Dynamic decoupling in reinforced concrete columns in structural core shape and applied to bridges

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Abstract—Bridges that are designed to transpose large valleys generally show low stiffness to the lateral bending and, therefore, it is not convenient to use the static modeling of the wind actuation. In this context, due to the large volume of concrete required for the molding of the pillars, it is fitting to design them with a cross section composed of thin walls, in order to confer economy, and with one of the faces open and braced by lintels to guarantee greater stiffness to the deformation by bending due to the performance of the shear forces. Such dynamic modeling of the pillars activates bending-torsion and presents stiffness to the structure bending divided into two matrices: \([J]\) and \([S]\). Due to the structural stiffness not being bonded in a single matrix, Rayleigh’s proportionality in the damping matrix assembly \([C]\) does not apply. Thus, the formulation for such an assembly of \([C]\) is proposed in this paper via the first three vibration modes of the structure. It models, with examples of application, a bridge with a deck backed on three pillars shaped as structural core braced by lintels and of thin-walled sections. Such pillars, with axes referenced in the torsional center of each, are inclined of \(\beta\) in relation to the global referential of the bridge coordinates, in order to apply the decoupling process of the differential equation system that governs the phenomenon and the resulting referential transformations. The vibration modes are verified by modeling with the ANSYS 2019 R1 software academic version.

Keywords—Dynamic decoupling, Bridge pillars, Wall panels, Proportional damping, Bending-Torsion.

I. INTRODUCTION

In the dynamic analysis of the usual structures, it is convenient to model the stiffness of the elements that compose them and group them into the unique stiffness matrix \([K]\) of the structure in the global coordinate reference. As for the pseudo-force due to the inertia that the structure exhibits when moving during the dynamic load, it is computed in the product of the mass matrix \([M]\) by the acceleration vector of the linear and angular displacements \(\{\ddot{\theta}\}\). Finally, the viscous damping of the structure is modeled by a damping matrix \([C]\) proportional to the others, where such proportionality is determined by modal analysis and the resulting first two vibration modes. Thus, it is carried out following the example of Blume [1], and Fleming and Romualdi [2].

On the other hand, the analysis of the pillar with cross section composed of thin walls results in two stiffness matrices. The first \([J]\) refers to the portion of the bending stiffness resulting from the pillar itself, and the second \([S]\) relates to the bending stiffness of the lintels that promote the pillar bracing along the height at the open end of the cross section. When highlighting the equation of the mentioned Wall Panels, Smith and Taranath [3] and also Stamato and Mancini [4] are mentioned. In such analyses, the static imposition of the loading is done. However, in the dynamic analysis, which will be carried out here, the inertial pseudo-force portion is added to the motion, and so the Partial Differential Equations system (PDE’s) is constituted by three matrices and, to that end, the proportionality procedure for the assembly of the damping matrix \([C]\) of the thin wall pillars is proposed. For such a proposition, the decoupling of the PDE system based on Rosman [5] and necessary transformations of referential via Glück [6] are used. Finally, the wind effect is adopted as a function of two variables \(q(x, t)\), in terms of the height \(x\) of the bridge pillar and under variation over time \(g(t)\). To that end, the positioning of the center of mass is needed for such
equation, as inspired by Hart [7], Vaicaitis [8], Laier [9] and Awruch [10].

The modeling of the wind action in the structures is processed, in the majority, from the perspective of the imposition of static loads. However, for the case of high-section thin-walled pillars (in works of art), lateral rigidity is reduced, making dynamic analysis of such a request important. This article proposes the systematization of the dynamic analysis of such requests in the pillars, thus providing a tool for manual calculation or even a mechanism of verification of the results obtained by modeling by commercial software.

II. THE WALL PANEL THEORY

The bridge pillar, considered of cross section made of thin and open walls, is conveniently analyzed through the Wall Panel Theory. Such technique consists of subdividing the core-shaped pillar into several walls and equating the balance of forces and moments for the internal forces and compatibilizing loads at the intersections of said walls. Finally, the differential equations of deflections and deformations are used. See Figure 1.

![Wall panel diagram](image)

**Fig. 1: Wall panels: (a) Drawing dimension notations, (b) Wall referential transformation**

Such equation results in Ordinary Differential Equations (ODEs) in terms of the rotation \( \nu' \), third derivative from the displacements \( \nu \), and the shear forces \( \{ V_f \} \) generated by external loads. The referred equations are written in terms of the center of gravity (CG) referential as:

\[
-J[\nu'_{CG}] + [S]\{ v_{CG}' \} = \{ V_f \}
\]

where: 
- \([J]\) - Bending stiffness matrix of the Wall Panels;
- \([S]\) - Lintel bending stiffness matrix;
- \(\{ v_{CG}' \}\) - Vector of linear displacements: \( \nu \) (towards \( y \)), \( \omega \) (towards \( z \)) and angulars \( \phi \)

(1)

The aforementioned pseudo-force is expressed as:

\[
F = \int_0^H (m \ddot{z}) \, dx
\]

(2 a)

\[
m = \frac{M}{H}
\]

(2 b)

where: \( m \) – distributed mass; \( M \) – total column mass; \( H \) – column height.

For the tridimensional structure, there is:

\[
\{dF\} = [M]\{ \ddot{\nu} \} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & l_p \end{bmatrix} \begin{bmatrix} \ddot{\nu} \\ \ddot{\omega} \\ \ddot{\phi} \end{bmatrix}
\]

(3)

where: \( l_p \) – Column inertia moment; \([M]\) – Mass matrix in the CM

Because the dynamic plot expressed in equation 3 is referenced in the center of mass (CM) and the static plot in the center of gravity (CG), the coordinate transformation of the (CM) into the (CG) turns necessary. See Figure 3.
Fig. 3: Mass and gravity centers: (a) Positioning, (b) Referential transformation of the CM into CG

The relation of the displacements in the Center of mass to the detriment of the Center of gravity is worth:

\[ v_{CM} = v_{CG} - d_{CMCG} \phi_{CG} \]  \hspace{1cm} (4 a)

\[ \omega_{CM} = \omega_{CG} + d_{CMCG} \phi_{CG} \]  \hspace{1cm} (4 b)

\[ \phi_{CM} = \phi_{CG} \]  \hspace{1cm} (4 c)

Matrically, the equations (4), expressed as law of coordinate transformation of the Center of Mass into the Center of Gravity, are expressed as:

\[ \{v_{CM}\} = [\tilde{T}_M].\{v_{CG}\} \]

\[ \omega_{CM} \]

\[ \phi_{CM} \]

where: \( \{v_{CM}\} \) – Vector of displacements in the Center of mass; \( \{v_{CG}\} \) – Vector of displacements in the center of gravity; \( [\tilde{T}_M] \) – Matrix of linear transformation from CM into CG.

Resulting from the transformation Law expressed in equation (5), the Wall Panel movement plot of equation (3) is rewritten as:

\[ dF_{CM} = [M].\{\ddot{v}_{CM}\} = [\tilde{T}_M]^T.[M].[\tilde{T}_M].\{\ddot{v}_{CG}\} \]  \hspace{1cm} (6)

Deriving equation (1) and adding equation (6), results in the ODE’s system that governs the dynamic problem of the Wall Panels, expressed by:

\[ -[J].\{\dddot{v}_{CG}\} + [S].\{\ddot{v}_{CG}\} + [\tilde{M}].\{\ddot{v}_{CG}\} = \{V_f\} \]  \hspace{1cm} (7)

where: \( [\tilde{M}] = [\tilde{T}_M]^T.[M].[\tilde{T}_M] \); \( [\tilde{M}] \) – Matrix of referenced mass in the CG, and: \( V'_{ext} = 2. D_1.x + D_2; \)

\[ V''_{ext} = \frac{q_1}{2H} \{V_f\} = V_{ext}.\{A'\}; \]

\[ V''_{ext} = D_1.x^2 + D_2.x + D_3; \]

\[ V''_{ext} = D_3 = Q = q_1.H. \]

IV. DYNAMIC DECOUPLING

The first procedure is to rewrite the system of ODEs (equation 7) in the main frame of the bending pillar stiffness \([J']\) by linear transformation by rotation matrix \([R_e]\). Then, \([J']\) is transformed into \([J''']\) which is equivalent to the matrix Identity \([I]\) via pre-multiplication and multiplication by \([J']^{-1/2}\) in order to maintain the symmetry of the other matrices and guarantee subsequent modal orthogonality and be able to use transposed matrices instead of inverse matrices. Finally, \([M''']\) and \([S''']\) are diagonalized through an iterative process until all terms outside the main diagonals of the matrices become of inferior module at a determined tolerance \(\varepsilon\). The set of decoupled PDE’s in the generalized referential, for the nondamped Wall Panels is expressed as:

\[ -q''''_{j}(x,t) + s_j.q''_{j}(x,t) + m_j.\ddot{q}_{j}(x,t) = V_{fj}^0(x,t) \]  \hspace{1cm} (8)

where: \( s_j \) is the \( j^{th} \) element of the main diagonal of the stiffness matrix of the diagonalized lintel bending \([s]\); \( m_j \) is the \( j^{th} \) element for the diagonalized mass matrix \([m] \); and \( j \) is the \( j^{th} \) vibration mode of the structure.

In the second procedure, the term of the damping \([\mathcal{C}].\{\ddot{v}''\} + [S].\{\ddot{v}'''\} \) is added with the damping matrix \([\mathcal{C}]\) constituted in terms proportional to the other matrices. Thus, the PDE system of viscously damped vibration for the Wall Panels is expressed by:

\[ [\tilde{M}].\{\ddot{v}_{CG}\} + [\mathcal{C}].\{\ddot{v}_{CG}''\} + [S].\{\ddot{v}_{CG}'''\} - [J].\{v_{CG}'\} = \{V_f\} \]  \hspace{1cm} (9)

where: \([\mathcal{C}] = \alpha_m.[\tilde{M}] + \alpha_s.[S] - \alpha_f.[J], \) \([\mathcal{C}] \) – Viscous damping matrix.

In the present decoupling, three referential transformations are made, where: the first transformation consists of diagonalizing the mass matrix \([\tilde{M}] \) referenced in the CG by means of diagonalization routine (for example the Jacobi Method); in the second transformation, \([m']\) becomes the identity matrix \([I]\) by pre-multiplication and multiplication by \([m']^{-1/2}\); and finally, in the third transformation, a looping process is applied for the joint diagonalization of \([S'']\) and \([J''']\) up to a certain tolerance adopted \(\varepsilon_S\) and \(\varepsilon_f\). See Flow Chart shown in Figure 4.

And resulting as a set of decoupled PDE’s, expressed as:

\[ \ddot{q}_{j}(x,t) + c_j.q''''_{j}(x,t) + s_j.q''_{j}(x,t) - j_{j}.q''_{j}(x,t) = V_{fj}^0(x,t) \]  \hspace{1cm} (10)
5. MODAL ANALYSIS

The ODE is written in the space x and temporal in t by means of the PDE’s resolution by the Method of Modal Superposition for the nondamped vibration as:

\[ j_{ij} u^{(j)}(x) - s_j u^{(i)}(x) - \omega^2 u_j(x) = 0 \]  
(11 a)

\[ \ddot{u}(t) - \omega^2 u(t) = 0 \]  
(11 b)

where: \( u(x, t) = u(x), g(t) \), \( \omega_j \) is the jth vibration frequency of the structure.

The analysis of equation (11 a) is carried out in order to determine the vibration modes and correlate \( s_j \) and \( f_j \). Then, \( u(x) = e^{\alpha x} \) is applied and the characteristic equation is expressed by:

\[ j_{ij} \alpha^4 - s_j \alpha^2 - \omega_j^2 = 0 \]  
(12)

Where the roots of the characteristic equation, equation (12), \( \alpha_1 \) and \( \alpha_2 \), and the relation \( j_{ij} \) and \( s_j \) in terms of the jth vibration frequency of the structure expressed by:

\[ \alpha_1 = \sqrt{\left( \frac{s_j}{2j_{ij}} \right)^2 + \frac{\omega_j^2}{j_{ij}}} + \frac{s_j}{2j_{ij}} \]  
(13 a)

\[ \alpha_2 = \sqrt{\left( \frac{s_j}{2j_{ij}} \right)^2 + \frac{\omega_j^2}{j_{ij}}} - \frac{s_j}{2j_{ij}} \]  
(13 b)

\[ j_{ij} = \frac{s_j \alpha_1^2 + \omega_j^2}{\alpha_1^4} \]  
(13 c)

The solution of the displacements \( u(x) \) is expressed as:

\[ u(x) = C_1 \sinh(\alpha_1 x) + C_2 \cosh(\alpha_2 x) + C_3 \sin(\alpha_3 x) + C_4 \cos(\alpha_4 x) \]  
(14)

Adopting the nondimensionalization of the roots \( \alpha \) into \( \lambda \), via \( \lambda_1 = \alpha_1 H \) and \( \lambda_2 = \alpha_2 H \), roots \( \lambda_1 \) and \( \lambda_2 \) are correlated by:

\[ \lambda_1^2 - \lambda_2^2 = \lambda^2 \]  
(15 a)

\[ \lambda = H \sqrt{\frac{s_j}{f_j}} \]  
(15 b)

Also, by applying the nondimensionalized roots \( \alpha_1 \) and \( \alpha_2 \) in the form of \( \lambda_1 \) and \( \lambda_2 \) and multiplying the latter, the frequencies of the structure are expressed as:

\[ \omega_j = \omega_0 \sqrt{\frac{f_j}{H^2}} \]  
(16 a)

\[ \omega_j^2 = \lambda_1 \lambda_2 \]  
(16 b)

The pillar analyzed here is of the fixed-in-the-base type, therefore, with displacement \( u(x = 0) = 0 \) null and rotation \( u'(x = 0) = 0 \) null. At the top, the end is free and, at a subsequent moment, bending and stress are shear and null, thus applying: \( u''(x = H) = u'''(x = H) = 0 \), respectively. Applying such boundary conditions in equation (14), after due nondimensionalization, the linear equation system results in:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \lambda_1 & 0 & -\lambda_2 \\
0 & \lambda_2 & \lambda_1 & 0 \\
0 & 0 & -\lambda_2 & \lambda_2
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4
\end{bmatrix}
= \begin{bmatrix}
0 \\
\lambda_2 \cosh(\lambda_1) \\
\lambda_1 \sinh(\lambda_1) \\
-\lambda_2 \cosh(\lambda_2) \\
\lambda_1 \sinh(\lambda_2) \\
-\lambda_2 \cosh(\lambda_2) \\
\lambda_2 \sinh(\lambda_2) \\
-\lambda_2 \cosh(\lambda_2)
\end{bmatrix}
\]

(17)

The nontrivial solution of the system, shown in equation (17), leads to the transcendental equation:

\[ f_1 \cosh(\lambda_1) \cos(\lambda_2) + f_2 \sinh(\lambda_1) \sin(\lambda_2) = -1 \]  
(18)
with: \[ f_1 = \frac{2. \lambda_1^3 \lambda_2^2}{\lambda_1 \lambda_2^2 + \lambda_1^2 \lambda_2}, \quad f_2 = \frac{\lambda_1^2 \lambda_2^2 - \lambda_1^4}{\lambda_1 \lambda_2^2 + \lambda_1^2 \lambda_2} \]

Highlighting that, the period will be:
\[ T_j = \frac{2. \pi}{\omega_j} = T_j^* \frac{H^2}{\sqrt{j_j}} \quad (19a) \]
\[ T_j^* = \frac{2. \pi}{\lambda_1 \lambda_2} \quad (19b) \]

In Figure 5, the change in the elastic line in the first four vibration modes is presented for \( \lambda^2 = 0 \). Figure 6 shows the ANSYS modeling, evidencing some vibration modes of the column with open cross section and thin walls.

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\[ T_j^* = \frac{2. \pi}{\lambda_1 \lambda_2} \]

| Mode | Mode | Mode | Mode | Mode | Mode | Mode |
|------|------|------|------|------|------|------|
| \( \lambda_1 \) | 1.87510 | 4.69409 | 7.85483 | 9.08911 | 10.02092 | 11.68901 | 14.62618 |
| \( \lambda_2 \) | 1.87510 | 4.69409 | 7.85483 | 9.08911 | 10.02092 | 11.68901 | 14.62618 |
| \( \omega_j \) | 3.51600 | 22.03448 | 61.69835 | 84.61192 | 105.97714 | 136.63295 | 213.92514 |
| \( T_j^* \) | 1.78703 | 0.28515 | 0.10144 | 0.07606 | 0.05929 | 0.04599 | 0.02937 |

\[ c_j = \alpha_1 \gamma_1 \frac{\lambda_1^2 \lambda_2}{\lambda_1 \lambda_2^2 + \lambda_1^2 \lambda_2} \]

\[ \omega_j = \omega_{J_1} \]

\[ \xi_j = \frac{\omega_j}{\gamma_1} \]

\[ \lambda_j = \alpha_2 \gamma_2 \frac{\lambda_1^2 \lambda_2}{\lambda_1 \lambda_2^2 + \lambda_1^2 \lambda_2} \]

\[ f_1 = \frac{2. \lambda_1^3 \lambda_2^2}{\lambda_1 \lambda_2^2 + \lambda_1^2 \lambda_2}, \quad f_2 = \frac{\lambda_1^2 \lambda_2^2 - \lambda_1^4}{\lambda_1 \lambda_2^2 + \lambda_1^2 \lambda_2} \]

\[ \xi_j = \frac{\omega_j}{\gamma_1} \]

\[ c_j^0 = 2. \xi_j \omega_j \]

VI. PROPORTIONAL DAMPING

Procedure 2 of diagonalization of the PDE’s system presented in equation (9) results in:
\[ c_j = \alpha_1 \gamma_1 \frac{\lambda_1^2 \lambda_2}{\lambda_1 \lambda_2^2 + \lambda_1^2 \lambda_2} \]

The analysis of the PDE shown in equation (10) and subsequent application of \( u(x) = e^{\alpha x} \) and \( g(t) = e^{\omega t} \), results as a characteristic equation of the damped free vibration:
\[ \omega^2 + (\xi_j \omega^4) \omega + (s_j \alpha^2 - j_j \alpha^4) = 0 \]

The resolution of the characteristic equation, equation (21), with the imposition of critical damping and subsequent \( u \) of the damping ratio \( \xi_j \) implies \( c_j^0 \) equals to:
\[ c_j^0 = 2. \xi_j \omega_j \]

where: \( \xi_j \) – damping ratio of the \( j \)-th vibration mode.

Equations 13 (c) and 15 (b) with due nondimensionalization of the roots \( \lambda_1 = \alpha_3 \gamma_1 \) and \( \lambda_2 = \alpha_2 \gamma_2 \), result in \( s_j \) and \( f_j \) in terms of the frequency \( \omega_j \), as:
\[ j_j = \frac{\omega_j^2}{(\lambda_1^2 - \lambda_j^2 \lambda_1^2)} \]  
\[ s_j = \lambda^2 j_j \]  
\[ (23a) \]
\[ (23b) \]

Finally, since it is only desired to establish proportions of the matrices \[ [M], [J] \] and \[ [S] \] that form the damping matrix \[ [C] \], the application of equations (22) and (23) in the condition expressed in equation (20) and using the frequency \[ \omega_j \] instead of \[ \omega_j \], since it is only desired to establish proportions of the matrices results in:

\[ \xi_j = \frac{\alpha_j}{2. \alpha_j} + \frac{\alpha_j - s_j}{2. \alpha_j} j_j \]  
\[ (24) \]

Imposing the first three vibration modes \( j = 1, 2 \) and \( 3 \) in equation (24), a system of equations is built, whose solution is:

\[ a_m = \frac{2(\omega_1^2 \xi_1(j_2-s_2-j_3-s_3) + \omega_2^2 \xi_2(j_3-s_3-j_1-s_1) + \omega_3^2 \xi_3(j_1-s_1-j_2-s_2))}{j_1(j_2-s_2) + j_2(j_3-s_3) + j_3(j_1-s_1)} \]  
\[ (25a) \]

\[ a_s = \frac{2(\omega_1^2 \xi_1(j_1-j_2) + \omega_2^2 \xi_2(j_1-j_2) + \omega_3^2 \xi_3(j_1-j_2))}{j_1(j_2-s_2) + j_2(j_3-s_3) + j_3(j_1-s_1)} \]  
\[ (25b) \]

\[ a_j = \frac{2(\omega_1^2 \xi_1(s_2-s_3) + \omega_2^2 \xi_2(s_1-s_3) + \omega_3^2 \xi_3(s_1-s_2))}{j_1(j_2-s_2) + j_2(j_3-s_3) + j_3(j_1-s_1)} \]  
\[ (25c) \]

In percentual terms there are:

\[ \mu_m = \frac{\alpha_m}{\alpha_m + \alpha_s + \alpha_j} \]  
\[ (26a) \]

\[ \mu_s = \frac{\alpha_s}{\alpha_m + \alpha_s + \alpha_j} \]  
\[ (26b) \]

\[ \mu_j = \frac{\alpha_j}{\alpha_m + \alpha_s + \alpha_j} \]  
\[ (26c) \]

**a) Center of Mass Positioning**

For the reinforced concrete, cross, and thin-walled open section, the determination of the center of mass coordinates (CM) is made necessary. To this end, the following definitions are used:

\[ x^*_{CM} = \frac{\sum_{i=1}^{n} x_{CM_i} \cdot m_i}{\sum_{i=1}^{n} m_i} \]  
\[ (27a) \]

\[ y^*_{CM} = \frac{\sum_{i=1}^{n} y_{CM_i} \cdot m_i}{\sum_{i=1}^{n} m_i} \]  
\[ (27b) \]

where: \( x_{CM_i} \) and \( y_{CM_i} \) - coordinates of the canonic center of mass \( i \) in relation to a generic referential \( x^* \) and \( y^* \) defined in the cross section; \( m_i \) - canonic element mass \( i \); \( n \) - number of canonic masses that formed in core; and \( x^*_{CM} \) and \( y^*_{CM} \) - the center of mass coordinates of the cross section in core shape in relation to referential \( x^* \) and \( y^* \).

The adoption of the subdivision presents in Figure 7, the concrete specific mass \( \rho_{CS} = 2400 \text{ kg/m}^3 \) steel \( \rho_S = 7860 \text{ kg/m}^3 \) results in:

\[ x^*_{CM} = \frac{x_{CM_1} \cdot m_1 - x_{CM_2} \cdot m_2 - x_{CM_3} \cdot m_3 + x_{CM_4} \cdot M_4 + x_{CM_5} \cdot M_5}{m_1 - m_2 - m_3 - M_4 - M_5} \]  
\[ (28a) \]

\[ y^*_{CM} = \frac{y_{CM_1} \cdot m_1 - y_{CM_2} \cdot m_2 - y_{CM_3} \cdot m_3 - y_{CM_4} \cdot M_4 + y_{CM_5} \cdot M_5}{m_1 - m_2 - m_3 - M_4 - M_5} \]  
\[ (28b) \]

**Fig. 7: Geometry of decomposition into canonic format masses**

Finally, the distances between the centers of mass (CM) and that of gravity (CG) are valid, according to what is shown in Figure 8:

\[ d_{YMC} = y^*_{CM} - y^*_{CG} \]  
\[ (29a) \]

\[ d_{XMC} = x^*_{CM} - x^*_{CG} \]  
\[ (29b) \]

**Fig. 8: Positioning of the CG and CM in the cross section**
VII. REFERENTIAL TRANSFORMATION

a) Transfer from the center of gravity CG to the torsional center D

According to the compatibilization of the rotations $\phi_D$ (in the torsional center) and $\phi_{CG}$ (in the center of gravity), the displacements $\omega_D$ via $\omega_{CG}$, and projection of the distance $d_{jCGD}$ from the center of gravity to the torsional center parallel to $z$ axis, procedures are the same for $v_D$ with $v_{CG}$ and $d_{zCGD}$. See Figure 9.

Matrically, the balance is expressed as:

$$\{v_D\} = [T_s]^{-1}.\{v_g\}$$

(32a)

$$\begin{pmatrix}
\cos \beta & \sin \beta & 0 \\
-\sin \beta & \cos \beta & 0 \\
0 & 0 & 1 \\
\end{pmatrix}^{-1}
\begin{pmatrix}
0 & 1 & -d_{zDOS} \\
0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
\omega_D \\
\phi_D \\
\end{pmatrix}
\begin{pmatrix}
v_D \\
\phi_g \\
\end{pmatrix}$$

(32b)

Combining equations (30) and (32), direct transfer from CG to Os is as follows:

$$\{v_g\} = [T_s]^{-1}.\{T_g\}.\{v_{CG}\}$$

(33)

b) Transfer from the torsional center D for the source of the coordinates $O_s$

Analogously as in (a) of this item, and considering the rotation $\beta$ between the referential $(y_{DL},z_{DL},x_{DL})$ in the torsional center and the global referential $(y_g,z_g,x_g)$, see Figure 10, there is:

$$v_D = v_{CG}$$

$$\omega_D = \omega_{CG}$$

$$\phi_D = \phi_g$$

(31c)

$$\phi_D = \phi_g$$

(31c)

$$v_D = v_{CG}$$

$$\omega_D = \omega_{CG}$$

$$\phi_D = \phi_g$$

(31c)

$$v_D = v_{CG}$$

$$\omega_D = \omega_{CG}$$

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(31c)

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(31c)

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$$\omega_D = \omega_{CG}$$

$$\phi_D = \phi_g$$

(31c)

$$v_D = v_{CG}$$

$$\omega_D = \omega_{CG}$$

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The shear force, over time, will be governed by a harmonic sinusoidal function \( V(t) = \sin(\omega t) \), with frequency \( \omega = 3 \text{ rad/s} \). In addition, the PDE system of equation (9) is expressed for the pillars and referenced in the center of gravity as:

\[
\begin{bmatrix}
7870.65 & 0 & 387.78 \\
0 & 7870.65 & 1506.08 \\
387.78 & 1506.08 & 319.73
\end{bmatrix} \begin{bmatrix}
\ddot{y} \\
\dot{\omega} \\
\phi
\end{bmatrix}
+ \begin{bmatrix}
120073 & 0 & -18909 \\
0 & 99853 & 5459 \\
-22378 & 0 & 174899
\end{bmatrix} \begin{bmatrix}
\frac{\ddot{\psi}}{} \\
\frac{\dot{\omega}}{} \\
\frac{\phi}{''''''}
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 6939 \\
0 & 0 & 10919 \\
0 & 0 & -12427
\end{bmatrix} \begin{bmatrix}
\frac{\ddot{\phi}}{} \\
\frac{\dot{\phi}}{} \\
\frac{\phi}{''''''}
\end{bmatrix}
- \begin{bmatrix}
240146 & 0 & -44757 \\
0 & 199706 & 0 \\
-44757 & 0 & 336224
\end{bmatrix} \begin{bmatrix}
\frac{\ddot{\phi}}{} \\
\frac{\dot{\phi}}{} \\
\frac{\phi}{''''''}
\end{bmatrix} = \{V\}
\]

The same PDE system, with the proportionality coefficients of the matrices \([M],[J] \text{ and } [S]\) to compose the viscous damping matrix \([C]\), is expressed as: \( \alpha_M = 0; \alpha_S = -3.80 \times 10^{16}; \alpha_J = -3.80 \times 10^{16}; \mu_M = 0; \mu_S = 0.5; \mu_J = 0.5 \).

And the set of decoupled PDE’s presented in equation (10) are:

\[
-\dddot{q}_1(x, t) + 0.03385.\dddot{q}_1(x, t) + 0.49519.\dddot{q}_1(x, t) - 0.00962.\dddot{q}_1(x, t) = V_P^x(x, t)
\]

\[
-\dddot{q}_2(x, t) + 0.04038.\dddot{q}_2(x, t) - 0.13363.\dddot{q}_2(x, t) - 1.26726.\dddot{q}_2(x, t) = V_P^x(x, t)
\]

\[
-\dddot{q}_3(x, t) + 0.00003.\dddot{q}_3(x, t) + 0.50243.\dddot{q}_3(x, t) + 0.00498.\dddot{q}_3(x, t) = V_P^x(x, t)
\]

(34)

In the set of EDP’s presented in equations (34), the first mode of vibration (mobilizing the lintels) is verified by crossing the equation (15 b) and the Figure 13.

Fig. 13: Column C in concrete C - 40, vibration modes activating the lintels for flexion: (a) 1st mode \((\omega_1 = 1,91493 \text{ rad/s})\), (b) 2nd mode and (c) 3rd mode via modeling in ANSYS

so, with the parameters \(s_j = -0.00962 \text{ and } j_j = -1\), and through equation (15 b) the parameter is determined \( \lambda = H.\sqrt{s_j^2/j_j^2} = 100 \text{ m},.\sqrt{-0.00962^2/-1} = 9.81, \text{ using Table 2 with the value immediately above } \lambda, \text{ then: } \omega_1^* = 2,54634 \text{ rad/s } \text{(for } \lambda = 10\).

Table 2: Vibrations modes of the Core Column by \( \lambda = 10 \)

| Mode | Mode | Mode | Mode | Mode | Mode |
|------|------|------|------|------|------|
| \(\lambda_1\) | 3.25746 | 5.0391 | 8.2839 | 9.64266 | 10.43138 | 11.87014 | 14.61565 |
| \(\lambda_2\) | 0.78169 | 4.50478 | 7.81267 | 9.10938 | 9.94051 | 11.44116 | 14.26945 |
| \(\omega_1^*\) | 2.54634 | 4.79389 | 65.84820 | 87.83870 | 103.69231 | 135.80821 | 208.55730 |
| \(T_i\) | 2.46754 | 0.25342 | 0.09542 | 0.07153 | 0.06059 | 0.04627 | 0.03013 |

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Finally, in Figure 13 (a) the first vibration mode of a single C-shaped pillar and braced with lintels is presented under the same distribution as this example, hence the reason for comparison, with an approximation of 2.24%. The frequency adjustment determined via TMC is used by the coefficient \(\alpha = \frac{1}{1.3}\), whose functionality is based on Dzięwolski [15].

\[
\Delta(\%) = \frac{[\omega_{\text{ANSYS}} - \omega_{\text{TMC}}]}{\omega_{\text{TMC}}} \cdot 100\% = \frac{|1.91493 - 1.95872| \text{rad s}^{-1}}{1.95872 \text{rad s}^{-1}} \cdot 100\% = 2.24\%
\]

The shear force \(V_f\) functions in the initial referential are presented in Table 3, as well as \(V_f^0\) in the generalized referential, and both for the columns \(P_1, P_2\) and \(P_3\).

**Table 3: Functions of the shear force per Column**

| Column | \(P_1\) | \(P_2\) | \(P_3\) |
|--------|--------|--------|--------|
| \(V_{f_1}\) | +119692.0 +4887.3 -1476.6 |
| \(V_{f_2}\) | +16023.0 -141.8 +26.4 |
| \(V_{f_3}\) | +44291.7 +24110.9 +728535.9 |
| \(V_f^0\) | +24.43 -13.0 -40.17 |
| \(V_{f_2}^0\) | +35.85 -42.31 +7.88 |
| \(V_{f_3}^0\) | +78.21 -42.57 -128.64 |

For the first vibration mode, the displacement function, \(u(x)\) and \(g(t)\), and the generalized and initial referential are, respectively:

\[
u_t(x) = 0.36275 \cdot \cosh(0.01983 \cdot x) \quad - \cos(0.01713 \cdot x) \\
+ 0.92535 \cdot \sin(0.01983 \cdot x) \\
+ 2.41421 \cdot \sin(0.01713 \cdot x)]
\]

\[
g_t(t) = 4.543 \cdot 10^{10} \cdot \sin(3.476 \cdot 10^{-3} \cdot t) \\
- 8.928 \cdot 10^6 \sin(3 \cdot t)
\]

\[
u_1(x) = 6.098 \cdot 10^{-4} \cdot (\cosh(1.847 \cdot 10^{-3} \cdot x) \\
- \cos(7.65 \cdot 10^{-4} \cdot x) \\
+ 1.556 \cdot 10^{-3} \cdot [-\sinh(1.847 \cdot 10^{-3} \cdot x) \\
+ 4.048 \cdot 10^{-3} \cdot \sin(7.65 \cdot 10^{-4} \cdot x)])
\]

\[
g_1(t) = 7.637 \cdot 10^{7} \cdot \sin(5.843 \cdot 10^{-6} \cdot t) \\
- 1.501 \cdot 10^6 \sin(3 \cdot t)
\]

Then, returning to the initial referential, the vector of displacements \(\{v_{CG}\}\) for the mode of larger deformations is written as:

\[
\{v_{CG}\} = [\Phi_R, \{q\}] = [\Phi_R, \{u_t(x), g_t(t)\}]
\]

where: \(\Phi_R = \{Re, [J']^{-1} \cdot [\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_{n-1}, \varphi_n]\}

And, finally, the rotation function \(\phi_{CG}(x)\) is expressed as:

\[
\phi_{CG}(x) = u_1(x) + u_2(x) + g_2(t) + u_3(x) + g_3(t)
\]

The relation between rotation \(\phi_{CG}(x)\) and the Bimoment \(B(x)\), originating from the Bending-Torsion Theory yields:

\[
B(x) = E \cdot I_{\omega^2} \phi_{CG}'(x) = E \cdot I_{\omega^2} \frac{d^2 [\phi_{CG}(x)]}{dx^2}
\]

After applying the referential transformations, we have the Bimoment in the Base and in the Top for the \(CG\), \(D\) and \(O_s\). See Table 4, as well as the rotation in Table 5.

**Table 4: Bimoment values per Pillar in the CG and D, and the set of pillars in \(O_s\)**

| \(x \cdot 10^7\) | \(CG_1\) | \(CG_2\) | \(CG_3\) | \(D_1\) | \(D_2\) | \(D_3\) |
|-----------------|----------|----------|----------|--------|--------|--------|
| \(B(x=0)\) | -3.99422 | 2.17432  | 6.56993  | -3.99422 | 2.17432 | 6.56993 |
| \(B(x=50)\) | -4.00790 | 2.18176  | 6.59242  | -4.00790 | 2.18176 | 6.59242 |
| \(B(x=H)\) | -4.04993 | 2.20464  | 6.66156  | -4.04993 | 2.20464 | 6.66156 |

**Table 5: Rotation values per Pillar in the CG and set of pillars in \(O_s\)**

| \(rad\) | \(CG_1\) | \(CG_2\) | \(CG_3\) | \(O_s\) |
|--------|----------|----------|----------|--------|
| \(\Phi(x=0)\) | 0 | 0 | 0 | Idem |
| \(\Phi(x=50m)\) | -57.42669 | 31.26114 | 94.45877 | Per |
| \(\Phi(x=H)\) | -233.82458 | 127.28615 | 384.60827 | Pillar |
EXAMPLE 2: In this second analysis, there is the braking force of the traffic and lateral cargo at the top. Thus, the shear force under the configuration shown in Figure 14 is presented as:

![Fig. 14: Angles formed between the external shear forces activated by braking in each Pillar and the axes y_{Di} in each of the three columns](image)

Adopting the same frequency \( \bar{\omega} = 3 \text{ rad/s} \) for the shear force, the values of the Top and Base Bimoments are expressed in Table 6.

| \( (x \times 10^5) \) | \( kN \cdot m^2 \) | In the Center of Gravity | In the Torsional Center | \( O_s \) |
|-----------------|----------------|------------------------|------------------------|-------|
| \( B(x = 0) \)  | 4.96373 | 2.70209 | 8.16464 | 4.96373 | 2.70209 | 8.16464 |
| \( B(x = \bar{x}) \) | 4.98075 | 2.71135 | 8.19264 | 4.98075 | 2.71135 | 8.19264 |
| \( B(x = H) \)  | 5.03302 | 2.73980 | 8.27860 | 5.03302 | 2.73980 | 8.27860 |

where the loadings are: \( q_1 = q_2 = 0 \text{ kN/m}, Q = 10 \text{ kN} \); the Top Bimoment \( B_H = 10 \text{ kN} \cdot m^2 \); and the torsional moment at the top is null \( M_{th} = 0 \text{ kN} \cdot m \).

IX. CONCLUSION

In this paper the implementation of the dynamic plots of acceleration and damping in the Wall Panel Theory was carried out, in order to promote the dynamic analysis of bridge pillars under the effect of Bending-Torsion and to provide the noncoincidence of the center of gravity with the torsional center, as a result of the geometry in thin-walled cross section. Moreover, the PDE system that governs the vibration of said Wall Panels has two bending stiffness matrices, being: \([L]\) for the pillar itself and \([S]\) for the lintels that promote bracing of said pillar. Thus, the Rayleigh proportional \([C]\) damping matrix assembly considers only a single stiffness matrix. At this point, an unpublished formulation is made for such a composition. Two procedures of diagonalization of the PDE system and transformations between the reference points of the center of gravity \((CG)\), the center of torsion \((D)\) and a global origin \((O_s)\) are discussed.

Such analyses encourage the of high columns in reinforced concrete subject of wind actuation, due to the large volume of concrete required for the molding of the pillars. It is fitting to design them with a cross section composed of thin walls, in order to confer economy, and with one of the faces open and braced by lintels.

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