Nef cycles on some hyperkähler fourfolds

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Abstract. We study the cones of surfaces on varieties of lines on cubic fourfolds and Hilbert schemes of points on K3 surfaces. From this we obtain new examples of nef cycles which fail to be pseudoeffective.

1 Introduction

The cones of curves and effective divisors are essential tools in the study of algebraic varieties. Here the intersection pairing between curves and divisors allows one to interpret these cones geometrically in terms of their duals; these are the cones of nef divisors and ‘movable’ curves respectively. In intermediate dimensions however, very little is known about the behaviour of the cones of effective cycles. Here a surprising feature is that nef cycles may fail to be pseudoeffective, showing that the usual geometric intuition for ‘positivity’ does not extend more generally. In the paper [8], Debarre–Ein–Lazarsfeld–Voisin presented examples of such cycles of codimension two on abelian fourfolds.

In this paper, we show that similar examples can also be found on certain holomorphic symplectic varieties (or ‘hyperkähler manifolds’). For these varieties, the cones of curves and divisors are already well-understood thanks to several recent advances in hyperkähler geometry [2–5,12,15]. Our main result is the following:

Theorem 1. Let \( Y \subset \mathbb{P}^5 \) be a very general cubic fourfold and let \( X \) be its variety of lines. Then the cone of pseudoeffective 2-cycles on \( X \) is strictly contained in the cone of nef 2-cycles. In fact, \( c_2(X) \) is positive on every surface, but has no effective multiple.

The special interest in the second Chern class \( c_2(X) \) here comes from the fact that it represents a positive rational multiple of the Beauville–Bogomolov form on \( H^2(X, \mathbb{Z}) \) (this fact in turn accounts for much of its positivity). As we will see in Section 4, \( c_2(X) \) is in addition ‘close’ to the boundary of the cone of 2-cycles on \( X \).

The main idea of the proof is to deform \( X \) to the Hilbert square of a Kummer surface. Here it is relatively easy to see that the second Chern class \( c_2(X) \) cannot be in the interior of the effective cone, because it has intersection number 0 with the fibers of any Lagrangian fibration. (On the other hand after going to such a deformation \( c_2(X) \) may no longer be nef). Then, using results of Voisin, the theorem will be a consequence of the following:

Proposition 2. Let \( X = S^{[2]} \) be the Hilbert square of a Kummer surface \( S \). Then no multiple of \( c_2(X) \) is numerically equivalent to an effective cycle.
Moreover, by another deformation argument, one finds that the same statement holds for a very general K3 surface.

In the last section we consider fourfolds of generalized Kummer type. Using results of Hassett–Tschinkel, we find that $3c_2(X)$ is in fact effective on every such fourfold.

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2 Preliminaries

We work over the complex numbers. For a projective variety $X$, let $N_k(X)$ denote the $\mathbb{R}$-vector space of dimension $k$-cycles on $X$ modulo numerical equivalence. In $N_k(X)$ we define the pseudoeffective cone $\overline{\text{Eff}}_k(X)$ to be the closure of the cone spanned by classes of $k$-dimensional subvarieties. A class $\alpha \in N_k(X)$ is said to be big if it lies in the interior of $\overline{\text{Eff}}_k(X)$. Note that this is equivalent to having $\alpha = ch^{\dim X-k} + e$ for $h$ the class of a very ample divisor; $\epsilon > 0$; and $e$ an effective $2$-cycle with $\mathbb{R}$-coefficients.

A codimension $k$-cycle is said to be nef if it has non-negative pairing with any $k$-dimensional subvariety. We let $\text{Nef}_k(X)$ denote cone spanned by nef cycles; this is the dual cone of $\overline{\text{Eff}}_k(X)$. For most of the varieties in this paper, it is known that numerical and homological equivalence coincide, so we may consider these as cones in $H^{2k}(X, \mathbb{R})$ and $H^{2k}(X, \mathbb{R})$ respectively. If $Y \subset X$ is a subvariety, we let $[Y] \in H^{*}(X, \mathbb{R})$ denote its corresponding cohomology class.

For a variety $X$, we denote by $X^{[n]}$ the Hilbert scheme parameterizing length $n$ subschemes of $X$.

2.1 Specialization of effective cycles

In the proof of Theorem 1, we will need a certain semi-continuity result for effective cycles. This result may be known to experts, but we include it here for the convenience of the reader and future reference.

**Proposition 3.** Let $f: \mathcal{X} \to T$ be a smooth family of projective varieties over a smooth variety $T$ and suppose that $\alpha \in H^{k,k}(\mathcal{X}, \mathbb{Z})$ is a class such that $\alpha|_{\mathcal{X}_t}$ is effective on the very general fiber. Then $\alpha|_{\mathcal{X}_t}$ is effective for every $t \in T$. 

**Proof.** This follows essentially from the theory of relative Hilbert schemes. For a given point $t_0 \in T$, we will show that effective algebraic cycles on nearby fibers extend to $\mathcal{X}_{t_0}$.

Choose a differential trivialization of the family $\sigma : \mathcal{X}_U \simeq X_{t_0} \times U$ in a neighbourhood around $t_0 \in T$. This induces a specialization map of cohomology groups

$$H^{2k}(\mathcal{X}_U, \mathbb{R}) \cong H^{2k}(X_{t_0}, \mathbb{R})$$

which is compatible with the restriction mapping. There are at most countably many components $\rho_i : H_i \to T$ of the Hilbert scheme parameterizing subschemes...
supported in the fibers $\mathcal{X}_t$ of $f$. For a component $H_i$, let $\pi_i : \mathcal{U}_i \to H_i$ denote the universal family of $H_i$. This fits into the diagram

$$
\begin{array}{ccc}
\mathcal{U}_i & \xrightarrow{\pi_i} & \mathcal{X} \\
| & | & | \\
H_i & \xrightarrow{\rho_i} & T
\end{array}
$$

where $\pi_i$ is flat. Let $T' = \bigcup_i \rho_i(H_i)$, where the union is taken over all indices $i$ such that $\rho_i$ is not surjective. This is a countable union of proper closed subsets of $T$.

If $t_1 \in U - T'$ and $\Gamma \subset \mathcal{X}_{t_1}$ is any subvariety of codimension $k$, then there exists a component $\pi : \mathcal{U} \to H$ of the Hilbert scheme such that $\Gamma$ is a fiber of $\pi$, and $\rho(H) = T$. Let $\phi : H' \to H$ be a desingularization of $H$ and define $\mathcal{U}' = H' \times_H \mathcal{U}$ and $\mathcal{X}' = H' \times_T \mathcal{X}'$, with the induced (smooth) morphism $\pi' : \mathcal{X}' \to H'$. We have $\mathcal{U}' \to \mathcal{X}'$ and a corresponding element in the local system $[\mathcal{U}'] \in H^0(H', R^{2n-2k}\pi'_t)$. Pick two points $t'_0, t'_1$ in the same connected component of $\pi'^{-1}(U)$ mapping to $t_0, t_1$ respectively. By construction, $[\Gamma] \in H^{2k}(\mathcal{X}'_{t'_0}, \mathbb{Z})$ is the restriction of $[\mathcal{U}']$ to the fiber over $t'_1$. Similarly, the restriction of $\mathcal{U}'$ to $\mathcal{X}'_{t'_0}$ is effective, and this is the image of $[\Gamma]$ via the specialization homomorphism. By linearity, it follows that if $\alpha|_{\mathcal{X}_{t_1}}$ represents an effective cycle on $\mathcal{X}_{t_1}$, then also $\alpha|_{\mathcal{X}_{t_0}}$ is effective on $\mathcal{X}_{t_0}$.

In other words, just like in the the case of divisors, the effective cones can only become larger after specialization. Note that the proposition also implies that a class $\alpha$ which is big on a very general fiber is also big on every fiber.

**Remark 4.** Essentially the same proof works in the Kähler setting using the relative Barlet space $\mathcal{B}(\mathcal{X}/S)$ parameterizing analytic cycles supported in the fibers of $f$. In fact, Greb–Lehn–Rollenske proved that for any smooth family $f : \mathcal{X} \to S$ of compact complex manifolds where a special fiber $\mathcal{X}_0$ is Kähler, then there is an open neighbourhood $U$ of $0 \in S$ over which every connected component of $\mathcal{B}(f^{-1}(U)/U)$ is proper over $U$.

### 3 Holomorphic symplectic fourfolds

We will study effective 2-cycles on certain holomorphic symplectic varieties (or hyperkähler manifolds). By definition, such a variety is a smooth, simply connected algebraic variety admitting a non-degenerate holomorphic two-form $\omega$ spanning $H^{2,0}(X)$. Even though hyperkähler manifolds are expected to play a fundamental role in the classification of projective varieties, there are currently just two known examples of such manifolds up to deformation: Hilbert schemes of points on a K3 surface and generalized Kummer varieties.

A hyperkähler manifold $X$ carries an integral, primitive quadratic form $q$ on the cohomology group $H^2(X, \mathbb{Z})$ called the Beauville–Bogomolov form. The signature of this quadratic form is $(3, b_2(X) - 3)$, and $(1, b_2(X) - 3)$ when restricted to the Picard group $\text{Pic}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z})$. For the hyperkähler fourfolds considered in this paper it is known that the second Chern class $c_2(X) = c_2(T_X)$ represents a positive rational multiple of the Beauville–Bogomolov form.
A subvariety $Y \subset X$ of dimension $\frac{1}{2}\dim X$ is said to be Lagrangian if the restriction of $\omega$ to the smooth part of $Y$ is trivial. It is well-known that in this case, the normal bundle $N_{Y|X}$ is isomorphic to the cotangent bundle $\Omega^1_Y$.

3.1 The Hilbert square of a K3

Let $S$ be a K3 surface and let $X = S^{[2]}$ be the Hilbert scheme of length 2 subschemes on $S$. Also, let $\widetilde{S} \times \widetilde{S}$ denote the blow-up of $S \times S$ along the diagonal $\Delta$. There is the usual diagram

\[
\begin{array}{ccc}
\widetilde{S} \times \widetilde{S} & \xrightarrow{\sigma} & S^{[2]} \\
\downarrow{\pi} & & \downarrow{\rho} \\
S \times S & \xrightarrow{\sigma} & S^{(2)}
\end{array}
\]

where $\sigma$ is the quotient by the involution; $\pi$ is the blow-up of the diagonal; and $\rho$ is the Hilbert–Chow morphism (which blows up the image of the diagonal in $S^{(2)}$). The Beauville–Bogomolov form has signature $(1, 20)$ on $\text{Pic}(X)$ and the diagram above yields decompositions $H^2(X, \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus \mathbb{Z} \delta$ and $\text{Pic}(S^{[2]}) = \text{Pic}(S) \oplus \mathbb{Z} \delta$ which are orthogonal with respect to $q$. Here $2\delta$ is the divisor corresponding to non-reduced subschemes. Moreover, for two divisor classes $\alpha, \beta \in \text{Pic}(X)$ we have

\[
c_2(X) \cdot \alpha \cdot \beta = 30q(\alpha, \beta).
\]

It follows from this and standard properties of the Beauville–Bogomolov form that

\[
c_2(X) \cdot D_1 \cdot D_2 \geq 0
\]

for any two distinct prime divisors $D_1, D_2$ on $X$.

Let $E$ be the exceptional divisor of the blow-up $\pi$. Using the exact sequence

\[
0 \rightarrow T_{\widetilde{S} \times \widetilde{S}} \rightarrow \sigma^*T_X \rightarrow \sigma^*O_\delta(\delta) \simeq O_E(2E) \rightarrow 0
\]

and the standard formulas for the Chern classes of a blow-up, one obtains the following expression for the pullback of $c_2(X)$ to $\widetilde{S} \times \widetilde{S}$:

\[
\sigma^*c_2(X) = \pi^*c_2(S \times S) - 3E^2.
\]

Note that $\pi^*c_2(S \times S) = 24o_1 + 24o_2$ where $o_i$ is the class of a fiber of the $i$-th projection $pr_i: \widetilde{S} \times \widetilde{S} \rightarrow S$. In particular, the first term in $[3]$ is effective.

3.2 Kummer K3 surfaces and the proof of Proposition 2

We will prove Theorem 1 by specializing to the Hilbert scheme of points of a Kummer surface. Here the presence of infinitely many Lagrangian planes is what forces $c_2(X)$ to be non-effective. This argument was inspired by Piatetski-Shapiro and Shafarevich’s proof of the Torelli theorem for K3 surfaces [18].

Recall that such K3 surfaces are constructed by taking the minimal desingularization of $A/i$ where $A$ is an abelian surface and $i$ is the involution $i(x) = -x$. 

4
The 16 two-torsion points give 16 smooth rational curves on $S$, which are $(-2)$-curves. It follows that the Picard number of $S$ is at least 17. The two main properties we need from $S$ are the following: $S$ can be embedded as a $(2,2,2)$-divisor in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$; and $S$ has infinitely many $(-2)$-curves. These facts are well-known for Kummer surfaces and not hard to prove directly. Note in particular that $S$ has three elliptic fibration structures $g_i : S \to \mathbb{P}^1$, $i = 1, 2, 3$. Here, for each elliptic fibration $g : S \to \mathbb{P}^1$ the Mordell-Weil group acts by translations on the sections of $g$ (all of which are smooth $(-2)$-curves) and we may assume that this group has have positive rank.

We will prove the following:

**Proposition.** Let $S$ be a Kummer surface and let $X = S^{[2]}$. Then no multiple of $c_2(X)$ represents an effective cycle.

**Proof.** Suppose to the contrary that on $X$ some multiple of $c_2(X)$ represents an effective cycle. Then, as $\sigma$ is a finite cover, also some multiple of

$$\sigma^* c_2(X) = \pi^* c_2(S \times S) - 3E^2$$

represents an effective cycle on $\tilde{S} \times \tilde{S}$. Let $Z$ denote this cycle.

Let $i : E \to X$ be the inclusion of the exceptional divisor. We may write

$$Z = i_* D + \Gamma$$

for $\Gamma$ an effective 2-cycle with no components contained in $E$, and $D$ an effective divisor in $E$. By construction $i_* \Gamma$ is an effective 1-cycle on $E$.

Regarding $E$ as a projective bundle $p : \mathbb{P}^1(\Omega^1_S) \to \Delta$ over $\Delta \simeq S$, we may write $D = O(a) + p^* M$ where $O(1)$ is the relative hyperplane bundle and $M$ is a line bundle on $S$. Here

$$O(1)^2 \cdot p^* L = c_1(\Omega^1_S) \cdot L = 0$$

(4)

for each $L \in \text{Pic}(S)$, as the canonical bundle of $S$ is trivial.

We have $\sigma^* c_2(X) \cdot E \cdot \pi^* L = 0$ for each line bundle $\mathcal{L} = L \boxtimes L$ on $S \times S$ (e.g., from equation (3)). In other words,

$$0 = Z \cdot \pi^* \mathcal{L} \cdot E = i_* D \cdot \pi^* \mathcal{L} \cdot E + \pi^* \mathcal{L} \cdot i_* \Gamma = (O(a) + p^* M) \cdot p^* (2L) \cdot O(-1) + \mathcal{L} \cdot \pi_* i^* \Gamma = -2M \cdot L + \mathcal{L} \cdot \pi_* i^* \Gamma.$$  

(5)

(Here we used (4) and the fact that $\mathcal{L}|_\Delta = 2L$.) We now consider two cases

(i) $\pi_* i^* \Gamma = 0$

(ii) $\pi_* i^* \Gamma \neq 0$

In the case (i), equation (5) shows that $M \cdot L = 0$ for all $L$, we find that $M \equiv 0$. The conclusion is that $D = O(a)$ for some $a \geq 0$. However, Kobayashi [13] showed that on a K3 surface there are no global symmetric differentials, i.e., for $a \geq 1$

$$H^0(E, O(a)) = H^0(S, S^a(\Omega^1_S)) = 0.$$
(In fact, by Boucksom–Demailly–Paun–Petermell [7], $O(1)$ is not even pseudoeffective). This contradicts the assumption that $D$ is an effective divisor, unless $a = 0$ and $D$ is trivial.

However, in this case, consider the 2-cycle $\gamma = \pi_s \Gamma = \pi_s Z$ on $S \times S$. Using the projection formula, we find that $\gamma$ is proportional to the cycle $c_2(S \times S) + 3\Delta$. In particular, $\gamma$ has negative intersection with all surfaces $C \times C \simeq \mathbb{P}^1 \times \mathbb{P}^1$, for $C$ a $(-2)$-curve on $S$ (of which there are infinitely many). From this we get that $\gamma \cap C \times C$ contains a 1-dimensional component for each $C$. Since the diagonal $\Delta_C \subset C \times C$ is an ample divisor, it has to meet this intersection. In particular, $\gamma \cap \Delta \cap C \times C \neq 0$ for every such $C$. This contradicts the assumption (i), which would imply that $\gamma \cap \Delta$ is a finite set of points.

In the case (ii), we consider again the non-zero 2-cycle $\gamma = \pi_s \Gamma$. As before, $\pi_s Z$ is a multiple of $c_2(S \times S) + 3\Delta$. Let $g : S \to \mathbb{P}^1$ be an elliptic fibration of $S$ and let $E$ be a general fiber of $g$. Also let

$$G = (g, g) : S \times S \to \mathbb{P}^1 \times \mathbb{P}^1.$$ 

We have $\Delta \cdot E \times E = E^2 = 0$. As $\pi_s Z = \pi_s (i_s D + \Gamma) = k\Delta + \gamma$ for some $k$, $\gamma$ is a linear combination of $\Delta$ and $c_2(S \times S)$ and so $\gamma \cdot E \times E = 0$. It follows that all of the components of $\gamma$ are contracted by $G$ to lower-dimensional varieties. However, in the case (ii), there must be an irreducible component of $\gamma$, say $\gamma_0$, which has 1-dimensional intersection with $\Delta$. It follows that $G(\gamma_0) = G(\Delta) = \Delta_{\mathbb{P}^1 \times \mathbb{P}^1}$. This holds for each of the three elliptic fibrations $g_i : S \to \mathbb{P}^1$, and so

$$\gamma_0 \subseteq \bigcap_{i=1}^3 (g_i, g_i)^{-1}(\Delta_{\mathbb{P}^1 \times \mathbb{P}^1}).$$

However, the map $(g_1, g_2, g_3) : S \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is injective. It follows that the intersection on the right-hand side is supported only on $\Delta$. Hence $\gamma_0 = \Delta$. This contradicts the assumption above that $\Gamma$ has no component supported in $E$.

Thus in both cases we obtain a contradiction, and so $c_2(X)$ cannot have an effective multiple.

Now the same conclusion holds on $S^{[2]}$ for $S$ a very general K3 surface. Indeed, let $S \to B$ be a smooth family of K3 surfaces over a smooth curve with special fiber $S_0$, a Kummer surface. Taking the relative Hilbert scheme of length-2 subschemes in the fibers of $f$ we obtain a smooth family $f : X \to B$ with fiber $X_t = S_t^{[2]}$. Now we apply Proposition 3 to the class $c_2(T_{X_t}) = c_2(T_{X_t})|_{X_t}$.

**Remark 5.** A priori, it could still be the case that $c_2(S^{[2]})$ is pseudoeffective, e.g., a limit of effective cycles, even for a general $S$. We do not know if the above argument can be used to show that this is not the case.

**Remark 6.** It is well-known that the naïve extension of the Hodge conjecture to non-projective Kähler manifolds fails in general. Such examples were first constructed by Zucker [24] on complex tori (see also [21]). Here we remark that one can obtain similar examples on hyperkähler manifolds of $K3^{[2]}$-type.

**Proposition 7.** Let $X$ be a very general hyperkähler manifold of $K3^{[2]}$-type. Then $X$ has no analytic 2-dimensional subvarieties.
To see this, recall that for a hyperkähler manifold of $K3^{[2]}$-type we have an isomorphism
\[ \text{Sym}^2(H^2(X, \mathbb{Q})) = H^4(X, \mathbb{Q}). \]
Hence for a general $X$, when $\text{Pic}(X) = 0$, the vector space of degree 4 Hodge classes $H^2,2(X, \mathbb{C}) \cap H^4(X, \mathbb{Q})$ is generated by a single element, namely $c_2(X)$. However, neither a multiple of $c_2$ or $-c_2$ represents an analytic subvariety on $X$. Indeed, if this was the case, then one can find a deformation of $X$ to $S^{[2]}$ for a projective Kummer K3 surface, and the class would stay effective (cf. Remark 5), contradicting Proposition 2.

The above result can also be deduced from the paper [20]. In fact, Verbitsky shows that $X$ does not have any analytic subvarieties of positive dimension.

### 3.3 Chern classes of $S^{[n]}$ for higher $n$

Using the same specialization argument, we can also prove that the low degree Chern classes $c_k(X)$ are not big on the generic $X = S^{[n]}$:

**Theorem 8.** Let $X = S^{[n]}$ be the Hilbert scheme of length $n$ subschemes on a $K3$ surface admitting an elliptic fibration. Then $c_k(X)$ is not big, for $1 \leq k \leq n$.

**Proof.** The elliptic fibration induces a Lagrangian fibration $f : X \to \text{Sym}^n(P^1) = \mathbb{P}^n$, where the general fiber $A$ of $f$ is an abelian variety. The restriction of the normal bundle sequence
\[ 0 \to T_A|_Y = \mathcal{O}_Y^k \to T_X|_A \to N_A|_Y = \mathcal{O}_Y^k \to 0 \]
to $Y = A \cap H_1 \cap \cdots \cap H_{n-k}$, shows that $c_k(X) \cdot A \cdot h^{n-k} = 0$ for $H_i \in |h|$, $h$ very ample on $X$. Now if $c_k$ is a big cycle, then it is numerically equivalent to $\epsilon h^k + e$, $\epsilon > 0$ and $e$ an effective cycle with $\mathbb{R}$-coefficients. However, the cycle $A \cdot h^{n-k}$ is nef, and so has strictly positive intersection number with $h^k$. Hence $c_k(X)$ cannot be big.

The fact that the same conclusion also holds for the general $S^{[n]}$ follows from Proposition 3 as before.

### 4 The variety of lines on a cubic fourfold

Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold. The variety of lines $X = F(Y)$ on $Y$ is a smooth 4-dimensional subvariety of the Grassmannian $Gr(2,6)$. A fundamental result due to Beauville and Donagi [10] says that $X$ is a holomorphic symplectic variety. Moreover, $X$ is deformation-equivalent to the Hilbert scheme $S^{[2]}$ of a $K3$ surface $S$ of degree 14.

Let $i : F \to Gr(2,6)$ be the natural embedding of $X$ in the Grassmannian. From the Plücker embedding, $X$ carries a polarization $g = i^*\mathcal{O}_{Gr(2,6)}(1)$ embedding $X$ into $\mathbb{P}^{14}$. Furthermore, if $U$ is the tautological rank two bundle on Grassmannian $Gr(2,6)$, then $c = c_2(U^\vee)$ is an effective 2-cycle on $X$ (see below). When $Y$ is very general, the vector space of Hodge classes $H^{2,2}(X) \cap H^4(X, \mathbb{Q})$
is a two-dimensional vector space, generated by $g^2$ and $c$. We have the following intersection numbers:

$$
g^4 = 108, \quad g^2 \cdot c = 45, \quad c^2 = 27. \tag{6}
$$

The cubic polynomial defining $Y$ shows that $X$ is the zero-set of a section of the vector bundle $S^3 U^\vee$ on $Gr(2, 6)$. Using this description, a standard Chern class computation shows that

$$c_2(X) = 5g^2 - 8c.$$

See for example [1] or [19] for detailed proofs of these statements.

4.1 Surfaces in the variety of lines

There are several interesting surfaces on $X$. For example, we may consider restrictions of codimension two Schubert cycles from the ambient Grassmannian $Gr(2, 6)$. In terms of $g^2$ and $c$, these cycles are given by $g^2 - c$ and $c$. Moreover, since on $Gr(2, 6)$ every effective cycle is nef (this is true on any homogeneous variety), also their classes remain nef when restricted to $X$.

There are two natural surfaces on $X$ with class proportional to $g^2 - c$. For example, fixing a general line $l \subset Y$, the surface parameterizing lines meeting $l$ represents the class $\frac{1}{3}(g^2 - c)$. Also, the variety of lines of ‘second type’ (that is, lines with normal bundle $O(1) \oplus O(-1)$ in $Y$) is a surface with corresponding class $5(g^2 - c)$ (see [6]).

The class $c$ is also represented by an irreducible surface. Indeed, for a general hyperplane $H = \mathbb{P}^4 \subset \mathbb{P}^5$ the surface

$$Z_H = \{ [l] \in X | l \subset H \}$$

is a smooth surface with $[Z_H] = c$, corresponding to the lines in the cubic threefold. It is known that this surface has general type.

We also have the following less obvious example: Suppose that $Y$ is the hyperplane section of some cubic fivefold $V \subset \mathbb{P}^6$. The variety of planes in $V$ is a smooth surface $F_2(V)$. If $Y$ is general, there is an embedding $F_2(V) \to F(Y)$ given by associating a plane to its intersection with the hyperplane section. The class of the image has class $63c$.

By the following result of Voisin [22], the class $c = c_2(U^\vee)$ is also extremal in the cone of effective 2-cycles on $X$:

**Lemma 9** (Voisin). Let $X$ be the variety of lines on a very general cubic fourfold and let $c = c_2(U^\vee)$. Then $c$ is effective but not big.

This result is quite surprising because the surface $Z_H \subset F$ behaves in many ways like a complete intersection: it deforms in a large family covering $X$ and its normal bundle is an ample vector bundle. For this reason, this surface was used as a counterexample to a question of Peternell in [16]. The proof is also interesting: it uses the fact that $Z_H$ is a Lagrangian submanifold of $X$. Now the fact that $Z_H$ is not big is essentially a consequence of the Hodge–Riemann relations (see Proposition [14]).

To bound the other half of the effective cone of $X$, we show the following:
Lemma 10. The second Chern class $c_2(X) = 5g^2 - 8c$ has no effective multiple.

Proof. This follows from Proposition 3 and Proposition 2, as we may specialize $Y$ so that $F(Y)$ is the Hilbert scheme of points on a Kummer surface.

The two previous lemmas together give a bound for the effective cone $\text{Eff}_2(X)$:

Corollary 11. Let $X$ be the variety of lines on a very general cubic fourfold. Then

$$\text{Eff}_2(X) = \mathbb{R}_{\geq 0}c + \mathbb{R}_{\geq 0}(g^2 - \lambda c)$$

for some $1 \leq \lambda \leq \frac{8}{5}$.

Proof of Theorem 1. From the intersection numbers (6), we find that no matter what the value of $\lambda$ is, the cone in (7) has a dual cone which strictly contains it. For example, $\text{Nef}_2(X)$ contains the classes $20c - g^2$ and $3g^2 - 5c$ for any $\lambda$ in the above interval. However neither of these classes are effective. It follows that $\text{Nef}_2(X) \supseteq \mathbb{R}_{\geq 0}(20c - g^2) + \mathbb{R}_{\geq 0}(3g - 5c) \supseteq \text{Eff}_2(X)$.

Remark 12. Although $c_2(X)$ is nef for very general cubics, there might be other cubics for which it is not. Indeed, taking $Y$ to contain a plane, we obtain a surface $P = \mathbb{P}^2 \subset F$, on which $c_2(X) \cdot P = -3$.

Remark 13. In his thesis [17], M. Rempel also considered examples of pseudo-effective cycles on hyperkähler fourfolds, in particular on the variety of lines on a cubic fourfold. Among his many results, he shows that the class $3g^2 - 5c$ (which is proportional to $c_2(X) - \frac{1}{5}g^2$) has no effective multiple. The proof involves a nice geometric argument using Voisin’s rational self-map $\varphi : F \dashrightarrow F$. He also conjectures that $\lambda = 1$ in the description of the effective cone above.

5 Lagrangian submanifolds

Using the same method as in Voisin’s proof of [22, Proposition 2.4], one obtains the following

Proposition 14. Let $X$ be an holomorphic symplectic variety of dimension four and let $Y \subset X$ be a Lagrangian surface. Then $[Y]$ is in the boundary of $\text{Eff}_2(X)$.

The classes of Lagrangian subvarieties of $X$ thus span a face of the effective cone $\text{Eff}_2(X)$ (which may have codimension $\geq 2$). Note that if $Y$ is not Lagrangian, then $\omega|_Y \wedge \overline{\omega}|_Y$ defines a volume form on $Y$ and so $[Y] \cdot \omega \wedge \overline{\omega} > 0$. It follows that this face can be described as the set of classes orthogonal to $\omega \wedge \overline{\omega}$.

Remark 15. Proposition [13] can be used to show that certain classes are extremal in other examples:

(i) If $S$ is a K3 surface and $C \subset S$ is a divisor, then $C^{[n]} \subset S^{[n]}$ is Lagrangian.

(ii) Any surface with $p_g = 0$ (e.g., a rational surface) in a hyperkähler fourfold $X$ is Lagrangian, hence extremal.
For a concrete example, if \( Y \subset \mathbb{P}^5 \) is a cubic fourfold containing a plane \( P \), then the dual plane \( P^\vee \subset X = F(Y) \) parameterizing lines in \( P \) is extremal. (See also [17, §3.2]). It was shown by Amerik in [1] that for \( Y \subset \mathbb{P}^5 \) a generic cubic fourfold, any Lagrangian subvariety of \( F(Y) \) must have a cohomology class proportional to \( c \).

6 Generalized Kummer varieties

Let \( X \) denote a generalized Kummer variety of dimension 4 of an abelian surface \( A \). Recall that these are defined as the fiber over 0 of the addition map \( A^3 \to A \). In this case, the group \( H^2(X, \mathbb{Z}) \) can be identified with \( H^2(A, \mathbb{Z}) \oplus \mathbb{Z} e \), where the class \( e \) is one half of the divisor corresponding to non-reduced subschemes in \( A^3 \). As before, \( c_2(X) \) represents a positive multiple of the Beauville–Bogomolov form [11] and the decomposition of \( H^2(X, \mathbb{Z}) \) is orthogonal with respect to this form. These varieties are however not deformation equivalent to Hilbert schemes of points on K3 surfaces (e.g., as their Betti numbers are different).

By [11, §4], we have a decomposition \( H^4(X) = S^2H^2(X) \oplus \perp 1_X \) where \( 1^n \) denotes a \( \mathbb{Z}^n \) with the intersection form represented by the identity matrix. There is a related decomposition

\[
H^4(X) = S^2H^2(X)^0 \oplus \perp 1_X^{81}
\]

where \( S^2H^2(X)^0 = c_2(X)^\perp \cap S^2H^2(X) \). On a general deformation of \( X \), we have \( S^2H^2(X)^0 = 0 \).

There are 81 distinguished rational surfaces on \( X \), whose classes are linearly independent in \( H^4(X, \mathbb{Q}) \). For each \( \tau \in A \) let \( W_\tau \) denote the locus in \( A^3 \) of subschemes supported at \( \tau \). As shown in [11], these are all isomorphic to a weighted projective space \( \mathbb{P}(1, 1, 3) \). Moreover, when \( \tau \in A^3 \) is a 3-torsion point on \( A \), \( W_\tau \) is a subvariety of \( X \).

Now, for \( p \in A \) there is another special surface in \( X \) given by the closure of the locus of subschemes \( (a_1, a_2, p) \in A^3 \) where \( a_1 + a_2 + p = 0 \in A \). This is isomorphic to the Kummer surface of \( A \) blown-up at a point. Now, Hassett–Tschinkel showed that as \( p \to \tau \in A^3 \), the flat limit of \( Y_p \) breaks into two pieces, \( W_\tau \) and a surface \( Z_\tau \). For example, \( Z_0 \) is the closure of the locus of points \( (0, a, -a) \) with \( a \in A - 0 \) (and the other \( Z_\tau \) are translates of this via the group \( A^3 \)). Moreover, [11, Proposition 5.1] says that

\[
c_2(X) = \frac{1}{3} \sum_{\tau \in A^3} |Z_\tau|.
\]

In particular, \( 3c_2(X) \) represents an effective cycle. As before, this class cannot be big, as it has intersection 0 with a Lagrangian fibration on a deformation of \( X \). Using Proposition [13] we get the following

**Proposition 16.** Let \( X \) be a very general projective generalized Kummer fourfold. The surfaces \( Z_\tau, W_\tau \) are all extremal, and span two 81-dimensional faces of the effective cone \( \text{Eff}_2(X) \). The second Chern class \( c_2(X) \) is effective, but is not big.
Interestingly, the surfaces $Z_\tau$ survive even when passing to a \textit{non-projective} deformation of $X$; they are trianalyic subvarieties in the sense of \cite{20}. In particular, $c_2$ is effective even on any smooth hyperkähler manifold of Kummer type.

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