CASTELNUOVO–MUMFORD REGULARITY OF
UNPROJECTIONS AND THE EISENBUD–GOTO REGULARITY
CONJECTURE

JUNHO CHOE

Abstract. McCullough and Peeva found sequences of counterexamples to the
Eisenbud–Goto conjecture on the Castelnuovo–Mumford regularity by using
Rees-like algebras, where entries of each sequence have increasing dimensions
and codimensions. In this paper we suggest another method to construct coun-
terexamples to the conjecture with any fixed dimension \( n \geq 3 \) and any fixed
codimension \( e \geq 2 \). Our strategy is an unprojection process and utilizes the
possible complexity of homogeneous ideals with three generators. Furthermore,
our counterexamples exhibit how singularities affect the Castelnuovo–Mumford
regularity.

1. Introduction

Throughout this paper, we work over an algebraically closed field \( \mathbb{k} \) of any char-
acteristic, every variety is always irreducible and reduced, and if its dimension is \( n \),
then it is called an \( n \)-fold for brevity. Let \( X \subseteq \mathbb{P}^r \) be a projective variety which is
nondegenerate, that is, it lies in no smaller subspaces. Write \( S(X) = S(\mathbb{P}^r)/I(X) \)
for the homogeneous coordinate ring and ideal of \( X \subseteq \mathbb{P}^r \).

The Castelnuovo–Mumford regularity, or simply the regularity, is one of the most
important invariants in projective algebraic geometry. It is used to express homo-
logical complexity as follows. Let \( S = S(\mathbb{P}^r) \) be the polynomial ring in \( r+1 \) variables
with the standard grading, and consider a finitely generated graded \( S \)-module \( M \)
with minimal (graded) free resolution

\[
0 \longrightarrow M \longrightarrow F_0 \longrightarrow F_1 \longrightarrow \cdots \longrightarrow F_i \longrightarrow \cdots
\]

where the graded \( S \)-free modules \( F_i \) can be written as

\[
F_i = \bigoplus_{j \in \mathbb{Z}} S^{\beta_{i,j}(M)}(-i-j).
\]

Then the regularity of \( M \) is defined to be

\[
\text{reg } M = \max \{ j \in \mathbb{Z} : \beta_{i,j}(M) \neq 0 \text{ for some } i \},
\]

and the regularity of \( X \subseteq \mathbb{P}^r \), denoted by \( \text{reg } X \), is defined to be that of \( I(X) \).

Note that the regularity also measures cohomological complexity in relation of
the Serre vanishing theorem. Refer to the well-known fact that for a coherent sheaf
$F$ on $\mathbb{P}^r$ if its section module
\[ M = \bigoplus_{j \in \mathbb{Z}} H^0(\mathbb{P}^r, F(j)) \]
with degree pieces $M_j = H^0(\mathbb{P}^r, F(j))$ is finitely generated over $S$, then an integer $q$ satisfies
\[ H^i(\mathbb{P}^r, F(q - i)) = 0 \quad \text{for every } i > 0 \]
precisely when $q \geq \text{reg } M$.

The *Eisenbud–Goto regularity conjecture*, or the *regularity conjecture* for short, says that the regularity of $X \subseteq \mathbb{P}^r$ has an upper bound in terms of the degree and codimension of $X$ as follows.

**Conjecture 1.1** (Eisenbud–Goto [EG84]). Let $X \subseteq \mathbb{P}^r$ be any nondegenerate projective variety. Then we have
\[ \text{reg } X \leq \deg X - \text{codim } X + 1. \]

This conjecture holds true for
1. any projective variety $X \subseteq \mathbb{P}^r$ such that $S(X)$ is Cohen–Macaulay [EG84],
2. any curves [Cas93, GLP83],
3. smooth or mildly singular surfaces over $\mathbb{C}$ [Pin86, Laz87, Niu15], and
4. smooth or mildly singular threefolds in $\mathbb{P}^5$ over $\mathbb{C}$ [Kwa99, NP22].

For other related results see for example [Ran90, Kwa00, Nom14, KP20].

But counterexamples to the regularity conjecture have been found by McCullough and Peeva [MP18]. Their method uses both the Rees-like algebra and the existence of homogeneous (nonprime) ideals having extremely big regularity. In fact, McCullough and Peeva showed that for any univariate (real) polynomial $p$, one can find a nondegenerate projective variety $X \subseteq \mathbb{P}^r$ of large dimension and codimension such that
\[ \text{reg } X > p(\deg X). \]

Moreover, they presented a computational example which is a nondegenerate singular projective threefold in $\mathbb{P}^5$ of degree 31 and regularity 38.

Our main result is an *unprojection* process that is able to construct a collection of counterexamples to the regularity conjecture with any fixed dimension $n \geq 3$ and any fixed codimension $e = r - n \geq 2$, based on Green’s partial elimination theory [Gre98]. Our method works due to the possible complexity of homogeneous ideals with three generators.

Now let us state our main theorem. We need the following asymptotic notation. If $(a_d)$ and $(b_d)$ are sequences of positive numbers, then we write
\[ a_d = \Omega(b_d) \quad \text{when} \quad 0 < \lim \inf_{d \to \infty} \frac{a_d}{b_d} \leq \infty. \]

**Theorem 1.2.** Fix integers $n \geq 3$ and $e \geq 2$. Then there exists a sequence
\[ (X_d \subseteq \mathbb{P}^{n+e} : d = 2, 3, \ldots) \]
of nondegenerate projective $n$-folds in $\mathbb{P}^{n+e}$ such that $\lim_{d \to \infty} \deg X_d = \infty$, and
\[ \text{reg } X_d = \Omega((\deg X_d)^k) \quad \text{with} \quad k = \left\lfloor \frac{n + 1}{2} \right\rfloor. \]
This paper is organized as follows. In Section 2 we introduce the partial elimination theory and the unprojection. In Section 3 we give preliminary results to compute significant invariants such as degree, regularity, and projective dimension. In Section 4 we list certain families of homogeneous ideals whose regularity values (seem to) rapidly increase. Some of them become ingredients for the proof of the main theorem. In Section 5 we finally prove the main theorem, and a specific counterexample to the regularity conjecture is given in detail.

The following assumptions and notations are used.

- $X \subseteq \mathbb{P}^r$ and $Y \subseteq \mathbb{P}^{r-1}$ are nondegenerate projective varieties.
- $S(X) = S(\mathbb{P}^r)/I(X)$ and $S(Y) = S(\mathbb{P}^{r-1})/I(Y)$ are homogeneous coordinate rings and homogeneous ideals of $X \subseteq \mathbb{P}^r$ and $Y \subseteq \mathbb{P}^{r-1}$, respectively.
- We simply write $S = S(\mathbb{P}^r)$ and $R = S(\mathbb{P}^{r-1})$.
- $V(I)$ is the projective scheme defined by a homogeneous ideal $I$.
- For a graded module or ring $M$ we denote by $M_j$ its piece in degree $j$.
- The missing entries in matrices and tables are usually assumed to be zero.

2. Projection and Unprojection

Unprojection is the reverse process of projection. In this section we outline projections from a point and associated unprojections from the perspective of Green’s partial elimination theory [Gre98]. For another approach to unprojections consult with [Rei00].

2.1. Partial elimination theory. Fix a point $z \in \mathbb{P}^r$ inside or outside $X \subseteq \mathbb{P}^r$, and consider the projection $\pi_z : \mathbb{P}^r \setminus z \to \mathbb{P}^{r-1}$ from $z$. In order to analyze geometry of the restricted map $\pi_z|_X : X \setminus z \to \mathbb{P}^{r-1}$ Green introduced partial elimination ideals [Gre98]. We collect some information on them below and refer to [CK22 Appendix] for their applications.

Taking suitable homogeneous coordinates $x_0, x_1, \ldots, x_r \in S_1$ in $\mathbb{P}^r$ we may assume that the point $z \in \mathbb{P}^r$ is given as

$$(x_0 : x_1 : \cdots : x_r) = (1 : 0 : \cdots : 0).$$

One thus has $\pi_z(x_0 : x_1 : \cdots : x_r) = (x_1 : \cdots : x_r) \in \mathbb{P}^{r-1}$. Set $R = \mathbb{k}[x_1, \ldots, x_r] \subseteq S$ so that it forms the homogeneous coordinate ring of $\mathbb{P}^{r-1}$ and satisfies $R[x_0] = S$. For an integer $i$ and the graded $R$-module

$$K_i(X, z) = \{f \in I(X) : f \text{ has degree at most } i \text{ in } x_0\}$$

the $i$-th partial elimination ideal of the pair $(X, z)$ is defined to be

$$K_i(X, z) = \{\text{coefficients of } x_0^i \text{ in any } f \in K_i(X, z)\} \subseteq R.$$

Note that $K_0(X, z)$ is nothing but the elimination ideal $I(X) \cap R$ of $I(X) \subseteq S$. Partial elimination ideals are homogeneous ideals that fit into both the short exact sequence

$$(2.1) \quad 0 \longrightarrow K_{i-1}(X, z) \longrightarrow K_i(X, z) \longrightarrow K_i(X, z)(-i) \longrightarrow 0$$

and the chain

$$(2.2) \quad K_0(X, z) \subseteq K_1(X, z) \subseteq \cdots \subseteq K_i(X, z) \subseteq \cdots.$$
We write \( X_z = V(K_0(X, z)) \subseteq \mathbb{P}^{r-1} \). Then since \( K_0(X, z) \) is prime without linear forms, the projective scheme \( X_z \subseteq \mathbb{P}^{r-1} \) is a nondegenerate projective variety with \( I(X_z) = K_0(X, z) \), and one can see that \( X_z \) has support
\[
X_z = \overline{\pi_z(X \setminus z)} \subseteq \mathbb{P}^{r-1}.
\]
The induced map
\[
\pi_z|_X : X \setminus z \to X_z
\]
is called the projection of \( X \) from \( z \).

On the other hand, we consider the homogeneous ideal
\[
K_\infty(X, z) = \bigcup_{i=0}^{\infty} K_i(X, z) \subseteq R
\]
to which the chain (2.2) stabilizes. If \( z \in X \), then \( \mathbb{P}C_z X = V(K_\infty(X, z)) \subseteq \mathbb{P}^{r-1} \) is called the projectivized tangent cone to \( X \) at \( z \), and if we forget its embedding in \( \mathbb{P}^{r-1} \), then it describes intrinsic and local geometry of \( X \) at \( z \). For example, one sees that \( \mathbb{P}C_z X = \text{Proj} \bigoplus_{i=0}^{\infty} m_z^i/m_z^{i+1} \) for the maximal ideal \( m_z \) of the local ring of \( X \) at \( z \), and if \( z \) is a smooth point of \( X \), then \( \mathbb{P}C_z X \) is just the projectivized tangent space to \( X \) at \( z \). Note that \( \mathbb{P}C_z X \) is known to be equidimensional with \( \dim \mathbb{P}C_z X = \dim X - 1 \). However, if \( z \notin X \), then we have \( K_\infty(X, z) = R \).

Partial elimination ideals have the following set-theoretic description.

**Proposition 2.1** ([Gre98 Proposition 6.2] and [CK22 Proposition A.2]). Let \( Z_i = V(K_i(X, z)) \) and \( Z_\infty = V(K_\infty(X, z)) \) in \( \mathbb{P}^{r-1} \). Then the support of \( Z_i \) is given as
\[
Z_i = \{ p \in \mathbb{P}^{r-1} : \text{length}(X \cap \pi_z^{-1}(p)) > i \} \cup Z_\infty \subseteq \mathbb{P}^{r-1},
\]
where \( \pi_z^{-1}(p) \cong \mathbb{A}^1 \) is the fiber of \( p \in \mathbb{P}^{r-1} \) under \( \pi_z : \mathbb{P}^r \setminus z \to \mathbb{P}^{r-1} \). In particular if \( z \) is not a vertex of \( X \), then \( \pi_z|_X : X \setminus z \to X_z \) is generically finite of degree
\[
\deg \pi_z|_X = \min\{ i \geq 1 : Z_i \neq X_z \}.
\]

This is related to the well-known degree formula
\[
\deg X = \deg \pi_z|_X \deg X_z + \begin{cases} \deg \mathbb{P}C_z X & \text{if } z \in X \\ 0 & \text{if } z \notin X \end{cases}
\]
for the case where \( \pi_z|_X : X \setminus z \to X_z \) is generically finite.

We specify the easiest but nontrivial case in this regard. First suppose that \( \pi_z|_X : X \setminus z \to X_z \) is birational so that \( K_0(X, z) \neq K_1(X, z) \). However the behavior of the partial elimination ideals \( K_1(X, z) \subseteq K_2(X, z) \subseteq \cdots \subseteq K_\infty(X, z) \) is still mysterious in general. So it would be desirable that they are the same. It is worth noting that this actually happens for interesting classes of nondegenerate projective varieties. See [HK15] and [CK22].

**Definition 2.2.** One says that \( \pi_z|_X \) is simple if it is birational together with \( K_1(X, z) = K_\infty(X, z) \).

Notice that if \( \pi_z|_X \) is simple, then \( z \in \mathbb{P}^r \) lies in \( X \subset \mathbb{P}^r \).
2.2. Unprojection. Now let us introduce the notion of unprojection. It depends on a given fake linear form that plays the role of the so-called unprojection variable.

Let \( Y \subseteq \mathbb{P}^{r-1} \) be a nondegenerate projective variety, and consider the fraction field of \( S(Y) \) that is equal to the function field \( k(\hat{Y}) \) of the affine cone \( \hat{Y} \subseteq \mathbb{A}^r \) of \( Y \subseteq \mathbb{P}^{r-1} \). Now we take account of an element \( \lambda \) in the algebraic closure of \( k(\hat{Y}) \), and assume that the minimal polynomial of \( \lambda \) over \( k(\hat{Y}) \) has the form

\[
\lambda^d + \frac{a_1}{f_1}\lambda^{d-1} + \cdots + \frac{a_d}{f_d},
\]

where \( d \geq 1 \) is the degree of \( \lambda \) over \( k(\hat{Y}) \), and the \( a_i \in S(Y) \) and \( f_i \in S(Y) \setminus 0 \) are homogeneous with \( \deg a_i = \deg f_i + i \) for all \( i = 1, \ldots, d \).

**Definition 2.3.** Let \( Y \) and \( \lambda \) be as above. If \( \lambda \notin S(Y)_1 \), then one says that \( \lambda \) is a fake linear form on \( Y \subseteq \mathbb{P}^{r-1} \). Two fake linear forms on \( Y \subseteq \mathbb{P}^{r-1} \) are called conjugate if so are they as algebraic elements over \( k(\hat{Y}) \).

Fake linear forms induce unprojections in the following sense.

**Theorem 2.4 (Unprojections).** Fix homogeneous coordinates \( x_0, x_1, \ldots, x_r \) on \( \mathbb{P}^r \) so that \( \mathbb{P}^{r-1} \) has homogeneous coordinates \( x_1, \ldots, x_r \) under the projection \( \pi_z : \mathbb{P}^r \setminus z \to \mathbb{P}^{r-1} \) from the point \( z = (1 : 0 : \cdots : 0) \in \mathbb{P}^r \). Then for any nondegenerate projective variety \( Y \subseteq \mathbb{P}^{r-1} \) and any integer \( d \geq 1 \) there is a one-to-one correspondence

\[
\{ X \subseteq \mathbb{P}^r : X_z = Y \subseteq \mathbb{P}^{r-1}, \text{ and } \pi_z|_X : X \setminus z \to Y \text{ has degree } d \}
\]

\[
\leftrightarrow \{ \text{fake linear forms } \lambda \text{ of degree } d \text{ on } Y \subseteq \mathbb{P}^{r-1} \text{ up to conjugacy} \},
\]

where \( X \subseteq \mathbb{P}^r \) stands for a nondegenerate projective variety. In addition under this correspondence if \( \pi_z|_X : X \setminus z \to Y \) is birational, that is, \( d = 1 \), then we have

\[
\frac{K_i(X, z)}{K_0(X, z)} = (f(a, f)^{i-1} : a^i) \subseteq S(Y)
\]

for each \( i \geq 1 \) and any fractional expression \( \lambda = a/f \in k(\hat{Y}) \).

**Proof.** The correspondence maps are sketched as follows.

\[
\begin{align*}
X \subseteq \mathbb{P}^r & \quad \leftrightarrow \quad x_0 \in k(\hat{X}) \\
\text{Proj} S(Y)[\lambda] \subseteq \mathbb{P}^r & \quad \leftrightarrow \quad \lambda
\end{align*}
\]

They are obviously inverses of each other.

Let \( X \subseteq \mathbb{P}^r \) be a nondegenerate projective variety being considered. We claim that the element \( x_0 \in k(\hat{X}) \) is a fake linear form of degree \( d \) on \( Y \subseteq \mathbb{P}^{r-1} \). Indeed since \( \pi_z|_X : X \setminus z \to Y \) is generically finite of degree \( d \), the function field \( k(\hat{X}) \) of the affine cone \( \hat{X} \subseteq \mathbb{A}^{r+1} \) is a degree \( d \) extension of \( k(\hat{Y}) \). Thus, \( x_0 \in k(\hat{X}) \) has degree \( d \) over \( k(\hat{Y}) \), for \( k(\hat{X}) = k(\hat{Y})(x_0) \). Note that \( I(Y) \neq K_d(X, z) \) by Proposition 2.1.

Pick a homogeneous form \( f \in K_d(X, z) \setminus I(Y) \) so that \( fx_0^d + a_1x_0^{d-1} + \cdots + a_d \in I(X) \) for some homogeneous forms \( a_i \in R \) with \( \deg a_i = \deg f + i \). Hence,

\[
x_0^d + \frac{a_1}{f}x_0^{d-1} + \cdots + \frac{a_d}{f}
\]

is the minimal polynomial of \( x_0 \in k(\hat{X}) \) over \( k(\hat{Y}) \). Also since \( X \subseteq \mathbb{P}^r \) is nondegenerate, one sees that \( x_0 \notin S(Y)_1 \).
On the other hand, let $\lambda$ be a fake linear form under discussion. Then consider a graded $R$-algebra $S(Y)[\lambda]$ in the algebraic closure of $k(\hat{Y})$, where $\lambda$ is regarded as a homogeneous element of degree 1 in it, and define a graded $R$-algebra epimorphism

$$\phi : S \to S(Y)[\lambda] \quad \text{by} \quad \phi(x_0) = \lambda.$$  

Notice that $\ker \phi \subset S$ is a homogeneous prime ideal and that it has no linear forms since $\lambda \notin S(Y)_1$. Take $X \subset P^r$ to be the nondegenerate projective variety defined by $\ker \phi$. Evidently we have $K_0(X, z) = I(Y)$, that is, $X_z = Y$. Also $\pi_z|_X : X \setminus z \to Y$ is generically finite of degree $d$, for $k(\hat{X}) = k(\hat{Y})(\lambda)$ is a degree $d$ extension of $k(\hat{Y})$.

For the last assertion let $g \in K_i(X, z)/K_0(X, z)$ be an element so that $g\lambda^i + b_1\lambda^{i-1} + \cdots + b_i = 0$ in $k(\hat{Y})[\lambda]$ for some homogeneous forms $b_i \in S(Y)$ with $\deg b_i = \deg g + i$. Substituting $a/f$ for $\lambda$ and clearing denominators we have $ga^i = -b_1a^{i-1}f \cdots b_i f^i \in f(a, f)^{i-1}$, hence $g \in (f(a, f)^{i-1} : a^i)$. So $K_i(X, z)/K_0(X, z) \subseteq (f(a, f)^{i-1} : a^i)$. Similarly, the reverse containment holds.

For convenience we use the following notation. It is well defined up to projective equivalence.

**Definition 2.5.** After choosing a point $z \in P^r$ if Theorem 2.4 associates a nondegenerate projective variety $X \subset P^r$ to a fake linear form $\lambda$ on $Y \subseteq P^{r-1}$, then we write

$$(X, z) = \text{Unpr}(Y, \lambda)$$

and call it the *unprojection* of the pair $(Y, \lambda)$.

On the other hand, to compute invariants of the unprojections we should usually determine when the partial elimination ideals stabilize, that is, for which values of $i$ the equality $K_i(X, z) = K_{i\infty}(X, z)$ holds. The following helps us to recognize whether an unprojection $(X, z)$ has $\pi_z|_X$ simple.

**Proposition 2.6.** Let $\lambda \in k(\hat{Y})$ be a fake linear form of degree 1 on $Y \subseteq P^{r-1}$ with unprojection $(X, z) = \text{Unpr}(Y, \lambda)$, and fix a fractional expression $\lambda = a/f \in k(\hat{Y})$ with $a, f$ homogeneous. Take $Z_i = V(K_i(X, z)) \subseteq P^{r-1}$ and $W_i = V((a, f)^i) \subseteq Y \subseteq P^{r-1}$ for each $i \geq 1$. With $n = \dim Y$ we set

$$\chi_{n-1}(W) = \begin{cases} \deg W & \text{if } \dim W = n - 1 \\ 0 & \text{if } \dim W < n - 1 \end{cases}$$

for any projective scheme $W \subseteq Y \subseteq P^{r-1}$, and consider the sequence

$$(d_0, d_1, d_2, \ldots) = (0, \chi_{n-1}(W_1), \chi_{n-1}(W_2), \ldots).$$

Then this sequence is convex and eventually linear with slope $\deg f \deg Y - \chi_{n-1}(Z_{\infty})$, where $Z_{\infty} = V(K_{\infty}(X, z)) \subseteq P^{r-1}$. Furthermore, if $S(Y)$ is Cohen–Macaulay, then the following are equivalent.

1. $\pi_z|_X$ is simple.
2. $Z_1$ and $Z_{\infty}$ have the same dimension and degree.
3. $(d_i)$ is linear.
4. $d_i \leq d_i \cdot i$ for all $i \geq 2$.
5. $z \in X$, and $\pi(X) = \pi(Y) + \pi(C_z X) + \deg C_z X - 1$, where $\pi$ means the sectional genus, and $C_z X$ is the cone over $\mathbb{PC}_z X$ with vertex $z$. 

Proof. The formula (2.4) induces the following diagram.

\[
0 \longrightarrow \frac{R}{K_1(X, z)}(-i\deg a) \xrightarrow{a^i} S(Y) \xrightarrow{f(a, f)^{i-1}} S(Y) \xrightarrow{(a, f)^i} 0 \\
0 \longrightarrow S(Y) \xrightarrow{(a, f)^i} S(Y) \xrightarrow{f} S(Y) \xrightarrow{(f)^i} 0
\]

From Hilbert polynomials of the graded \( R \)-modules in this diagram it follows that

\[
\chi_{n-1}(Z_i) - \deg f \deg Y - (d_i - d_{i-1}) = 0
\]

for every integer \( i \geq 1 \). Refer to the fact that \( \chi_{n-1}(Z_1) \geq \chi_{n-1}(Z_2) \geq \cdots \geq \chi_{n-1}(Z_\infty) \geq 0 \).

Therefore, what remains to check is the last assertion.

We prove the equivalence of (1) and (2). Let \( K_1 \) and \( K_\infty \) be the images of \( K_1(X, z) \) and \( K_\infty(X, z) \), respectively, in \( S(Y) \). Since \( K_1 = (f : a) \) by Theorem 2.4, it is unmixed in the Cohen–Macaulay ring \( S(Y) \). Consider the short exact sequence

\[
0 \longrightarrow K_\infty/K_1 \longrightarrow S(Y)/K_1 \longrightarrow S(Y)/K_\infty \longrightarrow 0
\]

and the containment

\[
\text{Ass}_{S(Y)}(K_\infty/K_1) \subseteq \text{Ass}_{S(Y)}(S(Y)/K_1)
\]

of sets of associated primes. One finds that \( K_\infty/K_1 = 0 \) if and only if \( \chi_{n-1}(Z_1) = \chi_{n-1}(Z_\infty) \).

Also, the statements (2), (3), and (4) are equivalent because of (2.5) and the convexity of \( (d_i) \). For the equivalence of (5) and the others refer to a formula [CK22, Theorem A.4(2)]. \( \square \)

3. Regularity and other invariants

This section is devoted to computing invariants by using the partial elimination theory. Let \( M \) be a finitely generated graded \( S \)-module with minimal free resolution

\[
F_0 \longrightarrow F_1 \longrightarrow \cdots \longrightarrow F_i \longrightarrow \cdots \bigoplus_{j \in \mathbb{Z}} S^{\beta_{i,j}(M)}(-i-j)
\]

The exponents \( \beta_{i,j}(M) \) are called graded Betti numbers of \( M \) and encode basic information of generators, relations, and syzygies of \( M \). Notice that by nature of Tor functors we have

\[
\beta_{i,j}(M) = \dim_k \text{Tor}_i^S(M, k)_{i+j},
\]

where \( k \) denotes the residue field of \( S \), and \( \text{Tor}_i^S(M, k) \) can be computed as the (co)homology group of the Koszul type complex

\[
\bigwedge^{i+1} S_1 \otimes M(-i-1) \longrightarrow \bigwedge^i S_1 \otimes M(-i) \xrightarrow{d} \bigwedge^{i-1} S_1 \otimes M(-i+1)
\]
Thus, it suffices to take account of the graded $R_0$-module $M_i(X, z)$ given by the short exact sequence induced by the filtered complex $C_i$. By a well-known Lefschetz type theorem we have

$$\text{Tor}_p^1(I(X), k) \cong \text{Tor}_p^R(I(X)/x_0I(X), k).$$

Thus, it suffices to take account of the graded $R$-module $I(X)/x_0I(X)$.

Now let us see the following Koszul type complex and its ascending filtration:

$$C_i = \bigwedge^i R_1 \otimes \frac{I(X)}{x_0I(X)}(-i) \quad \text{and} \quad F_pC_i = \bigwedge^i R_1 \otimes \frac{K_p(X, z)}{x_0K_{p-1}(X, z)}(-i).$$

This filtration is given by the short exact sequence

$$0 \to \frac{K_{p-1}(X, z)}{x_0K_{p-2}(X, z)} \to \frac{K_p(X, z)}{x_0K_{p-1}(X, z)} \to M_p(X, z) \to 0$$

that comes from the short exact sequence $\mathcal{E}$. Consider the spectral sequence $E$ induced by the filtered complex $C_*$. One has

$$E^0_{p,q} = G_pC_{p+q} \cong \bigwedge^p R_1 \otimes M_p(X, z)(-p-q),$$

$$E^1_{p,q} = H_q(G_pC_{p+q}) = \text{Tor}_p^R(M_p(X, z), k), \quad \text{and} \quad E^\infty_{p,q} = G_pH_{p+q}(C_*) \cong G_p\text{Tor}_p^R(I(X)/x_0I(X), k),$$

where $G$ means the associated graded object. We have constructed the desired spectral sequence.

**Remark 3.3.** Take integers $i$ and $k$, and consider the spectral sequence $E$ above. Then the differential

$$E^1_{k,i-k} = \text{Tor}_i^S(M_k(X, z), k) \to E^1_{k-1,i-k} = \text{Tor}_{i-1}^S(M_{k-1}(X, z), k)$$

is derived from a short exact sequence of the following form.

$$0 \to M_{k-1}(X, z) \to \frac{K_k(X, z)}{x_0K_{k-1}(X, z) + K_{k-2}(X, z)} \to M_k(X, z) \to 0$$
This is connected to the Griffiths–Harris second fundamental form for $X$ at $z$ [GH79]. Refer to [Gre98, Definition 6.12 and Example 6.13].

We suggest a simple trick for partial computation of the spectral sequence above. We first adopt some terminology.

**Definition 3.4.** A table is a $\mathbb{Z} \times \mathbb{Z}$ matrix of nonnegative integers. For instance the Betti table $B(M)$ of a finitely generated graded $S$-module $M$ is a table defined by

$$B(M)_{i,j} = \beta_{i,j}(M).$$

A table sequence is a collection of tables indexed by $\mathbb{Z}$. For a table sequence $B = (B_k : k \in \mathbb{Z})$ its cancellation is any table sequence that can be obtained from $B$ by repeating the following operation a finite number of times. For a tuple $(i, j, k, l)$ of integers with $l > 0$ subtract a common nonnegative integer from both $(B_k)_{i,j}$ and $(B_{k-l})_{i-1,j+1}$.

For a table $B$ its decomposition is a table sequence $(B_k : k \in \mathbb{Z})$ such that for each $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ the entries $(B_k)_{i,j}$ sum up to $B_{i,j}$.

Then the convergence of the spectral sequence above can be roughly described as follows.

**Proposition 3.5** (Cancellation principle). For any fixed point $z \in \mathbb{P}^r$ the Betti table $B(I(X))$ has a decomposition that is a cancellation of the table sequence $(B(M_k(X, z)) : k \in \mathbb{Z})$.

**Proof.** For convenience write

$$K^l_{i,j,k} = (E^l_{k,i-k})_{i+j} \quad \text{and} \quad K^\infty_{i,j,k} = (E^\infty_{k,i-k})_{i+j}$$

for the spectral sequence $E$ in Theorem 3.2. Now the differentials of $E$ induce the complexes

$$\cdots \to K^l_{i+1,j-1,k+l} \to K^l_{i,j,k} \to K^l_{i-1,j+1,k-l} \to \cdots$$

such that

1. the cohomology group at $K^l_{i,j,k}$ is $K^{l+1}_{i,j,k}$,
2. the $K^l_{i,j,k}$ stabilize to $K^\infty_{i,j,k}$ as $l$ increases,
3. $\dim_k K^l_{i,j,k} = \beta_{i,j}(M_k(X, z))$, and
4. $\sum_{k \in \mathbb{Z}} \dim_k K^\infty_{i,j,k} = \beta_{i,j}(I(X))$.

For the tables

$$B^l_k = (\dim_k K^l_{i,j,k} : (i, j) \in \mathbb{Z} \times \mathbb{Z}) \quad \text{and} \quad B^\infty_k = (\dim_k K^\infty_{i,j,k} : (i, j) \in \mathbb{Z} \times \mathbb{Z})$$

consider the table sequences

$$B^l = (B^l_k : k \in \mathbb{Z}) \quad \text{and} \quad B^\infty = (B^\infty_k : k \in \mathbb{Z}).$$

By the above properties of the $K^l_{i,j,k}$ the table sequence

$$B^l = (B(M_k(X, z)) : k \in \mathbb{Z})$$

is a cancellation of a decomposition $B^\infty$ of $B(I(X))$. □

For the cancellation principle the following can be regarded as the first nontrivial example.
Example 3.6. Let $E \subset \mathbb{P}^3$ be an elliptic normal curve of degree 4, and take a
(general) point $z \in E$. Then

1. $E_z \subset \mathbb{P}^2$ is a cubic plane curve, and
2. $K_1(E, z)$ is the homogeneous ideal of a point in $\mathbb{P}^2$ with $K_1(E, z) \supseteq K_0(E, z)$.

We have

\[
\mathbb{B}(M_0(E, z)) = \begin{bmatrix}
2 & 0 & 1 \\
3 & 1 \\
\end{bmatrix}, \quad \mathbb{B}(M_1(E, z)) = \begin{bmatrix}
2 & 0 & 1 \\
3 & 1 \\
\end{bmatrix}, \quad \text{and} \quad \mathbb{B}(M_k(E, z)) = 0
\]

for all $k \geq 2$. However, any two linearly independent quadratic forms in $I(E)$ do
not have nontrivial relations so that the cancellation in Proposition 3.5 must occur
between $\beta_{1,2}(M_1(E, z))$ and $\beta_{0,3}(M_0(E, z))$. Therefore, we have shown that the
decomposition of $\mathbb{B}(I(E))$ induced by the cancellation principle consists of $\mathbb{B}(I(E))$
itself and zero tables. Note that $E \subset \mathbb{P}^3$ is a complete intersection of two quadrics.

Let $M$ be a finitely generated graded $S$-module. Recall that the regularity of $M$
relates to the height of the Betti table $\mathbb{B}(M) = (\beta_{i,j}(M) : (i, j) \in \mathbb{Z} \times \mathbb{Z})$ when the
rows (resp. columns) are indexed by $j$ (resp. $i$), that is,

\[
\text{reg } M = \max\{j \in \mathbb{Z} : \beta_{i,j}(M) \neq 0 \text{ for some } i \in \mathbb{Z}\}.
\]

But the projective dimension of $M$, denoted by $\text{pd } M$, is known to measure the
width of $\mathbb{B}(M)$, which means that

\[
\text{pd } M = \max\{i \in \mathbb{Z} : \beta_{i,j}(M) \neq 0 \text{ for some } j \in \mathbb{Z}\}.
\]

This satisfies the Auslander–Buchsbaum formula

\[
\text{depth } M + \text{pd } M = \text{depth } S,
\]

where $\text{depth } M$ is the depth of the irrelevant ideal on $M$.

Definition 3.7. One says that $M_i(X, z)$ dominates the others

1. by the regularity if
   
   \begin{enumerate}
   \item[(a)] $\text{reg } M_i(X, z) \geq \text{reg } M_j(X, z)$ for every $j < i$, and if
   \item[(b)] $\text{reg } M_i(X, z) \geq \text{reg } M_j(X, z) + 2$ for every $j > i$, and
   \end{enumerate}

2. by the projective dimension if
   
   \begin{enumerate}
   \item[(a)] $\text{pd } M_i(X, z) \geq \text{pd } M_j(X, z) + 2$ for every $j < i$, and if
   \item[(b)] $\text{pd } M_i(X, z) \geq \text{pd } M_j(X, z)$ for every $j > i$.
   \end{enumerate}

With these conditions the cancellation principle directly implies the result below.

Proposition 3.8. Let $i \geq 0$ be an integer.

1. If $M_i(X, z)$ dominates the others by the regularity, then we have
   \[
   \text{reg } X = \text{reg } M_i(X, z).
   \]

2. If $M_i(X, z)$ dominates the others by the projective dimension, then we have
   \[
   \text{pd } S(X) = \text{pd } M_i(X, z) + 1.
   \]

The following tells us about invariants of an unprojection provided that the
associated projection is simple.
Corollary 3.9. Let $\lambda$ be a fake linear form on $Y \subseteq \mathbb{P}^{r-1}$, and suppose that $\pi_\ast |_X$ is simple for the unprojection $(X, z) = \text{Unpr}(Y, \lambda)$. Fix a fractional expression $\lambda = a/f \in \kappa(\hat{Y})$ with $a, f$ homogeneous, and put $T = I(Y) + (a, f) \subset R$.

1. For the projective scheme $W = V(T) \subseteq \mathbb{P}^{r-1}$ if $\dim W = \dim Y - 1$, then
   $$\deg X = (\deg f + 1) \deg Y - \begin{cases} 
   \deg W & \text{if } \dim W = \dim Y - 1 \\
   0 & \text{if } \dim W < \dim Y - 1.
   \end{cases}$$

2. If $\text{reg} T > \text{reg} Y + \deg f$, then
   $$\text{reg} X = \text{reg} T - \deg f + 1.$$

3. If $\text{pd} R/T > \text{pd} S(Y) + 2$, then
   $$\text{pd} S(X) = \text{pd} R/T - 1.$$

Proof. We begin with the following diagram with exact rows and column. For the column refer to the formula (2.4).

$$\begin{array}{ccccccccc}
0 & \rightarrow & M_1(X, z) & \rightarrow & S(Y)(-1) & \rightarrow & R_{K_1(X, z)}(-1) & \rightarrow & 0 \\
& & & & a & & & & \\
0 & \rightarrow & S(Y) & \rightarrow & S(Y)(\deg f) & \rightarrow & S(Y)_{(f)(\deg f)} & \rightarrow & 0 \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
R & \rightarrow & T_{(\deg f)} & \rightarrow & 0 \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & 0
\end{array}$$

(1) It is by the degree formula (2.3). Note that $K_1(X, z)$ is equal to $K_\infty(X, z)$ with $z \in X$, and so the column above computes the degree of $\mathbb{P}C_xX$. (2) Tensoring the diagram above with the residue field $k$ of $R$ we see that under the assumption $M_1(X, z)$ dominates the others by the regularity. Apply Proposition 3.8. (3) Similarly in this case $M_1(X, z)$ dominates the others by the projective dimension. □

Remark 3.10. Notice that if $Y \subseteq \mathbb{P}^{r-1}$ is a hypersurface, then the homogeneous ideal $T \subset R$ above becomes an ideal with three generators, namely a three-generated ideal. Such ideals have a wide range of complexity as shown in [Bru76]. This is a substantial basis for the construction of our counterexamples to the regularity conjecture.

A similar thing holds for a general fake linear form of degree 1.

Corollary 3.11. Suppose that depth $S(Y) \geq 2$. Take a regular sequence $f, a$ of homogeneous forms on $S(Y)$ with $\deg f = \deg a - 1 \geq 1$, and let $\lambda = a/f \in \kappa(\hat{Y})$.
be the induced fake linear form of degree 1 on \( Y \subseteq \mathbb{P}^{r-1} \). Then for the unprojection \( (X, z) = \text{Unpr}(Y, \lambda) \) the projection \( \pi_{z|X} \) is simple, and \( X \subseteq \mathbb{P}^r \) has invariants
\[
\text{deg } X = (\text{deg } f + 1) \text{ deg } Y, \quad \text{reg } X = \text{reg } Y + \text{deg } f, \quad \text{and} \quad \text{pd } S(X) = \text{pd } S(Y) + 1.
\]

**Proof.** The formula (2.4) says that
\[
\frac{K_i(X, z)}{K_0(X, z)} = (f(a, f)i^i - 1 : a^i) \subseteq (f) = (f)
\]
for every \( i \geq 1 \) with equality when \( i = 1 \). Thus, it follows that \( \pi_{z|X} : X \setminus z \to Y \) is simple together with
\[
\text{reg } M_1(X, z) \subseteq S(Y)(-\text{deg } f - 1).
\]
By Corollary 3.9 we have \( \text{deg } X = (\text{deg } f + 1) \text{ deg } Y \). Since \( \text{reg } M_1(X, z) = \text{reg } S(Y)(-\text{deg } f - 1) \neq \text{reg } S(Y) + \text{deg } f + 1 = \text{reg } Y + \text{deg } f \geq \text{reg } M_0(X, z) \), Proposition 3.8 tells us that \( \text{reg } X = \text{reg } M_1(X, z) = \text{reg } Y + \text{deg } f \). For the projective dimension part, we use the cancellation principle. Note that \( \text{pd } M_1(X, z) = \text{pd } S(Y) = \text{pd } I(Y) + 1 = \text{pd } M_0(X, z) + 1 \). Put \( p = \text{pd } M_1(X, z) \) and \( q = \max \{ j \in \mathbb{Z} : \beta_{p, j}(M_1(X, z)) \neq 0 \} \) so that \( \beta_{p, q}(M_1(X, z)) > 0 \). We however obtain \( \text{pd } M_0(X, z) = \text{pd } S(Y) - 1 = \text{pd } M_1(X, z) - 1 = p - 1 \) and \( \beta_{p, q+1}(M_0(X, z)) = \beta_{p, q}(S(Y)) = \beta_{p, q+\text{deg } f + 1}(M_1(X, z)) = 0 \). We are done. \( \square \)

4. Families of homogeneous ideals with high regularity

In this section we focus on sequences of homogeneous ideals whose values of the regularity grow fast.

**Example 4.1.** In [Cav04, Section 4.2] Caviglia considered homogeneous ideals
\[
I_d = (z_2^{d-1}y_1 - y_2^{d-1}z_1, y_1^d, z_1^d) \subseteq k[y_1, z_1, y_2, z_2].
\]
Finding its Gröbner basis with respect to the reverse lexicographic order he showed that
\[
\text{reg } I_d = d^2 - 1
\]
for every \( d \geq 2 \). Furthermore, for the homogeneous ideals
\[
I_{k, d} = (z_{i+1}^{d-1}y_i - y_{i+1}^{d-1}z_i : i = 1, \ldots, k - 1) + (y_1^d, z_1^d) \subseteq k[y_1, z_1, \ldots, y_k, z_k]
\]
he expected with computational evidence that for each \( k \geq 3 \) there exists some univariate (rational) polynomial \( p_k \) of order \( k \) such that
\[
\text{reg } I_{k, d} = p_k(d).
\]
Notice that in [BM15, Remark 2.7] Borna and Mohajer made the same claim for the homogeneous ideals
\[
J_{k, d} = (z_{i+1}^{d-1}y_i - y_{i+1}^{d-1}z_i, y_i^d, z_i^d : i = 1, \ldots, k - 1) \subseteq k[y_1, z_1, \ldots, y_k, z_k].
\]
They also observed that
\[
\text{reg } J_{3, d} = d^3 - 3d^2 + 5d - 3
\]
and presumed that
\[
\text{reg } J_{4, d} = d^4 - 4d^3 + 5d^2 + d - 3.
\]
Example 4.2. In [BMNB+11] Beder, McCullough, Núñez-Betancourt, Seceleanu, Snapp, and Stone constructed interesting three-generated homogeneous ideals denoted by $I_{2,(2,\ldots,2,1,d)}$ and raised the speculation that for the $k$-tuple $(2,\ldots,2,1,d)$ the estimate
\[
\text{reg } I_{2,(2,\ldots,2,1,d)} = \Omega(d^k)
\]
holds. With minor modifications these ideals are formed as follows.

Let $k \geq 2$ and $d \geq 0$ be integers. We first set
\[
F_{k-1} = y_{k-1}^{d+2}, \quad G_{k-1} = z_k \frac{d+1}{d} y_{k-1} - z_k^{d+2} \quad \text{and} \quad H_{k-1} = z_k^{d+2},
\]
and then inductively put
\[
F_i = y_i^{d+2k-2i}, \quad G_i = F_{i+1}y_i^2 + G_{i+1}y_{i+1}z_i + H_{i+1}z_i^2 \quad \text{and} \quad H_i = z_i^{d+2k-2i}
\]
for each $i = k-2, k-3, \ldots, 1$. Now the homogeneous ideal $I_{2,(2,\ldots,2,1,d)}$ is defined by
\[
I_{2,(2,\ldots,2,1,d)} = (F_1, G_1, H_1) \subset \mathbb{k}[y_1, z_1, \ldots, y_k, z_k].
\]
The following partially verifies Borna–Mohajer’s assertion in Example 4.1 and is a foundation for our counterexamples. Its proof is inspired by that of [CCM+19 Proposition 7.1].

**Theorem 4.3.** Fix an integer $k \geq 2$, and let $b_1, c_1, a_2, \ldots, b_{k-1}, c_{k-1}, a_k \geq 1$ be integers satisfying $1 \leq a_i < \min\{b_i, c_i\}$ for all $i = 2, \ldots, k-1$. Then the homogeneous ideal
\[
I = (z_i^{a_i+1} y_i - z_i^{a_i+1} z_i, y_i^{b_i}, z_i^{c_i} : i = 1, \ldots, k-1) \subset Q = \mathbb{k}[y_1, z_1, \ldots, y_k, z_k]
\]
has invariants
\[
(4.4) \quad \text{reg } I \geq (\min\{b_1, c_1\} - 1)a_2 \cdots a_k - \sum_{i=3}^k a_i \cdots a_k + \sum_{i=2}^k a_i + \max\{b_1, c_1\} - 1
\]
and $\text{pd } Q/I = 2k$.

**Proof.** Put $A = \mathbb{k}[y_k, z_k] \subset Q$, write $a_1 = \max\{b_1, c_1\}$, and let $M \subset Q/I$ be the $A$-submodule generated by all the monomials
\[
m(d_1, \ldots, d_{k-1}) = \prod_{i=1}^{k-1} y_i^{a_i-d_i} z_i^{d_i},
\]
where $0 \leq d_i \leq a_i$ for all $1 \leq i \leq k-1$. It is an $A$-direct summand of $Q/I$ since the monomials have the same degree in $y_i$ and $z_i$ for each $1 \leq i \leq k-1$. Note that $Q/I$ is finitely generated as an $A$-module, which implies that the $Q$- and $A$-module structures on $Q/I$ give the same local cohomology groups. Due to vanishing of local cohomology groups (see [Eis05 Corollary 4.5 and Proposition A1.16]) we have
\[
\text{reg } I = \text{reg } Q/I + 1 \geq \text{reg } M + 1 \quad \text{and} \quad \text{pd } Q/I \geq \text{pd } M + 2k - 2.
\]
Thanks to these inequalities it would be enough to compute $\text{reg } M$ and $\text{pd } M$.

Let us find the minimal free resolution of $M$. For integers $0 \leq d \leq \alpha$, where
\[
\alpha = a_1 \cdots a_{k-1} + a_2 \cdots a_{k-1} + \cdots + a_{k-1},
\]we set
\[
m_d = m(d_1, \ldots, d_{k-1})
\]when
\[
d = d_1 a_2 \cdots a_{k-1} + d_2 a_3 \cdots a_{k-1} + \cdots + d_{k-1}.
\]
with integers $0 \leq d_i \leq a_i$ for all $1 \leq i \leq k - 1$. They are well-defined by the “carry-borrow” rule

$$m(\ldots, d_i, d_{i+1}, \ldots) = m(\ldots, d_i + 1, 0, \ldots)$$

for the case where $d_{i+1} = a_{i+1}$ and $d_i < a_i$, which is induced by the generators $z_{i+1}^{a_{i+1}}y_i - y_{i+1}^{a_{i+1}}z_i \in I$ with $1 \leq i \leq k - 2$. For a similar reason we find that

$$(4.5) \quad m_dz_k^{a_k} = m_{d+1}y_k^{a_k}$$

for any $0 \leq d < \alpha$. Notice that if either $d \leq \alpha - \beta$ or $\gamma \leq d$ holds with

$$\beta = b_1a_2 \cdots a_{k-1} \quad \text{and} \quad \gamma = c_1a_2 \cdots a_{k-1},$$

then $m_d = 0$ in $M$, for $z_1^{a_1}$ and $y_1^{b_1}$ are members of $I$. We claim that $m_{\alpha-\beta+1}, \ldots, m_{\gamma-1} \in M$ are nonzero generators whose $A$-linear relations

$$f = \sum_{d=\alpha-\beta+1}^{\gamma-1} f_d m_d \in I$$

are spanned by the relations $(4.5)$.

Indeed, comparing the degrees of $f$ and the generators of $I$ in $y_i$ and $z_i$ for each $2 \leq i \leq k - 1$ one sees that

$$f \in (z_i^{a_{i+1}}y_i - y_{i+1}^{a_{i+1}}z_i : 1 \leq i \leq k - 1) + (y_1^{b_1}, z_1^{a_1}).$$

Plug

$$y_i = y_{k-1}^{a_{i+1}} \cdots a_{k-1} \quad \text{and} \quad z_i = z_{k-1}^{a_{i+1}} \cdots a_{k-1}$$

into the $m_d$ and $f$ for all $1 \leq i \leq k - 2$ so that we reach the images

$$\overline{m_d} = y_{k-1}^{a_{d-1}}z_{k-1}^{a_{d}} \quad \text{and} \quad \overline{f} = \sum_d f_d y_{k-1}^{a_{d-1}}z_{k-1}^{a_{d}} \in (z_k^{a_k}y_{k-1} - y_k^{a_k}z_{k-1}, y_k^{\beta}, z_k^{\gamma})$$

in $A[y_{k-1}, z_{k-1}]$. Our claim has been shown.

Consequently, by letting $N = \beta + \gamma - \alpha - 1$ the minimal free resolution of $M$ has differentials

$$-z_k^{a_k} \begin{pmatrix} y_k^{a_k} & y_k^{a_k} \\ -z_k^{a_k} & y_k^{a_k} \\ \vdots & \vdots \\ -z_k^{a_k} & y_k^{a_k} \end{pmatrix}$$

and

$$\begin{pmatrix} (y_k^{a_k})^N \\ (y_k^{a_k})^{N-1} z_k^{a_k} \\ (y_k^{a_k})^{N-2} (z_k^{a_k})^2 \\ \vdots \\ (z_k^{a_k})^N \end{pmatrix},$$

and the Betti table of $M$ looks like

$$\begin{array}{c|ccc} j \backslash i & 0 & 1 & 2 \\ \hline j_0 & N \\ j_1 & N + 1 \\ j_2 & 1 \end{array}$$

in the setting of

1. $j_0 = \sum_{i=1}^{k-1} a_i$,
2. $j_1 = j_0 + a_k - 1 = \sum_{i=1}^{k} a_i - 1$, and
3. $j_2 = j_1 + a_k N - 1 = (\beta + \gamma - \alpha - 1) a_k + \sum_{i=1}^{k} a_i - 2$.

We are done. \qed
Remark 4.4. If 
\[(b_1, c_1, a_2, \ldots, b_{k-1}, c_{k-1}, a_k) = (d, d, d - 1, \ldots, d, d - 1)\]
for an integer \(d \geq 2\), then the inequality (4.4) reads
\[\text{reg } I \geq (d - 1)^k - \sum_{i=1}^{k-2} (d - 1)^i + k(d - 1).\]

Therefore, we have verified the lower bound versions of (4.1), (4.2), and (4.3), which becomes evidence that the inequality (4.4) is sharp.

Generalizing a technique of Caviglia, Chardin, McCullough, Peeva, and Varbaro we establish sequences of three-generated homogeneous ideals with the same growth rates of regularity as those from Theorem 4.3.

Proposition 4.5 (cf. [CCM +19, Proposition 7.1]). Let \(Q \subset R\) be a polynomial ring obtained by eliminating two variables in \(R\), say \(y_0\) and \(z_0\), and take a homogeneous ideal \(U = (g_0, \ldots, g_s) \subset Q\) and positive integers \(\Delta\) and \(\delta\). Suppose that
\[(1) \ g_s \text{ is nonzero},
(2) \ \deg g_i = mi + \deg g_0 \text{ for a common integer } m \geq 0, \text{ and}
(3) \ the \ inequalities \ \Delta > s \text{ and } \delta > (m + 1)s \text{ hold.}\]

We put
\[G = g_s y_0^s + g_s - 1 y_0^{s-1} z_0^{m+1} + \cdots + g_0 z_0^{(m+1)s} \quad \text{and} \quad T = (G, y_0^\Delta, z_0^\delta).\]

Then we have
\[\deg R/T \leq \delta s, \quad \text{reg } T \geq \text{reg } U + \Delta + \delta - 2, \quad \text{and} \quad \text{pd } R/T \geq \text{pd } Q/U + 2.\]

Moreover, the following hold.
\[(1) \ If \ \(m + 1)(\Delta - s + 1) \geq \delta, \ then \ \deg R/T = \delta s.
(2) \ If \ g_s \text{ is irreducible, then so is } G \text{ provided that the } g_i \text{ are linearly independent over } k.\]

Proof. Mimicking the proof of [CCM +19, Proposition 7.1] with the monomial \(y^{\Delta-1}z^{\delta-1}\) we have the desired lower bounds of \(\text{reg } T\) and \(\text{pd } R/T\). Also, since the radical of \(T\) is \((y_0, z_0)\), we obtain
\[\deg R/T = \dim_K R/T \otimes_Q K,\]
where \(K\) is the fraction field of \(Q\). Let \(V\) be the \(K\)-vector space \(R/T \otimes_Q K\). By dividing elements of \(V\) by \(G\) the vector space \(V\) is shown to be generated by the monomials
\[y_0^{0, \delta-1}, \ldots, y_0^{s-1, \delta-1},
\]
(4.6)
\[
\vdots
\]
\[y_0^0 z_0^0, \ldots, y_0^{s-1} z_0^0.\]

Hence, the upper bound \(\deg R/T \leq \delta s\) holds.

Next assume that \((m + 1)(\Delta - s + 1) \geq \delta\). Introduce a new grading \(\deg'\) having values \(\deg' y_0 = m + 1, \deg' z_0 = 1, \text{ and } \deg' = 0\) on \(Q\). According to the values \(j\) of this grading we get the \(K\)-vector space decomposition
\[V = \bigoplus_{j \in \mathbb{Z}} V_j.\]
We divide into two cases: $j < \Delta(m+1)$ and $j \geq \Delta(m+1)$. If $j < \Delta(m+1)$, and if we express $j$ as $j = a(m+1) + b$ with integers $0 \leq b < m+1$ and $0 \leq a < \Delta$, then the summand $V_j$ is generated by the set \( \{ y_0^a z_0^b, y_0^{a-1} z_0^{b+1}, \ldots \} \) and becomes the cokernel of a matrix of the form
\[
\begin{pmatrix}
g_s \\
g_{s-1} & \ddots & \vdots \\
\vdots & \ddots & g_s \\
& \ddots & g_{s-1}
g_s \\
\end{pmatrix}
\]
However, in the range $j \geq \Delta(m+1)$ some matrix of the form
\[
\begin{pmatrix}
\cdots & g_s & g_s & \cdots \\
\ddots & \ddots & \ddots & \ddots \\
& \ddots & g_s & g_s & \cdots \\
\end{pmatrix}
\]
works for a presentation of $V_j$. It is due to the assumption $(m+1)(\Delta - s + 1) \geq \delta$. So one may observe that the monomials (4.6) form a basis of $V$. Finally, let us see the irreducibility of $G$. Refer to the fact that $G$ is homogeneous with respect to both the standard grading and deg'. Suppose that
\[
G = (a_t y_t^t + \cdots + a_{0} z_0^{(m+1)t})(b_u y_u^u + \cdots + b_{0} z_0^{(m+1)u})
\]
for integers $t, u \geq 0$ and homogeneous forms $a_i, b_j \in Q$ with deg $a_i = mi + \deg a_0$ and deg $b_j = mj + \deg b_0$. We find that $a_t b_u = g_s$, and so we may assume that $a_t$ is a nonzero constant. Since deg $a_i = mi + \deg a_0$, and since the $g_i$ are $k$-linearly independent, we have $t = 0$, and thus the first factor above is a nonzero constant, which means that $G$ is irreducible. \hfill \Box

We apply the proposition above to suitable homogeneous ideals in Theorem 4.3.

**Corollary 4.6.** Fix an integer $k \geq 2$, and let $d \geq 2$ be an integer. Then the polynomial ring $R = k[y_0, z_0, \ldots, y_k, z_k]$ admits an irreducible $(d + 6k - 8)$-form $G_{k,d} \in R$ and a three-generated homogeneous ideal
\[
T_{k,d} = (G_{k,d}, y_0^{6k-6}, z_0^{6k-7}) \subset R
\]
such that
\[
\text{reg}\, T_{k,d} = \Omega(d^k), \quad \text{and} \quad \text{pd}\, R/T_{k,d} = 2k + 2
\]
for all $i \geq 1$.

**Proof.** Write $Q = k[y_1, z_1, \ldots, y_k, z_k]$. By setting
\[
(b_i, c_i, a_{i+1}) = (d + 3i - 3, d + 3i - 2, d + 3i - 2), \quad 1 \leq i \leq k - 1,
\]
Theorem 4.3 produces a homogeneous ideal $U = (g_0, \ldots, g_{3k-4}) \subset Q$ whose generators are arranged as
\[
g_{3i} = y_{i+1}^{d+3i}, \quad g_{3i+1} = z_{i+1}^{d+3i+1}, \quad \text{and} \quad g_{3i+2} = y_{i+2}^{d+3i+1} z_i^{d+3i+1} - z_{i+2}^{d+3i+1} y_i + 1
\]
for integers $1 \leq i \leq k - 1$. Proposition 4.3 accomplishes the proof. \hfill \Box
5. PROOF OF THE MAIN THEOREM

In this section we show the main theorem.

Proof of Theorem 1.2. Fix an integer $k \geq 2$, and set $d \geq 2$. Consider the three-generated homogeneous ideal

$$T_{k,d} = (G_{k,d}, y_0^{6k-6}, z_0^{6k-7}) \subset R = k[y_0, z_0, \ldots, y_k, z_k]$$

formed in Corollary 4.6. Take

$$Y_d = V(G_{k,d}) \subset \mathbb{P}^{2k+1}$$

and

$$\lambda = \frac{y_0^{6k-6}}{z_0^{6k-7}} \in k(\hat{Y}_d)$$

to construct an unprojection $(X_d, z) = \text{Unpr}(Y_d, \lambda)$ in $\mathbb{P}^{2k+2}$. We claim that the sequence

$$(X_d \subset \mathbb{P}^{2k+2} : d \geq 2)$$

is the desired one with dimension $n = 2k$ and codimension $e = 2$.

Indeed, notice that part of Proposition 4.5 induces upper bounds

$$\deg W_i \leq \deg R/(G_{k,d}, y_0^{(6k-6)i}, z_0^{(6k-7)i}) \leq i \deg W_i$$

for the subschemes $W_i = V((a, f)^i) \subset Y_d \subset \mathbb{P}^{2k+1}$. By Proposition 2.6 and Corollary 3.9 we find that $\pi_z|_{X_d}$ is simple and that

$$\lim_{d \to \infty} \deg X_d = \infty, \quad \text{reg } X_d = \Omega((\deg X_d)^k), \quad \text{and } pd S(X_d) = 2k + 1$$

due to invariants of $T_{k,d}$.

For higher codimension cases $e > 2$ conduct the following process. Suppose given a counterexample $X_d \subset \mathbb{P}^{2k+e-1}$ of dimension $2k$ and codimension $e - 1 \geq 2$, and say that $S(X_d)$ has depth 2. Then pick a general fake linear form of degree 1 on $X_d \subset \mathbb{P}^{2k+e-1}$, and apply Corollary 3.11. We have therefore constructed a sequence

$$(X_d \subset \mathbb{P}^{2k+e} : d \geq 2)$$

of nondegenerate projective $2k$-folds satisfying

$$\lim_{d \to \infty} \deg X_d = \infty, \quad \text{reg } X_d = \Omega((\deg X_d)^k), \quad \text{and } pd S(X_d) = 2k + e - 1$$

for every $e \geq 2$. On the other hand for $(2k - 1)$-fold counterexamples consider a general hyperplane section of each $2k$-fold $X_d$. It is valid thanks to depth $S(X_d) = 2$. We are done. \hfill \Box

Remark 5.1. It can be said that our counterexamples work as they carry very bad singularities. For instance, let us look at the case $(X, z) = (X_d, z)$ of dimension $2k$ and codimension 2 in the proof above. Then the point $z \in \mathbb{P}^{2k+2}$ turns out to be a singular point of $X$, and its projectivized tangent cone $\mathbb{P}C_z X$ is cut out by $K_1(X, z)$, having exceedingly high regularity compared to the multiplicity of $X$ at $z$. Recall that the pathological syzygies of $I(X)$ immediately come from those of $M_1(X, z)$ in view of the cancellation principle.

Finally, we present in detail a counterexample to the regularity conjecture obtained from a homogeneous ideal in Example 4.2 by applying our method. Many of the computations below are done by the computer algebra system Macaulay2 [GS] over the field $\mathbb{Q}$ of rational numbers.
Example 5.2. Let $R = \mathbb{k}[y_0, z_0, y_1, z_1, y_2, z_2]$ be the homogeneous coordinate ring of $\mathbb{P}^5$, and take $Y \subset \mathbb{P}^5$ to be the hypersurface given by the irreducible homogeneous polynomial

$$G = y_1^4 y_0^2 + (z_2^3 y_1 - y_2^3 z_1)y_0 z_0 + z_1^4 z_0^2$$

that appears in Example 4.2. Notice that the three-generated homogeneous ideal

$$T = (G, y_1^4, z_0^3) \subset R$$

satisfies

$$\deg R/T = 6, \quad \reg T = 20, \quad \text{and} \quad \pd R/T = 6.$$ 

Now with the fake linear form

$$\lambda = y_1^4/z_0^3 \in \mathbb{k}(\hat{Y})$$

of degree 1 on $Y \subset \mathbb{P}^5$ consider the nondegenerate projective fourfold

$$(X, z) = \Unpr(Y, \lambda) \quad \text{in} \quad \mathbb{P}^6.$$ 

Then having

$$\deg X = 18, \quad \reg X = 18, \quad \text{and} \quad \pd S(X) = 5,$$

we have established a fourfold counterexample to the regularity conjecture. Here the regularity is exactly one more than the conjectural bound of Eisenbud and Goto. The Betti table $B(I(X))$ is computed to be the following.

$$\begin{array}{cccccccccccccc}
\begin{array}{l}
\end{array} \\
\begin{array}{cccccccccccccc}
\end{array} \\
\begin{array}{cccccccccccccc}
\end{array} \\
\begin{array}{cccccccccccccc}
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\begin{array}{cccccccccccccc}
\end{array} \\
\begin{array}{cccccccccccccc}
\end{array} \\
\begin{array}{cccccccccccccc}
\end{array} \\
\end{array}$$

Moreover, thanks to depth $S(X) = 2$ a general hyperplane section

$$X \cap \mathbb{P}^5 \subset \mathbb{P}^5$$

is a threefold counterexample with the same invariants as above. Compare it with the projective threefold in [MP18, Example 4.7].

Let us describe singularities of the fourfold $X$. The singular locus of $X$ is

$$\Sing X = C \cup \Lambda_1 \cup \cdots \cup \Lambda_4 \cup \mathbb{P}^3 \subset \mathbb{P}^6,$$

where $C \subset \mathbb{P}^6$ is a quartic curve in a 2-plane, the $\Lambda_i \subset \mathbb{P}^6$ are 2-planes in a certain order, and $\mathbb{P}^3 \subset \mathbb{P}^6$ is a 3-plane. Here the location of $z \in \mathbb{P}^6$ is obtained as

$$z \in (C \cap \Lambda_1 \cap \Lambda_2) \setminus (\Lambda_3 \cup \Lambda_4 \cup \mathbb{P}^3) \subset \Sing X.$$ 

The projectivized tangent cone $\mathbb{P}C_z X \subset \mathbb{P}^5$ is of degree 12, its homogeneous ideal has regularity 17, and its support is a 3-plane in $\mathbb{P}^5$.

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School of Mathematics, Korea Institute for Advanced Study (KIAS), 85 Hoegiro
Dongdaemun-gu, Seoul 02455, Republic of Korea

Email address: junhochoe@kias.re.kr