Harmonic Oscillator on the $SO(2, 2)$ hyperboloid

D.Petrosyan$^{(1)}$ and G.S.Pogosyan$^{(1,2,3)}$

(1) Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Moscow Region, 141980, Russia
(2) Physics Department, Yerevan State University, Yerevan, A.Manooogian 1, 0025, Armenia
(3) Departamento de Matematicas, CUCEI, Universidad de Guadalajara, Guadalajara, Jalisco, Mexico

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Abstract

In the present work the problem of the motion of the classical particle in the field of harmonic oscillator in the hyperbolic space $H_2^2: z_0^2 + z_1^2 - z_2^2 - z_3^2 = R^2$ has been solved. We have shown that all the finite classical trajectories are closed and periodic. The orbits of motion are ellipses or circles for bounded motion and parabolas and hyperbolas for infinite.

1 Introduction

In the recent articles [1, 2] we have discussed the classical and quantum Kepler-Coulomb problem on the hyperbolic configuration space (with the group of isometry $SO(2, 2)$), namely the hyperboloid $z_0^2 + z_1^2 - z_2^2 - z_3^2 = R^2; |z_0| \geq R^2$, where $z_i$ ($i = 0, 1, 2, 3$) are the Cartesian coordinates in the pseudo-euclidean ambient space $R_2;$. In particular, in [2] we have shown that for the classical motion, as in the case of flat Euclidean space $R_3$, all bounded trajectories are closed and periodic. In our previous article [3] we, on the same hyperbolic space, have introduced the quantum harmonic oscillator system. In the pseudo-spherical coordinate $(r, \tau, \varphi)$ [1, 4]

$$z_0 = \pm R \cosh r, \quad z_1 = R \sinh r \sin \tau, \quad z_2 = R \sinh r \cos \tau \cos \varphi, \quad z_3 = R \sinh r \cos \tau \sin \varphi,$$

where $r \geq 0$ is the “geodesic radial angle”, $\tau \in (-\infty, \infty)$, and $\varphi \in [0, 2\pi)$, this system is described by the Hamiltonian ($\hbar = m = 1$)

$$\mathcal{H} = -\frac{1}{2R^2} \left\{ \frac{1}{\sinh^2 r} \frac{\partial}{\partial r} \sinh^2 r \frac{\partial}{\partial r} - \frac{1}{\sinh^2 r} \left[ \frac{1}{\cosh \tau} \frac{\partial}{\partial \tau} \cosh \tau \frac{\partial}{\partial \tau} - \frac{1}{\cosh^2 \tau} \frac{\partial}{\partial \varphi} \right] \right\} + \frac{\omega^2 R^2}{2} \tanh^2 r. \quad (2)$$

The system of coordinate (1) is valid only for $|z_0| \geq R$ and the connection between two sets of coordinates $z_0 \to -z_0$ corresponds to the complex transformation of radial angle $r \to i\pi - r.$
The missing part of the surface for \(|z_0| < R\) may also be taken into account if we use another form of the pseudo-spherical coordinate, namely [4]

\[
\begin{align*}
    z_0 &= \pm R \cos \chi, \\
    z_1 &= R \sin \chi \cosh \mu, \\
    z_2 &= R \sin \chi \sinh \mu \cos \varphi, \\
    z_3 &= R \sin \chi \sinh \mu \sin \varphi,
\end{align*}
\] (3)

where now \(\chi \in (-\frac{\pi}{2}, \frac{\pi}{2})\), \(\mu \in (-\infty, \infty)\) and \(\varphi \in [0, 2\pi)\). It is also easy to see that the two pseudo-spherical system of coordinate (1) and (3) are connected by

\[
r \to i\chi, \quad \tau \to \mu - i\pi/2.
\] (4)

The purpose of this paper is to study the corresponding classical harmonic oscillator problem on the whole hyperboloid \(H_{2,2}: z_0^2 + z_1^2 - z_2^2 - z_3^2 = R^2\). We start from the Hamiltonian

\[
H = \frac{1}{2R^2} \left( p_r^2 - \frac{p_\tau^2}{\sinh^2 r} + \frac{p_\varphi^2}{\sinh^2 r \cosh^2 \tau} \right) + \frac{\omega^2 R^2}{2} \tanh^2 r,
\] (5)

described by the classical motion on the hyperboloid \(H_{2,2}\) for the region \(|z_0| \geq R\). To investigate the motion in the region \(|z_0| \leq R\), everywhere below, we will use the transformation (4).

We recall that the Kepler-Coulomb and harmonic oscillator systems belong to the well-known class of maximally superintegrable systems. The maximal superintegrability means that there are \((2n-1)\) functionally independent integrals of motion, including the Hamiltonian, which are well defined function on phase space in classical mechanics and respectively observables in quantum mechanics. The first search of superintegrable systems in two- and three-dimensional flat space was done in the pioneering works of Winternitz and Smorodinsky with co-authors in [5, 6], later the notion of superintegrability in the spaces of constant curvature has been introduced in the series of papers [7, 8, 9]. We can also mention some articles devoted to the investigation of various aspects of both classical and quantum superintegrable systems in the spaces of constant curvature, for instance [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] and very recent review [21] and references therein.

### 2 Integration of the Hamilton-Jacobi equation

The Hamilton-Jacobi equation associated with the Hamiltonian (5) is obtained after the substitution \(p_{\mu_i} \to \partial S/\partial \mu_i\), where \(\mu_i = (r, \tau, \varphi)\). Therefore we get

\[
H = \frac{1}{2R^2} \left\{ \left( \frac{\partial S}{\partial r} \right)^2 - \frac{1}{\sinh^2 r} \left( \frac{\partial S}{\partial \tau} \right)^2 + \frac{1}{\sinh^2 r \cosh^2 \tau} \left( \frac{\partial S}{\partial \varphi} \right)^2 \right\} + \frac{\omega^2 R^2}{2} \tanh^2 r = E. \tag{6}
\]

This equation is completely separable, and the coordinate \(\varphi\) is cyclic, we look for solutions for classical action \(S(r, \tau, \varphi, t)\) in form

\[
S(r, \tau, \varphi, t) = -Et + p_\varphi \varphi + S_1(r) + S_2(\tau), \tag{7}
\]
and obtain

\[ \left( \frac{\partial S_2}{\partial \tau} \right)^2 - \frac{p_\varphi^2}{\cosh^2 \tau} = -A, \]  
(8)

\[ \frac{1}{2R^2} \left( \frac{\partial S_1}{\partial r} \right)^2 + \frac{\omega^2 R^2}{2} \tanh^2 r + \frac{A}{2R^2 \sinh^2 r} = E, \]  
(9)

where \( A \) is the pseudo spherical separation constant. As it follows from the equation (8): \( (\partial S_2/\partial \tau)^2 = p_\varphi^2/\cosh^2 \tau - A \geq 0 \), the separation constant \( A \), in contrast to the motion in Euclidean space (or other spaces of constant curvature), can take not only the positive or zero but also the negative value. Another difference is that at the fixed values the constant \( A \): \( p_\varphi^2 \geq A \).

The "quasi - radial" equation (9) describes the motion in field of effective potential

\[ U_{\text{eff}}(r) = \frac{\omega^2 R^2}{2} \tanh^2 r + \frac{A}{2R^2 \sinh^2 r}. \]  
(10)

At the large \( r \sim \infty \) the effective potential \( U_{\text{eff}}(r) \) tends to a constant value equal to \( \omega^2 R^2 / 2 \), whereas the behavior at the point \( r = 0 \) is determined by the separation constant \( A \).

In case \( 0 \leq A < \omega^2 R^4 \) potential (10) has a minimum at \( r_0 = \tanh^{-1} \sqrt{A/\omega^2 R^4} \) (see Fig.1), and in this point

\[ 0 \leq U_{\text{eff}}(r_0) = \omega \sqrt{A} - \frac{A}{2R^2} < \frac{\omega^2 R^2}{2}, \]  
(11)

where equality is possible only in case of \( A = 0 \). For \( A \geq \omega^2 R^4 \) the potential \( U_{\text{eff}}(r) \) is repulsive on the whole semi-axis \( r \in [0, \infty) \) (see Fig.2). In the case of negative \( A \) the effective potential (11) is attractive and has a singularity for a small \( r \) as \( \sim r^{-2} \) (see Fig.3).

For the region \( |z_0| < R \) the differential equations (8) and (9) are transformed to the following ones

\[ \left( \frac{\partial S_2}{\partial \mu} \right)^2 + \frac{p_\varphi^2}{\sinh^2 \mu} = -A, \]  
(12)

\[ \frac{1}{2R^2} \left( \frac{\partial S_1}{\partial \chi} \right)^2 + \frac{\omega^2 R^2}{2} \tan^2 \chi + \frac{A}{2R^2 \sin^2 \chi} = -E, \]  
(13)

From the first equation it follows that the separation constant \( A \) takes the negative value \( A < 0 \). Therefore, we take into account the motion inside the region \( |z_0| < R \) when consider the cases of negative value of \( A \).

Integrating now equations (8) and (9) we get

\[ S_1(r) = \int \sqrt{2R^2E - \omega^2 R^4 \tanh^2 r - \frac{A}{\sinh^2 r}} dr, \]  
(14)

\[ S_2(\tau) = \int \sqrt{-A + \frac{p_\varphi^2}{\cosh^2 \tau}} d\tau. \]  
(15)
Since we are interested only the trajectories we will following the usual procedures [22] and consider the equations

$$\frac{\partial S}{\partial E} = \frac{\partial S_1}{\partial E} - t = -t_0, \quad \frac{\partial S}{\partial A} = \frac{\partial S_1}{\partial A} + \frac{\partial S_2}{\partial A} = \beta, \quad \frac{\partial S}{\partial \varphi} = \varphi + \frac{\partial S_2}{\partial \varphi} = \varphi_0$$

(16)

where $t_0, \varphi_0$ and $\beta$ are the constants.

Figure 1: Effective potential $U_{eff}(r)$ in case of $0 \leq A < \omega^2 R^4$ for value of $A = 0, 1/16, 1/8, 1/4$; ($\omega = R = 1$).

Figure 2: Effective potential $U_{eff}(r)$ in case of $A \geq \omega^2 R^4$ for value of $A = 2, 3, 4$; ($\omega = R = 1$).
2.1 Integration of quasi-radial part

From equations (14) and (16) we get that

\[ t - t_0 = \frac{1}{\omega} \int \frac{\tanh r dr}{\sqrt{-\tanh^4 r + 2(E/\omega^2 R^2 + A/2\omega^2 R^4) \tanh^2 r - A/\omega^2 R^4}}. \]  

(17)

Below we consider separately all four cases: \(0 < A < \omega^2 R^4\), \(A \geq \omega^2 R^4\), \(A < 0\) and \(A = 0\).

1. The case \(0 < A < \omega^2 R^4\). For the roots in the radical expression of denominator in (17) we have

\[ X_{1,2} = \frac{(2R^2 E + A) \pm \sqrt{(2R^2 E + A)^2 - 4A\omega^2 R^4}}{2\omega^2 R^4}, \]

(18)

where \(X = \tanh^2 r \in [0, 1]\). It’s obvious that the radicand in equation (18) is positive for any values of energy \( E > E_{\text{min}} = U_{\text{eff}}(r_0) \) and equal zero for \( E = E_{\text{min}} \). Thus the roots \(X_{1,2} \ (X_1 \leq X_2)\) are positive. It is easy to see that for \( E_{\text{min}} \leq E < \omega^2 R^2/2 \) both roots satisfy the inequality condition \( 0 < X_1 < X_2 < 1 \). At \( E \geq \omega^2 R^2/2 \): \( 0 < X_1 < 1 \leq X_2 \) and equality \( X_2 = 1 \) is possible only for \( E = \omega^2 R^2/2 \). The bounded motion exists exclusively for \( E_{\text{min}} \leq E < \omega^2 R^2/2 \). Below we will consider separately all possible cases, namely: \( E_{\text{min}} < E < \omega^2 R^2/2 \), \( E = E_{\text{min}} \), \( E > \omega^2 R^2/2 \) and \( E = \omega^2 R^2/2 \).

A. Performing the integration in formula (17) we get for \( E_{\text{min}} < E < \omega^2 R^2/2 \)

\[ 2\omega^2 R^2 \sinh^2 r = (1 - 2E/\omega^2 R^2)^{-1} \left\{ (2E - A/R^2) \right\} \]

\[ + \sqrt{(2E + A/R^2)^2 - 4A\omega^2 \sin[2\omega \sqrt{1 - 2E/\omega^2 R^2(t - t_0)}]} \].

(19)

Thus the motion is bounded and periodic with the period

\[ T = \frac{\pi}{\omega \sqrt{1 - 2E/\omega^2 R^2}}, \]

(20)
and unlike the motion in Euclidean space, depends on the energy of particle $E$. At the contraction limit $R \to \infty$ the “flat” trajectories for the energy value $E_{min} < E < \infty$

$$r^2 = \frac{E}{\omega^2} (1 + \sin[2\omega(t - t_0)]) ,$$  \hspace{1cm} (21)

are also periodic with the period $T = \frac{\pi}{\omega}$

**B.** In the case of minimum energy: $E = E_{min} = U_{eff}(r_0)$ or $E_{min} = \omega \sqrt{A} - A/2R^2$ the integral in (17) is not defined and we must solve directly the equation (9). From equation (9) we obtain

$$\left(\frac{\partial S_1}{\partial r}\right)^2 = - \left(\sqrt{A} \coth r - \sqrt{\omega^2 R^4 \tanh r}\right)^2 \geq 0,$$  \hspace{1cm} (22)

or $\partial S_1/\partial r = 0$ and $\tanh^2 r = \sqrt{A}/\omega^2 R^4$. Therefore

$$r = \tanh^{-1} \left(\sqrt{1 - \frac{1 - 2E}{\omega^2 R^2}}\right),$$  \hspace{1cm} (23)

i.e. the trajectories are circles. Here from two values of $\sqrt{A}$ allowed by equation $E = U_{eff}(r_0)$, we chose the smaller one $\sqrt{A} = \omega R^2 \left(1 - \sqrt{1 - 2E/\omega^2 R^2}\right)$ because it satisfies the condition $0 < A < \omega^2 R^4$. In case of contraction limit $R \to \infty$ we obtain $E = E_{min} = \omega \sqrt{A}$ and $r = \sqrt{E}/\omega$.

**C.** In case of $E > \omega^2 R^2/2$ we have after integration in (17)

$$2\omega^2 R^2 \sinh^2 r = (2E/\omega^2 R^2 - 1)^{-1} \left\{(A/R^2 - 2E) + \sqrt{(2E + A/R^2)^2 - 4A\omega^2 \cosh[2\omega \sqrt{2E/\omega^2 R^2 - 1(t(t - t_0))]}}\right\},$$  \hspace{1cm} (24)

i.e., the motion is not bounded.

**D.** For the limiting case of $E = \omega^2 R^2/2$ the roots of denominator are $X_1 = A/\omega^2 R^4$, $X_2 = 1$, thus $A/\omega^2 R^4 < \tanh^2 r < 1$ and motion is not bounded because of $\arctanh(A/\omega^2 R^4) < r < \infty$. The integration in (17) give us

$$\cosh^2 r = (1 - A/\omega^2 R^4)^{-1} + \omega^2(1 - A/\omega^2 R^4) (t - t_0)^2.$$  \hspace{1cm} (25)

**3.** Let us consider now the case of $A \geq \omega^2 R^4$ (see Fig. 2). From equation (17) we get that the only possible value for energy is $E > \omega^2 R^2/2$ and the roots satisfy the inequality $0 < X_1 < 1 < X_2$. Thereby, the equation of motion is determined by the formula (24). The motion of particle is limited only by the point $r_{min} = \tanh^{-1} \sqrt{X_1}$ i.e. it has the ability to go to infinity.
2. Let us consider finally the case of $A < 0$. From the equation (17) we have that the roots of denominator are

$$X_{1,2} = \frac{(2ER^2 - |A|) \pm \sqrt{(2ER^2 - |A|)^2 + 4|A|\omega^2R^4}}{2\omega^2R^4}$$

where again $X = \tanh^2 r \in [0, 1]$. It can be seen that $X_1 < 0 < X_2$ is independent of the value of $A$ and energy $E$. For the region $E \geq \omega^2R^2/2$ one of the roots is $X_2 > 1$, so the radicand is positive for any values of variable $r$, including the point $r = 0$: $r \in [0, \infty)$. The same situation develops for region $E < \omega^2R^2/2$, where $r \in [0, \tanh^{-1}\sqrt{X_2}]$. Therefore in case of negative $A$ the particle can penetrate from the region $z_0 \geq R$ to $0 \leq z_0 \leq R$.

Performing the integration in formula (17), we have for $E < \omega^2R^2/2$

$$\sinh^2 r = \frac{2R^2E + |A|}{2R^2(\omega^2R^2 - 2E)} + \frac{\sqrt{(2R^2E - |A|)^2 + 4|A|\omega^2R^4}}{2R^2(\omega^2R^2 - 2E)} \sin \left[ 2\omega\sqrt{1 - 2E/\omega^2R^2(t-t_0)} \right]$$

while for $E > \omega^2R^2$

$$\sinh^2 r = \frac{2R^2E + |A|}{2R^2(\omega^2R^2 - 2E)} + \frac{\sqrt{(2R^2E - |A|)^2 + 4|A|\omega^2R^4}}{2R^2(2E - \omega^2R^2)} \cosh \left[ 2\omega\sqrt{2E/\omega^2R^2 - 1(t-t_0)} \right]$$

From the formula (27) it follows that the motion at $E < \omega^2R^2/2$ is bounded and periodic with period (20). Below we will construct the bounded trajectories lying on the whole hyperboloïd, namely not only in the region $|z_0| \geq R$, but also $|z_0| \leq R$. In case when $E = \omega^2R^2$ the integration in (17) leads, up to a transformation $A \rightarrow -|A|$, to a result similar to the formula (25).

In the limiting case of $A = 0$ the formulas (27), (28) and (25) are simplified. For $0 < E < \omega^2R^2/2$ we get

$$\sinh^2 r = \frac{2E/\omega^2R^2}{1 - 2E/\omega^2R^2} \cos^2 \left( \omega\sqrt{1 - 2E/\omega^2R^2(t-t_0)} - \frac{\pi}{4} \right),$$

while in case of $E > \omega^2R^2/2$

$$\sinh r = \sqrt{\frac{2E/\omega^2R^2}{2E/\omega^2R^2 - 1}} \sinh \left( \omega\sqrt{2E/\omega^2R^2 - 1(t_0 - t)} \right).$$

Finally for $E = \omega^2R^2/2$ we obtain $\sinh r = \omega(t-t_0)$. 

7
2.2 Integration of the angular parts

1. Let us first consider the case when $A \geq 0$. From (14) and (15) we obtain

$$\frac{\partial S_1}{\partial A} = -\frac{1}{2} \int \frac{dr}{\sinh^2 r \sqrt{2R^2E - \omega^2 R^4 \tanh^2 r - A/\sinh^2 r}}, \tag{31}$$

$$\frac{\partial S_2}{\partial A} = -\frac{1}{2} \int \frac{d\tau}{\sqrt{-A + p^2_\varphi/\cosh^2 \tau}}. \tag{32}$$

The integrals can be easily calculated to give [23]

$$\frac{\partial S_2}{\partial A} = \frac{1}{\sqrt{A}} \arcsin \left[ \frac{\sinh \tau}{\sqrt{p^2_\varphi/A - 1}} \right], \tag{33}$$

$$\frac{\partial S_1}{\partial A} = \frac{1}{4\sqrt{A}} \arcsin \left[ \frac{2A \coth^2 r - (2ER^2 + A)}{\sqrt{(2ER^2 + A)^2 - 4A\omega^2 R^4}} \right]. \tag{34}$$

Here we require

$$-\sqrt{p^2_\varphi/A - 1} < \sinh \tau < \sqrt{p^2_\varphi/A - 1}, \tag{35}$$

and

$$|2A \coth^2 r - (2ER^2 + A)| < \sqrt{(2ER^2 + A)^2 - 4A\omega^2 R^4}. \tag{36}$$

The condition (36) is equivalent to $z_1 < \coth r < z_2$, where $z_{1,2}$ are the roots of denominator in integral (31):

$$z_{1,2} = \frac{(2ER^2 + A) \pm \sqrt{(2ER^2 + A)^2 - 4A\omega^2 R^4}}{2A}, \quad E \geq E_{\text{min}} = \omega \sqrt{A} - A/2R^2.$$

A final condition $z_2 > 1$ implies that $A > \omega^2 R^4$ and $E > \omega^2 R^2/2$ or $0 < A < \omega^2 R^4$ and $E > E_{\text{min}}$.

Therefore for $\partial S/\partial A$ we have

$$\frac{\partial S}{\partial A} = \frac{1}{4\sqrt{A}} \left\{ \arcsin \left[ \frac{2A \coth^2 r - (2ER^2 + A)}{\sqrt{(2ER^2 + A)^2 - 4A\omega^2 R^4}} \right] - 2 \arcsin \left[ \frac{\sinh \tau}{\sqrt{p^2_\varphi/A - 1}} \right] \right\} = \beta. \tag{37}$$

Next, from (15) and (16) we obtain

$$\frac{\partial S}{\partial p_\varphi} = \varphi + \int \frac{p_\varphi d\tau}{\cosh^2 \tau \sqrt{-A + p^2_\varphi/\cosh^2 \tau}} = \varphi + \arcsin \frac{\tanh \tau}{\sqrt{1 - A/p^2_\varphi}} = \varphi_0. \tag{38}$$
and hence
\[ \tanh \tau = \sqrt{1 - A/p^2} \sin(\phi_0 - \varphi). \]  \hspace{1cm} (39)

2. Let us consider the integration in formulas (31), (32) and (38) in the case \( A < 0 \). Instead of equation (37) we obtain [23]

\[ \frac{\partial S}{\partial A} = \frac{1}{4\sqrt{|A|}} \left\{ \arccosh \left[ \frac{2|A|\coth^2 r + (2ER^2 - |A|)}{\sqrt{(2ER^2 - |A|)^2 + 4|A|\omega^2R^4}} \right] \right. \\
- 2 \arcsinh \left[ \frac{\sinh \tau}{\sqrt{1 + p^2_e/|A|}} \right] \right\} = \beta, \]  \hspace{1cm} (40)

and
\[ \sin(\phi_0 - \varphi) = \frac{p_\varphi}{\sqrt{p^2_\varphi + |A|}} \tanh \tau, \]  \hspace{1cm} (41)

with the restriction for \( r \):
\[ \coth^2 r \geq \left( \frac{1}{2} - \frac{ER^2}{|A|} \right) + \sqrt{\left( \frac{1}{2} - \frac{ER^2}{|A|} \right)^2 + \omega^2R^4/|A|}. \]  \hspace{1cm} (42)

The limiting case of \( A = 0 \) could be easily calculated directly from equations (40) and (41). So, we get
\[ \frac{\partial S}{\partial A} \bigg|_{A=0} = \frac{\sqrt{2E\coth^2 r - \omega^2R^2}}{4ER} - \frac{\sinh \tau}{2p_\varphi} = \beta, \quad \sinh \tau = \tan(\phi_0 - \varphi). \]  \hspace{1cm} (43)

with the obvious restriction \( \coth^2 r \geq \omega^2R^2/2E \).

3 The trajectories for \( A > 0 \)

From (37) and (39) we have
\[ \coth^2 r = \left( \frac{ER^2}{A} + \frac{1}{2} \right) + \sqrt{\left( \frac{ER^2}{A} + \frac{1}{2} \right)^2 - \frac{\omega^2R^4}{A}} \sin(2\psi + 4\sqrt{A}\beta), \]  \hspace{1cm} (44)

where
\[ \psi = \arcsin \left( \frac{\sinh \tau}{\sqrt{p^2_\varphi/A - 1}} \right) = \arcsin \left( \frac{1}{\sqrt{1 + A/p^2_\varphi \cot^2(\phi_0 - \varphi)}} \right). \]  \hspace{1cm} (45)

Now we can rewrite the equation (44) in form
\[ \tanh^2 r = \frac{1}{\left( \frac{ER^2}{A} + \frac{1}{2} \right) + \sqrt{\left( \frac{ER^2}{A} + \frac{1}{2} \right)^2 - \frac{\omega^2R^4}{A}} \sin(2\psi + 4\sqrt{A}\beta)}. \]  \hspace{1cm} (46)
Thus we see from (45) that the dependence of angle $\tau$ in the equation of trajectories (46) can be eliminated. On the other hand from the formula (39) it follows that the motion of particle on the hyperboloid is restricted to the additional condition

$$\frac{z_1}{z_3} = \frac{\tanh \tau}{\sin \varphi} = \sqrt{1 - A/p^2}. \tag{47}$$

Therefore, without lost of generality we can choose $\tau = 0$ or $A = p^2$. Taking into account that the formula (46) is invariant about transformation $r \rightarrow i\pi - r$ we can conclude that all trajectories of motion, given by this formula, lie on the upper ($z_0 \geq R$) or lower ($z_0 \leq -R$) sheets of the two-sheeted hyperboloid: $z_0^2 - z_2^2 - z_3^2 = R^2$. Obviously they are symmetric with respect to transformation $z_0 \rightarrow -z_0$.

Putting now $A = p^2$ in (39) we obtain that $\psi = (\varphi_0 - \varphi)$ and the formula (46) gain the following form

$$\tanh^2 r = \frac{p}{1 + \varepsilon(R) \cos 2\varphi}, \tag{48}$$

where we use the notations

$$p = \left( \frac{ER^2}{A} + \frac{1}{2} \right)^{-1} > 0, \quad \varepsilon(R) = \sqrt{1 - \frac{4\omega^2R^4A}{(2ER^2 + A)^2}}; \tag{49}$$

and choose $\varphi_0 = -2\beta\sqrt{A} + \frac{\pi}{4}$ that the point $\varphi = 0$ will be the nearest to the center. It is clear that radicand is always positive because of $E > U_{eff}(r_0)$ for $0 < A < \omega^2R^4$ and $E > \omega^2R^2/2$ for $A \geq \omega^2R^4$.

**A.** Let us now consider the elliptic trajectory which is possible only for $0 < A < \omega^2R^4$ and $U_{eff}(r_0) < E < \omega^2R^2/2$ with

$$0 \leq \varepsilon(R) = \sqrt{1 - \frac{4\omega^2R^4A}{(2ER^2 + A)^2}} < 1. \tag{50}$$

Denote the minimum $r_{min}$ and maximum $r_{max}$ points on the orbit as a distance from the center of field. It is obvious that it corresponds to the angles $\varphi = 0$ and $\varphi = \pi/2$. Accordingly (48) we have

$$\tanh^2 r_{min} = \frac{p}{1 + \varepsilon(R)}, \quad \tanh^2 r_{max} = \frac{p}{1 - \varepsilon(R)}, \tag{51}$$

and correspondingly

$$r_{min} = \coth^{-1} \left\{ \sqrt{\left( \frac{ER^2}{A} + \frac{1}{2} \right) + \sqrt{\left( \frac{ER^2}{A} + \frac{1}{2} \right)^2 - \omega^2R^4/A}} \right\}, \tag{52}$$

$$r_{max} = \coth^{-1} \left\{ \sqrt{\left( \frac{ER^2}{A} + \frac{1}{2} \right) - \sqrt{\left( \frac{ER^2}{A} + \frac{1}{2} \right)^2 - \omega^2R^4/A}} \right\}. \tag{53}$$
Therefore the trajectories are the “ellipses” (see [1, 12, 13]) lying symmetrically of the point $z_0 = R$, $z_1 = z_2 = z_3 = 0$ on the upper sheet of the two-sheeted hyperboloid (see Fig. 4).

Figure 4: The figure shows the elliptic trajectories lying on the upper sheet of the two-sheeted hyperboloid $z_0^2 - z_2^2 - z_3^2 = R^2$ with the value $\varepsilon = 0.3$ and $p = 0.3, 0.4, 0.5$.

B. In the case of the minimum energy $E = E_{\text{min}} = U_{\text{eff}}(r_0)$ we have from (49) that $\varepsilon = 0$, $p = \omega R^2/\sqrt{A}$ and again as is mentioned in (23) the orbits are circles (see Fig. 5).

Figure 5: The cyclic trajectories for $\varepsilon = 0$ and $p = 0.2, 0.5, 0.8$.

C. For the case of energy values $E = \omega^2 R^2 / 2$ we get the analog of parabolic motion (see Fig. 6). The minimal distance $r_{\text{min}}$ from the center is given by the formula

$$ r_{\text{min}} = \coth^{-1} \left( \frac{\omega R^2}{\sqrt{A}} \right). $$
Figure 6: The figure shows the parabolic trajectories lying on the upper sheet of the two-sheeted hyperboloid $z_0^2 - z_2^2 - z_3^2 = R^2$ with the value of pairs $(p, \varepsilon) = (1/3, 2/3); (2/3, 1/3); (8/9, 1/9)$.

D. For the energy $E > \omega^2 R^2/2$ the trajectories of motion represent the hyperbolas with the minimal distance $r_{min}$ (see Fig. 7)

$$r_{min} = \coth^{-1} \left\{ \sqrt{\left( \frac{ER^2}{A} + \frac{1}{2} \right)} + \sqrt{\left( \frac{ER^2}{A} + \frac{1}{2} \right)^2 - \frac{\omega^2 R^4}{A}} \right\}. \quad (54)$$

Figure 7: The figure shows the hyperbolic trajectories lying on the upper sheet of the two-sheeted hyperboloid $z_0^2 - z_2^2 - z_3^2 = R^2$ with the value $\varepsilon = 0.8$ and $p = 0.2, 0.5, 0.8$. 
E. Finally we consider the case when \( A \geq \omega^2 R^4 \) and \( E > \omega^2 R^2 / 2 \). Taking into account that

\[
0 \leq \left( \frac{ER^2}{A} + \frac{1}{2} \right) - \sqrt{\left( \frac{ER^2}{A} + \frac{1}{2} \right)^2 - \frac{\omega^2 R^4}{A}} \leq 1,
\]

we obtain that the trajectories of motion are described by the hyperbolas having the similar form as in item D, with the minimal distance \( r_{\text{min}} \) as in formula (54), and \( r_{\text{max}} \to \infty \).

4 The trajectories for \( A \leq 0 \)

To simplify further formulas we set first \( p_\varphi = 0 \). Then, from equation (41) it follows that the motion occurs at a constant value of the azimuthal angle \( \varphi = \varphi_0 \) that is limited by the condition \( z_2 / z_0 = \tan \varphi_0 \). To further simplify it is enough to choose \( \varphi_0 = 0 \) or \( \varphi_0 = \pi \). Thus we get that trajectory of the motion (56) lies on the one-sheeted hyperboloid \( z_0^2 + z_1^2 - z_2^2 = R^2 \).

The formula (40) gives us the equation of the trajectory in the region \( z_0 > R \):

\[
\coth^2 r = \left( \frac{1}{2} - \frac{ER^2}{|A|} \right) + \sqrt{\left( \frac{1}{2} - \frac{ER^2}{|A|} \right)^2 + \frac{\omega^2 R^4}{|A|} \cosh(2\tau + 4\sqrt{|A|}|\beta)}.
\]

Performing the further transformation \( r \to i\chi \) and \( \tau \to \mu - i\pi/2 \) in formula (56), we obtain the equation of the trajectory in the region \( 0 < z_0 < R \):

\[
\cot^2 \chi = - \left( \frac{1}{2} - \frac{ER^2}{|A|} \right) + \sqrt{\left( \frac{1}{2} - \frac{ER^2}{|A|} \right)^2 + \frac{\omega^2 R^4}{|A|} \cosh(2\mu + 4\sqrt{|A|}|\beta)}.
\]

In the formula of trajectory (56) we must distinguish two cases, namely for the value of energy \( E < \omega^2 R^2 / 2 \) and \( E \geq \omega^2 R^2 / 2 \).

In the first case \( E < \omega^2 R^2 / 2 \) from equation (56) it follows that for any value of the variable \( \tau \in (-\infty, \infty) \) we have that \( \coth r > 1 \). Therefore, the trajectory of the motion extends from the point \( r = 0 \) at the \( \tau \to -\infty \) (\( z_0 = R, z_1 < 0, z_2 > 0 \)) to its maximum

\[
r_{\text{max}} = \coth^{-1} \sqrt{\left( \frac{1}{2} - \frac{ER^2}{|A|} \right) + \sqrt{\left( \frac{1}{2} - \frac{ER^2}{|A|} \right)^2 + \frac{\omega^2 R^4}{|A|}}},
\]

at the point \( \tau = -2\sqrt{|A|}|\beta \) and then goes back to the point \( r = 0 \) when \( \tau \to \infty \) (\( z_0 = R, z_1 > 0, z_2 > 0 \)). Further, on, the particle penetrate through the point \( z_0 = R \) from the region \( z_0 > R \) to the region \( 0 < z_0 < R \), which, as it follows from the equation (57), corresponds to the value of angles \( \mu \to \infty \) and \( \chi \to 0 \) (\( z_0 < R, z_1 > 0, z_2 > 0 \)). Further trajectory extends to the maximal value \( \chi_{\text{max}} \):

\[
\chi_{\text{max}} = \cot^{-1} - \sqrt{\left( \frac{1}{2} - \frac{ER^2}{|A|} \right) + \sqrt{\left( \frac{1}{2} - \frac{ER^2}{|A|} \right)^2 + \frac{\omega^2 R^4}{|A|}}} \leq \frac{\pi}{2}.
\]
at the point \( \mu = -2\sqrt{|A|}\beta \), and then continue to \( \mu \to -\infty, \chi \to 0 \) \((z_0 < R, z_1 > 0, z_2 < 0)\). After, the particle again passes the point \( z_0 = R \) and penetrates to the region \( z_0 \geq R \). Further using similar reasoning it can be shown that the trajectories in case of \( E < \omega^2 R^2 / 2 \), are a closed curve lying on the one-sheeted hyperboloid \( z_0^2 + z_1^2 - z_2 = R^2, z_0 > 0 \), so the motions are bounded and periodic. The same situation takes place for the case of \( z_0 < 0 \).

In the case of \( E \geq \omega^2 R^2 / 2 \) it is easy to see that the inequality

\[
\sqrt{\left(\frac{1}{2} - \frac{ER^2}{|A|}\right)^2 + \frac{\omega^2 R^4}{|A|}} \leq \frac{1}{2} + \frac{ER^2}{|A|},
\]

is valid. Thus the trajectory of the motion, depending on the sign of variable \( \tau \) is split into two paths. One of the paths begins from the large \( r \) at the minimal point

\[
\tau_{\min} = -2\sqrt{|A|}\beta + \frac{1}{2} \cosh^{-1} \frac{\left(\frac{1}{2} + \frac{ER^2}{|A|}\right)}{\sqrt{\left(\frac{1}{2} - \frac{ER^2}{|A|}\right)^2 + \frac{\omega^2 R^4}{|A|}}},
\]

and continues to the point \( r = 0 \) at \( \tau \to \infty \) \((z_0 = R, z_2 > 0)\). Then the trajectory passing the part of \( 0 < z_0 < R \) goes back from \((z_0 = R, z_2 < 0)\) at the point \( r = 0, \tau \sim \infty \) to \( r \in \infty \) at \( \tau_{\min} \). The second path is symmetric with respect to axis \( z_1 \). Thus the trajectories of motion in the case of \( E \geq \omega^2 R^2 / 2 \) are not bounded. Some examples of trajectories for the fixed negative \( A = -1 \) and various values of energy \( E \), are presented on the Fig. 8.

![Figure 8: The trajectories of motion in the case of \( A = -1; E = -3/2, -1/2, 1/4, 1/2, 3/2 \) \((\omega = R = 1)\)](image)

2. In the limiting case \( A = 0 \) it is easy to get from (43)

\[
\coth^2 r = \frac{\omega^2 R^2}{2E} + R\sqrt{E}(2\beta - \tan \varphi/p_\omega)^2,
\]

(62)
with \( \varphi_0 = 0 \). In the case of \( E < \omega^2 R^2 / 2 \) the bounded motion takes place \( r_{min} = 0 \) \((\varphi = \pi / 2)\) and \( r_{max} = \coth^{-1} \sqrt{\frac{\omega^2 R^2}{2E}} \) \((\varphi = \arctan 2 \beta \varphi)\), whereas for the \( E \geq \omega^2 R^2 / 2 \) the orbits are infinite: \( r \in [0, \infty) \). The trajectories of the motion can be presented on the hyperbolic cylinder \( z_0^2 - z_2^2 = R^2 \), \( z_1^2 = z_3^2 \), \( z_0 \geq R \) (see Fig. 9).

Figure 9: The bounded and infinite trajectories of the motion for \( A = 0 \) lying on the hyperbolic cylinder \( z_0^2 - z_2^2 = R^2 \), \( z_1^2 = z_3^2 \), and \( z_0 \geq R \). The figure show the cases \( E = 0.2, 0.5, 0.8 \) \((\omega = R = \rho \varphi = 1)\)

5 Conclusion

In this work the problem of the movement of classical particle in the harmonic oscillator field on hyperbolic space \( H_{2,2} \): \( z_0^2 + z_1^2 - z_2^2 - z_3^2 = R^2 \) is discussed. We have shown that in the case of positive values of hyper-spherical separation constant \( A \geq 0 \), the motion is possible only in region \( |z_0| \geq R \). We have found the conditions for the existence of bounded classical motion and shown that all the finite trajectories are closed and periodic. The period of the bounded orbit depends only (for fixed \( \omega \) and \( R \)) on the energy. The classical trajectories have the form of ellipses and circles for finite motion and parabolas and hyperbolas for infinite. All these trajectories lie on the upper or lower sheets of two dimensional two-sheeted hyperboloid \( z_0^2 - z_2^2 - z_3^2 = R^2 \). In the case of negative values of separation constant \( A < 0 \) the trajectories of particle lie on the one-sheeted hyperboloid \( z_0^2 + z_1^2 - z_2^2 = R^2 \) and are bounded and periodic for \( E < \omega^2 R^2 / 2 \) and infinite for \( E \geq \omega^2 R^2 / 2 \). The similar situation take place for \( A = 0 \), but in this case the trajectories of particle lie on the hyperbolic cylinder \( z_0^2 - z_2^2 = R^2 \), \( z_1^2 = z_3^2 \).

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