CR EIGENVALUE ESTIMATE AND KOHN-ROSSI COHOMOLOGY

ZHIWEI WANG & XIANGYU ZHOU

Abstract

Let $X$ be a compact connected CR manifold with a transversal CR $S^1$-action of real dimension $2n-1$, which is only assumed to be weakly pseudoconvex. Let $\Box_b$ be the $\overline{\partial}_b$-Laplacian, with respect to a $T$-rigid Hermitian metric (see Definition 3.2 of $T$-rigid Hermitian metric). Eigenvalue estimate of $\Box_b$ is a fundamental issue both in CR geometry and analysis. In this paper, we are able to obtain a sharp estimate of the number of eigenvalues smaller than or equal to $\lambda$ of $\Box_b$ acting on the $m$-th Fourier components of smooth $(n-1,q)$-forms on $X$, where $m \in \mathbb{Z}_+$ and $q \in \{0, 1, \cdots, n-1\}$. Here the sharp means the growth order with respect to $m$ is sharp. In particular, when $\lambda = 0$, we obtain the asymptotic estimate of the growth for $m$-th Fourier components $H^{n-1,q}_{b,m}(X)$ of $H^{n-1,q}_{b}(X)$ as $m \to +\infty$. Furthermore, we establish a Serre type duality theorem for Fourier components of Kohn-Rossi cohomology which is of independent interest. As a byproduct, the asymptotic growth of the dimensions of the Fourier components $H^{0,q}_{b,-m}(X)$ for $m \in Z_+$ is established. We also give applications of our main results, including Morse type inequalities, asymptotic Riemann-Roch type theorem, Grauert-Riemenschneider type criterion, and an orbifold version of our main results which provides an answer towards a folklore open problem informed to us by Hsiao.

1. Introduction

Let $(X, T^{1,0}X)$ be a compact connected CR manifold of real dimension $2n-1$, $n \geq 2$, where $T^{1,0}X$ is the given CR structure on $X$. For $p, q = 0, 1, \cdots, n-1$, let $\Omega^{p,q}(X)$ be the space of smooth $(p,q)$-forms on $X$. Associated to the CR structure, there naturally comes the tangential Cauchy-Riemann operator $\overline{\partial}_b$, which shares similar properties as $\overline{\partial}$-operator on complex manifolds. For example, $\overline{\partial}_b^2 = 0$. It induces

---

Key words and phrases. CR manifold, Kohn-Rossi cohomology, transversal CR $S^1$-action, Fourier decomposition, eigenvalue estimate.

This research is supported by National Key R&D Program of China (No. 2021YFA1002600 and No. 2021YFA1003100). The authors are partially supported respectively by NSFC grants (11701031, 12071035, 11688101, 11431013).
a complex $\{\Omega^{\cdot,\cdot}(X), \overline{\partial}_b\}$ which is called $\overline{\partial}_b$-complex, and an intrinsic cohomology theory known as Kohn-Rossi cohomology, say $H^{p,q}_b(X)$ defined as $H^{p,q}_b(X) := \text{Ker}\overline{\partial}_b|_{\Omega^{p,q}(X)}/\text{Im}\overline{\partial}_b|_{\Omega^{p,q-1}(X)}$. For references, we refer to [6, 10, 20, 24].

One of the most significant features of differential geometry is the input of a specific metric. CR geometry is not an exception. Putting a Hermitian metric into the CR structure, in a quite standard way, one gets the so-called $\square_b$-operator.

The mystery of the geometry of CR manifolds lies behind $\partial_b$ and $\square_b$-operators. One may expect many Riemannian geometric methods, which have shown great power in the study of fundamental problems in Riemannian geometry, such as gradient estimate, eigenvalue estimate, the analysis of heat kernel and so on, can be adapted into the CR picture.

However, both the $\partial_b$ and $\square_b$ operators are not elliptic operators, which put much more thorns on the way for CR geometers.

For the time being, there are many progress towards the study of $\partial_b$ and $\square_b$, or in other words, the study of Kohn-Rossi cohomology and CR eigenvalue estimate, most of which focus on strongly pseudoconvex CR manifolds. For example, the embedding problem of the CR manifolds (c.f. [1, 2, 14, 5, 7, 8, 13, 21, 23, 25, 26, 27, 28, 36, 37, 41, 42, 17], etc), and the study of the isolated singularities of complex hypersurfaces (c.f. [32, 33, 34, 35, 39, 40], etc). It is worth to mention that Stephen Yau and his coauthors have found many important and remarkable applications of the Kohn-Rossi cohomology to the theory of singularities, complex Plateau problem, the embedding problem of the CR manifolds, and rigidity problems on CR morphisms between compact strongly pseudoconvex CR manifolds ([12, 29, 43, 44, 46, 45, 47], etc).

Though there are important progress, the CR eigenvalue estimate and Kohn-Rossi cohomology of general CR manifold, for example, CR manifolds which are not necessarily strongly pseudoconvex, are not so well understood. This is the main concern of this paper.

In this paper, we take the task of studying the CR eigenvalue estimate and Kohn-Rossi cohomology of a compact connected CR manifold $X$ admitting a transversal CR $S^1$-action, which is only assumed to be weakly pseudoconvex.

Thanks to the transversal CR $S^1$-action, we have the Fourier decomposition $\Omega^{p,q}(X) = \oplus_{m \in \mathbb{Z}} \Omega^{p,q}_m(X)$, where $\Omega^{p,q}_m(X)$ is the $m$-th Fourier component of $\Omega^{p,q}(X)$ with respect to the $S^1$-action, and the $\overline{\partial}_b$ operator acts on the graded algebra $\oplus_q \Omega^{p,q}_m(X)$. One can thus define the $m$-th Fourier component $H^{p,q}_{b,m}(X)$ of $(p,q)$-th Kohn-Rossi cohomology group $H^{p,q}_b(X)$. Let $\square^{p,q}_{b,m}$ be the restriction of the $\overline{\partial}_b$-Laplace operator to the space $\Omega^{p,q}_m(X)$, which turns out to be a self-adjoint operator.
Let $\mathcal{H}_{b,m}^{p,q,\leq \lambda}$ be the linear span of the eigenforms of $\Box_{b,m}^{p,q}$ in $\Omega_{b,m}^{p,q}(X)$ with eigenvalues smaller than or equal to $\lambda$. By a Hodge type theory, $\mathcal{H}_{b,m}^{p,q,\leq 0}(X) := \mathcal{H}_{b,m}^{p,q}(X)$ is the space of $\Box_{b,m}^{p,q}$ harmonic forms, and isomorphic to $H_{b,m}^{p,q}(X)$. In particular, $H_{b,m}^{p,q}(X)$ is of finite dimension for every $m \in \mathbb{Z}$.

The first main result of this paper is the following asymptotic estimate for the distribution of eigenvalues of $\Box_{b,m}^{n-1,q}$.

**Theorem 1.1.** Let $(X, T^{1,0}X)$ be a compact connected CR manifold of dimension $2n - 1$, $n \geq 2$, where $T^{1,0}X$ is the given CR structure on $X$. Assume that $X$ admits a transversal CR $S^1$ action and $X$ is weakly pseudoconvex. Then for $m$ sufficiently large, if $0 \leq \lambda \leq m$,

\[ \dim \mathcal{H}_{b,m}^{n-1,q,\leq \lambda} \leq C(\lambda + 1)^q m^{n-1-q}, \]

and if $1 \leq m \leq \lambda$,

\[ \dim \mathcal{H}_{b,m}^{n-1,q,\leq \lambda} \leq C \lambda^{n-1}. \]

It is worth to mention that, examples are provided in Section 7 to show that the growth order, say $m^{n-1-q}$ in our Theorem 1.1 can not be improved in general.

Taking account of the transversal CR $S^1$-action, we want to ask more structures for the Kohn-Rossi cohomology. To this end, we get our second main result as follows.

**Theorem 1.2** (Serre type duality theorem). Let $X$ be a compact CR-manifold of real dimension $2n - 1$, which admits a transversal CR $S^1$-action which is locally free. Then we have the following conjugate linear isomorphism in the cohomological level

\[ H_{b,m}^{p,q}(X) \simeq H_{b,-m}^{n-1-p,n-1-q}(X). \]

With Serre type duality in hand, Theorem 1.1 gives us the following

**Theorem 1.3.** Let $X$ be a compact CR-manifold of real dimension $2n - 1$, which admits a transversal $S^1$-action which is locally free. Suppose that $X$ is a weakly pseudoconvex CR manifold. Then we have that

\[ \dim H_{b,-m}^{0,q}(X) \leq C m^q, \quad \text{as} \quad m \to +\infty. \]

It is worth to mention that Theorem 1.3 improves the corresponding estimates of Hsiao and Li in [19]. Namely, under the same assumption of Theorem 1.3, Hsiao and Li obtained the following estimate:

\[ \dim H_{b,-m}^{0,q}(X) = o(m^{n-1}) \quad \text{as} \quad m \to +\infty \quad \text{for} \quad 0 \leq q < n-1. \]

Several applications of our main result are in order. Firstly, combining Morse type inequalities in [19] with Theorem 1.2 there comes the following
**Theorem 1.4.** Let $X$ be a compact connected CR manifold with a transversal $CR S^1$-action. Assume that $\dim_{\mathbb{R}} X = 2n - 1$, $n \geq 2$. Then for every $q = 0, 1, \cdots, n - 1$, as $m \to +\infty$, we have

$$\dim H^{n-1,q}_{b,-m}(X) \leq \frac{m^{n-1}}{2\pi^n} \int_{X(n-1-q)} \det L_x |dv_X(x) + o(m^{n-1}),$$

$$\sum_{j=n-1-q}^{n-1} (-1)^j q^{-(n-1)} \dim H^{n-1,j}_{b,-m}(X)$$

$$\leq \frac{m^{n-1}}{2\pi^n} \sum_{j=n-1-q}^{n-1} (-1)^{q+j-(n-1)} \int_{X(n-1-j)} |\det L_x|dv_X(x) + o(m^{n-1}).$$

In particular, when $q = n - 1$, as $m \to +\infty$, we have the asymptotic Riemann-Roch theorem

$$\sum_{j=0}^{n-1} (-1)^{n-1-j} \dim H^{n-1,j}_{b,m}(X)$$

$$= \frac{m^{n-1}}{2\pi^n} \sum_{j=0}^{n-1} (-1)^{n-1-j} \int_{X(n-1-j)} |\det L_x|dv_X(x) + o(m^{n-1}).$$

**Theorem 1.5.** Let $X$ be a compact connected CR manifold with a transversal $CR S^1$-action. Assume that $\dim_{\mathbb{R}} X = 2n - 1$, $n \geq 2$. For every $q = 0, 1, 2, \cdots, n - 1$, as $m \to +\infty$, we have

$$\dim H^{n-1,q}_{b,m}(X) \leq \frac{m^{n-1}}{2\pi^n} \int_{X(q)} |\det L_x|dv_X(x) + o(m^{n-1}),$$

$$\sum_{j=n-1-q}^{n-1} (-1)^{q+j-(n-1)} \dim H^{n-1,j}_{b,m}(X)$$

$$\leq \frac{m^{n-1}}{2\pi^n} \sum_{j=n-1-q}^{n-1} (-1)^{q+j-(n-1)} \int_{X(j)} |\det L_x|dv_X(x) + o(m^{n-1}).$$

In particular, when $q = n - 1$, as $m \to +\infty$, we have the following asymptotic Riemann-Roch theorem

$$\sum_{j=0}^{n-1} (-1)^{n-1-j} \dim H^{n-1,j}_{b,m}(X)$$

$$= \frac{m^{n-1}}{2\pi^n} \sum_{j=0}^{n-1} (-1)^{n-1-j} \int_{X(j)} |\det L_x|dv_X(x) + o(m^{n-1}).$$

**Remark 1.1.** As a complement of Theorem 1.4 and Theorem 1.5, we point out that, by almost the same proof as in [19], under the same
assumptions, one can prove the corresponding strong Morse type inequalities for the complex $\Omega_{b,m}^{n-1} \cdot (X, \overline{\partial}_b)$ as follows. For every $q = 0, 1, \cdots, n - 1$, as $m \to +\infty$, we have

$$
\sum_{j=0}^{q} (-1)^{q-j} \dim H_{b,m}^{n-1,j}(X) \leq \frac{m^{n-1}}{2\pi^n} \sum_{j=0}^{q} (-1)^{q-j} \int_{X(j)} |\det L_x| dv_X(x) + o(m^{n-1});
$$

and as $m \to -\infty$, we have

$$
\sum_{j=0}^{q} (-1)^{q-j} \dim H_{b,m}^{n-1,j}(X) \leq \frac{|m|^{n-1}}{2\pi^n} \sum_{j=0}^{q} (-1)^{q-j} \int_{X(n-1-j)} |\det L_x| dv_X(x) + o(|m|^{n-1}).
$$

Combining Theorem 1.1 and Theorem 1.5, we can obtain the following Grauert-Riemenschneider type criterion

**Theorem 1.6.** Let $(X, T^{1,0}X)$ be a compact connected CR manifold of dimension $2n - 1$, $n \geq 2$, where $T^{1,0}X$ is the given CR structure on $X$. Assume that $X$ admits a transversal CR $S^1$-action. If $X$ is weakly pseudoconvex and strongly pseudoconvex at a point, then

$$
\dim H_{b,m}^{n-1,0}(X) \approx m^{n-1} \text{ as } m \to +\infty.
$$

That is to say, there are a lot of CR sections of the canonical bundle $K_X$ of $X$.

**Remark 1.2.** Theorem 1.6 can be used to study the embedding problem for weakly pseudoconvex CR manifolds with transversal CR $S^1$-action, which will be discussed in a future work.

Orbifold appears frequently when you do some quotients in algebraic geometry and reductions in mathematical physics, for example in the process of symplectic reduction. It is also a simplest case of singular space. For compact connected weakly pseudoconvex CR orbifolds with transversal CR $S^1$-actions, we establish the following

**Theorem 1.7.** Let $(X, T^{1,0}X)$ be a compact connected CR orbifold of dimension $2n - 1$, $n \geq 2$, where $T^{1,0}X$ is the given CR structure on $X$. Assume that $X$ admits a transversal CR $S^1$-action and $X$ is weakly pseudoconvex. Then for $m$ sufficiently large, if $0 \leq \lambda \leq m$,

$$
\dim H_{b,m}^{n-1,q} \leq C(\lambda + 1)^q m^{n-1-q},
$$

and if $1 \leq m \leq \lambda$,

$$
\dim H_{b,m}^{n-1,q} \leq C\lambda^{n-1}.
$$
Moreover, we establish the following isomorphism of Grauert type in the orbifold level, which we think is of independent interest.

**Theorem 1.8.** Let $M$ be a compact complex manifold and $G$ a compact Lie group. Suppose that $G$ acts on $M$ analytically, locally free and $\dim_{\mathbb{C}} M/G = n$. Let $L$ be a $G$-invariant holomorphic Hermitian line bundle over $M$. Suppose that $L$ admits a locally free $G$-action compatible with $M$. Take any orbifold Hermitian metric $h_L$ (i.e. $G$-invariant Hermitian metric) on $L$, set $\tilde{X} = \{ v \in L^* \mid |v|^2_{h_L} = 1 \}$ and $X = \tilde{X}/G$. Then for every $p,q = 0,1, \cdots, n$ and every $m \in \mathbb{Z}$, there is a bijective map $A_{m}^{(p,q)} : \Omega_{m}^{(p,q)}(X) \to \Omega_{m}^{(p,q)}(M/G, L^m/G)$ such that $A_{m}^{(p,q+1)} \bar{\partial}_{b,m} = \bar{\partial} A_{m}^{(p,q)}$ on $\Omega_{m}^{(p,q)}(X)$. Thus we have that

\[
\Omega_{m}^{p,q}(X) \simeq \Omega^{p,q}(M/G, L^m/G)
\]

\[
H^{p,q}_{b,m}(X) \simeq H^{p,q}(M/G, L^m/G).
\]

In particular, $\dim H_{b,m}^{p,q}(X) < \infty$.

It should be remarked that the case of $p = 0$ of Theorem 1.8 was established in [9]. From Theorem 1.7 and Theorem 1.8 we obtain the following

**Theorem 1.9.** Let $M$ be a compact complex manifold and $G$ a compact Lie group. Suppose that $G$ acts on $M$ analytically, locally free and $\dim_{\mathbb{C}} M/G = n$. Let $(L, h^L)$ be a $G$-invariant holomorphic Hermitian line bundle over $M$. Suppose that $L$ admits a locally free $G$-action compatible with $M$ and the curvature of $L$ is semi-positive. Then we have that for $m$ sufficiently large,

\[
\dim H^{n,q}(M/G, L^m/G) \leq C m^{n-q},
\]

where $C$ is a constant independent of $m$.

The above theorem corresponds to Berndtsson’s estimate in the orbifold case, which answers a folklore open question, say generalizing Berndtsson’s estimate to the orbifold setting, informed to us by Hsiao.

For reader’s convenient, we sketch the proof of Theorem 1.8. By Baouendi-Rothschild-Treves [3], we get a picture of the local structure of CR manifolds with transversal CR $S^1$ action. In fact, locally, it is a part of the a circle bundle, i.e. it can be decomposed to complex ball in $\mathbb{C}^{n-1}$ times a small angle, say $U =: B_{\varepsilon} \times (-\delta, \delta)$. Our first key observation is that, on $U$, $\bar{\partial}_b$-operator, $\Box_b$-operator, and any $\bar{\partial}_b$-closed $(n-1,q)$ form admit very good representations, say by ignoring some rotations, they coincide with the $\bar{\partial}$-operator, $\Box$-operator and $\bar{\partial}$-closed $(n-1,q)$-forms on the complex ball $B_{\varepsilon}$. Our second key observation is that the weakly pseudoconvexity condition provides us a local potential, which is plurisubharmonic on $B_{\varepsilon}$. All these observations inspire us to
construct trivial holomorphic line bundle over \( B_\varepsilon \), and let the local potential serve as a Hermitian metric of this line bundle, and then we can translate the CR eigenvalue estimate problem to a counterpart in the category of complex holomorphy. To do this, we modify Berndtsson’s trick, do careful analysis. Finally, Theorem 1.1 turns up in this paper.

The structure of this paper is organized as follows. In Section 2, we introduce the basics of CR manifolds with transversal CR \( S^1 \)-action. In Section 3, we introduce Hermitian CR geometry under transversal CR \( S^1 \)-action. In Section 4, we formulate the local picture of compact connected weakly pseudoconvex CR manifold with transversal CR \( S^1 \)-action by using Baouendi-Rothschild-Treves’ theory. In Section 5, we give a local representation of \( \partial \), \( \partial^* \) and \( \Box_{b,m} \). In Section 6, we prepare the scaling technique for the proof of Theorem 1.1. In Section 7, we give the proof of the Theorem 1.1, and explain why the estimate cannot be improved in general. In Section 8, we give a proof of a Serre type duality theorem, namely Theorem 1.2. In Section 9, we give some applications of our main results. Namely, we prove Theorem 1.4, Theorem 1.5, Theorem 1.6, Theorem 1.7, Theorem 1.8, Theorem 1.9.

2. CR manifold with transversal CR \( S^1 \)-action

Let \((X, T^{1,0}X)\) be a compact connected CR manifold of dimension \(2n - 1\), \(n \geq 2\), where \(T^{1,0}X\) is the given CR structure on \(X\). That is, \(T^{1,0}X\) is a sub-bundle of the complexified tangent bundle \(CTX\) of rank \(n - 1\), satisfying \(T^{1,0}X \cap T^{0,1}X = \{0\}\), where \(T^{0,1}X = \overline{T^{1,0}X}\), and \([\mathcal{V}, \mathcal{V}] \subset \mathcal{V}\), where \(\mathcal{V} = C^\infty(X, T^{1,0}X)\).

We assume throughout this paper that, \((X, T^{1,0}X)\) is a compact connected CR manifold with a transversal CR \(S^1\)-action.

Denote by \(e^{i\theta} (0 \leq \theta < 2\pi)\) the \(S^1\)-action: \(S^1 \times X \to X\), \((e^{i\theta}, x) \mapsto e^{i\theta} \circ x\). Set \(X_{\text{reg}} = \{x \in X : \forall e^{i\theta} \in S^1, \text{ if } e^{i\theta} \circ x = x, \text{ then } e^{i\theta} = \text{id}\}\). We call \(x \in X_{\text{reg}}\) a regular point of the \(S^1\)-action. It is proved in \([19]\) that \(X_{\text{reg}}\) is an open, dense subset of \(X\), and thus the measure of \(X \setminus X_{\text{reg}}\) is zero.

Let \(T \in C^\infty(X, TX)\) be the global real vector field induced by the \(S^1\)-action \(e^{i\theta} (\theta \in [0, 2\pi])\) given as follows

\[
(Tu)(x) = \frac{\partial}{\partial \theta}(u(e^{i\theta} \circ x))|_{\theta = 0}, u \in C^\infty(X).
\]

**Definition 2.1.** We say that the \(S^1\)-action is CR if

\[
[T, C^\infty(X, T^{1,0}X)] \subset C^\infty(X, T^{1,0}X),
\]

where \([\cdot, \cdot]\) is the Lie bracket between the smooth vector fields on \(X\). Furthermore, we say that the \(S^1\)-action is transversal if for each \(x \in X\),

\[
CT(x) \oplus T^{1,0}_x X \oplus T^{0,1}_x X = CT_x X.
\]
Denote by $\omega_0$ the global real 1-form determined by $\langle \omega_0, u \rangle = 0$, for every $u \in T^{1,0}X \oplus T^{0,1}X$ and $\langle \omega_0, T \rangle = -1$.

**Definition 2.2.** For $x \in X$, the Levi form $\mathcal{L}_x$ associated with the CR structure is the Hermitian quadratic form on $T^{1,0}X$ defined as follows. For any $U, V \in T^{1,0}X$, pick $U, V \in C^\infty(X, T^{1,0}X)$ such that $U(x) = U$, $V(x) = V$. Set

$$\mathcal{L}_x(U, V) = \frac{1}{2i} \langle [U, \overline{V}] (x), \omega_0(x) \rangle$$

where $[\cdot, \cdot]$ denotes the Lie bracket between smooth vector fields. Note that $\mathcal{L}_x$ does not depend on the choice of $U$ and $V$.

**Definition 2.3.** The CR structure on $X$ is called (weakly) pseudoconvex at $x \in X$ if $\mathcal{L}_x$ is positive semi-definite. It is called strongly pseudoconvex at $x$ if $\mathcal{L}_x$ is positive definite. If the CR structure is (strongly) pseudoconvex at every point of $X$, then $X$ is called a (strongly) pseudoconvex CR manifold.

Denote by $T^{*1,0}X$ and $T^{*0,1}X$ the dual bundle of $T^{1,0}X$ and $T^{0,1}X$ respectively. Define the vector bundle of $(p, q)$-forms by $T^{*p,q}X := \Lambda^p T^{*1,0}X \otimes \Lambda^q T^{*0,1}X$. Let $D \subset X$ be an open subset. Let $\Omega^{p,q}(D)$ denote the space of smooth sections of $T^{*p,q}X$ over $D$ and let $\Omega^0_0(D)$ be the subspace of $\Omega^{p,q}(D)$ whose elements have compact support in $D$.

Fix $\theta_0 \in [0, 2\pi)$. Let

$$de^{i\theta_0} : \mathbb{C}T_x X \to \mathbb{C}T_{e^{i\theta_0}x} X$$

denote the differential map of $e^{i\theta_0} : X \to X$. By the property of transversal CR $S^1$-action, one can check that

$$de^{i\theta_0} : T^{1,0}_x X \to T^{1,0}_{e^{i\theta_0}x} X,$$

$$de^{i\theta_0} : T^{0,1}_x X \to T^{0,1}_{e^{i\theta_0}x} X,$$

$$de^{i\theta_0}(T(x)) = T(e^{i\theta_0} \circ x).$$

Let $(de^{i\theta_0})^* : \Lambda^{p+q}(\mathbb{C}T^*X) \to \Lambda^{p+q}(\mathbb{C}T^*X)$ be the pull-back of $de^{i\theta_0}$, $p, q = 0, 1, \ldots, n - 1$. From (7), we can check that for every $p, q = 0, 1, \ldots, n - 1$,

$$(de^{i\theta_0})^* : T^{*p,q}_{e^{i\theta_0}x} X \to T^{*p,q}_x X.$$  

Let $u \in \Omega^{p,q}(X)$, define $Tu$ as follows. For any $X_1, \ldots, X_p \in T^{1,0}_x X$ and $Y_1, \ldots, Y_q \in T^{0,1}_x X$,

$$Tu(X_1, \ldots, X_p; Y_1, \ldots, Y_q) := \frac{\partial}{\partial \theta} ((de^{i\theta})^* u(X_1, \ldots, X_p; Y_1, \ldots, Y_q))|_{\theta=0}.$$  

From (1) and (2), we have that $Tu \in \Omega^{p,q}(X)$ for all $u \in \Omega^{p,q}(X)$.
Let $\overline{\partial}_b : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)$ be the tangential Cauchy-Riemann operator. For the definition of tangential Cauchy-Riemann operator, we refer to [6, 10, 20]. It is straightforward from (1) and (2) to see that

$$T \overline{\partial}_b = \overline{\partial}_b T$$

on $\Omega^{p,q}(X)$. For every $m \in \mathbb{Z}$, put $\Omega^{p,q}_m(X) := \{ u \in \Omega^{p,q}(X) : Tu = imu \}$. We have the $\overline{\partial}_b$-complex for every $m \in \mathbb{Z}$:

$$\cdots \to \Omega^{p,q-1}_m(X) \to \Omega^{p,q}_m(X) \to \Omega^{p,q+1}_m(X) \to \cdots$$

For every $m \in \mathbb{Z}$, the $(p,q)$-th $\overline{\partial}_b$ cohomology (or Kohn-Rossi cohomology) is given by

$$H^{p,q}_{b,m}(X) := \frac{\text{Ker} \overline{\partial}_b : \Omega^{p,q}_m(X) \to \Omega^{p,q+1}_m(X)}{\text{Im} \overline{\partial}_b : \Omega^{p,q-1}_m(X) \to \Omega^{p,q}_m(X)}.$$

For $m \in \mathbb{Z}$, when $q = 0$, $H^{0,0}_{b,m}(X)$ is the space of CR functions which lie in the eigenspace of $T$ and we call $H^{0,0}_{b,m}(X)$ the $m$-th Fourier component of CR functions. We say that a function $u \in C^\infty(X)$ is a Cauchy-Riemann (CR for short) function if $\overline{\partial}_b u = 0$ or in the other word, $\overline{\partial}_b u = 0$ for all $Z \in \mathcal{C}^\infty(X, T^{1,0}X)$.

3. Metric input: Hermitian CR geometry under transversal CR $S^1$-action

In this section, we collect facts we need on Hermitian CR geometry under transversal CR $S^1$-action. Lemmas and Theorems not specified are taken from [18, 19].

**Definition 3.1.** Let $D$ be an open set and let $V \in C^\infty(D, \mathbb{C}TX)$ be a vector field on $D$. We say that $V$ is $T$-rigid if

$$de^{i\theta}(V(x)) = V(e^{i\theta} \circ x)$$

for any $x, \theta \in [0, 2\pi)$ satisfying $x \in D$, $e^{i\theta} \circ x \in D$.

**Definition 3.2.** Let $\langle \cdot | \cdot \rangle$ be a Hermitian metric on $\mathbb{C}TX$. We say that $\langle \cdot | \cdot \rangle$ is $T$-rigid if for $T$-rigid vector fields $V, W$ on $D$, where $D$ is any open set, we have

$$\langle V(x) | W(x) \rangle = \langle (de^{i\theta}V)(e^{i\theta} \circ x) | (de^{i\theta}W)(e^{i\theta} \circ x) \rangle,$$

for any $x \in D$, $\theta \in [0, 2\pi)$ such that $e^{i\theta} \circ x \in D$.

**Lemma 3.1.** Let $X$ be a compact connected CR manifold with a transversal $S^1$-action. There is always a $T$-rigid Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$ such that $T^{1,0}X \perp T^{0,1}X$, $T \perp (T^{1,0}X \oplus T^{0,1}X)$, $\langle T | T \rangle = 1$ and $\langle u | v \rangle$ is real if $u, v$ are real tangent vectors.
From now on, we fix a $T$-rigid Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C} TX$ satisfying all the properties in Lemma 3.1. The Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C} TX$ induces by duality a Hermitian metric on $\mathbb{C} T^* X$ and also on the bundles of $(p,q)$-forms for $p,q = 0, 1, \cdots, n - 1$. We shall also denote all these induced metrics by $\langle \cdot | \cdot \rangle$. For every $v \in T^{*p,q} X$, we write $|v|^2 := \langle v|v \rangle$.

We have the pointwise orthogonal decompositions:

$$\mathbb{C} T^* X = T^{*1,0} X \oplus T^{*0,1} X \oplus \{ \lambda \omega_0 : \lambda \in \mathbb{C} \},$$

$$\mathbb{C} TX = T^{1,0} X \oplus T^{0,1} X \oplus \{ \lambda T : \lambda \in \mathbb{C} \}.$$

For any $p \in X$, locally there is an orthonormal frame $\{ U_1, \cdots, U_{n-1} \}$ of $T^{1,0} X$ with respect to the given $T$-rigid Hermitian metric $\langle \cdot | \cdot \rangle$ such that the Levi-form $L_p$ is diagonal in this frame, $L_p(U_i, U_j) = \lambda_j \delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$. The entries $\{ \lambda_1, \cdots, \lambda_{n-1} \}$ are called the eigenvalues of the Levi-form at $p$ with respect to the $T$-rigid Hermitian metric $\langle \cdot | \cdot \rangle$. Moreover, the determinant of $L_p$ is defined by $\det L_p = \lambda_1(p) \cdots \lambda_{n-1}(p)$.

Let $\langle \cdot | \cdot \rangle$ be the $L^2$ inner product on $\Omega^{p,q}(X)$ induced by $\langle \cdot | \cdot \rangle$ and let $\| \cdot \|$ denote the corresponding norm. Then for all $u,v \in \Omega^{p,q}(X)$

$$(u|v) = \int_X \langle u|v \rangle dv_X$$

where $dv_X$ is the volume form on $X$ induced by the $T$-rigid Hermitian metric. Let $L^2_{(p,q),m}(X)$ be the completion of $\Omega^{p,q}_m(X)$ with respect to $\langle \cdot | \cdot \rangle$. For every $u \in \Omega^{p,q}(X)$,

$$Q_{(p,q)}^m : L^2_{(p,q)}(X) \to L^2_{(p,q),m}(X)$$

be the orthogonal projection with respect to $\langle \cdot | \cdot \rangle$. Then for any $u \in \Omega^{p,q}(X)$,

$$Q_{(p,q)}^m u = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{i\theta} \circ x)e^{-im\theta} d\theta.$$

By using the elementary Fourier analysis, it is straightforward to see that for any $u \in \Omega^{p,q}(X)$,

$$\sum_{m=-N}^{N} Q_{(p,q)}^m u \to u$$

in $C^\infty$ topology as $N \to \infty$. For every $u \in L^2_{(p,q)}(X)$,

$$\sum_{m=-N}^{N} Q_{(p,q)}^m u \to u$$

in $L^2_{(p,q)}(X)$ as $N \to \infty$. If we denote the $\lim_{N \to \infty} \sum_{m=-N}^{N} Q_{(p,q)}^m u$ by $\sum_{m \in \mathbb{Z}} Q_{(p,q)}^m u$, then we write $u = \sum_{m \in \mathbb{Z}} Q_{(p,q)}^m u$. Thus we have the
following Fourier decomposition
\[ \Omega^{p,q}(X) = \oplus_{m \in \mathbb{Z}} \Omega_{m}^{p,q}(X), \]
\[ L_{(p,q)}^{2}(X) = \oplus_{m \in \mathbb{Z}} L_{(p,q),m}^{2}(X). \]

We have the following Fourier decomposition of the \((p, q)\)-th Kohn-Rossi cohomology
\[ H_{b}^{p,q}(X) = \oplus_{m \in \mathbb{Z}} H_{b,m}^{p,q}(X). \]

Let \( \overline{\partial}^{\ast}_{b} : \Omega^{p,q+1}(X) \to \Omega^{p,q}(X) \) be the formal adjoint of \( \overline{\partial}_{b} \) with respect to \( \langle \cdot | \cdot \rangle \). Since the Hermitian metrics \( \langle \cdot | \cdot \rangle \) are \( T \)-rigid, we can check that
\[ T \overline{\partial}^{\ast}_{b} = \overline{\partial}^{\ast}_{b} T \]
on \( \Omega^{p,q}(X) \) for \( p, q = 0, 1, \cdots, n - 1 \) and thus
\[ \overline{\partial}^{\ast}_{b} : \Omega^{p,q+1}(X) \to \Omega^{p,q}(X), \quad \forall m \in \mathbb{Z}. \]

Put
\[ \Box^{(p,q)}_{b} := \overline{\partial}_{b} \partial_{b} + \overline{\partial}_{b}^{\ast} \partial_{b} : \Omega^{p,q}(X) \to \Omega^{p,q}(X). \]

We also have that
\[ T \Box^{(p,q)}_{b} = \Box^{(p,q)}_{b} T, \]
thus
\[ \Box^{(p,q)}_{b} : \Omega_{m}^{p,q}(X) \to \Omega_{m}^{p,q}(X) \]

We will write \( \Box^{(p,q)}_{b,m} \) to denote the restriction of \( \Box^{(p,q)}_{b} \) on \( \Omega_{m}^{p,q}(X) \). For every \( m \in \mathbb{Z} \), we extend \( \Box^{(p,q)}_{b,m} \) to \( L_{(p,q),m}^{2}(X) \) by
\[ \Box^{(p,q)}_{b,m} : \text{Dom}(\Box^{(p,q)}_{b,m}) \subset L_{(p,q),m}^{2}(X) \to L_{(p,q),m}^{2}(X), \]
where \( \text{Dom}(\Box^{(p,q)}_{b,m}) = \{ u \in L_{(p,q),m}^{2}(X) : \Box^{(p,q)}_{b,m} u \in L_{(p,q),m}^{2}(X) \} \).

**Theorem 3.2.** For every \( s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), there exists a constant \( C_s \) such that
\[ \|u\|_{s+1} \leq C_s (\|\Box^{(p,q)}_{b} u\|_s + \|Tu\|_s + \|u\|_s), \quad \forall u \in \Omega^{p,q}(X), \]
where \( \| \cdot \|_s \) denotes the standard sobolev norm of order \( s \) on \( X \).

**Theorem 3.3.** Fix \( m \in \mathbb{Z} \), for every \( s \in \mathbb{N}_0 \), there is a constant \( C_{s,m} \)
such that
\[ \|u\|_{s+1} \leq C_{s,m} (\|\Box^{(p,q)}_{b,m} u\|_s + \|u\|_s), \quad \forall u \in \Omega_{m}^{p,q}(X). \]

**Theorem 3.4.** Fix \( m \in \mathbb{Z} \), \( \Box^{(p,q)}_{b,m} : \text{Dom}(\Box^{(p,q)}_{b,m}) \subset L_{(p,q),m}^{2}(X) \to L_{(p,q),m}^{2}(X) \), is a self-adjoint operator. The spectrum of \( \Box^{(p,q)}_{b,m} \) denoted
by \( \text{Spec}(\Box_{b,m}^{(p,q)}) \) is a discrete subset of \([0, +\infty)\). For every \( \lambda \in \text{Spec}(\Box_{b,m}^{(p,q)}) \) the eigenspace with respect to \( \lambda \)

\[
\mathcal{H}_{b,m,\lambda}^{p,q}(X) = \left\{ u \in \text{Dom}(\Box_{b,m}^{(p,q)}) : \Box_{b,m}^{(p,q)} u = \lambda u \right\}
\]

is finite dimensional with \( \mathcal{H}_{b,m,\lambda}^{p,q}(X) \subset \Omega_m^{p,q}(X) \) and for \( \lambda = 0 \) we denote by \( \mathcal{H}_{b,m}^{p,q}(X) \) the harmonic space \( \mathcal{H}_{b,m,0}^{p,q}(X) \) for brevity and then we have the Dolbeault isomorphism

\[
\mathcal{H}_{b,m}^{p,q}(X) \simeq H_{b,m}^{p,q}(X).
\]

In particular, we have

\[
\dim H_{b,m}^{p,q}(X) < \infty, \forall m \in \mathbb{Z}, \forall 0 \leq p, q \leq n - 1.
\]

For \( \lambda \geq 0 \), we collect the eigenspace of \( \Box_{b,m}^{(p,q)} \) whose eigenvalue is less than or equal to \( \lambda \) and define

\[
\mathcal{H}_{b,m,\leq \lambda}^{p,q} := \oplus_{\sigma \leq \lambda} \mathcal{H}_{b,m,\sigma}^{p,q}(X), \quad \mathcal{Z}_{b,m,\leq \lambda}^{p,q} := \text{Ker} \partial_{b} \cap \mathcal{H}_{b,m,\leq \lambda}^{p,q}.
\]

The Szegö kernel function of the space \( \mathcal{Z}_{b,m,\leq \lambda}^{p,q} \) is defined as

\[
\Pi_{m,\leq \lambda}^{p,q}(x) := \sum_{j=1}^{d_m} |g_j(x)|^2,
\]

where \( \{g_j\}_{j=1}^{d_m} \) is any orthonormal basis for the space \( \mathcal{Z}_{b,m,\leq \lambda}^{p,q} \).

It is easy to see that

\[
(3) \quad \dim \mathcal{Z}_{b,m,\leq \lambda}^{p,q} = \int_X \Pi_{m,\leq \lambda}^{p,q} dv_X.
\]

The extremal function \( S_{m,\leq \lambda}^{p,q}(y) \) for \( y \in X \) is defined by

\[
S_{m,\leq \lambda}^{p,q}(y) := \sup_{u \in \mathcal{Z}_{b,m,\leq \lambda}^{p,q}, \|u\|=1} |u(y)|^2.
\]

The next lemma is classical in Bergman’s theory of reproducing kernels.

**Lemma 3.5** (c.f. [4]). For any \( y \in X \),

\[
S_{m,\leq \sigma}^{p,q}(y) \leq \Pi_{m,\leq \sigma}^{p,q}(y) \leq \binom{n-1}{p} \binom{n-1}{q} S_{m,\leq \sigma}^{p,q}(y).
\]

In particular,

\[
\int_X S_{m,\leq \sigma}^{p,q}(y) dv_X \leq \dim \mathcal{Z}_{b,m,\leq \lambda}^{p,q} \leq \binom{n-1}{p} \binom{n-1}{q} \int_X S_{m,\leq \sigma}^{p,q}(y) dv_X.
\]

For the proof of the above Lemma, we refer to [4] Page 308, Lemma 4.1.
Remark 3.1. A typical example of compact CR manifold with a transversal CR $S^1$-action is the Grauert tube. Let $M$ be a compact Hermitian manifold of complex dimension $n$, $(L,h) \to M$ be a holomorphic line bundle. Denote by $\Theta$ the curvature of $(L,h)$. Let $X$ be the circle bundle $\{v \in L^* : |v|^2_{h^{-1}} = 1\}$ over $M$. $X$ is a real hypersurface in the complex manifold $L^*$ which is the boundary of the disc bundle $D = \{v \in L^* : |v|^2_{h^{-1}} < 1\}$, with the defining function $\rho = |v|^2_{h^{-1}} - 1$. The Levi form of $\rho$ restricted to the complex tangent plane of $X$ coincides with the pull-back of $\Theta$ through the canonical projection $\pi : X \to M$. It is a well-known fact to the expert (c.f. [9] Theorem 1.2) that

- the space $\Omega^0_m(X)$ can be identified with the space $\Omega^p,q(M,L^m)$,
- for each integer $m$, we get a subcomplex $(\Omega^p_m(X),\overline{\partial}_b)$ which is isomorphic to the Dolbeault complex $(\Omega^p,q(M,L^m),\overline{\partial})$, thus we get that the Kohn-Rossi cohomology group $H^p,q_0(X)$ is isomorphic to the Dolbeault cohomology group $H^p,q(M,L^m)$.

Grauert tube was first introduced by Grauert [15, 16]. Grauert established the identification of sections of line bundle $L$ over $M$ and CR functions on $X$. This identification was used by Zelditch [48] to study the asymptotic expansion of Bergman kernels. Further developments (e.g. the identification of $\Omega^0_m(X)$ with $\Omega^p,q(M,L^m)$ for $q \geq 0$), we refer to [9, 30].

It is worth to point out that, from almost the same proof of Theorem 1.2 in [9], one can get that $H^p,q_{b,m,\leq \lambda}(X) \simeq H^p,q_{\leq \lambda}(M,L^m)$.

Meanwhile, there are also many examples of compact CR manifolds with transversal CR $S^1$-action which are not tube type [19]. For example, let $X = \{(z_1,z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_1 + z_2|^2 + |z_2|^2 = 1\}$ which is a compact CR manifold with a transversal CR $S^1$-action defined by

$$X \times S^1 \to X,$$

$$(z_1,z_2) \mapsto (e^{i\theta}z_1,e^{2i\theta}z_2).$$

The $S^1$-action defined above is locally free and free on a dense, connected open subset $\{(z_1,z_2) \in X : z_1 \neq 0\}$. Note that the CR $S^1$ action on the boundary of a Grauert tube is globally free.

4. Local picture: canonical local coordinates

In this section, we draw the local picture for compact connected CR manifolds with transversal CR $S^1$-action. The following result is due to Baouendi-Rothschild-Treves [3].

Theorem 4.1 (c.f. [3]). Let $X$ be a compact CR manifold of dim$X = 2n - 1$, $n \geq 2$ with a transversal CR $S^1$-action. Let $\langle |\cdot| \rangle$ be the given $T$-rigid Hermitian metric on $X$. For any point $x_0 \in X$, there exists local coordinates $(x_1,\cdots,x_{2n-1}) = (z,\theta) = (z_1,\cdots,z_{n-1},\theta)$, $z_j = x_{2j-1} +
of T

\]

\( (z, \theta) : |z| < \varepsilon, |\theta| < \delta \) of \( x_0 \) such that

\[
T = \frac{\partial}{\partial \theta} \\
Z_j = \frac{\partial}{\partial z_j} + i \frac{\partial \varphi(z)}{\partial z_j} \frac{\partial}{\partial \theta}, j = 1, \ldots, n - 1,
\]

where \( \{Z_j(x)\}_{j=1}^{n-1} \) form a basis of \( T^{1,0}_x X \) for each \( x \in D \) and \( \varphi(z) \in C^\infty(D, \mathbb{R}) \) is independent of \( \theta \). Moreover, on \( D \) we can take \( (z, \theta) \) and \( \varphi \) so that \( (z(x_0), \theta(x_0)) = (0, 0) \) and \( \varphi(z) = \sum_{j=1}^{n-1} \lambda_j |z_j|^2 + O(|z|^3) \), \( \forall (z, \theta) \in D \), where \( \{\lambda_j\}_{j=1}^{n-1} \) are the eigenvalues of Levi-form of \( X \) at \( x_0 \) with respect to the given \( T \)-rigid Hermitian metric on \( X \).

**Remark 4.1.** It was proved in [19] that if \( x_0 \in X_{\text{reg}}, \delta \) can be taken to be \( \pi \), and if \( x_0 \) is not a regular point, say \( x_0 \in X_k, \delta \) can be taken to be any positive number smaller than \( \frac{\pi}{k} \).

From Definition 2.2 by easy computation, one can get that

**Proposition 4.2.** Let \( X \) be a compact CR manifold of dimension \( 2n - 1, n \geq 2 \) with a transversal CR \( S^1 \)-action. Let \( \langle \cdot | \cdot \rangle \) be the given \( T \)-rigid Hermitian metric on \( X \). Let \( D \) be a canonical local patch with canonical coordinates \( (z, \theta, \varphi) \) such that \( (z, \theta, \varphi) \) is trivial at \( x_0 \) as in Theorem 4.1. Suppose that \( X \) is weakly pseudoconvex, then \( i\partial \bar{\partial} \varphi \geq 0 \) as a \((1,1)\)-form on \( \bar{D} \).

**Lemma 4.3** ([19]). Fix \( x_0 \in X \) and let \( D = \bar{D} \times (-\delta, \delta) \subset \mathbb{C}^{n-1} \times \mathbb{R} \) be a canonical local patch with canonical coordinates \( (z, \theta, \varphi) \) such that \( (z, \theta, \varphi) \) is trivial at \( x_0 \). The \( T \)-rigid Hermitian metric on \( D \) induces an Hermitian metric on \( T^{1,0} \) in a standard way. Up to a coordinate transformation if necessary, we can find orthonormal frame \( \{e^j\}_{j=1}^{n-1} \) of \( T^{1,0} \) with respect to the fixed \( T \)-rigid Hermitian metric such that on \( D \), we have \( e^j(x) = e^j(z) = dz_j + O(|z|), \forall x = (z, \theta) \in D, j = 1, \cdots, n - 1 \). Moreover, if we denote by \( dv_X \) the volume form with respect to the \( T \)-rigid Hermitian metric on \( \text{CTX} \), then on \( D \) we have \( dv_X = \lambda(z)dv(z)d\theta \) with \( \lambda(z) \in C^\infty(\bar{D}, \mathbb{R}) \) which does not depend on \( \theta \) and \( dv(z) = 2^{n-1}dz_1 \cdots dz_{2n-2} \).

**Remark 4.2** (c.f. [19]). For any \( x_0 \in X \), let \( D = \bar{D} \times (-\delta, \delta) \subset \mathbb{C}^{n-1} \times \mathbb{R} \) be a canonical local patch with canonical coordinates \( (z, \theta, \varphi) \) such that \( (z, \theta, \varphi) \) is trivial at \( x_0 \). We identify \( \bar{D} \) with an open subset of \( \mathbb{C}^{n-1} \) with complex coordinates \( z = (z_1, \cdots, z_{n-1}) \). Since \( \{dz_j\}_{j=1}^{n-1} \) is a frame of \( T^{1,0}D \) over \( D \), we will treat them as the frame of \( T^{1,0} \bar{D} \) which is the bundle of \((1,0)\)-forms over the domain \( \bar{D} \). Let \( (g^{kj}(z)) \) be the Hermitian metric on \( T^{1,0} \bar{D} \) defined in the proof of Lemma 4.3. It
induces Hermitian metrics on $T^{1,0}\bar{D}$ and $T^{*,p,q}\bar{D}$ in a canonical way. We denote by the induced Hermitian metric on $T^{1,0}\bar{D}$ by $\omega$.

**Remark 4.3.** Under the local canonical coordinate and the metric chosen in Lemma 4.3, one can see that on $\tilde{D}$, $\omega = \sum_{j=1}^{n-1} e^j \wedge \bar{e}^j$, and $\omega(x_0) = \sum_{j=1}^{n-1} dz_j \wedge d\bar{z}_j$. Then the volume form on $\tilde{D}$ is given by $\omega = \sum_{j=1}^{n-1} dz_j \wedge d\bar{z}_j$. Here, $\lambda(z) \in C^\infty(\tilde{D}, \mathbb{R})$ is the function defined in Lemma 4.3.

**5. Local representations of $\partial_b$, $\partial^*_{\bar{b}}$ and $\Box_{b,m}^{(p,q)}$**

By the same calculations as in [19], we have the following local representation of the operators mentioned above.

**Lemma 5.1.** For all $u \in \Omega_{b,m}^{p,q}(X)$, on $D$ we have

$$\partial_b u = e^{im\theta} e^{-m\varphi} \overline{\partial}(e^{m\varphi} e^{-im\theta} u), \partial_{\bar{b}} u = e^{im\theta} e^{-m\varphi} \overline{\partial}^{1,2m\varphi}(e^{m\varphi} e^{-im\theta} u)$$

$$\Box_{b,m}^{(p,q)} u = e^{im\theta} e^{-m\varphi} \Box_{2m\varphi}^{(p,q)}(e^{m\varphi} e^{-im\theta} u).$$

Based on Lemma 5.1, we go a little bit further by direct computations to get the following

**Lemma 5.2.** Suppose that $u \in \Omega_{b,m}^{p,q}(X)$ satisfies $\Box_{b,m}^{(p,q)} u = \lambda u$. We define $\tilde{u} := e^{m\varphi} e^{-im\theta} u$, then $\tilde{u} \in \Omega_{b,m}^{p,q}(\tilde{D})$ and the following equality holds on $\tilde{D}$:

$$\Box_{2m\varphi}^{(p,q)} \tilde{u} = \lambda \tilde{u}.$$ 

Furthermore, for any $u \in \mathcal{H}_{b,m,\leq \sigma}^{p,q}(X)$, we get a form $\tilde{u} \in \mathcal{H}_{2m\varphi,\leq \sigma}^{p,q}(\tilde{D})$, where $\mathcal{H}_{b,m,\leq \sigma}^{p,q}(X)$ (resp. $\mathcal{H}_{2m\varphi,\leq \sigma}^{p,q}(\tilde{D})$) is the linear span of the eigenforms of $\Box_{b,m}^{(p,q)}$ (resp. $\Box_{2m\varphi}^{(p,q)}$) with eigenvalue less than or equal to $\sigma$ on $X$ (resp. on $\tilde{D}$).

Now we recall the so-called Siu’s $\partial\bar{\partial}$-formula. Let $(L, h)$ be a holomorphic Hermitian line bundle over a compact complex $n$-fold $(X, \omega)$ and $\alpha$ be a $L$-valued $(n, q)$-form. The Hodge-$*$ operator is defined by the formula

$$\alpha \wedge \overline{\ast \alpha} = |\alpha|^2 \omega_n,$$

where $\omega_n = \omega^n/n!$. We define an $(n - q, n - q)$-form $T_\alpha$ associated to $\alpha$ in a local trivialization as

$$T_\alpha = c_{n-q} \gamma \wedge \overline{\gamma} e^{-\psi},$$

where $\gamma = \ast \alpha$, $c_{n-q} = i^{(n-q)^2}$ and $\psi$ defines the metric of $L$. Note that the form $T_\alpha$ is well defined globally.
Lemma 5.3 (c.f. [4]). Let $\alpha$ be an $L$-valued $(n, q)$-form. If $\alpha$ is $\overline{\partial}$-closed

$$i\partial\overline{\partial}(T_\alpha \wedge \omega_{q-1}) \geq (-2Re(\square\alpha, \alpha) + \langle \Theta_L \wedge \Lambda\alpha, \alpha \rangle - c|\alpha|^2)\omega_n,$$

where $\Theta_L$ is the curvature of $(L, h)$ and locally $\Theta_L$ can be written as $\Theta_L = i\partial\overline{\partial}\psi$ if $\psi$ is the local potential of $h$, i.e. $h = e^{-\psi}$. The constant $c$ is equal to zero if $\overline{\partial}\omega_{q-1} = \overline{\partial}\omega_q = 0$.

Remark 5.1. The expressions of the operator $\overline{\Theta}$ might be different in the compact case and in the noncompact case. But if we consider it as the formal adjoint of the operator $\overline{\partial}$, the expressions should always be the same. If we consider the formal adjoint, the formula in Lemma 5.3 is pointwise in its nature.

6. A preparation for localization procedure: the scaling technique

In this section, we prepare necessary tools for the localization procedure needed in the proof of Theorem 1.1. This is a slight modification of results in [19], say from forms of $(0, q)$ type to forms of $p, q$ type.

Fix $x_0 \in X$, we take canonical local patch $D = \overline{D} \times (-\delta, \delta) = \{(z, \theta) : |z| < \varepsilon, |\theta| < \delta\}$ with canonical coordinates $(z, \theta, \varphi)$ such that $(z, \theta, \varphi)$ is trivial at $x_0$. In this section, we identify $D$ with an open subset of $\mathbb{C}^{n-1} = \mathbb{R}^{2n-2}$ with complex coordinates $z = (z_1, \cdots, z_{n-1})$. Let $L_1 \in T^{1,0}\overline{D}, \cdots, L_{n-1} \in T^{1,0}\overline{D}$ be the dual frame of $e^1, \cdots, e^{n-1}$ with respect to the $T$-rigid Hermitian metric $\langle \cdot | \cdot \rangle$ defined in Remark 4.2. Let $\omega$ be the induced Hermitian metric on $T^{1,0}\overline{D}$.

Let $\Omega^{p,q}(\overline{D})$ be the space of smooth $(p, q)$-forms on $\overline{D}$ and let $\Omega^{p,q}_0(\overline{D})$ be the subspace of $\Omega^{p,q}(\overline{D})$ whose elements have compact support in $\overline{D}$. Let $\langle \cdot | \cdot \rangle_{2\varphi}$ be the weighted inner product on the space $\Omega^{p,q}_0(\overline{D})$ defined as follows:

$$\langle f | g \rangle = \int_{\overline{D}} (f|g)e^{-2\varphi(z)}\lambda(z)dv(z)$$

where $f, g \in \Omega^{p,q}_0(\overline{D})$ and $\lambda(z)$ is as in Remark 4.2. We denote by $L^2_{(p, q)}(\overline{D}, 2\varphi)$ the completion of $\Omega^{p,q}_0(\overline{D})$ with respect to $\langle \cdot | \cdot \rangle_{2\varphi}$. For $r > 0$, let $\overline{D}_r = \{z \in \mathbb{C}^{n-1} : |z| < r\}$. Here $\{z \in \mathbb{C}^{n-1} : |z| < r\}$ means that $\{z \in \mathbb{C}^{n-1} : |z_j| < r, j = 1, \cdots, n-1\}$. For $m \in \mathbb{N}$, let $F_m$ be the scaling map $F_m(z) = (\frac{z_1}{\sqrt{m}}, \cdots, \frac{z_{n-1}}{\sqrt{m}}), z \in \overline{D}_{\log m}$. From now on, we assume $m$ is sufficiently large such that $F_m(\overline{D}_{\log m}) \subset \subset \overline{D}$. We define the scaled bundle $F_m^*T^{*p,q}\overline{D}$ on $\overline{D}_{\log m}$ to be the bundle whose fiber at
\[ z \in \tilde{D}_{\log m} \]
\[ F_m^* T_s^{p,q} \tilde{D} \big|_z = \left\{ \sum'_{|l|=p, |j|=q} a_{l,j} e^l \left( \frac{z}{\sqrt{m}} \right) \wedge \overline{e^j} \left( \frac{z}{\sqrt{m}} \right) : a_{l,j} \in \mathbb{C}, I, J \text{ strictly increasing} \right\}. \]

We take the Hermitian metric \( \langle \cdot | \cdot \rangle_{F_m^* T_s^{p,q} \tilde{D}} \) on \( F_m^* T_s^{p,q} \tilde{D} \) so that at each point \( z \in \tilde{D}_{\log m} \),
\[ \left\{ e^l \left( \frac{z}{\sqrt{m}} \right) \wedge \overline{e^j} \left( \frac{z}{\sqrt{m}} \right) : |l| = p, |j| = q, I, J \text{ strictly increasing} \right\} \]
is an orthonormal frame for \( F_m^* T_s^{p,q} \tilde{D} \) on \( \tilde{D}_{\log m} \).

Let \( F_m^* \Omega_0^{p,q}(\tilde{D}_r) \) denote the space of smooth sections of \( F_m^* \Omega_0^{p,q}(\tilde{D}_r) \) whose elements have compact support in \( \tilde{D}_r \). Given \( f \in \Omega_0^{p,q}(\tilde{D}_r) \), we write \( f = \sum'_{|j|=q} f_{l,j} e^l \wedge \overline{e^j} \). We define the scaled form \( F_m^* f \in F_m^* \Omega_0^{p,q}(\tilde{D}_{\log m}) \) by
\[ F_m^* f = \sum'_{I,J} f_{I,J} \left( \frac{z}{\sqrt{m}} \right) e^I \left( \frac{z}{\sqrt{m}} \right) \wedge \overline{e^J} \left( \frac{z}{\sqrt{m}} \right)p, z \in \tilde{D}_{\log m}. \]
For brevity, we denote \( F_m^* f \) by \( f(z/\sqrt{m}) \). Let \( P \) be a partial differential operator of order one on \( F_m \tilde{D}_{\log m} \) with \( C^\infty \) coefficients. We write \( P = \sum_{j=1}^{2n-2} a_j(z) \overline{\partial \overline{e}^j} \). The scaled partial differential operator \( P_{(m)} \) on \( \tilde{D}_{\log m} \)
is given by \( P_{(m)} = \sum_{j=1}^{2n-2} F_m^* a_j \overline{\partial \overline{e}^j} \). Let \( f \in C^\infty(F_m(\tilde{D}_{\log m})) \). We can check that
\[ P_{(m)}(F_m^* f) = \frac{1}{\sqrt{m}} F_m^* (P f). \]

Let \( \overline{\partial} : \Omega^{p,q}(\tilde{D}) \to \Omega^{p,q+1}(\tilde{D}) \) be the Cauchy-Riemann operator and we have
\[ \overline{\partial} = \sum_{j=1}^{n-1} e^j(z) \wedge L_j + \sum_{j=1}^{n-1} \langle \overline{\partial e^j}(z) \wedge (\overline{e^j}(z))^* \rangle \]
where \( (\overline{e^j}(z))^* : T_s^{p,q-1} \tilde{D} \to T_s^{p,q-1} \tilde{D} \) is the adjoint of \( \overline{e^j}(z) \wedge \) with respect to the Hermitian metric \( \langle \cdot | \cdot \rangle \) on \( T_s^{p,q-1} \tilde{D} \) given in Remark 4.2.

That is
\[ \langle e^j(z) \wedge u | v \rangle = \langle u | (e^j(z))^* \rangle \]
for all \( u \in T_s^{p,q-1} \tilde{D}, v \in T_s^{p,q} \tilde{D} \). The scaled differential operator \( \overline{\partial}_{(m)} : F_m^* \Omega^{p,q}(\tilde{D}_{\log m}) \to F_m^* \Omega^{p,q+1}(\tilde{D}_{\log m}) \) is given by
\[ \overline{\partial}_{(m)} = \sum_{j=1}^{n-1} e^j \left( \frac{z}{\sqrt{m}} \right) \wedge L_{j,(m)} + \sum_{j=1}^{n-1} \frac{1}{\sqrt{m}} \langle \overline{\partial e^j} \left( \frac{z}{\sqrt{m}} \right) \wedge (\overline{e^j} \left( \frac{z}{\sqrt{m}} \right))^* \rangle. \]
Similarly, \((e^{\lambda}(z)\sqrt{m})\ast : F^*_m T^{*p,q} \tilde{D} \rightarrow F^*_m T^{*p,q-1} \tilde{D}\) is the adjoint of \((e^{\lambda}(z)\sqrt{m})\ast \) with respect to \((\cdot, \cdot)_{F^*_m}, j=1, \cdots, n-1.\) From (6) and (7), \(\partial_{(m)}\) satisfies that

\[
\partial_{(m)} F^*_m f = \frac{1}{\sqrt{m}} F^*_m (\partial f), \ \forall f \in \Omega^{p,q}(F^*_m (\tilde{D}_{log m})).
\]

On the space \(F^*_m \Omega^{p,q}_0(\tilde{D}_{log m}),\) we define the weighted inner product \((\cdot, \cdot)_{2mF^*_m \varphi}\) as follows:

\[
(f | g)_{2mF^*_m \varphi} = \int_{\tilde{D}_{log m}} \langle f | g \rangle_{F^*_m} e^{-2mF^*_m \varphi} \lambda(z) \sqrt{m} dv(z).
\]

Let \(\partial^{*}_{(m)} : F^*_m \Omega^{p,q+1}(\tilde{D}_{log m}) \rightarrow F^*_m \Omega^{p,q}(\tilde{D}_{log m})\) be the formal adjoint of \(\partial_{(m)}\) with respect to the weighted inner product \((\cdot, \cdot)_{2mF^*_m \varphi}\). Let \(\partial^{*}_{=2m\varphi} : \Omega^{p,q+1} \tilde{D} \rightarrow \Omega^{p,q}(\tilde{D})\) be the formal adjoint of \(\partial\) with respect to the weighted inner product \((\cdot, \cdot)_{2m\varphi}.\) Then we also have

\[
\partial^{*}_{(m)} F^*_m f = \frac{1}{\sqrt{m}} F^*_m (\partial^{*}_{=2m\varphi} f), \ \forall f \in \Omega^{p,q}(F^*_m (\tilde{D}_{log m})).
\]

We now define the scaled complex Laplacian \(\Box^{\varphi}_{(m)} : F^*_m \Omega^{p,q}(\tilde{D}_{log m}) \rightarrow F^*_m \Omega^{p,q}(\tilde{D}_{log m})\) which is given by \(\Box^{\varphi}_{(m)} = \partial^{*}_{(m)} \partial_{(m)} + \partial_{(m)} \partial^{*}_{(m)}\). Then we can see that

\[
\Box^{\varphi}_{(m)} F^*_m f = \frac{1}{m} F^*_m (\Box^{\varphi}_{2m\varphi} f), \ \forall f \in \Omega^{p,q}(F^*_m (\tilde{D}_{log m})).
\]

Here

\[
\Box^{\varphi}_{2m\varphi} = \partial^{*}_{=2m\varphi} \partial^{*}_{=2m\varphi} + \partial^{*}_{=2m\varphi} \partial^{*}_{=2m\varphi} : \Omega^{p,q}(\tilde{D}) \rightarrow \Omega^{p,q}(\tilde{D})
\]

is the complex Laplacian with respect to the given Hermitian metric on \(T^{*p,q}(\tilde{D})\) and weight function \(2m\varphi(z)\) on \(\tilde{D}\).

Since \(2mF^*_m \varphi = 2\Phi_0(z) + \frac{1}{\sqrt{m}} O(|z|^3), \ \forall z \in \tilde{D}_{log m},\) where \(\Phi_0(z) = \sum_{j=1}^{n-1} \lambda_j |z_j|^2,\) we have

\[
\lim_{m \rightarrow \infty} \sup_{\tilde{D}_{log m}} |\partial^\alpha (2mF^*_m \varphi - 2\Phi_0)| = 0, \ \forall \alpha \in \mathbb{N}_0^{2n-2}.
\]

Consider \(\mathbb{C}^{n-1}\). Let \((\cdot, \cdot)_{\mathbb{C}^{n-1}}\) be the Hermitian metric with constant coefficients on \(T^{*p,q}\mathbb{C}^{n-1},\) such that at the origin, it is equal to \(\omega(0).\) Let \((\cdot, \cdot)_{2\Phi_0}\) be the \(L^2\) inner product on \(\Omega^{p,q}_0(\mathbb{C}^{n-1})\) given by

\[
(f | g)_{2\Phi_0} = \int_{\mathbb{C}^{n-1}} \langle f | g \rangle e^{-2\Phi_0(z)} \lambda(0) dv(z), \ f, g \in \Omega^{p,q}_0(\mathbb{C}^{n-1}),
\]

where \(\lambda(0)\) is the value of the function \(\lambda(z)\) given in Remark 4.2 at \(x_0.\)
Put
\[ \square^{(p,q)}_{2\Phi_0} = \overline{\partial}^*_{\partial} \cdot 2\Phi_0 + \partial^* \cdot 2\Phi_0 : \Omega^{p,q}(\mathbb{C}^{n-1}) \to \Omega^{p,q}(\mathbb{C}^{n-1}), \]
where \( \overline{\partial}^*_{\partial} \cdot 2\Phi_0 \) is the formal adjoint of \( \overline{\partial} \) with respect to \( \cdot \). From (11) and (12), one can identify
\[ \square^{(p,q)}_{2\Phi_0} \]
on \( \tilde{D}_{\log m} \), where \( \mathcal{P} \) is a second order partial differential operator and all the coefficients of \( \mathcal{P}_m \) are uniformly bounded with respect to \( m \) in \( \mathcal{C}^\mu(\tilde{D}_{\log m}) \) norm for every \( \mu \in \mathbb{N}_0 \) and \( \varepsilon_m \) is a sequence tending to zero as \( m \to \infty \).

From Gårding’s inequality together with Sobolev estimates for elliptic operators \( \square^{(p,q)}_{2\Phi_0} \), one can get the following

**Proposition 6.1** (c.f. [4]). Let \( u \in F_m^* \Omega^{p,q}(\tilde{D}_{\log m}) \). For every \( r > 0 \) with \( D_r \subset \subset \tilde{D}_{\log m} \), and every \( k \in \mathbb{N}^+ \) and \( k > \frac{2}{1 - 2\varepsilon} \), there is a constant \( C_{r,k} \) independent of \( m \) such that
\[ |u(0)|^2 \leq C_{r,k} \left( \|u\|_{2mF_m^{*\partial}, \tilde{D}_r}^2 + \|\square^{(p,q)}_{2\Phi_0} u\|_{2mF_m^{*\partial}, \tilde{D}_r}^2 \right). \]

7. Proof of the Theorem 1.1

Now, after a long way of preparing, we are on the way to prove Theorem 1.1 in the following three steps.

**Step 1.** Fix a point \( x_0 \in X \). From Lemma 4.1 and Remark 4.2 up to a coordinate transformation, we can choose a canonical local patch \( D = \tilde{D} \times (-\delta, \delta) = \{ (z, \theta) : |z| < \varepsilon, |\theta| < \delta \} \) with canonical coordinates \( (z, \theta, \varphi) \) such that \( (z, \theta, \varphi) \) is trivial at \( x_0 \) and the metric \( \omega \) induced by the \( T \)-rigid Hermitian metric on \( X \) be the Hermitian metric satisfies \( \omega = \frac{1}{2} \overline{\partial} \partial |z|^2 = \beta \) at \( x_0 \). Let \( u \in \mathcal{H}^{n-1,q}_{\omega, \beta} \) such that \( \|u\| = 1 \) and \( \overline{\partial}_b u = 0 \). Set \( \tilde{u} = e^{m\varphi} e^{-i m \theta} u \) on \( \tilde{D} \), then from Lemma 5.1 and Lemma 5.2 we know that \( \tilde{u} \in \mathcal{H}^{n-1,q}_{e^{m\varphi}, \beta} \) and \( \overline{\partial} \tilde{u} = 0 \). By the definition and Lemma 5.1 it is easy to show that
\[ |u|^2 = |\tilde{u}|^2 e^{-2m\varphi}, \]
\[ |\square^{(p,q)}_{2\Phi_0} u|^2 = |\square^{(p,q)}_{2\Phi_0} \tilde{u}|^2 e^{-2m\varphi}. \]

We construct a trivial holomorphic Hermitian line bundle \( (L := \tilde{D} \times \mathbb{C}, h := e^{-2m\varphi}) \) over \( \tilde{D} \). From (11) and (12), one can identify \( \tilde{u} \) with an \( L \)-valued \( (n-1,q) \) form on \( \tilde{D} \), i.e. a section of the bundle \( \Omega^{n-1,q} \otimes L \) over \( \tilde{D} \), and \( \square^{(n-1,q)}_{2m\varphi} \) with the formal \( \overline{\partial} \)-Laplacian operator on \( \tilde{D} \) with respect to the induced Hermitian metric \( \omega \) (see Remark 4.2) and the...
Hermitian metric $h$ of $L$ on $\tilde{D}$. For this consideration, we make the following notations throughout this section

\begin{equation}
[\tilde{u}]^2 := |\tilde{u}|^2 e^{-2m\varphi}
\end{equation}

\begin{equation}
[\square^{(n-1,q)}_{2m\varphi}\tilde{u}]^2 := [\square^{(n-1,q)}_{2m\varphi}\tilde{u}]^2 e^{-2m\varphi}.
\end{equation}

Now it is the right time for us to introduce the strategy of Berndtsson [4]. Since $X$ is pseudoconvex, then from Proposition 2.2 we have that $\Theta_L = i\partial\bar{\partial}\varphi \geq 0$.

From Lemma 5.3, we get that

\begin{equation}
i\partial\bar{\partial}(T_{\tilde{u}} \wedge \omega_{q-1}) \geq (-2Re(\square^{(n-1,q)}_{2m\varphi}\tilde{u}, \tilde{u}) - c[\tilde{u}]^2)\omega_{n-1}.
\end{equation}

For $r > 0$ small, we define

$$
\sigma(r) := \int_{|z|<r} [\tilde{u}]^2 \omega_{n-1} = \int_{|z|<r} T_{\tilde{u}} \wedge \omega_q =: s^2(r),
$$

$$
\lambda(r) := \left( \int_{|z|<r} [\square^{(n-1,q)}_{2m\varphi}\tilde{u}]^2 \right)^{1/2}.
$$

From Cauchy’s inequality, we get that

\begin{equation}
\int_{|z|<r} [\square^{(n-1,q)}_{2m\varphi}\tilde{u}] [\tilde{u}] \leq \lambda(r)\sigma(r)^{1/2}.
\end{equation}

Without loss of generality, we may assume that $\lambda \geq 1$.

From (15) we see that

\begin{equation}
\int_{|z|<r} (r^2 - |z|^2) i\partial\bar{\partial}(T_{\tilde{u}} \wedge \omega_{q-1}) \geq -cr^2\sigma(r) - 2r^2 \int_{|z|<r} [\square^{(n-1,q)}_{2m\varphi}\tilde{u}] [\tilde{u}] \omega_{n-1}.
\end{equation}

Applying Stokes’ formula to the left hand side of (16), we get that

\begin{equation}
2 \int_{|z|<r} iT_{\tilde{u}} \wedge \omega_{q-1} \wedge \beta \\
\leq \int_{|z|=r} iT_{\tilde{u}} \wedge \omega_{q-1} \wedge \partial|z|^2 + cr^2\sigma(r) + 2r^2\sigma(r)^{1/2}\lambda(r).
\end{equation}

Since $\omega$ is smooth and $\omega(0) = \beta$, up to shrinking the local patch if necessary, we have that

\begin{equation}
(1 - O(r))\omega \leq \beta \leq (1 - O(r))\omega.
\end{equation}

Note that if $\omega = \beta$, the boundary term in (17) can be estimated by an integral with respect to surface measure

$$r \int_{|z|=r} [\tilde{u}]^2 dS,$$
and this implies that in our case
\[
\int_{|z|=r} iT \wedge \omega q-1 \wedge \partial |z|^2 \leq r(1 - O(r)) \int_{|z|=r} \tilde{w}^2(\omega_{n-1}/\beta_{n-1})dS.
\]

However,
\[
\int_{|z|=r} \tilde{w}^2(\omega_{n-1}/\beta_{n-1})dS = \sigma'(r).
\]

From (17), (18) and (20), we get that
\[
2q(1 - O(1))\sigma(r) \leq r\sigma'(r) + 2r^2\sigma(r)^{1/2}\lambda(r),
\]
by incorporating the term \( cr^2\sigma(r) \) in \( O(r)\).

Dividing by \( 2rs \) to both sides of (21), we obtain
\[
q(1/r - O(1))s(r) \leq s'(r) + r\lambda(r).
\]

We are going to prove
\[
s(r) \leq Cr^k\lambda^{k/2}
\]
for \( k \leq q \) by induction over \( k \).

The statement is trivial for \( k = 0 \). In fact, from (11) and (13), we have that
\[
\sigma(r) = \int_{|z|<r} \tilde{w}^2\omega_{n-1} = \frac{1}{2\delta} \int_{|z|<r, -\delta \leq \theta \leq \delta} |w|^2dv_X \leq \frac{1}{2\delta},
\]
since we have assumed that \( \|u\| = 1 \).

Now we assume that it has been proved for a certain value of \( k < q \). Then (22) implies
\[
(k + 1)(1/r - O(1))s(r) \leq s'(r) + r\lambda(r).
\]

Since \( \tilde{w} \in H^{n-1,q}_{2m\phi;\lambda}(\tilde{D}) \), the form \( \Box_{2m\phi;\lambda} \tilde{w} \) also lies in \( H^{n-1,q}_{2m\phi;\lambda}(\tilde{D}) \), then by the induction hypothesis we get that
\[
\lambda(r) \leq Cr^k\lambda^{k/2+1}.
\]

From (23) and (24), we obtain that
\[
(k + 1)(1/r - O(1))s(r) \leq s'(r) + Cr^{k+1}\lambda^{k/2+1}.
\]

Set
\[
\Phi(r) = (k + 1) \int (1/r - O(1))dr \sim (k + 1)\log r
\]
and multiply (25) by the integrating factor \( e^{-\Phi(r)} \). The result is that
\[
(se^{-\Phi})' \geq -C\lambda^{k/2+1}.
\]

Integrate this inequality from \( r \) to \( \lambda^{-1/2} \). Since \( e^{-\Phi} \sim 1/r^{k+1} \), we get that
\[
r^{-(k+1)}s(r) \leq C\lambda^{k/2+1/2} + s(\lambda^{-1/2})\lambda^{k/2+1/2} \leq C\lambda^{k/2+1/2}.
\]
By induction, we obtain that
\[ s(r) \leq C r^q \lambda^q. \]

After squaring both sides, we obtain that
\[ \int_{|z| < r} \left[ \overline{u} \right]^2 \omega_{n-1} \leq C r^{2q} \lambda^q. \]

Go through the proof given above line by line, one can see that the constant \( C \) only depends on the local coordinate, \( c \) in Siu’s formula (which depends only on the metric \( \omega \)), \( O(1) \) and \( \delta \), but from the compactness of \( X \), one can get a uniform constant \( C \) independent of \( r \), \( m \), \( \lambda \) and the point \( x_0 \).

**Step 2.** In the sequel, we shall use the scaling technique in Section 6.

For any form \( u \in \Omega^{n-1,q}_{b,m}(X) \), we express \( u \) in terms of the trivialization and local canonical coordinates on \( D \) and write \( \overline{u} = e^{m\varphi}e^{-im\theta}u \) on \( \tilde{D} \) as before. Firstly we assume that \( \lambda \leq m \). Put \( \overline{u}^{(m)}(z) = F_m^* \overline{u}(z) = \overline{u}(\frac{z}{\sqrt{m}}) \), so that \( \overline{u}^{(m)} \) is defined for \( |z| < 1 \) if \( m \) is large enough.

We also have the scaled Laplacian \( \square^{(n-1,q)}(m) \), and from \( 8 \), it satisfies
\[ m \square^{(n-1,q)}(m) \overline{u}^{(m)} = F_m^* (\square^{(n-1,q)} \overline{u}) =: (\square^{(n-1,q)} \overline{u})^{(m)}. \]

From \( 10 \), \( \square^{(n-1,q)}(m) \) converges to a \( m \)-independent elliptic operator as \( m \to \infty \) on a neighborhood of \( |z| \leq 1 \).

Therefore, from Proposition 6.1 we obtain that
\[ |u(0)|^2 = (\overline{u})(0) \]
\[ \leq C_k \left( \int_{|z| < 1} \left[ \overline{u}^{(m)} \right]^2 \omega_{n-1}^{(m)} + \int_{|z| < 1} \left[ (\square^{(n-1,q)} k \overline{u}^{(m)}) \right]^2 \omega_{n-1}^{(m)} \right), \]
for \( m \) sufficiently large and \( k > \frac{n-1}{2} \), where \( C_{r,k} \) in Proposition 6.1 depends on \( r \) and \( k \), but here \( C_{r,k} = C_{1,k} =: C_k \) only depends on \( k \) since \( r = 1 \) in 27.

By coordinate transformation formula, we have that
\[ \int_{|z| < 1} \left[ \overline{u}^{(m)} \right]^2 \omega_{n-1}^{(m)} = m^{n-1} \int_{|z| < \frac{1}{\sqrt{m}}} \left[ \overline{u} \right]^2 \omega_{n-1}, \]
and
\[ \int_{|z| < 1} \left[ (\square^{(n-1,q)} \overline{u}^{(m)}) \right]^2 \omega_{n-1}^{(m)} = m^{n-1-2k} \int_{|z| < \frac{1}{\sqrt{m}}} \left[ (\square^{(n-1,q)} \overline{u}) \right]^2 \omega_{n-1}. \]
From (26) in Step 1, we get that

\[ m^{n-1} \int_{|z|<\frac{1}{\sqrt{m}}} |u|^2 \omega_{n-1} \leq Cm^{n-1-q}(\lambda + 1)^q, \]

and

\[ m^{n-1-2k} \int_{|z|<\frac{1}{\sqrt{m}}} [\left(\square_{2m^{q}}^{n-1,q}\right)^2 u]^2 \omega_{n-1} \leq Cm^{n-1-q}(\lambda + 1)^q(\lambda/m)^{2k}. \]

Combining (27), (28) and (29), we obtain that

\[ |u(0)|^2 \leq Cm^{n-1-q}(\lambda + 1)^q. \]

Secondly, if \( \lambda \geq m \), we apply the above procedure to the scaling \( \tilde{u}(\lambda) \) instead, and trivially get

\[ |u(0)|^2 \leq C\lambda^{n-1}. \]

**Step 3.** Since \( \partial_b \) commutes with \( \square_{b,m}^{(p,q)} \), we have the following exact sequence

\[ 0 \rightarrow \mathcal{F}_{b,m, \leq \lambda}^{n-1,q} \xrightarrow{\text{inclusion}} \mathcal{H}_{b,m, \leq \lambda}^{n-1,q} \xrightarrow{\partial_b} \mathcal{F}_{b,m, \leq \lambda}^{n-1,q+1}. \]

Thus we obtain that

\[ \dim \mathcal{H}_{b,m, \leq \lambda}^{n-1,q} \leq \dim \mathcal{F}_{b,m, \leq \lambda}^{n-1,q} + \dim \mathcal{F}_{b,m, \leq \lambda}^{n-1,q+1}. \]

From Lemma 3.5 we see that, for any \( y \in X \)

\[ \dim \mathcal{F}_{b,m, \leq \lambda}^{n-1,q} \leq \binom{n-1}{p} \binom{n-1}{q} \int_X S_{m, \leq \lambda}^{n-1,q}(y) d\nu_X \]

\[ \leq Cm^{n-1-q}(\lambda + 1)^q \]

with \( \lambda \leq m \), and

\[ \dim \mathcal{F}_{b,m, \leq \lambda}^{n-1,q} \leq \binom{n-1}{p} \binom{n-1}{q} \int_X S_{m, \leq \lambda}^{n-1,q}(y) d\nu_X \leq C\lambda^{n-1} \]

with \( \lambda \geq m \).

From (30), (31) and (32), we obtain that for \( \lambda \leq m \)

\[ \dim \mathcal{H}_{b,m, \leq \lambda}^{n-1,q} \leq C\left(m^{n-1-q}(\lambda + 1)^q + m^{n-2-q}(\lambda + 1)^{q+1}\right) \]

\[ \leq Cm^{n-1-q}(\lambda + 1)^q, \]

and for \( \lambda \geq m \),

\[ \dim \mathcal{H}_{b,m, \leq \lambda}^{n-1,q} \leq C\lambda^n. \]

In conclusion, we complete the proof of the Theorem 1.1.
Remark 7.1. In [4], Berndtsson constructed for any $0 \leq q \leq n$, a compact Kähler manifold $M$ of complex dimension $n$ and a semi-positive line bundle over $M$ such that for large $k$,
\[
\dim \mathcal{H}^{p,q}_{L^k} \geq C(\lambda + 1)^{q}k^{n-q}.
\]
Let $X$ be the circle bundle $X := \{ v \in L^1 : |v|_{L^1}^2 = 1 \}$ over $M$ which is of real dimension $2n + 1$. Then from the Theory of Grauert tube in Section 3.1, one can see that for large $m$,
\[
\dim \mathcal{H}^{p,q}_{L^m} \geq C(\lambda + 1)^{q}m^{n-q}.
\]
Thus we can see that in Theorem 1.1, the growth order is sharp.

8. Serre type duality theorem

Let $X$ be a compact CR-manifold of real dimension $2n - 1$, which admits a transversal CR $S^1$-action. Let $u \in \mathcal{H}^{p,q}_{m}(X)$, which means that $\square_{b,m} u = 0$, which is equivalent to the fact that $\overline{\partial}_b u = 0$ and $\partial^*_b u = 0$ from the Hodge theory of $\square_{b,m}$. We define the Hodge-* operator in the CR level by the following
\[
(u|v)dv_X = u \wedge \ast v \wedge \omega_0,
\]
where $u, v \in \Omega^p_m(X)$, $dv_X$ is the volume form on $X$ defined in Lemma 4.3, and $\omega_0$ is the global 1-form associated to the action of $S^1$.

Proposition 8.1. The Hodge-* operator is a complex linear operator
\[
*: \Omega^p_m(X) \to \Omega^{n-1-q,n-1-p}_m(X).
\]

Proof. For the proof, we follow the counterpart for complex manifold case in [22]. Suppose that $u, v \in \Omega^p_m(X)$. For simplicity we denote as follows
\[
A_p = \alpha_1 \cdots \alpha_p, \ B_q = \beta_1 \cdots \beta_q \text{ with } \alpha_1 < \cdots < \alpha_p, \ \beta_1 < \cdots < \beta_q
\]
\[
dz^{A_p} = dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p}, \ dz^{B_q} = dz^{\beta_1} \wedge \cdots \wedge dz^{\beta_q}.
\]
Moreover, for $A_p = \alpha_1 \cdots \alpha_p$, we put
\[
A_{n-1-p} = \alpha_{p+1} \cdots \alpha_{n-1},
\]
where $\alpha_{p+1} < \cdots < \alpha_n$ and $\{\alpha_1, \cdots, \alpha_p, \alpha_{p+1}, \cdots, \alpha_{n-1}\}$ is a permutation of $\{1, \cdots, n-1\}$. Similarly we define $B_{n-1-q}$ for a given $B_q = \beta_1 \cdots \beta_q$.

Then with this notation, we can write $u, v$ in local coordinates by
\[
u = \sum'_{A_p, B_q} u_{A_p, B_q} j(z, \theta)dz^{A_p} \wedge d\overline{z^{B_q}}
\]
\[
v = \sum'_{A_p, B_q} v_{A_p, B_q} j(z, \theta)dz^{A_p} \wedge d\overline{z^{B_q}},
\]
where $dz^A$ and $dz^B$ are $T$-invariant forms.

Set
\[
\text{sgn} \left( \begin{array}{cc} A_p & A_{n-1-p} \\ B_q & B_{n-1-q} \end{array} \right) = \text{sgn} \left( \begin{array}{cccc} \alpha_1 & \cdots & \alpha_p & \alpha_{p+1} & \cdots & \alpha_{n-1} \\ \beta_1 & \cdots & \beta_q & \beta_{q+1} & \cdots & \beta_{n-1} \end{array} \right),
\]

From the definition of $(\cdot|\cdot)$ and similar computations in [22], one can compute that $\ast \bar{v}$ can be represented by
\[
\ast \bar{v} = (i)^{n-1} (1)^{k'} \sum_{A_p, B_q} \text{sgn} \left( \begin{array}{cc} A_p & A_{n-1-p} \\ B_q & B_{n-1-q} \end{array} \right) \cdot g(z) v^A_p A_p (z, \theta) dz^{A_{n-1-p}} \wedge dz^{B_{n-1-q}},
\]
with $k' = (n-1)(n-2)/2 + (n-1)q$, $g(z) = \det(g_{i\overline{j}}(z))$ and
\[
\bar{v}^A_1 \cdots \bar{v}^A_k \ast \bar{v}(z, \theta) = \sum g_1^A_{\lambda_1} \cdots g_k^A_{\lambda_k} g_1^B_{\beta_1} \cdots g_k^B_{\beta_k} v_{\lambda_1} \cdots v_{\lambda_k} \bar{v}_{\beta_1} \cdots \bar{v}_{\beta_k} (z, \theta).
\]

Replacing $\bar{v}$ by $v$ and interchanging $A$ with $B$, we obtain
\[
\ast v = (i)^n (1)^k \sum_{A_p, B_q} \text{sgn} \left( \begin{array}{cc} A_p & A_{n-1-p} \\ B_q & B_{n-1-q} \end{array} \right) \cdot g(z) v^A_p B_q (z, \theta) dz^{B_{n-1-q}} \wedge dz^{A_{n-1-p}},
\]
with $k' = (n-1)(n-2)/2 + (n-1)p$, and
\[
v^A_p B_q (z, \theta) = \sum g_1^A_{\lambda_1} \cdots g_k^A_{\lambda_k} g_1^B_{\beta_1} \cdots g_k^B_{\beta_k} v_{\lambda_1} \cdots v_{\lambda_k} \bar{v}_{\beta_1} \cdots \bar{v}_{\beta_k} (z, \theta).
\]

It is easy to see that the map $v \rightarrow \ast v$ is linear, namely we have
\[
\ast (c_1 u + c_2 v) = c_1 \ast u + c_2 \ast v
\]
for $c_1, c_2 \in \mathbb{C}$.

From (35), it is easy to see that
\[
T(\ast v) = \text{im}(\ast v), \text{ i.e. } \ast v \in \Omega^{n-1-q, n-1-p}(X).
\]
q.e.d.

**Proposition 8.2.** We have $\ast \bar{v} = \ast v$.

**Proof.** It suffices to check this at arbitrarily fixed point $x_0 \in X$. Take canonical local coordinates $(z, \theta)$ so that $(z(x_0), \theta(x_0)) = (0, 0)$ and $g_{\alpha\overline{\beta}}(x_0) = \delta_{\alpha\beta}$. Then $g^A_{\alpha\overline{\beta}}(x_0) = \delta_{\alpha\beta}$, hence we have that $\overline{v^A_p \overline{B_q}(x_0)} = v^A_p \overline{B_q}(x_0)$ and $\overline{v^A_p B_q}(x_0) = v^A_p B_q(x_0)$. Moreover we have
\[
dz^{B_{n-1-q}} \wedge dz^{A_{n-1-p}} = (-1)^{(n-1)(n-1)-p} dz^{A_{n-1-p}} \wedge dz^{B_{n-1-q}}.
\]
Substituting these inequalities into (34) and (35), we get that
\[
\ast v(x_0) = (-1)^{n-1+(n-1)p+n-1-q+(n-1)-p} \ast \overline{v}(x_0) = \ast \overline{v}(x_0).
\]
q.e.d.
By standard arguments, we have the following

**Proposition 8.3.** If \( v \in \Omega^{p,q}_m(X) \), then \( \ast \ast v = (-1)^{p+q} v \).

In the sequel, we will deduce a formula for \( \overline{\partial}_b \).

Let \( u \in \Omega^{p,q-1}_m(X) \) and \( v \in \Omega^{p,q}_m(X) \). Recall that \( \overline{\partial}_b \) is defined by the following formula

\[
(\overline{\partial}_b u | v) = (u | \overline{\partial}_b v).
\]

(36)

By the definition of the \( \ast \)-operator, we have that \( (\overline{\partial}_b u | v) = \int_X \overline{\partial}_b u \wedge \ast \overline{\omega} \wedge \omega_0 \).

Since \( X \) is compact, we have that

\[
\int_X d(u \wedge \ast \overline{\omega} \wedge \omega_0) = 0.
\]

(37)

Since \( u \wedge \ast \overline{\omega} \) is an \((n-1, n-2)\) form, we have that

\[
\partial_b(u \wedge \ast \overline{\omega}) = 0 \quad \text{and} \quad \partial_b(u \wedge \ast \overline{\omega} \wedge \omega_0) = 0.
\]

Then we obtain that

\[
d(u \wedge \ast \overline{\omega} \wedge \omega_0) = \partial_b(u \wedge \ast \overline{\omega} \wedge \omega_0) + \overline{\partial}_b(u \wedge \ast \overline{\omega} \wedge \omega_0) = \overline{\partial}_b u \wedge \ast \overline{\omega} \wedge \omega_0 - (-1)^{p+q} u \wedge \overline{\partial}_b \ast \overline{\omega} \wedge \omega_0.
\]

(38)

\[
(\overline{\partial}_b u | v) = \int_X \overline{\partial}_b u \wedge \ast \overline{\omega} \wedge \omega_0 = (-1)^{p+q} \int_X u \wedge \overline{\partial}_b \ast \overline{\omega} \wedge \omega_0.
\]

(39)

By Proposition 8.3, we have that

\[
\int_X u \wedge \overline{\partial}_b \ast \overline{\omega} \wedge \omega_0 = - \int_X u \wedge \ast \overline{\partial}_b \ast \overline{\omega} \wedge \omega_0.
\]

(40)

Combining (36), (40) and (41), we have that

\[
(\overline{\partial}_b v) = - \partial_b \ast.
\]

(42)

Thus we have proved the following

**Proposition 8.4.** \( \overline{\partial}_b = - \ast \partial_b \ast \).

Now let \( u \in \mathcal{H}^{p,q}_{b,m}(X) \), i.e. \( \overline{\partial}_b u = 0 \) and \( \overline{\partial}_b u = 0 \). Set \( v = \ast \overline{\omega} \). By Proposition 8.4 and (34), we have that \( v \in \Omega^{n-1-p, n-1-q}_{-m}(X) \).

**Lemma 8.5.** \( \overline{\partial}_b v = 0 \) and \( \overline{\partial}_b v = 0 \), which means that

\[
v \in \mathcal{H}^{n-1-p, n-1-q}_{b,-m}(X).
\]
Proof. By direct computations, we have that
\[ \partial_b v = \bar{\partial}_b \ast \bar{u} = (-1)^{p+q+1} \ast \bar{\partial}_b \ast u = (-1)^{p+q} \ast \bar{\partial}_b u = 0. \]
\[ \bar{\partial}_b v = - \ast \partial_b \ast \bar{u} = (-1)^{p+q} \ast \bar{\partial}_b u = 0. \]
Thus we complete the proof of the Lemma. q.e.d.

Now we are on the way to get the following

**Theorem 8.6** (\(= \text{Theorem 1.2} \)). Let \(X\) be a compact CR-manifold of real dimension \(2n - 1\), which admits a transversal \(S^1\)-action. Then the Hodge \(*\)-operator defined by (33) induces a conjugate linear isomorphism
\[ * : H^{p,q}_{b,m}(X) \rightarrow H^{n-1-p,n-1-q}_{b,-m}(X) \]
\[ u \mapsto \ast \bar{u}. \]

In particular, by combining the Hodge theory for \(\Box_{b,m}^{p,q}\), we have the following conjugate line isomorphism in the cohomological level
\[ H^{p,q}_{b,m}(X) \simeq H^{n-1-p,n-1-q}_{b,-m}(X). \]

Proof. Based on Lemma 8.5 we only need to prove that if \(u \neq 0\), then \(\ast \bar{u} \neq 0\), which follows directly from the definition of the \(*\)-operator. q.e.d.

Combining Theorem 1.1 we obtain the following

**Theorem 8.7** (\(= \text{Theorem 1.3} \)). Let \(X\) be a compact CR-manifold of real dimension \(2n - 1\), which admits a transversal \(S^1\)-action. Suppose that \(X\) is a weakly pseudo-convex CR manifold. Then we have that for \(q \geq 1\),
\[ \dim H^{0,q}_{b,-m}(X) \leq Cm^q, \quad \text{as} \quad m \to +\infty. \]

9. Applications

In this section, we introduce some applications of our main results.

9.1. Morse inequalities and Grauert-Riemenschneider criterion. Let \(X\) be compact connected CR manifold of real dimension \(2n-1\), \(n \geq 2\) which admits a transversal CR \(S^1\)-action. Set \(X(q) := \{x \in X | L_x, \text{ has exactly } q \text{ negative eigenvalues and } n - 1 - q \text{ positive eigenvalues}\}\).

In [19], it is proved that the following Morse type inequalities hold.

**Theorem 9.1** (c.f. [19]). Let \(X\) be a compact connected CR manifold with a transversal CR \(S^1\)-action. Assume that \(\dim_{\mathbb{R}} X = 2n - 1\), \(n \geq 2\).
Then for every $q = 0, 1, \cdots, n - 1$, as $m \to +\infty$, we have
\[
\dim H^{0,q}_{b,m}(X) \leq \frac{m^{n-1}}{2\pi^n} \int_{X(q)} |\det L_x|dv_X(x) + o(m^{n-1}),
\]
\[
\sum_{j=0}^{q} (-1)^{q-j} \dim H^{0,j}_{b,m}(X)
\leq \frac{m^{n-1}}{2\pi^n} \sum_{j=0}^{q} (-1)^{q-j} \int_{X(j)} |\det L_x|dv_X(x) + o(m^{n-1}).
\]

In particular, when $q = n - 1$, as $m \to +\infty$, we have the asymptotic Riemann-Roch theorem
\[
\sum_{j=0}^{n-1} (-1)^j \dim H^{0,j}_{b,m}(X)
= \frac{m^{n-1}}{2\pi^n} \sum_{j=0}^{n-1} (-1)^j \int_{X(j)} |\det L_x|dv_X(x) + o(m^{n-1}).
\]

From Theorem 1.2, we get that

**Theorem 9.2** (= Theorem 1.4). Let $X$ be a compact connected CR manifold with a transversal CR $S^1$-action. Assume that $\dim_{\mathbb{R}} X = 2n - 1$, $n \geq 2$. Then for every $q = 0, 1, \cdots, n - 1$, as $m \to +\infty$, we have
\[
\dim H^{n-1,q}_{b,-m}(X) \leq \frac{m^{n-1}}{2\pi^n} \int_{X(n-1-q)} |\det L_x|dv_X(x) + o(m^{n-1}),
\]
\[
\sum_{j=n-1-q}^{n-1} (-1)^{j+q-(n-1)} \dim H^{n-1,j}_{b,-m}(X)
\leq \frac{m^{n-1}}{2\pi^n} \sum_{j=n-1-q}^{n-1} (-1)^{q+j-(n-1)} \int_{X(n-1-j)} |\det L_x|dv_X(x) + o(m^{n-1}).
\]

In particular, when $q = n - 1$, as $m \to +\infty$, we have the asymptotic Riemann-Roch theorem
\[
\sum_{j=0}^{n-1} (-1)^{n-1-j} \dim H^{n-1,j}_{b,m}(X)
= \frac{m^{n-1}}{2\pi^n} \sum_{j=0}^{n-1} (-1)^{n-1-j} \int_{X(n-1-j)} |\det L_x|dv_X(x) + o(m^{n-1}).
\]

The following is the corresponding Morse type inequalities for $m \leq 0$ in [19].
Theorem 9.3 (c.f. [19]). Let $X$ be a compact connected CR manifold with a transversal CR $S^1$-action. Assume that $\dim X = 2n - 1$, $n \geq 2$. For every $q = 0, 1, 2, \cdots, n - 1$, as $m \to -\infty$, we have
\[
\dim H_{b,m}^{0,q}(X) \leq \frac{|m|^{n-1}}{2\pi^n} \int_{X(n-1-q)} |\mathcal{L}_x|dv_x(x) + o(|m|^{n-1}),
\]
\[
\sum_{j=0}^{q} (-1)^{q-j} \dim H_{b,m}^{0,j}(X)
\leq \frac{|m|^{n-1}}{2\pi^n} \sum_{j=0}^{q} (-1)^{q-j} \int_{X(n-1-j)} |\mathcal{L}_x|dv_x(x) + o(|m|^{n-1}).
\]

In particular, when $q = n - 1$, as $m \to -\infty$, we have the following asymptotic Riemann-Roch theorem
\[
\sum_{j=0}^{n-1} (-1)^{j} \dim H_{b,m}^{0,j}(X)
\leq \frac{|m|^{n-1}}{2\pi^n} \sum_{j=0}^{n-1} (-1)^{j} \int_{X(n-1-j)} |\mathcal{L}_x|dv_x(x) + o(|m|^{n-1}).
\]

Applying our Theorem 1.2, we have that

Theorem 9.4 (=Theorem 1.5). Let $X$ be a compact connected CR manifold with a transversal CR $S^1$-action. Assume that $\dim X = 2n - 1$, $n \geq 2$. For every $q = 0, 1, 2, \cdots, n - 1$, as $m \to +\infty$, we have
\[
\dim H_{b,m}^{n-1,q}(X) \leq \frac{m^{n-1}}{2\pi^n} \int_{X(q)} |\mathcal{L}_x|dv_x(x) + o(m^{n-1}),
\]
\[
\sum_{j=n-1-q}^{n-1} (-1)^{q+j-(n-1)} \dim H_{b,m}^{n-1,j}(X)
\leq \frac{m^{n-1}}{2\pi^n} \sum_{j=n-1-q}^{n-1} (-1)^{q+j-(n-1)} \int_{X(j)} |\mathcal{L}_x|dv_x(x) + o(m^{n-1}).
\]

In particular, when $q = n - 1$, as $m \to +\infty$, we have the following asymptotic Riemann-Roch theorem
\[
\sum_{j=0}^{n-1} (-1)^{n-1-j} \dim H_{b,m}^{n-1,j}(X)
\leq \frac{m^{n-1}}{2\pi^n} \sum_{j=0}^{n-1} (-1)^{n-1-j} \int_{X(j)} |\mathcal{L}_x|dv_x(x) + o(m^{n-1}).
\]

From the asymptotic Riemann-Roch theorem in Theorem 9.4 and Theorem 1.1 we conclude that
Theorem 9.5 (=Theorem 1.6). Let \((X, T^{1.0}X)\) be a compact connected CR manifold of dimension \(2n - 1, n \geq 2\), where \(T^{1.0}X\) is the given CR structure on \(X\). Assume that \(X\) admits a transversal CR \(S^1\)-action. If \(X\) is weakly pseudoconvex and strongly pseudoconvex at a point, then

\[
\dim H^{n-1,0}_{b,m}(X) \approx n^{n-1} \text{ as } m \to +\infty.
\]

That is to say, there are a lot of CR sections of the canonical bundle \(K_X\) of \(X\).

By \(A \approx B\), we mean that there is a positive constant \(C\) such that \(C^{-1} \leq A/B \leq C\).

9.2. Application to orbifold. Let us first recall some basics of orbifold (c.f. [11, 31]). Let \(X\) be a Hausdorff space. An \((\mathcal{C}^\infty)\) orbifold chart for an open set \(U \subset M\) is a triple \((\tilde{U}, G, \varphi)\), where \(\tilde{U}\) is a domain in \(\mathbb{R}^n\), \(G\) is a finite group acting effectively as automorphisms of \(\tilde{U}\), and \(\varphi_U : \tilde{U} \to U\) is a continuous map such that \(\varphi \circ \sigma = \varphi\) for all \(\sigma \in G\), inducing a homeomorphism from the quotient space \(\tilde{U}/G\) onto \(U\). An injection between two charts \((\tilde{U}, G, \varphi)\) and \((\tilde{U}', G', \varphi')\) is a \((\mathcal{C}^\infty)\) embedding \(\lambda : \tilde{U} \to \tilde{U}'\) such that \(\varphi' \circ \lambda = \varphi\).

An orbifold atlas on \(X\) is a family \(\mathcal{V} = \{(U_i, G_i, \varphi_i)\}\) of orbifold charts such that \(\mathcal{U} = \{U_i\}\) is a covering of \(X\), where \(U_i = \varphi_i(\tilde{U}_i)\) and given two charts \((\tilde{U}_i, G_i, \varphi_i)\) and \((\tilde{U}_j, G'_j, \varphi'_j)\) and \(x \in U_i \cap U_j\), there exist a chart \((\tilde{U}_k, G_k, \varphi_k)\) with \(x \in U_k\) and injections \(\lambda_{jk} : (\tilde{U}_k, G_k, \varphi_k) \to (\tilde{U}_i, G_i, \varphi_i)\), \(\lambda_{jk} : (\tilde{U}_k, G_k, \varphi_k) \to (\tilde{U}_j, G_j, \varphi_j)\).

An orbifold atlas \(\mathcal{V}'\) is said to be a refinement of \(\mathcal{V}\) if there exists an injection of every chart of \(\mathcal{V}'\) into some chart of \(\mathcal{V}\). An orbifold \(\mathcal{X} = (X, \mathcal{V})\) is a Hausdorff space \(X\) with a (maximal) orbifold atlas \(\mathcal{V}\).

We can assume an additional structure such as orientation, Riemannian metric, almost-complex structure or complex structure, CR structure on every \(\tilde{V}\) in the orbifold atlas \(\mathcal{V}\). We understand the morphisms (and the groups) preserve the specified structure. Thus we can define oriented, Riemannian, almost-complex or complex and CR orbifolds.

Remark 9.1. Let \(X\) be a smooth manifold of real dimension \(m\), and \(G\) be a compact Lie group acting on \(X\) locally free. Then the quotient space \(X/G\) is an orbifold. Conversely, any orbifold \(X\) can be realized in this way, with \(G = O(m)\), the orthogonal group of degree \(m\) over \(\mathbb{R}\).

An orbifold vector bundle \(E\) over an orbifold \(\mathcal{X} = (X, \mathcal{V})\) is defined as follows: \(E\) is an orbifold and for \(U \in \mathcal{U}\), \((\tilde{E}_U, G^E_U, \varphi_U) : \tilde{E}_U \to \tilde{U}\) is a \(G^E_U\)-equivariant vector bundle and \((\tilde{E}_U, G^E_U, \varphi_U)\) (resp. \((\tilde{U}, G_U = G^E_U/K^E_U, \varphi_U)\), \(K^E_U = \text{Ker}(G^E_U \to \text{Diffeo}(\tilde{U}))\)) is the orbifold structure on \(E\) (resp. \(X\)). If \(G^E_U\) acts effectively on \(\tilde{U}\) for \(U \in \mathcal{U}\), we call \(E\) a proper orbifold vector bundle.
**Remark 9.2.** Let $E$ be an orbifold vector bundle on $(X, V)$. For $U \in \mathcal{U}$, let $\tilde{E}^{pr}_U$ be the maximal $K^E_U$-invariant subbundle of $\tilde{E}_U$ on $\tilde{U}$. Then $(G_U, \tilde{E}^{pr}_U)$ defines a proper orbifold vector bundle on $(X, V)$, denoted by $E^{pr}$.

From the above Remark, without loss of generality, we only consider proper orbifold vector bundle throughout this paper.

Let $E \to X$ be an orbifold vector bundle. A section $s : X \to E$ is called $C^k$ if for each $U \in \mathcal{U}$, $s|_U$ is covered by a $G^E_U$-invariant $C^k$ section $\tilde{s}_U : \tilde{U} \to \tilde{E}_U$.

If $X$ is an oriented orbifold, we define the integral $\int_X \Phi$ for a form $\Phi$ over $X$ as follows: if $\text{Supp}(\Phi) \subset U \in \mathcal{U}$, then

$$\int_X \Phi := \frac{1}{|G_U|} \int_{\tilde{U}} \tilde{\Phi}_U.$$

Now let $(X, T^{1.0}X)$ be a CR orbifold, i.e., there is an orbifold atlas $\mathcal{V}$ on $X$, such that for every $(\tilde{U}, G, \varphi) \in \mathcal{V}$, there is a CR structure $(\tilde{U}, T^{1.0}\tilde{U})$ so that the group $G$ acts on and preserves the CR structure on $\tilde{U}$.

In a similar way, one can define the $\overline{\partial}_b$-complex on $X$. Furthermore, Theorem 3.2, Theorem 3.3, Theorem 3.4 also hold for compact CR orbifold with transversal CR $S^1$-action. Since on every orbifold chart $(\tilde{U}, G, \varphi)$, the $S^1$-action generates a $G$-invariant vector field $T$ on $\tilde{U}$ which preserves the CR structure on $\tilde{U}$ and satisfies the transversal condition, then by almost the same arguments as in [3], we can see that Baouendi-Roth-Treves’ Theorem 4.1 on the existence of canonical local coordinates also holds on the CR orbifold setting. Then similarly, we have the following

**Theorem 9.6** (==Theorem 1.7). Let $(X, T^{1.0}X)$ be a compact connected CR orbifold of dimension $2n - 1$, $n \geq 2$, where $T^{1.0}X$ is the given CR structure on $X$. Assume that $X$ admits a transversal CR $S^1$ action and $X$ is weakly pseudoconvex. Then for $m$ sufficiently large, if $0 \leq \lambda \leq m$,

$$\dim \mathcal{H}^n_{b,m,\leq \lambda} \leq C(\lambda + 1)^q m^{n-1-q},$$

and if $1 \leq m \leq \lambda$,

$$\dim \mathcal{H}^n_{b,m,\leq \lambda} \leq C\lambda^{n-1}.$$

Here we omit the details and just give a sketch of the proof. We take the advantage that the key estimate in our proof of Theorem 1.1 is a pointwise estimate. We work on a local canonical orbifold chart $(\tilde{U}, G, \varphi)$. Note that everything on $\tilde{U}$ is invariant under the group action $G$, then the desired estimate on the orbifold chart above can be achieved by the same method as before, which can be naturally pushed-down to
by the invariance under the group action. Then from the compactness of the orbifold, we can complete the proof of Theorem 9.6.

**Remark 9.3.** Similarly, Theorem 8.6 and Theorem 1.3 can also be generalized to CR orbifold setting.

Let $M$ be a compact connected complex manifold and $G$ be a compact Lie group acts analytically on $M$. We assume that the action of $G$ on $M$ is locally free. Then $M/G$ is a complex orbifold. Suppose that $\dim C(M/G) = n$. Let $L \to M$ be a $G$-invariant holomorphic line bundle over $M$, i.e., the transition functions of $L$ are $G$-invariant. Suppose that $L$ admits a locally free $G$-action compatible with that on $M$, i.e., an action $(g, v)(\in G \times L) \mapsto g \circ v \in L$ with the property $\pi(g \circ v) = g \circ (\pi(v))$ where $\pi : L \to M$ is the bundle projection. Then $L/G$ is an orbifold holomorphic line bundle over $M/G$.

The $G$-action on $L$ can be naturally extended to $L^m := L^m \otimes m$ and $L^* = (L^m)^*$. Then $L^m/G$ and $L^*/G$ are also orbifold holomorphic line bundles over $M/G$. Put for $p, q = 0, \cdots, n$,

$$\Omega^{p,q}(M/G, L^m/G) := \{u \in \Omega^{p,q}(M, L^m) : g^* u = u, \forall g \in G\}.$$

Since the Cauchy-Riemann operator is $G$-invariant, we have the $\bar{\partial}$-complex

$$\bar{\partial}, \Omega^{p,\bullet}(M/G, L^m/G), p = 0, 1, \cdots, n,$$

and the $(p, q)$-th Dolbeault cohomology group:

$$H_{b,p,q}(X) := \ker \bar{\partial} : \Omega^{p,q}(M/G, L^m/G) \to \Omega^{p,q+1}(M/G, L^m/G) / \text{im} \bar{\partial} : \Omega^{p,q-1}(M/G, L^m/G) \to \Omega^{p,q+1}(M/G, L^m/G).$$

Take any $G$-invariant Hermitian metric $h_L$ on $L$, it induces a $G$-invariant Hermitian metric $h_{L^*}$ on $L^*$, set $\tilde{X} = \{v \in L^* | |v|^2_{h_{L^*}} = 1\}$. Then $X = \tilde{X}/G$ is a compact CR orbifold, and the natural $S^1$-action on $\tilde{X}$ induces a locally free $S^1$-action on $X$ which can be verified that the action is CR and transversal.

**Theorem 9.7 (= Theorem 1.8).** For every $p, q = 0, 1, \cdots, n$ and every $m \in \mathbb{Z}$, there is a bijective map $A_{m}^{(p, q)} : \Omega_{m}^{(p, q)}(X) \to \Omega^{(p, q)}(M, L^m)$ such that $A_{m}^{(p, q + 1)} - \bar{\partial} A_{m}^{(p, q)}$ on $\Omega_{m}^{(p, q)}(X)$. Thus we have that

$$\Omega^{p,q}_{m}(X) \simeq \Omega^{p,q}(M/G, L^m/G)$$

$$H_{b,p,q}(X) \simeq H_{b,p,q}(M/G, L^m/G).$$

In particular, $\dim H_{b,p,q}(X) < \infty$.

**Proof.** This is a small modification of the proof for the case $p = 0$ in [9]. The local orbifold structure of $L/G$ is the following commutative
diagram:

\[
\begin{array}{ccc}
\tilde{U}^* & \xrightarrow{\varphi^*} & U^* \\
\pi_{\tilde{U}^*} & & \pi_U \\
\tilde{U} & \xrightarrow{\varphi} & U,
\end{array}
\]

where \( \tilde{U}^* = \tilde{U} \times \mathbb{C} \), \( \pi_{\tilde{U}^*} \) and \( \pi_U \) are the projections, \( (\tilde{U}^*, G_{\tilde{U}^*}, \varphi^*) \) and \( (\tilde{U}, G_{\tilde{U}}, \varphi) \) are orbifold charts of \( L/G \) and \( M/G \) respectively. Let \( s \) be a local trivializing section of \( L/G \) defined on \( U \), it corresponds to a section \( \tilde{s} \) of \( \tilde{U}^* \) on \( \tilde{U} \) with the property \( (g \circ \tilde{s})(x) = \tilde{s}(g \circ x) \) for \( g \in G \) and \( x \in \tilde{U} \). Let \( |\tilde{s}|^2_{h^L} = e^{-\tilde{\psi}} \) on \( \tilde{U} \). Since \( h^L \) is a \( G \)-invariant Hermitian metric on \( L \), we have that \( \tilde{\psi} \) is a \( G \)-invariant function on \( \tilde{U} \) and \( G \) acts on \( \tilde{X} \) naturally. The local orbifold structure of \( X \) is the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{U} \times S^1 & \xrightarrow{\varphi^*} & (\tilde{U} \times S^1)/G_{\tilde{U}^*} \\
\pi_{\tilde{U}} & & \pi_U \\
\tilde{U} & \xrightarrow{\varphi} & U,
\end{array}
\]

where \( (\tilde{U} \times S^1)/G_{\tilde{U}^*} \subset X \) and \( U \subset M/G \).

We identify \( \tilde{U} \) with an open set of \( \mathbb{C}^n \), and introduce holomorphic coordinates \( z = (z_1, \cdots, z_n) \) on \( \tilde{U} \). We have the local diffeomorphism

\[
\tau: \tilde{U} \times (-\varepsilon_0, \varepsilon_0) \to X, \quad 0 < \varepsilon_0 \leq \pi,
\]

\[(z, \theta) \mapsto e^{-\tilde{\psi}} \tilde{s}^*(z)e^{-i\theta}.
\]

We understand the image of the map \( \tau \) is contained in an orbifold chart of \( X \).

Put \( D = \tilde{U} \times (-\varepsilon_0, \varepsilon_0) \) as a canonical coordinate patch with \( (z, \theta) \) canonical coordinates (with respect to the trivialization \( \tilde{s} \) on \( \tilde{U} \)) such that on \( D \), the global real vector field \( T \) induced by the \( S^1 \)-action is \( \frac{\partial}{\partial \theta} \) and

\[
T^{1,0}\tilde{U} = \{ \frac{\partial}{\partial z_j} - i \frac{\partial \tilde{\psi}^*}{\partial z_j}(z) \frac{\partial}{\partial \theta}; j = 1, 2, \cdots, n \},
\]

\[
T^{0,1}\tilde{U} = \{ \frac{\partial}{\partial z_j} + i \frac{\partial \tilde{\psi}^*}{\partial z_j}(z) \frac{\partial}{\partial \theta}; j = 1, 2, \cdots, n \},
\]

and

\[
T^{1,0}\tilde{U}^* = \{ dz_j; j = 1, 2, \cdots, n \}, T^{0,1}\tilde{U} = \{ d\overline{z}_j; j = 1, 2, \cdots, n \}.
\]
We first define a local map $A^{p,q}_m(D) : \Omega^{p,q}_m(D) \to \Omega^{p,q}(\tilde{U}, L^m)$ as follows

$$A^{p,q}_m(D) : \Omega^{p,q}_m(D) \to \Omega^{p,q}(\tilde{U}, L^m)$$

$$u(z, \theta) \mapsto \tilde{s}^m(z)e^{m\tilde{\psi}(z)}u(z, \theta)e^{-im\theta}.$$  

Note that $u(z, \theta)e^{-im\theta}$ is a form independent of $\theta$. It is easy to see that this map is a bijective. We will prove that $A^{p,q}_m(D)$ can be patched to a global operator $A^{(p,q)}_m : \Omega^{(p,q)}_m(X) \to \Omega^{(p,q)}(M/G, L^m/G)$.

Let $s$ and $s_1$ be local trivializing sections of $L/G$ on an open set $U \subset M/G$, and $\tilde{s}$ and $\tilde{s}_1$ be the corresponding sections on $\tilde{U}$. Let $(z, \theta) \in \mathbb{C}^n \times \mathbb{R}$ and $(z, \eta) \in \mathbb{C}^n \times \mathbb{R}$ be canonical coordinates of $D$ with respect to $\tilde{s}$ and $\tilde{s}_1$ respectively. Set $|\tilde{s}|^2_{h_L} = e^{-2\tilde{\psi}}$ and $|\tilde{s}_1|_{h_L}^2 = e^{-2\tilde{\psi}_1}$. It suffices to prove that

$$\tilde{s}^m(z)e^{m\tilde{\psi}(z)}u(z, \theta)e^{-im\theta} = \tilde{s}_1^m(z)e^{m\tilde{\psi}_1(z)}u_1(z, \eta)e^{-im\eta}$$

Observe that the above equality is invariant under the $G$-action.

Let $\tilde{s}_1 = g\tilde{s}$ for $g$ a unit on $\tilde{U}$. It is easy to find that

$$\tilde{\psi}_1 = \tilde{\psi} - \text{log}|g|.$$  

Note that if $\tau(z, \theta) = \tau_1(z, \eta)$, then with a certain branch of square root, we have

$$e^{-i\theta}\left(\frac{g(z)}{\overline{g}(z)}\right)^{\frac{1}{2}} = e^{-i\eta}.$$  

From (46) and (47), we can easily verify that (45) holds. Thus one proves that $A^{(p,q)}_m(D)$ can be patched to a global operator $A^{(p,q)}_m : \Omega^{(p,q)}_m(X) \to \Omega^{(p,q)}(M/G, L^m/G)$.

In the following, we prove that $A^{(p,q+1)}_m\overline{\partial}h_m = \overline{\partial}A^{(p,q)}_m$ on $\Omega^{(p,q)}_m(X)$. Denote by $\tilde{u}(z) = u(z, \theta)e^{-im\theta}$, which was previously known to be independent of $\theta$. From (43) and (44), by direct computations, one can see that

$$\overline{\partial}_b\tilde{u} = \sum_{j=1}^n e^{im\tilde{\psi}}d\overline{\psi}_j \wedge \left(\frac{\overline{\partial}\tilde{u}}{\overline{\partial}\overline{\psi}_j}(z) + m\frac{\partial\tilde{\psi}}{\partial\overline{\psi}_j}(z)\tilde{u}(z)\right).$$

Then we have

$$A^{p,q+1}_m(D)(\overline{\partial}_b u) = \tilde{s}^m(z)e^{m\tilde{\psi}(z)}\sum_{j=1}^n d\overline{\psi}_j \wedge \left(\frac{\overline{\partial}\tilde{u}}{\overline{\partial}\overline{\psi}_j}(z) + m\frac{\partial\tilde{\psi}}{\partial\overline{\psi}_j}(z)\tilde{u}(z)\right)$$

$$= \tilde{s}^m(z)\overline{\partial}(e^{m\tilde{\psi}}\tilde{u}(z)) = \overline{\partial}A^{(p,q)}_m u.$$  

The last equality follows from the fact that $\tilde{s}$ is holomorphic.

Since $u$ is $G_\tilde{s}$ invariant, $\tilde{s}$ and $\tilde{\psi}$ are $G_\tilde{U}$ invariant, and the actions commute with $\overline{\partial}$, we can see that $A^{p,q+1}_m(D)(\overline{\partial}_b u)$ is a well-defined local
section of $L_m$ over $\bar{U}$, which can be patched together to a global section of $(L^m/G)\bar{U}$ over $M/G$ by the previous proof.

The proof of Theorem 9.7 is completed. q.e.d.

Combining Theorem 9.6 and Theorem 9.7, we can get the following

**Theorem 9.8** (= Theorem 1.9). Let $M$ be a compact complex manifold and $G$ a compact Lie group. Suppose that $G$ acts on $M$ analytically, locally free and $\dim_{\mathbb{C}} M/G = n$. Let $(L, h_L)$ be a $G$-invariant holomorphic Hermitian line bundle over $M$. Suppose that $L$ admits a locally free $G$-action compatible with $M$ and the curvature of $L$ is semi-positive. Then we have that for $m$ sufficiently large,

$$\dim H^{n,q}(M/G, L^m/G) \leq C m^{n-q},$$

where $C$ is a constant independent of $m$.

**Remark 9.4.** The above theorem corresponds to Berndtsson’s estimate in the orbifold case, which answers a folklore open question, say generalizing Berndtsson’s estimate to the orbifold setting, informed to us by Hsiao.

**Acknowledgement.** The authors would like to thank Prof. Jih-Hsin Cheng and Prof. Chin-Yu Hsiao for their nice talks given at the Institute of Mathematics in Chinese Academy of Sciences and for their valuable comments on this paper. The first author would like to thank Dr. Xiaoshan Li for helpful discussions on the Serre-type duality theorem.

**References**

[1] T. Akahori, *A New Approach to the Local Embedding Theorem of CR-Structure for $n \geq 4$*, Memoirs of the A. M. S., No. 366, Providence, Rhode Island, 1987, MR 0888499, Zbl 0628.32025.

[2] A. Andreotti and Y.-T. Siu, *Projective embeddings of pseudoconcave spaces*, Ann. Sc. Norm. Super. Pisa, 24 (1970), 231–278, MR 0265633, Zbl 0195.36901.

[3] M. Baouendi, L. Rothschild and F. Treves, *CR structures with group action and extendability of CR functions*, Invent. Math., 83 (1985), 359–396, MR 0809720, Zbl 0598.32019.

[4] B. Berndtsson, *An eigenvalue estimate for the $\bar{\partial}$-Laplacian*, J. Differential Geom. 60, (2003), no. 2, 295–313, MR 1938114, Zbl 1042.58015.

[5] J. Bland, *Contact geometry and CR structures on $S^3$*, Acta Math. 172(1994), 1–49, MR 1263996, Zbl 0814.32002.

[6] A. Bogges, *CR manifolds and the Tangential Cauchy-Riemann Complex*, CRC Press, 1991, MR 1211412, Zbl 0760.32001.

[7] L. Boutet de Monvel, *Intégration des équations de Cauchy-Riemann induites formelles*, Séminaire Goulaouic-Lions-Schwartz, Exposé No. 9 (1974–1975), MR 0409893, Zbl 0317.58003.

[8] D. Burns, *Global behavior of some tangential Cauchy-Riemann equations*, In: Partial Differential Equations and Geometry (Proc, Conf., Park City, Utah, 1977), pp. 51–56, Lecture Notes in Pure and Applied Mathematics, 48, Dekker, New York (1979), MR 0535588, Zbl 0405.32006.
[9] J. Cheng, C. Hsiao and I. Tsai, *Index theorems on CR manifolds with $S^1$-action*, arXiv:1511.00063.

[10] S. Chen and M. Shaw, *Partial differential equations in several complex variables*, AMS/IP Studies in Advanced Mathematics, 19, American Mathematical Society, Providence, RI; International Press, Boston, MA, 2001, MR 1800297, Zbl 0963.32001.

[11] D. Coman and G. Marinescu, *Convergence of Fubini-Study currents for orbifold line bundles*. Internat. J. Math. 24, 1350051 (2013), MR 3084732, Zbl 1280.32011.

[12] R. Du and S. S.-T. Yau, *Kohn-Rossi cohomology and its application to the complex Plateau problem, III*. J. Differential Geom. 90 (2012), no. 2, 251–266, MR 2899875, Zbl 1254.32051.

[13] C. Epstein, *CR-structures on three dimensional circle bundles*. Invent. Math. 109 (1992), 351–403, MR 1172695, Zbl 0786.32013.

[14] G. Folland and J. J. Kohn, *The Neumann problem for the Cauchy-Riemann complex*, Ann. of Math. Studies, No. 75, Princeton University Press, New Jersey, 1972, MR 0461588, Zbl 0247.35093.

[15] H. Grauert, *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann., 146 (1962), 331–368, MR 0137127, Zbl 0178.42702.

[16] H. Grauert, *Theory of $q$-convexity and $q$-concavity*, Several complex variables VII. In: Grauert, H., Peternelle, Th, Remmert, R. (eds.) Encyclopedia of Mathematical Sciences, vol. 74. Springer, Berlin (1994), MR 1326623, Zbl 0806.32004.

[17] C. Hsiao, *Szegö kernel asymptotics for high power of CR line bundles and Kodaira embedding theorems on CR manifolds*, Mem. Amer. Math. Soc. 254 (2018), no. 1217, v+142 pp. ISBN: 978-1-4704-4101-2; 978-1-4704-4750-2, MR 3796426, Zbl 06951627.

[18] C. Hsiao and X. Li, *Szegö kernel asymptotics and Morse inequalities on CR manifolds with $S^1$-action*, Asian J. Math. 22 (2018), 413–450, MR 3845085, Zbl 1398.32002.

[19] C. Hsiao and X. Li, *Morse inequalities for Fourier components of Kohn-Rossi cohomology of CR manifolds with $S^1$-action*, Math. Z. 284 (2016), 441–468, MR 3545500, Zbl 1352.32013.

[20] H. Jacobowitz, *An Introduction to CR Structures*, vol. 32, Amer. Math. Soc., Providence, Rhode Island, 1990, MR 1067341, Zbl 0712.32001.

[21] H. Jacobowitz and F. Treves, *Non-realizable CR structures*, Invent. Math., 66 (1982), 231–249, MR 0656622, Zbl 0487.32015.

[22] K. Kodaira, *Complex manifolds and deformation of complex structures*. Translated from the Japanese by Kazuo Akao. With an appendix by Daisuke Fujikawa. Grundlehen der Mathematischen Wissenschaften, 283. Springer-Verlag, New York, 1986. x+465 pp., MR 2109686, Zbl 1058.32007.

[23] J. J. Kohn, *The range of Cauchy-Riemann operator*, Duke Math. J. 53(2) (1986), 525–545, MR 0850548, Zbl 0609.32015.

[24] J. J. Kohn and H. Rossi, *On the extension of holomorphic functions from the boundary of a complex manifold*, Ann. Math., 81 (1965), 451–472, MR 0177135, Zbl 0166.33802.

[25] M. Kuranishi, *Strongly pseudoconvex CR structures over small balls, Part I, An a priori estimate*, Ann. Math. , 115(1982), 451–500, MR 0657236, Zbl 0505.32018.

[26] M. Kuranishi, *Strongly pseudoconvex CR structures over small balls, Part II, A regularity theorem*, Ann. Math., 116(1982), 1–64, MR 0662117, Zbl 0505.32019.
[27] M. Kuranishi, *Strongly pseudoconvex CR structures over small balls, Part III, An embedding theorem*, Ann. Math., **116**(1982), 249–330, MR 0672837, Zbl 0576.32033.

[28] L. Lempert, *On three dimensional Cauchy-Riemann manifolds*, J. Amer. Math. Soc. **5**(1992), 923–969, MR 1157290, Zbl 0781.32014.

[29] H. Luk and S. S.-T. Yau, *Invariant Kohn-Rossi cohomology and obstruction to embedding of compact real $(2n-1)$-dimensional CR manifolds in $\mathbb{C}^N$*, J. Math. Soc. Japan **48** (1996), no. 1, 61–96, MR 1361547, Zbl 0855.32011.

[30] X. Ma and G. Marinescu, *The first coefficients of the asymptotic expansion of the Bergman kernel of the spin' Dirac operator*, Internat. J. Math. **17**(2006), 737–759, MR 2246888, Zbl 1106.58018.

[31] X. Ma and G. Marinescu, *Holomorphic Morse inequalities and Bergman kernels*, Progress in Math., vol. 254, Birkhäuser, Basel, 2007, 422 pp, MR 2339952, Zbl 1135.32001.

[32] J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Studies, No. 61, Princeton University Press, New Jersey, 1968, MR 0239612, Zbl 0184.48405.

[33] I. Naruki, *On Hodge structure of isolated singularity of complex hypersurfaces*, Proc. Japan Acad., **50**(1974), 334–336, MR 0463492, Zbl 0302.32014.

[34] I. Nakuri, *A note on isolated singularity. I*, Proc. Japan Acad. **51**(1975), 317–319, MR 0430305, Zbl 0333.32006.

[35] I. Nakuri, *A note on isolated singularity. II*, Proc. Japan Acad. **51**(1975), 380–383, MR 0430306, Zbl 0333.32007.

[36] L. Nirenberg, *On a question of Hans Lewy*, Russian Math. Surveys, **29**(1974), 251–262, MR 0410042, Zbl 0305.35017.

[37] H. Rossi, *Attaching analytic spaces to an analytic space along a pseudoconvex boundary*, In: Proceedings of Conference on Complex Manifolds, pp. 242–256. Springer, New York (1965), Zbl 0143.30301.

[38] Y.-T. Siu, *A vanishing theorem for semi-positive line bundles over non-Kähler manifolds*, J. Differential. Geom. vol. 19, (1984), 431–452, MR 0755233, Zbl 0577.32031.

[39] N. Tanaka, *On strongly pseudo-convex manifolds*, Proc. Japan Acad., **50**(1974), 416–419, MR 0372251, Zbl 0302.32019.

[40] N. Tanaka, *A differential geometric study on strongly pseudo-convex manifolds*. Lecture in Mathematics, Department of Mathematics, Kyoto University, No. 9, Kinokuniya Book-Store Co., Ltd., Tokyo, 1975, MR 0399517, Zbl 0331.53025.

[41] S. Webster, *On the local solution of the tangential Cauchy-Riemann equations*, Annals Inst. H. Poincaré (Anl), **6**(1989), 167–182, MR 0995503, Zbl 0679.32019.

[42] S. Webster, *On the proof of Kuranishi’s Embedding Theorem*, Annals Inst. H. Poincaré (Anl), **6**(1989), 183–207, MR 0995504, Zbl 0679.32020.

[43] S. S.-T. Yau, *Kohn-Rossi cohomology and its application to the complex Plateau problem. I*, Ann. of Math. (2) **113** (1981), no. 1, 67–110, MR 0604043, Zbl 0464.32012.

[44] S. S.-T. Yau, *Rigidity of CR morphisms between compact strongly pseudoconvex CR manifolds*. J. Eur. Math. Soc. (JEMS) **13**(2011), no. 1, 175–184, MR 2735079, Zbl 1202.32035.

[45] S. S.-T. Yau and H. Zuo, *Thom-Sebastiani properties of Kohn-Rossi cohomology of compact connected strongly pseudoconvex CR manifolds*. Sci. China Math. **60** (2017), no. 6, 1129–1136, MR 3647139, Zbl 1375.32057.
[46] S. S.-T. Yau and H. Zuo, *Recent results on rigidity of CR morphisms between compact strongly pseudoconvex CR manifolds*. Proceedings of the Sixth International Congress of Chinese Mathematicians. Vol. I, 173–197, Adv. Lect. Math. (ALM), 36, Int. Press, Somerville, MA, 2017, MR 3702075, Zbl 1379.32031.

[47] S. S.-T. Yau and H. Zuo, *Kohn-Rossi cohomology and nonexistence of CR morphisms between compact strongly pseudoconvex CR manifolds*. J. Diff. Geom., 111(2019), 567–580, MR 3934600, Zbl 07036516.

[48] S. Zelditch, *Szegö kernels and a theorem of Tian*, Internat. Math. Res. Notices. (1998), 317–331, MR 1616718, Zbl 0922.58082.

**School of Mathematical Sciences**  
**Beijing Normal University**  
**Laboratory of Mathematics and Complex Systems**  
**Ministry of Education**  
**Beijing 100875**  
**P. R. China**  

_E-mail address:_ zhiwei@bnu.edu.cn

**Institute of Mathematics**  
**Academy of Mathematics and Systems Sciences**  
**and Hua Loo-Keng Key Laboratory of Mathematics**  
**Chinese Academy of Sciences**  
**Beijing 100190**  
**P. R. China**

**School of Mathematical Sciences**  
**University of Chinese Academy of Sciences**  
**Beijing 100049**  
**P. R. China**  

_E-mail address:_ xyzhou@math.ac.cn