Non-uniform continuity of the Fokas-Olver-Rosenau-Qiao equation in Besov spaces

Xing Wu*

College of Information and Management Science, Henan Agricultural University,
Zhengzhou, Henan, 450002, China

Abstract: In this paper, we prove that the solution map of Fokas-Olver-Rosenau-Qiao equation (FORQ) is not uniformly continuous on the initial data in Besov spaces. Our result extends the previous non-uniform continuity in Sobolev spaces (Nonlinear Anal., 2014) [11] to Besov spaces and is consistent with the present work (J. Math. Fluid Mech., 2020) [17] on Novikov equation up to some coefficients when dropping the extra term $(\partial_x u)^3$ in FORQ.

Keywords: Fokas-Olver-Rosenau-Qiao equation, Non-uniform continuous dependence, Besov spaces

MSC (2010): 35B30; 35G25; 35Q53

1 Introduction

In this paper, we are concerned with the following Fokas-Olver-Rosenau-Qiao equation (FORQ)

\[ \begin{cases} 
  u_t - u_{xxt} + 3u^2u_x - u^3_x - 4uu_xu_{xx} + 2u_xu_{xx}^2 - u^2u_{xxx} + u_x^2u_{xxx} = 0, & t > 0, \quad x \in \mathbb{R}, \\
  u(0, x) = u_0, & x \in \mathbb{R}.
\end{cases} \tag{1.1} \]

Eq. (1.1) written in a slightly different form was first derived by Fokas [3] as an integrable generalisation of the modified KdV equation. Soon after, Fuchssteiner [5] and Olver-Rosenau [19] independently obtained similar versions of this equation by performing a simple explicit algorithm based on the bi-Hamiltonian representation of the classically integrable system. Several years later, the concise form written above was recovered by Qiao [20] from the two-dimensional Euler equations by using an approximation procedure.

The entire integrable hierarchy related to the FORQ equation was proposed by Qiao [21]. It also has bi-Hamiltonian structure, which was first derived in [19] and then in [20], admits Lax pair [20] and peakon travelling wave solutions that are orbitally stable [6, 22, 18]. For more discussion about Lax integrability and peakon solutions of FORQ we refer to [2], where this equation is also referred as the modified Camassa-Holm equation.

*E-mail:ny2008wx@163.com
The local well-posedness and ill-posedness for the Cauchy problem of the FORQ equation (1.1) in Sobolev spaces and Besov spaces were studied in the series of papers [11, 12, 13, 4]. It was showed by Himonas-Mantzavinos [11] that the FORQ is well-posed in Sobolev space $H^s$ with $s > \frac{5}{2}$ in the sense of Hadamard. Fu et al.[4] established the local well-posedness in Besov space $B^s_{p,r}$ with $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$, $1 \leq p, r \leq \infty$. After the non-uniform dependence for some dispersive equations was studied by Kenig et al. [14], the issue of non-uniform continuity of solutions on initial data has attracted much more attention, such as on classical Camassa-Holm equation [7, 8, 9, 16] and on famous Novikov equation [10, 17]. It was further proved in [11] that the dependence on initial data is sharp, i.e. the data-to-solution map is continuous but not uniformly continuous.

For studying the non-uniform continuity of the FORQ equation, it is more convenient to express (1.1) in the following equivalent nonlocal form

\[
\begin{align*}
\begin{cases}
  u_t + u^2 \partial_x u &= \frac{1}{3}(\partial_x u)^3 - \frac{4}{3}(1 - \partial_x^2)^{-1}[(\partial_x u)^3] - \partial_x(1 - \partial_x^2)^{-1}\left[\frac{2}{3} u^3 + u(\partial_x u)^2\right], \\
  u(0,x) &= u_0, \quad t > 0, \quad x \in \mathbb{R}.
\end{cases}
\end{align*}
\]

(1.2)

When removing $(\partial_x u)^3$ from (1.2), (1.2) becomes the following Novikov equation up to some coefficients

\[
\begin{align*}
\begin{cases}
  u_t + u^2 \partial_x u &= -\frac{4}{3}(1 - \partial_x^2)^{-1}[(\partial_x u)^3] - \partial_x(1 - \partial_x^2)^{-1}\left[\frac{2}{3} u^3 + u(\partial_x u)^2\right], \\
  u(0,x) &= u_0, \quad t > 0, \quad x \in \mathbb{R}.
\end{cases}
\end{align*}
\]

(1.3)

Recently, Li, Yu and Zhu [17] have proved that the solution map of Novikov equation is not uniformly continuous dependence on the initial data in the Besov spaces $B^s_{p,r}(\mathbb{R})$, $s > \max\{1 + \frac{3}{p}, \frac{5}{2}\}$, $1 \leq p, r \leq \infty$. It is noticed that well-posedness for Novikov equation holds for $s > \max\{1 + \frac{3}{p}, \frac{5}{2}\}$ while well-posedness for FORQ holds for $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$. This difference between the well-posedness index of these equations may be explained by the presence of the extra term $(\partial_x u)^3$ in FORQ, which is not quasi-linear and is absent from Novikov equation [11].

Up to the present, there is no result for the non-uniform continuous dependence of FORQ in Besov space and it seems more difficult due to the presence of the extra term $(\partial_x u)^3$, which elevates two regularities, compared with Novikov equation and the method developed for the Novikov equation in [17] will make it more complex. In this paper, we will follow a different route to bypass this problem. Firstly, consider a new system satisfied by $(1 - \partial_x)u \triangleq v$. Secondly, for any bounded set $v_0$ in working space, the corresponding solution $S_t(v_0)$ can be approximated by a function of first degree of time $t$ with convective term and nonlocal term being coefficients. With suitable choice of initial data, the difference between two solutions will produce a term from convective term which will not be small for small time, and thus we obtain the non-uniform continuous dependence of FORQ. These will be described in more detail later.

Now, we state our main result.

**Theorem 1.1** Let $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$, $1 \leq p, r \leq \infty$. The solution map $u_0 \to S_t(u_0)$ of the initial value problem (1.2) is not uniformly continuous from any bounded subset of $B^s_{p,r}(\mathbb{R})$ into $C([0,T]; B^s_{p,r}(\mathbb{R}))$. More precisely, there exist two sequences $u^{1,n}(0,x)$ and $u^{2,n}(0,x)$ such that

\[
\|u^{1,n}(0,x), u^{2,n}(0,x)\|_{B^s_{p,r}} \lesssim 1, \quad \lim_{n \to \infty} \|u^{1,n}(0,x) - u^{2,n}(0,x)\|_{B^s_{p,r}} = 0,
\]
but

\[ \liminf_{n \to \infty} \| S_t(u^{1,n}(0,x)) - S_t(u^{2,n}(0,x)) \|_{B^s_{p,r}} \geq t, \quad t \in [0,T_0], \]

with small positive time \( T_0 \) for \( T_0 \leq T \).

**Remark 1.1** Since \( B^s_{2,2} = H^s \) for any \( s \in \mathbb{R} \), our result extends the previous non-uniform continuity in Sobolev spaces [11] to Besov spaces.

**Remark 1.2** When dropping the extra term \( (\partial_x u)^3 \) in FORQ, we can get the same result on Novikov equation up to some coefficients, which is consistent with the present work [17] on Novikov equation. The method we use in proving Theorem 1.1 is different from [17] and is more general.

**Notations:** Given a Banach space \( X \), we denote the norm of a function on \( X \) by \( \| \cdot \|_X \), and

\[ \| \cdot \|_{L^\infty(T,X)} = \sup_{0 \leq t \leq T} \| \cdot \|_X. \]

The symbol \( A \lesssim B \) means that there is a uniform positive constant \( C \) independent of \( A \) and \( B \) such that \( A \leq CB \).

## 2 Littlewood-Paley analysis

In this section, we will review the definition of Littlewood-Paley decomposition and nonhomogeneous Besov space, and then list some useful properties. For more details, the readers can refer to [1].

There exists a couple of smooth functions \( (\chi, \varphi) \) valued in \([0,1]\), such that \( \chi \) is supported in the ball \( B \triangleq \{ \xi \in \mathbb{R}^d : |\xi| \leq \frac{4}{3} \} \), \( \varphi \) is supported in the ring \( C \triangleq \{ \xi \in \mathbb{R}^d : \frac{2}{3} \leq |\xi| \leq \frac{8}{3} \} \) and \( \varphi \equiv 1 \) for \( \frac{4}{3} \leq |\xi| \leq \frac{3}{2} \). Moreover,

\[
\forall \xi \in \mathbb{R}^d, \quad \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1,
\]

\[
\forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1,
\]

\[
|j - j'| \geq 2 \Rightarrow \text{Supp } \varphi(2^{-j} \cdot) \cap \text{Supp } \varphi(2^{-j'} \cdot) = \emptyset,
\]

\[
j \geq 1 \Rightarrow \text{Supp } \chi(\cdot) \cap \text{Supp } \varphi(2^{-j} \cdot) = \emptyset.
\]

Then, we can define the nonhomogeneous dyadic blocks \( \Delta_j \) and nonhomogeneous low frequency cut-off operator \( S_j \) as follows:

\[
\Delta_j u = 0, \text{ if } j \leq -2, \quad \Delta_{-1} u = \chi(D)u = \mathcal{F}^{-1}(\chi \mathcal{F} u),
\]

\[
\Delta_j u = \varphi(2^{-j} D)u = \mathcal{F}^{-1}(\varphi(2^{-j}) \mathcal{F} u), \text{ if } j \geq 0,
\]

\[
S_j u = \sum_{j' = -\infty}^{j-1} \Delta_{j'} u.
\]
Definition 2.1 ([1]) Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$. The nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^d)\) consists of all tempered distribution $u$ such that

$$||u||_{B^s_{p,r}(\mathbb{R}^d)} \triangleq \left( \sum_{j \in \mathbb{Z}} \left( 2^{js} \left\| \Delta_j u \right\|_{L^p(\mathbb{R}^d)} \right)^r \right)^{\frac{1}{r}} < \infty.$$ 

In the following, we list some basic lemmas and properties about Besov space which will be frequently used in proving our main result.

Lemma 2.1 ([1]) (1) Algebraic properties: \(\forall s > 0\), \(B^s_{p,r}(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)\) is a Banach algebra. \(B^s_{p,r}(\mathbb{R}^d)\) is a Banach algebra \(\iff B^s_{p,r}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d) \iff s > \frac{d}{p}\) or \(s = \frac{d}{p}\), \(r = 1\).

(2) For any \(s > 0\) and \(1 \leq p, r \leq \infty\), there exists a positive constant \(C = C(d, s, p, r)\) such that

$$\left\| uv \right\|_{B^s_{p,r}(\mathbb{R}^d)} \leq C \left( \left\| u \right\|_{L^\infty(\mathbb{R}^d)} \left\| v \right\|_{B^s_{p,r}(\mathbb{R}^d)} + \left\| v \right\|_{L^\infty(\mathbb{R}^d)} \left\| u \right\|_{B^s_{p,r}(\mathbb{R}^d)} \right).$$

(3) Let \(m \in \mathbb{R}\) and \(f\) be an \(S^m\) - multiplier (i.e., \(f : \mathbb{R}^d \rightarrow \mathbb{R}\) is smooth and satisfies that \(\forall \alpha \in \mathbb{N}^d\), there exists a constant \(C_\alpha\) such that \(\left| \partial^\alpha f(\xi) \right| \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}\) for all \(\xi \in \mathbb{R}^d\). Then the operator \(f(D)\) is continuous from \(B^s_{p,r}(\mathbb{R}^d)\) to \(B^{s-m}_{p,r}(\mathbb{R}^d)\).

(4) For any \(s \in \mathbb{R}\), \((1 - \partial_x)^{-1}\) is an isomorphic mapping from \(B^{s-1}_{p,r}(\mathbb{R}^d)\) into \(B^s_{p,r}(\mathbb{R}^d)\).

Lemma 2.2 ([1, 15]) Let \(1 \leq p, r \leq \infty\) and \(\sigma > -\min\{\frac{1}{p}, 1 - \frac{1}{p}\}\). There exists a constant \(C = C(p, r, \sigma)\) such that for any smooth solution to the following linear transport equation:

$$\partial_t f + v \partial_x f = g, \quad f|_{t=0} = f_0.$$ 

We have

$$\sup_{s \in [0, t]} \left\| f(s) \right\|_{B^s_{p,r}(\mathbb{R})} \leq C e^{CV_p(v, t)} \left( \left\| f_0 \right\|_{B^s_{p,r}(\mathbb{R})} + \int_0^t \left\| g(\tau) \right\|_{B^s_{p,r}(\mathbb{R})} d\tau \right),$$

with

$$V_p(v, t) = \begin{cases} \int_0^t \left\| \nabla v(s) \right\|_{B^{s-\frac{1}{p}}(\mathbb{R}^d)} ds, & \text{if } \sigma < 1 + \frac{1}{p}, \\ \int_0^t \left\| \nabla v(s) \right\|_{B^s_{p,r}(\mathbb{R})} ds, & \text{if } \sigma = 1 + \frac{1}{p} \text{ and } r > 1, \\ \int_0^t \left\| \nabla v(s) \right\|_{B^{s-1}_{p,r}(\mathbb{R})} ds, & \text{if } \sigma > 1 + \frac{1}{p} \text{ or } \{ \sigma = 1 + \frac{1}{p} \text{ and } r = 1 \}. \end{cases}$$

3 Reformulation of the System

Due to the presence of the extra term \((\partial_x u)^3\) in FORQ, it seems difficult to deal with Eq. (1.2) directly. Therefore, we shall first differentiate FORQ with respect to \(x\) and then simplify the resulting expression, we obtain

$$\partial_t (\partial_x u) = (\partial_x u)^2 \partial_x^2 u - 2u(\partial_x u)^2 - u^2 \partial_x^2 u + \left[ \frac{2}{3} u^3 + u(\partial_x u)^2 \right]$$

$$- (1 - \partial_x^2)^{-1} \partial_x \left[ \frac{1}{3} (\partial_x u)^3 \right] - (1 - \partial_x^2)^{-1} \left[ \frac{2}{3} u^3 + u(\partial_x u)^2 \right].$$

(3.1)
Let $v = (1 - \partial_x)u$, we have from (1.2) and (3.1) that

$$\begin{cases}
\partial_t v = (v^2 - 2uv)\partial_x v - \frac{1}{3}u^3 - \frac{1}{3}v^3 - \Phi_1(v) - \Phi_2(v), \\
u = (1 - \partial_x)^{-1}v, \\
v(0, x) = (1 - \partial_x)u_0(x) \triangleq v_0,
\end{cases}$$

(3.2)

where the nonlocal terms $\Phi_1(v), \Phi_2(v)$ are defined by

$$\Phi_1(v) = (1 - \partial_x^2)^{-1}\left[\frac{8}{3}u^3 - \frac{1}{3}v^3 - 3u^2v\right], \quad \Phi_2(v) = \partial_x(1 - \partial_x^2)^2\left[\frac{1}{3}v^3 - u^2v\right].$$

Since $(1 - \partial_x)^{-1}$ is an isomorphic mapping from $B^{s-1}_{p,r}(\mathbb{R})$ into $B^s_{p,r}(\mathbb{R})$, the non-uniform continuous dependence of $u$ in $B^s_{p,r}$ then can be transformed into that of $v$ in $B^{s-1}_{p,r}$.

### 4 Non-uniform continuous dependence

In this section, we will give the proof of Theorem 1.1. However, as explained above, we will directly consider Eq. (3.2) satisfied by $v$.

Firstly, we establish the estimates of the difference between the solution $S_t(v_0)$ and initial data $v_0$ in different Besov norms. That is

**Proposition 4.1** Assume that $\|v_0\|_{B^{s-1}_{p,r}} \lesssim 1$. Under the assumptions of Theorem 1.1, we have

$$\begin{align*}
\|S_t(v_0) - v_0\|_{B^{s-2}_{p,r}} &\lesssim t\|v_0\|_{B^{s-2}_{p,r}}\|v_0\|_{B^{s-1}_{p,r}}, \\
\|S_t(v_0) - v_0\|_{B^{s-1}_{p,r}} &\lesssim t(\|v_0\|_{B^{s-1}_{p,r}}^3 + \|v_0\|_{B^{s-2}_{p,r}}^2\|v_0\|_{B^{s}_{p,r}}), \\
\|S_t(v_0) - v_0\|_{B^{s}_{p,r}} &\lesssim t(\|v_0\|_{B^{s-1}_{p,r}}\|v_0\|_{B^{s}_{p,r}} + \|v_0\|_{B^{s-2}_{p,r}}\|v_0\|_{B^{s+1}_{p,r}}).
\end{align*}$$

**Proof** For simplicity, denote $v(t) = S_t(v_0)$. Firstly, according to the local well-posedness result [4, 11], there exists a positive time $T = T(||u_0||_{B^{s-1}_{p,r}}, s, p, r)$ such that the solution $u(t)$ belongs to $C([0, T]; B^{s}_{p,r})$. Moreover, by Lemmas 2.1-2.2, for all $t \in [0, T]$ and $\gamma \geq s - 2$, there holds

$$\|u(t)\|_{B^{\gamma}_{p,r}} \leq C\|u_0\|_{B^{\gamma}_{p,r}} \text{ or } \|v(t)\|_{B^{\gamma-1}_{p,r}} \leq C\|v_0\|_{B^{\gamma-1}_{p,r}}. \quad (4.1)$$

Now we shall estimate the different Besov norms of the term $v(t) - v_0$, which can be bounded by $t$ multiplying the corresponding Besov norms of initial data $v_0$.

It follows by differential mean value theorem and the Minkowski inequality that

$$\begin{align*}
\|v(t) - v_0\|_{B^{s-1}_{p,r}} &\lesssim \int_0^t \|\partial_t v\|_{B^{s-1}_{p,r}}d\tau \\
&\lesssim \int_0^t \|(v^2 - 2uv)\partial_x v\|_{B^{s-1}_{p,r}}d\tau + \int_0^t \left|\frac{1}{3}v^3\right|_{B^{s-1}_{p,r}}d\tau \\
&+ \int_0^t \left|\frac{1}{3}u^3\right|_{B^{s-1}_{p,r}}d\tau + \int_0^t ||\Phi_1(v)||_{B^{s-1}_{p,r}}d\tau + \int_0^t ||\Phi_2(v)||_{B^{s-1}_{p,r}}d\tau.
\end{align*}$$

Here, we shall only have to estimate $\|(v^2 - 2uv)\partial_x v\|_{B^{s-1}_{p,r}}$ and $\left|\frac{1}{3}v^3\right|_{B^{s-1}_{p,r}}$, since the other terms can be processed in a similar more relaxed way and have the same bound as $\|v^3\|_{B^{s-1}_{p,r}}$. 

5
Using the fact that $B^{s-2}_{p,r}$ is an Banach algebra with $s - 2 > \max\{\frac{1}{p}, \frac{1}{2}\}$, together with the product estimates (2) in Lemma 2.1, one has

\[
\|v^3\|_{B^{s+1}_{p,r}} \lesssim \|v\|_{B^{s+1}_{p,r}},
\]
\[
\|(v^2 - 2uv)\partial_x v\|_{B^{s+1}_{p,r}} \lesssim \|(v^2 - 2uv)\|_{L_\infty} \|\partial_x v\|_{B^{s+1}_{p,r}} + \|v\|_{L_\infty}^3 \|\partial_x v\|_{L_\infty}
\]
\[
\lesssim \|v\|_{B^{s+2}_{p,r}}^2 \|v\|_{B^{s+1}_{p,r}} + \|v\|_{B^{s+1}_{p,r}}^3,
\]
where we have used the relation $u = (1 - \partial_x)^{-1}v$, and $(1 - \partial_x)^{-1}$ is a $S^{-1}$--multiplier, which is continuous from $B^{s-1}_{p,r}(\mathbb{R})$ to $B^{s'}_{p,r}(\mathbb{R})$, thus $\|u\|_{B^{s'}_{p,r}} \lesssim \|v\|_{B^{s-1}_{p,r}}$ for any $s' \in \mathbb{R}$.

Therefore, in view of (4.1), for $t \in [0, T]$, we have

\[
\|v(t) - v_0\|_{B^{s+1}_{p,r}} \lesssim t\|v\|_{L^\infty_t(B^{s-2}_{p,r})}^2 \|v\|_{L^\infty_t(B^{s}_{p,r})} + \|v\|_{L^\infty_t(B^{s-2}_{p,r})}^3
\]
\[
\lesssim t(\|v_0\|_{B^{s+2}_{p,r}}^2 \|v_0\|_{B^{s}_{p,r}} + \|v_0\|_{B^{s+1}_{p,r}}).
\]

Following the same procedure of estimate as above, we have

\[
\|v(t) - v_0\|_{B^{s}_{p,r}} \lesssim \int_0^t \|\partial_x v\|_{B^{s-2}_{p,r}} d\tau
\]
\[
\lesssim \int_0^t \|(v^2 - 2uv)\partial_x v\|_{B^{s-2}_{p,r}} d\tau + \int_0^t \frac{1}{3} v^3 \|B^{s-2}_{p,r}\| d\tau
\]
\[
+ \int_0^t \frac{1}{3} u^3 \|B^{s-2}_{p,r}\| d\tau + \int_0^t \|\Phi_1(v)\|_{B^{s-2}_{p,r}} d\tau + \int_0^t \|\Phi_2(v)\|_{B^{s-2}_{p,r}} d\tau
\]
\[
\lesssim t\|v\|_{L^\infty_t(B^{s-2}_{p,r})}^2 \|v\|_{L^\infty_t(B^{s}_{p,r})} + \|v\|_{L^\infty_t(B^{s-2}_{p,r})}^2 \|v\|_{B^{s+1}_{p,r}} + \|v\|_{B^{s+1}_{p,r}}^2 \|v\|_{B^{s}_{p,r}},
\]

and

\[
\|v^3\|_{B^{s}_{p,r}} \lesssim \|v^2\|_{L_\infty} \|v\|_{B^{s}_{p,r}} + \|v\|_{B^{s+1}_{p,r}} \|v\|_{L_\infty}
\]
\[
\lesssim \|v\|_{B^{s+1}_{p,r}}^2 \|v\|_{B^{s}_{p,r}},
\]
\[
\|(v^2 - 2uv)\partial_x v\|_{B^{s}_{p,r}} \lesssim \|(v^2 - 2uv)\|_{L_\infty} \|\partial_x v\|_{B^{s}_{p,r}} + \|(v^2 - 2uv)\|_{B^{s}_{p,r}} \|\partial_x v\|_{L_\infty}
\]
\[
\lesssim \|v\|_{B^{s+2}_{p,r}}^2 \|v\|_{B^{s+1}_{p,r}} + \|v\|_{B^{s+1}_{p,r}}^2 \|v\|_{B^{s}_{p,r}},
\]
hence,

\[
\|v(t) - v_0\|_{B^{s}_{p,r}} \lesssim \int_0^t \|\partial_x v\|_{B^{s}_{p,r}} d\tau
\]
\[
\lesssim \int_0^t \|(v^2 - 2uv)\partial_x v\|_{B^{s}_{p,r}} d\tau + \int_0^t \frac{1}{3} v^3 \|B^{s}_{p,r}\| d\tau
\]
\[
+ \int_0^t \frac{1}{3} u^3 \|B^{s}_{p,r}\| d\tau + \int_0^t \|\Phi_1(v)\|_{B^{s}_{p,r}} d\tau + \int_0^t \|\Phi_2(v)\|_{B^{s}_{p,r}} d\tau
\]
\[
\lesssim t(\|v\|_{L^\infty_t(B^{s-2}_{p,r})}^2 \|v\|_{L^\infty_t(B^{s}_{p,r})} + \|v\|_{L^\infty_t(B^{s-2}_{p,r})}^2 \|v\|_{B^{s+1}_{p,r}} + \|v\|_{B^{s+1}_{p,r}} \|v\|_{B^{s}_{p,r}}),
\]

Thus, we finish the proof of Proposition 4.1.
With the different Besov norms estimates of \( v - v_0 \) at hand, we have the following core estimates, which implies that for any bounded initial data \( v_0 \) in \( B^{s-1}_{p,r} \), the corresponding solution \( S_t(v_0) \) can be approximated by \( v_0 + t(v_0^2 - 2u_0v_0)\partial_x v_0 + \frac{1}{3}tv_0^3 + t\left(\frac{1}{3}u_0^3 + \Phi_1(v_0) + \Phi_2(v_0)\right) \) near \( t = 0 \).

**Proposition 4.2** Assume that \( \|v_0\|_{B^{s-1}_{p,r}} \lesssim 1 \). Then under the assumptions of Theorem 1.1, there holds

\[
\|S_t(v_0) - v_0 - tv_0\|_{B^{s-1}_{p,r}} \lesssim t^2 (\|v_0\|_{B^{s-1}_{p,r}}^3 + \|v_0\|_{B^{s-2}_{p,r}}^2)\|v_0\|_{B^{s}_{p,r}} + \|v_0\|_{B^{s-2}_{p,r}}^4 \|v_0\|_{B^{s+1}_{p,r}})
\]

where \( v_0 = (v_0^2 - 2u_0v_0)\partial_x v_0 + \frac{1}{3}v_0^3 + \frac{1}{3}u_0^3 + \Phi_1(v_0) + \Phi_2(v_0) \).

**Proof** Using differential mean value theorem and the Minkowski inequality, we first arrive at

\[
\|v(t) - v_0 - tv_0\|_{B^{s-1}_{p,r}} \lesssim \int_0^t \|\partial_\tau v - v_0\|_{B^{s-1}_{p,r}} d\tau
\]

\[
\lesssim \int_0^t \|v^2 \partial_x v - v_0^2 \partial_x v_0\|_{B^{s-1}_{p,r}} d\tau + \int_0^t \|2uv \partial_x v - 2u_0v_0 \partial_x v_0\|_{B^{s-1}_{p,r}} d\tau
\]

\[
+ \int_0^t \|\frac{1}{3}v_0^3 - \frac{1}{3}u_0^3\|_{B^{s-1}_{p,r}} d\tau + \int_0^t \|\frac{1}{3}v^3 - \frac{1}{3}u^3\|_{B^{s-1}_{p,r}} d\tau
\]

\[
+ \int_0^t \|\Phi_1(v) - \Phi_1(v_0)\|_{B^{s-1}_{p,r}} d\tau + \int_0^t \|\Phi_2(v) - \Phi_2(v_0)\|_{B^{s-1}_{p,r}} d\tau. \quad (4.2)
\]

Using (3) in Lemma 2.1, it is sufficient to estimate \( \|v^2 \partial_x v - v_0^2 \partial_x v_0\|_{B^{s-1}_{p,r}} \), \( \|2uv \partial_x v - 2u_0v_0 \partial_x v_0\|_{B^{s-1}_{p,r}} \) and \( \|v^3 - v_0^3\|_{B^{s-1}_{p,r}} \), since the other terms can be processed in a similar more relaxed way and have the same bound as \( \|v^3 - v_0^3\|_{B^{s-1}_{p,r}} \).

It should be noticed that according to (4.1), \( \|v\|_{B^{s-1}_{p,r}} \lesssim \|v_0\|_{B^{s-1}_{p,r}} \lesssim 1 \), which will be frequently used later.

Due to the fact that \( B^{s-2}_{p,r} \) is an Banach algebra with \( s - 2 > \max\{\frac{1}{p}, \frac{1}{2}\} \), combining with the product estimates (2) in Lemma 2.1, we get

\[
\|v^2 \partial_x v - v_0^2 \partial_x v_0\|_{B^{s-1}_{p,r}} = \|(v^2 - v_0^2) \partial_x v + v_0^2 (\partial_x v - \partial_x v_0)\|_{B^{s-1}_{p,r}}
\]

\[
\lesssim \|(v^2 - v_0^2) \partial_x v\|_{B^{s-1}_{p,r}} + \|v_0^2 (\partial_x v - \partial_x v_0)\|_{B^{s-1}_{p,r}}
\]

\[
\lesssim \|v^2 - v_0^2\|_{L^\infty} \|\partial_x v\|_{B^{s-1}_{p,r}} + \|v^2 - v_0^2\|_{B^{s-1}_{p,r}} \|\partial_x v\|_{L^\infty}
\]

\[
+ \|v_0^2\|_{L^\infty} \|\partial_x v - \partial_x v_0\|_{B^{s-1}_{p,r}} + \|v_0\|_{B^{s-1}_{p,r}} \|\partial_x v - \partial_x v_0\|_{L^\infty}
\]

\[
\lesssim \|v - v_0\|_{B^{s-1}_{p,r}B^{s-1}_{p,r}} \|v\|_{B^{s}_{p,r}} + \|v - v_0\|_{B^{s}_{p,r}} + \|v\|_{B^{s}_{p,r}} \|v_0\|_{B^{s-2}_{p,r}}^2,
\]

and

\[
\|2uv \partial_x v - 2u_0v_0 \partial_x v_0\|_{B^{s-1}_{p,r}} = \|2(u - u_0) v \partial_x v_0 + 2u_0(v \partial_x v - v_0 \partial_x v_0)\|_{B^{s-1}_{p,r}}
\]

\[
\lesssim \|u - u_0\|_{B^{s-1}_{p,r}} \|v \partial_x v\|_{B^{s-1}_{p,r}} + \|u_0\|_{B^{s-1}_{p,r}} \|v \partial_x v - v_0 \partial_x v_0\|_{B^{s-1}_{p,r}}
\]

\[
\lesssim \|v - v_0\|_{B^{s-1}_{p,r}} \|v\|_{B^{s}_{p,r}} + \|v - v_0\|_{B^{s-1}_{p,r}} + \|v\|_{B^{s}_{p,r}} \|v_0\|_{B^{s-2}_{p,r}}^2,
\]

and

\[
\|v^3 - v_0^3\|_{B^{s-1}_{p,r}} = \|(v - v_0)(v^2 + v_0v + v_0^2)\|_{B^{s-1}_{p,r}} \lesssim \|v - v_0\|_{B^{s-1}_{p,r}}.
\]

Taking the above estimates into (4.2), which together with Proposition 4.1 yield

\[
\|v(t) - v_0 - tv_0\|_{B^{s-1}_{p,r}} \lesssim \int_0^t \|v - v_0\|_{B^{s}_{p,r}} + \|v - v_0\|_{B^{s}_{p,r}} \|v\|_{B^{s}_{p,r}} d\tau + \|v - v_0\|_{B^{s}_{p,r}} \|v_0\|_{B^{s-2}_{p,r}}^2 d\tau
\]

\[
\lesssim t^2 (\|v_0\|_{B^{s-1}_{p,r}}^3 + \|v_0\|_{B^{s-2}_{p,r}}^2 \|v_0\|_{B^{s}_{p,r}} + \|v\|_{B^{s}_{p,r}}^4 \|v_0\|_{B^{s+1}_{p,r}}).\]
Thus, we complete the proof of Proposition 4.2. □

Now, we move on the proof of Theorem 1.1.

Proof of Theorem 1.1 Let \( \hat{\phi} \in \mathcal{C}_0^\infty(\mathbb{R}) \) be an even, real-valued and non-negative function on \( \mathbb{R} \) and satisfy

\[
\hat{\phi}(x) = \begin{cases} 
1, & \text{if } |x| \leq \frac{1}{4}, \\
0, & \text{if } |x| \geq \frac{1}{2}.
\end{cases}
\]

Define the high frequency function \( f_n \) and the low frequency function \( g_n \) by

\[
f_n = 2^{-ns} \phi(x) \sin\left(\frac{17}{12}2^n x\right), \quad g_n = 2^{-\frac{n}{2}} \phi(x), \quad n \gg 1.
\]

It has been showed in [16] that \( \|f_n\|_{B^s_{p,r}} \lesssim 2^{n(\sigma-s)} \).

Let

\[
v^{1,n}(0, x) = (1 - \partial_x)(f_n + g_n), \quad v^{2,n}(0, x) = (1 - \partial_x)f_n.
\]

Consider Eq. (3.2) with initial data \( v^{1,n}(0, x) \) and \( v^{2,n}(0, x) \), respectively. Obviously, we have

\[
\|v^{1,n}(0, x) - v^{2,n}(0, x)\|_{B^s_{p,r}} = \|1 - \partial_x\|_{B^s_{p,r}} \leq C 2^{-\frac{n}{2}},
\]

which means that

\[
\lim_{n \to \infty} \|v^{1,n}(0, x) - v^{2,n}(0, x)\|_{B^s_{p,r}} = 0.
\]

It is easy to show that

\[
\|v^{1,n}(0, x)\|_{B^{s-2}_{p,r}} \lesssim \|f_n + g_n\|_{B^{s-1}_{p,r}} \lesssim \|f_n + g_n\|_{B^{s-\frac{1}{2}}_{p,r}} \lesssim 2^{-\frac{n}{4}},
\]

\[
\|v^{1,n}(0, x)\|_{B^{s+\tau}_{p,r}} \lesssim \|f_n + g_n\|_{B^{s+\sigma}_{p,r}} \lesssim 2^{n(\sigma+1)} \quad \text{for} \quad \sigma \geq -\frac{3}{2},
\]

\[
\|v^{2,n}(0, x)\|_{B^{s+t}_{p,r}} \lesssim \|f_n\|_{B^{s+t+1}_{p,r}} \lesssim 2^{n(t+1)} \quad \text{for} \quad t \in \mathbb{R},
\]

which imply

\[
\left(\|v^{1,n}(0, x)\|^3_{B^{s-1}_{p,r}} + \|v^{1,n}(0, x)\|^2_{B^{s-2}_{p,r}}\|v^{1,n}(0, x)\|_{B^{s}_{p,r}} + \|v^{1,n}(0, x)\|^4_{B^{s-2}_{p,r}}\|v^{1,n}(0, x)\|_{B^{s}_{p,r}}\right) \lesssim 1,
\]

\[
\left(\|v^{2,n}(0, x)\|^3_{B^{s-1}_{p,r}} + \|v^{2,n}(0, x)\|^2_{B^{s-2}_{p,r}}\|v^{2,n}(0, x)\|_{B^{s}_{p,r}} + \|v^{2,n}(0, x)\|^4_{B^{s-2}_{p,r}}\|v^{2,n}(0, x)\|_{B^{s}_{p,r}}\right) \lesssim 1.
\]

Furthermore, since \( v^{1,n}(0, x) \) and \( v^{2,n}(0, x) \) are both bounded in \( B^{s-1}_{p,r} \), according to Proposition 4.2, we deduce that

\[
\|S_t(v^{1,n}(0, x)) - S_t(v^{2,n}(0, x))\|_{B^{s-1}_{p,r}} \geq t \left|\left[\left(v^{1,n}(0, x)\right)^2 - 2u^{1,n}(0, x)v^{1,n}(0, x)\right] \partial_x v^{1,n}(0, x)\right.
\]

\[
-\left[\left(v^{2,n}(0, x)\right)^2 - 2u^{2,n}(0, x)v^{2,n}(0, x)\right] \partial_x v^{2,n}(0, x)
\]

\[
+ \frac{1}{3} \left[\left(v^{1,n}(0, x)\right)^3 - \left[v^{2,n}(0, x)\right]^3\right] + \frac{1}{3} \left[u^{1,n}(0, x)\right]^3 + \Phi_1(v^{1,n}(0, x)) + \Phi_2(v^{1,n}(0, x))
\]

\[
- \frac{1}{3} \left[u^{2,n}(0, x)\right]^3 + \Phi_1(v^{2,n}(0, x)) + \Phi_2(v^{2,n}(0, x))\right]\|_{B^{s-1}_{p,r}} - \|1 - \partial_x\|_{B^{s-1}_{p,r}} - Ct^2
\]

\[
\geq t \left|\left[\left(v^{1,n}(0, x)\right)^2 - 2u^{1,n}(0, x)v^{1,n}(0, x)\right] \partial_x v^{1,n}(0, x)\right.
\]

\[
-\left[\left(v^{2,n}(0, x)\right)^2 - 2u^{2,n}(0, x)v^{2,n}(0, x)\right] \partial_x v^{2,n}(0, x)\|_{B^{s-1}_{p,r}} - C 2^{-\frac{n}{2}} - Ct^2.
\]

(4.3)
For the sake of simplicity and convenience, in the following we denote
\[ v^{i,n}(0, x) \triangleq v_i, \quad u^{i,n}(0, x) \triangleq u_i, \quad i = 1, 2. \]

The coefficient of the first order term of \( t \) in the last inequality in (4.3) is simplified as
\[
(v_1^2 - 2u_1v_1)\partial_x v_1 - (v_2^2 - 2u_2v_2)\partial_x v_2
= (v_1^2\partial_x v_1 - v_2^2\partial_x v_2) - 2(u_1v_1\partial_x v_1 - v_2\partial_x v_2)
= I_1 - I_2.
\]

Bring in the concrete form of \( v_1 \) and \( v_2 \) where necessary, we have
\[
I_1 = (v_1^2 - v_2^2)\partial_x v_1 + v_2^2(\partial_x v_1 - \partial_x v_2)
= (v_1^2 - v_2^2)\partial_x (1 - \partial_x)(f_n + g_n) + v_2^2(\partial_x v_1 - \partial_x v_2)
= (v_1^2 - v_2^2)\partial_x (f_n + g_n) - (v_1^2 - v_2^2)\partial_x^2 g_n - (v_1^2 - v_2^2)\partial_x^2 f_n + v_2^2\partial_x (v_1 - v_2),
\]
\[
I_2 = 2u_1(v_1 - v_2)\partial_x v_1 + 2v_2(u_1\partial_x v_1 - u_2\partial_x v_2)
= 2u_1(v_1 - v_2)\partial_x (1 - \partial_x)(f_n + g_n) + 2v_2u_1\partial_x (v_1 - v_2) + 2v_2(u_1 - u_2)\partial_x v_2
= 2u_1(v_1 - v_2)\partial_x (f_n + g_n) - 2u_1(v_1 - v_2)\partial_x^2 g_n - 2u_1(v_1 - v_2)\partial_x^2 f_n
+ 2v_2u_1\partial_x (v_1 - v_2) + 2v_2(u_1 - u_2)\partial_x v_2.
\]

Using Lemma 2.1, after simple calculation, we obtain
\[
\|(v_1^2 - v_2^2)\partial_x (f_n + g_n)\|_{B^{p,r}_{-1}} \lesssim \|v_1^2 - v_2^2\|_L\|f_n + g_n\|_{B^{p,r}_{-1}} + \|v_1^2 - v_2^2\|_{B^{p,r}_{-1}}\|\partial_x (f_n + g_n)\|_{L^\infty} \lesssim 2^{-\frac{n}{4}},
\]
\[
\|(v_1^2 - v_2^2)\partial_x^2 g_n\|_{B^{p,r}_{-1}} \lesssim \|v_1^2 - v_2^2\|_{B^{p,r}_{-1}}\|\partial_x^2 g_n\|_{B^{p,r}_{-1}} \lesssim 2^{-\frac{n}{4}},
\]
\[
\|2u_1(v_1 - v_2)\partial_x (f_n + g_n)\|_{B^{p,r}_{-1}} \lesssim \|u_1\|_{B^{p,r}_{-1}}\|(v_1 - v_2)\|_{B^{p,r}_{-1}}\|\partial_x (f_n + g_n)\|_{B^{p,r}_{-1}} \lesssim 2^{-\frac{n}{4}},
\]
\[
\|2v_2u_1\partial_x (v_1 - v_2)\|_{B^{p,r}_{-1}} \lesssim \|u_1\|_{B^{p,r}_{-1}}\|(v_1 - v_2)\|_{B^{p,r}_{-1}}\|\partial_x (v_1 - v_2)\|_{B^{p,r}_{-1}} \lesssim 2^{-\frac{n}{4}},
\]
\[
\|2v_2(u_1 - u_2)\partial_x v_2\|_{B^{p,r}_{-1}} \lesssim \|v_2\|_{B^{p,r}_{-1}}\|(u_1 - u_2)\|_{B^{p,r}_{-1}}\|\partial_x v_2\|_{B^{p,r}_{-1}} \lesssim 2^{-\frac{n}{4}}.
\]

While
\[
[2u_1(v_1 - v_2) - (v_1^2 - v_2^2)]\partial_x^2 f_n = (1 - \partial_x)g_n(2f_n + g_n)\partial_x^2 f_n
= (1 - \partial_x)g_n(1 + \partial_x)g_n\partial_x^2 f_n + 2(1 - \partial_x)g_n\partial_x f_n\partial_x^2 f_n,
\]
using product law (2) in Lemma 2.1, we have that
\[
\|(1 - \partial_x)g_n\partial_x f_n\partial_x^2 f_n\|_{B^{p,r}_{-1}} \lesssim \|(1 - \partial_x)g_n\partial_x f_n\|_{B^{p,r}_{-1}}\|\partial_x^2 f_n\|_{L^\infty} + \|(1 - \partial_x)g_n\partial_x f_n\|_{L^\infty}\|\partial_x^2 f_n\|_{B^{p,r}_{-1}} \lesssim 2^{-\frac{n}{4}} \cdot 2^n \cdot 2^\alpha(2-s) + 2^{-\frac{n}{4}} \cdot 2^{n(1-s)} \cdot 2^n \lesssim 2^{-n(s-\frac{3}{4})}.
\]

Taking the above estimates into (4.3), we find that
\[
\|S_t(v^{1,n}(0, x)) - S_t(v^{2,n}(0, x))\|_{B^{p,r}_{-1}} \geq t\|(1 - \partial_x)g_n(1 + \partial_x)g_n\partial_x^2 f_n\|_{B^{p,r}_{-1}} - C2^{-\frac{n}{4}} - Ct^2. \quad (4.4)
\]
For the term \((1 - \partial_x)g_n(1 + \partial_x)g_n\partial_x^2 f_n\), it can be verified that \(\Delta_j((1 - \partial_x)g_n(1 + \partial_x)g_n\partial_x^2 f_n) = 0, j \neq n\) and \(\Delta_n((1 - \partial_x)g_n(1 + \partial_x)g_n\partial_x^2 f_n) = (1 - \partial_x)g_n(1 + \partial_x)g_n\partial_x^2 f_n\) for \(n \geq 5\). Direct calculation shows that

\[
\left\| (1 - \partial_x)g_n(1 + \partial_x)g_n\partial_x^2 f_n \right\|_{B^2_{p,r}} = 2^n(1 + 2^n x) + \frac{17}{12} n(1 + \partial_x)\phi \phi \cos(\frac{17}{12} 2^n x) + \frac{17}{12} (1 - \partial_x)\phi \phi \sin(\frac{17}{12} 2^n x)\right\|_{L^p},
\]

by the Riemann Theorem, where \((1 - \partial_x)\phi (1 + \partial_x)\phi \phi \phi \phi \triangleq \psi(x), which together with (4.4) yield

\[
\lim \inf_{n \to \infty} \left\| S_t'(v^{1,n}(0, x)) - S_t(v^{2,n}(0, x)) \right\|_{B^2_{p,r}} \geq t \quad for \ t \small{small enough.}
\]

That is to say, the solution map \(v_0 \to S_t(v_0)\) of the initial value problem (3.2) depends not uniformly continuous on initial data in \(B^2_{p,r}\).

Since \(u = (1 - \partial_x)^{-1}v\) and \((1 - \partial_x)^{-1}\) is an isomorphic mapping from \(B^2_{p,r}(\mathbb{R})\) into \(B^p_{p,r}(\mathbb{R})\), hence the non-uniform continuous dependence of \(v\) in \(B^2_{p,r}\) is consistent with that of \(u\) in \(B^p_{p,r}\).

Thus, this completes the proof of Theorem 1.1.

**Acknowledgments**

The author is very grateful to Dr. Jinlu Li for some useful suggestions. This work is partially supported by the National Natural Science Foundation of China (Grant No.11801090).

**References**

[1] H. Bahouri, J. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations. Springer-Verlag, Berlin, 2011.

[2] X. Chang, J. Szmigielski, Lax integrability and the peakon problem for the modified Camassa-Holm equation, Comm. Math. Phys. 358 (2018) 295-341.

[3] A. Fokas, On a class of physically important integrable equations, Physica D 87 (1995) 145-150.

[4] Y. Fu, G. Gui, Y. Liu, C. Qu, On the Cauchy problem for the integrable modified Camassa-Holm equation with cubic nonlinearity, J Differ Equ. 255 (2013) 1905-1938.

[5] B. Fuchssteiner, Some tricks from the symmetry toolbox for nonlinear equations: generalisations of the Camassa-Holm equation, Physica D 95 (1996) 229-243.

[6] G. Gui, Y. Liu, P. Olver, C. Qu, Wave-Breaking and Peakons for a Modified Camassa-Holm Equation, Commun. Math. Phys. 319 (2013) 731-759.
[7] A. Himonas, G. Misiołek, High-frequency smooth solutions and well-posedness of the Camassa-Holm equation, Int. Math. Res. Not. 51 (2005) 3135-3151.

[8] A. Himonas, C. Kenig, Non-uniform dependence on initial data for the CH equation on the line, Diff. Integr. Equ. 22 (2009) 201-224.

[9] A. Himonas, C. Kenig, G. Misiołek, Non-uniform dependence for the periodic CH equation, Commun. Partial Differ. Equ. 35 (2010) 1145-1162.

[10] A. Himonas, C. Holliman, The Cauchy problem for the Novikov equation, Nonlinearity 25 (2012) 449-479.

[11] A. Himonas, D. Mantzavinos, The Cauchy problem for the Fokas-Olver-rosenau-Qiao equation, Nonlinear Anal. 95 (2014) 499-529.

[12] A. Himonas, D. Matzavinos, Hölder continuity for the Fokas-Olver-Rosenau-Qiao equation, J Nonlinear Sci. 24 (2014) 1105-1124.

[13] A. Himonas, C. Holliman, Non-uniqueness for the Fokas-Olver-Rosenau-Qiao equation, J. Math. Anal. Appl. 470 (2019) 617-633.

[14] C. Kenig, G. Ponce, L. Vega, On the ill-posedness of some canonical dispersive equations, Duke Math. 106 (2001) 617-633.

[15] J. Li, Z. Yin, Well-posedness and analytic solutions of the two-component Euler-Poincaré system, Monatsh. Math. 183 (2017) 509–537.

[16] J. Li, Y. Yu, W. Zhu, Non-uniform dependence on initial data for the Camassa-Holm equation in Besov spaces, J. Differ. Equ. 269 (2020) 8686-8700.

[17] J. Li, M. Li, W. Zhu, Non-uniform dependence for Novikov equation in Besov spaces, 2020, J. Math. Fluid Mech. 22 (2020) 4:50.

[18] X. Liu, Y. Liu, C. Qu, Orbital stability of the train of peakons for an integrable modified Camassa-Holm equation, Adv Math. 255 (2014) 1-37.

[19] P.J. Olver, P. Rosenau, Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support, Phys. Rev. E 53 (1996) 1900-1906.

[20] Z. Qiao, A new integrable equation with cuspons and W/M-shape-peaks solitons, J. Math. Phys. 47 (2006) 112701.

[21] Z. Qiao, New integrable hierarchy, its parametric solutions, cuspons, one-peak solitons, and M/W shape peak solitons, J. Math. Phys. 48 (2007) 082701.

[22] C. Qu, X. Liu, Y. Liu, Stability of peakons for an integrable modified Camassa-Holm equation with cubic nonlinearity, Comm. Math. Phys. 322 (2013) 967-997.