Strong Griffiths singularities in random systems and their relation to extreme value statistics

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We consider interacting many particle systems with quenched disorder having strong Griffiths singularities, which are characterized by the dynamical exponent, z, such as random quantum systems and exclusion processes. In several d = 1 and d = 2 dimensional problems we have calculated the inverse time-scales, τ−1, in finite samples of linear size, L, either exactly or numerically. In all cases, having a discrete symmetry, the distribution function, P(τ−1, L), is found to depend on the variable, u = τ−1Lz/d, and to be universal given by the limit distribution of extremes of independent and identically distributed random numbers. This finding is explained in the framework of a strong disorder renormalization group approach when, after fast degrees of freedom are decimated out the system is transformed into a set of non-interacting localized excitations. The Fréchet distribution of P(τ−1, L) is expected to hold for all random systems having a strong disorder fixed point, in which the Griffiths singularities are dominated by disorder fluctuations.

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I. INTRODUCTION

In interacting many-particle systems quenched (i.e. time independent) disorder can induce unusual physical properties which do not exist in non-random systems. For example in a random ferromagnet the linear susceptibility, χ, is divergent in an extended part of the paramagnetic phase, which is called Griffiths phase[1]. This type of singular behavior is caused by rare regions of the sample, in which due to strong disorder fluctuations the local couplings are much stronger than their average and they are even stronger than the disordered field, such as the temperature in a classical system. Consequently these rare regions are locally in the ferromagnetic phase and the relaxation process associated to them involves a very large relaxation time, τ. In the thermodynamic limit the distribution of the τ-s has no upper limit which leads to singularities in several average physical quantities. One can say that these Griffiths singularities are controlled by a line of semi-critical fixed points, having a diverging relaxation time but a finite correlation length, ξ. In a classical disordered system in which the phase transition is triggered by the variation of the temperature these singularities are very weak[2] and often one needs astronomical times to be able to observe them. There are, however, another type of random systems in which the Griffiths effects are much stronger and one can measure them in conventional experiments.

Strong Griffiths effects can be observed, among others, in random quantum systems[3] at zero temperature, T = 0, in which a quantum phase transition can take place by varying a control parameter, δ. The prototype of random quantum systems is the random transverse-field Ising model (RTIM), which is defined in one dimension (1d) by the Hamiltonian,

$$ H = - \sum_{i=1}^{L-1} \lambda_i \sigma_i^x \sigma_{i+1}^z - \sum_{i=1}^{L} h_i \sigma_i^z, $$

(1)

in terms of the Pauli matrices, $\sigma_i^{x,z}$, at site i. The exchange couplings, $\lambda_i$, and the transverse fields, $h_i$, are both independent and identically distributed (iid) random variables and the control parameter is defined[4] as:

$$ \delta = \frac{[\ln h]_{av} - [\ln \lambda]_{av}}{\text{var}[\ln \lambda] + \text{var}[\ln h]}, $$

(2)

Here and in the following we use $[\ldots]_{av}$ to denote averaging over quenched disorder and $\text{var}[x]$ stands for the variance of x. The system is in the ferromagnetic (paramagnetic) phase for $\delta < 0$ ($\delta > 0$) and the quantum critical point is located at $\delta = 0$. It was first McCoy[5] who calculated exactly the average susceptibility of this model and found that $[\chi]_{av}$ is divergent in a finite regime of the paramagnetic phase. Later the excitation energy, $\epsilon \sim 1/\tau$, is shown[6] to vanish with the size of the system as:

$$ \epsilon \sim L^{-z}, $$

(3)

with the dynamical exponent, $z > 0$, which is a continuous function of $\delta$.

Another class of problems with strong Griffiths effects are stochastic many-particle systems[7] with quenched

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disorder \( \sigma \), such as the 1d partially asymmetric simple exclusion process (PASEP) with position or particle dependent hopping rates \( p_i \) or the zero range process (ZRP) with disorder. Also reaction-diffusion models with quenched disorder might show strong Griffiths effects, see, however, [13, 14]. As an example we consider here the PASEP with particle-wise (pw) disorder, in which the \( i \)-the particle, \( i = 1, 2, \ldots, N \), can hop to empty neighboring sites with a rate \( p_i \) to the right and \( q_i \) to the left in a 1d periodic lattice with \( L > N \) sites. Here the hop rates are \( \text{iid} \) random numbers and the control parameter is defined as:

\[
\delta_p = \frac{\langle \ln p \rangle_{\text{av}} - \langle \ln q \rangle_{\text{av}}}{\text{var}[\ln p] + \text{var}[\ln q]},
\]

so that the particles move to the right (to the left) for \( \delta_p > 0 \) (\( \delta_p < 0 \)). Here we restrict ourselves to the domain, \( \delta_p \geq 0 \). Strong Griffiths effects in the PASEP are caused by such rare particle clusters in which the local hopping rates of the particles prefer jumps against the global bias. As a result in the anomalous diffusion regime, which can be called as the Griffiths phase of the PASEP, the average displacement of particles grows in time as

\[
\langle x(t) \rangle \sim t^{1/(1 + z_p)},
\]

when the system undergoes a coarsening process. Here, the dynamical exponent of the PASEP, \( z_p \), is a continuous function of \( \delta_p \). Note that outside the Griffiths phase, \( \delta > \delta_G \), the stationary velocity of the particles is \( v = O(1) \), whereas in the Griffiths phase it vanishes as \( \delta = \delta_G \):

\[
v \sim L^{-z_p}.
\]

In general Griffiths singularities are characterized by the distribution of the relaxation times of slow processes or equivalently by the distribution of their inverse, \( 1/\tau \), which is just the excitation energy, \( \epsilon \), of the RTIM and the stationary velocity, \( v \), of the PASEP. In a finite system \( 1/\tau \) is just the smallest event associated to one of the rare regions and the occurrence of these rare regions as well as the distribution of \( 1/\tau \) depends on the form of the randomness in the dynamical model. In the mathematical literature such questions are studied in the frame of extreme value statistics (EVS) and rigorous results are known for \( \text{iid} \) random variables, which are distributed according to a parent distribution, \( \pi(y) (y \equiv 1/\tau) \). In this case the limit distributions are in a few different universal scaling forms depending on the asymptotic behavior of the parent distribution for large values of \( y \). If the decay of \( \pi(y) \) is faster than any power then it is the Gumbel distribution, for power-law tails it is the Fréchet distribution and for power-laws with an edge the Weibull distribution is the result. If the random variables are not \( \text{iid} \) there are no general mathematical results, however a few special cases have been studied and solved. Among others we can mention the statistics of extreme intensities of Gaussian interfaces which generally does not coincide with the known limit distributions of \( \text{iid} \) variables. Similarly, for hierarchically correlated variables the limit distribution deviates from the Gumbel form.

As we described above strong Griffiths affects are present in interacting particle systems having strong temporal and spatial correlations, therefore at first thought it seems unlikely that a simple connection with the theory of EVS exists. A closer look, however, shows that in a class of systems the rare events, which are the source of Griffiths singularities, are well localized and separated in such a way that they could be considered independent. For these systems then the distribution function, \( P_L(1/\tau) \), could correspond to the appropriate limit distribution of EVS.

In this paper we are going to study this issue in more details. For some random 1d models having a small correlation length we perform exact calculations. On the other hand at a general point of the Griffiths phase a renormalization group (RG) approach is applied, which is expected to give asymptotically exact dynamical singularities in the Griffiths phase. The RG equations can be analytically solved for some 1d random models, such as for the RFIM and for the PASEP, in other cases and in higher dimensions one resort to numerical calculations. For some 1d random models we also calculate numerically the different distribution functions.

The structure of the paper is the following. In Sec. II we consider Griffiths singularities in random quantum systems. In more details we study the 1d RTIM, but also the random quantum Potts chain, as well as the RTIM for ladders and in 2d and the random Heisenberg model in 1d and 2d are investigated. In Sec. III we study Griffiths singularities in random stochastic systems, such as in the PASEP with particle-wise and site-wise disorder. Our paper is closed by a discussion in Sec. IV.

II. GRIFFITHS SINGULARITIES IN RANDOM QUANTUM SYSTEMS

A. The RTIM in 1d

1. Exact results for extreme disorder

In order to gain some insight into the origin of Griffiths singularities we solve exactly the RTIM in Eq. (1) using the bimodal distribution: \( \lambda_i = \lambda \) with probability \( c \) and \( \lambda_i = \lambda^{-1} \) with probability \( 1 - c \) and \( h_i = 1 \). Furthermore we take the limit \( c \ll 1 \), when most of the couplings are weak and \( \lambda \gg 1 \), so that we are deeply in the paramagnetic phase. There are, however, rare regions in which all the couplings, say \( n \), are strong and appear with an exponentially small density, \( \rho(n) = e^n \). Such a cluster is typically embedded into a see of very weak couplings, thus the corresponding excitation energy is obtained by solving the gap in a cluster with free boundary conditions, leading to: \( \epsilon(n) \approx \lambda^{-n} \). Now we obtain that the distribution of the excitation energies in an infinite sys-
tem has a power-law tail:

\[ P(\epsilon) \approx \frac{1}{\ln \lambda} \epsilon^\omega, \quad \epsilon \to 0, \]  

and the gap exponent, \( \omega \), is given by:

\[ \omega = \frac{\ln(1/\epsilon)}{\ln \lambda} - 1. \]  

In a finite system of size, \( L \), the typical size of the largest cluster, \( n_1 \), is given by the condition, \( L \sum_{n \geq n_1} \rho(n) = 1 \), so that the typical value of the smallest gap, \( \epsilon_1 \), is given by: \( \epsilon_1 = L^{-z} \), with the dynamical exponent defined in Eq. (6):

\[ z = \frac{\ln \lambda}{\ln(1/\epsilon)}. \]  

Note that we obtain the relation, \( \omega = 1/z - 1 \), so that the distribution function for finite systems satisfies the scaling form (10):

\[ P_L(\epsilon_1) = L^z \tilde{P}_1(\epsilon_1 L^z), \]  

and the subscript of the scaling function, \( \tilde{P}_1(u) \), refers to the first gap. In the following we determine the scaling function. First, we note that the rare regions in the system are localized thus they could be placed at \( \sim L \) different positions of the chain. Furthermore the different rare regions are independent and their lengths, which are proportional to the log-gap, is distributed by the same exponential distribution. From these follows that \( \epsilon_1 \) is the smallest gap out of \( \sim L \) independent possible gaps, associated to the different rare regions, and the gaps are distributed by the same parent distribution given in a power-law form in Eq. (11). Consequently in the large-\( L \) limit the distribution function, \( \tilde{P}_1(u) \), is given by the standard Fréchet distribution (12):

\[ \tilde{P}_1(u) = \frac{1}{z} u^{1/z-1} \exp(-u^{1/z}), \]  

and \( u = u_0 \epsilon_1 L^z \), where the non-universal constant, \( u_0 \), depends on the amplitude of the tail in Eq. (7).

2. Analytical results of the strong disorder RG

The strong disorder RG method has been introduced by Ma and Dasgupta to study random antiferromagnetic Heisenberg chains. For the 1d RTIM Fisher has applied the method and solved analytically the RG equations in the vicinity of the random quantum critical point. During renormalization the strongest local term in the Hamiltonian, coupling or transverse field, is decimated out and between remaining sites new interactions are generated through perturbation calculation. If a strong bond, say \( A_2 \), connecting sites 1 and 2 is decimated out the effective transverse field, \( h \), acting on the two-site cluster is given by (4):

\[ h = \frac{h_1 h_2}{\kappa \lambda_2}, \]  

where \( \kappa = 1 \). On the contrary if a strong transverse field, say \( h_2 \) and thus the site 2 is decimated out the effective coupling, \( \tilde{\lambda} \), connecting the remaining sites, 1 and 3 is given by (4):

\[ \tilde{\lambda} = \frac{\lambda_1 \lambda_2}{nh_2}, \]  

with \( \kappa = 1 \). As renormalization goes on the energy scale, set by the strength of the strongest local term in the Hamiltonian, gradually decreases and the distributions of the renormalized couplings and transverse fields approach their fixed-point form. At the critical point couplings and transverse fields are decimated symmetrically and their fixed-points distribution functions are identical and have the so called infinite disorder property (4): the ratio of any two terms tends either to zero or to infinity. Consequently the decimation rules become exact and the critical singularities, both static and dynamical ones are asymptotically exact. In the disordered Griffiths phase, which is a part of the paramagnetic phase after a starting period of the RG when spin clusters of typical size, \( \xi \), are formed almost exclusively transverse fields are decimated out. At the fixed point the energy scale still goes to zero and the decimation equations are asymptotically exact leading to exact dynamical singularities, however the spatial correlations are short-ranged and correct only up to a range of \( \xi \). The analytical solution of the RG equations by Fisher is exact up to \( O(\delta) \) which is then extended into the complete Griffiths phase and (21) (22). The dynamical exponent, \( z \), is found to be the positive root of the equation (22):

\[ \left[ \left( \frac{\lambda}{T} \right)^{1/z} \right]_{\text{av}} = 1, \]  

which has also been derived by a mapping with the random random walk. For the binary disorder used in the previous section this leads to: \( \lambda^{1/z} + \lambda^{-1/z} (1 - c) = 1 \), having the solution in Eq. (9) in the limit \( c \ll 1 \). Note that this result is valid for any value of \( \lambda > 1 \), which is connected to the fact that the gap of a large \( n \)-bond cluster in a see of weaker bonds scales as: \( \epsilon(n) \sim \lambda^{-n} \). The dynamical exponent in Eq. (13) is a continuous function of \( \delta \), and in the vicinity of the critical point it is divergent as \( z \approx 1/2\delta \). It can be shown by the RG calculation that the low temperature behavior of the average susceptibility and that of the specific heat, \( c_V \), are given by (4) (22):

\[ \chi(T) \sim T^{-1+1/z}, \quad c_V(T) \sim T^{1/z}, \]  

thus the singularity involves the dynamical exponent. Note that at zero temperature \( \chi(0) \) is divergent for \( z > 1 \). The distribution of the smallest energy gap in a finite chain of length, \( L \gg \xi \), in the vicinity of the critical point: \( 0 < \delta \ll 1 \) is calculated by Fisher and Young. For the distribution of the log-gap, \( G = -\ln \epsilon_1 \), they obtained:

\[ d\text{Prob}(G|L) \approx 2\delta L n_G \exp(-L n_G) dG, \]  

where \( n_G \approx 1.58 \) and the distribution of critical points, \( n_1 \), is given by (21) (22). The typical value of the smallest gap, \( \epsilon_1 \), is given by (6):
where

\[ n_G \approx 4\delta^2 \exp(-2\delta G) \]  

(17)

is the density of remaining clusters. Here making use of the fact that in the vicinity of the critical point the correlation length is given by \[ \xi \sim \delta^{-2} \] and \[ z \approx 1/2\delta \] we obtain:

\[ n_G \sim \frac{1}{z} \xi^{1/z}. \]  

(18)

Inserting Eq. (18) into Eq. (16) we obtain that the scaling variable is \[ u = u_0\epsilon_1(L/\xi)^z \], in terms of which the distribution is in the Fréchet form in Eq. (11).

3. Phenomenological considerations

Results of the previous two subsections indicate that the distribution of the smallest gap of the RTIM in 1d is the Fréchet distribution, at least in the limiting cases i) extreme binary disorder with \( \xi \ll 1 \) but \( z \) is arbitrary (Sec. IIIA 1) ii) arbitrary form of disorder, but \( \delta \ll 1 \) (Sec. IIIA 2). Here we argue that the Fréchet form of the distribution should be generally valid in the disordered Griffiths phase, at least if the low-energy excitations in the system are localized. Here we note that performing the RG up to an energy scale, \( \Omega_0 \), which is much smaller than \( \Omega_\xi \sim \xi^{-z} \), we obtain an equivalent random spin chain with very weakly interacting effective spins. Typically, the log-fields are in the order of \( -\ln \lambda \sim \Omega_0^{-1/z} \), so they are small compared to the value of effective fields \( h\Omega_0 \).

Consequently, in the following RG steps almost exclusively transverse-fields are decimated out through the transformation in Eq. (19), which does not influence the transverse fields of the other active spins. At the energy scale, \( \Omega_0 \), the effective transverse fields, \( \tilde{h} \), have a power-law distribution \( P(h) \sim h^{-1+z}/z \), and the smallest one gives thus the smallest gap of the chain. Consequently the renormalized chain satisfies the conditions needed for the validity of the Fréchet limit distribution. Generally, for the \( n \)-th excitation energy the limit distribution is given by:

\[ \tilde{P}_n(u_n) = \frac{1}{z}u_n^{n/z-1}\exp(-u_n^{1/z}), \]  

(19)

in terms of \( u_n = u_0\epsilon_1(L/\xi)^z \). We note that the gap exponent is now \( \omega_n = n/z - 1 \), which has been obtained before through scaling considerations.

This type of reasoning for the smallest gap applies to another random systems, too, provided the decimation rules are analogous to those in Eqs. (12) and (13). In d-dimensions the only difference is that \( z \) is replaced by \( z/d \), provided the strong disorder RG approach leads to localized, non-interacting effective degrees of freedom.

Next, the distribution of excitation energies is analyzed in the ordered phase of the RTIM. In 1d due to duality the disordered and ordered phases are related to each other and in the RG procedure the role of the couplings and the transverse-fields are exchanged. Consequently, at the energy-scale, \( \Omega_0 \ll \Omega_\xi \), the typical log-fields are \( -\ln \tilde{h} \sim \Omega_0^{-1/z} \), whereas \( \lambda \sim \Omega_\xi \). Thus the renormalized chain is a classical random bond Ising chain with very small effective transverse fields having an exponentially vanishing first gap. The dynamical properties of the chain are governed by kink-like excitations. In an open chain the lowest excitation is one kink, which corresponds through duality to a spin flip in the disordered phase. Consequently the distribution of the second gap in an open chain follows the Fréchet distribution. On the contrary for a closed chain the lowest excitation involves two kinks and the corresponding excitation energy is the sum of the first two smallest effective couplings of the renormalized chain. Consequently the distribution of the second gap in a closed chain is related to the solution of the above extreme value problem. In higher dimensions the singularities in the ordered Griffiths phase have a more complicated structure. Here the renormalized system is a random bond Ising model with a complicated topology, and the (second) excitation, which is relevant in the dynamical properties corresponds to the creation of an oppositely magnetized domain. The excitation energy is just the sum of the effective couplings at the boundary of the domain.

In the following we check the above conjectures by numerical calculations.

4. Distribution of gaps from numerical diagonalization

The RTIM in 1d can be transformed into a problem of free fermions and the calculation of the gaps necessitates the diagonalization of a \( 2L \times 2L \) tridiagonal matrix the entries of which are the couplings and the transverse fields. In the numerical calculation we used a continuous, uniform distribution: \( \pi(\lambda) = 1 \) for \( 0 < \lambda \leq 1 \) and 0 otherwise; as well as \( \pi(h) = 1/h_0 \), for \( 0 < h < h_0 \) and 0 otherwise. The critical point is located at \( h_0 = 1 \), the disordered Griffiths-phase is in the region \( 1 < h_0 < \infty \), and the dynamical exponent is given by the solution of the equation: \( z\ln(1-z^{-2}) = -\ln h_0 \). We have calculated the first gaps for finite systems with \( L = 64, 128 \) and 256 and for two values of \( h_0 = 2 \) and 3. The probability distribution of the first gaps are shown in Fig. 1 which all fit well to the Fréchet distribution.

The same conclusion is obtained with the distribution of the second gap, \( \epsilon_2 \), which is presented in Fig. 2.

B. Numerical RG study of chains, ladders and higher dimensional systems

For more general models, in particular for a topology which is more complex than a linear chain to calculate the gaps one resorts to numerical implementation of the
1. Random quantum Potts chain

A simple generalization of the RTIM for $q$-state spin variables: $|s_i\rangle = |1\rangle = |2\rangle = \ldots |q\rangle$ is the random quantum Potts model defined in 1d by the Hamiltonian:

$$H_P = -\sum_{i=1}^{L-1} \lambda_i \delta(s_i, s_{i+1}) - \sum_{i=1}^{L} \frac{h_i}{q} \sum_{k=1}^{q-1} M_{ik},$$  \hspace{1cm} (20)$$

where: $M_{ik} |s_i\rangle = |s_i + 1, \text{mod } q\rangle$. The control parameter of the model is in the same form as for the RTIM in Eq.(2). The strong disorder RG approach is used for this model in Ref.[30] for the quantum critical point whereas in Ref.[21] for the Griffiths phase. The transformation rules for the bonds and external fields are of the form given in Eqs.(13) and (12) with $\kappa = 2/q$, thus the phenomenological argumentation directly apply here. The only difference comparing with the RTIM is the degeneracy of the excited states. The distribution of the first gap is thus still of Fréchet type, and gaps between subsequent multiplets behave as the higher excitations of the RTIM.

For the distribution of the gaps in the ordered Griffiths phase one can obtain similar conclusions as described in Sec. II A 3.

In the numerical application of the RG technique we have calculated the distribution of the gaps at a large finite system, $L = 2048$, for the uniform distribution with $h_0 = 3$ but for different values of $q = 2, 3, 4$ and 8. The dynamical exponent is $q$ dependent and calculated by a numerical integration of the analytical RG equations [32]. In Fig.3 the numerically obtained gap distributions are compared with the Fréchet distribution and excellent agreement is found.

strong disorder RG method. This type of method is first used in Ref.[28] and implemented for a finite system in Ref.[21]. In the finite lattice version we use here the decimation procedure is performed up to the last spin and the gap of the system is identified by the effective transverse field acting on that spin. By this procedure the first few RG steps are approximative, which will influence the value of the non-universal constant, $u_0$, but the later transformation steps close to the fixed point are presumably asymptotically exact. In the numerical calculations we used the uniform distribution and considered at least 10000 independent realizations of disorder.

FIG. 1: Scaling plot of the distribution of the first gap of the 1d RTIM for different sizes, $L = 64 (+, \square), 128 (\times, O)$ and 256 ($\ast, ■$) at two values of the uniform disorder. The full lines are the Fréchet distributions having the exact values of the dynamical exponent, $z$.

FIG. 2: The same as in Fig.1 for the second gap.

FIG. 3: Distribution of the log-gaps, identified as the transverse field of the last remaining spin in the RG procedure, for the the random quantum Potts chain with different values of $q$ having a uniform distribution with $h_0 = 3$. The Fréchet distributions are indicated by full lines for which the dynamical exponent is obtained by the solution of the analytical RG equations.
2. RTIM: ladder and 2d system

Here we consider the RTIM with a more complex topology, first a ladder composed of two chains and afterwards a large plaquette of a square lattice.

In Fig. 4 the distribution functions of the log-gaps of the ladder model with $h_0 = 2.5$ are presented in a log-log scale for different lengths up to $L = 1024$. The curves for different $L$-s are shifted to each other and a good scaling collapse can be obtained by using the scaling combination in Eq. (11) with a dynamical exponent, $z = 2.9$, as illustrated in the inset of Fig. 4. The (absolute value of the) asymptotic slope of the curves for small gaps is given by $\omega + 1$, which for localized excitations should be related to the dynamical exponent as:

$$\omega + 1 = \frac{d}{z}, \quad (21)$$

for a $d$-dimensional system. Here we obtained $\omega + 1 = 0.34$ so that the relation in Eq. (21) with $d = 1$ is very well satisfied. Finally, the scaled curve in the inset of Fig. 4 is very well described by the Fréchet distribution.

![FIG. 4: (Color online) Distribution of the log-gaps, identified as the transverse field of the last remaining spin in the RG procedure, for the ladder model with $h_0 = 2.5$ and for different lengths. In the inset a scaling collapse according to Eq. (11) with $z = 2.9$ is presented. The Fréchet distributions are indicated by full lines. The asymptotic slope of the curves is indicated by a dashed line and is given by $\omega + 1 = 0.34$, so that the relation in Eq. (21) is well satisfied.](image)

Results of the same analysis of the gaps of the square-lattice RTIM for $h_0 = 9$. are presented in Fig. 5 for $L = 32, 64$ and 128. Here the scaling collapse in the inset is obtained with $z = 2.7$, whereas the gap exponent is given by $\omega + 1 = 0.74$, so that the localization condition in Eq. (21) is satisfied. Also the Fréchet distribution fits very well the scaled curves.

![FIG. 5: (Color online) The same as in Fig. 4 for the square lattice RTIM with $h_0 = 9$ and for different sizes. In the inset the optimal scaling collapse is obtained with $z = 2.7$. The asymptotic slope of the curves is given by $\omega + 1 = 0.74$, so that the relation in Eq. (21) is well satisfied.](image)

3. Random Heisenberg models in 1d and 2d

The random quantum models studied so far have a discrete symmetry. In this section we are going to consider random Heisenberg models which have continuous symmetry and the renormalization procedure and the corresponding fixed points are somewhat different than for discrete symmetry models. To be specific the Heisenberg models we study here are defined by the Hamiltonian:

$$H_H = \sum_{i,j} J_{i,j} t_i t_j \vec{S}_i \cdot \vec{S}_j, \quad (22)$$

in terms of spin-$1/2$ variables. $\vec{S}_i$ at site, $i$, and the dilution variables: $t_i = 0$ with probability, $p$, and $t_i = 1$, otherwise. Here we consider two different types of models. i) For the non-diluted models with $p = 0$, the random couplings are both ferromagnetic and antiferromagnetic and their average is $[J]_{av} = 0$. ii) For the diluted models we have $0 < p < 1$ and the random couplings are only antiferromagnetic, $J_{i,j} > 0$.

We start with non-diluted models in 1d and 2d and calculate the smallest gap in the system by the numerical application of the RG method. As described in detail in Ref. 33 for these models the RG scales into a so called large spin fixed point [34], having an effective moment, $S_{eff} \sim L^{\zeta}$, with $\zeta \approx 1/2$. Due to the formation of large spins the low-temperature singularities of these models are also different than that of the RTIM, for example the average susceptibility has a Curie-like behavior.

The distribution of the log-gaps for the 1d non-diluted model is shown in Fig. 4 for different sizes, and in the inset the scaled curves are presented with $z = 3.3$. There are considerable deviations from the Fréchet distribution, in particular for small gaps. The relation in Eq. (21) is not satisfied, so that the excitations are non-localized.

The distribution of the log-gaps for the 2d system is
shown in Fig. 7 for three sizes, $L = 16$, 32 and 64 and in the inset the scaled curves are presented with $z = 2$. The Fréchet distribution appears to give a correct description and also the relation in Eq. (21) is satisfied. We note that in 2d there is frustration in this model, thus it is called as a spin glass.

Next, we consider diluted models on the square lattice for which the couplings are antiferromagnetic and distributed uniformly, $0 < J_{i,j} < 1$. For the dilution, $p$, we consider two values. In Fig. 8 we present the distribution of the gaps at $p = 0.125$, which is below the percolation threshold, whereas in Fig. we consider the dilution at $p = 0.45$, above the percolation threshold, when the system is broken into non-interacting finite clusters.

In the first case during renormalization the system scales into a large spin fixed point. As seen in Fig. 8 the distribution of the calculated gaps differs considerably from the Fréchet distribution and also the relation in Eq. (21) is not correctly satisfied.

For $p = 0.45$ the system is separated into independent parts and the smallest gap of the system is just the smallest gap of these clusters. Since the energy gaps of the clusters are expected to be distributed in identical power-law form, the applicability of the EVS is probable. Indeed, in Fig. 8 the numerically calculated gap distributions can be well described by the Fréchet form and also Eq. (21) is satisfied.

III. GRIFFITHS SINGULARITIES IN RANDOM STOCHASTIC SYSTEMS

A. PASEP with extreme disorder

First, we consider the PASEP with $p_w$ disorder as defined above Eq. (4) with a special form of the bimodal disorder, when the particles are of two kinds. For a fraction of $c \ll 1$ black particles the hopping rates are: $p_1 = 1$ and $q_1 = \lambda$, whereas the $(1-c)$ fraction of white particles have hopping rates: $p_2 = 1$ and $q_2 = \lambda^{-1}$. In order to obtain exact results we take the limit $\lambda \gg 1$. With this type of disorder the drift of the white particles to the right is slowed down by the black ones and the slowing down is more effective if two or more black particles happen
to stay behind each other. A cluster of $n$ black particles has a very small density: $\rho(n) = c^n$ and its speed to the right, in the absence of another black particles and clusters is given by, $v(n) \approx \lambda^{-n}$. Indeed in order to make the complete black cluster one step to the right all the $n$ particles should perform their hop to the unpreferential direction at the same timestep. At this point it is easy to notice the isomorphism of this problem with the RTIM in Sec. III A with the correspondence: $\epsilon \leftrightarrow v$. Thus we can immediately write for the distribution of the effective velocities of the black clusters:

$$P(v) = \frac{1}{\ln \lambda} c^v, \quad v \to 0,$$

with the exponent, $\omega$, defined in Eq. (3).

Evidently, the stationary velocity of the PASEP is given by the smallest effective speed of the clusters, $v_1$, and all the particles move behind that slowest black cluster. In a large finite system with a finite density of particles, the stationary velocity goes to zero as given in Eq. (3) with a dynamical exponent defined in Eq. (3). Consequently in a finite system the distribution of $v_1$ is given in the scaling form in Eq. (14), and the distribution function of the scaling variable: $u = u_0 v_1 L^z \gamma$, is given by the Fréchet distribution in Eq. (14).

The results obtained in this section can be easily generalized for the PASEP with site-wise (sw) disorder, in which case the hop rates depend on the position: for the site $i$ they are $p_i$ (right) and $q_i$ (left). The control parameter of the model is the same as for pw disorder in Eq. (14). For the extreme binary disorder considered above most of the sites are white and promote the movement of the particles to the right, but the few black sites and in particular the rare black clusters form barriers, which slow down the particle motion. This problem is studied in more details in Ref. [3]. Here we just note that the average velocity of a particle which goes through a single large barrier of size, $n$, is given by $\bar{v} \sim n^{-\lambda - 1/2}$, since due to particle-hole symmetry in the stationary state the barrier is filled up to $n/2$, so that the particles should make $n/2$ consecutive steps against the barrier. As a consequence the derivation in the previous paragraphs for pw disorder should by slightly modified, which leads to a dynamical exponent, $z_s$, given by:

$$z_s = \frac{z_p}{2}.$$  \hspace{1cm} (24)

In particular the distribution of the stationary velocity in a finite system is still given by the Fréchet form in terms of the scaling variable: $u = u_0 v_1 L^z \gamma$.

**B. Strong disorder RG and scaling results**

Here we show that the results in the previous section, i.e the relation between Griffiths singularities and EVS holds for a general form of disorder, too. We start with the PASEP with pw disorder for which a variant of the strong disorder RG approach has been applied [3]. For this model during renormalization the fastest hop rates are consecutively decimated out and new clusters of particles are created with effective hop rates obtained by a perturbation calculation. Without going to the details we mention that there is a one-to-one correspondence between the RG rules for the RTIM and that of the PASEP. In the Griffiths phase for $\delta_p > 0$ in the first part of the renormalization both left and right hop rates are decimated out until particle clusters of typical size, $\xi$, are created. (Here $\xi$ is the average correlation length, which behaves for small $\delta_p$ as $\xi \sim \delta_p^{-2}$.) At this scale of the RG the remaining effective particles have a practically vanishing left hop rate, $q_0 \approx 0$, and finite effective right hop rates, which have an asymptotic power-law distribution: $P(\tilde{p}) \sim \tilde{p}^{-1+1/z_p}$. Here the dynamical exponent, $z_p$, is given through the equation:

$$\left[ \left( \frac{q}{p} \right)^{1/z_p} \right]_{av} = 1,$$  \hspace{1cm} (25)

which makes the analogy with the RTIM complete, see in Eq. (14). The renormalized PASEP than consists of independent particles the velocity (right hop rate) of them has the same power-law distribution and the smallest of them is the stationary velocity of the finite system. Consequently the conditions of EVS are asymptotically satisfied, so that the distribution of the stationary velocity in a finite system is given by the Fréchet form in terms of the scaling variable: $u = u_0 v_1 (L/\xi)^z \gamma$.

For the PASEP with sw disorder one can not simply apply the strong disorder RG, so that we use here phenomenological, scaling considerations. As noticed in Sec. III A the rare regions are represented by large barriers, which in the length-scale, $\xi$, are expected to be
independent and well separated from each other. Scaling consideration in Ref. 8 show that the distribution of the velocities associated to large barriers is given by, \( P(v) \sim v^{-1+1/\nu} \), and for the dynamical exponent, \( \nu_z \), the relation in Eq. (24) holds in the entire Griffiths phase. Once more the stationary velocity in a finite system is given by the smallest velocity associated to the largest barrier, and its distribution is expected to be in the Fréchet form in terms of the scaling variable: \( u = u_0 v_1 (L/\xi)^z \).

Finally, we consider the 1d zero-range process (ZRP) with quenched disorder\[11\]. In this model the \( i \)-th lattice site can be occupied by \( n_i \geq 0 \) particles, from which the topmost one can hop to nearest neighbor sites with a position dependent rate: \( q_i \) to site \( i+1 \) and \( p_{i-1} \) to site \( i-1 \). It is known that the ZRP with this definition can be exactly mapped (up to translations of the configurations of the lattice) to the PASEP with \( pw \) disorder, as studied here. The sites of the ZRP are particles in the ASEP and the particle clusters in the ZRP correspond to holes in front of the particles in the PASEP. Then the stationary current in the ZRP is just the stationary particle velocity of the PASEP. From this mapping and the previous reasoning follows that the stationary current of the ZRP, \( J \), in a finite system scales as: \( J \sim L^{-\nu} \), and its distribution is the Fréchet distribution.

C. Numerical results

We start to analyze the distribution of the stationary velocity, \( v \), of the PASEP with uniform \( pw \) disorder, which is presented in Fig. 10. Here we have evaluated an exact expression for \( v \), which is described c.f. in Ref. 8 and in this way we have studied chains with up to \( N = 2048 \) particles over 100000 independent realizations of the disorder. Fig. 10 shows an excellent agreement with the Fréchet distribution having the exact dynamical exponent in Eq. (25).

For the PASEP with \( sw \) disorder there is no analytical expression for the stationary velocity so that \( v \) is calculated by simulation. We have considered 10000 random half-filled chains with binary disorder of different lengths up to \( L = 512 \). The results as presented in Fig. 11 are in good agreement with the Fréchet distribution, in which the exact dynamical exponent is taken from the scaling result in Eqs. (24) and (25).

IV. DISCUSSION

In this paper we have considered strong Griffiths effects in different interacting many particle systems and studied their possible relation with extreme value statistics. Our examples included random quantum systems in one and two dimensions as well as stochastic systems with quenched disorder in one dimension. Our exact, numerical and RG results indicate that for systems having a discrete symmetry the distribution of the inverse time scales (excitation energy for quantum systems, stationary velocity for exclusion models) in large finite samples has a universal form, which is the limit distribution of the extremes of \( iid \) random numbers. In these examples the Griffiths singularities are characterized by the dynamical exponent, \( z \), which is a continuous function of the control parameter, \( \delta \). However the distribution function depends on, \( u = L^{2/d_x} \), and given by the Fréchet distribution in Eq. (11). The physical picture behind this result is given by the strong disorder RG method: during renormalization fast degrees of freedom are gradually decimated out and the system finally transforms into a set of practically independent degrees of freedom. The characteristic time-scales of these localized units follow...
a power-law distribution with a $z$-dependent decay exponent. Since the physically relevant relaxation time is given by the largest time-scale we arrive to the results of EVS.

The universality of the distribution function of different problems is tested by numerical and RG calculations. For models with a discrete symmetry in all cases we obtained convincing evidence of universality. On the contrary for the random Heisenberg model, which has a continuous symmetry, the distribution function is found to depend on the specific form of the disorder. Interestingly, for the 2d spin-glass model as well as for the diluted 2d model above the percolation threshold the distribution function is found in universal Fréchet form.

One might ask the question how general these results are. On the basis of the RG approach we conjecture that for all interacting systems which have a disordered Griffiths phase the singular properties of which are controlled for all interacting systems which have a disordered Griffiths phase. On the basis of the RG approach we conjecture that function is found in universal Fréchet form.

For models with a discrete symmetry in all cases we obtain problems is tested by numerical and RG calculations.

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