Characterization of Relativistic MHD Turbulence

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Abstract. The objective of this work is to understand if and how the characteristics of relativistic MHD turbulence may differ from those of nonrelativistic MHD turbulence. We accomplish this by studying the invariants in the relativistic case and comparing them to what we know of nonrelativistic turbulence. Although much work has been done to understand the dynamics of nonrelativistic systems (mostly for ideal incompressible fluids), there is minimal literature explicitly describing the dynamics of relativistic MHD turbulence. Many authors simply assume that relativistic turbulence has the same invariants and obeys the same inverse energy cascade as non-relativistic systems.
1. Introduction

Many studies in numerical relativity and high-energy astrophysics depend on the dynamics of relativistic plasmas. These include phenomena such as primordial turbulence, neutron stars, active galactic nuclei, and accretion disks near black holes \[1, 2, 4, 7, 9, 17, 22, 23\]. Unfortunately, we do not know if the results of these studies are accurate because of approximations such as the use of non-relativistic fluid dynamics and the lack of a standard model to describe the dynamics of relativistic turbulence. In particular, very little is understood about the turbulent dynamics of a relativistic plasma or its effect on the evolution of magnetic fields. This can only effectively be studied through direct numerical simulation of the relativistic fluid.

In the following report we will first discuss what is currently known about the dynamics of nonrelativistic MHD systems. We introduce the standard nonrelativistic MHD evolution equations as well as the known invariants for the system. In section 3, we will introduce the relativistic MHD equations and the relativistic equivalents of the non-relativistic MHD invariants. We then present our results for a relativistic MHD code working in both a low energy (nonrelativistic) and high-energy (relativistic) regime. We conclude by discussing the similarities and differences between the two different systems.

2. Standard Nonrelativistic MHD

Work by Shebalin \[19, 20\], on homogeneous MHD turbulence best demonstrates how the dynamics of a magnetofluid can differ from that of a hydrodynamic fluid. Plasma can be accurately modeled as a fluid made up of charged particles that are therefore affected by magnetic fields as well as particle-particle interactions. Because of this, the magnetic field becomes a dynamic variable in addition to density, pressure and the velocity of particles. For example, in MHD turbulence, an equipartition occurs and we expect kinetic and magnetic energy fluctuations to become roughly equal. Shebalin modeled the magnetofluid as a homogenous system where the same statistics are considered valid everywhere in the computational domain. He utilized periodic boundary conditions and spectral methods in order to study how the dynamics of different scales interacted without the addition of boundary errors. Much of his work focused on an ideal MHD system, where the magnetic and fluid dissipation terms were excluded. Below are the evolution equations used by Shebalin to describe the non-compressible MHD system.

\[
\begin{align*}
\nabla \cdot \vec{v} &= 0 \\
\rho \frac{\partial \vec{v}}{\partial t} &= -\rho(\vec{v} \cdot \nabla \vec{v}) - \nabla p - \frac{1}{4\pi} \vec{B} \times \nabla \times \vec{B} \\
\frac{\partial \vec{B}}{\partial t} &= \nabla \times (\vec{v} \times \vec{B}).
\end{align*}
\]

By varying the mean magnetic field \((B_0)\) and angular velocity \((\Omega_0)\) of the system, Shebalin was able to define five different cases with different invariants as shown in
Table 1. MHD Turbulence and Invariants.

| Case | Mean Field \( B_0 \) | Angular Velocity | Invariants |
|------|-----------------|-----------------|------------|
| I    | 0               | 0               | \( E, H_C, H_M \) |
| II   | \( B_0 \neq 0 \) | 0               | \( E, H_C \) |
| III  | 0               | \( \Omega_0 \neq 0 \) | \( E, H_M \) |
| IV   | \( B_0 \neq 0 \) | \( \Omega_0 = \sigma B_0 \) | \( E, H_P \) |
| V    | \( B_0 \neq 0 \) | \( \Omega_0 \neq 0 \) (\( B_0 \times \Omega_0 \neq 0 \)) | \( E \) |

For an incompressible fluid \( u(k,t) \) is the Fourier coefficient of turbulent velocity and \( b(k,t) \) is the Fourier coefficient of the turbulent magnetic field. The energy, cross helicity and magnetic helicity can be expressed in terms of these as:

\[
E = \frac{1}{2N^3} \sum_k |u(k)|^2 + |b(k)|^2 \tag{4}
\]

\[
H_C = \frac{1}{2N^3} \sum_k u(k) \cdot b^*(k) \tag{5}
\]

\[
H_M = \frac{1}{2N^3} \sum_k \frac{i}{k^2} k \cdot b(k) \times b^*(k). \tag{6}
\]

This is assuming a cubic computational domain with \( N \) grid points in each direction. The statistical mechanics of the system is defined by the Gaussian canonical probability density function (PDF):

\[
D = \frac{1}{Z} exp(-\alpha E - \beta H_C - \gamma H_M) \tag{7}
\]

\[
Z = \int_\Gamma exp(-\alpha E - \beta H_C - \gamma H_M) d\Gamma \tag{8}
\]

\[
\hat{\Phi} = \int_\Gamma \Phi Dd\Gamma, \quad \bar{\Phi} = \frac{1}{T} \int_0^T \Phi dt. \tag{9}
\]

Where \( Z \) is the partition function and \( d\Gamma \) is the phase space volume. \( \hat{\Phi} \) shows how to calculate the ensemble averages using the PDF while \( \bar{\Phi} \) is the time average. If \( \hat{\Phi} = \bar{\Phi} \), the system is said to be ergodic but if \( \hat{\Phi} \neq \bar{\Phi} \), it is non-ergodic. Here \( \alpha, \beta \) and \( \gamma \) are inverse temperatures. The ensemble average magnetic energy (\( \hat{E}_M \)) is always greater than or equal to ensemble average kinetic energy (\( \hat{E}_K \)), and the inverse temperature terms can be found as a function of \( \hat{E}_M = \phi \).
Phase portraits resulting from computer simulations of Shebalin’s five cases show that coherent structures formed in many systems where the magnetofluid was experiencing turbulence [19, 20]. Coherent structure occurs when time-averaged physical variables in MHD turbulence have large mean values, rather than the zero mean values expected from theoretical ensemble predictions. MHD turbulence thus has broken ergodicity, which can be explained by finding the eigenmodes of the system. One out of the four eigenvalues associated with each of the lowest wavenumbers will be very much smaller than the others; the eigenvariables associated with these very small eigenvalues grow to have very large energies compared to other eigenvariables; when this happens, an almost force-free state occurs in which large-energy eigenmodes are quasistationary while low-energy eigenmodes remain turbulent; thus, the predicted ergodicity has been dynamically broken. This is observed to occur even in dissipative systems because broken ergodicity in MHD turbulence manifests itself at the smallest wavenumbers (largest length scales) where dissipation is negligible, resulting in the Kolmogorov spectrum. In the case of ideal hydrodynamic turbulence, broken ergodicity can occur in a finite model system, but only at the largest wavenumbers (smallest scales). When dissipation is added, the large wavenumber modes are most affected and their energy quickly decays away, so that broken ergodicity plays no role in decaying hydrodynamic turbulence.

In a compressible MHD system; Energy, Cross Helicity, Magnetic Helicity and therefore Parallel Helicity should all be conserved. Given that our relativistic system is by default a compressible system, we expect to see that this is true for the relativistic system as well. The equations for ideal compressible MHD are similar to the incompressible equations with the exception of the first equation.

\[
\frac{\partial \rho}{\partial t} = -\mathbf{\nabla} \cdot (\rho \mathbf{v}) \tag{13}
\]

\[
\rho \frac{\partial \mathbf{v}}{\partial t} = -\rho(\mathbf{v} \cdot \nabla) \mathbf{v} - \nabla p - \frac{1}{4\pi} \mathbf{B} \times \nabla \times \mathbf{B} \tag{14}
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \mathbf{\nabla} \times (\mathbf{v} \times \mathbf{B}). \tag{15}
\]
3. Relativistic MHD Systems

The fluid and electromagnetic fields of the GRMHD equations are developed from several well-known equations [6]. They include the conservation of particle number, the continuity equation, the conservation of energy-momentum, the magnetic constraint equation and the magnetic induction equation. A system consisting of a perfect fluid and an electromagnetic field, the ideal MHD stress-energy tensor is given by

\[ T^\mu_\nu = (\rho_0 h + b^2)u^\mu u^\nu + (P + \frac{b^2}{2})g^\mu_\nu - b^\mu b^\nu \]  

\[ h = 1 + \epsilon + \frac{P}{\rho_0} \]  

\[ b^\mu = \frac{1}{\sqrt{4\pi}} B^\mu_{(u)} \]  

\[ B^0_{(u)} = \frac{1}{\alpha} u_0 B^i \]  

\[ B^i_{(u)} = \frac{1}{u_0} \left( \frac{B^i}{\alpha} + B^0_{(u)} u^i \right) . \]  

Here, \( P \) is the fluid pressure, \( \rho_0 \) is density, \( B^i \) is magnetic field, \( u^\mu \) is four-velocity, \( h \) is the enthalpy, \( \epsilon \) is specific internal energy, and \( b^2 \) is the magnitude of the magnetic vector field squared. The evolution equations were given by Duez as [7]:

\[ \partial_t \rho_* = -\partial_j (\rho_* v^j) \]  

\[ \partial_t \tilde{\tau} = -\partial_i (\alpha^2 \sqrt{\gamma} T^0_i - \rho_* v^i) + s \]  

\[ \partial_t \tilde{S}_i = -\partial_j (\alpha \sqrt{\gamma} T^j_i) - \frac{1}{2} \alpha \sqrt{\gamma} T^{\alpha \beta}_i g_{\alpha \beta, i} \]  

\[ \partial_t \tilde{B}^i = -\partial_j (v^j \tilde{B}^i - v^i \tilde{B}^j) . \]  

Here, \( \rho_* \) is conserved mass density, \( \tilde{\tau} \) relates to energy density, \( \tilde{S}_i \) is momentum density, \( \tilde{B} \) is related to the magnetic field and \( s \) is the source term. \( \alpha \) is a lapse term related to the time evolution of the simulation. Notice that unlike the non relativistic system, we use the stress-energy tensor within the evolution equations so that 4-momentum conservation is built into the system. This results in a set of equations that look very different from that of the non relativistic system. We also had to modify the invariants studied in the non relativistic system as shown below.

\[ E = \frac{1}{2N^3} \sum_k T_{00} \]  

\[ P_4 = \frac{1}{2N^3} \sum_k \sqrt{T^2_{00} - T^2_{01} - T^2_{02} - T^2_{03}} \]  

\[ H_C = \frac{1}{2N^3} \sum_k \gamma \rho \ u(k) \cdot b^*(k) \]
Characterization of Relativistic MHD Turbulence

\[ H_M = \frac{1}{2N^3} \sum_k \frac{i}{k^2} k \cdot b(k) \times b^*(k) \]  
\[ H_P = H_C - \sigma H_M. \]  

Here we also listed Four Momentum, \( P_4 \) as an invariant and modified Cross Helicity to include relativistic mass increases. These are similar to the invariants used by Yokoi [?]. This all results in a system which may look different but should represent the same physics.

4. Results

The objective of this study is to test the relativistic MHD system in the “low-energy” regime in order to see if it can reproduce the accepted results from non-relativistic ideal turbulent MHD studies. After that, we ran the codes in the “high-energy” regime in order to see if the dynamics are the same as a non-relativistic plasma. Below is a matrix of the test runs.

### Table 2. Non Relativistic Numerical Simulations

| Variables          | Case 1 | Case 2 | Case 3 | Case 4 | Case 5 |
|--------------------|--------|--------|--------|--------|--------|
| Max Velocity       | 0.001  | 0.001  | 0.001  | 0.001  | 0.001  |
| \( \Omega_x \)     | 0      | 0      | 0      | 0      | 0      |
| \( \Omega_y \)     | 0      | 0      | 0      | 0      | 0      |
| \( \Omega_z \)     | 0      | 0      | 0.002  | 0.002  | 0.002  |
| Init Temperature   | 3000   | 3000   | 3000   | 3000   | 3000   |
| Init Density       | 1.0e-4 | 1.0e-4 | 1.0e-4 | 1.0e-4 | 1.0e-4 |
| Max Magnetic Field | 50     | 50     | 50     | 50     | 50     |
| \( B_x \)          | 0      | 100    | 0      | 0      | 100    |
| \( B_y \)          | 0      | 0      | 0      | 0      | 0      |
| \( B_z \)          | 0      | 0      | 0      | 100    | 0      |

Each data run utilized 4th order finite differencing on a grid with 64 x 64 x 64 internal data points. We ran these simulations for over 33,000 iterations to determine if certain variables were invariant.

5. Discussion

Our results show that in the low-energy regime: Energy, Cross Helicity, Magnetic Helicity and Parallel Helicity are all clearly conserved as expected. For the high-energy case, 4-Momentum, Cross Helicity, Magnetic Helicity and Parallel Helicity also appear to be conserved. Final variations in all cases, were found to be smaller than \( 10^{-10} \) of the total value so we can safely dismiss any changes in the conserved quantities as a result of numerical errors. In conclusion, we find that relativistic MHD turbulence has the same invariants as similar non relativistic systems.
Figure 1. Non Relativistic Energy Conservation

Figure 2. Non Relativistic Cross Helicity Conservation
Figure 3. Non Relativistic Magnetic Helicity Conservation

Figure 4. Non Relativistic Parallel Helicity Conservation
Figure 5. Relativistic 4 Momentum Conservation

Figure 6. Relativistic Cross Helicity Conservation
Figure 7. Relativistic Magnetic Helicity Conservation

Figure 8. Relativistic Parallel Helicity Conservation
Table 3. Relativistic Numerical Simulations

| Variables      | Case 1 | Case 2 | Case 3 | Case 4 | Case 5 |
|----------------|--------|--------|--------|--------|--------|
| Max Velocity   | 0.65   | 0.65   | 0.65   | 0.65   | 0.65   |
| $\Omega_x$     | 0      | 0      | 0      | 0      | 0      |
| $\Omega_y$     | 0      | 0      | 0      | 0      | 0      |
| $\Omega_z$     | 0      | 0      | 0.75   | 0.75   | 0.75   |
| Init Temperature| 2.8e15 | 2.8e15 | 2.8e15 | 2.8e15 | 2.8e15 |
| Init Density   | 9.7e29 | 9.7e29 | 9.7e29 | 9.7e29 | 9.7e29 |
| Max Magnetic Field | 1.0e16  | 1.0e16 | 1.0e16 | 1.0e16 | 1.0e16 |
| $B_x$          | 0      | 1.0e17 | 0      | 0      | 1.0e17 |
| $B_y$          | 0      | 0      | 0      | 0      | 0      |
| $B_z$          | 0      | 0      | 0      | 1.0e17 | 0      |

References

[1] Caprini C and Durrer R, “Gravitational waves from stochastic relativistic sources: Primordial turbulence and magnetic fields” Phys. Rev. D 74, 063521 (2006) [arXiv:astro-ph/0603476]
[2] Caprini C, Durrer R and Servant G, “The stochastic gravitational wave background from turbulence and magnetic fields generated by a first-order phase transition” JCAP 0912, 024 (2009) [arXiv:astro-ph/0909.0622]
[3] Cho J, “Simulations of Relativistic Force-Free Magnetohydrodynamic Turbulence” Astro. Phys. J. 621, 324-327 (2005)
[4] Cho J, “Relativistic Force-Free MHD Turbulence in a Black Hole’s Magnetosphere” J. of Korean Phys. Soc. 49, 4 (2006)
[5] Dolgov A D, Grasso D and Nicolis A, “Relic backgrounds of gravitational waves from cosmic turbulence” Phys. Rev. D 66 103505 (2002) [arXiv:astro-ph/0206461]
[6] Duez M D, Liu Y T, Shapiro S T and Stephens B C, “Relativistic magnetohydrodynamics in dynamical spacetimes: Numerical methods and tests” Phys. Rev. D 72 024028 (2005) [arXiv:astro-ph/0503420]
[7] Garrison D, “Numerical Relativity as a tool for studying the Early Universe” J. Grav 407197 (2014) [arXiv:gr-qc/1207.7097]
[8] Gogoberidze G, Kahniashvili T and Kosowsky A, “The Spectrum of Gravitational Radiation from Primordial Turbulence” Phys. Rev. D 76, 083002 (2007) [arXiv:astro-ph/0705.1733]
[9] Inoue T, Asano K and Ioka K, “Three-Dimensional Simulations of MHD Turbulence Behind Relativistic Shock Waves and Their Implications for GRBs” Astrophys. J. 734, 77 (2011) [arXiv:astro-ph.HE/1011.6350]
[10] Kahniashvili T, “Gravitational radiation from primordial helical turbulence” (2005) [arXiv:astro-ph/0508459]
[11] Kahniashvili T, Gogoberidze G and Ratra B, “Polarized cosmological gravitational waves from primordial helical turbulence” Phys. Rev. Lett. 95 151301 (2005) [arXiv:astro-ph/0505628]
[12] Kahniashvili T and Ratra B, “CMB anisotropies due to cosmological magnetosonic waves” Phys. Rev. D 75 023002 (2007) [arXiv:astro-ph/0611247]
[13] Kahniashvili T, Kosowsky A, Gogoberidze G and Maravin Y, “Detectability of Gravitational Waves from Phase Transitions” Phys. Rev. D 78, 043003 (2008) [arXiv:astro-ph/0806.0293]
[14] Kahniashvili T, Campanelli L, Gogoberidze G, Maravin Y and Ratra B, “Gravitational Radiation from Primordial Helical Inverse Cascade MHD Turbulence” Phys. Rev. D 78, 123006 (2008) [Erratum-ibid. D 79, 109901 (2009)] [arXiv:astro-ph/0809.1899]
Characterization of Relativistic MHD Turbulence

[15] Kahniashvili T, Kisslinger L and Stevens T, “Gravitational Radiation Generated by Magnetic Fields in Cosmological Phase Transitions” Phys. Rev. D 81, 023004 (2010) [arXiv:astro-ph/0905.0643].

[16] Kisslinger L S, “Astrophysical observations of early universe phase transitions” Mod. Phys. Lett. A19, 1179-1186 (2004) [arXiv:hep-ph/0402001].

[17] Melatos A and Peralta C, “Gravitational Radiation from Hydrodynamic Turbulence in a Differentially Rotating Neutron Star” Astrophys. J. 709, 77 (2010) [arXiv:astro-ph.HE/0911.1609].

[18] Nicolis A, “Relic gravitational waves from colliding bubbles and cosmic turbulence” Class. Quant. Grav. 21, L27 (2004) [arXiv:gr-qc/0303084].

[19] Shebalin J V and Montgomery D, “Turbulent magnetohydrodynamic density fluctuations” J. of Plasma Phys. 39, 339-367 (1988).

[20] Shebalin J V, “The Statistical Mechanics of Ideal Homogeneous Turbulence” NASA TP-2002-210783 (2002); “Theory and simulation of real and ideal magnetohydrodynamic turbulence” Discrete & Continuous Dynamical Systems B 5, 153-175 (2005); “Broken symmetries and magnetic dynamos” Physics of Plasmas 14, 102301 (2007); “Broken Ergodicity in Magnetohydrodynamic Turbulence” Geophys. Astrophys. Fluid Dyn., Online DOI:10.1080/03091929.2011.589385 (2011); “Broken ergodicity, magnetic helicity and the MHD dynamo” Geophys. Astrophys. Fluid Dyn., Online DOI:10.1080/03091929.2012.689299 (2012).

[21] Yokoi, N, “Cross Helicity and related dynamo” Geophys. Astrophys. Fluid Dyn., Online DOI:10.1080/03091929.2013.801564 (2013).

[22] Zhang W, MacFadyen A and Wang P, “Three-Dimensional Relativistic MHD Simulations of the Kelvin-Helmholtz Instability: Magnetic Field Amplification by a Turbulent Dynamo” (2009) [arXiv:astro-ph/0811.3638].

[23] Zrake J and MacFadyen A, “Numerical Simulations of Driven Relativistic MHD Turbulence” (2011) [arXiv:astro-ph.HE/1108.1991].