Global existence and optimal decay estimates of strong solutions to the compressible viscoelastic flows

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Abstract

This paper is dedicated to the global existence and optimal decay estimates of strong solutions to the compressible viscoelastic flows in the whole space $\mathbb{R}^n$ with any $n \geq 2$. We aim at extending those works by Qian & Zhang and Hu & Wang to the critical $L^p$ Besov space, which is not related to the usual energy space. With aid of intrinsic properties of viscoelastic fluids as in [30], we consider a more complicated hyperbolic-parabolic system than usual Navier-Stokes equations. We define “two effective velocities”, which allows us to cancel out the coupling among the density, the velocity and the deformation tensor. Consequently, the global existence of strong solutions is constructed by using elementary energy approaches only. Besides, the optimal time-decay estimates of strong solutions will be shown in the general $L^p$ critical framework, which improves those decay results due to Hu & Wu such that initial velocity could be large highly oscillating.

Keywords: compressible viscoelastic flows, critical Besov space, global existence, optimal decay.

Mathematical Subject Classification 2010: 35B40, 35C20, 35L60, 35Q35.

1 Introduction

We consider the following equations of multi-dimensional compressible viscoelastic flows:

$$
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) - \text{div}(2\mu D(u) + \lambda \text{div} \text{Id}) + \nabla P &= \alpha \text{div}(\rho F F^T), \\
\partial_t F + u \cdot \nabla F &= \nabla u F,
\end{align*}
$$

(1.1)

where $\rho \in \mathbb{R}_+$ is the density, $u \in \mathbb{R}^n$ stands for the velocity and $F \in \mathbb{R}^{n \times n}$ is the deformation gradient. $F^T$ means the transpose matrix of $F$. The pressure $P$ depends only upon the density and the function will be taking suitably smooth. Notations $\text{div}$, $\otimes$ and $\nabla$ denote the divergence operator, Kronecker tensor product and gradient operator, respectively. $D(u) = \frac{1}{2}(\nabla u + \nabla u^T)$.
is the strain tensor. The density-dependent viscosity coefficients $\mu$, $\lambda$ are assumed to be smooth and to satisfy $\mu > 0$, $\nu \triangleq \lambda + 2\mu > 0$. For simplicity, the elastic energy $W(F)$ in system (1.1) has been taken to be the special form of the Hookean linear elasticity:

$$W(F) = \frac{\alpha}{2}|F|^2, \quad \alpha > 0,$$

which, however, does not reduce the essential difficulties for analysis. The methods and results of this paper could be applied to more general cases.

In this paper, we focus on the Cauchy problem of system (1.1), so the corresponding initial data are supplemented by

$$(\rho, F; u)|_{t=0} = (\rho_0(x), F_0(x); u_0(x)), \quad x \in \mathbb{R}^n. \quad (1.2)$$

It is well known that there are some fluids do not satisfy the classical Newtonian law. Also, there have been many attempts to capture different phenomena for non-Newtonian fluids, see for example [13, 14, 23, 27] etc.. System (1.1) is compressible viscoelastic flow of the Oldroyd type exhibiting the elastic behavior, which is one of the non-Newtonian fluids. We are interested in the well-posedness and stability of solutions to the Cauchy problem (1.1)-(1.2), at least under the perturbation of constant equilibrium state $(1, I, 0)$.

Here let’s first recall previous efforts related to viscoelastic flows. For the incompressible viscoelastic flows, there has been much important progress on classical solutions. Lin-Liu-Zhang [23], Chen-Zhang [7], Lei-Liu-Zhou [24] and Lin-Zhang [28] established the local and global well-posedness with small data in Sobolev space $H^s$. Hu-Wu [21] proved the long-time behavior and weak-strong uniqueness of solutions. Chemin-Masmoudi [4] proved the existence of a local solution and a global small solution in critical Besov spaces, where the Cauchy-Green strain tensor is available in the evolution equation. Qian [29] proved the well-posedness of the incompressible viscoelastic system in critical spaces. Subsequently, Zhang-Fang [33] proved the global well-posedness in the critical $L^p$ Besov space. On the other hand, the global existence of weak solutions is still an open problem. Lions and Masmoudi [26] considered a special case that the contribution of the strain rate is neglected, and proved the global existence of a weak solution with general initial data.

For compressible viscoelastic flows, Lei-Zhou [28] proved the global existence of classical solutions for the two-dimensional Oldroyd model via the incompressible limit. The local existence of strong solutions was obtained by Hu-Wang [19]. Shortly, Hu-Wang [18] and Qian-Zhang [30] independently proved the global existence in the critical $L^2$ Besov space with initial data near equilibrium. For convenience of reader, we would like to state their results as follows.

**Theorem 1.1** ([18, 30]) Assume that $P'(1) > 0$. Then there exists two constant $\eta$ and $M$ such that if

$$(\rho_0 - 1, F_0 - I; u_0) \in \left(\dot{B}^{n/2-1,n/2}_{2,2}\right)^{1+n^2} \times \left(\dot{B}^{n/2-1}_{2,1}\right)^n.$$

and

$$\|(\rho_0 - 1, F_0 - I)\|_{\dot{B}^{n/2-1,n/2}_{2,2}} + \|u_0\|_{\dot{B}^{n/2-1}_{2,1}} \leq \eta,$$

and
then there exists a global unique solution \((\rho, F; u)\) of (1.1)–(1.2) such that

\[
\|(\rho - 1, F - I; u)\|_{\dot{B}^{n/2}_{2,1}} \leq M\left(\|(\rho_0 - 1, F_0 - I)\|_{\dot{B}^{n/2-1,n/2}_{2,1}} + \|u_0\|_{\dot{B}^{n/2-1}_{2,1}}\right),
\]

where

\[
\dot{B}^{n/2}_{2,1} \triangleq \left(\tilde{C}_b(\mathbb{R}_+; \dot{B}^{n/2-1,n/2}_{2,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{n/2+1,n/2}_{2,1})\right)^n \times \left(\tilde{C}_b(\mathbb{R}_+; \dot{B}^{n/2-1}_{2,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{n/2+1}_{2,1})\right)^n:
\]

with its norm, for \((a, O; v) \in \dot{E}^{n/2}\),

\[
\|(a, O; v)\|_{\dot{E}^{n/2}} = \|(a, O)\|_{L^\infty \dot{B}^{n/2-1,n/2}_2 \cap L^1 \dot{B}^{n/2+1,n/2}_2} + \|v\|_{L^\infty \dot{B}^{n/2-1}_2 \cap L^1 \dot{B}^{n/2+1}_2}.
\]

Concerning those norm notations for the hybrid Besov space \(\dot{L}^p\dot{B}^{s,\sigma}_{2,p}(p \geq 2)\) and \(\tilde{C}_b(\mathbb{B}^{s,\sigma}_{2,p})\), the reader is referred to Section 2 below. Theorem 1.1 stems from the scaling consideration. Note that system (1.1) is scaling invariant under the following transformation: for any constant \(\kappa\),

\[
(\rho_0(x), F_0(x); u_0(x)) \rightarrow (\rho_0(\kappa x), F_0(\kappa x); \lambda u_0(\kappa x)),
\]

\[
(\rho(t, x), F(t, x); u(t, x)) \rightarrow (\rho(\kappa^2 t, \kappa x), F(\kappa^2 t, \kappa x); \kappa u(\kappa^2 t, \kappa x)),
\]

up to changes of the Pressure \(P\) into \(\kappa^2 P\) and the constant \(\alpha\) into \(\kappa^2 \alpha\). This indicates the following definition of the critical Besov space.

**Definition 1.1** A functional space is called a critical space if the associated norm is invariant under the transformation \((\rho(t, x), F(t, x); u(t, x)) \rightarrow (\rho(\kappa^2 t, \kappa x), F(\kappa^2 t, \kappa x); \kappa u(\kappa^2 t, \kappa x))\) (up to a constant independent of \(\kappa\)).

Obviously, it is easy to see that \((\dot{B}^{n/2}_{2,1})^{1+n^2} \times (\dot{B}^{n/2-1}_{2,1})^n\) is the critical space according to Definition 1.1. It should be emphasized that such basic idea is motivated by the seminal paper [8], where the author first proved the global well-posedness for the compressible Navier-Stokes equations near equilibrium. Compared to [8], there is an outstanding difficulty for the compressible viscoelastic system, that is, how to capture the damping effect of the deformation tensor among more complicated coupling between the velocity, the density and the deformation tensor. Hu-Wang [18] and Qian-Zhang [30] independently explored some intrinsic properties of the viscoelastic system and established uniform estimate for more complicated linearized hyperbolic-parabolic systems, which eventually leads to Theorem 1.1.

The goal of this paper is twofold: firstly, we aim at extending the above statement (Theorem 1.1) to the critical \(L^p\) Besov space, which allows highly large oscillating initial velocity. Secondly, we shall exhibit the long time behavior of the constructed solution.

Denote

\[
E^{n/p} \triangleq \left\{(a, O; v) \mid (a, O; v) \in \left(\tilde{C}_b(\mathbb{R}_+; \dot{B}^{n/2-1,n/p}_{2,p}) \cap L^1(\mathbb{R}_+; \dot{B}^{n/2+1,n/p}_{2,p})\right)^n \right\}.
\]
\[ \times \left( \tilde{C}_b(\mathbb{R}_+; \dot{B}^{n/2-1,n/p-1}_{2,p}) \cap L^1(\mathbb{R}_+; \dot{B}^{n/2+1,n/p+1}_{2,p}) \right) \]

with its norm
\[ \| (\alpha, O; v) \|_{\mathcal{E}^{n/p}} = \| (\alpha, O) \|_{L^\infty \dot{B}^{n/2-1,n/p-1}_{2,p} \cap L^1 \dot{B}^{n/2+1,n/p+1}_{2,p}} + \| v \|_{L^\infty \dot{B}^{n/2-1,n/p-1}_{2,p} \cap L^1 \dot{B}^{n/2+1,n/p+1}_{2,p}}. \]

Now, we state the first result as follows.

**Theorem 1.2** Assume that \( P'(1) > 0 \). Let \( p \) satisfying \( 2 \leq p \leq \min \left( 4, \frac{2n}{n-2} \right) \) and, additionally, \( p \neq 4 \) if \( n = 2 \). If there exists two constant \( \eta \) and \( M \) such that if
\[
(\rho_0 - 1, F_0 - I; u_0) \in \left( \dot{B}^{n/2-1,n/p}_{2,p} \right)^{1+n^2} \times \left( \dot{B}^{n/2-1,n/p-1}_{2,p} \right)^n,
\]
and
\[
\| (\rho_0 - 1, F_0 - I) \|_{\dot{B}^{n/2-1,n/p}_{2,p}} + \| u_0 \|_{\dot{B}^{n/2-1,n/p-1}_{2,p}} \leq \eta,
\]
then the Cauchy problem (1.1)-(1.2) has a global unique solution \((\rho, F; u)\) such that \((\rho - 1, F - I; u) \in \mathcal{E}^{n/p} \) and
\[
\| (\rho - 1, F - I; u) \|_{\mathcal{E}^{n/p}} \leq M \left( \| (\rho_0 - 1, F_0 - I) \|_{\dot{B}^{n/2-1,n/p}_{2,p}} + \| u_0 \|_{\dot{B}^{n/2-1,n/p-1}_{2,p}} \right).
\]

In comparison with those results in critical \( L^p \) framework for compressible Navier-Stokes equations (see for example [3] and [5]), Theorem 1.2 is not so surprising. Let’s point out some new ingredients in the current proofs. To the best of our knowledge, there is a technical difficulty arising from a loss of one derivative for compressible N-S fluids, since there is no smoothing effect at high frequencies. To eliminate it, their proofs heavily rely on a paralinearized version combined with a Lagrangian change of variables, see [3, 5] for details.

To the compressible viscoelastic system, the situation becomes more complicated. As shown by [30], the damping effect of \( F \) can be produced by some intrinsic conditions (see Proposition 3.1), however, similar to the density, there is not any smoothing effect at high frequencies. Here, in order to solve (1.1) globally, we follow an elementary energy approach in terms of effective velocity rather than the elaborate Lagrangian change. The argument has been developed by Haspot [15, 16] for compressible Navier-Stokes equations, which is based on the use of Hoff’s viscous effective flux in [17]. Here, we introduce the following “two effective velocities”

\[
w = \nabla (-\Delta)^{-1} (2a - \text{div} v), \quad \Omega^{ij} = e^{ij} + \frac{1}{\mu_0} \Lambda (-\Delta)^{-1} O^{ij}.
\]

Indeed, the definition of \( w \) is almost the same as that in [15, 16]. A slight difference lies on the coefficient of \( a \), which comes from contribution of the deformation gradient \( F \). Another effective velocity with respect to \( \Omega^{ij} \) is new, which allows to cancel the coupling between \( e^{ij} \) and \( O^{ij} \) at high frequencies (see Sections 4 and 5 for more details). In physical dimensions \( n = 2, 3 \), the value of \( p \) enable us to consider the case \( p > n \) for which the velocity regularity exponent \( n/p - 1 \) becomes negative. Consequently, Theorem 1.2 applies to large highly oscillating initial velocities (see [3, 5] for more explanation).
An interesting question follows after gaining Theorem 1.2. One may wonder how the global strong solutions constructed above look like for large time. Although providing an accurate long-time asymptotic description is still out of reach, a number of results concerning the time decay rates of global solutions, sometimes referred to as $L^q - L^r$ decay rates are available. For example, Hu-Wu [20] proved the global existence of strong solutions to (1.1) as initial data are the small perturbation $(1, I; 0)$ in $H^2(\mathbb{R}^3)$. Furthermore, with the extra $L^1(\mathbb{R}^3)$ assumption, it was shown that those solutions converged to equilibrium state at the following way

$$\|((\rho - 1, F - I; u))_{L^p} \leq C(t)^{-\frac{3}{2}(1 - \frac{1}{p})}. \quad (1.4)$$

The decay rate in (1.4) turns out the same one for the heat kernel, which is sometime referred as the optimal time-decay rate. Next, we state a decay result for those solutions constructed in Theorem 1.2. Precisely, one has

**Theorem 1.3** Let $n \geq 2$ and $p$ satisfies $2 \leq p \leq \min\left(4, 2n/(n - 2)\right)$ and $p \neq 4$ if $n = 2$. Let $(\rho_0, u_0, F_0)$ fulfill the assumptions of Theorem 1.2 and $(\rho, u, F)$ be the global solution of System (1.1). Then there exists a constant $\sigma = \sigma(p, n, \lambda, \mu, \alpha, \rho)$ such that if additionally

$$\mathcal{G}_{p,0} \triangleq \|((\rho_0 - 1, F_0 - I; u_0))_{\dot{B}^{s_0}_{2,\infty}} \leq \sigma \quad \text{with} \quad s_0 \triangleq n(2/p - 1/2), \quad (1.5)$$

then we have for $t \geq 0$,

$$\mathcal{G}_p(t) \lesssim \left(\mathcal{G}_{p,0} + \|(\nabla \rho_0, \nabla F_0; u_0)_{\dot{B}^{n/p}_{p,1}}\right), \quad (1.6)$$

where $\mathcal{G}_p(t)$ is defined by

$$\mathcal{G}_p(t) \triangleq \sup_{s \in [\epsilon - s_0, \frac{3}{2} + 1]} \|((\tau)^{\frac{n+2}{2}}(\rho - 1, F - I; u))_{L_t^\infty \dot{B}^{s}_{2,1}} + \|((\tau)^{\alpha}(\nabla \rho, \nabla F; u))_{L_t^\infty \dot{B}^{n/p}_{p,1}}\|
+ \|\tau \nabla u\|_{L_t^\infty \dot{B}^{n/p}_{p,1}}, \quad (1.7)$$

with $\alpha := n/p + 1/2 - \epsilon$ and $\epsilon > 0$ is a sufficiently small constant.

Here $\|f\|_*$ and $\|f\|_h$ represent the low and high frequency part of some norm $\|f\|_*$ to a tempered distribution $f$ whose exact definition will be given in Section 2.

Some comments are in order.

1. Due to the Sobolev imbedding properties $L^1 \hookrightarrow \dot{B}^{0}_{1,\infty} \hookrightarrow \dot{B}^{-n/2}_{2,\infty}$, $\dot{H}^{-n/2} \hookrightarrow \dot{B}^{-n/2}_{2,\infty}$, our low-frequency assumption is less restrictive. Actually, the assumption is relevant in other contexts like the Boltzmann equation (see [31]), or hyperbolic systems with dissipation (see [32]).

2. The decay result remains true in the case of large highly oscillating initial velocities, since the case $p > n$ occurs in physical dimensions $n = 2, 3$, which was not shown by previous efforts (see for example [20]).
3. Likewise, “two effective velocities” play a key role in establishing the nonlinear time-weighted inequality (1.7). Furthermore, the optimal decay estimates of $L^a$-$L^r$ type can be derived from the definition of $G_p(t)$ by using interpolation tricks. The interested reader is referred to [6] for similar details.

The rest of this paper is arranged as follows: In Section 2, we first review the Littlewood-Paley theory and give definitions and estimates for the hybrid-Besov space. In Section 3, we reformulate our system into a hyperbolic-parabolic system coupled by the density, the velocity and the deformation gradient. Section 4 is devoted to presenting the proof of Theorem 1.2. In Section 5, we prove the decay estimate in Theorem 1.3. Some analysis properties in the hybrid Besov space are also given in the Appendix.

2 Littlewood-Paley Theory and the Hybrid Besov Space

Throughout the paper, we denote by $C$ a generic constant which may be different from line to line. The notation $A \lesssim B$ means $A \leq CB$ and $A \approx B$ indicates $A \leq CB$ and $B \leq CA$.

2.1 Littlewood-Paley decomposition

Let’s begin with the Littlewood-Paley decomposition. There exists two radial smooth functions $\varphi(x), \chi(x)$ supported in the annulus $\mathcal{C} = \{\xi \in \mathbb{R}^n : 3/4 \leq |\xi| \leq 8/3\}$ and the ball $B = \{\xi \in \mathbb{R}^n : |\xi| \leq 4/3\}$, respectively such that

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1 \quad \forall \xi \in \mathbb{R}^n.$$ $$\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$ 

The homogeneous dyadic blocks $\hat{\Delta}_j$ and the homogeneous low-frequency cut-off operators $\hat{S}_j$ are defined for all $j \in \mathbb{Z}$ by

$$\hat{\Delta}_j u = \varphi(2^{-j} D)f, \quad \hat{S}_j f = \sum_{k \leq j-1} \hat{\Delta}_k f = \chi(2^{-j} D)f.$$ 

The following Bernstein inequality will be repeatedly used throughout the paper.

Lemma 2.1 ([2]) Let $\mathcal{C}$ be an annulus and $B$ a ball. A constant $C$ exists such that for any nonnegative integer $k$, any couple $(p, q)$ in $[1, \infty]^2$ with $q \geq p \geq 1$, and any function $u$ of $L^p$, we have

$$\text{Supp} \hat{u} \subset \lambda B \Rightarrow \sup_{|\alpha| = k} \| \partial^\alpha u \|_{L^q} \leq C^{k+1} \lambda^{k+n(1 - \frac{1}{q})} \| u \|_{L^p},$$

$$\text{Supp} \hat{u} \subset \lambda \mathcal{C} \Rightarrow C^{-k-1} \lambda^k \| u \|_{L^p} \leq \sup_{|\alpha| = k} \| \partial^\alpha u \|_{L^p} \leq C^{k+1} \lambda^k \| u \|_{L^p}.$$
2.2 The hybrid Besov space

We denote by $\mathcal{Z}'(\mathbb{R}^n)$ the dual space of

$\mathcal{Z}(\mathbb{R}^n) \triangleq \{ f \in \mathcal{S}(\mathbb{R}^n) : \partial^\alpha \hat{f}(0) = 0, \forall \alpha \in (\mathbb{N} \cup 0)^n \}$.

Firstly, we give the definition of the homogeneous Besov space.

**Definition 2.1** Let $s$ be a real number and $(p, r)$ be in $[1, \infty]^2$. The homogeneous Besov space $\dot{B}_s^{1,1}$ consists of those distributions $u \in \mathcal{Z}'(\mathbb{R}^n)$ such that

$$\| u \|_{\dot{B}_s^{1,1}} \triangleq \left( \sum_{j \in \mathbb{Z}} 2^{jsr} \| \dot{\Delta}_j u \|_{L^p} \right)^{\frac{1}{r}} < \infty.$$

Secondly, we introduce the hybrid Besov space that will be used in this paper.

**Definition 2.2** Let $s, \sigma \in \mathbb{R}$, $1 \leq p \leq +\infty$. The hybrid Besov space $\dot{B}_s^{\infty,\sigma}$ is defined by

$$\dot{B}_s^{\infty,\sigma} \triangleq \{ f \in \mathcal{Z}'(\mathbb{R}^n) : \| f \|_{\dot{B}_s^{\infty,\sigma}} < \infty \},$$

with

$$\| f \|_{\dot{B}_s^{\infty,\sigma}} \triangleq \sum_{2^k \leq R_0} 2^{ks} \| \dot{\Delta}_k f \|_{L^2} + \sum_{2^k > R_0} 2^{k\sigma} \| \dot{\Delta}_k f \|_{L^p},$$

where $R_0$ is a fixed and sufficiently large constant which may depending on $\lambda(1), \mu(1), p$ and $n$.

Since we are concerned with time-dependent functions valued in Besov spaces, the space-time mixed norm is usually given by

$$\| u \|_{L^q_t \dot{B}_s^{\infty,\sigma}} := \left\| \| u(t, \cdot) \|_{\dot{B}_s^{\infty,\sigma}} \right\|_{L^q(0, T)}.$$

Here, we introduce another space-time mixed Besov norm, which is referred to Chemin-Lerner’s spaces. The definition is as follows.

$$\| u \|_{\dot{L}^q_t \dot{B}_s^{\infty,\sigma}} \triangleq \sum_{2^k \leq R_0} 2^{ks} \| \dot{\Delta}_k u \|_{L^q(0, T)L^2} + \sum_{2^k > R_0} 2^{k\sigma} \| \dot{\Delta}_k u \|_{L^q(0, T)L^2}.$$

The index $T$ will be omitted if $T = +\infty$ and we shall denote by $\tilde{C}_0(\dot{B}_s^{\infty,\sigma})$ the subset of functions $\tilde{L}^\infty(\dot{B}_s^{\infty,\sigma})$ which are continuous from $\mathbb{R}_+$ to $\dot{B}_s^{\infty,\sigma}$. It is easy to check that $\tilde{L}_T^1 \dot{B}_s^{\infty,\sigma} = L_T^1 \dot{B}_s^{\infty,\sigma}$ and $\tilde{L}_T^q \dot{B}_s^{\infty,\sigma} \subseteq L_T^q \dot{B}_s^{\infty,\sigma}$ for $q > 1$.

Also, for a tempered distribution $f$, we denote

$$f^\ell \triangleq \sum_{2^k \leq R_0} \dot{\Delta}_k f, \quad f^h \triangleq f - f^\ell,$$

and

$$\| f \|_{\dot{B}_s^{1,1}}^\ell = \sum_{2^k \leq R_0} 2^{ks} \| \dot{\Delta}_k f \|_{L^p}, \quad \| f \|_{\dot{B}_s^{1,1}}^h = \sum_{2^k > R_0} 2^{k\sigma} \| \dot{\Delta}_k f \|_{L^p}.$$
Then we have
\[ \|f\|_{\dot{B}^s_{p,q}} = \sum_{2^k \leq R_0} 2^{ks} \|\hat{\Delta}_k f\|_{L^q(0,T;L^p)}, \quad \|f\|_{\dot{B}^s_{p,\infty}} = \sum_{2^k > R_0} 2^{ks} \|\hat{\Delta}_k f\|_{L^q(0,T;L^p)}, \]
for \( s \in \mathbb{R} \).

**Lemma 2.2** For the Besov space, we have the following properties:

1. \( \dot{B}^s_{p,\sigma} \subseteq \dot{B}^s_{p,\varphi} \) for \( s_1 \geq s_2 \) and \( \dot{B}^s_{p,\sigma_2} \subseteq \dot{B}^s_{p,\sigma_1} \) for \( \sigma_1 \leq \sigma_2 \).

2. Interpolation: For \( s_1, s_2, \sigma_1, \sigma_2 \in \mathbb{R} \) and \( \theta \in [0,1] \), we have
\[ \|f\|_{\dot{B}^{s_1+\sigma_1,\sigma_2}_{p,\theta}} \leq \|f\|^{\theta}_{\dot{B}^{s_1,\sigma_1}_{p,1}} \|f\|^{1-\theta}_{\dot{B}^{s_2,\sigma_2}_{p,1}}. \]

3. Embedding: \( L^\infty \hookrightarrow \dot{B}^{s-n/2+n/p}_{2,p} \);
\[ \dot{B}^s_{2,1} \hookrightarrow \dot{B}^{s,n/2-n/2}_{2,p} \hookrightarrow \dot{B}^{s-n/2+n/p}_{p,1} \text{ for } p \geq 2. \]

**Lemma 2.3** ([11]) Let \( 1 \leq p, q, q_1, q_2 \leq \infty \) with \( \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q} \). Then we have the following:

1. If \( s_1, s_2 \leq n/p \) and \( s_1 + s_2 > n \max(0,2/p-1) \), then
\[ \|fg\|_{\dot{L}^{q_1}_{L}((\dot{B}^{s_1}_{p,1})^{s_2-n/p})} \leq C \|f\|_{\dot{L}^{q_1}_{L}(\dot{B}^{s_1}_{p,1})} \|g\|_{\dot{L}^{q_2}_{L}(\dot{B}^{s_2}_{p,1})}. \]

2. If \( s_1 \leq n/p, s_2 < n/p \) and \( s_1 + s_2 > n \max(0,2/p-1) \), then
\[ \|fg\|_{\dot{L}^{q_1}_{L}((\dot{B}^{s_1}_{p,1})^{s_2-n/p})} \leq C \|f\|_{\dot{L}^{q_1}_{L}(\dot{B}^{s_1}_{p,1})} \|g\|_{\dot{L}^{q_2}_{L}(\dot{B}^{s_2}_{p,1})}. \]

**Remark 2.1** Lemma 2.3 still remain true in the usual homogenous Besov spaces. For example the estimate in Lemma 2.3(1) becomes
\[ \|fg\|_{\dot{B}^{s_1+s_2-n/p}_{p,1}} \leq C \|f\|_{\dot{B}^{s_1}_{p,1}} \|g\|_{\dot{B}^{s_2}_{p,1}}. \]

**Lemma 2.4** ([6]) Let \( \sigma > 0 \) and \( 1 \leq p, r \leq \infty \). Then \( \dot{B}^\sigma_{p,r} \cap L^\infty \) is an algebra and
\[ \|fg\|_{\dot{B}^\sigma_{p,r}} \lesssim \|f\|_{\dot{B}^\sigma_{p,r}} \|g\|_{\dot{B}^\sigma_{p,r}} + \|g\|_{L^\infty} \|f\|_{\dot{B}^\sigma_{p,r}}. \]

Let \( \sigma_1, \sigma_2, p_1, p_2 \) satisfy
\[ \sigma_1 + \sigma_2 > 0, \sigma_1 \leq n/p_1, \sigma_2 \leq n/p_2, \sigma_1 \geq \sigma_2, \frac{1}{p_1} + \frac{1}{p_2} \leq 1. \]

Then we have
\[ \|fg\|_{\dot{B}^\sigma_{q,1}} \lesssim \|f\|_{\dot{B}^\sigma_{p_1,1}} \|f\|_{\dot{B}^\sigma_{p_2,1}} \text{ with } \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\sigma_1}{n}. \]
Finally for exponents \( \sigma > 0, \ 1 \leq p_1, p_2, q \leq \infty \) satisfying
\[
\frac{n}{p_1} + \frac{n}{p_2} - n \leq \sigma \leq \min\left(\frac{n}{p_1}, \frac{n}{p_2}\right) \quad \text{and} \quad \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\sigma}{n},
\]
we have
\[
\|fg\|_{\dot{B}^{-\sigma}_{q, \infty}} \lesssim \|f\|_{\dot{B}^\sigma_{p_1, 1}} \|g\|_{\dot{B}^{-\sigma}_{p_2, \infty}}.
\]

Lemma 2.5 ([6]) There exists a universal integer \( N_0 \) such that for any \( 2 \leq p \leq 4 \), and \( \sigma > 0 \), we have
\[
\|fg^h\|_{\dot{B}^{-s_0}_{2, \infty}} \lesssim \left(\|f\|_{\dot{B}^\sigma_{p, 1}} + \|S_{k_0 + N_0} f\|_{L^p}\right) \|g^h\|_{\dot{B}^{-\sigma}_{p, \infty}},
\]
\[
\|f^h g\|_{\dot{B}^{-s_0}_{2, \infty}} \lesssim \left(\|f^h\|_{\dot{B}^\sigma_{p, 1}} + \|S_{k_0 + N_0} f^h\|_{L^p}\right) \|g\|_{\dot{B}^{-\sigma}_{p, \infty}},
\]
with \( s_0 = n\left(\frac{2}{p} - \frac{1}{2}\right) \) and \( \frac{1}{p} = \frac{1}{2} - \frac{1}{p} \).

Lemma 2.6 ([6]) Let \( 1 \leq p, p_1 \leq \infty \) and
\[
-\min\left(\frac{n}{p_1}, \frac{n}{p}\right) < \sigma \leq 1 + \min\left(\frac{n}{p}, \frac{n}{p_1}\right).
\]
There exists a constant \( C > 0 \), depending only on \( \sigma \) such that for all \( j \in \mathbb{Z} \), we have
\[
\|\left[\nabla \cdot, \nabla \Delta_{\tilde{A}_j}\right] z\|_{L^p} \leq C j 2^{-j(\sigma - 1)} \|\nabla v\|_{\dot{B}^\sigma_{p_1, 1}} \|\nabla z\|_{\dot{B}^\sigma_{p_1, 1}},
\]
where \((c_j)_{j \in \mathbb{Z}}\) denotes a sequence such that \( \|c_j\|_{\ell^1} \leq 1 \).

### 3 Reformulation of System (1.1)

Here, we present intrinsic properties of compressible viscoelastic flows, which have been explored in [30].

**Proposition 3.1** The density \( \rho \) and the deformation gradient \( F \) of (1.1) satisfy the following relations:
\[
\nabla \cdot (\rho F^T) = 0 \quad \text{and} \quad F^{ik} \partial_i F^{ij} - F^{ij} \partial_i F^{ik} = 0,
\]
if the initial data \((\rho_0, F_0)\) satisfies
\[
\nabla \cdot (\rho_0 F_0^T) = 0 \quad \text{and} \quad F_0^{ik} \partial_i F_0^{ij} - F_0^{ij} \partial_i F_0^{ik} = 0.
\]

By Proposition 3.1, the \( i \)-th component of the vector \( \text{div}(\rho F^T) \) can be written as
\[
\partial_j(\rho F^{ik} F^{jk}) = \rho F^{jk} \partial_j F^{ik} + F^{ik} \partial_j (\rho F^{jk}) = \rho F^{jk} \partial_j F^{ik},
\]
where we used the first equality in (3.1).
Denote $\chi_0 = (P'(1))^{-1/2}$ and define
\[ a(t, x) = \rho_0(t, x), \quad v(t, x) = \chi_0 u(t, x), \quad O(t, x) = F(\chi_0^2 t, \chi_0 x) - I. \]
By using (3.3), we get
\[
\begin{aligned}
\partial_t a + v \cdot \nabla a + \nabla \cdot v &= -a\nabla \cdot v, \\
\partial_t v + v \cdot \nabla v - Av + \nabla a - \beta \nabla \cdot O = \beta O^{ij} \partial_j O^{ik} - I(a)Av - K(a)\nabla a \\
&\quad + \frac{1}{1+a} \text{div}(2\tilde{\mu}(a)D(v) + \tilde{\lambda}(a)\text{div}v\text{Id}), \\
\partial_t O + v \cdot \nabla O - \nabla v &= \nabla v O,
\end{aligned}
\]
where
\[
I(a) \triangleq \frac{a}{1+a}, \quad K(a) \triangleq \frac{P'(1+a)}{(1+a)P'(1)} - 1, \quad A = \mu(1)A + (\lambda(1) + \mu(1))\nabla \text{div},
\]
and
\[
\beta = \frac{\alpha}{P'(1)}, \quad \tilde{\mu}(a) = \mu(1+a) - \mu(1), \quad \tilde{\lambda}(a) = \lambda(1+a) - \lambda(1).
\]
$O^{jk}\partial_j O^{ik}$ is a vector function whose components are $(O^{jk}\partial_j O^{ik})_{i=1}^n$. For simplicity, we set $\lambda(1) = \lambda_0, \mu(1) = \mu_0$. Here and below, we normalize $\beta = 1$ and $\nu(1) := \lambda(1) + 2\mu(1) = 1$ without loss of generality.

For $s \in \mathbb{R}$, we denote
\[
\Lambda^s f \triangleq F^{-1}(|\xi|^s F(f)),
\]
and introduce two variables as in [30]:
\[
d = \Lambda^{-1}\text{div} v, \quad e^{ij} = \Lambda^{-1}\partial_j v^i. \tag{3.5}
\]
Using the second equality in (3.1), we have
\[
\Lambda^{-1}(\partial_j \partial_k O^{ik}) = -\Lambda \partial_j O^{ij} - \Lambda^{-1}\partial_k (O^{ij} \partial_l O^{lk} - O^{ik} \partial_l O^{ij}). \tag{3.6}
\]
Hence, with aid of (3.6), the system (3.4) can be reformulated as follows
\[
\begin{aligned}
\partial_t a + \Lambda d &= G_1, \\
\partial_t e^{ij} - \mu_0 \Delta e^{ij} - (\lambda_0 + \mu_0)\partial_i \partial_j d + \Lambda^{-1} \partial_i \partial_j a + \Lambda O^{ij} &= G_4^{ij}, \\
\partial_t O^{ij} - \Lambda e^{ij} &= G_3^{ij}, \\
d &= -\Lambda^{-2} \partial_i \partial_j e^{ij}, \quad v^i = -\Lambda^{-1} \partial_j e^{ij},
\end{aligned}
\tag{3.7}
\]
where $G_1 = -a\nabla \cdot v - v \cdot \nabla a$, $G_3^{ij} = \partial_k v^i O^{kj} - v \cdot \nabla O^{ij}$ and
\[
G_4^{ij} = -\Lambda^{-1} \partial_j \left(v \cdot \nabla v^i - O^{lk} \partial_l O^{ik} + I(a)(Av)^i + K(a)\partial_l a\right)
-\Lambda^{-1} \partial_k (O^{ij} \partial_l O^{lk} - O^{ik} \partial_l O^{ij})
+\Lambda^{-1} \partial_j \left(\frac{1}{1+a} \text{div}(2\tilde{\mu}(a)D(v) + \tilde{\lambda}(a)\text{div}v\text{Id})\right)^i.
\]
Additionally, we need the auxiliary equation in subsequent estimates
\[
\partial_t O^{ij} = -\partial_j a - G_0^{ij}, \quad G_0^{ij} = \partial_i (a O^{ij}), \tag{3.8}
\]
which is deduced from the first equality in (3.1).
4 Proof of Theorem 1.2

Inspired by [5], we may extend those results in [30] to the \(L^p\) critical framework. First of all, it is convenient to give the following interpolation inequalities

\[
\|f\|_{L^2_tB^{n/2-1,n/p}_{2,p}} \lesssim \|f\|_{L^2_t(0,T;B^{n/2-1,n/p-1}_{2,p})}^{1/2} \|f\|_{L^2_tB^{n/2+1,n/p+1}_{2,p}}^{1/2},
\]

(4.1)

The proof Theorem 1.2 is divided into several parts. The first one is to establish two a priori estimates.

4.1 Two a priori estimates

Let \(T > 0\). We denote the following functional space \(E^{n/p}_T\) by

\[
E^{n/p}_T \triangleq \{(a, O; v) \in (\tilde{L}^\infty(0, T; \dot{B}^{n/2-1,n/p}_{2,p}) \cap L^1(0, T; \dot{B}^{n/2+1,n/p}_{2,p}))^{1+n^2} \times (\tilde{L}^\infty(0, T; \dot{B}^{n/2-1,n/p-1}_{2,p}) \cap L^1(0, T; \dot{B}^{n/2+1,n/p+1}_{2,p}))^n \}
\]

with the norm

\[
\| (a, O; v) \|_{E^{n/p}_T} \triangleq \| (a, O) \|_{L^\infty(0, T; \dot{B}^{n/2-1,n/p}_{2,p}) \cap L^1(0, T; \dot{B}^{n/2+1,n/p+1}_{2,p})} + \| v \|_{L^\infty(0, T; \dot{B}^{n/2-1,n/p-1}_{2,p}) \cap L^1(0, T; \dot{B}^{n/2+1,n/p+1}_{2,p})}.
\]

Proposition 4.1 Let \(2 \leq p \leq \min(4, \frac{2n}{n-2})\) and \(p < 2n\). Assume that \((a, O; v)\) is a strong solution of system (3.4) on \([0, T] \) with

\[
\| a \|_{L^\infty([0, T] \times \mathbb{R}^n)} \leq \frac{1}{2}.
\]

Then we have

\[
\| (a, O; v) \|_{E^{n/p}_T} \leq C \{ \| (a_0, O_0; v_0) \|_{E^{n/p}_0} + \| (a, O; v) \|_{E^{n/p}_T}^2 (1 + \| (a, O; v) \|_{E^{n/p}_T}^{n+3}) \},
\]

(4.2)

where \( \| (a_0, O_0; v_0) \|_{E^{n/p}_0} \triangleq \| (a_0, O_0) \|_{B^{n/2-1,n/p}_{2,p}} + \| v_0 \|_{B^{n/2-1,n/p-1}_{2,p}} \).

We introduce another functional space \(E^{n/2}_T\) defined by

\[
E^{n/2}_T \triangleq \{(a, O; v) \in (\tilde{L}^\infty(0, T; \dot{B}^{n/2-1,n/2}_{2,2}) \cap L^1(0, T; \dot{B}^{n/2+1,n/2}_{2,2}))^{1+n^2} \times (\tilde{L}^\infty(0, T; \dot{B}^{n/2-1}_{2,1}) \cap L^1(0, T; \dot{B}^{n/2+1}_{2,1}))^n \}
\]

with the norm

\[
\| (a, O; v) \|_{E^{n/2}_T} \triangleq \| (a, O) \|_{L^\infty(0, T; \dot{B}^{n/2-1,n/2}_{2,2}) \cap L^1(0, T; \dot{B}^{n/2+1,n/2}_{2,2})} + \| v \|_{L^\infty(0, T; \dot{B}^{n/2-1}_{2,1}) \cap L^1(0, T; \dot{B}^{n/2+1}_{2,1})}.
\]
Proposition 4.2 Under the assumption of Proposition 4.1, we have

\[
\| \langle a, O ; v \rangle \|_{L^p_t} \leq C \left\{ \| \langle a_0, O_0 ; v_0 \rangle \|_{L^p_t} + 2 \right\},
\]

where \( \| \langle a, O ; v \rangle \|_{L^p_t} \triangleq \| \langle a_0, O_0 \rangle \|_{H^{n/2-1,n/2}} + \| v_0 \|_{H^{n/2-1}}. \)

The proof of Propositions 4.1-4.2 lie in the pure energy methods in terms of low-frequency and high-frequency decompositions.

**Step 1: Low-frequency estimates** \( (2^k \leq R_0) \).

Denote \( a_k = \Delta_k a \), \( O_k = \Delta_k O \) and \( d_k = \Delta_k d \), \( e_k = \Delta_k e \) for simplicity. By applying \( \Delta_k \) to (3.7), we have

\[
\begin{cases}
\partial_t a_k + \Delta d_k = \Delta_k G_1, \\
\partial_t e_k^{ij} - \mu_0 \Delta e_k^{ij} - (\lambda_0 + \mu_0) \partial_i \partial_j d_k + \Lambda^{-1} \partial_i \partial_j a_k + \Lambda O_k^{ij} = \Delta_k G_3^{ij}, \\
\partial_t O_k^{ij} - \Lambda e_k^{ij} = \Delta_k G_3^{ij}, \\
d_k = -\Lambda^{-2} \partial_i \partial_j e_k^{ij}.
\end{cases}
\]

Taking \( L^2 \) inner product of (4.4) with \( e_k^{ij} \), and then summing up the resulting equation with respect to indices \( i, j \), we arrive at

\[
\frac{1}{2} \| e_k \|_{L^2}^2 + \mu_0 \| \Delta e_k \|_{L^2}^2 + (\lambda_0 + \mu_0) \| \Delta d_k \|_{L^2}^2 - (a_k | \Delta d_k | + (\Lambda O_k | e_k |) = (\Delta_k G_4 | e_k |),
\]

where we have used the fact \( d_k = -\Lambda^{-2} \partial_i \partial_j e_k^{ij} \).

Taking \( L^2 \) inner product of (4.4)_1 and (4.4)_3 with \( a_k \) and \( O_k \), respectively, and then adding the resulting equations to (4.5) together, we obtain

\[
\frac{1}{2} (\| a_k \|_{L^2}^2 + \| O_k \|_{L^2}^2 + \| e_k \|_{L^2}^2 ) + \mu_0 \| \Delta e_k \|_{L^2}^2 + (\lambda_0 + \mu_0) \| \Delta d_k \|_{L^2}^2 = (\Delta_k G_1 | a_k | + (\Delta_k G_4 | e_k |) + (\Delta_k G_3 | O_k |).
\]

To capture the dissipation arising from \( (a, O) \), we next apply the operator \( \Lambda \) to (4.4)_1 and take the \( L^2 \) inner product of the resulting equation with \( -d_k \). Also, we take the \( L^2 \) inner product of (4.4)_2 with \( \Lambda^{-1} \partial_i \partial_j a_k \). Therefore, we add those resulting equations and get

\[
-\frac{d}{dt} (\Lambda a_k | d_k |) + \| \Lambda a_k \|_{L^2}^2 - \| \Lambda d_k \|_{L^2}^2 - (\Lambda^2 d_k | \Lambda a_k | + (O_k^{ij} | \partial_i \partial_j a_k |)
\]

\[
= - (\Delta_k G_1 | d_k | + (\Delta_k G_4^{ij} | \Lambda^{-1} \partial_i \partial_j a_k |).
\]
On the other hand, we apply $\Lambda$ to (4.4)$_3$ and then take the $L^2$ inner product of the resulting equation with $e^i_k$. We also take the $L^2$ inner product of (4.4)$_2$ with $\Lambda O^i_k$. By summing up those resulting equations, we obtain

\[
\frac{d}{dt}(\Lambda O_k|e_k) + \|\Lambda O_k\|_{L^2}^2 = \|\Lambda e_k\|_{L^2}^2 - \nu_1(\Lambda O_k|e_k) - 2\nu_1(\Lambda a_k|d_k)
\]

\[
+ (\nu_0 - \nu_1)(\Lambda e_k|O_k) + \nu_1(\Lambda e_k|O_k) + \nu_1(\Lambda \partial_i \partial_j a_k|O_k) + 2\nu_1(\partial_i \partial_j a_k|O_k).
\]

Consequently, we are led to the following inequality

\[
\frac{1}{2}(\|a_k\|_{L^2}^2 + \|O_k\|_{L^2}^2 + \|e_k\|_{L^2}^2) + \nu_1(\Lambda e_k|O_k)
\]

\[= \nu_1(\Lambda a_k|d_k) + \nu_1(\Lambda \partial_i \partial_j a_k|O_k) + \nu_1(\Lambda \partial_i \partial_j a_k|O_k).
\]

Now, we multiply a small constant $\nu_1 > 0$ (to be determined) to (4.7) and (4.8), respectively, and then add the resulting equations with (4.6) together. Consequently, we are led to the following inequality

\[
\frac{1}{2}(\|a_k\|_{L^2}^2 + \|O_k\|_{L^2}^2 + \|e_k\|_{L^2}^2) + \nu_1(\Lambda e_k|O_k)
\]

\[= \nu_1(\Lambda a_k|d_k) + \nu_1(\Lambda \partial_i \partial_j a_k|O_k) + \nu_1(\Lambda \partial_i \partial_j a_k|O_k).
\]

It follows from (3.8) that

\[
(\partial_i \partial_j a_k|O^i_k) = (a_k|\partial_i \partial_j O^i_k) = \nu_1(\Lambda a_k|d_k) + \nu_1(\Lambda \partial_i \partial_j a_k|O_k) + \nu_1(\Lambda \partial_i \partial_j a_k|O_k).
\]

Inserting (4.10) into (4.9), we can get

\[
\frac{d}{dt}f_{\ell,k} + \overline{f}_{\ell,k} = \nu_1(\Lambda a_k|d_k) + \nu_1(\Lambda \partial_i \partial_j a_k|O_k) + \nu_1(\Lambda \partial_i \partial_j a_k|O_k)
\]

where

\[
f_{\ell,k} \triangleq \|a_k\|_{L^2}^2 + \|O_k\|_{L^2}^2 + \|e_k\|_{L^2}^2 + 2\nu_1(\Lambda O_k|e_k) - 2\nu_1(\Lambda a_k|d_k),
\]

\[
\overline{f}_{\ell,k} \triangleq (\mu_0 - \nu_1)(\Lambda e_k|O_k) + \nu_1(\Lambda e_k|O_k) + 3\nu_1(\Lambda a_k|d_k)
\]

\[+ \nu_1(\Lambda O_k|e_k) + \nu_1(\Lambda \partial_i \partial_j a_k|O_k) - \nu_1(\Lambda a_k|d_k).
\]

For any fixed $R_0$, we choose $\nu_1 \sim \nu_1(\lambda_0, \mu_0, R_0)$ sufficiently small such that

\[
f_{\ell,k} \sim \|a_k\|_{L^2}^2 + \|O_k\|_{L^2}^2 + \|e_k\|_{L^2}^2, \quad \overline{f}_{\ell,k} \sim 2^{\nu_1}(\|a_k\|_{L^2}^2 + \|O_k\|_{L^2}^2). \quad (4.12)
\]
By using Cauchy-Schwarz inequality in (4.11), we can get the following equality owing to $2^k \leq R_0$,

$$\frac{d}{dt} f_{\ell,k} + 2^k f_{\ell,k} \lesssim \sum_{i=0,1,3,4} \| \tilde{\Delta}_k G_i \|_{L^2}, \quad (4.13)$$

which indicates that

$$\|(a, O; e)\|_{L^2_{T} B^{n/2-1}_{2,1}} + \|(a, O; e)\|_{L^2_{T} B^{n/2+1}_{2,1}} \lesssim \|(a_0, O_0; e_0)\|_{L^2_{T} B^{n/2-1}_{2,1}} + \sum_{i=0,1,3,4} \| G_i \|^2_{L^2_{T} B^{n/2+1}_{2,1}}. \quad (4.14)$$

Next we begin to bound those nonlinear terms arising in $G_i$ ($i = 0, 1, 3, 4$). Since the quadratic terms containing $a$ and $v$ have already been done in [5], it suffices to deal with different terms involving in $O$ as well as those cubic terms due to density-dependent viscosities. More precisely, we need to estimate the following terms according to the definitions of $G_i$,

$$G^i_0 := \partial_i (a O^{ij}), \quad G^i_3 := \partial_k v^i O^{kj} - v \cdot \nabla O^{ij}, \quad \Lambda^{-1} \partial_j (O^{ik} \partial_i O^{jk}), \quad \Lambda^{-1} \partial_k (O^{ik} \partial_i O^{jk}) \text{ in } G^i_4, \quad (4.15)$$

and

$$\Lambda^{-1} \partial_j \left( \frac{1}{1 + a} \text{div} (2\tilde{\mu}(a) D(v) + \tilde{\lambda}(a) \text{div} v) \right) \text{ in } G^i_4. \quad (4.16)$$

We write $G^i_0 = \partial_o O^{ij} + a \partial_i O^{ij}$. Regarding $\partial_o O^{ij}$, by taking $\gamma = -1, r_1 = \infty, r_2 = 1, r_3 = r_4 = 2, s_1 = s_2 = n/2 - 1, t_1 = t_2 = n/2$ in (A.2) and using (4.1), we arrive at

$$\sum_{2^k \leq R_0} 2^{k(n/2-1)} \| \Delta_k (\partial_o O^{ij}) \|_{L^2_T L^2} \lesssim \| O \|_{L^\infty_{T} B^{n/2-1,n/p-1}_{2,p}} \| \nabla a \|_{L^2_T B^{n/2+1,n/p+1}_{2,p}} + \| \nabla a \|_{L^2_T B^{n/2-1,n/p-1}_{2,p}} \| \nabla O \|_{L^2_T B^{n/2,n/p}_{2,p}} \lesssim \| (a, O; v) \|^2_{\mathcal{E}^p_T}. \quad (4.17)$$

The terms $a \partial_i O^{ij}, v \cdot \nabla O^{ij}$ in $G^i_3$ and (4.15) may be treated along the same lines as $\partial_o O^{ij}$, so we omit the details for brevity. In order to bound $\partial_k v^i O^{kj}$ in $G^i_3$, by taking $\gamma = 0, r_1 = \infty, r_2 = 1, r_3 = r_4 = 2, s_1 = s_2 = n/2 - 1, t_1 = t_2 = n/2$ in (A.2) and using (4.1), we have

$$\sum_{2^k \leq R_0} 2^{k(n/2-1)} \| \Delta_k (\partial_k v^i O^{kj}) \|_{L^2_T L^2} \lesssim \| O \|_{L^\infty_{T} B^{n/2-1,n/p-1}_{2,p}} \| \nabla v \|_{L^2_T B^{n/2+1,n/p+1}_{2,p}} + \| \nabla v \|_{L^2_T B^{n/2-1,n/p-1}_{2,p}} \| \nabla O \|_{L^2_T B^{n/2,n/p}_{2,p}} \lesssim \| (a, O; v) \|^2_{\mathcal{E}^p_T}. \quad (4.18)$$

Next we bound the cubic term (4.16) in $G^i_4$. Denote

$$I := \frac{1}{1 + a} \text{div} (2\tilde{\mu}(a) D(v))$$
To bound $I_1$, we have
\[
\sum_{2^k \leq R_0} 2^{k(n/2-1)} \| \hat{\Delta}_k \left( \frac{1}{1 + a} \tilde{\mu}(a) \nabla^2 v \right) \|_{L^1_T L^2} \\
\lesssim \sum_{2^k \leq R_0} 2^{k(n/2-1)} \left( \| \hat{\Delta}_k (I(a) \tilde{\mu}(a) \nabla^2 v) \|_{L^1_T L^2} + \| \hat{\Delta}_k (\tilde{\mu}(a) \nabla^2 v) \|_{L^1_T L^2} \right) \\
\lesssim \| I(a) \|_{L^\infty_T \mathfrak{b}^{s+2,1-n/p-1}_2} \| \tilde{\mu}(a) \|_{L^1_T \mathfrak{b}^{n+2,1-n/p-1}_2} + \| \tilde{\mu}(a) \|_{L^\infty_T \mathfrak{b}^{n+2,1-n/p-1}_2} \\
\lesssim \left( 1 + \| I(a) \|_{L^\infty_T \mathfrak{b}^{n+2,1-n/p-1}_2} \right) \| \tilde{\mu}(a) \|_{L^1_T \mathfrak{b}^{n+2,1-n/p-1}_2}. \tag{4.19}
\]
where we have applied $s_1 = s_2 = n/2 - 1, t_1 = t_2 = n/2, r_1 = r_2 = \infty, r_3 = 1, \gamma = -1$ in (A.2) of Proposition A.1 to deal with the term $I(a) \tilde{\mu}(a) \nabla^2 v$. Now we need to estimate $\| \tilde{\mu}(a) \nabla^2 v \|_{L^1_T \mathfrak{b}^{n+2,1-n/p-1}_2}$. From (A.2) and (A.1), we have
\[
\sum_{2^k \leq R_0} 2^{k(n/2-1)} \| \hat{\Delta}_k (\tilde{\mu}(a) \nabla^2 v) \|_{L^1_T L^2} \\
\lesssim \| \tilde{\mu}(a) \|_{L^\infty_T \mathfrak{b}^{n+2,1-n/p-1}_2} \| \nabla^2 v \|_{L^1_T \mathfrak{b}^{n+2,1-n/p-1}_2} + \| \nabla^2 v \|_{L^1_T \mathfrak{b}^{n+2,1-n/p-1}_2} \| \tilde{\mu}(a) \|_{L^\infty_T \mathfrak{b}^{n+2,1-n/p-1}_2} \\
\lesssim \| \tilde{\mu}(a) \|_{L^\infty_T \mathfrak{b}^{n+2,1-n/p-1}_2} \| v \|_{L^{1_T} \mathfrak{b}^{n+2,1-n/p-1}_2}. \tag{4.20}
\]
Inserting (4.20) and (4.21) into (4.19) and applying Proposition A.2, we can get
\[
\sum_{2^k \leq R_0} 2^{k(n/2-1)} \| \hat{\Delta}_k \left( \frac{1}{1 + a} \tilde{\mu}(a) \nabla^2 v \right) \|_{L^1_T L^2} \\
\lesssim \left( 1 + \| I(a) \|_{L^\infty_T \mathfrak{b}^{n+2,1-n/p-1}_2} \right) \| \tilde{\mu}(a) \|_{L^\infty_T \mathfrak{b}^{n+2,1-n/p+1}_2} \\
\lesssim \left( 1 + \| a \|_{L^\infty_T \mathfrak{b}^{n+2,n/p-1}_2} \right)^{n+3} \| a \|_{L^\infty_T \mathfrak{b}^{n+2,1-n/p+1}_2} \| v \|_{L^{1_T} \mathfrak{b}^{n+2,1-n/p+1}_2} \\
\lesssim \left( 1 + \| (a, O; v) \|_{\mathfrak{c}^{n+2}_T} \right)^{n+3} \| (a, O; v) \|_{\mathfrak{c}^{n+2}_T}. \tag{4.22}
\]
To bound $I_2$, we have
\[
\sum_{2^k \leq R_0} 2^{k(n/2-1)} \| \hat{\Delta}_k \left( \frac{1}{1 + a} \nabla \tilde{\mu}(a) \nabla v \right) \|_{L^1_T L^2}
\]
\[ \sum_{2^k \leq R_0} 2^{k(n/2-1)} \left( \| I(a) \nabla \bar{\mu}(a) \nabla v \|_{L_T^1 L^2} + \| \Delta_k (\nabla \bar{\mu}(a) \nabla v) \|_{L_T^1 L^2} \right) \]

\[ \lesssim \| I(a) \|_{L_T^\infty B_{2, p}^{n/2-1, n/p-1}} \| \nabla \bar{\mu}(a) \nabla v \|_{L_T^1 B_{2, p}^{n/2, n/p-1}} + \| \nabla \bar{\mu}(a) \nabla v \|_{L_T^1 B_{2, p}^{n/2-1, n/p-1}} \]

\[ \lesssim \left( 1 + \| I(a) \|_{L_T^\infty B_{2, p}^{n/2-1, n/p-1}} \right) \| \nabla \bar{\mu}(a) \nabla v \|_{L_T^1 B_{2, p}^{n/2-1, n/p-1}}. \quad (4.23) \]

From (A.2) and (A.1), the estimate of \( \| \nabla \bar{\mu}(a) \nabla v \|_{L_T^1 B_{2, p}^{n/2-1, n/p-1}} \) is

\[ \sum_{2^k \leq R_0} 2^{k(n/2-1)} \| \Delta_k (\nabla \bar{\mu}(a) \nabla v) \|_{L_T^1 L^2} \]

\[ \lesssim \| \nabla \bar{\mu}(a) \|_{L_T^2 B_{2, p}^{n/2, n/p}} \| \nabla v \|_{L_T^2 B_{2, p}^{n/2-1, n/p-1}} + \| \nabla \bar{\mu}(a) \|_{L_T^\infty B_{2, p}^{n/2-1, n/p-1}} \| \nabla v \|_{L_T^1 B_{2, p}^{n/2, n/p}} \]

\[ \lesssim \| \mu \|_{L_T^2 B_{2, p}^{n/2, n/p}} + \| \bar{\mu} \|_{L_T^\infty B_{2, p}^{n/2-1, n/p}} \| \nabla v \|_{L_T^1 B_{2, p}^{n/2+1, n/p+1}}. \quad (4.24) \]

\[ \sum_{2^k > R_0} 2^{k(n/2-1)} \| \Delta_k (\nabla \bar{\mu}(a) \nabla v) \|_{L_T^1 L^p} \]

\[ \lesssim \| \nabla \bar{\mu}(a) \|_{L_T^2 B_{2, p}^{n/2, n/p}} \| \nabla v \|_{L_T^2 B_{2, p}^{n/2-1, n/p-1}} \]

\[ \lesssim \| \mu \|_{L_T^2 B_{2, p}^{n/2, n/p}}. \quad (4.25) \]

Inserting (4.24) and (4.25) into (4.23) and applying PropositionA.2 and (4.1), we can get

\[ \sum_{2^k \leq R_0} 2^{k(n/2-1)} \| \Delta_k (\nabla \bar{\mu}(a) \nabla v) \|_{L_T^1 L^2} \]

\[ \lesssim \left( 1 + \| I(a) \|_{L_T^\infty B_{2, p}^{n/2-1, n/p}} \right) \left( \| \bar{\mu}(a) \|_{L_T^\infty B_{2, p}^{n/2-1, n/p}} \| v \|_{L_T^1 B_{2, p}^{n+1, n/p+1}} + \| v \|_{L_T^2 B_{2, p}^{n/2, n/p}} \| \bar{\mu}(a) \|_{L_T^2 B_{2, p}^{n/2, n/p}} \right) \]

\[ \lesssim \left( 1 + \| (a, O; v) \|_{\mathcal{E}_{T}^{(n/p)+3}} \right) \| (a, O; v) \|_{\mathcal{E}_{T}^{n/p}}. \quad (4.26) \]

Since bound of the cubic term \( \frac{1}{1+a} \text{div}(\lambda(a)\text{div} v) \) is the same as \( I \), we omit the details. Summing up all the estimates and remembering (4.14), we get

\[ \| (a, O; v) \|_{L_T^\infty B_{2, 1}^{n/2, n/p}} \lesssim \| (a, O; v) \|_{L_T^1 B_{2, 1}^{n/2+1, n/p}} \]

\[ \lesssim \| (a_0, O_0; e_0) \|_{L_T^\infty B_{2, 1}^{n/2-1}} + (1 + \| (a, O; v) \|_{\mathcal{E}_{T}^{(n/p)+3}}) \| (a, O; v) \|_{\mathcal{E}_{T}^{n/p}}. \quad (4.27) \]

**Step 2: High-frequency estimates** \( 2^k > R_0 \).

Inspired by [15, 16], we perform basic energy approaches in terms of effective velocities rather than the Lagrangian change as in [3, 5]. Denote by \( \tilde{d} = -\nabla(-\Delta)^{-1}\text{div} v \) the compressible part of \( v \). It is easy to see that \( \| \tilde{d} \|_{L_T^1 B_{2, p}^{n/2, n/p}} \approx \| d \|_{L_T^1 B_{2, p}^{n/2, n/p}} \). It follows from the first equality in (3.1) that

\[ -\nabla(-\Delta)^{-1}\text{div}(\nabla \cdot O) \]
Applying (A.6) to the above equations implies that
\[
\begin{align*}
&\nabla(-\Delta)^{-1} \left( \partial_t \partial_j [(1 + a)(\delta^{ij} + O^{ij})] \right) + \nabla(-\Delta)^{-1} \text{div} \text{div}(aI + aO) \\
&= \nabla(-\Delta)^{-1} \text{div} \text{div}(aI + aO) \\
&= -\nabla a + \nabla(-\Delta)^{-1} \text{div} \text{div}(aO).
\end{align*}
\]

Note that (4.28), we get the following equation for the compressible part of \( v \)
\[
\partial_t \tilde{a} - \Delta \tilde{a} + 2 \nabla a = G_2, \tag{4.29}
\]
where
\[
G_2 = -\nabla(-\Delta)^{-1} \text{div} \left( -v \cdot \nabla v + O^{jk} \partial_j O^{ik} - I(a)A v \\
-K(a) \nabla a - \text{div}(aO) + \frac{1}{1 + a} \text{div}(2\tilde{\mu}(a)D(v) + \tilde{\lambda}(a)\text{div}\text{Id}) \right). \tag{4.30}
\]

The motivation using the system (4.29) is to make a comparison with the usual compressible Navier-Stokes equations. Here, we consider more complicated hyperbolic-parabolic coupled system
\[
\begin{align*}
\left\{
\begin{array}{l}
\partial_t a + v \cdot \nabla a + \text{div} v = G_1, \\
\partial_t \tilde{a} - \Delta \tilde{a} + 2 \nabla a = G_2, \\
\partial_t \tilde{O}^{ij} + v \cdot \nabla O^{ij} - \Lambda e^{ij} = \tilde{G}_3^{ij}, \\
\partial_t e^{ij} - \mu_0 \Delta e^{ij} + \Lambda O^{ij} = \tilde{G}_4^{ij}
\end{array}
\right.
\end{align*}
\tag{4.31}
\]
where
\[
\begin{align*}
\tilde{G}_1 &= -a \nabla \cdot v, \quad \tilde{G}_3^{ij} = \partial_k v^i O^{kj} - a O^{ij}, \\
\tilde{G}_4^{ij} &= G_4^{ij} + (\lambda_0 + \mu_0) \partial_i \partial_j a - \Lambda^{-1} \partial_i \Delta a.
\end{align*}
\tag{4.32}
\]

Introduce two effective velocities as follows
\[
\begin{align*}
w &= \tilde{a} + 2 \nabla(-\Delta)^{-1} a = \nabla(-\Delta)^{-1}(2a - \text{div} v), \\
\Omega^{ij} &= e^{ij} + \frac{1}{\mu_0} \Lambda(-\Delta)^{-1} O^{ij}.
\end{align*}
\]

Noticing that the definition of \( w \) is almost the same as that in [15, 16]. The subtle difference lies on the coefficient of unknown \( a \), which comes from the contribution of deformation gradient \( F \), see (4.28). The new effective velocity \( \Omega^{ij} \) is used to cancel the coupling between \( e^{ij} \) and \( O^{ij} \) in the high-frequency estimate.

Firstly, we present those estimates for effective velocities. It follows from (4.31) that
\[
\begin{align*}
\begin{align*}
\left\{
\begin{array}{l}
\partial_t w - \Delta w = G_2 + 2 \nabla(-\Delta)^{-1} G_1 + 2w - 4 \nabla(-\Delta)^{-1} a, \\
\partial_t \Omega^{ij} - \mu_0 \Delta \Omega^{ij} = \tilde{G}_4^{ij} + \frac{1}{\mu_0} \Lambda^{-1} \tilde{G}_3^{ij} + \frac{1}{\mu_0} \Omega^{ij} - \frac{1}{\mu_0^2} \Lambda^{-1} O^{ij}.
\end{array}
\right.
\end{align*}
\tag{4.33}
\end{align*}
\]
Applying (A.6) to the above equations implies that
\[
\begin{align*}
\|w\|^h_{L^p_t H^{n/p-1}_x \cap L^q_t H^{n/q-1}_x} \lesssim \|w_0\|^h_{L^p_t H^{n/p-1}_x} + \|w\|^h_{L^p_t \dot{H}^{n/p-1}_x} + \|a\|^h_{L^p_t \dot{B}^{n/p-2}_x} \\
+ \|G_1\|^h_{L^1(\dot{B}^{n/p-2}_x)} + \|G_2\|^h_{L^1(\dot{B}^{n/p-1}_x)}, \tag{4.34}
\end{align*}
\]
and
\[
\|\Omega^{ij}\|_{L_T^\infty B_{p,1}^{n/p-1} \cap L_T^1 \dot{B}_{p,1}^{n/p+1}} \lesssim \|\Omega_0^{ij}\|_{B^{n/p-1}} + \|\Omega^{ij}\|_{L_T^1 \dot{B}_{p,1}^{n/p-1}} + \|O^{ij}\|_{L_T^1 \dot{B}_{p,1}^{n/p-2}} + \|G_3^{ij}\|_{L^1(B_{p,1}^{n/p-2})} + \|\tilde{G}_4^{ij}\|_{L^1(B_{p,1}^{n/p-1})}. \tag{4.35}
\]

Owing to the high frequency cut-off $2^k > R_0$, we have
\[
\|w\|_{L_T^1 \dot{B}_{p,1}^{n/p-1}} \lesssim R_0^{-2}\|w\|_{L_T^1 \dot{B}_{p,1}^{n/p-1}}, \quad \|a\|_{L_T^1 \dot{B}_{p,1}^{n/p-1}} \lesssim R_0^{-2}\|a\|_{L_T^1 \dot{B}_{p,1}^{n/p-1}},
\]
and
\[
\|\Omega^{ij}\|_{L_T^1 \dot{B}_{p,1}^{n/p-1}} \lesssim R_0^{-2}\|w^{ij}\|_{L_T^1 \dot{B}_{p,1}^{n/p-1}}, \quad \|O^{ij}\|_{L_T^1 \dot{B}_{p,1}^{n/p-2}} \lesssim R_0^{-2}\|O^{ij}\|_{L_T^1 \dot{B}_{p,1}^{n/p-2}}.
\]
Choosing $R_0 > 0$ sufficiently large, the terms $\|w\|_{L_T^1 \dot{B}_{p,1}^{n/p-1}}$ and $\|\Omega^{ij}\|_{L_T^1 \dot{B}_{p,1}^{n/p-1}}$ on the right-hand side of (4.34) and (4.35) can be absorbed by the corresponding parts in the left-hand side. Consequently, we conclude that
\[
\|w\|_{L_T^\infty B_{p,1}^{n/p-1} \cap L_T^1 \dot{B}_{p,1}^{n/p+1}} \lesssim \|w_0\|_{B^{n/p-1}} + R_0^{-2}\|a\|_{L_T^1 \dot{B}_{p,1}^{n/p}} + \|\tilde{G}_1\|_{L^1(B_{p,1}^{n/p-2})} + \|G_2\|_{L^1(B_{p,1}^{n/p-1})}, \tag{4.36}
\]
and
\[
\|\Omega^{ij}\|_{L_T^\infty B_{p,1}^{n/p-1} \cap L_T^1 \dot{B}_{p,1}^{n/p+1}} \lesssim \|\Omega_0^{ij}\|_{B^{n/p-1}} + R_0^{-2}\|O^{ij}\|_{L_T^1 \dot{B}_{p,1}^{n/p}} + \|\tilde{G}_3^{ij}\|_{L^1(B_{p,1}^{n/p-2})} + \|\tilde{G}_4^{ij}\|_{L^1(B_{p,1}^{n/p-1})}. \tag{4.37}
\]

Secondly, we see that $(a, O^{ij})$ satisfies the following damped equations in terms of effective velocities
\[
\begin{cases}
\partial_t a + v \cdot \nabla a + 2a = \tilde{G}_1 - \nabla \cdot w, \\
\partial_t O^{ij} + v \cdot \nabla O^{ij} + \frac{1}{\mu_0} O^{ij} = \tilde{G}_3^{ij} + \Lambda O^{ij}.
\end{cases}
\]
Applying $\hat{\Delta}_k$ to the above equations, we obtain
\[
\begin{cases}
\partial_t \hat{\Delta}_k a + v \cdot \nabla \hat{\Delta}_k a + 2\hat{\Delta}_k a = \hat{\Delta}_k \tilde{G}_1 - \hat{\Delta}_k \nabla \cdot w + R_k^1, \\
\partial_t \hat{\Delta}_k O^{ij} + v \cdot \nabla \hat{\Delta}_k O^{ij} + \frac{1}{\mu_0} \hat{\Delta}_k O^{ij} = \hat{\Delta}_k \tilde{G}_3^{ij} + \hat{\Delta}_k \Lambda O^{ij} + R_k^2,
\end{cases} \tag{4.38}
\]
where $R_k^1 := [v \cdot \nabla, \hat{\Delta}_k]a$ and $R_k^2 := [v \cdot \nabla, \hat{\Delta}_k]O^{ij}$. Multiplying (4.38)$_1$ by $\hat{\Delta}_k a \hat{\Delta}_j a$ and (4.38)$_2$ by $\hat{\Delta}_k O^{ij} \hat{\Delta}_k O^{ij}$, and then integrating over $\mathbb{R}^n \times [0, t]$, we can obtain
\[
\|\Delta_k a(t)\|_{L^p} + \int_0^t \|\Delta_k a\|_{L^p} d\tau \lesssim \|\Delta_k a_0\|_{L^p} + \int_0^t \|\nabla v\|_{L^\infty} \|\Delta_k a\|_{L^p} d\tau + \int_0^t \|\nabla v\|_{L^\infty} \|\Delta_k a\|_{L^p} d\tau
\]
\[
+ \int_0^t \|\Delta_k (\tilde{G}_1 - \Lambda w)\|_{L^p} d\tau + \int_0^t \|R_k^1\|_{L^p} d\tau \tag{4.39}
\]
and
\[\| \tilde{\Delta} O^{ij}(t) \|_{L^p} + \int_0^t \| \tilde{\Delta} O^{ij} \|_{L^p} d\tau \lesssim \| \tilde{\Delta} O_0^{ij} \|_{L^p} + \int_0^t \| \nabla v \|_{L^\infty} \| \tilde{\Delta} O^{ij} \|_{L^p} d\tau \]
\[+ \int_0^t \| \tilde{\Delta}_k (\tilde{G}_3^{ij} + \Lambda O^{ij}) \|_{L^p} d\tau + \int_0^t \| R_k^2 \|_{L^p} d\tau. \tag{4.40}\]

It follows from commutator estimates in [2] that
\[\sum_{j \in \mathbb{Z}} 2^j \| (R_j^1, R_j^2) \|_{L^p} \lesssim \| \nabla v \|_{L^p} (a, O^{ij}) \|_{\dot{H}^\frac{p}{p-1}}. \]

Now multiplying (4.39) and (4.40) by \(2^k \frac{n}{p} \), respectively, and then summing over the index \(k\) satisfying \(2^k > R_0\), we are led to
\[\| a \|_{L^p \dot{H}^{n/p}} \lesssim \| a_0 \|_{\dot{H}^{n/p}} + \| \nabla v \|_{L^p \dot{H}^{n/p}} \| a \|_{L^p \dot{H}^{n/p}} + \| \tilde{G}_1^h \|_{L^1 \dot{H}^{n/p}} + \| w \|_{L^1 \dot{H}^{n/p+1}} \]
\[\lesssim \| a_0 \|_{\dot{H}^{n/p}} + \| w \|_{L^1 \dot{H}^{n/p+1}} + \| (a, O; v) \|^{2}_{\dot{E}^{n/p}} + \| \tilde{G}_1^h \|_{L^1 \dot{H}^{n/p}}. \tag{4.41}\]

and
\[\| O^{ij} \|_{L^p \dot{H}^{n/p}} \lesssim \| O_0^{ij} \|_{\dot{H}^{n/p}} + \| \nabla v \|_{L^p \dot{H}^{n/p}} \| O^{ij} \|_{L^p \dot{H}^{n/p}} + \| \tilde{G}_3^h \|_{L^1 \dot{H}^{n/p+1}} + \| \Omega^{ij} \|_{L^1 \dot{H}^{n/p+1}} \]
\[\lesssim \| O_0^{ij} \|_{\dot{H}^{n/p}} + \| \Omega^{ij} \|_{L^1 \dot{H}^{n/p+1}} + \| (a, O; v) \|^{2}_{\dot{E}^{n/p}} + \| \tilde{G}_3^h \|_{L^1 \dot{H}^{n/p+1}}. \tag{4.42}\]

Multiply (4.41) and (4.42) by \(\delta > 0\) respectively, and then add two resulting inequalities to (4.36) and (4.37) together. By choosing \(R_0\) sufficiently large, we get
\[\| a \|_{L^p \dot{H}^{n/p}} + \| w \|_{L^p \dot{H}^{n/p-1} \cap L^1 \dot{H}^{n/p+1}} \lesssim \| a_0 \|_{\dot{H}^{n/p}} + \| w_0 \|_{\dot{H}^{n/p-1}} \]
\[+ \| (a, O; v) \|^{2}_{\dot{E}^{n/p}} + \| \tilde{G}_1^h \|_{L^1 \dot{H}^{n/p+1}} + \| G_2^h \|_{L^1 \dot{H}^{n/p+1}}, \]

and
\[\| O^{ij} \|_{L^p \dot{H}^{n/p}} + \| \Omega^{ij} \|_{L^p \dot{H}^{n/p-1} \cap L^1 \dot{H}^{n/p+1}} \lesssim \| O_0^{ij} \|_{\dot{H}^{n/p}} + \| \Omega_0^{ij} \|_{\dot{H}^{n/p-1}} \]
\[+ \| (a, O; v) \|^{2}_{\dot{E}^{n/p}} + \| \tilde{G}_3^h \|_{L^1 \dot{H}^{n/p+1}} + \| G_4^h \|_{L^1 \dot{H}^{n/p+1}}. \]

Keep in mind that \(w = \tilde{d} + 2\nabla(-\Delta)^{-1} a, \Omega^{ij} = e^{ij} + \frac{1}{\mu_0}(-\Delta)^{-1} \Lambda O^{ij}\), we arrive at
\[\| a \|_{L^p \dot{H}^{n/p}} + \| \tilde{d} \|_{L^p \dot{H}^{n/p-1} \cap L^1 \dot{H}^{n/p+1}} \lesssim \| a_0 \|_{\dot{H}^{n/p}} + \| d_0 \|_{\dot{H}^{n/p-1}} \]
\[+ \| (a, O; v) \|^{2}_{\dot{E}^{n/p}} + \| \tilde{G}_1^h \|_{L^1 \dot{H}^{n/p+1}} + \| G_2^h \|_{L^1 \dot{H}^{n/p+1}}. \tag{4.43}\]
and

\[ \|O_{ij}\|_{L^\infty_p B_{p,1}^n} + \|e_{ij}\|_{L^\infty_p B_{p,1}^n} \lesssim \|O_{ij}\|_{L^\infty_p B_{p,1}^n} + \|e_{ij}\|_{L^\infty_p B_{p,1}^n} \]

\[ + \|(a, O; v)\|^2_{e_{p}^n} + \|\tilde{G}_{ij}\|_{L^\frac{1}{4}_1(B_{p,1}^n)} + \|\tilde{G}_{ij}\|_{L^\frac{1}{4}_1(B_{p,1}^n)}. \tag{4.44} \]

In addition, remembering (4.32), we have

\[ \|\tilde{G}_{ij}\|_{L^\frac{1}{4}_1(B_{p,1}^n)} \lesssim \|G_{ij}\|_{L^\frac{1}{4}_1(B_{p,1}^n)} + \|\tilde{d}\|_{L^\frac{1}{4}_1(B_{p,1}^n)} + \|a\|_{L^\frac{1}{4}_1(B_{p,1}^n)}. \tag{4.45} \]

Hence, together with (4.43)-(4.45), we deduce that

\[ \|(a, O)\|_{L^\infty_p B_{p,1}^n} + \|e\|_{L^\infty_p B_{p,1}^n} \lesssim \|(a, O)\|_{L^\frac{1}{4}_1(B_{p,1}^n)} + \|\tilde{d}\|_{L^\frac{1}{4}_1(B_{p,1}^n)} + \|a\|_{L^\frac{1}{4}_1(B_{p,1}^n)} + \|(G_2, G_4)\|_{L^\frac{1}{4}_1(B_{p,1}^n)}. \tag{4.46} \]

Likely, we need to bound those different terms in \( \tilde{G}_i (i = 1, 3) \) and \( G_i (i = 2, 4) \) compared to [5], for example,

\[ \tilde{G}_{3ij} := \partial_k v^i O^{kj}, \]

\[ O \nabla O, \text{ div}(aO) \text{ in } G_2 \text{ and } G_4, \]

and

\[ \frac{1}{1 + a} \text{div}(2 \tilde{\mu}(a) D(v) + \tilde{\lambda}(a) \text{div} \text{Id}) \text{ in } G_2 \text{ and } G_4. \tag{4.47} \]

In order to bound \( \partial_k v^i O^{kj} \), from (A.1) of PropositionA.1 with \( r_1 = 1, r_2 = \infty, \sigma = \tau = n/p \), we have

\[
\sum_{2^k > R_0} 2^{k(n/p)} \|\tilde{\Delta}_k (\partial_k v^i O^{kj})\|_{L^1_p}\]

\[ \lesssim \|\nabla v\|_{L^1_p B_{2,p}^{n/2,n/p}} \|O\|_{L^\infty_p B_{2,p}^{n/2,n/p}} \lesssim \|v\|_{L^1_p B_{2,p}^{n/2,n/p}} \|O\|_{L^\infty_p B_{2,p}^{n/2,n/p}} \]

\[ \lesssim \|(a, O; v)\|^2_{e_{p}^n}. \]

For \( O \nabla O \), from (A.1) of PropositionA.1 with \( r_1 = r_2 = 2, \sigma = n/p, \tau = n/p - 1 \) and by applying interpolation (4.1), we have

\[
\sum_{2^k > R_0} 2^{k(n/p-1)} \|\tilde{\Delta}_k (O \nabla O)\|_{L^1_p}\]

\[ \lesssim \|O\|_{L^2_p B_{2,p}^{n/2,n/p}} \|\nabla O\|_{L^2_p B_{2,p}^{n/2,n/p-1}} \lesssim \|O\|^2_{L^2_p B_{2,p}^{n/2,n/p}} \]

\[ \lesssim \|(a, O; v)\|^2_{e_{p}^n}. \tag{4.48} \]
The estimate for \( \text{div}(aO) = a \nabla \cdot O + \nabla aO \) may be handled with at the same away as \( O \nabla O \). Next, we bound the cubic term \((4.47)\) in \( G_{ij}^r \). Following from the the same notation, we know that

\[
I = \frac{1}{1+a} \text{div}(2\tilde{\mu}(a)D(v)) = \frac{1}{1+a} \tilde{\mu}(a) \nabla^2 v + \frac{1}{1+a} \nabla \tilde{\mu}(a) \nabla v \triangleq I_1 + I_2.
\]

For \( I_1 \), we have

\[
\sum_{2^k > R_0} 2^{k(n/p-1)} \| \hat{\Delta}_k \left( \frac{1}{1+a} \nabla \tilde{\mu}(a) \nabla v \right) \|_{L^1_t L^p_x} \lesssim \sum_{2^k > R_0} 2^{k(n/p-1)} \left( \| \hat{\Delta}_k (I(\tilde{\mu}(a) \nabla^2 v) \|_{L^1_t B^{n/2-1,n/p}_x} + \| \hat{\Delta}_k (\nabla \tilde{\mu}(a) \nabla v) \|_{L^1_t B^{n/2-1,n/p}_x} \right) \lesssim \left( 1 + \| (a,O) \|_{C_T^p} \right) \| \nabla \tilde{\mu}(a) \nabla v \|_{L^1 T B^{n/2-1,n/p}_x} \lesssim \left( 1 + \| (a,O) \|_{C_T^p} \right) \| \nabla \tilde{\mu}(a) \nabla v \|_{L^1 T B^{n/2-1,n/p}_x}.
\]

where the third line is followed by taking \( \sigma = n/p, \tau = n/p - 1, r_1 = \infty, r_2 = 1 \) in \((A.1)\).

On the other hand, regarding \( I_2 \), we deduce that

\[
\sum_{2^k > R_0} 2^{k(n/p-1)} \| \hat{\Delta}_k \left( \frac{1}{1+a} \nabla \tilde{\mu}(a) \nabla v \right) \|_{L^1_t L^p_x} \lesssim \sum_{2^k > R_0} 2^{k(n/p-1)} \left( \| \hat{\Delta}_k (I(\tilde{\mu}(a) \nabla^2 v) \|_{L^1_t B^{n/2-1,n/p}_x} + \| \hat{\Delta}_k (\nabla \tilde{\mu}(a) \nabla v) \|_{L^1_t B^{n/2-1,n/p}_x} \right) \lesssim \left( 1 + \| (a,O) \|_{C_T^p} \right) \| \nabla \tilde{\mu}(a) \nabla v \|_{L^1 T B^{n/2-1,n/p}_x} \lesssim \left( 1 + \| (a,O) \|_{C_T^p} \right) \| \nabla \tilde{\mu}(a) \nabla v \|_{L^1 T B^{n/2-1,n/p}_x}.
\]

The computation for \( \frac{1}{1+a} \text{div}(\tilde{\lambda}(a) \text{div}v \text{Id}) \) totally follows from the same procedure as \( I \), so we omit details. By putting above estimates together, remembering \((4.46)\), we achieve that

\[
\|(a,O)\|_{L^1_T B^{n/p}_x}^{h} + \|e\|_{L^1_T B^{n/p-1}_x}^{h} + (1 + \|(a,O)\|_{C_T^p}^{h} \|(a,O)\|_{C_T^p}^{h}) \| (a,O) \|_{C_T^p}^{h}. \quad (4.49)
\]

**Step 3: Combination of two-step analysis.**
The inequality (4.2) is the consequence of (4.27) and (4.49), so the proof of Proposition 4.1 is finished. By using (A.3) in Proposition A.1, we can infer that

$$
\|(a, O; e)\|_{L_t^\infty \dot{B}_{2,1}^{n/2-1}} + \|(a, O; e)\|_{L_t^\infty \dot{B}_{2,1}^{n/2+1}} \\
\lesssim \|(a_0, O_0; c_0)\|_{L_t^\infty \dot{B}_{2,1}^{n/2-1}} \\
+ (1 + \|(a, O; v)\|_{\mathcal{E}_T^{n/p}})^{n+3}\|(a, O; v)\|_{\mathcal{E}_T^{n/p}} \|(a, O; v)\|_{\mathcal{E}_T^{n/p}}
$$

(4.50)

and

$$
\|(a, O)\|_{L_t^\infty \dot{B}_{2,1}^{n/2-1} \cap L_t^1 \dot{B}_{2,1}^{n/2}} + \|e\|_{L_t^\infty \dot{B}_{p,1}^{n/2-1} \cap L_t^1 \dot{B}_{p,1}^{n/2+1}} \\
\lesssim \|(a_0, O_0)\|_{L_t^\infty \dot{B}_{2,1}^{n/2-1}} + \|e_0\|_{L_t^\infty \dot{B}_{2,1}^{n/2-1}} \\
+ (1 + \|(a, O; v)\|_{\mathcal{E}_T^{n/p}})^{n+3}\|(a, O; v)\|_{\mathcal{E}_T^{n/p}} \|(a, O; v)\|_{\mathcal{E}_T^{n/2}}.
$$

(4.51)

The inequality (4.3) is followed by (4.50) and (4.51). Therefore, the proof of Proposition 4.2 is complete.

### 4.2 Approximate solutions and uniform estimates

The construction of approximate solutions is based on the following local-in-time existence.

**Theorem 4.1 ([30])** Assume \((\rho_0 - 1, F_0 - I) \in \dot{B}_{2,1}^{n/2} \cap L_t^1 \dot{B}_{2,1}^{n/2-1} \cap L_t^1 \dot{B}_{2,1}^{n/2+1} \cap L_t^1 \dot{B}_{2,1}^{n/2-1} \cap L_t^1 \dot{B}_{2,1}^{n/2+1} \cap L_t^1 \dot{B}_{2,1}^{n/2-1} \cap L_t^1 \dot{B}_{2,1}^{n/2+1})^n \) with \(\rho_0 \) bounded away from 0. There exists a positive time \(T\) such that system \((1.1)\) has a unique solution \((\rho, F; u)\) with \(\rho\) bounded away from 0 and

\[
(\rho - 1, F - I) \in \left( C([0, T); \dot{B}_{2,1}^{n/2}) \right)^{1+n^2}, \ u \in \left( C([0, T); \dot{B}_{2,1}^{n/2-1}) \right)^n.
\]

Additionally, if \((\rho_0 - 1, F_0 - I) \in \dot{B}_{2,1}^{n/2-1} \cap L_t^1 \dot{B}_{2,1}^{n/2+1} \cap L_t^1 \dot{B}_{2,1}^{n/2-1} \cap L_t^1 \dot{B}_{2,1}^{n/2+1})^n \), we have

\[
(\rho - 1, F - I) \in \left(C([0, T); \dot{B}_{2,1}^{n/2-1}) \right)^{1+n^2}.
\]

In order to apply Theorem 4.1, we need the following lemma, which could be shown by the proof of Lemma 4.2 in [1].

**Lemma 4.1** Let \(p \geq 2\). For any

\[
(\rho_0 - 1, F_0 - I; u_0) \in \left( \dot{B}_{2,p}^{n/2-1,n/p} \right)^{1+n^2} \times \left( \dot{B}_{2,p}^{n/2-1,n/p-1} \right)^n
\]

satisfying \(\rho_0 \geq c_0 > 0\), then there exists a sequence \(\{ (\rho_{0,k}; F_{0,k}; u_{0,k}) \}_{k \in \mathbb{N}} \) with \(\{ (\rho_{0,k} - 1, F_{0,k} - I; u_{0,k}) \}_{k \in \mathbb{N}} \) such that

\[
\|(\rho_{0,k} - \rho_0, F_{0,k} - F_0)\|_{\dot{B}_{2,p}^{n/2-1,n/p}} \to 0, \quad \|u_{0,k} - u_0\|_{\dot{B}_{2,p}^{n/2-1,n/p-1}} \to 0
\]

(4.52)

when \(k \to 0\). we also have \(\rho_{0,k} \geq c_0 \frac{T}{k} \) for any \(k \in \mathbb{N} \).
Let \((\rho_{0,k}, F_{0,k}; u_{0,k})\) be the sequence for initial data stated in Lemma 4.1. Then Theorem 4.1 indicates that there exists a maximal existence time \(T_k > 0\) such that system (1.1) with initial data \((\rho_{0,k}, F_{0,k}; u_{0,k})\) has a unique solution \((\rho_k, F_k; u_k)\) with \(\rho_k\) bounded away from 0, and satisfies
\[
\left(\rho_k - 1, F_k - I\right) \in \left(C([0, T_k); \dot{B}^{n/2-1}_{2,1} \cap \dot{B}^{n/2-1}_{2,1})\right)^{1+n^2},
\]
\[
u_k \in \left(C([0, T_k); \dot{B}^{n/2-1}_{2,1} \cap L^1(\dot{B}^{n/2+1}_{2,1}))\right)^n.
\]

Then using the definition of Hybrid Besov spaces and Bernstein inequality in Lemma 2.1, we have
\[
\left(\rho_k - 1, F_k - I\right) \in \left(C([0, T_k); \dot{B}^{n/2-1,n/p}_{2,p} \cap \dot{B}^{n/2-1,n/p}_{2,p} - 1)\right)^{1+n^2},
\]
\[
u_k \in \left(C([0, T_k); \dot{B}^{n/2-1,n/p}_{2,p} \cap L^1([0, T_k]; \dot{B}^{n/2+1,n/p+1}_{2,p})\right)^n.
\]

Set
\[
a_k(t, x) = \rho_k(\chi_0^2t, \chi_0x) - 1, \quad v_k(t, x) = \chi_0u_k(\chi_0^2t, \chi_0x), \quad O_k(t, x) = F_k(\chi_0^2t, \chi_0x) - I.
\]

From (1.3) and (4.52), we
\[
\|(a_0, O_0, v_0, k)\|_{E_0^{n/p}} \leq C_0\eta,
\]
for some constant \(C_0\). Given a constant \(M\) to be determined later on, we define
\[
T^*_k \triangleq \sup\{t \in [0, T_k] \mid \|(a_k, O_k, v_k)\|_{E_k^{n/p}} \leq M\eta\}.
\]

First we claim that
\[
T^*_k = T_k \quad \forall k \in \mathbb{N}.
\]

With aid of the continuity argument, it suffices to show for all \(k \in \mathbb{N}\),
\[
\|(a_k, O_k; v_k)\|_{E_k^{n/p}} \leq \frac{1}{2}M\eta.
\]

Indeed, noting that \(\|a_k\|_{L^\infty([0, T_k^*] \times \mathbb{R}^n)} \leq C_1\|a_k\|_{L^\infty_k(\dot{B}^{n/2-1,n/p}_{2,p})}\), we can choose \(\eta\) sufficiently small such that
\[
M\eta \leq \frac{1}{2C_1}.
\]

Then
\[
\|a_k\|_{L^\infty([0, T_k^*] \times \mathbb{R}^n)} \leq \frac{1}{2}.
\]

By applying Proposition 4.1, we obtain
\[
\|(a_k, O_k; v_k)\|_{E_k^{n/p}} \leq C\{C_0\eta + (M\eta)^2(1 + M\eta)^{n+3}\}.
\]
By choosing $M = 3CC_0$ and $\eta$ sufficient small enough such that
\[ C(M\eta)(1 + M\eta)^{n+3} \leq \frac{1}{6}, \]
so (4.53) is followed by (4.54) directly.

Therefore, we obtain a sequence of approximate solutions $(\rho_k, F_k; u_k)$ to the system (1.1) on $[0, T_k)$ satisfying
\[ \|(a_k, O_k; v_k)\|_{E_t^{n/p}} \leq M\eta, \]
for any $k \in \mathbb{N}$. From (4.3) and (4.55), we have
\[ \|(a_k, O_k; v_k)\|_{E_t^{n/2}} \leq C\left\{ \|(a_{0,k}, O_{0,k}; v_{0,k})\|_{E_0^{n/2}} + \|(a_k, O_k; v_k)\|_{E_t^{n/2}}(M\eta)(1 + M\eta)^{n+3} \right\}, \]
which implies
\[ \|(a_k, O_k; v_k)\|_{E_t^{n/2}} \leq C\|(a_{0,k}, O_{0,k}; v_{0,k})\|_{E_0^{n/2}}, \]
where we chose $\eta$ sufficiently small. Consequently, based on Proposition 4.2, the continuity argument ensures that $T_k = +\infty$ for any $k \in \mathbb{N}$.

### 4.3 Passing to the limit and existence

Next, the existence of the solution will be proved by the compact argument. We show that, up to an extraction, the sequence $(a_k, O_k; v_k)$ converges in the distributional sense to some function $(a, O; v)$ such that
\[ (a, O; v) \in \left( \tilde{L}^\infty B_{2,p}^{n/2-1,n/p} \cap \tilde{L}^1 B_{2,p}^{n/2+1,n/p} \right)^{1+n^2} \times \left( \tilde{L}^\infty B_{2,p}^{n/2-1,n/p-1} \cap \tilde{L}^1 B_{2,p}^{n/2+1,n/p+1} \right)^n. \]

Indeed, it follows from (4.55) that $(a_k, O_k)$ is uniformly bounded in $\tilde{L}^\infty(0, \infty; \dot{B}_{p,1}^{n/p})$ and $v_k$ is uniformly bounded in $\tilde{L}^\infty(0, \infty; \dot{B}_{p,1}^{n/p-1}) \cap L^1(0, \infty; \dot{B}_{p,1}^{n/p+1})$. By interpolation, we also deduce that $v_k$ is uniformly bounded in $\tilde{L}^{2+\varepsilon}(0, \infty; \dot{B}_{p,1}^{n/p+1-\varepsilon})$ for any $\varepsilon \in [0, 2]$. We claim that $(a_k, O_k; v_k)$ is uniformly bounded in
\[ \left( C_{loc}^{1/2}(\mathbb{R}^+; \dot{B}_{p,1}^{n/p-1}) \right)^{1+n^2} \times \left( C_{loc}^{2+\varepsilon}(\mathbb{R}^+; \dot{B}_{p,1}^{n/p-1-\varepsilon}) \right)^n \]
with $\zeta = \min\{\frac{2n}{p} - 1, 1\}$, which is a direct consequence of
\[ (\partial_t a_k, \partial_t O_k; \partial_t v_k) \in \left( \tilde{L}^{2}_{loc} \dot{B}_{p,1}^{n/p-1} \right)^{1+n^2} \times \left( \tilde{L}^{2+\varepsilon}_{loc} \dot{B}_{p,1}^{n/p-1-\varepsilon} \right)^n. \]

Recalling (3.4), we have
\[ \partial_t a_k = -v_k \cdot \nabla a_k - \nabla \cdot v_k - a_k \nabla \cdot v_k. \]
and
\[ \partial_t O_k = -v_k \cdot \nabla O_k + \nabla v_k + \nabla v_k O_k. \]

By interpolation and Lemma 2.3, it follows from (4.55) that \((\partial_t a_k, \partial_t O_k) \in \left( L^{2/\zeta}_{loc} \dot{B}^{n/p-1}_{p,1} \right)^{1+n^2} \), which implies that \((a_k, O_k)\) is uniformly bounded in \( \left( C^{1/2}_{loc} (\mathbb{R}^+; \dot{B}^{n/p-1}_{p,1}) \right)^{1+n^2} \). On the other hand,
\[
\partial_v k = -v_k \cdot \nabla v_k + A v_k - \nabla a_k + \nabla \cdot O_k + O^j_k \partial_j a^i_k
\]
\[
-I(a_k) A v_k - K(a_k) \nabla a + \frac{1}{1 + a_k} \operatorname{div} (2 \tilde{\mu}(a_k) D(v_k) + \tilde{\lambda}(a_k) \operatorname{div} v_k \mathbf{I}).
\]

It’s easy to see that
\[
\left\| A v_k \right\|_{L^{2/\zeta}_{p,1} \dot{B}^{n/p-1-\zeta}_{p,1}} \lesssim \left\| v_k \right\|_{L^{2/\zeta}_{p,1} \dot{B}^{n/p+1-\zeta}_{p,1}}. \tag{4.60}
\]

Thanks to Lemma 2.3 and Proposition A.2, we have
\[
\left\| (v_k \cdot \nabla v_k, I(a_k) A v_k) \right\|_{L^{2/\zeta}_{p,1} \dot{B}^{n/p-1-\zeta}_{p,1}} \lesssim \left\| v_k \right\|_{L^{2/\zeta}_{p,1} \dot{B}^{n/p-1-\zeta}_{p,1}} + \left\| a_k \right\|_{L^{\infty} \dot{B}^{n/p}_{p,1}} \left\| \nabla^2 v_k \right\|_{L^{2/\zeta}_{p,1} \dot{B}^{n/p-1-\zeta}_{p,1}}. \tag{4.61}
\]

Also, due to the embedding \( \dot{B}^{n/2-1,n/p}_{2,p} \hookrightarrow \dot{B}^{n/p-\zeta}_{p,1} \) and Proposition A.2, we arrive at
\[
\left\| (\nabla a_k, \nabla O_k) \right\|_{L^{\infty} \dot{B}^{n/p-1-\zeta}_{p,1}} + \left\| (K(a_k) \nabla a_k, O_k \nabla O_k) \right\|_{L^{\infty} \dot{B}^{n/p-1-\zeta}_{p,1}} \lesssim (1 + \left\| (a_k, O_k) \right\|_{L^{\infty} \dot{B}^{n/p}_{p,1}}) \left\| (a_k, O_k) \right\|_{L^{\infty} \dot{B}^{n/p-\zeta}_{p,1}}. \tag{4.62}
\]

As before, we write
\[
\frac{1}{1 + a_k} \operatorname{div} (2 \tilde{\mu}(a_k) D(v_k)) = \frac{1}{1 + a_k} \tilde{\mu}(a_k) \nabla^2 v_k + \frac{1}{1 + a_k} \nabla \tilde{\mu}(a_k) \nabla v_k.
\]

Then by applying Lemma 2.3 and Proposition A.2, we get
\[
\left\| \frac{1}{1 + a_k} \operatorname{div} (2 \tilde{\mu}(a_k) D(v_k)) \right\|_{L^{2/\zeta}_{p,1} \dot{B}^{n/p-1-\zeta}_{p,1}} \lesssim \left\| \frac{1}{1 + a_k} \tilde{\mu}(a_k) \nabla^2 v_k \right\|_{L^{2/\zeta}_{p,1} \dot{B}^{n/p-1-\zeta}_{p,1}} + \left\| \frac{1}{1 + a_k} \nabla \tilde{\mu}(a_k) \nabla v_k \right\|_{L^{2/\zeta}_{p,1} \dot{B}^{n/p-1-\zeta}_{p,1}} \lesssim (1 + \left\| I(a_k) \right\|_{L^{\infty} \dot{B}^{n/p}_{p,1}}) \left( \left\| \tilde{\mu}(a_k) \nabla^2 v_k \right\|_{L^{2/\zeta}_{p,1} \dot{B}^{n/p-1-\zeta}_{p,1}} + \left\| \nabla \tilde{\mu}(a_k) \nabla v_k \right\|_{L^{2/\zeta}_{p,1} \dot{B}^{n/p-1-\zeta}_{p,1}} \right)
\]
\[
\lesssim (1 + \left\| I(a_k) \right\|_{L^{\infty} \dot{B}^{n/p}_{p,1}}) \left( \left\| \tilde{\mu}(a_k) \right\|_{L^{\infty} \dot{B}^{n/p}_{p,1}} \left\| \nabla^2 v_k \right\|_{L^{2/\zeta}_{p,1} \dot{B}^{n/p-1-\zeta}_{p,1}} \right.
\]
\[
\left. + \left\| \nabla \tilde{\mu}(a_k) \right\|_{L^{\infty} \dot{B}^{n/p-1}_{p,1}} \left\| \nabla v_k \right\|_{L^{2/\zeta}_{p,1} \dot{B}^{n/p-\zeta}_{p,1}} \right) \tag{4.63}
\]

and \( \frac{1}{1 + a_k} \operatorname{div} (\tilde{\lambda}(a_k) \operatorname{div} v_k \mathbf{I}) \) may be treated along the same way. Consequently, combining (4.60) – (4.63), we conclude that
\[
\partial_v k \in \left( L^{2/\zeta}_{loc} \dot{B}^{n/p-1-\zeta}_{p,1} \right)^n,
\]
which implies that \( v_k \) is uniformly bounded in \( C_{loc}^{2+\frac{\nu}{p}}(\mathbb{R}_+; \dot{B}^{n/p-1}_p, \dot{B}^{n/p-1-\zeta}_p) \). Therefore the claim (4.58) is proved. Furthermore, we see that \((a_k, O_k; v_k)\) is equicontinuous on \( \mathbb{R}_+ \) valued in \((\dot{B}^{n/p-1}_p)_{1+n^2} \times (\dot{B}^{n/p-1-\zeta}_p)^n\). Let \( \{\phi_j\}_{j \in \mathbb{N}} \) be a sequence of smooth functions supported in the ball \( B(0, j + 1) \) and equal to 1 in \( B(0, j) \). It follows from (4.58) that \((\phi_j a_k, \phi_j O_k; \phi_j v_k)\) is uniformly bounded in

\[\left(C_{loc}^{1/2}(\mathbb{R}_+; \dot{B}^{n/p-1}_p)\right)^{1+n^2} \times \left(C_{loc}^{2+\frac{\nu}{p}}(\mathbb{R}_+; \dot{B}^{n/p-1}_p)\right)^n.\]

Observe that the map \((a_k, O_k; v_k) \mapsto \left(\phi_j a_k, \phi_j O_k; \phi_j v_k\right)\) is compact from

\[\left(\dot{B}^{n/p-1}_p \cap \dot{B}^{n/p}_p\right)^{1+n^2} \times \left(\dot{B}^{n/p-1-\zeta}_p \cap \dot{B}^{n/p}_p\right)^n\]

into

\[\left(\dot{B}^{n/p-1}_p\right)^{1+n^2} \times \left(\dot{B}^{n/p-1-\zeta}_p\right)^n.\]

By applying Ascoli’s theorem and Cantor’s diagonal process, there exist \((a, O; v)\) such that for any smooth function \( \phi \in C_0^\infty(\mathbb{R}^n)\),

\[
\begin{align*}
(\phi a_k, \phi O_k) &\to (\phi a, \phi O) \quad \text{in} \quad \left(L^\infty(\mathbb{R}_+; \dot{B}^{n/p-1}_p)\right)^{1+n^2}, \\
(\phi v_k) &\to \phi v \quad \text{in} \quad \left(L^\infty(\mathbb{R}_+; \dot{B}^{n/p-1-\zeta}_p)\right)^n,
\end{align*}
\]

(4.64) when \( k \to +\infty \) (up to an extraction). Actually, by interpolation, we also have

\[
\begin{align*}
(\phi a_k, \phi O_k) &\to (\phi a, \phi O) \quad \text{in} \quad \left(L^\infty(\mathbb{R}_+; \dot{B}^{n/p-s}_p)\right)^{1+n^2} \quad \forall \ 0 < s \leq 1, \\
(\phi v_k) &\to \phi v \quad \text{in} \quad \left(L^1(\mathbb{R}_+; \dot{B}^{n/p+s}_p)\right)^n \quad \forall \ -1 \leq s < 1.
\end{align*}
\]

(4.65)

Then, using the so-called Fatou property in Besov spaces and the uniform bound in (4.55), we conclude that (4.57) is fulfilled. It is a routine process to verify that \((a, O; v)\) satisfies the system (3.4) in the sense of distributions. Below is to check the desired regularity of solutions. Noticing that

\[
\begin{align*}
\partial_t a &+ v \cdot \nabla a = -\nabla \cdot v - a \nabla \cdot v \in L^1_{loc}(\dot{B}^{n/2-1,n/p}_{2,p} \cap L^1(\dot{B}^{n/2+1,n/p}_{2,p})), \\
\partial_t O &+ v \cdot \nabla O = \nabla v + \nabla v O \in L^1_{loc}(\dot{B}^{n/2-1,n/p}_{2,p} \cap L^1(\dot{B}^{n/2+1,n/p}_{2,p})),
\end{align*}
\]

since \((a_0, O_0) \in \dot{B}^{n/2-1,n/p}_{2,p}\), the classical result for transport equations indicates that

\[
(a, O) \in C(\mathbb{R}_+; \dot{B}^{n/2-1,n/p}_{2,p}).
\]

On the other hand,

\[
\begin{align*}
\partial_t v - \mathcal{A} v &= -v \cdot \nabla v - \nabla a + \nabla \cdot O + O^{ik} \partial_j O^{jk} - I(a) \mathcal{A} v - K(a) \nabla a, \\
&\quad + \frac{1}{1+\alpha} \text{div}(2\tilde{\mu}(a)D(v) + \tilde{\lambda}(a)\text{div}v Id), \\
&\in \dot{L}^1_{loc}(\dot{B}^{n/2-1,n/p-1}_{2,p}).
\end{align*}
\]

So the maximal regularity of heat equation enables us to get \( v \in C(\mathbb{R}_+; \dot{B}^{n/2-1,n/p-1}_{2,p})\).
4.4 Uniqueness

Due to techniques, allow us to only deal with the case 2 \( \leq p \leq n \) in the proof uniqueness of solutions. We will consider the remaining interval with respect to \( p \) in near future. The proof depends on a logarithmic inequality. For convenience of reader, we present it by a lemma.

**Lemma 4.2 ([10])** Let \( s \in \mathbb{R} \). Then for any \( 1 \leq p, r \leq +\infty \) and \( 0 < \varepsilon \leq 1 \), we have

\[
\| f \|_{\tilde{L}^r_t B^s_{p,1}} \leq C \frac{\| f \|_{\tilde{L}^r_t B^s_{p,\infty}}}{\varepsilon} \log \left( e + \frac{\| f \|_{\tilde{L}^r_t B^s_{p,\infty}}}{\varepsilon} \right).
\]

Assume that \((\rho_i, F_i; u_i)(i = 1, 2)\) are two solution to the system (1.1) with the same initial data. Without loss of generality, we may assume that

\[
\|(\rho_i - 1, F_i - I; u_i)\|_{\mathcal{E}^n/p} \leq M\eta, \quad \text{for} \quad i = 1, 2.
\] (4.66)

Using embedding and (4.66), we have

\[
\|\rho_i - 1\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^n)} \leq C\|\rho_i - 1\|_{\mathcal{E}^n/p} \leq CM\eta \leq \frac{1}{2}, \quad \text{for} \quad i = 1, 2
\]

for \( \eta > 0 \) sufficiently small. Set

\[
a_i(t, x) = \rho_i(\chi^2_0 t, \chi_0 x) - 1, \\
O_i(t, x) = F_i(\chi^2_0 t, \chi_0 x) - I, \\
v_i(t, x) = \chi_0 u_i(\chi^2_0 t, \chi_0 x),
\]

for \( i = 1, 2 \) and

\[
\delta a = a_1 - a_2, \delta O = O_1 - O_2; \delta v = v_1 - v_2.
\]

Thanks to (3.4), we find that \((\delta a, \delta v, \delta O)\) satisfies

\[
\begin{cases}
\partial_t \delta a + v_2 \cdot \nabla \delta a = \delta F, \\
\partial_t \delta v - A\delta v = \delta G, \\
\partial_t \delta O + v_2 \cdot \nabla \delta O = \delta H, \\
(\delta a, \delta O, \delta v) = (0, 0, 0),
\end{cases}
\] (4.67)

with

\[
\delta F = -\delta v \cdot \nabla a_1 - \nabla \cdot \delta v - a_1 \nabla \cdot \delta v - \delta a \nabla \cdot v_2, \\
\delta H = \delta v \cdot \nabla O_1 + \nabla \delta v + \nabla \delta v O_1 + \nabla v_2 \delta O, \\
\delta G = -\nabla \delta a + \nabla \cdot \delta O - (v_1 \cdot \nabla v_1 - v_2 \cdot \nabla v_2) + (O_{j\ell}^{jk} \partial_j O_1^{\ast k} - O_{j\ell}^{jk} \partial_j O_2^{\ast k}) - I(a_1)A v_1 + I(a_2)A v_2 - K(a_1)\nabla a_1 + K(a_2)\nabla a_2 \\
+ \frac{1}{1 + a_1} \text{div} (2\tilde{\mu}(a_1)D(v_1) + \tilde{\lambda}(a_1)\text{div} v_1 \text{Id})
\]
\[-\frac{1}{1 + a_2} \text{div}(2\tilde{\mu}(a_2)D(v_2) + \tilde{\lambda}(a_2)\text{div}v_2 \text{Id}). \quad (4.68)\]

In the following, we denote

\[V_i(t) = \int_0^t ||v_i(\tau)||_{\dot{B}_{p,i}^{n/p+1}} d\tau \quad \text{for } i = 1, 2 \quad (4.69)\]

and we denote by \(A\) a constant depending on \(\|a_i\|_{\dot{L}_t^\infty \dot{B}_{p,i}^{n/p}}\) for \(i = 1, 2\). Due to the embedding \(\mathcal{E}^{n/p} \subseteq \mathcal{E}^1(p \leq n)\), it is suffices to prove the uniqueness in \(\mathcal{E}^1\). So we take \(p = n\) in the subsequent process.

Applying Proposition A.3, we get

\[\| (\delta a(t), \delta O(t)) \|_{\dot{B}_{p,\infty}^0} \leq e^{CV_2(t)} \int_0^t \| (\delta F(\tau), \delta H(\tau)) \|_{\dot{B}_{p,\infty}^0} d\tau. \quad (4.70)\]

By Lemma 2.3, we have

\[\| (\delta F(\tau), \delta H(\tau)) \|_{\dot{B}_{p,\infty}^0} \lesssim \|v_2\|_{\dot{B}_{p,1}^2} \| (\delta a, \delta O) \|_{\dot{B}_{p,\infty}^0} + (1 + \|a_1, O_1\|_{\dot{B}_{p,1}^1}) \|\delta v\|_{\dot{B}_{p,1}^1}. \]

Inserting the equality into (4.70), we arrive at by Gronwall’s inequality

\[\| (\delta a(t), \delta O(t)) \|_{\dot{B}_{p,\infty}^0} \leq e^{CV_2(t)} \int_0^t (1 + \|a_1, O_1\|_{\dot{B}_{p,1}^1}) \|\delta v\|_{\dot{B}_{p,1}^1} d\tau. \quad (4.71)\]

Using Proposition A.4 to the second equation of (4.67) gives

\[\|\delta v\|_{\dot{L}_t^1 \dot{B}_{p,\infty}^1} + \|\delta v\|_{\dot{L}_t^2 \dot{B}_{p,\infty}^0} \lesssim \|\delta G(\tau)\|_{\dot{L}_t^1 \dot{B}_{p,\infty}^1}. \quad (4.72)\]

Furthermore, by Lemma 2.3 and Proposition A.2, it is shown that

\[\|\delta G(\tau)\|_{\dot{L}_t^1 \dot{B}_{p,\infty}^1} \lesssim \| (v_1, v_2) \|_{\dot{L}_t^2 \dot{B}_{p,1}^1} \|\delta v\|_{\dot{L}_t^2 \dot{B}_{p,\infty}^0} + A_t \|a_1\|_{\dot{L}_t^\infty \dot{B}_{p,1}^1} \|\delta v\|_{\dot{L}_t^1 \dot{B}_{p,\infty}^1}
+ A_t \int_0^t (1 + \|v_2\|_{\dot{B}_{p,1}^2}) \| (\delta a, \delta O) \|_{\dot{B}_{p,\infty}^0} d\tau. \quad (4.73)\]

According to our a priori estimates, by choosing \(\eta\) small, we have

\[A_t \|a_1\|_{\dot{L}_t^\infty \dot{B}_{p,1}^1} + \| (v_1, v_2) \|_{\dot{L}_t^2 \dot{B}_{p,1}^1} \lesssim M \eta \ll 1.\]

Consequently, inserting (4.73) into (4.72) to implies that

\[\|\delta v\|_{\dot{L}_t^1 \dot{B}_{p,\infty}^1} + \|\delta v\|_{\dot{L}_t^2 \dot{B}_{p,\infty}^0} \lesssim A_t \int_0^t (1 + \|v_2\|_{\dot{B}_{p,1}^2}) \| (\delta a, \delta O) \|_{\dot{B}_{p,\infty}^0} d\tau. \quad (4.74)\]

Combining (4.71) and (4.74), we get

\[\|\delta v\|_{\dot{L}_t^1 \dot{B}_{p,\infty}^1} \lesssim \int_0^t (1 + \|v_2\|_{\dot{B}_{p,1}^2}) \|\delta v\|_{\dot{L}_t^1 \dot{B}_{p,1}^1} d\tau. \quad (4.75)\]
Applying Lemma 4.2 with \( s = r = \varepsilon = 1 \) and \( f = \delta v \), we obtain
\[
\| \delta v \|_{\dot{L}^1_t \dot{B}^r_{p,1}} \leq C \| \delta v \|_{\dot{L}^1_t \dot{B}^r_{p,\infty}} \log \left( e + \frac{\| \delta v \|_{\dot{L}^1_t \dot{B}^r_{p,\infty}} + \| \delta v \|_{\dot{L}^1_t \dot{B}^2_{p,\infty}}}{\| \delta v \|_{\dot{L}^1_t \dot{B}^2_{p,\infty}}} \right),
\]
which together with (4.74) and (4.71) indicates that
\[
\| \delta v (t) \|_{\dot{L}^1_t \dot{B}^r_{p,\infty}} \leq e^{CV_2(t)} A_t \int_0^t (1 + \| v_2 \|_{\dot{B}^2_{p,1}}) \| \delta v \|_{\dot{L}^1_t \dot{B}^r_{p,\infty}} \times \log \left( e + C \tau \| \delta v \|_{\dot{L}^1_t \dot{B}^r_{p,\infty}}^{-1} \right) d\tau,
\]
where \( C \tau = \| \delta v \|_{\dot{L}^1_t \dot{B}^r_{p,\infty}} + \| \delta v \|_{\dot{L}^1_t \dot{B}^2_{p,\infty}} \). Noting \( \| v_2 \|_{\dot{B}^2_{p,1}} \) is integrable on \([0, \infty]\) and
\[
\int_0^1 \frac{dr}{r \log (r + C t r^{-1})} = +\infty,
\]
the Osgood lemma implies that \((\delta a, \delta O; \delta v) = 0\) on \([0, t]\). So a continuity argument ensures that \((a_1, O_1; v_1) = (a_2, O_2; v_2)\) for any \( t \in [0, \infty) \).

### 5 The proof of time-decay estimates

In this section, we aim at proving the time-weighted energy inequality (1.6) taking for granted Theorem 1.2. We will proceed the proof into the three subsections, according to three terms of the definition of \( G_p(t) \). Subsection 5.1 is devoted to the low-frequency estimates. In the spirit of [6], we need to perform nonlinear estimates in terms of deformation tensor in the Besov space with negative regularity. In Subsection 5.2, in order to overcome the technical difficulty that there is loss of one derivative for the density and deformation tensor at high frequencies, we develop “two effective velocities” to obtain the upper bound for the second term in \( G_p(t) \). To close the high-frequency estimates, in Subsection 5.3, a crucial observation enables us to establish gain of regularity and decay altogether for the velocity, which strongly depends on Proposition A.4.

For simplicity, we denote
\[
\mathcal{X}_p(t) \triangleq \| (a, O; v) \|_{E^{n,p}_t}.
\]
In what follows, we will use the two key lemmas repeatedly.

**Lemma 5.1** Let \( 0 \leq \sigma_1 \leq \sigma_2 \) with \( \sigma_2 > 1 \). It holds that
\[
\int_0^t \langle t - \tau \rangle^{-\sigma_1} \langle \tau \rangle^{-\sigma_2} d\tau \lesssim \langle t \rangle^{-\sigma_1}
\]
and
\[
\int_0^t \langle t - \tau \rangle^{-\sigma_1} \tau^{-\theta} \langle \tau \rangle^{\theta - \sigma_2} d\tau \lesssim \langle t \rangle^{-\sigma_1} \quad \text{if } 0 \leq \theta < 1.
\]
From (4.11) and (4.12), we have
\[ \frac{1}{p} \frac{d}{dt} X^p + BX^p \leq AX^{p-1} \]
for some constant \( B \geq 0 \) and measurable function \( A : [0, T] \to \mathbb{R}_+ \). Define \( X_\delta = (X^p + \delta^p)^{\frac{1}{p}} \) for \( \delta > 0 \). Then it holds that
\[ \frac{d}{dt} X_\delta + BX_\delta \leq A + B\delta. \quad (5.4) \]
For convenience, we denote by \( \| \cdot \|_{\delta,L^p} := (\| \cdot \|^p_L + \delta^p)^{1/p} \) for \( 1 \leq p < \infty \).

### 5.1 Bounds for the low frequencies

From (4.11) and (4.12), we have
\[ \frac{d}{dt} \left( \| (a_k, O_k; v_k) \|_{L^2}^2 \right) + 2^{2k} \| (a_k, O_k; v_k) \|_{L^2}^2 \lesssim \left( \sum_{i=0,1,3,4} \| \dot{\Delta}^i G_i \|_{L^2} \right) \| (a_k, O_k; v_k) \|_{L^2}. \]
It follows from Lemma 5.2 that
\[ \frac{d}{dt} \| (a_k, O_k; v_k) \|_{\delta,L^2} + 2^{2k} \| (a_k, O_k; v_k) \|_{\delta,L^2} \lesssim \sum_{i=0,1,3,4} \| \dot{\Delta}^i G_i \|_{L^2} + 2^{2k}\delta. \]
Then integrating in time and letting \( \delta \to 0 \), there exists a \( c_0 \) such that
\[ \| (a_k, O_k; v_k) \|_{L^2} \lesssim e^{-c_0 2^{2k}t} \| (\dot{\Delta}^i a_0, \dot{\Delta}^i O_0; \dot{\Delta}^i v_0) \|_{L^2} \]
\[ + \int_0^t e^{c_0 2^{2k}(\tau-t)} \sum_{i=0,1,3,4} \| \dot{\Delta}^i G_i \|_{L^2} d\tau. \]
Regarding the first term in (5.5), we multiply the factor \( \langle t \rangle^{\frac{s+s_0}{2}} 2^{ks} \) and sum up on \( 2^k \leq R_0 \) to get
\[ \langle t \rangle^{\frac{s+s_0}{2}} \sum_{2^k \leq R_0} 2^{ks} e^{-c_0 2^{2k}t} \| (\dot{\Delta}^i a_0, \dot{\Delta}^i O_0; \dot{\Delta}^i v_0) \|_{L^2} \]
\[ \lesssim \| (a_0, O_0; v_0) \|_{B_{2,\infty}}^{\epsilon} \sum_{2^k \leq R_0} \left( \sum_{2^k \leq R_0} \left( \langle t \rangle^{\frac{s+s_0}{2}} e^{-c_0 (2^k \sqrt{7})^2} + 2^{k_0 (s+s_0)} \right) \right) \]
\[ \lesssim \| (a_0, O_0; v_0) \|_{B_{2,\infty}}^{\epsilon} \]
where we have used the fact \( \sum_{2^k \leq R_0} \left( \langle t \rangle^{\frac{s+s_0}{2}} e^{-c_0 (2^k \sqrt{7})^2} + 2^{k_0 (s+s_0)} \right) \leq C \) when \( s + s_0 > 0 \). So we have
\[ \sum_{2^k \leq R_0} 2^{ks} e^{-c_0 2^{2k}t} \| (\dot{\Delta}^i a_0, \dot{\Delta}^i O_0; \dot{\Delta}^i v_0) \|_{L^2} \lesssim \langle t \rangle^{\frac{s+s_0}{2}} \| (a_0, O_0; v_0) \|_{B_{2,\infty}}^{\epsilon}. \]
Furthermore, the corresponding nonlinear term in (5.5) can be estimated as
\[
\sum_{2^{k_0} \leq R_0} 2^{k_0} \int_0^t e^{c_0 2^{k_0}(t-\tau)} \sum_{i=0,1,2,3} \| \hat{\Delta}_k G_i \|_{L^2} d\tau \lesssim \int_0^t \langle t-\tau \rangle^{-\frac{k_0}{2}} \sum_{i=0,1,2,3} \| G_i \|_{B^{-s_0}_{2,\infty}} \| t \rangle \lesssim (t)^{-\frac{k_0}{2}} \left( G_p(t) + \mathcal{X}_p(t) \right),
\]

(5.8)

We claim that if \( p \) fulfills the assumption as in Theorem 1.3, then we have for all \( t \geq 0 \),
\[
\int_0^t \langle t-\tau \rangle^{-\frac{k_0}{2}} \sum_{i=0,1,2,3} \| G_i \|_{B^{-s_0}_{2,\infty}} d\tau \lesssim \langle t \rangle^{-\frac{k_0}{2}} \left( G_p(t) + \mathcal{X}_p(t) \right),
\]

(5.9)

where \( G_p(t) \) and \( \mathcal{X}_p(t) \) are defined by (1.7) and (5.1).

Since those quadratic terms containing \( a \) and \( v \) in \( G_i(i = 0, 1, 2, 3) \) have already been done in [6], it suffices to give suitable decay estimates for some terms involving in \( O \). Precisely, we need to hand the following integral
\[
\int_0^t \langle t-\tau \rangle^{-\frac{k_0}{2}} \left( \partial_i(aO_i), \partial_k v^i O_i, v \cdot \nabla O_i, O^j \partial_j O^i, O^i \partial_i O^j, O^j \partial_j O^i \right) \| G_i \|_{B^{-s_0}_{2,\infty}} d\tau.
\]

As far as we know, the regularity level remains the same between the density and deformation tensor. Hence, these terms \( \partial_i a O_i, a \partial_i O_i, O^j \partial_j O^i, O^i \partial_i O^j, O^j \partial_j O^i \) can be treated along the same line. In principle, the above integral can be reduced to
\[
\int_0^t \langle t-\tau \rangle^{-\frac{k_0}{2}} \left( \nabla a \cdot O, \nabla v O, v \cdot \nabla O \right) \| G_i \|_{B^{-s_0}_{2,\infty}} d\tau.
\]

(5.10)

We decompose (5.10) as follows
\[
(5.10) \triangleq I^\ell + I^h,
\]
where
\[
I^\ell = \int_0^t \langle t-\tau \rangle^{-\frac{k_0}{2}} \left( O \cdot \nabla a^\ell, O \nabla v^\ell, v \cdot \nabla O^\ell \right) \| G_i \|_{B^{-s_0}_{2,\infty}} d\tau,
\]
and
\[
I^h = \int_0^t \langle t-\tau \rangle^{-\frac{k_0}{2}} \left( O \cdot \nabla a^h, O \nabla v^h, v \cdot \nabla O^h \right) \| G_i \|_{B^{-s_0}_{2,\infty}} d\tau.
\]

In order to handle \( I^\ell \) in terms of with \( a^\ell, O^\ell \) and \( v^\ell \), we use the following Lemma.

**Lemma 5.3** Let \( s_0 = n(2/p - 1/2) \) and \( p \) satisfy the assumption in Theorem 1.3. It holds that
\[
\| fg \|_{B^{-s_0}_{2,\infty}} \lesssim \| f \|_{\dot{B}^{-n/p}_{p,1}} \| g \|_{\dot{B}^{n/p-1}_{2,1}},
\]

(5.11)

and
\[
\| fg \|_{\dot{B}^{-n/p}_{2,\infty}} \lesssim \| f \|_{\dot{B}^{n/p-1}_{p,1}} \| g \|_{\dot{B}^{1-n/p}_{2,1}}.
\]

(5.12)
The reader is referred to [6] for the detailed proof. Owing to the embedding theorem and the definition of \( G_p(t) \), we shall often use the following inequalities

\[
\| (a, O; v) \|^\ell_{B^1_{p,1}} \lesssim \| (a, O; v) \|^\ell_{B^1_{p,1}} \lesssim \langle \tau \rangle^{-\frac{n}{2}} G_p(\tau), \tag{5.13}
\]

and

\[
\| (a, O) \|^\wedge_{B^1_{p,1}} \lesssim \langle \tau \rangle^{-\frac{n}{2}} G_p(\tau). \tag{5.14}
\]

Indeed, the above inequality is obvious for the high frequencies since \( \alpha \geq \frac{n}{p} \), and we have

\[
\| (a, O) \|^\ell_{B^1_{p,1}} \lesssim \| (a, O) \|^\ell_{B^1_{p,1}} \lesssim \langle \tau \rangle^{-\frac{1}{2}(s_0+n/2)} G_p(\tau) = \langle \tau \rangle^{-\frac{n}{p}} G_p(\tau). \tag{5.15}
\]

Notice that \( 1 - \frac{n}{p} \leq \frac{n}{p} \) and the definition of \( G_p(t) \), we arrive at

\[
\| v^h \|^\ell_{B^1_{p,1}} \lesssim \| v^h \|^\ell_{B^1_{p,1}} \lesssim \left( \| v^h \|^\ell_{B^1_{p,1}} \| \nabla v^h \|^\ell_{B^1_{p,1}} \right)^{\frac{1}{2}} \lesssim \tau^{-\frac{1}{2}} \langle \tau \rangle^{-\frac{n}{2}} G_p(\tau). \tag{5.16}
\]

Next, we begin with bound \( I^\ell \) and \( I^h \).

**Estimates for \( I^\ell \)**

Taking advantage of (5.11), (5.13), (5.14) and (5.16), we get

\[
\int_0^t \langle t - \tau \rangle^{-\frac{s_0+n}{2}} \langle v \cdot \nabla O^\ell \rangle_{B^1_{2,\infty}} d\tau \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s_0+n}{2}} \| v \|^\ell_{B^1_{p,1}} \| \nabla O^\ell \|^\ell_{B^1_{p,1}} d\tau \\
\lesssim G_p(t) \int_0^t \langle t - \tau \rangle^{-\frac{s_0+n}{2}} \left( \langle \tau \rangle^{-\frac{n}{2}} + \tau^{-\frac{n}{p}} \langle \tau \rangle^{-\frac{n}{2}} \right) \langle \tau \rangle^{-\frac{n}{p}} d\tau.
\]

Due to the fact that \( \frac{n}{p} + \frac{1}{2} > 1 \) and \( \frac{s_0+n}{2} \leq \frac{n}{p} + \frac{1}{2} \) for all \( s \leq 1 + \frac{n}{p} \), Lemma (5.1) implies that

\[
\int_0^t \langle t - \tau \rangle^{-\frac{s_0+n}{2}} \langle v \cdot \nabla O^\ell \rangle_{B^1_{2,\infty}} d\tau \lesssim \langle t \rangle^{-\frac{s_0+n}{2}} \| \nabla O^\ell \|^2_{B^1_{p,1}}(t). \tag{5.17}
\]

The terms \( O \cdot \nabla a^\ell \) and \( O \nabla v^\ell \) can be treated along with the same lines, so we feel free to skip the details.

**Estimates for \( I^h \)**

For the term \( I^h \) containing \( a^h, O^h \) and \( v^h \), as in [6], we proceed differently depending on whether \( p > n \) and \( p \leq n \). Let’s first consider the easy case \( 2 \leq p \leq n \). Applying (2.4) with \( \sigma = \frac{n}{p} - 1 \) yields

\[
\| f g^h \|^\ell_{B^1_{2,\infty}} \lesssim \| f \|^\ell_{B^1_{p,1}} \left( \left\| \hat{S}_{k_0+N_0} g^h \right\|_{L^p} + \| g^h \|^\ell_{B^1_{p,1}} \right) \lesssim \| f \|^\ell_{B^1_{p,1}} \| g^h \|^\ell_{B^1_{p,1}}, \tag{5.18}
\]

where we have used the Berstein inequality (\( p^* = \frac{2p}{p-2} \geq p \)) and the fact that only finite middle frequencies of \( g \) are involving in \( \hat{S}_{k_0+N_0} g^h \).

\footnote{The limit case \( p = n \) follows from \( \| f g^h \|^\ell_{B^1_{2,\infty}} \lesssim \| f \|^\ell_{L^\infty} \| g^h \|^\ell_{L^n} \lesssim \| f \|_{L^n} \| g^h \|_{L^n} \lesssim \| f \|^\ell_{B^1_{p,1}} \| g^h \|^\ell_{B^1_{p,1}}.}
Taking \( f = v \) and \( g = \nabla O \) in (5.18), we get
\[
\int_0^t \langle t - \tau \rangle^{-\frac{s + \rho_0}{2}} \| v \cdot \nabla O^h \|_{L^\infty_{t,s}} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s + \rho_0}{2}} \| v \|_{L^1_{t,s}} \| \nabla O^h \|_{L^2_{t,s}B_{p,1}} \, d\tau.
\] (5.19)

It follows from (5.13) and (5.16) that
\[
\| v \|_{L^1_{t,s}B_{p,1}} \lesssim \left( \langle \tau \rangle^{-\frac{1}{2}} + \tau^{-\frac{1}{2}} \langle \tau \rangle^{-\frac{\alpha}{2}} \right) G_p(\tau).
\] (5.20)

The definition of \( G_p(t) \) implies that
\[
\| \nabla O^h \|_{L^\infty_{t,s}B_{p,1}} \lesssim \langle \tau \rangle^{-\alpha} G_p(\tau) \quad \text{with} \quad \alpha = \frac{n}{p} + \frac{1}{2} - \varepsilon.
\] (5.21)

Inserting (5.20) and (5.21) into (5.19), we conclude that for \(-s_0 \leq s \leq \frac{d}{2} + 1 \),
\[
\int_0^t \langle t - \tau \rangle^{-\frac{s + \rho_0}{2}} \| v \cdot \nabla O^h \|_{L^\infty_{t,s}B_{2,\infty}} \, d\tau \\
\lesssim G_p^2(t) \int_0^t \langle t - \tau \rangle^{-\frac{s + \rho_0}{2}} \langle \tau \rangle^{-\alpha} \left( \langle \tau \rangle^{-\frac{1}{2}} + \tau^{-\frac{1}{2}} \langle \tau \rangle^{-\frac{\alpha}{2}} \right) \, d\tau \\
\lesssim \langle t \rangle^{-\frac{s + \rho_0}{2}} G_p^2(t).
\] (5.22)

Handling with the term \( O \cdot \nabla a^h \) is similar. With aid of (5.18), we have
\[
\int_0^t \langle t - \tau \rangle^{-\frac{s + \rho_0}{2}} \| O \cdot \nabla a^h \|_{L^\infty_{t,s}B_{2,\infty}} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s + \rho_0}{2}} \| O \|_{L^1_{t,s}B_{p,1}} \| \nabla a^h \|_{L^2_{t,s}B_{p,1}} \, d\tau \\
\lesssim G_p^2(t) \int_0^t \langle t - \tau \rangle^{-\frac{s + \rho_0}{2}} \langle \tau \rangle^{-\frac{1}{2} - \alpha} \, d\tau \\
\lesssim \langle t \rangle^{-\frac{s + \rho_0}{2}} G_p^2(t).
\] (5.23)

Regarding the term \( O \nabla v^h \), combining the embedding \( L^\infty_{t,s} \hookrightarrow B_{2,\infty}^{-s_0} \) and Hölder inequality, we obtain
\[
\int_0^t \langle t - \tau \rangle^{-\frac{s + \rho_0}{2}} \| O \nabla v^h \|_{L^\infty_{t,s}B_{2,\infty}} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s + \rho_0}{2}} \| O(\tau) \|_{L^p} \| \nabla v^h(\tau) \|_{L^p} \, d\tau.
\] (5.24)

By embedding, the definition of \( G_p(t) \) and the fact that \( \alpha \geq \frac{n}{2p} \) for sufficiently small \( \varepsilon > 0 \), we have
\[
\| O \|_{L^p} \lesssim \| O \|_{L^p}^{\alpha} + \| O^h \|_{L^p} \lesssim \| O \|_{L^p}^{\alpha} + \| O \|_{L^p}^{\alpha - \frac{n}{2p}} \\
\lesssim \left( \langle \tau \rangle^{-\frac{1}{2p}} + \langle \tau \rangle^{-\frac{\alpha}{2p}} \right) G_p(t) \lesssim \langle t \rangle^{-\frac{1}{2p}} G_p(t).
\] (5.25)
Arguing as for proving (5.16), it is easy to get for $2 \leq p \leq n$,
\[ \| \nabla v^h(\tau) \|_{L^p} \lesssim \| v^h(\tau) \|_{B_{p,1}^{\frac{n}{p}}} \lesssim \tau^{-\frac{1}{2}} \langle \tau \rangle^{-\frac{n}{2}} G_p(\tau). \] (5.26)

Furthermore, together with (5.25)-(5.26), we have
\[ \int_0^t \langle t - \tau \rangle^{-\frac{n}{2}} \| \nabla v^h(\tau) \|_{B_{2,0}^{\frac{n}{2}}} d\tau \lesssim G_p^2(t) \int_0^t \langle t - \tau \rangle^{-\frac{n}{2}} \langle \tau \rangle^{-\frac{n}{2}} d\tau \lesssim G_p^2(t) \langle t \rangle^{-\frac{n}{2}}. \] (5.27)

Let’s end that step by considering $f^h$ involving $a^h, O^h$ and $v^h$ in the case of $p > n$. Applying Inequality (2.3) with $\sigma = 1 - \frac{n}{p}$ and the embedding $B_{p,1}^{\frac{n}{p}} \hookrightarrow L^{p^*}$ give that
\[ \| fg^h \|_{B_{2,\infty}^{\frac{n}{p}}} \lesssim (\| f \|_{B_{p,1}^{1 - \frac{n}{p}}} + \| \dot{S}_{k_0 + N_0} f \|_{L^{p^*}}) \| g^h \|_{B_{p,1}^{\frac{n}{p}} - 1} \lesssim (\| f \|_{B_{2,1}^{\frac{n}{p}}} + \| f \|_{B_{p,1}^{1 - \frac{n}{p}}}) \| g^h \|_{B_{p,1}^{\frac{n}{p}} - 1}, \] (5.28)

where $\frac{1}{p^*} = \frac{1}{2} - \frac{1}{p}$. Taking $f = v$ and $g = \nabla O$, and then using (5.13), (5.16) as well as the definition of $G_p(t)$, we arrive at
\[ \int_0^t \langle t - \tau \rangle^{-\frac{n}{2}} \| (v \cdot \nabla O^h) \|_{B_{2,\infty}^{\frac{n}{2}}} d\tau \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{n}{2}} (\| v^\ell \|_{B_{2,1}^{\frac{n}{2}}} + \| v \|_{B_{p,1}^{1 - \frac{n}{2}}}) \| \nabla O^h \|_{B_{p,1}^{\frac{n}{2}} - 1} d\tau \lesssim G_p^2(t) \int_0^t \langle t - \tau \rangle^{-\frac{n}{2}} \langle \tau \rangle^{-\frac{n}{2}} d\tau \lesssim \langle t \rangle^{-\frac{n}{2}} G_p^2(t). \] (5.29)

Next by taking $f = O$ and $g = \nabla a$ in (5.28), we get
\[ \int_0^t \langle t - \tau \rangle^{-\frac{n}{2}} \| O \cdot \nabla a^h \|_{B_{2,\infty}^{\frac{n}{2}}} d\tau \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{n}{2}} (\| O^\ell \|_{B_{2,1}^{\frac{n}{2}}} + \| O \|_{B_{p,1}^{1 - \frac{n}{2}}}) \| \nabla a^h \|_{B_{p,1}^{\frac{n}{2}} - 1} d\tau. \] (5.30)

It follows from (5.13) and (5.14) that
\[ \| O^\ell \|_{B_{2,1}^{\frac{n}{2}}} + \| O \|_{B_{p,1}^{1 - \frac{n}{2}}} \lesssim (\langle \tau \rangle^{-\frac{n}{2p} - \frac{n}{2}} + \langle \tau \rangle^{-\frac{1}{2}} + \langle \tau \rangle^{-\alpha}) G_p(\tau). \] (5.31)

Consequently, we can get
\[ \int_0^t \langle t - \tau \rangle^{-\frac{n}{2}} \| O \nabla a^h \|_{B_{2,\infty}^{\frac{n}{2}}} d\tau \]
and (5.9), we deduce that we need to perform a suitable quasi-diagonalization, say to eliminate the technical difficulty, as the proof of global-in-time existence, to handle high frequencies. Let \( L \) energy methods of type, we obtain

\[
\| \nabla v^h \|_{B^{\frac{3}{2}}_{p,1}} \lesssim \left( \| v \|_{B^{\frac{3}{2}}_{p,1}} \| \nabla v \|_{B^{\frac{3}{2}}_{p,1}} \right)^{\frac{1}{2}} \lesssim \tau^{-\frac{1}{2}} \langle \tau \rangle^{-\frac{d}{2}} G_p(\tau).
\]

By interpolation, for all \( \tau \geq 0 \),

\[
\| \nabla v^h \|_{B^{3/2}_{p,1}} \lesssim \left( \| v \|_{B^{3/2}_{p,1}} \| \nabla v \|_{B^{3/2}_{p,1}} \right)^{\frac{1}{2}} \lesssim \tau^{-\frac{1}{2}} \langle \tau \rangle^{-\frac{d}{2}} G_p(\tau).
\]

Therefore, we are led to

\[
\begin{align*}
\int_0^t \langle t - \tau \rangle^{-\frac{s+\theta}{2}} \| O \nabla v^h \|_{B^{s}_{2,\infty}} d\tau \\
\lesssim G_p^2(t) \int_0^t \langle t - \tau \rangle^{-\frac{s+\theta}{2}} \langle \tau \rangle^{-\min\left(\frac{1}{2}, \frac{3a}{2p} - \frac{3}{d} \right) \tau^{-\frac{1}{2}} \tau^{-\frac{d}{2}} d\tau} \\
\lesssim G_p^2(t) \langle t \rangle^{-\frac{s+\theta}{2}}.
\end{align*}
\]

Putting together all the above estimates for those terms involving in \( O \) and those computations with respect to \( a \) and \( v \) (see [6]), we finish the proof of (5.9). Then by combining (5.7) and (5.9), we deduce that

\[
\langle t \rangle^{-\frac{s+\theta}{2}} \| (a, O; v)(t) \|_{B^{3/2}_{2,1}} \lesssim G_{p,0} + G_p^2(t) + A_p^2(t).
\]  

5.2 Decay estimates for the high frequencies of \((\nabla a, \nabla O; v)\)

This part is devoted to estimating the second term in \( G_p(t) \). The usual Duhamel principle cannot remain true, since there is a loss of one derivative for the density and deformation tensor at high frequencies. To eliminate the technical difficulty, as the proof of global-in-time existence, we need to perform a suitable quasi-diagonalization, say effective velocities, to handle high frequencies. Let \( \hat{\Delta}_k w = w_k \) and \( \hat{\Delta}_k \Omega^{ij} = \Omega_k^{ij} \). From (4.33) and (4.38), by employing the energy methods of \( L^p \) type, we obtain

\[
\frac{d}{dt} \| w_k \|^p_p + c_p 2^k \| w_k \|^p_p \lesssim \left\{ 2^{-k} \| a_k \|_p + 2^{-k} \| \hat{\Delta}_k \hat{G}_1 \|_p + \| \hat{\Delta}_k G_2 \|_p \right\} w_k \|_p^{p-1},
\]

\[
\frac{d}{dt} \| \Omega_k^{ij} \|^p_p + c_p 2^k \| \Omega_k^{ij} \|^p_p \lesssim \left\{ 2^{-k} \| O_k^{ij} \|_p + 2^{-k} \| \hat{\Delta}_k \hat{G}_3^{ij} \|_p + \| \hat{\Delta}_k \hat{G}_4^{ij} \|_p \right\} \| \Omega_k^{ij} \|_p^{p-1},
\]

\[
\lesssim G_p^2(t) \langle t \rangle^{-\frac{s+\theta}{2}}.
\]
\[
\frac{d}{dt} \|\Lambda a_k\|_p^p + c_p \|\Lambda a_k\|_p^p \\
\lesssim \left\{ \|\dot{\Lambda} \Delta_k \tilde{G}_1\|_p + 2^{2k} \|w\|_p + \|\text{div} v\|_\infty \|\Lambda a_k\|_p + \|\mathcal{R}_{1,k}\|_p \right\} \|\Lambda a_k\|_p^{p-1} \tag{5.38}
\]

and
\[
\frac{d}{dt} \|\Lambda O_{ij}^{ij}\|_p^p + c_p \|\Lambda O_{ij}^{ij}\|_p^p \\
\lesssim \left\{ \|\dot{\Lambda} \Delta_k \tilde{G}_3\|_p + 2^{2k} \|\Omega_{ij}\|_p + \|\text{div} v\|_\infty \|\Lambda O_{ij}^{ij}\|_p + \|\mathcal{R}_{2,k}\|_p \right\} \|\Lambda O_{ij}^{ij}\|_p^{p-1} \tag{5.39}
\]

with \(\mathcal{R}_{1,k} \triangleq [v \cdot \nabla, \Lambda \dot{\Delta}a] a\) and \(\mathcal{R}_{2,k} \triangleq [v \cdot \nabla, \Lambda \dot{\Delta}a] O_{ij}^{ij}\), where we chosen \(R_0\) sufficiently large such that \(2^k > R_0\). Furthermore, with aid of Lemma 5.2, there exists a constant \(c_p > 0\) such that
\[
\frac{d}{dt} \left( \|\Lambda a_k\|_{\delta,L_P} + \|d_k\|_{\delta,L_P} \right) + c_p \left( \|\Lambda a_k\|_{\delta,L_P} + 2^{2k} \|d_k\|_{\delta,L_P} \right) \\
\lesssim \|\dot{\Lambda} \Delta_k \tilde{G}_1\|_p + \|\dot{\Delta} G_2\|_p + \|\text{div} v\|_\infty \|\Lambda a_k\|_p + \|\mathcal{R}_{1,k}\|_p + 2^{2k} \delta, \tag{5.40}
\]

where we used the effective velocity in terms of \(a\) and \(d\). A similar estimate for \(O_{ij}^{ij}\) and \(e_{ij}^{ij}\) stems from (5.37) and (5.39):
\[
\frac{d}{dt} \left( \|\Lambda O_{ij}^{ij}\|_{\delta,L_P} + \|e_{ij}^{ij}\|_{\delta,L_P} \right) + \tilde{c}_p \left( \|\Lambda O_{ij}^{ij}\|_{\delta,L_P} + 2^{2k} \|e_{ij}^{ij}\|_{\delta,L_P} \right) \\
\lesssim \|\dot{\Lambda} \Delta_k \tilde{G}_3\|_p + \|\dot{\Delta} G_4\|_p + \|\text{div} v\|_\infty \|\Lambda O_{ij}^{ij}\|_p + \|\mathcal{R}_{2,k}\|_p + 2^{2k} \delta \tag{5.41}
\]

for some constant \(\tilde{c}_p > 0\). It’s easy to see that
\[
\|\dot{\Lambda} \Delta_k \tilde{G}_1\|_p \lesssim \|\dot{\Lambda} \Delta_k \tilde{G}_4\|_p + 2^{2k} \|d_k\|_p + \|\Lambda a_k\|_p \\
\lesssim \|\dot{\Lambda} \Delta_k \tilde{G}_4\|_p + 2^{2k} \|d_k\|_{\delta,L_P} + \|\Lambda a_k\|_{\delta,L_P}. \tag{5.42}
\]

Therefore, it follows from (5.40), (5.41) and (5.42) that
\[
\frac{d}{dt} \left( \|(\Lambda a_k, \Lambda O_k; v_k)\|_{\delta,L_P} + c_0 \|(\Lambda a_k, \Lambda O_k; v_k)\|_{\delta,L_P} \right) \\
\lesssim \|\text{div} v\|_\infty \|(\Lambda a_k, \Lambda O_k)\|_p + \|\mathcal{R}_{1,k}\|_p + \|\mathcal{R}_{2,k}\|_p \\
+ \|\dot{\Lambda} \Delta_k (\tilde{G}_1, \tilde{G}_3)\|_p + \|\dot{\Lambda} (G_2, G_4)\|_p + 2^{2k} \delta \tag{5.43}
\]

for some constant \(c_0 > 0\). Integrating in time on both sides and letting \(\delta \to 0\), we eventually get
\[
\|(\Lambda a_k, \Lambda O_k; v_k)(t)\|_p \leq e^{-c_0 t} \|(\Lambda a_k(0), \Lambda O_k(0), v_k(0))\|_p + C \int_0^t e^{c_0 (\tau-t)} g_k(\tau) d\tau, \tag{5.44}
\]

where
\[
g_k \triangleq \frac{\|\text{div} v\|_{\infty} \left( \|(\Lambda a_k)\|_p + \|\Lambda O_k\|_p \right)}{g_k}
\]
Multiplying (5.44) by \(\langle t \rangle^\alpha 2^{k(\frac{n}{p}-1)}\), taking the supremum on \([0, T]\) and summing up over \(k\) satisfying \(2^k > R_0\) yields

\[
\|\langle t \rangle^\alpha (\Lambda a, \Lambda O; v)\|_{L^\infty_t B_{p,1}^\frac{n}{p}-1} \lesssim \|\langle \Lambda a_0, \Lambda O_0; v_0 \rangle\|_{B_{p,1}^\frac{n}{p}-1} + \sum_{2^k > R_0} \sup_{0 \leq t \leq 2^k} \left( \langle t \rangle^\alpha \int_0^t e^{\epsilon_0(\tau-t)} 2^{k(\frac{n}{p}-1)} g_k(\tau) d\tau \right). \tag{5.45}
\]

Without loss of generality, we assume that \(T \geq 2\) and first bound the the supremum for \(0 \leq t \leq 2\). Notice that

\[
\sum_{2^k > R_0} \sup_{0 \leq t \leq 2} \left( \langle t \rangle^\alpha \int_0^t e^{\epsilon_0(\tau-t)} 2^{k(\frac{n}{p}-1)} g_k(\tau) d\tau \right) \lesssim \int_0^2 \sum_{2^k > R_0} 2^{k(\frac{n}{p}-1)} g_k(\tau) d\tau. \tag{5.46}
\]

Furthermore, it will be shown that the right side of (5.46) can be bounded by \(\mathcal{X}_p^2(2)\). Indeed, using Lemma 2.6 and the representation of \(\tilde{G}_i(i = 1, 3)\) and \(G_i(i = 2, 4)\), we get

\[
\int_0^2 \sum_{2^k > R_0} 2^{k(\frac{n}{p}-1)} g_k(\tau) d\tau \leq \int_0^2 \left\{ \|\text{div}v\|_{L^\infty} \left( \|\Lambda O\|_{B_{p,1}^\frac{n}{p}-1} + \|\Lambda a\|_{B_{p,1}^\frac{n}{p}-1} \right) \right. \]

\[
\left. + \|(\text{div}(aO), \Lambda(\partial_k v^i O^k)), O^ik \partial_l O^ik, O^ij \partial_l O^ik, O^ik \partial_l \partial_j O^k, O^ij \partial_l O^k)\|_{B_{p,1}^\frac{n}{p}-1} \right. \]

\[
\left. + \left\{ \|a \cdot \nabla v\|_{B_{p,1}^\frac{n}{p}}, \|\nabla v\|_{B_{p,1}^\frac{n}{p}}, \|O\|_{B_{p,1}^\frac{n}{p}}, \|\Lambda a\|_{B_{p,1}^\frac{n}{p}} \right\} d\tau. \tag{5.47}
\]

In comparison with [6], we focus on those terms only involving \(O\). For instance, we see that

\[
\int_0^2 \left( \|\text{div}v\|_{L^\infty} \|\Lambda O\|_{B_{p,1}^\frac{n}{p}-1} + \|\nabla v\|_{B_{p,1}^\frac{n}{p}}, \|O\|_{B_{p,1}^\frac{n}{p}} \right) d\tau \lesssim \|O\|_{L^\infty_t B_{p,1}^\frac{n}{p}} \int_0^2 \|v\|_{B_{p,1}^\frac{n}{p}} d\tau \lesssim \mathcal{X}_p^2(2).
\]
Owing to Lemma 2.3 and interpolation inequalities, we obtain

\[
\int_0^2 \left\| \left( \text{div}(aO), \Lambda(p_i O^{kj}), O^{lk} \partial_i O^{ik}, O^{lj} \partial_i O^{jk}, O^{lk} \partial_j O^{ij}, O^{jk} \partial_j O^* \right) \right\|_{\dot{B}^0_{p,1}}^h \, d\tau \\
\lesssim \int_0^2 \left\| a \right\|_{\dot{B}^0_{p,1}} \left\| O \right\|_{\dot{B}^0_{p,1}} \left\| v \right\|_{\dot{B}^{p+1}_{p,1}} + \left\| O \right\|_{\dot{B}^0_{p,1}} \left\| \nabla O \right\|_{\dot{B}^{0-1}_{p,1}} \, d\tau \\
\lesssim \left( \left\| a \right\|_{L^2 \dot{B}^0_{p,1}} + \left\| O \right\|_{L^2 \dot{B}^0_{p,1}} \right) \left\| O \right\|_{L^2 \dot{B}^{p+1}_{p,1}} + \left\| v \right\|_{L^1 \dot{B}^{p+1}_{p,1}} \left\| O \right\|_{L^\infty \dot{B}^{0-1}_{p,1}}.
\]

Hence, we can infer that

\[
\sum_{2^k > R_0} \sup_{0 \leq t \leq 2^k} \langle t \rangle^\alpha \int_0^t e^{c_0(\tau-t)} 2^{k(\frac{a}{p}-1)} g_k(\tau) \, d\tau \lesssim \mathcal{X}_p^2(2).
\] (5.48)

Let us now bound the supremum for \(2 \leq t \leq T\) in the last term of (5.45). To this end, we split the integral on \([0, t]\) into integrals on \([0, 1]\) and \([1, t]\). The integral on \([0, 1]\) is easy to handle: because \(e^{c_0(\tau-t)} \leq e^{-\frac{c_0 t}{2}}\) for \(2 \leq t \leq T\) and \(0 \leq \tau \leq 1\), so one can write

\[
\sum_{2^k > R_0} \sup_{2^k \leq t \leq T} \langle t \rangle^\alpha \int_0^1 e^{c_0(\tau-t)} 2^{k(\frac{a}{p}-1)} g_k(\tau) \, d\tau \lesssim \sum_{2^k > R_0} \sup_{2^k \leq t \leq T} \langle t \rangle^\alpha e^{-\frac{c_0 t}{2}} \int_0^1 2^{k(\frac{a}{p}-1)} g_k(\tau) \, d\tau \\
\lesssim \sum_{2^k > R_0} \int_0^1 2^{k(\frac{a}{p}-1)} g_k(\tau) \, d\tau.
\] (5.49)

Therefore, following the procedure leading to (5.48), we end up with

\[
\sum_{2^k > R_0} \sup_{2^k \leq t \leq T} \langle t \rangle^\alpha \int_0^t e^{c_0(\tau-t)} 2^{k(\frac{a}{p}-1)} g_k(\tau) \, d\tau \lesssim \mathcal{X}_p^2(1).
\] (5.50)

In order to bound the integral on \([1, t]\) for \(2 \leq t \leq T\), we notice that

\[
\sum_{2^k > R_0} \sup_{2^k \leq t \leq T} \left( \langle t \rangle^\alpha \int_1^t e^{c_0(\tau-t)} 2^{k(\frac{a}{p}-1)} g_k(\tau) \, d\tau \right) \lesssim \sum_{2^k > R_0} 2^{k(\frac{a}{p}-1)} \sup_{1 \leq t \leq T} (t^\alpha g_k(t)).
\] (5.51)

In nonlinear sources \(g^1_k, g^2_k\) and \(g^3_k\), the calculations for those terms with respect to \(O\) are totally similar, so we present them for \(\text{div}(aO)\) and \(\Lambda(p_i O^{kj})\) for brevity. We write

\[
\text{div}(aO) = a \nabla \cdot O + \nabla a \cdot O.
\]

Due to the same regularity level, it suffices to estimate the term \(a \nabla \cdot O\). Using Lemma 2.3, we deduce that

\[
\| t^\alpha (a \nabla \cdot O^h) \|_{L^\infty \dot{B}^{0-1}_{p,1}} \lesssim \| a \|_{L^\infty \dot{B}^0_{p,1}} \| t^\alpha \nabla O \|_{L^\infty \dot{B}^{0-1}_{p,1}} \leq \mathcal{X}_p(T) G_p(T),
\] (5.52)
It is obvious that
\[ \| t^\alpha (a \nabla \cdot O) \|_{L^p_t B^{n-1}_{p,1}} \lesssim \| t^{\frac{n}{2}} a \|_{L^\infty_t B^0_{p,1}} \| t^{\frac{n}{2}} O \|_{L^\infty_t B^0_{p,1}} \lesssim \left( \| t^{\frac{n}{2}} a \|_{L^\infty_t B^0_{p,1}} + \| t^{\frac{n}{2}} a \|_{L^\infty_t B^0_{p,1}} \right) \| t^{\frac{n}{2}} O \|_{L^\infty_t B^0_{p,1}} \leq G^2_p(T), \tag{5.53} \]

since the fact \( \frac{n}{2} \leq \frac{n_0}{2} + \frac{n}{4} - \varepsilon \) indicates that \( \| t^{\frac{n}{2}} z \|_{L^\infty_t B^0_{p,1}} \lesssim \| t^{\frac{n}{2}} z \|_{L^\infty_t B^0_{p,1}} \lesssim G_p(T) \) for \( z = a, O, v \). Combining (5.52) and (5.53), we get
\[ \| t^\alpha (a \nabla \cdot O) \|_{L^\infty_t B^{n-1}_{p,1}} \lesssim \mathcal{X}_p(T)G_p(T) + G^2_p(T). \tag{5.54} \]

In addition, it follows from (1.7) that
\[ \| \tau \nabla v \|_{L^\infty_t B^{n}_{p,1}} \lesssim G_p(t). \tag{5.55} \]

By Lemmas 2.1 and 2.3, we have
\[ \| t^\alpha \Lambda (\partial_k v^i O^{kj}) \|_{L^\infty_t B^{n-1}_{p,1}} \lesssim \| t^\alpha \left( O^{kj} \partial_k v^i \right) \|_{L^\infty_t B^{n}_{p,1}} \lesssim \| t^{\alpha-1} O \|_{L^\infty_t B^{n}_{p,1}} \| t \nabla v \|_{L^\infty_t B^{n}_{p,1}} \lesssim \left( \| t^{\alpha-1} O \|_{L^\infty_t B^{n}_{p,1}} + \| t^{\alpha-1} O \|_{L^\infty_t B^{n}_{p,1}} \right) \| t \nabla v \|_{L^\infty_t B^{n}_{p,1}}. \tag{5.56} \]

It is obvious that \( \| t^{\alpha-1} O \|_{L^\infty_t B^{n}_{p,1}} \leq G_p(T) \) according to the definition of \( G_p(T) \). On the other hand, we have the following estimates for \( z = a, O, v \),
\[ \| t^{\alpha-1} z \|_{L^\infty_t B^{n-1}_{p,1}} \lesssim \| t^{\alpha-1} z \|_{L^\infty_t B^{n-1-2\varepsilon}_{p,1}} \leq G_p(T), \tag{5.57} \]
as \( \alpha - 1 = \frac{1}{2}(s_0 + n/2 - 1 - 2\varepsilon) \) with enough small \( \varepsilon \). Consequently, we arrive at
\[ \| t^\alpha \Lambda (\partial_k v^i O^{kj}) \|_{L^\infty_t B^{n-1}_{p,1}} \lesssim G^2_p(T). \tag{5.58} \]

In a conclusion, by combining those estimates involving \( a \) and \( v \) in [6], we can conclude that
\[ \| (t)^\alpha (\Lambda a, \Lambda O; v) \|_{L^\infty_t B^{n-1}_{p,1}} \lesssim \| (\Lambda a_0, \Lambda O_0; v_0) \|_{B^{n-1}_{p,1}} + G^2_p(T) + \mathcal{X}_p^2(T). \tag{5.59} \]

### 5.3 Decay and gain of regularity for the high frequencies of \( v \)

In order to bound the last term in \( G_p(t) \), it is convenient to rewrite the velocity equation in the following way. First, it follows from (3.4) that
\[ \partial_t v - \mathcal{A} v = F \]
Hence, we have
\[
\partial_t(tAv) - A(tAv) = Av + tAF. \tag{5.61}
\]

We thus deduce from Proposition A.4 and the remark that follows, that
\[
\|\tau \nabla^2 v\|_{L_t^\infty B_{p,1}^{\frac{n}{p}+1}}^h \lesssim \|Av\|_{L_t^1 B_{p,1}^{\frac{n}{p}-1}}^h + \|\tau AF\|_{L_t^\infty B_{p,1}^{\frac{n}{p}-3}}^h \\
\lesssim \|v\|_{L_t^1 B_{p,1}^{\frac{n}{p}+1}}^h + \|\tau F\|_{L_t^\infty B_{p,1}^{\frac{n}{p}-1}}^h \\
\lesssim \chi_p(t) + \|\tau F\|_{L_t^\infty B_{p,1}^{\frac{n}{p}-1}}^h, \tag{5.62}
\]
where we have used the bounds given by Theorem 1.2. Secondly, we turn to bound the norm \(\|\tau F\|_{L_t^\infty B_{p,1}^{\frac{n}{p}-1}}^h\). Because \(\alpha \geq 1\), we have
\[
\|\tau(\nabla a, \nabla \cdot O)\|_{L_t^\infty B_{p,1}^{\frac{n}{p}}-1}^h \lesssim \|\tau^\alpha(a, O)\|_{L_t^\infty B_{p,1}^{\frac{n}{p}}}^h. \tag{5.63}
\]

Product and composition estimates indicate that
\[
\|\tau(K(a)\nabla a, O^{jk} \partial_j O^{*k})\|_{L_t^\infty B_{p,1}^{\frac{n}{p}}}^h \lesssim \|\tau^\alpha(a, O)\|_{L_t^\infty B_{p,1}^{\frac{n}{p}}}^h \lesssim \mathcal{G}_p^2(t). \tag{5.64}
\]
Together with those estimates for other nonlinear terms (see [6]), we can conclude that
\[
\|\tau \nabla v\|_{L_t^\infty B_{p,1}^{\frac{n}{p}}}^h \lesssim \chi_p^2(t) + \mathcal{G}_p^2(t) + \|\tau^\alpha(a, O)\|_{L_t^\infty B_{p,1}^{\frac{n}{p}}}^h. \tag{5.65}
\]
Finally, bounding the last term on the right-side of (5.65) according to (5.59), and adding up the obtained inequality to (5.35) and (5.65) yields for all \(t \geq 0\)
\[
\mathcal{G}_p(t) \lesssim \mathcal{G}_{p,0} + \|\tau^\alpha(a, O_0; v_0)\|_{B_{2,1}^{\frac{n}{p}-1}}^h + \|\tau(\nabla a_0, \nabla O_0; v_0)\|_{B_{2,1}^{\frac{n}{p}}}^h + \chi_p^2(t) \\
\lesssim \mathcal{G}_{p,0} + \mathcal{E}_{0}^{\frac{n}{p}} + \mathcal{G}_p^2(t) + \chi_p^2(t). \tag{5.66}
\]
It follows from Theorem 1.2 that \(\chi_p(t) \leq M\mathcal{E}_{0}^{\frac{n}{p}} \leq M\eta \ll 1\). On the other hand, \(\|\tau^\alpha(a, O_0; v_0)\|_{B_{2,\infty}^{\frac{n}{p}-1}}^h \lesssim \|\tau^\alpha(a, O_0; v_0)\|_{B_{2,\infty}^{\frac{n}{p}-1}}^h\), so one can conclude that (1.6) is fulfilled for all time if \(\mathcal{G}_{p,0}\) and \(\|\tau(\nabla a_0, \nabla O_0; v_0)\|_{B_{2,1}^{\frac{n}{p}}}^h\) are small enough. This finishes the proof of Theorem 1.3 eventually.

**Appendix: Some Estimates in the Hybrid Besov Space**

**Proposition A.1 ([5])** Let \(s_1, s_2, t_1, t_2, \sigma, \tau \in \mathbb{R}, 2 \leq p \leq 4\) and \(1 \leq r, r_1, r_2, r_3, r_4 \leq \infty\) with \(\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4}\). Then we have the following:
• If $\sigma, \tau \leq n/p$ and $\sigma + \tau > 0$, then

\[ \sum_{2^k > R_0} 2^{k(\sigma + \tau - n/p)} \| \hat{\Delta}_k (fg) \|_{L^p_t L^p_x} \leq C \| f \|_{L^{r_1}_t \dot{B}^{n/2-n/p+\sigma}_{p,q}} \| g \|_{L^{r_2}_t \dot{B}^{n/2-n/p+\sigma+\tau}_{p,q}}. \]

(A.1)

• If $s_1, s_2 \leq n/p$ and $s_1 + t_1 > n - \frac{2n}{p}$ with $s_1 + t_1 = s_2 + t_2$ and $\gamma \in \mathbb{R}$, then

\[ \sum_{2^k \leq R_0} 2^{k(s_1 + t_1 - n/2)} \| \hat{\Delta}_k (fg) \|_{L^p_t L^2_x} \leq C \left( \| f \|_{L^{r_1}_t \dot{B}^{s_1-n/2-n/p}_{p,q}} \| g \|_{L^{r_2}_t \dot{B}^{s_1-t_1-n/2+n/p+\gamma}_{p,q}} + \| g \|_{L^{r_3}_t \dot{B}^{s_2-n/2-n/p}_{p,q}} \| f \|_{L^{r_4}_t \dot{B}^{s_2-t_2-n/2+n/p}_{p,q}} \right). \]

(A.2)

• If $s_1, s_2 \leq n/2$ and $s_1 + t_1 > \frac{n}{2} - \frac{n}{p}$ with $s_1 + t_1 = s_2 + t_2$, then

\[ \sum_{k \in \mathbb{Z}} 2^{k(s_1 + t_1 - n/2)} \| \hat{\Delta}_k (fg) \|_{L^p_t L^2_x} \leq C \left( \| f \|_{L^{r_1}_t \dot{B}^{s_1-n/2-n/p}_{p,q}} \| g \|_{L^{r_2}_t \dot{B}^{1}_{p,q}} + \| g \|_{L^{r_3}_t \dot{B}^{s_2-n/2-n/p}_{p,q}} \| f \|_{L^{r_4}_t \dot{B}^{1}_{p,q}} \right). \]

(A.3)

**Proposition A.2 ([5])** Let $2 \leq p \leq 4$, $\sigma > 0$, and $s \geq \sigma - n/2 + n/p$, $r \geq 1$. Assume that $F \in W^{[s]+2,\infty}_{loc} \cap W^{[\sigma]+2,\infty}_{loc}$ with $F(0) = 0$. Then there has

\[ \| F(f) \|_{L^2_t \dot{B}^{s,\sigma}_{p,q}} \leq C(1 + \| f \|_{L^\infty_t \dot{B}^{n/p,n/p}_{p,q}})^{\max([s],[\sigma])+1} \| f \|_{L^2_t \dot{B}^{s,\sigma}_{p,q}}. \]

(A.4)

For any $s > 0$ and $p \geq 1$, there has

\[ \| F(f) \|_{L^p_t \dot{B}^{s}_{p,1}} \leq C(1 + \| f \|_{L^\infty_t \dot{B}^{n/p,n/p}_{p,q}})^{[s]+1} \| f \|_{L^p_t \dot{B}^{s}_{p,1}}. \]

(A.5)

**Proposition A.3 ([11])** Let $s \in (-n \min(1/p,1/p'), 1 + n/p)$ and $1 \leq p, q \leq \infty$. Let $v$ be a vector field such that $\nabla v \in L^1_t \dot{B}^{n/p}_{p,1}$. Assume that $f_0 \in \dot{B}^{s}_{p,q}, g \in L^1_t \dot{B}^{s}_{p,q}$, and $f$ is a solution of the transport equation

\[ \partial_t f + v \cdot \nabla f = g, \quad f|_{t=0} = f_0. \]

Then for $t \in [0, T]$, there holds

\[ \| f \|_{L^p_t \dot{B}^{s}_{p,q}} \leq \exp \left( C \int_0^t \| \nabla v(\tau) \|_{L^{n/p}_t \dot{B}^{n/p}_{p,1}} \, d\tau \right) \left( \| f_0 \|_{\dot{B}^{s}_{p,q}} + \int_0^t \| g(\tau) \|_{\dot{B}^{s}_{p,q}} \, d\tau \right). \]

For the heat equation, one has the following optimal regularity estimate.
Proposition A.4 Let $p, r \in [1, \infty]$, $s \in \mathbb{R}$, and $1 \leq \rho_2 \leq \rho_1 \leq \infty$. Assume that $u_0 \in \dot{B}^{s-1}_{p,r}$, $f \in \dot{L}^{\rho_2}_{T} \dot{B}^{s-3+\frac{2}{\rho_2}}_{p,r}$. Let $u$ be a solution of the equation

$$\partial_t u - \mu \Delta u = f, \quad u|_{t=0} = u_0.$$ 

Then for $t \in [0, T]$, there holds

$$\mu^{\frac{1}{\rho_1}} \|u\|_{\dot{L}^{\rho_1}_{T} \dot{B}^{s-1+2/\rho_1}_{p,r}} \leq C \left( \|u_0\|_{\dot{B}^{s-1}_{p,r}} + \mu^{1/\rho_2-1} \|f\|_{\dot{L}^{\rho_2}_{T} \dot{B}^{s-3+\frac{2}{\rho_2}}_{p,r}} \right). \quad (A.6)$$

Remark A.1 The estimate (A.6) is still hold for the following equation

$$\partial_t u - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u = f, \quad u|_{t=0} = u_0, \quad (A.7)$$

where $\lambda$ and $\mu$ are constants such that $\mu > 0$ and $\lambda + \mu > 0$ (up to the different dependence on the viscous coefficients). Indeed, both $\mathcal{P}u$ and $\mathcal{P}^\perp u$ satisfy the heat equation. We can apply $\mathcal{P}$ and $\mathcal{P}^\perp$ to (A.7) to get the heat estimate (A.6).

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