Nonabelian D-branes, Open Strings, and Gauge Theory ¹

Charles B. Thorn²

Institute for Fundamental Theory
Department of Physics, University of Florida, Gainesville FL 32611

Abstract

There is a subtle difference between the open string dynamics determined by the original dual resonance models and that determined by D-brane constructions within critical closed string theory. For instance, in contrast to the former, the latter have massless scalars in addition to the massless gluon shared by both. We introduce and explain the concept of nonabelian D-branes which illuminates this distinction. We employ this concept to offer new possibilities for string duals for large N QCD (pure Yang-Mills gauge theory).

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²E-mail address: thorn@phys.ufl.edu
1 Introduction

In the decade since Maldacena proposed the AdS/CFT correspondence, establishing the equivalence between $\mathcal{N} = 4$ supersymmetric Yang-Mills in 4 spacetime dimensions and type IIB superstring theory on an $\text{AdS}_5 \times S^5$ background [1], it has been a major challenge to find a string dual for QCD. Most attempts in this direction take the conformal $\mathcal{N} = 4$ case as a starting point and seek to break the symmetries down to those of QCD. In ’t Hooft’s $N \to \infty$ limit [2], which only involves the gluonic sector of QCD, this requires giving large masses to all gauginos and to the 6 unwanted scalars on the field side of the duality, with a corresponding modification of the background on the string side [3]. To describe QCD at finite $N$ one must also add new degrees of freedom to describe quarks [4].

A more economical approach, which has neither gauginos nor the 6 unwanted scalars from the start, is to base the string dual for QCD on subcritical string theory [5–7] in 4 space-time dimensions. Inspired by [8–10], we proposed the even G-parity sector [11, 12] of the Neveu-Schwarz open string model in 4 spacetime dimensions [13, 14] as the natural starting point [15]. For brevity we call this the NS+ model. Its low energy limit ($\alpha' \to 0$) is, in perturbation theory, precisely pure Yang-Mills gauge theory with no extra scalar fields and no tachyon [16]. It is probably the simplest open string model with this property. The idea for reaching large N QCD is to take the limit $\alpha' \to 0$ only after summing over all planar open string multi-loop diagrams. The hope is that the graph summation is more tractable in string theory than in field theory, because it can be interpreted as a tree-level shift in the closed string background.

The biggest technical challenge with this subcritical approach is that the perturbative closed string, interacting with the subcritical open string, is imperfectly understood, because it is so qualitatively different from all known (critical) closed string systems. One striking difference, seen in the open string nonplanar one loop diagram, is that the would be “graviton”, the lightest spin two state of the closed string, is massive. All known closed string tree amplitudes predict this state to be massless. Thus a new perturbative closed string theory, determined in principle by the multi-loop open string amplitudes, must be discovered and developed. This is a very interesting (and daunting) problem, which we do not address here.

Instead, our goal is to clarify and then exploit a subtle difference between what we call the true open string and the open string obtained as the T-dual of normal D-brane constructions [17] within critical closed string theory. The T-dual description of an open string with free ends, is an open string with its ends fixed to D-branes. In this way one can understand open strings as fluctuations of a closed string background. To describe a 4 dimensional field theory one introduces a stack of D3-branes on which open strings end. The spectrum of these open strings includes 4D massless vectors associated with open string vibrations parallel to the branes, as well as massless scalars corresponding to moduli for fluctuations of the branes into the extra dimensions transverse to the branes. In the 26 dimensional bosonic string model there would be 22 such massless scalars, whereas in the 10 dimensional NS or NS+ model there would be 6. The string dual to QCD cannot have any extra massless scalars. The fact that these extra massless scalars are not present in true open string theory in 4 dimensions motivated the proposal of [15].
The easiest way to understand this difference is to compare a 24-brane in the 26 dimensional bosonic string theory or an 8-brane in the 10 dimensional NS+ model with the open string in 25, 9 dimensions respectively. The massless level of 25 (9) dimensional bosonic (NS+) open string theory has only 23 (7) degrees of freedom (as a 25 (9) dimensional gluon should), whereas in the brane incarnation there are 24 (8) massless degrees of freedom, the 24th (8th) being a modulus for string fluctuations transverse to the brane. In particular, the usual brane realization of \( D = 25 \) open string has one extra scalar adjoint field in its low energy limit, which is absent in the low energy limit of the true open string theory. This is the reason why the low energy limit of open string models is pure Yang-Mills gauge theory, while in the D-brane interpretation the low energy limit has additional matter degrees of freedom.

In this paper we introduce the concept of nonabelian D-branes to describe true open strings as fluctuations of a closed string background. The usual (abelian) D-brane conditions only restrict the zero mode \( p_0 \), of the open string excitations transverse to the brane, to \( p_0 = 0 \). We propose a stronger nonabelian constraint that also knocks out the lowest non-zero modes \((a_{\pm 1}, b_{\pm 1/2})\). We construct operators \( J_{\pm} \) which commute with the (super) Virasoro algebra, in terms of which the nonabelian D-brane constraints are

\[
J_{\pm}|\text{Phys}\rangle = 0, \quad p_0|\text{Phys}\rangle = 0 \tag{1}
\]

When the dimension of space-time is one less than critical, that is when \( D = 25(9) \), \( J_3 \propto p_0 \), \( J_{\pm} \equiv J_1 \pm iJ_2 \), enjoy the Lie algebra of SU(2) (O(3)).

We develop these ideas in stages. In Section 2 we discuss the nonabelian D-brane construction which describes the \( D = 25 \) open bosonic string. Section 3 extends the construction to the \( D = 9 \) open NS+ string, and Section 4 discusses general subcritical \( D \). Finally in Section 5 we discuss various new options for a string dual of large N QCD.

## 2 Open bosonic string, \( D = 25 \)

When the physical state conditions are solved in \( D < 26 \) dimensional (subcritical) open string theory [18], the null space is (relatively) smaller than in \( D = 26 \) [19, 20]. In the latter case the null space is so big that fully two components of each oscillator decouple. This counting is reflected in the partition function

\[
\text{Tr}_{\text{Phys}}w^R = \prod_{n=1}^{\infty}(1 - w^n)^{-24}, \quad \text{for } D = 26, \tag{2}
\]

where \( R = \sum_{n=1}^{\infty}a_{-n} \cdot a_n \) is the (mass)\(^2\) level operator of bosonic open string theory. In contrast, for \( D < 26 \) one has

\[
\text{Tr}_{\text{Phys}}w^R = (1 - w)\prod_{n=1}^{\infty}(1 - w^n)^{-(D-1)}, \quad \text{for } D < 26. \tag{3}
\]
For $D = 25$ this last reads

$$\text{Tr}_{\text{Phys}} w^R = (1 - w) \prod_{n=1}^{\infty} (1 - w^n)^{-24}, \quad \text{for } D = 25. \quad (4)$$

In the D-brane interpretation of open strings, the ends of an open string are fixed to a D-brane, but its interior is allowed to vibrate in the full 26 dimensional space-time. For example, taking a 24-brane, its associated 25 dimensional open string would have a partition function identical to (2) not (4). The extra factor of $(1 - w)$ in the latter means, among other things, that the massless first excited open string state has 23 not 24 degrees of freedom. There is an extra massless scalar in the D-brane interpretation, compared to the open string model.

We ask therefore whether a more sophisticated brane construction can describe the true open string theory, without this extra massless scalar. For the bosonic string such a construction, which we briefly review in the following, has in essence been given before [8, 9] albeit in a different context. Consider a 24-brane at $x_{25} = 0$ in critical bosonic string theory. Then open strings are required to satisfy $x_{25}(\text{ends}) = 0$. It is convenient to do a T-duality transform from $x_{25}$ to $y_{25}(\sigma, \tau)$ satisfying Neumann conditions. Then the Dirichlet conditions on $x_{25}$ translate to the condition that $y_{25}$ carries zero momentum $p_0 = 0$. It is as though $y_{25}$ had been compactified and only the zero momentum mode kept. Had we kept all momentum modes after compactification of $y_{25}$ to a circle, the possible values of open string momenta would be $p_0 = 2\pi n/r$. For the special case $r = 4\pi \sqrt{\alpha'}$ it is known [21] that the full spectrum of this $c = 1$ conformal field theory supports an SU(2) symmetry. At the level of spectrum this can be seen by first identifying the SU(2) generator $J_3 = p_{25}^0 \sqrt{\alpha'}$ and then considering the SU(2) character of a single free boson including compactified momentum modes, $L_0 = R + \alpha' p_{0}^2$,

$$\text{Tr} w^L e^{-i\theta J_3} = \sum_{k=-\infty}^{\infty} w^{k^2/4} e^{i\theta/2} \prod_{n=1}^{\infty} \frac{1}{1 - w^n}. \quad (5)$$

The SU(2) character for irreducible representation $j$ is $\chi_j(\theta) = \sum_{m=-j}^{j} e^{im\theta}$ so we can write $e^{i\theta/2} + e^{-i\theta/2} = \chi_{k/2} - \chi_{(k-2)/2}$ from which we see

$$\text{Tr} w^L e^{-i\theta J_3} = \sum_{k=0}^{\infty} w^{k^2/4} \chi_{k/2}(\theta)(1 - w^{k+1}) \prod_{n=1}^{\infty} \frac{1}{1 - w^n} = \sum_{k=0}^{\infty} \chi_{k/2} w^{k^2/4} \prod_{n\neq k+1} \frac{1}{1 - w^n}. \quad (6)$$

This shows that the spectrum supports an SU(2) symmetry, because each irreducible representation occurs a positive number of times. Furthermore, it shows that the SU(2) invariant subspace, $k = 0$, has the partition function

$$\prod_{n=2}^{\infty} \frac{1}{1 - w^n} = (1 - w) \prod_{n=1}^{\infty} \frac{1}{1 - w^n}. \quad (7)$$
precisely the counting of longitudinal states needed to give the \( D = 25 \) open string. In this language the usual brane construction gives the \( J_3 = 0 \) subspace, the subspace invariant under rotations about the 3-axis. By strengthening the restriction to the full nonabelian SU(2) invariance we get the true open string spectrum. We call this a nonabelian D-brane.

There is also an explicit construction of the SU(2) generators, \( J_3, J_\pm = J_1 \pm iJ_2 \), based on vertex operators [21]:

\[
J_\pm = \int \frac{dz}{2\pi i z} : e^{\pm i y^{25}(z)/\sqrt{\alpha'}} : \quad J_3 = \sqrt{\alpha'} p_0^{25}
\]  

(8)

Because the vertex operator \( : e^{\pm i y^{25}(z)/\sqrt{\alpha'}} : \) has conformal weight 1, we have \([L_n, J_\pm] = 0\) (that \([L_n, J_3] = 0\) is trivial) provided the state space is restricted to states with momenta \( p_0^{25} = \mathbb{Z}/2\sqrt{\alpha'} \), i.e. the state space of the compactified \( y^{25} \). Thus the SU(2) invariance constraint can be consistently applied to the physical subspace, and we can then characterize the nonabelian D-brane physical open string states by the succinct conditions

\[
(L_0 - 1)|\text{Phys}\rangle = L_n|\text{Phys}\rangle = J^a|\text{Phys}\rangle = 0, \quad n > 0, \quad a = 1, 2, 3.
\]  

(9)

Here the \( L_n \) contain all 26 components of the \( a_n, p_0 \) operators. Note that the “normal” D-brane condition \( J_3 = \sqrt{\alpha'} p_0 = 0 \) ensures that the \( J_\pm \) are properly defined on the physical subspace. The non-null solutions of these conditions are in 1-1 correspondence with the non-null physical states of the open bosonic string in 25 space-time dimensions.

The SU(2) invariant subspace of the Fock space of \( y^{25} \) can also be identified with the conformal block of the \( c = 1 \) Virasoro generators \( L_n^{25} \equiv \ell_n \), built on the \( p_0^{25} = 0 \) primary state \(|0, 0\rangle\), for which it is immediate that \( J_a|0, 0\rangle = 0 \). Because it has zero momentum it also satisfies \( \ell_{-1}|0, 0\rangle = 0 \), so this conformal block is spanned by the states

\[
\ell_{-n}^{\lambda_n} \cdots \ell_{-2}^{\lambda_2} \ell_{-1}^{\lambda_1}|0, 0\rangle,
\]  

(10)

which are all SU(2) invariant because \([J^a, \ell_{-n}^{25}] = 0\). The partition function for this subspace is evidently \( \prod_{n=2}^{\infty} (1 - w^n)^{-1} \), precisely as needed for the longitudinal open string physical states.

3 Open Neveu-Schwarz string, D=9

The NS open string in \( D < 10 \) space-time dimensions [13, 14] has physical states, with counting analogous to the bosonic string, described by [20, 22] the partition functions\(^3\)

\[
\text{Tr}_{\text{Phys}} w^R(\pm)^{2R} = (1 \mp w^{1/2}) \prod_{r=1/2}^{\infty} (1 \pm w^r)^{(D-1)} \prod_{n=1}^{\infty} (1 - w^n)^{-(D-1)}, \quad \text{NS for } D < 10.
\]  

(11)

\(^3\)Here we use the Picture 2 Fock space [14] in which the massless gluon state is \( b_{-1/2}|0, p\rangle \).
One might think these are fermionic operators, but they are actually bosonic because we consider the integrated vertex operators $H$.

The $(\pm)$ case is needed in constructing the projector $(1 - (\pm)2R)/2$ onto even G-parity. Here $R = \sum_{n=1}^{\infty} a_{-n} \cdot a_n + \sum_{r=1/2}^{\infty} r b_{-r} \cdot b_r$. As in the bosonic theory the extra factor $(1 - w^{1/2})$ accounts for the transversality of the massless gluon state. For the critical dimension

$$\text{Tr}_{\text{Phys}} w^R (\pm)^{2R} = \prod_{r=1/2}^{\infty} (1 \pm w^r)^{(D-2)} \prod_{n=1}^{\infty} (1 - w^n)^{-1} - (D-2)$$

$$\rightarrow \prod_{r=1/2}^{\infty} (1 \pm w^r)^8 \prod_{n=1}^{\infty} (1 - w^n)^{-8}, \quad \text{NS for } D = 10. \quad (12)$$

Again for $D = 9$ (one less than critical) the counting is not quite the same as critical:

$$\text{Tr}_{\text{Phys}} w^R (\pm)^{2R} = (1 \mp w^{1/2}) \prod_{r=1/2}^{\infty} (1 \pm w^r)^8 \prod_{n=1}^{\infty} (1 - w^n)^{-8}$$

$$= \left( \frac{1 \pm w^{1/2}}{1 - w} \right) 7 \prod_{r=3/2}^{\infty} (1 \pm w^r)^8 \prod_{n=2}^{\infty} (1 - w^n)^{-8}, \quad \text{NS for } D = 9, \quad (13)$$

differing in the extra factor $1 \mp w^{1/2} = (1 - w)/(1 \pm w^{1/2})$ implying that the gluon state has only 7 degrees of freedom. For a normal 8-brane in 10 space-time dimensions this state would have 8 degrees of freedom.

So we should try to interpret the NS open string in 9 space-time dimensions as a non-abelian 8-brane. Accordingly, we impose $x^9(\text{ends}) = 0$, do a T-duality transformation to $y^9$ with momentum constraint $p_0^9 = 0$, and seek a nonabelian generalization of this condition. The nonabelian generators must commute with the super Virasoro algebra $G_r, L_n$, so we consider the integrated vertex operators $H(z) : e^{\pm i y^9(z)/\sqrt{2\alpha'}} :$, which have conformal weight 1,

$$J_\pm = \sqrt{2} \oint \frac{dz}{2\pi iz} H(z) : e^{\pm i y^9(z)/\sqrt{2\alpha'}} :. \quad (14)$$

One might think these are fermionic operators, but they are actually bosonic because $e^{iy^9(z)/\sqrt{2\alpha'}}$ is just a bosonized fermionic field when acting on the states with $p_0^9 \in \mathbb{Z}/\sqrt{2\alpha'}$. The $J_\pm$ commute with both $G_r$ and $L_n$ on this subspace of states. If we identify $J_3 = p_0 \sqrt{2\alpha'}$, so that $[J_3, J_\pm] = J_\pm$, we then find, with the above normalization, that $[J_+, J_-] = 2J_3$. The SU(2) character on this space of states is

$$\text{Tr} w^{L_0} (\pm)^{2R} e^{-i\theta J_3} = \sum_{k=0}^{\infty} w^{k^2/2} e^{i k \theta} \prod_{r=1/2}^{\infty} (1 \pm w^r) \prod_{n=1}^{\infty} \frac{1}{1 - w^n}$$

$$= \sum_{k=0}^{\infty} w^{k^2/2} \chi_k(\theta)(1 \mp w^{k+1/2}) \prod_{r=1/2}^{\infty} (1 \pm w^r) \prod_{n=1}^{\infty} \frac{1}{1 - w^n} \quad (15)$$

$$= \sum_{k=0}^{\infty} w^{k^2/2} \chi_k(\theta) \prod_{r \neq k+1/2}^{\infty} (1 \pm w^r) \prod_{n \neq 2k+1}^{\infty} \frac{1}{1 - w^n} \quad (16)$$

$$= \sum_{k=0}^{\infty} w^{k^2/2} \chi_k(\theta) \prod_{r \neq k+1/2}^{\infty} (1 \pm w^r) \prod_{n \neq 2k+1}^{\infty} \frac{1}{1 - w^n} \quad (17)$$
In the NS model we see that spinor representations of SU(2) are absent, so it is more accurate to say that the symmetry is O(3). This formula shows that the partition function for the O(3) invariant subspace is

\[
(1 \mp w^{1/2}) \prod_{r=1/2}^\infty (1 \pm w^r) \prod_{n=1}^\infty \frac{1}{1 - w^n} = \prod_{r=2/3}^\infty (1 \pm w^r) \prod_{n=2}^\infty \frac{1}{1 - w^n},
\]

which enumerates precisely the number of longitudinal states needed to describe the NS open string in 9 space-time dimensions. Again these states are in 1-1 correspondence with the superconformal block built on the O(3) invariant primary \( |0,0\rangle \):

\[
\ell_{-n} \cdots \ell_{-2} \gamma_{-r} \cdots \gamma_{-3/2} |0,0\rangle
\]

where \( \ell_n, g_r \) are the \( c = 1 \) super-Virasoro generators built out of \( a_n^0, b_r^0 \). Factors of \( \ell_{-1}, g_{-1/2} \) are absent because they kill \( |0,0\rangle \) since \( a_0^0 |0,0\rangle = 0 \).

Finally we mention that we could also describe this \( c = 1 \) system in a manifestly O(3) invariant way by replacing the bosonic \( y^0 \) with a pair of fermion fields \( H_1, H_2 \) and by identifying the original \( H \) as \( H^3 \). Then the three \( H^a \) transform as a vector under \( O(3) \) with generators

\[
J_a = \epsilon_{abc} \oint \frac{dz}{2\pi i z} H^b(z) H^c(z).
\]

Then the nonabelian D-brane condition would be \( J_a |\text{Phys} \rangle = 0 \).

4 General \( D \)

For the nonabelian D-brane construction to work for the open bosonic string in 25 space-time dimensions and the open NS string in 9 space-time dimensions, it was essential that the physical state conditions effectively removed two components from each mode. This will be the case if the (super)Virasoro central charge \( c \) is critical (\( c = 26, 10 \) respectively). In each case, adding the extra dimension promotes the \( c \) to critical. Imposing SU(2) (O(3)) invariance knocks out the first mode of the extra dimension and finally the critical (super) Virasoro conditions remove two components leading to (4) or (13) respectively.

This same construction doesn’t immediately work for other subcritical dimensions, because then adding the extra dimension only promotes \( c = D \) to \( c = D + 1 \) and, for \( D + 1 \) < critical, the physical state conditions only remove one component for \( n > 1 \) or \( r > 1/2 \) modes, though they do remove two from the \( n = 1 \) and \( r = 1/2 \) modes. Thus if we literally applied the construction to this case we would arrive at

\[
\text{Tr}_{\text{Phys}} w^R = (1 - w)^2 \prod_{n=1}^\infty (1 - w^n) \prod_{n=1}^\infty (1 - w^n)^{-D}
\]

\[
\text{Tr}_{\text{Phys}} w^{R(\pm)^2} = (1 \mp w^{1/2})^2 \prod_{r=1/2}^\infty (1 \pm w^r) \prod_{n=1}^\infty (1 - w^n)^{-D}
\]
instead of (3) or (11). However, notice that as far as the \(a_1\) and \(b_{1/2}\) modes are concerned the state counting does agree. In particular the massless states still have only transverse degrees of freedom.

Fortunately there is a deformation of the Virasoro generators, discovered by David Fairlie and me independently in 1971 (see [23]), with an easy extension to the NS super Virasoro generators:

\[
L_n^\alpha = i \alpha n a_n^{D+1} + \hat{L}_n, \quad G_r^\alpha = 2i \alpha r b_r^{D+1} + \hat{G}_r, \quad L_0^\alpha = \frac{\alpha^2}{2} + \hat{L}_0,
\]

(23)

where \(a_n^{D+1}, b_r^{D+1}\) are the bose and fermi oscillators associated with the extra dimension, and the hatted generators are the usual flat space generators in \(D+1\) dimensions. These modified operators satisfy the (super)Virasoro algebra with \(c = D + 1 + 12\alpha^2\) for the bosonic open string or \(c = D + 1 + 8\alpha^2\) for the NS open string. This deformation can be employed to obtain \(c = 26\) or \(c = 10\) which then determines \(\alpha^2 = (25 - D)/12\) or \(\alpha^2 = (9 - D)/8\). The conformal block built on the primary state \(|0,p\rangle\) will have the degeneracy of open string longitudinal modes, provided \(L_{-1}|0,p\rangle = 0\) (or \(G_{-1/2}|0,p\rangle = 0\) in the NS case). This determines \(\sqrt{2\alpha}p = i\alpha\). We must also deform the \(J_{\pm}\) in order that they commute with the deformed (super)Virasoro operators:

\[
J_{\pm} \to \int \frac{dz}{2\pi i z} : e^{i\gamma_{\pm} y^{D+1}(z)/\sqrt{\alpha'}} :,
\]

(24)

which is well defined on states with \(p_0 \in \left(\mathbb{Z}/\gamma_{\pm} - \gamma_{\pm}\right)/2\sqrt{\alpha'}\). The condition that \(J_{\pm}\) commute with the Virasoro algebra is that the vertex operator have conformal weight 1:

\[
\gamma^2 + i\sqrt{2}\alpha\gamma = 1, \quad \gamma_{\pm} = -i\frac{\alpha}{\sqrt{2}} \pm \sqrt{1 - \frac{\alpha^2}{2}} = -i\sqrt{\frac{25 - D}{24}} \pm \sqrt{\frac{D - 1}{24}}.
\]

(25)

The condition that \(J_{\pm}\) is well defined now reads

\[
p_0 \in -\frac{\frac{\gamma_{\pm}Z}{\gamma_{\pm}} + \gamma_{\pm}}{2\sqrt{\alpha'}}
\]

(26)

and can be met for both \(J_+\) and \(J_-\) only for the element \(1 \in \mathbb{Z}\), in which case \(p_0 = i\alpha/\sqrt{2\alpha'}\). It is then very easy to show that \(J_+|0,\frac{i\alpha}{\sqrt{2\alpha'}}\rangle = 0\). Then \(J_+\) kills the whole conformal block built on this ket, which as noted above is also killed by \(L_{-1}\). However the algebra of the \(J_{\pm},p_0\) is not the simple \(SU(2)\) we found for the undeformed case.

In the NS case the deformation that makes \(J_{\pm}\) commute with the super-Virasoro algebra reads

\[
J_{\pm} \to \sqrt{2} \int \frac{dz}{2\pi i z} H(z) : e^{i\gamma_{\pm} y^{D+1}(z)/\sqrt{\alpha'}} :,
\]

(27)

which is well defined on states with \(p_0 \in \left(\mathbb{Z} + 1/2)/\gamma_{\pm} - \gamma_{\pm}\right)/2\sqrt{\alpha'}\). The condition that \(J_{\pm}\) commute with the super Virasoro algebra is that the exponential vertex operator have
conformal weight 1/2:
\[ \gamma^2 + i \sqrt{2} \alpha \gamma = \frac{1}{2}, \quad \gamma_\pm = -i \frac{\alpha}{\sqrt{2}} \pm \sqrt{\frac{1}{2} - \frac{\alpha^2}{2}} = -i \sqrt{\frac{9 - D}{16} \pm \sqrt{\frac{D - 1}{16}}}. \] (28)

The condition that \( J_\pm \) is well defined now reads
\[ p_0 \in -\frac{2\gamma_\pm (Z + 1/2) + \gamma_\pm}{2\sqrt{\alpha'}} \] (29)
and can be met for both \( J_+ \) and \( J_- \) only for the element \( 0 \in \mathbb{Z} \), in which case \( p_0 = i\alpha/\sqrt{2\alpha'} \). It is then very easy to show that \( J_\pm|0, i\alpha/\sqrt{2\alpha'}\rangle = 0 \). Then \( J_\pm \) kills the whole superconformal block built on this ket, which as noted above is also killed by \( G_{-1/2} \). But again the algebra of the \( J_\pm, p_0 \) is not the simple O(3) we found for the undeformed NS case.

An unappealing feature of the \( D + 1 < \) critical case is the fact that the values of \( p_0 \) for which the \( J_\pm \) are well-defined are necessarily complex. This poses conceptual difficulties for the interpretation of the open string as a unitary worldsheet system with \( p_0^\dagger = p_0 \), because it forces an asymmetry between the kets and bras of the quantum mechanical interpretation:
\[ p_0|0, p\rangle = p|0, p\rangle \implies \langle 0, p|p_0 = p^\ast \langle 0, p| \] (30)
So the dual to \( |0, p\rangle \) must be \( \langle 0, p^\ast| \):
\[ \langle 0, p^\ast|0, p\rangle \neq 0, \quad \text{but} \quad \langle 0, p|0, p\rangle = 0. \] (31)

On the other hand, this asymmetry fits neatly in a string field [24] description as explained in [9]. This is because string fields are naturally associated with the kets, \( |A\rangle, |A\rangle \otimes |A\rangle, \ldots \) whereas the bras are used to define terms in the action.

5 String duals for gauge theory in 4 Dimensions

The concept of nonabelian D-branes extends the range of possible string duals for QCD. Since the \( \alpha' \to 0 \) limit removes all massive string states from the theory, it is obvious that there are many choices one can make for the \( \alpha' \neq 0 \) open string theory. The 4 dimensional NS+ model advocated in [15] has the simplest mass spectrum. For this model, though, we must await better understanding of the closed string system that couples to this subcritical open string system.

In the meantime, we can consider open string systems that couple to critical closed strings. Indeed the original \( \mathcal{N} = 4/\text{AdS}_5 \times S^5 \) duality is of this type. In this case the relevant \( \alpha' \neq 0 \) open string system is just the GSO [12] projected Neveu-Schwarz-Ramond [13, 14, 25] open string in 10 space-time dimensions with ends fixed to a stack of normal (abelian) D3-branes. We can also consider the NS+ open string model in 10 spacetime dimensions which, together with its corresponding closed string system, has been called the type 0 string theory [26]. Adding ordinary D3-branes to this type 0 theory provides a string dual for Yang-Mills
interacting with 6 adjoint massless scalars [27]. However, to reach the gluonic sector of QCD with this construction, one must find a mechanism to give large masses to these scalars.

Alternatively, the concept of nonabelian D-branes provides a way to completely remove these unwanted massless scalars from the beginning. By imposing nonabelian Dirichlet conditions, in all six extra dimensions, on open strings ending on the D3-branes, one removes (among other things) all 6 massless adjoint scalars from the type-0 theory. In this way, one produces a new candidate for a string dual of large N QCD, which avoids the conceptual difficulties of the 4 dimensional subcritical NS+ model. The price is a somewhat richer massive open string spectrum. However, all such string states become infinitely massive as $\alpha' \rightarrow 0$, and so the low energy limit will be pure Yang-Mills gauge theory with no extra matter fields. For describing QCD, for which the strong 't Hooft coupling limit is problematic because of asymptotic freedom, this model has the advantage of being cleanly defined as a string theory.

5.1 Scattering Amplitudes

The open string tree scattering amplitudes for this critical NS+ model are a subset of the critical NS tree amplitudes: one simply restricts the external strings to be in even G-parity states invariant under all 6 SU(2)'s associated with the 6 extra dimensions. Then internal poles of the tree amplitudes factorize only on this subset of invariant states. In particular, the massless state (in Picture 2 [14]) $b_{-1/2}^I [0, p]$, with $I = 4, 5, \ldots, 9$, $p^2 = 0$, transforms as a component\(^4\) of a vector under the $I$th SU(2) and so will decouple. Any number of strings in invariant states will never produce a single string in a noninvariant state.

Of course, NS+ trees with a noninvariant state on two or more external legs can be nonvanishing. This means that an amplitude with one or more loops will differ from the corresponding NS+ amplitude, because every line participating in a loop must include a projector onto invariant states. Dealing with these projectors at general loop order is a challenging unsolved problem. However, at one loop it is not difficult, especially if the external legs are states, such as the massless gluon, with no excitations in the 6 extra dimensions. In that case the modification resides solely in the partition function factor of the loop integrand. Only one of the lines in the loop needs to carry the projector, whose effect is simply to multiply the usual NS partition function by the factors $(1 \mp w^{1/2})^6$, one factor for each extra dimension (see Eq (18)). In addition, one must remember that, just as with normal D-branes, the loop momentum integral is only over the 4 components parallel to the D3-brane.

After the Jacobi transformation to cylinder variables $\ln q = 2\pi^2 / \ln w$, these differences produce the extra factors

$$
\left[ \sqrt{-\frac{\pi}{\ln q}} (1 \mp w^{1/2}) \right]^6 = \left[ \int d\mu q^{\mu^2/4} \sin^2 \frac{\mu}{2\sqrt{2}} \right]^6
\left[ \int d\mu q^{\mu^2/4} \cos^2 \frac{\mu}{2\sqrt{2}} \right]^6
$$

\(^4\)The other two components of the vector have $p' \neq 0$. 

relative to the usual one loop integrand in $D = 10$. On the right we have shown the interpretation of the extra factors as an integral over closed string “momenta” $\mu^I$ transverse to the D3-branes, $I = 4, 5, \ldots, 9$. Notice that the normal D-brane would not have the $(1 \mp w^{1/2})^6$ factor, so the $\sin^2$ or $\cos^2$ factor would be replaced by $1/2$. In these coordinates we see how the closed string couples to the nonabelian D3-brane. The difference from the normal D-brane is just the zero mode wave function factor $\psi_\pm^2(\mu^I)$ with $\psi_+(\mu^I) = 2^3 \prod_{I=4}^9 \sin(\mu^I/2\sqrt{2})$ ($\psi_-(\mu^I) = 2^3 \prod_{I=4}^9 \cos(\mu^I/2\sqrt{2})$). The $+$ and $-$ terms of the loop integrand describe the NS-NS and R-R sectors, respectively, of the type-0 closed string system. The presence of the factor $\psi_\pm$ for a nonabelian D-brane introduces an asymmetry in the coupling of these two sectors compared to the normal D-brane. For example at small $\mu$, corresponding to large distances transverse to the D3-branes, the NS-NS sector is relatively suppressed. The closed string system that propagates in the 10 dimensional bulk is just the critical type-0 closed string, which is well understood—at least in perturbation theory. However, as we have just seen, the coupling of these closed strings to a nonabelian D3-brane is significantly different from the coupling to the normal (abelian) D3-brane.

### 5.2 Lightcone gauge

We have described a number of open string models whose $\alpha' \to 0$ limit is Yang-Mills in weak coupling perturbation theory. To reach $N = \infty$ QCD, one next has to sum the planar diagrams. Lightcone quantization of the string [28] using the path integral description [29] provides one systematic approach to this problem, which promises to be very useful if ever a numerical assault on the problem is considered. We close by considering how the nonabelian D-brane concept fits into the lightcone description.

A typical multi-loop planar lightcone interacting string diagram [29] is shown in Fig. 1. The horizontal dimension of this two dimensional diagram is lightcone time $x^+ = (t + z)/\sqrt{2}$ and the vertical dimension is $p^+ = (p^0 + p^z)/\sqrt{2}$. The diagram describes the evolution in time of a system of open strings, breaking and rejoining as shown by the horizontal lines. For the critical open string, it represents a worldsheet path integral whose action is just the lightcone action for the free open string of interest. To sum all planar diagrams one simply

![Figure 1: A planar lightcone interacting open string diagram with 7 loops.](image-url)
sums over the number, length, and location of horizontal lines. For each beginning and end of a horizontal line there is a factor of string coupling $g$, and perhaps a prefactor (as needed for the NS string). As shown in [30] one can mark the presence or absence of a horizontal line at any point by an Ising spin variable $S(\sigma, \tau) = \pm 1$, and execute the sum as a sum over all spin configurations, with spin dependent terms in the action enforcing the necessary boundary conditions.

For this interpretation to succeed it is essential that the path integrand be local. At first glance incorporating D-brane constraints seems to spoil locality. However normal D-brane conditions are not a problem because they are applied locally on the worldsheet $x_{\text{ends}} = 0$. But this condition looks non-local in T-dual variables $\int d\sigma \dot{y} = 0$. Since we only know the nonabelian D-brane conditions in these T-dual coordinates, we need to consider them a little more carefully. Each intermediate open string must be projected onto states invariant under the SU(2)'s for all dimensions transverse to the brane. We can express the projector for the $I$th SU(2) as a group integral

$$P_I = \int d\Re e^{i\theta_a J^a_I}, \quad J^a_I = \int d\sigma J^a_I(\sigma)$$

(33)

For concreteness let’s focus on the critical NS+ open string ending on nonabelian D3-branes, so the symmetry is O(3), and $dR$ is the O(3) invariant Haar measure. The number of such projector $P = \prod_{I=1}^9 P_I$ insertions changes with the appearance or disappearance of a horizontal line. Since $P^2 = P$ we can put a projector on each time-slice of each individual string propagator, and introduce also independent $R$’s for each spacetime point $\sigma, t$:

$$P_I = \prod_t P_I = \int \prod_t dR(t) e^{i\theta_a(t) J^a_I} = \int \prod_\sigma \prod_t dR(\sigma, t) e^{i\int d\sigma d\theta_a(\sigma, t) J^a_I} \delta(R'(\sigma, t))$$

(34)

Then by deleting the $\delta(R'(\sigma, t))$ factor whenever $\sigma$ sits on a horizontal line, we arrive at a worldsheet local prescription for inserting projectors. In effect we are gauging each O(3) symmetry on the worldsheet, but replacing the factor $e^{-\int R^2/4}$ with $\prod_{\sigma,t} \delta(F(\sigma, \tau))$.

One application of this lightcone treatment of the NS+ open string ending on nonabelian D3-branes is to provide a good regularization for the lightcone worldsheet construction of the planar diagram sum in quantum field theory [30]. The one loop analysis of that construction for QCD, given in [31], showed the need for counterterms, whose coefficients could only be determined by hand. We hope that replacing the field theory planar graphs with $\alpha' \neq 0$ open string graphs as described here and using the GNS regularization [32] will systematically determine these counter terms.

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