WAVELET FILTER FUNCTIONS, THE MATRIX COMPLETION PROBLEM, AND PROJECTIVE MODULES OVER $C(\mathbb{T}^n)$

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Abstract. We discuss how one can use certain filters from signal processing to describe isomorphisms between certain projective $C(\mathbb{T}^n)$-modules. Conversely, we show how cancellation properties for finitely generated projective modules over $C(\mathbb{T}^n)$ can often be used to prove the existence of continuous high-pass filters, of the kind needed for multi-wavelets, corresponding to a given continuous low-pass filter. However, we also give an example of a continuous low-pass filter for which it is impossible to find corresponding continuous high-pass filters. In this way we give another approach to the solution of the matrix completion problem for filters of the kind arising in wavelet theory.

A key technique in wavelet theory is to use suitable low-pass filters to construct scaling functions and their multi-resolution analyses, and then to use corresponding high-pass filters to construct the corresponding wavelets. Thus once one has used a low-pass filter to construct a scaling function, it is important to be able to find associated high-pass filters. (They are not unique.) It has been pointed out several times in the literature that, given a continuous low-pass filter for dilation by $q > 2$, there is in general no continuous selection function for the construction of associated continuous high-pass filters, because of the existence of topological obstructions. (See the substantial discussion in section 2 of [6], and the references cited there.) It is known [13], [34], [4] that associated high-pass filters which are measurable will always exist. (See also the general Hilbert-space treatment of the existence of wavelets given in [3].) The problem of finding appropriate high-pass

1991 Mathematics Subject Classification. Primary 46L99; Secondary 42C40, 46H25.

Key words and phrases. finitely generated projective modules, $K$-theory, wavelets, filters, C*-algebras, Hilbert C*-module.

The first author was supported by a research grant from the National University of Singapore. The research of the second author was supported in part by National Science Foundation grant DMS99-70509.
filters given a low-pass filter is closely related to the more general problem of completing an $n \times n$ unitary matrix whose entries take values in $C(\mathbb{T}^n)$ when given its first row. Thus this problem is sometimes called the “matrix completion problem” for wavelets.

The main theorem in the literature concerning the existence of high-pass filters in the multivariate case was first given by Gröchenig [9] for dyadic dilation matrices. It is given a nice exposition in section 3.4 of [17]. The generalization to general dilation matrices is given an attractive treatment in theorem 5.15 of [34]. This theorem states the following. Let $A$ be an $n \times n$ dilation matrix with $|\det(A)| = q$. Let $m_0$ be a low-pass filter function on $\mathbb{T}^n$ associated with $A$ which is what is called $r$-regular. If $2q - 1 > n$, then there exist associated high-pass filters which are $r$-regular. (The case in which $q = 2$ and $n \geq 3$ is not covered by this theorem, but it is well-known that the case $q = 2$ can always be treated by simpler methods, which we indicate in Section 3 below.) The methods of proof are analytical. The condition of $r$-regularity implies that the corresponding filters (i.e. transfer functions) are infinitely differentiable. In Section 3 of this paper we show, using the methods of cancellation for projective modules (or vector bundles), that if we are given a \textbf{continuous} low-pass filter function $m_0$ on $\mathbb{T}^n$ corresponding to $A$, and if $2q - 1 > n$, then there will always exist \textbf{continuous} high-pass filters $m_1, m_2, \ldots, m_{q-1}$ associated to $m_0$.

For most applications one does want the filters to have a certain degree of smoothness. Our emphasis on only requiring that the filters be continuous is mainly to make clear that the issues discussed here are basically topological. Smoothness can always be restored, as discussed at the end of the proof of our main theorem.

We do not give an explicit algorithm for constructing the high-pass filters, which is not surprising since the proof of the cancellation theorem quoted from [11] is not algorithmic in nature, and we already know from the topological obstructions mentioned above that it is impossible to find an algorithm into which one can plug low-pass filters $m_0$ and read out in a continuous way corresponding continuous high pass filters. In both our situation and that of theorem 5.15 of [34] and the earlier references, the issue of constructing, for general dilation matrices, low-pass filters which generate continuous scaling functions is left open.

Finally in the fourth section of the paper we give a fairly explicit example, for $n = 5$ and $q = 3$, of a continuous low-pass filter for which it is \textbf{impossible} to construct a family of corresponding continuous high-pass filters. (Thus the condition $2q - 1 > n$ is best possible for $n = 5$.) Our construction uses classical facts about vector bundles over
spheres. We note that it is shown in [10] that if one is content with wavelet frames, and with the number of wavelets exceeding \( q \), then one can always find compactly supported wavelets of arbitrarily high smoothness.

The second author pointed out in a talk given in 1997 [25] that, at least for filters whose transfer functions are continuous, one can use “projective modules” over \( C(\mathbb{T}^n) \) to give an attractive framework for wavelet theory. In the present paper we make more explicit the relationship between filters and projective modules over \( C(\mathbb{T}^n) \). It is essential for our results that our filters have **continuous** transfer functions. (We will refer to such filters simply as “continuous filters”.) In a subsequent paper [18] we will directly use non-free projective modules to give a new variation on the construction of wavelets.

In the first two sections of the paper we establish our basic framework, and reformulate well-known facts to show that from a collection of continuous filters for an \( n \times n \) integer dilation matrix \( A \), one can construct an explicit isomorphism between a certain module over \( C(\mathbb{T}^n) \) determined by \( A \) and the free modules \( \bigoplus_{i=1}^{q}[C(\mathbb{T}^n)] \), where \( q = |\text{det}(A)| \).

We remark that we will make extensive use of “inner products” with values in algebras such as \( C(\mathbb{T}^n) \). Such inner products have been widely used in some related parts of harmonic analysis. See [14] and references therein. These inner products have also been used to some extent within the signal processing literature, often under the name of “bracket product”. See for example the discussion of bracket products for Weyl-Heisenberg frames in chapter 11 of [7]. These inner products are used intensively in the first sections of [24], which can appropriately be viewed as a discussion of windowed Fourier transforms for lattices in \( \mathbb{R}^n \) which are skew to the standard basis. These inner products seem to have been used in only a few places in the wavelet literature. See, for example, the bracket product defined in equation 2.6 of [26], and associated references. These inner products could profitably be used more widely, since many of the formulas in wavelet theory have attractive reformulations in terms of them, as we will see in part 2 below. These inner products will play an important role in [18].

The most rapidly accessible introduction to multivariate wavelets with which we are familiar is that given in [30], and we will have it in mind during our discussion below. Another attractive exposition is that given in section 5.2 of [34].

It seems likely that our techniques can be applied in the case of multiwavelets, as described for example in [1], but we have not explored this possibility.
The first author would like to thank Professors Lawrence Baggett and Dana Williams for many useful discussions on the topics discussed in this paper, and for their great hospitality towards her and her family during her visit to the University of Colorado at Boulder and Dartmouth College, respectively, during her sabbatical year.

1. Actions of finite subgroups on $C(\mathbb{T}^n)$, free modules, and module frames

Let $A$ be an $n \times n$ integer matrix, and set $q = |\det(A)|$. Assume that $q \geq 2$. (Eventually we will require that $A$ is a dilation matrix.) Let $\Gamma = AZ^n$. It is easily seen that $\Gamma$ is a subgroup of $G = \mathbb{Z}^n$ of finite index $q$. Any subgroup of $\mathbb{Z}^n$ of finite index arises in this way. For the next few paragraphs we can forget about $A$ and just keep $\Gamma$.

Let $B = C_f(G)$ denote the convolution algebra of complex-valued functions on $G$ of finite support. (This is where finite impulse response (FIR) filters reside.) Similarly we set $\mathcal{A} = C_f(\Gamma)$. In the evident way $\mathcal{A}$ is a subalgebra of $B$, and so we can view $B$ as a module over $\mathcal{A}$ using convolution. Let $\{p_1, \ldots, p_q\}$ be a set of coset representatives for the cosets of $\Gamma$ in $G$. For each $j$ let $e_j$ denote the delta-function at $p_j$. Each element, $f$, of $B$ is the sum of its restrictions to the cosets, and so is uniquely of the form $f = \sum h_j \ast e_j$ for $h_j$’s in $\mathcal{A}$. This means, by definition, that $B$ is a free $\mathcal{A}$-module of rank $q$, with the $e_j$’s serving as a module basis. The mapping $f \mapsto (h_j) \in \mathcal{A}^q$ (column vectors with entries in $\mathcal{A}$) defines an $\mathcal{A}$-module isomorphism from $B$ to $\mathcal{A}^q$.

In this kind of situation there is a natural $\mathcal{A}$-valued “inner product” (or “bracket product” as mentioned in the introduction) on $B$, whose use goes back at least to corollary 4.7 of [20]. We recall the natural involution on $B$ defined by $f^*(m) = \bar{f}(-m)$. For $f, g \in B$ we form the convolution $f \ast (g^*)$ and then restrict it to $\Gamma$ (i.e. by “down-sampling” or “decimating”). That is, we set

$$\langle f, g \rangle_{\mathcal{A}}(\gamma) = (f \ast (g^*)) (\gamma) = \sum_{G} f(m) \bar{g}(m - \gamma),$$

where $\gamma \in \Gamma$. (This inner product is $\mathcal{A}$-linear in its first variable.) It is easily checked that $\{e_j\}$ is an “orthonormal” basis for this inner product. All of this structure can be extended to various completions of $\mathcal{A}$ and $B$, such as those for the $\ell^1$-norm, and the $C^*$-norm. See [14] and references therein for the general theory of such inner products.

We will work mostly with the Fourier-transformed version of the above picture. Each of the dual groups $\hat{G}$ and $\hat{\Gamma}$ is isomorphic to $\mathbb{T}^n$, but we must distinguish carefully between these two dual groups, and it is only $\hat{G}$ which we will identify with $\mathbb{T}^n$. We view $\mathbb{T}^n$ as $\mathbb{R}^n/\mathbb{Z}^n$, and
as in [30] we use the exponentials $e^{2\pi in \cdot t}$ in defining Fourier series. Let $F$ be the subgroup of $G = \mathbb{T}^n$ consisting of the characters of $G$ which take value 1 on all of $\Gamma$. Then $F$ can be viewed as the dual group of $G/\Gamma$, and it is a finite group of order $q$. Its elements, viewed as characters of $G$, act by pointwise multiplication to give automorphisms of the convolution algebra $\mathcal{B}$. It is easily seen that $\mathcal{A}$ consists exactly of the elements of $\mathcal{B}$ which are left fixed by all the automorphisms from $F$. As the Fourier transform of $\mathcal{B}$ we take the completion $C(\mathbb{T}^n)$, consisting of the complex-valued continuous functions on $\mathbb{T}^n$, with pointwise multiplication, and adjoint given by pointwise complex conjugation. We will denote it again by $\mathcal{B}$. (This should not cause confusion.) In this picture the action of the group $F$ on $\mathcal{B}$ consists simply of translation by the elements of $F$. Then (the completion of) $\mathcal{A}$ becomes just the subalgebra of $\mathcal{B}$ consisting of the functions in $C(\mathbb{T}^n)$ which are invariant under translation by the elements of $F$. We denote it again by $\mathcal{A}$. In this picture the $\mathcal{A}$-valued inner product on $\mathcal{B}$ is then easily seen to be given by

$$\langle f, g \rangle_\mathcal{A}(x) = q^{-1} \sum_{w \in F} f(x-w)\overline{g(x-w)}.$$ 

That the factor $q^{-1}$ gives the correct normalization can be checked by considering the identity element of $\mathcal{B}$.

Of course, in the Fourier picture $\mathcal{B}$ is still free as an $\mathcal{A}$-module. To see this directly, consider the dual group, $\hat{F} (= G/\Gamma)$, of $F$. Extend each element of $\hat{F}$ to a character of $\hat{G}$. (This corresponds to choosing coset representatives for $\Gamma$ in $G$; there is no canonical choice of extensions in general). Let $\{e_j\}$ denote this collection of characters. A simple calculation shows that $\{e_j\}$ is orthonormal for the $\mathcal{A}$-valued inner product, and that

$$f = \sum \langle f, e_j \rangle_\mathcal{A} e_j$$

for every $f \in \mathcal{B}$. Thus $\{e_j\}$ is an $\mathcal{A}$-module basis for $\mathcal{B}$. It is very natural to define an $\mathcal{A}$-valued inner product on $\mathcal{A}^\oplus$ by

$$\langle (a_j), (e_j) \rangle_\mathcal{A} = \sum a_j \overline{e}_j.$$ 

The mapping $f \mapsto (\langle f, e_j \rangle_\mathcal{A})$ is “isometric” for this inner product. Of course, we should expect that there will be many other orthonormal $\mathcal{A}$-module bases for $\mathcal{B}$, and this will be the subject of the next section.

We summarize some of the above discussion with:

**Proposition 1.1.** Let $F$ be a finite subgroup of $\mathbb{T}^n$ of order $q$, and let $\mathcal{A}$ denote the subalgebra of $\mathcal{B} = C(\mathbb{T}^n)$ consisting of the functions
which are invariant under translation by the elements of \( F \). View \( B \) as a module over \( A \). Then \( B \) is a free module over \( A \) with \( q \) generators.

In Section 4 we will need to deal with \( A \)-modules which are not free, but are (finitely generated) projective. These will also be of central importance for a paper about wavelets presently under preparation [18]. By definition, a projective module is (isomorphic to) a direct summand of a free module. Let us view our free module as \( A^m \) for some integer \( m \), with “standard basis” \( \{ e_j \} \). Then this means that there is an \( m \times m \) matrix \( P \) with entries in \( A \) which is a projection, that is \( P^2 = P \), such that our projective module \( \Xi \) is of the form \( \Xi = P A^m \). In this (\( C^* \)-algebraic) setting a standard argument (see 5Bb in [33]) shows that \( P \) can be adjusted so that it is also “self-adjoint” in the evident sense. Then set \( \xi_j = Pe_j \) for each \( j \). For any \( \eta \in \Xi \) we have

\[
\eta = P\eta = P(\sum \langle \eta, e_j \rangle_A e_j) = \sum \langle \eta, e_j \rangle_A Pe_j = \sum \langle P\eta, e_j \rangle_A \xi_j
\]

\[
= \sum \langle \eta, Pe_j \rangle_A \xi_j = \sum \langle \eta, \xi_j \rangle_A \xi_j.
\]

There is no reason to expect that the \( \xi_j \)'s will be independent over \( A \), much less orthonormal. But anyone familiar with wavelets will feel comfortable about referring to the \( \xi_j \)'s as a “module frame” for \( \Xi \), as the second author did in [25]. In the more general setting of projective modules over \( C^* \)-algebras the reconstruction formula

\[
(1) \quad \eta = \sum \langle \eta, \xi_j \rangle_A \xi_j.
\]

appears, in different notation, in [21]. This indicates how the \( A \)-valued inner products are quite useful in discussing frames. In particular they are useful in discussing biorthogonal wavelet bases. The theory of module frames has been developed extensively in [8].

The \( A \)-valued inner products we defined earlier are a special cases of ones associated with “conditional expectations”, as noted in lemma 1.1 and proposition 4.17 of [20]. Frames for modules connected with conditional expectations have played an important role in other situations. See the extensive development by Watatani in [32], and the references therein. He uses the terminology “quasi-bases” instead of “frames”. Our situation is a special case of proposition 2.8.1 of [32], in which \( F \) is a finite group of cardinality \( q \) acting freely on a compact Hausdorff space \( X \), and the conditional expectation from \( C(X) \) onto the fixed point algebra is shown to be of the kind for which a quasi-basis will exist. But we remark that the projective module need not be free in this more general situation. For example, consider the two-element group acting on the two-sphere by the antipodal map. (See situation 2 in [22].)
2. Filter functions and free $C(𝕋^n)$ modules

In this section we briefly review the definition of filters, and their relationships to scaling functions and wavelets. We then use these filter functions to construct explicit isomorphisms between certain finitely generated projective modules over $C(𝕋^n)$ and free $C(𝕋^n)$-modules.

We now review the definition of filter functions corresponding to an arbitrary integer dilation matrix. In part we do this because it is important to be careful with the bookkeeping for the constants involved. We state everything in the Fourier transform picture. We also assume that our filters are continuous, so that the equations hold everywhere, and not just a.e. We will not state in detail the regularity conditions (“Cohen’s condition”, etc.) since we do not explicitly need them. We refer to [30] and [34] for precise statements.

**Definition 2.1.** Let $A$ be an integral dilation matrix, that is, an integer matrix all of whose eigenvalues have modulus strictly greater than 1, and let $q = |\det(A)|$. A continuous function $m_0 \in \mathcal{B} = C(𝕋^n)$ is called a low-pass filter (or “mask”) for dilation by $A$ if:

(i) $m_0(0) = q$,  
(ii) $\langle m_0, m_0 \rangle_A = q$  
(iii) $m_0(x) \neq 0$ for $x$ in a sufficiently large neighborhood of 0 well-related to $A$ ("Cohen’s condition", etc.).

It is shown in [30], [34] that if $m_0$ is a low-pass filter corresponding to dilation by $A$, then on setting $B = A^T$ and

$$Φ(x) = \Pi_{n=1}^{\infty}[q^{-1}m_0(B^{-n}(x))], \quad \text{(equation 1.47 of [30])}$$

we find that $Φ$ is the Fourier transform of a scaling function $φ \in L^2(ℝ^n)$ corresponding to dilation by $A$. In order to guarantee point-wise convergence of the product defining $Φ$ and to ensure that $Φ$ is continuous, we need only make the mild assumption that $m_0$ is $C^1$ at 0. The scaling function and its corresponding multi-resolution analyses are then used to construct a family of $q - 1$ wavelets corresponding to dilation by $A$. We note that it has recently been shown by L. Baggett and K. Merrill in [3] that multi-resolution analyses exist for every $n \times n$ integral dilation matrix $A$; and only very recently has it been shown by M. Bownik and D. Speegle in [4] that in the special $2 \times 2$ case, to every integer dilation matrix one can associate a scaling function whose Fourier transform is smooth and compactly supported.

In what follows we will never need to use condition (iii) of Definition 2.1. So when we say “low-pass filter” below we will mean just conditions (i) and (ii). But it is an open question, for general dilation matrices,
as to how often one can ensure that condition (iii) does hold in such a way that \( m_0 \) will yield a good scaling function.

**Definition 2.2.** Let \( A \) be an integral dilation matrix with \( |\det(A)| = q \), and let \( m_0 \in \mathcal{B} \) be a continuous low pass filter for \( A \). We say that a family \( m_1, \cdots, m_{q-1} \in \mathcal{B} \) is a high-pass filter family corresponding to the low-pass filter \( m_0 \), if

\[
\langle m_j, m_k \rangle_A = q \delta_{jk} \quad \text{(equation 1.24 of \[30\])}
\]

for \( 0 \leq j, k \leq q - 1. \)

For the reader’s reference we remark that the wavelets \( \psi_1, \cdots, \psi_{q-1} \in L^2(\mathbb{R}^n) \) associated to the filter functions \( \{ m_0, m_1, \cdots, m_{q-1} \} \) are given by the formula

\[
\hat{\psi}_i(x) = q^{-1}m_i(B^{-1}x)\Phi(B^{-1}x) \quad \text{(equation 1.39 of \[30\])}
\]

for \( 1 \leq i \leq q - 1 \). Thus if all of the filter functions are continuous, and if \( m_0 \) has sufficient regularity so that the (Fourier transform of the) scaling function \( \Phi \) is continuous, as mentioned in the sentence after the definition of \( \Phi \), then (the Fourier transforms of) the multiwavelets will be continuous too.

It is clear that if we rescale all of the \( m_j \)'s by \( q^{-1/2} \) then we obtain an orthonormal family in \( \mathcal{B} \). But it will be more convenient for us to rescale the inner product too, in a way which leads to some traditional formulas. We define our rescaled inner product by \( \langle \cdot, \cdot \rangle'_A = q \langle \cdot, \cdot \rangle_A \), so that

\[
\langle f, g \rangle'_A(x) = \sum_{w \in F} f(x - w)\bar{g}(x - w)
\]

for \( f, g \in \mathcal{B} \). Suppose now that we have a low-pass filter \( m_0 \) and a corresponding family \( m_1, \cdots, m_{q-1} \in \mathcal{B} \) of high-pass filters. We renormalize these by setting \( h_j = q^{-1}m_j \) for \( 0 \leq j \leq q - 1 \). This renormalization corresponds exactly to the renormalization factor in equation 1.35 of \[30\]. It is clear that the \( h_j \)'s then form an orthonormal set in \( \mathcal{B} \) for our new inner product \( \langle \cdot, \cdot \rangle'_A \), and that \( h_0(0) = 1 \). In all that follows we now use this new inner product.

We now use a traditional argument to show that such an orthonormal set will always be an \( A \)-module basis for \( \mathcal{B} \).

**Theorem 2.3.** Let \( A \) and \( \mathcal{B} \) be defined as above in terms of the finite subgroup \( F \) of order \( q \). Let \( h_0, \cdots, h_{q-1} \) be an orthonormal set in \( \mathcal{B} \). Then it is an \( A \)-basis for \( \mathcal{B} \).

**Proof.** The crux of the matter is to show that the \( h_j \)'s “span” \( \mathcal{B} \). Label the elements of \( F \) by integers \( i \) with \( 0 \leq i \leq q - 1 \). For each \( x \in \mathbb{T}^n \)
define a $q \times q$ matrix $U(x) = (u_{ji}(x))$ by $u_{ji}(x) = h_j(x - w_i)$. The orthogonality condition for the $h_j$'s says that for each $x$

$$\delta_{jk} = \sum_i h_j(x - w_i)\bar{h}_k(x - w_i) = \sum_i u_{ji}(x)\bar{u}_{ki}(x).$$

(See equation 1.41 of [30].) But this says exactly that $U(x)U(x)^* = I$. Because we are in finite dimensions, this implies that $U(x)$ is unitary, so that also $U(x)^*U(x) = I$. That is,

$$\sum_i h_i(x-w_j)h_i(x-w_k) = \delta_{jk}.$$ 

(See equation 1.43 of [30].) We apply this for $w_k = 0$ to obtain

$$\left(\sum_i \langle f, h_i \rangle A h_i \right)(x) = \sum_i \left(\sum_j f(x-w_j)\bar{h}_i(x-w_j)\right) h_i(x) = \sum_j f(x-w_j)\sum_i h_i(x-w_j)h_i(x) = f(x).$$

We remark that $h_0$ gives the first row of the unitary matrix $U$ above. Finding a corresponding set of high-pass filters, so $h_1, \ldots, h_{q-1}$, corresponds to finding the other rows of this matrix (with entries in $C(T^n)$) such that it is unitary. This is why the problem of finding high-pass filters is often referred to as the “matrix completion problem”.

3. When low-pass filters have high-pass filter families

As discussed in the introduction, it is important to find high-pass filters for a given low-pass filter. This is not always possible if one requires the filters to be continuous, as we will show in the next section. But here we give some positive results. We will use our Proposition 1.1 and the cancellation theorem for certain finitely generated projective modules over $C(T^n)$ to prove that if $n < 2q - 1$, then given any continuous low-pass-filter $m_0$ defined on $T^n$ corresponding to any integer $n \times n$ dilation matrix $A$ with $|\det(A)| = q$, it is possible to construct not only measurable, but in fact continuous high-pass filters corresponding to the low pass filter $m_0$. We do not need the condition of $r$-regularity, which implies that $m_0$ is infinitely differentiable. (See the third sentence of the proof of Theorem 5.15 of [34].) We now state the most general form of our theorem.
Theorem 3.1. Fix $n \in \mathbb{N}$, and let $F$ be a finite subgroup of $\mathbb{T}^n$ of order $q$. Let $A$ and $B$ be defined as done earlier, in terms of $F$. Suppose that $m_0 \in B$ and that $m_0$ satisfies conditions (i) and (ii) of Definition 2.1. Suppose that either $q = 2$, or $n < 2q - 1$. Then there exists a family of continuous high-pass filters $m_1, m_2, \ldots, m_{q-1} \in B$ corresponding to $m_0$. In particular, if $n \leq 4$ then for any $q$ it will be true that any continuous low-pass filter will have a corresponding family of continuous high-pass filters.

If a continuous low-pass filter $m_0$ has a corresponding family of continuous high-pass filters, and if $m_0$ is infinitely differentiable, then $m_0$ has a corresponding family of high-pass filters which are infinitely differentiable.

Proof. We use the notation of Proposition 1.1. Set $h_0 = q^{-1}m_0$, so that $\langle h_0, h_0 \rangle_A' = 1$. Viewing $A$ as an $A$-module over itself, we construct an $A$-module map $\sigma : A \to B$ by

$$\sigma(a) = ah_0.$$ 

We note that $\sigma$ preserves the $A$-valued inner products: for any $a, b \in A$ we have

$$\langle \sigma(a), \sigma(b) \rangle_A' = \langle ah_0, bh_0 \rangle_A' = a\langle h_0, h_0 \rangle_A h_0 = \langle a, b \rangle_A.$$ 

Hence $\sigma$ gives an $A$-module injection of $A$ into $B$. Thus $\sigma(A)$ is projective, and so it has a complementary module, $L$, such that

$$B = \sigma(A) \oplus L.$$ 

We can take $L$ to be orthogonal to $\sigma(A)$ with respect to the inner product described above. To see that this orthogonal complement $L$ exists, note that the “orthogonal” projection of $B$ onto $\sigma(A)$ is given by $f \mapsto \langle f, h_0 \rangle_A' h_0$, so that

$$L = \{ f - \langle f, h_0 \rangle_A' h_0 : f \in B \}.$$ 

Suppose that a high-pass filter family $m_1, m_2, \ldots, m_{q-1} \in B$ exists for $m_0$, and set $h_j = q^{-1}m_j$ for each $j$. Then from the equation of Definition 2.2 we see that the $h_j$’s for $j \geq 1$ will be an orthonormal family in $L$. From Theorem 2.3 we can deduce that they will actually form an orthonormal basis for $L$, so that $L$ must be a free $A$-module. Thus to show that a high-pass filter family exists, we need to show that $L$ is a free $A$-module, for then we can obtain an orthonormal basis. (Use, for example, the proof of proposition 2.1 of [21].) We can then multiply by $q$ to obtain the desired $m_j$’s. Note that we have

$$A \oplus L \cong A \oplus A^{q-1}.$$
Thus to show that $L$ is free we need to be able to “cancel” one copy of $\mathcal{A}$, so that $L \cong \mathcal{A}^{q-1}$.

We treat first the case in which $q = 2$. This has a simple solution, as is widely seen in the wavelet literature. In this case the group $F$ has only two elements. Let $w$ denote the non-identity element of $F$. Choose a continuous function $\tau$ on $\mathbb{T}^n$ such that $|\tau(x)| = 1$ and $\tau(x + w) = -\tau(x)$ for all $x \in \mathbb{T}^n$. This can be done, for example, by choosing $\tau$ to be a character on $\mathbb{T}^n$ such that $\tau(w) = -1$. Let $h_0$ as above be given. Define $h_1$ by $h_1(x) = \tau(x)\overline{h_0(x + w)}$. Then a simple standard calculation shows that the pair $h_0, h_1$ is an orthonormal set in $\mathbb{B}$. We can now apply Theorem 2.3 to conclude that $h_1$ is a basis for $L$. (We remark that this case is a special case of the fact that on any compact space any stably-free line bundle is free. This is because line-bundles are determined by their first Chern class; see Example 4.55 of [19]. But stably equivalent vector bundles have the same Chern class, by theorem 16.4.2 of [21].)

To treat the other case we use a theorem of Swan (theorem 1.6.3 of [27]) which shows that for any compact space $X$ the projective modules over $C(X)$ correspond to the complex vector bundles over $X$. One direction of this correspondence consists of assigning to a vector bundle its $C(X)$-module of continuous cross-sections. This enables us to use the facts about cancellation of vector bundles which are given in [21]. Our $\mathcal{A}$-valued inner products correspond to “Hermitian metrics” on vector bundles.

Suppose now that $n < 2q - 1$. Then $\left\lceil \frac{n}{2} \right\rceil$ denotes as in [21] the least integer greater than or equal to $n$. Then from theorem 8.1.5 of [21] we can deduce immediately that we can cancel $\mathcal{A}$, so that $L$ is a free module of rank $q - 1$. We have thus proved the existence of the desired family of continuous high-pass filters associated to $m_0$.

Suppose now that $m_0$ is infinitely differentiable, and that we have obtained a corresponding family $m_1, m_2, \ldots, m_{q-1}$ of continuous high-pass filters, perhaps by use of the first part of this theorem. We can uniformly approximate the $m_j$’s for $j \geq 1$ arbitrarily closely by infinitely differential functions, say $g_1, g_2, \ldots, g_{q-1}$. These functions need not be orthogonal. But we can try to apply to them a “Gram-Schmidt” process using $\langle \cdot, \cdot \rangle_\mathcal{A}$. The only care needed to make this work is that the approximations must be close enough so that, if $f_j$ denotes the orthogonal projection (defined much as in the early part of the proof of this theorem) of $g_j$ into the orthogonal complement of the span of the new $m_1, m_2, \ldots, m_{j-1}$, then $\langle f_j, f_j \rangle_\mathcal{A}$ must still be close enough to 1.
so that \(((f_j, f_j)_A)^{-1/2}\) exists and is an infinitely differentiable function. For then we can “normalize” \(f_j\) to obtain the new \(m_j\). □

On the other hand, if \(m_0\) is a FIR filter (i.e. a trigonometric polynomial) then it is quite another matter to determine whether corresponding high-pass filters can be found which are FIR filters. In this connection see the discussion of the matrix completion problem in the non-unitary case, but for Laurant polynomials, given in section 8 of [12]. It employs the Quillen-Suslin solution of the Serre conjecture. See also [13].

Of course, all of this discussion is not of much use except for those dilation matrices for which there exists a continuous low-pass filter which will produce a scaling function. As mentioned earlier, it does not seem to be known for which dilation matrices such a filter always exists, though one can always find a measurable such filter, [13].

4. A LOW-PASS FILTER WHICH DOES NOT HAVE A FULL SET OF HIGH-PASS FILTERS

We now construct an example of an integral dilation matrix \(A\) on \(\mathbb{R}^5\) with a given continuous, or even smooth, low-pass filter on \(\mathbb{R}^n\) which does not have a corresponding family of continuous high-pass filters. We emphasize that our essential assumption is that the filters which we consider, as functions on \(\mathbb{Z}^n\), decay at infinity fast enough that their Fourier transforms on \(\mathbb{T}^n\) are continuous. This condition is satisfied by almost all filters in practical use.

Our example is based on the fact that on \(\mathbb{T}^5\) there are complex vector bundles which are stably free but not free (see below). This does not happen on \(\mathbb{T}^n\) for \(n \leq 4\), since cancellation holds as discussed in the course of the proof of Theorem 3.2. (Our construction would also work for \(n > 5\).)

Accordingly, we consider \(\mathbb{Z}^5\). To be specific, and to make contact with wavelet theory, let \(A\) be the dilation matrix

\[
A = \begin{pmatrix} 0 & 3 \\ I_4 & 0 \end{pmatrix},
\]

so that \(\det(A) = 3\). Let \(\Gamma = A(\mathbb{Z}^5)\) so that \(\Gamma\) has as a generating set \(\{3e_1, e_2, e_3, e_4, e_5\}\), where the \(e_j\)'s are the standard generators for \(\mathbb{Z}^5\). We view the convolution \(C^*\)-algebra \(B = C^*(\mathbb{Z}^5)\) as a module over its subalgebra \(A = C^*(\Gamma)\). As a module, \(B\) is free, of rank 3, with (one possible) module basis given by the \(\delta\)-functions supported at 0, \(e_1\) and 2\(e_1\).
We view all of this in the Fourier picture. Thus we let $\mathcal{B} = C(\mathbb{T}^5)$. Of course $\mathcal{A} = C^*(\Gamma)$ is also isomorphic to $C(\mathbb{T}^5)$ but we must distinguish carefully between it and $C^*(\mathbb{Z}^5)$. We let $F$ be the subgroup of elements of $\mathbb{T}^5$ which, viewed as characters on $\mathbb{Z}^5$, have value 1 on $\Gamma$. So $F$ has 3 elements, and consists of the order-3 subgroup of the first copy of $\mathbb{T}$ in $\mathbb{T}^5$. Then $C^*(\Gamma)$ corresponds to the subalgebra, $\mathcal{A}$, of $\mathcal{B} = C(\mathbb{T}^5)$ consisting of functions invariant under translation by $F$. The action of $\mathcal{A}$ on $\mathcal{B} = C(\mathbb{T}^5)$ is by pointwise multiplication. We use the $\mathcal{A}$-valued inner product on $\mathcal{B}$ as before, defined by

$$\langle f, g \rangle_\mathcal{A}(x) = \sum_{w \in F} \bar{f}(x - w) g(x - w).$$

Of course $\mathcal{B}$ is free of rank 3 over $\mathcal{A}$, with orthonormal module basis obtained by renormalizing by $3^{-1/2}$ the set $\{1, \hat{e}_1, (\hat{e}_1)^2\}$, where $\hat{e}_1$ is the character of $\mathbb{T}^5$ corresponding to the character from $e_1$ on the first copy of $\mathbb{T}$ in $\mathbb{T}^5$.

As before, we take our low-pass filter to be renormalized, so given by a function $h_0 \in \mathcal{B}$ such that $\langle h_0, h_0 \rangle_\mathcal{A} = 1$, and $h_0(0) = 1$. One wants to find corresponding high-pass filters, $h_1$ and $h_2$, such that

$$\langle h_j, h_k \rangle_\mathcal{A} = \delta_{jk} \quad j, k = 0, 1, 2.$$

Then $\{h_0, h_1, h_2\}$ will be an “orthonormal” basis for $\mathcal{B}$ as $\mathcal{A}$-module by Theorem 2.3.

We now show how to construct an $h_0$ for which it is impossible to find corresponding continuous $h_1$ and $h_2$. To see what is involved, let $[h_0]$ denote the $\mathcal{A}$-submodule of $\mathcal{B}$ generated by $h_0$. Then $[h_0]$ is a free module of rank 1. We let $[h_0]^\perp$ denote the orthogonal complement of $[h_0]$ in $\mathcal{B}$ for the $\mathcal{A}$-valued inner product. Note that if $h_1$ and $h_2$ exist, then they form a module basis for $[h_0]^\perp$, and thus $[h_0]^\perp$ is a free $\mathcal{A}$-module of rank 2. Thus to find our desired example, it suffices to find $h_0$ such that $[h_0]^\perp$ is not a free module. (However $[h_0]^\perp \oplus [h_0] = \mathcal{B}$ will be free, so that $[h_0]^\perp$ will be “stably free”, and so will represent the same element of the $K$-group $K_0(\mathbb{T}^5)$ as the free module of rank 2. Thus the phenomenon we seek cannot be detected by the $K_0$-group.)

We base our construction of $h_0$ on classical facts about the homotopy groups of spheres and unitary groups, which show that on the 5-sphere $S^5$ one can construct complex vector-bundles having the properties we seek for $\mathbb{T}^5$. We view $S^5$ as the standard unit-sphere in $\mathbb{C}^3$. We will follow closely some constructions given in section 24.2 of [29] but we use slightly more concise notation. Following 24.2, we view $S^4$ as that equator of $S^5$ given as

$$S^4 = \{(v, ir) : v \in \mathbb{C}^2, \ r \in \mathbb{R}, \ ||v||^2 + r^2 = 1\}.$$
For simplicity we will label these points just by \((v, r)\). We view \(v\) as a column vector, and let \(v^*\) denote the corresponding row vector with complex conjugate entries. Then \(vv^*\) is a \(2 \times 2\) matrix, self-adjoint, of rank 1 (unless \(v = 0\)) with range in the subspace spanned by \(v\). Let \(U_2\) denote the \(2 \times 2\) unitary group. We let \(U_0\) be the function from \(S^4\) to \(U_2\) defined by

\[
U_0(v, r) = I_2 - 2(1 + ir)^{-2}vv^*.
\]

This is formula 5 of 24.2. Steenrod provides in 24.3 of [29] a proof that \(U_0\) is not path-connected through unitaries to a constant map from \(S^4\) to \(U_2\). The idea of the proof is basically that \(U_0\) is closely related to the suspension of the famous Hopf map from \(S^3\) to \(S^2\) which generates the homotopy group \(\pi_3(S^2) \cong \mathbb{Z}\). (Recall that \(SU_2\) is homeomorphic to \(S^3\).) Note that \(U_0\) can equally well be considered to be a unitary element of the \(C^*\)-algebra \(M_2(C(S^4))\) of \(2 \times 2\) matrices over \(C(S^4)\); we will frequently take this point of view.

There is a traditional way, described in section 18.1 of [29], to use a map such as \(U_0\) to construct a vector bundle over \(S^5\). Because we are aiming at \(T_5\), we use a simple variant of this construction to obtain instead a vector bundle over \(T \times S^4\). This variant is described in the course of the proof of Theorem 8.4 of [24], where, without additional complication, it is seen to work easily for non-commutative (unital) \(C^*\)-algebras. We identify vector bundles with their modules of continuous cross-sections, which are projective modules.

For the immediate purposes of this construction it is convenient to view \(T\) as the interval \([0, 1]\) with ends identified. We view \(C(T \times S^4)\) accordingly. Then the projective module (alias vector bundle) over \(C(T \times S^4)\) determined by \(U_0\), denoted \(X(U_0)\), is the vector space of continuous functions

\[
X(U_0) = \{F : [0, 1] \rightarrow (C(S^4))^2 : F(1) = U_0F(0)\},
\]

with the elements of \(C(T \times S^4)\) acting by pointwise multiplication. We define an inner product on \(X(U_0)\) with values in \(C = C(T \times S^4)\) by

\[
\langle F, G \rangle_C(s) = F(s)^*G(s),
\]

where \(G(s)\) is viewed as a column vector, etc. It follows immediately from lemma 8.10 of [24] that if \(X(U_0)\) were a free module, then \(U_0\) would be path-connected through unitaries to the constant map on \(S^4\) with value \(I_2\). Since this is not the case, the module \(X(U_0)\) is not free.

However, crucial to our purposes is the fact that the direct sum of \(X(U_0)\) with the free module of rank 1 is free (of rank 3). From lemma 8.6 of [24] the direct sum of \(X(U_0)\) with the free module of rank 1
comes by applying the above construction to
\[
\left( \begin{array}{cc} U_0 & 0 \\ 0 & I_1 \end{array} \right) = U_0 \oplus I_1
\]
instead of to \( U_0 \). From lemma 8.10 of [24] the fact that the direct sum is free is equivalent to the fact that \( U_0 \oplus I_1 \) is path-connected to \( I_3 \) through unitaries in \( M_3(C(S^4)) \). In order to try to obtain an explicit formula for our low-pass filter \( m_0 \), we need an explicit path. From the details given in [29] it is not difficult to see how to produce one. The key is that \( U_0 \oplus I_1 \) has the following factorization:
\[
\left( \begin{array}{cc} U_0(v, r) & 0 \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} I_2 - (1 - ir)^{-1}vv^* & v \\ -b(ir)v^* & ir \end{array} \right)^* \left( \begin{array}{cc} I_2 - (1 + ir)^{-1}vv^* & v \\ c(ir)v^* & ir \end{array} \right),
\]
where \( b \) and \( c \) will be defined below. Each of the two factors is in \( U_3 \), and so the \(^*\) applied to the first factor could equally well be \(-1\). The two factors extend to functions defined on all of \( S^5 \) except for one point.
Specifically, as in equation 2 of 24.2 of [29], we set
\[
\phi^+(v, \xi) = \left( \begin{array}{cc} I_2 - (1 + \bar{\xi})^{-1}vv^* & v \\ b(\xi)v^* & \xi \end{array} \right)
\]
for \((v, \xi) \in S^5\), with \( b(\xi) = (1 + \xi)(1 + \bar{\xi})^{-1} \). We note that \( \phi^+ \) is not defined at the south pole \((0, -1)\). We also see that the first factor above (before taking its inverse) is just \( \phi^+(v, ir) \). In the same way, following equation 4 of 24.2 of [29], we set
\[
\phi^-(v, \xi) = \left( \begin{array}{cc} I_2 - (1 - \bar{\xi})^{-1}vv^* & v \\ c(\xi)v^* & \xi \end{array} \right)
\]
for \((v, \xi) \in S^5\), with \( c(\xi) = (1 - \xi)(1 - \bar{\xi})^{-1} \). We note that \( \phi^- \) is not defined at the north pole \((0, 1)\). We also see that the second factor above is just \( \phi^-(v, ir) \). Each of \( \phi^+ \) and \( \phi^- \) has values in \( U_3 \).
We note that \( \phi^+(0, 1) = I_3 \), while \( \phi^-(0, -1) = I_2 \oplus (-I_1) \). This latter matrix is, of course, path-connected to \( I_3 \). If we adjust for this, and move \( S^4 \) gradually up to the north pole for \( \phi^+ \), and down to the south pole for \( \phi^- \), we obtain a path through unitaries from \( U_0 \oplus I_1 \) to \( I_3 \). We can explicitly implement this as follows. We let \( t \) denote the parameter for our path, ranging over \([0, 1]\). For \( \phi^+ \) we take the straight-line path \((1 - t)ir + t \) from \( ir \) to 1. Given \((v, ir)\), we must scale \( v \) accordingly so as to remain in \( S^4 \). We denote the scaling factor by \( k_t^+(r) \). It is easily computed. Then we define a map \( p_t^+ \) of \( S^4 \) into \( S^5 \) by
\[
p_t^+(v, r) = (k_t^+(r)v, (1 - t)ir + t).
\]
In a similar way we set
\[
p_t^-(v, r) = (k_t^-(r)v, (1 - t)ir - t).
\]
Then $t \mapsto \phi^+ \circ p_t^+$ is a path of elements of $U_3(C(S^4))$, as is $t \mapsto \phi^- \circ p_t^-$. We set

$$W_t = (\phi^+ \circ p_t^+)^*(\phi^- \circ p_t^-)c_t$$

where $c_t = \begin{pmatrix} I_2 & 0 \\ 0 & e^{\pi it} \end{pmatrix}$. Then $W_t$ is a path of elements of $U_3(C(S^4))$ which goes from $U_2 \oplus I_1$ to $I_3$.

As indicated in lemma 8.5 of [24], a simple calculation shows that we then obtain a $\mathcal{C}$-module isomorphism $\Phi$ from $X(U_0 \oplus I_1)$ to $X(I_3) = \mathcal{C}^3$ by setting

$$\Phi_F(t) = W_t W_0^{-1} F(t).$$

Because $W_t$ is unitary, $\Phi$ preserves the $\mathcal{C}$-valued inner products. In particular, $\Phi$ will carry $X(U_0)$ and $X(U_0)^\perp$ to orthogonal complementary submodules of $\mathcal{C}^3$. Thus $\Phi(X(U_0)^\perp)$ will be a free rank-one submodule whose orthogonal complement is not free, and so does not have a module basis. Now in $X(U_0 \oplus I_1)$ the module $X(U_0)^\perp$ has as one choice of (normalized) basis the evident constant function

$$E_3 = E_3(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in (C(S^4))^3.$$

Thus $\Phi(X(U_0)^\perp)$ will have as basis the $(C(S^4))^3$-valued function

$$N_0(t) = \Phi E_3(t) = W_t W_0^{-1} E_3(t) = W_t W_0^{-1} E_3,$$

which we can view as an element of $\mathcal{C}^3$. (This element is closely related to our desired filter $m_0$.) But $W_0 = U_0 \oplus I_1$, and so $W_0^{-1} E_3 = E_3$. Thus our desired function is simply

$$N_0(t) = W_t E_3.$$

Note that $N_0(1) = N_0(0)$ as needed, because $W_t$ goes from $U_2 \oplus I_1$ to $I_3$.

But if we look at the definition of $W_t$, we see that we should calculate

$$((\phi^- \circ p_t^-)c_t)(v, r) E_3 = \phi^-(p_t^-(v, r))c_t E_3$$

$$= \begin{pmatrix} k_t^-(r)v \\ e^{\pi it}((1-t)ir - t) \end{pmatrix}.$$

Then to obtain $N_0(t, v, r)$ we need only apply $(\phi^+ \circ p_t^+)^*$ to the above column vector. An explicit formula is not difficult to obtain, but we will not display it here. We note that $N_0(0, 0, 1) = (U_0(0, 1) \oplus I_1)e_3 = e_3$, where $e_3$ is the standard third basis vector of $\mathcal{C}^3$. This is related to the low-pass condition for our desired filter $m_0$.

Finally, we must relate all of the above to $\mathbb{T}^5$. We view $\mathbb{T}^5$ as $\mathbb{T} \times \mathbb{T}^4$, and we use the pinching map, $P$, from $\mathbb{T}^4$ onto $S^4$. To define it in a convenient way, we momentarily let $\mathbb{T}$ be the interval $I = [-1, 1]$.
with the ends identified. Then $I^4$ is just the unit ball in $\mathbb{R}^4$ for the usual supremum norm $\| \cdot \|_\infty$, and $T^4$ is $I^4$ with certain points of the boundary identified. The pinching map $P$ then identifies all the points of the boundary to just one point, forming $S^4$. A convenient formula for $P$, taking the boundary to $-e_5$, is

$$P(v) = (v \|v\|_2^{-1} \sin(\pi \|v\|_\infty), \cos(\pi \|v\|_\infty))$$

for $v \in I^4$, where $\|\cdot\|_2$ is the Euclidean norm on $\mathbb{R}^4$. The first component of the right-hand side must be viewed as an element of $C^2$ under the natural identification of $\mathbb{R}^4$ with $C^2$.

We now let $\tilde{P} = I \times P$ to obtain a map from $T^5$ onto $T \times S^4$. We use $\tilde{P}$ to pull back all of our earlier construction to $T^5$. In particular we set $H_0 = N_0 \circ \tilde{P}$. Then $H_0 \in (C(T^5))^3 = \mathcal{A}^3$, and $\langle H_0, H_0 \rangle_A = 1$. Furthermore, for $0 \in I^4$ we have $P(0) = (0, 1) \in S^4$, so that $\tilde{P}(0) = (0, 0, 1) \in T \times S^4$. We saw earlier that $N_0(0, 0, 1) = e_3$ and so $H_0(0) = e_3$ for $0 \in I^5$. We need to have made our identification of $T^5$ with $I^5$ in such a way that $0$ goes to $0$. Then $H_0$, as function on $T^5$, satisfies $H_0(0) = e_3$.

We must now compose $H_0$ with the $\mathcal{A}$-module isomorphism from $\mathcal{B} \cong C^*(\mathbb{Z}^5)$ to $\mathcal{A}^3 \cong (C^*(\Gamma))^3$ described early in this section. But we must do so in such a way that the third component in $\mathcal{A}^3$ corresponds to the submodule of $C^*(\mathbb{Z}^5)$ generated by the delta function at $0$ (discussed early in this section), so that the function $h_0$ to which $H_0$ is carried satisfies the low-pass condition $h_0(0) = 1$. Of course it will also satisfy $\langle h_0, h_0 \rangle_A = 1$. This is our desired filter.

We must check that this $h_0$ does not have two corresponding high-pass filters. If it did, then the (module-) orthogonal complement of the module generated by $h_0$ would be free. But this orthogonal complement corresponds, and is isomorphic to, the submodule of $\mathcal{A}^3$ which is the pull back of the module $X(U_0)$ over $C = C(T \times S^4)$ discussed earlier. Thus we need to know that this pulled-back module is not free. But it is not difficult to see that this module as essentially just the module constructed from pulling $U_0$ itself back to $T^4$. That is, we set $V_0 = U_0 \circ P$, so that $V_0 \in \mathcal{U}_2(C(T^4))$. Thus, just as argued above, we need to know that $V_0$ is not path connected to a constant unitary, as this will imply that the $C(T^4)$-module $X(V_0)$ is not free. For the proof of this fact given below we have received much help from Rob Kirby, from Elmer Rees through his reply to a query from Rob Kirby on our behalf, from Jon Berrick, and especially from Jie Wu, who gave helpful answers to many questions.

As is traditional, we let $[S^4, \mathcal{U}_2]$ denote the homotopy equivalence classes of continuous functions from $S^4$ to $\mathcal{U}_2$ which preserve base
points, and similarly, for \([\mathbb{T}^4, \mathcal{U}_2]\), etc. As discussed above, \(U_0\) represents an element of \([S^4, \mathcal{U}_2]\) which is not trivial, in the sense that it is not homotopic to a constant function. By composing with the pinch map \(P\) we obtain a mapping, \(P^*\), from \([S^4, \mathcal{U}_2]\) to \([\mathbb{T}^4, \mathcal{U}_2]\). We need to check that \(P^*(U_0)\) is not the trivial element of \([\mathbb{T}^4, \mathcal{U}_2]\). For this purpose it suffices to show that \(P^*\) is injective.

Now \(\mathcal{U}_2\) is a compact Lie group. It was shown by Milnor that any nice topological group \(G\) has a “classifying space”, \(Z\), such that \(G\) is homotopic to \(\Omega Z\), where \(\Omega Z\) is the loop-space of \(Z\) (see Theorem 9.2.2 and the paragraph before Theorem 9.2.4 of [28]). We now use \(Z\) to denote any classifying space for \(\mathcal{U}_2\) (the nature of the space will not be of importance in the proof). It is an elementary fact in homotopy theory that forming loop spaces is the adjoint of forming (reduced) suspensions (see Proposition 7.1.17 of [28]). We denote suspensions by \(\Sigma\). Thus

\[ [S^4, \mathcal{U}_2] = [S^4, \Omega Z] = [\Sigma S^4, Z], \]

and a similar equation holds with \(S^4\) replaced by \(\mathbb{T}^4\). To \(P^*\) there will then correspond the map \((\Sigma P)^*\), and our problem becomes that of showing that

\[ (\Sigma P)^* : [\Sigma S^4, Z] \to [\Sigma \mathbb{T}^4, Z] \]

is injective. For this, it clearly suffices to show that there is a mapping, \(f\), from \(\Sigma S^4\) to \(\Sigma \mathbb{T}^4\) such that \(\Sigma P \circ f\) is homotopic to the identity map on \(\Sigma S^4\).

We let \(\wedge\) denote the standard smash product of spaces, as in Definition 6.2.12 of [15], so that for any space \(X\) we have by definition \(\Sigma X = \mathbb{T} \wedge X\). For any two spaces \(X\) and \(Y\) we have that \(\Sigma(X \times Y)\) is homotopy equivalent to \(\Sigma(X \wedge Y) \vee \Sigma X \vee \Sigma Y\), where \(\vee\) denotes the reduced union, i.e. one-point union, or “wedge” (see Exercise 15b of [15]). When this exercise is applied several times to \(\Sigma \mathbb{T}^4\) we find that \(\Sigma \mathbb{T}^4\) is homotopy equivalent to a space of the form \(\Sigma S^4 \vee M\) for some space \(M\) which is a wedge of lower-dimensional spheres. We let \(f\) be the composition of the evident map from \(\Sigma S^4\) to \(\Sigma S^4 \vee M\) with a homotopy equivalence from \(\Sigma S^4 \vee M\) to \(\Sigma \mathbb{T}^4\). We must show that \(\Sigma P \circ f\) is homotopic to the identity map on \(\Sigma S^4\).

We first need some homology information about \(P\). Each of \(\mathbb{T}^4\) and \(S^4\) is an orientable compact manifold, so \(H_4(\mathbb{T}^4) \cong \mathbb{Z} \cong H_4(S^4)\). The cube \(I^4\) is homeomorphic with the 4-simplex, and the canonical map of \(I^4\) onto \(\mathbb{T}^4\) used above is a generator for \(H_4(\mathbb{T}^4)\). Similarly, the canonical map of \(I^4\) onto \(S^4\) which collapses the entire boundary is a generator of \(H_4(S^4)\). But \(P\) carries one of these canonical maps to the other. Thus viewed as a map in homology, \(H_4(P)\) is an isomorphism.
from $H_4(\mathbb{T}^4)$ onto $H_4(S^4)$. It follows from Theorem 4.4.10 of [13] that

$$H_5(\Sigma P) : H_5(\Sigma \mathbb{T}^4) \to H_5(\Sigma S^4)$$

is an isomorphism. Also, $H_5(\Sigma \mathbb{T}^4) \cong \mathbb{Z} \cong H_5(\Sigma S^4)$ because $\Sigma S^4 = S^5$.

Because $\Sigma \mathbb{T}^4$ is homotopy equivalent to $\Sigma S^4 \vee M$ where $M$ is a wedge of lower dimensional spheres, the map $f$ from $\Sigma S^4$ to $\Sigma \mathbb{T}^4$ must give an isomorphism from $H_5(\Sigma S^4)$ to $H_5(\Sigma \mathbb{T}^4)$. Hence $H_5(\Sigma P \circ f)$ is an isomorphism from $H_5(\Sigma S^4)$ to itself. Because $\Sigma S^4 = S^5$, we have $H_n(\Sigma S^4) = 0$ except when $n = 0$ or $n = 5$, where it is $\mathbb{Z}$ (see Theorem 4.6.6 of [31]). By the Hurewicz isomorphism theorem (Theorem 15.10 of [29]), $\pi_5(\Sigma S^4) = \pi_5(S^5) = H_5(S^5)$, with the isomorphism being natural by the discussion in [29] which precedes 15.10. Thus $\pi_5(\Sigma P \circ f)$ is an isomorphism. It preserves orientation, so $\Sigma P \circ f$ is homotopic to the identity map on $\Sigma S^4$. This implies, as described above, that $(\Sigma P)^*$ is injective, so that $P^*$ is injective. This in turn implies that $P^*(U_0)$ is non-trivial in $[\mathbb{T}^4, U_2]$, as we promised to show.

Finally, we remark that, much as discussed at the end of the proof of Theorem 3.1, $h_0$ can be approximated arbitrarily closely by low-pass filters which are smooth (i.e., infinitely differentiable). For sufficiently close approximates they will be connected to $h_0$ by paths through the space of low-pass filters. This can be seen directly, or proposition 5.1 of [23] can be used. Such smooth approximates will have complementary modules isomorphic to that of $h_0$, and so again not free. We suspect that in a similar way our specific $h_0$ can be adjusted so that it satisfies condition (iii) of Definition 2.1, but we have not worked out the details.

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