Another Chirality Quantum Phase Transition in the Atom-Cavity Interaction

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This paper is devoted to a study of a kind of chirality appearing in a model in cavity or circuit quantum electrodynamics. The model especially describes an artificial atom fully coupled to a 1-mode photon in a cavity when the interaction is in ultra-strong coupling regime. This chirality is played in the fully coupled (FC) Hamiltonian by two modified Jaynes-Cummings (JC) Hamiltonians. The modified JC Hamiltonian as well as the standard one has the superradiant ground state energy for sufficiently large coupling strength. The superradiance is caused by a phase transition found by Preparata, called the Dicke-Preparata (DP) superradiance. Using the one common Hamiltonian formalism with the chirality, we theoretically prove the followings: (i) when the coupling strength makes the weak or standard strong coupling regime, the ground state energy of the FC Hamiltonian can be well approximated with that of the standard and modified JC Hamiltonians since the chirality does not work in the regime; (ii) once the coupling strength plunges into the ultra-strong coupling regime, the effect of the chirality is turned on and works in the ground state energy of the FC Hamiltonian. That is, there is a kind of a phase transition concerning the chirality associated with the DP superradiance between the strong coupling regime and the ultra-strong coupling one.

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I. INTRODUCTION

The interaction between an atom and the light in nature follows the quantum electrodynamics (QED), and its coupling strength is governed by the fine structure constant. Thus, the perturbation theory is valid over the standard QED world. On the other hand, cavity QED supplies stronger interaction than the standard QED does. Namely, it can realize the strong coupling regime for the atom and the light in a cavity. Such a strong interaction is usually prepared with a two-level atom (or qubit) in a mirror cavity (i.e., a mirror resonator). Several solid-state analogues of strong coupling regime of cavity QED have been foreseen in superconducting systems. To sum up, we respectively replace the two-level atom, the laser, and the mirror resonator by an artificial atom, a microwave, and a microwave resonator on a superconducting circuit. This replaced cavity QED is called circuit QED. Actually, it has been demonstrated experimentally. It is remarkable that circuit QED is capable of realizing the so-called ultra-strong coupling regime for the interaction between the atom and the light. Namely, circuit QED has given us the technology which implements much stronger atom-light interaction than the standard QED does. Although the Jaynes-Cummings (JC) model, namely the rotating wave approximation (RWA), is useful to analyze several physical properties in the weak and the standard strong coupling regimes, it has been reported that JC model is no longer valid over the ultra-strong coupling regime and that the effect coming from the counter-rotating terms is important. Therefore, there is an urgent need to make an approximate Hamiltonian formalism which works as a non-perturbative theory for the ultra-strong coupling regime in circuit QED. Then, the points are that the theory should explain the effect of the counter-rotating terms and that the approximate Hamiltonians should be solvable.

This paper deals with a model in the theory of circuit QED, which describes the energy of a 2-level artificial atom fully coupled to a 1-mode photon in a cavity. We call its total Hamiltonian the fully coupled (FC) Hamiltonian in this paper, though it is also called the Rabi Hamiltonian. In Ref. Irish has proposed the generalized rotating wave approximation (GRWA). The GRWA comes up with the same expression of eigenstate energies as given by Feranchuk et al.. In numerical analysis, the GRWA is valid over even ultra-strong coupling regime. This expression suggests that a modified JC model should be useful for the eigenstate-energy problem for the FC model. Meanwhile, the modified JC Hamiltonian as well as the standard one has the superradiant ground state caused by the Dicke-type energy level crossings. These crossings take place in the case where the coupling strength is sufficiently large. This type of superradiance is the Dicke-Preparata’s. That is, it is a kind of phase transition so that the 2-level system coupled to a photon-field switches from the initial ground state to a new non-perturbative ground state. We decompose the FC Hamiltonian into a modified JC Hamiltonian and its chiral-counter Hamiltonian in the same way as in Ref. The former acts in the standard state space and the latter is among the modified JC Hamiltonians in the chiral state space. But, since the chiral-counter Hamiltonian controls the counter-rotating
terms in the standard state space, it breaks the properties special to the modified JC Hamiltonian. We call the pair of the former and latter the chiral pair. Using the mathematical structure of the chiral pair, in the same way that Bermudez et al. have studied a quantum phase transition in the Dirac oscillator from the point of the view of the chirality [43, 46], we will show that the FC Hamiltonian has the phase transition concerning the chirality associated with the Dicke-Preparata (DP) superradiance when the atom-cavity system plunges into the ultra-strong coupling regime from the standard strong coupling one. We also show how the effect of the chirality increases as the coupling strength grows large. In fact, the phase transition associated with the DP superradiance appears also in the standard state space when the coupling strength approaches the deep-strong coupling regime, but we realize that it takes place faster in the chiral state space than it does in the standard state space. In addition, by using our decomposition and estimating the non-commutativity between the two component Hamiltonians of the chiral pair, we will prove that the chirality associated with the DP superradiance does not work in the weak and standard strong coupling regimes, and thus, the ground state energy of the FC Hamiltonian is well approximated by that of the (modified) JC Hamiltonian. Some effective Hamiltonians are individually studied in Refs. [33, 34] to handle the weak, strong, and ultra-strong coupling regimes. We add another observation to them through our decomposition to handle all regimes in the one common Hamiltonian formalism.

Our paper is constructed as follows. In Sec. II we recall some facts on the ground state energy of the FC Hamiltonian, and formulate our problems. In Sec. III parameterizing the standard JC Hamiltonian, we define a modified JC Hamiltonian and its chiral-counter Hamiltonian to handle the counter-rotating terms. In Sec. IV we recall the Dicke-type energy level crossing and the superradiant ground state energy for the modified JC Hamiltonian. In Secs. V and VI we give answers to our problems formulated in Sec. II.

II. HAMILTONIAN OF FULLY COUPLED MODEL

In this section, we recall a mathematically exact expression of the ground state energy of the fully coupled (FC) Hamiltonian, and formulate our problems. Hamiltonians appearing in cavity QED and circuit QED fascinate some mathematicians. They belong to a class of matrix-valued Schrödinger operators [47].

A. Fully Coupled Hamiltonian

We consider the Hamiltonian describing the energy of the 2-level atom fully coupled to a 1-mode photon. We denote the creation (resp. annihilation) operator for the 1-mode photon by \( a^\dagger \) (resp. \( a \)). We use the standard notation for the the Pauli matrices as \( \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), \( \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \), and \( \sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). In addition, we denote the \( 2 \times 2 \) identity matrix by \( \sigma_0 \), i.e., \( \sigma_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

Our FC Hamiltonian \( H_{\text{FC}} \) reads:

\[
H_{\text{FC}} := \frac{\hbar \omega_a}{2} \sigma_z + \hbar \omega_c \left( a^\dagger a + \frac{1}{2} \right) + \hbar g \sigma_x \left( a^\dagger + a \right).
\]

This is the 1-mode photon version of the so-called spin-boson model [33, 40, 48, 51]. Here the constants \( \omega_a \) and \( \omega_c \) are respectively the atom transition frequency and the cavity resonance frequency, and the parameter \( g \geq 0 \) stands for the atom-photon coupling constant. Following Ref. [31], in this paper, we temporarily define the ultra-strong coupling regime by the region where the dimensionless coupling constant \( g/\omega_c \) satisfying the condition \( 0.1 \geq g/\omega_c \). Thus, we regard the region with the condition \( g/\omega_c \leq 0.1 \) as the weak and strong coupling regimes. In addition, we call the region with the condition \( 1 \geq g/\omega_c \) the deep-strong coupling regime.

We denote by \( E_{\text{FC}} \) the ground state energy of the FC Hamiltonian \( H_{\text{FC}} \). Since the FC model is the 1-mode photon version of the spin-boson model, we can apply several results on the spin-boson Hamiltonian to our FC Hamiltonian. In Ref. [50] we gave a strict expression of the ground state energy of the spin-boson model using the Bloch formula, the Duhamel formula, and the parity conservation. We apply the expression to the ground state energy \( E_{\text{FC}} \) now. To see the expression, we introduce some notations. First, we note that the FC Hamiltonian \( H_{\text{FC}} \) has the following parity conservation: \( [H_{\text{FC}}, \sigma_z (-1)^N] = 0 \). Using this parity conservation, there are many attempts to develop the accuracy of the numerical analysis for the eigenvalue problem for the FC Hamiltonian [33, 34, 52, 53]. The expression of the ground state energy \( E_{\text{FC}} \) has the two parts with even photon-numbers and the odd photon-numbers respectively. This division comes from the parity conservation. The method for the expressions in Ref. [50] reminds us of the computations of the transition amplitude of the so-called instanton gas by the Euclidean path integral [54].

We define functions \( I_{\text{even}}(\beta) \) and \( I_{\text{odd}}(\beta) \) of a variable \( \beta \geq 0 \) by

\[
I_{\text{even}}(\beta) := 1 + \sum_{\ell=1}^{\infty} \left( \frac{\omega_a}{2} \right)^{2\ell} \int_0^\beta \int_0^{\beta_1} \cdots \int_0^{\beta_{2\ell-1}} d\beta_{2\ell} \\
\times e^{-2(\sigma^2/\omega_c^2)(2G_{\beta_1} \cdots \beta_{2\ell}+2\ell)},
\]

\[
I_{\text{odd}}(\beta) := \beta \frac{\omega_a}{2} e^{-2(\sigma^2/\omega_c^2)} \\
+ \sum_{\ell=1}^{\infty} \left( \frac{\omega_a}{2} \right)^{2\ell+1} \int_0^\beta \int_0^{\beta_1} \cdots \int_0^{\beta_{2\ell+1}} d\beta_{2\ell+1} \\
\times e^{-2(\sigma^2/\omega_c^2)(2G_{\beta_1} \cdots \beta_{2\ell+1}+2\ell+1)},
\]

for the sequences \( \{G_{\beta_1, \cdots, \beta_{2\ell}}\}_{\ell=1}^{\infty} \) and \( \{F_{\beta_1, \cdots, \beta_{2\ell+1}}\}_{\ell=0}^{\infty} \).
given by

\[ G_{\beta_1, \ldots, \beta_{2\ell}} = -\sum_{p=1}^{\ell} e^{-(\beta_{2p-1} - \beta_{2p})\omega_c} + \sum_{p,q=1:p<q}^{\ell} (e^{-\beta_{2p-1} - \omega_c} - e^{-\beta_{2p}\omega_c}) \times (e^{\beta_{2q-1} - \omega_c} - e^{\beta_{2q}\omega_c}) \leq 0, \]

and

\[ F_{\beta_1, \ldots, \beta_{2\ell+1}} = e^{\beta_{2\ell+1}\omega_c} \sum_{p=1}^{\ell} (e^{-\beta_{2p-1}\omega_c} - e^{-\beta_{2p}\omega_c}) \leq 0. \]

Then, Theorem 1.3(i) of Ref. \[50\] says that the ground state energy \( E_{FC} \) is expressed as

\[ E_{FC} = \frac{\hbar \omega_c}{2} - \frac{\hbar^2}{\omega_c} - \lim_{\beta \to \infty} \frac{\hbar}{\beta} \ln \{ I_{even}(\beta) + I_{odd}(\beta) \} \]

for arbitrary coupling constant \( g \) provided that \( 1/2 \leq \omega_c/\omega_a \). We note that this expression was primarily proved without the assumption \( 1/2 \leq \omega_c/\omega_a \), but under the assumption that the vector \( \psi_+ \) defined below overlaps with the ground state \( \psi_{FC} \) of the FC Hamiltonian i.e., \( \langle \psi_{FC} | \psi_+ \rangle \neq 0 \). Here the vector \( \psi_+ \) is given by \( \psi_+ := \cos \{ gP \} | \uparrow, 0 \rangle + i \sin \{ gP \} | \downarrow, 0 \rangle \) and \( P = i(a^1 - a) / \omega_c \). The vectors \( | \downarrow, 0 \rangle \) and \( | \uparrow, 0 \rangle \) are respectively defined by \( | \downarrow, 0 \rangle := | \downarrow \rangle \otimes | 0 \rangle \) and \( | \uparrow, 0 \rangle := | \uparrow \rangle \otimes | 0 \rangle \) for spin states, \( | \downarrow \rangle \equiv (| 0 \rangle - | 1 \rangle) / \sqrt{2} \) and \( | \uparrow \rangle \equiv (| 0 \rangle + | 1 \rangle) / \sqrt{2} \), and the Fock vacuum \( | 0 \rangle \) of the photon-field. Theorem 1.3 (ii) of Ref. \[51\] says that this condition holds if \( 1 - e^{-2g^2/\omega_c^2} < 2\omega_c/\omega_a \). Then, it is always satisfied for all coupling constant \( g \) if \( 1/2 \leq \omega_c/\omega_a \). Thus, we assume the condition

\[ \frac{1}{2} \leq \frac{\omega_c}{\omega_a} \leq 1 \tag{2.2} \]

throughout this paper, where the latter inequality makes our arguments after Sec.111 simple.

Eq. (2.1) is strict, but the expression is very complicated because those of the functions \( I_{even}(\beta) \) and \( I_{odd}(\beta) \) are so. Thus, we can make it simpler with a constant. We modify the functions \( I_{even}(\beta) \) and \( I_{odd}(\beta) \) by replacing the constants \( G_{\beta_1, \ldots, \beta_{2\ell}} \) and \( F_{\beta_1, \ldots, \beta_{2\ell+1}} \) in them with \( \ell G \) and \( G/2 \), respectively, for an arbitrary constant \( G \) in the closed interval \([-1, 0] \):

\[ I_{even}^G(\beta) := \cosh \left[ (\beta \omega_a/2) e^{-2g^2(G+1)/\omega_c^2} \right] \] and

\[ I_{odd}^G(\beta) := \sinh \left[ (\beta \omega_a/2) e^{-2g^2(G+1)/\omega_c^2} \right]. \]

Then, Theorem 1.5 of Ref. \[51\] says that for every coupling constant \( g \) we can uniquely determine a constant \( G \) in the closed interval \([-1, 0] \) so that the ground state energy \( E_{FC} \) turns out to be a simple expression:

\[ E_{FC} = \frac{\hbar \omega_c}{2} - \frac{\hbar^2}{\omega_c} - \lim_{\beta \to \infty} \frac{\hbar}{\beta} \ln \left\{ \frac{I_{even}^G(\beta) + I_{odd}^G(\beta)}{r_{even}(\beta) - r_{odd}(\beta)} \right\} \]

\[ = \frac{\hbar \omega_c}{2} - \frac{\hbar^2}{\omega_c} - \frac{\hbar \omega_a}{2} \exp \left[ -\frac{2g^2}{\omega_c^2} (G + 1) \right]. \tag{2.3} \]

The constant \( G(\beta) \) is determined as a solution of the equation:

\[ \lim_{\beta \to \infty} \left( \frac{I_{even}^G(\beta) - I_{odd}^G(\beta)}{r_{even}(\beta) - r_{odd}(\beta)} \right)^{1/\beta} = 1. \]

In fact, we have the limit \( G(\beta) \to 0 \) as \( g \to \infty \) (See Sec.1A for its proof). We note that the constant \( hG(g) \) in Eq. (2.3) plays a role similar to the classical action associated with a single-instanton solution (see Eq.(3.6) of Ref. \[54\]) in the expression of the transition amplitude. Thus, \( r_{even}^G(\beta) \) and \( r_{odd}^G(\beta) \) correspond to individual Eq.(3.41) of Ref. \[54\].

By taking \(-1 \) and \( 0 \) as the constant \( G(g) \) in Eq. (2.3), we can derive the roughest estimates as in Fig.1(a):

\[ \epsilon_{low}(g) := \frac{\hbar \omega_c}{2} - \frac{\hbar^2}{\omega_c} - \frac{\hbar \omega_a}{2} \leq E_{FC} \leq \frac{\hbar \omega_c}{2} - \frac{\hbar^2}{\omega_c} - \frac{\hbar \omega_a}{2} e^{-2g^2/\omega_c^2} =: \epsilon_{upp}(g). \tag{2.4} \]

We note that \( E_{FC} \leq \hbar \omega_c/2 - \hbar^2/\omega_c + \hbar \omega_a/\omega_c \leq \hbar \omega_a/2 \) provided \( 3\omega_c/\omega_a < 4g^2 \) following the inequalities on p.161 of Ref. \[53 \[56\], but we use the upper bound \( \epsilon_{upp}(g) \) in this paper. We have the inequalities \( 0 \leq \epsilon_{upp}(g) - \epsilon_{low}(g) \leq \hbar \omega_a/2 \) and the limit as in Fig.1b): \( \lim_{g \to \infty} (\epsilon_{upp}(g) - \epsilon_{low}(g))/\hbar \omega_a = 1/2 \).

Namely, the difference between the two bounds, \( \epsilon_{upp}(g) \) and \( \epsilon_{low}(g) \), is the zero-point energy (i.e., the vacuum fluctuation) at most.

Let us point out two remarks here. The first one is that we have the asymptotic behaviors \[56\]:

\[ E_{FC} \approx -\hbar(\omega_a - \omega_c)/2 \]

if the coupling constant \( g \) is so small,

\[ E_{FC} \approx \frac{\hbar \omega_c}{2} - \frac{\hbar^2}{\omega_c} \]

if the coupling constant \( g \) is so large,

\[ E_{FC} \approx \epsilon_{upp}(g) \]

if the ratio \( g/\omega_a \) is so large.

(For the detailed proofs of them, see Sec.1A).

The other remark is concerned with the second asymptotic behavior of Eqs. (2.3). This limit is due to the fact that each energy of the Hamiltonian \( H_{FC} \) is dominated by that of the Hamiltonian \( H_{AA}^g \) when the ratio \( g/\omega_a \) is so large. Here the Hamiltonian \( H_{AA}^g \) is given by \( H_{AA}^g := \hbar \omega_c(a^+ a + \tfrac{1}{2}) + \hbar g \sigma_z (a^+ + a) \). This Hamiltonian \( H_{AA}^g \) is unitarily equivalent to the Hamiltonian

\[ \frac{1}{2} \leq \frac{\omega_c}{\omega_a} \leq 1 \tag{2.2} \]
state energy $E_{FC}^{1\text{st}}$ of the FC Hamiltonian, we have the asymptotic behavior:

$$E_{FC}^{1\text{st}} - E_{FC} = h g \times o(\omega_0^2/g^2) \quad \text{as } \omega_0^2/g \to 0 \quad (2.6)$$

(See Sec A for its proof). The degenerate ground state in circuit QED has been studied for a model of a chain of $N$ superconducting Josephson atoms in Ref. [51].

### B. Formulation of Our Problem

As far as the coupling constant $g$ makes the weak or the standard strong coupling regime of circuit QED, the Hamiltonian $H_{FC}$ is well approximated by that of the JC model: $H_{JC}^{(g)} := \hbar \omega_c \sigma_z + \hbar \omega_c (a^\dagger a + \frac{1}{2}) + h g (a^\dagger \sigma_- + \sigma_+ a)$, where $\sigma_- := (\sigma_x - i \sigma_y)/2 = (0 1 \ 1 0)$ and $\sigma_+ := (\sigma_x + i \sigma_y)/2 = (0 0 \ 0 1)$. This model is the 1-mode photon version of the Wigner-Weisskopf (WW) model called in mathematics and mathematical physics. That is, the JC (resp. WW) Hamiltonian is obtained by applying the RWA to the FC (resp. spin-boson) one. The WW Hamiltonian has a symmetry for a slightly modified photon-number operator (see, for instance, Eq.(6.2) of Ref.[49] or Eq.(2.21) of [38]). Mathematically, this symmetry reduces the eigenvalue problem for the JC Hamiltonian to the problem of solving a quadratic equation in each eigenspace of the modified photon number operator, so that the JC Hamiltonian becomes solvable. But the FC model does not have the symmetry. Thus, the eigenvalue problem for the FC model has not been solved exactly yet, or rather, even whether it is a solvable model has not been settled, though Reik and Doucha conjectured [58] that the solution of the eigenvalue problem would be exactly represented in Bargmann’s Hilbert space [59].

We denote the ground state energy of the JC Hamiltonian by $E_{JC}^{0\text{th}}$. As shown in Fig 3(a) (where we set $\omega_0$ and $\omega_c$ as $\omega_0 = \omega_c = \omega$), the ground state energy $E_{JC}^{0\text{th}}$ can supply a good approximation for the upper bound $\epsilon_{\text{upp}}(g)$ until about $g/\omega \approx 0.06$. Here, denote the sum of counter-rotating terms by $W(g) := h g (a^\dagger \sigma_- + \sigma_+ a)$. Then, employing the idea used in Refs. [46], we consider the Hamiltonian $H_{FC}(\gamma)$ with a dimensionless parameter $\gamma$ now: $H_{FC}(\gamma) := H_{JC}^{(g)} + \gamma W(g)$ for $0 \leq \gamma \leq 1$. We note that $H_{FC}(0) = H_{JC}$ and $H_{FC}(1) = H_{FC}$. When the parameter $\gamma$ is so small that the counter-rotating terms $\gamma W(g)$ is perturbative for the JC Hamiltonian $H_{JC}$, the Hamiltonian $H_{FC}(\gamma)$ has the same energy-spectral properties as the JC Hamiltonian does [41].

We note that the introduction of such a parameter was done in Ref. [46], too. On the other hand, when the parameter $\gamma$ is not so small, we cannot ignore the effect caused by the counter-rotating terms $\gamma W(g)$ as well as the importance of the effect is claimed in Refs. [24, 26, 29, 32]. For instance, let us take 1 as $\gamma$, then we have our desired Hamiltonian: $H_{FC}(1) = H_{FC}$. As in Figs 4, the approximation of the ground state energy $E_{FC}$ by the ground state energy $E_{JC}^{0\text{th}}$ does not function well any longer after

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**FIG. 1.** Set $\omega = \omega_0 = \omega_c$. (a) $\epsilon_{\text{upp}}(g)/\hbar \omega$ (red solid line), $\epsilon_{\text{low}}(g)/\hbar \omega$ (green solid line), numerically-calculated ground state energy $E_{FC}$ (blue dotted line); (b) $\epsilon_{\text{upp}}(g) - \epsilon_{\text{low}}(g))/\hbar \omega$ (black solid line).

**FIG. 2.** Energy Levels of $H_{FC}$ for $\omega = \omega_0 = \omega_c$. Each color indicates the $n$th level of the energy for $n = 0, 1, 2, \cdots$ from the bottom, where the 0th level energy means the ground state energy. We will make a remark on the accuracy of the numerical analysis for energy levels of the FC Hamiltonian at the tail end of Sec A.
a coupling regime. Therefore, we need another device to handle such a coupling regime. Meanwhile, Fig[3][b] says that the JC model has the phase transition at \( g = \omega \). That is, the superradiant ground state energy appears as follows: \( E^g_{JC} = 0 \) (a constant) if \( 0 \leq g/\omega \leq 1 \); \( E^g_{JC} = \hbar \omega - \hbar g \) (a first-degree polynomial function of \( g \)) if \( 1 \leq g/\omega \leq 1 + \sqrt{2} \). We note here that the ground state is dressed with no photon in the case \( 0 \leq g/\omega \leq 1 \), but it is dressed with a photon in the case \( 1 \leq g/\omega \leq 1 + \sqrt{2} \). We will precisely recall in Sec[IV] how the superradiant ground state energy appears.

![Fig. 3](image)

**FIG. 3.** Set \( \omega = \omega_c = \omega_{app}(g)/\hbar \omega(\text{red solid line}), \epsilon_{\text{low}}(g)/\hbar \omega(\text{green solid line}), \) and \( E^g_{JC}/\hbar \omega(\text{dark-blue dotted line}). \)

Introducing another dimensionless parameter \( \epsilon \) with \( 0 \leq \epsilon < 1 \), we parameterize the two frequencies \( \omega_a \) and \( \omega_c \), and then, denote the parameterized frequencies by \( \omega_a(\epsilon) \) and \( \omega_c(\epsilon) \). Replacing the original frequencies in the JC Hamiltonian \( H_{JC} \) with the parameterized ones, we obtain a family of the modified JC Hamiltonians \( \{H^g_{JC}(\epsilon)\}_{g,\epsilon} \):

\[
H^g_{JC}(\epsilon) := \frac{\hbar \omega_a(\epsilon)}{2} \sigma_z + \hbar \omega_c(\epsilon) \left( a^\dagger a + \frac{1}{2} \right) + \hbar g \left( a^\dagger \sigma_+ - \sigma_- a \right). \tag{2.7}
\]

In Sec[III] we will show that there is a parameterization so that the Hamiltonian \( H_{FC} \) is divided into the two parts:

\[
H_{FC} = H^g_{JC}(\epsilon) + \epsilon \sigma_x H^{g=0}_{JC}(0) \sigma_x, \tag{2.8}
\]

where the Hamiltonian \( H^{g=0}_{JC}(0) \) is also a modified JC one, given by replacing the parameters \( g \) and \( \epsilon \) in the modified JC Hamiltonian \( H^{g=0}_{JC}(\epsilon) \) with the scaled coupling constant \( g/\epsilon \) and the value 0 respectively. While the Hamiltonians \( H_{FC} \) and \( H^g_{JC}(\epsilon) \) act in the state space \( \mathcal{F} := \mathbb{C}^2 \otimes L^2(\mathbb{R}) \), the modified JC Hamiltonian \( H^{g=0}_{JC}(0) \) acts in the chiral state space \( \sigma_x \mathcal{F} \). Thus, we call the second term \( \epsilon \sigma_x H^{g=0}_{JC}(0) \sigma_x \) the chiral-counter Hamiltonian of the modified JC Hamiltonian \( H^g_{JC}(\epsilon) \), and then, the pair \( \{H^g_{JC}(\epsilon), \epsilon \sigma_x H^{g=0}_{JC}(0) \sigma_x \} \) the chiral pair for the decomposition (2.8).

We will show how to determine the parameterization for the modified JC Hamiltonian \( H^g_{JC}(\epsilon) \) as well as the naive meaning of the parameter \( \epsilon \) in Sec[III]. We should note that Bermudez et al. have handled such a chirality to study a quantum phase transition in the Dirac oscillator [13][14][15][16]. We will find another phase transition in the chiral-counter Hamiltonian.

We formulate our problem in the following. We denote each eigenstate of the modified JC Hamiltonian \( H^g_{JC}(\epsilon) \) by \( \varphi_{\nu}^{g}(\epsilon) \), and its eigenenergy by \( E^g_{\nu}(\epsilon) \), where the index \( \nu \) runs over all integers: \( H^g_{JC}(\epsilon) \varphi_{\nu}^{g}(\epsilon) = E^g_{\nu}(\epsilon) \varphi_{\nu}^{g}(\epsilon) \), \( \nu = 0, \pm 1, \pm 2, \cdots \), with \( \langle \varphi_{\nu}^{g}(\epsilon) | \varphi_{\tilde{\nu}}^{g}(\epsilon) \rangle = \delta_{\nu \tilde{\nu}} \). Here let us take eigenvalues \( E^g_{\nu}(\epsilon) \) so that the eigenvalue \( E^g_{\nu}(\epsilon) \) becomes the ground state energy when there is no interaction (i.e., \( g = 0 \)): \( E^g_{\nu}(\epsilon) \leq E^g_{0}(\epsilon) \) for \( \nu \neq 0 \). As shown in Sec[III] actually, we can take the index \( \nu \) so that the inequality, \( E^g_{\nu}(\epsilon) \leq E^g_{\nu}(\epsilon) \), holds for every coupling constant \( g \) and the parameter \( \epsilon \). As shown in Sec[IV] there is some chance that each of eigenenergies \( E^g_{\nu}(\epsilon) \) becomes the ground state energy when the coupling strength is large. Then, in the case where the eigenenergy \( E^g_{\nu}(\epsilon) \) with the index \( \nu < 0 \) becomes the ground state energy, it is called the superradiant ground state energy because of the physical mechanism found by Preparata[11][14]. We denote the (superradiant) ground state energy of the modified JC Hamiltonians \( H^g_{JC}(\epsilon) \) and \( H^{FC}_{JC}(0) \) by \( E^g_{JC}(\epsilon) \) and \( E^g_{FC}(0) \) respectively. For these ground state energies, we define a quantity \( E_{\text{low}}(g, \epsilon) \) by

\[
E_{\text{low}}(g, \epsilon) := E^{g}_{JC}(\epsilon) + \epsilon E^{g=0}_{JC}(0). \tag{2.9}
\]

Then, our main purpose is to approximate the ground state energy \( E_{FC} \) of the Hamiltonian \( H_{FC} \) by using this quantity \( E_{\text{low}}(g, \epsilon) \), and then, to study the role of the chiral part \( \epsilon E^{g=0}_{JC}(0) \) in the ground state energy \( E_{FC} \). If the modified JC Hamiltonian \( H^g_{JC}(\epsilon) \) and its chiral-counter Hamiltonian \( \epsilon \sigma_x H^{g=0}_{JC}(0) \sigma_x \) were commutable in the strong sense, i.e., \([\epsilon \hbar g \sigma_x H^{g=0}_{JC}(0) \sigma_x, \epsilon \hbar g \sigma_x H^{g=0}_{JC}(0) \sigma_x] = 0 \) for every \( t, s \in \mathbb{R} \), the ground state energy \( E_{FC} \) would be equal to the quantity \( E_{\text{low}}(g, \epsilon) \) which is then a constant function of the variable \( \epsilon \). But, unfortunately, they are not commutable in fact. Thus, we define the energy difference \( E_{\text{diff}}(\epsilon) \) between the energies \( E_{FC} \) and \( E_{\text{low}}(g, \epsilon) \) so that \( E_{FC} = E_{\text{low}}(g, \epsilon) + E_{\text{diff}}(\epsilon) \):

\[
E_{\text{diff}}(\epsilon) := E_{FC} - E_{\text{low}}(g, \epsilon). \tag{2.10}
\]

Thus, the energy difference \( E_{\text{diff}}(\epsilon) \) represents the non-commutativity between the two component Hamiltonians of the chiral pair. Our first problem consists of estimating the energy difference \( E_{\text{diff}}(\epsilon) \) and computing the energy \( \epsilon E^{g=0}_{JC}(0) \). The estimates of the energy difference \( E_{\text{diff}}(\epsilon) \) clarifies how the non-commutativity shifts the quantity \( E_{\text{low}}(g, \epsilon) \) from the true ground state energy \( E_{FC} \). The estimate of the value of the energy \( \epsilon E^{g=0}_{JC}(0) \) tells us how the effect of the chirality associated with the DP superradiance is turned on and works as the coupling strength grows.

For the normalized ground state \( \psi_{FC} \) of the Hamiltonian \( H_{FC} \) we set transition probability amplitudes \( A^g_{\nu}(\epsilon) \) and \( B^g_{\nu}(\epsilon) \) as \( A^g_{\nu}(\epsilon) := \langle \varphi_{\nu}^{g}(\epsilon) | \psi_{FC} \rangle \) and \( B^g_{\nu}(\epsilon) := \langle \varphi_{\nu}^{g}(\epsilon) | \sigma_x \psi_{FC} \rangle \) respectively. Here the vector \( \varphi_{\nu}^{g}(\epsilon) \) is the eigenstate of the modified JC Hamiltonian \( H^{g=0}_{JC}(0) \) with its eigenstate energy \( E^{g=0}_{\nu}(0) \). Since we can prove
that the orthonormal bases, \( \{ \phi^{F}_{\nu}(\varepsilon) \}_\nu \) and \( \{ \phi^{F\dagger}_{\nu}(0) \}_\nu \), are respectively complete in the state space \( F \) and the chiral state space \( \sigma_F \) in the same way as proving Propositions 3.1 and 3.2 of Ref. [11]. We can expand the vectors \( \psi_{E_{FC}} \) and \( \sigma_x \psi_{E_{FC}} \) as \( \psi_{E_{FC}} = \sum_\nu A^{\alpha}_{\nu}(\varepsilon) \phi^{F}_{\nu}(\varepsilon) \) and \( \sigma_x \psi_{E_{FC}} = \sum_\nu B^{\alpha}_{\nu}(0) \phi^{F\dagger}_{\nu}(0) \). Inserting these expansions into the expression: \( E_{FC} = \langle \psi_{E_{FC}} | H_{FC} | \psi_{E_{FC}} \rangle = \langle \psi_{E_{FC}} | H^{F\dagger}_{JC}(\varepsilon) \psi_{E_{FC}} \rangle + \varepsilon \langle \sigma_x \psi_{E_{FC}} | H^{F\dagger}_{JC}(\varepsilon) \sigma_x \psi_{E_{FC}} \rangle \), we reach the following expansion:

\[
E_{FC} = \sum_\nu |A^{\alpha}_{\nu}(\varepsilon)|^2 E^{\alpha}_{\nu}(\varepsilon) + \varepsilon \sum_\nu |B^{\alpha}_{\nu}(0)|^2 E^{\alpha}_{\nu}(0).
\]  

(2.11)

We are interested in which term mainly contributes in this expansion.

As precisely shown in Sec IV, the ground state energies \( E_{JC\nu}(\varepsilon) \) (resp. \( E^{\alpha\dagger}_{\nu}(0) \)) of the modified JC Hamiltonian becomes a superradiant ground state energy as the coupling constant g (resp. \( g/\varepsilon \)) gets large. Thus, we denote as \( E^{\alpha}_{\nu}(\varepsilon) \) := \( E_{JC\nu}(\varepsilon) \) and \( E^{\alpha\dagger}_{\nu}(0) \) := \( E^{\alpha\dagger}_{\nu}(0) \), respectively, with proper non-negative integers \( \nu_\alpha \) and \( \nu_{\alpha\dagger} \).

Thus, to solve our problem of computing the ground state energy \( E^{\alpha\dagger}_{\nu}(0) \) (resp. \( E^{\alpha\dagger}_{\nu}(\varepsilon) \)), we want to grasp the relation between the index \( \nu_{\alpha\dagger} \) (resp. \( \nu_\alpha \)) and the coupling constant g. By using the integers \( \nu_\alpha \) and \( \nu_{\alpha\dagger} \), we have another expression of the quantity \( E_{low}(g,\varepsilon) \) as \( E_{low}(g,\varepsilon) = E^{\alpha}_{\nu_\alpha}(\varepsilon) + \varepsilon E^{\alpha\dagger}_{\nu_{\alpha\dagger}}(0) \). Inserting Eq. (2.11) into Eq. (2.10) with this expression, and combining the newly obtained expression of the energy difference and the equations, \( \sum_\nu |A^{\alpha}_{\nu}(\varepsilon)|^2 = 1 \) and \( \sum_\nu |B^{\alpha}_{\nu}(0)|^2 = 1 \), we obtain the following expansion eventually:

\[
E_{diff}(\varepsilon) = \sum_{\nu \neq \nu_{\alpha\dagger}} |A^{\alpha}_{\nu}(\varepsilon)|^2 (E^{\alpha}_{\nu}(\varepsilon) - E^{\alpha\dagger}_{\nu}(0)) + \varepsilon \sum_{\nu \neq \nu_{\alpha\dagger}} |B^{\alpha}_{\nu}(0)|^2 (E^{\alpha\dagger}_{\nu}(0) - E^{\alpha\dagger}_{\nu}(\varepsilon))
\]  

(2.12)

with \( E^{\alpha}_{\nu}(\varepsilon) \geq E^{\alpha}_{\nu}(\varepsilon) \) and \( E^{\alpha\dagger}_{\nu}(0) \geq E^{\alpha\dagger}_{\nu}(\varepsilon) \). Hence it immediately follows from this that the energy difference is non-negative: \( E_{diff}(\varepsilon) \geq 0 \).

Define a lower bound \( E_{bud}(\varepsilon) \) and an upper bound \( E_{ubd}(\varepsilon) \) by \( E_{bud}(\varepsilon) := E_{low}(g) - E_{low}(g,\varepsilon) \) and \( E_{ubd}(\varepsilon) := E_{upp}(g) - E_{low}(g,\varepsilon) \), respectively. Then, rewriting the inequalities (2.3) as \( E_{low}(g,\varepsilon) \leq E_{low}(g) + E_{diff}(\varepsilon) \equiv E_{FC} \leq E_{upp}(g) \), we find that the estimates:

\[
\max \{ 0, E_{bud}(\varepsilon) \} \leq E_{diff}(\varepsilon) \leq E_{ubd}(\varepsilon).
\]  

(2.13)

Namely, we obtain the inequalities:

\[
\max \{ E_{low}(g,\varepsilon), E_{low}(g), E_{bud}(\varepsilon) \} \leq E_{FC} \leq E_{low}(g,\varepsilon) + E_{ubd}(\varepsilon).
\]  

(2.14)

Then, our problem of estimating the energy difference \( E_{diff}(\varepsilon) \) means to tune the parameter \( \varepsilon \) so that the upper bound \( E_{ubd}(\varepsilon) \) becomes small for every coupling constant g. We can make our estimate better if we find a better upper bound than the bound \( e_{upp}(g) \) and employ the better one instead of the bound \( e_{upper}(g) \).

We make a remark on the order with respect to the coupling constant g in the estimates (2.14). Actually, the order of the energy \( E_{low}(g,\varepsilon) \) is given as \( E_{low}(g,\varepsilon) \sim g^2 \) as \( g \to \infty \), and then, we have the order 2 as:

\[
E_{low}(g,\varepsilon) + E_{diff}(\varepsilon) \sim g^2 \quad \text{as} \quad g \to \infty,
\]

(2.15)

which is proved in Sec [13]. Thus, this is consistent with the order \( E_{FC} \sim g^2 \) (as \( g \to \infty \)) derived from the estimates (2.4).

Then, the following question arises: Which eigenenergy of the modified JC Hamiltonian has the order 0 or 1 (i.e., \( E^{\alpha}_{0}(\varepsilon) \sim g^0 \) and \( E_{low}(g) \sim \varepsilon g^2 \))? We can find an answer for this question in Preparata’s superradiance [38, 39, 43, 44]. Mathematically, this order comes from the fact that the ground state energy of the modified JC Hamiltonian \( H^{F\dagger}_{JC}(\varepsilon) \) (resp. \( H^{F\dagger}_{JC}(0) \)) is made by the envelope consisting of all energy levels \( E^{\alpha\dagger}_{\nu}(0) \) (resp. \( E^{\alpha\dagger}_{\nu}(\varepsilon) \) ), \( \nu = 0, -1, -2, \cdots \), as in Figs. [11] below. This envelope represents the appearance of the superradiant ground state energies caused by the Dicke-type energy level crossings. We use this fact in the ultra-strong coupling regime to approximate the ground state energy \( E_{FC} \) by the quantity \( E_{low}(g,\varepsilon) \). On the other hand, we cannot use this fact in the weak coupling regime because the Dicke-type energy level crossings is special to the sufficiently large coupling strength. So, we employ scaling transformation to argue the order in the weak coupling regime later.
Thus, the number of photons which the ground state of the FC Hamiltonian is dressed with is dominated by the states concentrating around the superradiant ground state. We will show at the tail end of Sec. V that the photon number is linked to the matrix size for the numerical computation of the ground state energy of the FC Hamiltonian.

### III. INTRODUCTION OF A KIND OF CHIRALITY

To handle the counter-rotating terms in the ultrastrong coupling regime as well as in the weak or strong coupling one, we determine our parameterization of the frequencies \( \omega_a \) and \( \omega_c \). The naive meaning of the introduction of the parameter \( \varepsilon \) is the following: Neither the rotating terms \( h g (a^\dagger \sigma_- + \sigma_+ a) \) nor the counter-rotating terms \( h g (a^\dagger \sigma_+ + \sigma_- a) \) can single-handedly make the system’s energy spectrum. They need the free Hamiltonian to pay off. Thus, the parameter \( \varepsilon \) represents how the rotating terms and the counter-rotating terms scramble for the free Hamiltonian. Adopting the idea in Ref. [38] into our argument, we give the parameterized frequencies \( \omega_a(\varepsilon) \) and \( \omega_c(\varepsilon) \) by

\[
\omega_a(\varepsilon) := (1 + \varepsilon)\omega_a \quad \text{and} \quad \omega_c(\varepsilon) := (1 - \varepsilon)\omega_c \quad (3.1)
\]

for \( 0 \leq \varepsilon < 1 \). We employ this parameterization from now on. Then, we realize that the definition (2.7) of the modified JC Hamiltonian with the parameterization (3.1) leads to the decomposition (2.8). Here, we note that both energy spectra of the Hamiltonians \( \varepsilon \sigma_x H_{JC}^{g/\varepsilon}(0)\sigma_x \) and \( \varepsilon H_{JC}^{g/\varepsilon}(0) \) are same, and that the equation, \( H_{JC}^{g/\varepsilon}(0) = H_{JC}^g \), holds.

We now recall the eigenstates of the modified JC Hamiltonian \( H_{JC}^g \) following the review in Ref. [12]. Define states \( |\uparrow, n\rangle \) and \( |\downarrow, n\rangle \) for \( n = 0, 1, 2, \cdots \) by \( |\uparrow, n\rangle := |\uparrow\rangle \otimes |n\rangle \) and \( |\downarrow, n\rangle := |\downarrow\rangle \otimes |n\rangle \), respectively, for spin states, \( |\uparrow\rangle \) and \( |\downarrow\rangle \), and the Fock states \( |n\rangle \), \( n = 0, 1, 2, \cdots \). The state \( |\uparrow, 0\rangle \) is an eigenstate of \( H_{JC}^g \), i.e.,

\[
H_{JC}^g(\varepsilon)|\uparrow, 0\rangle = E_{\uparrow,0}(\varepsilon)|\uparrow, 0\rangle
\]

with its eigenenergy

\[
E_{\uparrow,0}(\varepsilon) = -\frac{\hbar \Delta_c}{2} \quad (3.2)
\]

where \( \Delta_c := (1 + \varepsilon)\omega_a - (1 - \varepsilon)\omega_c = \Delta_0 + \varepsilon(\omega_a + \omega_c) \) and \( \Delta_0 := \omega_a - \omega_c \). We remember the assumption (2.2), which implies the condition \( \Delta_c \geq 0 \) for every \( \varepsilon \) with \( 0 \leq \varepsilon \leq 1 \).

For \( n = 0, 1, 2, \cdots \), set states \( |+, n\rangle_{g,\varepsilon} \) and \( |-, n\rangle_{g,\varepsilon} \) by

\[
|+, n\rangle_{g,\varepsilon} := \cos \theta^g_n(\varepsilon)|\uparrow, n\rangle + \sin \theta^g_n(\varepsilon)|\downarrow, n+1\rangle,
\]

\[
|-, n\rangle_{g,\varepsilon} := -\sin \theta^g_n(\varepsilon)|\uparrow, n\rangle + \cos \theta^g_n(\varepsilon)|\downarrow, n+1\rangle,
\]

where \( \theta^g_n(\varepsilon) := \frac{1}{2} \tan^{-1}(2g\sqrt{n+1}/\Delta_c) \) if \( \Delta_c \neq 0 \), and \( \theta^g_n(\varepsilon) = \pi/4 \) if \( \Delta_c = 0 \). Then, the states \( |+, n\rangle_{g,\varepsilon} \) and \( |-, n\rangle_{g,\varepsilon} \) are eigenstates of \( H_{JC}^g(\varepsilon) \):

\[
H_{JC}^g(\varepsilon)|\pm, n\rangle_{g,\varepsilon} = E_{\pm, n}(\varepsilon)|\pm, n\rangle_{g,\varepsilon}
\]

with individual eigenenergies

\[
E_{\pm, n}(\varepsilon) = (1 - \varepsilon)\hbar \omega_c(n+1) \pm \frac{\hbar}{2} \sqrt{\Delta_c^2 + 4g^2(n+1)} \quad (3.3)
\]

Therefore, we have the correspondence, \( E_{0,0}(\varepsilon) = E_{\uparrow,0}(\varepsilon) \), \( E_{+, n}(\varepsilon) = E_{\uparrow, n}(\varepsilon) \), and \( E_{-, n}(\varepsilon) = E_{\downarrow, n}(\varepsilon) \) with the relation \( |\nu| = n + 1 \) for \( n = 0, 1, 2, \cdots \). The set of all energies of \( H_{JC}^g(\varepsilon) \) is \( \{ E_{\uparrow,0}(\varepsilon), E_{\pm, n}(\varepsilon) | n = 0, 1, 2, \cdots \} \). For simplicity, here, we denote the state \( |\uparrow, 0\rangle \) by \( |-, 1\rangle_{g,\varepsilon} \) and \( |-, 1\rangle_{g,\varepsilon} \). Then, we recall the notation of the superradiance indices: For integers \( n_* := |\nu_*| - 1 \) and \( n_* := |\nu_*| - 1, \) the superradiance indices \( |\nu_*|, |\nu_*| \rangle \) can be rewritten, and then, the equations, \( E_{JC}^g(\varepsilon) = E_{\pm, n_*}(\varepsilon) \) and \( E_{JC}^g(0) = E_{\pm, n_*}(0) \), hold respectively.

### IV. SUPERRADIANT GROUND STATE ENERGY

In this section, we recall the superradiant ground state energy of the modified JC model. The DP superradiance is a kind of phase transition so that the system of the 2-level system coupled to a photon-field switches from the initial ground state to a new nonperturbative one [38, 39]. According to the results in Refs. [11, 12], there is the critical point of the coupling strength so that the following transition takes place: the energy \( E_{JC}^{g,-1}(\varepsilon) \equiv E_{\uparrow,0}(\varepsilon) \) is the ground state energy of the modified JC Hamiltonian \( H_{JC}^g(\varepsilon) \) for the coupling strength less than the critical point, but the energy \( E_{JC}^{g,0}(\varepsilon) \) replaces the old ground state energy \( E_{JC}^{g,-1}(\varepsilon) \) and becomes the new one for the coupling strength more than the critical point. At the critical point, the modified JC Hamiltonian has a degenerate ground state. The energy \( E_{JC}^{g,n}(\varepsilon), n = 1, 2, \cdots \), also becomes the ground state energy in turn as the coupling constant \( g \) grows larger and larger, though they are primarily an excited state energy when the coupling constant \( g \) is very small. The new ground state energy \( E_{JC}^{g,n}(\varepsilon) \) is called the superradiant ground state energy. The DP superradiance is a phenomenon special to very strong coupling strengths for the modified JC model as well as the standard JC model. The superradiant ground state energy \( E_{JC}^{g,n}(\varepsilon) \), \( n = 0, 1, 2, \cdots \), cannot be obtained by the perturbation theory.

We can see the phase transition if we regard the ground state energy as a function of the coupling strength \( g \): the energy \( E_{JC}^{g,-1}(\varepsilon) \) is a constant but the energy \( E_{JC}^{g,-n}(\varepsilon), n = 1, 2, \cdots \), is almost a first-degree polynomial function as in Eqs. (3.2) and (3.3). We can also explain this phase transition in terms of the photon number: the ground state is dressed with no photon in the case \( E_{JC}^{g}(\varepsilon) = E_{JC}^{g,-1}(\varepsilon) \), on the other hand, it is dressed with...
n + 1 photons in the case $E_{JC}^g(\varepsilon) = E_{-n}^g(\varepsilon)$ for each n with the condition $n \geq 0$. Because of the appearance of the superradiant ground state energy, there is surely an energy level crossing between the primary ground state energy and the superradiant ground state one, also, between the previous superradiant ground state energy and the new one. This energy level crossing is called the Dicke-type energy level crossing in Refs. [41, 42] because it is due to the mathematical property special to the Dicke model (for instance, see the remark at the tail end of Sec.4.3 of Ref. [27]).

We can find such superradiant ground states and energy level crossings for the standard and modified JC model by the numerical result in Figs. 4(a) and (b). Then, we can realize that the parameter $\varepsilon$ controls the curvature of the ground-state-energy curve. Moreover, the ground state energy $E_{JC}^g(\varepsilon)$ (i.e., the envelope consisting of $E_{-n}^g(\varepsilon)$, $n = -1, 0, 1, 2, \cdots$) makes the order: $E_{JC}^g(\varepsilon) \sim -g^2$ as $g \to \infty$ as in Eq. (2.13). It has been shown in Refs. [41, 42] how superradiant ground state energies of the JC model appear in turn. So, we recall the rigorous description of this phenomenon here.

To grasp superradiant ground state energy of our modified JC Hamiltonians $H_{JC}^{g/\varepsilon}$ and $H_{JC}^{g/\varepsilon}(0)$ we introduce some constants. We can find a quantity $G_n^g(\varepsilon)$ that completely controls the crossing between the energy $E_{1,0}^g(\varepsilon)$ and the energy $E_{-n}^g(\varepsilon)$:

$$G_n^g(\varepsilon) := (1 - \varepsilon)^2 \omega_c^2 \left\{ (n + 1) + \frac{\Delta_\varepsilon}{(1 - \varepsilon) \omega_c} \right\}$$

for $n = 0, 1, 2, \cdots$. Namely, it gives a necessary and sufficient condition for the crossing between the two energies. Then, in the same way as in Ref. [42], we can show the following facts:

$$\begin{align*}
E_{1,0}(\varepsilon) &< E_{-n}^g(\varepsilon) &\text{if and only if } g^2 > G_n^g(\varepsilon), \\
E_{1,0}(\varepsilon) &= E_{-n}^g(\varepsilon) &\text{if and only if } g^2 = G_n^g(\varepsilon), \\
E_{1,0}(\varepsilon) &> E_{-n}^g(\varepsilon) &\text{if and only if } g^2 < G_n^g(\varepsilon).
\end{align*}$$

for $n = 0, 1, 2, \cdots$. We here note the inequalities: $G_{n-1}^g(0) = 4 \varepsilon^2 \omega^2 n < (\sqrt{n} + 1 + \sqrt{n})^2 \varepsilon^2 \omega^2 < 4 \varepsilon^2 \omega^2 (n + 1) = G_n^g(0)$ for each $n = 1, 2, \cdots$. Thus, the following energy equations follow individually from the facts (4.1) and (4.4): $E_{JC}^{g/\varepsilon}(0) = E_{JC}^{g/\varepsilon}(n-1)(0)$ provided $\varepsilon^2 G_{n-1}^g(0) = 4 \varepsilon^2 \omega^2 n < g^2 < (\sqrt{n} + 1 + \sqrt{n})^2 \varepsilon^2 \omega^2$; and $E_{JC}^{g/\varepsilon}(0) = E_{JC}^{g/\varepsilon}(0)$ provided $(\sqrt{n} + 1 + \sqrt{n})^2 \varepsilon^2 \omega^2 < g^2 < 4 \varepsilon^2 \omega^2 (n + 1) = G_n^g(0)$, for each $n = 1, 2, \cdots$. Therefore, since $\varepsilon^2 G_n^g(0) = \epsilon^2 \omega^2 = (\sqrt{n} + 1 + \sqrt{n})^2 \varepsilon^2 \omega^2$, we can conclude the energy equation giving the answer to the above question:

$$E_{JC}^{g/\varepsilon}(0) = E_{JC}^{g/\varepsilon}(n-1)(0)$$

for $n = 0, 1, 2, \cdots$ if $(\sqrt{n} + 1 + \sqrt{n})^2 \varepsilon^2 \omega^2 < g^2 < (\sqrt{n} + 2 + \sqrt{n})^2 \varepsilon^2 \omega^2$. 

FIG. 4. Dicke-type energy level crossings among $E_{-n}^g(\varepsilon)$, $n = -1, 0, 1, 2, \cdots$ of the (modified) JC Hamiltonian. Each color indicates individual index $n$ of the energy $E_{-n}^g(\varepsilon)$. Here $\omega_a = \omega_c = \omega$. (a) standard JC model ($\varepsilon = 0$); (b) modified JC model ($\varepsilon = 0.50$).
It is not easy to compare the two energies \( E_{n-1}^g(\varepsilon) \) and \( E_{n-1}^g(\varepsilon) \) under the condition \( \varepsilon \neq 0 \). To see the crossing in such a case we introduce quantities \( G_{n}^{WC}(\varepsilon) \) and \( G_{n}^{SC}(\varepsilon) \) for \( n = 1, 2, \cdots \) by

\[
G_{n}^{WC}(\varepsilon) := 2(1-\varepsilon)^2\omega_c^2 \left\{ n + \frac{1}{4} \right. \\
+ \sqrt{ \left( n + \frac{1}{4} \right)^2 + \left( \frac{\Delta_\varepsilon}{2(1-\varepsilon)\omega_c} \right)^2 } \left. \right\}
\]

and

\[
G_{n}^{SC}(\varepsilon) := 2(1-\varepsilon)^2\omega_c^2 \left\{ n + \frac{1}{2} \right. \\
+ \sqrt{ \left( n(n+1) + \left( \frac{\Delta_\varepsilon}{2(1-\varepsilon)\omega_c} \right)^2 \right) } \left. \right\},
\]

respectively. Then, although we cannot give a necessary and sufficient condition for the crossing, we can give sufficient conditions for non-crossing and crossing. The same way as done in Ref. [12] brings us to the following facts:

\[
\begin{align*}
  g^2 < G_{n}^{WC}(\varepsilon) & \Rightarrow E_{n-1}^g(\varepsilon) < E_{n-1}^g(\varepsilon) , \\
  g^2 > G_{n}^{SC}(\varepsilon) & \Rightarrow E_{n+1}^g(\varepsilon) > E_{n-1}^g(\varepsilon).
\end{align*}
\]

The first implication of the facts (4.6) is an improvement of the result in Refs. [11, 12]. Its proof is in Sec. C. It is easy to check the inequalities:

\[
G_{n}^{WC}(\varepsilon) < G_{n}^{SC}(\varepsilon) < G_{n+1}^+(\varepsilon), \quad n = 1, 2, \cdots
\]

Since we have the inequalities, \( G_{n+1}^{WC}(\varepsilon) < G_{n+1}^{SC}(\varepsilon) \), for each natural number \( \ell \) with \( \ell \geq 2 \), if the condition \( g^2 < G_{n+1}^{WC}(\varepsilon) \) holds, then we have the inequalities \( E_{n+1}^g(\varepsilon) < E_{n+2}^g(\varepsilon) < \cdots < E_{n+\ell}^g(\varepsilon) < \cdots \) for each natural number \( \ell \). On the other hand, since \( G_{n}^{SC}(\varepsilon) > G_{n-\ell}^+(\varepsilon) \) for each natural number \( \ell = 1, \cdots, n-1 \), if the condition \( G_{n}^{SC}(\varepsilon) < g^2 \) holds, then we have the inequalities \( E_{n}^g(\varepsilon) < E_{n-\ell}^g(\varepsilon) < \cdots < E_{n-\ell}^g(\varepsilon) < \cdots < E_{n-1}^g(\varepsilon) \). We note that \( G_{n}^{SC}(\varepsilon) < G_{n+1}^{WC}(\varepsilon) \). Therefore, we finally obtain the energy equation:

\[
E_{n+1}^g(\varepsilon) = E_{n}^g(\varepsilon) \quad \text{if} \quad G_{n}^{SC}(\varepsilon) < g^2 < G_{n+1}^{WC}(\varepsilon).
\]

V. EFFECT CAUSED BY CHIRALITY AND NON-COMMUTATIVITY

In this section let us set the condition, \( \omega := \omega_\alpha = \omega_c \), to make our argument simple, though we can argue our problem under the condition \( \omega_\alpha \neq \omega_c \) with the assumption [22]. Then, we prove that for every coupling constant \( g \) with \( 0 \leq g < 1.193\omega \), there is a parameter \( \varepsilon \) \( 0 \leq \varepsilon \leq 0.5 \) so that

\[
E_{low}(g, \varepsilon) \leq E_{FC} \leq E_{low}(g, \varepsilon) + 0.56\hbar\omega,
\]

where the parameter \( \varepsilon \) is determined when the coupling constant \( g \) is arbitrarily given. In addition, we show that there is a phase transition concerning chirality associated with the DP superradiance between the strong coupling regime and the ultra-strong coupling one:

\[
\begin{align*}
  \varepsilon F_{JC}^g(0) = 0 & \quad \text{with} \quad \nu_{*} = 0, \\
  \text{in the weak or strong coupling regime}, \\
  \varepsilon F_{JC}^g(0) = \hbar\omega |\nu_{*}| - \hbarg\sqrt{|\nu_{*}|} & \quad \text{with} \quad \nu_{*} \neq 0, \\
  \text{in the ultra-strong coupling regime}.
\end{align*}
\]

More precisely, these statements can be proved as in Table I where the function \( F(x, \eta) \) of variables \( x \) and \( \eta \) is given by \( F(x, \eta) := \frac{1}{2}(1-e^{-2x^2}) + \eta \).

| coupling strength | superradiance indices | chiral part | upper bound of energy difference |
|-------------------|-----------------------|------------|---------------------------------|
| \( g/\omega \)    | \( [|\nu_{*}|, |\nu_{*}|] \) | \( E_{\text{upp}}^g(\varepsilon) \) | \( E_{\text{adj}}^g(\varepsilon) \leq E_{\text{adj}}^g(\varepsilon) \) |
| \( 0 \leq g/\omega \leq \varepsilon < 0.1 \) | \( [0, 0] \) \( (g/\omega) \) | \( 0 \) | \( E_{\text{adj}}(\varepsilon) = F(g/\omega, \varepsilon)\hbar\omega - \hbar^2/\omega \text{with} \quad \varepsilon = g/\omega \) |
| \( 0.1 \leq g/\omega \leq 0.244 \) | \( [0.1, 0.1] \) \( (0.1) \) | \( 0.10\hbar\omega - \hbarg \) | \( E_{\text{adj}}(0.1) = F(g/\omega, 0)\hbar\omega + \hbarg - \hbar^2/\omega < 0.25\hbar\omega \) |
| \( 0.244 \leq g/\omega \leq 0.4 \) | \( [0.1, 0.2] \) \( (0.2) \) | \( 0.20\hbar\omega - \hbarg \) | \( E_{\text{adj}}(0.2) = F(g/\omega, 0)\hbar\omega + \hbarg - \hbar^2/\omega < 0.4\hbar\omega \) |
| \( 0.4 \leq g/\omega \leq 0.9165 \) | \( [0.1, 0.3] \) \( (0.3) \) | \( 0.30\hbar\omega - \hbarg \) | \( E_{\text{adj}}(0.3) = F(g/\omega, 0)\hbar\omega + \hbarg - \hbar^2/\omega < 0.5\hbar\omega \) |
| \( 0.5 \leq g/\omega \leq 0.9168 \) | \( [0.1, 0.4] \) \( (0.4) \) | \( 0.40\hbar\omega - \hbarg \) | \( E_{\text{adj}}(0.4) = F(g/\omega, 0)\hbar\omega + \hbarg - \hbar^2/\omega < 0.53\hbar\omega \) |
| \( 0.9165 \leq g/\omega \leq 0.9659 \) | \( [1, 1] \) \( (0.4) \) | \( 0.40\hbar\omega - \hbarg \) | \( E_{\text{adj}}(0.4) = F(g/\omega, 0)\hbar\omega + \hbarg - \hbar^2/\omega < 0.53\hbar\omega \) |
| \( 0.9659 \leq g/\omega \leq 1.193 \) | \( [1, 1] \) \( (0.5) \) | \( 0.50\hbar\omega - \hbarg \) | \( E_{\text{adj}}(0.5) < 0.56\hbar\omega \) |

We here remember the estimate [2.14]: \( E_{FC} \leq E_{\text{low}}(g, \varepsilon) + E_{\text{adj}}(\varepsilon) = e_{\text{upp}}(g) \). The phase transition associated with the DP superradiance appears also for the modified JC Hamiltonian acting in the standard state space when the coupling strength approaches the deep-strong coupling regime, but Table I says that the phase
transition takes place faster in the chiral state space than it does in the standard state space. As for the coupling strength after \( g/\omega = 1.193 \), although we have not proved yet, the numerical analysis in Fig. 4(a) says the ground state energy \( E_{FC} \) is bounded by the energy \( E_{low}(g, 0.5) + 0.56\hbar \omega \) from above. Also see Fig. 4(b). The numerical analysis in Fig. 4(b) says that the difference between the numerically computed ground state energy \( E \) of the FC Hamiltonian and the energy \( E_{low}(g, 0.5) + 0.50\hbar \omega \) is less than 0.025\(\omega \). From these numerical analyses as in Figs. 4, the following problem arises: is the energy difference \( E_{diff}(\varepsilon) \) asymptotically equal to the vacuum fluctuation? We have not answered this question yet.

In the weak or strong coupling regime given by the region \( 0 \leq g/\omega \leq \varepsilon < 0.1 \), the superradiance indices \([|\nu_1|, |\nu_{**}|]|(\varepsilon)\) are \([0, 0]|(\varepsilon)\), and there is no effect of chirality associated with the DP superradiance, i.e., \( \varepsilon E_{JC}^{||}(0) = 0 \). Moreover, the ground state of the FC Hamiltonian is approximated as \( E_{FC} \approx E_{low}(g, \varepsilon) = \varepsilon E_{JC}^{||}(\varepsilon) \). Then, the ground state energy \( E_{FC} \) is well approximated by the ground state energy \( E_{JC} \) of the standard JC Hamiltonian \( H_{JC} \) because of the limit \( g \to 0 \). Here we note the following. The order of the energy \( E_{low}(g, \varepsilon) \) with respect to the coupling constant \( g \) is given by \( E_{low}(g, \varepsilon) \sim -g^0 \) as \( g \to 0 \). Thus, the energy \( E_{low}(g, \varepsilon) \) loses the dependence on the factor \( \hbar g^2/\omega_\text{c} \) in the weak coupling regime, though the ground state energy \( E_{FC} \) still keeps it as in Eq. (2.3). We need a device. So, we introduce a scaling transformation (i.e., a renormalization) to argue this problem in Sec. 7.

In the ultra-strong coupling regime, the index \( \nu_{**} \) in the superradiance indices \([|\nu_1|, |\nu_{**}|]|(\varepsilon)\) becomes non-zero (i.e., \( \nu_{**} < 0 \)), and the chiral part \( \varepsilon E_{JC}^{||}(0) \) is turned on and works in \( E_{low}(g, \varepsilon) \). Thus, the effect of the chirality associated with the DP superradiance appears. The effect of the chiral part increases more and more as the coupling strength grows large. In fact, Eqs. (4.5) and (4.8) say that as the coupling constant \( g \) grows large the index \( |\nu_{**}| \) increases faster than the index \( |\nu_1| \) in the superradiance indices \([|\nu_1|, |\nu_{**}|]|(\varepsilon)\) (See Tables II and VII).

These argument say that a phase transition takes place concerning chirality associated with the DP superradiance: the chiral part \( \varepsilon E_{JC}^{||}(0) \) is a constant in the weak or strong coupling regime, but it is almost a first-degree polynomial function of the coupling constant \( g \) in the ultra-string coupling regime. And moreover, in the ground state of the FC Hamiltonian \( H_{FC} \), there is no photon from the chiral-counter Hamiltonian \( \varepsilon H_{JC}^{||}(0) \) for the weak or strong coupling regime, on the other hand, some photons appear from the chiral-counter Hamiltonian for the ultra-strong coupling regime. In the case \( \omega_\text{a} \neq \omega_\text{c} \), we can also find the phase transition concerning the chirality associated with the DP superradiance because the chiral part \( \varepsilon E_{JC}^{||}(0) \) is a constant, \( \varepsilon E_{JC}^{||}(0) = -\varepsilon h(\omega_\text{a} - \omega_\text{c}) / 2 \), in the weak or strong coupling regime. On the other hand, it is almost a first-degree polynomial function, \( \varepsilon E_{JC}^{||}(n_{**} + 1) = \varepsilon h\omega_\text{c}(n_{**} + 1) - h\sqrt{n_{**} + 1} \), in the ultra-strong coupling regime.

A. Effect of chirality in \( E_{low}(g, \varepsilon) \)

Here we give some relations between the superradiance indices and the coupling strength. We pay our particular attention to how the index \( |\nu_{**}| = n_{**} + 1 \) (i.e., how the effect of the chirality associated with the DP superradiance) increases as the coupling constant \( g \) grows for an arbitrarily fixed parameter \( \varepsilon \). To see that we consider sufficient conditions with respect to the coupling constant \( g \) for the index \( |\nu_{**}| \).

1. In Case \([0, 0]|(\varepsilon)\)

We here consider the case where the superradiance indices are \([0, 0]|(\varepsilon)\). We note that there is no photon from the chiral-counter Hamiltonian in this case. The facts (4.5) and (4.8) tell us that the necessary and sufficient condition for the superradiance indices \([0, 0]|(\varepsilon)\) is equivalent to the condition, \( 0 \leq g^2 \leq G_0(\varepsilon) = (1 - \varepsilon^2)\omega^2 \) and \( 0 \leq (g/\varepsilon)^2 \leq G_0(0) = \omega^2 \). This is also equivalent to the condition,

\[
0 \leq g \leq \begin{cases} \varepsilon \omega & \text{if } 0 \leq \varepsilon \leq 1/\sqrt{2} \approx 0.7071, \\ \sqrt{1 - \varepsilon^2} \omega & \text{if } 1/\sqrt{2} < \varepsilon < 1. \end{cases} \tag{5.1}
\]

Therefore, we can obtain the weak and strong coupling regimes if we take the parameter \( \varepsilon \) less than 0.1.

When the integer \( n_{**} = |\nu_{**}| - 1 \) is equal to \(-1 \), a straightforward computation shows that \( E_{low}(g, \varepsilon) = E_{JC}^{||}(\varepsilon) \) (i.e., \( \varepsilon E_{JC}^{||}(0) = 0 \)) provided that \( \omega_\text{a} = \omega_\text{c} \). Therefore, there is no effect of the chirality associated with the DP superradiance in the weak or strong coupling regime. Actually, in the case \( \omega_\text{a} \neq \omega_\text{c} \), the chiral part depends only on the frequencies \( \omega_\text{a} \) and \( \omega_\text{c} \): \( \varepsilon E_{JC}^{||}(0) = -\varepsilon h(\omega_\text{a} - \omega_\text{c}) / 2 \). Thus, there is no effect.
of the chirality associated with the DP superradiance, either.

2. In Case $[0, \nu_+](\varepsilon)$ with $\nu_+ < 0$

We now consider the case where the superradiance indices are $[0, \nu_+](\varepsilon)$ with the condition $\nu_+ < 0$. We note that some photons appear from the chiral-counter Hamiltonian in this case. We see here that the index $|\nu_+| = n_+ + 1$ (i.e., the effect of the chirality associated with the DP superradiance) increases as the coupling constant $g$ grows. The facts (4.1) and (4.5) give us the sufficient condition for determining the superradiance indices.

The sufficient condition for the superradiance indices $[0, 1](\varepsilon)$ is implied by the condition:

$$\varepsilon \omega < g < \begin{cases} (\sqrt{2} + 1)\varepsilon \omega \\ \sqrt{1 - \varepsilon^2 \omega} \\ \sqrt{\frac{\sqrt{2} + 1}{\sqrt{2} + \sqrt{2}/2}} \end{cases}$$

Therefore, we can obtain the ultra-strong coupling regime if we take the parameter $\varepsilon$ more than or equal to 0.1.

More generally, for each integer $n_+ \geq 1$ (i.e., $|\nu_+| \geq 2$) we obtain the following fact. The sufficient condition for the superradiance indices $[0, n_+ + 1](\varepsilon)$ is implied by the condition:

$$\left\{ \begin{array}{ll} (\sqrt{n_+ + 1} + \sqrt{n_+}) \varepsilon \omega \\ \sqrt{1 - \varepsilon^2 \omega} \\ \sqrt{\frac{\sqrt{n_+ + 2} + \sqrt{n_+ + 1}}{\sqrt{2} + \sqrt{2}/2}} \end{array} \right\}$$

where $K(n) := \sqrt{\frac{(n + 1)^2 - 2(n + 1)\sqrt{n(n + 1)}}{2(n + 1)}}$, $n = 1, 2, \ldots$.

These results say that we cannot expect the superradiance indices $[0, n_+ + 1](\varepsilon)$ under the condition $1 \leq n_+$ and $\sqrt{2} - \sqrt{2}/2 \leq \varepsilon < 1$ since $K(n) \leq K(1) = \sqrt{2} - \sqrt{2}/2$ for each natural number $n$. On the other hand, we can expect the superradiance indices $[0, n_+ + 1](\varepsilon)$ with $n_+ \geq 1$ when parameter $\varepsilon$ are small enough. Let us arbitrarily take such a small parameter $\varepsilon$ that

$$\varepsilon^2 < \frac{1}{(\sqrt{n_+ + 2} + \sqrt{n_+ + 1})^2 + 1}.$$  \hspace{1cm} (5.3)

Then, the energy equation (4.5) says that a sufficient condition for the superradiance indices $[0, n_+ + 1](\varepsilon)$ is the condition:

$$\left\{ \begin{array}{l} (\sqrt{n_+ + 1} + \sqrt{n_+}) \varepsilon \omega \\ \sqrt{1 - \varepsilon^2 \omega} \\ \sqrt{\frac{\sqrt{n_+ + 2} + \sqrt{n_+ + 1}}{\sqrt{2} + \sqrt{2}/2}} \end{array} \right\}$$

$$< g < \left\{ \begin{array}{l} (\sqrt{n_+ + 2} + \sqrt{n_+ + 1}) \varepsilon \omega \\ \sqrt{1 - \varepsilon^2 \omega} \\ \sqrt{\frac{\sqrt{n_+ + 2} + \sqrt{n_+ + 1}}{\sqrt{2} + \sqrt{2}/2}} \end{array} \right\}.$$  \hspace{1cm} (5.4)

Meanwhile, for the superradiance indices $[1, 1](\varepsilon)$ we obtain a sufficient condition for it by the Eqs.(4.5) and (4.8) as:

$$\begin{cases} \sqrt{1 - \varepsilon^2 \omega} < g < (\sqrt{2} + 1)\varepsilon \omega \\ if \ 0.38268 \approx \sqrt{2} - \sqrt{2}/2 \leq \varepsilon \leq \varepsilon_0, \\ \sqrt{1 - \varepsilon^2 \omega} < g < C_{1}^{\text{WC}}(\varepsilon) \omega \\ if \ \varepsilon_0 < \varepsilon < 1/\sqrt{2} \approx 0.7071, \\ \varepsilon^2 \omega < g < C_{1}^{\text{WC}}(\varepsilon) \omega \\ if \ \varepsilon_0 < \varepsilon < \varepsilon_1, \end{cases}$$

where $C_{1}^{\text{WC}}(\varepsilon)$ is defined by:

$$C_{1}^{\text{WC}}(\varepsilon) := \sqrt{(1 - \varepsilon) \left\{ 5(1 - \varepsilon) + 41\varepsilon^2 - 50\varepsilon + 25 \right\} / 2},$$

and the numbers $\varepsilon_0$ and $\varepsilon_1$ with $0 < \varepsilon_0, \varepsilon_1 < 1$ are given by the solutions of the equations $(\sqrt{2} + 1)\varepsilon_0 = C_{1}^{\text{WC}}(\varepsilon_0)$ and $\varepsilon_1 = C_{1}^{\text{WC}}(\varepsilon_1)$ respectively. The solutions $\varepsilon_0$ and $\varepsilon_1$ are respectively given as $\varepsilon_0 \approx 0.496914$ and $\varepsilon_1 \approx 0.7500$. As for the superradiance indices $[1, 2](\varepsilon)$, the Eqs.(4.5) and (4.8) bring a sufficient condition for it:

$$\begin{cases} \sqrt{1 - \varepsilon^2 \omega} < g < (\sqrt{3} + \sqrt{2})\varepsilon \omega \\ if \ 0.302905 \approx \sqrt{18 - 6\varepsilon^2/6} \leq \varepsilon \leq 1 - \sqrt{2}/2, \\ (\sqrt{2} + 1)\varepsilon \omega < g < (\sqrt{3} + \sqrt{2})\varepsilon \omega \\ if \ \sqrt{2} - \sqrt{2}/2 < \varepsilon < \varepsilon_2, \\ (\sqrt{2} + 1)\varepsilon \omega < g < C_{1}^{\text{WC}}(\varepsilon) \omega \\ if \ \varepsilon_2 < \varepsilon < \varepsilon_3, \end{cases}$$

where the numbers $\varepsilon_2$ and $\varepsilon_3$ with $0 < \varepsilon_2, \varepsilon_3 < 1$ are respectively given by the solutions of the equations $(\sqrt{3} + \sqrt{2})\varepsilon_2 = C_{1}^{\text{WC}}(\varepsilon_2)$ and $(\sqrt{2} + 1)\varepsilon_3 = C_{1}^{\text{WC}}(\varepsilon_3)$. Thus, the solutions are $\varepsilon_2 \approx 0.424913$ and $\varepsilon_3 \approx 0.496914$.

In the same way, we can seek a sufficient condition for other superradiance indices by using the results in Sec. IV.

### B. Estimate of $E_{\text{diff}}(\varepsilon)$

Introducing the parameter $\varepsilon$ gives the degree of freedom of the curvature to the energy-curve of the modified JC Hamiltonian. For example, compare envelopes in Fig.4(a) and (b). In fact, we obtain too many degrees of freedom of the curvature to determine the best parameter $\varepsilon$. Thus, we determine here what is a better choice of the parameter $\varepsilon$. To do that, we propose the criterion for the parameter $\varepsilon$: when we fix the coupling constant $g$ arbitrarily, we chose such a parameter $\varepsilon$ as the energy difference $E_{\text{diff}}(\varepsilon)$ coming from the non-commutativity becomes as small as possible.
To determine the parameter $\varepsilon$ with this criterion, we employ the upper bound $E_{\text{ubd}}(\varepsilon)$ defined in the estimates (2.4): $E_{\text{diff}}(\varepsilon) \leq E_{\text{ubd}}(\varepsilon)$. For simplicity we consider and examine only the candidates $\varepsilon = 0.10, 0.20, 0.30, 0.40$ and 0.5 in this paper. The upper bound $E_{\text{ubd}}(\varepsilon)$ is numerically estimated for some parameters $\varepsilon$ as in Fig.6. According to these numerical analyses, we should take the parameter $\varepsilon$ as $\varepsilon = 0.10$ for $0 \leq g/\omega < 0.2$; $\varepsilon = 0.20$ for $0.2 \leq g/\omega < 0.4$; $\varepsilon = 0.30$ for $0.4 \leq g/\omega < 0.5$; $\varepsilon = 0.40$ for $0.5 \leq g/\omega < 1.06$; and $\varepsilon = 0.50$ for $1.06 \leq g/\omega$. We will prove this conjecture for the region $0 \leq g/\omega \leq 1.19$ (Tables II-VII). The results suggests that it should be a good way that the rotating terms and the counter-rotating terms fairly share in the free Hamiltonian with each other.

1. In Case $[0, 0](\varepsilon)$

   In the case $[0, 0](\varepsilon)$: First, we can easily estimate the energy difference $E_{\text{diff}}(\varepsilon)$ as $0 \leq E_{\text{diff}}(\varepsilon) \leq (\varepsilon + \frac{1}{2} - \frac{1}{\sqrt{2}} \varepsilon^2)\hbar \omega - \frac{\hbar^2}{\omega^2}$ in this case. We recall that the necessary and sufficient condition for the superradiance indices $[0, 0](\varepsilon)$ is equivalent to the condition (5.1). Thus, by using this estimate together with Eqs. (2.9) and (2.10), we obtain the approximation:

$$E_{\text{FC}} \approx E_{\text{low}}(g, \varepsilon), \quad \varepsilon \ll 1,$$

because of the condition (5.1). We remember that the weak and strong coupling regimes are obtained taking the parameter $\varepsilon$ less than 0.1 under the condition (5.1), and that the equation $\varepsilon E_{\text{J}\varepsilon}(0) = -\varepsilon \varepsilon h(\omega_a - \omega_c) / 2$ holds. Thus, in particular, the ground state energy $E_{\text{J}\varepsilon}$ has no effect of the chirality associated with the DP superradiance in these regimes provided that $\omega_a = \omega_c$. $E_{\text{FC}} \approx E_{\text{J}\varepsilon}(\varepsilon)$ by Eq.(2.9). Moreover, we note that the ground state energy of the modified JC Hamiltonian tends to that of the standard one as the parameter $\varepsilon$ goes to 0: $\lim_{\varepsilon \to 0} E_{\text{J}\varepsilon}(\varepsilon) = E_{\text{J}\varepsilon}$ because they are respectively given by $E_{\text{J}\varepsilon} = -\varepsilon h(\omega_a - \omega_c) / 2$ and $E_{\text{J}\varepsilon}(\varepsilon) = E_{\text{J}\varepsilon} - \varepsilon \varepsilon h(\omega_a + \omega_c) / 2$. Therefore, the ground state energy of the FC model is well approximated by that of the standard or modified JC model in the weak and strong coupling regimes.

   By the way, since the ground state energy of the modified JC Hamiltonian $H_{\text{J}\varepsilon}(\varepsilon)$ is a constant in the case $\nu_* = -1$, it does not take care of the change in the coupling strength in the weak and strong coupling regimes. We try to cope with this problem in the following: We introduce a renormalized coupling constant $g(\varepsilon, \alpha)$ for the parameter $\varepsilon$ and a new real parameter $\alpha$ as $g = \varepsilon^\alpha g(\varepsilon, \alpha)$ to restore the order with respect to the coupling constant $g$ in the weak or strong coupling regime. Forcibly adopting the analogy of the standard QED in the weak coupling regime, the parameter $\varepsilon$ should be regarded as the renormalization constant and the parameter $\alpha$ should be 1/2. So, we set the renormalized coupling constant $g_{\text{ws}}(\varepsilon)$ in the weak coupling regime as $g_{\text{ws}}(\varepsilon) := g(\varepsilon, 1/2)$. Although theory of QED for strong coupling regime has not been established yet, we employ $g_{\text{ws}}(\varepsilon)$ of the standard strong coupling regime based on the estimates (2.4).

Let us assume that we observe a coupling constant $g_{\text{ws}}$ with $(g_{\text{ws}}/\omega)^2 < 1/\sqrt{2}$ for the ground state energy $E_{\text{FC}}$ in the weak and strong coupling regimes. We define an energy $E_{\text{J}\varepsilon}^{\text{ws}}$ by $E_{\text{J}\varepsilon}^{\text{ws}} := E_{\text{J}\varepsilon}(g_{\text{ws}}^2/\omega^2)$. Then, we obtain the estimates of the ground state energy $E_{\text{FC}}$ in these regimes as:

$$- \frac{\hbar^2}{\omega^2} = E_{\text{J}\varepsilon}^{\text{ws}} \leq E_{\text{FC}} \leq 0. \quad (5.8)$$

We will prove this fact in Sec.2.1. The estimate (5.8) says that the ground state energy $E_{\text{FC}}$ of the FC model can be well approximated with the renormalized ground state energy $E_{\text{J}\varepsilon}^{\text{ws}}$ of the modified JC model for the weak and strong coupling regimes (i.e., $0 \leq g_{\text{ws}} < 1$). Thus, we should employ a sufficiently small parameter $\varepsilon = \varepsilon_{\text{ws}}$ in the weak and strong coupling regimes. Therefore, we obtain the following Table II.

| superradiance indices | region of coupling strength | estimate of the ground state energy |
|-----------------------|-----------------------------|-----------------------------------|
| $[0, 0](\varepsilon)$ | $0 \leq g/\omega \leq \varepsilon < 1/\sqrt{2}$ | $E_{\text{FC}} \leq 0$ |
|                        | $(0 \leq g/\omega < 0.1)$   | $E_{\text{FC}} \leq 0$ |
|                        | $-1/\sqrt{2} < E_{\text{FC}}/\hbar \omega \leq 0$ | $E_{\text{FC}}/\hbar \omega \leq 0$ |

| TABLE II. Ground state energy estimates. |

2. In Case $[0, \nu_*](\varepsilon)$ with $\nu_* < 0$

In the ultra-strong coupling regime, the chiral part $\varepsilon E_{\text{J}\varepsilon}(0)$ is turned on and works because the chirality associated with the DP superradiance appears. The effect of the chiral part $\varepsilon E_{\text{J}\varepsilon}(0)$ increases more and more as the coupling strength grows large because, as shown
We estimate the energy difference $E_{\text{diff}}(\epsilon)$ in the case where $n_s = -1$ and $n_{ss} = 0, 1, 2, \ldots$. To do that, we only have to estimate the lower bound $E_{\text{ubal}}(\epsilon)$ and the upper bound $E_{\text{ubal}}(\epsilon)$ with the help of the inequalities (2.13) and (2.2).

We define two functions $\mathcal{E}_{1,n_s}^{\text{low}}(x)$ and $\mathcal{E}_{1,n_s}^{\text{upp}}(x)$ by $\mathcal{E}_{1,n_s}^{\text{low}}(x) := \sqrt{n_{ss} + 1} x - x^2$ and $\mathcal{E}_{1,n_s}^{\text{upp}}(x) := \frac{1}{2} + \sqrt{n_{ss} + 1} x - x^2 - \frac{1}{2} e^{-2x^2}$, respectively, for each integer $n_s = 0, 1, 2, \ldots$. Applying the mean value theorem to the function $f(x) = x^2 + \frac{1}{2} e^{-2x^2}$, there is a point $x_0$ between the non-negative points 0 and $x$ so that $(f(x) - f(x_0))/(x - x_0) = f'(x_0)$. Thus, we know that the expression $\frac{1}{2} - x^2 + \frac{1}{2} e^{-2x^2}$ is positive:

$$\frac{1}{2} - x^2 + \frac{1}{2} e^{-2x^2} = -2x_0(1 - e^{-2x_0^2}) \leq 0,$$

which implies the estimate $\mathcal{E}_{1,n_s}^{\text{upp}}(x) \leq \sqrt{n_{ss} + 1} x$. Let $G_{1,n_s}$ be a positive constant given by a solution of the equation:

$$2x(1 - e^{-2x^2}) = \sqrt{n_{ss} + 1}.$$  

We note that the positive solution is uniquely determined.

In the case $[0, 1](\epsilon)$: For the region (5.2) we have to assume that the parameter $\epsilon$ satisfies $\epsilon < 1/\sqrt{2}$. We have the constant $G_{1,0} \approx 0.745363$ and the inequality $\mathcal{E}_{1,0}^{\text{upp}}(G_{1,0}) \leq 0.53$. Thus, the following estimates follow from the expressions (12):

$$h \omega \mathcal{E}_{1,0}^{\text{low}} \left( \frac{g}{\omega} \right)$$

$$\leq E_{\text{diff}}(\epsilon) \leq h \omega \mathcal{E}_{1,0}^{\text{upp}} \left( \frac{g}{\omega} \right) \leq \min\left\{ g/\omega, 0.53 \right\} h \omega =: \mathcal{E}_{\nu_s=1}(\epsilon)$$

in the region $\epsilon \leq g/\omega < \min\{\sqrt{2} + 1, \epsilon, \sqrt{1 - \epsilon^2}\}$ because of the condition (2.2). We note that $E_{\text{diff}}(\epsilon) \leq 0.5\hbar\omega$ for the coupling constant $g$ satisfying the condition $\epsilon < g/\omega < \min\{\sqrt{2} + 1, \epsilon, \sqrt{1 - \epsilon^2}, 0.6\}$ since $\mathcal{E}_{1,0}^{\text{upp}}(x) \leq 0.5$ for $0 \leq x \leq 0.6$.

In the case $[0, 2](\epsilon)$: We put the restriction caused by the condition (5.3) on the parameter $\epsilon$: $\epsilon < \{\sqrt{3} + \sqrt{2} + 1\}^{-1}$. We can compute the constant $G_{1,1}$ as $G_{1,1} \approx 0.88976$ and estimate the value $\mathcal{E}_{1,1}^{\text{upp}}(G_{1,1})$ as $\mathcal{E}_{1,1}^{\text{upp}}(G_{1,1}) \leq 0.87$. Thus, the expressions (12) come up with the following estimates:

$$- \epsilon h \omega + h \omega \mathcal{E}_{1,1}^{\text{low}} \left( \frac{g}{\omega} \right)$$

$$\leq E_{\text{diff}}(\epsilon) \leq - \epsilon h \omega + h \omega \mathcal{E}_{1,1}^{\text{upp}} \left( \frac{g}{\omega} \right) \leq \left( \min\{\sqrt{2g}/\omega, 0.87\} - \epsilon \right) h \omega$$

$$=: \mathcal{E}_{\nu_s=2}(\epsilon)$$

in the region $(\sqrt{2} + 1)\epsilon < g/\omega < (\sqrt{3} + \sqrt{2})\epsilon$, which is determined by the condition (5.4).

In the case $[0, 3](\epsilon)$: We put the restriction caused by the condition (5.3) on the parameter $\epsilon$: $\epsilon < \{2 + \sqrt{3}\}^{-1}$. The constant $G_{1,2}$ is obtained as $G_{1,2} = 1.000964$ and the value $\mathcal{E}_{1,2}^{\text{upp}}(G_{1,2})$ is estimated as $\mathcal{E}_{1,2}^{\text{upp}}(G_{1,2}) \leq 1.18$. Thus, we can derive the following estimates from the expressions (12):

$$- 2\epsilon \hbar \omega + \hbar \omega \mathcal{E}_{1,2}^{\text{low}} \left( \frac{g}{\omega} \right)$$

$$\leq E_{\text{diff}}(\epsilon) \leq - 2\epsilon \hbar \omega + \hbar \omega \mathcal{E}_{1,2}^{\text{upp}} \left( \frac{g}{\omega} \right) \leq \left( \min\{\sqrt{3g}/\omega, 1.18\} - 2\epsilon \right) \hbar \omega$$

$$=: \mathcal{E}_{\nu_s=3}(\epsilon)$$

in the region $(\sqrt{3} + \sqrt{2})\epsilon < g/\omega < (2 + \sqrt{3})\epsilon$ which is implied by the condition (5.3).

In the case $[0, 4](\epsilon)$: We put the restriction caused by the condition (5.3) on the parameter $\epsilon$: $\epsilon < \{\sqrt{3} + \sqrt{2} + 1\}^{-1}$. The constant $G_{1,3}$ is computed as $G_{1,3} = 1.09837$ and the value $\mathcal{E}_{1,3}^{\text{upp}}(G_{1,3})$ satisfies the inequality $\mathcal{E}_{1,3}^{\text{upp}}(G_{1,3}) \leq 1.47$. Thus, by the expressions (12), we obtain the following estimate:

$$- 3\epsilon \hbar \omega + \hbar \omega \mathcal{E}_{1,3}^{\text{low}} \left( \frac{g}{\omega} \right)$$

$$\leq E_{\text{diff}}(\epsilon) \leq - 3\epsilon \hbar \omega + \hbar \omega \mathcal{E}_{1,3}^{\text{upp}} \left( \frac{g}{\omega} \right) \leq \left( \min\{2g/\omega, 1.47\} - 3\epsilon \right) \hbar \omega$$

$$=: \mathcal{E}_{\nu_s=4}(\epsilon)$$

in the region $(2 + \sqrt{3})\epsilon < g/\omega < (\sqrt{5} + 2)\epsilon$ coming from the condition (5.3).

Next we estimate the energy difference $E_{\text{diff}}(\epsilon)$ in the case where $n_s, n_{ss} = 0, 1, 2, \ldots$ with the help of the inequalities (2.13) and (2.2).

Define two functions $\mathcal{E}_{n_s,n_{ss}}^{\text{low}}(x)$ and $\mathcal{E}_{n_s,n_{ss}}^{\text{upp}}(x)$ by $\mathcal{E}_{n_s,n_{ss}}^{\text{low}}(x) := \sqrt{(n_s + 1)x^2 + \epsilon^2} - (n_s + 1) + \mathcal{E}_{1,n_s}^{\text{low}}(x)$ and $\mathcal{E}_{n_s,n_{ss}}^{\text{upp}}(x) := \sqrt{(n_s + 1)x^2 + \epsilon^2} - (n_s + 1) + \mathcal{E}_{1,n_s}^{\text{upp}}(x)$ respectively. Then, we note $\mathcal{E}_{n_s,n_{ss}}^{\text{upp}}(x) \leq \sqrt{(n_s + 1)x^2 + \epsilon^2} - (n_s + 1) + \mathcal{E}_{1,n_s}^{\text{upp}}(G_{1,n_s})$.

In the case $[1,1](\epsilon)$: We put the restriction on $\epsilon$ as $0.38268 \approx \sqrt{2} - \sqrt{2}/2 \leq \epsilon < \epsilon_1 \approx 0.7500$ as in the region (5.3) of the coupling strength. Since there is a point $G_{0,0}$ so that $\mathcal{E}_{0,0}^{\text{upp}}(G_{0,0}) = \max_x \mathcal{E}_{0,0}^{\text{upp}}(x)$, the estimates (12) lead to

$$h \omega \mathcal{E}_{0,0}^{\text{low}} \left( \frac{g}{\omega} \right) \leq E_{\text{diff}}(\epsilon) \leq h \omega \mathcal{E}_{0,0}^{\text{upp}} \left( \frac{g}{\omega} \right)\quad (5.13)$$

$$\leq h \omega \mathcal{E}_{0,0}^{\text{upp}}(G_{0,0}).$$
We note here that $\mathcal{E}^{\text{supp}}(G_{0,0}) \leq 0.53$ for $\varepsilon = 0.40$ and $\mathcal{E}^{\text{supp}}(G_{0,0}) \leq 0.56$ for $\varepsilon = 0.50$.

In the case where the coupling constant $g$ is about in the region $0.1 \leq \omega < 1$, we obtain Tables III, IV by the estimates (5.11). Then, the index $\nu_4$ is just 0 but the index $\nu_3$ grows.

### TABLE III. Energy difference estimates ($\varepsilon = 0.10$).

| superradiance indices | region of coupling strength ($\varepsilon = 0.10$) | estimate of energy difference |
|-----------------------|--------------------------------|--------------------------------|
| [0, 1](0.10)          | $0.1 = g / \omega < (\sqrt{2} + 1) \varepsilon \approx 0.2414$ | $E_{\text{diff}}(0.10)/\hbar \omega \leq g / \omega < 0.2414$ |
| [0, 2](0.10)          | $0.2414 \approx (\sqrt{2} + 1) \varepsilon < g / \omega < (\sqrt{3} + \sqrt{7}) \varepsilon \approx 0.3146$ | $E_{\text{diff}}(0.10)/\hbar \omega \leq g / \omega < 0.3146$ |
| [0, 3](0.10)          | $0.3146 \approx (\sqrt{3} + \sqrt{7}) \varepsilon < g / \omega < (2 + \sqrt{3}) \varepsilon \approx 0.3772$ | $E_{\text{diff}}(0.10)/\hbar \omega \leq g / \omega < 0.3772$ |
| [0, 4](0.10)          | $0.3772 \approx (2 + \sqrt{3}) \varepsilon < g / \omega < (\sqrt{5} + 2) \varepsilon \approx 0.4236$ | $E_{\text{diff}}(0.10)/\hbar \omega \leq 2g / \omega - 0.3 < 0.5472$ |

### TABLE IV. Energy difference estimates ($\varepsilon = 0.20$).

| superradiance indices | region of coupling strength ($\varepsilon = 0.20$) | estimate of energy difference |
|-----------------------|--------------------------------|--------------------------------|
| [0, 1](0.20)          | $0.2 = g / \omega < (\sqrt{2} + 1) \varepsilon \approx 0.482843$ | $E_{\text{diff}}(0.20)/\hbar \omega \leq g / \omega < 0.482843$ |
| [0, 2](0.20)          | $0.482843 \approx (\sqrt{2} + 1) \varepsilon < g / \omega < (\sqrt{3} + \sqrt{7}) \varepsilon \approx 0.62925$ | $E_{\text{diff}}(0.20)/\hbar \omega \leq g / \omega < 0.62925$ |
| [0, 3](0.20)          | $0.62925 \approx (\sqrt{3} + \sqrt{7}) \varepsilon < g / \omega < (2 + \sqrt{3}) \varepsilon \approx 0.74641$ | $E_{\text{diff}}(0.20)/\hbar \omega \leq g / \omega < 0.74641$ |
| [0, 4](0.20)          | $0.74641 \approx (2 + \sqrt{3}) \varepsilon < g / \omega < (\sqrt{5} + 2) \varepsilon \approx 0.84721$ | $E_{\text{diff}}(0.20)/\hbar \omega \leq g / \omega < 0.84721$ |

### TABLE V. Energy difference estimates ($\varepsilon = 0.30$).

| superradiance indices | region of coupling strength ($\varepsilon = 0.30$) | estimate of energy difference |
|-----------------------|--------------------------------|--------------------------------|
| [0, 1](0.30)          | $0.3 = g / \omega < (\sqrt{2} + 1) \varepsilon \approx 0.724264$ | $E_{\text{diff}}(0.30)/\hbar \omega \leq g / \omega < 0.724264$ |
| [0, 2](0.30)          | $0.724264 \approx (\sqrt{2} + 1) \varepsilon < g / \omega < (\sqrt{3} + \sqrt{7}) \varepsilon \approx 0.8438793$ | $E_{\text{diff}}(0.30)/\hbar \omega \leq g / \omega < 0.8438793$ |

Following the numerical analyses in Fig III, the case where $\varepsilon = 0.40$ gives a good energy-difference estimate for the region $0.5 \leq \omega / \varepsilon < 1.06$ among the candidate $\varepsilon = 0.10, 0.20, 0.30, 0.40, 0.50$. But, as in Table VI, I our bound obtained is $0.9659$. It is caused by the fact that the bound $C_{\text{WC}}^1(\varepsilon)$ is not best possible.

Using the results in Tables III, IV, V, VI, VII, we can conclude the results on the estimates of the energy difference in Table II.

### VI. CONCENTRATION OF SUPERRADIANT GROUND STATE

We estimate the transition probabilities $|A_g^0(\varepsilon)|^2$ and $|B_{g_{\text{ref}}}^0(0)|^2$ under the condition $\omega = \omega_{\text{ref}} = \omega_1$ in this section. For simplicity, we reset the notations of the transition probability amplitudes $A_g(\pm, \varepsilon)$ and $B_{g_{\text{ref}}}^0(\pm, 0)$ as $A_g(\pm, \varepsilon) := \varepsilon \langle n, \pm | \psi_{E_{\text{FC}}}/g, \varepsilon \rangle$ and $B_{g_{\text{ref}}}^0(\pm, 0) := 0_{g/\varepsilon}(n, \pm | \sigma_x \psi_{E_{\text{FC}}}/g, 0, 0, 0)$, respectively, for $n = 0, 1, 2, \ldots$. Here we denoted the ground state of the FC model by the symbol $\psi_{E_{\text{FC}}}$. We meant the state $| \uparrow, 0 \rangle$ by $|\uparrow, 0\rangle$ and $| -1, 0 \rangle$ by $|\downarrow, 0\rangle$. So, for the state we set transition probability amplitudes $A_{g_{-1}}^0(\pm, \varepsilon)$ and $B_{g_{\text{ref}}}^{0, -1}(\pm, -0)$ as $A_{g_{-1}}^0(\pm, \varepsilon) := \varepsilon \langle n, -1 | -\sigma_x \psi_{E_{\text{FC}}}/g/\varepsilon, \psi_{E_{\text{FC}}}/g, 0 \rangle$ and $B_{g_{-1}}^{0, -1}(\pm, -0) := 0_{g/\varepsilon}(n, -1 | -\sigma_x \psi_{E_{\text{FC}}}/g, 0, 0, 0)$, respectively.

#### A. Weak and Strong Coupling Regimes

We will show the following estimates in Sec D.

$$1 - \frac{g^2}{\omega^2} \leq |A_{g_{-1}}^0(\pm, \varepsilon)|^2 + |B_{g_{-1}}^{0, -1}(\pm, -0)|^2 \leq 1.$$ (6.1)
Since the FC Hamiltonian $H_{FC}$ converges to the free Hamiltonian $(\hbar \omega/2)\sigma_z + h g (a\dagger a + 1/2)$ as $g \to 0$ in the norm resolvent sense, combining Lemma 4.9 of Ref. [61] or Theorem VIII.23 of Ref. [62] with the fact (E1), we have the limits:

$$\lim_{g \to 0} |A_n^{g}(\pm, \epsilon)|^2 = 1 \quad \text{and} \quad \lim_{g \to 0} |B_n^{g/\epsilon}(\pm, 0)|^2 = 0 \quad (6.2)$$

in the weak coupling regime determined by the condition (E1). Therefore, the estimates (6.1) and the limits (6.2) show the limit: $\lim_{g \to 0} \psi_{E_{FC}} = |\uparrow, 0\rangle$. Namely, we have proved that the ground state of the FC model is well approximated with that of the (modified) JC model in the weak and strong coupling regimes as well as the ground state energy.

### B. Ultra-Strong Coupling Regime

Eq. (2.12) leads to the inequalities:

$$\begin{align*}
0 \leq |A_n^{g}(\pm, \epsilon)|^2 &\leq \min \left\{ 1, \frac{E_{\text{diff}}^{g}(\epsilon)}{E_{\pm,n}(\epsilon) - E_{-n,n}(\epsilon)} \right\}, \\
0 \leq |B_n^{g/\epsilon}(\pm, 0)|^2 &\leq \min \left\{ 1, \frac{E_{\text{diff}}^{g}(\epsilon)}{\epsilon (E_{\pm,n}^{g/\epsilon}(0) - E_{-n,n}^{g/\epsilon}(0))} \right\}.
\end{align*}$$

Let us fix any superradiance indices $|{\nu}_n, {\nu}_*\rangle |(\epsilon) = |n + 1, n, n* + 1\rangle |(\epsilon)$ now. Then, we have the limit $\lim_{n \to \infty} |E - E_{\pm,n}^{g}(\epsilon)| = \lim_{n \to \infty} |E - E_{n,n}^{g}(\epsilon)| = \infty$ for energy $E = E_{\pm,n}^{\gamma}(\epsilon)$ by using the concrete expressions of the eigenstate energies $E_{\pm,0}(\epsilon)$ and $E_{\pm,n}(\epsilon)$. Similarly, we have the limit $\lim_{n \to \infty} |E - E_{\pm,n}^{g/\epsilon}(0)| = \lim_{n \to \infty} |E - E_{\pm,n}^{g/\epsilon}(0)| = \infty$ for energy $E = E_{\pm,n}^{g/\epsilon}(0)$. Since the energy difference $E_{\text{diff}}^{g}(\epsilon)$ is independent of the natural number $n$, we can conclude from inequalities (6.3) that the transition probabilities $|A_n^{g}(\pm, \epsilon)|^2$ and $|B_n^{g/\epsilon}(\pm, 0)|^2$ concentrate around the superradiant ground states:

$$\begin{align*}
\lim_{n \to \infty} |A_n^{g}(\pm, \epsilon)|^2 &= 0, \\
\lim_{n \to \infty} |B_n^{g/\epsilon}(\pm, 0)|^2 &= 0.
\end{align*}$$

Therefore, the number of photons that the ground state of the FC model is dressed with is determined by the number of photons that the states around the superradiant ground state of the modified JC model are.

At the tail end of this section we give a remark on the accuracy of the numerical analysis of the ground state energy of the FC model. The fact (1.2) says that the superradiant ground state energy of the modified JC Hamiltonian $H_{JC}^{\gamma}(\epsilon)$ appears for the coupling constant $g \sim \sqrt{n}$, as Preparata predicted in Ref. [43] (also see Ref. [44]). Combining inequalities (2.14) and (2.2), we can say that the superradiant ground state is the state of the quasi-particle dressed with about $n_\gamma$ photons when the coupling strength is so strong that the inequalities $G_{n-1}^{g}(\epsilon) < g^2 < G_{n+1}^{g}(\epsilon)$, hold. Let us fix any coupling constant $g$ satisfying these inequalities, and denote it by $g_{n\gamma}$. Namely, $G_{n-1}^{g}(\epsilon) < g_{n\gamma}^2 < G_{n+1}^{g}(\epsilon)$, which implies the equation:

$$\lim_{n \to \infty} \frac{g_{n\gamma}^2}{n} = 4(1 - \epsilon)^2 \omega_c^2$$

for the modified JC Hamiltonian $H_{JC}^{\gamma}(\epsilon)$. Similarly, we have the limit

$$\lim_{n \to \infty} \frac{g_{n\gamma}^2}{n} = 4\epsilon^2 \omega_c^2$$

for the modified JC Hamiltonian $H_{JC}^{\gamma}(0)$. Let us take 0.5 as the parameter $\epsilon$ and assume the condition $\omega_h = \omega_c$. Then, the concentration of the superradiant ground state together with the limits (6.4) and (6.5) says that for the given coupling constant $g$ we have to handle an $n \times n$-matrix with $n \approx g^2/\omega^2$ when we numerically calculate the ground state energy of the FC model. Namely, if we use the $n \times n$-matrix for the numerically-calculated ground state energy, the energy becomes inaccurate after the about dimensionless coupling strength $g/\omega = \sqrt{n}$. For example, see Figs 7(a) and (b).

#### FIG. 7. Set $\omega = \omega_h = \omega_c$, $e_{\text{upp}}(g)/\hbar \omega$ (red solid line), numerically-calculated ground state energy (blue dotted line). (a) $n = 100$ ($\sqrt{n} = 10$); (b) $n = 200$ ($\sqrt{n} = 14$).

the inequalities (2.5) the ground state energy $E_{\text{FC}}$ has to be less than or equal to the upper bound $e_{\text{upp}}(g)$. However, the numerically-calculated ground state energy has inaccuracy like $e_{\text{upp}}(g) < E_{\text{FC}}$ for the range of the coupling strength after the dimensionless coupling strength $g/\omega = \sqrt{n}$.

### VII. CONCLUSION

We have proposed a Hamiltonian formalism by decomposing the FC Hamiltonian into the modified JC Hamiltonian and its chiral-counter Hamiltonian. The two Hamiltonians play the chirality in the FC Hamiltonian. We have characterized the ground state energy of the FC Hamiltonian in all regimes by using our decomposition and estimating the non-commutativity between the two decomposed Hamiltonians. That is, we have shown that, in the weak or strong coupling regime, the ground state energy of the FC model can be well approximated by that of the (modified) JC Hamiltonian. In this process, there is no effect of the chirality associated with the DP.
superradiance. On the other hand, we have shown that, in the ultra-strong coupling regime, the effect of the chirality is turned on and works, and moreover, it increases as the coupling strength grows large. In this process, a phase transition concerning the chirality mathematically appears when the atom-cavity system plunges into the ultra-strong coupling regime from the strong coupling one. If the phase transition is experimentally observed, it means that the DP superradiance is indirectly observed.

We have employed the upper bound $e_{upp}(\omega)$ for the ground state energy $E_{FC}$ of the FC Hamiltonian in this paper to estimate the energy difference $E_{diff}(\varepsilon)$ between the energies $E_{FC}$ and $E_{low}(\varepsilon)$. The latter is given by the sum of the ground state energies of the modified JC Hamiltonian and its chiral-counter Hamiltonian. In our method, however, if we can find a better upper bound than the bound $e_{upp}(\omega)$, we can make the estimate better by using the better bound instead of the bound $e_{upp}(\omega)$.

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**Appendix A: Proof of Eqs. (2.5) and (2.6)**

The inequalities (2.4) immediately lead to the limits for the asymptotic behavior of the ground state $E_{FC}$: $\lim_{g \to 0} E_{FC} = -\hbar(\omega / \omega_c)/2$ and $\lim_{g \to \infty} E_{FC}/(\hbar^2/\omega_c) = 1$. The former and latter respectively imply the first and second of Eqs. (2.5).

To prove the third of Eqs. (2.5) and Eq. (2.6), we follow the method to prove Theorem 10 of Ref. [50] on the spin-boson Hamiltonian. As in Eq. (2.14) of Ref. [50], using the unitary operator $U_0 := (1/\sqrt{2})(\sigma_0 - i\sigma_y)$, we have the representation:

$$U_0^* H_{FC} U_0 = \hbar \omega_c \left( a^\dagger a + 1 \right) + \hbar g \sigma_z (a + a^\dagger) + \frac{\hbar \omega_c}{2} \sigma_x.$$

We apply the perturbation theory to the operator $U_0^* H_{FC} U_0/\hbar g$. Namely, we regard the term $-\hbar \omega_c/2g + \hbar g \sigma_z (a + a^\dagger)$ as a perturbation for the operator $(\omega / \omega_c)(a^\dagger a + 1/2) + \sigma_x (a + a^\dagger)$. Then, the infimum eigenvalue $\omega_c/2g - 1/(\omega_c/\hbar g)$ splits into two eigenvalues: $\omega_c/2g - 1/(\omega_c/\hbar g) \pm (\omega_c/2g)e^{-2(\omega_c/\hbar g)^2} + o(\omega_c/\hbar g)^2$. Multiplying this by $\hbar g$ and judging from the inequalities (2.4), we obtain the expressions:

$$E_{FC} = \frac{\hbar \omega_c}{2} - \frac{\hbar g^2}{\omega_c} e^{-2(\omega_c/\hbar g)^2} + \hbar g \times \frac{\hbar \omega_c}{2} \left( \frac{\omega_c}{\hbar g} \right)^2;$$

$$E_{FC}^{1st} = \frac{\hbar \omega_c}{2} - \frac{\hbar g^2}{\omega_c} + \frac{\hbar \omega_c}{2} e^{-2(\omega_c/\hbar g)^2} \left( \frac{\omega_c}{\hbar g} \right)^2 + \hbar g \times \frac{\hbar \omega_c}{2} \left( \frac{\omega_c}{\hbar g} \right)^2.$$

Therefore, we eventually reach the third of Eqs. (2.5) and Eq. (2.6). The former says that the limit $G(\omega) \to 0$ holds as $g \to \infty$.

**Appendix B: Proof of Inequality (2.15)**

Applying the inequalities (2.66) in Lemma 2.2 of Ref. [38], we have the following order estimates:

$$\begin{aligned}
- \frac{1}{1 - \varepsilon} &\leq \liminf_{g \to \infty} \frac{E_{FC}^{1st}(\varepsilon)}{\hbar^2 / \omega_c} \\
- \frac{1}{\varepsilon} &\leq \liminf_{g \to \infty} \frac{E_{FC}(\varepsilon)}{\hbar^2 / \omega_c}
\end{aligned}$$

(B1)

These estimates imply the following inequality:

$$0 \leq \limsup_{g \to \infty} \frac{E_{low}(\varepsilon)}{\hbar^2 / \omega_c} \leq \frac{1}{\varepsilon(1 - \varepsilon)}. \quad (B2)$$

Meanwhile, combining the definition of the upper bound $e_{upp}(\omega)$ in the inequality (2.4) with the inequalities (2.13) and (151), we reach the following bound:

$$0 \leq \limsup_{g \to \infty} E_{diff}(\varepsilon) \leq \frac{1}{\varepsilon(1 - \varepsilon)}. \quad (B3)$$

Thus, we know that the order of the energy difference $E_{diff}(\varepsilon)$ is given by $E_{diff}(\varepsilon) = O(\varepsilon^2)$ as $g \to \infty$. Therefore, we eventually obtain the order estimate (2.15).

**Appendix C: Proof of the 1st of the facts (4.6)**

We give a proof of the first statement of the facts (4.6) here. Set the number $\delta := (1 - \varepsilon)\omega_c$ for simplicity. Define a function $g(r)$ of a real variable $r$ by $g(r) := r^2 + (r - (4n + 1)r - (\Delta c / \delta)^2$. Then, the two solutions $r_{\pm}$ of $g(r) = 0$ are $r_{\pm} = \left(2n + \frac{1}{2} \pm \sqrt{(2n + \frac{1}{2})^2 + (\Delta c / \delta)^2} \right.$ Since one of them is negative (i.e., $r_ - < 0$), if $r$ satisfies the condition $0 \leq r < r_+$, then $g(r) < g(r_+) = 0$. Inserting $(g/\delta)^2$ into $r$ in the above, we have the following statement: the condition $(g/\delta)^2 < 2n + \frac{1}{2} + \sqrt{(2n + \frac{1}{2})^2 + (\Delta c / \delta)^2}$ implies the inequality $(g/\delta)^2 < 4(n + 1)(g/\delta)^2 + (\Delta c / \delta)^2$. 

Multiplying both sides of the second inequality in the above by $4(\delta/g)^2$, the second one implies $4(g/\delta)^2 < 4(4n + 1) + 4(\Delta_\varepsilon)^2$. The right hand side of this is decomposed and bounded as: $(\Delta_\varepsilon/g)^2 + 4n + 2\{ (\Delta_\varepsilon/g)^2 + 4n \} < (\Delta_\varepsilon/g)^2 + 4(n + 1) + (\Delta_\varepsilon/g)^2 + 4n + 2\sqrt{(\Delta_\varepsilon/g)^2 + 4n} = \left\{ \sqrt{(\Delta_\varepsilon/g)^2 + 4(n + 1)} + \sqrt{(\Delta_\varepsilon/g)^2 + 4n} \right\}^2$. Eventually, we obtain the mathematical statement:

$$\left( \frac{g}{\delta} \right)^2 < 2n + 1 + \sqrt{\left( 2n + 1 \right)^2 + \left( \frac{\Delta_\varepsilon}{g} \right)^2}$$

implies

$$2\left| \frac{g}{\delta} \right| < \sqrt{\left( \frac{\Delta_\varepsilon}{g} \right)^2 + 4(n + 1) + \left( \frac{\Delta_\varepsilon}{g} \right)^2 + 4n}.$$  

(C1)

Meanwhile, it is easy to show that the mathematical statement:

$$E_{\varepsilon,n}^- (\varepsilon) \leq E_{\varepsilon,n}^+ (\varepsilon)$$

is equivalent to

$$2\left| \frac{g}{\delta} \right| < \sqrt{\left( \frac{\Delta_\varepsilon}{g} \right)^2 + 4(n + 1) + \left( \frac{\Delta_\varepsilon}{g} \right)^2 + 4n},$$  

(C2)

where the symbol $\leq$ denotes the mathematical symbol $<, =$, or $>$. Combining the two statements (C1) and (C2), we can complete the proof of our desired result.

Appendix D: Proof of estimate (6.1)

In this appendix we give a mathematical proof of the estimate (6.1) from the below. We define the orthogonal projection operator $P_n$ by

$$P_n \psi := \langle n | \psi \rangle | n \rangle$$

for the normalized photon state $| n \rangle$. Then, we have the expression of the number operator $N := a^\dagger a$ as

$$N = \sum_{n=0}^\infty n P_n = \sum_{n=1}^\infty n P_n,$$

so that the equation $P_0 + N = P_0 + \sum_{n=1}^\infty n P_n$, which implies the operator inequality $P_0 + N = \sum_{n=0}^\infty n P_n = 1$. Thus, we reach the operator inequality:

$$P_0 \geq 1 - N.$$  

(D1)

By Lemma 4.3 of Ref. [61], we have the inequality:

$$\langle \psi_{EFC} | N | \psi_{EFC} \rangle \leq \frac{g^2}{\omega^2},$$  

(D2)

where the vector $\psi_{EFC}$ was the normalized ground state of the FC model. It is proved in the following. Using the commutator $[H_{FC}, a] \psi_{EFC} = (H_{FC} - E_{FC}) a \psi_{EFC}$ with $[H_{FC}, a] = -i \hbar \sigma_a$, we reach the so-called pull-through formula:

$$a \psi_{EFC} = -i \hbar (H_{FC} - E_{FC}) \sigma_a \psi_{EFC}$$

Applying this pull-through formula to the term $\langle \psi_{EFC} | a \dagger a \psi_{EFC} \rangle$ and using the Schwarz inequality, we obtain the inequality (D2). The two inequalities (D1) and (D2) imply the lower bound:

$$\langle \psi_{EFC} | \sigma_0 P_0 | \psi_{EFC} \rangle \geq 1 - \frac{g^2}{\omega^2}.$$  

(D3)

On the other hand, the Schwarz inequality brings the upper bound:

$$\langle \psi_{EFC} | \sigma_0 P_0 | \psi_{EFC} \rangle \leq \langle \psi_{EFC} | \psi_{EFC} \rangle^{1/2} \langle \sigma_0 P_0 \psi_{EFC} | \sigma_0 P_0 \psi_{EFC} \rangle^{1/2} = \langle \psi_{EFC} | \psi_{EFC} \rangle = 1.$$  

(D4)

Since the equation $| A_{\varepsilon,1}^\pm (\varepsilon) |^2 + | B_{\varepsilon,0}^\pm (\varepsilon) |^2 = \langle \psi_{EFC} | \sigma_0 P_0 | \psi_{EFC} \rangle$ follows from the straightforward computation, we are finally obtaining our desired estimate (6.1) by the bounds (D3) and (D4).

Appendix E: Proof of Estimates $E_{\text{dir}}(\varepsilon)$

1. In the Case $[0,0][\varepsilon)$

In the condition $[5.1]$, equivalent to the superradiance indices $[0,0][\varepsilon)$, the following estimates are derived from the inequalities (2.13) and (2.14):

$$E_{\text{dir}}^\varepsilon (\varepsilon) = E_{\text{low}}(\varepsilon, \varepsilon) \leq E_{\text{FC}} \leq 0,$$

for $0 \leq \varepsilon \leq \varepsilon_\omega$ with $0 \leq \varepsilon < 1/\sqrt{2}$.  

(E1)

Since the equations $E_{\text{low}}(\varepsilon, \varepsilon) = E_{\text{dir}}^\varepsilon (\varepsilon) = -\varepsilon \omega \hbar$ hold for $[0,0][\varepsilon)$, it is immediately follows form the definition of the upper bound $E_{\text{ubd}}(\varepsilon)$ and the estimates (2.13) that

$$0 \leq E_{\text{dir}}(\varepsilon) \leq E_{\text{ubd}}(\varepsilon) = (\varepsilon + \frac{1}{2} - \frac{1}{2} e^{-2\varepsilon^2/\omega^2}) \varepsilon \omega \hbar = -h^2 \omega.\hbar.$$  

The region of the bare coupling constant $g$ in the estimates (E1) leads to that of the renormalized coupling constant $g_{\text{WS}}(\varepsilon)$ as $0 \leq g_{\text{WS}}(\varepsilon) \leq \sqrt{\varepsilon} \omega \hbar$ with $0 \leq \varepsilon < 1/\sqrt{2}$. Since we assumed that the observed coupling constant $g_{\text{WS}}$ satisfies the condition $(g_{\text{WS}}/\omega)^2 < 1/\sqrt{2}$, the quantity $\varepsilon_{\text{WS}}$ satisfies the condition $0 \leq \varepsilon_{\text{WS}} < 1/\sqrt{2}$. Since $g_{\text{WS}}(\varepsilon_{\text{WS}}) = g_{\text{WS}}$, we have the equation $g = \sqrt{\varepsilon_{\text{WS}}^2 g_{\text{WS}}^2} = g_{\text{WS}}^2 / \omega$. Meanwhile, we have the equation $\varepsilon_{\text{WS}} g_{\text{WS}} = g_{\text{WS}}^2 / \omega$. Thus, the quantity $\varepsilon_{\text{WS}}$ satisfies the last condition of the fact (E1) and we can take $\varepsilon_{\text{WS}}$ as the parameter $\varepsilon$, i.e., $\varepsilon = \varepsilon_{\text{WS}}$. Then, the first part of the fact (E1) says that $-h^2 g_{\text{WS}}^2 / \omega = -\varepsilon_{\text{WS}} \hbar \omega = E_{\text{dir}}^\varepsilon (\varepsilon_{\text{WS}}) \leq E_{\text{FC}}$, which is the lower bound of the estimates (6.3).

2. In the Case $[0,\nu](\varepsilon)$ with $\nu_\varepsilon < 0$

In this case we have the value of integers $n_\varepsilon$ and $n_{\varepsilon_*}$ as:

$$n_{\varepsilon} = -1 \quad \text{and} \quad n_{\varepsilon_*} = 0, 1, 2, \ldots,$$

Using the concrete expressions of the lower bound $E_{\text{lb}}(\varepsilon)$ and the upper bound $E_{\text{ubd}}(\varepsilon)$, we have their other expressions:

$$E_{\text{lb}}(\varepsilon) = \hbar \omega \left[ -\varepsilon n_{\varepsilon_*} + \frac{g}{\omega} \sqrt{n_{\varepsilon_*} + 1} - \frac{g^2}{\omega^2} \right],$$

$$E_{\text{ubd}}(\varepsilon) = \hbar \omega \left[ \frac{1}{2} - \varepsilon n_{\varepsilon_*} + \frac{g}{\omega^2} \sqrt{n_{\varepsilon_*} + 1} - \frac{g^2}{\omega^2} \right. - \left. \frac{1}{2} e^{-2\varepsilon^2/\omega^2} \right],$$

$$n_{\varepsilon} = -1; n_{\varepsilon_*} = 0, 1, 2, \ldots.$$
with the help of the inequalities (2.13). Then, these equations lead to the inequalities:

\[
\begin{align*}
-\varepsilon \hbar \omega n_{**} + \hbar \omega \epsilon_{low}^{1-n_{**}} \left( \frac{g}{\omega} \right) &= E_{\text{lb}}(\varepsilon), \\
E_{\text{ub}}(\varepsilon) &= -\varepsilon \hbar \omega + \hbar \omega \epsilon_{upp}^{1-n_{**}} \left( \frac{g}{\omega} \right) \\
&\leq \varepsilon \hbar \omega \\
&+ \left[ \min \left\{ \sqrt{\frac{n_{**} + 1}{g} \omega}, \epsilon_{upp}^{1-n_{**}}(G_{-1,n_{**}}) \right\} \right] \hbar \omega.
\end{align*}
\] (E2)

The inequalities (E2) imply the results, (5.9), (5.10), (5.11), and (5.12).

In the case where \( n_{**} = 0, 1, 2, \cdots \), similarly to the above, we obtain the expressions of the lower bound \( E_{\text{lb}}(\varepsilon) \) and the upper bound \( E_{\text{ub}}(\varepsilon) \) as:

\[
\begin{align*}
E_{\text{lb}}(\varepsilon) &= \hbar \omega \left[ -(n_{*} + 1) - \varepsilon(n_{**} - n_{*}) \right] \\
&\quad + \frac{g}{\omega} \sqrt{n_{**} + 1 - \frac{g^2}{\omega^2}}, \\
E_{\text{ub}}(\varepsilon) &= \hbar \omega \left[ \frac{1}{2} - (n_{*} + 1) - \varepsilon(n_{**} - n_{*}) \right] \\
&\quad + \frac{g}{\omega} \sqrt{n_{**} + 1 - \frac{g^2}{\omega^2}} - \frac{1}{2} - \frac{2g^2}{\omega^2} \right],
\end{align*}
\] (E3)

\( n_{*, n_{**}} = 0, 1, 2, \cdots \).

Then, using the functions and Eqs. (12), we can estimate the lower bound \( E_{\text{lb}}(\varepsilon) \) and the upper bound \( E_{\text{ub}}(\varepsilon) \) as:

\[
\begin{align*}
\varepsilon \hbar \omega (n_{*} - n_{**}) + \hbar \omega \epsilon_{low}^{n_{*, n_{**}}} \left( \frac{g}{\omega} \right) \leq E_{\text{lb}}(\varepsilon), \\
E_{\text{ub}}(\varepsilon) &\leq \varepsilon \hbar \omega (n_{*} - n_{**}) + \hbar \omega \epsilon_{upp}^{n_{*, n_{**}}} \left( \frac{g}{\omega} \right)
\end{align*}
\] (E4)

for the superradiance indices \(||\nu_{*}|, \nu_{**}||(\varepsilon) = [n_{*} + 1, n_{**} + 1](\varepsilon)\), which implies the result (5.13).

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