Relaxing the Geodesic Rule in Defect Formation Algorithms

Levon Pogosian and Tanmay Vachaspati

Physics Department, Case Western Reserve University, Cleveland OH 44106-7079.

In studying the formation of topological defects, it is conventional to assume the “geodesic rule” which is equivalent to minimizing gradients of the order parameter. This assumption has been called into question in field-theoretic studies of first order phase transitions and in the case of local defects. We present a scheme for numerically investigating the formation of strings without assuming the geodesic rule. Our results show that the fraction of string in infinite strings grows as we deviate from the geodesic rule.

Within all known theoretical frameworks, the early universe must have been a close analog of condensed matter systems that we now study in the laboratory. In particular, the early universe must have undergone phase transitions during which it is very conceivable that topological defects would have been produced. The observation of any left-over defects today would directly provide us with a window to the universe when it was a mere fraction of a nanosecond old.

The study of topological defects has made long strides over the last twenty years [1]. Starting with order-of-magnitude estimates, we are now in a position to compare theory with laboratory experiments and astronomical observations. As our tools get sharper, finer details of our understanding of the properties of defects will be tested.

The properties of defects at formation, i.e. at the phase transition, are an important ingredient in their cosmology. For example, if the location of magnetic monopoles is very strongly correlated with the location of antimonopoles, rapid annihilation would ensue and the monopole over-abundance problem in cosmology would be absent. Similarly, the presence of infinite strings in cosmic string networks is expected to be very important in the cosmological role that they can play.

The properties of defects at formation have mostly been studied via lattice-based numerical simulations using the so-called Vachaspati-Vilenkin (VV) algorithm [2]. To describe these efforts on a concrete footing, consider the global $U(1)$ model

$$L = |\partial_\mu \phi|^2 - \frac{\lambda}{4} (|\phi|^2 - \eta^2)^2$$

(1)

where $\phi = |\phi| e^{i\alpha}$ is a complex scalar field. Apart from the discretization of space in the form of a specific lattice, the simulations also assume:

- Only the dynamics of the phase of $\phi$ is relevant. In other words, $|\phi|$ is set to be $\eta$ everywhere.
- The “Geodesic Rule”: The gradients of $\alpha$ cost a lot of energy and are minimized.

Then the simulation proceeds by laying down phases $\alpha$ at every lattice site, finding phase differences by minimizing gradients and then evaluating

$$\oint d\alpha$$

around all plaquettes of the lattice. The value of this integral yields the winding number and hence the number of strings passing through each plaquette. Then the strings are joined together and in this way the string network is constructed.

Very recently there has been a trend to study the formation of defects and, in particular, strings, from a more field-theoretic viewpoint [3,4,5,6,7,8] and without adopting a lattice [9,10]. Both efforts are very desirable and have shed further light on the problem. For example, the study of the evolution of fields during first order phase transitions (in two and three bubble collisions) has shown that the degree of freedom associated with $|\phi|$ can be important in the formation of defects. Generally, oscillations of $|\phi|$ lead to an excess of strings over a simple counting of the winding of $\alpha$ as in the VV algorithm. Also, it is often found that the number of defects is in excess of what the geodesic rule would imply. So, it seems that both assumptions of the VV algorithm might need to be relaxed.

Another situation where the geodesic rule has been called into question is in the formation of gauge (local) strings [1]. In this case, the model in eq. (1) is gauged. A consequence of this is that $\nabla \alpha$ is no longer a gauge invariant quantity. For example, the contribution to the energy density no longer comes from the ordinary gradient of $\alpha$ but from the covariant gradient:

$$(\nabla \alpha - eA)^2$$

where $A$ is the gauge field. And this questions the geodesic rule that is based on the ordinary gradient of $\alpha$.

A significant advance has been made in Ref. [12] where the collision of bubbles of true vacuum is studied analyti-
cally. The end result justifies the geodesic rule under certain circumstances. However, in the analysis, the gauge field is not provided with any thermal properties of its own and is a passive player during the phase transition.

In general, we would expect that the gauge field would have its own dynamics consistent with the thermal nature of the system. This could lead to the production of string in excess of what the geodesic rule would imply.

FIG. 1. The filled circles show the string density as a function of $\beta$ in the Monte Carlo simulation. The curve shows the results of a direct calculation of the string density where we evaluated the expected number of strings passing through a plaquette by summing over all phase distributions and values of $n$ with appropriate probability factors.

\[ P_n = \int_{n-0.5}^{n+0.5} dm \sqrt{\pi \beta} e^{-\beta(\delta \alpha + 2\pi m)^2} \]

where, $\beta \in (0, \infty)$ is a parameter.

The form of $P_n$ has two features which we think are desirable. First, for large $\beta$, the algorithm reduces to the VV algorithm with the geodesic rule. Secondly, the Gaussian form is reminiscent of the Boltzmann suppression factor with $\beta$ being related to the inverse temperature. On dimensional grounds, the connection of $\beta$ and the temperature $T$ is:

\[ \beta \sim \frac{\eta^2 \xi}{T} \]

where $\xi$ is the correlation length at the phase transition.

FIG. 2. The fraction of infinite string as a function of the total string density. The 1σ-error bars are found by running the simulation for each value of $\beta$ 100 times and then finding the spread in the fraction of infinite strings. The spread in values of the total string density is much smaller and has not been shown.

In this paper, we remedy the VV algorithm by relaxing the geodesic rule. Instead of calculating the gradient in the phase $\alpha$ on a lattice link by choosing the shortest path on the vacuum manifold, we choose the path using a probability distribution. More explicitly, if the phase at lattice site $A$ is $\alpha_A$ and that at lattice site $B$ is $\alpha_B$, then the difference in phase between sites $B$ and $A$ is:

\[ \Delta \alpha = \alpha_B - \alpha_A + 2\pi n \equiv \delta \alpha + 2\pi n \]

where, $n$ is any integer. The geodesic rule was that $n$ should be chosen so as to minimize $|\Delta \alpha|$. Here we take $n$ to be a random variable with the (Gaussian) probability distribution:

By changing the value of the parameter $\beta$, we can now study the effects of relaxing the geodesic rule. This is
equivalent to allowing for the production of extra strings in bubble collisions, say due to fluctuations of $|\phi|$, or, due to gauge field fluctuations.

We have performed the numerical simulations in this way on a cubic lattice together with the triangular discretization of phases described in [2]. Now it is possible for more than one string to pass through a plaquette in the lattice. This means that we can have a very high density of strings. Also, in connecting the string network, at every step there may be several choices to make. We make this choice randomly with equal probability assigned to every possible connection.

The total string density in the Monte Carlo simulation has been plotted as a function of the parameter $\beta$ in Fig. 1. On this figure we have also plotted the string density calculated by directly evaluating the expected number of strings passing through each plaquette. The two calculations are in agreement, giving us confidence in the Monte Carlo simulations.

FIG. 3. The loop length distribution for $\beta = 0.05$.

In Fig. 2, we show the string fraction in infinite string ($\rho_{inf}/\rho$) as a function of the total string density $\rho$ where an infinite string is defined to be one that is longer than the square of the lattice size. It is clear from the graph that departures from the geodesic rule - smaller values of $\beta$ - lead to an excess of infinite string fraction.

The graph in Fig. 2 can be fit by several different functional forms and the data does not convincingly point to any fit. In principal it should be possible to collect more data for higher values of the string density (lower values of $\beta$) and then fit to a power law in the limit $\rho_{inf}/\rho \rightarrow 1$. We have not been able to find the corresponding exponent since data collection rapidly becomes very expensive on computer resources as $\beta$ is reduced.

We have also plotted the number density of string loops as a function of the length ($l$) of the loop. For all values of $\beta$, the result is the usual scale invariant distribution where the number density falls off as $l^{-2.5}$. In Fig. 3 we show this length distribution (for $\beta = 0.05$) which is well fitted by:

$$dn = 0.5 \frac{dl}{l^{5/2}}.$$ 

To conclude, our results show that the implementation of the geodesic rule in the VV algorithm provides a lower bound on the fraction of infinite strings in lattice simulations. It also shows that any fluctuations in the gauge fields or magnitude of $\phi$ that produce more strings, are likely to increase the fraction of infinite string.

Acknowledgements: TV was supported by the Department of Energy.

[1] A. Vilenkin and E. P. S. Shellard, “Cosmic Strings and Other Topological Defects”, Cambridge University Press (1994).
[2] T. Vachaspati and A. Vilenkin, Phys. Rev. D24, 2036 (1984).
[3] S. Digal, S. Sengupta and A. M. Srivastava, Phys. Rev. D56, 2035 (1997).
[4] P. Saffin and E. Copeland, D54, 6088 (1996).
[5] A. Mello and L. Perivolaropoulos, Phys. Rev. D52, 992 (1995).
[6] A. Ferreira and A. Mello, Phys. Rev. D53, 6852 (1996).
[7] J. Ahonen and K. Enqvist, hep-ph/9704334.
[8] N. D. Antunes, L. M. A. Bettencourt and M. Hindmarsh, hep-ph/9708215.
[9] J. Borrill, T. W. B. Kibble, T. Vachaspati and A. Vilenkin, Phys. Rev. D52, 1934 (1995).
[10] J. Borrill, Phys.Rev.Lett. 76, 3255 (1996).
[11] S. Rudaz and A. Srivastava, Mod. Phys. Lett. A8, 1443 (1993).
[12] T. W. B. Kibble and A. Vilenkin, Phys. Rev. D52, 679 (1995).