Hamiltonicity of doubly semi-equivelar maps on the torus

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Abstract

The well-known twenty types of 2-uniform tilings of the plane give rise infinitely many doubly semi-equivelar maps on the torus. In this article, we show that every such doubly semi-equivelar map on the torus contains a Hamiltonian cycle. As a consequence, we establish the Nash-Williams conjecture for the graphs associated with these doubly semi-equivelar maps by showing that these graphs are either 3-connected or 4-connected.

Keywords: Doubly semi-equivelar maps, Hamiltonian cycles, connectivity.

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1 Introduction

Let $G$ denote a simple, connected graph. A cycle in $G$ is called a Hamiltonian cycle if it covers all the vertices of the graph. The existence of a Hamiltonian cycle is one of the important properties of a graph and the problem to find a Hamiltonian cycle is NP-complete.

Let $F$ denote a closed surface. An embedding of $G$ into $F$ is called a map $M$ on $F$ if the closure of each component of $F \setminus G$ is a $p$-gonal 2-disk $D_p, (p \geq 3)$, also called face, and the non-empty intersection of any such two disks is either a vertex or an edge. A map on $F$ is called a triangulation if each disk is triangular, i.e., $p = 3$. A cyclic sequence of faces around a vertex $v$ is called the face-cycle at $v$. In a map, if the face-cycle at each vertex is same, the map is called semi-equivelar.

A graph $G$ is $k$-connected if the removal of $k - 1$ vertices leaves a non-trivial connected graph. The connectivity $\kappa(G)$ is the maximum non-negative integer $k$ for which $G$ is $k$-connected. A map is called $k$-connected (resp. Hamiltonian) if the underlying graph is $k$-connected (resp. Hamiltonian).

The connectivity of a graph plays an important role to discuss Hamiltonicity. In the case of maps, significant progress has been made using the connectivity of the underlying graphs. It goes back to 1931 when Whitney [21] initiated the theory of Hamiltonian cycle in maps and he used the connectivity of the underlying graphs. He proved that every triangulation of the sphere is Hamiltonian if it is 4-connected. Tutte [20] generalized this result for an arbitrary 4-connected map on the sphere. Grunbaum [8] conjectured that every 4-connected map on the Projective plane is Hamiltonian, which was proved by Thomas and Yu [16]. Brunet et al. [7] showed that every 5-connected triangulation on the Klein bottle is Hamiltonian. For the torus, Grunbaum [8] and Nash-Williams [14] posed the following conjecture.

Conjecture 1.1 Every 4-connected map on the torus is Hamiltonian.
The above conjecture is yet to prove, however, several interesting related results have been established so far. Altshuler [1] established the conjecture for every 6-connected map on the torus. Brunet and Richter [2] established the conjecture for every 5-connected triangulation on the torus, which was further improved by Thomas and Yu [23] for any 5-connected map. Kawarabayashi and Ozeki [10] proved the conjecture for any 4-connected triangulation on the torus. The current progress on connectivity and Hamiltonicity of the underlying on the torus can be found in [24], [25], [22], [23].

Since the plane is the universal cover of the torus, it is natural to explore the maps on the torus that are associated with the k-uniform tilings. In this regard, the eleven 1-uniform tilings, also known as Archimedean tilings, provide semi-equivelar maps on the torus, see [19], [5], [12]. Altshuler [1] showed that the semi-equivelar maps on the torus of types [4^4] and [3^6] are Hamiltonian. Bouwer and Chernoff [4] proved that every semi-equivelar map on the torus of type [6^3] is Hamiltonian. Recently, Maity and Upadhyay [13] showed that every semi-equivelar map on the torus of the remaining eight types [3.12^2], [4.8^2], [4.6.12], [3^46], [3^3.4^2], [3^2.4.3.4], [3.6.3.6] and [3^46] are Hamiltonian. Thus, we have the following proposition.

**Proposition 1.1** All the semi-equivelar maps (corresponding to the 1-uniform tilings) on the torus are Hamiltonian.

A map with exactly two distinct face-cycles under certain conditions is called a doubly semi-equivelar map or DSEM (defined precisely in Section 2). There are exactly twenty distinct 2-uniform tilings of the plane [11] and every 2-uniform tiling gives doubly semi-equivelar maps on the torus. In fact, one can construct infinitely many doubly semi-equivelar maps on the torus by taking the quotient of 2-uniform tilings. In this article, we show that every doubly semi-equivelar map on the torus corresponding to a 2-uniform tiling is Hamiltonian. Here, we state our main results.

**Theorem 1.1** All the doubly semi-equivelar maps (corresponding to the 2-uniform tilings) on the torus are Hamiltonian.

**Theorem 1.2** Let M be a doubly semi-equivelar map corresponding to a 2-uniform tiling T.

1. If T is one of the following: [3^3.4^2 : 4^4], [3^3.4^2 : 4^4], [3^6 : 3^3.4^2], [3^6 : 3^3.4^2], [3^3.4^2 : 3^2.4.3.4], [3^3.4^2 : 3^2.4.3.4], [3^6 : 3^4.3.4], [3^4.2^6 : 3.6.3.6], [3^4.2^6 : 3.6.3.6], [3^6 : 3^4.6], [3^6 : 3^4.6], [3^6 : 3^4.6], [3^6 : 3^4.6], [3^6 : 3^4.6], [3^6 : 3^4.6], then M is 4-connected.

2. If T is [3^4.3.12 : 3.12^2] or [3.4.6.4 : 4.6.12], then M is 3-connected.

From Theorem 1.1 and Theorem 1.2, we establish Conjecture [1,4] for the doubly semi-equivelar maps. In other words:

**Corollary 1.1** All 4-connected doubly semi-equivelar maps (corresponding to 2-uniform tilings) on the torus are Hamiltonian.

This article is organized as follows. In Section 2 we give some basic definitions and notations. In Section 3 we discuss the connectivity of DSEMs (corresponding to the 2-uniform tilings) on the torus. In Section 4 we describe Hamiltonicity of such DSEMs. To explore a Hamiltonian cycle in a DSEM M, we construct a planar representation, called an M(i,j,k) representation, which is obtained by cutting M along two non-homologous cycles at a vertex. Clearly, finding a Hamiltonian cycle in M is equivalent to find a Hamiltonian cycle in its planar representation. We give a detailed description for the DSEMs corresponding to [3^6 : 3^3.4^2] and [3^6 : 3^3.4^2] and avoid a similar discussion for the remaining types. In support of our argument, we provide a diagram to describe a Hamiltonian cycle (indicated by thick black cycle) in a given DSEM (wherever it is required).
2 Definitions and notations

Let $G$ be the underlying graph of a map $M$ with the vertex set $V(G)$ and the edge set $E(G)$. Let $u$-$v$ denote the edge joining $u, v \in V(G)$. The notation $P = P(u_1, \ldots, u_n)$ denotes a path $u_1u_2\cdots u_n$. A path with a single vertex $u$ is referred to as a path of length zero, and is denoted as $P = P(u)$. In $P$, the vertices $u_1$ and $u_2$ are called the boundary vertices, and $u_i$, for $2 \leq i \leq n - 1$, are called the inner vertices. A path $P_1$ is an extension of another path $P_2$ if $P_2$ is a proper subgraph of $P_1$. A path $P = P(u_1, \ldots, u_n)$ is called a cycle, if it is closed, that is, $u_1 = u_n$. We denote a cycle by $C = C_n(u_1, \ldots, u_n)$. A cycle $C$ is contractible if it bounds a 2-disk $D_p$, otherwise called non-contractible.

The notations $v$ denote a cycle by $G$ exist distinct vertices $u, v \in V(G)$. A semi-equivelar of type $G$ there exist distinct vertices $z \in V(G)$. The geometric career $G$ of any two graphs $G_1$ and $G_2$ denote the usual union and intersection of two graphs.

Proof: Then removal of any two distinct vertices leaves the map connected. To establish the conjecture in our context, here, we determine the connectivity of the graphs associated with certain DSEMs.

Let $M$ be a map with precisely two distinct face-sequences $f_1$ and $f_2$. Then $M$ is called a doubly semi-equivelar map (or DSEM) if every vertex with the face sequence $(f_1, f_2)$ and every vertex with face sequence $f_2$ has the face-sequence of its link as $(f_1, f_2)$. Observe that $f_{ij}$ is either $f_1$ or $f_2$ for each $i, j$. We say $M$ is of type $(f_1, f_2)$. Clearly, if $f_1 = f_2$, then $M$ turns into a semi-equivelar map. As discussed earlier, the 2-uniform tilings of the plane induce the same type doubly semi-equivelar maps on the torus. We denote the types of such DSEMs by the same notation as used for the respective tilings.

3 Connectivity of doubly semi-equivelar maps

The Nash-Williams conjecture states about the Hamiltonicity of a map with respect to the $k$-connectivity of the underlying graph. To establish the conjecture in our context, here, we determine the $k$-connectivity of the graphs associated with certain DSEMs.

Lemma 3.0.1 Let $M$ be a DSEM (corresponding to a 2-uniform tiling of the plane) on the torus. Then removal of any two distinct vertices leaves the map connected.

Proof: Let $V(G)$ denote the vertex set of the underlying graph $G$ of $M$. Note that $G$ is connected. Suppose, if possible, there is a vertex $z_1 \in V(G)$ such that $G' = G - z_1$ is disconnected. Then, there exist distinct vertices $u, v \in V(G)$ such that every path between $u$ and $v$ passes through $z_1$. Choose such a path $P = u - a_1 - a_2 \cdots a_{n-1} - z_1 - a_{n+1} \cdots - v$. Let $\text{lk}(z_1) = C_i(u_1, w_2, \ldots, w_i)$. Choose a path $P_2$ between $a_{n-1}$ and $a_{n+1}$ in $\text{lk}(z_1)$. Now, observe that $P_2 = P_1 \cup P_2 \cup P_3$ is a connected sub graph of $G'$ such that $u, v \in V(P)$ and $z_1 \notin V(P)$. Therefore $G'$ is connected.

Suppose, if possible, there is a vertex $z_2 \in V(G)$ such that $G'' = G - z_2$ is disconnected. Then, there exist distinct vertices $u', v' \in G''$ such that $z_2$ is a vertex of any path connecting $u'$ and $v'$, say $P' = u' - b_1 - b_2 \cdots b_{n-1} - b_n - b_{n+1} \cdots - v'$. Then, either $z_2 \in \text{lk}(z_1)$ or $z_2 \notin \text{lk}(z_1)$.  

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First, suppose $z_2 \notin \text{lk}(z_1)$, then $G''$ is connected by a similar argument used for $G'$. Now suppose, $z_2 \in \text{lk}(z_1)$. Then $z_2 = w_k$ and $z_2-w_{k-1}$, $z_2-w_{k+1}$ are edges for some $1 \leq k \leq i$. Since $G'$ is connected, we get paths $P_1'$ from $u'$ to $w_{k+1}$ and $P_2'$ from $w_{k-1}$ to $u'$ which are not passing through $z_2$. Also, we get a path $P_3'$ from $w_{k-1}$ to $w_{k+1}$ in $\text{lk}(z_2)$. Note that $P_3'$ is a path in $G$ not passing through $z_1$. Thus we get a connected sub graph $P'' = P_1' \cup P_2' \cup P_3'$ such that $u', u' \in V(P'')$ and $z_2 \notin V(P'')$. Hence $G'''$ is connected.

The proof of the next theorem follows from Lemma 3.0.1

**Theorem 3.1** Let $M$ be a doubly semi-equivelar map of type $Y$, where $Y \in \{[[3,4.3.12 : 3.12]], [3,4.6.4 : 4.6.12]\}$. Then $M$ is 3-connected.

**Theorem 3.2** Let $M$ be a doubly semi-equivelar map of type $X$, where $X \in \{[[3^2.4^2 : 4^2]], [3^6 : 3^2.4^2], [3^6 : 3^2.4^{3.4}],[3^6 : 3^2.4^{3.4}],[3^2.6 : 3.6.3.6],[3^2.6 : 3^2.6^2],[3^2.4.3.4 : 3.4.6.4],[3^2.6 : 3^4.6],[3^6 : 3^4.6],[3^6 : 3^2.4.6],[3^2.4.6], [3^2.6 : 3^4.6] \}$. Then $M$ is 4-connected.

**Proof:** Let $V(G)$ denote the vertex set of the underlying graph $G$ of $M$, which is connected. Note that the degree of every vertex of $M$ is grater or equal to 4. Let $z_1, z_2 \in V(G)$. Then by Lemma 3.0.1 $G' = G - \{z_1\}$ and $G'' = G - \{z_1, z_2\}$ are connected. Suppose, if possible, there is a vertex $z_3 \in V(G)$ such that $G''' = G - \{z_1, z_2, z_3\}$ is disconnected. Then we get $u, v \in V(G)$ such that there is no path between $u$ and $v$ in $G'''$. Let $\text{lk}(z_1) = C_1(w_1, w_2, \ldots, w_l)$, $\text{lk}(z_2) = C_j(x_1, x_2, \ldots, x_j)$ and $\text{lk}(z_3) = C_k(y_1, y_2, \ldots, y_k)$. Now, as per the positions of $z_1, z_2, z_3$, we have the following cases:

1. $z_a \notin \text{lk}(z_b)$, where $a \neq b$ and $a, b \in \{1, 2, 3\}$.
2. $z_a \in \text{lk}(z_b)$ such that $z_c \notin \text{lk}(z_a)$ and $z_c \notin \text{lk}(z_b)$, where $a \neq b \neq c \neq a$ and $a, b, c \in \{1, 2, 3\}$.
3. $z_2, z_3 \in \text{lk}(z_1)$ and $z_2 \in \text{lk}(z_3)$.
4. $z_3, z_4 \in \text{lk}(z_2)$ such that $z_3 \notin \text{lk}(z_4)$, where $a \neq b \neq c \neq a$ and $a, b, c \in \{1, 2, 3\}$.

In Case 1 and 2, $G'''$ remains connected by Lemma 3.0.1 So we discuss the remaining cases.

**Case 3:** Since $z_2, z_3 \in \text{lk}(z_1)$, $z_2 = w_h$, $z_3 = w_l$ for some $1 \leq h, l \leq i$. Note that, $w_{h-1}, w_{h+1}$ are adjacent to $z_2$ and $w_{l-1}, w_{l+1}$ are adjacent to $z_3$. Hence, $w_{h-1}, w_{h+1} \in \text{lk}(z_2)$, $w_{l-1}, w_{l+1} \in \text{lk}(z_3)$ and $w_{h-1} = x_m, w_{h+1} = x_n, w_{l-1} = y_r, w_{l+1} = y_s$ for some $1 \leq m, n \leq j, 1 \leq r, s \leq k$. Let $R = x_{m-1} - x_{m-1}x_n - x_{n-1}x_n$ be the path in $\text{lk}(z_2)$ from $x_m$ to $x_n$ such that $z_1 \notin V(R)$ and $S = y_r - y_{r+1} - \cdots - y_{s-1} - y_s$ be the path in $\text{lk}(z_3)$ from $y_r$ to $y_s$ such that $z_1 \notin V(S)$.

**Subcase 3.1:** If $z_2, z_3$ are not adjacent such that $w_{h-1}, w_{l+1} \notin \{w_{h-1}, w_{l+1}\}$. Then $A_1 = w_{h-1}w_{h+1}x_{h+2} - \cdots - w_{h-2}w_{h-1}$ and $A_2 = w_{l-1}w_{l+1}x_{l-2} - \cdots - w_{l-2}w_{l-1}$ are paths in $\text{lk}(z_1)$ such that $z_2, z_3 \notin V(A_1) \cup V(A_2)$. Since $z_2 \in \text{lk}(z_3)$ and $z_3 \in \text{lk}(z_2)$, we have $z_2 = y_p, z_3 = x_q$ for some $1 \leq p \leq k, 1 \leq q \leq j$ and $z_2y_p+1, z_2y_p-1, z_3x_q-1, z_3x_q+1$ are edges. Consider the path $R = x_{m-1} - x_{m-1}x_{q} - x_{q-1}x_{q+1} - \cdots - x_{n-1}x_n$ in $\text{lk}(z_2)$, then we get sub paths $A_3 = x_{q+1} - x_{q+2} - \cdots - x_{n} (= w_{h-1})$ and $A_4 = x_{q-1} - x_{q-2} - \cdots - x_{m} (= w_{l-1})$. Similarly, the path $S = y_r - y_{r+1} - \cdots - y_{s-1} - y_s$ in $\text{lk}(z_3)$ gives sub paths $A_5 = y_{s+1} - y_{s+2} - \cdots - y_{l} (= w_{l+1})$ and $A_6 = y_{s-1} - y_{s-2} - \cdots - y_{r} (= w_{h-1})$. In this manner, we get connected sub graphs $W_1 = A_3 \cup A_2 \cup A_6$ and $W_2 = A_5 \cup A_1 \cup A_4$ of $G'''$. If $V(W_1) \cap V(W_2) \neq \phi$, then $W_1 \cup W_2$ is connected, so $G'''$ is connected. If $V(W_1) \cap V(W_2) = \phi$, then the links of all the vertces of $G'''$ remain complete except for the vertices of $W_1$ and $W_2$ as they belong to the links of $z_1, z_2$ or $z_3$. Note that there exists either a vertex or an edge in $W_1$ contained in a face $F_1$, where $F_1$ is a face in $M$ that does not contain the vertices $z_1, z_2$ or $z_3$. Similarly, for a vertex or an edge in $W_2$, we get a face $F_1$. Corresponding to the faces $F_1, F_2$, we get faces $F_2, H_2$ such that $F_1 \cap F_2 \neq \phi, H_1 \cap H_2 \neq \phi$, otherwise links of some vertices of these faces are not complete. In this manner, we get finite sequences of faces $F_1, F_2, \cdots, F_e$ and $H_1, H_2, \cdots, H_f$ corresponding to $W_1$ and $W_2$, respectively. Since $M$ is a map on the finite number of vertices, after some stage, we get $F_u \cap H_v \neq \phi$, for some $1 \leq u \leq e, 1 \leq v \leq f$. Therefore, every vertex of $W_1$ is connected to every vertex of $W_2$. Hence, $G'''$ is connected.
If $z_2, z_3$ are not adjacent such that either $w_{l-1} = w_{h+1}$ or $w_{l+1} = w_{h-1}$, then proceeding similarly, we get connected sub graphs $W_1'$ and $W_2'$ of $G''$. By using the similar argument as above, $G''$ is connected.

**Subcase 3.2:** Let $z_2, z_3$ are adjacent. As $z_2, z_3 \in \mathrm{lk}(z_1), z_2 = w_h, z_3 = w_{h+1}$ for some $1 \leq h \leq i$ and $z_2, w_{h-1}, z_2, z_3 = w_{h+1}$ are edges. Clearly, there is a path $B_1 = w_{h+2}-w_{h+3} \cdots -w_h-2-w_h-1$ between $w_{h+2}$ and $w_{h-1}$ such that $z_1, z_2, z_3 \notin V(B_1)$. Since $z_3 \in \mathrm{lk}(z_1)$ and $z_3-w_h+2$ is an edge, so $w_{h+2} \in \mathrm{lk}(z_3)$. Now $z_2(= w_h)-z_3(= w_{h+1})$ is an edge, so $z_2 \in \mathrm{lk}(z_3)$. If $z_2 = y_p$ for some $1 \leq p \leq k$ in $\mathrm{lk}(z_3)$, then $B_2 = y_{p+1}-y_{p+2} \cdots -w_h+2$ is a path between $y_{p+1}-w_h+2$ such that $z_1, z_2, z_3 \notin V(B_2)$. As $w_{h-1}-y_2$ and $y_{p+1}-z_2$ are edges, this gives $w_{h-1}, y_{p+1} \in \mathrm{lk}(z_2)$. Hence, we have a path $B_3$ between $w_{h-1}$ and $y_{p+1}$. Thus, from above, we get a connected sub graph $W = B_1 \cup B_3 \cup B_2$ of $G''$. Since $G''$ is connected, every vertices of $G''$ is connected with some vertices of $W$, therefore $G''$ is connected.

**Case 4:** Without loss of generality, suppose that $z_2, z_3 \in \mathrm{lk}(z_1)$ and $z_2 \notin \mathrm{lk}(z_3)$. As discussed in Case 3, $z_2, z_3 \in \mathrm{lk}(z_1)$ gives $z_2 = w_h, z_3 = w_l$ for some $1 \leq h, l \leq i$, $w_{h-1}, w_{h+1} \in \mathrm{lk}(z_2), w_{l-1}, w_{l+1} \in \mathrm{lk}(z_3)$ and $w_{h-1} = x_m, w_{h+1} = x_n, w_{l-1} = y_r, w_{l+1} = y_s$ for some $1 \leq m, n, l, j \leq h, s, k$. Also, we get a path $B = x_m-x_{m+1} \cdots -x_{n-1}-x_n$ in $\mathrm{lk}(z_2)$ from $x_m$ to $x_n$ such that $z_1 \notin V(B)$ and a path $S = y_r-y_{r+1} \cdots -y_{s-1}-y_s$ in $\mathrm{lk}(z_3)$ from $y_r$ to $y_s$ such that $z_1 \notin V(S)$.

If $z_2, z_3$ are such that $w_{l-1}, w_{l+1} \notin \{w_{h-1}, w_{h+1}\}$. Then $A_1 = w_{l+1}-w_{l+2} \cdots -w_{h-2}-w_{h-1}$ and $A_2 = w_{h+1}-w_{h+2} \cdots -w_{l-2}-w_{l-1}$ are paths in $\mathrm{lk}(z_1)$ such that $z_2, z_3 \notin V(A_1) \cup V(A_2)$. Clearly, $Z = R \cup A_2 \cup S \cup A_3$ is a connected sub graph of $G''$. Therefore $G''$ is connected.

If $z_2, z_3$ are such that either $w_{l-1} = w_{h+1}$ or $w_{l+1} = w_{h-1}$, then proceeding similarly, we get a connected sub graph $Z'$ of $G''$. By using the similar argument as above, $G''$ is connected.

Hence, by the Cases 1-4, $M$ is 4-connected.

**Proof of Theorem 1.2** The proof follows from Theorem 3.1 and Theorem 3.2.

### 4 Hamiltonicity of doubly semi-equivelar maps on the torus

Throughout this section by a DSEM, we mean a doubly semi-equivelar map on the torus. Let $M$ be a DSEM of type $[f_1 : f_2]$ with vertex set $V(M)$. For $j = 1, 2$, let $V_j$ denote the set of vertices with face-sequence $f_j$. The notations $|V(M)|$, $|V_j|$ denote the cardinality of the vertex set $V(M)$, $V_j$ respectively.

#### 4.1 DSEMs of type $[3^6 : 3^2 4^2]_r$ and $[3^6 : 3^2 4^2]_2$

Let $M'$ be a DSEM of type $[3^6 : 3^2 4^2]_r$ with the vertex set $V(M')$, where $r \in \{1, 2\}$. For $M'$, we see that the number of triangular faces is $4|V(3^6)|$ or $2|V(3^3 4^2)|$. Thus, if $M'$ exists, then $2|V(3^6)| = |V(3^3 4^2)|$. Similarly, for the existence of $M^2$, we get $|V(3^6)| = |V(3^3 4^2)|$. Let $u$ be a vertex with face-sequence $(3^6)$ or $(3^3 4^2)$. We denote their respective links by $\mathrm{lk}(u) = C_6(u_1, u_2, u_3, u_4, u_5, u_6)$ or $\mathrm{lk}(u) = C_7(u_1, u_2, u_3, u_4, u_5, u_6, u_7)$ (labeling of vertices in the links is considered anti-clockwise).

The bold appearance of some $u_t$ means $u$ is not adjacent with $u_t$ by an edge.

Considering Figures 4.1.1 and 4.1.2 as the planar drawings of the maps $M^1$ and $M^2$ respectively, we define certain types of paths as follows.

**Definition 4.1.1** A path $P_1 = P(\ldots, y_{i-1}, y_i, y_{i+1}, \ldots)$ in $M'$ is of type $A_1$ if, either (i) every vertex of $P_1$ has the face-sequence $(3^6)$ or (ii) each vertex of $P_1$ has the face-sequence $(3^3 4^2)$ such that all the triangles (square) which incident on its inner vertices lie on the one side of the path $P_1$.

See thick black paths in Figure 4.1.1 and Figure 4.1.2.
Definition 4.1.2 A path $P_2 = P(\ldots, z_{i-1}, z_i, z_{i+1}, \ldots)$ in $M^1$, such that $z_{i-1}, z_i, z_{i+1}$ are inner vertices of $P_2$ or an extended path of $P_2$, is of type $A_2$ (shown by green colored paths in Figure 4.1.1), if either of the following three conditions follows for each vertex of the path.

1. If $\text{lk}(z_i) = C_7(m, z_{i-1}, n, o, p, z_{i+1}, q)$ and $\text{lk}(z_{i-1}) = C_7(q, m, r, z_{i-2}, n, o, z_i)$, then $\text{lk}(z_{i+1}) = C_6(s, q, z_i, p, t, z_{i+2})$.
2. If $\text{lk}(z_i) = C_7(m, z_{i+1}, n, o, p, z_{i-1}, q)$ and $\text{lk}(z_{i-1}) = C_6(r, z_{i-2}, s, q, z_i, p)$, then $\text{lk}(z_{i+1}) = C_7(o, z_i, q, m, t, z_{i+2}, n)$.
3. If $\text{lk}(z_i) = C_6(z_{i-1}, m, n, z_{i+1}, o, p)$ and $\text{lk}(z_{i-1}) = C_7(r, z_{i-2}, s, t, m, z_i, p)$, then $\text{lk}(z_{i+1}) = C_7(u, z_{i+2}, v, w, o, z_i, n)$.

Definition 4.1.3 A path $P_3 = P(\ldots, z_{i-1}, z_i, z_{i+1}, \ldots)$ in $M^2$, such that $z_{i-1}, z_i, z_{i+1}$ are inner vertices of $P_3$ or an extended path of $P_3$, is of type $A_2$ (shown by green colored paths in Figure 4.1.2), if either of the following four conditions follows for each vertex of the path.

1. If $\text{lk}(z_i) = C_7(m, z_{i-1}, n, o, p, z_{i+1}, q)$ and $\text{lk}(z_{i-1}) = C_7(r, z_{i-2}, n, o, z_i, q, m)$, then $\text{lk}(z_{i+1}) = C_6(z_{i+1}, s, u, z_{i+3}, v, t)$.
2. If $\text{lk}(z_i) = C_7(m, z_{i+1}, n, o, p, z_{i-1}, q)$ and $\text{lk}(z_{i-1}) = C_6(z_{i-2}, s, q, z_i, p, r)$, then $\text{lk}(z_{i+1}) = C_7(o, z_i, q, m, t, z_{i+2}, n)$, $\text{lk}(z_{i+2}) = C_6(z_{i+1}, t, u, z_{i+3}, v, n)$.
3. If $\text{lk}(z_i) = C_6(z_{i-1}, m, n, z_{i+1}, o, p)$ and $\text{lk}(z_{i-1}) = C_7(q, z_{i-2}, r, s, m, z_i, p)$, then $\text{lk}(z_{i+1}) = C_6(z_{i+1}, t, u, o, z_{i+3}, w, x, u, z_{i+1}, t)$.
4. If $\text{lk}(z_i) = C_6(z_{i-1}, m, n, z_{i+1}, o, p)$ and $\text{lk}(z_{i-1}) = C_6(z_{i-2}, r, m, z_i, p, q)$, then $\text{lk}(z_{i+1}) = C_7(s, z_{i+2}, t, u, o, z_i, n)$, $\text{lk}(z_{i+2}) = C_7(u, z_{i+1}, n, s, v, z_{i+3}, t)$.

Remark 1 Recall that $M^r$ is a map with a finite vertex set. Hence, for a given maximal path $P = (v_1 \cdot v_2 \cdot \cdots \cdot v_r \cdot v_r)$ of type $A_\alpha$, $\alpha \in \{1, 2\}$, there is an edge $e = v_1 \cdot v_r$ in $M^r$ such that $P \cup e$ is a cycle $Q$. Depending on $A_\alpha$, the corresponding cycle is called a cycle of type $A_\alpha$. Then we see that such cycles are non-contractible.

Lemma 4.1.1 The cycle $Q$ of type $A_\alpha$, for $\alpha \in \{1, 2\}$, is non-contractible.

Proof. On the contrary, suppose $Q$ is a contractible cycle of type $A_1$. Then $Q$ is the boundary of a 2-disk, say $D$. Let $v$, $e$, and $f$ denote the number of vertices, edges, and faces of $D$ respectively. Let $m$ and $n$ represent the inner and boundary vertices of $D$, respectively. Here, $m = m_1 + m_2$, in which $m_1$ and $m_2$ represent the number of vertices with face-sequences of $(3^6)$ and $(3^3, 4^2)$, respectively. Then we have two cases depending on $Q$. In the first case, if the incident faces on $Q$ are quadrangles, then $v = n + m_1 + m_2$, $e = 3n/2 + 6m_1/2 + 3m_2/2$, and $f = 2n/4 + 6m_1/3 + 3m_2/2 + 3m_2/2$. In the second case, if the incident faces on $Q$ are triangles, then $v = n + m_1 + m_2$, $e = 4n/2 + 6m_1/2 + 5m_2/2$, and $f = 3n/3 + 6m_1/3 + 2m_2/4 + 3m_2/2$. For both cases, we obtain the Euler characteristic $\chi(D) = v - e + f = 0$. We get a contradiction, as the Euler characteristic of a 2-disk is 1. Hence, $Q$ is not contractible. Similarly, if $Q$ is of type $A_2$, then $Q$ is non-contractible.

Let $Q$ be a cycle of type $A_1$ with the vertex set $V(Q)$. Let $S_Q$ denote the set of all the faces incident at $v$ for all $v \in V(Q)$. Then the geometric carrier $|S_Q|$ is a cylinder, as $Q$ is non-contractible. Let $\partial|S_Q|$ denote the boundary of $|S_Q|$.

Let $Q_1$ and $Q_2$ be cycles of the same type in $M^r$. We say that cycles $Q_1$ and $Q_2$ are homologous if there is a cylinder whose boundary is $\{Q_1, Q_2\}$.
Lemma 4.1.2 Let $Q$ be a cycle of type $A_1$ such that $\partial|S_Q| = \{Q_1, Q_2\}$. Then $Q$, $Q_1$, and $Q_2$ are of the same type with equal length.

Proof. Let $Q_\alpha \in \{Q_1, Q_2\}$. Without loss of generality, let $Q_\alpha = Q_1$, consider the faces that are incident with both $Q$ and $Q_1$. Now, depending on $Q$, we see that if the faces incident with both $Q$ and $Q_1$ are quadrangles (respectively triangles), then the faces on the other side of $Q_1$ must be triangles (respectively triangles or quadrangles). This shows that $Q_1$ is of type $A_1$. Similarly, $Q_2$ is also of type $A_1$.

Let $Q = C_l(v_1, \ldots, v_l)$, $Q_1 = C_m(u_1, \ldots, u_m)$, and $Q_2 = C_n(w_1, \ldots, w_n)$. Now we show $l = m = n$ by contradiction. For this, it is enough to deal with the case, when $l < m < n$. Following the definition of type $A_1$, the face-sequences of $v_1, v_2, \ldots, v_{l-1}, v_l$ are same throughout the cycle $Q$. Since $l < m$, $\text{lk}(v_l)$ contains the vertices $u_1, u_{l+i}$, and $w_i$ for some $i > 0$. Then, the face-sequences of $v_l$ and $v_{l-1}$ are not same. This is not possible. Therefore, $l = m = n$. \qed

$M(i,j,k)$-representation of $M^r$: Let $u \in V(M^r)$ and $Q_\alpha$ be cycles of type $A_\alpha$ through $u$, where $\alpha \in \{1, 2\}$. Let $Q_1 = C_i(u_1, u_2, \ldots, u_i)$. We cut $M^r$ first along the cycle $Q_1$. This gives a cylinder, say $R_1$, bounded by identical cycle $Q_1$. We say that a cycle is horizontal (resp. vertical) if it is $Q_1$ or homologous to $Q_1$ (resp. $Q_2$ or homologous to $Q_2$). In $R_1$, starting from the vertex $u$, make another cut along the path $P \subset Q_2$, until it reaches $Q_1$ again for the first time. As a result, we get the unfold torus (a planer representation), say $R_2$. The idea of $M(i,j,k)$-representation is taken from [12] and a detailed description of such representation for the given type DSEM is available in [15].

Without loss of generality, suppose that the quadrangular faces are incident on $Q_1$. In $R_2$, let there are $j$ cycles, as shown in Fig. 4.1.3 and Fig. 4.1.4, which are homologous to $Q_1$ along $P$. Since $\text{length}(Q_1) = i$ and the number of horizontal cycles along $P$ is $j$. So, we denote $R_2$ by $(i, j)$ representation. To reconstruct the map $M^r$ from its $(i, j)$ representation, one gets a natural way of identification of the vertical cycle of $R_2$, but identification of the horizontal cycle generally needs some shifting so that the lower horizontal cycle is identified with the upper horizontal cycle. Suppose $k$ is such shifting, i.e., $k + 1$ is the starting vertex of the upper horizontal cycle. This gives another representation say $M^r(i,j,k)$ representation of the $(i,j)$ representation of $M^r$. The admissible relations among $i, j, k$ of the $M^r(i,j,k)$ is given in Lemma 4.1.3

![Figure 4.1.3: M(i,j,k) of DSEM of type [3^6: 3^3.4^2]_1](image1)

![Figure 4.1.4: M(i,j,k) of DSEM of type [3^6: 3^3.4^2]_2](image2)

Lemma 4.1.3 Let $r \in \{1, 2\}$. Let $M^r$ be a DSEM of type $[3^6 : 3^3.4^2]_r$. Then $M^r$ admits an $M^r(i,j,k)$ representation, if the following holds: (i) $i \geq 3$ and $j = (r + 2)m$, where $m \in \mathbb{N}$, (ii) $ij \geq 3(r + 2)$, (iii) $0 \leq k \leq i - 1$.

Proof. Note that an $M^r(i,j,k)$ of $M^r$ contains $j$ number of $A_1$ type disjoint horizontal cycles of length $i$. Since these cycles cover all the vertices of $M^r$, the number of vertices in $M^r$ is $n = ij$. Clearly if $i \leq 2$, $M^r$ is not a map. So $i \geq 3$. If $j = 1$ then $M^r$ is not a map and if $j = 2$ then $M^r$ has
no vertices of face-sequence \((3^i)\). If \(j = (r + 2)m + 1\) or \((r + 2)m + 2\), then \(2|V(3^i)| \neq |V(3^i, 4^2)|\) for \(M^1\) and \(|V(3^i)| \neq |V(3^i, 4^2)|\) for \(M^2\). So \(j = (r + 2)m\). Thus \(n = ij \geq 3(r + 2)\). Since the length of the horizontal cycle is \(i\), we get \(k \in \{0, 1, \ldots, i - 1\}\). The converse part follows directly by constructing \(M^r(i, j, k)\) representation for the given values of \(i, j,\) and \(k\).

In the upcoming subsections, we proceed in a similar way for the remaining types DSEMs. We determine the conditions among \(i, j, k\) to construct its \(M(i, j, k)\) representation. For this, we consider fixed types of paths on maps, such paths can be defined using the definition of link (as in Subsection 4.1). Since these DSEMs are on the finite vertex set, given every maximal path \(P\) of such types there is an edge \(e\) such that \(P \cup \{e\}\) is a non-contractible, as in Lemma 4.1.3

### 4.2 DSEMs of types \([3^3, 4^2 : 4^1]\) and \([3^3, 4^2 : 4^1]\)

Let \(M^r\) be a DSEM of type \([3^3, 4^2 : 4^1]\), where \(r \in \{1, 2\}\). Then for the existence of \(M^1\) and \(M^2\) we have \(|V(3^3, 4^2)| = 2|V(4^1)|\) and \(|V(3^3, 4^2)| = |V(4^1)|\) respectively. Now, consider the following paths in \(M^r\) as follows.

A path \(P_1 = P(\ldots, y_{i-1}, y_i, y_{i+1}, \ldots)\) in \(M^r\), say of type \(B_1\), indicated by thick black paths. The vertices \(y_i\)'s have the face-sequence either \(4^1\) or \((3^3, 4^2)\).

A path \(P_2 = P(\ldots, y_{i-2}, z_i-2, z_i-1, y_i, z_i, z_{i+1}, y_{i+1}, \ldots)\) in \(M^1\), say of type \(B_2\), indicated by green paths. The vertices \(y_i\)'s and \(z_i\)'s have the face-sequences \(4^1\) and \((3^3, 4^2)\) respectively.

A path \(P_3 = P(\ldots, y_{i-1}, z_i-2, z_i-1, y_i, y_{i+1}, z_i, z_{i+1}, y_{i+2}, \ldots)\) in \(M^2\), say of type \(B_2\), indicated by green paths. The vertices \(y_i\)'s and \(z_i\)'s have the face-sequences \((4^1)\) and \((3^3, 4^2)\) respectively.

![Figure 4.2.1: Paths of type B1, B2 in DSEM of types [3^3, 4^2 : 4^1] and [3^3, 4^2 : 4^1]](image)

![Figure 4.2.2: M(i, j, k) of DSEM of type [3^3, 4^2 : 4^1]](image)

![Figure 4.2.3: M(i, j, k) of DSEM of type [3^3, 4^2 : 4^1]](image)

Now, first cutting \(M^r\) along a cycle of type \(B_1\) and then along a cycle of type \(B_2\), we get \(M^r(i, j, k)\)-representation of \(M^r\), shown in Figures 4.2.2 and 4.2.3. To get the relation among \(i, j, k\), we proceed similarly as in Lemma 4.1.3 This gives the following result.

**Lemma 4.2.1** Let \(r \in \{1, 2\}\). The DSEM \(M^r\) of type \([3^3, 4^2 : 4^1]\) admits an \(M^r(i, j, k)\)-representation iff the following holds: (i) \(i \geq 3\) and \(j = (r + 2)m\), where \(m \in \mathbb{N}\), (ii) \(ij \geq 3(r + 2)\), (iii) \(0 \leq k \leq i - 1\).
**4.3 DSEMs of types \([3^3.4^2 : 3^2.4.3.4]\)_1**

Let \(M\) be a DSEM of type \([3^3.4^2 : 3^2.4.3.4]\)_1. Then, for the existence of \(M\), we have \(2|V_{(3^3.4^2)}| = |V_{(3^2.4.3.4)}|\). Now consider the following path through a vertex in Figure 4.3.1. One can define such a path using the definition of link as in Subsection 4.1.

A path \(P_1 = P(\ldots, w_i, z_j, z_{j+1}, z_{j+2}, z_{j+3}, w_{i+1}, \ldots)\) in \(M\), say of type \(C_1\), indicated by thick black paths in Figure 4.3.1. The vertices \(w_i\)'s and \(z_j\)'s have face-sequences \((3^3, 4^2)\) and \((3^2, 4, 3, 4)\) respectively.

Note that, through a vertex \(v\) with the face-sequence \((3^2, 4, 3, 4)\), we see two paths of type \(C_1\). Now as in the previous section, cutting \(M\) along these paths one by one, we get its \(M(i, j, k)\) representation. The relations among \(i, j, k\) are given by Lemma 4.3.1.

![Figure 4.3.1: Paths of type \(C_1\)](image)

![Figure 4.3.2: \(M(i, j = 4m + 4, k)\) of DSEM of type \([3^3.4^2 : 3^2.4.3.4]\)_1](image)

![Figure 4.3.3: \(M(i, j = 4m + 2, k)\) of DSEM of type \([3^3.4^2 : 3^2.4.3.4]\)_1](image)

**Lemma 4.3.1** A DSEM \(M\) of type \([3^3.4^2 : 3^2.4.3.4]\)_1 admits an \(M(i, j, k)\) representation iff the following holds: (i) \(i = 5m, m \in \mathbb{N}\) and \(j\) is even, (ii) number of vertices of \(M(i, j, k)\) is \(6ij/5 \geq 12\), (iii) if \(j = 4m + 2, m \in \mathbb{N} \cup \{0\}\), then \(k \in \{5r + 3 : 0 \leq r < i/5\}\), and if \(j = 4m, m \in \mathbb{N}\), then \(k \in \{5r : 0 \leq r < i/5\}\).

**Proof.** A representation \(M(i, j, k)\) of a DSEM \(M\) of type \([3^3.4^2 : 3^2.4.3.4]\)_1 has \(j\) disjoint horizontal cycles of \(C_1\) type having length \(i\). Let \(Q_0, Q_1, \ldots, Q_{j-1}\) be the horizontal cycles of type \(C_1\). The number of vertices with face-sequence \((3^3, 4^2)\) lying between horizontal cycles \(Q_{(2s+1) (mod j)}\) and \(Q_{(2s+2) (mod j)}\), for \(0 \leq s \leq j - 1\), is \(2i/5 \cdot j/2\). Thus, the total number of vertices in \(M\) is \(n = ij + ij/5 = 6ij/5\). If \(j = 1\), then \(M(i, 1, k)\) has no vertex with face-sequence \((3^3, 4^2)\) or \((3^2, 4, 3, 4)\). So \(j \geq 2\). If \(j \geq 2\) and \(j\) is not an even integer then we get some vertices in the base horizontal cycle which does not have the face-sequence \((3^3, 4^2)\) or \((3^2, 4, 3, 4)\). So, \(j\) is even.

If \(j\) is even and \(i < 5\) then again the representation \(M(i, j, k)\) has some vertices which do not have the face-sequence \((3^3, 4^2)\) or \((3^2, 4, 3, 4)\). So, \(i \geq 5\). If \(i \geq 5\) and not a multiple of 5, then
2|V(3^3, 4^2)| \neq |V(3^2, 4, 3, 4)|. This is not possible. So i = 5m, where m ∈ N and n = 6ij/5 ≥ 12.

If j = 4m + 2, m ∈ N ∪ {0} and k ∈ \{r : 0 ≤ r ≤ i − 1\} \{5r + 3 : 0 ≤ r < i/5\} then we get some vertices which do not have the face-sequence (3^3, 4^2) or (3^2, 4, 3, 4). So, k ∈ \{5r + 3 : 0 ≤ r < i/5\} for j = 4m + 2, m ∈ N ∪ {0}. Similarly if j = 4m, m ∈ N, then k ∈ \{5r : 0 ≤ r < i/5\}. This completes the proof. □

4.4 DSEM of types \([3^3, 4^2 : 3^2, 4.3.4]_2\)

Let M be a DSEM of type \([3^3, 4^2 : 3^2, 4.3.4]_2\). It is easy to see that, if M exists then |V(3^3, 4^2)| = |V(3^2, 4, 3, 4)|. We consider following types of paths in M as follows.

A path \(P_1 = P(\ldots, u_i, v_j, u_{i+1}, v_{j+1}, \ldots)\) in M, say of type \(Y_1\), indicated by thick black paths in Figure 4.4.1. Here the vertices \(u_i\)'s and \(v_j\)'s have face-sequences \((3^3, 4^2, 3, 4)\) and \((3^2, 4, 3, 4)\) respectively.

A path \(P_2 = P(\ldots, u_{i-1}, u_i, u_{i+1}, \ldots)\) in M, say of type \(Y_2\), indicated by green paths in Figure 4.4.1. Here the vertices \(u_i\)'s of the path has face-sequence either \((3^3, 4^2)\) or \((3^2, 4, 3, 4)\).

To construct \(M(i, j, k)\), we cut \(M\) along a cycle of type \(Y_1\) and then, take second cut along a cycle of type \(Y_2\), assume without loss of generality \(Y_2\) whose all the vertices have the face-sequence \((3^2, 4, 3, 4)\). Following, a similar argument given in Lemma 4.4.1 we obtain Lemma 4.4.2.

**Lemma 4.4.1** A DSEM M of type \([3^3, 4^2 : 3^2, 4.3.4]_2\) admits an \(M(i, j, k)\) representation iff the following holds: (i) \(j \geq 2\) and \(j\) even, (ii) if \(j = 2\) then \(i \geq 8\), and if \(j \geq 4\) then \(i \geq 4\), also \(i = 4m\), where \(m \in \mathbb{N}\), (iii) \(ij \geq 16\), (iv) if \(j = 2\) then \(k \in \{4r : 0 < r < i/4\}\), and if \(j \geq 4\) then \(k \in \{4r : 0 \leq r < i/4\}\).

4.5 DSEM of type \([3^6 : 3^2, 4.3.4]_2\)

Let M be a DSEM of type \([3^6 : 3^2, 4.3.4]_2\). We consider the following types of paths in M as follows. A path \(P_1 = P(\ldots, u_{i-1}, u_i, u_{i+1}, \ldots)\) in M, say of type \(E_1\), indicated by thick black paths in Figure 4.5.1. The vertices \(u_i\)'s have the face-sequence \((3^2, 4, 3, 4)\).

A path \(P_2 = P(\ldots, u_i, v_j, v_{j+1}, v_{j+2}, v_{j+3}, u_{i+1}, \ldots)\) in M, say of type \(E_2\), indicated by green paths in Figure 4.5.1. The vertices \(u_i\)'s and \(v_j\)'s have the face-sequences \((3^6)\) and \((3^2, 4, 3, 4)\) respectively.

An \(M(i, j, k)\) representation of M follows by we first cutting M along an \(E_1\) type cycle and then along an \(E_2\) type cycle. Now, proceeding similarly, as in Lemma 4.3.1 we get the following lemma.
Lemma 4.5.1 A DSEM $M$ of type $[3^6 : 3^2.4.3.4]$ admits an $M(i, j, k)$ representation iff the following holds: (i) $j \geq 2$ even, (ii) if $j = 2$ then $i \geq 9$ and if $j \geq 4$ then $i \geq 6$, also $i = 3m$, $m \in \mathbb{N}$, (iii) number of vertices of $M(i, j, k) = 7ij/6 \geq 21$, (iv) if $j = 2$ then $k \in \{3r + 2 : 0 < r < (i - 3)/3\}$, if $j = 4m + 2$ then $k \in \{3r + 2 : 0 \leq r < i/3\}$, and if $j = 4m$ then $k \in \{3r : 0 \leq r < i/3\}$, $m \in \mathbb{N}$.

4.6 DSEMs of type $[3.4^2.6 : 3.6.3.6]_1$ and $[3.4^2.6 : 3.6.3.6]_2$

Let $M'$ be a DSEM of type $[3.4^2.6 : 3.6.3.6]_r$, where $r \in \{1, 2\}$. In $M'$, consider following types of paths as follows.

A path $P_1 = P(\ldots, y_{i-1}, y_i, y_{i+1}, \ldots)$ in $M'$, say of type $D_1$, indicated by thick black paths in Figure 4.6.1. The vertices $y_i$’s have the face sequence $(3, 4^2, 6)$.

A path $P_2 = P(\ldots, y_i, y_{i+1}, z_i, y_{i+2}, y_{i+3}, z_{i+1}, \ldots)$ in $M'$, say of type $D_2$, indicated by green paths in Figure 4.6.1. The vertices $y_i$’s and $z_i$’s have the face-sequences $(3, 4^2, 6)$ and $(3, 6, 3, 6)$ respectively.

To construct an $M'(i, j, k)$ representation of $M'$, we cut $M'$ along a cycle of type $D_1$ and then cut along a cycle of type $D_2$. Without loss of generality, let the starting adjacent face to the base horizontal cycle $D_1$ is a 3-gon.
Lemma 4.6.1 A DSEM $M^r$ of type $[3.4^2.6 : 3.6.3.6]^r$, for $r \in \{1, 2\}$, admits an $M^r(i, j, k)$-representation iff the following holds: (i) $i \geq 6$ and $i, j$ even, (ii) number of vertices of $M(i, j, k) = 5ij/4 \geq 15$, (iii) if $j = 2$ then $4 \leq k \leq i - 2$, and if $j \geq 4$ then $0 \leq k \leq i - 1$.

Proof. Let $M^r$ be a DSEM of type $[3.4^2.6 : 3.6.3.6]^r$ with $n$ vertices. Its $M^r(i, j, k)$ representation has $j$ disjoint horizontal cycles of $D_1$ type, say $Q_0, Q_1, \ldots, Q_{j-1}$, of length $i$. Note that the number of vertices between the horizontal cycles $Q_{2s(mod j)}$ and $Q_{(2s+1)(mod j)}$ for $0 \leq s \leq j - 1$ with the face-sequence $(3, 6, 3, 6)$ is $i/2 \cdot j/2$. Therefore, the number of vertices in $M^r$ is $n = ij + ij/4 = 5ij/4$.

If $j = 1$, then $M^r(i, 1, k)$ has no vertex with face-sequence $(3, 4^2, 6)$. This is not possible. Thus, $j \geq 2$. If $j \geq 2$ and $j$ is not an even integer, then we see that there is no vertex in the base horizontal cycle which has face-sequence $(3, 4^2, 6)$ after identifying the boundaries of $M^r(i, j, k)$. Thus $j$ is even.

For $i \leq 4$, $M^r$ is not a map. So $i > 4$. If $i > 4$ and not an even integer, $M^r$ is not a map. So $i \geq 6$, and $i, j$ are even. Thus $n = 5ij/4 \geq 15$.

If $j = 2$ and $k \in \{r : 0 \leq r \leq i - 1\} \setminus \{0, 1, 2, 3, i - 1\}$, then we get some vertices whose link cannot be constructed. So, for $j = 2$, we get $4 \leq k \leq i - 2$. Similarly, if $j \geq 4$, then $0 \leq k \leq i - 1$. Thus the proof.

4.7 DSEMs of type $[3^2.6^2 : 3.6.3.6]$

Let $M$ be a DSEM of type $[3^2.6^2 : 3.6.3.6]$. We consider the following types of paths in $M$ as follows.

A path $P_1 = (\ldots, y_i, y_i, y_{i+1,} \ldots)$ in $M$, say of type $F_1$ indicated by thick black paths, shown in Figure 4.7.1. The vertices $y_i$’s have the face-sequence $(3^2, 6^2)$.

A path $P_2 = (\ldots, y_i, z_i, y_{i+1}, z_{i+1}, \ldots)$ in $M$, say of type $F_2$, indicated by green paths, shown Figure 4.7.1. The vertices $y_i$’s and $z_i$’s have the face-sequences $(3^2, 6^2)$ and $(3, 6, 3, 6)$ respectively.
A construction of an $M(i, j, k)$ representation for $M$ follows by first cutting $M$ along a cycle of type $F_1$ and then cutting it along a cycle of type $F_2$. This gives the following result.

**Lemma 4.7.1** A DSEM $M$ of type $[3^2, 6^2 : 3.6.3.6]$ admits an $M(i, j, k)$-representation iff the following holds: (i) $j \geq 1$ and $i$ even, (ii) number of vertices of $M(i, j, k) = 3ij/2 \geq 15$, (iii) $i \geq 10$ if $j = 1$, and $i \geq 6$ if $j \geq 2$, (iv) if $j = 1$ then $k \in \{2r + 5 : 0 \leq r < (i - 8)/2\} \setminus \{(i - 8)/2 + 5\}$, if $j = 2$ then $k \in \{2r : 0 < r < i/2\}$, if $j = 2m + 1$, where $m \in \mathbb{N}$ then $k \in \{2r + 1 : 0 \leq r < i/2\}$, and if $j = 2m + 2$, where $m \in \mathbb{N}$ then $k \in \{2r : 0 < r < i/2\}$.

**Proof.** Let $M$ be a DSEM of type $[3^2, 6^2 : 3.6.3.6]$ with $n$ vertices. Then $M(i, j, k)$ of $M$ has $j$ disjoint horizontal cycles of type $F_1$, say $Q_0, Q_1, \ldots, Q_{j-1}$, of length $i$. Note that the number of adjacent vertices with face-sequence $(3, 6, 3, 6)$ which are lying on one side of horizontal cycles $Q_s$ and not belonging to any horizontal cycles $Q_s$ for $0 \leq s \leq j - 1$ is $i/2$ - $j$. So, $n = ij + ij/2 = 3ij/2$.

If $j = 1$ and $i$ is not an even integer, then some vertex in the base horizontal cycle do not have face-sequence $(3^2, 6^2)$, which is not possible. So, if $j = 1$, then $i$ is an even integer. Similarly, if $j > 1$, then $i$ is an even integer.

If $j = 1$ and $i < 10$, then we get some vertex in $M(i, 1, k)$ whose link can not be constructed. So, for $j = 1$, we get $i \geq 10$. Similarly, as above, we get that $i \geq 6$ if $j \geq 2$. Thus $n = 3ij/2 \geq 15$.

If $j = 1$ and $k \in \{r : 0 \leq r \leq i - 1\} \setminus \{(2r + 5 : 0 \leq r < (i - 8)/2\} \setminus \{(i - 8)/2 + 5\}$, then we get some vertices whose link can not be constructed. So, for $j = 1$, we get $k \in \{2r + 5 : 0 \leq r < (i - 8)/2\} \setminus \{(i - 8)/2 + 5\}$. If $j = 2$ and $k \in \{r : 0 \leq r \leq i - 1\} \setminus \{2r : 0 < r < i/2\}$, then some vertices in the lower horizontal cycle do not follow the face-sequence $(3^2, 6^2)$ after identifying the boundaries of $M(i, j, k)$. Which is a contradiction. So, for $j = 2$, we get $k \in \{2r : 0 < r < i/2\}$. Similarly as above, we see that if $j = 2m + 1$, where $m \in \mathbb{N}$, then $k \in \{2r + 1 : 0 \leq r < i/2\}$, and if $j = 2m + 2$, where $m \in \mathbb{N}$, then $k \in \{2r : 0 \leq r < i/2\}$. This completes the proof. \hfill $\square$

### 4.8 DSEM of type $[3^6 : 3^2, 6^2]$

Let $M$ be a DSEM of type $[3^6 : 3^2, 6^2]$. In $M$, we consider a path as follows.

A path $P_1 = P(\ldots, y_i, z_i, z_{i+1}, y_{i+1}, \ldots)$ in $M$, say of type $H_1$, indicated by thick black or green paths, shown in Figure 4.8.1. Here the vertices $y_i$'s and $z_i$'s have the face-sequences $(3^6)$ and $(3^2, 6^2)$ respectively.

![Figure 4.8.1: Paths of type $H_1$](image)

![Figure 4.8.2: $M(i, j, k)$](image)

Now, construct $M(i, j, k)$ representation of $M$ by first cutting $M$ along a black colored cycle of type $H_1$ through a vertex $v$ with face-sequence $(3^6)$ and then cutting it along a green colored cycle of type $H_1$. Note that both these cycles are non-homologous. Then we have the following lemma.

**Lemma 4.8.1** A DSEM $M$ of type $[3^6 : 3^2, 6^2]$ admits an $M(i, j, k)$-representation iff the following holds: (i) $i \geq 9$ and $i = 3m$, $m \in \mathbb{N}$ for $j = 1$, (ii) $i \geq 6$ and $i = 3m$, $m \in \mathbb{N}$ for $j > 1$, (iii) number of vertices of $M(i, j, k) = 7ij/3$, (iv) if $j = 1$ then $k \in \{3r : 1 < r < i/3\}$, and if $j > 1$ then $k \in \{3r : 0 \leq r < i/3\}$.
**Proof.** Let $M$ be a DSEM of type $[3^6 : 3^2.6^2]$ with $n$ vertices. Clearly $M(i, j, k)$ of $M$ has $j$ disjoint horizontal cycles of $H_1$ type, say $Q_0, Q_1, \ldots, Q_{j-1}$, of length $i$. Note that the number of vertices with face-sequence $(3^2, 6^2)$ which are above the horizontal cycles $Q_s$ and not belonging to any horizontal cycles $Q_s$ for $0 \leq s \leq j - 1$ is $4i/3 \cdot j$. In $M$, therefore $n = ij + 4ij/3 = 7ij/3$.

If $j = 1$ and $i < 9$, then the link of some vertices in $M(i, 1, k)$ can not be completed. This is not possible. So, for $j = 1$, we get $i \geq 9$. Also, if $i \geq 9$ and $i$ is not a multiple of 3, then after identifying the boundaries of $M(i, j, k)$, there is no vertex in the horizontal base cycle having face-sequences $(3^6)$ and $(3^2, 6^2)$. Which is not possible. So, $i \geq 9$ and $i = 3m, m \in \mathbb{N}$ for $j = 1$. Similarly, as above, for $j > 1$ we get that $i \geq 6$ and $i = 3m$, where $m \in \mathbb{N}$. Thus $n = 7ij/3 \geq 21$.

If $j = 1$ and $k \in \{r : 0 \leq r \leq i - 1\} \setminus \{3r : 1 < r < i/3\}$, then we get some vertices whose link can not be completed or does not follow the face-sequences $(3^6)$ and $(3^2, 6^2)$. So, for $j = 1$, we get $k \in \{3r : 1 < r < i/3\}$. Similarly, as above, we see that if $j > 1$, then $k \in \{3r : 0 \leq r < i/3\}$. Thus the proof.

4.9 DSEMs of type $[3^4.6 : 3^2.6^2]$

Let $M$ be a DSEM of type $[3^4.6 : 3^2.6^2]$. Consider the following types of paths as follows.

A path $P_1 = P(\ldots, y_i, z_i, y_{i+1}, z_{i+1}, \ldots)$ in $M$, say of type $I_1$, indicated by thick black paths, shown in Figure 4.9.1. Here, the vertices $y_i$’s and $z_i$’s have the face-sequences $(3^2, 6^2)$ and $(3^4, 6)$ respectively.

A path $P_2 = P(\ldots, y_i, y_{i+1}, z_i, z_{i+1}, y_{i+2}, y_{i+3}, \ldots)$ in $M$, say of type $I_2$, indicated by green paths, shown in Figure 4.9.1. Here the vertices $y_i$’s and $z_i$’s have the face-sequences $(3^2, 6^2)$ and $(3^4, 6)$ respectively.

As in previous sections, we construct $M(i, j, k)$ representation of $M$ by first cutting $M$ along a cycle of type $I_1$ and then cutting it along a cycle of type $I_2$ where the beginning adjacent face to the cycle $I_1$ is a hexagon. This gives the following lemma.

![Figure 4.9.1: Paths of type $I_1$, $I_2$](image1)

![Figure 4.9.2: $M(i, 2, k)$](image2)

![Figure 4.9.3: $M(i, 4, k)$](image3)

![Figure 4.9.4: $M(i, j = 4m + 2, k)$](image4)

![Figure 4.9.5: $M(i, j = 4m + 4, k)$](image5)
Lemma 4.9.1. A DSEM $M$ of type $[3^4.6 : 3^2.6^2]$ admits an $M(i,j,k)$-representation iff the following holds: (i) $i \geq 6$ and $i,j$ even, (ii) number of vertices in $M(i,j,k) = ij \geq 12$, (iii) if $j = 2$ then $k \in \{2r + 3 : 0 \leq r \leq (i-6)/2\}$, and if $j \geq 4$ then $k \in \{2r + 1 : 0 \leq r \leq (i-2)/2\}$.

Proof. Let $M$ be a DSEM of type $[3^4.6 : 3^2.6^2]$ with $n$ vertices. Then its $M(i,j,k)$ has $j$ disjoint horizontal cycles of $I_1$ type having length $i$. Thus the number of vertices in $M$ is $n = ij$.

If $j = 1$, then $M(i,1,k)$ has no vertex with face-sequence $(3^4,6)$. It is not possible. Therefore, $j \geq 2$. If $j \geq 2$ and $j$ is not an even integer, then no vertex in the base horizontal cycle follows the face-sequence $(3^4,6)$ after identifying the boundaries of $M(i,j,k)$. Therefore, $j$ is even.

If $j$ is an even integer and $i < 6$, then we get some vertex in $M(i,1,k)$ whose link can not be constructed. So, $i \geq 6$. Also, if $i \geq 6$ and $i$ is not an even integer, then similarly as above, there is no vertex in the horizontal base cycle having face-sequences $(3^4,6)$ and $(3^2,6^2)$. Which is not possible. So, $i \geq 6$ and $i,j$ are even integers. Thus, $n = ij \geq 12$.

If $j = 2$ and $k \in \{r : 0 \leq r \leq (i-6)/2\}$, then some vertices in the lower horizontal cycle do not follow the face-sequences $(3^4,6)$ and $(3^2,6^2)$. Which is a contradiction. So, for $j = 2$, we get $k \in \{2r + 3 : 0 \leq r \leq (i-6)/2\}$. Similarly, as above, we see that if $j \geq 4$, then $k \in \{2r + 1 : 0 \leq r \leq (i-2)/2\}$. This completes the proof. \[\square\]

4.10 DSEMs of type $[3^2.4.3.4 : 3.4.6.4]$ Let $M$ be a DSEM of type $[3^2.4.3.4 : 3.4.6.4]$. We consider a path in $M$ as follows. A path $P_i = P(\ldots, y_i, y_{i+1}, z_i, z_{i+1}, \ldots)$ in $M$, say of type $J_1$, indicated by thick black or green paths, shown in Figure 4.10.1. The inner vertices $y_i$’s and $z_i$’s have the face-sequences $(3,4,6,4)$ and $(3^2,4,3,4)$ respectively.

![Figure 4.10.1: Path of type $J_1$](image)

![Figure 4.10.2: $M(i,j,k)$](image)

Now, construct $M(i,j,k)$ representation of $M$ by first cutting $M$ along a black colored cycle of type $J_1$ and then cutting it along the green colored cycle of type $J_1$, where without loss of generality, let the beginning adjacent face to the base horizontal cycle is a quadrangle.

Lemma 4.10.1 A DSEM $M$ of type $[3^2.4.3.4 : 3.4.6.4]$ admits an $M(i,j,k)$-representation iff the following holds: (i) $j$ even and $i = 4m$, $m \in \mathbb{N}$, (ii) number of vertices of $M(i,j,k) = 3ij/2 \geq 12$, (iii) $k \in \{4r : 0 \leq r < i/4\}$.

Proof. Let $M$ be a DSEM of type $[3^2.4.3.4 : 3.4.6.4]$ having $n$ vertices. Then its $M(i,j,k)$ has $j$ disjoint horizontal cycles of $J_1$ type, say $Q_0, Q_1, \ldots, Q_{j-1}$, of length $i$. Let $Q_0 = C(w_{0,0}, w_{0,1}, \ldots, w_{0,i-1})$, $Q_1 = C(w_{1,0}, w_{1,1}, \ldots, w_{1,i-1}), \ldots, Q_{j-1} = C(w_{j-1,0}, w_{j-1,1}, \ldots, w_{j-1,i-1})$ be the list of horizontal cycles. Observe that the vertices having face-sequence $(3^2,4,3,4)$ lying between horizontal cycles $Q_{2s(mod j)}$ and $Q_{(2s+1)(mod j)}$ for $0 \leq s \leq j-1$ is $4i/4 \cdot j/2$. Therefore, $n = ij + ij/2 = 3ij/2$.

If $j = 1$, then no vertex in the base horizontal cycle follow the face-sequence $(3^2,4,3,4)$ after identifying the boundaries of $M(i,1,k)$. Therefore, $j \geq 2$. If $j \geq 2$ and $j$ is not an even integer, then as above no vertex in the base horizontal cycle follows the face-sequence $(3^2,4,3,4)$. So $j$ is
an even integer. If \( j \) is even and \( i \neq 4m \), where \( m \in \mathbb{N} \), then the \( \text{lk}(w_{0,0}) \) is not of type \((3, 4, 6, 4)\). Which is not possible. So, \( j \) is an even integer and \( i = 4m \), where \( m \in \mathbb{N} \). Thus, \( 3ij/ \geq 12 \).

If \( j \) is an even integer and \( k \in \{r : 0 \leq r \leq i - 1\} \setminus \{k \in \{4r : 0 \leq r < i/4\}\} \), then some vertex in \( M(i, j, k) \) do not follow the face-sequences \((3^2, 4, 3, 4)\) and \((3, 4, 6, 4)\). So, \( k \in \{4r : 0 \leq r < i/4\} \). This completes the proof. \( \square \)

### 4.11 DSEM of type \([3^6 : 3^4.6]_1\)

Let \( M \) be a DSEM of type \([3^6 : 3^4.6]_1\). Then for the existence of the map \( M \), we see that \( |V_{(3^6)}| = |V_{(3^4,6)}| \). Now, consider a fixed type path in \( M \) as follows. A path \( P_i = P(\ldots, z_i, y_i, y_{i+1}, y_{i+2}, z_{i+1}, \ldots) \) in \( M \), say of type \( K_1 \), indicated by thick black or green paths, shown in Figure 4.11.1. The vertices \( z_i \)'s and \( y_i \)'s have the face-sequences \((3^6)\) and \((3^4, 6)\) respectively or vertices \( z_i \)'s and \( y_i \)'s have the face-sequence \((3^6)\).

![Figure 4.11.1: Paths of type \( K_1 \)](image)

![Figure 4.11.2: \( M(i, j = 6m + 3, k) \)](image)

![Figure 4.11.3: \( M(i, j = 6m + 6, k) \)](image)

As in previous sections, \( M \) has an \( M(i, j, k) \) representation. The admissible relations among \( i, j, k \) of \( M(i, j, k) \) are given below.

**Lemma 4.11.1** A DSEM \( M \) of type \([3^6 : 3^4.6]_1\) admits an \( M(i, j, k) \)-representation iff the following holds: (i) \( j = 3m \) and \( i = 4m \), where \( m \in \mathbb{N} \), (ii) number of vertices of \( M(i, j, k) \) is \( ij \geq 12 \), (iii) \( k \in \{4r + 1 : 0 \leq r < i/4\} \).

**Proof.** Let \( M \) be a DSEM of type \([3^6 : 3^4.6]_1\) having \( n \) vertices. Its \( M(i, j, k) \) has \( j \) disjoint horizontal cycles of \( K_1 \) type, say \( Q_0, Q_1, \ldots, Q_{j-1} \), of length \( i \). Since the vertices of \( M \) are in these cycles, the vertices in \( M \) is \( n = ij \). If \( j = 1, 2 \) then \( M \) is not a map. So \( j \geq 3 \). If \( j \geq 3 \) and \( j \neq 3m \), \( m \in \mathbb{N} \), then no vertex in the base horizontal cycle follows the face-sequence \((3^6)\) after identifying the boundaries of \( M(i, j, k) \). So \( j = 3m \), where \( m \in \mathbb{N} \). Clearly if \( i \leq 3 \), \( M \) is not a map. So \( i \geq 4 \). If \( j = 3m \) and \( i \neq 4m \), where \( m \in \mathbb{N} \), then either \( |V_{(3^6)}| \neq |V_{(3^4,6)}| \) or \( M \) is not a map. So, \( i = 4m \), where \( m \in \mathbb{N} \). Thus \( n = ij \geq 12 \).

If \( k \in \{r : 0 \leq r \leq i - 1\} \setminus \{4r + 1 : 0 \leq r < i/4\} \), then the link of some vertex in \( M(i, j, k) \) can not be completed. So, \( k \in \{4r + 1 : 0 \leq r < i/4\} \). This completes the proof. \( \square \)

### 4.12 DSEM of type \([3^6 : 3^4.6]_2\)

Let \( M \) be a DSEM of type \([3^6 : 3^4.6]_2\). We consider a fixed type path as follows.

A path \( P_i = P(\ldots, y_i, z_i, z_{i+1}, y_{i+1}, \ldots) \) in \( M \), say of type \( K_1' \), indicated by thick black or green paths, shown in Figure 4.12.1. The vertices \( y_i \)'s and \( z_i \)'s have the face-sequences \((3^6)\) and \((3^4, 6)\) respectively.
Now, construct $M(i, j, k)$ representation of $M$ by first cutting $M$ along a black colored cycle of type $K_1^i$ through a vertex with face-sequence $(3^6)$ and then cutting it along the green colored cycle of type $K_1^i$.

![Figure 4.12.1: Paths of type $K_1^i$](image1)

![Figure 4.12.2: $M(i, j, k)$](image2)

We get admissible relations among $i, j, k$ of $M(i, j, k)$ in the next lemma and its proof follows from the similar arguments as in Lemma 4.10.1

**Lemma 4.12.1** A DSEM $M$ of type $[3^6 : 3^1]_2$ admits an $M(i, j, k)$-representation iff the following holds: (i) $j$ even and $i = 3m$, $m \in \mathbb{N}$, (ii) $i \geq 9$ for $j = 2$ and $i \geq 6$ for $j \geq 4$, (iii) number of vertices of $M(i, j, k) = 4ij/3 \geq 24$, (iv) if $j = 2$ then $4 \leq k \leq i - 1$, and if $j \geq 4$ then $0 \leq k \leq i - 1$.

### 4.13 DSEMs of type $[3^6 : 3^2]_2$

Let $M$ be a DSEM of type $[3^6 : 3^2]_2$. Consider a fixed type path $P_1 = P(\ldots, y_i, z_i, z_{i+1}, z_{i+2}, z_{i+3}, y_{i+1}, \ldots)$ in $M$, say of type $L_1$, indicated by thick black or green paths, shown in Figure 4.13.1. The vertices $y_i$'s and $z_i$'s have the face-sequences $(3^6)$ and $(3^2, 4, 12)$ respectively.

An $M(i, j, k)$ representation of $M$ follows by first cutting $M$ along a black colored cycle of type $L_1$ through a vertex with face-sequence $(3^6)$ and then cutting it along a green colored cycle of type $L_1$.

![Figure 4.13.1: Paths of type $L_1$](image3)

![Figure 4.13.2: $M(i, j, k)$](image4)

**Lemma 4.13.1** A DSEM $M$ of type $[3^6 : 3^2]_2$ admits an $M(i, j, k)$-representation iff the following holds: (i) $j$ even and $i = 5m$, $m \in \mathbb{N}$, (ii) $i \geq 15$ for $j = 2$, (ii) $i \geq 10$ for $j \geq 4$, (iii) number of vertices of $M(i, j, k) = 7ij/5 \geq 42$, (iv) if $j = 2$ then $k \in \{5r : 1 < r < i/5\}$, and if $j \geq 4$ then $k \in \{5r : 0 \leq r < i/5\}$.

**Proof.** Let $M$ be a DSEM of type $[3^6 : 3^2]_2$ having $n$ vertices. An $M(i, j, k)$ has $j$ disjoint horizontal cycles of $L_1$ type, say $Q_0, Q_1, \ldots, Q_{j-1}$, of length $i$. Let $Q_0 = C(w_0, w_1, \ldots, w_{i-1}), Q_1 = C(w_1, w_1, \ldots, w_{i-1}), \ldots, Q_{j-1} = C(w_{j-1}, w_{j-1}, \ldots, w_{j-1})$ be the list of horizontal cycles. Note that the vertices with face-sequence $(3^2, 4, 12)$ lying between cycles $Q_{2s(mod j)}$ and $Q_{2s+1(mod j)}$ for $0 \leq s \leq j - 1$ is $4i/5 \cdot j/2$. Therefore, $n = ij + 2ij/5 = 7ij/5$.

If $j = 1$, then no vertex in the base horizontal cycle follows the face-sequence $(3^2, 4, 12)$ after identifying the boundaries of $M(i, 1, k)$. Therefore, $j \geq 2$. If $j \geq 2$ and $j$ is not an even integer,
then similarly as above no vertex in the base horizontal cycle follows the face-sequence \((3^2, 4, 12)\). So \(j\) is an even integer. If \(j\) is even and \(i \neq 5m\), where \(m \in \mathbb{N}\), then the \(\text{lk}(w_{0,0})\) is not of type \((3^6)\). Which is not possible. So, \(i = 5m\), where \(m \in \mathbb{N}\).

If \(j = 2\) and \(i < 15\), then we get some vertex in \(M(i, 2, k)\) whose link can not be constructed. This is not possible. So, for \(j = 2\), we get \(i \geq 15\). Similarly, \(i \geq 10\) for \(j \geq 4\). Thus \(n = 7ij/5 \geq 42\).

If \(j = 2\) and \(k \in \{r : 0 \leq r \leq i - 1\} \setminus \{5r : 1 < r < i/5\}\), then we get some vertex in \(M(i, 2, k)\) whose link can not be constructed. So, if \(j = 2\), then \(k \in \{5r : 1 < r < i/5\}\). Similarly, as above, we see that if \(j \geq 4\), then \(k \in \{5r : 0 \leq r < i/5\}\). This completes the proof. \(\square\)

### 4.14 DSEM of type \([3.4.3.12 : 3.12^2]\)

Let \(M\) be a DSEM of type \([3.4.3.12 : 3.12^2]\). In \(M\), consider a fixed type path \(P_1 = P(\ldots, y_i, z_i, z_{i+1}, y_{i+1}, \ldots)\), say of type \(N_1\), indicated by thick black or green paths, shown in Figure 4.14.1. The vertices \(y_i\)’s and \(z_i\)’s have the face-sequences \((3, 4, 3, 12)\) and \((3, 12^2)\) respectively.

![Figure 4.14.1: Paths of type \(N_1\)](image)

Now construct an \(M(i, j, k)\) representation of \(M\), as in previous subsections, by cutting \(M\) through a vertex \(v\) with the face-sequence \((3, 12^2)\) one by one along the cycles of type \(N_1\). Assume that the beginning adjacent face to the base horizontal black cycle is 3-gon.

**Lemma 4.14.1** A DSEM \(M\) of type \([3.4.3.12 : 3.12^2]\) admits an \(M(i, j, k)\)-representation iff the following holds: (i) \(j \geq 1\) and \(i = 6m\), \(m \in \mathbb{N}\), (ii) \(i \geq 24\) for \(j = 1\), and \(i \geq 18\) for \(j = 2\), (iii) \(i \geq 12\) for \(j > 2\), (iv) number of vertices of \(M(i, j, k) = 8ij/6 \geq 32\), (v) if \(j = 1\) then \(k \in \{6r + 3 : 1 \leq r < (i - 6)/6\}\), if \(j = 2\) then \(k \in \{6r + 5 : 1 \leq r < (i - 12)/6\}\), if \(j = 2m + 1\) then \(k \in \{6r + 3 : 0 \leq r \leq (i - 6)/6\}\), and if \(j = 2m + 2\) then \(k \in \{6r + 5 : 0 \leq r \leq (i - 6)/6\}\).

**Proof.** Let \(M\) be a DSEM of type \([3.4.3.12 : 3.12^2]\) having \(n\) vertices. An \(M(i, j, k)\) of \(M\) has \(j\) disjoint horizontal cycles of \(N_1\) type, say \(Q_0, Q_1, \ldots, Q_{j-1}\), of length \(i\). Let \(Q_0 = C(w_{0,0}, w_{0,1}, \ldots, w_{0,i-1}), Q_1 = C(w_{1,0}, w_{1,1}, \ldots, w_{1,i-1}), \ldots, Q_{j-1} = C(w_{j-1,0}, w_{j-1,1}, \ldots, w_{j-1,i-1})\) be the list of horizontal cycles. Observe that the vertices having face-sequence \((3, 4, 3, 12)\) which are lie above the cycles \(Q_s\) and not belonging to any of these cycles \(Q_s\) for \(0 \leq s \leq j - 1\) is \(2i/6 \cdot j\). So, the total number of vertices in \(M\) is \(n = ij + 2ij/6 = 8ij/6\).
If \( j = 1 \) and \( i \neq 6m \), where \( m \in \mathbb{N} \), then the face-sequence of \( w_{0,0} \) is not of the type \((3,12^2)\).

This is not possible. So, if \( j = 1 \), then \( i = 6m \), where \( m \in \mathbb{N} \). Similarly, if \( j > 1 \), then \( i = 6m \).

If \( j = 1 \) and \( i < 24 \), then link of some vertices cannot be completed. So, for \( j = 1 \), we get \( i \geq 24 \). Similarly, as above, we get that \( i \geq 18 \) if \( j = 2 \) and \( i \geq 12 \) for \( j > 2 \). Thus, \( n = 8ij/6 \geq 32 \).

If \( j = 1 \) and \( k \in \{r : 0 \leq r \leq i - 1\} \setminus \{k \in \{6r+3 : 1 \leq r < (i - 6)/6\}\} \) then some vertices do not have the given face-sequences. So, \( k \in \{6r+3 : 1 \leq r < (i - 6)/6\} \) for \( j = 1 \). Proceeding similarly, we see if \( j = 2 \) then \( k \in \{6r+5 : 1 \leq r < (i - 12)/6\} \) if \( j = 2m+1 \) then \( k \in \{6r+3 : 0 \leq r < (i - 6)/6\} \), and if \( j = 2m+2 \) then \( k \in \{6r+5 : 0 \leq r < (i - 6)/6\} \). Thus the proof.

\[ \blacksquare \]

### 4.15 DSEM of type \([3.4.6.4 : 4.6.12]\)

Let \( M \) be a DSEM of type \([3.4.6.4 : 4.6.12]\). In \( M \), consider a fixed type path \( P_1 = P(\ldots , y_i , y_{i+1} , y_{i+2} , y_{i+3} , z_i , z_{i+1} , \ldots ) \) in \( M \), say of type \( O_1 \), indicated by thick black or green paths, shown in Figure 4.15.1. The vertices \( y_i \)'s and \( z_i \)'s have the face-sequences \((4,6,12)\) and \((3,4,6,4)\) respectively.

We construct \( M(i,j,k) \) representation of \( M \) by first cutting \( M \) along a black coloured cycle of type \( O_1 \) through a vertex \( v \) having face-sequence \((4,6,12)\) and then cutting it along a green coloured cycle of type \( O_1 \) where the beginning adjacent face to black coloured cycle is a 4-gon. The admissible relations among \( i,j,k \) of \( M(i,j,k) \) are given in the next lemma, and its proof follows from the similar arguments as in Lemma [4.13.1]

**Lemma 4.15.1** A DSEM \( M \) of type \([3.4.6.4 : 4.6.12]\) admits an \( M(i,j,k) \)-representation iff the following holds: (i) \( j \) even, (ii) \( i \geq 12 \) and \( i = 6m \), \( m \in \mathbb{N} \), (iii) number of vertices of \( M(i,j,k) \) is \( 3ij/2 \geq 36 \), (iv) if \( j = 2 \) then \( k \in \{3r + 2 : 1 < r < (i - 3)/3\} \), and if \( j \geq 4 \) then \( k \in \{3r + 2 : 0 \leq r < i/3\} \).

### 4.16 DSEM of type \([3.4^2.6 : 3.4.6.4]\)

Let \( M \) be a DSEM of type \([3.4^2.6 : 3.4.6.4]\). In \( M \), consider a fixed type path \( P_1 = P(\ldots , y_i , y_{i+1} , y_{i+2} , z_i , z_{i+1} , y_{i+3} , \ldots ) \), say of type \( R_1 \), indicated by thick black or green paths, shown in Figure 4.16.1. The vertices \( y_i \)'s and \( z_i \)'s have the face-sequences \((3,4,6,4)\) and \((3,4^2,6)\) respectively.
We construct $M(i,j,k)$ representation of $M$ by first cutting $M$ along a black colored cycle of type $R_1$ through a vertex $v$ with the face-sequence $(3,4,6,4)$ and then cutting it along a green colored cycle of type $R_1$, assume that the beginning adjacent face to base horizontal cycle is a 3-gon. The admissible relations among $i,j,k$ are given by the next lemma, and its proof follows from the similar arguments as in Lemma 4.10.1

Lemma 4.16.1 A DSEM $M$ of type $[3.4.2.6 : 3.4.6.4]$ admits an $M(i,j,k)$-representation iff the following holds: (i) $j$ even, (ii) $i \geq 5$ and $i = 5m$, $m \in \mathbb{N}$, (iii) number of vertices of $M(i,j,k) = 9ij/5 \geq 18$, (iv) $k \in \{5r : 0 \leq r < i/5\}$.

4.17 DSEMs of type $[3.3.4.2 : 3.4.6.4]$

Let $M$ be a DSEM of type $[3.3.4.2 : 3.4.6.4]$. In $M$, consider a path $P_1 = P(\ldots, y_i, z_i, z_{i+1}, z_{i+2}, y_{i+1}, \ldots)$, say of type $S_1$, indicated by thick black or green paths shown in Figure 4.17.1. The vertices $y_i$’s and $z_i$’s have the face-sequences $(3^3,4^2)$ and $(3,4,6,4)$ respectively or $y_i$’s and $z_i$’s have the face-sequence $(3^3,4^2)$.

An $M(i,j,k)$ representation of $M$ follows by cutting $M$ one by one along a black and green colored cycles of type $S_1$ through a vertex $v$ with face-sequence $(3,4,6,4)$. Without loss of generality, let the beginning adjacent face to the base horizontal cycle is a 4-gon.

Lemma 4.17.1 The DSEM $M$ of type $[3.3.4.2 : 3.4.6.4]$ admits an $M(i,j,k)$-representation iff the following holds: (i) $j = 3m$, where $m \in \mathbb{N}$, (ii) $i \geq 4$ and $i = 4m$, $m \in \mathbb{N}$, (iii) number of vertices of $M(i,j,k) = ij \geq 12$, (iv) $k \in \{4r + 3 : 0 \leq r < i/4\}$.

Proof. Let $M$ be a DSEM of type $[3.3.4.2 : 3.4.6.4]$ having $n$ vertices. Clearly, its representation $M(i,j,k)$ has $j$ disjoint horizontal cycles of $S_1$ type, say $Q_0, Q_1, \ldots, Q_{j-1}$, of length $i$. Let $Q_0 = C(w_{0,0}, w_{0,1}, \ldots, w_{0,i-1}), Q_1 = C(w_{1,0}, w_{1,1}, \ldots, w_{1,i-1}), \ldots, Q_{j-1} = C(w_{j-1,0}, w_{j-1,1}, \ldots, w_{j-1,i-1})$ be the list of horizontal cycles. Since the vertices of $M$ are in these cycles, the vertices in $M$ is $n = ij$. If $j = 1, 2$ then $M$ is not a map. So $j \geq 3$. If $j \geq 3$ and $j \neq 3m, m \in \mathbb{N}$, then no vertex in the base horizontal cycle follow the face-sequences $(3^3,4^2)$ and $(3,4,6,4)$ after identifying the boundaries of $M(i,j,k)$. So $j = 3m$, where $m \in \mathbb{N}$. Clearly if $i \leq 3$, $M$ is not a map. So $i \geq 4$. If $j = 3m$ and $i \neq 4m$, where $m \in \mathbb{N}$, then the lk$(w_{0,0})$ is not of type $(3,4,6,4)$. Which is not possible. So, $i = 4m$, where $m \in \mathbb{N}$. Thus, $n = ij \geq 12$.

If $k \in \{r : 0 \leq r \leq i - 1\} \setminus \{4r + 3 : 0 \leq r < i/4\}$, then link of some vertices in $M(i,j,k)$ can not be completed. So, $k \in \{4r + 3 : 0 \leq r < i/4\}$. This completes the proof. 

Proof of Theorem 1.1 Let $M$ be a DSEM of type $Y_1$, where $Y_1 \in \{[3^6 : 3.3.4^2], [3^6 : 3.4.2^4], [3.3.4^2 : 4^4], [3.3.4^2 : 4^4], [3.3.4^2 : 3^2.4^3.4.2]\}$ with its $M(i,j,k)$ representation (shown in Figures 4.1.3, 4.1.4,
4.2.2, 4.2.3 and 4.4.2). Now, if we remove the diagonal edges (indicated by thick black edges), we get a map, say $M'$, on the torus of type $[4^4]$. Clearly, $V(M') = V(M)$ and $E(M') \subset E(M)$. By [11] Theorem 4], the map $M'$ contains a Hamiltonian cycle, say $H$. The cycle $H$ is also a Hamiltonian cycle in $M'$ as $E(M') \subset E(M)$. Hence, $M$ is Hamiltonian.

Let $M$ be a DSEM of the type $[3.4^2.6 : 3.6.3.6]_2$, $[3^6 : 3^2.6^2]$, $[3^2.4.3.4 : 3.6.4.6]$, $[3^6 : 3^4.6]_2$, $[3^6 : 3^2.4.12]$, $[3.4^2.6 : 3.6.4.6]$, or $[3.4^4.6 : 3.6.4.6]$. Then the thick black cycle in its $M(i,j,k)$, shown respectively in Figures 4.6.4, 4.10.2, 4.12.2, 4.13.2, 4.15.2 or 4.16.2, is Hamiltonian. Thus $M$ is Hamiltonian.

Similarly, if $M$ is one of the types, $[3^3.4^2 : 3^2.4.3.4]_1$, $[3^6 : 3^2.4.3.4]$, $[3^2.6^2 : 3.6.3.6]$, $[3^2.6^2 : 3^4.6]$, $[3^6 : 3^2.6^2]$, $[3^6 : 3^4.12]$, $[3.4^2.6 : 3.6.4.6]$, or $[3^2.4^2 : 3.4.6.4]$, then depending on $j$, we get its $M(i,j,k)$ representation, shown in respective subsection. Note that the black thick cycle drawn in such representation is Hamiltonian. Hence $M$ is Hamiltonian.

$$\blacksquare$$

5 Conclusion

In this article, we have discussed the connectivity of DSEMs on the torus corresponding to the twenty 2-uniform tilings of the plane and shown that every such DSEM is either 3-connected or 4-connected. Using this, we have established the Nash-Williams conjecture for such maps. Recall that, in a map $M$, the combinatorial curvature of a vertex $v$ with the face-sequence $(p_1, \ldots, p_k)$ is given by $\phi(v) = 1 - \left(\sum_{i=1}^{k} n_i\right)/2 + \left(\sum_{i=1}^{k} n_i\right)/p_i$. Note that the DSEMs, discussed here, have $\phi(v) = 0$ for all the vertices. However, one can construct many more doubly semi-equivelar maps which may not have such curvature restriction. For instance, if we stack (subdividing a face by introducing a new vertex inside the face and joining this new vertex to each vertex of the face by an edge) all the 4-gonal faces of any semi-equivelar map of type $(4.8^2)$, then the resulting map is a DSEM with two types face-sequences $(4^4)$ and $(3^2.8^2)$, that is other than the twenty types. However, the idea used for the connectivity of DSEMs can be applied to arbitrary DSEMs on the torus. One can also explore similarly the Hamiltonicity of a class of maps on the torus corresponding to the remaining types $k$-uniform tilings for $3 \leq k \leq 7$. Thus a natural question arises for the readers to determine the complete classification of DSEMs on the torus in terms of types and to check their Hamiltonicity.

References

[1] A. Altshuler, Hamiltonian circuits in some maps on the torus, Discrete Math. 4 (1972) 299-314.

[2] A. Altshuler, Construction and enumeration of regular maps on the torus, Discrete Math. 4 (1973) 201-217.

[3] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer-Verlag London, 2008.

[4] I.Z. Bouwer, W.W. Chernoff, The toroidal graphs $\{6,3\}_{b,c}$ are Hamiltonian, Ars Combin. 25A (1988) 179-186.

[5] U. Brehm, W. Kühnel, Equivelar maps on the torus, Eur. J. Combin. 29 (2008) 1843-1861.

[6] R. Burnet, R.B. Richter, Hamiltonicity of 5-connected toroidal triangulations, J. Graph Theory 20 (1995) 267-286.

[7] R. Burnet, A. Nakamoto, S. Negami, Every 5-connected triangulations of the Klein bottle is Hamiltonian, Yokohama Math. J. 47 (1999), 239-244.

[8] B. Grünbaum, Polytopes, graphs, and complexes, Bull. Amer. Math. Soc. 76 (1970) 1131-1201.
[9] B. Grünbaum, G.C. Shephard, Tilings and Patterns. New York: W. H. Freeman and com. 1987.

[10] K. Kawarabayashi, K. Ozeki, Hamiltonian cycles in 4-connected toroidal triangulations, Electron. Notes Discrete Math. 38 (2011) 493-498.

[11] O. Krötenheerdt, Die homogenen Mosaike n-ter Ordnung in der euklidischen Ebene I. Wissenschaftliche Zeitschrift der Martin-Luther-Universität Halle-Wittenberg. 18 (1969) 273-290.

[12] D. Maity, A.K. Upadhyay, On enumeration of a class of toroidal graphs, Contrib. to Disc. Math. 13 (2018) 79-119.

[13] D. Maity, A.K. Upadhyay, Hamiltonicity of a class of toroidal graphs, Math. Slovaca 70 (2020) 497-503.

[14] C.St.J.A. Nash-Williams, Unexplored and semi-explored territories in graph theory. In: New directions in the theory of graphs, Academic Press, New York, 1973, pp. 169-176.

[15] Y. Singh, A.K. Tiwari, Enumeration of some doubly semi-equivelar maps on torus. arXiv:2005.00332

[16] R. Thomas, X. Yu, 4-connected projective planar graphs are Hamiltonian, J. Combin. Theory, Ser. B 62 (1994) 114-132.

[17] R. Thomas, X. Yu, 5-connected toroidal graphs are Hamiltonian, J. Combin. Theory, Ser. B 69 (1997) 79-96.

[18] A.K. Tiwari, A. Tripathi, Y. Singh, P. Gupta, Doubly semi-equivelar maps on torus and Klein bottle, Journal of Mathematics, 2020.

[19] A.K. Tiwari, A.K. Upadhyay, Semi-equivelar maps on the torus and the Klein bottle with few vertices, Math. Slovaca 67 (2017) 519-532.

[20] W.T. Tutte, A theorem on planar graphs, Trans. Amer. Math. Soc. 82 (1956) 99-116.

[21] H. Whitney, A theorem on graphs, Ann. of Math. 32 (1931) 378-390.

[22] R. Thomas, X. Yu and W. Zang, Hamilton paths in toroidal graphs, J. Combin. Theory Ser. B 94 (2005), 214-236.

[23] A. Nakamoto and K. Ozeki, Hamiltonian cycles in bipartite quadrangulations on the torus, J. Graph Theory 69 (2012) 143-151.

[24] K. Kawarabayashi and K. Ozeki, 5-connected toroidal graphs are Hamiltonian-connected, SIAM J. Discrete Math. 30(1) (2016) 112-140.

[25] K. Ozeki and C. T. Zamfirescu, Every 4-connected graph with crossing number 2 is Hamiltonian, SIAM J. Discrete Math. 32(4) (2018) 2783-2794.