We propose a system of functional relations having a universal form connected to the $U_q(X_r^{(1)})$ Bethe ansatz equation. Based on the analysis of it, we conjecture a new sum formula for the Rogers dilogarithm function in terms of the scaling dimensions of the $X_r^{(1)}$ parafermion conformal field theory.
1. Introduction

It has been noticed for some time that the Rogers dilogarithm function curiously emerges in the calculation of physical quantities in various integrable systems. Besides the outstanding result in three dimensions [1], one finds extensive examples in two dimensions such as in finite size corrections to the ground state energy [2,3], low-temperature asymptotics of the specific heat capacity [4-8] and ultraviolet Casimir energies [9-13] in perturbed conformal field theories (CFTs) [14]. All these quantities are essentially identifiable with the central charges of the relevant CFTs [4,15] and have often been evaluated in terms of the Rogers dilogarithm function [16]. It is therefore tempting to seek a universal background for the common appearance of the dilogarithm in the various calculational techniques exploited in these works, e.g., functional relations (FRs) among row-to-row transfer matrices, string hypotheses, thermodynamic Bethe ansatz (TBA) [17] and so forth.

This Letter is our first step toward understanding the interrelation among these approaches. We shall propose a FR having a universal form related to the $U_q(X_r^{(1)})$ Bethe ansatz equation [18]. It may be viewed as a “spectral parameter” dependent version of the recursion relation among the “Yangian characters” [19] whose combinations appear in the arguments of the dilogarithm [7,8,20]. When $q = \exp(\frac{2\pi i}{\ell+g})$, we especially consider a restricted version of the FR which closes among finitely many unknown functions. Here $\ell$ is a positive integer and $g$ is the dual Coxeter number. The restricted FR originates in the equilibrium condition on the ratio of the string and hole densities in the thermodynamic analysis of the $U_q(X_r^{(1)})$ Bethe equation in [7,8]. On the other hand, the special case $X_r^{(1)} = A_1^{(1)}$ of the FR possesses essentially the same form as that for certain combinations of the actual transfer matrices [3,6] in the restricted solid-on-solid (RSOS) models [21]. Although the full content of the FR is yet under investigation, our heuristic analysis extending the $A_1^{(1)}$ case [3] leads to a new conjecture that connects an analytically continued Rogers dilogarithm function and the scaling dimensions in parafermion CFTs [22]. It is basically labeled by a triad $(X_r^{(1)}, \ell, \Lambda)$ where $\Lambda$ is the classical part of a level $\ell$ dominant integral weight of $X_r^{(1)}$. The earlier conjecture [20] (see also [8]) involving the central charges concerns the special case $(X_r^{(1)}, \ell, 0)$
in our new one. We expect further extensions should be possible related to various coset CFTs by using our FR. The common structure of the FRs between the RSOS transfer matrices and the TBA has also been observed in [3, 23] for $X_r^{(1)} = A_1^{(1)}$.

2. $U_q(X_r^{(1)})$ functional relation

Let $X_r^{(1)}$ denote one of the rank $r$ non-twisted affine Lie algebras [24] $A_r^{(1)} (r \geq 1), B_r^{(1)} (r \geq 2), C_r^{(1)} (r \geq 1), D_r^{(1)} (r \geq 3), E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, F_4^{(1)}$ and $G_2^{(1)}$ and $U_q(X_r^{(1)})$ be the $q$–deformation of its universal enveloping algebra [25]. Let $\alpha_a, \Lambda_a (1 \leq a \leq r)$ be the simple roots and the fundamental weights of the classical part $X_r$, respectively. We set $\rho = \sum_{a=1}^r \Lambda_a, Q = \sum_{a=1}^r Z\alpha_a, P = \sum_{a=1}^r Z\Lambda_a, P_\ell = \{ \Lambda \in P | 0 \leq \langle \Lambda | \text{maximal root} \rangle \leq \ell \}, \mathcal{A}_* = \sum_{a=1}^r C\alpha_a$ and $\mathcal{A}_R = \sum_{a=1}^r R\Lambda_a$. Introduce the bilinear form $\langle \cdot | \cdot \rangle$ on $\mathcal{A}_*$ in a standard way and normalize the roots as $| \text{| long root} |^2 = 2$. Fix an integer $\ell \geq 1$ and put $\ell_a = t_a \ell, G = \{(a, m) | 1 \leq a \leq r, 1 \leq m \leq \ell_a - 1\}$ throughout. The following notation will be used in the sequel $1 \leq a, b \leq r$.

\begin{align}
t_a &= \frac{2}{\alpha_a | \alpha_a |}, \quad t_{ab} = \max(t_a, t_b), \quad (1a) \\
A_{ab}^{mk} &= t_{ab}(\min(m/k, k/m) - mk/t_{ab}) \quad \text{for } (a, m), (b, k) \in G, \quad (1b) \\
B_{ab} &= \frac{t_b}{t_{ab}} C_{ab}, \quad C_{ab} = \frac{2(\alpha_a | \alpha_b)}{(\alpha_a | \alpha_a |)}, \quad I_{ab} = 2\delta_{ab} - B_{ab}. \quad (1c)
\end{align}

Here $C_{ab}$ and $B_{ab} = B_{ba}$ are the Cartan and the symmetrized Cartan matrix, respectively. The nodes in the Dynkin diagrams are numerated according to [24].

Let $Y_{m}^{(a)}(u), 1 \leq a \leq r, m \geq 1$ be functions of a complex parameter $u$ obeying

\begin{align}
Y_{m}^{(a)}(u + \frac{i}{t_a})Y_{m}^{(a)}(u - \frac{i}{t_a}) &= \frac{\prod_{b=1}^r \prod_{k=1}^3 F_{k}^{I_{ab}\delta_{a_k, t_{ab}}}}{(1 + Y_{m-1}^{(a)}(u)^{-1})(1 + Y_{m+1}^{(a)}(u)^{-1})}, \quad (2a) \\
F_{k} &= \prod_{j=1}^{k-1} \prod_{n=0}^{k-1-|j|} \left( 1 + Y_{k,m/t_a+j}^{(b)}(u + i(k - 1 - |j| - 2n)/t_b) \right), \quad (2b)
\end{align}

for $1 \leq a \leq r, m \geq 1$, where by convention $Y_{0}^{(a)}(u)^{-1} = 0$ and $Y_{m}^{(a)}(u) = 0$ if $m \notin \mathbb{Z}$. We call (2) the (unrestricted) $U_q(X_r^{(1)})$ functional relation. If $Y_{t_a}^{(a)}(u)^{-1} = 0$ is further imposed, the denominator of (2a) becomes $\prod_{j=1}^{t_a-1} (1 + Y_{j}^{(a)}(u)^{-1})^{T_{jm}}$ with $T$ being the matrix $I$ of (1c) for $X_r = A_{\ell_a - 1}$. In this case, (2) closes among...
\{Y^{(a)}_m(u)|(a,m) \in G\} and will be referred as the restricted $U_q(X^{(1)}_r)$ functional relation. The next section will concern this situation (see (6) and (15)). Several remarks are in order:

(i) The restricted FR (2) is closely related to the $U_q(X^{(1)}_r)$ Bethe equation \[18\] and its thermodynamic treatment using the string hypothesis in \[8\]. Under the identification $Y^{(a)}_m(u) = \exp(-\beta \epsilon_m^{(a)}(u))$, it actually corresponds to the thermal equilibrium condition (2.22) of \[8\] in the high temperature limit.

(ii) In the simply laced cases $X_r = A_r, D_r$ and $E_{6,7,8}$, the restricted FR (2) takes a simple form

$$Y^{(a)}_m(u+i)Y^{(a)}_m(u-i) = \frac{\prod_{b=1}^{r}(1 + Y^{(b)}_m(u))^{I_{ab}}}{\prod_{j=1}^{r-1}(1 + Y^{(a)}_j(u)-1)^{\overline{T}_{jm}}},$$

where $\overline{T}$ denotes the $I$ for $A_{\ell-1}$.

(iii) Besides the trivial $\ell = 1$ case, (3) essentially coincides with the FR in \[13,26\] for $\ell = 2$ and the FR in \[3,6\] for $X_r = A_1$ with $\ell$ general. The former appeared in the TBA approach to factorized scattering theories whilst the latter is known to hold \[3,6\] for certain combinations of the actual transfer matrices in the RSOS models \[21\]. In these cases, the FRs have been successfully used to determine the central charges by combining them with the data from actual physical systems such as “bulk behavior” \[3\], “mass term” \[9,13\], etc.

(iv) The FR (2) for $Y^{(a)}_m(u)$’s can be viewed as a generalization of the recursion relation among the quantity $Q^{(a)}_m$ in \[19\]. To see this, recall the definitions

$$Q^{(a)}_m = \sum_Z Z(a,m,n) \chi_\omega(a,m,n), \quad n = (n_1, \ldots, n_r) \in \mathbb{Z}_{\geq 0}^r, \quad (4a)$$

$$\omega(a,m,n) = m\Lambda_a - \sum_{b=1}^r n_b \alpha_b, \quad Z(a,m,n) = \sum_{\nu} \prod_{b=1}^r \prod_{k=1}^{\infty} \left( \mathcal{P}^{(b)}_k(\nu) + \nu^{(b)}_k \right), \quad (4b)$$

$$\mathcal{P}^{(b)}_k(\nu) = \min(m,k)\delta_{ab} - 2 \sum_{j \geq 1} \min(k,j)\nu^{(b)}_j - \sum_{c \neq b}^{r} \sum_{j \geq 1} \max(kC_c, jC_b)\nu^{(c)}_j, \quad (4c)$$

for $1 \leq a \leq r, \ m \geq 0$. Here the symbol $\binom{\quad}{\quad}$ in (4b) is the binomial coefficient and the sum extends over all possible decompositions $\{\nu^{(b)}_k | n_b = \sum_{k=1}^{\infty} k\nu^{(b)}_k, \nu^{(b)}_k \in \mathbb{Z}_{\geq 0}, 1 \leq b \leq r, \ k \geq 1\}$ such that $\mathcal{P}^{(b)}_k(\nu) \geq 0$ for $1 \leq b \leq r, \ k \geq 1$. The quantity $\chi_\omega$
in (4a) is the character of the irreducible $X_r$ module $V_\omega$ with highest weight $\omega$, i.e.,
\[ \chi_\omega = \chi_\omega(z) = \Tr_{V_\omega} \exp\left(-\frac{2\pi i}{\ell+g}(z+\rho)\right), \]
where we have exhibited the dependence on its argument $z \in \mathfrak{h}^*$. We shall simply write $Q_m^{(a)}$ to mean the corresponding $Q_m^{(a)}(z)$. Then the following recursion relation is known among the $Q_m^{(a)}$’s [19],

\[ Q_m^{(a)} = Q_{m-1}^{(a)}Q_{m+1}^{(a)} + Q_m^{(a)} \prod_{b=1}^{r} \prod_{k=0}^{\infty} Q_k^{-2J_{b,a}^k} \quad \text{for } 1 \leq a \leq r, m \geq 0, \tag{5a} \]

\[ 2J_{a,b}^n = B_{a,b}(\frac{t_{a,b}}{t_a} \delta_{t_b n, t_a k} + \frac{t_{a,b}}{t_b} \sum_{j=1}^{t_b - t_a} j(\delta_{t_b(n+1)-t_a j, t_a k} + \delta_{t_b(n-1)+t_a j, t_a k})) \tag{5b} \]

for $1 \leq a, b \leq r$, $n, k \geq 0$.

with the initial condition $Q_{-1}^{(a)} = 0$, $Q_0^{(a)} = 1$. In (5b), the sum $\sum_{j=1}^{t_b - t_a}$ is zero unless $t_b > t_a$ and $J_{a,b}^n$ is a natural extension of the $\hat{J}_{a,b}^n(0)$ defined in [8] for the range $0 < n < \ell_a$, $0 < k < \ell_b$. Now consider the limit $u \to \infty$ in (2). From (5) the resulting algebraic equation admits a constant solution

\[ Y_m^{(a)}(\infty) = \frac{Q_m^{(a)} \prod_{b=1}^{r} \prod_{k=0}^{\infty} Q_k^{-2J_{b,a}^k}}{Q_{m+1}^{(a)}Q_{m-1}^{(a)}}, \tag{6} \]

which implies that (2) is a “$u$–version” of (5).
3. Dilogarithm conjecture

Here we formulate our dilogarithm conjecture firstly then discuss its physical background in the light of the FR (2). Let \( \log x \) signify the logarithm in the branch \(-\pi < \text{Im}(\log x) \leq \pi \) for \( x \neq 0 \). Consider a contour on a complex \( x \)-plane \( \mathcal{C} = \mathcal{C}[f|\mathcal{M}, \mathcal{N}] \), \( (f \in \mathbb{C}, \mathcal{M}, \mathcal{N} \in \mathbb{Z}) \) from 0 to \( f \) which intersects the branch cut of \( \log x \) for \( M \) times and that of \( \log(1-x) \) for \( N \) times in total. Here the intersection number is counted as +1 when \( \mathcal{C} \) goes across the cut of \( \log x \) (resp. \( \log(1-x) \)) from the upper (resp. lower) half plane to the lower (resp. upper) and as -1 if opposite. Let \( \text{Log}_C(f) \) denote the analytic continuation of \( \log f \) along the contour \( \mathcal{C} \), namely,

\[
\text{Log}_C(f) = \log f + 2\pi i M, \quad \text{Log}_C(1-f) = \log(1-f) + 2\pi i N.
\]

Define the multivalued and single-valued Rogers dilogarithms \( L_C(f) \) and \( L(f) \) by

\[
L_C(f) = -\frac{1}{2} \int_{\mathcal{C}} \left( \frac{\text{Log}_C(1-x)}{x} + \frac{\text{Log}_C x}{1-x} \right) dx,
\]

\[
L(f) = -\frac{1}{2} \int_{\mathcal{C}_0} \left( \frac{\log(1-x)}{x} + \frac{\log x}{1-x} \right) dx,
\]

where \( \mathcal{C}_0 \) is a contour which does not go across the branch cuts of \( \log x \) and \( \log(1-x) \). It follows from the definitions that

\[
L(f) = \begin{cases} 
\frac{\pi^2}{3} - \frac{\pi i}{2} \log f - L(f^{-1}) & \text{if } f \in \mathbb{R}_{>1},
\end{cases}
\]

\[
-\frac{\pi^2}{6} + \frac{\pi i}{2} \log(1-f) + L(\frac{1}{1-f}) & \text{if } f \in \mathbb{R}_{<0},
\]

\[
L_C(f) = L(f) + \pi i M \log(1-f) - \pi i N \log f + 2\pi^2 M N.
\]

We shall call an element \( z \in \mathcal{F} \) *regular* if it satisfies \( Q_m^{(a)}(z) \neq 0 \) for all \( 1 \leq a \leq r, 1 \leq m \leq \ell_a \) and *singular* otherwise. Suppose \( z \) is regular and put

\[
f_m^{(a)}(z) = 1 - \frac{Q_m^{(a)}(z)Q_{m+1}^{(a)}(z)}{Q_m^{(a)}(z)^2} \quad (a, m) \in G.
\]

Then \( f_m^{(a)}(z) \neq 0, 1, \infty \) for \( \forall (a, m) \in G \). Given a set of integers \( S = \{M_m^{(a)}, N_m^{(a)} \in \mathbb{Z} | (a, m) \in G\} \), let \( \mathcal{C}_{a,m} = \mathcal{C}[f_m^{(a)}(z)|M_m^{(a)}, N_m^{(a)}] \) be the contour as specified above.

Motivated by the study of the FR (2) which will be discussed later, we define

\[
\frac{\pi^2}{6} c(z, S) = \sum_{(a,m) \in G} \left( L_{\mathcal{C}_{a,m}}(f_m^{(a)}(z)) - \frac{\pi i}{2} D_{\mathcal{C}_{a,m}}(z) \text{Log}_{\mathcal{C}_{a,m}}(1-f_m^{(a)}(z)) \right),
\]

\[
\pi i D_{\mathcal{C}_{a,m}}(z) = \text{Log}_{\mathcal{C}_{a,m}}(f_m^{(a)}(z)) - \sum_{(b,k) \in G} A_{b,k}^{m} B_{ab} \text{Log}_{\mathcal{C}_{b,k}}(1-f_k^{(b)}(z)).
\]
By applying (7), (9b) and (11b) to (11a), one can split $c(z, S)$ into $S$–dependent and independent parts as

$$c(z, S) = c_0(z) - 24T(z, S),$$

$$\frac{\pi^2}{6} c_0(z) = \sum_{(a, m) \in G} \left( L(f_m^{(a)}(z)) - \frac{\pi i}{2} d_m^{(a)}(z) \log(1 - f_m^{(a)}(z)) \right),$$

$$\pi i d_m^{(a)}(z) = \log f_m^{(a)}(z) - \sum_{(b, k) \in G} A_{ab}^{m k} B_{ab} \log(1 - f_k^{(b)}(z)),$$

$$T(z, S) = \frac{1}{2} \sum_{(a, m) \in G} A_{ab}^{m k} B_{ab} N_m^{(a)} N_k^{(b)} - \sum_{(a, m) \in G} \left( \frac{1}{2} d_m^{(a)}(z) + M_m^{(a)} \right) N_m^{(a)}.$$

Next we introduce an element $\lambda(z) \in \hat{H}^*$ by

$$\lambda(z) = \frac{1}{2\pi i} \sum_{a=1}^{r} \sum_{b=1}^{r} C_{ab} \left( \sum_{j=1}^{\ell_b - 1} j \log(1 - f_j^{(b)}(z)) + \ell_b \left( \log Q_{\ell_b - 1}(z) - \log Q_{\ell_b}(z) \right) \right) \Lambda_a.$$  

From now on, we will mainly concern with the specialization $z = \Lambda \in P_\ell$, which is relevant to our restricted FR (2) (see (6) and (15) below).

**Conjecture.** Let $z \in \hat{H}^*$ be regular and put

$$c_0(z) = \frac{\ell \dim X_r}{\ell + g} - r - 24(\Delta_x^{\Lambda}(z) + N(z)),$$

$$\Delta^{\Lambda}_y = \frac{(y|z + 2\rho)}{2(\ell + g)} - \frac{|y|^2}{2\ell} \text{ for } y, z \in \hat{H}^*.$$  

Suppose $\Lambda \in P_\ell$. Then $N(\Lambda) \in \mathbb{Z}$ if $\Lambda$ is regular. In case $\Lambda$ is singular, $N(z)$ converges to finitely many integers depending on the ways $z \in \hat{H}^*_R$ approaches $\Lambda$.

This conjecture has been supported by numerical experiments for $X_r = A_r, B_r, C_r$ and $D_r$ with small values of the level $\ell$ and rank $r$. It is not difficult to prove $N(z) \in \mathbb{R}$ for any regular $z \in \hat{H}^*_R$. Because of the discontinuity of the log function, $c_0(z), \lambda(z)$ and therefore $N(z)$ assume finitely many values when $z$ approaches a singular $\Lambda$. Hereafter we shall arbitrarily fix one way of letting $z \to \Lambda$ (regular, $z \in \hat{H}^*_R$) and all the formulas involving $\Lambda \in P_\ell$ should be understood as defined by this limit whenever $\Lambda$ is singular. From numerical tests,
\[ Q_m^{(a)}(\Lambda) = Q_{\ell_a}^{(a)}(\Lambda)Q_{\ell_a-m}^{(a)}(\Lambda)^* \] seems valid for any \(-1 \leq m \leq \ell_a + 1\) and \(\Lambda \in P_\ell\), where \(^*\) denotes complex conjugation. Note in particular that it implies

\[ Q_{\ell_a+1}^{(a)}(\Lambda) = 0. \] (15)

Assuming this and \((A.10,11)\) in \([8]\), one can show that \(\lambda(\Lambda)\) is finite and satisfies

\[ \Lambda \equiv \lambda(\Lambda) \mod Q, \] (16a)

\[ T(\Lambda, S) = -\frac{1}{2\ell}|\lambda(\Lambda) + \beta(S)|^2 + \frac{1}{2\ell}|\lambda(\Lambda)|^2 + \text{integer}, \] (16b)

\[ \beta(S) = \sum_{a=1}^{r} \sum_{m=1}^{\ell_a-1} mN_m^{(a)} \alpha_a \in Q, \] (16c)

where the integer part is dependent on \(\Lambda\) and \(S\). Combining \((16b)\) with \((12a)\) and admitting the conjecture, we have

\[ c(\Lambda, S) = \frac{\ell \dim X}{\ell + g} - r - 24(\Delta^\Lambda_{\lambda(\Lambda)+\beta(S)} + \text{integer}) \text{ for any } \Lambda \in P_\ell. \] (17)

The RHS of \((17)\) with the congruence properties \((16a,c)\) is well known as the central charge \(-24(\text{scaling dimension mod } \mathbb{Z})\) of the level \(\ell X^{(1)}_r\) parafermion CFT \([22]\). As the integer set \(S\) is chosen variously, \(\beta(S)\) \((16c)\) ranges over the root lattice \(Q\). Thus all the spectra in the parafermion CFTs seem to come out from the Rogers dilogarithm function through the quantity \(c(\Lambda, S)\) \((11a)\). This is our main observation in this Letter. We note that in the case \(\Lambda = 0 \in P_\ell, \lambda(0) = 0\) holds and the earlier conjecture in \([20,8]\) corresponds to \(N(0) = 0\).

There are actually two sources for considering the quantity \(c(\Lambda, S)\) \((11a)\) in connection to the parafermion CFTs, which we shall now explain. The physical meaning of the integer set \(S\) is yet to be clarified in the light of these connections. The first source is to calculate finite-size corrections to the ground state energies for \(X^{(1)}_{1}\) fusion RSOS models. In \([3]\) Klümper and Pearce evaluated them for \(X^{(1)}_{1} = A^{(1)}_1\) via the Rogers dilogarithm and showed that the resulting exponents in regime I/II become the parafermion values in agreement with the earlier result in \([21]\). They started from the FR essentially identical to the restricted \(U_q(\mathfrak{a}^{(1)}_1)\) FR \((2)\), where \(Y_m^{(1)}(u)\) indeed is the finite-size correction part of the row-to-row transfer matrices. Such an interpretation is not known so far for the \(U_q(X^{(1)}_r)\) FR.
In general. However we have found that the $c(\Lambda, S)$ naturally arises as the “$c$ for excitations” in their sense through a heuristic calculation extending the $A_1^{(1)}$ case. Although the treatment in [3] is not necessarily identical to our formulation here, the regime I/II result therein effectively confirms our conjecture for the simplest case $X_r^{(1)} = A_1^{(1)}$. We shall emphasize that for higher rank algebras, the present $c(\Lambda, S)$ only corresponds to “regime I/II-like region” with a special “fusion type $\forall s_a > 1$” in the sense of section 4.2 in [8]. Thus we expect further generalizations should be possible related to various coset CFTs. See for example the regime III/IV result in [3] for $A_1^{(1)}$.

The second source is to study integrable perturbations [14] of parafermion CFTs. In [9-13], ultraviolet Casimir energies in various perturbed CFTs have been obtained by using integral equations of the form

$$Rm_j^{(a)} \text{ch} u = \epsilon_j^{(a)}(u) + \sum_{(b,k) \in G} \int_{-\infty}^{\infty} dv \Psi_{ab}^{jk}(u - v) \log \left( 1 + \exp(-\epsilon_k^{(b)}(v)) \right), \quad (18)$$

for $(a, m) \in G$, which originates in the TBA. Here $m_j^{(a)}$ denotes mass, $\epsilon_j^{(a)}(u)$ is the energy of the physical excitation with rapidity $u$, $\Psi_{ab}^{jk}(u)$ is some kernel and $R$ is the system size corresponding to the inverse temperature in the TBA. In addition, there are some observations that (18) also yields low-lying excitation energies by introducing “imaginary chemical potentials” [23,27,28]. Our $c(\Lambda, S)$ stems from a modification of (18) so as to match the logarithm of (2) under the identification $Y_j^{(a)}(u) = e^{-\epsilon_j^{(a)}(u)}$ up to the LHS which is tending to zero in the ultraviolet (or high temperature) limit. In particular, we choose $\forall m_j^{(a)} > 0$ as $j$–independent, add the imaginary term $\pi i D_{c,a,j}(\Lambda)$ (11b) on the RHS and take $\Psi_{ab}^{jk}(u)$ to be the universal kernel occurring in the $U_q(X_r^{(1)})$ Bethe equations, i.e.,

$$2\pi \Psi_{ab}^{jk}(u) = \int dx e^{iux} (\delta_{ab} \delta_{jk} - \hat{M}_{ab}(\pi x/2) \hat{A}^{(l)}_{ab}(\pi x/2))$$

using (2.10) of [8]. Then an analogous calculation to [11] amounts to considering the $c(\Lambda, S)$ as the “$c$ for excitations”.

These observations have led to the quantity $c(\Lambda, S)$ which has been conjectured to possess the remarkable property (17). Our $U_q(X_r^{(1)})$ FR (2) lies as their common background. It is to be analyzed further to actually yield the physical consequences in the $U_q(X_r^{(1)})$ Bethe ansatz systems. The details omitted here will
appear elsewhere.

The authors thank P.A.Pearce for supporting their visit to University of Melbourne, kind hospitality and explaining the work [3]. They also thank M.T.Batchelor, R.J.Baxter, V.V.Bazhanov, J.Suzuki, M.Wakimoto and R.B.Zhang for valuable comments. This work is supported by the Australian Research Council.

References

[1] R.J.Baxter, Physica 18D (1986) 321
[2] A.Klümper, M.T.Batchelor and P.A.Pearce, J.Phys.A 24 (1991) 3111
[3] A.Klümper and P.A.Pearce, “Conformal weights of RSOS lattice models and their fusion hierarchies”, preprint No.23-1991 (1991)
[4] I.Affleck, Phys.Rev.Lett. 56 (1986) 746
[5] H.M.Babujan, Nucl.Phys. B215[FS7] (1983) 317;
    H.J.de Vega and M.Karowski, Nucl.Phys. B285[FS19] (1987) 619;
    A.N.Kirillov and N.Yu.Reshetikhin, J.Phys.A 20 (1987) 1587
[6] V.V.Bazhanov and Yu.N.Reshetikhin, Int.J.Mod.Phys.A 4 (1989) 115
[7] V.V.Bazhanov and Yu.N.Reshetikhin, J.Phys.A 23 (1990) 1477
[8] A.Kuniba, “Thermodynamics of the $U_q(X^{(1)}_r)$ Bethe Ansatz System With $q$ A Root of Unity”, ANU preprint (1991)
[9] Al.B.Zamolodchikov, Nucl.Phys. B342 (1990) 695; B358 (1991) 497
[10] V.V.Bazhanov and N.Yu.Reshetikhin, Prog.Theor.Phys.Suppl. 102 (1990) 301
[11] M.J.Martins, Phys.Rev.Lett. 65 (1990) 2091
[12] T.R.Klassen and E.Melzer, Nucl.Phys. B338 (1990) 485; B350 (1991) 635
[13] V.A.Fateev and Al.B.Zamolodchikov, Phys.Lett. B271 (1991) 91
[14] A.B.Zamolodchikov, Int.J.Mod.Phys. A4 (1989) 4235
[15] H.W.J.Blöte, J.L.Cardy and M.P.Nightingale, Phys.Rev.Lett. 56 (1986) 742
[16] L.Lewin, Polylogarithms and associated functions, (North-Holland, 1981)
[17] C.N.Yang and C.P.Yang, J.Math.Phys. 10 (1969) 1115
[18] N.Yu.Reshetikhin and P. B. Wiegmann, Phys.Lett B189 (1987) 125
[19] A.N.Kirillov and N.Yu.Reshetikhin, Zap.Nauch.Semin.LOMI160 (1987) 211
[20] A.N.Kirillov, Zap.Nauch.Semin.LOMI164 (1987) 121 and private communications
[21] G.E.Andrews, R.J.Baxter and P.J.Forrester, J.Stat.Phys.**35** (1984) 193; 
    E.Date, M.Jimbo, A.Kuniba, T.Miwa and M.Okado, Nucl.Phys.**B290** 
    [FS20] (1987) 231; Adv.Stud.in Pure Math.**16** (1988) 17 
[22] V.A.Fateev and A.B.Zamolodchikov, Sov.Phys.JETP **62** (1985) 215; 
    D.Gepner, Nucl.Phys.**B290**[FS20] (1987) 10 
[23] T.R.Klassen and E.Melzer, Nucl.Phys.**B370** (1992) 511 
[24] V.G.Kac, *Infinite dimensional Lie algebras*, (Cambridge University Press, 1990) 
[25] M.Jimbo, Lett.Math.Phys. **10** (1985) 63; 
    V.G.Drinfel’d, ICM proceedings, Berkeley (1987) 798 
[26] A.B.Zamolodchikov, Phys.Lett.**B253** (1991) 391 
[27] M.J.Martins, Phys.Rev.Lett. **67** (1991) 419 
[28] P.Fendley, Nucl.Phys.**B372** (1992) 533