Versal Deformations of a Dirac Type Differential Operator

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Abstract

If we are given a smooth differential operator in the variable \( x \in \mathbb{R}/2\pi\mathbb{Z} \), its normal form, as is well known, is the simplest form obtainable by means of the \( \text{Diff}(S^1) \)-group action on the space of all such operators. A versal deformation of this operator is a normal form for some parametric infinitesimal family including the operator. Our study is devoted to analysis of versal deformations of a Dirac type differential operator using the theory of induced \( \text{Diff}(S^1) \)-actions endowed with centrally extended Lie-Poisson brackets. After constructing a general expression for tranversal deformations of a Dirac type differential operator, we interpret it via the Lie-algebraic theory of induced \( \text{Diff}(S^1) \)-actions on a special Poisson manifold and determine its generic moment mapping. Using a Marsden-Weinstein reduction with respect to certain Casimir generated distributions, we describe a wide class of versally deformed Dirac type differential operators depending on complex parameters.

1 Introduction

Suppose we are given the linear 2-vector first order Dirac differential operator on the real axis \( \mathbb{R} \):

\[
L_\lambda f := -\frac{df}{dx} + l_\lambda[u, v; z] f, \quad l_\lambda[u, v; z] := \begin{pmatrix} z - \lambda & u \\ v & \lambda - z \end{pmatrix}
\]  

(1.1)

acting on the Sobolev space \( W^{(1)}_{2,\text{loc}}(\mathbb{R}; \mathbb{C}^2) \) and depending on \( 2\pi \)-periodic coefficients \( u, v, z \in C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{C}) \) and a complex parameter \( \lambda \in \mathbb{C} \). The variety of all operators (1.1), parametrized by \( \lambda \), will be denoted by \( \mathcal{L}_\lambda \).

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Let $A := \text{Diff}(S^1)$ be the group of orientation preserving diffeomorphisms of the circle $S^1$. A group action of $A$ on $L_\lambda$ can be defined as follows: Fixing a parametrization of $S^1$, i.e., a $C^\infty$ covering $p : \mathbb{R} \to S^1$ such that the mapping $p : [a, a + 2\pi) = S^1$ is one-to-one for every real $a$ and $p(x + 2\pi) = p(x)$ for all $x \in \mathbb{R}$, each $\phi \in A$ can obviously be represented by a smooth mapping $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\phi(\xi + 2\pi) = \phi(\xi) + 2\pi \quad \text{and} \quad \phi'(\xi) > 0$$

for all $\xi \in \mathbb{R}$. Upon making the change of variables

$$x = \phi(\xi), \quad f(\phi(\xi)) = \Phi(\xi)\tilde{f}(\xi),$$

with $\phi \in A$, $\Phi \in G := C^\infty(\mathbb{R}/2\pi\mathbb{Z}; SL(2; \mathbb{C}))$ and $x, \xi \in \mathbb{R}$, in (1.1), it is easy to see that the differential operator $L_\lambda$ transforms into $L_\lambda^{(\phi, \Phi)} : W_2^{(1)} \to W_2^{(1)}$ defined as

$$L_\lambda^{(\phi, \Phi)}\tilde{f}(\xi) := -\frac{d\tilde{f}}{d\xi} + l_\lambda^{(\phi, \Phi)}[u, v; z]\tilde{f},$$

where

$$l_\lambda^{(\phi, \Phi)}[u, v; z] := -\Phi^{-1}(\xi)\frac{d\Phi(\xi)}{d\xi} + \phi'(\xi)\Phi^{-1}(\xi)l_\lambda[u, v; z]\Phi(\xi).$$

We assume now that the matrix $\Phi(\xi)$ is chosen so that $l_\lambda^{(\phi, \Phi)}[u, v; z] = l_\lambda[\tilde{u}, \tilde{v}; \tilde{z}]$ for all $\lambda \in \mathbb{C}$ and some mapping $(\tilde{u}, \tilde{v}; \tilde{z})^T \in C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{C}^2 \times \mathbb{C})$. Whence we obtain an induced nonlinear transformation $A^*(\phi, \Phi) : L_\lambda \to L_\lambda$, $(\phi, \Phi) \in A \times G$, where

$$A^*(\phi, \Phi)l_\lambda[u, v; z] := l_\lambda^{(\phi, \Phi)}[u, v; z]$$

for all mappings in $C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{C}^2 \times \mathbb{C})$. This together with expression (1.5) determines an automorphism $A^*$ of $A$, for a fixed $\Phi$, that we shall study in detail. We are primarily interested in describing normal forms and versal deformations of (1.1) with respect to the automorphism $A^*$.

As is well known (see [1, 2, 5]), a normal form of the operator (1.1) is the simplest (in some sense) representative of its orbit under the group action of $A$ on the space $L_\lambda$. A versal deformation of (1.1) is a normal form for a stable parametric infinitesimal family including (1.1). As will be shown below, all such deformations can be described by means of Lie-algebraic analysis of this group action on $L_\lambda$ and an associated momentum mapping reduced on certain invariant subspaces.

## 2 Lie-algebraic structure of the $A$-action

Let us consider the loop group $G := G_{S^1}(SL(2; \mathbb{C}))$ of all smooth mappings $S^1 \to SL(2; \mathbb{C})$ and its corresponding group $A$-action on a functional manifold $M \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{C}^3)$, which is assumed to be equivariant; that is, the diagram

$$
\begin{align*}
M & \xrightarrow{\Phi} G^* \\
A \Phi & \downarrow \quad \downarrow \text{Ad}_{\Phi^{-1}}^* \\
M & \xrightarrow{\Phi} G^*
\end{align*}
$$

(2.1)
commutes for all \( l \) in the adjoint \( \mathcal{G}^* \) of the loop Lie algebra and \( \Phi \in G \). Whence we can define on \( M \) a natural Poisson structure that induces the following canonical Lie-Poisson structure on \( \mathcal{G}^* \): for any \( \gamma, \mu \in D(\mathcal{G}^*) \),

\[
\{ \gamma, \mu \} := (l, [\nabla \gamma(l), \nabla \mu(l)]).
\]  

(2.2)

Here \((\cdot,\cdot)\) is the usual Killing type nondegenerate, symmetric, invariant scalar product on the loop Lie algebra \( \mathcal{G} = C_{S^1}(sl(2;\mathbb{C})) \), i.e. for any \( a,b \in \mathcal{G} \),

\[
(a, b) := \int_0^{2\pi} dx \, Sp(ab)
\]

(2.3)

and \( \nabla : D(\mathcal{G}^*) \to \mathcal{G} \) is defined as \( (\nabla \gamma(l), \delta l) := \frac{d}{d\epsilon}\gamma(l + \epsilon \delta l) \big|_{\epsilon=0} \) for any \( \delta l \in \mathcal{G}^*, \gamma \in D(\mathcal{G}^*) \).

In order to address the problems posed in Section 1, we need to centrally extend the group action \( A_{\Phi} : M \to M, \Phi \in G \), as follows: for \( \hat{\Phi} := (\Phi, c) \in \hat{G} := G \times \mathbb{C} \) the corresponding action \( A_{\hat{\Phi}} : M \to M \) is defined so that the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{i} & \hat{G}^* \\
A_{\hat{\Phi}} \downarrow & & \downarrow Ad^*_{A_{\hat{\Phi}}^{-1}} \\
M & \xrightarrow{i} & \hat{G}^*
\end{array}
\]

(2.4)

commutes for all \( \hat{\Phi} \in \hat{G} \) and \( i = (l, c) \in \hat{G}^* \). This leads to the following (unique!) choice of the extended \( Ad^* \)-action in (2.4):

\[
Ad^*_{A_{\hat{\Phi}}^{-1}} : (l, c) \in \mathcal{G}^* \to \left( \phi'(\xi)Ad_{A_{\hat{\Phi}}^{-1}}l(x) - c\Phi^{-1}\frac{d\Phi}{d\xi}, c \right)
\]

(2.5)

for all \( \hat{\Phi} \in \hat{G}, l \in \mathcal{G}^* \) at \( \xi \in \mathbb{R} \), \( x = \phi(\xi) \) and \( c \in \mathbb{C} \). This expression follows from the fact that the loop Lie algebra \( \mathcal{G} \) admits only the central extension \( \mathcal{G} \oplus \mathbb{C} \). As the homology groups \( H^1(\mathcal{G}) = 0 \) and \( H^2(\mathcal{G}) = 1 \), it is represented as

\[
[(a, \alpha), (b, \beta)] := ((a, b), (a, db/dx))
\]

(2.6)

for any \( a, b \in \mathcal{G} \) and \( \alpha, \beta \in \mathbb{C} \). Taking \( c \) to be unity and defining an appropriate diffeomorphism \( x \to \phi(x) = \xi \) of \( \mathbb{R} \), it is easy to see that \( Ad^*_{A_{\hat{\Phi}}^{-1}} \) has the same structure element as that of the action \( A^*(\phi, \Phi) \) on \( L_{\lambda} \) defined above. Whence it is clear that our Lie-algebraic analysis is intimately connected with the structure of the \( G \)-orbits induced by the diffeomorphism group \( A = Diff(S^1) \).

We define a natural Lie-Poisson bracket on the adjoint space \( \hat{G}^* \) as follows: for any \( \gamma, \mu \in D(\hat{G}) \subset \hat{G}^* \),

\[
\{ \gamma, \mu \}_0 := (l, [\nabla \gamma(l), \nabla \mu(l)]) + \left( \nabla \gamma(l), \frac{d\nabla \mu(l)}{dx} \right),
\]

(2.7)

and deform it into a brackets pencil using a constant parameter \( \lambda \in \mathbb{C} \) via

\[
\{ \gamma, \mu \}_0 \xrightarrow{\lambda} \{ \gamma, \mu \}_\lambda := (\nabla \gamma(l), \frac{d}{dx} \nabla \mu(l)) + (l + \lambda J, [\nabla \gamma(l), \nabla \mu(l)]),
\]

(2.8)

where \( J \in sl^*(2;\mathbb{C}) \) is chosen here to be the constant matrix

\[
J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(2.9)
The following compatibility condition is almost obvious [8, 10].

**Lemma 2.1.** A pencil of brackets (2.8) is a Poisson brackets pencil for each \( \lambda \in \mathbb{C} \) and \( J \in \mathfrak{sl}^*(2; \mathbb{C}) \), i.e. it is compatible.

**Proof.** It is well known that the Lie derivative of a Poisson bracket is also a Poisson bracket if and only if

\[
\{ \gamma, \mu \}_{1} := \mathfrak{L}_{K} \{ \gamma, \mu \}_{0} - \{ \mathfrak{L}_{K} \gamma, \mu \}_{0} - \{ \gamma, \mathfrak{L}_{K} \mu \}_{0}
\]  

(2.10)

satisfies the Jacobi identity for all \( \gamma, \mu \in \mathfrak{d}_{\mathbb{C}}(\mathfrak{g}^*) \), where \( \mathfrak{L}_{K} \) is the Lie derivative with respect to a vector field \( K : \mathbb{G}^* \to T(\mathbb{G}^*) \). Choosing \( K(l) := J \), it is easy to verify that the bracket (2.10) satisfies the Jacobi identity and is the usual Poisson bracket on \( \mathbb{G}^* \). Consequently, the Poisson bracket (2.10) is also a Poisson bracket along a generic orbit of the vector field \( dl/d\lambda = J \), hence the deformation (2.8) is also Poisson, as was to be proved.

### 3 \hspace{1em} Casimir functionals and reduction problem

A Casimir functional \( h \in I_{\lambda}(\mathbb{G}^*) \) is defined, as usual, as a functional \( h \in \mathfrak{d}_{\mathbb{C}}(\mathbb{G}^*) \) that is invariant with respect to the following \( \lambda \)-deformed \( Ad_{\Phi^{-1}}^{\ast} \)-action:

\[
Ad_{\Phi^{-1}}^{\ast} : (l, 1) \in \mathbb{G}^* \to \left( Ad_{\Phi^{-1}}^{\ast}(l + \lambda J) - \Phi^{-1} \frac{d\Phi}{dx}, 1 \right)
\]  

(3.1)

for any \( \Phi \in \mathbb{G}, l \in \mathbb{G}^* \) and \( \lambda \in \mathbb{C} \). It is easy to see from this definition that \( h \in I_{\lambda}(\mathbb{G}^*) \) if the equation

\[
\frac{d\nabla h(l)}{dx} = [l + \lambda J, \nabla h(l)]
\]  

(3.2)

is satisfied for all \( \lambda \in \mathbb{C} \). Assuming further that there exists an asymptotic expansion of the form

\[
h(\lambda) \sim \sum_{j \in \mathbb{Z}_{+}} h_{j} \lambda^{-j}
\]  

(3.3)

as \( |\lambda| \to \infty \), one can readily verify that \( h_{0} \in I_{1}(\mathbb{G}^*) \) and that for all \( j, k \in \mathbb{Z}_{+} \)

\[
\{ h_{j}, h_{k} \}_{0} = 0 = \{ h_{j}, h_{k} \}_{1}, \quad \{ \gamma, h_{j} \}_{0} = \{ \gamma, h_{j+1} \}_{1},
\]  

(3.4)

where \( \gamma \in D(\mathbb{G}^*) \) is arbitrary.

Let us now consider the action (2.1) at a fixed \( l = l[u, v; z] \in \mathbb{G}^* \). It is easy to see that this action does not necessarily preserve the form of the element \( l \). Thus we must reduce the initial \( \mathbb{G} \)-action on \( \mathbb{G}^* \) to an appropriate subgroup; for this we develop the reduction procedure employed in [8–10].

Define the distribution

\[
D_{1} := \left\{ K \in T(\mathbb{G}^*) : K(l) = [J, \nabla \gamma(l)], l \in \mathbb{G}^*, \gamma \in D(\mathbb{G}^*) \right\}.
\]  

(3.5)
$D_1$ is integrable, that is $[D_1, D_1] \subset D_1$, since the bracket $\{\cdot, \cdot\}_1$ is Poisson. Now define another distribution

$$D_0 := \left\{ K \in T(\hat{G}^*) : K(l) = [l - \frac{d}{dx} \nabla h_0], h_0 \in I_1(\hat{G}^*) \right\}, \quad (3.6)$$

which is clearly also integrable on $\hat{G}^*$, since $[D_0, D_0] \subset D_0$. The set of maximal integral submanifolds of (3.6) generates the foliation $\hat{G}^*_J \setminus D_0$ whose leaves are the intersections of fixed integral submanifolds $\hat{G}^*_J \subset \hat{G}^*$ passing through an element $l[u, v; z] \in \hat{G}^*$. If the foliation $\hat{G}^*_J \setminus D_0$ is sufficiently smooth, one can define the quotient manifold $\hat{G}^*_\text{red} := \hat{G}^*_J / (\hat{G}^*_J \setminus D_0)$ with its associated projection mapping $\hat{G}^*_J \to \hat{G}^*_\text{red}$. To continue this line of reasoning, we shall obtain explicit constructions of the objects introduced.

$D_1$ is obviously generated by the vector fields

$$\frac{dl}{dt} = \left( \begin{array}{cc} 0 & 2b \\ -2c & 0 \end{array} \right), \quad \nabla \gamma(l) = \left( \begin{array}{cc} a & b \\ c & -a \end{array} \right), \quad (3.7)$$

where $t$ is a complex evolution parameter and $l \in \hat{G}^*_J$, where $\hat{G}^*_J \subset \hat{G}^*$ is the isotropy Lie subalgebra of the element $J \in \hat{G}^*$. Hence the integral submanifold $\hat{G}^*_J \subset \hat{G}^*$ consists of orbits of an element $l = l[u, v; z] \in \hat{G}^*$, with $z \in C$, with respect to the vector fields (3.7). The distribution $D_0$ on $T(\hat{G}^*)$ is generated by the vector fields

$$\frac{dl}{d\tau} = \left( \begin{array}{cc} -\chi_x & -2u\chi \\ 2v\chi & \chi_x \end{array} \right), \quad \nabla h_0(l) = \left( \begin{array}{cc} \chi & 0 \\ 0 & -\chi \end{array} \right), \quad (3.8)$$

where $\tau$ is a complex evolution parameter and $l = l[u, v; z] \in \hat{G}^*$. It follows immediately from (3.8) that

$$\frac{dz}{d\tau} = -\chi_x, \quad \frac{du}{d\tau} = -2u\chi \quad \text{and} \quad \frac{dv}{d\tau} = 2v\chi \quad (3.9)$$

for all $\tau \in \mathbb{R}$ along $D_0$. Eliminating the variable $\chi$ from (3.9), we obtain

$$\frac{d}{d\tau} \left[ \frac{d}{dx} (\ln u) - 2z \right] = 0 = \frac{d}{d\tau} \left[ \frac{d}{dx} (\ln v) + 2z \right]; \quad (3.10)$$

that is, the mapping

$$\hat{G}^* \ni l = \left( \begin{array}{cc} z & u \\ v & -z \end{array} \right) \mapsto \left( \begin{array}{cc} 0 & \exp(\partial^{-1}\alpha) \\ \exp(\partial^{-1}\beta) & 0 \end{array} \right) \rightarrow \hat{G}^*_\text{red}, \quad (3.11)$$

where

$$\alpha := u_x u^{-1} - 2z, \quad \beta := v_x v^{-1} + 2z, \quad (3.12)$$

explicitly determines the reduction $\nu : \hat{G}^* \to \hat{G}^*_\text{red}$ discussed above. We are now in a position to compute the bracket (2.8) reduced upon the submanifold $\hat{G}^*_\text{red}$ by defining the functionals $\lambda, \mu \in D(\hat{G}^*)$ to be constant along the distribution $D_0$, that is

$$\gamma := \tilde{\gamma} \circ \nu, \quad \mu := \tilde{\mu} \circ \nu, \quad (3.13)$$
for any $\tilde{\gamma}, \tilde{\mu} \in D(\hat{G}_\text{red}^*)$. From (3.12) one readily obtains the expressions

$$\nabla \gamma(l)|_{l \in \hat{G}_\text{red}^*} = \begin{pmatrix} \frac{\delta \tilde{\gamma}}{\delta \beta} - \frac{\delta \tilde{\gamma}}{\delta \alpha} & -1 \frac{\delta \tilde{\gamma}}{\delta \beta} \frac{\partial}{\partial x} \\ -\frac{1}{u} \frac{\delta \tilde{\gamma}}{\delta \alpha} \frac{\partial}{\partial x} & \frac{\delta \tilde{\gamma}}{\delta \alpha} - \frac{\delta \tilde{\gamma}}{\delta \beta} \end{pmatrix},$$

(3.14)

$$\nabla \mu(l)|_{l \in \hat{G}_\text{red}^*} = \begin{pmatrix} \frac{\delta \tilde{\mu}}{\delta \beta} - \frac{\delta \tilde{\mu}}{\delta \alpha} & -1 \frac{\delta \tilde{\mu}}{\delta \beta} \frac{\partial}{\partial x} \\ -\frac{1}{u} \frac{\delta \tilde{\mu}}{\delta \alpha} \frac{\partial}{\partial x} & \frac{\delta \tilde{\mu}}{\delta \alpha} - \frac{\delta \tilde{\mu}}{\delta \beta} \end{pmatrix},$$

which satisfy the desired identities

$$(\nabla \gamma(l), dl/d\tau) = 0 = (\nabla \mu(l), dl/d\tau)$$

(3.15)

for all $l \in \hat{G}_\text{red}^* \subset \hat{G}^*$. Substituting now (3.14) into (2.8), we obtain

$$\{\tilde{\gamma}, \tilde{\mu}\}_\lambda := \{\gamma, \mu\}_\lambda |_{l \in \hat{G}_\text{red}^*} = (\nabla \tilde{\gamma}, (\eta + \lambda \theta) \nabla \tilde{\mu}),$$

(3.16)

where we have used the obvious relationship

$$\{\tilde{\gamma}, \tilde{\mu}\}_\lambda \circ \nu = \{\tilde{\gamma} \circ \nu, \tilde{\mu} \circ \nu\}_\lambda,$$

(3.17)

and where

$$\eta := \begin{pmatrix} 2\partial \\ -\partial \exp[-\partial^{-1}(\alpha + \beta)] \partial^2 - 2\partial - \partial \cdot \alpha \exp[-\partial^{-1}(\alpha + \beta)] \cdot \partial \\ -\partial \exp[-\partial^{-1}(\alpha + \beta)] \partial^2 - 2\partial - \partial \cdot \beta \exp[-\partial^{-1}(\alpha + \beta)] \cdot \partial \\ 2\partial \end{pmatrix},$$

(3.18)

$$\theta := \begin{pmatrix} 0 & 2\partial \exp[-\partial^{-1}(\alpha + \beta)] \partial \\ -2\partial \exp[-\partial^{-1}(\alpha \beta)] \partial & 0 \end{pmatrix}.$$

It is straightforward to verify that these integro-differential, implictic (=co-symplectic= Poisson) operators are compatible [4] (see also [11] for a general theory of iso-symplectic structures on functional manifolds) on the reduced submanifold $\hat{G}_\text{red}^*$ and define a bi-Hamiltonian structure on it.

4 Diff($S^1$) action, associated momentum mapping and versal deformations

Let us introduce some additional notation concerning versal deformations [1, 7]. By a deformation of the operator (1.1) we shall mean an operator of the same form with a matrix $l_\lambda(\epsilon)$ whose entries are analytic in $\epsilon$ in a neighborhood of $\epsilon = 0$ in $\mathbb{C}^n$ and satisfies $l_\lambda(0) = l_\lambda$ for all $\lambda \in \mathbb{C}$. The coordinates $\epsilon_i \in \mathbb{C}, 1 \leq i \leq n$, of $\epsilon$ are called the deformation parameters and the space of these parameters is called the base of the deformation.
Two deformations \( l'_\lambda(\epsilon) \) and \( l''_\lambda(\epsilon) \) of a matrix \( l_\lambda \) will be called equivalent if there exists a deformation \( A'(\phi_\epsilon): l'_\lambda(\epsilon) \to l''_\lambda(\epsilon) \) generated by a diffeomorphism \( \phi_\epsilon \in \text{Diff}(S^1) \) satisfying \( \phi_\epsilon |_{\epsilon=0} = \text{id} \).

From a given deformation \( l_\lambda(\epsilon) \) one can obtain a new deformation \( \tilde{l}_\lambda(\tilde{\epsilon}) \) by setting \( \tilde{l}_\lambda(\tilde{\epsilon}) := l_\lambda(\epsilon(\tilde{\epsilon})) \), where \( \epsilon: \mathbb{C}^m \to \mathbb{C}^n \) is an analytic mapping in a neighborhood of \( \tilde{\epsilon} = 0 \) in \( \mathbb{C}^m \) and satisfies the condition \( \epsilon(0) = 0 \). The deformation \( \tilde{l}_\lambda(\tilde{\epsilon}) \) is said to be induced from \( l_\lambda(\epsilon) \) by the mapping \( \epsilon: \mathbb{C}^m \to \mathbb{C}^n \).

A deformation \( \tilde{l}_\lambda(\epsilon), \epsilon \in \mathbb{C}^n \), is called versal if every one of its deformations \( l_\lambda(\epsilon) \), \( \epsilon \in \mathbb{C}^m \), is equivalent to a deformation induced from it. A versal deformation is said to be universal if the induced deformation described in the definition of versality is unique.

Before we give a definition of a transversal deformation for the induced group \( \hat{G}_{\text{red}} \) orbits, let us consider a family of smooth induced transformations \( \phi_\sigma(x) \in \hat{G}_{\text{red}}, \sigma \in \mathbb{R}, \) where \( \phi_\sigma(x) = 1 + O(\sigma) \) as \( \sigma \to 0 \). Each such transformation generates (via formula (1.5)) a new matrix \( l_\lambda(\sigma), \sigma \to 0 \), that obviously belongs to the orbit space associated to the \( \hat{G}_{\text{red}} \) action. The set of matrices

\[
\frac{dl_\lambda(\sigma)}{d\sigma} \bigg|_{\sigma=0} \in \hat{G}_{\text{red}}^* \tag{4.1}
\]

spans a linear subspace \( \tilde{V}_\lambda \subset \hat{G}_{\text{red}}^* \) of finite codimension. Consider an arbitrary deformation \( l_\lambda(\epsilon), \epsilon \in \mathbb{C}^n \), of a given matrix \( l_\lambda \in \hat{G}_{\text{red}}^* \) and denote by \( \hat{E}_\lambda \) the linear span in \( \hat{G}_{\text{red}}^* \) over the matrices \( \partial l_\lambda(\epsilon)/\partial \epsilon_i |_{\epsilon=0}, 1 \leq i \leq n \). The above deformation is said to be transverse to the induced \( \hat{G}_{\text{red}} \) orbit if the subspaces \( \hat{E}_\lambda \) and \( \tilde{V}_\lambda \) together span their ambient space, that is

\[
\hat{E}_\lambda + \tilde{V}_\lambda = \hat{G}_{\text{red}}^*. \tag{4.2}
\]

The following general theorem [1] holds for versal deformations of the Dirac operator (1.1).

**Theorem 4.1.** A deformation \( l_\lambda(\epsilon), \epsilon \in \mathbb{C}^n \), is versal if and only if it is transverse to the induced group \( \hat{G} \) orbit.

This theorem can be proved by applying standard perturbation theory techniques to the Dirac type operator (1.1).

We are now ready to make use of the results of Section 3 to describe the spaces \( \hat{E}_\lambda \) and \( \tilde{V}_\lambda \) analytically. Let \( \tilde{\gamma} \in D(\hat{G}_{\text{red}}^*) \) be any smooth functional on \( \hat{G}_{\text{red}}^* \); it generates a flow on the loop group \( \hat{G}_{\text{red}}^* \) orbit via the \((\sigma, x)\)-evolutions

\[
\frac{dl}{d\sigma} := \{\tilde{\gamma}, l\}_\lambda, \quad \frac{dl}{dx} := \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} l \tag{4.3}
\]

with respect to the Poisson bracket (3.16). In view of (4.3), (3.16) implies that the subspace \( \tilde{V}_\lambda \) is isomorphic to the following subspace of vector functions in \( T^*(M) \):

\[
V_\lambda := \{ \Lambda_\lambda \psi := (\eta + \lambda \theta) \psi : \nabla \tilde{\gamma} = \psi \in T^*(M) \} \tag{4.4}
\]

Theorem 4.1 suggests the following construction of versal deformations for the Dirac type operator (1.1): As \( \Lambda_\lambda \) is skew-symmetric, the operator \( i\Lambda_\lambda \) is formally selfadjoint in the space \( L_2(\mathbb{R}/2\pi \mathbb{Z}; \mathbb{C}^2) \). Therefore, the orthogonal complement to the subspace \( V_\lambda \) with respect to the natural scalar product in \( L_2(\mathbb{R}/2\pi \mathbb{Z}; \mathbb{C}^2) \) consists of \( 2\pi \)-periodic solutions to the equation

\[
\Lambda_\lambda \psi = 0. \tag{4.5}
\]
Whence we have the following characterization of versal deformations of the operator (1.1).

**Theorem 4.2.** *The prolongation of the matrix $l_\lambda \in \hat{G}^*_\text{red}^\ast$ defined as*

$$
\bar{l}_\lambda(\epsilon) := \begin{pmatrix}
\lambda & \exp(\partial^{-1}\beta) \\
\exp(\partial^{-1}\alpha) & -\lambda
\end{pmatrix}
+ \sum_{i,j=1}^{2} \epsilon_{ij} \bar{f}_i \otimes \bar{f}_j
$$

*generates a versal deformation of the Dirac type operator (1.1). Here $\otimes$ is the usual Kronecker tensor product in $\mathbb{C}^2$, $\epsilon_{ij} \in \mathbb{C}, 1 \leq i,j \leq 2$, $\epsilon_{12} = -\epsilon_{21}$ are any deformation constants, and $\bar{f}_i \in W_2^{(1)}(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{C}^2), i = 1,2$, are two linearly independent, normalized solutions to the Dirac equations*

$$
\frac{d\bar{f}_i}{dx} + \bar{l}_\lambda \bar{f}_i = 0, \quad \|\bar{f}_i; \bar{f}_j\|_{x=0} = 1,
$$

*with spectral parameter $\lambda \in \mathbb{C}$.*

**Proof.** It is easy to verify that the set of solutions to equation (4.5) is isomorphic to the set of functions

$$
\hat{\psi} = \sum_{i,j=1}^{2} \epsilon_{ij} \bar{f}_i \otimes \bar{f}_j,
$$

and these functions satisfy the canonical Casimir equation

$$
\left[l_\lambda, \hat{\psi}\right] - \frac{d\hat{\psi}}{dx} = 0,
$$

which is equivalent to equation (4.5). Owing to the fact that any matrix $l_\lambda \in \hat{G}^*_\text{red}^\ast$ in (1.1) can be transformed into the expression $\bar{l}_\lambda(0) \in \hat{G}^*_\text{red}^\ast$ with functional parameters $\alpha, \beta$ given by (3.12), this leads to the general form (4.6) for versal deformations of (1.1). This ends the proof.

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