Physics-inspired forms of the Bayesian Cramér-Rao bound

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Using differential geometry, I derive a form of the Bayesian Cramér-Rao bound that remains invariant under reparametrization. With the invariant formulation at hand, I find the optimal and naturally invariant bound among the Gill-Levit family of bounds. By assuming that the prior probability density is the square of a wavefunction, I also express the bounds in terms of functionals that are quadratic with respect to the wavefunction and its gradient. The problem of finding an unfavorable prior to tighten the bound for minimax estimation is shown, in a special case, to be equivalent to finding the ground state of a Schrödinger equation, with the Fisher information playing the role of the potential. To illustrate the theory, two quantum estimation problems, namely, optomechanical waveform estimation and subdiffraction in coherent optical imaging, are discussed.

I. INTRODUCTION

Differential geometry has been useful in the study of statistical measures, Cramér-Rao bounds, and asymptotic statistics [1–3], but its usefulness for Bayesian and minimax statistics is less clear. The Bayesian Cramér-Rao bounds [4, 5], pioneered by Schützenberger [6] and Van Trees [7], may serve as a bridge.

To be specific, consider a p-dimensional parameter \( \theta = (\theta^1, \ldots, \theta^p) \in \Theta \subseteq \mathbb{R}^p \), a scalar parameter of interest \( \beta(\theta) \in \mathbb{R} \), and an estimator \( \hat{\beta}(X) \), where \( X \) is a set of \( n \) independent and identically distributed (i.i.d.) observation random variables with a family of probability densities \( \{ f^{(n)}(x|\theta) = \prod_{j=1}^{n} f(x_j|\theta) : \theta \in \Theta \} \) and a reference measure \( d\mu^{(n)}(x) = \prod_{j=1}^{n} d\mu(x_j) \). Generalization of the theory for a vectorial \( \beta \) is straightforward but tedious and not included here. Define the mean-square risk as

\[
R(\theta) \equiv \int \left[ \hat{\beta}(x) - \beta(\theta) \right]^2 f^{(n)}(x|\theta) d\mu^{(n)}(x). \tag{1.1}
\]

The Cramér-Rao bound for any unbiased estimator is given by

\[
R(\theta) \geq \frac{C(\theta)}{n}, \tag{1.2}
\]

\[
C(\theta) \equiv u_{a}(\theta) \left[ F(\theta)^{-1} \right]^{ab} u_{b}(\theta), \tag{1.3}
\]

where

\[
u_{a} \equiv \partial_{\theta a} \beta, \quad \partial_{a} \equiv \partial_{\theta_{a} \theta}, \tag{1.4}\]

Einstein summation is assumed, \( F \) is the Fisher information matrix defined as

\[
F_{ab} \equiv \int (\partial_{a} \ln f) (\partial_{b} \ln f) f d\mu, \tag{1.5}
\]

and \( F^{-1} \) is its matrix inverse, with entries denoted by \( (F^{-1})^{ab} \). For simplicity, I call Eqs. (1.2) and (1.3) the local bound hereafter.

The restriction to unbiased estimators is one of the biggest shortcomings of the local bound. A fruitful remedy is to consider bounds on the Bayesian risk

\[
\langle R \rangle = \mathbb{E} \left[ (\hat{\beta} - \beta)^2 \right] = \int R(\theta)\pi(\theta)d\theta, \tag{1.6}
\]

where \( \mathbb{E} \) now denotes the expectation over both the observation and the parameter as random variables and \( \pi \) is a prior probability density [4]. In particular, Gill and Levit proposed a general family of Bayesian Cramér-Rao bounds, valid for any biased or unbiased estimator, given by [5]

\[
\langle R \rangle \geq B \equiv \frac{\langle A \rangle^2}{n \langle F \rangle + \langle P \rangle}, \tag{1.7}
\]

\[
A \equiv v^a u_a, \tag{1.8}
\]

\[
F \equiv v^a F_{ab} v^b, \tag{1.9}
\]

\[
P \equiv \left[ \frac{1}{\pi} \partial_a (\pi v^a) \right]^2, \tag{1.10}
\]

where \( v, A, F, \) and \( P \) are all functions of \( \theta, \pi \) is assumed to vanish on the boundary of \( \Theta \), and \( \langle \cdot \rangle \) denotes the prior expectation, as in Eq. (1.6). \( v \in \mathbb{R}^p \) is a free term, and by choosing it judiciously, many useful forms of \( B \) can be obtained [5]. An arbitrarily chosen \( v \), however, may lead to a \( B \) that varies if the parametrization of the underlying model \( \{ f^{(n)}(x|\theta) \} \) is changed. This property is unpleasant, as there can be infinitely many parametrizations for the same model and it is not clear which parametrization leads to the tightest bound. In Sec. II, I propose a condition on \( v \) that makes \( B \) invariant. I also derive an invariant form of \( B \) using the language of differential geometry [8]. With the invariant form, \( B \) is guaranteed to give the same value for a model, regardless of the parametrization.

A related question is how \( v \) should be chosen. Although Gill and Levit suggested a few options based on prior works or convenience, it is unclear which is better, or if there exists an optimal choice. In Sec. III, I show that there is indeed an optimal choice, and it agrees with a couple of popular options in special cases. The inspiration comes from the geometric picture of \( v \) as a vector field, which generalizes the role of a tangent vector in the local theory [2, 9]. By virtue of the invariant formalism, the resultant bound is naturally invariant.
Bayesian bounds are also useful for minimax statistics [10] by providing lower bounds on the worst-case risk via
\[ \sup_{\theta \in \Theta} R(\theta) \geq \langle R \rangle \] (1.11)
for any prior. In this context, an unfavorable prior should be chosen to tighten a lower bound. Given Eqs. (1.7)–(1.10), it is unclear how the prior should be chosen, as \( \langle P \rangle \) is highly non-linear with respect to \( \pi \). To help with this problem, in Sec. IV I rewrite Eqs. (1.7)–(1.10) in a form that looks more familiar, at least to physicists. To be specific, I identify the prior density with the square of a wavefunction, such that \( \langle A \rangle, \langle F \rangle \), and most importantly \( \langle P \rangle \) all become quadratic functionals of the wavefunction and its gradient. In a special case, \( n(F) + \langle P \rangle \) becomes the average energy of a wave that obeys a Schrödinger equation. Finding the tightest bound for minimax estimation then becomes equivalent to finding the ground-state energy of the wave, and insights from quantum mechanics turn out to be handy.

In terms of other prior works, Refs. [11, 12] also study Bayesian Cramér-Rao bounds in geometric terms, but do not discuss the question of invariance or find the optimal Gill-Levit bound. References [13, 14] derive the asymptotically optimal form of the Gill-Levit bounds, but do not find the exact optimal form. Regarding the wave picture, the fact that \( F \) is quadratic with respect to \( \partial \sqrt{\tilde{f}} \) is well known in statistics [1], and Frieden even claimed that it serves as a fundamental principle for physics [15]. He assumed that \( f \) is the square of a wavefunction and derived wave equations from this fact, but had to introduce further creative assumptions. He also did not consider Bayesian bounds. To my knowledge, the wave picture of a Bayesian Cramér-Rao bound was first proposed in Ref. [16], which considers the special case \( \beta = \theta \) with a scalar \( \theta \) and uses the wave picture as a trick to solve a parameter-estimation problem in optical imaging. Here, as before [16], I do not claim that my results have any foundational implications for physics, merely that the correspondence is interesting and useful for statistics problems.

Section V comes full circle and applies the statistical theory to quantum estimation [17, 18], where actual quantum systems are considered. I consider two important problems in quantum optics, namely, optomechanical waveform estimation [19, 20] and subdiffraction incoherent imaging [17, 21]. The first problem is relevant to gravitational-wave detectors, where quantum noise is now playing a major role [22]; I show the importance of including prior information in deriving a meaningful quantum limit in terms of spectral quantities, following Ref. [20]. The second problem is, of course, a fundamental one in optics and relevant to both fluorescence microscopy and observational astronomy. Recent studies, based on quantum estimation theory, have shown that judicious measurements can substantially improve the imaging of subdiffraction objects [21], although most prior works are based on the local bound, which is valid for unbiased estimators only. By considering the minimax perspective, the Bayesian bound, and the wave picture, I discuss the implication of a zero information for the estimator convergence rate for the multi-source localization problem studied in Ref. [23].

II. INVARIANCE

To model reparametrization, consider a bijective differentiable map \( \tilde{\theta}(\theta) \). The transformation laws are
\[ \partial_{\tilde{\theta}} a = J_{ab}^b \partial_b a, \quad \tilde{\theta} = \partial_{\tilde{\theta}} a \] (2.1)
\[ \frac{d\pi}{\|J\|} = \pi = \|J\| \tilde{\pi}, \] (2.2)
\[ u_a = J_{ab}^b \tilde{u}_b, \quad F_{ab} = J_{ab}^c \tilde{F}_{cd} J_{bd}^d, \] (2.3)
where
\[ J_{ab}^b \equiv \partial_b \tilde{\theta}^b \] (2.4)
is the Jacobian matrix. \( |J| \) denotes its determinant, and \( \|J\| \) denotes the absolute value of the determinant. Equations (2.3) imply that the components of \( u \) are covariant and \( F \) is a \((0, 2)\) tensor [8]. On the other hand, \( \beta, \beta, J, \mu, R, \) and \( \langle \cdot \rangle \) remain invariant, as these quantities depend on the statistical problem and should not depend on the parametrization of the underlying model.

It is well known that the local bound is invariant under reparametrization [9], in the sense of
\[ u_a \left(F^{-1}\right)^{ab} u_b = u_a \left(\tilde{F}^{-1}\right)^{ab} \tilde{u}_b. \] (2.5)
The Gill-Levit bounds can also be made invariant.

Proposition 1. B is invariant under reparametrization if \( v \) obeys the transformation law
\[ v^a J_b^a = v^b. \] (2.6)

Proof. Given Eq. (2.6), it is obvious that
\[ A = \tilde{v}^a \tilde{a}_a, \quad F = \tilde{v}^a \tilde{F}_{ab} \tilde{b}^b \] (2.7)
remain invariant upon reparametrization. To deal with \( P \), define the inverse Jacobian matrix as
\[ J_a^b \equiv \partial_a \theta^b, \] (2.8)
which obey
\[ J_a^b J_b^c = J_a^c J_b^c = \delta^c_a, \quad |\tilde{J}| = \frac{1}{|J|}, \] (2.9)
where \( \delta \) is the Kronecker delta. Consider
\[ \frac{1}{\pi} \partial_a (\pi v^a) = \frac{J_a^b}{|J|} \partial_b \left( |J| \tilde{\pi} \tilde{v}^c \tilde{J}_c^a \right) \] (2.10)
\[ = \tilde{\pi}^{c} \frac{J_a^b}{|J|} \partial_b \left( |J| \tilde{J}_c^a \right) + \frac{1}{\pi} \partial_a \left( \tilde{\pi} \tilde{v}^b \right). \] (2.11)
The first term can be shown to vanish as follows:
\[ \frac{J_a^b}{|J|} \partial_b \left( |J| \tilde{J}_c^a \right) = \partial_a \ln |J| + J_a^b \partial_b \tilde{J}_c^a \] (2.12)
\[ = -\partial_a \ln |\tilde{J}| + J_a^b \partial_b J_c^a \] (2.13)
\[ = -J_a^b \partial_a \partial_c \tilde{J}_c^a + J_a^b \partial_b \tilde{J}_c^a \] (2.14)
\[ = -J_a^b \left( \partial_a \partial_b \theta^a - \partial_b \partial_a \theta^a \right) = 0, \] (2.15)
where Eq. (2.14) uses Jacobi’s formula to simplify $\tilde{\partial}_c \ln |\tilde{J}|$. Hence

$$\frac{1}{\pi} \partial_a (\pi v^a) = \frac{1}{\pi} \partial_b (\pi \tilde{\partial}^b),$$  \hspace{1cm} (2.16)$$
and $P$ is invariant. As the prior expectation $\langle \cdot \rangle$ is also invariant, $B$ is invariant. \hfill \Box

In the language of differential geometry, Eq. (2.6) means that the components of $v$ are contravariant [8]. In other words, $v$ defines a vector field in the parameter space, with components $(v^1, \ldots, v^p)$ with respect to a parametrization. If one does not transform the components as per Eq. (2.6) upon reparametrization, $B$ changes—the reason, from the geometric perspective, is that it has become a bound for a different vector field.

A “natural” choice of $v$ according to Gill and Levit is [5]

$$v^a = (F^{-1})^{ab} u_b.$$  \hspace{1cm} (2.17)

It obeys Eq. (2.6), and also leads to the simplification

$$A = F = u_a (F^{-1})^{ab} u_b = C,$$  \hspace{1cm} (2.18)

which coincides with the local bound given by Eq. (1.3), and the resultant Bayesian bound is

$$B = \frac{(C)^2}{n(C) + \langle P \rangle}.$$  \hspace{1cm} (2.19)

This form becomes an inequality of Borovkov and Sakhanenko for a scalar $\theta$ [24]; see also Ref. [25]. Most importantly, Eq. (2.19) agrees with some classic “no-go” theorems in the asymptotic local theory by Hájek and Le Cam that generalize the Cramér-Rao bound but are much more sophisticated [5, 26]. Equation (2.17) is not the only contravariant choice, however. It does not even exist if $u$ is not in the range of the $F$ matrix [27]. It is also not the optimal choice for the Gill-Levit bounds in general, as Sec. III later shows.

Assuming $g_{ab} = \delta_{ab}$, another useful choice of $v$ is

$$v^a = \left[ (n \langle F \rangle + \langle G \rangle)^{-1} \right]^{ab} \langle u_b \rangle,$$  \hspace{1cm} (2.20)

$$G_{ab} = \frac{1}{\pi} (\partial_a \pi) - \frac{1}{\pi} (\partial_b \pi),$$  \hspace{1cm} (2.21)

leading to

$$B = \langle u_a \rangle \left[ (n \langle F \rangle + \langle G \rangle)^{-1} \right]^{ab} \langle u_b \rangle.$$  \hspace{1cm} (2.22)

If $u$ is $\theta$-independent, Eq. (2.22) coincides with the original version by Schützenberger and Van Trees [6, 7]. $\langle G \rangle$ plays the role of prior information and can regularize the inverse when $\langle F \rangle$ is ill-conditioned. The regularization is especially important for waveform-estimation problems [7, 20]. If $u$ is $\theta$-independent, $f$ is Gaussian, $\theta$ is the location parameter, and $\pi$ is also Gaussian, then $u$, $F$, and $G$ are $\theta$-independent, and Eq. (2.22) is equal to the optimal Bayesian risk [7]. This version is usually not invariant, however.

I assume hereafter that $v$ is a vector field with contravariant components. The formalism can then be made more elegant by defining the invariant quantities

$$\epsilon = \sqrt{|g|} d^p \theta, \quad \rho = \frac{\pi}{\sqrt{|g|}}, \quad \pi d^p \theta = \rho \epsilon,$$  \hspace{1cm} (2.23)

where $|g|$ is the determinant of a Riemannian (positive-definite) metric $g_{ab}$. While many have argued that the Fisher information is a natural choice for the metric [1], it may cause problems if $|F| = 0$, so I keep the metric unspecified here for generality. The divergence term in Eq. (1.10) becomes

$$\frac{1}{\pi} \partial_a (\pi v^a) = \frac{1}{\sqrt{|g|} \rho} \partial_a \left( \sqrt{|g|} \rho v^a \right) = \frac{1}{\rho} \nabla_a (\rho v^a),$$  \hspace{1cm} (2.24)

where $\nabla_a$ is the Riemannian covariant derivative [8]. With these suggestive expressions at hand, I propose the following.

**Proposition 2 (Invariant Gill-Levit bounds).** If the parameter space has no boundary, or if the prior density $\rho$ vanishes on the boundary when there is one, the Bayesian mean-square risk has a lower bound given by Eq. (1.7), where

$$\langle A \rangle = \int \langle v^a u_a \rangle \rho \epsilon,$$  \hspace{1cm} (2.25)

$$\langle F \rangle = \int \langle v^a F_{ab} v^b \rangle \rho \epsilon,$$  \hspace{1cm} (2.26)

$$\langle P \rangle = \int \left[ \frac{1}{\rho} \nabla_a (\rho v^a) \right]^2 \rho \epsilon.$$  \hspace{1cm} (2.27)

**Proof.** For completeness, I provide a proof that proceeds in a manifestly invariant way, so that the proposition is proved also for a curved metric. Define the bias as

$$b = \int \left( \beta - \beta \right) f^{(n)} d\mu^{(n)},$$  \hspace{1cm} (2.28)

and write, via the Leibniz rule for the covariant derivative [8],

$$\int \nabla_a (b \rho v^a) \epsilon = \int \left( \beta - \beta \right) \nabla_a \left( f^{(n)} \rho v^a \right) d\mu^{(n)} \epsilon$$

$$- \int \langle v^a \nabla_a \beta \rangle \rho \epsilon.$$  \hspace{1cm} (2.29)

It can be shown that the left-hand side of Eq. (2.29) is zero by applying the Stokes theorem [8, 28] and requiring that $\rho$ vanishes on the boundary if there is one. With $\nabla_a \beta = \partial_a \beta$ when $\nabla_a$ acts on a scalar, the last term in Eq. (2.29) is precisely $\langle A \rangle$ in Eq. (2.25). I obtain

$$\langle A \rangle = \int \int \left( \beta - \beta \right) \nabla_a \left( f^{(n)} \rho v^a \right) d\mu^{(n)} \epsilon$$

$$= \mathbb{E} \left[ \left( \beta - \beta \right) s \right],$$  \hspace{1cm} (2.31)

where $s$ is a generalized score function given by

$$s = \frac{1}{f^{(n)} \rho} \nabla_a \left( f^{(n)} \rho v^a \right)$$

$$= \frac{1}{f^{(n)} \rho} v^a \nabla_a f^{(n)} + \frac{1}{\rho} \nabla_a (\rho v^a).$$  \hspace{1cm} (2.33)
The expectation can be regarded as an inner product. The Cauchy-Schwarz inequality then gives
\[
\langle A \rangle^2 \leq \mathbb{E} \left[ (\hat{\beta} - \beta)^2 \right] \mathbb{E} \left( s^2 \right).
\] (2.34)

With the usual premise
\[
\int \nabla_a f d\mu = \int \partial_a f(x|\theta) d\mu(x) = \partial_a \int f d\mu = 0,
\] (2.35)

it can be shown that
\[
\mathbb{E} \left( s^2 \right) = n \langle F \rangle + \langle P \rangle,
\] (2.36)

with \( \langle F \rangle \) given by Eq. (2.26) and \( \langle P \rangle \) given by Eq. (2.27). Hence, Eq. (2.34) leads to Eq. (1.7), together with Eqs. (2.25)–(2.27).

If one assumes \( g_{ab} = \delta_{ab} \) everywhere in the parameter space, Proposition 2 is, of course, equivalent to the Gill-Levit bounds given by Eqs. (1.7)–(1.10). If the Riemann curvature tensor with respect to the metric is zero everywhere, one can always find a parametrization for which \( g_{ab} = \delta_{ab} \) [8], and the two formulations are equivalent. But if not, the metric is said to be curved, and Proposition 2 is more general. Proposition 2 may also be regarded as a special case of Theorem 2.1 in Ref. [11], although the latter is so general that the bound there may depend on the estimator.

While it is unclear whether curved metrics are useful for the kind of problems considered here, one immediate advantage of the invariant formulation is that all the ensuing results are guaranteed to be invariant.

III. OPTIMAL GILL-LEVIT BOUND

To derive the optimal Gill-Levit bound, it is illuminating to first recall the concept of least favorable submodels in the local theory [9]. Pick a curve in the parameter space that passes through the true value and denote the tangent vector there as \( v \). The local bound for the one-dimensional submodel is given by
\[
C(v) = \frac{(v^a u_a)^2}{v^a F_{ab} v^b}.
\] (3.1)

Define an inner product between two vectors as
\[
\langle v, w \rangle_g = v^a w_a = \theta^a g_{ab} w^b,
\] (3.2)

where the usual convention of index lowering and raising via \( g_{ab} \) and its inverse \( g^{ab} \) in differential geometry is assumed [8]. Let \( F \) be an operator that obeys \( (F v)_a = F_{ab} v^b \). If \( F \) is positive-definite, \( F^{-1} \) and the square roots \( F^{1/2} \) and \( F^{-1/2} \) exist [29]. The Cauchy-Schwarz inequality gives
\[
C(v) = \frac{(v, v)_g}{(v, F v)_g} \leq \frac{\langle F^{1/2} v, F^{-1/2} u \rangle_g^2}{(v, F v)_g} \leq \langle u, F^{-1} u \rangle_g = u_a \left( F^{-1} \right)^{ab} u_b,
\] (3.3)

which coincides with Eq. (1.3) for the full model. A least favorable tangent vector that attains the equality must satisfy
\[
v^a \propto \left( F^{-1} \right)^{ab} u_b.
\] (3.5)

Thus, Eq. (1.3) can be evaluated by considering the tangent space at the true parameter and picking the worst direction. Remarkably, this concept can be generalized for an infinite-dimensional parameter space [2, 3], although this case is outside the scope of this work.

For the Gill-Levit bounds, the “natural” choice of \( v \) given by Eq. (2.17) is a least favorable choice in the local theory. Thus, one may intuit that \( v \) plays an analogous role of picking out directions in the Bayesian bound, except that \( v \) should now be considered as a vector field, as depicted in Fig. 1. In differential geometry, a vector field can generate a family of integral curves, called a flow, in the manifold, and vice versa [28]. In the context of statistics, each curve corresponds to a one-dimensional submodel, so the concept of locally least favorable submodels may be generalized to a concept of least favorable flows. Following this intuition, I can generalize the strategy of optimizing over \( v \) to obtain the tightest bound, as follows.

FIG. 1. Left: a geometric picture of a one-dimensional submodel as a curve in the manifold and its tangent vector at the true parameter value \( \theta \) in the local theory. Right: a picture of \( v \) as a vector field in the Bayesian theory.

Theorem 1 (Optimal Gill-Levit bound).
\[
\max_v B = \langle u, L^{-1} u \rangle_{\rho} \equiv B_{\max},
\] (3.6)

where the inner product between two vector fields is defined as
\[
\langle v, u \rangle_{\rho} \equiv \int v^a u_a \rho \, \delta,
\] (3.7)

the linear, self-adjoint, and positive-semidefinite operator \( L \) is defined as
\[
(L v)_a = n F_{ab} v^b - \nabla_a \left[ \frac{1}{\rho} \nabla_b \left( \rho v^b \right) \right],
\] (3.8)

and \( u \) is assumed to be in the range of \( L \), such that \( L^{-1} u \) exists. A least favorable vector field, defined as a \( v \) that maximizes \( B \), must satisfy
\[
v \propto L^{-1} u.
\] (3.9)
Proof. In terms of the inner product given by Eq. (3.7), Eqs. (2.25)–(2.27) can be expressed as

\[
\langle A \rangle = \langle v, u \rangle_\rho, \quad \langle F \rangle = \langle v, Fv \rangle_\rho, \quad \langle P \rangle = \int [\nabla_a (\rho v^a)] \frac{1}{\rho} \nabla_b (\rho v^b) \epsilon \quad \text{(3.12)}
\]

\[
= - \int \rho v^a \nabla_a \left[ \frac{1}{\rho} \nabla_b (\rho v^b) \right] \epsilon \quad \text{(3.13)}
\]

\[
= \langle v, Pv \rangle_\rho, \quad (Pv)_a = - \nabla_a \left[ \frac{1}{\rho} \nabla_b (\rho v^b) \right], \quad \text{(3.15)}
\]

where Eq. (3.13) comes from integration by parts, as enabled by the Leibniz rule and the Stokes theorem [8], and the fact that \( \rho \) vanishes on the boundary if there is one. One can check that \( F \) and \( P \) are linear, self-adjoint, and positive-semidefinite operators. Furthermore,

\[
n \langle F \rangle + \langle P \rangle = \langle v, Lv \rangle_\rho, \quad L = nF + P. \quad \text{(3.16)}
\]

As \( L^{-1}u \) is assumed to exist, the Cauchy-Schwarz inequality yields

\[
B = \frac{\langle v, u \rangle_\rho^2}{\langle v, Lv \rangle_\rho} = \frac{(L^{1/2}v, L^{-1/2}u)_\rho^2}{\langle v, Lv \rangle_\rho} \leq \langle u, L^{-1}u \rangle_\rho, \quad \text{(3.17)}
\]

and the equality is attained if and only if \( v \) obeys Eq. (3.9). \( \square \)

Within the Gill-Levit family, \( B_{\text{max}} \) is not only the maximum but also the closest in spirit to the local bound given by Eq. (1.3), with the \( L^{-1} \) operator generalizing the role of \( F^{-1} \). Furthermore, note that \( B_{\text{max}} \) is naturally invariant—this fact would have been tedious to prove, with a proliferation of Jacobians, if the invariant approach were not adopted from the start.

The most difficult part of computing \( B_{\text{max}} \) is solving for \( L^{-1}u \). Let \( v = L^{-1}u \), which is a least favorable field. It obeys the second-order field equation

\[
(Lv)_a = nF_{ab}v^b - \nabla_a \left[ \frac{1}{\rho} \nabla_b (\rho v^b) \right] = u_a. \quad \text{(3.18)}
\]

The solution, expressible in terms of an impulse-response (Green) function, can be substituted into Eq. (3.6) to give \( B_{\text{max}} \). For large \( n \), Eq. (3.18) can be simplified to

\[
nF_{ab}v^b \approx u_a, \quad B_{\text{max}} \approx \frac{\langle \psi \rangle}{n}, \quad \text{(3.19)}
\]

so Eq. (3.5) is asymptotically least favorable to the Gill-Levit family, in nice agreement with the local theory [26] and earlier results [13, 14]. Note, however, that the exact optimal choice according to Eq. (3.18) also depends on the prior and some derivatives. The correction to the local theory becomes especially important if \( u \) is not in the range of \( F \) and Eq. (3.5) has no solution. The question of what to do when \( u \) is not even in the range of \( L \), and Eq. (3.18) has no solution, remains open.

Another special case is when \( g_{ab} = \delta_{ab} \), \( u \) and \( F \) are \( \theta \)-independent, and \( \pi \) is Gaussian, viz.,

\[
\pi(\theta) \propto \exp \left[ -\frac{1}{2} (\theta^a - \bar{\theta}^a) G_{ab} (\theta^b - \bar{\theta}^b) \right], \quad \text{(3.20)}
\]

where \( G \) is a positive-semidefinite tensor and \( \bar{\theta} \) is the prior location. Then the solution to Eq. (3.18) is

\[
v^a = \left[ (nF + G)^{-1} \right]_{ab} u_b, \quad \text{(3.21)}
\]

and \( B_{\text{max}} \) becomes

\[
B_{\text{max}} = u_a \left[ (nF + G)^{-1} \right]_{ab} u_b, \quad \text{(3.22)}
\]

which coincides with the Schützenberger-Van Trees version given by Eq. (2.22), since \( \langle u \rangle = u \), \( \langle F \rangle = F \), and \( \langle G \rangle = G \) in this case.

IV. WAVE PICTURE

I now switch gears and make the substitution

\[
\rho = \psi^2, \quad \text{(4.1)}
\]

where \( \psi \) is a real function of the parameter. I call \( \psi \) a wavefunction. Like \( \rho \), \( \psi \) should also vanish on the boundary of the parameter space for the Gill-Levit bounds to hold. All the functionals in Eqs. (2.25)–(2.27) turn out to be quadratic with respect to \( \psi \) and \( \nabla \psi \), given by

\[
\langle A \rangle = \int \psi^a u_a \psi^2 \epsilon, \quad \text{(4.2)}
\]

\[
\langle F \rangle = \int (\psi^a F_{ab} \psi^b) \psi^2 \epsilon, \quad \text{(4.3)}
\]

\[
\langle P \rangle = \int (D\psi)^2 \epsilon, \quad \text{(4.4)}
\]

\[
D \equiv \nabla_a \psi^a + 2u^a \nabla_a. \quad \text{(4.5)}
\]

The problem of choosing an unfavorable prior to tighten the bound for minimax estimation now becomes a problem of finding the wavefunction that maximizes \( B \). To simplify, I define yet another inner product as

\[
\langle \psi, \phi \rangle \equiv \int \psi \phi \epsilon. \quad \text{(4.6)}
\]

The normalization condition for the prior density becomes

\[
\int \rho \epsilon = \langle \psi, \psi \rangle = 1. \quad \text{(4.7)}
\]

It can be shown that

\[
\langle A \rangle = \langle \psi, A\psi \rangle, \quad \text{(4.8)}
\]

\[
\langle F \rangle = \langle \psi, F\psi \rangle, \quad \text{(4.9)}
\]

\[
\langle P \rangle = \langle D\psi, D\psi \rangle = \langle \psi, D^\dagger D\psi \rangle, \quad \text{(4.10)}
\]

\[
D^\dagger = \nabla_a \psi^a - 2u^a \nabla_a, \quad \text{(4.11)}
\]

\[
B = \langle \psi, A\psi \rangle^2, \quad \text{(4.12)}
\]

\[
H \equiv nF + D^\dagger D. \quad \text{(4.13)}
\]
Note that D may be a nonlinear operator, if the choice of \( v \), such as Eq. (3.18), depends on the prior. To proceed, I assume that \( v \) does not depend on \( \psi \) and D is linear. Then I can follow the approach in Sec. III to obtain

\[
B = \frac{\langle H^{1/2}\psi, H^{-1/2}A\psi \rangle^2}{\langle \psi, H\psi \rangle} \leq \langle A\psi, H^{-1}A\psi \rangle .
\]  

(4.14)

The equality is attained if and only if

\[
H\psi = (nF + D^1D)\psi = \lambda A\psi,
\]

(4.15)

where \( \lambda \) is an arbitrary nonzero real number. Let \( \psi_\lambda \) be a solution of Eq. (4.15) as a function of \( \lambda \), subject to the normalization constraint given by Eq. (4.7). Then

\[
B = \frac{1}{\lambda} \langle \psi_\lambda, A\psi_\lambda \rangle ,
\]

(4.16)

and this expression should be maximized with respect to \( \lambda \) to obtain the tightest lower bound on \( \sup_\theta R(\theta) \).

A substantial simplification can be made if \( g_{ab} = \delta_{ab} \) and \( u, v \), and therefore \( A \) are \( \theta \)-independent. Equation (4.15) becomes

\[
\left[ nF(\theta) - 4(v^a\partial_a)^2 \right] \psi(\theta) = \lambda A\psi(\theta),
\]

(4.17)

which is a time-independent Schrödinger equation. The Fisher information \( F = v^aF_{ab}v^b \), evaluated in the direction of \( v \), plays the role of the potential, while the directional derivative \( v^a\partial_a \) plays the role of the momentum. The bound becomes

\[
B = \frac{\Lambda^2}{\langle \psi, H\psi \rangle} .
\]

(4.18)

To maximize \( B \), one should therefore solve for

\[
B_{\text{worst}} \equiv \sup_{\psi:|\langle \psi, \psi \rangle| = 1} B = \frac{\Lambda^2}{E_{\text{min}}} ,
\]

(4.19)

\[
E_{\text{min}} \equiv \inf_{\psi:|\langle \psi, \psi \rangle| = 1} \langle \psi, H\psi \rangle ,
\]

(4.20)

that is, the ground-state energy. Adding a phase to the wavefunction cannot reduce the energy, so the consideration of only real wavefunctions is justified here.

The wave correspondence makes sense, as intuition suggests that an unfavorable prior should be concentrated near the minimum of the Fisher information, just as the ground state should be concentrated near the bottom of the potential. If the prior density is made too sharp, however, the prior information \( \langle P \rangle \) would become large, and therefore a balance between \( n\langle F \rangle \) and \( \langle P \rangle \) should be struck to minimize their sum, just as the ground state achieves the optimal balance between the potential and kinetic energies.

In the asymptotic limit of infinite \( n \), the ground-state energy is the classical limit given by

\[
E_{\text{min}} = n \inf_{\theta \in \Theta} F(\theta) + o(n) ,
\]

(4.21)

where \( o(n) \) denotes terms asymptotically smaller than \( n \). If the infimum of \( F(\theta) \) is strictly positive, \( B_{\text{worst}} \) obeys the parametric rate \( 1/n \) asymptotically. A more interesting case is when the infimum is zero, \( E_{\text{min}} = o(n) \), and the bound mandates a convergence rate slower than the parametric rate. If \( \theta \) is a scalar, \( \beta = \theta \), and

\[
F(\beta) \leq A|\beta - \beta_0|^m , \quad m > 0 ,
\]

(4.22)

where \( A \) is a positive constant and \( \beta_0 \) is the minimum point, then it is not difficult to show by variational arguments that

\[
E_{\text{min}} \leq Bn^{2/(m+2)} ,
\]

(4.23)

where \( B \) is another positive constant.

\section{Quantum Estimation Theory}

\subsection{Basics}

In the following, I assume \( n = 1 \) without loss of generality. Let \( \{ \rho(\theta) : \theta \in \Theta \} \) be a family of density operators that model a quantum system. The generalized Born’s rule states that the statistics of any measurement of the system can be modeled by a positive operator-valued measure (POVM) \( E \) \cite{18} via

\[
f(x|\theta)d\mu(x) = \text{tr} \left[ dE(x)\rho(\theta) \right] ,
\]

(5.1)

where \( \text{tr} \) denotes the operator trace. For any POVM, an upper bound on the Fisher information is given by \cite{18, 30}

\[
F = v^aF_{ab}v^b \leq v^aK_{ab}v^b \equiv K
\]

(5.2)

for any vector \( v \), where \( K \) is the Helstrom information matrix \cite{17} defined as

\[
K_{ab}(\theta) \equiv \text{tr} \left[ \rho(\theta)S_a(\theta) \circ S_b(\theta) \right] ,
\]

(5.3)

\( A \circ B \equiv (AB + BA)/2 \) denotes the Jordan product, and \( S_a \), a score operator, is a solution to

\[
\partial_a \rho(\theta) = \rho(\theta) \circ S_a(\theta) .
\]

(5.4)

While there exist other quantum versions of the Fisher information and the Cramér-Rao bound that are of interest when \( \beta \) is vectorial \cite{18, 31, 32, 33}, they turn out to offer, at best, a factor-of-two improvement over the Helstrom version only \cite{3, 34}. I focus on the Helstrom information hereafter.

With Eq. (5.2), a quantum lower bound on \( B \) for any POVM can be obtained simply by replacing \( F \) with \( K \). To be explicit,

\[
\langle R \rangle \geq B \geq Q \equiv \frac{\langle A \rangle^2}{\langle K \rangle + \langle P \rangle} .
\]

(5.5)

As \( K \) is also a positive-semidefinite \((0, 2)\) tensor, all the results in the previous sections apply to the quantum bound as well. In particular, following Theorem 1, the optimal quantum bound is

\[
Q_{\text{max}} \equiv \max_v Q = \langle u, R^{-1}u \rangle ,
\]

(5.6)

\[
(Rv)_a \equiv K_{ab}v^b - \nabla_a \left[ \frac{1}{\rho} \nabla_b \left( \rho v^b \right) \right] .
\]

(5.7)
B. Waveform estimation

Consider a quantum dynamical system, such as the optomechanical force sensor depicted in Fig. 2, under the influence of a classical waveform \(\theta(t)\). Using the principles of larger Hilbert space and deferred measurements [35], the statistics of a sequentially measured quantum system can be modeled by a POVM at the final time and a density-operator family given by

\[
\hat{q}(\theta) = U(\theta) |\Psi\rangle \langle\Psi| U(\theta)^\dagger, \tag{5.8}
\]

\[
U(\theta) = T \exp \left\{ \frac{1}{i\hbar} \int_{-T/2}^{T/2} [H_0(t) - q\theta(t)] \, dt \right\}, \tag{5.9}
\]

where \(|\Psi\rangle\) is the initial state of the quantum system, \(q\) is a position operator, \(H_0(t)\) is the rest of the Hamiltonian, \(T\) is the total observation time, and \(T\) denotes time ordering of the operator exponential.

It can be shown that

\[
u_a \approx \hbar_a \delta t, \tag{5.15}
\]

\[
K_{ab} \approx \frac{4\delta t^2}{\hbar^2} C_q(t_a, t_b), \tag{5.16}
\]

\[
C_q(t_a, t_b) \equiv \langle \Psi | \hat{q}(t_a) \circ \hat{q}(t_b) |\Psi\rangle - \langle \Psi | \hat{q}(t_a) |\Psi\rangle \langle \Psi | \hat{q}(t_b) |\Psi\rangle, \tag{5.17}
\]

where

\[
\hat{q}(t_a) \equiv U(t_{a-1}, t_1)^\dagger qU(t_{a-1}, t_1) \tag{5.18}
\]

is the Heisenberg picture of \(q\) and \(C_q\) is its covariance function. If \(\hat{q}(t)\) is stationary, the covariance can be written in terms of a power spectral density \(S_q(\omega)\) as [19]

\[
C_q(t_a, t_b) = \int_{-\infty}^{\infty} S_q(\omega) \exp\left[ i\omega(t_b - t_a) \right] \frac{d\omega}{2\pi}. \tag{5.19}
\]

With the assumption of stationary processes and long observation time (SPLOT) [7], \(K\) can be approximated as a circulant matrix [36] and expressed as

\[
K_{ab} \approx \frac{\delta t}{p} \sum_{j=0}^{p-1} 4S_q(\omega_j) \exp\left[ i\omega_j(t_a - t_b) \right], \tag{5.20}
\]

where \(\omega_j = \omega_0 + 2\pi j/T\) and \(\omega_0 = -\pi/\delta t\). Similarly, if \(\theta(t)\) is a stationary Gaussian random process with power spectral density \(S_\theta(\omega)\),

\[
G_{ab} \approx \frac{\delta t}{p} \sum_{j=0}^{p-1} \frac{1}{S_\theta(\omega_j)} \exp\left[ i\omega_j(t_a - t_b) \right]. \tag{5.21}
\]

As \(V_{ja} = \exp(-i\omega_j t_a)/\sqrt{p}\) is a unitary matrix, the inverse of \(K + G\) can be computed analytically to give

\[
[(K + G)^{-1}]^{ab} \approx \frac{1}{T} \sum_{j=0}^{p-1} \frac{\exp\left[ i\omega_j(t_a - t_b) \right]}{4S_q(\omega_j)/\hbar^2 + 1/S_\theta(\omega_j)}. \tag{5.22}
\]

\(u\), as given by Eq. (5.15), does not depend on \(\theta\). If the dynamics of the system is linear [19], \(K\) also does not depend on \(\theta\). Thus, the same argument that leads to Eq. (3.22) can be used to give

\[
Q_{\text{max}} = u_a \left( (K + G)^{-1} \right)^{ab} u_b, \tag{5.23}
\]

\[
\approx \frac{1}{T} \sum_{j=0}^{p-1} \frac{\delta t^2 \hbar_a \hbar_b \exp\left[ i\omega_j(t_a - t_b) \right]}{4S_q(\omega_j)/\hbar^2 + 1/S_\theta(\omega_j)}. \tag{5.24}
\]

Taking the continuous and long time limit with \(\delta t \to 0, T \to \infty\), and \(d\omega = 2\pi/T\) hence results in

\[
Q_{\text{max}} \to \int_{-\infty}^{\infty} \frac{|\tilde{h}(\omega)|^2}{4S_q(\omega)/\hbar^2 + 1/S_\theta(\omega)} \frac{d\omega}{2\pi}. \tag{5.25}
\]

\[
\tilde{h}(\omega) \equiv \int_{-\infty}^{\infty} h(t) \exp(-i\omega t) \, dt. \tag{5.26}
\]
If $\beta = \theta(\tau)$ with $\tilde{h}(t) = \delta(t - \tau)$ and $|\tilde{h}(\omega)| = 1$, Eq. (5.25) agrees with the result in Ref. [20]. Compared with Ref. [20], which derives a quantum bound on $\langle R \rangle$ directly, the derivation here clarifies the relation of Eq. (5.25) to the Helstrom information and the Gill-Levit formalism.

While Eq. (5.25) holds for any measurement, it can say something more about measurements in the linear form of

$$X(t) = \int_{-\infty}^{\infty} h_X(t-t')\theta(t')dt' + Z(t), \quad (5.27)$$

where $h_X$ is an impulse-response function of the system and $Z$ is a stationary noise process that is uncorrelated with $\theta$. In optomechanics, such a process can be obtained by homodyne detection of the output light. Let the estimator be

$$\tilde{\beta} = \int_{-\infty}^{\infty} \tilde{h}(t)X(t)dt, \quad (5.28)$$

where $\tilde{h}(t)$ is a linear filter, or more precisely a smoother in control-theoretic terminology, as it is applied to the whole observation record to estimate the waveform at an intermediate time [37]. By standard Wiener filtering theory [7], the minimum mean-square risk in the SPLOT limit is

$$\langle R \rangle \rightarrow \int_{-\infty}^{\infty} \frac{|\tilde{h}(\omega)|^2}{h_X(\omega)^2/S_Z(\omega) + 1/S_\theta(\omega)} d\omega, \quad (5.29)$$

$$\tilde{h}_X(\omega) \equiv \int_{-\infty}^{\infty} h_X(t) \exp(-iwt)dt, \quad (5.30)$$

where $S_Z$ is the power spectral density of $Z$. Comparing Eqs. (5.25) and (5.29), one sees that $\langle R \rangle \geq Q_{\text{max}}$ implies

$$\frac{S_Z(\omega)}{|\tilde{h}_X(\omega)|^2} \geq \frac{h^2}{4S_\theta(\omega)}, \quad (5.31)$$

which serves as a fundamental quantum limit on the noise floor. To reach this limit for an optomechanical system, back-action evasion and quantum-limited measurements are necessary [20]. It is possible to derive alternative quantum limits in terms of the optics by appealing to the interaction picture and tighter limits that account for loss by choosing the purification of the quantum state judiciously [38]. Reference [39] reports an experimental demonstration of mirror-motion estimation close to such quantum limits.

It is noteworthy that, prior to Ref. [20], Braginsky and Khalili derived an expression similar to Eq. (5.16) by optimizing a signal-to-noise ratio (SNR) in terms of an observable [19]. A spectral form of their optimal SNR, derived from a heuristic energy-time uncertainty relation, can be found in Ref. [40]. They called the results the energetic quantum limit. The similarity is not a coincidence, as the Helstrom information can also be expressed as the solution to an optimization problem [3, 30]. While their results are seminal and capture the basic physics, the results here and in Ref. [20] are more precise in terms of its meaning. The SNR does not have a direct operational meaning in statistics, whereas here the statistical problem is clearly defined in terms of a mean-square risk, and the bound is proven to hold for any POVM and any biased or unbiased estimator, not just observables. The clear definition of a risk is important, as different problems have different types of risk and different optimal measurements, and no single SNR-based treatment can deal with all of them. For example, while a linear measurement in the form of Eq. (5.27) can achieve the optimal SNR and also optimal waveform estimation, more careful studies reveal that it is suboptimal with respect to the quantum limits for waveform detection [41] and spectrum parameter estimation [42] and inferior to photon-counting measurements for those problems.

Equation (5.25) demonstrates the importance of prior information in the form of $1/S_\theta(\omega)$, as the integral may not converge without it; see Ref. [43] for an example in optical phase estimation. If $\beta = \theta(\tau)$, Eqs. (5.25) and (5.29) are steady-state values that do not scale with $T$. This is an extreme example where the i.i.d. condition does not hold, the standard asymptotic theory [18, 26] fails, the convergence rate is slower than the parametric rate, and prior information is indispensable. The information that can be acquired in one time slot with duration $\delta t$ is infinitesimal, but a finite risk can still be achieved because there exist prior correlations in $\theta(t)$ across different times before and after $t = \tau$, meaning that information over multiple time slots can contribute to the estimation of each $\theta(\tau)$. This intuition explains why the optimal estimator is a smoother.

### C. Subdiffraction incoherent optical imaging

For another application of quantum estimation theory, consider the far-field paraxial imaging of $p$ spatially incoherent and equally bright point sources [44], as depicted in Fig. 3. On the image plane, the density operator of each photon can be modeled as [21]

$$\rho(\theta) = \frac{1}{p} \sum_{a=1}^{p} \exp(-ik\theta^a) |\Psi(\theta)|^2 \rho(\theta) \exp(i\theta^a), \quad (5.32)$$

$$|\Psi(\theta)| = \int_{-\infty}^{\infty} dx \Psi(x) |x\rangle, \quad (5.33)$$

where $\theta$ is a vector of the unknown source positions on the object plane that is assumed to be one-dimensional for simplicity, $|x\rangle$ is the Dirac eigenket for the image-plane photon position that obeys $\langle x|x'\rangle = \delta(x - x')$, $\Psi$ is the point-spread function of the imaging system for the optical field, and $k$ is the momentum operator. $x$ is normalized with respect to the magnification factor.

Direct imaging can be modeled as a measurement of each photon in the position basis [21]. The probability density of each observed position is then

$$f(x|\theta) = \langle x| \rho(\theta) |x\rangle = \frac{1}{p} \sum_{a=1}^{p} h(x - \theta^a), \quad (5.34)$$

$$h(x) \equiv |\Psi(x)|^2. \quad (5.35)$$

The Fisher information is

$$v^a F_{ab}(\theta)v^b = \int_{-\infty}^{\infty} dx \frac{v^a \partial_a h(x - \theta^a)^2}{p^2 f(x|\theta)}. \quad (5.36)$$
In particular, at $\theta = 0$,

$$v^{a} F_{ab}(0) v^{b} = (v^{a} w_{a})^{2} \int_{-\infty}^{\infty} dx \frac{1}{h(x)} \left[ \frac{\partial h(x)}{\partial x} \right]^{2}, \quad (5.37)$$

$$w_{a} = \frac{1}{p}, \quad a = 1, \ldots, p. \quad (5.38)$$

The kernel of $F(0)$ is then the $(p - 1)$-dimensional space

$$\ker [F(0)] = \{ v \in \mathbb{R}^{p} : v^{a} w_{a} = 0 \}, \quad (5.39)$$

while the range is the one-dimensional space

$$\text{ran} [F(0)] = \{ cw : c \in \mathbb{R} \}. \quad (5.40)$$

Assume hereafter that $\beta$ is a linear function of $\theta$, such that $u$ is $\theta$-independent. For any $\beta$ with $u \notin \text{ran}[F(0)]$, a $v \in \ker [F(0)]$ can always be found such that $v^{a} w_{a} \neq 0$ but $v^{a} F_{ab}(0) v^{b} = 0$. Section IV then implies that, from the minimax perspective, any estimator of this $\beta$ must have a convergence rate slower than the parametric rate with respect to $n$ detected photons. Only a $\beta$ with $u \in \text{ran}[F(0)]$ has a nonzero information at $\theta = 0$ for any $v$ with $v^{a} u_{a} \neq 0$. This $\beta$ is proportional to the object centroid $w_{a} \theta^{a} = (\sum \theta^{a})/p$, and the parametric rate is indeed possible by taking the sample mean of the photon positions, provided that $h$ has a finite variance [45].

The Helstrom information turns out to be much higher [21, 23]. For $n$ detected photons and i.i.d. quantum states, the Helstrom information is simply $n$ times that for one photon [18]. For $p = 2$, $K(\theta)$ turns out to be full-rank [21], and estimation of not only the centroid but also the separation $|\theta^{2} - \theta^{1}|$ at the parametric rate is possible via spatial-mode demultiplexing [16]. For $p \geq 2$, Bisketzi and coworkers found that $K(\theta)$ has a rank of two as $\theta \to 0$ [23]. Then Sec. IV implies that any $\beta$ with $u \notin \text{ran}[K(0)]$ cannot be estimated at the parametric rate by any measurement, and only a $\beta$ with $u$ in the two-dimensional range may be estimated at the parametric rate.

Computing the Bayesian bounds and finding the measurements and estimators to achieve them in general will require a much more involved analysis that is outside the scope of this work.

Note that the mean-square risk is usually not invariant with respect to a reparametrization of $\beta$, so a zero of the information may become positive for a reparametrized $\beta$ and the convergence rate may change [5]. For example, if $\beta$ is a moment of the object distribution, the parametric rate becomes possible [45].

The take-away message is this: Whereas a vanishing information means that the local bound for unbiased estimators is infinite, from the minimax perspective, it implies merely that the estimator convergence rate must be slower than the parametric rate. With the wave picture, a limit on the convergence rate can be derived by considering the behavior of the information near the minimum more carefully, and more precise bounds can also be derived by quantum-inspired techniques.

As the quantum and microscopy communities have so far focused on the local theory [18, 21, 46] while the statistics and machine-learning communities have focused on the minimax perspective [47], the Bayesian bound may provide a bridge between the two lines of research.

VI. CONCLUSION

Compared with the local theory, the use of Bayesian Cramér-Rao bounds has been less systematic in the literature and often relied on the ingenuity of the researcher to pick the appropriate form. This work resolves some of the ambiguities and hopefully inspires further progress via the physics connections.

The formalism here looks ripe for a generalization for infinite-dimensional parameter spaces in a manner similar to the local theory [2, 3, 9]. An important application would be to derive semiparametric bounds with slow convergence rates [5] in a more systematic fashion.

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[1] Shun-ichi Amari and Hiroshi Nagaoka, *Methods of Information Geometry* (American Mathematical Society/Oxford University Press, Providence, 2000); Shun-ichi Amari, *Information Geometry and Its Applications* (Springer Japan, Tokyo, 2016).

[2] Peter J. Bickel, Chris A. J. Klaassen, Ya'acov Ritov, and John A. Wellner, *Efficient and Adaptive Estimation for Semiparametric Models* (Springer, New York, 1993).

[3] Mankei Tsang, Francesco Albarelli, and Animesh Datta, “Quantum Semiparametric Estimation,” *Physical Review X* 10, 031023 (2020).

[4] Harry L. Van Trees and Kristine L. Bell, eds., *Bayesian Bounds for Parameter Estimation and Nonlinear Filtering/Tracking*.
(Wiley-IEEE, Piscataway, 2007).

[5] Richard D. Gill and Boris Y. Levit, “Applications of the Van Trees inequality: A Bayesian Cramér-Rao bound,” Bernoulli 1, 59–79 (1995).

[6] M. P. Schützenberger, “A generalization of the Fréchet-Cramér inequality to the case of Bayes estimation,” Bull. Amer. Math. Soc. 63, 142 (1957).

[7] Harry L. Van Trees, Detection, Estimation, and Modulation Theory, Part I (John Wiley & Sons, New York, 2001).

[8] Sean M. Carroll, Spacetime and Geometry: An Introduction to General Relativity (Cambridge University Press, Cambridge, 2019).

[9] Charles Stein, “Efficient Nonparametric Testing and Estimation,” in Proc. Third Berkeley Symp. on Math. Statist. and Prob., Vol. 1: Contributions to the Theory of Statistics (University of California Press, Berkeley, 1956) pp. 187–195.

[10] Alexandre B. Tsybakov, Introduction to Nonparametric Estimation (Springer, New York, 2009).

[11] P. E. Jupp, “A van Trees inequality for estimators on manifolds,” Journal of Multivariate Analysis 101, 1814–1825 (2010).

[12] M. Ashok Kumar and Kumar Vijay Mishra, “Information Geometric Approach to Bayesian Lower Error Bounds,” in 2018 IEEE International Symposium on Information Theory (ISIT) (IEEE, Piscataway, NJ, 2018) pp. 746–750.

[13] R. Abu-Shanab and A. Veretennikov, “On asymptotic Borovkov-Sakhnenko inequality with unbounded parameter set,” Theory of Probability and Mathematical Statistics 90, 1–12 (2015).

[14] Ken-ichi Koike, “Asymptotic comparison of some Bayesian information bounds,” Communications in Statistics - Theory and Methods (2020), 10.1080/03610926.2020.1752722.

[15] B. Roy Frieden, Physics from Fisher Information: A Unification (Cambridge University Press, Cambridge, 1998).

[16] Mankei Tsang, “Conservative classical and quantum resolution limits for incoherent imaging,” Journal of Modern Optics 65, 1385–1391 (2018).

[17] Carl W. Helstrom, Quantum Detection and Estimation Theory (Academic Press, New York, 1976).

[18] Masahito Hayashi, Quantum Information Theory: Mathematical Foundation (Springer, Berlin, 2017).

[19] Vladimir B. Braginsky and David V. Klyshko, Quantum Measurement (Cambridge University Press, Cambridge, 1992).

[20] Mankei Tsang, Howard M. Wiseman, and Carlton M. Caves, “Fundamental quantum limit to waveform estimation,” Physical Review Letters 106, 090401 (2011).

[21] Mankei Tsang, Ranjith Nair, and Xiao-Ming Lu, “Quantum Theory of Superresolution for Two Incoherent Optical Point Sources,” Physical Review X 6, 031033 (2016); Mankei Tsang, “Resolving starlight: a quantum perspective,” Contemporary Physics 60, 279–298 (2019).

[22] B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) (LIGO Scientific Collaboration and Virgo Collaboration), “GW150914: The advanced LIGO detectors in the era of first discoveries,” Physical Review Letters 116, 131103 (2016); Haixing Miao, Rana X Adhikari, Yiqiu Ma, Belinda Pang, and Yanbei Chen, “Towards the Fundamental Quantum Limit of Linear Measurements of Classical Signals,” Physical Review Letters 119, 050801 (2017); M. Tse et al., “Quantum-Enhanced Advanced LIGO Detectors in the Era of Gravitational-Wave Astronomy,” Physical Review Letters 123, 231107 (2019); F. Aernesen et al. (Virgo Collaboration), “Increasing the Astrophysical Reach of the Advanced Virgo Detector via the Application of Squeezed Vacuum States of Light,” Physical Review Letters 123, 231108 (2019); H. Yu et al. (LIGO Scientific Collaboration), “Quantum correlations between light and the kilogram-mass mirrors of LIGO,” Nature 583, 43–47 (2020).

[23] Evangelia Bisketi, Dominic Branford, and Animesh Datta, “Quantum limits of localisation microscopy,” New Journal of Physics 21, 123032 (2019).

[24] A. A. Borovkov, Mathematical Statistics (Gordon and Breach, Amsterdam, 1998).

[25] A. W. van der Vaart, Asymptotic Statistics (Cambridge University Press, Cambridge, UK, 1998).

[26] P. Stoica and T. L. Marzetta, “Parameter estimation problems with singular information matrices,” IEEE Transactions on Signal Processing 49, 87–90 (2001).

[27] John M. Lee, Introduction to Smooth Manifolds (Springer-Verlag, New York, 2003).

[28] Roger A. Horn and Charles R. Johnson, Matrix Analysis (Cambridge University Press, Cambridge, 1985).

[29] Hiroshi Nagaoka, “A New Approach to Cramér-Rao Bounds for Quantum State Estimation,” IEICE Technical Report IT 89-42, 9–14 (1989), reprinted in Asymptotic Theory of Quantum Statistical Inference, edited by Masahito Hayashi (World Scientific, Singapore, 2005) Chap. 8, pp. 100–112.

[30] Richard D. Gill and Mădălîn Țuțu, “On asymptotic quantum statistical inference,” in From Probability to Statistics and Back: High-Dimensional Models and Processes – A Festschrift in Honor of Jon A. Wellner, Collections, Vol. 9, edited by M. Banerjee, F. Bunea, J. Huang, V. Kolchinskii, and M. H. Maathuis (Institute of Mathematical Statistics, Beachwood, Ohio, USA, 2013) pp. 105–127.

[31] Rafal Demkowicz-Dobrzański, Wojciech Górecki, and Mădălîn Țuțu, “Multi-parameter estimation beyond quantum Fisher information,” Journal of Physics A: Mathematical and Theoretical 53, 363001 (2020).

[32] Jun Suzuki, Yuxiang Yang, and Masahito Hayashi, “Quantum state estimation with nuisance parameters,” Journal of Physics A: Mathematical and Theoretical (2020), 10.1088/1751-8121/abb878.

[33] Angelo Carollo, Bernardo Spagnolo, Alexander A. Dubkov, and Davide Valenti, “On quantumness in multi-parameter quantum estimation,” Journal of Statistical Mechanics: Theory and Experiment 2019, 094010 (2019); “Erratum: On quantumness in multi-parameter quantum estimation (2019 J. Stat. Mech. 094010),” Journal of Statistical Mechanics: Theory and Experiment 2020, 029902 (2020).

[34] Michael A. Nielsen and Isaac L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2011).

[35] Robert M. Gray, “Toeplitz and circulant matrices: A review,” Asymptotic Statistics 119, 87–143 (2010).

[36] Mankei Tsang, “Quantum metrology with open dynamical systems,” New Journal of Physics 15, 073005 (2013).
[39] Kohjiro Iwasawa, Kenzo Makino, Hidehiro Yonezawa, Mankei Tsang, Aleksandar Davidovic, Elanor Huntington, and Akira Furusawa, “Quantum-limited mirror-motion estimation,” Physical Review Letters 111, 163602 (2013).

[40] Vladimir B. Braginsky, Mikhail L. Gorodetsky, Farid Ya. Khalili, and Kip S. Thorne, “Energetic quantum limit in large-scale interferometers,” AIP Conference Proceedings 523, 180–190 (2000).

[41] Mankei Tsang and Ranjith Nair, “Fundamental quantum limits to waveform detection,” Physical Review A 86, 042115 (2012).

[42] Shilin Ng, Shan Zheng Ang, Trevor A. Wheatley, Hidehiro Yonezawa, Akira Furusawa, Elanor H. Huntington, and Mankei Tsang, “Spectrum analysis with quantum dynamical systems,” Physical Review A 93, 042121 (2016).

[43] Dominic W. Berry, Michael J. W. Hall, and Howard M. Wiseman, “Stochastic Heisenberg Limit: Optimal Estimation of a Fluctuating Phase,” Physical Review Letters 111, 113601 (2013).

[44] Joseph W. Goodman, Introduction to Fourier Optics (McGraw-Hill, New York, 2004).

[45] Mankei Tsang, “Semiparametric estimation for incoherent optical imaging,” Physical Review Research 1, 033006 (2019).

[46] Jerry Chao, E. Sally Ward, and Raimund J. Ober, “Fisher information theory for parameter estimation in single molecule microscopy: tutorial,” Journal of the Optical Society of America A 33, B36 (2016).

[47] David L. Donoho, Iain M. Johnstone, Jeffrey C. Hoch, and Alan S. Stern, “Maximum Entropy and the Nearly Black Object,” Journal of the Royal Statistical Society. Series B (Methodological) 54, 41–81 (1992); Sitan Chen and Ankur Moitra, “Algorithmic Foundations for the Diffraction Limit,” arXiv:2004.07659 [physics, stat] (2020).