Persistence of hydrodynamic envelope solitons: detection and rogue wave occurrence

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The observation of a wave group persisting for more than 200 periods in the direct numerical simulation of nonlinear unidirectional irregular water waves in deep water is discussed. The simulation conditions are characterized by parameters realistic for broad-banded waves in the sea. The group is identified by the solution of the associated scattering problem for the nonlinear Schrödinger equation as the intense envelope soliton with remarkably stable parameters. Most of extreme waves occur on top of this group, resulting in higher and longer rogue wave events.

I. INTRODUCTION

The intriguing problem of anomalously high sea waves (rogue waves) is in the focus of numerous multifaceted studies, see e.g. reviews [1–3]. The approach to study this complicated natural phenomenon by means of the direct numerical simulation (DNS) of the irregular wave evolution in time has become popular, thanks to the high performance of modern computers and improved numerical algorithms. The potential Euler equations serve well when simulating the wind-generated gravity waves on the sea surface. They may be solved by modern codes incomparably quicker than the Navier – Stokes equations and the spectral Zakharov equations [4, 5], what allows the simulation of large wave ensembles within strongly- or fully nonlinear frameworks. A series of the research works based on the DNS of the Euler equations in 2D and 3D geometries has been performed in the recent years, see e.g. [6–9] among many others.

In our previous research we have performed the stochastic numerical simulation of waves on the water surface with spectral parameters typical for oceanic wind waves [10–12] using the High Order Spectral Method [13]. These simulations were performed with the purpose to better understand the mechanisms of generation of abnormally large waves, and to evaluate rogue wave characteristics, including the probability distribution function (PDF). In the simulations of unidirectional irregular deep-water waves extremely long-living wave groups were observed, which were related in [12] to the strongly nonlinear counterpart of the envelope solitons of the nonlinear Schrödiner equation. No firm justification of this conjecture was provided though.

Very short long-living wave groups in infinitely deep water with the maximum steepness close to the breaking onset were first observed in the numerical simulations of the primitive hydrodynamic equations in [14], and later in [15]. Such strongly nonlinear hydrodynamic envelope solitons were later on reproduced in laboratory conditions: single solitons first [16] and then interacting pairs of solitons [17]. However, these soliton groups were propagating on the surface of calm water and were not surrounded by other waves. Meanwhile the concept of soliton gas in hydrodynamics is discussed in recent publications with different level of rigor [18, 19].

In the present work we reconsider our observation of long-living wave groups in the numerical simulation of irregular sea waves and discuss their nature on a more solid background. We apply the nonlinear analysis based on the Inverse Scattering Transform (IST) [20, 21] which helps to reveal the soliton-type nonlinear groups and to estimate them quantitatively.

The IST was probably first applied to the analysis of the soliton content of water waves in the works [22, 23] for the experimental data under the conditions of shallow water. The potential of application of the IST to oceanic waves was discussed in detail in the book by A. Osborne [24], and in the series of publications [25–28]; it was also applied to optical pulses, see e.g. [29, 30]. The IST formulated in periodic interval was used in all the mentioned above studies except [30]. In all these works the location of a particular soliton may be in principle found by analyzing the eigenfunctions of the associated scattering problem, what is a demanding task. Following our method [20], the application of a sliding window transformation allows to better adjust the local carrier, to locate the solitons and to consider the scattering problem in infinite line for compact potentials. As we show below, these features provide with surprisingly robust results of the IST-based analysis when applied to evolving strongly nonlinear strongly modulated wave trains.

The paper is organized as follows. The conditions of the numerical simulation when the long-living wave groups were observed are discussed in Sec. II. The principle scheme of the IST-based nonlinear analysis is explained in Sec. III; it is applied to a toy example of a single hydrodynamic envelope soliton. The main part of the work is presented in Sec. IV, where the numerically simulated evolution of strongly nonlinear unidirectional water waves is interpreted with the help of the windowed IST analysis. We conclude the paper with a discussion.
of the significance of the results and of the perspectives in Sec. V.

II. DIRECT NUMERICAL SIMULATION OF IRREGULAR UNIDIRECTIONAL WAVES IN DEEP WATER

The simulations of planar surface waves with one horizontal axis $Ox$ is discussed in this paper. The initial condition is specified according to the JONSWAP-shape power spectrum

$$S(\omega) \sim \left(\frac{\omega}{\omega_p}\right)^{-5} \exp \left[ -\frac{5}{4} \left( \frac{\omega}{\omega_p} \right)^{-4} \right] r^\gamma, \quad (1)$$

$$r = \exp \left[ -\frac{1}{2\delta^2} \left( \frac{\omega - \omega_p}{\omega_p} \right)^2 \right],$$

$$\delta = \begin{cases} 0.07 & \text{if } \omega < \omega_p, \\ 0.09 & \text{if } \omega > \omega_p, \end{cases}$$

with the prescribed peak wave period, $T_p = 2\pi/\omega_p = 10$ s, significant wave height, $H_{1/3} \approx 4\sigma \approx 3.5$ m (where $\sigma$ is the standard deviation of the surface displacement), and the spectrum peakedness parameter $\gamma = 3$. The frequency spectrum (1) is transformed to the wavenumber spectrum $S(k)$ using the dispersion law for infinitely deep water,

$$\omega^2 = gk, \quad (2)$$

where $g$ is the acceleration due to gravity.

The wave evolution in time is calculated by means of the High Order Spectral Method (HOSM, [13]). Basically, the evolution of the wave sequence is solved for $120T_p$ in a 10-km domain, what corresponds to 60 dominant wave lengths $\lambda_p = 2\pi/k_p$, where $k_p$ is related to $\omega_p$ according to (2). Periodic boundary conditions apply. The statistical database is represented by many ($N_R > 100$) repetitions of the numerical experiment with the same initial power spectrum $S(k)$, each corresponds to a realization of random wave phases of the initial condition. In our numerical experiments we store the simulated water data with high resolution in both, space and time, and hence are able to examine the spatio-temporal wave evolution in detail.

The time series at a few ($N_L \geq 1$) equally spaced locations within the simulated domain are retrieved and used for the further statistical analysis. Accordingly, the total number of the processed time series is $N_R N_L$. The number $N_L$ may take the value from 1 up to the number of grid points along the $Ox$ axis, which is as much as $N_x = 2048$. One should understand that in the case of too dense locations of the ”measurement” points (i.e., when $N_L$ is too large), the time series become correlated what can lead to spurious artifacts in the statistics.

When analyzing the collected data, we noticed in [12] that though the wave height probability distribution function averaged over the large number of realizations agreed rather well with the celebrated Rayleigh distribution (see Fig. 1), the PDFs plotted for some particular realizations demonstrated an extraordinary behavior (see realization No 295 in Fig. 1). In the figure $H$ denotes the wave height (calculated from the time series according to the zero-crossing method). The significant wave height $H_{1/3}$ is defined as the mean of the one-third of the highest waves.

The curves in Fig. 1 represent the wave height exceedance probability distributions for $N_R = 100$, $N_R = 300$ and $N_R = 999$ realizations (see the legend), when the time series from all the grid points are used (i.e., $N_L = N_x$). Note that the curves for $N_R = 300$ and $N_R = 999$ practically coincide within the range $H/H_{1/3} < 3$ and clearly exhibit increase of the probability above the Rayleigh reference curve starting from the wave height excess of about $H/H_{1/3} = 2$. The filled areas show the estimate of the confidence interval calculated as the standard deviation among the PDFs obtained for the time series retrieved from different $N_x$ locations (i.e., $N_L = 1$ for the every subset), see details in [12]. The PDF for the particular realization No 295 is shown with the black dash-dotted curve; it is well above the averaged distribution functions, demonstrating much higher probability of large waves.

The thorough investigation of the realization No 295

FIG. 1. Exceedance probability distribution functions calculated in the direct numerical simulation for different number of realizations (see the numbers $N_R$ in the legend). The PDF for a particular realization No 295 is plotted by the dedicated curve. The Rayleigh function is given by the dotted line (note that the horizontal axis corresponds to the squared scaled wave height).
reveals the mechanism of occurrence of the abnormal PDF. In this and a few other realizations intense wave groups are formed at the early stage of the evolution. They preserve their group structure for surprisingly long. The sea surface simulated in the series No 295 for a longer period of 240$T_p$ is shown in Fig. 2. A few samples of the momentary surface displacements are also plotted by the white curves. The intense group (close to the coordinate origin) may be easily discerned for more than 200 dominant wave periods $T_p$. The reference moving with the group velocity of the linear waves, $C_{gr} = \omega_p/k_p/2$, is used in the figure.

It was speculated in [12] that such a group should be a strongly nonlinear analogue of the envelope soliton known within the framework of the nonlinear Schrödinger (NLS) equation. It is well known that within the framework of the nonlinear Schrödinger theory solitons can emerge at the late stage of development of the modulational (Benjamin–Feir) instability. The windowed IST analysis which may be employed to study time series or spatial series was outlined in [20]. In the present paper we apply it to the spatial series (snapshots) of unidirectional waves in deep water. To this end we assume that the focusing nonlinear Schrödinger equation

$$i \left( \frac{\partial A}{\partial t} + C_{gr} \frac{\partial A}{\partial x} \right) + \frac{\omega_0}{8 k_0^2} \frac{\partial^2 A}{\partial x^2} - \frac{\omega_0 k_0^2}{2} |A|^2 A = 0$$

can serve as the local first approximation to real water waves. Here $k_0 > 0$ stands for the carrier wavenumber and $\omega_0$ is the cyclic frequency; the latter are linked according to (2); $C_{gr} = \omega_0/k_0/2$ is the linear group velocity of the carrier. The complex amplitude $A(x, t)$ is related to the surface displacement $\eta(x, t)$ according to the formulas which take into account three asymptotic orders,

$$\eta(x, t) = \eta + \eta^{(1)} + \eta^{(2)} + \eta^{(3)},$$

$$\eta^{(1)} = \text{Re} \left( AE \right), \quad \eta^{(2)} = \frac{k_0}{2} \text{Re} \left( A^2 E^2 \right),$$

$$\eta^{(3)} = -\frac{1}{2} \text{Im} \left( A^2 E \right) + \frac{3 k_0^2}{8} \text{Re} \left( AE \right),$$

$$\eta = \frac{1}{4} \mathcal{H} \frac{\partial |A|^2}{\partial x}, \quad E = \exp \left( i \omega_0 t - ik_0 x \right).$$

Here $\mathcal{H}$ denotes the Hilbert transform, see details in [36, 37]. The NLS theory implies the assumption of a small wave steepness, $k_0|\eta| \ll 1$, and of a narrow spectrum, $\delta_k \ll 1$, where $\delta_k$ is the dimensionless width of the wavenumber spectrum.

The equation (4) may be further reduced to the dimensionless form

$$\frac{\partial Q}{\partial T} + \frac{\partial^2 Q}{\partial X^2} + 2|Q|^2 Q = 0$$

FIG. 2. False color representation of the evolution of the sea surface in the experiment No 295, and a few snapshots of the surface (white curves). The comoving frame of reference is used.
after the change of variables

\[ X = \sqrt{2}s_0k_0 \left( x - C_{gr}t \right) , \quad T = s_0^2\omega_0 t , \quad Q = \frac{k_0}{s_0} A(X,t) \]  

where we choose the reference steepness \( s_0 \) to be specified by the maximum amplitude of the complex envelope, \( s_0 = k_0 \max \{|A|\} \). Then the modulus of the new function \( Q(X,T) \) will be limited by one.

The celebrated envelope solitons, which are eternally stable localized wave groups, solutions of the equation (6), read

\[ Q(X,T) = a_s \frac{\exp \left( i\phi_s + i \left( \frac{a_s^2 - (\frac{s}{2})^2}{T} + \frac{s}{2} X \right) \right)}{\cosh \left[ a_s (X - v_s T - X_s) \right]} . \]  

Here \( a_s \) and \( v_s \) are the dimensionless amplitude and velocity of the soliton, \( X_s \) and \( \phi_s \) are the location and complex phase of the soliton respectively.

If envelope solitons are present in the given wave field \( Q(X) \), then their amplitudes and velocities may be recovered from the solution of the boundary-value problem on the eigenfunctions \( \psi_1(X) \) and \( \psi_2(X) \) [38],

\[ \frac{d}{dX} \left( \psi_1 \psi_2 \right) = \left( \frac{\mu}{-Q^*(X)} - \mu \right) \left( \frac{\psi_1}{\psi_2} \right) , \]  

\[ \mu = \frac{a_s}{2} + i\frac{v_s}{4} . \]  

The discrete eigenvalues \( \mu \) specify amplitudes and velocities of the envelope solitons, though their locations and phases are determined by the eigenfunctions.

Restricting our attention to finite potentials \( Q(X) \) with compact support and to the discrete eigenspectrum (which determines solitons), the functions \( \psi_1(X) \) and \( \psi_2(X) \) should be finite in the infinite line \( -\infty < X < \infty \). This restriction allows to approximate the problem in infinite line by the one in a finite interval, and to formulate the corresponding boundary conditions. As a result, the boundary-value problem may be efficiently solved numerically (we employ the shooting method), and the eigenvalues \( \{\mu\} \) (one or several) specify the envelope solitons. The dimensional amplitudes, \( A_s \), and velocities, \( V_s \), of the envelope solitons are obtained by inverting the transformations (7),

\[ A_s = \frac{2s_0}{k_0} \Re(\mu) , \quad V_s = C_{gr} + \frac{s_0\omega_0}{\sqrt{2k_0}} \Im(\mu) . \]  

As was mentioned above, the NLS theory implies the narrowband condition, what is hardly fulfilled in the real sea. At the same time waves in deep water are known to form groups, therefore it may be expected that within some small spatial areas the local wavelengths of energetic waves are relatively similar. Then the NLS model should be more efficient in approximating the wave surface within shorter spatial intervals.

At the same time, it may be realized, that the estimation of the soliton content strongly depends on the choice of the carrier wavenumber \( k_0 \). The similarity parameter of the NLS equation (4) is in fact proportional to \( BF1^2 \), which for the given wavenumber spectrum width \( \Delta k \) and standard deviation \( \sigma \) reads \( BF1 = 2\sqrt{2k_0^2 \sigma / \Delta k} \). Here \( \delta_1 = \Delta k / \kappa_0 = 2\delta_0 \) due to the dispersion law (2). Hence, \( BF1 \) is proportional to the squared carrier wavenumber.

The carrier wavenumber is not well-defined in the situation of a broad wave spectrum. On the other hand, the accuracy of estimation of the carrier wavenumber \( k_0 \) in shorter samples decreases. This problem was in the focus of our paper [21], where different ways to estimate the parameter \( k_0 \) were examined. In the present work we choose \( k_0 \) to be equal to the mean wavenumber calculated for the power Fourier transform of the given short sample of the spatial series. The intrinsic wavenumber of soliton may differ from \( k_0 \) due to the imaginary part of the corresponding eigenvalue \( \mu \), see (10) and (11).

The use of a sliding window along the \( x \) coordinate (boxcar transform) allows us to locate the detected soliton groups as described below. From the practical point of view, only solitary groups which possess relatively large amplitudes are of the interest. Envelope solitons with larger amplitudes are narrower (see (8)), therefore the minimum width of an intense soliton may be calculated based on the characteristic intensity of the wave sample, what allows us to estimate the necessary width of the sliding window \( L_w \) from above. In the analysis we define \( L_w \) according to the formula

\[ L_w = \frac{W}{\sqrt{2s_0k_0}} , \]  

where \( W \) is a number of the order of 20 (see details in [21]), and \( s_0 = k_0 \max \{|A|\} \) is the maximum wave steepness as discussed above.

In Fig. 3 the windowed IST procedure is applied to a snapshot of an envelope soliton. The surface displacement \( \eta(x) \) (shown with the solid line in Fig. 3) is calculated from \( Q(X) \) (8) according to (7) and (5). The maximum steepness of group may be roughly estimated based on the complex envelope \( A(x) \) as \( s_0 = 0.3 \), though the amplitude of the wave crest \( (k_0 \max \eta) \) is in fact larger due to the nonlinear harmonics introduced by the reconstruction formulas (5).

![FIG. 3. Envelope soliton with the steepness \( k_0 \max |A| = 0.3 \) (the solid line) and the soliton amplitudes \( A_s \) detected using the windowed IST-based procedure with (empty red circles) and without (full blue circles) the Hanning mask, \( W = 20 \).](image-url)
The windowed transform splits the long spatial series into overlapping shorter samples of the length \( L_w \). A sample \( \eta(x) \) is the only input data which is used by the procedure of the IST analysis. At first, the complex amplitude \( A_w(x) \) is calculated with the help of an iterative procedure which inverts the relation (5), similar to [37, 39] using the local mean wavenumber as the value of \( k_0 \). Then \( A_w(x) \) is transformed to \( Q_w(X) \) using (7), and after that the eigenvalue problem (9) with appropriate boundary conditions is solved to obtain the discrete eigenvalues \( \{\mu\} \). The soliton parameters \( \{A_s\} \) and \( \{V_s\} \) are calculated for the given sample as a result.

The values of \( A_s \) obtained in the sliding window are shown with symbols in Fig. 3 as the function of location of the window \( x_w \). The estimated soliton amplitude reaches the maximum when the window covers the entire soliton group. The soliton amplitudes decay when the lesser part of the soliton is captured by the window. Consequently, the soliton amplitude as the function of coordinate, \( A_s(x_w) \), forms an arc, see Fig. 3. When the sliding window is significantly longer than the soliton width, the estimated soliton amplitudes in the neighboring samples have similar values and form a plateau of the arc (the full blue circles in Fig. 3). The value of \( A_s \) achieved in the central part of the ‘arc’ is used as the estimator of the true amplitude of the envelope soliton. The location of the soliton may be better estimated using a slightly smaller value of \( W \) or by applying the Hanning mask which makes the procedure more stable (the empty red circles in Fig. 3).

The actual procedure of the IST analysis which is applied in this work consists of two steps. A shorter window with the Hanning mask is used at the first stage to locate envelope solitons. At the second stage the problem is solved more precisely for the selected solitons among the largest revealed soliton groups. At the second stage a longer window of a rectangular shape is used. The windowed IST procedure was tested in [21] on the example of soliton-type solutions of the primitive Euler equations. It was shown that the soliton amplitudes \( A_s \) obtained using this procedure are equal to the actual envelope amplitudes, \( A_{env} = (A_{cr} + A_{tr})/2 \), with the accuracy of about 10-15% for the range of steepness \( k_p A_{env} < 0.3 \). Here \( A_{cr} \) and \( A_{tr} \) are the maximum amplitudes of the crest and of the trough respectively of the waves which belong to the soliton group. The soliton group shown in Fig. 3 consists of about two wave cycles; its steepness \( k_p A_{env} \) is about 0.3. Therefore the result of the tests reported in [21] seems to be unexpectedly good bearing in mind the implied assumptions and approximations. The advantage of the IST-based analysis is that it can disclose envelope solitons even when they are completely disguised by surrounding random waves.

In the next section the described nonlinear analysis is applied to the snapshots of evolving irregular waves shown in Fig. 2. The procedure is fully automated; it uses the spatial series of the surface displacement as the only input data.

IV. REVEALED HYDRODYNAMIC ENVELOPE SOLITONS

The water surface evolution which is analyzed using the IST-based procedure is shown in Fig. 2; details of the numerical simulation are described in Sec. II, see also [12]. Under the selected conditions the wave variance and the ensemble averaged instant wavenumber spectrum do not change significantly in the course of the wave evolution.

A snapshot of the surface at the moment not long after the beginning of the simulation, \( t \approx 10T_p \), is shown in Fig. 4 by the solid curve. Symbols in Fig. 4a provide with the result of the first step of the windowed IST procedure, which seeks for the solitons in a short sliding window, similar to Fig. 3. One may see that many groups in the series are recognized as solitons, though there are only few of them which possess significant amplitudes. Two ‘arcs’ which reveal soliton-type groups may be seen in the interval between about 5500 m and 8000 m. At the second step, the ‘arcs’ are analyzed by a dedicated subroutine, and the locations of the maximum solitons are determined (see the blue curve with the red dot in Fig. 4b). Then, the eigenvalue problem is solved once again in a larger interval around the selected solitons, what gives more accurate estimates (the stars in Fig. 4b).

The parameters of the revealed large-amplitude solitons (amplitude \( A_s \) and velocity \( V_s \)) are indicated in Fig. 4b. At this instant, two large-amplitude soliton groups are recognized as the result. The largest has the amplitude \( A_{env} \approx A_s \approx 3.44 \text{ m} \) (which is close to the significant wave height \( 3.5 \text{ m} \)) and the velocity is \( V_s \approx 7.66 \text{ m/s} \) (which is close to the estimation of the wave group velocity \( C_g \approx 7.81 \text{ m/s} \) for 10-s waves). Though the second arc in Fig. 4a indicates the presence of another soliton to the right from the maximum one (at approximately \( x = 7000 \text{ m} \)), it is not recognized by the automated procedure. According to Fig. 4b, the second revealed soliton at \( x \approx 2900 \text{ m} \) is of much smaller amplitude and propagates significantly slower. This estimate is consistent with the appearance of the waves, which look significantly shorter than at the location of the maximum soliton.

Two other examples how the IST-based procedure performs are shown in Fig. 5 and Fig. 6 for much later instants, after about 100 periods of the evolution. The case shown in Fig. 5 corresponds to the moment of one of the greatest elevations observed during the evolution. The case in Fig. 6 is about a half period earlier, when a very deep trough occurs. Note that in contrast to the case shown in Fig. 4, single soliton groups are found in the other examples at the first stage of the IST analysis. Hence, small-scale solitons seem to disappear in the course of the wave evolution. At the same time, the amplitudes and velocities of the maximum solitons in Figs. 4, 5, 6 are very close. In Fig. 6 the automated procedure has recognized two solitons with very similar parameters in close vicinity to each other. It is obviously the result of insufficient capability of the subroutine which determines...
The procedure is applied to the frequent sequence of the surface snapshots, what allows to trace the evolution of the soliton groups. For simplicity, we consider only the soliton with the largest amplitude among all recognized solitons at a given time. The evolution of the maximum soliton parameters are shown in Fig. 7. The amplitudes of the largest soliton are shown with circles in Fig. 7a. Note that (i) the values of the soliton amplitude at the neighboring instants do not scatter much, what confirms robustness of the result of the analysis, and (ii) the soliton amplitude is slowly varying, but does not change much during about 240 wave periods (varies within about 17%). These two observations are in accord with the data of the soliton velocity in Fig. 7b; the velocity of the maximum soliton varies within about 2%. The locations of the maximum soliton at each instant are shown in Fig. 8 with blue symbols. They correspond to the position of the intense wave group which is formed at the early stage of the evolution, thus this group remains the maximum soliton throughout the entire simulation period.

In Fig. 7a the solid curve shows the instant maxima of the modulus of the scaled surface displacement. It follows, that though within the first $50T_p$ the values of maxima of the surface displacement were similar to the solitary group amplitude (about the significant wave height $4\sigma$), later on about twice larger waves occur. The locations of the instant maximum displacements max $|\eta|$ are shown in Fig. 8 with white symbols. One may see that the absolute majority of the extreme waves occur on top of the soliton group. Thus, the soliton group plays decisive role in the formation of extreme waves in this simulation.

In Fig. 7b the linear wave group velocities calculated in the entire simulation domain basing on the wavenumber of the peak value of the momentary Fourier transform, and on the mean wavenumber, are shown by magenta crosses and black dots respectively. Due to the discrete-
ness of the resolved wavenumbers, the peak wavenumber may jump between several competing Fourier modes. The mean wavenumber is stable during the evolution, it is significantly larger than the peak value, and hence the corresponding velocity is smaller than others. As noted above, the soliton possesses almost constant velocity all the time; its wavelength approximately corresponds to the peak of the spectrum. Meanwhile, groups moving with different velocities may be readily seen in Fig. 8.

The evolution of the spatial Fourier transform of the surface within a short interval of $20\lambda_p$ following the maximum soliton is plotted in Fig. 7c. The intensification and fading of a few different wave modes may be observed. The carrier wavenumbers $k_0$ used by the IST procedure are shown with the green circles; they correspond to the local mean wavenumbers. The intrinsic wavenumber of the soliton $k_s$ may be calculated from the soliton velocity $V_s$, according to the dispersion relation (2), $k_s = g/(2V_s)^2$. These values are plotted in Fig. 7c by blue circles. One may see that the series of $k_s$ is much more stable than the local carrier wavenumbers $k_0$ and are slightly smaller in value. Note that the soliton wavenumber may correspond to none of the instant spectral peaks.

V. CONCLUSIONS

In this work we trace the evolution of a group of intense waves in the field of irregular deep-water waves with the help of the method based on the Inverse Scattering Transform. The method allows us to interpret the observed waves in terms of envelope solitons of the nonlinear Schrödinger equation. The soliton parameters estimated in close time instants do not show significant scatter; the soliton parameters are determined equally accurately at the instants of appearance of huge crests (Fig. 5) or deep troughs (Fig. 6) despite the fact that the wave conditions are obviously beyond the formal limits of the NLS theory.
waves are steep and the soliton group consists of just a few wave cycles). At the moment, the fully automated procedure may sometimes misinterpret the primary data of the windowed IST-based analysis, if several solitons interact or when the soliton group is extremely short. This issue requires a further improvement.

The wave evolution is calculated using the strongly nonlinear solver of the primitive potential water equations, hence is believed to be quite realistic. Surprisingly, an intense group is formed by chance from irregular waves and persists for remarkably long time. The IST-based analysis reveals the soliton nature of this group, and evaluates the key parameters of it, which are the intrinsic amplitude, velocity and location. The estimated amplitude and velocity of the soliton group vary a little during about 240 periods of the wave evolution, hence the soliton persists through nonlinear interactions with surrounding waves for so long. At the same time, the effect of a growing soliton which is acquiring the wave energy from smaller ones, known for nonintegrable systems (e.g. [40]), has not been observed.

Since the intrinsic amplitude of the soliton group is about the significant wave height and remains approximately constant, it is not much surprising that most of the extreme events occur on top of the soliton group and lead to heavier wings of the PDF. The maximum surface displacement observed in the simulation is about 2 significant wave heights, $A_{cr}/H_{1/3} \approx 1.9$, which is an extraordinary value. It may be interpreted as the superposition of the soliton group with the remaining irregular wave background. This idea is in line with the discussion of in-situ rogue wave registrations in [20], where a significant part of the recorded rogue waves was explained as the combination of soliton-like groups with the random background. In the present work we confirm possibility of this scenario of rogue wave generation using the direct numerical simulation.

The occurrence of long-living soliton-type patterns which facilitate the generation of extremely high waves should also lead to longer rogue wave events, in agreement with the direct evaluation of 3D rogue wave lifetimes reported in [49, 50].

Soliton-type groups and Peregrine breather-type wave patterns were found in water wave sequences by eye or by fitting in, e.g. [34, 41–43], and also through the analysis of the associated scattering problem for the entire wave record, e.g. [18, 44, 45]. There is also significant amount of relevant research in the field of nonlinear optics [46–48]. However, to the best of our knowledge no one has observed truly soliton groups propagating through intense irregular broad-banded water waves for so long. Hence in this work we underpin the interpretation of nonlinear modulated oceanic waves in terms of a soliton gas by the solid basis.

The planar geometry seems to be the major restriction of the considered wave simulation. However, in our preliminary study of three-dimensional waves a soliton-like wave pattern is able to survive for a few tens of wave periods if the angle spectrum is not too broad (these results will be reported elsewhere). Thus, a wave group dynamics similar to the one presented in the this work can occur in realistic long-crested oceanic waves.

![FIG. 6. Same as in Fig. 4, but for the instant $t = 1035$ s.](image)

![FIG. 7. Evolution of the amplitude (a) and the velocity (b) of the maximum revealed soliton group. The soliton velocity $V_s$ is compared with the peak and mean wave velocities calculated for the entire domain. The carrier wavenumbers $k_0$ used by the IST procedure and the soliton wavenumbers $k_s$ are shown over the instantaneous spatial Fourier transform of the short interval taken at the current locations of the soliton](image)
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