We introduce simulated packing and cracking as a technique for evaluating partisan-gerrymandering measures. We apply it to historical congressional and legislative elections to evaluate four measures: partisan bias, declination, efficiency gap, and mean-median difference. While the efficiency gap recognizes simulated packing and cracking in a completely predictable manner (a fact that follows immediately from the efficiency gap’s definition) and the declination does a very good job of recording simulated packing and cracking, we conclude that both of the other two measures record it poorly. This deficiency is especially notable given the frequent use of such measures in outlier analyses.

1. INTRODUCTION

A gerrymander is a district plan that (dis)advantages a particular group. Commonly, the group in question is a racial minority, a group of incumbents or a political party. In this article we will focus on cases in which the group is a political party — partisan gerrymanders. The clean separation between the various types of gerrymanders given above is frequently not clear in practice. For example, in Radogno v. Illinois State Board of Elections\(^1\) the 2011 legislative district plan was challenged (unsuccessfully) as both a racial and a partisan gerrymander. Similarly, in Cooper v. Harris, the 2011 North Carolina congressional plan was challenged as a racial gerrymander\(^2\) the resulting remedial plan was then challenged (unsuccessfully) on remand as a partisan gerrymander\(^3\) Regardless, we are still able to ask to what extent a district plan qualifies as a partisan gerrymander, whether or not it might qualify as a gerrymander of another type as well. For the remainder of the article, unless otherwise noted, by “gerrymander” we mean “partisan gerrymander”.

This article is predicated on the notion that quantifying gerrymanders is an important task. A proper discussion justifying this claim is too lengthy for this introduction, so we defer it to Section 2. For similar reasons, we defer an exploration of the various methods for quantifying gerrymanders to Section 3. For this introduction, we restrict ourselves to observing that two general categories of gerrymandering measures have been proposed: “compactness measures” evaluate the shapes of districts while “partisan-asymmetry measures” evaluate the distribution of votes among districts.\(^4\) The main goals of this article are to

\(^1\)Radogno v. Illinois State Board of Elections, NO. 1:11-cv-04884 (N.D. Ill. Oct. 21, 2011).
\(^2\)Cooper v. Harris, No. 15-1262 (S. Ct.).
\(^3\)Cooper v. Harris, No. 16-166 (S. Ct.).
\(^4\)The utilization of computer-generated district plans is symbiotic with both classes of measures. The scant attention we pay to computer-generated plans should not be read as an attempt to minimize their importance.
(1) Introduce a new technique, *simulated packing and cracking (SPC)*\(^5\) for evaluating the ability of the partisan-asymmetry measures to detect gerrymandering and

(2) Apply SPC to four gerrymandering measures using a large collection of historical elections in order to evaluate their ability to record the primary signal of partisan gerrymandering.

Comparisons and evaluations of partisan-asymmetry measures already exist in the literature. We mention some recent instances. In (Stephanopoulos & McGhee, May 2018) the authors explore the extent to which several measures adhere to the *Efficiency Principle*, which is a monotonicity principle relating partisan advantage, seats and votes.\(^6\) In (Best et al., 2017), the authors evaluate a number of measures on two specific elections based primarily on their stability under uniform vote swings. A similar approach to the prior studies is taken in (Nagle, December 2015, 2017), although with different theoretical considerations and different measures considered. The authors in (Campisi et al., Dec 2019) focus on the ability of a specific measure to deal with unequal turnout among districts. In (Warrington, 2019), the author considers how various measures evaluate a number of hypothetical elections. Finally, in (Katz et al., 2020), the authors evaluate measures based on what they contend are universal assumptions.

Unfortunately, there is essentially no agreement among the above analyses as to which measures are the most effective. For some situations, this appears not to matter. For instance, in (Warshaw & Stephanopoulos, 2019) the authors attempt to quantify the political effects of gerrymandering. For the four measures they consider — the same four we consider here — there is general agreement as to the small but measurable effects as averaged over a population of plans. Whatever their strengths and weaknesses, each of the four measures comes to the same general conclusion when considering plans en masse. On the other hand, there are many examples of specific elections for which specific measures disagree strongly (see, e.g., (Warrington, 2019; Best et al., 2017)). The existence of such discrepancies is an important matter to consider more carefully.

Measures have been frequently used to evaluate specific plans, most notably as part of the “(extreme) outlier approach” (see, e.g., (Duchin, 2018; Counsel for Appellees Common Cause, et al, 2019). As we discuss in Section 2, we believe it likely that partisan-asymmetry measures will continue to be applied in the future in situations in which the efficacy of individual (as opposed to aggregate) measures really does matter. This article is an attempt to provide additional evidence regarding which of the individual measures should or should not be used in such contexts. In short, we propose the following criterion as one that should be satisfied by any useful partisan-gerrymandering measure:

**SPC Criterion:** A partisan-gerrymandering measure should reliably detect packing and cracking due to one or more applications of SPC.

The organization of this article is as follows. We begin, in Section 2, with arguments for why it is worthwhile to quantify partisan gerrymandering. Simulated packing and cracking, which consists simply in modifying the vote distributions of an election in a manner consistent with how someone aiming to create a gerrymandering would do so, is introduced in Section 3.3. An analysis of the

\(^5\)McGhee (McGhee, 2014, Figs. 1 and 2) uses the same underlying idea to investigate how the efficiency gap, partisan bias and responsiveness respond to the flipping of seats in a handful of constructed examples. An early version of this technique (applied to the declination only) appears in the preprint (Buzas & Warrington, 2017).

\(^6\)The efficiency principle is closely related to the SPC Criterion introduced below; see (Veomett, 2018) for further explorations of the efficiency principle in the context of the efficiency gap.
various partisan-asymmetry measures appears in Section 4. We close in Sections 5 and 6 with a discussion and summary of our results; we conclude, in part, that the mean-median difference and partisan bias do a very poor job of recognizing the signal of SPC.

2. WHY SHOULD GERRYMANDERS BE QUANTIFIED?

We divide the reasons for attempting to identify gerrymanders into three broad categories. These categories are followed by a brief introduction to the “outlier method”, a natural place to apply partisan-asymmetry measures.

2.1. Litigation. The possible utility of quantitative social science in partisan-gerrymandering litigation was crystallized in *Davis v. Bandemer,* which established that district plans could be challenged as partisan gerrymanders. Quantitative tools were incorporated in successive cases in different ways. For example, in *Whitford v. Gill,* the plaintiffs suggested that raw values of the efficiency gap measure be used as a means of demonstrating discriminatory effect (Peter G. Earle and J. Gerald Hebert and Ruth Greenwood and Annabelle Harless and Danielle Lang and Michele Odorizzi and Nicholas O. Stephanopoulos and Douglas M. Poland, 2016). In *Rucho v. League of Women Voters,* three different measures (partisan bias, mean-median difference, and the efficiency gap) were used to compare the challenged plan to computer-generated plans in order to support vote dilution claims. Ultimately, the US Supreme Court decided in *Rucho v. Common Cause* that partisan gerrymandering claims are non-justiciable, thereby closing off the federal courts to partisan gerrymandering claims. However, the role of this aspect of quantitative social science in such cases in state courts is still undetermined.

In *League of Women Voters of Pennsylvania v. Commonwealth of Pennsylvania,* the plaintiffs used the efficiency gap measure in a manner similar to how it was used in *Whitford v. Gill,* showing the historical plan as an outlier in the context of historical plans. Similarly, in *Common Cause v. Lewis,* computer simulations and the efficiency gap were combined by the plaintiffs in a very similar manner to that used in *Rucho v. Common Cause.*

We expect quantitative measures to continue to be used in future state cases as one way of substantiating the effects of a given district plan. *Exactly* how they will be used remains unclear, and indeed, they may be used in different states in different ways. In some situations it might make sense to use measures to compare a challenged plan to historical plans while in other situations the use of computer-generated plans may be preferable. The importance of such measures in a given case may range from non-existent to central. And for the most extreme gerrymanders, there may be unanimity among all measures considered. In less clear-cut examples the different measures may give very different answers. Regardless, the bar for measures used in the legal context is potentially very high as the ramifications of false positives/negatives may be significant.

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7478 U.S. 109 (1986)
8*Whitford v. Gill,* No. 15-cv-421, F. Supp. 3d (2016).
9*League of Women Voters of North Carolina v. Rucho,* No. 1:16-CV-1164-textscWO-JEP, 2016.
10Petitioner’s Opening Brief (Public Verions), No. 159 MM 2017, Jan. 15, 2018. Available at [https://www.brennancenter.org/sites/default/files/legal-work/LWV_v_PA_Petitioners-Brief.pdf](https://www.brennancenter.org/sites/default/files/legal-work/LWV_v_PA_Petitioners-Brief.pdf).
11Complaint, Wake County Superior Court, No. 18-cv-014001, [https://www.brennancenter.org/sites/default/files/legal-work/Common-Cause-v.-Lewis-Complaint-FILED-Nov-13-2018%20%281%29.pdf](https://www.brennancenter.org/sites/default/files/legal-work/Common-Cause-v.-Lewis-Complaint-FILED-Nov-13-2018%20%281%29.pdf).
2.2. **Evaluating actual and proposed plans.** Generally speaking, partisan asymmetry measures can be used to flag plans as potentially unfair (although one must expect false positives) or fair (although one must expect false negatives). These measures could be applied to maps proposed by a redistricting committee or to maps proposed by citizens. Functionality in this vein is central to the purpose of PlanScore (Migurski et al., 2018). Such measures could be useful in citizen-focused redistricting tools such as Districtr (Hully et al., 2019) as well. In fact, a number of measures are now available in Dave’s Redistricting App (Bradlee et al., 2020).

2.3. **Education and research.** The final category is a catch-all for uses that are essentially independent of the redistricting process. The media may want to use compactness measures to illustrate in a memorable way to their readers the districts with the most contorted boundaries (see, for example, (Ingraham, May 15, 2014)). Researchers may wish to study the evolution of gerrymandering over the decades or to measure the impacts of gerrymandering on political parties such as in (Warnshaw & Stephanopoulos, 2019). The characteristics needed of the measures will vary widely in this category of uses.

2.4. **The outlier method.** The partisan-asymmetry measures we consider in this article all associate a single number to a given election/district plan. There are two basic approaches to interpreting the resulting number. The first, most natural approach is to consider its value in absolute terms. For example, (McGhee & Stephanopoulos, 2015) proposed that in a legal test for gerrymanders, a threshold of 0.08 be set for the efficiency gap for state house plans: plans with an efficiency gap greater than 0.08 in absolute value would be considered gerrymandered. While appealing in its simplicity, this approach leaves no room for confounding factors such as the geographic distribution of voters.

In several recent cases that have been litigated (for example, (League, 2018)), the plaintiffs have focused on how the score on the plan of interest compares to the distribution of scores arising from computer-generated plans. The idea in this approach is that the computer-generated plans incorporate the realities that mapmakers have to take into account when drawing maps. If, say, the distribution of efficiency gap scores is a bell curve with 95 percent of the values between -0.12 and 0.12, then a score of 0.20 is an “extreme outlier”; one would expect to see such a large value arise from a computer-generated plan only rarely. From this point of view, the efficiency gap serves as a marker for putative gerrymanders in the same way that contorted boundaries do. The logic of this approach depends on the relative magnitude of the efficiency gap score (or whatever measure is being used) being a good proxy for the severity of the gerrymander. In other words, high values of measure must be strongly correlated with severe gerrymanders. As we show in Section 4, two of the measures we consider fail this requirement to the extent that they fail to record SPC.

3. **How can gerrymanders be quantified?**

3.1. **Partisan-asymmetry measures studied.** We focus in this article on four measures that have been proposed to quantify partisan gerrymandering:

1. *(Partisan) bias* (Gelman & King, 1994). Bias, compares the predicted share of seats a party would win if they won fifty percent of the statewide vote (determined using a uniform vote swing) to fifty percent of the seats.

2. *Declination* (Warrington, Mar 2018). Dec, is a measure of differential responsiveness derived from the average winning margins of each party and the fractions of seats won.
(3) **Efficiency gap** (McGhee [2014]; McGhee & Stephanopoulos [2015]), EG, measures the relative number of votes “wasted” by each party. In this article we will assume there is equal turnout in each district.

(4) **Mean-median difference** (McDonald & Best, Dec 2015), MM, a standard technique for evaluating the asymmetry of a distribution, encodes the difference between the mean of the Democratic vote shares among districts and the corresponding median value.

Each measure associates a number to an $N$-district election, defined as a weakly increasing sequence $\ell = (\ell_1, \ell_2, \ldots, \ell_N)$ in which $\ell_i$ indicates the Democratic fraction of the two-party (legislative) vote in district $i$. Partisan bias and the efficiency gap associate a number between $-1/2$ and $1/2$ while the other two measures output a number between $-1$ and $1$. It is important to note that the measures we consider invariably output different values from election to election; the “fairness” of each district plan is filtered through the lens of individual elections.

Our choice of measures to consider is guided by two considerations. The efficiency gap, mean-median difference, and partisan bias are the most widely used and discussed of the measures that have been proposed. For example, these are the three measures implemented by PlanScore (Migurski et al., 2018). We include the declination along with these three because, as discussed in (Warrington, 2019), we believe it to be at least as efficacious, if not more so, than these other three measures.

Each of the measures listed above has either minor or major variants that could be considered as well. The partisan bias measure considers the seats allocated to each party at the 50%-vote level on the seats-votes curve. One variant compares the seats that would be won by each party at the observed statewide vote level $V$ as compared to $1 - V$ (see (Katz et al., 2020) for a more extensive description of these interconnected notions). In (Warrington, Mar 2018), a minor variant of the declination is introduced with the goal of attaining a muted response in elections in which one party wins the vast majority of the seats. The original version of the efficiency gap, defined in terms of *wasted votes* applies to elections in which turnout is allowed to vary among districts. See (Veomett, 2018) for a discussion of some of the ramifications of assuming equal turnout for the efficiency gap as we do. See (Nagle, December 2015) for a discussion of the mean-median difference and some related measures.

As mentioned in the introduction, partisan-asymmetry measures are not the only approach to identifying possible gerrymanders. Historically, partisan gerrymanders have been recognized through their contorted geographical boundaries. However, contorted shapes, while often arising as a symptom of gerrymandering, are by no means a necessary characteristic. So while contorted shapes may raise red flags, their absence is not particularly informative. Nonetheless, compactness metrics continue to play a very important role in identifying gerrymanders. They can be used to support allegations of intent by, for example, showing that the challenged plan is significantly less compact than expected based on a suite of computer-generated plans. Perhaps more crucially, the creation of computer-generated district plans requires some means of filtering out plans in which one or more districts is sufficiently contorted — a purpose for which compactness metrics are well suited.

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12To our knowledge, these are the only partisan-asymmetry measures that have played a role in litigation, although for the declination it is only through expert witness testimony in the federal district court cases in Michigan (*League of Women Voters of Michigan v. Benson*, Case 2:17-cv-14148-ELC-DPH-GJQ) and Ohio (*Ohio A. Philip Randolph Institute v. Householder*, Case: 1:18-cv-00357-TSB-KNM-MHW, Plaintiff’s Proposed Finding of Fact, IV.A.5.); the others feature in multiple additional cases.
The reason compactness measures are not a reliable means of detecting gerrymanders is that gerrymanders are created by intentionally allocating the voters of each party to the various districts in an advantageous manner. This may require contorted district boundaries or may not. We now discuss the precise manner by which map drawers create advantageous allocations of voters.

3.2. **Packing and cracking.** The “packing and cracking” used to allocate voters advantageously works as follows. Suppose the Democrats are in control of redistricting and the Republicans are poised to win district $X$. In packing, Republicans are moved from $X$ to other districts in which the Republicans already have enough strength to win. These votes are effectively wasted in the new districts while district $X$ falls to the Democrats. Cracking works similarly, except now the Republicans are spread among districts that they have no chance of winning. Once the cracking occurs, the recipient districts are lost by the Republicans by smaller margins, but they are still lost by them.

In discussing packing and cracking, it is helpful to be able to visualize how the voters of each party are distributed among districts. We visualize the vector of district-level Democratic vote shares, $\ell$, by plotting a point $v_i = (i/N - 1/2N, \ell_i)$ for each $i$. Figure 1 illustrates plots of $\ell$ for three hypothetical elections. Figure 2 illustrates the effect of pro-Republican packing and cracking on the vote distribution illustrated in Figure 1A. Note that in Figure 2B, Republicans have won two additional districts and the Democratic majorities in the three districts they win have increased slightly; in Figure 2C, the Republican majorities are where votes get redistributed to.
While packing and cracking is the general technique by which gerrymanders are created, in reality there will be ambiguities and unknowables. For example, it is unreasonable to suppose that there is a single, well-defined “starting” plan to which the packing and cracking are applied. Certainly the map drawers may use the plan currently in use as a starting point, but they may work from other plans as well. Additionally, there is no reason to suppose that the final map is drawn all in one step. A much more natural approach is to iteratively progress towards a map that balances reward with the map drawers’ tolerance for risk and other priorities.\footnote{See, for example, the sequence of maps proposed for the North Carolina Superior Court, \url{https://www.ncleg.gov/Legislation/SupplementalDocs/2017/H717maps/H717maps}.}

3.3. Simulated packing and cracking. In order to validate the measures described in Section \ref{sec:metrics}, we will examine how perturbations of a vote distribution by packing and/or cracking affect the value of each individual measure. The technique we will use is that of simulated packing and cracking (SPC). In short, we manually modify a vote distribution by packing or cracking so as to flip a single district from one party to the other. There are four possible composite choices: whether we are packing or cracking and whether it is the Republicans or the Democrats who are in charge of the gerrymander. In reality, the votes from the flipped district could be distributed among other districts by a combination of packing and cracking, but we do not attempt to model this. We focus in the below explanation on the case in which the Republicans are flipping a single Democratic district to Republican control using cracking. The other three cases are treated similarly.

In practice, the details of a gerrymander will depend on many factors. One such factor will be the geography of the state. If a given district is being cracked so as to turn it from a Democratic district to a Republican district, the surplus Democratic voters will have to be allocated to adjacent districts. Of course, this process can be iterated by swapping other Democratic voters for Republican voters in second-order neighbors of the original district. Nonetheless, there are obvious geographic constraints that may be significant. But the measures we consider in this article should, if they are effective measures, record this particular packing and cracking regardless of whether or not it geographically feasible. Said another way, while the vote distributions we work with in this article are taken from historical election data (and hence derive from underlying physical district plans), mathematically the geography is irrelevant to the questions we ask.

Another factor that affects the details of a gerrymander is how risk averse the gerrymandering party is. For example, if the Democrats wish to maximize their potential gain in seats (albeit at a high risk of the plan backfiring) they can crack Republican districts by creating districts that are (say) 49\% Republican. On the other hand, if the Democrats feel the political winds will be against them in the upcoming decade, they may prefer to pack Republicans into districts so that the Democratic districts are no more than (say) 35\% Republican.

As stated above, SPC amounts to flipping one district by redistributing some of the voters from the given district. In the default case, we move some of the Democratic voters out from a district so that the Republicans win that district without giving up their hold on any other. To operationalize this idea, we make the following conventions for how the gerrymander is achieved. It is important to note that in some situations it will not be possible to follow the conventions. In such situations, the attempted application of SPC fails.

(1) The district selected to be flipped is the Democratic district that is won by the narrowest margin.

\footnote{We assume in this article that each district has the same number of total votes.}
(2) The gerrymander does not create any new Republican-majority districts with a Democratic vote fraction of greater than 0.45. We choose this value on the basis that a 45–55 split is frequently considered the threshold for a race to be competitive (see, for example, (Abramowitz et al., 2006)). Any Republican-majority district with a Democratic vote fraction higher than this before the cracking is allowed to remain at such a level.

(3) The modified Democratic vote fraction in the flipped district is set to be 0.45.

(4) The Democratic votes shifted from the flipped district are distributed evenly among the Republican-majority districts with a Democratic vote fraction of at most 0.45. In order to avoid violating the Convention 2, this process may need to be iterated (see Example 1 below).

In order to illustrate the method in practice we present the following example of flipping a district from Democratic to Republican by cracking. While we use hypothetical data in this example, all subsequent applications of simulated packing and cracking in this article begin with vote distributions from actual elections.

Example 1. Consider a 10-district election
\[ \ell = (0.37, 0.40, 0.43, 0.46, 0.60, 0.63, 0.66, 0.69, 0.72, 0.75). \]

By Convention 1, we flip the fifth district. By Convention 3, the Democratic vote fraction in this district gets changes from 0.6 to 0.45. In order to maintain the same statewide Democratic vote fraction, there must be a net increase of 0.15 among the first three districts (note that the fourth district is not included since its Democratic vote fraction already exceeds 0.45). Convention 4 instructs us to distribute these Democratic votes evenly among the three districts. The resulting vote distribution is
\[ (0.42, 0.45, 0.48, 0.46, 0.45, 0.63, 0.66, 0.69, 0.72, 0.75). \]

However, following Convention 4, we iterate the process by redistributing the excess fraction of 0.03 = 0.48 – 0.45 from the third district evenly among the first two districts. Since the second district is already at a value of 0.45, the amount is entirely distributed to the first district. This yields a final vote distribution of
\[ \ell^* = (0.45, 0.45, 0.45, 0.46, 0.45, 0.63, 0.66, 0.69, 0.72, 0.75). \]

4. Evaluations of measures using SPC applied to historical elections

We now examine how the values of the measures introduced in Section 3.1 change in response to simulated packing and cracking.

The starting vote distributions, to which SPC will be applied, are taken from two different collections of historical election data, both described in detail in (Warrington, Mar 2018). The first collection consists of congressional elections since 1972 while the second collection consists of elections to state lower-house legislatures since 1972. For each such election we attempt the four possible types of SPC: packing in favor of the Democrats, packing in the favor of the Republicans, cracking in favor of the Democrats and cracking in favor of the Republicans.

Our first exploration is of how the value of each measure changes under a single application of SPC. Ideally, flipping a single seat from (say) Democratic to Republican would lead to a consistent vote distribution.
Figure 3. Plot of change in measures (each scaled by $N$) due to SPC. For each subplot, the 961 red squares represent single applications of pro-Republican SPC to House elections since 1972; the 939 blue circles represent single applications of pro-Democratic SPC. Measures depicted are partisan bias (A), declination (B), efficiency gap (C) and mean-median difference (D). Horizontal and vertical coordinates have been jittered to reduce overlap.

increase in the measure value, regardless of the starting distribution. Empirically, however, there is an inverse dependency on the number of districts $N$.

In fact, it follows from its definition that flipping a single seat for an $N$-district election will lead to a change of exactly $1/N$ in the efficiency gap. While the partisan bias need not change by exactly $1/N$, it will change by a multiple of $1/N$. Empirical data (see [Buzas & Warrington, 2017] as well as the figures in this article) suggest that the declination changes by approximately $1/2N$. Given these relationships, we have chosen to focus on the change in value of each measure multiplied by the number of districts, $N$. This exactly removes the dependency on $N$ for the efficiency gap and partisan bias and appears to approximately remove it for the declination.

Figure 3 displays four plots, one for each of the measures we consider. Each point in each plot corresponds to a single successful (see below) application of SPC to one of the elections in our data set. Red squares denote pro-Republican packing/cracking while blue circles denote pro-Democratic packing/cracking. Packing versus cracking are not distinguished in Figure 3. The horizontal coordinate of each SPC application is given by the number of districts $N$ in the underlying plan to which SPC is being applied; the vertical coordinate is the change in measure, scaled by $N$.

Table 1 contains details about the data of Figure 3 as well as for the analogous figure (not shown) for legislative elections.
Table 1. Success of each measure at moving the expected direction under one application of SPC for House and state-legislative elections.

|       | pro-Democratic |         | pro-Republican |         |
|-------|----------------|---------|----------------|---------|
|       | Fraction correct | Decrease | Increase | Decrease | Increase |
| **House** | | | | | |
| Bias  | 0.531 | 508 | 431 | 460 | 501 |
| Dec   | 0.999 | 939 | 0 | 2 | 959 |
| EG    | 1.000 | 939 | 0 | 0 | 961 |
| MM    | 0.455 | 422 | 517 | 518 | 443 |
| **Legislative** | | | | | |
| Bias  | 0.340 | 538 | 752 | 951 | 338 |
| Dec   | 1.000 | 1290 | 0 | 0 | 1289 |
| EG    | 1.000 | 1290 | 0 | 0 | 1289 |
| MM    | 0.308 | 428 | 862 | 923 | 366 |

The instances of SPC plotted in Figure 3 and the subsequent figures are restricted in two ways. First, we require that each party win at least one seat both before and after the packing/cracking. This is necessary for the declination of each distribution to be defined. Second, we require that there be at least three districts into which to distribute the votes from the flipped district for each application of SPC. For example, when a seat is being flipped from Democratic to Republican by cracking, we require that there be at least three Republican seats in the original distribution. Together, these restrictions exclude all elections from states in which there are four or fewer congressional districts. In the current apportionment cycle there are 21 such states:

AK, AR, DE, HI, IA, ID, KS, ME, MS, MT, ND,
NE, NH, NM, NV, RI, SD, UT, VT, WV, and WY.

There were 676 state-year pairs in which there were at least five Congressional districts and each party won at least one seat. Given the four possible combinations of packing/cracking and pro-Republican/pro-Democratic, this offers 2704 possible applications of SPC. However, for 804 of these, either there was not enough room for the chosen packing/cracking or one of the constraints was not satisfied. Remaining are the 1900 simulations illustrated in each subplot of Figure 3.

Our second exploration is of how the measures change under the flipping of multiple seats. Figure 4 considers the net change after three such flips. Axes are as in Figure 3. In Figure 5, we consider the incremental effects. For example, in Figure 5B we show one line plot for each legislative election considered. For each election, we successively apply SPC, evaluating the resulting vote distribution under the declination after each application. Each election results in up to four line plots, one for each of the four possible combinations of packing/cracking and pro-Republican/pro-Democratic. In subplot B, the pro-Republican line plots are increasing and the pro-Democratic ones are decreasing. This indicates that larger changes in the declination generally correspond to greater numbers of flipped seats under SPC.

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16We have chosen three for the congressional elections as representative of what might happen in a severe gerrymander such as in North Carolina.

17We have only shown the line plots for a sample of 12 legislative elections (about 2% of the data set) in order to avoid clutter.
Figure 4. Plot of net change in measures (each scaled by $N$) due to three applications of SPC. For each subplot, the 540 red squares represent triple applications of pro-Republican SPC to House elections since 1972; the 529 blue circles represent single applications of pro-Democratic SPC. Measures depicted are partisan bias (A), declination (B), efficiency gap (C) and mean-median difference (D). Horizontal and vertical coordinates have been jittered to reduce overlap.

Our third exploration looks at the extent to which our results depend on the implementation choices we have made for SPC. Two fundamental choices we make are how close to parity we allow a modified district to get and how votes from the flipped district are reallocated to the receiving districts. Below we list the alternatives we consider (phrased relative to a pro-Republican SPC).

1. Maximum competitiveness: The maximum Democratic vote share allowed in a modified district.
   (a) $0.45$: Default.
   (b) $0.40$: Increased risk aversion relative to default.
   (c) $0.49$: Decreased risk aversion relative to default.
2. Algorithm: How voters from the flipped district are reallocated to the receiving districts.
   (a) Even: Default. Distribute new Democratic voters evenly among the receiving districts, iterating as necessary.
   (b) Equalization: Distribute new Democratic voters preferentially to the least Democratic districts. This has the effect of equalizing Democratic support among the Republican districts (when cracking) or among Democratic districts (when packing).
   (c) Concentration: Distribute new Democratic voters preferentially to the most Democratic districts. This has the effect of accentuating differences in Democratic support.
FIGURE 5. Plot of incremental change in measures under successive applications of SPC. Measures depicted are partisan bias (A), declination (B), efficiency gap (C) and mean-median difference (D).

among the Republican districts (when cracking) or among Democratic districts (when packing).

When flipping a single seat, each of the nine combinations of choices generates data analogous to that illustrated in Figure 3. As the numerous individual subplots would be difficult to compare even qualitatively, we summarize the results as a scatter plot in Figure 6A. This figure contains 36 data points, one for each of the nine SPC variations paired with each of the four measures we consider. The horizontal axis records the fraction of time the measure changes in the expected direction while the vertical axis records the variation in the change, appropriately scaled. Figure 6B contains analogous data restricted to elections where each party enjoys at least 45% of the vote. We now describe in more detail how these summary statistics are computed.

To determine the horizontal coordinate for a given SPC-measure pair, we determine the fraction of successful applications of that variation of SPC that move the associated measure in the expected direction: Under a pro-Republican packing/cracking, we expect a given measure value to increase in value (i.e., become more positive if initially positive and less negative if initially negative). For example, for the default SPC variation used for Figure 3C, the efficiency gap increases for every single pro-Republican packing/cracking and decreases for every single pro-Democratic packing/cracking. This pair has a horizontal coordinate of 1 in Figure 6. For the declination, the fact that in Figure 3B, 959 out of 961 of the red squares lie strictly above 0 while all 939 of the blue circles lie strictly below 0, yields a horizontal coordinate of 0.999.

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We are not distinguishing in this figure between pro-Republican/pro-Democratic or packing/cracking
Figure 6. Fraction of elections for which each measure moves in the expected direction plotted against the uniformity of the change (as measured by the coefficient of variation) for each variation of the standard SPC algorithm. The sizes of markers denote the corresponding maximum competitiveness as described in the text; the redistribution algorithm used is not displayed graphically. Subplot (A) is based on all House elections in our data set, subplot (B) on those “competitive” ones for which the statewide Democratic votes was between 45% and 55%.

The vertical coordinate quantifies the consistency of each measure in recording packing and cracking, as measured by the coefficient of variation, defined as the sample standard deviation of the measure divided by its sample mean. Dividing by the sample mean accounts for the fact that the scalings of the original measures are essentially arbitrary; this normalization allows us to compare them on an equal footing. As an example, consider again the declination as shown in Figure 3.B. The standard deviation of the 1900 data points is 0.428 while the corresponding mean is 2.36. The ratio of 0.428/2.36 yields a coefficient of variation of 0.181.

In Figure 6, color and shape denote measure as indicated in legend. Small, medium and large markers denote maximum competitiveness of 0.40, 0.45 and 0.49, respectively. The redistribution algorithm used is not indicated graphically.

5. Discussion

Our starting point for this article is that any effective partisan-asymmetry measure should be able to consistently recognize packing and cracking, the accepted means by which partisan-gerrymanders are created. As a proxy for recognizing packing and cracking in historical election data, which can be difficult to separate from social and political factors, we have proposed the SPC Criterion. Below we explain our understanding of why the measures behave as shown in this article under applications of SPC. For clarity of exposition, we describe the behavior in the context of pro-Republican applications of SPC.

Efficiency gap: As illustrated in Figures 3, 4 and 5, the efficiency gap captures each seat flip with a change of exactly $1/N$. It is important to note here that this depends in part on our assumption of equal turnout in each district: As shown in McGhee (2014), if $V$ denotes the statewide Democratic vote, the efficiency gap is $1/N V$. In these computations, we multiply the differences for all the pro-Democratic applications of SPC by $-1$ so that the expected differences are always positive.

For example, “For packing and cracking are the ways in which a partisan gerrymander dilutes votes,” Gill v. Whitford, concurring opinion (J. Kagan), https://www.law.cornell.edu/supct/pdf/16-1161.pdf, page 4.
vote share and $S$ denotes the fraction of Democratic seats, then the value of the efficiency gap reduces to $(S - 1/2) - 2(V - 1/2)$. Since SPC does not change $V$, flipping a seat (i.e., changing $S$ by $\pm 1/N$) changes the efficiency gap in a completely predictable manner. This is arguably the ideal outcome. If turnout is not assumed to be equal and one computes the efficiency gap according to the more general notion of wasted votes (see (McGhee, 2014)), its value will likely change by amounts close to $1/N$, but not exactly equal to it.

Because of the strict relationship among seats, votes and the value of the (equal-turnout) efficiency gap, SPC does not actually provide any new insights in this case. However, we have included the efficiency gap in this article for two reasons. First, it has been the most widely used measure over the past several years. Second, its behavior illustrates any arguably ideal way a measure should behave under SPC. It therefore serves as a useful benchmark for the evaluating the performance of the other measures considered.

Declination: As proved in Theorem 1 of (Warrington, Mar 2018), the declination will increase under a pro-Republican application of SPC as long as the modified Democratic vote fraction in the flipped district is greater than or equal to the average Democratic vote fraction in the other Republican districts. That this is the case follows from the geometric definition of the declination as an angle along with three facts. First, that we are flipping the most competitive (Democratic) district. Second, that the average Democratic vote share will stay the same or increase in both the Democratic districts and the Republican districts. And third, that the Republicans gain one seat.

The declination does a very good job of responding to SPC. In Figures 3 and 4 it shows a almost entirely clean separation between positive values for pro-Republican packing/cracking and negative values for pro-Democratic packing/cracking (note the two negative values for pro-Republican packing/cracking in Figure 3.B). And when change in declination is considered incrementally as in Figure 5, we see that it increases in a regular manner for the sample shown although at slightly different rates for different individual elections.

Having discussed both declination and efficiency gap individually, it is useful to contrast their behaviour. The efficiency gap changes by a fixed, predictable amount for every seat flip. The declination changes consistently, but with some variation. Delving into the pros and cons of each measure is beyond the scope of this article. However, we do wish to point out that an argument can be made that a measure should change by varying amounts depending on the particulars of how the vote distribution has changed. As just one example, consider two applications of pro-Republican SPC that differ only in whether the redistributed voters are packed or cracked. Under cracking, these voters would be redistributed to Republican districts, thereby making these districts slightly more Democratic. If these districts are already reasonably competitive, the addition of Democratic voters might tip them over into being highly competitive. In such a scenario, it might be better for the Republicans to redistribute these voters through packing to districts that are already safely Democratic. The latter case could lead to a “safer” gerrymander that could reasonably be measured as being more advantageous to the Republicans than the former version.

Partisan bias: The partisan bias measure compares the seat share each party would earn were the statewide vote shifted to 50%. Suppose, as above that the statewide Democratic vote share is

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21If we flip the second-most competitive district, the fraction of House elections for which the declination moves in the expected direction decreases from 0.999 (see Table 1) to 0.989.
22This is true regardless of whether voters in the flipped district are redistributed through packing or through cracking.
23This is true qualitatively for the entire data set as well.
The value of partisan bias is then determined by the fraction of districts with a Democratic vote share of less than \( V \). When SPC is applied, the change in bias is therefore determined by the fraction of districts whose Democratic vote share switches from one side of \( V \) to the other. If the state is evenly split, the flipped district will typically flip from above \( V \) to below, increasing the bias by \( 1/N \). The increased likelihood of this with decreased values of the maximum competitiveness can be seen in Figure 6.A. In this figure, the sizes of the squares indicate the final Democratic vote share in the flipped district. Each of the large, medium and small squares are cleanly separated from the other two groups. Furthermore, the smaller the square (and hence the greater the swing in the flipped-district Democratic vote share) the more likely the partisan bias increases.

The redistribution of votes can cause other districts to move back across \( V \) in the opposite direction. This accounts for the applications of SPC illustrated in Figure 3.A that move in the wrong direction. As a concrete example consider a seven-district election with Democratic votes shares of 0.203, 0.294, 0.347, 0.394, 0.417, 0.607 and 0.672. The statewide average is 0.419. When the statewide average is shifted to 0.5, the number of Republican districts remains at five. Under a pro-Republican, cracking application of SPC, the original district values become 0.234, 0.325, 0.378, 0.425, 0.448, 0.450 and 0.672. As the statewide average doesn’t change, only three Republican districts remain when the statewide average is shifted to 0.5, leading to a change in partisan bias of \(-2/7\).

Mean-median difference:

When SPC is applied to a vote distribution, the statewide Democratic vote share does not change. Any change to the measure therefore results from a new value for the Democratic vote share in the median district. While its value often increases as expected, as seen in Figure 3.D it frequently decreases, though only by small amounts. This can happen when some of the Democratic voters moved out of the flipped district are redistributed to the median district. The same figure also shows high variability when it does move in the expected direction.

Figure 6 depicts two additional facets of measures under SPC. First, the vertical coordinate is a measure of how uniform the change to a given measure is under SPC as the underlying election varies. In general, lower coefficients of variation are preferable. Second this figure summarizes how dependent the results for each measure are on the structure of the SPC model. As shown by the tight clustering for the declination and efficiency gap, these two measures are insensitive to the implementation details of SPC. The clouds for partisan bias and mean-median difference show there is a reasonable amount of dependency, yet these two measures do not do particularly well for any of the SPC variations.

One commonality between the efficiency gap and the declination is that they are averaging vote shares over districts. This explains the stability for these measures illustrated in Figure 6 under varying the redistribution algorithm. These measures are, of course, still sensitive to other features such as the fraction of seats won by each party. Partisan bias and the mean-median difference are more sensitive to what happens in specific districts; this feature appears to be responsible at least in part to the lower success rates of these measures.

6. Conclusion

We have introduced simulated packing and cracking as a technique for examining how partisan-gerrymandering measures respond to the type of vote-distribution modifications that occur in partisan gerrymandering. Partisan bias and mean-median difference are unable to consistently record
simulated packing and cracking, even when restricted to competitive elections, while the declina-
tion and (by definition) the efficiency gap do well. As a result, we recommend that neither partisan
bias nor the mean-median difference be used for the “outlier” or “ensemble” method, where it is
crucial that more extreme values of the measure indicate more extreme levels of partisan gerry-
mandering.

7. ACKNOWLEDGMENTS

The US congressional data through 2014 was provided by (Jacobson 2017). The election data
was analyzed using the python-based SageMath (Stein et al. 2016). See (Warrington Mar 2018)
for packages used for, and details of, the imputation of votes. Additional Python packages em-
ployed in this article were Matplotlib (Hunter 2007) for plotting and visualization and SciPy (Jones
et al. 2001–) for statistical methods. All non-library code may be found at (Warrington 2020).
We thank Jordan Ellenberg for suggesting one of the SPC redistribution algorithms for (Buzas &
Warrington 2017) and to Eric McGhee, John Nagle, Nick Stephanopoulos and Ellen Veomett for
helpful comments on a draft of this article.

REFERENCES

Abramowitz, A. I., Alexander, B., & Gunning, M. (2006). Incumbency, redistricting, and the
decline of competition in U.S. House elections. Journal of Politics, 68(1), 75–88.
URL http://dx.doi.org/10.1111/j.1468-2508.2006.00371.x

Best, R. E., Donahue, S. J., Krasno, J., Magleby, D. B., & McDonald, M. D. (2017). Considering
the prospects for establishing a packing gerrymandering standard. Elect. Law J., 17(1).

Bradlee, D., Crowley, T., & Ramsay, A. (2020). Dave’s redistricting app.
URL http://davesredistricting.org/

Buzas, J. S., & Warrington, G. S. (2017). Gerrymandering and the net number of US House
seats won due to vote-distribution asymmetries. ArXiv e-prints, http://arxiv.org/abs/1707.08681v2.

Campisi, M., Padilla, A., Ratliff, T. C., & Veomett, E. (Dec 2019). Declination as a metric to detect
partisan gerrymandering. Election Law Journal, 18(4), 371–387.

Counsel for Appellees Common Cause, et al (2019). Brief for Common Cause appellees. (No. 18-
422). Available at https://www.brennancenter.org/our-work/court-cases/rucho-v-common-cause

Duchin, M. (2018). Outlier analysis for pennsylvania congressional redistricting. Available at https://www.governor.pa.gov/wp-content/uploads/2018/02/md-report.pdf

Gelman, A., & King, G. (1994). Enhancing democracy through legislative redistricting. Am.
Political Sci. Rev., 88, 541–559.

Hully, M., Buck, R., Pizzimenti, A., Schall, A., & Blanchard, E. (2019). Districtr.
URL https://districtr.org

Hunter, J. D. (2007). Matplotlib: A 2D graphics environment. Computing In Science & Engineer-
ing, 9(3), 90–95.

Ingramah, C. (May 15, 2014). Wonkblog: America’s most gerrymandered congressional districts.
The Washington Post. Accessed online, February 07, 2020.
URL https://www.washingtonpost.com/news/wonk/wp/2014/05/15/
Jacobson, G. C. (2017). Private communication.

Jones, E., Oliphant, T., Peterson, P., et al. (2001–). SciPy: Open source scientific tools for Python. Accessed online, February 07, 2020.
URL http://www.scipy.org/

Katz, J. N., King, G., & Rosenblatt, E. (2020). Theoretical foundations and empirical evaluations of partisan fairness in district-based democracies. American Political Science Review, 114, 164–178.

League, 2018 (2018). League of Women Voters of Pennsylvania v. The Commonwealth of Pennsylvania, 178 A.3d 737 (2018).

McDonald, M. D., & Best, R. E. (Dec 2015). Unfair partisan gerrymanders in politics and law: A diagnostic applied to six cases. Elect. Law J., 14(4), 312–330.

McGhee, E. (2014). Measuring partisan bias in single-member district electoral systems. Legis. Stud. Q., 39, 55–85.

McGhee, E., & Stephanopoulos, N. (2015). Partisan gerrymandering and the efficiency gap. 82 University of Chicago Law Review, 831. 70 pages. U of Chicago, Public Law working Paper No. 493. Available at SSRN: https://ssrn.com/abstract=2457468.

Migurski, M., McGhee, E., Jackman, S., Stephanopoulos, N., & Greenwood, R. (2018). Planscore. Accessed online Feb 07, 2020.
URL http://www.planscore.org

Nagle, J. F. (2017). How competitive should a fair single member districting plan be? Elect. Law J., 16(1), 196–209.

Nagle, J. F. (December 2015). Measures of partisan bias for legislating fair elections. Election Law Journal, 14(4), 346–360.

Peter G. Earle and J. Gerald Hebert and Ruth Greenwood and Annabelle Harless and Danielle Lang and Michele Odorizzi and Nicholas O. Stephanopoulos and Douglas M. Poland (2016). Plaintiff’s trial brief. (No. 15-cv-421-bc). Available at https://www.brennancenter.org/sites/default/files/legal-work/Whitford-PlaintiffsTrialBrief051616.pdf

Stein, W., et al. (2016). Sage Mathematics Software (Version 7.1). The Sage Development Team. http://www.sagemath.org.

Stephanopoulos, N. O., & McGhee, E. M. (May 2018). The measure of a metric: The debate over quantifying partisan gerrymandering. Stanford Law Review, 70(5), 1503–1568.
URL https://www.stanfordlawreview.org/print/article/the-measure-of-a-metric/

Veomett, E. (2018). The efficiency gap, voter turnout, and the efficiency principle. ArXiv e-prints. http://arxiv.org/abs/1801.05301

Warrington, G. S. (2019). A comparison of partisan-gerrymandering measures. Election Law Journal, 18, 262–281.

Warrington, G. S. (2020). Election data and computer code. http://www.cems.uvm.edu/~gswarrin/gerrymandering/ [Online; accessed 18-June-2020].

Warrington, G. S. (Mar 2018). Quantifying gerrymandering using the vote distribution. Election Law Journal, 17(1).
Warshaw, C., & Stephanopoulos, N. (2019). The impact of partisan gerrymandering on political parties. Available at SSRN: https://ssrn.com/abstract=3330695

DEPARTMENT OF MATHEMATICS & STATISTICS, UNIVERSITY OF VERMONT, 16 COLCHESTER AVE., BURLINGTON, VT 05401, USA
E-mail address: jeff.buzas@uvm.edu

DEPARTMENT OF MATHEMATICS & STATISTICS, UNIVERSITY OF VERMONT, 16 COLCHESTER AVE., BURLINGTON, VT 05401, USA
E-mail address: gswarrin@uvm.edu