Bound for DP color function of 2-connected graphs

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Abstract

The chromatic polynomial of a graph $G$ is the polynomial function $P(G, m)$ which counts the number of proper $m$-colorings of $G$. One classical problem in chromatic polynomials theory is to estimate bounds for the chromatic polynomials of graphs. In this note we consider the same problems in the context of DP-coloring which introduced by Dvořák and Postle, obtain tight upper bounds for the DP color function of $n$-vertex 2-connected graphs.

Keywords: DP-coloring, DP color function; 2-connected graph; ear decomposition.

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1 Introduction

All graphs $G = (V, E)$ considered in this note are finite and simple. The set of natural numbers is $\mathbb{N} = \{1, 2, 3, \ldots \}$. For $m \in \mathbb{N}$, let $[m] = \{1, \ldots, m\}$. A proper $m$-coloring of $G$ is a mapping $c : V \to [m]$ such that $c(u) \neq c(v)$ whenever $uv \in E$. In 1912, Birkhoff [3] introduced a function $P(G, m)$ which counts the number of proper $m$-colorings of $G$, it is a polynomial in $m$ and called chromatic polynomial of $G$. The book by Dong, Koh and Teo [4] gives an overview for chromatic polynomial problems.

Since it is difficult to get a simple expression for the chromatic polynomial of an arbitrary graph, the bounds for the chromatic polynomials of graphs are of particular interest. For connected graphs of order $n$, the upper bound for their chromatic polynomials can be found in [4].

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Theorem 1.1 ([1], Theorem 15.3.2). Let $G$ be a connected graph of order $n$. Then for all $m \in \mathbb{N}$,
\[ P(G, m) \leq m(m - 1)^{n-1}, \]
where equality holds for $m \geq 3$ if and only if $G$ is a tree.

For the family of 2-connected graphs, Tomescu obtained the following result.

Theorem 1.2 ([20], Theorem 2.1). Let $G$ be a 2-connected graph of order $n \geq 3$. Then for all $m \in \mathbb{N}$ with $m \geq 3$,
\[ P(G, m) \leq (m - 1)^n + (-1)^n(m - 1), \]
where equality holds if and only if $G \cong C_n$; or $G \cong K(2, 3)$ for the case that $n = 5$ and $m = 3$.

In [10], Felix gave a survey on the upper bounds for the chromatic polynomials of graphs of given order and size. Recently, some authors focus on the upper bounds for the chromatic polynomials of graphs on $n$ vertices with chromatic number $k$, they obtained some inspired results, see [7, 8, 9, 11, 16, 17] for example.

In this note we obtain analogous results to that in Theorems 1.1 and 1.2, in the context of DP-coloring. The DP-coloring (also called corresponding coloring) is a generalization of the list coloring, introduced by Dvořák and Postle [6].

Definition 1.3 ([6]). Let $G = (V, E)$ be a graph. If $X, Y \subseteq V(G)$, we use $G[X]$ for the subgraph of $G$ induced by $X$, and we use $E_G(X, Y)$ for the subset of $E(G)$ with one endpoint in $X$, and one endpoint in $Y$. Given a set $S$, $\mathcal{P}(S)$ is the power set of $S$.

- A cover of a graph $G$ is a pair $\mathcal{H} = (L, H)$ consisting of a graph $H$ and a function $L : V(G) \rightarrow \mathcal{P}(V(H))$ satisfying the following four requirements:
  (1) the set $\{L(u) : u \in V(G)\}$ is a partition of $V(H)$;
  (2) for every $u \in V(G)$, the graph $H[L(u)]$ is complete;
  (3) if $E_H(L(u), L(v))$ is nonempty, then $u = v$ or $uv \in E(G)$;
  (4) if $uv \in E(G)$, then $E_H(L(u), L(v))$ is a matching (the matching may be empty).

- A cover $\mathcal{H}$ is $m$-fold if $|L(u)| = m$ for each $u \in V(G)$, and $\mathcal{H}$ is full if for each $uv \in E(G)$, $E_H(L(u), L(v))$ is a perfect matching.

- An $\mathcal{H}$-coloring of $G$ is an independent set in $H$ of size $|V(G)|$. 

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• The \textit{DP-chromatic number} of \( G \), denoted by \( \chi_{DP}(G) \), is the smallest \( m \in \mathbb{N} \) such that \( G \) admits an \( \mathcal{H} \)-coloring for every \( m \)-fold cover \( \mathcal{H} \) of \( G \).

In 2021, Kaul and Mudrock \cite{13} gave the definition of DP color function.

\textbf{Definition 1.4} (\cite{13}). Let \( \mathcal{H} = (L, H) \) be a cover of graph \( G \). We denote \( P_{DP}(G, \mathcal{H}) \) the number of \( \mathcal{H} \)-colorings of \( G \). The \textit{DP color function} of \( G \), denoted by \( P_{DP}(G, m) \), is the minimum value of \( P_{DP}(G, \mathcal{H}) \) where the minimum is taken over all possible \( m \)-fold cover \( \mathcal{H} \) of \( G \).

In \cite{13}, the authors obtained an upper bound for the DP color function of an arbitrary graph, by using a probabilistic argument.

\textbf{Lemma 1.5} (\cite{13}). For any graph \( G \) and all \( m \in \mathbb{N} \),

\[ P_{DP}(G, m) \leq \frac{m|V(G)|}{m|E(G)|} \cdot \frac{(m-1)^{|E(G)|}}{|E(G)|}. \]

For a connected graph, Kaul and Mudrock \cite{13} gave the following result.

\textbf{Lemma 1.6} (\cite{13}). For any connected graph \( G \) and all \( m \in \mathbb{N} \),

\[ P_{DP}(G, m) = \frac{m|V(G)|}{m|E(G)|} \cdot \frac{(m-1)^{|E(G)|}}{|E(G)|} \]

if and only if \( G \) is a tree.

Combining Lemmas 1.5 and 1.6 one can obtain the following result easily.

\textbf{Theorem 1.7}. Let \( G \) be a connected graph with order \( n \). Then for all \( m \in \mathbb{N} \),

\[ P_{DP}(G, m) \leq m(m-1)^{n-1} \]

with equality holds if and only if \( G \) is a tree.

In the next section, we will obtain tight upper bounds for the DP color function of 2-connected graphs with order \( n \), and give a new proof of Theorem 1.7. We also note that Kaul, Mudrock, and their coauthors obtain lots of results on DP color function, see \cite{2, 12, 14, 15, 18} for example, they study the asymptotics of \( P(G, m) - P_{DP}(G, m) \) for a fixed graph \( G \), they develop techniques to evaluate \( P_{DP}(G, m) \) for some classes of graphs such as chordal graphs, unicyclic graphs, theta graphs, Cartesian product graphs, joint graphs, vertex-gluing graphs, clique-gluing graphs, etc. Zhang and Dong \cite{22} give some sufficient conditions for graphs belong to \( DP_{\approx} \) (\( DP_{<} \), respectively) where \( DP_{\approx} \) (\( DP_{<} \), respectively) is the set of graphs \( G \) for which there exists an \( M \in \mathbb{N} \) such that \( P_{DP}(G, m) = P(G, m) \) \((P_{DP}(G, m) = P(G, m), \) respectively) holds for all \( m \geq M \). Their results extend Dong and Yang’s results in \cite{5}.
2 Bound for DP color function of 2-connected graphs

In order to get the upper bound for the DP color function of 2-connected graphs, we first consider the DP color function of a graph obtained by adding an ear to a graph, then combine it with the ear decomposition of 2-connected graphs, the result follows.

An ear of a graph $G$ is a maximal path whose internal vertices have degree 2 in $G$. An ear decomposition of $G$ is a decomposition $Q_0, \ldots, Q_k$ such that $Q_0$ is a cycle and $Q_i$ for $i \geq 1$ is an ear of $Q_0 \cup \cdots \cup Q_{i-1}$. It is well known that every 2-connected graph has an ear decomposition.

**Theorem 2.1** ([21], Theorem 4.2.8). A graph is 2-connected if and only if it has an ear decomposition. Furthermore, every cycle in a 2-connected graph is the initial cycle in some ear decomposition.

**Theorem 2.2.** Let $G$ be a graph with $n$ vertices, $u, v$ be two distinct vertices in $V(G)$. If $G'$ is a graph obtained by adding an ear $uw_1w_2 \ldots w_lv$ of length $l + 1$ ($l \geq 1$) to $G$, then for all $m \in \mathbb{N}$,

$$P_{DP}(G', m) \leq P_{DP}(G, m)(m - 1)^{l+1} \bigg/ m.$$

**Proof.** Suppose that $\mathcal{H} = (L, H)$ is an arbitrary full $m$-fold cover of $G$ in which $L(x) = \{(x, i) : i \in [m]\}$ for each $x \in V(G)$. Let $L'(x) = \{(x, i) : i \in [m]\}$ for each $x \in V(G')$, and

$$H' = H + E_{H'}(L(u), L(w_1)) \sum_{i=1}^{l-1} E_{H'}(L(w_i), L(w_{i+1})) + E_{H'}(L(w_l), L(v))$$

where $E_{H'}(L(u), L(w_1))$ ($E_{H'}(L(w_i), L(w_{i+1}))$, $E_{H'}(L(w_l), L(v))$ respectively) is a perfect matching between $L(u)$ and $L(w_1)$ ($L(w_i)$ and $L(w_{i+1})$, $L(w_l)$ and $L(v)$ respectively) chosen uniformly at random from all possible perfect matchings, then $\mathcal{H}' = (L', H')$ is a full $m$-fold cover of $G'$.

Suppose $I$ is an $\mathcal{H}$-coloring of $G$, we can get an $\mathcal{H}'$-coloring $I' = I \cup \{x_1, \ldots, x_l\}$ of $G'$, by choosing $x_i \in L(w_i)$ such that $x_i$ ($1 \leq i \leq l$) is not adjacent to any vertex of $I \cup \{x_1, \ldots, x_{i-1}\}$ in $H'$. We choose each $x_i$ in the order $x_1$ to $x_l$.

For $1 \leq i \leq l - 1$, $w_i$ is adjacent to only one vertex of $V(G) \cup \{w_1, \ldots, w_{l-1}\}$ in $G'$, so there are exactly $m - 1$ ways to choose the vertex $x_i \in L(w_i)$. Then we consider the choice for $x_l$. Assume that $I \cap L(v) = \{(v, i)\}$ and $x_{l-1} = (w_{l-1}, j)$, $i, j \in [m]$, there are two cases,

- **Case 1:** If $(v, i)$ and $(w_{l-1}, j)$ have a common neighbor in $L(w_l)$, then there are exactly $m - 1$ ways to choose $x_l$.
- **Case 2:** If $(v, i)$ and $(w_{l-1}, j)$ have different neighbors in $L(w_l)$, then there are exactly $m - 2$ ways to choose $x_l$.
Notice that
\[ Pr[(v, i) \text{ and } (w_{l-1}, j) \text{ have a common neighbor in } \mathcal{H}'] = \frac{1}{m}, \]
and
\[ Pr[(v, i) \text{ and } (w_{l-1}, j) \text{ have different neighbors in } \mathcal{H}'] = 1 - \frac{1}{m}. \]
Now, let \( X \) be the random variable equal to the number of \( \mathcal{H}' \)-colorings of \( G' \), then from the discussion above, the value of \( X \) could be \( P_{DP}(G, \mathcal{H})(m-1)^l \) or \( P_{DP}(G, \mathcal{H})(m-1)^{l-1}(m-2) \). Furthermore, the expectation of \( X \) is
\[ E(X) = P_{DP}(G, \mathcal{H})(m-1)^l \frac{1}{m} + P_{DP}(G, \mathcal{H})(m-1)^{l-1}(m-2)(1 - \frac{1}{m}). \]
Then we have
\[ P_{DP}(G', m) \leq P_{DP}(G, \mathcal{H})(m-1)^l \frac{1}{m} + P_{DP}(G, \mathcal{H})(m-1)^{l-1}(m-2)(1 - \frac{1}{m}) = P_{DP}(G, \mathcal{H}) \frac{(m-1)^{l+1}}{m} \leq P_{DP}(G, m) \frac{(m-1)^{l+1}}{m}. \]
The proof is completed.

Theorem 2.3. Let \( G \) be a graph with \( n \) vertices, \( u, v \) be two distinct vertices in \( V(G) \) and \( uv \notin E(G) \). If \( G' = G + uv \), then for all \( m \in \mathbb{N} \),
\[ P_{DP}(G', m) \leq P_{DP}(G, m) \frac{m-1}{m}. \]

Proof. Suppose that \( \mathcal{H} = (L, H) \) is an arbitrary full \( m \)-fold cover of \( G \). Let \( L' = L \) and \( H' = H + E_{H'}(L(u), L(v)) \) where \( E_{H'}(L(u), L(v)) \) is a perfect matching between \( L(u) \) and \( L(v) \) chosen uniformly at random from the \( m! \) possible perfect matchings, then \( \mathcal{H}' = (L', H') \) is a full \( m \)-fold cover of \( G' \). Let \( t = P_{DP}(G, \mathcal{H}) \) and \( \mathcal{I} = \{I_1, \ldots, I_t\} \) be the set of all \( \mathcal{H} \)-colorings of \( G \).

For each \( i \in [t] \), let \( E_i \) be the event that \( I_i \) is also an \( \mathcal{H}' \)-coloring of \( G' \). When \( I_i \cap L(u) \) is not adjacent to \( I_i \cap L(v) \) in \( H' \), the event \( E_i \) occur, so
\[ Pr[E_i] = 1 - \frac{1}{m}. \]

Let \( X_i \) be the random variable that is one if \( E_i \) occurs and zero otherwise. Let \( X = \sum_{i=1}^{t} X_i \), then \( X \) is the random variable which equals \( P_{DP}(G', \mathcal{H}') \). By the linearity of expectation, the expectation of \( X \) is
\[ E[X] = \sum_{i=1}^{t} E[X_i] = t(1 - \frac{1}{m}) = P_{DP}(G, \mathcal{H})(1 - \frac{1}{m}). \]
Then, combining the arbitrariness of $H = (L, H)$, we have

$$P_{DP}(G', m) \leq E[X] \leq P_{DP}(G, m)(1 - \frac{1}{m}).$$

The proof is complete.

**Corollary 2.4.** Let $G$ be a graph with $n$ vertices, $u, v$ be two distinct vertices in $V(G)$ and $uv \not\in E(G)$. If $G' = G + uv$, then for all $m \in \mathbb{N}$ and $m \geq \max\{2, \chi_{DP}(G)\}$,

$$P_{DP}(G', m) < P_{DP}(G, m).$$

**Proof.** When $m \geq \max\{2, \chi_{DP}(G)\}$, we have $1 − \frac{1}{m} < 1$ and $P_{DP}(G, m) > 0$, the corollary is straightforward from Theorem 2.3.

By using Corollary 2.4, we give another proof of Theorem 1.7 as follow.

**Proof.** (proof of Theorem 1.7) Let $T$ be a spanning tree of $G$, then $P_{DP}(T, m) = m(m - 1)^{n - 1}$ for all $m \in \mathbb{N}$. From Proposition 2.3 in [1], $\chi_{DP}(G) \leq 2$ if and only if $G$ is a tree. We discuss the two cases as follow.

**Case 1** $|E(T)| = |E(G)|$. In this case, $G \cong T$, $P_{DP}(G, m) = m(m - 1)^{n - 1}$ for all $m \in \mathbb{N}$.

**Case 2** $|E(T)| < |E(G)|$. In this case $G$ is not a tree, so $\chi_{DP}(G) \geq 3$.

When $m \geq \max\{2, \chi_{DP}(G)\}$, then $P_{DP}(G, m) < P_{DP}(T, m)$ from Corollary 2.4.

When $m < \max\{2, \chi_{DP}(G)\}$, then $P_{DP}(G, m) = 0 \leq m(m - 1)^{n - 1}$.

Summarizing the above, the theorem follows.

In [13], Kaul and Mudrock computed the DP color function of the unicyclic graph (i.e., a connected graph containing exactly one cycle) with $n$ vertices, so the DP color function of the cycle with $n$ vertices can be deduced.

**Lemma 2.5** ([13], Theorem 11). Let $C_n$ be the cycle with $n$ vertices.

(i) If $n$ is odd, then for all $m \in \mathbb{N}$,

$$P_{DP}(C_n, m) = (m - 1)^n - (m - 1).$$

(ii) If $n$ is even, then for all $m \in \mathbb{N}$ and $m \geq 2$,

$$P_{DP}(C_n, m) = (m - 1)^n - 1.$$
Theorem 2.6. Let $G$ be a 2-connected graph with $n$ vertices, $G_0$ be a cycle of length $l_0$ in $G$.

(i) If $G_0$ is an odd cycle, then for all $m \in \mathbb{N}$,

$$P_{DP}(G, m) \leq (m - 1)^n - (m - 1)^{n-l_0+1},$$

where equality holds if and only if $G \cong G_0$.

(ii) If $G_0$ is an even cycle, then for all $m \in \mathbb{N}$ and $m \geq 2$,

$$P_{DP}(G, m) \leq (m - 1)^n - (m - 1)^{n-l_0},$$

where equality holds if and only if $G \cong G_0$.

Proof. From Theorem 2.1, $G$ has an ear decomposition $Q_0, \ldots, Q_k$ such that $Q_0 \cong G_0$ is the cycle of length $l_0$ and $Q_i$ is an ear of $Q_0 \cup \cdots \cup Q_{i-1}$ for $i \geq 1$. Suppose that ear $Q_i$ has length $l_i + 1$ ($l_i \geq 0$) for $1 \leq i \leq k$, then we have $\sum_{i=0}^k l_i = n$.

By Theorems 2.2, 2.3 and Lemma 2.5, if $G_0$ is an odd cycle,

$$P_{DP}(G, m) \leq P_{DP}(G_0, m) \prod_{i=1}^k \frac{(m - 1)^{l_i+1}}{m} \leq \frac{(m - 1)^{l_0} - (m - 1)^{n-l_0+k}}{m^k} = \frac{(m - 1)^{n+k} - (m - 1)^{n-l_0+k+1}}{(m - 1)^k} \leq (m - 1)^n - (m - 1)^{n-l_0+1},$$

where the next to the last equality holds if and only if $k = 0$, i.e., $G \cong G_0$ is an $n$-vertex odd cycle. With a similar argument, if $G_0$ is an even cycle,

$$P_{DP}(G, m) \leq P_{DP}(G_0, m) \prod_{i=1}^k \frac{(m - 1)^{l_i+1}}{m} \leq \frac{(m - 1)^{l_0} - (m - 1)^{n-l_0+k}}{m^k} = \frac{(m - 1)^{n+k} - (m - 1)^{n-l_0+k}}{(m - 1)^k} \leq (m - 1)^n - (m - 1)^{n-l_0},$$

where equalities holds if and only if $G \cong G_0$ is an $n$-vertex even cycle. The proof is completed. □
Theorem 2.7. Let $G$ be a 2-connected graph with $n$ vertices.

(i) If $n$ is odd, then for all $m \in \mathbb{N}$,

$$P_{DP}(G,m) \leq (m-1)^n - (m-1),$$

where equality holds if and only if $G$ is an odd cycle with $n$ vertices.

(ii) If $n$ is even, then for all $m \in \mathbb{N}$ and $m \geq 2$,

$$P_{DP}(G,m) \leq (m-1)^n - 1,$$

where equality holds if and only if $G$ is an even cycle with $n$ vertices.

Proof. Because every 2-connected graph contains a cycle, the theorem follows from Theorem 2.6.

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