Improved Inference for Checking Type Annotations

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Abstract. We consider type inference in the Hindley/Milner system extended with type annotations and constraints with a particular focus on Haskell-style type classes. We observe that standard inference algorithms are incomplete in the presence of nested type annotations. To improve the situation we introduce a novel inference scheme for checking type annotations. Our inference scheme is also incomplete in general but improves over existing implementations as found e.g. in the Glasgow Haskell Compiler (GHC). For certain cases (e.g. Haskell 98) our inference scheme is complete. Our approach has been fully implemented as part of the Chameleon system (experimental version of Haskell).

1 Introduction

Type inference for the Hindley/Milner system [Mil78] and extensions [Rem93, Pot98, OSW99] of it is a heavily studied area. Surprisingly, little attention has been given to the impact of type annotations (a.k.a. user-provided type declarations) and user-provided constraints on the type inference process. For concreteness, we assume that the constraint domain is described in terms of Haskell type classes [Jon92, HHPW94]. Type classes represent a user-programmable constraint domain which can be used to code up almost arbitrary properties. Hence, we believe that the content of this paper is of importance for any Hindley/Milner extension which supports type annotations and constraints. The surprising observation is that even for “simple” type classes type inference in the presence of type annotations becomes a hard problem.

Example 1. The following program is a variation of an example from [JN00].

Note that we make use of nested type annotations.\(^1\)

```haskell
class Foo a b where foo :: a->b->Int
instance Foo Int b
instance Foo Bool b
```

\(^1\)For concreteness, we annotate 1 with \texttt{Int} because in Haskell 1 is in general only a number.
We introduce a two-parameter type class `Foo` which comes with a method `foo` which has the constrained type `∀a, b. Foo a b ⇒ a → b → Bool`. The two instance declarations state that `Foo Int b` and `Foo Bool b` hold for any `b`. Consider functions `p` and `q`. In each case the subexpression `y+(1::Int)` forces `y` to be of type `Int`. The body of function `f x = foo y x` generates the constraint `Foo Int t_x` where `t_x` is the type of `x`. Note that this constraint is equivalent to `True` due to the instance declaration.

We find that `f` has the inferred type `∀t_x. Foo t_y t_x ⇒ t_x → Int`. We need to verify that this type subsumes the annotated type `f :: c -> Int` which is interpreted as `∀c. c → Int`. More formally, we write $C_g \vdash \sigma_i \leq \sigma_a$ to denote that the inferred type $\sigma_i$ subsumes the annotated type $\sigma_a$ under some constraint $C_g$. Suppose $\sigma_i = (\forall a. C_i \Rightarrow t_i)$ and $\sigma_a = (\forall b. C_a \Rightarrow t_a)$ where there are no name clashes between $a$ and $b$. Then, the subsumption condition is (logically) equivalent to $C_g \vdash (\exists a. C_i \land t_i = t_a)$. In this statement, we assume that $\vdash$ refers to the model-theoretic entailment relation and $\Rightarrow$ refers to Boolean implication. Outermost universal quantifiers are left implicit. Note that in our system, we only consider type equality rather than the more general form of subtyping. For our example, we find that the subsumption condition holds. Hence, expressions `p` and `q` are well-typed.

Let’s see what some common Haskell implementations such as Hugs [HUG] and GHC [GHC] say. Expression `p` is accepted by Hugs but rejected by GHC whereas GHC accepts `q` which is rejected by Hugs! Why?

In a traditional type inference scheme [DM82], constraints are generated while traversing the abstract syntax tree. At certain nodes (e.g. let) the constraint solver is invoked. Additionally, we need to check for correctness of type annotations (a.k.a. subsumption check). The above examples show that different traversals of the abstract syntax tree yields different results. E.g. Hugs seems to perform a right-first traversal. We visit `f :: c -> Int; f x = foo y x` first without considering the constraints arising out of the left tuple component. Hence, we find that `f` has the inferred type `∀t_x. Foo t_y t_x ⇒ t_x → Int` where `y` has type `t_y`. This type does not subsume the annotated type. Therefore, type inference fails. Note that GHC seems to favor a left-first traversal of the abstract syntax tree.

The question is whether there is an inherent problem with nested type annotations, or whether it’s simply a specific problem of the inference algorithms implemented in Hugs and GHC.

**Example 2.** Here is a variation of an example mentioned in [Fax03]. We make use of the class and instance declarations from the previous example.
test y = let f :: c->Int
     f x = foo y x
     in f y

Note that test may be given types \( \text{Int} \rightarrow \text{Int} \) and \( \text{Bool} \rightarrow \text{Int} \). The constraint \( \text{Foo} \ t_y \ t_x \) arising out of the program text can be either satisfied by \( t_y = \text{Int} \) or \( t_y = \text{Bool} \). However, the “principal” type of test is of the form \( \forall t_y. (\forall t_x. \text{Foo} \ t_y \ t_x) \Rightarrow t_y \rightarrow \text{Int} \). Note that the system we are considering does not allow for constraints of the form \( \forall t_x. \text{Foo} \ t_y \ t_x \). Hence, the above example has no (expressible) principal type. We note that the situation is different for (standard) Hindley/Milner with type annotations. As shown by Odersky and Läufer \[OL96\], the problem of finding a solution such that \( \sigma_i \) (the inferred type) is an instance of \( \sigma_a \) (the annotated type) can be reduced to unification under a mixed prefix \[MH92\]. Hence, we either find a principal solution \( \phi \) such that \( \vdash \phi(\sigma_i) \leq \phi(\sigma_a) \) or no solutions. Hence, inference for Hindley/Milner with type annotations is complete.

We conclude that type inference for Hindley/Milner with constraints and (nested) type annotations is incomplete. The incompleteness arises because the subsumption check not only involves a test for correctness of annotations, but may also need to find a solution. The above example shows that in our general case there might not necessarily be a principal solution.

In order to resurrect completeness we could impose a syntactic restriction on the set of programs. E.g., we could simply rule out type annotations for “nested” let-definitions, or require that the types of all lambda-bound variables occurring in the scope of a nested annotation must be explicitly provided (although it’s unclear whether this is a sufficient condition). In any case, we consider these as too severe restrictions.

In fact, the simplest solutions seems to be to enrich the language of constraints. Note that the subsumption condition itself is a solution to the subsumption problem. Effectively, we add constraints of the form \( \forall b_i. (C_a \supset (\exists a. C_i \land t_i = t_a)) \) to our language of constraints. Then, \( \forall t_y. (\forall t_x. \text{Foo} \ t_y \ t_x) \Rightarrow t_y \rightarrow \text{Int} \) will become a valid type of test in the above example. This may be a potential solution for some cases, but is definitely undesirable for Haskell where constraints are attached to dictionaries. It is by no means obvious how to construct dictionaries \[HHPW94\] for “higher-order” constraints. Furthermore, we believe that type inference easily becomes undecidable depending on the underlying primitive constraint domain.

In this paper, we settle for a compromise between full type inference and full type checking. We only check for the correctness of type annotations. But before checking we infer as much as possible. Our contributions are:

- We introduce a novel formulation of improved inference for checking annotations in terms of Constraint Handling Rules (CHR) \[Fru95\]. While inferring the type of some inner expression we can reach the result of inference for some outer expression.
- We can identify a class of programs for which inference is complete.
- Our approach is fully implemented as part of the Chameleon system \[SW\]. We can type a much larger class of programs compared to Hugs and GHC. E.g., Example \[I\] is typable in our system. We refer to \[SW\] for more examples.
We write $\forall \alpha. D \Rightarrow t \in \Gamma$ for $x : \forall \alpha. D \Rightarrow t \in \Gamma$ and $\bar{a} = f(s(C_1,t_1)) - f(s(\Gamma))$ for $\bar{a} = f(s(C_1,t_1)) - f(s(\Gamma))$.

2 Types and Constraints

We present an extension of the Hindley/Milner system with constraints and type annotations.

Expressions $\quad e ::= x \mid \lambda x.e \mid e \ e \mid \text{let } f = e \text{ in } e \mid \text{let } f :: C \Rightarrow t \text{ in } e$

Types $\quad t ::= a \mid t \rightarrow t \mid T \bar{t}$

Type Schemes $\quad \sigma ::= t \mid \forall \alpha. C \Rightarrow t$

Constraints $\quad C ::= t \mid t \mid U \bar{t} \mid C \land C$

CHRs $\quad R ::= U \bar{t} \leftrightarrow C \mid U_1 \bar{t}_1 \ldots U_n \bar{t}_n \Rightarrow C$

We write $\bar{o}$ to denote a sequence of objects $o_1, \ldots, o_n$ and $\bar{a} \bar{t}$ to denote $o_1 : t_{11}, \ldots, o_n : t_n$. W.l.o.g., we assume that lambda-bound and let-bound variables have been $\alpha$-renamed to avoid name clashes. We record these variables in some environment $\Gamma$. Note that we consider $\Gamma$ as an (ordered) list of elements, though we commonly use set notation. We denote by $\{x_1 : \sigma_1, \ldots, x_n : \sigma_n\}$. We make use of pairs, integers, booleans etc. in examples.

We find two kinds of constraints. Equations $t_1 = t_2$ among types $t_1$ and $t_2$ and user-defined constraints $U \bar{t}$. We assume that $U$ refers to type classes such as $Foo$. For our purposes, we restrict ourselves to single-headed simplification CHRs $U \bar{t} \iff C$ and multi-headed propagation CHRs $U_1 \bar{t}_1, \ldots, U_n \bar{t}_n \Rightarrow C$.

We note that CHRs describe logic formula. E.g. $U \bar{t} \iff C$ can be interpreted as $\forall \alpha. U \bar{t} \iff (\exists b. C)$ where $\bar{a} = f(s(\Gamma))$ and $\bar{b} = f(s(C)) - \bar{a}$, and $U_1 \bar{t}_1, \ldots, U_n \bar{t}_n \Rightarrow C$ can be interpreted as $\forall \alpha. (U_1 \bar{t}_1 \land \ldots \land U_n \bar{t}_n) \supset (\exists b. C)$ where $\bar{a} = f(s(\bar{t}_1), \ldots, \bar{t}_n)$ and $\bar{b} = f(s(C)) - \bar{a}$. Via CHRs we can model most known type class extensions.

We refer the interested reader to [SST04] for a detailed account of translating...
classes and instances to CHRs. We claim that using CHRs we can cover a sufficiently large range of Hindley/Milner type systems with constraints such as functional dependencies, records etc. Note that CHRs additionally offer multi-headed simplification CHRs which in our experience so far do not seem to be necessary in the type classes context. Due to space limitations, we only give one simple example showing how to express Haskell 98 type class relations in terms of CHRs.

We assume that the meaning of user-defined constraints (introduced by class and instance declarations) has already been encoded in terms of some set \( P_p \) of CHRs. User-defined functions are recorded in some initial environment \( \Gamma_{init} \).

**Example 3.** Consider

```haskell
class Eq a where (==) :: a->a->Bool
instance Eq a => Eq [a]
class Eq q => Ord a where (<) :: a->a->Bool
class Foo a b where foo :: a->b->Int
instance Foo Int b
instance Foo Bool b
```

Note that the declaration `class Eq a => Ord a` introduces a new type class `Ord` and imposes the additional condition that `Ord a` implies `Eq a` (which is sensible assuming that an ordering relation assumes the existence of an equality relation). We can model such a condition via a propagation rule. Hence, \( P_p \) consists of

\[
\begin{align*}
\text{(Super)} & \quad \text{Ord} a \implies Eq a \quad (F1) \\
\text{(Eq1)} & \quad Eq a \iff Eq [a] \quad (F2) \\
\end{align*}
\]

and \( \Gamma_{init} = \{ (==) : \forall a.Eq a \Rightarrow a \rightarrow a \rightarrow Bool, (<) : \forall a.Ord a \Rightarrow a \rightarrow a \rightarrow Bool, foo : \forall a,b.Foo a b \Rightarrow a \rightarrow b \rightarrow Int \} \).

We introduce judgments of the form \( C, \Gamma \vdash e : t \) where \( C \) is a constraint, \( \Gamma \) refers to the set of lambda-bound variables, predefined and user-defined functions, \( e \) is an expression and \( t \) is a type. We leave the type class theory \( P_p \) implicit. We say a judgment is valid iff there is a derivation w.r.t. the rules found in Figure 1. Commonly, we require that constraints appearing in judgments are satisfiable. We say that a valid judgment is satisfiable iff all constraints appearing in the derivation are satisfiable. A constraint is satisfiable w.r.t. a type class theory iff we find some model satisfying the theory and constraint. We say a theory \( P_p \) is satisfiable iff we find some model for \( P_p \).

In rule (Var-\forall E), we assume that \( x \) either refers to a lambda- or let-bound variable. Note that only let-bound variables and primitives can be polymorphic. For convenience, we combine variable introduction with quantifier elimination. We can build an instance of a type scheme if the instantiated constraint is entailed by the given constraint w.r.t. type class theory \( P_p \).

In rule (Let) we couple the quantifier introduction rule with the introduction of user-defined functions. In our formulation, \( C_2 \) does not necessarily guarantee that \( C_1 \) is satisfiable. However, our rule (Let) is sound for a lazy semantics which applies to Haskell.
Rule (LetA) introduces a type annotation. Note that we assume that type annotations are closed, i.e. all variables appearing in \( C_1 \Rightarrow t_1 \) are assumed to be universally quantified. This is the assumption made for Haskell 98 [Has]. Note that the environment for typing the function body includes the binding \( g : \forall a.C_1 \Rightarrow t_1 \). Hence, we allow for polymorphic recursive functions.

The other rules are standard. Note that we left out the rule for monomorphic recursive functions for simplicity.

3 Type Inference via CHRs

We introduce our improved inference scheme first by example. Then, we show how to map the typing problem to a set of CHRs. We give a description of the semantics of CHRs adapted to our setting. Finally, we show how to perform type inference in terms of CHR solving.

3.1 Motivating Examples

The following examples give an overview of the process by which we abstract the typing problem in terms of constraints and CHRs.

Example 4. Consider the following program.

\[
\text{let } f \text{ x } = \text{x in } (f \text{ True}, f \text{ y})
\]

We introduce new predicates, (special-purpose) user constraints, \( g(t) \) and \( f(t) \) to constrain \( t \) to the types of functions \( g \) and \( f \) respectively. It is necessary for us to provide a meaning for these constraints, which we will do in terms of CHR rules. The body of each rule will contain all constraints arising from the definition of the corresponding function, which represent that function’s type.

For the program above we may generate rules similar to the following.

\[
g(t) \iff t = \text{t}_y \to (\text{t}_1, \text{t}_2), f(\text{t}_f1), \text{t}_f1 = \text{Bool} \to \text{t}_1, f(\text{t}_f2), \text{t}_f2 = \text{t}_y \to \text{f}_2
\]

\[
f(t) \iff t = \text{t}_x \to \text{t}_x
\]

The arrow separating the rule head from the rule body can be read as logical equivalence. Variables mentioned only in a rule’s body are implicitly existentially quantified.

In the \( g \) rule we see that \( g \)’s type is of the form \( \text{t}_y \to (\text{t}_1, \text{t}_2) \), where \( \text{t}_1 \) and \( \text{t}_2 \) are the results of applying function \( f \) to a \( \text{Bool} \) and a \( \text{t}_y \). We represent \( f \)’s type, at both call sites in the program, by the \( f \) user constraint.

The \( f \) rule is much more straightforward. It simply states that \( t \) is \( f \)’s type if \( t \) is the function type \( \text{t}_x \to \text{t}_x \), for some \( \text{t}_x \), which is clear from the definition of \( f \).

We can infer \( g \)’s type by performing a CHR derivation, solving the constraint \( g(t) \) by applying CHRs (removing the constraint matching the lhs with the rhs). Note that we avoid renaming variables where unnecessary.

\[
g(t) \Rightarrow g \text{ t } = \text{t}_y \to (\text{t}_1, \text{t}_2), f(\text{t}_f1), \text{t}_f1 = \text{Bool} \to \text{t}_1, f(\text{t}_f2), \text{t}_f2 = \text{t}_y \to \text{f}_2
\]

\[
\Rightarrow f \text{ t } = \text{t}_y \to (\text{t}_1, \text{t}_2), \text{t}_f1 = \text{t}_x \to \text{t}_x, \text{t}_f1 = \text{Bool} \to \text{t}_1, f(\text{t}_f2), \text{t}_f2 = \text{t}_y \to \text{f}_2
\]

\[
\Rightarrow f \text{ t } = \text{t}_y \to (\text{t}_1, \text{t}_2), \text{t}_f1 = \text{t}_x \to \text{t}_x, \text{t}_f1 = \text{Bool} \to \text{t}_1, \text{t}_f2 = \text{t}_x \to \text{t}_x', \text{t}_f2 = \text{t}_y \to \text{f}_2
\]

If we solve the resulting constraints for \( t \), we see that \( g \)’s type is \( \forall \text{t}_y, \text{t}_y \to (\text{Bool}, \text{t}_y) \).
Example 5. The program below is a slightly modified version of the program presented in Example 4.

\[
g \ y = \text{let } f \ x = (y, x) \text{ in } (f \ True, f \ y)
\]

The key difference is that \(f\) now contains a free variable \(y\). Since \(y\) is monomorphic within the scope of \(g\) we must ensure that all uses of \(y\) in all definitions, are consistent, i.e. each CHR rule which makes mention of \(t_y\), \(y\)'s type, must be referring to the same variable. This is important since the scope of variables used in a CHR rule is limited to that rule alone.

In order to enforce this, we perform a transformation akin to lambda-lifting, but at the type level. Instead of user constraints of form \(f(t)\) we now use binary constraints \(f(t, l)\) where the \(l\) parameter represents \(f\)'s environment.

We would generate rules like the following from this program.

\[
g(t, l) \iff t = t_y \to (t_1, t_2), f(t_{f_1}, t_{f_1} = \text{Bool} \to t_1, f(t_{f_2}, \langle t_x \rangle), t_{f_2} = t_y \to f_2, l = ls \\
f(t, l) \iff t = t_x \to (t_y, t_x), f = \langle \langle t_y \rangle \rangle)
\]

We write \((t_1, ..., t_n)\) to indicate a type-level list containing \(n\) types. A list with an \(n\)-element prefix but an unbounded tail is denoted by \(\langle t_1, ..., t_n \mid l \rangle\). When unifying such a type against another list, \(t\) will be bound to some sublist containing all elements after the \(n\)th.

As mentioned above, we now use binary predicates to represent the type of a function. The first argument, which we commonly refer to as the \(t\) component, will still be bound to the function’s type. The second component, which we call \(l\), represents a list of unbound variables in scope of that function. We have ensured that whenever the \(f\) constraint is invoked from the \(g\) rule that \(t_y\), the type of \(y\), is made available to it. So, in essence, the \(t_y\) that we use in the \(f\) rule will have the same type as the \(t_y\) in \(g\), rather than simply being a fresh variable known only in \(g\).

Example 6. We now return to the program first introduced in Example 1 and generate the CHR rules corresponding to the function \(p\), which is repeated below.

For simplicity we will assume that (+) is defined only on \(\text{Ints}\), i.e. (+) : \(\text{Int} \to \text{Int} \to \text{Int}\).

\[
p \ y = (\text{let } f : c \to \text{Int} \\
\quad f \ x = \text{foo} \ y \ x \\
\quad \text{in } f, y + (1: \text{Int}))
\]

We generate the following CHRs from this fragment of the program. We also include the rule which represents \(\text{foo}\)'s type, and the rule which corresponds to the instance \(\text{Foo} \ \text{Int} \ b\).

\[
p(t, l) \iff t = t_y \to (t_1, t_2), f(t_{f_1}, t_{f_1} = \text{Int} \to \text{Int}, t_{plus} = t_y \to \text{Int} \to t_r, l = (ls, \langle t_y, t_x \rangle) \\
f_a(t, l) \iff p(t', l), t = c \to \text{Int}, l = (\langle t_y \mid ls \rangle, \langle t_y, t_x \rangle) \\
f(t, l) \iff t = t_x \to t_r, f_o(t_{f_o}, ls'), t_{f_o} = t_y \to t_x \to t_r, f_a(t, l), l = (\langle t_y \mid ls \rangle, \langle t_y, t_z \rangle) \\
\text{foo}(t, l) \iff t = a \to b \to \text{Int}, \text{Foo} a b \\
\text{Foo \ Int} \ b \iff \text{True}
\]
Here we have extended the scheme which we used to generate the constraints in the previous example. We stick to binary predicates, but have expanded the \( l \) component to include two lists. The first list, which we refer to as the \emph{local} \( l \) component contains, as before, a type-level list of all unbound lambda variables in scope of the function. The second list, which we will often denote \( LT \) simply contains all of the lambda-bound variables from the top-level definition down, in a fixed order.

We introduce a symbol \( f_a \) and generate a new rule to represent \( f \)'s annotated type. Note also that we add a call to the \( f \) rule to unify \( f \)'s inferred type with the declared type.

As demonstrated earlier, in order to check \( f \) it is necessary to consider all of the type information available in \( f \)'s context. In particular, for this program, it is critical that we know \( y \) has type \( \text{Int} \), in order to reduce the \( \text{Foo Int} \) \( b \) constraint which arises from the use of \( \text{foo} \), but is absent from the annotation.

The way we introduce \( f \)'s context into \( f \) is by adding a call from the \( f_a \) rule to the immediate parent definition, which in this case is represented by \( p \). In this instance we are not interested in \( p \)'s type, only the effect it has on lambda-bound variables, and any type class constraints which may arise. Note that if \( p \) were itself embedded within a function definition, then it too would have such a call (to its own parent), and so \( f \) would indirectly inherit \( p \)'s context.

We perform the following simplified derivation to demonstrate that the CHR formulation above captures the necessary context information within \( f \).

\[
\begin{align*}
f(t, l) \rightarrow_f & f_0(l_{foo}, l'), t_{foo} = t_y \rightarrow t_x \rightarrow t_r, f_a(t, l), \\
l &= ((t_y), (t_y,t_z)), \ldots \\
r_{foo} & t_{foo} = a \rightarrow b \rightarrow \text{Int}, \text{Foo } a \ b, t_{foo} = t_y \rightarrow t_x \rightarrow t_r, f_a(t, l), \\
l &= ((t_y), (t_y,t_z)), \ldots \\
\end{align*}
\]

Through the call to \( p \) from \( f_a \) we introduce sufficient context information to determine that \( t_y \) is an \( \text{Int} \), and to consequently reduce away the \( \text{Foo} \) constraint using the instance rule. Note that the \( LT \) component is necessary here because \( p \) is not aware of its own lambda-bound variables. Without the \( LT \) component, \( p \) would not be able to “export” the required information about \( t_y \) to \( f \).

### 3.2 Constraint and CHR Generation

In detail, we show how to map expressions to constraints and CHRs. Lambda-abstractions such as \( \lambda x.e \) are preprocessed and turned into \( \lambda x : t_x.e \) where \( t_x \) is a fresh type variable. We assume that \( LT \) contains all such type variables \( t_x \) attached to lambda-abstractions.

For each function definition \( f = e \) we generate a CHR of the form \( f(t, l) \iff C \) where \( l \) refers to a pair \((l_l, l_q)\). The constraint \( C \) is generated out of the program text of \( e \). We maintain that \( l_l \) denotes the set of types of lambda-bound variables in the environment and \( l_q \) refers to \( LT \) the types of all lambda-bound variables.

\[
\begin{align*}
f(t, l) \rightarrow_f & f_0(l_{foo}, l'), t_{foo} = t_y \rightarrow t_x \rightarrow t_r, f_a(t, l), \\
l &= ((t_y), (t_y,t_z)), \ldots \\
r_{foo} & t_{foo} = a \rightarrow b \rightarrow \text{Int}, \text{Foo } a \ b, t_{foo} = t_y \rightarrow t_x \rightarrow t_r, f_a(t, l), \\
l &= ((t_y), (t_y,t_z)), \ldots \\
\end{align*}
\]
Constraint Generation:

\[
\begin{align*}
\text{(Var-x)} & \quad (x : t_1) \in \Gamma \quad t_2 \text{ fresh} \quad \frac{\Gamma, E, x \vdash t_2 (t_2 = t_1 \parallel t_2)}{
\text{(Var-f)} & \quad t, l, l_g \text{ fresh} \quad (f \in E \text{ or } f_a \in E \text{ non-recursive}) \quad C = \{f((t, l), l = ((t_2), l_0))\} \quad (x : t_2), E, f \vdash_C (C \parallel t)\\
\text{(VarA-f)} & \quad f_a \in E \text{ recursive} \quad t, l, l_g \text{ fresh} \quad C = \{f_a((t, l), l = ((t_2), l_0))\} \quad \{x : t_2\}, E, f \vdash_C (C \parallel t)\\
\text{(Abs)} & \quad C' = \{C, t_3 = t_1 \rightarrow t_2\} \quad t_3 \text{ fresh} \quad \frac{\Gamma, E, \lambda x : t_1.E \vdash_C (C \parallel t_3)}{
\text{(App)} & \quad \frac{\Gamma, E, e_1 \vdash_C (C \parallel t_1)}{
\text{(App)}} & \quad \frac{\Gamma, E, e_2 \vdash_C (C \parallel t_2)}{
\text{(AppA)}} & \quad \frac{\Gamma, E, \{g_a\}, e_2 \vdash_C (C \parallel t)}{
\text{(Let)}} & \quad \frac{\Gamma, E, \{g\}, e_2 \vdash_C (C \parallel t)}{
\text{(LetA)}} & \quad \frac{\Gamma, E, \{g\}, e_1 \vdash_C (C \parallel t)}{
\text{CHR Generation:}}
\end{align*}
\]

\[
\begin{align*}
\text{(Var)} & \quad h, \Gamma, E, v \vdash_R \emptyset \quad (\text{App}) \quad \frac{h, \Gamma, E, e_1 \vdash_R P_1}{h, \Gamma, E, e_2 \vdash_R P_2} \quad \frac{h, \Gamma, E, \lambda x : t.E \vdash_R P_1 \cup P_2}{h, \Gamma, E, \lambda x : t.E \vdash_R P_1 \cup P_2} \quad (\text{Abs}) \quad \frac{h, \Gamma, x : t.E \vdash_R P}{h, \Gamma, E, \lambda x : t.E \vdash_R P}\\
\text{(Let)} & \quad \Gamma = \{x_1 : t_1, \ldots, x_n : t_n\} \quad t, t', l, l_r, l_g \text{ fresh} \quad \frac{g, \Gamma, E, \{g\}, e_2 \vdash_R P_1 \quad h, \Gamma, E, \{g\}, e_2 \vdash_R P_2}{g, \Gamma, E, \{g\}, e_1 \vdash_C (C' \parallel t')} \quad (\text{LetA}) \quad \frac{\{g_a(t, l) \iff t = t'_l, C'_r, l = (\{l_1, \ldots, l_n\}|r), LT, h(t'_l, l)\} \cup \{g(t, l) \iff l = (\{l_1, \ldots, l_n\}|r), LT, g_a(t, l), C'_l, t = t'_l\}}{h, \Gamma, E, \{g\}, e_1 \vdash_C (C' \parallel t')} \quad \frac{g \vdash_C (C' \parallel t')}{g \vdash_C (C' \parallel t')} \quad (\text{LetA)} \quad \frac{\Gamma = \{x_1 : t_1, \ldots, x_n : t_n\} \quad t, t', l, l_r \text{ fresh}}{g, \Gamma, E, \{g\}, e_1 \vdash_C (C' \parallel t')}
\end{align*}
\]

\[\text{Fig. 2. Constraint and CHR Generation}\]

We make use of list notation (on the level of types) to refer to the types of λ-bound variables. In order to avoid confusion with lists of values, we write \(\langle l_1, \ldots, l_n \rangle\) to denote the list of types \(l_1, \ldots, l_n\). We write \(\langle l|l'\rangle\) to denote the list of types with head \(l\) and tail \(r\).

For constraint generation, we employ judgments \(\Gamma, E, e \vdash_C (C \parallel t)\) where environment \(\Gamma\), set of predicate symbols \(E\) and expression \(e\) are input values and constraint \(C\) and type \(t\) are output parameters. Note that \(\Gamma\) consists of lambda-bound variables only whereas \(E\) holds the set of predicate symbols referring to primitive and let-defined functions. Initially, we assume that \(E_{init}\) holds all the
symbols defined in $P_{init}$ which is the CHR representation of all functions in $\Gamma_{init}$. The rules can be found in Figure 2.

Consider rule (Var-f). If the function does not carry an annotation or the function is not recursive \(^2\) we make use of the definition CHR. However, strictly making use of the definition CHR might introduce cycles among CHRs, e.g. consider polymorphic recursive functions. In such cases we make use of the annotation CHR, see rule (VarA-f). In both rules we set $l$ to the sequence of types of all lambda-variables in scope. Note that we might pass in more types of lambda-bound variables than expected by that function. This is safe because we leave the first component of $l$ “open” at definition sites. That is, we expect at least the types of lambda-bound variables in scope at the definition site and possibly some more. The second component which refers to the sequence of all types of lambda-bound variables appearing in the entire program is left unconstrained. This component will be only constrained at definition sites (see CHR generation rules (Let) and (LetA)). Note that in rule (Abs) the order of lambda-bound variables added to type environment matters. Hence, we silently treat $\Gamma$ as a list rather than a set. In rules (Let) and (LetA) the constraints arising out of $e_1$ might not appear in $C$ unless we use function $g$ in $e_2$. Note that we do not generate a constraint for the subsumption condition which will be checked separately.

For rule generation, we employ judgments of the form $h, \Gamma, E, e \vdash_R P$ where CHR $h$, environment $\Gamma$, set of predicate symbols $E$ and expression $e$ are input values and the set $P$ of CHRs is the output value. As an invariant we maintain that $h$ refers to the surrounding definition of expression $e$. Initially, we assume that $h$ refers to some trivial CHR $h(t, l) \iff True$ and $E$ refers to the set of primitive functions. We refer to Figure 2 for details. There are two interesting rules.

Rule (Let) deals with unannotated functions. Note that we do not add $g$ to $E$ when generating constraints and rules from $e_1$. Hence, we assume for simplicity that unannotated functions are not allowed to be recursive. Of course, our system \(^3\) handles unannotated, recursive functions. Their treatment is described in a forthcoming report. The novel idea of our inference scheme is that we reach surrounding constraints within the definition of $g$ via the constraint $h(t_2, l)\triangledown$. The $\triangledown$ marker (left out in Example 6 for simplicity) serves two purposes: (1) We potentially create cycles among CHRs because we might reintroduce renamed copies of surrounding constraints. Markers will allow us to detect such cycles to avoid non-termination of CHRs. (2) Variables occurring in marked constraints are potentially part of the environment. Hence, we should not quantify over those variables. Examples will follow shortly to highlight these points.

Rule (LetA) is similar to rule (Let). Here, the annotation CHR includes the surrounding definition $h$. The actual inference result is reported in the definition CHR.

3.3 CHR Solving

We introduce the marked CHR semantics. We assume that each constraint is attached with either a $\triangledown$ marker or a $\epsilon$ (pronounced empty) marker. The empty marker is commonly left implicit. We refer to constraints carrying a $\triangledown$ marker as marked constraints. A constraint carrying the empty marker is unmarked. In

\(^2\) We can easily check whether a function is recursive or not by a simple dependency analysis.
A derivation step from global set of constraints \( C \) to \( C' \). A derivation step using either rules in \( P \) is a sequence of derivation steps such that no further derivation step is \( C \) →\( C' \). A derivation, denoted \( C \rightarrow^* \), is a sequence of derivation steps using either rules in \( P \) such that no further derivation step is applicable to \( C' \). The operational semantics of CHRs exhaustively apply rules to the global set of constraints, being careful not to apply propagation rules twice on the same constraints (to avoid infinite propagation). We say a set of CHRs is terminating if for each \( C \) there exists \( C' \) such that \( C \rightarrow^* \).

**Example 7.** Consider

```haskell
class Erk a where erk :: a
class Foo a where foo :: a
f = (erk, let g :: Foo a => a; g = foo in g)
```

Here is a sketch of the translation to CHRs.

\[
g_a(t) \iff f(t) \oplus \text{Foo } t \\
g(t) \iff g_a(t) \oplus \text{Foo } t
\]

Consider the derivation

\[
g(t_1) \rightarrow_g g_a(t_1) \oplus \text{Foo } t_1 \rightarrow g_a f(t'_1) \oplus \text{Foo } t_1 \\
\rightarrow_f t' = (a, b), (Erk \ a) \oplus g(b) \oplus \text{Foo } t_1 \\
\rightarrow_g t' = (a, b), (Erk \ a) \oplus g_a(b) \oplus (\text{Foo } b) \oplus \text{Foo } t_1
\]

In step \( \rightarrow_f \) we propagate \( \oplus \) to all new constraints. Note that we encounter a cycle among CHRs (see underlined constraints). Indeed, CHRs may be “non-terminating” because we introduce repeated duplicates of surrounding constraints. To avoid non-termination we introduce an additional CHR Cycle Removal step

\[
\rightarrow_g t' = (a, b), (Erk \ a) \oplus g_a(b) \oplus (\text{Foo } b) \oplus \text{Foo } t_1 \\
\rightarrow_{\text{CCR}} t' = (a, b), (Erk \ a) \oplus (\text{Foo } b) \oplus \text{Foo } t_1
\]

which is defined as follows.
Consider type inference for an expression and an environment \( \Gamma \) models \( E \) (CHRs and \( h \)). Let \( \Gamma, e \) a typing problem such that \( h, \Gamma, E, e \vdash_R P_e \) and \( (\Gamma_{init}, P_{init}) \) models \( E_{init} \) for some \( \Gamma_{init} \). Then, \( P_p \cup P_e \cup P_{init} \) is terminating.

### Definition 2 (CHR Cycle Removal)

Let \( f(t, l) \in C \) and \( f(t', l') \in C' \) and a derivation \( \vdash C \vdash ... \vdash C' \). Then, \( \vdash C \vdash ... \vdash C' \vdash_{CCR} C' \vdash f(t', l') \).

We assume that \( \vdash_{CCR} \) is applied aggressively.

We argue that this derivation step is sound because any further rule application on \( g_a(b) \in \) will only add renamed copies of constraints already present in the store.

### Lemma 1 (CCR Soundness)

Let \( P \) be a set of CHRs and \( C \) and \( C' \) two constraints such that \( C \vdash \Gamma \vdash C' \). Then \( P \models C \vdash \exists_f(C)C' \).

We can also argue that we break any potential cycle among predicate symbols referring to function symbols. Note that we do not consider breaking cycles among two unmarked constraints. Such cases will only occur in case of unannotated, recursive functions which are left out for simplicity. A detailed description of such cases will appear in a forthcoming report. Also note that we never remove cycles in case of user-defined constraints. In such a case, the type class theory might be non-terminating. Hence, we state the type class theory is terminating.

### Lemma 2 (CCR Termination)

Let \( P_p \) be a terminating type class theory, \( h(t, l) \iff \text{True} \) a CHR, \( E_{init} \) a set of primitive predicate symbols, \( P_{init} \) a set of CHRs and \( \Gamma, e \) a typing problem such that \( h, \Gamma, E_{init}, e \vdash_R P_e \) and \( (\Gamma_{init}, P_{init}) \) models \( E_{init} \) for some \( \Gamma_{init} \). Then, \( P_p \cup P_e \cup P_{init} \) is terminating.

### 3.4 Type Inference via CHR Solving

Consider type inference for an expression \( e \) w.r.t. an environment \( \Gamma \) of lambda-bound variables and an environment \( \Gamma_{init} \) of primitive functions and type class theory \( P_p \). We assume \( (P_{init}, E_{init}) \) model \( \Gamma_{init} \) such that for each \( f : \forall a.C \Rightarrow t' \) we find \( f(t, l) \iff C, t = t' \in P_{init} \) and \( f \in E_{init} \). Then, we generate \( \Gamma, e \vdash C \vdash (C\Gamma t) \) and \( h, \Gamma, E_{init}, e \vdash R P_e \). We generally assume that \( P \) denotes \( P_p \cup P_e \cup P_{init} \).

For typability we need to check that (1) constraint \( C \) is satisfiable, and (2) all type annotations in \( e \) are correct. We are now in the position to describe CHR-based satisfiability and subsumption check procedures.

### Definition 3 (Satisfiability Check)

Let \( P \) be a set of CHRs and \( C \) a constraint such that \( C \vdash \Gamma \vdash C' \) for some constraint \( C' \). We say that \( C \) is satisfiable iff the unifier of all equations in \( C' \) exists.

Soundness of the above definition follows from results stated in [SS02] in combination with Lemma 1. Of course, decidability of the satisfiability check depends on whether CHRs are terminating.

To check for correctness of type annotations we first need to calculate the set of all subsumption problems. Let \( E_{sub(e)} \) be the set of all predicate symbols \( g_a \) where each \( g_a \) refers to some subexpression (let \( g : C_1 \Rightarrow t_1 \in e_1 \) in \( e_2 \)) in \( e \). Let \( F_{sub(e)} \) be a formula such that \( \forall t, l, g_a(t, l) \iff g(t, l) \in F_{sub(c)} \) for all \( g_a \in E_{sub(e)} \). It remains to verify that the type annotation is correct under the abstraction of type inference in terms of \( P \). Formally, we need to verify that
\[ P \models F_{\sub(c)} \] where \( P \) refers to first-order logic interpretation of the set of CHRs \( P \). In [SS02], we introduced a CNF (Canonical Normal Form) procedure to test for equivalence among constraints \( \forall t, l. (g_a(t, l) \leftrightarrow g(t, l)) \) w.r.t. some set of CHR by executing \( g_a(t, l) \) and \( g(t, l) \) and verify that the resulting final stores are equivalent modulo variables in the initial store (here \( \{t, l\} \)). Thus, we can phrase the subsumption check as follows. We write \( \exists_{t, l}.C \) to denote \( \exists \forall \text{vars}(C) - \{t, l\}.C \).

**Definition 4 (Subsumption Check).** Let \( g_a \in E_{\sub(c)} \) and \( P \) be a set of CHRs. We say that \( g \)'s annotation is correct iff (1) we execute \( g_a(t, l) \Rightarrow^*_P C_1 \) and \( g(t, l) \Rightarrow^*_P C_2 \), (2) we have that \( \models (\exists_{t, l}.C_1) \Rightarrow (\exists_{t, l}.C_2) \).

Soundness of the above definition follows from results stated in [SS02] in combination with Lemma 1.

**Example 8.** Recall Example 7 and the derivation \( g(t_1) \Rightarrow^* t' = (a, b), (\text{Erk } a)_{\oplus} , \text{Foot } t_1 \).

A similar calculation shows that \( g_a(t_1) \Rightarrow^* s' = (c, d), (\text{Erk } c)_{\oplus} , \text{Foot } t_1 \). Note that the resulting constraints are logically equivalent modulo variable renamings. Hence, \( g \)'s annotation is correct.

Note that in our formulation of type inference, the type of an expression is described by a set of constraints w.r.t. a set of CHRs. The following procedure describes how to build the associated type scheme. Markers attached to constraints provide important information which variables arise from the surrounding scope. Of course, we need to be careful not to quantify over those variables.

**Definition 5 (Building of Type Schemes).** Let \( P \) be a set of CHRs, \( g \) a function symbol. We say function \( g \) has type \( \forall a.C' \Rightarrow t' \) w.r.t. \( P \) iff (1) \( g(t, l) \Rightarrow^* C, l = (l_1, l_2) \) for some constraint \( C \), (2) \( \phi \), the m.g.u. of \( C, l = (l_1, l_2) \) exists, (3) let \( D \subseteq \phi(C) \) such that \( D \) is maximal and \( D \) consists of unmarked user-defined constraints only, (4) let \( \bar{a} = ft(D, \phi t) - ft(\phi l) \), (5) let \( C' = \phi(C) \) and \( t' = \phi(t) \).

**Example 9.** According to Example 8, we find that \( g \) has type \( \forall t_1 . (\text{Erk } a, \text{Foot } t_1) \Rightarrow t_1 \). Note that \( \text{Erk } a \) arises from \( f \)'s program text.

We are able to state soundness of our approach.

**Theorem 1 (Soundness).** Let \( P_e, P_p \) and \( P_{\init} \) be three sets of CHRs, \( h \) a CHR in \( P_{\init} \), \( \Gamma \) an environment of simply-typed bindings, \( \Gamma_{\init} \) an environment of primitive functions, \( e \) an expression, \( E_{\init} \) a set of predicate symbols, \( C \) a constraint and \( t \) a type such that \( \langle P_{\init}, E_{\init} \rangle \) models \( \Gamma_{\init} \) and \( \Gamma, E_{\init}, e \vdash_C (C \downarrow t) \) and \( h, \Gamma, E_{\init}, e \vdash_{R \in P_e} P_e \) and type checking of all annotations in \( e \) is successful. Let \( C \vdash_{P_p \cup P_{\init} \cup P_e} C' \) for some constraint \( C' \). Let \( \phi \) be the m.g.u. of \( C' \) where we treat all variables in \( \Gamma \) as Skolem constants. Then, \( \phi(C'), \phi(\Gamma) \cup \Gamma_{\init} \vdash e : \phi(t) \).

The challenge is to identify some sufficient criteria under which our type inference method is complete. Because we only check for subsumption we need to guarantee that each subsumption condition will be either true or false. E.g. in Example 8, the subsumption condition boils down to the constraint \( \forall t_x . \text{Foot } t_y t_x \).

Note that we can satisfy this constraint by setting \( t_y \) to either \( \text{Int} \) or \( \text{Bool} \). Hence,
our task is to prevent such situations from happening. In fact, such situations can never happen for single-parameter type classes. But what about multi-parameter type classes? The important point is to ensure that fixing one parameter will immediately fix all the others. That is, in case of \(\forall t_x.\text{Foo } t_y \ t_x\) we know that \(t_x\) is uniquely determined by \(t_y\). We can enforce such conditions in terms of functional dependencies.

**Definition 6.** We say a type class \(TC\) is fully functional iff we find a class declaration
\[
\text{class } TC \ a_1 \ldots \ a_n \ | \ fd_1, \ldots, fd_n \text{ where } fd_i = a_i \rightarrow a_1 \ldots a_{i-1} a_{i+1} \ldots a_n.
\]

We argue that for fully functional dependencies solutions (if they exist) must be unique. In fact, this is not sufficient because there are still cases where we need guess.

**Example 10.** Consider the following simplified representation of the Show class.

```haskell
class Show a where
  show :: a->String
  read :: String->a
  f :: Show a => String->String
  f x = show (read x)
```

The subsumption check boils down to the formula \(\forall a.\text{Show } a \supset \exists a'.\text{Show } a'\) which is obviously a true statement (take \(a' = a\)). However, in our translation to CHRs we effectively check for \(\text{Show } a \leftrightarrow \text{Show } a'\) which obviously does not hold.

There are further sources where we need to take a guess.

**Example 11.** Consider

```haskell
class Foo
instance Show Int
instance Foo a => Show a
```

where \(P_p = \{(S1) \text{Show Int } \leftrightarrow \text{True} | (S2) \text{Show } a \leftrightarrow \text{Foo } a\}\). We have that \(P_p \models \text{Show Int}\). However, \(\text{Show Int } \rightarrow_{S2} \text{Foo Int}\) where \(\models \text{Foo Int } \leftrightarrow \text{True}\) which suggests that \(P_p \models \text{Show Int}\) might not hold. Clearly, by guessing the right path in the derivation we find that \(\text{Show Int } \models_{S1} \text{True}\).

To ensure that our subsumption check (Definition 4) is complete we need to rule out ambiguous types and require that the type class theory is complete. A type is ambiguous iff we can not determine the variables appearing in constraints by looking at the types alone. The annotation \(f :: \text{Show } a \Rightarrow \text{String} \Rightarrow \text{String}\) in Example 10 is ambiguous. A type class theory \(P_p\) is complete iff \(P_p\) is confluent, terminating and range-restricted (i.e. grounding the lhs of CHRs grounds the rhs) and all simplification rules are single-headed. The type class theory \(P_p\) in Example 11 is non-confluent. In [SS02] we have identified these conditions as sufficient to ensure completeness of the Canonical Normal Form procedure to test for equivalence among constraints.

**Theorem 2.** Let \(P_p\) a complete and fully functional type class theory. Then our CHR-based inference scheme infers principal types if the types arising are unambiguous.
4 Related Work and Conclusion

Simonet and Pottier [SP04] introduce HMG(X), a refined version of HM(X) [OSW99, Sul00], which includes among others type annotations. Their type inference approach is based on the “allow for more solutions” philosophy. Hence, they achieve complete type inference immediately. However, they only consider tractable type inference for the specific case of equations as the only primitive constraints.

An approach in the same spirit is considered by Hinze and Peyton-Jones [HJ00]. They sketch an extension of Haskell to allow for “higher-order” instances which logically correspond to nested equivalence relations. As pointed out by Faxén [Fax03], in such an extended Haskell version it would be possible to type the program in Example 2. We believe this is an interesting avenue to pursue. We are not aware of any formal results nor a concrete implementation of their proposal.

Pierce and Turner [PT00] develop a local type inference scheme where user-provided type information is propagated inwards to nodes which are below the annotation in the abstract syntax tree. Their motivation is to remove redundant annotations. Note that Peyton-Jones and Shields [PJS04] describe a particular instance of local type inference based on the work by Odersky and Läufer [OL96]. In our approach we are able to freely distribute type information across the entire abstract syntax tree. Currently, we only distribute information about the types of lambda-bound variables and type class constraints. We believe that our approach can be extended to a system with rank-k types. We plan to pursue this topic in future work.

In this paper, we have presented a novel inference scheme where the entire type inference problem is mapped to a set of CHRs. Due to the constraint-based nature of our approach, we are able to make available the results of inference for outer expressions while inferring the type of inner expressions. We have fully implemented the improved CHR-based inference system as part of the Chameleon system [SW]. Our system improves over previous implementations such as Hugs and GHC. For some cases, e.g. unambiguous Haskell 98 programs, we can even state completeness. We note that our improved inference scheme can host the type debugging techniques described in [SSW03, SSW04].

In future work, we plan to follow the path of Odersky and Läufer [OL96] and compute (non-principal in general) solutions to subsumption problems. We strongly believe that our improved CHR-based inference will be of high value for such an attempt. Another alternative inference approach not mentioned so far is to only generate all necessary subsumption problems $\sigma_i \leq \sigma_a$ and wait for the “proper” moment to solve or check them. Of course, we still need to process them in a certain order and might fail for the same reason we failed in Example 1. Clearly, our constraint-based approach allows us to “exchange” intermediate results among two subsumption problems which may be crucial for successful inference.

References

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A Variations

Our existing translation to CHRs is slightly lazier than most inference algorithms in the sense that we do not infer the types of let-bound variables which are never called.

Example 12. Consider:

\[
\begin{align*}
\text{f} \ x &= \text{let} \ g = x \ x \\
\quad &\text{in} \ x
\end{align*}
\]

We generate CHR rules that look like the following:

\[
\begin{align*}
f(t) &\iff t = t_x \to t_x \\
g(t) &\iff t_x = t_x \to t
\end{align*}
\]

Since the \textit{f} rule never calls \textit{g}, the unsatisfiable constraint is not introduced, and this program is considered well-typed.

This “laziness” can be problematic whenever we need to compare the inferred type of some function with its declared type. Consider an annotated function \textit{g}, nested within the definition of function \textit{f}, from which we generate an inference rule, \textit{g}(\textit{t}, \textit{l}) \iff \textit{a}(\textit{t}, \textit{l}), \textit{C}_i, and an annotation rule, \textit{a}(\textit{t}, \textit{l}) \iff \textit{f}(\textit{t}', \textit{l}'), \textit{C}_a.

It’s possible that the types of some global variables are affected in \textit{g}, but not in \textit{a}. In order for \textit{g}(\textit{t}, \textit{l}) and \textit{a}(\textit{t}, \textit{l}) to be equivalent, we depend on \textit{g}’s context, as called in the \textit{a} rule, to in turn call \textit{g} and introduce those missing constraints.

Example 13. Consider the following program.

\[
\begin{align*}
f \ y &= \text{let} \ g :: \text{Bool} \\
\quad &\text{g} = y \\
\quad &\text{in} \ 'a'
\end{align*}
\]

We generate the following (simplified) CHR rules:

\[
\begin{align*}
f(\textit{t}, \textit{l}) &\iff \textit{t} = \text{Char}, \textit{l} = (\langle\rangle, (\textit{t}_y)) \\
g(\textit{t}, \textit{l}) &\iff \textit{t} = \textit{t}_y, \textit{l} = ((\textit{t}_y), (\textit{t}_y)), \textit{a}(\textit{t}, \textit{l}) \\
a(\textit{t}, \textit{l}) &\iff \textit{t} = \text{Bool}, \textit{l} = ((\textit{t}_y), (\textit{t}_y)), \textit{f}(\textit{t}', \textit{l})
\end{align*}
\]

Even though \textit{g}’s type annotation is acceptable, our subsumption check would fail, because \textit{g}(\textit{t}, \textit{l}) and \textit{a}(\textit{t}, \textit{l}) are not equivalent wrt the \textit{l} component. Clearly, the \textit{g} rule implies \textit{t}_y = \text{Bool}, but the \textit{a} rule does not.

We can remedy this situation by ensuring that all nested functions are called by their parent function. In this way, when we consider a function annotation, the definition of the function becomes part of its context.

Example 14. We modify the program of Example 13, forcing \textit{f} to call \textit{g}, but disregard its value.

\[
\begin{align*}
f \ y &= \text{let} \ g :: \text{Bool} \\
\quad &\text{g} = y \\
\quad &\text{in} \ \text{const} \ 'a' \ g
\end{align*}
\]
The CHR rule associated with \( f \) would now look something like:

\[
\begin{align*}
  f(t, l) & \iff t_{\text{const}} = a \to b \to a, a = \text{Char}, t = t_y \to a, \\
  l & = (\langle \rangle, \langle t_y \rangle), \ g(b, l')
\end{align*}
\]

This solves our immediate problem, in that the constraints arising from \( g(t, l) \) and \( g_a(t, l) \) are now equivalent wrt \( t \) and \( l \).

Unfortunately this does not work in the case where the function we call is recursive. Consider:

\[
f y = \text{let } g :: \text{Bool} \\
  g = \text{const y g}
\]

Here, if \( f \) were to call \( g \), the constraint generated to represent \( g \)'s type would be \( g_a(t', l') \). We then face the same problem, that from within \( g_a \) we have no association between \( t_y \) and \( \text{Bool} \).

Clearly, a syntactic transformation of the source program to introduce calls to otherwise uncalled functions is not sufficient. We must modify the CHR generation process to directly insert calls to the inference constraints of functions which are not already called.

Example 15. We return to the rules generated in Example 13. The following CHR rule is a modified version of the \( f \) rule above which now contains a call to \( g \), further constraining the lambda-bound type variables.

\[
\begin{align*}
  f(t, l) & \iff t = \text{Char}, \ l = (\langle \rangle, \langle t_y \rangle), \ g(t', l')
\end{align*}
\]

Using this modified rule, we see now that \( g(t, l) \) and \( g_a(t, l) \), are equivalent, since \( t_y = \text{Bool} \) in both. The \( \ominus \) mark on the \( g \) constraint is not significant here, though it does accomodates the simpler form of cycle breaking (by simply removing the repeated constraint) than the equivalent unmarked constraint would.

B Monomorphic Recursive Functions

In case of monomorphic recursive functions (i.e. recursive functions with no type annotation) we need to update our strategy for breaking cycles among CHRs (Definition 2). We denote by \( \text{NRF} \) the set of all non-recursive functions, by \( \text{MRF} \) the set of all recursive functions which carry no type annotations and by \( \text{ARF} \) the set of all annotated recursive functions.

**Definition 7 (CHR Cycle Removal).** Let \( f(t, l) \in C \) and \( f(t', l') \in C' \) where \( f \in \text{NRF} \cup \text{ARF} \) and a derivation \( \ldots \Rightarrow C \Rightarrow \ldots \Rightarrow C' \). Then, \( \ldots \Rightarrow C \Rightarrow \ldots \Rightarrow C' \Rightarrow \ominus \text{CCR} \ C' \ominus f(t', l') \).

Let \( f(t, l) \in C \) and \( f(t', l') \in C' \) where \( f \in \text{MRF} \) and a derivation \( \ldots \Rightarrow C \Rightarrow \ldots \Rightarrow C' \). Then, \( \ldots \Rightarrow C \Rightarrow \ldots \Rightarrow C' \Rightarrow \ominus \text{CCR} \ C' \ominus f(t', l') \ominus t' = t \).

Let \( f(t, l) \in C \) and \( f(t', l') \in C' \) where \( f \in \text{MRF} \) and \( f(t', l') \) is known to arise from the exact same program source location as \( f(t, l) \), and given a derivation \( \ldots \Rightarrow C \Rightarrow \ldots \Rightarrow C' \). Then, \( \ldots \Rightarrow C \Rightarrow \ldots \Rightarrow C' \Rightarrow \ominus \text{Mono} \ C' \ominus f(t', l'), t' = t \).

We assume that \( \ominus \text{CCR} \) and \( \ominus \text{Mono} \) are applied aggressively.
Note that according to the definition of a $\rightarrow_{\text{Mono}}$ step, we should only ever break cycles amongst constraints representing unannotated, recursive function types when they arise from the same program location. Unifying the types of different calls to the same function is overly restrictive, as the following example illustrates.

**Example 16.** Consider the following program.

```plaintext
e = \text{let } f = f \\
in (f::\text{Int}, f::\text{Bool})
```

The two $f$s in the body of $e$ are calls to the same recursive, unannotated function, i.e. $f \in \text{MRF}$. It would be unnecessarily restrictive, however, to aggressively apply $\rightarrow_{\text{Mono}}$ here and unify the types of the two $f$s, resulting in a type error. Indeed, these two $f$s are not even part of the same cycle.

We note that breaking cycles among constraints arising from the exact same program source location is sufficient.

**Example 17.** Consider

$$f = \ldots f_1 \ldots f_2$$

where we added numbers 1 and 2 to different uses sites of $f$. Here is a sketch of type inference where we annotate $\rightarrow$ to refer to the number of CHR steps applied so far. $f \rightarrow^n f_1, f_2 \rightarrow^j f_1.1, f_1.2, f_2$. In the last step we reduce a call to $f$ at location 1. Note that we make use of a refined marking scheme. We keep track of the original source location and add the location of the constraint introduced to the store to the existing locations. We refer to [SSW03] for the details of such a “location-history” aware refinement of the CHR semantics. Hence, applying rule (Mono) twice will remove $f_{1.1}$ and $f_{1.2}$ (and equate the types of $f_{1.1}$ and $f_{1.2}$ with $f$). Similarly, we remove the cycles created by $f_2$.

We note that for $\rightarrow_{\text{CCR}}$ there is no need to equate the $l$ component in case of $f \in \text{MRF}$. The $l$ component is always set at the use site of functions (see constraint generation rules (Var-f)). Equating the $l$ component would yield a strictly weaker system.

**Example 18.** Consider $f x = (f_1 \ldots, \text{let } g y = \ldots f_2 \ldots \text{in } g)$ where we added numbers 1 and 2 to different (monomorphic) uses sites of $f$. Here is a sketch of type inference $f \rightarrow^n \ldots f_1, \ldots, g \rightarrow^{n+m} \ldots f_1, \ldots, f_2$ where natural numbers $n$ and $m$ refer to the number of CHR steps applied so far. Assume we fully equate $f_1$ and $f_2$. Note that their local $l_i$ components differ (because this component is always exact!) Hence, type inference fails.

Note that (as before in case of $\text{ARF}$ and $\text{NRF}$) we only break cycles among constraints which carry the same marker. This yields a more precise method.

**Example 19.** Assume we have situation where

$$\ldots \Rightarrow f(\text{Int}), \ldots \Rightarrow \ldots \Rightarrow f(t)_{\ominus}, \ldots$$

Removing $f(t)_{\ominus}$ and adding $t = \text{Int}$ might be too restrictive.
Note that by construction $\rightarrow_{Mono}$ never applies to $ARF$ and $NRF$. Any potential cycle will be eventually broken. We can re-establish Lemmas 2 and state a slightly stronger Lemma 1 which guarantee soundness of our CHR-based inference approach.

**Lemma 3 (CCR-Mono Soundness).** Let $P$ be a set of CHRs and $C$ and $C'$ two constraints such that $C \Rightarrow_p^* C'$. Then $P \models \exists_{fv(C)} C' \supset C$.

Note that $P \models C \supset \exists_{fv(C)} C'$ does not hold anymore. We may reject typable programs because we strictly enforce the (Mono) rule.

**Example 20.** Consider

\[ h \times = (h_1 \ 'a') \&\& (h_2 \ True) \]

Here is a sketch of type inference.

\[
\begin{align*}
\color{red}h(t) &\quad \Rightarrow t = tx \rightarrow \text{Bool}, h_1(\text{Char} \rightarrow \text{Bool}), h_2(\text{Bool} \rightarrow \text{Bool}) \\
\color{red}h(t) &\quad \Rightarrow t = tx \rightarrow \text{Bool}, \text{Char} \rightarrow \text{Bool} = tx' \rightarrow \text{Bool}, h_1(\text{Char} \rightarrow \text{Bool}), \\
&\quad h_{1,2}(\text{Bool} \rightarrow \text{Bool}), h_2(\text{Bool} \rightarrow \text{Bool}) \\
\color{red}h(t) &\quad \Rightarrow_{Mono} t = tx \rightarrow \text{Bool}, \text{Char} \rightarrow \text{Bool} = tx' \rightarrow \text{Bool}, \text{Char} \rightarrow \text{Char} = \text{Char} \rightarrow \text{Bool}, \\
&\quad h_{1,2}(\text{Bool} \rightarrow \text{Bool}), h_2(\text{Bool} \rightarrow \text{Bool}) \\
\color{red}h(t) &\quad \Rightarrow_{Mono} t = tx \rightarrow \text{Bool}, \text{Char} \rightarrow \text{Bool} = tx' \rightarrow \text{Bool}, \text{Char} \rightarrow \text{Char} = \text{Char} \rightarrow \text{Bool}, \\
&\quad \text{Char} \rightarrow \text{Bool} = \text{Bool} \rightarrow \text{Bool}, h_2(\text{Bool} \rightarrow \text{Bool}) \\
\color{red}&\quad \Leftarrow \text{False}
\end{align*}
\]

It is interesting to note is that our type inference scheme for recursive function is more relaxed compared to the one found in some other established type checkers.

**Example 21.** Consider the following program.

\[
\begin{align*}
e &:: \text{Bool} \\
e &\ = g \\
f &:: \text{Bool} \rightarrow a \\
f &\ = g \\
g &\ = f \ e
\end{align*}
\]

In the case of GHC, the following error reported is:

\[
\text{mono-rec.hs:5:} \\
\text{Couldn't match 'Bool} \rightarrow a \text{' against 'Bool'} \\
\text{Expected type: Bool} \rightarrow a \\
\text{Inferred type: Bool}
\]

In the definition of 'f': $f = g$

The problem reported here stems from the fact that within the mutually recursive binding group consisting of $e$, $f$ and $g$, $f$ is assigned two ununifiable types, $\text{Bool}$ and $\text{Bool} \rightarrow a$: the first because it must have the same type as $g$, which according to $e$ must be $\text{Bool}$; and the second because of its type declaration.
Our translation scheme is more liberal than this, in that \( g \)'s type within \( e \) and \( f \) may be different. Essentially, we only require that the type of a variable be identical at all locations within the mutually recursive subgroup if a type declaration has been provided for that variable.

(Simplified) Translation of the above program to CHRs yields.

\[
\begin{align*}
  e_a(t) & \iff t = \text{Bool} \\
  e(t) & \iff g(t) \\
  f_a(t) & \iff t = \text{Bool} \to a \\
  f(t) & \iff g(t) \\
  g(t) & \iff e_a(t_e), f_a(t_f), t_f = t_e \to \text{Bool}
\end{align*}
\]

It is clear from the above that there are no cycles present amongst these rules. We can use them to successfully infer a type for any of the variables in the program.

In fact, our handling of binding groups is similar to [Jon99] where type inference of binding groups proceeds as follows:

1. Extend the type environment with the type signatures. In this case \( f :: \forall a. \text{Bool} \to a \) and \( e :: \text{Bool} \).
2. Do type inference on the bindings without type signatures, in this case \( g = f \cdot e \). Do generalisation too, and extend the environment, giving \( g :: \forall a. a \).
3. Now, and only now, do type inference on the bindings with signatures.