POLISH SPACES OF BANACH SPACES.
COMPLEXITY OF ISOMETRY CLASSES AND GENERIC PROPERTIES

MAREK CÚTH, MARTIN DOLEŽAL, MICHAL DOUCHA, AND ONDŘEJ KURKA

Abstract. We present and thoroughly study natural Polish spaces of separable Banach spaces. These spaces are defined as spaces of norms, resp. pseudonorms on the countable infinite-dimensional rational vector space. We provide an exhaustive comparison of these spaces with the admissible topologies recently introduced by Godefroy and Saint-Raymond and show that Borel complexities differ little with respect to these two different topological approaches.

We then focus mainly on the Borel complexities of isometry classes of the most classical Banach spaces. We prove that the infinite-dimensional Hilbert space is characterized as the unique separable infinite-dimensional Banach space whose isometry class is closed, and also as the unique separable infinite-dimensional Banach space whose isomorphism class is $F_{σG}$. For $p \in [1, 2) \cup (2, \infty)$, we show that the isometry classes of $L_p[0, 1]$ and $L_q$ are $G_{δ}$-complete and $F_{σδ}$-complete, respectively. Then we show that the isometry class of the Gurariĭ space is $G_{δ}$-complete and the isometry class of $c_0$ is $F_{σδ}$-complete. The isometry class of the former space is moreover proved to be dense $G_{δ}$. Additionally, we compute the complexities of many other natural classes of Banach spaces; for instance, $L_{p, λ+}$-spaces, for $p, λ \geq 1$, are shown to be $G_{δ}$, superreflexive spaces are shown to be $F_{σδ}$, and spaces with local II-basis structure are shown to be $Σ_0^6$. The paper is concluded with many open problems and suggestions for a future research.

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Introduction

Banach spaces and descriptive set theory have a long history of mutual interactions. An explicit use of descriptive set theory to Banach space theory can be traced back at least to the seminal papers of Bourgain ([6], [5]), where it has become apparent that descriptive set theory is an indispensable tool for universality problems. That is a theme that has been investigated by researchers working with Banach spaces ever since (see e.g. [2] and [12] and references therein).

As it eventually turned out, ‘Descriptive set theory of Banach spaces’ is an interesting and rich subject of its own and it has received some considerable attention in the recent years. One of the starting points was the idea of Bossard of coding separable Banach spaces in [3], [4]. His approach was, which can be considered standard, to choose some universal separable Banach space $X$, e.g. $C(2^\mathbb{N})$, and consider the Effros-Borel space $F(X)$. Recall that this is the set of closed subsets of $X$ equipped with a certain $\sigma$-algebra which makes $F(X)$ a standard Borel space, i.e. a measurable space which is isomorphic, as a measurable space, to a Polish space equipped with the $\sigma$-algebra of Borel sets. It is then not too difficult to show that a subset $SB(X) \subseteq F(X)$ consisting of all closed linear subspaces is a Borel subset, and therefore a standard Borel space itself.

A drawback of this idea is that there is no canonical or natural (Polish) topology on $SB(X)$. So although one can ask whether a given class of Banach spaces is Borel or not, the question about the exact complexity of that particular class is meaningless. A recent work [23] of Godefroy and Saint-Raymond addresses this issue. They still work with the space $SB(X)$, but among the many Polish topologies on $SB(X)$ giving the Effros-Borel structure, they select some particular subclass which is called admissible topologies. Although no particular admissible topology is canonical, the set of requirements put on this class guarantees that the exact Borel complexities vary little.

One of the aims of this paper is to present a concrete and natural Polish space (and some variants of it) of separable Banach spaces, which is convenient to work with and in which the computations of Borel complexities are usually as straightforward as they could be. Informally, it is the space of all norms, resp. pseudonorms, on the space of all finitely supported sequences of rational numbers - the unique infinite-dimensional vector space over $\mathbb{Q}$ with a countable Hamel basis. We note that this is, in spirit, similar to how (for instance) Vershik topologized the space of all Polish metric spaces ([48]), or how Grigorchuk topologized the space of all $n$-generated, resp. finitely generated, groups ([25]).

This space has already appeared in the previous work of the authors in [10] as a useful coding of Banach spaces. Here we investigate it further. Our goals are twofold:

(a) to show that the spaces of norms, resp. pseudonorms, have all the advantages of admissible topologies on $SB(X)$,

(b) to demonstrate the strength of this new approach by computing (in many cases, these computations are sharp) complexities of various classes of Banach spaces; this includes improving some estimates of Godefroy and Saint-Raymond and addressing some of their problems, as well as initiating the research of computing the complexities of isometry classes of Banach spaces.

The goal (a) is achieved mainly by the following theorem.

Theorem A. There is a $\Sigma^0_2$-measurable mapping from the space of norms to any admissible topology of Godefroy and Saint-Raymond that associates to each norm a space isometric to the space which the norm defines, and vice versa.
Additionally, while the exact Borel complexities are more or less independent of the choice of the admissible topology, some finer topological properties such as being meager or comeager (this is mentioned below), or the description of the topological closures are not. We obtain a neat characterization of topological closures in the spaces of norms and pseudonorms in terms of finite representability, we refer the reader to Proposition 1.9.

The rest of the paper focuses on goal (b). Let us describe some of the main results here.

First we focus on computing complexities of isometry classes of Banach spaces, i.e. how easy/difficult it is to define a concrete Banach space uniquely up to isometry. There is an active ongoing research whether for a particular Banach space its isomorphism class is Borel or not (see e.g. [21], [33], [17], [22]) while it is known that isometry classes of separable Banach spaces are always Borel (note that the linear isometry relation is Borel bi-reducible with an orbit equivalence relation [11] and orbit equivalence relations have Borel equivalence classes [29, Theorem 15.14]). Having a topology at our disposal we compute complexities of isometry classes of several classical Banach spaces.

**Theorem B.** (1) The infinite-dimensional separable Hilbert space is characterized as the unique infinite-dimensional separable Banach space whose isometry class is closed (see Theorem 3.4).

(2) For $p \in [1, 2) \cup (2, \infty)$, the isometry class of $L_p([0, 1], \lambda)$ is $G_\delta$-complete. Moreover, for every $\lambda \geq 1$, the class of separable $L_{p, \lambda^+}$-spaces is $G_\delta$ and the class of separable $L_p$-spaces is $G_\delta^*$, improving the estimate from [23] (see Theorems 4.4 and 4.6 and Corollary 4.7).

(3) For $p \in [1, 2) \cup (2, \infty)$, the isometry class of $\ell_p$ is $F_{\sigma\delta}$-complete (see Theorem 5.1).

(4) The isometry class of $c_0$ is $F_{\sigma\delta}$-complete (see Theorem 5.1).

(5) The isometry class of the Gurari˘ı space is $G_\delta$-complete (see Corollary 4.2).

Regarding the Hilbert space, we also uniquely characterize it by the complexity of its isomorphism class.

**Theorem C.** The infinite-dimensional separable Hilbert space is characterized as the unique, up to isomorphism, infinite-dimensional separable Banach space whose isomorphism class is $F_\sigma$ (see Theorem 6.1).

Let us also present here few sample results which involve complexities of more general classes of Banach spaces.

**Theorem D.** (1) The class of all superreflexive spaces is $F_{\sigma\delta}$ (see Theorem 7.3).

(2) The class of all spaces with local $\Pi$-basis structure is $\Sigma^0_\infty$ (see Theorem 7.13).

(3) The class of spaces whose Szlenk index is bounded by $\omega^\alpha$ is $\Pi^{0}_{\omega^\alpha+1}$ (see Theorem 7.7).

Next we consider various ‘genericity’ or ‘Baire category’ problems. For the space of norms and pseudonorms we have a definitive solution.

**Theorem E.** The isometry class of the Gurari˘ı space is dense $G_\delta$ in the space of norms and pseudonorms, i.e. the Gurari˘ı space is the generic separable Banach space (see Theorem 2.1).

We then consider Baire category problems also for admissible topologies, thus addressing Problem 5.5 of Godefroy and Saint-Raymond from [23]. In particular, we confirm their suspicion that being meager is not independent of the choice of the admissible topology.
**Theorem F.** For any universal Banach space $X$, any admissible topology $\tau$ on $SB(X)$ and infinite-dimensional Banach spaces $Y$ and $Z$ such that $Y \hookrightarrow Z$ and $Z \not\hookrightarrow Y \oplus F$ for every finite-dimensional space $F$, there exists a finer admissible topology $\tau' \supseteq \tau$ such that the class of Banach spaces isomorphic to $Z$ is nowhere dense in $(SB(X), \tau')$. In particular, there exists an admissible topology in which the Gurari˘ı space is meager (see Theorem 2.12).

On the other hand, the Gurari˘ı space is dense $G_\delta$ in the Wijsman topology (see Theorem 2.17).

The paper is organized as follows. Section 1 first introduces the Polish spaces of norms and pseudonorms and investigates their basic topological properties. The rest of the section is then devoted to the proof of Theorem F. This is a very technical part that is independent on the rest of the paper, so readers who are satisfied with our definitions of Polish spaces of norms/pseudonorms without objections can safely skip it and proceed immediately to Section 2. There we prove Theorems E and F. That part can be also read independently, so readers interested only in descriptive complexities can focus on Sections 3, 4, 5, 6, and 7 where Theorems B, C, and D are proved.

Final Section 8 presents several open problems and directions for a further research.

Let us conclude the introduction by setting up some notation that will be used throughout the paper.

**Notation:** Throughout the paper we usually denote the Borel classes of low complexity by the traditional notation such as $F_\sigma$ and $G_\delta$, or even $F_{\sigma\delta}$ (countable intersection of $F_\sigma$ sets) and $G_{\delta\sigma}$ (countable union of $G_\delta$ sets). However whenever it is more convenient or necessary we use the notation $\Sigma^0_{\alpha}$, resp. $\Pi^0_{\alpha}$, where $\alpha < \omega_1$ (we refer to [29, Section 11] for this notation). We emphasize that open sets, resp. closed sets, are $\Sigma^0_1$, resp. $\Pi^0_1$, by this notation.

In a few occasions, for a Borel class $\Gamma$ we will use the notion of $\Gamma$-hard and $\Gamma$-complete sets. We refer the reader to [29, Definition 22.9] for this notion. For a reader not familiar with this notion, let us emphasize that a set $A$ is $\Gamma$-hard, for a Borel class $\Gamma$, in particular implies that $A$ is not of lower complexity than $\Gamma$. Thus results stating that some set is $\Sigma^0_\alpha$-complete mean that the set is $\Sigma^0_\alpha$ and not simpler.

Moreover, given a class $\Gamma$ of sets in metrizable spaces, we say that $f : X \to Y$ is $\Gamma$-measurable if $f^{-1}(U) \in \Gamma$ for every open set $U \subseteq Y$.

Given Banach spaces $X$ and $Y$, we denote by $X \equiv Y$ (resp. $X \simeq Y$) the fact that those two spaces are linearly isometric (resp. isomorphic). We denote by $X \hookrightarrow Y$ the fact that $Y$ contains a subspace isomorphic to $X$. For $K \geq 1$, a $K$-isomorphism $T : X \to Y$ is a linear map with $K^{-1}\|x\| \leq \|Tx\| \leq K\|x\|$, $x \in X$. If $\{x_1, \ldots, x_n\}$ are linearly independent elements of $X$ and $y_1, \ldots, y_n \in Y$, we write $(Y, y_1, \ldots, y_n) \overset{K}{\sim} (X, x_1, \ldots, x_n)$ if the linear operator $T : \text{span}\{x_1, \ldots, x_n\} \to \text{span}\{y_1, \ldots, y_n\}$ sending $x_i$ to $y_i$ satisfies $\|T\|, \|T^{-1}\| < K$. If $X$ has a canonical basis $(x_1, \ldots, x_n)$ which is clear from the context, we just write $(Y, y_1, \ldots, y_n) \overset{K}{\sim} X$ instead of $(Y, y_1, \ldots, y_n) \overset{K}{\sim} (X, x_1, \ldots, x_n)$. Moreover, if $Y$ is clear from the context we write $(y_1, \ldots, y_n) \overset{K}{\sim} X$ instead of $(Y, y_1, \ldots, y_n) \overset{K}{\sim} X$.

Finally, in order to avoid any confusion, we emphasize that throughout the text $\ell^p_n$ denotes the $n$-dimensional $\ell^p$-space, i.e. the upper index denotes dimension.
1. **The Polish spaces of separable Banach spaces**

We begin with formalizing the class of all separable (infinite-dimensional) Banach spaces as a Polish space. The main outcomes of this section are the following.

(1) In the first subsection, we introduce the main notions of this paper: the Polish spaces of pseudonorms $\mathcal{P}$ (and $\mathcal{P}_\infty$) representing separable (infinite-dimensional) Banach spaces, and we recall the space of norms $\mathcal{B}$ that has appeared in our previous work [10]. We show some interesting features of these space, e.g. the neat relation between finite representability and topological closures in these space; see Proposition 1.9 and its corollaries.

(2) In the second subsection, we recall the coding $\mathcal{SB}(X)$ (and $\mathcal{SB}_\infty(X)$) of separable (infinite-dimensional) Banach spaces. We recall the notion of an admissible topology introduced in [23], which is a Polish topology corresponding to the Effros-Borel structure of $\mathcal{SB}(X)$. We explore some basic relations between codings $\mathcal{P}$, $\mathcal{P}_\infty$, $\mathcal{B}$, $\mathcal{SB}(X)$ and $\mathcal{SB}_\infty(X)$. We show there is a continuous reduction from $\mathcal{SB}(X)$ to $\mathcal{P}$, a $\Sigma^0_2$-measurable reduction from $\mathcal{P}_\infty$ to $\mathcal{B}$, and a $\Sigma^0_4$-measurable reduction from $\mathcal{P}$ to $\mathcal{SB}(X)$, see Theorem 1.17, Proposition 1.20 and Theorem 1.24. Here by a ‘reduction’, we mean a Borel map $\Phi$ such that the code and its image are both codes of the same (up to isometry) Banach space.

(3) The third subsection is devoted to the proof of Theorem 1.25, by which there is a $\Sigma^0_2$-measurable reduction from $\mathcal{B}$ to $\mathcal{SB}_\infty(X)$. Further, we note that the developed techniques also lead to a $\Sigma^0_3$-measurable reduction from $\mathcal{P}$ to $\mathcal{SB}(X)$, which is an improvement of the result mentioned above.

The meaning of the reductions above is that there is not a big difference between Borel ranks when considered in any of the Polish spaces mentioned above. However, it seems that considering $\mathcal{P}_\infty$ and $\mathcal{B}$ has some pleasant additional features, see e.g. Proposition 1.13 and its corollaries.

Let us emphasize that the existence of a Borel reduction from $\mathcal{B}$ to $\mathcal{SB}_\infty(X)$ has been essentially proved in [32, Lemma 2.4]. Going through the proof of [32, Lemma 2.4], one may obtain a reduction which is $\Sigma^0_3$-measurable; however, the proof does not seem to give a $\Sigma^0_2$-measurable reduction (which is the optimal result). In order to obtain this improvement, see Theorem 1.25, we have to develop a whole machinery of new ideas in combination with very technical results, and this is the reason why we devote a whole subsection to the proof.

Since our reductions from $\mathcal{SB}_\infty(X)$ to $\mathcal{P}_\infty$ and from $\mathcal{B}$ to $\mathcal{SB}_\infty(X)$ are optimal, it seems to be a very interesting open problem of whether there exists a continuous reduction from $\mathcal{P}_\infty$ to $\mathcal{B}$ or at least a $\Sigma^0_2$-measurable reduction from $\mathcal{P}_\infty$ to $\mathcal{SB}_\infty(X)$, see Question 1 and Question 2.

1.1. **Introduction of the spaces $\mathcal{P}$, $\mathcal{P}_\infty$ and $\mathcal{B}$, and their topological features.**

Let us start with the following idea of coding the class of separable (infinite-dimensional) Banach spaces. It is based on the idea presented already in our previous paper [10], where the space $\mathcal{B}$ was defined.

By $V$, we shall denote the vector space over $\mathbb{Q}$ of all finitely supported sequences of rational numbers; that is, the unique infinite-dimensional vector space over $\mathbb{Q}$ with a countable Hamel basis $(e_n)_{n \in \mathbb{N}}$.

**Definition 1.1.** Let us denote by $\mathcal{P}$ the space of all pseudonorms on the vector space $V$. Since $\mathcal{P}$ is a closed subset of $\mathbb{R}^V$, this gives $\mathcal{P}$ the Polish topology inherited from $\mathbb{R}^V$. The subbasis of this topology is given by sets of the form $U[v, I] := \{\mu \in \mathcal{P} : \mu(v) \in I\}$, where $v \in V$ and $I$ an open interval.
We often identify \( \mu \in \mathcal{P} \) with its extension to the pseudonorm on the space \( c_{00} \), that is, the vector space over \( \mathbb{R} \) of all finitely supported sequences of real numbers.

For every \( \mu \in \mathcal{P} \) we denote by \( X_\mu \) the Banach space given as the completion of the quotient space \( X/N \), where \( X = (c_{00}, \mu) \) and \( N = \{ x \in c_{00} : \mu(x) = 0 \} \). In what follows we often consider \( V \) as a subspace of \( X_\mu \), that is, we identify every \( v \in V \) with its equivalence class \([v]_N \in X_\mu\).

By \( \mathcal{P}_\infty \) we denote the set of those \( \mu \in \mathcal{P} \) for which \( X_\mu \) is infinite-dimensional Banach space. As we did in [10], by \( \mathcal{B} \) we denote the set of those \( \mu \in \mathcal{P}_\infty \) for which the extension of \( \mu \) to \( c_{00} \) is an actual norm, that is, the vectors \( e_1, e_2, \ldots \) are linearly independent in \( X_\mu \).

We endow \( \mathcal{P}_\infty \) and \( \mathcal{B} \) with topologies inherited from \( \mathcal{P} \).

Our first aim is to show that the topologies on \( \mathcal{P}_\infty \) and \( \mathcal{B} \) are Polish (see Corollary 1.5). This can be easily verified directly, here we obtain it as a corollary of the fact that the relation \( \sim \) defined before is open in \( \mathcal{P} \), a very useful fact that will prove important many times in the paper.

We need the following background first. Given a metric space \((M, d), \varepsilon > 0 \) and \( N, S \subseteq M \) we say that \( N \) is \( \varepsilon \)-dense for \( S \) if for every \( x \in S \) there is \( y \in N \) with \( d(x, y) < \varepsilon \) (let us emphasize that we do not assume \( N \subseteq S \)). For further references, we recall the following well-known approximation lemma, for a proof see e.g. [1, Lemma 12.1.11].

**Lemma 1.2.** There is a function \( \phi_1 : [0, 1) \to [0, 1) \) continuous at zero with \( \phi_1(0) = 0 \) such that whenever \( T : E \to X \) is a linear operator between Banach spaces, \( \varepsilon \in (0, 1), M \subseteq E \) is \( \varepsilon \)-dense for \( S_E \) and

\[
\forall m \in M : \|Tm\| - 1 < \varepsilon,
\]

then \( T \) is \((1 + \phi_1(\varepsilon))\)-isomorphism between \( E \) and \( T(E) \).

The following definition precises the notation \( \overset{K}{\sim} \) defined in the introduction.

**Definition 1.3.** If \( v_1, \ldots, v_n \in V \) are given, for \( \mu \in \mathcal{P} \), instead of \((X_\mu, v_1, \ldots, v_n) \overset{K}{\sim} X \), we shall write \((\mu, v_1, \ldots, v_n) \overset{K}{\sim} X \).

**Lemma 1.4.** Let \( X \) be a Banach space with \( \{x_1, \ldots, x_n\} \subseteq X \) linearly independent and let \( v_1, \ldots, v_n \in V \). Then for any \( K > 1 \) the set

\[
\mathcal{N}(\{(x_1, K, (v_i)_i)\}) = \{ \mu \in \mathcal{P} : (\mu, v_1, \ldots, v_n) \overset{K}{\sim} (X, x_1, \ldots, x_n) \}
\]

is open in \( \mathcal{P} \).

In particular, the set of those \( \mu \in \mathcal{P} \) for which the set \( \{v_1, \ldots, v_n\} \) is linearly independent in \( X_\mu \) is open in \( \mathcal{P} \).

**Proof.** Pick some \( \mu \in \mathcal{N}(\{(x_1, K, (v_i)_i)\}) \). By definition, the linear map \( T \) sending \( v_i \) to \( x_i \in X, i \leq n \), is a linear isomorphism satisfying \( \max\{\|T\|, \|T^{-1}\|\} < L \) for some \( L < K \). Let \( \phi_1 \) be the function provided by Lemma 1.2 and pick \( \varepsilon > 0 \) such that \( L(1 + \phi_1(2\varepsilon)) < K \). Let \( N \subseteq V \) be a finite \( \varepsilon \)-dense set for the sphere of \( \text{span}\{v_1, \ldots, v_n\} \subseteq X_\mu \) such that \( \mu(v) \in (1 - \varepsilon, 1 + \varepsilon) \) for every \( v \in N \). Then

\[
U := \{ \nu \in \mathcal{P} : \forall v \in N : |\nu(v) - \mu(v)| < \varepsilon \}
\]

is an open neighborhood of \( \mu \) and \( U \subseteq \mathcal{N}(\{(x_1, K, (v_i)_i)\}) \). Indeed, for any \( \nu \in U \) we have that \( \text{id} : (\text{span}\{v_1, \ldots, v_n\}, \mu) \to (\text{span}\{v_1, \ldots, v_n\}, \nu) \) is \((1 + \phi_1(2\varepsilon))\)-isomorphism; hence, the linear map \( T \) considered as a map between \( (\text{span}\{v_1, \ldots, v_n\}, \nu) \) and \( \text{span}\{x_1, \ldots, x_n\} \) satisfies \( \|T\| < L(1 + \phi_1(2\varepsilon)) < K \), and similarly \( \|T^{-1}\| < K \); hence, \( \nu \in \mathcal{N}(\{(x_1, K, (v_i)_i)\}) \).

The “in particular” part easily follows, because \( v_1, \ldots, v_n \in V \) are linearly independent if and only if there exists \( K > 1 \) with \( (\mu, v_1, \ldots, v_n) \overset{K}{\sim} \ell^1 \). \( \square \)
Corollary 1.5. Both $\mathcal{P}_\infty$ and $\mathcal{B}$ are $G_\delta$ sets in $\mathcal{P}$.

Since we are interested mainly in subsets of $\mathcal{P}$ closed under isometries, we introduce the following notation.

Notation 1.6. Let $Z$ be a separable Banach space and let $\mathcal{I}$ be a subset of $\mathcal{P}$. We put
\[
\langle Z \rangle_\mathcal{I} := \{ \mu \in \mathcal{I} : X_\mu \equiv Z \} \quad \text{and} \quad \langle Z \rangle_\mathcal{I}_\equiv := \{ \mu \in \mathcal{I} : X_\mu \simeq Z \}.
\]
If $\mathcal{I}$ is clear from the context we write $\langle Z \rangle_\equiv$ and $\langle Z \rangle_\equiv_\equiv$ instead of $\langle Z \rangle_\mathcal{I}_\equiv$ and $\langle Z \rangle_\mathcal{I}_\equiv_\equiv$ respectively.

The important feature of the topology of the spaces $\mathcal{P}$, $\mathcal{P}_\infty$ and $\mathcal{B}$ is that basic open neighborhoods are defined using finite data, i.e. finitely many vectors. That suggests that topological properties of the aforementioned spaces should be closely related to the local theory of Banach spaces. This is certainly a point that could be investigated further in a future research. Here we just observe how topological closures are related to finite representability, see Proposition 1.9. In order to formulate our results, let us consider the following generalization of the classical notion of finite representability.

Definition 1.7. Let $\mathcal{F}$ be a family of Banach spaces. We say that a Banach space $X$ is finitely representable in $\mathcal{F}$ if given any finite-dimensional subspace $E$ of $X$ and $\varepsilon > 0$ there exists a finite-dimensional subspace $F$ of some $Y \in \mathcal{F}$ which is $(1 + \varepsilon)$-isomorphic to $E$.

If the family $\mathcal{F}$ consists of one Banach space $Y$ only, we say rather that $X$ is finitely representable in $Y$ than in $\{Y\}$.

If $\mathcal{F} \subseteq \mathcal{P}$, by saying that $X$ is finitely representable in $\mathcal{F}$ we mean it is finitely representable in $\{X_\mu : \mu \in \mathcal{F}\}$.

The following is an easy observation which we will use further, the proof follows e.g. immediately from [11] Lemma 12.1.7 in the case that $\mathcal{F}$ contains one Banach space only. For the more general situation the proof is analogous.

Lemma 1.8. Let $\mathcal{F}$ be a family of infinite-dimensional Banach spaces and $\mu \in \mathcal{P}_\infty$. Let $\{k(n)\}_{n=1}^{\infty}$ be a sequence such that $\{e_{k(n)} : n \in \mathbb{N}\}$ is a linearly independent set in $X_\mu$ and $\text{span}\{e_{k(n)} : n \in \mathbb{N}\} = X_\mu$. Then $X_\mu$ is finitely representable in $\mathcal{F}$ if and only if for every $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists a finite dimensional subspace $F$ of some $Y \in \mathcal{F}$ which is $(1 + \varepsilon)$-isomorphic to $\text{span}\{e_{k(1)}, \ldots, e_{k(n)}\}$.

Proposition 1.9. Let $\mathcal{F} \subseteq \mathcal{B}$ be such that $\langle X_\mu \rangle_\mathcal{B}_\equiv \subseteq \mathcal{F}$ for every $\mu \in \mathcal{F}$. Then
\[
\{ \nu \in \mathcal{B} : X_\nu \text{ is finitely representable in } \mathcal{F} \} = \overline{\mathcal{F} \cap \mathcal{B}}.
\]
The same holds if we replace $\mathcal{B}$ with $\mathcal{P}_\infty$ or with $\mathcal{P}$.

In particular, if $X$ is a separable infinite-dimensional Banach space, then
\[
\{ \nu \in \mathcal{B} : X_\nu \text{ is finitely representable in } X \} = \overline{\langle X \rangle_\mathcal{B}_\equiv \cap \mathcal{B}}
\]
and similarly also if we replace $\mathcal{B}$ with $\mathcal{P}_\infty$ or with $\mathcal{P}$.

Proof. “$\subseteq$”: Fix $\nu \in \mathcal{B}$ such that $X_\nu$ is finitely-representable in $\mathcal{F}$. Pick $v_1, \ldots, v_n \in V$ and $\varepsilon > 0$. We shall show there is $\mu_0 \in \mathcal{F}$ with $|\mu_0(v_i) - \nu(v_i)| < \varepsilon$, $i \leq n$. Let $m \in \mathbb{N}$ be such that $\{v_1, \ldots, v_n\} \subseteq \text{span}_{\mathbb{Q}}\{e_j : j \leq m\}$. Put $C := \max\{\nu(v_i) : i = 1, \ldots, n\}$ and $Z := \text{span}\{e_1, \ldots, e_m\} \subseteq X_\nu$. Since $X_\nu$ is finitely representable in $\mathcal{F}$, there is $\mu \in \mathcal{F}$ and an $(1 + \frac{\varepsilon}{C})$-isomorphism $T : Z \to X_\mu$. Set $x_i := T(e_i), i \leq m$, and extend $x_1, \ldots, x_m$ to a linearly independent sequence $(x_i)_{i=1}^{\infty}$ whose span is dense in $X_\mu$. Consider $\mu_0 \in \mathcal{P}$ given by setting $\mu_0(\sum_{i \in I} \alpha_i e_i) = \mu(\sum_{i \in I} \alpha_i x_i)$, where $I \subseteq \mathbb{N}$ is finite and $(\alpha_i)_{i \in I} \subseteq \mathbb{Q}$. Clearly, $X_{\mu_0} \equiv X_\mu$ and $\mu_0 \in \mathcal{B}$, so $\mu_0 \in \mathcal{F}$. 

Proof. “$\supseteq$”: Fix $\mu \in \mathcal{F}$. Since $\mu$ is finitely representable in $\mathcal{F}$, there is $\mu_0 \in \mathcal{F}$ with $|\mu_0(v_i) - \mu(v_i)| < \varepsilon$, $i \leq n$. Let $m \in \mathbb{N}$ be such that $\{v_1, \ldots, v_n\} \subseteq \text{span}_{\mathbb{Q}}\{e_j : j \leq m\}$. Put $C := \max\{\nu(v_i) : i = 1, \ldots, n\}$ and $Z := \text{span}\{e_1, \ldots, e_m\} \subseteq X_\nu$. Since $X_\nu$ is finitely representable in $\mathcal{F}$, there is $\mu \in \mathcal{F}$ and an $(1 + \frac{\varepsilon}{C})$-isomorphism $T : Z \to X_\mu$. Set $x_i := T(e_i), i \leq m$, and extend $x_1, \ldots, x_m$ to a linearly independent sequence $(x_i)_{i=1}^{\infty}$ whose span is dense in $X_\mu$. Consider $\mu_0 \in \mathcal{P}$ given by setting $\mu_0(\sum_{i \in I} \alpha_i e_i) = \mu(\sum_{i \in I} \alpha_i x_i)$, where $I \subseteq \mathbb{N}$ is finite and $(\alpha_i)_{i \in I} \subseteq \mathbb{Q}$. Clearly, $X_{\mu_0} \equiv X_\mu$ and $\mu_0 \in \mathcal{B}$, so $\mu_0 \in \mathcal{F}$.
Corollary 1.10. Let

\[ \mu_0(v_i) = \mu(\sum_{j=1}^{m} \alpha_j x_j) \leq (1 + \frac{1}{\delta^2})\nu(\sum_{j=1}^{m} \alpha_j e_j) = (1 + \frac{1}{\delta^2})\nu(v_i) \]

and similarly \( \mu_0(v_i) \geq (1 + \frac{1}{\delta^2})^{-1}\nu(v_i) \geq (1 - \frac{1}{\delta^2})\nu(v_i) \). Thus, \( |\mu_0(v_i) - \nu(v_i)| \leq \frac{\delta}{\delta^2} < \varepsilon \) for every \( i \leq n \).

The case when we replace \( B \) with \( \mathcal{P}_\infty \) or \( \mathcal{P} \) is analogous, this time we only do not require \( (x_i)_{i=1}^\infty \) to be linearly independent.

"\( \subset \)"; Fix \( \nu \in \mathcal{F} \cap B \). In order to see that \( X_{\nu} \) is finitely representable in \( \mathcal{F} \), we will use Lemma 1.8. Pick \( n \in \mathbb{N} \) and \( \varepsilon > 0 \). Let \( \phi_1 \) be the function from Lemma 1.2. Let \( \delta > 0 \) be such that \( \phi_1(2\delta) < \varepsilon \) and let \( N \subseteq V \) be a finite set which is \( \delta \)-dense for the sphere of \( \langle \text{span}\{e_1, \ldots, e_n\}, \nu \rangle \) and \( \nu(v) \in (1 - \delta, 1 + \delta) \) for every \( v \in N \).

Pick \( \mu \in \mathcal{F} \) such that \( |\mu(v) - \nu(v)| < \delta \), \( v \in N \). Then \( \text{id} : \langle \text{span}\{e_1, \ldots, e_n\}, \nu \rangle \to \langle \text{span}\{e_1, \ldots, e_n\}, \mu \rangle \) is \((1 + \phi_1(2\delta))\)-isomorphism. Thus, \( X_{\nu} \) is finitely representable in \( \mathcal{F} \). The case when we replace \( B \) with \( \mathcal{P}_\infty \) or \( \mathcal{P} \) is similar.

This result will have many interesting consequences. Let us state one of them, of a general nature, here. Other will appear in appropriate sections when needed.

**Corollary 1.10.** Let \( X \) be a separable infinite-dimensional Banach space. Then

\[ \{ \mu \in B : X_\mu \text{ is finitely representable in } X \} \]

is a closed set in \( B \). The same holds if we replace \( B \) with \( \mathcal{P}_\infty \) or with \( \mathcal{P} \).

We also consider a kind of an opposite notion where instead of finitely representing one fixed Banach space in a class of Banach spaces, we represent a class of Banach spaces in one fixed Banach space.

**Definition 1.11.** Let \( \mathcal{F} \) be a class of finite dimensional Banach spaces and let \( X \) be a Banach space. We say that \( \mathcal{F} \) is **representable in** \( X \) if for every \( F \in \mathcal{F} \) and \( \varepsilon > 0 \) there exists a subspace of \( X \) that is \((1 + \varepsilon)\)-isomorphic to \( F \).

Analogously, we say that \( \mathcal{F} \) is **crudely representable in** \( X \) if there is \( \lambda \geq 1 \) such that for every \( F \in \mathcal{F} \) and \( \varepsilon > 0 \) there exists a subspace of \( X \) that is \((\lambda + \varepsilon)\)-isomorphic to \( F \). If the family \( \mathcal{F} \) consists of all finite-dimensional subspaces of a (possibly infinite-dimensional) Banach space \( Y \), we say that \( Y \) is crudely finitely representable in \( X \).

**Proposition 1.12.** Let \( \mathcal{F} \) be an arbitrary class of finite dimensional Banach spaces. Then the set of those \( \mu \in \mathcal{P} \) such that \( \mathcal{F} \) is representable in \( X_\mu \) is \( G_\delta \). Analogously, the set of those \( \mu \in \mathcal{P} \) such that \( \mathcal{F} \) is crudely representable in \( X_\mu \) is \( G_{\delta\sigma} \).

In particular, for a fixed Banach space \( Y \), the set of those \( \mu \in \mathcal{P} \) such that \( Y \) is finitely representable, resp. crudely finitely representable, in \( X_\mu \) is \( G_\delta \), resp. \( G_{\delta\sigma} \).

**Proof.** For an arbitrary finite dimensional Banach space \( F \) (with a fixed basis \( b_F \)) and \( K > 1 \), we set

\[ R(F, K) := \{ \mu \in \mathcal{P} : \exists v_1, \ldots, v_n \in V \ (\mu, v_1, \ldots, v_n) \overset{K}{\sim} F \} \]

By Lemma 1.4, \( R(F, K) \) is open.

Now since for each \( n \) the space of all isometry classes of \( n \)-dimensional Banach spaces with the Banach-Mazur distance is compact, in particular it is separable, we can without loss of generality assume that the class \( \mathcal{F} \) is countable. As a consequence, we get that the set of those \( \mu \in \mathcal{P} \) such that \( \mathcal{F} \) is representable in \( X_\mu \) is equal to

\[ \bigcap_{F \in \mathcal{F}, n \in \mathbb{N}} R(F, 1 + 1/n), \]
which is $G_d$.

Analogously, we get that the set of those $\mu \in \mathcal{P}$ such that $\mathcal{F}$ is crudely representable in $X_\mu$ is equal to

$$\bigcup_\lambda \bigcap_{F \in \mathcal{F}} \bigcap_{n \in \mathbb{N}} R(F, \lambda + 1/n),$$

which is $G_{ds}$.

The ‘In particular’ part follows immediately. □

We conclude this subsection by showing another nice feature of the above topologies on examples. We can show that the natural maps $K \mapsto C(K)$ and $\lambda \mapsto L_\mu(\lambda)$, where $K$ is a compact metrizable space and $\lambda$ is a Borel probability measure on a fixed compact metric space, are continuous.

**Example 1.13.**  
(a) Let $\mathcal{K}([0,1]^\mathbb{N})$ denote the space of all compact subsets of the Hilbert cube $[0,1]^\mathbb{N}$ endowed with the Vietoris topology. Then there exists a continuous mapping $\rho: \mathcal{K}([0,1]^\mathbb{N}) \to \mathcal{P}$ such that $X_{\rho(K)} \equiv C(K)$ for every $K \in \mathcal{K}([0,1]^\mathbb{N})$.

(b) Let $L$ be a compact metric space, let $p \in [1, \infty)$ be fixed and let $\mathcal{Prob}(L)$ denote the space of all Borel probability measures on $L$ endowed with the weak* topology (generated by elements of the Banach space $C(L)$). Then there exists a continuous mapping $\sigma: \mathcal{Prob}(L) \to \mathcal{P}$ such that $X_{\sigma(\lambda)} \equiv L_\mu(\lambda)$ for every $\lambda \in \mathcal{Prob}(L)$.

**Proof.**  
(a) Let $\{f_i: i \in \mathbb{N}\}$ be a linearly dense subset of $C([0,1]^\mathbb{N})$. For every compact subset $K$ of $[0,1]^\mathbb{N}$, we define $\rho(K) \in \mathcal{P}$ by

$$\rho(K) \left( \sum_{i=1}^n r_i e_i \right) = \sup_{x \in K} \left| \sum_{i=1}^n r_i f_i(x) \right|, \quad \sum_{i=1}^n r_i e_i \in V.$$

It is clear that $X_{\rho(K)} \equiv C(K)$, so we only need to check the continuity of $\rho$. It is enough to show that $\rho^{-1}(U[v,I])$ is an open subset of $\mathcal{K}([0,1]^\mathbb{N})$ for every $v \in V$ and every open interval $I$ (recall that $U[v,I] = \{\mu \in \mathcal{P}: \mu(v) \in I\}$). So let us fix $\tilde{K} \in \rho^{-1}(U[v,I])$, and assume that $v = \sum_{i=1}^n r_i e_i$. Fix $x_0 \in \tilde{K}$ such that

$$\left| \sum_{i=1}^n r_i f_i(x_0) \right| = \sup_{x \in \tilde{K}} \left| \sum_{i=1}^n r_i f_i(x) \right|.$$

Fix also $\varepsilon > 0$ such that both numbers $|\sum_{i=1}^n r_i f_i(x_0)| + \varepsilon$ belong to $I$. Now find open subsets $U, V$ of $[0,1]^\mathbb{N}$ such that $x_0 \in U$ and $\tilde{K} \subseteq V$, and such that

$$\inf_{x \in U} \left| \sum_{i=1}^n r_i f_i(x) \right| > \left| \sum_{i=1}^n r_i f_i(x_0) \right| - \varepsilon$$

and

$$\sup_{x \in V} \left| \sum_{i=1}^n r_i f_i(x) \right| < \sup_{x \in \tilde{K}} \left| \sum_{i=1}^n r_i f_i(x) \right| + \varepsilon.$$

Then

$$U := \{K \in \mathcal{K}([0,1]^\mathbb{N}): K \cap U \neq \emptyset \text{ and } K \subseteq V\}$$

is an open neighborhood of $\tilde{K}$ such that $\rho(U) \subseteq U[v,I]$. 


(b) This is similar to (a) but even easier. Let \( \{g_i : i \in \mathbb{N}\} \) be a linearly dense subset of \( C(L) \). For every Borel probability measure \( \lambda \) on \( L \), we define \( \sigma(\lambda) \) by

\[
\sigma(\lambda) \left( \sum_{i=1}^{n} r_i e_i \right) = \left( \int L \left| \sum_{i=1}^{n} r_i g_i \right|^p d\lambda \right)^{\frac{1}{p}}, \quad \sum_{i=1}^{n} r_i e_i \in V.
\]

It is clear that \( X_{\sigma(\lambda)} \equiv L_\mu(\lambda) \), so we only need to check the continuity of \( \sigma \).

Remark 1.14. After the introduction of the spaces \( \mathcal{P}, \mathcal{P}_\infty, \) and \( \mathcal{B} \), one faces the question which of them is ‘the right one’, with which to work. For now, we leave the question undecided. Since we are mainly interested in infinite-dimensional Banach spaces, we prefer to work mainly with \( \mathcal{P}_\infty \) and \( \mathcal{B} \), and indeed the space \( \mathcal{P} \) has a certain pathological property when computing the closed isometry classes (we refer to Remark 1.2). On the other hand, it turns out that at least as far as one wants to transfer some computations performed in the space of pseudo norms directly to \( \mathcal{P} \), it holds true also in any admissible topology.

Regarding \( \mathcal{P}_\infty \) and \( \mathcal{B} \), in most of the arguments there is no difference whether we work with the former or the latter space. However, there are few exceptions when it seems to be convenient to work with the assumption that the sequence of vectors \( \{e_n : n \in \mathbb{N}\} \subseteq V \) is linearly independent, and then we work with \( \mathcal{B} \).

1.2. Relations between codings \( \mathcal{P}, \mathcal{P}_\infty, \mathcal{B}, SB(X) \) and \( SB_\infty(X) \). Here we recall the approach to topologizing the class of all separable (infinite-dimensional) Banach spaces by Godefroy and Saint-Raymond from [23] which was a partial motivation for our research.

Definition 1.15. Let \( X \) be a Polish space and let us denote by \( \mathcal{F}(X) \) the set of all the closed subsets of \( X \). For an open set \( U \subseteq X \) we put \( E^+(U) = \{ F \in \mathcal{F}(X) : U \cap F \neq \emptyset \} \). Following [23], we say that a Polish topology \( \tau \) on the set \( \mathcal{F}(X) \) is admissible if it satisfies the following two conditions:

(i) For every open subset \( U \) of \( X \), the set \( E^+(U) \) is \( \tau \)-open.

(ii) There exists a subbasis of \( \tau \) such that every set from this subbasis is a countable union of sets of the form \( E^+(U) \setminus E^+(V) \), where \( U \) and \( V \) are open in \( X \).

We note that Godefroy and Saint-Raymond also suggest the following optional condition that is satisfied by many natural admissible topologies.

(iii) The set \( \{(x, F) \in X \times \mathcal{F}(X) : x \in F\} \) is closed in \( X \times \mathcal{F}(X) \).

If \( X \) is a separable Banach space, we denote by \( SB(X) \subseteq \mathcal{F}(X) \) the set of closed vector subspaces of \( X \). We denote by \( SB_\infty(X) \) the subset of \( SB(X) \) consisting of infinite-dimensional spaces. We say that a topology on \( SB(X) \) or \( SB_\infty(X) \) is

\footnote{note that the condition (ii) is different from what is mentioned in [23], however, as the authors have confirmed, there is a typo in the condition from [23] which makes it wrong (otherwise no single one of the topologies mentioned in [23] would be admissible)}
admissible if it induced by an admissible topology on \( F(X) \). Both \( SB(X) \) and \( SB_\infty(X) \) are Polish spaces when endowed with an admissible topology; see Remark 1.16.

If \( Z \) is a separable Banach space, we put, similarly as in Notation 1.6,

\[
\langle Z \rangle_\approx := \{ F \in SB(X) : F \cong Z \} \quad \text{and} \quad \langle Z \rangle_{\preceq} := \{ F \in SB(X) : F \preceq Z \}.
\]

It will be always clear from the context whether we work with subsets of \( \mathcal{P} \), or \( SB(X) \).

**Remark 1.16.** If \( \tau \) is an admissible topology on a separable Banach space \( X \) then \( SB(X) \) is a \( G_\delta \)-subset of \((\mathcal{F}(X), \tau)(\text{see } [23 \text{ Section 3}])\). Moreover, by [23 Corollary 4.2], \( SB_\infty(X) \) is \( G_\delta \)-subset of \((SB(X), \tau)\). (In fact, the paper [23] deals only with the case \( X = C(2^\omega) \) but the generalization to any separable Banach space is easy.)

Certain connection between codings \( SB(X) \) and \( \mathcal{P} \) of separable Banach spaces might be deduced already from [23].

**Theorem 1.17.** Let \( X \) be isometrically universal separable Banach space and let \( \tau \) be an admissible topology on \( SB(X) \). Then there is a continuous mapping \( \Phi : (SB(X), \tau) \to \mathcal{P} \) such that for every \( F \in SB(X) \) we have \( F \equiv X_\Phi(F) \).

**Proof.** By [23 Theorem 4.1], there are continuous functions \( (f_n)_{n \in \mathbb{N}} \) on \( SB(X) \) with values in \( X \) such that for each \( F \in SB(X) \) we have \( \{ f_n(F) : n \in \mathbb{N} \} = F \).

Consider the mapping \( \Phi \) given by \( \Phi(F)(\sum_{n=1}^{k} a_n e_n) := \| \sum_{n=1}^{k} a_n f_n(F) \|_X \) for every \( F \in SB(X) \) and \( a_1, \ldots, a_k \in \mathbb{Q} \). Then it is easy to see that \( \Phi \) is the mapping we need. \( \square \)

The following relation between various codings of Banach spaces as \( SB(X) \) is easy.

**Observation 1.18.** Let \( X,Y \) be isometrically universal separable Banach spaces and let \( \tau_1 \) and \( \tau_2 \) be admissible topologies on \( SB(X) \) and \( SB(Y) \), respectively. Then there is a \( \Sigma^0_2 \)-measurable mapping \( f : (SB(X), \tau_1) \to (SB(Y), \tau_2) \) such that for every \( F \in SB(X) \) we have \( F \equiv f(F) \). Moreover, \( f \) can be chosen such that for every open set \( U \subseteq Y \) there is an open set \( V \subseteq X \) such that \( f^{-1}(E^+(U)) = E^+(V) \).

**Proof.** Let \( j : X \to Y \) be an isometry (not necessarily surjective). Then the mapping \( f \) given by \( f(F) := j(F) \), \( F \in SB(X) \), does the job, because \( f^{-1}(E^+(U)) = E^+(j^{-1}(U)) \) for every open set \( U \subseteq X \). \( \square \)

Let us note the following easy fact which we record here for a later reference.

The proof is easy and so it is omitted.

**Lemma 1.19.** Let \( X \) be isometrically universal separable Banach space, \( \tau \) be an admissible topology on \( SB(X) \), \( Y \) be a Polish space, \( f : Y \to SB(X) \) be a mapping and \( n \in \mathbb{N} \), \( n \geq 2 \), be such that \( f^{-1}(E^+(U)) \) is a \( \Delta^0_n \) set in \( Y \) for every open set \( U \subseteq X \). Then \( f \) is \( \Sigma^0_n \)-measurable.

A straightforward idea leads to the following relation between \( \mathcal{P}_\infty \) and \( \mathcal{B} \).

**Proposition 1.20.** There is a \( \Sigma^0_2 \)-measurable mapping \( \Phi : \mathcal{P}_\infty \to \mathcal{B} \) such that for every \( \mu \in \mathcal{P}_\infty \) we have \( X_\mu = X_\Phi(\mu) \).

Moreover, \( \Phi \) can be chosen such that \( \Phi^{-1}(U[v, I]) \in \Delta^0_2(\mathcal{P}_\infty) \) for each \( v \in V \) and open interval \( I \).

**Proof.** For each \( \mu \in \mathcal{P}_\infty \) let us inductively define natural numbers \( (n_k(\mu))_{k \in \mathbb{N}} \) by

\[
n_1(\mu) := \min \{ n \in \mathbb{N} : \mu(e_n) \neq 0 \},
\]

\[
n_{k+1}(\mu) := \min \{ n \in \mathbb{N} : e_{n_{k+1}}(\mu), \ldots, e_{n_k}(\mu), e_n \text{ are linearly independent} \}.
\]
Consider the mapping Φ given by Φ(µ)(∑_{k=1}^{∞} a_n e_n) := µ(∑_{i=1}^{k} a_i e_{n_i(µ)}) for every µ ∈ ℙ_∞ and a_1, ..., a_k ∈ Q. It is easy to see that Φ(µ) ∈ B and that X_µ is isometric to X_{Φ(µ)} for every µ ∈ ℙ_∞.

For all natural numbers \( N_1 < \ldots < N_k \) the set \( \{ µ ∈ ℙ_∞ : n_1(µ) = N_1, \ldots, n_k(µ) = N_k \} \) is a \( \Delta^0_1 \) set in ℙ_∞. Indeed, we may prove it by induction on k because for each \( k ∈ \mathbb{N} \) and each µ ∈ ℙ_∞ we have that \( n_1(µ) = N_1, \ldots, n_{k+1}(µ) = N_{k+1} \) if

\[
\forall n ∈ \mathbb{N} : e_n, e_{N_k}, e_n \text{ are linearly dependent}
\]

which is an intersection of a \( \Delta^0_1 \)-condition (by the inductive assumption) with a closed and open condition (by Lemma 1.4).

Let us pick \( v = \sum_{i=1}^{k} a_i e_n ∈ V \) and an open interval \( I \). Then

\[
Φ^{-1}(U[v,I]) = \{ µ ∈ ℙ_∞ : µ(∑_{i=1}^{k} a_i e_{n_i(µ)}) ∈ I \},
\]

which is a \( \Delta^0_1 \) set in ℙ_∞. Indeed, on one hand we have \( µ ∈ Φ^{-1}(U[v,I]) \) iff there are natural numbers \( N_1 < N_2 < \ldots < N_k < N_k \) such that \( n_1(µ) = N_1, \ldots, n_k(µ) = N_k \) and \( µ(∑_{i=1}^{k} a_i e_N) \in I \), which witnesses that \( Φ^{-1}(U[v,I]) ∈ Σ^0_2(ℙ_∞) \) as it is a countable union of \( \Delta^0_1 \) sets. On the other hand, we have that \( µ ∈ Φ^{-1}(U[v,I]) \) iff for each \( l ∈ \mathbb{N} \) we have that either \( n_k(µ) > l \) or there are natural numbers \( N_1 < N_2 < \ldots < N_k < N_k \leq l \) such that \( n_1(µ) = N_1, \ldots, n_k(µ) = N_k \) and \( µ(∑_{i=1}^{k} a_i e_N) \in I \), which witnesses that \( Φ^{-1}(U[v,I]) ∈ Π^0_3(ℙ_∞) \) as it is a countable intersection of \( \Delta^0_1 \) sets.

This proves the “Moreover” part from which it easily follows that Φ is \( \Sigma^0_2 \)-measurable.

**Remark 1.21.** For \( d ∈ \mathbb{N} \), let us consider the sets \( ℙ_d := \{ µ ∈ ℙ : \dim X_µ = d \} \) and

\[
B_d := \{ µ ∈ ℙ : e_1, \ldots, e_d \text{ is a basis of } X_µ \text{ and } µ(e_i) = 0 \text{ for every } i > d \}.
\]

A similar argument as in Proposition 1.20 shows that for every \( d ∈ \mathbb{N} \) there is a \( \Sigma^0_2 \)-measurable mapping Φ : ℙ_d → B_d such that for every µ ∈ ℙ_d we have \( X_µ ≡ X_{Φ(µ)} \).

Finally let us consider the reduction from ℙ to SB(X). An optimal result would be to have a \( \Sigma^0_2 \)-reduction. This is because, as was already observed in [23], the identity map between two admissible topologies is only \( \Sigma^0_2 \)-measurable in general. Using the ideas of the proof of [22] Lemma 2.4 we obtain Theorem 1.21. This result is improved in the next subsection, see Theorem 1.28 but since some steps remain the same, let us give a sketch of the argument (we will be a bit sketchy at the places which will be modified later).

**Lemma 1.22.** Let \( n ∈ \mathbb{N} \), \( X \) be isometrically universal separable Banach space and let \( τ \) be an admissible topology on SB(X). Let there exist \( \Sigma^0_{n+1} \)-measurable mappings \( χ_k : B → X, k ∈ \mathbb{N}, \) such that \( X_µ ≡ \text{span}\{ χ_k(µ) : k ∈ \mathbb{N} \} \) for every µ ∈ B.

Then there exists a \( \Sigma^0_{n+1} \)-measurable mapping Φ : B → (SB(X), τ) such that for every µ ∈ B we have \( X_µ ≡ Φ(µ) \).

**Proof.** Consider the mapping Φ : B → (SB(X), τ) defined as

\[
Φ(ν) := \text{span}\{ χ_k(ν) : k ∈ \mathbb{N} \}, \quad ν ∈ B.
\]

We have \( X_ν ≡ Φ(ν) \). For every open set \( U ⊆ X \), using the \( \Sigma^0_{n+1} \)-measurability of \( χ_k \)'s, it is easy to see that \( Φ^{-1}(E^+(U)) \) is \( \Sigma^0_2 \) in B. Thus, by Lemma 1.19 the mapping Φ is \( \Sigma^0_{n+1} \)-measurable.
Proposition 1.26. For any isometrically universal separable Banach space \( \Phi : B \to a \sum \Phi : P \to \mathbb{R} \) there exist continuous mappings \( \chi \) be an admissible topology on \( P \) such that for every \( \mu \in \mathcal{P} \) we have \( X_\mu \equiv \Phi(\mu) \).

Sketch of the proof. By Remark 1.21 it suffices to find, for every \( d \in \mathbb{N} \cup \{ \infty \} \), a \( \Sigma_3^0 \)-measurable reduction from \( B_d \) to \( SB(X) \), where \( B_\infty = B \). This is done for every \( d \in \mathbb{N} \cup \{ \infty \} \) in a similar way. Let us concentrate further only on the case of \( d = \infty \), the other cases are similar. From the proof of [32, Lemma 2.4], it follows that there are Borel measurable mappings \( \chi_k : B \to X, k \in \mathbb{N} \), such that \( X_\mu \equiv \{ \chi_k(\mu) : k \in \mathbb{N} \} \) for every \( \mu \in \mathcal{B} \). A careful inspection of the proof actually shows that the mappings \( \chi_k \) are \( \Sigma_3^0 \)-measurable (since this part is improved in the next subsection, see Proposition 1.25, we do not give any more details here). Thus, an application of Lemma 1.22 finishes the proof. \( \square \)

1.3. An optimal reduction from \( B \) to \( SB(X) \). This subsection is devoted mainly to the proof of the following result.

Theorem 1.24. Let \( X \) be isometrically universal separable Banach space and let \( \tau \) be an admissible topology on \( SB(X) \). Then there is a \( \Sigma_3^0 \)-measurable mapping \( \Phi : \mathcal{P} \to (SB(X), \tau) \) such that for every \( \mu \in \mathcal{P} \) we have \( X_\mu \equiv \Phi(\mu) \).

The main ingredient of the proof is the following.

Proposition 1.26. For any isometrically universal separable Banach space \( X \), there exist continuous mappings \( \chi_k : B \to X, k \in \mathbb{N} \), such that

\[
\left\| \sum_{k=1}^n a_k \chi_k(\nu) \right\| = \nu \left( \sum_{k=1}^n a_k e_k \right)
\]

for every \( \sum_{k=1}^n a_k e_k \in c_00 \) and every \( \nu \in B \).

Remark 1.27. Similarly as in Remark 1.21 we may easily obtain a variant of Proposition 1.26 for \( B_d, d \in \mathbb{N} \). Indeed, let \( d \in \mathbb{N} \) be given. For \( \nu \in B_d \), let us define \( \tilde{\nu} \in B \) by

\[
\tilde{\nu} \left( \sum_{i=1}^d a_i e_i \right) := \nu \left( \sum_{i=1}^d a_i e_i \right) + \sum_{i=d+1}^\infty |a_i|, \quad \sum_{i=1}^d a_i e_i \in c_00.
\]

If \( \chi_k, k \in \mathbb{N} \), are as in Proposition 1.26, then we may consider mappings \( \tilde{\chi}_k : B_d \to X, k \leq d \), defined by \( \tilde{\chi}_k(\nu) = \chi_k(\tilde{\nu}), \nu \in B_d \).

We postpone the proof of Proposition 1.26 to the very end of this subsection.

Proof of Theorem 1.24. Follows immediately from Lemma 1.22 and Proposition 1.26. \( \square \)

Similarly as above, we obtain also the following.

Theorem 1.28. Let \( X \) be isometrically universal separable Banach space and let \( \tau \) be an admissible topology on \( SB(X) \). Then there is a \( \Sigma_3^0 \)-measurable mapping \( \Phi : \mathcal{P} \to (SB(X), \tau) \) such that for every \( \mu \in \mathcal{P} \) we have \( X_\mu \equiv \Phi(\mu) \).

Proof. This is similar to the proof of Theorem 1.24, the only modification is that we use Proposition 1.26 and Remark 1.27 instead of the reference to the proof of [32, Lemma 2.4]. \( \square \)

The aim of the remainder of this subsection is now to prove Proposition 1.26.
Proposition 1.29. For each $A \subseteq \mathcal{P}$, the following conditions are equivalent:

1. There are a separable Banach space $U$ and continuous mappings $\chi_k : A \to U$, $k \in \mathbb{N}$, such that
   \[
   \left\| \sum_{k=1}^{n} a_k \chi_k(\nu) \right\| = \nu \left( \sum_{k=1}^{n} a_k e_k \right)
   \]
   for every $\sum_{k=1}^{n} a_k e_k \in c_{00}$ and every $\nu \in A$.

2. There are continuous functions $\alpha_k : \mathcal{A}^2 \to [0, \infty)$, $k \in \mathbb{N}$, such that
   \[
   \alpha_k(\nu, \nu) = 0
   \]
   for every $\nu$ and $k$ and the following property is satisfied: If $\nu \in A$ and $z^* \in (c_{00})^*$
   satisfy $|z^*(x)| \leq \nu(x)$ for every $x \in c_{00}$, then there is a mapping $\Gamma : A \to (c_{00})^*$
   such that $\Gamma(\nu) = z^*$, $|\Gamma(\mu)(x)| \leq \mu(x)$ for every $\mu \in A$ and $x \in c_{00}$, and
   \[
   |\Gamma(\mu)(e_k) - \Gamma(\lambda)(e_k)| \leq \alpha_k(\mu, \lambda)
   \]
   for every $\mu, \lambda \in A$ and every $k$.

   Moreover, if $A$ consists only of pseudonorms $\nu$ with $\nu(e_k) \leq 1$ for every $k$, then these conditions are equivalent with:

   (2') For every $\eta \in [0, 1)$, there are continuous functions $\beta_k : \mathcal{A}^2 \to [0, \infty)$, $k \in \mathbb{N}$, such that
   \[
   \beta_k(\eta, \nu) = 0
   \]
   for every $\nu$ and $k$ and the following property is satisfied: If $\nu \in A$ and $z^* \in (c_{00})^*$
   satisfy $|z^*(x)| \leq \nu(x)$ for every $x \in c_{00}$, then there is a mapping $\Gamma : A \to (c_{00})^*$
   such that $\Gamma(\nu) = \eta \cdot z^*$, $|\Gamma(\mu)(x)| \leq \mu(x)$ for every $\mu \in A$ and $x \in c_{00}$, and
   \[
   |\Gamma(\mu)(e_k) - \Gamma(\lambda)(e_k)| \leq \beta_k(\mu, \lambda)
   \]
   for every $\mu, \lambda \in A$ and every $k$.

Remark 1.30. The conditions (1) and (2) from Proposition 1.29 are equivalent also with the following one:

3. There are continuous functions $\alpha_k : \mathcal{A}^2 \to [0, \infty)$, $k \in \mathbb{N}$, such that
   \[
   \alpha_k(\nu, \nu) = 0
   \]
   for every $\nu$ and $k$ and
   \[
   \sum_{\mu, \lambda, k} |\alpha_{\mu, \lambda, k}(\mu, \lambda)| \geq \nu \left( \sum_{\lambda, k} (a_{\mu, \lambda, k} - a_{\lambda, \nu, k}) e_k \right) - \sum_{\mu \neq \nu} \mu \left( \sum_{\lambda, k} (a_{\mu, \lambda, k} - a_{\lambda, \mu, k}) e_k \right)
   \]
   for every $\nu \in A$ and every system $(a_{\mu, \lambda, k})_{\mu, \lambda \in A, k \in \mathbb{N}}$ of real numbers with finite support.

The proof is similar to the proof of (1) $\iff$ (2) below (for (1) $\Rightarrow$ (3) the choice of $\alpha_k$’s is the same as in the proof of (1) $\Rightarrow$ (2), for (3) $\Rightarrow$ (1) the construction of the space $U$ is the same as in (2) $\Rightarrow$ (1)). We omit the full proof, because the details are technical and we do not use the condition (3) any further. Let us note that even though we tried to find an application of the condition (3), we did not find it and this is basically the reason why we had to develop conditions (2) and (2').

Proof of Proposition 1.29 (1) $\Rightarrow$ (2): Given such $U$ and $\chi_k : A \to U$, $k \in \mathbb{N}$, we put
\[
\alpha_k(\nu, \mu) = \| \chi_k(\nu) - \chi_k(\mu) \|, \quad \nu, \mu \in A, k \in \mathbb{N}.
\]
Denote by $I_{\mu} : (c_{00}, \mu) \to U$ the isometry given by $e_k \mapsto \chi_k(\mu)$. Let $\nu \in A$ and
$z^* \in (c_{00})^*$ satisfying $|z^*(x)| \leq \nu(x)$ be given. For $x, y \in c_{00}$ with $I_{\nu}x = I_{\nu}y$, we have $|z^*(y - x)| \leq \nu(y - x) = \| I_{\nu}(y - x) \| = 0$, and so $z^*(x) = z^*(y)$. Thus, the formula
\[
u^*(I_{\nu}x) = z^*(x), \quad x \in c_{00},
\]
defines a functional on $L_p(c_{00})$ such that $|u^*(I_p x)| = |z^*(x)| \leq \nu(x) = \|I_p x\|$. By the Hahn-Banach theorem, we can extend $u^*$ to the whole $U$ in the way that

$$|u^*(u)| \leq \|u\|, \quad u \in U.$$

For every $\mu \in \mathcal{A}$, let us put

$$\Gamma(\mu)(x) = u^*(I_p x), \quad x \in c_{00}.$$

We obtain $|\Gamma(\mu)(x)| = |u^*(I_p x)| \leq \|I_p x\| = \mu(x)$ and $|\Gamma(\mu)(e_k) - \Gamma(\lambda)(e_k)| = |u^*(I_p e_k) - u^*(I_p e_k)| = |u^*(\chi_k(\mu) - \chi_k(\lambda))| \leq \|\chi_k(\mu) - \chi_k(\lambda)\| = \alpha_k(\mu, \lambda)$ for $\mu, \lambda \in \mathcal{A}$ and $k \in \mathbb{N}$.

$(2) \Rightarrow (1)$: Given such $\alpha_k : \mathcal{A} \rightarrow [0, \infty), k \in \mathbb{N}$, we define a subset of $c_{00}(\mathcal{A} \times \mathbb{N})$ by

$$\Omega = \co\left\{ \sum_k a_k e_{\mu,k} : \mu \left( \sum_k a_k e_k \right) \leq 1 \right\} \cup \left\{ c \cdot (e_{\mu,k} - e_{\lambda,k}) : |c| \cdot \alpha_k(\mu, \lambda) \leq 1 \right\},$$

and denote the corresponding Minkowski functional by $g$. Let $U$ be the completion of the quotient space $X/N$, where $X = (c_{00}(\mathcal{A} \times \mathbb{N}), g)$ and $N = \{ x \in c_{00}(\mathcal{A} \times \mathbb{N}) : \rho(x) = 0 \}$. In what follows, we identify every $x \in c_{00}(\mathcal{A} \times \mathbb{N})$ with its equivalence class $[x]_N \in U$. Let us define

$$\chi_k : \mathcal{A} \rightarrow U, \quad \nu \mapsto e_{\nu,k}.$$

As $e_{\nu,k} = (e_{\nu,k} - e_{\lambda,k}) \in \Omega$ whenever $|c| \cdot \alpha_k(\mu, \lambda) \leq 1$, we obtain $g(e_{\nu,k} - e_{\lambda,k}) \leq \alpha_k(\mu, \lambda)$, that is, $g(\chi_k(\mu) - \chi_k(\lambda)) \leq \alpha_k(\mu, \lambda)$. For a fixed $\mu$, we have $\alpha_k(\mu, \lambda) \rightarrow \alpha_k(\mu, \mu) = 0$ as $\lambda \rightarrow \mu$, and consequently $g(\chi_k(\mu) - \chi_k(\lambda)) \rightarrow 0$ as $\lambda \rightarrow \mu$. Therefore, $\chi_k$ is continuous on $\mathcal{A}$. It follows that the image of $\chi_k$ is separable. As these images contain all basic vectors $e_{\nu,k}$, the space $U$ is separable.

We need to show that

$$\nu(x) = g(\mathcal{F})$$

for fixed $\nu \in \mathcal{A}$, $x = \sum_{k \in \mathbb{N}} a_k e_k \in c_{00}$ and its image $\mathcal{F} = \sum_{k \in \mathbb{N}} a_k e_{\nu,k}$. The inequality $\nu(x) \geq g(\mathcal{F})$ follows immediately from the definition of $\Omega$ (for any $c \geq \nu(x)$ with $c > 0$, we have $\nu(\frac{1}{c}x) \leq 1$, and so $\frac{1}{c}x \in \Omega$, hence $g(\frac{1}{c}x) \leq 1$ and $g(\mathcal{F}) \leq c$).

Let us show the opposite inequality $\nu(x) \leq g(\mathcal{F})$. Using the Hahn-Banach theorem, we can pick $z^* \in (c_{00})^\#$ satisfying $z^*(x) = \nu(x)$ and $|z^*(y)| \leq \nu(y)$ for every $y \in c_{00}$. Let $\Gamma : \mathcal{A} \rightarrow (c_{00})^\#$ be the mapping provided for $\nu$ and $z^*$, and let $u^*(c_{00}(\mathcal{A} \times \mathbb{N}))$ be given by

$$u^*(e_{\nu,k}) = \Gamma(\mu)(e_k), \quad \mu \in \mathcal{A}, k \in \mathbb{N}.$$

Then $u^*(\mathcal{F}) = u^*(\sum_k a_k e_{\nu,k}) = \sum_k a_k u^*(e_{\nu,k}) = \sum_k a_k \Gamma(\nu)(e_k) = \Gamma(\nu)(\sum_k a_k e_k) = z^*(x) = \nu(x)$. It is sufficient to show that $u^* \leq 1$ on $\Omega$ (equivalently $|u^*(y)| \leq g(y)$ for every $y \in c_{00}(\mathcal{A} \times \mathbb{N})$), since it follows that $\nu(x) = u^*(\mathcal{F}) \leq g(\mathcal{F})$.

To show that $u^* \leq 1$ on $\Omega$, we need to check that

$$\mu\left( \sum_k b_k e_{\mu,k} \right) \leq 1 \quad \Rightarrow \quad u^*\left( \sum_k b_k e_{\mu,k} \right) \leq 1$$

and

$$|c| \cdot \alpha_k(\mu, \lambda) \leq 1 \quad \Rightarrow \quad u^*(c \cdot (e_{\mu,k} - e_{\lambda,k})) \leq 1.$$

Concerning the first implication, we compute $u^*(\sum_k b_k e_{\mu,k}) = \sum_k b_k u^*(e_{\mu,k}) = \sum_k b_k \mu(\sum_k b_k e_k) \leq \mu(\sum_k b_k e_k) \leq 1$. Concerning the second implication, we compute $u^*(c \cdot (e_{\mu,k} - e_{\lambda,k})) = \mu((e_{\nu,k} - e_{\lambda,k})) = \mu(e_{\nu,k} - e_{\lambda,k}) = c \mu(\mu(\sum_k b_k e_k))$. We have $c \mu(\mu(\sum_k b_k e_k)) \leq |c| \cdot \alpha_k(\mu, \lambda) \leq 1$.

$(2) \Rightarrow (2')$: The choice $\beta_k = \alpha_k$ works. Indeed, if $\Gamma$ is provided by $(2)$, we can take $\eta \cdot \Gamma$.

$(2') \Rightarrow (2)$: For every $n \in \mathbb{N}$, let $\beta_n^\eta : \mathcal{A}^2 \rightarrow [0, \infty), k \in \mathbb{N}$, be provided by $(2')$ for $\eta = (1 - 2^{-n})$. We can assume that each $\beta_n^{\eta}$ is a pseudometric. Indeed, instead
of $\beta^n_k$, we can take the maximal minorizing pseudometric $\tilde{\beta}^n_k$ (in such a case, $\tilde{\beta}^n_k$ is continuous and, since the function $(\mu, \lambda) \mapsto |\Gamma(\mu)(e_k) - \Gamma(\lambda)(e_k)|$ is a pseudometric, if it minorizes $\beta^n_k$, then it minorizes $\tilde{\beta}^n_k$ as well). Moreover, we can assume that $\beta^1_\ast$ is a metric (it is possible to add a compatible metric on $A$ to $\beta^1_\ast$).

Let us define

$$\alpha(\mu, \lambda) = \max_{n,k} \min \{\beta^n_k(\mu, \lambda), 2^{-\max\{n,k\}}\}, \quad \mu, \lambda \in A.$$  

It is easy to check that $\alpha$ is continuous. Due to our additional assumptions, $\alpha$ is a metric on $A$. We want to show that there are some constants $c_k$ such that the choice $\alpha_k = c_k \cdot \alpha$ works.

Let $\nu \in A$ and $z^* \in (e_{00})^\#$ satisfy $|z^*(x)| \leq \nu(x)$ for every $x \in c_{00}$. For every $n \in \mathbb{N}$, there is a mapping $\Gamma^n : A \to (e_{00})^\#$ such that

$$\Gamma^n(\nu) = (1 - 2^{-n}) \cdot z^*,$$

and, if we denote

$$\gamma^n_k(\mu) = \Gamma^n(\mu)(e_k),$$

then

$$|\gamma^n_k(\mu) - \gamma^n_k(\lambda)| \leq \beta^n_k(\mu, \lambda)$$

for every $\mu, \lambda \in A$ and every $k$. Let us note that

$$|\gamma^n_k(\mu)| \leq 1,$$

as $|\gamma^n_k(\mu)| = |\Gamma^n(\mu)(e_k)| \leq \mu(e_k) \leq 1$ by the assumption on $A$.

Now, we define the desired mapping $\Gamma$. For practical purposes, we first define

$$\gamma^n_k(\mu) = 0 \quad \text{for } n \in \mathbb{Z}, n \leq 0.$$

For every $n \in \mathbb{Z}$, let $f_n$ denote the piecewise linear function supported by $[2^{-n-3}, 2^{-n-1}]$ which is linear on $[2^{-n-3}, 2^{-n-2}]$ and $[2^{-n-2}, 2^{-n-1}]$, and for which $f_n(2^{-n-2}) = 1$.

In this way, we have $\sum_{n \in \mathbb{Z}} f_n = 1$ on $(0, \infty)$. We define

$$\gamma_k(\nu) = z^*(e_k)$$

and

$$\gamma_k(\mu) = \sum_{n \in \mathbb{Z}} f_n(\alpha(\mu, \nu)) \gamma^n_k(\mu), \quad \mu \neq \nu.$$

Finally, we put $\Gamma(\mu)(e_k) = \gamma_k(\mu)$, so $\Gamma(\nu) = z^*$ and $\Gamma(\mu) = \sum_{n \in \mathbb{Z}} f_n(\alpha(\mu, \nu)) \Gamma^n(\mu)$ for $\mu \neq \nu$. In both cases $\mu = \nu$ and $\mu \neq \nu$, it follows that $|\Gamma(\mu)(x)| \leq \mu(x)$ for every $x \in c_{00}$. It remains to prove the inequality

$$|\gamma_k(\mu) - \gamma_k(\lambda)| \leq c_k \cdot \alpha(\mu, \lambda)$$

for some suitable constants $c_k$.

Let us show that the implication

(1) \hspace{1cm} \alpha(\mu, \lambda) < 2^{-n} \Rightarrow |\gamma^n_k(\mu) - \gamma^n_k(\lambda)| \leq 2^{k+1} \alpha(\mu, \lambda)

holds. Clearly, we can suppose that $n \geq 1$. If $\alpha(\mu, \lambda) \geq 2^{-k}$, then $2^{k+1} \alpha(\mu, \lambda) \geq 2 \geq |\gamma^n_k(\mu) - \gamma^n_k(\lambda)|$. So, let us assume that $\alpha(\mu, \lambda) < 2^{-k}$. Since $\min\{\beta^n_k(\mu, \lambda), 2^{-\max\{n,k\}}\} \leq \alpha(\mu, \lambda) < 2^{-\max\{n,k\}}$, we have $\min\{\beta^n_k(\mu, \lambda), 2^{-\max\{n,k\}}\} = \beta^n_k(\mu, \lambda)$, and so $|\gamma^n_k(\mu) - \gamma^n_k(\lambda)| \leq \beta^n_k(\mu, \lambda)$.

Next, we show that

(2) \hspace{1cm} 2^{-n-4} \leq \alpha(\mu, \nu) < 2^{-n} \Rightarrow |\gamma^n_k(\mu) - \gamma_k(\nu)| \leq (2^{k+1} + 16) \alpha(\mu, \nu).
If \( n \geq 1 \), then \( \gamma_k(\nu) - \gamma_k^n(\nu) = z^*(e_k) - (1 - 2^{-n})z^*(e_k) = 2^{-n}z^*(e_k) \). If \( n \leq 0 \), then \( \gamma_k(\nu) - \gamma_k^0(\nu) = z^*(e_k) \). In both cases, \( |\gamma_k(\nu) - \gamma_k^n(\nu)| \leq 2^{-n}|z^*(e_k)| \leq 2^{-n}\nu(e_k) \leq 2^{-n} \leq 2^k\alpha(\mu, \nu) \). Using (1), we can compute

\[
|\gamma_k^n(\mu) - \gamma_k(\nu)| \leq |\gamma_k^n(\mu) - \gamma_k(\nu)| + |\gamma_k^n(\nu) - \gamma_k(\nu)| \leq (2^{k+1} + 2^4)\alpha(\mu, \nu).
\]

Further, it follows from (2) that

\[
|\gamma_k(\mu) - \gamma_k(\nu)| \leq (2^{k+1} + 16)\alpha(\mu, \nu).
\]

Indeed, since \( f_n \) is supported by \( [2^{-n-3}, 2^{-n-1}] \), we have always

\[
f_n(\alpha(\mu, \nu))|\gamma_k^n(\mu) - \gamma_k(\nu)| \leq f_n(\alpha(\mu, \nu))(2^{k+1} + 16)\alpha(\mu, \nu),
\]

and it is sufficient to use that \( \gamma_k(\mu) - \gamma_k(\nu) = \sum_{n \in \mathbb{Z}} f_n(\alpha(\mu, \nu))(\gamma_k^n(\mu) - \gamma_k(\nu)) \) for \( \mu \neq \nu \).

Now, we are going to investigate the value \( |\gamma_k(\mu) - \gamma_k(\lambda)| \). First, we have

\[
\alpha(\lambda, \nu) \geq 2\alpha(\mu, \nu) \quad \Rightarrow \quad |\gamma_k(\mu) - \gamma_k(\lambda)| \leq 3 \cdot (2^{k+1} + 16)\alpha(\mu, \lambda).
\]

Indeed, as \( \alpha(\mu, \lambda) \geq \alpha(\lambda, \nu) - \alpha(\mu, \nu) \geq 2\alpha(\mu, \nu) - \alpha(\mu, \nu) = \alpha(\mu, \nu) \), we can apply (3) and write

\[
|\gamma_k(\mu) - \gamma_k(\lambda)| \leq |\gamma_k(\mu) - \gamma_k(\nu)| + |\gamma_k(\lambda) - \gamma_k(\nu)| \\
\leq (2^{k+1} + 16)(\alpha(\mu, \nu) + \alpha(\lambda, \nu)) \\
= (2^{k+1} + 16)(\alpha(\lambda, \nu) - \alpha(\mu, \nu) + 2\alpha(\mu, \nu)) \\
\leq (2^{k+1} + 16)(1 + 2)\alpha(\mu, \lambda).
\]

Now, we prove the last but the most challenging implication

\[
(5) \quad \alpha(\mu, \nu) \leq \alpha(\lambda, \nu) < 2\alpha(\mu, \nu) \quad \Rightarrow \quad |\gamma_k(\mu) - \gamma_k(\lambda)| \leq [12(2^{k+1} + 16) + 2^{k+1}]\alpha(\mu, \lambda).
\]

Let us compute

\[
\gamma_k(\mu) - \gamma_k(\lambda) = \sum_{n \in \mathbb{Z}} [f_n(\alpha(\mu, \nu))\gamma_k^n(\mu) - f_n(\alpha(\lambda, \nu))\gamma_k^n(\lambda)] \\
= \sum_{n \in \mathbb{Z}} [f_n(\alpha(\mu, \nu))\gamma_k^n(\mu) - f_n(\alpha(\lambda, \nu))\gamma_k^n(\mu) + f_n(\alpha(\lambda, \nu))\gamma_k^n(\mu) - f_n(\alpha(\lambda, \nu))\gamma_k^n(\lambda)] \\
= \sum_{n \in \mathbb{Z}} [f_n(\alpha(\mu, \nu)) - f_n(\alpha(\lambda, \nu))][\gamma_k^n(\mu) - \gamma_k^n(\mu)] \\
+ \sum_{n \in \mathbb{Z}} f_n(\alpha(\lambda, \nu))[\gamma_k^n(\mu) - \gamma_k^n(\lambda)].
\]

Hence, \( |\gamma_k(\mu) - \gamma_k(\lambda)| \) is less than or equal to

\[
\sum_{n \in \mathbb{Z}} |f_n(\alpha(\mu, \nu)) - f_n(\alpha(\lambda, \nu))|(|\gamma_k^n(\mu) - \gamma_k(\nu)| + \sum_{n \in \mathbb{Z}} f_n(\alpha(\lambda, \nu))[\gamma_k^n(\mu) - \gamma_k^n(\lambda)].
\]

Let us notice that

- \( f_n(\alpha(\mu, \nu)) \neq 0 \) iff \( 2^{-n-3} < \alpha(\mu, \nu) < 2^{-n-1} \),
- \( f_n(\alpha(\lambda, \nu)) \neq 0 \) iff \( 2^{-n-3} < \alpha(\lambda, \nu) < 2^{-n-1} \), and \( 2^{-n-4} < \alpha(\mu, \nu) < 2^{-n-1} \) in this case,
- the function \( f_n \) is Lipschitz with the constant \( 2^{n+3} \).
So, if \( f_n(\alpha, \nu) \neq 0 \) or \( f_n(\lambda, \nu) \neq 0 \), then \( 2^{-n-4} < \alpha(\mu, \nu) < 2^{-n-1} \), and (2) can be applied. We obtain for the first sum that

\[
\sum_{n \in \mathbb{Z}} |f_n(\alpha, \nu) - f_n(\lambda, \nu)||\gamma_n(\mu) - \gamma_n(\nu)|
\leq \sum_{2^{-n-4} < \alpha(\mu, \nu) < 2^{-n-1}} 2^{n+3}|\alpha(\mu, \nu) - \alpha(\lambda, \nu)|(2^k+1 + 16)\alpha(\mu, \nu)
\leq \sum_{2^{-n-4} < \alpha(\mu, \nu) < 2^{-n-1}} 2^{n+3}\alpha(\mu, \lambda)(2^k+1 + 16)2^{n-1}
\leq 3 \cdot 2^2(2^k+1 + 16)\alpha(\mu, \lambda).
\]

Concerning the second sum, we notice that if \( f_n(\alpha, \nu) \neq 0 \), then \( \alpha(\lambda, \nu) < 2^{-n-1} \), and so \( \alpha(\mu, \lambda) \leq \alpha(\mu, \nu) + \alpha(\lambda, \nu) \leq 2\alpha(\lambda, \nu) < 2^{-n} \). Applying (1), we obtain

\[
\sum_{n \in \mathbb{Z}} f_n(\alpha(\lambda, \nu)||\gamma_n(\mu) - \gamma_n(\lambda)| = \sum_{n \in \mathbb{Z}} f_n(\alpha(\lambda, \nu))2^{k+1}\alpha(\mu, \lambda) = 2^{k+1}\alpha(\mu, \lambda),
\]

and (5) follows.

Finally, we finish the proof with the observation that (4) and (5) provide

\[
|\gamma_k(\mu) - \gamma_k(\nu)| \leq | \gamma_k(\mu) - \gamma_k(\lambda) | + | \gamma_k(\lambda) - \gamma_k(\nu) | \leq 12(2^k+1 + 16) + 2^k+1|\alpha(\mu, \lambda)|.
\]

We can suppose that \( \alpha(\mu, \nu) \leq \alpha(\lambda, \nu) \). If \( \alpha(\lambda, \nu) \geq 2\alpha(\mu, \nu) \), we use (4), and if \( \alpha(\lambda, \nu) < 2\alpha(\mu, \nu) \), we use (5). \( \square \)

**Definition 1.31.** Let \( B_{(1)} \) denote the set of all norms \( \nu \in B \) such that \( \nu(e_k) = 1 \) for each \( k \in \mathbb{N} \).

**Lemma 1.32.** The condition \( (2') \) from Proposition 1.29 is valid for \( A = B_{(1)} \).

**Proof.** Let \( \eta \in [0, 1) \) be given. We fix numbers \( \kappa_k < 1 \) such that \( \eta \leq \kappa_1 < \kappa_2 < \kappa_3 < \ldots \) For every \( \mu, \lambda \in B_{(1)} \), we define recursively

\[
\beta_1(\mu, \lambda) = 0
\]

and

\[
\beta_{k+1}(\mu, \lambda) = \sup \left\{ \mu \left( e_{k+1} + \sum_{i=1}^k a_i e_i \right) - \lambda \left( e_{k+1} + \sum_{i=1}^k a_i e_i \right) + \sum_{i=1}^k |a_i| \beta_i(\mu, \lambda) : a_1, \ldots, a_k \in \mathbb{R}, \min \left\{ \mu \left( \sum_{i=1}^k a_i e_i \right), \lambda \left( \sum_{i=1}^k a_i e_i \right) \right\} < \frac{2\kappa_{k+1}}{\kappa_{k+1} - \kappa_k} \right\}.
\]

Clearly, \( \beta_k(\nu, \nu) = 0 \) for every \( \nu \in B_{(1)} \). Let us sketch a proof of continuity of the functions \( \beta_k \). The function \( \beta_1 = 0 \) is obviously continuous. Assuming that \( \beta_i \) is continuous for every \( i \leq k \), we consider for \( \delta > 0 \) the set

\[
U_{\delta}^{k+1}(\mu) = \left\{ \mu' \in B_{(1)} : \{ \forall x \in \text{span} \{e_1, \ldots, e_{k+1}\} \setminus \{0\} : (1+\delta)^{-1} < \frac{\mu'(x)}{\mu(x)} < 1+\delta \} \right\}.
\]

Using Lemma 1.2, it is easy to see that \( U_{\delta}^{k+1}(\mu) \) is an open neighborhood of \( \mu \) in \( B_{(1)} \). Given \( \varepsilon > 0 \), we can find \( \delta > 0 \) such that \( |\beta_{k+1}(\mu', \lambda') - \beta_{k+1}(\mu, \lambda)| < \varepsilon \) for every \( (\mu', \lambda') \in U_{\delta}^{k+1}(\mu) \times U_{\delta}^{k+1}(\lambda) \). The details are left to the reader.

Let us prove that the functions \( \beta_k \) work. Given \( \nu \in B_{(1)} \) and \( z^* \in (c_{00})^\# \) satisfying \( |z^*(x)| \leq \nu(x) \) for every \( x \in c_{00} \), we define first

\[
\gamma_1(\mu) = \eta \cdot z^*(e_1), \quad \mu \in B_{(1)}.
\]
Recurrsively, we define for every \( k \in \mathbb{N} \) functions
\[
\begin{align*}
u_{k+1}(\mu) &= \sup_{a_1, \ldots, a_k} \left[ -\kappa_{k+1}\mu \left( -e_{k+1} + \sum_{i=1}^{k} a_i e_i \right) + \sum_{i=1}^{k} a_i \gamma_i(\mu) \right], \\
\gamma_{k+1}(\mu) &= \inf_{a_1, \ldots, a_k} \left[ \kappa_{k+1}\mu \left( e_{k+1} + \sum_{i=1}^{k} a_i e_i \right) - \sum_{i=1}^{k} a_i \gamma_i(\mu) \right]
\end{align*}
\]
and
\[
\gamma_{k+1}(\mu) = p_{k+1} \nu_{k+1}(\mu) + q_{k+1} \nu_{k+1}(\mu),
\]
where numbers \( p_{k+1} \geq 0, q_{k+1} \geq 0 \) with \( p_{k+1} + q_{k+1} = 1 \) are chosen in the way that
\[
\gamma_{k+1}(\nu) = \eta \cdot z^*(e_{k+1}).
\]
Let us check that it is possible to choose such numbers. Note first that \( \gamma_i(\nu) = \eta \cdot z^*(e_i) \). Assuming that the functions \( \gamma_i \) are already defined and satisfy \( \gamma_i(\nu) = \eta \cdot z^*(e_i) \) for \( i \leq k \), we notice that, for every \( a_1, \ldots, a_k \in \mathbb{R} \),
\[
\pm \eta \cdot z^*(e_{k+1}) + \sum_{i=1}^{k} a_i \gamma_i(\nu) = \eta \cdot z^* \left( \pm e_{k+1} + \sum_{i=1}^{k} a_i e_i \right) \leq \kappa_{k+1} \nu \left( \pm e_{k+1} + \sum_{i=1}^{k} a_i e_i \right),
\]
and consequently
\[
\eta \cdot z^*(e_{k+1}) \geq -\kappa_{k+1} \nu \left( -e_{k+1} + \sum_{i=1}^{k} a_i e_i \right) + \sum_{i=1}^{k} a_i \gamma_i(\nu),
\]
\[
\eta \cdot z^*(e_{k+1}) \leq \kappa_{k+1} \nu \left( e_{k+1} + \sum_{i=1}^{k} a_i e_i \right) - \sum_{i=1}^{k} a_i \gamma_i(\nu).
\]
This gives
\[
u_{k+1}(\nu) \leq \gamma_{k+1}(e_{k+1}) \leq \nu_{k+1}(\nu),
\]
and it follows that suitable \( p_{k+1} \) and \( q_{k+1} \) do exist.

Let us prove that
\[
\sum_{i=1}^{k} a_i \gamma_i(\mu) \leq \kappa_{k+1} \mu \left( \sum_{i=1}^{k} a_i e_i \right)
\]
for every \( \mu \in \mathcal{B}(\nu), k \in \mathbb{N} \) and \( a_1, \ldots, a_k \in \mathbb{R} \). For \( k = 1 \), we just write \( a_1 \gamma_1(\mu) = a_1 \eta \cdot z^*(e_1) \leq |a_1| |\kappa_1 \nu(e_1) = |a_1| |\kappa_1 | \mu(e_1) = \kappa_1 \mu(a_1 e_1) \). Assume that (4) is valid for \( k \). We show first that
\[
u_{k+1}(\mu) \leq \gamma_{k+1}(\mu) \leq \nu_{k+1}(\mu).
\]
Clearly, it is sufficient to show just that \( \nu_{k+1}(\mu) \leq v_{k+1}(\mu) \). Given \( b_1, \ldots, b_k \) and \( c_1, \ldots, c_k \), we need to check that
\[
-\kappa_{k+1} \mu \left( -e_{k+1} + \sum_{i=1}^{k} b_i e_i \right) + \sum_{i=1}^{k} b_i \gamma_i(\mu) \leq \kappa_{k+1} \mu \left( e_{k+1} + \sum_{i=1}^{k} c_i e_i \right) - \sum_{i=1}^{k} c_i \gamma_i(\mu).
\]
But this is easy, as
\[
\sum_{i=1}^{k} b_i \gamma_i(\mu) + \sum_{i=1}^{k} c_i \gamma_i(\mu) = \sum_{i=1}^{k} (b_i + c_i) \gamma_i(\mu) \leq \kappa_{k+1} \mu \left( \sum_{i=1}^{k} (b_i + c_i) e_i \right)
\]
\[
\leq \kappa_{k+1} \left[ \mu \left( -e_{k+1} + \sum_{i=1}^{k} b_i e_i \right) + \mu \left( e_{k+1} + \sum_{i=1}^{k} c_i e_i \right) \right]
\]
Now, let us verify (7) for \( k + 1 \). We can suppose that \( a_{k+1} = \pm 1 \). For \( a_{k+1} = 1 \), it is enough to use

\[
\gamma_{k+1}(\mu) \leq v_{k+1}(\mu) \leq \kappa_{k+1}\mu(e_{k+1} + \sum_{i=1}^{k} a_i e_i) - \sum_{i=1}^{k} a_i \gamma_i(\mu),
\]

and for \( a_{k+1} = -1 \), it is enough to use

\[
\gamma_{k+1}(\mu) \geq u_{k+1}(\mu) \geq -\kappa_{k+1}\mu(-e_{k+1} + \sum_{i=1}^{k} a_i e_i) + \sum_{i=1}^{k} a_i \gamma_i(\mu).
\]

Next, let us prove that

\[
|\gamma_k(\mu) - \gamma_k(\lambda)| \leq \beta_k(\mu, \lambda)
\]

for every \( \mu, \lambda \in B(1) \) and \( k \in \mathbb{N} \). This is clear for \( k = 1 \), as \( \gamma_1 \) is constant. Assume that (8) is valid for \( i \leq k \). To prove it for \( k+1 \), it is sufficient to show the inequalities

\[
|u_{k+1}(\mu) - u_{k+1}(\lambda)| \leq \beta_{k+1}(\mu, \lambda) \quad \text{and} \quad |v_{k+1}(\mu) - v_{k+1}(\lambda)| \leq \beta_{k+1}(\mu, \lambda).
\]

We consider the function \( v_{k+1} \) only, since the inequality for \( u_{k+1} \) can be shown in the same way. Let us note first that, in the definition of \( v_{k+1}(\mu) \), it is possible to take the infimum only over \( k \)-tuples with

\[
\mu \left( \sum_{i=1}^{k} a_i e_i \right) \leq \frac{2\kappa_{k+1}}{\kappa_{k+1} - \kappa_k}.
\]

Indeed, for \( a_1, \ldots, a_k \) which do not satisfy this condition, using (7), we obtain

\[
\kappa_{k+1}\mu(e_{k+1} + \sum_{i=1}^{k} a_i e_i) - \sum_{i=1}^{k} a_i \gamma_i(\mu)
\]\n
\[
\geq \kappa_{k+1}\mu \left( \sum_{i=1}^{k} a_i e_i \right) - \kappa_{k+1}\mu(e_{k+1} + \sum_{i=1}^{k} a_i e_i)
\]

\[
= (\kappa_{k+1} - \kappa_k)\mu \left( \sum_{i=1}^{k} a_i e_i \right) - \kappa_{k+1}
\]

\[
\geq 2\kappa_{k+1} - \kappa_{k+1} = \kappa_{k+1} = \kappa_{k+1}\mu(e_{k+1} + \sum_{i=1}^{k} 0 \cdot e_i) - \sum_{i=1}^{k} 0 \cdot \gamma_i(\mu).
\]
There are a separable Banach space $U_k$ for each $k$ for every $\sum k$. For every $\sum k$, the functions $\beta_k$ of $U_k$ and continuous mappings $\chi_k : B_{(1)} \to U, k \in \mathbb{N}$, such that

$$\| \sum_{k=1}^{n} a_k \chi_k(\nu) \| = \nu \left( \sum_{k=1}^{n} a_k e_k \right)$$

for every $\sum_{k=1}^{n} a_k e_k \in c_{00}$ and every $\nu \in B_{(1)}$. Since $X$ contains an isometric copy of $U$, we can suppose that $U \subseteq X$.

Let us consider the continuous mapping $\Psi : B \to B_{(1)}$ given by

$$\Psi(\mu) \left( \sum_{k=1}^{n} a_k e_k \right) = \mu \left( \sum_{k=1}^{n} a_k e_k \right).$$

If we define $\tilde{\chi}_k(\mu) = \mu(e_k) : \chi_k(\Psi(\mu))$, $\mu \in B$, then we get

$$\left\| \sum_{k=1}^{n} b_k \tilde{\chi}_k(\mu) \right\| = \left\| \sum_{k=1}^{n} b_k \mu(e_k) \chi_k(\Psi(\mu)) \right\| = \Psi(\mu) \left( \sum_{k=1}^{n} b_k \mu(e_k)e_k \right) = \mu \left( \sum_{k=1}^{n} b_k e_k \right)$$

for every $\sum_{k=1}^{n} b_k e_k \in c_{00}$ and every $\mu \in B$. \hfill \Box

2. Generic properties

As soon as one has a Polish space, or more generally a Baire space, of some objects it is natural and often useful to find properties (of these objects) that are generic; that is, the corresponding subset of the space is comeager. In the case of the spaces $P, P_\infty$ and $B$ we resolve this problem completely, see Theorem 2.1.
In the case of the spaces $SB(X)$ with the admissible topology, this is the content of Problem 5.5 from [23]. We show that in that case the situation is more complicated. In particular, we confirm the suspicion of Godefroy and Saint-Raymond that being meager in $SB(X)$ is not independent of the chosen admissible topology.

2.1. Generic objects in $\mathcal{P}$. The main result of this subsection is the following.

**Theorem 2.1.** Let $\mathcal{G}$ be the Gurarii space. The set $(\mathcal{G})^I_\infty$ is dense $G_\delta$ in $I$ for any $I \in \{\mathcal{P}, \mathcal{P}_\infty, \mathcal{B}\}$.

Let us recall what the Gurarii space is. One of the characterizations of the Gurarii space is the following, for more details we refer the interested reader e.g. to [8] (the characterization below is provided by [8, Lemma 2.2]).

**Definition 2.2.** The Gurarii space is the unique (up to isometry) separable Banach space such that for every $\varepsilon > 0$ and every isometric embedding $g : A \to B$, where $B$ is a finite-dimensional Banach space and $A$ is a subspace of $\mathcal{G}$, there is a $(1 + \varepsilon)$-isomorphism $f : B \to \mathcal{G}$ such that $\|f \circ g - id_A\| \leq \varepsilon$.

In the remainder of this subsection we prove Theorem 2.1. Let us start with the most technical part, namely that $(\mathcal{G})^I_\infty$ is $G_\delta$ in $\mathcal{P}_\infty$.

We need two technical lemmas first.

**Lemma 2.3.** For every $\mu \in \mathcal{P}$, finite set $A \subseteq V$ and $\varepsilon > 0$ there exists $\nu \in \mathcal{B}$ with $|\mu(x) - \nu(x)| < \varepsilon$ and $\nu(x) \in \mathcal{Q}$ for every $x \in A$.

**Proof.** It suffices to define such norm $\nu$ on span $A$ since then we can easily find some extension to the whole $V$. We assume that $0 \notin A$ and moreover we can assume that no two elements of $A$ lie in the same one-dimensional subspace, i.e. are scalar multiples of each other. Indeed, otherwise we would find a subset $A' \subseteq A$ where no elements are scalar multiples of each other and every element of $A$ is a scalar multiple, necessarily rational scalar multiple, of some element from $A'$. Then proving the fact for $A'$ for sufficiently small $\delta$ automatically proves it for $A$ and $\varepsilon$.

We enumerate $A$ as $\{a_1, \ldots, a_n\}$ and so that the first $k$ elements $a_1, \ldots, a_k$, for some $k \leq n$, are linearly independent and form a basis of span $A$.

**Claim.** By perturbing $\mu$ on $A$ by an arbitrarily small $\delta > 0$ we can without loss of generality assume that for every $i \leq n$, $\mu(a_i) < K_i := \inf\{\sum_{j \in J} \mu(a_j a_j) : i \notin J \subseteq \{1, \ldots, n\}, a_i = \sum_{j \in J} a_j\}$.

Suppose the claim is proved. Then for every $i \leq n$ we set $\nu'(a_i)$ to be an arbitrary positive rational number in the interval $[\mu(a_i), \min\{K_i, \mu(a_i) + \varepsilon\}]$. From the assumption it is now clear that for all $i \leq n$, we have

$$\nu'(a_i) \leq \inf\{\sum_{j=1}^{n} |a_j| \nu'(a_j) : a_i = \sum_{j=1}^{n} a_j a_j\}.$$

We extend $\nu'$ to a norm $\nu$ on span $A$ by the formula

$$\nu(v) := \inf\{\sum_{i=1}^{n} |a_i| \nu'(a_i) : v = \sum_{i=1}^{n} a_i a_i\},$$

for $v \in$ span $A$. From the previous assumption, it follows that $\nu(a_i) = \nu'(a_i)$, for all $i \leq n$. Moreover, $\nu$ is indeed a norm since $\nu(a_i) > 0$ for all $i \leq n$, and the infimum in the definition of $\nu$ is, by compactness, always attained.

It remains to prove the claim. Let $\| \cdot \|_2$ be the $\ell_2$ norm on span $A$ with $a_1, \ldots, a_k$ the orthonormal basis. For each $m \in \mathbb{N}$ set $\mu_m := \mu + \frac{\varepsilon}{m^2}$. Clearly $\mu_m \to \mu$, so it
suffices to show that each \( \mu_m \) satisfies the condition from the claim. Suppose that for some \( m \in \mathbb{N} \) and \( i \leq n \) we have

\[
\mu_m(a_i) = \inf \{ \sum_{j \in J} \mu_m(\alpha_j a_j) : i \notin J \subseteq \{1, \ldots, n\}, a_i = \sum_{j \in J} \alpha_j a_j \}.
\]

By compactness, the infimum is attained, i.e., there exists \( (\alpha_j)_{j \leq n} \), with \( \alpha_i = 0, a_i = \sum_{j \leq n} \alpha_j a_j \) and \( \mu_m(a_i) = \sum_{j \leq n} \mu_m(\alpha_j a_j) \). Indeed, if the infimum is approximated by a sequence \( (\alpha_i', \ldots, \alpha_n')_{i \in \mathbb{N}} \subseteq \mathbb{R}^n \), then since each coordinate is bounded (because up to finitely many \( i \)'s we have \( \sum_{j=1}^n \mu_m(\alpha_j a_j) \leq 2\mu_m(a_i) \)), we may pass to a converging subsequence and attain the infimum at the limit. The \( \ell_2 \) norm \( \| \cdot \| \) is strictly convex, so \( \| a_i \|_2 < \sum_{j \leq n} \| \alpha_j a_j \|_2 \), while \( \mu \) by triangle inequality satisfies \( \mu(a_i) \leq \sum_{j \leq n} \mu(\alpha_j a_j) \). Since \( \mu_m \) is the sum of \( \mu \) and a positive multiple of the \( \ell_2 \) norm, we must have \( \mu_m(a_i) < \sum_{j \leq n} \mu_m(\alpha_j a_j) \), a contradiction. \( \square \)

Let us emphasize that if we write that a mapping is an “isometry” or an “isomorphism”, we do not mean it is surjective if this is not explicitly mentioned.

**Lemma 2.4.** (i) Given a basis \( b_k = \{e_1, \ldots, e_n\} \) of a finite-dimensional Banach space \( E \), there is \( C > 0 \) and a function \( \phi^E_2 : [0, C) \rightarrow [0, \infty) \) continuous at zero with \( \phi^E_2(0) = 0 \) such that whenever \( X \) is a Banach space with \( E \subseteq X \) and \( \{x_i : i \leq n\} \subseteq X \) are such that \( \|x_i - e_i\| < \varepsilon, i \leq n \), for some \( \varepsilon < C \), then the linear operator \( T : E \rightarrow X \) given by \( T(e_i) := x_i \) is \((1 + \phi^E_2(\varepsilon)))\)-isomorphism and \( \|T - Id_E\| \leq \phi^E_2(\varepsilon) \).

(ii) Let \( \varepsilon \in (0, 1) \), \( T : X \rightarrow Y \) be a surjective \((1 + \varepsilon)\)-isomorphism between Banach spaces \( X \) and \( Y \), \( N \) be \( \varepsilon \)-dense for \( SX \). Then \( T(N) \) is \( 3\varepsilon \)-dense for \( SY \).

**Proof.** (i) Pick \( C > 0 \) such that \( C \sum_{i=1}^n |\lambda_i| \leq \| \sum_{i=1}^n \lambda_i e_i \| \) for every \( (\lambda_i)_{i=1}^n \in \mathbb{R}^n \). Then for any \( x = \sum_{i=1}^n \lambda_i e_i \) we have

\[
\|Tx - x\| \leq \sum_{i=1}^n |\lambda_i| \|x_i - e_i\| < \frac{\varepsilon}{C} \|x\|.
\]

Thus, \( \|T - Id_E\| < \frac{\varepsilon}{C} \), \( \|T\| \leq 1 + \frac{\varepsilon}{C} \) and \( \|Tx\| \geq (1 - \frac{\varepsilon}{C})\|x\| = (1 + \frac{\varepsilon}{C} - 1)\|x\| \).

Thus, we may put \( \phi^E_2(\varepsilon) := \frac{\varepsilon}{C} \) for \( \varepsilon \in [0, C) \).

(ii) Let \( \varepsilon > 0 \), \( T : X \rightarrow Y \) and \( N \) be as in the assumptions. Then for every \( y \in SY \) there is \( x \in N \) with \( \|x - T^{-1}(y)\| < \varepsilon \). \( y \) thus, we have

\[
\|y - T^*\| \leq \|y - \frac{y}{\|T^{-1}(y)\|} \| + \|T\| \cdot \|x - T^{-1}(y)\| < 1 - \frac{1}{\|T^{-1}(y)\|} + (1 + \varepsilon) \varepsilon \leq \varepsilon + 2\varepsilon = 3\varepsilon.
\]

\( \square \)

**Notation 2.5.** For a finite set \( A \subseteq V \) and \( P, P' \) partial functions on \( V \) (i.e., functions whose domains are subsets of \( V \)), with \( A \subseteq \text{dom}(P), \text{dom}(P') \), we put \( d_A(P, P') := \max_{a \in A} |P(a) - P'(a)| \).

Let \( T \) be the countable set of tuples \( (n, n', P, P', g) \) such that:

(a) \( n, n' \in \mathbb{N} \);
(b) \( P \in \mathbb{Q}^{\text{dom}(P)}, P' \in \mathbb{Q}^{\text{dom}(P')} \) where \( \text{dom}(P) \) and \( \text{dom}(P') \) are finite subsets of \( V \);
(c) there exists \( \nu \in \mathbb{B} \) such that \( P' = \nu |_{\text{dom}(P')} \);
(d) \( g : \text{dom}(P) \rightarrow \text{dom}(P') \) is a one-to-one mapping;
(e) \( P = P' \circ g \);
Whenever \( \mu' \in \mathcal{P} \), \( \nu' \in \mathcal{B} \) are such that \( \mathcal{P}' \subseteq \nu' \), \( \delta_{\text{dom}(\mathcal{P})}(\mathcal{P}, \mathcal{P}') < \frac{1}{n} \) and \( \mu' \) restricted to \( \text{span}(\text{dom}(\mathcal{P})) \subseteq c_{00} \) is a norm then there exists \( T_g : \text{span}(\text{dom}(\mathcal{P})), \mathcal{P}' \rightarrow \text{span}(\text{dom}(\mathcal{P}')), \nu' \) which is \((1 + \frac{1}{n})\)-isomorphism and \( T_g \supseteq T_0 \).

For \( (n, n', \mathcal{P}, \mathcal{P}', g) \in T \), we get \( G(n, n', \mathcal{P}, \mathcal{P}', g) \) be the set of \( \mu \in \mathcal{P}_{\infty} \) such that
- whenever \( \delta_{\text{dom}(\mathcal{P})}(\mathcal{P}, \mu) < \frac{1}{n} \) and \( \mu \) restricted to \( \text{span}(\text{dom}(\mathcal{P})) \subseteq c_{00} \) is a norm, there is a \( \mathbb{Q} \)-linear mapping \( \Phi : \mathcal{V} \cap \text{span}(\text{dom}(\mathcal{P}')) \rightarrow \mathcal{V} \) such that \( \mu(\Phi(gx) - x) < \frac{2}{n} \mu(x) \) for every \( x \in \text{dom}(\mathcal{P}) \) and \( |\mathcal{P}'(x) - \mu(\Phi(x))| < \frac{1}{n} |\mathcal{P}'(x)\) for every \( x \in \text{dom}(\mathcal{P})\).

**Proposition 2.6** Let \( \mu \in \mathcal{P}_{\infty} \). Then \( X_\mu \) is isometric to the Gurarii space if and only if \( \mu \in G(n, n', \mathcal{P}, \mathcal{P}', g) \) for every \( (n, n', \mathcal{P}, \mathcal{P}', g) \in T \).

**Proof.** In order to prove the first implication, let \( \mu \in \mathcal{P}_{\infty} \) be such that \( X_\mu \) is isometric to the Gurarii space and let \( (n, n', \mathcal{P}, \mathcal{P}', g) \in T \) be such that \( \delta_{\text{dom}(\mathcal{P})}(\mathcal{P}, \mu) < \frac{1}{n} \) and \( \mu \) restricted to \( \text{span}(\text{dom}(\mathcal{P})) \subseteq c_{00} \) is a norm. Consider the finite dimensional space \( A := (\text{span}(\text{dom}(\mathcal{P})), \mu) \). Let \( \nu \in \mathcal{B} \) be as in (6). Put \( B := (\text{span}(\text{dom}(\mathcal{P}')), \nu) \) and pick a basis \( b \subseteq \mathcal{V} \) of \( B \). By (31) Lemma 2.2, there is a \((1 + \frac{1}{n})\)-isomorphism \( S : B \rightarrow X_\mu \) such that \( \|S(T_g) - I_A\| < \frac{1}{n} \). By Lemma 2.4(1) we may for every \( b \in \mathcal{B} \) find \( x_0 \in V \) such that the linear mapping \( Q : S(B) \rightarrow X_\mu \) given by \( Q(S(b)) = x_b \), \( b \in \mathcal{B} \), is \((1 + \frac{1}{n})\)-isomorphism with \( \|Q - I_d\| < \frac{1}{n} \). Consider \( \Phi = QS|_{\mathcal{V} \cap \text{span}(\text{dom}(\mathcal{P}'))} \). This is indeed a \( \mathbb{Q} \)-linear map and since \( QS \) is \((1 + \frac{1}{n})^2\)-isomorphism and \( (1 + \frac{1}{n})^2 < 1 + \frac{1}{n} \), we have \( |\mu(\Phi(x)) - \nu(x)| < \frac{1}{n} \nu(x) \) for \( x \in \text{dom}(\mathcal{P}) \). Moreover, for every \( x \in \text{dom}(\mathcal{P}) \) we have
\[
\mu(\Phi(gx) - x) = \mu(QST_gx - x) \leq \mu(QST_gx - ST_gx) + \mu(ST_gx - x) < \frac{1}{n} \|S(T_g)\| \mu(x) + \frac{1}{n} \mu(x) \leq \frac{1}{n} \mu(x).
\]
This shows that \( \mu \in G(n, n', \mathcal{P}, \mathcal{P}', g) \).

In order to prove the second implication, let \( \mu \in \mathcal{P}_{\infty} \) be such that \( \mu \in G(n, n', \mathcal{P}, \mathcal{P}', g) \) whenever \( (n, n', \mathcal{P}, \mathcal{P}', g) \in T \). In what follows for \( x \in c_{00} \) we denote by \( [x] \in X_\mu \) the equivalence class corresponding to \( x \). Pick a finite dimensional space \( A \subseteq X_\mu \), and an isometry \( G : A \rightarrow B \), where \( B \) is a finite dimensional Banach space, we may without loss of generality assume \( B \subseteq X_{\mu B} \) for some \( \mu_B \in \mathcal{B} \). Let \( b_A := \{a_1, \ldots, a_k\} \) be a normalized basis of \( A \) and extend \( G(b_A) = \{G(a_1), \ldots, G(a_k)\} \) to a normalized basis \( b_B = \{b_1, \ldots, b_k\} \) of \( B \). Fix \( \eta > 0 \). It suffices to find \((1 + \eta)\)-isomorphism \( \Psi : B \rightarrow X_\mu \) with \( \|\Psi G - I_A\| \leq \eta \). Consider the functions \( \phi_t \) and \( \phi_t^2 \) from Lemma 1.2 and Lemma 2.1(1). Pick \( \delta \in (0, 1) \) such that \( \max\{\phi_t(1), \phi_t^2(1)\} > \eta \) whenever \( t < \delta \) and \( \varepsilon \in (0, \frac{1}{2\delta}) \) such that \( \phi_\varepsilon(5\varepsilon) < \frac{1}{2\delta} \) and \( \varepsilon + 72 \max\{\varepsilon, \phi_\varepsilon(5\varepsilon)\} < \delta \).

**Claim 1.** There are \( M, N \in V \) finite sets such that \( \mu \) restricted to \( \text{span} \subseteq c_{00} \) is a norm and surjective \((1 + \varepsilon)\)-isomorphism \( T_A : A \rightarrow \text{span}(\text{dom}(\mu)) \) and \( T_B : B \rightarrow \text{span}(\text{dom}(\mu_B)) \) such that:
- \( N \) and \( M \) are \( \varepsilon \)-dense sets for \( S_{T_A(A)} \) and \( S_{T_B(B)} \) respectively.
- We have \( \|T_Aa_i - a_i\| < \varepsilon \) for every \( a_i \in b_A \) and \( \|(T_A)^{-1}x - [x]\|_{X_\mu} < \varepsilon, \|\mu(x) - 1\| < \varepsilon \) for every \( x \in N \).
- \( (T_B)^{-1}(M) \) is \( \frac{1}{2} \)-dense for \( S_B \) and max\{\( \|\mu_B((T_B)^{-1}x) - 1\|, \|\mu_B(x) - 1\|\} < \frac{1}{2} \) for every \( x \in M \).
- \( (T_B G(T_A)^{-1})(N) \subseteq M \).

**Proof of Claim 1** By Lemma 2.1(1) we may pick \( \{f_1, \ldots, f_n\} \subseteq V \) such that the linear operator \( T_A : A \rightarrow X_\mu \) given by \( T_A(a_i) = [f_i], i \leq j \), is \((1 + \frac{\varepsilon}{2})\)-isomorphism and \( \|T_Ax - x\|_{X_\mu} < \frac{\varepsilon}{2} \|x\|_{X_\mu}, x \in A \). This implies that \( \mu \) restricted
to \( \text{span}\{f_1, \ldots, f_j\} \) is a norm and since \( T_A(A) \) is isometric to \( (\text{span}\{f_1, \ldots, f_n\}, \mu) \) we consider \( T_A \) as a \((1 + \frac{\varepsilon}{2})\)-isomorphism between \( A \) and \((\text{span}\{f_1, \ldots, f_n\}, \mu)\). Now, pick \( N' \subseteq A \) a finite \( \frac{\varepsilon}{6}\)-dense set for \( S_A \) consisting of rational linear combinations of points from \( b_A \) with \( b_A \subseteq N' \) such that \( ||x||_{X_n} - 1 < \frac{\varepsilon}{2} \) for every \( x \in N'. \) Then \( ||[T_A x] - x||_{X_n} < \frac{\varepsilon}{2} ||x||_{X_n} < \frac{1}{2}(1 + \frac{\varepsilon}{6}) < \frac{\varepsilon}{6} \) for every \( x \in N'. \) Put \( N := T_A(N') \subseteq V. \) Then we easily obtain \( ||[\mu(x) - 1] < \frac{\varepsilon}{2} \) for every \( x \in N \) and, by Lemma 2.3, \( N \) is \( \frac{\varepsilon}{6}\)-dense in \( S_{T_A(A)}. \) Similarly as above, we may pick \( \{g_1, \ldots, g_k\} \subseteq V \) such that the linear operator \( T_B : B \to X_{\mu_B} \) given by \( T_B(b_i) = g_i, i \leq k, \) is \((1 + \frac{\varepsilon}{6})\)-isomorphism and we find \( M' \subseteq B \) a finite \( \frac{\varepsilon}{6}\)-dense set for \( S_B \) consisting of rational linear combinations of points from \( b_B \) with \( M' \supseteq \{G(x) : \ x \in N'\} \cup b_B \) and \( ||[\mu_B(x) - 1] < \frac{\varepsilon}{6} \) for \( x \in M'. \) Put \( M := T_B(M') \), then similarly as above \( ||[\mu_B(x) - 1] < \frac{\varepsilon}{6} \) for every \( x \in M \) and \( M \) is \( \frac{\varepsilon}{6}\)-dense in \( S_{T_B(B)}. \) Finally, we obviously have \((T_BG(T_A)^{-1})(N) = T_B(G(N')) \subseteq T_B(M') = M. \) \( \square \)

By Lemma 2.3 there is \( \nu \in B \) having rational values on \( M \) with \( d_M(\nu, \mu_B \circ (T_B)^{-1}) < \frac{\varepsilon}{6} \). Put \( P' = \nu|_M \), consider the one-to-one map \( g : N \to M \) given by \( g := T_BG(T_A)^{-1}|_N \) and put \( P = P' \circ g. \) Let \( n \in \mathbb{N} \) be the integer part of \( \frac{\varepsilon}{2} \) and \( n' \in \mathbb{N} \) be the integer part of \( \frac{1}{3n} \). Easy computations show that \( \frac{3}{2} \varepsilon \leq \frac{1}{n} < 2 \varepsilon \) and \( 9 \max\{\varepsilon, \phi_1(5\varepsilon)\} \leq \frac{1}{n} < 18 \max\{\varepsilon, \phi_1(5\varepsilon)\} \) (in the last inequality we are using that \( \max\{\varepsilon, \phi_1(5\varepsilon)\} < \frac{1}{24} \)).

Note that for every \( x \in M \) we have

\[ \max\{||[\nu(T_B x) - 1]|, ||[\nu(x) - 1]|\} \leq \frac{\varepsilon}{2} + \max\{||[\mu_B(x) - 1]|, ||[\mu_B((T_B)^{-1} x) - 1]|\} < \varepsilon. \]

**Claim 2.** We have \((n, n', P, P', g) \in T \) and \( d_N(P, \mu) < \frac{1}{4}. \)

**Proof of Claim 2.** In order to see that \( d_N(P, \mu) < \frac{1}{4} \), pick \( x \in N. \) Then

\[ |P(x) - \mu(x)| \leq \frac{\varepsilon}{2} + ||[\mu_B(G(T_A)^{-1} x)] - \mu(x)||_N = \frac{\varepsilon}{2} + \max\{||[\mu_B((T_B)^{-1} x)]|\}_{X_n} ||x||_{X_n} < \frac{\varepsilon}{2}. \]

In order to see that \((n, n', P, P', g) \in T \), let us verify the condition \((1)\). Let \( \nu' \in B \) be such that \( P' \subseteq \nu' \), \( d_N(P, \mu') < \frac{1}{4} < 2 \varepsilon \) and \( \nu' \) restricted to \( \text{span} N \subseteq c_{00} \) is a norm. Note that \( ||[\mu'(x) - 1]| < 5\varepsilon \) for every \( x \in N \) and so, since \( N \) is \( \varepsilon\)-dense for the sphere of \( T_A(A) = (\text{span} N, \mu) \), the mapping \( id : (\text{span} N, \mu) \to (\text{span} N, \mu') \) is \((1 + \phi_1(5\varepsilon))\)-isomorphism. Further, \( ||[\nu'(x) - 1]| = ||[\nu(x) - 1]| < \varepsilon \) for every \( x \in M \) and so the mapping \( id : (\text{span} M, \nu') \to (\text{span} M, \nu') \) is \((1 + \phi_1(5\varepsilon))\)-isomorphism as well. Finally, since \( T_BG(T_A)^{-1} \) is \((1 + \varepsilon)^2\)-isomorphism between \((\text{span} N, \mu) \) and \((\text{span} g(N), \mu_B) \) and

\[ (1 + \phi_1(5\varepsilon))2(1 + \varepsilon)^2 \leq (1 + 3\phi_1(5\varepsilon))(1 + \varepsilon) \leq 1 + 9 \max\{\varepsilon, \phi_1(5\varepsilon)\} \leq 1 + \frac{1}{n'}, \]

we have that \( T_B := id \circ T_B \circ G \circ (T_A)^{-1} \circ id : (\text{span} N, \mu') \to (\text{span} M, \nu') \) is \((1 + \frac{1}{n'})\)-isomorphism. \( \square \)

Since \( \mu \in G(n, n', P, P', g) \), there is a \( \Omega\)-linear mapping \( \Phi : V \cap (\text{span} M, \nu) \to V \) such that \( \mu(\Phi(g x) - x) < \frac{1}{4} \mu(x) \) for every \( x \in N \) and \( ||[\nu(x) - \mu(\Phi(x))]||_N < \frac{1}{4} \nu(x) \) for every \( x \in M \). It is easy to see that \( \Phi \) extends to a bounded linear operator \( \Phi' : (\text{span} M, \nu) \to X_{\mu}. \) Finally, consider \( \Psi := \Phi' \circ T_B : B \to X_{\mu}. \)

For every \( x \in M \) we have

\[ ||[\mu(\Phi(x) - 1)] \leq ||[\mu(\Phi(x) - \nu(x))] + ||[\nu(x) - 1]| \leq \frac{1}{n'} \nu(x) + \varepsilon \leq \frac{1}{n'} (1 + \varepsilon) + \varepsilon < \delta; \]

thus, \( ||[\Psi(x)]|_{X_\mu} - 1| < \delta \) for every \( x \in (T_B)^{-1}(M) \) and \( \Psi \) is \((1 + \eta)\)-isomorphism. Further, we have

\[ ||[\Psi g(a_i) - a_i]|_{X_n} \leq ||[\Phi g(T_A a_i) - [T_A a_i]|_{X_n} + ||[T_A a_i] - a_i]|_{X_n} \leq \frac{1}{n'} (1 + \varepsilon) + \varepsilon < \delta; \]
hence, by Lemma 2.7(1) we have \( \|\Phi'T_B G - I_A\| \leq \frac{\phi_{b,A}}{\alpha}(1 + \varepsilon) + \varepsilon < \eta \). \qed

**Theorem 2.7.** Let \( G \) be the Gurari˘ı space. The set \( \langle G \rangle_{\mathcal{P}_{\infty}} \) is \( G_\delta \) in \( \mathcal{P}_{\infty} \).

**Proof.** By Proposition 2.6 we have for the countable set \( T \) defined before Proposition 2.6 that
\[
\langle G \rangle_{\mathcal{P}_{\infty}} = \bigcap_{(n, n', P, P', g) \in T} G(n, n', P, P', g),
\]
where \( G(n, n', P, P', g) \) is a union of a closed and an open set in \( \mathcal{P}_{\infty} \) (here we use the observation that the set \( \{\mu \in \mathcal{P}_{\infty} : \mu \text{ restricted to } \text{span}(\text{dom} P) \subseteq c_0 \text{ is a norm} \} \) is open due to Lemma 1.4); thus it is a countable intersection of \( G_\delta \) sets. \qed

**Proof of Theorem 2.7** Let us recall that \( \mathcal{P}_{\infty} \) and \( \mathcal{B} \) are \( G_\delta \) in \( \mathcal{P} \), see Corollary 1.5. Thus, since we have \( \langle G \rangle_{\mathcal{P}_{\infty}} = \langle G \rangle_{\mathcal{P}_{\infty}} \cap \mathcal{B} \), it follows from Proposition 2.6 that \( \langle G \rangle_{\mathcal{P}_{\infty}} \) is \( G_\delta \) in any \( I \in \{\mathcal{P}, \mathcal{P}_{\infty}, \mathcal{B}\} \).

By Proposition 1.3 we also have that \( \langle G \rangle_{\mathcal{I}} \) is dense in \( I \) for every \( I \in \{\mathcal{P}, \mathcal{P}_{\infty}, \mathcal{B}\} \). \qed

2.2. **Generic objects in \( SB(X) \).** In this subsection, we address Problem 5.5 from [23] which suggests to investigate generic properties of admissible topologies. We have both positive and negative results. The positive result is Theorem 2.10 which shows that the Gurari˘ı space is dense \( G_\delta \) in the Wijsman topology. The negative results are Propositions 2.11 and 2.13 and Theorem 2.12.

**Definition 2.8.** Given a closed set \( H \) in \( X \) we denote by \( E^-(H) \) the set \( SB(X) \setminus E^+(X \setminus H) \), that is, \( E^-(H) = \{ F \in SB(X) : F \subseteq H \} \). Obviously, this is a closed set in any admissible topology on \( SB(X) \).

**Definition 2.9.** Let \( X \) be isometrically universal separable Banach space. By \( \tau_W \) we denote the Wijsman topology on \( SB(X) \), that is, the minimal topology such that the mappings \( SB(X) \ni F \mapsto \text{dist}_X(x, F) \) are continuous for every \( x \in X \). Note that \( \tau_W \) is admissible, see [23] Section 2.

**Theorem 2.10.** \( \langle G \rangle_{\mathcal{P}_{\infty}} \) is dense \( G_\delta \) in \( (SB(G), \tau_W) \).

**Proof.** The class is \( G_\delta \) since it is \( G_\delta \) in \( \mathcal{P} \) by Theorem 2.1 and there is a continuous reduction \( (SB(G), \tau_W) \) to \( \mathcal{P} \) by Theorem 1.1. So we must show that it is dense.

Choose a basic open set \( N \) in \( \tau_W \) which is given by some closed subspace \( X \subseteq G \), finitely many points \( x_1, \ldots, x_n \in G \) and \( \varepsilon > 0 \) so that
\[
N = \{ Z \in SB(G) : \forall i \leq n (|\text{dist}_G(x_i, X) - \text{dist}_G(x_i, Z)| < \varepsilon) \}.
\]
Let us find a space \( G \) isometric to \( G \) such that \( G \in N \). Let \( Y \) be span\( \{ X \cup \{ x_i : i \leq n \} \} \). Since \( X \) embeds into both \( Y \) and \( G \) we can consider the push-out of that diagram, i.e. the amalgamated sum of \( Y \) and \( G \) along the common subspace \( X \). Recall this is nothing but the quotient \( (\mathbb{G} \oplus_1 Y)/Z \), where \( Z = \{ (z, -z) : z \in X \} \). Denote this space by \( G' \) and notice that \( \mathbb{G} \) is naturally embedded into \( G' \). It is straightforward to verify that for each \( i \leq n \), \( \text{dist}_{G'}(x_i, G) = \text{dist}_{G}(x_i, X) \). Since \( G \) is universal, there is a linear isometric embedding \( \iota : G' \to \mathbb{G} \). As there is a linear isometry \( \phi : \text{span}\{ X \cup \{ x_i : i \leq n \} \} \to \text{span}\{ x_i : i \leq n \} \), by [31] Theorem 1.1 there is a bijective linear isometry \( \Phi : \mathbb{G} \to G \) such that \( \| \Phi \circ \iota(x_i) - x_i \| < \varepsilon \), for each \( i \leq n \). By triangle inequality, it follows that \( G := \Phi \circ \iota[G \subseteq \mathbb{G}] \) satisfies for each \( i \leq n \), \( |\text{dist}_G(x_i, G) - \text{dist}_G(x_i, X)| < \varepsilon \), so it is the desired space isometric to \( G \) lying in the open set \( N \). \qed

The rest of the section is devoted to negative results. They show that the definition of admissible topology allows a lot of flexibility by which one can alter which properties should be meager or not.
Proposition 2.11. Let $X$ be an isometrically universal separable Banach space and let $\tau$ be an admissible topology on $SB(X)$. Then there exists an admissible topology $\tau' \supseteq \tau$ on $SB(X)$ such that the set $\langle G \rangle_\sim$ is nowhere dense in $(SB_\infty(X), \tau')$.

Proof. By the definition of an admissible topology, we may pick $(U_n)_{n \in \mathbb{N}}$, a basis of the topology $\tau$, such that for every $n \in \mathbb{N}$ there are nonempty open sets $V_n^k$, $k = 1, \ldots, N_n$ and $W_n$ in $X$ such that the set $U'_n$ defined by

$$U'_n = \bigcap_{k=1}^{N_n} E^+(V_n^k) \setminus E^+(W_n)$$

is a nonempty subset of $U_n$.

We claim that for every $n \in \mathbb{N}$ there is $F_n \in U'_n$ such that $G \not
rightarrow F_n$. Indeed, pick an arbitrary $Z \subseteq U'_n$. We may without loss of generality assume there is $H_0 \subseteq Z$ with $H_0 \simeq \mathbb{G}$ and since $\mathbb{G}$ is isometrically universal, there is $H_1 \subseteq H_0$ with $H_1 \approx \ell_2$. Now, pick points $v_k \in Z \cap V_k^n$, $k = 1, \ldots, N_n$. Then we put $F_n := \text{span}\{v_1, \ldots, v_{N_n}, u : u \in H_1\}$. Since $F_n$ is a subset of $Z$, we have $F_n \not
rightarrow E^+(W_n)$ and since it contains the points $v_1, \ldots, v_{N_n}$, we have $F_n \in U'_n$. Moreover, it is a space isomorphic to $\ell_2$ and so $\mathbb{G} \not
rightarrow F_n$.

Thus, for every $n \in \mathbb{N}$ there is a closed subspace $F_n$ of $X$ such that $U_n \cap E^-(F_n)$ is a nonempty set disjoint from $\langle G \rangle_\sim$.

It is a classical fact, see e.g., [29, Lemma 13.2 and Lemma 13.3], that the topology $\tau'$ generated by $\tau \cup \{E^-(F_n) : n \in \mathbb{N}\}$ is Polish. It is easy to check it is admissible. Moreover, for every $n \in \mathbb{N}$ we have that $U_n \cap E^-(F_n)$ is a nonempty $\tau'$-open set in $U_n$ disjoint from $\langle G \rangle_\sim$. It follows that nonempty sets of the form $U_n \cap \bigcap_{m \in I} E^-(F_m)$, for finite $I \subseteq \mathbb{N}$, give us a $\pi$-basis of $\tau'$. Since obviously each element of the form $U_n \cap \bigcap_{m \in I} E^-(F_m)$ is disjoint from $\langle G \rangle_\sim$, the set $\langle G \rangle_\sim$ is $\tau'$-nowhere dense.

□

Actually, one may observe that the same proof gives the following more general result, where the pair $(\mathbb{G}, \ell_2)$ is replaced by a more general pair of Banach spaces.

Theorem 2.12. Let $X$ be an isometrically universal separable Banach space and let $\tau$ be an admissible topology on $SB(X)$. Let $Y$ and $Z$ be infinite-dimensional Banach spaces such that $Y \rightarrow Z$ and $Z \not
rightarrow Y \oplus F$ for every finite-dimensional space $F$.

Then there exists an admissible topology $\tau' \supseteq \tau$ on $SB(X)$ such that the set $\langle Z \rangle_\sim$ is nowhere dense in $(SB_\infty(X), \tau')$.

It is even possible to find an admissible topology $\tau$ such that $\langle \ell_2 \rangle_\sim$ is not a meager set in $(SB_\infty(X), \tau)$ which is an immediate consequence of the following more general observation (the property (P) below would be “$X$ is isometric to $\ell_2$”).

Proposition 2.13. Let $X$ be isometrically universal separable Banach space and $\tau$ be an admissible topology on $SB_\infty(X)$. Let (P) be a non-void property (i.e. there are spaces with such a property) of infinite-dimensional Banach spaces closed under taking subspaces. Then there is an admissible topology $\tau' \supseteq \tau$ such that the set $\{Y \in SB_\infty(X) : Y$ has (P) $\}$ has non-empty interior in $(SB_\infty(X), \tau')$.

Proof. Pick $F \in SB_\infty(X)$ with (P). Using again the classical fact, see e.g. [29, Lemma 13.2], that the topology $\tau'$ generated by $\tau \cup \{E^-(F)\}$ is Polish, it is easy to check it is admissible. Then the $\tau'$-open set $E^-(F)$ is a subset of $\{Y \in SB_\infty(X) : Y$ has (P) $\}$.

□
3. Spaces with closed isometry classes

From this section on, we start our investigation of descriptive complexity of isometry classes. Let us first observe that no isometry class can be open as every isometry class actually has an empty interior. Indeed, it follows from Proposition 1.9 that the isometry class of every isometrically universal separable Banach space is dense. Since there are obviously many pairwise non-isometric universal Banach spaces we get that every open set (in all \( \mathcal{P} \), \( \mathcal{P}_\infty \) and \( \mathcal{B} \)) contains norms, resp. pseudonorms, defining different Banach spaces. The same argument can be also used to show that every isomorphism class has an empty interior.

**Lemma 3.1.** \( \langle \ell_2 \rangle_\equiv \) is closed in \( \mathcal{B} \) and \( \mathcal{P}_\infty \).

*Proof.* Hilbert spaces are characterized among Banach spaces as those Banach spaces whose norm satisfies the parallelogram law, i.e. \( \|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \) for any pair of elements \( x, y \). It is clear that a norm satisfies the parallelogram law if and only if it satisfies it on a dense set of vectors, therefore every norm, resp. pseudonorm, from \( \mathcal{B} \), resp. \( \mathcal{P}_\infty \), satisfying the parallelogram law on \( V \) defines a Hilbert space. Since norms, resp. pseudonorms, from \( \mathcal{B} \), resp. \( \mathcal{P}_\infty \), define only infinite-dimensional spaces, they define spaces isometric to \( \ell_2(\mathbb{N}) \). Since the parallelogram law is clearly a closed condition, we are done.

**Remark 3.2.** We note that here we need to work with the spaces \( \mathcal{B} \) or \( \mathcal{P}_\infty \), since in \( \mathcal{P} \) the only space with closed isometry class is the trivial space. To show it, first notice that the trivial space is indeed closed. Next we show that any open neighborhood of a pseudonorm defining trivial space contains a pseudonorm defining arbitrary Banach space, which will finish our claim. Let such an open neighborhood be fixed. We may assume that it is of the form \( \{ \mu \in \mathcal{P} : \mu(v_i) < \varepsilon, i \leq n \} \), where \( v_1, \ldots, v_n \in V \) and \( \varepsilon > 0 \). Let \( m \) be such that all \( v_i, i \leq n \), are in \( \text{span}_\mathbb{Q}\{e_j: j \leq m\} \). Let \( X \) be an arbitrary separable Banach space and let \( (f_i)_{i \in \mathbb{N}} \subseteq X \) be a sequence whose span is dense in \( X \). We define \( \mu \in \mathcal{P} \) by \( \mu(e_j) = 0 \), for \( j \leq m \), and \( \mu(\sum_{i \in I} \alpha_i e_{m+i}) = \| \sum_{i \in I} \alpha_i f_i \|_X \), where \( I \subseteq \mathbb{N} \) is finite and \( (\alpha_i)_{i \in I} \subseteq \mathbb{Q} \). This defines \( \mu \) separately on \( \text{span}_\mathbb{Q}\{e_i: i \leq m\} \) and \( \text{span}_\mathbb{Q}\{e_i: i > m\} \), however the extension to the whole \( V \) is unique. It is clear that \( \mu \) is in the fixed open neighborhood and that \( X_\mu \equiv X \).

One may be interested whether there are other Banach spaces whose isometry class is closed. The answer is negative. First, let us state another corollary of Proposition 1.9.

**Lemma 3.3.** Let \( X \) be a separable infinite-dimensional Banach space. Then \( \langle \ell_2 \rangle_\equiv \subseteq \langle X \rangle_\equiv \cap \mathcal{B} \). The same holds if we replace \( \mathcal{B} \) with \( \mathcal{P}_\infty \) or \( \mathcal{P} \).

*Proof.* By the Dvoretzky’s theorem, \( \ell_2 \) is finitely representable in every separable infinite-dimensional Banach space (see e.g. [1] Theorem 13.3.7)). So we are done by applying Proposition 1.9.

The following theorem is now an immediate consequence of Lemmas 3.1 and 3.3.

**Theorem 3.4.** \( \ell_2 \) is the only separable infinite-dimensional Banach space whose isometry class is closed in \( \mathcal{B} \). The same holds if we replace \( \mathcal{B} \) by \( \mathcal{P}_\infty \).

On the other hand, no isomorphism class can be closed. We show something stronger. Let us first start with the following simple lemma from descriptive set theory. Although it should be well known, we could not find a proper reference, so we provide a sketch of the proof.

**Lemma 3.5.** Suppose that \( X \) is a Polish space and \( B \subseteq X \) is a Borel set which is not \( G_\delta \). Then \( B \) is \( F_\sigma \)-hard. The same with the roles of \( G_\delta \) and \( F_\sigma \) interchanged.
Proof. By Hurewicz theorem (see e.g. [23] Theorem 21.18), there is a set $C \subseteq X$ homeomorphic to the Cantor space such that $C \cap B$ is countable dense in $C$. Then $C \cap B$ is $F_\sigma$ but not $G_\delta$ in the zero-dimensional Polish space $C$, and so it is $F_\sigma$-complete in $C$ by Wadge’s theorem (see e.g. [23] Theorem 22.10). So for any zero-dimensional Polish space $Y$ and any $F_\sigma$-subset $A$ of $Y$, there is a Wadge reduction of $A \subseteq Y$ to $C \cap B \subseteq C$. But any such reduction is also a reduction of $A \subseteq Y$ to $B \subseteq X$, and so $B$ is $F_\sigma$-hard.

The argument with the roles of $F_\sigma$ and $G_\delta$ interchanged is similar. \end{proof}

Proposition 3.6. No isomorphism class can be closed in $\mathcal{P}_\infty$, $\mathcal{B}$ and $\mathcal{P}$, and with the possible exception of spaces isomorphic to $G$ for which we do not know the answer, no isomorphism class can even be $G_\delta$.

Moreover, $\langle \ell_2 \rangle_\sim$ is $F_\sigma$-complete in both $\mathcal{P}_\infty$ and $\mathcal{B}$.

Proof. Let $X$ be a separable infinite-dimensional Banach space. We show that $\langle X \rangle_\sim$ is dense (we show the argument only for $\mathcal{P}_\infty$, the other cases are analogous). Let $F$ be a finite-dimensional Banach space. It is well known that every finite-dimensional space is dense (we show the argument only for separable Banach space is finitely representable in the possible exception of spaces isomorphic to $G$). So for any zero-dimensional Polish space $X$ and we have $X \approx F \oplus Y$ for some Banach space $Y$. Since $F$ was arbitrary, it follows that every separable Banach space is finitely representable in $\langle X \rangle_\sim$, so by Proposition 1.9 $\langle X \rangle_\sim = \mathcal{P}_\infty$, hence $\langle X \rangle_\sim$ is dense.

It follows that $\langle X \rangle_\sim$ cannot be closed for any $X$ because it is dense and there are obviously two non-isomorphic spaces. Moreover, if $X$ is not isomorphic to the Gurarii space then $\langle X \rangle_\sim$ cannot be $G_\delta$ since by Theorem 2.1, the isometry class of the Gurarii space is dense $G_\delta$, so it would have non-empty intersection with $\langle X \rangle_\sim$ otherwise.

Finally suppose that $X = \ell_2$. The isomorphism class of $\ell_2$ is proved to be $F_\sigma$ in an admissible topology on $SB_\infty$ in [23] Theorem 4.3. The same proof, which we briefly sketch, works also for $\mathcal{P}_\infty$ and $\mathcal{B}$. By Kwapień’s theorem (see e.g. [1] Theorem 7.4.1) a separable infinite-dimensional Banach space is isomorphic to $\ell_2$ if and only if it is of type 2 and of cotype 2. It is clear from the definition of type and cotype (see e.g. [1] Definition 6.2.10) that these properties are $F_\sigma$. So to show that $\langle \ell_2 \rangle_\sim$ is $F_\sigma$-complete, by Lemma 3.4 it suffices to show that $\langle \ell_2 \rangle_\sim$ is not $G_\delta$, which we have already proved. \end{proof}

Later, in Theorem 5.1 we prove that $\ell_2$ is actually the unique, up to isomorphism, separable infinite-dimensional Banach space whose isomorphism class is $F_\sigma$.

Remark 3.7. An alternative proof showing that the isomorphism class $\langle \ell_2 \rangle_\sim$ is $F_\sigma$ follows from [33] Theorem 2’ (see also Remark 4 therein) which provides a formula defining spaces isomorphic to $\ell_2$ and which obviously defines an $F_\sigma$ set in $(\mathcal{P}_\infty$ and $\mathcal{B})$.

3.1. $QSL_p$-spaces. We finish the section by considering some natural closed subspaces of $\mathcal{P}$, $\mathcal{P}_\infty$ and $\mathcal{B}$.

In [33], Kwapień denotes by $S_p$, resp. $SQ_p$, for $1 \leq p < \infty$, the class of all Banach spaces isometric to a subspace of $L_p(\mu)$, resp. to a subspace of some quotient of $L_p(\mu)$, for some measure $\mu$.

Let us address the class $S_p$ first. We have the following simple lemma.

Lemma 3.8. Let $1 \leq p < \infty$. Put

$$M := \{\mu \in \mathcal{B} : X_\mu \text{ is isometric to a subspace of } L_p[0, 1]\}.$$  

Then $M$ is a closed set in $\mathcal{B}$ and we have

$$M = \langle \ell_p \rangle_\sim \cap \mathcal{B} = \{\mu \in \mathcal{B} : X_\mu \text{ is } L_{p,1+} \text{ space} \cap \mathcal{B}.$$
The same holds if we replace $B$ with $\mathcal{P}_\infty$.

**Proof.** We recall the fact that a separable infinite-dimensional Banach space is isometric to a subspace of $L_p[0,1]$ if and only if it is finitely representable in $\ell_p$, see e.g. [1, Theorem 12.1.9]. The rest follows from Proposition 1.9. We refer the reader to Section 4 for a definition of the class $\mathcal{L}_{p,1+}$. □

In the rest, we focus on the class $SQ_p$. Notice that for $p = 1$, this class coincides with the class of all Banach spaces, and for $p = 2$, this class consists of Hilbert spaces.

These Banach spaces are also called $QSL_p$-spaces in literature and since it seems this is the more recent terminology, this is what we will use further. It seems to be well known, see e.g. [19, Theorem 12.1.9], that this class of spaces is characterized by Proposition 5.9 below. This result was probably essentially proved by Kwapien [34] (however, in his paper he considered the isomorphic variant only), for a more detailed explanation of the proof (and even for a generalization) one may consult e.g. the proof in [37, Theorem 3.2] which uses the ideas from [15] and [26]. Let us note that, by Proposition 5.9 and [27, Proposition 0], the class of $QSL_p$ spaces coincides with the class of $p$-spaces considered already in 1971 by Herz [27].

**Proposition 3.9.** A Banach space $X$ is a $QSL_p$-space, if and only if for every real valued $(m, n)$-matrix $M$ satisfying

$$\sum_{i=1}^{n} \sum_{j=1}^{m} |M(i,j)r_j|^p \leq \sum_{k=1}^{m} |r_k|^p,$$

for all $m$-tuples $r_1, \ldots, r_m \in \mathbb{R}$, we have

$$\sum_{i=1}^{n} \left\| \sum_{j=1}^{m} M(i,j)x_j \right\|^p \leq \sum_{k=1}^{m} \|x_k\|^p_{X},$$

for all $m$-tuples $x_1, \ldots, x_m \in X$.

Since it is clear that it suffices to verify the condition from Proposition 5.9 only on dense tuples of vectors, and that this condition is closed, we immediately obtain the following.

**Proposition 3.10.** For every $1 < p < \infty$, the set

$$\{ \mu \in \mathcal{P}_\infty : X_\mu \text{ is } QSL_p \}$$

is closed in $\mathcal{P}_\infty$.

The same is true if $\mathcal{P}_\infty$ is replaced by $B$.

Denote now the set $\{ \mu \in \mathcal{P}_\infty : X_\mu \text{ is } QSL_p \}$ by $QSL_p$. By Lemma 5.9 for $1 \leq p < \infty$, the set $M_p := \{ \mu \in \mathcal{P}_\infty : X_\mu \text{ is isometric to a subspace of } L_p[0,1] \}$ is closed. Clearly, $M_p \subseteq QSL_p$ (and for $p = 2$ there is an equality).

If $p \neq 2$ then $M_p \neq QSL_p$ because there exists an infinite-dimensional separable Banach space which is isomorphic to a quotient of $L_p[0,1]$ but not to its subspace. Indeed, if $p = 1$ this is easy since every separable Banach space is isomorphic to a quotient of $\ell_1$, see e.g. [11, Theorem 2.3.1]. If $2 < q < p < \infty$ then $\ell_q$ is isometric to a quotient of $L_p[0,1]$ (because its dual $\ell_q$ embeds isometrically into $L_p'[0,1]$) but is not isomorphic to a subspace of $L_p[0,1]$, see e.g. [11, Theorem 6.4.18]. Finally, if $1 < p < 2$ then by [15, Corollary 2] there exists a subspace $X$ of $\ell_p \subseteq L_p'$ which
is not isomorphic to a quotient of \( L_p \) and so \( X^* \) is isometric to a quotient of \( L_p \), which is not isomorphic to a subspace of \( L_p \). We would like to thank Bill Johnson for providing us those examples.

Moreover, we have the following.

**Proposition 3.11.** For \( p \in [1, 2) \cup (2, \infty) \), the set \( M_p \) has an empty interior in \( \text{QSL}_{L_p} \).

**Proof.** Fix \( p \in [1, 2) \cup (2, \infty) \). Pick \( \mu \in \text{QSL}_p \) such that \( X_\mu \) does not isometrically embed as a subspace into \( L_p[0, 1] \) (such a space exists, see the examples above).

Let \( U \) be now a basic open neighborhood of some \( \nu \in \text{QSL}_{L_p} \). Since the class of \( \text{QSL}_{L_p} \)-spaces is clearly closed under taking \( \ell_p \)-sums (see e.g. [16]), \( X_\nu \oplus_p X_\mu \) is still a \( \text{QSL}_{L_p} \)-space. It is easy to define \( \nu' \in U \) so that \( X_{\nu'} \) is isometric to \( X_\nu \oplus_p X_\mu \).

Now since \( X_\mu \) does not isometrically embed as a subspace into \( L_p[0, 1] \), neither \( X_{\nu'} \) does. By [1] Theorem 12.1.9, \( X_{\nu'} \) is not finitely representable in \( \ell_p \), so also not in \( L_p[0, 1] \) (by [1] Proposition 12.1.8). It follows from Proposition 3.10 that there exists a basic open neighborhood \( U' \) of \( \nu' \) avoiding \( M_p \). Now \( U \cap U' \) is a non-empty open subset of \( U \) avoiding \( M_p \) and we are done. \( \square \)

**Corollary 3.12.** For \( p \in [1, 2) \cup (2, \infty) \), \( L_p[0, 1] \) is not a generic \( \text{QSL}_{L_p} \)-space.

4. **Spaces with \( G_\delta \) isometry classes**

In this section, we investigate Banach spaces whose isometry classes are \( G_\delta \), or even \( G_\delta \)-complete. Besides \( \ell_2 \), whose isometry class is actually closed, we have already proved in Theorem 2.11 that the isometry class of the Gurari\u0103 space is \( G_\delta \) in \( P_\infty \) and \( B \). We start the section with some basic corollaries of that result; in particular, that the isometry class of \( G \) is even \( G_\delta \)-complete. The main results of the section however concern the Lebesgue spaces \( L_p([0, 1], \lambda) \), for \( 1 \leq p < \infty \).

Since for any separable infinite-dimensional Banach space \( X \) we obviously have \( \langle X \rangle_p^B = \langle X \rangle_p^B \cap B \), it is sufficient to formulate our positive result in the coding of \( P_\infty \) and negative results in the coding of \( B \).

**Lemma 4.1.** Let \( X, Y \) be separable infinite-dimensional Banach spaces such that \( X \) is finitely representable in \( Y \) and \( Y \) is finitely representable in \( X \). If \( \langle X \rangle_p^B \) is \( G_\delta \) in \( B \) and \( X \not\equiv Y \), then

\( (i) \) \( \langle Y \rangle_p^B \) is not \( G_\delta \) in \( B \).

\( (ii) \) \( \langle X \rangle_p^B \) is \( G_\delta \)-complete in \( B \).

**Proof.** Recall that by Proposition 3.9 we have that both \( \langle X \rangle_p^B \) and \( \langle Y \rangle_p^B \) are dense in \( N := \{ \nu \in B : X_\nu \text{ is finitely representable in } X \} \).

\( (i) \): If both \( \langle X \rangle_p^B \) and \( \langle Y \rangle_p^B \) are \( G_\delta \), by the Baire theorem we have that \( \langle X \rangle_p^B \cap \langle Y \rangle_p^B \) is comeager in \( N \). Thus, the intersection cannot be an empty set and we obtain \( X \equiv Y \).

\( (ii) \): Since \( X \not\equiv Y \), we have that \( \langle X \rangle_p^B \) has empty interior in \( N \). But it is also

\footnote{More precisely, by [18] Theorem 1 (see also e.g. [14] Corollary 3.2) for every \( n \in \mathbb{N} \) there exists a subspace \( E_n \) of \( \ell_p^\infty \) such that \( g(E_n) \geq K/\sqrt{n} \) where \( K > 0 \) is a constant independent of \( n \) and \( g(E_n) \) is a quantity related to the notion of a “GL-space” (or space with the “Gordon-Lewis property”). This implies that if we denote by \( E_p^n \) the space \( E_n \) endowed with the \( \ell_p \)-norm, we obtain \( g(E_p^n) \geq K/\sqrt{n} \) \( (\ell_p^\infty \oplus_p \ell_p^\infty)^{-1} = K2^{-1/p}n^{1/2-1/p'} \to \infty \); hence, \( X := \langle \bigoplus E_p^n \rangle \), the \( \ell_p \)-sum of spaces \( E_p^n \), is isometric to a subspace of \( \ell_p \) but it is not a GL-space. If \( X \) was isomorphic to a quotient of \( L_p \), then \( X^* \) would be isomorphic to a subspace of \( L_p \), which would imply that \( X^* \) and \( X \) are GL-spaces (see e.g. [14] Proposition 17.9 and Proposition 17.10), a contradiction.}
Corollary 4.2. \( G \) is the only isometrically universal separable Banach space whose isometry class is \( G_{\delta} \) in \( B \). The same holds if we replace \( B \) by \( P_\infty \).

Moreover, \( \langle G \rangle_\infty \) is \( G_{\delta} \)-complete in both \( P_\infty \) and \( B \).

Proof. By Theorem 2.4, the isometry class of \( G \) is \( G_{\delta} \). Let \( X \) be an isometrically universal separable Banach space. By Lemma 4.1, if \( X \not\equiv G \) then \( \langle X \rangle_\equiv \) is not \( G_{\delta} \) in \( B \) (and so not in \( P_\infty \), either).

For the “moreover” part we use Lemma 4.1 and any Banach space \( X \) not isometric to \( G \) that is finitely representable in \( G \) and vice versa (e.g. any other universal separable Banach space or \( c_0 \)).

The same proof gives us actually the following strengthening. Let us recall that by Maurey–Pisier theorem, see [40] or [1, Theorem 12.3.14], a Banach space \( G \) whose isometry class is \( G_{\delta} \) is finitely-representable in \( G \) if and only if \( \ell_\infty \) is finitely-representable in \( G \).

Theorem 4.3. \( G \) is the only separable Banach space with no nontrivial cotype whose isometry class is \( G_{\delta} \) in \( B \). The same holds if we replace \( B \) by \( P_\infty \).

Proof. Any separable Banach space is finitely representable in \( c_0 \), so by Lemma 4.1 there is at most one Banach space \( X \) such that \( c_0 \) is finitely representable in \( X \) and \( \langle X \rangle_\equiv \equiv G_{\delta} \). By Theorem 2.4 \( \langle G \rangle_\infty \equiv G_{\delta} \).

4.1. \( L_p \)-spaces. Let us recall that a Banach space \( X \) is said to be an \( L_p,\lambda \)-space (with \( 1 \leq p \leq \infty \) and \( \lambda \geq 1 \)) if every finite-dimensional subspace of \( X \) is contained in another finite-dimensional subspace of \( X \) whose Banach-Mazur distance \( d_{BM} \) to the corresponding \( L_p^n \) is at most \( \lambda \). A space \( X \) is said to be an \( L_p \)-space, resp. \( L_p,\lambda \)-space, if it is an \( L_p,\lambda' \)-space for some \( \lambda' \geq 1 \), resp. for every \( \lambda' > \lambda \).

The main result of this subsection is the following.

Theorem 4.4. For every \( 1 \leq p < \infty \), \( p \not= 2 \), the isometry class of \( L_p[0,1] \) is \( G_\delta \)-complete in \( B \) and \( P_\infty \).

Moreover, \( L_p[0,1] \) is the only separable \( L_p,1^+ \) space whose isometry class is \( G_{\delta} \) in \( B \), and the same holds if we replace \( B \) by \( P_\infty \).

The next theorem is a crucial step in proving Theorem 4.4. However, it is also of independent interest and its corollary improves the related result from [23].

Remark 4.5. It is easy to see (e.g. using [35] Section 17, Theorem 6]) that for every \( p \in [1,\infty] \) and \( \lambda \geq 1 \) we have that a separable infinite-dimensional Banach space \( Y \) is a \( L_p,\lambda \) space if and only if for every \( \varepsilon > 0 \) there is an increasing sequence \( \{F_k\}_{k=1}^\infty \) of finite-dimensional subspaces whose union is dense in \( Y \) such that \( d_{BM}(\ell_p^n, F_k, F_k) \leq \lambda + \varepsilon \) for every \( k \in \mathbb{N} \).

Let us note that the following result admits a generalization (see Proposition 6.3). This is the reason why we use in the proof the characterization of \( L_p,\lambda \) spaces mentioned in Remark 4.5.

Theorem 4.6. Let \( 1 \leq p \leq \infty \) and \( \lambda \geq 1 \). The class of separable \( L_p,\lambda \) spaces is \( G_\delta \) in \( P \). In particular, the class of separable infinite-dimensional \( L_p,\lambda \) spaces is \( G_\delta \) in \( P_\infty \).

Proof. For each finite tuple \( \vec{v} \) of elements from \( V \) we set \( S_{\vec{v}} \) to be the set of all finite tuples \( \vec{w} \), \( \mathbb{Q} \)-linearly independent in \( V \), such that each element of \( \vec{v} \) (considered as
an element of \(c_{00}\) lies in \(\operatorname{span} \vec{w}\). For \(m \in \mathbb{N}\) and a finite tuple \(\vec{v}\) of elements from \(V\) we set
\[
P(\vec{v}, m) := \{ \mu \in \mathcal{P}: \text{if } \mu \text{ restricted to } \operatorname{span}\{v_1, \ldots, v_{|\vec{v}|}\} \subseteq c_{00} \text{ is a norm, then there exist } \vec{w} \in S_{\vec{v}} \text{ and } n \in \mathbb{N} \text{ such that}
\]
\[
\mu \text{ restricted to } \operatorname{span}\{w_1, \ldots, w_{|\vec{w}|}\} \subseteq c_{00} \text{ is a norm and}
\]
\[
\left(\left(\operatorname{span}\{w_1, \ldots, w_{|\vec{w}|}\}, \mu\right) \right) \sim \left(\ell_p^{\dim Y}\right).
\]
Then, using the observation that \(\{ \mu \in \mathcal{P} : \mu \text{ restricted to } \operatorname{span}\{v_1, \ldots, v_{|\vec{v}|}\} \subseteq c_{00} \text{ is a norm}\} \) is open due to Lemma 1.4, \(P(\vec{v}, m)\) is a union of a closed and an open set, so it is \(G_\delta\).

Denote by \(LI\) the set of all finite tuples \(\vec{v} = (v_1, \ldots, v_{|\vec{v}|})\) of elements from \(V\) which are linearly independent in \(c_{00}\). We now set
\[
\mathcal{G} := \bigcap_{m \in \mathbb{N}} \bigcap_{\vec{v} \in LI} P(\vec{v}, m),
\]
which is clearly \(G_\delta\). We shall prove that it defines the class of separable \(\mathcal{L}_{p,\lambda^+}\) spaces.

If \(\mu \in \mathcal{G}\), it is clear that for every \(m\), we can recursively build an increasing sequence \(\{F_k\}_{k=1}^\infty\) of finite-dimensional subspaces whose union is dense in \(X_\mu\) such that we have \(d_{BM}(F_k, \ell_p^{\dim F_k}) \leq \lambda + \frac{1}{m}\) for every \(k \in \mathbb{N}\). It follows that \(X_\mu\) is \(\mathcal{L}_{p,\lambda^+}\) space.

On the other hand, let \(\mu \in \mathcal{P}\) be such that \(X_\mu\) is \(\mathcal{L}_{p,\lambda^+}\) space. In what follows for \(x \in c_{00}\) we denote by \([x] \in X_\mu\) the equivalence class corresponding to \(x\). Pick some \(m \in \mathbb{N}\) and an \(n\)-tuple \(\vec{v} \in LI\) such that \(\mu\) restricted to \(\operatorname{span}\{v_1, \ldots, v_n\} \subseteq c_{00}\) is a norm, so \(\{v_1, \ldots, v_n\}\) is a basis of \(\operatorname{span}\{v_1, \ldots, v_n\}\).

Pick \(\lambda' \in (\lambda, \lambda + \frac{1}{m})\) and \(\delta > 1\) with \(\lambda' < \lambda + \frac{1}{m}\). Since \(X_\mu\) is \(\mathcal{L}_{p,\lambda^+}\) space, there is an increasing sequence \(\{F_k\}_{k=1}^\infty\) of finite-dimensional subspaces whose union is dense in \(X_\mu\) such that \(\sup_{k \in \mathbb{N}} d_{BM}(F_k, \ell_p^{\dim F_k}) \leq \lambda'\). By Section 17, Theorem 6, we can find a finite dimensional subspace \(\operatorname{span}\{v_1, \ldots, v_n\} \subseteq Y \subseteq X_\mu\) and \(k \in \mathbb{N}\) such that \(d_{BM}(Y, F_k) \leq \delta\) so \(d_{BM}(Y, \ell_p^{\dim Y}) \leq \delta\lambda'\). Select \(y_{n+1}, \ldots, y_{\dim Y} \in Y\) such that \(b = \{v_1, \ldots, v_n, y_{n+1}, \ldots, y_{\dim Y}\}\) is a basis of \(Y\). Let \(\phi_2\) be the function from Lemma 2.4(1) and let \(\eta > 0\) be such that \(\lambda' < (1 + \phi_2(\eta)) - \frac{\lambda}{m}\). Further, for every \(n + 1 \leq i \leq \dim Y\) pick \(v_i \in V\) with \(\|v_i - y_i\|_{X_\mu} < \eta\). Then \(\mu\) restricted to \(\operatorname{span}\{v_1, \ldots, v_{\dim Y}\} \subseteq c_{00}\) is a norm and \(\operatorname{span}\{v_1, \ldots, v_{\dim Y}\} \subseteq X_\mu\) is isometric to \(\operatorname{span}\{v_1, \ldots, v_{\dim Y}\}, \mu\) such that \(d_{BM}(\operatorname{span}\{v_1, \ldots, v_{\dim Y}\}, \mu) \leq \lambda'\). Then \(\mu\) restricted to \(\operatorname{span}\{v_1, \ldots, v_{\dim Y}\} \subseteq c_{00}\) is a norm, \(\vec{w} \in S_{\vec{v}}\) and
\[
((\mu, \vec{w}) \leq \lambda + \frac{1}{m}).
\]

Note that for \(1 \leq p \leq \infty\) the class of \(\mathcal{L}_p\) spaces is obtained as the union \(\bigcup_{\lambda \geq 1} \mathcal{L}_{p,\lambda^+}\). It is shown in [23, Proposition 4.5] that \(\mathcal{L}_p\) is \(\Sigma_1^0\) in an admissible topology. It is immediate from Theorem 1.6 and (using Theorem 1.17) that we have a better estimate.

**Corollary 4.7.** For every \(1 \leq p \leq \infty\) the class of separable \(\mathcal{L}_p\) spaces is \(G_\delta\) in \(\mathcal{P}\) and any admissible topology.

Let us recall the following classical result.
are \( f, g \in L_p(\mu) \) small enough (to be specified later). Suppose to the contrary that there are \( a \in c_0^N \) such that \( (z_n) \not\overset{\sim}{\to} (\ell_p^N) \) for every \( a \in c_0^N \). If \( (z_n) \not\overset{\sim}{\to} (\ell_p^N) \) for every \( \varepsilon > 0 \), we write \( (z_n) \equiv \ell_p^N \).

**Theorem 4.9.** Let \( 1 \leq p < \infty \), \( p \neq 2 \), and let \( X \) be a separable infinite-dimensional \( L_{p,1+} \) space. Then the following assertions are equivalent.

(i) \( X \) is isometric to \( L_p[0,1] \).

(ii) For every \( x \in S_X \) the following condition is satisfied

\[
\forall N \in \mathbb{N} \\exists x_1, \ldots, x_N \in X : \ (x_i)_{i=1}^N \equiv \ell_p^N \text{ and } N^{1/p} \cdot x = \sum_{i=1}^N x_i.
\]

(iii) For every \( x \in S_X \) the following condition is satisfied

\[
\forall \varepsilon > 0 \exists x_1, x_2 \in X : \ (x_1, x_2) \overset{\sim}{\to} (\ell_p^2) \text{ and } 2^{1/p} \cdot x = x_1 + x_2.
\]

(iv) For every \( x \in S_X \) the following condition is satisfied

\[
\forall \varepsilon > 0 \forall \delta > 0 \exists x_1, x_2 \in X : \ (x_1, x_2) \overset{\sim}{\to} (\ell_p^2) \text{ and } \|2^{1/p} \cdot x - x_1 - x_2\| < \delta.
\]

**Proof.** (i) \( \Rightarrow \) (ii): Pick \( f \in S_{L_p[0,1]} \) and \( N \in \mathbb{N} \). Then, using the continuity of the mapping \([0,1] \ni x \mapsto f_0^x[f]\), we find 0 = \( x_0 < x_1 < \ldots < x_N = 1 \) such that \( f_0^i[f] = \sum_{i=1}^N [f] \) for every \( i = 1, \ldots, N \). We put \( f_i := N^{1/p} \cdot f \cdot \chi_{[x_{i-1}, x_i]} \), \( i = 1, \ldots, N \). Then, since the supports of \( f_i \) are disjoint and since \( f_i \) are normalized, we have \( f_i \equiv \ell_p^N \). Further, we obviously have \( N^{1/p} \cdot f = \sum_{i=1}^N f_i \).

Obviously, we have (ii) \( \Rightarrow \) (iii) and (iii) \( \Rightarrow \) (iv).

(iii) \( \Rightarrow \) (i): In order to get a contradiction, let us assume that \( X \) is not isometric to \( L_p[0,1] \) which, by Theorem 4.8, implies that \( X \) is isometric to \( L_p(\mu) \), where \( (\Omega, S, \mu) \) is a measure space for which there is \( \omega \in \Omega \) with \( \mu(\omega) = 1 \). Fix \( \varepsilon > 0 \) small enough (to be specified later). Suppose to the contrary that there are \( f, g \in L_p(\mu) \) such that \( (f, g) \overset{\sim}{\to} (\ell_p^2) \) and \( 2^{1/p} \cdot \delta_\omega = f + g \), where \( \delta_\omega \) is the Dirac function supported by the point \( \omega \). For \( \mu \)-a.e. \( x \in \Omega \setminus \{\omega\} \), we have \( f(x) + g(x) = 0 \), so we assume this holds for all \( x \in \Omega \setminus \{\omega\} \). We without loss of generality assume that \( f(\omega) \geq g(\omega) \).

We claim that both \( f(\omega) \) and \( g(\omega) \) are positive and \( \|f(\omega) - g(\omega)\| < \frac{1}{2} \) if \( \varepsilon > 0 \) is chosen sufficiently small. Indeed, we have

\[
(1 + \varepsilon)p - \frac{1}{(1 + \varepsilon)^p} \geq \left\| |f(\omega)| - |g(\omega)| \right\| = \left| |f(\omega)| - |g(\omega)| \right|^p,
\]

which implies \( |f(\omega)| - |g(\omega)| \| < 2^{-1/p} \) for sufficiently small \( \varepsilon > 0 \). The claim follows since if both \( f(\omega) \) and \( g(\omega) \) were not positive we would have \( 2^{1/p} > 2^{-1/p} > \|f(\omega)| - |g(\omega)|\| = \|f(\omega)| - |g(\omega)|\| = 2^{1/p} \), a contradiction.
First, let us handle the case when $1 \leq p < 2$. We have
\[
\|2f\|_p^p = \int_{\Omega(\omega)} |2f|^p \, d\mu + (2f(\omega))^p = \int_{\Omega(\omega)} |f - g|^p \, d\mu + (2f(\omega))^p
\]
\[
= \|f - g\|_p^p + (2f(\omega))^p - ((f - g)(\omega))^p
\]
\[
\geq \|f - g\|_p^p + ((f + g)(\omega))^p = \|f - g\|_p^p + \|f + g\|_p^p;
\]
where in the inequality we used superadditivity of the function $[0, \infty) \ni t \mapsto t^p$. Thus, $(f, g) \overset{1+\varepsilon}{\sim} \ell_p^2$ implies
\[
(1 + \varepsilon)^p \geq \|f\|_p^p + \|f + g\|_p^p \geq \frac{4}{2p};
\]
hence, if $1 \leq p < 2$ we get a contradiction for sufficiently small $\varepsilon > 0$.

Finally, let us handle the case when $p > 2$. Note that since $f(\omega) \geq g(\omega) \geq 0$ and $f(\omega) + g(\omega) = 2^{1/p}$, we have $g(\omega) \leq 2^{1/p-1}$. Further, we have
\[
\|2g\|_p^p = \int_{\Omega(\omega)} |f - g|^p \, d\mu + (2g(\omega))^p \leq \|f - g\|_p^p + 2.
\]
Thus, $(f, g) \overset{1+\varepsilon}{\sim} \ell_p^2$ implies
\[
\frac{1}{(1 + \varepsilon)^p} \leq \|g\|_p^p \leq \frac{\|f - g\|_p^p + 2}{2p} \leq \frac{2(1 + \varepsilon)^p + 2}{2p};
\]
hence, if $p > 2$ we get a contradiction for sufficiently small $\varepsilon > 0$.

$(iv) \Rightarrow (iii)$: Fix $x \in S_X$ and $\varepsilon > 0$. Pick $\delta > 0$ small enough (to be specified later). Applying the condition $(iv)$ we obtain $x_1', x_2' \in X$ such that $(x_1', x_2') \overset{1+\varepsilon}{\sim} \ell_p^2$ and $\|2^{1/p} \cdot x - x_1' - x_2'\| < \delta$. Now set $x_1 = x_1' + (2^{1/p} \cdot x - (x_1' + x_2'))/2$ and $x_2 = x_2' + (2^{1/p} \cdot x - (x_1' + x_2'))/2$. If $\delta$ was chosen sufficiently small, we have $(x_1, x_2) \overset{1+\varepsilon}{\sim} \ell_p^2$ and clearly $2^{1/p} \cdot x = x_1 + x_2$.

Let us note the following easy observation. The proof is easy and so omitted.

Fact 4.10. Let $v, w \in V$, $v \neq 0$ and $a, b \in \mathbb{R}$. Then the set
\[
\left\{ \mu \in \mathcal{P} : \mu(v) \neq 0 \text{ and } \mu(a \cdot \frac{v}{\mu(v)} - w) < b \right\}
\]
is open in $\mathcal{P}$.

Proof of Theorem 4.4: Let $\mathcal{F}$ be the set of those $\nu \in \mathcal{P}_\infty$ for which $X_{\nu}$ is a $\mathcal{L}_{p,1}$ space. By Theorem 4.6, $\mathcal{F} \subseteq \mathcal{P}_\infty$ is a $G_\delta$ set. By Theorem 4.9 using the obvious observation that condition $(iv)$ may be verified on a dense subset, we have
\[
\langle L_p[0, 1] \rangle_{=,p} = \mathcal{F} \cap \bigcap_{v \in V \cap k \in \mathbb{N}} U_{v, n, k},
\]
where $U_{v, n, k}$ are open sets (using Fact 4.10 and Lemma 1.3) defined as
\[
U_{v, n, k} := \left\{ \mu \in \mathcal{P}_\infty : \exists v_1, v_2 \in V : (v_1, v_2) \overset{1+\varepsilon}{\sim} \ell_p^2 \text{ and } \mu(2^{1/p} \cdot \frac{v}{\mu(v)} - v_1 - v_2) < \frac{1}{k} \right\}.
\]
Thus, $\langle L_p[0, 1] \rangle_{=,p}$ is a $G_\delta$ set.

On the other hand, since any $L_p(\mu)$ is finitely representable in $\ell_p$ and vice versa (see e.g. [11 Proposition 12.1.8]), from Lemma 1.1 and Theorem 4.8 we obtain that there is at most one (up to isometry) $\mathcal{L}_{p,1}$ space $X$ such that $(X)_{=}$ is $G_\delta$ in $\mathcal{B}$ and that $\langle L_p[0, 1] \rangle_{=} = G_\delta$-complete.
5. Spaces with $F_{σδ}$ isometry classes

In this section we focus on another classical Banach spaces, namely $ℓ_p$ spaces, for $p ∈ [1, 2) ∪ (2, ∞)$, and $c_0$. The main result of this section is the following.

**Theorem 5.1.** The sets $⟨c_0⟩_∞$ and $⟨ℓ_p⟩_∞$ (for $p ∈ [1, 2) ∪ (2, ∞)$) are $F_{σδ}$-complete in both $P_∞$ and $B$

Note that in order to obtain that result we prove Proposition 5.6 and Theorem 5.13 which are of independent interest and where the “easiest possible” isometric characterizations of the Banach spaces $ℓ_p$, resp. $c_0$, among $L_{p,1}$ spaces, resp. $L_{∞,1}$ spaces are given. The proof of Theorem 5.1 follows immediately from Proposition 5.4, Proposition 5.9, and Proposition 5.11.

Let us emphasize that in subsection 5.2 we compute the Borel complexity of the operation assigning to a given Banach space a Szlenk derivative of its dual unit ball, which could be of an independent interest as well. See e.g. subsection 7.2 for some consequences. The reason why we need to do it here is obviously that our isometric characterization of the space $c_0$ involves Szlenk derivatives.

We start with the part which is common for both cases – that is, for $⟨c_0⟩_∞$ and $⟨ℓ_p⟩_∞$.

**Lemma 5.2.** Let $p ∈ [1, ∞)$ and let $X = (⟨X_n⟩_{n∈N})_p$ be the $ℓ_p$-sum of the family $(X_n)_{n∈N}$ of separable infinite-dimensional Banach spaces. Then $X ≡ ℓ_p$ if and only if $X_n ≡ ℓ_p$ for every $n ∈ N$.

Similarly, let $X = (⟨X_n⟩_{n∈N})_0$ be the $c_0$-sum of the family $(X_n)_{n∈N}$ of separable infinite-dimensional Banach spaces. Then $X ≡ c_0$ if and only if $X_n ≡ c_0$ for every $n ∈ N$.

**Proof.** It is easy and well-known that the $ℓ_p$-sum of countably many $ℓ_p$ spaces is isometric to $ℓ_p$, and that the $c_0$-sum of countably many $c_0$ spaces is isometric to $c_0$. The opposite implications follow from the facts that every 1-complemented infinite-dimensional subspace of $ℓ_p$ is isometric to $ℓ_p$, and that every 1-complemented infinite-dimensional subspace of $c_0$ is isometric to $c_0$, see [39] page 54.

**Proposition 5.3.** Let $X$ be one of the spaces $ℓ_p$, $p ∈ [1, 2) ∪ (2, ∞)$, or $c_0$. Then the set $(X)_∞ ≡ F_{σδ}$-hard in $B$.

**Proof.** Our plan is to find a Wadge reduction of a known $F_{σδ}$-hard set to $(X)_∞^B$. For this purpose we will use the set

$$P_3 = \{x ∈ 2^{N×N} : \forall m \text{ there are only finitely many } n \text{ with } x(m,n) = 1\}$$

(see e.g. [29] Section 23.A) for the fact that $P_3$ is $F_{σδ}$-hard in $2^{[N×N]}$. But before we start to construct the reduction of $P_3$ to $(X)_∞^B$ we need to do some preparation.

By Theorem 1.3 (in case $X = ℓ_p$) and Theorem 1.3 (in case $X = c_0$) we know that $(X)_∞^B$ is not $G_δ$ in $B$. Therefore it is $F_{σδ}$-hard in $B$ by Lemma 5.3. Now as the set

$N_2 = \{x ∈ N^2 : \text{there are only finitely many } n \text{ with } x(n) = 1\}$

is $F_σ$ in $2^N$, it is Wadge reducible to $(X)_∞^B$, so there is a continuous function $f : 2^N → B$ such that

$$x ∈ N_2 ⇔ f(x) ∈ (X)_∞^B.$$ 

We fix a bijection $b : N^2 → N$. For every $x ∈ 2^N$ and every $m ∈ N$ we define $g_m(x) ∈ P_∞$ as follows. Suppose that $v = \sum_{n∈N} α_n e_n$ is an element of $V$ (i.e., $α_n$ is a rational number for every $n$, and $α_n \neq 0$ only for finitely many $n$’s), then we put

$$g_m(x)(v) = g(x) \left(\sum_{n∈N} α_b(m,n)e_n\right).$$
Let \( c_0 = 0 \) and if \( zw \) we have
\[
f(x)(v) = \sqrt{\sum_{m \in \mathbb{N}} (q_m(x^{(m)}))(v))^p}, \quad v \in V,
\]
and if \( X = c_0 \) we put
\[
f(x)(v) = \sup\{ (q_m(x^{(m)}))(v): m \in \mathbb{N}\}, \quad v \in V.
\]
This formula, together with the preceding considerations, easily imply that \( f(x) \in \mathcal{B} \) and that \( X \) is isometric to the \( \ell_p \)-sum, or to the \( c_0 \)-sum (depending on whether \( X = \ell_p \) or \( X = c_0 \)), of the spaces \( X_{\ell_p} \), \( m \in \mathbb{N} \). Continuity of the functions \( q_m \) and \( x \mapsto x^{(m)} \), \( m \in \mathbb{N} \), immediately implies continuity of \( f \). By Lemma 5.2, \( f(x) \in \langle X \rangle^g \) if and only if \( g(x^{(m)}) \in \langle X \rangle^g \) for every \( m \in \mathbb{N} \). Hence,
\[
x \in P_3 \Leftrightarrow \forall m \in \mathbb{N}: x^{(m)} \in N_2 \Leftrightarrow f(x) \in \langle X \rangle^g.
\]
5.1. The spaces \( \ell_p \). The purpose of this subsection is to prove the following result.

**Proposition 5.4.** For every \( p \in [1, 2) \cup (2, \infty) \) we have that \( \ell_p \) is \( F_{\sigma \delta} \) in \( P_\infty \).

We start with the following classical result, which is sometimes named the Clarkson’s inequality. The proof may be found on various places, the original one is in the paper by Clarkson, see [1]. In fact, we use only a very special case of the Clarkson’s inequality where \( z, w \) are required to be elements of the real line instead of an \( L_p \) space (and this case is rather straightforward to prove).

**Lemma 5.5** (Clarkson’s inequality). Let \( 1 \leq p < \infty \), \( p \neq 2 \). If \( p > 2 \), then for every \( z, w \in \mathbb{R} \) we have
\[
|z + w|^p + |z - w|^p - 2|z|^p - 2|w|^p \geq 0.
\]
If \( p < 2 \) then reverse inequality holds. Moreover, the equality holds if and only if \( zw = 0 \).

**Proposition 5.6.** Let \( 1 \leq p < \infty \), \( p \neq 2 \), and let \( X \) be a separable infinite-dimensional \( L_{p,1+} \) space. Let \( D \) be a dense subset of \( X \). Then the following assertions are equivalent.

(i) \( X \) is isometric to \( \ell_p \).

(ii) For every \( x \in S_X \) and every \( \delta \in (0, 1) \) the following condition is satisfied:
\[
\exists N \in \mathbb{N} \exists \varepsilon > 0 \forall x_1, \ldots, x_N \in X: (N^{1/p} \cdot x_i)_{i=1}^N \stackrel{1+\varepsilon}{\sim} \ell_p^N \Rightarrow \|x - \sum_{i=1}^N x_i\| > \delta.
\]

(iii) For every \( x \in S_X \) the following condition is satisfied:
\[
\exists N \in \mathbb{N} \forall x_1, \ldots, x_N \in X: (N^{1/p} \cdot x_i)_{i=1}^N \equiv \ell_p^N \Rightarrow x \neq \sum_{i=1}^N x_i.
\]

(iv) For every \( x \in D \setminus \{0\} \) and every \( \delta \in (0, 1) \) the following condition is satisfied:
\[
\exists N \in \mathbb{N} \exists \varepsilon > 0 \forall x_1, \ldots, x_N \in D: (N^{1/p} \cdot x_i)_{i=1}^N \stackrel{1+\varepsilon}{\sim} \ell_p^N \Rightarrow \left\| \frac{x}{\|x\|} - \sum_{i=1}^N x_i \right\| \geq \delta.
\]

Note that the set \( \{e_k(n, m): n \in \mathbb{N}\} \) is both linearly independent and linearly dense in \( X_{\ell_p} \), and that \( q_m(x)(e_k) = 0 \) if \( k \not\in \{b(m, n): n \in \mathbb{N}\} \). Also, \( X_{\ell_p} \) is isometric to \( X_{\ell_p} \), where the isometry is induced by the operator
\[
e_k \mapsto \begin{cases} e_n & k = b(m, n), \\ 0 & k \not\in \{b(m, n): n \in \mathbb{N}\}. \end{cases}
\]
Proof. (i) $\implies$ (ii): Fix $x \in S_{l_p}$ and $\delta \in (0, 1)$. Pick $l \in \mathbb{N}$ with $\sum_{k=1}^{l} |x(k)|^p > \delta^p$ and $N \in \mathbb{N}$ such that $\sum_{k=1}^{l} |x(k)| - \frac{1}{N} \|x\|_p^p > \delta^p$. Fix a sequence $(\varepsilon_m)_{m \in \mathbb{N}} \in (0, 1)^N$ with $\varepsilon_m \to 0$. In order to get a contradiction, for every $m \in \mathbb{N}$, pick $x_1^m, \ldots, x_N^m \in \ell_p$ such that $(N^{1/p} \cdot x_i^m)_{i=1}^N \parallel x_i^m \parallel_{\ell_p}^p$ and $\|x - \sum_{i=1}^{N} x_i^m\| \leq \delta$ for every $m \in \mathbb{N}$.

We claim that there is $m \in \mathbb{N}$ such that $|x_i^m(k)x_j^m(k)| < \eta := N^{-(2+2/p)} \varepsilon_m$ for every $i, j \in \{1, \ldots, N\}$, $i \neq j$, and $k \in \{1, \ldots, l\}$. Indeed, otherwise there are $i, j, k$ such that $|x_i^m(k)x_j^m(k)| \geq \eta$ for infinitely many $m$'s. By passing to a subsequence, we may assume that this holds for every $m \in \mathbb{N}$. Since the sequences $(|x_i^m(k)|)_{m}$ and $(|x_j^m(k)|)_{m}$ are bounded, by passing to a subsequence we may assume there are numbers $a, b \in \mathbb{R}$ with $x_i^m(k) \to a$, $x_j^m(k) \to b$ and $|ab| \geq \eta > 0$. Since $(N^{1/p} \cdot x_i^m, N^{1/p} \cdot x_j^m) \parallel x_i^m \parallel_{\ell_p}^p$ for $p > 2$ we obtain

$$0 \leq |a+b|^p + |a-b|^p = \lim_{m} (|x_i^m(k) + x_j^m(k)|^p + |x_i^m(k) - x_j^m(k)|^p - 2|x_i^m(k)|^p - 2|x_j^m(k)|^p)$$

$$\quad \leq \lim_{m} (\|x_i^m + x_j^m\|^p + \|x_i^m - x_j^m\|^p - 2\|x_i^m\|^p - 2\|x_j^m\|^p) = 0;$$

hence, $|a+b|^p + |a-b|^p = 2|a|^p + 2|b|^p = 0$ which, by Lemma 5.5, is in contradiction with $|ab| > 0$. The case when $p < 2$ is similar.

From now on, we write $x_i$ instead of $x_i^m$, where $m \in \mathbb{N}$ is chosen to satisfy the claim above. Fix $k \leq l$. By the claim above, there is at most one $i_0 \in \{1, \ldots, N\}$ with $|x_{i_0}(k)| \geq \sqrt{\eta}$ and for this $i_0$ we have $|x_{i_0}(k)| \leq \|x_{i_0}\| \leq 2N^{-1/p}$. Consequently, we have

$$\sum_{i=1}^{N} |x_i(k)| \leq \frac{2}{N^{1/p}} + \sum_{i \in \{\ldots, N\} \setminus \{x_{i_0}(k)\}} |x_i(k)| \leq \frac{2}{N^{1/p}} + N \cdot \sqrt{\eta} = \frac{3}{N^{1/p}}.$$ 

Thus, we have

$$\|x - \sum_{i=1}^{N} x_i\| = \sum_{k=1}^{l} \left( |x(k)| \right) \geq \sum_{k=1}^{l} \left( |x_{i_0}(k)| \right) \|x - \sum_{i=1}^{N} x_i^m\| \leq \delta.$$  

which is in contradiction with $\|x - \sum_{i=1}^{N} x_i\| = \|x - \sum_{i=1}^{N} x_i^m\| \leq \delta$.

(iii) $\implies$ (i) is obvious.

(iii) $\implies$ (i): suppose that $X$ is not isometric to $\ell_p$. By Theorem 1.2, $X$ is isometric to $L_{p}[0,1] \oplus Y$ for some (possibly trivial) Banach space $Y$. By abusing the notation, we may assume that $X = L_{p}[0,1] \oplus Y$. Let $1 \in L_{p}[0,1]$ be the constant 1 function, and define $x \in X = L_{p}[0,1] \oplus Y$ by $x = (1, 0)$. Now fix $N \in \mathbb{N}$ arbitrarily. Define $x_1, \ldots, x_N \in X$ by $x_i = (\chi_{[\frac{i-1}{N} \leq \cdot \leq \frac{i}{N})}, 0)$. Clearly $(N^{1/p} \cdot x_i)_{i=1}^N \equiv \ell_p$ and we have $x = \sum_{i=1}^{N} x_i$.

(ii) is obvious, so it only remains to show that (iii) $\implies$ (ii). For every $x \in X \setminus \{0\}$, $\delta \in \mathbb{N}$, $\varepsilon > 0$, and $x_1, \ldots, x_N \in X$ we denote by $V(x, \delta, N, \varepsilon, (x_i)_{i=1}^{N})$ the assertion that if $(N^{1/p} \cdot x_i)_{i=1}^{N} \parallel x_i \parallel_{\ell_p}^p$ then $\|x - \sum_{i=1}^{N} x_i\| \|x\|_p > \delta$. The desired implication straightforwardly follows by the following two easy observations. First, if $x \in D \setminus \{0\}$, $\delta$, $N$ and $\varepsilon$ are given such that $V(x, \delta, N, \varepsilon, (x_i)_{i=1}^{N})$ holds for every $x_1, \ldots, x_N \in D$ then $V(x, \delta, N, \varepsilon, (x_i)_{i=1}^{N})$ holds for every $x_1, \ldots, x_N \in X$. Second, if for every $x \in D \setminus \{0\}$ and $\delta$ there are $N$ and $\varepsilon$ such that $V(x, \frac{1+\varepsilon}{N}, N, \varepsilon, (x_i)_{i=1}^{N})$ holds for every $x_1, \ldots, x_N \in X$, then for every $x \in X \setminus \{0\}$ and $\delta$ there are $N$ and $\varepsilon$ such that $V(x, \delta, N, \varepsilon, (x_i)_{i=1}^{N})$ holds for every $x_1, \ldots, x_N \in X$.

Proof of Proposition 5.4 Let $\mathcal{F}$ be the set of those $\nu \in \mathcal{P}_{\infty}$ for which $X_{\nu}$ is an $\mathcal{L}_{p,1+}$ space. By Theorem 1.7, $\mathcal{F} \subseteq \mathcal{P}_{\infty}$ is a $G_{\delta}$ set. By Proposition 5.3 (i) $\iff$ (iv),
we have
\[ \langle \ell_p \rangle_{m,\infty} = \mathcal{F} \cap \bigcap_{v \in V(\{0\})} \bigcap_{m \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} V_{c,m,n,k}, \]
where the closed (see Fact 4.10 and Lemma 1.4) sets $V_{c,m,n,k}$ are given by
\[ V_{c,m,n,k} := \left\{ \mu \in \mathcal{P}_\infty : \mu(v) = 0 \text{ or for every } (v_i)_{i=1}^n \in V^n \text{ we have} \right\} \]
\[ - \left( \left( \sum_{i=1}^n |\frac{v_i}{\mu(v)}|^n \right) \right) \text{ or } \mu\left( \frac{v}{\mu(v)} - \sum_{i=1}^n v_i \right) \geq \frac{1}{m}. \]
Thus, $\langle \ell_p \rangle_{m,\infty}$ is an $F_{\sigma,\delta}$ set.

5.2. Dual unit balls and the Szlenk derivative. The purpose here is to show that mappings which assign a dual unit ball and its Szlenk derivative to a separable Banach space may be realized as a Borel map, see Lemma 5.10 and Lemma 5.9.

This will be later used in order to estimate the Borel complexity of the isometry class of the space $c_0$ because the isometric characterization of the space $c_0$ we use involves Szlenk derivatives, see Theorem 5.11. Note that the issue of handling Szlenk derivations as Borel maps was previously considered also by Bossard in [1], but our approach is slightly different as we prefer to work with coding $\mathcal{P}$ and we also need to obtain an estimate on the Borel class of the mapping.

Let us recall that given a real Banach space $X$, $w^*$-compact set $F \subseteq X^*$ and $\varepsilon > 0$, the Szlenk derivative is given as
\[ F'_\varepsilon = \{ x^* \in F : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \text{diam}(U \cap F) \geq \varepsilon \}. \]

We start by defining dual unit balls as closed subsets of $B_{c_0}$ equipped with the weak* topology, i.e., the topology generated by elements of the unique predual $\ell_1$.

Lemma 5.7. Let $X$ be a separable Banach space and let $\{ x_n : n \in \mathbb{N} \}$ be a dense set in $B_X$. Then the mapping $B_X^* \ni x^* \mapsto (x^*(x_n))_{n=1}^\infty \in B_{c_0}$ is $\| \cdot \| - \| \cdot \| \text{ isometry and } w^*,w^*\text{-homeomorphism onto the set}$
\[ \Omega(X) := \left\{ (a_n)_{n=1}^\infty \in B_{c_0} : M \subseteq \mathbb{N} \text{ finite} \Rightarrow \left| \sum_{n \in M} a_n \right| \leq \left\| \sum_{n \in M} x_n \right\| \right\}. \]

Proof. That the mapping is $w^*,w^*$ homeomorphism onto its image follows from the fact that $B_X^*$ is $w^*$-compact and the mapping is one-to-one (because $(x_n)$ separate the points of $B_X^*$) and $w^*,w^*$ continuous (because on $B_{c_0}$ the $w^*$-topology coincides with the topology of pointwise convergence). It is also straightforward to see that the mapping is isometry. Thus, it suffices to proof that
\[ \{ (x^*(x_n))_{n=1}^\infty : x^* \in B_X^* \} = \Omega(X). \]
The inclusion $\subseteq$ is easy, let us prove $\supseteq$. Given numbers $a_1,a_2,\ldots$ satisfying $|\sum_{n \in M} a_n| \leq \left\| \sum_{n \in M} x_n \right\|$ for any finite $M \subseteq \mathbb{N}$, we need to find $x^* \in B_X^*$ such that $x^*(x_n) = a_n$ for each $n$.

Let us realize first that
- $|a_n - a_m| \leq \|x_n - x_m\|$ for every $n,m$,
- $|a_n + a_m - a_l| \leq \|x_n + x_m - x_l\|$ for every $n,m,l$.

We check the first inequality only, the second inequality can be checked in the same way. Given $\varepsilon > 0$, let $n'$ different from $n$ and $m$ be such that $\|x_{n'} - x_{n}\| < \varepsilon$. We obtain $|a_n - a_{n'}| = \|(a_n - a_{n'}) + (a_{n'} - a_{n'})\| \leq |a_n - a_{n'}| + |a_{n'} - a_{n'}| \leq \|x_n + x_{n'}\| + \|x_{n'} + x_{n'}\| \leq 2\|x_n + x_{n'}\| < 2\|x_{n'} + x_{n'}\|$. Since $\varepsilon > 0$ was chosen arbitrarily, we arrive at $|a_n - a_n| \leq \|x_n - x_{n'}\|$.

It follows that there is a function $f : B_X \rightarrow \mathbb{R}$ with the Lipschitz constant 1 such that $f(x_n) = a_n$ for each $n$. We claim that $f(u + v) = f(u) + f(v)$ and $f(\alpha u) = \alpha f(u)$, whenever $u,\alpha,u,v,u+v \in B_X$. Given $\varepsilon > 0$, we pick $n,m,l$ such
Lemma 5.9. For every $\nu \in \mathcal{P}$ we can choose a countable dense subset $\{x^\nu_n : n \in \mathbb{N}\}$ of $B_{X_\nu}$ in such a way that the mapping $\Omega : \mathcal{P} \to \mathcal{K}(B_{l_\infty}, w^*)$ given by

$$\Omega(\nu) = \left\{ (a_n)_{n=1}^\infty \in B_{l_\infty} : M \subseteq \mathbb{N} \text{ finite } \Rightarrow \left| \sum_{n \in M} a_n \right| \leq \nu \left( \sum_{n \in M} x^\nu_n \right) \right\}$$

is continuous.

Proof. First of all, we describe the choice of the sets $\{x^\nu_n : n \in \mathbb{N}\}$, $\nu \in \mathcal{P}$. Let $g : [0, \infty) \to [1, \infty)$ be given by $g(t) = 1$ for $t \leq 1$ and $g(t) = t$ for $t > 1$. Let $\{v_n : n \in \mathbb{N}\}$ be an enumeration of all elements of the vector space $V$ (which is naturally embedded into all Banach spaces $X_\nu$, $\nu \in \mathcal{P}$). Now for every $\nu \in \mathcal{P}$ and every $n \in \mathbb{N}$ we define $x^\nu_n \in B_{X_\nu}$ by $x^\nu_n = \frac{v_n}{g^{-1}(\nu(n))}$. Then for every $\nu \in \mathcal{P}$ we have that $\{x^\nu_n : n \in \mathbb{N}\}$ is a dense subset of $B_{X_\nu}$. Note also that the set

$$\left\{ (\nu, (a_n)_{n=1}^\infty) \in \mathcal{P} \times B_{l_\infty} : \left| \sum_{n \in M} a_n \right| > \nu \left( \sum_{n \in M} x^\nu_n \right) \right\}$$

is open in $\mathcal{P} \times (B_{l_\infty}, w^*)$ for every $M \subseteq \mathbb{N}$ finite (the proof is easy and, similarly as the proof of Fact 4.10, it is omitted).

Proof. The sets of the form

$$\{ F \in \mathcal{K}(X) : F \subseteq W \} \quad \text{and} \quad \{ F \in \mathcal{K}(X) : F \cap W \neq \emptyset \},$$

where $W$ ranges over all open subsets of $X$, form a subbasis of the topology of $\mathcal{K}(X)$. So we only need to check that $f^{-1}(U)$ is an $\Sigma^0_{\alpha+1}$ set for every open set $U$ of one of these forms. For the first case this follows immediately from the assumptions and for the second case, if $\{U_n : n \in \mathbb{N}\}$ is an open basis for the topology of $X$, we have

$$f^{-1}(\{ F \in \mathcal{K}(X) : F \cap W \neq \emptyset \}) = \bigcup_{n \in \mathbb{N} \text{ such that } U_n \subseteq W} \left( P \setminus \{ p \in P : f(p) \subseteq X \setminus \overline{U_n} \} \right),$$

which, by the assumptions, is countable union of sets from $\Sigma^0_{\alpha}(P) \cup \Pi^0_{\alpha}(P)$. 

Lemma 5.8. Let $P$ be a Polish space, $X$ a metrizable compact, $\alpha \in [1, \omega_1)$ and $f : P \to \mathcal{K}(X)$ a mapping such that $\{ p \in P : f(p) \subseteq U \} \in \Sigma^0_{\alpha}(P) \cup \Pi^0_{\alpha}(P)$ for every open $U \subseteq X$. Then $f$ is $\Sigma^0_{\alpha+1}$-measurable.

Proof. Now, it is easy to see that $f$ uniquely extends to a linear functional on $X$. □

By the above, every dual unit ball of a separable Banach space may be realized as a subset of the unit ball of $\ell_\infty$. Thus, in what follows we use the following convention.

Convention. Whenever we talk about open (closed, $F_\sigma$, etc.) subsets of $B_{l_\infty}$ we always mean open (closed, $F_\sigma$, etc.) subsets in the weak* topology. On the other hand, whenever we talk about the diameter of a subset of $B_{l_\infty}$, or about the distance of two subsets of $B_{l_\infty}$, we always mean the diameter, or the distance, with respect to the metric given by the norm of $\ell_\infty$. Also, we write only $\mathcal{K}(B_{l_\infty})$ instead of $\mathcal{K}(B_{l_\infty}, w^*)$.
Pick an open subset $U$ of $B_{l_1}$. We have
\[
\Omega^{-1} \left( \{ F \in \mathcal{K}(B_{l_1}) : F \subseteq U \} \right)
= \left\{ \nu \in \mathcal{P} : \forall (a_n)_{n=1}^\infty \in B_{l_1} \left( \sum_{n=1}^\infty a_n \right) > \nu \left( \sum_{n=1}^\infty x_n \right) \lor ((a_n)_{n=1}^\infty \subseteq U) \right\}.
\]

The complement of the last set is the projection of a closed subset of $\mathcal{P} \times (B_{l_1}, w^*)$ onto the first coordinate. As the space $(B_{l_1}, w^*)$ is compact, the complement is a closed subset of $\mathcal{P}$.

It remains to show that the set $\{ \nu \in \mathcal{P} : \Omega(\nu) \cap U \neq \emptyset \}$ is open. Pick $\nu \in \mathcal{P}$ with $\Omega(\nu) \cap U \neq \emptyset$. By Lemma 5.10, there exists $x^* \in B_{N}$ such that the sequence $(a_n)_{n=1}^\infty$ given by $a_n = x^*(x_n^*)$, $n \in \mathbb{N}$, satisfies $(a_n)_{n=1}^\infty \subseteq \Omega(\nu) \cap U$. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ be such that $(b_n)_{n=1}^\infty \subseteq l_\infty$ is an element of $U$ whenever $|b_n - a_n| < \varepsilon$ for every $1 \leq n \leq N$. Let us consider subspaces of $c_0$ given as $E = \text{span}\{v_1, \ldots, v_N\}$ and $F = \{ x \in E : (x, v) = 0 \}$. Let $G$ be such that $F + G = E$ and $G \cap V = G$ (it is enough to pick a basis of $E$ consisting of vectors from $V$, and using the Gauss elimination to determine which vectors from the basis generate the algebraic complement to $F$). Let $P_F : E \to F$ and $P_G : E \to G$ be linear projections onto $F$ and $G$, respectively. Pick $\delta < \min\{1, \frac{\varepsilon}{3}\}$ such that $\delta \cdot |x^*(P_G v_n)| < \varepsilon/3$ for every $1 \leq n \leq N$. Finally, put
\[
\mathcal{O} := \{ \nu' \in \mathcal{P} : \frac{1}{1 - \delta} \nu(x) > \nu'(x) > (1 - \delta)\nu(x) \text{ for every } x \in G \setminus \{0\} \} \cap \bigcap_{n=1}^N \{ \nu' \in \mathcal{P} : \nu'(P_F v_n) < \delta, |\nu'(v_n) - \nu(v_n)| < \delta \}.
\]
Then $\mathcal{O}$ is an open neighborhood of $\nu$, which easily follows from Lemma 5.4 and the fact that $G \cap V$ is dense in $G$.

We will show that $\mathcal{O} \subseteq \{ \nu' \in \mathcal{P} : \Omega(\nu') \cap U \neq \emptyset \}$. Pick $\nu' \in \mathcal{O}$. If we put $y^*(x) := (1 - \delta)x^*(x)$ for $x \in G$, then $|y^*(x)| = (1 - \delta)|x^*(x)| \leq (1 - \delta)\nu(x) \leq \nu'(x)$ for every $x \in G$ and so by the Hahn-Banach theorem we may extend $y^*$ to a functional (denoted again by $y^*$) from the dual unit ball of $X^\nu$. By Lemma 5.7 the sequence $(b_n)_{n=1}^\infty \subseteq \Omega(\nu')$, $n \in \mathbb{N}$, is in $\Omega(\nu')$. Moreover, for every $1 \leq n \leq N$ we have
\[
|b_n - a_n| = \left| \frac{1}{g(\nu'(v_n))} y^*(v_n) - \frac{1}{g(\nu(v_n))} x^*(v_n) \right|
\leq \left| \frac{1}{g(\nu'(v_n))} y^*(P_F v_n) + \frac{1}{g(\nu'(v_n))} y^*(P_G v_n) - \frac{1}{g(\nu(v_n))} x^*(P_F v_n + P_G v_n) \right|
\leq \frac{1}{g(\nu'(v_n))} |y^*(P_F v_n)| + \frac{1}{g(\nu'(v_n))} (1 - \delta)\nu(x^*(P_G v_n)) + \frac{1}{g(\nu(v_n))} |x^*(P_G v_n)|
\leq \delta + (g(\nu'(v_n)) - g(\nu(v_n))) + \delta |x^*(P_G v_n)|
\leq \delta + 2\delta |x^*(P_G v_n)| < \varepsilon,
\]
and so $(b_n)_{n=1}^\infty \subseteq \Omega(\nu') \cap U$. Hence, $\mathcal{O} \subseteq \{ \nu' \in \mathcal{P} : \Omega(\nu') \cap U \neq \emptyset \}$, so $\nu \in \mathcal{P} : \Omega(\nu) \cap U \neq \emptyset$ is open and $\Omega$ is a continuous mapping. 

We close the first part of the subsection by realizing that the mapping which assigns to every compact subset of $B_{l_1}$ its Szlenk derivative is Borel. Let us note that the result is almost optimal as the mapping from Lemma 5.10 is not $F_{\sigma}$-measurable, see Corollary 5.13.

**Lemma 5.10.** For every $\varepsilon > 0$, the function $s_\varepsilon : \mathcal{K}(B_{l_1}, w^*) \to \mathcal{K}(B_{l_1}, w^*)$ given by $s_\varepsilon(F) = F + \varepsilon$ is $\Sigma_{\lambda}^0$-measurable.
Proof. First, we claim that the set
\[ \{ F \in K(B_{l_1}) : \text{diam}(U \cap F) < \varepsilon \} \]
is an \( F_\sigma \) set for every open subset \( U \) of \( B_{l_1} \). Indeed, the set above equals
\[ \bigcup_{k=1}^{\infty} \{ F \in K(B_{l_1}) : \text{diam}(U \cap F) \leq \varepsilon - \frac{1}{k} \} \]
\[ = \bigcap_{k=1}^{\infty} \bigcup_{O_1, O_2 \text{ open subsets of } U, \text{dist}(O_1, O_2) \geq \varepsilon - \frac{1}{k}} \{ F \in K(B_{l_1}) : F \cap O_1 = \emptyset \text{ or } F \cap O_2 = \emptyset \}, \]
and our claim immediately follows.

Now let \( W \) be an open subset of \( B_{l_1} \). Let \( \{ U_n : n \in \mathbb{N} \} \) be an open basis for the weak* topology of \( B_{l_1} \). Then we have (using a compactness argument in the last equality) that
\[ s_\varepsilon^{-1}(\{ F \in K(B_{l_1}) : F \subseteq W \}) \]
\[ = \{ F \in K(B_{l_1}) : \exists M \subseteq \mathbb{N} \left( \left( \bigcup_{n \in M} \text{diam}(U_n \cap F) < \varepsilon \right) \text{ and } (F \subseteq W \cup \bigcup_{n \notin M} U_n) \right) \} \]
\[ = \{ F \in K(B_{l_1}) : \exists M \subseteq \mathbb{N} \left( \left( \bigcup_{n \in M} \text{diam}(U_n \cap F) < \varepsilon \right) \text{ and } (F \subseteq W \cup \bigcup_{n \notin M} U_n) \right) \}, \]
and our previous claim implies that the last set is \( F_\sigma \).

Thus, by Lemma 5.3 the mapping \( s_\varepsilon \) is \( \Sigma^0_1 \)-measurable. \( \square \)

5.3. The space \( c_0 \). The main goal of this subsection is to prove the following.

Proposition 5.11. \( \langle c_0 \rangle_\infty \) is an \( F_\sigma \theta \) set in \( P_\infty \).

Our estimate on the Borel complexity of the isometry class of \( c_0 \) is based on an isometric characterization of \( c_0 \) among \( \mathcal{L}_{\infty,1} \) spaces. Let us recall that \( \mathcal{L}_{\infty,1} \) spaces are often called the Lindenstrauss spaces or \( L_1 \)-predual spaces. There are many different characterizations of this class of spaces. Let us recall one which we will use further, see e.g. [35, p. 232] (the “in particular” part follows from the easy part of Theorem 4.8 applied to \( X^* \) and the fact that \( L_1[0, 1] \) is not isomorphic to a subspace of a separable dual Banach space, see e.g. [11, Theorem 6.3.7]).

Theorem 5.12. Let \( X \) be a Banach space. Then the following conditions are equivalent:

(i) \( X \) is a \( \mathcal{L}_{\infty,1} \) space.

(ii) \( X^* \) is isometric to \( L_1(\mu) \) for some measure \( \mu \).

In particular, if \( X \) is a \( \mathcal{L}_{\infty,1} \) space with \( X^* \) separable then \( X^* \) is isometric to \( l_1 \).

The isometric characterization of \( c_0 \) which we use for our upper estimate follows.

Theorem 5.13. Let \( X \) be a separable \( \mathcal{L}_{\infty,1} \)-space and let \( 0 < \varepsilon < 1 \). Then \( X \) is isometric to \( c_0 \) if and only if
\[ (B_{X^*})_{2\varepsilon} = (1 - \varepsilon)B_{X^*}. \]

Proof. First, we show that \( (B_{c_0})_{2\varepsilon} = (1 - \varepsilon)B_{c_0} \) (this must be known but we were unable to find any reference). By a standard argument, \( (1 - \varepsilon)B_{X^*} \subseteq (B_{X^*})_{2\varepsilon} \) for any infinite-dimensional \( X \). (Let \( x^* \in (1 - \varepsilon)B_{X^*} \). Any \( w^* \)-open set \( U \) containing \( 0 \) contains also both \( y^* \) and \( -y^* \) for some \( y^* \in S_{X^*} \), and so \( \text{diam}(U \cap B_{X^*}) = 2 \). For this reason, any \( w^* \)-open set \( V \) containing \( x^* \) fulfills \( \text{diam}(V \cap (x^* + \varepsilon B_{X^*})) = 2\varepsilon \), in particular, \( \text{diam}(V \cap B_{X^*}) \geq 2\varepsilon \). This proves that \( x^* \in (B_{X^*})_{2\varepsilon} \).
Let us show that the opposite inclusion takes place for $X = c_0$. Assuming $1 - \varepsilon < \|x^*\| \leq 1$, we need to check that $x^* \notin (B_{X'})'_{2\varepsilon}$. Let $e_1, e_2, \ldots$ be the canonical basis of $c_0$. Let $n$ be large enough that $\sum_{i=1}^{n} |x^*(e_i)| > 1 - \varepsilon$ and let $\delta > 0$ satisfy $2\delta n < (\sum_{i=1}^{n} |x^*(e_i)|) = (1 - \varepsilon)$. Let

$$U = \{y^* \in c_0^* : 1 \leq i \leq n \Rightarrow |y^*(e_i) - x^*(e_i)| < \delta\}.$$

For $y^*, z^* \in U \cap B_{c_0^*}$, we have

$$\sum_{i=n+1}^{\infty} |y^*(e_i)| = \|y^*\| - \sum_{i=1}^{n} |y^*(e_i)| \leq 1 - \sum_{i=1}^{n} |x^*(e_i)| + \delta n,$$

and the same for $z^*$, thus

$$\|y^* - z^*\| \leq \sum_{i=1}^{n} |y^*(e_i) - z^*(e_i)| + \sum_{i=n+1}^{\infty} |y^*(e_i)| + \sum_{i=n+1}^{\infty} |z^*(e_i)|$$

$$\leq 2\delta n + 2[1 - \sum_{i=1}^{n} |x^*(e_i)|] + 2\delta n.$$

We get $\text{diam}(U \cap B_{c_0^*}) \leq 4\delta n + 2[1 - \sum_{i=1}^{n} |x^*(e_i)|] < 2\varepsilon$.

Now, let us assume that $X$ satisfies $(B_{X^*})'_{2\varepsilon} = (1 - \varepsilon)B_{X^*}$. Clearly, $X$ is infinite-dimensional, as $(B_{X^*})'_{2\varepsilon}$ is non-empty. Moreover, $X^*$ is separable because the Szlenk index of $X$ is $\omega$, see e.g. [36] Proposition 3 and Theorem 1). Thus, by Theorem 5.12, the dual $X^*$ is isometric to $\ell_1$. Let $e_1^*, e_2^*, \ldots$ be a basis of $X^*$ that is 1-equivalent to the canonical basis of $\ell_1$, and let $e_1^{**}, e_2^{**}, \ldots$ be the dual basic sequence in $X^{**}$. We claim that the functionals $e_n^{**}$ are $w^*$-continuous.

Suppose that $e_n^{**}$ is not $w^*$-continuous for some $n$. It means that $\{x^* \in X^* : e_n^{**}(x^*) = 0\}$ is not $w^*$-closed. By the Banach-Dieudonné theorem, the set $\{x^* \in B_{X^*} : e_n^{**}(x^*) = 0\}$ is not $w^*$-closed, too. The space $(B_{X^*}, w^*)$ is metrizable, so there is a sequence $x_k^*$ in $B_{X^*}$ with $e_n^{**}(x_k^*) = 0$ which $w^*$-converges to some $x^*$ with $e_n^{**}(x^*) \neq 0$. Without loss of generality, let us assume that $e_n^{**}(x^*) > 0$ and that $e_n^{**}(x_k^*)$ converges to some $a_i$ for every $i$. Then clearly $a_n = 0$. Note that $\sum_{i=1}^{\infty} |a_i| \leq 1$, which follows from the fact that $\sum_{i=1}^{\infty} |e_i^{**}(x_k^*)| = \|x_k^*\| \leq 1$ for every $k$. Let us put $a^* = \sum_{i=1}^{\infty} a_i e_i^*$, $y_k^* = x_k^* - a^*$ and $y^* = x^* - a^*$. Then $e_n^{**}(y_k^*) = 0$, $e_n^{**}(y^*) > 0$, the sequence $y_k^*$ is $w^*$-convergent to $y^*$ and, moreover, $e_{n}^{**}(y_k^*)$ converges to 0 for every $i$. Choosing a subsequence and making a small perturbation, we can find a sequence $z_i^*$ which is a block sequence with respect to the basis $e_i^*$ and which still $w^*$-converges to $y^*$. Without loss of generality, let us assume that $\|z_i^*\|$ converges to some $\lambda$, clearly with $\lambda \geq \|y^*\| > 0$, and let us consider $u_i^* = \frac{1}{\lambda^{1/2}} z_i^*$ and $u^* = \frac{1}{\lambda} y^*$.

So, we have seen that there is a normalized block sequence $u_i^*$ in $X^*$ which $w^*$-converges to some $u^*$ with $e_n^{**}(u^*) > 0$. We put

$$u_i^* = (1 - \varepsilon)e_n^* + \varepsilon u_i^*, \quad u^* = (1 - \varepsilon)e_n^* + \varepsilon u^*.$$

Then $u_i^*$ is a sequence in $B_{X^*}$, that $w^*$-converges to $u^*$. Since $\|v_l^* - v_{l'}^*\| = 2\varepsilon$ for $l \neq l'$, any $w^*$-open set $U$ containing $v^*$ fulfills $\text{diam}(U \cap B_{X^*}) \geq 2\varepsilon$. It follows that $v^* \in (B_{X^*})'_{2\varepsilon}$ and, by our assumption, $v^* \in (1 - \varepsilon)B_{X^*}$. At the same time,

$$\|v^*\| \geq e_n^{**}(u^*) = (1 - \varepsilon) + \varepsilon e_n^{**}(u^*) > 1 - \varepsilon,$$

which is not possible.

Hence, the functionals $e_n^{**}$ are $w^*$-continuous indeed. Every $e_n^{**}$ is therefore the evaluation of some $e_n \in X$. Finally, it is easy to check that $e_1, e_2, \ldots$ is a basis of $X$ that is 1-equivalent to the canonical basis of $c_0$. □
Proof of Proposition 5.11. Let $F$ be the set of those $\mu \in \mathcal{P}_\infty$ for which $X_\mu$ is $\mathcal{L}_{1,1}$-space. By Theorem 4.12, $F$ is $G_\delta$ in $\mathcal{P}_\infty$. Let $\Omega$ be the mapping from Lemma 5.9 and let us denote by $\triangle$ the closed set $\{ (x, x) : x \in K(\ell_\infty^\sigma) \}$ in $K(\ell_\infty^\sigma) \times K(\ell_\infty^\sigma)$. By Lemma 5.10 and Theorem 5.13, we have that $(\langle e_0 \rangle) = F \cap \{ \nu \in \mathcal{P} : (\frac{1}{2} \Omega(\nu), \Omega_1(\nu)) \in \triangle \}$. By Lemma 5.9 and Lemma 5.10 the mapping $\mathcal{P} \ni \nu \mapsto (\frac{1}{2} \Omega(\nu), \Omega_1(\nu)) \in K(\ell_\infty^\sigma) \times K(\ell_\infty^\sigma)$ is $\Sigma^0_3$-measurable, so we obtain that $(\langle e_0 \rangle) = F_\sigma$ in $\mathcal{P}_\infty$. □

Corollary 5.14. Let $\varepsilon > 0$. Then the mapping $s_\varepsilon$ from Lemma 5.11 is not $\Sigma^0_2$-measurable.

Proof. Otherwise, similarly as in the proof of Proposition 5.11 we would prove that $(\langle e_0 \rangle) = G_\delta$ in $\mathcal{P}_\infty$, which is not possible due to Theorem 4.13. □

6. Spaces with descriptively simple isomorphism classes

While there are several Banach spaces whose isometry classes have low complexity, there are reasons to suspect that isomorphism classes are rather complicated in general. The main result of this section is the following.

Theorem 6.1. The Hilbert space $\ell_2$ is characterized as the unique, up to isomorphism, infinite-dimensional separable Banach space $X$ such that $\langle X \rangle_\infty$ is $F_\sigma$ in $\mathcal{B}$. The same holds if we replace $\mathcal{B}$ with $\mathcal{P}_\infty$.

Recall that the isomorphism class of $\ell_2$ is $F_\sigma$, see Proposition 3.6. Besides this space, it is proved in [23, Theorem 4.12] that separable Banach spaces determined by their pavings have $\Sigma^0_3$ isomorphism classes in any admissible topology. We refer the interested reader to the text below for a definition of spaces determined by their pavings. Here we just briefly note that this class of spaces was introduced by Johnson, Lindenstrauss and Schechtman in [28] and that there are known examples of separable Banach spaces determined by their pavings not isomorphic to $\ell_2$ (e.g. certain $\ell_2$-sums of finite-dimensional spaces are such). The second main result of this section is the following: we mention above.

Theorem 6.2. Let $X$ be a separable infinite-dimensional Banach space that is determined by its pavings. Then $\langle X \rangle_\infty$ is $G_{4\sigma}$ in $\mathcal{P}_\infty$. In particular, it is $G_{4\sigma}$ in $\mathcal{P}$ and in any admissible topology.

Let us start with the proof of Theorem 6.1.

Proof of Theorem 6.1. We only need to show that if a separable infinite-dimensional Banach space $X$ is not isomorphic to $\ell_2$, then $\langle X \rangle_\infty$ is not $F_\sigma$ in $\mathcal{B}$. In what follows, we denote by $\mathcal{T}$ the set of finite tuples (including empty) of natural numbers without repetition. The length of $\gamma \in \mathcal{T}$ is denoted by $|\gamma|$ and its range by $\text{rng}(\gamma)$. Moreover, for every $\gamma \in \mathcal{T}$ and every $\mu \in \mathcal{B}$ we put

$$M^\gamma_\mu := \{ \nu \in \mathcal{B} : \text{ for every } (a_i)_{i=1}^{\gamma} \in \mathbb{Q}^{\gamma} \text{ we have } \nu \left( \sum_{i=1}^{\gamma} a_i e_i \right) = \mu \left( \sum_{i=1}^{\gamma} a_i e_{\gamma(i)} \right) \}.$$  

In order to get a contradiction assume that $(F_n)_{n=1}^\infty$ are closed sets in $\mathcal{B}$ such that $\langle X \rangle_\infty = \bigcup_{n=1}^\infty F_n$.

Claim. For every $\mu \in \mathcal{B}$ with $\langle X_\mu \rangle_\infty \subseteq \bigcup_{n=1}^\infty F_n$ there exist $\gamma \in \mathcal{T}$ and $m \in \mathbb{N}$ such that we have $M^{\gamma'}_\mu \cap F_m \neq \emptyset$ for every $\gamma' \in \mathcal{T}$ with $\gamma' \geq \gamma$.
Proof of the claim. Suppose the statement is not true. In particular, it does not hold for \( \gamma = \emptyset \) and \( m = 1 \). That is, there is some \( \gamma_1' \in T \) so that \( M_{\mu_0}^{\gamma_1'} \cap F_1 = \emptyset \). If \( 1 \in \text{rng}(\gamma_1') \), we set \( \gamma_1 = \gamma_1' \). Otherwise, we set \( \gamma_1 = \gamma_1' = 1 \).

In the next step, we use that the statement is not true for \( \gamma_1 \) and \( m = 2 \) to obtain \( \gamma_2' \in T, \gamma_2' \supseteq \gamma_1 \) so that \( M_{\mu_0}^{\gamma_2'} \cap F_2 = \emptyset \). If \( 2 \in \text{rng}(\gamma_2') \), we set \( \gamma_2 = \gamma_2' \). Otherwise, we set \( \gamma_2 = \gamma_2' = 2 \).

We continue analogously. At the end of the recursion, we obtain a bijection \( \pi : \mathbb{N} \to \mathbb{N} \) such that \( \pi \supseteq \gamma_n \) for every \( n \in \mathbb{N} \) and \( M_{\mu_0}^{\gamma_n} \cap F_n = \emptyset \) for every \( n \in \mathbb{N} \). Consider \( \mu_0 \in B \) given as

\[
\mu_0 \left( \sum_{i=1}^k a_i e_i \right) := \mu \left( \sum_{i=1}^k a_i e_{\pi(i)} \right), \quad k \in \mathbb{N}, \ (a_i)_{i=1}^k \in \mathbb{Q}^k.
\]

Then the linear mapping given by \( e_i \mapsto e_{\pi(i)}, i \in \mathbb{N} \), witnesses that \( X_{\mu_0} \equiv X_{\mu} \) and \( \mu_0 \in M_0^{\gamma_n} \) for every \( n \in \mathbb{N} \). Thus, \( \mu_0 \notin \bigcup_{n=1}^{\infty} F_n \), which is in contradiction with \( \mu_0 \in \bigcup_{n=1}^{\infty} F_n \). \( \square \)

Since \( X \notin \ell_2 \), by the celebrated solution to the homogeneous subspace problem following from the results of Komorowski and Tomczak-Jaegermann (30) and of Gowers (24), it must contain an infinite-dimensional closed subspace \( Y \subseteq X \) that is not isomorphic to \( X \). Let \( I \subseteq \mathbb{N} \) be an infinite subset and \( \{x_n\}_{n \in \mathbb{N}} \) a sequence of linearly independent vectors in \( X \) so that \( \text{span}\{x_n\}_{n \in \mathbb{N}} = X \) and \( \text{span}\{x_n : n \in I\} = Y \). We define \( \mu \in B \) as

\[
\mu \left( \sum_{i=1}^k a_i e_i \right) := \left\| \sum_{i=1}^k a_i x_i \right\|_X, \quad k \in \mathbb{N}, \ (a_i)_{i=1}^k \in \mathbb{Q}^k.
\]

Then \( \langle X_{\mu} \rangle = \langle X \rangle \subseteq \bigcup_{n=1}^{\infty} F_n \) and so, by the claim above, there exist \( \gamma \in T \) and \( m, n \in \mathbb{N} \) with \( M_{\mu}^{\gamma'} \cap F_{m,n} = \emptyset \) for every \( \gamma' \in T \) with \( \gamma' \supseteq \gamma \). Consider now the space \( Z := \langle \text{span}\{e_i : i \in I \cup \text{rng}(\gamma)\}, \mu \rangle \subseteq X_{\mu} \) which is isomorphic to \( Y \oplus F \) for some finite-dimensional Banach space \( F \). Fix some \( \tilde{I} \subseteq I \) such that \( |I \setminus \tilde{I}| = \dim F \) and such that \( (I \setminus \tilde{I}) \cap \text{rng}(\gamma) = \emptyset \). Denote by \( Y' \) the subspace \( \text{span}\{e_i : i \in \tilde{I}\} \) and by \( E \) the finite dimensional subspace \( \text{span}\{e_i : i \in (I \setminus \tilde{I})\} \) isomorphic to \( F \). It is easy to check that we have \( Y \simeq Y' \oplus E \). Define \( \tilde{Z} := \langle \text{span}\{e_i : i \in \tilde{I} \cup \text{rng}(\gamma)\}, \mu \rangle \). Then we have

\[
\tilde{Z} \simeq Y' \oplus F \simeq Y' \oplus E \simeq Y,
\]

and so \( \tilde{Z} \not\equiv X \). Let \( \varphi : \mathbb{N} \to \text{rng}(\gamma) \cup \tilde{I} \) be a bijection with \( \varphi \supseteq \gamma \). We define \( \nu \in B \) by

\[
\nu \left( \sum_{i=1}^k a_i e_i \right) := \mu \left( \sum_{i=1}^k a_i e_{\varphi(i)} \right), \quad k \in \mathbb{N}, \ (a_i)_{i=1}^k \in \mathbb{Q}^k.
\]

Clearly, \( X_{\nu} \equiv \tilde{Z} \not\equiv X \).

We claim that \( \nu \in F_m \). This will be in contradiction with the fact that \( F_m \subseteq \langle X \rangle \). Since \( F_m \) is closed, it suffices to check that each basic open neighborhood of \( \nu \) intersects \( F_m \). Pick \( v_1, \ldots, v_l \in V \) and \( \varepsilon > 0 \). We need to find \( \nu' \in F_m \) so that \( |\nu'(v_j) - \nu(v_j)| < \varepsilon \) for every \( j \leq l \).

Let \( L \in \mathbb{N}, L \geq |\gamma| \), be such that \( v_1, \ldots, v_l \in \text{span}\{e_i : i \leq L\} \). Since \( \varphi|_{\{1, \ldots, L\}} \supseteq \gamma \), we may pick \( \nu' \in M_{\mu}^{\varphi(1, \ldots, L)} \cap F_m \). Then

\[
\nu' \left( \sum_{i=1}^L a_i e_i \right) = \mu \left( \sum_{i=1}^L a_i e_{\varphi(i)} \right) = \nu \left( \sum_{i=1}^L a_i e_i \right), \quad (a_i)_{i=1}^L \in \mathbb{Q}^L.
\]

In particular, \( \nu'(v_j) = \nu(v_j), j \leq l \), as desired. \( \square \)
Proposition 6.3. Let $X$ be a separable infinite-dimensional Banach space, \{E_n\}_{n=1}^\infty paving of $X$ and $\lambda \geq 1$. Then the set
\[
\{ \mu \in \mathcal{P}_\infty : \text{for every } \varepsilon > 0 \text{ there is a paving } \{F_k\}_{k=1}^\infty \text{ of } X_\mu \text{ and an increasing sequence } (u_k)_{k=1}^\infty \in \mathbb{N}^N \text{ with sup}_{k \in \mathbb{N}} d_{BM}(F_k, E_{u_k}) \leq \lambda + \varepsilon \}
\]
is $G_\delta$ in $\mathcal{P}_\infty$.

Proof. The proof is verbatim the same as the proof of Theorem 4.6 with the only exception that instead of $l^p_0$ we write $E_n$ and we suppose we have some fixed basis of each $E_n$. \hfill \square

Proof of Theorem 6.2. Pick a paving $\{E_n\}_{n=1}^\infty$ of $X$. It is easy to see that for every $\mu \in \mathcal{P}_\infty$ the Banach space $X_\mu$ is isomorphic to $X$ if and only if $\mu$ belongs to the set from Proposition 6.3 for some $\lambda \geq 1$. The “In particular” part follows since $\mathcal{P}_\infty$ is $G_\delta$ in $\mathcal{P}$, see Corollary 1.3. For admissible topologies, the result follows by applying Theorem 1.17. \hfill \square

7. Miscellaneous

7.1. Superreflexive spaces. Recall that a map $f : M \to N$ between metric spaces is called a $C$-bilipschitz embedding if
\[
\forall x \neq y \in M : C^{-1}d_M(x, y) < d_N(f(x), f(y)) < Cd_M(x, y).
\]

Lemma 7.1. Let $M$ be a finite metric space and $C > 0$. The set $E(M, C)$ consisting of those $\mu \in \mathcal{P}$ such that $M$ admits a $C$-bilipschitz embedding into $X_\mu$ is open in $\mathcal{P}$.

Proof. Let $\mu \in E(M, C)$. Thus, there is a $C$-bilipschitz embedding $f : M \to X_\mu$. By perturbing the image of $f$ if necessary, we may without loss of generality assume that $f(M) \subseteq V$.

Consider $\varepsilon > 0$ and the open neighborhood $U_\varepsilon$ consisting of those $\mu' \in \mathcal{P}$ for which $|\mu(f(x) - f(y)) - \mu'(f(x) - f(y))| < \varepsilon$ for every $x, y \in M$. Then $U_\varepsilon \subseteq E(M, C)$ for $\varepsilon > 0$ small enough. Indeed, it suffices to choose $\varepsilon$ smaller than
\[
\min\{ \min_{x \neq y \in M} C_{d_M}(x, y) - C_{d_M}(f(x) - f(y)), \min_{x \neq y \in M} C_{d_M}(f(x) - f(y)) - C^{-1}d_M(x, y) \}.
\]
The easy verification is left to the reader. \hfill \square

Proposition 7.2. Let $(M_n)_{n \in \mathbb{N}}$ be a sequence of finite metric spaces and let $\mathcal{X}$ be the class of those Banach spaces $X$ for which there exists a constant $C$ such that for every $n \in \mathbb{N}$, $M_n$ admits a $C$-bilipschitz embedding into $X$.

Then $\mathcal{F} := \{ \mu \in \mathcal{P}_\infty : X_\mu \text{ is in } \mathcal{X} \}$ is $G_\delta$ in $\mathcal{P}_\infty$.

Proof. Follows immediately from Lemma 7.1 because we have
\[
\mathcal{F} = \mathcal{P}_\infty \cap \bigcup_{C > 0} \bigcap_{n \in \mathbb{N}} E(M_n, C).
\]
Bourgain in his seminal paper [7] found a sequence of finite metric spaces \((M_n)_{n \in \mathbb{N}}\) such that a separable Banach space is not superreflexive if and only if there exists a constant \(C\) such that for every \(n \in \mathbb{N}\), \(M_n\) admits \(C\)-bilipschitz embedding into \(X\). We refer the interested reader to [13] Section 9 for some more related facts and results. Thus, combining this result with Proposition 7.2 we obtain the following.

**Theorem 7.3.** The class of all superreflexive spaces is \(F_{\omega_\alpha}\) in \(\mathcal{P}_\infty\).

A metric space \(M\) is called locally finite if it is uniformly discrete and all balls in \(M\) are finite sets (in particular, every such \(M\) is at most countable). Let us mention a result by Ostrovskii by which a locally finite metric space bilipschitz embeds into a Banach space \(X\) if and only if all of its finite subsets admit uniformly bilipschitz embeddings into \(X\), see [14] or [14] Theorem 2.6. Thus, from Proposition 7.2 we obtain also the following.

**Corollary 7.4.** Let \(M\) be a locally finite metric space. Then the set of those \(\mu \in \mathcal{P}_\infty\) for which \(M\) admits a bilipschitz embeddings into \(X_{\mu}\) is \(G_{\omega_\alpha}\) in \(\mathcal{P}_\infty\).

It is well-known that many important classes of separable Banach spaces are not Borel. This concerns e.g. reflexive spaces, spaces with separable dual, spaces containing \(\ell_1\), spaces with the Radon-Nikodým property, spaces isomorphic to \(L_p[0,1]\) for \(p \in (1,2) \cup (2,\infty)\), or spaces isomorphic to \(c_0\). We refer to [4] page 130 and Corollary 3.3 and [33] Theorem 1.1 for papers which contain the corresponding results and to the monograph [12] and the survey [21] for some more information. Thus, e.g. in combination with Corollary 7.4 we see that none of those classes might be characterized as a class into which a given locally finite metric space bilipschitz embeds. Let us give an example of such a result which is related to [12] Problem 12.5(b)]. This is an elementary, but interesting application of the whole theory.

**Corollary 7.5.** There does not exist a locally finite metric space \(M\) such that any separable Banach space \(X\) is not reflexive if and only if \(M\) admits a bilipschitz embeddings into \(X\).

**Remark 7.6.** Let us draw attention of the reader once more to the remarkable paper [44], where the authors found a metric characterization of reflexivity even though such a condition is necessarily non Borel (as mentioned above).

### 7.2. Szlenk indices

In this subsection we give estimates on the Borel classes of spaces with Szlenk index less than or equal to a given ordinal number. Note that it is a result by Bossard, see [4] Section 4, that those sets are Borel and their Borel classes are unbounded. So our contribution here is that we provide certain quantitative estimates from above. Similarly, we give an estimate on the Borel class of spaces with summable Szlenk index, which is a quantitative improvement of the result mentioned in [20] page 367. Let us start with the corresponding definitions. Let \(X\) be a real Banach space and \(K \subseteq X^*\) a \(w^*\)-compact set. Following [10], for \(\varepsilon > 0\) we define \(s_\varepsilon(K)\) as the Szlenk derivative of the set \(K\) (see Subsection 7.2) and then we inductively define \(s^n_\varepsilon(K)\) for an ordinal \(\alpha\) by \(s^{n+1}_\varepsilon(K) := s_\varepsilon(s^n_\varepsilon(K))\) and \(s^n_\varepsilon(K) := \bigcap_{\beta < \alpha} s^n_\varepsilon(K)\) if \(\alpha\) is a limit ordinal. Given a real Banach space \(X\), \(Sz(X,\varepsilon)\) is the least ordinal \(\alpha\) such that \(s^n_\varepsilon(B_X) = \emptyset\), if such an ordinal exists (otherwise we write \(Sz(X,\varepsilon) = \infty\)). The Szlenk index is defined by \(Sz(X) = \sup_{\varepsilon > 0} Sz(X,\varepsilon)\).

Recall that for a separable infinite-dimensional Banach space \(X\) the Szlenk index is either \(\infty\) or \(\omega^\alpha\) for some \(\alpha \in [1,\omega_1]\), see [80] Section 3.

**Theorem 7.7.** Let \(\alpha \in [1,\infty)\) be an ordinal. Then \[
\{\mu \in \mathcal{P}_\infty : Sz(X_{\mu}) \leq \omega^\alpha\}
\]
is a $\Pi^0_{\omega+1}$ set in $\mathcal{P}_\infty$.

**Proof.** Using Lemma 5.10 it is easy to prove by induction on $n$ that the mapping $K(B_{c_0}) \ni F \mapsto s^n_\alpha(F) \in K(B_{c_0})$ is $\Sigma^0_{n+1}$-measurable for every $n \in \mathbb{N}$. Further, the mapping $K(B_{c_0}) \ni F \mapsto s_\alpha(F) \in K(B_{c_0})$ is $\Sigma^0_{\omega+1}$-measurable. Indeed, for every open $V \subseteq B_{c_0}$, by compactness argument, we have
\[
\{ F : s_\alpha(F) \subseteq V \} = \bigcup_{n=1}^\infty \{ F : s^n_\alpha(F) \subseteq V \}
\]
which is a $\Sigma^0_1$ set, so by Lemma 5.8 the mapping $s^n_\alpha$ is $\Sigma^0_{\omega+1}$-measurable. Similarly, we prove by transfinite induction that $s^n_\beta$ is $\Sigma^0_{\beta+1}$-measurable whenever $\beta \in [\omega, \omega_1)$ is a limit ordinal.

Let $\Omega$ be the mapping from Lemma 5.9. Then by Lemma 5.7 we have
\[
\{ \mu \in \mathcal{P}_\infty : Sz(X_\mu) \leq \omega^n \} = \bigcap_{k \in \mathbb{N}} \{ \mu \in \mathcal{P}_\infty : Sz(X_\mu, \frac{1}{k}) \leq \omega^n \}
\]
which, by the above and Lemma 5.9, is a countable intersection of preimages of closed sets under $\Sigma^0_{\omega+1}$-measurable mapping, so it is a $\Pi^0_{\omega+1}$ set in $\mathcal{P}_\infty$. □

Let us recall that a Banach space $X$ has **summable Szlenk index** if there is a constant $M$ such that for all positive $\varepsilon_1, \ldots, \varepsilon_n$ with $s_1, \ldots, s_n B_{X^*} \neq \emptyset$ we have $\sum_{i=1}^n \varepsilon_i \leq M$.

**Proposition 7.8.** The set $\{ \mu \in \mathcal{P}_\infty : X_\mu$ has a summable Szlenk index$\}$ is a $\Sigma^0_{\omega+2}$ set in $\mathcal{P}_\infty$.

**Proof.** Let $\Omega$ be the mapping from Lemma 5.9. It is easy to see that the set $\{ \mu \in \mathcal{P}_\infty : X_\mu$ has a summable Szlenk index$\}$ is equal to
\[
\bigcup_{M \in \mathbb{N}} \bigcap_{\sum_{i=1}^n \varepsilon_i \geq M} \{ \mu \in \mathcal{P}_\infty : s_{\varepsilon_1} \ldots s_{\varepsilon_n} \Omega(\mu) = \emptyset \},
\]
which by Lemma 5.9 and Lemma 5.10 is a $\Sigma^0_{\omega+2}$ set in $\mathcal{P}_\infty$. □

Finally, let us note that similarly one can of course estimate Borel complexity of various other classes of spaces related to Szlenk derivations, e.g. spaces with Szlenk power type at most $p$ etc.

### 7.3. Spaces having Schauder basis-like structures.

It is an open problem whether the class of spaces with Schauder basis is a Borel set in $\mathcal{B}$ (see e.g. [12], Problem 8) and note that by the results from Section 1 it does not matter whether we use the coding $SB(C([0,1]))$ or $\mathcal{B}$. However, it was proved by Ghawadrah that the class of spaces with $\pi$-property is Borel (actually, it is $\Sigma^0_1$ in $\mathcal{P}_\infty$ which follows immediately from [16], Lemma 2.1), see also [19]) and that the class of spaces with the bounded approximation property (BAP) is Borel (actually, it is $\Sigma^0_1$ in $\mathcal{P}_\infty$ which follows immediately from [15], Lemma 2.1) and this estimate has recently been improved to $\Sigma^0_0$ in any admissible topology (see [19]).

One is therefore led to the question of finding examples of Banach spaces having BAP but not the Schauder basis. Such an example was constructed by Szarek [17]. Actually, Szarek considered classes of separable spaces with local basis structure (LBS) and local II-basis structure (LIIBS) for which we have
\[
basis \implies (\text{LIIBS}) \implies (\text{(LBS) and (BAP)}) \implies (\text{BAP})
\]
Then for every $\nu$, $\Psi(\ldots)$ and Definition 7.10. A Banach space $E$ have dimension $\leq \sup \|x\|$ whenever $n, m \in \mathbb{N}$, $n \leq m \leq d$ and $a_1, \ldots, a_m \in \mathbb{R}$. The basis constant of $(x_i^d)_{i=1}^n$ is denoted by $bc((x_i^d)_{i=1}^n)$. We further denote $bc(X) = \inf \{bc((x_i^d)_{i=1}^n) : (x_i^d)_{i=1}^n$ is a basis of $X\}$.

**Definition 7.10.** By the basis constant of a basis $(x_i^d)_{i=1}^n$ of a Banach space $X$ of dimension $d \in [0, \infty)$ we mean the least number $C \geq 1$ such that $C \sup_{i \geq 1} a_i x_i$ whenever $n, m \in \mathbb{N}$, $n \leq m \leq d$ and $a_1, \ldots, a_m \in \mathbb{R}$. The basis constant of $(x_i^d)_{i=1}^n$ is denoted by $bc((x_i^d)_{i=1}^n)$. We further denote $bc(X) = \inf \{bc((x_i^d)_{i=1}^n) : (x_i^d)_{i=1}^n$ is a basis of $X\}$.

**Definition 7.10.** A Banach space $X$ is said to have the local basis structure (LBS) if $X = \bigcup_{n=1}^{\infty} E_n$ where $E_1 \subseteq E_2 \subseteq \ldots$ are finite-dimensional subspaces satisfying $\sup_{n \in \mathbb{N}} bc(E_n) < \infty$.

Further, $X$ is said to have the local $\Pi$-basis structure (LIBS) if $X = \bigcup_{n=1}^{\infty} E_n$, where $E_1 \subseteq E_2 \subseteq \ldots$ are finite-dimensional subspaces satisfying $\sup_{n \in \mathbb{N}} bc(E_n) < \infty$ for which there are projections $P_n : X \to E_n$ such that $P_n(X) = E_n$ and $\sup_{n \in \mathbb{N}} \|P_n\| < \infty$.

**Lemma 7.11.** Whenever $E$ is a finite-dimensional subspace of a Banach space $X$, $\delta \in (0, 1)$, $K > 0$, $T : E \to X$ is a $(1+\delta)$-isomorphism (not necessarily surjective) with $\|T - I\| < \delta$ and $P : X \to E$ is a projection with $P(X) = E$ and $\|P\| \leq K$, then for every subspace $F$ of $E$ we have $\|TP_T^2 - IT(E)\| \leq 4\delta K$.

Moreover, whenever $\|TP_T^2 - IT(E)\| < \delta$ then $T(E)$ is $(\frac{1+\delta)K}{1-\delta}$ complemented in $X$.

**Proof.** Let $f_1, \ldots, f_n$ be a basis of $F$. Then for every $x = \sum_{i=1}^n a_i T(f_i) \in T(F)$ we have $\|TP x - x\| = \|TP(\sum_{i=1}^n a_i T(f_i) - f_i))\| \leq (1+\delta)K \|T - I\| T^{-1}x \| \leq (1+\delta)^2 \delta K \|x\|$.

Moreover, if $\|TP_T^2 - IT(E)\| < \delta$ then the mapping $TP_T^2$ is an isomorphism with $\|TP_T^2 - IT(E)\| \leq \sum_{i=0}^{\infty} q^i = \frac{1}{1-q}$. It is now straightforward to prove that $P' := (TP_T^2)^{-1}TP : X \to T(E)$ is a projection onto $T(E)$ with $\|P'\| \leq \frac{(1+\delta)K}{1-\delta}$.

**Lemma 7.12.** For every $\mu \in B$, $K, l \in \mathbb{N}$ and $v_1, \ldots, v_m \in V$, let us denote by $\Phi(\mu, K, v_1, \ldots, v_m)$ and $\Psi(\mu, K, l, v_1, \ldots, v_m)$ the formulae

$$
\Phi(\mu, K, v_1, \ldots, v_m) = \exists a_1, \ldots, a_m \in \mathbb{R} : \max_{1 \leq k \leq m} \mu \left( \sum_{i=1}^k a_i v_i \right) \leq K \mu \left( \sum_{i=1}^m a_i v_i \right)
$$

and

$$
\Psi(\mu, K, l, v_1, \ldots, v_m) = \exists u_1, \ldots, u_l \in Q \text{-span}\{v_1, \ldots, v_m\} \forall a_1, \ldots, a_m, b_1, \ldots, b_l \in \mathbb{R} : \mu \left( \sum_{i=1}^m a_i v_i + \sum_{i=1}^l b_i u_i \right) \leq K \mu \left( \sum_{i=1}^m a_i v_i + \sum_{i=1}^l b_i u_i \right).
$$

Then for every $\nu \in B$ the following holds.
(a) The space $X_\nu$ has LBS if and only if
\[
\exists K \in \mathbb{N} \forall n \in \mathbb{N} \exists m \in \mathbb{N} \exists v_1, \ldots, v_m \in V, \{e_1, \ldots, e_n\} \subseteq \text{span}\{v_1, \ldots, v_m\}
\]
\[
\Phi(\nu, K, v_1, \ldots, v_m).
\]

(b) The space $X_\nu$ has LIIBS if and only if
\[
\exists K \in \mathbb{N} \forall n \in \mathbb{N} \exists m \in \mathbb{N} \exists v_1, \ldots, v_m \in V, \{e_1, \ldots, e_n\} \subseteq \text{span}\{v_1, \ldots, v_m\}
\]
\[
\Phi(\nu, K, v_1, \ldots, v_m) \land \forall l \in \mathbb{N} \Psi(\nu, K, l, v_1, \ldots, v_m).
\]

Proof. We prove only the more difficult part (b). Since $\nu \in \mathcal{B}$, the space $X_\nu$ is just the completion of $(c_0, \nu)$ (it is not necessary to consider a quotient). So, the notions of linear span and of linear independence have the same meaning in $c_0$ and in $X_\nu$, if performed on subsets of $c_0$.

Let us suppose that $\nu \in \mathcal{B}$ satisfies the formula in (b) for some $K \in \mathbb{N}$. We put $E_0 = \{0\}$ and choose recursively subspaces $E_1 \subseteq E_2 \subseteq \cdots$ of $X_\nu$, each of which is generated by a finite number of elements of $V$, in the following way.

Assuming that $E_j$ has been already chosen, we pick first $n_{j+1} \geq j + 1$ such that $E_j \subseteq \text{span}\{e_1, \ldots, e_{n_{j+1}}\}$. Then we can pick $m_{j+1} \in \mathbb{N}$ and $v_1^{j+1}, \ldots, v_{m_{j+1}}^{j+1} \in V$ with $\{e_1, \ldots, e_{n_{j+1}}\} \subseteq \text{span}\{v_1^{j+1}, \ldots, v_{m_{j+1}}^{j+1}\}$ such that $\Phi(\nu, K, v_1^{j+1}, \ldots, v_{m_{j+1}}^{j+1})$ and for every $l \in \mathbb{N}$, $\Psi(\nu, K, l, v_1^{j+1}, \ldots, v_{m_{j+1}}^{j+1})$ hold.

We put $E_{j+1} = \text{span}\{v_1^{j+1}, \ldots, v_{m_{j+1}}^{j+1}\}$. In this way, we obtain $E_j \subseteq E_{j+1}$. Also, $X_\nu = \bigcup_{n=1}^{\infty} E_n$ (we have $e_j^{n+1} \in E_{j+1}$, as $n_{j+1} \geq j + 1$). If we take all non-zero vectors $v_i^{j+1}, 1 \leq i \leq m_{j+1}$, we obtain a basis of $E_{j+1}$ with the basis constant at most $K$.

To show that the sequence $E_1 \subseteq E_2 \subseteq \cdots$ witnesses that $X_\nu$ has LIIBS, it remains to find a projection $P_{j+1}$ of $X_\nu$ onto $E_{j+1}$ such that $\|P_{j+1}\| \leq K$.

Let us pick some $l \in \mathbb{N}$ and put $E^{(l)} = \text{span}\{v_1^{j+1}, \ldots, v_{m_{j+1}}^{j+1}, e_1, \ldots, e_l\}$. By $\Psi(\nu, K, l, v_1^{j+1}, \ldots, v_{m_{j+1}}^{j+1})$, there exists a projection $P^{(l)}$ of $E^{(l)}$ onto $E_{j+1}$ with $\|P^{(l)}\| \leq K$. Since the norms of $P^{(l)}$, for $l \in \mathbb{N}$, are uniformly bounded and have a fixed finite-dimensional range, there exists their accumulation point in SOT which is a projection $P_{j+1} : X_\nu \to E_{j+1}$ of norm bounded by $K$ as desired.

Conversely, suppose that $X_\nu$ has LIIBS as witnessed by some $C > 1$ and a sequence $(E_n)_{n \in \mathbb{N}}$ of finite-dimensional subspaces satisfying $X_\nu = \bigcup_{n=1}^{\infty} E_n$ and sup_{n \in \mathbb{N}} \nu(E_n) < C$, for which there are projections $P_n : X_\nu \to E_n$ such that $P_n(X_\nu) = E_n$ and sup_{n \in \mathbb{N}} \|P_n\| < C$. Pick $D > 0$ such that $H_n := (\text{span}\{e_1, \ldots, e_n\}, \nu)$ is $D$-complemented in $X_\nu$ and let $\phi_1 := \phi_{e_1, \ldots, e_n}$ be the function from Lemma 2.4\{4\}.

Fix $\varepsilon > 0$ such that $\phi_1(t)$ is small enough (to be specified later) whenever $t < \varepsilon$. Fix $k \in \mathbb{N}$ such that there are $h_1, \ldots, h_k \in E_k$ with $\nu(e_i - h_i) < \varepsilon$. If $\phi_1(\varepsilon)$ is small enough we have \[
\frac{(1 + \phi_1(\varepsilon))}{1 - \phi_1(\varepsilon)} \cdot \frac{1}{D} \leq 2D \text{ (this value refers to the "Moreover" part in Lemma 7.11).}
\]

By Lemma 7.11, span\{h_i : i \leq n\} is $2D$-complemented in $X_\nu$, so let $Q : X_\nu \to \text{span}\{h_i : i \leq n\}$ be the corresponding projection. Pick a basis $h_{n+1}, \ldots, h_{\dim E_k}$ of the space $E_k \cap Q^{-1}(0)$ which is $(2D + 1)$-complemented in $E_k$. Let $\phi_2 := \phi_{h_{n+1}, \ldots, h_{\dim E_k}}$ be the function from Lemma 2.4\{4\}.

Fix $\delta > 0$ such that $\phi_2(t)$ is small enough (to be specified later) whenever $t < \delta$. Finally, find $f_{n+1}, \ldots, f_{\dim E_k} \in V$ with $\nu(f_j - h_j) < \delta$ for $j = n + 1, \ldots, \dim E_k$.

We claim that the space $F_n := (\text{span}\{e_1, \ldots, e_n, f_{n+1}, \ldots, f_{\dim E_k}\}, \nu)$ is $2C$-complemented in $X_\nu$ and $d_{BM}(F_n, E_k) < 2$. If we denote by $T : E_k \to F_n$ the linear mapping given by $h_i \mapsto e_i, i \leq n$, and $h_j \mapsto f_j, n + 1 \leq j \leq \dim E_k$, then for
every \( y \in \text{span}\{h_i \colon i \leq n\} \) and \( z \in \text{span}\{h_j \colon j = n + 1, \ldots, \dim E_k\} \) we have
\[
\nu(T(y + z) - y - z) \leq \nu(Ty - y) + \nu(Tz - z) \leq \phi_1(\varepsilon)\nu(y) + \phi_2(\delta)\nu(z)
\]
\[
\leq \left( \phi_1(\varepsilon)2D + \phi_2(\delta)(2D + 1) \right)\nu(y + z);
\]
hence, if \( \eta := \left( \phi_1(\varepsilon)2D + \phi_2(\delta)(2D + 1) \right) < 1 \), we obtain \( \|T\| \leq 1 + \|I - T\| \leq 1 + \eta \)
and \( \|Tx\| \geq \|x\| - \|(I - T)x\| \geq (1 - \eta)\|x\| \) for every \( x \in E_k \) so \( T \) is an isomorphism
with \( \|T\|^{-1} \leq (1 - \eta)^{-1} \). Thus, by Lemma 7.11 if \( \phi_1(\varepsilon) \) and \( \phi_2(\delta) \) are small enough
(and so \( \eta \) is small enough), we obtain \( \|T\|\|T^{-1}\| < 2 \) and \( F_n \) is 2C-complemented
in \( X_\mu \).

Thus, \( \text{bc}(F_n) \leq \text{bc}(E_k)d_{BM}(E_k, F_n) < 2C \) which is witnessed by some basis
\( v_1, \ldots, v_m \in V \) of \( F_n \). This shows that \( \Phi(\nu, 2C, v_1, \ldots, v_m) \) holds. Let \( P : X_\mu \to F_n \)
be a projection with \( P[X_\mu] = F_n \) and \( \|P\| \leq 2C \). Given \( l \in \mathbb{N} \), let \( T \subseteq \{1, \ldots, l\} \)
be a set such that \( (e_i)_{i \in T} \) together with \( (v_i)_{i=1}^m \) form a basis of \( \text{span}\{(v_1, \ldots, v_m) \cup \{e_1, \ldots, e_l\}\} \).
Pick \( A > 0 \) such that \( (v_i)_{i=1}^m \cup (e_i)_{i \in T} \sim \ell_1^{m+|T|} \). For \( i \in T \) pick
\( u_i \in \text{span}_Q\{v_1, \ldots, v_m\} \) such that \( \nu(u_i - P(e_i)) < \frac{C_1}{A} \). Then for every \( a_1, \ldots, a_m \in \mathbb{R} \)
and every \( (b_i)_{i \in T} \in \mathbb{R}^T \) we have
\[
\nu\left( \sum_{i=1}^m a_i v_i + \sum_{i \in T} b_i e_i \right) \leq 2C\nu\left( \sum_{i=1}^m a_i v_i + \sum_{i \in T} b_i e_i \right) + \nu\left( \sum_{i \in T} b_i (u_i - P(e_i)) \right)
\]
\[
\leq 3C\nu\left( \sum_{i=1}^m a_i v_i + \sum_{i \in T} b_i e_i \right).
\]
Thus, the linear mapping \( O : \text{span}\{(v_1, \ldots, v_m) \cup \{e_1, \ldots, e_l\}\} \to \text{span}\{v_1, \ldots, v_m\} \)
given by \( v_i \mapsto v_i, i \leq m, \) and \( e_i \mapsto u_i, i \in T, \) is a linear projection, and if we put
\( u_i := O(e_i) \in V \) for every \( i \in \{1, \ldots, l\} \), we see that \( \Psi(\nu, 3C, l, v_1, \ldots, v_m) \)
holds and the formula in (b) is satisfied with \( K = 3C \). \( \square \)

**Theorem 7.13.** (a) The class of spaces which have LBS is \( \Sigma^0_4 \) in \( \mathcal{B} \).
(b) The class of spaces which have L\(\text{L}\)BS is \( \Sigma^0_4 \) in \( \mathcal{B} \).

**Proof.** This follows from Lemma 7.12 because the conditions given by formulas \( \Phi \) and \( \Psi \) are obviously closed and \( F_\sigma \), respectively. \( \square \)

8. Open questions and remarks

In Section 7 we investigated three ways of formalizing the class of all separable
infinite-dimensional Banach spaces as a Polish space. Those were denoted by \( \mathcal{P}_\infty \),
\( \mathcal{B} \) and \( \text{SB}_\infty(X) \). We obtained an optimal reduction from \( \mathcal{B} \) to \( \text{SB}_\infty(X) \) and from
\( \text{SB}_\infty(X) \) to \( \mathcal{P}_\infty \). However, our reduction from \( \mathcal{P}_\infty \) to \( \mathcal{B} \) seems not to be optimal,
so one is tempted to ask the following.

**Question 1.** Does there exist a continuous mapping \( \Phi : \mathcal{P}_\infty \to \mathcal{B} \) such that for
every \( \mu \in \mathcal{P}_\infty \) we have \( X_\mu \equiv X_{\Phi(\mu)} \)?

Note that a positive answer to Question 1 would imply a positive answer to
Question 2 and that a sufficient condition for a positive solution of Question 2 is
provided by Proposition 6.28.

**Question 2.** Let \( X \) be an isometrically universal separable Banach space and let \( \tau \) be
an admissible topology on \( \text{SB}(X) \). Does there exist a \( \Sigma^0_\tau \)-measurable mapping
\( \Phi : \mathcal{P}_\infty \to (\text{SB}(X), \tau) \) such that for every \( \mu \in \mathcal{P}_\infty \) we have \( X_\mu \equiv \Phi(\mu) \)?

In Theorem 6.1 we proved that \( \ell_2 \) is the unique separable infinite-dimensional
Banach space (up to isomorphism) whose isomorphism class is \( F_\sigma \). Following 28,
we say that a separable infinite-dimensional Banach space \( X \) is determined by its
finite dimensional subspaces if it is isomorphic to every separable Banach space \( Y \) which is finitely crudely representable in \( X \) and for which \( X \) is finitely crudely representable in \( Y \). Note that \( \ell_2 \) is determined by its finite dimensional subspaces and that if a separable infinite-dimensional Banach space is determined by its finite dimensional subspaces then it is obviously determined by its pavings and so, by Theorem 6.1 it is isomorphic to every separable infinite-dimensional Banach space which is determined by its finite dimensional subspaces. We believe that Theorem 6.1 could be instrumental for proving this conjecture, since it follows from this theorem that the conjecture is equivalent to the positive answer to the following question. We thank Gilles Godefroy who suggested to us that there might be a relation between having \( F_\sigma \) isomorphism class and being determined by finite dimensional subspaces.

**Question 3.** Let \( X \) be a separable infinite-dimensional Banach space determined by its finite dimensional subspaces. Is \( \langle X \rangle \thicksim F_\sigma \) in \( B \)?

It would be interesting to know whether there is a separable infinite-dimensional Banach space \( X \) such that \( \langle X \rangle \thicksim G_\delta \) in \( B \) or in \( P_\infty \). Note that the only one possible candidate is the Gurari˘ı space, see Section 8 for more details. One of the possible strategies to answer Question 4 in negative for \( P_\infty \) would be to find an admissible topology \( \tau \) on \( SB(X) \) such that \( \langle G \rangle \thicksim \) is a dense and meager set in \( (SB(X), \tau) \). However, we do not even know whether \( \langle G \rangle \thicksim \) is Borel.

**Question 4.** Is \( \langle G \rangle \thicksim a G_\delta \) set in \( P_\infty \) or in \( B \)? Is it at least Borel?

Solving the homogeneous Banach space problem, Komorowski and Tomczak-Jaegermann (30), and Gowers (24) proved that if a separable infinite-dimensional Banach space is isomorphic to all of its closed infinite-dimensional subspaces, then it is isomorphic to \( \ell_2 \). It seems that the isometric variant of this result is open; that is, whether \( \ell_2 \) is the only separable infinite-dimensional Banach space that is isometric to all of its infinite-dimensional closed subspaces. We note that any Banach space satisfying this criterion must be, by the Gowers’ result, isomorphic to \( \ell_2 \). Our initial interest in this problem was that we observed that a positive answer implies that whenever \( \langle X \rangle \equiv \) is closed in \( P_\infty \) then \( X \equiv \ell_2 \). Eventually we found another argument (see Section 3), but the question is clearly of independent interest.

**Question 5.** Let \( X \) be a separable infinite-dimensional Banach space which is isometric to all of its closed infinite-dimensional subspaces. Is then \( X \) isometric to \( \ell_2 \)?

In Sections 2, 3 and 4 we proved that \( \langle G \rangle \equiv \), resp. \( \langle L_p[0,1] \rangle \equiv \), for \( p \in [1, \infty) \), are \( G_\delta \); we even proved that they are dense \( G_\delta \) in \( P_\infty \), resp. in \( L_{p,1+} \cap P_\infty \). Coincidentally, all these spaces are Fraïssé limits (we refer to [14] Proposition 3.7) for this statement about \( L_p[0,1] \). According to [15], no other examples of separable Banach spaces which are Fraïssé limits seem to be known. This motivates us to ask the following.

**Question 6.** Does there exist a separable infinite-dimensional Banach space \( X \) which is not isometric to \( L_p[0,1] \), for \( p \in [1, \infty) \), and to the Gurari˘ı space, and \( \langle X \rangle \equiv \) is \( G_\delta \) in \( P_\infty \) or in \( B \)?

It also follows that for \( 1 \leq p < \infty \), \( L_p[0,1] \) is a generic \( L_{p,1+} \)-space. On the other hand, by Corollary 5.12 for \( p \in [1,2) \cup (2, \infty) \), \( L_p[0,1] \) is not a generic \( QSL_p \)-space. For \( p = 2 \), \( \ell_2 \) is obviously the generic \( QSL_2 \)-space, and since \( QSL_{1+} \)-spaces coincide with the class of all Banach spaces, for \( p = 1 \), \( G \) is the generic \( QSL_1 \)-space. This leaves open the next question.
Question 7. For $p \in (1, 2) \cup (2, \infty)$, does there exist a generic $QSL_p$-space in $\mathcal{B}$ or $\mathcal{P}_\infty$?

In Theorem 7.3 we have computed that the class of superreflexive spaces is $F_{\sigma\delta}$. It is easy to check that the class of superreflexive spaces is dense in $\mathcal{P}_\infty$ and $\mathcal{B}$, so it cannot be $G_\delta$ as then this class would have a non-empty intersection with the isometry class of $G$ which is not superreflexive. However, the following is not known to us.

Question 8. Is the class of all superreflexive spaces $F_{\sigma\delta}$-complete in $\mathcal{P}_\infty$ or $\mathcal{B}$?

Taking into account that spaces with summable Szlenk index form a class of spaces which is $\Sigma^0_{\omega+2}$, see Proposition 7.5, the following seems to be an interesting problem.

Question 9. Is the set $\{\mu \in \mathcal{P}_\infty : X_\mu$ has a summable Szlenk index$\}$ of a finite Borel class?

Even though we do not formulate it as a numbered question, a natural project to consider is to determine at least upper bounds for isometry classes of other (classical or less classical) separable infinite-dimensional Banach spaces, such as $C[0,1]$, $C([0,\alpha])$ with $\alpha$ countable ordinal, Orlicz sequence spaces, Orlicz function spaces, spaces of absolutely continuous functions, Tsirelson’s space, etc.

Kechris in [29, page 189] mentions that there are not known any natural examples of Borel sets from topology or analysis that are $\Pi^0_{\xi}$ or $\Sigma^0_{\xi}$, for $\xi \geq 5$, and not of lower complexity. We think that the area of research investigated in this paper is a good one to find such examples.

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