ON THE FEKETE-SZEGÖ PROBLEM FOR CONCAVE UNIVALENT FUNCTIONS

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Abstract. We consider the Fekete-Szegö problem with real parameter \( \lambda \) for the class \( Co(\alpha) \) of concave univalent functions.

1. Introduction

Let \( S \) denote the class of all univalent (analytic) functions

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

defined on the unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \). Then the classical Fekete-Szegö inequality, presented by means of Loewner’s method, for the coefficients of \( f \in S \) is that

\[
|a_3 - \lambda a_2^2| \leq 1 + 2 \exp(-2\lambda/(1 - \lambda)) \quad \text{for} \; \lambda \in [0, 1).
\]

As \( \lambda \to 1^- \), we have the elementary inequality \( |a_3 - a_2^2| \leq 1 \). Moreover, the coefficient functional

\[
\Lambda_\lambda(f) = a_3 - \lambda a_2^2
\]
on the normalized analytic functions \( f \) in the unit disk \( \mathbb{D} \) plays an important role in function theory. For example, the quantity \( a_3 - a_2^2 \) represents \( S_f(0)/6 \), where \( S_f \) denotes the Schwarzian derivative \( (f''/f')' - (f''/f')^2/2 \) of locally univalent functions \( f \) in \( \mathbb{D} \). In the literature, there exists a large number of results about inequalities for \( \Lambda_\lambda(f) \) corresponding to various subclasses of \( S \). The problem of maximizing the absolute value of the functional \( \Lambda_\lambda(f) \) is called the Fekete-Szegö problem. In [8], Koepf solved the Fekete-Szegö problem for close-to-convex functions and the largest real number \( \lambda \) for which \( \Lambda_\lambda(f) \) is maximized by the Koebe function \( z/(1 - z)^2 \) is \( \lambda = 1/3 \), and later in [9] (see also [10]), this result was generalized for functions that are close-to-convex of order \( \beta \). In [12], Pfuger employed the variational method to give another treatment of the Fekete-Szegö inequality which includes a description of the image domains under extremal functions. Later, Pfuger [13] used Jenkin’s method to show that

\[
|\Lambda_\lambda(f)| \leq 1 + 2|\exp(-2\lambda/(1 - \lambda))|, \; f \in S,
\]

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holds for complex \( \lambda \) such that\( \Re \left( \frac{1}{1 - \lambda} \right) \geq 1 \). The inequality is sharp if and only if \( \lambda \) is in a certain pear shaped subregion of the disk given by

\[
\lambda = 1 - (u + itv)/(u^2 + v^2), \quad 1 \leq t \leq 1,
\]

where \( u = 1 - \log(\cos \phi) \) and \( v = \tan \phi - \phi, \quad 0 < \phi < \pi/2 \).

In this paper, we solve the Fekete-Szegö problem for functions in the class \( Co(\alpha) \) of concave univalent functions, with real parameter \( \lambda \).

2. Preliminaries

A function \( f : \mathbb{D} \to \mathbb{C} \) is said to belong to the family \( Co(\alpha) \) if \( f \) satisfies the following conditions:

(i) \( f \) is analytic in \( \mathbb{D} \) with the standard normalization \( f(0) = f'(0) - 1 = 0 \). In addition it satisfies \( f(1) = \infty \).

(ii) \( f \) maps \( \mathbb{D} \) conformally onto a set whose complement with respect to \( \mathbb{C} \) is convex.

(iii) the opening angle of \( f(D) \) at \( \infty \) is less than or equal to \( \pi \alpha \), \( \alpha \in (1, 2] \).

This class has been extensively studied in the recent years and for a detailed discussion about concave functions, we refer to [1, 2, 6] and the references therein. We note that for \( f \in Co(\alpha), \ \alpha \in (1, 2] \), the closed set \( \mathbb{C} \setminus f(\mathbb{D}) \) is convex and unbounded.

We recall the analytic characterization for functions in \( Co(\alpha) \), \( \alpha \in (1, 2] \): \( f \in Co(\alpha) \) if and only if \( \Re P_f(z) > 0 \) in \( \mathbb{D} \), where

\[
P_f(z) = \frac{2}{\alpha - 1} \left[ \frac{(\alpha + 1) 1 + z}{2} - 1 - z \frac{f''(z)}{f'(z)} \right].
\]

In [4], we have used this characterization and proved the following theorem which will be used to prove our result.

Theorem A. Let \( \alpha \in (1, 2] \). A function \( f \in Co(\alpha) \) if and only if there exists a starlike function \( \phi \in \mathcal{S}^* \) such that \( f(z) = \Lambda_\phi(z) \), where

\[
\Lambda_\phi(z) = \int_0^z \frac{1}{(1 - t)^{\alpha+1}} \left( \frac{t}{\phi(t)} \right)^{(\alpha-1)/2} dt.
\]

We also recall a lemma due to Koepf [8, Lemma 3].

Lemma A. Let \( g(z) = z + b_2z + b_3z^2 + \cdots \in \mathcal{S}^* \). Then \( |b_3 - \lambda b_2^2| \leq \max \{1, |3 - 4\lambda| \} \) which is sharp for the Koebe function \( k \) if \( |\lambda - 3/4| \geq 1/4 \) and for \( (k(z^2))^{1/2} = \frac{z}{1-z} \) if \( |\lambda - 3/4| \leq 1/4 \).

Here \( \mathcal{S}^* \) denote the family of functions \( g \in \mathcal{S} \) that map \( \mathbb{D} \) into domains that are starlike with respect to the origin. Each \( g \in \mathcal{S}^* \) is characterized by the condition \( \Re (zg'(z)/g(z)) > 0 \) in \( \mathbb{D} \). Ma and Minda [11] presented the Fekete-Szegö problem for more general classes through subordination, which includes the classes of starlike and convex functions, respectively. In a recent paper, the authors in [5] obtained a new method of solving the Fekete-Szegö problem for classes of close-to-convex functions defined in terms of subordination.
3. Main Result and its Proof

We recall from Theorem A that \( f \in Co(\alpha) \) if and only if there exists a function \( \phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n \in \mathcal{S}^* \) such that

\[
(3.1) \quad f'(z) = \frac{1}{(1-z)^{\alpha+1}} \left( \frac{z}{\phi(z)} \right)^{\frac{\alpha-1}{2}},
\]

where \( f \) has the form given by (1.1). Comparing the coefficients of \( z \) and \( z^2 \) on the both sides of the series expansion of (3.1), we obtain that

\[
a_2 = \frac{\alpha + 1}{2} - \frac{\alpha - 1}{4} \phi_2, \quad \text{and} \quad a_3 = \frac{(\alpha + 1)(\alpha + 2)}{6} - \frac{\alpha^2 - 1}{6} \phi_2 - \frac{\alpha - 1}{6} \phi_3 + \frac{\alpha^2 - 1}{24} \phi_2^2,
\]

respectively. A computation yields,

\[
a_3 - \lambda a_2^2 = \frac{(\alpha + 1)^2}{4} \left( \frac{2(\alpha + 2)}{3(\alpha + 1)} - \lambda \right) + \frac{\alpha^2 - 1}{4} \left( \frac{2}{3} - \frac{\lambda}{3} \right) \phi_2
\]

\[
- \frac{\alpha - 1}{6} \left[ \phi_3 - \left( \frac{2(\alpha + 1) - 3\lambda(\alpha - 1)}{8} \right) \phi_2^2 \right].
\]

**Case (1):** Let \( \lambda \in \left(-\infty, \frac{2(\alpha - 3)}{3(\alpha - 1)}\right) \). We observe that the assumption on \( \lambda \) is seen to be equivalent to

\[
\frac{2(\alpha + 1) - 3\lambda(\alpha - 1)}{8} \geq 1
\]

and the first term in the last expression is nonnegative. Hence, using Lemma A for the last term in (3.2), and noting that \( |\phi_2| \leq 2 \), we have from the equality (3.2),

\[
|a_3 - \lambda a_2^2| \leq \frac{(\alpha + 1)^2}{4} \left( \frac{2(\alpha + 2)}{3(\alpha + 1)} - \lambda \right) + \frac{\alpha^2 - 1}{4} \left( \frac{2}{3} - \frac{\lambda}{3} \right) |\phi_2|
\]

\[
+ \frac{\alpha - 1}{6} \left| \phi_3 - \left( \frac{2(\alpha + 1) - 3\lambda(\alpha - 1)}{8} \right) \phi_2^2 \right|
\]

\[
\leq \frac{(\alpha + 1)^2}{4} \left( \frac{2(\alpha + 2)}{3(\alpha + 1)} - \lambda \right) + \frac{\alpha^2 - 1}{2} \left( \frac{2}{3} - \frac{\lambda}{3} \right)
\]

\[
+ \frac{\alpha - 1}{6} \left( \frac{2(\alpha + 1) - 3\lambda(\alpha - 1)}{2} - 3 \right).
\]

Thus, simplifying the right hand expression gives

\[
(3.3) \quad |a_3 - \lambda a_2^2| \leq \frac{2\alpha^2 + 1}{3} - \lambda \alpha^2, \quad \text{if} \quad \lambda \in \left(-\infty, \frac{2(\alpha - 3)}{3(\alpha - 1)}\right).
\]

**Case (2):** Let \( \lambda \geq \frac{2(\alpha + 2)}{3(\alpha + 1)} \) so that the first term in (3.2) is nonpositive. The condition on \( \lambda \) in particular gives \( \lambda > 2/3 \) and therefore, our assumption on \( \lambda \)
implies that
\[
\frac{2(\alpha + 1) - 3\lambda(\alpha - 1)}{8} < \frac{1}{2}
\]
Again, it follows from Lemma A that
\[
\left| \phi_3 - \left( \frac{2(\alpha + 1) - 3\lambda(\alpha - 1)}{8} \right) \phi_2^2 \right| \leq 3 - \frac{2(\alpha + 1) - 3\lambda(\alpha - 1)}{2}
\]
In view of these observations and an use of the inequality \(|\phi_2| \leq 2\), the equality (3.2) gives
\[
|a_3 - \lambda a_2^2| \leq \frac{(\alpha + 1)^2}{4} \left( \frac{2(\alpha + 2)}{3(\alpha + 1)} - \lambda \right) - \frac{\alpha^2 - 1}{2} \left( \frac{2}{3} - \lambda \right)
\]
\[
+ \frac{\alpha - 1}{6} \left( 3 - \frac{2(\alpha + 1) - 3\lambda(\alpha - 1)}{2} \right)
\]
Thus, simplifying the right hand expression gives
\[
|a_3 - \lambda a_2^2| \leq \frac{2\alpha^2 + 1}{3}, \quad \text{if} \quad \lambda \geq \frac{2(\alpha + 2)}{3(\alpha + 1)}.
\]
The inequalities in both cases are sharp for the functions
\[
f(z) = \frac{1}{2\alpha} \left[ \left( \frac{1 + z}{1 - z} \right)^{\alpha} - 1 \right].
\]
**Case (3):** To get the complete solution of the Fekete-Szegö problem, we need to consider the case
\[
\lambda \in \left( \frac{2(\alpha - 3)}{3(\alpha - 1)}, \frac{2(\alpha + 2)}{3(\alpha + 1)} \right).
\]
Now, we deal with this case by using the formulas (3.1) and (3.2) together with the representation formula for \( \phi \in S^* \):
\[
z\phi'(z) \phi(z) = \frac{1 + z\omega(z)}{1 - z\omega(z)},
\]
where \( \omega : \mathbb{D} \to \mathbb{D} \) is a function analytic in \( \mathbb{D} \) with the Taylor series
\[
\omega(z) = \sum_{n=0}^{\infty} c_n z^n.
\]
Inserting the resulting formulas
\[
\phi_2 = 2c_0 \quad \text{and} \quad \phi_3 = c_1 + 3c_0^2
\]
into (3.2) yields
\[
a_3 - \lambda a_2^2 = \frac{(\alpha + 1)^2}{4} \left( \frac{2(\alpha + 2)}{3(\alpha + 1)} - \lambda \right) + \frac{\alpha^2 - 1}{2} \left( \frac{\lambda - 2}{3} \right) c_0
\]
\[
- \frac{\alpha - 1}{6} \left[ c_1 + \frac{4 - 2\alpha + 3\lambda(\alpha - 1)}{2} c_0 \right]
\]
\[
=: A + Bc_0 + Cc_0^2 + Dc_1,
\]
Fekete-Szegö problem

where
\[
\begin{aligned}
A &= \frac{(\alpha + 1)(\alpha + 2)}{6} - \frac{\lambda(\alpha + 1)^2}{4}, \\
B &= (\alpha^2 - 1) \left(\frac{\lambda}{2} - \frac{1}{3}\right), \\
C &= -\frac{(\alpha - 1)(4 - 2\alpha + 3\lambda(\alpha - 1))}{12}, \\
D &= -\frac{\alpha - 1}{6}.
\end{aligned}
\]

It is well-known that $|c_0| \leq 1$ and $|c_1| \leq 1 - |c_0|^2$. Using this we obtain,
\[
|a_3 - \lambda a_2^2| = |A + Bc_0 + Cc_0^2 + Dc_1| \\
\leq |A + Bc_0 + Cc_0^2| + |D||c_1| \\
\leq |A + Bc_0 + Cc_0^2| + |D|(1 - |c_0|^2).
\]

Let $c_0 = re^{i\theta}$. First we search for the maximum of $|A + Bc_0 + Cc_0^2|$ where we fix $r$ and vary $\theta$. To this end, we consider the expression
\[
|A + Bc_0 + Cc_0^2|^2 = |A + Br e^{i\theta} + Cr^2 e^{2i\theta}|^2 = (A - Cr^2)^2 + B^2 r^2 + (2ABr + 2BCr^3)\cos \theta + 4ACr^2(\cos \theta)^2 =: f(r, \theta).
\]
Afterwards, we have to find the biggest value of the maximum function, if $r$ varies in the interval $(0,1]$.

We need to deal with several subcases of (3.6).

**Case A:** Let $\lambda \in \left(\frac{2(\alpha - 3)}{3(\alpha - 1)}, \frac{2(\alpha - 2)}{3(\alpha - 1)}\right)$. We observe that $C > 0$, $B < 0$, and $A + Cr^2 > 0$ for $r \in [0, 1]$. Hence the corresponding quadratic function
\[
h(x) = (A - Cr^2)^2 + B^2 r^2 + 2Br(A + Cr^2)x + 4ACr^2x^2; \ x \in [-1, 1],
\]
attains its maximum value for any $r \in (0, 1]$ at $x = -1$. Therefore, our task is to find the maximum value of
\[
g(r) = A - Br + Cr^2 + \frac{\alpha - 1}{6}(1 - r^2).
\]
The inequalities $g'(0) = -B$ and
\[
g'(1) = -B + 2C - \frac{\alpha - 1}{3} = \frac{\alpha - 1}{6}(-6\lambda + 4(\alpha - 1)) > 0
\]
for $\lambda < \frac{2(\alpha - 1)}{3\alpha}$ imply
\[
g(r) \leq g(1) = A - B + C = \frac{2\alpha^2 + 1}{3} - \lambda\alpha^2.
\]

**Case B:** If $\lambda = \frac{2(\alpha - 2)}{3(\alpha - 1)}$, then $C = 0$ and $h$ is a linear function that has its maximum value at $x = -1$. The considerations of Case A apply and again we get the maximum value $g(1)$ as above.
**Case C:** Let $\lambda \in \left( \frac{2(\alpha-2)}{3(\alpha-1)}, \frac{2(\alpha-1)}{3\alpha} \right)$. Firstly, we prove that in this interval the quadratic function $h$ is monotonic decreasing for $x \in [-1, 1]$. Since the function $h : \mathbb{R} \to \mathbb{R}$ has its maximum at

$$x(r) = \frac{-B(A + Cr^2)}{4ACr} = \frac{-B}{4} \left( \frac{1}{Cr} + \frac{r}{A} \right),$$

it is sufficient to show that $x(r)$ is monotonic increasing and $x(1) < -1$. The first assertion is trivial and the second one is equivalent to

$$j(\lambda) = \alpha^2(3\lambda - 2)^2 - 4 + 3\lambda > 0$$

for the parameters $\lambda$ in question. This inequality is easily proved. Hence, we get the same upper bound as in Cases A and B. In conclusion, Cases A, B and C give,

$$|a_3 - \lambda a_2^2| \leq \frac{2\alpha^2 + 1}{3} - \lambda\alpha^2, \text{ if } \lambda \in \left( \frac{2(\alpha-3)}{3(\alpha-1)}, \frac{2(\alpha-1)}{3\alpha} \right).$$

**Case D:** Let $\lambda \in \left[ \frac{2(\alpha-1)}{3\alpha}, \frac{2}{3} \right)$ and we may factorize $j(\lambda)$ as

$$j(\lambda) = 9\alpha^2(\lambda - \lambda_1)(\lambda - \lambda_2),$$

where

$$\lambda_1 = \frac{4\alpha^2 - 1 - \sqrt{8\alpha^2 + 1}}{6\alpha^2} \quad \text{and} \quad \lambda_2 = \frac{4\alpha^2 - 1 + \sqrt{8\alpha^2 + 1}}{6\alpha^2}.$$ 

We observe that $\lambda_2 > \lambda_1$. For $\lambda \in \left[ \frac{2(\alpha-1)}{3\alpha}, \lambda_1 \right)$, the function $h$ has its maximum value at $x = -1$ and the function $g$ has its maximum value at

$$r_m = \frac{-B}{-2C + \alpha - \frac{1}{3}} \in (0, 1].$$

Hence, the maximum of the Fekete-Szegő functional is

$$g(r_m) = \frac{\alpha(10 - 9\lambda) - (3\lambda - 2)}{9(2 - \lambda) + 3\alpha(3\lambda - 2)}.$$ 

For $\lambda \in [\lambda_1, \frac{2}{3})$, the number

$$r_0 = \frac{B}{2C \left( 1 + \sqrt{1 - \frac{B^2}{4AC}} \right)} \in (0, 1]$$

is the unique solution of $x(r) = -1$ in the interval $(0, 1]$. It is easily seen that $r_m < r_0$ for $\lambda < \frac{2}{3}$. Further,

$$k(r) = \sqrt{h(x(r))} + \frac{\alpha - 1}{6}(1 - r^2) = (A - Cr^2)\sqrt{1 - \frac{B^2}{4AC} + \alpha - 1}(1 - r^2)$$

is monotonic decreasing for $r \geq r_0$. Hence, the maximum value of $|a_3 - \lambda a_2^2|$ is $g(r_m)$ in this part of the interval in question, too. The extremal function maps $D$ onto a wedge shaped region with an opening angle at infinity less than $\pi\alpha$ and one finite vertex as in Example 3.12 in \cite{3}. 


**Case E:** For $\lambda = \frac{2}{3}$, we have $B = 0$ and $C = -\frac{\alpha - 1}{6}$. Hence the maximum is attained for $\cos \theta = 0$ and any $r \in (0, 1]$. In all these cases, we get

$$|a_3 - \lambda a_2^2| = \frac{\alpha}{3}$$

as the sharp upper bound. The extremal functions map $\mathbb{D}$ onto a region with an opening angle at infinity equal to $\pi \alpha$ and two finite vertices as in Example 3.13 in [3].

In conclusion, Cases D and E give,

$$|a_3 - \lambda a_2^2| \leq \frac{\alpha(10 - 9\lambda) - (3\lambda - 2)}{9(2 - \lambda) + 3\alpha(3\lambda - 2)}, \text{ if } \lambda \in \left(\frac{2(\alpha - 1)}{3\alpha}, \frac{2}{3}\right].$$

**Case F:** Let $\lambda \in (\frac{2}{3}, \lambda_2]$. Since $B > 0$, the function $x(r)$ is monotonic decreasing now. The number

$$r_1 = \frac{B}{-2C \left(1 + \sqrt{1 - \frac{B^2}{4AC}}\right)} \in (0, 1]$$

is the unique solution of the equation $x(r) = 1$ lying in (0,1]. For $r < r_1$, we have $h(x) \leq h(1)$. We consider the function

$$l(r) = A + Br + Cr^2 + \frac{\alpha - 1}{6}(1 - r^2)$$

and determine the maximum of this function to be attained at

$$r_n = \frac{B}{-2C + \frac{\alpha - 1}{3}}.$$ 

It is easily proved that $r_n > r_1$. Since $k(r)$ is monotonic increasing, we get the maximum value of the Fekete-Szegő functional in this case as

$$k(1) = (A - C)\sqrt{1 - \frac{B^2}{4AC}} = \alpha(1 - \lambda)\sqrt{\frac{12(1 - \lambda)}{(4 - 3\lambda)^2 - \alpha^2(3\lambda - 2)^2}},$$

which is attained for $c_0 = e^{i\theta_0}$, where

$$\cos \theta_0 = \frac{-B(A + C)}{4AC}.$$

In this case, the extremal function $f$ is defined by the solution of the following complex differential equation

$$f'(z) = \frac{(1 - ze^{i\theta_0})^{\alpha - 1}}{(1 - z)^{\alpha + 1}}.$$

In conclusion, in this case, we have,

$$|a_3 - \lambda a_2^2| \leq \alpha(1 - \lambda)\sqrt{\frac{12(1 - \lambda)}{(4 - 3\lambda)^2 - \alpha^2(3\lambda - 2)^2}}, \text{ if } \lambda \in (2/3, \lambda_2].$$
Case G: Let $\lambda \in \left(\lambda_2, \frac{2(\alpha+2)}{3(\alpha+1)}\right)$. Since $x(1) < -1$ for these $\lambda$, the number

$$r_2 = \frac{B}{-2C \left(1 - \sqrt{1 - \frac{B^2}{4AC}}\right)}$$

satisfies $x(r_2) = -1$ and $r_2 \in (0, 1)$. For $r \leq r_2$, we can make similar considerations as in the preceding case, i.e. for $r \leq r_1$ the function $l(r)$ takes the maximum value, and for $r \in (r_1, r_2]$, the function $k(r)$ plays this role. For $r > r_2$, the point $x(r)$ does not lie in the interval $[-1, 1]$. Hence, the maximum in question is attained for $x = -1$ or $x = 1$. We see that $A + C < 0$ and $-A - Cr^2 > 0$ for the values of $\lambda$ that we are considering now, the maximum of (3.6) is attained for $x = -1$, i.e. for $c_0 = -r$. Hence, for $r \in (r_2, 1]$ the maximum function is

$$n(r) = -A + Br - Cr^2 + \frac{\alpha - 1}{6}(1 - r^2).$$

Since

$$-C > \frac{\alpha - 1}{6} \quad \text{and} \quad B > 0,$$

we get $n(r) \leq n(1)$ in the interval in question and hence

$$|a_3 - \lambda a_2^2| \leq n(1) = -A + B - C = \lambda \alpha^2 - \frac{2\alpha^2 + 1}{3}$$

whenever $\lambda \in \left(\lambda_2, \frac{2(\alpha+2)}{3(\alpha+1)}\right)$.

Equations (3.3), (3.4), (3.7), (3.9), (3.10) and Case G give

**Theorem.** For $\alpha \in (1, 2]$, let $f \in \text{Co}(\alpha)$ have the expansion (1.1). Then, we have

$$|a_3 - \lambda a_2^2| \leq \begin{cases} 
\frac{2\alpha^2 + 1}{3} - \lambda \alpha^2 & \text{for } \lambda \in (-\infty, \frac{2(\alpha - 1)}{3\alpha}] \\
\frac{\alpha(10 - 9\lambda) - (3\lambda - 2)}{9(2 - \lambda) + 3\alpha(3\lambda - 2)} & \text{for } \frac{2(\alpha - 1)}{3\alpha} \leq \lambda \leq \frac{2}{3} \\
\alpha(1 - \lambda)\sqrt{\frac{12(1 - \lambda)}{(4 - 3\lambda)^2 - \alpha^2(3\lambda - 2)^2}} & \text{for } \frac{2}{3} \leq \lambda \leq \lambda_2 \\
\lambda \alpha^2 - \frac{2\alpha^2 + 1}{3} & \text{for } \lambda \in [\lambda_2, \infty),
\end{cases}$$

where $\lambda_2$ is given by (3.8). To emphasize the fact that the bound is a continuous function of $\lambda$ for any $\alpha$ we mention two different expressions for the same bound for some values of $\lambda$. The inequalities are sharp.

The Fekete-Szeg"{o} inequalities for functions in the class $\text{Co}(\alpha)$ for complex values of $\lambda$ remain an open problem.

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