Weak Hopf Algebras unify the Hennings–Kauffman–Radford and the Reshetikhin–Turaev invariant

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Abstract

We present an invariant of connected and oriented closed 3-manifolds based on a coribon Weak Hopf Algebra $H$ with a suitable left-integral. Our invariant can be understood as the generalization to Weak Hopf Algebras of the Hennings–Kauffman–Radford evaluation of an unoriented framed link using a dual quantum-trace. This quantum trace satisfies conditions that render the link evaluation invariant under Kirby moves. If $H$ is a suitable finite-dimensional Hopf algebra (not weak), our invariant reduces to the Kauffman–Radford invariant for the dual of $H$. If $H$ is the Weak Hopf Algebra Tannaka–Kreˇın reconstructed from a modular category $\mathcal{C}$, our invariant agrees with the Reshetikhin–Turaev invariant. In particular, the proof of invariance of the Reshetikhin–Turaev invariant becomes as simple as that of the Kauffman–Radford invariant. Modularity of $\mathcal{C}$ is only used once in order to show that the invariant is non-zero; apart from this, a fusion category with ribbon structure would be sufficient. Our generalization of the Kauffman–Radford invariant for a Weak Hopf Algebra $H$ and the Reshetikhin–Turaev invariant for its category of finite-dimensional comodules $\mathcal{C} \simeq \mathcal{M}^H$ always agree by construction. There is no need to consider a quotient of the representation category modulo ‘negligible morphisms’ at any stage, and our construction contains the Reshetikhin–Turaev invariant for an arbitrary modular category $\mathcal{C}$, whether its relationship with some quantum group is known or not.

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1 Introduction

In some special cases, the following two quantum invariants of connected and oriented closed (smooth) 3-manifolds are related: the invariant of Reshetikhin–Turaev [1] and the invariant of Hennings [2]. Since the latter has been studied mainly for finite-dimensional unimodular ribbon Hopf algebras, we focus on the reformulation of the Hennings invariant according to Kauffman and Radford [3]. This reformulation exploits the fact that these special Hopf algebras have a unique cointegral with suitable properties, and this results in a substantial simplification of Hennings’ original construction.

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It has already been demonstrated by Hennings [2] that the Reshetikhin–Turaev invariant for the modular categories associated with $U_q(\mathfrak{sl}_2)$ at suitable roots of unity $q$ appears as a special case of his construction. In order to show this, one needs to understand the quotient of the category of tilting modules over $U_q(\mathfrak{sl}_2)$ modulo so-called negligible morphisms. This quotient is finitely semisimple and has the structure of a modular category.

This example of Hennings raises the question of whether the agreement of the Hennings invariant with the Reshetikhin–Turaev invariant is a coincidence or whether there is an explanation in conceptual terms. One answer to this question was given by Lyubashenko [4]. Starting from the fusion category with ribbon structure $\mathcal{C}$ used in the Reshetikhin–Turaev invariant, he uses Majid’s [5] universal coend $F = \text{coend}(\mathcal{C}, 1_\mathcal{C})$ over the identity functor $1_\mathcal{C}: \mathcal{C} \to \mathcal{C}$ which forms a Hopf algebra object $F \in |\mathcal{C}|$ (braided group in Majid’s terminology). If $\mathcal{C} \cong H_M$ is the category of modules over a finite-dimensional Hopf algebra $H$, Lyubashenko’s invariant for $F \in |\mathcal{C}|$ coincides with the Kauffman–Radford invariant for $H$.

Let us summarize the idea of Lyubashenko’s unification of the invariants as follows: He forces the Hopf algebra $H$ of the Kauffman–Radford invariant into the language of the Reshetikhin–Turaev invariant by transmuting [5, Section 4] it into a Hopf algebra object $F \in |\mathcal{C}|$.

The purpose of the present article is to reverse this approach and to present a second way of unifying the Reshetikhin–Turaev with the Kauffman–Radford invariant, this time by forcing the fusion category with ribbon structure $\mathcal{C}$ into the language of the Kauffman–Radford invariant. Thanks to the recent generalization of Tannaka–Kreĭn reconstruction to fusion categories [6, 7], we know that each fusion category $\mathcal{C}$, in particular each modular category, is equivalent to the category of finite-dimensional comodules $\mathcal{C} \cong \mathcal{M}^H$ over a Weak Hopf Algebra (WHA) $H = \text{coend}(\mathcal{C}, \omega)$, the universal coend over the long canonical functor $\omega: \mathcal{C} \to \text{Vect}_k$. We are therefore able to recover $\mathcal{C}$ from $H$ and can therefore express the Reshetikhin–Turaev invariant entirely in terms of $H$, i.e. in the language of the Kauffman–Radford invariant, something that is not possible in Lyubashenko’s approach. One of the advantages of this point of view is that it renders the proof of invariance of the Reshetikhin–Turaev invariant as easy as that of the Kauffman–Radford invariant whereas Lyubashenko’s approach renders the proof of invariance of the Kauffman–Radford invariant as difficult as that of the Reshetikhin–Turaev invariant.

Let us now sketch how one can find a (co)algebra with extra structure $H$ such that the Reshetikhin–Turaev invariant for $\mathcal{C}$ agrees with the Kauffman–Redford invariant for $H$. It appears that all modular categories that yield interesting 3-manifold invariants, i.e. invariants that are stronger than invariants of homotopy type, have objects of non-integer Frobenius–Perron dimension [1]. These categories therefore do not form the categories of modules over any Hopf algebra, see, for example [8, Theorem 8.33]. The naive conjecture that the Hennings invariant for a Hopf algebra $H$ might agree with the Reshetikhin–Turaev invariant for the category $H_M$ of modules over $H$, is therefore not even well phrased.

Any conjecture on a coincidence of the Hennings with the Reshetikhin–Turaev invariant has a chance of being true only if it is not $H_M$ itself, but rather a quotient of $H_M$ modulo suitable negligible morphisms, that forms the modular category. This is how Hennings’ original example works. But since there exist modular categories for which no relationship with one of the standard quantum groups is known [9], any conceptual approach to relating the Hennings with the Reshetikhin–Turaev invariant needs to avoid taking a quotient of the representation category.

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1 This is an observation about the existing literature, and we are not aware of any counter-example.
Although the topologically interesting modular categories are not the categories of modules over any Hopf algebra, every modular category is the category of (co)modules over a Weak Hopf Algebra (WHA) \[6\] — no quotient required: for each modular category \(\mathcal{C}\), one can Tannaka–Kreĭn reconstruct a WHA \(H\) whose category of finite-dimensional comodules \(\mathcal{M}^H\) is equivalent as a \(k\)-linear additive ribbon category to the original modular category \(\mathcal{C}\).

In the present article, we define an invariant of 3-manifolds for a suitable class of WHAs (see Theorem \(5.1\) below). Our invariant can be understood as the generalization of the Kauffman–Radford invariant from Hopf algebras to WHAs. In the special case in which the WHA \(H\) is a Hopf algebra, our invariant reduces to the Kauffman–Radford invariant (for the dual \(\mathfrak{sl}_2\) of \(H\)). In the special case in which \(H\) is the WHA reconstructed from a modular category \(\mathcal{C}\), our invariant agrees with the Reshetikhin–Turaev invariant for \(\mathcal{C}\).

In fact, by construction, our generalization of the Kauffman–Radford invariant for a WHA \(H\) always agrees with the Reshetikhin–Turaev invariant for the ribbon category \(\mathcal{M}^H\). Modularity of \(\mathcal{M}^H\) is sufficient for the invariant to be non-zero. As a by-product, we obtain a new proof of invariance of the Reshetikhin–Turaev invariant for an arbitrary modular category \(\mathcal{C}\) using computations in \(H\) rather than computations in \(\mathcal{C}\). This new proof is substantially shorter than the original proof presented in \([10\) Section II.3.\]

In the near future, there will be a companion article relating the Turaev–Viro invariant \([11]\) with the Kuperberg invariant \([12]\) along the same lines. Apart from tidying up some twenty-year-old results, these identities between quantum invariants can be expected to prove useful if one tries to categorify these invariants. Whereas categorifying a modular category to some 2-category with extra structure is still far beyond reach, categorifying the reconstructed WHAs appears to be much more promising. In particular, the WHAs reconstructed from the modular categories associated with \(U_q(\mathfrak{sl}_2)\) at roots of unity itself together with the relevant quotient of its category of modules.

The present article is structured as follows. In Section \(2\) we briefly summarize some background material on WHAs and on Tannaka–Kreĭn reconstruction. Section \(3\) contains results on integrals and cointegrals in the WHA reconstructed from a fusion category with ribbon structure. We then define various ways of evaluating unoriented framed links in \(S^3\) in Section \(4\). Our new invariant is presented in Section \(5\) in which we also show that it encompasses both the Reshetikhin–Turaev invariant and the Kauffman–Radford invariant. Appendix \(A\) contains more details on WHAs with extra structure, and Appendix \(B\) on their Tannaka–Kreĭn reconstruction from fusion categories with extra structure.

## 2 Preliminaries

In this section, we fix our notation and sketch some background material on Weak Hopf Algebras, their categories of finite-dimensional comodules, and on the canonical Weak Hopf Algebra associated to each modular category via generalized Tannaka–Kreĭn reconstruction.

We use the following notation. If \(\mathcal{C}\) is a category, we write \(X \in |\mathcal{C}|\) for the objects \(X\) of \(\mathcal{C}\), \(\operatorname{Hom}(X,Y)\) for the collection of all morphisms \(f: X \to Y\) and \(\operatorname{End}(X) = \operatorname{Hom}(X,X)\). We denote the identity morphism of \(X\) by \(\operatorname{id}_X: X \to X\) and the composition of morphisms \(f: X \to Y\) and \(g: Y \to Z\) by \(g \circ f: X \to Z\). If two objects \(X,Y \in |\mathcal{C}|\) are isomorphic,

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\(^2\)For technical reasons, we work with comodules rather than modules, and so our WHA \(H\) corresponds to the dual of the Hopf algebra featuring in the Kauffman–Radford invariant, and it is the category of finite-dimensional comodules \(\mathcal{M}^H\) of \(H\) that forms the modular category.
we write $X \cong Y$. If two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent, we write $\mathcal{C} \simeq \mathcal{D}$. The identity functor on $\mathcal{C}$ is denoted by $1_{\mathcal{C}}$. The category of vector spaces over a field $k$ is denoted by $\text{Vect}_k$ and its full subcategory of finite-dimensional vector spaces by $\text{vect}_k$. Both are $k$-linear, abelian and symmetric monoidal, and $\text{vect}_k$ is autonomous. The $n$-fold tensor power of some object $X \in |\mathcal{C}|$ of a monoidal category $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)$ is denoted by $X^{\otimes n}$, $n \in \mathbb{N}_0$. We set $X^{\otimes 0} := 1$. We use the notation $\mathbb{N}$ and $\mathbb{N}_0$ for the positive integers and the non-negative integers, respectively.

2.1 Weak Hopf Algebras and their corepresentations

For the basics of Weak Bialgebras (WBAs) and Weak Hopf Algebras (WHAs), we refer to [13, 14] and to Appendix A. In a WHA $H$ over some field $k$, we denote by $\mu: H \otimes H \to H$, $\eta: k \to H$, $\Delta: H \to H \otimes H$, $\varepsilon: H \to k$ and $S: H \to H$ the multiplication, unit, comultiplication, counit and antipode, respectively. The source and target counital maps are denoted by $\varepsilon_s$ and $\varepsilon_t$, and the source and target base algebras by $H_s$ and $H_t$, respectively. The opposite comultiplication is given by $\Delta^{\text{op}} = \tau_{H,H} \circ \Delta$ where $\tau_{H,H}(x \otimes y) = y \otimes x$ for all $x, y \in H$.

The category of right $H$-comodules that are finite-dimensional over $k$, is denoted by $\mathcal{M}^H$. It is a $k$-linear abelian and left-autonomous monoidal category equipped with a $k$-linear, faithful and exact forgetful functor $U^H: \mathcal{M}^H \to \text{vect}_k$. This functor is in general not strong monoidal and thereby not a fibre functor in the technical sense, but it is equipped with a separable Frobenius structure.

A copivotal form $w: H \to k$ for $H$ is a dual group-like linear form such that $S^2(x) = w(x)x^n \overline{w}(x^m)$ for all $x \in H$, i.e. one that implements the square of the antipode by dual conjugation. Here $\overline{w}: H \to k$ denotes the convolution inverse of $w$. We call such a linear form $w$ copivotal because it is this structure that renders the category $\mathcal{M}^H$ a pivotal category.

The universal $r$-form of a coquasi-triangular WHA and its weak convolution inverse are denoted by $r: H \otimes H \to k$ and $\overline{r}: H \otimes H \to k$, respectively. Similarly, $\nu: H \to k$ and $\overline{\nu}: H \to k$ denote the universal ribbon form and its convolution inverse in a coribbon WHA. Recall that each coribbon WHA is copivotal with $w(x) = v(x')\nu(x'')$ for all $x \in H$, involving the second dual Drinfel’d element $v: H \to k$, $x \mapsto r(S(x') \otimes x'')$ and the universal ribbon form $\nu$.

For the convenience of the reader, we have collected in Appendix A the basic definitions and more detailed references to the literature as well as the basic facts about the relevant additional structure on $\mathcal{M}^H$. For example, if $H$ is coribbon, then $\mathcal{M}^H$ is a ribbon category. If $H$ is finite-dimensional, split cosemisimple and pure, then $\mathcal{M}^H$ is fusion, and if $H$ is in addition coribbon and weakly cofactorizable, then $\mathcal{M}^H$ forms a modular category.

2.2 Tannaka–Krein reconstruction

For every multi-fusion category $\mathcal{C}$ that is $k$-linear over some field $k$, there is a canonical functor

$$\omega: \mathcal{C} \to \text{Vect}_k, \quad X \mapsto \text{Hom}_k(\hat{V}, \hat{V} \otimes X).$$

(2.1)

Here we use the small progenerator

$$\hat{V} = \bigoplus_{j \in I} V_j,$$

(2.2)

where the biproduct is over a set $I$ of one representative $V_j$, $j \in I$, for each isomorphism class of simple objects of $\mathcal{C}$. The functor $\omega$ is known as the long canonical functor [15,16] and can be used in order to Tannaka–Krein reconstruct a finite-dimensional split cosemisimple
coassociative counital coalgebra $H$ over $k$. It is given by the universal coend $H = \text{coend}(\mathcal{C}, \omega)$ of $\omega$, and $\mathcal{M}^H \simeq \mathcal{C}$ are equivalent as $k$-linear additive categories. Since $\omega$ has a separable Frobenius structure \cite{[6,7]}, $H$ forms a WHA, and $\mathcal{M}^H \simeq \mathcal{C}$ are equivalent as monoidal categories as well.

If $\mathcal{C}$ carries a pivotal, spherical or ribbon structure, then $H$ is copivotal, cospherical or coribbon, respectively. If $\mathcal{C}$ is fusion, then $H$ is copure, and if $\mathcal{C}$ is modular, then $H$ is weakly cofactorizable \cite{[6,17]}. In all of these cases, the equivalence $\mathcal{M}^H \simeq \mathcal{C}$ is compatible with the extra structure. In Appendix B, we have compiled more details on how the extra structure of $H$ is related to that of $\mathcal{C}$ and on how to perform computations in $H$.

Note in particular that $\omega_X = \text{Hom}_k(\widehat{V}, \widehat{V} \otimes X)$ and $\text{Hom}_k(\widehat{V} \otimes X, \widehat{V})$ are dually paired for all $X \in |\mathcal{C}|$, see (B.2), and that most computations can be conveniently phrased in terms of pairs of dual bases $(e_m^{(X)})_m$ and $(e_m^{(\omega_X)})_m$ of $\omega X$ and $(\omega X)^*$, respectively. The vector space underlying the reconstructed WHA $H$ is given in (B.4). Also note the reconstruction of the copivotal form $w$ of (B.16) and the isomorphism $D_C \in \text{End}(\widehat{V})$ of (B.18) involved.

### 3 The reconstructed Weak Hopf Algebra

In order to see how our invariant encompasses the Reshetikhin–Turaev invariant \cite{[1]}, we need to develop some of the integral theory of the WHA reconstructed from a modular category $\mathcal{C}$. This is done in the present section. For background material on the integral theory of WHAs, we refer to [13].

Let $H$ be a coribbon WHA. A dual trace is an element $\chi \in H$ such that $\Delta^\text{op}(\chi) = \Delta(\chi)$. It is called $S$-invariant if $S(\chi) = \chi$. A dual quantum trace is an element $t \in H$ such that $\Delta^\text{op}(t) = (\text{id}_H \otimes S^2) \circ \Delta(t)$. A dual quantum trace $t \in H$ is called $S$-compatible if

$$\overline{w}(t \prime)S(t \prime) = \overline{w}(t)\overline{t}.$$  \hfill (3.1)

Observe that $t = w(\chi')\overline{\chi''}$ is a dual quantum trace if and only if $\chi$ is a dual trace. In this situation, $t$ is $S$-compatible if and only if $\chi$ is $S$-invariant.

Note that for each $V \in |\mathcal{M}^H|$, its dual character $\chi_V \in H$ (see (A.47)) forms a dual trace, and its dual quantum character $T_V \in H$ (see (A.48)) is a dual quantum trace. Observe that $S(\chi_V) = \chi_V^*$ and if $V^* \cong V$ in $\mathcal{M}^H$, then we have in addition that $\chi_V^* = \chi_V$, i.e. the dual character $\chi_V$ is $S$-invariant.

An element $t \in H$ of a finite-dimensional WHA is called non-degenerate if $H^* \rightarrow k, \varphi \mapsto \varphi(t)$ is non-degenerate as a functional on the dual WHA $H^*$, i.e. its kernel does not contain any non-zero left-ideal of $H^*$. This holds if and only if the bilinear form $H^* \otimes H^* \rightarrow k, \varphi \otimes \psi \mapsto \varphi(t')\psi(t'')$ is non-degenerate. A left-integral $\ell \in H$ is an element that satisfies $x\ell = \varepsilon_\\ell(x)\ell$ for all $x \in H$. A right-integral $r \in H$ is an element that satisfies $rx = r\varepsilon_s(x)$ for all $x \in H$. A two-sided integral is both a left- and a right-integral. A [left-,right-]cointegral of $H$ is a [left-,right-]integral of $H^*$.

A linear form $\zeta: H \rightarrow k$ is called dual central if $\zeta(x')x'' = x'\zeta(x'')$ for all $x \in H$. If $H$ is the WHA reconstructed from a multi-fusion category $\mathcal{C}$, then it is split cosemisimple, and so its dual central linear forms can be computed as follows.

**Proposition 3.1.** Let $\mathcal{C}$ be a multi-fusion category over the field $k$ and $H = \text{coend}(\mathcal{C}, \omega)$ be the finite-dimensional and split cosemisimple WHA reconstructed from $\mathcal{C}$ using the long
Then a basis for the vector space of all dual central linear forms is given by \( \{ \zeta_j \}_{j \in I} \) where

\[
\zeta_j([\vartheta|v]_X) = \begin{cases} 
\varepsilon([\vartheta|v]_X), & \text{if } X \cong V_j, \\
0, & \text{else,} 
\end{cases}
\]

for all simple \( X \in |\mathcal{C}| \). A dual central linear form \( \zeta = \sum_{j \in I} c_j \zeta_j \) with coefficients \( c_j \in \mathbb{k} \) is convolution invertible if and only if \( c_j \neq 0 \) for all \( j \in I \).

The following theorem demonstrates that the canonical WHA reconstructed from a spherical multi-fusion category \( \mathcal{C} \) contains a very special left-integral \( \ell \in H \). This integral will feature in the construction of the invariant below.

**Theorem 3.2.** Let \( \mathcal{C} \) be a spherical multi-fusion category over \( \mathbb{k} \) and \( H = \text{coend}(\mathcal{C}, \omega) \) be the finite-dimensional and split cosemisimple cospherical WHA reconstructed from \( \mathcal{C} \) using the long canonical functor (2.1).

1. The following element \( \ell \in H \) is a left-integral:

\[
\ell = \sum_{j \in I} \dim V_j \sum_{m} [D_{\tilde{V}}^{-1} \circ e_{m}(V_j) \circ (D_{\tilde{V}} \otimes \text{id}_{V_j})e_{m}(V_j)]_{V_j}.
\]

2. The following linear form \( c: H \to \mathbb{k} \) is a two-sided cointegral:

\[
c([\vartheta|v]_X) = \begin{cases} 
\varepsilon([\vartheta|\rho^{-1}]_1)\varepsilon([\rho \hat{V}|v]_1), & \text{if } X \cong 1, \\
0, & \text{else,} 
\end{cases}
\]

for all simple \( X \in |\mathcal{C}| \).

3. Both \( \ell \) and \( c \) are non-degenerate.

4. The cointegral \( c \) is \( S \)-invariant, and \( \ell \) is a dual quantum trace.

5. The integral can be expressed as \( \ell = \zeta(t_{\tilde{V}})t_{\tilde{V}}'' = w(x'_{\tilde{V}})\zeta(x''_{\tilde{V}})x''_{\tilde{V}} \) where \( x'_{\tilde{V}} \in H \) is the dual character \( (A.47) \) associated with the small progene \( \hat{V} \) of (2.2); \( t_{\tilde{V}} = w(x'_{\tilde{V}})x''_{\tilde{V}} \in H \) is its dual quantum character \( (A.48) \); and \( \zeta: H \to \mathbb{k} \) is the convolution invertible and dual central linear form given by

\[
\zeta([\vartheta|v]_X) = (\dim X) \varepsilon([\vartheta|v]_X)
\]

for all simple \( X \in |\mathcal{C}| \).

6. The integral \( \ell \) is \( S \)-compatible.

**Proof.** 1. Note that the canonical left-integral \( \ell_{\text{can}} \in H \) that exists in every WHA, turns out to be a multiple of our \( \ell \):

\[
\ell_{\text{can}} = \sum_{j} b_j \beta_j (S^2(b'_j)) = |I| \ell.
\]

Here we have written \( \sum_{j} b_j \otimes \beta_j \) for the canonical element in \( H \otimes H^* \). Therefore, \( \ell_{\text{can}} = 0 \) whenever the characteristic of \( \mathbb{k} \) divides the number \( |I| \) of isomorphism classes of the simple objects of \( \mathcal{C} \). Our integral \( \ell \in H \) avoids this problem and never vanishes as we show in Part (3) below. The proof that it indeed forms a left-integral is by a direct calculation and is most transparent if one uses the isomorphism \( H \cong \text{End}(\hat{V}^* \otimes \hat{V}) \) of \([17]\) Section 4.1.
2. By the result dual to [13] Lemma 3.3, the set of right-cointegrals of $H$ is isomorphic as a left-$H$-comodule to $\text{Hom}_{\mathcal{M}^H}(H, H_s)$ where both $H$ and $H_s \cong 1$ are viewed as right-$H$-comodules. Since $H$ is split cosemisimple, the set $\text{Hom}_{\mathcal{M}^H}(H, H_s)$ and thereby the set of right-cointegrals is known explicitly. A direct computation shows that such a right-cointegral is two-sided if and only if it is a scalar multiple of $c$ of (3.4).

3. A direct computation shows that $c \mapsto \ell = 1$. By [13] Theorem 3.18, both $\ell$ and $c$ are therefore non-degenerate. In particular, $\ell \neq 0$. The theorem also implies that $\ell \mapsto c = 1^*$, where $1^* \in H^*$ is the unit of the dual WHA, a result that is needed in Part (4) below.

4. Since there exists the two-sided non-degenerate cointegral $c$ of $H$, by the result dual to [13] Lemma 3.21, all two-sided cointegrals are $S$-invariant. In particular, $c$ is. Since $c$ is a non-degenerate left-cointegral and $\ell \mapsto c = 1^*$, $\ell$ is its dual left-integral, and we can apply the result dual to [13] Theorem 3.20. Since $c$ is $S$-invariant, this theorem implies that $\ell$ is a dual quantum trace.

5. Since the coalgebra underlying $H$ is finite-dimensional and split cosemisimple, we know the coefficients of the right-$H$-comodule $\hat{V} \in \mathcal{M}^H$ and can compute its dual character (A.47):

$$
\chi_{\hat{V}} = \sum_{j \in I} \chi_{V_j} = \sum_{j \in I} \sum_m [e_m^{(V_j)}|e_m^{(V_j)}]_{V_j} \in H.
$$

(3.7)

A direct computation using the copivotal form (B.16) and equation (3.5), proves the claim.

6. Since $\hat{V}^* \cong \hat{V}$, its dual character is $S$-invariant, i.e. $S(\chi_{\hat{V}}) = \chi_{\hat{V}}$. In order to prove the claim, we use that $\zeta \circ S = \zeta$.

\[\square\]

4 Evaluation of (unoriented) framed links

4.1 Ribbon diagrams

Let $\mathcal{C}$ be a ribbon category (see Appendix A.6). Every morphism of $\mathcal{C}$ can be represented by a composition of tensor products of the following string diagrams,
in which the components of the ribbon are labeled by objects \( X, Y, \ldots \in |\mathcal{C}| \). The monoidal unit \( 1 \in |\mathcal{C}| \) is invisible in these diagrams. If an object label \( X \in |\mathcal{C}| \) is replaced by its dual \( X^* \), the arrow is reversed. Note that we read composition from top to bottom and the tensor product from left to right. This agrees with about half of the literature, but notably differs from Turaev [10] who reads composition from the bottom up and who calls the right-handed rather than the left-handed twist \( \nu_X \). Our choice of diagrams turns out to be convenient in the present context as we study corepresentations rather than representations of (Weak) Hopf Algebras.

By a result of Reshetikhin–Turaev [18] which can be viewed as a coherence theorem for ribbon categories, every plane projection of an oriented framed tangle in \( S^3 \) can be arranged to agree with such a diagram, but without labels. If one now labels the components of the tangle with objects of \( \mathcal{C} \), the resulting morphism of \( \mathcal{C} \) can be shown to be independent of the chosen projection [18]. Therefore, a given ribbon category \( \mathcal{C} \) associates morphisms of \( \mathcal{C} \) with labeled oriented framed tangles and, more specially, endomorphisms of the monoidal unit \( 1 \in |\mathcal{C}| \) with labeled oriented framed links.

4.2 Reshetikhin–Turaev evaluation for ribbon categories

Let \( \mathcal{C} \) be a ribbon category, \( V \in |\mathcal{C}| \) be an object of \( \mathcal{C} \) such that \( V^* \cong V \), and \( \zeta : 1_{\mathcal{C}} \Rightarrow 1_{\mathcal{C}} \) be a natural isomorphism of the identity functor such that \( \zeta^* = \zeta \). Given a plane projection of an (unoriented) framed link \( L \) in \( S^3 \), we label all components by \( V \) and insert \( \zeta_V \) somewhere (anywhere) into each component of the link. We call the resulting morphism

\[
\langle L \rangle^{(\mathcal{C})}_{V, \zeta} : 1 \rightarrow 1,
\]

the Reshetikhin–Turaev evaluation of the link \( L \). Notice that (4.1) is independent of the orientation of each component of \( L \) and therefore well defined. The link evaluation (4.1) for \( V = \hat{V} \) of (2.2) and \( \zeta \) the natural transformation associated with the linear form \( \zeta \) of (3.5) is the one that features in the Reshetikhin–Turaev invariant [1].

4.3 Evaluation for coribbon Weak Hopf Algebras

In this section, we evaluate an (unoriented) framed link in the ribbon category \( \mathcal{M}^H \) for a suitable coribbon WHA \( H \).
Remark 4.1 (Summary of results from [8]). Let $H$ be a coribbon WHA over some field $k$. Then the category $\mathcal{M}^H$ of finite-dimensional right $H$-comodules is a ribbon category with

$$
ev_V: V^* \otimes V \to H, \quad \vartheta \otimes v \to \vartheta(v_0)e_s(v_1), \tag{4.2}$$

$$\coev_V: H \to V \otimes V^*, \quad x \mapsto ((e_j)_0 \otimes e^j)\varepsilon(x(e_j)_1), \tag{4.3}$$

$$\nu_V: V \to V, \quad v \mapsto v \nu(v_1), \tag{4.4}$$

$$\nu^{-1}_V: V \to V, \quad v \mapsto v_0 \nu(v_1), \tag{4.5}$$

$$\sigma_{V,W}: V \otimes W \to W \otimes V, \quad v \otimes w \mapsto (w_0 \otimes v_0)r(w_1 \otimes v_1), \tag{4.6}$$

$$\sigma^{-1}_{V,W}: W \otimes V \to V \otimes W, \quad w \otimes v \mapsto (v_0 \otimes w_0)\varepsilon(w_1 \otimes v_1), \tag{4.7}$$

where we have used Sweedler notation for comodules (see Appendix A.1), $e_j \otimes e^j \in V \otimes V^*$ denotes the canonical element, and $H_s$ plays the role of the monoidal unit of $\mathcal{M}^H$.

The right evaluation and coevaluation $\ev_V$ and $\coev_V$ can be computed as in (A.43) and (A.44), respectively. Replacing one object $V \in |\mathcal{M}^H|$ by its dual is done as in (A.13).

In the remainder of this section, we show how the Reshetikhin–Turaev invariant $(L_V^\mathcal{M}_\zeta)$ can be computed using only the WHA $H$. In this computation, the dual quantum character $T_V$ appears once for each component of $L$, in conjunction with the linear form associated with $\zeta$. The element

$$\ell = \zeta(T_V^\mathcal{M})T_V^\mathcal{M} \in H \tag{4.9}$$

turns out to be an $S$-invariant dual quantum trace.

Proposition 4.2. Let $H$ be a WBA and $U^H: \mathcal{M}^H \to \Vect_k$ be the usual forgetful functor. Every natural transformation $f: 1_{\mathcal{M}^H} \Rightarrow 1_{\mathcal{M}^H}$ is of the form

$$f_V(v) = v_0\alpha^{(f)}(v_1) \tag{4.10}$$

for all $V \in |\mathcal{M}^H|$ and $v \in V$. Here $\alpha^{(f)}: H \to k$ is a uniquely determined dual central linear form. In addition, $f: 1_{\mathcal{M}^H} \Rightarrow 1_{\mathcal{M}^H}$ is a natural equivalence if and only if $\alpha^{(f)}$ is convolution invertible.

Proof. By the universal property of the universal coend $H \cong \coend(\mathcal{M}^H, U^H)$, condition (4.10) defines a unique linear form $\alpha^{(f)}$ for the natural transformation $\alpha$. Since each $f_V: V \to V$ is a morphism, $\alpha^{(f)}$ is dual central. The convolution inverse $\alpha^{(f)^{-1}}$ is given by $f_V^{-1}(v) = v_0\alpha^{(f)}(v_1)$ if it exists, again using the universal property of the coend. \qed

Proposition 4.3. Let $H$ be a WBA and $U^H: \mathcal{M}^H \to \Vect_k$ be the usual forgetful functor. Every natural transformation $f: - \otimes - \Rightarrow - \otimes -$ of the functor $- \otimes -: \mathcal{M}^H \times \mathcal{M}^H \to \mathcal{M}^H \times \mathcal{M}^H$ is of the form

$$f_{V,W}(v \otimes w) = v_0 \otimes w_0 \alpha^{(f)}(v_1 \otimes w_1) \tag{4.11}$$

for all $V, W \in |\mathcal{M}^H|$ and $v \in V$, $w \in W$. Here $\alpha^{(f)}: H \otimes H \to k$ is a uniquely determined linear form that satisfies

$$\varepsilon(x'y')\alpha(x'' \otimes y'') = \alpha(x \otimes y) = \alpha(x' \otimes y')\varepsilon(x''y''), \tag{4.12}$$

$$x'y'\alpha(x'' \otimes y'') = \alpha(x' \otimes y')x''y'', \tag{4.13}$$

for all $x, y \in H$. 
Proof. By the universal property of the universal coend $H \otimes H \cong \text{coend}(\mathcal{M}^H \times \mathcal{M}^H, U^H \otimes U^H)$, (4.11) defines a unique linear form $\alpha(f)$ for the natural transformation $\alpha$, and (4.12) is satisfied. Since each $f_{V,W} : V \otimes W \to V \otimes W$ is a morphism, condition (4.13) holds.

In the following, we use the same symbol for the natural transformation and for the associated linear form, for example, $\alpha : 1_{\mathcal{M}^H} \Rightarrow 1_{\mathcal{M}^H}$ and $\beta : H \otimes H \to k$.

**Lemma 4.4.** Let $H$ be a coribbon WHA and $f : X \otimes V \to X \otimes V$ be a morphism in $\mathcal{M}^H$. Then

$$X \xrightarrow{f} V \quad (x) = \sum_{i,j} g_i(x) \otimes e^j((h_i(e_j)_0)w((h_i(e_j)))_1) \quad (4.14)$$

for all $x \in X$ where we have written $f = \sum_i g_i \otimes h_i$ with $g_i \in \text{End}(X)$ and $h_i \in \text{End}(V)$, $\text{coev}_V = \sum_j e_j \otimes e^j$, and where $w : H \to k$ denotes the copivotal form. The diagram is in the ribbon category $\mathcal{M}^H$ and is drawn in blackboard framing.

Proof. Use the definitions in Remark 4.1 as well as

$$(x_0 \otimes v_0)\varepsilon(\varepsilon_s(x_1)v_1) = x \otimes v \quad (4.15)$$

and

$$(x_0 \otimes \vartheta_0)\varepsilon(x_1\varepsilon_s(\vartheta_1)) = x \otimes \vartheta \quad (4.16)$$

for all $x \in X$, $v \in V$, and $\vartheta \in V^*$.

**Proposition 4.5.** Let $H$ be a coribbon WHA and $\alpha : - \otimes - \Rightarrow - \otimes -$ be a natural transformation. Then for all $X, V \in |\mathcal{M}^H|$,

$$X \xrightarrow{\alpha_{X,V}} V \quad (x) = x_0 \alpha(x_1 \otimes T_V) \quad (4.17)$$

for all $x \in X$. The diagram is in $\mathcal{M}^H$ and drawn in blackboard framing, and $T_V \in H$ denotes the dual quantum character of (A.48).

Proof. Apply Lemma 4.4 to $f_{X,V} = \alpha_{X,V}$.

**Lemma 4.6.** Let $H$ be a coribbon WHA and $\alpha : 1_{\mathcal{M}^H} \Rightarrow 1_{\mathcal{M}^H}$ be a natural transformation. Then for $V \in |\mathcal{M}^H|$,\n
$$\xrightarrow{\alpha_V} V \quad (h) = \alpha(T^*_V)\varepsilon_s(h\varepsilon_s(T^*_V))) \quad (4.18)$$

for all $h \in H_s$. 
Proof. Recall that the trace (4.18) is a linear map $H_s \to H_s$ where $H_s \cong 1 \in \mathcal{M}^H$ is the monoidal unit. Using the definitions in Remark 4.1, we obtain

\[(4.18) = \alpha(T'_V) \varepsilon(hT''_V) \varepsilon_s(S(T''_V)) \] (4.19)

which can be shown to agree with the right hand side of (4.18).

Theorem 4.7. Let $H$ be a coribbon WHA, $L$ be an (unoriented) framed link in $S^3$ with $m$ components, $V \in \mathcal{M}^H$ and $\zeta : 1_{\mathcal{M}^H} \Rightarrow 1_{\mathcal{M}^H}$ be a natural equivalence such that $\zeta* = \zeta$. Then the Reshetikhin–Turaev evaluation is of the form

\[\langle L \rangle_{V,\zeta}^{(\mathcal{M}^H)}(h) = \varphi(L)(h \otimes \ell \otimes \cdots \otimes \ell)\] (4.20)

for all $h \in H_s$ with a linear map $\varphi(L) : H_s \otimes H \otimes \cdots \otimes H \to H$. Here, the element

\[\ell = \zeta(T'_V)T'_V \in H\] (4.21)

is an $S$-compatible dual quantum trace.

Proof. We prove a slightly stronger claim in which we insert into each component $L_j$, $1 \leq j \leq m$, of $L$ a natural transformation $\gamma(j) : V \to V$ that is of the form $\gamma(j) = \xi(j) \circ \zeta_V$ with arbitrary $\xi(j) : V \to V$. The proof proceeds by induction on the number of components $m$.

If $m = 1$, the link evaluation is of the form

\[\langle L \rangle_{V,\zeta}^{(\mathcal{M}^H)}(h) = \nu_V(1)\] (4.22)

where $\nu_V$, $z \in \mathbb{Z}$, is the appropriate power of the twist. Putting $\alpha_V = \nu_V \circ \gamma_V(1) = \nu_V \circ \xi_V(1) \circ \zeta_V$ in Lemma 4.6 shows that

\[\langle L \rangle_{V,\zeta}^{(\mathcal{M}^H)}(h) = \zeta(T'_V)\psi(1)(T''_V)\varepsilon_s(h\varepsilon_t(T''_V)) = \psi(1)(\ell')\varepsilon_s(h\varepsilon_t(\ell''))\] (4.23)

for all $h \in H_s$ where the linear form $\psi(1) : H \to k$ implements the natural transformation $\psi(1) = \nu_V \circ \xi_V(1)$. This proves our stronger proposition. For $\xi_V(1) = \text{id}_V$, we obtain the claim of the theorem for $m = 1$ as a special case.

We now assume that our stronger assumption holds for some $m \in \mathbb{N}$ and consider a link $L$ with $m + 1$ components. If we select one of the components, labeled $V$ and, without loss of generality, numbered $m + 1$, the diagram of $L$ can be arranged in such a way that the selected component $V$ appears in the following fashion:

\[f_{V^\otimes m} = \] (4.24)
with some natural transformation \( \beta: - \otimes - \Rightarrow - \otimes - \) and some \( z \in \mathbb{Z} \). We now apply Proposition 4.5 with \( \alpha_{V \otimes m, V} = (\id_{V \otimes m} \otimes (\nu^V_\ell \circ \gamma^{(m+1)}_V)) \circ \beta_{V \otimes m, V} \) and \( X = V^{\otimes m} \). This proposition shows that

\[
f_{V \otimes m}(x) = x_0 \omega(x_1),
\]

for all \( x \in V^{\otimes m} \) where

\[
\omega(y) = \zeta(T^V_\ell) \psi^{(m+1)}(T^m_\ell) \alpha(y \otimes T^m_\ell) = \psi^{(m+1)}(\ell') \alpha(y \otimes \ell')
\]

for all \( y \in H \). Here, \( \psi^{(m+1)}: H \to k \) is the linear form that implements the natural transformation \( \psi^V_{V \otimes (m+1)} = \nu^V_\ell \circ \xi^{(m+1)}_\ell \). In particular, the map \( f_{V \otimes m} \) is natural in \( V^{\otimes m} \). Even stronger, we can split \( X = V^{\otimes m} = Y \otimes V \) with \( Y = V^{\otimes (m-1)} \) and see that \( f_{Y \otimes V}(y \otimes v) = y_0 \otimes v_0 \omega(y_1 v_1) \) for all \( y \in Y \) and \( v \in V \) which shows that \( f_{Y \otimes V} \) is natural both in \( Y \) and in \( V \). By induction, we see that \( f_{V \otimes m} \) is natural in each component \( V \) of the tensor power \( V^{\otimes m} \).

The evaluation \( \langle L \rangle^{(M^H)}_{V, \xi} \) for the \((m+1)\)-component link is therefore equal to the evaluation for an \( m \)-component link in which we have inserted different natural transformations into every component: the composition of \( f_{V \otimes m} \) which is natural in each tensor factor \( V \) with the \( \zeta^V \) from the original claim. By the assumption of our induction, \( \langle L \rangle^{(M^H)}_{V, \xi} \) is of the form of our stronger claim. Note that for each component of \( L \), (4.26) is applied once and yields one tensor factor \( \ell \).

Depending on the order in which we consider the components of \( L \), we may obtain different linear maps \( \varphi^{(L)} \). The right hand side of (4.20), however, always agrees with the Reshetikhin–Turaev evaluation. 

### 4.4 Generalized dual Hennings–Kauffman–Radford evaluation

The idea of Hennings [2] in the case of Hopf algebras (not weak) is that the \( S \)-compatible dual quantum trace \( \ell \in H \) that features in Theorem 4.7 can be replaced by an arbitrary \( S \)-compatible dual quantum trace and still yields a well-defined link evaluation, i.e. an evaluation that gives the same value irrespective of the diagram that is used in order to represent the link. The same holds for WHAs. We summarize this result in the following.

**Definition 4.8.** Let \( H \) be a coribbon WHA, \( \ell \in H \) be an \( S \)-compatible dual quantum trace, and \( L \) be an \( m \)-component (unoriented) framed link in \( S^3 \). The generalized Hennings–Kauffman–Radford invariant \( \langle L \rangle^{(H)}_\ell \) is defined as follows. Consider a diagram of \( L \). Label each component of \( L \) with a formal symbol \( X \) which stands for a finite-dimensional right \( H \)-comodule with coaction \( \beta_X: X \to X \otimes H \). We impose only the relations that \( X \) be a rigid object of \( \text{Vect}_k \) and that \( \beta_X \) be a right-\( H \) comodule, i.e.

\[
\alpha_{X, H, H} \circ (\beta_X \otimes \id_H) \circ \beta_X = (\id_X \otimes \Delta) \circ \beta_X,
\]

\[
\rho_X \circ (\id_X \otimes \varepsilon) \circ \beta_X = \id_X.
\]

Theorem 4.7 applies, and so

\[
\langle L \rangle^{(M^H)}_{X, \id_H}(h) = \varphi^{(L)}(h \otimes \underbrace{\ell \otimes \cdots \otimes \ell}_{m})
\]

for all \( h \in H_s \) with a linear map \( \varphi^{(L)}: H_s \otimes H^{\otimes m} \to H_s \). The theorem computes the \( S \)-compatible dual quantum trace as \( \ell = T_X \in H \), the dual quantum character of the formal
comodule $\mathcal{X}$. The Hennings–Kauffman–Radford evaluation is then defined by replacing this element $\tilde{\ell}$ with the given $S$-compatible dual quantum trace $\ell \in H$:

$$\langle L \rangle_{\ell}^{(H)}(h) = \varphi^{(L)}(h \otimes \ell \otimes \cdots \otimes \ell).$$

(4.30)

for all $h \in H_s$.

**Theorem 4.9.** Let $H$ be a coribbon WHA, $\ell \in H$ be an $S$-compatible dual quantum trace, and $L$ be an $m$-component (unoriented) framed link in $S^3$. The Hennings–Kauffman–Radford evaluation $\langle L \rangle_{\ell}^{(H)}$ is well defined, i.e. it is independent of the diagram used to represent $L$.

**Proof.** The proof is dual of the proof of Hennings [2] or the proof of Kauffman–Radford [3]. The idea is that in (4.30), the linear map $\varphi^{(L)}: H_s \otimes H^\otimes m \rightarrow H_s$ may initially depend on the diagram that represents $L$ and on the order of the components of $L$ used in the proof of Theorem 4.7, but the linear map

$$\varphi^{(L)}(- \otimes \ell \otimes \cdots \otimes \ell): H_s \rightarrow H_s$$

(4.31)

is independent of the diagram that represents the link $L$. Kauffman–Radford [3] show that the condition that $\ell \in H$ be an $S$-compatible dual quantum trace is sufficient in order to establish independence. Since they work with left modules whereas we use right-comodules, we can simply rotate all their diagrams by $180^\circ$ in order to prove our claim. The coherence theorem for the monoidal category $\mathcal{M}^H$ ensures that the monoidal unit $H_s \cong 1$ can be inserted at an arbitrary position in all tensor products.

Finally, note that since $\ell$ is $S$-compatible, the dual trace $\chi = \overline{\varphi}(\ell')\ell''$ is $S$-invariant, and so the evaluation is independent of the orientation of each individual component of $L$.

The following corollary to Theorem 4.7 establishes the relation between the two link evaluations.

**Corollary 4.10.** Let $\mathcal{C}$ be a multi-fusion category over $k$ which has a ribbon structure, and let $H = \text{coend}(\mathcal{C}, \omega)$ be the finite-dimensional and split cosemisimple coribbon WHA reconstructed from $\mathcal{C}$ using the long canonical functor (2.1). Then the Reshetikhin–Turaev and the Hennings–Kauffman–Radford evaluations agree for every (unoriented) framed link $L$ in $S^3$:

$$\langle L \rangle_{\mathcal{C}}^{(\mathcal{C})} = \langle L \rangle_{\ell}^{(H)}$$

(4.32)

with $\widehat{V}$ of (2.2), $\zeta$ of (3.5) and $\ell$ of (3.3).

**Proof.** Theorem 4.7 and Theorem 3.2(5).

\[ \Box \]

5 Invariants of 3-manifolds

5.1 The invariant for a Weak Hopf Algebra

We proceed in analogy to the work of Hennings [2] and Kauffman–Radford [3] and show that if the $S$-compatible dual quantum trace $\ell$ is a left-integral, then the link evaluation can be made invariant under Kirby moves.
Theorem 5.1. Let $H$ be a coribbon WHA over some field $k$ and $\ell \in H$ be a left-integral which is an $S$-compatible dual quantum trace. Let $L$ be an (unoriented) framed link in $S^3$ with components $L_1, \ldots, L_m$, $m \in \mathbb{N}$. If there exist $\beta, \gamma \in k \setminus \{0\}$ such that the following two conditions,

$$\nu(x')\overline{\nu}(\varepsilon_t(x'')\ell) = \frac{\gamma^2}{\beta} \nu(x), \quad (5.1)$$

$$\overline{\nu}(x') \nu(\varepsilon_t(x'')\ell) = \beta \overline{\nu}(x), \quad (5.2)$$

hold for all $x \in H$, then

$$I(M_L) = \beta^\sigma \gamma^{-\sigma - m - 1} \langle L \rangle^{(H)}_\ell \in \text{End}(\mathbb{1}) \quad (5.3)$$

forms an invariant of connected and oriented closed (smooth) 3-manifolds. Here, $\mathbb{1} \cong H_s$ denotes the monoidal unit of $\mathcal{M}_H$.

In (5.3), $M_L$ is the 3-manifold obtained from $S^3$ by surgery along $L$; $\langle L \rangle^{(H)}_\ell$ denotes the generalized Hennings–Kauffman–Radford evaluation of the link $L$ using the dual quantum trace $\ell$; and $\sigma$ is the signature of the linking matrix of $L$ with framing numbers on its diagonal.

Proof. By the theorems of Wallace and Lickorish [19,20], of Kirby [21] and of Fenn–Rourke [22], $M_L$ is diffeomorphic as an oriented manifold to $\tilde{M}_L$ if and only if the (unoriented) framed link $L$ can be transformed into $\tilde{L}$ using a finite sequence of Kirby-(+1)- and Kirby-(−1)-moves. Note that $I(M_L)$ is independent of the numbering of the components of the link, and so $I(M_L)$ is well-defined for each given Kirby diagram $L$ of some connected and oriented closed 3-manifold.

We first show that (5.1) and (5.2) imply that $\langle L \rangle^{(H)}_\ell$ is invariant up to the specified scalar factors $\gamma^2/\beta$ and $\beta$ under Kirby-(+1)-moves and under Kirby-(−1)-moves, respectively. For the Kirby-(+1)-move, we show that

$$\left\langle \left( \bigotimes_{n=0}^{\infty} \mathcal{X} \right) \right\rangle^{(H)}_\ell = \frac{\gamma^2}{\beta} \left\langle \left( \bigotimes_{n=0}^{\infty} \mathcal{X} \right) \right\rangle^{(H)}_\ell, \quad (5.4)$$

for all $n \in \mathbb{N}_0$. The left-hand side evaluates to $(\text{id}_\mathcal{X} \otimes f) \circ \beta_X : \mathcal{X} \to \mathcal{X}$ with

$$f(x) = \overline{\nu}(\ell') \overline{\nu}(x \otimes \ell'')$$

$$= \nu(x') \overline{\nu}(x'') \overline{\nu}(\ell') \overline{\nu}(x'' \otimes \ell'')$$

$$= \nu(x') \overline{\nu}(x'' \ell)$$

$$= \nu(x') \overline{\nu}(\varepsilon_t(x'')\ell)$$

$$= \frac{\gamma^2}{\beta} \nu(x), \quad (5.5)$$

for all $x \in H$ with $\overline{\nu}(\varepsilon_t(x') \otimes \varepsilon_t(\ell))$ of (A.33). We have used convolution invertibility of $\nu$, a consequence of equation (A.38), that $\ell$ is a left-integral, and equation (5.1). The result agrees
with the evaluation of the right-hand side. Recall that with our definition of a coribbon WHA $H$, the universal ribbon form $\nu: H \to k$ gives rise to the isomorphisms $\nu_X : X \to X$ in $\mathcal{M}^H$ which represent the left-handed (!) twist. For the Kirby-$(−1)$-move, we show that

$$\langle X^\otimes n \rangle \bigg|_\ell \mathcal{M}^H = \beta \langle X^\otimes n \rangle \bigg|_\ell \mathcal{M}^H$$

(5.6)

for all $n \in \mathbb{N}_0$. The left-hand side evaluates to $(\text{id}_X \otimes g) \circ \beta_X : X \to X$ with

$$g(x) = \nu(\ell')q(x \otimes \ell'')$$

$$= \nu(x')\nu(x'')\nu(\ell')q(x'' \otimes \ell'')$$

$$= \nu(x')\nu(x''\ell)$$

$$= \nu(x')\nu(\epsilon_1(x'')\ell)$$

$$= \beta \nu(x')$$

(5.7)

for all $x \in H$ with $q(x \otimes y) = \text{of (A.32)}$. We have used convolution invertibility of $\nu$, equation (A.38), that $\ell$ is a left-integral, and equation (5.2). The result agrees with the evaluation of the right-hand side.

Since the Kirby-$(+1)$-move decreases both $m$ and $\sigma$ by one whereas the Kirby-$(−1)$-move decreases $m$ by one and increases $\sigma$ by one, the expression $I(M_L)$ in (5.3) is invariant under both moves.

The above argument does not simplify much if we restrict the proof to special Kirby-$(−1)$-moves. The special Kirby-$(−1)$-move is obtained for $n = 0$ in (5.6), i.e. by inserting the monoidal unit for $X^\otimes n$ which is an invisible line. Note that the coribbon WHA we use is not required to be copure, and so the invariant takes its values in $\text{End}(\mathbb{1})$ which need not be isomorphic to $k$.

### 5.2 The Hennings–Kauffman–Radford invariant

In this section, we show that if $H$ is a finite-dimensional unimodular ribbon Hopf algebra, then its dual $H^*$ satisfies the assumptions of Theorem 5.1. In this case, our invariant (5.3) for $H^*$ agrees up to a factor with the Kauffman–Radford formulation [3] of the Hennings invariant [2] for $H$.

First, if we work with a Hopf algebra (not weak), then Theorem 5.1 reduces to the following

**Corollary 5.2.** Let $H$ be a coribbon Hopf algebra over some field $k$ and $\ell \in H$ be a left-integral which is an $S$-compatible dual quantum trace. Let $L$ be an (unoriented) framed link in $S^3$ with components $L_1, \ldots, L_m$, $m \in \mathbb{N}$. If there exist $\beta, \gamma \in k \setminus \{0\}$ such that $\nu(\ell) = \frac{\gamma^2}{\beta}$ and $\nu(\ell) = \beta$, then

$$I(M_L) = \beta^\sigma \gamma^{-\sigma-m-1} \langle L_\ell \rangle \in k$$

(5.8)

forms an invariant of connected and oriented closed (smooth) 3-manifolds.
Proof. In a Hopf algebra, we have $\varepsilon_i = \eta \circ \varepsilon$. This simplifies \((5.1)\) and \((5.2)\). Our assumptions $\nu(\ell) = \gamma^2/\beta$ and $\nu(\ell) = \beta$ then imply these two conditions. Finally, $1 \cong H_s = k$, and so $\text{End}(1) = k$.

The following proposition shows that this corollary applies to the Hopf algebra dual to the one featuring in Kauffman–Radford [3]:

**Proposition 5.3.** Let $H$ be a finite-dimensional unimodular ribbon Hopf algebra over some field $k$ and $H^*$ be its dual Hopf algebra. Then $H^*$ is coribbon and has a unique (up to a scalar) non-zero left-integral $\ell \in H^*$ which forms an $S$-compatible dual quantum trace. Furthermore, the invariant $I(M_L)$ of \((5.8)\) is $\gamma$ times the invariant $\text{INV}(K)$ of [3, page 147].

Proof. The ribbon Hopf algebra $H$ is pivotal with some pivotal element $\mu \in H$, i.e. $\mu$ is group-like, the ribbon element is given by $r = \mu^{-1}u$ where $u = \sum S(b_i) a_i$ is the first Drinfel’d element and $R = \sum a_i \otimes b_i$ denotes the universal $R$-matrix, and we have $S^2(x) = \mu x \mu^{-1}$ for all $x \in H$. Note that in [3], our $\mu$ and $r$ are called $G$ and $\nu$, respectively.

By [3, 23], there exists a right-cointegral $\rho: H \to k$, unique up to a scalar, such that

$$
\rho(xy) = \rho(S^2(y)x),
$$

$$
\rho(\mu^2 x) = \rho(S(x)),
$$

for all $x, y \in H$. Note that in [3], our $\rho$ is called $\lambda$.

We pair $H$ with its dual $H^*$ using the evaluation map $H^* \otimes H \to k$, $\varphi \otimes x \mapsto \langle \varphi, x \rangle = \varphi(x)$. The ribbon structure of $H$ then gives rise to a coribbon structure on $H^*$. In order to match the ribbon structure of $H$ in the terminology of [3] with our definition of a coribbon (weak) Hopf algebra, we extend the canonical pairing to tensor products such that $\langle \varphi \otimes \psi, x \otimes y \rangle = \langle \varphi, y \rangle \langle \psi, x \rangle$ for all $\varphi, \psi \in H^*$ and $x, y \in H$. Note that when writing down formulas, this is the uncommon choice, but when drawing string diagrams in $\text{vect}_k$, these can now be drawn without crossings. Also, with this choice, a left-integral of $H$ is related to a right-cointegral of $H^*$.

The right-cointegral $\rho: H \to k$ then gives rise to a left-integral $\ell \in H^*$, defined by $\ell = \sum_i \rho(b_i) \beta_i$ where we have used the canonical element $\sum_i b_i \otimes \beta_i \in H \otimes H^*$. The ribbon element $r \in H$ gives rise to a universal ribbon form $\nu: H^* \to k$ such that $\nu(\varphi) = \varphi(r^{-1})$ for all $\varphi \in H^*$, and the pivotal element $\mu \in H$ is related to the copivotal form $w: H^* \to k$ by $w(\varphi) = \varphi(\mu^{-1})$ for all $\varphi \in H^*$.

The condition \((5.9)\) implies that $\ell$ is a dual quantum-trace, and condition \((5.10)\) ensures that $\ell(\nu(\ell')) = S(\ell)$ which, given that $\ell$ is a dual quantum trace, can be shown to be equivalent to $\ell$ being $S$-compatible.

Whereas [3] uses left-modules of $H$, we work with right-comodules of $H^*$. As in the proof of Theorem 4.9, our diagrams are obtained by rotating the diagrams of [3] by $180^\circ$. Also note that in [3, page 147], $\lambda(\nu) = \gamma^2/\beta$ and $\lambda(\nu^{-1}) = \beta$ which shows precisely how our invariant is related with that of Kauffman–Radford.

\[\square\]

### 5.3 The Reshetikhin–Turaev invariant

Let $\mathcal{C}$ be a modular category. We now specialize our invariant \((5.3)\) to the case in which $H$ is the canonical WHA obtained from $\mathcal{C}$ by Tannaka–Krein reconstruction. First, we show that the conditions \((5.1)\) and \((5.2)\) are almost satisfied as soon as $\mathcal{C}$ is a multi-fusion category with a ribbon structure.
Proposition 5.4. Let $\mathcal{C}$ be a multi-fusion category over $k$ which has a ribbon structure, and let $H = \text{coend}(\mathcal{C}, \omega)$ be the finite-dimensional and split cosemisimple coribbon WHA reconstructed from $\mathcal{C}$ using the long canonical functor (2.1).

Let $V \in |\mathcal{M}|$ be such that $V^* \cong V$ and $\zeta: H \to k$ be a convolution invertible and dual central linear form that satisfies $\zeta \circ S = \zeta$. Let finally $\ell = \mu(\chi_V')\zeta(\chi_V')\chi''_V$ where $\chi_V$ is the dual character of $V$. Then

$$
\nu(x')\varphi(\varepsilon(x''))\ell = \alpha \nu(x),
$$

and

$$
\varphi(x')\nu(\varepsilon(x''))\ell = \beta \varphi(x),
$$

for all $x \in H$. Here the $c_j$ are the coefficients of $\zeta$ as in Proposition 3.1, the $m_j \in \mathbb{N}_0$ are the multiplicities of the simple objects in $V$, i.e. $V \cong \bigoplus_{j \in I} m_j V_j$, and the $\nu_j$ are the eigenvalues of the ribbon twist $\nu: V_j \to V_j$ (the left-handed one) on the simple objects, respectively.

Proof. First, from

$$
\chi_V = \sum_{j \in I} m_j \sum_{\ell} [e_{(V_j)}^{(\nu_j)}]_{V_j},
$$

we compute

$$
\ell = \sum_{j \in I} c_j m_j \sum_{\ell} [D_{V_j}^{-1} \circ e_{(V_j)}^{(\nu_j)} \circ (D_{V_j} \otimes \text{id}_{V_j}) | e_{(\ell)}^{(V_j)}]_{V_j}.
$$

Then we evaluate the left-hand-side of (5.11) for arbitrary $x = \vartheta[|v|]_X \in H$, $X \in |\mathcal{C}|$, $\vartheta \in (\omega X)^*$, $v \in \omega X$:

$$
(\varphi \circ \mu) \circ (\nu \otimes \varepsilon_X \otimes \text{id}_H) \circ (\Delta \otimes \text{id}_H)((\vartheta[|v|]_X \otimes \ell)

= \sum_{j \in I} c_j m_j \sum_{\ell} \nu([\vartheta[|v|]_X]_X) \varphi(\mu(\varepsilon_X[|v|]_X \otimes \ell) [D_{V_j}^{-1} \circ e_{(V_j)}^{(\nu_j)} \circ (D_{V_j} \otimes \text{id}_{V_j}) | e_{(\ell)}^{(V_j)}]_{V_j}).
$$

We now use the expressions for $\nu$ and $\varphi$ of (5.26) and (5.27), and that $g_Y(\xi \otimes w) = \text{tr}_{\hat{\varphi}}(D_{\hat{\varphi}} \otimes \xi \otimes w)$ for all $Y \in |\mathcal{C}|$, $\xi \in (\omega Y)^*$, $w \in \omega Y$, because $\mathcal{C}$ is spherical (c.f. (B.20)), as well as $\varepsilon_X$ from (B.23) and obtain

$$
(5.17) = \sum_{j \in I} c_j m_j \sum_{\ell} \text{tr}_{\hat{\varphi}}\left( D_{\hat{\varphi}} \otimes \vartheta \circ ((\text{id}_{\hat{\varphi}} \otimes \nu_X) \circ e_{p}^{(X)}) \right)
$$

$$
\cdot \text{tr}_{\hat{\varphi}}\left( e_{(V_j)}^{(\nu_j)} \circ (D_{\hat{\varphi}} \otimes \text{id}_{V_j}) \circ ((\Phi_X(v) \otimes \Psi_X(e_{(X)}^{p})) \otimes \text{id}_{V_j}) \circ (\text{id}_{\hat{\varphi}} \otimes \nu_{V_j}^{-1}) \circ e_{(\ell)}^{(V_j)} \right).
$$

Using (B.3) and the fact that the trace is cyclic and multiplicative for tensor products of morphisms, we see that

$$
(5.17) = \sum_{j \in I} c_j m_j \sum_{m} g_X(\vartheta \otimes ((\text{id}_{\hat{\varphi}} \otimes \nu_X) \circ e_{m}^{(X)}) \otimes \text{id}_{V_j}) \text{tr}_{\hat{\varphi}}(\nu_{V_j}^{-1}).
$$

(5.19)
With the dual basis lemma and exploiting that the $V_j$ are simple, we arrive at the right hand side of (5.11):

$$ (5.17) = \nu((\bar{\theta}|v)_X) \sum_{j \in I} c_j m_j \nu_j^{-1} \dim V_j, $$

The proof of (5.12) is identical except that $\nu_X$ and $\nu_X^{-1}$ as well as $\nu$ and $\nu$ are interchanged. \qed

**Corollary 5.5.** Let $\mathcal{C}$ be a multi-fusion category over $k$ which has a ribbon structure, and let $H = \text{coend}(\mathcal{C}, \omega)$ be the finite-dimensional and split cosemisimple coribbon WHA reconstructed from $\mathcal{C}$ using the long canonical functor (2.1). If $V = \hat{V}$ and $c_j = \dim V_j$, the element $\ell$ in Proposition 5.4 agrees with the left-integral (3.3) which is known to be an $S$-compatible dual quantum trace. In this case, in (5.3),

$$ \langle - \rangle^{(H)}_\ell = \langle - \rangle^{(\mathcal{C})}_{\hat{V}_\ell} $$

is the Reshetikhin–Turaev evaluation. Furthermore, we compute that

$$ \alpha = \sum_{j \in I} \nu_j^{-1}(\dim V_j)^2, \quad \beta = \sum_{j \in I} \nu_j(\dim V_j)^2. $$

Finally, the following proposition shows that if $\mathcal{C}$ is modular, then $\alpha \neq 0$ and $\beta \neq 0$.

**Proposition 5.6.** Let $\mathcal{C}$ be a modular category, linear over the field $k$, and let $H = \text{coend}(\mathcal{C}, \omega)$ be the finite-dimensional and split cosemisimple coribbon WHA reconstructed from $\mathcal{C}$ using the long canonical functor (2.1). Assume that in Proposition 5.4, we have $V = \hat{V}$ and $c_j = \dim V_j$. Then $\alpha \neq 0$ and $\beta \neq 0$ in that proposition.

**Proof.** Let $i, j \in I$, and denote by $S_{ij}$ the coefficients of the $S$-matrix. Then

$$ \alpha S_{ij} = \alpha \left\langle \begin{array}{ccc} i & \bar{\theta} & j \\ \ell & \ell & \ell \end{array} \right\rangle^{(H)} = \sum_{p \in I} \dim V_p \left\langle \begin{array}{ccc} i \bar{\theta} & \bar{\theta} & j \\ p & p & p \end{array} \right\rangle^{(H)} $$

$$ = \sum_{p \in I} \nu^{-1}_i \nu^{-1}_p \nu^{-1}_j S_{ip} S_{pj}, $$

where we have used that the $V_i, i \in I$ are simple; a Kirby-(+1)-move; and again that the $V_i$ are simple. This implies that

$$ \alpha S_{ij} \nu_j \nu_j = \sum_{p \in I} \nu^{-1}_p S_{ip} S_{pj}. $$

If $\alpha = 0$, then the right hand side would vanish for all $i, j \in I$, i.e. the matrix product $ST = 0$ where $T_{pj} = \nu^{-1}_p S_{pj}$. This contradicts the invertibility of $S$. Interchanging $\nu$ and $\nu^{-1}$ as well as $\alpha$ and $\beta$ in the above argument establishes that $\beta = 0$. \qed

**Corollary 5.7.** Let $\mathcal{C}$ be a modular category, linear over the field $k$, such that its global dimension has a square root $D \in k$, i.e.

$$ D^2 = \sum_{j \in I} (\dim V_j)^2. $$

(5.25)
Let $H = \text{coend}(C, \omega)$ be the finite-dimensional and split cosemisimple coribbon WHA reconstructed from $C$ using the long canonical functor (2.1). In Proposition 5.4, assume that $V = \hat{V}$ and $c_j = \dim V_j$. Then (5.1) and (5.2) hold for $\gamma = D$, and the invariant (5.3) agrees with the Reshetikhin–Turaev invariant.

**Proof.** Recall from Corollary 5.5 that with these choices of $V$ and $c_j$, $\langle - \rangle^H_\ell = \langle - \rangle^C_{\hat{V}, \zeta}$.

In [10, Section II.3], our $\beta$ is called $\Delta$, and our $\alpha$ corresponds to $d_0^{-1}$ (recall that our $\nu$ is the left-handed ribbon twist, but the $\nu$ in [10] is the right-handed one). Furthermore, in [10, Section II.3], it is shown that $\alpha \beta = \Delta / d_0 = D^2$, i.e. with our choice of $\gamma = D$, we arrive at $\alpha = \gamma^2 / \beta$ as required in (5.1). □

The preceding corollary shows in particular that our proofs of Theorem 5.1 and Proposition 5.4 can be combined to a new proof of invariance of the Reshetikhin–Turaev invariant, c.f. [10, Section II.3].

### 5.4 Contrast with the Lyubashenko invariant

Let us finally point out what is the difference between our invariant $I(M_L)$ (Theorem 5.1) for the WHA $H = \text{coend}(C, \omega)$ reconstructed from a modular category $C$ and Lyubashenko’s invariant [4] that uses Majid’s coend $F = \text{coend}(C, 1_C)$. Although these two invariants take the same numerical value whenever each of them agrees with the Reshetikhin–Turaev invariant, their computation is rather different.

Whereas our $H$ is a vector space equipped with linear structure maps that make it a WHA, Lyubashenko’s $F$ is a Hopf algebra object $H \in |C|$, i.e. an object of $C$ whose structure maps are morphisms in $C$.

Let, for example, $C_3$ be a modular category associated with $U_q(sl_2)$ with 3 simple objects up to isomorphism. We denote representatives of the classes of simple objects by $X_0 \cong 1$, $X_1$ and $X_2$. The fusion rules are given by $X_1 \otimes X_1 \cong X_0 \oplus X_2$, $X_1 \otimes X_2 \cong X_1$ and $X_2 \otimes X_2 \cong X_0$. We know that $C_3 \simeq \mathcal{M}^H$ for our reconstructed WHA.

Lyubashenko’s coend is the $H$-comodule

$$F \cong (X_0 \otimes X_0^*) \oplus (X_1 \otimes X_1^*) \oplus (X_2 \otimes X_2^*) \cong 3X_0 \oplus X_2,$$

where we write $nX = X \oplus \cdots \oplus X$ (direct sum of $n$ terms). Note that its decomposition does not contain $X_1$ and that the $X_1$-term is included as $X_1 \otimes X_1^* \hookrightarrow X_0 \oplus X_2$ (as $H$-comodules). My coend, however, is $H$ itself with the regular coaction, i.e. the following $H$-comodule:

$$H \cong 3X_0 \oplus 4X_1 \oplus 3X_2.$$

Therefore, obviously, the two coends differ as objects of $\mathcal{M}^H$ and also have different $k$-dimensions. Note that $\dim_k X_0 = 3$, $\dim_k X_1 = 4$ and $\dim_k X_2 = 3$ in $C_3 \simeq \mathcal{M}^H$.

### A Weak Hopf Algebras and their corepresentations

In this appendix, we collect the relevant definitions on WHAs with additional structure.
A.1 Weak Bialgebras

Given a WHA $H$ over some field $k$, the source and target counital maps are given by

$$
\varepsilon_s := (\text{id}_H \otimes \varepsilon) \circ (\text{id}_H \otimes \mu) \circ (\tau_{H,H} \otimes \text{id}_H) \circ (\text{id}_H \otimes \Delta) \circ (\text{id}_H \otimes \eta): H \to H, \quad (A.1)
$$

$$
\varepsilon_t := (\varepsilon \otimes \text{id}_H) \circ (\mu \otimes \text{id}_H) \circ (\text{id}_H \otimes \tau_{H,H}) \circ (\Delta \otimes \text{id}_H) \circ (\eta \otimes \text{id}_H): H \to H. \quad (A.2)
$$

Here $\tau_{V,W}(v \otimes w) = w \otimes v$, $v \in V$, $w \in W$, denotes the symmetric braiding in $\text{Vect}_k$. The mutually commuting subalgebras $H_s := \varepsilon_s(H)$ and $H_t := \varepsilon_t(H)$ are called the source and target base algebra of $H$, respectively.

A WBA [WHA] is a bialgebra [Hopf algebra] if and only if $\varepsilon_s = \eta \circ \varepsilon$, if and only if $\varepsilon_t = \eta \circ \varepsilon$, and if and only if $H_s \cong k$. If $H$ is a finite-dimensional WBA [WHA], then so is its dual vector space $H^*$. Every finite-dimensional WHA $H$ has an invertible antipode.

We abbreviate $\eta(1) = 1$ and use Sweedler’s notation $\Delta(x) = x' \otimes x''$ for the comultiplication of $x \in H$ and $\beta(v) = v_0 \otimes v_1$ for the coaction $\beta: V \to V \otimes H$ of a right $H$-comodule $V$. If $H$ is finite-dimensional, we also use the Sweedler arrows $x \to \varphi = \varphi'(x) \varphi''$ and $x \to x' \varphi(x'')$ for $x \in H$, $\varphi \in H^*$.

The category $\mathcal{M}^H$ of finite-dimensional right $H$-comodules of a WBA forms a $k$-linear abelian monoidal category $(\mathcal{M}^H, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ as follows [6]. The source base algebra $H_s$ is a right $H$-comodule with

$$
\beta_{H_s}: H_s \to H_s \otimes H, \quad x \mapsto x' \otimes x''. \quad (A.3)
$$

It forms the monoidal unit object, $\mathbb{1} = H_s$ of $\mathcal{M}^H$. Given $V, W \in |\mathcal{M}^H|$, their tensor product is the vector space

$$
V \otimes W := \{ v \otimes w \in V \otimes W \mid v \otimes w = (v_0 \otimes w_0)\varepsilon(v_1w_1) \} \quad (A.4)
$$

with the coaction

$$
\beta_{V \otimes W}: V \otimes W \to (V \otimes W) \otimes H, \quad v \otimes w \mapsto (v_0 \otimes w_0) \otimes (v_1w_1). \quad (A.5)
$$

The associator $\alpha_{U,V,W}: (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ is induced from that of $\text{Vect}_k$. The left and right unit constraints are given by

$$
\lambda_V: H_s \otimes V \to V, \quad h \otimes v \mapsto v_0\varepsilon(hv_1), \quad (A.6)
$$

$$
\rho_V: V \otimes H_s \to V, \quad v \otimes h \mapsto v_0\varepsilon(v_1h). \quad (A.7)
$$

For convenience, we list their inverses, too:

$$
\lambda_V^{-1}: V \to H_s \otimes V, \quad v \mapsto (1' \otimes v_0)\varepsilon(1''v_1), \quad (A.8)
$$

$$
\rho_V^{-1}: V \to V \otimes H_s, \quad v \mapsto v_0 \otimes \varepsilon_s(v_1). \quad (A.9)
$$

A.2 Weak Hopf Algebras

A left-autonomous [23] category $\mathcal{C}$ is a monoidal category in which every object $X \in |\mathcal{C}|$ is equipped with a left-dual $(X^*, \text{ev}_X, \text{coev}_X)$, i.e. an object $X^* \in |\mathcal{C}|$ and morphisms $\text{ev}_X: X^* \otimes$
for all \( \text{vect} \in |M| \), the dual vector space \( V^* \) forms a right \( H \)-comodule with

\[
\beta_{V^*} : V^* \rightarrow V^* \otimes H, \quad \partial \mapsto (v \mapsto \partial(v_0) \otimes S(v_1)).
\] (A.13)

The left-dual of \( V \) is given by \((V^*, \text{ev}_V, \text{coev}_V)\) where

\[
\text{ev}_V : V^* \otimes V \rightarrow H_s, \quad \partial \otimes v \mapsto \partial(v_0)\varepsilon_s(v_1),
\] (A.14)

\[
\text{coev}_V : H_s \rightarrow V \otimes V^*, \quad x \mapsto \sum ((e_j)_0 \otimes e^j)\varepsilon(x(e_j)_1).
\] (A.15)

Here we have used the evaluation and coevaluation maps that turn \( V \) into a left-dual of \( V \) in \( \text{vect}_k \):

\[
\text{ev}^{(\text{vect}_k)}_V : V^* \otimes V \rightarrow k, \quad \partial \otimes v \mapsto \partial(v),
\] (A.16)

\[
\text{coev}^{(\text{vect}_k)}_V : k \rightarrow V \otimes V^*, \quad 1 \mapsto \sum_j e_j \otimes e^j.
\] (A.17)

### A.3 Copivotal Weak Hopf Algebras

Let \( H \) be a WBA. A linear form \( f : H \rightarrow k \) is said to be \textit{convolution invertible} if there exists some linear form \( \overline{f} : H \rightarrow k \) such that \( f(x')\overline{f}(x'') = \varepsilon(x) = \overline{f}(x')f(x'') \) for all \( x \in H \). The linear form \( f \) is called \textit{dual central} if \( f(x')x'' = x'f(x'') \) for all \( x \in H \). It is called \textit{dual group-like} if \( \varepsilon(x'y')f(x'')f(y'') = f(xy) = f(x')f(y')\varepsilon(x'y'') \) for all \( x, y \in H \) and \( f(\varepsilon_t(x)) = \varepsilon(x) = f(\varepsilon_s(x)) \) for all \( x \in H \). Note that in a WBA, every dual group-like linear form is convolution invertible with \( \overline{f}(x) = f(S(x)) \).

A WHA \( H \) is called \textit{copivotal} [17] if there exists a dual group-like linear form \( w : H \rightarrow k \) such that \( S^2(x) = w(x'x''w(x'')) \) for all \( x \in H \).

A \textit{pivotal category} \([21] C \) is a left-autonomous category with a monoidal natural equivalence \( \tau : 1_C \Rightarrow \star \circ \star \) such that \( (\tau_X)^* = \tau_X^{-1} \) for all \( X \in |C| \).

Given a copivotal WHA \( H \), the category \( M^H \) is pivotal with \( \tau_V : V \rightarrow V^{**} \) given by

\[
\tau_V(v) = \tau_V^{(\text{vect}_k)}(v_0)w(v_1)
\] (A.18)

for all \( V \in |M^H| \) and \( v \in V \). Here we denote by \( \tau_V^{(\text{vect}_k)} : V \rightarrow V^{**} \) the pivotal structure of \( \text{vect}_k \) which is just the usual canonical identification \( V \cong V^{**} \).
A right-autonomous category $\mathcal{C}$ is a monoidal category in which every object $X \in |\mathcal{C}|$ is equipped with a right-dual $(\overline{X}, \overline{ev}_X, \overline{coev}_X)$, i.e. an object $\overline{X} \in |\mathcal{C}|$ with morphisms $\overline{ev}_X : X \otimes \overline{X} \to 1$ (right evaluation) and $\overline{coev}_X : 1 \to X \otimes X$ (right coevaluation) that satisfy the triangle identities

$$\lambda_X \circ (\overline{ev}_X \otimes \text{id}_X) \circ \alpha^{-1}_{X,X,X} \circ (\text{id}_X \otimes \overline{coev}_X) \circ \rho^{-1}_X = \text{id}_X,$$  \hspace{1cm} (A.19)  

$$\rho_X \circ (\text{id}_X \otimes \overline{ev}_X) \circ \alpha^{-1}_{X,X,X} \circ (\overline{coev}_X \otimes \text{id}_X) \circ \lambda^{-1}_X = \text{id}_X.$$ \hspace{1cm} (A.20) 

Note that every pivotal category $\mathcal{C}$ is not only left-, but also right-autonomous with $\overline{X} = X^*$ and

$$\overline{ev}_X = ev_{X^*} \circ (\tau_X \otimes \text{id}_{X^*}),$$ \hspace{1cm} (A.21)  

$$\overline{coev}_X = (\text{id}_{X^*} \otimes \tau^{-1}_X) \circ coev_{X^*}$$ \hspace{1cm} (A.22) 

for all $X \in |\mathcal{C}|$. We can therefore define the right-dual of a morphism $f : X \to Y$ as

$$\overline{f} := \overline{\rho}_X \circ (\overline{id}_X \otimes \overline{ev}_Y) \circ \alpha_{X,Y,Y}^{-1} \circ ((\overline{id}_X \otimes f) \otimes \text{id}_Y) \circ (\overline{coev}_X \otimes \text{id}_Y) \circ \overline{\lambda}_X^{-1}.$$ \hspace{1cm} (A.23) 

It can be shown to agree with the left-dual, i.e. $\overline{f} = f^*$. 

Using both left- and right-duals, we can define two traces of a morphism $f : X \to Y$, the left-trace

$$\text{tr}^{(L)}_X(f) = ev_X \circ (\text{id}_{X^*} \otimes f) \circ \overline{coev}_X : 1 \to 1$$ \hspace{1cm} (A.24) 

and the right-trace

$$\text{tr}^{(R)}_X(f) = \overline{ev}_X \circ (f \otimes \text{id}_{X^*}) \circ coev_X : 1 \to 1.$$ \hspace{1cm} (A.25) 

Both left- and right-traces are cyclic, i.e. $\text{tr}^{(L)}_X(g \circ f) = \text{tr}^{(L)}_Y(f \circ g)$ for all $f : X \to Y$ and $g : Y \to X$ and similarly for the right-trace. In general, however, left- and right-traces need not agree.

### A.4 Cospherical Weak Hopf Algebras

A spherical category [25] is a pivotal category in which $\text{tr}^{(L)}_X(f) = \text{tr}^{(R)}_X(f)$ for all morphisms $f : X \to X$ in $\mathcal{C}$. In this case, the above expression is just called the trace of $f$ and denoted by $\text{tr}(f)$, and $\text{dim}(X) = \text{tr}(\text{id}_X)$ is called the dimension of $X$. Note that in a spherical category, $\text{tr}(f) = \text{tr}(f^*)$ for every morphism $f : X \to X$ and thus $\text{dim}(X) = \text{dim}(X^*)$. Finally, $\text{tr}_{X_1 \otimes X_2}(h_1 \otimes h_2) = \text{tr}_{X_1}(h_1) \text{tr}_{X_2}(h_2)$ for all $h_j : X_j \to X_j$, $j \in \{1, 2\}$.

A cospherical WHA [17] $H$ is a copivotal WHA for which $\text{tr}^{(L)}_V(f) = \text{tr}^{(R)}_V(f)$ for all morphisms $f : V \to V$ of $\mathcal{M}^H$. If $H$ is a cospherical WHA, then $\mathcal{M}^H$ is therefore spherical.

### A.5 Coquasitriangular Weak Hopf Algebras

A coquasitriangular WHA [6] is a WHA with a linear form $r : H \otimes H \to k$, the universal $r$-form, that satisfies the following conditions:

1. For all $x, y \in H$,

$$\varepsilon(x'y')r(x'' \otimes y'') = r(x \otimes y) = r(x' \otimes y')\varepsilon(y''x'').$$ \hspace{1cm} (A.26)
2. There exists some linear $\tau: H \otimes H \to k$ that is a weak convolution inverse of $r$, i.e.
\[ \tau(x' \otimes y') r(x'' \otimes y'') = \varepsilon(xy), \] (A.27)
\[ r(x' \otimes y') \tau(x'' \otimes y'') = \varepsilon(xy). \] (A.28)

3. For all $x, y, z \in H$,
\[ x' y' r(x'' \otimes y'') \equiv r(x' \otimes y') y'' x', \] (A.29)
\[ r((xy) \otimes z) \equiv r(y \otimes z') r(x \otimes z''), \] (A.30)
\[ r(x \otimes (yz)) \equiv r(x' \otimes y) r(x'' \otimes z). \] (A.31)

Note that $\tau$ in (2) is uniquely determined by $r$ if one imposes (A.26), (A.27) and (A.28).

In a coquasitriangular WHA $H$, we define the linear form $q: H \otimes H \to k$ by
\[ q(x \otimes y) = r(x' \otimes y') r(y'' \otimes x''), \] (A.32)
for all $x, y \in H$. Its weak convolution inverse $\overline{\tau}: H \otimes H \to k$ is then given by
\[ \overline{\tau}(x \otimes y) = \tau(y' \otimes x') \tau(x'' \otimes y''), \] (A.33)
for all $x, y \in H$. The dual Drinfel’d elements are the linear forms $u: H \to k$ and $v: H \to k$ given by $u(x) = r(S(x') \otimes x')$ and $v(x) = r(S(x') \otimes x')$ for all $x \in H$.

A braided monoidal category $C$ is a monoidal category with natural isomorphisms $\sigma_{X,Y,Z} : X \otimes Y \otimes Z \to Y \otimes X \otimes Z$ that satisfy the two hexagon axioms
\[ \sigma_{X,Y,Z} = \alpha_{X,Z,Y} \circ (\sigma_{X,Z} \otimes \text{id}_Y) \circ \alpha_{X,Y,Z}^{-1} \circ (\text{id}_X \otimes \sigma_{Y,Z}) \circ \alpha_{X,Y,Z}, \] (A.34)
\[ \sigma_{X,Y,Z} = \alpha_{Y,Z,X}^{-1} \circ (\text{id}_Y \otimes \sigma_{X,Z}) \circ \alpha_{Y,X,Z} \circ (\sigma_{X,Y} \otimes \text{id}_Z) \circ \alpha_{X,Y,Z}^{-1}, \] (A.35)
for all $X, Y, Z \in |C|$.

If $H$ is a coquasitriangular WHA, then the category $\mathcal{M}^H$ is braided monoidal with braiding $\sigma_{V,W}: V \otimes W \to W \otimes V$ given by
\[ \sigma_{V,W}(v \otimes w) = (w_0 \otimes v_0) r(w_1 \otimes v_1) \] (A.36)
for all $V, W \in |\mathcal{M}^H|$ and $v \in V$, $w \in W$. Note that
\[ Q_{V,W} = \sigma_{W,V} \circ \sigma_{V,W} \] (A.37)
can be computed as $Q_{V,W}(v \otimes w) = (v_0 \otimes w_0) q(v_1 \otimes w_1)$ for all $v \in V$, $w \in W$, and similarly $Q_{V,W}^{-1}(v \otimes w) = (v_0 \otimes w_0) \overline{q}(v_1 \otimes w_1)$.

### A.6 Coribbon Weak Hopf Algebras

A coribbon WHA is a coquasitriangular WHA with a convolution invertible and dual central linear form $\nu: H \to k$, the universal ribbon twist, such that
\[ \nu(xy) = \nu(x') \nu(y') q(x'' \otimes y''), \] (A.38)
\[ \nu(S(x)) = \nu(x) \] (A.39)
all $x, y \in H$. 

---

The text above is a detailed explanation of certain algebraic structures and their properties, focusing on the concepts of coquasitriangular WHAs, braiding, and coribbons. The notation and mathematical expressions are typical of advanced algebraic theory, involving tensors, monoidal categories, and linear forms. The examples given illustrate how these concepts are applied in specific algebraic contexts, such as within a coquasitriangular WHA, where the relationships between elements and operations are rigorously defined and analyzed.
A ribbon category is a braided monoidal category that is left-autonomous with natural isomorphisms \( \nu_X : X \to X \), the ribbon twist, such that
\[
\nu_X \otimes Y = \sigma_{Y,X} \circ \sigma_{X,Y} \circ (\nu_X \otimes \nu_Y)
\]
and
\[
(\nu_X \otimes \text{id}_{X^*}) \circ \text{coev}_X = (\text{id}_X \otimes \nu_{X^*}) \circ \text{coev}_X
\]
for all \( X,Y \in |C| \).

Note that every ribbon category is pivotal with \( \tau_X : X \to X^{**} \) given by
\[
\tau_X = \lambda_{X^{**}} \circ (\text{ev}_X \otimes \text{id}_{X^{**}}) \circ (\sigma_{X,X^*} \otimes \text{id}_{X^{**}}) \circ (\nu_X \otimes \text{coev}_{X^*}) \circ \rho_X^{-1},
\]
and furthermore spherical. For convenience, we give the right evaluation and coevaluation of (A.21) and (A.22):
\[
ev_X = \text{ev}_X \circ \sigma_{X,X^*} \circ (\nu_X \otimes \text{id}_{X^*}),
\]
\[
\text{coev}_X = (\text{id}_{X^*} \otimes \nu_X) \circ \sigma_{X,X^*} \circ \text{coev}_X.
\]

If \( H \) is a coribbon WHA, then \( M^H \) is a ribbon category with ribbon twist
\[
\nu_V : V \to V, \quad v \mapsto v_0 \nu(v_1),
\]
for all \( V \in |M^H| \) and \( v \in V \). Every coribbon WHA is cospherical with the copivotal form \( w(x) = v(x') \nu(x'') \) for all \( x \in H \), involving the second Drinfel’d element and the universal ribbon form.

Using the definition of a ribbon category as in this Appendix, we can draw the corresponding string diagrams. If we draw composition from top to bottom and the tensor product from left to right, we arrive at the diagrams shown in Section 4.1.

A.7 Cosemisimple Weak Hopf Algebras and fusion categories

A monoidal category \( C \) is called \( k \)-linear over some field \( k \) if the underlying category is \( k \)-linear, i.e. enriched in the category \( \text{Vect}_k \), and the tensor product of morphisms is \( k \)-bilinear. A \( k \)-linear category is called additive if it has a terminal object and all binary products. Such a category automatically has all finite biproducts. A \( k \)-linear category is abelian if it is additive, has all finite limits, and if every monomorphism is a kernel and every epimorphism a cokernel. Note that in \( k \)-linear pivotal categories, the traces \( \text{tr}_X^L, \text{tr}_X^R : \text{End}(X) \to \text{End}(1) \) are \( k \)-linear.

In a \( k \)-linear additive category \( C \), we call an object \( X \in |C| \) simple if \( \text{End}(X) \cong k \). A \( k \)-linear additive monoidal category is called pure if the monoidal unit \( 1 \) is simple. A \( k \)-linear additive category is called split semisimple if every object \( X \in |C| \) is isomorphic to a finite biproduct of simple objects. It is called finitely split semisimple if it is split semisimple and there exist only a finite number of simple objects up to isomorphism. If \( C \) is a \( k \)-linear additive category that is split semisimple, we denote by \( \{ V_j \}_{j \in I} \) a family of representatives of the isomorphism classes of the simple objects \( V_j, j \in I \), of \( C \), indexed by the set \( I \).

A multi-fusion category \( C \) over \( k \), see, for example [8], is a finitely split semisimple \( k \)-linear additive autonomous monoidal category such that \( \text{Hom}(X,Y) \) is finite-dimensional over \( k \) for all \( X,Y \in |C| \). A fusion category over \( k \) is a pure multi-fusion category. Note that every
multi-fusion category is abelian and essentially small, and that in every multi-fusion category, if \( X \in |C| \) is simple, then so is \( X^* \).

If \( C \) is a coalgebra and \( V \) a finite-dimensional right \( C \)-comodule with coaction \( \beta: V \to V \otimes C \) and basis \( (v_j)_j \), then there are elements \( c_{ij}^{(V)} \in C \) uniquely determined by the condition that \( \beta_V(v_j) = \sum_k v_k \otimes c_{ij}^{(V)} \). They are called the coefficients of \( V \) with respect to that basis. They span the coefficient coalgebra \( C(V) = \text{span}_k \{ c_{ij}^{(V)} \} \), a subcoalgebra of \( C \).

Let \( W \) be a finite-dimensional vector space over \( k \) with dual space \( W^* \) and a pair of dual bases \( (e_j)_j \) and \( (e^j)_j \) of \( W \) and \( W^* \), respectively. We abbreviate \( c_{jk}^{(W)} = e^j \otimes e_k \in W^* \otimes W \).

The coalgebra \( (W^* \otimes W, \Delta, \varepsilon) \) with \( \Delta(c_{jk}^{(W)}) = \sum_k c_{jk}^{(W)} \otimes c_{jk}^{(W)} \) and \( \varepsilon(c_{jk}^{(W)}) = \delta_{jk} \) is called the matrix coalgebra associated with \( W \). In this case, \( W \) is a right \( W^* \otimes W \)-comodule, and \( W^* \otimes W \) is its coefficient coalgebra.

A coalgebra \( C \) is called cosimple if \( C \) has no subcoalgebras other than \( C \) and \( \{0\} \). The coalgebra \( C \) is called cosemisimple if it is a coproduct in \( \text{Vect}_k \) of cosimple coalgebras. The coalgebra \( C \) is called split cosemisimple if it is cosemisimple and every cosimple subcoalgebra is a matrix coalgebra. A right \( C \)-comodule \( V \) of some coalgebra \( C \) is called irreducible if \( V \neq \{0\} \) and \( V \) has no subcomodules other than \( V \) and \( \{0\} \).

If \( H \) is a WHA over some field \( k \), then \( \mathcal{M}^H \) is a \( k \)-linear abelian autonomous monoidal category such that \( \text{Hom}(X,Y) \) is finite-dimensional over \( k \) for all \( X,Y \in |C| \). If \( H \) is in addition [finite-dimensional and] split cosemisimple, then \( \mathcal{M}^H \) is [finitely] split semisimple.

A WHA is called copure if its base algebras intersect trivially, i.e. if \( H_s \cap H_t \cong k \), see, for example [6]. In this case, \( \mathcal{M}^H \) is pure.

A WHA over \( k \) is said to be multi-fusion if it is finite-dimensional and split cosemisimple. It is called fusion if it is in addition copure. In these cases, \( \mathcal{M}^H \) is multi-fusion or fusion over \( k \), respectively.

### A.8 Comodular Weak Hopf Algebras

Let \( \mathcal{C} \) be a ribbon category and \( V, W \in |\mathcal{C}| \). We call the evaluation of the Hopf link with components labeled by \( V \) and \( W \),

\[
S_{V,W} := \text{tr}_{V \otimes W}(Q_{V,W}) \in \text{End}(I).
\] (A.46)

If \( \mathcal{C} \) is in addition multi-fusion with a family \( \{ V_j \}_{j \in I} \) of representatives of the isomorphism classes of objects, we write \( S_{j\ell} := S_{V_jV_{\ell}} \) if \( j, \ell \in I \). If \( \mathcal{C} \) is fusion, \( \text{End}(I) \cong k \), and so \( S_{j\ell} \in k \).

A modular category is a fusion category that has the structure of a ribbon category and for which the \( |I| \times |I| \)-matrix with coefficients \( S_{j\ell}; j, \ell \in I \), is non-degenerate.

Let \( H \) be a copure coribbon WHA with copivotal form \( w: H \to k \) and \( V \in |\mathcal{M}^H| \), \( n := \dim_k V \). The dual character of \( V \) is the element

\[
\chi_V = \sum_{j=1}^n c_{jj}^{(V)} \in H,
\] (A.47)

and the dual quantum character the element

\[
T_V = \sum_{j,\ell=1}^n c_{j\ell}^{(V)} w(c_{j\ell}^{(V)}) \in H.
\] (A.48)
We denote the space of dual quantum characters of $H$ by

$$T(H) = \text{span}_k \{ T_V : V \in |\mathcal{M}^H| \}.$$  

(A.49)

If $H$ is a copure coribbon WHA and the morphism $\hat{f}^{(\gamma)}_{V,W} : V \otimes W \to V \otimes W$ is of the form

$$\hat{f}^{(\gamma)}_{V,W} = (\text{id}_V \otimes \gamma) \circ (\text{id}_V \otimes \eta_{H,W} \otimes \text{id}_H) \circ (\beta_V \otimes \beta_W)$$

(A.50)

with a linear form $\gamma : H \otimes H \to k$ that satisfies

$$x'y'(x'' \otimes y'') = \gamma(x' \otimes y')x''y'',$$

(A.51)

$$\varepsilon(x'y')\gamma(x'' \otimes y'') = \gamma(x \otimes y)$$

(A.52)

for all $x, y \in H$, then

$$\text{tr}_{V \otimes W}(\hat{f}^{(\gamma)}_{V,W}) = c^{(\gamma)}_{V,W} \text{id}_{Hs},$$

(A.53)

where the element $c^{(\gamma)}_{V,W} \in k$ is determined by

$$\gamma(T'_V \otimes T'_W)\varepsilon_s(S(T''_V T''_W)) = c^{(\gamma)}_{V,W} \eta(1).$$

(A.54)

Given a copure coribbon WHA, we can therefore define a linear form $\tilde{q} : T(H) \otimes T(H) \to k$, $T_V \otimes T_W \to \tilde{q}_{V,W}$ where the $\tilde{q}_{V,W} \in k$ are determined by

$$\tilde{q}(T'_V \otimes T'_W)\varepsilon_s(S(T''_V T''_W)) = \tilde{q}_{V,W} \eta(1).$$

(A.55)

$H$ is called weakly cofactorizable if every linear form $\varphi : T(H) \to k$ can be written as $\varphi(-) = \tilde{q}(- \otimes x)$ for some $x \in T(H)$.

A comodular WHA [5] is a coribbon WHA that is fusion and weakly cofactorizable. If $H$ is a comodular WHA, then $\mathcal{M}^H$ is a modular category [5].

B Tannaka–Kreĭn reconstruction for a fusion categories

B.1 Fusion categories

Let $\mathcal{C}$ be a multi-fusion category (see Appendix A.7) over some field $k$. By $\{V_j\}_{j \in I}$ we denote a (finite) set of representatives of the isomorphism classes of simple objects of $\mathcal{C}$. We use the small progenerator $\hat{V} = \bigoplus_{j \in I} V_j$. The long canonical functor

$$\omega : \mathcal{C} \to \textbf{Vect}_k, \quad X \mapsto \text{Hom}(\hat{V}, \hat{V} \otimes X),$$

$$f \mapsto (\text{id}_\hat{V} \otimes f) \circ -,$$

(B.1)

is $k$-linear, faithful and exact [15], takes values in $\textbf{vect}_k$, and has a separable Frobenius structure [5, 7, 16].

The algebra $\mathcal{R} := \text{End}(\hat{V}) \cong \omega 1 \cong k^{1|l}$ has a basis $(\lambda_j)_{j \in I}$ of orthogonal idempotents given by $\lambda_j = \text{id}_{V_j} \in \mathcal{R}$, $j \in I$. It forms a Frobenius algebra $(\mathcal{R}, \circ, \text{id}_\mathcal{R}, \Delta_\mathcal{R}, \varepsilon_\mathcal{R})$ with comultiplication $\Delta_\mathcal{R} : \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}$ and counit $\varepsilon_\mathcal{R} : \mathcal{R} \to k$ given by $\Delta(\lambda_j) = \lambda_j \otimes \lambda_j$ and $\varepsilon(\lambda_j) = 1$ for all $j \in I$. The element $\Delta(\text{id}_\mathcal{R})$ is a separability idempotent. Such a Frobenius algebra is called index one or Frobenius separable [26].
The category $\mathcal{C}$ is equipped with a family of non-degenerate bilinear forms
\[
g_X : \text{Hom}(\hat{V} \otimes X, \hat{V}) \otimes \text{Hom}(\hat{V}, \hat{V} \otimes X) \to k, \quad \vartheta \otimes v \mapsto \varepsilon_R(\vartheta \circ v),
\] (B.2)
which is associative, i.e. compatible with composition, $g_X((\vartheta \circ \omega f) \otimes v) = g_Y(\vartheta \otimes (\omega f \circ v))$ for all $v : \hat{V} \to \hat{V} \otimes X, \vartheta : \hat{V} \otimes Y \to \hat{V}$ and $f : X \to Y$. We use $g_X$ in order to identify $(\omega X)^* \cong \text{Hom}_k(\hat{V} \otimes X, \hat{V})$.

By $(e^m_{(X)})_m$ and $(e^m_{(X)})_m^*$ we denote a pair of dual bases of $\omega X = \text{Hom}(\hat{V}, \hat{V} \otimes X)$ and $\text{Hom}(\hat{V} \otimes X, \hat{V})$ with respect to $g_X$. In particular, we can choose $e^{(1)}_j = \rho_{\hat{V}}^{-1} \circ \lambda_j$ and $e^{(1)}_j = \lambda_j \circ \rho_{\hat{V}}$. Many computations are particularly convenient if performed in these bases.

In addition to the dual basis lemma, i.e. the triangle identities for evaluation with the bilinear form $g_X$, the pair of dual bases satisfies
\[
\sum_j e^{(X)}_j \circ e^{(X)}_j = \text{id}_{\hat{V}} \otimes \text{id}_X
\] (B.3)
for all simple $X \in |\mathcal{C}|$.

By a generalization of Tannaka–Kreˇın reconstruction from strong monoidal functors to functors with separable Frobenius structure, we obtain a finite-dimensional, split cosemisimple WHA
\[
H = \text{coend}(\mathcal{C}, \omega) = \bigoplus_{j \in I} (\omega V^j)^* \otimes \omega V^j,
\] (B.4)
such that $\mathcal{C} \cong \mathcal{M}^H$ are equivalent as $k$-linear additive monoidal categories. If $\mathcal{C}$ is fusion, i.e. pure, then in addition $H$ is copure, i.e. $H_s \cap H_t \cong k$. The operations of $H$ are given as follows [6],
\[
\mu([\vartheta]|v)_{X} \otimes [\zeta]|w|_{X^Y} = ([\vartheta \circ (\vartheta \otimes \text{id}_Y) \circ \alpha_{\hat{V},X,Y}^{-1} \circ (\vartheta \otimes \text{id}_Y) \circ w]_{X^Y},
\] (B.5)
\[
\eta(1) = [\rho_{\hat{V}}^{-1}]_1, \quad (B.6)
\]
\[
\Delta([\vartheta]|v)_{X} = \sum_j [\vartheta]|e^{(X)}_j|_{X} \otimes [e^{(X)}_j]|v|_{X},
\] (B.7)
\[
\varepsilon([\vartheta]|v)_{X} = \varepsilon_R(\vartheta \circ v),
\] (B.8)
\[
S([e^{(X)}_j]|e^{(X^*)}_j|_{X^*} = [\varepsilon_R(e^{(X)}_j)|e^{(X^*)}_j]_{X^*},
\] (B.9)
where we write $[\vartheta]|v|_{X} \in (\omega X)^* \otimes \omega X$ with $v \in \omega X$, $\vartheta \in (\omega X)^*$ and simple $X \in |\mathcal{C}|$ for the homogeneous elements of $H$. The precise form of the universal coend as a colimit also allows us to use the same expression for arbitrary objects of $\mathcal{C}$, but subject to the relations that $[\zeta]|(\omega f)(v)|_Y = [(\omega f)^*|\zeta|v|_X$ for all $v \in \omega X$, $\zeta \in (\omega Y)^*$ and for all morphisms $f : X \to Y$ of $\mathcal{C}$. Recall that $(\omega f)(v) = (\text{id}_{\hat{V}} \otimes f) \circ v$ and $(\omega f)^*|\zeta = \zeta \circ (\text{id}_{\hat{V}} \otimes f)$. Furthermore, by $(e^{(X^*)}_j)_j$, we denote the basis of $\omega(X^*)$ defined by
\[
\tilde{e}^{(X^*)}_j = \Psi_X(e^{(X)}_j),
\] (B.10)
where
\[
\Psi_X(\vartheta) = (\vartheta \otimes \text{id}_{X^*}) \circ \alpha_{\hat{V},X,X^*}^{-1} \circ (\text{id}_{\hat{V}} \otimes \text{coev}_{X}) \circ \rho_{\hat{V}}^{-1}.
\] (B.11)
By $(\tilde{e}^{(X^*)}_j)_j$ we denote its dual basis with respect to the bilinear form $g_{X^*}$, c.f. (B.2).
The source and target counital maps are given by
\[
\varepsilon_s(\theta[v]_X) = \left[\rho_{\tilde{V}}^{-1} \circ \theta \circ v_1\right],
\]
\[
\varepsilon_t(\theta[v]_X) = \sum_{\ell} \left[\varepsilon^{\ell}_{(1)}\right] \rho_{\tilde{V}}^{-1} g_X(\theta \otimes ((\rho_{\tilde{V}} \circ \varepsilon^{(1)}_{\ell}) \otimes \text{id}_X) \circ v),
\]
for all simple \(X \in |C|\) and \(\theta \in (\omega X)^*, v \in \omega X\).

Note that if \(X \in |C|\) is an arbitrary object, then \(\omega X\) forms a right-\(H\) comodule with the coaction
\[
\beta_{\omega X} : \omega X \to \omega X \otimes H, \quad v \mapsto \sum_j e^{(x)}_j \otimes [e^{(x)}_j]_X v.
\]

Its coefficient coalgebra is given by \(C(X) = ((\omega X)^* \otimes \omega X)/N_X \subseteq H\) where the subspace \(N_X \subseteq (\omega X)^* \otimes \omega X\) is generated by the elements
\[
[\theta(\omega f)(v)]_X - [(\omega f)^*(\theta)]_X
\]
for all \(v \in \omega X, \theta \in (\omega X)^*\) and \(f \in \text{End}(X)\).

### B.2 Additional structure

If \(C\) is pivotal with the monoidal natural isomorphism \(\tau_X : X \to X^{**}\), then \(H = \text{coend}(C, \omega)\) is copivotal \[^{[17]}\] with copivotal form and its convolution inverse given by
\[
w(\theta[v]_X) = g_X((D_{\tilde{V}}^{-1} \circ \theta \circ (D_{\tilde{V}} \otimes \text{id}_X)) \circ v),
\]
\[
\overline{w}(\theta[v]_X) = g_X(\theta \otimes ((D_{\tilde{V}} \otimes \text{id}_X) \circ \text{id} \circ D_{\tilde{V}}^{-1})),
\]
for all \(v \in \omega X, \theta \in (\omega X)^*\) and \(X \in |C|\). Here
\[
D_{\tilde{V}} = \sum_{j \in I} (\dim V_j)^{-1} \text{id}_{V_j} : \tilde{V} \to \tilde{V}.
\]

More generally, there is a natural equivalence \(D : 1_C \Rightarrow 1_C\) given by isomorphisms \(D_X : X \to X\) for all \(X \in |C|\) as follows. If
\[
X \cong \bigoplus_{j \in I} m_j V_j
\]
with multiplicities \(m_j \in \mathbb{N}_0\), then \(D_X(v) = (\dim V_j) \text{id}_{V_j}\) for all homogeneous \(v \in m_j V_j\).

If \(C\) is spherical, then \(H\) is cospherical \[^{[17]}\]. In this case, the bilinear forms \(g_X\) are related to the traces in \(C\) as follows,
\[
g_X(\theta \otimes v) = \varepsilon_R(\theta \circ v) = \text{tr}_{\tilde{V}}(D_{\tilde{V}} \circ \theta \circ v),
\]
for all \(v \in \omega X, \theta \in (\omega X)^*, X \in |C|\).

Furthermore, the basis dual to the \(e^{(X^*)}_{j}\) of \[^{[B.10]}\] can be computed as
\[
\overline{e}^{(x^*)}_{j} = \Phi X(e^{(x)}_{j}),
\]
where
\[
\Phi X(v) = D_{\tilde{V}}^{-1} \circ \rho_{\tilde{V}} \circ (\text{id}_{\tilde{V}} \otimes \varepsilon_{X} X^*) \circ \alpha_{\tilde{V}, X^*} \circ (v \otimes \text{id}_{X^*}) \circ (D_{\tilde{V}} \otimes \text{id}_{X^*}),
\]
for all \(v \in \omega X, \theta \in (\omega X)^*, X \in |C|\).
and \((B.13)\) simplifies to

\[
\varepsilon_t([\vartheta|v]_X) = [\Phi_X(v) \circ \Psi_X(\vartheta) \circ \rho_V^{-1}]_I
\]

(B.23)

with \(\Psi_X\) as in \((B.11)\).

If \(C\) is braided with braiding \(\sigma_{X,Y}: X \otimes Y \rightarrow Y \otimes X\), then \(H\) is coquasi-triangular \([6]\) with universal \(r\)-form and its weak convolution inverse

\[
r([\vartheta|v]_X \otimes [\zeta|w]_Y) = g_{X \otimes Y}((\vartheta \otimes (\vartheta \otimes \text{id}_Y) \circ \alpha_{V,Y,X}^{-1}) \otimes ((\text{id}_V \otimes \sigma_{Y,X}) \circ \alpha_{V,Y,X} \circ (w \otimes \text{id}_X) \circ v)),
\]

(B.24)

\[
\varpi([\vartheta|x]_X) = g_{X \otimes Y}((\vartheta \otimes (\vartheta \otimes \text{id}_X) \circ \alpha_{V,Y,Z}^{-1} \circ (\text{id}_V \otimes \sigma_{Y,X}^{-1})) \otimes (\alpha_{V,Y,X} \circ (v \otimes \text{id}_Y) \circ w)),
\]

(B.25)

for all \(v \in \omega_X, w \in \omega_Y, \vartheta \in (\omega X)^*, \zeta \in (\omega Y)^*\) and \(X,Y \in |C|\).

If \(C\) is ribbon with twist \(\nu_X: X \rightarrow X\), then \(H\) is coribbon \([6]\) with universal ribbon form and its convolution inverse given by

\[
\nu([\vartheta|v]_X) = g_X(\vartheta \circ ((\text{id}_V \otimes \nu_X) \circ v))
\]

(B.26)

\[
\varpi([\vartheta|x]_X) = g_X(\vartheta \circ ((\text{id}_V \otimes \nu_X^{-1}) \circ v))
\]

(B.27)

for all \(v \in \omega X, \vartheta \in (\omega X)^*, X \in |C|\).

Every ribbon category is pivotal, and in this case, the copivotal form is given by \(w(x) = v(x') \nu(x'')\) where \(v: H \rightarrow k, x \mapsto v(S(x') \otimes x'')\) denotes the second dual Drinfel’d element. If \(C\) is modular, then \(H\) is fusion, coribbon and weakly cofactorizable \([6]\).

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