Superconducting islands with semiconductor-nanowire-based topological Josephson junctions

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We theoretically study superconducting islands based on semiconductor-nanowire Josephson junctions and take into account the presence of subgap quasiparticle excitations in the spectrum of the junction. Our method extends the standard model Hamiltonian for a superconducting charge qubit and replaces the Josephson potential by the Bogoliubov-de Gennes Hamiltonian of the nanowire junction, projected onto the relevant low-energy subgap subspace. This allows to fully incorporate the coherent dynamics of subgap levels in the junction. The combined effect of spin-orbit coupling and Zeeman energy in the nanowires forming the junction triggers a topological transition, where the subgap levels evolve from finite-energy Andreev bound states into near-zero energy Majorana bound states. The interplay between the microscopic energy scales governing the nanowire junction (the Josephson energy, the Majorana coupling and the Majorana energy splitting), with the charging energy of the superconducting island, gives rise to a great variety of physical regimes. Based on this interplay of different energy scales, we fully characterize the microwave response of the junction, from the Cooper pair box to the transmon regimes, and show how the presence of Majoranas can be detected through distinct spectroscopic features.

I. INTRODUCTION

Josephson junctions (JJ) involving mesoscopic superconducting islands are one of the most versatile platforms for quantum state engineering and solid-state qubit implementation.¹² Their physics is governed by the competition between two energy scales: the charging energy $E_C$ of the island and the Josephson coupling $E_J$ across the junction. This competition is described by the Hamiltonian

$$H = 4E_C(\hat{N} - n_g)^2 + V_J(\hat{\varphi}),$$

$$V_J(\hat{\varphi}) = -E_J \cos \hat{\varphi},$$

where $V_J(\hat{\varphi})$ is the Josephson potential, $\hat{N}$ is the number of Cooper pairs in the island, conjugate to the junction superconducting phase difference $\hat{\varphi}$, and $n_g = Q_g/2e = V_g/(2eC_g)$ is a gate-induced charge offset in the island in units of a Cooper pair. The latter is controlled by a gate at potential $V_g$ with gate-island capacitance $C_g$. Equation 1 can be simply interpreted as the energy stored in a LC oscillator where the standard (linear) inductance $L$ is replaced by the (nonlinear) Josephson inductance

$$L_J^{-1}(\varphi) = (2e^2/\hbar)^2 d^2V_J(\varphi)/d\varphi^2 = (2e^2/\hbar)^2E_J \cos \varphi.$$  

In the limit $E_J \ll E_C$, charge quantization is strong, which manifests as Coulomb Blockade oscillations in units of $2e$. At points with half-integer $n_g = m + 1/2$, $N$ and $N + 1$ states become nearly degenerate, defining a charge qubit. In this so-called Cooper pair box (CPB) regime, the charge dispersion of the qubit frequency (i.e., its variation as a function of the gate-induced offset charge) is large, since charge eigenenergies depend strongly on gate $V_g$, making the qubit very susceptible to charge noise. In the opposite $E_J \gg E_C$ so-called transmon regime,⁵ quantum fluctuations suppress charge quantization and charge dispersion is exponentially-suppressed. As a result, the qubit susceptibility to noise is strongly suppressed and quantum coherence is correspondingly enhanced. This comes, however, at the cost of reduced anharmonicity (the transmon spectrum is almost harmonic with a frequency given by the Josephson plasma frequency $\omega_{pl} = \sqrt{8E_JE_C}/\hbar$), which reduces the operation time due to leakage out of the qubit subspace.

The above discussion assumes a a sinusoidal current-phase relation which gives a Josephson relation of the form $V_J(\varphi) = -E_J \cos(\varphi)$. This is an excellent description of a superconductor-insulator-superconductor (SIS) tunnel junction, which forms the basis of almost all state-of-the-art superconducting qubits. More recently, alternative technologies are sought in order to replace the weak link in the JJ and reach further operational functionalities. Such alternatives include semiconducting nanowires (NWs) — also known as gatemons —, two-dimensional gases and van der Waals heterostructures.⁶ Arguably, their main goal is to have compatibility with large magnetic fields and tunability by means of gate voltages, both of which are key requirements to reach a topological superconductor state, as predicted in many platforms.¹⁷—²³ This opens the possibility of using standard circuit QED techniques for microwave (MW) readout of topological qubits based on Majorana bound states in such platforms.²⁴—²⁶

The physics of most of the alternative weak link junctions cited above differ considerably from standard SIS
tunnel junctions. In particular, the Josephson effect in NW junctions is typically dominated by a small number of highly transmitting channels, see e. g. Refs. 8, 36 and 37. This implies that the current-phase relation is no longer sinusoidal and thus $V_J(\phi) \neq -E_J \cos(\phi)$ in Eq. 1. A proper description of superconducting islands presenting such non-sinusoidal Josephson potentials thus needs a correct treatment of the microscopic mechanisms governing the subgap spectrum (Andreev levels) of the weak link, which in turn dictates the final form of $V_J(\phi)$.

We focus here on a specific proposal where the weak link is based on a semiconducting NW which is proximitized by a superconductor in its left and right regions, thus forming a superconductor-normal-superconductor (SNS) junction, see Fig. 1. For the purposes of this work, the two regions are viewed as two Josephson-coupled superconducting islands. Interestingly, an intrinsic Rashba spin-orbit (SO) coupling in the NW combined with an external Zeeman field $B$ generates, for a small chemical potential $\mu$ in the NW, helical bands with spin-momentum locking similar to that of topological insulators. As demonstrated by Lutchyn et al. and Oreg et al., when proximitizing such helical bands with a standard s-wave superconductor, this system is a physical realization of the Kitaev model for one-dimensional p-wave superconductivity. Similar to the Kitaev model, these Lutchyn-Oreg wires possess phases with non-trivial electronic topology. In particular, they can be driven into a topological superconductor phase when the external Zeeman field $B$ exceeds a critical value $B_c = \sqrt{\Delta^2 + \mu^2}$, where $\Delta$ is the superconducting pairing term induced in the semiconducting NW owing to proximity effect.

In NWs with finite length $L_S$, this topological superconductor phase is characterized by Majorana bound states (MBSs) emerging in pairs, one at either end of the wire. One pair of Majoranas states forms a non-local fermion. The occupation of two such fermions, like in e.g. a SNS junction with two topological NW segments, defines the elementary qubit in proposals of topological quantum computers.

The goal of this paper is to present a comprehensive study of the Josephson-coupled superconducting islands described by a generalization of Eq. 1 that incorporates the dynamics of Majoranas in the junction if present. The resulting Hamiltonian, which we will present in Eq. 16, is derived as a low-energy projection of the full microscopic Hamiltonian for the two coupled islands,

$$H = 4E_C(\hat{N} - n_g)^2 + V_J(\hat{\phi})$$

(2)

$$V_J(\hat{\phi}) = \frac{1}{2} \hat{c}^\dagger \hat{c} H_{BdG}(\hat{\phi}) \hat{c},$$

where $\hat{N}$ is now the relative Cooper-pair number operator and its conjugate $\hat{\phi}$ is the island superconducting phase difference. $E_C = e^2/2C_C$ is the relative charging energy, written in terms of a total capacitance that we denote generically as $C_C = C_J + C_g$ (with $C_J$ and $C_g$ the shunting and gate capacitances, respectively), see schematic circuit in Fig. 1. This charging energy results from a combination of on-site charging energies of each island and the mutual charging energy between the islands. $V_J$ is the full, microscopic, non-interacting Bogoliubov-de Gennes (BdG) Hamiltonian of the junction modeled as two Lutchyn-Oreg segments coupled through a weak link. Orange circles with $\gamma_i$ represent Majorana bound states. (b) Spectrum of the island in the charging regime and for $B \sim B_c$, showing all the competing energy scales in the problem. Blue/orange dashed curves denote even/odd parity Coulomb parabolas in the absence of tunneling coupling across the junction. A finite coupling generates both standard Josephson coupling (avoided crossings $\sim E_J$ between same-color parabolas with minima differing by two electron charges $2e$ in gate space) and Majorana coupling (avoided crossings $\sim E_M$ between different-color parabolas with minima differing by one electron charge $e$ in gate space). The Majorana energy splitting $\delta$ changes with $B$ field, when it becomes smaller than $E_C$ (as in the case shown here) the ground state of the island around $n_g = 0.5$ becomes odd.

Our discussion is based on the simplest model that covers all these relevant regimes: two segments of a single-mode semiconductor NW that are proximitized by a con-
ventional s-wave superconductor separated by a short normal region, thus forming a SNS junction with a weak link of normal transparency $T_N$. The BdG spectrum of such weak link creates the Josephson potential $V_J(\varphi)$ that enters the superconducting island Hamiltonian in Eq. (2). This model adds another important energy scale $E_M$ corresponding to the junction coupling between MBSs localized at either side of the weak link, that may or may not dominate over $E_J$, see Fig. 1. For the single-channel short junction case considered in this work, the Josephson coupling $E_J$ may easily be smaller than $E_M$, since $E_M \sim \sqrt{T_N \Delta_T}$, while $E_J \sim T_N \Delta_{\scriptscriptstyle \text{Majorana}}$ with $T_N$ denoting the normal transmission of the junction and $\Delta_T$ the so-called “topological minigap” separating MBSs from the rest of quasiparticle excitations in the system, see Fig. 2. Thus, the Majorana-mediated Josephson energy $E_M$ introduces another important ratio $E_M/E_C$ into the problem. Finally, the spatial overlap between Majoranas belonging to the same proximitized portions of the NW also introduces a new energy scale $\delta$, representing their hybridization splitting, which depends on microscopic parameters of the NW such as length, magnetic field, chemical potential, etc. The interplay of all these energy scales gives rise to a rich variety of novel regimes and physical phenomena, well beyond that of standard superconducting islands, as we shall describe.

The paper is organized as follows. Section II is devoted to various relevant aspects of the NW-based Josephson junctions that we analyze here. After presenting the BdG model for the junction in subsection II A we discuss the basic phenomenology regarding the subgap spectrum and the relevant energy scales of the NW junction in subsection II B with emphasis on $E_M$ and $\delta$ which, as argued above, give rise to novel regimes not discussed before. Section III is devoted to the complete superconducting island problem. In subsection III A we discuss in detail our projection method that allows us to simplify the full island hamiltonian in Eq. 2 and keep only the relevant, subgap, degrees of freedom. Subsection III B discusses the island hamiltonian in tight-binding form, while we discuss the dependence of the superconducting island parameters on the microscopic parameters of the NW in subsection III C. Section IV sets the stage before discussing the main results of the paper. Here, we include a benchmark of the method against well-known limits (e.g. Majorana island limit) in subsection IV A and a discussion about the Josephson inductance and anharmonicity of the junction in subsection IV B. A detailed discussion about the ratio $E_M/E_J$ is included in subsection IV C. We finally present the main results of our work regarding the microwave spectroscopy of a NW-based superconducting island in section V. Results for different regimes are discussed in detail. This includes the $E_M/E_C < 1$ regime (subsection VA) and the opposite regime of non-negligible $E_M/E_C$ ratios, subsection VB which is of relevance to the experiments with junctions in the few-channel NW regime. In subsection VC we also discuss a regime with $E_M/E_C \lesssim 1$ and $E_J \to 0$, which is relevant to the recent experimental observation of parity mixing in superconducting islands owing to zero modes. Finally, we conclude the paper with some final remarks in section VI.

We finish this section by mentioning that, in parallel to this work, we present a companion paper, Ref. 51 with emphasis on the transmon limit with $E_M/E_C > 1$ and the role that parity crossings have on the transmon microwave spectrum owing to Majorana oscillations (oscillations of $\varphi$ as a function of magnetic field).

II. NW-BASED JOSEPHSON JUNCTION

A. Model

With full generality, the Josephson potential is given by the BdG Hamiltonian. In a short SNS NW we can write it as

$$H_{\text{BdG}}(\varphi) = \left(\frac{H_{\text{NW}}}{\Delta(x, \varphi)} + \frac{H_{\text{N}}}{\Delta(x, \varphi)}\right),$$

(3)

where $H_{\text{NW}} = H_L + H_R + V_{T_N}$ is the normal NW Hamiltonian. $H_{\text{NW}}$ consists of the Hamiltonians for the two (left/right) segments $H_{L/R}$, coupled across a short weak link of transparency $T_N \in [0, 1]$ by a $V_{T_N}$. Each segment contains all the microscopic NW details (Rashba spin-orbit coupling $\alpha_{\text{SO}}$, Zeeman field $B$ and chemical potential $\mu$) and is described by a single-band Lutchyn-Oreg model. Each segment $H_{L/R}$ contains all the microscopic NW details (Rashba spin-orbit coupling $\alpha_{\text{SO}}$, Zeeman field $B$ and chemical potential $\mu$) and is described by a single-band Lutchyn-Oreg model. The BdG spectrum of $H_{\text{BdG}}(\varphi)$ is given by

$$\hat{H}_{L/R} = \hat{p}_x \frac{\hat{p}_x}{2m} - \mu - \frac{\alpha_{\text{SO}}}{\hbar} \hat{\sigma}_y \hat{p}_x + B \hat{\sigma}_z,$$

(4)

with $\hat{p}_z = -i\hbar \partial_z$ the momentum operator and $\hat{\sigma}_i$ Pauli matrices in spin space. $\Delta(\varphi) = \hat{\sigma}_y \Delta e^{\pm i\varphi/2}$ (where the $\pm$ corresponds to $x \in L/R$, respectively) is the induced pairing term. The discretized version of the above model reads:

$$H_{\text{BdG}} = \frac{1}{2} \sum_i \hat{c}_i \hat{c}_i + \frac{1}{2} \sum_{\langle ij \rangle} \hat{c}_i \hat{v}_{ij} \hat{c}_j,$$

(5)

where $\hat{c}_i = (c_{i\uparrow}, c_{i\downarrow}, c_{i\uparrow}^\dagger, c_{i\downarrow}^\dagger)$ are Nambu spinors in spin ($\sigma_i$) and particle-hole ($\zeta_i$) sectors, $\langle ij \rangle$ means sum over nearest neighbours, and $h_i$, $v_{ij}$ are onsite and hopping parts of the Hamiltonian:

$$h_i = (2t - \mu) \zeta_i \sigma_0 + B \zeta_i \sigma_x + \Delta(\varphi) \zeta_i \sigma_y,$$

$$v_{ij} = -t_{ij} \zeta_i \sigma_0 - \frac{\alpha_{\text{SO}}}{2a} \zeta_i \sigma_y,$$

(6)

$$t_{ij} = \begin{cases} t & \text{within the same region} \\ \tau t & \text{at interface} \end{cases}.$$

The tight-binding hopping parameter above is $t = \hbar^2/2ma_0^2$, where $m = 0.015m_e$ in InSb NWs, $m_e$ is the
the BdG spectrum different NWs for increasing lengths $L_S = 2.2\mu m$ (a), $L_S = 3\mu m$ (b) and $L_S = 5\mu m$ (c), as a function of the ratio $B/B_c$. As the magnetic field increases, the energy of the lowest mode (orange line) decreases until it reaches zero at $B \sim B_c$ (with finite-$L_S$ corrections). This smooth cross-over is the finite-size version of the predicted $L_S \rightarrow \infty$ topological transition at exactly $B = B_c$, where the Zeeman-dominated gap at zero momentum $\Delta_{\mu = 0} = \Delta_0$ closes and reopens again. For $B > B_c$ and finite $L_S$, the lowest energy mode is the superposition of two weakly overlapping MBSSs, which endows it with a finite energy $\delta$ due to Majorana hybridization. This Majorana splitting is of order $\delta \approx \frac{\hbar v_F}{\alpha} e^{-2L_S/\xi_M} \cos(p_F L_S)$, where $p_F$ is the Fermi momentum (that grows with $\mu$ and/or $B$), $\xi_M = \hbar v_F/\Delta$ is the Majorana superconducting coherence length, and $v_F$ denotes the Fermi velocity. For small-to-moderate magnetic fields and small chemical potentials, the Fermi velocity is well approximated by $v_F \sim \alpha_{SO}/\hbar$ and thus $\xi_M \sim \alpha_{SO}/\Delta$. For larger magnetic fields, the Majorana length acquires a prefactor that depends on the ratio between the Zeeman energy and the SO energy $E_{SO} = m\alpha_{SO}/2\hbar^2$, resulting in a parametric dependence $\xi_M \sim 2(B/\Delta) l_{SO}$, with the SO length given by $l_{SO} = \hbar^2/(\alpha_{SO})$. As it becomes evident from this discussion, the energy splitting $\delta$ of Majoranas has a rich dependence on the microscopic details of the NW and, importantly, oscillates around zero with an amplitude that grows with $B$, see e.g. Fig. 2(a). For fields sufficiently above $B_c$, the near-zero Majorana mode becomes separated from the rest of excitations by a topological gap given by the superconducting pairing term at the Fermi momentum $\Delta_{\mu = 0} = \Delta_T$. The topological gap $\Delta_T$, the zero-momentum $\Delta_0$ and the Majorana splitting $\delta$ are all marked by arrows in Fig. 2. At $\mu = 0$ we can write $\Delta_T$ analytically as:

$$\Delta_T = \frac{2\Delta E_{SO}}{\sqrt{E_{SO}(2E_{SO} + \sqrt{B^2 + 4E_{SO}^2})}}.$$  (7)

It has a maximum value $\Delta_T = \Delta$ in the SO-dominated limit $E_{SO} \gg \hbar v_F$ and can be much smaller in the opposite Zeeman-dominated limit $B \gg E_{SO}$, where it decreases with $B$ as $\Delta_T \sim 2\sqrt{E_{SO}/B}$ (of order $2\sqrt{\Delta E_{SO}}$ near the $\mu = 0$ critical Zeeman field $B_\mu \approx \Delta$). The value is the topological gap $\Delta_T$ is particularly important for this paper since it governs the Majorana Josephson coupling term $E_M$. Its dependence on microscopic parameters is therefore very relevant in superconducting island qubits based on topological NWs with MBSSs, and will be discussed in detail throughout this paper.

In Figs. 3 and 4 we show various examples of typical $\varphi$-dispersing subgap spectra of a short NW SNS junction in the topological $B > B_c$ regime. We focus in particular on how the subgap spectrum (orange lines) of the SNS junction changes for decreasing transparency factor $\tau$. Quite generically, the short-junction subgap spectrum can be expressed as an effective model of four Majorana

**FIG. 2. BdG spectrum as a function of Zeeman field $B$ and Majorana oscillations.** BdG spectrum of NWs of increasing lengths $L_S = 2.2\mu m$ (a), $L_S = 3\mu m$ (b) and $L_S = 5\mu m$ (c), as a function of the ratio $B/B_c$. For $B > B_c$ the lowest mode (orange line), corresponding to weakly overlapping MBSSs, shows clear oscillations of amplitude $\delta$ around zero energy in (a) and (b). These oscillations become progressively reduced as $L_S$ increases, (c). For $B \gg B_c$, the Majorana mode around zero energy is separated from the quasi-continuum formed by the rest of BdG excitations (grey lines) by a so-called topological gap $\Delta_T$, which varies with the $B$ field and depends on microscopic parameters of the wire ($\mu$ and $\alpha_{SO}$). Near $B_c$, the relevant gap is the one that closes and reopens at the topological transition (the zero momentum gap $\Delta_0$).

**B. Subgap spectrum and relevant energy scales**

To set the stage and for the sake of completeness, we now discuss some well-known results about the subgap spectrum of both single NWs and NW SNS junctions (for a recent review, see Ref. [52]). The aim of this subsection is to provide some estimates (notably of $E_M$), based on the single-band NW model in Eq. [4], which will be of relevance to our superconducting island results in the next subsections. As we already mentioned, a proximitized NW undergoes a topological phase transition at $B_c = \sqrt{\Delta^2 + \mu^2}$ simultaneous with the appearance of MBSSs at their edges, see sketch in Fig. 1. This is illustrated in Fig. 2 which plots the electron’s mass and $a_0$ a lattice discretization parameter. The finite weak link transparency $T_N$ is modeled through a renormalization of $t$ by a “transparency factor” $\tau \in [0,1]$ that is monotonous (although non-linear) with $T_N$ [19]. Some other material parameters are fixed according to typical experimental values: $\mu = 0.5meV$, $\Delta = 0.25meV$, $\alpha_{SO} = 20meVA$. For simplicity, in what follows both NW segments are assumed to be equal and of the same length $L_S$. 

$$\Delta_T = \frac{2\Delta E_{SO}}{\sqrt{E_{SO}(2E_{SO} + \sqrt{B^2 + 4E_{SO}^2})}}.$$  (7)
The four finite-energy subgap eigenstates (Andreev modes) are originated from the overlap four MBSs as discussed in the main text. For \( \tau = 1 \), they touch the quasi-continuum formed by the rest of BdG levels (gray lines) at the so-called topological gap, namely \( E_M = \Delta_T \). In this limit, the \( \varphi = \pi \) crossing of the inner Majoranas gives rise to the so-called 4\( \pi \) Josephson effect. This anomalous Josephson effect is destroyed by finite \( L_S \) corrections, however, due to a lifting of the \( \varphi = \pi \) crossing by the remaining \( \lambda \) terms, as shown in Figs. 3 and 4.

The value of the inner Majorana coupling \( E_M \) can be estimated from the above plots to be in the approximate range of a few tenths of \( \Delta \) (depending on the tunneling coupling). The upper bound for \( E_M \) is reached in the transparent limit (\( \tau = 1 \)), where \( \lambda_{23} = 2T_m = E_M \) touches the quasi-continuum formed by the rest of BdG levels (gray lines), see Fig. 3 (a). For \( B \gg B_c \), this happens at the topological gap, namely \( E_M = \Delta_T \) (for the particular microscopic parameters of this plot \( \Delta_T \sim 0.5\Delta \)). For \( \tau \neq 0 \), the inner Majorana coupling is always \( E_M < \Delta_T \), Fig. 3 (b), and can be approximated as \( E_M \sim \sqrt{T_N \Delta_T} \), with \( T_N \) the normal transmission of the junction. This reflects the fact that, at high \( B \) fields, and still neglecting the role of the outer Majorana modes, the physics governing the NW low-energy subgap spectrum is that of a (single) proximitized helical channel, as described by the Fu-Kane model for a quantum spin Hall edge (8).

By considering the critical current supported by a single channel \( I_N^0 = eT_N \Delta T/2 \mu \), and comparing it with the critical current resulting from the 4\( \pi \) Majorana Josephson effect \( I_M^0 = e\sqrt{T_N \Delta_T} /2 \mu \), the ratio between both couplings can be estimated as

\[
E_M / E_J \sim \eta \Delta_T / \Delta \sim 2 \eta \sqrt{E_{SO} / B} + O(\mu),
\]

with a prefactor \( \eta = \sqrt{T_N} / T_N > 1 \). We will come back with more precise estimations of the ratio \( E_M / E_J \) in subsection BVC.

### III. NW-BASED SUPERCONDUCTING ISLANDS

#### A. Effective low-energy model and projection

Our first goal is to derive a quantitative but simple low-energy description of a short SNS NW junction that extends Eq. (2) by taking into account both standard Josephson events due to Cooper pair tunneling, as well as anomalous Majorana-mediated events where a single electron is transferred across the junction. In order to do this, it is convenient to distinguish two contributions,
\[ V_J = V_J^{\text{bulk}} + \hat{H}_{\text{BdG}}^{\text{sub}}. \]

The first one takes into account the bulk of the BdG levels above the gap, whose occupation is assumed in thermal equilibrium. We write this contribution as:

\[ V_J^{\text{bulk}}(\varphi) = - \sum_{\epsilon_p > \Delta} \epsilon_p(\varphi). \]  \hspace{1cm} (11)

The second contribution corresponds to the subgap sector. As in the preceding subsection, we assume there are only two independent spin-resolved fermionic subgap states (short junction). Unlike for the states above the gap, we do not make further assumptions about them and instead treat their dynamics as fully coherent, governed by the \( \hat{H}_{\text{BdG}}(\varphi) \) Hamiltonian introduced in Eq. [8]. By extending \( V_J \) with a contribution \( \hat{H}_{\text{BdG}}^{\text{sub}} \) in this way, we are supplementing our relevant quantum degrees of freedom \( N, \varphi \) with the \( \gamma_i \) Majorana operators. The challenge remains of relating \( \hat{H}_{\text{BdG}}^{\text{sub}}(\varphi) \) to the microscopic Hamiltonian \( H_{\text{BdG}} \) by projecting the latter onto the low-energy subspace of Majorana operators. The procedure, described above, starts by defining the basis of left and right low-energy fermions \( c_{L/R} \) and \( c^\dagger_{L/R} \) of the decoupled NWs. These states are a basis to the four lowest BdG eigenstates of the microscopic \( H_{\text{BdG}} \) with \( \tau = 0 \), and are related to the \( \gamma_i \) operators by a simple rotation

\[ \sqrt{2} \gamma_1 = c_L + c^\dagger_R, \quad \sqrt{2} \gamma_2 = i(c_L - c^\dagger_R), \]
\[ \sqrt{2} \gamma_3 = c_R + c^\dagger_L, \quad \sqrt{2} \gamma_4 = i(c_R - c^\dagger_L). \]  \hspace{1cm} (12)

In terms of these operators, the fermion numbers on each segment are simply \( \tilde{n}_L = c^\dagger_L c_L = (1 + i\gamma_1 \gamma_2)/2 \) and \( \tilde{n}_R = c^\dagger_R c_R = (1 + i\gamma_3 \gamma_4)/2 \). Next, we integrate out all states outside this low-energy decoupled subspace. This is done by computing the matrix elements of the resolvent of \( H_{\text{BdG}} \), \( G(\omega) = (\omega + i\varepsilon - H_{\text{BdG}})^{-1} \) at \( \omega = 0 \) on the \( \psi^0 = (c_L, c^\dagger_L, c_R, c^\dagger_R) \) state basis. This defines a 4 × 4 matrix, whose inverse is the matrix \( \hat{H}_{ij} \) of the \( H_{\text{BdG}} \) projection we are after,

\[ (\hat{H}^{-1})_{ij} = \langle \psi^0_i | \hat{G}(\omega = 0) | \psi^0_j \rangle, \]
\[ \hat{H} = \frac{1}{2} \sum_{ij} \psi^0_i \hat{H}_{ij} \psi^0_j. \]  \hspace{1cm} (13)

Finally, we identify the above \( \hat{H} \) as the \( \hat{H}_{\text{BdG}}^{\text{sub}} \) in Eq. [8], from which we extract the dependence of \( \lambda_{ij}(\varphi) \) on all microscopic parameter in \( H_{\text{BdG}} \) using Eq. [12]. This identification is an approximation, although we have checked that it is a very accurate one in practice.

We can now write the matrix elements of \( V_J = V_J^{\text{bulk}} + \hat{H}_{\text{BdG}}^{\text{sub}} \) in the parity basis \( |n_L n_R\rangle \). The effective Josephson coupling reads:

\[ V_J(\varphi) = \]
\[ \begin{pmatrix}
V_J^{\text{bulk}}(\varphi) + \langle 00 | \hat{H}_{\text{BdG}}^{\text{sub}}(\varphi) | 00 \rangle & \langle 00 | \hat{H}_{\text{BdG}}^{\text{sub}}(\varphi) | 11 \rangle \\
\langle 11 | \hat{H}_{\text{BdG}}^{\text{sub}}(\varphi) | 00 \rangle & V_J^{\text{bulk}}(\varphi) + \langle 11 | \hat{H}_{\text{BdG}}^{\text{sub}}(\varphi) | 11 \rangle
\end{pmatrix}. \]

The final low-energy Hamiltonian is thus a generalization of Eq. [1] to a 2 × 2 operator with the above \( V_J \)

\[ \hat{H} = [4E_C (i\partial_\varphi - n_\gamma)^2] \mathbb{I} + V_J(\varphi). \]  \hspace{1cm} (15)

The eigenstates of Eq. [16] are defined as a two component spinor \( \Psi_k = (f_k(\varphi), g_k(\varphi))^T \), owing to the pseudospin structure in the parity basis. The components of this spinor have not the same periodicity (while \( f(\varphi) \) is just \( 2\pi \) periodic, \( g(\varphi) \) displays antiperiodicity). To make the Hamiltonian fully periodic, it is rotated according to \( \hat{H}(\varphi) \rightarrow U \hat{H}(\varphi) U^\dagger \), with \( U = \text{diag}(1, e^{i\varepsilon/2}) \). Their fermionic \( n_{L/R} \) parity content can be calculated by projecting each eigenstate onto the parity axis defined by \( \tau_2 \equiv |00\rangle - |11\rangle \). Henceforth, we plot energy levels with a well-defined even/odd parity using blue/orange lines, while mixed parities are encoded using gradient colours between blue and orange, with a light-green midpoint color denoting a 50% parity mixture.
B. NW-based superconducting islands model in tight-binding form

Next we want to solve the superconducting island Hamiltonian of Eq. (16). This is accomplished by discretizing the phase space as \( \varphi_j = 2\pi j/\ell^2 \), \( j = 1, 2, \ldots, \ell^2 \). In so doing, the Hamiltonian acquires a tight-binding form, where the discretized phase may be seen as a set of sites arranged into a circular chain. This discretization defines a finite fermionic Hilbert space and operators \( b_i^{\dagger} \), whose action on the ground state is defined as \( b_i^{\dagger}|0\rangle = \Psi(\varphi_i) \), where \( \Psi(\varphi) \) is the Hamiltonian eigenstate at phase \( \varphi \). The derivative \( N = -i\partial_\varphi \) translates in this language into the usual hopping term \(-i\partial_\varphi = -(b_{i+1} - b_{i-1})b_i/(2a_\varphi) \), where \( a_\varphi = 2\sin(\pi/\ell^2) \) is the phase lattice constant. Using this tight-binding language, the Hamiltonian (16) reads

\[
H(\varphi) = \sum_i b_i^{\dagger} h_i^\varphi b_i + \sum_{ij} b_i^{\dagger} v_{ij}^\varphi b_j,
\]

\[
h_i^\varphi = 4E_C(2a_\varphi^{-2} + n_g^2) + V_j(\varphi),
\]

\[
v_{ij}^\varphi = 4E_C \left[ \text{sgn}(j-i) \right] n_g a_\varphi^{-1} - a_\varphi^{-2} \right].
\]

Each site element \( h_i^\varphi \), \( v_{ij}^\varphi \) is a \( 2 \times 2 \) matrix, owing to the pseudospin structure from even-odd projection, Eq. (15). This tight-binding model is numerically solved by means of the MathQ software.

C. Dependence of the superconducting islands parameters on microscopic parameters of the NW

The NW microscopic details enter this problem through the effective Josephson potential \( V_J(\varphi) \). In particular, the three relevant NW energy scales that govern the superconducting island Hamiltonian in Eq. (16) (the Josephson coupling \( E_J \), the energy splitting between different fermionic parities \( \delta \), and the single-electron contribution to the Josephson coupling \( E_M \)) can be defined in terms of the projected Hamiltonian as:

\[
E_J = \int_0^{2\pi} \frac{d\varphi}{\pi} \left[ V_{J,\text{bulk}}(\varphi) + \langle 00|\hat{H}_{\text{BDG}}^\text{sub}(\varphi)|00\rangle \right] \cos(\varphi),
\]

\[
\delta = \langle 11|\hat{H}_{\text{BDG}}^\text{sub}(\varphi = 0)|11\rangle - \langle 00|\hat{H}_{\text{BDG}}^\text{sub}(\varphi = 0)|00\rangle,
\]

\[
E_M = \int_0^{2\pi} \frac{d\varphi}{\pi} \langle 00|\hat{H}_{\text{BDG}}^\text{sub}(\varphi)|11\rangle \cos(\varphi).
\]

All these parameters depend on relevant quantities such as e.g. NW length and magnetic field. As defined above, \( E_J \) refers to the energy contribution associated to Cooper-pair transfers across the junction, and hence to the critical current of the system. \( E_M \) on the other hand accounts for single-quasiparticle transfer through the subgap states of the spectrum, either Andreev states (trivial) or Majorana states (topological phase). \( \delta \) is the minimal energy cost for exciting one quasiparticle on each NW segment above the ground state. Importantly, the effective models in Refs. 28 and 29 assume a simplified Josephson term of the form

\[
V_J(\varphi) = \begin{pmatrix} -E_J \cos(\varphi) & E_M \cos(\varphi/2) \\ E_M \cos(\varphi/2) & -E_J \cos(\varphi) \end{pmatrix},
\]

which is not able to capture the full \( \varphi \)-anharmonicity, or the various parameter regimes and their associated phenomenology inherent to the microscopic description employed here. This includes the trivial (Andreev) regime, the topological (Majorana) regime and the crossover/transition between the two with \( B \) field. Our approach also yields the detailed dependence of the junction \( \delta \), \( E_M \) and \( E_J \) on various NW parameters (such as e.g. \( \alpha_{SO} \)) and transparency of the junction, see subsection IV.C. These three quantities are plotted in Fig. 5 as a function of \( B \) before and after the topological transition. They inherit some important features of the NW behavior for finite \( B \) fields. These include the closing and reopening of the gap and the characteristic oscillatory pattern due to finite-length splitting of Majorana excitations.

IV. BENCHMARK RESULTS

A. Known limits

As first checks of our procedure, we benchmark our method against well-known limits. This includes the standard superconducting island behavior in the \( B \to 0 \), \( \tau \to 0 \) limits, Fig. 3. As expected, the charge dispersion \( \partial E_n/\partial n_g \) of all energy levels \( E_n \) gets progressively reduced by increasing the ratio \( E_J/E_C \), and the island crosses over from the CPB to the transmon regime. The latter is characterized by a spectrum of a slightly anharmonic oscillator with frequency given by the plasma frequency

\[
\omega_{pl} = \sqrt{8E_JE_C}/h.
\]
FIG. 7. Spectrum of NW-based superconducting islands with magnetic field (basic phenomenology). The parity content of energy levels is calculated by projecting each eigenstate onto the parity axis defined by $\hat{\tau}_z \equiv \langle 00 \rangle - \langle 11 \rangle$ (see main text). The even sector is represented by blue parabola with minima at $n_g = m \in \mathbb{Z}$, while the odd sector is represented by orange parabola with minima at $n_g = m - 1/2$. Top panels (a–d) (Coulomb island): charging regime with $E_C = 0.5\Delta$ and $E_J/E_C = 10^{-4}$. Transparency factor $\tau = 0.01$. For zero magnetic field, odd parabola are shifted in the energy axis by exactly $\delta = 2\Delta$, panel (a). The spectrum is 2$\pi$-periodic. This energy shift $\delta$ decreases for increasing $B$ fields, see panel (b) with $B = 0.7B_c$, and vanishes exactly at the topological transition $B = B_c$, panel (c), where the spectrum becomes $\pi$-periodic. Panel (d) corresponds to $B = 1.5B_c$. Lower panels (e–h) (finite Josephson and Majorana couplings): same as top panels but with $\tau = 0.8$. The finite Josephson coupling (here $E_J = 0.8E_C$) results in avoided crossings between parabolae of the same parity. Panel (h): For $B > B_c$, there is also a finite Majorana coupling that induces avoided crossings around $n_g = 0.25$ between parabolae of opposite parity. Note that the Majorana-induced avoided crossing ($\sim E_M$) is non-negligible with respect to the maximum at $n_g = 0.5$ ($\sim E_C$). Rest of parameters: same as NW of Fig. 2(c).

We can reach by drastically reducing both the Josephson and Majorana couplings in the $\tau \to 0$ limit (i.e. for two isolated NWs with charging energy $E_C$). For $B = 0$ (orange lines), odd parabolae are energy shifted from even ones (blue lines) by exactly $\delta = 2\Delta$, panel (a). As the Zeeman field increases, $\delta$ becomes progressively reduced until it becomes of the order of $E_C$ (panel (b)) or smaller, which results in a transition from an even to an odd ground state around half-integer $n_g = m + 1/2$, with $m \in \mathbb{Z}$. For $B = B_c$, panel (c), both parity sectors have minima at zero energy and the periodicity becomes $4\pi$-periodic as opposed to the $2\pi$-periodicity of the standard superconducting island at $B = 0$. Increasing $\tau$ (lower panels), results in finite Josephson coupling which leads to avoided crossings between parabolae of the same parity due to $E_J$. For $B > B_c$, panel (h), there appear also avoided crossings between parabolae of the opposite parity owing to a finite Majorana coupling $E_M$.

FIG. 8. Deviation of SNS junctions from standard Josephson behavior. Josephson inductance $L_J$ of SNS junctions (solid lines) provides an useful tool to measure deviation from idealised conditions ($L_J^{-1} \sim \cos \varphi$, dashed lines). Panels describe phase dependence of $L_J^{-1}$ for increasing magnetic fields. Parameters: $L_S = 2.2\mu m$, $\tau = 0.8$, $B/B_c = 0, 0.8, 1, 1.2$.

FIG. 9. Anharmonicity $\alpha$ of a NW-based superconducting qubit at $n_g = 0.5$. Solid (dashed) curves show transmon dependence on $B$ for short (long) wires. Both CPB (panel a) and transmon limits (panel b) are displayed. Anharmonicity provides a precise smoking gun to detect topological transitions and Majorana oscillations, specially for the transmon limit. Parameters: $L_S = 2.2, 5\mu m$, $\tau = 0.8$, $E_J/E_C = 0.5, 25$.

B. Josephson inductance and anharmonicity

In a standard superconducting island, as mentioned in the introduction, the SIS Josephson junction is well described by an energy-phase relation of the form $V_{J}^{SIS}(\varphi) = -E_J \cos(\varphi)$. Its corresponding inverse Josephson inductance reads $L_J^{-1}(-\varphi) = (2e^2/h)^2 V_{J}^{SIS}(\varphi)/d\varphi^2 = (2e^2/h)^2 E_J \cos(\varphi)$. The NW-based JJ that we discuss here strongly differs from this cosine form (which is only valid in the tunneling limit and in the absence of external magnetic fields, $\tau \to 0, B = 0$). These deviations from a cosine form have relevant consequences when e.g. using superconducting islands as qubits. An important figure of merit is the anharmonic-
Majorana versus standard Josephson coupling. Both panels show the evolution of the ratio $E_M/E_J$ as the transparency factor $\tau \in [0,1]$ is increased for different $B$ fields in the topological regime. For clarity, both linear (panel (a)) and logarithmic (panel (b)) plot scales are provided. For small transparencies below $\tau \approx 0.2$, the Majorana coupling $E_M$ becomes larger than $E_J$. Rest of parameters: same as NW of Fig. 2 (a).

For low magnetic fields, it becomes manifest that a standard, cosine-like SIS inductance (dashed line) is by no means sufficient to study the qubit evolution with magnetic field and across topological transitions in NW-based islands in the few-channel regime.

As for the anharmonicity parameter $\alpha \equiv E_{12} - E_{01}$, where $E_{mn}$ is the energy difference between $m$ and $n$ energy states, which controls the leakage rate into noncomputational states (the high-energy states out of the two-level qubit Hilbert space), Figure 9 illustrates this by looking at the Josephson inductance $L_J$ for NWs with transparent links at several $B$ values, before and after the topological transition. Even for low magnetic fields, it becomes manifest that a standard, cosine-like SIS inductance (dashed line) is by no means sufficient to study the qubit evolution with magnetic field and across topological transitions in NW-based islands in the few-channel regime.

### C. Ratio $E_M/E_J$ for NW-based single channel Josephson junctions in the topological regime.

An important figure of merit that governs the different physical regimes of Eq. 10 is the ratio $E_M/E_J$, which controls the relative amplitude of different-parity and same-parity anticrossings in the CPB/transmon spectrum (Fig. 7), as well as the ratio $E_M/E_C$ at fixed $E_J/E_C$. This subsection elaborates on this aspect of the problem, and shows that while $E_M/E_J$ depends on various model parameters, it is in general not small.

In Fig. 10, we plot the dependence of the ratio $E_M/E_J$ with $\tau$ for different $B$ fields in the topological regime. For small $\tau$, this ratio can be much larger than unity, while it is of order $E_M/E_J \sim 0.1$ for $\tau \to 1$. This behavior is consistent with the different expected dependence on transparency $T_N$ of $E_J$ and $E_M$, as discussed before. In Fig. 11 we further plot the ratio $E_M/E_J$ against $\sqrt{E_{SO}/B}$ at fixed, finite $\mu$. As expected, it deviates from the $\mu = 0$ estimations of Eq. 10. Panel (a) illustrates the deviation by plotting results for $\mu = 2\Delta$. In contrast, this ratio starts to approach the $E_M/E_J \propto \sqrt{E_{SO}/B}$ estimation as $\mu$ and $E_{SO}/B$ decrease; panel (b). The main conclusion that can be drawn from this discussion is that, in general, $E_M/E_J$ is not a small number. For typical islands in the $E_J/E_C > 1$ regime, this also implies that $E_M$ is not small as compared with $E_C$.

These calculations and estimations are based on the single channel junction. We can expect that multichannel systems will show an overall increase of the Josephson coupling $E_J$. Considering a simple formula for the multichannel Josephson potential $V_J(\varphi) = -\Delta \sum_{i=1}^{M} \sqrt{1 - T_i \sin^2(\varphi/2)}$ (which just assumes a short junction in the Andreev limit containing $M$ channels with normal transmission probabilities $T_i = 1,...,M$), the Josephson coupling is $E_J = \Delta/4 \sum_{i=1}^{M} T_i$. If the junction contains $m$ highly transmitting channels, we can expect an overall reduction of the above estimation for $E_M/E_J$ of order $\sim 1/m$. Even in these multichannel cases, we argue that the parameter regimes explored in previous papers using low-energy effective toy model$^{28,29}$ (with very small ratios $E_M/E_J \sim 10^{-3}$) are somewhat unphysical since this would imply hundreds of highly transmitting channels (i.e., hundreds of trivial subbands contributing to $E_J$ on top of a topological single band contributing to $E_M$). Another limiting case in which $E_M/E_J$ is small, is the $E_{SO}/B \ll 1$ limit in a few-channel junction. This case, however, corresponds to a very small topological gap $\Delta_T \sim 2\Delta \sqrt{E_{SO}/B} \ll \Delta$, which would naturally hinder the observation of any Majorana-related physics in the superconducting island properties.

Even in the opposite strong CPB regime with $E_J/E_C \ll 1$, the first experimental evidence of hybridization between different parity sectors owing to the $E_M$ coupling$^{30}$ gives estimated ratios of the order $E_M/E_C \approx 0.25$, which, again, is much larger than the previously explored regimes in Refs. 28 and 29. This regime seems to suggest that the superconducting islands used in the experiments of Ref. 30 are based on few-channel junctions in the small transparency regime (see Fig. 10).

We will discuss in full the implications of the different ratios $E_M/E_C$ in the next subsections. The novel regime
Having discussed various relevant aspects of NW-based superconducting islands we are now ready to analyze in detail the microwave response of such junctions. Using the solutions of Eq. (1) in terms of Mathieu functions, the microwave absorption spectrum of the island can be written in linear response as

\[ S(\omega) = \sum_k \left| \langle k | \hat{N} | 0 \rangle \right|^2 \delta(\omega - (\omega_k - \omega_0)). \]  (21)

This expression measures the energy transitions \( \omega_{0k} = \omega_k - \omega_0 \) between the \( k = 0 \) ground state \( E_0 = \hbar \omega_0 \) and the excited states \( E_k = \hbar \omega_k \) of the junction with a probability weighted by the matrix elements of the relative number operator

\[ \langle k | \hat{N} | 0 \rangle = \int_0^{2\pi} d\varphi \Psi_k^\dagger \begin{pmatrix} -i\partial_{\varphi} & 0 \\ 0 & -i\partial_{\varphi} + 1/2 \end{pmatrix} \Psi_0. \]  (22)

A detailed discussion about the spectral weights of relevant microwave transitions in terms of the above matrix elements, and their dependence on various island parameters, is included in subsections \( \text{V A 2} \) and \( \text{V B 2} \).

We note that the above notation \( \omega_{0k} = \omega_k - \omega_0 \) in terms of energy differences between the ground state and the excited states, with index \( k \) ordered in terms of increasing energies, may induce to some confusion in the context of this paper since parity conservation (i.e. negligible \( E_M \)) can render some of these transitions invisible. For example, at \( B = 0 \) the first transition \( \omega_0 \) is even-even (allowed), see e.g. Fig. 7(a), and hence a standard qubit transition (assuming \( 2\Delta \gg \hbar \omega_{pl} \), so that odd states are at higher energies). However, for \( B = B_c \), the equivalent parity-preserving transition is now \( \omega_{03} \), see e.g. Fig. 7(c), with a strongly suppressed \( \omega_0 \) and \( \omega_2 \).

When \( B > B_c \) and \( E_M \) is finite, we can have other situations, such as \( \omega_{01} \) (a transition within the ground state manifold) becoming visible thanks to the Majorana-induced parity mixing, particularly close to the \( n_g = 0.25 \) and \( n_g = 0.75 \) anticrossings, see e.g. Fig. 7(h).

When needed, and to avoid ambiguities, we will use, together with the above notation, a notation drawn from the superconducting qubit literature, based on the solutions of Eq. (1) in terms of Mathieu functions. This notation assumes decoupled even-odd sectors, essentially the diagonal part of Eq. (16), whose eigenstates are labelled as \( |n, e/o\rangle \), with \( m \) denoting the bosonic mode index of the island and \( e/o \) denoting even/odd parity. For example, using this notation, a transition \( \omega_{01} \) at \( B = 0 \) corresponds to a standard interband qubit transition \( |0, e\rangle \rightarrow |1, e\rangle \), with \( \omega_{01} = 4E_C/\hbar \) at \( n_g = 0 \), or \( \omega_{01} = E_J/\hbar \) at \( n_g = 0.5 \), see Fig. 7(e). For \( B > B_c \), the transition \( \omega_{01} \) corresponds now to an intraband transition between two states (of approximately well defined parity) within the ground state manifold, namely \( |0, e\rangle \rightarrow |0, o\rangle \) at \( n_g = 0 \), or viceversa \( |0, o\rangle \rightarrow |0, e\rangle \) at \( n_g = 0.5 \). These intraband transitions depend on the charge dispersion of the island and are of order \( \hbar \omega_{01} \approx E_C (E_C/E_J)^{1/4} \exp(-\sqrt{8E_J/E_C}) \) in the \( E_J/E_C \gg 1 \) limit. Obviously, this notation in terms of well defined parity is strictly valid only in the \( E_M \rightarrow 0 \) limit (namely in the basis of \( \tilde{z} \equiv |00\rangle - |11\rangle \). In the opposite limit, parity is not well defined and we will rather use \( |m, \pm\rangle \), denoting the two mixed-parity eigenstates that diagonalize Eq. (16). Using the previous example for \( B > B_c \), the intraband transition within the ground state manifold around \( n_g = 0.25 \) is better described by the notation \( |0, -\rangle \rightarrow |0, +\rangle \) and is of order \( \omega_{01} \approx E_M/\hbar \).

A. NW-based superconducting islands in the \( E_M/E_C \ll 1 \) regime

1. Microwave spectroscopy in the \( E_M/E_C \ll 1 \) regime

To make connection with published literature, we first analyze the \( E_M/E_C \ll 1 \) regime. In order to artificially enhance the ratio \( E_J/E_M \), which models a multichannel situation, as discussed before, we add by hand a Josephson term \(-E_J \cos \varphi \rightarrow V_J(\varphi)\), such that the total \( E_J \) used in the calculations is much larger than the one we obtain from the single band NW calculation (the BdG...
spectrum employed here corresponds to the NW in Fig. 2(a). The microwave spectra of a paradigmatic case with $E_J/E_C \approx 5$ in this $E_M/E_C \ll 1$ regime are shown in Fig. 12. We first plot the overall magnetic field dependence of transitions $\omega_{0n}$ and energy levels $E_n$ at fixed $n_g = 0$, panels (a-c). The $B > B_c$ microwave spectrum in this limit is just that of a transmon with a split line: owing to the Majorana coupling $E_M$, the original ground state splits into a doublet $|0, \pm\rangle$, while there appear two possible interband qubit transitions from the $|0, -\rangle$ ground state to two excited states of approximately good parity $|1, 0/c\rangle$ (Fig. 12(c)). These split lines give rise to two possible transitions $\omega_{02}$ and $\omega_{03}$. We also show the corresponding matrix elements, shown as the thickness of the transition frequencies, in panel (b). Apart from the odd state that goes down in energy and reaches zero at $B \sim B_c$, an important aspect of the overall magnetic field dependence of the three panels in Figs. 12 (a-c) is the complete absence of $B > B_c$ parity crossings (originating from the Majorana oscillations in the NW spectrum of Fig. 1(a)). This can be understood as a consequence of the large $E_C$, as compared to $E_M$, which largely prevents the changes in the ground state fermionic parity that are associated to Majorana oscillations. As a result, all the lines for $B > B_c$ are almost independent of $B$-field (including both curvature and parity, see colors of the lines in (c)). Importantly, the intraband transition within the ground state doublet $\omega_{01} (|0, -\rangle \rightarrow |0, +\rangle)$ has no spectral weight in this regime. Thus, it is not visible in the absorption spectrum of (a). Different magnetic field cuts (colored bars in (a)) are shown in panels (d-f), with the corresponding energy states in (g-i). The splitting of the lines shows dispersion as a function of $n_g$ (as expected for this particular $E_J/E_C$ ratio) while having very little dependence on $B$ field (the three $B$-field cuts are essentially the same). The visible effect of Majoranas in the NW is the appearance of “spectral holes” in the $\omega_{03}$ transition near $n_g = 0.25$ and $n_g = 0.75$, (namely, a zero of the transition matrix element at that point). This happens as the small $E_M$ coupling weakly removes the degeneracy of the even and odd parity sectors in the $E_M \rightarrow 0$ limit (occurring at $n_g = 0.25$ and $n_g = 0.75$). All the above results are in full agreement with Refs. 28, 29 and 35.

2. **Dependence of the spectral weights on the ratio $E_J/E_C$ for different $n_g$**

Before proceeding to the discussion of the $E_M/E_C \gtrsim 1$ regime, we will analyze the above $E_M/E_C \ll 1$ results in terms of the spectral weights of the involved transitions. Fig. 13 shows these matrix elements as we cross over from the CPB to the transmon regime by increasing $E_J/E_C$. It also shows the corresponding spectra versus $n_g$ at specific values of $E_J/E_C$. Different columns represent magnetic field configurations in the NW (before, at and after a minimum of a Majorana oscillation). The top left panels (a-c), show the spectral weights of the first transitions as a function of $E_J/E_C$ and fixed $B/B_c = 1.2$ (before the first parity crossing in the NW spectrum of Fig. 2(a)), and for different $n_g = 0$ (a), $n_g = 0.25$ (b) and $n_g = 0.5$ (c). The overall behavior changes very little for different gates, with a dominant $\omega_{02}$ transition and a weaker $\omega_{03}$ transition. These transitions can be understood by looking at the spectra for different $E_J/E_C$ cuts, which are shown in the lower left panels (d-g). For increasing ratios $E_J/E_C (E_J/E_C = 2, 5, 10, 25$ from (d) to (g), corresponding to the colored bars at the top), the spectra evolve from the CPB to the transmon regimes, reaching an almost doubly degenerate transmon spectrum, as expected for $E_M \ll E_C$. By changing the magnetic field right at a parity crossing ($B/B_c = 1.256$, shown in the top center panels (h-j), the behavior is very similar. This comes at no surprise since the spectra are essentially the same as for $B/B_c = 1.2$ (compare the bottom center panels (k-n) with the lower left panels (d-g)). The
FIG. 13. Spectral weights and spectra in the $E_M/E_C \ll 1$ regime for increasing $E_J/E_C$ ratios. The $E_J/E_C$ ratio is tuned by artificially increasing the $E_J$ amplitude, keeping $E_C$ constant. This new regime imposes $E_C > E_M$, similarly to the regimes considered in previous references 28,29,35. Top left panels (a-c): spectral weights of the first transitions as a function of $E_J/E_C$ and fixed $B/B_c = 1.2$ (before a parity crossing) at different $n_g = 0$ (a), $n_g = 0.25$ (b) and $n_g = 0.5$ (c). Lower left panels (d-g): different spectra at this magnetic field $B/B_c = 1.256$ for increasing ratios $E_J/E_C$ from the CPB to the transmon regime ($E_J/E_C = 2, 5, 10, 25$ from (d) to (g), corresponding to the colored bars at the top). Top center panels (h-j) and bottom center panels (k-n): same as (a-c) and (d-g) but for $B/B_c = 1.256$ (right at a parity crossing). Top right panels (o-q) and bottom right panels (r-u): same as before but for $B/B_c = 1.3$ (after a parity crossing). For transmon regimes $E_J/E_C \gg 1$, we recover the almost doubly degenerate transmon spectrum that should be expected for $E_C > E_M$. Besides, only the first transmon transitions $\omega_02$, $\omega_03$ are allowed. The superconducting islands are formed with two NW segments like the one in Fig. 2 (a) and $\tau = 0.8$.

same happens after a parity crossing in the NW spectrum ($B/B_c = 1.3$), shown in the top right panels (o-q) and bottom right panels (r-u)). The overall $E_M/E_C \ll 1$ behaviour therefore shows little dependence with $B$, except for the split transmon lines for $B > B_c$, as discussed in Fig. 12. As we discuss in the next subsections, a larger $E_M/E_C$ ratio completely changes this picture.

B. NW-based superconducting islands with non-negligible $E_M/E_C$ ratios

1. Microwave spectroscopy in the $E_M/E_C \gtrsim 1$ regime

As soon as the ratio $E_M/E_C$ becomes non-negligible, the overall microwave spectral response becomes completely different from, and substantially more complex than that of the preceding subsection, exhibiting a stronger dependences with gate and Zeeman fields. In Fig. 14 we plot the microwave spectra of a superconducting island nominally in an intermediate CPB-transmon regime with $E_J/E_C = 2$, but with a larger $E_M/E_C$ ratio ($E_M/E_C \approx 0.56$, to be compared to the $E_M/E_C \approx 0.17$ case shown in Fig. 12).

In Figs. (a-f), we plot the $B$-dependence of the microwave response for different gates voltages. Fig. 14 (a) shows this magnetic field dependence at $n_g = 0$ (with the corresponding transition peaks widened by their corresponding spectral weights, as represented in Fig. 14 (b)). The overall response is seemingly similar to the one discussed in Fig. 12 including the split lines for $B > B_c$. Note, however, that the splitting in Fig. 12 comes from
interband transitions, as we mentioned, while here the lowest line corresponds to a $\omega_{01}$ transition ($|0, e\rangle \rightarrow |0, o\rangle$, namely, an intraband transition flipping parity). This microwave resonance directly reflects Majorana coupling within the lowest energy doublet. This explains why this lowest line lies near zero frequency and shows a sizable modulation with $B$-field (compare with panel Fig. 12 (a)). The upper line here is a standard qubit transition $\omega_{03}$ ($|0, e\rangle \rightarrow |1, e\rangle$) which conserves parity.

At $n_g = 0.5$, Figs. 14(c-d), the microwave spectrum is similar to the previous case but with all transitions with inverted parities respect $n_g = 0$ (namely, the $\omega_{01}$ transition corresponds now to the process $|0, o\rangle \rightarrow |0, e\rangle$, while the $\omega_{03}$ to the process $|0, o\rangle \rightarrow |1, o\rangle$). This is expected since the ground state at $n_g = 0.5$ is now odd (there is a 1e shift with respect to the previous $n_g = 0$ case, see the spectra in Figs. 14(j-l). Interestingly, the magnetic field dependence of the $\omega_{01}$ transition is the opposite to the one at $n_g = 0$, with exchanged maxima and minima.

At $n_g = 0.25$, Fig. 14(e-f), the response is richer with more transition lines becoming visible. This originates from the strong parity mixing induced by the $E_M$ Majorana coupling at this gate value. Apart from the previous lines, the spectrum now shows another transition originating from an allowed interband transition between states of mixed parity $\omega_{02}$ ($|0, -\rangle \rightarrow |1, -\rangle$). Note that, as opposed to Fig. 12(a), the $\omega_{02}$ transition shows spectral holes precisely at $B$ fields where the $\omega_{01}$ transition has minima. This phenomenon is related to parity crossings in the NW spectrum owing to Majorana oscillations and can be understood by looking at the $n_g$ dependence at different magnetic fields across one of such minima (red, yellow and blue bars). The lower bottom panels of Fig. 14 show this gate dependence (both for the microwave spectra, (g-i), and for the energy spectrum, (j-l)). If we compare the spectra for $B$ fields before and after a parity crossing, panels (j) and (l), the two lowest energy states $E_0$ and $E_1$ are shifted in gate voltage by exactly one electron (a shift $n_g \rightarrow n_g + 0.5$) while flipping parity. We explain in full this phenomenon in the next subsection.

2. Dependence of the spectral weights on the ratio $E_I/E_C$ for different $n_g$

Our previous results for non-negligible $E_M/E_C$ ratios can be fully understood by analyzing in detail the corresponding spectral weights for increasing $E_I/E_C$. Our results are summarized in Fig. 15. They clearly demonstrate that the phenomenology in this $E_M/E_C \gtrsim 1$ regime is completely different from the one shown before in Fig. 13. In the top left panels Fig. 15(a-c), we present the spectral weights as a function of $E_I/E_C$ and fixed $B/B_c = 1.2$ (namely, before the first parity crossing in the NW spectrum of Fig. 2(a)). The different panels show different gates, $n_g = 0$ (a), $n_g = 0.25$ (b) and $n_g = 0.5$ (c). The lower left panels (d-g) show the full $n_g$ dependence of the spectrum at specific values of $E_I/E_C$.

FIG. 14. Microwave spectroscopy of a NW-based superconducting islands in the $E_M \approx E_C$ regime (with $E_I/E_C \approx 2$). The precise ratios used in the plots are $E_M/E_C \approx 0.56$ and $E_M/E_I \approx 0.28$. The superconducting islands are formed with two NW segments like the one in Fig. 2(a) and $\tau = 0.8$. Panel (a): contour plot of microwave absorption spectrum $S_N(\omega)$ versus $\omega$ and $B/B_c$ at $n_g = 0$. (b): Transition frequencies and spectral weights (shadowed widths). Panels (c-d): same as (a-b) but at $n_g = 0.5$. Panels (e-f): same as (a-b) but at $n_g = 0.25$. Panels (g-i): gate dependence of $S_N(\omega)$ at the three magnetic fields (color bars) marked in (a), (c) and (d). Panels (j-l): spectra corresponding to panels (g-i).
FIG. 15. Spectral weights and spectra for non-negligible $E_M/E_C \gtrsim 1$ ratios and increasing $E_J/E_C$ ratios. Top left panels (a-c): spectral weights of the first transitions as a function of $E_J/E_C$ and fixed $B/B_c = 1.2$ (before a parity crossing) at different $n_g = 0$ (a), $n_g = 0.25$ (b) and $n_g = 0.5$ (c). Lower left panels (d-g): different spectra at this magnetic field $B/B_c = 1.256$ (right at a parity crossing). Top right panels (o-q) and bottom right panels (r-u): same as before but for $B/B_c = 1.3$ (after a parity crossing). In the CPB regime, the $\omega_{01}$ transition has some weight but deep in the transmon regime the only allowed transition is a transmon line $\omega_{03}$. Note how the matrix elements are fully exchanged between $n_g = 0$ and $n_g = 0.5$ after a parity crossing (i.e. compare panel (a) with (q) and (c) with (o)). The exact cancellation at $E_J/E_C = 5$ of the $\omega_{01}$ transition (panel (c)) results from parity degeneracy at $n_g = 0.5$ (panel (e)). After a parity crossing, the full spectrum is shifted by one $e$ unit, while flipping parity, and the parity degeneracy occurs now at $n_g = 0$ and $n_g = 1$ (panel (l)). Consequently, the exact cancellation at $E_J/E_C = 5$ of the $\omega_{01}$ occurs now at $n_g = 0$ (panel (h)).

marked by colored bars in the upper panels. The spectral weights of different transitions have now a strong dependence on $n_g$ (as opposed to the results in Figs. 12 and 13). For $n_g = 0$, the dominant transition is the standard qubit transition in the even parity sector ($\omega_{03}$, corresponding to $|0,e\rangle \rightarrow |1,e\rangle$, see e.g Fig. 15 (d)). At $n_g = 0.25$, Fig. 15 (b), and for large $E_J/E_C \gtrsim 5$, the transitions $\omega_{02}$ and $\omega_{04}$, which signal Majorana-mediated parity mixing, are dominant. At $n_g = 0.5$, Fig. 15 (c), we find the same tendency (a large spectral weight for $\omega_{02}$ and $\omega_{04}$ for large $E_J/E_C \gtrsim 5$). Notably, the $\omega_{03}$ transition is now completely absent (compare with the $n_g = 0$ panel in Fig. 15 (a)). Importantly, the transfer of spectral weight between the $\omega_{03}$ and the $\omega_{02}$ transition occurs precisely at $E_J/E_C = 5$, where $\omega_{01}$ has an exact minimum. Since this transition corresponds to an intraband transition within the lowest energy manifold (i.e. the transition $|0,e\rangle \rightarrow |0,o\rangle$ between the lowest energy states with opposite fermionic parity), this minimum should be related to a parity crossing. Indeed, for $E_J/E_C = 5$ there is an exact parity crossing at $n_g = 0.5$, Fig. 15 (e), which occurs as $\delta$ becomes of order $E_C$. Other representative $E_J/E_C$ ratios are shown in Fig. 15 (d-g). Before and after the $n_g = 0.5$ parity crossing at $E_J/E_C = 5$, the ground state changes parity from odd, Fig. 15 (d), to even, Fig. 15 (f), which explains the transfer of spectral weights discussed above. All this phenomenology depends on magnetic field. The top right panels (o-q) show
the same matrix elements as in (a-c) but after the minimum of the Majorana oscillation in the NW spectrum (here at $B/B_c = 1.3$). Remarkably, at this magnetic field all the matrix elements at $n_\delta$ and $n_\delta + 0.5$ gate voltages become interchanged, relative to those at fields before the minimum of the Majorana oscillation. Namely, all the matrix elements that we find for $n_\delta = 0$ correspond now to $n_\delta = 0.5$, and viceversa (compare panel (a) with (q) and (c) with (o)). If we now compare the spectra at this magnetic field, panels (r-u), with the ones corresponding to the magnetic field before the minimum, panels (d-g), we find that there is an exact shift of one electron in the low energy sector recall that a 0.5 shift in $n_\delta$ corresponds to a single electron. This is consistent with our explanation of the results of Fig. 14 and demonstrates that, indeed, the microwave response of NW-based superconducting islands is sensitive to the underlying Majorana physics (including finite-length Majorana oscillations and the resulting fermion parity crossings of the ground state energy). This novel $E_M/E_C \gtrsim 1$ result, with emphasis on the deep transmon regime, is the focus of a companion paper in Ref. 51.

C. Microwave spectroscopy in the regime $E_J \ll E_C, E_M$

In this subsection we explore another novel regime relevant for the experimental data reported in Ref. 50. In these experiments, avoided crossings between even and odd parity sectors at high magnetic fields are estimated to be in the $E_M \approx 10$GHz range. Considering that the charging energies of the superconducting islands are of order $E_C \approx 40$GHz, this gives a ratio $E_M/E_C \approx 0.25$. Interestingly, these islands are in a very strong charging regime with negligible Josephson coupling $E_J$, which defines a completely new operation regime $E_J \ll E_C, E_M$. We study this novel regime in Figs. 16 and 17. In the first case, we concentrate on the microwave response of a long NW (see Fig. 2 (c)) such that the energy splitting $\delta$ owing to Majorana overlap is always $\delta \lesssim E_C$ for all magnetic fields. In this case, the main transition line for the three relevant gates is always the intraband transition $\omega_{01}$ within the ground state manifold, Figs. 16 (a-f). The full gate dependence of $\omega_{01}$ for the three magnetic fields marked in the upper panels is shown in Figs. 16 (g-i). Again, a clear $n_\delta - n_\delta + 0.5$ shift occurs as magnetic field increases. Panels (j-l) show the corresponding spectra. Considering that $\delta \lesssim E_C$ for all gates, this parameter regime is optimal for Majorana detection, since $\omega_{01}$ faithfully maps Majorana oscillations for all $B$. This is no longer the case for shorter wires, where we can find realistic situations with $\delta \gtrsim E_C$. In such cases, Majorana hybridization does not always occur within the ground state manifold, which gives rise to rather involved spectra. We illustrate one of this cases in Fig. 17. Similar to the previous figures, we also plot the magnetic field dependence for the three relevant gates of the problem.

FIG. 16. Microwave spectroscopy of a NW-based superconducting islands in the $E_J \to 0$ regime, for long wires. In contrast to previous cases, here we consider a tunnel junction for the wire, $\tau \ll 1$, so that the Josephson term $E_J$ almost vanish. This in turn makes the Majorana coupling $E_M$ much larger that $E_J$ (see Fig. 10 for $E_M/E_J$ vs. $\tau$ dependence). We use, in particular, $\tau \approx 0.01$, which translates into $E_M/E_J \sim 20$. This results in $E_C$ being the dominant energy scale of the island, as can be seen from ratios $E_M/E_C \approx 0.2$ and $E_J/E_C \approx 0.01$ (this regime is relevant for the experiments reported in Ref. 50). Contour plots of $S_N(\omega, B)$ and transition frequencies are alternatively shown for different gates: $n_\delta = 0$ (panels (a-b)), $n_\delta = 0.5$ (panels (c-d)), and $n_\delta = 0.25$ (panels (e-f)). Transition frequency lines are shadowed according to their spectral weight. Panels (g-i) render gate dependence of $S_N(\omega)$ (g-i) and spectra (j-l) before, at and after a parity crossing at $n_\delta = 0.25$ (marked by coloured bars). The superconducting islands are formed with two NW segments like the one in Fig. 2 (c) with $L_S = 5 \mu$m.
In this regime, the magnetic field dependence is patchy, with large regions in magnetic field where a sharp resonance in the microwave response at a given $n_g$ implies no response at the others. This is clearly seen in Figs. (a) and (b), corresponding to $n_g = 0$ where no low-frequency response is observed until we reach the magnetic field marked with the yellow bar (where $\delta \approx 0$). At lower magnetic fields, $\delta$ is typically larger than $E_C$, which prevents from having Majorana-induced parity mixing in the ground state manifold (hence the absence of low $\omega$ response). After the magnetic field marked with the blue bar, the response is flat when $\omega_{01} \approx 0$. At $n_g = 0.5$, Figs. (c) and (d), we obtain an approximate mirror image of the previous case: here, the only response occurs for fields below the yellow bar, saturating to $\omega_{01} \approx 0$ before the red bar. At $n_g = 0.25$, Figs. (e) and (f), the only finite response occurs within the narrow field window between red and blue bars. This peculiar microwave response can be fully understood by analyzing the $n_g$ dependence of the energy spectra at these three magnetic fields, see Figs. (j-l) (the corresponding microwave responses are plotted in (g-i)). The magnetic field at the red bar corresponds to a situation with $\delta > E_C$. In this case, the ground state has well-defined even parity and the only possible transition is a standard interband qubit transition (at $\omega \approx 4E_C$ at $n_g = 0$ and $\omega \approx 0$, owing to $E_J \rightarrow 0$, at $n_g = 0.5$). At $n_g = 0.25$, the only parity mixing occurs at higher bands, but not within the ground state manifold. This residual mixing is weakly visible as a small splitting of the main qubit transition, see Fig. (g). The central panels, Figs. (h) and (k), correspond to a $\delta \approx 0$ situation (yellow bar). This is the only magnetic field region where Majorana-mediated mixing within the ground state manifold is possible for all $n_g$. Larger magnetic fields where $-\delta > E_C$ (blue bar) induce a change of ground state parity, which is now odd for all $n_g$, Fig. (l). The only allowed transitions occur now near $n_g = 0$ and $n_g = 1$ and correspond to a $\omega_{01} \approx 0$ within the odd parity sector. Again, weak Majorana mixing occurs for higher bands and is seen as faint splittings near $n_g = 0$ and $n_g = 1$, Fig. (i). This gate dependence explains the peculiar microwave response as a function of increasing magnetic fields. We finish by noting that this seemingly extreme regime with a $2\pi$-periodic odd-parity ground state has been reported in the experiments discussed in Ref. 66.

VI. FINAL REMARKS AND CONCLUSIONS

We have presented a detailed analysis of the microwave response of superconducting islands where the weak link in the Josephson element is formed by a proximitized semiconducting NW. Specifically, we describe the Josephson junction as two segments of a single-mode semiconductor NW that are proximitized by a conventional s-wave superconductor (the so-called Lutchyn-Oreg model) separated by a short normal region. The BdG spectrum of such weak link creates the Josephson potential $V_J(\phi)$ that enters the superconducting island Hamiltonian substituting the standard $V_J(\phi) = -E_J \cos(\phi)$ in conventional superconducting islands. Our description allows to uncover all the relevant regimes (from the trivial to the topological one) as the external Zeeman field increases. It takes into account both standard Josephson events due to Cooper pair tunneling, as well as anomalous Majorana-mediated events where a single electron is transferred across the junction. This anomalous single-electron Josephson tunneling is governed by the subgap excitations of the BdG Hamiltonian, whose dynamics are fully taken into account by means of a projection technique. Quite naturally, the superconducting island properties depend on important microscopic NW parameters,
notably the energy splitting between different fermionic parities on each NW segment, \( \delta \) (the so-called Majorana splitting due to finite length), and the single-electron contribution to the Josephson coupling, \( E_M \). These new scales in the problem, together with \( E_J \) and \( E_C \), define novel regimes such as e.g. \( E_M/E_C \gtrsim 1 \) and/or \( \delta/E_C \gtrsim 1 \), hitherto unexplored in the literature and relevant for current experiments using NW Josephson junctions.

Our results demonstrate that the microwave response is a very useful tool to study Majorana physics in such junctions. Being a global measurement, it allows to avoid some of the issues that challenge the interpretation of zero-bias anomalies in tunneling spectroscopy. As we discuss, the typical experimental knobs in standard transport experiments for Majorana detection (i.e. the external Zeeman field), can be supplemented with other knobs that characterize the island (\( n_q, E_C, E_J \)) in order to unveil Majorana physics in the junction. This, in particular, allows to fully characterize Majorana oscillations and their concomitant fermion parity crossings.

The discussion presented in this paper is based on the simplest model that can describe all the relevant regimes. The analysis performed here may be readily extended to other relevant NW regimes not discussed here, like multiband NWs, the role of the electrostatic environment and orbital effects. Other geometries currently under intensive experimental study, including junctions with quantum dots, superconducting islands in the fluxonium regime, and gate-monolayers based on full-shell NWs can be also studied using our method. While the focus of the paper is on semiconductor-nanowire junctions, our procedure is general and can be applied to other weak links and gate-tunable Josephson junctions where the subgap BdG spectrum is a crucial contribution to the Josephson potential. Novel systems where our method could be extremely useful include two-dimensional semiconductor gases proximitized by superconductors and van der Waals heterostructures.

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