Corrigendum for the comparison theorems in ”A new definition of viscosity solutions for a class of second-order degenerate elliptic integro-differential equations”.

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In this note, we shall present the correction of the proofs of the comparison results in the paper [1]. In order to show clearly the correct way of the demonstration, we shall simplify the problem to the following.

(Problem (I)) : \[ F(x, u, \nabla u, \nabla^2 u) - \int_{\mathbb{R}^N} u(x + z) - u(x) \]
\[-1_{|z| \leq 1}(z, \nabla u(x))q(dz) = 0 \quad \text{in} \quad \Omega, \quad (1)\]

(Problem (II)) : \[ F(x, u, \nabla u, \nabla^2 u) - \int_{\{z \in \mathbb{R}^N | x + z \in \Omega\}} u(x + z) - u(x) \]
\[-1_{|z| \leq 1}(z, \nabla u(x))q(dz) = 0 \quad \text{in} \quad \Omega, \quad (2)\]

where \( \Omega \subset \mathbb{R}^N \) is open, and \( q(dz) \) is a positive Radon measure such that \( \int_{|z| \leq 1} |z|^2 q(dz) + \int_{|z| > 1} 1q(dz) < \infty \). Although in [1] only (II) was studied, in order to avoid the non-essential technical complexity, here, let us give the explanation mainly for (I). For (I), we consider the Dirichlet B.C.:
\[ u(x) = g(x) \quad \forall x \in \Omega^c, \quad (3) \]
where $g$ is a given continuous function in $\Omega^c$. For (II), we assume that $\Omega$ is a precompact convex open subset in $\mathbb{R}^N$ with $C^1$ boundary satisfying the uniform exterior sphere condition, and consider either the Dirichlet B.C.:

$$u(x) = h(x) \quad \forall x \in \partial \Omega,$$

(4)

where $h$ is a given continuous function on $\partial \Omega$, or the Neumann B.C.:

$$\langle \nabla u(x), n(x) \rangle = 0 \quad \forall x \in \partial \Omega,$$

(5)

where $n(x) \in \mathbb{R}^N$ the outward unit normal vector field defined on $\partial \Omega$. The above problems are studied in the framework of the viscosity solutions introduced in [1]. Under all the assumptions in [1], for (I) the following comparison result holds, and for (II), although the proofs therein are incomplete, the comparison results stated in [1] hold, and we shall show in a future article.

**Theorem 1.1 (Problem I with Dirichlet B.C.)** Assume that $\Omega$ is bounded, and the conditions for $F$ in [1] hold. Let $u \in USC(\mathbb{R}^N)$ and $v \in LSC(\mathbb{R}^N)$ be respectively a viscosity subsolution and a supersolution of (1) in $\Omega$, which satisfy $u \leq v$ on $\Omega^c$. Then, $u \leq v$ in $\Omega$.

To prove Theorem 1.1, we approximate the solutions $u$ and $v$ by the sup-convolution: $u^r(x) = \sup_{y \in \mathbb{R}^N} \{u(y) - \frac{1}{2r^2}|x - y|^2\}$ and the infconvolution: $v^r(x) = \inf_{y \in \mathbb{R}^N} \{v(y) + \frac{1}{2r^2}|x - y|^2\} \ (x \in \mathbb{R}^N)$, where $r > 0$.

**Lemma 1.2 (Approximation for Problem (I))** Let $u$ and $v$ be respectively a viscosity subsolution and a supersolution of (1). For any $\nu > 0$ there exists $r > 0$ such that $u^r$ and $v^r$ are respectively a subsolution and a supersolution of the following problems.

$$F(x, u, \nabla u, \nabla^2 u) - \int_{\mathbb{R}^N} u(x + z) - u(x) - 1_{|z| < 1} \langle z, \nabla u(x) \rangle q(dz) \leq \nu,$$

(6)

$$F(x, v, \nabla v, \nabla^2 v) - \int_{\mathbb{R}^N} v(x + z) - v(x) - 1_{|z| < 1} \langle z, \nabla v(x) \rangle q(dz) \geq -\nu,$$

(7)

in $\Omega_r = \{x \in \Omega | \ dist(x, \partial \Omega) > \sqrt{2Mr}\}$, where $M = \max\{\sup_{\Omega}|u|, \sup_{\Omega}|v|\}$. 

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Remark that $u'$ is semiconvex, $v_r$ is semiconcave, and both are Lipschitz continuous in $\mathbb{R}^N$. We then deduce from the Jensen’s maximum principle and the Alexandrov’s theorem (deep results in the convex analysis, see [2] and [3]), the following lemma, the last claim of which is quite important in the limit procedure in the nonlocal term.

**Lemma 1.3** Let $U$ be semiconvex and $V$ be semiconcave in $\Omega$. For \( \phi(x, y) = \alpha|x - y|^2 \) (\( \alpha > 0 \)) consider $\Phi(x, y) = U(x) - V(y) - \phi(x, y)$, and assume that $(x, y)$ is an interior maximum of $\Phi$ in $\overline{\Omega} \times \overline{\Omega}$. Assume also that there is an open precompact subset $O$ of $\Omega \times \Omega$ containing $(x, y)$, and that $\mu = \sup_{O} \Phi(x, y) - \sup_{\partial O} \Phi(x, y) > 0$. Then, the following holds.

(i) There exists a sequence of points $(x_m, y_m) \in O$ (\( m \in \mathbb{N} \)) such that $\lim_{m \to \infty} (x_m, y_m) = (x, y)$, and $(p_m, X_m) \in J_{\Omega}^+ U(x_m)$, $(p'_m, Y_m) \in J_{\Omega}^- V(y_m)$ such that $\lim_{m \to \infty} p_m = \lim_{m \to \infty} p'_m = 2\alpha(x_m - y_m) = p$, and $X_m \leq Y_m \quad \forall m$.

(ii) For $P_m = (p_m - p, -(p'_m - p))$, $\Phi_m(x, y) = \Phi(x, y) - \langle P_m, (x, y) \rangle$ takes a maximum at $(x_m, y_m)$ in $O$.

(iii) The following holds for any $z \in \mathbb{R}^N$ such that $(x_m + z, y_m + z) \in O$.

\[
U(x_m + z) - U(x_m) - \langle p_m, z \rangle \leq V(y_m + z) - V(y_m) - \langle p'_m, z \rangle. \tag{8}
\]

By admitting these lemmas here, let us show how Theorem 1.1 is proved.

**Proof of Theorem 1.1.** We use the argument by contradiction, and assume that $\max_{\overline{\Omega}} (u - v) = (u - v)(x_0) = M_0 > 0$ for $x_0 \in \Omega$. Then, we approximate $u$ by $u'$ (supconvolution) and $v$ by $v_r$ (infconvolution), which are a subsolution and a supersolution of (6) and (7), respectively. Clearly, $\max_{\overline{\Omega}} (u' - v_r) \geq M_0 > 0$. Let $x \in \Omega$ be the maximizer of $u' - v_r$. In the following, we abbreviate the index and write $u = u'$, $v = v_r$ without any confusion. As in the PDE theory, consider $\Phi(x, y) = u(x) - v(y) - \alpha|x - y|^2$, and let $(\hat{x}, \hat{y})$ be the maximizer of $\Phi$. Then, from Lemma 1.3 there exists $(x_m, y_m) \in \Omega$ (\( m \in \mathbb{N} \)) such that $\lim_{m \to \infty} (x_m, y_m) = (\hat{x}, \hat{y})$, and we can take $(\epsilon_m, \delta_m)$ a pair of positive numbers such that $u(x_m + z) \leq u(x_m) + \langle p_m, z \rangle + \frac{1}{2} \langle X_m z, z \rangle + \delta_m |z|^2$, $v(y_m + z) \geq v(y_m) + \langle p'_m, z \rangle + \frac{1}{2} \langle Y_m z, z \rangle - \delta_m |z|^2$, for $\forall |z| \leq \epsilon_m$. From the definition of the viscosity solutions, we have

\[
F(x_m, u(x_m), p_m, X_m) - \int_{|z| \leq \epsilon_m} \frac{1}{2} \langle X_m + 2\delta_m I \rangle z \rangle dq(z)
- \int_{|z| \geq \epsilon_m} u(x_m + z) - u(x_m) - 1_{|z| \leq \epsilon_m} (z, p_m) q(dz) \leq \nu,
\]
\[
F(y_m, v(y_m), p'_m, Y_m) - \int_{|z| \leq \varepsilon_m} \frac{1}{2} \langle (Y_m - 2\delta_m I) z, z \rangle dq(z) \\
- \int_{|z| \geq \varepsilon_m} v(y_m + z) - v(y_m) - 1_{|z| \leq 1} \langle z, p'_m \rangle q(dz) \geq -\nu.
\]

By taking the difference of the above two inequalities, by using (8), and by passing \( m \to \infty \) (thanking to (8), it is now available), we can obtain the desired contradiction. The claim \( u \leq v \) is proved.

**Remark 1.1.** To prove the comparison results for (II) (in [1]), we do the approximation by the supconvolution: \( u_r(x) = \sup_{y \in \Omega} \{ u(y) - \frac{1}{2r^2} |x - y|^2 \} \), and the inconvolution: \( v_r(x) = \inf_{y \in \Omega} \{ v(y) + \frac{1}{2r^2} |x - y|^2 \} \) as in Lemma 1.2. Because of the restriction of the domain of the integral of the nonlocal term and the Neumann B.C., a slight technical complexity is added. The approximating problem for (2)-(5) in \( \overline{\Omega} \) is as follows.

\[
\min \{ F(x, u(x), \nabla u(x), \nabla^2 u(x)) + \min_{y \in \Omega, |x - y| \leq \sqrt{2Mr}} \{ -\int \langle z, \nabla u(x) \rangle q(dz), \min_{y \in \partial \Omega, |x - y| \leq \sqrt{2Mr}} \{ \langle n(y), \nabla u(x) \rangle + \rho \} \leq \nu \} \}
\]

\[
\max \{ F(x, v(x), \nabla v(x), \nabla^2 v(x)) + \max_{y \in \Omega, |x - y| \leq \sqrt{2Mr}} \{ -\int \langle z, \nabla v(x) \rangle q(dz), \max_{y \in \partial \Omega, |x - y| \leq \sqrt{2Mr}} \{ \langle n(y), \nabla v(x) \rangle - \rho \} \geq -\nu \} \}
\]

We deduce the comparison result from this approximation and Lemma 1.3, by using the similar argument as in the proof of Theorem 1.1.

**References**

[1] M. Arisawa, A new definition of viscosity solutions for a class of second-order degenerate elliptic integro-differential equations.

[2] M.G. Crandall, H. Ishii, and P.-L. Lions, User’s guide to viscosity solutions of second order partial differential equations. Bulletin of the AMS, vol.27, no. 1 1992.

[3] W.H. Fleming and H.M. Soner, Controlled Markov processes and Visco-sity solutions, Springer-Verlag 1992.