Simulating the Lipkin-Meshkov-Glick model in a hybrid quantum system

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We propose an efficient scheme for simulating the Lipkin-Meshkov-Glick (LMG) model with nitrogen-vacancy (NV) center ensembles in diamond magnetically coupled to superconducting coplanar waveguide cavities. With the assistance of external microwave driving fields, we show that the interaction of the NV spins can be easily controlled, and several types of the LMG model can be realized by tuning the different parameters. Under the thermal dynamical limit, the distinct nonequilibrium second order quantum phase transition of the spin ensemble can be achieved at the critical point. Furthermore, we show that the spin squeezed state can be generated by tailoring the LMG Hamiltonian to possess the two-axis counter-twisting form in this hybrid quantum system.

I. INTRODUCTION

Manipulating the couplings of collective particles has been a fascinating subject with the development of new technologies for ultracold atoms, trapped ions, and solid-state spins [1–11]. One of the most important applications is to simulate the Lipkin-Meshkov-Glick (LMG) model in these different systems [12–14]. This model, which was first proposed in nuclear physics [15], has become a hot issue in the field of quantum information and quantum simulation [16–18]. Because of utilizing this model, we can not only manipulate the special quantum states such as the coherent spin state or spin squeezed state [19–22], but also “tailor” the microscopic interaction between the particles to mimic quantum phase transitions of the macroscopic system [16–18, 23]. In spite of many outstanding investigations for simulating the LMG model, it remains a challenge to realize the general LMG model in laboratory [20–23]. Therefore, it is appealing to present an experimentally feasible scheme for realizing the LMG model.

Recently, much attention has been paid to manipulating nitrogen-vacancy (NV) center ensembles in hybrid quantum systems [24–39]. NV centers in diamond have exhibited the excellent features such as fast microwave manipulation, optical preparation, and detection, and long coherence time even at room temperature [40–48]. Besides, we can directly combine NV centers with other quantum systems without requiring sophisticated trapping techniques [40, 41]. Therefore, hybrid quantum systems composed of superconducting circuits and NV centers have been extensively investigated [41–56]. Utilizing these hybrid systems, one can prepare fantastic quantum states, design quantum logic gates, store or transfer quantum states [24–29, 31–35, 44–49, 51, 56]. Furthermore, we can perform some quantum simulating tasks with this spin-photon system [33, 57, 58]. In a recent paper, a protocol for simulating the Dicke model and Dicke Lattice Model is proposed with the isotropy and anisotropy

NV center ensembles in the periodic superconducting microwave cavities respectively [59]. This prominent work demonstrates that hybrid quantum systems provide a realistic platform for studying characteristic phenomena of nonequilibrium quantum systems in various configurations.

In this work, we propose an experimentally feasible scheme for simulating the LMG model in a hybrid quantum system with an NV center ensemble in diamond coupled to superconducting coplanar waveguide cavities. Under the condition of large detunings as well as the bad cavity limit, we can obtain the generalized LMG model in this hybrid system. We discuss several forms of the LMG model by adjusting the parameters such as detunings, Rabi frequencies, coupling coefficients and so on. In particular, we focus on the positive field case $h > 0$ of the $\chi = 0$ LMG model with ferromagnetic interactions $\lambda > 0$. Under the thermal dynamical limit, the distinct non-equilibrium second order quantum phase transition of the coupled spins can be achieved at the critical point, as the magnitude of the interaction strength varies. On the other hand, the LMG model with the form of $H \sim J_{x}^{2}, J_{y}^{2}, J_{z}^{2}$ or $H \sim (J_{x}^{2} - J_{y}^{2})$, corresponding to the one-axis twisting or two-axis counter-twisting Hamiltonian, can also be utilized for generating the spin squeezed state [19–23, 31, 51, 58, 61]. Therefore, by tailoring the LMG model with the two-axis counter-twisting form, we can prepare the spin squeezed state with high degree of squeezing based on this kind of interaction. Our work provides a realistic platform for implementing LMG-type models and for studying characteristic phenomena of nonequilibrium quantum systems with hybrid quantum systems.

II. THE MODEL

As illustrated in Fig. 1(a), in this hybrid quantum system two superconducting coplanar waveguide cavities are strongly coupled together with the coefficient $\epsilon \sim 10 \text{ MHz}$. Meanwhile one of them is magnetically coupled to an NV center ensemble [40, 41, 55–59]. The energy level structure of the single NV center is shown

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the Bohr magneton), which results in a three-level system. The random frequency splitting in the center of one cavity and this ensemble is magnetically coupled to this cavity. (b) Level diagram of the NV center ground triplet state denoted as \(|\psi_0\rangle\). The electronic ground state is the spin degenerate states \(|\psi_0\rangle\) and \(|\psi_1\rangle\), and we use the same symbols to implement Raman transitions between the two excited spin states \(|+\rangle\) and \(|-\rangle\).

In Fig. 1(b), the electronic ground state is the spin triplet state denoted as \(|m_s = 0, \pm 1\rangle\), and the zero-field splitting between the degenerate sublevels \(|m_s = \pm 1\rangle\) and \(|m_s = 0\rangle\) is \(D = 2\pi \times 2.88\) GHz. Then we apply a homogeneous static magnetic field \(B_1\) to remove the degenerate states \(|m_s = \pm 1\rangle\) with the Zeeman splitting \(\delta B/2\pi = 2g_e\mu_B B_1 \approx 100\) MHz \((g_e \approx 2\) is the nitrogen-vacancy Landé factor, \(\mu_B = 14\) MHz m\(^{-1}\) is the Bohr magneton), which results in a three-level system denoted as \(|m_s = 0\rangle \equiv |0\rangle, |m_s = +1\rangle \equiv |+\rangle,\) and \(|m_s = -1\rangle \equiv |-\rangle\).

In this solid-state system, the presence of local strain and the coupling of the NV spins to other electronic or nuclear spins in the surrounding can substantially generate the random frequency splitting \(\delta_j\) between the states \(|+\rangle\) and \(|-\rangle\) for each NV center [59]. As a result, the coherence time is limited by this inhomogeneous broadening, resulting in an ensemble dephasing time \(T_2^*\), which is about one microsecond according to the current experiments [62–64]. However, one can extend this coherence time from \(T_2^*\) to the value \(T_2\) (close to the intrinsic spin coherence time) through the spin echo technology [62–64]. It has been reported that, the NV spin coherence time in an ensemble is comparable to that of single NV center, with \(T_2 > 600\)\(\mu\)s for a sample with natural abundance of \(^{13}\)C and paramagnetic impurity density \(\varrho \sim 10^{15}\) cm\(^{-3}\) [65, 66]. In this work, we have neglected these terms in the Hamiltonian and included their major influence via the dephasing rate \(\gamma_{\text{dep}}\) to the master equation equivalently.

Therefore, the Hamiltonian for describing these two cavities and the homogeneous NV center ensemble is (\(h = 1\))

\[
H_1 = H_{\text{NV}} + H_{\text{10}} + H_{\text{NC}} + H_{\text{CC}},
\]

where \(H_{\text{NV}} = \sum_{j=1}^{N} \omega_+ |+\rangle_j \langle +| + \omega_- |\rangle_j \langle -|\) is the free Hamiltonian for the homogeneous NV ensemble, \(H_{\text{10}} = \omega_1 c_1^\dagger c_1 + \omega_2 c_2^\dagger c_2\) is the free Hamiltonian for these two cavities with the destruction operators \(c_i\), (\(i = 1, 2\)), \(H_{\text{NC}} = \sum_{j=1}^{N} [\eta_1 c_1^\dagger |+\rangle_j \langle 0| + \eta_2 c_1 |\rangle_j \langle 0| + \text{h.c.}]\) is the interaction between cavity 1 and the NV centers, and \(H_{\text{CC}} = \epsilon (c_1^2 + c_1^2)\) is the interaction between these two cavities. For the related parameters, \(N\) is the number of the available NV centers, \(\omega_\pm = D \pm B/2\) the transition frequencies between the states \(|\pm\rangle\) and \(|0\rangle\), \(\omega_1\) and \(\omega_2\) the cavity mode frequencies, and \(\eta_1\) and \(\eta_2\) are the coupling strengths between the cavity mode \(c_1\) and the \(j\)th NV center for the transitions \(|0\rangle_j \rightarrow |+\rangle_j\) and \(|0\rangle_j \rightarrow |\rangle_j\).

In order to acquire four distinct Raman transitions between the two spin states \(|m_s = +1\rangle\) and \(|m_s = -1\rangle\), we utilize the canonical transformation \(c_1 = (\hat{a} + \hat{b})/\sqrt{2}, c_2 = (\hat{a} - \hat{b})/\sqrt{2}, c_1^\dagger = (\hat{a} + \hat{b})/\sqrt{2},\) and \(c_2^\dagger = (\hat{a} - \hat{b})/\sqrt{2}\). As a result, we can get the equivalent form of the above Hamiltonian

\[
H_1' = \nu_a \hat{a}^\dagger \hat{a} + \nu_b \hat{b}^\dagger \hat{b} + H_{\text{NV}}
\]

\[
+ \sum_{j=1}^{N} (g_1 \hat{a} |+\rangle_j \langle 0| + g_2 \hat{a} |\rangle_j \langle 0| + \text{h.c.})
\]

\[
+ g_3 (\hat{b}^\dagger |+\rangle_j \langle 0| + \hat{b}^\dagger |\rangle_j \langle 0| + \text{h.c.}),
\]

where \(\hat{a}(\hat{a}^\dagger)\) and \(\hat{b}(\hat{b}^\dagger)\) are destruction (creation) operators for these two supermodes. \(g_k (k = 1, 2, 3, 4)\) are the relative average coupling coefficients between the supermodes and NV centers \((g_1 = g_3 = \eta_1/\sqrt{2}, g_2 = g_4 = \eta_2/\sqrt{2})\), and \(\nu_a = \epsilon (\omega_+ + \omega_-)/2,\) and \(\nu_b = -\epsilon (\omega_1 + \omega_2)/2\) are the relative frequencies of these two supermodes respectively. Meanwhile, we introduce four microwave classical fields with frequencies \(\omega_k (k = 1, 2, 3, 4)\) to the NV center ensemble, and neglect the NV centers’ anisotropy and the deference from locations.

As a result, the total Hamiltonian for this hybrid quantum system can be expressed as

\[
H_{\text{total}} = H_0 + H_{\text{int}},
\]

where

\[
H_0 = \nu_a \hat{a}^\dagger \hat{a} + \nu_b \hat{b}^\dagger \hat{b} + \sum_{j=1}^{N} (\omega_+ |+\rangle_j \langle +| + \omega_- |\rangle_j \langle -|),
\]
where \( \sigma = \Omega / \delta_B \), the coefficients for the non-linear terms of these two supermodes have no contributions in the adiabatic elimination course. According to the definition of \( \hat{T}_a \) and \( \hat{T}_b \), we transform the equation (10) into the generalized LMG model Hamiltonian [15-23, 61],

\[
H_{LMG} = -2h\hat{J}_z - 2\lambda \left( \hat{J}_x + \hat{\chi} \hat{J}_y \right),
\]

where the parameters are given by

\[
-2h = \mu_0 - \frac{\zeta_a \left( (L_a^0)^2 - (L_b^0)^2 \right)}{NK_a} - \frac{\zeta_b \left( (L_b^0)^2 - (L_a^0)^2 \right)}{NK_b},
\]

\[
2\lambda = \frac{\zeta_a \left( L_a^0 + \bar{L}_a^0 \right)^2}{K_a} + \frac{\zeta_b \left( L_b^0 + \bar{L}_b^0 \right)^2}{K_b},
\]

\[
\chi = \frac{K_b \zeta_a \left( L_a^0 + \bar{L}_a^0 \right)^2 + K_a \zeta_b \left( L_b^0 + \bar{L}_b^0 \right)^2}{K_a K_b},
\]

\[
\lambda_a = \kappa_a (L_a^0)^2 + \sigma_a \beta_a L_a^0, \quad \lambda_b = \kappa_b (L_b^0)^2 + \sigma_b \beta_b L_b^0.
\]

These parameters can be controlled by adjusting the relevant parameters such as the detunings, Rabi frequencies, and coupling coefficients.
In addition, we consider several forms of the LMG model by tuning these parameters. In order to describe them more clearly, we list three groups of parameters in TABLE. I. Then we can get several different forms of the LMG Hamiltonian via choosing the suitable parameters. When we set the parameters as those in the first column of TABLE. I, we can get the conventional LMG Hamiltonian with the expression

$$H = -2h \hat{J}_z - \frac{2\lambda}{N}(\hat{J}_x^2 - \hat{J}_y^2),$$

(12)

where $h = -\mu_0/2$, $\chi = 1$, $\alpha_a = \alpha_b = \alpha = \sqrt{2}/2$, $\beta_a = \beta_b = \beta = \sqrt{2}/2$, $\lambda = \Lambda_a = \Lambda_b$, and $\sigma_a \simeq \sigma_b$. The master equation reduces to the form

$$\dot{\rho} = -i[H, \rho] + \frac{2\Gamma_a \alpha^2}{N}D[\hat{J}_-]\rho + \frac{2\Gamma_b \beta^2}{N}D[\hat{J}_+]\rho + \gamma_{dep} D[\hat{J}_z]\rho.$$  

(13)

This kind of LMG model has been investigated for phase transitions and multiparticle entanglement [12-17]. One can generate the spin squeezed state by the two-axis counter-twisting interactions utilizing this kind of Hamiltonian [19-21].

Secondly, we can get the isotropic Hamiltonian when choosing the second column parameters in TABLE. I.

$$H = -2h \hat{J}_z - \frac{2\lambda}{N}(\hat{J}_x^2 + \hat{J}_y^2),$$

(14)

where $h = -\mu_0/2$, $\chi = 1$, $\Lambda_a = \Lambda_b \equiv \lambda$, $\alpha_a = \beta_b = 1$, $\alpha_b = \beta_a = 0$ and $\sigma_a \simeq \sigma_b$. Then the master equation is

$$\dot{\rho} = -i[H, \rho] + \frac{\Gamma_a}{N}D[\hat{J}_-]\rho + \frac{\Gamma_b}{N}D[\hat{J}_+]\rho + \gamma_{dep} D[\hat{J}_z]\rho.$$  

(15)

This is an isotropic LMG Hamiltonian which can be solved exactly because of $H = -2h \hat{J}_z - 2\lambda \hat{J}_x^2/N + 2\lambda \hat{J}_y^2/N$, where $\hat{J}_x^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$. We can get the double degenerate ground states for this Hamiltonian when $h = 0$. Otherwise, for the symmetry breaking case $h \neq 0$, we will get an unique ground state ($h > 0$, $|↑↑↑\cdots\rangle = |m_z = N/2\rangle$) or $h < 0$, $|↓↓↓\cdots\rangle = |m_z = -N/2\rangle$).

In addition there will be the transition between the ferromagnetic and antiferromagnetic interactions by tuning the sign of $\lambda$ when we set $h = 0$, and so on.

Finally, we can obtain the simple Hamiltonian according to the third column in TABLE. I,

$$H = -2h \hat{J}_z - \frac{2\lambda}{N} \hat{J}_x^2,$$

(16)

and achieve the master equation

$$\dot{\rho} = -i[H, \rho] + \frac{\Gamma_a}{N} D[\hat{J}_-] \rho + \frac{\Gamma_b}{N} D[\hat{J}_+] \rho + \gamma_{dep} D[\hat{J}_z] \rho,$$

(17)

where $\lambda = 2\Lambda_a$, $\chi = 0$, $\alpha_a = \beta_a = 1$, $\beta_b = \sqrt{2}/2$, and $\alpha_b = 0$. There will be many other applications of this kind of interactions, e.g., simulating the first order and second order phase transitions, preparing multiparticle entanglement and generating spin squeezed state by the one-axis twisting interactions [12-14]. In the following, we will investigate this kind of phase transition in this hybrid quantum system for the Hamiltonian with the form given by equation (16).

### III. THE SECOND-ORDER PHASE TRANSITION

Under the thermodynamic limit corresponding to that $N \rightarrow \infty$, $V \rightarrow \infty$, and the density $n = N/V$ keeps finite, we can neglect the quantum fluctuation for this spin-spin interaction system, for example, $\langle \hat{J}_k \hat{J}_l \rangle \rightarrow \langle \hat{J}_k \rangle \langle \hat{J}_l \rangle$, where $k,l \in \{x,y,z\}$. Considering the ferromagnetic interactions ($\lambda > 0$) from equation (16), we will apply the method of semiclassical equations of motion to simulate the second-order quantum phase transition. The differential equations of motion for the expectation values of collective spin components $\langle \hat{J}_i \rangle = \text{Tr}(\rho \hat{J}_i)$ are readily derived from the master equation (17). Utilizing the relations $d \langle \hat{J}_i \rangle /dt = \text{Tr}(\dot{\rho} \hat{J}_i)$ and the definition of

| Parameters (MHz) | Two-axis counter-twisting LMG model (Equation (12)) | Isotropic LMG model (Equation (14)) | One-axis twisting LMG model (Equation (16)) |
|------------------|---------------------------------------------------|-----------------------------------|-------------------------------------------|
| $\sqrt{N} g_k (N \approx 10^{12})$ | 12 | 12 | 12 |
| $|\Omega_a|/2\pi$ | 4 | 0 | 7 |
| $|\Omega_b|/2\pi$ | 1 | 1 | 3 |
| $|\Omega_c|/2\pi$ | 1 | 1 | 0.77 |
| $|\Delta_{a1}|/2\pi$ | 4 | 0 | 0 |
| $|\Delta_{a2}|/2\pi$ | 20 | 20 | 30 |
| $|\Delta_{b1}|/2\pi$ | 80 | 80 | 70 |
| $|\Delta_{b2}|/2\pi$ | 80 | 80 | 50 |
| $\sigma_a \alpha_a /2\pi$ | 0.3 | 0.3 | 0.6 |
| $\sigma_a \beta_a /2\pi$ | 0.3 | 0 | 0.6 |
| $\sigma_b \alpha_b /2\pi$ | 0.3 | 0 | 0 |
| $\sigma_b \beta_b /2\pi$ | -0.3 | 0.3 | 0.092 |
NV center ensemble in this system can be interpreted as the normal phase and the broken phase, we can apply the solutions of motion to exhibit a bifurcation at the critical point of the coupling strength $\lambda$. We then derive the steady-state conditions for the normal and broken phases.

The set of coupled equations, which are given by

$$
\dot{X} = 2hY - \Gamma_b ZX - \gamma_{dep} X/2
$$

$$
\dot{Y} = -2hX + 2\lambda ZX - \Gamma_b ZY - \gamma_{dep} Y/2
$$

(18)

$$
\dot{Z} = -2\lambda XY + \Gamma_b (X^2 + Y^2).
$$

(19)

$$
2hY - \Gamma_b ZX - \gamma_{dep} X/2 = 0,
$$

(20)

$$
-2hX + 2\lambda ZX - \Gamma_b ZY - \gamma_{dep} Y/2 = 0,
$$

(21)

$$
-2\lambda XY + \Gamma_b (X^2 + Y^2) = 0,
$$

(22)

$$
X^2 + Y^2 + Z^2 = 1.
$$

(23)

These three formulas cannot form a closed set of group, and the constraint $X^2 + Y^2 + Z^2 = 1$ corresponds to the conservation of angular momentum. Therefore, we can get a closed set of group with these four equations, and obtain the steady-state analytical solutions or numerical solutions from these equations. The numerical solutions for the infinite $N$ will lead to $\langle J_x \rangle = \langle J_y \rangle = 0$ for all $\lambda$. Then we derive the steady-state analytical solutions of motion from these equations

$$
2hY - \Gamma_b ZX - \gamma_{dep} X/2 = 0,
$$

(24)

$$
-2hX + 2\lambda ZX - \Gamma_b ZY - \gamma_{dep} Y/2 = 0,
$$

(25)

$$
-2\lambda XY + \Gamma_b (X^2 + Y^2) = 0,
$$

(26)

$$
X^2 + Y^2 + Z^2 = 1.
$$

(27)

We solve the equations (21)-(24) and find the critical point of the coupling strength $\lambda$,

$$
\lambda_c = h + \frac{\Gamma_b^2}{4h}.
$$

(28)

It is obviously that the critical point $\lambda_c$ is varying with $\Gamma_b$ and $h$, but immune to the dephasing $\gamma_{dep}$.

When the coupling strength satisfies $\lambda < \lambda_c$, the steady-state belongs to the normal phase and the analytical solutions of motion are given by

$$
Z_{np} = 1, X_{np} = Y_{np} = 0.
$$

(29)

While for $\lambda > \lambda_c$, which is the region for the second-order phase transition, the steady-state corresponds to the broken phase and the analytical solutions are given by

$$
Z_{bp} = Z_0 - r_0,
$$

(30)

$$
X_{bp} = \pm \sqrt{\frac{1 - (Z_0 - r_0)^2}{1 + \Gamma_b Z_0^2/2h}},
$$

(31)

$$
Y_{bp} = \pm \frac{\Gamma_b}{2h} Z_0 (\sqrt{1 - (Z_0 - r_0)^2} + \Gamma_b Z_0^2/2h),
$$

(32)

where $Z_0 = 2h/\Gamma_b (\lambda - \sqrt{\lambda^2 - \Gamma_b^2 (1 + r_0) / h})$, and the dimensionless parameter $r_0 = \gamma_{dep} / 2\Gamma_b$. The analytical solutions of motion exhibit a bifurcation at the critical coupling strength $\lambda_c$.

In order to exhibit the macroscopic difference between the normal phase and the broken phase, we can apply the Bloch vector to describe the average collective spins. The NV center ensemble in this system can be interpreted as the collective spin-1/2 particle system. According to the definition of spin vacuum state and spin coherent state [19], the average value of the spin ensemble can be expressed as $X = \sin \theta \cos \phi$, $Y = \sin \theta \sin \phi$, and $Z = \cos \theta$, where the vector $(1, \theta, \phi)$ means the average-spin direction for this spin ensemble. According to this definition, we can describe the macroscopic eigenstate as $(\theta, \phi)$. When we set $\theta = 0$, the average value is $Z = 1$, $X = 0$, and $Y = 0$, which means the average-spin direction is along $+z$ axis and the spin ensemble belongs to the normal phase area. On the other hand, if $\theta \neq 0$, the average value is $Z = \cos \theta$, $X = \sin \theta \cos \phi$, and $Y = \sin \theta \sin \phi$, which means the Bloch vector is rotated away from $+z$ axis and then the system falls in the broken phase area. The second-order phase transition will occur in this area when $\lambda = \lambda_c$.

In this setup, we assume that the density of the NV centers is about $\rho \sim 10^{15}$ cm$^{-3}$ with the total number $N \sim 10^{12}$, and we set the parameters as the last column for the simple LMG model in TABLE I. We can calculate the expectation values of the spin components of the Bloch vector numerically from the equations (21)-(24), where the effective coupling strengths $\lambda \sim 2\pi \times 0.25$ MHz and $h \sim 2\pi \times 0.25$ MHz, the effective dissipation rate $\Gamma_b \sim 2\pi \times 0.05$ MHz and the dephasing rate $\gamma_{dep} \sim 2\pi \times 0.02$ MHz. The scaling factor is $\gamma \sim 2\pi \times 0.25$ MHz.
As shown in Fig. 2(a), (b), (c) and (d), we set $\gamma_{\text{dep}} = 0.2\gamma$ and $h = \gamma$, and get the curve for the average values of the first-order $\langle J_i \rangle / j$ and the quadratic $\langle J^2_i \rangle / j^2$ spin components varying with $\lambda / \gamma$, where $\gamma_{\text{dep}} = 0.2\gamma$ for (a) - (b) and $\gamma_{\text{dep}} = 0.4\gamma$ for (c) - (d). (e) - (f) The three dimensional plots of $\langle J_i \rangle / j$ and $\langle J^2_i \rangle / j^2$ versus $(\lambda / \gamma; \gamma_{\text{dep}} / \gamma)$.

In order to illustrate the effects of the coupling $\lambda$, the dissipation $\Gamma_b$ and the dephasing $\gamma_{\text{dep}}$ on phase transitions, in Figs. 2 and 3 we present the average values of the first-order spin components and the quadratic spin components under different conditions from equations (26)-(29).

As shown in Fig. 2(a), (b), (c) and (d), we set $\gamma_{\text{dep}} = 0.2\gamma$ and $h = \gamma$, and get the curve for the average values of the first-order $\langle J_i \rangle / j$ and the quadratic $\langle J^2_i \rangle / j^2$ spin components varying with $\lambda / \gamma$, where $\Gamma_b = 0.2\gamma$ (Fig. 2 (a) and (b)) and $\Gamma_b = 0.8\gamma$ (Fig. 2(c) and (d)). In Fig. 2(e) and (f), we also display the three dimensional surface of these average values varying with the two parameters $(\lambda / \gamma; \Gamma_b / \gamma)$ when $\gamma_{\text{dep}} = 0.2\gamma$ and $h = \gamma$. One can find that the second-order phase transition occurs at the point near $\lambda_c$. It is shown that the average value of $\langle J_x \rangle / j$ or $\langle J^2_x \rangle / j^2$ is always 1 when $\lambda < \lambda_c$, and then displays a discontinuous transition as $\lambda / \gamma$ increases to a value larger than 1. As for the $\langle J_y \rangle / j$ or $\langle J^2_y \rangle / j^2$ component, it exhibits a reversed behavior compared to the $\langle J_x \rangle / j$ or $\langle J^2_x \rangle / j^2$ component. However, both display the discontinuous behaviour at the point $\lambda_c$, which signifies quantum phase transition occurs. In addition, as shown in Fig. 2(e) and (f), when the dissipation $\Gamma_b$ increases, the critical point $\lambda_c$ will be slightly shifted. The value of $\langle \tilde{J}_y \rangle / j$ or $\langle \tilde{J}^2_y \rangle / j^2$ in the vicinity of $\lambda_c$ also increases a little. Comparing Fig. 2(a) or (b) with Fig. 2(c) or (d), the critical point has changed from about $\lambda_c = 1.01\gamma$ to $\lambda_c = 1.16\gamma$. Moreover, in the vicinity of $\lambda_c$, the value of $\langle \tilde{J}_y \rangle / j$ has changed from a value less than 0.1 to one larger than 0.2.

In Fig. 3(a), (b), (c) and (d), we set $\Gamma_b = 0.5\gamma$ and $h = \gamma$, and can obtain the illustrations for the values of the first-order $\langle \tilde{J}_i \rangle / j$ and the quadratic $\langle \tilde{J}^2_i \rangle / j^2$ spin components varying with $\lambda / \gamma$, where $\gamma_{\text{dep}} = 0.2\gamma$ (Fig. 3(a) and (b)) or $\gamma_{\text{dep}} = 0.4\gamma$ (Fig. 3(c) and (d)). Moreover, as shown in Fig. 3(e) and (f), we also plot the three dimensional surface of these average values varying with the parameters $(\lambda / \gamma; \gamma_{\text{dep}} / \gamma)$ when $\Gamma_b = 0.5\gamma$ and $h = \gamma$. We can obtain the critical point for the phase transition $\lambda_c = 1.0625\gamma$ from Fig. 3, which is immune to the dephasing $\gamma_{\text{dep}}$. Figure 3 displays the same behavior of the average spin components as in Fig. 2. We find a discontinuous behavior as $\lambda$ increases to a value larger than $\lambda_c$, and $\langle \tilde{J}_y \rangle / j$ or $\langle \tilde{J}^2_y \rangle / j^2$ exhibits a reversed behavior compared to $\langle \tilde{J}_x \rangle / j$ or $\langle \tilde{J}^2_x \rangle / j^2$. Although the dephasing will not affect the critical point, we can not neglect its effect on the phase transition. As shown in Fig. 3(e) and (f), when increasing $\lambda$ and $\gamma_{\text{dep}}$, the values of $\langle \tilde{J}_y \rangle / j$ will be enlarged continuously. Meanwhile $\langle \tilde{J}_x \rangle / j$ will be suppressed continuously too. When $\lambda / \gamma \sim 10$ and $\gamma_{\text{dep}} / \gamma \sim 1$, $\langle \tilde{J}_y \rangle / j$ will keep the value of 0.24, while $\langle \tilde{J}_x \rangle / j$ will be about 0.87. Therefore, we can simulate the second-order quantum phase transition with this hybrid quantum system, which provides the very convincing evidence for manipulating the spin ensembles realistically in our scheme.

### IV. THE SPIN SQUEEZED STATE

It is known that spin squeezed states can be prepared efficiently by one-axis twisting or two-axis counter-twisting interactions [19]. These kinds of interactions are equivalent to the LMG model with the form of equation (16) or (12). Although the two-axis counter-twisting Hamiltonian is superior to the one-axis twisting one, the spin-spin interaction with the form of two-axis counter-twisting has not been realized in any experiments due to the demanding requirements [19–22, 61]. We will discuss how to prepare spin squeezed states by simulating the two-axis counter-twisting interaction with equation (12) in our scheme. In this system, the coherent coupling strength between a single NV center and the cavity is much less than the cavity dissipation rates $|g_i| < \kappa_{a,b}$, but this collective coupling can be enhanced by increasing the number of the NV centers. Therefore, we can neglect the dissipations as long as the condition $\sqrt{N}|g_i| > \kappa_{a,b}$ is satisfied.

In order to describe the degree of spin squeezing, we introduce the definition [61]

$$\xi^2 = \frac{4 \min(\Delta \tilde{J}_x^2)}{N},$$

### (30)
where \( \vec{n}_\perp \) refers to an axis perpendicular to the mean-spin direction, and the term “min” is the minimization over all directions \( \vec{n}_\perp \). The first step is to determine the mean-spin direction \( \vec{n}_0 \) by the expectation values \( \langle \hat{J}_\alpha \rangle \), with \( \alpha \in \{x, y, z\} \). We write \( \vec{n}_0 \) with spherical coordinates \( \vec{n}_0 = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \), and this description is equivalent to the coherent spin state \( |\theta, \phi\rangle \). We can get the other two orthogonal bases which are perpendicular to \( \vec{n}_0 \),

\[
\vec{n}_1 = (-\sin \phi, \cos \phi, 0), \\
\vec{n}_2 = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta).
\]

(31)

(32)

Hence, \( \vec{n}_\perp = \vec{n}_1 \cos \beta + \vec{n}_2 \sin \beta \) is the arbitrary direction vector perpendicular to \( \vec{n}_0 \), and we can find a pair of optimal quadrature operators by tuning \( \beta \). Then we obtain two components of the angular momentum,

\[
\hat{J}_{\vec{n}_1} = -\sin \phi \hat{J}_x + \cos \phi \hat{J}_y, \\
\hat{J}_{\vec{n}_2} = \cos \theta \cos \phi \hat{J}_x + \cos \theta \sin \phi \hat{J}_y - \sin \theta \hat{J}_z.
\]

(33)

(34)

As a result, we acquire the expression of the optimal squeezing parameter

\[
\xi^2 = \frac{2}{N} \langle |\hat{J}_{\vec{n}_1}^2 + \hat{J}_{\vec{n}_2}^2| - \sqrt{\langle |\hat{J}_{\vec{n}_1}^2 - \hat{J}_{\vec{n}_2}^2|\rangle^2 + 4 \text{Cov}(\hat{J}_{\vec{n}_1}, \hat{J}_{\vec{n}_2})}\rangle,
\]

(35)

where

\[
\text{Cov}(\hat{J}_{\vec{n}_1}, \hat{J}_{\vec{n}_2}) = \frac{1}{2} \langle \hat{J}_{\vec{n}_1} \hat{J}_{\vec{n}_2} + \hat{J}_{\vec{n}_2} \hat{J}_{\vec{n}_1} \rangle.
\]

We can distinguish between spin coherent states and spin squeezed states distinctly for this NV center ensemble according to \( \xi^2 = 1 \) or \( \xi^2 < 1 \). Therefore, it is imperative for us to carry out some numerical calculations for the squeezing parameter in this system.

It is evident that one can strengthen the spin squeezing degree by increasing the total number of the spins [20–22, 45, 61]. As discussed above, we can get the relations \( \sigma_\alpha \sigma_\alpha \simeq \sigma_\beta \sigma_\beta \simeq -\sigma_\alpha \sigma_\beta \) by tuning the collective couplings, the Rabi frequencies, and the detunings. Then we can get \( \alpha = \beta = \sqrt{2}/2 \) when setting \( \sigma_\alpha \simeq \sqrt{2} \sigma_\beta \), \( \sigma_\beta \simeq \sqrt{2} \sigma_\alpha \). We investigate equations (12) and (13) and assume that the coefficients are \( \alpha \equiv |\Lambda_\alpha| = |\Lambda_\beta| \), \( \sigma_\alpha \simeq \sigma_\beta \equiv \sigma \), \( \zeta_\alpha \equiv \zeta_\beta \equiv \zeta \), and the dissipations of the supermodes are \( \kappa_\alpha = \kappa_\beta \equiv \kappa \).

We choose the first column parameters for the conventional LMG model in TABLE. I and assume the dissipations \( \kappa \sim 2 \pi \times 0.1 \text{ MHz} \). We choose the number of NV centers \( N \simeq 10^6 \) and get the collective coupling \( \sqrt{N} g_1 \simeq 2 \pi \times 12 \text{ kHz} \). Then the effective coupling coefficient is \( \lambda = \sigma^2 \zeta/\langle \kappa^2 + \zeta^2 \rangle \simeq \sigma^2 \zeta/\kappa^2 \simeq 0 \text{ Hz} \) and the effective dissipations are \( \Gamma_{a,b} = \sigma^2 \kappa/\langle \kappa^2 + \zeta^2 \rangle \simeq 2 \pi^2/\kappa \sim 2 \pi \times 1.8 \text{ Hz} \). So we cannot prepare the spin squeezed state when \( N \simeq 10^6 \), because \( \lambda \) and \( \Gamma_{a,b} \) are too weak. When we set \( N \simeq 10^{10} \) and the collective coupling as \( \sqrt{N} g_1 \simeq 2 \pi \times 1.2 \text{ MHz} \), utilizing the same parameters above, we can get the effective coupling coefficient as \( \lambda = \sigma^2 \zeta/\langle \kappa^2 + \zeta^2 \rangle \simeq 2 \pi \times 4.43 \text{ kHz} \) and the effective dissipations as \( \Gamma_{a,b} = \sigma^2 \kappa/\langle \kappa^2 + \zeta^2 \rangle \simeq 2 \pi \times 16.4 \text{ kHz} \). It is evident that we can not prepare the spin squeezed state unless \( N > 10^{10} \). On the other hand, if we choose \( N \simeq 10^{12} \) and utilize the same parameters above, we find that the effective coupling \( \lambda \sim 2 \pi \times 65.2 \text{ kHz} \) is much stronger than the effective dissipations \( \Gamma_{a,b} \sim 2 \pi \times 2.4 \text{ kHz} \). Therefore, one can not prepare the spin squeezed state unless \( N > 10^{10} \) in this setup. These estimations above also show that the appropriate choice for simulating spin squeezed state is \( N \simeq 10^{12} \).

In the condition of weak excitations and \( N \gg 1 \), we map the collective spin operators \( \hat{J}_+ \) into the boson operators \( \hat{d}^\dagger \) in the Holstein-Primakoff representation,

\[
\begin{align*}
\hat{J}_+ &= \sqrt{N} \hat{d}^\dagger, \\
\hat{J}_- &= \sqrt{N} \hat{d}, \\
\hat{J}_z &= (\hat{d}^\dagger \hat{d} - \frac{N}{2}),
\end{align*}
\]

(36)

where the operators \( \hat{d}^\dagger \) and \( \hat{d} \) obey the standard boson commutator \([\hat{d}, \hat{d}^\dagger] = 1 \). The equations (12) and (13) can be transformed as

\[
\dot{\rho} = -i[H_T, \rho] + \Gamma_a D[\hat{d}] \rho + \Gamma_b D[\hat{d}^\dagger] \rho + \gamma_{\text{dep}} D[\hat{d}^\dagger \hat{d}] \rho,
\]

\( H_T = -2 \hbar \omega \hat{d}^\dagger \hat{d} - \lambda \hat{d}^2 - \lambda \hat{d}^\dagger \hat{d}^\dagger. \)

(37)

(38)

We assume the collective spins are initially prepared in the state \( |\theta = 0\rangle \) \((Z = 1)\) and the dissipations satisfy \( \Gamma_a = \Gamma_b \). Then we solve the master equation (37) numerically and present the simulations in Fig. 4. In this numerical simulation, the parameters are set as those in the first column of TABLE. I. With these parameters, we can obtain the effective coupling coefficient \( \lambda \simeq 2 \pi \times 65 \text{ kHz} \) and the corresponding effective dissipations \( \Gamma_{a,b} \simeq 2 \pi \times 0.65 \text{ kHz} \). The dephasing rates are \( \gamma_{\text{dep}} \simeq 2 \pi \times 1.3 \text{ kHz} \), \( 2 \pi \times 1.95 \text{ kHz} \), and \( 2 \pi \times 2.6 \text{ kHz} \). Therefore, the scaling factor in Fig. 4 is \( \gamma \sim 2 \pi \times 65 \text{ kHz} \).

As shown in Fig. 4(a), we can definitely get spin squeezed for a relatively short evolution time. The system can always be in the spin squeezed state when \( 0 < t < \frac{\hbar}{\Gamma_{a,b}} \), and the squeezing parameter is about \( \xi^2 \sim 10 \text{ dB} \) when \( \Gamma_{a,b} \simeq 2 \pi \times 0.065 \text{ kHz} \). Nevertheless, this nonclassical state will be destroyed because of the dissipation and dephasing. In Fig. 4(b), it is evident that the dissipation and dephasing will reduce the squeezing degree, and it finally evolves into a thermal state.

To examine the feasibility of our scheme in realistic experiment, we now discuss the relevant available experimental parameters. The magnetic coupling strength between the cavity and a single NV center is about \( g_1 \sim 2 \pi \times 10 \text{ Hz} \). The dissipation rate of microwave cavities is about \( \kappa_a, \kappa_b > 2 \pi \times 1 \text{ kHz} \). In the practical situation, we consider a spin ensemble of \( N \sim 10^{12} \) NV centers coupled to the cavity, and the collective coupling strength satisfies \( \sqrt{N} g_1 \sim 2 \pi \times 10 \text{ MHz} \) [52].
other hand, the relaxation time of the NV spin triplet is much longer than the relative smaller number of the excitations. On the other hand, we can obtain the degree of squeezing in a shorter time $t \leq 1.5/\gamma$. (b) The evolution for a longer time $0 \leq t \leq 3/\gamma$.

Rabi frequency is about $|\Omega^\ast| \sim 2\pi \times 1$ MHz. Based on the chosen parameters, the time for maintaining the spin squeezed state is about $0 < t < 2 \mu$s in this system, and we can obtain the degree of squeezing $\xi^2 \sim -10$ dB with the relative smaller number of the excitations. On the other hand, the relaxation time of the NV spin triplet ranging from milliseconds at room temperature to several seconds at low temperature has been reported [53]. In addition, the dephasing time ($T_2 \propto 1/\gamma_{\text{dep}}$) more than 400 $\mu$s for spin ensemble has been demonstrated, and can be raised to about 2 ms with an isotopically pure diamond sample [62–66]. Thus, the coherence time is sufficient for achieving the desired spin squeezed state.

V. CONCLUSION

In summary, with the assistance of classical microwave fields and superconducting coplanar waveguide superconducting quantum interference devices, we implement an exquisite setup for simulating LMG model with an NV center ensemble in diamond. Utilizing this hybrid quantum system, we can not only achieve the second order quantum phase transition when the Hamiltonian and the dephasing rates are $\gamma_{\text{dep}} = 0.02\gamma$, $0.03\gamma$, $0.04\gamma$ respectively. (a) The evolution for a shorter time $0 \leq t \leq 1.5/\gamma$. (b) The evolution for a longer time $0 \leq t \leq 3/\gamma$.

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