ON MARSHAK’S AND CONNES’ VIEWS OF CHIRALITY

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ABSTRACT

I render the substance of the discussions I had with Robert E. Marshak shortly before his death, wherein the kinship between the “neutrino paradigm” —espoused by Marshak— and the central notion of K-cycle in noncommutative geometry (NCG) was found. In that context, we give a brief account of the Connes–Lott reconstruction of the Standard Model (SM).

1. Bob’s last adventure

I met Bob Marshak at a Texan barbecue. It was mid-September of 1991. There was a workshop to celebrate the sixtieth anniversary of his closest disciple, E. C. G. Sudarshan of the Center for Particle Physics of the University of Texas at Austin. As the physicists formed in line for the barbecue, I happened to fill the place just before Bob. I turned to congratulate him on his moving speech of the previous evening. Minutes later we were fast and deep in conversation (whenever Bob heartily attacked the “good stuff” at the barbecue). It was a friendship, of sorts, à première vue.

Although an indefatigable traveller, Bob had never visited Spain. We started together thinking about a trip that would allow him to sample the Spanish cultural diversity and to meet some of the Spanish particle physicists. We agreed that October 1992 would be a good time for travelling to Spain. He would be going first to the World Fair in Seville, spending a week in Andalucía. Afterwards he was to make a tour of several Spanish universities. On November 27 of 1991 he wrote me from Virginia: “My wife and I think that this Spanish trip will compare in excitement with our first trip long ago (in 1953) arranged by Amaldi for visits to the excellent Italian universities and cultural treasures”.

Bob and Ruth Marshak indeed flew into Madrid on October the first of 1992. Then they departed for Granada and Seville. I met them in Madrid on the 8th, at the station, upon the arrival of the bullet train from Seville. I could not avoid noticing that Bob was in worse physical condition that he had seemed to be in Texas one year before. Nevertheless, he kept in reasonably good health and high spirits during the trip. I believe he enjoyed it immensely. The old Spanish and Flemish masters, Miró and Picasso gave special pleasure to him and Ruth. I witnessed his childlike gaiety and was enchanted by his love of life and physics. Bob lectured on the triumphs of the SM of particle interactions and gave us his personal recollections of the startup period in Particle Physics. He was never far from a fax machine, meanwhile, as he was giving the finishing touches to Conceptual Foundations of Modern Particle Physics,¹ fated to be his posthumous book.
2. Bob’s theoretical concerns at the end of his life

During two unforgettable weeks in October 1992, Bob showered on me his intimate knowledge of all theoretical aspects of particle physics. We talked over breakfast, we talked on the trains, we talked over late Spanish dinners. Some of the things he tried to explain to me I understood only when I got Conceptual Foundations... in my hands. Others I will never fully grasp. Marshak’s book is indeed a superb conceptual legacy. All the challenging problems associated with the SM are expounded with penetrating detail and grouped in a coherent whole.

To report that Robert’s scientific interests in the last period of his life turned around the themes of his book will surprise no one. However, there were favourites. He stressed how the original Marshak–Sudarshan version of V–A invariance (in contrast to the Feynman–Gell-Mann one) was based on the principle of chirality invariance, and he tried to impress on me the importance of chirality and chiral gauge anomaly-free constraints in modern particle dynamics. He explained to me at length the origins of the $U(1)$ and the “strong CP” problems in QCD and his solution (proposed together with S. Okubo) to the latter. He was eloquent on the advantages of the grand unification model based on $SO(10)$. I cherish a very lucid account by Bob’s own hand of the earlier lepton-baryon symmetry, leading to the concepts of weak hypercharge and weak isospin.

Bob also talked to me about his work as deputy leader of one of the “theoretical” groups during the atom bomb project at Los Alamos. Robert and Ruth shared with me vivid memories about Klaus Fuchs, who passed to the Soviets the secrets of the bomb. I came to a measure of understanding and respect for the ethical convictions that led him to assume the perilous presidency of New York City College. And this is how I came into the privilege of being almost the last person to learn from Bob.

In exchange, Bob asked me to report to him on the reconstruction of the SM in the non-commutative geometry approach pioneered by Connes and Lott. He was fascinated by NCG. During those lively discussions, we realized that the “neutrino paradigm” that pervades Marshak’s view of the SM and Connes’ key concept of $K$-cycle are like two sides of the same coin.

On the 25th of October I wished Robert and Ruth good travel on their departure from Spain. Bob was contented and in an expansive mood. Some time later I got a last letter from Bob. Little did I suspect that we would not meet again by shade or sunlight.

3. Chirality invariance and Noncommutative geometry

I can do no better to pay homage to Bob than to deliver the substance of the conversations we had on Connes’ generalized geometry and the Standard Model. Marshak contends that the chiral invariance of the Weyl fermions plays a key rôle in the SM. Because of the large scale of the spontaneous symmetry breaking mechanism that gives masses to the fermions, it is expected that the departures from the “neutrino paradigm” are small, except perhaps for the top quark. I will introduce
noncommutative geometry by considering a seemingly unrelated question: the possibility of deriving the motion of a classical particle on a manifold from the motion of quantum particles.

On a Riemannian manifold free particles move along geodesics. A few years ago, Connes realized that the simplest way to obtain geodesic motion from quantum motion was to use neutrinos. Connes’ argument goes as follows. Let $H := L^2(S)$ be the space of square integrable sections of the irreducible spinor bundle $S$ over the compact spin manifold $M$, and $D$ the corresponding Dirac operator. Recall that the algebra $A = C^\infty(M)$ of smooth (complex) functions over the manifold acts on $H$ by multiplication operators, i.e., multiplication by scalars on each fibre of $S$. The densely defined operator $[D, f]$, for $f \in A$, is bounded. Indeed, we have immediately $D(fs) - fDs = c(df)s$, where $c(df)$ means Clifford multiplication of the spinor $s$ by $df$ and $d$ denotes the ordinary differential of $f$. This operator is majorized by the supremum norm of $df$, which equals the Lipschitz norm of $f$, i.e., $\|f\|_{\text{Lip}} := \sup_{p \neq q} |f(p) - f(q)|/d(p, q)$, with $d(p, q)$ denoting the geodesic distance. The geodesic distance is defined conventionally as the minimum path length from $p$ to $q$, but we can now turn the procedure around and recover the metric on $M$ from the Dirac operator and the algebra of functions directly:

$$d(p, q) := \sup \{ |f(p) - f(q)| : f \in A, \|[D, f]\| \leq 1 \}. \quad (1)$$

Is it possible to derive the classical action from the kinematics of quantum scalar particles? Indeed it should be, as the Laplacian encodes the Riemannian geometry of the manifold. The formula is:

$$d(p, q) := \sup \{ |f(p) - f(q)| : f \in A, \frac{1}{2}(\Delta f^2 + f^2 \Delta) - f \Delta f \leq 1 \}.$$ 

The previous formula is given by Fröhlich and Gawędzki, who credit it to J. Dereziński. The proof is the same, once one realizes that left hand side of the inequality is the multiplication operator by $\|df\|^2$. However, this is considerably more complicated.

Next, one can formalize the above into the key concept for integrodifferential calculus in noncommutative geometry. By definition, a $K$-cycle $(\mathcal{H}, D)$ on the $\ast$-algebra $A$ consists of a unitary representation of $A$ on a Hilbert space $\mathcal{H}$, together with an (unbounded) selfadjoint operator $D$ on $\mathcal{H}$ with compact resolvent, such that $[D, a]$ is bounded for all $a \in A$. We also assume $\mathcal{H}$ is a $\mathbb{Z}_2$-graded Hilbert space, equipped with a grading operator $\Gamma$ such that $\Gamma^2 = 1$, that $A$ acts on $\mathcal{H}$ by even operators, and that $D$ is an odd operator (i.e., $a\Gamma = \Gamma a$ for $a \in A$, and $D\Gamma = -\Gamma D$). Then the right hand side of equation (1) defines also a distance on the space of states of the algebra (equipped with a $K$-cycle), so it admits a natural noncommutative generalization.
There is much more to it, from the physical point of view. Noncommutative geometry comes into its own when we consider \( K \)-cycles associated to finite algebras. Nothing more natural in the sequel than to take up those finite algebras that give rise to the gauge groups of particle physics and “Dirac operators” relating the left- and right-handed representations of these algebras, just like the standard Dirac operator relates the left- and right-handed spinor representations. The matrix elements of these Dirac operators are given by the Yukawa couplings among the fermions. It is possible to combine both constructions to yield a Dirac–Yukawa operator, that contains (in NCG) all the relevant information pertaining to the SM. There is no way to figure out the mentioned parameters a priori. Nevertheless, in contrast to the conventional version, the Higgs sector (thus the boson mass matrix) is at the output end of Connes’ machine and the properties of the symmetry-breaking sector are entirely determined. Indeed, the existence of the Higgs sector is a consequence of chirality: it is the gauge field associated to the intrinsic “discreteness” of the space that results from the existence of left- and right-handed representations. This helps to explain some characteristics of the Higgs field that are analogous to those of nonabelian Yang–Mills fields. In particular, in that reconstruction of the SM, the masses of the intermediate vector bosons and of the Higgs particle are calculated, at least at the tree level, in terms of the Yukawa couplings. They must be of the same order of the top quark mass. We give some more details of the Connes–Lott setup in the next two Sections.

4. Connes’ mathematical machine

A “noncommutative space” is just a noncommutative algebra \( \mathcal{A} \) (of operators on a Hilbert space). To get differential calculus on such a space, one embeds \( \mathcal{A} \) in a universal graded differential algebra \( \Omega^* \mathcal{A} = \bigoplus_{n \geq 0} \Omega^n \mathcal{A} \) generated by symbols \( a_0 \, da_1 \ldots da_n \) with a derivation \( d \) satisfying \( d(a_0 \, da_1 \ldots da_n) = da_0 \, da_1 \ldots da_n \), \( d(1) = 0 \), \( d^2 = 0 \). This is an \( \mathcal{A} \)-bimodule: we multiply \( a_0 \, da_1 \ldots da_n \) by \( b \in \mathcal{A} \) on the right by applying the rule \((da)b = d(ab) - adb \) repeatedly.

For the commutative case \( \mathcal{A} = C^\infty(M) \), the smooth sections of a hermitian vector bundle on \( M \) form a (right) module \( \mathcal{E} \) over the algebra \( \mathcal{A} \), which is of the form \( p\mathcal{A}^m \) with \( p^2 = p = p^* \) in some \( m \times m \) matrix algebra over \( \mathcal{A} \); moreover, \( \mathcal{E} \) carries a positive hermitian form \((\cdot, \cdot)\) with values in \( \mathcal{A} \). Such modules, over more general algebras, are “noncommutative vector bundles”. A compatible connection on \( \mathcal{E} \) is then a linear map \( \nabla: \mathcal{E} \to \mathcal{E} \otimes \mathcal{A} \Omega^1 \mathcal{A} \) satisfying \( \nabla(sa) = \nabla(s)a + s \otimes da \) and \( d(s, s') = (\nabla s, s') + (s, \nabla s') \), for \( s, s' \in \mathcal{E} \), \( a \in \mathcal{A} \). Its curvature is the matrix-valued 2-form \( \theta \) given by \( \nabla^2(s) = \theta s \). Gauge transformations \( \nabla \mapsto u \nabla u^* \) are given by unitary matrices \( u \) over \( \mathcal{A} \) satisfying \( up = pu \); thus the utility of the vector bundle \( \mathcal{E} \) is to specify the gauge group.

Integration over a noncommutative space is given by the “Dixmier trace” of compact operators on a Hilbert space \( \mathcal{H} \). A positive compact operator \( \mathcal{A} \) with eigenvalues \( \lambda_1 \geq \lambda_2 \geq \ldots \) lies in the Dixmier trace class if and only if \( \lambda_1 + \cdots + \lambda_n = \ldots \)
$O(\log n)$. The Dixmier trace is a generalized limit of the form
\[
\text{Tr}^+ A := \lim_{n \to \infty} \frac{\lambda_1 + \cdots + \lambda_n}{\log n},
\]
which can be extended to a linear functional on the full Dixmier trace class; we have $\text{Tr}^+ T = 0$ for $T$ in the ordinary trace class.

When $\mathcal{H} = L^2(S)$ is a spinor bundle over a compact Riemannian manifold $M$ of even dimension $n = 2m$, and $D = \gamma^\mu \partial_\mu$ is the Dirac operator, then $|D|^{-n}$ lies in the Dixmier trace class, and a fundamental trace theorem of Connes\textsuperscript{7,8} yields the following integral formula, for $a \in C^\infty(M)$:
\[
\text{Tr}^+ (a|D|^{-n}) = \frac{1}{m!(2\pi)^m} \int_M a(x) \, d\text{vol}(x).\tag{2}
\]

This is how the Dixmier trace, in the presence of a $K$-cycle, gives a precise generalization of integration over a manifold.

The $K$-cycle also allows us to refine the “differential calculus” by reducing the large differential algebra $\Omega^\bullet \mathcal{A}$ to a more useful one. We can represent $\Omega^\bullet \mathcal{A}$ on $\mathcal{H}$ by taking
\[
\pi(a_0 \, da_1 \ldots da_n) := i^n a_0 \, [D, a_1] \ldots [D, a_n].
\]
One can have $\pi(b) = 0$ with $\pi(db) \neq 0$. We must factor out the differential ideal of “junk” $J := \{ b' + db'' \in \Omega^\bullet \mathcal{A} : \pi b' = \pi b'' = 0 \}$, thereby obtaining a new graded differential algebra of “$D$-forms” by
\[
\Omega^\bullet D\mathcal{A} := \pi(\Omega^\bullet \mathcal{A})/J \equiv \pi_D(\Omega^\bullet \mathcal{A}).
\]

For the Dirac $K$-cycle, the quotient algebra $\Omega^\bullet D_{C^\infty}(M)$ is the usual algebra of differential forms on $M$.

“Universal” connections and curvatures on $\Omega^\bullet \mathcal{A}$ pass to connections and curvatures on $\Omega^\bullet D\mathcal{A}$. We can integrate the square of the curvature $\theta$ to get a gauge-invariant and nonnegative functional $I(\nabla) := \text{Tr}^+ (\pi(\theta)^2 |D|^{-n})$ on (universal) connections. Factoring out the unwanted junk $J$ is accomplished by a certain orthogonal projection $P$; if we start from a connection $\tilde{\nabla}$ defined with $D$-forms, we can set
\[
YM(\tilde{\nabla}) := \|P\pi(\theta)\|^2 = \inf \{ I(\nabla) : \pi_D(\nabla) = \tilde{\nabla} \}.\tag{3}
\]

In the commutative Riemannian case, $\tilde{\nabla}$ is given by an ordinary 1-form $\omega$ on $M$, and the trace theorem (2) gives
\[
\|P\pi(\theta)\|^2 = \frac{(2\pi)^{-n/2}}{(n/2)!} \int_M \|d\omega\|^2 \, d\text{vol},
\]
which is the classical Yang–Mills action.
5. Reconstructing the Standard Model

We take, as algebras and Hilbert space for the model:

\[ A := C^\infty(M, \mathbb{R}) \otimes \mathbb{R}(\mathbb{C} \oplus \mathbb{H}) \cong C^\infty(M, \mathbb{C}) \oplus C^\infty(M, \mathbb{H}); \]
\[ B := C^\infty(M, \mathbb{R}) \otimes \mathbb{R}(\mathbb{C} \oplus M_3(\mathbb{C})) \cong C^\infty(M, \mathbb{C}) \oplus M_3(C^\infty(M, \mathbb{C})); \]
\[ \mathcal{H} := L^2(S) \otimes \mathcal{H}_F, \]

where \( \mathcal{H}_F \) is a finite dimensional Hilbert space carrying commuting representations of the “finite-part” algebras

\[ A_F := \mathbb{C} \oplus \mathbb{H}, \quad B_F := \mathbb{C} \oplus M_3(\mathbb{C}). \]

The representation \( \pi \) of \( A \) on \( \mathcal{H} \) decomposes into representations \( \pi_\ell \oplus \pi_q \) on the lepton and quark sectors: \( \mathcal{H} = \mathcal{H}_\ell \oplus \mathcal{H}_q \). Likewise, the representation \( \sigma \) of \( B \) on \( \mathcal{H} \) decomposes into \( \sigma_\ell \oplus \sigma_q \). We take \( \sigma_\ell(\mu, B) := \mu I \) on \( \mathcal{H}_\ell \), for \( (\mu, B) \in B \) (no colouring of leptons), and \( \sigma_q(\mu, B) = \sigma'(B) \), where \( \sigma' \) is a faithful representation of \( M_3(C^\infty(M, \mathbb{C})) \). Thus \( \mathcal{H}_q \) splits as \( \mathcal{H}_q = \mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_1 = \mathcal{H}_1 \otimes \mathbb{C}^3 \). Since \( \pi_q(A) \) must commute with \( \sigma'(B) \), we have \( \pi_q = \pi_1 \otimes I_3 \) where \( \pi_1 \) is a representation of \( A \) on \( \mathcal{H}_1 \). Writing \( \pi_0 = \pi_\ell \), we arrive at

\[ \pi(\lambda, q) = \pi_0(\lambda, q) \oplus \pi_1(\lambda, q) \oplus \pi_1(\lambda, q) \oplus \pi_1(\lambda, q), \quad \text{for } (\lambda, q) \in A. \]

Here \( \pi_0, \pi_1 \) are independent real representations of \( A \).

\( \mathcal{H} \) can be graded so that both \( A \) and \( B \) act by even operators. The grading operator is \( \Gamma := \pi(1, -1) \), so \( \pi(\lambda, q) \) has a block matrix form over \( \mathcal{H} = \mathcal{H}_R \oplus \mathcal{H}_L \).

We take

\[ \pi_0(\lambda, q) := \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta^* & \alpha^* \end{pmatrix} \otimes I_{N_G}, \quad \pi_1(\lambda, q) := \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta^* & \alpha^* \end{pmatrix} \otimes I_{N_G}, \]

where \( N_G \) is the number of particle generations.

The operator \( D \) which gives the \( K \)-cycle must act independently on each of \( \mathcal{H}_\ell \) and \( \mathcal{H}_q \); otherwise, the matrix \( [D, \pi(\lambda, q)] \) will contain cross-terms not commuting with all \( \sigma(\mu, B) \). This condition forces \( D \) to be of the form \( D = D_0 \oplus D_1 \oplus D_1 \oplus D_1 \), where \( D_0, D_1 \) are odd operators on \( \mathcal{H}_0, \mathcal{H}_1 \) respectively.

If we apply this scheme to the “finite-part” algebras only, we retrieve matrix operators \( D_{F0} \) on \( \mathcal{H}_{F0} \) and \( D_{F1} \) on \( \mathcal{H}_{F1} \), of the form

\[ D_{Fj} = \begin{pmatrix} 0 & G_j^\dagger \\ G_j & 0 \end{pmatrix} \]
with respect to the right-left splitting, where $G_0$, $G_1$ are suitable complex matrices. Specifically, we have

$$G_1 = \begin{pmatrix} g_d & 0 \\ 0 & g_u \end{pmatrix}, \quad G_0 = \begin{pmatrix} g_e & 0 \end{pmatrix},$$

where $g_d, g_u, g_e \in M_{N_C}(\mathbb{C})$.

We now take the graded tensor product of the $K$-cycles $(C^\infty(M, \mathbb{R}), L^2(S), \emptyset)$ and $(\mathcal{A}_F, \mathcal{H}_F, D_F)$. The $K$-cycle $(\mathcal{A}, \mathcal{H}, D)$, with $\mathcal{A} := C^\infty(M, \mathbb{R}) \otimes \mathcal{A}_F$, $\mathcal{H} := L^2(S) \otimes \mathcal{H}_F$, is given by:

$$D := (\emptyset \otimes I) \oplus (1 \otimes D_F),$$

and we stipulate that the graded differential algebra $\Omega_D(A)$ be defined as the graded tensor product of algebras:

$$\Omega^*_D(A) := \Omega^*_g(C^\infty(M, \mathbb{R})) \otimes \Omega^*_D(A_F).$$

This amounts to the rule that, for $f \in C^\infty(M, \mathbb{C})$ and $(\lambda, q) \in \mathcal{A}_F$, $c(df) = \gamma^\mu \partial_\mu f$ anticommutes with $\delta(\lambda, q) := [D_F, \pi_F(\lambda, q)] = \begin{pmatrix} 0 & G^\dagger(q - \lambda) \\ (\lambda - q)G & 0 \end{pmatrix}$.

We have $\Omega^0_D(A) \simeq A \simeq C^\infty(M, \mathbb{C}) \oplus C^\infty(M, \mathbb{H})$. Next, $\Omega^1_D(A)$ is generated by elements of the form $(g_0c(df_1), r_0c(dr_1)) + (f_2, r_2) \delta(\lambda, q_1)$, where $f_j \in C^\infty(M, \mathbb{C})$, $r_j \in C^\infty(M, \mathbb{H})$. Schematically, we may write

$$\Omega^1_D(A) = \begin{pmatrix} \mathcal{E}^1(M, \mathbb{C}) & C^\infty(M, \mathbb{H}) \\ C^\infty(M, \mathbb{H}) & \mathcal{E}^1(M, \mathbb{H}) \end{pmatrix},$$

where $\mathcal{E}^k$ denotes (ordinary) $k$-forms. To determine $\pi(\Omega^2(A))$, we notice that

$$\begin{pmatrix} df_1 & G^\dagger s_1 \\ r_1 G & dq_1 \end{pmatrix} \begin{pmatrix} df_2 & G^\dagger s_2 \\ r_2 G & dq_2 \end{pmatrix} = \begin{pmatrix} df_1 \cdot df_2 + G^\dagger s_1 r_2 G & G^\dagger(s_1 dq_2 - df_1 s_2) \\ (dq_1 r_2 - r_1 df_2)G & dq_1 \cdot dq_2 + r_1 G G^\dagger s_2 \end{pmatrix},$$

with the dot denoting Clifford multiplication; the anticommutation rule enables the $df_j$ to slip past the matrices $G$ or $G^\dagger$ with a change of sign.

To find $\Omega^2_D(A)$, we must identify and factor out the junk subspace $J^2$. Two independent scalar terms in this subspace drop from $\mathbf{df}_1 \cdot \mathbf{df}_2$ and $\mathbf{dq}_1 \cdot \mathbf{dq}_2$; another term arises from the relation

$$GG^\dagger \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \otimes (GG^\dagger)_+ + \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \otimes (GG^\dagger)_-,$$

with $(G_0 G_0^\dagger)_\pm = \frac{1}{2}(g_e g_e^\dagger)$. $(G_0 G_1^\dagger)_\pm = \frac{1}{2}(g_d g_d^\dagger \pm g_u g_u^\dagger)$. The “antiquaternionic” second term on the right lives in $J^2$. A full computation shows that the elements of $J^2$ are

$$\begin{pmatrix} \psi \otimes I & 0 \\ 0 & \chi \otimes I + \tau \otimes (GG^\dagger)_- \end{pmatrix}$$
where $\psi, \chi, \tau$ are respectively complex, quaternionic and antiquaternionic-valued functions on $M$. We can identify $\Omega^2_D(A)$ with the orthogonal complement of $J^2$, i.e., we can “subtract off” the junk terms, and express an element of $\pi_D(\Omega^2(A))$ as

$$
\left( df_1 \wedge df_2 + (G^1 s_1 r_2 G) \downarrow (dq_1 r_2 - r_1 df_2) G \quad G^1 (s_1 dq_2 - df_1 s_2) \right),
$$

where the subindex $\perp$ on a matrix indicates that its trace has been subtracted out.

We may express $\Omega^2_D(A)$ schematically as:

$$
\begin{pmatrix}
\mathcal{E}^2(M, \mathbb{C}) \oplus C^\infty(M, \mathbb{H}) & \mathcal{E}^1(M, \mathbb{H}) \\
\mathcal{E}^1(M, \mathbb{H}) & \mathcal{E}^2(M, \mathbb{H}) \oplus C^\infty(M, \mathbb{H})
\end{pmatrix},
$$

with the following multiplication rule for $\Omega^1_D(A) \times \Omega^1_D(A) \to \Omega^2_D(A)$:

$$
\begin{pmatrix}
A_1 & s_1 \\
r_1 & V_1
\end{pmatrix}
\begin{pmatrix}
A_2 & s_2 \\
r_2 & V_2
\end{pmatrix}
= \begin{pmatrix}
A_1 \wedge A_2 \oplus s_1 r_2 & s_1 V_2 - A_1 s_2 \\
V_1 r_2 - r_1 A_2 & V_1 \wedge V_2 \oplus r_1 s_2
\end{pmatrix}.
$$

The differentials $d: \Omega^0_D(A) \to \Omega^1_D(A)$ and $d: \Omega^1_D(A) \to \Omega^2_D(A)$ are given by:

$$d(f, q) := \begin{pmatrix}
df \\
f - q
\end{pmatrix}, \quad d\begin{pmatrix}
A & s \\
r & V
\end{pmatrix} = \begin{pmatrix}
dA \oplus (r + s) & -ds - A + V \\
dr - A + V & dV \oplus (r + s)
\end{pmatrix}.
$$

Similar arguments apply to the algebra $B$; we have $\Omega^0_D(B) \simeq B$ and $\Omega^1_D(B) \simeq \sigma(\Omega^1(B))$. Since $1 \otimes D_F$ commutes with $\sigma(B)$, the only junk arises from the scalar level in Clifford algebra; its removal yields:

$$\Omega^2_D(B) \simeq \mathcal{E}^2(M, \mathbb{C}) \oplus M_3(\mathcal{E}^2(M, \mathbb{C})).$$

We can add skewsymmetric 1-forms $\alpha \in \Omega^1_D(A)$ and $\beta \in \Omega^1_D(B)$ by identifying these modules with their (faithful) representations on $\mathcal{H}$. Thus we consider $\alpha + \beta$ as a connection form. The total curvature is given by

$$\theta = (d\alpha + \alpha^2) + (d\beta + \beta^2) = \theta_\alpha + \theta_\beta,$$

since the cross-terms cancel. Take

$$\alpha = \begin{pmatrix}
A & r^* \\
r & V
\end{pmatrix}, \quad \beta = \begin{pmatrix}
A' & 0 \\
0 & K
\end{pmatrix},$$

where $A, A', V, K$ are antisymmetric 1-forms with respective values in $\mathbb{C}, \mathbb{C}, \mathbb{H}$ and $M_3(\mathbb{C})$.

Reduction of the gauge group from $U(1) \times SU(2) \times U(1) \times U(3)$ to $SU(2) \times U(1) \times SU(3)$ is effected by the algebraic chirality condition:

$$\text{Tr}_{\mathcal{H}_L}(\alpha + \beta) = 0, \quad \text{Tr}_{\mathcal{H}_R}(\alpha + \beta) = 0.$$
Now \( V^* = -V \) means \( V \) is a zero-trace quaternion, so \( \text{Tr}_{H^1}(\alpha) = 0 \) automatically; thus \( \text{Tr}_{H^2}(\beta) = 0 \), which yields the condition \( A' = -(K_{11} + K_{22} + K_{33}) \). Moreover,

\[
\text{Tr}_{H^r}(\alpha + \beta) = N_G(A + A') + 3N_G(A - A) + 2N_G(K_{11} + K_{22} + K_{33}),
\]
on separating the lepton and quark sectors; thus \( A + A' + 2(K_{11} + K_{22} + K_{33}) = 0 \). Combining both conditions, we get the chirality reduction rule:

\[
A = A' = -(K_{11} + K_{22} + K_{33}).
\]

The bosonic Yang–Mills functional (3) for the model may now be computed, yielding the Lagrangian \( \mathcal{L} \) from the trace theorem: \( I(\nabla) = \int_M \mathcal{L} \). The several components of \( \theta \) contribute various terms of the Lagrangian; two-forms on \( M \) yield the pure-gauge part, one-forms yield the kinetic term for the Higgs field and its coupling with the flavour gauge bosons, and functions on \( M \) give the Higgs self-interaction term.

The appearance of the Higgs terms in the Lagrangian happens as follows. We write \( q = 1 + r \) where \( r \) is the quaternionic function in equation (4). It turns out that \( \theta_\alpha \) depends on \( q \) only through the expressions \( Dq = dq - qA + Vq \) (a covariant derivative) and \( (qq^* - 1) \). We then interpret \( q \) as a Higgs doublet:

\[
q = \left( \begin{array}{cc}
\Phi_1 & -\Phi_2^* \\
\Phi_2 & \Phi_1^*
\end{array} \right) = \Phi_1 - \Phi_2^* j.
\]

Then \( \mathcal{L} \) is of the general form: pure-gauge part + \( C_1(D_\mu \Phi)(D^\mu \Phi) + C_0(||\Phi_1||^2 + ||\Phi_2||^2 - 1)^2 \).

Two aspects of this Lagrangian must be remarked. Firstly, the reduction rules affect mainly the coefficients of the pure gauge terms. Secondly, there is some freedom in selecting the exact form of the Dixmier trace one must use. We can use \( \text{Tr}^+ = \alpha_\ell \text{Tr}_\ell^+ + \alpha_q \text{Tr}_q^+ \), with \( \alpha_\ell + \alpha_q = 1 \). These coefficients enter the junk components \( \psi, \chi \) of \( \theta_\alpha \), and thereby enter the Lagrangian in a nonlinear way. The result of this computation has been given by Kastler and Schücker. After identification with the usual notations for the gauge fields, it is:

\[
\mathcal{L} = -N_G(3\alpha_\ell + \frac{11}{3} \alpha_q)F_{\mu\nu}F^{\mu\nu} - N_G(\frac{1}{4} \alpha_\ell + \frac{2}{3} \alpha_q)H_{\mu\nu}^a H^{\mu\nu}_a - N_G\alpha_q G^a_{\mu\nu} G^{\mu\nu}_a \\
+ 2(\alpha_\ell \text{tr}(g^\dagger_\ell g_e) + 3\alpha_q \text{tr}(g^\dagger_d g_d + g^\dagger_u g_u))(D_\mu \Phi)(D^\mu \Phi) + (||\Phi_1||^2 + ||\Phi_2||^2 - 1)^2 \times \\
\times \left[ \frac{3}{2} \alpha_\ell \text{tr}(g^\dagger_\ell g_e)^2 + \frac{9}{2} \alpha_q \text{tr}((g^\dagger_d g_d)^2 + (g^\dagger_u g_u)^2) + 3\alpha_q \text{tr}(g^\dagger_d g_d g^\dagger_u g_u) \\
- \frac{1}{N_G} \left( \frac{1}{\alpha_\ell + 6\alpha_q} + \frac{1}{2\alpha_\ell + 6\alpha_q} \right) (\alpha_\ell \text{tr}(g^\dagger_\ell g_e) + 3\alpha_q \text{tr}(g^\dagger_d g_d + g^\dagger_u g_u))^2 \right].
\]

There seems to be no reason, at present, to take \( \alpha_\ell \) and \( \alpha_q \) different from \( \frac{1}{2} \).
Therefore, NCG suggests values for the masses of undiscovered particles. From the expression (5), we obtain immediately \( m_W = \sqrt{C_1/N_G} \), from which the mass of the top quark is estimated to be \( m_t = 160.4 \) GeV. Also, the mass of the Higgs particle would be given by \( m_H = 2\sqrt{C_0/C_1} \); one gets \( m_H = 251.7 \) GeV.

It has been shown that parameter restrictions like the above (coming from non-commutative geometry models) do not survive quantum corrections.\(^{10}\) On the other hand, if one adopts the point of view that these restrictions are to be interpreted as tree-level constraints, and as such are implemented in a mass-independent scheme at a given energy scale, it is found that the physical predictions on the top and Higgs masses depend fairly weakly on the aforementioned energy scale.\(^{11}\)

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