Poynting vector in stationary axisymmetric electrovacuum spacetimes reexamined

V S Manko\textsuperscript{1}, E D Rodchenko\textsuperscript{2}, B I Sadovnikov\textsuperscript{2} and J Sod–Hoffs\textsuperscript{1}

\textsuperscript{1} Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN, A.P. 14-740, 07000 México D.F., Mexico
\textsuperscript{2} Department of Quantum Statistics and Field Theory, Lomonosov Moscow State University, Moscow 119899, Russia

Abstract. A simple formula, invariant under the duality rotation $\Phi \rightarrow \text{e}^{i\alpha}\Phi$, is obtained for the Poynting vector within the framework of the Ernst formalism, and its application to the known exact solutions for a charged massive magnetic dipole is considered.

PACS numbers: 04.20.Jb, 04.20.Cv, 04.40.Nr
1. Introduction

In a recent paper [1] the vorticity tensor and Poynting vector have been analyzed in the context of Bonnor’s frame–dragging effect [2] occurring in the field of a static mass endowed with electric charge and magnetic dipole moment. In spite of presenting several useful general formulae, the authors of [1], in our opinion, failed to achieve one of the main goals of their research which was the demonstration in the explicit form, using a particular exact solution, the appearance of the predicted by Bonnor factor \( q_b \), \( q \) being the electric charge and \( b \) the magnetic dipole moment, in the expression for the component \( \varphi \) of Poynting vector. This may have the following two explanations: firstly, the formula (28) of [1] defining \( S^\varphi \) is not quite appropriate for concrete applications, even in the simplest cases such as Manko’s solution [3] for a magnetized Kerr–Newman mass utilized in [1]. And, secondly, the determinantal form of writing specific exact solutions employed in [1] only permits to arrive at some formal expressions which need to be further simplified by expanding the determinants.

A detailed examination of the paper [1] reveals that it contains an intrinsic inconsistency which is the attribution to the same function — the electric component of the electromagnetic 4–potential — different signs in the field equations and during the calculation of the Poynting vector. To make things worse, it appears that the determinantal expressions of [1] which are entirely taken over from the paper [4] devoted to the multisoliton electrovac solution, are all presented with errors, including the definitions of the quantities \( h_l(\alpha_n) \). Such distorted formulae can be neither reproduced nor used in any physical analysis; neither they can be considered as a substitute to the elegant original formulae defining Manko’s electrovac solution [3].

The objective of the present paper is twofold: (i) the derivation of a simple formula for the component \( \varphi \) of the Poynting vector which would be consistent with the Ernst formalism and based on it, and (ii) its subsequent application to two exact solutions for a charged massive magnetic dipole for providing an explicit demonstration of Bonnor’s frame–dragging effect via the Poynting vector. The point (i) of our objective will be achieved by first reviewing the Ernst formalism of complex potentials, paying attention to the historical notations used in the original Ernst’s article [5], and then deriving the expression for \( S^\varphi \) in which the simplifications will be possible thanks to the use of the differential relations provided by the Ernst formalism. The main advantage of the new formula for \( S^\varphi \) thus obtained will consist in the absence in it of the metric function \( \omega \) which is normally the most complicated metric coefficient in the electrovac spacetimes. The formula will also permit us to draw an important conclusion about the invariance of \( S^\varphi \) under the duality rotation \( \Phi \rightarrow e^{i\alpha}\Phi, \alpha = \text{const} \). The point (ii) will be realized on the way of applying the formula for \( S^\varphi \) to Manko’s electrovac solution [3], and to the rational function solution constructed in the paper [6]. Since the latter solution is written in the spheroidal coordinates \((x, y)\), a corresponding formula for \( S^\varphi \) in these coordinates will be also worked out.
2. Brief review of the Ernst formalism

In his famous paper [5] Ernst reduced the problem of finding stationary axisymmetric electrovacuum solutions of the Einstein–Maxwell equations to solving a concise system of two differential equations for the complex potentials $E$ and $\Phi$:

\[
(\text{Re} E + \Phi \bar{\Phi}) \Delta E = (\nabla E + 2 \bar{\Phi} \nabla \Phi) \nabla E,
\]
\[
(\text{Re} E + \Phi \bar{\Phi}) \Delta \Phi = (\nabla E + 2 \bar{\Phi} \nabla \Phi) \nabla \Phi,
\]

(1)

where a bar over a symbol denotes complex conjugation, $\Delta$ and $\nabla$ are the three-dimensional Laplacian and gradient operators, respectively (in the Weyl–Papapetrou cylindrical coordinates $(\rho, z)$)

\[
\Delta A := A_{,\rho,\rho} + \rho^{-1} A_{,\rho} + A_{,z,z}, \quad \nabla A \cdot \nabla B := A_{,\rho} B_{,\rho} + A_{,z} B_{,z}.
\]

(2)

The relation of the potentials $E$ and $\Phi$ to the coefficients in the stationary axisymmetric line element

\[
ds^2 = g_{ik} dx^i dx^k = f^{-1} [e^{2\gamma} (d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f (dt - \omega d\varphi)^2
\]

(3)

($f$, $\omega$ and $\gamma$ are functions of $\rho$ and $z$ only; $x^1 = \rho$, $x^2 = z$, $x^3 = \varphi$, $x^4 = t$) and to the $\varphi$ and $t$ components of the electromagnetic 4–potential

\[
A_i = (0, 0, A_3, -A_4)
\]

(4)

is defined by the following equations:

\[
\mathcal{E} = f - \Phi \bar{\Phi} + i\chi, \quad \Phi = A_4 + iA_3',
\]

(5)

\[
\chi_{,\rho} = \rho^{-1} f^2 \omega_{,z} - 2 \text{Im}(\bar{\Phi} \Phi_{,\rho}), \quad \chi_{,z} = -\rho^{-1} f^2 \omega_{,\rho} - 2 \text{Im}(\bar{\Phi} \Phi_{,z}),
\]

(6)

\[
A_{3,\rho} = \rho^{-1} f (A_{3,z} - \omega A_{4,,z}), \quad A'_{3,z} = -\rho^{-1} f (A_{3,\rho} - \omega A_{4,\rho}),
\]

(7)

and these do not involve the metric function $\gamma$ which can be found, for known $\mathcal{E}$ and $\Phi$, from the system of the first order differential equations

\[
\gamma_{,\rho} = \frac{1}{4} \rho f^{-2} [((\mathcal{E}_{,\rho} + 2 \bar{\Phi} \Phi_{,\rho})(\bar{\mathcal{E}}_{,\rho} + 2 \bar{\Phi} \bar{\Phi}_{,\rho})
\]
\[
- (\mathcal{E}_{,z} + 2 \bar{\Phi} \Phi_{,z})(\bar{\mathcal{E}}_{,z} + 2 \bar{\Phi} \bar{\Phi}_{,z})] - \rho f^{-1} (\Phi_{,\rho} \bar{\Phi}_{,\rho} - \Phi_{,z} \bar{\Phi}_{,z}),
\]

\[
\gamma_{,z} = \frac{1}{2} \rho f^{-2} \text{Re} [(\mathcal{E}_{,\rho} + 2 \bar{\Phi} \Phi_{,\rho})(\bar{\mathcal{E}}_{,z} + 2 \bar{\Phi} \bar{\Phi}_{,z})] - 2 \rho f^{-1} \text{Re}(\bar{\Phi}_{,\rho} \Phi_{,z}),
\]

(8)

the integrability condition of which is the system (1).

Once the Ernst equations (1) are solved, the metric function $f$, the electric potential $A_4$ and the functions $\chi$ and $A_3'$ can be found from the formulae

\[
f = \text{Re}(\mathcal{E}) + \Phi \bar{\Phi}, \quad \chi = \text{Im}(\mathcal{E}), \quad A_4 = \text{Re}(\Phi), \quad A_3' = \text{Im}(\Phi),
\]

(9)

which, in turn, permit to obtain the metric coefficient $\omega$ by integrating equations (6), and subsequently the magnetic potential $A_3$ by integrating equations (7). Lastly, the function $\gamma$ is obtainable from the system (8).

Therefore, the knowledge of the Ernst potentials $\mathcal{E}$ and $\Phi$ is sufficient for the reconstruction of the whole metric and of the electromagnetic field outside the sources.
It is worth pointing out that historically the sign ‘−’ in the formula (4) was chosen for convenience, with the idea to avoid the minus sign in the formula (5) defining the potential \( \Phi \). It is the conventional character of this choice which later allowed some researchers to formally redefine at their own taste the \( t \)-component of the potential \( A_t \), so that the Ernst potential \( \Phi \) may be written for instance in the form \( \Phi = -A_t + i A'_\varphi \), \( A_t := -A_4, \ A'_\varphi := A'_3 \).

The only thing one should be aware of if one wants to redefine the original Ernst’s electric potential \( A_4 \), is not to forget to carry out the respective changes in equations (5) and (7). The paper [1] is very instructive in this connection. Its authors first wrote out (up to a couple misprints) the original Ernst’s formulae, but then, probably not knowing or having forgotten formula (4), they carried out the calculation of the components of the electromagnetic energy–momentum tensor as if they were working not with the potential \( A_4 \) but with \( -A_4 \). Such inconsistency did not permit them to take any advantage of the Ernst formalism and carry out further simplifications of the formula for \( S^\varphi \).

3. Poynting vector in the Ernst formalism

The Poynting vector \( S^\alpha \) (in this paper the Greek indices take the values 1, 2 and 3) which was considered in application to the stationary axisymmetric electrovacuum case in [1] is defined by the formula\( \dagger \)

\[
S^\alpha = T^{\alpha i}u_i, \tag{11}
\]

where \( T^{ik} \) is the energy–momentum tensor of the electromagnetic field

\[
T_{ik} = \frac{1}{4\pi}(F_iF_k^l - \frac{1}{4}g_{ik}F_{lm}F^{lm}'), \tag{12}
\]

\( F_{ik} \) being the electromagnetic field tensor

\[
F_{ik} = A_{k,i} - A_{i,k}, \tag{13}
\]

and \( u^i \) is the 4–velocity vector with the components

\[
u^i = (0, 0, f^{-1/2}), \quad u_i = (0, 0, f^{1/2} \omega, -f^{1/2}), \quad u^iu_i = -1. \tag{14}
\]

Taking into account that

\[
g^{11} = g^{22} = ve^{-2\gamma}, \ g^{33} = \rho^{-2}f, \ g^{34} = \rho^{-2}f\omega, \ g^{44} = -f^{-1} + \rho^{-2}f\omega^2;
\]

\[
F_{13} = A_{3,\rho}, \quad F_{14} = -A_{4,\rho}, \quad F_{23} = A_{3,z}, \quad F_{24} = -A_{4,z},
\]

\[
T_{11} = \frac{1}{4\pi}[\rho^{-2}fA_{3,\rho}^2 + (\rho^{-2}f\omega^2 - f^{-1})A_{4,\rho}^2 - 2\rho^{-2}f\omega A_{3,\rho}A_{4,\rho} - \frac{1}{2}A],
\]

\[
T_{12} = \frac{1}{4\pi}[\rho^{-2}f(A_{3,\rho} - \omega A_{4,\rho})(A_{3,z} - \omega A_{4,z}) - f^{-1}A_{4,\rho}A_{4,z},
\]

\[
T_{22} = \frac{1}{4\pi}[\rho^{-2}fA_{3,z}^2 + (\rho^{-2}f\omega^2 - f^{-1})A_{4,z}^2 - 2\rho^{-2}f\omega A_{3,z}A_{4,z} - \frac{1}{2}A],
\]

\( \dagger \) In comparison with the paper [1] we have changed the sign in the definition of \( T_{ik} \) following the book [8], and consequently changed the sign in the original formula for \( S^\alpha \) used by Herrera et al.
\[ T_{33} = \frac{1}{4\pi} f e^{-2\gamma} [\left( \nabla A_3 \right)^2 - \frac{1}{2} (\rho^2 f^{-1} - f \omega^2) A], \]
\[ T_{34} = -\frac{1}{4\pi} f e^{-2\gamma} (\nabla A_3 \nabla A_4) + \frac{1}{2} f \omega A, \]
\[ T_{44} = \frac{1}{4\pi} f e^{-2\gamma} [\left( \nabla A_4 \right)^2 + \frac{1}{2} f A], \]
\[ A := \rho^{-2} f (\nabla A_3)^2 + (\rho^2 f \omega^2 - f^{-1})(\nabla A_4)^2 - 2 \rho^{-2} f \omega \nabla A_3 \nabla A_4, \tag{15} \]

it is straightforward to verify that \( S^\rho = S^z = 0 \) identically, while the calculation of the non–zero component \( S^\phi \) leads to the expression

\[ S^\phi = \frac{f^{3/2} e^{-2\gamma}}{4\pi \rho^2} \left[ \omega (\nabla A_4)^2 - \nabla A_3 \nabla A_4 \right]. \tag{16} \]

Formula (16) differs from the formula (28) of [1] in the sign of the second term in brackets, and it also contains the factor 4 in the denominator on the right–hand side which is missing in [1]. The difference in sign is explained by the already mentioned inconsistency in the use of the \( t \)–component of the potential \( A_i \) taking place in [1].

Formula (16) is congruent with the equations (5) and (7) and, as a result, can be further simplified. Indeed, by passing in (16) from \( A_3 \) to \( A_3' \) with the aid of equations (7), we obtain

\[ S^\phi = \frac{\sqrt{f} e^{-2\gamma}}{4\pi \rho^2} (A_{1,\rho} A_{3,z}' - A_{1,z} A_{3,\rho}'), \tag{17} \]

thus getting rid of the metric function \( \omega \). Then, making use of (6), we arrive at the final result

\[ S^\phi = \frac{\sqrt{f} e^{-2\gamma}}{4\pi \rho^2} \text{Im}(\Phi, \Phi, z). \tag{18} \]

Formula (18) is by far more suitable for the use in concrete applications than formula (16) since in order to see whether or not \( S^\phi \) is zero, the knowledge of only one Ernst’s potential \( \Phi \) is sufficient. From (18) also follows that \( S^\phi \) is invariant under the duality rotation transformation

\[ \Phi \rightarrow e^{i\alpha} \Phi, \quad \alpha = \text{const} \tag{19} \]

which means in particular that \( S^\phi \) will be equal to zero (and consequently no frame–dragging will occur) in the electrovacuum spacetimes representing a static mass endowed with both electric and magnetic dipole moments if the corresponding exact solution was obtained from the magnetostatic (or electrostatic) solution by means of a duality rotation. More generally, \( S^\phi \) of some electrovac solution will be equal to zero if the potential \( \mathcal{E} \) of that solution is a real function, and its potential \( \Phi \) can be made a real or pure imaginary function via exclusively an appropriate duality rotation.

We shall now illustrate the use of formula (18) with two examples.

4. Examples

As the first example we will consider Manko’s solution [3] for a charged, magnetized, rotating mass, restricting ourselves to the case when the total angular momentum is
equal to zero (together with the whole set of the rotational multipole moments). The Ernst potentials $E$ and $\Phi$ of this solution have the form [3]

\[
E = \frac{A - B}{A + B}, \quad \Phi = \frac{C}{A + B},
\]

\[
A = \kappa_+^2 [(m^2 - q^2 - b)(R_+r_+ + R_-r_-) + iq\kappa_+(R_+r_+ - R_-r_-)]
+ \kappa_-^2 [(m^2 - q^2 + b)(R_+r_+ + R_-r_-) - iq\kappa_+(R_+r_+ - R_-r_-)]
- 4b^2 (R_+r_- + r_+r_-),
\]

\[
B = m\kappa_+\kappa_- \{ \kappa_+\kappa_-(R_+ + R_- + r_+ + r_-) - (m^2 - q^2)(R_+ + R_- - r_+ - r_-) + iq[(\kappa_+ - \kappa_-)(R_+ - R_-) - (\kappa_+ + \kappa_-)(r_+ - r_-)] \},
\]

\[
C = \kappa_+\kappa_- \{ q\kappa_+\kappa_-(R_+ + R_- + r_+ + r_-) - q(m^2 - q^2)
\times (R_+ + R_- - r_+ - r_-) + i[\kappa_+(q^2 + b)(R_+ - R_- - r_+ + r_-)
- \kappa_-(q^2 - b)(R_+ - R_- + r_+ - r_-)] \},
\]

\[
R_\pm = \sqrt{\rho^2 + [z \pm \frac{1}{2}(\kappa_+ + \kappa_-)]^2}, \quad r_\pm = \sqrt{\rho^2 + [z \pm \frac{1}{2}(\kappa_+ - \kappa_-)]^2},
\]

\[
\kappa_\pm = \sqrt{m^2 - q^2 \pm 2b}, \tag{20}
\]

where the arbitrary real parameters $m$, $q$, and $b$ are, respectively, the total mass, total charge and magnetic dipole moment of the source. On the upper part of the symmetry axis ($\rho = 0$, $z > (\kappa_+ + \kappa_-)/2$) the potentials [20] take the form

\[
E(\rho = 0, z) = \frac{z - m}{z + m}, \quad \Phi(\rho = 0, z) = \frac{qz + ib}{z(z + m)}, \tag{21}
\]

and these axis data are sufficient for the construction of $E$ and $\Phi$ in the whole space (i.e., for arriving at formulae [20]) with the aid of Sibgatullin’s method [9]. The metric functions $f$ and $\gamma$ which enter the expression for $S^\varphi$ are defined for the Manko solution in terms of $A$, $B$ and $C$ by the formulae

\[
f = \frac{A\hat{A} - B\hat{B} + C\hat{C}}{(A + B)(A - B)}, \quad e^{-2\gamma} = \frac{16\kappa_+^4 \kappa_-^4 R_+ R_- r_+ r_-}{A\hat{A} - B\hat{B} + C\hat{C}}. \tag{22}
\]

Since the metric coefficient $\omega$ which has a rather complicated form is not needed for the calculation of $S^\varphi$, it is not given here.

In the absence of one of the parameters $q$ or $b$ the potential $E$ becomes a real function, while $\Phi$ becomes a real ($b = 0$) or pure imaginary ($q = 0$) function. During the reduction to the magnetostatic case the disappearance of the imaginary part of $E$ is obvious because of the presence of the factor $iq$ in $A$ and $B$. At the same time, in the electrostatic limit ($b = 0$) the vanishing of imaginary quantities and the reduction to the well-known Reissner–Nordström solution is not that trivial, and we find it instructive to show how this limit can be performed (we observe that in [1] the electrostatic limit of Manko’s solution was accomplished in an absolutely erroneous way, see formulae (39) of [1] where each term is artificially multiplied by $b$ for getting the factor $iqb$). Setting $b = 0$ in [20] leads to

\[
\kappa_+ = \kappa_- = \sqrt{m^2 - q^2}, \quad r_+ = r_- = \sqrt{\rho^2 + z^2}, \tag{23}
\]
and then it is easy to see that the sum of the imaginary terms in the first and second lines of the expression for \( A \) will be equal to zero and, besides, the imaginary terms in \( B \) will cancel out because \( r_- - r_+ = 0 \) and \( \kappa_+ - \kappa_- = 0 \). In the analogous way \( C \) also becomes a real function in the limit \( b = 0 \).

The calculation of the component \( S^\varphi \) of the Poynting vector with the aid of formulae (18), (20) and (22) is straightforward and does not exhibit any difficulty; the resulting expression is

\[
S^\varphi = \frac{128 q b \kappa^6 F(\rho, z, R_-, r_-)}{\pi (A + B)^{5/2}(A + B)^{5/2}(AA - BB + CC)^{1/2}},
\]

where \( F(\rho, z, R_-, r_-) \) is some coefficient which is not written down explicitly here because of its cumbersome form. The appearance of the factor \( qb \) in the numerator of \( S^\varphi \) is the desired result at which the above calculation was aimed. It clearly demonstrates that the non–vanishing of the Poynting vector in the solution for a static mass possessing an electric charge and magnetic dipole moment is due to the coexistence of the electric and magnetic fields.

Another solution appropriate for the description of the exterior field of a static mass endowed with both the electric charge and magnetic dipole moment is the three–parameter specialization of the electrovac solution constructed in the paper [6]. Its characteristic feature is that it admits a rational functions representation in the ellipsoidal coordinates. The Ernst potentials \( E \) and \( \Phi \) of the MSM solution have the form

\[
E = \frac{A - B}{A + B}, \quad \Phi = \frac{C}{A + B},
\]

\[
A = 2\left[(\kappa^2 x^2 - \delta y^2)^2 - d^4\right] - 2i\kappa q b x y (1 - y^2),
\]

\[
B = m [2\kappa^3 x (x^2 - 1) + (1 - y^2)(2\kappa \delta x - i q b y)],
\]

\[
C = 2\kappa^2 (x^2 - 1) (\kappa q x + i b y) + (1 - y^2)[2\kappa \delta x - i b y (q^2 - 2\delta)],
\]

where

\[
x = \frac{1}{2\kappa}(r_+ + r_-), \quad y = \frac{1}{2\kappa}(r_+ - r_-), \quad r_\pm = \sqrt{\rho^2 + (z \pm \kappa)^2},
\]

\[
\kappa = \sqrt{d + \delta}, \quad d = \frac{1}{4}(m^2 - q^2), \quad \delta = \frac{b^2}{m^2 - q^2},
\]

the interpretation of the arbitrary real parameters \( m, q \) and \( b \) being exactly the same as in the previous example, i.e., mass, charge and magnetic dipole moment of the source, respectively.

Both the magnetostatic and electrostatic limits of the solution (25) are classical: in the absence of the electric charge it reduces to Bonnor’s solution for a massive magnetic dipole [10], while in the absence of the magnetic field it represents the charged Darmois solution [11].

In the spheroidal coordinates \((x, y)\) the line element (3) assumes the form

\[
ds^2 = \kappa^2 f^{-1}\left[\varepsilon^2 (x^2 - y^2)\left(\frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2}\right) + (x^2 - 1)(1 - y^2) d\varphi^2\right] - f(dt - \omega d\varphi)^2,
\]

(27)
and the metric functions \( f, \gamma \) and \( \omega \) of the MSM solution are defined by the following expressions:

\[
\begin{align*}
    f &= E/D, \quad e^{-2\gamma} = 16\kappa^8 (x^2 - y^2)^4/E, \quad \omega = qb(1 - y^2)F/E, \\
    E &= 4[\kappa^2(x^2 - 1) + \delta(1 - y^2)]^4 - 4\kappa^2 q^2 b^2 y^4 (x^2 - 1)(1 - y^2), \\
    D &= 4[(\kappa^2 x^2 - \delta y^2)^2 + \kappa^3 m x(x^2 - 1) + \kappa m \delta x(1 - y^2) - d^2]^2 \\
    &+ q^2 b^2 y^2 (2\kappa x + m)^2 (1 - y^2)^2, \\
    F &= [\kappa^2(x^2 - 1) + \delta(1 - y^2)]^2 [4[\kappa^2(x^2 - 1) + \delta(1 - y^2)] + (1 - y^2) \\
    &\times (4\kappa m x + 2m^2 - q^2) + 2\kappa^2 y^2 (x^2 - 1) \\
    &\times \{\kappa m x[(2\kappa x + m)^2 - 4\delta y^2 - q^2] - 2\kappa^2 q^2 x^2 - 8d\delta y^2\}. \\
\end{align*}
\]

For the calculation of \( S^\varphi \) in the spheroidal coordinates \((x, y)\) it is necessary to carry out an appropriate coordinate change in (17) and (18). The formulae relating \( \rho \) and \( z \) to \( x \) and \( y \) are

\[
\rho = \kappa\sqrt{(x^2 - 1)(1 - y^2)}, \quad z = \kappa xy, \tag{29}
\]

and the partial derivatives with respect to \( \rho \) and \( z \) can be changed to the derivatives with respect to \( x \) and \( y \) with the aid of the formulae

\[
\frac{\partial}{\partial \rho} = \frac{\sqrt{(x^2 - 1)(1 - y^2)}}{\kappa(x^2 - y^2)} \left( x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right), \\
\frac{\partial}{\partial z} = \frac{y(x^2 - 1)}{\kappa(x^2 - y^2)} \cdot \frac{\partial}{\partial x} + \frac{x(1 - y^2)}{\kappa(x^2 - y^2)} \cdot \frac{\partial}{\partial y}. \tag{30}
\]

Then for the terms containing derivatives of the potentials \( A_4 \) and \( A'_4 \) with respect to \( \rho \) and \( z \) we get the expression

\[
A_{4, \rho} A'_{3, z} - A_{4, z} A'_{3, \rho} = \frac{\sqrt{(x^2 - 1)(1 - y^2)}}{\kappa (x^2 - y^2)} (A_{4, x} A'_{3, y} - A_{4, y} A'_{3, x}), \tag{31}
\]

so that formula \( 18 \) in the coordinates \((x, y)\) finally rewrites as

\[
S^\varphi = \frac{\sqrt{f} e^{-2\gamma}}{4\pi \kappa \sqrt{(x^2 - y^2)}} \text{Im}(\Phi_x \Phi_y). \tag{32}
\]

The substitution of (25) and (28) into (32) yields the following result:

\[
S^\varphi = \frac{qb\kappa^5 (x^2 - y^2)^3 F(x, y)}{64\pi \sqrt{ED} 5/2 (m^2 - q^2)^6}, \tag{33}
\]

where \( F(x, y) \) in the numerator of \( S^\varphi \) is some function of the coordinates \( x \) and \( y \) which is not given here explicitly because of its complicated form. However, in order the reader could have an idea of how the function \( F(x, y) \) looks like, below we give its expression in the equatorial plane \((y = 0)\):

\[
F(x, y = 0) = \{x(\kappa x + m)[(m^2 - q^2)^2 + 4b^2] + \kappa (m^4 - q^4)\} \\
\times [(m^2 - q^2)^2(1 - x^2) - 4b^2 x^2]^5. \tag{34}
\]

In the above example the numerator of \( S^\varphi \), as expected, contains the factor \( qb \), and vanishing of either the electric charge \( q \) or magnetic dipole moment \( b \) causes \( S^\varphi \) to vanish.
too. When both parameters \( q \) and \( b \) have non-zero values, the frame-dragging effect takes place which gives birth to the flow of electromagnetic energy in the \( \varphi \)-direction predicted by Bonnor.

5. Conclusion

In this paper we have succeeded in demonstrating that the use of Ernst’s complex potentials formalism simplifies considerably the study of Poynting vector in stationary electrovacuum spacetimes with axial symmetry. The formulae obtained by us for the only non-zero component of Poynting vector require exclusively the knowledge of the electromagnetic Ernst potential \( \Phi \) for establishing the presence or absence of the azimuthal electromagnetic energy flows in a given spacetime. The component \( S^\varphi \) turns out to be invariant under the duality rotations of the potential \( \Phi \), which helps to single out special cases where the presence of both the electric and magnetic fields does not produce the frame-dragging effect. By direct calculation we have confirmed Bonnor’s prediction that \( S^\varphi \) does not vanish in spacetimes representing a static mass endowed with both the electric charge and magnetic dipole moment.

Acknowledgments

This work was partially supported by Project 45946–F from CONACyT of Mexico.

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