ZERO COUNTING AND INVARIANT SETS OF DIFFERENTIAL EQUATIONS

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Abstract. Consider a polynomial vector field $\xi$ in $\mathbb{C}^n$ with algebraic coefficients, and $K$ a compact piece of a trajectory. Let $N(K,d)$ denote the maximal number of isolated intersections between $K$ and an algebraic hypersurface of degree $d$. We introduce a condition on $\xi$ called constructible orbits and show that under this condition $N(K,d)$ grows polynomially with $d$.

We establish the constructible orbits condition for linear differential equations over $\mathbb{C}(t)$, for planar polynomial differential equations and for some differential equations related to the automorphic $j$-function.

As an application of the main result we prove a polylogarithmic upper bound for the number of rational points of a given height in planar projections of $K$ following works of Bombieri-Pila and Masser.

1. Introduction

Let $\xi$ be a polynomial vector field in $\mathbb{C}^n$ defined over an algebraically closed field $K \subset \mathbb{C}$,

$$\xi = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i}, \quad \xi_i \in K[x_1, \ldots, x_n].$$  \hspace{1cm} (1)

We denote by $\text{Sing} \xi$ the singular locus of $\xi$, i.e. the set of common zeros of $\xi_i$. For $p \in \mathbb{C}^n$, we define the orbit of $p$, denoted $O_p$, to be the leaf of the (singular) complex foliation determined by $\xi$. Equivalently $O_p$ is the minimal $\xi$-invariant set containing $p$. We remark that if $p \in \text{Sing} \xi$ then $O_p = \{p\}$. A subset of $\mathbb{C}^n$ is said to be $\xi$-invariant if it is a union of orbits. We define the orbit closure at $p$, denoted $\overline{O}_p$, to be the Zariski closure of $O_p$. Equivalently, $\overline{O}_p$ is the minimal Zariski closed set containing $O_p$.

Definition 1. Let $V \subset \mathbb{C}^n$ be a $\xi$-invariant variety defined over $K$. We say that $\xi$ admits (or has) constructible orbits in $V$ if the relation

$$E \subset V \times V, \quad E := \{(p,q) \in V \times V : q \in \overline{O}_p\}$$  \hspace{1cm} (2)

is $K$-constructible\footnote{See Remark concerning constructibility over $K$ vs. $\mathbb{C}$.} (below constructible set always means over $K$ unless otherwise stated).

The notion of differential equations with constructible orbits is motivated by Nesterenko’s work on E-functions and in particular by the paper [25]. For more background see [5]. In [3] we establish the constructible orbits condition for systems of linear differential equations over $\mathbb{C}(t)$; for planar polynomial differential equations; and for some differential equations related to the automorphic $j$-function.

We now turn to the description of our main result. Let $D_r$ (resp. $\overline{D}_r$) denote the open (resp. closed) disc of radius $r$ centered around the origin in $\mathbb{C}$. If $r$ is
omitted $r = 1$ is assumed. When we speak of holomorphic functions on non-open set we always mean that the function is holomorphic in a neighborhood of the set.

**Definition 2.** A holomorphic map $\phi : \bar{D} \to V$ is said to be a parametrized (singular) trajectory of $\xi$ if $\phi(\bar{D}) \not\subset \text{Sing} \xi$ and if for every $z \in \bar{D}$ the complex vectors $\phi'(z)$ and $\xi(\phi(z))$ are complex proportional whenever they are both non-zero.

Note that we do not require $z$ to act as the natural time parameter with respect to the $\xi$-flow in the definition of parameterized trajectories. In particular we allow $\phi(\bar{D})$ to pass through singular points of $\xi$ as long as it remains holomorphic. We define the transcendence degree $\kappa(\phi)$ to be the dimension of the Zariski closure of $\phi(\bar{D})$. The following is our main result.

**Theorem 1.** Let $V, \xi$ be defined over $\mathbb{K} = \mathbb{Q}^{\text{alg}}$ and $V$ invariant under $\xi$. Suppose that $\xi$ admits constructible orbits in $V$. Let $\phi : \bar{D} \to V$ be a parametrized trajectory of $\xi$. Then there exists a constant $C_\phi$ with the following property: For every $P \in \mathbb{C}[x_1, \ldots, x_n]$ such that $P \circ \phi \neq 0$,

$$\#\{z \in \bar{D} : P(\phi(z)) = 0\} \leq C_\phi \cdot d^{2\kappa(m+1)} \log d$$

where $d = \deg P$, $\kappa = \kappa(\phi)$ and $m = \dim V$.

For example, whenever a system of a type considered in [15] is defined over $\mathbb{K} = \mathbb{Q}^{\text{alg}}$, Theorem 1 applies to produce a polynomial estimate (in $d$) for the number of intersections between a parametrized trajectory of the system and an algebraic hypersurface of degree $d$.

For linear differential equations over $\mathbb{C}(t)$, Novikov-Yakovenko [29] give a single exponential (in $d$) upper bound for left hand side of (3). Since linear differential equations always admit constructible orbits, Theorem 1 improves this to a polynomial asymptotic whenever the equation is defined over $\mathbb{Q}^{\text{alg}}$. Similarly, [30] gives an iterated exponential bound for trajectories of arbitrary polynomial vector fields, and we see that this can be improved to a polynomial estimate when the vector field has constructible orbits and is defined over $\mathbb{Q}^{\text{alg}}$. Polynomial zero estimates have significant applications that do not follow from the previously known estimates. We demonstrate one such application in the following subsection 1.1.

1.1. Application: density of rational points on parametrized trajectories. For a reduced quotient $\mathbb{K}$ we introduce the *height* $H(\mathbb{K}) = \max(|a|, |b|)$. To avoid technicalities we set $H(0) = 1$. For a vector $v \in \mathbb{Q}^k$ we let $H(v)$ denote the maximal height of its components. For a set $X \subset \mathbb{C}^k$ we denote

$$X(\mathbb{Q}, H) := \{v \in X \cap \mathbb{Q}^k : H(v) \leq H\}, \quad N(X, H) := \#X(\mathbb{Q}, H).$$

The problem of estimating the number of integers or rational points of bounded height for various types of sets $X$ has been considered by numerous authors, starting with the work of Jarník [18]. From the seminal works of Bombieri-Pila [5] and Pila-Wilkie [31] it is known that an asymptotic estimate $N(X, H) = O(H^\varepsilon)$ holds for any $\varepsilon > 0$ if $X$ is a transcendental curve definable in any o-minimal structure (and an appropriate generalization holds for higher dimensional sets as well). In this and all other asymptotics discussed in this section, the asymptotic is taken with respect to $H$ for a fixed $X$.

In some cases one may hope to improve the estimate $O(H^\varepsilon)$ to a polylogarithmic estimate $O(\log^\gamma H)$ for some constant $\gamma$ (depending on $X$). Such results have been
obtained by Masser \cite{23} for $X$ given by a (compact piece of) the graph of the Riemann zeta function, and subsequently by Besson \cite{2}, Boxall-Jones \cite{6} and Jones-Thomas \cite{19} for graphs of other special functions; and by Pila \cite{32} for arithmetic orbits. In this subsection we show that a similar polylogarithmic estimate holds for parametrized trajectories of differential equations with constructible orbits.

We recall the following proposition from \cite{23}. For a finite set $S \subset \mathbb{C}^2$ we denote by $\omega(S)$ the degree of the minimal algebraic curve containing $S$. Note that the original proposition allows for more refined control over various parameters, and holds for more general number fields in place of $\mathbb{Q}$. We present a simplified version sufficient for our purposes.

**Proposition 3** (\cite{23} Proposition 2). Let $r > 0$ and let $f_1, f_2 : \bar{D}_2 \to \mathbb{C}$ be two holomorphic functions and denote $\Phi := (f_1, f_2)$. Suppose $Z \subset D_r$ is a finite set of complex numbers such that $\Phi(z) \in \mathbb{Q}^2$ for $z \in Z$ and denote $H := \max_{z \in Z} |\Phi(z)|$. Then

$$\omega(\Phi(Z)) = O(\log H). \quad (5)$$

The following is a direct corollary.

**Corollary 4.** Let $V, \xi$ be defined over $K = \mathbb{Q}^{alg}$ and $V$ invariant under $\xi$. Suppose that $\xi$ admits constructible orbits in $V$. Let $\phi : \bar{D} \to V$ be a parametrized trajectory of $\xi$. Let $P_1, P_2 \in \mathbb{C}[x_1, \ldots, x_n]$ and set

$$\Phi : \bar{D} \to \mathbb{C}^2, \quad \Phi := (P_1 \circ \phi, P_2 \circ \phi). \quad (6)$$

If $\text{Im} \Phi$ is not contained in an algebraic curve then

$$N(\text{Im} \Phi, H) = O(\log^{2n(m+1)} H \cdot \log \log H) \quad (7)$$

where $\kappa = \kappa(\phi)$ and $m = \dim V$.

**Proof.** Let $\phi$ be holomorphic in a disc $D_r$ for some $r > 1$. Covering $\bar{D}$ by $N$ discs of radius $(r - 1)/2$ and applying Proposition 3 we see that

$$\{ v \in (\text{Im} \Phi) \cap \mathbb{Q}^2 : H(v) \leq H \} \subset \bigcup_{j=1, \ldots, N} C_j \cap \text{Im} \Phi. \quad (8)$$

where each $C_j \subset \mathbb{C}^2$ is an algebraic curve $C_j = \{Q_j = 0\}$ and $\deg Q_j = O(\log H)$. By Theorem 1 since we assume that $\text{Im} \Phi$ is not contained in $C_j$, we have

$$\# [C_j \cap \text{Im} \Phi] \leq \# \{ z \in \bar{D} : [Q_j(P_1, P_2) \circ \phi](z) = 0 \} = O(d^2 \kappa^{1+\rho}) \log d \quad (9)$$

where $d = \deg Q_j(P_1, P_2) = O(\log H)$. The statement of the corollary follows immediately. \hfill \qed

1.2. **Outline of the proof of Theorem 1** Let $\xi$ be defined over $\mathbb{Q}^{alg}$. For simplicity we consider the case $V = \mathbb{C}^n$. Fix $p \in \mathbb{C}^n$ a non-singular point of $\xi$ and consider a parametrized trajectory $\phi : \bar{D} \to \mathbb{C}^n$ with $\phi(0) = p$. To simplify the exposition suppose further that the Zariski closure of $\text{Im} \phi$ is $\mathbb{C}^n$. Our goal is to estimate, for a given polynomial $P$, the number of zeros of $f := P \circ \phi$ in $\bar{D}$. We may assume without loss of generality that $P$ has unit norm (for instance $L_2$-norm) in the space of polynomials of degree $d$.

Our basic zero-counting argument follows a familiar approach using the Jensen inequality to compare the growth of $f$ to the number of its zeros. The precise statement used is given in Proposition 24. Roughly, the number of zeros is estimated
(up to a multiplicative constant) by \( \log M - \log m \) where \( m \) denotes the maximum of \( |f| \) on \( D \), and \( M \) denotes the maximum of \( |f| \) on some slightly larger disc.

Since \( \phi \) is bounded on \( D \) as well as a slightly larger disc, it is easy to see that \( \log M \) grows at most polynomially in \( d \) (in fact \( \log M = O(d \log d) \)). The main difficulty is thus in producing a lower estimate for \( \log m \) which is polynomial in \( d \). By a Cauchy estimate argument the maximum of \( |f| \) over a disc can be estimated from below in terms of its derivatives at the origin. Since these derivatives can be computed in terms of \( \xi \), we reduce the problem to the following statement (see Lemma 23): there exists \( k = O(d^n) \) such that \( \log |\xi^k P(p)| \) admits a lower estimate polynomial in \( d \).

Let \( \mu := \mu(d) \) be some polynomial function of \( d \) that will be determined later. We consider the linear map \( T \) from the space of polynomials of degree \( d \) to \( \mathbb{C}^n \), taking a polynomial \( P \) to the vector of its first \( \mu \) derivatives at \( p \). The existence of a non-trivial kernel for \( T \) is equivalent to the vanishing of certain minors, which we call \( \mu \)-elimination minors. Each minor \( M \) is a polynomial function in \( \mathbb{C}^n \) with explicitly bounded degrees and coefficients. We show (see Lemma 22) that if \( \epsilon := |M(p)| \) is non-zero for one of these minors then there exists \( k < \mu \) with \( \log |\xi^k P(p)| \geq \log \epsilon - \text{poly}(d) \). It thus remains to prove that for some suitable \( M \) we have \( \log \epsilon \geq -\text{poly}(d) \).

Using a diophantine Lojasiewicz inequality due to Brownawell (see §3.2) we show that for some minor \( M \), the value \( \log \epsilon \) is, up to polynomial factors in \( d \), bounded from below by the logarithm of the distance from \( p \) to the zero locus \( V^d \mu \) of the set of minors (i.e. if all minors are small then there must in fact be a nearby common zero). It remains to give a polynomial lower bound for this logarithmic distance.

Results known as multiplicity estimates (see §2.1) imply that if \( \mu = C d^n \) for a sufficiently large constant \( C \in \mathbb{N} \) then any polynomial having the first \( \mu \) derivatives vanishing at a point \( q \in \mathbb{C}^n \) must in fact be identically vanishing on the trajectory through \( q \). It follows that for this choice of \( \mu \) the set \( V^d \mu \), i.e. the set of points for which \( T \) admits a non-trivial kernel, is contained in the union of all the trajectories that satisfy some non-trivial polynomial relation. Call this set \( B \). In particular, \( p \notin B \) by assumption. In general the set \( B \) may be extremely complicated. However, we claim that for systems with constructible orbits \( B \) is a Zariski closed set. In particular it follows that since \( p \notin B \) the distance between \( p \) and \( B \), and hence the distance between \( p \) and \( V^d \mu \), is lower bounded by a constant independent of \( d \). This concludes the proof.

To see that \( B \) is closed, we note that since the orbit-closure relation \( q \in \overline{\mathcal{O}_p} \) is constructible its \( p \)-fibers have uniformly bounded degrees. It follows that there is some constant \( N \) such that any trajectory satisfying some non-trivial polynomial relation must in fact satisfy a polynomial relation of degree at most \( N \). The set of points where the trajectory satisfies such a relation for a fixed degree \( N \) is Zariski closed, and is in fact given by \( V^d \mu \) for \( d = N \) and \( \mu \) suitably chosen as above.

To conclude we make some remarks concerning the general case. Replacing \( \mathbb{C}^n \) by a \( \mathbb{Q}^{\text{alg}} \)-variety \( V \) does not introduce essential difficulties: one simply replaces the general polynomial ring by the coordinate ring of \( V \). However, if the trajectory through \( p \) satisfies an algebraic relation then the construction outlined above cannot be carried out verbatim (as we would have \( p \in V^d \mu \) for any sufficiently large \( d \)). One may be tempted to replace \( V \) in this case by the Zariski closure of the trajectory,
but since this variety is not necessarily defined over $\mathbb{Q}^{\text{alg}}$ this would preclude our use of a diophantine \text{Lojasiewicz} inequality.

Instead, we rely on the theory of Gröbner basis. Namely we show that one can restrict to a variety $V'$ defined over $\mathbb{Q}^{\text{alg}}$ such that the ideals of definition of the orbits $\mathcal{U}_q$ all share the same Gröbner diagram, for $q$ in some open neighborhood $U$ of $p$. This essentially uniformizes the orbits in a neighborhood of $p$. One can then construct a set of $\mu$-eliminating minors analogous to those considered above, whose set of common zeros does not intersect $U$. The rest of the proof then proceeds essentially unchanged.

2. Preliminaries and constructible orbits

In this section we recall some preliminary results and prove some technical statements needed for the main argument. We fix $\xi, V$ as in $[1]$ When we use the asymptotic notation $O(\cdot)$ the constants may depend on $\xi, V$.

We introduce some notations. We denote by $\mathcal{P}_d$ (resp. $\mathcal{P}_{d, \ell}$) the set (resp. vector space) of polynomials of (total) degree $d$ (resp. at most $d$) in $\mathbb{C}[x_1, \ldots, x_n]$. If $W \subset \mathbb{C}^n$ is an algebraic variety we denote its ideal of definition by $I_W \subset \mathbb{C}[x_1, \ldots, x_n]$. If $S$ is a collection of polynomials we denote its common zero locus by $Z(S)$. We summarize some basic properties of $\xi$-invariance below.

- An algebraic variety $W$ is $\xi$-invariant if and only if its ideal $I_W$ is stable under $\xi$, where we view $\xi$ as a derivative of $\mathbb{C}[x_1, \ldots, x_n]$. Indeed, suppose $W$ is $\xi$-invariant and let $P \in I_W$. Then $P$ vanishes identically on $W$, and hence so does its derivative along $\xi$, i.e. $\xi P \in I_W$. Conversely, suppose that $I_W$ is stable under $\xi$, fix $p \in W$ and let $P \in I_W$. Then $\xi^k P \in I_W$ and in particular $\xi^k P(p) = 0$ for every $k \in \mathbb{N}$. By analytic continuation we deduce that $P$ vanishes identically on $\mathcal{O}_p$, i.e. $\mathcal{O}_p \subset W$ and hence $W$ is $\xi$-invariant.
- The Zariski closure $\bar{X}$ of a $\xi$-invariant set $X$ is $\xi$-invariant. Indeed, if $P \in I_{\bar{X}} = I_X$ then $P$ vanishes identically on $X$, and since $X$ is invariant under $\xi$ it follows that $\xi P$ also vanishes identically on $X$, i.e. $\xi P \in I_X = I_{\bar{X}}$.
- If $V$ is a $\xi$-invariant variety and $V_1, \ldots, V_k$ are its irreducible components then each $V_j$ is $\xi$-invariant. For instance let $V'_1 = V_1 \setminus (V_2 \cup \cdots \cup V_k)$, which is a Zariski-dense subset of $V_1$. Then $V'_1$ is open in $V$ and it follows that for every $p \in V'_1$, the germ of its $\xi$-orbit is contained in $V'_1 \subset V$. Thus if $P \in I_{V_1}$ then $\xi P$ vanishes on $V'_1$ and hence also on its Zariski closure $V_1$, i.e. $\xi P \in I_{V_1}$. In particular it follows that the orbit closures $\mathcal{O}_p$ are irreducible.

2.1. Multiplicity estimates. We recall some known estimates on the multiplicity of a polynomial restricted to the trajectory of a polynomial vector field. The following estimate from $[4]$ (improving upon similar results of $[13]$) is sharp with respect to $d$ and has explicit constants.

**Theorem 2** ($[4]$ Corollary 1). Let $p \in \mathbb{C}^n \setminus \text{Sing} \xi$ and denote by $\gamma_p$ the (germ of the) trajectory of $\xi$ through $p$. Let $P \in \mathcal{P}_{d, \ell}$. If $P|_{\gamma_p} \neq 0$ then
\[
\text{mult}_p P|_{\gamma_p} \leq 2^{n+1}(d + (n - 1)\delta)^n
\]
where $\delta = \deg \xi$.

However, if one allows the constants to depend on the trajectory $\gamma_p$ then essentially better estimates (with respect to $d$) can be obtained in the case that $\gamma_p$...
satisfies some non-trivial polynomial relations. The following result appeared in [26] (and later [3, 4]).

**Theorem 3** ([26, Theorem 1]). Let \( p \in \mathbb{C}^n \setminus \text{Sing} \) and denote by \( \gamma_p \) the (germ of the) trajectory of \( \xi \) through \( p \). Let \( \kappa \) denote the dimension of the Zariski closure \( \overline{\gamma_p} \) of \( \gamma_p \). There exists a constant \( C_{\gamma_p} \) such that for every \( P \in \mathcal{P} \leq d \), if \( P|_{\gamma_p} \not\equiv 0 \) then

\[
\text{mult}_p P|_{\gamma_p} \leq C_{\gamma_p} d^\kappa. \tag{11}
\]

Moreover, \( C_{\gamma_p} \) can be chosen to depend only on \( \deg \gamma_p \).

### 2.2. Constructible orbits.

The following lemma is an easy consequence of Theorem 2.

**Lemma 5.** Let \( p \in V \) and \( P \in \mathcal{P} \leq d \). Then \( P \in \mathcal{I}_{p} \) if and only if \( P, \xi P, \ldots, \xi^\nu P \) vanish at \( p \) where \( \nu = Cd^n \) and \( C \) is a constant depending only on \( n \).

**Proof.** If \( p \in \text{Sing} \xi \) then \( \overline{\mathcal{I}_p} = \{p\} \) and the claim is obvious. Otherwise, in the notation of Theorem 2 we have \( P \in \mathcal{I}_{p} \) if and only if \( P|_{\gamma_p} \equiv 0 \) and the result follows from the theorem. \( \square \)

If \( I \subset \mathbb{C}[x_1, \ldots, x_n] \) is an ideal, we say that \( I \) is generated in degree \( N \) if \( I \) is generated as an ideal by \( I \cap \mathcal{P} \leq N \). If \( V \subset \mathbb{C}^n \) is an irreducible algebraic variety \( \deg V \) will denote its degree, i.e. the number of intersections with a generic affine plane of complementary dimension. If \( V \) is reducible we define \( \deg V \) as the sum of the degrees of its irreducible components.

**Lemma 6.** The ideal of an algebraic variety \( V \subset \mathbb{C}^n \) is generated in some degree \( N = N(\deg V) \) depending only on \( \deg V \). Conversely, if \( I \) is generated in degree \( N \) then \( \deg \mathcal{Z}(I) \) is bounded by some \( D = D(N) \) depending only on \( N \).

**Proof.** The second statement is a consequence of the Bezout theorem: since \( \mathcal{Z}(I) \) is cut-out by equations of degree at most \( N \), its degree can be estimated in terms of \( N \). For the first statement, we first note that \( V \) is set-theoretically defined by a collection of equations of degree at most \( \deg V \). If \( V \) is irreducible (or pure-dimensional) then such a collection is given, for instance, by the canonical equations of Chow and van der Waerden [14, Corollary 3.2.6]. If \( V = V_1 \cup \cdots \cup V_k \) is a decomposition into irreducible components and \( I_1, \ldots, I_k \) denote the corresponding ideals constructed above, then the ideal \( I_1 \cdots I_k \) defines \( V \) set-theoretically and is generated in degree \( \sum_j \deg V_j = \deg V \).

Let \( I \) denote the ideal generated by equations as above. Then by the Nullstellensatz we have \( I_V = \sqrt{I} \). The radical \( \sqrt{I} \) can be explicitly computed from the equations generating \( I \) (see e.g. [22]), and in particular the degrees of the generators of \( \sqrt{I} \) can be estimated in terms of \( \deg V \) as claimed ([22] also gives an explicit estimate, which we do not require but may be of use for deriving an effective version of some of the results in this paper). \( \square \)

We have the following alternative characterization of the notion of constructible orbits.

**Proposition 7.** The following are equivalent:

1. \( \xi \) has constructible orbits in \( V \).
2. The ideals \( \mathcal{I}_p \) for \( p \in V \) are generated in some degree \( N \) independent of \( p \).
3. The degrees \( \deg \overline{\mathcal{I}_p} \) for \( p \in V \) are uniformly bounded.
Proof. The equivalence of conditions 2 and 3 follows from Lemma \[6\].

Next we show that 1 implies 3. Suppose that $\xi$ has constructible orbits in $V$. Then \{\overline{O}_p : p \in V\} are the fibers of the constructible set $E$ as in \[2\]. This already implies that the degrees of $\overline{O}_p$ are uniformly bounded. Indeed, by standard finiteness properties of the constructible class, there is a uniform upper bound $\nu_k$ for the number of isolated intersections between any affine-linear plane $L$ of codimension $k$ and any of the fibers $\overline{O}_p$ of $E$. For any fixed fiber $\overline{O}_p$ one can choose the affine planes $L_k$ such that the number of isolated intersections between $\overline{O}_p$ and $L_k$ is the degree of the $k$-dimensional part \[14\] of $\overline{O}_p$, and thus $\nu_1 + \cdots + \nu_n$ is a uniform upper bound for the degrees of $\overline{O}_p$. We remark that the same proof holds if we assume that \[2\] is $C$-constructible instead of $K$-constructible.

Finally we show that 2 implies 1. Suppose the ideal $I_{\overline{O}_p}$ is generated in degree $N$ for every $p \in V$. Then we have

$$q \in \overline{O}_p \iff \forall p \in \mathbb{P}^\infty : [P \in I_{\overline{O}_p} \implies P(q) = 0].$$

(12)

We note that the right hand side of \[12\] (with $N$ fixed as above) is a first order $K$-formula in $p,q$ and hence defines a constructible set, assuming we show that the condition $P \in I_{\overline{O}_p}$ (for $P$ of degree at most $N$) can be expressed by a $K$-formula. For this, observe that by Lemma \[5\]

$$P \in I_{\overline{O}_p} \iff P(p) = 0, [\xi P](p) = 0, \ldots, [\xi^n P](p) = 0.$$

(13)

where $\nu$ is a constant depending only on $N$ and $n$, which is indeed a (quantifier-free) $K$-formula in $p$ and the coefficients of $P$. \[\square\]

Remark 8. The proof of Proposition \[7\] shows that if the orbit closure relation is constructible over $C$ then it is constructible over $K$ (since the implication $1 \implies 3$ works over $C$ as well, and condition 3 does not depend on the field of definition).

Therefore in Definition \[7\] it does not matter if we require constructibility over $K$ or $C$.

2.3. Gr"obner bases. Let $\prec$ denote the degree-lexicographic ordering on the monomials in $\mathbb{C}[x_1, \ldots, x_n]$. For $P \in \mathbb{C}[x_1, \ldots, x_n]$ we let $\text{LT}(P) \in \mathbb{N}^n$ denote the index of its leading (i.e. highest with respect to $\prec$) monomial and $\text{supp} P \subset \mathbb{N}^n$ denote the set of indices of all monomials with non-zero coefficients in $P$. If $S$ is a set of polynomials we denote $\text{LT}(S) := \{\text{LT}(s) : s \in S\}$. We recall the following multi-variate division with remainder theorem.

Proposition 9 ([8] Theorem 2.3]). Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be an ideal and $P \in P_{\leq d}$. Then $P$ can be represented in the form $P = QA + R$ where

$$Q \in I, \quad A \in \mathbb{C}[x_1, \ldots, x_n], \quad \text{supp} R \subset \text{LT}(P_{\leq d}) \setminus \text{LT}(I).$$

(14)

If $I \subset \mathbb{C}[x_1, \ldots, x_n]$ is an ideal and $S \subset I$, we say that $S$ is a Gr"obner basis if $\text{LT}(I)$ is generated by $\text{LT}(S)$ (as an ideal in the semigroup $\mathbb{N}^n$).

Proposition 10. Suppose that $\xi$ has constructible orbits in $V$. Then the set of diagrams $\mathcal{D} := \{\text{LT}(I_{\overline{O}_p}) : p \in V\}$ is finite, and for each $\mathcal{D} \in \mathcal{D}$ the set $\mathcal{J}_\mathcal{D} \subset V$ given by

$$\mathcal{J}_\mathcal{D} := \{p \in V : \text{LT}(I_{\overline{O}_p}) = \mathcal{D}\}$$

(15)

\[2\]In fact $\overline{O}_p$ is irreducible and in particular pure-dimensional, but we prove the more general statement for completeness.
is constructible.

If \( \xi, V \) are defined over \( \mathbb{Q} \) then each \( \mathcal{D} \) is invariant under the action of \( \text{Gal}(K/\mathbb{Q}) \).

**Proof.** By Proposition 7 the ideals \( I_{\overline{\sigma_\mathcal{D}}} \) are generated in some degree \( N \) independent of \( p \). Then there exists exists some \( N' \), depending only on \( N \), such that for every \( p \) the ideal \( I_{\overline{\sigma_\mathcal{D}}} \) has a Gröbner basis \( S_p \subset \mathbb{P}_{\leq N'} \) (in fact \( N' \) can be explicitly estimated from \( N \), for instance using Buchberger’s algorithm, though we shall not use this fact). Since \( \text{LT}(I_{\overline{\sigma_\mathcal{D}}}) \) is generated by \( \text{LT}(S_p) \) and the latter varies over finitely many values, we see that \( \mathcal{D} \) is finite. Moreover, given \( \mathcal{D} \in \mathcal{D} \) we have for every \( p \in V \)

\[
\text{LT}(I_{\overline{\sigma_\mathcal{D}}}) = \mathcal{D} \iff \text{LT}(I_{\overline{\sigma_\mathcal{D}}} \cap \mathbb{P}_{\leq N'}) = \mathcal{D} \cap \text{LT}(\mathbb{P}_{\leq N'}). 
\]  

(16)

The right hand side of (16) can be expressed as follows

\[
\left[ \forall (\alpha \in \mathcal{D}, |\alpha| \leq N') \exists (P = x^\alpha + \sum_{\beta < \alpha} c_\beta x^\beta) : P \in I_{\overline{\sigma_\mathcal{D}}} \right] \land
\left[ \forall (\alpha \notin \mathcal{D}, |\alpha| \leq N') \neg \exists (P = x^\alpha + \sum_{\beta < \alpha} c_\beta x^\beta) : P \in I_{\overline{\sigma_\mathcal{D}}} \right] 
\]  

(17)

Since it was shown in the proof of Proposition 7 that the condition \( P \in I_{\overline{\sigma_\mathcal{D}}} \) is expressible by a \( \mathbb{K} \)-formula, we see that indeed the set \( \mathcal{D} \) of \( p \) satisfying (16) for a fixed \( \mathcal{D} \) is indeed a constructible set.

Suppose now that \( \xi, V \) are defined over \( \mathbb{Q} \) and let \( \sigma \in \text{Gal}(K/\mathbb{Q}) \). A polynomial \( P \in \mathbb{K}[x_1, \ldots, x_n] \) satisfies \( P \in I_{\overline{\sigma_\mathcal{D}}} \) if and only if \( P, \xi P, \ldots \) all vanish at \( p \). Applying \( \sigma \) (which commutes with \( \xi \) by our assumption) we see that this occurs if and only if \( \sigma P, \xi(\sigma P), \ldots \) all vanish at \( \sigma(p) \), i.e. if and only if \( \sigma P \in I_{\overline{\sigma_\mathcal{D}}} \). Thus \( I_{\overline{\sigma_\mathcal{D}}} = \sigma I_{\overline{\sigma_\mathcal{D}}} \). Since the Gröbner diagram is invariant under automorphisms of the field (depending only on the leading term with a non-zero coefficient) we see that \( \text{LT}(I_{\overline{\sigma_\mathcal{D}}}) = \text{LT}(\sigma I_{\overline{\sigma_\mathcal{D}}}) \) so that indeed each strata \( \mathcal{D} \) is invariant under \( \sigma \). \( \square \)

**Proposition 11.** Suppose that \( \xi \) has constructible orbits in \( V \) and let \( p \in V \). There exists a \( \xi \)-invariant \( \mathbb{K} \)-variety \( V' \subset V \) and a Zariski open dense subset \( U \subset V' \) such that \( \mathcal{O}_p \subset U \) and \( \text{LT}(I_{\overline{\sigma_q}}) \) is constant for \( q \in U \).

If \( \xi, V \) are defined over \( \mathbb{Q} \) then \( V' \) can be chosen to be defined over \( \mathbb{Q} \).

**Proof.** We may without loss of generality replace \( V \) by one of its irreducible components. Recall that the subsets \( \mathcal{D} \subset V \) of Proposition 11 form a finite partition of \( V \) into constructible sets. Moreover, each of the sets \( \mathcal{D} \) is \( \xi \)-invariant. Indeed, suppose \( q \in \mathcal{D} \) and \( q' \in \mathcal{O}_q \). Then \( \mathcal{O}_{q'} = \mathcal{O}_q \), and therefore \( \text{LT}(I_{\overline{\sigma_q}}) = \text{LT}(I_{\overline{\sigma_{q'}}}) = \mathcal{D} \), so \( q' \in \mathcal{D} \). It follows that the Zariski closures \( \mathcal{D} \) are \( \xi \)-invariant as well.

Since \( V \) is irreducible, precisely one of the sets \( \mathcal{D}_0 \) is dense. If \( p \notin \mathcal{D} \) for any \( \mathcal{D} \neq \mathcal{D}_0 \) then one can take

\[
V' = V, \quad U = V \setminus \bigcup_{\mathcal{D} \neq \mathcal{D}_0} \overline{\mathcal{D}} .
\]  

(18)

Clearly \( p \in U \) and the \( \xi \)-invariance of \( \overline{\mathcal{D}} \) implies that \( \mathcal{O}_p \subset U \) as well. By definition for any \( q \in U \) we have \( \text{LT}(I_{\overline{\sigma_q}}) = \mathcal{D}_0 \).

If \( p \in \mathcal{D} \) for some \( \mathcal{D} \neq \mathcal{D}_0 \) we replace \( V \) by \( \overline{\mathcal{D}} \), which is a \( \xi \)-invariant \( \mathbb{K} \)-variety of dimension strictly smaller than \( \dim V \). The proof is concluded by induction on \( \dim V \).
If \( \xi, V \) are defined over \( \mathbb{Q} \) then we can essentially repeat the proof above, replacing the Zariski topology over \( K \) by \( \mathbb{Q} \). We may assume without loss of generality that \( V \) is irreducible over \( \mathbb{Q} \). Then \( V \) may be decomposed into finitely many \( K \)-irreducible components \( V_j \), which are transitively permuted by \( \text{Gal}(K/\mathbb{Q}) \) [21, Theorem III.4.10]. Since according to Proposition 11 each of the strata \( \mathcal{S}_{2\nu} \) are invariant under \( \text{Gal}(K/\mathbb{Q}) \), it follows that precisely one of the sets \( \mathcal{S}_{2\nu} \) is Zariski dense in \( V \) over \( \mathbb{Q} \). The rest of the proof can be concluded verbatim (using the Zariski topology over \( \mathbb{Q} \)). □

3. Diophantine geometry of \( \xi \)

In this section we recall some preliminary results on diophantine geometry and apply them to study the geometry of \( \xi \). We fix \( \xi, V \) as in [11] and in this section we also assume that they are defined over \( \mathbb{Z} \). When we use the asymptotic notation \( O(\cdot) \) the constants may depend on \( \xi, V \).

3.1. Heights. Let \( P \in \mathbb{Z}[x_1, \ldots, x_n] \) be non-zero. We define the height \( H(P) \) to be the maximum of the absolute values of the coefficients of \( P \), and the logarithmic height \( h(P) := \log H(P) \). We denote by \( \| \cdot \| \) the standard \( L_2 \)-norm on \( \mathbb{Q}[x_1, \ldots, x_n] \) with respect to the standard monomial basis, and the standard Hermitian norm on \( \mathbb{C}^n \). We require some basic inequalities concerning heights as follows.

**Lemma 12** (cf. [27, Lemma 1.2]). Let \( P_1, \ldots, P_s \in \mathbb{Z}[x_1, \ldots, x_n] \) and \( P = P_1 \cdots P_s \). Then

\[
h(P) \leq h(P_1) + \cdots + h(P_s) + O(\deg P) \tag{19}
\]

**Proof.** We remark that in [27] a slightly different notion of height it used: if \( c(Q) \) denotes the greatest common divisor of the coefficients of \( Q \) then Nesterenko uses the height function \( h(Q) := h(Q/c(Q)) \). The conclusion follows from [27, Lemma 1.2] since

\[
h(P) = \log c(P) + \hat{h}(P) \leq \sum_j \log c(P_j) + \sum_j \hat{h}(P_j) + O(\deg P) \\
\leq h(P_1) + \cdots + h(P_s) + O(\deg P). \tag{20}
\]

**Lemma 13.** Let \( P \in \mathbb{Z}[x_1, \ldots, x_n] \). Then

\[
h(\xi^k P) \leq h(P) + O(k \log(\deg P + k)). \tag{21}
\]

**Proof.** We first note that for every \( Q \in \mathbb{Z}[x_1, \ldots, x_n] \) and for \( i = 1, \ldots, n \) we have

\[
H\left(\frac{\partial}{\partial x_i}, Q\right) \leq (\deg Q) H(Q) \tag{22}
\]

\[
H(\xi_i Q) = O(H(Q)) \tag{23}
\]

\[
h(\xi Q) \leq h(Q) + O(1 + \log \deg Q). \tag{24}
\]

where for the second estimate we note that each monomial in \( \xi_i Q \) is given by a sum of \( O(1) \) terms, each bounded by \( O(H(Q)) \), and the third estimate follows from the previous two. Finally, since \( \deg \xi^k P = \deg P + O(k) \) we have by repeated application of (24) that

\[
h(\xi^k P) \leq h(P) + kO(1 + \log(\deg P + O(k))) \\
= h(P) + O(k \log(\deg P + k)). \tag{25}
\]
Lemma 14. Let $A$ be a $\rho \times \rho$ matrix with entries $A_{ij} \in \mathbb{Z}[x_1, \ldots, x_n]$ satisfying $\deg(A_{ij}) \leq d$ and $h(A_{ij}) \leq h$. Write $M = \det A$. Then

$$\deg M \leq \rho d, \quad h(M) \leq \rho h + O(\rho d) + \rho \log \rho.$$  \hfill (26)

Proof. We expand $M = \sum_{\alpha} S_{\alpha}$ as a sum of $\rho!$ summands $S_{\alpha}$, each a product of $\rho$ entries of $A$. Then $\deg M \leq \rho d$ is clear, and $h(S_{\alpha}) \leq \rho h + O(\rho d)$ according to Lemma 15. Finally, by the subadditivity of $H(\cdot)$ we have

$$h(M) = \log H(\sum_{\alpha} S_{\alpha}) \leq \log(\rho e^{\rho h + O(\rho d)}) \leq \rho h + O(\rho d) + \rho \log \rho.$$  \hfill (27)

□

Lemma 15. Let $P \in \mathbb{Z}[x_1, \ldots, x_n]$ with $\deg P = d$ and $h(P) = h$. Let $p \in \mathbb{C}^n$. Then $|P(p)| \leq (d + 1)^n e^{h(P)} \max(\|p\|^d, 1)$.

Proof. If $P(x) = \sum_{|\alpha| \leq d} c_{\alpha} x^\alpha$ then $c_{\alpha} \leq e^{h(P)}$ and

$$|P(p)| \leq \sum_{|\alpha| \leq d} e^{h(P)}|p^\alpha| \leq (d + 1)^n e^{h(P)} \max(\|p\|^d, 1).$$  \hfill (28)

□

3.2. A diophantine Lojasiewicz inequality. In this section we recall a result of Brownawell [8] showing that if a collection of polynomials of bounded height and degree are small at a point $p \in \mathbb{C}^n$ then $p$ is close to a common zero of these polynomials. This is the only diophantine-type estimate that will be used in the sequel. We remark that an earlier slightly weaker estimate of [7] would also suffice for our purposes and is presented with fully explicit constants (which may be of interest in deriving an effective version of the results of this paper). We present a formulation which is somewhat weaker than the precise one given in [8] Theorem 2.1 but sufficient for our purposes. Since this result plays a key role in our considerations, we present a proof for the convenience of the reader in §3.3.

Let $\omega_1, \omega_2 \in \mathbb{C}P^n$ and $\bar{\omega}_1, \bar{\omega}_2 \in \mathbb{C}P^{n+1}$ respective representatives chosen such that their maximal coordinate has modulus 1. We define the projective distance between $\omega_1, \omega_2$ to be

$$\text{dist}(\omega_1, \omega_2) := \max_{0 \leq i, j \leq n} |\bar{\omega}_i^1 \bar{\omega}_j^2 - \bar{\omega}_i^2 \bar{\omega}_j^1|.$$  \hfill (29)

If $W \subset \mathbb{C}P^n$ is a projective variety we define $\text{dist}(\omega, W)$ to be the minimal distance between $\omega$ and a point of $W$.

We denote by $\psi : \mathbb{C}^n \to \mathbb{C}P^n$ the standard embedding defined by $(x_1, \ldots, x_n) \to (1 : x_1 : \cdots : x_n)$ and by $H_\infty := \{x_0 = 0\}$ the hyperplane at infinity.

Theorem 4 ([8] Theorem 2.1]). Let $\mathcal{P} := \{P_\alpha\} \subset \mathbb{Z}[x_1, \ldots, x_n]$ be a collection of polynomials satisfying $\deg P_\alpha \leq d$ and $h(P_\alpha) \leq h$ for every $\alpha$. Denote by $W$ their common zero locus. Let $p \in \mathbb{C}^n$ and suppose $|P_\alpha(p)| \leq \epsilon$ for every $\alpha$. Then

$$\log \epsilon \geq d^n [n \log \text{dist}(\psi(p), \psi(W) \cup H_\infty) - O(d + h)].$$  \hfill (30)

If $V \subset \mathbb{C}^n$ is a fixed variety of dimension $m$ defined over $\mathbb{Q}$ and $W$ denotes the common zero locus of $\mathcal{P}$ in $V$ then one may replace $d^n$ above by $d^m$ (where the asymptotic constants can depend on $V$).
3.3. **Proof of Theorem 4.** We recall some results on diophantine geometry with projective ideals following [27]. We present all of the results in the context of the field \( \mathbb{Q} \), although the material can be developed for a finite extension with minor differences. Unlike the rest of this paper, polynomials and ideals considered in this section are projective. Let \( \mathbb{Q}[X] := \mathbb{Q}[x_0, \ldots, x_n] \) and similarly for \( \mathbb{Z}[X] \). Following [27], for any unmixed homogeneous ideal \( I \subset \mathbb{Q}[X] \) one can associate the following quantities:

1. The (projective) dimension of \( I \) denoted \( \dim I \).
2. The degree of \( I \) denoted \( \deg I \in \mathbb{N} \).
3. The (logarithmic) height of \( I \) denoted \( h(I) \in \mathbb{R}_{\geq 0} \).
4. For each \( \omega \in CP^n \) the absolute value of \( I \) at \( \omega \), denoted \( |I(\omega)| \in \mathbb{R}_{\geq 0} \).

We recall also that the exponent of a primary ideal \( I \) with associated prime \( p = \sqrt{I} \) is the minimal \( k \in \mathbb{N} \) such that \( p^k \subset I \).

We note that [27] normalizes the height \( h(P) \) of a polynomial by first normalizing the coefficients to have no common factor. We denote this normalized height by \( \hat{h}(P) \) as in the proof of Lemma [12] and cite the results from [27] using this notation to avoid confusion. Note that we always have \( h(P) \leq \hat{h}(P) \).

The following proposition shows that the latter three quantities (the third taken under logarithm) are essentially linear under primary decomposition.

**Proposition 16 ([27] Proposition 1.2).** Let \( I \subset \mathbb{Q}[X] \) be a homogeneous unmixed ideal with \( \dim I \geq 0 \). Suppose \( I = I_1 \cap \cdots \cap I_s \) is the reduced primary decomposition of \( I \) with \( p_j = \sqrt{I_j} \) and \( k_j \) the exponent of \( I_j \). Let \( \omega \in CP^n \). Then

1. \( \sum_{j=1}^s k_j \deg p_j = \deg I \).
2. \( \sum_{j=1}^s k_j h(p_j) \leq h(I) + O(\deg I) \).
3. \( \sum_{j=1}^s k_j \log |p_j(\omega)| \leq \log |I(\omega)| + O(\deg I) \).

Let \( P \in \mathbb{Z}[X] \) be homogeneous and \( \omega \in CP^n \). We define the normalized value of \( P \) at \( \omega \) to be \( \|P\|_{\omega} := |P(\bar{\omega})| \cdot H(P)^{-1} \) where \( \bar{\omega} \in \mathbb{C}^{n+1} \) is a representative of \( \omega \) chosen such that its maximal coordinate has modulus 1. The following proposition provides estimates on the degree, height and absolute value at a point \( \omega \) of a projective hypersurface in terms of its equation.

**Proposition 17 ([27] Proposition 1.3).** Let \( P \in \mathbb{Z}[X] \) be homogeneous and non-zero and set \( J := (P) \). Let \( \omega \in CP^n \). Then

1. \( \deg J = \deg P \).
2. \( h(J) \leq \hat{h}(P) + O(\deg P) \).
3. \( \log |J(\omega)| \leq \log \|P\|_{\omega} + O(\deg P) \).

The following proposition allows us to intersect an irreducible projective variety with a divisor with explicit estimates on the degree, height and absolute value at a point \( \omega \).

**Proposition 18 ([27] Proposition 1.4).** Let \( p \subset \mathbb{Q}[X] \) be a homogeneous prime ideal with \( \dim p \geq 1 \). Suppose \( Q \in \mathbb{Z}[X] \) is homogeneous and \( Q \not\subset p \). Then there exists a homogeneous unmixed ideal \( J \subset \mathbb{Q}[X] \) with \( \dim J = \dim p - 1 \) and \( Z(J) = Z((p, Q)) \) such that

1. \( \deg J \leq \deg p \cdot \deg Q \).
2. \( h(J) \leq \deg p \cdot \hat{h}(Q) + h(p) \cdot \deg Q + O(\deg p \cdot \deg Q) \).
(3) For any \( \omega \in \mathbb{C}P^n \) we have
\[
\log |J(\omega)| \leq \log \delta + \deg p \cdot \tilde{h}(Q) + h(p) \cdot \deg Q + O(\deg p \cdot \deg Q)
\]
where
\[
\delta = \begin{cases} 
\|Q\|_\omega & \text{if } \text{dist}(\omega, Z(p)) < \|Q\|_\omega \\
\|p\|_\omega & \text{otherwise} 
\end{cases}
\]
The inequality (31) remains true also in the case \( \dim p = 0 \) if we formally set \( J \) to be the irrelevant ideal \((X)\) and formally set \( |J(\omega)| = 1 \).

The following proposition establishes a relation between the absolute value of an ideal \( I \) at \( \omega \) and \( \text{dist}(\omega, Z(I)) \).

**Proposition 19** ([27, Proposition 1.5]). Let \( I \subset \mathbb{Q}[X] \) be a homogeneous unmixed ideal and \( r := \dim I + 1 \). Let \( \omega \in \mathbb{C}P^n \). Then
\[
\deg I \cdot \log \text{dist}(\omega, Z(I)) \leq \frac{1}{r} \log |I(\omega)| + \frac{1}{r} h(I) + O(\deg I).
\]

We are now ready to give the proof of Theorem 4.

**Proof of Theorem 4.** For simplicity of the presentation we will prove the first statement of the theorem. For the second statement one can reduce to the case that \( V \) is irreducible over \( \mathbb{Q} \), and the proof is essentially the same as presented below except that we begin with the ideal \( q_0 = I(V) \). We leave the details to the reader.

Denote by \( \tilde{P}_\alpha \in \mathbb{Z}[x_0, \ldots, x_n] \) the homogenization of \( P_\alpha \) and let \( \widehat{\mathcal{P}} := \{ \tilde{P}_\alpha \} \). Note that
\[
\tilde{h}(\tilde{P}_\alpha) = \tilde{h}(P_\alpha) \leq h.
\]
Let \( \omega = \psi(p) \) and denote by \( \tilde{W} \) the Zariski closure of \( \psi(W) \). We remark that the common zeros of \( \tilde{\mathcal{P}} \) lie in \( \tilde{W} \cup H_\infty \).

Let \( q_0 := (0) \). If \( W = Z(q_0) \) there is nothing to prove. Otherwise we choose a polynomial \( P_1 \in \tilde{\mathcal{P}} \setminus q_0 \). Applying Proposition 17 we obtain an ideal \( J_1 \) satisfying
\[
\deg J_1 \leq O(d), \quad h(J_1) \leq O(d + h), \quad \log |J_1(\omega)| \leq \log \|P_1\|_\omega + O(d).
\]
Suppose there exists an associated prime \( q_1 \) of \( J_1 \) and a polynomial \( P_2 \in \tilde{\mathcal{P}} \setminus q_1 \) such that
\[
\text{dist}(\omega, Z(q_1)) < \|P_2\|_\omega.
\]
Then applying Proposition 18 to \( q_1, P_2 \) and using Proposition 16 we obtain an ideal \( J_2 \) satisfying
\[
\deg J_2 \leq O(d^2), \quad h(J_2) \leq O(d^2 + hd), \quad \log |J_2(\omega)| \leq \log \|P_2\|_\omega + O(d^2 + hd).
\]
Continuing in this manner we finally obtain a polynomial \( P_s \in \tilde{\mathcal{P}} \setminus q_{s-1} \) and \( J_s \) with \( \dim J_s = n - s \) such that
\[
\deg J_s \leq d^s, \quad h(J_s) \leq O(d^s + hd^{s-1}), \quad \log |J_s(\omega)| \leq \log \|P_s\|_\omega + O(d^s + hd^{s-1})
\]
and such that for any of the associated primes \( q \) of \( J_s \), and any \( P \in \tilde{\mathcal{P}} \setminus q \) we have
\[
\text{dist}(\omega, Z(q)) \geq \|P\|_\omega.
\]
We construct a rooted tree \( T \) whose vertices \( v \) are homogeneous prime ideals \( p_v \subset \mathbb{Q}[x_0, \ldots, x_n] \) with associated multiplicities \( k_v \in \mathbb{N} \). The root is \( p_{s-1} \), and its children are the primary components of \( J_s \) with their associated multiplicities.
(or just $J_s$ with multiplicity 1 if $J_s$ is the irrelevant ideal). The tree is constructed recursively as follows.

- If $v$ is a vertex and $Z(p_v) \subseteq W \cup H_\infty$ or $p_v$ is the irrelevant ideal we declare it a leaf.
- Otherwise we choose an arbitrary polynomial $P_v \in \tilde{D}$ satisfying $P_v \notin p_v$.
  We remark that by construction we automatically have
  \[ \text{dist}(\omega, Z(p_v)) \geq \| P_v \|_\omega. \tag{39} \]
  We let $J_v$ denote the ideal constructed in Proposition 18 and $p_j, k_j$ the components of its primary decomposition. We define the children of $v$ to be the $p_j$ with associated multiplicities $k_v k_j$. If $J_v$ is the irrelevant ideal we define $J_v$ itself to be the single child of $v$, with multiplicity 1.

Denote by $T_r$ the set of vertices whose ideals have dimension $n - r$ in $T$ and by $\mathcal{L}$ the set of leaves, excluding irrelevant ideals. A simple induction starting with (38) and using Propositions 16 and 18 gives for $r = s, \ldots, n$
  \[ \sum_{v \in T_r} k_v \deg p_v \leq d^r \]  
  \[ \sum_{v \in T_r} k_v h(p_v) \leq O(hd^{r-1} + d^r). \]  

Moreover, recursive expansion of $|J_s(\omega)|$ in (38) using Propositions 16 and 18 taking into account (39), gives
  \[ \log \| P_s \|_\omega \geq \sum_{v \in \mathcal{L}} k_v \log |p_v(\omega)| - O(d^{n+1} + hd^n). \]  

We note that for any $v \in \mathcal{L}$ we have $\text{dist}(\omega, Z(p_v)) \leq \text{dist}(\omega, \hat{W} \cup H_\infty)$. Thus applying Proposition 18 we have
  \[ \log \| P_s \|_\omega \geq nd^n \cdot \log \text{dist}(\omega, \hat{W} \cup H_\infty) - O(d^{n+1} + hd^n). \]  

Finally noting that $\| P_s \|_\omega \leq \varepsilon$ gives the statement of the theorem. \qed

3.4. The $\mu$-elimination minors. Let $D \subseteq \mathbb{N}^n$ be an ideal. We denote by $\mathcal{R}_D \subseteq \mathbb{Q}[x_1, \ldots, x_n]$ the $\mathbb{Q}$-span of all monomials lying outside $D$, and by $\mathcal{R}_{\leq d} := \mathcal{R}_D \cap \mathcal{R}_{\leq d}$. We write $\rho = \rho(d) := \dim \mathcal{R}_{\leq d}$. It is known that $\rho \sim d^k$ where $k$ is a natural number (see [11] Proposition 12.2) and the paragraph that follows it; in fact $\rho$ agrees with a polynomial in $d$ for sufficiently large $d$). In particular if $D = \text{LT}(I)$ for an ideal $I \subseteq \mathbb{Q}[x_1, \ldots, x_n]$ then $k = \dim Z(I)$ (see [11] Theorem 12.4). We assume that $k \geq 1$.

Let $\mu := \mu(d) \in \mathbb{N}$ be an arbitrary function satisfying $\mu(d) > \rho(d)$. Below when $d$ is clear from the context we write $\mu$ for $\mu(d)$ to simplify the notation. We define a linear map
  \[ T_d^{\rho} : \mathcal{R}_{\leq d}^{\rho} \rightarrow \mathbb{Q}[x_1, \ldots, x_n]^{\oplus (\mu+1)}, \quad T_d^{\rho}(P) = (P, \xi P, \ldots, \xi^\mu P). \]  

We represent $T_d^{\rho}$ with respect to the monomial basis of $\mathcal{R}_{\leq d}$ as a $\rho \times (\mu + 1)$ matrix with entries in $\mathbb{Z}[x_1, \ldots, x_n]$. We let $\mathcal{M}_d^{\rho}$ denote the set of all top-dimensional minors of this matrix, called $\mu$-elimination minors. The following is obvious.

**Lemma 20.** Let $p \in \mathbb{C}^n$. Then $p$ is a common zero of $\mathcal{M}_d^{\rho}$ if and only if there exists a non-zero $P \in \mathcal{R}_{\leq d}^{\rho}$ such that $P, \ldots, \xi^\mu P$ vanish at $p$. 

Next, we estimate the degree and height of the minors in $\mathcal{M}_d^D$.

**Lemma 21.** For every $M \in \mathcal{M}_d^D$, we have

$$\deg M \leq O(d^\kappa \mu)$$

$$h(M) \leq O(d^\kappa \mu \log \mu)$$

(45)

**Proof.** Recall the $M$ is the determinant of a $\rho \times \rho$ matrix $A$ whose entries are of the form $A_{\alpha,k} := \xi^k x^\alpha$ where $k \leq \mu$ and $x^\alpha$ is a monomial of degree bounded by $d$. Using Lemma 13 we have

$$\deg A_{\alpha,k} \leq O(\mu)$$

$$h(A_{\alpha,k}) \leq O(\mu \log \mu).$$

(46)

Recalling that $\rho \sim d^\kappa$ and using Lemma 14 gives the statement of the lemma.

Finally we show that if a minor in $\mathcal{M}_d^D$ is large at a point, then every polynomial from $\mathcal{R}_d^D$ has a large $\xi$-derivative there. Below we use the notation $\log_0 x := \max(\log x, 0)$.

**Lemma 22.** Let $p \in \mathbb{C}^n$, $M \in \mathcal{M}_d^D$ and suppose that $|M(p)| = \varepsilon > 0$. Then for every polynomial $P \in \mathcal{R}_d^D$ there exists an index $0 \leq k \leq \mu$ such that

$$\log |\xi^k P(p)| \geq \log \|P\| + \log \varepsilon - O(d^\kappa \mu (\log \mu + \log_0 \|p\|))$$

(47)

**Proof.** We use the notations of the proof of Lemma 21. Denote by $A_p$ the matrix obtained by evaluating the entries of $A$ at $p$, so that $|\det A_p| = \varepsilon$. We may suppose that $\|P\| = 1$. Let $\delta = \max_{k_j} |\xi^{k_j} P(p)|$ where $\{k_j\}$ is the set of derivatives appearing in the matrix $A$ defining $M$. Then $\|A_p P\| \leq \sqrt{\rho_0 \delta}$ so $\|A_p^{-1}\| \geq (\sqrt{\rho_0 \delta})^{-1}$.

On the other hand using (46) and Lemma 15 we have

$$\|A_p\| \leq e^{O(\mu \log \mu)} \max(\|p\|, 1)^{O(\mu)}.$$  

(48)

The following inequality (see e.g. [34, (1.1)]) follows easily from the singular value decomposition of $A$

$$\|A_p^{-1}\| |\det A_p| \leq \|A_p\|^{\dim A - 1}. \hspace{1cm} (49)$$

Using $\dim A = \rho \sim d^\kappa$ we have

$$\rho^{-1/2} \delta^{-1} \varepsilon \leq \|A_p^{-1}\| |\det A_p| \leq \|A_p\|^{\dim A - 1} \leq e^{O(d^\kappa \mu \log \mu)} \max(\|p\|, 1)^{\dim A} \leq O(d^\kappa \mu)$$

(50)

from which it follows that

$$\delta > e^{-O(d^\kappa \mu \log \mu)} \max(\|p\|, 1)^{-O(d^\kappa \mu)} \varepsilon$$

(51)

as claimed.

**4. Proof of the main theorem**

In this section we give a proof of the main theorem. We fix $\xi, V$ as in [41] with $K = \mathbb{Q}^{\text{alg}}$. 
4.1. Universal lower bounds for $\xi$-derivatives. The following theorem is the main technical ingredient in the proof of the main theorem.

**Theorem 5.** Assume $\xi$ has constructible orbits in $V$. Fix $p \in V$ and denote $m := \dim V$ and $\kappa := \dim \mathcal{O}_p$. There exists a constant $C_p$ such that for any polynomial $P \in \mathcal{P}_{\leq d}$ there exists a polynomial $R \in \mathcal{P}_{\leq d}$ satisfying $R|_{\mathcal{O}_p} \equiv P|_{\mathcal{O}_p}$ and an index $0 \leq k \leq C_p d^\kappa$ such that

$$\log |\xi^k R(p)| \geq \log \|R\| - O(d^{2\kappa(m+1)} \log d).$$

Here the asymptotic constants may depend on $p$ as well.

**Proof.** Recall the notations of $[3,4]$. Denote $D := \text{LT}(I_{p\mathcal{O}_p})$. We will in fact prove a somewhat refined conclusion: that one can choose $R$ in the conclusion of the theorem such that $R \in \mathcal{P}_{\leq d} \subset \mathcal{P}_{\leq d}$.

We first reduce to the case where $\xi$ is defined over $\mathbb{Q}$. Suppose that $\xi$ is defined over a finite extension $L$ of $\mathbb{Q}$. By the primitive element theorem [14] Theorem 14] we may have $L = \mathbb{Q}(\alpha)$ for some $\alpha \in L$. We introduce an extra variable $y$ and let $\xi_0$ denote the $\mathbb{Q}$-vector field obtained from $\xi$ by expressing each $L$-coefficient of $\xi$ as a $\mathbb{Q}$-rational function in $\alpha$ and substituting $y$ for $\alpha$, and letting $\xi_0(y) = 0$. By construction $\{y = \alpha\}$ is invariant under $\xi_0$ and $\xi_0|_{\{y = \alpha\}} \equiv \xi$, so that the orbit structure of $\xi_0$ on $\{y = \alpha\}$ is the same as that of $\xi$.

We let $D \in \mathbb{Z}[y]$ denote the common denominator of the coefficients of $\xi_0$ so that $\xi_1 = D \cdot \xi_0$ has polynomial coefficients over $\mathbb{Z}$. Note that $D(\alpha) \neq 0$ by construction, and hence $\xi_0, \xi_1$ are non-zero constant multiples of each other on $\{y = \alpha\}$ and thus have the same orbit structure there. We set $p_1 := (p, \alpha)$ and $V_1 := V \times \{\alpha\}$. In particular, $p_1 \in V_1$ and $\xi_1$ has constructible orbits in $V_1$.

We claim that it will suffice to prove the claim for $\xi_1, V_1$. We first note that $m, \kappa$ are the same in this case. We choose the monomial ordering $<$ such that $y$ precedes $x_1, \ldots, x_n$. Then since $I_{\mathcal{O}_p}$ contains the polynomial $y - \alpha$, its Gröbner diagram $D_1$ contains all monomials containing $y$. It also contains the Gröbner diagram of $I_{\mathcal{O}_p} \subset I_{\mathcal{O}_{p_1}}$. Let $R_1 \in \mathcal{P}_{\leq d}^{D_1}$ and $k$ be as in the conclusion of the theorem. Then in fact $R_1 \in \mathcal{P}_{\leq d}^{D_1} \cap \mathbb{C}[x_1, \ldots, x_n]$, and

$$\log |\xi^k R_1(p)| = \log |\xi_1^k R_1(p, \alpha)| = \log |D(\alpha)| - k \log |\xi_1^k R_1(p, \alpha)| \geq -k \log |D(\alpha)| + \log \|R_1\| - O(d^{2\kappa(m+1)} \log d).$$

Using $k = O(d^n)$ and $|D(\alpha)| = O(1)$ we get the conclusion of the theorem for $\xi_1, V_1$.

Next, we reduce to the case that $V$ is also defined over $\mathbb{Q}$. Suppose $V$ is defined over a finite extension of $\mathbb{L}$ of $\mathbb{Q}$. The Galois conjugates $\sigma V$ for $\sigma \in \text{Gal}(\mathbb{L}/\mathbb{Q})$ are also $\xi$-invariant,

$$\xi(I_{\sigma V}) = \xi(\sigma I_V) = \sigma(\xi I_V) \subset \sigma I_V = I_{\sigma V}$$

where we used our assumption that $\xi$ is defined over $\mathbb{Q}$ and hence commutes with $\sigma$. The union of the (finitely many) Galois conjugates of $V$ is defined over $\mathbb{Q}$ [21 III.4, Theorem 10] and $\xi$-invariant, and it will suffice to prove the conclusion with $V$ replaced by this union.

By Proposition 11, after replacing $V$ by its $\mathbb{Q}$-subvariety $V'$, we may assume without loss of generality that there exists a Zariski open $U \subset V$ containing $\mathcal{O}_p$ such that the Gröbner diagram $\text{LT}(I_{\mathcal{O}_p}) = D$ for $q \in U$. Since the dimension and degree of an affine variety are determined by its Gröbner diagram, we see that
dim $\overline{U}_q = \kappa$ and deg $\overline{U}_q$ is constant for $q \in U$. In the notations of Theorem 3 we have $\overline{U}_q = \overline{q}$ and we may therefore choose a constant $C$ such that for every $q \in U$ and every $Q \in \mathcal{P}_{\leq d}$, if $Q|_{\overline{q}} \neq 0$ then $\text{mult}_{\overline{q}} Q \leq Cd^\kappa$. Set $\mu(d) := Cd^\kappa$.

If $Q \in \mathcal{P}_{\leq d}$ is non-zero and $q \in U$ then since $\text{supp} Q \cap \overline{D} = \emptyset$ we have $Q \notin \mathcal{I}_{\overline{q}}$. Thus by the above $\text{mult}_{\overline{q}} Q \leq \mu$. If we denote $W := Z(\mathcal{M}_d^D)$ then it follows from Lemma 20 that $W \cap U = \emptyset$. Since $U$ is open, we deduce that $\text{dist}(\psi(p), \psi(W) \cup H_\infty)$ is bounded from below by a constant independent of $d$. By Lemma 21 we also have for every $M \in \mathcal{M}_d^D$

$$\text{deg} M \leq O(d^{2\kappa})$$

Applying now Corollary 4 we see that there exists a minor $M \in \mathcal{M}_d^D$ such that $\varepsilon := \log |M(p)|$ satisfies

$$\log \varepsilon \geq -O(d^{2\kappa(m+1)} \log d).$$

Recall that by Proposition 11 we may choose a polynomial $R \in \mathcal{P}_{\leq d}$ satisfying $\text{supp} R \subset \text{LT}(\mathcal{P}_{\leq d}) \setminus D$, i.e., $R \in \mathcal{P}_{\leq d}$, and $R|_{\overline{\mathcal{P}}_p} \equiv P|_{\overline{\mathcal{P}}_p}$. Lemma 22 now implies that there exists $0 \leq k \leq \mu$ such that

$$\log \left| \xi^k R(p) \right| \geq \log \|R\| + \log \varepsilon - O(d^{2\kappa} \log d + \log \|p\|) \geq \log \|R\| - O(d^{2\kappa(m+1)} \log d)$$

as claimed. \hfill \square

4.2. Lower bounds and zero counting for parametrized $\xi$-trajectories. We begin with a simple lower bound for the maximum modulus of a polynomial on a parametrized trajectory in terms of its $\xi$-derivatives.

**Lemma 23.** Let $\phi : \widetilde{D} \to V$ be a parametrized $\xi$-trajectory and $z_0 \in D$, and set $p := \phi(z_0)$. Then for any polynomial $P \in \mathcal{P}$ and any $k \in \mathbb{N}$ we have

$$\log \max_{\widetilde{D}} |P \circ \phi| \geq \log \left| \xi^k P(p) \right| - O(k \log k)$$

where the asymptotic constants may depend on $p$.

**Proof.** If $p \in \text{Sing} \xi$ there is nothing to prove. Otherwise one can solve the differential equation $\dot{x} = \xi(x)$ and obtain a holomorphic function

$$\hat{\phi} : D_r \to V, \quad \hat{\phi}'(z) = \xi(\phi(z)), \quad \hat{\phi}(0) = p. \quad (59)$$

Restricting $r$ further we may assume that $\hat{\phi}(D_r) \subset \phi(D)$. It will thus suffice to prove \eqref{55} with $\hat{\phi}$ and $D_r$ in place of $\phi$ and $D$.

Let $P \in \mathcal{P}$ and denote $f = P \circ \hat{\phi}$ and $M_P := \max_{D_r} |f|$. By definition we have $f^{(k)}(0) = \xi^k P(p)$. Thus by the Cauchy estimate

$$\log \left| \xi^k P(p) \right| = \log \left| f^{(k)}(0) \right| \leq \log M + \log(k!) - k \log r$$

which, since $r$ is fixed, implies

$$\log M \geq \log \left| \xi^k P(p) \right| - O(k \log k). \quad (61)$$

We recall the following consequence of Jensen’s formula \cite{16}. \hfill \square
Proposition 24. Let $r > 1$ and let $f : \bar{D}_r \to \mathbb{C}$ be a holomorphic function. Denote by $M$ (resp. $m$) the maximum of $|f|$ on $\bar{D}_r$ (resp. $\bar{D}$). There exists a constant $C_r$ such that
\[
\# \{ z \in \bar{D} : f(z) = 0 \} \leq C_r \log \frac{M}{m}.
\] (62)

We are now ready to complete the proof of Theorem 1.

Proof of Theorem 1. Fix a point $p \in \phi(D) \setminus \text{Sing} \xi$. Then $\phi(\bar{D}) \subset \Omega_p$, and by analytic continuation along the trajectory of $\xi$ we see that a polynomial vanishes on $\phi(\bar{D})$ if and only if it vanishes on $\Omega_p$. Thus the Zariski closure of $\phi(\bar{D})$ is $\Omega_p$ and in particular $\kappa(\phi) = \dim \Omega_p$. Let $P \in \mathcal{P}_d$. According to Theorem 5 there exists a polynomial $R \in \mathcal{P}_d$ satisfying $R|_{\Omega_p} \equiv P|_{\Omega_p}$ and an index $0 \leq k \leq C_d \kappa$ such that
\[
\log \max_{D} |\xi^k R(p)| \geq \log \|R\| - O(d^{2\kappa(m+1)} \log d).
\] (63)

In particular, we have $R \circ \phi \equiv P \circ \phi$, and it will thus suffice to count the zeros of $R \circ \phi$ in $\bar{D}$. According to Lemma 24 we have
\[
\log \max_{\bar{D}} |R \circ \phi| \geq \log |\xi^k R(p)| - O(k \log k)
\] (64)

\[
\geq \log \|R\| - O(d^{2\kappa(m+1)} \log d).
\]

On the other hand since $\phi$ is holomorphic in a neighborhood of $\bar{D}$ it is certainly holomorphic in a slightly larger disk $D_c \supset D$. In particular $\phi$ is bounded in $D_c$ and
\[
\log \max_{D_c} |R \circ \phi| \leq \log \|R\| + O(d \log d).
\] (65)

Finally, by Proposition 24 we have
\[
\# \{ z \in \bar{D} : R \circ \phi = 0 \} \leq C_r \left[ \log \|R\| + O(d \log d) - \log \|R\| + O(d^{2\kappa(m+1)} \log d) \right] \leq O(d^{2\kappa(m+1)} \log d)
\] (66)
as claimed.

5. Systems with constructible orbits

In this section we give several examples of systems of differential equations having constructible orbits (and some that do not).

5.1. Linear systems.

5.1.1. Background. Our main motivating example for the notion of constructible orbits comes from the work of Nesterenko [24] on transcendence properties of $E$-functions. A key element of Nesterenko’s approach is the following result ([24, Theorem 3] and the corollary following it): if a linear differential equation admits a solution satisfying no non-trivial algebraic relations, then all solutions in a Zariski open neighborhood enjoy the same property. In other words, after removing a proper closed set, the orbit closure relation becomes trivial (every two points are related). The general notion of constructible orbits introduced in this paper is meant to provide a more uniform approach for the case where solutions may satisfy certain algebraic relations.
Nesterenko’s proof of the aforementioned result is based on linear differential Galois theory. Nesterenko [25, Theorem 2] establishes a bijection between the algebraic $\xi$-invariant sets and varieties in a certain projective space which are invariant under the action of the differential Galois group. One can then reduce the study of $\xi$-orbits to the study of orbits of this algebraic group.

Below we essentially follow Nesterenko’s approach to prove that any linear system of differential equations over $\mathbb{C}(t)$ admits constructible orbits (see Theorem 6 for the precise statement). However to treat this more general case we find it technically convenient to replace Nesterenko’s use of homogeneous unmixed ideals by the related concept of Chow varieties and Chow forms. We recall the necessary background on the Galois theory of linear differential equations and on Chow varieties in §5.1.3 and §5.1.4 respectively.

5.1.2. Setup. Consider a system of linear differential equations over $\mathbb{K}(t)$,

$$ \dot{y} = A(t)y, \quad A \in \text{Mat}_{n \times n}(\mathbb{K}(t)). $$

To fit this into our general framework we consider the ambient space $\mathbb{C}^{n+1} \cong \mathbb{C}_{t} \times \mathbb{C}_{y}^n$ where $\mathbb{C}_t$ denotes a copy of $\mathbb{C}$ with coordinate $t$ and $\mathbb{C}_y^n$ a copy of $\mathbb{C}^n$ with coordinates $y = (y_1, \ldots, y_n)$. We sometimes embed $\mathbb{C}_y^n$ into a projective space $\mathbb{C}P^n_y$ with homogeneous coordinates $(y_0 : \cdots : y_n)$.

Let $q(t)$ denote a polynomial of minimal degree such that $\tilde{A}(t) := q(t)A(t)$ has polynomial entries and $\tilde{A}(t) = 0$ whenever $t$ is a pole of $A(t)$. Let

$$ \xi = q(t)\frac{\partial}{\partial t} + A(t)\frac{\partial}{\partial y} = q(t)\frac{\partial}{\partial t} + \tilde{A}(t)\frac{\partial}{\partial y}. $$

(68)

Then $\xi$ corresponds to (67) in the sense that the graph of any (local) solution of (67) is a trajectory of $\xi$. The requirement that $\tilde{A}(t)$ vanishes at the poles of $A(t)$ implies that $\text{Sing } \xi$ agrees with the polar locus of $A(t)$. Without imposing this condition, $\xi$ could admit non-constant trajectories with $t \equiv \text{const.}$ While the following Theorem 6 remains true in this case as well, such trajectories do not correspond to solutions of (67) and we therefore prefer to rule them out as a matter of convenience.

Our main goal in this section is the following theorem.

**Theorem 6.** The vector field (68) has constructible orbits in $\mathbb{C}_t \times \mathbb{C}_y^n$.

We first give an example illustrating that the conclusion of Theorem 6 fails if we allow the linear system (67) to depend on additional parameters.

**Example 25.** Consider the differential equation and corresponding vector field in the ambient space $\mathbb{C}^3 \cong \mathbb{C}_t \times \mathbb{C}_y \times \mathbb{C}_a$,

$$ \dot{y} = \frac{a}{t}y, \quad \xi = t\frac{\partial}{\partial t} + a\frac{\partial}{\partial y}. $$

(69)

For every point $p = (1, 1, m/n)$ with $m/n$ a reduced fraction we have $\overline{\mathbb{V}}_p = \{a = m/n, y^n = t^n\}$. The degrees of these varieties are not uniformly bounded over $p \in \mathbb{C}^3$, and it follows from Proposition 7 that $\xi$ does not have constructible orbits in $\mathbb{C}^3$. 

5.1.3. Galois theory of linear differential equations. We recall a few basic facts concerning linear differential equations and their Galois groups. We fix a small disk $U \subset \mathbb{C}$ not containing any singular points of (67). The matrix equation

$$F = A(t)F, \quad F(t) \in \text{GL}_n(\mathbb{C})$$

(70)

admits a solution $\Phi(t)$ whose entries are holomorphic functions in $U$ known as a fundamental solution. The extension $L \supset \mathbb{C}(t)$ generated over $\mathbb{C}(t)$ by the entries of $\Phi$ is a differential field called a Picard-Vessiot extension. The differential Galois group $G := \text{Gal}(L/\mathbb{C}(t))$ is defined to be the set of field automorphisms of $L$ over $\mathbb{C}(t)$ which commute with derivation.

If $\sigma \in G$ then $\sigma(\Phi)$ is another fundamental solution of (70) and this readily implies that $\sigma(\Phi) = \Phi \cdot A_\sigma$ where $A_\sigma \in \text{GL}_n(\mathbb{C})$. We will require the following fundamental theorem from the Galois theory of linear differential equations.

**Theorem 7 (15 Theorem 1.27).** In the notations above,

1. The map $\sigma \rightarrow A_\sigma$ is a representation which embeds $\text{Gal}(L/\mathbb{C}(t))$ as a linear algebraic subgroup of $\text{GL}_n(\mathbb{C})$.

2. The field of elements in $L$ invariant under $\text{Gal}(L/\mathbb{C}(t))$ is $\mathbb{C}(t)$.

5.1.4. Chow varieties. Recall that there exists a quasi-projective variety $\mathcal{G}_{k,d}(\mathbb{P}^n)$ called the open Chow variety, parameterizing subvarieties (reduced, possibly reducible) of dimension $k$ and degree $d$ in $\mathbb{P}^n$ [15 Theorem 21.2]. That is, there is a bijective correspondence between the varieties $X \subset \mathbb{P}^n$ (of dimension $k$ and degree $d$) and points of $\mathcal{G}_{k,d}(\mathbb{P}^n)$, which we denote $X \rightarrow \mathcal{R}_X$ (the point $\mathcal{R}_X$ is called the Chow form of $X$). If $X$ is defined over a field $K$ then the point $\mathcal{R}_X$ is $K$-rational [10, p.40]. There is a natural action of $\text{GL}_{n+1}$ on $\mathcal{G}_{k,d}(\mathbb{P}^n)$, which we denote by $g$, satisfying $g \circ \mathcal{R}_X = \mathcal{R}_{gX}$ for $g \in \text{GL}_{n+1}$ [10 Proposition 2.3].

The projective closure $\overline{\mathcal{G}}_{k,d}(\mathbb{P}^n)$ of $\mathcal{G}_{k,d}(\mathbb{P}^n)$ parameterizes all effective cycles of dimension $k$ and degree $d$ on $\mathbb{P}^n$ [15 p.272], i.e. formal linear combinations $\sum a_i X_i$ where $a_i$ are positive integers, $X_i \subset \mathbb{P}^n$ are varieties of dimension $k$ and $\sum a_i \deg(X_i) = d$. There is a projective map

$$\text{prod} : \mathcal{G}_{k,d_1}(\mathbb{P}^n) \times \mathcal{G}_{k,d_2}(\mathbb{P}^n) \rightarrow \mathcal{G}_{k,d_1+d_2}(\mathbb{P}^n)$$

(71)

mapping two effective cycles to their formal sum (in the Chow form coordinates, this corresponds to the product of the corresponding forms [15, p.272]).

We fix projective coordinates $y_0 : \cdots : y_n$ on $\mathbb{P}^n$, and consider the affine space $\mathbb{A}^n$ defined by $y_0 \neq 0$ and the corresponding hyperplane at infinity $H_\infty = \{y_0 = 0\}$. For us it will be convenient to consider a quasi-projective subvariety $\mathcal{G}_{k,d}^n \subset \mathcal{G}_{k,d}(\mathbb{P}^n)$ parameterizing subvarieties (reduced, possibly reducible) of $\mathbb{A}^n$. To show that this is indeed a quasi-projective variety, let $B(k,d) \subset \mathcal{G}_{k,d}(\mathbb{P}^n)$ denote the set of cycles having a non-trivial component on $H_\infty$. Then

$$B(k,d) = \cup_{j=1}^d \text{prod}(\mathcal{G}_{k,d-j}(\mathbb{P}^n) \times \mathcal{G}_{k,j}(H_\infty))$$

(72)

where $\mathcal{G}_{k,j}(H_\infty) \subset \mathcal{G}_{k,j}(\mathbb{P}^n)$ denotes the set of cycles supported at $H_\infty$, which is closed by [15 Exercise 21.4]. Each set in the union above is closed, being the image of a projective variety [14 Theorem 4.1.7], so $B(k,d)$ is closed, and we finally deduce that

$$\mathcal{C}_{k,d} = \mathcal{G}_{k,d}(\mathbb{P}^n) \setminus B(k,d)$$

(73)
is quasi-projective as claimed. If \( X \subset \mathbb{A}^n \) then we will write \( \mathcal{R}_X \) as a shorthand for \( \mathcal{R}_{\bar{X}} \), where \( \bar{X} \) is the projective closure of \( X \).

The embedding \( \text{GL}_n \to \text{GL}_{n+1} : g \mapsto \text{diag}(1,g) \) defines an action of \( \text{GL}_n \) on \( \mathbb{P}^n \) which preserves \( \mathbb{A}^n \) and restricts to the usual action of \( \text{GL}_n \) in the standard chart \( y_0 = 1 \). By definition \( \mathcal{C}_{k,d}^n \) is invariant under the \( \circ \)-action of \( \text{GL}_n \), and again we have \( g \circ \mathcal{R}_X = \mathcal{R}_{gX} \) where \( g \in \text{GL}_n \) and \( X \subset \mathbb{A}^n \) is an affine variety of degree \( d \).

5.1.5. The Galois group action and \( \xi \)-invariant varieties. We define an injection

\[
\iota : \mathcal{C}_{k,d}^n(\mathbb{C}) \to \mathcal{C}_{k,d}^n(L), \quad \iota(\mathcal{R}_X) := \Phi \circ \mathcal{R}_X. \tag{74}
\]

If \( X \subset \mathbb{C}^n \) is a variety of degree \( d \) then \( \iota(\mathcal{R}_X) \) represents the variety whose underlying set consists of the graphs of solutions \( \Phi \cdot v \) of (67) for \( v \in X \). In particular we have the following equivalence.

Lemma 26. Let \( X \subset \mathbb{A}^n(\mathbb{C}(t)) \) be a subvariety of dimension \( k \) and degree \( d \). Then \( X \) is \( \xi \)-invariant if and only if \( \mathcal{R}_X \) is in \( \text{Im} \iota \).

Proof. Let \( t_0 \in U \) be generic and denote by \( X_{t_0} \subset \mathbb{A}^n(\mathbb{C}) \) the fiber of \( X \) at \( t_0 \). Then \( X \) is invariant (in a neighborhood of \( X_{t_0} \) and then by analytic continuation everywhere) if and only if for every \( t \in U \) the fiber \( X_t \) is obtained by parallel transport from \( X_{t_0} \),

\[
X_t = \Phi(t) \cdot \Phi(t_0)^{-1} : X_{t_0}, \tag{75}
\]

or in other words, if and only if

\[
\mathcal{R}_X = \mathcal{R}_{\Phi(t) \cdot \Phi(t_0)^{-1} : X_{t_0}} = \Phi(t) \circ \mathcal{R}_{\Phi(t_0)^{-1} : X_{t_0}} = \iota(\mathcal{R}_{\Phi(t_0)^{-1} : X_{t_0}}). \tag{76}
\]

This occurs if and only if \( \mathcal{R}_X \in \text{Im} \iota \). \( \square \)

Recall that by Theorem 4 we have an action of \( G \) on \( \mathbb{A}^n \) defined by the embedding \( G \ni \sigma \to A_\sigma \in \text{GL}_n(\mathbb{C}). \)

Proposition 27. The map \( \iota \) induces an inclusion-preserving bijection between

1. the set of subvarieties of \( \mathbb{A}^n(\mathbb{C}(t)) \) of dimension \( k \) and degree \( d \) that are invariant under the action of \( G \) on \( \mathbb{A}^n(\mathbb{C}). \)
2. the set of subvarieties of \( \mathbb{A}^n(\mathbb{C}(t)) \) of dimension \( k \) and degree \( d \) that are \( \xi \)-invariant.

Proof. Let \( V \subset \mathbb{A}^n(\mathbb{C}). \) By Lemma 26 we need to show that \( \iota(\mathcal{R}_V) \in \mathcal{C}_{k,d}^n(\mathbb{C}(t)) \) if and only if \( V \) is invariant under the action of \( G \). To see this we note that for every \( \sigma \in G \),

\[
\sigma(\iota(\mathcal{R}_V)) = \sigma(\Phi \circ \mathcal{R}_V) = (\Phi \cdot A_\sigma) \circ \mathcal{R}_V = \Phi \circ \mathcal{R}_{A_\sigma V} = \iota(\mathcal{R}_{A_\sigma V}). \tag{77}
\]

Thus \( V \) is invariant under \( G \) if and only if \( \iota(\mathcal{R}_V) \) is invariant under \( G \). But the latter is equivalent, by Theorem 4(b) applied to coordinates in some affine chart, to \( \iota(\mathcal{R}_V) \in \mathcal{C}_{k,d}^n(\mathbb{C}(t)). \) \( \square \)

A variety \( X \subset C_t \times C_d^u \) whose components project dominantly to \( C_t \) may naturally be viewed as a subvariety of \( \mathbb{A}^n(\mathbb{C}(t)) \). We make this identification below. Similarly, every subvariety \( X \subset \mathbb{A}^n(\mathbb{C}(t)) \) naturally corresponds to a subvariety \( \overline{X} \subset C_t \times C_d^u \) (\( X \) is originally defined as a subset of \( U \times C_d^u \) for some open dense subset \( U \subset C \), and \( \overline{X} \) is the Zariski closure of this set).

We now turn to the description of the orbit closures \( \overline{\mathcal{O}}_p \) for \( p \in C_t \times C_d^u \).
Proposition 28. For every \( p \in \mathbb{C}_t \times \mathbb{C}_y^n \) the orbit closure \( \overline{O}_p \) takes one of the following two forms:

1. The singleton \( \{ p \} \) if \( p \in \text{Sing} \xi \).
2. The variety \( \tilde{X} \) where \( \mathcal{R}_X = \iota(\mathcal{R}_V) \) and \( V \subset \mathbb{A}^n(\mathbb{C}) \) is the Zariski closure of an orbit \( G \cdot y \) for some \( y \in \mathbb{A}^n(\mathbb{C}) \).

Proof. Let \( p = (t_0, y_0) \). If \( p \in \text{Sing} \xi \) then certainly \( \mathcal{O}_p = \overline{O}_p = \{ p \} \). Otherwise \( t_0 \) is a non-singular point of \( \mathcal{O}_p \) and thus \( \overline{O}_p \) is an irreducible \( \xi \)-invariant variety projecting dominantly onto \( \mathbb{C}_t \) and may thus be viewed as a \( \mathbb{C}(t) \)-subvariety of \( \mathbb{A}^n(\mathbb{C}(t)) \). Moreover \( \overline{O}_p \) is the minimal such variety containing \( (t_0, y_0) \). Since the bijection of Proposition 27 is inclusion-preserving, we see that \( \mathcal{R}_{\overline{O}_p} = \iota(\mathcal{R}_V) \) where \( V \) is the smallest \( G \)-invariant variety containing \( y = \Phi(t_0)^{-1}y_0 \), namely the Zariski closure of \( G \cdot y \).

Let the Chow coordinates of \( \mathcal{R}_X \) in \( \mathcal{C}_{k,d}(\mathbb{C}(t)) \) be given as \( (C_0 : \cdots : C_N) \) where \( C_k \in \mathbb{C}[t] \) are chosen to have no common factor. Then we define the height \( ht \) of \( X \) denoted \( htX \) to be the maximum among the \( t \)-degrees of \( C_k \).

Lemma 29. Let \( \mathcal{R}_X \in \mathcal{C}_{k,d}(\mathbb{C}(t)) \). Then \( \deg \tilde{X} \) admits an upper bound depending only on \( d = \deg_{\mathbb{C}(t)} \mathcal{X} \) and \( ht \mathcal{X} \).

Proof. A classical construction of Chow and van der Waerden [14] Corollary 3.2.6] gives a canonical expression for the equations of the projective closure \( \tilde{X} \subset \mathbb{P}^n(\mathbb{C}(t)) \) in terms of its Chow coordinates. These equations are homogeneous of degree \( d \) in the homogeneous coordinates on \( \mathbb{P}^n \). Each coefficient of these equations is given by some (fixed) polynomial combination of the Chow coordinates of \( \mathcal{R}_X \). Choosing the Chow coordinates to be polynomial and coprime as above, we obtain a system of polynomial equations in \( t \) with degrees bounded in terms of \( \deg X \).

Since the equations above define \( \tilde{X} \) over \( \mathbb{C}(t) \), passing to the affine chart by setting \( y_0 = 1 \) we obtain a system of equations (with degrees depending only on \( d, \deg \mathcal{X} \)) for \( X \) over \( \mathbb{C}(t) \). We denote by \( Z \) the zero locus of these equations in \( \mathbb{C}_t \times \mathbb{C}_y^n \). Then \( Z \) consists of \( \tilde{X} \) and possibly extra components projecting non-dominantly to \( \mathbb{C}_t \). The degree of \( Z \), and hence of \( \tilde{X} \), can be estimated from above using the Bezout theorem in terms of the degrees of the defining equations (which depend only on \( d, \deg X \)).

We require one more lemma concerning the orbits of \( G \). For \( y \in \mathbb{A}^n(\mathbb{C}) \) we denote by \( V_y \) the Zariski closure of the orbit \( G \cdot y \).

Lemma 30. The set \( \{ \mathcal{R}_V : y \in \mathbb{A}^n(\mathbb{C}) \} \) is a \( \mathbb{C} \)-constructible subset of the disjoint union of finitely many Chow varieties \( \mathcal{C}_{k,d} \).

Proof. Since \( G \) acts as an algebraic group on \( \mathbb{A}^n \), the relation \( \{ (y, z) : z \in G \cdot y \} \) is constructible, being expressible by the \( \mathbb{C} \)-formula \( \exists g \in G : z = g \cdot y \). It follows by an argument similar to the one given in the proof of Proposition 7 that the degrees of \( V_y \) are uniformly bounded by a constant \( D \) independent of \( y \), and by Lemma 6 their ideals are therefore generated in some uniformly bounded degree \( N \). It then follows that the relation \( z \in V_y \) is constructible as well: it can be expressed in the form “every polynomial \( P \) of degree at most \( N \) vanishing on \( G \cdot y \) also vanishes on \( z \)”, which is readily translated into a \( \mathbb{C} \)-formula.
Let \( \mathcal{C} \) denote the disjoint union of \( \mathcal{C}^n_{k,d} \) for \( k = 0, \ldots, n \) and \( d = 1, \ldots, D \). The relation
\[
\{(z, \mathcal{R}_V) : z \in V \} \subset \mathbb{C}^n \times \mathcal{C}
\] (78)
is constructible (in fact Zariski closed) by [13, Theorem 21.2]. Then \( \{\mathcal{R}_V_y : y \in \mathbb{A}^n(\mathbb{C})\} \) is given by
\[
\{\mathcal{R}_V \in \mathcal{C} : \exists y \in \mathbb{C}^n \forall z \in \mathbb{C}^n : z \in V_y \iff z \in V \}
\] (79)
which is expressible by a \( \mathbb{C} \)-formula hence \( \mathbb{C} \)-constructible as claimed.

We are now ready to complete the proof of Theorem 6.

Proof of Theorem 6. Let \( p \in \mathbb{C}_p \times \mathbb{C}^n_{k,d} \). According to Proposition 25 it suffices to show that \( \deg \overline{\mathcal{R}}_p \) is uniformly bounded independent of \( p \). In the case that \( p \in \text{Sing} \xi \) this is obvious. By Proposition 28 it remains to consider the case \( \overline{\mathcal{R}}_p = \mathcal{X}_V \) where \( \mathcal{R}_{X_V} = \imath(\mathcal{R}_V) \) and \( V_y \subset \mathbb{A}^n(\mathbb{C}) \) is the Zariski closure of an orbit \( G \cdot y \) for some \( y \in \mathbb{A}^n(\mathbb{C}) \).

By Lemma 29 \( \{\mathcal{R}_V_y : y \in \mathbb{A}^n(\mathbb{C})\} \) is a \( \mathbb{C} \)-constructible subset of the disjoint union of finitely many Chow varieties \( \mathcal{C}^n_{k,d} \). It will be enough to show that \( \deg \mathcal{X}_V \) is uniformly bounded for \( \mathcal{R}_V \) in this set. Moreover since every constructible set can be stratified into finitely many smooth strata [1] Theorem 5.38 it will be enough to consider \( \mathcal{R}_V \in M \) for some complex connected manifold \( M \subset \mathcal{C}^n_{k,d} \) with fixed \( k,d \). By Lemma 29 it will suffice to show that \( \text{ht} \mathcal{X}_V \) is bounded uniformly over \( \mathcal{R}_V \in M \).

We may assume that \( M \) is contained in some affine chart \( c_k \neq 0 \) where \( c_k \) is one of the homogeneous Chow coordinates (otherwise cover \( M \) by finitely many such charts). Without loss of generality \( k = 0 \) and we consider the affine coordinates \( b_k = c_k/c_0 \) on \( \mathcal{C}^n_{k,d} \). It will suffice to show that \( b_k(\mathcal{R}_{X_V}) \) are all rational functions in \( t \) of degree bounded uniformly over \( \mathcal{R}_V \in M \). Recall that
\[
\mathcal{R}_{X_V} = \imath(\mathcal{R}_V) = \Phi \circ \mathcal{R}_V \in \mathcal{C}^n_{k,d}(\mathbb{C}(t))
\] (80)
Assume for simplicity of the notation that \( t_0 \in U \) has been chosen so that \( \Phi(t_0) \) is the identity matrix. Then
\[
f_k : M \times U \to \mathbb{C}, \quad f_k(\mathcal{R}_V, t) := b_k(\mathcal{R}_{X_V}) = b_k(\Phi(t) \circ \mathcal{R}_V)
\] (81)
is holomorphic in a neighborhood of \( M \times \{t_0\} \). Moreover since \( \mathcal{R}_{X_V} \), is a \( \mathbb{C}(t) \)-rational point of \( \mathcal{C}^n_{k,d} \) for every \( \mathcal{R}_V \in M \) and \( b_k \) are affine coordinates, we see that \( f_k(\mathcal{R}_V, t) = b_k(\mathcal{R}_{X_V}) \in \mathbb{C}(t) \) is rational of some degree \( \delta(\mathcal{R}_V) \) as a function of \( t \) for each fixed \( \mathcal{R}_V \in M \). By Corollary 35 we deduce that the degrees \( \delta(\mathcal{R}_V) \) are uniformly bounded over \( \mathcal{R}_V \in M \) thus concluding the proof.

5.2. Planar differential equations. Consider a planar differential equation
\[
\xi = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y}, \quad P, Q \in \mathbb{C}[x, y]
\] (82)
and denote \( m = \max(\deg P, \deg Q) \). For \( p \in \mathbb{C}^2 \), the orbit closure \( \overline{\mathcal{R}}_p \) may be
- the singleton \( \{p\} \), if \( p \) is a singular point;
- a curve, if \( p \) is non-singular but lies on an invariant algebraic curve of \( \xi \), i.e. an algebraic curve invariant under the flow of \( \xi \);
- or the whole plane otherwise.

Theorem 8. The vector field (82) has constructible orbits in \( \mathbb{C}^2 \).
Proof: If \((52)\) has finitely many invariant algebraic curves then we are done by Proposition 7. On the other hand, according to a theorem of Jouanolou [20] (following work of Darboux) if \((52)\) admits at least \(2 + m(m + 1)/2\) invariant algebraic curves then it admits a rational first integral \(R\) (i.e. a rational function invariant under the flow of \(\xi\)). Then every orbit closure \(\overline{\mathcal{O}_p}\) is either a singleton (if \(p \in \text{Sing} \xi\)) or an algebraic curve contained in the level set \(\{R(x, y) = R(p)\}\), in which case \(\deg \overline{\mathcal{O}_p} \leq \deg R\). Thus we are done by Proposition 7.

\[\square\]

5.3. The \(j\)-function and related systems.

5.3.1. Preliminaries on \(j\) and its differential equation. Recall that \(j : \mathbb{H} \rightarrow \mathbb{C}\) is a surjective holomorphic function invariant under the action of \(\text{SL}(2, \mathbb{Z})\) on \(\mathbb{H}\). We note also that the set of critical values of \(j\) is \(\{0, 1728\}\). It is known that \(j\) satisfies a third-order differential equation. To introduce this equation, we recall first the notion of the Schwartzian derivative \(S(f)\) of a holomorphic function \(f\), defined as

\[S(f) := \left(\frac{f''}{f'}\right)' - \frac{1}{2} \left(\frac{f''}{f'}\right)^2.\]  \((83)\)

The Schwartzian derivative satisfies the following chain-rule type relation

\[S(f \circ g) = (g')^2[S(f) \circ (g)] + S(g).\]  \((84)\)

The solutions of the differential equation \(S(f) = 0\) are given precisely by the fractional linear transformations \(f(z) = g \cdot z\) for \(g \in \text{SL}(2, \mathbb{C})\).

The \(j\) function satisfies the non-linear differential equation \(\chi(j) = 0\) where (see [24], page 20)

\[\chi(f) := S(f) + R(f)(f')^2, \quad R(f) := \frac{f^2 - 1968f + 2654208}{2f^2(f - 1728)^2}.\]  \((85)\)

As observed in [12], the general solution for the equation \(\chi(f) = 0\) takes the form \(f(z) = j(g \cdot z)\) for \(g \in \text{SL}(2, \mathbb{C})\). Indeed, let \(f(z)\) be any solution. Pick some point \(z_0\) where \(f(z_0) \notin \{0, 1728\}\), i.e. a non-critical value of \(j\). Then locally around \(z_0\) one can write \(f(z) = j \circ \phi(z)\) for some (locally) holomorphic lifting \(\phi(z)\). But then

\[0 = \chi(f) = \chi(j \circ \phi) = S(j \circ \phi) + R(j \circ \phi)(j' \circ \phi \cdot \phi')^2\]

\[= (\phi')^2[\chi(j) \circ \phi] + S(\phi) = S(\phi)\]  \((86)\)

which implies that \(\phi(z) = g \cdot z\) for \(g \in \text{SL}(2, \mathbb{C})\).

5.3.2. Constructible orbits for \(\chi = 0\). The equation \(\chi(f) = 0\) can be re-written in the form

\[f''' = A(f, f', f''), \quad A(f, f', f'') := R(f)(f')^3 + \frac{3(f'')^2}{2f'}.\]  \((87)\)

We note that no solution of the original equation \(\chi(f) = 0\) belongs to the polar locus \(\{f = 0, 1728\} \cup \{f' = 0\}\) since the equation admits no constant solutions. We let \(q(f, f', f'') := f^3(f - 1728)^3(f')^2\), so that \(A = qA\) is polynomial in \(f, f', f''\) and vanishes on the polar locus of \(A\).

We choose the ambient space to be \(\mathbb{C}^3 \times \mathbb{C}^3\) with the coordinates \(t, y, \tilde{y}, \tilde{y}\). Let

\[\xi := q(y, \tilde{y}, \tilde{y}) \frac{\partial}{\partial x} + \tilde{y} \frac{\partial}{\partial y} + A(y, \tilde{y}, \tilde{y}) \frac{\partial}{\partial \tilde{y}}.\]  \((88)\)

Then \(\text{Sing} \xi = \{y = 0\}\). Thus \(\xi\) corresponds to the equation \(\chi(f) = 0\) in the sense that for any solution \(f(z)\) the map \(z \rightarrow (z, f, f', f'')\) forms a parametrized
trajectory of $\xi$, and every trajectory through a non-singular point is described in this way.

We recall the following result of Nishioka [28].

**Theorem 9** ([28] Theorem on page 1). Let $G \subset \text{SL}(2, \mathbb{C})$ be a Zariski dense subgroup and $D \subset \mathbb{C}P^1$ a $G$-invariant domain. Let $f : D \to \mathbb{C}$ be a non-constant holomorphic functions and suppose that it is $G$-automorphic, i.e. $f(z) = f(g \cdot z)$ for any $g \in G$. Then $f$ satisfies no second order algebraic differential equation over $\mathbb{C}(t)$.

The following is a direct corollary.

**Corollary 31.** The vector field (88) is defined over $\mathbb{Q}$ and has constructible orbits in $\mathbb{C}_t \times \mathbb{C}^3$.

**Proof.** Let $p \in \mathbb{C}_t \times \mathbb{C}^3$. If $p \in \text{Sing} \xi$ then $\hat{O}_p = \{p\}$. Otherwise the trajectory through $p$ is locally given as the image of the map $z \to (z, f(t, f', f''))$ where $f$ is a solution of $\chi(f) = 0$, i.e. $f = j(g \cdot z)$ for some $g \in \text{SL}(2, \mathbb{C})$. We claim that in this case $\hat{O}_p = \mathbb{C}_t \times \mathbb{C}^3$. Otherwise, there would exist a non-zero polynomial $P \in \mathbb{C}[t, y, \dot{y}, \ddot{y}]$ vanishing identically on $\hat{O}_p$, and in particular $P(t, f(t), f'(t), f''(t)) = 0$. This possibility is ruled out by Theorem 9, since $f = j(g \cdot z)$ is automorphic with respect to $g^{-1}\text{SL}(2, \mathbb{Z})g$ on the domain $g^{-1}\mathbb{H}$. □

**Remark 32.** Nishioka [28] Theorem on page 1] proves that Theorem 9 remains true also if one considers equations over the field $\mathbb{C}(t, e^{at})$ where $a$ is any fixed complex number. One can similarly construct a vector field with trajectories of the form $(z, e^{az}, j(z), j'(z), j''(z))$ and show that it has constructible orbits in $\mathbb{C}^5$.

5.3.3. Geodesically independent $j$-translates. Let $r_1, \ldots, r_n \in \mathbb{Q}^{\text{alg}}(t)$ be $n$ non-constant rational functions with algebraic coefficients and suppose that there exists some $t_0 \in \mathbb{C}$ such that $r_i(t_0) \in \mathbb{H}$ for $r = 1, \ldots, n$. Following Pila [33], we will say that $r_1, \ldots, r_n$ are $\mathbb{L}$-geodesically independent for a field $\mathbb{L} \subset \mathbb{C}$ if there exists no relation of the form

$$r_i \equiv gr_j, \quad g \in \text{GL}(2, \mathbb{L}), \quad i \neq j.$$  

(89)

**Proposition 33.** There exists a vector field $\xi$ on $\mathbb{C}_t \times \mathbb{C}^{3n}$ defined over $\mathbb{Q}^{\text{alg}}$ whose trajectories (except the singular points) locally coincide with the images of the maps

$$t \to (t, j(a_1), j'(a_1), j''(a_1), \ldots, j(a_n), j'(a_n), j''(a_n)).$$  

(90)

where $a_k(t) = g_k \cdot r_k(t)$ and $g_1, \ldots, g_n \in \text{GL}(2, \mathbb{C})$. Moreover none of these maps have image contained in $\text{Sing} \xi$.

**Proof.** The proof of the Proposition is similar to the construction of [88]. Namely, making a change of variable $w = r_k(t)$ and $\frac{dt}{dz} = \frac{1}{(1(t))^2}$ we transforms the equation $\chi(f(w)) = 0$ into an equation $\chi_k(f(t)) = 0$ such that $\chi_k \in \mathbb{C}(t, f, f', f'')$ and the solutions of $\chi_k(f(t)) = 0$ are precisely the functions of the form $j(g \cdot r_k(t))$ for any $g \in \text{GL}(2, \mathbb{C})$. We then construct a corresponding differential equation

$$f''' = A_k(t, f, f', f''), \quad A_k \in \mathbb{C}(t, f, f', f'')$$  

(91)

as in [88] (whose singular locus may also contain the set $\{r_k'(t) = 0\}$). Finally we construct the vector field $\xi$ on $\mathbb{C}_t \times \mathbb{C}^{3n}$ by taking $n$ independent copies of the

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3Pila uses “geodesically independent” for what we call $\mathbb{Q}$-geodesically independent.
corresponding vectors fields \( f \) (but all sharing the same time variable \( t \)). We leave the detailed derivation to the reader.

We note in particular that the map

\[
t \rightarrow (t, j(r_1), j'(r_1), j''(r_1), \ldots, j(r_n), j'(r_n), j''(r_n))
\]

defined in a neighborhood of \( t_0 \) forms a parametrized trajectory of the vector field \( \xi \) defined in Proposition 33.

We recall the following theorem from [33].

**Theorem 10** ([33, Theorem 1.1]). Suppose that \( r_1, \ldots, r_n \) are Q-geodesically independent. Then the \( 3n \) functions

\[
j(r_1), j'(r_1), j''(r_1), \ldots, j(r_n), j'(r_n), j''(r_n)
\]

(defined in a neighborhood of \( t_0 \)) are algebraically independent over \( \mathbb{C}(t) \).

The following is a direct corollary.

**Corollary 34.** Suppose that \( r_1, \ldots, r_n \) are C-geodesically independent. Then the vector field defined in Proposition 33 has constructible orbits in \( \mathbb{C}_t \times \mathbb{C}^{3n} \).

**Proof.** Let \( p \in \mathbb{C}_t \times \mathbb{C}^{3n} \). If \( p \in \text{Sing} \xi \) then \( \mathcal{O}_p = \{p\} \). Otherwise the trajectory through \( p \) is locally given as the image of a map \( \Phi \). We claim that in this case \( \mathcal{O}_p = \mathbb{C}_t \times \mathbb{C}^{3n} \). Otherwise, there would exist a non-zero algebraic relation over \( \mathbb{C}(t) \) between the functions

\[
j(a_1), j'(a_1), j''(a_1), \ldots, j(a_n), j'(a_n), j''(a_n)
\]

where \( a_k(t) = g_k \cdot r_k(t) \) and \( g_1, \ldots, g_n \in \text{GL}(2, \mathbb{C}) \). But these functions are C-geodesically independent (because \( r_1, \ldots, r_n \) are) so this contradicts Theorem 10.

We remark that while Theorem 10 requires only the assumption of Q-geodesic independence of the functions \( r_1, \ldots, r_n \), in Corollary 34 we must require C-geodesic independence.

5.4. **Concluding remarks.** The examples presented in this section are not meant to give an exhaustive list of the classical differential equations satisfying the constructible orbits hypothesis. For instance, the Weierstrass \( \wp \) and \( \zeta \) functions satisfy differential equation with constructible orbits. Pila’s paper [33] contains a functional independence result involving the \( j, \wp \) and exponential functions, making it possible to extend the results of §5.3.3 to deal with these functions as well.

We remark also that while we focus in this paper on holomorphic solutions of differential equations, it is often interesting to consider the behavior of solutions in domains whose boundary contain a singularity. In a paper to appear separately we show that if a linear differential equation admits a regular singular point with quasi-unipotent monodromy then Theorem 1 can be extended to domains whose boundary contains the singular point. The result applies in particular to the Gauss-Manin connection (or Picard-Fuchs system) of algebraic families defined over \( \mathbb{Q}^{\text{alg}} \) and their sections (e.g. abelian integrals), showing that such functions satisfy an analog of Theorem 1 in essentially arbitrary domains. It is reasonable to expect that a result analogous to Corollary 4 would follow in such extended domains as well.
APPENDIX A. TAYLOR COEFFICIENTS OF RATIONAL FUNCTIONS

We consider a power series \( f(t) \in \mathbb{C}[[t]] \),
\[
f(t) = a_0 + a_1 t + \cdots \quad a_k \in \mathbb{C}.
\] (95)
If \( f \) is not clear from the context we will write \( a_k = a_k(f) \). We say that \( f \) is rational of degree \( d \) if it is the Taylor series at \( t = 0 \) of a rational function \( R(t) \) of degree \( d \).

**Theorem 11.** There exists a set of polynomials \( \mathcal{R}_d \) in the variables \( \{a_k\}_{k \in \mathbb{N}} \) such that \( f \) is rational of degree at most \( d \) if and only if \( C(a_0(f), a_1(f), \ldots) = 0 \) for every \( C \in \mathcal{R}_d \).

**Proof.** Denote
\[
P(t) = p_0 + \cdots + p_d t^d, \quad Q(t) = q_0 + \cdots + q_d t^d.
\] (96)
For \( N > d \) we consider the system of equations
\[
P(t) = f(t)Q(t) + O(t^N).
\] (97)
This is a system of \( N \) linear homogeneous conditions on the variables \( p_i, q_i \) with coefficients depending on \( a_i \). In particular, one can write a set of minors \( C^N_\alpha \in \mathbb{C}[a_0, \ldots, a_{N-1}] \) such that (97) admits a non-zero solutions \( P, Q \) if and only if \( C^N_\alpha \) vanish for every \( \alpha \). We take the set \( \mathcal{R}_d \) to be the union of the sets \( \{C^N_\alpha\} \) for every \( N \). If \( f \) is rational of degree at most \( d \) then the system (97) is solvable for every \( N \), so all the conditions \( C^N_\alpha \) vanish. Conversely, we suppose that these conditions all vanish and prove that \( f \) is rational of degree at most \( d \).

Let \( N > d \). Since the minors \( C^N_\alpha \) vanish, (97) admits a non-zero solution \( P_N, Q_N \). Then \( Q_N \neq 0 \), because otherwise \( P_N = O(t^N) \) which is impossible since \( P_N \) must be a non-zero polynomial of degree \( d < N \). Set \( R_N := P_N/Q_N \). Since \( Q_N \) has order at most \( d \) at \( t = 0 \) we have from (97)
\[
R_N(t) - f(t) = O(t^{N-d})
\] (98)
We claim that for \( i, j \geq 3d + 1 \) we have \( R_i \equiv R_j \). Indeed, \( R_i - R_j \) is a rational function of order at most \( 2d \) and by (98) we have
\[
R_i(t) - R_j(t) = O(t^{3d+1-d}) = O(t^{2d+1})
\] (99)
which is possible only if \( R_i - R_j \equiv 0 \). To conclude, if we set \( R := R_{3d+1} \) then (98) shows that \( R \) approximates \( f \) to every order, hence \( f \equiv R \) proving that \( f \) is rational of degree at most \( d \).

**Corollary 35.** Let \( M \) be a complex connected manifold, \( t_0 \in \mathbb{C} \) and \( W \) an open neighborhood of \( M \times \{t_0\} \) in \( M \times \mathbb{C} \). Suppose that \( f : W \to \mathbb{C} \) is holomorphic, and for every \( p \in M \) the function \( f_p(\cdot) := f(p, \cdot) \) is rational. Then the degrees of \( f_p \) are uniformly bounded over \( p \in M \).

**Proof.** We write a Taylor expansion \( f_p(t) = \sum a_k(p)(t - t_0)^k \) where \( a_k \) are holomorphic functions given by
\[
a_k(p) : M \to \mathbb{C}, \quad a_k(p) = \frac{1}{k!} \frac{\partial^k}{\partial t^k} f(p, t)|_{t=t_0}.
\] (100)
By Theorem 11 it is enough to prove that for some \( d \), the conditions from \( \mathcal{R}_d \) vanish identically in \( p \). Assume to the contrary that for every \( d \) there exists a condition \( C_d \in \mathcal{R}_d \) such that \( C_d := C_d(f_p) \) is not identically vanishing as a function of \( p \). Then the zero locus of \( C_d(f_p) \) is a proper analytic subset \( V_d \) of \( M \), and in particular
is nowhere dense and closed. On the other hand, by assumption $f_p$ is rational for every $p$, and Theorem 11 implies that $M = \bigcup_d V_d$ contradicting the Baire category theorem. □

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