Poisson structure on moduli of flat connections on Riemann surfaces and $r$-matrix

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Abstract

We consider the space of graph connections (lattice gauge fields) which can be endowed with a Poisson structure in terms of a _ciliated fat graph._ (A ciliated fat graph is a graph with a fixed linear order of ends of edges at each vertex.) Our aim is however to study the Poisson structure on the moduli space of locally flat vector bundles on a Riemann surface with holes (i.e. with boundary). It is shown that this moduli space can be obtained as a quotient of the space of graph connections by the Poisson action of a lattice gauge group endowed with a Poisson-Lie structure. The present paper contains as a part an updated version of a 1992 preprint [1] which we decided still deserves publishing. We have removed some obsolete inessential remarks and added some newer ones.
1 Introduction

The moduli space of flat $G$-bundles on a Riemann surface is the classical phase space for the Chern-Simons gauge theory and, thus, it is in a sense the classical limit of the WZNW conformal field theory. This means that quantizing it one can get a space of quantum states which turns out to be isomorphic to the space of conformal blocks of the corresponding WZNW theory. This statement has been checked by several authors (cf. [8], [20]) with help of different quantization methods. On the other hand, the moduli spaces of flat bundles as well as closely related to them moduli spaces of holomorphic bundles (cf. [17]) attracted much attention from a purely mathematical point of view (cf. [3], [15]).

In section 3 we discuss in detail the canonical Poisson structure on the moduli space of flat bundles on Riemann surfaces with holes (i.e. with boundary). In section 4 we construct a Poisson structure on the space of graph connections in such a way that the action of the graph gauge group is Poissonian with respect to an appropriate nontrivial Poisson-Lie structure. The considerations of this section are inspired mainly by constructions of refs. [19], [2], [1] where a discrete analogue of current algebra was suggested and investigated. Then we prove that the quotient of the space of graph connections by the gauge group coincides with the moduli space of flat connections on a Riemann surface determined by the graph.

One of the main aims of our preprint [11] (see also ref. [12]) was to give a description of the moduli space of flat connections in the form ready for quantisation. We are not going to discuss this problem here for since that time some progress towards this direction was made, see refs. [3], [9], [13]. The interested reader can find details and discussions in those papers. Note only that the result of quantisation is a noncommutative algebra having WZNW model conformal blocks as its representation space and being functorial with respect to the imbeddings of surfaces.

For the readers interested in a more understandable and detailed presentation we would like to recommend a very good review by M.Audin [7].

2 Ciliated fat graphs and Poisson manifolds

The moduli space of flat connections on a compact Riemann surface is by definition a subquotient of a topologically trivial space of all connections. This description is useful also since a nontrivial Poisson manifold (which is the moduli space, or an orbifold, to be more precise) is represented as a result of a reduction of a trivial symplectic manifold (see sect. 3 for details). Unlike the former, the latter has plenty of convenient parameterizations. The only disadvantage of this description is that the space of connections is infinite dimensional. In this paper (sect. 4) we consider an alternative description of the moduli space in which the role of the space of all connections on a Riemann surface is played by a finite dimensional manifold. The idea is quite familiar both from lattice gauge theory and from Čech cohomology. Namely, consider a triangulation of a compact

\footnote{It worths to look through ref. [7] not only because of a very transparent presentation, but also because of very nice pictures there.}
Riemann surface $S$ (with boundary, in general). Then we get a graph formed by the vertices and the edges of this triangulation. By a graph connection (or lattice gauge field) we mean an assignment of a group element of the gauge group $G$ to each (oriented) edge. The group of lattice gauge transformations $\mathcal{G}^l$ acting on the space of graph connections in a natural way is simply a product of several copies of $G$, one copy for each vertex of the graph. A flat graph connection satisfies the condition that the monodromies around all the faces of the triangulation are equal to $1 \in G$. (The monodromy is the product of group elements corresponding to the consecutive edges of a face, whatever shape of faces is used. One has to take account only of the orientations of the edges in an obvious way.) Now, it is a standard assertion that the moduli space of (smooth) flat connections on $S$ is isomorphic to the space of flat graph connections modulo graph gauge transformations. (This is in fact nothing but the statement in Čech cohomology that this space is represented by $H^1(S, G)$.) Dealing with a surface with holes amounts to saying that some faces of the triangulation are left empty and one does not have to require anything about the corresponding monodromies. It is important to note that if a graph $l$ is obtained from a triangulation of a surface $S$ it can be endowed with an additional structure which (together with the graph itself) contains all the information about the topology of the surface. We suppose that $S$ is oriented. The orientation of $S$ induces a cyclic order of the ends of edges incident to each vertex. A graph $l$ with a given cyclic order at each vertex is called a fat graph. If $S$ has at least one hole the most economical way is to consider a fat graph with all the faces empty, what is always possible. Conversely, given a fat graph $l$ the corresponding surface can be restored by replacing edges of $l$ by strips glued together at vertices respecting the cyclic order (cf. fig. 1). Summarizing, in order to describe the moduli space $\mathcal{M}$ of flat connections on a surface $S$ with holes we choose a fat graph corresponding to $S$ (this choice is not unique) and consider the quotient of the space of graph connections $\mathcal{A}^l$ by the action of graph gauge transformations, $\mathcal{M} = \mathcal{A}^l/\mathcal{G}^l$.

Having described the moduli space as a manifold we are interested now in describing its Poisson structure. Let us forget for a moment that we can define a Poisson structure on $\mathcal{M}$ by reduction of the space of all (smooth) connections on $S$ and try instead to define a Poisson structure on $\mathcal{A}^l$ in such a way that it can be pulled down on $\mathcal{M}$. We would like to have such a Poisson structure on $\mathcal{A}^l$ that the projection $\mathcal{A}^l \to \mathcal{M}$ will be a Poisson map. This can be achieved if $\mathcal{G}^l$ will act on $\mathcal{A}^l$ in a Poisson way (see ref. [19] for the definition of Poisson group actions on Poisson manifolds). For this aim we have to define first a Poisson-Lie structure on $\mathcal{G}^l$ itself. The group of graph gauge transformations $\mathcal{G}^l$ is the direct product of several copies of $G$, with one copy per each vertex of $l$. Let us define the Poisson structure on $\mathcal{G}^l$ as a direct product of Poisson structures on each copy of $G$ in $\mathcal{G}^l$. The latter can be defined independently at each vertex. (To define a Poisson structure on $G$ one has to choose a classical $r$-matrix.) Now we look for a Poisson structure on $\mathcal{A}^l$. The requirement that the action of $\mathcal{G}^l$ is a Poisson one is almost sufficient to determine the Poisson structure on $\mathcal{A}^l$. The ambiguity amounts in fact to choosing a linear order of ends of edges at each vertex. Therefore, instead of fat graphs we have to deal with graphs with linear order. Let us call such graphs *ciliated fat graphs*. A ciliated fat graph

\[\text{As it will be proved in sect. [we obtain in this way the same Poisson structure as defined by the reduction procedure from smooth connections.}\]
can be considered as a fat graph with an additional structure (the fat graph underlying a given ciliated fat one is restored uniquely). This additional structure (linear order at each vertex) can be represented by picturing the underlying fat graph on a sheet of paper in such a way that the cyclic order is everywhere, say, counterclockwise and by placing at each vertex a small cilium separating the minimal and the maximal end incident to that vertex. As it was mentioned, a fat graph defines a surface, that is an oriented surface with holes (fig. 1); a ciliated fat graph, similarly, defines a ciliated surface, that is an oriented surface with holes and with some points marked on the boundary (fig. 2). Thus for every ciliated fat graph we have an associated Poisson manifold, namely the space of graph connection endowed with an $r$-matrix Poisson structure. It may happen of course that two different ciliated graphs give isomorphic Poisson manifolds of graph connections. In particular, one can show that the isomorphism class of the arising Poisson manifold depends only on the diffeomorphism class of the corresponding ciliated surface.

It may be worth mentioning some distinguished examples of graphs and corresponding Poisson manifolds. The Poisson manifold corresponding to the graph consisting only of two vertices and one edge (fig. 3a) coincides with the Poisson-Lie group $G$ provided the $r$-matrices chosen at the vertices are related by the operation of permutation of tensor factors ($r = r_{12}$ $\mapsto$ $r_{21}$). With the same condition on $r$-matrices, the graph consisting of two vertices and two edges connecting them (fig. 3b) yields the manifold $G \times G$ endowed with a Poisson-Lie structure coinciding with that of the double $D \simeq G \times G$. If we take the same $r$-matrices at two vertices we get $D^+$ as our Poisson manifold (see ref. [19] for the definitions of doubles). Finally, the graph consisting of one vertex and one edge (fig. 3c) corresponds to the Poisson manifold $G^*$, the dual Poisson-Lie group.

The following operations with graphs are important to discuss: i) erasing an edge (fig. 4), ii) contracting an edge (fig. 5), iii) gluing graph(s) (fig. 6), and iv) adding a loop (see sect. 4). The linear orders at the vertices touched by such an operation descend from those of the original graph in a more or less obvious way (cf. figs. 4,5,6). We have to mention only that there are in fact two ways to contract an edge which differ in what happens to the cilia. The operation of gluing deserves some explanation. Given two vertices on a graph with the same number $N$ of ends of edges incident to them we can form a new graph by erasing both vertices and gluing together thus liberated edges. (The $k$-th end liberated at one vertex is to be glued to the $(N-k)$-th end at the other vertex.) Note that with help of this operation one can glue together two different graphs obtaining a single new one.

For the operations on graphs just described there exist natural maps between the corresponding spaces of graph connections. These maps are in fact projections in directions shown by the arrows in figs. 4,5,6. A pleasant feature is that these maps turn out to be Poisson maps. More precisely, in case of gluing one has to require that the $r$-matrices at two vertices to be glued are related by permutation of tensor factors. Consider, for instance, a map corresponding to gluing together two simplest graphs (fig. 7a) each of which represents the Poisson-Lie group $G$ (an edge with two vertices). The result of gluing is again a graph of the same shape while the corresponding map of graph connections, $G \times G \to G$, is simply the group product which is known to be a Poisson map. Similarly, gluing together the graphs representing $D$ gives the Poisson map $D \times D \to D$ (fig. 7b) corresponding to the group multiplication. Contracting one of two edges of the
$D$ graph (fig. 7c) one obtains the Poisson map $D \rightarrow G^*$. As a Poisson manifold, the dual group $G^*$ can be identified with the coset $D/G_\Delta$ where $G_\Delta$ is the diagonal subgroup in $D \simeq G \times G$ (cf. ref. 13). The isomorphism of $G^*$ with the coset $D/G_\Delta$ shows that there is a Poisson action of $D$ on $G^*$, i.e. a Poisson map $D \times G^* \rightarrow G^*$ which again can be described by gluing graphs (as shown in fig. 7d). Looking at the pictures above suggests the following generalisation of the notion of a double. Namely, we can define a Poisson-Lie group, called in general a polyuble 3 by the ciliated fat graph consisting of two vertices and several edges connecting them (analogously to the case of the double, the $r$-matrices at two vertices should be related by the operation of permutation of tensor factors, while the order of ends should be opposite; fig. 7e). An immediate observation is that on the space of graph connections $\mathcal{A}^l$ for an arbitrary ciliated fat graph $l$ there is a Poisson action of a polyuble, $P_n(G)$, adjusted to each vertex, where $n$ is the number of legs at that vertex (see, fig. 7f). Thus the space $\mathcal{A}^l$ is a homogeneous space for the group $P^l$ which is a direct product (in the sense of Poisson groups) of $P_n(G)$’s. Note also that the group of graph gauge transformations $\mathcal{G}^l$ which gives us the moduli space $\mathcal{M} = \mathcal{A}^l/G^l$ is a Poisson subgroup in $P^l$. (Any individual polyuble, disregarding for the moment the Poisson structure, is a product $G \times \ldots \times G$ and contains the diagonal subgroup which turns out to be a Poisson subgroup.)

Finally, it is worth mentioning that some particular cases of Poisson manifolds defined by graphs have been considered in literature. Namely the Poisson manifold of graph connections on a graph corresponding to the boundary of a polygon was suggested in ref. 13 as a discrete approximation of current algebra coadjoint space. (See also refs. 2, 1 where this discrete approximation was used to investigate WZNW conformal model.)

### 3 Poisson structure of moduli spaces

In this section we shall describe a Poisson structure on the space of flat connections modulo gauge transformations on Riemann surfaces with holes by means of a reduction of the space of all smooth connections on them.

Let $S$ be an oriented compact Riemann surface with holes. Let $\mathcal{A}$ be the space of smooth $G$-connections on it, where $G$ is a reductive complex Lie group with the Lie algebra $\mathfrak{g}$ with a chosen nondegenerate invariant quadratic form which we denote by $\text{tr}$. The space $\mathcal{A}$ is in a natural way a symplectic manifold with the symplectic structure

$$\Omega = \int_S \text{tr} (\delta A \wedge \delta A),$$

where $A \in \mathcal{A}$ is a $\mathfrak{g}$-valued 1-form on $S$, $\delta$ is the external differential on $\mathcal{A}$, and $\wedge$ is a shorthand way to denote the wedge product both on $\mathcal{A}$ and on $S$. This symplectic structure is well known to be invariant with respect to the gauge transformations

$$A \mapsto g^{-1}Ag + g^{-1}dg,$$

where $g$ is a $G$-valued function on $S$.

\footnote{We dedicate the Poisson-Lie groups of this type to I.V.Polyubin}
Now let us try to define the momentum mapping for this action. One can easily check that the infinitesimal gauge transformation $\epsilon$ is generated by the Hamiltonian function

$$H_\epsilon = \int_S \text{tr} (\epsilon (dA + A \wedge A)) + \int_{\partial S} \text{tr} (\epsilon A).$$  \hfill (3)

The Hamiltonian generating a given transformation is defined only up to an additive constant and therefore the Poisson brackets between them, in general, reproduce the commutation relations between the elements of the gauge algebra only up to a cocycle:

$$\{H_{\epsilon_1}, H_{\epsilon_2}\} = H_{[\epsilon_1, \epsilon_2]} + c(\epsilon_1, \epsilon_2).$$  \hfill (4)

In our case

$$c(\epsilon_1, \epsilon_2) = \int_{\partial S} \text{tr} (\epsilon_1 \epsilon_2).$$  \hfill (5)

One can prove that this cocycle is nontrivial and therefore we can define the momentum mapping not for the algebra of gauge transformations itself, but only for its central extension by the 2-cocycle eq.(5).

Let $\mathfrak{g}^S$ denote the algebra of gauge transformations centrally extended by (5) and let $G^S$ be the corresponding group. The space $\mathfrak{g}^S$ is the space of pairs $(\epsilon, z)$, where $\epsilon$ is a $\mathfrak{g}$-valued function and $z$ is a complex number. Let us consider the space $(\mathfrak{g}^S)^*$ consisting of triples $(R, B, x)$ where $R$ is a $\mathfrak{g}$-valued two form on $S$, $B$ is a $\mathfrak{g}$-valued 1-form on the boundary of $S$ and $x$ is a complex number. There is a nondegenerate pairing $\langle, \rangle$ between $\mathfrak{g}^S$ and $(\mathfrak{g}^S)^*$,

$$\langle (R, B, x), (\epsilon, z) \rangle = \int_S \text{tr} (\epsilon R) + \int_{\partial S} \text{tr} (\epsilon B) + zx.$$  \hfill (6)

The momentum map for the action of $\mathfrak{g}^S$ can be defined now as a mapping $A \mapsto (\mathfrak{g}^S)^*$, given by the curvature and by the restriction of the connection form to the boundary.

$$A \mapsto (dA + A \wedge A, A|_{\partial S}, 1)$$  \hfill (7)

Now consider the Hamiltonian reduction of $A$ with respect to $G^S_0$, the group of gauge transformations equal to the identity on the boundary which yields the space of flat connections on $S$ modulo gauge transformations from $G^S_0$,

$$\mathcal{M}_0 = \{A \in \mathcal{A} \mid dA + A \wedge A = 0\} / G^S_0.$$  \hfill (8)

The space $\mathcal{M}_0$ can also be considered as the space of values of flat connections restricted to the boundary. It is well known that the space of $G$-connections on a circle can be identified with the coadjoint space of the affine Kac-Moody algebra with the standard Kirillov-Kostant Poisson structure. The following proposition shows that these two Poisson structures are related:

**Proposition 1** The mapping from the space $\mathcal{M}_0$ to the Kac-Moody coadjoint representation space sending a flat connection on the Riemann surface $S$ to its restriction to a component of the boundary is a Poisson mapping.
Proof. This mapping is essentially the momentum mapping for the action of gauge transformations. □

Now let us consider the quotient of the space $\mathcal{M}_0$ by the whole group $\mathcal{G}^S$ (the group $\mathcal{G}^S$ acts on $\mathcal{M}_0$ because the group $\mathcal{G}^S_0$ of gauge transformations equal to the identity on the boundary is normal in $\mathcal{G}^S$). The quotient space,

$$\mathcal{M} = \{ A \in \mathcal{A} \mid dA + A \wedge A = 0 \} / \mathcal{G}^S_1,$$

is a finite dimensional Poisson manifold. Its symplectic leaves are in one-to-one correspondence with the coadjoint orbits of the centrally extended group of gauge transformations which in turn are parameterized by the conjugacy classes of monodromies around the holes. Thus we have

**Proposition 2** The space of flat $G$-connections modulo gauge transformations, $\mathcal{M}$, on a Riemann surface with holes inherits a Poisson structure from the space of all (smooth) $G$-connections. The symplectic leaves of this structure are parameterised by the conjugacy classes of monodromies around holes.

### 4 Graph connections

In this section we shall construct a Poisson structure on the space of graph connections, $\mathcal{A}^l$, in such a way that the lattice gauge group, $\mathcal{G}^l$, endowed with a nontrivial $r$-matrix Lie-Poisson structure acts on $\mathcal{A}^l$ in a Poisson way.

Let $l$ be a ciliated fat graph homotopically equivalent to a Riemann surface $S$ with holes. Denote by $E(l)$ the set of ends of edges of $l$ and by $N(l)$ the set of its vertices. Each element of $N(l)$ corresponds to the subset of $E(l)$ of ends of edges incident to a given vertex. In what follows we shall identify elements of $N(l)$ with the corresponding subsets. A mapping which sends an end of an edge $\alpha$ to the opposite end of the same edge $\alpha^\vee$ is an involution of the set $E(l)$. The ciliated fatness of $l$ defines an ordering inside each $n \in N(l)$. One can easily see that such data – a set divided into ordered subsets and an involution of it without fixed points – unambiguously define a ciliated fat graph. Let $[\alpha]$ be the vertex containing $\alpha$ and $[\alpha, \alpha^\vee]$ be the edge linking $\alpha$ and $\alpha^\vee$.

Let us call a graph connection on a graph $l$ an assignment of an element $A_\alpha$ of the group $G$ to each $\alpha \in E(l)$ such that

$$A_\alpha^{\vee} = A_\alpha^{-1}. \tag{10}$$

The lattice gauge group $\mathcal{G}^l$ is a product of finite dimensional groups $G$ — one copy for each vertex of the graph. The group $\mathcal{G}^l$ acts on $\mathcal{A}^l$ in a natural way:

$$A_\alpha \mapsto g_\alpha^{-1} A_\alpha g_\alpha. \tag{11}$$

\footnote{Perhaps it would be more natural to assign a group element to each edge, as we did in sect. \ref{sect2} above, rather than to each end of an edge. However, in this case we would have to choose some orientations of the edges. Then we would have to have a definition of the Poisson manifold $\mathcal{A}^l$ which would depend on an oriented ciliated fat graph. In such a case it would be possible to prove that two Poisson manifolds corresponding to two graphs differing only by their orientations are isomorphic. We prefer to get rid of this complication at the price of a slightly more complicated notation.}
The space of graph connections can be considered as a quotient space of the space of flat connections on a surface $S$. Indeed, let us take the surface $S$ corresponding to the graph $l$ and imbed the graph into it in a way such that $S$ is contractable to the image of $l$. Then for a (smooth) connection $A$ on $S$ we can construct a graph connection on $l$ assigning to $\alpha \in E(l)$ the parallel transport operator along the edge linking $\alpha^\vee$ and $\alpha$. This graph connection does not change if we transform the connection $A$ by a gauge transformation equal to the identity at the vertices. It is clear that every graph connection can be continuously extended to the surface and therefore the space of graph connections $A^l$ can be represented as a quotient,

$$A^l \cong \{ A \in \mathcal{A} \mid dA + A \wedge A = 0 \} / \mathcal{G}^l_1,$$

(12)

where $\mathcal{G}^l_1$ is the group of gauge transformations equal to the identity at the vertices. Of course, this representation is defined only up to the action of the graph gauge group and, therefore, the isomorphism between the spaces $\mathcal{M}$ and $A^l / \mathcal{G}^l_1$ is canonical.

This isomorphism shows that although the space $A^l$ has so far no a priori Poisson structure, the space $A^l / \mathcal{G}^l_1$ has one. Our aim is to introduce a Poisson structure on $A^l$ compatible with that on $A^l / \mathcal{G}^l_1$ and with the graph gauge group action.

Let us fix for each vertex $n$ of the graph a classical $r$-matrix $r(n) \in \mathfrak{g} \otimes \mathfrak{g}$, that is to say, a solution of the classical Yang-Baxter equation:

$$[r_{12}(n), r_{13}(n)] + [r_{12}(n), r_{23}(n)] + [r_{13}(n), r_{23}(n)] = 0 \quad (13)$$

such that

$$\frac{1}{2} (r_{12}(n) + r_{21}(n)) = t, \quad (14)$$

where $t \in \mathfrak{g} \otimes \mathfrak{g}$ is a quadratic Casimir element:

$$t = \sum e_i \otimes e_i, \quad (15)$$

where $\{e_i\}$ is an orthonormal basis in $\mathfrak{g}$.

Let us define a bivector field $B$ on $A^l$ as

$$B = \sum_{n \in N(l)} \left( \sum_{\alpha, \beta \in \eta; \alpha < \beta} r^{ij}(n) X^\alpha_i \wedge X^\beta_j + \frac{1}{2} \sum_{\alpha \in \eta} r^{ij}(n) X^\alpha_i \wedge X^\alpha_j \right), \quad (16)$$

where $X^\alpha_i = L^\alpha_i - R^\alpha_i$, $L^\alpha_i$ and $R^\alpha_i$ are, respectively, the left- and right-invariant vector fields corresponding to the element $e_i \in \mathfrak{g}$ on the group assigned to $\alpha \in E(l)$ and $r^{ij}(n)$ is the $r$-matrix at the vertex $n$ written in the basis $\{e_i\}$. Note that the vector fields $X^\alpha_i$ are chosen to be consistent with eq.(10).

**Proposition 3** a) The bivector $B$ defines a Poisson structure on $A^l$. b) The group $\mathcal{G}^l_1$ endowed with the direct product Poisson-Lie structure acts on $A^l$ in a Poisson way.

Note that although the $r$-matrix, $r(n)$, is allowed to differ for different vertices, its symmetric part, $t$, is required to be the same everywhere.
The proof can be obtained by a straightforward check.

Sometimes it is however more convenient to use other ways of presenting the Poisson bivector \( \Omega \). If one separates explicitly the symmetric, \( t \), and skew-symmetric, \( r_a = \frac{1}{2}(r_{12} - r_{21}) \), parts of the \( r \)-matrix, so that \( r = r_a + t \), one gets

\[
B = \sum_n \left( r_a^{ij}(n) X_i^\alpha(n) \otimes X_j^\alpha(n) + \sum_{\alpha, \beta \in n} (n, \alpha, \beta) \sum_i X_i^\alpha \otimes X_i^\beta \right),
\]

where \( X_i^\alpha(n) = \sum_{\alpha \in n} X_i^\alpha \) and

\[
(n, \alpha, \beta) = \begin{cases} 
1 & \alpha > \beta \\
0 & \alpha = \beta \\
-1 & \alpha < \beta 
\end{cases}, \quad \text{for } \alpha, \beta \in n. \quad (17)
\]

Since the vectors \( X_i^\alpha(n) \) are tangent to \( G^l \)-orbits, one sees that the Poisson bracket induced by eq.(16) on the quotient \( M = A^l/G^l \) does not change if the skew-symmetric part \( r_a \) of the \( r \)-matrix is changed.

Another way of defining the Poisson structure is to give explicit expressions for the Poisson brackets between matrix elements of \( A_\alpha \) in some representation of the group \( G \); we consider these matrix elements as functions on \( A^l \). (We shall denote matrices representing \( A_\alpha \) and \( r \) by the same symbols, \( A_\alpha \) and \( r \), respectively.)

\[
\{A_\alpha \otimes A_\alpha \} = r_a(1) (A_\alpha \otimes A_\alpha) + (A_\alpha \otimes A_\alpha) r_a(2)
\]

for the case \([\alpha] \neq [\alpha^\vee] \). Here \( r(1) = r([\alpha]) \), \( r(2) = r([\alpha^\vee]) \).

\[
\{A_\alpha, A_\alpha \} = r_a (A_\alpha \otimes A_\alpha) + (A_\alpha \otimes A_\alpha) r_a + (1 \otimes A_\alpha) r_{21} (A_\alpha \otimes 1) - (A_\alpha \otimes 1) r (1 \otimes A_\alpha) \quad (19)
\]

for the case \([\alpha] = [\alpha^\vee], \alpha < \alpha^\vee \).

\[
\{A_\alpha \otimes A_\beta \} = r (A_\alpha \otimes A_\beta)
\]

for the case \([\alpha] = [\beta] \neq [\alpha^\vee] \neq [\beta^\vee] \neq [\alpha], \alpha < \beta \).

\[
\{A_\alpha, A_\beta \} = r(1) (A_\alpha \otimes A_\beta) + (A_\alpha \otimes A_\beta) r(2) \quad (20)
\]

for the case \([\alpha] = [\beta] \neq [\alpha^\vee] = [\beta^\vee], \alpha < \beta, \alpha^\vee < \beta^\vee \), \( r(1) = r([\alpha]) \), \( r(2) = r([\alpha^\vee]) \).

\[
\{A_\alpha, A_\beta \} = r (A_\alpha \otimes A_\beta) + (A_\alpha \otimes A_\beta) r(1 \otimes A_\beta) r_{21} (A_\alpha \otimes 1) - (A_\alpha \otimes 1) r (1 \otimes A_\beta). \quad (21)
\]

for the case \([\alpha] = [\beta] = [\alpha^\vee] = [\beta^\vee] \) and \( \alpha < \beta \). Unfortunately the complete list of all possible configurations of one or two edges and cilia is rather long (there are fourteen of them) and we stop here. The reader can easily observe how one can write down expressions for other configurations by analogy.

As it was described in sect. 2, there exist such operations on ciliated fat graphs as erasing an edge, contracting an edge towards a vertex, gluing two vertices of the same
valence and adding a loop. One can also change a graph to another one corresponding to the same ciliated surface. All these transformations induce mappings between the corresponding spaces of graph connections. Let us now describe them explicitly.

**Erasing an edge (fig. 4).** This operation is the most obvious one. The mapping between graph connections is just given by forgetting the group element assigned to the edge to be erased.

**Contracting an edge (fig. 5).** This operation can be applied to an edge with distinct ends (i.e. $[\alpha] \neq [\alpha^\vee]$). Let $\alpha$ be an end of such an edge. (In fig. 5, it is the right one for the projection $R$ and the left one for $L$.) Make $A_\alpha$ be equal to identity by applying a gauge transformation (that is the action of one copy of $G$) at the vertex $[\alpha^\vee]$. Erase the cilium at the vertex $[\alpha^\vee]$. Then contract the edge $[\alpha, \alpha^\vee]$ leaving the group elements on the other edges unchanged (as they were after the above gauge transformation).

Note that, as it is shown in fig. 5, this operation depends not only on the edge but also on the choice of a particular end of it. To emphasize this we say that we contract the edge towards a vertex, in our case $[\alpha]$.

**Gluing two vertices (fig. 6).** This operation can be applied to two vertices $n$ and $n'$ having the same valence, i.e. $|n| = |n'|$, and such that their $r_\alpha$-matrices are opposite, i.e. $r_\alpha(n) = -r_\alpha(n')$. Disconnect the ends of edges at the vertices and connect them in the order prescribed by gluing (fig.6) inserting an arbitrarily ciliated 2-valent vertex at each connection. Until now we left the group elements on the edges unchanged. Now take each inserted 2-vertex and contract towards it one of the two incident edges.

**Adding a loop.** One can add a loop (an edge $[\alpha, \alpha^\vee]$ with $\alpha = \alpha^\vee$) to a vertex between two consecutive ends of edges. Assign the unit group element to the new loop.

**Ciliated graphs and ciliated surfaces.** As it was mentioned several times above, a graph imbedded into an oriented surface inherits a fatness (cyclic order of ends of edges at vertices). Assume now that we have a graph imbedded into a surface in such a way that the vertices are mapped into the boundary. This graph inherits a ciliated fatness since there is a canonical linear order of the ends of edges meeting at a boundary point. On the other hand, given a ciliated fat graph imbedded into the corresponding surface (that is, we assume that the surface is retractable to the image of the graph) there exists a unique up to the isotopy way to move its vertices to the boundary reproducing the given ciliation. We have just to move each vertex to the boundary component which the cilium looks onto. If we now erase the edges of the graph and leave the cilia stuck out off the boundary components we get a ciliated surface (e.g., figs. 2b, 2d).

Suppose now we have two ciliated fat graphs $l$ and $l'$ corresponding to the same ciliated surface. (This means, in particular, that their vertices are identified.) We are going to construct an isomorphism $\mathcal{A}^l \xrightarrow{\sim} \mathcal{A}^l'$ between the spaces of graph connections on them. Let $\alpha \in E(l')$ be an end of an edge of $l'$. Take the edge $[\alpha, \alpha^\vee]$ of $l$ and retract it to the graph $l'$. We obtain a path on the graph $l$ connecting the vertices $[\alpha]$ and $[\alpha^\vee]$ and isotopic to the edge $[\alpha, \alpha^\vee]$. Assign to $\alpha$ the monodromy of the graph connection $\mathcal{A}^l$ along this path. Carrying out this procedure for all $\alpha \in E(l')$ we get the desired isomorphism.

Now let us summarize some properties of the spaces of graph connections equipped with the Poisson bracket eq. (16).
Proposition 4
1) The mappings between graph connections corresponding to erasing an edge, contracting an edge towards a vertex and gluing two vertices are the Poisson projections onto the image.
2) The isomorphism of the spaces of graph connections for two graphs corresponding to isomorphic ciliated surfaces is an isomorphism of Poisson manifolds.
3) Adding a loop is a Poisson imbedding.

The proof of the proposition is a straightforward and not very complicated explicit calculation that we omit here. Let us mention only the following statement useful for the proof as well as by itself.

Let \( f \) be a face of a ciliated fat graph \( l \) such that there are no cilia looking into \( f \). Let \( A^l(h, f) \) be the set of graph connections with the monodromy around the face \( f \) of \( l \) conjugated to \( h \in G \). Then \( A^l(h, f) \) is a Poisson submanifold in \( A^l \).

Let us proceed now to the relation between the space of graph connections and the space of ordinary connections.

Proposition 5 The quotient of the space of graph connections by the graph gauge group is isomorphic as a Poisson manifold to the quotient of the space of flat connections on the corresponding Riemann surface by the gauge group, i.e.,

\[
A^l/G^l \cong \mathcal{M}. \tag{24}
\]

Remark 1. Let us note that this statement shows that all the ambiguities in the construction of the space \( A^l \) – such as choices of ordering and of \( r \)-matrices – do not influence the Poisson structure of its quotient by the gauge group. The latter depends only on the cyclic order and on the symmetric part, \( t \), of the \( r \)-matrices (cf. eq.(14)). This could not be otherwise, because these are just the data defining the Poisson manifold \( \mathcal{M} \) by eq.(11), provided the surface \( S \) there is defined by the ciliated fat graph \( l \) here and the invariant scalar product \( \text{tr} \) there is defined by the Casimir element \( t \) here. However it is impossible to introduce a Poisson structure on \( A^l \) compatible with that on the gauge quotient without fixing nontrivial \( r \)-matrices. Note also that topologically these moduli spaces are always isomorphic to products of several copies of the group \( G \) modulo the overall \( G \)-conjugation, although they are not isomorphic to each other as Poisson manifolds. For example a sphere with three holes and a torus with one hole give topologically the same spaces, \((G \times G)/Ad G\), while the Poisson structure for, e.g., \( G = SL(2) \) is trivial in the first case and nontrivial in the second one.

Remark 2. The description of the moduli space \( \mathcal{M} \) of flat connections in the graph language has an advantage that this language allows us to describe rather explicitly the space of functions on \( \mathcal{M} \) using representation theory. In particular one can construct a linear basis in the space of regular functions on \( \mathcal{M} \) in the following way.

Assign an irreducible representation \( \pi_\alpha \) of \( G \) in a space \( V_\alpha \) to each \( \alpha \in E(l) \) in such a way that \( \pi_\alpha \circ = \pi_\alpha \) and assign an intertwiner \( C_n \in \text{Inv}(\otimes_{\alpha \in n} V_\alpha^*) \) to each vertex \( n \). We can consider matrices from \( \text{End} V_\alpha \) as belonging to \( V = \otimes_{\alpha \in E(l)} V_\alpha \) and the intertwiners \( C_n \) as belonging to its dual, \( V^* \).
For each such data \((l, C_\bullet, \pi_\bullet)\) we can define a function \(\psi(l, C_\bullet, \pi_\bullet)\) on \(A^l\)

\[
\psi(l, C_\bullet, \pi_\bullet)(\{A_\alpha\}) = \bigotimes_n C_n \bigotimes_{\alpha \in E_1(l)} \pi_\alpha(A_\alpha)
\]

where \(E_1(l) \subset E(l)\) is a set of ends of edges containing exactly one end of each edge. The ambiguity in the choice of this set is inessential because \(\pi_\psi(A_\psi) = \pi_\alpha(A_\alpha)\) as an element of \(V_\alpha \otimes V_\alpha^*\).

One can easily verify that all such functions are \(G^l\) invariant and that they indeed form a complete set of functions on \(\mathcal{M}\). The latter is an obvious consequence of the Peter-Weyl theorem.

**Proof of the proposition 5.** The Poisson bracket of two functions \(\Psi\) and \(\Phi\) on the space \(A\) of smooth connections on \(S\) can be written as

\[
\{\Psi, \Phi\}_S = \int_S \text{tr} \left( \frac{\delta \Psi}{\delta A} \wedge \frac{\delta \Phi}{\delta A} \right),
\]

To prove the proposition we need to compute the Poisson bivector on \(\mathcal{M}\) induced from eq.(26) by the reduction procedure described in the sect. 3 and compare the result with the bivector induced by eq.(16).

In order to be able to work with the bracket eq.(26) and build a bridge between the smooth and the combinatorial approaches to the Poisson brackets on flat connections let us first compute the Poisson bracket using the formula (26) in one particular case. Let \(I_1\) and \(I_2\) be two oriented intervals imbedded into \(S\) and intersecting transversally. Let us compute using eq.(26) the Poisson bracket between two arbitrary functions \(\Psi\) and \(\Phi\) of the corresponding monodromies considered as functions on \(A\).

The result of the computation is a function on the space \(A\). However, for our further purposes, we need to compute only the restriction of the result to the connections such that their restrictions to the intervals vanishes everywhere except for two subintervals containing the ends of \(I_1\) and \(I_2\) and none of their intersection points. In this case the expression for the bracket is especially simple, it depends only on the monodromies along the segments and can be straightforwardly computed from eq.(26):

\[
\{\Psi, \Phi\} = t^{ij}(R_i \Psi)(R_j \Phi) \sum_{k \in I_1 \cap I_2} \varepsilon(k).
\]

Here \(k\) runs over the intersection points, \(\varepsilon(k)\) is 1 or \(-1\) if the first segment crosses the second one from the left or from the right, respectively, \(\{R_i\}\) is a basis of the left-invariant vector fields on \(G\) and \(t^{ij}\) is the matrix of the quadratic Casimir \(t \in \mathfrak{g} \otimes \mathfrak{g}\) (e.g., eq. (14)).

Note that this formula does not give Poisson brackets between functions of monodromy along a single segment. Moreover the formula (26) is not applicable to compute such brackets.

Let us recall the definition of the Hamiltonian reduction in the language of Poisson brackets. Let \(\mathcal{M}\) be a symplectic manifold with symplectic action of the group \(G\), \(\mu\) be a
momentum map (corresponding to the action of $G$ or of a subgroup of it, $M_0 = \mu^{-1}(0)$, $N = M_0 \times G$ be the reduced space and $\pi : M_0 \to N$ be the canonical projection. The Poisson bracket of two functions $\Psi$ and $\Phi$ on $N$ is defined as follows. Let $\Psi^*$ and $\Phi^*$ be any two functions on $M$ such that their restrictions to $M_0$ coincide with $\pi^*\Psi$ and $\pi^*\Phi$ respectively. Then $\{\Psi, \Phi\}(x) := \{\Psi^*, \Phi^*\}(y)$, for any $x \in N$ and any $y \in \pi^{-1}(x)$. (Note that this procedure includes at least three arbitrary choices: the choice of functions $\Psi^*$ and $\Phi^*$ for given $\Psi$ and $\Phi$ and the choice of $y$ for given $x$. We are going to make these choices in a way maximally simplifying the calculations.)

This definition can be applied to our situation. We have the space of all connections, $A$, as $M$, the space of flat connections as $M_0$ and the moduli space, $M$, as $N$. Our task is to compute Poisson bracket on $M$ or, equivalently, between $G$-invariant functions on $A$ and, then, compare the result with the one given by eq.(16).

Let $\Psi$ and $\Phi$ be arbitrary two such functions and let $I$ and $I'$ be two imbeddings of the graph $I$ into the surface such that the images of vertices are disjoint and the images of edges are transversal. Using the mappings $A \to A'$ given by monodromies along the edges we can lift $\Psi$ and $\Phi$ using, respectively, $I$ and $I'$ to a $G^S$-invariant functions $\Psi^*$ and $\Phi^*$ on $A$.

Now to compute the bracket between $\Psi^*$ and $\Phi^*$ we need only to apply the lemma to all intersecting edges of $I$ and $I'$. To simplify the computations one can choose a convenient pair of graph imbeddings as well as a convenient flat connection within the given $G^I$-orbit. (In fact, we need two imbeddings $I$ and $I'$ since the formula (26) is not applicable for computing brackets between functions given by one imbedding.)

Fix a ciliation on $l$ and imbed $l$ in the surface in the way such that all vertices map to the boundary and all cilia look outside the surface. Thus we get our first embedding $l$. To get the second imbedding $I'$ deform the imbedding $l$ in order to make the edges of $I$ and $I'$ transversal and the formula (27) applicable. Fix a point at the middle of each edge. Then move each vertex along the boundary component a little to the left (if viewed from outside) together with incident edges keeping the middle points stable and making the number of the intersection points between deformed and initial edges as low as possible. Such a deformation is illustrated in fig. 8.

We have one intersection point for any two ends of edges $\alpha \in E(l)$ and $\beta \in E(I')$ belonging to the same vertex. Let us say that these intersection points are associated to this vertex. There is also one intersection point at the middle of each edge which we associate in an arbitrary way to one of the vertices of the corresponding edge.

Let us choose now a convenient connection within a given $G^I$ orbit. One can fix a disjoint collection of patches around each vertex of $l$ in such a way that each patch contains the corresponding vertex of $I'$ as well as all the segments of edges between these vertices and the intersection points associated to them. Since the patches are disjoint and topologically trivial, one can make the connection on them to be zero.

Note that since we have chosen the connection to be trivial around the vertices, we can apply the formula (27). Note also that the intersection points at the middle of the edges give trivial contribution. Finally we get an expression for a bivector giving
Poisson bracket of $\Psi$ and $\Phi$ as the following sum over all other intersection points

$$B = \sum_n \left( \sum_{i: \alpha, \beta \in n; \alpha < \beta} X_\alpha^i \wedge X_\beta^i \right),$$  

which coincides with eq.(17) up to terms vanishing on $G^l$-invariant functions. □

In this section we described the Poisson structure on $\mathcal{A}$ which gave us a description of the Poisson structure on $\mathcal{M}$ as well. As we mentioned in the Proposition 2 above, it is also possible to characterize the symplectic leaves in $\mathcal{M}$. It might be, however, useful to have an explicit description of the symplectic structure on those leaves. For such a description we refer to the paper by A.Alekseev and A.Malkin, ref. [5], see also their work ref. [4] where a useful description of the symplectic structure on the symplectic leaves in Poisson-Lie groups is given.

### Appendix. Ruijsenaars equations

In this Appendix we describe the geometric meaning of the trigonometric Ruijsenaars Hamiltonian integrable system [18], see eq. (A24) below. This system is a generalization of several integrable systems such as rational and trigonometric Calogero system, rational Ruijsenaars system and finite Toda chains. All those systems can be obtained from the trigonometric Ruijsenaars system by suitable limiting procedures. Another aspect which makes this system very interesting is its duality property, what means that coordinates and Hamiltonians enter this system symmetrically, i.e., there exists an involution of the phase space interchanging them. This property fails to be present in all the above listed limiting cases but the rational Calogero one, where this duality is well know even in the quasiclassical case. The quantum version of the trigonometric Ruijsenaars system is the system of MacDonald difference operators [16] and the duality between coordinates and hamiltonians appears there in disguise of MacDonald’s conjecture recently proved by Cherednik [10] by the methods quite different from those described in the present paper. However we shall not discuss the quantum aspects of this problem here.

We show here that one can interpret the phase space of the trigonometric Ruijsenaars system as a symplectic leaf of the lowest dimension in the moduli space $\mathcal{M}$ of flat $G = SL(k)$ connections on a once holed torus $\mathbb{T}$. The commuting Hamiltonians described in [15] are certain conjugation invariant functions of one of monodromies, the monodromy around one of the cycles of the torus, while the coordinates are the eigenvalues of the other. This picture shows that the duality is nothing but the action of the element of the mapping class group of the torus interchanging these two cycles.

As a by-product, we introduce a Poisson bracket, as well as a set of commuting Hamiltonians, on an auxiliary space $G \times G$. The flows generated by the Hamiltonians are particularly simple and the corresponding Hamiltonian equations can be easily integrated. The projection of the Poisson structure and the Hamiltonians on the quotient

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6The relation between the Ruijsenaars system and moduli of flat connections on the torus was found by Gorsky and Nekrasov in ref. [14].
$G \times G/\text{Ad } G$ exists and gives exactly the Ruijsenaars Hamiltonian system upon restriction to a certain symplectic leaf. This procedure gives a way to solve the Ruijsenaars equation explicitly. Algorithmically, it is of course just the same as it was proposed by S.N.M.Ruijsenaars and H.Shneider [18]; we just give a natural geometric meaning to it.

Our aim is now to prove the above statement. For this purpose we have to do the following.

1. Compute the canonical Poisson bracket on $\mathcal{M} = G \times G/\text{Ad } G$ (using the technique developed in the main part of the paper).

2. Choose coordinates on $\mathcal{M}$ canonically conjugated with respect to the Poisson bracket to the eigenvalues of one of the monodromy operators.

3. Compute a certain function of the other monodromy conjugacy class and verify that this gives exactly the trigonometric Ruijsenaars Hamiltonian.

To describe the symplectic structure on $\mathcal{M}$, choose the ciliated fat graph $l$ consisting of two edges and one vertex with the ciliated fat graph structure as shown in fig. 9 corresponding to the once holed torus (fig. 10).

The space of graph connections, $\mathcal{A}^l$, for such a graph is just a product of two copies of the group $G$,

$$\mathcal{A}^l = G \times G = \{(A, B)\}, \quad \text{(A1)}$$

where $A$ and $B$ are assigned to the edges of the graph as indicated in fig. 9.

The Poisson brackets on $\mathcal{A}^l$ are given by the relations following from the definition, eq.(16):

$$\{A, A\} = r_a (A \otimes A) + (A \otimes A) r_a + (1 \otimes A) r_21 (A \otimes 1) - (A \otimes 1) r (1 \otimes A), \quad \text{(A2)}$$

$$\{B, B\} = r_a (B \otimes B) + (B \otimes B) r_a + (1 \otimes B) r_21 (B \otimes 1) - (B \otimes 1) r (1 \otimes B), \quad \text{(A3)}$$

$$\{A, B\} = r (A \otimes B) + (A \otimes B) r + (1 \otimes B) r_21 (A \otimes 1) - (A \otimes 1) r (1 \otimes B), \quad \text{(A4)}$$

where $r_a = \frac{1}{2}(r - r_21)$; $A$ and $B$ are the matrix functions on $G \times G$ corresponding to $A$ and $B$, respectively, in the standard $k$-dimensional representation.

Introduce the standard notation, $G^*$, for the group $G$ equipped with the Poisson structure given by eq.(A2) and corresponding to the graph consisting of just one loop. The relation eq.(A2), which coincides, of course with eq.(20), is called sometimes the reflection equation.

The projections $p_1$ and $p_2$ of $\mathcal{A}^l = G \times G$ onto the first and the second factor, respectively, are obviously Poisson maps $p_{1,2} : \mathcal{A}^l \to G^*$

Now let us restrict ourselves to the case of the standard $r$-matrix,

$$r = \sum_{\alpha>0} E_{\alpha} \otimes E_{-\alpha} + \frac{1}{2} \sum_i H_i \otimes H_i. \quad \text{(A5)}$$

In this case one can easily derive from eqs.(A2, A4) the following commutation relations

$$\{\text{tr } A^n, A\} = 0, \quad \{\text{tr } B^n, B\} = 0, \quad \text{(A6)}$$

$$\{\text{tr } A^n, B\} = n(A^n)_0, \quad \text{(A7)}$$

$$\{\text{tr } B^n, A\} = nA(B^n)_0, \quad \text{(A8)}$$

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where \((X)_0\) denotes the traceless part of the matrix \(X\). Therefore, the functions \(\text{tr} B^n\) for \(n = 1, \ldots, k - 1\) considered as Hamiltonians generate commuting flows on \(\mathcal{A}^l\):

\[
B(t_1, \ldots, t_{k-1}) = B(0, \ldots, 0), \quad (A9)
\]

\[
A(t_1, \ldots, t_{k-1}) = A(0, \ldots, 0) e^{(t_1 B + \cdots + t_{k-1} B^{k-1})_0}. \quad (A10)
\]

As it was shown in the main part of the paper the lattice gauge group \(G^l\) acts on \(\mathcal{A}^l\) in a Poisson way, and the quotient Poisson manifold coincides with the moduli space \(\mathcal{M}\) of smooth flat connections on the Riemann surface corresponding to the fat graph \(l\). In our case, the group \(G^l\) is just \(G\) itself (since the graph has only one vertex) which acts on \(A\) and \(B\) by a simultaneous conjugation:

\[
g : (A, B) \mapsto (g A g^{-1}, g B g^{-1}). \quad (A11)
\]

The functions \(\text{tr} A^n\) and \(\text{tr} B^n\) are invariant under this action, and therefore they descend to the moduli space \(\mathcal{M}\) and generate commuting flows there as well, the trajectories on \(\mathcal{M}\) being just the projections of those given in eqs.(A9), (A10).

However the moduli space \(\mathcal{M}\) is in our case a Poisson manifold with a degenerate Poisson bracket. The symplectic leaves in \(\mathcal{M}\) correspond to connections having a fixed conjugacy class of the monodromy around the hole. In our case, the latter is just the matrix \(ABA^{-1}B^{-1}\).

Let us recall now that we are actually dealing with the case of \(G = SL(k)\). Different symplectic leaves in \(\mathcal{M}\) have different dimensions and the lowest dimension among them, in this case, is \(2(k - 1)\). Those leaves correspond to the monodromy around the hole being conjugated to a matrix \(x 1 + P\), where \(1\) is the unit matrix, \(\text{rk} P \leq 1\), and \(x \neq 0\) is a number which parameterises this set of symplectic leaves of the lowest dimension. (Indeed, the only conjugation invariant of an operator of rank not greater than one is its trace. The latter is \(\text{tr} P = x^{1-k} - x\), since \(\det(x 1 + P) = 1\).)

On such leaves, the family of functions \(\text{tr} A^n, n = 1, \ldots, k - 1\), form a full set of commuting variables. Let us introduce local coordinates on these symplectic leaves in the following way. Let \(\lambda_1, \ldots, \lambda_k\) be the eigenvalues of the matrix \(A\) and \(q_1, \ldots, q_k\) be the diagonal matrix elements of \(B\) in the basis in which \(A\) is diagonal. Imposing the condition \(\text{rk} P \leq 1\) and conjugating \(B\) by a diagonal matrix one can make \(B\) take the form

\[
\text{B}^i_j = \frac{\sqrt{q_i q_j (1 - x)}}{\lambda_i / \lambda_j - x}. \quad (A12)
\]

The functions \(\lambda_i \) and \(q_j \) are locally well defined functions on the symplectic leaves and the Poisson brackets between them are

\[
\{\lambda_i, \lambda_j\} = 0, \quad (A13)
\]

\[
\{q_i, q_j\} = q_i q_j \frac{(\lambda_i + \lambda_j)}{(\lambda_i / \lambda_j - x)(\lambda_j / \lambda_i - x)(\lambda_i - \lambda_j)}, \quad (A14)
\]

\[
\{\lambda_i, q_j\} = \lambda_i q_j \delta_{i,j}. \quad (A15)
\]

**Proof of the formulas eqs. (A13–15).** To simplify calculations we assume for a moment that we are working with the group \(GL(k)\) rather than \(SL(k)\). Having computed the
Poisson structure on the quotient space by the lattice gauge group we can then restrict it to the subspace corresponding to $SL(k)$-connections since the latter space is a Poisson submanifold in the whole quotient space. The bivector defining the Poisson structure on $A^l$ for the group $GL(k)$ can be rewritten in the form

$$B = \frac{1}{2} \sum_{i,j,u,v \in \{1,...,4\}} E_j^{i(u)} \otimes E_i^{j(v)} (\epsilon(u,v) + \epsilon(i,j)),$$  \hspace{1cm} (A16)

where $\epsilon(i,j)$ is $-1, 0,$ or $1$ if $i$ is less, equal, or greater than $j,$ respectively, and $E_j^{i(u)}$ are the standard $gl(k)$ generators acting on the $u$-th end of an edge. (In our case $E_i^{j(1)}$ acts on $A$ from the left, $E_i^{j(2)}$ acts on $B$ from the left, $E_i^{j(3)}$ acts on $A$ from the right and $E_i^{j(4)}$ acts on $B$ from the right.)

It is not practical, however, to compute the Poisson brackets between $\lambda_i$ and $q_j$ using this bivector as it is, because it does not agree with the diagonal of the matrix $A$. In order to avoid this difficulty, since we are interested only in computing Poisson brackets of gauge invariant functions, we may change the bivector (A16) in such a way that it still defines the same Poisson bracket on the space of gauge invariant functions. In other words, there are different ways to write down Poisson brackets on the coset space $G \times G/AdG$ in terms of a bivector on the space $G \times G$. On the other hand, since we know that the projection $\pi : G \times G \to G \times G/AdG$ is a Poisson map, i.e. the bracket of gauge invariant functions is gauge invariant, it suffices to compute the value of the bracket of two such functions on a submanifold $F \subset G \times G$ which intersects each gauge orbit (i.e. each $AdG$-orbit) at least once. In doing this way, we can simplify computations by changing the bivector (A16) by terms vanishing on $F$. As a prescription, one can formulate the following rule of allowed modifications of the bivector. One can add to any vector $E_j^{i(\alpha)}$ a vector which is tangent to gauge orbits or vanishes on $F$. The vectors tangent to the gauge orbits are just generators of the gauge transformations, in our case $\sum_{u=1}^4 E_j^{i(u)}$. The vectors vanishing on $F$ (which is in our case the space of connections with $A$ diagonal) are, for example, $\lambda_i E_j^{i(1)} + \lambda_j E_j^{i(3)}$. Using these rules one can replace:

$$E_j^{i(1)} \leadsto \frac{\lambda_i}{\lambda_i - \lambda_j} (E_j^{i(2)} + E_j^{i(4)}),$$  \hspace{1cm} (A17)

$$E_j^{i(3)} \leadsto \frac{\lambda_j}{\lambda_i - \lambda_j} (E_j^{i(2)} + E_j^{i(4)}).$$  \hspace{1cm} (A18)

By this trick the bivector $B$ can be transformed to the form

$$B' = \sum_{i>j} E_j^{i(2)} \wedge E_i^{j(4)} \frac{\lambda_i + \lambda_j}{2(\lambda_i - \lambda_j)} + \frac{1}{2} \sum_i E_i^{i(2)} \wedge E_i^{i(1)} + E_i^{i(3)} \wedge E_i^{i(4)}.$$  \hspace{1cm} (A19)

Applying this bivector (which now leaves $A$ diagonal) we get the desired Poisson brackets. $\square$

The form of the brackets eqs.(A13, A14) is such that in order to define the variables canonically conjugated to $\lambda_i$ we can just multiply $q_i$ by factors not depending on $q_i$. For
example, one can take the variables

\[ s_i = q_i x^{n-1} \left( \prod_{k,k \neq i} \frac{(\lambda_k - \lambda_i)(\lambda_i - \lambda_k)}{(\lambda_k - x\lambda_i)(\lambda_i - x\lambda_k)} \right)^{\frac{1}{2}}. \]  

(A20)

One can check by an explicit computation that these new variables, \( s_i \), have Poisson brackets

\[ \{ s_i, s_j \} = 0, \]  

(A21)

\[ \{ \lambda_i, s_j \} = \lambda_i s_j \delta_{i,j}. \]  

(A22)

Using the formula

\[ \det B = x^{-n(n-1)/2} \prod_i q_i \prod_{i \neq j} \frac{(\lambda_i - \lambda_j)}{(x\lambda_i - \lambda_j)}, \]  

(A23)

which can be easily proved by induction, one can express the function \( H = \text{tr} (B + B^{-1}) \) in terms of \( \lambda_i \) and \( s_i \);

\[ H = \sum_i (s_i + s_i^{-1}) x^{n-1} \left( \prod_{k,k \neq i} \frac{(\lambda_k - \lambda_i)(\lambda_i - \lambda_k)}{(\lambda_k - x\lambda_i)(\lambda_i - x\lambda_k)} \right)^{\frac{1}{2}} \]  

(A24)

that turns out to be just the Ruijsenaars hamiltonian.

Note, that the Poisson structure on \( A^l = G \times G \) is nice from various points of view. In particular, it is nondegenerate close to the identity. The action of the group on this space by conjugation is a Poisson one and has a well defined momentum map (in the sense of Poisson-Lie groups) \( \mu : G \times G \to G^* \) such that \( \mu : (A, B) \mapsto ABA^{-1}B^{-1} \).

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Fig. 1
Examples of fat graphs and surfaces corresponding to them.
The cyclic orders at vertices are understood to be counterclockwise.
The graph (a) gives a disk with two holes (b).
The graph (c) gives a torus with one hole (d).
Fig. 2
Examples of ciliated fat graphs and corresponding ciliated surfaces.
Cilia are indicated by small strokes at the vertices.
The graph \((a)\) gives a disk with two holes \((b)\).
The graph \((c)\) gives a torus with one hole \((d)\).
Fig. 3
The graphs corresponding to
(a) the Poisson-Lie group $G$,
(b) its double $D \cong G \times G$,
(c) its dual Poisson-Lie group $G^*$. 

Fig. 4
Operation of erasing an edge.
The shaded region represents the remainder of the graph.
Fig. 5

Operations of contractions of an edge.
$L$ and $R$ are the two different ways of contraction.
$L$ corresponds to factoring by gauge transformations at the vertex $n_R$.
$R$ corresponds to factoring by gauge transformations at the vertex $n_L$.

Fig. 6

Operation of gluing graphs.
Some particular cases of gluing graphs which correspond to natural operations in Poisson-Lie groups:

(a) multiplication in $G$,
(b) multiplication in $D$,
(c) projection $D \to G^*$,
(d) action of $D$ on $G^*$,
(e) multiplication in the 5-uble,
(f) action of the 5-uble on a space of graph connections.
The special way of deforming a graph drawn on a surface which gives two transversal graphs; the deformed graph is shown by the broken line.
Fig. 9
The ciliated graph corresponding to a holed torus.

Fig. 10
The same graph drawn on the holed torus.