DIVERGENCE FUNCTION OF THE BRAIDED THOMPSON GROUP

YUYA KODAMA

Abstract. We prove that the braided Thompson group $BV$ has a linear divergence function. By the work of Druţu, Mozes, and Sapir, this implies none of asymptotic cones of $BV$ has a cut-point.

1. Introduction

R. Thompson groups $F$, $T$, and $V$ are defined by Richard Thompson in 1965. These groups have many interesting properties. For instance, $F$ is the first example of a torsion-free group of type $F_\infty$ but not of type $F$, by Brown and Geoghegan [6, Theorems 5.3 and 7.2]. $T$ and $V$ are also group of type $F_\infty$, by Brown [5, Theorem 4.17], and known as the first examples of infinite simple group with finite presentation, by Thompson. These groups have been studied using not only algebra but also analysis and geometry.

On the other hand, various “Thompson-like” groups have been considered to study the relationship with Thompson groups and their own interesting properties. In this paper, we focus on the generalization of $V$, braided Thompson group $BV$ (sometimes this group is written as $V_{br}$). This group is defined independently by Brin [4] and Dehornoy [11]. It is known that $BV$ has similar properties to those of $V$. For instance, Brin [3, Theorem 5.1] showed $BV$ is finitely presented, where the generators and relations are similar to those of $V$, and Bux, Fluch, Marschler, Witzel, and Zaremsky [9, Main Theorem] proved that this group is also of type $F_\infty$. Zaremsky [18] suggests the relationship between $BV$ and metric spaces being CAT(0) or hyperbolic.

Golan and Sapir [16, Theorem 1.1] showed that Thompson groups $F$, $T$, and $V$ have linear divergence functions. Roughly speaking, the divergence function of a finitely generated group $G$ is a function given by the length of the path connecting two points at the same distance from the origin while avoiding a small ball with the center at the origin in the Cayley graph. Gersten [15, section 2] introduced divergence of connected geodesic metric spaces as collections of such functions. We focus on each function rather than a collection, since it corresponds to the topological characterization of the asymptotic cones of the group [12, Lemma 3.17]. Since braid groups have linear divergence functions (Proposition 3.3), it is natural to expect that so does braided Thompson group $BV$. In fact,
Golan and Sapir posed a question whether their proof can be extended to Thompson-like groups. In this paper, we give a partial answer to this question.

**Theorem 1.1.** Braided Thompson group $BV$ has a linear divergence function.

This paper is organized as follows. In Section 2, we summarize definitions of Thompson groups, braid groups, braided Thompson group, and in Section 3, we define the divergence functions of finitely generated groups. In Section 4, first we prepare some lemmas on the number of carets of elements in $BV$. Then we construct a path which satisfies the requirement for the definition of the divergence function. This path is connecting two points $g$ in $BV$ and the point $v(|g|)$ in $F < BV$ which only depend the word length of $g$. This is achieved in the following way: For $g$, we construct the element $h$ (denoted by $w_1 w_2 w_3$ in Section 4) in $BV$ such that $gh$ and $v(|g|)$ commute. Then, we move $g \to gh \to ghv(|g|) = v(|g|)gh \to v(|g|)$. We remark that the above paths do not work for elements having less than three carets. For those elements $g$, we consider $gx_1$, a multiplication by a generator $x_1$, instead of $g$ itself.

It is also interesting to study divergence functions of other Thompson-like groups, similar to $BV$. For example, Brady, Burillo, Cleary, and Stein [2] defined $BF$ (sometimes denoted by $F_{br}$), which is braided version of Thompson group $F$. Acora and Cumplido [1] defined a family of infinitely braided Thompson’s groups, which contains $BV$ as a special case. Another example is the Higman-Thompson groups, for instance.

**Acknowledgements.** I appreciate the referee for his or her close reading and precious comments. I would like to thank my supervisor, Professor Tomohiro Fukaya for his guidance.

## 2. Background

### 2.1. Finitely generated groups and binary words.** A group $G$ is said to be **finitely generated** if there exists a subset $X$ such that every element of $G$ can be written as a product of finitely many elements in $X \cup X^{-1}$, where $X^{-1} := \{x^{-1} \mid x \in X\}$. We call such a product a **word** in $X$. We use “$\equiv$” and “$=$” to express equalities as words in $X$ and as elements of $G$, respectively. Let $x \equiv x_1 x_2 \cdots x_n$ be a word in $X$. A word $x'$ is said to be a **prefix** of $x$, denoted by $x' \leq x$, if $x' \equiv \emptyset$ or $x' \equiv x_1 \cdots x_k$ for some $1 \leq k \leq n$, where $\emptyset$ denotes the empty word. A word $x'$ is said to be a **strict prefix** of $x$ if $x'$ is a prefix of $x$ and $x' \neq x$.

Let $w$ be a finite character string that consists of 0 and 1. We call such a character string a **binary word** and we also use “$\equiv$” to express equality. By the similar way above, we define a **prefix** and **strict prefix** of $w$. For every two binary words $w_1 \equiv a_1 a_2 \cdots a_j$ and $w_2 \equiv b_1 b_2 \cdots b_k$ where $a_i$ and $b_i \in \{0, 1\}$ for every $i$, $w_1 w_2$ denotes the concatenation $a_1 a_2 \cdots a_j b_1 b_2 \cdots b_k$. 
2.2. Thompson groups. A rooted binary tree is a tree with a distinguished vertex (root) that has 2 edges, and vertices with either degree 1 (leaves) or degree 3. We think of a rooted binary tree as a descending tree with the root as the only top vertex (level 0) and vertices of different levels. We define a caret of a rooted binary tree to be a subtree of the tree that consists a vertex together with two downward-directed edges. We write all-right tree $T_n$ for the rooted binary tree that is constructed by attaching a caret to the right edge of a caret $n$ times. Thus $T_1$ is a caret. See Figure 1. The number of carets play important role to estimate the word lengths of elements of Thompson groups.

Let $(T_+, \sigma, T_-)$ be a triplet where $T_+$ and $T_-$ be finite rooted binary trees with $n$ caret, $L$ be the set of $(n + 1)$ leaves and $\sigma$ be a permutation of $L$. We order the leaves of $T_+$ and $T_-$ from left to right from 0 to $n$, respectively and use the numbers to represent the permutation $\sigma$. We call this tree diagram. For example, see Figure 2.

Let $(T_+, \sigma, T_-)$ be the above tree diagram. We define a reduction of carets of a tree diagram as follows. We assume that two leaves $i, i + 1$ have the same parent in $T_+$, two leaves $\sigma(i), \sigma(i + 1)$ have the same parent in $T_-$, and $\sigma(i + 1) = \sigma(i) + 1$ holds. In that case, each pair of the leaves forms carets. Then, we get the trees $T'_+$ and $T'_-$ by removing those carets. We regard the roots of the above carets as new leaves of the new trees, and we write $i_+$ and $i_-$ for the new leaves of $T'_+$ and $T'_-$, respectively. By sending $i_+$ to $i_-$ and sending other leaves by $\sigma$, we also get the permutation $\sigma'$ on the set of $n$ leaves. This
operation and its inverse are called reduction and attachment of carets, respectively. For example, see Figure 3 and 4.

Using these operations, we define the equivalence relation on the set of tree diagrams as follows. Two tree diagrams $(T_+, \sigma, T_-)$ and $(T'_+, \sigma', T'_-)$ are equivalent if $(T_+, \sigma, T_-)$ is obtained from $(T'_+, \sigma', T'_-)$ by a finite number of reductions and attachments. The Thompson group $V$ consists of all equivalence classes of tree diagrams. The product on $V$ is defined in the following way.

For every two elements $a, b \in V$ represented by tree diagrams $(A_+, \alpha, A_-)$ and $(B_+, \beta, B_-)$, by successive attachments of carets, we get diagrams $(A'_+, \alpha', A'_-)$ and $(B'_+, \beta', B'_-)$ representing the same elements and such that $A'_- = B'_+$. Then the product $ab \in V$ is the equivalence class of $(A'_+, \alpha' \beta', B'_-)$, where the permutation $\alpha' \beta'$ is composed from left to right. For example, see Figure 5 and 6.

The group $T$ is a subgroup of $V$ consists of equivalence classes of tree diagrams $(T_+, \sigma, T_-)$ where $\sigma$ is a cyclic permutation, and the group $F$ is a subgroup of $T$ consists of equivalence classes of tree diagrams $(T_+, \sigma, T_-)$ where $\sigma$ is the identity.

For every caret of a rooted binary tree, we label its left edge by 0 and the right edge by 1. Since every leaf $o$ of such a tree $T$ corresponds to a unique path $s_T(o)$ from the root to the leaf.
the leaf, every leaf \( o \) corresponds to a binary word \( \ell_T(o) \) labeling the path from the root to \( o \). We identify the path \( s_T(o) \) with the binary word \( \ell_T(o) \).

By identifying the Cantor set \( C \) with the set of infinite binary words, we can associate each tree diagram \( (T_+, \sigma, T_-) \) to a homeomorphism from \( C \) to itself. Indeed, for every leaf \( o \) of \( T_+ \) and infinite binary word \( w \), by mapping \( \ell_{T_+}(o)w \) to \( \ell_{T_-}(\sigma(o))w \), we get a homeomorphism. By the definition of this homomorphism \( V \to \text{Homeo}(C) \), the homeomorphism coming from \( (T_+, \sigma, T_-) \) is the identity if and only if \( \sigma \) is the identity and \( T_+ = T_- \), so the homomorphism is injective. Hence, \( V \) is a subgroup of the homeomorphism group of \( C \).

See [10] for details of the properties of Thompson groups.

2.3. Braid groups. Let \( n \in \mathbb{N} \). We briefly review the definition of geometric braid groups \( B_n \). See [17, Section 1.2] for details. Let \( I \) be the closed interval \( [0, 1] \subset \mathbb{R} \). We call a topological space which is homeomorphic to \( I \) a topological interval.

**Definition 2.1 ([17, Definition 1.4]).** A geometric braid on \( n \) strings is a set \( b \subset \mathbb{R}^2 \times I \) formed by \( n \) disjoint topological intervals called the strings of \( b \) such that the projection \( \mathbb{R}^2 \times I \to I \) maps each string homeomorphically onto \( I \) and

\[
\begin{align*}
b \cap (\mathbb{R}^2 \times \{0\}) &= \{(0, 0, 0), (1, 0, 0), \ldots, (n-1, 0, 0)\}, \\
b \cap (\mathbb{R}^2 \times \{1\}) &= \{(0, 0, 1), (1, 0, 1), \ldots, (n-1, 0, 1)\}.
\end{align*}
\]

We assume that every string goes from the bottom to up.

By the definition, every string of \( b \) meets each plane \( \mathbb{R}^2 \times \{t\} \) with \( t \in I \) in exactly one point and connects a point \( (i, 0, 0) \) to a point \( (\sigma(i), 0, 1) \), where \( \sigma \) is a permutation of \( \{0, 1, \ldots, n-1\} \). We call the both points endpoints of the string, and call \( \sigma \) the underlying permutation of the braid.

**Definition 2.2 ([17]).** Two geometric braids \( b \) and \( b' \) on \( n \) strings are isotopic if there exists a continuous map \( F: b \times I \to \mathbb{R}^2 \times I \) such that for each \( s \in I \), the map \( F_s: b \to \mathbb{R}^2 \times I; x \mapsto F(x, s) \) is an embedding whose image is a geometric braid on \( n \) strings, \( F_0 = \text{Id}: b \to b \), and \( F_1(b) = b' \). Both the map \( F \) and the family of geometric braids \( \{F_s(b)\}_{s \in I} \) are called an isotopy of \( b \) to \( b' \).

The relation of isotopy is an equivalence relation on the class of geometric braids on \( n \) strings. We call the equivalence classes and each string of an equivalence class braid (on \( n \) strands) and strand, respectively. We write \( B_n \) for the set of braids on \( n \) strands.

For every two geometric braids \( b_1 \) and \( b_2 \), we define their product \( b_1b_2 \) to be the set of points \( (x, y, t) \in \mathbb{R}^2 \times I \) such that

\[
(x, y, 2t) \in b_1 \text{ if } 0 \leq t \leq \frac{1}{2},
\]

and

\[
(x, y, 2t - 1) \in b_2 \text{ if } \frac{1}{2} \leq t \leq 1.
\]
It is clear that if $b_1$ and $b_2$ are isotopic to geometric braids $b'_1$ and $b'_2$, respectively, then the product $b_1b_2$ is isotopic to the product $b'_1b'_2$. Hence the product of $B_n$ is defined by the equivalence class of products of geometric braids.

A braid can be projected onto $\mathbb{R} \times \{0\} \times I$ along the second coordinate with “crossing information” at each crossing point. Indeed, if necessary, by appropriate isotopies, we can assume that the number of strands involved in any intersection is two, every two strands meet transversely at each intersection point of the two strands, and there are only a finite number of such intersections. We call the intersections crossing points. For each crossing point, the one with the lesser $y$-coordinate is denoted by over crossing, and the other is denoted by the corresponding under crossing. Then, we draw each over crossing by a continuous line, and each under crossing by a broken line. For example, Figure 7 are the projection of the elements in $B_4$ and $B_5$. In this paper, we identify braids with projected braids equipped with crossing information.

We introduce an operation for braids which we use to define the product of elements of braided Thompson group.

**Definition 2.3 (splitting).** Let $0 \leq i \leq n - 1$. Let $B$ be a braid on $n$ strands, \( \{b_k \mid k = 0, 1, \ldots, n - 1\} \) be the set of strands of $B$, $\sigma$ be the underlying permutation of $B$, and \((k, 0, 0)\) and \((\sigma(k), 0, 1)\) be endpoints of each strand $b_k$. We define a braid on $(n + 1)$ strands $B'$ to be the following: $B'$ is obtained by adding strand $b'_i$ to $B$ such that it satisfies the following:

1. Endpoints of $b'_i$ are $(i + 1/2, 0, 0)$ and $(\sigma(i) + 1/2, 0, 0)$, then shift all endpoints appropriately so that they have integer $x$-values.
2. The strand $b'_i$ does not cross with $b_i$.
3. The strand $b'_i$ intersects with strands other than $b_i$ in the same way that $b_i$ intersects with strands in braid $B$.

In other words, $B'$ is a braid such that $b'_i$ is to the right of $b_i$ and the braid obtained from $B'$ by removing $b_i$ is equal to $B$.

We say that $b_i$ and $b'_i$ are parallel, and we call $B'$ the splitting of the strand $b_i$.

For example, see Figure 7.

2.4. **Braided Thompson group.** Elements of Thompson groups $V$ can be seen as pairs of finite rooted binary trees, with permutations from leaves to itself. Roughly speaking, by replacing permutations with braids, we get elements of $BV$.

Let $T_+$ and $T_-$ be finite rooted binary trees with $n$ carets and $br$ be a braid on $n + 1$ strands from bottom to up. \((T_+, br, T_-)\) denotes a diagram where the leaves of both trees are joined by the braid with $T_+$ positioned upside down. We call this tree-braid-tree diagram. For example, see Figure 8.

Let \((T_+, br, T_-)\) be the above tree-braid-tree diagram. Similar to Thompson groups, we define a reduction of carets of a tree-braid-tree diagram as follows. We assume that two
strands $b_i$ and $b'_i$ are parallel (cf. Definition 2.3) and each endpoints of $b_i$ and $b'_i$ have the same parent in $T_+$ and $T_-$. In that case, each pair of the endpoints (leaves) forms carets. Then, we get the trees $T'_+$ and $T'_-$ by removing those carets. We regard the roots of the above carets as new leaves of the new trees, and we write $i_+$ and $i_-$ for the new leaves of $T'_+$ and $T'_-$, respectively. By removing the strand $b'_i$, letting the endpoints of $b_i$ be the new leaves, and keeping the other strands, we also get the braid $br'$ on $n$ strands from $br$. This operation and its inverse operation are called reduction of carets and splitting of a strand, respectively. For example, see Figure 9.

Using these operations, we define the equivalence relation on the set of tree-braid-tree diagrams as follows. Two tree-braid-tree diagrams $(T_+, br, T_-)$ and $(T'_+, br', T'_-)$ are equivalent if $(T_+, br, T_-)$ is obtained from $(T'_+, br', T'_-)$ by finite number of reductions and
Figure 9. An example of reduction

splittings. Each equivalence class has a unique representative with minimal number of carets. We call this diagram a reduced tree-braid-tree diagram.

The braided Thompson group $BV$ consists of all equivalence classes of tree-braid-tree diagrams. The product on $BV$ is defined in the following way.

For every two elements $a, b \in BV$ represented by tree-braid-tree diagrams $(A_+, br_A, A_-)$ and $(B_+, br_B, B_-)$, by successive splittings of strands, we get diagrams $(A'_+, br'_A, A'_-)$ and $(B'_+, br'_B, B'_-)$ representing the same elements and such that $A'_- = B'_+$. Hence $br'_A$ and $br'_B$ are braids from the same braid group. Then the product $ab \in BV$ is the equivalence class of $(A'_+, br'_Abr_B, A'_-)$, where $br'_A br'_B$ is the braid that $br'_A$ and $br'_B$ connected from the bottom to the top, in this order. Figure 10 shows an example of a multiplication of elements of $BV$.

It is known that $BV$ has the following infinite presentation.

**Theorem 2.4** ([2 Theorem 2.4]). The group $BV$ admits a presentation with generators:

- $x_i$, for $i \geq 0$,
- $\sigma_i$, for $i \geq 1$,
- $\tau_i$, for $i \geq 1$.

and relators

A $x_j x_i = x_i x_{j+1}$, for $j > i$
B1 $\sigma_i \sigma_j = \sigma_j \sigma_i$, for $j - i \geq 2$
B2 $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
B3 $\sigma_i \tau_j = \tau_j \sigma_i$ for $j - i \geq 2$
B4 $\sigma_i \tau_{i+1} \sigma_i = \tau_{i+1} \sigma_i \tau_{i+1}$
C1 $\sigma_i x_j = x_j \sigma_i$, for $i < j$
C2 $\sigma_i x_i = x_{i-1} \sigma_i \sigma_{i+1}$
C3 $\sigma_i x_j = x_j \sigma_{i+1}$, for $i \geq j + 2$
Figure 10. An example of the product $ab$ in $BV$

Figure 11. The infinite generators $x_i$

\[\begin{align*}
C4 & \quad \sigma_{i+1}x_i = x_{i+1}\sigma_{i+1}\sigma_{i+2} \\
D1 & \quad \tau_i x_j = x_j\tau_{i+1}, \text{ for } i - j \geq 2 \\
D2 & \quad \tau_i x_{i-1} = \sigma_i\tau_{i+1} \\
D3 & \quad \tau_i = x_{i-1}\tau_{i+1}\sigma_i.
\end{align*}\]

The reduced diagrams of generators $x_i$ are in Figure 11. The reduced diagrams of generators $\sigma_i$ and $\tau_i$ are in Figure 12. We note that a set of the generators $x_i$ corresponds to the standard infinite generating set of Thompson group $F$. Indeed, each $x_i$ is regarded as two rooted binary trees and identical permutation (see the upper low of Figure 2). Hence, $BV$ contains $F$ as a subgroup. Incidentally, in some papers, Thompson groups are defined by “tree-permutation-tree diagrams” similar to tree-braid-tree diagrams.

Moreover, it is also known that $BV$ has the following finite presentation.
Theorem 2.5 ([2, Theorem 3.1]). The group BV admits a finite presentation with generators $x_0, x_1, \sigma_1, \tau_1$ and relators

\begin{itemize}
  \item[a] $x_2x_0 = x_0x_3, \ x_3x_1 = x_1x_4$
  \item[c1] $\sigma_1x_2 = x_2\sigma_1, \ \sigma_1x_3 = x_3\sigma_1, \ \sigma_2x_3 = x_3\sigma_2, \ \sigma_2x_4 = x_4\sigma_2$
  \item[c3] $\sigma_2x_0 = x_0\sigma_3, \ \sigma_3x_1 = x_1\sigma_4$
  \item[c4] $\sigma_1x_0 = x_1\sigma_1\sigma_2, \ \sigma_2x_1 = x_2\sigma_2\sigma_3$
  \item[d1] $\tau_2x_0 = x_0\tau_3, \ \tau_3x_1 = x_1\tau_4$
  \item[d2] $\tau_1x_0 = \sigma_1\tau_2, \ \tau_2x_1 = \sigma_2\tau_3$
  \item[b1] $\sigma_1\sigma_3 = \sigma_3\sigma_1$
  \item[b2] $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$
  \item[b3] $\sigma_1\tau_3 = \tau_3\sigma_1$
  \item[b4] $\sigma_1\tau_2\sigma_1 = \tau_2\sigma_1\tau_2$
\end{itemize}

where the letters in the relators not in the set of 4 generators are defined inductively by $x_{i+2} = x_i^{-1}x_{i+1}x_i$ for $i \geq 0$, $\sigma_{i+1} = x_{i-1}^{-1}\sigma_i x_i\sigma_i^{-1}$ for $i \geq 1$, and $\tau_{i+1} = x_{i-1}^{-1}\tau_i x_i^{-1}$ for $i \geq 1$. 

Figure 12. The infinite generators $\sigma_i$ and $\tau_i$
We call \( \{x_0, x_1, \sigma_1, \tau_1\} \) standard generating set of \( BV \).

As well as Thompson groups, for every caret of a rooted binary tree, we label its left edge by 0 and the right edge by 1. Since every leaf \( o \) of such a tree \( T \) corresponds to a unique path \( s_T(o) \) from the root to the leaf, every leaf \( o \) corresponds to a binary word \( \ell_T(o) \) labeling the path from the root to \( o \). We identify the path \( s_T(o) \) with the word \( \ell_T(o) \). The path \( \ell_T(o) \) will be called a branch of \( T \). Let \( (T_+, br, T_-) \) be a tree-braid-tree diagram of \( g \in BV \), \( o \) be a leaf of \( T_+ \), and \( o' \) be the corresponding leaf of \( T_- \). We say that \( \ell_{T_+}(o) \to \ell_{T_-}(o') \) is a branch of the tree-braid-tree diagram \( (T_+, br, T_-) \).

Let \( T \) be a rooted binary tree with \( n \) carets. Recall that \( T_n \) denotes an all-right tree. Then \( (T, \text{Id}, T_n) \in F \) is termed positive element. Because there exist \( 0 \leq i_1 < i_2 < \cdots < i_k \) and \( r_1, r_2, \ldots, r_k > 0 \) such that

\[
x_{i_1}^{r_1} x_{i_2}^{r_2} \cdots x_{i_k}^{r_k} = (T, \text{Id}, T_n)
\]

holds, where each \( x_{i_j} \) is given by a diagram in Figure 11 (See [10, Theorem 2.5]). Similarly, \( (T_n, \text{Id}, T) \in F \) is termed negative element. Since every element in \( F \) is rewritten as a product of positive element and negative element, we call the product seminormal form. For non-trivial element, let

\[
x_{i_1}^{r_1} x_{i_2}^{r_2} \cdots x_{i_k}^{r_k} x_{j_1}^{-s_1} \cdots x_{j_2}^{-s_2} x_{j_1}^{-s_1}
\]

be a seminormal form, where \( 0 \leq i_1 < i_2 < \cdots < i_k \neq j_1 > \cdots j_2 > j_1 \geq 0 \) and \( r_1, r_2, \ldots, r_k, s_1, \ldots, s_t > 0 \). This form is unique if we require the following condition: if \( x_i \) and \( x_i^{-1} \) exist in this form, then \( x_{i+1} \) or \( x_{i+1}^{-1} \) also exists. We call the unique form normal form. By using the relation A, we can always get the normal form from the above seminormal form. Furthermore, every such normal form represents non-identity element of \( F \). See [10, Corollary-Definition 2.7]. We call the part with positive exponents the one with negative exponents in the normal form positive part and negative part, respectively.

3. Divergence functions of finitely generated groups

Let \( G \) be a finitely generated group, \( X \) be a finite generating set of \( G \), and \( \Gamma \) be the Cayley graph \( \text{Cay}(G, X) \). We will define the divergence functions of \( G \). Since we consider asymptotic behavior of functions, we introduce a relation on the set of functions \( \mathbb{R}_+ \to \mathbb{R}_+ \) as follows. For such \( f \) and \( g \), we define \( f \preceq g \) if

\[
f(x) \leq Ag(Bx + C) + Dx + E
\]

for some \( A, B, C, D, E \geq 0 \) and all \( x \). This defines an equivalence relation on the set of functions \( \mathbb{R}_+ \to \mathbb{R}_+ \), by saying \( f \approx g \) if \( f \preceq g \) and \( g \preceq f \). We note that all linear functions and constant functions are equivalent.

Let \( \delta \in (0, 1) \). Then the \( \delta \)-divergence function of \( \Gamma \) is the smallest function \( f_\delta(x) \) such that every two vertices of \( \Gamma \) at distance \( x \) from the identity can be connected by a path in
Γ of length less than \( f_\delta(x) \) and avoiding the ball of radius \( \delta x \) with a center at the identity. If no such path exists, take \( f_\delta(x) = \infty \). For each \( \delta \in (0,1) \), the equivalence class of \( f_\delta(x) \) is invariant under quasi-isometries, especially, it does not depend on the choice of finite generating set \( X \). Hence the \( \delta \)-divergence function of \( G \) is defined to be the equivalence class of the \( \delta \)-divergence function of \( \Gamma \).

**Definition 3.1.** We say that the group \( G \) has a *linear divergence function* if there exists \( \delta \in (0,1) \) such that the \( \delta \)-divergence function of \( G \) is equivalent to a linear function.

By definition, if \( f_\delta \) is equivalent to a linear function, then for every \( 0 < \delta' < \delta \), \( f_{\delta'} \) is equivalent to a linear function. Indeed, since a path that avoids the ball of radius \( \delta x \) also avoids the ball of radius \( \delta' x \), \( f_\delta(x) \geq f_{\delta'}(x) \) holds for every \( x \).

Druțu, Mozes and Sapir showed that having a linear divergence function is equivalent to the following topological property of asymptotic cones.

**Theorem 3.2 ([12, correct version of Lemma 3.17; 13]).** The following are equivalent.

1. \( G \) has a linear divergence function.
2. For every \( \delta \in (0, \frac{1}{54}) \), \( f_\delta \) is equivalent to a linear function.
3. None of asymptotic cones of \( G \) has a cut-point.

We believe that the following result is well known. However, for reader’s convenience, we give a sketch of a proof.

**Proposition 3.3.** For all \( n \geq 3 \), the braid group \( B_n \) has a linear divergence function.

**Sketch of proof.** First, we note that the center of \( B_n \) is isomorphic to \( \mathbb{Z} \) ([17, Theorem 1.24]). Secondly, we also note that \( B_n \) is not virtually cyclic, since \( B_n \) has a subgroup which is isomorphic to \( \mathbb{Z}^2 \). Indeed, \( B_3 \) is a subgroup of \( B_n \), and \( B_3 \) has a subgroup isomorphic to \( \mathbb{Z}^2 \) which is generated by commutative elements \( p \) and \( q \) (Figure 13, 14). By combining the above two notes, we have that none of asymptotic cones of \( B_n \) has a cut-point ([14, Theorem 6.5]). By Theorem 3.2 this is equivalent to the property that \( B_n \) has a linear divergence function. \( \square \)
Remark 3.4. The above argument can work for pure braid groups as well.

4. Proof of Theorem 1.1

4.1. Number of carets for elements of $BV$. Let $X = \{x_0, x_1, \sigma_1, \tau_1\}$ be the standard generating set of $BV$. For an element $g \in BV$, $|g|$ denotes the word length of $g$ with respect to the generating set $X$, and $N(g)$ denotes the number of carets in one of the trees in the reduced tree-braid-tree diagram of $g$. We will use the following estimate.

Theorem 4.1 ([7, Theorem 3.6]). For an element $g$ of $BV$ in tree-braid-tree diagram with $k$ total crossings, there exists a constant $C_1$ for which the word length satisfies the following inequalities:

$$C_1 \max\{N(g), \sqrt[3]{k}\} \leq |g|.$$

Here we can assume that $0 < C_1 < 1$ without loss of generality.

Let $g \in BV$ with a reduced tree-braid-tree diagram $(T_+(g), br(g), T_-(g))$. We call $T_+(g)$ the domain-tree of $g$, $T_-(g)$ the range-tree of $g$ and $br(g)$ the braid of $g$. Let $T$ be a rooted binary tree. Then, $\ell_0(T)$ denotes the length of the left most branch of $T$, that is, $\ell_0(T) = \ell$ if and only if $0^\ell$ is a branch of $T$, where we define

$$i^\ell \equiv i \cdots i.$$

for $i = 0, 1$. Similarly, $\ell_1(T)$ denotes the length of the right most branch of $T$. For an element $g \in BV$, we define $\ell_i(g) := \ell_i(T_-(g))$, $i = 0, 1$.

We will need the following lemmas. Although the proofs of them are almost the same as in [16], we write down the proofs for reader’s convenience. Note that the definition of $N(g)$ in this paper is different from that of $N'(g)$ in [16]. The former denotes the number of carets and the latter the number of leaves, respectively.

The following lemma corresponds to [16, Lemma 2.2].

Lemma 4.2. Let $g$ be an element in $BV$ with reduced tree-braid-tree diagram $(T_+(g), br(g), T_-(g))$ and assume that $N(g) \geq 3$. Then

$$N(g) - 1 \leq N(gx_0) \leq N(g) + 1.$$

In addition,

1. If $\ell_0(g) = 1$ then $N(gx_0) = N(g) + 1$ and $\ell_0(gx_0) = 1$.
2. If $\ell_0(g) \neq 1$ then $N(gx_0) = N(g)$ or $N(gx_0) = N(g) - 1$. Moreover, $\ell_0(gx_0) = \ell_0(g) - 1$.
3. If $\ell_0(g) \neq 1$ and either 1 or 01 is a strict prefix of some branch of $T_-(g)$, then $N(gx_0) = N(g)$ and 1 is a strict prefix of some branch of $T_-(gx_0)$, where $T_-(gx_0)$ is a range-tree of reduced tree-braid-tree diagram of $gx_0$. 
Proof. We start by proving part (2). Assume that \( \ell_0(g) \neq 1 \). To multiply \( g \) by \( x_0 \), we replace the reduced tree-braid-tree diagram of \( x_0 \) by an equivalent tree-braid-tree diagram \((R_+, \text{Id}, R_-)\) where \( R_+ = T_- (g) \). Then \((T_+ (g), \text{br}(g), R_-)\) is a tree-braid-tree diagram of the product \( gx_0 \). Let \( u \to wv \) be a branch of \((T_+ (g), \text{br}(g), T_- (g))\) where \( w \equiv 00 \), \( w \equiv 01 \), or \( w \equiv 1 \). By the construction of \((R_+, \text{Id}, R_-)\), \( u \to wv \) is a branch of \((T_+ (g), \text{br}(g), R_-)\) where \( w' \equiv 0 \), \( w' \equiv 10 \), or \( w' \equiv 11 \), respectively. All branches of \((T_+ (g), \text{br}(g), R_-)\) can be written in this way.

If \((T_+ (g), \text{br}(g), R_-)\) is reduced diagram, then all assertions of part (2) hold, by the relation of \( w \equiv 00 \) and \( w' \equiv 0 \). Indeed, it follows from the domain-tree that \( N(gx_0) = N(g) \). Moreover, since the reduced diagram of \( x_0 \) has a branch \( 00 \to 0 \), \( \ell_0(R_-) = \ell_0(R_+)-1 = \ell_0(T_- (g)) - 1 = \ell_0(g) - 1 \). Hence, we can assume that \((T_+ (g), \text{br}(g), R_-)\) is not reduced, that is, this diagram has a pair of branches \( x0 \to y0 \) and \( x1 \to y1 \) such that corresponding strands are parallel. Then \( y \equiv 1 \). Indeed, if \( y \) is an empty word, \( R_- \) has a branch 1. This contradicts the fact that \( \ell_1(x_0) \neq 1 \). If \( 0 \) is a prefix of \( y \), then the diagram \((T_+ (g), \text{br}(g), T_- (g))\) of \( g \) has the pair of branches \( x0 \to 0y0 \) and \( x1 \to 0y1 \) such that corresponding strands are parallel, in contradiction to \((T_+ (g), \text{br}(g), T_- (g))\) being reduced. If \( 10 \) or \( 11 \) is a prefix of \( y \), the assumption that \((T_+ (g), \text{br}(g), T_- (g))\) is reduced yields a contradiction in a similar way. Hence, \( y \equiv 1 \). So \( R_- \) has branches of form 10 and 11. Now, we reduce the carets corresponding to \( x0 \to 10 \) and \( x1 \to 11 \) of the diagram \((T_+ (g), \text{br}(g), R_-)\). Then the obtained diagram \((T'_+ (g), \text{br}'(g), R'_-)\) of \( gx_0 \) has a branch \( x \to 1 \) and this diagram is reduced. Indeed, if not reduced, there exists a pair of branches \( x'0 \to y'0 \) and \( x'1 \to y'1 \) such that corresponding strands are parallel. If \( y' \) is not an empty word, \( 0 \) is a prefix of \( y' \). This contradicts with the same way as in branches \( y0 \) and \( y1 \). If \( y' \) is an empty word, then the obtained diagram has a branch \( x'0 \to 0 \). Since \( N(g) \geq 3 \) and \( R_- \) has branches 10 and 11, 0 is a strict prefix of some branch of \( R_- \). Hence, \( 0 \) is a strict prefix of some branch of \( R'_- \). This is a contradiction. Since the reduction is replacing branches 10 and 11 with 1, we have \( \ell_0(R'_-) = \ell_0(R_-) \). Hence, part (2) holds.

In the conditions of part (3) of the lemma, the tree-braid-tree diagram \((T_+ (g), \text{br}(g), R_-)\) of \( gx_0 \) is reduced. Indeed, since \( 1 \) or 01 is a strict prefix of some branch of \( T_- (g) \), by the relation of \( w \) and \( w' \), either 11 or 10 is a strict prefix of some branch of \( R_- \). It follows that \((T_+ (g), \text{br}(g), R_-)\) is reduced, because, if not, as noted above, \( R_- \) has branches both 11 and 10. In particular, 1 is a strict prefix of some branch of \( R_- \). Hence, part (3) holds.

Now assume the condition of part (1). To multiply \( g \) by \( x_0 \), we replace the tree-braid-tree diagram \((T_+ (g), \text{br}(g), T_- (g))\) by an equivalent diagram \((T'_+ (g), \text{br}'(g), T'_- (g))\) by attaching carets to the leaf of the branch 0 of \( T_- (g) \) and to the corresponding leaf of \( T'_+ (g) \), and splitting of the corresponding strand. Let \((R_+, \text{Id}, R_-)\) be the tree-braid-tree diagram of \( x_0 \) such that \( R_+ = T'_- (g) \). Then we get the diagram \((T'_+ (g), \text{br}'(g), R_-)\) of \( gx_0 \) and we can proceed as part (2). To complete the proof it suffices to prove that \((T'_+ (g), \text{br}'(g), R_-)\) is reduced. Indeed, in that case, by the construction, \( N(gx_0) = N(g) + 1 \) and since
\( \ell_0(R_+) = 2 \), we have \( \ell_0(gx_0) = \ell_0(R_-) = 1 \). If \((T'_+(g), br'(g), R_-)\) is not reduced, there exists a pair of branches \(x_0 \to y_0\) and \(x_1 \to y_1\) such that corresponding strands are parallel. By the construction, the left most and second from the left branches of \(T'_-(g)\) are 00 and 01. Hence the left most and second from the left branches of \(R_-\) are 0 and 10. This means \(y\) is not empty and 0 and 10 are not prefixes of \(y\). Then we first assume that 11 is a prefix of \(y\) and let \(y \equiv 11y'\) (\(y'\) is probably an empty word). From the construction of \(R_-\), \((T'_+(g), br'(g), T'_-(g))\) has a pair of branch \(x_0 \to 1y'0\) and \(x_1 \to 1y'1\) such that corresponding strands are parallel. On the other hand, 00 and 01 are only branches of \(T'_-(g)\) that can be reduced. This is a contradiction. Finally, we assume that \(y \equiv 1\). Then, \((R_+, \text{Id}, R_-)\) is the reduced tree-braid-tree diagram of \(x_0\). Since \(R_+ = T'_+(g)\) and \(N(x_0) = 2\) (cf. Figure 11) hold, by the construction of \(T'_-(g)\), \(N(g) = 2 - 1 = 1\). This contradicts the assumption of the lemma. Hence, part (1) holds. \(\square\)

The following corollary corresponds to Corollary 2.3. The proof given here is slightly modified.

**Corollary 4.3.** Let \(g\) be an element in \(BV\) with a reduced tree-braid-tree diagram \((T_+(g), br(g), T_-(g))\) such that \(N(g) \geq 3\). Let \(\ell := \ell_0(g)\). Then the following assertions hold.

1. If \(N(g) \geq 3 + (\ell - 1)\) then for every \(i \geq 0\) we have
   \[
   N(gx_i^\ell) \geq N(g) + i - 2(\ell - 1).
   \]

2. If either 1 or 01 is a strict prefix of some branch of \(T_-(g)\) then for every \(i \geq 0\) we have
   \[
   N(gx_i^\ell) = \max\{N(g), N(g) + i - (\ell - 1)\}.
   \]

**Proof.** To prove part (1), we first assume that \(\ell = 1\). By applying Lemma 4.2 \((1)\) to \(g\) iteratively, we have that for every \(i \geq 0\),
   \[
   N(gx_i^1) = N(g) + i.
   \]

Thus, we can assume that \(\ell > 1\). Since \(N(g) \geq 3 + (\ell - 1)\), we can apply Lemma 4.2 \((2)\) to \(g\) at least \((\ell - 1)\) times. Then we have \(\ell_0(gx_0^{\ell - 1}) = 1\) and for every \(i \leq \ell - 1\)
   \[
   N(gx_i^{\ell - 1}) \geq N(g) - i \geq N(g) - (\ell - 1) \geq N(g) + i - 2(\ell - 1). \tag{4.1}
   \]

Since \(N(gx_0^{\ell - 1}) \geq 3\) and \(\ell_0(gx_0^{\ell - 1}) = 1\), by applying Lemma 4.2 \((1)\) to \(gx_0^{\ell - 1}\) iteratively, we have
   \[
   N(gx_0^{\ell - 1}x_0^j) = N(gx_0^{\ell - 1}) + j \geq N(g) - (\ell - 1) + j, \tag{4.2}
   \]

for every \(j \geq 0\). By substituting \(j = i - (\ell - 1)\) in inequality \((4.2)\), we have that for every \(i \geq \ell - 1\),
   \[
   N(gx_i^\ell) \geq N(g) + i - 2(\ell - 1). \tag{4.3}
   \]
It follows from inequalities (4.1) and (4.3) that for every \( i \geq 0 \),
\[
N(gx_0^i) \geq N(g) + i - 2(\ell - 1),
\]
as required.

In the condition of part (2), we first assume that \( \ell = 1 \), again. By Lemma 4.4 (1) (applying iteratively), we have
\[
N(gx_0^i) = N(g) + i = \max\{N(g), N(g) + i\},
\]
for every \( i \geq 0 \). Thus, we can assume that \( \ell > 1 \). Since \( N(g) \geq 3 \) and either 1 or 01 is a strict prefix of some branch of \( T_-(g) \), by Lemma 4.2 (2) and (3), \( N(gx_0^i) = N(g) \geq N(g) + i - (\ell - 1) \) for every \( i \in \{0, \ldots, \ell - 1\} \) and \( \ell_0(gx_0^{\ell-1}) = 1 \). Thus, it suffices to prove that for every \( i \geq \ell - 1 \) we have \( N(gx_0^i) = N(g) + i - (\ell - 1) \geq N(g) \). Since \( \ell_0(gx_0^{\ell-1}) = 1 \) and \( N(gx_0^{\ell-1}) = N(g) \geq 3 \), by Lemma 4.2 (1), we have
\[
N(gx_0^{\ell-1}) = N(gx_0^{\ell-1}) + j = N(g) + j,
\]
for every \( j \geq 0 \). Substituting \( j = i - (\ell - 1) \) in inequality (4.4) gives that for every \( i \geq \ell - 1 \),
\[
N(gx_0^i) = N(g) + i - (\ell - 1),
\]
as required. \( \square \)

The proofs of the following lemma and corollary are symmetric to those of Lemma 4.2 and Corollary 4.3. We only need to switch 0 and 1.

**Lemma 4.4.** Let \( g \) be an element in \( BV \) with reduced tree-braid-tree diagram \( (T_+(g), br(g), T_-(g)) \) and assume that \( N(g) \geq 3 \). Then
\[
N(g) - 1 \leq N(gx_0^{-1}) \leq N(g) + 1.
\]

In addition,

(1) If \( \ell_1(g) = 1 \) then \( N(gx_0^{-1}) = N(g) + 1 \) and \( \ell_1(gx_0^{-1}) = 1 \).

(2) If \( \ell_1(g) \neq 1 \) then \( N(gx_0^{-1}) = N(g) \) or \( N(gx_0^{-1}) = N(g) - 1 \). Moreover, \( \ell_1(gx_0^{-1}) = \ell_1(g) - 1 \).

(3) If \( \ell_1(g) \neq 1 \) and either 0 or 10 is a strict prefix of some branch of \( T_-(g) \), then
\[
N(gx_0^{-1}) = N(g) \text{ and 0 is a strict prefix of some branch of } T_-(gx_0^{-1}), \text{ where } T_-(gx_0^{-1}) \text{ is a range-tree of reduced tree-braid-tree diagram of } gx_0^{-1}.
\]

**Corollary 4.5.** Let \( g \) be an element in \( BV \) with reduced tree-braid-tree diagram \( (T_+(g), br(g), T_-(g)) \) such that \( N(g) \geq 3 \). Let \( \ell := \ell_1(g) \). Then the following assertions hold.

(1) If \( N(g) \geq 3 + (\ell - 1) \) then for every \( i \geq 0 \) we have
\[
N(gx_0^{-i}) \geq N(g) + i - 2(\ell - 1).
\]
The next lemma describes the result of multiplying an element of $BV$ on the right by an element of $F$ with a following specific form. Let $u$ be a finite binary non-empty word and $h \in F$ be an non-identity element with reduced tree-braid-tree diagram $(T_+(h), \text{Id}, T_-(h))$. Let $T$ be a minimal finite rooted binary tree which contains the branch $u$. We take two copies of the tree $T$. To the first copy, we attach the tree $T_+(h)$ at the end of the branch $u$, and we write $R_+$ for this tree. To the second copy, we attach the tree $T_-(h)$ at the end of the branch $u$, and we write $R_-$ for this tree. Then the element $h_{[u]}$ is the one represented by the tree-braid-tree diagram, where domain-tree is $R_+$, range-tree is $R_-$ and braid is the “identity”, that is, all strands are straight. It is clear from the definition that $h_{[u]} \in F < BV$. For example, $x_0[0]$ and $x_0[1]$ are elements corresponding to the diagrams in Figure 15 or 16. Note that $x_0^2x_1^{-1}x_0^{-1} = x_0[0]$ holds, see Figure 17.

The following lemma corresponds to [16, Lemma 2.6].

**Lemma 4.6.** Let $g \in BV$ be a non-identity element, $u \to v$ be a branch of $g$, $h$ be a non-identity element of $F$. Let $h' = h_{[v]}$. Then

$$N(gh') = N(g) + N(h).$$
Moreover, if \( h \) consists of branches \( w_i \to z_i, i = 1, \ldots, k \) and \( B \) is the set of branches of \( g \) which are not equal to \( u \to v \), then \( gh' \) consists of branches \( uw_i \to vz_i, i = 1, \ldots, k \) along with all branches in \( B \).

**Proof.** Let \((T_+(g), br(g), T_-(g)), (T_+(h), \text{Id}, T_-(h))\) and \((T_+(h'), \text{Id}, T_-(h'))\) be the reduced tree-braid-tree diagrams of \( g, h \) and \( h' \), respectively. To multiply \( g \) by \( h' \), we note that the minimal refinement of \( T_-(g) \) and \( T_+(h') \) is the tree obtained from \( T_-(g) \) by attaching the tree \( T_+(h) \) at the bottom of the branch \( v \), since \( T_-(g) \) has a branch \( v \) and \( T_+(h') \) is constructed from the minimal tree which has a branch \( v \). Let \( S \) denote the described tree and \((R_1, br'(g), S)\) be an equivalent tree-braid-tree diagram of \( g \). We note that \( R_1 \) is obtained from \( T_+(g) \) by attaching a copy of \( T_+(h) \) to the bottom of the branch \( u \). If \((S, \text{Id}, R_2)\) is a tree-braid-tree diagram of \( h' \), we also note that \( R_2 \) can be obtained from \( T_-(g) \) by attaching \( T_-(h) \) at the bottom of the branch \( v \). The product of the tree-braid-tree diagrams \((R_1, br'(g), S)\) and \((S, \text{Id}, R_2)\) is \((R_1, br'(g), R_2)\). Since \((R_1, br'(g), S)\) has branches \( uw_i \to vw_i \) and branches in \( B \) (\( x \to y \) denotes these one), and \((S, \text{Id}, R_2)\) has branches \( vw_i \to vz_i \) and \( y \to y \), it follows that \((R_1, br'(g), R_2)\) has branches \( uw_i \to vz_i \) and \( x \to y \). To finish the proof, it remains to prove that \((R_1, br'(g), R_2)\) is reduced. Since \((T_+(h), \text{Id}, T_-(h))\) is reduced, \((T_+(h), \text{Id}, T_-(h))\) has no pair of branches of the form \( p0 \to q0 \) and \( p1 \to q1 \) where \( p0, p1 \equiv w_i \) for some \( i \), respectively and \( q0, q1 \equiv z_i \) for some \( i \), respectively. Hence, \((R_1, br'(g), R_2)\) has no pair of branches of the form \( up0 \to vq0 \) and \( up1 \to vq1 \), that is, \((R_1, br'(g), R_2)\) has no pair of the branches of the form \( uz_i \to vq_i \).
such that reducible. Similarly, since $(T_+(g),br(g),T_-(g))$ is reduced, $(R_1,br'(g),R_2)$ has no pair of branches of the form $x \rightarrow y$ such that reducible.

Recall that $R_1$ is obtained from $T_+(g)$ by attaching a copy of $T_+(h)$. Then it is clear that $N(gh') = N(g) + N(h)$ holds, and the proof is complete. □

4.2. Construction of the path. If $w$ is a word over the alphabet $X$, $\|w\|$ denotes the length of $w$. Note that any word $w$ over the alphabet $X$ can be regarded as an element of $BV$, then we have $|w| \leq \|w\|$.

**Remark 4.7.** Golan-Sapir constructed a path between elements whose number of carets is greater than or equal to three in [16, Proposition 2.7]. Linear divergence of Thompson groups $F$, $T$, $V$ follow immediately from this path. However, in the case of the braided Thompson group $BV$, we need a little more discussion. Because the number of $g \in BV$ such that $N(g) \leq 2$ is infinite. For example, $\tau_1, \tau_1^2, \tau_1^3, \ldots$ all have one caret.

First, we consider elements in $BV$ whose number of carets are greater than or equal to three (Proposition 4.8). Next, we construct paths between elements whose number of carets are less than three and others.

The following proposition corresponds to [16, Proposition 2.7]. In [16], they constructed the path that consists of five subpaths, subpath 1, ..., subpath 5. In this paper, we will take a similar process, but our subpath 3 (and therefore also the path $w$) is different from the original one. Our subpath 3 does not work for Thompson group $T$, but an almost similar approach works for Thompson groups $F$ and $V$.

**Proposition 4.8.** There exist constants $\delta, D > 0$ and a positive integer $Q$ such that the following holds. Let $g \in BV$ be an element with $N(g) \geq 3$. Then there exists a path of length at most $D|g|$ in the Cayley graph $\Gamma = \text{Cay}(BV,X)$ which avoids a $\delta|g|$-neighborhood of the identity and which has initial vertex $g$ and terminal vertex $x_0^{Q|g|} x_1^{-1} x_0^{-Q|g|+1}$.

In other words, there exists a word $w$ in the alphabet $X$ such that $\|w\| < D|g|$; for any prefix $w'$ of $w$, we have $|gw'| > \delta|g|$ and such that 

$$gw = x_0^{Q|g|} x_1^{-1} x_0^{-Q|g|+1}.$$ 

**Proof.** Let $C_1$ be the constant from Theorem 4.1. We give 5 subwords $w_1, \ldots, w_5$ and then let $w \equiv w_1 \cdots w_5$. Let $(T_+(g),br(g),T_-(g))$ be the reduced tree-braid-tree diagram of $g$.

**Subpath 1.** If 0 is not a branch of $T_-(g)$ we let $w_1 \equiv \emptyset$ and let $g_1 = g$. Otherwise, we let $w_1 \equiv x_0^2 x_1^{-1} x_0^{-1}$ and let $g_1 = gw_1$.

Let $(T_+(g_1),br_1,T_-(g_1))$ be the reduced tree-braid-tree diagram of $g_1$. The following lemma corresponds to [16, Lemma 2.8].

**Lemma 4.9.** We have that 0 is not a branch of $T_-(g_1)$. Moreover, $N(g) \leq N(g_1) \leq N(g) + 2$ hold, and for every prefix $w'$ of $w_1$, we have $N(gw') \geq N(g)$. 

Thus, we can assume that $0$ is a branch of $T_{-}(g)$. Let $u$ be the binary word such that $(T_{+}(g), br(g), T_{-}(g))$ has the branch $u \to 0$. We recall that $w_1 = x_0^2 x_1^{-1} x_0^{-1} = x_0[0]$ (cf. Figure 17). Hence, by Lemma 4.6, $(T_{+}(g), br(g), T_{-}(g))$ has the branch $u \to 0_2$ is a branch of the reduced tree-braid-tree diagram of $g_1 = gw_1 = gx_0[0]$ for each branch $v_1 \to v_2$ of $x_0$. Therefore, 0 is not a branch of $T_{-}(g_1)$ since it is a strict prefix of some branch.

For the second claim, by Lemma 4.6, we have

$$N(g_1) = N(gx_0[0]) = N(g) + N(x_0) = N(g) + 2.$$  

For the last claim, we will consider the number of carets of $gx_0$, $gx_0^2$ and $gx_0^2 x_1^{-1}$. Since $0$ is a branch of $T_{-}(g)$, we have $\ell_0(g) = \ell_0(T_{-}(g)) = 1$. Hence, by Lemma 4.2 (1), $N(gx_0) = N(g) + 1$ and $\ell_0(gx_0) = 1$. Again, by applying Lemma 4.2 (1) to $gx_0$, $N(gx_0^2) = N(gx_0) + 1 = N(g) + 2$. Finally, we note that $gx_0^2 x_1^{-1} = g_1 x_0$ and $N(g_1) = N(g) + 2$. By applying the inequality in Lemma 4.2 to $g_1$, we have

$$N(gx_0^2 x_1^{-1}) = N(g_1 x_0) \geq N(g_1) - 1 = N(g) + 1,$$

and the proof is complete. \qed

Subpath 2. We fix an integer $M \geq 100/C_1$. Then we define a word $w_2$ by

$$w_2 \equiv x_0^{-M(N(g_1)+1)} x_1 x_0^{M(N(g_1)+1)}$$

and we let $g_2 = g_1 w_2$.

Let $(T_{+}(g_2), br_2, T_{-}(g_2))$ be the reduced tree-braid-tree diagram of $g_2$. The following lemma corresponds to \[16\] Lemma 2.9.

**Lemma 4.10.** The following assertions hold.

1. For every prefix $w'$ of $w_2$, we have $N(g_1 w') \geq N(g_1)$.
2. $N(g_2) \geq MN(g_1)$.

**Proof.** We first prove part (2). It follows from the relation A in Theorem 2.4 that, as an element of $BV$, we have $w_2 = x_m$ where $m = M(N(g_1) + 1) + 1$. Let $\ell_1 = \ell_1(g_1)$ and $u$ be a finite binary word such that $u \to 1^{\ell_1}$ is a branch of $(T_{+}(g_1), br_1, T_{-}(g_1))$. By considering the minimum tree-braid-tree diagram where some branch is $1^{\ell_1}$, $\ell_1 \leq N(g_1)$ is clear. We note that $x_m = x_{m-\ell_1}$. Hence, by Lemma 4.6, we have

$$N(g_2) = N(g_1 w_2) = N(g_1 x_m) = N(g_1 x_{m-\ell_1}) = N(g_1) + N(x_{m-\ell_1}).$$

By the definition of standard infinite generating set of $F$, we have $N(x_{m-\ell_1}) = m - \ell_1 + 2$. Hence,

$$N(g_2) = N(g_1) + m - \ell_1 + 2 = N(g_1) - \ell_1 + MN(g_1) + 3 \geq MN(g_1),$$

as $\ell_1 \leq N(g_1)$. Thus, part (2) holds.
Before proceeding the proof of part (1), we note that if \( z_j \to q_j, j = 1, \ldots, n \) are the branches of \( x_{m-t_1} \), then, by Lemma 4.6, the branches of the diagram \((T_+(g_2), br_2, T_-(g_2))\) of \( g_2 = g_1 x_m \equiv uz_j \to 1^t q_j, j = 1, \ldots, n \) as well as all branches \( a_k \to b_k \) which are branches of the diagram \((T_+(g_1), br_1, T_-(g_1))\), other than \( u \to 1^t \). By Lemma 4.9, 0 is a strict prefix of some branch of \( T_-(g_1) \). Hence, 0 is a strict prefix of some branch of \( T_-(g_2) \).

Now, let \( w' \) be a prefix of \( w_2 \). Then either (a) \( w' \equiv x_0^{-i} \) for some \( 0 \leq i \leq M(N(g_1) + 1) \), or (b) \( w' \equiv x_0^{-M(N(g_1)+1)} x_1 x_0^i \) for some \( 0 \leq i \leq M(N(g_1) + 1) \).

By Lemma 4.9, \( N(g_1) \geq N(g) \) and 0 is a strict prefix of some branch of \( T_-(g_1) \). Hence, by Corollary 4.5 (2), for \( g_1 \) and any \( i \geq 0 \), \( N(g_1 x_0^{-i}) \geq N(g_1) \). Hence, part (1) of the lemma holds for prefixes \( w' \) of type (a). Next, we consider the element \( g_1 w' \) for \( w' \) of type (b). As an element of \( BV \), we have

\[
g_1 w' = g_1 x_0^{-M(N(g_1)+1)} x_1 x_0^i = g_1 x_0^{-M(N(g_1)+1)} x_1 x_0^{M(N(g_1)+1)} x_0^{-M(N(g_1)+1)} = g_2 x_0^{-M(N(g_1)+1)}.
\]

From the note above, 0 is a strict prefix of some branch of \( T_-(g_2) \), and we have already shown that \( N(g_2) \geq MN(g_1) \geq N(g) \). Hence, by Corollary 4.5 (2), for \( g_2 \) and any \( s \geq 0 \), we have

\[
N(g_2 x_0^{-s}) \geq N(g_2) \geq MN(g_1).
\]

We also note that \( i - M(N(g_1) + 1) \leq 0 \). By substituting \(-s = i - M(N(g_1) + 1)\), we have

\[
N(g_1 w' = N(g_2 x_0^{-i-M(N(g_1)+1)}) \geq MN(g_1),
\]

as required. \( \square \)

**Subpath 3.** For reduced tree-braid-tree diagram \((T_+(g_1), br_1, T_-(g_1))\) of \( g_1 \), we assume that some strand of \( br_1 \) connects 0th leaf to kth leaf, where \( 0 \leq k \leq N(g_1) \). Let \( h \) be the element of \( BV \) given by the (maybe not reduced) tree-braid-tree diagram \((T_-(g_1), br_h, T_+(g_1))\), where \( br_h \) is following. If \( k > 0 \), the kth strand goes over the \((k-1)\)-th strand, \((k-2)\)-th strand, \ldots, 0th strand, in order, and other strands are straight. If \( k = 0 \), all strands are straight. For example, Figure 18 illustrates the construction of braid for \( N(g_1) = 5 \) and \( k = 4 \). Now, we let \( w_3 \) be the minimal word over \( X \) such that \( h = w_3 \) in \( BV \) and let \( g_3 = g_2w_3 \).

Part (2) and (3) of the following lemma correspond to [16, Lemma 2.10].

**Lemma 4.11.** The following assertions hold.

1. \( \|w_3\| \leq 14N(g_1) \).
(2) $N(g_3) \geq (M - 1)N(g_1) + M + 3$.

(3) $\ell_0(g_3) \leq N(g_1) + 1$ and $0^\ell_0(g_3) \rightarrow 0^\ell_0(g_3)$ is a branch of $g_3$.

**Proof.** Let $n = N(g_1)$. To prove part (1), we split $(T_-(g_1), br_h, T_+(g_1))$ into three diagrams by using all-right trees $T_n$, giving a split into a positive element $(T_n, \text{Id}, T_n) \in F$, a braid element $(T_n, br_h, T_n)$, and a negative element $(T_n, \text{Id}, T_+(g_1)) \in F$, where $T_n$ has $n$ carets (recall Figure 1). Let $p$, $Br_h$ and $q$ be the minimal words over $X$ such that $p = (T_-(g_1), \text{Id}, T_n)$, $Br_h = (T_n, br_h, T_n)$ and $q = (T_n, \text{Id}, T_+(g_1))$ in $BV$, respectively. We identify these words with elements of $BV$.

First, we prove that $|p| \leq 6n$ holds. Let $A$ be the standard generating set of $F$ such that $A \subset X$ and we note that $|p|_A \leq 6N(p)$ (see [8, Theorem 1 and Proposition 2]). We also note that the tree-braid-tree diagram $(T_-(g_1), \text{Id}, T_n)$ might not be reduced. Hence, we have

$$|p| \leq |p|_A \leq 6N(p) \leq 6n.$$  

Similarly, we have $|q| \leq 6n$.

Next, we prove that $|Br_h| \leq 2n$ holds. To get this upper bound, we rewrite the word $Br_h$ by elements of infinite generator of $BV$. The following rewritings are obvious.

$$
k = n \quad \Rightarrow \quad Br_h = \tau_n \sigma_{n-1} \ldots \sigma_1,
$$

$$
k = n - 1 \quad \Rightarrow \quad Br_h = \sigma_{n-1} \ldots \sigma_1,
$$

$$
k = n - 2 \quad \Rightarrow \quad Br_h = \sigma_{n-2} \ldots \sigma_1,
$$

$$
\vdots
$$

$$
k = 1 \quad \Rightarrow \quad Br_h = \sigma_1,
$$

$$
k = 0 \quad \Rightarrow \quad Br_h = \emptyset.
$$
It suffices to consider only the case $k = n$, as we can get the following estimation. Indeed, for $n \geq 4$, we have

$$Br_h = \tau_n \sigma_{n-1} \cdots \sigma_2 \sigma_1$$

$$= (x_0^{(n-2)} \tau_2 x_0^{n-2})(x_0^{(n-3)} \sigma_2 x_0^{n-3}) \cdots (x_0^{-1} \sigma_2 x_0)(\sigma_2 \sigma_1)$$

$$= (x_0^{(n-2)} \tau_2 x_0)(\sigma_2 x_0) \cdots (\sigma_2 x_0)(\sigma_2 \sigma_1)$$

$$= (x_0^{(n-2)} \sigma_1^{-1} \tau_1 x_0 x_0)(x_0^{-1} \sigma_1 x_1 \sigma_1^{-1} x_0) \cdots (x_0^{-1} \sigma_1 x_1 \sigma_1^{-1} x_0)(x_0^{-1} \sigma_1 x_1 \sigma_1^{-1} \sigma_1)$$

$$= (x_0^{(n-2)} \sigma_1^{-1} \tau_1 x_0)(\sigma_1 x_1 \sigma_1^{-1}) \cdots (\sigma_1 x_1 \sigma_1^{-1})(\sigma_1 x_1)$$

$$= (x_0^{(n-2)} \sigma_1^{-1} \tau_1 x_0)\sigma_1 x_1^{-2}$$

where we rewrite $\tau_n = x_0^{-(n-2)} \tau_2 x_0^{n-2}$, $\sigma_i = x_0^{-(i-2)} \sigma_2 x_0^{i-2}$ for each $i \geq 3$, $\tau_2 = \sigma_1^{-1} \tau_1 x_0$, and $\sigma_2 = x_0^{-1} \sigma_1 x_1 \sigma_1^{-1}$ by the relations D1($j = 0$), C3($j = 0$), D2($i = 1$), and C2($i = 1$) in Theorem 2.4, respectively. Hence, we have

$$|Br_h| \leq n - 2 + 3 + 1 + n - 2 = 2n.$$

When $n = 3$, we have

$$Br_h = \tau_3 \sigma_2 \sigma_1$$

$$= (x_0^{-1} \tau_2 x_0)(\sigma_2 \sigma_1)$$

$$= (x_0^{-1} \sigma_1^{-1} \tau_1 x_0 x_0)(x_0^{-1} \sigma_1 x_1)$$

$$= x_0^{-1} \sigma_1^{-1} \tau_1 x_0 \sigma_1 x_1.$$

Hence, we have

$$|Br_h| \leq 6 = 2 \times 3 = 2n.$$

Therefore, we have

$$\|w_3\| = |h| \leq |p| + |Br_h| + |q| \leq 14n,$$

as required.

For part (2) and (3), we recall the form of branches of the reduced tree-braid-tree diagram $(T_+(g_2), br_2, T_-(g_2))$ of $g_2$ as described in the proof of Lemma 4.10. Let $\ell_1 = \ell_1(T_-(g_1))$ and let $m = M(N(g_1) + 1) + 1$. Then

$$g_2 = g_1 x_m = g_1 x_{m-\ell_1 [1]}.$$

Now, let $u$ be such that $u \to 1^\ell_1$ is a branch of $(T_+(g_1), br_1, T_-(g_1))$. If $z_j \to q_j$, $j = 1, \ldots, n$ are the branches of reduced tree-braid-tree diagram of $x_{m-\ell_1}$ then the branches of $(T_+(g_2), br_2, T_-(g_2))$ are $uz_j \to 1^\ell_1 q_j$, $j = 1, \ldots, n$ as well as all the branches $a_k \to b_k$ of $(T_+(g_1), br_1, T_-(g_1))$, other than $u \to 1^\ell_1$.

Let $v$ be such that $1^\ell_1 \to v$ is a branch of the tree-braid-tree diagram $(T_-(g_1), br_h, T_+(g_1))$ of $h$. Then $uz_j \to v q_j$, $j = 1, \ldots, n$ are all branch of reduced tree-braid-tree diagram of
where \(T\) replace tree \(T\) at the end of branch \(1^\ell_1\) of the \(T_-(g_1)\), and so \(T_-(g_1)\) is a rooted subtree of \(T_-(g_2)\). Then to multiply \(g_2\) by \(h\), we replace \((T_-(g_1), br_h, T_+(g_1))\) by an equivalent tree-braid-tree diagram \((T_+(h), br'_h, T_-(h))\) where \(T_+(h) = T_-(g_2)\). By construction, \(1^\ell_1q_j \rightarrow vq_j\) are branches of \((T_+(h), br'_h, T_-(h))\). Hence,

\[
(T_+(g_2), br_2, T_-(g_2)) \cdot (T_+(h), br'_h, T_-(h)) = (T_+(g_2), br_2br'_h, T_-(h))
\]

has branches \(uz_j \rightarrow vq_j, j = 1, \ldots, n\). We recall that \(z_j \rightarrow q_j, j = 1, \ldots, n\) are branches of a reduced tree-braid-tree diagram. Hence, we can not reduce the caret formed by the \(uz_j \rightarrow vq_j\) of the tree-braid-tree diagram \((T_+(g_2), br_2br'_h, T_-(h))\).

Since \(\ell_1 \leq N(g_1)\), we have

\[
N(g_3) = N(g_2h) \geq N(x_{m-\ell_1}) \\
= m - \ell_1 + 2 \\
\geq m - N(g_1) + 2 \\
= M(N(g_1) + 1) + 1 - N(g_1) + 2 \\
= (M - 1)N(g_1) + M + 3,
\]

as required.

For part (3), let \(r = \ell_0(T_+(g_1))\). We note that \(r \leq N(g_1)\) holds by the same reason as \(\ell_1 \leq N(g_1)\). Let \(s\) be a binary word such that \(0^r \rightarrow s\) is a branch of \((T_+(g_1), br_1, T_-(g_1))\) of \(g_1\). By the definition of \(br_h\), \(s \rightarrow 0^r\) is a branch of the diagram \((T_-(g_1), br_h, T_+(g_1))\) of \(h\). Recall that \(u \rightarrow 1^{\ell_1}\) is a branch of \((T_+(g_1), br_1, T_-(g_1))\) of \(g_1\). We consider two cases: (a) \(u \neq 0^r\), and (b) \(u \equiv 0^r\).

In case (a), \(0^r \rightarrow s\) is a branch of \((T_+(g_2), br_2, T_-(g_2))\). Indeed, every branch of \((T_+(g_1), br_1, T_-(g_1))\) of \(g_1\), other than \(u \rightarrow 1^{\ell_1}\), is also a branch of \((T_+(g_2), br_2, T_-(g_2))\). Then since \((T_+(g_2), br_2, T_-(g_2))\) has the branch \(0^r \rightarrow s\) and \((T_+(g_1), br_h, T_+(g_1))\) of \(h\) has the branch \(s \rightarrow 0^r\), by adding some minimal number of caret if necessary, the tree-braid-tree diagram of the product \(g_2h\) has the branch \(0^r \rightarrow 0^r\). Since this diagram might be not reduced, \(\ell_0(g_3) \leq r\) holds. Hence, by \(r \leq N(g_1)\) holds, we have \(\ell_0(g_3) \leq N(g_1)\) and \(0^{\ell_0(g_3)} \rightarrow 0^{\ell_0(g_1)}\) is reduced tree-braid-tree diagram of \(g_2h = g_3\), as required. We illustrate a sketch of the product \(g_2h\) in Figure [19].

In case (b), \(u \equiv 0^r\), so \(1^{\ell_1} \equiv s\). Since the diagram \((T_-(g_1), br_h, T_+(g_1))\) of \(h\) has the branches \(1^{\ell_1} \rightarrow v\) and \(s \rightarrow 0^r\), we have \(v \equiv 0^r \equiv u\). Since the reduced tree-braid-tree diagram of \(g_2h = g_3\) has the branches \(uz_j \rightarrow vq_j\), this diagram has the branches \(0^r z_j \rightarrow 0^r q_j\). Since \(m - \ell_1 > 0\) holds, \(x_{m-\ell}\) has a branch \(0 \rightarrow 0\). Hence, reduced tree-braid-tree diagram of \(g_3\) has a branch \(0^{\ell_1+1} \rightarrow 0^{\ell_1+1}\), as required. \(\square\)
Figure 19. A rough sketch of a tree-braid-tree diagram of the product $g_1 x_{m-\ell_1[1 \ell_1]} h$ with only the important branches

Subpath 4. We fix an integer $Q \geq 12M/C_1^2$ and let

$$w_4 \equiv x_0^{Q|g|} x_1^{-1} x_0^{-Q|g|+1}.$$  

We also let $g_4 = g_3 w_4$. We note that $w_4$ is a word representing the terminal vertex of Proposition 4.8.

Part (1) of the following lemma corresponds to [16, Lemma 2.11].

Lemma 4.12. The following assertions hold.

(1) For every prefix $w'$ of $w_4$ we have

$$N(g_3 w') \geq N(g_3) + \frac{1}{2} \|w'\| - 2N(g_1) - 1.$$  

(2) As elements in $BV$, $g_3$ and $w_4$ commute.
Proof. To prove part (1), let \( \ell = \ell_0(g_3) \). We first consider prefixes \( w' \) of \( w_4 \) which are positive power of \( x_0 \). We note that by Lemmas 4.11 (2), 4.9 and 4.11 (3), we have

\[
N(g_3) \geq (M - 1)N(g_1) + M + 3 \\
\geq N(g_1) + N(g_1) + M + 3 \\
\geq N(g) + \ell + M + 2 \\
\geq 3 + \ell - 1.
\]

Hence, we can apply Corollary 4.3 (1) to \( g_3 \). Again we note that \( \ell - 1 \leq N(g_1) \) holds by Lemma 4.11 (3). Then we have

\[
N(g_3w') = N(g_3x_0^i) \geq N(g_3) + i - 2(\ell - 1) \\
\geq N(g_3) + i - 2N(g_1) \\
= N(g_3) + \|w'\| - 2N(g_1) \\
\geq N(g_3) + \frac{1}{2}\|w'\| - 2N(g_1).
\]

Thus, to finish the proof of part (1), it suffices to show that for every prefix \( w' \) of \( w_4 \) which contains the letter \( x_1^{-1} \), we have

\[
N(g_3w') \geq N(g_3x_0^{Q|g|}) - 1.
\]

Indeed, in that case, by inequality (4.5),

\[
N(g_3w') \geq N(g_3x_0^{Q|g|}) - 1 \\
\geq N(g_3) + Q|g| - 2N(g_1) - 1 \\
= N(g_3) + \frac{1}{2}\|w_4\| - 2N(g_1) - 1 \\
\geq N(g_3) + \frac{1}{2}\|w'\| - 2N(g_1) - 1,
\]

as desired. To show inequality (4.6), we first consider the following prefix

\[
p \equiv x_0^{Q|g|}x_1^{-1}x_0^{-1} \equiv x_0^{Q|g|-2} \cdot x_0^{-1}x_1^{-1}x_0^{-1}.
\]

Since \( C_1 Q \geq 1200 \) and \( N(g) \geq 3 \) hold, by Theorem 4.1, we note that we have

\[
\frac{1}{C_1}|g| - Q|g| \leq \frac{|g|}{C_1} - \frac{1200|g|}{C_1} = \frac{-1199}{C_1}|g| \leq -1199N(g) < -5.
\]

By Lemmas 4.11 (3), 4.9 and Theorem 4.1, we have

\[
\ell \leq N(g_1) + 1 \leq N(g) + 3 \leq \frac{1}{C_1}|g| + 3 < Q|g| - 2.
\]

Since \( N(g_3) \geq 3 + \ell - 1 \) holds, by Lemma 4.2 (1) and (2) (if necessary, apply them repeatedly), we have \( \ell_0(g_3x_0^{Q|g|-2}) = 1 \). In other words, the range-tree of the reduced tree-braid-tree diagram of \( g_3x_0^{Q|g|-2} \) has a branch \( 0 \). By applying Lemma 4.2 (1) to \( g_3x_0^{Q|g|-2} \),
twice, we have
\[ N(g_3 x_0 ^{Q|g|}) = N(g_3 x_0 ^{Q|g|-2} \cdot x_0 ^2) = N(g_3 x_0 ^{Q|g|-2} \cdot x_0 ) + 1 = N(g_3 x_0 ^{Q|g|-2}) + 2. \]
Since \( x_0 ^2 x_1 ^{-1} x_0 ^{-1} = x_0 |g| \) and \( \ell_0 (g_3 x_0 ^{Q|g|-2}) = 1 \) hold, by Lemma 4.6 we have
\[ N(g_3 p) = N(g_3 x_0 ^{Q|g|-2} \cdot x_0 ^2 x_1 ^{-1} x_0 ^{-1}) = N(g_3 x_0 ^{Q|g|-2} \cdot x_0 |g|) = N(g_3 x_0 ^{Q|g|}) + N(x_0) = N(g_3 x_0 ^{Q|g|}) - 2 + 2 = N(g_3 x_0 ^{Q|g|}), \]
so inequality (4.6) holds for the prefix \( p \). We also note that by Lemma 4.6, \( \ell_0 (g_3 p) \neq 1 \) and \( N(g_3 p) \geq 3 \).

To finish the proof, it remains to prove that inequality (4.6) holds for the prefix (a) \( w' \equiv x_0 ^{|g|} x_1 ^{-1} \) and prefixes of the form (b) \( w' \equiv x_0 ^{|g|} x_1 ^{-1} x_0 ^{-i} \), for \( 1 < i \leq Q|g| - 1 \). For the case (a), we note that \( \ell_0 (g_3 p) \neq 1 \) and \( g_3 w' = g_3 p x_0 \). Hence, by applying Lemma 4.2 (2) to \( g_3 p \), we have
\[ N(g_3 w') = N(g_3 p x_0) \geq N(g_3 p) - 1 = N(g_3 x_0 ^{|g|}) - 1, \]
as required. Finally, we note again that \( \ell_0 (g_3 p) \neq 1 \) and prefixes of the form (b) can be written as \( w' \equiv p x_0 ^{-i} x_0 ^{-1} \). Hence, by Corollary 4.5 (2) we have
\[ N(g_3 w') = N(g_3 p x_0 ^{-i} x_0 ^{-1}) \geq N(g_3 p) = N(g_3 x_0 ^{|g|}), \]
as required.

For part (2), we first note that reduced tree-braid-tree diagram of \( g_3 \) has a “same length” branch \( 0' \to 0' \). By calculating \( x_0 ^{i} (x_0 ^{i} x_1 ^{-1} x_0 ^{-1} x_0 ^{-i}) \), for \( i = 1, 2, \ldots, Q|g| - 2 \) inductively, we get tree-braid-tree diagram of \( w_4 \) as Figure 20. Since \( \ell - 1 < Q|g| - 2 \), holds, we can calculate \( w_4 g_3 \) and \( g_3 w_4 \) as Figure 21 and Figure 22 respectively. Hence, we have \( w_4 g_3 = g_3 w_4 \), as required.

**Subpath 5.** Let \( w_5 \) be a minimal word in the alphabet \( X \) such that \( w_5 = g_3 ^{-1} \) in \( BV \). Let \( g_5 = g_4 w_5 \).

It follows from Lemma 4.12 (2) that
\[ g w = g w_1 w_2 w_3 w_4 w_5 = g_5 = g_3 w_4 g_3 ^{-1} = w_4. \]
Hence, \( g w = x_0 ^{|g|} x_1 ^{-1} x_0 ^{-Q|g|+1} \) for \( w \equiv w_1 w_2 w_3 w_4 w_5 \), as required.

It remains to prove that one can choose constants \( \delta, D \) (independently of \( g \)), so that path \( w \) satisfies the conditions in the Proposition 4.8. First, by definitions of subpaths, we have the following.
\[ \| w \| \leq \| w_1 w_2 w_3 \| + \| w_4 \| + \| w_5 \| \]
Figure 20. A tree-braid-tree diagram of $w_4$

\[
= \|w_1w_2w_3\| + 2Q|g| + |g_3| \\
= \|w_1w_2w_3\| + 2Q|g| + |gw_1w_2w_3| \\
\leq \|w_1w_2w_3\| + 2Q|g| + |g| + \|w_1w_2w_3\| \\
= 2\|w_1w_2w_3\| + (2Q + 1)|g| \\
\leq 2\|w_1w_2w_3\| + 3Q|g|.
\]

Furthermore, we have a upper bound of $\|w_1w_2w_3\|$ as follows.

\[
\|w_1w_2w_3\| \leq \|w_1\| + \|w_2\| + \|w_3\| \\
\leq 4 + 2M(N(g_1) + 1) + 1 + 14N(g_1) \\
= 2MN(g_1) + 14N(g_1) + 5 + 2M \\
\leq 2M(N(g) + 2) + 14(N(g) + 2) + 5 + 2M \\
= 2MN(g) + 14N(g) + 33 + 6M \\
< 2MN(g) + 14N(g) + 33N(g) + 2M \times 3 \\
= 2MN(g) + 47N(g) + 2M \times 3 \\
\leq 2MN(g) + MN(g) + 2MN(g) \\
= 5MN(g) \\
\leq \frac{5M}{C_1}|g|, \quad (4.7)
\]
Figure 21. Calculation of $w_4g_3$

Figure 22. Calculation of $g_3w_4$
where these inequalities follow from the definition of the subpaths, Lemmas 4.11 (1), 4.9, the definition of $M$ and Theorem 4.1. Therefore, we have $\|w\| \leq D|g|$ where $D = 10M/C_1 + 3Q$, as required. Now, let $\delta = C_1/10M$. The following lemma corresponds to [16, Lemma 2.12] and completes the proof of Proposition 4.8.

**Lemma 4.13.** Let $w'$ be a prefix of $w$. Then $|gw'| > \delta|g|$.

**Proof.** First, we note that by Lemma 4.11 (2),

$$N(g_3) \geq (M - 1)N(g_1) + M + 3.$$  

Then for each prefix $\tilde{w} \leq w_4$ we have by Lemma 4.12,

$$N(g_3\tilde{w}) \geq N(g_3) + \frac{1}{2}\|\tilde{w}\| - 2N(g_1) - 1 \geq \frac{1}{2}\|\tilde{w}\| + (M - 3)N(g_1) + M + 2 \geq \frac{1}{2}\|\tilde{w}\|. \quad (4.8)$$

We separate the proof into two cases depending on the length of $g$.

Case (1): $|g| < 10MN(g)$.

It follows from Lemmas 4.9 and 4.10 (1) that for every prefix $w' \leq w_1w_2$, we have $N(gw') \geq N(g)$. Then, by applying Theorem 4.1 to $gw'$, we have

$$|gw'| \geq C_1N(gw') \geq C_1N(g) > \frac{C_1}{10M}|g| = \delta|g|,$$

as required. Next, we consider a prefix $w' \leq w_3$. By Theorem 4.1, Lemma 4.10 (2) and $M \geq 100/C_1$,

$$|g_2| \geq C_1N(g_2) \geq C_1MN(g_1) \geq 100N(g_1).$$

Since we already know that $\|w_3\| \leq 14N(g)$ (Lemma 4.11 (1)), $N(g_1) \geq N(g)$ (Lemma 4.9) and $N(g) > |g|/10M$ (assumption of case (1)) hold, we have

$$|g_2w'| \geq |g_2| - \|w'\| \geq |g_2| - \|w_3\| \geq 100N(g_1) - 14N(g_1) = 86N(g_1) \geq 86N(g) > \frac{86}{10M}|g| > \frac{C_1}{10M}|g| = \delta|g|,$$
as required. Now, let $w'$ be a prefix of $w_4$. By Theorem 4.1 Lemmas 4.12 (1), 4.11 (2), 4.9 and assumption of case (1), we have

$$|g_3w'| \geq C_1 N(g_3w')$$
$$\geq C_1(N(g_3) - 2N(g_1) - 1)$$
$$\geq C_1((M - 1)N(g_1) + M + 3 - 2N(g_1) - 1)$$
$$= C_1((M - 3)N(g_1) + M + 2)$$
$$> C_1 N(g)$$
$$> \frac{C_1}{10M} |g| = \delta |g|,$$

as required. Finally, we consider a prefix $w' \leq w_5$. Since $\|w_1w_2w_3\| \leq (5M/C_1)|g|$ (inequality (4.7)'') and $Q \geq 12M/C_1^2$, we have

$$\|w_5\| = |g_3| = |gw_1w_2w_3| \leq |g| + \|w_1w_2w_3\|$$
$$\leq |g| + \frac{5M}{C_1} |g|$$
$$< \frac{M}{C_1} |g| + \frac{5M}{C_1} |g|$$
$$= \frac{6M}{C_1} |g|$$
$$\leq \frac{C_1 Q}{2} |g|.$$

By Theorem 4.1 inequality (4.8) for $\tilde{w} = w_4$ (then $\|w_4\| = 2Q|g|$) and the definition of $Q$, we have

$$|g_4w'| \geq |g_4| - \|w'\| \geq |g_4| - \|w_5\|$$
$$> C_1 N(g_4) - \frac{C_1 Q}{2} |g|$$
$$\geq C_1 Q|g| - \frac{C_1 Q}{2} |g|$$
$$= \frac{C_1 Q}{2} |g|$$
$$> \frac{C_1}{10M} |g| = \delta |g|,$$

as required. Hence, the lemma holds in case (1).

Case (2): $|g| \geq 10MN(g)$.

Since $\|w_1w_2w_3\| \leq 5MN(g)$ (inequality (4.7)) and $|g|/2 \geq 5MN(g)$ (assumption of case (2)), for any prefix $w' \leq w_1w_2w_3$ we have

$$|gw'| \geq |g| - \|w'\|$$
$$\geq |g| - \|w_1w_2w_3\|$$
\[
\begin{align*}
\geq |g| - 5MN(g) \\
\geq |g| - \frac{1}{2}|g| \\
= \frac{1}{2}|g| \\
> \frac{C_1}{10M}|g| = \delta|g|,
\end{align*}
\]

as required. In particular, we note that \( |g_3| \geq |g|/2 \) holds where \( g_3 = gw_1w_2w_3 \). Let \( w' \leq w_4 \). If \( \|w'\| \leq |g|/5 \), then we have
\[
|g_3w'| \geq |g_3| - \|w'\| \geq \frac{1}{2}|g| - \frac{1}{5}|g| = \frac{3}{10}|g| > \frac{C_1}{10M}|g| = \delta|g|,
\]
as desired. Hence, we can assume that \( \|w'\| > |g|/5 \). In that case, by Theorem 4.1 and inequality (4.8) we have
\[
|g_3w'| \geq C_1N(g_3w') > \frac{C_1}{2}\|w'\| > \frac{C_1}{10}|g| > \frac{C_1}{10M}|g| = \delta|g|,
\]
as required. To finish the proof, we note that by Theorem 4.1 and inequality (4.8) for \( \tilde{w} = w_4 \) (then \( \|w_4\| = 2Q|g| \)), we have
\[
|g_4| = |g_3w_4| \geq C_1N(g_3w_4) > \frac{C_1}{2}\|w_4\| = C_1Q|g|.
\]
By inequality (4.7) and assumption of case (2), we also note that we have
\[
\|w_5\| = |g_3| = |gw_1w_2w_3| \\
\leq |g| + \|w_1w_2w_3\| \\
\leq |g| + 5MN(g) \\
\leq |g| + \frac{1}{2}|g| \\
= \frac{3}{2}|g|.
\]
Hence, since \( Q \geq 12M/C_1^2 \), we have
\[
|g_4w'| \geq |g_4| - \|w'\| \geq |g_4| - \|w_5\| \\
> C_1Q|g| - \frac{3}{2}|g| \\
= (C_1Q - \frac{3}{2})|g| \\
> |g| \\
> \frac{C_1}{10M}|g| = \delta|g|,
\]
as required. □
By multiplying $x_1$ on the right to an element having one or two carets, we get the element of $BV$ satisfying the assumption of Proposition 4.8.

**Lemma 4.14.** Let $g \in BV$ be such that $N(g) \leq 2$. Then $N(gx_1) \geq 3$.

**Proof.** By regarding each braid as just a permutation, it can be shown by finite number of direct calculations. Indeed, each tree-braid-tree diagram is reduced if there exists no strands pair such that they have a same parent. Hence, if it is reduced when considering $g$ as the element of $V$, then it is also reduced in $BV$. For example, see Figure 23. The endpoints of each strand are represented by the same number, with a blank representing some braid. □

**Lemma 4.15.** Let $g \in BV$. Then

$$|g| - 1 \leq |gx_1| \leq |g| + 1.$$ 

**Proof.** The first inequality follows from $|g| \leq |gx_1| + |x_1^{-1}| = |gx_1| + 1$. The second inequality follows from $|gx_1| \leq |g| + |x_1| = |g| + 1$. □

The following proposition immediately implies that braided Thompson group has liner divergence, completing the proof of Theorem 1.1. See Figure 24 for the overview of the paths. The idea of paths (corresponding two vertical lines in the middle) in the following proposition comes from [16, Theorem 2.13].

**Proposition 4.16.** There exist constants $\delta_{BV}$ and $D_{BV} > 0$ such that the following holds. Let $g \in BV$ be an element with $|g| \geq 2$. Then there exists a path of length at most $D_{BV}|g|$ in the Cayley graph $\Gamma = Cay(BV, X)$ which avoids a $\delta_{BV}|g|$-neighborhood of the identity and which has initial vertex $g$ and terminal vertex $x_0^Q|g|^{-1}x_1^{-1}x_0^{-Q|g|+1}$. 

**Figure 23.** Calculating example of $gx_1$
In other words, there exists a word $w_{BV}$ in the alphabet $X$ such that $||w|| < D_{BV}|g|$; for any prefix $w'$ of $w_{BV}$, we have $|gw'| > \delta_{BV}|g|$ and such that

$$gw_{BV} = x_0^{|g|}x_1^{-1}x_0^{-Q|g|+1}.$$ 

Proof. For each natural number $k > 0$, let

$$v(k) = x_0^{Qk}x_1^{-1}x_0^{-Qk+1}.$$ 

First, if $N(g) \geq 3$ then the proposition follows from Proposition 4.8. Hence, we can assume that $N(g) \leq 2$. Then by Lemma 4.15, $|gx_1| = |g| - 1$, $|g|$ or $|g| + 1$ and by Lemma 4.14, $N(gx_1) \geq 3$. Let

$$D_{BV} = 2D + 4Q + 1$$ 

$$\delta_{BV} = \min\left\{\frac{1}{2}\delta, \frac{1}{2}C_1Q\right\}.$$ 

In the following, we will use Proposition 4.8 to construct the path connecting $gx_1$ and $v(|g|)$. We consider three cases depending on the length of $gx_1$.

Case (1): $|gx_1| = |g| - 1$.

By Proposition 4.8, there exists a path of length at most $D(|g| - 1)$ which avoids a $\delta(|g| - 1)$-neighborhood of identity and which has initial vertex $gx_1$ and terminal vertex $v(|g| - 1)$. Since $|g| \geq 2$, we have $\delta(|g| - 1) \geq (\delta/2)|g|$. Hence this path avoids a $(\delta/2)|g|$-neighborhood of identity. Thus, we construct a path connecting $v(|g| - 1)$ and $v(|g|)$. Let

$$p(|g| - 1) \equiv x_0^{Q(|g| - 1)-1}x_1x_0^{-1}x_1^{-Q|g|+1}.$$ 

It is clear that $p(|g| - 1)$ labels a path from $v(|g| - 1)$ to $v(|g|)$ and the length of $p(|g| - 1)$ is at most $2Q|g|$. In the following, we prove that for any prefix $p'$ of $p(|g| - 1)$, we have

**Figure 24.** The path connecting $g_1$ and $g_2$ where $|g_1|, |g_2| \geq 2$
$|v(|g| - 1)p'| > C_1Q|g|$.
Indeed, it is easy to see that the positive part of the normal form of the element $v(|g| - 1)p'$ is $x_0^i$ for $i \geq Q(|g| - 1)$. Hence, by [8, Theorem 3], we have

$$N(v(|g| - 1)p') \geq N(x_0^i) \geq Q(|g| - 1) + 1 > Q(|g| - 1),$$

where we note that this theorem claims only the relationship between the number of carets of elements in $F$ and their exponents, so it can be applied to $BV$. Hence, by Theorem 4.1 we have

$$|v(|g| - 1)p'| \geq C_1N(v(|g| - 1)p') > C_1Q(|g| - 1).$$

Since $|g| \geq 2$, we have $C_1Q(|g| - 1) \geq (C_1Q/2)|g|$, as required.

Case (2): $|gx_1| = |g|$.

By Proposition 4.8 all assertions follow.

Case (3): $|gx_1| = |g| + 1$.

By Proposition 4.8 there exists a path of length at most $D(|g| + 1)$ which avoids a $\delta(|g| + 1)$-neighborhood of identity and which has initial vertex $gx_1$ and terminal vertex $v(|g| + 1)$. Since $|g| \geq 2$, we have $D(|g| + 1) \leq 2D|g|$. Thus, we construct a path connecting $v(|g|)$ and $v(|g| + 1)$. Let

$$p(|g|) \equiv x_0^{Q(|g|)-1}x_1x_0^{Q}x_1^{-1}x_0^{-Q(|g|+1)+1}.$$  

It is clear that $p(|g|)$ labels a path from $v(|g|)$ to $v(|g| + 1)$ and the length of $p(|g|)$ is at most $2Q(|g| + 1)$. Since $|g| \geq 2$, we have $2Q(|g| + 1) \leq 4Q|g|$. By the almost same argument as case (1), we have

$$|v(|g|)p'| \geq C_1N(v(|g|)p') \geq C_1N(x_0^i) > C_1Q|g|,$$

for any prefix $p' \leq p(|g|)$ and corresponding $i \geq Q|g|$, as required. \hfill \Box

References

1. J. Aroca and M. Cumplido, A new family of infinitely braided Thompson’s groups, arXiv preprint arXiv:2005.09593 (2020).
2. T. Brady, J. Burillo, S. Cleary, and M. Stein, Pure braid subgroups of braided Thompson’s groups, Publicacions matematiques 52 (2008), no. 1, 57–89.
3. M. G Brin, The algebra of strand splitting II: A presentation for the braid group on one strand, International Journal of Algebra and Computation 16 (2006), no. 01, 203–219.
4. ______, The algebra of strand splitting. I. A braided version of Thompson’s group V, Journal of Group Theory 10 (2007), no. 6, 757–788.
5. K. S Brown, Finiteness properties of groups, Journal of Pure and Applied Algebra 44 (1987), no. 1-3, 45–75.
6. K. S Brown and R. Geoghegan, An infinite-dimensional torsion-free $FP_\infty$ group, Inventiones mathematicae 77 (1984), no. 2, 367–381.
7. J. Burillo and S. Cleary, Metric properties of braided Thompson’s groups, Indiana University mathematics journal 58 (2009), no. 2, 605–615.
[8] J. Burillo, S. Cleary, and M. Stein, *Metrics and embeddings of generalizations of Thompson’s group F*, Transactions of the American Mathematical Society **353** (2001), no. 4, 1677–1689.

[9] K.-U. Bux, M. G. Fluch, M. Marschler, S. Witzel, and M. C. Zaremsky, *The braided Thompson’s groups are of type $F_\infty$*, Journal für die reine und angewandte Mathematik **2016** (2016), no. 718, 59–101.

[10] J. W Cannon, W. J Floyd, and W. R Parry, *Introductory notes on Richard Thompson’s groups*, Enseignement Mathématique **42** (1996), 215–256.

[11] P. Dehornoy, *The group of parenthesized braids*, Advances in Mathematics **205** (2006), no. 2, 354–409.

[12] C. Druţu, S. Mozes, and M. Sapir, *Divergence in lattices in semisimple Lie groups and graphs of groups*, Transactions of the American Mathematical Society **362** (2010), no. 5, 2451–2505.

[13] ______., *Corrigendum to “Divergence in lattices in semisimple Lie groups and graphs of groups”*, Transactions of the American Mathematical Society **370** (2018), no. 1, 749–754.

[14] C. Druţu, M. Sapir, et al., *Tree-graded spaces and asymptotic cones of groups*, Topology **44** (2005), no. 5, 959–1058.

[15] S. M Gersten, *Quadratic divergence of geodesics in CAT(0) spaces*, Geometric & Functional Analysis GAFA **4** (1994), no. 1, 37–51.

[16] G. Golan and M. Sapir, *Divergence functions of Thompson groups*, Geometriae Dedicata **201** (2019), no. 1, 227–242.

[17] C. Kassel and V. Turaev, *Braid groups*, Vol. 247, Springer Science & Business Media, 2008.

[18] M. C. Zaremsky, *Geometric structures related to the braided Thompson groups*, arXiv preprint arXiv:1803.02717 (2018).

Department of Mathematical Sciences, Tokyo Metropolitan University, Minami-osawa Hachioji, Tokyo, 192-0397, Japan

E-mail address: kodama-yuya@ed.tmu.ac.jp