Scaling behavior of the four-point renormalized coupling constant in the two dimensional O(2) and O(3) non-linear sigma models

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We report thermodynamic values of four-point renormalized coupling constant calculated by Monte Carlo simulations in the continuum limits of the lattice versions of the two-dimensional O(2) and O(3) non-linear sigma models. In each case the critical index of the coupling constant vanishes, which leads to hyperscaling (non-triviality).

A general problem of (Euclidean) quantum field theory (QFT) is whether the n-point \((n > 2)\) correlation functions, which are constructed so as to satisfy Osterwalder-Schrader axioms [1], are those of the Gaussian model in the continuum limit. If so this QFT is non-interacting, and is referred to be a trivial theory [2]. It is rigorously proven that one cannot construct an interacting continuum limit of the \(\lambda \phi^4\) theory in the symmetric phase in dimensions larger than 4 \((D > 4)\) [3], and is widely conjectured that this is the case even in four dimensions [4]. A trivial theory, which might be interacting only when a finite cutoff is imposed in the theory, cannot be a candidate of a genuine QFT, so it is very important to find out whether a theory is trivial or not. Without any proof, it is generally believed that a QFT which is not asymptotic free in 4D would be trivial. An important example of such a theory is the 4D non-compact QED, and the issue of triviality in this theory remains unresolved with variant conclusions of studies.
As far as the issue of the triviality is concerned, studying the scaling behavior of the four-point renormalized coupling constant \( g^{(4)} \) is primarily important. A rigorous theorem \[5\] states that vanishing of \( g^{(4)} \) is sufficient for triviality in the Ising like ferromagnetic systems, and it is conjectured that the same holds for other type of ferromagnetic lattice models \[2\]. While much (numerical) studies on the scaling behavior of \( g^{(4)}_R \) have been done for the \( \lambda \phi^4 \) theories in various dimensions \[6\] \[7\] and for the Yukawa type interaction \[8\], little has been studied for the 2D O(N) models for \( N \geq 2 \) \[9\], which share many similarities with 4D lattice gauge theories. In particular, the 2D O(2) non-linear sigma model is not asymptotic free in the context of perturbation theory, while O(\(N \geq 3\)) non-linear sigma models are asymptotic free \[10\].

In this article, we investigate the scaling behaviors of \( g^{(4)} \) for the 2D O(N) (\( N=2,3 \)) non-linear sigma models on a square lattice of linear size \( L \) with periodic boundary condition, based on Monte Carlo simulations. The model is defined by the action

\[
A = -\beta \sum_{<i,j>} \sigma_i \cdot \sigma_j, \tag{1}
\]

where \( \beta \) is the inverse temperature (\( \beta = 1/T \)), and \( \sigma_i \) is an \( N \)-dimensional unit vector at lattice site \( i \). We report compelling numerical evidences that the continuum limits of both models are not trivial, irrespective of the property of perturbative asymptotic freedom.

The four-point renormalized coupling constant \( g^{(4)} \) which is basically the connected four-point function defined at zero-momentum, and which is constructed so as to be independent of rescaling of field strength, can be written in a system with translation symmetry as

\[
g^{(4)} = \lim_{L \to \infty} g^{(4)}_L \equiv \lim_{L \to \infty} U_L \ (\xi_L/L)^D, \tag{2}
\]

Here, \( g^{(4)}_L \) and \( \xi_L \) are respectively the renormalized coupling constant and correlation length \[11\] defined on a finite lattice of linear size \( L \), and \( U_L \) is a modified (\( N \) component) Binder’s cumulant ratio defined as \( U_L \equiv [(1+2/N) < S^2 >^2 - < S^4 >]/ < S^2 >^2 \), with \( S^2 \equiv |\sum_i \sigma_i|^2 \) \[12\].
For a system displaying power-law type scaling behavior, \( g^{(4)} \) scales as \( g^{(4)}(t) \sim t^{-2\Delta + \gamma + D\nu} \), where the notations are standard (\( t \), for example, is the deviation of the dimensionless temperature from the coupling defined as \( t \equiv (\beta_c - \beta)/\beta_c \)). The inequality, 
\[ 2\Delta \leq \gamma + D\nu \]
holds in general for ferromagnetic systems with power-law singularities. For the \( \lambda \phi^4 \) models, the inequality has been rigorously proven for \( D > 4 \) so that \( \lim_{t \to 0} g^{(4)}(t) \to 0 \), whereas the equality holds for \( D \leq 4 \) (hyperscaling). In 4D, it is rigorously proven under some mild assumptions that the scaling behavior of \( g^{(4)}(t) \) has such a multiplicative logarithmic correction that \( \lim_{t \to 0} g^{(4)}(t) \to 0 \).

For the 2D O(N) (N=2,3) models, we have calculated \( g^{(4)} \) by employing Wolff’s single cluster algorithm on square lattice with periodic boundary condition. For each model, our continuum limit is achieved by adjusting the value of \( \beta \) (\( T \)) so that the correlation length in unit of lattice spacing starts to be diverging; at the same time, the lattice spacing scales with \( \beta \) so that the correlation length multiplied by the lattice spacing remains a constant. In general the auto-correlation time for \( g^{(4)} \) is much larger than that of \( \chi \) or \( \xi \), so much more computational efforts are required for the precise determination of \( g^{(4)} \). For a given \( \beta \) and \( L \), 20-80 different bins were obtained for our calculations, with each bin being composed of 10 000 measurements each of which was separated by 8-15 consecutive one cluster updating. The statistical errors were estimated by jack-knife method.

In order to monitor the effect of finite size in the measurements of \( g^{(4)} \), for each model at an arbitrary temperature in the scaling region, we measured \( g^{(4)}_L \) by varying \( L \) by 10 from \( L=10 \) until \( g^{(4)}_L \) did not vary with further increasing of \( L \). For each model, we conclude that \( g^{(4)}_L \) (as well as \( \chi \) and \( \xi \)) converges to its thermodynamic value (within the statistical error of very precise Monte Carlo data) on the condition that \( L/\xi_\infty \geq 7 \) (Figure(1)). According to the theory of finite size scaling, the following relation holds for the renormalized four-point coupling:

\[
g^{(4)}_L(\beta) = g^{(4)}(\beta)f_g(s), \quad s \equiv L/\xi_\infty(\beta),
\]

with \( f_g \) representing a scaling function characterized by \( g^{(4)} \). \( f_g(s) \) has no explicit tempera-
ture dependence so that the thermodynamic condition holds for any temperature.

We thus chose L such that \( L/\xi_\infty \approx 7 \) for the direct measurement of the thermodynamic \( g^{(4)} \) (\( T=1.19, 1.10, 1.04, \) and \( 1.02 \) for the O(2) model; \( \beta = 1.5, 1.6, \) and \( 1.7 \) for the O(3) model). For the temperatures where the corresponding \( \xi \) becomes very large, we fix \( L/\xi_\infty \) to be a value much smaller than 7 so that, according to Eq.(3), \( g_L^{(4)} \) thus obtained is exactly proportional to its corresponding thermodynamic value. When the corresponding \( \xi_\infty(\beta) \) is known at a certain \( \beta \), this procedure enables one to obtain accurate thermodynamic values of other physical quantities without using sufficiently large L required in direct measurements \[15\]. Let us illustrate this for the 2D O(2) model, where very accurate values of \( \xi_\infty \) are already available up to \( T = 0.98 \) \[16\]. At \( T = 1.02 \) (\( \xi_\infty = 26.20(20) \)), we obtained \( g^{(4)} = 8.86(13) \) using \( L= 184 \) and \( g_L^{(4)} = 4.79(2) \), so \( f_g(s) \) is calculated to be 0.541(10) for \( s = L/\xi_\infty = 2.10(2) \). At \( T = 0.98 \) (\( \xi \simeq 70.5(7) \)), choosing \( L=148 \) makes the value of \( s \) the same as at \( T = 1.02 \) within the statistical errors, so with our Monte Carlo data, \( g_{148}^{(4)} = 4.81(2) \), we extract \( g^{(4)} = 8.89(20) \) for \( T = 0.98 \). For the O(3) model, we fixed the value of \( s = 1.165(21) \) by choosing \( L=40, 75, \) and \( 142 \) for \( \beta = 1.7, 1.8, \) and \( 1.9 \) respectively. Our final results are summarized in Table (1).

For such broad ranges of correlation length (\( 5.01(2) \leq \xi \leq 70.5(9) \) for the O(2); \( 11.1(1) \leq \xi \leq 121.9(7) \) for the O(3)) we observe that \( g^{(4)}(t) \sim t^0 \). These behaviors are qualitatively the same as those in the 2D and 3D Ising models \[8\]. We thus conclude that the continuum limits are non-trivial for both models. In particular, the perturbative property of the non-asymptotic freedom in the 2D O(2) model and that of the asymptotic freedom in the 2D O(3) model do not affect the non-trivialities in these models. Especially, because of the asymptotic freedom in the 2D O(3) model, the value of the renormalized coupling constant defined at any other value of the momentum must be smaller than our value, \( g^{(4)} = 6.6(1) \). Another remarkable conclusion of the current study is the existence of hyperscaling in these models. Hyperscaling relations cannot be defined in our models due to the exponential singularities in the critical behaviors of \( \xi \) and \( \chi \); nevertheless, the fundamental claim of the hyperscaling in the sense that the thermodynamic correlation length is the only relevant
scale in the scaling region, still holds in two dimensions irrespective of the type of critical behavior.

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TABLE I. $g(4)$ for the 2D O(2) and O(3) non-linear sigma models. For the O(3) model, we fixed $s = L/\xi_{\infty} = 1.165$ by choosing $L=40, 75, \text{ and } 142$ for $\beta = 1.7, 1.8, \text{ and } 1.9$ respectively. The corresponding values of $g(4)_L$ are 2.56(1), 2.56(1), and 2.59(2) respectively, showing the constancy of $g(4)$ in this range of $\beta$ as well. $f_g(s)$ is calculated to be 0.394(5). $\xi_{\infty}(\beta) = 64.6(5) \text{ and } 121.7(8)$ for $\beta = 1.8 \text{ and } 1.9$ [17].

| $g(4)$ | & | & | & |
|-------|---|--------|--------|--------|
| O(2) | 1.19 | 1.10 | 1.04 | 1.02 | 0.98 |
| | 8.70(16) | 8.80(12) | 8.71(10) | 8.86(13) | 8.89(20) |
| O(3) | 1.5 | 1.6 | 1.7 | 1.8 | 1.9 |
| | 6.58(13) | 6.70(15) | 6.50(6) | 6.50(11) | 6.57(13) |
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Figure Caption

Figure(1): $g_L^{(4)}/g^{(4)}$ as a function of $L/\xi$ at $T = 1.10$ ($\xi = 9.32(2)$) for the 2D O(2) model, and at $\beta = 1.5$ ($\xi = 11.05(2)$) for the 2D O(3) model. The range of L is [10,80] for the O(2), and is [10,100] for the O(3), varied by 10.