Stability of dark solitons in three dimensional dipolar Bose-Einstein condensates

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The dynamical stability of dark solitons in dipolar Bose-Einstein condensates is studied. For standard short-range interacting condensates dark solitons are unstable against transverse excitations in two and three dimensions. On the contrary, due to its non local character, the dipolar interaction allows for stable 3D stationary dark solitons, opening a qualitatively novel scenario in nonlinear atom optics. We discuss in detail the conditions to achieve this stability, which demand the use of an additional optical lattice, and the stability regimes.

The physics of Bose-Einstein condensates (BECs) is, due to the interatomic interactions, inherently nonlinear, closely resembling the physics of other nonlinear systems, and in particular non linear optics. Non linear atom optics [1] has indeed attracted a major attention in the last years, including phenomena like four-wave mixing [2] and condensate collapse [3]. One of the major consequences of nonlinearity is the possibility of achieving solitons, i.e. non-dispersive self-bound solutions, in quasi-one dimensional BECs. Bright solitons have been reported in BECs with negative s-wave scattering length $a < 0$ (the equivalent of self-focusing nonlinearity) [4]. Dark solitons (DSs) have been realized as well in BECs with $a > 0$ (self-defocusing nonlinearity) [5]. In addition, optical lattices have allowed for the observation of gap solitons [6].

The stability of solitons depends crucially on the quasi-one dimensionality of the systems, which for the case of BEC solitons demands a sufficiently strong transversal confinement of the condensates [7]. For the particular case of dark solitons, if the transversal size of the system becomes comparable to the width of the dark soliton (typically provided by the healing length of the system, as discussed below), then the dark soliton plane becomes dynamically unstable. This dynamical instability (so-called snake instability) has been previously studied in the context of non linear optics [8]. In the context of BEC, it has been shown that this instability leads to a strong bending of the nodal plane, which breaks down into vortex rings and sound excitations [3], as experimentally observed in Ref. [10].

Nonlinear phenomena constitute an excellent example of the crucial role played by interactions in quantum gases. Until recently, typical experiments involved particles interacting dominantly via short-range isotropic potentials, which, due to the very low energies involved, are fully determined by the corresponding s-wave scattering length. However, recent experiments on cold molecules [11], Rydberg atoms [12], and atoms with large magnetic moment [13], open a fascinating new research area, namely that of dipolar gases, for which the dipole-dipole interaction (DDI) plays a significant or even dominant role. The DDI is long-range and anisotropic (partially attractive), and leads to fundamentally new physics in ultra cold gases [14]. Time-of-flight experiments in Chromium have allowed for the first observation of DDI effects in cold gases [15], which have been remarkably enhanced recently by means of Feshbach resonances [16].

Dipolar gases present a rich non-linear physics, since the DDI leads to non-local non-linearity, similar as that encountered in plasmas [17], nematic liquid crystals [18], thermo-optical materials [19] and photorefractive crystals [20]. Nonlocality leads to a wealth of novel phenomena in nonlinear physics, as the modification of modulation instability [21], the change of the soliton interaction [22], and the stabilization of azimuths [23]. Particularly interesting is the possibility of stabilization of localized waves in cubic nonlinear materials with a symmetric nonlocal nonlinear response [24]. Multidimensional solitons have been experimentally observed in nematic liquid crystals [25] and in photorefractive screening solitons [26].

In this letter, we show that the long-range character of the DDI may have striking consequences for the stability of dark solitons in dipolar BECs. Contrary to usual BECs, for which, as mentioned above, dark nodal planes become unstable when departing from the one-dimensional condition, the DDI may stabilize dark nodal planes even if the transversal size of the condensate becomes arbitrarily much wider than the condensate healing length. This stabilization is purely due to the long-range character of the DDI. We study in detail the conditions for this stabilization, and the stabilization regimes.

In the following, we consider a dipolar BEC of particles with mass $m$ and electric dipole $d$ (the results are equally valid for magnetic dipoles) oriented in the $z$-direction by a sufficiently large external field, and that hence interact via a dipole-dipole potential: $V_d(\vec{r}) = \frac{\mu_0 d^2}{4\pi\epsilon_0} \sum_{i<j} \frac{\vec{d}_i \cdot \vec{d}_j}{r_{ij}^3}$, where $r_{ij}$ is the distance between particles $i$ and $j$. The interaction potential $V_d(\vec{r})$ is long-range and anisotropic, and leads to fundamentally new physics in ultra cold gases. Time-of-flight experiments in Chromium have allowed for the first observation of DDI effects in cold gases, which have been remarkably enhanced recently by means of Feshbach resonances.

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\[ \alpha d^2 (1 - 3 \cos^2(\theta))/r^3 , \] where \( \theta \) is the angle formed by the vector joining the interacting particles and the dipole interaction. The coefficient \( \alpha \) can be tuned within the range \(-1/2 \leq \alpha \leq 1\) by rotating the external field that orients the dipoles much faster than any other relevant time scale in the system [29]. At sufficiently low temperatures the physics of the dipolar BEC is provided by a non-local non-linear Schrödinger equation (NLSE) of the form:

\[ i\hbar \frac{\partial}{\partial t} \Psi(\vec{r},t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{dip}}(x,y) + g |\Psi(\vec{r},t)|^2 \right] \Psi(\vec{r},t) + \int d^2 \vec{r}' V_{\text{dip}}(\vec{r} - \vec{r}') |\Psi(\vec{r}',t)|^2 \right] \Psi(\vec{r},t) , \quad \text{(1)} \]

where \( g = 4\pi \hbar^2 a/m \), with \( a \) the s-wave scattering length (we consider \( a > 0 \)) and \( m \) the particle mass. For reasons that will become clear below, the BEC is assumed to be in a 2D optical lattice, \( V_{\text{dip}}(x,y) = s E_R (\sin^2(qx) + \sin^2(qy)) \), where \( E_R = h^2 q^2 /2m \) is the recoil energy, \( q \) is the laser wave vector and \( s \) is a dimensionless parameter providing the lattice depth. In the tight-binding regime (i.e. for a sufficiently strong lattice but still maintaining coherence), we may write \( \Psi(\vec{r},t) = \sum_{i,j} f_{ij}(x,y) \psi_{i,j}(z,t) \), where \( f_{ij}(x,y) \) is the Wannier function associated to the lowest energy band and the site located at \((b_i, b_j)\), with \( b = \pi/q \). Substituting this ansatz in Eq. (1) we obtain a discrete NLSE [30]. We may then return to a continuous equation, where the lattice is taken into account in an effective mass along the lattice directions and in the renormalization of the coupling constant [31].

\[ i\hbar \frac{\partial}{\partial t} \Psi(\vec{r},t) = \left[ -\frac{\hbar^2}{2m^*} \nabla^2 - \frac{\hbar^2}{2m} \nabla_z^2 + \tilde{g} |\Psi(\vec{r},t)|^2 \right] \Psi(\vec{r},t) + \int d^2 \vec{r}' V_{\text{dip}}(\vec{r} - \vec{r}') |\Psi(\vec{r}',t)|^2 \right] \Psi(\vec{r},t) , \quad \text{(2)} \]

where \( \tilde{g} = b^2 g \int f(x,y)^3 dx dy + g_0 C \) [32], with \( g_0 = \alpha s \pi \hbar^2 /3 \), \( m^* = \hbar^2 /2D \) is the effective mass, and \( J = \int d^2 x dy f_{ij}(x,y) - (\hbar^2 /2m) \nabla_i f_{ij}(x,y) f_{ij}(x,y) \), for neighboring sites \((i,j)\) and \((i',j')\). The validity of Eq. (2) is limited to radial momenta \( k_{\rho} \ll 2\pi/b \), in which one can ignore the discreteness of lattice. In the following we use the convenient dimensionless parameter \( \beta = g_0 / \tilde{g} \), that characterizes the strength of the DDI compared to the short range interaction. The Fourier transform of the DDI: \( V_{\text{dip}}(\vec{k}) = g_0 [3 \cos^2 \theta_k - 1] /2 \), with \( \cos^2 \theta_k = k_z^2 / |\vec{k}|^2 \), is needed later for the calculations.

Due to its partially attractive character, the stability of a dipolar BEC is a matter of obvious concern [14]. A Bogoliubov analysis of an homogeneous dipolar condensate reveals that the dispersion relation for quasiparticles is of the form \( \epsilon(\vec{k}) = [E_{\text{kin}}(\vec{k}) |E_{\text{int}}(\vec{k})|^2]^{1/2} \), where \( E_{\text{kin}}(\vec{k}) = \hbar^2 k_\rho^2 /2m^* + \hbar^2 k_z^2 /2m \) is the kinetic energy, and \( E_{\text{int}}(\vec{k}) = 2(g + \tilde{V}_{\text{dip}}(\vec{k}))n_0 \) is the interaction energy, which includes both short-range and dipolar parts. Note that \( \tilde{V}_{\text{dip}}(\vec{k}) \) may be positive or negative, and hence for low momenta (phonon excitations) the dynamical instability (phonon instability) is just prevented if \(-1 < \beta < 2 \). If \( g_0 > 0 \), phonons with \( \vec{k} \) lying on the \( xy \) plane are unstable if \( \beta > 2 \), while for \( g_0 < 0 \) phonons with \( \vec{k} \) along \( z \) are unstable if \( \beta < -1 \) [33].

In this paper, we are particularly concerned about the stability of a dark soliton in a three dimensional dipolar BEC. We assume that the dark soliton lies on the \( xy \) plane, hence the solution can be written in the following form: \( \Psi(\vec{r},t) = \psi_0(z) \exp[-i\mu t / \hbar] \), where \( \mu \) is the chemical potential. Introducing this expression into Eq. (1) we obtain a one-dimensional NLSE in \( z \) of the form:

\[ \mu \psi_0(z) = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + \tilde{g} |\psi_0(z)|^2 \right] \psi_0(z) . \quad \text{(3)} \]

Note that, since \( \psi_0 \) is independent of \( x \) and \( y \), in this equation the DDI interaction just regularizes the value of the local coupling constant \( \tilde{g} = \tilde{g} + g_0 \). This equation allows for a simple solution describing a dark soliton, \( \psi_0 = \sqrt{n_0} \tanh(z/\zeta) \), where \( \zeta = \hbar / \sqrt{mg n_0} \) is the corresponding healing length and \( n_0 \) is the bulk density. Note that, interestingly, due to the modification of the local coupling constant the size \( \zeta \) of the dark soliton depends on the DDI.

We are especially interested in the dynamical stability of these nodal planes. To study this stability we perform a Bogoliubov analysis, considering a transversal perturbation of the dark soliton planes: \( \Psi(\vec{r},t) = \Psi_0(\vec{r},t) + \chi(\vec{r},t) \exp(-i\mu t / \hbar) \), where \( \chi(\vec{r},t) = u(z) \exp[i(qx - \epsilon t / \hbar)] + v(z) \exp[-i(qx - \epsilon^* t / \hbar)] \), where \( q \) is the momentum of the transverse modes with energy \( \epsilon \). Introducing this ansatz into Eq. (4) and linearizing in \( \epsilon \), one obtains the corresponding Bogoliubov-de Gennes (BdG) equations for the excitation energies \( \epsilon \) and the corresponding eigenfunctions \( f_{\pm} = u \pm v \):

\[ \epsilon f_-(z) = \left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial z^2} - \frac{m}{m^*} q^2 \right) - \mu + 3\tilde{g} |\psi_0(z)|^2 \right] f_+(z) \]

\[ -\frac{3}{2} g_0 q \psi_0(z) \int dz' \exp(-q|z-z'|) \psi_0(z') f_+(z') , \quad \text{(4)} \]

\[ \epsilon f_+(z) = \left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial z^2} - \frac{m}{m^*} q^2 \right) - \mu + \tilde{g} |\psi_0(z)|^2 \right] f_-(z) . \quad \text{(5)} \]

Note that the dipole interaction has two main effects. On one side, as mentioned above, it leads to a regularized \( g \). On the other side, it introduces a qualitatively new term in the second line of Eq. (4). Whereas the first effect just leads to a quantitative modification of the dark soliton width, the second effect is a purely dipole-induced non local effect, which, as we show below, may lead to remarkable consequences for the dark soliton stability. For every transverse momentum \( q \) we determine the lowest eigenenergy, which provides the dispersion law \( \epsilon(q) \).
When $\beta = 0$ (no DDI) and $m/m^* = 1$ (no lattice), we recover the same BdG equations discussed in the context of standard short-range interacting condensates [7]. It has been shown that in that case the dispersion law $\epsilon(q)$ is purely imaginary for $q<1$ (Fig.1). Hence dark-soliton planes in homogeneous three-dimensional short-range interacting condensates, are dynamically unstable against transverse modulations. This so-called snake instability has been experimentally observed in non linear optics [8] and recently in the context of BEC [10]. In the latter case, the dark-soliton plane bends and decays into more stable structures such as vortex rings and sound excitations [9]. The stabilization of the soliton demands a strong radial confinement [7], i.e. the dark-solitons cannot be considered any longer as three-dimensional, but on the contrary acquire a one-dimensional character.

In the presence of DDI ($\beta \neq 0$) but without lattice ($m/m^* = 1$), the transverse instability persists, since $\epsilon(q)$ remains purely imaginary for $q<1$. For $\beta > 0 (\beta < 0)$, $|\epsilon(q)|$ decreases (increases) when $|\beta|$ grows (Fig.1). The situation changes completely when $m/m^*$ is sufficiently small since, as discussed below, $\epsilon(q)$ is purely real and, as a consequence, the dark soliton becomes dynamically stable (see Fig.3). As we detail in the following, this remarkable fact can be clearly understood by means of a transparent physical picture. First we notice that for low momenta the spectrum is always linear in $q$, suggesting the idea that for low momenta the system may be described by an elastic model of the two dimensional nodal plane of the dark soliton. The Lagrangian density for the nodal plane reads

$$\mathcal{L} = \frac{M}{2} \frac{d^2 \Phi}{dt^2} - \frac{\sigma}{2} |\nabla \Phi|^2$$

where $\Phi$ is the displacement field of the nodal plane from the ground state, $M$ is the mass for unit area and $\sigma$ plays the role of a surface tension. The mass $M$ of the soliton can be easily calculated expanding the energy of a moving soliton up to second order in the velocity. We obtain $M = -4 \hbar n_0/c$, where $c = \sqrt{\gamma n_0/m}$ is the sound velocity. Notice that $M < 0$ since the dark soliton represents an absence of atoms. The surface tension can be calculated inserting a suitable variational ansatz $\Psi_{\text{var}}(\vec{r}) = \sqrt{n_0} \tanh((z - \sqrt{2} \alpha \cos(qx))/\zeta)$ (describing a transverse modulation of the nodal plane with amplitude $\alpha$ and momentum $q$) in the energy functional and expanding up to second order in $\alpha$ and $q$:

$$\sigma = \frac{4n_0\hbar^2}{3m^*} - 2gM^2n^2_0(1 + \beta)$$

This expression can be considered as one of the main results of this Letter. The nature of the eigenmodes $\omega^2 = (\sigma/M)q^2$ crucially depends on the sign of $\sigma/M$. In the absence of DDI ($\beta = 0$), $\sigma$ is always positive and hence the modes are purely imaginary. The dark soliton is dynamically unstable against the above mentioned snake instability. Note that for $\beta = 0$ and $m/m^* = 1$ our result coincides with the one found in Ref. [7]. In the absence of an additional optical lattice, the dynamical instability of the DS at low $q$ dissipates for $\beta > 2$, i.e. for situations for which the homogeneous dipolar BEC as a whole is itself, as commented above, unstable against local collapses. Increasing the depth of the lattice potential reduces the role of the kinetic energy term $(m/m^*)q^2$ in Eqs. (4) and (5) (or equivalently reduces the first term in Eq. (7)) and hence enhances the role of the DDI. A sufficiently large DDI or small $m/m^*$ such that

$$m/m^* < \frac{3(1 + \beta)}{2(1 + \beta)}$$

leads to stable low-energy phonons with $q \to 0$. We have evaluated from a direct numerical calculation of the BdG equations (11) and (12) the stability threshold at which $\epsilon(q \to 0)$ becomes real. Fig.2 compares our numerical and analytical results, which are in excellent agreement.

When $m/m^*$ decreases further or $\beta$ grows, a wider regime of low momenta is stabilized (Fig.3). Note in
Figure 3: Real part of the excitation energies of a DS for $m/m^* = 1$, and $\beta = 1.6$. Solid line corresponds to the analytical result for low momenta while empty circles correspond to numerical results. The imaginary part of the excitation energies is equal to zero for the range of momenta considered in the figure.

Fig. 3 that the dispersion law at low momenta is very accurately described by our analytical results. A sufficiently strong optical lattice and large DDI can stabilize all the modes with momenta up to $q \sqrt{m/m^* \zeta} \sim 1$. Although we observe instabilities for momenta $q \sqrt{m/m^* \zeta} \sim 1$, this large-momentum instability is typically irrelevant, since for sufficiently small $m/m^*$ it concerns momenta much larger than the lattice momentum. Although our effective mass theory breaks down for such momenta, it becomes clear that such high momentum instabilities are physically prevented by the zero point oscillations at each lattice site.

Summarizing, contrary to short-range interacting BECs, where stable dark solitons demand a sufficiently strong transverse confinement, dipolar BECs allow for stable dark solitons of arbitrarily large transversal sizes. We have obtained the stability conditions, which demand a sufficiently large dipole and a sufficiently deep optical lattice in the nodal plane. We presented analytical results for the lowest part of the spectrum which agree with those obtained numerically. The stabilization of nodal planes is purely linked to the long-range nature of the DDI, opening a qualitatively new scenario in non linear atom optics.

We acknowledge fruitful discussion with Lev P. Pitaevskii, L. Pricoupenko and G. V. Shlyapnikov. This work was supported by the DFG (SFB-TR21, SFB407, SPP1116), by the Ministère de la Recherche (grant ACI Nanoscience 201), by the ANR (Grants Nos. NT05-2.42103 and 05-Nano-008-02), and by the IFRAF Institute. LPTMC is UMR 7600 of CNRS.

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[32] $C \simeq \sum_{n=0,j \neq 0} |\tilde{n}(2\pi j/b, 2\pi j/b)|^2$ where $\tilde{n}$ is the Fourier transform of $n$ and $n(x, y) = j(x, y)$.