ON A POSITIVITY PROPERTY OF THE REAL PART OF LOGARITHMIC DERIVATIVE OF THE RIEMANN $\xi$–FUNCTION

EDVINAS GOLDSSTEIN AND ANDRIUS GRIGUTIS

Abstract. In this paper we investigate the positivity property of the real part of logarithmic derivative of the Riemann $\xi$–function for $1/2 < \sigma < 1$ and sufficiently large $t$. We give an explicit upper and lower bounds for $\Re\sum_{\rho} 1/(s-\rho)$, where the sum runs over the zeros of $\xi(s)$ on the line $1/2+it$. We also check the positivity of $\Re\xi'/\xi(s)$ for $1/2 < \sigma < 1$ assuming that there occur a non-trivial zeros of $\zeta(s)$ off the critical line.

1. Introduction

For the complex $s = \sigma + it$ the Riemann $\xi$–function is defined by

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s),$$

where $\zeta(s)$ is Riemann $\zeta$–function. The functions $\xi(s)$ and $\zeta(s)$ have the same zeros in the strip $0 < \sigma < 1$ and the famous Riemann hypothesis states that they all are located on the line $1/2+it$ - called the critical line. Zeros in the strip $0 < \sigma < 1$ are known as non-trivial zeros of $\zeta(s)$. The Riemann $\zeta$–function also has zeros at each even negative integer $s = -2n$ - these are known as the trivial zeros of $\zeta(s)$. The function $\xi(s)$ also satisfies $\xi(s) = \xi(1-s)$ and $\overline{\xi(s)} = \xi(\overline{s})$. From this, it is clear that $\xi(\sigma + it) = 0$ iff $\xi(1-\sigma + it) = 0$. Also, if $s$ is a non-trivial zero of $\xi(s)$ off the critical line then the four numbers $\{s, \overline{s}, 1-s, 1-\overline{s}\}$ would all be non-trivial zeros off the line.

By $\rho = \beta + i\gamma$ we denote a non-trivial zero of $\zeta(s)$, i.e. $\zeta(\rho) = 0$. The function $\xi(s)$ can be expanded as an infinite product by $\rho$, see Edwards [4, p. 39] and Wolfram MathWorld [23],

$$\xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right) = \frac{1}{2} \prod_{\rho} \left(1 - \frac{s}{\rho}\right),$$

where the product is taken in an order which pairs each root $\rho$ with the corresponding root $1-\rho$. The logarithmic derivative of $\xi(s)$ is

$$\frac{\xi'}{\xi}(s) = \sum_{\rho} \frac{1}{s-\rho},$$
where the summation is understood the same way as defining the product (1). There is a direct relation between location of zeros of complex function \( f \) and behavior of its modulus or real part of logarithmic derivative. Matiyasevich, Saidak and Zvengrowski \cite{14} note that “...strict decrease of the modulus of any continuous complex function \( f \) along any curve in the complex plane clearly implies that \( f \) can have no zero along that curve.” The relation between monotonicity of modulus of complex function \( |f| \) and sign of its real part of logarithmic derivative \( \Re f' \) is provided in Lemma 6.

It is known that (see for example Hinkkanen \cite{9})

\[
\Re \frac{\xi'}{\xi}(s) > 0 \text{ when } \Re s > 1
\]

and the Riemann hypothesis is equivalent to

\[
\Re \frac{\xi'}{\xi}(s) > 0 \text{ when } \Re s > \frac{1}{2}.
\]

Lagarias \cite{10} proved that

\[
\inf \left\{ \Re \frac{\xi'}{\xi}(s) : -\infty < t < \infty \right\} = \frac{\xi'}{\xi}(\sigma) \tag{3}
\]

for \( \sigma > 10 \) and Garunkštis \cite{6} later improved (3) for \( \sigma > a \), where \( \sigma > a \) is a zero-free region of \( \xi(s) \). See also Broughan \cite{2} on the subject.

In the paper by Sondow and Dumitrescu \cite{19} there was given the following reformulation of the Riemann hypothesis.

**Theorem 1.** (Sondow, Dumitrescu) The following statements are equivalent.

I. If \( t \) is any fixed real number, then \( |\xi(\sigma + it)| \) is increasing for \( 1/2 < \sigma < \infty \).

II. If \( t \) is any fixed real number, then \( |\xi(\sigma + it)| \) is decreasing for \( -\infty < \sigma < 1/2 \).

III. The Riemann hypothesis is true.

Also, in the same paper it was proved the following theorem.

**Theorem 2.** (Sondow, Dumitrescu) The \( \xi \)-function is increasing in modulus along every horizontal half-line lying in any open right half-plane that contains no \( \xi \) zeros. Similarly, the modulus decreases on each horizontal half-line in any zero-free, open left half-plane.

Matiyasevich, Saidak and Zvengrowski \cite{14} slightly reformulated the Theorem 2.

**Theorem 3.** (Matiyasevich, Saidak, Zvengrowski) Let \( \sigma_0 \) be greater than or equal to the real part of any zero of \( \xi \). Then \( |\xi(s)| \) is strictly increasing\(^1\) in the half-plane \( \sigma > \sigma_0 \).

\(^1\)With respect to \( \sigma \).
In this paper we further investigate the function $\xi'/\xi(s)$. We set

$$\frac{\xi'}{\xi}(s) = \sum_{\rho=1/2+i\gamma} \frac{1}{s-\rho} + \sum_{\rho \neq 1/2+i\gamma} \frac{1}{s-\rho} =: \Sigma_1 + \Sigma_2,$$

(4)

where the summation again is understood as defining (1). This ensures an absolute convergence of the series in (4) for $s : \xi(s) \neq 0$, see Edwards [4, p. 42]. Obviously, the sum $\Sigma_1$ exists, while $\Sigma_2$ might be vacuous as the Riemann hypothesis is unsolved.

For $1/2 < \sigma < 1$ and sufficiently large $t$, in Theorem 4 below, we give an explicit lower and upper bounds for $\Re\Sigma_1$. The lower bound of $\Re\Sigma_1$ in Theorem 4 suggests that $\Re\xi'/\xi(s)$ may remain positive asymptotically close to the critical line despite that $\Re\Sigma_2$ might occur if the Riemann hypothesis fails. In Section 4 we test the positivity of $\Re(\Sigma_1 + \Sigma_2)$ assuming that a certain versions of $\Sigma_2$ exist - an obtained results widen Theorems 2 and 3, see Figures 1 and 2 in Section 4.

We start the investigation of $\Sigma_1$ by an observation that there are infinitely many zeros of $\zeta(s)$ lying on the line $1/2 + it$ (see Hardy [7]), however we do not know the quantity of zeros of $\zeta(s)$ lying in the strip $1/2 < \sigma < 1$. The initial result on the part of non-trivial zeros on the critical line of the Riemann zeta-function was obtained by Selberg [18]. There was proved that at least a positive proportion of all non-trivial zeros lie on the critical line. Later this result was improved by several authors, see for example Levinson [11], Conrey [3], Feng [5], Pratt et al. [17]. Based on the mentioned facts, we formulate the following theorem for $\Re\Sigma_1$.

**Theorem 4.** Let $1/2 < \sigma < 1$. Let $c$ be the part of non-trivial zeros of $\zeta(s)$ lying on the line $1/2 + it$ and

$$A(t) = 0.12\log \frac{t}{2\pi} - 2.32\log\log t - 18.432 - \epsilon_1(t),$$

$$B(t) = 0.49\log \frac{t}{2\pi} + 0.58\log\log t + 4.603 + \epsilon_2(t),$$

where $\epsilon_1(t)$ and $\epsilon_2(t)$ are known explicit $t$ functions (see (14) and (15) below) both vanishing as $t^{-1}\log t, t \to \infty$.

Then

$$0 < c\left(\sigma - \frac{1}{2}\right) A(t) < \Re\sum_{\rho=1/2+i\gamma} \frac{1}{s-\rho}, t > 1.984 \times 10^{114},$$

$$\Re\sum_{\rho=1/2+i\gamma} \frac{1}{s-\rho} < \frac{cB(t)}{\sigma - 1/2}, t > 14.635.$$

We prove Theorem 4 in Section 3. This theorem leads to the following corollary.

**Corollary 5.** The function

$$\Re\frac{\xi'}{\xi}(s) = -\Re\frac{\xi'}{\xi}(1-s) > 0$$
if
\[ \Re \sum_{\rho \neq 1/2+iy} \frac{1}{s-\rho} + c \left( \sigma - \frac{1}{2} \right) A(t) > 0. \quad (5) \]

The remaining structure of this article is: in Section 2 we formulate an auxiliary statements, while in the last Section 4 we depict the condition (5) assuming that the Riemann hypothesis fails.

2. Lemmas

In this section we formulate a several auxiliary lemmas, which are needed for the proof of Theorem 4.

**Lemma 6.** (a) Let \( f \) be holomorphic in an open domain \( D \) and not identically zero. Let us also suppose \( \Re \left( \frac{f'(s)}{f(s)} \right) < 0 \) for all \( s \in D \) such that \( f(s) \neq 0 \). Then \( |f(s)| \) is strictly decreasing with respect to \( \sigma \) in \( D \), i.e. for each \( s_0 \in D \) there exists a \( \delta > 0 \) such that \( |f(s)| \) is strictly monotonically decreasing with respect to \( \sigma \) on the horizontal interval from \( s_0 - \delta \) to \( s_0 + \delta \).

(b) Conversely, if \( |f(s)| \) is decreasing with respect to \( \sigma \) in \( D \), then \( \Re \left( \frac{f'(s)}{f(s)} \right) \leq 0 \) for all \( s \in D \) such that \( f(s) \neq 0 \).

**Proof.** See Matiyasevich, Saidak, Zvengrowski [14] for the proof.

**Note 1:** Of course, the analogous results hold for monotone increasing \( |f(s)| \) and \( \Re \left( \frac{f'(s)}{f(s)} \right) > 0 \).

**Lemma 7.** Let \( N(T) \) be the number of zeros of \( \xi(s) \) in the rectangle \( 0 < \sigma < 1, \) \( 0 < t < T \). If \( T \geq e \), then
\[ \left| N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi e} - \frac{7}{8} \right| \leq 0.110 \log T + 0.290 \log \log T + 2.290 + \frac{25}{48\pi T}. \quad (6) \]

**Proof.** In the paper by Trudgian [22, p. 283] it is derived that, for \( T \geq 1 \)
\[ \left| N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi e} - \frac{7}{8} \right| \leq |S(T)| + \frac{1}{4\pi} \arctan \left( \frac{1}{2T} \right) + \frac{T}{4\pi} \log \left( 1 + \frac{1}{4T^2} \right) + \frac{1}{3\pi T}, \]
where \( \pi S(T) \) is the argument of the Riemann zeta-function along the critical line. From the paper by Platt and Trudgian [16, Cor. 1] (see also Hasanalizade, Shen, Wong [8])
\[ |S(T)| \leq 0.110 \log T + 0.290 \log \log T + 2.290, \ T \geq e \]
and, using inequalities,
\[ \arctan \frac{1}{t} = \int_0^{1/t} \frac{dx}{1+x^2} \leq \frac{1}{t}, \ t > 0 \]
and
\[
\log(1 + t) \leq t, t > -1,
\]
we get a desired result.

**Lemma 8.** If \( a, b, \alpha > 0 \), then the following inequality holds
\[
\int_{\alpha}^{t} \log \frac{u}{2\pi} \frac{du}{a^2 + b^2(u-t)^2} \geq \frac{1}{ab} \log \left( \frac{t}{2\pi} \right) \arctan \left( \frac{b(t-\alpha)}{a} \right) - \kappa,
\]
when \( t > t_0 \geq \alpha \), where \( t_0 \) and constant \( \kappa > 0 \) are both sufficiently large and \( \kappa \) is independent on \( t \).

In particular, if \( a = 1/2, b = 1 \) and \( \alpha = 14.134725 \ldots \), then the provided inequality holds if \( t > 23 \) and \( \kappa = 0.135 \).

**Proof.** We set up the function
\[
F(t) = \int_{\alpha}^{t} \log \frac{u}{2\pi} \frac{du}{a^2 + b^2(u-t)^2} - \frac{1}{ab} \log \left( \frac{t}{2\pi} \right) \arctan \left( \frac{b(t-\alpha)}{a} \right) + \kappa
\]
and show that \( t \) derivative \( F'(t) \geq 0 \) for \( t > t_0 \geq \alpha \). Indeed, according to the Leibniz integral rule (see for example Mackevičius [12] or Spivak [20])
\[
F'(t) = 2b^2 \int_{\alpha}^{t} \frac{(u-t) \log u/2\pi \, du}{(a^2 + b^2(u-t)^2)^2} + \left( \frac{1}{a^2} - \frac{1}{a^2 + b^2(t-\alpha)^2} \right) \log \frac{t}{2\pi} \frac{1}{abt} \frac{b(t-\alpha)}{a}.
\]
The last integral is
\[
2b^2 \int_{\alpha}^{t} \frac{(u-t) \log u/2\pi \, du}{(a^2 + b^2(u-t)^2)^2} = -\int_{\alpha}^{t} \log \frac{u}{2\pi} \frac{d}{a^2 + b^2(u-t)^2} \frac{1}{a^2 + b^2(u-t)^2} - \int_{\alpha}^{t} \log(t/2\pi) \frac{du}{a^2 + b^2(t-\alpha)^2} + \int_{\alpha}^{t} \frac{du}{u(a^2 + b^2(t-\alpha)^2)},
\]
where
\[
\int_{\alpha}^{t} \frac{du}{u(a^2 + b^2(t-\alpha)^2)} = \frac{b^2}{b^2t^2 + a^2} \int_{\alpha}^{t} \left( \frac{1}{b^2u} + \frac{2t-u}{a^2 + b^2(u-t)^2} \right) \, du
\]
\[
= \frac{\log(t/\alpha)}{b^2t^2 + a^2} + \frac{b}{a} \frac{t}{b^2t^2 + a^2} \arctan \left( \frac{b(t-\alpha)}{a} \right) + \frac{1}{2} \cdot \frac{1}{b^2t^2 + a^2} \log \left( 1 + \frac{b^2(t-\alpha)^2}{a^2} \right).
\]
Therefore

\[ F'(t) = \frac{1}{2b^2t^2 + a^2} \log \left( \left( \frac{t}{\alpha} \right)^2 + \left( \frac{bt}{a\alpha} \right)^2 \right) - \frac{\log(t/\alpha)}{a^2 + b^2(t - \alpha)^2} \]

\[ - \frac{a}{b} \cdot \frac{1}{t} \cdot \frac{1}{b^2t^2 + a^2} \arctan \left( \frac{b(t - \alpha)}{a} \right). \]

For \( t \geq \alpha + a/b \), it holds that

\[ \frac{bt(t - \alpha)}{a\alpha} \geq \frac{t}{\alpha}, \]

and

\[ F'(t) \geq \frac{\log \sqrt{2}}{a^2 + b^2t^2} - \frac{(\alpha(2t - \alpha))\log(t/\alpha)}{(a^2 + b^2t^2)(a^2 + b^2(t - \alpha)^2)} \]

\[ - \frac{a}{b} \cdot \frac{1}{t} \cdot \frac{1}{b^2t^2 + a^2} \arctan \left( \frac{b(t - \alpha)}{a} \right). \]

The positive term of the right-hand side of inequality (7) vanishes as \( t^{-2} \) while the two negative terms as \( t^{-3} \log t \), which means that \( F'(t) > 0 \) if \( t > t_0 \geq \alpha \) and \( t_0 \) is sufficiently large.

We next check whether \( F(t_0) \geq 0 \). It is easy to see that

\[ \lim_{t \to \alpha^+} F(t) = \kappa > 0. \]

Therefore, due to continuity of \( F(t) \), \( F(t) > 0 \) for at least \( t \in (\alpha, t_0] \) if \( \kappa \) is large enough and \( t_0 \) is dependent on \( \kappa \).

For the particular case \( a = 1/2, b = 1 \) and \( \alpha = 14.134725 \ldots \) we check with Mathematica \(^{[13]}\) that \( F'(t) > 0 \), when \( t > 23 \) and \( F(23) = 0.00092 \ldots \) if \( \kappa = 0.135 \).

**Lemma 9.** If \( t > 1 \), then

\[ \frac{\pi}{2} - \frac{1}{t} < \arctan t < \frac{\pi}{2} - \frac{1}{2t}. \] (8)

**Proof.** The first inequality of (8) follows from

\[ \frac{\pi}{2} = \int_0^\infty \frac{dx}{1 + x^2} = \int_0^t \frac{dx}{1 + x^2} + \int_t^\infty \frac{dx}{1 + x^2} \leq \arctan t + \int_t^\infty \frac{dx}{x^2} = \arctan t + \frac{1}{t}, \]

and the second

\[ \frac{\pi}{2} = \int_0^\infty \frac{dx}{1 + x^2} = \int_0^t \frac{dx}{1 + x^2} + \int_t^\infty \frac{dx}{1 + x^2} > \arctan t + \int_t^\infty \frac{dx}{x^2 + x^2} = \arctan t + \frac{1}{2t}. \]

**Note 2:** The first inequality in (8) holds for \( t > 0 \) also.

**Note 3:** The function \( \arctan \) is an odd function and for \( t < -1 \) the estimates are
\[-\frac{\pi}{2} - \frac{1}{2}\alpha < \arctan(t) < -\frac{\pi}{2} - \frac{1}{t} .\]

**Lemma 10.** Let \( \alpha > 0 \) and \( b > a > 0 \) be a constants. For \( t > t_0 \geq \alpha + a/b \), let

\[
\tilde{A}(t) := \frac{\pi}{ab} \log \left( \frac{t}{2\pi} \right) - \frac{\log \frac{t}{2\pi}}{b^2(t-\alpha)} - \kappa
\]

and

\[
\tilde{B}(t) := \left( \frac{\pi}{ab} + \frac{1}{b^2t} \right) \log \frac{t+1}{2\pi} + \frac{\log(t+1)}{b^2t} ,
\]

where \( \kappa > 0 \) is a constant from Lemma 8 and \( t_0 \) is sufficiently large.

Then

\[
\tilde{A}(t) < \int_{\alpha}^{\infty} \frac{\log(u/2\pi) du}{a^2 + b^2(u-t)^2} < \tilde{B}(t).
\]

**Proof.** For the lower bound, by elementary calculation and Lemmas 8 and 9 we obtain

\[
\int_{\alpha}^{\infty} \frac{\log(u/2\pi) du}{a^2 + b^2(u-t)^2} = \left( \int_{\alpha}^{t} + \int_{t}^{\infty} \right) \frac{\log(u/2\pi) du}{a^2 + b^2(u-t)^2}
\]

\[
> \int_{t}^{\infty} \frac{\log(u/2\pi) du}{a^2 + b^2(u-t)^2} + \log \left( \frac{t}{2\pi} \right) \int_{t}^{\infty} \frac{du}{a^2 + b^2(u-t)^2}
\]

\[
> \frac{1}{ab} \log \left( \frac{t}{2\pi} \right) \arctan \left( \frac{b(t-\alpha)}{a} \right) - \kappa + \frac{\pi}{2ab} \log \left( \frac{t}{2\pi} \right)
\]

\[
> \frac{\pi}{ab} \log \left( \frac{t}{2\pi} \right) - \frac{\log \frac{t}{2\pi}}{b^2(t-\alpha)} - \kappa = \tilde{A}(t).
\]

By the same thoughts for the upper bound we get

\[
\int_{\alpha}^{\infty} \frac{\log(u/2\pi) du}{a^2 + b^2(u-t)^2} = \left( \int_{\alpha}^{t+1} + \int_{t+1}^{\infty} \right) \frac{\log(u/2\pi) du}{a^2 + b^2(u-t)^2}
\]

\[
< \log \left( \frac{t+1}{2\pi} \right) \int_{\alpha}^{t+1} \frac{du}{a^2 + b^2(u-t)^2} + \frac{1}{b^2} \int_{t+1}^{\infty} \frac{\log(u/2\pi) du}{(u-t)^2}
\]

\[
= \frac{1}{ab} \log \left( \frac{t+1}{2\pi} \right) \left( \arctan \left( \frac{b}{a} \right) + \arctan \left( \frac{t-\alpha}{a/b} \right) \right) + \frac{(1 + \frac{1}{t}) \log(t+1) - \log 2\pi}{b^2}
\]

\[
< \left( \frac{\pi}{ab} - \frac{t-\alpha+1}{2b^2(t-\alpha)} \right) \log \left( \frac{t+1}{2\pi} \right) + \frac{(1 + \frac{1}{t}) \log(t+1) - \log 2\pi}{b^2}
\]

\[
< \left( \frac{\pi}{ab} + \frac{1}{2b^2} \right) \log \left( \frac{t+1}{2\pi} \right) + \frac{\log(t+1)}{b^2t} = \tilde{B}(t).
\]

**Lemma 11.** Let \( \alpha > 0 \) and \( b > a > 0 \) be a constants. For \( t > \alpha + a/b \), let

\[
\tilde{C}(t) := \frac{1}{4b^2t} \log \left( \frac{t}{2\pi} \right) - \frac{\alpha}{b^2t^2} \log \left( \frac{\alpha}{2\pi} \right)
\]
and
\[ \tilde{D}(t) := \frac{1}{2b^2t} \log \left( \frac{2t^3}{4\pi^3} \right). \]

Then
\[ \tilde{C}(t) < \int_{\alpha}^{\infty} \frac{\log(u/2\pi)}{a^2 + b^2(u+t)^2} du < \tilde{D}(t). \]

**Proof.** We do the same as in the proof of the previous lemma. For the lower bound
\[
\int_{\alpha}^{\infty} \frac{\log(u/2\pi)}{a^2 + b^2(u+t)^2} du = \left( \int_{\alpha}^{t} + \int_{t}^{\infty} \right) \frac{\log(u/2\pi)}{a^2 + b^2(u+t)^2} \\
> \frac{1}{ab} \log \left( \frac{\alpha}{2\pi} \right) \left( \arctan \frac{2t}{a/b} - \arctan \frac{t+\alpha}{a/b} \right) + \frac{1}{ab} \log \left( \frac{t}{2\pi} \right) \left( \frac{\pi}{2} - \arctan \frac{2t}{a/b} \right) \\
> \frac{1}{ab} \log \left( \frac{\alpha}{2\pi} \right) \left( \frac{\pi}{2} - \frac{a/b}{2t} - \frac{\pi}{2} + \frac{a/b}{2(t+\alpha)} \right) + \frac{1}{ab} \log \left( \frac{t}{2\pi} \right) \left( \frac{\pi}{2} - \frac{\pi}{2} + \frac{a/b}{4t} \right) \\
> \frac{1}{ab} \log \left( \frac{\alpha}{2\pi} \right) - \frac{\alpha}{b^2t^2} \log \left( \frac{\alpha}{2\pi} \right) = \tilde{C}(t).
\]

And for the upper bound
\[
\int_{\alpha}^{\infty} \frac{\log(u/2\pi)}{a^2 + b^2(u+t)^2} du = \left( \int_{\alpha}^{t} + \int_{t}^{\infty} \right) \frac{\log(u/2\pi)}{a^2 + b^2(u+t)^2} \\
< \log \left( \frac{t}{2\pi} \right) \int_{\alpha}^{t} \frac{du}{a^2 + b^2(u+t)^2} + \int_{t}^{\infty} \frac{\log(u/2\pi)}{b^2(u+t)^2} du \\
= \frac{1}{ab} \log \left( \frac{t}{2\pi} \right) \left( \arctan \left( \frac{2t}{a/b} \right) - \arctan \left( \frac{t+\alpha}{a/b} \right) \right) + \frac{1}{2b^2t} \log \left( \frac{2t}{\pi} \right) \\
< \frac{1}{ab} \log \left( \frac{t}{2\pi} \right) \left( \frac{\pi}{2} - \frac{a/b}{4t} - \frac{\pi}{2} + \frac{a/b}{t+\alpha} \right) + \frac{1}{2b^2t} \log \left( \frac{2t}{\pi} \right) \\
< \frac{1}{b^2t} \log \left( \frac{2t}{\pi} \right) + \frac{1}{b^2t} \log \left( \frac{t}{2\pi} \right) = \tilde{D}(t).
\]

The next lemma we need is well known as a summation by parts.

**Lemma 12.** Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence of complex numbers and \( G(u) \) a continuously differentiable function on \([1, x]\). If \( A(u) = \sum_{n \leq u} a_n \), then
\[
\sum_{n \leq x} a_n G(n) = A(x) G(x) - \int_{1}^{x} A(u) G'(u) \, du.
\]

**Proof.** See for example Murty [15, p. 18] or Apostol [1, p. 54] for the proof.
In the below met inequalities numbers are rounded up to two or three decimal places.

**Lemma 13.** Let \( \rho = \beta + i \gamma \) denote a non-trivial zero of \( \zeta(s) \). Let \( a, b > 0 \) and \( \gamma := 14.134725 \ldots \) (\( \zeta(1/2 + i \gamma) = 0 \)). If \( t > \gamma \), then

\[
\sum_{\rho = \beta + i \gamma} \frac{1}{a^2 + b^2(t - \gamma)^2} = \sum_{\gamma > 0} \frac{1}{a^2 + b^2(t - \gamma)^2} + \sum_{\gamma > 0} \frac{1}{a^2 + b^2(t + \gamma)^2} =: S_1 + S_2,
\]

where

\[
\left| S_1 - \frac{1}{2\pi} \int_{\gamma}^{\infty} \frac{\log(u/2\pi)}{a^2 + b^2(u-t)^2} du \right| < \frac{0.22 \log t + 0.58 \log \log t + 4.58}{a^2} + \frac{0.166}{a^2 t} \left( 1 + \frac{2.411a}{b} \right)
\]

and

\[
\left| S_2 - \frac{1}{2\pi} \int_{\gamma}^{\infty} \frac{\log(u/2\pi)}{a^2 + b^2(u+t)^2} du \right| < \frac{3.811}{a^2 + b^2(\gamma_1 + t)^2} + \frac{0.045}{ab}.
\]

**Proof.** Since \( \zeta(\rho) = \zeta(\bar{\rho}) = 0 \) we have that

\[
\sum_{\rho = \beta + i \gamma} \frac{1}{a^2 + b^2(t - \gamma)^2} = \sum_{\gamma > 0} \frac{1}{a^2 + b^2(t - \gamma)^2} + \sum_{\gamma > 0} \frac{1}{a^2 + b^2(t + \gamma)^2} = S_1 + S_2.
\]

For \( S_1 \), by Lemma 12

\[
S_1 = - \int_{\gamma}^{\infty} N(u) f'(u) du,
\]

where \( f(u) := 1/(a^2 + b^2(t - u)^2) \) and the step function \( N(u) \) is defined in Lemma 7. Let \( N_{up}(u) \) and \( N_{low}(u) \) be the corresponding continues upper and lower bounds of \( N(u) \). From Lemma 7,

\[
N_{up}(u) = \frac{u}{2\pi} \log \frac{u}{2\pi e} + 0.11 \log u + 0.29 \log \log u + 3.165 + \frac{25}{48\pi u},
\]

\[
N_{low}(u) = \frac{u}{2\pi} \log \frac{u}{2\pi e} - 0.11 \log u - 0.29 \log \log u - 1.415 - \frac{25}{48\pi u}.
\]

Let us observe that \( u \) derivative \( f'(u) \) is non-negative for \( u \leq t \) and \( f'(u) \) is negative for \( u > t \). As \( N_{up}(u), N_{low}(u) \) are continues functions, then
\[ S_1 \leq -\int_{\gamma_1}^t N_{low}(u)f'(u)du - \int_{\gamma_1}^\infty N_{up}(u)f'(u)du = -\int_{\gamma_1}^\infty \frac{u}{2\pi} \log \frac{u}{2\pi e} f'(u)du \\
+ \int_{\gamma_1}^t \left(0.11 \log u + 0.29 \log \log u + 1.415 + \frac{25}{48\pi u}\right) df(u) \\
- \int_{t}^\infty \left(0.11 \log u + 0.29 \log \log u + 3.165 + \frac{25}{48\pi u}\right) df(u) \\
\leq \frac{1}{2\pi} \int_{\gamma_1}^\infty \frac{\log(u/2\pi)}{a^2 + b^2(u-t)^2} + \frac{\gamma_1}{2\pi} \log \left(\frac{\gamma_1}{2\pi e}\right) f'(\gamma_1) \\
+ (f(t) - f(\gamma_1)) \left(0.11 \log t + 0.29 \log \log t + 1.415 + \frac{25}{48\pi \gamma_1}\right) \\
+ f(t) \left(3.165 + \frac{25}{48\pi t}\right) - 0.11 \int_{t}^\infty \log u df(u) - 0.29 \int_{t}^\infty \log \log u df(u). \tag{9} \]

For the integrals in (9) it holds that

\[-\int_{t}^\infty \log u df(u) = f(t) \log t + \int_{t}^\infty \frac{f(u)du}{u} < f(t) \log t + \frac{\pi/2}{ab} \cdot \frac{1}{t},\]
\[-\int_{t}^\infty \log \log u df(u) < f(t) \log \log t + \frac{\pi/2}{ab} \cdot \frac{1}{t \log t}.\]

Therefore

\[ S_1 < \frac{1}{2\pi} \int_{\gamma_1}^\infty \frac{\log(u/2\pi)}{a^2 + b^2(u-t)^2} + \frac{0.220 \log t + 0.580 \log \log t + 4.580}{a^2} \\
+ \frac{0.166}{a^2 t} \left(1 + \frac{2.413a}{b}\right). \]

By the similar arguments, the lower bound of \( S_1 \) is

\[ S_1 > \frac{1}{2\pi} \int_{\gamma_1}^\infty \frac{\log(u/2\pi)}{a^2 + b^2(u-t)^2} - \frac{0.220 \log t + 0.580 \log \log t + 4.580}{a^2} \\
- \frac{0.166}{a^2 t} \left(1 + \frac{2.413a}{b}\right). \]

The upper bound of

\[ S_2 = -\int_{\gamma_1}^\infty N(u)g'(u)du, \ g(u) := 1/(a^2 + b^2(t+u)^2), \]
observing that $g(u)$ is decreasing for $u \geq 0$, is

$$S_2 < - \int_{\gamma}^{\infty} N_{up}(u)g'(u) \, du = - \int_{\gamma}^{\infty} \frac{u}{2\pi} \log \frac{u}{2\pi e} \, dg(u)$$

$$- \int_{\gamma}^{\infty} \left( 0.11 \log u + 0.29 \log \log t + 3.165 + \frac{25}{48\pi u} \right) g'(u) \, du$$

$$= \frac{1}{2\pi} \int_{\gamma}^{\infty} \frac{\log (u/2\pi)}{a^2 + b^2(u+t)^2} + \frac{\gamma_1}{2\pi} \log \left( \frac{\gamma_1}{2\pi e} \right) g(\gamma_1)$$

$$- 0.11 \int_{\gamma}^{\infty} \log u g'(u) \, du - 0.29 \int_{\gamma}^{\infty} \log \log u g'(u) \, du$$

$$- \int_{\gamma}^{\infty} \left( 3.165 + \frac{25}{48\pi u} \right) g'(u) \, du. \quad (10)$$

The integrals in (10) and (11) evaluate to

$$- \int_{\gamma}^{\infty} \log u g'(u) \, du = g(\gamma_1) \log \gamma_1 + \int_{\gamma}^{\infty} \frac{du}{u(a^2 + b^2(t+u)^2)} < g(\gamma_1) \log \gamma_1 + \frac{\pi/2}{\gamma_1 \rho b},$$

$$- \int_{\gamma}^{\infty} \log \log u g'(u) \, du < g(\gamma_1) \log \gamma_1 + \frac{\pi/2}{\gamma_1 \rho b},$$

$$- \int_{\gamma}^{\infty} \left( 3.165 + \frac{25}{48\pi u} \right) g'(u) \, du < g(\gamma_1) \left( 3.165 + \frac{25}{48\pi \gamma_1} \right).$$

Therefore

$$S_2 < \frac{1}{2\pi} \int_{\gamma}^{\infty} \frac{\log (u/2\pi)}{a^2 + b^2(u+t)^2} + 3.811g(\gamma_1) + \frac{0.045}{ab}.$$

Arguing the same, the lower bound of $S_2$ is

$$S_2 > \frac{1}{2\pi} \int_{\gamma}^{\infty} \frac{\log (u/2\pi)}{a^2 + b^2(u+t)^2} - 3.811g(\gamma_1) - \frac{0.045}{ab}.$$

The proof follows by collecting the upper and lower bounds of $S_1$ and $S_2$.

3. Proof of Theorem

In this section we prove the Theorem.

Proof. [Proof of Theorem] Let $1/2 < \sigma < 1$. Since $0 < (\sigma - 1/2)^2 < 1/4$, we have that

$$\sum_{\rho=1/2+i\gamma} \frac{\sigma - 1/2}{1/4 + (t-\gamma)^2} < \Re \sum_{\rho=1/2+i\gamma} \frac{1}{s-\rho} < \sum_{\rho=1/2+i\gamma} \frac{(\sigma - 1/2)^{-1}}{1 + 4(t-\gamma)^2}. \quad (12)$$
Recall that \( c \) denotes the part of zeros of \( \zeta(s) \) on the line \( 1/2 + it \). Then, the total quantity \( N(T) \) of non-trivial zeros of \( \zeta(s) \) in the rectangle \( 0 < \sigma < 1, 0 < t < T \) can be expressed as \( N(T) = cN(T) + (1 - c)N(T) \). Then, by Lemma 13 and Lemma 7 with \( cN(T) \)

\[
\Re \sum_{\rho=1/2+i\gamma} \frac{1}{s-\rho} = \sum_{\rho=1/2+i\gamma} \frac{\sigma - 1/2}{(\sigma - 1/2)^2 + (t - \gamma)^2} \quad (13)
\]

where \( M(t) = O(\log t) \) as \( t \to \infty \) and the explicit lower and upper bounds of \( M(t) \) for \( t > 14.134725 \ldots \) are given in Lemma 13.

Combining (12) and (13) and applying Lemmas 10, 11 and 13 with \( a = 1/2, b = 1 \) and \( \alpha = \gamma_1 = 14.134725 \ldots \) for the lower bound we get

\[
\Re \sum_{\rho=1/2+i\gamma} \frac{1}{s-\rho} > c(\sigma - 1/2) \int_{\gamma}^{\infty} \left( \frac{\log(2\pi u)}{1/4 + (u-t)^2} + \frac{\log(2\pi u)}{1/4 + (u+t)^2} \right) du
\]

\[
+ c(\sigma - 1/2) \left(-0.88 \log t - 2.32 \log \log t - 18.41 - \frac{1.465}{t} - \frac{3.811}{0.25 + (\gamma_1 + t)^2} \right)
\]

\[
> c(\sigma - 1/2) \left(0.12 \log \frac{t}{2\pi} - 2.32 \log \log t - 18.432 - \varepsilon_1(t) \right),
\]

where

\[
\varepsilon_1(t) = \left(\frac{1}{8\pi t} - \frac{1}{2\pi(t - \gamma_1)}\right) \log \frac{t}{2\pi} - \frac{1.465}{t} - \frac{\gamma_1 \log (2\pi t)}{2\pi t^2} - \frac{3.811}{0.25 + (\gamma_1 + t)^2}. \quad (14)
\]

We check with Mathematica 13 that

\[
0.12 \log \frac{t}{2\pi} - 2.32 \log \log t - 18.432 \geq 49 \times 10^{-6}, |\varepsilon_1(t)| \leq 1.65 \times 10^{-113},
\]

when \( t \geq 1.984 \times 10^{114} \).

By the same arguments, with \( a = 1 \) and \( b = 2 \), for the upper bound we get

\[
\Re \sum_{\rho=1/2+i\gamma} \frac{1}{s-\rho} < c(\sigma - 1/2) \int_{\gamma}^{\infty} \left( \frac{\log(2\pi u)}{1+4(u-t)^2} + \frac{\log(2\pi u)}{1+4(u+t)^2} \right) du
\]
+ c(\sigma - 1/2) \left( 0.22 \log t + 0.58 \log \log t + 4.603 + \frac{0.367}{t} + \frac{3.811}{1 + 4(\gamma + t)^2} \right) \\
< c(\sigma - 1/2) \left( 0.49 \log \frac{t}{2\pi} + 0.58 \log \log t + 4.603 + \epsilon_2(t) \right),

where

\[
\epsilon_2(t) = \frac{0.637}{t} + \frac{3.811}{1 + 4(t + \gamma)^2} + \frac{\log(t + 1) + \frac{1}{2} \log \frac{2\pi^3}{4\pi^2}}{8\pi t}.
\]

(15)

4. Can \(\mathcal{R} \frac{\xi'}{\xi}(s)\) remain positive if there are zeros off the critical line?

In this section we assume that the Riemann hypothesis fails by three different scenarios:
I. there is only one zero in the region \(1/2 < \sigma < 1, \ t > 0\),
II. there is a finite number \(n \geq 2\) of zeros off the critical line,
III. there are infinitely many of zeros off the critical line.

I. Assume that there is one point \(\tilde{\beta} + i\tilde{\gamma}\) such that \(\zeta(\tilde{\beta} + i\tilde{\gamma}) = 0\) for \(1/2 < \tilde{\beta} < 1, \ t > 0\). Then, by Theorem 4 with \(c = 1\) and estimation,

\[
\mathcal{R} \frac{\xi'}{\xi}(s) = \left( \sigma - \frac{1}{2} \right) \sum_{\rho = 1/2 + i\gamma} \frac{1}{(\sigma - 1/2)^2 + (t - \gamma)^2}
\]

\[
+ \frac{\sigma - \tilde{\beta}}{(\sigma - \tilde{\beta})^2 + (t - \gamma)^2} + \frac{\sigma - \tilde{\beta}}{(\sigma - \tilde{\beta})^2 + (t + \gamma)^2}
\]

\[
+ \frac{\sigma - (1 - \tilde{\beta})}{(\sigma - (1 - \tilde{\beta}))^2 + (t - \gamma)^2} + \frac{\sigma - (1 - \tilde{\beta})}{(\sigma - (1 - \tilde{\beta}))^2 + (t + \gamma)^2}
\]

\[
> 0.11 \left( \sigma - \frac{1}{2} \right) \log \frac{t}{2\pi} + \frac{\sigma - \tilde{\beta}}{(\sigma - \tilde{\beta})^2 + (t - \gamma)^2} + O \left( \frac{\log \log t}{\log t} \right) > 0
\]

if

\[
(\sigma, t) \in \left\{ \frac{\sigma - \tilde{\beta}}{(\sigma - \tilde{\beta})^2 + (t - \gamma)^2} > -0.11 \left( \sigma - \frac{1}{2} \right) \log \frac{t}{2\pi} \right\}
\]

(16)

and \(t\) is sufficiently large that \(\log \log t / \log t\) is negligible. The region of \((\sigma, t)\) given by (16) might have the following gray view given in Figure 1 below. Figure 1 was obtained by Mathematica [13] with some chosen point \(\tilde{\beta} + i\tilde{\gamma}\).

II. Assume that there is a finite number \(n \geq 2\) of points \(\tilde{\beta}_k + i\tilde{\gamma}_k, \ k = 1, \ldots, n\) such that \(\zeta(\tilde{\beta}_k + i\tilde{\gamma}_k) = 0\) for \(1/2 < \tilde{\beta}_k < 1, \ t > 0\). Then, by Theorem 4 with \(c = 1\) and
Figure 1: Whole gray region satisfies inequality (16). Theorem 2 or 3 gives a dashed gray strip, where $\Re \xi'/\xi(s) > 0$.

Figure 2: Whole gray region satisfies inequality (17). Theorem 2 or 3 gives a dashed gray strip, where $\Re \xi'/\xi(s) > 0$.

Previous means,

$$\Re \frac{\xi'}{\xi}(s) > 0.11 \left( \sigma - \frac{1}{2} \right) \log \frac{t}{2\pi} + \sum_{k=1}^{n} \frac{\sigma - \bar{\beta}_k}{(\sigma - \bar{\beta}_k)^2 + (t - \bar{\gamma}_k)^2} + O \left( \frac{\log \log t}{\log t} \right) > 0$$

if

$$(\sigma, t) \in \left\{ \sum_{k=1}^{n} \frac{\sigma - \bar{\beta}_k}{(\sigma - \bar{\beta}_k)^2 + (t - \bar{\gamma}_k)^2} > -0.11 \left( \sigma - \frac{1}{2} \right) \log \frac{t}{2\pi} \right\}$$

(17)

and $t$ is sufficiently large that $\log \log t / \log t$ is negligible. The region of $(\sigma, t)$ given by (17) might have the following gray view given in Figure 2 above. Figure 2 was obtained by Mathematica [13] too with some chosen $\bar{\beta}_k$ and $\bar{\gamma}_k$, where the black points are $\bar{\beta}_k + i\bar{\gamma}_k$.

III. Assume that there are infinitely many points $\bar{\beta}_k + i\bar{\gamma}_k$, such that $\zeta(\bar{\beta}_k + i\bar{\gamma}_k) = 0$ for $1/2 < \bar{\beta}_k < 1, \ t > 0$. 

14
Then, by the same arguments as in I. and II.,

\[ \Re \frac{\xi'(s)}{\xi(s)} > c \cdot 0.11 \left( \sigma - \frac{1}{2} \right) \log \frac{t}{2\pi} \]

\[ + \sum_{\rho = \beta_k + i\gamma_k} \frac{\sigma - \bar{\beta}_k}{(\sigma - \bar{\beta}_k)^2 + (t - \gamma_k)^2} - \sum_{\gamma_k > 0} \frac{1/2}{(t + \gamma_k)^2} + O\left( \frac{\log \log t}{\log t} \right) > 0 \quad (18) \]

if

\[ (\sigma, t) \in \left\{ \sum_{\rho = \beta_k + i\gamma_k} \frac{\sigma - \bar{\beta}_k}{(\sigma - \bar{\beta}_k)^2 + (t - \gamma_k)^2} > -c \cdot 0.11 \left( \sigma - \frac{1}{2} \right) \log \frac{t}{2\pi} \right\} \]

and \( t \) is sufficiently large. We note that \( \sum_{\gamma_k > 0} \frac{1/2}{(t + \gamma_k)^2} = O\left( \frac{\log t}{t} \right), t \to \infty \) in (18), see Lemmas [11] and [13].

REFERENCES

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, 1998.

[2] K. Broughan, *Extension of the Riemann \( \xi \)-Function’s Logarithmic Derivative Positivity Region to Near the Critical Strip*, Canad. Math. Bull., 52(2), 186–194, 2009, doi:10.4153/CMB-2009-021-3.

[3] J. B. Conrey, *More than two fifths of the zeros of the Riemann zeta function are on the critical line*, J. Reine Angew. Math., 399 (1989) 1-26.

[4] H. M. Edwards, *Riemann’s Zeta Function*, Academic Press, New York (1974). Reprinted by Dover Publications, Mineola, N.Y. (2001).

[5] S. Feng, *Zeros of the Riemann zeta function on the critical line*, J. Number Theory, 132(4), 2012, 511-542.

[6] R. Garunkštis, *On a positivity property of the Riemann \( \xi \)-function*, Liet. matem. rink. 42 (2002), 179–184.

[7] G. H. Hardy, *Sur les zéros de la fonction \( \zeta(s) \).* Comptes Rendus 158(1914) 1012–1014.

[8] E. Hasanalizade, Q. Shen, P. J. Wong, *Counting zeros of the Riemann zeta function*, J. Number Theory, 2021.

[9] A. Hinkkanen, *On functions of bounded type*, Complex Variables Theory Appl. 34 (1997), 119-139.

[10] J. C. Lagarias, *On a positivity property of the Riemann \( \xi \) function*, Acta Arith. 89 (1999), 217–234.

[11] N. Levinson, *More than one third of the zeros of Riemann’s \( z \)-function are on \( \sigma = 1/2 \)*, Adv. Math., 13 (4) (1974), 383–436.

[12] V. Mackevičius, *Integralas ir matas*, TEV, 1998, ISBN 9986-546-47-8.

[13] Mathematica (Version 9.0), Wolfram Research, Inc., Champaign, Illinois, 2012.

[14] Yu. Matiyasevich, F. Saidak, P. Zvengrowski, *Horizontal monotonicity of the modulus of the zeta function, L-functions, and related functions*, Acta Arith., 166.2 (2014), 189–200.

[15] M. R. Murty, *Problems in Analytic Number Theory*, Grad. Texts in Math. 206, Springer, New York, 2001.

[16] D. J. Platt, T. S. Trudgian, *An improved explicit bound on \( |\zeta(1/2+i t)| \)*, J. Number Theory, 147, 2015, 842–851.

[17] K. Pratt, N. Robles, A. Zaharescu et al., *More than five-twelfths of the zeros of \( \zeta \) are on the critical line*, Res. Math. Sci. 7(2), (2020).

[18] A. Selberg, *On the zeros of Riemann’s zeta-function*, Skr. Norske Vid. Akad. Oslo I., 10 (1942), 59.
[19] J. Sondow, C. Dumitrescu, A monotonicity property of Riemann’s xi function and a reformulation of the Riemann hypothesis, Period. Math. Hungar. 60 (2010), 37–40.

[20] M. Spivak, Calculus (3 ed.), Houston, Texas: Publish or Perish, 1994, ISBN 978-0-914098-89-8.

[21] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function, 2nd edition, Oxford Univ. Press, 1986.

[22] T.S. Trudgian, An improved upper bound for the argument of the Riemann zeta-function on the critical line II, Journal of Number Theory, 134 (2014), 280-292.

[23] Wolfram MathWorld, https://mathworld.wolfram.com/Xi-Function.html

Andrius Grigutis, Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University, Naugarduko 24, LT-03225 Vilnius, Lithuania, e-mail: andrius.grigutis@mif.vu.lt

Edvinas Goldstein, Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University, Naugarduko 24, LT-03225 Vilnius, Lithuania, e-mail: edvinasgoldstein@gmail.com

Corresponding Author: Andrius Grigutis