Explicit Formulæ for Computing Euler Polynomials in Terms of Stirling Numbers of the Second Kind

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Abstract. In the paper, the author elementarily unifies and generalizes eight identities involving the functions \( \frac{1}{e^t - 1} \) and their derivatives. By one of these identities, the author establishes two explicit formulæ for computing Euler polynomials and two-parameter Euler polynomials, which are a newly introduced notion, in terms of Stirling numbers of the second kind.

1. Introduction

In [3], the following eight identities were elementarily and inductively established.

**Theorem 1.1** ([3, Theorems 2.1 to 2.4 and Corollaries 2.1 to 2.4]). For \( k \in \mathbb{N} \), we have

\[
\left( \frac{1}{e^t - 1} \right)^{(k)} = \sum_{m=1}^{k+1} \lambda_{k,m} \left( \frac{1}{e^t - 1} \right)^m, \quad \left( \frac{1}{1 - e^{-t}} \right)^{(k)} = \sum_{m=1}^{k+1} \mu_{k,m} \left( \frac{1}{1 - e^{-t}} \right)^m, \tag{1.1}
\]

\[
\left( \frac{1}{1 - e^{-t}} \right)^k = \sum_{m=1}^{k} a_{k,m-1} \left( \frac{1}{1 - e^{-t}} \right)^{(m-1)}, \quad \left( \frac{1}{e^t - 1} \right)^k = \sum_{m=1}^{k} b_{k,m-1} \left( \frac{1}{e^t - 1} \right)^{(m-1)}, \tag{1.2}
\]

\[
\left( \frac{1}{1 - e^{-t}} \right)^k = 1 + \sum_{m=1}^{k} a_{k,m-1} \left( \frac{1}{e^t - 1} \right)^{(m-1)}, \quad \left( \frac{1}{e^t - 1} \right)^k = 1 + \sum_{m=1}^{k} b_{k,m-1} \left( \frac{1}{1 - e^{-t}} \right)^{(m-1)}, \tag{1.3}
\]

where

\[
\lambda_{k,m} = (-1)^k (m-1)! S(k+1, m), \quad \mu_{k,m} = (-1)^{m-1} (m-1)! S(k+1, m), \tag{1.5}
\]

\[
a_{k,m-1} = (-1)^{m^2+1} M_{k-m+1}(k, m), \quad b_{k,m-1} = (-1)^{k-m} a_{k,m-1}, \tag{1.6}
\]

\[
M_j(k, i) = \begin{vmatrix}
\frac{1}{(i+1)!} & S(i+1, i) & S(i+2, i) & \cdots & S(i+j-1, i) \\
\frac{1}{(i+1)!} & S(i+1, i+1) & S(i+2, i+1) & \cdots & S(i+j-1, i+1) \\
\frac{1}{(i+2)!} & 0 & S(i+2, i+2) & \cdots & S(i+j-1, i+2) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{(i+j-2)!} & 0 & 0 & \cdots & S(i+j-1, i+j-1)
\end{vmatrix}, \quad j \in \mathbb{N}, \tag{1.7}
\]

and

\[
S(k, m) = \frac{1}{m!} \sum_{\ell=1}^{m} (-1)^{m-\ell} \binom{m}{\ell} \ell^k, \quad 1 \leq m \leq k \tag{1.8}
\]

are Stirling numbers of the second kind which may be generated by

\[
\frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n, k) \frac{x^n}{n!}, \quad k \in \mathbb{N}. \tag{1.9}
\]
It was pointed out in [3, Remark 5.3] that the above eight identities involving the functions \( \frac{1}{e^t - 1} \) and their derivatives are equivalent to each other.

By virtue of the first identity in (1.1), among other things, an explicit formula for computing Bernoulli numbers \( B_{2k} \), which are defined by the power series expansion

\[
\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} B_i \frac{t^i}{i!} = 1 - \frac{t}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{t^{2k}}{(2k)!}
\]  

for \(|t| < 2\pi\), in terms of Stirling numbers of the second kind \( S(n, k) \), was discovered in [3] as follows.

**Theorem 1.2** ([3, Theorem 3.1]). For \( k \in \mathbb{N} \), Bernoulli numbers \( B_{2k} \) may be computed by

\[
B_{2k} = 1 + \sum_{m=1}^{2k-1} \frac{S(2k + 1, m + 1) S(2k; 2k - m)}{(2k)!} - \frac{2k}{2k + 1} \sum_{m=1}^{2k} S(2k, m) S(2k + 1, 2k - m + 1).
\]  

In [8], making use of Faà di Bruno’s formula, combinatorial techniques, and much knowledge on Bell polynomials and Stirling numbers of the first and second kinds, the above eight identities were generalized and unified as follows.

**Theorem 1.3** ([8, Theorems 3.1 and 3.2]). For \( \alpha, \lambda \in \mathbb{R} \),

1. when \( n \in \mathbb{N} \), we have

\[
\left( \frac{1}{1 - \lambda e^{\alpha t}} \right)^{(n)} = (-1)^n \alpha^n \sum_{k=1}^{n+1} (-1)^{k-1} (k-1)! S(n+1, k) \left( \frac{1}{1 - \lambda e^{\alpha t}} \right)^k;
\]  

2. when \( n \in \mathbb{N} \), we have

\[
\left( \frac{1}{1 - \lambda e^{\alpha t}} \right)^n = \frac{(-1)^{n-1}}{(n-1)!} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{\alpha^{k-1}} s(n, k) \left( \frac{1}{1 - \lambda e^{\alpha t}} \right)^{(k-1)},
\]  

where \( s(n, k) \) for \( n \geq k \geq 1 \) denote Stirling numbers of the first kind which may be generated by

\[
\frac{[\ln(1 + x)]^m}{m!} = \sum_{k=m}^{\infty} s(k, m) \frac{x^k}{k!}, \quad |x| < 1.
\]  

As a consequence of comparing the identity (1.13) for \( \alpha = -1 \) and \( \lambda = 1 \) with the first identity in (1.3), it was derived in [8, Corollary 3.2] that, when \( n \geq k \),

\[
s(n, k) = (-1)^{n+k^2} (n-1)! M_{n-k+1}(n, k),
\]  

where \( M_{n-k+1}(n, k) \) is defined by (1.7).

Employing the identity (1.12) for \( \alpha = 1 \), among other things, an explicit representation for calculating Apostol-Bernoulli numbers \( B_n(\lambda) \) for \( n \in \{0\} \cup \mathbb{N} \), which may be defined by

\[
\frac{t}{\lambda e^t - 1} = \sum_{k=0}^{\infty} B_k(\lambda) \frac{t^k}{k!}
\]  

for \( \lambda \in \mathbb{R} \) and \(|t| < 2\pi\), in terms of Stirling numbers of the second kind \( S(n, k) \), was obtained in [8] as follows.

**Theorem 1.4** ([8, Theorem 4.1]). For \( \lambda \neq 1 \) and \( n \in \mathbb{N} \), we have

\[
B_n(\lambda) = (-1)^{n-1} n \sum_{k=1}^{n} \frac{(k-1)!}{(\lambda - 1)^k} S(n, k).
\]  

The first aim of this paper is to supply, just basing on the first identity in (1.1) and the second identity in (1.3), without using the abstruse Faà di Bruno’s formula, deep techniques in combinatorics, and any recondite knowledge on Bell polynomials and Stirling numbers of the first and second kinds, an elementary proof for the identities (1.12) and (1.13) in Theorem 1.3.

The second aim of this paper is to find explicit formulae for computing Euler numbers and polynomials in terms of Stirling numbers of the second kind \( S(n, k) \).

The third aim of this paper is to introduce a notion “two-parameter Euler polynomials”, a generalization of the classical Euler polynomials, and establish an explicit formula for computing the newly defined two-parameter Euler polynomials in terms of Stirling numbers of the second kind \( S(n, k) \).
2. AN ELEMENTARY PROOF OF THEOREM 1.3

In order to elementarily prove Theorem 1.3, we would like to rewrite it as Theorem 2.1 below.

**Theorem 2.1.** Let \( \lambda \neq 0 \) and \( \alpha \neq 0 \) be real constants and \( k \in \mathbb{N} \). When \( \lambda > 0 \) and \( t \neq -\frac{\ln \lambda}{\alpha} \) or when \( \lambda < 0 \) and \( t \in \mathbb{R} \), we have

\[
\frac{d^k}{dt^k}\left(\frac{1}{\lambda e^{\alpha t} - 1}\right) = (-1)^k \alpha^k \sum_{m=1}^{k+1} (m-1)! S(k+1, m) \left(\frac{1}{\lambda e^{\alpha t} - 1}\right)^m
\]

(2.1)

and

\[
\left(\frac{1}{\lambda e^{\alpha t} - 1}\right)^k = \frac{1}{(k-1)!} \sum_{m=1}^{k} \frac{(-1)^{m-1}}{\alpha^{m-1}} s(k, m) \frac{d^{m-1}}{dt^{m-1}} \left(\frac{1}{\lambda e^{\alpha t} - 1}\right).
\]

(2.2)

where \( s(k, m) \) and \( S(k + 1, m) \) represent Stirling numbers of the first and second kinds.

**Proof.** For \( \lambda > 0 \), let

\[ F(u) = \frac{1}{e^u - 1} \quad \text{and} \quad u = u(t) = \ln \lambda + \alpha t. \]

(2.3)

From \( \lambda e^{\alpha t} = e^{\ln \lambda + \alpha t} = e^{u(t)} \) and \( u'(t) = \alpha \), it follows that for \( 1 < \ell < k \)

\[
\frac{d^k}{dt^k}\left(\frac{1}{\lambda e^{\alpha t} - 1}\right) = \frac{d^k}{dt^k}\left(\frac{1}{e^{\ln \lambda + \alpha t} - 1}\right) = \frac{d^k F(u(t))}{dt^k} = \alpha \frac{d^{k-1} F'(u(t))}{dt^{k-1}} = \alpha^k F^{(k)}(\ln \lambda + \alpha t).
\]

Combining this with the first identity (1.1) yields that

\[
\frac{d^k}{dt^k}\left(\frac{1}{\lambda e^{\alpha t} - 1}\right) = \alpha^k F^{(k)}(\ln \lambda + \alpha t) = \alpha^k \sum_{m=1}^{k+1} \lambda_{k, m} \left(\frac{1}{e^{\ln \lambda + \alpha t} - 1}\right)^m
\]

\[
= \alpha^k \sum_{m=1}^{k+1} \lambda_{k, m} \left(\frac{1}{\lambda e^{\alpha t} - 1}\right)^m = (-1)^k \alpha^k \sum_{m=1}^{k+1} (m-1)! S(k+1, m) \left(\frac{1}{\lambda e^{\alpha t} - 1}\right)^m.
\]

The identity (2.1) for \( \lambda > 0 \) is thus proved.

By the second identity in (1.3) and the equalities in (1.6) and (1.15), it follows that for \( 1 < \ell < k \)

\[
\left(\frac{1}{\lambda e^{\alpha t} - 1}\right)^k = \left(\frac{1}{e^{\ln \lambda + \alpha t} - 1}\right)^k = \left(\frac{1}{e^u - 1}\right)^k = \sum_{m=1}^{k} \frac{d^{m-1}}{du^{m-1}} \left(\frac{1}{e^u - 1}\right)
\]

\[
= \sum_{m=1}^{k} \frac{b_{k, m-1}}{\alpha} \frac{d^{m-2}}{du^{m-2}} \left(\frac{1}{e^u - 1}\right) = \sum_{m=1}^{k} \frac{b_{k, m-1}}{\alpha^\ell \left(\frac{1}{e^u - 1}\right)^\ell} = \sum_{m=1}^{k} \frac{1}{\alpha^{m-1}} \frac{d^{m-1}}{t^{m-1}} \left(\frac{1}{e^{\ln \lambda + \alpha t} - 1}\right)
\]

\[
= \sum_{m=1}^{k} \frac{(-1)^{k-m} \alpha_{k,m-1}}{\alpha^{m-1}} \frac{d^{m-1}}{t^{m-1}} \left(\frac{1}{\lambda e^{\alpha t} - 1}\right)
\]

\[
= \frac{1}{(k-1)!} \sum_{m=1}^{k} \frac{(-1)^{m-1}}{\alpha^{m-1}} s(k, m) \frac{d^{m-1}}{t^{m-1}} \left(\frac{1}{\lambda e^{\alpha t} - 1}\right).
\]

The identity (2.2) for \( \lambda > 0 \) is thus proved.

Let

\[ H(t) = \frac{1}{t+1}, \quad t \in \mathbb{R}, \]

(2.4)

It is clear that \( F(t) \), defined in (2.3), and \( H(t) \) may be regarded as composite functions of \( u \) with \( u = u(t) = e^t + 1 \) respectively and that \( u'(t) = e^t \). Further by implications of the first identity in (1.1) and the second identity
respectively defined by the power expansions
\[
\left( \frac{1}{e^t + 1} \right)^{(k)} = \sum_{m=1}^{k+1} (-1)^{m-1} \lambda_{k,m} \left( \frac{1}{e^t + 1} \right)^m \quad \text{and} \quad \left( \frac{1}{e^t + 1} \right)^k = (-1)^{k-1} \sum_{m=1}^{k} b_{k,m-1} \left( \frac{1}{e^t + 1} \right)^{(m-1)}.
\] \tag{2.5}

For \( \lambda < 0 \), let \( v = v(t) = \ln |\lambda| + \alpha t \). As the above arguments, from \( \lambda e^{\alpha t} = -e^{\ln |\lambda| + \alpha t} \) and \( v'(t) = \alpha \), it follows that for \( 1 < t < k \)
\[
\frac{d^k}{dt^k} \left( \frac{1}{\lambda e^{\alpha t} - 1} \right) = -\frac{d^k}{dt^k} \left( \frac{1}{e^{\ln |\lambda| + \alpha t} + 1} \right) = -\frac{d^k}{dt^k} H(v(t)) = -\alpha^k H^{(k)}(v(t)) = -\alpha^k H^{(k)}(\ln |\lambda| + \alpha t).
\]

Combining this with the first identity in (2.5) and the first formula in (1.5) gives that
\[
\frac{d^k}{dt^k} \left( \frac{1}{\lambda e^{\alpha t} - 1} \right) = -\alpha^k H^{(k)}(\ln |\lambda| + \alpha t) = -\alpha^k \sum_{m=1}^{k+1} (-1)^{m-1} \lambda_{k,m} \left( \frac{1}{e^{\ln |\lambda| + \alpha t} + 1} \right)^m = (-1)^k \alpha^k \sum_{m=1}^{k+1} (m-1)! S(k+1,m) \left( \frac{1}{\lambda e^{\alpha t} - 1} \right)^m.
\]

The identity (2.1) for \( \lambda < 0 \) is thus proved. By the second identity in (2.5) and the equalities in (1.6) and (1.15), it follows that for \( \lambda < 0 \)
\[
\left( \frac{1}{\lambda e^{\alpha t} - 1} \right)^k = \left( \frac{1}{e^{\ln |\lambda| + \alpha t} - 1} \right)^k = \left( \frac{1}{e^v + 1} \right)^k = -\sum_{m=1}^{k} b_{k,m-1} \frac{d^{m-1}}{dt^{m-1}} \left( \frac{1}{e^v + 1} \right) = -\sum_{m=1}^{k} b_{k,m-1} \frac{d^{m-1}}{dt^{m-1}} \left( \frac{1}{\lambda e^{\alpha t} - 1} \right) = \sum_{m=1}^{k} \frac{(-1)^{k-m} M_{k-m+1}(k,m)}{\alpha^{m-1}} \frac{d^{m-1}}{dt^{m-1}} \left( \frac{1}{\lambda e^{\alpha t} - 1} \right) = \frac{1}{(k-1)!} \sum_{m=1}^{k} \frac{(-1)^{m-1}}{\alpha^{m-1}} s(k,m) \frac{d^{m-1}}{dt^{m-1}} \left( \frac{1}{\lambda e^{\alpha t} - 1} \right).
\]

The identity (2.2) for \( \lambda < 0 \) is thus proved. The proof of Theorem 2.1 is complete. \( \square \)

3. Explicit Formulae for Euler Numbers and Polynomials

It is general knowledge that Euler numbers and polynomials may be generated as follows.

**Definition 3.1** ([1, p. 804]). For \( x \in \mathbb{R} \) and \( k \in \{0\} \cup \mathbb{N} \), Euler numbers \( E_k \) and Euler polynomials \( E_k(x) \) are respectively defined by the power expansions
\[
\frac{2e^{t/2}}{e^t + 1} = \sum_{k=0}^{\infty} E_k \left( \frac{t}{2} \right)^k \quad \text{and} \quad \frac{2e^{xt}}{e^t + 1} = \sum_{k=0}^{\infty} E_k(x) \frac{t^k}{k!} \] \tag{3.1}

which converge uniformly with respect to \( t \in (-\pi, \pi) \).

By definition, it is clear that
\[
E_n = 2^n E_n \left( \frac{1}{2} \right). \tag{3.2}
\]

Since the generating function \( \frac{2e^{t/2}}{e^t + 1} \) of Euler numbers \( E_k \) in (3.1) is even on \( \mathbb{R} \), then \( E_{2k-1} = 0 \) for all \( k \in \mathbb{N} \).

**Theorem 3.1.** For \( n \in \mathbb{N} \), Euler polynomials \( E_n(x) \) may be calculated by
\[
E_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \left[ \sum_{\ell=1}^{n-k+1} \frac{(-1)^{\ell-1} (\ell - 1)! S(n - k + 1, \ell)}{2^{\ell-1}} \right] x^k. \tag{3.3}
\]
where $S(n-k+1, \ell)$ are Stirling numbers of the second kind. Consequently, Euler numbers $E_{2n}$ for $n \in \mathbb{N}$ may be calculated by

$$E_{2n} = 4^n \sum_{k=0}^{2n} \frac{\sum_{\ell=1}^{2n-k+1} (-1)^{\ell-1}(\ell-1)!}{2^{\ell-1}S(2n-k+1, \ell)} \frac{(-1)^k}{2^k} \binom{2n}{k}. \quad (3.4)$$

Moreover, Stirling numbers $S(n, k)$ satisfy

$$\sum_{k=0}^{2n-1} \sum_{\ell=1}^{2n-k} \frac{(-1)^{\ell-1}(\ell-1)!}{2^{\ell-1}S(2n-k, \ell)} \frac{(-1)^k}{2^k} \binom{2n-1}{k} = 0. \quad (3.5)$$

**Proof.** By Leibniz's theorem for differentiation and the first identity in (2.5), we have

$$\frac{d^n}{dt^n} \left( \frac{2e^{xt}}{e^t + 1} \right) = 2^n \sum_{i=0}^{n} \binom{n}{i} t^i \frac{d^{n-i}}{dt^{n-i}} \left( \frac{1}{e^t + 1} \right) = 2e^{xt} \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \lambda_{n-i,j} \left( \frac{1}{e^t + 1} \right)^j x^i. \quad (3.6)$$

Combining this with the $n$-th differentiation on both sides of the second generating function in (3.1) reveal that

$$\sum_{k=m}^{\infty} E_k(x) \frac{t^{k-n}}{(k-n)!} = 2e^{xt} \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \lambda_{n-i,j} \left( \frac{1}{e^t + 1} \right)^j x^i. \quad (3.7)$$

Further taking $t \to 0$ and employing the first equality (1.5) give

$$E_n(x) = \sum_{i=0}^{n} \sum_{j=1}^{n-i+1} \frac{(-1)^{j-1}(-1)^i}{2^{j-1}j!} \binom{n}{i} \lambda_{n-i,j} x^i = (-1)^n \sum_{i=0}^{n} \binom{n}{i} \left[ \sum_{j=1}^{n-i+1} \frac{(-1)^{j-1}}{2^{j-1}(j-1)!} S(n-i+1, j) \right] x^i. \quad (3.8)$$

The proof of the formula (3.3) is complete.

Replacing $n$ by $2n$ and $x$ by $\frac{x}{2}$ in (3.3) and using (3.2) produce the formula (3.4).

The equality (3.5) follows from the property $E_{2k-1} = 0$ for all $k \in \mathbb{N}$ and the relation (3.2). The proof of Theorem 3.1 is complete. \qed

### 4. Two-parameter Euler polynomials and their explicit formula

Euler numbers and polynomials mentioned in the above section may be generalized as follows.

**Definition 4.1.** For $x \in \mathbb{R}$, $\alpha \neq 0$, and $\lambda > 0$, the quantity $E_k(x; \alpha, \lambda)$ generated by

$$\frac{2e^{xz}}{\lambda e^{\alpha z} + 1} = \sum_{k=0}^{\infty} E_k(x; \alpha, \lambda) \frac{z^k}{k!}, \quad |\alpha z + \ln \lambda| < \pi \quad (4.1)$$

are called two-parameter Euler polynomials.

It is easy to deduce that two-parameter Euler polynomials $E_k(x; \alpha, \lambda)$ satisfy

$$E_k(x; 1, 1) = E_k(x) \quad \text{and} \quad E_k(x; \alpha, \lambda) = \alpha^k E_k \left( \frac{x}{\alpha}; 1, \lambda \right) = x^k E_k \left( 1; \frac{\alpha}{x}, \lambda \right). \quad (4.2)$$

This shows that the introduction of two-parameter Euler polynomials $E_k(x; \alpha, \lambda)$ is not trivial.

**Theorem 4.1.** For $x \in \mathbb{R}$, $\alpha \neq 0$, $\lambda > 0$, and $n \in \{0\} \cup \mathbb{N}$, two-parameter Euler polynomials $E_n(x; \alpha, \lambda)$ may be computed in terms of Stirling numbers of the second kind $S(n, k)$ by

$$E_n(x; \alpha, \lambda) = 2 \sum_{k=0}^{n} (-\alpha)^{n-k} \binom{n}{k} \left[ \sum_{m=1}^{n-k+1} (-1)^{m-1} (m-1)! S(n-k+1, m) \left( \frac{1}{\lambda+1} \right)^m \right] x^k. \quad (4.3)$$

**Proof.** In light of Leibniz's theorem for differentiation and the formula (2.1), we have

$$\frac{d^n}{dt^n} \left( \frac{2e^{xt}}{\lambda e^{\alpha t} + 1} \right) = 2^n \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dt^k} \left( \frac{1}{\lambda e^{\alpha t} + 1} \right) \frac{d^{n-k} e^{xt}}{dt^{n-k}}. \quad (4.4)$$
\[= 2e^{xt} \sum_{k=0}^{n} \sum_{m=1}^{k+1} \left( \begin{array}{c} n \\ k \end{array} \right) x^{n-k} (-1)^{k+m-1} \alpha^k (m-1)! S(k+1, m) \left( \frac{1}{\lambda e^{xt} + 1} \right)^m.\]

As a result, it follows that
\[\sum_{k=n}^{\infty} E_k(x; \alpha, \lambda) \left( \frac{k-n}{k-n} \right)! = 2e^{xt} \sum_{k=0}^{n} \sum_{m=1}^{k+1} \left( \begin{array}{c} n \\ k \end{array} \right) x^{n-k} (-1)^{k+m-1} \alpha^k (m-1)! S(k+1, m) \left( \frac{1}{\lambda e^{xt} + 1} \right)^m.\]

Further taking the limit \(t \to 0\) leads to
\[E_n(x; \alpha, \lambda) = 2 \sum_{k=0}^{n} \sum_{m=1}^{k+1} \left( \begin{array}{c} n \\ k \end{array} \right) x^{n-k} (-1)^{k+m-1} \alpha^k (m-1)! S(k+1, m) \left( \frac{1}{\lambda + 1} \right)^m.\]

The proof of Theorem 4.1 is complete.

\textbf{Remark 4.1.} The second equality in (1.6) corrects equations (2.23) and (2.26) in [3, p. 573].

\textbf{Remark 4.2.} The functions \(\frac{1}{e^{xt} - 1}\) and their derivatives have also been investigated in the paper [7] in a different direction.

\textbf{Remark 4.3.} In [4, 6], several formulae for computing Stirling numbers of the first kind were discovered. By these formulae, some properties of Stirling numbers of the first kind were found in [4, 6] and closely related references therein.

\textbf{Remark 4.4.} After completing this paper, the author discovers that the first identity in (1.1) and the special case \(\lambda = \alpha = 1\) of the identity (2.2) were listed, but without proof, in [2, p. 559].

\textbf{Remark 4.5.} This paper is a slightly modified version of the preprint [5].

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