Optimal Control of Thermostatic Loads for Planning Aggregate Consumption: Characterization of Solution and Explicit Strategies

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Abstract—We consider the problem of planning the aggregate energy consumption for a set of thermostatically controlled loads for demand response, accounting price forecast trajectory and thermal comfort constraints. We address this as a continuous-time optimal control problem, and analytically characterize the structure of its solution in the general case. In the special case, when the price forecast is monotone and the loads have equal dynamics, we show that it is possible to determine the solution in an explicit form. Taking this fact into account, we handle the non-monotone price case by considering several subproblems, each correspond to a time subinterval where the price function is monotone, and then allocating to each subinterval a fraction of the total energy budget. This way, for each time subinterval, the problem reduces to a simple convex optimization problem with a scalar decision variable, for which a descent direction is also known. The price forecasts for the day-ahead energy market typically have no more than four monotone segments, so the resulting optimization problem can be solved efficiently with modest computational resources.

I. INTRODUCTION

Thermostatically controlled loads (TCLs), such as air conditioners (ACs), are valuable as flexible resources to elicit demand response, i.e., for actively controlling the loads to offset intermittency in the generation side (e.g., due to renewables) [1], [2], [10], [11], [14]. Utilities or load serving entities (LSEs) can dynamically exploit the thermal inertia of the population of TCLs to strategically plan and control the aggregate consumption in a desired manner. In this paper, we consider the optimal planning problem for an LSE wherein the objective is to plan the power consumption trajectory over a time horizon to minimize the total purchase cost of energy (e.g., from a day-ahead market) while adhering to the individual thermal comfort limits and TCL dynamics constraints, given that a forecasted price trajectory is available over the planning horizon. Here, we restrict the planning problem to single horizon case, although one can envisage solving the same in a sliding time-window manner.

Since TCLs allow discrete ON-OFF (i.e., non-convex) controls, a direct application of standard optimal control tools such as Pontryagin’s Maximum Principle (PMP) subject to state (here, temperature) inequality constraints becomes non-trivial. For example, physical TCLs have maximum switching frequency constraints which do not allow “holding” the TCLs in ON or OFF mode over an interval of time even though the price forecast over that interval is extremal. This suggests accounting the switching frequency constraint explicitly in control design, and that the solution structure for optimal control trajectory may not be trivial even for simple cases (e.g., monotone price forecast). On the other hand, the computational challenge in solving mixed integer control problems brings forth the question: is it possible to recover the non-convex optimal control from the simpler (albeit numerical) optimal control solution?

In Section II, we outline the optimal control problem accounting switching constraints, and describe the convex relaxation. Sections III and IV characterize structural results for the solution for general and monotone price forecasts, respectively. These results motivate a decomposition strategy (Section V) allowing us to solve simpler subproblems over monotone price segments. This paper extends our earlier results [9] to apply PMP for the planning problem accounting switching constraints. Section VI concludes the paper.

II. OPTIMAL PLANNING PROBLEM

For specificity, hereafter we refer TCLs as ACs. We consider an optimal consumption planning problem over time $t \in [0, T]$ for $N$ ACs with respective (indoor) temperature states $\{x_i(t)\}_{i=1}^N$, thermal coefficients $\{\alpha_i, \beta_i\}_{i=1}^N$, ON-OFF controls $\{u_i(t)\}_{i=1}^N$, and initial conditions $\{x_{i0}\}_{i=1}^N$. We suppose that the ACs have upper and lower thermal comfort levels $\{L_i, U_i\}_{i=1}^N$, the ambient temperature trajectory is $\hat{x}(t) = \max_i U_i$, and a total energy budget for the LSE is $E$. Let $[N] := \{1, 2, \ldots, N\}$, and $\mathbf{u} := (u_1, \ldots, u_N) \in \{0,1\}^N$. Given a price forecast $\pi(t)$, assuming Newtonian thermal dynamics for indoor temperature trajectories, and that an ON AC draws power $P$, the planning problem is to minimize the energy procurement cost, i.e., to minimize

$$J(\mathbf{u}) = \int_0^T \pi(t) (u_1(t) + u_2(t) + \ldots + u_N(t)) \, P \, dt,$$

subject to

$$\dot{x}_i(t) = -\alpha_i(x_i(t) - \hat{x}(t)) - \beta_i u_i(t), \text{ a.e. } t \in [0, T], \quad i \in [N],$$

$$L_i \leq x_i(t) \leq U_i \quad \text{for all } t \in [0, T], \quad i \in [N],$$

$$\int_0^T \sum_{i=1}^N u_i(t) \, dt = E,$$

$$\mathbf{u} \in \{0,1\}^N \quad \text{a.e. } t \in [0, T].$$

*We use the abbreviation a.e. to mean “almost everywhere”.

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In order to apply standard optimal control tools to characterize the solution of this planning problem, namely necessary conditions of optimality in the form of the PMP, we consider a modification to this problem, analyze its solution, and then relate the solution of the modified problem to the solution of the original one.

The main difficulties in analyzing the planning problem in its original form are constraints (1) and (2). By introducing an additional state variable $x_{N+1}$, constraint (1), known as an isoperimetric constraint, can be rewritten as a dynamical constraint

$$\dot{x}_{N+1}(t) = u_1(t) + u_2(t) + \ldots + u_N(t), \quad \text{a.e. } t \in [0, T],$$

with end-point conditions $x_{N+1}(0) = 0$, $x_{N+1}(T) = E$. The difficulty with constraint (2) is the fact that it makes the set of possible control values non-convex and an optimal solution to this continuous-time problem might not exist. In fact, when the optimal solution would be to maintain the temperature constant, e.g., along the thermal limits $U_i$ or $L_i$, the corresponding control would have to chatter between 0 and 1 at infinite frequency. Such a solution would not be defined when the trajectories are assumed to be measurable functions (we would have to enlarge the space of trajectories to include the so-called Young measures [13]). Also, such solution would not be practically implementable in ACs.

In order to guarantee that the optimal solution is not a chattering solution, we relax the admissible control values set to its convex hull, allowing intermediate control values,

$$u_i(t) \in [0, 1] \quad \text{a.e. } t \in [0, T], \quad i \in [N].$$

Natural question arises: if the ACs only have ON-OFF control, how do we interpret and implement a solution that cannot be implementable in ACs?

Assuming $\dot{x}$ is constant in that period, we have that $\ddot{u}_i$ is also constant and

$$\ddot{u}_i \int_0^{T_m} e^{-\alpha_i(T_m-s)} ds = \int_0^T e^{-\alpha_i(T_m-s)} ds,$$

implying that $\lambda$ is given explicitly by

$$\lambda = \frac{1}{\alpha_i} \log(1 + (e^{\alpha_i T_m} - 1)\dot{u}_i).$$

The state trajectory resulting from $\ddot{u}_i$ will over-approximate the trajectory resulting from $u_i$. When the state $x_i(t)$ is at the lower limit $L_i$, we should instead use a control starting with OFF segment, i.e.,

$$\ddot{u}_i(t) = \begin{cases} 0 & \text{for } t \in [0, T_m - \lambda), \\ 1 & \text{for } t \in [T_m - \lambda, T_m), \end{cases}$$

For simplicity, omitting the scaling factor $P$, the modified problem, then, is to minimize

$$J(u) = \int_0^T \pi(t) (u_1(t) + u_2(t) + \ldots + u_N(t)) dt$$

subject to

$$\dot{x}_i(t) = -\alpha_i(x_i(t) - \dot{x}(t)) - \beta_i u_i(t), \quad \text{a.e. } t \in [0, T], \quad i \in [N],$$

$$\ddot{x}_{N+1}(t) = \sum_{i=1}^N u_i(t), \quad \text{a.e. } t \in [0, T],$$

$$x_i(0) = x_{i0}, \quad i \in [N],$$

$$x_{N+1}(0) = 0, \quad x_{N+1}(T) = E,$$

$$u_i(t) \in [0, 1], \quad \text{a.e. } t \in [0, T], \quad i \in [N],$$

$$L_i \leq x_i(t) \leq U_i, \quad \text{for all } t \in [0, T], \quad i \in [N].$$

Here, the minimization is to be carried out over functions $u \in \mathcal{U}$, with $\mathcal{U}$ being the set of measurable functions $u : [0, T] \to \mathbb{R}^N$. In the next section, we analyze and characterize the solution to this problem, which we will henceforth refer as problem (P).

III. CHARACTERIZATION OF THE SOLUTION: GENERAL CASE

We start by defining two control values $\ddot{u}_i, \dot{u}_i$ given by

$$\ddot{u}_i := \frac{\alpha_i}{\beta_i} (\dot{x} - U_i), \quad \dot{u}_i := \frac{\alpha_i}{\beta_i} (\dot{x} - L_i),$$

that are used in the results below. These controls lead to “zero” dynamics when the state is on each boundary, thereby permitting it to slide along the same. The control $\ddot{u}_i$ permits the state to slide along the upper boundary $U_i$, while $\dot{u}_i$ permits to slide the state along the lower boundary $L_i$.

The main result in this section is valid under the following assumptions.

**Assumption AI:**

1) The initial states are admissible, i.e.,

$$L_i \leq x_{i0} \leq U_i, \quad \text{for all } i \in [N].$$

2) The total energy preserved, $E$, can be spent respecting the limits $L_i, U_i$ for all initial states, that is

$$E \in [E, \bar{E}],$$
where $\bar{E} = \int_0^T \sum_{i=1}^N \max\{0, \bar{u}_i(t)\} dt$ and $\underline{E} = \int_0^T \sum_{i=1}^N \min\{\bar{u}_i(t), 1\} dt$.

3) When the states are on the boundary of the admissible region, there is a control that drives the states into the interior of the admissible region, i.e., the values of $\alpha_i$ and $\beta_i$ are such that for all $i$, and for all possible $\hat{x}$,

\[- \alpha_i(L_i - \hat{x}) > 0, \quad (9)\]

and

\[- \alpha_i(U_i - \hat{x}) - \beta_i < 0. \quad (10)\]

\[\text{Assumption A2: The functions } \pi(\cdot) \text{ and } \hat{x}(\cdot) \text{ are differentiable. The function } \pi(t) \text{ does not take the specific form}
\]

\[\pi(t) = Ae^{\alpha t} + B\]

for some $i$, and some constant values $A$ and $B$, on some subinterval of $[0, T]$ of nonzero measure.

Assumption A1 guarantees the existence of at least one admissible control-state pair satisfying the constraints. It imposes the requirement that power of the AC unit is capable of overcoming the losses for the range of outside temperatures considered. Assumption A2 is of a technical nature. If a certain very specific growth of the price would be allowed, some algebraic coincidences might lead to singular controls which are much more difficult to analyse. Assumption A2 rules out the singular control scenario.

Assumptions A1 and A2 are imposed throughout the paper. In addition to these assumptions, for some results it is also useful to consider the following equal dynamics hypothesis H1, which permits us to deduce further relevant properties for homogeneous population of ACs.

\[\text{Hypothesis H1: There exist constants } \alpha, \beta, L, U \text{ such that for all } i \in [N], \text{ we have } \alpha_i = \alpha, \beta_i = \beta, L_i = L, U_i = U.\]

With these assumptions, using results from optimal control theory (see e.g. [12]), in particular applying and analysing necessary conditions of optimality in the form of a normal maximum principle in [8], we can establish that the optimal solution has the following characteristics.

\[\text{Theorem 1: Each component } u_i^* \text{ of the optimal control is piecewise constant, and at each time it can assume only one of the 4 values: 0, 1, } \bar{u}_i, \text{ or } u_i. \text{ The value } \bar{u}_i \text{ occurs only when the corresponding component of the trajectory is on the upper boundary, } x_i^* = U_i, \text{ and the value } u_i \text{ occurs only when the corresponding component of the trajectory is on the lower boundary, } x_i^* = L_i.\]

Moreover, if H1 holds then the transitions to the values 0 or 1 occur simultaneously for all components of the control.

A. Proof of the characterization result

First, we guarantee the existence of an admissible solution. Then we guarantee that the PMP can be written in normal form. We note that for each $(t, x)$ the set

\[\{(v, \ell) \in \mathbb{R}^n \times \mathbb{R} : v = (v_1, v_2, \ldots, v_N, v_{N+1}), v_i = -\alpha_i(x_i - \hat{x}(t)) - \beta u_i, v_{N+1} = u_1 + u_2 + \ldots + u_i + \ell \geq \pi(t)(u_1 + u_2 + \ldots + u_N), u_i \in [0, 1], t = 1, 2, \ldots, N\}\]

is convex. This can be seen by noting that this set is simply a Cartesian product of convex sets, and is therefore convex. Combining this with assumptions A1, the problem data satisfy the conditions to guarantee the existence of an optimal solution [3, Thm. 23.11].

That the PMP is satisfied in normal form, can be checked by verifying that certain inward–pointing conditions are satisfied along the trajectory when the state constraint is active (see [7], [8]). In this case, the inequalities (9) directly imply the inward-pointing constraints qualifications that guarantee normality. Therefore, we can apply the strengthened version of maximum principle in Thm. 3.2 [8] and obtain the following conditions involving a scalar $\pi^*$ that can be interpreted as an intermediate price.

\[\text{Proposition 1: If } (x^*, u^*) \text{ is a local minimizer for problem (P), then there exist a scalar } \pi^*, \text{ absolutely continuous functions } p_i, q_i : [0, T] \to \mathbb{R}^n, \text{ and positive Radon measures } \mu_i, \ell_i \text{ on } [0, T], \text{ for } i \in [N], \text{ satisfying}\]

\[\begin{align*}
\dot{p}_i(t) &= \alpha_i q_i(t), \quad \text{a.e. } t \in [0, T], \\
q_i(t) &= p_i(t) + \mu_i\{[0, t]\} - \ell_i\{[0, t]\}, \quad t \in [0, T], \\
q_i(T) &= p_i(T) + \mu_i\{[0, T]\} - \ell_i\{[0, T]\} = 0,
\end{align*}\]

\[\supp\{\ell_i\} \subset \{t : x_i(t) = L_i\},\]

\[\supp\{\mu_i\} \subset \{t : x_i(t) = U_i\}.\]

and for a.e. $t \in [0, T]$, the control $u^*$ maximizes

\[u \to \sum_{i=1}^N (\pi^* - \pi(t) - \beta q_i) u_i.\]

Moreover, defining $H[t]$ to be the Hamiltonian evaluated at time $t$,

\[H[t] = \sum_{i=1}^N \alpha_i q_i(x_i^*(t) - \hat{x}(t))\]

\[+ \sum_{i=1}^N (\pi^* - \pi(t) - \beta q_i) u_i^*,\]

we have

\[H[t] = H[0] + \int_0^t H_1[s] ds.\]

We now deduce a few lemmas, that combined yield the result in the theorem.

\[\text{Lemma 1: Consider the control values}\]

\[u_i = \frac{\alpha_i}{\beta_i}(\hat{x} - U_i), \text{ and } u_i = \frac{\alpha_i}{\beta_i}(\hat{x} - L_i).\]

The optimal control satisfies

\[u_i^*(t) = \begin{cases} 
0 & \text{if } \pi^* - \pi(t) - \beta q_i > 0, \\
1 & \text{if } \pi^* - \pi(t) - \beta q_i < 0, \\
\bar{u}_i & \text{if } \pi^* - \pi(t) - \beta q_i = 0, x_i(t) = L_i, \\
u_i & \text{if } \pi^* - \pi(t) - \beta q_i = 0, x_i(t) = U_i.
\end{cases}\]

It remains to analyze whether with $\pi^* - \pi(t) - \beta q_i = 0$, other intermediate values of control could be achieved. The following lemma establishes that no intermediate control values are attained when the state is strictly inside the boundaries.

\[\text{Lemma 2: If Assumption 2 holds, then for any time interval } I \subset [0, T] \text{ in which } x_i(t) \in [L_i, U_i), \text{ the control}\]

\[\text{We use the symbol } \supp \text{ to denote support of a function.}\]
is a piecewise constant function taking values from the set \{0, 1\}.

At this point, it remains to prove the last assertion of the theorem, concerning the synchronization of the controls when H1 holds. To this end, we proceed as follows.

Lemma 3: Assume that H1 holds. Consider two trajectories $x_i$ and $x_j$ ending in the interior of the admissible state set, i.e., $x_i(T), x_j(T) \in (L_i, U_i)$, with control $u_i = u_j = \nu_{\text{end}}$ and $u_{\text{end}}$ being either the value 0 or 1. Let $\tilde{t}_i$ and $\tilde{t}_j$ be the initial instants of the maximum time interval ending in $T$ with control $u_{\text{end}}$, that is

\[
\tilde{t}_i := \inf\{t \in [0, T] : u_i(s) = u_{\text{end}}, s \in [t, T]\}
\]
\[
\tilde{t}_j := \inf\{t \in [0, T] : u_j(s) = u_{\text{end}}, s \in [t, T]\}.
\]

We have that $\tilde{t}_i = \tilde{t}_j$.

The following lemma, whose proof is standard using dynamic programming arguments, enables us to generalize the last assertions.

Lemma 4: Consider the optimal control problem $(P^\pi)$ defined in (3)-(8) with solution $(x^\pi, u^\pi)$. For some time $T^\pi$ consider the following optimal control problem $(P_{\text{mon}})$ having solution $(x^{\pi_{\text{mon}}}, u^{\pi_{\text{mon}}})$:

Minimize

\[
J(x, u) = \int_0^T \pi(t)(u_1(t) + u_2(t) + \ldots + u_N(t)) \, dt
\]

subject to

\[
x_i(T^\pi) = x_i^\pi(T^\pi), \quad \text{all } i
\]

We have that $x_i^{\pi_{\text{mon}}}(t) = x_i^\pi(t)$ for all $t \in [0, T^\pi]$, all $i$.

Combining Lemma 3 and Lemma 4 by placing $T^\pi$ at any instant of time for which the trajectories are in the interior of the admissible state constraint set, we can establish that all transitions of the control function to 0 or to 1 are synchronized.

IV. CHARACTERIZATION OF THE SOLUTION: THE MONOTONE PRICE CASE

Consider first the case when the function $\pi$ is monotonically increasing.

Proposition 2: Assume that the function $\pi$ is monotonically increasing and $\tilde{x}$ is constant. Then there exist time instants $t_i^\pi$, for $i = 1, 2, \ldots, N$, such that the optimal control is

\[
u_i^\pi(t) = \begin{cases} 1 & \text{if } t < t_i^\pi, x_i(t) \in (L_i, U_i) \\ u_i & \text{if } t = t_i^\pi, x_i(t) = L_i \\ 0 & \text{if } t > t_i^\pi, x_i(t) \in (L_i, U_i) \\ \bar{u}_i & \text{if } t < t_i^\pi, x_i(t) = U_i. \end{cases}
\]

Moreover, if H1 holds then for all $i$ we have $t_i^\pi = t^\pi$.

In the case in which the all dynamics are equal, i.e., H1 holds, we can find the solution in explicit form.

Theorem 2: Assume H1 holds. Then, the function $\pi$ is monotonically increasing and $\tilde{x}$ is constant.

Then the times $t_i^{L, \text{in}}$ (entry time on $L_i$) are given by

\[
t_i^{L, \text{in}} = \frac{1}{\alpha_i} \ln \frac{x_{i0} + \beta_i/\alpha_i - \tilde{x}}{L_i + \beta_i/\alpha_i - \tilde{x}},
\]

and the time $t^0$ (time to go from $L_i$ to $U_i$ with zero control) is

\[
t_i^0 = \frac{1}{\alpha_i} \ln \frac{\tilde{x} - L_i}{\tilde{x} - U_i}
\]

the time $t^*$ solves

\[
\sum_{i=1,2,...,N} \min\{t_i^{L, \text{in}}, t^*\} + [t^* - t_i^{L, \text{in}}]^+ u_i + [T - t^* - t^0]^+ \bar{u}_i = E
\]

(in the case where $t_i^{L, \text{in}} \leq t^* < t_i^{U, \text{in}} \leq T$ for all $i$, $t^*$ is given by the simpler expression)

\[
t_{\text{out}} = E - (1 - \bar{u}) \sum_{i=1,2,...,N} t_i^{L, \text{in}} - N\bar{u}(T - t^0).
\]

and

\[
t_i^{U, \text{in}} = t_i^0 + t^*.
\]

The controls are

\[
u_i^\pi(t) = \begin{cases} 1 & \text{if } t \in [0, \min\{t_i^{L, \text{in}}, t^*\}) \\ u_i & \text{if } t \in [\min\{t_i^{L, \text{in}}, t^*\}, t^*) \\ 0 & \text{if } t \in [t^*, \min\{t_i^{U, \text{in}}, T\}) \\ \bar{u}_i & \text{if } t \in [\min\{t_i^{U, \text{in}}, T\}, T] \end{cases}
\]

the trajectories are

\[
x_i^\pi(t) = \begin{cases} e^{-\alpha_i t} \frac{x_{i0}}{1 - e^{-\alpha_i t}} & \text{if } t \in [0, \min\{t_i^{L, \text{in}}, t^*\}) \\ L_i & \text{if } t \in [\min\{t_i^{L, \text{in}}, t^*\}, t^*) \\ e^{-\alpha_i(t-t^*)} L_i & \text{if } t \in [t^*, \min\{t_i^{U, \text{in}}, T\}) \\ U_i & \text{if } t \in [\min\{t_i^{U, \text{in}}, T\}, T] \end{cases}
\]

the multipliers $\pi^*$ and $q$ satisfy

\[
\dot{q}_i(t) = \alpha_i q_i(t) \quad t \in [0, \min\{t_i^{L, \text{in}}, t^*\})
\]
\[
q_i(t) = \frac{1}{\beta_i} (\pi^* - \pi(t_i^{L, \text{in}}))
\]
\[
q_i(t) = \frac{1}{\beta_i} (\pi^* - \pi(t_i^{L, \text{in}})) \quad t \in [\min\{t_i^{L, \text{in}}, t^*\}, t^*]
\]
\[
\dot{q}_i(t) = \alpha_i q_i(t) \quad t \in [t^*, \min\{t_i^{U, \text{in}}, T\])
\]
\[
q_i(t) = \frac{1}{\beta_i} (\pi^* - \pi(t_i^{U, \text{in}})) \quad t \in [\min\{t_i^{U, \text{in}}, T\}, T]
\]

\[
q_i(T) = 0
\]

and

\[
\pi^* > \pi(t) + \beta_i q_i(t) \quad t \in [0, \min\{t_i^{L, \text{in}}, t^*\})
\]
\[
\pi^* = \pi(t) + \beta_i q_i(t) \quad t \in [\min\{t_i^{L, \text{in}}, t^*\}, t^*]
\]
\[
\pi^* < \pi(t) + \beta_i q_i(t) \quad t \in [t^*, \min\{t_i^{U, \text{in}}, T\})
\]

\[
\pi^* = \pi(t) + \beta_i q_i(t) \quad t \in [\min\{t_i^{U, \text{in}}, T\}, T]
\]

In case $x_i(T) \in (L_i, U_i)$, then $\pi^*$ is given by

\[
\pi^* = \pi(t^*),
\]
The case when the function $\pi$ is monotonically decreasing can be analyzed likewise.

**A. Example**

Consider a planning problem with the following data

$N = 2; T = 24; L_i = 1; U_i = 22; \bar{x} = 30; \alpha = 0.1; \beta = 20\alpha; P = 1; \tau = 0.5 \times N \times T; X_0 = [L_i+1, U_i-1, 0]; \pi(t) = 1 + t$.

From Thm. 2 we obtain $t_i^{L,i,n} = [1.17783, 1.845], t_0 = 4.0547, t^* = 15.7469, t_i^{L,i,n} = 19.8016$, minimum value of cost $J = 245.9712$, and $\pi^* = 8.6376$.

The results are illustrated in Fig. 1 where, in solid red, we show the optimal states $x_i^*(t)$ obtained using numerical optimal control solver ICLOCS [5], and in dotted blue the solution constructed with the values obtained from Thm 2.

**V. A SIMPLE EXPLICIT STRATEGY**

In the previous section we have determined the problem solution in explicit form in the case where the price function is monotone. In this section we address the general price function problem by considering $m$ subproblems, each in a time subinterval where the price function is monotone and allocating to each subinterval a fraction of the total energy spent.

Below, we describe an algorithm to iteratively find the value of the energy fraction in each monotone interval.

This is a simple optimization problem with just $m$ scalar optimization variables. Also, there are additional features that make this problem simple. First, it is a convex problem with respect to the $m$ scalar decision variables, as we show below. Secondly, the multipliers $\pi^*$, which can also be determined explicitly, define a descent direction and also an optimality criterion. The usefulness of the multipliers in optimum allocation of common resources has a long tradition in optimization [4].

The multiplier $\pi^*$ in each segment is precisely the multiplier associated with each isoperimetric constraint and so acts as a marginal cost of the energy fractions in each segment.

The optimal solution is obtained when the $\pi^*$’s in each segment are all equal. So, when the values of the $\pi^*$’s are different, they provide information on the descent direction of each energy fraction, contributing to rapidly determining a solution to the problem. In addition they provide a stopping criterion by detecting optimality.

**A. The parametric problem and its convexity**

Let the price curve have $M$ monotone segments $T_1, T_2, \ldots T_M$, with $T_1 + T_2 + \ldots + T_M = T$, and let $E = (E_1, E_2, \ldots E_M)$, with $E_1 + E_2 + \ldots + E_M = E$, be a possible distribution of the total energy by the segments.

Consider the set of admissible distributions $E$ to be

$E = \{(E_1, E_2, \ldots E_M) \in \mathbb{R}_+^M : 
E_1 + E_2 + \ldots + E_M = E,
E_m \in [E_m - \Delta E_m, E_m] \text{ for } m = 1, 2, \ldots M\}$,

where $\Delta E_m = T_m \Delta \bar{u}$ and $E_m = T_m N_u$.

For some $E = (E_1, E_2, \ldots E_M) \in E$ consider the parametric problem

$P(E) : V(E) = M \min_{\mathbb{N} \in \mathbb{U}}
J(u) = \int_0^T \pi(t) (u_1(t) + u_2(t) + \ldots + u_N(t)) \, dt
\text{subject to}
\dot{x}_i(t) = -\alpha_i (x_i(t) - \bar{x}(t)) - \beta_i u_i(t) \text{ a.e. } t \in [0, T],
\dot{x}_{N+1}(t) = \sum_{i=1}^N u_i(t) \text{ a.e. } t \in [0, T],
x_i(0) = x_{i0}, \quad i = 1, 2, \ldots, N,
x_{N+1}(0) = 0,
x_{N+1}(T_1) = E_1,
x_{N+1}(T_1 + T_2) = E_1 + E_2,
\ldots
x_{N+1}(T_1 + T_2 + \ldots + T_M) = E_1 + E_2 + \ldots + E_M,
u_i(t) \in [0, 1] \text{ a.e. } t \in [0, T], \quad i = 1, 2, \ldots, N,
L_i \leq x_i(t) \leq U_i \text{ all } t \in [0, T], \quad i = 1, 2, \ldots, N.

We have the following result

**Proposition 3:** The parametric problem $P(E)$ is a convex program.

**B. Algorithm**

1. Divide the price curve into $M$ monotone segments $T_1, T_2, \ldots T_M$.
2. Distribute the total energy consumed by each of the $M$ segments such that they total $E$

$E_1 + E_2 + \ldots E_M = E$
and $E_m \in [\hat{E}_m, E_m]$ for $m = 1, 2, \ldots, M$.

3) Determine the solution in each segment, as well as $\pi^*$, using the explicit formulas in Thm. 5.1 or 5.2.

4) Compare the multipliers $\pi^*$ in each segment.

If the multipliers $\pi^*$ in each segment are all equal then Stop.

Else reduce $E_i$ in segments with higher $\pi_i^*$, and augment $E_i$ in segments with lower $\pi_i^*$.

5) Repeat from 3.

Step 4 can be implemented using the simple rule: In iteration $k$

$$E_i^{(k)} = E_i^{(k-1)} - \gamma \frac{\pi_i^* - \hat{\pi}^*}{\hat{\pi}^*} \hat{E},$$

where $\hat{E}$ and $\hat{\pi}^*$ are average values of $E_i$ and $\pi_i^*$ respectively, and $\gamma$ is a positive parameter (we have selected $\gamma = 0.5$ in the example below). If $E_i \notin [\hat{E}_i, E_i]$, then select the nearest extremum in this interval and rescale the remaining non-saturated $E_i$’s so that the sum of all still totals $E$.

**Theorem 3:** If the multipliers $\pi^*$ in each segment are all equal in the solution resulting from the previous algorithm, then the solution is optimal.

Proof. The concatenation of the controls, trajectories and multipliers satisfy the optimality conditions of Proposition 1.

**C. Example**

We consider an example with price function $\pi(t) = 5 - \sin(2\pi t/24)$, and remaining data equal to Example 1.

The price function is decreasing in the segment $[0, 6]$, increasing in $[6, 18]$ and again decreasing in $[18, 24]$. Using the optimal control solver we obtain $\text{cost} = 112.6562; \pi^* = 5.3090$. From the algorithm above we obtain in each segment $E = [6.8054, 12.1189, 5.0757]; \pi^* = [4.3165, 4.4029, 4.0591]; Cost = 112.6750$.

The resulting optimal states $x^*_i(t)$ are illustrated in Fig. 2; we use solid red for the numerical solution obtained using the solver ICLOCS, and dotted blue for the solution constructed with the algorithm above.

![Optimal state trajectories](image)

**Fig. 2:** Optimal state trajectories for the non-monotone price case, $x^*_i(t)$, $i = 1, 2, 3$, with $x^*_i(0) = [19, 21, 0]$: Numerical solution obtained using an optimal control solver (solid red) and analytical solution using Thm. 2 (dotted blue)

**VI. CONCLUSIONS**

In this paper, we considered an optimal planning problem for demand response from the perspective of a utility or LSE, where the objective is to compute aggregate consumption for a population of thermostatic loads, conditioned on a forecasted price trajectory, that incurs the minimum cost of energy over the planning horizon. Solution of this problem can be used by the LSE for purchasing energy from the day-ahead market. A natural optimal control formulation is given that is non-convex in controls, and accounts practical switching constraints for thermostatic loads. We showed that solution of a convex relaxation can be used to recover the optimal (non-convex) solutions compliant with the switching constraints. Structural results for this relaxed problem are then exploited to further decompose this problem to subproblems over time-intervals corresponding to monotone segments of the price forecast trajectory, which are shown to be computationally malleable than the original mixed-integer optimal control problem.

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APPENDIX
PROOF OF INTERMEDIATE RESULTS

Proof of intermediate results, not present in the version submitted to L-CSS+CDC for lack of space, are place here for completeness and for reviewing purposes.

A. Proof of Proposition 1

Noticing that \( x_{N+1} \) is unconstrained and that \( q \) does not depend on the state, the application of the maximum principle yields \( q_{N+1} = p_{N+1} \). That is, \( q_{N+1} \) has a constant value. Denote such a value by \( \pi^* \). To obtain condition (16), we transform the problem into an autonomous one by considering an additional state with dynamics \( \dot{t} = 1 \) and with initial state equal to zero. From the constancy of the Hamiltonian in the autonomous case the result (16) follows. The remaining conditions follow from direct application of Thm. 3.2 in [8].

B. Proof of Proposition 2

Assume the price is monotonically increasing. Then, the optimal strategy is to consume energy as early as possible while respecting the constraints. That, combined with Theorem 1 suggests the control function described as a candidate to optimal. With the control function and the initial state defined, we can compute the trajectories and the adjoint vectors and show that the candidate solution, in fact satisfies the optimality conditions in Proposition 1.

The computations are done explicitly below, in the proof of Thm. 2, for the case H1 is satisfied.

C. Proof of Theorem 2

Assuming \( H1 \), by Thm 1 we have \( t_i^* = t^* \) for all \( i \in \{1, 2, \ldots, N\} \). Let \( t^*_{L,in} \) be the entry time on the boundary \( L_i \) (the first instant \( t \) for which \( x_i(t) = L_i \)). The first control switch might occur at \( t^*_{L,in} \) or \( t^* \), depending which occurs first. Therefore we have

\[
u^*_i(t) = \begin{cases} 1 & \text{if } t \in [0, \min\{t^*_{L,in}, t^*\}] \\ \hat{u}_i & \text{if } t \in [\min\{t^*_{L,in}, t^*\}, T]\end{cases} \]

Similarly, we define \( t^*_{U,in} \) to be the entry time on the boundary \( U_i \) (the first instant \( t \) for which \( x_i(t) = U_i \)), and

\[
u^*_i(t) = \begin{cases} 0 & \text{if } t \in [t^*, \min\{t^*_{U,in}, T\}] \\ \bar{u}_i & \text{if } t \in [\min\{t^*_{U,in}, T\}, T]\end{cases} \]

This defines the trajectories

\[
x_i^*(t) = \begin{cases} e^{-\alpha_i t} x_{i0} & \text{if } t \in [0, \min\{t^*_{L,in}, t^*\}] \\ (\bar{x} - \beta_i/\alpha_i)(1 - e^{-\alpha_i t}) & \text{if } t \in [\min\{t^*_{L,in}, t^*\}, t^*] \\ e^{-\alpha_i(t-t^*)} L_i & \text{if } t \in [t^*, \min\{t^*_{U,in}, T\}] \\ U_i & \text{if } t \in [\min\{t^*_{U,in}, T\}, T]\end{cases} \]

and, using also the optimality conditions, the adjoint multipliers satisfy

\[
\dot{q}_i(t) = \alpha_q_i(t) \quad t \in [0, \min\{t^*_{L,in}, t^*\})
\]

\[
q_i(t) = \frac{1}{\beta_i}(\pi^* - \pi(t)) \quad t \in [\min\{t^*_{L,in}, t^*\}, t^*]
\]

\[
\dot{q}_i(t) = \alpha_q(t) \quad t \in [t^*, \min\{t^*_{U,in}, T\}]
\]

\[
q_i(t) = \frac{1}{\beta_i}(\pi^* - \pi(t)) \quad t \in [\min\{t^*_{U,in}, T\}, T)
\]

\[
q_i(T) = 0
\]

where \( \pi^* \) also satisfies

\[
\pi^* > \pi(t) + \beta q_i(t) \quad t \in [0, \min\{t^*_{L,in}, t^*\})
\]

\[
\pi^* = \pi(t) + \beta q_i(t) \quad t \in [\min\{t^*_{L,in}, t^*\}, t^*]
\]

\[
\pi^* < \pi(t) + \beta q_i(t) \quad t \in [t^*, \min\{t^*_{U,in}, T\}]
\]

\[
\pi^* = \pi(t) + \beta q_i(t) \quad t \in [\min\{t^*_{U,in}, T\}, T]
\]

In case \( x_i(t) \in (L_i, U_i) \), then \( \pi^* \) is given by

\[
\pi^* = \pi(t^*),
\]

else

\[
\pi^* = \frac{\pi(t^* + t^0) - \pi(t^*) e^{\alpha t^0}}{1 - e^{\alpha t^0}}.
\]

The knowledge of the trajectories enables us to compute the times \( t^*_{L,in} \) and \( t^*_{U,in} \) explicitly. The time to go from \( x_{i0} \) to \( L_i \) with control \( u^*_i(t) = 1 \) is

\[
t^*_{L,in} = \frac{1}{\alpha_i} \ln \frac{x_{i0} + \beta_i/\alpha_i - \bar{x}}{L_i + \beta_i/\alpha_i - \bar{x}},
\]

the time \( t^0 \) (time to go from \( L_i \) to \( U_i \) with zero control) is

\[
t^0 = \frac{1}{\alpha_i} \ln \frac{\bar{x} - L_i}{\bar{x} - U_i},
\]

the isoperimetric constraints impose that the time \( t^* \) solves

\[
\sum_{i=1,2,\ldots,N} \min\{t^*_{L,in}, t^*\} + [t^* - t^*_{L,in}]^+ u_i
\]

\[
+ [T - t^* - t^0]^+ \bar{u}_i = E
\]

(35)

(36)

(in the case where \( t^*_{L,in} \leq t^* < t^*_{U,in} \leq T \) for all \( i \), \( t^* \) is given by the simpler expression

\[
t^*_{out} = E - (1 - u) \sum_{i=1,2,\ldots,N} t^*_{L,in} - N(\bar{u})(T - t^0)
\]

and

\[
t^*_{U,in} = t^0 + t^*.
\]

(37)
D. Proof of Proposition 3

We start by noting that each state component $x_i$ can be written as an affine functional of the function $u$, since

$$x_i(t) = e^{-\alpha t}x_{i0} + \int_0^t e^{-\alpha(t-s)}(\alpha_i\dot{x}(s) - \beta_iu_i(s))ds.$$ 

Therefore, all constraints of problem $\mathcal{P}(\mathcal{E})$, can be written in the form

$$g_i(u, \mathcal{E}) = 0,$$

and $h_i(u, \mathcal{E}) \leq 0,$

with $g_i$ and $h_i$ affine functions of $u$ and $\mathcal{E}$, defining a jointly convex domain in $(\mathcal{E}, u)$. As a consequence, the set-valued map $R: \mathbb{R}^M \rightarrow \mathcal{U}$

$$R(\mathcal{E}) := \{ u \in \mathcal{U} : g_i(\mathcal{E}, u) \leq 0, h_i(\mathcal{E}, u) \leq 0 \}$$

is convex on $\mathcal{E}$. That is, the set

$$\text{Graph}(R) := \{ (\mathcal{E}, u) : \mathcal{E} \in \mathcal{E}, u \in R(\mathcal{E}) \}$$

is convex, or equivalently [6], for all $\lambda \in [0, 1]$, all $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{E}$

$$\lambda R(\mathcal{E}_1) + (1 - \lambda)R(\mathcal{E}_2) \subseteq R(\lambda \mathcal{E}_1 + (1 - \lambda)\mathcal{E}_2).$$

We note also that the set $\mathcal{E}$ is convex. Therefore, using the arguments in Prop. 2.1, in [6], we can show that for all $\lambda \in [0, 1]$, all $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{E}$

$$V(\lambda \mathcal{E}_1 + (1 - \lambda)\mathcal{E}_2) = \min_{u \in R(\lambda \mathcal{E}_1 + (1 - \lambda)\mathcal{E}_2)} J(u)$$

$$\leq \min_{u_1 \in R(\mathcal{E}_1), u_2 \in R(\mathcal{E}_2)} J(\lambda u_1 + (1 - \lambda)u_2)$$

$$= \lambda \min_{u_1 \in R(\mathcal{E}_1)} J(u_1) + (1 - \lambda)\min_{u_2 \in R(\mathcal{E}_2)} J(u_2)$$

$$= \lambda V(\mathcal{E}_1) + (1 - \lambda)V(\mathcal{E}_2).$$

That is, $\mathcal{E} \mapsto V(\mathcal{E})$ is convex on $\mathcal{E}$.

E. Proof of Lemma 1

The maximization of the Hamiltonian condition directly yields the case when $u^*_i(t) = 0$ and when $u^*_i(t) = 1$. Other intermediate values can only occur if $\pi^* - \pi(t) - \beta_iq_i = 0$.

When the trajectory is on the boundary $L_i$ for some interval of time, we must have $\dot{x}_i(t) = 0$. The dynamic equation equal to zero immediately yields $u^*_i(t) = u^*_i = \alpha_i(\dot{x}_i - L_i)$.

The same argument can be used on the boundary $U_i$ to obtain $u_i$.

F. Proof of Lemma 2

For the function $u_i$ to assume some intermediate values not in the set \{0, 1\}, by the maximization of the Hamiltonian we would have to have in that time interval

$$\pi^* - \pi(t) - \beta_iq_i = 0$$

and

$$\frac{d}{dt}(\pi^* - \pi(t) - \beta_iq_i) = 0$$

Developing this last equation and substituting $q_i$ from the previous equation, we obtain

$$\frac{d}{dt}\pi(t) = \alpha_i\pi(t) - \alpha_i\pi^*.$$ 

However, the solution of this equation is precisely of the structure that Assumption 2 rules out.

G. Proof of Lemma 3

Assume in contradiction to what we wish to prove that $\tilde{t}_i < \tilde{t}_j$, and, without loss of generality, that $u_{end} = 1$. By (14)-(16), we have

$$q_i(T) = q_j(T) = 0,$$

$$\dot{q}_i(t) = \alpha q_i(t), \quad t \in [\tilde{t}_j, T],$$

$$\dot{q}_j(t) = \alpha q_j(t), \quad t \in [\tilde{t}_j, T].$$

Therefore

$$q_i(t) = q_j(t), \quad t \in [\tilde{t}_j, T].$$

By the way $\tilde{t}_j$ is defined, we have

$$\pi^* - \pi(t) - \beta_iq_i(t) = 0, \quad t = \tilde{t}_j, \tag{39}$$

but also

$$\pi^* - \pi(t) - \beta_iq_i(t) > 0, \quad t \in [\tilde{t}_j, T] \tag{40}$$

Since $\tilde{t}_i < \tilde{t}_j$, the last two equations are a contradiction. Repeating the same argument when $u_{end} = 0$, we prove the lemma.