On radiuses of convergence of $q$-metallic numbers and related $q$-rational numbers

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Abstract

The $q$-rational numbers and the $q$-irrational numbers were introduced by Morier-Genoud and Ovsienko. In this paper, we focus on $q$-real quadratic irrational numbers, especially $q$-metallic numbers and $q$-rational sequences which converge to $q$-metallic numbers, and consider the radiuses of convergence of them when we assume that $q$ is a complex number. We construct two sequences given by recurrence formula as a generalization of the $q$-deformation of the Fibonacci and Pell numbers which are introduced by Morier-Genoud and Ovsienko. For these two sequences, we prove a conjecture of Leclere, Morier-Genoud, Ovsienko and Veselov concerning the expected lower bound of the radiuses of convergence. In addition, we obtain a relationship between the radius of convergence of these two sequences in two special cases.

Keywords: Continued fractions, Metallic numbers, $q$-Deformed real numbers, Zeros of palindromic polynomial, Rouché theorem

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1 Introduction

It is well-known that the Euler $q$-integer is a polynomial with integer coefficients of $q$, which is a kind of quantization of integers. Morier-Genoud and Ovsienko define $q$-rationals [11] and $q$-irrationals [12] based on some combinatorial properties of rational numbers and number-theoretic properties of irrational numbers. The $q$-rationals and the $q$-irrationals are related to mathematical physics, combinatorics, number theory, quantum algebra, and knot theory, such as the Jones polynomial and Alexander polynomial of rational knots (see [5,9,13], and the connection between $q$-rationals to knot theory is discussed in Appendix A of [11]), quantum Teichmüller spaces [2], and the Markov-Hurwitz approximation theory (see [4,8]).

We know that a positive rational number $\frac{r}{s} > 1$ with coprime integers $r$ and $s \neq 0$ has regular and negative continued fraction expansions as follows:

$$\frac{r}{s} = a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}} = c_1 - \frac{1}{c_2 - \frac{1}{\ddots - \frac{1}{c_k}}}$$
with \( a_i \geq 1 \) and \( c_j \geq 2 \). These expansions are denoted by \([a_1, a_2, \ldots, a_n]\) and \([\llbracket c_1, c_2, \ldots, c_k\rrbracket]\), respectively. Since \([a_1, a_2, \ldots, a_n] = [a_1, a_2, \ldots, a_n - 1, 1]\), we always assume that \( n = 2m \) is even. In this way, we have the unique regular and negative continued fraction expansion of \( \sqrt{5} \). Morier-Genoud and Ovsienko [11] defined the \( q \)-deformations \([a_1, \ldots, a_{2m}]_q\) and \([\llbracket c_1, \ldots, c_k\rrbracket]_q\), respectively. The \( q \)-deformations of the fractions are rational expressions in the variable \( q \) with integer coefficients. These \( q \)-deformations are equal, and thus one can define \( \llbracket \sqrt{5} \rrbracket_q := [a_1, \ldots, a_{2m}]_q = [\llbracket c_1, \ldots, c_k\rrbracket]_q \) as \( q \)-rationals.

On the other hand, the \( q \)-rationals can also be considered as a formal power series in \( q \). Furthermore, given an arbitrary rational sequence \((x_k)_{k \geq 1}\) that converges to an irrational number \( x > 1 \), one can consider the \( q \)-rational sequence \((\llbracket x_k \rrbracket_q)_{k \geq 1}\). Morier-Genoud and Ovsienko proved that the Taylor series of \( [x_k]_q \) stabilizes, as \( k \) grows. This is the limit series of the series \([x_k]_q\) that does not depend of the choice of the sequence \((\llbracket x_k \rrbracket_q)_{k \geq 1}\). Hence, for an irrational \( x > 1 \), the \( q \)-deformation of \( x \) is defined as the following power series in \( q \):

\[
[x]_q = \sum_{s=0}^{\infty} x_s q^s
\]

where \( x_s \) are integers [12].

In the study of \( q \)-deformation of real quadratic irrational numbers, Leclere and Morier-Genoud [7] proved some properties corresponding to classical real quadratic irrational numbers. On the other hand, since the \( q \)-real numbers are defined as power series in \( q \), under the assumption that \( q \) is a complex number, Leclere, Morier-Genoud, Ovsienko, and Veselov [8] study its radiuses of convergence and give the following conjecture which can be viewed as a \( q \)-deformation of Hurwitz’s Irrational Number Theorem.

**Conjecture 1.1** ([8]) For any real number \( x \geq 1 \), the radius of convergence of the series \([x]_q\) is greater than or equal to \( R_{(1)} = \frac{3 - \sqrt{5}}{2} \).

In the case where \( x = [a_1, 1, a_3, 1, a_5, 1, \ldots] \) \((a_1 \geq 1, a_{2m-1} \geq 2)\) and \( x = [n, n, \ldots, n] \) \((n = 1 \text{ or } 2)\), Conjecture 1.1 has been proved in [8]. The \( x = [n, n, \ldots, n] = \frac{n + \sqrt{n^2 + 4}}{2} \) is called the \((n)\)th metallic number or the \((n)\)th metallic mean. The \( n \)th \( q \)-metallic number is defined as the \( q \)-deformation of \( x \) and is denoted by \([x]_q\). When \( n = 1 \), \([x]_q\) is called the first \( q \)-metallic number or \( q \)-golden number. For convenience, we use the notation \( R_{(1)} \) instead of \( R_n \) in [8]. It is shown that \( R_{(1)} \) is the radius of convergence of the \( q \)-golden number \( \left[ \frac{1 + \sqrt{5}}{2} \right]_q \). In this paper we prove Conjecture 1.1 for the \( n \)th metallic numbers \((n \geq 3)\) and its convergence rational sequence. More precisely, the following two theorems are proved:

**Theorem 1.2** For the \( n \)th metallic number \( x = [n, n, \ldots, n] \) with \( n \geq 3 \), the radius of convergence of the series \([x]_q\) is greater than or equal to \( R_{(1)} \).

**Theorem 1.3** If \( x = [n, n, \ldots, n] \), then the radius of convergence of the series \([x]_q\) is greater than or equal to \( R_{(1)} \).

It is worth noting that the ideas for the proof of Theorems 1.2 and 1.3 are different. We convert the problem of finding the radius of convergence of the \( q \)-metallic number
to considering the zeros of its discriminant. The palindrome of the discriminant of the $q$-metallic number is the key to proving Theorem 1.2. For the case of $[n, \ldots, n]_q$, its radius of convergence can also be converted to the zeros of a polynomial by considering the denominator part. On the other hand, Morier-Genoud and Ovsienko define the $q$-deformation of Fibonacci numbers and Pell numbers (see [11]). Inspired by their study, we construct two sequences $\{M_k(q,n)\}_{k \geq 0}$ and $\{\widehat{M}_k(q,n)\}_{k \geq 0}$ as a generalization of the $q$-deformation of Fibonacci numbers and Pell numbers. Different from the case of the $q$-metallic number, we mainly use the Rouché theorem (see [1,14]) to complete the proof of Theorem 1.3 by considering the sequence $\{M_k(q,n)\}_{k \geq 0}$.

A relationship between the radius of convergence of the $q$-metallic number $[n, n, \ldots, n]_q$ and its truncated $q$-rational number $[n, n, \ldots, n]_q$ was examined in [8] in the cases $n = 1$ and 2. This means that for each $q$-rational in the $q$-rational sequence that converges to the $q$-silver number $[2, 2, \ldots, 2]_q$, its radius of convergence is not only greater than or equal to $R(1)$ but also greater than or equal to the radius of convergence of the $q$-silver number. Motivated by these results, we prove the following statement.

**Theorem 1.4** For $n = 3, 4$, the radius of convergence of the series $[n, n, \ldots, n]_q$ is greater than the radius of convergence of the series $[n, n, \ldots, n]_q$.

The proof of Theorem 1.4 is similar to that of Theorem 1.3. We still need to the Rouché theorem (see [1,14]). However, for the general case $n \geq 5$, there are limitations to this approach as one can not apply Rouché theorem (see [1,14]).

This paper is organized into the following sections. In Sect. 2, we give definitions and some properties about $q$-rational numbers and $q$-irrational numbers. Then we consider the $q$-metallic number and its discriminant. In Sect. 3, we consider the radius of convergence of the $q$-metallic number. Firstly, we know that finding the radius of convergence can be transformed into finding the zeros of the discriminant. In particular, we deal with the discriminant appropriately to obtain a palindromic polynomial $P_n(q)$. We use its related properties to prove that $P_n(q)$ does not have a zero in the disk with radius $R(1)$ and center at the origin. From this fact we prove Theorem 1.2.

In Sect. 4, we generalize the $q$-deformation of Fibonacci numbers and Pell numbers and define the recurrence formula of the sequence $\{M_k(q,n)\}_{k \geq 0}$. We also give an important property about them and prove Theorems 1.3 and 1.4 by this property and the Rouché theorem (see [1,14]).

## 2 The definition and some properties of $q$-real numbers

At the beginning, we briefly describe the definitions of the $q$-rational number and the $q$-irrational number. For details, see [11,12]. Then we consider the $q$-metallic number and its discriminant. The discriminant of the $q$-metallic number has palindromic properties.

### 2.1 $q$-continued fractions and $q$-rational numbers

Let $q$ be a formal parameter. For a non-negative integer $n$ we denote the Euler $q$-integer by
\[ [n]_q := \begin{cases} 
0 & \text{for } n = 0, \\
\frac{1 - q^n}{1 - q} & \text{for } n \geq 1.
\end{cases} \quad (2.1) \]

We note that \( q^n [n]_q^{-1} = \frac{q^n(1 - q^{-n})}{1 - q^{-1}} = q[n]_q. \)

The \( q \)-deformations of regular and negative continued fraction expansions are defined by Morier-Genoud and Ovsienko \([11]\) as follows:

\[ [a_1, a_2, \ldots, a_{2m}]_q := [a_1]_q + \frac{q^{a_1}}{[a_2]_q^{-1} + \frac{q^{a_2}}{[a_3]_q + \frac{q^{a_3}}{[a_4]_q^{-1} + \cdots}}}. \quad (2.2) \]

and

\[ [c_1, c_2, \ldots, c_k]_q := [c_1]_q - \frac{q^{c_1-1}}{[c_2]_q - \frac{q^{c_2-1}}{[c_3]_q - \frac{q^{c_3-1}}{[c_4]_q - \cdots}}} \quad (2.3) \]

**Theorem 2.1** (Morier-Genoud and Ovsienko \([11]\)) *If a positive rational number \( \frac{r}{s} \) is given in the form \( \frac{r}{s} = [a_1, \ldots, a_{2m}] = \{c_1, \ldots, c_k\} \), then*

\[ [a_1, \ldots, a_{2m}]_q = [[c_1, \ldots, c_k]]_q. \quad (2.4) \]

By Theorem 2.1, the \( q \)-rational number \( \left[\frac{r}{s}\right]_q \) is defined as

\[ \left[\frac{r}{s}\right]_q := [a_1, \ldots, a_{2m}]_q = [[c_1, \ldots, c_k]]_q. \]

**2.2 \( q \)-irrationals**

Let \( x > 1 \) be an irrational number. Let \( (x_k)_{k \geq 1} \) be a rational number sequence that converges to \( x \), and consider the sequence \((x_k]_q)_{k \geq 1}\) of \( q \)-deformations. For \( k \geq 1 \), we express \([x_k]_q\) as the formal power series:

\[ [x_k]_q = \sum_{s=0}^{\infty} x_{ks} q^s, \quad (2.5) \]
where $x_{k,s}$ are integers. Then the $q$-deformed irrational number $x$ is defined as the following formal power series in $q$:

$$[x]_q = \sum_{s=0}^{\infty} x_s q^s \quad (x_s = \lim_{k \to \infty} x_{k,s}).$$  \hspace{1cm} (2.6)

The existence of the limit and its independence of the choice of the converging sequence $(x_k)_{k \geq 1}$ is guaranteed by the following proposition.

**Proposition 2.2** (Morier-Genoud and Ovsienko [12][Theorem 1]) Given an irrational real number $x \geq 1$, for every $k \geq 0$, the coefficients $x_{k,s}$ of (2.5) are stabilize as $k$ grows. Moreover, the limit $x_s$ of coefficients (2.6) are integers, and independent of the choice of the converging sequence $(x_k)_{k \geq 1}$.

By Proposition 2.2, we may construct $q$-irrational numbers using continued fractions as follows: for $x = [a_1, a_2, a_3, \ldots] > 1$, the sequence of rational numbers which converges to $x$ can be chosen as $x_k = [a_1, a_2, \ldots, a_k]$.

### 2.3 $q$-metallic numbers and its discriminant

For a $q$-deformation of real quadratic irrational number, we have the following theorem which proved by Leclere and Morier-Genoud [7].

**Theorem 2.3** (Leclere and Morier-Genoud [7][Theorem 2.(i),(ii)]) Let $x = \frac{r + \sqrt{p}}{s}$ be a real quadratic irrational. Its $q$-deformation $[x]_q$ satisfy the following

(i) $[x]_q = \frac{R + \sqrt{P}}{S}$, with $R, P, S \in \mathbb{Z}[q]$, and $P$ a palindrome;

(ii) $[x]_q$ is solution of an equation $AX^2 + BX + C = 0$, with $A, B, C \in \mathbb{Z}[q]$.

Suppose that $x = \frac{n + \sqrt{n^2 + 4}}{2}$ is a metallic number. Note that $x$ is a real quadratic irrational number. Following the Example 4.5 of [7], we obtain the formula

$$[x]_q = [n]_q + \frac{q^n}{[n]_{q^{-1}} + \frac{q^{-n}}{[x]_q}}$$  \hspace{1cm} (2.7)

By part (ii) of Theorem 2.3 the $q$-metallic number is a solution of the algebraic equation

$$q[x]_q^2 - ((q - 1)(q^n + 1) + q[n]_q)[x]_q - 1 = 0,$$  \hspace{1cm} (2.8)

which is a $q$-deformation of $x^2 - nx - 1 = 0$. This can be deduced from (2.7).
Definition 2.4 We denote by $D_n(q)$ the discriminant which is satisfied as the quadratic Eq. (2.8) for $[x]_q$. More precisely, it is given by

$$D_n(q) = \begin{cases} 
1 + 2q - q^2 + 2q^3 + q^4 & \text{for } n = 1, \\
1 + 2q + 3q^2 + 3q^4 + 2q^5 + q^6 & \text{for } n = 2, \\
1 + 2q^2 + \sum_{t=3}^{n-1} (t-1)q^t + (n+1)q^n + (n-4)q^{n+1} & \text{for } n \geq 3, \\
+ (n+1)q^{n+2} + \sum_{t=n+3}^{2n-1} (2n-t+1)q^t + 2q^{2n} + q^{2n+2} & 
\end{cases} 
$$

(2.9)

Then $D_n(q)$ is said to be the discriminant of the $q$-metallic number $[x]_q$.

We note that all of the coefficients of $D_n(q)$ are positive integers for $n \geq 4$. By part (i) of Theorem 2.3, the discriminant $D_n(q)$ is a palindromic polynomial. This property is very helpful for us to prove Theorem 1.2.

3 The radius of convergence of $q$-metallic numbers
Since $q$-irrational is defined as power series in $q$, we can consider its radiuses of convergence. In this section, we always assume that $q$ is a complex number.

3.1 Some observations
For $q$-metallic numbers

$$[x]_q = \left[ \frac{n + \sqrt{n^2 + 4}}{2} \right]_q = [n, n, \ldots]_q,$$

with $n \geq 1$, we denote the radiuses of convergence of $[x]_q$ by $R_{(n)}$. In the cases of $n = 1$ and $n = 2$ the following has already been known.

Proposition 3.1 ([8][Propositions 3.1 and 4.1])

$$R_{(1)} = \frac{3 - \sqrt{5}}{2} = 1 - \frac{1}{1 + \sqrt{5}} \approx 0.38197, \quad R_{(2)} = \frac{1 + \sqrt{2} - \sqrt{2\sqrt{2} - 1}}{2} \approx 0.53101.$$

In complex analysis, it is well-known that the radius of convergence of a power series $[x]_q$ centered on a point 0 is equal to the distance from 0 to the nearest point so that $[x]_q$ cannot be defined in a way that makes it holomorphic. Since

$$[x]_q = \frac{q[n]_q + (q^n + 1)(q-1) + \sqrt{D_n(q)}}{2q},$$

we can calculate $R_{(n)}$ by finding the roots of $D_n(q)$.

Since the discriminant $D_n(q)$ is a palindromic polynomial, we can find the roots of it by using the next lemma.

Lemma 3.2 ([6, Corollary 1]) Let $P(q) \in \mathbb{Q}[q]$ be a palindromic polynomial given by

$$P(q) = a_0 + a_1q + a_2q^2 + \cdots + a_{n-2}q^{n-2} + a_{n-1}q^{n-1} + a_nq^n,$$
having even degree $n$. If
\[ \max\{|a_k| : k \in \{0, 1, 2, \ldots, \frac{n}{2} - 1\}\} \geq |a_{\frac{n}{2}}|, \]
then $P(q)$ have at least two roots on the unit circle.

Since $D_n(q)$ satisfies Lemma 3.2, it have at least two roots on the unit circle. Indeed, for any positive integer $n$, we have
\[ D_n(q) = (1 - q + q^2)P_n(q), \]
where
\[ P_n(q) := \begin{cases} 1 + 3q + q^2 & \text{for } n = 1, \\ 1 + q + 4q^2 + q^3 + q^4 & \text{for } n = 2, \\ 1 + \sum_{t=1}^{n-1} tq^t + (n + 2)q^n + \sum_{t=n+1}^{2n-1} (2n-t)q^t + q^{2n} & \text{for } t = n \geq 3. \end{cases} \]

Hence, we obtain $R_{(n)} \leq 1$, and $R_{(n)} = \min\{|q| : P_n(q) = 0\}$.

**Lemma 3.3** A polynomial $P(q) \in \mathbb{Q}[q]$ is palindromic if and only if the following two conditions are satisfied:

(i) $1$ is a root of even multiplicity (possibly zero),
(ii) if $q \neq \pm 1$ is a root, then $\frac{1}{q}$ is a root with the same multiplicity.

For a proof of Lemma 3.3, see [10].

### 3.2 Proof of Theorem 1.2.

We prove Theorem 1.2 by contradiction. Let $n \geq 3$, and assume that $q = re^{i\theta}$ ($0 < r \leq R_{(1)}$, $0 \leq \theta \leq 2\pi$) is a root of $P_n(q)$. By Lemma 3.3, (ii), since $P_n(q)$ is palindrome, it follows that $q^{-1} = r^{-1}e^{-i\theta}$ is also a root of $P_n(q)$. Hence, $P_n(q)$ has a root that can be expressed by $q = rz$ ($r \geq \frac{1}{R_{(1)}} > \frac{5}{2}$, $|z| = 1$). Then, $q$ is a zero of the polynomial
\[ \sum_{t=0}^{2n} \mu_t z^t, \quad (3.1) \]

where $\mu_t = \begin{cases} 1 & \text{for } t = 0, \\ tr^t & \text{for } 0 < t < n, \\ (n + 2)r^n & \text{for } t = n, \\ (2n-t)r^n & \text{for } n < t < 2n, \\ r^{2n} & \text{for } t = 2n. \end{cases}$

**Lemma 3.4** For $n \geq 3$, the sequence $\{\mu_t\}_{t=0}^{2n}$ with $\mu_t > 0$ for $t = 0, 1, 2, \ldots, 2n$, is strictly increasing and $\mu_{2n-1} < \frac{2}{5} \mu_{2n}$.

**Proof** Since $r > \frac{5}{2}$, then $\mu_{2n-1} = r^{2n-1} < \frac{2}{5} r^{2n} = \frac{2}{5} \mu_{2n}$. \qed

We now turn to the proof of Theorem 1.2.
**Proof of Theorem 1.2:** Let us consider the polynomial

\[ F(z) = (1 - z) \sum_{t=0}^{2n} \mu_t z^t = -\mu_{2n} z^{2n+1} + (\mu_{2n} - \mu_{2n-1}) z^{2n} + \cdots + (\mu_1 - \mu_0) z + \mu_0. \]

By Lemma 3.4 and the fact that \(|z| = 1\), one has

\[
|F(z)| = \left| -\mu_{2n} z^{2n+1} + \mu_{2n} z^{2n} - \frac{2}{5} \mu_{2n} z^{2n} + \left( \frac{2}{5} \mu_{2n} \right) z^{2n} + \cdots + (\mu_1 - \mu_0) z + \mu_0 \right| \\
\geq |\mu_{2n}| |z|^{2n} \left| z - \frac{3}{5} \right| \\
- |z|^{2n} \left( \frac{2}{5} \mu_{2n} - \mu_{2n-1} \right) + \frac{(\mu_{2n-1} - \mu_{2n-2})}{z} + \cdots + \frac{(\mu_1 - \mu_0)}{z^{2n-1}} + \frac{\mu_0}{z^{2n}} \\
\geq |\mu_{2n}| |z|^{2n} \left| z - \frac{3}{5} \right| \\
- |z|^{2n} \left( \frac{2}{5} \mu_{2n} - \mu_{2n-1} \right) + \frac{(\mu_{2n-1} - \mu_{2n-2})}{|z|} + \cdots + \frac{|\mu_1 - \mu_0|}{|z|^{2n-1}} + \frac{|\mu_0|}{|z|^{2n}} \\
\geq \mu_{2n} \left| z - \frac{3}{5} \right| \left( \frac{2}{5} \right) = r^{2n} \left| z - \frac{3}{5} \left( \frac{2}{5} \right) \right| \geq 0.
\]

Note that the equality in the last inequality holds if and only if \(z = 1\). But \(z = 1\) is not a zero of (3.1). It turns out that (3.1) has no zero on the unit circle. This leads to a contradiction. \(\square\)

By part (ii) of Lemma 3.3 and the proof of Theorem 1.2, it follows that \(P_n(q)\) has no zeros inside \(\left\{ q \in \mathbb{C} : |q| > \frac{1}{R(1)} = \frac{3 + \sqrt{5}}{2} \right\}\). Then we have the following corollary.

**Corollary 3.5** All zeros of \(P_n(q)\) lie in the annulus

\[ A_1 := \left\{ q \in \mathbb{C} : \frac{3 - \sqrt{5}}{2} \leq |q| \leq \frac{3 + \sqrt{5}}{2} \right\}. \]

**Remark 3.6** For \(1 \leq n \leq 48\), we calculate \(R(n)\) to six decimal places using Maple (see Table 1). We observe that \(R(n)\) is not monotone increasing (see \(n = 20, 26, 32, 33, 39\) and \(45\) in Table 1). Also, the values for \(n = 25\) and \(n = 27\) seem to be equal. By calculating more precisely, we have \(R(25) = 0.895047428487267 \ldots\) and \(R(27) = 0.895046958159118 \ldots\).

### 4 q-rational number sequence with convergence on q-metallic numbers

#### 4.1 Two sequences \(\{M_t(q,n)\}_{k \geq 0}\) and \(\{\widetilde{M}_t(q,n)\}_{k \geq 0}\)

Since the first metallic number can be represented by the limit of the quotient of two consecutive terms of the Fibonacci sequence, the \(n\)th metallic number satisfies the same property for an appropriately defined sequence. In this section, we first consider such a sequence and its \(q\)-deformation.

Following [3], fix a natural number \(n\), we consider the rational fractions \(\frac{\overline{q_k \ldots q_0}}{\overline{q_k \ldots q_0}}\) \((k \geq 1)\) which is called the “convergents” of the \(n\)th metallic number. The numerators and the denominators in the “convergents” can be determined by the sequence \(\{A_k(n)\}_{k \geq 0}\) which defined by \(A_0(n) = 0, A_1(n) = 1,\) and \(A_{k+2}(n) = nA_{k+1}(n) + A_k(n)\). Then the
Morier-Genoud and Ovsienko introduced two sequences which converge to number and q-
The sequence \( \{A_{k+1}(n)\} \) converges the \( n \)th metallic number. The sequence \( \{A_k(n)\}_{k \geq 0} \) satisfies the following recurrence formula:

\[
A_{k+4}(n) = (n^2 + 2)A_{k+2}(n) - A_k(n) \quad (k = 0, 1, 2, \ldots).
\]

For the sequence \( \left\{ \frac{A_{k+1}(n)}{A_k(n)} \right\}_{k \geq 1} \), by Proposition 2.2, one can consider their \( q \)-deformations \( \left\{ \left[ \frac{A_{k+1}(n)}{A_k(n)} \right]_q \right\}_{k \geq 1} \), which converges the \( n \)th \( q \)-metallic number. Following [11], we know that \( q \)-rationals can be written as the quotient of two coprime polynomials in \( q \) with integer coefficients. Therefore, for any \( k \geq 1 \), we denote the numerator and the denominator of \( \left[ \frac{A_{k+1}(n)}{A_k(n)} \right]_q \) by \( M_k(q, n) \) and \( \widetilde{M}_k(q, n) \), respectively, i.e.,

\[
\left[ \frac{A_{k+1}(n)}{A_k(n)} \right]_q := \frac{M_{k+1}(q, n)}{M_k(q, n)}.
\]

Morier-Genoud and Ovsienko introduced two sequences which converge to \( q \)-golden number and \( q \)-silver number [11]. These sequences can also be viewed as a \( q \)-deformation of Fibonacci numbers and Pell numbers, respectively. Inspired by their study, we introduce two sequences which are generalizations.

**Proposition 4.1** The sequences \( \{M_k(q, n)\}_{k \geq 0} \) and \( \{\widetilde{M}_k(q, n)\}_{k \geq 0} \) satisfy the following recurrence formulas:

\[
M_{2l+1}(q, n) = q[n]_q M_{2l}(q, n) + M_{2l-1}(q, n),
\]

\[
M_{2l+2}(q, n) = [n]_q M_{2l+1}(q, n) + q^{2n} M_{2l}(q, n),
\]

\[
\widetilde{M}_{2l+1}(q, n) = [n]_q \widetilde{M}_{2l}(q, n) + q^{2n} \widetilde{M}_{2l-1}(q, n),
\]

\[
\widetilde{M}_{2l+2}(q, n) = q[n]_q \widetilde{M}_{2l+1}(q, n) + \widetilde{M}_{2l}(q, n),
\]

with \( l \geq 1 \), and

\[
M_0(q, n) = 0, \quad M_1(q, n) = 1, \quad M_2(q, n) = [n]_q, \quad M_3(q, n) = 1 + q([n]_q)^2,
\]

\[
\widetilde{M}_0(q, n) = 0, \quad \widetilde{M}_1(q, n) = 1, \quad \widetilde{M}_2(q, n) = [n]_q, \quad \widetilde{M}_3(q, n) = ([n]_q)^2 + q^{2n-1}.
\]
Proof We show this proposition by using weighted triangulations. For a detailed definition of weighted triangulation, see Sects. 2.1 and 2.3 in [11]. Any $q$-rational number corresponds to a weighted triangulation. When the initial vertex is given, the remaining vertices can be calculated by weighted Farey sum (see Sect. 2.5 in [11]). Then $[n, n, \ldots, n]_q$ corresponds to the weighted triangulation depicted in Fig. 1, where the weights of the unmarked edges are 1 and the convergents of the continued fraction correspond to the black vertices. Therefore, the numerator and denominator of $\left[ \frac{A_{k+1}(n)}{A_k(n)} \right]_q$ are satisfy (4.3) and (4.4), respectively.

Both $\{M_k(q, n)\}_{k \geq 0}$ and $\{\hat{M}_k(q, n)\}_{k \geq 0}$ can be viewed as the $q$-deformation of the sequence $\{A_k(n)\}_{k \geq 0}$. By Corollary 1.7 of [11], when $k$ is fixed, the polynomials $\{M_k(q, n)\}_{k \geq 0}$ and $\{\hat{M}_k(q, n)\}_{k \geq 0}$ are of degree $kn - (n + 1)$, and the coefficients of $\{M_k(q, n)\}_{k \geq 0}$ and $\{\hat{M}_k(q, n)\}_{k \geq 0}$ are mirror-symmetric to each other:

$$M_k(q, n) = q^{kn-(n+1)}\hat{M}_k(q^{-1}, n).$$

From (4.3) and Proposition 4.1, we can obtain the following proposition, which can be viewed as a $q$-deformation of (4.1).

**Proposition 4.2** The sequence $\{M_k(q, n)\}_{k \geq 1}$ is determined by the recurrence formula

$$M_{k+4}(q, n) = f(q, n)M_{k+2}(q, n) - q^{2n}M_k(q, n),$$

where

$$f(q, n) := 1 + q([n]_q)^2 + q^{2n} = 1 + \sum_{t=1}^{n} t q^t + \sum_{t=n+1}^{2n-1} (2n - t) q^t + q^{2n},$$

and the initial data

$$M_1(q, n) = 1, M_2(q, n) = [n]_q,$$

$$M_3(q, n) = 1 + q([n]_q)^2, M_4(q, n) = (q^{2n} + 1)[n]_q + q([n]_q)^3.$$  

Note that $f(q, n) + 2q^n = P_n(q)$, and so $f(q, n)$ is also a palindromic polynomial.
4.2 Proof of Theorem 1.3.

For any \( n \in \mathbb{N} \), the equation

\[
\frac{M_{k+2}(q, n)}{M_k(q, n)} = f(q, n) - q^{2n} \frac{M_{k-2}(q, n)}{M_k(q, n)}
\]  

(4.6)
is implied by (4.5).

We set \( C_n = \{ q \in \mathbb{C} : |q| = R(n) \} \) and \( D_n = \{ q \in \mathbb{C} : |q| < R(n) \} \). For any function \( f(q) \) of one complex variable \( q \), we use the notation \( [f(q)]_{C_n} \) to denote the restriction of \( |f(q)| \) of the circle \( C_n \). Before starting the proof, we consider the following three lemmas.

Lemma 4.3 On the circle \( C_1 \), one has

\[ [f(q, n)]_{C_1} > 1 - R_{(1)} \]

Proof On the circle \( C_1 \), one has

\[ [f(q, n)]_{C_1} \geq f(-R_{(1)}; n) = 1 + \sum_{t=1}^{n} (-1)^t R_{(1)}^t + \sum_{t=n+1}^{2n-1} (-1)^t (2n-t)R_{(1)}^t + R_{(1)}^{2n} \]

Furthermore, the \( f(-R_{(1)}; n) \) in the above inequality can also be written as

\[
f(-R_{(1)}; n) = 1 + \sum_{t \in [0, \ldots, n]} R_{(1)}^t (t - (t + 1)R_{(1)}) \\
+ \sum_{t \in [n+1, \ldots, 2n-2]} R_{(1)}^t ((2n-t) - (2n-t-1)R_{(1)}) + R_{(1)}^{2n}.
\]

Note that for \( t \geq 1 \), it follows that \( t - (t + 1)R_{(1)} > 0 \) and \( t - (t-1)R_{(1)} > 0 \). Then, we have

\[ f(-R_{(1)}, n) > 1 - R_{(1)}. \]

Hence, the lemma is proved. \( \square \)

Lemma 4.4 If

\[ \left| \frac{M_{k-2}(q, n)}{M_k(q, n)} \right|_{C_1} < \frac{1}{R_{(1)}}, \]

then

\[ \left| \frac{M_k(q, n)}{M_{k+2}(q, n)} \right|_{C_1} < \frac{1}{R_{(1)}}. \]

Proof From (4.6), one has

\[
\left| \frac{M_{k+2}(q, n)}{M_k(q, n)} \right|_{C_1} \geq \left| f(q, n) \right|_{C_1} - R_{(1)}^{2n} \left| \frac{M_{k-2}(q, n)}{M_k(q, n)} \right|_{C_1}.
\]

By Lemma 4.3 and

\[
\left| \frac{M_{k-2}(q, n)}{M_k(q, n)} \right|_{C_1} < \frac{1}{R_{(1)}},
\]

it follows that

\[
\left| \frac{M_{k+2}(q, n)}{M_k(q, n)} \right|_{C_1} \geq \left| f(q, n) \right|_{C_1} - R_{(1)}^{2n} \left| \frac{M_{k-2}(q, n)}{M_k(q, n)} \right|_{C_1} > 1 - R_{(1)} - R_{(1)}^{2n-1} > R_{(1)} \text{ with } n \geq 2.
\]

On the other hand, the case of \( n = 1 \), has been proved (see [8, Proposition 3.1]). \( \square \)

Lemma 4.5 \( f(q, n) \) has no zeros inside \( D_1 \).

Proof The proof is similar to that of Theorem 1.2. Instead of (3.1), we consider

\[
\sum_{t=0}^{2n} \mu_t z^t,
\]

(4.7)
Lemma 4.7

On the circles \( C_n \)

By using experimental values of Table 1, one has

\[ \left| f(q, n) \right|_{C_1} > 2R_n^4. \]

Proof of Theorem 1.3: We prove that \( \mathcal{M}_k(q, n) \) has no zeros inside \( D_1 \) by induction on \( k \).

Assume that \( \mathcal{M}_k(q, n) \) has no zeros inside \( D_1 \). By Lemmas 4.3 and 4.4, the following inequality

\[ \left| f(q, n) \right|_{C_1} \geq R_n^{2q} \left| \frac{\mathcal{M}_{k-2}(q, n)}{\mathcal{M}_k(q, n)} \right|_{C_1} \]

holds. Since \( f(q, n) \) has no zeros inside \( D_1 \), by the Rouché theorem (see [1,14]), \( \mathcal{M}_{k+2}(q, n) \) has no zeros inside \( D_1 \). Hence, by induction on \( k \), the proof can be completed. \( \Box \)

Since \( \widehat{\mathcal{M}}_k(q, n) \) also satisfies (4.5), by the proof of Theorem 1.3, we have the following corollary.

**Corollary 4.6** The zeros of the polynomials \( \widehat{\mathcal{M}}_k(q, n) \) and \( \mathcal{M}_k(q, n) \) lie in the annulus \( A_1 \).

### 4.3 Proof of Theorem 1.4.

In order to prove Theorem 1.4, we need the following two lemmas.

**Lemma 4.7** On the circles \( C_3 \) and \( C_4 \), one has

\[ \left| f(q, 3) \right|_{C_3} > R_3^6 + R_3^3 + R_3^2, \]

\[ \left| f(q, 4) \right|_{C_4} > R_4^6 + R_4^5 + R_4^4 + R_4^3 + R_4^2. \]

Proof By using experimental values of Table 1, one has

\[ R_3^6 - R_3^5 + R_3^4 - 4R_3^3 + R_3^2 - R_3 + 1 > 0 \]

\[ R_4^6 - R_4^7 + R_4^6 - 4R_4^5 + 3R_4^4 - 4R_4^3 + R_4^2 - R_4 + 1 > 0. \]

Then

\[ \left| f(q, 3) \right|_{C_3} \geq f(-R_3, 3) = R_3^6 - R_3^5 + 2R_3^4 - 3R_3^3 + 2R_3^2 - R_3 + 1 \]

\[ > R_3^6 - R_3^5 + 2R_3^4 - 3R_3^3 - (R_3^6 - R_3^5 + R_3^4 - 4R_3^3 + R_3^2 - R_3 + 1) \]

\[ = R_3^4 + R_3^3 + R_3^2, \]

\[ \left| f(q, 4) \right|_{C_4} \geq f(-R_4, 4) = R_4^6 - R_4^7 + R_4^6 - 3R_4^5 + 4R_4^4 - 3R_4^3 + 2R_4^2 - R_4 + 1 \]

\[ > R_4^6 - R_4^7 + 2R_4^6 - 3R_4^5 + 4R_4^4 - 3R_4^3 + 2R_4^2 - R_4 + 1 \]

\[ - (R_4^6 - R_4^7 + R_4^6 - 4R_4^5 + 3R_4^4 - 4R_4^3 + R_4^2 - R_4 + 1) \]

\[ = R_4^4 + R_4^5 + R_4^4 + R_4^3 + R_4^2. \]

\( \Box \)
Lemma 4.8 For $n = 3, 4$, if \( \left| \frac{M_{k-2}(q, n)}{M_k(q, n)} \right|_{C_n} < \frac{1}{R(n)} \), then \( \left| \frac{M_k(q, n)}{M_{k+2}(q, n)} \right|_{C_n} < \frac{1}{R(n)} \).

Proof From (4.6), one has
\[
\left| \frac{M_{k-2}(q, n)}{M_k(q, n)} \right|_{C_n} \geq \left| f(q, n) \right|_{C_n} - R_{(n)}^2 \left| \frac{M_{k-2}(q, n)}{M_k(q, n)} \right|_{C_n}.
\]
By Lemma 4.7 and \( \left| \frac{M_{k-2}(q, n)}{M_k(q, n)} \right|_{C_n} < \frac{1}{R(n)} \), it follows that
\[
\begin{align*}
\left| \frac{M_{k+2}(q, 3)}{M_k(q, 3)} \right|_{C_3} &> R_4^3 + R_4^3 + R_4^3 - R_4^5 > R_3^5 > R_3^5, \\
\left| \frac{M_{k+2}(q, 4)}{M_k(q, 4)} \right|_{C_4} &> R_4^5 + R_4^5 + R_4^3 + R_4^3 + R_4^3 + R_4^3 - R_4^5 > R_4^3.
\end{align*}
\]
Hence the lemma is proved. \( \square \)

We now turn to the proof of Theorem 1.4.

Proof of Theorem 1.4: By induction on $k$, we prove that $M_k(q, n)$ has no zeros inside $D_n$ for $n = 3, 4$. Assume that $M_k(q, n)$ has no zeros inside $D_n$. Since $R_{(n)} \leq 1$, Lemma 4.7 implies that
\[
\left| f(q, 3) \right|_{C_3} > R_3^4 + R_3^4 + R_3^4 > 2R_3^4 > 2R_3^4 \geq \left| 2q^3 \right|_{C_3},
\]
\[
\left| f(q, 4) \right|_{C_4} > R_4^5 + R_4^5 + R_4^3 + R_4^3 + R_4^3 + R_4^3 > 2R_4^3 \geq \left| 2q^3 \right|_{C_4}.
\]
Since $f(q, n) + 2q^n = P_n(q)$, and $P_n(q)$ has no zeros inside $D_n$, we see that $f(q, n)$ has no zeros inside $D_n$, by using the Rouché theorem (see [1,14]).

On the other hand, by Lemmas 4.7 and 4.8, the inequality
\[
\left| f(q, n) \right|_{C_n} \geq R_{(n)}^2 \left| \frac{M_{k-2}(q, n)}{M_k(q, n)} \right|_{C_n}
\]
holds. Again, by the Rouché theorem (see [1,14]), $M_{k+2}(q, n)$ has no zeros inside $D_n$. By induction on $k$, the proof can be completed. \( \square \)

Since $\widetilde{M}_k(q, n)$ also satisfies (4.5), by the proof of Theorem 1.4, we have the following corollary which improves Corollary 4.6 for $n = 3$ and 4.

Corollary 4.9 For $n = 3, 4$, the zeros of the polynomials $\widetilde{M}_k(q, n)$ and $M_k(q, n)$ lie in the annulus
\[
\mathcal{A}_n := \{ q \in \mathbb{C} : R_{(n)} \leq |q| \leq R_{(n)}^{-1} \}.
\]

Remark 4.10 For $n \in \mathbb{N}$, we can estimate $\left| f(q, n) \right|_{C_n}$ by a direct calculation as in the proof of Lemma 4.7. However, for $n \geq 5$, there are some limitations to this approach to construct an inequality of $\left| f(q, n) \right|_{C_n}$ to check the following conditions.

(1) $\left| f(q, n) \right|_{C_n} \geq 2q^n|C_n|$;
(2) If $\left| \frac{M_{k-2}(q, n)}{M_k(q, n)} \right|_{C_n} < \frac{1}{R(n)}$, then $\left| \frac{M_k(q, n)}{M_{k+2}(q, n)} \right|_{C_n} < \frac{1}{R(n)}$. 

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References
1. Conway, J.B.: Functions of One Complex Variable I. Springer-Verlag, New York (1978)
2. Fock, V.V., Chekhov, L.: Quantum Teichmüller spaces. Theor. Math. Phys. 120, 1245–1259 (1999)
3. Frame, J.S.: Continued fractions and matrices. Am. Math. Monthly 56(2), 96–103 (1949)
4. Kogiso, T.: q-deformations and t-deformations of Markov triples. arXiv:2008.12913 (2020)
5. Kogiso, T., Wakui, M.: A bridge between Conway-Coxeter friezes and rational tangles through the Kauffman bracket polynomials. J. Knot Theory Ramif. 28(14), 1950083 (2019)
6. Konvalina, J., Matache, V.: Palindrome-polynomials with roots on the unit circle. C.R. Math. Acad. Sci. Soc. R. Can. 26, 39–44 (2004)
7. Leclere, L., Morier-Genoud, S.: q-deformations in the modular group and of the real quadratic irrational numbers. Adv. Appl. Math. 130, 28 (2021)
8. Leclere, L., Morier-Genoud, S., Ovsienko, V., Veselov, A.: On radius of convergence of q-deformed real numbers. arXiv:2102.00891 (2021)
9. Lee, K., Schiffler, R.: Cluster algebras and Jones polynomials, Selecta Math. (N.S.) 25, Paper No.58, 41pp (2019)
10. Markovsky, I., Rao, S.: Palindromic polynomials, time-reversible systems, and conserved quantities. In: Proceedings of the 16th Mediterranean Conference on Control and Automation. Congress Centre, Ajaccio, France. 25–27 June, pp. 125–130 (2008)
11. Morier-Genoud, S., Ovsienko, V.: q-deformed rationals and q-continued fractions. Forum Math. Sigma 8, e13 (2020)
12. Morier-Genoud, S., Ovsienko, V.: On q-deformed real numbers. Exp. Math. https://doi.org/10.1080/10586458.2019.1671922 (2019)
13. Nagai, W., Terashima, Y.: Cluster variables, ancestral triangles and Alexander polynomials. Adv. Math. 363, 106965 (2020)
14. Titchmarsh, E.C.: The Theory of Functions. Oxford University Press, Oxford (1958)

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