Group Orders That Imply a Nontrivial $p$-Core

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Abstract. Given a prime number $p$ and a natural number $m$ not divided by $p$, we propose the problem of finding the smallest number $r_0$ such that for $r \geq r_0$, every group $G$ of order $p^r m$ has a non-trivial normal $p$-subgroup. We prove that we can explicitly calculate the number $r_0$ in the case where every group of order $p^r m$ is solvable for all $r$, and we obtain the value of $r_0$ for a case where $m$ is a product of two primes.

1. Introduction

Throughout this note, $p$ will be a fixed prime number. We use $O_p(G)$ to denote the $p$-core of $G$, that is, its largest normal $p$-subgroup.

We propose the following optimization problem: Given a number $m$ not divisible by $p$, find the smallest $r_0$ such that every group having order $n = p^r m$, with $r \geq r_0$, has a nontrivial $p$-core $O_p(G)$. Denote such number $r_0$ by $\Lambda(p, m)$. In Theorem 2.1 we will prove that $\Lambda(p, m)$ is well-defined for any prime $p$ and number $m$ (with $p \nmid m$). In Theorem 2.3 we explicitly determine the value of $\Lambda(p, m)$ in the case that all groups whose order have the form $p^r m$ are solvable (for example, if $m$ is prime or if both $p$ and $m$ are odd).

Finally, in Section 3 we calculate $\Lambda(2, 15)$, a case that is not covered by the previous theorem.

We remark that the motivation for this research came from the search for examples of finite groups $G$ such that the Brown complex $S_p(G)$ of nontrivial $p$-subgroups of $G$ (see for example [5] for the definition and properties) is connected but not contractible. It is known that $S_p(G)$ is contractible when $G$ has a nontrivial normal $p$-subgroup, and Quillen conjectured in [8] that the converse is also true.
2. Theorems

**Theorem 2.1.** For any prime number $p$ and natural number $m$ such that $p \nmid m$, there is a number $\Lambda(p, m)$ such that if $r \geq \Lambda(p, m)$, any group of order $p^r m$ has a non-trivial $p$-core $O_p(G)$.

*Proof.* Let $G$ be a group of order $p^r m$ with $O_p(G) = 1$. Let $P$ be a Sylow $p$-subgroup of $G$. Since the kernel of the action of $G$ on the set of cosets of $P$ is precisely $O_p(G)$, we obtain that $G$ embeds in $S_m$, and so $p^r$ divides $(m - 1)!$. Hence, if $p^{r_0}$ is the largest power of $p$ dividing $(m - 1)!$, we obtain that $\Lambda(p, m) \leq r_0 + 1$.

For $t, q$ natural numbers, let $\gamma(t, q)$ be the product

$$\gamma(t, q) = (q^t - 1)(q^{t-1} - 1)\cdots(q^2 - 1)(q - 1),$$

and if $m = q_1^{t_1} q_2^{t_2} \cdots q_k^{t_k}$ is a prime factorization of $m$, with the $q_i$ pairwise distinct and $t_i > 0$ for each $i$, we let $\Gamma(m) = \gamma(t_1, q_1)\cdots\gamma(t_k, q_k)$. We prove that if $p^{r_0}$ is the largest power of $p$ dividing $\Gamma(m)$, then $\Lambda(p, m) \geq r_0 + 1$.

**Theorem 2.2.** Let $n = p^s m$ where $p \nmid m$ and $s > 0$. If $p^s \mid \Gamma(m)$, then there is a group of order $n$ with $O_p(G) = 1$.

*Proof.* Let $K$ be the group $C_{q_1}^{t_1} \times \cdots \times C_{q_k}^{t_k}$, that is, a product of elementary abelian groups, where $m = q_1^{t_1} \cdots q_k^{t_k}$ and $q_1, \ldots, q_k$ are distinct primes and $C_q$ denotes the cyclic group of order $q$. Then $\Gamma(m)$ divides the order of $\text{Aut}(K)$, and hence so does $p^s$. Let $H$ be a subgroup of $\text{Aut}(K)$ of order $p^s$. For every $S \in H$ and $k \in K$ define the map $T_{S,k} : K \rightarrow K$ by $T_{S,k}(x) = Sx + k$. Then $G = \{ T_{S,k} \mid S \in H, k \in K \}$ is also a subgroup of $\text{Aut}(K)$. If we identify $H$ with the subgroup of maps of the form $T_{S,0}$ and $K$ with the subgroup of maps of the form $T_{1,k}$, then $G$ is just the semidirect product of $K$ by $H$. Hence $|G| = n$. We have that $G$ acts transitively on $K$ in a natural fashion, and the stabilizer of $0 \in K$ is $H$, a $p$-Sylow subgroup of $G$. Hence the stabilizers of points in $K$ are precisely the Sylow subgroups of $G$, so their intersection $O_p(G)$ contains only the identity $K \rightarrow K$, as we wanted to prove.

The next theorem will show that the lower bound given by Theorem 2.2 is tight in some cases.

**Theorem 2.3.** Let $n = p^s m$, where $p \nmid m$. If $G$ is a group of order $n$ and $p^s$ does not divide $\Gamma(m)$ then either:

1. $O_p(G) \neq 1$, or
2. $G$ is not solvable.

*Proof.* Let $G$ be solvable with order $n = p^s m$ and $O_p(G) = 1$. Let $F(G)$ be the Fitting subgroup of $G$. Consider the map $c : G \rightarrow \text{Aut}(F(G))$, sending $g$ to $c_g : F(G) \rightarrow F(G)$ given by conjugation by $g$. The restriction of $c$ to $P$, a $p$-Sylow subgroup of $G$, has kernel $P \cap C_G(F(G))$. Since $C_G(F(G)) \leq F(G)$ (Theorem 7.67 from [4]), and $F(G)$ does not contain elements of order $p$ by our assumption on $O_p(G)$, we have $P \cap C_G(F(G)) = 1$ and so $P$ acts faithfully on $F(G)$. If $m = q_1^{t_1} \cdots q_k^{t_k}$ is the prime factorization of $m$, we
have that $F(G)$ is the direct product of the $O_{q_i}(G)$ for $i = 1, \ldots, k$. Hence $P \leq \text{Aut}(F(G)) \cong \text{Aut}(O_{q_1}(G)) \times \cdots \times \text{Aut}(O_{q_k}(G))$. Let $g \in P$ such that the action induced by $c_g$ on $\prod_i O_{q_i}(G)/\Phi(O_{q_i}(G))$, is the identity. Since $c_g$ acts on each factor $O_{q_i}(G)/\Phi(O_{q_i}(G))$ as the identity, then by Theorem 5.1.4 from [2], we have that it acts as the identity on each $O_{q_i}(G)$. By the faithful action of $P$ on $F(G)$, we have that $g = 1$. This implies that $P$ acts faithfully on $\prod_i O_{q_i}(G)/\Phi(O_{q_i}(G))$. But then $|P|$ divides the order of the automorphism group of $\prod_i O_{q_i}(G)/\Phi(O_{q_i}(G))$, which is a product of elementary abelian groups of respective orders $q_i^{s_i}$ with $s_i \leq t_i$ for all $i$. Hence $p^s = |P|$ divides $\Gamma(m)$. □

Corollary 2.4. Let $p^s$ be the largest power of $p$ that divides $\Gamma(m)$. If $m$ is prime, or if both $p, m$ are odd, then $\Lambda(p, m) = s + 1$.

Proof. By Burnside’s $p,q$-theorem, and the Odd Order Theorem, we have that all groups that have order of the form $p^r m$ for some $r$ are solvable. Therefore, for all $r > s$, by Theorem 2.3 we have that all groups of order $p^r m$ have non-trivial $p$-core. □

At this moment, we can prove that in some cases, the group constructed in 2.2 is unique.

Theorem 2.5. Let $n = p^s m$ where $p \nmid m$ and $s > 0$. If $p^s \nmid \Gamma(m)$, but $p^s \nmid \Gamma(m')$ for all proper divisors $m'$ of $m$, then up to isomorphism, the group constructed in the proof of Theorem 2.2 is the only solvable group of order $n$ with $O_p(G) = 1$.

Proof. With the notation of the argument of the proof of 2.3, if $G$ is a solvable group of order $n$ with $O_p(G) = 1$, we must have that $|O_{q_i}(G)| = q_i^{t_i}$ and $\Phi(O_{q_i}(G)) = 1$ for all $i$ in order to satisfy the divisibility conditions. Hence $O_{q_i}(G)$ is elementary abelian and a $q_i$-Sylow subgroup for all $i$, and so $G$ is the semidirect product of a $p$-Sylow subgroup $P$ of $F(G) = C_{q_1}^{t_1} \times \cdots \times C_{q_k}^{t_k}$ with $F(G)$, where the action of $P$ on $F(G)$ by conjugation is faithful. Hence $G$ is isomorphic to the group constructed in the proof of Theorem 2.3. □

One case in that we may apply Theorem 2.5 is when $n = 864$. There are 4725 groups of order $864 = 2^5 3^3$, but only one of them has the property of having a trivial 2-core.

3. An example

An example that cannot be tackled with the previous results is the case $p = 2$, $m = 3 \cdot 5 = 15$. In this case, $\Gamma(15) = (3 - 1)(5 - 1) = 2^2$. Not all groups with order of the form $2^r \cdot 3 \cdot 5$ are solvable, however, we will prove that $\Lambda(2, 15)$ is actually 4. (The group $S_5$ attests that $\Lambda(2, 15) > 3$.)

Theorem 3.1. Every group $G$ of order $2^r \cdot 3 \cdot 5$ for $r \geq 4$ is such that $O_2(G) \neq 1$. 
Proof. Let $G$ be a group of order $2^r \cdot 3 \cdot 5$ for $r \geq 4$. Suppose that $O_2(G) = 1$. From Theorem 2.3 we obtain that $G$ is not solvable. We will prove then that $O_3(G) = 1$. Suppose otherwise, and let $T = O_3(G)$. Then $|G/T| = 2^r \cdot 5$, and so $G/T$ is solvable. Since $2^r \nmid \Gamma(5)$, from Theorem 2.3 we have that $O_2(G/T) \neq 1$. Let $L \triangleleft G$ such that $O_2(G/T) = L/T$. Suppose $|L/T| = 2^j$. Since $O_2(G/L) = 1$, $|G/L| = 2^{r-j} \cdot 5$ and $G/L$ is solvable, we have that $2^{r-j}$ divides $\Gamma(5) = 2^2$, that is, $r - j \leq 2$. Now, $L$ is also solvable and $\Gamma(3) = 3 - 1 = 2$, hence if we had $j \geq 2$ we would have $O_2(L) \neq 1$, and $G$ would have a non-trivial subnormal 2-subgroup, which contradicts our assumption that $O_2(G) = 1$. Hence $j = 1$. But then $r - 1 \leq 2$, which contradicts that $r \geq 4$. Hence $O_3(G) = 1$. By a similar argument, we get that $O_5(G) = 1$.

From [1] we obtain that $G$ is not simple. Hence $G$ has a proper minimal normal subgroup $M$. From the previous paragraph, we obtain that $M$ is not abelian, since in that case we would have that $M \leq F(G)$. The only possibility is that $M = A_5$. We have then a morphism $c: G \to \text{Aut}(A_5)$ sending $g$ to $c_g$, the conjugation by $g$. Since $\text{Aut}(A_5) = S_5$, and $|c(G)| = |\text{Inn}(G)| \geq |\text{Inn}(A_5)| = 60$, in any case the kernel of $c$ is a nontrivial normal 2-subgroup.

□

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