Classical lifting processes and multiplicative vector fields *

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Abstract

We extend the calculus of multiplicative vector fields and differential forms and their intrinsic derivatives from Lie groups to Lie groupoids; this generalization turns out to include also the classical process of complete lifting from arbitrary manifolds to tangent and cotangent bundles. Using this calculus we give a new description of the Lie bialgebroid structure associated with a Poisson groupoid.

1 Introduction

Multiplicative multivector fields and forms play an important technical role in Poisson group theory [10], [18]: most basically, the Poisson tensor is itself multiplicative and its intrinsic derivative gives the bracket on the Lie algebra dual. If one extends this approach to Poisson groupoids—an extension we did not contemplate in [14], [15]—it turns out to also generalize the classical concept of complete lift.

Complete and vertical lifts of vector fields and differential forms from a manifold to its tangent and cotangent bundles are dealt with comprehensively in the work of Yano and Ishihara [20]. The existence of the complete lifting processes is one of the fundamental features which set the geometry

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of tangent and cotangent bundles apart from that of general vector and principal bundles, and it is consequently reasonable to expect that it extend to Lie algebroids and their duals. In fact it extends only when the Lie algebroid is integrable: the complete lift of a vector field on a manifold \( M \) to \( TM \) is effectively the intrinsic derivative of an associated multiplicative vector field on the pair groupoid \( M \times M \). In general, a multiplicative vector field on a groupoid \( G \) induces vector fields on its Lie algebroid \( AG \) and its dual \( A^*G \). We characterize the former in terms of the Lie algebroid structure of \( TAG \to TM \), and the latter in terms of the Poisson structure on \( A^*G \) (Theorem \([4,4]\)). Multiplicative forms on \( G \) likewise induce forms on \( AG \) and \( A^*G \); in this paper, where we are concerned with applications to Poisson groupoids, we deal only with 1-forms.

The importance of these processes is that, by slightly generalizing the notion of multiplicative vector field, and incorporating the vertical lifts, we obtain a complete description, well adapted to the geometry of the groupoid, of the vector fields on the Lie algebroid of a Lie groupoid. In particular, we give in \( \S 7 \) a new description of the Poisson structure on the Lie algebroid of a Poisson groupoid, and thus of the Lie algebroid structure on the dual.

We begin in \( \S 2 \) with the case of vector fields on a vector bundle \((A,q,M)\). Here it is natural to consider those vector fields \( X:A \to TA \) which are vector bundle morphisms with respect to the vector bundle structure of \( TA \) over \( TM \). Such linear vector fields correspond to covariant differential operators \( D:GA \to GA \) in the same way that, given a connection \( \nabla \) in \( A \), the horizontal lift of a vector field \( x \) on \( M \) corresponds to \( \nabla_x:GA \to GA \). Secondly, any section \( X \) of \( A \) induces a vector field \( X^\uparrow \) on \( A \); if \( A = TM \) then \( X^\uparrow \) is the vertical lift of \( X \); in the general case we call it the core vector field corresponding to \( X \). The core vector fields and the linear vector fields together generate \( \mathcal{X}(A) \). We believe that the material of this section is mostly folklore.

In \( \S 3 \) we give a similar calculus on general Lie groupoids. Given a multiplicative vector field \( \xi \) on a groupoid \( G \to M \), application of the Lie functor leads to a linear vector field \( \xi \) on \( AG \) for which the corresponding differential operator \( D_\xi \) is a derivation of the Lie algebroid bracket. In the case where \( G \) is a pair groupoid \( M \times M \), the process \( \xi \mapsto \xi \) is the complete lifting of \( [20] \). For a Lie group \( G \) it is linearization at the identity. In order to obtain a complete description of the vector fields on \( AG \) we weaken the multiplicativity condition to what we call a star vector field; for any Lie groupoid \( G \) the \( \bar{\xi} \), for \( \xi \) a star vector field, and the core vector fields \( X^\uparrow \), for \( X \in \Gamma AG \), generate \( \mathcal{X}(AG) \).

For any vector bundle \( A \), there is a bijective correspondence between the linear vector fields on \( A \) and the linear vector fields on the dual \( A^* \). Thus for a Lie algebroid dual \( A^*G \) we obtain in \( \S 4 \) a description of the vector fields on \( A^*G \) in terms of the “duals” of the \( \bar{\xi} \), for \( \xi \) a star vector field on \( G \), and the core vector fields \( \varphi^\uparrow \) for \( \varphi \in \Gamma A^*G \). More generally, we show in Theorem \([4,4]\) that for any abstract Lie algebroid \( A \), a linear vector field \( \xi \) is a Lie algebroid morphism \( A \to TA \) if and only if the corresponding vector field on \( A^* \) is Poisson (with respect to the dual Poisson structure on \( A^* \)), and this is so if and only if \( D_\xi \) is a derivation of the Lie algebroid bracket.

In \( \S 5 \) and \( \S 6 \) we give a comparable analysis for differential 1–forms on Lie algebroids and their duals.

For a Poisson groupoid \( G \rightarrow P \), the 1–forms have a bracket structure reflecting the fact that \( T^*G \to G \) is a Lie algebroid. In the final \( \S 7 \) we show that 1–forms on a Poisson groupoid admit a calculus similar to that of \( \S 3 \), with the differential operators on \( AG \) now replaced by operators \( \Omega^1(P) \to \Omega^1(P) \); for example, if \( \Phi \in \Omega^1(G) \) is multiplicative, then \( D_\Phi \) is a derivation of the Poisson
bracket on $\Omega^1(P)$. In these terms we obtain in Theorem 7.3 a complete description of the bracket structure on $\Omega^1(G)$.

It is worth noting that the treatment we give here is entirely coordinate free, and so may offer something new even in the classical case. In the early stages of the work we were unaware of [20] and our approach was chiefly influenced by the few brief remarks in [3] and [1].

Some of the material of this paper was announced in [13].

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2 Linear vector fields

We first need some preliminaries concerning linear vector fields on vector bundles. These are, in effect, homogeneous vector fields of degree 1 in the sense of [3], but the point of view we adopt here will be important in what follows. Consider a fixed vector bundle $(A,q,M)$. Recall from [14] and references given there the tangent double vector bundle

$$
\begin{array}{c}
TA \\
p_A \\
A
\end{array} \xrightarrow{T(q)} \begin{array}{c}
T(q) \\
p \\
q
\end{array} \xrightarrow{p} TM
$$

(1)

We recall the notation of [14]. In $A$ and $TM$, we use standard notation, with the zero of $A$ over $m \in M$ being $0^A_m$, and the zero of $TM$ over $m$ being $0^T_m$.

We denote elements of $TA$ by $\xi, \eta, \zeta \ldots$, and we write $(\xi; X, x; m)$ to indicate that $X = p_A(\xi)$, $x = T(q)(\xi)$, and $m = p(T(q)(\xi)) = q(p_A(\xi))$. With respect to the tangent bundle structure $(TA, p_A, A)$, we use standard notation: + for addition, $-$ for subtraction and juxtaposition for scalar multiplication. The notation $T_X(A)$ will always denote the fibre $p^{-1}_A(X)$, for $X \in A$, with respect to this bundle. The zero element in $T_X(A)$ is denoted $0_X$. We refer to this bundle structure as the $p_A$-bundle structure.

With respect to the $T(q)$-bundle structure, $(TA, T(q), TM)$, we use $+$ for addition, $-$ for subtraction, and \cdot for scalar multiplication. This addition and scalar multiplication on $TA$ are precisely the tangents of the addition and scalar multiplication in $A$. The fibre over $x \in TM$ will always be denoted $T(q)^{-1}(x)$, and the zero element of this fibre is $T(0)(x)$.

For each $m \in M$, the tangent space $T_{0^m}(A_m)$ identifies canonically with $A_m$; we denote the element of $T_{0^m}(A_m)$ corresponding to $X \in A_m$ by $\f X$. The elements $\f X$, $X \in A$, form the core of $TA$. Note that, for $X, Y \in A_m$ and $t \in \mathbb{R}$,

$$
\f X + \f Y = \f X + \f Y = \f X + \f Y, \quad t\f X = t\f X = t \cdot \f X.
$$
Given a morphism of vector bundles \( \varphi: A' \to A, \ f: M' \to M \), we denote the pullback of \( A \) across \( f \) by \( f^! A \), and the induced morphism \( A' \to f^! A \) over \( M' \) by \( \varphi^! \). Associated to the double vector bundle structure on \( TA \) are the two core sequences:

\[
\begin{align*}
q^! A & \xrightarrow{\tau} TA \xrightarrow{T(q)^!} q^! TM, \\
p^! A & \xrightarrow{\nu} TA \xrightarrow{p^!} p^! A,
\end{align*}
\]

over \( A \) and \( TM \) respectively. Here \( \tau \) and \( \nu \) are the maps \( \tau(X,Y) = \tilde{0}X + Y \), \( \nu(x,Y) = T(0)(x) + Y \); we call \( \tau \) and \( \nu \) the translation maps. \( \tau \) assigns to \((X,Y) \in A_m \times A_m \) the element of \( TX(A_m) \) which has its tail at \( X \), and is parallel to \( Y \).

Given \( X \in \Gamma(A) \) define a vector field \( X^\uparrow \) on \( A \) by \( X^\uparrow(Y) = \tau(Y,X(qY)) \), \( Y \in A \). (In \cite{14} we used the notation \( \tilde{X} \).) Then

\[
X^\uparrow(F)(Y) = \left. \frac{d}{dt} F(Y + tX(qY)) \right|_0
\]

for \( F \in C^\infty(A), \ Y \in A \), and so

\[
X^\uparrow(f \circ q) = 0, \quad X^\uparrow(\ell_\varphi) = \langle \varphi, X \rangle \circ q, \quad [X^\uparrow, Y^\uparrow] = 0,
\]

for \( f \in C^\infty(M), \ X,Y \in \Gamma(A), \ \varphi \in \Gamma(A^*) \). Here \( \ell_\varphi \in C^\infty(A) \) is the fibrewise linear function determined by \( \varphi \), namely \( X \mapsto \langle \varphi(qX), X \rangle \). Note also that \( (fX)^\uparrow = (f \circ q)X^\uparrow \). We call \( X^\uparrow \) the core vector field corresponding to \( X \in \Gamma A \); if \( A = TM \), it is the vertical lift of \( X \).

A section \( X \in \Gamma(A) \) also induces a section \( \tilde{X} \) of \( T(q) \) by \( \tilde{X}(x) = \nu(x,X(px)) \). Note that

\[
(X + Y)\ = \tilde{X} + \tilde{Y}, \quad (fX)\ = (f \circ p) \cdot \tilde{X},
\]

for \( X,Y \in \Gamma(A), f \in C^\infty(M) \).

**Definition 2.1** A linear vector field on \( A \) is a pair \((\xi,x)\), where \( \xi \) is a vector field on \( A \), and \( x \) is a vector field on \( M \), such that

\[
\begin{array}{ccc}
A & \xrightarrow{\xi} & TA \\
q & \downarrow & T(q) \\
M & \xrightarrow{x} & TM,
\end{array}
\]

is a morphism of vector bundles.

In particular, \( \xi \) projects under \( q \) to \( x \), and

\[
\xi(X + Y) = \xi(X) + \xi(Y), \quad \xi(tX) = t \cdot \xi(X),
\]

Proposition 2.2 Let \( A \) on \( \ell \) fibrewise linear functions

maps \( C \). From this, and the similar result for scalar multiplication, it follows that a linear vector field

Proof: (i)\( \Rightarrow \) (ii). Suppose first that \( \xi, \eta \in T(A) \) have \( T(q)(\xi) = T(q)(\eta) \), and that \( F \in C^\infty(A) \). Write \( \xi = \frac{d}{dt}X_t \big|_0 \), \( \eta = \frac{d}{dt}Y_t \big|_0 \), where \( q(X_t) = q(Y_t) \) for \( t \) close to \( 0 \in \mathbb{R} \). Then

\[
(\xi + \eta)(F)(X_0 + Y_0) = \frac{d}{dt}F(X_t + Y_t) \bigg|_0 = \frac{d}{dt}F(X_t) \bigg|_0 + \frac{d}{dt}F(Y_t) \bigg|_0 = \xi(F)(X_0) + \eta(F)(Y_0).
\]

From this, and the similar result for scalar multiplication, it follows that a linear vector field \( \xi \) maps \( C^\infty(A) \) into \( C^\infty(A) \). Since \( \xi \) is projectable under \( q \), it is clear that \( \xi \) maps \( q^*C^\infty(M) \) into \( q^*C^\infty(M) \).

(ii)\( \Rightarrow \) (iii). From the assumption that \( \xi \) sends \( q^*C^\infty(M) \) into itself, and a standard result, it follows that \( \xi \) projects to a vector field \( x \) on \( M \), and hence \( q \circ \varphi_t = f_t \circ q \) for all \( t \), where \( \varphi_t \) and \( f_t \) are flows for \( \xi \) and \( x \) respectively. It remains to prove that the \( \varphi_t \) are linear and since this is a local question, we may assume that \( A = M \times V \) and that \( \xi \) and \( x \) have global flows. The details are straightforward.

(iii)\( \Rightarrow \) (i). Suppose, for simplicity, that \( \xi \) has a global flow \( \varphi_t \) by vector bundle morphisms over a global flow \( f_t \) on \( M \). Then \( \xi \) certainly projects under \( q \) to the vector field \( x \) generated by \( f_t \). For \( X, Y \in A \) with \( q(X) = q(Y) \),

\[
\xi(X + Y) = T(t \mapsto \varphi_t(X + Y))_0(1) = T(t \mapsto \varphi_t(X) + \varphi_t(Y))_0(1) = T(+(T(t \mapsto \varphi_t(X))_0(1), T(t \mapsto \varphi_t(Y))_0(1)) = T(+(\xi(X), \xi(Y)) = \xi(X) + \xi(Y),
\]

and similarly one proves that \( \xi(tX) = t \cdot \xi(X) \) for \( t \in \mathbb{R} \).
Corollary 2.3 If \((\xi, x)\) and \((\eta, y)\) are linear vector fields, then \(\langle [\xi, \eta], [x, y] \rangle\) is also.

Now consider a linear vector field \((\xi, x)\). Since \(C^\infty_{\text{lin}}(A)\) is canonically isomorphic (as a \(C^\infty(M)\) module) to \(\Gamma A^*\), condition (ii) above shows that \(\xi\) induces a map

\[ D^{(s)}_\xi: \Gamma A^* \to \Gamma A^*, \quad \text{such that} \quad \xi D^{(s)}_\xi(\varphi) = \xi(\varphi). \tag{5} \]

Clearly, \(D^{(s)}_\xi\) is additive, and it is easily checked that \(D^{(s)}_\xi(f \varphi) = f D^{(s)}_\xi(\varphi) + x(f)\varphi\) for \(f \in C^\infty(M)\). Thus \(D^{(s)}_\xi\) is a covariant differential operator on \(A^*\) ([11, III §2]), and we have a map, which is linear over \(C^\infty(M)\) and bracket-preserving,

\[ (\xi, x) \mapsto D^{(s)}_\xi, \quad \Gamma^{LIN}TA \to \Gamma \text{CDO}(A^*), \tag{6} \]

where \(\Gamma^{LIN}TA\) is the set of linear vector fields on \(A\) and \(\text{CDO}(A^*)\) is the vector bundle on \(M\) whose sections are \(\mathbb{R}\)-linear maps \(D: \Gamma A^* \to \Gamma A^*\) such that \(D(f \varphi) = f D(\varphi) + x(f)\varphi\) for all \(f \in C^\infty(M)\), \(\varphi \in \Gamma A^*\), and some fixed \(x \in \mathcal{X}(M)\). With respect to the commutator bracket, \(\text{CDO}(A^*)\) is a Lie algebroid on \(M\) with anchor \(D \mapsto x\).

By Corollary 2.3, \(\Gamma^{LIN}TA\) is closed under the bracket on \(\mathcal{X}(A)\). Let \(a\) denote the map \((\xi, x) \mapsto x\). Then the kernel of \(a\) consists of those vertical vector fields on \(A\) which are linear. Any vertical vector field \(\xi\) can be written as \(\xi(X) = \tau(X, X(X))\), where \(X: A \to A\) has \(q \circ X = X\), and \(\xi\) is linear iff \(X\) is a vector bundle morphism. Thus the kernel of \(a\) can be identified with \(\Gamma \text{End}(A)\). Since this is the module of sections of a vector bundle over \(M\), it follows that \(\Gamma^{LIN}TA\) is also the module of sections of a vector bundle on \(M\); we denote this by \(T^{LIN}A\). We can now regard \(\xi\) as a morphism of Lie algebroids from \(T^{LIN}A\) to \(\text{CDO}(A^*)\).

Proposition 2.4 The morphism \(T^{LIN}A \to \text{CDO}(A^*)\) just defined is an isomorphism of Lie algebroids.

Proof: It suffices to prove (see [11, III 2.8]) that the restriction of this map to the kernels of the anchor maps (the adjoint bundles) is an isomorphism, and this restriction is the canonical identification of \(\text{End}(A)\) with \(\text{End}(A^*)\).

Now each covariant differential operator \(D^{(s)}\) on \(A^*\) corresponds to a covariant differential operator \(D\) on \(A\) by

\[ \langle D^{(s)}(\varphi), X \rangle = a(D)(\langle \varphi, X \rangle) - \langle \varphi, D(X) \rangle, \]

where \(\varphi \in \Gamma A^*\), \(X \in \Gamma A\). This defines an isomorphism of Lie algebroids \(\text{CDO}(A) \cong \text{CDO}(A^*)\). In sum, we have obtained a correspondence between linear vector fields on \(A\), and covariant differential operators on either \(A\) or \(A^*\). Letting \(D_\xi\) denote the element of \(\Gamma \text{CDO}(A)\) corresponding to \(D^{(s)}_\xi\), we have

\[ \langle \varphi, D_\xi(X) \rangle = x(\langle \varphi, X \rangle) - \xi(\ell_\varphi) \circ X. \tag{7} \]
Proposition 2.5 Let \((\xi, x)\) be a linear vector field on \(A\) and let \(D = D_\xi\) be the corresponding element of \(\Gamma \text{CDO}(A)\). Then, for all \(X \in \Gamma A\) and \(m \in M\),
\[
\tau(X(m), D(X)(m)) = T(X)(x(m)) - \xi(X(m)).
\]
If \(\varphi_t\) is a flow for \(\xi\) near \(X(m)\) and \(f_t\) the corresponding flow for \(x\) near \(m\), then
\[
v(x(m), D(X)(m)) = \frac{d}{dt}(X(f_t(m)) - \varphi_t(X(m)))\bigg|_0.
\]

Proof. Let \(Y = D(X)(m)\) and let \(Z\) be the RHS of the first equation. It suffices to prove that
\[
\tau(X(m), Y) = \tau(X(m), Z),
\]
both of which are vertical tangent vectors to \(A\) at \(X(m)\). The functions on \(A\) are generated by those of the form \(\ell_\varphi\) for \(\varphi \in \Gamma A^*\) and those of the form \(f \circ q\) for \(f \in C^\infty(M)\). Since both vectors are vertical, they coincide on all \(f \circ q\). Now, for any \(\varphi \in \Gamma A^*\),
\[
\tau(X(m), Y)(\ell_\varphi) = \langle \varphi, Y \rangle = x(m)(\langle \varphi, X \rangle) - \xi(\ell_\varphi)(X(m))
\]
and
\[
\tau(X(m), Z)(\ell_\varphi) = T(X)(x(m))(\ell_\varphi) - \xi(X(m))(\ell_\varphi),
\]
whence the result.

For the second equation, note first that \(T(X)(x(m))\) and \(\xi(X(m))\) have the same two projections, and therefore
\[
v(x(m), D(X)(m)) = T(X)(x(m)) - \xi(X(m)).
\]
From this the second equation follows.

If, in Proposition 2.3, \(X(m) = 0\) for a specific \(m \in M\), then \(D(X)(m)\) depends only on \(x, X\) and \(m\) and can be identified with \(T(X)(x(m))\); the map \(T_m(M) \to A_m, x \mapsto T(X)(x)\), is the intrinsic derivative of \([\mathbb{1}]\). If, in addition, \(A = TM\), this map is the linearization of a vector field at a singularity in the sense of \([\mathbb{1}]\, p.72\).

Notice that there is now a bijective correspondence between linear vector fields on \(A\) and linear vector fields on \(A^*\). It follows from Proposition 2.3 below that if \((\xi, x) \in \mathcal{X}^{\text{LIN}}(A)\) and \((\xi^*, x) \in \mathcal{X}^{\text{LIN}}(A^*)\) correspond in this way, and if \(\varphi_t\) is a (local) flow for \(\xi\), then \(\varphi^*_{-t}\) is a (local) flow for \(\xi^*\).

Next recall the tangent pairing between \(TA \to TM\) and \(T(A^*) \to TM\) of \([\mathbb{14}]\, 5.3\): given \(\xi \in T(A^*)\) and \(\xi \in TA\) with \(T(q)(\xi) = T(q_*)(\xi)\) we can write \(\xi = \frac{d}{dt}\varphi_t \bigg|_0 \in T(A^*)\) and \(\xi = \frac{d}{dt}X_t \bigg|_0 \in TA\) where \(X_t \in A\) and \(\varphi_t \in A^*\) have \(q_*(\varphi_t) = q(X_t)\) for \(t\) near zero. Now the tangent pairing \(\langle \xi, \epsilon \rangle\) is defined by
\[
\langle \xi, \epsilon \rangle = \frac{d}{dt}\langle \varphi_t, X_t \rangle \bigg|_0.
\]

Proposition 2.6 \([\mathbb{14}]\, 6.3\) Given \((\xi; X_m, x; m) \in TA\) and \((\xi; \varphi_m, x; m) \in T(A^*)\), let \(X \in \Gamma(A)\) and \(\varphi \in \Gamma(A^*)\) be any sections taking the values \(X_m\) and \(\varphi_m\) at \(m\). Then
\[
\langle \xi, \epsilon \rangle = \xi(l_x) + \xi(\ell_\varphi) - x(\langle \varphi, X \rangle).
\]
From this the following characterization of the dual of a linear vector field follows easily.

**Proposition 2.7** Let \((\xi, x)\) be a linear vector field on \(A\). For \(\varphi \in A^*_m, m \in M\), the value \(H = \xi(\varphi)\) of the corresponding linear vector field on \(A^*\) is the unique element of the form \((H; \varphi, x(m); m) \in T(A^*)\) for which

\[
\langle \langle H, \xi(X) \rangle \rangle = 0
\]

for all \(X \in A_m\).

### 3 Vector fields on Lie groupoids

Now consider a Lie groupoid \(G\) on base \(M\), and its Lie algebroid \(AG\); see [11] for the basic facts concerning Lie groupoids and Lie algebroids. Applying the tangent functor to the operations in \(G \rightarrow M\) yields the tangent groupoid \(TG \rightarrow TM\). If \(\kappa: G \ast G \rightarrow G\) is the multiplication, then the multiplication in \(TG\) is \(X \ast Y = T(\kappa)(X, Y)\).

**Proposition 3.1** [19, 2.6] Let \(X \in T_g(G)\) and \(Y \in T_h(G)\) have \(T(\alpha)(X) = T(\beta)(Y) = x\). Then

\[
X \ast Y = T(L_X)(Y) + T(R_Y)(X) - T(L_X)T(R_Y)(T(1)(x))
\]

where \(X, Y\) are any bisections of \(G\) with \(X(\alpha g) = g, Y(\alpha h) = h\).

Here \(1\) is the identity map embedding the base manifold \(M\) into \(G\). A *bisection* \([3]\) of \(G\) is a submanifold \(X\) of \(G\) such that \(\alpha: X \rightarrow M\) and \(\beta: X \rightarrow M\) are both diffeomorphisms. Bisections are naturally identified with admissible sections \([11, II \S 5]\) and we will use both formulations without comment. (If \(G\) is locally trivial, the bisections are the gauge transformations of the corresponding principal bundle.)

The tangent bundle projection \(p_G: TG \rightarrow G\) is a groupoid morphism over \(p: TM \rightarrow M\) and applying the Lie functor yields a morphism of Lie algebroids

\[
\begin{array}{ccc}
ATG & \xrightarrow{q_{TG}} & TM \\
\downarrow & & \downarrow pM \\
AG & \xrightarrow{q_G} & M,
\end{array}
\]

where \(q_{TG}\) and \(q_G\) are the bundle projections of the Lie algebroids. In fact \(ATG \rightarrow AG\) can be given a vector bundle structure, by applying the Lie functor to the operations in \(TG \rightarrow G\) \([14, \S 7]\). This gives \([3]\) the structure of a double vector bundle with core \(AG\).

**Theorem 3.2** [14, 7.1] Let \(G\) be a Lie groupoid on base \(M\). Then there is a canonical isomorphism of double vector bundles \(j_G: TAG \rightarrow ATG\), where \(ATG\) is as above and \(TAG\) is the tangent double vector bundle of \(AG \rightarrow M\), which induces the identities on the side bundles \(AG\) and \(TM\) and on the cores \(AG\).
Definition 3.3 A multiplicative vector field on $G$ is a pair of vector fields $(\xi, x)$ where $\xi \in \mathcal{X}(G)$, $x \in \mathcal{X}(M)$, such that $\xi; G \to TG$ is a morphism of groupoids over $x; M \to TM$.

For a group $G$, we must of course have $x = 0$ and so 3.3 coincides with the definition in [10, 1.3]. For a pair groupoid $M \times M$, the multiplicative vector fields are those of the form $x \times x$, where $x \in \mathcal{X}(M)$.

Example 3.4 Given $X \in \Gamma AG$, the right- and left-invariant vector fields corresponding to $X$ are defined by $\tilde{X}(g) = T(R_g)(X(\beta g))$ and $\tilde{X}(g) = T(L_g)T(i)(X(\alpha g))$, where $i: G \to G$ is the inversion in $G$. Write $\xi = \tilde{X} + \tilde{X}$. Then $\xi$ is a multiplicative vector field over $a(X)$, the anchor of $X$.

Proposition 3.5 Let $\xi$ be a vector field on a Lie groupoid $G$ and $x$ a vector field on its base $M$. Then the following are equivalent:

(i). $(\xi, x)$ is a multiplicative vector field on $G$;

(ii). The flows $\varphi_t$ of $\xi$ are (local) Lie groupoid automorphisms over the flows $f_t$ of $x$.

Proof. $(\Rightarrow)$ Assume for simplicity that $\varphi_t$ and $f_t$ are global flows for $\xi$ and $x$. Since $\xi$ projects to $x$ under the groupoid source projection $\alpha$, it follows that $\alpha \circ \varphi_t = f_t \circ \alpha$. Similarly $\beta \circ \varphi_t = f_t \circ \beta$.

Denote $\{(h, g) \in G \times G | \alpha(h) = \beta(g)\}$ by $G \ast G$. Define a vector field $\xi \ast \xi$ on $G \ast G$ by $\xi \ast \xi(h, g) = (\xi(h), \xi(g))$; since $T(\alpha)(\xi(h)) = x(\alpha(h)) = x(\beta(g)) = T(\beta)(\xi(g))$, $\xi \ast \xi$ is tangent to $G \ast G$. Evidently $\xi \ast \xi$ has flow $\psi_t(h, g) = (\varphi_t(h), \varphi_t(g))$. Denoting the groupoid composition by $\kappa: G \ast G \to G$, we know that $\xi \ast \kappa$ projects to $\xi$ under $\kappa$. It follows that $\varphi_t(h) \varphi_t(g) = \varphi_t(hg)$ for all $(h, g) \in G \ast G$, and so $\varphi_t$ is a Lie groupoid automorphism.

The converse is established by retracing these steps in the reverse order.

\[\square\]

Call $F: G \to \mathbb{R}$ a multiplicative function if it is a groupoid morphism from $G$ into the abelian group $\mathbb{R}$.

Corollary 3.6 Let $(\xi, x)$ be a multiplicative vector field on $G$, and let $F: G \to \mathbb{R}$ be a multiplicative function. Then $\xi(F)$ is also a multiplicative function.

It follows from Proposition 3.3 or from [12, 4.3], that if $(\xi, x)$ and $(\eta, y)$ are multiplicative vector fields, then $([\xi, \eta], [x, y])$ is also.

Proposition 3.7 Let $(\xi, x)$ be a multiplicative vector field and take $X \in \Gamma AG$ with corresponding right- and left-invariant vector fields $\tilde{X}$ and $\tilde{X}$. Then $[\xi, \tilde{X}]$ is right-invariant and $[\xi, \tilde{X}]$ is left-invariant.
Proof. Because $\xi$ is multiplicative we have, as in $\text{[3, §3]}$, that $\xi \ast \xi \sim \xi$. Further, if $Z \in \mathcal{X}(G)$ is $\alpha$-vertical, it is right-invariant iff $Z \ast 0 \sim Z$ (see $\text{[12, §4]}$). So we have $[\xi \ast \xi, \overline{X} \ast 0] \sim [\xi, \overline{X}]$, whence $[\xi, \overline{X}] \ast 0 \sim [\xi, \overline{X}]$, and so $[\xi, \overline{X}]$ is right-invariant.

A similar proof applies in the left-invariant case.

For a multiplicative vector field $(\xi, x)$ there is now a map

$$D_{\xi}: \Gamma AG \to \Gamma AG,$$

$$D_{\xi}(X) = [\xi, X].$$

The proof of the following result is straightforward.

**Proposition 3.8**

(i). The map $D_{\xi}: \Gamma AG \to \Gamma AG$ is a derivation of the bracket structure.

(ii). For $X \in \Gamma AG$ and any vector field $\overline{X}$ on $G$ with $\overline{X}|_M = X$, we have $D_{\xi}(X) = [\xi, \overline{X}]|_M$.

(iii). If $(\eta, y)$ is a second multiplicative vector field, then $D_{[\xi, \eta]} = [D_{\xi}, D_{\eta}]$.

Given a multiplicative vector field $(\xi, x)$, applying the Lie functor produces a section

$$\begin{array}{ccc}
AG & \xrightarrow{\tau} & M \\
\uparrow A(\xi) & & \uparrow x \\
\uparrow q_G & & \uparrow \overline{x} \\
\end{array}$$

(10)

Define $\tilde{\xi} = (j_G)^{-1} \circ A(\xi)$; it is clear that $(\tilde{\xi}, x)$ is a linear vector field on $AG$. If $G = M \times M$ and $\xi = x \times x$ where $x \in \mathcal{X}(M)$, then $\tilde{\xi}$ is the complete lift $\tilde{x}$ of $x$ to $TM$ in the sense of $\text{[20]}$. The linear vector field $(\tilde{x}, x)$ corresponds to the Lie derivative $L_x: \mathcal{X}(M) \to \mathcal{X}(M)$, $y \mapsto [x, y]$. From Proposition 2.5 we therefore have

$$\tau(x(m), [x, y](m)) = \overline{y}(x(m)) - T(x)(y(m)).$$

(11)

If $\varphi_t$ is a (local) flow for $\xi$, then $A(\varphi_t)$ is a (local) flow for $\tilde{\xi}$; this follows from the classical result for complete lifts to tangent bundles. Since $a_{TG} \circ j_G = J_M \circ T(a_G)$, the vector field $\tilde{\xi}$ on $AG$ projects under $a_G$ to $\tilde{x}$ on $TM$.

Since $(\tilde{\xi}, x)$ is linear, it induces a covariant differential operator $D_{\tilde{\xi}}: \Gamma AG \to \Gamma AG$ as in $\text{[7]}$.

**Theorem 3.9** For any multiplicative vector field $(\xi, x)$ on $G$, $D_{\tilde{\xi}} = D_{\xi}$.

Proof. Applying (11) to $\xi$ and $\overline{X}$ for $X \in \Gamma AG$, at $1_m \in G$, and recalling that $\xi(1_m) = T(1)(x(m))$, we have

$$\tau(X(m), [\xi, \overline{X}](1_m)) = T(\overline{X})(T(1)(x(m)) - \tilde{\xi}(X(m)).$$
Note that \( \tau \) and the tildes here refer to the double vector bundle \( T^2G \). On the other hand, in \( TAG \) we have, from Proposition 2.5,
\[
\tau(X(m), D_{\xi}(X)(m)) = T(X)(x(m)) - \tilde{\xi}(X(m)).
\]
Note that \( \tilde{\xi} \) as a vector field on \( AG \) is the restriction of \( \tilde{\xi} \) as a vector field on \( TG \).

From [14, §7] we know that
\[
\tilde{X} = j_G \circ T(X),
\]
where the tilde on the left refers to \( T^2G \) and the arrow on the right refers to the tangent groupoid \( TG \). So
\[
T(\tilde{X})(m) = J_G(T(X))(m).
\]
Since \( j_G: TAG \to ATG \) is a restriction of \( J_G: T^2G \to T^2G \), it follows that
\[
T(\tilde{X})(T(1)(x(m))) = T(X)(x(m)).
\]
Regarding \( ATG \subseteq T^2G \) and noting that the \( \tau \) for \( TAG \) is then the restriction of the \( \tau \) for \( T^2G \), we have
\[
D_{\tilde{\xi}}(X)(m) = [\xi, \tilde{X}](1_m).
\]
\[\blacksquare\]

For any multiplicative vector fields \((\xi, x)\) and \((\eta, y)\) on \( G \) and \( f \in C^\infty(M) \), \( \varphi \in \Gamma A^*G \), we now have
\[
\tilde{\xi} + \tilde{\eta} = \tilde{\xi} + \tilde{\eta}, \quad \tilde{\xi}(f \circ q) = x(f) \circ q, \quad \tilde{\xi}(\ell_\varphi) = \ell_{D_\xi(\varphi)}.
\]
Proposition 3.10 For \((\xi, x), (\eta, y)\) multiplicative vector fields on \( G \), \( X, Y \in \Gamma AG \), and \( F \) a multiplicative function on \( G \),
\[
[\tilde{\xi}, \tilde{\eta}] = [\tilde{\xi}, \tilde{\eta}], \quad [\tilde{\xi}, X^\dagger] = D_\xi(X)^\dagger, \quad [X^\dagger, Y^\dagger] = 0, \quad \tilde{\xi}(F) = \tilde{\xi}(F).
\]

Proof. The third equation is known from [3], and the last follows from Proposition 3.5.

For the first and second equations, it suffices to verify equality on functions of the forms \( \ell_\varphi \), \( \varphi \in \Gamma A^*G \), and \( f \circ q \), \( f \in C^\infty(M) \). For the second equation and \( \varphi \in \Gamma A^*G \) we have
\[
[\tilde{\xi}, X^\dagger](\ell_\varphi) = \tilde{\xi}(\langle \varphi, X \rangle \circ q) - X^\dagger(\ell_{D_\xi(\varphi)})
\]
\[
= x(\langle \varphi, X \rangle) \circ q - \langle D_\xi(\varphi), X \rangle \circ q
\]
\[
= \langle \varphi, D_\xi(X) \rangle \circ q
\]
\[
= D_\xi(X)^\dagger(\ell_\varphi),
\]
whilst for \( f \in C^\infty(M) \),
\[
[\tilde{\xi}, X^\dagger](f \circ q) = 0 = D_\xi(X)^\dagger(f \circ q).
\]
The first equation is proved in a similar way.
\[\blacksquare\]
Taking $G$ to be the pair groupoid $M \times M$ we of course recover the classical description of vector fields on a tangent bundle $[20]$. Multiplicative vector fields are rather special and we now briefly consider two more general types of vector field on Lie groupoids. In what follows we omit most proofs, which are similar to those given above.

**Definition 3.11** A star vector field on $G$ is a pair of vector fields $(\xi, x) \in \mathcal{X}(G) \times \mathcal{X}(M)$, such that $T(\alpha) \circ \xi = x \circ \alpha$ and $\xi \circ 1 = T(1) \circ x$.

The terminology comes from [8], where a star map of groupoids is a map which preserves the $\alpha$-fibration and the identities.

**Lemma 3.12** Given any vector field $x$ on $M$, there is a star vector field $(\xi, x)$ on $G$.

**Proof.** Define a vector field $\eta$ on $G$ by setting $\eta(m) = T(1)(x(m))$ for $m \in M$, and extending over $G$. Then $\mu: G \to TM$ defined by $\mu(g) = T(\alpha)(\eta(g)) - x(\alpha g)$ is a section of the pullback bundle $\alpha^*TM$. Since $\alpha$ is a surjective submersion, there is a vector field $\zeta$ on $G$ with $T(\alpha) \circ \zeta = \mu$; we can also require that $\zeta$ vanish on all $1_m \in G$. Now $\xi = \eta - \zeta$ is a star vector field over $x$.

**Proposition 3.13** Let $\xi$ be a vector field on a Lie groupoid $G$ and $x$ a vector field on its base $M$. Then the following are equivalent:

(i) $(\xi, x)$ is a star vector field on $G$;

(ii) The flows $\varphi_t$ of $\xi$ are (local) star maps over the flows $f_t$ of $x$.

It follows as before that if $(\xi, x)$ and $(\eta, y)$ are star vector fields, then $([\xi, \eta], [x, y])$ is also. If $(\xi, x)$ is a star vector field and $X \in \Gamma A^*G$, then it is clear that $[\xi, X]$ is $\alpha$-vertical. We now define $D_\xi: \Gamma A^*G \to \Gamma A^*G$ by

$$D_\xi(X) = [\xi, X] \circ 1.$$  

Again, $D_\xi(X) = [\xi, X] \circ 1$ for any vector field $X$ on $G$ such that $X \circ 1 = X$.

Given a star vector field $(\xi, x)$, we can still apply the Lie functor and obtain a linear vector field $\tilde{\xi} = (j_G)^{-1} \circ A(\xi)$ on $AG$. If $\varphi_t$ is a (local) flow for $\xi$, then $A(\varphi_t)$ is still a (local) flow for $\tilde{\xi}$. The proof of the following result follows as for Theorem $3.9$.

**Theorem 3.14** For any star vector field $(\xi, x)$ on $G$, $D_\xi = D_\xi$.

Equations (12) continue to hold for star vector fields $(\xi, x)$ and $(\eta, y)$ and $f \in C^\infty(M)$, $\varphi \in \Gamma A^*G$. 

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Proposition 3.15 For \((\xi, x), (\eta, y)\) star vector fields on \(G\), and \(X, Y \in \Gamma AG\),
\[
[\tilde{\xi}, \tilde{\eta}] = [\tilde{\xi}, \tilde{\eta}], \quad [\tilde{\xi}, X^\uparrow] = D_\xi(X)^\uparrow, \quad [X^\uparrow, Y^\uparrow] = 0.
\]
In view of the following result, these equations determine the bracket structure for all vector fields on \(AG\).

Proposition 3.16 The vector fields of the form \(\tilde{\xi}\), where \((\xi, x)\) is a star vector field, together with those of the form \(X^\uparrow\), where \(X \in \Gamma AG\), generate \(X(AG)\).

Proof. Take \(\Xi \in TAG\) with \(T(q)(\Xi) = x(m)\) and \(p_{AG}(\Xi) = X\). Extend \(x(m) \in TM\) to a vector field \(x\) on \(M\). By Lemma 3.12, there is a star vector field \((\xi, x)\) on \(G\). We now have \(T(q)(\tilde{\xi}(X)) = x(m)\) and so, by the first sequence in (2), we have
\[
\Xi = \tilde{\xi}(X) + Y^\uparrow(X)
\]
for some \(Y \in \Gamma AG\).

Lastly in this section, we briefly consider the notion of affine vector field on a Lie groupoid.

Definition 3.17 A vector field \(\xi\) on \(G\) is affine if for all \(g, h \in G\) with \(\alpha(g) = \beta(h) = m\) we have
\[
\xi(gh) = T(L_\mathcal{X})(\xi(h)) + T(R_\mathcal{Y})(\xi(g)) - T(L_\mathcal{X})T(R_\mathcal{Y})(\xi(1_m)),
\]
where \(\mathcal{X}, \mathcal{Y}\) are bisections with \(\mathcal{X}(\alpha g) = g, \mathcal{Y}(\alpha h) = h\).

A multiplicative vector field is affine (see Proposition 3.1) and for any \(X \in \Gamma AG\), both \(\tilde{X}\) and \(\bar{X}\) are affine vector fields. It is clear that the sum and scalar multiples of affine vector fields are affine. Further, affine vector fields are closed under the bracket of vector fields.

Any affine vector field on a Lie group is a sum of a multiplicative vector field and a right-invariant vector field (see [18, 4.11] for the case of bivector fields). Since multiplicative and right-invariant vector fields on a groupoid are always both \(\alpha\)- and \(\beta\)-projectable, the following result is the best that can be expected in the general case. It is easy to construct affine vector fields on a pair groupoid \(M \times M\) which are not projectable.

Proposition 3.18 Let \(\xi\) be an affine vector field on the Lie groupoid \(G\), and suppose that \(\xi\) is both \(\alpha\)- and \(\beta\)-projectable. Then \(\xi\) is the sum of a multiplicative and a right-invariant (or left-invariant) vector field.

Proof. Define \(X \in \Gamma AG\) by \(X(m) = \xi(1_m) - T(1)T(\alpha)(\xi(1_m))\), where \(m \in M\), and write \(\eta = \xi - \bar{X}\). We prove that \(\eta\) is multiplicative. Let \(x, y\) be the vector fields on \(M\) for which \(\xi \sim x, \xi \sim y\). Then \(\bar{X} \sim y - x\) and so \(\eta \sim x\) and \(\eta \sim y\). Clearly \(\eta(1_m) = T(1)(x(m))\) for \(m \in M\).

Since \(\xi\) and \(\bar{X}\) are affine, it follows that \(\eta\) is affine. In this case, the affine condition implies that \(\eta(gh) = \eta(g) \cdot \eta(h)\). Thus \(\eta\) is multiplicative.

\[\square\]
4 Vector fields on Lie algebroid duals

Now consider the Lie algebroid dual \( q: A^* G \to M \) for a Lie groupoid \( G \) on base \( M \). The tangent double vector bundle \((T(A^* G); A^* G, TM; M)\) has core \( A^* G \) and so for each \( \varphi \in \Gamma A^* G \) there is a core vector field \( \varphi^\dagger \in \mathcal{X}(A^* G) \) such that

\[
\varphi^\dagger (\ell X) = \langle \varphi, X \rangle \circ q_*, \quad \varphi^\dagger (f \circ q_*) = 0,
\]

for all \( X \in \Gamma AG, \ f \in C^\infty(M) \). Thus \([\varphi^\dagger, \psi^\dagger] = 0\) for all \( \varphi, \psi \in \Gamma A^* G \).

Let \((\xi, x)\) be a star vector field on \( G \). Then the covariant differential operators \( D_\xi \in \Gamma \text{CDO}(AG) \) and \( D_\xi^{(s)} \in \Gamma \text{CDO}(A^* G) \) induce a linear vector field \((H_\xi, x)\) on \( A^* G \). From (5) and the fact that \( H_\xi \sim x \) we have

\[
H_\xi (\ell X) = \ell_{D_\xi(X)}, \quad H_\xi (f \circ q_*) = x(f) \circ q_*,
\]

for \( X \in \Gamma AG, \ f \in C^\infty(M) \). The proof of the following result follows as for Proposition 3.10.

**Proposition 4.1** For \((\xi, x), (\eta, y)\) star vector fields on \( G \) and \( \varphi, \psi \in \Gamma A^* G \),

\[
[H_\xi, H_\eta] = H_{[\xi, \eta]}, \quad [H_\xi, \varphi^\dagger] = D_\xi^{(s)}(\varphi^\dagger), \quad [\varphi^\dagger, \psi^\dagger] = 0.
\]

The next result shows in particular that for \((\xi, x)\) a multiplicative vector field on \( G \), the vector field \((H_\xi, x)\) is a Poisson vector field (called a Poisson infinitesimal automorphism in [9]) with respect to the dual Poisson structure \([4, 5]\) on \( A^* G \).

**Proposition 4.2** Let \((H, x)\) be a linear vector field on a Lie algebroid dual \( A^* \), with corresponding covariant differential operator \( D: \Gamma A \to \Gamma A \). Then \( H \) is a Poisson vector field (that is, a derivation of the Poisson bracket of functions on \( A^* \)) if and only if \( D \) is a derivation of the bracket in \( \Gamma A \).

**Proof.** Note first that if \( D \in \Gamma \text{CDO}(A) \) is a derivation of the bracket in \( \Gamma A \) then \( a(D(X)) = [\bar{a}(D), a(X)] \) for all \( X \in \Gamma A \), where \( \bar{a} \) is the anchor on \( \text{CDO}(A) \); this follows by expanding the expression \( D([X, fY]) \) in two ways.

Now the result is a straightforward calculation using the bracket relations for the Poisson structure on \( A^* \) ([14, equation (39)]).

\[\square\]

**Definition 4.3** Let \( A \) be a Lie algebroid on base \( M \). Then a morphic vector field on \( A \) is a pair \((\Xi, x)\) where \( \Xi \in \mathcal{X}(A), \ x \in \mathcal{X}(M) \), such that \( \Xi: A \to TA \) is a Lie algebroid morphism over \( x: M \to TM \).

Here we take \( TA \) as equipped with the tangent Lie algebroid structure on base \( TM \) described in [14, §5]. Using [14, 7.1] it follows from the definition of \( \xi \) that if \((\xi, x)\) is a multiplicative vector field on a Lie groupoid \( G \), then \((\xi, x)\) is a morphic vector field on \( AG \).
Theorem 4.4 Let $A$ be a Lie algebroid on base $M$, and let $(\Xi, x)$ be a linear vector field on $A$. The following are equivalent:

(i). $(\Xi, x)$ is a morphic vector field;

(ii). The dual vector field $(\Xi^*, x)$ on $A^*$ is Poisson;

(iii). $D_{\Xi} : \Gamma A \to \Gamma A$ is a derivation;

(iv). The flows of $\Xi$ are (local) Lie algebroid automorphisms over the flows of $x$.

Proof. The equivalence of (ii) and (iii) is Proposition 4.2 above. The equivalence of (ii) and (iv) follows from the facts that a vector field is Poisson iff its flows are (local) Poisson automorphisms [1, III §10], and that a vector bundle isomorphism of Lie algebroids is a Lie algebroid isomorphism iff its dual is a Poisson isomorphism. To prove the equivalence of (i) and (ii) we first need some results of independent interest.

Proposition 4.5 [19, 4.6] Let $B \to S$ be a vector subbundle of a Lie algebroid $A \to M$. Then $B$ is a Lie subalgebroid of $A$ if and only if $B^\perp$ is coisotropic in $A^*$ with the dual Poisson structure.

Proposition 4.6 Let $A$ be any Lie algebroid, and consider $\varphi \in \Gamma A^*$. Then $d\varphi = 0$ if and only if $\text{im}(\varphi)$ is a coisotropic submanifold of $A^*$ with the dual Poisson structure.

Proof. For any $X \in \Gamma(A)$, let $f_X$ be the function $\langle \varphi, X \rangle$ on the base manifold $M$. It is clear that $\ell_X - q^* f_X$ vanishes on the graph $\text{im} \varphi$. In fact, the space of functions vanishing on $\text{im} \varphi$ is spanned by all such functions for $X \in \Gamma(A)$. Therefore, to show that $\text{im} \varphi$ is coisotropic, it suffices to prove that $\{\ell_X - q^* f_X, \ell_Y - q^* f_Y\}$ vanishes on $\text{im} \varphi$ for any $X, Y \in \Gamma(A)$.

Now,

$$\{\ell_X - q^* f_X, \ell_Y - q^* f_Y\} = \{\ell_X, \ell_Y\} - \{q^* f_X, q^* f_Y\} = \ell_{[X,Y]} + q^* (a(Y)f_X) - q^*(a(X)f_Y),$$

where the last step follows directly from the definition of the Poisson structure on $A^*$. By evaluating at $\varphi(m)$, one obtains that

$$\{\ell_X - q^* f_X, \ell_Y - q^* f_Y\}(\varphi(m)) = \langle[X,Y], \varphi\rangle(m) + (a(Y)(X, \varphi))(m) - (a(X)(Y, \varphi))(m) = -(d\varphi)(X, Y)(m).$$

This leads to our conclusion: $\text{im} \varphi$ is coisotropic iff $\varphi$ is closed.

$\square$
Remark 4.7 When $A$ is a tangent bundle Lie algebroid $TM$, $A^*$ is the cotangent bundle $T^*M$ with the canonical symplectic structure. In this case Proposition 4.6 is just the well-known fact, that the graph of a one-form is Lagrangian iff the form is closed. On the other hand, if $A$ is a Lie algebra $\mathfrak{g}$, a one-form $\varphi$ is just a point in $\mathfrak{g}^*$. It is coisotropic iff the Poisson tensor vanishes at the point, which is exactly what $d\varphi = 0$ means.

Proposition 4.8 Let $X$ be a vector field on a Poisson manifold $P$. Then $X$ is Poisson if and only if $\text{im}(X)$ is a coisotropic submanifold of $TP$ with the tangent Poisson structure.

Proof. Take $A = T^*P$ in the preceding Proposition. Then for $X \in \Gamma TP = \Gamma A^*$ we have $dX = [\pi, X] = L_X(\pi)$, where $\pi$ is the Poisson tensor. Thus $dX = 0$ if and only if $X$ is a Poisson vector field. The result now follows from the fact that the dual Poisson structure on $TP$ from the Lie algebroid structure on $T^*P$ is the tangent Poisson structure.

We can now complete the proof of Theorem 4.4. Firstly, since $\Xi$ and $x$ are embeddings, $\Xi: A \rightarrow TA$ is a morphism over $x: M \rightarrow TM$ if and only if $\text{im}(\Xi) \rightarrow \text{im}(x)$ is a Lie subalgebroid of $TA \rightarrow TM$.

Next recall from [14, §5] the dual $T^*(A) \rightarrow TM$ of the vector bundle $TA \rightarrow TM$. The tangent pairing (8) induces an isomorphism $I: T(A^*) \rightarrow T^*(A)$, and this is also an isomorphism of the Poisson structures on $T(A^*)$ and $T^*(A)$ [14, 5.6]. It now follows easily from Proposition 2.7 that $I$ maps $\text{im}(\Xi_*)$ isomorphically onto $\text{im}(\Xi)^\perp$. Using Propositions 4.5 and 4.8, the proof of Theorem 4.4 is complete.

Theorem 4.9 Let $G$ be an $\alpha$-simply connected Lie groupoid on base $M$. Then if $(\Xi, x)$ is a morphic vector field on $AG$, there is a unique multiplicative vector field $(\xi, x)$ on $G$ such that $\tilde{\xi} = \Xi$.

Proof. This is an immediate consequence of the above and the integrability of Lie algebroid morphisms [15].

5 Forms on Lie groupoids

We begin by considering any vector bundle $(A, q, M)$, and the double vector bundle structure $(T^*(A); A, A^*; M)$ described in [14, §5]. Using the structure of $T^*A$, one can develop a concept of linear differential form on $A$ corresponding to the concept of linear vector field in [3]. Here we will only present the few facts which we need.
Definition 5.1 A linear 1-form on $A$ is a pair $(\Upsilon, \varphi)$ where $\Upsilon \in \Omega^1(A)$ and $\varphi \in \Gamma A^*$ such that

$$
\begin{array}{ccc}
T^*A & \xrightarrow{r} & A^* \\
\Upsilon & \xrightarrow{\uparrow} & \varphi \\
A & \xrightarrow{\longrightarrow} & M
\end{array}
$$

is a morphism of vector bundles.

Here the map $r$ is defined by $\langle r(\Phi), X \rangle = \langle \Phi, X^\uparrow(Y) \rangle$ for $\Phi \in T^*_Y A$.

Note that the base map of a linear 1-form is not a 1-form on the base but rather a section of the dual bundle. The sum of linear 1-forms is a linear 1-form and scalar multiples of linear 1-forms are linear 1-forms. The pairing of a linear vector field with a linear 1-form is a fibrewise linear function.

Now consider a Lie groupoid $G$ on $M$. The cotangent bundle $T^*G$ has a natural Lie groupoid structure on base $A^*G$ defined in [3] and [10]. As in [14, §7], we take the source $\tilde{\alpha}$ and target $\tilde{\beta}$ to be given by

$$
\tilde{\alpha}(\omega)(X) = \omega(T(L_g)(X - T(1)(a(X)))), \quad \tilde{\beta}(\omega)(Y) = \omega(T(R_g)(Y)),
$$

where $\omega \in T^*_g G$, $X \in A_{ag}G$ and $Y \in A_{bg}G$. If $\theta \in T^*_h G$ and $\bar{\alpha}(\theta) = \bar{\beta}(\omega)$ then $\alpha h = \beta g$ and we define $\theta \bullet \omega \in T^*_h G$ by

$$(\theta \bullet \omega)(Y \bullet X) = \theta(Y) + \omega(X),$$

where $Y \in T^*_h G$, $X \in T^*_g G$. The identity element $\tilde{1}_\varphi \in T^*_1 G$ corresponding to $\varphi \in A^*_G$ is defined by $\tilde{1}_\varphi(T(1)(x) + X) = \varphi(X)$ for $X \in A_m G$, $x \in T_m(M)$.

Definition 5.2 A multiplicative 1-form on $G$ is a pair $(\Phi, \varphi)$ where $\Phi \in \Omega^1(G)$ and $\varphi \in \Gamma A^*G$ such that

$$
\begin{array}{ccc}
T^*G & \xrightarrow{\Phi} & A^*G \\
\varphi & \xrightarrow{\uparrow} & \\
G & \xrightarrow{\longrightarrow} & M
\end{array}
$$

is a morphism of Lie groupoids.

A star 1-form on $G$ is similarly a pair $(\Phi, \varphi)$ where $\Phi \in \Omega^1(G)$ and $\varphi \in \Gamma A^*G$ such that $\tilde{\alpha} \circ \Phi = \varphi \circ \alpha$ and $\Phi \circ 1 = \tilde{1} \circ \varphi$.

Suppose $(\Phi, \varphi)$ is a multiplicative 1-form and $(\xi, x)$ is a multiplicative vector field on $G$. Writing $F = \langle \Phi, \xi \rangle: G \to \mathbb{R}$, we have, for $h, g \in G$ compatible,

$$
F(hg) = \langle \Phi(h) \bullet \Phi(g), \xi(h) \bullet \xi(g) \rangle.
$$
and it follows directly from the definition of the multiplication \( \bullet \) in \( T^*G \) that this is equal to \( \langle \Phi(h), \xi(h) \rangle + \langle \Phi(g), \xi(g) \rangle \). Thus \( F \) is a multiplicative function.

Put differently, the standard pairing \( T^*G \times TG \to \mathbb{R} \) is a groupoid morphism with respect to the pullback groupoid structure with base \( A^*G \times TM \). Applying the Lie functor as in [14, 7.2], we obtain a pairing \( \langle \langle \ , \ \rangle \rangle : AT^*G \times ATG \to \mathbb{R} \). This yields an isomorphism of double vector bundles \( i_G : AT^*G \to A^*TG \) defined by

\[
\langle i_G(x), \xi \rangle^* = \langle x, \xi \rangle
\]

for \( x \in AT^*G, \xi \in ATG \). Here \( A^*TG \) is the dual of \( ATG \) over \( AG \) and \( \langle \ , \ \rangle^* \) is the standard pairing of \( A^*TG \) and \( ATG \). Still following [14], take the dual of \( j_G : TAG \to ATG \) over \( AG \) and define \( j'_G = j^*_G \circ i_G : AT^*G \to T^*AG \); this \( j'_G \) is an isomorphism of double vector bundles preserving \( AG, A^*G \) and the cores. The following result is now immediate.

**Proposition 5.3** For \( x \in AT^*G \) and \( \xi \in TAG \),

\[
\langle x, j_G(\xi) \rangle^* = \langle x, j'_G(x, \xi) \rangle.
\]

Now consider a star 1-form \((\Phi, \varphi)\) on \( G \). Applying the Lie functor gives a vector bundle morphism \( A(\Phi) : AG \to AT^*G \). Define \( \tilde{\Phi} = j'_G \circ A(\Phi) \); since \( j'_G \) is an isomorphism of double vector bundles, \((\tilde{\Phi}, \varphi)\) is a linear 1-form on \( AG \).

**Proposition 5.4** (i). Let \((\Phi, \varphi)\) be a star 1-form and \( X \in \Gamma AG \). Then

\[
\langle \tilde{\Phi}, X^\dagger \rangle = \langle \varphi, X \rangle \circ q.
\]

(ii). Let \((\Phi, \varphi)\) be a star 1-form and \((\xi, x)\) a star vector field on \( G \). Then

\[
\langle \tilde{\Phi}, \tilde{\xi} \rangle = \langle \tilde{\Phi}, \xi \rangle,
\]

where for any function \( F : G \to \mathbb{R} \), we denote \( A(F) : AG \to \mathbb{R} \) by \( \tilde{F} \).

**Proof.** (i) follows from the fact that \( r \circ \tilde{\Phi} = \varphi \circ q \), as in [13]. For (ii), since \( \langle \ , \ \rangle = A(\langle \ , \ \rangle) \), we have

\[
\langle \tilde{\Phi}, \tilde{\xi} \rangle = \langle j'_G \circ A(\Phi), j^{-1}_G \circ A(\xi) \rangle = \langle A(\Phi), A(\xi) \rangle = A(\langle \Phi, \xi \rangle) = \langle \tilde{\Phi}, \xi \rangle.
\]

These two equations describe the behaviour of the forms \((\tilde{\Phi}, \varphi)\) on \( \mathcal{X}(AG) \). To complete the description of the 1-forms on \( AG \) we need to include the pullbacks of forms on the base manifold.

Given \( \omega \in \Omega^1(M) \) there is the 1-form \( q^* \omega \in \Omega^1(AG) \). Here it is immediate that

\[
\langle q^* \omega, \tilde{\xi} \rangle = \langle \omega, x \rangle \circ q, \quad \langle q^* \omega, X^\dagger \rangle = 0,
\]

for star vector fields \((\xi, x)\) on \( G \) and \( X \in \Gamma AG \), since \( \tilde{\xi} \) projects to \( x \) under \( q \) and \( X^\dagger \) projects to 0.
Lemma 5.5 \textit{Given any } \varphi \in \Gamma A^*G, \text{ there is a star 1-form } (\Phi, \varphi) \text{ on } G. \\

\textbf{Proof.} Define a 1-form } \Psi \text{ on } G \text{ by setting } \Psi(1_m) = \tilde{1}(\varphi(m)) \text{ for } m \in M, \text{ and extending over } G. \text{ Then } \mu : G \to A^*G \text{ defined by } \mu(g) = \tilde{\alpha}(\Psi(g)) - \varphi(\alpha g) \text{ is a section of the pullback bundle } \alpha^!A^*G. \text{ Since } \tilde{\alpha} : T^*G \to A^*G \text{ is a fibrewise surjection, there is a 1-form } \Upsilon \text{ on } G \text{ with } \tilde{\alpha} \circ \Upsilon = \mu; \text{ we can also require that } \Upsilon \text{ vanish on all } 1_m \in G. \text{ Now } \Phi = \Psi - \Upsilon \text{ is a star 1-form over } \varphi. \quad \blacksquare

Proposition 5.6 \textit{The 1-forms } \tilde{\Phi}, \text{ where } (\Phi, \varphi) \text{ is a star 1-form on } G, \text{ together with the pullbacks } q^*\omega, \text{ where } \omega \in \Omega^1(M), \text{ generate } \Omega^1(AG). \\

\textbf{Proof.} \text{Take } \Upsilon \in T^*_X AG \text{ with } r(\Upsilon) = \varphi(m). \text{ Extend } \varphi(m) \text{ to a section } \varphi \text{ of } \Gamma A^*G. \text{ By Lemma 5.5, there is a star 1-form } (\Phi, \varphi) \text{ on } G. \text{ So at } X \text{ we have } r(\tilde{\Phi}(X)) = \varphi(m) \text{ and so,}
\begin{align*}
\Upsilon &= \tilde{\Phi}(X) + q^*\omega
\end{align*}
\text{for some } \omega \in \Omega^1(M). \quad \blacksquare

Lastly in this section, we briefly indicate how to characterize the } \tilde{\Phi} \text{ corresponding to multiplicative 1–forms. For any Lie algebroid } A \text{ on } M, \text{ the dual } A^* \text{ has its Poisson structure and } T^*A^* \to A^* \text{ has therefore a cotangent Lie algebroid structure (see [14] for detail and references). For any vector bundle } A \text{ there is a canonical map } R : T^*A^* \to T^*A \text{ which is an isomorphism of double vector bundles preserving } A \text{ and } A^* \text{ [14]. Using } R \text{ we transfer the Lie algebroid structure of } T^*A^* \to A^* \text{ to } T^*A \to A^*. \\

\text{Now consider a multiplicative 1–form } (\Phi, \varphi) \text{ on a Lie groupoid } G \Rightarrow M. \text{ We have } \tilde{\Phi} = j'_{\tilde{G}} \circ A(\Phi) \text{ and by [14], 7.3] we know } j'_{\tilde{G}} = R \circ s^{-1}, \text{ where } s : T^*A^*G \to AT^*G \text{ is the canonical isomorphism between the Lie algebroid of the symplectic groupoid } T^*G \text{ and the cotangent Lie algebroid of its base. Since } A(\Phi) \text{ and } s \text{ are Lie algebroid morphisms, it follows that } \tilde{\Phi} \text{ is a morphic 1–form on } AG \text{ in the sense of the following definition.}

Definition 5.7 \textit{Let } A \text{ be a Lie algebroid on } M. \text{ A linear 1–form } (\Upsilon, \varphi) \text{ on } A \text{ is a morphic 1–form if } \Upsilon : A \to T^*A \text{ is a Lie algebroid morphism over } \varphi : M \to A^*. \\

\text{The next result now follows just as in Theorem [1,9].}

Theorem 5.8 \textit{Let } G \text{ be an } \alpha \text{–simply connected Lie groupoid on base } M. \text{ Then if } (\Upsilon, \varphi) \text{ is a morphic 1–form on } AG, \text{ there is a unique multiplicative 1–form } (\Phi, \varphi) \text{ on } G \text{ such that } \tilde{\Phi} = \Upsilon. 

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6 Forms on Lie algebroid duals

Before we consider forms, notice that there is another way in which one may extend the calculus of vertical and complete lifts to a Lie algebroid. Consider any Lie algebroid $A$ on base $M$. Each $X \in \Gamma A$ defines an element $L_X$ of $\Gamma \text{CDO}(A)$ by $L_X(Y) = [X,Y]$ for which the corresponding element of $\Gamma \text{CDO}(A^*)$ is the Lie derivative, also denoted $L_X$, defined by

$$L_X(\varphi)(Y) = a(X)(\varphi(Y)) - \varphi([X,Y])$$

for $\varphi \in \Gamma A^*$. There is a corresponding morphic vector field $(\widetilde{X},a(X))$ on $A$. By definition, we now have

$$\widetilde{X}(\ell_\varphi) = \ell_{L_X(\varphi)}, \quad \widetilde{X}(f \circ q) = a(X)(f \circ q),$$

for $\varphi \in \Gamma A^*$, $f \in C(M)$. Similarly, there is a linear Poisson vector field $(H_X,a(X))$ on $A^*$ such that

$$H_X(\ell_Y) = \ell_{[X,Y]}, \quad H_X(f \circ q_*) = a(X)(f \circ q_*),$$

for $Y \in \Gamma A$, $f \in C(M)$. The following result is now immediate.

**Proposition 6.1** For $X,Y \in \Gamma A$, and $\varphi,\psi \in \Gamma A^*$,

$$[\widetilde{X},\widetilde{Y}] = [X,Y], \quad [\widetilde{X},\varphi^\dagger] = [X,\varphi]^\dagger, \quad [X^\dagger,\varphi^\dagger] = 0,$$

$$[H_X,H_Y] = H_{[X,Y]}, \quad [H_X,\varphi^\dagger] = L_X(\varphi)^\dagger, \quad [\varphi^\dagger,\psi^\dagger] = 0.$$

Although this formalism is valid for any Lie algebroid, not necessarily integrable, the vector fields $\widetilde{X}$ and $X^\dagger$ do not generally generate the module of vector fields on $A$. For example, if $A$ is totally intransitive and abelian (that is, a vector bundle) then the vector fields $\widetilde{X}$ are all zero.

We now describe forms on a Lie algebroid dual in terms of forms of two specific types. Whereas linear vector fields on the dual of a vector bundle $A$ correspond to linear vector fields on $A$, there is no such result for 1–forms, and the description we give here is not related to that for 1–forms on $A$ in the same way that our descriptions of vector fields on $AG$ and $A^*G$ are related. In fact, the following description applies to any vector bundle.

Firstly, given any $\omega \in \Omega^1(M)$, there is the corresponding 1-form $q_*^*\omega \in \Omega^1(A^*)$. Evidently, for $\varphi \in \Gamma A^*$ and $X \in \Gamma A$,

$$\langle q_*^*\omega,\varphi^\dagger \rangle = 0, \quad \langle q_*^*\omega,H_X \rangle = \langle \omega,a(X) \circ q_* \rangle.$$

If $A = AG$ is integrable and $(\xi,x)$ is a star vector field on $G$, we also have

$$\langle q_*^*\omega,H_\xi \rangle = \langle \omega,x \rangle \circ q_*.$$

Secondly, for each $X \in \Gamma A$, there is the form $\delta\ell_X \in \Omega^1(A^*)$. From (20) it follows that, for $X,Y \in \Gamma A$ and $\varphi \in \Gamma A^*$,

$$\langle \delta\ell_X,H_Y \rangle = \ell_{[Y,X]}, \quad \langle \delta\ell_X,\varphi^\dagger \rangle = \langle \varphi,X \rangle \circ q_*.$$
If $A = AG$ is integrable and $(\xi, x)$ is a star vector field on $G$, we also have

$$\langle \delta \ell_X, H_\xi \rangle = \ell_{D_\xi(X)}.$$

By a result similar to Proposition 5.6, forms of these two types generate $\Omega^1(A^*)$.

Now $A^*$ has the Poisson structure dual to its Lie algebroid structure. By [14, 5.7], the Poisson anchor $\pi^\# : T^*(A^*) \to T(A^*)$ is a morphism of double vector bundles, as in Figure 1, and the induced map of the cores is $-a^*: T^* M \to A^*$.

For $\mathfrak{z} \in \Omega^1(A^*)$, denote the vector field on $A^*$ corresponding under $\pi^\#$ by $\mathfrak{z}^\#$. Then we have, in particular, that for $\omega \in \Omega^1(M)$,

$$(q^*\omega)^\# = -a^*(\omega)^\dagger;$$

that is, the Hamiltonian vector field corresponding to the pullback form $q^*\omega$ is the negative of the vertical lift corresponding to $a^*(\omega) \in \Gamma A^*$.

Further, it is easy to check that, for $X \in \Gamma A$,

$$(\delta \ell_X)^\# = H_X.$$

Thus $H_X$ is the Hamiltonian vector field with energy $\ell_X$.

### 7 Fields and forms on Poisson groupoids

In [17], Weinstein showed that the Lie algebroid dual of a Poisson groupoid $G$ has itself a Lie algebroid structure; he obtained this by means of a general linearization result along an arbitrary coisotropic submanifold of any Poisson manifold. In [14] we obtained this structure by applying the
Lie functor to the Poisson tensor \( \pi^\#: T^*G \to TG \) and showing that, after certain canonical isomorphisms, this defines a Poisson structure on \( AG \) whose dual is the desired Lie algebroid structure on \( A^*G \). Here we provide a particularly concrete and explicit description of this structure. Firstly, in Theorem 7.5 we show how the bracket of 1-forms on the Poisson manifold \( AG \) is determined in terms of the bracket of 1-forms of \( G \); this uses the process \( \Phi \mapsto \tilde{\Phi} \) of §5 and a family of covariant differential operators \( D_\Phi \) in \( T^*P \), induced by the star 1-forms \( \Phi \) on \( G \). Secondly, equation (23) is an explicit formula for the bracket on \( \Gamma A^*G \) in terms of the anchor of \( A^*G \) and the Poisson tensor \( T^*G \to TG \) (note that, by Lemma 7.3, every section of \( A^*G \) is the base of some star 1-form on \( G \)). If \( G \) is a pair (or coarse) Poisson groupoid \( \mathcal{P} \times P \), then (23) reduces to the usual formula for the bracket of 1-forms on a Poisson manifold.

Throughout this section, \( G \) denotes a Poisson groupoid on base \( P \). We follow the notation and conventions of [14].

Consider a star 1-form \( (\Phi, \varphi) \) on \( G \). Analogously to the construction in §3 we define an operator \( D_\Phi \) on \( T^*P \). Note first that for any \( \omega \in \Omega^1(P) \), the pullback \( \beta^\# \omega \in \Omega^1(G) \) projects under \( \tilde{\alpha}: T^*G \to A^*G \) to the zero section of \( A^*G \). Since \( \Phi \) projects to \( \varphi \) under \( \tilde{\alpha} \) by assumption, and \( \tilde{\alpha} \) is a Lie algebroid morphism, it follows that \( [\Phi, \beta^\# \omega] \) also projects to zero; that is,

\[
\langle [\Phi, \beta^\# \omega], T(L_g)(X - T(1)(aX)) \rangle = 0
\]

for all \( g \in G \) and \( X \in A_{\alpha g}G \). Now every \( \beta \)-vertical vector is the left-translate of some \( Y \in T_{1_m}(\beta^{-1}(m)) \) and every such \( Y \) is equal to \( T(i)(X) = -X + T(1)(aX) \) for some \( X \in A_mG \). It therefore follows from (21) that \( [\Phi, \beta^\# \omega] \) annuls all \( \beta \)-vertical vectors. We can therefore define \( D_\Phi(\omega) \in \Omega^1(P) \) by

\[
\langle D_\Phi(\omega), T(\beta)(Y) \rangle = \langle [\Phi, \beta^\# \omega], Y \rangle
\]

(22)

where \( Y \in T_{1_m}G \).

**Proposition 7.1**

(i). For \( (\Phi, \varphi) \) a star 1-form, \( D_\Phi: \Omega^1(P) \to \Omega^1(P) \) is a covariant differential operator over \( a_\star(\varphi) \); that is, it is \( \mathbb{R} \)-linear and \( D_\Phi(f \omega) = fD_\Phi(\omega) + a_\star(\varphi)(f) \omega \) for all \( f \in C^\infty(P) \), \( \omega \in \Omega^1(P) \);

(ii). for any \( Y \in T_{1_m}G \) and any \( \theta \in \Omega^1(G) \) such that \( \theta(1_m) = \beta^\# \omega(1_m) \) for all \( m \in P \),

\[
\langle D_\Phi(\omega), T(\beta)(Y) \rangle = \langle [\Phi, \theta], Y \rangle;
\]

(iii). if \( (\Psi, \psi) \) is another star 1-form, then \( D_{[\Phi, \Psi]} = [D_\Phi, D_\Psi] \).

**Proof.** The proof of (i) is straightforward, and (iii) follows directly from (ii). To prove (ii), it is sufficient to show that if \( \theta \in \Omega^1(G) \) vanishes on the identity elements of \( G \), then \( [\Phi, \theta] \) does also. But \( \tilde{1}: A^*G \to T^*G \) is known to be a Lie algebroid morphism, so \( 0 \sim \theta \) and \( \varphi \sim \Phi \) imply that \( 0 = [\varphi, 0] \sim [\Phi, \theta] \).

\( \Box \)

**Theorem 7.2** Let \( (\Phi, \varphi) \) be a multiplicative 1-form on \( G \). Then for all \( \omega \in \Omega^1(P) \),
(i) \( a^*D_{\Phi}(\omega) = [\varphi, a^*\omega] \),
(ii) \([\Phi, \beta^*\omega] = \beta^*(D_{\Phi}(\omega))\),
(iii) \(D_{\Phi}\) is a derivation of the Poisson bracket on \(\Omega^1(P)\).

**Proof.** (i) For any \(X \in \Gamma(AG)\), using \([23]\),
\[
\langle a^*D_{\Phi}\omega, X \rangle = \langle D_{\Phi}\omega, aX \rangle = \langle D_{\Phi}\omega, T\beta X \rangle = \langle [\Phi, \beta^*\omega], X \rangle,
\]
where we consider \(X\) as a section of \(T^*G\).

As a section of \(A^*G\), \(a^*\omega\) can be considered as a conormal one-form along the base space \(P\). Let \(\tilde{\omega}\) be a one-form on \(G\) extending \(a^*\omega\). Now it suffices to show that \(\langle [\Phi, \beta^*\omega], X \rangle = \langle [\Phi, \tilde{\omega}], X \rangle\), since the latter is, by definition (see \([17]\)), \(\langle \varphi, a^*\omega \rangle, X \rangle\). Let \(\tilde{\omega} = \beta^*\omega - a^*\omega\). Then it is clear that \(\tilde{\omega} = 0\) when restricted to \(T^*\).

As a section of \(A^*G\), \(a^*\omega\) can be considered as a conormal one-form along the base space \(P\). Let \(\tilde{\omega}\) be a one-form on \(G\) extending \(a^*\omega\). Now it suffices to show that \(\langle [\Phi, \beta^*\omega], X \rangle = \langle [\Phi, \tilde{\omega}], X \rangle\), since the latter is, by definition (see \([17]\)), \(\langle \varphi, a^*\omega \rangle, X \rangle\). Let \(\tilde{\omega} = \beta^*\omega - a^*\omega\). Then it is clear that \(\tilde{\omega} = 0\) when restricted to \(T^*\). Therefore, \(\beta(\tilde{\omega})|_P = 0\). Since \(\tilde{\beta} : T^*G \rightarrow A^*G\) is a Lie algebroid morphism over \(\beta : G \rightarrow P\), we have \(\beta(\tilde{\omega})|_P = 0\). This is to say that \([\Phi, \beta^*\omega] = [\Phi, a^*\omega]\) when restricted to \(T^*G\). This concludes the proof for (i).

(ii) We need a lemma before proving (ii). Let \(\Lambda \subset G \times G \times G\) be the graph of the groupoid multiplication.

**Lemma 7.3** Suppose that \(\theta\) is a one-form on \(G\). Then \(\theta\) is the pull back of a one-form on the base manifold \(P\) iff \((0, \theta, -\theta)\) is conormal to \(\Lambda\).

**Proof.** Suppose that \((0, \theta, -\theta)\) is conormal to \(\Lambda\). Let \(\delta_y\) and \(\delta_z\) be any tangent vectors on \(G\) such that \(\langle \beta_y, \delta_y \rangle = \beta_z, \delta_z \rangle\). This condition, of course, implies that \(\beta(y) = \beta(z)\). Let \(x = zy^{-1}\). Then there exists a vector \(\delta_x \in T_xG\) such that \((\delta_x, \delta_y, \delta_z)\) is tangent to the graph \(\Lambda\). It thus follows, by assumption, that \(\langle \theta, \delta_y \rangle = \langle \theta, \delta_z \rangle\), which implies immediately that \(\theta = \beta^*\psi\) for some one-form \(\psi\) on the base manifold. The other direction is obvious.

Since \(\Phi : G \rightarrow T^*G\) is a groupoid morphism, the triple product \(\Phi \times \Phi \times \Phi\) maps \(\Lambda\) into the graph of the groupoid multiplication of \(T^*G\). The latter is the space of all \((\xi, \eta, \zeta) \in T^*G \times T^*G \times T^*G\) such that \((\xi, \eta, -\zeta)\) is conormal to \(\Lambda\). In other words, this is equivalent to saying that \((\Phi, \Phi, -\Phi)\), as a one-form on \(G \times G \times G\), is conormal to \(\Lambda\). On the other hand, we know that \((0, \beta^*\omega, -\beta^*\omega)\) is conormal to \(\Lambda\) according to the lemma above. Since \(\Lambda\) is a coisotropic submanifold of \(G \times G \times \mathbb{G}\), we have that \((0, [\Phi, \beta^*\omega], -[\Phi, \beta^*\omega])\) is conormal to \(\Lambda\). It thus follows from Lemma 7.3 that \([\Phi, \beta^*\omega] = \beta^*\psi\) for some one-form \(\psi\) on the base \(P\). Since \(\beta\) is a submersion, it is now clear that \(\psi = D_{\Phi}(\omega)\).

(iii) Let \(\omega, \theta\) be any one-forms on \(P\), and take \(Y \in T_{1m}G\). Using (ii),
\[
\langle D_{\Phi}[\omega, \theta], T\beta Y \rangle = \langle \beta^*D_{\Phi}[\omega, \theta], Y \rangle = \langle \Phi, \beta^*\omega, \theta, Y \rangle = -\langle \Phi, [\beta^*\omega, \beta^*\theta], Y \rangle \quad \text{(recall \(\beta : G \rightarrow P\) is anti-Poisson)}
\]
\[
= -\langle \Phi, [\beta^*\omega, \beta^*\theta], Y \rangle - \langle \beta^*\omega, [\Phi, \beta^*\theta], Y \rangle
\]

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\[\begin{align*}
&= -\langle [\beta^* D_\Phi \omega, \beta^* \theta], Y \rangle - \langle [\beta^* \omega, \beta^* D_\Phi \theta], Y \rangle \\
&= \langle [D_{\Phi \omega}, \theta], T \beta Y \rangle + \langle [\omega, D_{\Phi \theta}], T \beta Y \rangle.
\end{align*}\]

It thus follows that \(D_\Phi [\omega, \theta] = [D_\Phi \omega, \theta] + [\omega, D_\Phi \theta]\).

\[\square\]

Since \(\pi^\#: T^*G \to TG\) is a groupoid morphism over \(a_*: A^*G \to TP\), it follows that if \((\Phi, \varphi)\) is a star 1-form, then \((\Phi^\#, a_*(\varphi))\) is a star vector field, and if \((\Phi, \varphi)\) is multiplicative, then \((\Phi^\#, a_*(\varphi))\) is also. Further, it is easily seen that

\[a_*(D_\Phi(\omega)) = D_{\Phi^\#}(a_*(\omega))\]

for all \(\omega \in \Omega^1(P)\).

**Lemma 7.4** Let \((\Phi, \varphi)\) be a star 1-form on \(G\), and write \(\xi = \Phi^\#\). Then \(\Phi^\# = \tilde{\xi}\) with respect to the Poisson structure on \(AG\) induced by the Lie algebroid structure on \(A^*G\).

**Proof.** Apply \(A\) to \(\xi = \pi^\# \circ \Phi\) and recall that \(\pi^\#_{AG} \circ j_G^* = j_G^{-1} \circ A(\pi^\#)\).

\[\square\]

**Theorem 7.5** Let \((\Phi, \varphi)\) and \((\Psi, \psi)\) be star 1-forms on \(G\), and let \(\omega, \theta \in \Omega^1(P)\). Then

\[\begin{align*}
[\Phi, \Psi] &= \overline{[\Phi, \Psi]}, \\
[\Phi, q^* \omega] &= q^*(D_\Phi(\omega)), \\
[q^* \theta, q^* \omega] &= 0.
\end{align*}\]

**Proof.** The last equation is of course known. We verify the first two by evaluating them on all \(\tilde{\zeta}\), where \((\zeta, z)\) is a star vector field on \(G\), and all \(Z^\uparrow\), where \(Z \in \Gamma AG\). By Proposition 3.16, this suffices.

For any star vector field \((\zeta, z)\), we have

\[\langle [\Phi, \Psi], Z^\uparrow \rangle = \overline{\langle \Phi, \Psi \rangle} = \tilde{\xi}(\langle \overline{\Phi}, \tilde{\zeta} \rangle) - \overline{\tilde{\eta}}(\langle \overline{\Phi}, \tilde{\zeta} \rangle) - \overline{\langle \Psi, \tilde{\zeta} \rangle} - \overline{\tilde{\xi}(\langle \overline{\Phi}, \tilde{\zeta} \rangle)}\]

where \(\xi = \Phi^\#, \eta = \Psi^\#\). Using (17) and Proposition 3.16, it is straightforward to verify that this is the tilde of \(\langle [\Phi, \Psi], \zeta \rangle\), and is therefore equal to \(\langle [\Phi, \Psi], \zeta \rangle\).

For any \(Z \in \Gamma AG\), (16) gives \(\langle [\Phi, \Psi], Z^\uparrow \rangle = \langle [\varphi, \psi], Z \rangle \circ q\). Similarly expanding out the left hand side, the equation reduces to

\[\langle [\varphi, \psi], Z \rangle = a_*(\varphi)\langle [\psi, Z] \rangle - a_*(\psi)\langle [\varphi, Z] \rangle + \langle \psi, D_q(Z) \rangle - \langle \varphi, D_q(Z) \rangle - Z(\langle \Psi, \xi \rangle).\]

Now the equality of these two functions on \(P\) follows from similarly expanding out \(\langle [\Phi, \Psi], Z^\uparrow \rangle\) on \(G\) and restricting the result to the identity elements of \(G\).

For the second equation, we expand out in the same way, and use the lemma below.

Lastly, it is easy to see that both sides of

\[\langle [\Phi, q^* \omega], Z^\uparrow \rangle = \langle q^* D_\Phi(\omega), Z^\uparrow \rangle\]

are zero, for all \(Z \in \Gamma AG\).
Lemma 7.6 Let \((\Phi, \varphi)\) be a star 1-form, \((\zeta, z)\) a star vector field, and \(\omega \in \Omega^1(P)\). Then

\[
\langle D_\Phi(\omega), z \rangle - \langle \varphi, D_\zeta(a^*_\omega) \rangle = a^*_\omega(\langle \Phi, \zeta \rangle) + \delta \omega(a^*_\varphi, z).
\]

Proof. Let \(X = a^*_\omega \in \Gamma(AG)\) and let \(\vec{X}\) be the corresponding right invariant vector field. Thus

\[
(a^*_\omega)\langle \Phi, \zeta \rangle = \langle L_{\vec{X}} \Phi, \zeta \rangle|_P + \langle \Phi, L_{\vec{X}} \zeta \rangle|_P.
\]

(24)

The second term here is

\[
\langle \Phi, L_{\vec{X}} \zeta \rangle|_P = -\langle \Phi, [\zeta, \vec{X}] \rangle|_P = -\langle \varphi, D_\zeta X \rangle = -\langle \varphi, D_\zeta(a^*_\omega) \rangle.
\]

On the other hand,

\[
\langle D_\Phi \omega, z \rangle = \langle [\Phi, \beta^* \omega], \zeta \rangle|_P = \langle L_{\Phi^#} \beta^* \omega - L(\beta^* \omega)^# \Phi - \delta(\pi(\Phi, \beta^* \omega)), \zeta \rangle|_P.
\]

Clearly, \((\beta^* \omega)^#\) is tangent to the \(\alpha\)-fibers and is right invariant. In fact, \((\beta^* \omega)^# = -(a^*_\omega)^\rightarrow = \vec{X}\). Next,

\[
\langle L_{\Phi^#} \beta^* \omega, \zeta \rangle|_P = \langle \delta \pi(\Phi, \beta^* \omega), \zeta \rangle|_P + \langle (\delta \omega)(\beta^* \Phi^#, \beta^* \zeta) \rangle|_P
\]

(25)

Here, in the last step, we used \(\beta^* \Phi^# = a^*_\varphi\). This is because on \(P\), \(\Phi = \varphi\) is conormal to \(P\) and then \(\Phi^# = \varphi^# = a^*_\varphi\) is tangent to \(P\).

Hence,

\[
\langle D_\Phi \omega, z \rangle = \langle \delta \omega(a^*_\varphi, z) + \langle L_{\vec{X}} \Phi, \zeta \rangle|_P.
\]

Subtracting Equation (25) from (24), one gets

\[
a^*_\omega(\langle \Phi, \zeta \rangle) - \langle D_\phi \omega, z \rangle = \langle \Phi, L_{\vec{X}} \zeta \rangle|_P - (\delta \omega(a^*_\varphi, z) = -\langle \varphi, D_\zeta(a^*_\omega) \rangle - (\delta \omega)(a^*_\varphi, z).
\]

This concludes the proof of the lemma.

\[\square\]

By Proposition 5.6, the three equations in Theorem 7.5 completely determine the bracket structure in \(\Omega^1(AG)\). We could, in fact, define the Poisson structure on \(AG\) by means of them, and obtain the Lie algebroid structure of \(A^*G\) as its dual. In any case, equation (23) now gives an explicit expression for the Lie bialgebroid bracket.
Given a Poisson Lie group \( G \) with dual \( G^* \) it is elementary that there is a bijective correspondence between right–invariant vector fields on \( G^* \) and right–invariant 1–forms on \( G \). For general Poisson groupoids, such a result is not possible.

However, suppose that \( G \to P \) is a Poisson groupoid with dual \( G^* \to P \) in the sense of \([17]\); thus there is an isomorphism of Lie algebroids \( K: A(G^*) \to A^*G \) whose dual \( K^*: AG \to A^*(G^*) \) is also an isomorphism of Lie algebroids. Let \((\xi, x)\) be a multiplicative vector field on \( G \). Then \( H_\xi \in \mathcal{X}(A^*G) \), the dual of \( \xi \in \mathcal{X}(AG) \), induces the morphic vector field \( T(K)^{-1} \circ H_\xi \circ K \) on \( A(G^*) \). If \( G^* \) is \( \alpha \)–simply connected, applying Theorem 4.9 gives a multiplicative vector field \( \eta \in \mathcal{X}(G^*) \) such that \( T(K) \circ \eta = H_\xi \circ K \). Reversing this argument gives the following result.

**Proposition 7.7** Let \( G \to P \) and \( G^* \to P \) be \( \alpha \)–simply connected Poisson groupoids in duality. Then the above gives a bijective correspondence between multiplicative vector fields on \( G \) and multiplicative vector fields on \( G^* \).

For example, let \( \Gamma \to P \) be an \( \alpha \)–simply connected symplectic groupoid realizing a simply–connected Poisson manifold \( P \). Then \( \Gamma \to P \) and \( P \times P \to P \) are dual Poisson groupoids \([17]\). From 7.7 it therefore follows that every vector field on \( P \) induces a multiplicative vector field on \( \Gamma \), and every multiplicative vector field on \( \Gamma \) arises this way.

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