Involution on surfaces with $p_g = q = 0$ and $K^2 = 3$

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Abstract

We study surfaces of general type $S$ with $p_g = q = 0$ and $K^2 = 3$ having an involution $i$ such that the bicanonical map of $S$ is not composed with $i$. It is shown that, if $S/i$ is not rational, then $S/i$ is birational to an Enriques surface or it has Kodaira dimension 1 and the possibilities for the ramification divisor of the covering map $S \to S/i$ are described. We also show that these two cases do occur, providing an example. In this example $S$ has a hyperelliptic fibration of genus 3 and the bicanonical map of $S$ is of degree 2 onto a rational surface.

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1 Introduction

Minimal surfaces $S$ of general type with $p_g = q = 0$ have been studied by several authors in the last years, but a classification is still missing. For these surfaces the canonical divisor $K$ satisfies $1 \leq K^2 \leq 9$ and there are examples for all values of $K^2$ (see e.g. [BHPV]). The study of the bicanonical map $\phi_2$ of $S$, and in particular the case where $\phi_2$ is composed with an involution of $S$, has also provided some examples (cf. [CFM], [MP1], [MP2], [MP3], [MP4], [MP5]).

For the case $K^2 = 3$, there are examples with bicanonical map of degree 2 onto a nodal Enriques surface (see [MP3], [MP4]) and with bicanonical map of degree 4 onto a rational surface (see [Bu], [In], [Ca], [Ke], [Na]). In these constructions with $\deg(\phi_2) = 4$ the surface $S$ has involutions $i_j$, $j = 1, 2, 3$, such that $\phi_2$ is composed with $i_j$ and such that $S/i_j$ is birational to an Enriques surface or $S/i_j$ is a rational surface.

There are also the constructions given in [PPS1] and [PPS2], obtained using Q-Gorenstein smoothing theory, and the recent construction given in [BP], but we have no information about the bicanonical map or the existence of an involution in these cases.

In this paper we want to study the case where $K^2_S = 3$ and $S$ has an involution $i$ such that the bicanonical map of $S$ is not composed with $i$. We show that, if $S/i$ is not rational, then $S/i$ is birational to an Enriques surface or it has Kodaira dimension 1 and we describe the possibilities for the ramification divisor of the covering map $S \to S/i$. We also show that these two cases do occur, providing an example. In this example $S$ has a hyperelliptic genus 3 fibration and the bicanonical map of $S$ is of degree 2 onto a rational surface.

The paper is organized as follows. In Sections 2 and 3 we recall some facts about involutions on surfaces and about the possibilities for the branch locus
(the projection of the ramification divisor) in the quotient surface $S/i$. This is used to prove our main results in Section 4, Theorems 4 and 5. Section 5 contains the construction of an example, which is obtained as a $\mathbb{Z}_2^2$ cover of $\mathbb{P}^2$. The ramification divisor of this covering is computed using the Computational Algebra System Magma [BCP]. The corresponding code lines are given in the Appendix.

**Notation**

We work over the complex numbers; all varieties are assumed to be projective algebraic.

An *involution* of a surface $S$ is an automorphism of $S$ of order 2. We say that a map is *composed with an involution* $i$ of $S$ if it factors through the double cover $S \to S/i$.

An $(-2)$-curve $N$ on a surface is a curve isomorphic to $\mathbb{P}^1$ such that $N^2 = -2$.

An $(m_1, m_2, \ldots)$-point of a curve, or point of type $(m_1, m_2, \ldots)$, is a singular point of multiplicity $m_1$, which resolves to a point of multiplicity $m_2$ after one blow-up, etc.

The rest of the notation is standard in Algebraic Geometry.

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2 **General facts on involutions**

The following is according to [CM].

Let $S$ be a smooth minimal surface of general type with an involution $i$. Since $S$ is minimal of general type, this involution is biregular. The fixed locus of $i$ is the union of a smooth curve $R''$ (possibly empty) and of $t \geq 0$ isolated points $P_1, \ldots, P_t$. Let $S/i$ be the quotient of $S$ by $i$ and $p : S \to S/i$ be the projection onto the quotient. The surface $S/i$ has nodes at the points $Q_i := p(P_i)$, $i = 1, \ldots, t$, and is smooth elsewhere. If $R'' \neq \emptyset$, the image via $p$ of $R''$ is a smooth curve $B''$ not containing the singular points $Q_i$, $i = 1, \ldots, t$. Let now $h : V \to S$ be the blow-up of $S$ at $P_1, \ldots, P_t$ and set $R' = h^*(R'')$. The involution $i$ induces a biregular involution $\tilde{i}$ on $V$ whose fixed locus is $R := R' + \sum_i h^{-1}(P_i)$. The quotient $W := V/\tilde{i}$ is smooth and one has a commutative diagram:

\[\begin{array}{ccc}
V & \xrightarrow{h} & S \\
\downarrow{\pi} & & \downarrow{p} \\
W & \xrightarrow{g} & S/i
\end{array}\]

where $\pi : V \to W$ is the projection onto the quotient and $g : W \to S/i$ is the minimal desingularization map. Notice that

\[A_i := g^{-1}(Q_i), \quad i = 1, \ldots, t,\]
are \((-2)\)-curves and \(\pi^*(A_i) = 2 \cdot h^{-1}(P)\).

Set \(B' := g^*(B'')\). Since \(\pi\) is a double cover with branch locus \(B' + \sum A_i\), it is determined by a line bundle \(L\) on \(W\) such that

\[ 2L \equiv B := B' + \sum A_i. \]

It is well known that (cf. [BHPV] V. 22):

\[ p_g(S) = p_g(V) = p_g(W) + h^0(W, \mathcal{O}_W(K_W + L)), \]
\[ q(S) = q(V) = q(W) + h^1(W, \mathcal{O}_W(K_W + L)), \]
\[ K_S^2 - t = K_V^2 = 2(K_W + L)^2 \]

and

\[ \chi(\mathcal{O}_S) = \chi(\mathcal{O}_V) = 2\chi(\mathcal{O}_W) + \frac{1}{2}L(K_W + L). \]

Lemma 1 ([CM], [CCM]) The bicanonical map \(\phi_2\) of \(S\) (given by \(|2K_S|\)) is composed with \(i\) if and only if \(h^0(W, \mathcal{O}_W(2K_W + L)) = 0\).

3 Numerical restrictions

Let \(P\) be a minimal model of the resolution \(W\) of \(S/i\) and \(\rho : W \to P\) be the corresponding projection. Denote by \(\overline{B}\) the projection \(\rho(B)\) and by \(\delta\) the "projection" of \(L\).

Remark 2 If \(\overline{B}\) is singular, there are exceptional divisors \(E_i\) and numbers \(r_i \in 2\mathbb{N}\) such that

\[ E_i^2 = -1, \]
\[ K_W \equiv \rho^*(K_P) + \sum E_i, \]
\[ 2L \equiv B = \rho^*(\overline{B}) = \sum r_i E_i \equiv \rho^*(2\delta) = \sum r_i E_i. \]

The next result follows from Propositions 2, 3 a) and 4 b) of [Ri1].

Proposition 3 (cf. [CM], [Ri1]) Let \(S\) be a smooth minimal surface of general type with \(p_g = 0\) and \(K^2 = 3\) having an involution \(i\). With the previous notation, we have:

a) \(K_P(K_P + \delta) + \frac{1}{2}\sum (r_i - 2) = h^0(W, \mathcal{O}_W(2K_W + L));\)

b) \(\delta^2 = -2K_P^2 - 3K_P\delta + \frac{1}{2}\sum (r_i - 2)(r_i - 4) + 2h^0(W, \mathcal{O}_W(2K_W + L)) - 2;\)

c) the number of isolated fixed points of \(i\) is \(t = 7 - 2h^0(W, \mathcal{O}_W(2K_W + L));\)

d) \(K_W^2 \geq 2h^0(W, \mathcal{O}_W(2K_W + L)) - 4.\)
4 Possibilities

If \( p_\varphi(S) = 0 \) and the bicanonical map of \( S \) is composed with the involution \( i \), then it is known that \( S/i \) is birational to an Enriques surface or \( S/i \) is a rational surface (cf. [MP3, MP5]). This follows easily from Proposition 3 a), b): we have \( K_P(K_P + \delta) + \frac{1}{2} \sum (r_i - 2) = 0 \), thus \( K_P \) nef implies \( K_P^2 = K_P \delta = 0 \). Hence \( P \) is birational to an Enriques surface or it has Kodaira dimension 1. In this last case \( K_P \delta = 0 \) implies the existence of an elliptic fibration in \( S \), which is impossible because \( S \) is of general type.

Consider the branch divisor \( \overline{B} = B' + \sum A_i \subset W \) as above and let \( \overline{B}, \overline{B}' \) be the projection of \( B, B' \) on the minimal model \( P \) of \( W \). We have the following:

**Theorem 4** Let \( S \) be a smooth minimal surface of general type with \( p_\varphi = 0 \) and \( K^2 = 3 \) having an involution \( i \) such that the bicanonical map of \( S \) is not composed with \( i \).

Then the number of isolated fixed points of \( i \) is \( t = 5 \) and, if \( S/i \) is not rational, one of the following holds:

a) \( P \) is an Enriques surface and

\[
\begin{align*}
(i) & \quad \overline{B}'^2 = 10, \overline{B}' \text{ has a quadruple point and at most one double point } (\text{thus } p_a(B') = 0 \text{ or } -1), \text{ or} \\
(ii) & \quad \overline{B}'^2 = 8, \overline{B}' \text{ has a } (3, 3)\text{-point and no other singularities } (\text{thus } p_a(B') = -1); \\
\end{align*}
\]

b) \( \text{Kod}(P) = 1, \quad p_\varphi(P) = q(P) = 0, \overline{B}'^2 = -2, \quad p_a(\overline{B}') = 1 \) and \( \overline{B}' \) has at most two double points.

Moreover, cases a) (i) and b) do occur; there is an example with bicanonical map of degree 2 onto a rational surface.

**Proof**:

Proposition 3 c) of [BHPV] gives \( h^0(W, \mathcal{O}_W(2K_W + L)) \leq 1 \). Since \( \phi_2 \) is not composed with \( i \), we have \( h^0(W, \mathcal{O}_W(2K_W + L)) = 1 \). Then, from Proposition 3 \( \varphi_1 = 5 \) and \( K_W^2 \geq -2 \).

**Case 1**: \( \text{Kod}(P) = 0 \).

We have \( p_\varphi(P) \leq p_\varphi(S) = q(S) \), thus \( p_\varphi(P) = q(P) = 0 \) and then, from the classification of surfaces (see e.g. [Be] or [BHPV]), \( P \) is an Enriques surface. We obtain from Proposition 3 that \( \sum (r_i - 2) = 2 \) and \( \overline{B}'^2 = (2\delta)^2 = 0 \). Moreover, since \( K_W A_i = 0 \), each \((-2)\)-curve \( A_i \subset B \) is contracted to a singular point of \( \overline{B}' \) or is mapped onto a \((-2)\)-curve of the Enriques surface \( P \).

**Case 2**: \( \text{Kod}(P) = 1 \).

In this case \( K_P \) is numerically equivalent to a rational multiple of a fibre of an elliptic fibration of \( P \) (see e.g. [BHPV, V, 12]). This implies \( K_P^2 \neq 0 \), because otherwise \( \overline{B}' \) is contained in the elliptic fibration of \( P \) and then \( S \) has an elliptic fibration, which is impossible since \( S \) is of general type. From Proposition 3 a) and b) we get \( \sum (r_i - 2) = 0, \quad K_P \overline{B}' = 2K_P \delta = 2 \) and \( \overline{B}'^2 = (2\delta)^2 = -12 \).

**Case 3**: \( \text{Kod}(P) = 2 \).
Claim: If $K_P B = 0$, then $\overline{B} = B$ is a disjoint union of $(-2)$-curves.

Proof: Since $P$ is minimal of general type, $K_P$ is nef and big and then every component of $\overline{B}$ is a $(-2)$-curve and the intersection form on the components of the reduced effective divisor $\overline{B}$ is negative definite by the Algebraic Index Theorem (see e.g. [HHPV, IV. 2.16]). The claim is true if each connected component $C$ of $\overline{B}$ is irreducible. If $C$ is not irreducible, there is one component $\theta$ of $C$ such that $\theta(C - \theta) = 1$ and this implies that $B$ has a $(-3)$-curve, contradicting $B \equiv 0 \pmod{2}$. $\Diamond$

Since $P$ is of general type, $K_P^2 \geq 1$. Hence Proposition 3 implies $K_P \delta = 0$, $\sum (r_i - 2) = 0$ and $\delta^2 = -2$. Therefore $\overline{B}^2 = -8$ and then $B$ is a disjoint union of four $(-2)$-curves. But Proposition 3(c) gives $t = 5 \neq 4$.

An example for a) (i) and b) is given in Section 5.

In the conditions of Theorem 4 S/i is a rational surface, or $S/i$ is birational to an Enriques surface or Kod($S/i$) = 1. We have no example for $S/i$ rational (and $\phi_2$ not composed with $i$) but there is at least one possibility that may occur: $S$ is the smooth minimal model of a double cover of $\mathbb{P}^2$ ramified over a reduced plane curve of degree 16 with a quadruple point and five $(5,5)$-points. The construction of such a curve seems to be a nontrivial computational problem.

We can be more precise about the components of the branch locus $B' + \sum A_i \subset W$.

Theorem 5 Let $S$ be a smooth minimal surface of general type with $p_g = 0$ and $K^2 = 3$ having an involution $i$ such that the bicanonical map of $S$ is not composed with $i$.

With the previous notation, one of the following holds (here $\Gamma_{a,b}$ denotes a smooth irreducible curve with self-intersection $a$ and genus $b$):

a) $B' = \Gamma_{-6,0}$, $K^2_W = -1$, or

b) $B' = \Gamma_{-6,0} + \Gamma_{-4,0}$, $K^2_W = -2$, or

c) $B' = \Gamma_{-2,1} + \Gamma_{-4,0}$, $K^2_W = -1$, or

d) $B' = \Gamma_{-2,1} + \Gamma_{-4,0} + \Gamma'_{-4,0}$, $K^2_W = -2$, or

e) $B' = \Gamma_{-2,1}$, $K^2_W = 0$ (and Kod($W$) = 1).

Moreover, if Kod($W$) = 1, the possibilities for the multiple fibres $m, F_i$ of the elliptic fibration of $W$ are:

$(m_1, m_2, m_3) = (2, 2, 2)$ or $(m_1, m_2) = (2, 3), (2, 4)$ or $(3, 3).

The corresponding fibrations in $S$ are of genus $3, 7, 5$ or $4$, respectively.

There is an example for a) and c).

Remark 6 It is immediate from this theorem that the surface $S$ contains at least a smooth rational curve or a smooth elliptic curve. In cases b), c) and d), $K_S$ is not ample.
Proof of Theorem 5
We have \((2K_W + B')B' = 4K_W L + 4L^2 + 10 = 4L(K_W + L) + 10 = 2\). Since \(B \equiv 0 \pmod{2}\) and \(2K_W + B'\) is nef (because \(2K_S\) is nef), \(B'\) contains an irreducible component \(\Gamma\) such that \((2K_W + B')\Gamma = 2\) and possibly some components \(\Gamma_1, \ldots, \Gamma_l\) such that \((2K_W + B')\Gamma_i = 0, i = 1, \ldots, l\). These components are \((-4)\)-curves, because \(K_V \pi^*(\Gamma_i) = 0\) implies that the support of \(\pi^*(\Gamma_i)\) is a \((-2)\)-curve.

Denote by \(\Gamma_V\) the support of \(\pi^*(\Gamma)\). One has \(K_V \Gamma_V = 1\) and then \(2g(\Gamma_V) = 3 + \Gamma_V \geq 0\). The fact \((2K_W + B')(2K_W + B' - 3\Gamma) = 0\) implies, from the Algebraic Index Theorem, that \((2K_W + B' - 3\Gamma)^2 \leq 0\). This gives \(\Gamma^2 \leq 0\), thus \(\Gamma_V \leq -1\) or \(-3\) (equivalently \(\Gamma^2 = -2\) or \(-6\)).

We have seen above that \(K_V^2 \geq -2\) (Proposition 3 d)) and that, if \(W\) is birational to an Enriques surface, \(K_W^2 \leq -1\) (\(\overline{B}\) is singular). Now we claim that \(K_W^2 \leq -1\) if \(W\) is rational. In fact, \(-K_W(2K_W + B') = -2\) and \(2K_W + B'\) is nef, hence \(h^0(W, \mathcal{O}_W(-K_W)) = 0\) and then \(K_W^2 \leq -1\) from the Riemann-Roch Theorem.

Now from \(2 - 2K_W^2 = K_W B' = K_W \Gamma + 2l = 2g(\Gamma) - 2 + \Gamma^2 + 2l\) one gets
\[-2K_W^2 + 4 + \Gamma^2 = 2g(\Gamma) + 2l.

The possibilities allowed by this equation are:
- \(g(\Gamma) = 0, \Gamma^2 = -6\) and \((K_W^2, l) = (-1, 0)\) or \((-2, 1)\);
- \(g(\Gamma) = 1, \Gamma^2 = -2\) and \((K_W^2, l) = (0, 0), (-1, 1)\) or \((-2, 2)\).

Finally we prove the assertion about the multiple fibres of the elliptic fibration of \(W\), in the case \(\text{Kod}(W) = 1\). The canonical bundle formula (see e.g. [BHPV] V, 12.3) gives \(K_P \equiv -F + \sum_i^m (m_i - 1)F_i\), where \(m_i F_i \equiv F\) is a multiple fibre of the elliptic fibration of the minimal model \(P\) of \(W, i = 1, \ldots, n\). Since \(K_P B = 2\), we have then
\[
B F \left( -1 + \sum_{i=1}^n \frac{m_i - 1}{m_i} \right) = 2 \quad \text{and} \quad B F \geq 2m_i, \quad i = 1, \ldots, n.
\]

This immediately yields \(n \leq 3\) and \(n = 3 \Rightarrow m_1 = m_2 = m_3 = 2\). It is not difficult to see that if \(n = 2\), then \((m_1, m_2) = (2, 3), (2, 4)\) or \((3, 3)\).

The example for cases a) and c) is given in Section 5.

5 Example

5.1 Bidouble covers
A bidouble cover is a finite flat Galois morphism with Galois group \(\mathbb{Z}_2^2\). Following [Ca] or [Pa], to define a bidouble cover \(V \to X\), with \(V, X\) smooth surfaces, it suffices to present:
- smooth divisors \(D_1, D_2, D_3 \subset X\) with pairwise transverse intersections and no common intersection;
· line bundles $L_1, L_2, L_3$ such that $2L_g \equiv D_j + D_k$ for each permutation $(g, j, k)$ of $(1, 2, 3)$.

If Pic$(X)$ has no 2-torsion, the $L_i$'s are uniquely determined by the $D_i$'s.

Let $N := 2K_X + \sum L_i$. One has $2K_V \equiv \psi^*(N)$ and

$$H^0(V, \mathcal{O}_V(2K_V)) \simeq H^0(X, \mathcal{O}_X(N)) \oplus \bigoplus_{i=1}^3 H^0(X, \mathcal{O}_X(N - L_i)).$$

The bicanonical map of $V$ is composed with the involution $i_g$, associated to $2L_g \equiv D_j + D_k$ for each permutation $(g, j, k)$ of $(1, 2, 3)$. We verify below that the quotients $W_g := V/i_g$ satisfy:

· $W_1$ is birational to an Enriques surface;
· Kod$(W_2) = 1$, $p_g(W_2) = q(W_2) = 0$;
· $W_3$ is rational.

Moreover, the surface $S$ has an hyperelliptic fibration of genus 3 and the bicanonical map $\phi_2$ of $S$ is of degree 2 onto a rational surface. The map $\phi_2$ is composed with the involution induced by $i_3$ and is not composed with the involutions induced by $i_1$ and $i_2$.

### Step 1: Construction of $S$

Let $p_0, p_1, p_2 \in \mathbb{P}^2$ be distinct points and $T_1, T_2$ be the lines $p_0p_1, p_0p_2$, respectively. In the Appendix we use the Magma functions LinSys and ParSch given in [Ri2] to compute plane curves $C_6$ of degree 6 and $C_5$ of degree 5 such that:

· the singularities of $C_6$ are a double point at $p_0$, $(2, 2)$-points at $p_1, p_2$ tangent to $T_1, T_2$ and a triple point $p_3$ which resolves to a $(2, 2)$-point after one blow-up;
· the singularities of $C_5$ are a $(2, 2)$-point at $p_1$ tangent to $T_1$ and a $(2, 2, 2, 2)$-point at $p_2$ tangent to $T_2$ such that the intersection number of $C_5$ and $C_6$ at $p_2$ is 12;
· $C_5$ contains $p_0$ and intersects $C_6$ with multiplicity 7 at $p_3$.

Let $\mu : X \to \mathbb{P}^2$ be the map which resolves the singularities of $C_5 + C_6$ and let $E_i, E'_i, \ldots$ be the exceptional divisors (with self-intersection $(-1)$) corresponding to the blow-ups at $p_i$, $i = 0, \ldots, 3$. Let $T$ denote a general line in $\mathbb{P}^2$ and let the notation $\tilde{\cdot}$ denote the total transform $\mu^*(\cdot)$ of a curve.
Let $V \to X$ be the bidouble cover determined by the divisors
\[
D_1 := \tilde{C}_5 - E_0 - (E_1 + 2E'_1) - (2E_2 + 2E'_2 + 2E'''_2) - (E_3 + E'_3 + E'''_3),
\[
D_2 := \tilde{T}_1 - E_0 - 2E'_1 + E_3 - E'_3 + E'''_3,
\[
D_3 := \tilde{C}_6 + \tilde{T}_2 - 3E_0 - (2E_1 + 2E'_1) - (2E_2 + 4E'_2 + 0E''_2 + 2E'''_2) - (3E_3 + E'_3 + 3E''_3)
\]
and let $S$ be the minimal model of $V$.

**Step 2: Invariants of $S$.**

We have
\[
L_1 \equiv 4\tilde{T} - 2E_0 - (E_1 + 2E'_1) - (E_2 + 2E'_2 + 2E'''_2) - (E_3 + E'_3 + E'''_3),
\]
\[
L_2 \equiv 6\tilde{T} - 2E_0 - (2E_1 + 2E'_1) - (2E_2 + 3E'_2 + 2E''_2) - (2E_3 + E'_3 + 2E''_3),
\]
\[
L_3 \equiv 3\tilde{T} - E_0 - (E_1 + 2E'_1) - (E_2 + 2E'_2 + E''_2 + E'''_2) - E_3
\]
and
\[
K_X + L_1 \equiv \tilde{T} - E_0 - E'_1 - E'_2 + E''_2,
\]
\[
K_X + L_2 \equiv 3\tilde{T} - E_0 - (E_1 + E'_1) - (E_2 + 2E'_2 + E''_2) - (E_3 + E''_3),
\]
\[
K_X + L_3 \equiv -E'_1 + E'_3 + E''_3.
\]

One has
\[
\chi(O_S) = 4\chi(O_X) + \frac{1}{2} \sum_{i=1}^{3} L_i(K_X + L_i) = 4 - 1 - 1 = 1
\]
and
\[
p_g(S) = p_g(X) + \sum_{i=1}^{3} h^0(X, O_X(K_X + L_i)) = 0
\]
(see the Appendix for the computation of $h^0(X, O_X(K_X + L_3))$).

**Step 3: Calculation of $K_S^2$.**

Let $N = 2K_X + \sum_{i=1}^{3} L_i$. From the computations in the Appendix we get
\[
h^0(V, O_V(2K_V)) = h^0(X, O_X(N)) + \sum_{i=1}^{3} h^0(X, O_X(N - L_i)) = 4.
\]
The surface $V$ contains at least eight $(-1)$-curves (in $\tilde{T}_1 + \tilde{T}_2$), hence $K_S^2 \geq K_V^2 + 8 = N^2 + 8 = 1$ and then $S$ is of general type. Since $h^0(V, O_V(2K_V)) = h^0(S, O_S(2K_S)) = K_S^2 + 1$ (see e.g. [BHPV], VII. 5.), then $K_S^2 = 3$.

**Step 4: The surface $W_1$.**

Let $W_1$ be the double cover of $X$ with branch locus $D_2 + D_3$. It is well known...
that the smooth minimal model of $W_1$ is an Enriques surface (see e.g. [CD]).

Step 5: The surface $W_2$.
Let $W_2$ be the double cover of $X$ with branch locus $D_1 + D_3$. One has

$$\chi(O_{W_2}) = 2\chi(O_X) + \frac{1}{2}L_2(K_X + L_2) = 2 - 1 = 1$$

and

$$p_g(W_2) = p_g(X) + h^0(X, O_X(K_X + L_2)) = 0.$$  

We show in the Appendix that

$$h^0(X, O_X(K_X + 2L_2)) = 1 \quad \text{and} \quad h^0(X, O_X(6K_X + 6L_2)) = 2.$$  

This implies $\text{Kod}(W_2) > 0$ and, since

$$h^0(W_2, O_{W_2}(2K_{W_2})) = h^0(X, O_X(2K_X + L_2)) + h^0(X, O_X(2K_X + 2L_2)) = 1,$$

$W_2$ is not of general type (see e.g. [BHPV] VII. 5.). This way $\text{Kod}(W_2) = 1$.

Step 6: The surface $W_3$.
Let $\rho : W_3 \to X$ be the double cover with branch locus $D_1 + D_2$. The pencil of conics tangent to the lines $T_1, T_2$ at $p_1, p_2$ lifts to a rational fibration of $W_3$ (and lifts to a genus 3 fibration of $S$). Since

$$\chi(O_{W_3}) = 2\chi(O_X) + \frac{1}{2}L_3(K_X + L_3) = 2 - 1 = 1$$

and

$$p_g(W_3) = p_g(X) + h^0(X, O_X(K_X + L_3)) = 0,$$

then $W_3$ is a rational surface.

Step 7: Bicanonical map.
As computed in the Appendix, one has

$$h^0(X, O_X(2K_X + L_1 + L_2)) = 1,$$

$$h^0(X, O_X(2K_X + L_1 + L_3)) = 0,$$

$$h^0(X, O_X(2K_X + L_2 + L_3)) = 0,$$

hence the bicanonical map $\phi_2'$ of $V$ is not composed with the involutions $i_1$ and $i_2$ and is composed with $i_3$. Let $\psi_1 : V \to W_3$ be the double cover corresponding to $i_3$ and let $\psi_2 : W_3 \to \mathbb{P}^3$ be the map induced by

$$H^0(X, O_X(p^*(2K_X + L_1 + L_2 + L_3))) \oplus H^0(X, O_X(p^*(2K_X + L_1 + L_2 + R))),$$

where $R$ is the ramification divisor of the map $\rho : W_3 \to X$ defined above. We have

$$\phi_2' = \psi_1 \circ \psi_2.$$  

It is shown in the Appendix that the degree of $\psi_2(W_3)$ is 6. Since $(2K_S)^2 = 12$, this implies that the bicanonical map of $S$ is of degree 2.
Appendix: Magma code

Here the Computational Algebra System Magma ([BCP]) is used to perform some calculations.
We use the following Magma functions, given in [Ri2]: LinSys, which computes linear systems of plane curves with non-ordinary singularities, and ParSch, whose output is a scheme which parametrizes given degree plane curves with given singularities.

1) First we compute the curves $C_5$ and $C_6$ referred in Section [5.2].

```magma
K:=Rationals();
A<x,y>:=AffineSpace(K,2);
L:=\{LinearSystem(A,6),LinearSystem(A,5),LinearSystem(A,3),
    LinearSystem(A,2),LinearSystem(A,2),LinearSystem(A,1),LinearSystem(A,1)\};
P:=\{A![0,0],A![0,1],A![1,0],A![1,1]\};
M:=\{[[2],[2,2],[2,2,2],[1,1,1]],
    [[1],[1,1],[1,1,1],[1,1,1]],
    [[0],[1,1],[1,0,0],[1,0,0]],
    [[0],[1,1],[1,0,0],[0,0,0]],
    [[0],[0,0],[1,1,1],[0,0,0]],
    [[0],[0,0],[0,0,0],[1,1,1]]\};
T:=\{[],[0,1],[1,0],[1,15/61*r1 + 443/2745],[1,-21465/95648*r1 - 23559/59780]\};
S:=ParSch(L,P,M,T,[],[],5);
```

We want to compute points (some infinitely near) such that:
- the sets of elements of $L[1], L[2]$ which have singularities, at those points, of multiplicities given by $M[1], M[2]$ are non-empty;
- the five sets of elements of $L[3], \ldots, L[7]$ of curves with singularities, at those points, of multiplicities given by $M[3], \ldots, M[7]$ are empty.

This last step is needed in order to obtain a non-reduced curve. The following gives a scheme which parametrizes such curves.

```magma
S:=ParSch(L,P,M,T,[],[],5);
```

This scheme is zero dimensional. We compute a point in $S$

```magma
PointsOverSplittingField(S);
```

and we use the function LinSys to compute the reduced curves $C_5$ and $C_6$.

```magma
R<r1>:=PolynomialRing(Rationals());
K<r1>:=NumberField(r1^2 + 1496/675*r1 + 10976/625);
A<x,y>:=AffineSpace(K,2);
L5:=LinearSystem(A,5);L6:=LinearSystem(A,6);
P:=\{A![0,0],A![0,1],A![1,0],A![1,1]\};
M5:=\{[[2],[2,2],[2,2,2],[1,1,1]],
    [[0],[1,1],[1,0,0],[0,0,0]],
    [[0],[1,1],[0,0,0],[1,1,1]],
    [[1],[],[1,0,0],[1,1,1]],
    [[0],[],[0,0,0],[1,1,1]]\};
T:=\{[],[0,1],[1,0],[1,15/61*r1 + 443/2745],[1,-21465/95648*r1 - 23559/59780]\};
J5:=LinSys(L5,P,M5,T);C_5:=Curve(A,Sections(J5)[1]);
J6:=LinSys(L6,P,M6,T);C_6:=Curve(A,Sections(J6)[1]);
```

The equations of $C_5$ and $C_6$ are, in affine space:
\[ 3660x^5y^5(-900x+1341)x^4y-16640x^4y^4(-3550x-12858)x^3y^2+4500x-300)x^3y^2 +21960x^3y^{12}(1500x+14313)x^{12}y+23780x^2y^3+700x+3412)x^{12}y +3660x-915y^2+3660y^4-5690y^{12}+3660y^2-915y \]

and
\[ 35882945x^6y^2-12929700x+26583872)x^4y^2 \]

with \( x^2 + 1496/675x + 10976/625 = 0 \).

2) From Section 5.2 one has:
\[ 2K_X + \sum L_i \equiv 7\overline{E} - 3E_0 - (2E_1 + 4E_1') - (2E_2 + 4E_2' + 2E_2'') - \cdots \]

Below we compute the dimension of the first cohomology group \( h^0(\mathcal{O}_X(\cdot)) \) for each of these divisors (it is immediate that \( h^0(2K_X + L_1 + L_3) = 0 \)). We obtain 3, 0, 1, 2, 1, 0, respectively.

\[ M := \left[ \begin{array}{c} [3], [2, 4], [2, 4, 0, 2], [1, 1, 1], \\ [1, 1, 1], [1, 2, 0, 1], [1, 0, 1], \\ [2], [2, 2], [2, 4, 0, 2], [2, 0, 2], \\ [6], [6, 6], [6, 12, 0, 6], [6, 0, 6], \\ [2], [1, 2], [1, 3, 0, 0], [1, 0, 1], \\ [1], [1, 2], [1, 2, 0, 1], [0, 0, 0] \end{array} \right] \]

\[ d := [7, 3, 6, 18, 4, 3] \]

3) Now we describe how to compute the degree of the scheme \( \psi_2(W_3) \) referred in Section 5.2 Step 7. The complete code is available at http://home.utad.pt/~crito/magma_code.html

Let \( f_0 \) be the defining equation of the curve \( D_1 + D_2 \), \( f_1 \) be the equation of the unique effective plane curve corresponding to \( 2K_X + L_1 + L_2 \) and let \( J_7 \) be the linear system of plane curves corresponding to \( 2K_X + L_1 + L_2 + L_3 \). We define (a singular model of) \( W_3 \) in a weighted projective space and we define the map \( \psi_2 : W_3 \to \mathbb{P}^3 \).

\[ WP<x, y, z> := \text{ProjectiveSpace}(K, [3, 1, 1, 1]) \]
\[ W3 := \text{Scheme}(WP, w^2-f6) \]
\[ P3 := \text{ProjectiveSpace}(K, 3) \]
\[ psi2 := \text{map}(W3 \to P3 | \text{(Sections}(J7) \text{ div } (x*y)) \text{ cat } [w*(f4 div (x*y))]) \]

We want to compute
but 6 GB of computer memory are not enough for this task. Thus we compute
the degree of the intersection of two hyperplane sections of $\psi_2(W_3)$.

We obtain degree 6.

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