partial^1 f^\nu \in C([0,T]; \mathbb{L}^2_\psi) \cap L^2(0,T; \mathbb{H}^1_\psi).
\]

As a consequence \( \partial^1 M^\nu \in L^2(0,T; H^1) \), so \( \partial^1 \varphi^\nu \in L^2(0,T; H^1) \) and if in addition, \( H_\nu \in L^2(0,T; H^2) \) then we deduce that \( \partial^1 \varphi^\nu \in L^2(0,T; H^2) \). Analogous results hold for \( w^\nu := D^1 f^\nu \) which satisfies the equation

\[
\partial_t w^\nu + \mathcal{D}^{\nu}(w^\nu) + \mathcal{T}^{\nu}(w^\nu, H^\nu) + \mathcal{A}(w^\nu, \mathcal{M}(w^\nu)) + \frac{\gamma}{\tau} w^\nu = W^\nu \quad \text{in} \quad (0,T),
\]

\[
u(0) = D^1 f_0,
\]
where
\[ W' = \nu D^1 f' + \alpha (D^1(\psi' v) \times (\rho_\nu \ast H')) \cdot \nabla_v f' - D^1(\psi' v) \cdot \nabla_x f + \frac{\chi_0}{\tau} D^1 \psi H' \in L^2(0, T; \mathbb{L}_\nu^2). \]

Therefore
\[ D^1 f' \in C([0, T]; \mathbb{L}_\nu^2) \cap L^2(0, T; \mathbb{H}_\nu^1), \]
and coming back to the equation satisfied by \( f' \), we see that all the terms of the time derivative belong to \( L^2(0, T; \mathbb{L}_\nu^2) \) except for \( \nu(v \cdot \nabla_v) f' \) which belongs to \( L^2(0, T; L^2) \) and so \( \partial_t f' \in L^2(0, T; L^2) \). This completes the proof of the lemma.

**Step 2.** Now we will prove the uniform bounds given in (82) and (83). For simplicity, we omit the superscript \( \nu \) on the functions \( f' \), \( H' \) and \( M' = M(f') \).

Let \( e, d \in \mathbb{N} \) such that \( d + e \leq 3 \), we set \( g = D^d \varphi f \). Then, \( g \) satisfies the equation
\[ \partial_t g + \mathcal{D}'(g) + T'(g, H) + \mathcal{A}'(f, M) + \mathcal{B}'(f, H) = \sum_{i=1}^4 F_i^{(d,e)} \quad \text{in } (0, T), \tag{87} \]
see (30) for the definition of \( \mathcal{A}'(f, M) \) and \( \mathcal{B}'(f, H) \) and where
\[ F_1^{(0,e)} = F_2^{(0,e)} = F_3^{(0,0)} = F_4^{(d,0)} = 0, \]
\[ F_1^{(d,e)} = \nu \sum_{0 \leq d_1 < d} \binom{d}{d_1} (D^{d-d_1} v \cdot \nabla_v) D^{d_1} \varphi f \quad \text{if } d \geq 1, \]
\[ F_2^{(d,e)} = -\sum_{0 \leq d_1 < d} \binom{d}{d_1} D^{d-d_1} (\psi' v) \cdot \nabla_x D^{d_1} \varphi f \quad \text{if } d \geq 1, \]
\[ F_3^{(d,e)} = \alpha \sum_{(d_1, e_1) \in E^{(d,e)}} \binom{e_1}{e_1} \left( \left( D^{d-d_1} (\psi' v) \times (\rho_\nu \ast \varphi^{e-e_1} H) \right) \cdot \nabla_v \right) D^{d_1} \varphi f \]
\[ \hspace{1cm} \text{if } (d, e) \neq (0, 0), \]
\[ F_4^{(d,e)} = -\beta \sum_{0 \leq e_1 < e} \binom{e_1}{e_1} (\rho_\nu \ast \varphi^{e-e_1} H) \times D^d \varphi f \quad \text{if } e \geq 1, \tag{88} \]
here we set
\[ E^{(d,e)} = \{(d_1, e_1); \; d_1 \leq d, \; e_1 \leq e, \; (d_1, e_1) \neq (d, e)\}. \tag{89} \]

Since \( (T'(g, H), g)_{\mathbb{L}_\nu^2} = 0 \), then using entropy inequalities (33) and (37) (with \( 1 \leq \theta \leq 6 \) for \( \theta \in \mathbb{N}^d, |\theta| = d \leq 3 \)), we obtain
\[ \begin{aligned}
\frac{1}{2} \frac{d}{dt} \| g \|_{\mathbb{L}_\nu^2}^2 + \frac{\gamma}{\tau} \| g \|_{\mathbb{L}_\nu^2}^2 + \frac{\chi_0}{2\tau \gamma} \| \varphi H \|_2^2 + \nu \left( \frac{\chi_0}{\tau \gamma} \| \varphi H \|_2^2 + \| \nabla x, v g \|_{\mathbb{L}_\nu^2}^2 \right) \\
\leq \frac{3\chi_0}{\tau \gamma} \| \varphi H \|_2^2 + \| g \|_{\mathbb{L}_\nu^2}^2 + \sum_{i=1}^4 \| F_i^{(d,e)} \|_{\mathbb{L}_\nu^2}. \tag{90} 
\end{aligned} \]

Next we will estimate the norms \( \| F_i^{(d,e)} \|_{\mathbb{L}_\nu^2} \) appearing in the right hand side of inequality (90). From now on, \( C > 0 \) denotes various constants which are independent of \( \nu \) and \( \varepsilon \).
1. **Estimation of** $\|F^{(d,e)}_1\|_{L^2_v}$. For $d \geq 1$

$$F^{(d,e)}_1 = \nu C^{d-1}_d (D^3 v \cdot \nabla v) D^{d-1} \partial f = \nu C^{d-1}_d (D^{d-1} \partial f),$$

so for $\nu > 0$ small

$$\|F^{(d,e)}_1\|_{L^2_v} \leq C \|f\|_{H^3_v}. \tag{91}$$

2. **Estimation of** $\|F^{(d,e)}_2\|_{L^2_v}$. Since $\|D^{d-d_1} (\psi v)\|_{L^\infty} \leq C$ for $\nu > 0$ small, we get

$$\|F^{(d,e)}_2\|_{L^2_v} \leq C \|f\|_{H^3_v}. \tag{92}$$

3. **Estimation of** $\|F^{(d,e)}_3\|_{L^2_v}$. Since $d + e \leq 3$ then

$$d_1 + e_1 \leq 2, \quad \forall \ (d_1, e_1) \in E^{(d,e)}. \tag{93}$$

Therefore using Sobolev embedding theorems, we can bound each term of $F^{(d,e)}_3$, according to the different values of $e - e_1$, as follows:

3.1. *If $e - e_1 \leq 1$* then since $\|\rho_v\|_{L^1} = 1$, using Young’s inequality and the Sobolev embedding $H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$ we get

$$\|\rho_v \ast \partial^{e-e_1} H\|_{L^\infty} \leq \|\partial^{e-e_1} H\|_{L^\infty} \leq C \|\partial^{e-e_1} H\|_{H^2} \leq C \|H\|_{H^3},$$

and the inequality

$$\|((D^{d-d_1} (\psi v) \ast (\rho_v \ast \partial^{e-e_1} H)) \cdot \nabla v) D^{d_1} \partial f\|_{L^2_v} \leq C \|\partial^{e-e_1} H\|_{L^\infty} \|\nabla_v D^{d_1} \partial f\|_{L^2_v},$$

leads by using (93) to

$$\|((D^{d-d_1} (\psi v) \ast (\rho_v \ast \partial^{e-e_1} H)) \cdot \nabla v) D^{d_1} \partial f\|_{L^2_v} \leq C \|H\|_{H^3} \|f\|_{H^3_v}. \tag{94}$$

3.2. *If $e - e_1 = 2$*, (then $e \geq 2$), according to the Sobolev embedding $H^1(\mathbb{R}^3) \subset L^4(\mathbb{R}^3)$, we get

$$\|\rho_v \ast \partial^{e-e_1} H\|_{L^4} \leq \|\partial^2 H\|_{L^4} \leq C \|\partial^2 H\|_{H^1} \leq C \|H\|_{H^3},$$

and

$$\left\| \nabla_v D^{d_1} \partial f \right\|_{L^4(L^2_v)} \leq C \left\| \nabla_v D^{d_1} \partial f \right\|_{H^1(L^2_v)} \leq C \|f\|_{H^3_v},$$

because $2 + e_1 + d_1 = e + d_1 \leq e + d \leq 3$, so inequality (94) is satisfied.

3.3. *If $e - e_1 = 3$* then ($e = 3, \ e_1 = 0$ and $d = 0 = d_1$)

$$\|\rho_v \ast \partial^{e-e_1} H\| \leq \|\partial^3 H\| \leq \|H\|_{H^3},$$

and

$$\left\| \frac{\nabla_v D^{d_1} \partial f}{\sqrt{\psi}} \right\|_{L^\infty(L^2_v)} = \left\| \frac{\nabla_v f}{\sqrt{\psi}} \right\|_{L^\infty(L^2_v)} \leq C \left\| \frac{\nabla_v f}{\sqrt{\psi}} \right\|_{H^1(L^2_v)} \leq C \|f\|_{H^3_v},$$

and we get again inequality (94).

Gathering all the previous results, we conclude that

$$\|F^{(d,e)}_3\|_{L^2_v} \leq C \|H\|_{H^3} \|f\|_{H^3_v}. \tag{95}$$
4. Estimation of $\|F_4^{(d,e)}\|_{L^2_v}$. Since $e_1 + d \leq e + d \leq 3$, we can bound each term of $F_4^{(d,e)}$ according to the different values of $e - e_1$ as we have done to estimate $F_3^{(d,e)}$ and we get the same result

$$
\|F_4^{(d,e)}\|_{L^2_v} \leq C \|H\|_{H^3} \|f\|_{H^3}.
$$

(96)

Gathering inequalities (91), (92), (95) and (96) we arrive at

$$
\sum_{i=1}^{4} \|F_i^{(d,e)}\|_{L^2_v} \leq C \|f\|_{H^3} (1 + \|H\|_{H^3}),
$$

thus

$$
\|g\|_{L^2_v} \sum_{i=1}^{4} \|F_i^{(d,e)}\|_{L^2_v} \leq \frac{\gamma}{2T} \|g\|_{L^2_v}^2 + C \|f\|_{H^3}^2 + C \|f\|_{H^3} \|H\|_{H^3} \|g\|_{L^2_v}.
$$

(97)

Inserting inequality (97) into (90) leads to the following inequality for $\nu$ small enough

$$
\frac{1}{2} \frac{d}{dt} \|g\|_{L^2_v}^2 + \frac{\gamma}{2T} \|g\|_{L^2_v}^2 + \frac{\chi_0}{\tau \gamma} \|H\|_{H^3}^2 + \frac{1}{\tau \gamma} \|\nabla (x,v) f\|_{L^2_v}^2 + \nu \left( \|\Delta \varphi\|^2 + \|\nabla (x,v) g\|_{L^2_v}^2 \right)
$$

$$
\leq \frac{3 \chi_0}{\tau \gamma} \|H\|_{H^3}^2 + C \|f\|_{H^3}^2 + C \|H\|_{H^3} \|f\|_{H^3} (1 + \|H\|_{H^3}),
$$

(98)

We recall that $g = D^d \varphi f$ so summing up inequalities (98) for $d + e \leq 3$ we get

$$
\frac{1}{2} \frac{d}{dt} \|f\|_{H^3}^2 + \frac{\gamma}{2T} \|f\|_{H^3}^2 + \frac{\chi_0}{\tau \gamma} \|H\|_{H^3}^2 + \frac{1}{\tau \gamma} \|\nabla (x,v) f\|_{H^3}^2 + \nu \left( \|\Delta \varphi\|^2 + \|\nabla (x,v) f\|_{H^3}^2 \right)
$$

$$
\leq \frac{3 \chi_0}{\tau \gamma} \|H\|_{H^3}^2 + C \|f\|_{H^3}^2 + C \|H\|_{H^3} \|f\|_{H^3} (1 + \|H\|_{H^3}),
$$

(99)

and then

$$
\frac{d}{dt} \|f\|_{H^3}^2 + \frac{\gamma}{\tau} \|f\|_{H^3}^2 + 2 \nu \|\nabla (x,v) f\|_{H^3}^2 + \frac{\chi_0}{\tau \gamma} \|\varphi\|_{H^3}^2 + \frac{\chi_0}{\tau \gamma} \|H\|_{H^3}^2
$$

$$
\leq C (\|H\|_{H^3}^2 + \|f\|_{H^3}^2 + \|f\|_{H^3}^2),
$$

where the constant $C > 0$ is independent of $\nu$ and $\varepsilon$.

We set $g(t) = \|f(t)\|_{H^3}^2$, $y_0 = \|f_0\|_{H^3}$ and $z(t) = C \|H_e(t)\|_{H^3}$ so $y$ satisfies the inequality

$$
y'(t) \leq z(t) + C y(t) + C y^2(t)
$$

in $(0, T)$,

(100)

with $y(0) = y_0$. Then we can use the comparison with the solution $u$ of the Riccati equation

$$
u u'(t) = z(t) + C u(t) + C u^2(t)
$$

in $(0, T)$, $u(0) = y_0,$

to deduce that there exists a time $T^* \in [0, T]$ and a constant $C^* > 0$ which depend on $\|f_0\|_{H^3}$ and $\|H_e\|_{L^2(0,T; H^3)}$ but not of $\nu$ and $\varepsilon$ such that on $[0, T^*]$, we get

$y(t) \leq C^*$.

Therefore from inequality (99) we deduce that

$$
\|f\|_{C([0,T^*];H^3)} + \|H\|_{L^2(0,T^*;H^3)} \leq C^*.
$$

where from now on $C^* > 0$ denotes various constants independent of $\nu$ and $\varepsilon$, and so

$$
\|M\|_{C([0,T^*];H^3)} \leq C^*,
$$

$$
\sqrt{\nu} \left( \|\nabla (x,v) f\|_{L^2(0,T^*;H^3)} + \|\varphi\|_{L^2(0,T^*;H^3)} \right) \leq C^*.
$$
Step 3. Let us prove estimate (83) for $\partial_t f''$. We split $\partial_t f''$ into three parts
\[
\partial_t f'' = P_1'' + P_2'' + P_3'',
\]
where
\[
P_1'' = \nu(\Delta_x f'' + \Delta_v f'') - \beta (\rho_v \ast H'') \times f'' - \frac{1}{\tau} (\gamma f'' - \chi_0 \psi H''),
\]
\[
P_2'' = \nu(v \cdot \nabla_v) f'' - \psi'' v \cdot \nabla_v f'' + \alpha \left( \psi'' v \times (\rho_v \ast H'') \cdot \nabla \right) f'',
\]
\[
P_3'' = -\frac{1}{\varepsilon\tau} (\gamma f'' - \psi M'').
\]

In view of the previous estimates, we see that $P_1''$ is uniformly bounded with respect to $\nu$ and $\varepsilon$ in $L^2(0, T; L^2)$, $P_2''$ is uniformly bounded in $L^2(0, T; L^2)$ and regarding to the last term $P_3''$, we get the result stated in (83). This ends the proof of Theorem 5.1.

6. Proof of Theorem 1.1. Now, we are in position to prove the local-in-time existence result for $\varepsilon > 0$ fixed, to problem $(P_\varepsilon)$ given in Theorem 1.1. Let for $\nu > 0$ fixed, the solution $(f'', \varphi'')$ of problem $(Q'')$ satisfying (81). We consider the following weak formulation of the regularized problem $(Q''')$ with the aim to pass to the limit as $\nu \to 0$. For test functions $\Phi \in D([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3)$ and $\zeta \in L^2(0, T; H^1)$

\[
- \int_0^T f'' \cdot (\partial_t \Phi + \varphi'' v \cdot \nabla_x \Phi) \, dx \, dv \, dt + \nu \int_0^T \int \nabla_x f'' \cdot \nabla_x \Phi \, dx \, dv \, dt - \beta \int_0^T \int (\psi'' v \times (\rho_v \ast H'') \cdot \nabla \Phi) \cdot f'' \, dx \, dv \, dt
\]
\[
+ \alpha \int_0^T \int (\psi'' v \times (\rho_v \ast H'') \cdot \nabla \Phi) \cdot f'' \, dx \, dv \, dt
\]
\[
+ \frac{1}{\varepsilon \tau} \int_0^T \int (\gamma f'' - \psi M'') \cdot \Phi \, dx \, dv \, dt = \int_0^T \int \nabla \varphi'' \cdot \nabla \zeta \, dx \, dt + \beta \int_0^T \int (\rho_v \ast H'') \times f'' \cdot \Phi \, dx \, dv \, dt + \frac{1}{\tau} \int_0^T \int (\gamma f'' - \chi_0 \psi H'') \cdot \Phi \, dx \, dv \, dt
\]
\[
+ \frac{1}{\varepsilon \tau} \int_0^T \int (\gamma f'' - \psi M'') \cdot \Phi \, dx \, dv \, dt = \int_0^T \int (M'' + H_\varepsilon) \cdot \nabla \zeta \, dx \, dt,
\]
where $M'' = M(f'')$ and $H'' = \nabla \varphi''$.

The uniform estimates given in Theorem 5.1 imply that:

**Lemma 6.1.** For $\varepsilon > 0$ fixed, there exist functions $f_\varepsilon$ and $H_\varepsilon$ such that for subsequences, we get as $\nu \to 0$:
\[
f'' \to f_\varepsilon \quad \text{weakly-* in } L^\infty(0, T^*; \mathbb{R}^3),
\]
\[
\nu \nabla_x f'' \to 0 \quad \text{strongly in } L^2(0, T^*; \mathbb{R}^3),
\]
\[
\nu \nabla_v f'' \to 0 \quad \text{strongly in } L^2(0, T^*; \mathbb{R}^3),
\]
\[
M'' \to M_\varepsilon \quad \text{weakly-* in } L^\infty(0, T^*; \mathbb{R}^3),
\]
\[
H'' \to H_\varepsilon \quad \text{weakly in } L^2(0, T^*; H^3),
\]
\[
\nu \varphi'' \to 0 \quad \text{strongly in } L^2(0, T^*; H^3),
\]

(103)
where \( f_\varepsilon \in L^\infty(0, T^*; \mathbb{H}^3) \), \( M_\varepsilon \in L^\infty(0, T^*; H^3) \) and \( H_\varepsilon \in L^2(0, T^*; H^3) \) satisfy (17). Moreover \( M_\varepsilon = M(f_\varepsilon) \) and

\[
f^\nu \to f_\varepsilon \text{ strongly in } L^2(0, T^*; L^2_{loc}),
\]

(104)

\[
\rho_\nu \ast H^\nu \to H_\varepsilon \text{ weakly in } L^2(0, T^*; L^2).
\]

(105)

Proof. We have only to justify the two last convergences. Knowing that \( L^2(0, T^*; H^1) \cap H^1(0, T^*; L^2) \) is compactly embedded into \( L^2(0, T^*; L^2_{loc}) \), we deduce (104). Let us prove (105). Using a density argument, we can consider test functions of the form \( \omega(t) \xi(x) \) where \( \omega \in L^2(0, T^*) \) is a scalar function and \( \xi \in L^2(\mathbb{R}^3) \) is vectorial.

We write setting \( \tilde{\rho}_\nu(x) = \rho_\nu(-x) \)

\[
\int_0^T \int (\tilde{\rho}_\nu \ast H^\nu - H_\varepsilon) \cdot \omega \, dx dt = \int_0^T \int (H^\nu \cdot (\tilde{\rho}_\nu \ast \xi) - H_\varepsilon \cdot \xi) \omega \, dx dt
\]

= \int_0^T \int H^\nu \cdot (\tilde{\rho}_\nu \ast \xi - \xi) \omega \, dx dt + \int_0^T \int (H^\nu - H_\varepsilon) \cdot \xi \omega \, dx dt,

and clearly the last integral tends towards 0 as \( \nu \to 0 \) and the following bound leads to the result

\[
| \int_0^T \int H^\nu \cdot (\tilde{\rho}_\nu \ast \xi - \xi) \omega \, dx dt | \leq \| H^\nu \|_{L^2(0, T^*; L^2)} \| \tilde{\rho}_\nu \ast \xi - \xi \|_{L^2} \| \omega \|_{L^2(0, T^*)}.
\]

In fact, using the derivative of a convolution product, we prove similarly the weak convergence of \( \rho_\nu \ast H^\nu \) in \( L^2(0, T^*; H^1) \).

With the results of Lemma 6.1, we easily perform the limit when \( \nu \to 0 \) in equation (102) to obtain

\[
\int_0^T \int H_\varepsilon \cdot \nabla \xi \, dx dt = - \int_0^T \int M_\varepsilon \ast H_\varepsilon \cdot \nabla \xi \, dx dt,
\]

(106)

for all \( \xi \in L^2(0, T^*; H^1(\mathbb{R}^3)) \). Hence \( H_\varepsilon \) satisfies the equation

\[
\text{div}(H_\varepsilon + M_\varepsilon) = -\text{div} H_\varepsilon, \quad \text{a.e. in } (0, T^*) \times \mathbb{R}^3,
\]

and since \( \text{curl} H^\nu = 0 \), we get \( \text{curl} H_\varepsilon = 0 \).

Next we will perform the limit in (101). All the terms factored by \( \nu \) converge towards 0 by (103). Let us examine the nonlinear terms of the equation.

1. First the convergence (104) implies in particular that

\[
\psi^\nu f^\nu \to f_\varepsilon \text{ strongly in } L^2(0, T^*; L^2_{loc}),
\]

(107)

so for \( \Phi \in \mathcal{D}([0, T^*] \times \mathbb{R}^3) \times \mathbb{R}^3) \)

\[
\int_0^T \int \psi^\nu f^\nu \cdot (v \cdot \nabla_x \Phi) \, dx dt = \int_0^T \int f_\varepsilon \cdot (v \cdot \nabla_x \Phi) \, dx dt.
\]

2. Consider now the nonlinear terms appearing in front of \( \alpha \) and \( \beta \). For \( \Phi \in \mathcal{D}([0, T^*] \times \mathbb{R}^3 \times \mathbb{R}^3) \), convergence (104) implies that

\[
f^\nu \times \Phi \to f_\varepsilon \times \Phi \text{ strongly in } L^2(0, T^*; L^2),
\]

and writing

\[
\int_0^T \int (\rho_\nu \ast H^\nu) \times f^\nu \cdot \Phi \, dx dt = \int_0^T \int (\rho_\nu \ast H^\nu) \cdot (f^\nu \times \Phi) \, dx dt,
\]
we see that as $\nu \to 0$
\[ \int_0^T \left( \rho_v \ast H' \right) \times f'' \cdot \Phi \, dx \, dv \, dt \to \int_0^T H_x \times f_x \cdot \Phi \, dx \, dv \, dt. \]

Now convergence (107) implies that
\[ \psi'' f'' \times \Phi \to f_x \times \Phi \text{ strongly in } L^2(0,T^*; L^2), \]

so we get as previously, the following limit
\[ \int_0^T \left( \psi'' \nu \times (\rho_v \ast H') \cdot \nabla v \right) \cdot f'' \cdot \Phi \, dx \, dv \, dt \to \int_0^T (v \times H_x \cdot \nabla v) \Phi \cdot f_x \, dx \, dv \, dt. \]

Gathering the previous results, we infer that for all $\Phi \in \mathcal{D}([0,T^*] \times \mathbb{R}^3 \times \mathbb{R}^3)$, we have
\[ - \int_0^T f_x \cdot (\partial_t \Phi + v \cdot \nabla_x \Phi) \, dx \, dv \, dt + \alpha \int_0^T (v \times H_x \cdot \nabla v) \Phi \cdot f \, dx \, dv \, dt \]
\[ + \beta \int_0^T H_x \times f_x \cdot \Phi \, dx \, dv \, dt + \frac{1}{\tau} \int_0^T (\gamma f_x - \chi_0 \psi H_x) \cdot \Phi \, dx \, dv \, dt \]
\[ \cdots \]

so $f_x$ satisfies the weak formulation of the kinetic equation (6). Choosing $\Phi \in \mathcal{D}((0,T^*) \times \mathbb{R}^3 \times \mathbb{R}^3)$, we see that $f_x$ satisfies the kinetic equation
\[ \partial_t f_x = - (v \cdot \nabla_x) f_x + \alpha (v \times H_x \cdot \nabla v) f_x - \beta H_x \times f_x \]
\[ - \frac{1}{\tau} (\gamma f_x - \chi_0 \psi H_x) - \frac{1}{\tau} (\gamma f_x - \psi M_x), \]

a.e. in $(0, T^*) \times \mathbb{R}^3 \times \mathbb{R}^3$ and $\partial_t f_x \in L^2(0, T; H^2)$ so $f_x(0)$ makes a sense and taking $\Phi \in \mathcal{D}((0,T^*) \times \mathbb{R}^3 \times \mathbb{R}^3)$ in the weak formulation, we deduce that $f_x(0) = f_0$ a.e. in $\mathbb{R}^3 \times \mathbb{R}^3$.

To end the proof of Theorem 1.1, let us verify estimate (18). The time derivative of $M_x$ satisfies in the sense of distributions the equation
\[ \partial_x M_x = \gamma \int \partial_t f_x \, dv = -\gamma \left( \int \nabla_x \cdot (v \otimes f_x) \, dv - \alpha \int (v \times H_x) \cdot \nabla v \right) \]
\[ + \beta \int H_x \times f_x \, dv + \frac{1}{\tau} \int (\gamma f_x - \chi_0 \psi H_x) \, dv + \frac{1}{\tau} \int (\gamma f_x - \psi M_x) \, dv \]
\[ = - \nabla_x \cdot J_x - \beta H_x \times M_x - \frac{\gamma}{\tau} (M_x - \chi_0 H_x), \]

so $M_x$ satisfies equation (9) where the current $J_x$ is defined by (10) and since for all $\delta \in \mathbb{N}^3, |\delta| \leq 3$, we have
\[ \int |\partial^\delta J_x|^2 \, dx \leq \gamma^2 (\int |v|^2 \psi \, dv) (\int |\partial^\delta f| \psi \, dv) \]
then we get
\[ \|J_x\|_{L^\infty((0,T^*);H^3)} \leq C \|f_x\|_{L^\infty((0,T^*); \mathbb{R}^3)} \leq C^*. \]
Next for $\delta \in \mathbb{N}^3$, $|\delta| \leq 2$

$$\int |\partial^\delta (H_\varepsilon \times M_\varepsilon)|^2 \, dx \leq 2 \sum_{\delta_1 + \delta_2 = \delta} \int |\partial^{\delta_1} H_\varepsilon|^2 |\partial^{\delta_2} M_\varepsilon|^2 \, dx$$

$$\leq 2 \sum_{\delta_1 + \delta_2 = \delta} \|\partial^{\delta_1} H_\varepsilon\|_{L^4}^2 \|\partial^{\delta_2} M_\varepsilon\|_{L^4}^2 \leq C \|H_\varepsilon\|_{H^3}^2 \|M_\varepsilon\|_{H^3}^2,$$

therefore $H_\varepsilon \times M_\varepsilon$ is uniformly bounded in $L^2(0, T^*; H^2)$ and so is $\partial_t M_\varepsilon$. This ends the proof of Theorem 1.1.

7. Proof of Theorem 1.2. In this section we shall prove our second main result Theorem 1.2. We consider for $\varepsilon > 0$ fixed, the solution $(f_\varepsilon, H_\varepsilon)$ of problem $(P_\varepsilon)$ provided by Theorem 1.1, we hope to pass to the limit as $\varepsilon \to 0$, in problem $(B_\varepsilon)$ satisfied by the magnetization $M_\varepsilon = \gamma \int f_\varepsilon \, dv$ and the magnetic field $H_\varepsilon$. For convenience, we recall the equations under consideration

$$\partial_t M_\varepsilon + \nabla_x \cdot J_\varepsilon + \beta H_\varepsilon \times M_\varepsilon + \frac{\gamma}{\tau} (M_\varepsilon - \chi_0 H_\varepsilon) = 0 \quad \text{in} \ (0, T^*) \times \mathbb{R}^3,$$
$$M_\varepsilon(0) = M_0 \quad \text{in} \ \mathbb{R}^3,$$  \hspace{1cm} (109)

where the current is given by $J_\varepsilon = \gamma \int v \otimes f_\varepsilon \, dv$. The weak formulation of equation of $M_\varepsilon$ writes as

$$- \int_0^{T^*} \int M_\varepsilon \cdot \partial_t \phi \, dx \, dt - \int_0^{T^*} \int J_\varepsilon \cdot \nabla \phi \, dx \, dt + \beta \int_0^{T^*} \int H_\varepsilon \times M_\varepsilon \cdot \phi \, dx \, dt$$
$$+ \frac{\gamma}{\tau} \int_0^{T^*} (M_\varepsilon - \chi_0 H_\varepsilon) \cdot \phi \, dx \, dt = \int_0^{T^*} M_0 \cdot \phi(0) \, dx,$$  \hspace{1cm} (110)

for all test functions $\phi \in \mathcal{D}([0, T^*] \times \mathbb{R}^3)$.

The uniform estimates (17) and (18) allow to get the convergences stated in (19) for subsequences. We have shown in section 3, see (48) that $J_\varepsilon \rightharpoonup 0$ weakly-* in $L^\infty(0, T^*; L^2)$ and since it is uniformly bounded in $L^\infty(0, T^*; H^3)$, we get (21).

Next we infer that for $\phi \in \mathcal{D}([0, T^*] \times \mathbb{R}^3)$

$$M_\varepsilon \times \phi \to M_\varepsilon \times \phi \quad \text{strongly in} \ L^2(0, T^*; L^2),$$

and consequently

$$\int_0^{T^*} \int H_\varepsilon \times M_\varepsilon \cdot \phi \, dx \, dt \to \int_0^{T^*} \int H \times M \cdot \phi \, dx \, dt.$$

This implies in particular the convergence of $H_\varepsilon \times M_\varepsilon$ towards $H \times M$ in the sense of distributions and since this sequence is bounded in $L^2(0, T; H^2)$, we deduce the convergence stated in (20).

Now it is easy to pass to the limit as $\varepsilon \to 0$ in the magnetostatic equations to get

$$\text{div} (H + M) = - \text{div} H_\varepsilon, \quad \text{curl} H = 0 \quad \text{in} \ (0, T^*) \times \mathbb{R}^3,$$
and the previous convergences imply that when letting $\varepsilon \to 0$ in (110), we get for all $\phi \in D([0, T^\ast[ \times \mathbb{R}^3)$

\[- \int_0^{T^\ast} \int M \cdot \partial_t \phi \, dx \, dt + \beta \int_0^{T^\ast} H \times M \cdot \phi \, dx \, dt + \frac{\gamma}{\tau} \int_0^{T^\ast} (M - \chi_0 H) \cdot \phi \, dx \, dt = \int M_0 \cdot \phi(0) \, dx.\]

Choosing $\phi \in D((0, T^\ast) \times \mathbb{R}^3)$ we deduce that $(M, H)$ satisfies the Bloch equation

\[\partial_t M + \beta H \times M + \frac{\gamma}{\tau} (M - \chi_0 H) = 0,\]

first in the sense of distributions then almost everywhere. Since $M \in H^1([0, T^\ast]; H^2)$ ⊂ $C([0, T^\ast]; H^2)$, $M(0)$ is well defined and taking $\phi \in D([0, T^\ast[ \times \mathbb{R}^3)$ we get

\[\int M(0) \cdot \phi(0) \, dx = \int M_0 \cdot \phi(0) \, dx\]

so that $M(0) = M_0$. This ends the proof of Theorem 1.2.

**Remark 1.** In [3] (see also [8]), the authors considered diffusion models which are close to the ones discussed in our work. The question we shall discuss in a forthcoming paper is the derivation of Bloch-Torrey equation as the diffusive limit of the kinetic Bloch equation.

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