Gradient Descent in the Absence of Global Lipschitz Continuity of the Gradients

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Abstract. Gradient descent (GD) is a collection of continuous optimization methods that have achieved immeasurable success in practice. Owing to data science applications, GD with diminishing step sizes has become a prominent variant. While this variant of GD has been well-studied in the literature for objectives with globally Lipschitz continuous gradients or by requiring bounded iterates, objectives from data science problems do not satisfy such assumptions. Thus, in this work, we provide a novel global convergence analysis of GD with diminishing step sizes for differentiable nonconvex functions whose gradients are only locally Lipschitz continuous. Through our analysis, we generalize what is known about gradient descent with diminishing step sizes including interesting topological facts; and we elucidate the varied behaviors that can occur in the previously overlooked divergence regime. Thus, we provide the most general global convergence analysis of GD with diminishing step sizes under realistic conditions for data science problems.

Key words. Gradient Descent, Diminishing Step Sizes, Convergence, Divergence

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1. Introduction. Proposed nearly two centuries ago [11, 12, 25, 33], gradient descent is a set of canonical continuous optimization methods that have achieved immeasurable success in a plethora of applications (e.g., [7, 16, 26]). Owing to their prominence and utility in data science, gradient descent methods have continued to grow in variety, and their theory has received renewed interest by the optimization and data science communities for problems in this area (e.g., [17, 19, 31, 39]). In particular, gradient descent with pre-scheduled step sizes has become popular owing to the additional expense of using line search techniques for data science problems. Correspondingly, the theory of gradient descent with pre-scheduled step sizes has grown in a number of interesting directions including new local convergence rate analyses (e.g., [17, 29]) and saddle-point avoidance analyses (e.g., [18, 27, 32]).

That said, the more fundamental global convergence analysis of gradient descent with pre-scheduled step sizes has lagged owing to two challenges. First, gradient descent with pre-scheduled step sizes does not guarantee a monotonic reduction in the objective function (c.f., Armijo’s Method [1]), which is the key ingredient used to analyze such methods via Zoutendijk’s approach [50]. Second, because of the nonconvexity of common data science problems, the analysis of gradient descent cannot leverage uniform continuity of the gradient.

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1Canonical data science problems such as Poisson regression, linear three-or-more layer feed forward networks, and linear three-or-more time horizon recurrent networks fail to possess globally Lipschitz continuous gradients or uniformly continuous gradients when trained using standard loss functions [42, §1].
function, global Lipschitz continuity of the gradient function or presuppose that its iterates remain in a bounded region for a function with a locally Lipschitz continuous gradient,\(^2\) which are instrumental assumptions for overcoming the previous challenge \([44, 45]\). As a result of the latter challenge, typical analysis approaches for global convergence of gradient descent fall short (see subsection 2.3 for an overview). In fact, even the new vogue for analysis in machine learning, the continuous approach \([3, 4, 34, 35]\), falls short, as this approach requires compactness of the image space of the iterates in \([3, \text{Theorem 3.2}]\), boundedness of iterates \([20, \text{Theorem 2}]\), or global Lipschitz continuity of the gradient of the objective function \([35, \text{Assumption 1}]\). To summarize, to the best of our knowledge, existing global convergence analyses of gradient descent with diminishing step sizes do not apply to canonical, nonconvex, differentiable data science problems.

To address this shortcoming, we generalize recently developed techniques for the analysis of stochastic gradient descent \([41–43]\) to analyze gradient descent with diminishing step sizes for nonconvex optimization problems that are bounded from below and whose gradient is \textit{locally Lipschitz continuous}, which are more realistic assumptions for canonical data science problems \([42, \S A]\). Our analysis has several important contributions.

1. First, we present a novel upper-bound model, which can be used under milder assumptions that are appropriate for data science problems (see subsection 2.3 for a discussion, and Lemma 3.1 for the result). This upper-bound model is directly useful in analyzing many other algorithms for unconstrained optimization, and the strategies used to prove the result seem useful for analyzing algorithms for constrained optimization.

2. Second, our analysis provides counterexamples to what is known about gradient descent with diminishing step sizes. Specifically, previous results (e.g., \([5, \text{Proposition 1.2.4}]\)) showed that, under a global Lipschitz continuity assumption on the gradient, the iterates tend to a region where the gradient is zero; the objective function converges to a finite limit; and, if the iterates remain bounded, then the iterates converge to a stationary point. Our analysis, under the more realistic local Lipschitz continuity assumption on the gradient, offers a correction to this view—that the gradient function can remain bounded away from zero and the objective function can diverge (see explicitly constructed examples in section 4).

3. Our analysis addresses a preliminary question about gradient descent and nonconvexity: \textit{given a relatively arbitrary objective function, can its nonconvexity cause gradient descent to behave erratically in a region?} Despite the general nonconvexity allowed by our assumptions, we show that the limit supremum and limit infimum of the objective function evaluated at the iterates must tend to each other if the iterates remain in a region for long enough—even if they eventually escape; we also show that the limit of the gradient function evaluated at the iterates must tend to zero if the iterates remain in a region for long enough—even if they eventually escape (see Theorem 3.9). A more interesting question is whether such a statement holds uniformly over important subsets of nonconvex objective functions of the ones considered here.\(^3\) Our analysis at least gives hope that such a statement may be true.

\(^2\)If the iterates remain in a bounded region, then compactness and the local Lipschitz continuity condition would imply a global Lipschitz continuous constant in the bounded region.

\(^3\)Such functions must be beyond those which have globally Lipschitz continuous gradients, and still be valid for data science problems. To the best of our knowledge, the appropriate notion of continuity of the gradients has not been determined.
4. Our analysis adds several topological insights to what is known (e.g., [5, Proposition 1.2.4]). Primarily, we show, the subsequential limit points of the iterates are a connected set that is either a singleton or infinite. Moreover, if the set is infinite, we conclude that it cannot contain an open set.

Thus, to the best of our knowledge, our results provide the most general and complete global convergence analysis of gradient descent with diminishing step sizes under realistic assumptions for nonconvex, differentiable optimization problems that arise in data science.

The remainder of this work is organized as follows. In section 2, we specify the class of nonconvex optimization problems of interest and the precise form of gradient descent with diminishing step sizes. In section 3, we analyze the behavior of gradient descent with diminishing step sizes. In section 4, we construct examples that elucidate the possible behaviors of gradient descent with diminishing step sizes in the divergence regime. Final remarks are given in section 5.

2. Gradient Descent. We begin by introducing the general class of optimization problems that we consider in this work. Then, we specify the precise form of gradient descent with diminishing step sizes. With the problem class and procedure specified, we describe relevant analysis approaches in the literature.

2.1. Optimization Problem. To cover a variety of canonical problems in data science [42, §A], consider the optimization problem

\[
\min_{x \in \mathbb{R}^p} F(x),
\]

under the following assumptions.

**Assumption 2.1.** The objective function, \( F : \mathbb{R}^p \to \mathbb{R} \), is bounded from below by a constant \( F_{l.b.} \).

**Assumption 2.2.** The gradient function, \( \hat{F}(x) = \nabla F(z)|_{z=x} \), exists \( \forall x \in \mathbb{R}^p \), and is locally Lipschitz continuous.

For our context, we use the following definition of local Lipschitz continuity.

**Definition 2.3.** A function \( G : \mathbb{R}^p \to \mathbb{R}^p \) is locally Lipschitz continuous if for every \( x \in \mathbb{R}^p \) there exists an open ball of \( x, N \), and there exists \( L \geq 0 \), such that, \( \forall y, z \in N \),

\[
\frac{\|G(y) - G(z)\|_2}{\|y - z\|_2} \leq L.
\]

Equivalently, \( G \) is locally Lipschitz continuous if for every compact set \( C \subset \mathbb{R}^p \), there exists \( L \geq 0 \) such that (2.2) holds for all \( y, z \in C \). This well-known statement is shown in Lemma A.1.

To give an example of the broad applicability of Assumption 2.2, any optimization problem whose objective function is twice continuously differentiable immediately satisfies Assumption 2.2. This well-known statement is given formally in Lemma B.1.
2.2. Gradient Descent with Diminishing Step Sizes. Now, suppose we apply gradient
descent with diminishing step-sizes to solve (2.1). Specifically, given $x_0 \in \mathbb{R}^p$, we generate a
sequence $\{x_k : k \in \mathbb{N}\}$ according to

$$x_{k+1} = x_k - M_k \dot{F}(x_k),$$

where $M_k$ satisfies some of the following properties:

Property 2.4. $\{M_k : k + 1 \in \mathbb{N}\} \subset \mathbb{R}^{p \times p}$ are symmetric positive definite matrices;

Property 2.5. $\sum_{k=0}^{\infty} \lambda_{\text{min}}(M_k)$ diverges, where $\lambda_{\text{min}}(M_k)$ denotes the smallest eigenvalue
of $M_k$; and

Property 2.6. $\lim_{k \to \infty} \lambda_{\text{max}}(M_k) = 0$ where $\lambda_{\text{max}}(M_k)$ denotes the largest eigenvalue of $M_k$.

Properties 2.4 to 2.6 are a matrix-valued generalization of classical diminishing step size
requirements [5, Proposition 1.2.4]. Moreover, Properties 2.4 to 2.6 are enough to show
that the objective function evaluated at the iterates converges, and to show that the limit
infimum of the norm of the gradient function evaluated at the iterates converges to zero (see
Theorem 3.5). To show that the gradient function converges to zero, these properties will be
augmented with the following:

Property 2.7. There exists $\kappa \geq 1$ such that $\lambda_{\text{max}}(M_k)/\lambda_{\text{min}}(M_k) \leq \kappa$ for all $k + 1 \in \mathbb{N}$.

Of interest, Properties 2.4 to 2.6 can potentially account for adaptive step size selection
procedures that exist in the literature, namely those that do not make use of objective function
information. For example, Properties 2.4 to 2.6 can apply to the method of [5] (with $\lambda = 1$), which combines incremental nonlinear least squares, the Gauss-Newton method, and the
Extended Kalman Filter. However, especially in the nonlinear case, Property 2.5 would be
difficult to verify without assuming something akin to what is called persistent excitation
in the control literature [6, 8, 28, 40]. Indeed, in the objective-free first-order optimization
(e.g., AdaGrad-type methods), this persistent excitation condition often manifests through a
combination of assumptions about the optimization problem (e.g., bounded gradients) and
the diagonal or identity-scaling choice of $\{M_k : k + 1 \in \mathbb{N}\}$ [15, 21–24, 46, 47].

2.3. Important Analysis Approaches in the Literature. With the problem and algorithm
established, we briefly review two important analysis frameworks in the literature with respect
to simple objective functions satisfying Assumptions 2.1 and 2.2: $|x|^3$ and $\exp(x)$. Note, these
two examples are essential components in verifying that canonical data science problems
neither have globally Lipschitz continuous gradients nor uniformly continuous gradients [see
42, §A].

In one analysis framework for trust region methods [e.g., 30, 36], continuity of the gradient
function, properties of the algorithm, and evaluations of the objective function are needed to
show that the limit infimum of the gradient function evaluated at the iterates goes to zero.
Furthermore, assuming uniform continuity of the gradient function allows for the conclusion:
the limit of the gradient function evaluated at the iterates goes to zero [9, 10, 30, 36, 48, 49].

\footnote{In [9, AF.2], uniform continuity implies the needed property.}
While continuity of the gradient function certainly holds for our two example objectives, neither of them satisfy uniform continuity of the gradient function. Moreover, in our context, gradient function information is not combined with objective function information to ensure sufficient decay at each step, which limits our ability to use the assumption of continuity of the gradient in place of Assumption 2.2.\textsuperscript{5}

In the other analysis framework espoused by [5, Proposition 1.2.4], [38, Theorem 3.2], and [2, Lemma 10.4], the essential ingredient is a global upper-bound model for the objective function,

\begin{equation}
F(y) \leq F(x) + \dot{F}(x)^\top(y - x) + \frac{L}{2} \|y - x\|_2^2, \forall y, x \in \mathbb{R}^p,
\end{equation}

where $L$ is a fixed constant that arises from the assumption that the gradient function is globally Lipschitz continuous (i.e., $L$ is the same regardless of $x \in \mathbb{R}^p$ and $N$ in Definition 2.3). Indeed, this global upper-bound model is commonly used in recent analyses, both deterministic and stochastic [13–15, 21–24, 46, 47]. This global upper-bound model is actualized by replacing $y$ with $x_{k+1}$, $x$ with $x_k$, and rewriting the right-hand-side strictly in terms of quantities depending on $x_k$. Then, the upper-bound model is manipulated to show that the objective function is decreasing. Unfortunately, such a global upper-bound model does not apply to the two simple example objective functions, which renders such analyses inapplicable to common data science problems.

In [5, Exercise 1.2.5], this global upper-bound model is relaxed to the case where such an $L$ exists for every level set of the objective function and assumes every level set is bounded. In this case, this relaxed upper-bound model can then be used to establish that, if a gradient descent procedure remains in a level set, then the objective function converges to a finite value and the gradient function converges to zero. Indeed, this relaxed upper-bound model can account for $|x|^3$, but it cannot account for $\exp(x)$ nor our example in section 4, which has bounded level sets yet the iterates never remain in any level set. Hence, even this relaxation cannot account for the types of problems that satisfy Assumptions 2.1 and 2.2.

Our approach can be viewed as a generalization of [5, Exercise 1.2.5] as we can use Assumption 2.2 to write a valid upper-bound model for any two points in $\mathbb{R}^p$, even though we only assume local Lipschitz continuity of the gradient (see Lemma 3.1 and Example 3.2). We now introduce this analysis approach.

3. Global Convergence Analysis. Here, we study the global convergence of gradient descent, (2.3), with diminishing step sizes satisfying Properties 2.4 to 2.6 on a general class of nonconvex functions as defined by Assumptions 2.1 and 2.2. Our main conclusion is that, despite the allowed nonconvexity of a problem, the objective function and gradient function at the iterates are either stabilizing or the iterates must continually tend further away from the origin. Thus, if we somehow know that the iterates remain bounded, then they must converge to a stationary point.

To prove these claims, our main innovation is to analyze the gradient descent procedure under a stopping time framework, which is a theoretical construction that allows us to analyze

\textsuperscript{5}This raises the question of how much objective function information is really needed in order to ensure similar results as trust-region without substantially increasing computational costs for data science problems.
the without modifying it. We enumerate the steps in our analysis here.

1. In subsection 3.1, we establish a novel upper bound model based on stopping times to relate the optimality gaps of two arbitrary points even under local Lipschitz continuity of the gradient function (see Lemma 3.1). We then simplify this statement when we substitute the two arbitrary points with consecutive iterates generated by the gradient descent procedure with diminishing step sizes (see Corollary 3.4).

2. In subsection 3.2, we apply Zoutendijk’s analysis approach [50]. We show that the limit supremum and limit infimum of the objective function evaluated at the iterates must tend to each other if the iterates remain in a region for long enough (even if they eventually escape). We also show that the limit infimum of the gradient function evaluated at the iterates must tend to zero if the iterates remain in a region for long enough (even if they eventually escape).

3. In subsection 3.3, we strengthen the preceding statement: we show that the limit of the gradient function evaluated at the iterates tends to zero if the iterates remain in a region for long enough (even if they eventually escape).

4. In subsection 3.4, we establish topological properties of the iterates when their subse-

...quent limits are a bounded set. In particular, we establish the well-known results that the limit points of the iterates converge to a closed set where the gradient function is zero, and we establish—to the best of our knowledge—the novel result that this set is connected and cannot contain an open set (see Theorem 3.9). In other words, when it converges, gradient descent with diminishing step sizes either tends to a single point or an infinite set that must, in a sense, lack volume. Moreover, gradient descent with diminishing step sizes cannot have a cycle nor can it converge to a limit cycle with a finite number of points.

We turn our attention to the divergence regime in section 4.

3.1. A Relationship for the Optimality Gap. We now establish an upper-bound inequality for the optimality gap between two points in \( \mathbb{R}^p \) under local Lipschitz continuity (see subsection 2.3). To establish this result, we make use of a technique from probability theory that analyzes stochastic processes under stopping times. For the deterministic equivalent, we define for an arbitrary point \( x \in \mathbb{R}^p \) and \( R \geq 0 \),

\[
\pi_x(R) = \begin{cases} 
1 & \|x\|_2 \leq R \\
0 & \text{otherwise}.
\end{cases}
\]

**Lemma 3.1.** Suppose \( F : \mathbb{R}^p \to \mathbb{R} \) satisfies Assumptions 2.1 and 2.2. Then, for all \( R \geq 0 \) there exists a constant \( C_R > 0 \) such that, for all \( x, y \in \mathbb{R}^p \),

\[
[F(y) - F_{l,b}] \pi_y(R) \pi_x(R) \leq \left[ F(x) - F_{l,b} - \hat{F}(x)^\top (y - x) + C_R \|y - x\|_2^2 \right] \pi_x(R).
\]

At first glance, we might think that Lemma 3.1 can be proved by combining (2.4) with \( L \geq 0 \) specific to the radius \( R > 0 \) of interest to show

\[
[F(y) - F_{l,b}] \pi_y(R) \pi_x(R) \leq \left[ F(x) - F_{l,b} - \hat{F}(x)^\top (y - x) + \frac{L}{2} \|y - x\|_2^2 \right] \pi_y(R) \pi_x(R),
\]
and then using \( \pi_y(R) \pi_x(R) \leq \pi_x(R) \) to upper bound the right hand side to conclude,

\[
[F(y) - F_{l,b}] \pi_y(R) \pi_x(R) \leq \left[ F(x) - F_{l,b} - \dot{F}(x)^T (y - x) + \frac{L}{2} \| y - x \|_2^2 \right] \pi_x(R).
\]

Unfortunately, it is the last step that can be problematic as the the right hand side can become negative, which produces a false inequality. The following example illustrates the issue.

**Example 3.2.** Consider

\[
F(x) = \begin{cases} 
10(1 - x) & x \leq 1 \\
\frac{1}{10x - 9} - 1 & x > 1, 
\end{cases} \quad \text{for which } \dot{F}(x) = \begin{cases} 
-10 & x \leq 1 \\
-\frac{10}{(10x - 9)^2} & x > 1,
\end{cases}
\]

which is bounded from below and for which \( \dot{F}(x) \) is globally Lipschitz continuous. If we now set \( R = 1, x = 1 \), then we see that \( L = 0 \) on \([-1, 1]\) and \( \pi_x(1) = 1 \). If we now select \( y = 11 \) (which would be the iterate generated by a gradient descent procedure at \( x = 1 \)) then \( \pi_y(1) = 0 \). Plugging this into (3.4), \( 0 = (F(11) + 1)0 \leq (0 + 1 - 100)1 = -99 \), which is false. Hence, proving Lemma 3.1 requires a little more care as we show below.

**Proof.** First, for any \( R \geq 0 \) define \( L_R \) to be the Lipschitz constant for the gradient in the closed ball of radius \( R \) around the point \( 0 \in \mathbb{R}^p \), which is well defined by Assumption 2.2 and Lemma A.1. For any fixed \( \delta > 0 \), it readily follows that \( L_R \leq L_{R + \delta} \). Second, let \( L(y, x) \) be the Lipschitz constant of the gradient in a closed ball of radius \( \| y - x \|_2 \) around the point \( x \). Finally, for any \( R \geq 0 \), define \( G_R \) to be the maximum \( \| \dot{F}(x) \|_2 \) for all \( x \) in a closed ball of radius \( R \) around the point \( 0 \in \mathbb{R}^p \). Now, let \( y, x \in \mathbb{R}^p \) be arbitrary.

By Taylor’s remainder theorem,

\[
F(y) - F_{l,b} = F(x) - F_{l,b} + \dot{F}(x)^T (y - x)
+ \int_0^1 \left[ \dot{F}(x + t(y - x)) - \dot{F}(x) \right]^T (y - x) dt.
\]

By applying Assumption 2.2 to the last term,

\[
F(y) - F_{l,b} \leq F(x) - F_{l,b} + \dot{F}(x)^T (y - x) + \frac{L(y,x)}{2} \| y - x \|_2^2.
\]

Note, to understand why we must keep going at this point in the proof, see Remark 3.6. We now introduce \( \pi_y(R) \) and \( \pi_x(R) \) into (3.7). That is,

\[
[F(y) - F_{l,b}] \pi_y(R) \pi_x(R) \leq \left[ F(x) - F_{l,b} - \dot{F}(x)^T (y - x) + \frac{L(y,x)}{2} \| y - x \|_2^2 \right] \pi_y(R) \pi_x(R).
\]

If \( \pi_y(R) \pi_x(R) = 1 \), then \( \| y \|_2 \leq R \) and \( \| x \|_2 \leq R \). Thus, \( L(y,x) \leq L_R \leq L_{R + \delta} \). When \( \pi_y(R) \pi_x(R) = 0 \), then both sides are trivially zero. Therefore,

\[
[F(y) - F_{l,b}] \pi_y(R) \pi_x(R) \leq \left[ F(x) - F_{l,b} - \dot{F}(x)^T (y - x) + \frac{L_{R + \delta}}{2} \| y - x \|_2^2 \right] \pi_y(R) \pi_x(R).
\]
Now, we want $\pi_x(R)$ alone on the right hand side. So, we simply add and subtract a term involving $\pi_x(R)$, and study the difference term. That is,

$$
\begin{align*}
[F(y) - F_{l.b.}]\pi_y(R)\pi_x(R) \\
\leq \left[ F(x) - F_{l.b.} - \hat{F}(x)^\top(y - x) + \frac{L_{R+\delta}}{2} \| y - x \|_2^2 \right] \pi_x(R) \\
+ \left[ F(x) - F_{l.b.} - \hat{F}(x)^\top(y - x) + \frac{L_{R+\delta}}{2} \| y - x \|_2^2 \right] [\pi_y(R)\pi_x(R) - \pi_x(R)].
\end{align*}
$$

(3.10)

We now have two cases to upper bound the last term of (3.10). Note, $\pi_y(R)\pi_x(R) - \pi_x(R) \leq 0$.

Case 1. If

$$
F(x) - F_{l.b.} - \hat{F}(x)^\top(y - x) + \frac{L_{R+\delta}}{2} \| y - x \|_2^2 \geq 0,
$$

(3.11)

then

$$
\begin{align*}
\left[ F(x) - F_{l.b.} - \hat{F}(x)^\top(y - x) + \frac{L_{R+\delta}}{2} \| y - x \|_2^2 \right] [\pi_y(R)\pi_x(R) - \pi_x(R)] \leq 0.
\end{align*}
$$

(3.12)

Hence, in this case, we can upper bound the last term in (3.10) by any non-negative term.

Case 2. If

$$
\begin{align*}
F(x) - F_{l.b.} - \hat{F}(x)^\top(y - x) + \frac{L_{R+\delta}}{2} \| y - x \|_2^2 < 0,
\end{align*}
$$

(3.13)

then, using $\pi_y(R)\pi_x(R) \leq \pi_x(R)$,

$$
\begin{align*}
\left[ F(x) - F_{l.b.} - \hat{F}(x)^\top(y - x) + \frac{L_{R+\delta}}{2} \| y - x \|_2^2 \right] [\pi_y(R)\pi_x(R) - \pi_x(R)] \geq 0.
\end{align*}
$$

(3.14)

Thus, we need only to find a lower bound for the first term in the product to upper bound the entire term. Specifically,

$$
- \| \hat{F}(x) \|_2 \| y - x \|_2 \leq F(x) - F_{l.b.} - \hat{F}(x)^\top(y - x) + \frac{L_{R+\delta}}{2} \| y - x \|_2^2.
$$

(3.15)

Now, when $\pi_y(R)\pi_x(R) < \pi_x(R)$, $\| x \|_2 \leq R$. Moreover, if (3.13) holds, then $R + \delta < \| y \|_2$. To see this, suppose (3.13) holds and $\| y \|_2 \leq R + \delta$. Then $L(y, x) \leq L_{R+\delta}$. If we now apply (3.7) and this inequality,

$$
0 \leq F(y) - F_{l.b.} \leq F(x) - F_{l.b.} - \hat{F}(x)^\top(y - x) + \frac{L_{R+\delta}}{2} \| y - x \|_2^2,
$$

(3.16)

which contradicts (3.13). Hence, in this case, $R + \delta < \| y \|_2$.

Using the triangle inequality, $R + \delta < \| y \|_2 \leq \| x \|_2 + \| y - x \|_2 \leq R + \| y - x \|_2$. That is, $1 \leq \| y - x \|_2/\delta \leq \| y - x \|_2^2/\delta^2$. 


Hence,

\[
\begin{align*}
\left[F(x) - F_{l.b.} - \dot{F}(x)^\top (y - x) + \frac{L_{R+\delta}}{2} \|y - x\|_2^2 \right] \left[\pi_y(R)\pi_x(R) - \pi_x(R)\right] \\
\leq \delta \left\|\dot{F}(x)\right\|_2 \|y - x\|_2 \left[\pi_x(R) - \pi_y(R)\pi_x(R)\right] \\
\leq \frac{G_R}{\delta} \|y - x\|_2 \pi_x(R),
\end{align*}
\]

(3.17)

(3.18)

(3.19)

where in the last line we have used \(\pi_x(R) - \pi_y(R)\pi_x(R) \leq \pi_x(R)\) as these are \(\{0,1\}\)-valued quantities.

Putting these two cases together in (3.10), we conclude,

\[
\left[F(y) - F_{l.b.}\right] \pi_y(R) \pi_x(R) \\
\leq \left[F(x) - F_{l.b.} - \dot{F}(x)^\top (y - x) + \left(\frac{L_{R+\delta}}{2} + \frac{G_R}{\delta}\right) \|y - x\|_2^2 \right] \pi_x(R).
\]

(3.20)

Letting \(C_R = L_{R+\delta}/2 + G_R/\delta\), the conclusion follows.

\textit{Remark 3.3.} If we replace \((y, x)\) with \((x_{k+1}, x_k)\) in the preceding result, it might be tempting to choose a \(\delta\) that minimizes \(C_R\) and then to use a standard approach to find a complexity result. However, this complexity result would only hold if all of the iterates remained within a radius \(R\) of 0, which, under Assumptions 2.1 and 2.2, cannot be guaranteed a priori as shown by our construction in section 4. Thus, a complexity result would only be appropriate if some additional information is known to guarantee a single Lipschitz constant (e.g., by knowing that the iterates remain bounded), in which case we would directly make use of (2.4) and we would have no use for Lemma 3.1.

We now apply Lemma 3.1 to the iterate sequence generated by gradient descent. To do so, we will make use of the following notation

\[
\chi_0^k(R) = \begin{cases} 1 & \|x_j\|_2 \leq R, \ j = 0, \ldots, k \\ 0 & \text{otherwise.} \end{cases}
\]

(3.21)

That is, \(\chi_0^k(R) = \pi_{x_0}(R)\pi_{x_1}(R) \cdots \pi_{x_k}(R)\). With this notation, we have the following simplification of Lemma 3.1 when applied to gradient descent.

\textbf{Corollary 3.4.} Suppose \(F : \mathbb{R}^p \to \mathbb{R}\) satisfies Assumptions 2.1 and 2.2. Let \(x_0 \in \mathbb{R}^p\) and let \(\{x_k : k \in \mathbb{N}\}\) be generated by (2.3) satisfying Properties 2.4 and 2.6. Then \(\forall R \geq 0, \ \exists K \in \mathbb{N}\) such that for all \(k \geq K\),

\[
\left[F(x_{k+1}) - F_{l.b.}\right] \chi_{k+1}^0(R) \leq \left[F(x_k) - F_{l.b.} - \frac{1}{2} \lambda_{\min}(M_k) \left\|\dot{F}(x_k)\right\|_2^2 \right] \chi_k^0(R).
\]

(3.22)
Proof. By Lemma 3.1, \( \exists C_R > 0 \) such that, for any \( k + 1 \in \mathbb{N} \),
\[
[F(x_{k+1}) - F_{lb}] \pi_{x_{k+1}}(R) \pi_{x_k}(R)
\]
(3.23)
where we have made use of (2.3) to replace \( x_{k+1} - x_k \). If we now multiply both sides by the non-negative quantity \( \pi_{x_0}(R) \cdots \pi_{x_{k-1}}(R) \), then
\[
[F(x_{k+1}) - F_{lb}] \lambda^0_{k+1}(R)
\]
(3.24)
\[
\leq \left[ F(x_k) - F_{lb} - \hat{F}(x_k)^T M_k \hat{F}(x_k) + C_R \left\| M_k \hat{F}(x_k) \right\|_2^2 \right] \pi_{x_k}(R),
\]
where we have used (3.23) to replace \( x_{k+1} - x_k \). If we now multiply both sides by the non-negative quantity \( \pi_{x_0}(R) \cdots \pi_{x_{k-1}}(R) \), then
\[
[F(x_{k+1}) - F_{lb}] \lambda^0_{k+1}(R)
\]
(3.25)
\[
\leq \left[ F(x_k) - F_{lb} - \hat{F}(x_k)^T M_k \hat{F}(x_k) + C_R \left\| M_k \hat{F}(x_k) \right\|_2^2 \right] \pi_{x_k}(R).
\]

The result follows if we show that \( \exists K \in \mathbb{N} \) such that \( \forall k \geq K \),
\[
\hat{F}(x_k)^T M_k \hat{F}(x_k) + C_R \left\| M_k \hat{F}(x_k) \right\|_2^2 \leq -\frac{1}{2} \lambda_{\min}(M_k) \left\| \hat{F}(x_k) \right\|_2^2.
\]
(3.26)
To this end, we prove, if \( M \) is symmetric positive definite with \( \lambda_{\max}(M) < 1/(2C_R) \) then, for any \( v \in \mathbb{R}^p \) with unit norm, \(-v^T M v + C_R v^T M M v \leq -\frac{1}{2} \lambda_{\min}(M) \). Let \( 0 < \lambda_{\min}(M) = \lambda_0 \leq \lambda_{p-1} \cdots \leq \lambda_2 \leq \lambda_1 = \lambda_{\max}(M) < 1/(2C_R) \). Using the Schur Decomposition, there exists an orthogonal matrix \( Q \) such that \(-v^T M v + C_R v^T M M v = \sum_{\ell=1}^p (-\lambda_\ell + C_R \lambda_\ell^2) w_\ell^2 \), where \( w_\ell \) is the \( \ell \)th component of \( Q v \) (note, \( \| w \|_2 = \| Q v \|_2 = \| v \|_2 = 1 \)). Since \( \lambda_\ell < 1/(2C_R) \), it follows that \( C_R \lambda_\ell^2 < \lambda_\ell/2 \). Subtracting \( \lambda_\ell \) from both sides, \(-\lambda_\ell + C_R \lambda_\ell^2 < -\lambda_\ell/2 \leq -\lambda_{\min}(M)/2 \). Thus,
\[
-v^T M v + C_R v^T M M v \leq -\sum_{\ell=1}^p \frac{\lambda_\ell}{2} w_\ell^2 = -\frac{1}{2} v^T M v \leq -\frac{\lambda_{\min}(M)}{2}.
\]
Since \( \lambda_{\max}(M_k) \rightarrow 0 \), there exists a \( K \in \mathbb{N} \) such that \( \forall k \geq K \), \( \lambda_{\max}(M_k) \leq 1/(2C_R) \). Hence, there exists a \( K \) such that \( \forall k \geq K \), (3.25) holds.

3.2. Applying Zoutendijk’s Analysis Approach. We now apply the recursive relationship established in Corollary 3.4 to study the objective and gradient using Zoutendijk’s analysis method [50]. Recall, our main conclusion from the next result is that the limit supremum and limit infimum of the objective function evaluated at the iterates must tend to each other if the iterates persist in a region for long enough (even if they eventually escape), and the limit infimum of the gradient function evaluated at the iterates must tend to zero under similar circumstances. We stress that these conclusions are not the same as presupposing that the iterates remain in a bounded region.

Theorem 3.5. Suppose \( F : \mathbb{R}^p \rightarrow \mathbb{R} \) satisfies Assumptions 2.1 and 2.2. Let \( x_0 \in \mathbb{R}^p \) and let \( \{x_k : k \in \mathbb{N} \} \) be generated by (2.3) satisfying Properties 2.4 to 2.6. Then, for all \( R \geq 0 \),
\[
\lim_{k \rightarrow \infty} F(x_k) \chi_k^0(R) \text{ exists and is finite, and } \lim_{k \rightarrow \infty} \left\| \hat{F}(x_k) \right\|_2 \chi_k^0(R) = 0.
\]
(3.27)
If \( \sup_k \|x_k\|_2 < \infty \) then \( \lim_{k \rightarrow \infty} F(x_k) \) exists and is finite, and \( \lim_{k \rightarrow \infty} \left\| \hat{F}(x_k) \right\|_2 = 0 \).
Proof. Let $R \geq 0$. The conditions of Corollary 3.4 are satisfied, and its conclusion is used freely herein. For the objective function, $\exists K \in \mathbb{N}$ such that for all $k \geq K$, $|F(x_{k+1}) - F_{l.b.}| \chi^0_{k+1}(R) \leq |F(x_k) - F_{l.b.}| \chi^0_k(R)$. As $\{[F(x_k) - F_{l.b.}] \chi^0_k(R) : k \geq K\}$ is a nonincreasing sequence bounded from below, it converges. Now, if we further assume that $\sup_k \|x_k\|_2 < \infty$, then there exists an $R > 0$ such that $\chi^0_k(R) = 1 \forall k + 1 \in \mathbb{N}$. Hence, $\lim_{k \to \infty} F(x_k) - F_{l.b.}$ exists and is finite.

For the gradient function, applying the conclusion of Corollary 3.4 and rearranging terms, for all $k \geq K$,

$$\frac{1}{2} \lambda_{\min}(M_k) \left\| \dot{F}(x_k) \right\|_2^2 \chi^0_k(R) \leq [F(x_k) - F_{l.b.}] \chi^0_k(R) - [F(x_{k+1}) - F_{l.b.}] \chi^0_{k+1}(R).$$

(3.28)

Letting $j \geq K$ and using $F(x_{j+1}) - F_{l.b.} \geq 0$,

$$\sum_{k=K}^{j} \frac{1}{2} \lambda_{\min}(M_k) \left\| \dot{F}(x_k) \right\|_2 \chi^0_k(R) \leq [F(x_K) - F_{l.b.}] \chi^0_K(R).$$

(3.29)

Now for a contradiction, suppose $\exists c > 0$ such that $\liminf_{k \to \infty} \left\| \dot{F}(x_k) \right\|_2^2 \chi^0_k(R) > c$. Then, there exists a $K' > K$ such that

$$\frac{c}{2} \sum_{k=K'}^{j} \lambda_{\min}(M_k) \leq \sum_{k=K'}^{j} \frac{1}{2} \lambda_{\min}(M_k) \left\| \dot{F}(x_k) \right\|_2 \chi^0_k(R) \leq [F(x_K) - F_{l.b.}] \chi^0_K(R) < \infty.$$  

(3.30)

By Property 2.5, we have a contradiction. This part of the result follows for any $R \geq 0$.

Now, if $\sup_k \|x_k\|_2 < \infty$ then there exists an $R > 0$ such that $\sup_k \|x_k\|_2 < R$. Therefore, $\chi^0_k(R) = 1$ for all $k + 1 \in \mathbb{N}$, and, thus, the final part of the result follows.

Remark 3.6. Suppose we directly attempt to use Zoutendijk’s analysis approach in (3.7) with $y = x_{k+1}$ and $x = x_k$. We begin by rearranging (3.7) and summing up to $j \in \mathbb{N}$ to conclude,

$$\sum_{k=0}^{j} \dot{F}(x_k)^\intercal M_k \dot{F}(x_k) - \frac{L(x_{k+1}, x_k)}{2} \left\| M_k \dot{F}(x_k) \right\|_2^2 \leq F(x_0) - F_{l.b.}.$$  

(3.31)

Thus, we conclude

$$\lim_{k \to \infty} \dot{F}(x_k)^\intercal M_k \dot{F}(x_k) - \frac{L(x_{k+1}, x_k)}{2} \left\| M_k \dot{F}(x_k) \right\|_2^2 = 0.$$  

(3.32)

Unfortunately, this conclusion does not imply that $\dot{F}(x_k) \to 0$ as $k \to \infty$. For instance, suppose as $k \to \infty$, $L(x_{k+1}, x_k) \to \infty$. If $M_k = 2L(x_{k+1}, x_k)^{-1}I$ for all $k$, then a straightforward substitution will show that the limit is satisfied yet $\dot{F}(x_k)$ does not have to be zero.

Hence, using Zoutendijk’s analysis method on this line of logic would not produce the desired conclusion. However, as shown in Theorem 3.5, using Zoutendijk’s analysis method on the conclusion of Lemma 3.1 is fruitful.
\section{Convergence of the Gradient.}

One limitation of Theorem 3.5 is that it only provides for the limit infimum of the gradient function to be zero. Here, we will use Property 2.7 to conclude that the limit of the gradient function is zero.

\textbf{Theorem 3.7.} Suppose $F : \mathbb{R}^p \to \mathbb{R}$ satisfies Assumptions 2.1 and 2.2. Let $x_0 \in \mathbb{R}^p$ and let $\{x_k : k \in \mathbb{N}\}$ be generated by (2.3) satisfying Properties 2.4 to 2.7. Then, for all $R \geq 0$,

\begin{equation}
\lim_{k \to \infty} F(x_k)\chi^0_k(R) \text{ exists and is finite, and } \lim_{k \to \infty} \|\dot{F}(x_k)\|_2 \chi^0_k(R) = 0.
\end{equation}

If $\sup_k \|x_k\|_2 < \infty$ then $\lim_{k \to \infty} F(x_k)$ exists and is finite, and $\lim_{k \to \infty} \|\dot{F}(x_k)\|_2 = 0$.

\textbf{Proof.} By Theorem 3.5, we need only prove, for any $R \geq 0$, $\lim \sup_{k \to \infty} \|\dot{F}(x_k)\|_2 \chi^0_k(R) = 0$. Fix $R \geq 0$. There are two cases.

\textbf{Case 1.} For some $K + 1 \in \mathbb{N}$, $\chi^0_k(R) = 0$. Then, $\chi^0_k(R) = 0$ for all $k \geq K$. The result follows.

\textbf{Case 2.} For all $k + 1 \in \mathbb{N}$, $\chi^0_k(R) = 1$. In this case, $\|x_k\|_2 \leq R$ for all $k + 1 \in \mathbb{N}$. Let $L_R$ be the Lipschitz constant in the closed ball of radius $R$ around 0 (see Lemma A.1), and let $G_R$ be the supremum of $\|\dot{F}(x)\|_2$ over all $x$ in the closed ball of radius $R$ around 0.

We now proceed in two steps. First, we show that for any $\epsilon > 0$, there exists a $K' \in \mathbb{N}$ such that $\forall k \geq K'$,

\begin{equation}
\|\dot{F}(x_{k+1})\|_2 - \|\dot{F}(x_k)\|_2 < \frac{\epsilon}{4}.
\end{equation}

Then, we use a proof-by-contradiction to show that the $\lim \sup_{k \to \infty} \|\dot{F}(x_k)\|_2 \neq \epsilon$.

For the first part, let $\epsilon > 0$. Now,

\begin{align}
\|\dot{F}(x_{k+1})\|_2 - \|\dot{F}(x_k)\|_2 &\leq \|\dot{F}(x_{k+1}) - \dot{F}(x_k)\|_2 \\
&\leq L_R \|x_{k+1} - x_k\|_2 \\
&\leq L_R \|M_k \dot{F}(x_k)\|_2 \\
&\leq L_R G_R \lambda_{\text{max}}(M_k).
\end{align}

By Property 2.6, $\exists K' \in \mathbb{N}$ such that, $\forall k \geq K'$, $L_R G_R \lambda_{\text{max}}(M_k) < \epsilon/4$.

Suppose now $\lim \sup_{k \to \infty} \|\dot{F}(x_k)\|_2 > \epsilon$. Let $u_0 = \min\{k > \max\{K, K'\} : \|\dot{F}(x_k)\|_2 > \epsilon\}$, where $K$ is given by Corollary 3.4. By Theorem 3.5, we can now define the following three subsequences of $\mathbb{N}$ for all $i \in \mathbb{N}$:

1. $j_i = \min\{t > u_{i-1} : \|\dot{F}(x_t)\|_2 < \epsilon/2\}$.
2. $u_i = \min\{t > j_i : \|\dot{F}(x_t)\|_2 > \epsilon\}$.
3. $\ell_i = \min\{t \in [j_i, u_i) : \|\dot{F}(x_s)\|_2 > \epsilon/2, s = t + 1, \ldots, u_i\}$.

Important, since $\ell_i > K'$ for all $i \in \mathbb{N}$ and $\|\dot{F}(x_{\ell_i+1})\|_2 > \epsilon/2$, $\|\dot{F}(x_{\ell_i})\|_2 > \epsilon/4$. Using these
subsequences and the same steps from the first part,

\begin{align}
\frac{\epsilon}{2} &< \left\| \dot{F}(x_{u_i}) \right\|_2 - \left\| \dot{F}(x_{\ell_i}) \right\|_2 \\
&< \sum_{t=\ell_i}^{u_i-1} \left\| \dot{F}(x_{t+1}) \right\|_2 - \left\| \dot{F}(x_t) \right\|_2 \\
&< \sum_{t=\ell_i}^{u_i-1} L_R \lambda_{\text{max}}(M_k) \left\| \dot{F}(x_t) \right\|_2 \\
&< \frac{\epsilon}{4} \sum_{t=\ell_i}^{u_i-1} L_R \lambda_{\text{max}}(M_k) \left( \frac{4 \left\| \dot{F}(x_k) \right\|_2}{\epsilon} \right)^2 \\
&< \frac{\epsilon}{4} \sum_{t=\ell_i}^{u_i-1} L_R \lambda_{\text{max}}(M_k) \left( \frac{4 \left\| \dot{F}(x_k) \right\|_2}{\epsilon} \right)^2,
\end{align}

where we made use of \( \epsilon/4 < \left\| \dot{F}(x_s) \right\|_2 \) for \( s = \ell_i, \ldots, u_i - 1 \) in the ultimate line. Simplifying and applying Property 2.7, \( \forall i \in \mathbb{N} \),

\begin{equation}
\frac{\epsilon^2}{8 L_R \kappa} < \sum_{t=\ell_i}^{u_i-1} \lambda_{\text{min}}(M_k) \left\| \dot{F}(x_t) \right\|_2^2.
\end{equation}

Summing both sides over \( i \in \mathbb{N} \), the left hand side diverges while the right hand side is bounded by (3.29). Hence, we have a contradiction and the conclusion follows.

From this proof, we might question whether it is necessary to use Property 2.7 in order to replace \( \lambda_{\text{max}}(M_k) \) with \( \lambda_{\text{min}}(M_k) \) in (3.44). We provide a concrete example where our reasoning faces difficulty if Property 2.7 is not used. As the example below shows, it is possible to relax Property 2.7 if the sequence \( \{M_k\} \) eventually has common invariant subspaces, but we do not pursue this here.

**Example 3.8.** Let \( F : \mathbb{R}^2 \to \mathbb{R} \) be

\begin{equation}
F(x) = \frac{1}{2}(x^{(1)})^2 + \frac{1}{10}(x^{(2)})^2,
\end{equation}

where \( x^{(i)} \) is the \( i \)th component of \( x \). Consider now \( x_0 \) such that \( x^{(1)}_0 = 0 \) and \( x^{(2)}_0 = 1 \). In order to violate Property 2.7, let

\begin{equation}
M_k = \frac{1}{5} \begin{bmatrix}
(k+1)^{-1/2} & 0 \\
0 & (k+1)^{-1}
\end{bmatrix}, \quad k + 1 \in \mathbb{N}.
\end{equation}

When we apply gradient descent, \( x^{(1)}_k = 0 \) for all \( k \in \mathbb{N} \), and \( x^{(2)}_k > 0.8(k+1)^{-1/5} \) [37, p.1578]. Then, \( \left\| \dot{F}(x_k) \right\|_2 > 0.16(k+1)^{-1/5} \). Now, we have, \( \lambda_{\text{max}}(M_k) \left\| \dot{F}(x_k) \right\|_2 > 0.01(k+1)^{-9/10}, \)
which produces a divergent series, whereas \( \lambda_{\min}(M_k) \) in place of \( \lambda_{\max}(M_k) \) would produce a convergent series.\(^6\)

3.4. Topological Properties of the Iterates. We now turn our attention to the asymptotic behavior of the iterates in the bounded regime. We will make use of the closure of subsequential limits (see Lemma C.1) and a fact about the density of subsequential limits of a decaying sequence (see Lemma C.2). We state the main result in Theorem 3.9.

**Theorem 3.9.** Suppose \( F : \mathbb{R}^p \to \mathbb{R} \) satisfies Assumptions 2.1 and 2.2. Let \( x_0 \in \mathbb{R}^p \) and let \( \{x_k : k \in \mathbb{N}\} \) be generated by (2.3) satisfying Properties 2.4 to 2.7. If \( \sup_k \|x_k\|_2 < \infty \) and we let \( \mathcal{C} \) denote the subsequential limits of \( \{x_k : k + 1 \in \mathbb{N}\} \), then

1. \( \mathcal{C} \) is closed;
2. \( \forall z \in \mathcal{C}, \hat{F}(z) = 0; \)
3. \( \mathcal{C} \) is connected;
4. \( \mathcal{C} \) does not contain an open set; and
5. Either \( |\mathcal{C}| = 1 \) or \( |\mathcal{C}| = \infty \).

**Proof.** The first statement follows from Lemma C.1. For the second statement, if \( z \in \mathcal{C} \) then there is a subsequence \( \{x_{k_j} : j \in \mathbb{N}\} \) such that \( \lim_j x_{k_j} = z \). By the continuity of \( x \mapsto \hat{F}(x) \) (see Assumption 2.2), \( \hat{F}(z) = \lim_j \hat{F}(x_{k_j}) \). The limit on the right hand side is zero by Theorem 3.7.

For the third statement, recall that \( \mathcal{C} \) is bounded by hypothesis and \( \mathcal{C} \) is closed by the first statement. Hence, \( \mathcal{C} \) is compact. Suppose that \( \mathcal{C} \) is not connected. Then, there are two disjoint open sets, \( O_1 \) and \( O_2 \), whose union contains \( \mathcal{C} \) and whose individual intersections with \( \mathcal{C} \) are non-empty. We denote the intersections of \( O_1 \) and \( O_2 \) with \( \mathcal{C} \) by \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \), respectively. We now proceed in three steps. First, we verify that \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are closed, and, consequently, compact. Second, we use compactness to show that the distance between \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) is strictly larger than zero. Third, we use the diminishing step sizes and Lemma C.2 to derive a contradiction.

Suppose \( \mathcal{C}_1 \) is not closed. Let \( z \) be a limit point of \( \mathcal{C}_1 \) that is not in \( \mathcal{C}_1 \). Then \( z \in \mathcal{C} \), which implies that \( z \in \mathcal{C}_2 \subset O_2 \). There is a sequence of points in \( \mathcal{C}_1 \) contained in an arbitrarily small neighborhood of \( z \), which implies \( \mathcal{C}_1 \cap O_2 \neq \emptyset \), which is a contradiction. Hence, \( \mathcal{C}_1 \) is closed. The same argument shows \( \mathcal{C}_2 \) is closed.

Since \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are closed and bounded, they are compact. Now, \( (z_1, z_2) \mapsto \|z_1 - z_2\|_2 \) is a continuous function. Hence, this function applied to \( \mathcal{C}_1 \times \mathcal{C}_2 \) must achieve its minimum at some points \( z_1^* \in \mathcal{C}_1 \) and \( z_2^* \in \mathcal{C}_2 \). If \( z_1^* = z_2^* \), then \( O_1 \cap O_2 \neq \emptyset \) which is a contradiction. Hence, \( z_1^* \neq z_2^* \) so the distance between any points in \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) is at least \( \|z_1^* - z_2^*\|_2 > 0 \).

Define a function \( g : \mathbb{R}^p \to \mathbb{R}_{\geq 0} \) such that \( g(x) = \inf_{w \in \mathcal{C}_1} \|x - w\|_2 \). Then, \( g(z) = 0 \) for any \( z \in \mathcal{C}_1 \) and \( g(z) \geq \|z_1^* - z_2^*\|_2 \) for \( z \in \mathcal{C}_2 \). Hence, \( \lim \inf_k g(x_k) = 0 \) and \( \lim \sup_k g(x_k) \geq \|z_1^* - z_2^*\|_2 \). We now verify, \( \lim_k g(x_{k+1}) - g(x_k) = 0 \), and apply Lemma C.2 to derive a contradiction.

---

\(^6\)To show convergence, we need an upper bound on the rate of convergence of \( x_k^{(2)} \), which is on the order of \( (k + 1)^{-1/5} \) [see 37, p.1578].
the globally Lipschitz continuous function, $F$quantity and the gradient function converges to zero \cite[Proposition 1.2.4]{5}. For example, of the gradient, when the iterates diverge, the objective function still converges to a finite value of global Lipschitz continuity of the gradient. Importantly, under global Lipschitz continuity $\nabla F$ may diverge. Of course, this divergence regime is possible even under the stricter assumption of local Lipschitz continuity. Under our more realistic assumption of local Lipschitz continuity, will the objective function still converge to a finite value when the iterates diverge? Unfortunately, the answer is no. In this section, we will construct several examples that show the extreme behaviors which can occur in the divergence regime when only local Lipschitz continuity is assumed. Of note, we construct an example in which catastrophic divergence can occur: the iterates diverge, the objective function diverges to infinity, and the gradient norm remains uniformly bounded away from zero. We show this construction here. Our remaining constructions are found as specified in the following Table 1. We underscore that our constructions can be used to generate objective functions on which gradient descent can have other interesting behaviors that we do not explicitly construct here (e.g., the limit infimum and limit supremum of the gradient function being distinct).

\[ g(x_{k+1}) - g(x_k) = \inf_{w \in C_1} \| x_{k+1} - w \|^2 - \| x_k - w \|^2 \]

\[ \leq \| x_{k+1} - w_k \|^2 - \| x_k - w_k \|^2 \]

\[ \leq \| x_{k+1} - x_k \|^2 \]

\[ \leq \| M_k \hat{F}(x_k) \|^2. \]

Note, $\| M_k \hat{F}(x_k) \|^2 \leq \lambda_{\max}(M_k) G_R$, where $G_R = \sup_{x:||x||_2 \leq R} \| \hat{F}(x) \|^2$ and $R = \sup_k \| x_k \|^2 < \infty$. Since $\lambda_{\max}(M_k) \to 0$, then $g(x_{k+1}) - g(x_k) \to 0$. Hence, by Lemma C.2, there is a subsequence, $\{g(x_{k_j}) : j \in \mathbb{N}\}$ that converges to, say, $\| z^*_1 - z^*_2 \|^2/2$. Consequently, $\{x_{k_j} : j \in \mathbb{N}\}$ is a bounded sequence, and it has a subsequence that converges to a point $z^*$ such that $\inf_{w \in C_1} \| z^* - w \|^2 = \| z^*_1 - z^*_2 \|^2/2$. Hence, $z^*$ is a subsequential limit but it is not in either $C_1$ nor $C_2$, which is a contradiction. Thus, $C$ is connected.

For the fourth statement, suppose $C$ contains an open set $O$. Let $z \in O$. Since $z$ is a limit point of a subsequence, there exists an $k_0 \in \mathbb{N}$ such that $x_{k_0} \in O$. By the first statement, $\hat{F}(x_{k_0}) = 0$, which, by (2.3), implies $x_j = x_{k_0}$ for all $j \geq k_0$. Hence, $C$ is the singleton $\{x_{k_0}\}$, which is a contradiction. Thus, $C$ cannot contain an open set.

For the final statement, recall that $C$ is connected. This implies that $C$ cannot contain a finite number of points other than a single point. So, either $|C| = 1$ or $|C| = \infty$.

4. The Divergence Regime. Theorem 3.5 leaves open the possibility that the iterates can diverge. Of course, this divergence regime is possible even under the stricter assumption of global Lipschitz continuity of the gradient. Importantly, under global Lipschitz continuity of the gradient, when the iterates diverge, the objective function still converges to a finite quantity and the gradient function converges to zero \cite[Proposition 1.2.4]{5}. For example, the globally Lipschitz continuous function, $F(x) = \exp(-x^2)$, achieves its minimum as the iterates converge; and, in this divergence regime, the objective function converges to zero and the gradient function converges to zero.

Under our more realistic assumption of local Lipschitz continuity, will the objective function converge to a finite quantity and will the gradient function converge to zero when the iterates diverge? Unfortunately, the answer is no. In this section, we will construct several examples that show the extreme behaviors which can occur in the divergence regime when only local Lipschitz continuity is assumed. Of note, we construct an example in which catastrophic divergence can occur: the iterates diverge, the objective function diverges to infinity, and the gradient norm remains uniformly bounded away from zero. We show this construction here. Our remaining constructions are found as specified in the following Table 1. We underscore that our constructions can be used to generate objective functions on which gradient descent can have other interesting behaviors that we do not explicitly construct here (e.g., the limit infimum and limit supremum of the gradient function being distinct).
Table 1  
Summary of Counterexamples for the Divergence Regime.

| Reference  | Summary                                                                 |
|------------|-------------------------------------------------------------------------|
| This section | A case for which the iterates of gradient descent will produce objective function values that diverge and gradient function values that are uniformly bounded away from zero. |
| Appendix D  | A case for which the iterates of gradient descent will produce objective function values whose limit supremum is infinity and whose limit infimum is zero, while the gradient function values remain bounded away from zero. |
| Appendix E  | A case for which the iterates of gradient descent will produce objective function values that diverge and the gradient function tends to zero. |

4.1. Construction of the Objective Function. Let \( \{m_k : k + 1 \in \mathbb{N}\} \) be a sequence of scalars such that \( m_k > 0, \sum_{k} m_k = \infty, \) and \( m_k \to 0 \) as \( k \to \infty \). Define \( S_0 = 0 \) and \( S_{k+1} = \sum_{j=0}^{k} m_k \) for all integers \( k \geq 0 \).

For the objective function, define \( F : \mathbb{R} \to \mathbb{R} \) by

\[
F(x) = \begin{cases} 
-x & x \leq 0 \\
 f_j(x) & x \in (S_j, S_{j+1}], \forall j + 1 \in \mathbb{N},
\end{cases}
\]

where \( \{f_j : (S_j, S_{j+1}] \to \mathbb{R} : j + 1 \in \mathbb{N}\} \) are defined iteratively as follows. Let

\[
f_0(x) = \begin{cases} 
-x & x \in (0, \frac{m_0}{16}) \\
\frac{8}{m_0} (x - \frac{m_0}{8})^2 - \frac{3m_0}{16} & x \in \left(\frac{m_0}{16}, \frac{3m_0}{16}\right) \\
-\frac{5m_0}{16} \exp \left(\frac{5m_0/16}{x - m_0/2} + 1\right) + \frac{m_0}{4} & x \in \left[\frac{3m_0}{16}, \frac{m_0}{2}\right)
\end{cases}
\]

which is plotted for a particular choice of \( m_0 \) in Figure 1. Now, for \( j \in \mathbb{N} \), let \( x' = x - S_j \) and
let

\[
\begin{align*}
\text{(4.3)} 
\quad f_j(x) &= \begin{cases}
-\frac{m_j}{4} + f_{j-1}(S_j) & x' \in (0, \frac{m_j}{16}) \\
\frac{8}{m_j}(x' - \frac{m_j}{8})^2 - \frac{3m_j}{8} + f_{j-1}(S_j) & x' \in \left[\frac{m_j}{16}, \frac{3m_j}{8}\right] \\
-\frac{5m_j}{16} \exp\left(\frac{5m_j/16}{x' - m_j/2} + 1\right) + \frac{m_j}{4} + f_{j-1}(S_j) & x' \in \left[\frac{3m_j}{8}, \frac{m_j}{2}\right] \\
-\frac{8}{m_j}x' - \frac{7m_j}{8})^2 + \frac{19m_j}{32} + f_{j-1}(S_j) & x' = \frac{m_j}{2} \\
-\frac{5m_j}{16} \exp\left(\frac{-5m_j/16}{x' - m_j/2} + 1\right) + \frac{m_j}{4} + f_{j-1}(S_j) & x' \in \left[\frac{m_j}{2}, \frac{13m_j}{16}\right] \\
-\frac{8}{m_j}x' - \frac{7m_j}{8})^2 + \frac{19m_j}{32} + f_{j-1}(S_j) & x' \in \left[\frac{13m_j}{16}, \frac{15m_j}{16}\right] \\
-x' + \frac{3m_j}{2} + f_{j-1}(S_j) & x' \in \left[\frac{15m_j}{16}, m_j\right].
\end{cases}
\end{align*}
\]

Figure 1. Plot of \(f_0(x)\) with \(m_0 = 8.0\) with each component shown in a different color.

4.2. Properties of the Objective Function. We show that \(F: \mathbb{R} \to \mathbb{R}\) as defined in (4.1)–(4.3) satisfies Assumptions 2.1 and 2.2. We begin by proving that each component, \(f_j: (S_j, S_{j+1}] \to \mathbb{R}\), satisfies Assumptions 2.1 and 2.2 on its domain.

Remark 4.1. Below, we define the continuous extension of \(f_j\) on \([S_j, S_{j+1}]\) by the value of \(f_j(x)\) on \((S_j, S_{j+1}]\) and by \(\lim_{x\downarrow S_j} f_j(x)\) for the point \(x = S_j\). Moreover, at the ends of the interval, we use differentiability and the corresponding notation, \(\dot{f}_j\), to mean the one-sided derivatives.

Proposition 4.2. The continuous extension \(f_0: (S_0, S_1] \to \mathbb{R}\) (as defined in (4.2)) to \([S_0, S_1]\) is continuous on its domain, bounded from below by \(-3m_0/32\), differentiable on its domain with \(\dot{f}_0(S_0) = \dot{f}_0(S_1) = -1\), and its derivative is locally Lipschitz continuous. Similarly, the continuous extension of \(f_j: (S_j, S_{j+1}] \to \mathbb{R}\) (as defined in (4.3)) to \([S_j, S_{j+1}]\) is continuous on its domain, bounded from below by \(f_{j-1}(S_j) - 3m_j/32\), differentiable on its domain with \(\dot{f}_j(S_j) = -1\) and \(\dot{f}_j(S_{j+1}) = -1\), and its derivative is locally Lipschitz continuous.
**Proof.** We only look at an arbitrary \( j \in \mathbb{N} \) as the proof is identical for \( f_0 \). Moreover, since \( f_j \) is equal to its reflection across the vertical axis \( x = S_j + m_j/2 \) followed by a reflection over the horizontal axis \( y = m_j/4 + f_{j-1}(S_j) \), it is enough to show continuity and differentiability on \([S_j, S_j + m_j/2]\).

To establish continuity, we need to show that the left sided limits of \( f_j \) agree with the function value at \( S_j + \delta m_j/16 \) for \( \delta = 1,3,8 \). Starting with \( \delta = 1 \), \( \lim_{x \to S_j+m_j/16} x + S_j + f_{j-1}(S_j) = -m_j/16 + f_{j-1}(S_j) \). By direct substitution,

\[
(4.4) \quad f_j(S_j + m_j/16) = \frac{8}{m_j} \left( \frac{m_j}{16} \right)^2 - \frac{3m_j}{32} + f_{j-1}(S_j) = \frac{m_j}{32} - \frac{3m_j}{32} + f_{j-1}(S_j).
\]

Hence, the left limit agrees with the function value at \( \delta = 1 \). For \( \delta = 3 \), the symmetry and continuity of the quadratic function implies \( \lim_{x \to S_j+3m_j/16} f_j(x) = -m_j/16 + f_{j-1}(S_j) \). By direct substitution,

\[
(4.5) \quad f_j(S_j + 3m_j/16) = -\frac{5m_j}{16} \exp \left( \frac{5m_j/16}{x - S_j - m_j/2} + 1 \right) + \frac{4m_j}{16} + f_{j-1}(S_j) = -\frac{m_j}{4} + f_{j-1}(S_j).
\]

Hence, the left limit agrees with the function value at \( \delta = 3 \). For \( \delta = 8 \),

\[
(4.6) \quad \lim_{x \to S_j+m_j/2} -\frac{5m_j}{16} \exp \left( \frac{5m_j/16}{x - S_j - m_j/2} + 1 \right) + \frac{m_j}{4} + f_{j-1}(S_j) = \frac{m_j}{4} + f_{j-1}(S_j),
\]

which is just \( f_j(S_j + m_j/2) \). Hence, the continuous extension of \( f_j \) is continuous on \([S_j, S_{j+1}]\).

We now compute the derivatives of the components of \( f_j \) on \([S_j, S_{j+1}]\) with the convention of assigning the one-sided derivative to the component function that includes its end point. Let \( x' = x - S_j \).

\[
(4.7) \quad \dot{f}_j(x) = \begin{cases} 
\frac{16}{m_j} (x' - \frac{m_j}{8}) & x' \in \left[ 0, \frac{m_j}{16} \right) \\
\left( \frac{5m_j/16}{x' - m_j/2} \right)^2 \exp \left( \frac{5m_j/16}{x' - m_j/2} + 1 \right) & x' \in \left( \frac{m_j}{16}, \frac{3m_j}{16} \right) \\
\left( \frac{5m_j/16}{x' - m_j/2} \right)^2 \exp \left( -\frac{5m_j/16}{x' - m_j/2} + 1 \right) & x' \in \left( \frac{3m_j}{16}, \frac{m_j}{2} \right) \\
-\frac{16}{m_j} (x' - \frac{7m_j}{8}) & x' \in \left( \frac{m_j}{2}, \frac{13m_j}{16} \right) \\
-1 & x' \in \left( \frac{13m_j}{16}, \frac{15m_j}{16} \right) \\
-1 & x' \in \left( \frac{15m_j}{16}, m_j \right].
\end{cases}
\]

In order to extend these component derivatives to the continuous extension of \( f_j \), we need to verify that the left hand limits of the derivatives agree with the right side derivatives at \( S_j + \delta m_j/16 \) for \( \delta = 1,3 \) of (4.7), and we need to verify that the left hand limit of the derivative is 0 at \( \delta = 8 \) of (4.7). Starting with \( \delta = 1 \), the left hand limit is \(-1\) and, by direct calculation,

\[
(4.8) \quad \lim_{x \to S_j+m_j/16} \frac{16}{m_j} \left( S_j + \frac{m_j}{16} - S_j - \frac{m_j}{8} \right) = -1.
\]

For \( \delta = 3 \), the left hand limit is

\[
(4.9) \quad \lim_{x \to S_j+3m_j/16} \frac{16}{m_j} \left( x - S_j - \frac{m_j}{8} \right) = 1,
\]
and a direct evaluation of the third component \((4.7)\) is
\[
(4.10) \quad \left(\frac{5m_j/16}{S_j + 3m_j/16 - S_j - m_j/2}\right)^2 \exp\left(\frac{5m_j/16}{S_j + 3m_j/16 - S_j - m_j/2} + 1\right) = 1.
\]
For \(\delta = 8\), we need to check that the left hand limit is zero, which can be confirmed by checking that the argument of the exponential term goes to \(-\infty\) as \(x \uparrow S_j + 8m_j/16\).

Overall, the derivative of the continuous extension of \(f_j\) is well defined at every point on its interval, is continuous, and is given by \((4.11)\) with \(x' = x - S_j\)
\[
(4.11) \quad \hat{f}_j(x) = \begin{cases} 
-1 & x' \in [0, \frac{m_j}{16}) \\
\frac{16}{m_j} (x' - \frac{m_j}{8}) & x' \in \left(\frac{m_j}{16}, \frac{3m_j}{16}\right) \\
\left(\frac{5m_j/16}{x' - m_j/2}\right)^2 \exp\left(\frac{5m_j/16}{x' - m_j/2} + 1\right) & x' \in \left(\frac{3m_j}{16}, \frac{m_j}{2}\right) \\
\frac{16}{m_j} (x' - \frac{7m_j}{8}) & x' = m_j/2 \\
0 & x' \in \left(\frac{m_j}{2}, \frac{13m_j}{16}\right) \\
\left(-\frac{5m_j/16}{x' - m_j/2}\right)^2 \exp\left(-\frac{5m_j/16}{x' - m_j/2} + 1\right) & x' \in \left(\frac{13m_j}{16}, \frac{15m_j}{16}\right) \\
-1 & x' \in \left(\frac{15m_j}{16}, m_j\right].
\end{cases}
\]

Using this derivative, we can calculate the lower bound for the function. By the derivative of the extension of \(f_j\), \((4.11)\), we see that the function is decreasing only on \([S_j, S_j + m_j/8]\) and \([S_j + 7m_j/8, S_{j+1}]\). Moreover, \(f_j(S_j + m_j/8) = -3m_j/32 + f_{j-1}(S_j)\) and \(f_j(S_{j+1}) = m_j/2 + f_{j-1}(S_j)\). Thus, the lower bound of the extension of \(f_j\) is as stated.

Our last step is to verify the local Lipschitz continuity of the derivative. It is easy to verify that within its interval, the components are twice continuously differentiable. As a result, we can use Lemma B.1. Similarly, if we define the second derivative at \(S_j + m_j/2\) to be 0, we can verify that the objective is twice continuously differentiable at \(S_j + m_j/2\). Then, we can use Lemma B.1 again. To conclude, we need to examine what happens around the points \(S_j + \delta m_j/16\) for \(\delta = 1, 3\).

Starting with \(\delta = 1\), consider the points \(S_j + m_j/16 - \epsilon_1 m_j/16\) and \(S_j + m_j/16 + \epsilon_2 m_j/16\) for \(\epsilon_1, \epsilon_2 > 0\) sufficiently small. Then the difference in the derivatives at these points divided by the distance between the points is
\[
(4.12) \quad \frac{16}{m_j} \left(\frac{m_j}{16} + \epsilon_2 m_j/16\right) + 1 = \frac{\epsilon_2}{\epsilon_2 + \epsilon_1 m_j/16} \leq \frac{16}{m_j}.
\]
Therefore, we conclude that the derivative is locally Lipschitz near \(S_j + m_j/16\). For \(\delta = 3\), we compute the same ratio at the points \(S_j + 3m_j/16 - \epsilon_1 m_j/16\) and \(S_j + 3m_j/16 + 5\epsilon_2 m_j/16\) for \(\epsilon_1, \epsilon_2 \in [0, 1/4]\) where at most either \(\epsilon_1\) or \(\epsilon_2\) is zero. The ratio of the difference in the derivatives and the points is
\[
(4.13) \quad \frac{\left|\frac{\epsilon_2}{\epsilon_2 - 1} \exp\left(\frac{\epsilon_2}{\epsilon_2 - 1}\right) - 1 + \epsilon_1\right|}{\left(5\epsilon_2 + \epsilon_1 m_j/16\right)} \leq \frac{\epsilon_2 + \epsilon_2^2/2 + \epsilon_1}{5\epsilon_2 + \epsilon_1 m_j/16} \leq \frac{16}{m_j}.
\]
Therefore, we conclude that the derivative is locally Lipschitz near \(S_j + 3m_j/16\).
With this calculation complete, we can now verify that \( F \) satisfies Assumptions 2.1 and 2.2.

**Proposition 4.3.** The function \( F : \mathbb{R} \to \mathbb{R} \) is continuous and differentiable on its domain; the function \( F \) is lower bounded; the derivative of the function \( F \) is locally Lipschitz continuous; \( F(S_j) = S_j/2 \); and \( \dot{F}(S_j) = -1 \) for all \( j + 1 \in \mathbb{N} \). And, the derivative of the function \( F \) is not globally Lipschitz continuous.

**Proof.** Using Proposition 4.2, we can verify continuity of \( F \) by checking the continuity of \( F \) at \( x = S_j \) for all \( j \). Note, \( F(S) = f_{j-1}(S_j) \). We need to verify that the right side limit converges to \( f_{j-1}(S_j) \). That is,

\[
\lim_{x \uparrow S_j} F(x) = \lim_{x \downarrow S_j} f_j(x) = \lim_{x \downarrow S_j} -x + S_j + f_{j-1}(S_j) = f_{j-1}(S_j).
\]

Thus, \( F \) is continuous at all \( S_j \) for \( j \in \mathbb{N} \). We check \( x = S_0 = 0 \) as well. \( F(0) = 0 \) by definition. Moreover,

\[
\lim_{x \uparrow 0} F(x) = \lim_{x \downarrow 0} f_0(x) = \lim_{x \downarrow 0} -x = 0.
\]

Hence, \( F \) is continuous on its domain.

For differentiability, we similarly need only verify the differentiability of \( F \) at \( x = S_j \) for all \( j \). By Proposition 4.2, it follows that the derivative at each \( S_j \) is \(-1\) from the left and the right for \( j \in \mathbb{N} \). Hence, the derivative exists at each \( S_j \) for \( j \in \mathbb{N} \). Moreover, the left hand derivative at \( S_0 \) is \(-1\), which agrees with the right hand derivative. Hence, \( F \) is continuously differentiable on its domain. Since the derivative is constant in a small neighborhood of \( S_j \), it is Lipschitz continuous in this region. Finally, \( \dot{F}(S_j) = -1 \) for all \( j + 1 \in \mathbb{N} \).

To show that \( F \) is lower bounded, we will first calculate the values of \( F(S_j) \). We proceed by induction. \( F(S_0) = 0 = S_0/2 \). Now suppose the statement holds up to \( j \). Then \( F(S_j) = S_j/2 \). By construction, \( f_{j-1}(S_j) = F(S_j) = S_j/2 \). Now,

\[
F(S_{j+1}) = f_j(S_{j+1}) = -S_{j+1} + S_j + \frac{3m_j}{2} + f_{j-1}(S_j)
\]

\[
= -S_j - m_j + S_j + \frac{3m_j}{2} + \frac{S_j}{2} = \frac{S_{j+1}}{2}.
\]

To show the lower bound property, recall that \( f_j(x) \geq f_{j-1}(S_j) - 3m_j/32 = F(S_j) - 3m_j/32 = S_j/2 - 3m_j/32 \). Since \( S_j \to \infty \) and \( m_j \to 0 \) by construction, \( F(x) \geq \inf_j S_j/2 - 3m_j/32 > -\infty \).

Finally, we verify that \( F \) is not globally Lipschitz continuous. For a contradiction, suppose there exists an \( L \) such that for any \( x, x' \in \mathbb{R} \), \( |\dot{F}(x) - \dot{F}(x')| \leq L|x - x'| \). Since \( m_j \to 0 \), there exists \( j \in \mathbb{N} \) such that \( Lm_j/2 < 1 \). Then, \( |\dot{F}(S_j + m_j/2) - \dot{F}(S_j)| \leq Lm_j/2 < 1 \). However, by Proposition 4.2, \( \dot{F}(S_j + m_j/2) = \dot{f}_j(S_j + m_j/2) = 0 \) and \( F(S_j) = \dot{f}_j(S_j) = -1 \), which implies \( |\dot{F}(S_j + m_j/2) - \dot{F}(S_j)| = 1 \), which is a contradiction.

**4.3. Properties of Gradient Descent on the Objective Function.** We are now ready to show that gradient descent with diminishing step sizes generates iterates that diverge, and whose objective function diverges and gradient function remains bounded away from zero.

**Proposition 4.4.** Let \( \{m_k : k + 1 \in \mathbb{N}\} \) be any positive sequence such that \( \sum_k m_k \) diverges and \( m_k \to 0 \). Define \( F : \mathbb{R} \to \mathbb{R} \) as in (4.1). Suppose \( x_0 = 0 \) and let \( \{x_k : k \in \mathbb{N}\} \) be generated
according to (2.3) with $M_k = m_k I$ for all $k + 1 \in \mathbb{N}$. Then, $\{M_k\}$ satisfies Properties 2.4 to 2.6. Moreover, (a) $\lim_k x_k = \infty$, (b) $\lim_k F(x_k) = \infty$, and (c) $\lim_k |\dot{F}(x_k)| = 1$.

Proof. To prove the result, we need only show that $x_k = S_k$ where we recall that $S_0 = 0$ and $S_k = \sum_{j=0}^{k-1} m_j$. For $k = 0$, $x_0 = 0 = S_0$. Suppose this holds up to $k$. Then, by Proposition 4.3,

(4.18) \[ x_{k+1} = x_k - M_k \dot{F}(x_k) = S_k - m_k(-1) = S_{k+1}. \]

Now, since $S_k$ diverges, the iterates diverge (part (a)). Moreover, by Proposition 4.3, since $F(x_k) = F(S_k) = S_k/2$, the objective function also diverges (part (b)). Finally, by Proposition 4.3, $\dot{F}(x_k) = \dot{F}(S_k) = -1$ (part (c)).

In summary, as the example from Proposition 4.4 and the example of $F(x) = \exp(-x^2)$ show, under our assumptions about the objective function and properties of gradient descent, we cannot conclude anything additional about the objective behavior of the function or the gradient in the regime where the iterates generated by gradient descent with diminishing step sizes diverge.

5. Conclusion. In this paper, we have analyzed the global behavior of gradient descent with diminishing step sizes for differentiable nonconvex functions whose gradients are only locally Lipschitz continuous. To the best of our knowledge, we have provided the most general convergence analysis of gradient descent with diminishing step sizes. Specifically, we have shown that the iterates cannot produce erratic behavior in the objective function nor gradient function when they persist in a region for sufficiently long, even if they eventually escape. We also construct specific examples to show the types of erratic behaviors which can occur when the iterates escape off to infinity. Our analysis has also raised a number of interesting questions with varying degrees of practical interest.

1. Is there a notion of continuity on the gradients that is appropriate for data science yet more restrictive than Assumption 2.2 for which Theorem 3.5 or Theorem 3.7 hold uniformly over the family of functions specified by this notion of continuity?
2. Is there a choice of step sizes that ensures the subsequential limit points of the iterates is a set that is a singleton?
3. Is there a function class that is necessary and sufficient to avoid the divergence regime and the corresponding erratic behaviors for gradient descent with diminishing step size?

References.
[1] L. Armijo, Minimization of functions having lipschitz continuous first partial derivatives, Pacific Journal of mathematics, 16 (1966), pp. 1–3.
[2] A. Beck, First-order methods in optimization, SIAM, 2017.
[3] M. Benaïm, Dynamics of stochastic approximation algorithms, in Seminaire de probabilités XXXIII, Springer, 1999, pp. 1–68.
[4] A. Benveniste, M. Métivier, and P. Priouret, Adaptive algorithms and stochastic approximations, vol. 22, Springer Science & Business Media, 2012.
[5] D. Bertsekas, Nonlinear Programming, vol. 4, Athena Scientific, 2016.
[6] S. Bittanti, P. Bolzern, and M. Campi, Convergence and exponential convergence of identification algorithms with directional forgetting factor, Automatica, 26 (1990), pp. 929–932.

[7] L. Bottou, Large-scale machine learning with stochastic gradient descent, in Proceedings of COMPSTAT’2010, Springer, 2010, pp. 177–186.

[8] L. Cao and H. M. Schwartz, Exponential convergence of the kalman filter based parameter estimation algorithm, International Journal of Adaptive Control and Signal Processing, 17 (2003), pp. 763–783.

[9] C. Cartis, N. I. Gould, and P. L. Toint, Adaptive cubic regularisation methods for unconstrained optimization. part i: motivation, convergence and numerical results, Mathematical Programming, 127 (2011), pp. 245–295.

[10] C. Cartis, N. I. Gould, and P. L. Toint, Adaptive cubic regularisation methods for unconstrained optimization. part ii: worst-case function-and derivative-evaluation complexity, Mathematical programming, 130 (2011), pp. 295–319.

[11] A. Cauchy et al., Méthode générale pour la résolution des systemes d’équations simultanées, Comp. Rend. Sci. Paris, 25 (1847), pp. 536–538.

[12] H. B. Curry, The method of steepest descent for non-linear minimization problems, Quarterly of Applied Mathematics, 2 (1944), pp. 258–261.

[13] F. E. Curtis and K. Scheinberg, Adaptive stochastic optimization: A framework for analyzing stochastic optimization algorithms, IEEE Signal Processing Magazine, 37 (2020), pp. 32–42.

[14] F. E. Curtis, K. Scheinberg, and R. Shi, A stochastic trust region algorithm based on careful step normalization, Informs Journal on Optimization, 1 (2019), pp. 200–220.

[15] A. Défossez, L. Bottou, F. Bach, and N. Usunier, A simple convergence proof of adam and adagrad, arXiv preprint arXiv:2003.02395, (2020).

[16] H. W. Dommel and W. F. Tinney, Optimal power flow solutions, IEEE Transactions on power apparatus and systems, (1968), pp. 1866–1876.

[17] S. Du, J. Lee, H. Li, L. Wang, and X. Zhai, Gradient descent finds global minima of deep neural networks, in International conference on machine learning, PMLR, 2019, pp. 1675–1685.

[18] S. S. Du, C. Jin, J. D. Lee, M. I. Jordan, A. Singh, and B. Poczos, Gradient descent can take exponential time to escape saddle points, Advances in neural information processing systems, 30 (2017).

[19] S. S. Du, X. Zhai, B. Poczos, and A. Singh, Gradient descent provably optimizes over-parameterized neural networks, arXiv preprint arXiv:1810.02054, (2018).

[20] J.-C. Fort and G. Pages, Convergence of stochastic algorithms: From the kushner–clark theorem to the lyapounov functional method, Advances in applied probability, 28 (1996), pp. 1072–1094.

[21] G. N. Grapiglia and G. F. Stella, An adaptive trust-region method without function evaluations, Computational Optimization and Applications, 82 (2022), pp. 31–60.

[22] S. Gratton, S. Jerad, and P. L. Toint, Convergence properties of an objective-function-free optimization regularization algorithm, including an o(ε^{3/2}) complexity bound, arXiv preprint arXiv:2203.09947, (2022).

[23] S. Gratton, S. Jerad, and P. L. Toint, First-order objective-function-free optimiza-
tion algorithms and their complexity, arXiv preprint arXiv:2203.01757, (2022).
[24] S. Gratton, S. Jerad, and P. L. Toint, Parametric complexity analysis for a class of first-order adagrad-like algorithms, arXiv preprint arXiv:2203.01647, (2022).
[25] J. Hadamard, Mémoire sur le problème d’analyse relatif à l’équilibre des plaques élastiques encastrées, vol. 33, Imprimerie nationale, 1908.
[26] G. Iyengar and A. K. C. Ma, Fast gradient descent method for mean-cvar optimization, Annals of Operations Research, 205 (2013), pp. 203–212.
[27] C. Jin, P. Netrapalli, and M. I. Jordan, Accelerated gradient descent escapes saddle points faster than gradient descent, in Conference On Learning Theory, PMLR, 2018, pp. 1042–1085.
[28] R. M. Johnstone, C. R. Johnson Jr, R. R. Bitmead, and B. D. Anderson, Exponential convergence of recursive least squares with exponential forgetting factor, Systems & Control Letters, 2 (1982), pp. 77–82.
[29] H. Karimi, J. Nutini, and M. Schmidt, Linear convergence of gradient and proximal-gradient methods under the polyak-foisieiewicz condition, in Joint European Conference on Machine Learning and Knowledge Discovery in Databases, Springer, 2016, pp. 795–811.
[30] X. Ke and J. Han, A class of nonmonotone trust region algorithms for unconstrained optimization problems, Science in China Series A: Mathematics, 41 (1998), pp. 927–932.
[31] J. Lee, L. Xiao, S. Schoenholz, Y. Bahri, R. Novak, J. Sohl-Dickstein, and J. Pennington, Wide neural networks of any depth evolve as linear models under gradient descent, Advances in neural information processing systems, 32 (2019).
[32] J. D. Lee, M. Simchowitz, M. I. Jordan, and B. Recht, Gradient descent only converges to minimizers, in Conference on learning theory, PMLR, 2016, pp. 1246–1257.
[33] C. Lémaréchal, Cauchy and the gradient method, Doc Math Extra, 251 (2012), p. 10.
[34] L. Ljung, Analysis of recursive stochastic algorithms, IEEE transactions on automatic control, 22 (1977), pp. 551–575.
[35] P. Mertikopoulos, N. Hallak, A. Kavis, and V. Cevher, On the almost sure convergence of stochastic gradient descent in non-convex problems, arXiv preprint arXiv:2006.11144, (2020).
[36] J. J. Moré, Recent developments in algorithms and software for trust region methods, Springer, 1983.
[37] A. Nemirovski, A. Juditsky, G. Lan, and A. Shapiro, Robust stochastic approximation approach to stochastic programming, SIAM Journal on optimization, 19 (2009), pp. 1574–1609.
[38] J. Nocedal and S. Wright, Numerical optimization, Springer Science & Business Media, 2006.
[39] S. Oymak and M. Soltanolkotabi, Overparameterized nonlinear learning: Gradient descent takes the shortest path?, in International Conference on Machine Learning, PMLR, 2019, pp. 4951–4960.
[40] J. Parkum, N. K. Poulsen, and J. Holst, Recursive forgetting algorithms, International Journal of Control, 55 (1992), pp. 109–128.
[41] V. Patel, Stopping criteria for, and strong convergence of, stochastic gradient descent on bottou-curtis-nocedal functions, Mathematical Programming, (2021), pp. 1–42.
[42] V. Patel, B. Tian, and S. Zhang, Global convergence and stability of stochastic
gradient descent, arXiv preprint arXiv:2110.01663, (2021).

[43] V. Patel and S. Zhang, Stochastic gradient descent on nonconvex functions with general noise models, arXiv preprint arXiv:2104.00423, (2021).

[44] S. Reddi, S. Sra, B. Poczos, and A. J. Smola, Proximal stochastic methods for nonsmooth nonconvex finite-sum optimization, Advances in neural information processing systems, 29 (2016), pp. 1145–1153.

[45] S. J. Reddi, A. Hefny, S. Sra, B. Poczos, and A. Smola, Stochastic variance reduction for nonconvex optimization, in International conference on machine learning, PMLR, 2016, pp. 314–323.

[46] R. Ward, X. Wu, and L. Bottou, Adagrad stepsizes: Sharp convergence over nonconvex landscapes, The Journal of Machine Learning Research, 21 (2020), pp. 9047–9076.

[47] X. Wu, R. Ward, and L. Bottou, Wngrad: Learn the learning rate in gradient descent, arXiv preprint arXiv:1803.02865, (2018).

[48] J. Zhang, Y. Wang, and X. Zhang, Superlinearly convergent trust-region method without the assumption of positive-definite hessian, Journal of optimization theory and applications, 129 (2006), pp. 201–218.

[49] J. Zhang, L. Wu, and X. Zhang, A trust region method for optimization problem with singular solutions, Applied Mathematics and Optimization, 56 (2007), pp. 379–394.

[50] G. Zoutendijk, Methods of feasible directions: a study in linear and non-linear programming, Elsevier, 1960.

Appendix A. Equivalent Definitions for Local Lipschitz Continuity.

Lemma A.1. A function $G : \mathbb{R}^p \to \mathbb{R}^p$ is locally Lipschitz continuous if and only if for every compact set $C \subset \mathbb{R}^p$, there exists an $L \geq 0$ such that

$$\frac{\|G(y) - G(z)\|_2}{\|y - z\|_2} \leq L, \forall y, z \in C. \tag{A.1}$$

Proof. Suppose $G$ is locally Lipschitz continuous. Suppose for a contradiction, there exists a compact set for which no such $L$ exists. Then for every $\ell \in \mathbb{N}$, we can find a pair $y_\ell, z_\ell \in C$ such that

$$\frac{\|G(y_\ell) - G(z_\ell)\|_2}{\|y_\ell - z_\ell\|_2} > \ell. \tag{A.2}$$

By compactness, there exists a subsequence $\{\ell_k : k \in \mathbb{N}\}$ and $y, z \in C$ such that $y_{\ell_k} \to y$ and $z_{\ell_k} \to z$ as $k \to \infty$. If $\|y - z\|_2 > 0$, then, for $k \in \mathbb{N}$ sufficiently large,

$$\frac{\|G(y_{\ell_k}) - G(z_{\ell_k})\|_2}{\|y_{\ell_k} - z_{\ell_k}\|_2} \leq \frac{2\sup_{x \in C} \|G(x)\|_2}{0.5 \|y - z\|_2} < \infty, \tag{A.3}$$

which is a contradiction. Hence, $\|y - z\|_2 = 0$; that is, $y = z$. This also provides a contradiction as $G$ is locally Lipschitz continuous at $y = z$ and so for $k \in \mathbb{N}$ sufficiently large, $y_{\ell_k}$ and $z_{\ell_k}$ would be inside of $\mathcal{N}$ from Definition 2.3.

For the other direction of the result: for any point $x \in \mathbb{R}^p$ and any open ball containing $x$, we can take the closure of this open ball to generate a compact set $C$. The result follows.
Appendix B. Continuous Hessians Implies Local Lipschitz Continuity.

Lemma B.1. Suppose $F$ is twice continuously differentiable for all $x \in \mathbb{R}^p$. Then $F(x)$ is locally Lipschitz continuous.

Proof. Let $\tilde{F}(x)$ denote the Hessian of $F$. Then, by assumption, $\|\tilde{F}(x)\|_2$ is a continuous function and it is bounded over any compact region. By Taylor’s theorem, for any $x, y \in \mathbb{R}^p$, $\tilde{F}(x) - \tilde{F}(y) = \int_0^1 \frac{d}{dt}\tilde{F}(x + t(x-y))(x-y)dt$. Let $K \subset \mathbb{R}^p$ be compact. By continuity and compactness, there exists an $L$ for $K$ such that $\|\tilde{F}(x)\|_2 \leq L$ for all $x \in K$. Hence, by Hölder’s inequality, for any $x, y \in K$, $\|F(x) - F(y)\| \leq L\|x-y\|_2$. As $K$ is arbitrary, the result follows. ■

Appendix C. Some Properties of Subsequential Limits.

Lemma C.1. Let $\{a_n : n \in \mathbb{N}\} \subset \mathbb{R}$. Let $C$ be the set of its subsequential limits. Then $C$ is closed.

Proof. Let $z$ be a limit point of $C$. Then, we can construct a sequence $\{z_k : k \in \mathbb{N}\} \subset C$ such that for every $K \in \mathbb{N}$ and for all $k \geq K$, $\|z_k - z\|_2 \leq 2^{-K-1}$. Moreover, since $z_k \in C$, $\exists n_k \in \mathbb{N}$ such that $\|a_{n_k} - z\|_2 \leq 2^{-K-1}$. Let $\epsilon > 0$ and let $K \in \mathbb{N}$ such that $2^{-K} < \epsilon$. Then, $\forall k \geq K$, $\|a_{n_k} - z\|_2 \leq \|a_{n_k} - z_k\|_2 + \|z_k - z\|_2 \leq 2^{-K} < \epsilon$. Hence, $z = \lim_{k} a_{n_k} \in C$. ■

Lemma C.2. Let $\{a_n : n \in \mathbb{N}\} \subset \mathbb{R}$ such that $\lim inf_n a_n$ and $\lim sup_n a_n$ are finite. If $\lim_{n} a_{n+1} - a_n = 0$, then for any $z \in [\lim inf_n a_n, \lim sup_n a_n]$, there is a subsequence of $\{a_n : n \in \mathbb{N}\}$ that converges to $z$.

Proof. We begin by showing that any closed interval strictly between the limit infimum and limit supremum contains a subsequential limit. Let $r_1 < r_2$ such that $\lim inf_n a_n < r_1$ and $r_2 < \lim sup_n a_n$. If there exists an infinite subsequence $\{a_{n_k} : k \in \mathbb{N}\} \subset [r_1, r_2]$, then sequential compactness implies that $\{a_{n_k} : k \in \mathbb{N}\}$ has a subsequence which converges in $[r_1, r_2]$. Suppose now, $\exists K \in \mathbb{N}$ such that $\forall n \geq K$, $a_n \notin [r_1, r_2]$. Since the $\lim inf_n a_n < r_1 < r_2 < \lim sup_n a_n$, there exists a subsequence $\{a_{n_k} : k \in \mathbb{N}\}$ such that $a_{n_k} < r_1$ and $r_2 < a_{n_k+1}$. However, this is a contradiction since $a_{n_k+1} - a_n \to 0$ as $k \to \infty$. Hence, there is always a subsequence in any closed interval between $\lim inf_n a_n$ and $\lim sup_n a_n$.

We have that if $z$ is either the limit infimum or limit supremum then there is a subsequence of $\{a_n : n \in \mathbb{N}\}$ that converges to this value. So take $\lim inf_n a_n < z < \lim sup_n a_n$. We now proceed by induction. Let $z_0 = \lim inf_n a_n$. There is a subsequence that converges to a point in $[0.5(z + z_0), z]$. Let $z_1$ be this limit. If $z \neq z_1$, then $|z_1 - z| \leq 2^{-1}(z - z_0)$ and we define $z_2$ as the subsequential limit in $[0.5(z + z_1), z]$. If $z = z_1$ then we stop. Suppose we proceed by induction such that $\{z_j : j = 1, \ldots, k\}$ are subsequential limits such that $|z_j - z| \leq 2^{-j}(z - z_0)$. If $z \neq z_k$, then we can find $z_{k+1}$ as the limit of a subsequence in $[0.5(z + z_k), z]$, which we denote $z_{k+1}$. Moreover, $|z_{k+1} - z| \leq 2^{-k-1}(z - z_0)$. If we never terminate at $z$ for some $k \in \mathbb{N}$, then $\{z_k : k \in \mathbb{N}\}$ is a sequence of subsequential limits converging to $z$. By Lemma C.1, $z$ is a subsequential limit. ■

Appendix D. Divergence Regime: Nonexistence of Objective Function Limit. Here, we use as similar construction for Proposition 4.4 to construct an objective function $F$ such that when gradient descent is applied to this objective function with a specific initialization, $\lim sup_k F(x_k) = \infty$, $\lim inf_k F(x_k) = 0$ and $|\dot{F}(x_k)| = 1$ for all $k$. We proceed in three general steps corresponding to each subsection below.
D.1. Objective Function Target Values. Let \( \{m_k : k + 1 \in \mathbb{N}\} \) be a sequence of scalars such that \( m_k > 0, \sum_k m_k = \infty \), and \( m_k \to 0 \) as \( k \to \infty \). Define \( S_0 = 0 \) and \( S_{k+1} = \sum_{j=0}^k m_k \) for all integers \( k \geq 0 \). We will now construct a sequence \( \{O_k : k + 1 \in \mathbb{N}\} \) which will serve as target values for each iterate of our objective function.

1. Let \( O_0 = 0 \). For convenience, let \( u_0 = \ell_0 = 0 \).
2. Let \( \ell_1 = 1 + \min\{k \geq 0 : O_0 + \frac{1}{2} \sum_{j=0}^k m_j > 1\} \). From the divergence of \( \sum_k m_k \), it is clear that such an \( \ell_1 \) is finite. Define \( O_k = O_0 + \frac{1}{2} \sum_{j=0}^{k-1} m_j \) for \( k \in [1, \ell_1] \cap \mathbb{N} \).
3. Let \( u_1 = 1 + \min\{k \geq \ell_1 : O_{\ell_1} - \sum_{j=\ell_1}^k m_j < 0\} \). Again, from the divergence of \( \sum_k m_k \), \( u_1 \) is finite. Define \( O_k = O_{\ell_1} - \sum_{j=\ell_1}^{k-1} m_j \) for \( k \in [\ell_1 + 1, u_1] \cap \mathbb{N} \).
4. For \( t \in \mathbb{N} \), let \( \ell_{t+1} = 1 + \min\{k \geq u_t : O_{u_t} + \frac{1}{2} \sum_{j=u_t}^k m_j > t+1\} \). From the divergence of \( \sum_k m_k \), \( \ell_{t+1} \) is finite if \( u_t \) is finite. Define \( O_k = O_{u_t} + \frac{1}{2} \sum_{j=u_t}^{k-1} m_j \) for \( k \in [u_t + 1, \ell_{t+1}] \cap \mathbb{N} \).
5. For \( t \in \mathbb{N} \), let \( u_{t+1} = 1 + \min\{k \geq \ell_{t+1} : O_{\ell_{t+1}} - \sum_{j=\ell_{t+1}}^{k-1} m_j < 0\} \). From the divergence of \( \sum_k m_k \), \( u_{t+1} \) is finite if \( \ell_{t+1} \) is finite. Define \( O_k = O_{\ell_{t+1}} - \sum_{j=\ell_{t+1}}^{k-1} m_j \) for \( k \in [\ell_{t+1} + 1, \ldots, u_{t+1}] \cap \mathbb{N} \).

We point out several facts about the sequence \( \{O_k : k + 1 \in \mathbb{N}\} \). First, \( \lim t O_t = \infty \) by construction. Second, we verify, \( \lim u_t O_t = 0 \). By construction, \( O_{u_t - 1} > 0 \) and \( 0 > O_{u_t} = O_{u_t - 1} - m_{u_t - 1} \geq -m_{u_t - 1} \). Since \( m_{u_t - 1} \to 0 \) as \( t \to \infty \), \( \lim \inf_t O_{u_t} = 0 \). In turn, the limit of the sequence exists and is zero. Third, we verify, \( \lim \sup_k O_k = \infty \) and \( \lim \inf_k O_k = 0 \). For any \( k \in \mathbb{N}_{>\ell_1} \), there exists a \( t \in \mathbb{N} \) such that \( k \in [\ell_t + 1, u_t] \) or \( k \in [u_t + 1, \ell_{t+1}] \). If \( k \in [\ell_t + 1, u_t] \), then \( O_k \in [O_{u_t}, O_{\ell_t}] \). If \( k \in [u_t + 1, \ell_{t+1}] \), then \( O_k \in [O_{u_t}, O_{\ell_{t+1}}] \). Hence, the third fact holds because of the first two.

D.2. Construction of the Objective Function. With these sequences established, we now state our objective function.

\[
F(x) = \begin{cases} 
-x & x \leq 0, \\
\tilde{f}_0(x) & x \in (0, S_{\ell_1}], \\
\tilde{f}_t(x) & x \in (S_{\ell_t}, S_{\ell_{t+1}}], \forall t \in \mathbb{N},
\end{cases}
\]

where

\[
\tilde{f}_0(x) = \begin{cases} 
-f_j(x) & x \in (S_j, S_{j+1}], \quad j \in \{0, \ldots, \ell_1 - 1\};
\end{cases}
\]

\[
\tilde{f}_t(x) = \begin{cases} 
O_{\ell_t} - (x - S_{\ell_t}) & x \in (S_{\ell_t}, S_{u_t}) \\
f_j(x) & x \in (S_j, S_{j+1}], \quad j \in \{u_t, \ldots, \ell_{t+1} - 1\};
\end{cases}
\]
and

\[
f_j(x) = \begin{cases} 
-x' + O_j & x' \in (0, \frac{m_j}{16}) \\
\frac{8}{m_j} (x' - \frac{m_j}{8})^2 - \frac{3m_j}{32} + O_j & x' \in \left[\frac{m_j}{16}, \frac{3m_j}{32}\right] \\
-5m_j \exp\left(\frac{5m_j/16}{x' - m_j/2} + 1\right) + \frac{m_j}{4} + O_j & x' \in \left[\frac{3m_j}{16}, \frac{m_j}{2}\right] \\
-\frac{8}{m_j} (x' - \frac{7m_j}{8})^2 + \frac{19m_j}{32} + O_j & x' = \frac{m_j}{2} \\
-\frac{8}{m_j} (x' - \frac{m_j}{8})^2 - \frac{3m_j}{32} + O_j & x' \in \left[\frac{13m_j}{16}, \frac{15m_j}{16}\right] \\
-x' + \frac{3m_j}{2} + O_j & x' \in \left[\frac{15m_j}{16}, m_j\right].
\end{cases}
\]

with \(x' = x - S_j\).

**D.3. Properties of the Objective Function.** Here, we verify (D.1) satisfies Assumptions 2.1 and 2.2. We need to verify certain properties of \(\tilde{f}_t(x)\), which we do now.

**Proposition D.1.** Let \(t + 1 \in \mathbb{N}\). The continuous extension of \(\tilde{f}_t : (S_{\ell_t}, S_{\ell_{t+1}}) \to \mathbb{R}\), (D.3), to \([S_{\ell_t}, S_{\ell_{t+1}}]\) is

1. continuous on \([S_{\ell_t}, S_{\ell_{t+1}}]\) with values \(O_{\ell_t}, O_{u_t}\) and \(O_{\ell_{t+1}}\) at points \(S_{\ell_t}, S_{u_t}\) and \(S_{\ell_{t+1}}\), respectively;
2. bounded from below by \(\min\{O_j - \frac{3m_j}{32} : j = u_t, \ldots, s_{\ell_{t+1}}-1\}\);
3. differentiable on \([S_{\ell_t}, S_{\ell_{t+1}}]\) with the one-sided derivatives being \(-1\) at the end points of the interval;
4. locally Lipschitz continuous.

**Proof.** We note that (D.3) has several components. The \(f_j(x)\) are the same as those defined by (4.2) but shifted vertically by a constant. Hence, by Proposition 4.2, the continuous extension of \(f_j(x)\) to \([S_j, S_{j+1}]\) is continuous; bounded from below by \(O_j - \frac{3m_j}{32}\); differentiable with the one-sided derivatives being \(-1\) on the end points of the interval; and locally Lipschitz continuous.

We use these facts to show the remaining properties of \(\tilde{f}_t(x)\). First, to verify continuity, we need only verify that the components agree at the points \(x \in \{S_{u_t}, S_{u_t+1}, \ldots, S_{\ell_{t+1}}-1\}\). When \(x = S_{u_t}\),

\[
\tilde{f}_t(S_{u_t}) = O_{\ell_t} + (S_{u_t} - S_{\ell_t}) = O_{\ell_t} + \left(\sum_{k=0}^{u_t-1} m_k - \sum_{k=0}^{\ell_t-1} m_k\right) = O_{\ell_t} + \sum_{k=\ell_t}^{u_t-1} m_k = O_{u_t}.
\]

Moreover,

\[
\lim_{x \downarrow S_{u_t}} \tilde{f}_t(x) = \lim_{x \downarrow S_{u_t}} f_{u_t}(x) = \lim_{x \downarrow S_{u_t}} -(x - S_{u_t}) + O_{u_t} = O_{u_t}.
\]

Hence, the evaluation of \(\tilde{f}_t(x)\) at \(S_{u_t}\) agrees with its limit from the right. For the remaining points, let \(j \in \{u_t + 1, \ldots, \ell_{t+1} - 1\}\). Then,

\[
\tilde{f}_t(S_j) = f_{j-1}(S_j) = -(S_j - S_{j-1}) + \frac{3m_{j-1}}{2} + O_{j-1} = \frac{m_{j-1}}{2} + O_{u_t} + \frac{1}{2} \sum_{k=u_t}^{j-1} m_k = O_j.
\]
Moreover,

\[(D.8) \quad \lim_{x \downarrow S_j} \tilde{f}(S_j) = \lim_{x \uparrow S_j} f_j(S_j) = \lim_{x \downarrow S_j} -(x - S_j) + O_j = O_j.\]

Hence, \(\tilde{f}(x)\) is continuous. Moreover, we have also shown that the continuous extension of \(\tilde{f}(x)\) has the stated values at \(x \in \{S_{\ell_t}, S_{\ell_t+1}, \ldots, S_{\ell_{t+1}-1}\}\).

For the lower bound, we have that \(\tilde{f}(x) \geq O_{u_t}\) for \(x \in (S_{\ell_t}, S_{u_t}]\). By Proposition D.1, each \(f_j(x) \geq O_j - \frac{3m_j}{32}\). Hence, the lower bound follows.

We now verify differentiability. By the properties of a linear function and Proposition 4.2, each component of \(\tilde{f}(x)\) is differentiable on its domain; it is lower bounded; its derivative is locally Lipschitz continuous; \(f(x)\) is bounded from below. By Proposition D.1, the component \(\tilde{\ell}(x)\) is differentiable and the one-sided derivatives are \(-1\) at the end of the interval on which it is defined.

To check local Lipschitz continuity of \(\tilde{f}(x)\), we note that each component of \(\tilde{f}(x)\) is locally Lipschitz continuous in its domain either because it is a linear function or by Proposition D.1. Hence, we need to only check that local Lipschitz continuity holds for each \(x \in \{S_{u_t}, S_{u_t+1}, \ldots, S_{\ell_{t+1}-1}\}\). For \(j \in \{u_t, \ldots, \ell_{t+1}-1\}\), the derivative of \(\tilde{f}(x)\) is \(-1\) in \((S_j - m_j/32, S_j + m_j/32)\). Hence, the derivative is locally Lipschitz continuous at the stated values of \(x\).

Proposition D.2. The function \(F : \mathbb{R} \to \mathbb{R}\) as defined in (D.1) is continuous and differentiable on its domain; it is lower bounded; its derivative is locally Lipschitz continuous; \(F(S_{\ell_t}) = O_{\ell_t}, \forall t \in \mathbb{N}\); \(F(S_{u_t}) = O_{u_t} \forall t \in \mathbb{N}\); and \(F\)'s derivative is not globally Lipschitz continuous.

Proof. The proof is similar to Proposition 4.3. Hence, we will only verify that \(F\) is lower bounded. By Proposition D.1, the component \(\tilde{f}(x)\) of \(F\) for some \(t+1 \in \mathbb{N}\) is bounded from below by some \(O_j - \frac{3m_j}{32}\) for some choice of \(j\). So it is enough for us to show, \(\{O_j - \frac{3m_j}{32}\} \) is bounded from below. By construction, \(\liminf_j O_j = 0\) and \(\lim_j m_j = 0\). Hence, \(\{O_j - \frac{3m_j}{32}\} \) is bounded from below. Thus, \(F\) is bounded from below.

D.4. Properties of Gradient Descent on the Objective Function. We now show that when gradient descent is applied to the constructed problem, the objective function’s limit supremum is infinite and limit infimum is zero, all while the gradient function remains bounded away from 0.

Proposition D.3. Let \(\{m_k : k+1 \in \mathbb{N}\}\) be any positive sequence such that \(\sum_k m_k\) diverges and \(m_k \to 0\). Define \(F: \mathbb{R} \to \mathbb{R}\) as in (D.1). Suppose \(x_0 = 0\) and let \(\{x_k : k \in \mathbb{N}\}\) be generated according to (2.3) with \(M_k = m_k I\) for all \(k+1 \in \mathbb{N}\). Then, \(\{M_k\}\) satisfies Properties 2.4 to 2.6. Moreover, (a) \(\lim_k x_k = \infty\); (b) \(\limsup_k F(x_k) = \infty\); (c) \(\liminf_k F(x_k) = 0\); and (d) \(\lim_k |F(x_k)| = -1\).

Proof. We first show, \(x_k = S_k\) for all \(k \in \mathbb{N}\). \(0 = x_0 = S_0\). Suppose the claim is true up to \(k \in \mathbb{N}\). Then, \(\exists t+1 \in \mathbb{N}\) such that \(\dot{F}(x_{k+1}) = \dot{f}(x_k)\). Using Proposition D.1 or properties of
a linear function, \( \dot{F} (x) = \dot{f} (S) = -1 \). Therefore,

\[(D.9) \quad x_{k+1} = x_k - M_k \dot{F} (x_k) = S_k - m_k \dot{f} (S_k) = S_k + m_k = S_{k+1}. \]

Thus, as \( k \to \infty \), the iterates diverge and \( \dot{F} (x_k) = -1 \) for all \( k + 1 \in \mathbb{N} \). Now, \( F(x_k) = F(S_k) = O_k \) for every \( k + 1 \in \mathbb{N} \). By properties of \( \{ O_k \} \), the limit supremum and limit infimum of this sequence is \( \infty \) and 0, respectively. The result follows.

We stress that the choice of the limit supremum and limit infimum can be readily modified by choosing a different definition for \( \{ \ell_i \} \) and \( \{ u_i \} \). Hence, the limit supremum can be made to be finite and even agree with the limit infimum. Moreover, the limit infimum can be set larger than 0.

**Appendix E. Divergence Regime: Objective Function Diverges, Gradient Function Converges to Zero.** Here, we construct an objective function that is bounded below and has locally Lipschitz continuous gradients. Importantly, when we apply gradient descent with diminishing step sizes to this objective function, the iterates of the procedure diverge, the objective function evaluated at the iterates will diverge, and the gradient function will converge to zero. This objective function will be constructed in a similar fashion to our other divergence regime examples.

**E.1. Construction of the Objective Function.** Let \( \{ m_k : k + 1 \} \) be a positive sequence such that \( \sum_k m_k \) diverges and \( m_k \to 0 \). Let \( S_0 = 0 \) and \( S_{k+1} = \sum_j m_j \). Then, \( \sum_k \frac{m_k}{S_{k+1}} \) diverges. Let \( K = \min \{ k > 0 : S_k \geq 1 \} \) and define

\[(E.1) \quad T_k = \begin{cases} S_k & k = 0, \ldots, K, \\ T_K + \sum_{j=K}^k \frac{m_j}{S_{j+1}} & k > K. \end{cases} \]

Moreover, define

\[(E.2) \quad d_k = \begin{cases} 1 & k = 0, \ldots, K, \\ \frac{1}{S_{K+1}} & k > K. \end{cases} \]

Finally, let

\[(E.3) \quad F(x) = \begin{cases} -x & x \leq 0, \\ f_j(x) & x \in (T_j, T_{j+1}], \ j + 1 \in \mathbb{N}, \end{cases} \]
where \( f_0(x), \ldots, f_{K-1}(x) \) are identical to (4.3); and, letting \( x' = x - T_j \),
\[
(E.4) \quad f_j(x) = \begin{cases} 
-d_j x' + f_{j-1}(T_j) & x' \in \left[ 0, \frac{(2-d_j)m_j}{16S_{j+1}} \right] \\
\frac{8S_{j+1}}{m_j} x' - \frac{m_j}{S_{j+1}} \left( \frac{32}{4d_j} \right) + f_{j-1}(T_j) & x' \in \left[ \frac{(2-d_j)m_j}{16S_{j+1}}, \frac{3m_j}{16S_{j+1}} \right] \\
-\frac{5m_j}{16S_{j+1}} \exp \left( \frac{5}{16} \frac{m_j}{S_{j+1}x'/m_j} + 1 \right) + \frac{m_j}{S_{j+1}} \left( \frac{32}{4d_j} f_{j-1}(T_j) \right) & x' \in \left[ \frac{3m_j}{16S_{j+1}}, \frac{5m_j}{16S_{j+1}} \right] \\
\frac{5m_j}{16S_{j+1}} \exp \left( \frac{5}{16} \frac{m_j}{S_{j+1}x'/m_j} + 1 \right) + \frac{m_j}{S_{j+1}} \left( \frac{32}{4d_j} f_{j-1}(T_j) \right) & x' \in \left[ \frac{5m_j}{16S_{j+1}}, \frac{13m_j}{16S_{j+1}} \right] \\
-\frac{8S_{j+1}}{m_j} x' - \frac{m_j}{S_{j+1}} \left( \frac{32}{4d_j} f_{j-1}(T_j) \right) & x' \in \left[ \frac{13m_j}{16S_{j+1}}, \frac{(d_j+14)m_j}{16S_{j+1}} \right] \\
-\frac{d_j+1}{m_j} x' + \frac{m_j}{2S_{j+1}} \left( \frac{32}{4d_j} f_{j-1}(T_j) \right) & x' \in \left[ \frac{(d_j+14)m_j}{16S_{j+1}}, \frac{m_j}{S_{j+1}} \right],
\end{cases}
\]
for \( j \geq K \).

**E.2. Properties of the Objective Function.** Here, we verify, (E.3) satisfies Assumptions 2.1 and 2.2. We begin by studying the properties of \( f_j(x) \) for \( j \geq K \). Note, we already know the properties of \( f_j(x) \) for \( j < K \) by Proposition 4.2.

**Proposition E.1.** Let \( j > K \). The continuous extension of \( f_j : (T_j, T_{j+1}) \rightarrow \mathbb{R} \), (E.4), to \([T_j, T_{j+1}]\) is continuous on its domain; bounded from below by \( f_{j-1}(T_j) - m_j/(8S_{j+1})\); differentiable on its domain with \( \dot{f}_j(T_j) = -d_j \) and \( \dot{f}_j(T_{j+1}) = -d_j+1\); its derivative is locally Lipschitz continuous; and \( f_j(T_{j+1}) \geq f_j(T_j) + 7m_j/(16S_{j+1}) \).

**Proof.** The proof of this result is similar to that of Proposition 4.2. Hence, we only produce the values of \( f_j(x) \) and \( \dot{f}_j(x) \) at key points.

1. At \( x = T_j \), \( f_j(T_j) = f_{j-1}(T_j). \) \( \dot{f}_j(T_j) = -d_j. \)
2. At \( x = (2-d_j)m_j/(16S_{j+1}) + T_j \),
\[
(E.5) \quad f_j(x) = \frac{m_j}{S_{j+1}} \left( \frac{d_j^2 - 2d_j}{16} \right) + f_{j-1}(T_j),
\]
and \( \dot{f}_j(x) = -d_j. \)
3. At \( x = T_j + 3m_j/(16S_{j+1}) \),
\[
(E.6) \quad f_j(x) = \frac{m_j}{S_{j+1}} \left( \frac{1 + d_j^2 - 4d_j}{32} \right) + f_{j-1}(T_j),
\]
and \( \dot{f}_j(x) = 1. \)
4. At \( x = T_j + m_j/(2S_{j+1}) \),
\[
(E.7) \quad f_j(x) = \frac{m_j}{S_{j+1}} \left( \frac{11 + d_j^2 - 4d_j}{32} \right) + f_{j-1}(T_j),
\]
and \( \dot{f}_j(x) = 0. \)
\[
(E.8) \quad f_j(x)
\]
5. At $x = T_j + 13m_j/(16S_{j+1})$, 

$$f_j(x) = \frac{m_j}{S_{j+1}} \left( \frac{21 + d_j^2 - 4d_j}{32} \right) + f_{j-1}(T_j),$$  

(E.9) and $\dot{f}_j(x) = 1$.

6. At $x = T_j + (d_{j+1} + 14)m_j/(16S_j)$, 

$$f_j(x) = \frac{m_j}{S_{j+1}} \left( \frac{22 + d_j^2 - d_{j+1}^2 - 4d_j}{32} \right) + f_{j-1}(T_j),$$  

(E.10) and $\dot{f}_j(x) = -d_{j+1}$.

7. At $x = T_j + m_j/S_{j+1}$, 

$$f_j(x) = \frac{m_j}{S_{j+1}} \left( \frac{22 + d_j^2 + d_{j+1}^2 - 4d_j - 4d_{j+1}}{32} \right) + f_{j-1}(T_j),$$  

(E.11) and $\dot{f}_j(x) = -d_{j+1}$.

Note, $f_j(T_{j+1}) = f_j(T_j + m_j/S_{j+1}) \geq (22 - 8)m_j/(32S_{j+1}) + f_{j-1}(T_j)$.

Proposition E.2. The function $F : \mathbb{R} \rightarrow \mathbb{R}$ as defined in (E.3) is continuous and differentiable on its domain; it is lower bounded; its derivative is locally Lipschitz continuous; $F(T_j) \geq 7T_j/16$ for $j + 1 \in \mathbb{N}$; $\dot{F}(T_j) = -d_j$ for all $j + 1 \in \mathbb{N}$; and $F$‘s derivative is not globally Lipschitz continuous.

Proof. As the proof of this statement is similar to the other constructions, we only verify the values of the objective and the derivative at $\{T_j\}$. For $j = 0$, $F(T_0) = 0$. For $j = 1, \ldots, K$, $F(T_j) = F(S_j) = f_{j-1}(S_j) = S_j/2 \geq 7S_j/16 = 7T_j/16$ by Proposition 4.2. For $j > K$, $F(T_j) = f_{j-1}(T_j) \geq f_{j-1}(T_{j-1}) + \frac{m_j}{16S_{j+1}} = F(T_{j-1}) + \frac{m_j}{16S_{j+1}}$ by Proposition E.1. By induction, for $j + 1 \in \mathbb{N}$, $F(T_j) \geq 7T_j/16$. Similarly, either by Proposition 4.2 or Proposition E.1, $\dot{F}(T_j) = \dot{f}_j(T_j) = -d_j$.

E.3. Properties of Gradient Descent on the Objective. We now show that when gradient descent is applied to the constructed problem, the objective function diverges, and the gradient function converges to zero.

Proposition E.3. Let $\{m_k : k + 1 \in \mathbb{N}\}$ be any positive sequence such that $\sum_k m_k$ diverges and $m_k \rightarrow 0$. Define $F : \mathbb{R} \rightarrow \mathbb{R}$ as in (E.3). Suppose $x_0 = 0$ and let $\{x_k : k \in \mathbb{N}\}$ be generated according to (2.3) with $M_k = m_kI$ for all $k + 1 \in \mathbb{N}$. Then, $\{M_k\}$ satisfies Properties 2.4 to 2.6. Moreover, (a) $\lim_k x_k = \infty$; (b) $\lim_k F(x_k) = \infty$; and (c) $\lim_k |\dot{F}(x_k)| = 0$.

Proof. We show that $x_k = T_k$ for all $k + 1 \in \mathbb{N}$. For the base case, $x_0 = 0 = T_0$. Suppose $x_k = T_k$ for some $k < K$. Then, 

$$x_{k+1} = x_k - M_k\dot{F}(x_k) = T_k + m_kd_k = T_k + m_k = T_{k+1}.$$  

(E.12) This implies that $x_K = T_K$. Now, suppose $x_k = T_k$ for some $k > K$. Then, 

$$x_{k+1} = x_k - M_K\dot{F}(x_k) = T_k + m_kd_k = T_k + \frac{m_k}{S_{k+1}} = T_{k+1}.$$  

(E.13)
Hence, $F(x_k) = F(T_k) \geq 7T_k/16$, which diverges to infinity. Moreover, for $k > K$, $|\dot{F}(x_k)| = |\dot{F}(T_k)| = d_k = 1/S_{k+1}$ which tends to zero.