Theorem 1

Let \(\psi\) be the entangled, pure state \(\sum_{j,k=1}^{n} v_j v_k^* |\psi_1^j \cdots \psi_p^j \rangle \langle \psi_1^k \cdots \psi_p^k|\). The closest, pure, product states to \(E_\psi\) are a distance \(\sqrt{2(1-|v_{j_0}|^2)}\) away from \(E_\psi\), where \(|v_{j_0}| = \max \{|v_1|, \ldots, |v_p|\}\). An example of such a closest, pure product state is the projection, \(S_\psi = |\psi_1^o \cdots \psi_p^o \rangle \langle \psi_1^o \cdots \psi_p^o|\).
We shall give the proof of this theorem in the next section. For now, let us describe how it is used to construct $G_\psi$, which is quite simple. Take $C_\psi$ to be the $N-1$ dimensional hyperplane which contains $S_\psi$ and is perpendicular to the line, $L_\psi$, connecting $E_\psi$ with $\frac{1}{\sqrt{n}}I$, the totally mixed state. Similarly, let $F_\psi$ be the parallel hyperplane, which contains any projection which is furthest away from $E_\psi$. (These are precisely the projections which commute with $E_\psi$, a separable example of which is $R = |\psi_1^+\cdots\psi_{p-1}^+\psi_p^-\psi_{p+1}^-\cdots\psi_{n}^-|^2$. Then we have the following theorem.

**Theorem 2** All separable states either lie on one of the hyperplanes $C_\psi$ or $F_\psi$ or lie between them. Thus every state outside the sandwich formed by $C_\psi$ and $F_\psi$ is entangled. This region, $G_\psi$, outside the $C_\psi$, $F_\psi$ sandwich is an open, connected neighborhood of $E_\psi = \sum_{j,k=1}^{n} v_j^* v_k |\psi_1^j\cdots\psi_p^k|^2$. A state $Q$ is in $G_\psi$ if and only if $\langle Q, E_\psi \rangle = \text{Trace}(Q E_\psi) > |v_{j_0}|^2$ where, as before, $|v_{j_0}| = \max \{|v_1|, \ldots, |v_p|\}$.

Again we shall postpone the proof until the next section and instead shall now make a few remarks and give one last theorem.

**Remark 3** $G_\psi$ will be largest when $|v_{j_0}|^2$ is smallest. Since $\sum_{i=1}^{n} |v_i|^2 = 1$, this occurs when $|v_1|^2 = \cdots = |v_p|^2 = \frac{1}{n^2}$. Thus it is these states, which are the maximally entangled ones when $n_1 = \cdots = n_p$, that can withstand the greatest amount of decoherence and still be in $G_\psi$ and so remain entangled.

**Remark 4** Since, the plane $C_\psi$ contains the separable state $S_\psi$, there is no larger neighborhood of $E_\psi$, consisting solely of entangled states, given by an inequality, $\langle Q, E_\psi \rangle > K$, than $G_\psi$. In this sense $G_\psi$ is the largest neighborhood of $E_\psi$ consisting solely of entangled states. It may, however, not contain the largest ball of entangled states centered at $E_\psi$.

**Remark 5** It is well known [16], [17], [18], [19] that if $E_\psi$ is a maximally entangled state on $\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n$, then the separable state on the line $L_\psi$, which connects $E_\psi$ with $\frac{1}{\sqrt{n}}I$, that is closest to $E_\psi$ is $W(s) = (1 - s)\frac{1}{\sqrt{n}}I + sE_\psi$, where $s = (1 + n^{p-1})^{-1}$. When $p = 2$, it is easy to compute that this state lies in the hyperplane $C_\psi$. This has two important consequences: a) all separable states, not just those on $L_\psi$, the state $W(s)$ is the closest to $E_\psi$, and b) the neighborhood, $G_\psi$, contains the largest open ball of entangled states centered at $E_\psi$. Thus in this case $G_\psi$ is the largest physically usable neighborhood of $E_\psi$ consisting solely of entangled states. When $p > 2$, the state $W(s)$ lies inside the sandwich formed by $C_\psi$ and $F_\psi$. This means $G_\psi$ might not, in this case, contain the largest ball of entangled states centered at $E_\psi$. It also means that, in this case, $W(s)$ is not the closest separable state to $E_\psi$. Indeed, simple geometry shows the line which contains $E_\psi$ and intersects the line connecting $W(s)$ with $S_\psi$ perpendicularly, intersects that line at a separable state which is closer to $E_\psi$ than $W(s)$.

**Remark 6** From the last example given in the remark just made, it should be clear that we do not claim all states between $C_\psi$ and $F_\psi$ are separable. Many are entangled. In fact, numerical simulation for low dimensional bipartite cases indicates that a large percentage of the states inside the sandwich are entangled. However, there are no separable states outside the sandwich.

To finish this introduction, we shall state our last theorem. Basically it says that for a system modelled on $\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_p}$ with $n_1 \leq \cdots \leq n_p$, the thickness of the thinnest sandwich which contains all separable states goes to $0$ as $n_{p-1}$ or $p$ increases to infinity. This means that for systems with a large number of particles, or with at least two particles modelled on large dimensional Hilbert spaces, all separable states cluster near a hyperplane which contains the totally mixed state. Before stating the theorem we have to make the following definition.

**Definition 7** For the set of integers $\{n_1, \ldots, n_p\}$, and any partition $\pi$ of the set into two subsets, $\{\{n_{1_1}, \ldots, n_{s_1}\}, \{n_{s_1+1}, \ldots, n_{s_2}\}\}$, let $f(\pi) = \min(s_1, \ldots, s_{s_2}, n_{s_1+1} \cdots n_{s_2})$. Then $\kappa(n_1, \ldots, n_p)$ is the maximum over all partitions of $\{n_1, \ldots, n_p\}$ into two subsets of $1/f(\pi)$.

For example, if the system consists of $p$-qubits, then $\kappa = 2^{-m}$ if $p = 2^m$ and $\kappa = 2^{-(m-1)}$ if $p = 2^{m-1}$. On the other hand if, for instance, the system is modelled on $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4 \otimes \mathbb{C}^{30}$, then $\kappa = \frac{1}{27}$. In all cases, $\kappa \leq (n_1 n_3 \cdots n_{p-1})^{-1}$ if $p$ is odd and $\kappa \leq \max((n_{p-1} n_3 \cdots n_{p-3})^{-1}, (n_2 n_4 \cdots n_{p-2})^{-1})$ if $p$ is even.

**Theorem 8** Consider a quantum system modelled on $\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_p}$. There exist parallel hyperplanes which are a distance $\kappa \sqrt{\frac{N}{N-1}}$ apart and which have the property that all separable states either lie on one of the planes or lie between them. In particular for every separable state $T$ the largest ball of separable states centered at $T$ must have a radius no bigger than $\kappa \sqrt{\frac{N}{N-1}}$. 


II. PROOFS OF THEOREMS

In this section we prove our theorems, starting with the first. For \( \psi = \sum_{j=1}^{n_1} v_j | \psi_j \cdot \psi_j \rangle \), the associated projection is

\[
E_\psi = \sum_{j,k=1}^{n_1} v_j \| \psi_k \rangle \langle \psi_k | \psi_j \rangle \|_2 \sum_{j,k=1}^{n_1} \| \psi_k \rangle \langle \psi_k | \psi_j \rangle \|_2 \| \psi_j \rangle \langle \psi_j | \psi_j \rangle \|_2.
\]

For \( \mu = 1, \ldots, p \), take \( A_\mu \) to be the projection

\[
A = \sum_{j,k=1}^{n_1} a_{jk} \gamma \mu | \psi_j \rangle \langle \psi_k | \psi_k \rangle \|_2 | \psi_j \rangle \langle \psi_j | \psi_j \rangle \|_2.
\]

Then we want to find the smallest distance from such an \( A \) to \( E_\psi \). To do so, first note the square of the distance from \( A \) to \( E_\psi \) is

\[
\| E_\psi - A \|^2 = \langle E_\psi - A, E_\psi - A \rangle = \langle E_\psi, E_\psi \rangle - 2 \Re \langle E_\psi, A \rangle + \langle A, A \rangle.
\]

Since \( E_\psi \) and \( A \) are positive semi-definite Hermitian operators, their inner product is real and equals \( Tr(E_\psi A) \). Hence \( \| E_\psi - A \|^2 = 2(1 - Tr(E_\psi A)) \). This will be minimum when \( Tr(E_\psi A) = \sum_{j=1}^{n_1} v_j \| \psi_j \rangle \langle \psi_j | \psi_j \rangle \|_2 \| \psi_j \rangle \langle \psi_j | \psi_j \rangle \|_2 \| \psi_j \rangle \langle \psi_j | \psi_j \rangle \|_2 \), which is maximum.

Setting \( \Phi = \sum_{j=1}^{n_1} v_j | \psi_j \rangle \langle \psi_j | \psi_j \rangle \|_2 \), we see that \( Tr(E_\psi A) = \Phi \bar{\Phi} = |\Phi|^2 \). In turn \( |\Phi|^2 = \sum_{j=1}^{n_1} v_j | \psi_j \rangle \langle \psi_j | \psi_j \rangle \|_2 \|^2 \leq (\sum_{j=1}^{n_1} v_j | \psi_j \rangle \langle \psi_j | \psi_j \rangle \|_2)^2 \). This last expression is equivalent to \( \langle \rho, V \beta \rangle ^2 \), where \( V \) is the \( n_1 \times n_1 \) diagonal matrix with the \( | \psi_j \rangle \) as the diagonal entries and \( \rho \) and \( \beta \) are the \( n_1 \) dimensional vectors with the components \( \rho_j = v_j \) and \( \beta_j = v_j \). Using the Cauchy-Schwarz inequality and the definition of the operator norm of a matrix, we obtain the inequality \( |\langle \rho, V \beta \rangle|^2 \leq |\| \rho \||^2 |V|^2 \|\beta\| ^2 \). Since \( V \) is a diagonal matrix, \( |\|V\||^2 = \max | \psi_j \rangle \langle \psi_j | \psi_j \rangle \|_2 \). Furthermore, by assumption \( \sum_{j=1}^{n_1} v_j | \psi_j \rangle \langle \psi_j | \psi_j \rangle \|_2 = 1 \) and so \( |\|\rho\||^2 = 1 \) and \( |\|\beta\||^2 \leq 1 \). Thus \( Tr(E_\psi A) = |\langle \rho, V \beta \rangle|^2 \leq |\|v_{j_0}\||^2 \). Noting that if \( S_\psi = | \psi_1 \cdot \psi_1 \rangle \langle \psi_1 \cdot \psi_1 | \rangle \), then \( Tr(E_\psi S_\psi) = |\|v_{j_0}\||^2 \), we obtain the proof of the first theorem.

The proof of the second, basically, uses simple vector operations and facts from trigonometry. For two states \( K \) and \( Q \), take \( V(K, Q) \) to be the vector with tail at \( K \) and head at \( Q \). As above, take \( S_\psi \) to be any of the closest pure product states to \( E_\psi \) and consider the triangle whose sides are \( V(\frac{1}{N} I, E_\psi), V(\frac{1}{N} I, S_\psi), \) and \( V(S_\psi, E_\psi) \). Since \( E_\psi \) and \( S_\psi \) are rank 1 projections, the length of the first side is \( \sqrt{\frac{N}{N-1}} \). Moreover, we have just proven the length of the third side is \( \sqrt{2(1 - |\|v_{j_0}\||^2)} \). Since this is true regardless of \( S_\psi \), the projection of \( V(\frac{1}{N} I, S_\psi) \) onto \( V(\frac{1}{N} I, E_\psi) \) will be the same for all \( S_\psi \). This means that all \( S_\psi \) lie in the hyperplane, \( C_\psi \), which is perpendicular to \( V(\frac{1}{N} I, E_\psi) \). This hyperplane divides the set of states into two regions: i) one which contains the plane and all states on the \( \frac{1}{N} I \) side of the plane and ii) \( G_\psi \), which is the open, connected set which includes \( E_\psi \) and all states on that side of \( C_\psi \). A state, \( Q \), is in \( G_\psi \) if and only if the projection of \( V(\frac{1}{N} I, Q) \) onto \( V(\frac{1}{N} I, E_\psi) \), is longer than the projection of \( V(\frac{1}{N} I, S_\psi) \) onto \( V(\frac{1}{N} I, E_\psi) \), i.e. \( \langle Q - \frac{1}{N} I, E_\psi - \frac{1}{N} I \rangle > \langle S_\psi - \frac{1}{N} I, E_\psi - \frac{1}{N} I \rangle = |\|v_{j_0}\||^2 - \frac{1}{N} \). Expanding the left hand side of this inequality and using the fact that if \( P \) is rank 1, then \( \langle P, \frac{1}{N} I \rangle = Tr(\frac{1}{N} IP) = \frac{1}{N} \), we get \( \langle Q, E_\psi - \frac{1}{N} I \rangle > |\|v_{j_0}\||^2 - \frac{1}{N} \). This proves the inequality in the theorem and it also shows why there are no separable states in \( G_\psi \). Indeed due to the convexity of the set of separable states, if there were a separable state in \( G_\psi \), then there would have to be a pure, separable state in \( G_\psi \). But this last inequality shows that such a state would be closer to \( E_\psi \) than is possible by theorem (1).

The same reasoning can be applied to \( F_\psi \). By (2) we see that the projections (separable or entangled) which are furthest from \( E_\psi \) are those whose inner product with \( E_\psi \) is 0. These are precisely the ones which commute with \( E_\psi \). Since they all must lie on \( F_\psi \), it follows that there can be no states (separable or entangled) on the side of \( F_\psi \) that does not contain \( E_\psi \). Hence all separable states either lie on \( C_\psi \) or \( F_\psi \) or lie between them.

As for the last theorem, first note that the projection of \( V(\frac{1}{N} I, S_\psi) \) onto \( V(\frac{1}{N} I, E_\psi) \) has length \( |\|v_{j_0}\||^2 - \frac{1}{N} \) \( \sqrt{\frac{N}{N-1}} \). Thus the distance between \( C_\psi \) and \( F_\psi \) is \( |\|v_{j_0}\||^2 \). Since \( \sum_{j=1}^{n_1} | v_j |^2 = 1 \), the minimum \( |v_{j_0}| \) is \( \frac{1}{n_1} \). To obtain the last theorem we express \( C^N \) as the tensor product of \( C^{N_1} \) and \( C^{N_2} \) where \( N_1 = n_1 \cdot n_1 \cdots n_1 \) and \( N_2 = n_{\pi+1} \cdot n_{\pi+1} \cdots n_{\pi+1} \), with \( \pi \) being the partition which makes \( \frac{1}{|\|v_{j_0}\||^2} \) maximum. Hence \( C^{N_1} = C^{n_1} \otimes \cdots \otimes C^{n_1} \) and \( C^{N_2} = C^{n_{\pi+1}} \otimes \cdots \otimes C^{n_{\pi+1}} \). Any state which is separable in \( C^{N_1} \otimes \cdots \otimes C^{N_2} \) is also separable in \( C^{N_1} \otimes C^{N_2} \). However, these latter states are all sandwiched between hyperplanes \( C_\psi \) and \( F_\psi \) which are associated with an entangled state for which \( |v_j|^2 = \frac{1}{N_1} \). It follows from what we said a moment ago that for such a state the distance between \( C_\psi \) and \( F_\psi \) is \( \frac{1}{\sqrt{\sqrt{\frac{N}{N-1}}}} = \kappa \frac{\sqrt{N}}{N} \).

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