ON THE MALLE CONJECTURE AND THE
SELF-TWISTED COVER

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Abstract. We show that for a large class of finite groups $G$, the number of Galois extensions $E/Q$ of group $G$ and discriminant $|d_E| \leq y$ grows like a power of $y$ (for some specified exponent). The groups $G$ are the regular Galois groups over $Q$ and the extensions $E/Q$ that we count are obtained by specialization from a given regular Galois extension $F/Q(T)$. The extensions $E/Q$ can further be prescribed any unramified local behavior at each suitably large prime $p \leq \log(y)/\delta$ for some $\delta \geq 1$. This result is a step toward the Malle conjecture on the number of Galois extensions of given group and bounded discriminant. The local conditions further make it a notable constraint on regular Galois groups over $Q$. The method uses the notion of self-twisted cover that we introduce.

1. Main results

Given a finite group $G$ and a real number $y > 0$, there are only finitely many Galois extensions $E/Q$ (inside a fixed algebraic closure $\overline{Q}$ of $Q$) of group $G$ and discriminant $|d_E| \leq y$ (Hermite’s theorem). Estimating their number $N(G, y)$ is a classical topic ($\S 1.1$). Here we consider the extensions $E/Q$ obtained by specialization from a given Galois function field extension $F/Q(T)$ of group $G$ ($\S 1.2$). We obtain estimates for the number of those which satisfy the above group and ramification conditions. Our lower bound (obviously also a lower bound for $N(G, y)$) already has the conjectural growth for $N(G, y)$: a power of $y$ ($\S 1.3$). Furthermore the extensions $E/Q$ we produce satisfy some additional local conditions at a finite but growing with $y$ set of primes. This provides noteworthy constraints (though unknown yet not to be non-vacuous) on regular Galois groups over $Q$, related to analytic issues around the Tchebotarev theorem ($\S 1.4$). The role of the self-twisted cover from the title is explained in $\S 1.5$.

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1.1. The Malle conjecture is a classical landmark in this context. It predicts that for some constant $a(G) \in [0,1]$, specifically defined by Malle (recalled in §1.3), and for all $\varepsilon > 0$,

\begin{equation}
(\ast) \quad c_1(G) y^{a(G)} \leq N(G,y) \leq c_2(G,\varepsilon) y^{a(G)+\varepsilon}
\end{equation}

for some positive constants $c_1(G)$, $c_2(G,\varepsilon)$ and $y_0(G,\varepsilon)$ [Mal02]. A more precise asymptotic for $N(G,Y)$ (as $y \to \infty$) is even offered in [Mal04], namely $N(G,y) \sim c(G) y^{a(G)(\log(y))^{b(G)}}$, for some other specified constant $b(G) \geq 0$, and an another (unspecified) constant $c(G) > 0$.

The lower bound in (\ast) is a strong statement; it implies in particular that $G$ is the Galois group of at least one extension $E/\mathbb{Q}$, which is an open question for many groups – the so-called Inverse Galois Problem. Relying on the Shafarevich theorem solving the IGP for solvable groups, Klüners and Malle proved the conjecture (\ast) for nilpotent groups [KM04]. Klünners also established the lower bound for dihedral groups of order $2p$ with $p$ an odd prime [Klü06]. As to the more precise asymptotic for $N(G,Y)$, it has only been proved for abelian groups [Wri89], [Mäk85], for $S_3$ [BF10] [BW08] and for generalized quaternion groups [Klü05b, ch.7, satz 7.6].

We point out that there is a more general form of the conjecture for not necessarily Galois extensions $E/\mathbb{Q}$ for which there are further significant results, notably in the case the group (of the Galois closure) is $S_n$ (with $n = |E : \mathbb{Q}|$): results of Davenport-Heilbronn [DH71] andDatškovskij-Wright [DW88] ($n = 3$), Bhargava [Bha05], [Bha10] ($n = 4,5$), Ellenberg-Venkatesh [EV06] (upper bounds). There is also a counter-example to this more general form of the conjecture [Klü05a]. Finally there are quite interesting investigations on analogs of the problem over function fields of finite fields [EV05], [VE10], [EVW].

1.2. Our specialization approach. Besides solvable groups, there is another classical class of finite groups known to be Galois groups over $\mathbb{Q}$: those groups $G$ which are regular Galois groups over $\mathbb{Q}$, i.e., such that there exists a Galois extension $F/\mathbb{Q}(T)$ of group $G$ with $F \cap \overline{\mathbb{Q}} = \mathbb{Q}$. In addition to abelian and dihedral groups, this class includes many non solvable groups, e.g. all symmetric and alternating groups and many simple groups. We obtain for all these groups a lower bound in $y^\alpha$ for $N(G,y)$, as predicted by the Malle estimate (\ast).

To our knowledge, this is a new step toward the conjecture. Expectedly our exponent $\alpha$ is smaller than its Malle counterpart $a(G)$: our approach only takes into account those extensions which are specializations of a geometric extension $F/\mathbb{Q}(T)$. Our result is rather a specialization result (as presented in §2). Still it is interesting to already get the right growth for $N(G,y)$ from a single extension $F/\mathbb{Q}(T)$.
In this geometric situation, Hilbert’s irreducibility theorem classically produces “many” \( t_0 \in \mathbb{Q} \) such that the corresponding specialized extensions \( F_{t_0}/\mathbb{Q} \) are Galois extensions of group \( G \). Beyond making more precise these “many \( t_0 \in \mathbb{Q} \)” and controlling the corresponding discriminants, our goal requires a further step which is to show that many of these extensions are distinct. It is for this part that the self-twisted cover, a novel tool that we construct in §3, will be used (we say more in §1.5).

1.3. Statement of the main result. In addition to being of group \( G \) and discriminant \( \leq y \), we will be able to prescribe their local behavior at many primes to the Galois extensions \( E/\mathbb{Q} \) that we will produce. The following notation helps phrase these “local conditions”.

Given a finite group \( G \), a finite set \( S \) of primes and for each \( p \in S \), a subset \( F_p \subseteq G \) consisting of a non-empty union of conjugacy classes of \( G \), the collection \( F = (F_p)_{p \in S} \) is called a Frobenius data for \( G \) on \( S \). The number of Galois extensions \( E/\mathbb{Q} \) of group \( G \), of discriminant \( |d_E| \leq y \) and which are unramified with Frobenius \( \text{Frob}_p(E/\mathbb{Q}) \in F_p \) \((p \in S)\) is denoted by \( N(G, y, F) \).

The parameter \( \delta(G) \) that appears below in the definition of our exponent is the minimal affine branching index of regular realizations of \( G \) over \( \mathbb{Q} \), i.e., the minimal degree of the discriminant \( \Delta_P(T) \) of a polynomial \( P \in \mathbb{Q}[T, Y] \), monic in \( Y \), such that \( \mathbb{Q}(T)[Y]/\langle P \rangle \) is a regular Galois extension of \( \mathbb{Q}(T) \) of group \( G \).

**Theorem 1.1.** Let \( G \) be a regular Galois group over \( \mathbb{Q} \), non trivial. There exists a constant \( p_0(G) \) with the following property. For every \( \delta > \delta(G) \), for every suitably large \( y \) (depending on \( G \) and \( \delta \)) and every Frobenius data \( F_y \) on \( S_y = \{ p_0(G) < p \leq \log(y)/\delta \} \), we have

\[
N(G, y, F_y) \geq y^{\alpha(G, \delta)} \quad \text{with} \quad \alpha(G, \delta) = (1 - 1/|G|)/\delta.
\]

Furthermore, the desired extensions \( E/\mathbb{Q} \) can be obtained by specializing some regular realization \( F/\mathbb{Q}(T) \) of \( G \).

§2 says more on \( \delta(G) \). If a regular realization \( F/\mathbb{Q}(T) \) of \( G \) is given by a polynomial \( P \in \mathbb{Q}[T, Y] \), monic in \( Y \), then \( \delta(G) < 2|G|\deg_T(P) \). One can then take \( \delta = 2|G|\deg_T(P) \) in theorem 1.1 or the more intrinsic value \( \delta = 3r|G|^3\log |G| \) with \( r \) the branch point number of \( F/\mathbb{Q}(T) \). By comparison, Malle’s exponent is \( a(G) = (|G|(1 − 1/\ell))^{-1} \) where \( \ell \) is the smallest prime divisor of \( |G| \); inequality \( a(G) \geq 1/\delta(G) \) is proved in general in lemma 2.1, following a suggestion of G. Malle.

A more precise form of theorem 1.1 is stated in §2 which starts from any given regular realization \( F/\mathbb{Q}(T) \) of \( G \) and which shows other
features of our result: ramification can also be prescribed at any finitely many suitably large primes under the assumption that $F/\mathbb{Q}(T)$ has at least one $\mathbb{Q}$-rational branch point; the exponent $\alpha(G, \delta)$ can be replaced by $1/\delta$ under Lang’s conjecture; the Hilbert irreducibility aspect is expanded; and there is an upper bound part; see theorem 2.3.

Remark 1.2. Extending theorem 1.1 (and theorem 2.3) to arbitrary number fields (instead of $\mathbb{Q}$) seems to present no major obstacles, only requiring to extend some ingredients we use, which are only available in the literature over $\mathbb{Q}$ but should hold over number fields. As each finite group is known to be a regular Galois group over some suitably big number field, we could deduce that the same is true for the lower bound part in the Malle conjecture: given any finite group, there is a number field $k_0$ such that a lower bound like in (*) (appropriately generalized) holds over every number field containing $k_0$.

1.4. On the local conditions. (a) Regarding this aspect, theorem 1.1 improves on our previous work, with N. Ghazi, about the Grunwald problem. From [DG12], if $G$ is a regular Galois group over $\mathbb{Q}$, then every unramified Grunwald problem for $G$ at some finite set $S$ of primes $p \geq p_0(G)$ can be solved, i.e. every collection of unramified extensions $E_p/\mathbb{Q}_p$ of group $H_p \subset G$ ($p \in S$) is induced by some Galois extension $E/\mathbb{Q}$ of group $G$. Theorem 1.1 does more: it provides, for every given discriminant size, a big number of such extensions $E/\mathbb{Q}$, a number that grows as in Malle’s predictions.

Malle had suggested that his estimates should hold with some local conditions [Mal04, Remark 1.2]. However, unlike his, ours have a set of primes, $S_y$, which grows with $y$. We focus below on this.

(b) First we note that the set of primes where the local behavior can be prescribed as in theorem 1.1 cannot be expected to be much bigger than the set $S_y$:

- indeed, that every possible Frobenius data on $S_y$ occurs in at least one extension $E/\mathbb{Q}$ counted by $N(G, y)$ already gives $N(G, y) \geq c^{u(y)}$, with $c$ the number of conjugacy classes of $G$ and $u(y)$ the number of primes in $S_y = \{p_0(G) < p \leq \log(y)/\delta\}$. Now $c^{u(y)}$ compares to the conjectural upper bounds for $N(G, y)$: $\log c^{u(y)} \sim \log(y) / \log(\log(y))$ and $\log(y^{u(G)+\varepsilon}) \sim \log(y)$ (up to multiplicative constants).

- the restriction that the primes $p$ be suitably large ($p > p_0(G)$) cannot be removed either as the famous Wang’s counter-example shows [Wan69]: no Galois extension $E/\mathbb{Q}$ of group $\mathbb{Z}/8\mathbb{Z}$ is unramified at 2 with Frobenius of order 8. Other counter-examples with other primes than 2 have been recently produced by Neftin [Nef13].
(c) There is a further connection of our result with the Tchebotarev density theorem. The following definition helps explain it.

**Definition 1.3.** Given a real number \( \ell \geq 0 \), we say that a finite group \( G \) is of **Tchebotarev exponent** \( \leq \ell \), which we write \( \text{tch}(G) \leq \ell \), if there exist real numbers \( m, \delta > 0 \) such that for every \( x > m \) and every Frobenius data \( F_x = (F_p)_{m < p \leq x} \) for \( G \), there exists at least one Galois extension \( E/\mathbb{Q} \) of group \( G \) such that these two conditions hold:

1. for each \( m < p \leq x \), \( E/\mathbb{Q} \) is unramified and \( \text{Frob}_p(E/\mathbb{Q}) \in F_p \),
2. \( \log |d_E| \leq \delta x^\ell \).

Fix \( \delta > \delta(G) \) and \( m \) suitably large (in particular \( m \geq p_0(G) \)). Theorem 1.1 for \( y = e^{\delta x} \) with \( x > m \) provides many\(^1 \) extensions \( E/\mathbb{Q} \) satisfying conditions of definition 1.3 with \( \ell = 1 \).

**Corollary 1.4.** If a finite group \( G \) is a regular Galois group over \( \mathbb{Q} \), then \( \text{tch}(G) \leq 1 \).

On the other hand there is a universal lower bound for \( \text{tch}(G) \). Some famous estimates on the Tchebotarev theorem [LMO79] (see also [LO77], [Ser81]) show that, under the General Riemann Hypothesis, for every finite group \( G \), we have

\[
\text{tch}(G) > (1/2) - \varepsilon, \quad \text{for every } \varepsilon > 0. \quad (**) 
\]

(More precisely, they show that if a Galois extension \( E/\mathbb{Q} \) is of group \( G \) and \( \log |d_E| \leq x^{1/2}/\log x \), there are at least \( \pi(x) - 2x/(|G| \log x) \) non totally split primes \( p \leq x \) in \( E/\mathbb{Q} \) (with \( \pi(x) \) is the number of primes \( p \leq x \)). As \( \pi(x) - 2x/(|G| \log x) \to +\infty \), the trivial totally split behavior — \( F_p = \{1\} \) for each \( m < p \leq x \) — does not occur if \( x \gg 1 \).

Corollary 1.4 raises the question of whether \( \text{tch}(G) > 1 \) for some group \( G \), in which case \( G \) could not be a regular Galois group over \( \mathbb{Q} \). Such a group may not exist (if the so-called Regular Inverse Galois Problem is true), while at the other extreme it cannot be ruled out at the moment that \( \text{tch}(G) = \infty \) for some group \( G \). Many possibilities exist in between for Galois groups \( G \) over \( \mathbb{Q} \): that realizations exist that satisfy the local conditions of definition 1.3 (1) or not, that the corresponding discriminants can be bounded as in definition 1.3 (2), for some \( \ell \in [1/2, \infty[ \) or not. Somehow the Tchebotarev exponent provides a measure of the gap (possibly empty) between the classical and regular Inverse Galois Problems.

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\(^1\)at least \( e^{\gamma x} \) with \( \gamma = 1 - 1/|G| \) (for \( x \gg 1 \)).

\(^2\)[LMO79] also has an unconditional conclusion, which, using our terminology, leads to \( \text{tch}(G) \geq (\log \log x)/(2 \log x) \) (with definition 1.3 extended to allow \( \ell \) to be a function of \( x \)).
Gaining information on Tchebotarev exponents however seems difficult. Even for $G = \mathbb{Z}/2\mathbb{Z}$ and the case of the totally split behavior, for which the problem amounts to bounding the least square-free integer $d_m(x)$ that is a quadratic residue modulo each prime $m < p \leq x$. Changing 1/2 to 1 in (**) the remaining possible improvement (as $\mathbb{Z}/2\mathbb{Z}$ is a regular Galois group over $\mathbb{Q}$), is plausible as some easy heuristics show but relates to deep questions in number theory (e.g. [Ser81, §2.5]).

1.5. Role of the self-twisted cover. Our method starts with a regular realization $F/\mathbb{Q}(T)$ of $G$. The extensions $E/\mathbb{Q}$ that we wish to produce will be specializations $F_{t_0}/\mathbb{Q}$ of it at some integers $t_0$. A key tool is the twisting lemma from [DG12], which reduces the search of specializations of a given type to that of rational points on a certain twisted cover. We use it twice, first over $\mathbb{Q}_p$ as in [DG12], to construct specializations $F_{t_0}/\mathbb{Q}$ with a specified local behavior. A main ingredient for this first stage is the Lang-Weil estimate for the number of rational points on a curve over a finite field. We obtain many good specialisations $t_0 \in \mathbb{Z}$ and a lower bound for their number.

The next question is to bound the number of the corresponding specializations $F_{t_0}/\mathbb{Q}$ that are non-isomorphic. First we reduce it to counting integral points of a given size on certain twisted covers. This is our second use of the twisting lemma, over $\mathbb{Q}$ this time. For the count of the integral points, we use a result of Walkowiak [Wal05]$^3$, based on a method of Heath-Brown [HB02]. However the bounds from [Wal05] involve the height of the defining polynomials, which here depend on the specializations $F_{t_0}/\mathbb{Q}$. We have to control the dependence in $t_0$. This is where enters the self-twisted cover, which as we will see, is a family of covers, depending only on the original extension $F/\mathbb{Q}(T)$ and which has all the twisted covers among its fibers. As a result, a bound of the form $c_1 t_{\delta}^2$ for the height of the polynomials above will follow with $c_1$ and $c_2$ depending only on $F/\mathbb{Q}(T)$.

In §2 below we present theorem 2.3, the more precise version of theorem 1.1. §3 is the construction of the self-twisted cover. §4 gives the proof of theorem 2.3.

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$^3$We need to slightly improve Walkowiak's result to get the right exponent $\alpha(G, \delta)$ in theorem 1.1; see §4.2.3.
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2. The specialization version of the result

2.1. Basics. Given a field \( k \), we work without distinction with a regular extension \( F/k(T) \) or with the associated \( k \)-cover \( f : X \to \mathbb{P}^1 \): \( f \) is the normalization of \( \mathbb{P}^1 \) in \( F \) and \( F \) is the function field \( k(X) \) of \( X \).

We assume that \( k \) is of characteristic 0; \( k = \mathbb{Q} \) in most of the paper.

Recall that a (regular) \( k \)-cover of \( \mathbb{P}^1 \) is a finite and generically unramified morphism \( f : X \to \mathbb{P}^1 \) defined over \( k \) with \( X \) a normal and geometrically irreducible variety. The \( k \)-cover \( f : X \to \mathbb{P}^1 \) is said to be Galois if the field extension \( k(X)/k(T) \) is Galois; if in addition \( f : X \to \mathbb{P}^1 \) is given together with an isomorphism \( G \to \text{Gal}(k(X)/k(T)) \), it is called a (regular) \( k \)-Galois cover of group \( G \).

By group and branch point set of a \( k \)-cover \( f \), we mean those of the \( \overline{k} \)-cover \( \overline{f} \otimes_k \overline{k} \): the group of a \( \overline{k} \)-cover \( X \to \mathbb{P}^1 \) is the Galois group of the Galois closure of the extension \( \overline{k}(X)/\overline{k}(T) \). The branch point set of \( \overline{f} \otimes_k \overline{k} \) is the (finite) set of points \( t \in \mathbb{P}^1(\overline{k}) \) such that the associated discrete valuations are ramified in the extension \( \overline{k}(X)/\overline{k}(T) \).

Given a regular Galois extension \( F/k(T) \) and \( t_0 \in \mathbb{P}^1(k) \), the specialization of \( F/\mathbb{Q}(T) \) at \( t_0 \) is the residue extension of an (arbitrary) prime above \( (T - t_0) \) in the integral closure of \( \mathbb{Q}[T]_{(T-t_0)} \) in \( F \) (as usual use \( \mathbb{Q}[1/T]_{(1/T)} \) instead if \( t_0 = \infty \)). We denote it by \( F_{t_0}/k \).

Given a regular Galois extension \( F/k(T) \), we say a prime \( p \) is good for \( F/\mathbb{Q}(T) \) if \( p \nmid |G| \), the branch divisor \( t = \{t_1, \ldots, t_r\} \) is étale at \( p \) and there is no vertical ramification at \( p \); and it is bad otherwise. We refer to [DG12] for the precise definitions. We only use here the standard fact that there are only finitely many bad primes.

2.2. The minimal affine branching index \( \delta(G) \). Given a regular extension \( F/\mathbb{Q}[T] \), we call the irreducible polynomial \( P(T,Y) \) of a primitive element, integral over \( \mathbb{Z}[T] \), an integral affine model of \( F/\mathbb{Q}(T) \); \( P(T,Y) \in \mathbb{Z}[T] \) and is monic in \( Y \). Denote the discriminant of \( P \) relative to \( Y \) by \( \Delta_P(T) \in \mathbb{Z}[T] \) and its degree by \( \delta_P \). The minimal degree \( \delta_P \) obtained in this manner is called the minimal affine branching index of \( F/\mathbb{Q}(T) \) and denoted by \( \delta_{F/\mathbb{Q}(T)} \). For any integral affine model \( P(T,Y) \) of \( F/\mathbb{Q}(T) \), we have

\[
\delta_{F/\mathbb{Q}(T)} \leq \delta_P < 2|G| \deg_T(P).
\]

If \( G \) is a regular Galois group over \( \mathbb{Q} \), the parameter \( \delta(G) \) involved in theorem 1.1 is the minimum of all \( \delta_{F/\mathbb{Q}[T]} \) with \( F/\mathbb{Q}[T] \) running over all regular realizations of \( G \).
Lemma 2.1. Let $G$ be a non trivial regular Galois group over $\mathbb{Q}$.

(a) If $F/\mathbb{Q}(T)$ is a regular realization of $G$ with $r$ branch points and $g$ is the genus of $F$, then
\[
\delta(G) < 3(2g + 1)|G|^2 \log |G| \leq 3r|G|^3 \log |G|.
\]

(b) Furthermore we have $\delta(G) \geq 1/a(G)$.

Proof of lemma 2.1. (a) The first inequality follows from a result of Sadi [Sad99, §2.2] which provides an affine model $P(T, Y)$ of $F/\mathbb{Q}(T)$ such that
\[
\deg_T(P) \leq (2g + 1)|G| \log |G|/\log 2.
\]
The second inequality follows from the Riemann-Hurwitz formula.

(b) Let $F/\mathbb{Q}(T)$ be a regular realization of $G$, $d_F \in \mathbb{Q}[T]$ be the absolute discriminant of $F/\mathbb{Q}(T)$ (the discriminant of a $\mathbb{Q}[T]$-basis of the integral closure of $\mathbb{Q}[T]$ in $F$) and $P(T, Y)$ be an integral affine model of $F/\mathbb{Q}(T)$. Inequality (b) follows from the following ones:
\[
\delta_p \geq \deg(d_F) \geq |G|(1 - 1/\ell) = a(G).
\]
where $\ell$ is as before the smallest prime divisor of $|G|$. The first inequality $\deg(d_F) \leq \delta_p$ is standard. Classically the polynomial $d_F$ is a generator of the ideal $N_{F/\mathbb{Q}(T)}(D_{F/\mathbb{Q}(T)})$ where $D_{F/\mathbb{Q}(T)}$ is the different and $N_{F/\mathbb{Q}(T)}$ is the norm relative to the extension $F/\mathbb{Q}(T)$ [Ser62, III, §3]. From [Ser62, III, §6], in the prime ideal decomposition $D_{F/\mathbb{Q}(T)} = \prod \mathfrak{p}^{u_p}$, we have $u_p \geq e_p - 1$ for each prime $\mathfrak{p}$, where $e_p = e_p$ is the corresponding ramification index, which only depends on the prime $\mathfrak{p}$ below $\mathfrak{p}$. The following inequalities, where $f_{\mathfrak{p}}$ denotes the residue degree of $\mathfrak{p}$, finish the proof:
\[
\deg(d_F) \geq \sum \sum_{\mathfrak{p}|\mathfrak{p}} f_{\mathfrak{p}}(e_p - 1) = \sum |G| - |G|/e_p \geq |G|(1 - 1/\ell).
\]

Remark 2.2. Our parameter $\delta(G)$ can also be compared to the minimum, say $\rho(G)$, of all branch point numbers $r$ of regular realizations $F/\mathbb{Q}(T)$ of $G$: for such an extension $F/\mathbb{Q}(T)$ we have $\deg(d_F) \geq r - 1$ whence $\delta(G) \geq \rho(G) - 1$. But the inequality $a(G) \geq 1/(\rho(G) - 1)$ does not hold in general. For example the symmetric group $S_n$ can be regularly realized over $\mathbb{Q}$ with 3 branch points so $\rho(S_n) = 3$ while $a(S_n) = 2/n!$ The analog of theorem 1.1 with $r - 1$ replacing $\delta_{F/\mathbb{Q}(T)}$ is false if the upper bound part of Malle’s conjecture is true.

2.3. The specialization result. Theorem 2.3 is a more precise version of our main result. It gives explicit estimates from which the asymptotic estimates of theorem 1.1 can easily be deduced (as explained in §2.3.4). Another difference is that it starts from a given regular
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Galois extension $F/\mathbb{Q}(T)$ and the extensions $E/\mathbb{Q}$ we count are specializations of it. Below are the necessary additional notation and data.

2.3.1. Notation. The following notation is used throughout the paper:
- for a Frobenius data $\mathcal{F}$ on a set of primes $S$, the product of all ratios $|\mathcal{F}_p|/|G|$ with $p \in S$, i.e. the density of $\mathcal{F}$, is denoted by $\chi(\mathcal{F})$,
- for a finite set $S$ of primes, set $\Pi(S) = \prod_{p \in S} p$,
- we also use the classical functions $\pi(x)$ and $\Pi(x)$ to denote respectively the number of primes $\leq x$ and the product of all primes $\leq x$. We have the classical asymptotics at $\infty$: $\pi(x) \sim x / \log(x)$ and $\log \Pi(x) \sim x$,
- the height of a polynomial $F$ with coefficients in $\mathbb{Q}$ is the maximum of the absolute values of its coefficients and is denoted by $H(F)$.

2.3.2. Data. Fix the following for the rest of the paper:
- $G$ is a non trivial finite group,
- $F/\mathbb{Q}(T)$ is a regular Galois extension of group $G$,
- $f : X \to \mathbb{P}^1$ is the corresponding $\mathbb{Q}$-cover,
- $t = \{t_1, \ldots, t_r\} \subset \mathbb{P}^1(\overline{\mathbb{Q}})$ is the branch point set of $F/\mathbb{Q}(T)$,
- $g$ is the genus of the curve $X$,
- $p_0(F/\mathbb{Q}(T))$ is the prime defined as follows. Let $p_{-1}$ be the biggest prime $p$ such $p$ is bad for $F/\mathbb{Q}(T)$ or $p < r^2|G|^2$. Then $p_0(F/\mathbb{Q}(T))$ is the smallest prime $p$ such that the interval $[p_{-1}, p_0]$ contains as many primes as there are non-trivial conjugacy classes of $G$. For the rest of the paper we fix a prime $p_0 \geq p_0(F/\mathbb{Q}(T))$.
- $\delta_{F/\mathbb{Q}(T)}$ or $\delta_F$ for short is the minimal affine branching index of $F/\mathbb{Q}(T)$,
- $P(T, Y)$ is an integral affine model of $F/\mathbb{Q}(T)$ such that $\delta_P = \delta_F$ (with $\delta_P$ the degree of the discriminant $\Delta_P$) and which is primitive, i.e. has relatively prime coefficients (an assumption which one can always reduce to),
- $S$ is a finite set of primes subject to these conditions:
  (a) if no branch point of $f$ is in $\mathbb{Z}$ then $S = \emptyset$.
  (b) if at least one of the branch points of $f$, say $t_1$ is in $\mathbb{Z}$, then $S$ is a finite set of good primes $p$, not dividing $t_1$ and not in $[p_{-1}, p_0]$.
  (If at least one branch point is $\mathbb{Q}$-rational, one can reduce to the assumption in (b) via a simple change of the variable $T$).
- for $x > p_0$, $S_x$ is the set of primes $p$ such that $p_0 < p \leq x$ and $p \notin S$.
- for technical reasons we change the condition “$|d_E| \leq y$” from theorem 1.1 to the more complicated one “$|d_E| \leq \rho(x)$” where
  $$\rho(x) = (1 + \delta_P)H(\Delta_P)[\Pi(S)\Pi(x)]^{\delta_P}$$
(we have \( \log \rho(x) \sim \delta_P x \) and so a simple change of variable leads back to the original condition).

2.3.3. **Statement.** For \( x > p_0 \) let \( \mathcal{F}_x \) be a Frobenius data on \( S_x \). Theorem 2.3 is about the number \( N_F(x, S, \mathcal{F}_x) \) of distinct specializations \( F_{t_0}/\mathbb{Q} \) at points \( t_0 \in \mathbb{Z} \) that satisfy

(i) \( \text{Gal}(F_{t_0}/\mathbb{Q}) = G \),

(ii) for each \( p \in S_x \), \( F_{t_0}/\mathbb{Q} \) is unramified and \( \text{Frob}_p(F_{t_0}/\mathbb{Q}) \in \mathcal{F}_p \),

(iii) for each \( p \in S \), \( F_{t_0}/\mathbb{Q} \) is ramified at \( p \),

(iv) \( |d_{F_{t_0}}| \leq \rho(x) \).

**Theorem 2.3.** (a) There exist constants \( C_1, C_2, C_3, C_4 \) only depending on \( P(T, Y) \) such that for every \( x > p_0 \), we have

\[
N_F(x, S, \mathcal{F}_x) \geq C_1 \frac{\chi(\mathcal{F}_x)}{\Pi(S)^2} \frac{\Pi(x)^{1-1/|G|}}{(\log \Pi(x))^{C_2} C_3^{\sigma(x)}} - C_4
\]

so

\[
\log(N_F(x, S, \mathcal{F}_x)) \text{ is bigger than a function } \lambda(x) \sim (1-1/|G|)x.
\]

(b) Furthermore the specializations \( F_{t_0}/\mathbb{Q} \) counted by the lower bound can be taken to be specializations at integers \( t_0 \in [1, \Pi(S) \Pi(x)] \).

(c) Under Lang’s conjecture on rational points on a variety of general type and if \( g \geq 2 \), we have this better inequality

\[
N_F(x, S, \mathcal{F}_x) \geq C_5 \frac{\chi(\mathcal{F}_x)}{\Pi(S)^2} \frac{\Pi(x)}{C_6^{\sigma(x)}} - C_7
\]

for constants \( C_5, C_6, C_7 \) only depending on \( P \), so then

\[
\log(N_F(x, S, \mathcal{F}_x)) \text{ is bigger than a function } \lambda(x) \sim x.
\]

(d) We have the following upper bound for the number \( N_F(x, \mathcal{F}_x) \) of integers \( t_0 \in [1, \Pi(S_x)] \) such that condition (ii) above holds:

\[
N_F(x, \mathcal{F}_x) \leq \chi(\mathcal{F}_x) \frac{\Pi(S_x)}{\beta} (2 - \lambda)^{|S_x|}
\]

where \( \lambda = (r|G| - 1)/r^2|G|^2 \in [0, 1/4] \) and \( \beta \) depends only on \( F/\mathbb{Q}(T) \).

2.3.4. **Proof of theorem 1.1 assuming theorem 2.3.** Pick a regular realization \( F/\mathbb{Q}(T) \) of \( G \) and an integral affine model \( P(T, Y) \) such that \( \delta_P = \delta_F = \delta(G) \). Set \( p_0(G) = p_0(F/\mathbb{Q}(T)) \). Fix \( \delta > \delta(G) \) and set \( \delta^- = (\delta + \delta(G))/2 \). Let \( y > 0 \) and \( x = \log(y)/\delta^- \).

As \( \delta^- < \delta \) we have \( S_y \subset S_x \). Complete the given Frobenius data \( \mathcal{F}_x \) on \( S_y \) in an arbitrary way to make it a Frobenius data \( \mathcal{F}_x \) on \( S_x \). Apply theorem 2.3 to \( \mathcal{F}_x \) and \( S = \emptyset \). As \( \log \rho(x) \sim (\delta_P / \delta^-) \log y \), if \( y \) is suitably large, \( \rho(x) \leq y \). It follows that \( N(G, y, \mathcal{F}_y) \geq N_F(x, 0, \mathcal{F}_x) \) and so \( N(G, y, \mathcal{F}_y) \) can be bounded from below by the right-hand side.
term of the inequality from theorem 2.3 (a) with \( x = \log(y)/\delta^- \). The logarithm of this term is asymptotic to \((1 - 1/|G|) \log(y)/\delta^- \). Conclude that for suitably large \( y \), this term is bigger than \( y^{(1 - 1/|G|)/\delta} \).

2.3.5. Remarks. (a) In the situation \( S \neq \emptyset \), for which it is possible to prescribe ramification at some primes, the assumption that at least one branch point is \( \mathbb{Q} \)-rational cannot be removed, as explained in Legrand’s paper [Leg13], which is devoted to this situation. Many groups have a regular realization \( F/\mathbb{Q}(T) \) satisfying this assumption, although being of even order is a necessary condition [Leg13, §3.2]: abelian groups of even order, symmetric groups \( S_n \) (\( n \geq 2 \)), alternating groups \( A_n \) (\( n \geq 4 \)), many simple groups (including the Monster), etc.

For these groups, theorem 2.3 leads to a generalized version of the inequality from theorem 1.1 where the left-hand term \( N(G, y, F_y) \) is replaced by the (smaller) number, say \( N(G, S, y, F_y) \), of extensions \( E/\mathbb{Q} \) which, in addition to the conditions prescribed in theorem 1.1, are required to ramify at every prime from a finite set \( S \) of suitably big primes (and where the set \( S_y \) is of course replaced by \( S_y \setminus S \)).

(b) The upper bound in theorem 2.3 (d) concerns extensions \( E/\mathbb{Q} \) that are specializations of a given regular extension \( F/\mathbb{Q}(T) \) (at integers \( t_0 \)) and so does not directly lead to upper bounds for \( N(G, y, F_y) \). A natural hypothesis to make in this context is that \( G \) has a generic extension \( F/\mathbb{Q}(T) \) (or more generally a parametric extension, as defined in [Leg13]): indeed then all Galois extensions \( E/\mathbb{Q} \) of group \( G \) are specializations of \( F/\mathbb{Q}(T) \) (at points \( t_0 \in \mathbb{Q} \)). But only the four groups \{1\}, \( \mathbb{Z}/2\mathbb{Z} \), \( \mathbb{Z}/3\mathbb{Z} \), \( S_3 \) have a generic extension \( F/\mathbb{Q}(T) \) [JLY02, p.194].

2.3.6. An application of the upper bound part from theorem 2.3. For every \( x > p_0 \), let \( N_{\text{tot.split}}(x) \) be the set of all integers \( t_0 \geq 1 \) such that the specialization \( F_{t_0}/\mathbb{Q} \) is totally split at each prime \( p_0 < p \leq x \), i.e. which satisfy condition (ii) from theorem 2.3 with \( F_p \) taken to be the trivial conjugacy class for each \( p \in S_x \).

**Corollary 2.4.** For every \( x > p_0 \), \( N_{\text{tot.split}}(x) \) is a union of (many) cosets modulo \( \Pi(S_x) \) but its density decreases to 0 as \( x \to +\infty \).

**Proof.** We anticipate on §4 to say that \( N_{\text{tot.split}}(x) \) is a union of cosets modulo \( \Pi(S_x) \) (see proposition 4.3 (c)) and focus on the density part of the statement. Every integer \( t_0 \in N_{\text{tot.split}}(x) \) writes \( t_0 = u + k \Pi(S_x) \) with \( u \) one of the elements in \([1, \Pi(S_x)]\) counted by \( N_F(x, F_x) \) and \( k \in \mathbb{Z} \). Let \( N \geq 1 \) be any integer. If \( 1 \leq t_0 \leq N \), then \( k \leq N/\Pi(S_x) \).
It follows then from theorem 2.3 (d) that

\[ |N_{\text{tot.split}}(x) \cap [1, N]| \leq \frac{N}{\Pi(S_x)} \times N_F(x, F_x) \leq \frac{N}{\beta} \times \left( \frac{2 - \lambda}{|G|} \right)^{|S_x|} \]

which divided by \( N \) tends to 0 as \( x \to +\infty \). \( \square \)

Similar density conclusions can be obtained for other local behaviors for which the sets \( F_p \) are not too big compared to \( G \).

3. The self-twisted cover

In \( \S 3.1 \), we recall the twisting operation on covers and the twisting lemma (\( \S 3.1.2 \)) and explain the motivation for introducing the self-twisted cover (\( \S 3.1.3 \)). \( \S 3.2 \) is devoted to its construction. Covers are viewed here as fundamental group representations. The correspondence is briefly recalled in \( \S 3.1.1 \).

3.1. Twisting G-Galois covers. For the material of this subsection, we refer to [DG12].

3.1.1. Fundamental groups representations of covers. Given a field \( k \), denote its absolute Galois group by \( G_k \).

If \( E/k \) is a Galois extension of group \( G \), an epimorphism \( \varphi : G_k \to G \) such that \( E \) is the fixed field of \( \ker(\varphi) \) in \( \overline{k} \) is called a \( G_k \)-representation of \( E/k \).

Given a finite subset \( t \subset \mathbb{P}^1(\overline{k}) \) invariant under \( G_k \), the \( k \)-fundamental group of \( \mathbb{P}^1 \setminus t \) is denoted by \( \pi_1(\mathbb{P}^1 \setminus t, t) \); here \( t \) denotes the fixed base point, which corresponds to choosing an embedding of \( k(T) \) in \( \Omega \). The \( k \)-fundamental group \( \pi_1(\mathbb{P}^1 \setminus t, t)_{\overline{k}} \) is defined as the Galois group of the maximal algebraic extension \( \Omega_{t,k}/\overline{k}(T) \) (inside \( \Omega \)) unramified above \( \mathbb{P}^1 \setminus t \) and the \( k \)-fundamental group \( \pi_1(\mathbb{P}^1 \setminus t, t)_{k} \) as the group of the Galois extension \( \Omega_{t,k}/k(T) \).

Degree \( d \) \( k \)-covers of \( \mathbb{P}^1 \) (resp. \( k \)-G-Galois covers of \( \mathbb{P}^1 \) of group \( G \)) with branch points in \( t \) correspond to transitive homomorphisms \( \pi_1(\mathbb{P}^1 \setminus t, t)_{k} \to S_d \) (resp. to epimorphisms \( \pi_1(\mathbb{P}^1 \setminus t, t)_{k} \to G \)), with the extra regularity condition that the restriction of \( \phi \) to \( \pi_1(\mathbb{P}^1 \setminus t, t)_{\overline{k}} \) remains transitive (resp. remains onto). These corresponding homomorphisms are called the fundamental group representations (or \( \pi_1 \)-representations for short) of the cover \( f \) (resp the G-cover \( f \)).

Each \( k \)-rational point \( t_0 \in \mathbb{P}^1(k) \setminus t \) provides a section \( s_{t_0} : G_k \to \pi_1(\mathbb{P}^1 \setminus t, t)_{k} \) to the exact sequence

\[ 1 \to \pi_1(\mathbb{P}^1 \setminus t, t)_{\overline{k}} \to \pi_1(\mathbb{P}^1 \setminus t, t)_{k} \to G_k \to 1 \]
which is uniquely defined up to conjugation by an element in the fundamental group \( \pi_1(\mathbb{P}^1 \setminus \{t, t\}) \).

If \( \phi : \pi_1(\mathbb{P}^1 \setminus \{t, t\}) \rightarrow G \) represents a \( k \)-Galois cover \( f : X \rightarrow \mathbb{P}^1 \), the morphism \( \phi \circ s_{t_0} : G_k \rightarrow G \) is the specialization representation of \( \phi \) at \( t_0 \). The fixed field in \( \overline{k} \) of \( \ker(\phi \circ s_{t_0}) \) is the specialization \( k(X)_{t_0}/k(T) \) of \( k(X)/k(T) \) at \( t_0 \) (defined in §2.1).

3.1.2. The twisting lemma. We recall how a regular \( k \)-Galois cover \( f : X \rightarrow \mathbb{P}^1 \) of group \( G \) can be twisted by some Galois extension \( E/k \) of group \( H \subset G \). Formally this is done in terms of the associated \( \pi_1 \)- and \( G_k \)-representations.

Let \( \phi : \pi_1(\mathbb{P}^1 \setminus \{t, t\}) \rightarrow G \) be a \( \pi_1 \)-representation of the regular \( k \)-G-cover \( f : X \rightarrow \mathbb{P}^1 \) and \( \varphi : G_k \rightarrow G \) be a \( G_k \)-representation of the Galois extension \( E/k \).

Denote the right-regular (resp. left-regular) representation of \( G \) by \( \delta : G \rightarrow S_d \) (resp. by \( \gamma : G \rightarrow S_d \)) where \( d = |G| \). Define \( \varphi^* : G_k \rightarrow G \) by \( \varphi^*(g) = \varphi(g)^{-1} \). Consider the map \( \tilde{\phi}^\varphi : \pi_1(\mathbb{P}^1 \setminus \{t, t\}) \rightarrow S_d \) defined by the following formula, where \( r \) is the restriction map \( \pi_1(\mathbb{P}^1 \setminus \{t, t\}) \rightarrow G_k \) and \( \times \) is the multiplication in the symmetric group \( S_d \):

\[
\tilde{\phi}^\varphi(\theta) = \gamma(\phi(\theta)) \times \delta^{\varphi^* r(\theta)} \quad (\theta \in \pi_1(\mathbb{P}^1 \setminus \{t, t\}).
\]

The map \( \tilde{\phi}^\varphi \) is a group homomorphism with the same restriction on \( \pi_1(\mathbb{P}^1 \setminus \{t, t\}) \) as \( \phi \). It is called the twisted representation of \( \phi \) by \( \varphi \).

The associated regular \( k \)-cover is a \( k \)-model of the cover \( f \otimes_k \overline{E} \). It is denoted by \( \tilde{f}^\varphi : \tilde{X}^\varphi \rightarrow \mathbb{P}^1 \) and called the twisted cover of \( f \) by \( \varphi \). The following statement is the main property of the twisted cover.

**Twisting lemma 3.1.** Let \( t_0 \in \mathbb{P}^1(k) \setminus \{t\} \). Then the specialization representation \( \phi \circ s_{t_0} : G_k \rightarrow G \) is conjugate in \( G \) to \( \varphi : G_k \rightarrow G \) if and only if there exists \( x_0 \in \tilde{X}^\varphi(k) \) such that \( \tilde{f}^\varphi(x_0) = t_0 \).

3.1.3. The motivation for the self-twisted cover. As explained in §1.5, we will have to control the height of some polynomials defining some twisted covers. These twisted covers are obtained by twisting the given \( G \)-Galois cover \( f : X \rightarrow \mathbb{P}^1 \) by its own specializations \( \mathbb{Q}(X)_{u_0}/\mathbb{Q} \) (\( u_0 \in \mathbb{Q} \)); we call them the fiber-twisted covers. §3.2 shows that the fiber-twisted covers are all members of an algebraic family of covers: the self-twisted cover. The practical use for the end of the paper is the following result. It is a consequence of lemma 3.4.

**Theorem 3.2.** Given a regular \( k \)-G-cover \( f : X \rightarrow \mathbb{P}^1 \), there exists a polynomial \( \tilde{P}(U, T, Y) \in k[\mathbb{P}^1] \) irreducible in \( k(U)(T)[Y] \), monic in \( Y \), and a finite set \( \mathcal{E} \subset k \) such that for every \( u_0 \in k \setminus \mathcal{E} \),
3.2. The self-twisted cover. Let \( U \) be a new indeterminate (algebraically independent from \( T \) and \( Y \)). Fix an algebraically closed field \( \Omega \) containing \( k(T, U) \), which we will use as a common base point \( t \) for all fundamental groups involved. The algebraic closures of \( k(T, U), k(T), k(U) \) and \( k \) should be understood as the ones inside \( \Omega \).

3.2.1. A \( \pi_1 \)-representation of \( f \otimes_k k(U) \). As the compositum \( \Omega_{t,k} \cdot k(U) \) is contained in \( \Omega_{t,k(U)} \), there is a restriction morphism

\[
\text{res}_{k(U)/k} : \pi_1(\mathbb{P}^1 \setminus t, t)_{k(U)} \to \pi_1(\mathbb{P}^1 \setminus t, t)_k,
\]

which induces a map between the geometric parts of the fundamental groups:

\[
\text{res}_{k(U)/k} : \pi_1(\mathbb{P}^1 \setminus t, t)_{k(U)} \to \pi_1(\mathbb{P}^1 \setminus t, t)_k.
\]

We also use the notation \( \text{res}_{k(U)/k} \) for the map \( G_{k(U)} \to G_k \) induced on the absolute Galois groups.

**Lemma 3.3.** \( \text{res}_{k(U)/k} : \pi_1(\mathbb{P}^1 \setminus t, t)_{k(U)} \to \pi_1(\mathbb{P}^1 \setminus t, t)_k \) is surjective and \( \text{res}_{k(U)/k} : \pi_1(\mathbb{P}^1 \setminus t, t)_{k(U)} \to \pi_1(\mathbb{P}^1 \setminus t, t)_k \) is an isomorphism.

**Proof.** Every \( \sigma \in \pi_1(\mathbb{P}^1 \setminus t, t)_k \) extends to an element of \( G_{k(T)} \), which extends naturally to an automorphism of \( k(T)(U) \) fixing \( U \) (and \( k(T) \)), which in turn extends to an element \( \tilde{\sigma} \in G_{k(T,U)} \). As \( t \) is \( G_k \)-invariant, \( \tilde{\sigma} \) permutes the extensions \( F/k(U)(T) \) that are unramified above \( \mathbb{P}^1 \setminus t \). Conclude that \( \tilde{\sigma} \) factors through \( \pi_1(\mathbb{P}^1 \setminus t, t)_{k(U)} \) to provide a preimage of \( \sigma \) via the map \( \text{res}_{k(U)/k} \), as desired in the first statement.

To show that \( \text{res}_{k(U)/k} : \pi_1(\mathbb{P}^1 \setminus t, t)_{k(U)} \to \pi_1(\mathbb{P}^1 \setminus t, t)_k \) is surjective, it suffices to show that the following morphism is:

\[
\text{Gal}(\Omega_{t,k} \cdot k(U)/k(U)(T)) \to \pi_1(\mathbb{P}^1 \setminus t, t)_k = \text{Gal}(\Omega_{t,k}/k(T)).
\]

This morphism is in fact an isomorphism: indeed extending the base field from \( k \) to \( k(U) \) (over which \( T \) is transcendental) does not change the group of regular Galois extensions.

As \( k \) is of characteristic 0, the morphism \( \text{res}_{k(U)/k} : \pi_1(\mathbb{P}^1 \setminus t, t)_{k(U)} \to \pi_1(\mathbb{P}^1 \setminus t, t)_k \) is even an isomorphism. More precisely, it follows from [Ser92, theorem 6.3.3] that \( \Omega_{t,k(U)} = \Omega_{t,k} \cdot k(U) \).

Set \( \phi \otimes_k k(U) = \phi \circ \text{res}_{k(U)/k} \). The epimorphism \( \phi \otimes_k k(U) : \pi_1(\mathbb{P}^1 \setminus t, t)_{k(U)} \to G \)
is a $\pi_1$-representation of the regular $G$-Galois cover $f \otimes_k k(U)$.

3.2.2. A $G_{k(U)}$-representation. Composing $\phi \otimes_k k(U)$ with the section $s_U : G_{k(U)} \to \pi_1(\mathbb{P}^1 \setminus t, t)_{k(U)}$ associated with the point $U \in \mathbb{P}^1(k(U))$ provides a $G_{k(U)}$-representation

$$\phi_U : G_{k(U)} \to G$$

which is the specialization representation of $\phi \otimes_k k(U)$ at $t = U$. It corresponds to the generic fiber of $F/k(T)$. Denote it by $F_U/k(U)$.

3.2.3. The self-twisted cover. Twist the representation $\phi \otimes_k k(U)$ by the epimorphism $\phi_U$ to get the self-twisted representation

$$\tilde{\phi} \phi_U : \pi_1(\mathbb{P}^1 \setminus t, t)_{k(U)} \to S_d.$$

We call the corresponding cover

$$\tilde{f} \phi_U : \tilde{X} \otimes_k k(U) \phi_U \to \mathbb{P}^1_{k(U)}$$

the self-twisted cover of $f$.

3.2.4. The fiber-twisted cover at $t_0$. Let $t_0 \in \mathbb{P}^1(k) \setminus t$. Twist the representation $\phi$ by the specialization representation $\phi \circ s_{t_0} : G_k \to G$ to get the twisted representation

$$\tilde{\phi} \phi_{s_{t_0}} : \pi_1(\mathbb{P}^1 \setminus t, t)_{k} \to S_d$$

which corresponds to a cover

$$\tilde{f} \phi_{s_{t_0}} : \tilde{X} \phi_{s_{t_0}} \to \mathbb{P}^1_k.$$

We call them respectively the fiber-twisted representation and the fiber-twisted cover at $t_0$.

3.2.5. Description of the self-twisted cover. Set $\Psi_U = \phi \otimes_k k(U) \phi_U$. For every $\Theta \in \pi_1(\mathbb{P}^1 \setminus t, t)_{k(U)}$, we have

$$\Psi_U(\theta) = \gamma(((\phi \otimes_k k(U))(\Theta)) \times \delta(\phi_U(R(\Theta))^{-1})$$

where $R : \pi_1(\mathbb{P}^1 \setminus t, t)_{k(U)} \to G_{k(U)}$ is the natural surjection. The element $\Theta$ uniquely writes $\Theta = \chi s_U(\sigma)$ with $\chi \in \pi_1(\mathbb{P}^1 \setminus t, t)_{k(U)}$ and $\sigma \in G_{k(U)}$. Whence

$$(\phi \otimes_k k(U))(\Theta) = (\phi \otimes_k k(U))(\chi)(\phi \otimes_k k(U))(s_U(\sigma))$$

and, using that $\phi_U = \phi \otimes_k k(U) \circ s_U$,

$$\phi_U(R(\Theta)) = (\phi \otimes_k k(U))(s_U(\sigma)).$$
Finally we obtain the following formula, where, by $\text{conj}(g)$ ($g \in G$), we denote the permutation of $G$ induced by the conjugation $x \rightarrow gxg^{-1}$:

$$\Psi_U(\theta) = \gamma((\phi \otimes_k k(U))(\chi)) \times \text{conj}((\phi \otimes_k k(U))(s_U(\sigma))).$$

Denote the field extension corresponding to the $\pi_1$-representation $\Psi_U$ by $F_k(U)/k(U)(T)$. The field $F_k(U)$ is the fixed field in $\Omega_{t,k(U)}$ of the subgroup $\Gamma_U \subset \pi_1(\mathbb{P}^1 \setminus t, t)_{k(U)}$ of all elements $\Theta$ such that $\Psi_U(\theta)$ fixes the neutral element of $G$. We obtain

$$\Gamma_U = \ker(\phi \otimes_k k(U)) \cdot s_U(G_{k(U)})$$

and $F_k(U)$ is the fixed field in $F_k(U)$ of all elements in $s_U(G_{k(U)})$.

3.2.6. Description of the fiber-twisted covers. Let $t_0 \in \mathbb{P}^1(k) \setminus t$ and set $\phi_{t_0} = \phi \circ s_{t_0}$ and $\Psi_{t_0} = \phi_{t_0}^{\phi_{t_0}}$. Every element $\theta \in \pi_1(\mathbb{P}^1 \setminus t, t)_{k_0}$ uniquely writes $\theta = x s_{t_0}(\tau)$ with $x \in \pi_1(\mathbb{P}^1 \setminus t, t)_{k_0}$ and $\tau \in G_k$. Proceeding exactly as above but with $U$ replaced by $t_0$, $\phi \otimes_k k(U)$ by $\phi$ and $\Theta = \chi s_U(\sigma)$ by $\theta = x s_{t_0}(\tau)$, we obtain that

$$\Psi_{t_0}(\theta) = \gamma(\phi(x)) \times \text{conj}(\phi(s_{t_0}(\tau)))$$

and if $\widetilde{F}_{t_0} / k(T)$ is the field extension corresponding to the $\pi_1$-representation $\Psi_{t_0}$, $\widetilde{F}_{t_0}$ is the fixed field in $F$ of all elements in $s_{t_0}(G_k)$.

3.2.7. Comparison.

**Lemma 3.4.** There is a finite subset $\mathcal{E} \subset k$ such that for each $t_0 \in k \setminus \mathcal{E}$, the fiber-twisted cover $f_{\otimes s_{t_0}} : \widetilde{X} \otimes_{s_{t_0}} \phi_U \rightarrow \mathbb{P}^1_k$ is $k$-isomorphic to the specialization of the self-twisted cover $f \otimes_k k(U) : X \otimes_k k(U) \rightarrow \mathbb{P}^1_{k(U)}$ at $U = t_0$.

**Proof.** Set $d = |G|$. By construction, the extension $(\widetilde{F}_k(U))/(k(U)(T)$ is regular over $k(U)$. From the Bertini-Noether theorem, for every $t_0 \in k$ but in a finite subset $\mathcal{E}$, which we possibly enlarge to contain the branch point set $t$, the specialized extension at $U = t_0$ is regular over $k$ and is of degree

$$[\widetilde{F}_k(U) : k(U)(T)] = [F_k(U) : k(U)(T)] = [F : k(T)] = d.$$ 

Up to enlarging again $\mathcal{E}$, one may also assume that the genus of this specialization is the same as the genus of the function field $\widetilde{F}_k(U)$, which equals $g$, the genus of $F$. The rest of the proof shows that this specialization is the extension $\widetilde{F}_{t_0} / k(T)$.

\(^4\)Taking any other element of $G$ gives the same field up to $k(U)(T)$-isomorphism.
Set $d = |G|$. Pick primitive elements $Y$ and $\tilde{Y}_U$ of the two extensions $F/k(T)$ and $\widetilde{F k(U)}^{\phi_U}/k(U)(T)$, integral over $k[T]$ and $k[U,T]$ respectively. As $\widetilde{F k(U)}^{\phi_U} \subset Fk(U)$, one can write

$$\tilde{Y}_U = \sum_{i=0}^{d-1} a_i(U) Y^i$$

with $a_0(U), \ldots, a_{d-1}(U) \in k(U)$. Enlarge the set $\mathcal{E}$ to contain the poles of $a_0(U), \ldots, a_{d-1}(U)$. Fix $t_0 \in k \setminus \mathcal{E}$. Consider the corresponding specialization $\tilde{Y}_{t_0} = \sum_{i=0}^{d-1} a_i(t_0) Y^i$. The associated extension $k(T, \tilde{Y}_{t_0})/k(T)$ is the specialization of $\widetilde{F k(U)}^{\phi_U}/k(U)(T)$ at $U = t_0$. By construction $\tilde{Y}_{t_0} \in F$. The last paragraph of the proof below shows that $\tilde{Y}_{t_0}$ is fixed by all elements in $s_{t_0}(G_k)$. We will then be able to conclude that $k(T, \tilde{Y}_{t_0}) \subset \tilde{F}^{\phi_{t_0}}$ and finally that these two fields are equal since $[k(T, \tilde{Y}_{t_0}) : k(T)] = [\tilde{F}^{\phi_{t_0}} : k(T)] = d$.

As $U \notin t$, there exists an embedding

$$\widetilde{F k(U)}^{\phi_U} \to k(U)((T - U))$$

which maps $Y_U$ to a formal power series

$$\tilde{Y}_U = \sum_{n=0}^{\infty} b_n(U)(T - U)^n \quad \text{with } b_n(U) \in \overline{k(U)} \quad (n \geq 0).$$

Furthermore, $\widetilde{F k(U)}^{\phi_U}$ is fixed by all elements $s_U(\sigma) \in s_U(G_{k(U)})$, which, by definition of $s_U$, act via the action of $\sigma \in G_{k(U)}$ on the coefficients $b_n(U)$; conclude that $b_n(U) \in k(U)$ $(n \geq 0)$. Finally from the Eisenstein theorem\footnote{This classical result is often stated for formal power series $\sum_{n \geq 0} b_n T^n$, algebraic over $\mathbb{Q}(T)$ and with coefficients $b_n \in \overline{\mathbb{Q}}$, but is true in a bigger generality including the situation where $\mathbb{Q}$ and $\mathbb{Z}$ are respectively replaced by $k(U)$ and $k[U]$. For example, the proof given in [DR79] easily extends to this situation.}, there exists a polynomial $E(U) \in k[U]$ such that $E(U)^n b_n(U) \in k[U]$ for every $n \geq 0$. Enlarge again the set $\mathcal{E}$ to contain the roots of $E(U)$. For $t_0 \in k \setminus \mathcal{E}$, specializing $U$ to $t_0$ in the displayed formula above produces $\tilde{Y}_{t_0}$ as a formal power series in $k[[T - t_0]]$, which amounts to saying that, up to some $k$-isomorphism, $\tilde{Y}_{t_0}$ and so $\tilde{F}^{\phi_{t_0}}$ are fixed by all elements in $s_{t_0}(G_k)$. \hfill $\square$

Let $\tilde{P}(U, T, Y) \in k[U, T, Y]$ be the irreducible polynomial of $\tilde{Y}_U$ over $k[U, T]$. Theorem 3.2 holds for this polynomial $\tilde{P}(U, T, Y)$ (up to enlarging again the finite set $\mathcal{E}$). When $k = \mathbb{Q}$ we may and will choose
the element \( \tilde{Y}_U \) integral over \( \mathbb{Z}[T,Y] \) (and not just \( \mathbb{Q}(T,Y) \)) so that \( \tilde{P}(U,T,Y) \) lies in \( \mathbb{Z}[U,T,Y] \) and will assume further that the coefficients of \( P(U,T,Y) \) are relatively prime.

4. Proof of theorem 2.3

We retain the notation and data introduced in §2.1.

Fix a real number \( x > p_0 \) and a Frobenius data \( \mathcal{F}_x \) on \( S_x \).

Fix also a subset \( S_0 \) of primes \( p \in [p_1,p_0] \), with as many elements as there are non-trivial conjugacy classes in \( G \). Associate then in a one-one way a non-trivial conjugacy class \( F_p \) to each prime \( p \in S_0 \). Set \( S_0_x = S_0 \cup S_x \) and denote the Frobenius data \( (\mathcal{F}_p)_{p \in S_{0x}} \) by \( \mathcal{F}_{0x} \).

4.1. First part: many good specializations \( t_0 \in \mathbb{Z} \).

The goal of the first part is proposition 4.3 which shows that there are “many” \( t_0 \in \mathbb{Z} \) such that conditions (i)-(iv) of theorem 2.3 are satisfied. The goal of the second part will be to show that there are “many” distinct corresponding extensions \( F_{t_0}/\mathbb{Q} \).

We use the method of [DG12] for this first part. We re-explain it in the special context of this paper and make the adjustments that we will need for the rest of the proof. We refer to [DG12] for more details on the main arguments and for references. Working over number fields and even over \( \mathbb{Q} \), we can give improved quantitative conclusions (compared to the existence statements of [DG12]). As in [DG12], there is first a local stage followed by a globalization argument.

4.1.1. Local stage. Below, given \( t_0 \in \mathbb{Q}_p \) we say that \( t_0 \notin \mathfrak{t} \) modulo \( p \) if \( t_0 \) does not meet any of the branch points of \( F/\mathbb{Q}(T) \) modulo \( p \).

Proposition 4.1. Given our regular \( \mathbb{Q}-G \)-Galois cover \( f : X \to \mathbb{P}^1 \), a prime \( p \) and a subset \( \mathcal{F}_p \subset G \) consisting of a non-empty union of conjugacy classes of \( G \), consider the set

\[
\mathcal{T}(\mathcal{F}_p) = \left\{ t_0 \in \mathbb{Z} \mid t_0 \notin \mathfrak{t} \text{ modulo } p, \text{ Frobp}(F_{t_0}/\mathbb{Q}) \in \mathcal{F}_p \right\}.
\]

If \( p \) is a good prime for \( f \), the set \( \mathcal{T}(\mathcal{F}_p) \) is the union of cosets modulo \( p \). Furthermore, the number \( \nu(\mathcal{F}_p) \) of these cosets satisfies

\[
\nu(\mathcal{F}_p) \geq \frac{|\mathcal{F}_p|}{|G|} \times (p + 1 - 2g\sqrt{p} - |G|(r + 1))
\]

Recall that for two points \( t, t' \in \overline{\mathbb{Q}}_p \cup \{\infty\} \), meeting modulo \( p \) means that \( |t|_p \leq 1, |t'|_p \leq 1 \) and \( |t - t'|_p < 1 \), or else \( |t|_p \geq 1, |t'|_p \geq 1 \) and \( |t^{-1} - (t')^{-1}|_p < 1 \), where \( \overline{\mathbb{Q}}_p \) is some arbitrary prolongation of the \( p \)-adic absolute value \( v \) to \( \overline{\mathbb{Q}}_p \).
and $\nu(F_p) \leq \frac{|F_p|}{|G|} \times (p + 1 + 2g\sqrt{p})$.

Proof. We follow the method from [DG12]. Similar estimates though not in this explicit form can also be found in [Eke90].

We may and will assume that the subset $F_p$ consists of a single conjugacy class.

Set $f_p = f \otimes \mathbb{Q} \mathbb{Q}_p$ and denote the corresponding $\pi_1$-representation by $\phi_p : \pi_1(\mathbb{P}^1 \setminus t, t)_{\mathbb{Q}_p} \to \mathbb{G}$. Pick an element $g_p \in F_p$ and consider the unique unramified epimorphism $\phi_p : G_{\mathbb{Q}_p} \to \langle g_p \rangle$ that sends the Frobenius of $\mathbb{Q}_p$ to $g_p$. The condition "$t_0 \notin t$ modulo $p$" implies that $p$ is unramified in the specialization $F_{t_0}/\mathbb{Q}$. Then $t_0 \in T(F_p)$ if and only if the specialization representation $\phi \circ s_{t_0} : G_{\mathbb{Q}} \to G$ of $F/\mathbb{Q}(T)$ at $t_0$ is conjugate in $G$ to $\phi_p : G_{\mathbb{Q}_p} \to \langle g_p \rangle$. From the twisting lemma 3.1, this is equivalent to the existence of a $\mathbb{K}$-rational point above $t_0$ in the covering space of the twisted cover $\tilde{f}_p^{\phi_p} : \tilde{X}_p^{\phi_p} \to \mathbb{P}^1$. As $p$ is a good prime, this last cover has good reduction; denote the special fiber by $\tilde{f}_p : \tilde{X}_p \to \mathbb{P}^1_{\mathbb{F}_p}$. The last existence condition is then equivalent to the existence of some point $\mathbf{p} \in \tilde{X}_p(\mathbb{F}_p)$ above the coset $t_0 \in \mathbb{P}^1(\mathbb{F}_p)$ of $t_0$ modulo $p$: the direct part is clear while the converse follows from Hensel’s lemma.

From the Lang-Weil bound, the number of $\mathbb{F}_p$-rational points on $\tilde{X}_p$ is $\geq p + 1 - 2g\sqrt{p}$. Removing the points that lie above a branch point or the point at infinity leads to the announced first estimate, a final observation for this calculation being that for $t_0 \notin t$ modulo $p$, the number of $\mathbb{F}_p$-rational points $\mathbf{p} \in \tilde{X}_p(\mathbb{F}_p)$ above $t_0$ is $|\text{Cen}_G(g_p)| = |G|/|F_p|$: this number is indeed the same as the number of $\omega \in G$ such that $\phi \circ s_{t_0} = \omega \varphi_p \omega^{-1}$ (as the proof of the twisting lemma in [DG12] shows). Using the upper bound part of Lang-Weil leads to the second estimate. \qed

If in addition $p \geq r^2|G|^2$ (in particular if $p \in S_{0x}$), then the right-hand side term in the inequality of proposition 4.1 is $> 0$ (use that $g < r|G|/2 - 1$ if $|G| > 1$, which follows from Riemann-Hurwitz).

**Proposition 4.2.** Assume the branch point $t_1$ of the $\mathbb{Q}$-$G$-Galois cover $f : X \to \mathbb{P}^1$ is in $\mathbb{Z}$. Given a prime $p$, consider the set $T(ra/p) = \{t_0 \in \mathbb{Z} \mid F_{t_0}/\mathbb{Q} \text{ is ramified at } p\}$.

If $p$ is a good prime for $f$, the set $T(ra/p)$ contains the coset of $t_1 + p \in \mathbb{Z}$ modulo $p^2$. 
Proof. Let \( t_0 \in \mathbb{Z} \) such that \( t_0 \equiv t_1 + p \) modulo \( p^2 \). Then \( t_0 - t_1 \) is of \( p \)-adic valuation 1. As \( p \) is good, it follows that \( F_{t_0}/\mathbb{Q} \) is ramified at \( p \). This last conclusion is part of the “Grothendieck-Beckmann theorem” for which we refer to [Gro71] and [Bec91, proposition 4.2]; see also [Leg13] where this result is discussed together with further developments in the spirit of proposition 4.2. \( \square \)

4.1.2. Globalization. Set
\[
\begin{align*}
\beta &= \Pi(S_0) \\
B(x) &= \beta \Pi(S)^2 \Pi(S_x)
\end{align*}
\]
and consider the intersection
\[
\bigcap_{p \in S_{0x}} T(F_p) \cap \bigcap_{p \in S} T(ra/p).
\]
From proposition 4.1, proposition 4.2 and the Chinese remainder theorem, this set contains
\[
\mathcal{N}(S, F_{0x}) = \prod_{p \in S_{0x}} \nu(F_p)
\]
cosets modulo \( B(x) \). Denote the set of their representatives in \([1, B(x)]\) by \( T(S, F_{0x}) \); the cardinality of this set is \( \mathcal{N}(S, F_{0x}) \).

Proposition 4.3. (a) For every integer \( t_0 \in T(S, F_{0x}) \), the extension \( F_{t_0}/\mathbb{Q} \) satisfies the four conditions (i)-(iv) from theorem 2.3, with (ii) even replaced by the following sharper version (ii+) of (ii), that is
(i) \( \text{Gal}(F_{t_0}/\mathbb{Q}) = G \),
(ii+) \( F_{t_0}/\mathbb{Q} \) is unramified and \( \text{Frob}_p(F_{t_0}/\mathbb{Q}) \in \mathcal{F}_p \) for every \( p \in S_{0x} \) (and not just for every \( p \in S_x \)),
(iii) \( F_{t_0}/\mathbb{Q} \) is ramified at \( p \) for every \( p \in S \),
(iv) \( |d_{F_{t_0}}| \leq \rho(x) \).

(b) We have
\[
\mathcal{N}(S, F_{0x}) \geq \chi(F_x) \frac{B(x)}{\beta \Pi(S)^2} \left( \frac{1}{2r|G|} \right)^{|S_x|}
\]
(c) The set of integers \( t_0 \in \mathbb{Z} \) such that for each \( p \in S_x \), \( F_{t_0}/\mathbb{Q} \) is unramified and \( \text{Frob}_p(F_{t_0}/\mathbb{Q}) \in \mathcal{F}_p \) consists of cosets modulo \( \Pi(S_x) \) and the set \( T(\emptyset, F_x) \) of their representatives in \([1, \Pi(S_x)]\) is of cardinality
\[
\mathcal{N}(\emptyset, F_x) = \prod_{p \in S_x} \nu(F_p) \leq \chi(F_x) \frac{\Pi(S_x)}{\beta} (2 - \lambda)^{|S_x|}
\]
where \( \lambda = (r|G| - 1)/r^2|G|^2 \).

Conclusion (c) proves conclusion (d) of theorem 2.3.
Proof. (a) Fix \( t_0 \in T(S, F_{0x}) \) (or more generally congruent modulo \( B(x) \) to some element in \( T(S, F_{0a}) \)).

Conditions (ii+), (iii) hold by definition of the sets \( T(F_p) \) and \( T(ra/p) \).

A classical argument then shows that (i) follows from (ii+): indeed because of the Frobenius condition at the primes \( p \in S_0 \), the subgroup \( \text{Gal}(F_{t_0}/\mathbb{Q}) \subset G \) meets every conjugacy class of \( G \); from a lemma of Jordan [Jor72], it is all of \( G \).

From (i), the polynomial \( P(t_0, Y) \) is irreducible in \( \mathbb{Q}[Y] \). As it is monic and with integral coefficients, its discriminant, which is \( \Delta P(t_0) \), is a multiple of the absolute discriminant \( d_{F_{t_0}} \) of the extension \( F_{t_0}/\mathbb{Q} \).

Conjoined with \( 1 \leq t_0 \leq B(x) \), this leads to
\[
|d_{F_{t_0}}| \leq (1 + \delta P) H(\Delta P) B(x)^\delta P
\]
and conclusion (iv) follows from the definition of \( \rho(x) \) (given in §2.3.2).

(b) Using proposition 4.1, we obtain
\[
\mathcal{N}(S, \mathcal{F}_{0x}) \geq \prod_{y \in S_x} \frac{|F_p|}{|G|} \times (p + 1 - 2g\sqrt{p} - |G|(r + 1)) \geq \chi(F_x) \times \prod_{p \in S_x} \frac{1}{p} \times \prod_{p \in S_x} \left( 1 + \frac{1}{p} - \frac{2g\sqrt{p}}{p} - \frac{(r + 1)|G|}{p} \right)
\]
Using again that \( g < r|G|/2 - 1 \) (if \( |G| > 1 \)) and that \( p \geq r^2|G|^2 \) for each \( p \in S_x \), we have
\[
1 + \frac{1}{p} - \frac{2g\sqrt{p}}{p} - \frac{|G|(r + 1)}{p} > 1 - \frac{r|G| - 2}{r|G|} - \frac{(r + 1)|G|}{r^2|G|^2} = \frac{2}{r|G|} - \frac{(r + 1)|G|}{r^2|G|^2} = \frac{(r - 1)|G|}{r^2|G|^2} \geq \frac{1}{2r|G|}
\]
which yields the announced first estimate.

(c) Here we use the conclusion from proposition 4.1 that for each \( p \in S_x \), the set \( T(F_p) \) consists exactly of \( \nu(F_p) \) cosets modulo \( p \). Combined with the Chinese remainder, this gives that the set \( T(\emptyset, F_x) \) consists of exactly \( \mathcal{N}(\emptyset, F_x) = \prod_{p \in S_x} \nu(F_p) \) elements. Proceed then similarly as in (b) but using the upper bound part of proposition 4.1 to obtain the desired estimate. \( \square \)

Remark 4.4. Consider the situation with \( S = \emptyset \) and allowing no local condition at some primes \( p \in S_x \) — no restriction on \( \text{Frob}_p(F_{t_0}/\mathbb{Q}) \) and no unramified condition —. We have \( \nu(F_p) = p \) for such primes and
obtain this generalized lower bound: if \( S'_x \subset S_x \) is the subset of primes where there is a local condition, then

\[
N(\emptyset, \mathcal{F}_{0x}) \geq \chi(\mathcal{F}_x) \frac{B(x)}{\beta} \left( \frac{1}{2r|G|} \right)^{|S'_x|}.
\]

In particular, the number of integers \( t_0 \in [1, B(x)] \) where conditions (i) \( \text{Gal}(F_{t_0}/\mathbb{Q}) = G \) and (iv) \( |d_{F_{t_0}}| \leq \rho(x) \) hold (i.e. no local condition at any prime) is \( \geq B(x)/\beta \).

4.2. Second part: many good specializations \( F_{t_0}/\mathbb{Q} \).

4.2.1. Reduction to counting integral points on curves. We will now estimate the number, say \( N(S, \mathcal{F}_{0x}) \), of distinct specializations \( F_{t_0}/\mathbb{Q} \) with \( t_0 \in \mathcal{T}(S, \mathcal{F}_{0x}) \). We will give a lower bound for the number of non conjugate specialization representations \( \phi \circ s_{t_0} : \text{G}_\mathbb{Q} \rightarrow G \) with \( t_0 \in \mathcal{T}(S, \mathcal{F}_{0x}) \). Given two such representations, the associated field extensions are equal if and only if the representations have the same kernel, or, equivalently, if they differ by some automorphism of \( G \).

Dividing the previous bound by \( |\text{Aut}(G)| \) will thus yield the desired bound for \( N(S, \mathcal{F}_{0x}) \).

Consider the polynomial \( \tilde{P}(U, T, Y) \in \mathbb{Z}[U, T, Y] \) given by theorem 3.2 and its discriminant \( \Delta_{\tilde{P}} \in \mathbb{Z}[U, T] \) (relative to \( Y \)). As \( \tilde{P}(U, T, Y) \) is irreducible in \( \mathbb{Q}(U, T)[Y] \), \( \Delta_{\tilde{P}}(U, T) \neq 0 \). Write it as a polynomial in \( T \) of degree \( N \) and denote its leading coefficient by \( \Delta_{\tilde{P}, N}(U) \); we have \( \Delta_{\tilde{P}, N}(U) \in \mathbb{Z}[U] \) and \( \Delta_{\tilde{P}, N}(U) \neq 0 \).

Drop from the set \( \mathcal{T}(S, \mathcal{F}_{0x}) \) the finitely many integers \( u_0 \) for which \( \Delta_{\tilde{P}, N}(u_0) = 0 \) or which are in the exceptional set \( \mathcal{E} \) from theorem 3.2. Denote the resulting set by \( \mathcal{T}(S, \mathcal{F}_{0x})' \) and the number of dropped elements by \( E \).

Fix \( u_0 \in \mathcal{T}(S, \mathcal{F}_{0x})' \) and consider the fiber-twisted cover at \( u_0 \):

\[
\tilde{f}_{\phi s_{u_0}} : \tilde{X}^{\phi s_{u_0}} \rightarrow \mathbb{P}^1_{\mathbb{Q}}.
\]

Let \( t_0 \in \mathcal{T}(S, \mathcal{F}_{0x})' \). From the twisting lemma 3.1, the two representations \( \phi \circ s_{u_0} \) and \( \phi \circ s_{t_0} \) are conjugate in \( G \) if and only if there exists \( x_0 \in \tilde{X}^{\phi s_{u_0}}(\mathbb{Q}) \) such that \( \tilde{f}_{\phi s_{u_0}}(x_0) = t_0 \).

We have \( \Delta_{\tilde{P}}(u_0, t_0) \neq 0 \) except for at most \( N \) integers \( t_0 \). For the non-exceptional \( t_0 \), the polynomial \( \tilde{P}(u_0, t_0, Y) \) has only distinct roots \( y \in \overline{\mathbb{Q}} \) and, using theorem 3.2, the corresponding points \( (t_0, y) \) on the affine curve \( \tilde{P}(u_0, t, y) = 0 \) exactly correspond to the points \( x \) on the smooth projective curve \( \tilde{X}^{\phi s_{u_0}} \) above \( t_0 \). Furthermore, in this correspondence, the \( \mathbb{Q} \)-rational points \( x \) correspond to the couples \( (t_0, y) \) with \( y \in \mathbb{Q} \).

Conclude that up to some term \( \leq N \), the number of \( t_0 \) for which \( \phi \circ s_{u_0} \)
and $\phi \circ s_{t_0}$ are conjugate in $G$ is equal to the number of $\mathbb{Q}$-rational points $(t_0, y)$ on the affine curve $\tilde{P}(u_0, t, y) = 0$.

Note further that such a $\mathbb{Q}$-rational point $(t_0, y)$ has necessarily integral coordinates as $t_0 \in \mathbb{Z}$ and $\tilde{P}(u_0, T, Y) \in \mathbb{Z}[T, Y]$ and is monic in $Y$. Therefore we are reduced to counting the integers $t_0 \in [1, B(x)]$ such that there is an integral point $(t_0, y) \in \mathbb{Z}^2$ on the curve $\tilde{P}(u_0, t, y) = 0$.

4.2.2. Diophantine estimates. The constants $c_i, i > 0$ that will enter depend only on the polynomial $P(T, Y)$.

The curve $\tilde{P}(u_0, t, y) = 0$ is of genus $g$ (theorem 3.2 (c)) and we have

$$\begin{cases} 
\deg(\tilde{P}(u_0, T, Y)) \leq \deg(\tilde{P}(U, T, Y)) = c_1 \\
\deg_Y(\tilde{P}(u_0, T, Y)) = \deg_Y(\tilde{P}(U, T, Y)) = |G| \\
H(\tilde{P}(u_0, T, Y)) \leq c_2u_0^{c_3} \leq c_2B(x)^{c_3}
\end{cases}$$

For real numbers $\gamma, D, H, B \geq 0$ and $d_Y \geq 2$, consider all polynomials $F \in \mathbb{Z}[T, Y]$, primitive, monic in $Y$, irreducible in $\mathbb{Q}(T)[Y]$, such that $\deg_Y(F) = d_Y$, of total degree $\leq D$, of height $\leq H$ and such that the affine curve $P(t, y) = 0$ is of genus $\leq \gamma$. For each such polynomial, the number of integers $t \in [1, B]$ such that there exists $y \in \mathbb{Z}$ such that $F(t, y) = 0$ is a finite set. Denote by $Z(\gamma, D, d_Y, H, B)$ the maximal cardinality of all these finite sets.

Using the diophantine parameter $Z(\gamma, D, d_Y, H, B)$, conclude that the number of $t_0 \in T(S, \mathcal{F}_{0x})'$ such that the two representations $\phi \circ s_{u_0}$ and $\phi \circ s_{t_0}$ are conjugate in $G$ is less than or equal to

$$Z(g, c_1, |G|, c_2B(x)^{c_3}, B(x)).$$

Thus we obtain

$$N(S, \mathcal{F}_{0x}) \geq \frac{N(S, \mathcal{F}_{0x}) - E}{|\text{Aut}(G)| \cdot [Z(g, c_1, |G|, c_2B(x)^{c_3}, B(x)) + N]}$$

Next take into account proposition 4.3 and note that $N_F(x, S, \mathcal{F}_x) \geq N(S, \mathcal{F}_{0x})$ to write

$$N_F(x, S, \mathcal{F}_x) \geq \frac{\chi(\mathcal{F}_x)(2r|G|)^{-|S_x|}B(x)}{\beta \Pi(S)^2} - E$$

Assume that the genus $g$ of $X$ is $\geq 2$ and that Lang’s conjecture holds. This conjecture is that if $V$ is a variety of general type defined over a number field $K$ then the set $V(K)$ of $K$-rational points is not Zariski-dense in $V$ [Lan86]. We will use it through the following consequence proved by Caporaso, Harris and Mazur [CHM97]: they showed that Lang’s conjecture implies that for every number field $K$
and every integer \( g \geq 2 \) there exists a finite integer \( B(g, K) \) such that
\[ \text{card}(\mathcal{C}(K)) \leq B(g, K) \]
for every curve \( C \) of genus \( g \) defined over \( K \).

Under this conjecture we obtain
\[ Z(g, c_1, |G|, c_2B(x)^{c_3}, B(x)) + N \leq c_4. \]

In the general case \( g \geq 0 \) we use an unconditional result of Walkowiak [Wal05, §2.4] which shows that if \( d_y \geq 2 \) then
\[ Z(\gamma, D, d_y, H, B) \leq a_1 D^2 (\log H^+)^{a_4} B^{1/d_y} (\log B)^{a_4} \]
where \( H^+ = \max(H, e^c) \) and \( a_1, \ldots, a_4 \) are absolute constants. See §4.2.3 for more on this result. We deduce
\[ Z(g, c_1, |G|, c_2B(x)^{c_3}, B(x)) + N \leq c_5 B(x)^{1/|G|} \log(B(x))^{c_6}. \]

Conclude that unconditionally:
\[ N_F(x, S, \mathcal{F}_x) \geq c_7 \frac{\chi(\mathcal{F}_x)}{\Pi(S)^2} \frac{B(x)^{1-1/|G|}}{(\log B(x))^{c_9} c_8^{[S_2]}} - c_{10} \]
and, under Lang’s conjecture:
\[ N_F(x, S, \mathcal{F}_x) \geq c_7 \frac{\chi(\mathcal{F}_x)}{\Pi(S)^2} \frac{B(x)}{c_8^{[S_2]}} - c_{10}. \]

Note that \( c_{11} \Pi(x) \leq B(x) \leq \Pi(x) \Pi(S)^2 \), that \( |S_2| \leq \pi(x) \) and \( 0 < c_8 < 1 \) to obtain the estimates of theorem 2.3 (a) and (c). Theorem 2.3 (b) follows from the containment \( \mathcal{T}(S, \mathcal{F}_x) \subset [1, B(x)] \subset [1, \Pi(S) \Pi(x)] \).

4.2.3. Walkowiak’s result. Let \( F \in \mathbb{Z}[T, Y] \) be a polynomial, irreducible in \( \mathbb{Z}[T, Y] \). Set \( D = \deg(F) \) and \( H^+ = \max(H(F), e^c) \). The result we use in the proof above is the following.

**Theorem 4.5 (Walkowiak).** Assume \( \deg_Y F \geq 2 \). There exist absolute constants \( a_1, \ldots, a_4 \) such that for every real number \( B > 0 \), the number of integers \( t_0 \in [1, B] \) such that \( F(t_0, Y) \) has a root in \( \mathbb{Z} \) is less than
\[ a_1 D^{a_2} (\log H^+)^{a_4} B^{1/\deg_Y(F)} (\log B)^{a_4}. \]

This result is proved in [Wal05, §2.4] but with \( B^{1/2} \) instead of \( B^{1/\deg_Y(F)} \). We explain here how to modify Walkowiak’s arguments to obtain the better exponent \( 1/\deg_Y(F) \). The only change to make is in the final stage of the proof in [Wal05, §§2.2-2.3].

**Proof.** Walkowiak’s central result is the following bound for the number
\[ N(F, B) \text{ of } (t, y) \in \mathbb{Z}^2 \text{ such that } \max(|t|, |y|) \leq B \text{ and } F(t, y) = 0: \]
\[ N(F, B) \leq 2^{36} D^5 \log^3(1250d^{11} B^{5D-1}) \log^2(B) B^{1/D}. \]

To prove theorem 4.5, his basic idea is to use Liouville’s inequality to get upper bounds \( |y| \leq B' \) for roots \( y \in \mathbb{Z} \) of polynomials \( F(t_0, Y) \).
with \( t_0 \in [1, B] \); the bound above for \( N(F, B) \) with \( B \) taken to be \( B' \) provides then a bound for the desired set. The main terms that appear in the resulting bound come from \((B')^{1/D}\). They may be too big however in some cases and Walkowiak uses a trick to obtain his final bound in \( B^{1/2} \). In order to obtain \( B^{1/n} \) instead, Walkowiak’s trick should be modified as follows.

Set \( L_1 = \log(H^+) \), \( L_2 = \log(\log(H^+)) \), \( m = \deg_T F \) and \( n = \deg_Y F \); one may assume \( m > 0 \). Let \( t_0 \in [1, B] \) such that \( F(t_0, Y) \) has a root \( y \in \mathbb{Z} \). Liouville’s inequality gives

\[ |y| \leq 2(m + 1)H^+B^m = B'. \]

The main terms in \((B')^{1/D}\) are \((H^+)^{1/D}\) and \((B^m)^{1/D}\).

**Case 1:** \( mnL_1/L_2 \leq D \). On the one hand, we have \( 1/D \leq L_2/L_1 \) and so \((H^+)^{1/D} \leq (H^+)^{\frac{L_2}{L_1}} = \log(H^+)\). On the other hand \( m/D \leq 1/n \) and so \( B^{m/D} \leq B^{1/n} \). The upper bound for \( N(F, B') \) is indeed as announced in the statement of theorem 4.5.

**Case 2:** \( mnL_1/L_2 > D \). Set \( E = \lceil mnL_1/L_2 \rceil + 1 \) and consider the polynomial \( G \in \mathbb{Z}[T, Y] \) defined by \( G(T, Y) = F(T, T^E + Y) \). For \( y' = y - t_0^E \) we have \( G(t_0, y') = 0 \) and

\[ |y'| \leq 2(m + 1)H^+B^m + B^E \leq 2(m + 1)H^+B^E = B''. \]

Use then the upper bound for \( N(F, B) \) with \( F \) and \( B \) respectively taken to be \( G \) and \( B'' \). As \( \deg_Y G = \deg_Y F = n \) and \( nE \leq \deg G \leq nE + m \), the main terms are in this case

\[
(H^+)^{1/\deg(G)} \leq (H^+)^{1/nE} \leq (H^+)^{L_1/L_2} = \log(H^+)
\]

and \( B^E/\deg G \leq B^{1/n} \).

Again the upper bound for \( N(G, B'') \) is as announced. \( \square \)

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