GALOIS REPRESENTATIONS AND TORSION IN THE COHERENT 
COHOMOLOGY OF HILBERT MODULAR VARIETIES

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Abstract. Let \( F \) be a totally real number field and let \( p \) be a prime unramified in \( F \). We prove the existence of Galois pseudo-representations attached to mod \( p^m \) Hecke eigenclasses of paritious weight occurring in the coherent cohomology of Hilbert modular varieties for \( F \) of level prime to \( p \).

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1. Introduction

In [Ash92], A. Ash conjectured that mod \( p \) representations of the absolute Galois group of \( \mathbb{Q} \) can be associated to Hecke eigenclasses in the cohomology of a congruence subgroup \( \Gamma \) of \( GL_n(\mathbb{Z}) \), with coefficient in a representation of \( \Gamma \) over a finite field of characteristic \( p \). Using some reductions and Eichler-Shimura theory, this conjecture is proved for \( n = 1, 2 \) in [Ash97]. An analogous conjecture for the group \( GL_2/K \) when \( K \) is a quadratic imaginary field appeared in work of L.M. Figueiredo ([Fig99]). Earlier conjectures on the existence of Galois representations attached to torsion cohomology classes go back to works F. Grunewald ([Gru72], [GHM78]).

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In [CG12a] and [CG12b], F. Calegari and D. Geraghty showed how to generalize the Taylor-Wiles modularity lifting results to prove modularity lifting theorems over general fields, contingent on a conjecture asserting the existence of Galois representations attached to certain torsion cohomology classes. Their conjecture (Conjecture A of [CG12a] and [CG12b]) predicts moreover that these representations should have the “expected” local properties.

In this paper, motivated by the work of Calegari and Geraghty, we solve the problem of existence of Galois representations attached to mod $p^m$ Hilbert modular eigenclasses for a totally real field $F$ in which $p$ is unramified. We work with the coherent cohomology of Hilbert modular varieties, so that we can see in particular the contribution of irregular weight forms. In [Sch13], P. Scholze proves the existence of Galois representations arising from the torsion singular cohomology of locally symmetric varieties over a totally real or a CM field.

We remark that a proof of Conjecture A of [CG12a] and [CG12b] in our context would moreover require to investigate some local properties of the Galois representations that we obtain: we do not address this issue here, though we plan to come back to it in later work.

We now state our main result. Let $F$ be a totally real field of degree $g > 1$ over $\mathbb{Q}$ in which a fixed prime $p$ is unramified, and let $N \geq 4$ be an integer coprime with $p$. Let $\mathcal{M}_{\text{tor}}^\text{tor}$ denote a toroidal compactification of the Hilbert modular variety for a fixed prime $p$. Let $\mathcal{M}_{\text{tor}}$ be a scheme over $R_m := \mathcal{O}_E/ (\varpi_E)^m$, where $E$ is a sufficiently large extension of $\mathbb{Q}_p$ with ring of integer $\mathcal{O}_E$ and uniformizer $\varpi_E$. We denote by $\partial$ the boundary divisor of $\mathcal{M}_{\text{tor}}$ and by $\omega$ the Hodge bundle on $\mathcal{M}_{\text{tor}}$ (it is a locally free sheaf of rank $g$, cf. [2.1]). Assuming $E$ is large enough, we say that an algebraic character $\kappa : (\text{Res}_{\mathbb{Q}^F_p}^F \mathbb{G}_m)_{R_m} \to \mathbb{G}_m_{R_m}$ is a paritious weight if the $g$ integers canonically attached to $\kappa$ have the same parity (cf. Definition 2.2.1). Let $\mathcal{S}$ be a finite set of places of $F$ containing the infinite places together with the places dividing $pN$, and let $G_{F, \mathcal{S}}$ be the Galois groups of the maximal extension of $F$ inside $\overline{\mathbb{Q}}$ that is unramified outside $\mathcal{S}$. We prove the following (cf. Corollary 4.4.7):

**Theorem.** Let $\kappa$ be a paritious weight. For any $i \geq 0$ and for any Hecke eigenclass $c \in H^i(\mathcal{M}_{\text{tor}}^\text{tor}, \omega^{\kappa}(-\mathcal{D}))$ there is a continuous $R_m$-linear two-dimensional pseudo-representation $\tau_c$ of the Galois group $G_{F, \mathcal{S}}$ such that

$$\tau_c(\text{Frob}_\lambda) = a_\lambda(c)$$

for all places $\lambda$ of $F$ outside $\mathcal{S}$. Here $a_\lambda(c)$ is the eigenvalue of the Hecke operator $T_\lambda$ acting on $c$.

We remark that the semisimple Galois representation that we attach to $c \in H^i(\mathcal{M}_{\text{tor}}^\text{tor}, \omega^{\kappa}(-\mathcal{D}))$ ultimately arises by taking the semisimplification of the reduction modulo $\varpi_E^m$ of an integral model of the $p$-adic Galois representation attached – by work of Carayol ([Car86]), Taylor ([Tay89]), and Blasius-Rogawski ([BR89]) – to a complex Hilbert cuspidal eigenform $f \in H^0(\mathcal{M}_{\text{tor}}^\text{tor}, \omega^{\kappa'}(-\mathcal{D}))$ of some regular weight $\kappa'$.

We also remark that beginning with § 3.3 we assume that $p$ is inert in $F$ (rather than merely unramified), since this achieves significant notational simplifications. Thus, strictly speaking, our results are only proved under this additional hypothesis. However, at the expense of complicating the notation, the proofs as written extend immediately to the unramified case. We have preferred to postpone introducing these notational complications to future work, in which we intend to deal also with the case when $p$ is allowed to be ramified in $F$ (a case which introduces much more substantial complications of its own).
Assume for simplicity that $R_m = F$ is a field. While the congruences produced via multiplication by the partial Hasse invariants of $\text{Cor01}$ and $\text{AG05}$ allow one to attach Galois representations to mod $\pi_F$ Hilbert modular eigenforms of any paritious weight – for example, when these forms do not lift to characteristic zero – the same method does not work well when considering cohomology of higher degree. To prove the theorem we then construct (cf. Theorem 4.4.4) a Hecke-equivariant resolution of $\omega^\kappa(-D)$:

\[ 0 \to \omega^\kappa(-D) \to F_1 \to F_2 \to \ldots \to F_t \to 0 \]

such that each sheaf $F_i$ is favorable in the sense of Definition 4.2.1, i.e., (1) the cohomology group $H^j(M_{\text{tor}}, F_i)$ vanishes for $j > 0$ and any $i$; (2) each Hecke eigensystem in $\Gamma(M_{\text{tor}}, F_i)$ has attached a Galois representation whose Frobenii traces match the Hecke eigenvalues away from $S$.

Each sheaf $F_i$ is built as a finite direct sum of suitable sheaves of the form $\iota^* \omega^\kappa'(-D)$, where $\kappa'$ is a paritious weight, and $\iota : Z \to M_{\text{tor}}$ is the closed embedding associated to a stratum $Z$ of $M_{\text{tor}}$ defined by the vanishing of some partial Hasse invariants. The determination of the weights $\kappa'$ and the strata $Z$ appearing in each $F_i$ is a result of combinatorial considerations and of an inductive process (cf. Lemma 4.4.5 and see below).

Two ingredients that go into the construction of the resolution (1.0.0.1) are: (i) the fact that $\det \omega$ is ample on the minimal compactification $M^*$ of the Hilbert modular variety (cf. Lemma 4.2.2); (ii) the construction of a canonical Hecke-equivariant trivialization:

\[ b_\tau : \left( \omega^{\otimes p}_{\sigma \tau} \otimes \omega_\tau \right)|_{Z_\tau} \xrightarrow{\simeq} O_{Z_\tau}[1] \]

induced by the Kodaira-Spencer isomorphism and by the partial Hasse invariant $h_\tau$ (cf. 3.2). Here $Z_\tau$ denotes the zero locus of the partial Hasse invariant $h_\tau$ associated to the infinite place $\tau$ of $F$, and $\sigma$ is the arithmetic Frobenius of $F$. The operator $b_\tau$ is closely related to the $\tau$-partial theta operator of $\text{Kat78}$ and $\text{AG05}$. It can be seen as a generalization of the operator $B$ considered by G. Robert in $\text{Rob80}$, by J.-P. Serre in $\text{Ser96}$, and by B. Edixhoven in $\text{Edi92}$ (cf. Remark 3.2.5).

To illustrate how the resolution (1.0.0.1) is constructed, let us assume for simplicity that $g = 2$, $p$ is inert in $F$, $R_m = F$ is a field, and $\kappa$ is the weight attached to the pair $1 = (1, 1)$. The two partial Hasse invariants $h_1$ and $h_2$ available under these assumptions shift weights by $(-1, p)$ and $(p, -1)$ respectively (cf. 3.1). The canonical trivializations $b_1$ and $b_2$ of (1.0.0.2) shift weights by $(1, p)$ and $(p, 1)$ respectively. By the ampleness of $\det \omega = \omega^1$ on $M^*$ we can find a large positive integer $N$ such that $\omega^{1+N\cdot(p-1)}(-D)$ is a favorable sheaf on $M_{\text{tor}}$. We consider the exact sequence:

\[ 0 \to \omega^1(-D) \xrightarrow{(h_1h_2)^N} \omega^{1+N\cdot(p-1)}(-D) \to \omega_1^{1+N\cdot(p-1)} \oplus \omega_2^{1+N\cdot(p-1)} \to \omega|_{Z_{1,N} \cap Z_{2,N}} \to 0, \]

where $Z_{i,N}$ denotes the zero locus of $h^N_i$. Using the isomorphism

\[ \omega|_{Z_{1,N} \cap Z_{2,N}} \simeq \omega|_{Z_{1,N} \cap Z_{2,N}} \]

induced by the operator $(b_1b_2)^S$ for some large positive integer $S$, it is not hard to see (cf. Lemma 4.2.2) that $\omega^{1+N\cdot(p-1)}|_{Z_{1,N} \cap Z_{2,N}}$ is a favorable sheaf – notice that the scheme $Z_{1,N} \cap Z_{2,N}$ is zero-dimensional. Therefore the second and the fourth non-zero terms of sequence (1.0.0.3) are favorable, while the third non-zero term – whose support has dimension one – might not be

\[ \omega^{1+N\cdot(p-1)} \]
favorable. For any positive integer $M$ we can write a resolution of $\omega^{1+N,(p-1)}_{Z_{1,N}}$ as:

$$0 \to \omega^{1+N,(p-1)}_{Z_{1,N}} \xrightarrow{h^M} \omega^{1+N,(p-1)+M,(p-1)}_{Z_{1,N}} \to \omega^{1+N,(p-1)+M,(p-1)}_{Z_{1,N}\cap Z_{2,M}} \to 0,$$

where the last non-zero term is favorable. For any positive integer $M'$ the operator $b^M_{M'}$ gives a Hecke twist-equivariant isomorphism of the middle term of the above sequence with the sheaf

$$\omega^{1+N,(p-1)+M,(p-1)+M', (1,p)}_{Z_{1,N}}.$$

We can choose $M$ and $M'$ so that (1.0.0.5) is a favorable sheaf (cf. Lemma 4.2.2): this follows from the fact that the interior of the positive cone spanned in $\mathbb{R}^2$ by the weights $(p, -1)$ and $(1, p)$ of the operators $h_2$ and $b_1$ contains the ample weight $(1, 1)$. We conclude that also the middle term of (1.0.0.4) is favorable. Repeating this argument for $\omega^{1+N,(p-1)}_{Z_{2,N}}$ we obtain by an induction process the desired resolution of $\omega^4(-D)$ by favorable sheaves. (We remark that for this algorithm to work, we also need to construct suitable resolutions for the sheaves with zero-dimensional support appearing in (1.0.0.3) and (1.0.0.4). We are avoiding this issue here).

The paper is organized as follows: in section 2 we recall a few facts about geometric Hilbert modular forms; in section 3 we briefly recall the definition of the partial Hasse invariants, and then we construct the operators $b_g$ and suitable liftings of them; in section 4 we introduce favorable weights and some weight shifting tricks that allow us to construct favorable resolutions.

We plan to address the following in future work:

- Assume $p$ is unramified in $F$ and let $\mathfrak{p}$ be a prime of $F$ above $p$. If $c$ is a non-zero cuspidal Hecke eigenclass of weight $\kappa$, and if the entries of $\kappa$ relative to the prime $\mathfrak{p}$ are all equal to one, then the Galois representation attached to $c$ is expected to be unramified at $\mathfrak{p}$. Proving unramifiedness of the representation is the main step necessary to make the modularity lifting results of [CG12a] and [CG12b] unconditional in the context of coherent cohomology of Hilbert modular varieties (cf. [CG12a], 3, for the case of modular curves).

- We plan to drop the condition that $p$ is unramified in the totally real field $F$, extending the results of this paper to the case in which $p$ is an arbitrary prime number not dividing the level $N$. This can be done by working with Pappas-Rapoport splitting models for Hilbert modular varieties, and suitably defined partial Hasse invariants and $b$-like operators.

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2. HILBERT-BLUMENTHAL MODULAR SCHEMES

We denote by $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$. We fix a rational prime $p$ and a field isomorphism $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$: this defines an embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_p$. Base changes of algebras and schemes will often be denoted by a subscript, if no confusion arises.

Let $F$ be a totally real subfield of $\overline{\mathbb{Q}}$ of degree $g > 1$, with ring of integers $\mathcal{O}_F$, unit group $U_F$, and totally positive units $U_F^+$. Denote by $\mathfrak{d}_F$ the different ideal of $F/\mathbb{Q}$ and by $d_F$ its norm.
Denote by \( \Sigma \) the set of embeddings of \( F \) in \( \overline{\mathbb{Q}} \). Fix a prime number \( p \) unramified in \( F \). Let \( \mathcal{C} := \{ c_1 = (1), c_2, \ldots, c_h, \ldots \} \) be a fixed set of representatives for the elements of the narrow class group of \( F \). We assume that all these fractional ideals are prime to \( p \).

2.1. Moduli spaces of HBAS. We gather here some facts about Hilbert modular schemes, following [KL05] and [DT04]. Cf. also [Gor02] and [Kat78].

Let \( S \) be a locally noetherian scheme. A Hilbert-Blumenthal abelian \( S \)-scheme (HBAS) with real multiplication by \( O_F \) is the datum of an abelian \( S \)-scheme \( X \) of relative dimension \( g \), together with a ring embedding \( O_F \to \text{End}_S X \). If \( X \) is a HBAS over \( S \) with real multiplication by \( O_F \), the dual abelian scheme \( X^\vee \) with its induced \( O_F \)-action is also a HBAS over \( S \) with real multiplication by \( O_F \) ([Rap78], 1.2).

Let \( c \in \mathcal{C} \) be a fractional ideal of \( F \), with cone of positive elements \( c^+ \). If \( X \) is a HBAS over \( S \) with real multiplication by \( O_F \), there is a natural injective \( O_F \)-linear map \( c \to \text{Hom}_{O_F}(X, X \otimes_{O_F} c) \). A \( c \)-polarization of \( X \) is an \( S \)-isomorphism \( \lambda : X^\vee \to X \otimes_{O_F} c \) of HBAS’s under which the symmetric elements (resp. the polarizations) of \( \text{Hom}_{O_F}(X, X^\vee) \) corresponds to the elements of \( c \) (resp. of \( c^+ \)) in \( \text{Hom}_{O_F}(X, X \otimes_{O_F} c) \).

For a positive integer \( N \), a \( \Gamma_{00}(N) \)-level structure on a HBAS \( X \) over \( S \) is an \( O_F \)-linear embedding of \( S \)-schemes \( i : \partial_F^{-1} \otimes_{\mathbb{Z}} \mu_N \to X \).

Assume \( N \geq 4 \) and denote by \( \mathcal{M}_\ell = \mathcal{M}_{\ell, N} \) the functor assigning to a locally noetherian scheme \( S \) over \( \text{Spec} \mathbb{Z}[(Nd_F)^{-1}] \) the set of isomorphism classes of triples \( (X, \lambda, i) \) consisting of a \( c \)-polarized HBAS \( (X, \lambda) \) over \( S \) with real multiplication by \( O_F \), together with a \( \Gamma_{00}(N) \)-level structure \( i \). The functor \( \mathcal{M}_\ell \) is represented by a scheme over \( \text{Spec} \mathbb{Z}[(Nd_F)^{-1}] \), also denoted \( \mathcal{M}_\ell \), which is smooth of relative dimension \( g \). By a result of Ribet, the fibers of \( \mathcal{M}_\ell \) over \( \text{Spec} \mathbb{Z}[(Nd_F)^{-1}] \) are geometrically irreducible ([Gor02], Ch. 3, 6.3). Notice that for any tuple \( (X, \lambda, i) / S \) as above, the sheaf \( \mathcal{L}ie(X/S) \) is a locally free \( O_F \otimes_{\mathbb{Z}} O_S \)-module of rank one, since \( d_F \) is invertible in \( S \) ([DP94], Corollary 2.9).

We define the \( \text{Spec}(\mathbb{Z}[(Nd_F)^{-1}]) \)-scheme

\[
\mathcal{M} := \bigsqcup_{c \in \mathcal{C}} \mathcal{M}_c,
\]

and we denote by \( A \to \mathcal{M} \) the universal HBAS over \( \mathcal{M} \).

2.1.1. Toroidal compactifications. For any ideal class \( c \in \mathcal{C} \) fix a rational polyhedral admissible cone decomposition \( \Phi_c \) for the isomorphism classes of \( \Gamma_{00}(N) \)-cusps of the \( \text{Spec}(\mathbb{Z}[(Nd_F)^{-1}]) \)-scheme \( \mathcal{M}_c \) ([DT04], 5). By loc.cit., Théorème 5.2, there exists a smooth proper scheme \( \mathcal{M}^\text{tor}_{c, \Phi_c} \) over \( \text{Spec}(\mathbb{Z}[(Nd_F)^{-1}]) \) containing \( \mathcal{M}_c \) as a fiberwise dense open subscheme. We shall abbreviate \( \mathcal{M}^\text{tor}_{c, \Phi_c} \) with \( \mathcal{M}^\text{tor}_c \), and we set \( \mathcal{M}^\text{tor} := \bigsqcup_{c \in \mathcal{C}} \mathcal{M}^\text{tor}_c \). The boundary \( D := \mathcal{M}^\text{tor} - \mathcal{M} \) is a relative simple normal crossing divisor on \( \mathcal{M}^\text{tor} \).

There exists a semi-abelian scheme \( \pi : A^\text{tor} \to \mathcal{M}^\text{tor} \) extending the universal abelian scheme \( A \to \mathcal{M} \) over \( \text{Spec}(\mathbb{Z}[(Nd_F)^{-1}]) \), which is unique up to isomorphisms restricting to the identity on \( A \); it is endowed with an \( O_F \)-action and an embedding \( \partial_F^{-1} \otimes_{\mathbb{Z}} \mu_N \to A^\text{tor} \) extending the corresponding data on \( A \). If \( e : \mathcal{M}^\text{tor}_c \to A^\text{tor}_c \) denotes the unit section of the semi-abelian scheme \( A^\text{tor} \) over \( \mathcal{M}^\text{tor}_c \), we set:

\[
\omega := e^* \Omega^1_{A^\text{tor} / \mathcal{M}^\text{tor}_c}.
\]

This is a locally free \(( O_F \otimes_{\mathbb{Z}} O_{\mathcal{M}^\text{tor}_c}) \)-module of rank one over \( \mathcal{M}^\text{tor}_c \). Its restriction to \( \mathcal{M}_c \) coincides with \( e^* \Omega^1_{A^\text{tor} / \mathcal{M}^\text{tor}_c} \), which we also denote by \( \omega \) if no confusion arises.
2.2. Geometric Hilbert modular forms. Let $F'$ denote the Galois closure of $F$ inside $\mathbb{Q}$, and let $\mathcal{O}_{F'}$ be its ring of integers. Fix an $\mathcal{O}_{F'}$-algebra $R$ and let

$$\kappa : (\text{Res}_{\mathbb{Z}}^R \mathbb{G}_m)_R \to \mathbb{G}_{m,R}$$

be an algebraic character defined over $R$. If $\kappa$ is the character induced by the norm map of $F/\mathbb{Q}$ we write $\kappa = \text{Nm}$. Since $R$ is an $\mathcal{O}_{F'}$-algebra, the group $(\text{Res}_{\mathbb{Z}}^R \mathbb{G}_m)_R$ is a split torus, and its $R$-rational character group has canonical generators $\{\chi_\tau : \tau \in \Sigma\}$ indexed by the set $\Sigma$ of embeddings of $F$ in $\mathbb{Q}$. The vector bundle $\omega_R$ decomposes over $\mathcal{M}_{R}^{\text{tor}}$ as $\omega_R = \bigoplus_{\tau \in \Sigma} \omega_{R,\tau}$, where $\omega_{R,\tau}$ is the invertible subsheaf of $\omega_R$ on which $\mathcal{O}_F$ acts via the composition of $\tau$ with the structure morphism $\mathcal{O}_{F'}(\mathcal{P}) \to R$. If $\kappa = \prod_{\tau \in \Sigma} \chi_{k_{\tau}}$ for some integers $k_{\tau}$, we write $\kappa = (k_{\tau})_{\tau \in \Sigma}$ and we define:

$$\omega_{R,\tau}^\kappa := \bigotimes_{\tau \in \Sigma} \omega_{R,\tau}^{\otimes k_{\tau}};$$

where the tensor product is over $\mathcal{O}_{M_{R}^{\text{tor}}}$. In particular $\omega_{R,\tau}^{\text{Nm}} = \text{det} \omega_{R} = \bigotimes_{\tau \in \Sigma} \omega_{R,\tau}$.

A (geometric) Hilbert modular form over $R$ of level $\Gamma_0(N)$ and weight $\kappa$ is an element of the $R$-module $H^0(\mathcal{M}_{R}^{\text{tor}}, \omega_{R,\tau}^\kappa)$. A (geometric) cuspidal Hilbert modular form over $R$ of level $\Gamma_0(N)$ and weight $\kappa$ is an element of the $R$-module $H^0(\mathcal{M}_{R}^{\text{tor}}, \omega_{R,\tau}^\kappa (-\Delta))$.

Notice that $H^0(\mathcal{M}_{R}^{\text{tor}}, \omega_{R,\tau}^\kappa)$ decomposes as the direct sum of the spaces $H^0(\mathcal{M}_{\tau}^{\text{tor}}, \omega_{\tau}^\kappa)$ for $\tau \in \mathcal{C}$, and we have an obvious notion of $\mathcal{C}$-polarized Hilbert modular forms. The K"ocher principle ([DT04], Théorème 7.1) guarantees that, if $g > 1$, we have:

$$(2.2.0.2) \quad H^0(\mathcal{M}_{R}^{\text{tor}}, \omega_{R,\tau}^\kappa) = H^0(\mathcal{M}_{\tau}^{\text{tor}}, \omega_{\tau}^\kappa).$$

In particular, the $R$-module $H^0(\mathcal{M}_{R}^{\text{tor}}, \omega_{R,\tau}^\kappa)$ is independent on the choice of toroidal compactification of $\mathcal{M}_{R}$.

Let $R'$ be an $R$-algebra and denote by $(X, \lambda, i, \eta)$ a tuple consisting of a $\mathcal{C}$-polarized HBAS $(X, \lambda, i)/R'$ with $\Gamma_0(N)$-level structure, together with the choice of a generator $\eta$ for the free rank one $\mathcal{O}_F \otimes_{\mathbb{Z}} R'$-module $H^0(X, \Omega^1_{X/R'})$. If no confusion arises, we will call such a tuple a (c-polarized) test object over $R'$. A $\mathcal{C}$-polarized Hilbert modular form over $R$ of level $\Gamma_0(N)$ and weight $\kappa$ can be interpreted as a rule $f$ which assigns to any $R$-algebra $R'$ and to any test object $(X, \lambda, i, \eta)$ over $R'$ an element $f(X, \lambda, i, \eta) \in R'$ in such a way that this assignment depends only on the isomorphism class of $(X, \lambda, i, \eta)$, is compatible with base change, and satisfies

$$f(X, \lambda, i, a^{-1}\eta) = \kappa(a) \cdot f(X, \lambda, i, \eta)$$

for all $a \in (\mathcal{O}_F \otimes_{\mathbb{Z}} R')^\times$ (cf. [Kat78]).

**Definition 2.2.1.** Assume that $R$ is an $\mathcal{O}_{F'}(\mathcal{P})$-algebra. We say that the weight $\kappa = (k_{\tau})_{\tau \in \Sigma}$ defined over $R$ is *paritious* if all the integers $k_{\tau}$ have the same parity; in this case, this common parity is called the *parity* of the weight. We say that $\kappa$ is *regular* if it is paritious and all the integers $k_{\tau}$ are larger than 1.

2.2.2. Hecke operators. Fix an $\mathcal{O}_{F'}(\mathcal{P})$-algebra $R$ and a weight $\kappa : (\text{Res}_{\mathbb{Z}}^R \mathbb{G}_m)_R \to \mathbb{G}_{m,R}$. Let $a$ be an integral ideal of $\mathcal{O}_F$ coprime with $pN$. Denote by $\mathcal{M}_R(a)$ the $R$-scheme representing the functor that takes a locally noetherian $R$-scheme $S$ to the set of isomorphism classes of tuples $(X, \lambda, i, [\varepsilon], C)$ satisfying the following conditions: there are $\varepsilon, \varepsilon' \in \mathcal{C}$ such that $(X, \lambda, i)$ is a $\mathcal{C}$-polarized HBAS over $S$ with $\Gamma_0(N)$-level structure; $[\varepsilon]$ is the $U^2_{F,N}$-orbit of an isomorphism $\varepsilon : a\mathcal{C} \to \varepsilon'$ identifying the positive cones on both sides (here $U_{F,N}$ denote the subgroup of $U_F$ consisting of units congruent to one modulo $N$); $C$ is an $\mathcal{O}_F$-stable closed subgroup $S$-scheme of $X$ such that
C1: \( i(b^{-1} \otimes_{\mathbb{Z}} \mu_N) \) is disjoint from \( C \), and

C2: étale locally on \( S \), the group scheme \( C \) is \( \mathcal{O}_F \)-linearly isomorphic to the constant group scheme \( \mathcal{O}_F/\mathfrak{a} \).

There are two finite maps \( \pi_1 : \mathcal{M}_R(\mathfrak{a}) \to \mathcal{M}_R \) and \( \pi_2 : \mathcal{M}_R(\mathfrak{a}) \to \mathcal{M}_R \) defined respectively by forgetting \( ([\varepsilon], C) \) and by quotienting \( (X, \lambda, i) \) by \( C \) ([KL05], 1.9-10). We obtain as in loc.cit., 1.11, a map \( T_{\mathfrak{a}} : \pi_2^*\omega_{R}^\kappa \to \pi_1^*\omega_{R}^\kappa \).

If \( \kappa' \) is another weight and \( \xi : \omega_{R}^\kappa \to \omega_{R}^{\kappa'} \) is a homomorphism of sheaves of \( (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{M}_R}) \)-modules, we say that \( \xi \) is equivariant with respect to the action of the Hecke operator \( T_{\mathfrak{a}} \) if the following diagram commutes:

\[
\begin{array}{ccc}
\pi_2^*\omega_{R}^\kappa & \xrightarrow{T_{\mathfrak{a}}} & \pi_1^*\omega_{R}^\kappa \\
\pi_2^*\xi & \uparrow \ & \uparrow \pi_1^*\xi \\
\pi_2^*\omega_{R}^{\kappa'} & \xrightarrow{T_{\mathfrak{a}}} & \pi_1^*\omega_{R}^{\kappa'}
\end{array}
\]

Applying \( \pi_{1*} \) to \( \pi_2^*\omega_{R}^\kappa \xrightarrow{T_{\mathfrak{a}}} \pi_1^*\omega_{R}^\kappa \) and taking the trace, we obtain for any integer \( j \geq 0 \) an endomorphism \( T_{\mathfrak{a}} \) : \( H^j(\mathcal{M}_R, \omega_{R}^\kappa) \to H^j(\mathcal{M}_R, \omega_{R}^{\kappa'}) \). The endomorphism \( T_{\mathfrak{a}} \) extends to \( \mathcal{M}_{tor}^\kappa \) and we will use the same symbol to denote its action on \( H^j(\mathcal{M}_{tor}^\kappa, \omega_{R}^\kappa) \).

Let \( S \) denote the set of maximal ideals of \( \mathcal{O}_F \) dividing \( pN \), and let \( T_S := R[T_{\mathfrak{a}} : \mathfrak{a} \text{ coprime with the primes in } S] \) be the commutative subalgebra of \( End_R(H^j(\mathcal{M}_{tor}^\kappa, \omega_{R}^\kappa)) \) generated by the Hecke operators \( T_{\mathfrak{a}} \) when \( \mathfrak{a} \) varies over the integral ideals of \( \mathcal{O}_F \) coprime with \( pN \). We call \( T_S \) the Hecke algebra in weight \( \kappa \) and of level \( \Gamma_0(N) \). Notice that there is no reference in our notation to the cohomological degree \( j \) at which the Hecke algebra is acting, or to the weight \( \kappa \): the context will make this clear to the reader. We will also denote by the symbol \( T_S \) the Hecke algebra acting on cohomology with coefficients in \( \omega_{R}^\kappa(-D) \).

For later use, we now describe the action of the Hecke operator \( T_{\mathfrak{a}} \) on \( H^0(\mathcal{M}_R, \omega_{R}^\kappa) \) (cf. [Hid06], 4.2.9). Fix a fractional ideal \( \mathfrak{c} \in \mathfrak{C} \) and let \( (X, \lambda, i, \eta) \) be a \( \mathfrak{c} \)-polarized test object as in 2.2 defined over an \( R \)-algebra \( R' \). Fix an \( \mathcal{O}_F \)-stable closed subgroup scheme \( C \) of \( X \) which is defined over \( R' \) and satisfies conditions C1 and C2 given above. The corresponding isogeny of abelian schemes \( \pi : X \to X' := X/C \) is étale. We let \( (X', \pi_0 \lambda, \pi_0 i) \) be the \( \mathfrak{c} \)-polarized HBAS obtained by quotienting \( (X, \lambda, i) \) by \( C \). Since \( \pi \) is an étale isogeny there is a canonical isomorphism \( \pi_0^*\Omega_{X'/R'} \simeq \Omega_{X/R'} \) whose inverse induces an \( \mathcal{O}_F \otimes_{\mathbb{Z}} R' \)-linear identification \( \pi_0^* : H^0(X, \Omega_{X/R'}) \to H^0(X', \Omega_{X'/R'}) \).

Let \( \mathfrak{c}' \) be the unique fractional ideal in \( \mathfrak{C} \) for which there is an \( \mathcal{O}_F \)-linear isomorphism \( \mathfrak{c} \mathfrak{a} \simeq \mathfrak{c}' \) preserving the positive cones on both sides, and let \( f \in H^0(\mathcal{M}_{\mathfrak{c}', R}, \omega_{R}^\kappa) \) be a \( \mathfrak{c}' \)-polarized Hilbert modular form. For any \( R \)-algebra \( R' \) and any \( \mathfrak{c} \)-polarized test object \( (X, \lambda, i, \eta) \) defined over \( R' \) we have:

\[
(2.2.2.1) \quad (T_{\mathfrak{a}} f)(X, \lambda, i, \eta) = \frac{1}{Nm^0_{\mathfrak{a}}(\mathfrak{a})} \sum_{\mathfrak{c}} f(X/C, \pi_0 \lambda, \pi_0 i, \pi_0^0 \eta)
\]

where \( C \) varies over the closed \( \mathcal{O}_F \)-stable subgroups of \( X \) satisfying conditions C1 and C2.

2.2.3. Comparison with Hilbert modular forms for \( \text{Res}^F_{Q} \mathcal{G}_{2/F} \). The scheme \( \mathcal{G}_{(1), Q} \) is the Shimura variety of \( \Gamma_{00}(N) \)-level associated to the subgroup \( \mathcal{G}^* \) of \( \text{Res}^F_{Q} \mathcal{G}_{2/F} \) consisting of matrices whose determinant lies in \( \mathbb{G}_{m/Q} \). A \( Q \)-form \( \text{Sh} \) of the Shimura variety of \( \Gamma_{00}(N) \)-level associated to \( \text{Res}^F_{Q} \mathcal{G}_{2/F} \) (over which classical Hilbert modular forms are defined) is identified with the quotient of \( \mathcal{M}_Q \) by the action of the group \( H := U_F^+/U_{F,N}^2 \). Here \( U_{F,N} \) denote the subgroup of
$U_F$ consisting of units congruent to one modulo $N$, and $H$ acts on $\mathcal{M}_\mathbb{Q}$ via its natural action on the polarizations of HBAS. If $A$ is an $F'$-algebra containing the square roots of the elements of $U_F^+$, then $H^0(\text{Sh}_A^{\text{tor}}, \omega_A^\epsilon) = H^0(\mathcal{M}_A^{\text{tor}}, \omega_A^\epsilon)^H$ (cf. [DT04] and [KL05] for more details). Following [KL05], 1.11.8, the Hecke operator $T_\sigma$ defined in 2.2.2 coincides with the projection
\[
H^0(\mathcal{M}_A^{\text{tor}}, \omega_A^\epsilon) \to H^0(\text{Sh}_A^{\text{tor}}, \omega_A^\epsilon) \quad f \mapsto \frac{1}{|H|} \sum_{\epsilon \in H} \epsilon \cdot f
\]
followed by the classical Hecke operator acting on spaces of Hilbert modular forms for $\text{Res}^F_{\mathbb{Q}} GL_2/F$. In particular, by work of Carayol ([Car86]), Taylor ([Tay89]), and Blasius-Rogawski ([BR89]) there exist Galois representations attached to geometric complex Hilbert cuspidal eigenforms of regular weight. Rogawski-Tunnel ([RT83]) and Ohta ([Oht84]) proved the existence of such representations in the case the weight is $Nm$, and Jarvis ([Jar97]) in the partial weight one case.

3. Weight shifting operators

We keep the notation introduced in the previous section. In particular, we maintain the assumptions that $N \geq 4$ and that $p$ is a prime unramified in $F$ and not dividing $N$.

3.1. Partial Hasse invariants. We start by recalling the definition of the partial Hasse invariants in the unramified case ([Gor01], [AG05]) and we present some properties of these operators.

Let $\mathbb{F}$ be a finite field of characteristic $p$ which is also an $O_{F',(p)}$-algebra, and denote by $\sigma : \mathbb{F} \to \mathbb{F}$ its absolute Frobenius automorphism. The chosen embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ allows us to identify $\Sigma$ with the set of $p$-adic embeddings $F \to \overline{\mathbb{Q}}_p$, and hence with the set of ring homomorphisms from $O_F/(p)$ to $\mathbb{F}$.

We use the subscript $\mathbb{F}$ to denote base change to $\text{Spec} \mathbb{F}$, e.g. $\mathcal{A}_{\mathbb{F}}^{\text{tor}}, \mathcal{M}^{\text{tor}}_{\mathbb{F}}$, etc.

The Verschiebung map $V : \mathcal{A}_{\mathbb{F}}^{\text{tor},(p)} \to \mathcal{A}_{\mathbb{F}}^{\text{tor}}$ preserves the $O_F$-action and it induces a morphism $\omega_{\mathbb{F}} \to \omega_{\mathbb{F}}^{(p)}$ of $O_F \otimes \mathbb{Z} \mathcal{M}^{\text{tor}}_{\mathbb{F}}$-modules. For each $\tau \in \Sigma$ this gives rise to an $O_{\mathcal{M}^{\text{tor}}_{\mathbb{F}}}$-linear map $\omega_{\mathbb{F},\tau} \to \omega_{\mathbb{F},\sigma^{-1}\tau}^{\otimes p}$ and therefore to a canonical section
\[
h_\tau \in H^0(\mathcal{M}^{\text{tor}}_{\mathbb{F}}, \omega_{\mathbb{F},\sigma^{-1}\tau}^{\otimes p} \otimes \omega_{\mathbb{F},\tau}^{\otimes -1}).
\]
The modular form $h_\tau$ is called the partial Hasse invariant at the place $\tau$. The product $h = \prod_{\tau \in \Sigma} h_\tau$ is a modular form of weight $Nm^{p-1}$ and it is called the total Hasse invariant.

Lemma 3.1.1. Let $R$ be an $\mathbb{F}$-algebra and suppose we are given:

1. A test object $(X, \lambda, i, \eta)$ defined over $R$;
2. An $O_F$-stable closed subgroup scheme $C$ of $X$ which is defined over $R$ and satisfies conditions C1 and C2 of paragraph 2.2.2.

Denote by $\pi : X \to X' := X/C$ the corresponding isogeny and let $(X', \pi_* \lambda, \pi_* i, \pi_*^0 \eta)$ be the associated test object, constructed as in 2.2.2. Then:
\[
h_\tau(X, \lambda, i, \eta) = h_\tau(X', \pi_* \lambda, \pi_* i, \pi_*^0 \eta).
\]

Proof. The partial Hasse invariant $h_\tau$ is characterized by the following property:
\[
V(\eta_\tau) = h_\tau(X, \lambda, i, \eta) \cdot \eta_\tau^{\otimes -1}\tau,
\]
where we denoted by $\eta_\tau$ the $\tau$-component of $\eta$ under the natural isomorphism $\omega_R = \bigoplus_{\tau \in \Sigma} \omega_{R,\tau}$.
Consider the natural commutative diagram:

\[
\begin{array}{ccc}
X(p) & \xrightarrow{V} & X \\
\pi(p) \downarrow & & \downarrow \pi \\
X'(p) & \xrightarrow{V} & X'
\end{array}
\]

Both vertical arrows are étale maps. We obtain therefore the commutative diagram:

\[
H^0(X, \Omega^1_{X/R}) \xrightarrow{V} H^0(X(p), \Omega^1_{X(p)/R}) \\
\simeq \downarrow \pi_*^0 & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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The Hodge filtration implies therefore the existence of a canonical trivialization over $\mathcal{M}$.

By passing to the $\tau$-isotypical parts, and using the fact that $\mathcal{O}_{\mathcal{M}_0/\mathcal{F}}$ is coprime with $\mathcal{O}_{\mathcal{M}_0}$, we obtain a nowhere vanishing section:

$$\mathcal{O}_{\mathcal{M}_0/\mathcal{F}, \tau} \cong \mathcal{O}_{\mathcal{M}_0/\mathcal{F}} \otimes \mathcal{O}_{\mathcal{M}_0/\mathcal{F}, \tau}.$$
Assume now that $R = \mathbb{F}$ is a finite field of characteristic $p$. Recall that we denoted by $i : Z_\tau \to \mathcal{M}_p$ the closed embedding of the divisor $Z_\tau$. Since $\omega_{\mathcal{F},\tau}$ is canonically a subsheaf of $\mathcal{H}_{\text{dr},\mathcal{F},\tau}$ we obtain an isomorphism of sheaves over $Z_\tau$:

$$i^* \left( \mathcal{H}_{\text{dr},\mathcal{F},\tau}^1/\omega_{\mathcal{F},\tau} \right) \simeq i^* \omega_{\mathcal{F},\sigma-1,\tau}^{\sigma_p}.$$

Using (3.2.1.2) we obtain a canonical trivialization:

$$(3.2.2.2) \quad i^* \left( \omega_{\mathcal{F},\sigma-1,\tau}^{\sigma_p} \otimes \omega_{\mathcal{F},\tau} \right) = \mathcal{O}_{Z_\tau}.$$

For simplicity of notation, we now drop the inverse image functor $i^*$ from the cohomology groups we consider, and this shall not cause any ambiguity.

**Definition 3.2.3.** We denote by $b_\tau$ the canonical nowhere vanishing section of the sheaf $\omega_{\mathcal{F},\sigma-1,\tau}^{\sigma_p} \otimes \omega_{\mathcal{F},\tau}$ over $Z_\tau$ induced by the trivialization (3.2.1.2) or (3.2.2.2):

$$b_\tau \in H^0(Z_\tau, \omega_{\mathcal{F},\sigma-1,\tau}^{\sigma_p} \otimes \omega_{\mathcal{F},\tau}).$$

We remark that since each stratum $Z_\tau$ is disjoint from the boundary $D$ of the toroidal compactification of $\mathcal{M}_\mathbb{F}$, we have $H^i(Z_\tau, \omega_\mathbb{F}^\kappa) = H^i(Z_\tau, \omega_\mathbb{F}(D))$ for any weight $\kappa$.

**Corollary 3.2.4.** For any weight $\kappa$ and for any integer $i \geq 0$ there is a Hecke-equivariant isomorphism of $\mathbb{F}$-vector spaces:

$$b_\tau : H^i(Z_\tau, \omega_\mathbb{F}^\kappa)[1] \xrightarrow{\cong} H^i(Z_\tau, \omega_\mathbb{F}^{\sigma_p} \otimes \omega_{\mathcal{F},\tau}).$$

**Proof.** Recall that the action of the Hecke algebra on $H^0(Z_\tau, \omega_{\mathcal{F},\sigma-1,\tau}^{\sigma_p} \otimes \omega_{\mathcal{F},\tau})$ is defined by virtue of the second part of Proposition 3.1.3.

By (3.2.1.2) the operator $b_\tau$ is obtained by applying the Kodaira-Spencer isomorphism to the image in $H^0(Z_\tau, \omega_{\mathcal{F},\sigma-1,\tau}^{\sigma_p} \otimes \omega_{\mathcal{F},\tau}^{\sigma_p-1} \otimes J_\tau / J_\tau^2)$ of the partial Hasse invariant $h_\tau$. Consequently, the result follows from Corollary 3.1.2 combined with (3.2.1.1). Since $b_\tau$ is nowhere vanishing on $Z_\tau$, the map induced by multiplication is an isomorphism. \hfill \Box

**Remark 3.2.5.** The operators $b_\tau$ can be interpreted as analogues of the operator induced on mod $p$ elliptic modular forms by multiplication by the Eisenstein series $E_{p+1}$ (cf. [Ser73], 1.4, and [SD73], 3). Assume that $p > 3$ and denote by $X(N)$ the compactified modular curve over $\mathbb{F}$ of principal level $N \geq 4$ prime to $p$. Denote by $X_{ss}$ its supersingular locus and let $k > p + 1$. In [Rob80], Robert shows that multiplication by the Eisenstein series $E_{p+1}$ induces a Hecke equivariant isomorphism:

$$B : H^0(X_{ss}, \omega^{\otimes k})[1] \to H^0(X_{ss}, \omega^{\otimes k+(p+1)}).$$

Notice that $B$ does not coincide with the theta operator $\theta = \frac{d}{dq}$, but we have the relation:

$$\theta|_{H^0(X_{ss}, \omega^{\otimes k})[1]} = \frac{k}{12} \cdot B.$$

In particular the restriction of the theta operator to $H^0(X_{ss}, \omega^{\otimes k})$ is identically zero when the weight $k$ is divisible by $p$. The operators $b_\tau$ that we have constructed are closely related to the partial theta operators considered in [Kat78] and [AG05], but have the crucial property of being nowhere vanishing over suitable strata of the Hilbert modular variety.
3.3. Construction of liftings. We assume from now on that $p$ is inert in $F$: this additional assumption is only meant to simplify the notation.

As above, we let $\mathbb{F}$ be a finite field of characteristic $p$ which is also an $O_{E^\flat, (p)}$-algebra, and denote by $\sigma : \mathbb{F} \to \mathbb{F}$ its absolute Frobenius automorphism. We continue to identify $\Sigma$ with the set of embeddings of $O_{E^\flat}/(p)$ into $\mathbb{F}$. Via this identification, we can label the elements of $\Sigma$ as $\tau_1, ..., \tau_g$ so that

$$\sigma^{-1} \circ \tau_i = \tau_{i-1}$$

for all $i$, with the convention that $\tau_1$ stands for $\tau_{1 \pmod{g}}$. We will often identify the set $\Sigma$ with the set $\{1, ..., g\}$.

Fix a (large enough) finite extension $E$ of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$. Denote by $\varpi_E$ a choice of uniformizer of the ring of integers $O_E$ of $E$ and assume that $O_E/(\varpi_E) = \mathbb{F}$. Fix an integer $m \geq 1$ and set

$$R_m := O_E/(\varpi_E^m).$$

Let $N \geq 4$ be an integer not divisible by $p$ and assume as before that each fractional ideal in the set $\mathcal{C} = \{c_1, ..., c_k\}$ is prime to $p$.

In this section, $\mathcal{M}$ will denote the base change of the the Hilbert modular scheme $(\mathbb{2.1.0.6})$ to $\text{Spec } R_m$. If $\pi : \mathcal{A} \to \mathcal{M}$ is the universal family over $\mathcal{M}$, we set $\omega = \pi^* \Omega^1_{\mathcal{A}/\mathcal{M}}$ and $\omega_i := \omega_{\tau_i}$. The base changes of these objects to $\text{Spec } \mathbb{F}$ are denoted by $\overline{\mathcal{M}}, \overline{\mathcal{A}}, \overline{\varpi},$ and $\overline{\varpi}_i$ respectively. The partial Hasse invariant for the place $\tau_i$ will be denoted $h_i$: it is an element of $H^0(\overline{\mathcal{M}}, \overline{\varpi}_i^{\otimes p} \otimes \overline{\varpi}_i^{\otimes -1})$ and its zero locus is the reduced divisor $Z_i \subset \overline{\mathcal{M}}$. The symbols $\mathcal{M}_{\text{tor}}, \overline{\mathcal{M}}_{\text{tor}}, \mathcal{A}_{\text{tor}}, \overline{\mathcal{A}}_{\text{tor}}$ have the obvious meanings.

Let $\{e_1, ..., e_g\}$ be the standard $\mathbb{Z}$-basis of $\mathbb{Z}^g$ and set, for any integer $i$ such that $1 \leq i \leq g$:

$$p_i := pe_{i-1} - e_i, \quad q_i := pe_{i-1} + e_i,$$

where all the subscripts are taken modulo $g$. If $J$ is a subset of $\{1, ..., g\}$ we also set:

$$p_J = \sum_{i \in J} p_i, \quad q_J = \sum_{i \in J} q_i,$$

with the convention that the sum is the zero tuple $0$ if $J$ is empty.

Since $(\text{Res}_{\mathbb{Z}}^O_O G_m)_{R_m}$ is a split torus, the weight of a Hilbert modular form over $R_m$ is of the form $\kappa = \prod_{i=1}^g \lambda_i^{k_i}$ (cf. $(\mathbb{2.2})$). We set in this case $\mathbb{K} = \sum_i k_i e_i \in \mathbb{Z}^g$ and we denote the invertible sheaf $\omega^\kappa$ on $\mathcal{M}$ (or on $\mathcal{M}_{\text{tor}}$) by $\omega^\mathbb{K}$, so that

$$\omega^\mathbb{K} = \bigotimes_{i=1}^g \omega_i^{\otimes k_i}.$$

Similarly we denote $\overline{\omega}^\kappa$ by $\overline{\omega}^\mathbb{K}$.

3.3.1. Liftings of the partial Hasse invariants. Fix a positive integer $M$ divisible by $p^{m-1}$. Let $\mathcal{U} \subset \mathcal{M}_{\text{tor}}$ be an open affine subscheme of $\mathcal{M}_{\text{tor}}$. The restriction $h_{i, M}$ of the $i$th partial Hasse invariant to $\mathcal{U} \subset \text{Spec } R_m$ can be lifted to an element $\tilde{h}_{i, M}$ in $H^0(\mathcal{U}, \omega^{\mathbb{K}})$. Since $p^{m-1}$ divides $M$, the $M$th power of $h_{i, M}$ is independent on the choice of the particular lift $\tilde{h}_{i, M}$.

We deduce that the sections $\{\tilde{h}_{i, M}\}_{\mathcal{U} \subset \mathcal{M}_{\text{tor}}}$, when $\mathcal{U}$ varies over an open affine covering $\mathcal{U}$ of $\mathcal{M}_{\text{tor}}$, glue together into a global section

$$\tilde{h}_{i, M} \in H^0(\mathcal{M}_{\text{tor}}, \omega^{MP_i}).$$

Notice that $\tilde{h}_{i, M}$ does not depend on the choice of affine covering $\mathcal{U}$, and it is the only lift of $h_{i, M}$ to $H^0(\mathcal{M}_{\text{tor}}, \omega^{MP_i})$ which locally is the $M$th power of a lift of $h_i$. We clearly have $\tilde{h}_{i, M_1} \cdot \tilde{h}_{i, M_2} = \tilde{h}_{i, M_1 + M_2}$ for any positive integers $M_1, M_2$ divisible by $p^{m-1}$.
Lemma 3.3.2. Fix a positive integer $M$ divisible by $p^{m-1}$. For any weight $k \in \mathbb{Z}^g$, multiplication by $\tilde{h}_{i,M}$ induces a Hecke-equivariant embedding:

$$\tilde{h}_{i,M} : H^0(\mathcal{M}^{\text{tor}}, \omega^M) \hookrightarrow H^0(\mathcal{M}^{\text{tor}}, \omega^{k+Mp_i}).$$

Proof. It is enough to work over $\mathcal{M}$. Let $R$ be an $R_{tor}$-algebra and fix a test object $(X, \lambda, i, \eta)$ defined over $R$. If $C \subset X$ is an $\mathcal{O}_F$-stable closed subgroup scheme of $X$ satisfying (C1) and (C2) of paragraph 2.2.2, denote by $\pi : X \to X' := X/C$ the corresponding étale isogeny, and by $\pi^0_\eta : H^0(X, \Omega^1_{X/R}) \to H^0(X', \Omega^1_{X'/R})$ the (inverse of the) induced isomorphism on the differentials.

Working locally on $\text{Spec } R$, we may assume that $(X/C, \pi_*\lambda, \pi_*\eta)$ (resp. $(X, \lambda, i)$) is classified by a map from $\text{Spec } R$ to an open affine subscheme of $\mathcal{M}$. If we let $\tilde{h}_i$ (resp. $\tilde{h}_i$) denote a local lift of $h_i$ over such an open affine subscheme of $\mathcal{M}$, then we have:

$$\tilde{h}_{i,M}(X/C, \pi_*\lambda, \pi_*\eta, \pi_*\eta^0) = \left[\tilde{h}_i(X/C, \pi_*\lambda, \pi_*\eta, \pi_*\eta^0)\right]^M,$$

$$\tilde{h}_{i,M}(X, \lambda, i, \eta) = \left[\tilde{h}_i(X, \lambda, i, \eta)\right]^M.$$ By Lemma 3.1.1 we also have:

$$\tilde{h}_i(X/C, \pi_*\lambda, \pi_*\eta, \pi_*\eta^0) \equiv \tilde{h}_i(X, \lambda, i, \eta) \pmod{p}.$$ Since $p^{m-1}$ divides $M$, this implies

(3.2.1) $$\tilde{h}_{i,M}(X/C, \pi_*\lambda, \pi_*\eta, \pi_*\eta^0) = \tilde{h}_{i,M}(X, \lambda, i, \eta).$$

The result of the lemma follows from the definition of the Hecke operators. $\square$

Notation 3.3.3. We introduce some notation for later convenience. If $M = (M_1, ..., M_g)$ is a $g$-tuple of non-negative integers all divisible by $p^{m-1}$, define:

$$\tilde{h}_M := \prod_{i=1}^g \tilde{h}_{i, M_i},$$

with the convention that $\tilde{h}_{i,0} := 1$. We also define:

$$M_p := \sum_{i=1}^g M_i p_i.$$ Then $\tilde{h}_M$ is a modular form of weight $M_p$.

If $M_i > 0$, denote by $Z_{M_i, e_i}$ the closed subscheme of $\mathcal{M}$ (or of $\mathcal{M}^{\text{tor}}$) defined by the vanishing of $\tilde{h}_{M_i, e_i} = \tilde{h}_{i, M_i}$; we set $Z_0 e_i := \mathcal{M}$ (or $\mathcal{M}^{\text{tor}}$). Define:

$$Z_M := \bigcap_{i=1}^g Z_{M_i, e_i}.$$ If $\mathcal{F}$ is a sheaf on $\mathcal{M}^{\text{tor}}$, we denote its restriction to $Z_M$ by $\mathcal{F}|_{Z_M}$ or, if no confusion arises, by $\mathcal{F}$ again.

Corollary 3.3.4. Let $M = (M_1, ..., M_g)$ be a $g$-tuple of non-negative integers all divisible by $p^{m-1}$. For any weight $k \in \mathbb{Z}^g$, multiplication by $\tilde{h}_M$ induces a Hecke-equivariant embedding:

$$\tilde{h}_M : H^0(\mathcal{M}^{\text{tor}}, \omega^k) \hookrightarrow H^0(\mathcal{M}^{\text{tor}}, \omega^{k+M_p}).$$

We give the following:

Definition 3.3.5. For $M = (M_1, ..., M_g) \in \mathbb{Z}^g_{\geq 0}$, the support of $M$ is the subset

$$|M| := \{i \mid M_i = 0\}$$

of $\{1, \ldots, g\}$. The dimension of $M$ is defined to be $\dim(M) := \#|M|$.
The reason for this definition of support is that \( \dim(M) \) equals the dimension of the reduced subscheme of \( Z_M \), or equivalently the dimension of the set theoretical support of the sheaf \( \omega_{|Z_M}^k \). By abuse of language, we also say that \( \omega_{|Z_M}^k \) has dimension \( \dim(M) \).

3.3.6. Liftings of the operators \( b_p \). Similarly to what we have done for powers of the partial Hasse invariants, lifts of some powers of the operators \( b_i := b_{r_i} \in H^0(\mathbb{Z}_i, \mathcal{O}^q) \) defined in \( 3.2 \) can be constructed.

Fix two positive integers \( M \) and \( T \) divisible by \( p^{m-1} \) and such that \( T > M + p^{m-1} \). The closed embedding \( \mathbb{Z}_e \to \mathbb{Z} \to Z_{Me_0} \) over \( \text{Spec} \, R_m \) induces a map in cohomology:

\[
H^0(Z_{Me_0}, \omega_q^T) \to H^0(\mathbb{Z}_{e_0}, \mathcal{O}^q).
\]

By covering \( Z_{Me_0} \) with affine open subschemes, we deduce as in \( 3.3.1 \) the existence of an element

\[
\tilde{b}_{i,M,T} \in H^0(Z_{Me_0}, \omega_q^T)
\]

uniquely characterized by the following properties: \( \tilde{b}_{i,M,T} \) lifts \( b_i^T \in H^0(\mathbb{Z}_{e_0}, \mathcal{O}^q) \), and locally on \( Z_{Me_0} \) it is the \( T \)th power of a lift of \( b_i \). Notice that \( \tilde{b}_{i,M,T} \) is also nowhere vanishing on \( Z_{Me_0} \).

Using the construction of the operator \( b_i \) given in \( 3.2.1 \), we see that if \( (X, \lambda, i, \eta) \) is a test object defined over an \( \mathbb{F}_p \)-algebra \( R \) such that the triple \( (X, \lambda, i) \) determines a point in \( Z_i(R) \), and if \( C \) is an \( \mathcal{O}_F \)-stable closed subgroup scheme of \( X \) that satisfies conditions \( C1 \) and \( C2 \) of paragraphs \( 2.2.2 \) for some prime-to-\( pN \) integral ideal \( \mathfrak{a} \) of \( \mathcal{O}_F \), then

\[
b_i(X', \pi, \lambda, \pi, \eta) = \frac{1}{\text{Nm}_{\mathfrak{a}}^F} : b_i(X, \lambda, i, \eta),
\]

where \( \pi : X \to X' := X/C \) denotes the corresponding isogeny (cf. Lemma \( 3.1.1 \)). Therefore multiplication by \( \tilde{b}_{i,M,T} \) induces a Hecke-equivariant isomorphism:

\[
H^j_{\mathfrak{a}}(Z_{Me_0}, \omega_q^k)[T] \to H^j(Z_{Me_0}, \omega_{T\mathfrak{a}}^k)
\]

for all \( j \geq 0 \).

Notation 3.3.7. Analogously to what we have done for lifts of the partial Hasse invariants, we introduce some simplifying notation. Let \( M = (M_1, ..., M_g) \) and \( T = (T_1, ..., T_g) \) be two \( g \)-tuples of non-negative integers all divisible by \( p^{m-1} \). Assume that if \( M_i = 0 \) then \( T_i = 0 \), and that if \( M_i > 0 \), then either \( T_i = 0 \) or \( T_i > M_i + p^{m-1} \). We set:

\[
\tilde{b}_{M,T} := \prod_{i=1}^g \tilde{b}_{i,M_i,T_i},
\]

with the convention:

\[
\tilde{b}_{i,M_i,T_i} := 1, \quad \text{if} \ M_i T_i = 0.
\]

Under our assumptions, \( \tilde{b}_{M,T} \) is a nowhere vanishing section of

\[
H^0(Z_M, \omega_{\sum T_i \mathfrak{a}_i}).
\]

When no ambiguity arises, we write \( \tilde{b}_T := \tilde{b}_{M,T} \).

We remark that when \( M = 0 \) our conventions imply that \( \tilde{b}_{M,T} \) is the identity function.

Corollary 3.3.8. Let \( M, T \in (p^{m-1}Z_{\geq 0})^g \) be such that if \( M_i = 0 \) then \( T_i = 0 \), and if \( M_i > 0 \), then either \( T_i = 0 \) or \( T_i > M_i + p^{m-1} \). For any weight \( k \in \mathbb{Z}_g \) and any \( j \geq 0 \) there is a Hecke-equivariant isomorphism of \( R_m \)-modules:

\[
\tilde{b}_T : H^j(Z_M, \omega^k(-D))[T] \cong H^j(Z_M, \omega_{k+\sum T_i \mathfrak{a}_i}(-D))
\]
for some integer \( t \).

4. Pseudo-representations attached to torsion Hilbert modular forms

4.1. Hecke modules of Galois type. We give a general framework for the Hecke actions on a module to give rise to pseudo-representations. Pseudo-representations were introduced by A. Wiles in the two-dimensional case ([Wiles]), and by R. Taylor in general settings ([Tay]).

Let \( G \) be a topological group and \( R \) a topological ring. Fix a positive integer \( d \) and denote by \( \mathfrak{S}_{d+1} \) the symmetric group on \( d+1 \) letters, and by \( \mathfrak{S} \) its signature character. An \( R \)-valued pseudo-representation of \( G \) of dimension \( d \) is a continuous function \( \tau : G \to R \) such that:

1. \( \tau(1) = d \),
2. \( \tau(g_1 g_2) = \tau(g_2 g_1) \) for all \( g_1, g_2 \in G \), and
3. \( d \) is the smallest positive integer such that for all \( g_1, \ldots, g_{d+1} \in G \) we have

\[
\sum_{\sigma \in \mathfrak{S}_{d+1}} \text{sign}(\sigma) \cdot \tau_\sigma(g_1, \ldots, g_{d+1}) = 0,
\]

where \( \tau_\sigma : G^{d+1} \to R \) is the function defined as follows: if \( \sigma \in \mathfrak{S}_{d+1} \) has disjoint cycle decomposition \( \sigma = (i_1^{(1)} \ldots i_{r_1}^{(1)}) \cdots (i_1^{(s)} \ldots i_{r_s}^{(s)}) \), then:

\[
\tau_\sigma(g_1, \ldots, g_{d+1}) := \tau(g_1^{(1)} \ldots g_{i_1}^{(1)} \ldots g_1^{(s)} \ldots g_{i_s}^{(s)}).
\]

Construction 4.1.1. Let \( R \) be a topological ring and \( G \) a finite group. Let \( \mathcal{R}_G^{\text{ps}} \) denote the universal ring for the two-dimensional pseudo-representations of \( G \) with values in an \( R \)-algebra: it is the quotient of the polynomial ring \( R[t_g : g \in G] \) by the ideal generated by

\[
t_1 - 2, \ t_1 g_1 - t_2 g_1, \ t_1 g_2 - t_2 g_2, \ \text{for } g_1, g_2 \in G, \text{ and } \]

\[
t_1 t_1 g_1 + t_2 g_1 g_2 + t_3 g_1 g_2 - t_1 t_1 g_1 g_2 - t_2 t_1 g_1 g_2 - t_3 t_1 g_1 g_2 \ \text{for } g_1, g_2, g_3 \in G.
\]

The natural map \( G \to \mathcal{R}_G^{\text{ps}} \) given by \( g \mapsto t_g \) is the universal two-dimensional pseudo-representation of \( G \) with values in an \( R \)-algebra. The construction of \( \mathcal{R}_G^{\text{ps}} \) is functorial in \( G \): if \( \phi : G \to G' \) is a group homomorphism, the \( R \)-algebra map \( R[t_g : g \in G] \to R[t_{g'} : g' \in G'] \) induced by the assignment \( t_g \mapsto t_{\phi(g)} \) factors via a homomorphism \( \phi_\mathcal{R}^{\text{ps}} : \mathcal{R}_G^{\text{ps}} \to \mathcal{R}_{G'}^{\text{ps}}. \) In particular, if \( \phi \) is surjective so is \( \phi_\mathcal{R}^{\text{ps}}. \)

Assume now that \( G \) is a profinite group and that \( G = \varprojlim \{G_i\} \) for some projective system \( \{G_i\} \) of finite groups. We define the universal ring for the continuous two-dimensional pseudo-representations of \( G \) with values in topological \( R \)-algebras to be:

\[
\mathcal{R}_G^{\text{ps}} := \varprojlim_i \mathcal{R}_{G_i}^{\text{ps}},
\]

where the maps in the inverse system arise from functoriality. The ring \( \mathcal{R}_G^{\text{ps}} \) is a topological \( R \)-algebra endowed with the projective limit topology; in general it is not Noetherian.

Let \( K \subset \mathbb{Q} \) be a number field and fix a finite set \( S \) of places of \( K \). We assume that \( S \) contains all the archimedean places of \( K \). Let \( G_{K,S} \) denote the Galois group of the maximal extension of \( K \) inside \( \mathbb{Q} \) that is unramified outside \( S \). The profinite group \( G_{K,S} \) satisfies Mazur’s finiteness condition.

Let \( \mathbb{T}_{univ} = R[T_\lambda : \lambda \notin S] \) denote the universal Hecke algebra, i.e., the polynomial algebra over \( R \) whose variables are indexed by the places of \( K \) outside \( S \). There is a natural homomorphism of \( R \)-algebras:

\[
\mathbb{T}_{univ} \to \mathcal{R}_{G_{K,S}}^{\text{ps}}, \ T_\lambda \mapsto t_{\text{Frob}_\lambda}, \ \text{for } \lambda \notin S.
\]

This homomorphism has dense image by Chebotarev density theorem.
Definition 4.1.2. A bounded complex in the category of $T_{S}^\text{univ}$-modules is said to be of Galois type if the action of $T_{S}^\text{univ}$ on each term of the complex factors through the image of $T_{S}^\text{univ} \to R_{G,K,S}^{ps}$ and extends by continuity to an action of $R_{G,K,S}^{ps}$.

For any integer $n$ and any $T_{S}^\text{univ}$-module $M$ define the $n$th Tate twist $M[n]$ of $M$ as follows: the underlying $R$-modules of $M$ and $M[n]$ coincide, but the action $\ast$ of $T_{S}^\text{univ}$ on $m \in M[n]$ is defined as:

$$T_{\lambda} \ast m := (\text{Nm}_{Q}^{K}\lambda)^{n} \cdot T_{\lambda}m, \quad \lambda \notin S,$$

where $T_{\lambda}m$ denotes the action of $T_{\lambda}$ on $m \in M$, and $\text{Nm}_{Q}^{K}\lambda$ is the norm of $\lambda$ with respect to the extension $K/Q$.

Proposition 4.1.3. Let $M$ be a $T_{S}^\text{univ}$-module of Galois type and denote by $\tau_{M}^{ps} : G_{K,S} \to \text{End}_{R}(M)$ the attached two-dimensional pseudo-representation.

1. If $\text{End}_{R}(M)$ is an algebraically closed field of characteristic zero, or of characteristic larger than 2, then $\tau_{M}^{ps}$ is the trace of a uniquely determined semisimple continuous representation $\rho_{M} : G_{K,S} \to GL_{2}(\text{End}_{R}(M))$.
2. If $n$ is any integer, then $M[n]$ is of Galois type.
3. Let $N$ be another $T_{S}^\text{univ}$-module of Galois type and let $f : M \to N$ be a continuous $T_{S}^\text{univ}$-linear homomorphism. Then the kernel and the cokernel of $f$ are of Galois type.

In particular, all cohomology groups of a complex of $T_{S}^\text{univ}$-modules of Galois type are of Galois type.

Proof. A proof of the first statement can be found in [Tay91] (for the characteristic zero case) and in [Rou96] (for the general case). The other statements are straightforward.

In general, an extension of $T_{S}^\text{univ}$-modules of Galois type is not of Galois type. This is the essential difficulty we will need to overcome.

4.2. Favorable weights. We maintain the assumptions and the notation from sections 2 and 3. In particular we have fixed an integer $N \geq 4$ and a prime $p$ inert in the totally real field $F$ and not dividing $N$. We let $S$ be the set consisting of the archimedean places of $F$ together with the places above $pN$. In this paragraph, by a Hecke module we mean a finitely generated $S$-module over $S$ univ endowed with an action of the universal Hecke algebra $T_{S}^\text{univ}$. A Hecke module is said to be of Galois type if it satisfies the condition of Definition 4.1.2 relatively to the number field $F$.

For any $k \in \mathbb{Z}^{g}$, the $R_{m}$-modules $H^{i}(M_{\text{tor}}, \omega^{k}(-D))$ are Hecke modules, with $T_{S}^\text{univ}$ acting via the Hecke algebra $T_{S}$ defined in 2.2.2. If $p^{m-1}$ divides an integer $M \geq 0$, the map induced by multiplication by $h_{i,M}$ is Hecke equivariant (Lemma 3.3.2), so that for any $M \in (p^{m-1}\mathbb{Z}_{\geq 0})^{g}$ each $H^{i}(Z_{M}, \omega^{k}(-D))$ is also a Hecke module.

We give the following important definition:

Definition 4.2.1. For $M \in (p^{m-1}\mathbb{Z}_{\geq 0})^{g}$ and $k \in \mathbb{Z}^{g}$, we say that $k$ is a favorable weight with respect to $M$ if:

- $H^{0}(Z_{M}, \omega^{k}(-D))$ is of Galois type, and
- $H^{i}(Z_{M}, \omega^{k}(-D)) = 0$ for all $i > 0$.

In this case we also say that $\omega^{k}(-D)|Z_{M}$ is a favorable sheaf.
**Lemma 4.2.2.** For any weight $k \in \mathbb{Z}^g$, there is an integer $n_0 = n_0(k)$ such that for any $n \geq n_0$ and any $i > 0$ we have:

$$
H^i(\mathcal{M}_{\text{tor}}^*, \omega^{k+n-1}(-D)) = 0,
$$

$$
H^i(\mathcal{M}_{\text{tor}}^*, \omega^{k+n-1}_E(-D)) = 0.
$$

**Proof.** Let $\mathcal{M}_t$ be the arithmetic minimal compactification of $\mathcal{M}$ (cf. [Cha90] and the modifications described in [KL05] or [DT04] to work with $\Gamma_0(N)$ level structure). Denote by $\pi$ the natural proper morphism $\mathcal{M}^\text{tor} \to \mathcal{M}$ of $\text{Spec} \mathbb{F}$-schemes. We denote as usual by $\omega$ the sheaf of invariant 1-differentials of $\mathcal{M}^\text{tor}$; its determinant $\det \omega = \omega^1$ descends to an ample invertible sheaf $\pi_*\omega^1$ on $\mathcal{M}$, which we denote by $\omega^1_\text{min}$, and which satisfies $\pi_*\omega^1_\text{min} = \omega^1$. The sheaf $\omega^1_\text{min}$ is independent on the choice of toroidal compactification ([KL05], 1.8.2). (We remark en passant that in general the invertible sheaf $\omega^k$ on $\mathcal{M}^\text{tor}$ might not descend to an invertible sheaf on $\mathcal{M}$.) We set $\omega^{m-1}_\text{min} := (\omega^1_\text{min})^\otimes m$.

By Theorem 8.2.1.3 of [Lan13], for any $q > 0$ and any weight $k' \in \mathbb{Z}^g$ we have $R^q\pi_*(\omega^{k'}(-D)) = 0$. For any integers $i, m > 0$ we have therefore:

$$
H^i(\mathcal{M}^\text{tor}, \omega^k \otimes \omega^{m-1}(-D)) \simeq H^i(\mathcal{M}, \pi_*(\omega^k(-D) \otimes \pi_*\omega^{m-1}_\text{min})) \simeq H^i(\mathcal{M}^*, \pi_*(\omega^k(-D) \otimes \omega^{m-1}_\text{min})).
$$

Since $\pi$ is proper, $\pi_*(\omega^k(-D))$ is coherent and therefore the ampleness of $\omega^1_\text{min}$ implies that there exists an integer $n_0 > 0$ (depending on $k$) such that for all $i > 0$ and for all $n \geq n_0$

$$
H^i(\mathcal{M}^*, \pi_*(\omega^k(-D) \otimes \omega^{m-1}_\text{min})) = 0.
$$

We conclude that $H^i(\mathcal{M}^\text{tor}, \omega^{k+n-1}(-D)) = 0$. The two statements in the lemma then follow from Nakayama’s lemma. \(\square\)

### 4.3. Weight shifting tricks.

We define a set $\Delta$ of weights that will play a special role in the rest of the paper.

**Definition 4.3.1.** We denote by $\Delta$ the set of tuples $k \in (\mathbb{Z}_{>0})^g$ for which $H^{>0}(\mathcal{M}^\text{tor}, \omega^k(-D)) = 0$.

Lemma 4.2.2 states that for any weight $k \in \mathbb{Z}^g$ there is a positive integer $n_0 = n_0(k)$ such that for all $n \geq n_0$ we have $k + n \cdot 1 \in \Delta$.

In order to guarantee the existence of Galois representations associated to Hilbert modular forms in characteristic zero, we will often require that the weights we consider are paritious, and sometimes even regular (cf. Definition 2.2.1).

We introduce the following notation: if $M = (M_1, ..., M_g)$ is a $g$-tuple of integers and $J$ is a subset of $\Sigma = \{1, ..., g\}$, we set

$$M_J := \sum_{i \in J} M_i e_i,$$

with the convention that a summation over the empty set equals the zero tuple.

**Lemma 4.3.2.** Let $M = (M_1, ..., M_g)$ be a $g$-tuple of non-negative integers all divisible by $p^{m-1}$. Let $k = (k_1, ..., k_g) \in \mathbb{Z}^g$ be a regular weight and fix a subset $J$ of $\{1, ..., g\}$. Assume that the triple $(M, k, J)$ satisfies the following condition:

(*) if $k' \in \mathbb{Z}^g$ is such that $|k'_i - k_i| \leq p \cdot \#J \cdot \max\{M_1, ..., M_g\}$ for all $i = 1, ..., g$, then $k' \in \Delta$. 

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Then \( k \) is favorable with respect to \( M_j \), i.e., \( H^0(Z_{M_j}, \omega^k(-D)) \) is of Galois type, and for any \( j > 0 \) we have \( H^1(Z_{M_j}, \omega^k(-D)) = 0 \).

**Proof.** We can assume that \( M_j \) is non-zero. Assume first \( J = \{ j \} \) and \( M_j > 0 \). Multiplication by \( h_{j,M_j} \) induces a Hecke equivariant exact sequence of sheaves of \( \mathcal{O}_{\mathcal{M}^\text{tor}} \)-modules:

\[
(4.3.2.1) \quad 0 \to \omega^{k-M_j p_j}(-D) \to \omega^k(-D) \to \iota_* \omega^k_{|Z_{M_j} e_j} \to 0,
\]

where \( \iota \) denotes the closed embedding \( Z_{M_j} e_j \hookrightarrow \mathcal{M}^\text{tor} \).

By condition (*) the weights \( k \) and \( k-M_j p_j \) belong to \( \Delta \), so that \( H^0(\mathcal{M}^\text{tor}, \omega^k(-D)) = 0 \) and \( H^0(\mathcal{M}^\text{tor}, \omega^{k-M_j p_j}(-D)) = 0 \). Considering the long exact sequence in cohomology associated to \( (4.3.2.1) \), we see that:

\[
H^0(Z_{M_j} e_j, \omega^k) = H^0(\mathcal{M}^\text{tor}, \iota_* \omega^k_{|Z_{M_j} e_j}) = 0,
\]

and the natural map

\[
H^0(\mathcal{M}^\text{tor}, \omega^k(-D)) \to H^0(Z_{M_j} e_j, \omega^k)
\]

is surjective. Since \( k \) belongs to \( \Delta \) and it is regular, \( H^0(\mathcal{M}^\text{tor}, \omega^k(-D)) \) is of Galois type, and therefore also \( H^0(Z_{M_j} e_j, \omega^k) \) is of Galois type by Proposition 4.1.3.

Assume now that the lemma is true for any subset of \( \{ 1, ..., g \} \) containing \( r \) elements, where \( r \) is fixed and \( 1 \leq r \leq g - 1 \). Assume \( J \) is a subset of \( \{ 1, ..., g \} \) containing \( r + 1 \) elements and write \( J \) as the disjoint union of some \( J' \) and the singleton \( \{ l \} \); we can assume that \( M_l > 0 \) and that \( M_{J'} \) is non-zero. We have the following Hecke equivariant exact sequence of sheaves of \( \mathcal{O}_{Z_{M_{J'}}} \)-modules:

\[
0 \to \omega^{k-M_l p_l} \to \omega^k_{|Z_{M_{J'}}} \to \iota_* \omega^k_{|Z_{M_J}} \to 0,
\]

where \( \iota \) now denotes the closed immersion \( Z_{M_J} \hookrightarrow Z_{M_{J'}} \) and the injective non-zero morphism is induced by multiplication by \( h_{l,M_l} \).

If condition (*) is true for \( J \), it is also true for \( J' \). Since \( k-M_l p_l \) and \( k \) are favorable weights with respect to \( M_{J'} \), we deduce that \( H^0(Z_{M_J}, \omega^k) \) is the quotient of a Hecke module of Galois type and hence it is itself of Galois type. Finally, as the positive degree cohomology of \( \omega^{k-M_l p_l}_{|Z_{M_{J'}}} \) and \( \omega^k_{|Z_{M_{J'}}} \) vanishes, so does the positive degree cohomology of \( \iota_* \omega^k_{|Z_{M_J}} \). \( \square \)

**Lemma 4.3.3.** Let \( M \in (p^m-1Z_{>0})^g \) be a \( g \)-tuple having support \( J \subseteq \{ 1, ..., g \} \), and let \( k_1, ..., k_r \in \mathbb{Z}^g \) be paritious weights. Then there is a tuple

\[
\sum_{j \in J} N_j e_j \in (2p^m-1Z_{>0})^J,
\]

such that the weight \( k_\alpha + \sum_{j \in J} N_j p_j \) is favorable with respect to \( M \) for every \( \alpha = 1, ..., r \).

**Proof.** Let \( \ell \) denote the line spanned by the vector \( 1 = (1, ..., 1) \) in the Euclidean space \( \mathbb{R}^g \), and denote by \( E : \mathbb{R}^g \to \mathbb{R}^g \) the orthogonal projection onto \( \ell \). For any positive integers \( C, D \) we define the following set of integral points in a truncated cylinder of \( \mathbb{R}^g \):

\[
\Gamma_{C,D} = \{ a \in (Z_{>0})^g : \text{dist}(a, \ell) < C, \| E(a) \| > D \}.
\]

We fix a positive integer \( \bar{C} \), whose value we will increase during the course of the proof as needed. By Lemma 4.2.2, there is an integer \( \bar{D} = D(\bar{C}) > 0 \) such that \( \Gamma_{\bar{C}, \bar{D}} \) is entirely contained in \( \Delta \).

Since \( 1 \) belongs to the interior of the positive cone spanned in \( \mathbb{R}^g \) by the set \( \{ p_j, q_i : j \in J, i \in J^c \} \), by increasing \( \bar{C} \) if necessary we can find tuples

\[
\sum_{j \in J} N_j e_j \in (2p^m-1Z_{>0})^J, \quad \text{and} \quad \sum_{j \in J^c} N_j e_j \in (2p^m-1Z_{>0})^{J^c}
\]
such that:

$$\sum_{j \in J} N_j p_j + \sum_{j \in J^c} N_j q_j \in \Gamma \bar{C}, \bar{D}.$$  

By increasing the $N_j$'s (and consequently $\bar{C}$) if necessary, we can moreover assume that

$$N_j > M_j + p^{m-1} \quad \text{for all } j \in J^c,$$

and that for every $\alpha = 1, \ldots, r$ the paritious weight

$$h_\alpha := k_\alpha + \left( \sum_{j \in J} N_j p_j + \sum_{j \in J^c} N_j q_j \right)$$

is regular and belongs to $\Gamma_{C,D}$.

Modulo further increasing $\bar{C}$, we can also assume that the weight $h_\alpha$ satisfies condition (\*) of Lemma 4.3.2 with respect to $M$. It follows that $H^0(Z_M, \omega^{h_\alpha}(-D))$ is of Galois type and $H^{>0}(Z_M, \omega^{h_\alpha}(-D))$ is trivial. The lemma now follows as, by Corollary 3.3.8 multiplication by

$$\tilde{b} \sum_{j \in J^c} N_j e_j \in H^0(Z_M, \omega^{\sum_{j \in J^c} N_j q_j}(-D))$$

induces for any $i \geq 0$ an isomorphism of $R_m$-modules

$$H^i(Z_M, \omega^{k_\alpha + \sum_{j \in J^c} N_j p_j}(-D)) \simeq H^i(Z_M, \omega^{h_\alpha}(-D))$$

which is compatible with the Hecke action up to a twist. $\Box$

### 4.4. Favorable resolutions and pseudo-representations.

In this section, we prove that all cohomology groups of $\omega^k(-D)$ over $\mathcal{M}^{tor}$ are of Galois type, if $k$ is paritious. In fact, we will construct a resolution of the sheaf $\omega^k(-D)$ by favorable sheaves in the sense of Definition 4.2.1. Hence the cohomology groups of $\omega^k(-D)$ are computed by the complex consisting of $H^0$ of each term in the resolution.

We keep the notation introduced earlier. In particular recall that $R_m = \mathcal{O}_E/\bar{\pi}^m_E$ and the Hilbert modular variety $\mathcal{M}^{tor}$ is defined over $R_m$.

**Definition 4.4.1.** Let $M, M' \in (p^m-1\mathbb{Z}_{>0})^g$ and let $k, k' \in \mathbb{Z}^g$ be two paritious weights with the same parity. A homomorphism of sheaves of $\mathcal{O}_{\mathcal{M}^{tor}}$-modules

$$\xi : \omega^k(-D)|_{Z_M} \to \omega^{k'}(-D)|_{Z_{M'}}$$

is called **admissible** either if it is the zero homomorphism, or if the following three conditions are satisfied:

- $k' - k = N_p$ for some $N \in (2p^m-1\mathbb{Z}_{>0})^g$ (cf. 3.3.3 for the meaning of this notation),
- $\xi$ is induced by multiplication by $\alpha \tilde{h}_N$ for some $\alpha \in R_m$, and
- for each $i$ such that $M_i > 0$, we have $M'_i > 0$ and $M_i + (k'_i - k_i) \geq M'_i$.

The first condition in the definition ensures that $\tilde{h}_N$ is defined; the last condition ensures that $\xi$ is a well-defined map. Notice also that if an admissible homomorphism $\omega^k(-D)|_{Z_M} \to \omega^{k'}(-D)|_{Z_{M'}}$ is nontrivial, the last condition for admissibility implies that $|M| \geq |M'|$; in particular, $\dim(M) \geq \dim(M')$. When $\dim(M) = \dim(M')$, the condition will force $|M| = |M'|$.

We remark that the composition of admissible homomorphisms is an admissible homomorphism.

As a consequence of our discussion in section 3.3.1 we have:

**Lemma 4.4.2.** An admissible homomorphism $\omega^k(-D)|_{Z_M} \to \omega^{k'}(-D)|_{Z_{M'}}$ is $T_3$-equivariant.

An **admissible complex** (resp. admissible double complex) is a bounded complex (resp. bounded double complex) $C^\bullet$ (resp. $C^{**}$) of sheaves of coherent $\mathcal{O}_{\mathcal{M}^{tor}}$-modules, satisfying the following three conditions:
• each $C^i$ (resp. $C^{ij}$) is a finite direct sum of sheaves of the form $\omega^k(-D)|_{Z^M}$, which we call terms,
• the weights of the terms of the complex are all paritious, they all have a common parity, and
• each differential in the complex (resp. double complex) can be represented by a matrix whose entries are admissible homomorphisms between terms of the complex.

An admissible morphism between two admissible complexes is a morphism of complexes for which the homomorphisms in each degree can be represented by matrices whose entries are admissible homomorphisms between terms of the two complexes. (This is equivalent to requiring that the cone of the morphism is an admissible complex). Also, the total complex of an admissible double complex is an admissible complex.

The dimension of an admissible complex is the maximal dimension of the set theoretical support of its terms, or equivalently the maximal dimension of the $M$'s appearing in the complex (cf. Definition 3.3.5).

**Lemma 4.4.3.** An admissible complex $C^\bullet$ of dimension $r$ can be written as:

$$\text{Cone} \left[ C^\bullet_{\dim=r} \to C^\bullet_{\dim<r} \right] [-1],$$

where:

• the two complexes $C^\bullet_{\dim=r}$ and $C^\bullet_{\dim<r}$ are admissible and the morphism between them is also admissible,
• $C^\bullet_{\dim<r}$ has dimension strictly less than $r$, and
• each term in $C^\bullet_{\dim=r}$ has dimension exactly $r$.

Moreover $C^\bullet_{\dim=r}$ is a direct sum $\bigoplus J C^\bullet_J$ of admissible complexes, where the sum is taken over all subsets $J \subseteq \{1, \ldots, g\}$ (of cardinality $r$), and each $C^\bullet_J$ consists of sheaves with support $J$.

**Proof.** The admissibility condition implies that in the complex $C^\bullet$ any map from a term of dimension strictly less than $r$ to a term of dimension $r$ is zero. Let $C^\bullet_{\dim<r}$ denote the shift by $[1]$ of the complex consisting of all terms of $C^\bullet$ with dimension strictly less than $r$, together with all morphisms among them. Let $C^\bullet_{\dim=r}$ be the complex consisting of all terms of $C^\bullet$ with dimension exactly $r$, together with all morphisms among them. The morphism $C^\bullet_{\dim=r} \to C^\bullet_{\dim<r}$ is also taken from $C^\bullet$. One checks immediately that $C^\bullet$ is nothing but $\text{Cone} \left[ C^\bullet_{\dim=r} \to C^\bullet_{\dim<r} \right] [-1]$. The second part of the lemma follows from the same argument, as the admissibility condition implies that there is no nontrivial map between terms of dimension $r$ with different supports (cf. remarks after Definition 4.4.1). \[\square\]

Our goal is the following theorem:

**Theorem 4.4.4.** Let $k \in \mathbb{Z}^g$ be a paritious weight. There exists an admissible complex $C^\bullet$ quasi-isomorphic to $\omega^k(-D)$ such that all terms of $C^\bullet$ are favorable. Hence there exists a bounded complex of $\mathcal{T}_S$-modules of Galois type whose cohomology groups are $H^\bullet(\mathcal{M}^{\text{tor}}, \omega^k(-D))$.

**Proof.** We construct the admissible complex $C^\bullet$ inductively, starting from $r = g$ and proceeding downwards to $r = 0$, on the following statement: for any $r = 0, \ldots, g$, the sheaf $\omega^k(-D)$ is quasi-isomorphic to the total complex of the following admissible double complex:

$$C^\bullet_{\dim\geq r} \xrightarrow{\eta_r} C^\bullet_{\dim<r},$$

where $\eta_r$ is the admissibility condition.
where $C_{\dim \geq r}^\bullet$ is some admissible complex consisting of favorable sheaves of dimension $\geq r$, $C_{\dim < r}^\bullet$ is some admissible complex of dimension strictly less than $r$, and $\eta_r$ is an admissible morphism.

The proof of the theorem is thus reduced to Lemma 4.4.5 below. Indeed, granting the lemma and assuming that we know the above statement for $r$, we can construct a new double complex as follows. Lemma 4.4.5 implies that there is an admissible morphism of admissible complexes which is a quasi-isomorphism

$$C_{\dim < r}^\bullet \xrightarrow{\cong} \text{Cone } [D_{\dim = r-i}^\bullet \rightarrow D_{\dim < r-i}^\bullet] [-1]$$

(here $r - i$ is the dimension of $C_{\dim < r}^\bullet$), such that $D_{\dim = r-i}^\bullet$ consists of favorable sheaves of dimension $r - i$, and $D_{\dim < r-i}^\bullet$ has dimension strictly less than $r - i$. Hence $\omega^k(-D)$ is quasi-isomorphic to the total complex of

$$C_{\dim \geq r}^\bullet \xrightarrow{\eta_r} \text{Cone } [D_{\dim = r-i}^\bullet \rightarrow D_{\dim < r-i}^\bullet] [-1].$$

Rearranging the terms, we see that $\omega^k(-D)$ is quasi-isomorphic to the total complex of

$$\text{Cone } [C_{\dim \geq r}^\bullet \rightarrow D_{\dim = r-i}^\bullet] [-1] \rightarrow D_{\dim < r-i}^\bullet [-1],$$

finishing the inductive proof.

The final statement in the theorem follows since $0 \rightarrow \omega^k(-D) \rightarrow C^\bullet$ is an acyclic resolution of $\omega^k(-D)$.  

We are then left with proving the following:

**Lemma 4.4.5.** Let $C^\bullet$ be an admissible complex of dimension $r$. Then there exists an admissible complex $D^\bullet$ of dimension $r$ and an admissible morphism $C^\bullet \rightarrow D^\bullet$ which is a quasi-isomorphism, such that all the $r$-dimensional terms of $D^\bullet$ are favorable. Combining this with Lemma 4.4.3 we obtain a quasi-isomorphism

$$C^\bullet \xrightarrow{\eta_r} \text{Cone } [D_{\dim = r}^\bullet \rightarrow D_{\dim < r}^\bullet] [-1],$$

where the two complexes on the right hand side are admissible and connected by an admissible morphism, the complex $D_{\dim = r}^\bullet$ consists of favorable sheaves of dimension $r$, and $D_{\dim < r}^\bullet$ has dimension strictly less than $r$.

**Proof.** We first apply Lemma 4.4.3 to write $C^\bullet$ as $C_{\dim = r}^\bullet$ $\left[ \bigoplus_J C_J^\bullet \rightarrow C_{\dim < r}^\bullet \right] [-1]$, where each $C_J^\bullet$ is an admissible complex consisting of sheaves with support $J$, and the direct sum is taken over all subsets $J \subseteq \{1, \ldots, g\}$ of cardinality $r$.

For each subset $J$ of $\{1, \ldots, g\}$ of cardinality $r$, we choose a tuple $N^{(J)} \in (2p^{m-1}\mathbb{Z}_{>0})^J$ such that for each term $\omega^k(-D)|_M$ of $C_{\dim = r}$ with support $J$, the weight $k + N^{(J)}_p$ is favorable with respect to $M$. The existence of such $N^{(J)}$ follows from Lemma 4.3.3 since $C_J^\bullet$ has only finitely many terms. We also choose $N \in 2p^{m-1}\mathbb{Z}_{>0}$ such that $N > N^{(J)}$ for all subsets $J$ of $\{1, \ldots, g\}$ of cardinality $r$, and all $j \in J$.

We now define an admissible double complex $A^{\bullet \bullet}$ that resolves term-by-term the complex $C^\bullet$ as follows:

1. Let $\omega^k(-D)|_M$ be a term of the complex $C_J^\bullet$. Consider the following complex (Koszul complex):
\[ (4.4.5.1) \quad \omega^k(-D)|_{Z_M} \xrightarrow{\sim} \left[ \omega^{k+N_p}(-D)|_{Z_M} \rightarrow \bigoplus_{j \in J} \omega^{k+N_p}(-D)|_{Z_{M+N[j]_{e_j}}} \rightarrow \cdots \rightarrow \omega^{k+N_p}(-D)|_{Z_{M+N(\mathbb{N})}} \right] , \]

where the homomorphisms between the terms inside the brackets are induced by restrictions and come from the Čech formalism. We point out that all the terms inside the brackets have dimension strictly less than \( r \) except the first one, which has dimension \( r \) and is favorable.

Observe that (4.4.5.1) is a resolution of \( \omega^k(-D)|_{Z_M} \). This is because its completion at each closed point can be identified with the completion at an appropriate point of the following resolution (whose exactness is proved in Lemma 4.4.6):

\[ (4.4.5.2) \quad \frac{R[x_1, \ldots, x_g]}{(x_{i_1}^{M_1}; l \in J^c)} \xrightarrow{\sim} \left[ \frac{R[x_1, \ldots, x_g]}{(x_{i_1}^{M_1}; l \in J^c)} \rightarrow \cdots \rightarrow \frac{R[x_1, \ldots, x_g]}{(x_{i_1}^{M_1}, x_{i_j}^{N(j)}; j \in J, l \in J^c)} \right] . \]

We will refer to (4.4.5.2) as the “toy model” of (4.4.5.1). (Here \( R \) denotes a finite \( R_m \)-module).

(2) Denote by \( N \) the \( g \)-tuple \((N, \ldots, N)\). To each term \( \omega^k(-D)|_{Z_M} \) of \( C_{\dim < r}^* \), we associate the following complex:

\[ (4.4.5.3) \quad \omega^k(-D)|_{Z_M} \xrightarrow{\sim} \left[ \omega^{k+N_p}(-D)|_{Z_{[M+N]|M^c}} \rightarrow \cdots \rightarrow \bigoplus_{J} \omega^{k+N_p}(-D)|_{Z_{N_J+(M+N)|M^c-J}} \rightarrow \cdots \rightarrow \omega^{k+N_p}(-D)|_{Z_N} \right] , \]

where the direct sum in the \( s \)th term inside the brackets is taken over all subsets \( J \) of \( \{1, \ldots, g\} \) of cardinality \( s - 1 \). We point out that all terms of (4.4.5.3) have dimension strictly smaller than \( r \).

Similar to the argument above, (4.4.5.3) is a resolution of \( \omega^k(-D)|_{Z_M} \) because its completion at each closed point can be identified with the completion at an appropriate point of the following resolution:

\[ (4.4.5.4) \quad \frac{R[x_1, \ldots, x_g]}{(x_{i_1}^{M_1}; l \in |M|^c)} \xrightarrow{\sim} \left[ \frac{R[x_1, \ldots, x_g]}{(x_{i_1}^{M_1+N}; l \in |M|^c)} \rightarrow \cdots \rightarrow \bigoplus_{j} \frac{R[x_1, \ldots, x_g]}{(x_{i_1}^{N_j}, x_{i_1}^{M+N}; j \in J, l \in |M|^c-J)} \rightarrow \cdots \rightarrow \frac{R[x_1, \ldots, x_g]}{(x_{i_1}^{N_1}, \ldots, x_{i_1}^{N_g})} \right] . \]

\[ ^1 \text{For example, when } g = 3, J = \{1\}, \text{ and } n_j = N^{(j)}_j \text{ the resolution is the following long exact sequence:} \]

\[ 0 \rightarrow \frac{R[x_1, x_2, x_3]}{(x_1^{M_1})} \xrightarrow{x_2 x_3} \frac{R[x_1, x_2, x_3]}{(x_1^{M_1})} \rightarrow \frac{R[x_1, x_2, x_3]}{(x_1^{M_1}, x_2^{n_2})} \oplus \frac{R[x_1, x_2, x_3]}{(x_1^{M_1}, x_3^{n_3})} \rightarrow \frac{R[x_1, x_2, x_3]}{(x_1^{M_1}, x_2^{n_2}, x_3^{n_3})} \rightarrow 0. \]
The exactness of (4.4.5.4) is proved in the Lemma 4.4.6.\footnote{For example, when \( g = 3 \) and \( \mathbf{M} = (m_1, m_2, 0) \), the resolution is the following long exact sequence:

\[
0 \to \frac{R[x_1, x_2, x_3]}{(x_1^{m_1}, x_2^{m_2})} \xrightarrow{\cdot x_3} \frac{R[x_1, x_2, x_3]}{(x_1^{m_1}, x_2^{m_2})} \to \frac{R[x_1, x_2, x_3]}{(x_1^{m_1+N}, x_2^{m_2+N})} \oplus \frac{R[x_1, x_2, x_3]}{(x_1^{m_1+N}, x_2^{m_2+N}, x_3^{N})} \to \frac{R[x_1, x_2, x_3]}{(x_1^{m_1+N}, x_2^{m_2+N}, x_3^{N})} \to 0.
\]

(3) We now specify the morphisms among the resolutions constructed above. For an admissible homomorphism \( \omega^k(-D)|_{Z_M} \to \omega^k(-D)|_{Z_{M'}} \) arising from \( C_{\chi}^* \) and given by multiplication by \( h_{R} \), the corresponding morphism between the associated resolutions is given by multiplication by \( \alpha h_{R} \). For an admissible morphism \( \omega^k(-D)|_{Z_M} \to \omega^k(-D)|_{Z_{M'}} \) coming from \( C_{\dim<r}^* \) and given by multiplication by \( h_{R} \), the corresponding morphisms between the associated resolutions is given by multiplication by \( \alpha h_{R} \). It is clear that the morphisms are admissible in these two cases.

We now consider more carefully the case of a (nontrivial) admissible morphism \( \omega^k(-D)|_{Z_M} \to \omega^k(-D)|_{Z_{M'}} \) coming from \( C_{\cdot}^* \to C_{\dim<r}^* \) and given by multiplication by \( h_{R} \) (so that \( k' = k + R_p \)). The admissibility condition implies that \( J = |\mathbf{M}| \geq |\mathbf{M}'| \). The morphism between the corresponding resolutions is obtained by taking the direct sum of morphisms

\[
\alpha h_{R_+N-N}(j) : \omega^{k+N_p}(j)|_{Z_{M+N}(J)} \to \omega^{(k+R_p)+N_p}(j)|_{Z_{M+N}(J)}(-D) \in J.
\]

As before, by considering completions at closed points, the admissibility of this morphism follows from the fact that the morphism from the toy model (4.4.5.1) to (4.4.5.3) given by the formula below is well-defined for all subsets \( I \subseteq J \):

\[
\frac{R[x_1, \ldots, x_g]}{(x_j^M)} \xrightarrow{\alpha x^{R-N}(J)} \frac{R[x_1, \ldots, x_g]}{(x_j^M, x_i^N) : j \in J, i \in I}.
\]

This is proved in Lemma 4.4.6.

Connecting the complexes of sheaves in (4.4.5.1) and (4.4.5.3) via the above morphisms, we obtain a first quadrant, admissible double complex \( A^{ullet \circ} \) together with an admissible morphism of complexes \( C^* \to A^\circ \) such that \( 0 \to C^i \to A^i \) is a resolution of \( C^i \) for all \( i \geq 0 \). This produces a quasi-isomorphism

\[
C^* \to D^* := \text{Tot}(A^\circ).
\]

\[\square\]

**Lemma 4.4.6.** The morphisms (4.4.5.3) and (4.4.5.4) are quasi-isomorphisms. The map defined by (4.4.5.6) from the right hand side of (4.4.5.2) to the right hand side of (4.4.5.4) is a well-defined morphism of complexes.

**Proof.** The morphisms in (4.4.5.2) can be rewritten as tensor products over \( R_m \) of the following quasi-isomorphisms of complexes of \( R_m \)-modules:

\[
R[x_l]/(x_l^M) \xrightarrow{\sim} R[x_l]/(x_l^{M_J}) \quad \text{for } l \in J^c;
\]

\[
R[x_j] \xrightarrow{\sim} R[x_j]/(x_j^{N_J}) \quad \text{for } j \in J.
\]
(In this proof, we adopt the convention of not labeling a morphism induced by the identity function). The morphisms in (4.4.5.4) can be rewritten as the tensor product over $R_m$ of the following quasi-isomorphisms:

$$R[x_i]/(x_i^{M_i}) \xrightarrow{\cong} R[x_i]/(x_i^{M_i+N}) \rightarrow R[x_i]/(x_i^N)$$

for $l \in |M|^c$;

$$R[x_j] \xrightarrow{\cong} R[x_j]/(x_j^N)$$

for $j \in |M|$.

Finally, the map (4.4.5.6) can be written as the tensor product over $R_m$ of the following morphisms:

$$R[x_i] \xrightarrow{\alpha} R[x_i]/(x_i^{N_1}) \xrightarrow{\beta} R[x_i]/(x_i^{N_1})$$

for $l \in |M'|$;

$$R[x_i]/(x_i^{M_i'}) \xrightarrow{\gamma} R[x_i]/(x_i^{M_i'+N})$$

for $l \in |M| - |M'|$;

$$R[x_i]/(x_i^{M_i}) \rightarrow R[x_i]/(x_i^{M_i}) \rightarrow 0$$

for $l \in |M|^c$.

Here in each diagram, the morphism $\alpha$ is induced by multiplication by $x_i^{k_i-k_i}$, $\beta$ is induced by multiplication by $x_i^{(k_i' - k_i) + (N - N_1)}$, and $\gamma$ is induced by multiplication by $x_i^{(k_i' - k_i) + N}$.

**Corollary 4.4.7.** Let $k \in \mathbb{Z}^g$ be a paritious weight. Any cuspidal Hecke eigenclass $\tau \in H^*(\mathcal{M}^{\text{tor}}, \omega^k(-D))$ has canonically attached a continuous, $R_m$-linear, two-dimensional pseudo-representation $\tau_c$ of the Galois group $G_{F,S}$ such that

$$\tau_c(\text{Frob}_\lambda) = a_\lambda$$

for all finite primes $\lambda$ of $F$ outside $S$, where $T_\lambda c = a_\lambda c$.

**Proof.** Let $c \in H^*(\mathcal{M}^{\text{tor}}, \omega^k(-D))$ be a Hecke eigenclass and set $M := R_m f$. By Theorem 4.4.4, $M$ is a Hecke module of Galois type, and there is a continuous pseudo-representation

$$\tau_c : G_{F,S} \rightarrow \text{End}_{R_m}(M) = R_m$$

satisfying the conditions stated above. Notice that $\tau_c$ has dimension two as ultimately it is obtained by reducing modulo $\pi_E^{\text{tor}}$ an integral model of the $p$-adic Galois representation attached to a characteristic-zero Hilbert modular eigenform of some weight $k'$ (via our fixed embedding $\mathbb{Q} \rightarrow \overline{\mathbb{Q}}_p$). In general $k \neq k'$.

$\square$
References

[AG05] F. Andreatta and E. Z. Goren, Hilbert Modular Forms: mod p and p-adic Aspects, vol. 173, Memoirs of the American Mathematical Society, no. 819, American Mathematical Society, 2005.

[Ash92] A. Ash, Galois Representations Attached to mod p Cohomology of GL(n, Z), Duke Math. J. 65 (1992), 253–295.

[Ash97] ______, Galois Representations and Hecke Operators Associated with the mod p Cohomology of GL(1, Z) and GL(2, Z), Proc. Amer. Math. Soc. 125 (1997), 3209–3212.

[BR89] D. Blasius and J. Rogawski, Galois Representations for Hilbert Modular Forms, Bull. A.M.S. 21 (1989), 65–69.

[Car86] H. Carayol, Sur les Représentations l-adiques Associées aux Formes Modulaires de Hilbert, Ann. Sci. Ec. Norm. Sup. 19 (1986), 409–468.

[CG12a] F. Calegari and D. Geraghty, Modularity Liftings beyond the Taylor-Wiles Method, preprint, ArXiv (2012).

[CG12b] ______, Modularity Liftings beyond the Taylor-Wiles Method II, preprint, ArXiv (2012).

[Cha90] C.-L. Chai, Arithmetic Minimal Compactification of the Hilbert-Blumenthal Moduli Spaces, Appendix to C.-L. Chai, Geometric Aspects of Dwork Theory, A. Adolphson, F. Baldassarri, P. Berthelot, N. Katz and F. Loeser eds., Walter de Gruyter (2012), 65–69.

[CG12c] ______, Lectures on Hilbert Modular Varieties and Modular Forms, CRM Monograph Series, vol. 14, American Mathematical Society, 2002.

[Hid06] H. Hida, Hilbert Modular Forms and Iwasawa Theory, Oxford University Press, 2006.

[Jar97] F. Jarvis, On Galois Representations Associated to Hilbert Modular Forms, J. Reine Angew. Math. 491 (1997), 199–216.

[Kat78] N. M. Katz, p-adic L-functions for CM Fields, Invent. Math. 49 (1978), 199–297.

[KL06] M. Kisin and K. F. Lai, Overconvergent Hilbert Modular Forms, Amer. J. Math. 127 (2005), 735–783.

[Lan13] K.-W. Lan, Compactifications of PEL-type Shimura Varieties and Kuga Families with Ordinary Loci, preprint (2013).

[Oht84] M. Ohta, Hilbert Modular Forms of Weight One and Galois Representations, Progr. Math. 46 (1984), 333–353.

[Rap78] M. Rapoport, Compactifications de l’Espace de Modules de Hilbert-Blumenthal, Compositio Math. 36 (1978), 255–335.

[Rob80] G. Robert, Congruences entre Séries d’Eisenstein, dans le Cas Supersingular, Inventiones Math. 61 (1980), 103–158.

[Rou96] R. Rouquier, Caractérisation des Caractères et Pseudo-caractères, J. of Algebra 180 (1996), 571–586.

[Ri83] J. Rogawski and J. Tunnell, On Artin l-functions Associated to Hilbert Modular Forms of Weight 1, Invent. Math. 74 (1983), 1–42.

[Sch13] P. Scholze, On Torsion in the Cohomology of Locally Symmetric Varieties, preprint, ArXiv (2013).

[SD73] H.P.F. Swinnerton-Dyer, On l-adic Representations and Congruences for Coefficients of Modular Forms, Proceedings of the 1972 Antwerp International Summer School on Modular Forms, Springer Lecture Notes in Mathematics 350 (1973), 1–55.

[Ser73] J.-P. Serre, Congruences et Formes Modulaires (d’après H.P.F. Swinnerton-Dyer), Esposó 416, Séminaire N. Bourbaki 1971/72, Springer Lecture Notes in Mathematics 317 (1973), 319–338.

[Ser96] J.-P. Serre, Two Letters on Quaternions and Modular Forms (mod p), Israel J. Math. 95 (1996), 281–299.

[Tay89] R. Taylor, On Galois Representations Associated to Hilbert Modular Forms, Invent. Math. 98 (1989), 265–280.
[Tay91] ______, *Galois Representations Associated to Siegel Modular Forms of Low Weight*, Duke Math. J. 63 (1991), 281–332.

[Wil88] A. Wiles, *On Ordinary \( \lambda \)-adic Representations Associated to Modular Forms*, Invent. Math. 94 (1988), 529–573.

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