Stabilization of 2-D Mindlin-Timoshenko Plates with Localized Acoustic Boundary Feedback

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Abstract. In this paper, we investigate the well-posedness and the asymptotic stability of a two dimensional Mindlin-Timoshenko plate imposed the so-called acoustic control by a part of the boundary and a Dirichlet boundary condition on the remainder. We first establish the well-posedness results of our model based on the theory of linear operator semigroup and then prove that the system is not exponentially stable by using the frequency domain approach. Finally, we show that the system is polynomially stable with the aid of the exponential or polynomial stability of a system with standard damping acting on a part of the boundary.

1. Introduction. The stabilization of boundary value problem of deformed structures has been considerably stimulated during the past few decades by the wide applications of flexible materials in space technology and robot components (see [12, 13, 17, 19, 22, 33]). In particular, structures of interest are largely carried out in problems which require appropriate feedback mechanisms to stabilize the vibrating materials that may be inherently unstable without external control or that only have very weak natural damping. We refer the reader to ([6, 8, 10, 16, 24, 35, 36]) for some classic results on the well posedness and stabilizability by feedback controls. As a class of more comprehensive distributed parameter systems (Mindlin-Timoshenko Plate), the model contains shear effects in addition to displacement and rotational inertia effects, which reflects the vibration characteristics of thin plates.

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in general case more accurately. Therefore, there is an extensive literature (such as Lagnese [20], Sare [34], Dalsen [11] and Messaoudi [25]) on the well-posedness and the asymptotic stabilization of Mindlin-Timoshenko beam or plate systems. The conservative two-dimensional model is described as follows

\[ \begin{bmatrix} \psi_t, \varphi_t, \omega_t \end{bmatrix}^T - \left[ \rho_1 \ell_1, \rho_1 \ell_2, \rho_2 \ell_3 \right]^T = 0, \quad \text{in } \Omega \times (0, +\infty), \]

(1)

where \(\Omega \subset \mathbb{R}^2\) is an open bounded set and 0 denotes a three-dimensional zero column vector. \(\Gamma\) stands for the transpose symbol. \(\ell_1 := \ell_1(\psi, \varphi, \omega), \ell_2 := \ell_2(\psi, \varphi, \omega)\), and \(\ell_3 := \ell_3(\psi, \varphi, \omega)\) are given by

\[
\begin{aligned}
\ell_1(\psi, \varphi, \omega) &= D \left[ \frac{\partial \psi}{\partial x} \left( \frac{\partial \psi}{\partial x} + \mu \frac{\partial \varphi}{\partial y} \right) + \frac{1 - \mu}{2} \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial x} \right) \right] - K \left( \psi + \frac{\partial \omega}{\partial x} \right), \\
\ell_2(\psi, \varphi, \omega) &= D \left[ \frac{\partial \varphi}{\partial y} \left( \frac{\partial \psi}{\partial y} + \mu \frac{\partial \varphi}{\partial x} \right) + \frac{1 - \mu}{2} \frac{\partial}{\partial x} \left( \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y} \right) \right] - K \left( \varphi + \frac{\partial \omega}{\partial y} \right), \\
\ell_3(\psi, \varphi, \omega) &= K \left[ \frac{\partial}{\partial x} \left( \psi + \frac{\partial \omega}{\partial y} \right) + \frac{\partial}{\partial y} \left( \varphi + \frac{\partial \omega}{\partial x} \right) \right].
\end{aligned}
\]

(2)

\(\rho_1 := \frac{12}{\rho h}, \rho_2 := \frac{1}{\rho h}\), and the positive constants \(\rho, h, K, D\) denote the density of the thin plate, thickness of the thin plate, the shear modulus and the modulus of flexural rigidity, respectively. \(\mu\) represents the Poisson’s ratio \((0 < \mu < \frac{1}{2})\). The functions \(\omega, \psi, \varphi\), depending on \((x, y, t) \in \Omega \times [0, +\infty]\), symbolize the transversal displacement of the midplane of the thin plate and two shear angles respectively, (see [20], [21] for details).

In [20], Lagnese assumed the domain \(\Omega\) satisfies Lipschitz continuity condition on the boundary \(\Gamma\) such that \(\Gamma = \Gamma_0 \cup \Gamma_1\) for two disjoint and relatively open sets \(\Gamma_0, \Gamma_1\), where \(\Gamma_0, \Gamma_1\) are the closure of \(\Gamma_0, \Gamma_1\), respectively. In this regard, the system (1) with the following boundary conditions was mainly discussed provided that \(\Gamma_1 \neq \emptyset\), i.e.,

\[
\begin{aligned}
\psi = \varphi = \omega &= 0, \quad \text{on } \Gamma_0 \times (0, +\infty), \\
\left[ D \left[ \frac{\partial \psi}{\partial x} + \mu \frac{\partial \varphi}{\partial y} - \frac{1 - \mu}{2} \left( \frac{\partial \psi}{\partial x} + \frac{\partial \varphi}{\partial y} \right) \right] \cdot \nu \right] &= m_1, \\
\left[ D \left[ \frac{1 - \mu}{2} \left( \frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial x} \right) \frac{\partial \varphi}{\partial y} + \mu \frac{\partial \psi}{\partial x} \right] \cdot \nu \right] &= m_2, \\
\left[ K \left[ \frac{\partial \psi}{\partial x} + \frac{\partial \omega}{\partial y}, \varphi + \frac{\partial \omega}{\partial y} \right] \cdot \nu \right] &= m_3, \quad \text{on } \Gamma_1 \times (0, +\infty).
\end{aligned}
\]

(3)

Here, \(\nu := (\nu_1, \nu_2)^T\) is referred to as the unit outward normal vector of the boundary \(\Gamma\) and the linear boundary feedbacks were defined as

\[
\begin{bmatrix} m_1, m_2, m_3 \end{bmatrix}^T = -F \begin{bmatrix} \psi_t, \varphi_t, \omega_t \end{bmatrix}^T,
\]

with \(F = [f_{ij}]_{3 \times 3}\), a matrix of real \(L^\infty(\Gamma_1)\) functions, which is symmetric and positive semidefinite on \(\Gamma_1\). Based on the above conditions, it was proved that the system (1) is exponentially stable in the absence of any restrictions on the system coefficients. The same conclusion was obtained by Dalsen in [11], where the stability of the magnetoelastic Mindlin-Timoshenko plate model was investigated by introducing nonlinear locally supported damping \(p(x, y, \psi_t, \varphi_t, \omega_t) = \begin{bmatrix} p^1(x, y, \psi_t), p^2(x, y, \varphi_t), p^3(x, y, \omega_t) \end{bmatrix}^T\) into the interior of the plate, viz.,
Provided that the nonlinear function $p(x, y, \psi, \varphi, \omega)$ satisfying the so-called “dissipativity assumptions” [11], the exponential stability of the energy of system (4) was obtained by means of Nakao’s lemma [28]. However, this model is polynomially stable rather than exponentially stable, proved by Sare [34] who mainly considered putting the terms with frictional dissipation effects $d_1 \psi_t, d_2 \varphi_t$ into rotation angle equations of this model with Dirichlet boundary conditions, namely,

$$
\begin{aligned}
&[[\psi_t, \varphi_t, \omega_t]]^T - [\rho_1 \ell_1, \rho_1 \ell_2, \rho_2 \ell_3]^T - \frac{1}{\rho} [\nabla \times [\phi^1, \phi^2, \phi^3]^T \times [m_1, m_2, 0]^T]^T + p(x, y, \psi, \varphi, \omega) = 0, \\
&\text{in } \Omega \times (0, +\infty), \\
&[[h^3_{12} \phi_1, h^3_{12} \phi_2, h \phi_3]]^T + \nabla \times \left(\nabla \times \left[[h^3_{12} \phi_1, h^3_{12} \phi_2, h \phi_3]]^T\right)\right) = 0, \\
&\text{in } \Omega \times (0, +\infty), \\
&\nabla \cdot [\phi^1, \phi^2]^T = 0, \ \psi = \varphi = \omega = 0, \ [\phi^1, \phi^2]^T \cdot [\nu \times \left(\nabla \times [\phi^1, \phi^2]^T\right)] = 0, \ \partial \phi^3 \partial [\nu \times [\phi^1, \phi^2]^T] = 0, \\
&\text{on } \Gamma \times (0, +\infty).
\end{aligned}
$$

(4)

Since Russell and Zhang [32] first proposed the concepts of indirect damping mechanisms into the research on system stability, a large number of academic literatures have published in various journals (see [1, 2, 5, 23, 26, 27, 31, 37] and the references therein). However, the models discussed in the previous literatures are either a one spatial dimension or a relatively simple one such that a spectrum method or Lyapunov method can be relatively easy to obtain the desired stability. Hence, the research method used in the above literatures is mainly based on the precise eigenvalue calculations or the concepts in system theory together with the theoretical achievements proposed by Huang, Prüss and Gearhart independently (see [15, 29, 30]). As we can see from (1) and (2), the Mindlin-Timoshenko plate system, a highly coupled partial differential equations (PDEs), is more complex than previous models. The stability of system (1) can hardly be obtained by the spectral method resulting from the impossibility of decoupling for this model.

In recent years, with the increasing demand of modern vehicles in the speed and comfort of riding, the vibration and acoustic radiation of plates and shell structures has attracted the widespread attention in engineering fields, such as vehicle, ship, aircraft cabin structure, etc. Therefore, the design of acoustic materials or structures for noise absorption, noise elimination and noise isolation has become an important necessity. Acoustic boundary controls may consist of a large number of acoustic point sources such as speakers, or a distributed acoustic actuation mechanism made of polyvinylidene fluoride (PVDF) or other advanced smart film materials. As a special kind of indirect dampings, they have been widely studied by Barucq et al. [4], Rivera [31] et al. and Beale [5] and others. To our knowledge, the stability of Mindlin-Timoshenko plate systems with such boundary conditions has not been studied by some author. Based on this case, the main goal of this
paper is to apply the acoustic boundary conditions to the two dimensional Mindlin-
Timoshenko plate and investigates its stability. In other words, we consider the
following initial-boundary-value problem
\[
\begin{align*}
[\psi_t, \varphi_t, \omega_t]^T &- [\rho_1 \ell_1, \rho_1 \ell_2, \rho_2 \ell_3]^T = 0, \quad \text{in } \Omega \times (0, +\infty), \\
\psi = \varphi = \omega = 0, & \quad \text{on } \Gamma_0 \times (0, +\infty), \\
\left[ \delta^{(1)}, \delta^{(2)}_t, \delta^{(3)}_t \right] &\left[ B_1 \delta^{(1)} - C_1 \psi_t, B_2 \delta^{(2)} - C_2 \varphi_t, B_3 \delta^{(3)} - C_3 \omega_t \right], \\
D \left[ \frac{\partial \psi}{\partial x} + \mu \frac{\partial \varphi}{\partial y} \right] + \frac{1 - \mu}{2} \left( \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} \right) &\cdot \nu = (\delta^{(1)}, C_1)_{\mathbb{C}^n}, \\
\left[ \frac{1 - \mu}{2} \left( \frac{\partial \psi}{\partial x} + \frac{\partial \varphi}{\partial y} \right), \frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x} \right] &\cdot \nu = (\delta^{(2)}, C_2)_{\mathbb{C}^n}, \\
K \left[ \psi + \frac{\partial \omega}{\partial x}, \varphi + \frac{\partial \omega}{\partial y} \right] &\cdot \nu = (\delta^{(3)}, C_3)_{\mathbb{C}^n}, \\
\left[ \psi_t, \varphi_t, \omega_t \right]^T \big|_{t=0} & = \left[ \psi_0, \varphi_0, \omega_0 \right]^T, \\
\left[ \psi_t, \varphi_t, \omega_t \right]^T \big|_{t=0} & = \left[ \psi_0, \varphi_0, \omega_1 \right]^T, \quad (x, y) \in \Omega, \\
\left[ \delta^{(1)}, \delta^{(2)}, \delta^{(3)} \right]^T \big|_{t=0} & = \left[ \delta^{(1)}_0, \delta^{(2)}_0, \delta^{(3)}_0 \right], \quad (x, y) \in \Gamma_1,
\end{align*}
\]
and

\[
\begin{align*}
    h_1(\psi, \varphi) &= D \left[ \frac{\partial \psi}{\partial x} + \mu \frac{\partial \varphi}{\partial y} + \frac{1 - \mu}{2} \left( \frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial x} \right) \right] \cdot \nu, \\
    h_2(\psi, \varphi) &= D \left[ \frac{1 - \mu}{2} \left( \frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial x} \right), \frac{\partial \varphi}{\partial y} + \mu \frac{\partial \psi}{\partial x} \right] \cdot \nu, \\
    h_3(\psi, \varphi, \omega) &= K \left[ \psi + \frac{\partial \omega}{\partial x}, \varphi + \frac{\partial \omega}{\partial y} \right] \cdot \nu.
\end{align*}
\]

For all sufficiently smooth \([\psi, \varphi, \omega]\) and \([\psi^*, \varphi^*, \omega^*]\), the following equation holds using integration by parts

\[
\int_{\Omega} \left[ \bar{\psi}^* \ell_1 + \bar{\varphi}^* \ell_2 + \bar{\omega}^* \ell_3 \right] dxdy + a(\psi, \varphi, \omega, [\psi^*, \varphi^*, \omega^*]) = \int_{\Gamma} \left[ \bar{\psi}^* h_1(\psi, \varphi) + \bar{\varphi}^* h_2(\psi, \varphi) + \bar{\omega}^* h_3(\psi, \varphi) \right] d\Gamma.
\]

Based on the complexity of the model studied in this paper, the spectral analysis method cannot be applied to the following discussion. Therefore, our approach, avoiding spectrum calculations, is based on a contradiction argument occurring in a special analysis for the resolvent on the imaginary axis. Furthermore, we deduce some estimations of energy components from the resolvent equation of system (1). Combining the equivalent conditions for the polynomial stabilization theorem of semigroup given by Borichev et al. [7] and the above estimates, the polynomial decay of energy (8) can be proved. As far as we know these estimates were not given in the literature. Compared with other related works (see [6, 10, 16, 24, 34, 35, 37]), our main contributions are summarized as follows:

1) A class of acoustic boundary controls derived from theoretical acoustics are applied to the stability research of a Mindlin-Timoshenko plate. This acoustic boundary controller introduces extra degrees of freedom in designing controllers which could be exploited in solving a variety of control problems, such as disturbance rejection, pole assignment, etc.

2) Based on the high degree of coupling of this model, we only use the frequency domain methods rather than Lyapunov methods or spectral methods to obtain many intensive estimations which are applied to analyse the stability of Mindlin-Timoshenko plate.

3) With some reasonable hypotheses, we prove that the Mindlin-Timoshenko plate system with a localized acoustic boundary condition is not exponentially stable but polynomially stable.

The remaining part of this paper is organized as follows: the existence and uniqueness of the solution of system (5) using the contraction semigroup theory are given in Section 2. In Section 3, we shall prove the non-exponential stability of system (5) via the theory originated from Huang [18] and Prüss [30] in their research of the generalized first order linear evolution equation. Finally, the system (5) is demonstrated to be polynomial stability subject to acoustic control conditions in Section 4. In section 5, we present some concluding remarks.

2. Statement of the well-posedness of global solution. In this section we study the existence and uniqueness of strong and global solutions for the system (5) by using the semigroup theories. We use the usual notation \(H^k(\Omega)\) or \(H^k_0(\Omega)\) as indication of general Sobolev spaces of order \(k\) on a regular domain. For the
purpose of endowing with a norm associated with the energy (8), we use the following estimates derived from Korn's inequality in [20], i.e.,

\begin{equation}
\int_{\Omega} \left[ \left( \frac{\partial \psi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 + \mu \frac{\partial \psi}{\partial x} \frac{\partial \phi}{\partial y} + \mu \frac{\partial \psi}{\partial y} \frac{\partial \phi}{\partial x} + \frac{1}{2} \left( \frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial x} \right)^2 \right] \, dx \, dy \geq \varrho_0 \left[ \| \psi \|_{H_0^1(\Omega)}^2 + \| \phi \|_{H_0^1(\Omega)}^2 \right].
\end{equation}

Moreover, for every $\theta_0 > 0$, there exists $\zeta := \zeta(\theta_0) > 0$ such that for all $\theta > \theta_0$ and for every $[\psi, \phi, \omega]^T \in [H_0^1(\Omega)]^3$,

\begin{equation}
\int_{\Omega} \left[ \left( \frac{\partial \psi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 + \mu \frac{\partial \psi}{\partial x} \frac{\partial \phi}{\partial y} + \mu \frac{\partial \psi}{\partial y} \frac{\partial \phi}{\partial x} + \frac{1}{2} \left( \frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial x} \right)^2 \right] \, dx \, dy
+ \theta \int_{\Omega} \left[ \left( \frac{\partial \omega}{\partial x} \right)^2 + \left( \frac{\partial \omega}{\partial y} \right)^2 \right] \, dx \, dy
\geq \zeta \left[ \| \nabla \psi \|_{L^2(\Omega)}^2 + \| \nabla \phi \|_{L^2(\Omega)}^2 + \| \nabla \omega \|_{L^2(\Omega)}^2 \right] + \| \omega \|_{L^2(\Omega)}^2.
\end{equation}

**Remark 1.** From the results of Lemma 2.1, we can define a complex Hilbert space $W$ whose norm is equivalent to the usual norm in $[H^1(\Omega)]^3$, that is

\[ W := \left\{ [\psi, \phi, \omega]^T \in [H^1(\Omega)]^3 \mid \psi = \phi = \omega = 0, \text{ on } \Gamma_0 \right\}, \]

with the norm

\[ \| [\psi, \phi, \omega]^T \|_W^2 = D \int_{\Omega} \left[ \left( \frac{\partial \psi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 + \mu \frac{\partial \psi}{\partial x} \frac{\partial \phi}{\partial y} + \mu \frac{\partial \psi}{\partial y} \frac{\partial \phi}{\partial x} + \frac{1}{2} \left( \frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial x} \right)^2 \right] \, dx \, dy \]
\[ + K \int_{\Omega} \left[ \left( \frac{\partial \omega}{\partial y} \right)^2 + \left( \frac{\partial \omega}{\partial x} \right)^2 \right] \, dx \, dy, \]

and the inner product

\[ \left( [\psi^{(1)}, \phi^{(1)}, \omega^{(1)}]^T, [\psi^{(2)}, \phi^{(2)}, \omega^{(2)}]^T \right)_W = D \int_{\Omega} \left[ \left( \frac{\partial \psi^{(1)}}{\partial x} \frac{\partial \psi^{(2)}}{\partial x} + \frac{\partial \phi^{(1)}}{\partial x} \frac{\partial \phi^{(2)}}{\partial x} + \mu \frac{\partial \psi^{(1)}}{\partial x} \frac{\partial \phi^{(2)}}{\partial x} + \mu \frac{\partial \psi^{(2)}}{\partial x} \frac{\partial \phi^{(1)}}{\partial x} \right) \, dx \, dy \right] \]
\[ + \frac{1}{2} \left( \frac{\partial \psi^{(1)}}{\partial y} + \frac{\partial \phi^{(1)}}{\partial y} \right) \left( \frac{\partial \psi^{(2)}}{\partial y} + \frac{\partial \phi^{(2)}}{\partial y} \right) \, dx \, dy \]
\[ + K \int_{\Omega} \left[ \left( \frac{\partial \omega^{(1)}}{\partial x} \right) \left( \frac{\partial \omega^{(2)}}{\partial x} \right) + \left( \frac{\partial \omega^{(1)}}{\partial y} \right) \left( \frac{\partial \omega^{(2)}}{\partial y} \right) \right] \, dx \, dy, \]

for all $[\psi^{(1)}, \phi^{(1)}, \omega^{(1)}]^T, [\psi^{(2)}, \phi^{(2)}, \omega^{(2)}]^T \in W$. In the sequel, we omit the transposed mark for simplicity of notation if there is no ambiguity in calculation.

We first need to establish the following function spaces and partial derivative operators for investigating the well-posedness and stability of the solution of system (5).

Let $\mathcal{H} = W \times V \times U$ be a product normed space, endowed with the norm

\[ \| [\psi, \phi, \omega; \delta^{(1)}, \delta^{(2)}, \delta^{(3)}] \|_\mathcal{H}^2 = \| [\psi, \phi, \omega] \|_W^2 + \| [\phi, \delta^{(1)}, \delta^{(2)}, \delta^{(3)}] \|_V^2. \]
where,

\[ V := \left\{ [φ, v, η] \mid [φ, v, η] \in [L^2(Ω)]^3 \right\}, \]

\[ U := \left\{ [δ(1), δ(2), δ(3)] \mid δ(j) \in L^2(Γ_j), j = 1, 2, 3 \right\}, \]

are endowed with the following norms respectively, namely,

\[ \| [φ, v, η]\|_V^2 = ρh \int_Ω \left[ \frac{h^2}{12} (|φ|^2 + |v|^2) + |η|^2 \right] dx dy, \]

(16)

with the inner product

\[ \left( [φ(1), v(1), η(1)], [φ(2), v(2), η(2)] \right)_V = ρh \int_Ω \left[ \frac{h^2}{12} (φ(1)φ(2) + v(1)v(2)) + η(1)η(2) \right] dx dy, \]

for all \([φ(1), v(1), η(1)], [φ(2), v(2), η(2)] \in V\), and

\[ \| [δ(1), δ(2), δ(3)]\|_Ω^2 = \int_{Γ_1} \left[ \| δ(1) \|^2_{n} + \| δ(2) \|^2_{n} + \| δ(3) \|^2_{n} \right] dΓ_1. \]

(17)

Set \( U = [ψ, φ, ω; φ, v, η; δ(1), δ(2), δ(3)] \) and define a linear operator \( A : H \subset H \)

\[ \mathcal{A}U = [φ, v, η; ρ_1ℓ_1, ρ_1ℓ_2, ρ_2ℓ_3; B_1δ(1) - C_1γ_0φ, B_2δ(2) - C_2γ_0v, B_3δ(3) - C_3γ_0η], \]

(18)

with

\[ \mathcal{D}(\mathcal{A}) = \left\{ U ∈ H \mid [φ, v, η] ∈ W, [ρ_1ℓ_1, ρ_1ℓ_2, ρ_2ℓ_3] ∈ V, ψ = φ = ω = 0, \text{ on } Γ_0, \right. \]

\[ h_1(ψ, φ) = (δ(1), C_1)_C^n, \quad h_2(ψ, φ) = (δ(2), C_2)_C^n, \]

\[ h_3(ψ, φ, ω) = (δ(3), C_3)_C^n, \quad \text{on } Γ_1 \}, \]

where \( γ_0φ, γ_0v, γ_0η \) symbolize the trace of \( φ, v, η \) on \( Γ_1 \), respectively. Equation (5) can be thus rewritten as an evolution equation

\[ \mathcal{U}_0 = \mathcal{A}U, \quad \mathcal{U}(0) = \mathcal{U}_0, \]

(19)

where \( \mathcal{U}_0 = [ψ_0, φ_0, ω_0; ψ_1, φ_1, ω_1; δ^{(1)}_0, δ^{(2)}_0, δ^{(3)}_0]. \)

**Definition 2.2.** (Weak solution) Let \( ([ψ_0, φ_0, ω_0]; [ψ_1, φ_1, ω_1]) \in W × V \) denote by \( W^* \) the dual space of \( W \). Then \( ([ψ, φ, ω]; [ψ_t, φ_t, ω_t]) \in C^0((0, +∞); W × V) \)

with \([ψ_t, φ_t, ω_t] \in (L^2((0, +∞)); W^*)^3 \) is referred to as the weak solution of (5), if the following equality holds

\[ \left\langle [ψ_t, φ_t, ω_t], [ψ^*, φ^*, ω^*] \right\rangle_{W^*, W} + a([ψ, φ, ω], [ψ^*, φ^*, ω^*]) = \int_{Γ_1} \left[ (δ^{(1)}, C_1)_C^n γ_0ψ^* + (δ^{(2)}, C_2)_C^n γ_0φ^* + (δ^{(3)}, C_3)_C^n γ_0ω^* \right] dΓ_1, \]

(20)

for every \([ψ^*, φ^*, ω^*] \in W. \)

**Proposition 1.** The operator \( \mathcal{A} \) is \( m \)-dissipative.
Let $YUBIAO LIU, CHUNGUO ZHANG AND TEHUAN CHEN$

and (18), we obtain

$$
\begin{aligned}
\text{such that for every } u \in D
\end{aligned}
$$

which is equivalent to

$$
\int \left[ \left( \phi + \frac{\partial \eta}{\partial x} \right) \left( \psi + \frac{\partial \varphi}{\partial x} \right) + \left( \nu + \frac{\partial \eta}{\partial y} \right) \left( \varphi + \frac{\partial \varphi}{\partial y} \right) \right] dx dy
$$

$$
\begin{pmatrix}
0 \\
1 \\
3
\end{pmatrix} = \left[ \begin{pmatrix}
\gamma_0 \phi, \delta^{(1)} \\
\gamma_0 \varphi, \delta^{(2)} \\
\gamma_0 \psi, \delta^{(3)}
\end{pmatrix} \right] \in C_\mathcal{H}
$$

$$
\int \left( (B_1 \delta^{(1)} - C_1 \gamma_0 \phi, \delta^{(1)})_{C_\mathcal{H}} + (B_2 \delta^{(2)} - C_2 \gamma_0 \psi, \delta^{(2)})_{C_\mathcal{H}} + (B_3 \delta^{(3)} - C_3 \gamma_0 \eta, \delta^{(3)})_{C_\mathcal{H}} \right) dx dy
$$

$$
\begin{pmatrix}
0 \\
1 \\
3
\end{pmatrix} = \left[ \begin{pmatrix}
\gamma_0 \phi, \delta^{(1)} \\
\gamma_0 \varphi, \delta^{(2)} \\
\gamma_0 \psi, \delta^{(3)}
\end{pmatrix} \right] \in C_\mathcal{H}
$$

which implies that $\mathbf{S}$ is dissipative.

In the sequel, we prove that there exists a positive number $\lambda$ such that $\lambda I - \mathbf{S}$ is surjective. That is to say that there is $U = [\psi, \varphi, \omega; \phi, v, \eta; \delta^{(1)}, \delta^{(2)}, \delta^{(3)}] \in \mathcal{D}(\mathbf{S})$ such that for every $U = [f_1, f_2, f_3; g_1, g_2, g_3; u^{(1)}, u^{(2)}, u^{(3)}] \in \mathcal{H},$

$$(\lambda I - \mathbf{S})U = U_1,$$

which is equivalent to

$$\begin{cases}
\lambda[\psi, \varphi, \omega] - [\phi, v, \eta] = [f_1, f_2, f_3], \\
\lambda[\phi, v, \eta] - [\rho_1 \ell_1, \rho_1 \ell_2, \rho_2 \ell_3] = [g_1, g_2, g_3], \\
([\lambda I - B_1] \delta^{(1)} + C_1 \gamma_0 \phi, \delta^{(1)} + C_2 \gamma_0 \psi, \delta^{(3)} + C_3 \gamma_0 \eta)
\end{cases}
$$

Suppose that there exists a $U \in \mathcal{D}(\mathbf{S})$ such that (22) holds. Hence, when $\lambda \notin \sigma(B_j)$ ($j = 1, 2, 3$), then we successively obtain

$$\begin{cases}
\delta^{(1)} = (\lambda I - B_1)^{-1} (u^{(1)} + C_1 \gamma_0 f_1 - \lambda C_1 \gamma_0 \psi), \\
\delta^{(2)} = (\lambda I - B_2)^{-1} (u^{(2)} + C_2 \gamma_0 f_2 - \lambda C_2 \gamma_0 \varphi), \\
\delta^{(3)} = (\lambda I - B_3)^{-1} (u^{(3)} + C_3 \gamma_0 f_3 - \lambda C_3 \gamma_0 \omega).
\end{cases}
$$

Hence, we only look for $[\psi, \varphi, \omega] \in W$ satisfying

$$\lambda^3[\psi, \varphi, \omega] - [\rho_1 \ell_1, \rho_1 \ell_2, \rho_2 \ell_3] = \lambda[f_1, f_2, f_3] + [g_1, g_2, g_3]
$$
and the boundary conditions defined in the domain $\mathcal{D}(\mathcal{A})$.

To complete the proof of Proposition 1, it remains to show that the equation (24) has a unique weak solution.

For any $[\psi^{(1)}, \varphi^{(1)}, \omega^{(1)}] \in W$, by taking the inner product of $[\psi^{(1)}, \varphi^{(1)}, \omega^{(1)}]$ with (24) in $V$, it yields

$$J_{\lambda}([\psi, \varphi, \omega], [\psi^{(1)}, \varphi^{(1)}, \omega^{(1)}]) = L(\psi^{(1)}, \varphi^{(1)}, \omega^{(1)}),$$

(25)

where

$$J_{\lambda}([\psi, \varphi, \omega], [\psi^{(1)}, \varphi^{(1)}, \omega^{(1)}]) = \lambda^2 \int_{\Omega} \left[ \frac{\rho h^3}{12} \left( (\psi^{(1)} + \varphi^{(1)}) \bar{\omega}^{(1)} + \rho h \bar{\omega}^{(1)} \right) dx dy + a([\psi, \varphi, \omega], [\psi^{(1)}, \varphi^{(1)}, \omega^{(1)}]) \right] + \int_{\Gamma_1} \left[ \lambda \left( \lambda I - B_1 \right)^{-1} C_1, C_1 \right]_{\mathbb{C}^n} \gamma_0 \psi \bar{\psi}^{(1)} + \lambda \left( \lambda I - B_2 \right)^{-1} C_2, C_2 \right]_{\mathbb{C}^n} \gamma_0 \varphi \bar{\varphi}^{(1)} + \lambda \left( \lambda I - B_3 \right)^{-1} C_3, C_3 \right]_{\mathbb{C}^n} \gamma_0 \omega \bar{\omega}^{(1)} \right] d\Gamma_1,

L(\psi^{(1)}, \varphi^{(1)}, \omega^{(1)})

Applying Hölder’s inequality and Sobolev trace theorem, we know that $L$ is a bounded linear functional on $W$. Moreover, $J_{\lambda}([\psi, \varphi, \omega], [\psi^{(1)}, \varphi^{(1)}, \omega^{(1)}])$ has both conjugate bilinear form and coerciveness on $W$. In fact, for any $[\psi, \varphi, \omega] \in W$, taking $\psi^{(1)} = \psi, \varphi^{(1)} = \varphi, \omega^{(1)} = \omega$ results in

$$J_{\lambda}([\psi, \varphi, \omega], [\psi, \varphi, \omega]) = \lambda^2 \int_{\Omega} \left[ \frac{\rho h^3}{12} \left( |\psi|^2 + |\varphi|^2 \right) + \rho h |\omega|^2 \right] dx dy + \|[\psi, \varphi, \omega]\|^2_W + \int_{\Gamma_1} \left[ \lambda \left( \lambda I - B_1 \right)^{-1} C_1, C_1 \right]_{\mathbb{C}^n} |\gamma_0 \psi|^2 + \lambda \left( \lambda I - B_2 \right)^{-1} C_2, C_2 \right]_{\mathbb{C}^n} |\gamma_0 \varphi|^2 + \lambda \left( \lambda I - B_3 \right)^{-1} C_3, C_3 \right]_{\mathbb{C}^n} |\gamma_0 \omega|^2 \right] d\Gamma_1.

(26)

By letting $r_j = (\lambda I - B_j)^{-1} C_j$ ($j = 1, 2, 3$), it follows from (7) that

$$\Re((\lambda I - B_j)^{-1} C_j, C_j)_{\mathbb{C}^n} = \Re(r_j, (\lambda I - B_j)r_j)_{\mathbb{C}^n} = \lambda (r_j, r_j)_{\mathbb{C}^n} - \Re(r_j, B_j r_j)_{\mathbb{C}^n} \geq 0.

(27)

Hence, (26) and (27) indicate that $\Re J_{\lambda}([\psi, \varphi, \omega], [\psi, \varphi, \omega]) \geq \|[\psi, \varphi, \omega]\|^2_W$, which states clearly that $J_{\lambda}$ is coercive.

Lax-Milgram lemma [3] shows that (25) has a unique solution $[\psi, \varphi, \omega] \in W$. Furthermore, letting $[\psi^{(1)}, \varphi^{(1)}, \omega^{(1)}] \in (\mathcal{D}(\Omega))^3$ in (25) and defining $[\delta^{(1)}, \delta^{(2)}, \delta^{(3)}]$ by (23), we obtain

$$\lambda^2 [\psi, \varphi, \omega] = [\rho_1 \ell_1, \rho_1 \ell_2, \rho_2 \ell_3] = \lambda [f_1, f_2, f_3] + [g_1, g_2, g_3], \text{ in } (\mathcal{D}'(\Omega))^3.

(28)
By substituting $\lambda^2[\psi, \varphi, \omega]$ of (28) into (25), this equation becomes
\[
\int_\Omega \left[ \bar{\psi}^{(1)} \ell_1 + \bar{\phi}^{(1)} \ell_2 + \bar{\omega}^{(1)} \ell_3 \right] dx dy + a([\psi, \varphi, \omega], [\psi^{(1)}, \varphi^{(1)}, \omega^{(1)}]) = 0.
\]
By combining (10) and (11), it is easy to verify that
\[
\begin{cases}
h_1(\psi, \varphi) = (\delta^{(1)}, C_1)_{C_n}, \\
h_2(\psi, \varphi) = (\delta^{(2)}, C_2)_{C_n}, \\
h_3(\psi, \varphi, \omega) = (\delta^{(3)}, C_3)_{C_n},
\end{cases}
\]
and
\[
\begin{aligned}
\left[ [\psi_t, \varphi_t, \omega_t], [\psi^{(1)}, \varphi^{(2)}, \omega^{(3)}] \right]_{W^*, W} + a([\psi, \varphi, \omega], [\psi^{(1)}, \varphi^{(2)}, \omega^{(3)}]) \\
= \int_{\Gamma_1} \left[ (\delta^{(1)}, C_1)_{C_n} \gamma_0 \bar{\psi}^{(1)} + (\delta^{(2)}, C_2)_{C_n} \gamma_0 \bar{\phi}^{(2)} + (\delta^{(3)}, C_3)_{C_n} \gamma_0 \bar{\omega}^{(3)} \right] d\Gamma_1.
\end{aligned}
\]
Hence, we conclude that $\lambda I - A'$ is surjective.

**Remark 2.** From the above proof, it can be seen that if 0 is not included in the eigenvalue of $B_1$ for all $(x, y) \in \Gamma_1$, then $A'$ is a one-to-one mapping and $A'^{-1}$ is bounded which implies that $A'$ is a closed operator. Thus, we reach the conclusion that $A'$ generates a $C_0$-semigroup $e^{tA'}$ of contractions on $\mathcal{H}$ by using the Lumer-Phillips theorem [29].

**Corollary 1.** For an initial value $U_0 \in \mathcal{H}$, the system (5) has an unique weak solution $U \in C([0, +\infty); \mathcal{H})$. Moreover, if $U_0 \in \mathcal{D}(A')$, then $U \in C([0, +\infty); \mathcal{D}(A') \cap C^1([0, +\infty); \mathcal{H})$.

3. **The Lack of Exponential Stability.** In this section, we show that the system (5) is not exponentially stable by using the equivalent conditions for exponential stability of $C_0$-semigroups in a Hilbert space. This correlative results were mainly described as follows (see Huang [18], Prüss [30] and Gearhart [15]).

**Theorem 3.1.** A $C_0$-semigroup $e^{tA'}$ of contractions on Hilbert space is exponentially stable if and only if
\[
\varphi(A') \supset \{i\lambda | \lambda \in \mathbb{R} \},
\]
and
\[
\sup \{ ||(i\lambda I - A')^{-1}|| | \lambda \in \mathbb{R} \} < \infty.
\]

Consider the problem without control input
\[
\begin{aligned}
\left[ [\psi_t, \varphi_t, \omega_t], [\psi_1, \varphi_1, \omega_1] \right] - [\rho_1 \ell_1, \rho_1 \ell_2, \rho_2 \ell_3] = 0, &\quad \text{in } \Omega \times (0, +\infty), \\
[\psi = \varphi = \omega = 0, &\quad \text{on } \Gamma_0 \times (0, +\infty), \\
D \left[ \frac{\partial \psi}{\partial x} + \mu \frac{\partial \varphi}{\partial y} - \frac{1 - \mu}{2} \left( \frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial x} \right) \right] \cdot \nu = 0, &\quad \text{on } \Gamma_0 \times (0, +\infty), \\
D \left[ \frac{1 - \mu}{2} \left( \frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial x} \right) \right] \cdot \nu = 0, &\quad \text{on } \Gamma_1 \times (0, +\infty), \\
K \left[ \psi + \frac{\partial \omega}{\partial x} \right] \cdot \nu = 0, &\quad \text{on } \Gamma_1 \times (0, +\infty), \\
[\psi, \varphi, \omega]_{t=0} = [\psi_0, \varphi_0, \omega_0], &\quad [\psi_t, \varphi_t, \omega_t]_{t=0} = [\psi_1, \varphi_1, \omega_1], \quad (x, y) \in \Omega,
\end{aligned}
\]
and define linear operators \( \mathcal{A}_1, \mathcal{A}_2 \) and \( A \) as follows, namely,
\[
\mathcal{A}_1 \mathbf{U} = \begin{bmatrix} \phi, v, \eta; \rho_1 \ell_1, \rho_1 \ell_2, \rho_2 \ell_3; 0 \end{bmatrix},
\]
with the domain
\[
\mathcal{D}(\mathcal{A}_1) = \left\{ \mathbf{U} \in \mathcal{H} \mid [\phi, v, \eta] \in \mathfrak{W}, [\rho_1 \ell_1, \rho_1 \ell_2, \rho_2 \ell_3] \in \mathcal{V}, \psi = \varphi = \omega = 0, \text{ on } \Gamma_0, \right. \\
h_1(\psi, \varphi) = h_2(\psi, \varphi) = h_3(\psi, \varphi, \omega) = 0, \text{ on } \Gamma_1 \right\}, \tag{32}
\]
and
\[
\mathcal{A}_2 \mathbf{U} = \begin{bmatrix} 0; 0; B_1 \delta^{(1)} - C_1 \gamma_0 \varphi, B_2 \delta^{(2)} - C_2 \gamma_0 v, B_3 \delta^{(3)} - C_3 \gamma_0 \eta \end{bmatrix},
\]
with the domain
\[
\mathcal{D}(\mathcal{A}_2) = \left\{ \mathbf{U} \in \mathcal{H} \mid [\phi, v, \eta] \in \mathfrak{W}, [\rho_1 \ell_1, \rho_1 \ell_2, \rho_2 \ell_3] \in \mathcal{V}, \psi = \varphi = \omega = 0, \text{ on } \Gamma_0, \right. \\
h_1(\psi, \varphi) = (\delta^1, C_1)_{\mathcal{C}^0}, \, h_2(\psi, \varphi) = (\delta^2, C_2)_{\mathcal{C}^0}, \\
h_3(\psi, \varphi, \omega) = (\delta^3, C_3)_{\mathcal{C}^0}, \text{ on } \Gamma_1 \} \right\}, \tag{33}
\]
and
\[
A[\psi, \varphi, \omega] = -[\rho_1 \ell_1, \rho_1 \ell_2, \rho_2 \ell_3], \tag{34}
\]
with the domain
\[
\mathcal{D}(A) = \left\{ [\psi, \varphi, \omega] \in (H^2(\Omega))^3 \mid \psi = \varphi = \omega = 0, \text{ on } \Gamma_0, \right. \\
h_1(\psi, \varphi) = h_2(\psi, \varphi) = h_3(\psi, \varphi, \omega) = 0, \text{ on } \Gamma_1 \right\}. \tag{35}
\]

Therefore, the system (31) is equivalent to
\[
\begin{aligned}
[\dot{\psi}, \dot{\varphi}, \dot{\omega}] + A[\psi, \varphi, \omega] &= 0, \quad \text{ in } \Omega \times (0, +\infty), \\
[\psi(0, \varphi, \omega)|_{t=0} = [\psi_0, \varphi_0, \omega_0], \quad [\psi(t, \varphi, \omega)|_{t=0} = [\psi_1, \varphi_1, \omega_1], \quad (x, y) \in \Omega. \tag{36}
\end{aligned}
\]

Clearly, from the definition of (18), (32) and (33), we have
\[
A = \mathcal{A}_1 + \mathcal{A}_2, \tag{36}
\]
and
\[
A_1 = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}. \tag{37}
\]

**Lemma 3.2.** The operator \( A \) defined in (34) is positive definite and self-adjoint. Moreover, the resolvent of \( A \) is compact in \( \mathcal{V} \).

**Proof.** Let \([\psi, \varphi, \omega], [\dot{\psi}, \dot{\varphi}, \dot{\omega}] \in \mathcal{D}(A)\). By the formula of (11), we obtain
\[
\left( A[\psi, \varphi, \omega], [\dot{\psi}, \dot{\varphi}, \dot{\omega}] \right)_V \\
= D \int_\Omega \left[ \begin{array}{c} \frac{\partial \psi}{\partial x} \frac{\partial \tilde{\varphi}}{\partial y} + \frac{\partial \varphi}{\partial y} \frac{\partial \tilde{\psi}}{\partial x} + \mu \frac{\partial \tilde{\psi}}{\partial x} \frac{\partial \tilde{\varphi}}{\partial y} + \mu \frac{\partial \tilde{\psi}}{\partial y} \frac{\partial \tilde{\varphi}}{\partial x} + 2 \frac{\partial \tilde{\psi}}{\partial y} \frac{\partial \tilde{\varphi}}{\partial x} + \frac{\partial \tilde{\psi}}{\partial y} \frac{\partial \tilde{\varphi}}{\partial x} \end{array} \right] dxdy + K \int_\Omega \left[ \begin{array}{c} \tilde{\psi} + \frac{\partial \omega}{\partial x} \end{array} \right] \left( \begin{array}{c} \tilde{\varphi} + \frac{\partial \tilde{\psi}}{\partial x} \end{array} \right) dxdy = \left( [\psi, \varphi, \omega], A[\dot{\psi}, \dot{\varphi}, \dot{\omega}] \right)_V. \tag{38}
\]
This implies that \( A \) is a symmetric operator in \( V \). In particular, taking \( \tilde{\psi} = \psi, \tilde{\varphi} = \varphi, \omega = \omega \) in (38) together with (13) we can get

\[
\begin{align*}
(A[\psi, \varphi, \omega], [\psi, \varphi, \omega])_V &= D \int_{\Omega} \left[ \frac{\partial^2 \tilde{\psi}}{\partial x^2} + \frac{\partial^2 \tilde{\varphi}}{\partial y^2} + \mu \frac{\partial \tilde{\psi}}{\partial x} \frac{\partial \tilde{\varphi}}{\partial y} + \mu \frac{\partial \tilde{\psi}}{\partial x} + \frac{\partial \tilde{\varphi}}{\partial y} \right]^2 + \frac{1}{2} \left( \frac{\partial \tilde{\psi}}{\partial x} + \frac{\partial \tilde{\varphi}}{\partial y} \right)^2 \right] dxdy \\
+ \frac{1}{2} \left( \frac{\partial \tilde{\psi}}{\partial x} + \frac{\partial \tilde{\varphi}}{\partial y} \right)^2 \right] dxdy \geq C||[\psi, \varphi, \omega]||^2_V.
\end{align*}
\]

(39)

Hence, combining (38) and (39), we have proved the positive definiteness of \( A \) in \( V \).

Denote by \( A^* \) an adjoint operator of \( A \). For every \([\tilde{\psi}, \tilde{\varphi}, \tilde{\omega}] \in \mathcal{D}(A^*), [\psi, \varphi, \omega] \in \mathcal{D}(A)\), then there exists \([\psi^0, \varphi^0, \omega^0] \in V\) such that \( A^*[\tilde{\psi}, \tilde{\varphi}, \tilde{\omega}] = [\psi^0, \varphi^0, \omega^0]\). Moreover,

\[
(A[\psi, \varphi, \omega], [\tilde{\psi}, \tilde{\varphi}, \tilde{\omega}])_V = ([\psi, \varphi, \omega], [\psi^0, \varphi^0, \omega^0])_V,
\]

i.e.,

\[
-\int_{\Omega} \left[ \tilde{\psi} \ell_1 + \tilde{\varphi} \ell_2 + \tilde{\omega} \ell_3 \right] dxdy = \rho h \int_{\Omega} \left[ \frac{h^2}{12} (\tilde{\psi} \tilde{\varphi}^0 + \tilde{\varphi} \tilde{\psi}^0) + \omega \tilde{\omega}^0 \right] dxdy.
\]

(40)

By using the formula (11) and integrating by parts, we calculate from (40) to get

\[
\int_{\Omega} \left\{ \psi \left[ D \left( \frac{\partial \tilde{\psi}}{\partial x} + \frac{1}{2} \frac{\partial \tilde{\psi}}{\partial y} \right) + \frac{\partial \tilde{\varphi}}{\partial x} \right] + \varphi \right\} \left[ D \left( \frac{\partial \tilde{\varphi}}{\partial y} + \frac{1}{2} \frac{\partial \tilde{\varphi}}{\partial x} \right) + \frac{\partial \tilde{\psi}}{\partial y} \right] + \omega \left[ K \left( \frac{\partial \tilde{\psi}}{\partial x} + \frac{\partial \tilde{\varphi}}{\partial y} \right) + \frac{\partial \tilde{\varphi}}{\partial y} \right] dxdy = 0.
\]

Because of the arbitrariness of \([\psi, \varphi, \omega]\), we further obtain the following equations

\[
\begin{align*}
D \left( \frac{\partial \tilde{\psi}}{\partial x} + \frac{1}{2} \frac{\partial \tilde{\psi}}{\partial y} \right) + \frac{\partial \tilde{\varphi}}{\partial x} \right] - K \left( \frac{\partial \tilde{\psi}}{\partial x} + \frac{\partial \tilde{\varphi}}{\partial y} \right) = -\rho h^3 \psi^0, \quad & \text{in } \Omega \times (0, +\infty), \\
D \left( \frac{\partial \tilde{\varphi}}{\partial y} + \frac{1}{2} \frac{\partial \tilde{\varphi}}{\partial x} \right) + \frac{\partial \tilde{\psi}}{\partial y} \right] - K \left( \frac{\partial \tilde{\varphi}}{\partial y} + \frac{\partial \tilde{\psi}}{\partial x} \right) = -\rho h^3 \varphi^0, \quad & \text{on } \Gamma_0 \times (0, +\infty), \\
K \left( \frac{\partial \tilde{\psi}}{\partial x} + \frac{\partial \tilde{\varphi}}{\partial y} \right) + \frac{\partial \tilde{\varphi}}{\partial y} \right) = -\rho h \omega^0, \quad & \text{on } \Gamma_1 \times (0, +\infty), \\
\tilde{\psi} = \tilde{\varphi} = \tilde{\omega} = 0, & \text{on } \Gamma_0 \times (0, +\infty), \\
h_1(\tilde{\psi}, \tilde{\varphi}) = h_2(\tilde{\psi}, \tilde{\varphi}) = h_3(\tilde{\psi}, \tilde{\varphi}, \tilde{\omega}) = 0, & \text{on } \Gamma_1 \times (0, +\infty).
\end{align*}
\]

(41)

The uniqueness theorem of elliptic boundary-value problems in [14] shows that there exists a unique solution \([\tilde{\psi}, \tilde{\varphi}, \tilde{\omega}] \in \mathcal{D}(A) \) of (41) such that \(||[\tilde{\psi}, \tilde{\varphi}, \tilde{\omega}]||_{H^2(\Omega)}^3 \leq M ||[\psi^0, \varphi^0, \omega^0]||^2_\mathcal{D}\) for some \(M > 0\). It follows that \(\mathcal{D}(A) = \mathcal{D}(A^*)\) and \(A = A^*\).

Finally, for \([\psi, \varphi, \omega] \in \mathcal{D}(A), [\psi^0, \varphi^0, \omega^0] \in V\), we solve the equation \(A[\psi, \varphi, \omega] = [\psi^0, \varphi^0, \omega^0]\). Proceeding as in the deduction of (40), we have \(||[\psi, \varphi, \omega]||_{H^2(\Omega)}^3 \leq M ||[\psi^0, \varphi^0, \omega^0]||^2_\mathcal{D}\), which means that \(A^{-1}\) is compact. By the continuity of the resolvent, we know that \(A\) has a compact resolvent. The proof is complete.

\[\square\]

**Corollary 2.** According to the self-adjointness of \( A \), it is easy to verify, by (37), that \( \mathcal{A}_1 \) is skew-self-adjoint in \( W \times V \), i.e.,

\[
\mathcal{A}_1 = -\mathcal{A}_1^*.
\]

(42)
Lemma 3.3. The resolvent of $\mathcal{A}$ is compact.

Proof. By Corollary 2, we know that $\mathcal{A}_1 = -\mathcal{A}_2^*$, which implies that $1 \in \rho(\mathcal{A}_1)$. Here, $\rho(\mathcal{A}_1)$ denotes the resolvent set of $\mathcal{A}_1$. Suppose that $\{X_n\}_{n \geq 1} \subset W \times V$ is a bounded sequence, that is, there exists a positive constant $M$ such that $\|X_n\|_{W \times V} \leq M$. Let $Z_n = (I - \mathcal{A}_1)X_n$, then we thus have

$$
\left\| (I - \mathcal{A}_1)Z_n \right\|_{W \times V}^2 \\
= (Z_n - \mathcal{A}_1Z_n, Z_n - \mathcal{A}_1Z_n)_{W \times V} \\
= \|Z_n\|_{W \times V}^2 - (Z_n, \mathcal{A}_1Z_n)_{W \times V} - (\mathcal{A}_1Z_n, Z_n)_{W \times V} + \|\mathcal{A}_1Z_n\|_{W \times V}^2 \\
= \|Z_n\|_{W \times V}^2 + \|\mathcal{A}_1Z_n\|_{W \times V}^2 \leq M.
$$

By applying Sobolev embedding theorem, we know that $W \hookrightarrow V$ is a compact embedding. Hence, $Z_n$ has convergent subsequences, which indicates the compactness of $(I - \mathcal{A}_1)^{-1}$.

For every $Z \in \mathcal{D}(\mathcal{A}_1)$ and $\lambda \in \rho(\mathcal{A}_1)$, we apply (42) to get

$$
\left\| (\lambda I - \mathcal{A}_1)Z \right\|_{W \times V}^2 = \lambda^2 \|Z\|_{W \times V}^2 + \|\mathcal{A}_1Z\|_{W \times V}^2 + (\lambda - \lambda)(\mathcal{A}_1Z, Z)_{W \times V}.
$$

Thus, as $\Re(\lambda) > 0$, we obtain

$$
\left\| (\lambda I - \mathcal{A}_1)^{-1} \right\| \leq \frac{1}{\Re(\lambda)}.
$$

Because of

$$
(\lambda I - \mathcal{A}_1)^{-1} = [I - (1 - \lambda)(\lambda I - \mathcal{A}_1)^{-1}]^{-1}(I - \mathcal{A}_1)^{-1},
$$

and

$$
\lambda I - \mathcal{A} = (\lambda I - \mathcal{A}_1)[I - (\lambda I - \mathcal{A}_1)^{-1}\mathcal{A}_2],
$$

therefore, for every $\lambda \in \rho(\mathcal{A}_1)$, $(I - (\lambda I - \mathcal{A}_1)^{-1}\mathcal{A}_2)^{-1}$ exists and bounded provided that $\Re(\lambda) \geq 2\|\mathcal{A}_2\|$. It follows from (44), (45) and (46) that

$$
(\lambda I - \mathcal{A})^{-1} = [I - (\lambda I - \mathcal{A}_1)^{-1}\mathcal{A}_2]^{-1}(\lambda I - \mathcal{A}_1)^{-1},
$$

which implies that the resolvent of $\mathcal{A}$ is compact. \hfill \Box

Lemma 3.4. The resolvent set $\rho(\mathcal{A}) \supset \{i\lambda|\lambda \in \mathbb{R}\}$.

Proof. Suppose Lemma 3.4 is not true. Then there exists a nonzero constant $\lambda \in \mathbb{R}$ such that $i\lambda \notin \rho(\mathcal{A})$. From Lemma 3.3, we know that $\mathcal{A}$ only has point spectrum, which implies that $i\lambda \in \sigma_p(\mathcal{A})$. This shows that there exists a nonzero vector $U = [\psi, \varphi, \omega; \phi, v, \eta; \delta^{(1)}; \delta^{(2)}; \delta^{(3)}] \in \mathcal{D}(\mathcal{A})$ satisfying

$$
(i\lambda I - \mathcal{A})U = 0,
$$

that is

$$
\begin{cases}
(i\lambda \psi, \varphi, \omega) - [\phi, v, \eta] = 0, \\
i\lambda [\phi, v, \eta] - [\rho_1 \ell_1, \rho_1 \ell_2, \rho_2 \ell_3] = 0, \\
\left[(i\lambda I - B_1)\delta^{(1)} + C_1 \gamma_0 \phi, (i\lambda I - B_2)\delta^{(2)} + C_2 \gamma_0 v, (i\lambda I - B_3)\delta^{(3)} + C_3 \gamma_0 \eta\right] = 0.
\end{cases}
$$

(48)
Since
\[
0 = \Re((i\lambda I - \mathcal{A})\mathcal{U},\mathcal{U})_{\mathcal{H}} = -\Re(\mathcal{A}\mathcal{U},\mathcal{U})_{\mathcal{H}}
\]
\[
= -\int_{\Gamma_1} \left[ \Re(B_1\delta^{(1)},\delta^{(1)})_{C^n} + \Re(B_2\delta^{(2)},\delta^{(2)})_{C^n} + \Re(B_3\delta^{(3)},\delta^{(3)})_{C^n} \right] d\Gamma_1,
\]
by the assumption of (7), we thus obtain
\[
\delta^{(1)} = \delta^{(2)} = \delta^{(3)} = 0, \quad \text{on } \Gamma_1.
\]
(50)

By substituting (50) into (48), we have
\[
\left[ \psi,\varphi,\omega; \phi,v,\eta \right] = 0, \quad \text{on } \Gamma_1.
\]
From (48), we have
\[
-\lambda^2[\psi,\varphi,\omega] - [\rho_1\ell_1,\rho_1\ell_2,\rho_2\ell_3] = 0.
\]
On the basis of the uniqueness theorem of elliptic boundary-value problems, it leads to
\[
[\psi,\varphi,\omega; \phi,v,\eta; \delta^{(1)},\delta^{(2)},\delta^{(3)}] = 0 \quad \text{in } \Omega,
\]
which contradicts \( \mathcal{U} \neq 0 \). This completes the proof of Lemma 3.4.

\[\square\]

From Lemma 3.4 and Theorem 3.1, we know that system (5) satisfies one of two equivalent conditions for exponential stability. To complete the proof of non-exponential stability of system (5), what we need to do next is to verify that (30) does not hold. In order to achieve this purpose, we first introduce a positive self-adjoint operator \( A_1 \) given by
\[
\mathcal{D}(A_1) = \left\{ [\psi,\varphi,\omega] \in (H^2(\Omega))^3 \cap W | \psi = \varphi = \omega = 0, \text{ on } \Gamma_0, \right. \\
\left. (h_1(\psi,\varphi),h_2(\psi,\varphi),h_3(\psi,\varphi,\omega)) + Q[\psi,\varphi,\omega] = 0, \text{ on } \Gamma_1 \right\},
\]
where
\[
Q = \begin{pmatrix} (C_1,C_1)_{C^n} \\
(C_2,C_2)_{C^n} \\
(C_3,C_3)_{C^n} \end{pmatrix},
\]
and
\[
A_1[\psi,\varphi,\omega] = A[\psi,\varphi,\omega], \quad \text{for every } [\psi,\varphi,\omega] \in \mathcal{D}(A_1).
\]
(52)
Denote by \( \{\lambda_n^2\}_{n \geq 1} \) the (discrete) spectral set of \( A_1 \) and let \( \Phi_n = [\psi_n,\varphi_n,\omega_n] \) be the corresponding orthogonal eigenvectors. By applying Lemma 3.2, it can be seen that \( \lambda_n \) is positive and goes to +\( \infty \), as \( n \) approaches infinity.

**Proposition 2.** Let \([\psi,\varphi,\omega] \in W\), then there exists a positive constant \( \bar{c} \) depending only on \( \Omega \) such that
\[
\| [\gamma_0\psi,\gamma_0\varphi,\gamma_0\omega] \|_{V(\Gamma_1)}^2 \leq \bar{c} \| [\psi,\varphi,\omega] \|_{V} \| [\psi,\varphi,\omega] \|_{W}.
\]
(53)

**Proof.** It can be seen from the trace inequality in [3] that there exists a constant \( c_1 > 0 \) only related to \( \Omega \) such that
\[
\| \gamma_0 u \|_{L^1(\Gamma_1)} \leq c_1 \| u \|_{W^{1,1}(\Omega)}, \quad \text{for every } u \in W^{1,1}(\Omega).
\]
(54)
By replacing \( u \) by \( \psi^2,\varphi^2,\omega^2 \), respectively, we have
By the Cauchy-Schwarz inequality and Poincaré's inequality, we obtain

\[ \int_{\Gamma_1} |\gamma_0 \psi|^2 d\Gamma_1 \leq c_1 \int_{\Omega} \left[ |\psi|^2 + 2|\psi||\nabla \psi| \right] d\Omega, \]
\[ \int_{\Gamma_1} |\gamma_0 \varphi|^2 d\Gamma_1 \leq c_1 \int_{\Omega} \left[ |\varphi|^2 + 2|\varphi||\nabla \varphi| \right] d\Omega, \]
\[ \int_{\Gamma_1} |\gamma_0 \omega|^2 d\Gamma_1 \leq c_1 \int_{\Omega} \left[ |\omega|^2 + 2|\omega||\nabla \omega| \right] d\Omega. \]  

(55)

By the Cauchy-Schwarz inequality and Poincaré's inequality, we obtain

\[ \int_{\Gamma_1} \left[ |\gamma_0 \psi|^2 + |\gamma_0 \varphi|^2 + |\gamma_0 \omega|^2 \right] d\Gamma_1 \]
\[ \leq c_2 (\|\psi\|^2_{L^2(\Omega)} + \|\varphi\|^2_{L^2(\Omega)} + \|\omega\|^2_{L^2(\Omega)}) \frac{1}{2} (\|\nabla \psi\|^2_{L^2(\Omega)} + \|\nabla \varphi\|^2_{L^2(\Omega)} + \|\nabla \omega\|^2_{L^2(\Omega)}) \frac{1}{2}, \]  

(56)

where \( c_2 \) only depends on \( \Omega \). By combining (13), (16) and (56) we get (53). \( \square \)

**Proposition 3.** For all natural number \( n \), we let \( \beta_n = \lambda_n \), then there exist a set of sequences \( \Pi_n \in \mathcal{D}(A) \) and a constant \( \tilde{c} > 0 \) such that

\[ ||\Pi_n||_{\mathcal{H}} \geq 1, \]  

(57)

\[ ||(i\beta_n - A')\Pi_n||_{\mathcal{H}} \leq \tilde{c}\beta_n^{-\frac{1}{2}}. \]  

(58)

**Proof:** Assuming that \( \Pi_n = \beta_n^{-1}[\Phi_n, i\beta_n \Phi_n, \delta_n] \), where \( \Phi_n = [\psi_n, \varphi_n, \omega_n], \delta_n = [\delta_n^{(1)}, \delta_n^{(2)}, \delta_n^{(3)}] = [-C_1 \gamma_0 \psi_n, -C_2 \gamma_0 \varphi_n, -C_3 \gamma_0 \omega_n] \) on \( \Gamma_1 \). One can thus assert that \( \Pi_n \) belongs to \( \mathcal{D}(A') \). Indeed,

\[ \left[ (\delta_n^{(1)}, C_1)_C^n, (\delta_n^{(2)}, C_2)_C^n, (\delta_n^{(3)}, C_3)_C^n \right] \]
\[ = - \left[ (C_1, C_1)_C^n \gamma_0 \psi_n, (C_2, C_2)_C^n \gamma_0 \varphi_n, (C_3, C_3)_C^n \gamma_0 \omega_n \right] \]
\[ = -Q[\gamma_0 \psi_n, \gamma_0 \varphi_n, \gamma_0 \omega_n] \]
\[ = [\delta_1(\psi_n, \varphi_n), \delta_2(\psi_n, \varphi_n), \delta_3(\psi_n, \varphi_n, \omega_n)], \quad \text{on} \ \Gamma_1. \]  

(59)

Clearly, it follows from the definition of \( \Pi_n \) that

\[ ||\Pi_n||_{\mathcal{H}} \geq ||\Phi_n||_{\mathcal{V}} = 1. \]

Moreover, we also obtain

\[ (i\beta_n - A')\Pi_n \]
\[ = \beta_n^{-1}[0, 0, (i\beta_n - B_1)\delta_n^{(1)} + i\beta_n C_1 \gamma_0 \psi_n, (i\beta_n - B_2)\delta_n^{(2)} + i\beta_n C_2 \gamma_0 \varphi_n, \]
\[ (i\beta_n - B_3)\delta_n^{(3)} + i\beta_n C_3 \gamma_0 \omega_n] \]
\[ = \beta_n^{-1}[0, 0, B_1 C_1 \gamma_0 \psi_n, B_2 C_2 \gamma_0 \varphi_n, B_3 C_3 \gamma_0 \omega_n]. \]

Therefore, by using the boundness of \( B_j \) and \( C_j \), we have

\[ ||(i\beta_n - A')\Pi_n||_{\mathcal{H}}^2 \]
\[ = \beta_n^{-2} \int_{\Gamma_1} \left[ ||B_1 C_1 \gamma_0 \psi_n||_C^n + ||B_2 C_2 \gamma_0 \varphi_n||_C^n \right. \]
\[ + ||B_3 C_3 \gamma_0 \omega_n||_C^n \left. \right] d\Gamma_1 \]
\[ \leq \beta_n^{-2} \tilde{c}_1 \int_{\Gamma_1} \left[ ||\gamma_0 \psi_n||^2 + ||\gamma_0 \varphi_n||^2 + ||\gamma_0 \omega_n||^2 \right] d\Gamma_1, \]  

(60)

where \( \tilde{c}_1 > 0 \) is independent of \( \Phi_n \) and \( \Gamma_1 \).
From (53), the estimate (60) is simplified into
\[ \| (i\beta_n - \mathcal{A}) \Pi_n \|_H \leq \tilde{c} \beta_n^{\frac{1}{2}} \| \psi_n, \varphi_n, \omega_n \|_V \| \psi_n, \varphi_n, \omega_n \|_W \leq \tilde{c} \beta_n^{-\frac{1}{2}} \]
which completes the proof. \( \square \)

**Theorem 3.5.** The \( C_0 \)-semigroup generated by \( \mathcal{A} \) in \( \mathcal{H} \) is not exponentially stable.

**Proof.** We let \((i\beta_n - \mathcal{A}) \Pi_n = \Psi_n\). According to Proposition 3, we get
\[
\| (i\beta_n - \mathcal{A})^{-1} \|_{L(\mathcal{H})} = \sup_{\Psi \neq 0} \frac{\| (i\beta_n - \mathcal{A})^{-1} \Psi \|_H}{\| \Psi \|_H}
\geq \frac{\| (i\beta_n - \mathcal{A})^{-1} \Psi_n \|_H}{\| \Psi_n \|_H} = \frac{\| \Pi_n \|_H}{\| (i\beta_n - \mathcal{A}) \Pi_n \|_H}
\geq \frac{1}{c} \rho_n \to +\infty, \quad (n \to +\infty).
\]
This contradicts (30). We thus complete the proof of Theorem 3.5. \( \square \)

4. **Polynomial stability.** In this section, the polynomial stability of the semigroup solution of system (5) can be obtained by using a part of fundamental theories in [7].

**Theorem 4.1.** Let \( T(t)_{t \geq 0} \) be a bounded \( C_0 \)-semigroup on a Hilbert space \( \mathcal{H} \) with generator \( A \) such that \( \varrho(A) \supset i\mathbb{R} \). Then for a fixed \( \alpha > 0 \), the following conditions are equivalent:

(i) \( \| (isI - A)^{-1} \| = O(|s|^\alpha), \quad s \to \infty, \)

(ii) \( \| T(t)A^{-1} \| = O(t^{-\frac{1}{\alpha}}), \quad t \to \infty. \)

We consider the auxiliary system
\[
\begin{align*}
\begin{cases}
\psi_{tt} + \varphi_{tt} + \omega_{tt} = 0, & \text{in } \Omega \times (0, +\infty), \\
\psi = \varphi = \omega = 0, & \text{on } \Gamma_0 \times (0, +\infty), \\
D \left[ \frac{\partial \psi}{\partial x} + \mu \frac{\partial \varphi}{\partial y}, \begin{cases}
1 - \mu \left( \frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial x} \right)
\end{cases}, \begin{cases}
\frac{\partial \varphi}{\partial y} + \mu \frac{\partial \psi}{\partial x}
\end{cases} \right] \cdot \nu = -\psi_t, \\
D \left[ \frac{1}{2} \left( \frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial x} \right), \begin{cases}
\frac{\partial \varphi}{\partial y} + \mu \frac{\partial \psi}{\partial x}
\end{cases} \cdot \nu = -\varphi_t, \\
K \left[ \psi + \frac{\partial \omega}{\partial x}, \varphi + \frac{\partial \omega}{\partial y} \right] \cdot \nu = -\omega_t, & \text{on } \Gamma_1 \times (0, +\infty),
\end{cases}
\end{align*}
\]
\[
[\psi, \varphi, \omega]_{t=0} = [\psi_0, \varphi_0, \omega_0], \quad [\psi_t, \varphi_t, \omega_t]_{t=0} = [\psi_1, \varphi_1, \omega_1], \quad (x, y) \in \Omega.
\]

We let the space \( \mathcal{H}_1 = W \times V \) and introduce a new operator \( \mathcal{A}_3 \) given by
\[
\mathcal{A}_3[\psi, \varphi, \omega; \phi, v, \eta] = [\phi, v; \rho_1, \rho_1, \rho_2 \ell_3],
\]
with the domain
taking the modulus, it follows from Cauchy-Schwarz’s inequality that

$$\mathcal{D}(\mathcal{H}_3) = \left\{ [\psi, \varphi, \omega; \phi, v, \eta] \in \mathcal{H}_1 | [\phi, v, \eta] \in W, \ [\rho_1 \ell_1, \rho_1 \ell_2, \rho_2 \ell_3] \in V, \right.$$ 

$$\psi = \varphi = \omega = 0, \text{ on } \Gamma_0, \ h_1(\psi, \varphi) = -\psi_t, \ h_2(\psi, \varphi) = -\varphi_t,$$

$$h_3(\psi, \varphi, \omega) = -\omega_t, \text{ on } \Gamma_1 \right\}.$$

In what follows, we prove that the system (5) is polynomially stable with the aid of the exponential or polynomial stability of system (63). Moreover, the decay rate of system (5) depends on the type of stability of (63).

**Proposition 4.** Suppose that the energy of system (63) is exponentially stable and there exist constants \( p > 0 \) and \( l > 0 \) such that for \( s \in \mathbb{R} \) with \( |s| \) large enough satisfying

$$\Re((isI - B_j)^{-1}C_j, C_j)_{C_n} \geq \frac{l}{|s|^{2p}}, \ j = 1, 2, 3,$$  \hfill (65)

then the energy of system (5) has a polynomial decay estimation, i.e.,

$$E(t) \leq \tilde{C} \frac{1}{t^{\frac{p+1}{2}}} \|U_0\|_{\mathcal{G}(\mathcal{H})}^2, \quad \forall t > 0,$$  \hfill (66)

where \( \tilde{C} \) is independent of \( U_0 \).

**Proof.** For every \( s \in \mathbb{R} \) and \( \mathcal{U}_1 = [f_1, f_2, f_3; g_1, g_2, g_3; u^{(1)}, u^{(2)}, u^{(3)}] \in \mathcal{H} \), we denote

$$\mathcal{U} = [\psi, \varphi, \omega; \phi, v, \eta; \delta^{(1)}, \delta^{(2)}, \delta^{(3)}] = (isI - \mathcal{A})^{-1}\mathcal{U}_1.$$  \hfill (67)

By replacing \( \lambda, \psi^{(1)}, \varphi^{(2)}, \omega^{(3)} \) in (25) by \( is, \psi, \varphi, \omega \), respectively, we obtain

$$-s^2 \int_\Omega \left[ \frac{\rho h^3}{12} (|\psi|^2 + |\varphi|^2) + \rho h |\omega|^2 \right] dxdy + a([\psi, \varphi, \omega], [\psi, \varphi, \omega])$$

$$+ \int_{\Gamma_1} \left[ is \left((isI - B_1)^{-1} C_1, C_1\right)_{C_n} |\gamma_0|^2 + is \left((isI - B_2)^{-1} C_2, C_2\right)_{C_n} |\gamma_0|^2 + is \left((isI - B_3)^{-1} C_3, C_3\right)_{C_n} |\gamma_0|^2 \right] d\Gamma_1,$$

$$= \int_\Omega \left[ \frac{\rho h^3}{12} ((isf_1 + g_1)\psi + (isf_2 + g_2)\varphi) + \rho h (isf_3 + g_3)\omega \right] dxdy$$

$$+ \int_{\Gamma_1} \left[ \left((isI - B_1)^{-1}(u^{(1)} + C_1 f_1), C_1\right)_{C_n} \gamma_0 \psi + \left((isI - B_2)^{-1}(u^{(2)} + C_2 f_2), C_2\right)_{C_n} \gamma_0 \varphi + \left((isI - B_3)^{-1}(u^{(3)} + C_3 f_3), C_3\right)_{C_n} \gamma_0 \omega \right] d\Gamma_1.$$  \hfill (68)

By taking the imaginary part for both sides of (68) in the first instance and then taking the modulus, it follows from Cauchy-Schwarz’s inequality that

$$|s| \int_{\Gamma_1} \left[ \Re\left((isI - B_1)^{-1} C_1, C_1\right)_{C_n} |\gamma_0|^2 + \Re\left((isI - B_2)^{-1} C_2, C_2\right)_{C_n} |\gamma_0|^2 \right] d\Gamma_1,$$

$$+ \Re\left((isI - B_3)^{-1} C_3, C_3\right)_{C_n} |\gamma_0|^2 \right] d\Gamma_1,$$

$$\leq \tilde{C} \left( |s| \|\mathcal{U}_1\|_{\mathcal{H}} [\psi, \varphi, \omega] + \|\mathcal{U}_1\|_{\mathcal{H}} [\psi, \varphi, \omega] \right)$$

$$+ \frac{1}{|s|} \|\mathcal{U}_1\|_{\mathcal{H}} [\gamma_0 \psi, \gamma_0 \varphi, \gamma_0 \omega] \right)_{V(\Gamma_1)}.$$  \hfill (69)
From (65), we can deduce that for \( |s| \) large enough,
\[
\left| s \right| \frac{1}{|s|^{2p}} \int_{\Gamma_1} \left[ \frac{\rho h^3}{12} \left( |\gamma_0 \psi|^2 + |\gamma_0 \varphi|^2 \right) + \rho h|\gamma_0 \omega|^2 \right] d\Gamma_1,
\]
\[
\leq \tilde{C}_1 \left( |s| \|\mathbf{U}_t\|_H \|\psi, \varphi, \omega\|_V + \frac{1}{|s|} \|\mathbf{U}_t\|_H \|\gamma_0 \psi, \gamma_0 \varphi, \gamma_0 \omega\|_{V(\Gamma_1)} \right).
\]
This implies that
\[
\|\gamma_0 \psi, \gamma_0 \varphi, \gamma_0 \omega\|_{V(\Gamma_1)} \leq \tilde{C}_1 \left( |s|^{2p} \|\mathbf{U}_t\|_H \|\psi, \varphi, \omega\|_V + |s|^{2p-2} \|\mathbf{U}_t\|_H \|\gamma_0 \psi, \gamma_0 \varphi, \gamma_0 \omega\|_{V(\Gamma_1)} \right).
\]
By using Cauchy inequality with \( \varepsilon > 0 \), we further obtain
\[
\|\gamma_0 \psi, \gamma_0 \varphi, \gamma_0 \omega\|_{V(\Gamma_1)} \leq \tilde{C}_2 \left( |s|^{2p} \|\mathbf{U}_t\|_H \|\psi, \varphi, \omega\|_V + |s|^{4(p-1)} \|\mathbf{U}_t\|_H^4 \right).
\]
(70)
For every \( s \in \mathbb{R} \) and \( [r_1, r_2, r_3] \in V \), we can prove that there exists a solution \([\psi, \varphi, \omega] \in W\) belonging to the following equation
\[
\begin{cases}
(-s^2 I + A)[\psi, \varphi, \omega] = [r_1, r_2, r_3], & \text{in } \Omega \times (0, +\infty), \\
[\psi, \varphi, \omega] = 0, & \text{on } \Gamma_0 \times (0, +\infty), \\
h_1[\psi, \omega], h_2[\psi, \varphi], h_3[\psi, \varphi, \omega] = -is[\psi, \varphi, \omega], & \text{on } \Gamma_1 \times (0, +\infty),
\end{cases}
\]
and satisfying
\[
\begin{aligned}
\|\psi, \varphi, \omega\|_W + |s| \|\psi, \varphi, \omega\|_V &\leq \tilde{C}_3 \|[r_1, r_2, r_3]\|_V, \\
|s| \|\gamma_0 \psi, \gamma_0 \varphi, \gamma_0 \omega\|_{V(\Gamma_1)} &\leq \tilde{C}_4 \|[r_1, r_2, r_3]\|_V.
\end{aligned}
\]
(72)
In fact, from the theory of Prüss, Huang and Gearhart (see [15],[18],[30]) on the equivalent conditions of exponential stability for linear dynamical systems, we have that there exists a constant \( M > 0 \) such that \( \| (is I - \alpha_\delta)^{-1} \|_{C(H_1)} \leq M < +\infty \) for all \( s \in \mathbb{R} \). For any \( s \in \mathbb{R} \) and \([r_1, r_2, r_3] \in V\), we thus find \([\psi, \varphi, \omega; \psi_1, \varphi_1, \omega_1] = [0; r_1, r_2, r_3]\) and satisfying
\[
\|[\psi, \varphi, \omega; \psi_1, \varphi_1, \omega_1]\|_{H_1} \leq M \|[0; r_1, r_2, r_3]\|_{H_1} = M \|[r_1, r_2, r_3]\|_V.
\]
(73)
We thus obtain that
\[
\begin{cases}
is[\psi, \varphi, \omega] - [\psi_1, \varphi_1, \omega_1] = 0, \\
is[\psi_1, \varphi_1, \omega_1] + A[\psi, \varphi, \omega] = [r_1, r_2, r_3].
\end{cases}
\]
(74)
From (73) and (74), we can deduce that (71) holds. Moreover,
\[
\|\psi, \varphi, \omega\|_W + |s| \|\psi, \varphi, \omega\|_V \leq \tilde{C}_5 \|[r_1, r_2, r_3]\|_V,
\]
(75)
which implies that the first estimate of (72) holds. In order to obtain the second inequality of (72), by taking the inner product of \([\psi, \varphi, \omega]\) with the first equation of (71) in \( V \), it yields
\[
-s^2 \int_{\Omega} \left[ \frac{\rho h^3}{12} \left( |\psi|^2 + |\varphi|^2 \right) + \rho h|\omega|^2 \right] dxdy
\]
\[
- \int_{\Omega} \left[ \psi \ell_1[\psi, \varphi, \omega] + \varphi \ell_2[\psi, \varphi, \omega] + \omega \ell_3[\psi, \varphi, \omega] \right] dxdy
\]
\[
= \int_{\Omega} \left[ \frac{\rho h^3}{12} (r_1 \tilde{\psi} + r_2 \tilde{\varphi}) + \rho hr_3 \tilde{\omega} \right] dxdy.
\]
(76)
By integrating by parts, we have

\[- s^2 \int_\Omega \left[ \frac{\rho h^3}{12} (|\psi|^2 + |\phi|^2) + \rho h |\omega|^2 \right] dxdy + a([\psi, \phi, \omega], [\dot{\psi}, \dot{\phi}, \dot{\omega}])
+ is \int_{\Gamma_1} \left[ |\gamma_0 \dot{\psi}|^2 + |\gamma_0 \dot{\phi}|^2 + |\gamma_0 |\omega|^2 \right] d\Gamma_1
= \int_\Omega \left[ \frac{\rho h^3}{12} (r_1 \ddot{\psi} + r_2 \ddot{\phi}) + \rho hr_3 \ddot{\omega} \right] dxdy.\] (77)

By taking the imaginary part of (77), we arrive at

\[|s| \int_{\Gamma_1} \left[ |\gamma_0 \dot{\psi}|^2 + |\gamma_0 \dot{\phi}|^2 + |\gamma_0 |\omega|^2 \right] d\Gamma_1
= \Im \left\{ \int_\Omega \left[ \frac{\rho h^3}{12} (r_1 \ddot{\psi} + r_2 \ddot{\phi}) + \rho hr_3 \ddot{\omega} \right] dxdy \right\} \leq \tilde{C}_5 \| [\dot{\psi}, \dot{\phi}, \dot{\omega}] \|_V \| [r_1, r_2, r_3] \|_V.\] (78)

By combining (72) and (78), we obtain

\[|s| \int_{\Gamma_1} \left[ |\gamma_0 \dot{\psi}|^2 + |\gamma_0 \dot{\phi}|^2 + |\gamma_0 |\omega|^2 \right] d\Gamma_1 \leq \tilde{C}_3 \tilde{C}_5 \frac{\| [r_1, r_2, r_3] \|_V^2}{|s|},\]

which proves the second estimate of (72).

By replacing \(\lambda, \psi^{(1)}, \phi^{(2)}, \omega^{(3)}\) in (25) by \(s, \dot{\psi}, \dot{\phi}, \dot{\omega}\), respectively, we have

\[- s^2 \int_\Omega \left[ \frac{\rho h^3}{12} (\psi \ddot{\psi} + \phi \ddot{\phi}) + \rho h \omega \ddot{\omega} \right] dxdy + a([\psi, \phi, \omega], [\dot{\psi}, \dot{\phi}, \dot{\omega}])
+ \int_{\Gamma_1} \left[ is \left( (isI - B_1)^{-1} C_1, C_1 \right)_{\mathcal{C}_n} \gamma_0 \gamma_0 \ddot{\psi} \right.
+ is \left( (isI - B_2)^{-1} C_2, C_2 \right)_{\mathcal{C}_n} \gamma_0 \phi \gamma_0 \ddot{\phi} \n+ is \left( (isI - B_3)^{-1} C_3, C_3 \right)_{\mathcal{C}_n} \gamma_0 \omega \gamma_0 \ddot{\omega} \right] d\Gamma_1,
= \int_\Omega \left[ \frac{\rho h^3}{12} \left( (isf_1 + g_1) \ddot{\psi} + (isf_2 + g_2) \ddot{\phi} \right) + \rho h (isf_3 + g_3) \ddot{\omega} \right] dxdy
+ \int_{\Gamma_1} \left[ \left( (isI - B_1)^{-1} (u^{(1)} + C_1 f_1), C_1 \right)_{\mathcal{C}_n} \gamma_0 \ddot{\psi} \right.
+ \left( (isI - B_2)^{-1} (u^{(2)} + C_2 f_2), C_2 \right)_{\mathcal{C}_n} \gamma_0 \ddot{\phi} \n+ \left( (isI - B_3)^{-1} (u^{(3)} + C_3 f_3), C_3 \right)_{\mathcal{C}_n} \gamma_0 \ddot{\omega} \right] d\Gamma_1.\] (79)

As the functional \(a([\psi, \phi, \omega], [\dot{\psi}, \dot{\phi}, \dot{\omega}])\) defined in the identity (9) with the property

\[a([\psi, \phi, \omega], [\dot{\psi}, \dot{\phi}, \dot{\omega}]) = a([\ddot{\psi}, \ddot{\phi}, \ddot{\omega}], [\ddot{\psi}, \ddot{\phi}, \ddot{\omega}]),\]
then by using (71) together with the definition of the operator $A$ in (34) and the formula (11), we get

$$a([\psi, \varphi, \omega], [\bar{\psi}, \bar{\varphi}, \bar{\omega}])$$

$$= - \int_{\Omega} \left[ \psi \ell_1 [\bar{\psi}, \bar{\varphi}, \bar{\omega}] + \varphi \ell_2 [\bar{\psi}, \bar{\varphi}, \bar{\omega}] + \omega \ell_3 [\bar{\psi}, \bar{\varphi}, \bar{\omega}] \right] dxdy$$

$$+ \int_{\Gamma_1} \left[ \gamma_0 \psi \bar{h}_1[\gamma_0 \bar{\psi}, \gamma_0 \bar{\varphi}] + \gamma_0 \varphi \bar{h}_2[\gamma_0 \bar{\psi}, \gamma_0 \bar{\varphi}] + \gamma_0 \omega \bar{h}_3[\gamma_0 \bar{\psi}, \gamma_0 \bar{\varphi}, \gamma_0 \bar{\omega}] \right] d\Gamma_1$$

$$= \int_{\Omega} \left[ \frac{\rho h^3}{12} (\bar{\psi} + s^2 \bar{\psi}) + \frac{\rho h^3}{12} \psi (\bar{\varphi} + s^2 \bar{\varphi}) + \rho h (\bar{\varphi} + s^2 \bar{\omega}) \right] dxdy$$

By replacing $r_1, r_2, r_3$ in (81) by $\psi, \varphi, \omega$, respectively and using Cauchy-Schwarz inequality, we deduce that

$$\| [\psi, \varphi, \omega] \|_V^2 \leq \tilde{C}_0 \left( |s| \| U_1 \|_H [\psi, \varphi, \omega] \|_V + \| U_1 \|_H \| [\psi, \varphi, \omega] \|_V \right)$$

$$+ \frac{1}{|s|} \| U_1 \|_H \| [\gamma_0 \psi, \gamma_0 \varphi, \gamma_0 \omega] \|_{V(\Gamma_1)} + \| [\gamma_0 \psi, \gamma_0 \varphi, \gamma_0 \omega] \|_{V(\Gamma_1)} \| [\gamma_0 \psi, \gamma_0 \varphi, \gamma_0 \omega] \|_{V(\Gamma_1)}.$$
and

$$\|\psi, \varphi, \omega\|_V = C_1|\psi| + C_2|\varphi| + C_3|\omega|.$$

(83)

By removing $|\psi, \varphi, \omega\|_V$ on both sides of (83), we get

$$|\psi, \varphi, \omega\|_V \leq \tilde{C}_7\left(\tilde{C}_3\|U_1\|_{\mathcal{H}} + \tilde{C}_4|\gamma_0\psi, \gamma_0\varphi, \gamma_0\omega|\right).$$

(84)

It follows from (70) and (84) that

$$|\psi, \varphi, \omega\|_V \leq \tilde{C}_7\left(\tilde{C}_3\|U_1\|_{\mathcal{H}} + \tilde{C}_4|\gamma_0\psi, \gamma_0\varphi, \gamma_0\omega|\right).$$

(85)

By using Young's inequality again for (85), we deduce that for $|s|$ large enough

$$|\psi, \varphi, \omega\|_V^2 \leq \tilde{C}_9|s|^{4p}\|U_1\|_{\mathcal{H}}^2.$$

(86)

We note (84) to obtain

$$|\psi, \varphi, \omega\|_V^2 \leq \tilde{C}_{10}\left(\|U_1\|_{\mathcal{H}}^2 + |\gamma_0\psi, \gamma_0\varphi, \gamma_0\omega|\right).$$

(87)

Similarly to (22) and (23) but replacing $\lambda$ by $is$, it follows that

$$\|\psi, \varphi, \omega\|_V^2 = \|is\psi - f_1, is\varphi - f_2, is\omega - f_3\|_V^2$$

$$= \int_{\Omega} \left[\int_{\Omega} \frac{\partial^3}{\partial x^3} (|is\psi - f_1|^2 + |is\varphi - f_2|^2 + |is\omega - f_3|^2) dx dy \right]$$

$$\leq \tilde{C}_{12}|s|^{2p}\|\psi, \varphi, \omega\|_V^2 + \|U_1\|_{\mathcal{H}}^2 \leq \tilde{C}_{13}|s|^{4p+2}\|U_1\|_{\mathcal{H}}^2,$$

and

$$\|\delta(1), \delta(2), \delta(3)\|_{\mathcal{H}}^2$$

$$\leq \tilde{C}_{14}\left(\frac{1}{|s|^2}\|U_1\|_{\mathcal{H}}^2 + |\gamma_0\psi, \gamma_0\varphi, \gamma_0\omega|\right).$$

Finally, we can also obtain the estimation

$$\|\psi, \varphi, \omega\|_V^2 \leq \tilde{C}_{16}|s|^{4p+2}\|U_1\|_{\mathcal{H}}^2.$$
Indeed, by applying the expression of norm of $W$ in (14) together with (68), we have

$$
\| [\psi, \varphi, \omega] \|_{W}^{2} = D \int_{\Omega} \left[ \frac{\partial \psi}{\partial x}^{2} + \left| \frac{\partial \varphi}{\partial y} \right|^{2} + \mu \frac{\partial \psi}{\partial x} \frac{\partial \varphi}{\partial y} + \mu \frac{\partial \psi}{\partial y} \frac{\partial \varphi}{\partial x} + \frac{1 - \mu}{2} \left| \frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial x} \right|^{2} \right] dx dy \\
+ K \int_{\Omega} \left[ \left| \psi + \frac{\partial \omega}{\partial y} \right|^{2} + \left| \varphi + \frac{\partial \omega}{\partial y} \right|^{2} \right] dx dy
$$

$$
= s^{2} \int_{\Omega} \left[ \frac{\rho_{h}^{3}}{12} (|\psi|^{2} + |\varphi|^{2}) + \rho_{h}|\omega|^{2} \right] dx dy \\
+ \int_{\Gamma_{1}} \left[ \left( (isI - B_{1})^{-1} (u^{(1)} + C_{1} f_{1}) \right)_{C_{s}} \gamma_{0} \psi \right] \\
+ \left( (isI - B_{2})^{-1} (u^{(2)} + C_{2} f_{2}) \right)_{C_{s}} \gamma_{0} \varphi \\
+ \left( (isI - B_{3})^{-1} (u^{(3)} + C_{3} f_{3}) \right)_{C_{s}} \gamma_{0} \omega \right] d\Gamma_{1} \\
- is \int_{\Gamma_{1}} \left[ \left( (isI - B_{1})^{-1} C_{1}, C_{1} \right)_{C_{s}} \gamma_{0} \psi \right]^{2} \\
+ \left( (isI - B_{2})^{-1} C_{2}, C_{2} \right)_{C_{s}} \gamma_{0} \varphi^{2} + \left( (isI - B_{3})^{-1} C_{3}, C_{3} \right)_{C_{s}} \gamma_{0} \omega^{2} \right] d\Gamma_{1}
$$

$$
\leq \tilde{C}_{17} \left( \| s \|^{2} \| [\psi, \varphi, \omega] \|_{V}^{2} + |s| \| U_{1} \|_{\mathcal{H}} \| [\psi, \varphi, \omega] \|_{V} + \| U_{1} \|_{\mathcal{H}} \| [\psi, \varphi, \omega] \|_{V} \\
+ \frac{1}{|s|} \| U_{1} \|_{\mathcal{H}} \| [\gamma_{0} \psi, \gamma_{0} \varphi, \gamma_{0} \omega] \|_{V(\Gamma_{1})} + \| [\gamma_{0} \psi, \gamma_{0} \varphi, \gamma_{0} \omega] \|_{V(\Gamma_{1})}^{2} \right) \\
\leq \tilde{C}_{18} \left( |s|^{4p+2} \| U_{1} \|_{\mathcal{H}}^{2} + |s|^{2p+1} \| U_{1} \|_{\mathcal{H}}^{2} + |s|^{2p-1} \| U_{1} \|_{\mathcal{H}}^{2} + |s|^{4p} \| U_{1} \|_{\mathcal{H}}^{2} \right) \\
\leq \tilde{C}_{19} |s|^{4p+2} \| U_{1} \|_{\mathcal{H}}^{2},
$$

By combining (88), (89) and (90), we thus obtain that for $|s|$ large enough

$$
\| U \|_{\mathcal{H}}^{2} = \| [\psi, \varphi, \omega] \|_{V}^{2} + \| [\phi, \psi, \omega] \|_{V}^{2} + \| [\delta^{(1)}, \delta^{(2)}, \delta^{(3)}] \|_{V}^{2} \\
\leq \tilde{C}_{16} |s|^{4p+2} \| U_{1} \|_{\mathcal{H}}^{2} + \tilde{C}_{17} |s|^{2p+1} \| U_{1} \|_{\mathcal{H}}^{2} + \| \phi(s, \omega) \|_{\mathcal{H}}^{2} \\
\leq \tilde{C}_{19} |s|^{4p+2} \| U_{1} \|_{\mathcal{H}}^{2},
$$

which means, from (67), that

$$
\| (isI - \mathcal{A})^{-1} \|_{\mathcal{L}(\mathcal{H})} = O(|s|^{2p+1}), \quad s \to \infty.
$$

According to the Theorem 4.1, we reach the conclusion that

$$
\| U(t) \|_{\mathcal{H}} \leq \tilde{C}_{2} \frac{1}{t^{\frac{1}{2p+1}}} \| U_{0} \|_{\mathcal{H}},
$$

which proves (66).

Proposition 5. Assume that the energy of system (63) is polynomially stable and there exist constants $p > 0$ and $l > 0$ such that for $s \in \mathbb{R}$ with $|s|$ large enough, (65) holds. Then, the energy of the solution of system (5) satisfies a polynomial decay,
where \( C > 0 \) and \( \eta > 0 \) are all independent of \( U_0 \).

**Proof.** When the energy of system (63) is polynomially stable, we apply the Theorem 4.1 to obtain that there are constants \( \eta > 0 \) and \( M > 0 \) such that

\[
\| \hat{\psi}, \hat{\varphi}, \hat{\omega}; \hat{\psi}_t, \hat{\varphi}_t, \hat{\omega}_t \| \leq M \| s \|^{3/2} \| r_1, r_2, r_3 \|_V.
\]

Proceeding as in the proof of Proposition 4, we obtain

\[
\begin{align*}
\| \hat{\psi}, \hat{\varphi}, \hat{\omega} \|_V + \| s \| \| \hat{\psi}, \hat{\varphi}, \hat{\omega} \|_V &\leq \tilde{C}_3 \| s \|^{2/3} \| r_1, r_2, r_3 \|_V, \\
\| \hat{\psi}, \hat{\varphi}, \hat{\omega} \|_V &\leq \tilde{C}_4 \| s \|^{2/3} \| r_1, r_2, r_3 \|_V.
\end{align*}
\]

With a slight modification of the estimate in (83) and (85), we can show that

\[
\begin{align*}
\| \hat{\psi}, \hat{\varphi}, \hat{\omega} \|_V^2 &\leq \tilde{C}_{20} \| s \|^{4p+2\eta} \| U_1 \|_H^2, \\
\| \hat{\psi}, \hat{\varphi}, \hat{\omega} \|_V^2 &\leq \tilde{C}_{21} \| s \|^{4p+\eta} \| U_1 \|_H^2.
\end{align*}
\]

Hence, (88), (89) and (90) successively become

\[
\begin{align*}
\| \hat{\psi}, \hat{\varphi}, \hat{\omega} \|_V^2 &\leq \tilde{C}_{12} \| s \|^{2p+2\eta} \| U_1 \|_H^2 + \| \hat{U}_1 \|_H^2, \\
\| \delta^{(1)}, \delta^{(2)}, \delta^{(3)} \|_V^2 &\leq \tilde{C}_{14} \left( \frac{1}{s^{p+1}} \| \hat{U}_1 \|_H^2 + \| \delta^{(1)} \|_V^2 \right), \\
\| \hat{\psi}, \hat{\varphi}, \hat{\omega} \|_W^2 &\leq \tilde{C}_{24} \| s \|^{4p+2+2\eta} \| U_1 \|_H^2.
\end{align*}
\]

It is easy to deduce from the above estimates and (67) that

\[
\| (isI - \mathcal{A})^{-1} \|_{\mathcal{L}(H)} = O\left( \| s \|^{-2p+1+\eta} \right), \quad s \to \infty.
\]

This implies that the estimate (92) is proved based on the results of Theorem 4.1. \(\square\)

Finally, we will clarify the sufficient conditions for \( B_j \) and \( C_j \) such that (65) is tenable.

**Proposition 6.** Assume that \( B_j \in C^{0,1}(\Gamma_1, M_{n \times n}(\mathbb{C})) \), \( C_j \in C^{0,1}(\Gamma_1, \mathbb{C}^n) \) and \( M(x, y) \in M_{n \times n}(\mathbb{C}) \) have constant scalar entries. Denote by \( B_j^* \) the adjoint of \( B_j \) with respect to \((\cdot, \cdot)_\mathbb{C}^n\), and we let \( S_j = \frac{B_j + B_j^*}{2} \). Moreover, suppose that there exist natural numbers \( d_1, d_2, d_3 \) satisfying

\[
\begin{align*}
\{ d_1 &\in \mathbb{N} | \mathcal{F}_1(\mathcal{B}_1^{m} C_1) \neq 0 \}, \\
\{ d_2 &\in \mathbb{N} | \mathcal{F}_2(\mathcal{B}_2^{m} C_2) \neq 0 \}, \\
\{ d_3 &\in \mathbb{N} | \mathcal{F}_3(\mathcal{B}_3^{m} C_3) \neq 0 \},
\end{align*}
\]

where \( \mathcal{F}_j \) (\( j = 1, 2, 3 \)) denote the projection from \( \mathbb{C}^n \) into \((\ker S_j)^\perp\). Then there is \( l > 0 \) such that for \( |s| \) large enough, we obtain

\[
\Re \left( (isI - B_j)^{-1} C_j, C_j \right)_{\mathbb{C}^n} \geq \frac{l}{|s|^{2(d+1)}}, \quad j = 1, 2, 3, \quad (94)
\]

where \( d = \min\{d_1, d_2, d_3\} \).
Proof. In fact, by the conditions of (93), we can deduce that \( \mathcal{F}_j(B_j^* C_j) = 0 \) for all \( r < d \). On the other hand, by the definition of \( S_j \), we know that \( S_j \) is self-adjoint and \( S_j C_j = S_j \mathcal{F}_j C_j \). Moreover, there exists \( \hat{C}_{25} > 0 \) such that
\[
-(S_j \mathcal{F}_j q, \mathcal{F}_j q)_{\mathbb{C}^n} \geq \hat{C}_{25} \| q \|_{\mathbb{C}^n}^2, \quad \text{for all } q \in \mathbb{C}^n.
\] (95)
Indeed, by the continuity of \( B_j \) together with (6) and (7), we obtain
\[
-(S_j q, q)_{\mathbb{C}^n} = -\left( \frac{B_j + B_j^*}{2}, q \right)_{\mathbb{C}^n} = -\Re (B_j q, q)_{\mathbb{C}^n} \geq \hat{C}_{25} \| q \|_{\mathbb{C}^n}^2.
\]
Since
\[
-(S_j(isI - B_j)^{-1} C_j, (isI - B_j)^{-1} C_j)_{\mathbb{C}^n}
= -(S_j \sum_{k=0}^{\infty} (-i)^{k+1} \frac{B_j^k}{s^{k+1}} C_j, \sum_{m=0}^{\infty} (-i)^{m+1} \frac{B_j^m}{s^{m+1}} C_j)_{\mathbb{C}^n}
= -(\sum_{m=0}^{\infty} (-i)^{m+1} \frac{B_j^m}{s^{m+1}} C_j, S_j(-i)^{d+1} \frac{B_j^d}{s^{d+1}} C_j)_{\mathbb{C}^n} + O(|s|^{-(2d+3)})
= -(\mathcal{F}_j B_j^d C_j, S_j \mathcal{F}_j B_j^d C_j)_{\mathbb{C}^n} + O(|s|^{-(2d+3)}),
\] (96)
from (95) and (96), we thus have
\[
\Re \left( (isI - B_j)^{-1} C_j, C_j \right)_{\mathbb{C}^n}
= \Re \left( (isI - B_j)^{-1} C_j, (isI - B_j)(isI - B_j)^{-1} C_j \right)_{\mathbb{C}^n}
= -(S_j(isI - B_j)^{-1} C_j, (isI - B_j)^{-1} C_j)_{\mathbb{C}^n}
= -(S_j(isI - B_j)^{-1} C_j, (isI - B_j)^{-1} C_j)_{\mathbb{C}^n}
= -(\mathcal{F}_j B_j^d C_j, S_j \mathcal{F}_j B_j^d C_j)_{\mathbb{C}^n} + O(|s|^{-(2d+3)})
\]
(97)
\[
\geq \hat{C}_{25} \frac{\| B_j^d C_j \|_{\mathbb{C}^n}^2}{|s|^{2(d+1)}} + O(|s|^{-(2d+3)}) \geq \frac{l}{|s|^{2(d+1)}}.
\]
The proof is complete. \(
\square \)

5. Conclusion. In this paper, we have presented the nonuniform stability in a 2-D Mindlin-Timoshenko plate with acoustic boundary control conditions. In this case, we have further discussed and proved that the energy of this model is polynomially stable under this boundary conditions. The continuing work is focused on the numerical simulation to verify the validity of the above conclusions or on the stability of the Mindlin-Timoshenko plate with acoustic boundary conditions of thermal effects. Future work will deal with other forms of thin plates, cylindrical shells and beams, with this type of control applied on a part of the boundary.

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