Mathematical Analysis of the TB Model with Treatment via Caputo-Type Fractional Derivative

Xiao-Hong Zhang,1 Aatif Ali,2 Muhammad Altaf Khan,3 Mohammad Y. Alshahrani,4 Taseer Muhammad,5 and Saeed Islam

1College of Science, Hunan City University, Yiyang 413000, China
2Department of Mathematics, Abdul Wali Khan University, Mardan 23200, Khyber Pakhtunkhwa, Pakistan
3Institute for Ground Water Studies, Faculty of Natural and Agricultural Sciences, University of the Free State, Bloemfontein, South Africa
4Department of Clinical Laboratory Sciences, College of Applied Medical Sciences, King Khalid University, P.O. Box 61413, Abha 9088, Saudi Arabia
5Department of Mathematics, College of Sciences, King Khalid University, Abha 61413, Saudi Arabia

Correspondence should be addressed to Muhammad Altaf Khan; altafdir@gmail.com

Received 3 June 2021; Accepted 27 October 2021; Published 16 November 2021

In this study, we formulate a noninteger-order mathematical model via the Caputo operator for the transmission dynamics of the bacterial disease tuberculosis (TB) in Khyber Pakhtunkhwa (KP), Pakistan. The number of confirmed cases from 2002 to 2017 is considered as incidence data for the estimation of parameters or to parameterize the model and analysis. The positivity and boundedness of the model solution are derived. For the dynamics of the tuberculosis model, we find the equilibrium points and the basic reproduction number. The proposed model is locally and globally stable at disease-free equilibrium, if the reproduction number $R_0 < 1$. Furthermore, to examine the behavior of the various parameters and different values of fractional-order derivative graphically, the most common iterative scheme based on fundamental theorem and Lagrange interpolation polynomial is implemented. From the numerical result, it is observed that the contact rate and treatment rate have a great impact on curtailing the tuberculosis disease. Furthermore, proper treatment is a key factor in reducing the TB transmission and prevalence. Also, the results are more precise for lower fractional order. The results from various numerical plots show that the fractional model gives more insights into the disease dynamics and on how to curtail the disease spread.

1. Introduction

Tuberculosis (TB) usually caused by a bacterium called Mycobacterium tuberculosis bacterium (MTB) is a contagious infectious disease. This life-threatening disease is still imposing an alarming situation and health challenge across the globe, specifically for developing countries. TB is listed in the topmost death-causing diseases due to the high rate of mortality. Tuberculosis spread from MTB infects healthy people through the air, when they sneeze, spit, speak, or cough. Common symptoms of this disease are high fever with chills, chronic cough, night sweats, nail clubbing, weight loss, and fatigue [1]. The total number of people who died from TB in the year 2019 is 1.4 million, among which 208000 were reported HIV-positive. According to the WHO, around 10 million TB cases were estimated worldwide, in which over 95% has occurred in developing countries [2].

Several mathematical models for the transmission dynamics and on how to curtail the disease were developed. In the study of infectious diseases, the mathematical models play a key role and provide helpful information on how to control the spread and disease. There are a lot of models developed by researchers for TB dynamics. In 1962, a TB model was proposed by Waaler et al. [3], and Revelle et al.
formulated a TB disease compartmental model [4]. Liu and Zhang provided a TB model and discussed the effect of vaccination and treatment [5]. Liu et al. formulated another model and used statistical data of TB to check the seasonal effects [6]. The rabies transmission and control analysis were proposed in [7]. Wallis explored a TB model with reaction, while Kim et al. presented a model to reduce the spread of TB with optimal control strategies in the Philippines [8, 9]. Recently, a TB model with slow and fast exposed classes using real data of Pakistan was proposed by Khan et al. [10].

All the abovementioned models are formulated via integer-order derivatives. These classical models do not provide information about the learning mechanism and memory effect and thus have some limitations, while fractional models provide more compatible and realistic results. Many fractional-order derivatives were proposed in [11, 12] and have a wide range of applications in the fields of epidemiology, physics, engineering, fluid dynamics, and many others [13–16]. The fractional TB mathematical model for children and adults is investigated in [17]. In recent years, fractional-order models gained much attention due to vast applications. In biological systems, fractional models are used for better understanding of the dynamics of various diseases. Recently, in the current situation of the pandemic, many fractional-order COVID-19 models and dengue transmission models were developed in [18–20]. The fractional calculus using the series approach of the type \((p, q)\)-Mathieu-type series has been suggested in [21].

Tuberculosis disease is a main cause of mortality and morbidity and so is a massive health challenge in Pakistan. Pakistan is at the fifth position in the list of high-burden TB countries [22]. In Pakistan, the current incidence rate is more than 0.5 million and more than 50,000 die annually [23, 24]. TB is considered as a massive burden in the province Khyber Pakhtunkhwa (KP), Pakistan. A report issued by the National TB Control Programme showed that an estimated total of 462920 new cases were reported and treated in KP, Pakistan, from 2002 to 2017.

Motivated by the abovementioned work and the previous literature in view, we study the dynamical TB model with standard incidence rate explored in [10] by considering Caputo fractional derivatives for more insights into the disease. We also used the real data from year 2002 to year 2017 of Khyber Pakhtunkhwa (KP) for the parametrization of the model [25]. The remaining work is organized as follows: Section 2 contains preliminaries, and model description is given in Section 3. The analysis of the model and estimation of parameters are given in Section 4, and Section 5 contains the numerical simulation. Finally, the study is concluded in Section 6.

2. Fractional Basics Concepts

The basic definitions are presented related to fractional calculus.

**Definition 1.** The derivative for the function \(w \in \mathbb{C}^p\) having order \(\alpha\) in the Caputo operator is defined as [26]

\[
C_{D_t}^\alpha (w(t)) = \frac{1}{\Gamma(p - \alpha)} \int_0^t (t - \xi)^{p - \alpha - 1} w' \; d\xi. \tag{1}
\]

**Definition 2.** For the given function \(w : \mathbb{R}^+ \rightarrow \mathbb{R}\) having fractional order \(\alpha > 0\), the fractional integral is

\[
I_t^\alpha (w(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha - 1} w(\xi)\; d\xi, \tag{2}
\]

where \(\Gamma(\cdot)\) denotes the gamma function.

**Definition 3.** For the Caputo fractional dynamical system, let \((p^*)\) be the equilibrium point. Then,

\[
C_{D_t}^\alpha (p(t)) = w(t, p(t)), \quad \alpha \in (0, 1), \tag{3}
\]

iff \(w(t, p^*) = 0\).

3. Description of the Model

To analyze the transmission dynamics of TB disease, we consider the tuberculosis model proposed in [10] for more insight into the disease dynamics studied via the Caputo fractional operator. The model is divided into six compartments: the susceptible compartment \(S(t)\), slow and fast exposed compartments \(E_1(t)\) and \(E_2(t)\), and \(I(t)\), \(T(t)\), and \(R(t)\) representing the infected, treated, and recovered compartments, respectively. When the susceptible person after getting infection by interacting with infected individual remains in the incubation period, then it is due to the nature of infection that the individual goes to fast or slow exposed class. We assumed that the slow exposed individuals before entering into the infected compartment must join the fast exposed individuals. Then, the sum of all compartments is \(N(t)\), that is,

\[
N(t) = S(t) + E_1(t) + E_2(t) + I(t) + T(t) + R(t). \tag{4}
\]

The nonlinear fractional differential system governed by these assumptions is described as follows [10]:

\[
\begin{align*}
C_{D_t}^\alpha S &= \Delta \frac{BSI}{N} - \nu S, \\
C_{D_t}^\alpha E_1 &= \omega \frac{BSI}{N} - (\nu + \varsigma_1) E_1 + (1 - \rho)\delta T, \\
C_{D_t}^\alpha E_2 &= (1 - \omega) \frac{BSI}{N} + \varsigma_1 E_1 - (\nu + \varsigma_2) E_2, \\
C_{D_t}^\alpha I &= \varsigma_2 E_2 + \rho \delta T - (\nu + \gamma + \varsigma_1) I, \\
C_{D_t}^\alpha T &= \gamma I - (\nu + \delta + \varsigma_2 + \phi) T, \\
C_{D_t}^\alpha R &= \phi T - \nu R.
\end{align*}
\]

In model (5), after interaction between susceptible and infected persons, a ratio \(\omega (0 < \omega < 1)\) of susceptible individuals \(S(t)\) joins \(E_1(t)\) and a fraction \((1 - \omega)\) enters into fast exposed class \(E_2\) directly. \(\Delta\) and \(\nu\) denote the birth and death
rates, respectively, while β represents the successful transmission coefficient. The compartments $I$- and $T$-induced rates due to disease are given by $\sigma_1$ and $\sigma_2$, respectively; the rate of progression from $E_1$ to $E_2$ is $c_1$ and from $E_2$ to $I$ is $c_2$. For the infected individuals, the per capita treatment rate is represented by $\gamma$ and $\delta$ is the rate, where individuals quit the $T(t)$ compartment due to incomplete treatment. A fraction of those that reenter the infected class $I$ denoted by $\rho \delta T$ and the remaining $\delta T(1-\rho)$ rejoins the slow exposed class depending on treatment state of the individuals. The evolution rate is $\phi$, where treated $T$ becomes recovered $R$. The parameter $\rho (0<\rho<1)$ in $(1-\rho)\delta$ represents the ratio of drug defiance people in compartment $T$; let

$$
m_1 = (\nu + c_1),
$$
m_2 = (\nu + c_2),
$$
m_3 = (\nu + \gamma + \sigma_1),
$$
m_4 = (\nu + \delta + \sigma_2 + \phi).
$$

Model (5) can be written as

$$
\begin{align*}
C D_t^a S &= \Delta - \frac{BSI}{N} - \nu S, \\
C D_t^a E_1 &= \omega \frac{BSI}{N} - m_1 E_1 + (1-\rho)\delta T, \\
C D_t^a E_2 &= (1-\omega) \frac{BSI}{N} + c_1 E_1 - m_2 E_2, \\
C D_t^a I &= c_2 E_2 + \rho \delta T - m_3 I, \\
C D_t^a T &= \gamma I - m_4 T, \\
C D_t^a R &= \phi T - \nu R,
\end{align*}
$$

subjected to appropriate nonnegative conditions

$$
S(0) \geq 0, \\
E_1(0) \geq 0, \\
E_2(0) \geq 0, \\
I(0) \geq 0, \\
T(0) \geq 0, \\
R(0) \geq 0.
$$

In model (7), the fractional derivatives $\alpha \in (0,1]$ considered as in Caputo sense and with biological parameter values, both estimated and fitted, are displayed in Table 1.

### 4. Fractional Model Analysis

#### 4.1. Invariant Region and Attractivity

Model (7) is explored in a feasible region $\Xi \subset \mathbb{R}_+^6$, such that

$$
\Xi = \{(S(t), E(t), E_1(t), T(t), R(t)) \in \mathbb{R}_+^6 \mid S(t) \leq \frac{\Delta}{\nu}\}. 
$$

**Lemma 1.** $\Xi \subset \mathbb{R}_+^6$ is a positively invariant region with nonnegative initial conditions for model (7) in $\mathbb{R}_+^6$.

**Proof.** The net population becomes

$$
C D_t^a \Xi(t) = C D_t^a S(t) + C D_t^a E_1(t) + C D_t^a I(t) + C D_t^a E_2(t) + C D_t^a T(t) + C D_t^a R(t),
$$

and then we have

$$
C D_t^a \Xi(t) + \nu \Xi(t) \leq \Delta.
$$

Now by Laplace transform, we obtained

$$
\Xi(t) \leq \Xi(0) E_{a,1}(\nu^\alpha) + \Delta t^\alpha E_{a,a+1}(\nu^\alpha),
$$

$$
\Xi(t) \leq \frac{\Delta}{\nu}.
$$

Thus, the model solution with nonnegative conditions in $\Xi$ remains in $\Xi$. So, $\Xi$ is positive invariant and hence attracts all solutions in $\mathbb{R}_+^6$. □

Now, for the model solution positivity,

$$
R_+^6 = \{u \in \mathbb{R}^6 \mid u \geq 0\} \text{ and } u(t) = [S(t), E(t), E_1(t), E_2(t), T(t), R(t)]^T.
$$

**Corollary 1** (see [27]). We assumed that $j(t) \in \mathcal{C}[c,d]$ and $C D_t^a j(t) \in (c,d]$, where $\alpha \in (0,1]$. If
$$C^a D_t^α j(t) ≥ 0, ∀ u ∈ (c, d), \text{ then } j(t) \text{ is nondecreasing}$$

$$C^a D_t^α j(t) ≤ 0, ∀ u ∈ (c, d), \text{ then } j(t) \text{ is nonincreasing}$$

4.2. Positivity and Boundedness

**Proposition 1.** Model (7) solution is nonnegative and bounded by $∀ (S(0), E_1(0), E_2(0), I(0), T(0), R(0)) ∈ R^6_+$, for $t > 0$.

**Proof.** Using the result given in [28], we show the non-negativity of the proposed model $R^6_+$. System (7) gives

$$C^a D_t^α S|_{t=0} = Δ > 0,$$

$$C^a D_t^α E_1|_{t=0} = \frac{βSI}{S + E_2 + I + T + R} + (1 - ρ)δT ≥ 0,$$

$$C^a D_t^α E_2|_{t=0} = (1 - ω)\frac{βSI}{S + E_2 + I + T + R} + ςE_1 ≥ 0,$$

$$C^a D_t^α I|_{t=0} = ςE_2 + ρδT ≥ 0,$$

$$C^a D_t^α T|_{t=0} = γI ≥ 0,$$

$$C^a D_t^α R|_{t=0} = φT ≥ 0.$$

(14)

So, Corollary 1 gives the required result, and we can say that the solution is in $R^6_+$.

$$Ξ = \{(S, E_1, E_2, I, T, R) ∈ R^6_+: (S, E_1, E_2, I, T, R) ≥ 0\}. $$

(15)

The sum of all terms is positive; thus, the solution of model (7) is bounded.

4.3. Parameter Estimation. In this section, the method of least-square curve fitting is used to estimate the parameters from the confirmed cases of tuberculosis in Khyber Pakhtunkhwa, Pakistan, since 2002 – 2017. The birth rate $Δ$ and natural death rate $γ$ are estimated from the literature. The other biological parameters are fitted from incidence data. Figure 1 illustrates the model’s best fitted curve, and the values of parameter with description are tabulated in Table 1. The value of control basic reproductive number is $R_0 = 1.38$, estimated via fitted parameters. The curve fitting is summarized in a few steps for model (7) as follows:

$$\frac{dw}{dt} = f(t, w, Ξ), \quad w(t_0) = w_0. \quad (16)$$

Let $Ξ$ denote the unknown parameters and $w$ be the vector-dependent variables. The objective function is taken for better possible fit and is given as

$$Ξ = \sum_{k=1}^n (w_{t_k} - \bar{w}_{t_k})^2, \quad (17)$$

where $w_{t_k}$ and $\bar{w}_{t_k}$ are considered as the model solution and actual data points at time $t_k$. To obtain model parameters by minimizing the objective function and for better agreement, we follow the optimization algorithm as well [29].

4.4. Model Equilibria and Reproduction Number. To get the equilibria for fractional-order TB model (5), we have

$$C^a D_t^α S = C^a D_t^α E_1 = C^a D_t^α E_2 = C^a D_t^α I = C^a D_t^α T = C^a D_t^α R = 0. \quad (18)$$

Model (7) has two equilibrium points:

1. The risk-free or DFE $Ξ_0$ is

$$Ξ_0 = \left(\frac{Δ}{γ}, 0, 0, 0, 0, 0\right). \quad (19)$$

By using the next generation technique given in [30], we have

| Parameter | Description | Value (years$^{-1}$) | Reference |
|-----------|-------------|----------------------|-----------|
| $Δ$       | Birth rate  | $γ \times N(0)$      | Estimated |
| $γ$       | Natural death rate | $1/67.7$ | [23] |
| $β$       | Rate of contact | $0.6001$ | Fitted |
| $γ$       | Treatment rate | $0.1500$ | Fitted |
| $ρ$       | Failure of treatment | $0.2959$ | Fitted |
| $σ_1$     | Disease-induced death rate in $I$ | $0.2738$ | Fitted |
| $σ_2$     | Disease-induced death rate during treatment | $0.1000$ | Fitted |
| $ς$       | Rate of moving from $E_1$ to $E_2$ | $0.2738$ | Fitted |
| $ς$       | Transfer rate from $E_1$ to $I$ | $0.1000$ | Fitted |
| $δ$       | Leaving rate of $T$ reentering to $I$ or $E$ | $0.0649$ | Fitted |
| $ω$       | Fraction of $S$ becoming $I$ | $0.5259$ | Fitted |
| $ϕ$       | Recovery rate | $0.0100$ | Fitted |
The reproductive number $R_0$ is given by

$$R_0 = \frac{\beta \zeta_1 \omega}{m_1 m_3 m_4} + \frac{\beta \zeta_2 (1 - \omega)}{m_2 m_3}$$

(21)

Thus, the reproductive number $R_0$ is given by

$$R_0 = \rho (FV^{-1}) = \frac{\beta \zeta_1 \omega}{m_1 m_2 m_3} + \frac{\beta \zeta_2 (1 - \omega)}{m_2 m_3}$$

(20)

(21)

Endemic equilibrium (EE) $E_1$ is given by $E_1 = (S^*, E_1^*, I^*, T^*, R^*)$, where

$$S^* = \frac{N^*}{R_0},$$

$$E_1^* = \frac{1}{m_1 m_4} \left( \frac{m_4 \omega \beta}{R_0} + (1 - \rho) \gamma \delta \right) I^*,$$

$$E_2^* = \frac{1}{\zeta_2 m_4} (m_4 m_3 - \rho \gamma \delta) I^*,$$

$$T^* = \frac{V}{m_4},$$

$$R^* = \frac{\phi \gamma}{\varphi m_4} I^*,$$

$$I^* = \frac{m_1 m_4 \zeta_2 (R_0 - 1) N^*}{m_4 \zeta_2 \omega \beta + (\gamma m_1 (m_3 m_4 - \rho \gamma \delta) + \gamma \zeta_2 (m_1 + (1 - \rho) \delta)) + m_1 \zeta_2 (\phi \gamma + m_4 \gamma)/R_0}.$$
It is clear from above that $\mathcal{E}_1$ exists only if $R_0 > 1$.

4.5. Stability of the DFE

**Theorem 1.** The DFE $(\mathcal{E}_0)$ is LAS if the eigenvalues $\lambda_i$ of model (7) satisfy $|\arg(\lambda_i)| > \alpha \pi/2$.

**Proof.** The Jacobian measure of system (7) evaluated at $\mathcal{E}_0$ is given by

$$J(\mathcal{E}_0) = \begin{pmatrix} -\nu & 0 & 0 & -\beta & 0 & 0 \\ 0 & -m_1 & 0 & \beta \omega & \delta (1-\rho) & 0 \\ 0 & c_1 & -m_2 & (1-\omega)\beta & 0 & 0 \\ 0 & 0 & c_2 & -m_3 & \rho \delta & 0 \\ 0 & 0 & 0 & \gamma & -m_4 & 0 \\ 0 & 0 & 0 & 0 & \phi & -\nu \end{pmatrix}. $$

(23)

We have $-\nu$ (twice), while for the others, the following is shown:

$$P(\lambda) = \lambda^4 + c_1 \lambda^3 + c_2 \lambda^2 + c_3 \lambda + c_4,$$

(24)

where

$$
\begin{align*}
    c_1 &= m_1 + m_2 + m_3 + m_4, \\
    c_2 &= m_2 m_4 + m_1 (m_2 + m_3 + m_4) + m_2 m_3 (1-\mathcal{R}_2) + m_4 (1-\mathcal{R}_4), \\
    c_3 &= (m_1 + m_2)_m m_4 (1-\mathcal{R}_4), \\
    c_4 &= m_1 m_2 m_3 m_4 (1-\mathcal{R}_0).
\end{align*}
$$

(25)

It can be observed that the coefficients shown above are positive, i.e., $c_i > 0$, for $i = 1, \ldots, 4$; furthermore, the Routh–Hurwitz criteria can be satisfied easily, $c_1 c_2 c_3 - 4 c_4^2 > 0$. So, the TB model is locally asymptotic at the DFE when $R_0 < 1$. Thus, $|\arg(\lambda)| > \alpha \pi/2$ for all $\alpha \in [0, 1)$, i.e., local asymptotic stability.

4.6. Global Stability of DFE

**Theorem 2.** The fractional tuberculosis model is GAS at risk-free equilibrium $(\mathcal{E}_0)$ if $R_0 < 1$.

**Proof.** The appropriate Lyapunov function $(\mathcal{L}(t))$ for global stability of model (7) is defined as

$$\mathcal{L}(t) = l_1 E_1 + l_2 E_2 + l_3 I + l_4 T,$$

(26)

where $l_n > 0$, with $n = 1, \ldots, 4$, are positive constants and the time fractional derivative of $\mathcal{L}(t)$ is

$$^{C}D_t^{\alpha} \mathcal{L}(t) = l_1^{C}D_t^{\alpha} E_1 + l_2^{C}D_t^{\alpha} E_2 + l_3^{C}D_t^{\alpha} I + l_4^{C}D_t^{\alpha} T.$$

(27)

Considering fractional system (7), we obtain

$$^{C}D_t^{\alpha} \mathcal{L}(E_1, E_2, I, T) = l_1 \left( \frac{\beta SI}{N} - m_1 E_1 + (1-\rho) \delta T \right) + l_2 \left( 1 - \omega \frac{\beta SI}{N} + \delta E_1 - m_2 E_2 \right) + l_3 \left( \frac{\beta SI}{N} + \delta E_1 - m_3 E_3 \right) + l_4 \left( \gamma I - m_4 T \right),$$

\[ \leq l_1 \left( \omega \beta I - m_1 E_1 + (1-\rho) \delta T \right) + l_2 \left( 1 - \omega \beta I + \delta E_1 - m_2 E_2 \right) + l_3 \left( \frac{\beta SI}{N} + \delta E_1 - m_3 E_3 \right) + l_4 \left( \gamma I - m_4 T \right), \]

$$\text{where } S \leq N.$$

(28)

Now, we choose

$$
\begin{align*}
l_1 &= c_1, \\
l_2 &= m_1, \\
l_3 &= \frac{m_1 m_3}{c_2}, \\
l_4 &= \frac{c_1 c_2 (1-\rho) \delta + m_1 m_2 \rho \delta}{c_2 m_4}.
\end{align*}
$$

(29)

Thus, $^{C}D_t^{\alpha} \mathcal{L}(t) \leq l_3 m_3 (R_0 - 1) I$. Therefore, the variables and parameters are nonnegative with $^{C}D_t^{\alpha} \mathcal{L}(t) \leq 0$ iff $E_1 = E_2 = I = T = 0$. Hence, $(E_1, E_2, I, T) \to (0,0,0,0)$ as $t \to \infty$. We get $S \to S_0$ and $R \to R_0$ as $t \to \infty$ from system (7). Thus, the solution of model (7) with nonnegative initial conditions as $t \to \infty$ approaches $\mathcal{E}_0$ according to the fractional case developed in [31], in the feasible region. Hence, it complies that the disease-free equilibrium of system (7) is GAS.

**Lemma 2.** Model (7) is unstable at DFE if $R_0 > 1$. 

□
Figure 2: Numerical visualization of cumulative infected compartments of the TB model when $R_0 > 1$ (a) and $R_0 < 1$ (b).

Figure 3: Continued.
5. Numerical Scheme and Simulation

The numerical iterative method [32] is used for the numerical solution of model (7). A differential system is given as

\[
\frac{D_{0}^{\alpha} (w - w_{0}) (t)}{D_{0}^{\alpha} (w - w_{0}) (t)} = F (t, w(t)),
\]

\[
w_{0} (t_{0}) = w_{0}.
\]

By applying the fractional integral operator, we obtained

\[
w(t) - w(t_{0}) = \frac{1}{\Gamma (\alpha)} \int_{t_{0}}^{t} (t - h)^{\alpha - 1} F (h, w(h)) \, dh.
\]

By using the trapezoidal quadrature formula,

\[
\int_{t_{0}}^{t_{n+1}} (t_{n+1} - h)^{\alpha - 1} j(h) \, dh = \int_{t_{0}}^{t_{n+1}} (t_{n+1} - h)^{\alpha - 1} j_{n+1} (h) \, dh.
\]

The right-hand side of (32) yields

\[
\int_{t_{0}}^{t_{n+1}} (t_{n+1} - h)^{\alpha - 1} j_{n+1} (h) \, dh = \sum_{i=0}^{n+1} a_{i+1} j(t_{i}),
\]

where

Figure 3: Numerical results of model (7) for different fractional orders \(\alpha\).
Figure 4: Continued.
Figure 4: Behavior of the TB model with an impact of contact rate $\beta$ at $\alpha = 1$.

Figure 5: Continued.
When nodes $t_i = t_0 + i_h$ are equispaced, then (34) reduces to

$$a_{i,n+1} = \int_{t_i}^{t_{i+1}} (t_{i+1} - h)^{\alpha-1} \phi_{j,x+1}(h)dh$$  \hspace{1cm} (34)$$

$$X_{i,n+1}(h) = \begin{cases} 
(h - t_{i-1})/(t_i - t_{i-1}) & \text{if } t_{i-1} < h < t_i, \\
(t_{i+1} - h)/(t_{i+1} - t_i) & \text{if } t_i < h < t_{i+1}, \\
0 & \text{otherwise.}
\end{cases}$$  \hspace{1cm} (35)$$

The one-step fractional Adams–Moulton equation is
Figure 6: The treatment rate γ impact on cumulative infected persons (a–c) and variation of ρ on infectious individuals (d–f) for α = 1, α = 0.95, and α = 0.90.
\( \Gamma(\alpha) \)

\[
\begin{align*}
\Gamma(\alpha) & = \sum_{i=0}^{n} a_{i,n+1} F(t_i, w_i) + a_{n+1,n+1} F(t_{n+1}, w_{n+1}) \\
\end{align*}
\]

(37)

We have a simplified expression as follows:

\[
\begin{align*}
  b_{i,n+1} &= \frac{g^i}{\alpha} \left( (n+1-i)^\alpha - (n-i)^\alpha \right) \\
  w_{n+1} &= w_0 + \frac{1}{\Gamma(\alpha)} \sum_{i=0}^{n} b_{i,n+1} F(t_i, w_i). \\
\end{align*}
\]

(40)

Thus, the predictor \( w_n^P + 1 \) is determined by

\[
\begin{align*}
  w_n^P &= w_n + \frac{1}{\Gamma(\alpha)} \sum_{i=0}^{n} b_{i,n+1} F(t_i, w_i). \\
\end{align*}
\]

(41)

We explore the dynamics of fractional TB model (7) numerically by the generalized predictor-corrector of Adams–Bashforth–Moulton method mentioned above. To control the disease up to some extent and for memory effect, we analyze the effect of different parameters by varying some sensitive parameters and fractional order \( \alpha \).
The infection must be reduced with the help of control strategies and treatment. To illustrate the effects of sensitive parameters, the population in 2017 of Khyber Pakhtunkhwa was 30,523,371 [33], which is to be considered as \( N(0) \). The state variables assumed as \( E_1(0), E_2(0), \) and \( I(0) \) are 83000, 50000, and 8010, respectively. Moreover, no treated or recovered cases are considered initially, that is, \( T(0) = R(0) = 0 \). Then, the susceptible individuals can be obtained easily as \( S(0) = 30 \). The behaviors of cumulative infected people for \( S \) individuals can be obtained easily as \( S(t) \). Moreover, no treated or recovered cases are considered initially.

Values, there were observed decreases in the infective rate and infectious people with variation of failure of treatment are discussed and shown graphically, which show the impact of contact rate \( \beta \) on model (7) for \( \alpha = 1 \) and \( \alpha = 0.85 \), respectively. The decrease is observed in infected compartments with a decrease in \( \beta \), and for smaller order, the impact of \( \alpha \) is more pronounced. Also, the behaviors of cumulative infected people with an impact of treatment rate and infectious people with variation of failure of treatment \( \rho \) are plotted in Figure 6 for various fractional orders \( \alpha \). So, by increasing treatment rate \( \gamma \), a decay in cumulative infected individuals is observed and vice versa for infective compartments in case of \( \rho \). Finally, the behavior of total infected people with and without treatment is depicted in Figure 7, and results become more precise for smaller values of fractional order \( \alpha \). From numerical results, we can say that the tuberculosis infection can be controlled with proper treatment.

6. Conclusion

Tuberculosis has resulted in a lot of infected cases and deaths in Pakistan. The government of Pakistan and particularly the province KP have put many efforts for its minimization by treating the infected cases and also the relapse cases. To understand the TB infections in the KP province, we studied a mathematical model with slow and fast exposed cases and its effect on the model dynamics. We studied the essential mathematics involved in the modeling of the fractional-order model. Then, we investigated the stability of the model and proved the TB model to be locally as well as globally asymptotically stable. The stability results were obtained in the context of Caputo operator. Some discussion on the estimation procedures for the investigations of the model parameters is considered. We utilized the realistic parameters and obtained results graphically. We provided an efficient numerical scheme based on the Adams–Bashforth–Moulton method and obtained the graphical results. Numerical results were achieved by considering different parameters and fractional order values, and we discussed its impact on disease eradications. The effect of treatment rate and its related results was explored. Some results with treatment and without treatment are discussed and shown graphically, which show the disease can be eliminated by treating the infected people. While considering the effect of fractional-order parameter values, there were observed decreases in the infective compartments. From the analysis, the work shown suggests that the disease can be minimized more efficiently if the government takes serious actions by educating the people, making awareness, etc., and provides better treatment at doorsteps. Regarding the future of modeling of TB in Pakistan, we will explore and extend the results by considering the cases across the country with vaccinations and relapse model.

Data Availability

Data are available on the reasonable request to the corresponding author.

Conflicts of Interest

The authors declare that no conflicts of interest.

Acknowledgments

The authors extend their appreciation to King Khalid University, Saudi Arabia, for its support under number 78.

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