Congruence relations satisfied by quaternionic modular forms

Shoyu Nagaoka

Received: 4 November 2021 / Accepted: 7 February 2023 / Published online: 20 March 2023
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Abstract
The theory of quaternionic modular forms has been studied for decades as an example of the modular forms of many variables. The purpose of this study is to provide some congruence relations satisfied by such quaternionic modular forms.

Keywords Quaternionic modular forms · Eisenstein series

Mathematics Subject Classification 11F33 · 11F55

1 Introduction
The basis of the theory of quaternionic modular forms can be found in a study by Krieg [5], who developed the theory. For example, he analyzed the structure of the (quaternionic) Maass space and succeeded in obtaining an explicit formula of the Fourier coefficient of the quaternionic Eisenstein series of degree 2. Furthermore, he determined the structure of the graded ring of quaternionic modular forms, also in the case of degree 2. By contrast, except for [3], the $p$-adic and mod $p$ theories have not been fully studied.

In this paper, we refer to the arithmetic properties of quaternionic modular forms and provide the following results on the mod $p$ theory of quaternionic modular forms:

- Ramanujan-type congruences (Theorem 3.2)
- Congruences mod 23 (Theorem 3.5)
- Congruences of Eisenstein series (Theorem 3.8)

This work was supported by JSPS KAKENHI: Grants-in-Aid (C)(No. 20K03547).

Shoyu Nagaoka
shoyu122.sn@gmail.com

1 Department of Mathematics, Yamato University, Suita, Osaka 564-0082, Japan
2 Preliminaries

The notations used in this study are from Krieg’s study [7].

We denote by \( \mathcal{O} \) the order of Hurwitz quaternions, i.e.,

\[
\mathcal{O} = \mathbb{Z}e_0 + \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3,
\]

where \( e_0 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4) \) and \( 1 = e_1 e_2 e_3 e_4 \) is the standard basis for real quaternions. Let \( \mathcal{O}^\sharp \) be the dual lattice, that is,

\[
\mathcal{O}^\sharp = \mathbb{Z}2e_1 + \mathbb{Z}(e_1 + e_2) + \mathbb{Z}(e_1 + e_3) + \mathbb{Z}(e_1 + e_4).
\]

If \( \tau \) denotes the reduced trace form, then the elements of the dual lattice of \( \text{Sym}(2; \mathcal{O}) = \{ S \in \text{Mat}(2; \mathcal{O}) \mid S = t^S \} \) with respect to \( \tau \) is given in the following form:

\[
T = \begin{pmatrix} n \frac{1}{2} t \\ \frac{1}{2} m \end{pmatrix} \in \text{Sym}^\tau(2; \mathcal{O}),
\]

where \( n, m \in \mathbb{Z}, t \in \mathcal{O}^\sharp \). Given \( 0 \neq T \in \text{Sym}^\tau(2; \mathcal{O}) \), the “greatest common divisor” of \( T \) is defined by

\[
\varepsilon(T) := \max \{ d \in \mathbb{N} \mid d^{-1} T \in \text{Sym}^\tau(2; \mathcal{O}) \}.
\]

Let \( [\Gamma_2(\mathcal{O}), k] \) denote the vector space consisting of modular forms of weight \( k \) for the quaternionic modular group \( \Gamma_2(\mathcal{O}) := \text{Sp}(2, \mathcal{O}) \) and \( [\Gamma_2(\mathcal{O}), k]_0 \) denote the subspace of the cusp forms.

Each quaternionic modular form \( f \in [\Gamma_2(\mathcal{O}), k] \) \((k \equiv 0 \pmod{2})\) possesses a Fourier expansion of the form

\[
f(Z) = \sum_{T \in \text{Sym}^\tau(2; \mathcal{O})} a(f; T) e^{2\pi i \tau(T, Z)}, \quad Z \in H(2; \mathbb{H}),
\]

where \( H(2; \mathbb{H}) \) is the quaternionic upper-half space of degree 2. For a subring \( R \subset \mathbb{C} \), we denote by \( [\Gamma_2(\mathcal{O}), k; R] \) the set consisting of \( f \in [\Gamma_2(\mathcal{O}), k] \) with \( a(f; T) \in R \) for all \( T \).

A modular form \( f \in [\Gamma_2(\mathcal{O}), k] \) belongs to the Maass space \( \mathcal{M}(k, \mathbb{H}) \) of weight \( k \) if and only if a function \( a_f^* : \mathbb{N} \cup \{0\} \to \mathbb{C} \) exists such that all \( 0 \neq T \in \text{Sym}^\tau(2; \mathcal{O}) \) \( T \geq 0 \) satisfy

\[
a(f; T) = \sum_{d \in \mathbb{N}} d^{-k-1} a_f^*(2\text{det}(T)/d^2).
\]
According to Andrianov [1], this condition can be equivalently replaced by the Maass relation

\[ a(f; T) = a\left(f; \left(\frac{m}{t/2}, \frac{n}{t/2}\right)\right) = \sum_{\substack{d \in \mathbb{N} \\mid \epsilon(T) \\backslash d}} d^{k-1} a\left(f; \left(\frac{1}{t/2d}, \frac{mn}{d^2}\right)\right). \]

We set \( \mathcal{M}(k, \mathbb{H}; R) = \mathcal{M}(k, \mathbb{H}) \cap [\Gamma_2(\mathcal{O}), k; R] \) and \( \mathcal{M}(k, \mathbb{H}; R)_0 = \mathcal{M}(k, \mathbb{H}) \cap [\Gamma_2(\mathcal{O}), k; R]_0 \).

A typical example of an element in \( \mathcal{M}(k, \mathbb{H}) \) is the (quaternionic) Eisenstein series \( E_{k, \mathbb{H}} \) \((k > 6, \text{ even})\).

**Theorem 2.1** (Krieg [7, Theorem 3]) Let \( k > 6 \) be an even integer. The Eisenstein series \( E_{k, \mathbb{H}} \) is an element of \( \mathcal{M}(k, \mathbb{H}; \mathbb{Q}) \) and has the Fourier expansion

\[ E_{k, \mathbb{H}}(Z) = 1 + \sum_{T \in \text{Sym}^\tau(2; \mathcal{O}), T \geq 0, T \neq 0} a(E_{k, \mathbb{H}}; T) e^{2\pi i T Z}, \]

where

\[ a(E_{k, \mathbb{H}}; T) = \sum_{\substack{d \in \mathbb{N} \\mid \epsilon(T) \\backslash d}} d^{k-1} a^*(2\det(T)/d^2) \]

for \( 0 \neq T \in \text{Sym}^\tau(2, \mathcal{O}), T \geq 0, \) and

\[ a_k^*(\ell) = \begin{cases} -\frac{2k}{B_k} & \text{if } \ell = 0 \\ \frac{4k(k-2)}{(2^{k-2} - 1) B_k B_{k-2}} [\sigma_{k-3}(\ell) - 2^{k-2} \sigma_{k-3}(\ell/4)] & \text{if } \ell \in \mathbb{N}. \end{cases} \]

Here, \( B_m \) is the \( m \)-th Bernoulli number and

\[ \sigma_m(\ell) = \begin{cases} \sum_{d | \ell} d^m & \text{if } \ell \in \mathbb{N}, \\ 0 & \text{otherwise}. \end{cases} \]

**Remark 2.2** The condition \( "k > 6" \) is necessary for the convergence of the Eisenstein series. However, it is known that the above formula for the Fourier coefficients holds when \( k \) is 4 or 6. This is justified by the so-called Hecke trick. See also [7, Remark 1].

Let \( E_k \) denote the elliptic Eisenstein series of weight \( k \) whose Fourier expansion is

\[ E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi inz}, \quad z \in \mathbb{H}_1, \]

where \( \mathbb{H}_1 \) is the upper-half plane.
For the Siegel operator $\Phi$, we obtain

$$
\Phi(E_{k,\mathbb{H}}) = E_k
$$

when $k \geq 4$ is even.

Herein, we introduce examples of cusp forms.

We set

$$
X_{10} := \frac{17}{161280} \left( E_{4,\mathbb{H}}E_{6,\mathbb{H}} - E_{10,\mathbb{H}} \right),
$$

(2.1)

$$
X_{12} := \frac{21421}{203212800} \left( \frac{441}{691} E_{4,\mathbb{H}}^3 + \frac{250}{691} E_{6,\mathbb{H}}^2 - E_{12,\mathbb{H}} \right).
$$

(2.2)

Herein, these cusp forms are normalized as

$$
a \left( X_{10}; \left( \frac{1}{2} \begin{array}{c} e_1 + e_2 \\ e_1 - e_2 \end{array} \right) \right) = a \left( X_{12}; \left( \frac{1}{2} \begin{array}{c} e_1 + e_2 \\ e_1 - e_2 \end{array} \right) \right) = 1
$$

and

$$
X_k \in \mathcal{M}(k, \mathbb{H}; \mathbb{Z}_0) \quad (k = 10, 12).
$$

These forms are part of the set of generators of the graded ring of the quaternionic modular forms of degree 2 constructed by Krieg [8].

### 3 Congruence properties of quaternionic modular forms

In this section, we provide some results regarding the congruence properties of quaternionic modular forms of degree 2.

To study the congruence properties of the Eisenstein series, we consider a constant multiple of $E_{k,\mathbb{H}}$:

$$
G_{k,\mathbb{H}} := -\frac{(2^{k-2} - 1) B_k B_{k-2}}{4k(k-2)} E_{k,\mathbb{H}}.
$$

(3.1)

The following result is a simple consequence of Theorem 2.1 and Remark 2.2.

**Proposition 3.1** Assume that $k \geq 4$ is even.

(1) We thus assume that $T \neq O_2$. Then,

$$
a(G_{k,\mathbb{H}}: T) = \sum_{d|\varepsilon(T)} d^{k-1} b_k^* (2\text{det}(T)/d^2)
$$

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and

\[ b_k^*(\ell) = \begin{cases} 
\frac{(2^{k-2} - 1) B_{k-2}}{2(k-2)} & \text{if } \ell = 0 \\
\frac{\sigma_{k-3}(\ell) - 2^{k-2} \sigma_{k-3}(\ell/4)}{2k} & \text{if } \ell \in \mathbb{N}
\end{cases} \]

(2) \( a(G_k, H; O_2) = -\frac{(2^{k-2} - 1) B_k B_{k-2}}{4k(k-2)} \).

### 3.1 Ramanujan-type congruence

Let

\[ \Delta(z) = q \prod_{n \geq 1} (1 - q^n)^{24} \quad (q = e^{2\pi iz}) \]

be the delta function, which is an elliptic cusp form of weight 12. The Ramanujan tau function \( \tau(n) \ (n \in \mathbb{N}) \) is defined by

\[ \Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n. \]

Ramanujan’s congruence

\[ \tau(n) \equiv \sigma_{11}(n) \pmod{691} \]

is interpreted as a congruence relation between the weight 12 elliptic Eisenstein series \( G_{12} \) and \( \Delta \) as follows:

\[ G_{12} \equiv \Delta \pmod{691}, \quad (3.2) \]

where \( G_k \) is defined as

\[ G_k := -\frac{B_k}{2k} E_k. \]

The following provides some results, which are the quaternionic version of (3.2).

**Theorem 3.2** Let \( G_{k, H} \) be the Eisenstein series defined in (3.1). If a prime number \( p \geq 5 \) satisfies the conditions

\[ \text{ord}_p \left( \frac{(2^{k-2} - 1) B_{k-2}}{k-2} \right) > 0, \quad \text{ord}_p \left( \frac{B_k}{k} \right) \geq 0, \]

then there is a quaternionic cusp form

\[ \chi_k \in [\Gamma_2(O), k; \mathbb{Z}(p)]_0 \]
such that

\[ G_{k, \mathbb{H}} \equiv \chi_k \pmod{p}, \]

where \( \mathbb{Z}_{(p)} \) denotes the ring of \( p \)-integral rational numbers.

**Proof** We apply the Siegel \( \Phi \)-operator to \( G_{k, \mathbb{H}} \). Based on the assumption (*) and Proposition 3.1, \( \Phi(G_{k, \mathbb{H}}) \) is an elliptic modular form of weight \( k \), whose Fourier coefficients are all divisible by \( p \). Hence, we can write

\[ \Phi(G_{k, \mathbb{H}}) = p \cdot f, \]

where \( f \) is an elliptic modular form of weight \( k \) with \( p \)-integral Fourier coefficients. Because \( f \) has \( p \)-integral Fourier coefficients, we can write

\[ f = P(E_4, E_6) \quad \text{with} \quad P(X_1, X_2) \in \mathbb{Z}_{(p)}[X_1, X_2]. \]

We then set

\[ F := P(E_{4, \mathbb{H}}, E_{6, \mathbb{H}}). \]

Herein, we note that both \( E_{4, \mathbb{H}} \) and \( E_{6, \mathbb{H}} \) have \( p \)-integral Fourier coefficients and \( \Phi(E_{k, \mathbb{H}}) = E_k \) (\( k = 4, 6 \)).

If we set

\[ \chi_k := G_{k, \mathbb{H}} - p \cdot F, \]

by the above construction, we then obtain \( \Phi(\chi_k) = 0 \) and \( G_{k, \mathbb{H}} \equiv \chi_k \pmod{p} \).

**Example 3.3** (1) When \( k = 10 \), we can take \( p = 17 \) as a prime number \( p \) that satisfies the condition (*) and \( \chi_{10} = X_{10} \), where \( X_{10} \) is defined in (2.1). We then obtain

\[ G_{10, \mathbb{H}} \equiv X_{10} \pmod{17}. \]

As numerical examples,

\[
\begin{align*}
1 &= a \left( G_{10, \mathbb{H}}; \left( \frac{e_1+e_2}{2} \right) \right) \equiv a \left( X_{10}; \left( \frac{e_1+e_2}{2} \right) \right) = 1 \pmod{17} \\
129 &= a \left( G_{10, \mathbb{H}}; \left( 0 \right) \right) \equiv a \left( X_{10}; \left( 0 \right) \right) = -24 \pmod{17} \\
2188 &= a \left( G_{10, \mathbb{H}}; \left( \frac{\epsilon_1+\epsilon_2}{2} \right) \right) \equiv a \left( X_{10}; \left( \frac{\epsilon_1+\epsilon_2}{2} \right) \right) = 12 \pmod{17} \\
&\vdots
\end{align*}
\]
(2) When \( k = 14 \), we can take \( p = 691 \). We then set

\[
X_{14} := E_{4, \mathbb{H}} X_{10}. \tag{3.3}
\]

We thus obtain

\[
G_{14, \mathbb{H}} \equiv X_{14} \pmod{691}. \tag{3.4}
\]

As numerical examples,

\[
1 = a \left( G_{14, \mathbb{H}}; \left( \frac{1}{e_1 - e_2^2} \right) \right) \equiv a \left( X_{14}; \left( \frac{1}{e_1 - e_2^2} \right) \right) = 1 \pmod{691}
\]

\[
2049 = a \left( G_{14, \mathbb{H}}; \left( \frac{1}{0} \right) \right) \equiv a \left( X_{14}; \left( \frac{0}{0} \right) \right) = -24 \pmod{691}
\]

\[
177148 = a \left( G_{14, \mathbb{H}}; \left( \frac{1}{e_1 - e_2^2} \right) \right) \equiv a \left( X_{14}; \left( \frac{1}{e_1 - e_2^2} \right) \right) = 252 \pmod{691}
\]

(See [9] for more numerical examples.)

As we will show later (Lemma 3.4), the Fourier coefficient \( a(X_{14}; T) \) is given as

\[
a(X_{14}; T) = \sum_{d \mid \varepsilon(T)} d^{13} \tau^*(2 \det(T)/d^2),
\]

\[
\tau^*(\ell) = \tau(\ell) - 2^{12} \tau(\ell/4),
\]

if \( \text{rank}(T) = 2 \). By contrast,

\[
a(G_{14, \mathbb{H}}; T) = \sum_{d \mid \varepsilon(T)} d^{13} b_{14}^*(2 \det(T)/d^2),
\]

\[
b_{14}^*(\ell) = \sigma_{11}(\ell) - 2^{12} \sigma_{11}(\ell/4).
\]

Therefore, the fact (3.4) reproduces the congruence relation \( \tau(n) \equiv \sigma_{11}(n) \pmod{691} \).

(3) Table 1 lists the prime numbers \( p \) that satisfy the condition (*)..

| Table 1 Primes that satisfy (*) |
|---|---|---|---|---|---|---|---|---|---|
| \( k \) | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| \( p \) | 17 | 31 | 691 | 43, 127 | 257, 3617 | 73, 43867 |

(See Springer)
3.2 Congruence mod 23

In the process of determining the ring structure of degree 2 Siegel modular forms, Igusa constructed a Siegel cusp form $X_{35}$ of odd weight 35. In [4], the congruence relation

$$\Theta(X_{35}) \equiv 0 \pmod{23}$$

was reported where $\Theta$ is the theta operator on Siegel modular forms defined by

$$\Theta(F) := \sum_T a(T) \cdot \det(T) e^{2\pi i \text{tr}(TZ)}$$

for $F = \sum_T a(T) e^{2\pi i \text{tr}(TZ)}$. This fact means that, if $\det(T) \not\equiv 0 \pmod{23}$, then the corresponding Fourier coefficient $a(X_{35}; T)$ is divisible by 23. After that, such Siegel modular forms were found one after another. For example,

$$\Theta(E_{12,S}) \equiv \Theta(\vartheta_L) \equiv 0 \pmod{23}$$

was proved, where $E_{12,S}$ is the weight 12 Eisenstein series for the Siegel modular group of degree 2, and $\vartheta_L$ is the Siegel theta series associated with the Leech lattice $L$, which is also a Siegel modular form of weight 12.

In this section, we provide some congruence relation mod 23 concerning quaternionic modular forms.

In (3.3), we consider a cusp form $X_{14}$ defined by $X_{14} = E_4, H X_{10}$.

**Lemma 3.4** The modular form $X_{14}$ is an element of the Maass space $\mathcal{M}(14; \mathbb{H})_0$, and the Fourier coefficient $a(X_{14}; T)$ is given as follows:

$$a(X_{14}; T) = \sum_{d \mid \epsilon(T)} d^{13} \tau^*(2\det(T)/d^2)$$

$$\tau^*(\ell) = \tau(\ell) - 2^{12} \tau(\ell/4), \quad (3.5)$$

where $\tau(n)$ is the Ramanujan tau function. In particular, $X_{14} \in \mathcal{M}(14; \mathbb{H}; \mathbb{Z})_0$.

**Proof** In [7, Proposition 3], Krieg constructed an isomorphism

$$\Omega : \mathcal{M}(k, \mathbb{H}) \to \mathcal{M}_{k-2},$$

where $\mathcal{M}_k$ is a subspace of $[\Gamma_0(4), k]$ defined by certain conditions on Hecke operators (cf. [7]). Under this isomorphism, we see the following:

$$\Omega(X_{14}) = \Delta(z) - 2^{12} \Delta(4z).$$

If we translate this fact into Fourier coefficients, we obtain (3.5). The second main result is as follows. 

\[ \square \]

\[ \square \] Springer
Theorem 3.5 Let $X_{14}$ be the cusp form defined above. If a matrix $T$ satisfies $\chi_{-23}(2\det(T)) = -1$, then the corresponding Fourier coefficient $a(X_{14}, T)$ satisfies

$$a(X_{14}; T) \equiv 0 \pmod{23},$$

where $\chi_{-23}(n) = \left(\frac{-23}{n}\right)$ is the Kronecker symbol.

Proof First, we note that the following fact regarding $\tau(n)$ holds: If a prime number $p$ satisfies $\left(\frac{p}{23}\right) = -1$, then $\tau(p) \equiv 0 \pmod{23}$.

We will now start the proof. Based on the previous Lemma, $a(X_{14}; T)$ is given as

$$a(X_{14}; T) = \sum_{d | \varepsilon(T)} d^{13} \tau^*(2\det(T)/d^2).$$

We show that $\tau^*(2\det(T)/d^2) \equiv 0 \pmod{23}$ for any $d$ with $d | \varepsilon(T)$.

The assumption $\chi_{-23}(2\det(T)) = -1$ implies $\chi_{-23}(2\det(T)/d^2) = -1$, and thus it is sufficient to show the following:

(**) If $\ell \in \mathbb{N}$ satisfies $\chi_{-23}(\ell) = -1$, then $\tau^*(\ell) \equiv 0 \pmod{23}$.

When $\ell = 4\ell'$, $\chi_{-23}(\ell) = -1$ is equivalent to $\chi_{-23}(\ell') = -1$. Hence, the following should be proved:

$$\tau(\ell) \equiv 0 \pmod{23} \text{ if } \chi_{-23}(\ell) = -1. \quad (3.6)$$

We take the prime decomposition of $\ell$: $\ell = \prod p_i^{e_i}$ (where $p_i$ is prime). From the assumption that $\chi_{-23}(\ell) = -1$, there is a factor $p_i^{e_i}$ such that

$$\chi_{-23}(p_i) = -1 \text{ and } e_i : \text{odd}.$$

In general, the following recursion formula holds for any prime $p$:

$$\tau(p^{n+1}) = \tau(p^n)\tau(p) - p^{11}\tau(p^{n-1}) \quad (n \geq 1).$$

This fact implies that, if $\tau(p) \equiv 0 \pmod{23}$, then $\tau(p^{\text{odd}}) \equiv 0 \pmod{23}$.

We apply this fact to the above prime number $p_i$. Because $\chi_{-23}(p_i) = \left(\frac{-23}{p_i}\right) = \left(\frac{p_i}{23}\right) = -1$, we have $\tau(p_i) \equiv 0 \pmod{23}$, and necessarily, $\tau(p_i^{e_i}) \equiv 0 \pmod{23}$. Consequently

$$\tau(\ell) = \tau(p_i^{e_i}) \tau\left(\prod_{j \neq i} p_j^{e_j}\right) \equiv 0 \pmod{23}.$$

This shows (3.6), and thus (**), which completes the proof of Theorem 3.5.
Corollary 3.6

\[ \Theta_{X_{-23}}(X_{14}) - \Theta(X_{14}) \equiv 0 \pmod{23}, \]

where \( \Theta \) and \( \Theta_\chi \) are the theta operators on quaternionic modular forms defined by

\[ \Theta(F) = \sum_T a(F; T) \cdot (2\det(T)) e^{2\pi i \tau(T, Z)}, \]

\[ \Theta_\chi(F) = \sum_T a(F; T) \cdot (2\det(T)) \cdot \chi(2\det(T)) e^{2\pi i \tau(T, Z)} \]

for \( F = \sum T a(F; T) e^{2\pi i \tau(T, Z)} \).

Example 3.7

(1) If \( T = \left( \frac{1}{e_1-e_2} \frac{e_1+e_2}{2} \right) \), then \( \chi_{-23}(2\det(T)) = \chi_{-23}(5) = -1 \) and

\[ a(X_{14}; T) = 4830 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 23 \equiv 0 \pmod{23}. \]

(2) If \( T = \left( \frac{6}{3(e_1-e_2)} \frac{3(e_1+e_2)}{18} \right) = 6 \cdot \left( \frac{1}{e_1-e_2} \frac{e_1+e_2}{2} \right) \), and thus

\[ \varepsilon(T) = 6 \quad \text{and} \quad \chi_{-23}(2\det(T)) = \chi_{-23}(2^2 \cdot 3^2 \cdot 5) = -1. \]

Then,

\[ a(X_{14}; T) = \sum_{d|6} d^{13} \tau^*(2\det(T)/d^2) \]

\[ = \tau^*(2^2 \cdot 3^2 \cdot 5) + 2^{13} \tau^*(3^2 \cdot 5) + 3^{13} \tau^*(2^2 \cdot 5) + 6^{13} \tau^*(5), \]

and

\[ \tau^*(2^2 \cdot 3^2 \cdot 5) = \tau(2^2 \cdot 3^2 \cdot 5) - 2^{12} \tau(3^2 \cdot 5) \equiv 0 \pmod{23}, \]

\[ \tau^*(3^2 \cdot 5) = \tau(3^2 \cdot 5) \equiv 0 \pmod{23}, \]

\[ \tau^*(2^2 \cdot 5) = \tau(2^2 \cdot 5) - 2^{12} \tau(5) \equiv 0 \pmod{23}, \]

\[ \tau^*(5) = \tau(5) \equiv 0 \pmod{23}. \]

Therefore, we obtain \( a(X_{14}; T) \equiv 0 \pmod{23} \).

3.3 Congruences for Eisenstein series

In Sect. 2, we introduced the Eisenstein series \( G_{k, \mathbb{H}} \) of weight \( k \). We provide some congruence relations for \( G_{k, \mathbb{H}} \):
Theorem 3.8 Let $G_{k,\mathbb{H}}$ be the quaternionic Eisenstein series of weight $k$ defined in (3.1). Assume that $p = 2k - 5$ is a prime number.

If $T \in \text{Sym}^T(\mathcal{O})$ satisfies $\chi_{-p}(2\det(T)) = -1$, then the corresponding Fourier coefficient $a(G_{k,\mathbb{H}}, T)$ is divisible by $p$, namely,

$$a(G_{k,\mathbb{H}}; T) \equiv 0 \pmod{p},$$

where $\chi_{-p}$ is the Kronecker symbol.

**Proof** We may assume that $T > 0$. We recall the formula for $a(G_{k,\mathbb{H}}; T)$:

$$a(G_{k,\mathbb{H}}; T) = \sum_{d|\det(T)} d^{k-1}b_k^*(2\det(T)/d^2)$$

$$b_k^*(\ell) = \sigma_{k-3}(\ell) - 2^{k-2}\sigma_{k-3}(\ell/4).$$

We show that

$$b_k^*(2\det(T)/d^2) \equiv 0 \pmod{p}. \quad (3.7)$$

Since $\chi_{-p}(2\det(T)) = -1$ is equivalent to $\chi_{-p}(2\det(T)/d^2) = -1$, it is sufficient to show that, if $\chi_{-p}(\ell) = -1$, then

$$b_k^*(\ell) \equiv 0 \pmod{p}. \quad (3.8)$$

Since $\chi_{-p}(\ell') = \chi_{-p}(\ell)$ when $\ell = 4\ell'$, the proof of (3.8) is reduced to showing the following:

$$\sigma_{k-3}(\ell) = \sigma_{p-1}(\ell/2) \equiv 0 \pmod{p} \quad \text{if} \quad \chi_{-p}(\ell) = -1. \quad (3.9)$$

Similar to the proof of Theorem 3.5, we take the prime decomposition $\ell = \prod_{i=1}^{n} p_i^{e_i}$.

Then, by assumption, there is a factor $p_j$ ($1 \leq j \leq n$) satisfying $\chi_{-p}(p_j) = -1$ and $e_j : \text{odd}$.

We obtain

$$\sigma_{p-1}(\ell) = \sum_{d|\ell} d^{p-1} = \sum_{d|\ell} \left(\frac{d}{p}\right) = \sum_{d|\ell} \chi_{-p}(d) = \prod_{i=1}^{n} \sum_{k_i=0}^{e_i} \chi_{-p}(p_i^{k_i}) \pmod{p}.$$
This proves (3.9) and completes the proof of Theorem 3.8. □

**Corollary 3.9** If \( T \in \text{Sym}^r(\mathcal{O}) \) satisfies \( \chi_{-23}(2\det(T)) = -1 \), then

\[
a(G_{14,\mathbb{H}}; T) \equiv 0 \pmod{23}.
\]

Compare with Theorem 3.5.

### 3.4 Remark on theta operator

In Sect. 3.2, we mentioned the theta operator. As an analogue of the results obtained in these cases, the following results are expected to hold.

\( (\natural) \) Let \( p \geq 5 \) be a prime number. Then, for any \( F \in \Gamma_2(\mathcal{O}), k; \mathbb{Z}_p \), there is a cusp form \( X \in \Gamma_2(\mathcal{O}), k + p + 1; \mathbb{Z}_p \) satisfying

\[
\Theta(F) \equiv X \pmod{p}.
\]

We have the following numerical evidence.

**Example 3.10** (1) \( \Theta(G_{4,\mathbb{H}}) \equiv X_{10} \pmod{5} \)

Numerical examples:

\[
1 = a \left( \Theta(G_{4,\mathbb{H}}); \left( \begin{array}{c} \frac{e_1 + e_2}{2} \\ 1 \end{array} \right) \right) \equiv a \left( X_{10}; \left( \begin{array}{c} \frac{e_1 + e_2}{2} \\ 1 \end{array} \right) \right) = 1 \pmod{5}
\]

\[
6 = a \left( \Theta(G_{4,\mathbb{H}}); \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right) \equiv a \left( X_{10}; \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right) = -24 \pmod{5}
\]

\[
12 = a \left( \Theta(G_{4,\mathbb{H}}); \left( \begin{array}{c} \frac{e_1 + e_2}{2} \\ 2 \end{array} \right) \right) \equiv a \left( X_{10}; \left( \begin{array}{c} \frac{e_1 + e_2}{2} \\ 2 \end{array} \right) \right) = 12 \pmod{5}
\]

\[\vdots\]

(2) \( \Theta(G_{6,\mathbb{H}}) \equiv X_{14} \pmod{7} \)

Numerical examples:

\[
1 = a \left( \Theta(G_{6,\mathbb{H}}); \left( \begin{array}{c} \frac{e_1 + e_2}{2} \\ 1 \end{array} \right) \right) \equiv a \left( X_{14}; \left( \begin{array}{c} \frac{e_1 + e_2}{2} \\ 1 \end{array} \right) \right) = 1 \pmod{7}
\]

\[
18 = a \left( \Theta(G_{6,\mathbb{H}}); \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right) \equiv a \left( X_{14}; \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right) = -24 \pmod{7}
\]

\[
84 = a \left( \Theta(G_{6,\mathbb{H}}); \left( \begin{array}{c} \frac{e_1 + e_2}{2} \\ 2 \end{array} \right) \right) \equiv a \left( X_{14}; \left( \begin{array}{c} \frac{e_1 + e_2}{2} \\ 2 \end{array} \right) \right) = 252 \pmod{7}
\]

\[\vdots\]

**Remark 3.11** The modular form \( X \) in \( (\natural) \) is expected to construct as follows:

\[
X = \text{(const.)} \cdot [F, F_{p-1}],
\]

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where \([ \ast, \ast ]\) is the Rankin-Cohen bracket and \(F_{p-1}\) is a modular form satisfying
\[
F_{p-1} \in \left[ \Gamma_2(O), p - 1; \mathbb{Z}(p) \right] \quad \text{and} \quad F_{p-1} \equiv 1 \pmod{p}.
\]
However, the theory of Rankin-Cohen bracket and the existence of modular form \(F_{p-1}\) has not yet been established in the case of quaternionic modular forms.

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