Space-time Curvature of Classical Electromagnetism

R. W. M. Woodside (rob.woodside@ucfv.ca)
Physics Dept., University College of the Fraser Valley
33844 King Rd., Abbotsford, B.C., Canada, V2S 7M8

Abstract.

The space-time curvature carried by electromagnetic fields is discovered and a new unification of geometry and electromagnetism is found. Curvature is invariant under charge reversal symmetry. Electromagnetic field equations are examined with De Rham co homology theory. Radiative electromagnetic fields must be exact and co exact to preclude unobserved massless topological charges. Weyl’s conformal tensor, here called “the gravitational field”, is decomposed into a divergence-free non-local piece with support everywhere and a local piece with the same support as the matter. By tuning a local gravitational field to a Maxwell field the electromagnetic field’s local gravitational field is discovered. This gravitational field carries the electromagnetic field’s polarization or phase information, unlike Maxwell’s stress-energy tensor. The unification assumes Einstein’s equations and derives Maxwell’s equations from curvature assumptions. Gravity forbids magnetic monopoles! This unification is stronger than the Einstein-Maxwell equations alone, as those equations must produce the electromagnetic field’s local gravitational field and not just any conformal tensor. Charged black holes are examples. Curvature of radiative null electromagnetic fields is characterized.

Keywords: Curvature, Electromagnetism, De Rham Cohomology, Conformal Tensor, Duality Rotation, Magnetic Monopole

Dedicated to the memory of Prof. M. H. L. Pryce (January 24, 1913–July 24, 2003)

1. Introduction

This work discovers the space-time curvature carried by the electromagnetic field and provides a new unification of geometry and classical electromagnetism. The new unification contains the Einstein equations to handle the mechanics and permits the derivation of the Maxwell equations from the full second Bianchi identities. This is a purely classical work and quantum considerations are merely mentioned.

Central to this work are the requirements that the electromagnetic field be expressed as a two form \( \mathbf{F} \) and fit into general relativity under the demand that the total stress-energy tensor used in the Einstein equations contain the Maxwell stress-energy tensor \( T_F \).
the notation with the conventions of \[1\] and in S.I. units $T_F$ is

$$T_F^{\alpha \beta} = \frac{\varepsilon_0}{2} (F^{\mu \alpha} F_{\mu \beta} + *F^{\mu \alpha} *F_{\mu \beta}), \quad (1.0.1)$$

where $\varepsilon_0 = 8.85418782 \cdot 10^{-12}$ farad/meter is the electric vacuum permittivity.

Originally \[2\] general relativity was conceived as a unification of mechanics and geometry that explained gravitation. It was just a bonus \[3\] that electromagnetism also entered the unification via equation \[1.0.1\]. If the Maxwell stress-energy tensor carried all the properties of the electromagnetic field, showing electromagnetism to be entirely reducible to mechanics, that would have been the end of the story.

However, the electromagnetic field has polarization or phase information that is not contained in the Maxwell stress-energy tensor \[4\]. Since Weyl’s conformal tensor, the totally traceless piece of Riemann curvature, is supposed to contain the phase or polarization information carried by gravitational radiation, one should expect it to do the same for electromagnetic radiation.

This is born out by the discovery of a piece of the Weyl conformal tensor that depends explicitly on the electromagnetic field and contains this polarization or phase information. It is denoted by $C_F$, called “the local gravitational field of the electromagnetic field”, and given by:

$$C_F^{\alpha \beta} = \frac{8\pi}{c^4} \frac{G\varepsilon_0}{4} \left\{ \frac{3}{2} \left( F^{\alpha \beta} F_{\gamma \delta} - *F^{\alpha \beta} *F_{\gamma \delta} \right) + \right. \\
- \frac{1}{4} \delta^{\alpha \beta} F^{\mu \nu} F_{\mu \nu} + \frac{1}{4} \eta^{\alpha \beta \gamma \delta} *F^{\mu \nu} F_{\mu \nu} \right\}, \quad (1.0.2)$$

where $G = 6.6762 \cdot 10^{-11}$ Newton-meter$^2$/kilogram$^2$ is Newton’s gravitational constant, $c = 2.99792458 \cdot 10^8$ meter/second is the speed of light, $\delta$ is a fully antisymmetric tensor, and $\eta$ is the permutation tensor. The traces in the expression \[1.0.2\] for $C_F$ are the Lorentz invariants of the electromagnetic field

$$F^{\mu \nu} F_{\mu \nu} = -2(E^2 - c^2 B^2) \quad (1.0.3a)$$

and

$$*F^{\mu \nu} F_{\mu \nu} = 4cE \cdot B, \quad (1.0.3b)$$

where $\vec{E}$ is the electric field strength in Volt/meter and $\vec{B}$ is the magnetic field strength in Tesla.

The remaining piece of curvature $M_F$ carried by the electromagnetic field is determined by the Einstein equations and as suggested
by equation (1.0.1) is

$$M_{\gamma\delta}^{\alpha\beta} = 8\pi \frac{G\epsilon_0}{c^4} \frac{1}{2} \left( F^{\alpha\beta} F_{\gamma\delta} + *F^{\alpha\beta} *F_{\gamma\delta} \right).$$  \hspace{1cm} (1.0.4)$$

The space-time curvature $R_F$ carried by the electromagnetic field is then

$$R_F = M_F + C_F.$$  \hspace{1cm} (1.0.5)$$

The second section of this paper examines various “electromagnetic” field equations from a topological viewpoint. In addition to the Maxwell equations, this section considers other “electromagnetic” field equations consistent with duality rotations [1, 5], classical mechanics, and null fields [6]. It also gives an attempt to derive the Maxwell equations from the Maxwell stress-energy tensor and Lorentz force density in general relativity, without the benefit of the new unification. This old derivation requires a trivial space-time topology to make all closed forms exact and recover the global vector potential in Maxwell’s equations. The non-trivial topology of the black hole solutions vitiates this derivation. However, the new unification precisely selects the exact Maxwell fields. This global vector potential in the Maxwell equations precludes the unobserved magnetic monopoles [5, 7, 8]. Thus a major conclusion of this work is that the unification of electromagnetism and geometry also forbids magnetic monopoles.

The second section also finds that “electromagnetic” null fields allow the possibility of massless topological magnetic or electric charges that have never been observed. The new unification removes the magnetic charges, but the removal of topological massless electric charges requires the introduction of a second vector potential making $F$ both exact and co exact for radiative fields. The Maxwell equations will admit this second global vector potential, but as they stand they also permit massless electric charges.

In the third section on space-time curvature the conformal tensor is identified as “the gravitational field”. It is invariantly split into a divergence free piece called “the non local gravitational field” with support everywhere; and a piece, usually with non zero divergence, called “the local gravitational field”. Most importantly, the local gravitational field has the same support as the matter. Also the Riemann curvature is decomposed by trace and split with an old technique [9] that is further developed here. This greatly simplifies calculations. It also permits the duality rotation of curvature, which shows that curvature is invariant under charge reversal symmetry. Any quantity explicitly dependant on the electromagnetic field that...
remains invariant under charge reversal symmetry must be quadratic
in the electromagnetic field.

In the fourth section, these curvature decompositions make it a
simple matter to find a tensor quadratic in \( F \) and with the sym-
metries of a conformal tensor. This defines \( C_F \) up to an overall
numerical factor. Specific examples can then be checked to find this
factor. For an electrically charged spherical black hole the value is
+3. For a magnetically charged spherical black hole the value is −3.
Of course conformally flat electro vac solutions give a value of zero.
By picking the value +3 the local gravitational field is tuned precisely
to the physical Maxwell fields. Thus the electrically charged spherical
black hole provides a physical example showing that \( C_F \) exists and
is given by \( \text{(1.0.2)} \).

The fifth section presents this new curvature based unification.
Here the full second Bianchi identities require the introduction of a
third rank tensor as a generalized Lorentz force density that traces
down to the usual Lorentz force density. The traceless piece of this
generalized Lorentz force density requires that a local gravitational
piece of the current’s curvature leak out beyond the support of the
currents and have the same support as the electromagnetic field.

The sixth section gives the charged spherical and charged rotating
black holes as examples of the new unification. While a radiative null
field example is not given, the curvature of such expected physical
solutions is characterized.

The concluding section points out some new directions that this
work suggests.

2. Classical Electromagnetism

Classical electromagnetism is a well verified theory in the limit that
the currents and fields are continuous matter distributions. It fails
miserably with granularities in the field (photons) and the currents
(electrons). Similarly, general relativity deals with continuous cur-
vature and continuous matter distributions described by the stress-
energy tensor and cannot deal with the granularity of matter. Nearly
a century of work trying to find this quantum limit of general rel-
ativity has shown that radically new ideas are required. The effort
continues to be hampered by the lack of experimental results. One
radical idea suggested \( \text{(10)} \) by general relativity is that the topology
of space-time is not trivial, as in the black hole solutions. In the
quantum processes of emission and absorption the spatial support
of the energy tensor changes from connected to disconnected and
vice versa. Although the curvature carried by electrons and photons will be experimentally inaccessible for a very long time, this curvature must be in the quantum limit of general relativity. While awaiting new experimental directions, it is useful to examine classical electromagnetism from the topological perspective of De Rham’s cohomology theory [11].

2.1. The Maxwell Equations

The Maxwell equations relate the electromagnetic field $F$, the vector potential $A$, and the electric current $J$ by

$$\mathbf{F} = d\mathbf{A}, \quad d \ast \mathbf{F} = \frac{1}{\varepsilon_0} \ast \mathbf{J}. \quad (2.1.1)$$

The first Maxwell equation says that the cohomology class of the global vector potential $\mathbf{A}$ determines the electromagnetic field $\mathbf{F}$. This demands a Maxwell field to be exact and forbids the existence of magnetic monopoles [5, 7, 8]. The second says that the cohomology class of the form dual to $\mathbf{F}$ determines the electric current three-form $\ast \mathbf{J}$. Superficially this permits two kinds of Maxwell fields, those with sources where $\mathbf{J}$ is not zero satisfying equations (2.1.1) and the source free Maxwell fields obeying the source free Maxwell equations

$$\mathbf{F} = d\mathbf{A}, \quad d \ast \mathbf{F} = 0. \quad (2.1.2)$$

Topologically there are two types of source free Maxwell fields. Only topologically non-trivial space-times will permit non-trivial solutions to equations (2.1.2) where $\ast \mathbf{F}$ is closed and not exact. There are also the trivial solutions to equations (2.1.2) where $\ast \mathbf{F}$ is exact

$$\mathbf{F} = d\mathbf{A}, \quad \ast \mathbf{F} = dB \quad (2.1.3)$$

and $\mathbf{B}$ is a global one form.

The source free equations (2.1.2) were introduced to handle the radiation fields and are very successful in the classical regime. However, all observed classical radiation fields obey equations (2.1.3) and since all exact forms are closed, they trivially solve the source free equations (2.1.2). A non-trivial radiative solution to the source free equations would correspond to a topological charge travelling at light speed. No charges have ever been observed to travel at the speed of light and quantum theoretic arguments have been made against them [12].

All exact forms are closed and Poincaré’s lemma [11] gives the difference between them. This states that on any region smoothly
contractible to a point, all closed forms are exact. So the existence of a closed, not exact form is the hallmark of a topological hole [11] in space-time. Long ago it was pointed out [10] that one can take any space-time that has Maxwell fields with sources and commit space-time surgery to remove all those regions with current. The result is a source free Maxwell field with $\ast \mathbf{F}$ being closed and not exact in a space-time with horrible boundaries. The charged black holes are an example. They are non-trivial solutions to the source free equations (2.1.2) with topological charge $Q$ found with

$$Q = \varepsilon_0 \int_S \ast \mathbf{F},$$

(2.1.4)

where $S$ is a two sphere enclosing the topological hole that by De Rham’s theorem [11] belongs to the closed and not exact $\ast \mathbf{F}$. It was also thought [10] that the continuous currents might be some smooth average over topological charges, making the source free equations (2.1.2) more fundamental than the Maxwell equations (2.1.1).

It is tempting to think that the source free equations (2.1.2) are the fundamental ones with the topological charges possessing asymptotic rest frames and the radiative solutions obeying equations (2.1.3) with no asymptotic rest frame. If this were the case and there was some reason to believe that only a certain number of field lines could thread a topological hole, then one would have the granularity of charge arising in a believable way. The atomicity of charge is such a deep problem that it led to the introduction of magnetic monopoles [5, 8] and the abandonment of the global vector potential $\mathbf{A}$.

## 2.2. Duality Rotations

The finishing touch on the Maxwell equations was the requirement of symmetry between the source free electric and magnetic fields. Analysing the fields in a charging capacitor physically demanded this aesthetic argument. On purely aesthetic grounds, one could imagine a complete symmetry between electric and magnetic currents $\mathbf{J}_D$:

$$d \mathbf{F}_D = \frac{1}{\varepsilon_0} \ast \mathbf{J}_D, \quad d \ast \mathbf{F}_D = \frac{1}{\varepsilon_0} \ast \mathbf{J}. \quad (2.2.1)$$

These “electromagnetic” fields $\mathbf{F}_D$ trivially contain the Maxwell fields. However, their non-trivial solutions are not Maxwell fields as they have no global vector potential $\mathbf{A}$. No such magnetic currents, even the topological ones [7, 8], have ever been observed. The topological magnetic currents arise as non-trivial solutions to

$$d \mathbf{F}_D = 0, \quad d \ast \mathbf{F}_D = \frac{1}{\varepsilon_0} \ast \mathbf{J}. \quad (2.2.2)$$
The magnetically charged black holes are a special case and obey

\[ dF_D = 0, \quad *F_D = -dA, \]  

(2.2.3)

where \( A \) is a global potential for the Maxwell field of the corresponding electrically charged black hole. The topological magnetic charge \( Q_M \) is found with

\[ Q_M = \varepsilon_0 \int_S F_D, \]  

(2.2.4)

where \( S \) is a two sphere enclosing the topological hole that is associated with the closed and not exact \( F_D \).

All these \( F_D \) come from duality rotated Maxwell fields [1, 5]. Given any two form \( F \), a duality rotated two form \( e^{*\alpha} F \) is defined by

\[ e^{*\alpha} F = F \cos \alpha + *F \sin \alpha, \]  

(2.2.5)

where \( \alpha \) is a real number. Only when \( F \) and its dual have the same topology will the duality rotated \( e^{*\alpha} F \) have the same topology as \( F \). The only Maxwell fields with this property occur in equations (2.1.3) where \( F \) and \(*F\) are both exact. Regardless of the differential properties of \( F \) the duality rotation (2.2.5) maps two components from two ‘planes’, one in \( F \) and one in \(*F\), into one ‘plane’ in \( e^{*\alpha} F \). On every quarter turn the mapping is one to one and invertible. These quarter turns correspond to potential symmetries that \( F \) might enjoy.

When \( \alpha = \frac{\pi}{2} \), the duality rotated field \( e^{*\alpha} F \) is precisely \(*F\), the dual of the field \( F \). The dual operation on a two form shuffles space-time components into space-space components and vice versa, interchanging the roles of the purely electric fields and purely magnetic fields. This interchange symmetry must turn electric currents into magnetic currents and vice versa as the currents produce the fields. Interchange symmetry requires the electromagnetic field to obey the field equations (2.2.1).

When \( \alpha = \pi \), the new \( e^{*\alpha} F \) is merely the field \( F \) with a sign reversal. This occurs when the sign of the currents or vector potentials is reversed. All the electromagnetic field equations presented here enjoy this charge reversal symmetry. To respect this symmetry, any quantity explicitly dependant on an electromagnetic field must be quadratic in the field.

It is well known [1, 4, 10] that any duality rotated field \( e^{*\alpha} F \) gives rise to the same Maxwell stress-energy tensor as \( F \) itself in equation (1.0.1). Looking only at the mechanical properties \( T_F \), one could not tell them apart! This is disastrous in general relativity where, in the electro vac case, one inserts \( T_F \) made from a source free Maxwell field \( F \) into the Einstein equations and solves for the metric.
tensor. How different is the space-time when it carries a duality-rotated electromagnetic field with the same $T_F$? Thinking physically about the difficulty, one realizes that the gravitational fields of these duality-rotated electromagnetic fields should be different.

2.3. Requirements of Classical Mechanics

The connection between electromagnetism and classical mechanics is made with the Maxwell stress-energy tensor $T_F$ given in equation (1.0.1). In classical mechanics the external force density on a fluid, whose mechanical properties are specified by a stress-energy tensor, is given by the divergence of that stress-energy tensor. The electric currents also have mechanical properties given by a stress-energy tensor $T_J$ and, in the absence of other external agents, its divergence is the Lorentz force density $f_L$. This Lorentz force density is exerted on the currents by the electromagnetic field and has the component expression

$$T_J^{\mu \alpha}_{\mu} = f_L^{\alpha} = J^{\mu}F^{\alpha}_{\mu}. \quad (2.3.1)$$

Assuming the Maxwell equations, the Lorentz force law, and classical mechanics, an old argument \cite{3} permits the derivation of the Maxwell stress-energy tensor. Given two of the three electromagnetic items and classical mechanics, one can usually recover the third. Here the argument is adapted to find the electromagnetic field equations required by classical mechanics, the Maxwell stress-energy tensor and the Lorentz force law.

When no other agents are present the total stress-energy tensor $T$ is the point wise sum of $T_J$ and $T_F$

$$T = T_J + T_F. \quad (2.3.2)$$

With nothing else to push on the system, $T$ must be divergence free and obey the component equation

$$T^{\mu}_{\alpha,\mu} = 0. \quad (2.3.3)$$

Combining the last three equations yields

$$f_L^{\alpha} = J^{\mu}F^{\alpha}_{\mu} = T_J^{\mu}_{\alpha,\mu} = -T_F^{\mu}_{\alpha,\mu}. \quad (2.3.4)$$

Now, only using tensor identities including

$$*F^{\alpha}_{\beta} = \frac{1}{2}\eta^{\alpha\beta\mu\nu}F^{\mu\nu}, \quad F^{\alpha}_{\beta} = -\frac{1}{2}\eta^{\alpha\beta\mu\nu}*F^{\mu\nu}, \quad (2.3.5)$$

$$F^{\mu\nu}_{\beta,\alpha} = *F^{\alpha\beta}_{\gamma} + *F^{\gamma\alpha}_{\beta} + *F^{\beta\gamma}_{\alpha},$$

$$- *F^{\mu\nu}_{\beta,\alpha} = F^{\alpha\beta}_{\gamma} + F^{\gamma\alpha}_{\beta} + F^{\beta\gamma}_{\alpha}, \quad (2.3.6)$$
and without using any electromagnetic field equations the divergence of equation (1.0.1) can be beaten into

\[ T_{F}^{\mu \alpha; \mu} = \varepsilon_{0}(F^{\mu \nu}_{\alpha \mu} F_{\mu \alpha} + *F^{\mu \nu}_{\alpha \mu} *F_{\mu \alpha}). \]  

(2.3.7)

Comparing the equations (2.3.4) and (2.3.7), one finds

\[ J^{\mu}F_{\mu \alpha} = \varepsilon_{0}(F^{\mu \nu}_{\mu \nu} F_{\mu \alpha} + *F^{\mu \nu}_{\mu \nu} *F_{\mu \alpha}). \]  

(2.3.8)

Generally, this can be true only when

\[ F^{\alpha \mu}_{\mu} = \frac{1}{\varepsilon_{0}}J^{\alpha}, \quad *F^{\alpha \mu}_{\mu} = 0. \]  

(2.3.9)

Rewriting these equations in differential forms returns equations (2.2.2) that are not the Maxwell equations. In the early part of the last century it was assumed that space-time topology was trivial and under that assumption equations (2.3.9) do become the Maxwell equations.

It could be argued that the existence of magnetic currents requires the Lorentz force law to be taken as

\[ T_{J}^{\mu \alpha; \mu} = f_{L \alpha} = J^{\mu}F_{D \alpha \mu} - J_{D \mu}^{\mu} *F_{D \alpha \mu}. \]  

(2.3.10)

Where upon the previous argument requires the field equations to be

\[ F_{D}^{\alpha \mu}_{\mu} = \frac{1}{\varepsilon_{0}}J^{\alpha}, \quad *F_{D}^{\alpha \mu}_{\mu} = -\frac{1}{\varepsilon_{0}}J_{D}^{\alpha}. \]  

(2.3.11)

These give rise to the field equations (2.2.1) when written as differential forms. Thus with the modified Lorentz force density (2.3.10), classical mechanics can accommodate both types of magnetic currents occurring in the field equations (2.2.2) and (2.2.1).

2.4. Null Electromagnetic Fields

The null electromagnetic fields [6] occur when the Lorentz invariants (1.0.3) vanish everywhere. This algebraic constraint requires the electric and magnetic fields at any point to be perpendicular and the magnitude of the electric field to be the speed of light times the magnitude of the magnetic field. The plane spanned by the electric and magnetic field vectors is perpendicular to the momentum density or Poynting vector of the electromagnetic field at that point. A duality rotation of \( \alpha \) merely rotates both electric and magnetic field vectors by an angle of \( \alpha \) about an axis along the Poynting vector, while leaving them perpendicular and with the same magnitudes. If
the null field were a plane polarized electromagnetic wave, a duality rotation of $\alpha$ would rotate the plane of polarization by $\alpha$. If the null field were a circularly polarized plane wave then the duality rotation of $\alpha$ would give the wave a phase shift $\alpha$. Thus the duality angle, which is overlooked by the Maxwell stress-energy tensor, fixes the polarization or phase information of the electromagnetic field.

Null fields arise when $F$ and $*F$ share a common direction $L$ that must be null. In geometric units the fields are

$$F = L \wedge P, \quad *F = L \wedge Q,$$

(2.4.1)

where $P$ and $Q$ must be perpendicular spacelike one forms. This expression and the four dimensionality of space-time require

$$g(L, L) = g(L, P) = g(L, Q) = g(P, Q) = 0, \quad g(P, P) = g(Q, Q) > 0.$$  \hspace{1cm} (2.4.2)

The last equality follows from the vanishing of the Lorentz invariant in the well-known identity

$$F^{\mu\alpha} F^{\mu\beta} - *F^{\mu\alpha} *F^{\mu\beta} = \frac{1}{2} \delta^{\alpha}_{\beta} F^{\mu\nu} F_{\mu\nu}.$$  \hspace{1cm} (2.4.3)

Notice that since $L$ is null, the first of equations (2.4.2) fixes $P$ and $Q$ up to a constant multiple of $L$.

The currents are divergences of the fields and here one has

$$\text{Div}(F) = [P, L] + (\text{Div} P) L - (\text{Div} L) P,$$

(2.4.4a)

$$\text{Div}(*F) = [Q, L] + (\text{Div} Q) L - (\text{Div} L) Q.$$  \hspace{1cm} (2.4.4b)

The $L$, $P$, and $Q$ can always be chosen so that

$$[P, L] = [Q, L] = 0, \quad \text{Div} L = \text{Div} P = \text{Div} Q = 0,$$

(2.4.5)

forcing the currents to vanish. Such null fields obey the field equations

$$dF_{D} = 0, \quad d *F_{D} = 0.$$  \hspace{1cm} (2.4.6)

So these null fields permit topological electric and magnetic currents where either $F_{D}$ or $*F_{D}$ are closed and not exact. Like the Maxwell fields of equations (2.1.3), these $F_{D}$ have the same topology under duality rotation. The difference between the field equations (2.1.3) and (2.4.6) are the topological currents permitted by equations (2.4.6). These topological currents correspond to magnetic or electric charges travelling at light speed. These unobserved
charges can be removed and the field equations (2.4.6) reduced to the Maxwell fields (2.1.3) by requiring the null fields to have the potentials

\[ A = \lambda P, \quad B = \lambda Q, \]  

(2.4.7)

where \( P \) and \( Q \) are closed or exact

\[ dP = 0, \quad dQ = 0, \]  

(2.4.8a)

and \( \lambda \) is a function, such that

\[ L = d\lambda. \]  

(2.4.8b)

With the currents vanishing, the divergence of the Maxwell stress-energy tensor also vanishes by equation (2.3.7). Applying this condition to equation (1.0.1) for the Maxwell stress-energy tensor built with the null fields (2.4.1) yields the following additional constraints on \( L, P, \) and \( Q \)

\[ \nabla_L L = \nabla_L P = \nabla_L Q = 0. \]  

(2.4.9)

So \( L \) is tangent to a null geodesic along which \( P \) and \( Q \) are parallel.

There are some very unphysical null fields [13] and one wonders how well they model the physical radiation fields. By radiation fields one has in mind beams or pulses of light travelling in otherwise empty space. Such objects have well measured mechanical properties [14, 15] and should obey the source free Maxwell equations (2.1.3).

### 3. Space-time Curvature

The full Riemann curvature \( R \) of space-time decomposes by trace into a piece \( M \) consisting only of curvature traces, and the totally traceless remains \( C \) of the full curvature, known as Weyl’s conformal tensor

\[ R = M + C. \]  

(3.0.1)

The Einstein equations give the components of \( M \) explicitly in terms of the total stress-energy tensor of the matter present in the space-time. Only by solving the Einstein equations for the metric tensor can one directly find the full curvature \( R \) and, consequently, \( C \). Since \( C \) is the only piece of the curvature that survives outside the matter distribution \( M \), it merits being named “the gravitational field” of the matter described by \( M \). Further, when freely falling towards \( M, C \) is the only one of the obvious candidates for “the gravitational field” that does not adopt its flat space-time values. So far no explicit
conditions have been placed on C. It is whatever it has to be to permit a solution of the Einstein equations. At worst it vanishes in conformally flat solutions; so that whatever bizarre matter or boundary conditions obtain, this matter has no gravitational field. The hallmark for the physical existence of any material thing is its gravitational field. This forces consideration of the gravitational field of the electromagnetic field and removes conformally flat solutions as unphysical.

3.1. Splitting the Conformal Tensor

The full second Bianchi identity is conveniently written as the divergence on the first index of \( \ast R \ast \) and denoted by

\[
\text{Div} \ast R \ast = 0, \tag{3.1.1}
\]

where \( \ast R \ast \) is the double dual of the full curvature defined by

\[
\ast R \ast^{\alpha\beta \gamma\delta} = \frac{1}{4} \eta^{\alpha\beta\mu\nu} R_{\mu\nu \rho\lambda} \eta_{\rho\lambda\gamma\delta}. \tag{3.1.2}
\]

The double dual of any fourth rank tensor may be analogously defined. The decomposition (3.1.1) and the identity (3.1.1) allows the invariant decomposition of the gravitational field into

\[
C = C_0 + C_1, \tag{3.1.3}
\]

where \( C_0 \) and \( C_1 \) respectively obey

\[
\text{Div} \ast C_0 \ast = 0, \quad \text{Div} \ast C_1 \ast = -\text{Div} \ast M \ast \neq 0. \tag{3.1.4}
\]

The homogeneous piece \( C_0 \) is the non-local part of the gravitational field and depends on the arrangement of distant matter containing fields, currents, and neutral matter. The inhomogeneous piece \( C_1 \) is the local gravitational field of the matter \( M \) and has the same support as \( M \). Although algebraically and geometrically very different than \( M \), \( C_1 \) will contain the same arbitrary functions that comprise \( M \), ensuring their common support.

3.2. Splitting the Riemann Tensor

Decomposing the curvature \( R \) by trace yields three terms

\[
R = M_2 + M_1 + C, \tag{3.2.1}
\]

where \( M_2 \) depends on the curvature scalar \( R \)

\[
M_2^{\alpha\beta \gamma\delta} = \frac{1}{12} \delta^{\alpha\beta}_\gamma \delta^{\gamma\delta}_\delta R. \tag{3.2.2}
\]
and $M_1$ depends on the trace free part of the Ricci tensor

$$M_1^{\alpha\beta\gamma\delta} = -\frac{1}{2}\delta^{\alpha\beta\rho\lambda}\left(R_\rho^\lambda - \frac{1}{4}\delta^\lambda_{\rho\delta}R\right). \quad (3.2.3)$$

This decomposition is unique in that the subscripts on $M_2$ and $M_1$ refer to the number of non-zero traces in each term. The term $M$ in equation (3.0.1) is the sum of $M_2$ and $M_1$. The trace of the Einstein equations determines the curvature scalar in $M_2$, and $M_1$ is found with the trace free part of the Einstein equations.

Just as one splits second rank tensors into symmetric and anti-symmetric pieces. One can define \([9]\) for fourth rank tensors

$$R^+ = \frac{1}{2}(R + *R^*), \quad (3.2.4a)$$

$$R^- = \frac{1}{2}(R - *R^*). \quad (3.2.4b)$$

Since this application of the Hodge star obeys $** = -1$, one discovers \([9]\)

$$*R^+* = +R_+, \quad *R_+ + R_+* = 0, \quad (3.2.5a)$$

$$*R^-* = -R_-, \quad *R_- - R_-* = 0. \quad (3.2.5b)$$

Analogous relations to (3.2.4) and (3.2.5) hold for any fourth rank tensor. It is well known that $C$ behaves like $R_-$, as does $M_2$ by equation (3.2.2). The traceless Ricci piece $M_1$ behaves like $R_+$ according to equation (3.2.3). Using this to find the double dual of equation (3.2.1) yields

$$*R^* = -M_2 + M_1 - C, \quad (3.2.6)$$

which allows the second Bianchi identity (3.1.1) to be written

$$\text{Div} *R^* = -\text{Div} M_2 + \text{Div} M_1 - \text{Div} C = 0. \quad (3.2.7)$$

All the terms in the middle expression are of the form

$$D^\alpha_{\beta\gamma} = T^{\mu\alpha}_{\beta\gamma\mu}, \quad (3.2.8)$$

where $D$ is antisymmetric on its lower two indices. Such a third rank tensor can be decomposed by trace, into a trace free piece and a piece with trace. Since $C$ is totally traceless, its divergence must also be traceless. Examining equation (3.2.2) shows that the divergence of $M_2$ can have no traceless piece. Thus it is left to the divergence of $M_1$ to provide both trace and traceless pieces to annihilate the other divergences.
Applying this to the electromagnetic field with a traceless $T_F$, there is only an $M_1$ component for the mechanical contribution of $F$ to the curvature. This is the $M_F$ reported in equation (1.0.4). For the second Bianchi identity (3.1.4) to work in general there must also be a gravitational piece $C_F$ of the curvature also carried by the electromagnetic field. Thus the second Bianchi identity and the Einstein equations with $T_F$, generally require the existence of the local gravitational field carried by the electromagnetic field.

3.3. Duality Rotation of Curvature

Duality rotation is mere index shuffling and must not be restricted to electromagnetic fields. With the electromagnetic field carrying a curvature, one must shuffle the curvature indices in the same way to get the curvature carried by the duality-rotated electromagnetic field. Applying it to curvature would give

$$e^{*\alpha}R = R \cos \alpha + *R \sin \alpha. \quad (3.3.1)$$

Since all the ‘planes’ in $F$ are duality rotated, the same should be done for curvature, naturally giving

$$e^{*\alpha}Re^{*\alpha} = R \cos^2 \alpha + *R \sin^2 \alpha + (*R + R*) \cos \alpha \sin \alpha. \quad (3.3.2)$$

This expression simplifies considerably when the components $R_+$ and $R_-$ from equation (3.2.4) are used. With equation (3.2.5) one gets

$$e^{*\alpha}R_+e^{*\alpha} = R_+ \quad (3.3.3a)$$

$$e^{*\alpha}R_-e^{*\alpha} = e^{*2\alpha}R_- \quad (3.3.3b)$$

So $M_1$ and $M_F$, behaving like $R_+$, are invariant under duality rotation, but $M_2$, $C$, and $C_F$, behaving like $R_-$ are only invariant under a duality rotation of $n\pi$ with integer $n$. One can confirm these results by inserting $e^{*\alpha}F$ for $F$ in equation (1.0.4) for $M_F$, and in equation (1.0.2) for $C_F$.

A duality rotation of $\pi$ corresponds to charge reversal symmetry. Thus the mechanical properties, the gravitational fields and curvature of generic space-times will be the same for charge reversed Maxwell fields and currents. The mechanical properties $M$ of most matter will fix the duality angle up to a half turn with a non-vanishing $M_2$ component, which requires a non-vanishing trace of the stress-energy tensor. This luxury is unavailable with the traceless Maxwell stress-energy tensor and so it is left to the gravitational field to set the duality angle up to a half turn. Note that a quarter turn
producing magnetic monopoles reverses the sign of $C_F$. If magnetic monopoles existed, it would be impossible to consistently define a $C_F$.

### 4. The Local Gravitational Field Carried by the Electromagnetic Field

The foregoing makes the determination of $C_F$ very straightforward. Charge reversal symmetry requires it to be quadratic in $F$ and the symmetries of a gravitational field require it to obey the analogue of equation (3.2.4b). With $M_F$ given by equation (1.0.4) a natural candidate for $C_F$ is

$$F \otimes F - *F \otimes *F.$$  

(4.0.1)

This expression has two symmetric blocks of antisymmetric indices. But unlike a gravitational field, it can have a non-zero trace and can span 4-volume. To restore these indicial symmetries, one simply removes these traces with the help of the identity (2.4.5) for the Lorentz invariant (1.0.3a) and the 4-volume containing the Lorentz invariant (1.0.3b)

$$\frac{1}{24} \eta_{\mu\nu\rho\lambda} (F^{\mu\nu} F^{\rho\lambda} - *F^{\mu\nu} *F^{\rho\lambda}) = \frac{1}{6} *F^{\mu\nu} F_{\mu\nu},$$  

(4.0.2)

to get

$$C_F^{\alpha\beta}_{\gamma\delta} = 8\pi G \frac{\varepsilon_0}{c^4} \left( \frac{k}{2} \right) (F^{\alpha\beta} F_{\gamma\delta} - *F^{\alpha\beta} *F_{\gamma\delta} +$$

$$- \frac{1}{6} \delta^{\alpha\beta}_{\gamma\delta} F^{\mu\nu} F_{\mu\nu} + \frac{1}{6} \eta^{\alpha\beta}_{\gamma\delta} *F^{\mu\nu} F_{\mu\nu}),$$  

(4.0.3)

where $k$ is an undetermined constant. Fortunately $\delta$ spans no 4-volume, $\eta$ has no trace, and both obey the analogue of equation (3.2.5b). Thus equation (4.0.3) has all the symmetries of a gravitational field. All attempts to determine the constant $k$ from the second Bianchi identities failed. However, an example with a Maxwell field will at once show the existence of $C_F$ and give the constant $k$. Amazingly this will smuggle in a global vector potential, as this choice for $k$ will globally set the duality angle in $C_F$ to that of a Maxwell field.

The simplest example, although algebraically special, is an electrically charged spherical black hole whose metric in Schwarzschild
coordinates is
\[ ds^2 = -\left(1 - 2\frac{M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - 2\frac{M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2, \] (4.0.4)

where \( M \) is the singularity’s mass and \( Q \) its charge, both in geometric units. The electromagnetic field, also in geometric units is
\[ F = -\frac{Q}{r^2}dt \wedge dr. \] (4.0.5)

This field solves the source free Maxwell equations (2.1.2). The fact that \( *F \) is closed and not exact gives rise \([10]\) to the topological charge \( Q \). In these coordinates there is no magnetic field and so only the first Lorentz invariant (1.0.3a) is not zero. Using the metric (4.0.4) to calculate the full curvature, one finds
\[ R = C_0 + M_F + C_F. \] (4.0.6)

Here \( C_0 \) is divergence free and proportional to \( \frac{M}{r} \) and \( M_F \) is found by inserting the \( F \) from equation (4.0.5) into the general expression (1.0.4) for \( M_F \). Both \( M_F \) and \( C_F \) in equation (4.0.6) are proportional to \( \frac{Q^2}{r^2} \). To get agreement with this \( C_F \) and the expression (4.0.3), with expression (4.0.5) for \( F \) inserted, requires
\[ k = +3. \] (4.0.7)

Putting this value of \( k \) in the expression (4.0.3), it becomes the expression for \( C_F \) reported in equation (1.0.2).

5. New Unification of Relativity and Classical Electromagnetism

The first assumption of this curvature based unification is that the total curvature of the electrified matter is broken into a current piece \( R_J \) and a field piece \( R_F \)
\[ R = R_J + R_F. \] (5.0.1a)

By the Einstein equations this implies the splitting of the total stress-energy tensor into a current and field piece as in equation (2.3.2). The second assumption is the component expression for the curvature
carried by the classical electromagnetic field

\[ R_F^{\alpha\beta}_{\gamma\delta} = 8\pi \frac{G\varepsilon_0}{c^4} \times \]
\[ \left( 2F^{\alpha\beta}F_{\gamma\delta} - \ast F^{\alpha\beta} \ast F_{\gamma\delta} - \frac{1}{4}\delta^{\alpha\beta}_{\gamma\delta}F^{\mu\nu}F_{\mu\nu} + \frac{1}{4}\eta^{\alpha\beta}_{\gamma\delta} \ast F^{\mu\nu}F_{\mu\nu} \right) \]

(5.0.1b)

Applying the Einstein equations to the trace of this expression returns equation (1.0.1) for \( T_F \). Finally to introduce the current and permit the derivation of the Maxwell equations a generalized Lorentz force \( F_L \) is introduced. In components it is

\[ F_L^{\alpha\beta}_{\gamma\delta} = \delta^{\mu\alpha}_{\beta\gamma}J^\nuF_{\mu\nu} - J^\alphaF_{\beta\gamma} - \delta^{\mu\nu}_{\beta\gamma}J_\muF^\alpha_{\nu} + \]
\[ -\varepsilon_0 \left( \ast F^{\alpha\beta} \ast F_{\gamma\delta} + \delta^{\beta\gamma}_{\gamma\delta} \ast \ast F^\alpha_{\mu} \ast F^\mu_{\nu\rho} \right). \]

(5.0.1c)

Tracing this expression on \( \alpha \) and \( \gamma \) gives the Lorentz force density (2.3.1). This new unification contains all the ingredients for deriving the field equations (2.2.2) required by classical mechanics. The reduction of a closed \( F \) to an exact one as required by the Maxwell equation happens because the duality angle of \( R_F \) has been set to a Maxwell field by the choice of \( k \) in equation (4.0.7).

The generalized Lorentz force \( F_L \) contains all possible third rank tensors of the form ‘current times field’. It is important to observe that the terms with field derivatives in them can be non-zero outside the current distributions. This leads one to expect that a ‘local’ piece of the current’s gravitational field \( C_{J1} \), whose divergence is not zero, leaks out with the electromagnetic field into the ‘vacuum’ beyond the support of \( J \). This is required as the assumed curvature split (5.0.1a) into a current piece \( R_J \) and a field piece \( R_F \) can cut across both \( M \) and \( C \) in the curvature split (3.0.1).

The generalized Lorentz force \( F_L \) was found analogously to deriving the Lorentz force from the Maxwell equations, the Maxwell stress-energy tensor and classical mechanics. To start one uses the full second Bianchi identity (3.1.1) applied to equation (5.0.1a)

\[ \text{Div} \ast R = \text{Div} \ast R_J \ast + \text{Div} \ast R_F^\ast = 0. \]

(5.0.2)

In analogy with equation (2.3.4), the generalized Lorentz force \( F_L \) is defined by

\[ 8\pi \frac{G}{c^4}F_L = \text{Div} \ast R_J^\ast = -\text{Div} \ast R_F^\ast. \]

(5.0.3)

Remembering that the components \( M_F \) and \( C_F \) behave respectively like the expression (3.2.5a) and (3.2.5b), this last equation becomes

\[ 8\pi \frac{G}{c^4}F_L = -\text{Div} M_F + \text{Div} C_F. \]

(5.0.4)
The helpful identities

\[
\frac{1}{2} \eta^{\alpha \gamma} \gamma \delta \ast F_{\mu \nu} F_{\mu \nu} = \delta^{\mu \nu} \left( F_{\mu}^{\alpha} F_{\nu}^{\beta} - \ast F_{\mu}^{\alpha} \ast F_{\nu}^{\beta} \right) + \\
- \left( F^{\alpha \beta} F_{\gamma \delta} - \ast F^{\alpha \beta} \ast F_{\gamma \delta} \right),
\]

(5.0.5)

and

\[
\frac{1}{4} \delta^{\alpha \beta} F_{\mu \nu} F_{\mu \nu} = - \frac{1}{2} \delta^{\alpha \beta} \left( F_{\mu \nu} - \ast F_{\mu \nu} \right),
\]

(5.0.6)

are inserted into \( \mathbf{C}_F \) in equation (1.0.2). This \( \mathbf{C}_F \), and \( \mathbf{M}_F \) from equation (1.0.4) are put into equation (5.0.4). This allows the right hand side of equation (5.0.3) to be beaten into divergences of \( \mathbf{F} \) and \( \ast \mathbf{F} \) and the field derivatives in (5.0.1c). This calculation also involves the identities (2.3.5) and (2.3.6). Finally substituting the currents for the divergences according to equation (2.3.9), then gives the expression reported in assumption (5.0.1c).

6. Examples

Here the charged black holes are presented as examples of the unification (5.0.1). Both of these black hole solutions are examples of electro vac universes with topological electric currents arising from the fact that \( \ast \mathbf{F} \) is closed and not exact.

Although no example for physical radiative null fields is given, the curvature of such expected solutions is characterized with the new unification.

6.1. \( \mathbf{F}_{\mu \nu} \mathbf{F}_{\mu \nu} \neq 0 \), \( \ast \mathbf{F}_{\mu \nu} \mathbf{F}_{\mu \nu} \neq 0 \)

A formidable physical example is the rotating charged black hole whose metric in Boyer Lindquist coordinates and geometrised units is

\[
ds^2 = - \left( 1 - \frac{2Mr - Q^2}{r^2 + u^2} \right) dt^2 - 2 \frac{(a^2 - u^2)}{a (r^2 + u^2)} d\phi dt + \\
\frac{1}{a^2} \left( a^2 - u^2 \right) \left( r^2 + a^2 + \frac{(a^2 - u^2)}{r^2 + u^2} \frac{2Mr - Q^2}{r^2 + u^2} \right) d\phi^2 + \\
\frac{r^2 + u^2}{r^2 - 2Mr + a^2 + Q^2} dr^2 + \frac{r^2 + u^2}{a^2 - u^2} du^2,
\]

(6.1.1)
where $M$ is the mass, $a$ is the angular momentum per unit mass, $Q$ is the charge, and $u = r \cos \theta$. The electromagnetic field for this space-time is

$$\begin{align*}
\mathbf{F} &= Q \left( \frac{r^2 - u^2}{(r^2 + u^2)^2} \right) dr \wedge dt + 2Q \frac{ru}{(r^2 + u^2)^2} du \wedge dt + \\
&- Q \left( \frac{a^2 - u^2}{a(r^2 + u^2)^2} \right) dr \wedge d\phi - 2Q \frac{ru}{a(r^2 + u^2)^2} du \wedge d\phi.
\end{align*}$$

(6.1.2)

Although this space-time is still algebraically special, both Lorentz invariants $\left[ 1.0.3 \right]$ are not zero here. With $R_F$ found from inserting equation (6.1.2) into equation (5.0.1b), the full curvature is found to be

$$R = C_0 + C_{J1} + R_F,$$

(6.1.3)

where this $C_0$ is proportional to $M$, $C_{J1}$ and $R_F$ are proportional to $Q^2$. The gravitational field $C_0$ has terms with factors that contain $Q^2$ and if $Q$ is set to zero, then $C_0$ becomes exactly the curvature for an uncharged rotating black hole. The second Bianchi identity for this space-time breaks into

$$\begin{align*}
\text{Div} * C_0 * &= 0, \\
8\pi Gc^4 F_L &= -\text{Div} * R_F * = \text{Div} * C_{J1} * \neq 0.
\end{align*}$$

(6.1.4)

The divergence free $C_0$ is a non-local gravitational field proportional to $M$. Each of the three non-zero terms can be calculated independently, showing their equality. Although the Lorentz force for this space-time vanishes, the generalized Lorentz force does not by virtue of the field terms in assumption (5.0.1c). The local gravitational field $C_{J1}$ is the gravitational field of the topological charge and the last equation shows that it is required to sustain the curvature of the electromagnetic field. Thus the charged rotating black hole is an example of the new unification with a non-zero generalized Lorentz force.

6.2. $F^{\mu\nu}F_{\mu\nu} \neq 0$, $*F^{\mu\nu}F_{\mu\nu} = 0$

The charged spherical black hole of equation (4.0.5) is a physical example of this unification, but the spherical symmetry is so restrictive that the fields are essentially cut off from the sources. The generalized Lorentz force vanishes for this space-time. Calculating it explicitly with equation (5.0.1c) and (4.0.6), one finds that the two field derivatives in equation (5.0.1c) cancel each other. As in equation
(3.1.4) for the splitting of the gravitational field into non-local and local pieces, one finds

\[ \text{Div} \ast C_0 = 0, \quad \text{Div} \ast C_F = \text{Div} \ast M_F \neq 0. \] (6.2.1)

The non-local piece \( C_0 \) is proportional to the mass of the black hole and is exactly Weyl’s conformal tensor for an uncharged spherical black hole.

6.3. \( F^\mu\nu F_{\mu\nu} = 0, \quad *F^\mu\nu F_{\mu\nu} = 0 \)

Presumably null fields describe the physical radiation fields that are pulses or beams of electromagnetic radiation. The principle of mass energy equivalence and Newtonian gravitation requires the existence of tidal forces outside the light beam. Any such gravitational effects beyond the support of the beam require a non-local gravitational field \( C_0 \). The radiation fields must possess a nonzero non-local gravitational field,

\[ C_0 \neq 0. \] (6.3.1)

Inside the beam described by null fields the gravitational field of the electromagnetic field reduces to

\[ C_F = 8\pi \frac{G_0}{c^4} \left\{ \frac{3}{2} \left( (L \wedge P) \otimes (L \wedge P) - (L \wedge Q) \otimes (L \wedge Q) \right) \right\}. \] (6.3.2)

There is a large body of literature on the gravitational properties of light, but to the author’s knowledge there are no exact solutions to the Einstein-Maxwell equations found so far that satisfy equations (6.3.1) and (6.3.2). Without a known physical solution, one can still use the unification (5.0.1) to characterize the expected physical solutions.

Even though the unification (5.0.1) gives back the Maxwell equations, there is still the problem of massless charges \[12\]. Some assumption is required to reduce the full Maxwell equations (2.1.1) to the Maxwell equations (2.1.3) for radiation. Null fields obeying the differential relations (2.4.8) supply the two vector potentials required for equations (2.4.4). An \( F \) and \( *F \) satisfying these equations can be duality rotated into each other and so one should expect symmetry between \( P \) and \( Q \). This is the case with equations (2.4.2), (2.4.5) and (2.4.9) that the source free null fields must obey.

However, the duality angle of \( R_F \) is fixed up to a half turn with \( C_F \) from equation (6.3.2) and both show no such symmetry. Inserting the null field (2.4.1) into equation (5.0.1b) gives

\[ R_F = 8\pi \frac{G_0}{c^4} \left\{ 2 (L \wedge P) \otimes (L \wedge P) - (L \wedge Q) \otimes (L \wedge Q) \right\}. \] (6.3.3)
Using this and the last of equation (5.0.3) to calculate the generalized Lorentz force density, one finds

$$F_L^\alpha_{\beta \gamma} = -\varepsilon_0 g^{\mu \nu} L_\mu L^\alpha \left( P_{\nu,\lambda} P^\lambda + 2 Q_{\nu,\lambda} Q^\lambda \right)$$

(6.3.4)

on applying the constraints (2.4.5) and (2.4.9). One would expect the radiation fields to be completely cut off from their sources. This requires setting \( R_\mathbf{J} \) to zero. By the first of equation (5.0.3), the vanishing of \( R_\mathbf{J} \) forces the generalized Lorentz force density (6.3.4) to be zero. Respecting the symmetry between \( \mathbf{P} \) and \( \mathbf{Q} \), that will happen when they are tangent to geodesics

$$\nabla_\mathbf{P} \mathbf{P} = \nabla_\mathbf{Q} \mathbf{Q} = 0.$$  

(6.3.5)

Remembering the non-local gravitational field (6.3.1), the full curvature for a null radiative space-time is

$$R = 8\pi \frac{G\varepsilon_0}{c^4} \{ 2 (\mathbf{L} \wedge \mathbf{P}) \otimes (\mathbf{L} \wedge \mathbf{P}) - (\mathbf{L} \wedge \mathbf{Q}) \otimes (\mathbf{L} \wedge \mathbf{Q}) \} + C_0,$$  

(6.3.6)

with \( \mathbf{L}, \mathbf{P} \) and \( \mathbf{Q} \) satisfying equations (2.4.2), (2.4.5), (2.4.8), (2.4.9), and (6.3.5).

Further classical mechanics and classical electromagnetism on a trivial space-time topology cannot account for a light beam or pulse with a finite cross sectional area and an intrinsic rotational angular momentum \[16\]. Essentially, the argument is that travelling any internal motion is frozen out when travelling at the speed of light. The momentum circulation that could produce rotational angular momentum is prohibited and any angular momentum can only be gained at the price of an infinite moment arm. The only remaining possibilities to explain finite light beams with measured internal angular momentum are the introduction of quantum spin or a non-trivial topology that can produce rotational angular momentum as occurs with rotating black holes. Perhaps quantum spin is topological in origin and results from a closed two form that is not exact.

Regarding curvature as various kinds of fluid, one can interpret the second Bianchi identity as regulating their flows. The special relativistic condition of no flow at light speed then suggests that the curvature for radiation fields obey

$$\text{Div} \mathbf{C}_\mathbf{F} = \text{Div} \mathbf{M}_\mathbf{F} = 0.$$  

(6.3.7)

These conditions do follow from equations (2.4.5), (2.4.9) and (6.3.5). In fact these equations permit

$$\text{Div} (\mathbf{F} \otimes \mathbf{F}) = \text{Div} (\ast \mathbf{F} \otimes \ast \mathbf{F}) = 0.$$  

(6.3.8)
7. Conclusions

This work was based on splitting the full curvature in various ways and applying the full second Bianchi identity to these splits. Mechanically the Riemann tensor decomposes into

\[ R = M_2 + M_1 + C_1 + C_0. \] (7.0.1)

This permitted the application of duality rotations to the components of the curvature tensor. The \( M_1 \) component is invariant under duality rotations, but the \( M_2 \) and \( C \) components are only invariant under a duality half turn. Such a half turn on a Maxwell field corresponds to charge reversal symmetry and so space-time curvature is invariant under charge reversal symmetry. Applying the second Bianchi identity to this decomposition required that the trace and traceless pieces of the divergence of \( M_1 \) annihilate those of \( M_2 \) and \( C_1 \) respectively, while the divergence of \( C_0 \) vanished. When \( M_1 \) has non zero trace and traceless divergences, then \( M_2, M_1, \) and \( C_1 \) share a common support with the total stress-energy tensor \( T \), while the divergence free \( C_0 \) extends beyond the support of \( T \). Beyond \( T \), \( C_0 \) becomes the total curvature. Splitting Weyl’s conformal tensor into distant \( C_0 \) and local \( C_1 \) gravitational fields hinges more on the question of support than of divergence. There may be applications of this splitting elsewhere. When the total stress-energy tensor is made of simple functions, as with an ideal fluid, there may be local gravitational fields made of the same functions, ensuring their common support. Further work will tell.

Turning to electromagnetism, the radiative fields are quite different than the non-radiative fields. Since the absence of magnetic charges suggested the introduction of the global vector potential \( A \), then the absence of lightlike electric charges should also suggest modifying the Maxwell equations by the introduction of a second vector potential \( B \) for \( \ast F \) in the radiative solutions.

The curvature split between current and field cuts across both the mechanical and gravitational components giving

\[ R = M_{2J} + M_{1J} + C_{1J} + M_F + C_F + C_0. \] (7.0.2)

This is the first assumption \([5.0.1a]\) in the new curvature based unification of classical electromagnetism and general relativity. The expression \([5.0.1b]\) for \( R_F \) is an extension of the usual Einstein equations from the traces of curvature to the full curvature by specifying \( C_F \). For the geometry to contain the polarization or phase information of the electromagnetic field, that information must occur in Weyl’s conformal tensor. By requiring the Einstein-Maxwell
equations to produce a curvature containing $R_F$, instead of any curvature, one is hopefully limiting the Einstein-Maxwell solutions to just the physical ones. Applying the second Bianchi identity to this current and field split gives rise to the assumption (5.0.1c) for the generalized Lorentz force $F_L$. The non-radiative fields will, like the rotating charged black hole, in general have a non-zero $F_L$. Radiative fields, being cut off from their sources, will have a vanishing $F_L$.

The trace of $F_L$ is the Lorentz force density and, outside the currents, its traceless piece ties $R_F$ to the gravitational field $C_{J1}$ of the currents that produce $F$. $C_{J1}$ has the same support as $F$ and like $F$ has leaked out beyond the support of the currents. The matter here is current and field. So even though $C_{J1}$ occurs beyond the support of the currents, it is still a local field occurring within the support of the matter. The charged rotating black hole shows that even topological currents carry this local gravitational field in addition to their non-local gravitational fields $C_0$. One wonders about the physical meaning of the traceless piece of $F_L$.

Essentially the non-local gravitational field extends beyond the support of the matter that generates it and the local gravitational field has the same support as the matter that produces it. Since radiative solutions have an independent existence from the sources that produce them and soon leave these sources behind, the radiative solutions are in this sense non-local. In a region distant from the emitting sources the curvature (7.0.2) reduces to

$$ R = M_F + C_F + C_0, \quad (7.0.3) $$

where $C_0$ is a distant gravitational field allowing for gravitational effects beyond the support of $F$. A vanishing $F_L$ requires

$$ \text{Div}C_F = \text{Div}M_F. \quad (7.0.4) $$

This is trivially satisfied with both divergences being zero. In this case $C_F$ has the same support as $F$, so it is a local field. However, being radiation, $C_F$ has the non-local characteristic of a vanishing divergence.

From an aesthetic point of view one ought to replace the assumption (5.0.1c) for the generalized Lorentz force $F_L$ with the Maxwell equations (2.1.1) and be rid of the magnetic monopoles from the outset. One could then still produce the $F_L$ by the argument given at the end of section 5. However, from a logical point of view it is sweeter to show that the curvature assumptions (5.0.1) require the Maxwell equations and have the geometry eliminate the magnetic monopoles. It is surprising that seemingly local demands (5.0.1) can
give rise to the global result that \( \mathbf{F} \) is a Maxwell field with a global vector potential. Although equation (1.0.2) is a local definition of \( \mathbf{C}_F \), matching the duality angle of \( \mathbf{C}_F \) to a Maxwell field is required everywhere and is a global demand. This theoretical success indicates the physical significance of \( \mathbf{C}_F \).

The major discovery of this work is the expression (1.0.2) for \( \mathbf{C}_F \). The arguments that led to that expression are quite general and should defeat the criticism that \( \mathbf{C}_F \) was built on algebraically special black holes and will fail elsewhere. It would be useful to have a physical solution to the Einstein-Maxwell equations with non-zero currents that were not overwhelmed by symmetry. Then one could extend this analysis into the currents and see how the full second Bianchi identity works there. Further successful examples will give knowledge and comfort; but will not prove the generality for \( \mathbf{C}_F \) that is claimed here. However, a single credible counterexample or the observation of a magnetic monopole will vitiate this work.

The small coupling constant required by the Einstein equations,

\[
8\pi G \varepsilon_0 c^4 = 1.8382 \cdot 10^{-54} \text{Volt}^{-2},
\]

permits the superposition of electromagnetic fields. It has also led many to believe that the gravitational consequences of electromagnetism are insignificant. Nothing could be further from the truth. It is a matter of principle to unify classical electromagnetism and gravitation and the curvature-based unification presented here allows the electromagnetic field to appear as an algebraically special piece of curvature. This fulfills the nineteenth century speculation that gravity and electromagnetism are both aspects of Riemann curvature.

This theory is not experimentally vacuous. The smallness of the coupling constant merely means that it could be a long time before curvature detectors are sufficiently sensitive while withstanding an intense electromagnetic field; or sufficiently sensitive over very long distances having less intense fields. One wonders about the consequences of \( \mathbf{C}_F \) in the environment around very strongly magnetised neutron stars [17]. Further, what are its consequences in the Jacobi equation for geodesic separation that might apply to trans galactic travel? When two electromagnetic fields are superposed could the interaction terms in the curvature have any bearing on the problem of emission or absorption?

The physical geometry of space-time is determined by specifying the metric tensor or the full curvature tensor [11, 18]. The Einstein equations, which link classical mechanics to physical geometry, may
be written as

$$M_1^{\alpha\beta}_{\gamma\delta} = \frac{8\pi G}{c^4} \left( -\frac{1}{2} \delta^{\alpha\gamma\lambda}_{\delta\rho} \left( T^\rho_\lambda - \frac{1}{4} \delta^\rho_\lambda T \right) \right)$$  \hspace{1cm} (7.0.6a)$$

and

$$M_2^{\alpha\beta}_{\gamma\delta} = \frac{8\pi G}{c^4} \left( -\frac{1}{12} \delta^{\alpha\beta}_{\gamma\delta} T \right),$$  \hspace{1cm} (7.0.6b)$$

where $T$ is the total stress-energy tensor and $T$ its trace. There is no mention of Weyl’s conformal tensor that would complete the specification of the physical geometry.

Placing constraints on Weyl’s conformal tensor is the novel feature of this work. Such constraints are meant to limit the solutions to those with a physical gravitational field. If the constraints are too limiting and they forbid physical solutions, then they will have to be altered. Similar constraints might deal with the embarrassing number of Ricci flat universes, which may or may not describe gravitational radiation. It is an open question whether the Einstein equations will have to be extended to the full curvature to handle gravitational radiation.

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