Correlation functions with multiple boundaries in JT gravity and resolvents

Yusuke Kimura

1 KEK Theory Center, Institute of Particle and Nuclear Studies, KEK, 1-1 Oho, Tsukuba, Ibaraki 305-0801, Japan
E-mail: kimurayu@post.kek.jp

Abstract

We compute the higher-genus contributions to the correlation functions in Jackiw–Teitelboim gravity, by utilizing the large-$g$ asymptotics of the Weil–Petersson volumes as functions of genus $g$ and the number of boundaries, $n$, obtained in a previous study. We also compute the higher-genus contributions to resolvents on the matrix-integral side for any $n$ number of boundaries of the Riemann surfaces. Further, we compute the intersection numbers on the moduli space of the Riemann surfaces of large genera with any $n$ number of boundaries, focusing on the cases with two or more boundaries ($n \geq 2$).
Contents

1 Introduction 1

2 Summary 4

3 Correlation functions with multiple boundaries in JT gravity and resolvent in the dual matrix integral 5
   3.1 Correlation functions with \( n \) number of boundaries in JT gravity . . . . . . . . . . 5
   3.2 Resolvents on the matrix-integral side . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 8

4 Intersection numbers with multiple boundaries 9

5 Concluding remarks and unresolved problems 14

1 Introduction

Jackiw–Teitelboim (JT) gravity \cite{1,2} is a two-dimensional (2D) topological quantum gravitational theory that is considered on Riemann surfaces, possibly with boundaries. When the Riemann surfaces have asymptotic boundaries, the quantum fluctuations or “wiggles” along the asymptotic boundaries are controlled by the Schwarzian theory \cite{3,4,5}. As the one-dimensional Schwarzian theory describes the low-energy limit of the Sachdev–Ye–Kitaev (SYK) model \cite{6,3,5}, it relates JT gravity to SYK models. Moreover, JT gravity has been studied in the context of AdS/CFT correspondence \cite{7}.

Recently, the duality of JT gravity and the matrix integral has been discussed in \cite{8}. Mirzakhani’s recursion relation \cite{9} that computes the volume of the moduli of hyperbolic Riemann surfaces on the JT gravity side corresponds to the Eynard–Orantin topological recursion \cite{11} on the matrix-integral side; this correspondence is important in the discussion of the duality of the two theories. Recent studies of JT gravity include \cite{13,8,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29}.

The path integral in the JT gravity on Riemann surfaces with asymptotic boundaries is obtained by computing the path integral over the wiggles along the asymptotic boundaries of the surfaces \cite{30,31,32}, together with the path integral over the moduli of the Riemann surfaces with the geodesic boundaries and the path integral over the “trumpets” connecting an asymptotic boundary (with which a boundary wiggle is associated) and a geodesic boundary \cite{8}. The volume of the moduli of hyperbolic Riemann surfaces with geodesic boundaries is known as the “Weil–Petersson volume.”

The path integral over a “trumpet” connecting an asymptotic boundary and a geodesic boundary of a Riemann surface was computed in \cite{8}. Thus, the correlation function in JT gravity is

\[ \text{The genus expansion of a Hermitian-matrix integral } \text{is obtained via the topological recursion of Eynard and Orantin.} \]
obtained when the Weil–Petersson volumes are successfully computed. Although the computation of the Weil–Petersson volumes for any genus \((g)\) and for any number \((n)\) of the geodesic boundaries is possible in principle, it is not simple, as we discuss briefly.

One of the goals in this study is to compute the higher-genus correlation functions with any number of the boundaries in JT gravity. Utilizing this result, we determine the higher-genus contributions to the resolvents on the matrix-integral side. We also compute the intersection numbers on the moduli of the hyperbolic Riemann surfaces of the large genera \((g)\), with any number of boundaries.

The computation results of the resolvents in this study might be a useful tool for confirming the duality of JT gravity and the matrix integral as discussed in [8].

The genus expansion of the correlation function in JT gravity is as follows [8]:

\[
<Z(\beta_1)\ldots Z(\beta_n)>_c \approx \sum_{g=0}^{\infty} Z_{g,n}(\beta_1,\ldots,\beta_n) \cdot e^{(-2g+2-n)S_0}.
\] (1)

Here, \(Z_{g,n}(\beta_1,\ldots,\beta_n)\) denotes the JT path integral for the Riemann surfaces with a topology of genus \(g\) with \(n\) asymptotic boundaries. The genus-\(g\) partition function with \(n\) boundaries, \(Z_{g,n}(\beta_1,\ldots,\beta_n)\), can be expressed as follows [8]:

\[
Z_{g,n}(\beta_1,\ldots,\beta_n) = \prod_{i=1}^{n} \int_{0}^{\infty} b_i \, db_i \, V_{g,n}(b_1,\ldots,b_n) \prod_{j=1}^{n} Z_{\text{trumpet}}^{\text{Sch}}(\beta_j,b_j),
\] (2)

where \(Z_{\text{trumpet}}^{\text{Sch}}\) denotes the contribution from the path integral over the trumpet connecting an asymptotic boundary and a geodesic boundary. \(V_{g,n}(b_1,\ldots,b_n)\) denotes the Weil–Petersson volume of the moduli of the Riemann surfaces of genus \(g\) with \(n\) geodesic boundaries. \(b_1,\ldots,b_n\) denote the lengths of the geodesic boundaries. The contribution from the trumpet, \(Z_{\text{trumpet}}^{\text{Sch}}\), was computed in [8]. As stated previously, one can compute the correlation function with \(n\) boundaries by evaluating the values of the Weil–Petersson volumes, \(V_{g,n}(b_1,\ldots,b_n)\). However, computing the higher-genus contributions of the correlation function by directly evaluating the Weil–Petersson volumes for large genera \(g\) is difficult.

Mirzakhani discovered a method to compute the Weil–Petersson volumes, \(V_{g,n}\), for any genus, \(g\), recursively [9]. However, as genus \(g\) becomes large, performing the actual computation becomes difficult. Although the evaluation of the Weil–Petersson volume is, in principle, possible for any genus \(g\) by utilizing Mirzakhani’s recursion formula [9] iteratively, an expression of the Weil–Petersson volume, \(V_{g,n}(b_1,\ldots,b_n)\), cannot be obtained as a function of genus \(g\) and the \(n\) number of the geodesic boundaries. Obtaining the expression of the Weil–Petersson volume as a function of \(g\) and \(n\) is generally not considered a straightforward problem [3].

\textsuperscript{2}There is a normalization constant, \(\alpha\), in the genus-\(g\) partition functions [8]. We adopt the convention to set \(\alpha = 1\) in this study.

\textsuperscript{3}Several efforts have been made to obtain the asymptotic expressions [33, 34, 22] of the Weil–Petersson volumes, \(V_{g,n}\), as functions of \(g\) when \(g\) is large.

---

2
The asymptotic expressions for the Weil–Petersson volumes, \( V_{g,n}(b_1, \ldots, b_n) \), as functions of \( g \) and \( n \) were obtained when \( g \) was large in [22] by utilizing partial differential equations [35, 36] that hold for the moduli of the hyperbolic Riemann surfaces. The use of the expressions of the Weil–Petersson volumes in [22] enables the direct evaluation of the higher-genus contributions to the correlation functions in JT gravity, avoiding the iterative steps to recursively compute genus-\( g \) partition functions, which becomes difficult once \( g \) becomes large.

We compute the higher-genus contributions to the correlation function in JT gravity with any number of the asymptotic boundaries by utilizing the volume evaluated in [22]. We also evaluate the intersection numbers of the line bundles on the moduli of the Riemann surfaces of a large genus. The results obtained in this study determine the expressions of the correlation functions and the intersection numbers with large genus as explicit functions of genus \( g \) of the surface and the number of boundaries, \( n \); this may provide a clear outlook for studying their behavior.

The intersection numbers of the line bundles on the moduli of the Riemann surfaces were discussed in the context of topological gravity in [37, 38]. The Witten conjecture [37] was used for the equivalence of two models of 2D quantum gravitational theory, and the statement of the conjecture involved the intersection numbers of the line bundles on the moduli of the Riemann surfaces [37]. A proof of the Witten conjecture was given in [40]. Proofs of this conjecture can also be found in [41, 42, 43].

We also estimate the higher-genus contributions to the resolvent on the matrix-integral side utilizing large-\( g \) asymptotics of the Weil–Petersson volumes in [22], as stated earlier.

Additional theme of this study is to analyze the mathematical structure of the asymptotic behaviors of the intersection numbers of the line bundles on the moduli space of the hyperbolic Riemann surfaces with boundaries. The partition function of the generating function of intersection indices of line bundles on the moduli of hyperbolic Riemann surfaces satisfy the Virasoro constraints [37, 40, 42], and this is essentially related to the Witten conjecture. We deduce large-\( g \) asymptotic intersection numbers from the large-\( g \) asymptotics of the Weil–Petersson volumes. It seems natural to expect that the partition function of the generating function of such intersection numbers also satisfy constraints analogous to the Virasoro constraints.

The intersection numbers with one boundary were explicitly computed when the genus is large in [17], and correlation functions with two boundaries were explicitly computed in the low-temperature limit in [20]. The authors in [17, 20] used the Korteweg–de Vries (KdV) hierarchy approach to calculate these; this is considerably different from the approach used in this study.

The remainder of this note is structured as follows. We outline the main results of this work in section 2. In section 3, we compute the higher-genus contributions to the correlation function with any number of the boundaries in JT gravity and discuss the physical consequences. Furthermore, we discuss an application to higher-order spectral form factor in section 3.1. Studies of the spectral form factor in JT gravity can be found, e.g., in [41, 45, 8, 46, 47]. In section 3.2, we also evaluate the resolvent on the matrix-integral side, which may provide a tool for confirming the duality [8] of JT gravity and the matrix integral.

---

4The Virasoro conjecture proposed by the authors in [39] yields a generalization of the Witten conjecture.
In section 4, we evaluate the intersection numbers of the moduli of the Riemann surfaces of large genera with any number of boundaries larger than or equal to two. We also demonstrate that the intersection indices obtained from the deduced large-$g$ asymptotic intersection numbers satisfy variants of the string and dilaton equations. In section 5, we present our concluding remarks and the unresolved problems.

2 Summary

According to the property of JT gravity, the correlation functions can be computed from the Weil–Petersson volume [8]. Based on this property, we computed the higher-genus contributions to the correlation functions with any number of boundaries in section 3.1. These are obtained as functions of genus $g$, the number of the boundaries $n$ of the Riemann surface, and the lengths of the asymptotic boundaries $\beta_i$, $i = 1, \ldots, n$.

In the region where the genus $g$ of Riemann surface is large, when the parameters that parameterize the geodesic boundary lengths are denoted as $b_i$, $i = 1, \ldots, n$, then the large-$g$ asymptotic Weil–Petersson volume of the moduli space of the hyperbolic Riemann surfaces is given as product of hyperbolic functions, $\frac{\sinh\left(\frac{\beta_i}{2}\right)}{b_i}$ [22].

Therefore, higher-genus contributions to the correlation functions are given as a sum of products of functions in the variable of $\beta_i$ over the indices $i$ times a coefficient that depends on genus $g$ and the number of the boundaries $n$, but does not depend on $\beta_i$. The computational results of the higher-genus contributions to the correlation functions are presented in Section 3.1. Furthermore, we discussed an application to a higher-order spectral form factor.

Higher-genus contributions to resolvents on the matrix-integral side are computed in Section 3.2.

In this study, we focused on the region of the large genera $g$ of the Riemann surfaces to obtain computational results that are applicable to any number of boundaries. The focus on such regions provided clear arguments. In section 4, we determined that large-$g$ asymptotic intersection numbers exhibited a rich mathematical structure.

We computed the intersection numbers of line bundles associated with the cotangent spaces to marked points of the Riemann surface and the Weil–Petersson symplectic form $\omega$ for the Riemann surfaces of large genera by using large-genus asymptotics of the Weil–Petersson volumes deduced in [22]. These intersection numbers may provide useful information that can be compared with previously reported results.

The large-$g$ intersection numbers are functions of genus $g$ and the number of boundaries $n$. Generally, obtaining such a presentation of intersection numbers of line bundles on the moduli space of Riemann surfaces for any number of boundaries is difficult. Therefore, we restricted the discussion to the hyperbolic Riemann surfaces of large genera. The large-$g$ asymptotic intersection numbers are given by equation (21).

Furthermore, we observe that the large-$g$ asymptotic intersection indices (that we compute in Section 4 as (24)) deduced from the large-$g$ asymptotic intersection numbers satisfy variants of dilaton and string equations in the large-$g$ limit, given as (27) and (32). This implies that the
asymptotic intersection indices that we obtained exhibit, to some degree, a rich mathematical structure.

By proving the Witten conjecture, Mirzakhani [42] expressed the coefficients of the Weil–Petersson volume in the variables $b_i$’s as intersections numbers of line bundles on the moduli space of the hyperbolic Riemann surfaces. The partition function, $\exp(F)$, of the generating function, $F$, of the intersection indices satisfies the Virasoro constraints, $L_n\exp(F) = 0$ [37 40 42].

String and dilaton equations precisely correspond to the constraints $L_{-1}\exp(F) = 0$ and $L_0\exp(F) = 0$ [42]. These correspondences imply that because the asymptotic intersection indices deduced in section 4 satisfy variants of string and dilaton equations, the partition function $\exp(\tilde{F})$ of the generating function $\tilde{F}$ of the asymptotic intersection indices deduced in this study satisfy a variant of the two constraints. Thus, we postulate that the partition function $\exp(\tilde{F})$ of the generating function of the asymptotic intersection indices satisfies a variant of the Virasoro constraints. This seems to suggest that the symmetry of the partition function of the generating function is deformed, as a result of restriction to the large-genus asymptotic region.

3 Correlation functions with multiple boundaries in JT gravity and resolvent in the dual matrix integral

3.1 Correlation functions with $n$ number of boundaries in JT gravity

We compute the correlation functions in JT gravity with any number of boundaries. To facilitate the analysis, we limit the discussion to the higher-genus contributions to the correlation functions. An advantage of imposing the constraint is as follows: The correlation functions in JT gravity can be computed by evaluating the Weil–Petersson volume, which is the volume of the moduli of the hyperbolic Riemann surfaces, as discussed in [3]. Mirzakhani established a method to recursively compute the Weil–Petersson volumes for any genus, $g$, of the Riemann surfaces [9]; however, the difficulty of the computation using Mirzakhani’s recursive method [9] increases as genus $g$ becomes large. Thus, precisely expressing the Weil–Petersson volume, $V_{g,n}$, as a function of genus $g$ and the number asymptotic boundaries ($n$) is presently considerably difficult. However, when the discussion is limited to the region where genus $g$ of the Riemann surface is large, the Weil–Petersson volume, $V_{g,n}$, can be expressed as a function of $g$ and $n$, as obtained in [22]. The result in [22] can be utilized to compute the higher-genus contributions to the correlation function in JT gravity for any number of asymptotic boundaries.

Under conditions $g >> b_i$, $i = 1, \ldots, n$, and $g >> 1$, where $b_i$’s denote the lengths of the geodesic boundaries of a Riemann surface, the Weil–Petersson volume, $V_{g,n}(b_1, \ldots, b_n)$, of the moduli of the Riemann surface with genus $g$ and $n$ geodesic boundaries was evaluated to have the
Following expression to the leading order \[22\]:

\[
V_{g,n}(b_1, \ldots, b_n) \sim \sqrt{\frac{2}{\pi}} 2^n (4\pi^2)^{2g+n-3} \Gamma(2g+n-\frac{5}{2}) \cdot \prod_{i=1}^{n} \frac{\sinh(b_i^2)}{b_i^2}.
\]

(3)

Applying expression (3), we compute the higher-genus correlation functions in JT gravity with any number of boundaries.

The connected correlation function of JT gravity on a Riemann surface of genus \(g\) with \(n\) asymptotic boundaries has genus expansion \[8\], as noted in the introduction. As previously mentioned, \(Z_{\text{trumpet}}^{\text{Sch}}(\beta, b)\) denotes the contribution from the path integral over the “trumpet” connecting an asymptotic wiggly boundary and a geodesic boundary of length \(b\). The contribution, \(Z_{\text{trumpet}}^{\text{Sch}}(\beta, b)\), was computed in \[8\] as

\[
\sqrt{\frac{\gamma^2}{2\pi \beta}} e^{-\frac{\gamma b^2}{2 \beta}}.
\]

First, we calculate the higher-genus correlation for \(n = 2\) to illustrate the proposed method for computing the higher-genus correlations in JT gravity for \(n\) number of boundaries.

For \(n = 2\), the genus expansion of the correlation function becomes as follows \[8\] :

\[
< Z(\beta_1)Z(\beta_2) > \simeq \sum_{g=0}^{\infty} Z_{g,2}(\beta_1, \beta_2) \cdot e^{-2gS_0},
\]

(4)

Where the genus-\(g\) path integral, \(Z_{g,2}\), is given as \[8\]

\[
Z_{g,2}(\beta_1, \beta_2) = \int_0^{\infty} b_1 db_1 \int_0^{\infty} b_2 db_2 V_{g,2}(b_1, b_2) Z_{\text{trumpet}}^{\text{Sch}}(\beta_1, b_1) Z_{\text{trumpet}}^{\text{Sch}}(\beta_2, b_2).
\]

(5)

We apply the expression for the Weil–Petersson volume, \(V_{g,2}(b_1, b_2)\), deduced in \[22\], which holds when genus \(g\) is large to path integral \[6\] to obtain the higher-genus contributions to the correlation function with two boundaries in JT gravity.

The Weil–Petersson volume, \(V_{g,2}(b_1, b_2)\), is given as follows under conditions \(g \gg b_1, b_2\) and \(g \gg 1 \) \[22\] :

\[
V_{g,2}(b_1, b_2) \sim \sqrt{\frac{2}{\pi}} 4(4\pi^2)^{g-1} \frac{\sinh(b_1^2/2) \sinh(b_2^2/2)}{b_1 b_2}.
\]

(6)

We substitute expression (6) into (5) to compute the genus-\(g\) partition function, \(Z_{g,2}\), for a

\[5\] For \(n = 1\), the expression of the Weil–Petersson volume, \(V_{g,1}(b)\), was obtained in \[8\] under conditions \(g \gg b\) and \(g \gg 1\).
large genus, \( g \) \((g >> 1)\), as follows\(^6\):

\[
Z_{g,2}(\beta_1, \beta_2) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2\pi}} \left(\int_0^\infty db_1 \sinh\left(\frac{b_1}{2}\right) e^{-\frac{\gamma b_1^2}{2\pi\beta_1}}\right) \left(\int_0^\infty db_2 \sinh\left(\frac{b_2}{2}\right) e^{-\frac{\gamma b_2^2}{2\pi\beta_2}}\right)
\]

\[
= \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2\pi}} \left(\int_0^\infty db_1 \sinh\left(\frac{b_1}{2}\right) e^{-\frac{\gamma b_1^2}{2\pi\beta_1}}\right) \left(\int_0^\infty db_2 \sinh\left(\frac{b_2}{2}\right) e^{-\frac{\gamma b_2^2}{2\pi\beta_2}}\right)
\]

\[
= \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2\pi}} \left(\int_0^\infty db_1 \sinh\left(\frac{b_1}{2}\right) e^{-\frac{\gamma b_1^2}{2\pi\beta_1}}\right) \left(\int_0^\infty db_2 \sinh\left(\frac{b_2}{2}\right) e^{-\frac{\gamma b_2^2}{2\pi\beta_2}}\right)
\]

because \( \int_0^\infty db \sinh\left(\frac{b}{2}\right) e^{-\frac{\gamma b^2}{2\pi}} = \sqrt{\frac{\pi}{2\gamma}} e^{-\frac{\gamma b^2}{2\pi}} \).

Thus, we deduce that the higher-genus contributions to the correlation function with two boundaries can be expressed as follows:

\[
\sqrt{\frac{2}{\pi}} e^{\frac{\beta_1 + \beta_2}{8\gamma}} \text{erf}\left(\sqrt{\frac{\beta_1}{8\gamma}}\right) \text{erf}\left(\sqrt{\frac{\beta_2}{8\gamma}}\right) \sum_{g>>1} \left(4\pi^2\right)^{2g-1} \Gamma(2g - \frac{1}{2}) \cdot e^{-2gS_0}.
\]  

(8)

The computation of the genus-\( g \) partition function with general \( n \) boundaries is analogous to the computation of the genus-\( g \) partition function with two boundaries that we discussed earlier. Utilizing expression (3), we find that the genus-\( g \) partition function with \( n \) boundaries can be derived as follows when \( g >> 1 \):

\[
Z_{g,n}(\beta_1, \ldots, \beta_n) = \sqrt{\frac{2}{\pi}} 2^n \left(4\pi^2\right)^{2g+n-3} \Gamma(2g + n - \frac{5}{2}) \prod_{i=1}^n \frac{\gamma}{2\pi\beta_i} \prod_{j=1}^n \int_0^\infty db_j \sinh\left(\frac{b_j}{2}\right) e^{-\frac{\gamma b_j^2}{2\pi\beta_j}}
\]

\[
= \sqrt{\frac{2}{\pi}} \left(4\pi^2\right)^{2g+n-3} \Gamma(2g + n - \frac{5}{2}) e^{\sum_{i=1}^n \frac{\beta_i}{8\gamma}} \prod_{j=1}^n \text{erf}\left(\sqrt{\frac{\beta_j}{8\gamma}}\right).
\]  

(9)

Similar to the case with two boundaries, the resulting function is the product of a coefficient that depends on genus \( g \), an exponential function, and the error functions.

We deduce that the higher-genus contributions to the correlation function with \( n \) boundaries are given as follows:

\[
\sqrt{\frac{2}{\pi}} e^{\sum_{i=1}^n \frac{\beta_i}{8\gamma}} \prod_{j=1}^n \text{erf}\left(\sqrt{\frac{\beta_j}{8\gamma}}\right) \sum_{g>>1} \left(4\pi^2\right)^{2g+n-3} \Gamma(2g + n - \frac{5}{2}) \cdot e^{(-2g+2-n)S_0}.
\]  

(10)

\(^6\)Owing to the presence of factors \( Z_{\text{Sch}}^{\text{trumpet}}(\beta_1, b_1) \) and \( Z_{\text{Sch}}^{\text{trumpet}}(\beta_2, b_2) \) in the integrand of \(\ref{Z_{g,n}}\), we expect that the integral over the region where \( g >> b_1, b_2 \) makes the dominant contribution. Given this observation, we expect that the use of expression \(\ref{Z_{g,2}}\) for \( V_{g,2}(b_1, b_2) \) over the entire region in integral \(\ref{Z_{g,2}}\) approximates the exact result with high precision.
From the computed correlation functions, we learn that the nonperturbative correction takes the form of $e^{-\frac{3\beta}{4\pi^2}}$. This agrees with the results of the correction and the effect of the ZZ brane [48], as discussed in [17].

The proposed method applies to cases with two or more boundaries, i.e. $n \geq 2$. The higher-genus contributions to the correlation function with one boundary can also be deduced in a similar manner using the large-$g$ asymptotic of the Weil–Petersson volume with one boundary, $V_{g,1}(b)$, as deduced in [8].

The lower-genus contributions to the correlation functions can be obtained from the Weil–Petersson volumes of the moduli of the Riemann surfaces of lower genera, which can be computed by directly applying Mirzakhani’s recursion relation. For higher-genus contributions, the direct computation by applying Mirzakhani’s recursion relation is challenging; therefore, the proposed method for computing the higher-genus contributions is important.

An application to the higher-order spectral form factor is also presented in this section. Substituting $\beta \pm iT$ for $\beta$ in (10), we obtained the higher-genus contributions to the higher-order spectral form factor $< \prod_n Z(\beta + iT)Z(\beta - iT) >_c$ as

$$\sqrt{\frac{2}{\pi}} e^{\frac{2n}{\sqrt{2\gamma}}} (\text{erf}(\sqrt{\frac{\beta + iT}{8\gamma}}))^n (\text{erf}(\sqrt{\frac{\beta - iT}{8\gamma}}))^n \cdot \sum_{g \gg 1} (4\pi^2)^{2g+2n-3} \Gamma(2g + 2n - \frac{5}{2}) \cdot e^{-(2g+2-2n)S_0}. \quad (11)$$

The leading-order term in the genus expansion of the higher-order spectral form factor $< \prod_n Z(\beta + iT)Z(\beta - iT) >_c$ was computed in [47], and the result in [47] revealed that the leading-order term is proportional to $(\beta^2 + T^2)^{\frac{3}{2}}$. From the obtained higher-genus contribution (11), we determine that this property is persistent in the higher-genus terms when $\beta$ is small because $\text{erf}(z) \sim \frac{2}{\sqrt{\pi}} z$ when $z$ is small, and $e^{\frac{2n}{\sqrt{2\gamma}}} (\text{erf}(\sqrt{\frac{\beta + iT}{8\gamma}}))^n (\text{erf}(\sqrt{\frac{\beta - iT}{8\gamma}}))^n$ in (11) is approximated by $\frac{1}{(8\gamma)^n \pi^n} (\beta^2 + T^2)^{\frac{3}{2}}$ when $\beta$ is small. Thus, the higher-genus terms are still proportional to $(\beta^2 + T^2)^{\frac{3}{2}}$ when $\beta$ is small.

As $\beta$ grows large, higher-order terms in $\beta$ in the exponential function and the error functions in (11) result in large contributions, and the value of the higher-genus contribution (11) to the higher-order spectral form factor $< \prod_n Z(\beta + iT)Z(\beta - iT) >_c$ is not proportional to $(\beta^2 + T^2)^{\frac{3}{2}}$.

### 3.2 Resolvents on the matrix-integral side

The correlation functions of the resolvents in the double-scaled matrix integral as dual of JT gravity have the following genus expansion [8]:

$$< R(E_1) \ldots R(E_n) >_c = \sum_{g=0}^{\infty} R_{g,n}(E_1, \ldots, E_n) \cdot e^{-(2g-n+2)S_0}. \quad (12)$$

Genus expansion (12) defines the multi-resolvent correlators, $R_{g,n}$, as discussed in [8]. We compute the multi-resolvent correlators, $R_{g,n}$, when genus $g$ is large. This yields the large-genus contributions to the correlation functions of the resolvents, $< R(E_1) \ldots R(E_n) >_c$. 


Functions $W_{g,n}$ are defined as follows, as discussed in [49, 8]:

$$W_{g,n}(z_1, \ldots, z_n) = (-2)^n \cdot \prod_{i=1}^{n} z_i \cdot R_{g,n}(-z_1^2, \ldots, -z_n^2). \quad (13)$$

As proved in [10], function $W_{g,n}$ is given in terms of the Weil–Petersson volume, $V_{g,n}$, as follows:

$$W_{g,n}(z_1, \ldots, z_n) = \prod_{i=1}^{n} \int_0^\infty b_i \, db_i \, e^{-b_i z_i} \cdot V_{g,n}(b_1, \ldots, b_n). \quad (14)$$

For the region where $g \gg 1$, using the equation (14), we can compute $W_{g,n}$ by utilizing the large-$g$ asymptotic of the Weil–Petersson volume, $V_{g,n}$, deduced in [22]. Based on the definition of $W_{g,n}$ (13), the multi-resolvent correlator, $R_{g,n}$, is deduced from the computed $W_{g,n}$.

Applying the large-$g$ asymptotic Weil–Petersson volume (3) into (14), we obtain $W_{g,n}$ as follows:

$$W_{g,n}(z_1, \ldots, z_n) \sim \sqrt{2 \pi} \frac{2^n (4 \pi^2)^{2g+n-3} \Gamma(2g+n-\frac{5}{2})}{\prod_{i=1}^{n} \int_0^\infty db_i \sinh(b_i) e^{-b_i z_i}} \quad (15)$$

According to this result, in region $g \gg 1$, the multi-resolvent correlator, $R_{g,n}$, is determined as follows:

$$R_{g,n}(E_1, \ldots, E_n) \sim 2^n \sqrt{\frac{2}{\pi}} \frac{(4 \pi^2)^{2g+n-3} \Gamma(2g+n-\frac{5}{2})}{\prod_{i=1}^{n} \frac{1}{(4E_i + 1)}} \frac{1}{\sqrt{(-1)^n \prod_{i=1}^{n} E_i}}. \quad (16)$$

Therefore, we deduce that the higher-genus contributions to the correlation functions of the resolvents of the double-scaled matrix integral with $n$ boundaries can be given as follows:

$$\sum_{g \gg 1} 2^n \sqrt{\frac{2}{\pi}} \frac{(4 \pi^2)^{2g+n-3} \Gamma(2g+n-\frac{5}{2})}{\prod_{i=1}^{n} \frac{1}{(4E_i + 1)}} \frac{1}{\sqrt{(-1)^n \prod_{i=1}^{n} E_i}} \cdot e^{(-2g-n+2)S_0}. \quad (17)$$

Similar to the statement presented in section 3.1, the proposed method applies to cases with two or more boundaries. Using the large-$g$ asymptotic of the Weil–Petersson volume with one boundary, $V_{g,1}(b)$, as deduced in [8], we can obtain the higher-genus contributions to the resolvent with one boundary.

### 4 Intersection numbers with multiple boundaries

The 2D topological gravity is an intersection theory of line bundles on the moduli space, and the correlation functions are determined by the intersection theory. The Weil–Petersson volume contains the information of the correlation functions [38], and the intersection numbers of the
cohomology classes associated with the first Chern classes of line bundles and the Weil–Petersson symplectic form can also be extracted from the Weil–Petersson volume.

We determine the intersection numbers of the cohomology classes for the moduli of the Riemann surfaces with boundaries, under conditions we compute the intersection numbers of the cohomology classes associated with the line bundles based on the large moduli of the Riemann surfaces with boundaries, under conditions with large genera with

As discussed in [38], the following equality holds owing to a result in [42]:

\[ V_{g,n}(b_1, \ldots, b_n) = \frac{\exp(2\pi^2 \kappa_1 + \frac{1}{2} \sum_{i=1}^{n} b_i^2 \psi_i)}{\prod_{i=1}^{g+n-3} (3g-3+n-\sum_{i=1}^{n} m_i)!} \prod_{i=1}^{n} \frac{b_i^{2m_i}}{2^m (2^m - 1)^m_{1}} < \kappa_1^{3g-3+n-\sum_{i=1}^{n} m_i} \prod_{i=1}^{n} \psi_i^{m_i}. \]

\[ \prod_{i=1}^{n} \psi_i^{m_i} = \psi \prod_{i=1}^{n} \mathcal{L}_i \]

We use \( \mathcal{L}_i \) to represent the cotangent spaces to the marked points in a closed Reimann surface of genus \( g \). We use \( \psi_i \) to denote the first Chern class of the cotangent space, \( \mathcal{L}_i \), namely \( \psi_i = c_1(\mathcal{L}_i) \). The \( m_i \)'s are non-negative integers that satisfy condition \( 3g-3+n \geq \sum_{i=1}^{n} m_i \geq 0 \). Utilizing equality (18) and based on the large \( g \) asymptotics of the Weil–Petersson volumes deduced in [22], we compute the intersection numbers of the cohomology classes associated with the line bundles on the moduli of the Riemann surfaces with boundaries, under conditions \( g >> b_i, i = 1, \ldots, n \), and \( g >> 1 \). This is the capacity to which our argument applies in this section.

Applying expression (3) for the large \( g \) asymptotic of the Weil–Petersson volume obtained in [22] into equality (18), we obtain the following relation:

\[ \sqrt{\frac{2}{\pi}} 2^n (4\pi^2)^{g+n-3} \Gamma(2g+n-\frac{5}{2}) \prod_{i=1}^{n} \frac{\sinh(b_i^2)}{b_i} \sim (19) \]

\[ \sum_{3g-3+n \geq \sum_{i=1}^{n} m_i \geq 0} (2\pi)^{3g-3+n-\sum_{i=1}^{n} m_i} (3g-3+n-\sum_{i=1}^{n} m_i)! \prod_{i=1}^{n} m_i! \prod_{i=1}^{n} b_i^{2m_i} < \kappa_1^{3g-3+n-\sum_{i=1}^{n} m_i} \prod_{i=1}^{n} \psi_i^{m_i}. \]

Because \( \frac{\sinh(b_i^2)}{b_i} \) has the following Taylor expansion:

\[ \frac{\sinh(b_i^2)}{b_i} = \sum_{n=0}^{\infty} \frac{b_i^{2n}}{2^{2n+1} \cdot (2n+1)!}, \]

the coefficient of \( \prod_{i=1}^{n} b_i^{2m_i} \) on the LHS of (19) can be determined from (20). By comparing the coefficients of \( \prod_{i=1}^{n} b_i^{2m_i} \) on both sides of (19), we determine the intersection number as follows under \( g >> b_i, i = 1, \ldots, n \), and \( g >> 1 \):

\[ < \kappa_1^{3g-3+n-\sum_{i=1}^{n} m_i} \prod_{i=1}^{n} \psi_i^{m_i} > \sim \sqrt{\frac{2}{\pi}} 2^{g+n-\frac{5}{2}} \prod_{i=1}^{n} b_i^{2m_i} < \kappa_1^{3g-3+n-\sum_{i=1}^{n} m_i} \prod_{i=1}^{n} \psi_i^{m_i} >. \]

\[ \kappa_1 \] denotes the first Miller–Morita–Mumford class, which is cohomologous to the Weil–Petersson symplectic form \( \omega \) times \( \frac{1}{\kappa_1} \), that is, \( \kappa_1 = \omega \cdot [50], [51] \).

Relation (19) holds only under \( g >> b_i, i = 1, \ldots, n \), and \( g >> 1 \). The right-hand side contains finitely many terms in \( b_i \), whereas the left-hand side (LHS) contains infinitely many terms in \( b_i \); the relation does not hold when \( b_i >> g \).
Thus, we obtain the intersection numbers with \( n \) number of boundaries where \( n \geq 2 \) under conditions \( g >> b_i, \ i = 1, \ldots, n, \) and \( g >> 1 \). We expect that this study can provide useful information that is comparable to previously known results.

We would like to investigate the structures of the large-\( g \) asymptotic intersection numbers \([21]\). Specifically, we deduce that the intersection numbers \([21]\) satisfy two equations that are relevant to two-dimensional topological gravity.

Before stating the two equations that the intersection numbers satisfy, we introduce two equations, namely string and dilaton equations. We then demonstrate that the asymptotic intersection numbers \([21]\) satisfy the variants of the two equations. We use \(<\tau_{d_1}, \ldots, \tau_{d_n}>_g\) to represent the intersection number of powers of the line bundles \(\psi_i\) as follows:

\[
<\tau_{d_1}, \ldots, \tau_{d_n}>_g = \int_{\mathcal{M}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}. \tag{22}
\]

\(d_i\) are non-negative integers and satisfy the relation \(\sum_{i=1}^{n} d_i = 3g - 3 + n\).

The intersection indices \(<\tau^1_{d_1} \cdots \tau^n_{d_n}>_g\) satisfy the string and dilaton equations \([37]\) as follows:

\[
<\tau_0, \tau_{d_1}, \ldots, \tau_{d_n}>_g = \sum_{d_i \neq 0} <\tau_{d_1}, \ldots, \tau_{d_i-1}, \ldots>_g, \tag{23}
\]

\[
<\tau_1, \tau_{d_1}, \ldots, \tau_{d_n}>_g = (2g - 2 + n) <\tau_{d_1}, \ldots, \tau_{d_n}>_g.
\]

The first equation in \((23)\) is the string equation, and the second equation in \((23)\) is the dilaton equation.

Because the large-\( g \) asymptotic intersection numbers \([21]\) are obtained as approximate values that are valid in the region with the large genus \( g \), we do not expect that the intersection indices obtained from the asymptotic expressions \([21]\) exactly satisfy the two equations \((23)\). We compared the ratio of the intersection indices \(<\tau_1, \tau_{d_1}, \ldots, \tau_{d_n}>_g^{\text{asymp.}}\) to \((2g - 2 + n) <\tau_{d_1}, \ldots, \tau_{d_n}>_g^{\text{asymp.}}\) and the ratio \(<\tau_0, \tau_{d_1}, \ldots, \tau_{d_n}>_g^{\text{asymp.}}\) to \(\sum_{d_i \neq 0} <\tau_{d_1}, \ldots, \tau_{d_i-1}, \ldots>_g^{\text{asymp.}}\). We used the large-\( g \) asymptotic formula \([21]\) and determined that the ratios were not equal to 1. We placed \(\text{asymp.}\) in the superscript to emphasize that these values are asymptotic approximations obtained using \([21]\). However, they satisfy the two equations up to finite factors, as demonstrated in this study.

Considering the situation where \(\sum_{i=1}^{n} m_i = 3g - 3 + n\) in the equation \([21]\), we obtain the large-\( g \) asymptotic of the intersection index, \(<\tau_{d_1}, \ldots, \tau_{d_n}>_g^{\text{asymp.}}\), as follows:

\[
<\tau_{d_1}, \ldots, \tau_{d_n}>_g^{\text{asymp.}} = \prod_{i=1}^{n} \psi_i^{d_i} \sim \sqrt{\frac{2}{\pi}} \frac{2^{g+n-3} \pi^{2(2g-3+n)}}{\Gamma(2g + n - \frac{5}{2})} \frac{\prod_{i=1}^{n} d_i!}{\prod_{i=1}^{n}(2d_i + 1)!}. \tag{24}
\]

The special intersection index \(<\tau_{d_1}, \ldots, \tau_{d_n}>_{g>0}^{\text{asymp.}}\), where \(d_{n+1} = 1\), is used to obtain the intersection index \(<\tau_1, \tau_{d_1}, \ldots, \tau_{d_n}>_g\). Therefore, we obtain the intersection index \(<\tau_1, \tau_{d_1}, \ldots, \tau_{d_n}>_g^{\text{asymp.}}\) as follows:

\[
<\tau_1, \tau_{d_1}, \ldots, \tau_{d_n}>_g^{\text{asymp.}} \sim \sqrt{\frac{2}{\pi}} \frac{2^{g+n-2} \pi^{2(2g-2+n)}}{\Gamma(2g + n - \frac{3}{2})} \frac{\prod_{i=1}^{n} d_i!}{3! \prod_{i=1}^{n}(2d_i + 1)!}. \tag{25}
\]
Therefore, we deduce that the ratio $< \tau_1, \tau_{d_1}, \ldots, \tau_{d_n} >_g^{\text{asymp.}}$ to $(2g - 2 + n) < \tau_{d_1}, \ldots, \tau_{d_n} >_g^{\text{asymp.}}$ can be expressed as follows:

$$\frac{< \tau_1, \tau_{d_1}, \ldots, \tau_{d_n} >_g^{\text{asymp.}}}{(2g - 2 + n) < \tau_{d_1}, \ldots, \tau_{d_n} >_g^{\text{asymp.}}} = \frac{2\pi^2}{3!} \cdot \frac{2g + n - \frac{5}{2}}{2g + n - 2}. \quad (26)$$

The computational result [26] revealed that the ratio is not 1; however, the ratio (26) tends toward $\frac{2\pi^2}{3}$ as the genus $g$ tends toward infinity. Therefore, we deduce that the large-$g$ asymptotic intersection indices computed from the asymptotic expression (21) satisfy a variant of the dilaton equation when the genus $g$ is large:

$$< \tau_1, \tau_{d_1}, \ldots, \tau_{d_n} >_g^{\text{asymp.}} = \frac{\pi^2}{3} (2g - 2 + n) < \tau_{d_1}, \ldots, \tau_{d_n} >_g^{\text{asymp.}}. \quad (27)$$

Next, we demonstrate that the large-$g$ asymptotic intersection indices satisfy a variant of the string equation. The LHS term of the string equation, $< \tau_0, \tau_{d_1}, \ldots, \tau_{d_n} >_g$, is a special case of the intersection index $< \tau_{d_1}, \ldots, \tau_{d_n}, \tau_{d_{n+1}} >_g$, where $d_{n+1} = 0$. Because there are $n + 1$ marked points for this equation, we have $\sum_{i=1}^{n+1} d_i = 3g - 3 + n + 1 = 3g - 2 + n$, and for the special situation $d_{n+1} = 0$, we have $\sum_{i=1}^{n} d_i = 3g - 2 + n$. Large-$g$ asymptotic intersection index $< \tau_0, \tau_{d_1}, \ldots, \tau_{d_n} >_g^{\text{asymp.}}$ is then expressed as follows:

$$< \tau_0, \tau_{d_1}, \ldots, \tau_{d_n} >_g^{\text{asymp.}} \sim \sqrt{\frac{2\pi}{2g + n - 2}} \cdot \pi^2 (2g - 2 + n) \cdot \Gamma(2g + n - \frac{3}{2}) \cdot \frac{\prod_{i=1}^{n} d_i !}{(2d_i + 1) !}. \quad (28)$$

For simplicity, we focus on the cases where $d_i, i = 1, \ldots, n$, are nonzero. The large-$g$ asymptotic intersection index $< \tau_{d_1}, \ldots, \tau_{d_{n-1}}, \ldots >_g^{\text{asymp.}}$ is expressed as follows:

$$< \tau_{d_1}, \ldots, \tau_{d_{n-1}}, \ldots >_g^{\text{asymp.}} \sim \sqrt{\frac{2\pi}{2g + n - 3}} \cdot \pi^2 (2g - 3 + n) \cdot \Gamma(2g + n - \frac{5}{2}) \cdot \frac{\prod_{j \neq i} d_j ! \cdot (d_i - 1) !}{\prod_{j \neq i} (2d_j + 1) ! \cdot (2d_i - 1) !}. \quad (29)$$

We then find that the sum $\sum_{d_i \neq 0} < \tau_{d_1}, \ldots, \tau_{d_{n-1}}, \ldots >_g^{\text{asymp.}}$ is expressed as follows:

$$\sum_{d_i \neq 0} < \tau_{d_1}, \ldots, \tau_{d_{n-1}}, \ldots >_g^{\text{asymp.}} \sim \sqrt{\frac{2\pi}{2g + n - 3}} \cdot \pi^2 (2g - 3 + n) \cdot \Gamma(2g + n - \frac{5}{2}) \cdot \prod_{j \neq i} d_j ! \cdot (d_i - 1) ! \cdot \sum_{i=1}^{n} \frac{\prod_{j \neq i} d_j !}{\prod_{j \neq i} (2d_j + 1) ! \cdot (2d_i - 1) !}. \quad (30)$$

Therefore, the ratios $< \tau_0, \tau_{d_1}, \ldots, \tau_{d_n} >_g^{\text{asymp.}}$ to $\sum_{d_i \neq 0} < \tau_{d_1}, \ldots, \tau_{d_{n-1}}, \ldots >_g^{\text{asymp.}}$ can be expressed as follows:

$$\frac{< \tau_0, \tau_{d_1}, \ldots, \tau_{d_n} >_g^{\text{asymp.}}}{\sum_{d_i \neq 0} < \tau_{d_1}, \ldots, \tau_{d_{n-1}}, \ldots >_g^{\text{asymp.}}} = \frac{2\pi^2}{2(6g - 4 + 3n)}. \quad (31)$$
In the limit at which the genus $g$ goes to infinity, the ratio (31) tends toward $\frac{\pi^2}{3}$. Thus, we find that the large-$g$ asymptotic intersection indices deduced from (21) satisfy a variant of the string equation in the large-$g$ limit as follows:

$$<\tau_0, \tau_{d_1}, \ldots, \tau_{d_n}>_{g}^{\text{asympt}} = \frac{\pi^2}{3} \sum_{d_l \neq 0} <\tau_{d_1}, \ldots, \tau_{d_l-1}, \ldots>_{g}^{\text{asympt}}.$$  \hfill (32)

The Witten conjecture [37] is used to determine the equivalence of two versions of 2D quantum gravitational theory. We state the relation of large-$g$ asymptotic intersection numbers (21) that we deduced to the Witten conjecture.

The relation of the large-$g$ asymptotic intersection numbers (21) to the Witten conjecture can be explicitly stated using Mirzakhani’s method to prove the Witten conjecture in [42]. Several proofs of the Witten conjecture have been provided. Kontsevich provided a proof of the Witten conjecture in [40]. Mirzakhani proved the Witten conjecture using a different approach in [42]. The recursion relation of the Weil–Petersson volumes was effectively used in the proof in [42], and as proved in [9], which revealed that the Weil–Petersson volume of the moduli space of the Riemann surface with boundaries can be expressed as a polynomial in the geodesic boundary lengths $b_i$. The main aspect in the proof of the Witten conjecture given in [42] was to express the coefficients of the Weil–Petersson volume as a polynomial in $b_i$’s using the intersection numbers of the line bundles associated with the cotangent spaces to the marked points of the Riemann surface. Using these expressions, Mirzakhani revealed [42] that the partition function $\exp(F)$ of the generating function $F$ of the intersection indices satisfy the Virasoro constraints as follows:

$$L_n \exp(F) = 0.$$ \hfill (33)

The generating function $F$ is defined as follows: one considers the formal sum of the intersection indices, $F_g$:

$$F_g(t_0, t_1, \ldots) = \sum_{\{d_i\}} \prod_{l>0} t_l^{n_l} \cdot \prod <\tau_{d_i}>_{g}.$$ \hfill (34)

The summation is performed over all sequences of non-negative integers, $\{d_i\}$, with finitely many nonzero terms. $n_l$ in the formal sum (34) denotes the number of indices $i$ of $d_i$ such that $d_i = l$. Subsequently, the generating function $F$ is defined as:

$$F = \sum_{g=0}^{\infty} F_g \zeta^{2g-2}.$$ \hfill (35)

As we mentioned previously, by expressing the coefficients of the Weil–Petersson volumes in $b_i$ in terms of the intersection numbers of the line bundles over the moduli space of the hyperbolic Riemann surfaces, Mirzakhani [42] revealed that the partition function $\exp(F)$ satisfies the Virasoro constraints: $L_n \exp(F) = 0$, $n \geq -1$. 


The sequence of the differential operators, $L_{-1}$, $L_0$, $L_1$, $\ldots$, $L_n$, $\ldots$, is defined as follows:

\[
L_{-1} = \partial_t + \sum_{i=1}^{\infty} t_{i+1} \partial_{t_i} + \frac{c^2}{2} t_0^2 \tag{36}
\]

\[
L_0 = \sum_{i=1}^{\infty} \frac{2i+1}{2} \partial_{t_i} + \frac{3}{2} \partial_t + \frac{1}{16} \tag{37}
\]

\[
L_n = \sum_{i=0}^{\infty} \frac{(2n+2i+1)!!}{2^{2n+1}(2i+1)!!} t_i \partial_{t_{n+i}} - \frac{(2n+3)!!}{2^{2n+1}} \partial_{t_{n+1}} + \frac{c^2}{2} \sum_{i=0}^{n-1} \frac{(2i+1)(2n-2i-1)!!}{2^{2n+1}} \partial_{t_i} \partial_{t_{n-i}} \quad (n \geq 1).
\]

In (36), $(2i+1)!! = (2i+1) \cdot (2i-1) \cdot \ldots \cdot 3 \cdot 1$. The operators $L_n$, $n \geq -1$, are referred to as Virasoro operators. They satisfy the following relationship:

\[
[L_n, L_m] = (n - m) L_{n+m}.
\]

The two constraints, $L_{-1} \exp(F) = 0$ and $L_0 \exp(F) = 0$, precisely correspond to the string equation and the dilaton equation \cite{23,42}. We observed that the large-$g$ asymptotic intersection indices \cite{24} obtained from the asymptotic intersection numbers \cite{21} satisfy variants of dilaton and string equations in the large-$g$ limit, as in \cite{27} and \cite{32}. These results imply that the partition function of the generating function of the large-$g$ asymptotic intersection indices satisfies the variants of the two constraints $L_{-1} \exp(F) = 0$ and $L_0 \exp(F) = 0$. We denote the generating function of the large-$g$ asymptotic intersection indices $< \tau_{d_1}, \ldots, \tau_{d_n}>_{g \text{asymp}}$ as $\tilde{F}$. Namely, $\tilde{F}$ is defined as:

\[
\tilde{F} = \sum_{g=0}^{\infty} \tilde{F}_g \zeta^{2g-2};
\]

where

\[
\tilde{F}_g(t_0, t_1, \ldots) = \sum_{\{d_i\}} \prod_{t_i>0} \frac{t_i^{n_i}}{n_i!} < \prod \tau_{d_i}>_{g \text{asymp}}.
\]

(The values of the large-$g$ asymptotic intersection indices $< \prod \tau_{d_i}>_{g \text{asymp}}$ in the sum (39) are given as (24).) Equations (27) and (32) imply that there exist variants of operators $\tilde{L}_0$ and $\tilde{L}_{-1}$, which we denote as $\tilde{L}_0$ and $\tilde{L}_{-1}$, and the partition function $\exp(\tilde{F})$ satisfies the constraints $\tilde{L}_0 \exp(\tilde{F}) = 0$ and $\tilde{L}_{-1} \exp(\tilde{F}) = 0$.

These observations lead us to postulate that the large-$g$ asymptotic partition function $\exp(\tilde{F})$ satisfies a variant of Virasoro constraints in the large-$g$ limit. We expect that the variants of Virasoro operators, which we denote as $\tilde{L}_n$, $n \geq -1$, exist such that the partition function $\exp(\tilde{F})$ satisfies the constraints $\tilde{L}_n \exp(\tilde{F}) = 0$. The construction of all such operators can be undertaken in a future study.

5 Concluding remarks and unresolved problems

In this study, we discussed the physical applications of the large $g$ asymptotics of Weil–Petersson volumes to JT gravity. We computed the higher-genus contributions to the correlation functions with multiple boundaries and deduced the resolvent on the matrix integral side. We obtained results for any number of boundaries of the Riemann surfaces. We also evaluated the intersection
numbers of line bundles on the moduli of the Riemann surfaces of large genera with general number of boundaries \((n \geq 2)\).

As the resolvent is a fundamental quantity in the matrix integral, our results may provide information that can be utilized to confirm the duality of JT gravity and the matrix integral, as discussed in [8].

In general, the determination of the asymptotic expressions of the Weil–Petersson volumes is not simple. The large \(g\) asymptotics of the Weil–Petersson volumes [22] that were used in this study to compute the contributions to the correlation functions, resolvent, and intersection numbers involved the leading terms in \(g\). A future study can focus on the calculation of the subleading terms and investigating the effects of their inclusion.

The nonperturbative effects of the matrix integral might be observed from the higher-genus contributions that were computed in this study. This can be another potential direction of future research.

According to a result in [9], the Weil–Petersson volume, \(V_{g,n}(b_1, \ldots, b_n)\), is a polynomial in \(b\). In the region where \(b_i > g\), terms \(\prod_{i=1}^{n} b_i^{2m_i}\), where \(\sum_{i=1}^{n} m_i = 3g - 3 + n\) makes the dominant contribution. In the region where \(b_i > g\), the expressions of the intersection numbers should be significantly different from those deduced in section 4. It might be interesting to deduce the expressions for the intersection numbers in the region where \(b_i > g\).

In section 4, we deduced large-\(g\) asymptotic intersection numbers of line bundles on the moduli space of the Riemann surfaces with boundaries and discussed the mathematical structure of the partition function of the generating function of the intersection indices, which are related to the Witten conjecture. We observed that the asymptotic intersection indices computed in Section 4 satisfy variants of the string and dilaton equations in the large-\(g\) limit. We postulated that the partition function \(\exp(\tilde{F})\) of the generating function of the large-\(g\) asymptotic intersection indices satisfies a variant of the Virasoro constraints.

Further study is necessary to determine whether the partition function of the generating function of large-\(g\) asymptotic intersection indices indeed satisfies the deformation of the Virasoro constraints in the large-\(g\) limit. Another unresolved problem is that whether there is a method to refine the asymptotic intersection numbers [21] for the resulting asymptotic partition function of the generating function of the refined intersection indices that satisfies the exact Virasoro constraints.

**Acknowledgments**

We would like to thank Shun’ya Mizoguchi and Kazuhiro Sakai for discussions.

---

\(^9\)The authors in [21] evaluated the asymptotic Weil–Petersson volume for any genus with one geodesic boundary length becoming large.
References

[1] C. Teitelboim, “Gravitation and Hamiltonian Structure in Two Space-Time Dimensions,” *Phys. Lett.* B126, 41–45 (1983).

[2] R. Jackiw, “Lower Dimensional Gravity,” *Nucl. Phys.* B252, 343–356 (1985).

[3] A. Kitaev, “A simple model of quantum holography talk1 and talk2,” Talks at KITP on April 7, 2015 and May 27, 2015.

[4] J. Maldacena and D. Stanford, “Remarks on the Sachdev-Ye-Kitaev model,” *Phys. Rev.* D94, no.10, 106002 (2016) [arXiv:1604.07818 [hep-th]].

[5] A. Kitaev and S. J. Suh, “The soft mode in the Sachdev-Ye-Kitaev model and its gravity dual,” *JHEP* 05 (2018) 183 [arXiv:1711.08467 [hep-th]].

[6] S. Sachdev and J. Ye, “Gapless spin fluid ground state in a random, quantum Heisenberg magnet,” *Phys. Rev. Lett.* 70, 3339 (1993) [arXiv:cond-mat/9212030 [cond-mat]].

[7] A. Almheiri and J. Polchinski, “Models of AdS$_2$ backreaction and holography,” *JHEP* 11, 014 (2015) [arXiv:1402.6334 [hep-th]].

[8] P. Saad, S. H. Shenker and D. Stanford, “JT gravity as a matrix integral,” [arXiv:1903.11115 [hep-th]].

[9] M. Mirzakhani, “Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces,” *Invent. math.* 167 (2007) 179–222.

[10] B. Eynard and N. Orantin, “Weil-Petersson volume of moduli spaces, Mirzakhani’s recursion and matrix models,” [arXiv:0705.3600 [math-ph]].

[11] B. Eynard and N. Orantin, “Invariants of algebraic curves and topological expansion,” *Commun. Num. Theor. Phys.* 1, 347–452 (2007) [arXiv:math-ph/0702045 [math-ph]].

[12] B. Eynard, “Topological expansion for the 1-Hermitian matrix model correlation functions,” *JHEP* 11 (2004) 031 [arXiv:hep-th/0407261 [hep-th]].

[13] A. Blommaert, T. G. Mertens and H. Verschelde, “Clocks and Rods in Jackiw-Teitelboim Quantum Gravity,” *JHEP* 09 (2019) 060 [arXiv:1902.11194 [hep-th]].

[14] J. Cotler, K. Jensen and A. Maloney, “Low-dimensional de Sitter quantum gravity,” *JHEP* 06 (2020) 048 [arXiv:1905.03780 [hep-th]].

[15] U. Moitra, S. K. Sake, S. P. Trivedi and V. Vishal, “Jackiw-Teitelboim Gravity and Rotating Black Holes,” *JHEP* 11 (2019) 047 [arXiv:1905.10378 [hep-th]].

[16] D. Stanford and E. Witten, “JT Gravity and the Ensembles of Random Matrix Theory,” [arXiv:1907.03363 [hep-th]].

[17] K. Okuyama and K. Sakai, “JT gravity, KdV equations and macroscopic loop operators,” *JHEP* 01 (2020) 156 [arXiv:1911.01659 [hep-th]].

[18] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian and A. Tajdini, “Replica Wormholes and the Entropy of Hawking Radiation,” *JHEP* 05 (2020) 013 [arXiv:1911.12333 [hep-th]].
D. Marolf and H. Maxfield, “Transcending the ensemble: baby universes, spacetime wormholes, and the order and disorder of black hole information,” JHEP 08, 044 (2020) [arXiv:2002.08950 [hep-th]].

K. Okuyama and K. Sakai, “Multi-boundary correlators in JT gravity,” JHEP 08, 126 (2020) [arXiv:2004.07555 [hep-th]].

H. Maxfield and G. J. Turiaci, “The path integral of 3D gravity near extremality; or, JT gravity with defects as a matrix integral,” JHEP 01, 118 (2021) [arXiv:2006.11317 [hep-th]].

Y. Kimura, “JT gravity and the asymptotic Weil–Petersson volume,” Phys. Lett. B811, 135989 (2020) [arXiv:2008.04141 [hep-th]].

D. Momeni and P. Channuie, “Exact Solutions of (deformed) Jackiw-Teitelboim Gravity,” [arXiv:2009.03723 [hep-th]].

D. Momeni, “Real classical geometry with arbitrary deficit parameter(s) $\alpha(\rho)$ in deformed Jackiw–Teitelboim gravity,” Eur. Phys. J. C81, no.3, 202 (2021) [arXiv:2010.00377 [hep-th]].

M. Alishahiha, A. Faraji Astaneh, G. Jafari, A. Naseh and B. Taghavi, “Free energy for deformed Jackiw-Teitelboim gravity,” Phys. Rev. D103, no.4, 046005 (2021) [arXiv:2010.02016 [hep-th]].

K. Narayan, “On aspects of 2-dim dilaton gravity, dimensional reduction and holography,” [arXiv:2010.12955 [hep-th]].

U. Moitra, S. K. Sake and S. P. Trivedi, “Jackiw-Teitelboim Gravity in the Second Order Formalism,” [arXiv:2101.00596 [hep-th]].

L. Griguolo, J. Papalini and D. Seminara, “On the perturbative expansion of exact bi-local correlators in JT gravity,” JHEP 05, 140 (2021) [arXiv:2101.06252 [hep-th]].

L. Griguolo, R. Panerai, J. Papalini and D. Seminara, “Nonperturbative effects and resurgence in JT gravity at finite cutoff,” [arXiv:2106.01375 [hep-th]].

K. Jensen, “Chaos in AdS$_2$ Holography,” Phys. Rev. Lett. 117, no.11, 111601 (2016) [arXiv:1605.06098 [hep-th]].

J. Maldacena, D. Stanford and Z. Yang, “Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space,” PTEP 2016, no.12, 12C104 (2016) [arXiv:1606.01857 [hep-th]].

J. Engelsöy, T. G. Mertens and H. Verlinde, “An investigation of AdS$_2$ backreaction and holography,” JHEP 07 (2016) 139 [arXiv:1606.03438 [hep-th]].

R. C. Penner, “Weil-Petersson volumes,” J. Diff. Geom. 35 (1992) 559–608.

P. Zograf, “On the large genus asymptotics of Weil-Petersson volumes,” [arXiv:0812.0544 [math.AG]].

N. Do and P. Norbury, “Weil-Petersson volumes and cone surfaces,” Geom. Dedicata 141 (2009) 93 [arXiv:math/0603406 [math.AG]].
[36] N. Do, “Intersection theory on moduli spaces of curves via hyperbolic geometry,” Ph.D. Thesis at the University of Melbourne (2008).

[37] E. Witten, “Two-dimensional gravity and intersection theory on moduli space,” Surveys Diff. Geom. 1, 243–310 (1991).

[38] R. Dijkgraaf and E. Witten, “Developments in Topological Gravity,” Int. J. Mod. Phys. A33, no.30, 1830029 (2018) [arXiv:1804.03275 [hep-th]].

[39] T. Eguchi, K. Hori and C. S. Xiong, “Quantum cohomology and Virasoro algebra,” Phys. Lett. B402, 71–80 (1997) [arXiv:hep-th/9703086 [hep-th]].

[40] M. Kontsevich, “Intersection theory on the moduli space of curves and the matrix Airy function,” Commun. Math. Phys. 147, 1–23 (1992).

[41] A. Okounkov and R. Pandharipande, “Gromov-Witten theory, Hurwitz numbers, and matrix models,” Proc. Symp. Pure Math. 80 (2009) 325 [arXiv:math/0101147 [math.AG]].

[42] M. Mirzakhani, “Weil-Petersson volumes and intersection theory on the moduli space of curves,” J. Amer. Math. Soc. 20 (2007) 1–23.

[43] M. Kazarian and S. Lando, “An algebro-geometric proof of Witten’s conjecture,” J. Am. Math. Soc. 20 (2007) 1079–1089.

[44] J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher and M. Tezuka, “Black Holes and Random Matrices,” JHEP 05, 118 (2017) [arXiv:1611.04650 [hep-th]].

[45] P. Saad, S. H. Shenker and D. Stanford, “A semiclassical ramp in SYK and in gravity,” arXiv:1806.06840 [hep-th].

[46] P. Saad, “Late Time Correlation Functions, Baby Universes, and ETH in JT Gravity,” arXiv:1910.10311 [hep-th].

[47] T. G. Mertens and G. J. Turiaci, “Liouville quantum gravity – holography, JT and matrices,” JHEP 01, 073 (2021) [arXiv:2006.07072 [hep-th]].

[48] A. B. Zamolodchikov and A. B. Zamolodchikov, “Liouville field theory on a pseudosphere,” arXiv:hep-th/0101152 [hep-th].

[49] B. Eynard, “A short overview of the “Topological recursion”,” arXiv:1412.3286 [math-ph].

[50] S. Wolpert, “On the homology of the moduli space of stable curves,” Ann. of Math. 118 (1983) 491–523.

[51] S. Wolpert, “Chern forms and the Riemann tensor for the moduli space of curves,” Invent. math. 85 (1986) 119–145.