Analytical mesoscale modeling of aeolian sand transport

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We analyze the mesoscale structure of aeolian sand transport, based on a recently developed two-species continuum model. The calculated sand flux and important average characteristics of the grain trajectories are found to be in remarkable agreement with field and wind-tunnel data. We conclude that the essential mesoscale physics is insensitive to unresolved details on smaller scales and well captured by the coarse-grained analytical model, thus providing a sound basis for precise and numerically efficient mesoscale modeling of aeolian structure formation.

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Aeolian sand transport is the process of erratic grain hopping and irregular meandering of diffuse sand clouds occasionally observed on a windy day at the beach. It remains perplexing how the wide variety of distinctive aeolian sand patterns, from tiny ripples to huge dunes, emerges from such seemingly chaotic dynamics. Attempts to derive coarse-grained mathematical models for aeolian transport started with Bagnold’s seminal work in the 1930s [1] and are still the subject of ongoing research [2, 3], for good conceptual and practical reasons. With average grain trajectories exceeding the sub-millimeter grain scale by orders of magnitude, aeolian transport is a mesoscale phenomenon that should be amenable to such coarse-grained modeling. Moreover, faithful grain-scale simulations remain forbiddingly expensive, even with modern computers, so that even numerical approaches cannot avoid fairly drastic idealizations [4–7].

A radically coarse-grained mean-field model [8] that maps the whole mobilized grain population onto a single effective grain trajectory has been successful in explaining desert dune formation [9–11]. While such approaches roughly account for the more energetic saltating grains, they fail to explicitly resolve the substantial fraction of grains that only perform very small hops and do not eject other grains from the bed. This less spectacular transport mode, conventionally referred to as bedload, reptation, or creep, is however thought to be largely responsible for ripple and megaripple formation [1, 12], which therefore eludes the mean-field approaches. Also ecologically important processes, such as dust emission and vegetation invasion, are sensitive to the detailed mesoscale structure of aeolian transport [3, 13].

An improved “two-species” formalism parametrizes the mobilized sand cloud in terms of “saltons” and “reptons”, loosely corresponding to the phenomenologically observed fast and slow subpopulations. The pertinence of such a reduced scheme has recently been substantiated through numerical studies based on a multi-species formalism [14]. Further support comes from the success of an analytically tractable two-species continuum model [15] in accounting for extensive sand flux data comprising a wide range of grain sizes and wind speeds [16, 17]. Taken together, these studies suggested that only a slight extension of the simple mean-field picture is needed to arrive at accurate macroscopic transport laws. But they left largely unanswered the question how faithfully the two-species scheme represents the actual mesoscale structure of aeolian transport. Indeed, it has become a widespread belief that this cannot be captured by analytical models, at all, and that one rather has to resort to more laborious and less practical grain-scale numerical modeling [3]. In the following, we provide a comprehensive comparison of analytical predictions and experimental data that challenges this view.

As a major culprit for the perceived shortcomings of analytical models, many authors [3, 14, 18] identified Owen’s hypothesis [19], which provides an effective hydrodynamic boundary condition widely used in analytical approaches. It states that the shear stress \( \tau_a \) exerted by the air on the ground (at height \( z = 0 \)) is near the threshold \( \tau_t \), below which aeolian transport dies out,

\[
\tau_a(z = 0) \approx \tau_t. \tag{1}
\]

This relation naturally holds for wind speeds near the threshold and is considered a fair approximation at higher wind speeds, where the increasing amount of saltating grains screens the bed from the wind.

In this debate, it is crucial to distinguish hydrodynamic boundary conditions from microscopic conditions at the boundary, i.e., the two questions: does Eq. (1) accurately describe the actual conditions at the bed? And, is it a viable hydrodynamic boundary condition for accurate coarse-grained models? Clearly, only the latter question (and its denial by several authors) concerns us, here. Durán et al. [18] criticized that Eq. (1) gives rise to wind profiles without a focal point, at variance with wind-tunnel observations by Bagnold. (Bag-}

nold had found that aeolian transport renders the wind velocity at a certain height above the ground independent of the overall wind strength [1].) Further, Kok et al. [3] compared results from numerical computations [5], performed with and without Owen’s approximation, to experimental data. They concluded that, instead of using Eq. (1), \( \tau_a(z = 0) \) needs to be computed directly from
a detailed numerical modeling of the grain trajectories.

However, as demonstrated in Ref. [15], the results from recent comprehensive wind-tunnel investigations of Bagnold’s focus [20] can very reasonably be reproduced by a simple analytical model that uses Eq. (1) and nevertheless provides a fully satisfactory description of comprehensive flux data. Moreover, as shown below, even for the mesoscale structure of the aeolian transport layer remarkably accurate results can be obtained that way. In fact, our analytical predictions based on Eq. (1) turn out to be fully compatible with the experimental data and numerical results put forward by Kok et al. [3] in order to invalidate it. This underscores that both Bagnold’s focus and Owen’s hypothesis are convenient notions to characterize the mesoscopic structure of the transport layer, approximately, on a hydrodynamic level. They are, however, not supposed (or required) to provide a microscopically precise description of the experimental conditions. Neither should this be a main criterion for judging the viability of mesoscale models.

Our following quantitative analysis is based on the analytical two-species continuum model by Lämmel et al. [15]. It consists of hydrodynamic equations for the aeolian flux based on conservation laws and closure relations. The closures rely on a simplified representation of the grain-scale physics, in particular of the grain hopping, the dissipative grain-bed and grain-air interactions, and the feedback of the grain hopping onto the wind. The model is thus constructed in the same spirit as previous mean-field models, but accounts for additional mesoscale structure in a simple and effective way. Crucially, the gained flexibility is due to an improved representation of the grain-scale physics, not to new free parameters. Key features are the balance between saltons and reptons, and their dissimilar interactions with (and feedback on) the wind velocity field. The model was previously validated against extensive flux data $q(d, u_*)$ [16, 17] covering a wide range of grain sizes $d$ and wind strengths $\tau \equiv g_{air} u_*^2$. (Here, $u_*$ is the shear velocity and $g_{air}$ the air density.) In the following, we address for the first time the model’s mesoscale properties and compare them to available data.

We start by checking the model prediction for the transport velocity $v$, which together with the mobile grain density $\rho$ makes up the flux $q = \rho v$ (both height-integrated). In the two-species model, $v$ is obtained as a weighted average over the repton and salton speeds, $v^{rep}$ and $v^{sal}$, respectively. Their relative weight $\rho^{rep}/\rho^{sal} \propto 1 - v_{c}^{sal}/v_{c}^{sal}$ is itself dependent on $v^{sal}$ and vanishes if $v_{c}^{sal}$ falls below a certain minimum value $v_{c}^{sal}$ necessary to eject grains upon impact. Now, following Ref. [3], the experimental value of $v$ is estimated by linearly extrapolating measured grain speed profiles down to the bed, where trajectories of any length contribute to the average. In Fig. 1 we reproduce a large data set obtained that way from wind tunnel observations published by various research teams using different devices and measurement techniques. The data invariably indicate that the mean grain speed at the ground is almost independent of the overall wind strength, as also corroborated by recent grain-scale simulations [6]. While this observation may seem somewhat unintuitive, at first sight, it is perfectly reproduced by the two-species model. We even find a remarkable agreement in the absolute values, using the same set of model parameters employed in fitting flux data $q(d, u_*)$ by Creyssels et al. [17], in Ref. [15] and
FIG. 3. The mean height $z_m$ of the saltation layer as function of the wind strength. To estimate $z_m$ from the field data by Greeley et al. [24] and Namikas [22] (filled squares), grain speed profiles were fitted by an exponential distribution. The fit to the data from Ref. [22] was alternatively also performed after conversion to a cumulative distribution (open squares). The theoretical prediction, Eq. (3), was evaluated for $v_{50}^{\text{sal}}/v^{\text{sal}} = 0.66$ (solid) and 0.41 (dashed line). We also included the prediction of the numerical COMSALT model [5] (dotted line, taken from Ref. [3]).

in Fig. 5 (inset), below.

Another interesting mesoscale property, closely related to the transport velocity $v$, is the average length of the grain trajectories, or “hop length” $l$. Within the two-species model, the trajectories are, for a given grain, approximated as parabolic with hop length $l = 2v_{50}/g$ and hop height $h = v_{50}^2/(2g)$, where $g$ is the gravitational acceleration and $v_{50}$ the initial vertical speed upon ejection from the bed. For reptons, the latter is on the order of the threshold shear velocity $u_{\tau} \equiv (\tau/\rho_{\text{air}})^{1/2}$ [25]. For saltons, $v_{50}$ has not been calculated within the two-species continuum approach, before, but using it as a free parameter, the calculated species-averaged hop length turns out to be in excellent agreement with the measured data, for a constant $v_{50}^{\text{sal}}/v^{\text{sal}} = 0.27$ (Fig. 2). We can also compute this ratio, which can be interpreted as an effective restitution coefficient for rebounding saltons, within the two-species framework. A numerically computed saltation trajectory is required to reproduce the trajectory-averaged horizontal grain speed $v$ (discussed above) and the restitution coefficient $\alpha$ employed in the model [25]. This gives $v_{50}^{\text{sal}}/v^{\text{sal}} \approx 0.23$ for $\tau \approx \tau_{\text{t}}$, close to the fit result.

Next, we turn to the characteristic height of the transport layer. Experiments usually find that the flux of grains above the ground can be approximated by an exponential height distribution, with fair accuracy. The characteristic height $z_m$ of the transport deduced from flux data by such an exponential fit is on the order of a few centimeters and almost independent of the wind strength. But there seems to be some uncertainty, depending on whether the distribution is sampled directly or cumulatively. Also, there is apparently no clear consensus about possible deviations from the exponential form due to a possible deficit [26] or excess [22] of grains near the ground; maybe because of difficulties in determining the exact bed level and variable sampling efficiencies [27]. These open issues notwithstanding, we have gathered experimental estimates for $z_m$ from (cumulative) height distributions extracted from literature data in Figs. 3, 4.

In the two-species framework, the mean saltation height $z_m$ is implicitly defined in terms of the horizontal saltation and reptation fluxes $\partial_z q^{\text{rep}}, \partial_z q^{\text{sal}}$ and their height-integrated values $q^{\text{rep}}, q^{\text{sal}}$, respectively,

$$\frac{q^{\text{sal}} + q^{\text{rep}}}{2} = \int_0^{z_m} dz \left[ \partial_z q^{\text{rep}}(z) + \partial_z q^{\text{sal}}(z) \right].$$

The detailed distributions of grain trajectories are not explicitly resolved in the model. But since $z_m$ turns out to lie above the reptation layer, the integration over $\partial_z q^{\text{rep}}$ simply yields $q^{\text{rep}}$. To perform the remaining integral over the saltation fraction, a specific form of $\partial_z q^{\text{sal}}(z)$ has to be postulated. Assuming a simple exponential for $h^{\text{sal}} \partial_z q^{\text{sal}}(z)/h^{\text{sal}}$ (and checking that the numerical value of $z_m$ is insensitive to this particular choice [25]), we obtain

$$z_m = -(v_{50}^{\text{sal}})^2 \ln\left(1 + q^{\text{rep}}/q^{\text{sal}}\right)/2g.$$

This is plotted in Figs. 3 and 4. As in Fig. 2, $v_{50}^{\text{sal}}/v^{\text{sal}}$ was used as a free fit parameter. In view of the experimental uncertainty in the absolute values of $z_m$, the fit results seem reasonably consistent.

Finally, we want to address what is probably the most enigmatic and most debated mesoscale property of aeolian sand transport, the so-called saturation length $\ell_{\text{sat}}$. 

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**FIG. 4.** The mean transport layer height $z_m$ as function of the grain size $d$ for a fixed shear velocity $u_\tau = 0.5$ m s$^{-1}$ (left panel) and as a function of the wind strength (right panel). Results deduced from wind tunnel data by Liu and Dong [28] (symbols) are compared to the two-species result, Eq. (3) (solid lines) using $v_{50}^{\text{sal}}/v^{\text{sal}} = 0.50$ as a global fit parameter and subtracting 3.7 cm from the raw data for $z_m$ to enforce mutual consistency of the experimental data in Refs. [22, 24, 28].
This central notion was originally introduced by Sauer- man et al. [8] to quantify how the aeolian sand transport adapts to changes in the wind over uneven topographies. They considered flux transients on the upwind slope of a sand dune and in the downwind wake region as two pertinent instances giving rise to quite diverse numerical values and parameter dependencies for $L_{\text{dune}}$ [10]. As an emergent mesoscale concept, $L_{\text{sat}}$ is thus intrinsically context-dependent, and attempts to promote narrower definitions of the saturation length (such as the distance needed for a hopping grain to be accelerated to the fluid velocity [29] or the distance over which the sand flux saturates at the entrance of a sand bed [30]) seem counterproductive.

Arguably the most interesting saturation transients are those near the crest of small dunes, due to changes in the wind speed (rather than sand coverage). They are responsible for the emergence of the relevant mesoscale $L_{\text{sat}}$ with respect to which aeolian dunes may be considered large or small, and which gives rise to a minimum dune size $L_{\text{min}} \propto \ell_{\text{dune}}$ [9, 10, 32, 33]. As demonstrated in Fig. 5, the original attempt to calculate this specific saturation length was only partly successful. The prediction [8]

$$L_{\text{sat}} = \frac{2(v_{\text{sat}})^2}{g}/(u_t^2/u_{\text{sat}}^2 - 1), \quad (4)$$

with the reduced gravitational acceleration $\tilde{g} \approx 0.6g$, fits field data for $L_{\text{min}}$ [30] reasonably well, but the mean-field model cannot, with the same parameter values, account for the flux data (inset). One should not hastily dismiss Eq. 4 on account of this failure, though. The problem is caused by the mean-field evaluation of Eq. 4 and is easily cured by a straightforward reevaluation within the two-species framework (solid lines in Fig. 5). For computational details and further theoretical and experimental support for an essentially constant value of $L_{\text{sat}}/\ell_{\text{dune}}$ (rather than a constant $L_{\text{sat}}$ [30]), we refer the reader to Ref. [25].

Summarizing, we scrutinized the mesoscale predictions of an analytical two-species continuum model for aeolian sand transport. The model parametrizes the mobilized sand cloud in terms of two idealized representative transport modes, so-called saltons and reptons. It was found to consistently account for an extensive compilation of independently generated field and wind-tunnel data, using a consistent set of model parameters. Our results challenge widespread beliefs that the mesoscale properties of aeolian sand transport are inaccessible to analytical approaches. They recommend the two-species model as a sound basis for a precise and highly efficient mesoscale modeling of aeolian structure formation, dust emission, and desertification.

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