A Perturbative Improvement of the Hierarchical Approximation

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ABSTRACT

We propose a perturbative improvement of the hierarchical approximation for gaussian models. The procedure is based on a relabeling of the momenta which allows one to express the symmetries of the hierarchical model using a simple multiplication group. The representations of this group are used to expand the action. The perturbative expansion is treated as a problem of symmetry breaking using Ward identity techniques.
1. Introduction

In many respects, gauge theories provide a satisfactory framework to describe the interactions among elementary particles at energies accessible with existing colliders. Nevertheless, the perturbative methods which are spectacularly accurate for QED at low energy are not adequate to take into account large distance effects among strongly interacting particles. The renormalization group (RG) method is an essential tool to understand and describe these phenomena and, more generally, the infra-red behavior of a field theory. However, the practical implementation of this method requires approximations which are usually difficult to control or improve. The goal of this paper is to discuss the improvement of such an approximation in a very simple scalar theory.

In his remarkable analysis of the partition function of the scalar field theory, K. Wilson \cite{1} has shown that by replacing some of the contributions by order of magnitude estimations, it was possible to obtain a very simple renormalization group transformation called the approximate recursion formula. This recursion formula played an important role in the development of the RG method because the basic ideas (fixed points, relevant directions..) were not too difficult to work out in detail\cite{1,2} in this simplified RG transformation. Similar recursion formulas hold exactly for hierarchical models.\cite{3} For this reason, we call the approximation made by Wilson the hierarchical approximation.

An interesting feature\cite{3} of the hierarchical models is their large group of symmetry. An appropriate use of these symmetries allows, in the case of Ising spins, to cut down the time necessary to calculate numerically\cite{4} the partition function with $2^n$ sites, from $T = 2^{2n}$ to $(\log(T))^2$, without making any approximations. In our attempt to improve the hierarchical approximation, we would like to keep the largest possible subgroup of unbroken symmetries at each stage of the calculation.

The improvement of the hierarchical approximation is in principle straightforward: we have to restore the details erased in Wilson’s discussion.\cite{1} However there are several practical problems which will be encountered. First, there is a book-
keeping problem: we need to write all the corrections. Second, there is a priority problem: we would like to know which corrections are the most important in order to calculate them first. Finally, there is the feasibility of the calculation itself.

We present here a method which overcomes these problems in a natural way. For simplicity, we have restricted the discussion to a gaussian model on a finite one-dimensional lattice with $2^n$ sites. The Fourier modes of the scalar field are denoted $\Phi_k$ where $\Phi^*_k = \Phi_{-k}$ and $k$ are integers used to express the momenta in $\frac{2\pi}{2^n}$ units. These integers are understood modulo $2^n$ in the following (periodicity in momentum space). The action reads

$$S = \frac{1}{2^{n+1}} \sum_{k=1}^{2^n} g(k) \Phi_k \Phi_{-k}$$

where $g(k)$ is even, real and positive.

We proceed in three steps. First, we relabel the momenta in a way which is convenient to read their “shell” assignment in Wilson’s discussion (section 2). This relabeling allows us to introduce a group of transformation whose orbits are precisely these shells. The representations of this group are known and can be used to expand the kinetic term, i.e., the function $g(k)$ in Eq. (1), for each of the shells. This solves the bookkeeping problem. Nicely enough, the classification of the representations of the group mentioned above comes with an index indicating its resolution power (called degree of ramification). This naturally provides the successive orders of our perturbative expansion. We then show that if we only retain the trivial representation in the expansion, we obtain the hierarchical approximation (section 3). In this limit, the group of transformation is a symmetry of the action which can be identified with the symmetry group of the hierarchical model mentioned above. The improvement of the the hierarchical approximation can thus be treated as a symmetry breaking problem. We describe the first corrections using the familiar apparatus of the Ward identities in section 4.

We emphasize that this procedure takes advantage of the symmetry present at order zero in an optimal way. We have checked in simple examples that the largest
corrections correspond to the smallest breaking of the symmetry. The extension of this construction for $D$-dimensional models is rather straightforward and mentioned at the end of the paper. The use of this scheme for interactive theories (Landau-Ginzburg or Ising type) is presently under study and briefly discussed at the end of section 3.

2. A Convenient Relabeling of the Momenta

In this section, we introduce a relabeling of the momenta $k$ appearing in Eq. (1). For this purpose, we use a set orthonormal functions which is a discrete version of the Walsh system\textsuperscript{[6]} motivated by Wilson shell decomposition (see section II of Ref.[1] for detail). We first define

$$\Psi_0(k) \equiv \begin{cases} 
1 & \text{if } k = -2^{n-2} + 1, \ldots, 2^{n-2} - 1 \\
\omega & \text{if } k = 2^{n-2} \\
\omega^* & \text{if } k = -2^{n-2} \\
0 & \text{otherwise}
\end{cases} \quad (2)$$

and

$$\Psi_1(k) \equiv 1 - \Psi_0(k) \quad (3)$$

with the notation $\omega \equiv \frac{1+i}{2}$. 
For a given integer $a = a_0 + a_1 2^1 + a_2 2^2 + \ldots \ldots + a_{n-1} 2^{n-1}$ with $a_l = 0$ or 1 we define

$$f_a(k) \equiv \prod_{l=0}^{n-1} \Psi_{a_l}(2^l k). \quad (4)$$

It is clear that $f_a^*(k) = f_a(-k)$ and we can check that

$$\sum_k f_a(k) f_b^*(k) = \delta_{a,b}. \quad (5)$$

A more detailed analysis shows that $f_a(k)$ is non-zero only when $k = \pm k[a]$ for a function $k[a]$ which will be specified. More precisely, it is possible to write

$$f_a(k) = \omega \delta_{k, k[a]} + \omega^* \delta_{k, -k[a]} \quad (6)$$

This relation defines a one-to-one map $k[a]$. In the following, $a$ will also be treated as an integer modulo $2^n$. We can now expand

$$\Phi_k = \sum_{a=1}^{2^n} c_a f_a(k). \quad (7)$$

With this and Eqs. (5) and (6) we can rewrite

$$S = \frac{1}{2^{n+1}} \sum_{a=1}^{2^n} \tilde{g}(a) c_a^2 \quad (8)$$

where $\tilde{g}(a) \equiv g(k[a])$. By construction, the $c_a$ are real field variables. For convenience we shall also use their complex form

$$\sigma_a \equiv \frac{1}{2} (c_a + c_{-a}) + \frac{i}{2} (c_a - c_{-a}) \quad (9)$$

We can now explain the correspondence between this relabeling and Wilson’s cell decomposition. Clearly, if $a_0 = 1$, $f_a(k)$ is supported in the high momentum region. More precisely, the 0-th shell, i.e., the one integrated first in the RG procedure, consists in configurations which can be expanded in terms of the $f_{1+a_12+\ldots}(k)$. Similarly, the modes corresponding tho the $l$-th shell are made out of the the $f_{2^l+a_{l+1}2^{l+1}+\ldots}(k)$. 

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3. The Hierarchical Approximation
and its Systematic Improvement

In the previous section, we have introduced new field variables \( c_a \) corresponding to the \( l \)-th shell when \( a \) can be divided by \( 2^l \) but not by \( 2^{l+1} \). This property is not affected if \( a \) is multiplied (modulo \( 2^n \)) by any odd number. Note also that the odd numbers form an abelian group with respect to the multiplication modulo \( 2^n \). The orbit of this group within the integers modulo \( 2^n \) are precisely the sets of numbers that we have put in correspondence with the shells. The representations of this group have been studied and classified. The results can be found in a book by Taibleson.\(^5\)

In order to use Taibleson results, we have to embed the labels introduced above and denoted \( a \), in the 2-adic integers. Such a technique\(^7\) has already been used, for instance, to discuss the random walk representation of the hierarchical model.\(^8\) When \( a \) can be divided by \( 2^l \) but not by \( 2^{l+1} \), we say that the 2-adic norm, noted \( |a|_2 \), is \( 2^{-l} \). In the infinite volume limit, or in other words when \( n \) tends to infinity, the multiplicative group of the odd numbers is called the 2-adic units. The representations of this group will be denoted \( \Pi_s \). This means that if \( u_1 \) and \( u_2 \) are 2-adic units, then \( \Pi_s(u_1 u_2) = \Pi_s(u_1)\Pi_s(u_2) \). The label \( s \) specifies the representation in a way which will be explained below.

It is easy to construct explicitly the representations \( \Pi_s \). A 2-adic unit can be written\(^9\) as \( u = \pm \text{Exp}(4z) \) where \( z \) is a (2-adic) integer and \( \text{Exp} \) the 2-adic exponential. \( \Pi_s(u) \) is even or odd under multiplication by \(-1\). On the other hand, \( z \) is an additive parametrization and the \( z \) dependence of \( \Pi_s \) will be of the form \( e^{i2\pi qa} \) where \( q \) is an odd integer and \( r \) a positive integer. Taibleson calls \( r+2 \) the degree of ramification. In summary, the label \( s \) is a short notation for the parity, \( r \) and \( q \) an odd integer modulo \( 2^r \).

We can now use these representations to expand the the kinetic term function \( \tilde{g}(a) \) in each shell. For a given shell \( l \), the \( a \) have the form \( 2^l u \) (so \( |a|_2 = 2^{-l} \)) and
we can write,
\[
\tilde{g}(2^lu) = \sum_s g_{l,s} \Pi_s(u)
\]  \hspace{1cm} (10)

At finite volume, i.e., at finite \( n \), the units are understood modulo \( 2^{n-l} \) and consequently the sum over the representations \( s \) is restricted to \( r \leq n - l - 2 \). The numerical coefficients \( g_{l,s} \) are easily calculable using the orthogonality relations among the representation.

The hierarchical approximation is obtained by retaining only the trivial representation in the expansion (10). In this approximation, and using the definition introduced in Eq.(9), the action reads
\[
S = \frac{1}{2^{n+1}} \sum_{l=0}^{n-1} g_{l,+0} \sum_{a: \sigma_a = 2^{-l}} \sigma_a \sigma_{-a}
\]  \hspace{1cm} (11)

After a Fourier transform, we obtain a hierarchical model having the general form written by Dyson in Ref.[3], as discussed below.

The classification of the representations of the 2-adic units suggest that we improve the hierarchical approximation by taking into account the additional terms in the expansion (10) \textit{order by order in the degree of ramification}. Intuitively, this corresponds to the fact that the degree of ramification measures the “power of resolution” of the representation. Numerically, this works quite well. For instance, for \( g(k) = 1 - \cos(\frac{4k\pi}{2^l}) \), we obtain with \( n = 4 \) and \( l = 0 \), \( g_{+,0} = 1.63 \), \( g_{-,0} = 0.31 \), \( g_{+,1} = 0.25 \) and \( g_{-,1} = -0.05 \). The subscripts denote the parity and \( r \) respectively. Obvious indices have been skipped.

The actions (8) and (11) take a more familiar form after Fourier transformation in \( a \). In general, the \( l \)-th momentum shell is responsible for interactions among the averages of the Fourier transformed fields inside boxes of size \( 2^l \). In the hierarchical approximation, the interactions depend only on the position of these boxes inside boxes of size \( 2^{l+1} \). When the corrections are introduced up to \( r = r_{\text{max}} \), the interactions depend on the position inside boxes of size \( 2^{l+2+r_{\text{max}}} \). If local interactions
in these new fields are introduced, then it is easy to generalize the numerical\textsuperscript{[4]} or diagrammatic\textsuperscript{[10]} treatment for small $r_{\text{max}}$.

4. The Improvement of the Hierarchical Approximation As a Symmetry Breaking Problem

It is important to realize that the hierarchical approximation of $S$ given in Eq.(11) is invariant under the transformation

$$\sigma_a \longrightarrow \sigma_{ua}$$

(12)

for any odd number $u$. When the other terms of the expansion are incorporated, this symmetry is broken by each term in a definite way. This allows us to use Ward identities techniques. We discuss the simplest case below.

Suppose we want to calculate the 2-point function, using the perturbative expansion described in the previous section. First we use the new variables $c_a$ and the inverse of the map $k[a]$ defined in section 2 to write

$$\langle \Phi_k \Phi_{-k} \rangle = \left\langle c_{a|k|}^2 \right\rangle .$$

(13)

In the hierarchical approximation, the value of this expression depends only on the momentum shell specified by $|a|/2$. In other words,

$$\langle c_{ua}^2 \rangle_0 = \langle c_a^2 \rangle_0$$

(14)

where $\langle \rangle_0$ means that the quantity is evaluated, at order zero, or in other words, in the hierarchical approximation.

Suppose that we now include a correction $\delta \tilde{g}(a)$ to the approximation of $\tilde{g}(a)$. Then in first order in this perturbation, we recover the a momentum dependence
within the shells given by
\[
\langle c_{ua}^2 \rangle_1 = \langle c_{a}^2 \rangle_1 - \frac{1}{2^{n+1}} \sum_b (\delta \tilde{g}(ub) - \delta \tilde{g}(b)) \langle c_{b}^2 c_{a}^2 \rangle_0
\]  
(15)

In the model considered here, the \( \langle c_{b}^2 c_{a}^2 \rangle_0 \) contribution can be evaluated straightforwardly and we can check that we recover the first term in the expansion of \( (1/\tilde{g}(ua)) - (1/\tilde{g}(a)) \). The important point is that the corrections are evaluated using the unperturbed action. It is clear that similar methods can be used for higher point functions and in the interacting case.

5. Conclusions and Perspectives

We have proposed a systematic improvement of the hierarchical approximation for the gaussian model in one dimension. This procedure exploits in an optimal way the symmetries of the approximation and make sense as a perturbative expansion.

An extension to \( D \)-dimension can be constructed easily by noticing the approximate correspondence between the integration over successive shells and the block-spin method. On a \( D \)-dimensional cubic lattice, we can decompose the block-spin procedure into \( D \) steps (one in each directions). This suggests the introduction of functions
\[
\Psi^j_0(k_1, ..., k_D) \equiv \Psi_0(k_j)
\]  
(16)

and the replacement of the definition (4) by
\[
f_a(k_1......k_D) = \Psi^1_{a_0}(k_1)......\Psi^D_{a_{D-1}}(k_D)\Psi^1_{a_{D}}(2k_1)......\Psi^D_{a_{2D-1}}(2k_D).....
\]  
(17)

This procedure keeps the computational advantages of the \( D = 1 \) case.

Our general objective is the construction of a perturbative expansion for which the RG method and the \( \epsilon \)-expansion are very well understood in the 0-th order approximation. A first test of the reliability of the method will be the calculation of the first corrections to the critical exponents. In the longer term, we also expect the method to shed some light on Polyakov’s conjecture for the 3D Ising model.\footnote{11}
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