Abundance of independent sequences in compact spaces and Boolean algebras

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Abstract
It follows from a theorem of Rosenthal that a compact space is ccc if and only if its every Eberlein continuous image is metrizable. Motivated by this result, for a class of compact spaces \(C\) we define its orthogonal \(C^\perp\) as the class of all compact spaces for which every continuous image in \(C\) is metrizable. We study how this operation relates classes where centeredness is scarce with classes where it is abundant (like Eberlein and ccc compacta), and also classes where independence is scarce (most notably weakly Radon-Nikodým compacta) with classes where it is abundant. We study these problems for zero-dimensional compact spaces with the aid of Boolean algebras, and show the main difficulties that arise when we deal with general settings. Our main results are the constructions of several relevant examples.

Keywords Boolean algebra · Stone space · Independent sequence · Weakly Radon-Nikodym compact

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1 Introduction

The class of weakly Radon-Nikodým (WRN) compact spaces was introduced by Glasner and Megrelishvili [11] as a natural superclass of the well-studied Radon-Nikodým compacta. A compact space is weakly Radon-Nikodým if it is homeomorphic to a weak∗ compact subset of X∗, where X is a Banach space containing no isomorphic copy of ℓ1. There is a useful purely combinatorial characterization of WRN (see [22]) and it reads as follows.

Theorem 1 A compact space K is WRN if and only if K can be embedded into some cube [0, 1]κ in such a way that for every p < q the family of pairs

$$\{(x \in K : x(\alpha) \leq p), (x \in K : x(\alpha) \geq q)\}_\alpha$$

contains no infinite independent sequence.

Recall that a sequence of pairs of disjoint sets (A0, A1) is said to be independent if Aε1 \cap Aε2 \cap \cdots \cap Aεm \neq \emptyset for every m and any choice of ε1, ..., εm ∈ {0, 1}. Accordingly, a sequence of subsets An of a space X is independent if the pairs (An, X \ An) are independent.

If K is Eberlein compact (that is, K is homeomorphic to a weakly compact subset of some Banach space) then, by the classical Amir-Lindenstrauss theorem, K embeds into some cube [0, 1]κ in such a way that the family of sets \{x \in K : x(\alpha) \geq q\} contains no infinite centered sequence whenever q > 0. Recall that a family of sets is said to be centered if any finite intersection of its elements is nonempty.

From this point of view, there is certain parallelism between Eberlein compacta, in which centered sequences are scarce, and weakly Radon-Nikodým compacta, in which independent sequences are scarce. This analogy can be made more transparent when we confine ourselves to the zero-dimensional case. Let us say that a Boolean algebra A is weakly Radon-Nikodým (Eberlein) if the Stone space of A is weakly Radon-Nikodým compact (Eberlein compact, respectively).

Theorem 2 A Boolean algebra A is weakly Radon-Nikodým if and only if A is generated by a family G ⊂ A which can be written as G = \bigcup_n G_n so that no G_n contains an infinite independent sequence [4, Proposition 3.2].

A Boolean algebra A is Eberlein if and only if A is generated by a family G ⊂ A which can be written as G = \bigcup_n G_n so that no G_n contains an infinite centered sequence [29, Remark 2, p. 107].

In a sense, weakly Radon-Nikodým compacta and Eberlein compacta are closely related to the scarcity of independence or centeredness; how do we describe the abundance of those two properties? In the latter case the answer is the well known world of chain conditions, the most basic of which is the countable chain condition (ccc). A Boolean algebra is ccc if it contains no uncountable pairwise disjoint family, which is equivalent to saying that every uncountable subfamily contains an infinite centered family. A similar definition applies to compact spaces by considering families of open sets. Playing a little with the definitions and using a classical theorem of Rosenthal (see Sect. 3), a kind of duality becomes apparent.

Theorem 3 A Boolean algebra A is ccc if and only if every Eberlein subalgebra of A is countable. A compact space K is ccc if and only if every Eberlein continuous image of K is metrizable.

This inspires a research program around the following items.
(1) There is an analogy between conditions involving independence and conditions involving centeredness. While the abundance of centeredness (chain conditions) is well studied, its parallel world of abundance of independence is not so much.

(2) The abundance and scarcity classes can be studied in duality: Given a class $C$ of one kind, we can create the orthogonal class $C^\perp$ of the other kind. In the topological setting, $C^\perp$ would be the class of compact spaces $K$ whose all continuous images in $C$ are metrizable. In the Boolean setting, the class of Boolean algebras whose all subalgebras in $C$ are countable.

(3) To what extent the Boolean and the topological settings differ? Is it possible to reduce the study of compact spaces in these classes to the case of zero-dimensional compacta through taking subspaces or continuous images?

In this paper we make first steps towards exploring these ideas. Section 2 shows some basic facts about the behavior of the operation of taking orthogonals of a class. In Sect. 3 we stay in the centered world, examining the orthogonal classes there. Using the concept of orthogonality, Theorem 3 above states that $\text{Eberlein}^\perp = \text{ccc}$. Indeed, the orthogonal class of many important classes of compact spaces studied in the Banach Space setting coincides with the class of $\text{ccc}$ compact spaces (see Proposition 9). In general, as we shall see in Sect. 3, one can interpret a large number of (essentially) known results in this manner.

In the subsequent sections we investigate the class of compact spaces $\text{WRN}^\perp$ and the related class $\text{WRN(B)}^\perp$ of Boolean algebras. In particular, in Sect. 4 we provide a simple characterization of Boolean algebras in $\text{WRN(B)}^\perp$, whereas in Sect. 5 we construct, under $MA_{\omega_1}$, an example of a nonmetrizable zero-dimensional Corson compact space whose clopen algebra belongs to $\text{WRN(B)}^\perp$. The difficulties in checking whether this example belongs to $\text{WRN}^\perp$ motivate us to study the relation between $\text{WRN}^\perp$ and $\text{WRN(B)}^\perp$. Section 6 is mainly devoted to the construction of a zero-dimensional nonmetrizable compact space whose clopen algebra belongs to $\text{WRN(B)}^\perp$ but having a nonmetrizable WRN continuous image (so not in $\text{WRN}^\perp$).

One can also consider a natural stronger version of orthogonality, requiring compact spaces to be hereditarily orthogonal to the original class. In the final section we deal with the strong orthogonal of the class of zero-dimensional compacta, previously investigated by Piotr Koszmider. Here the basic problem is whether such an orthogonal class is nontrivial, i.e. if there is a nonmetrizable compactum $K$ such that every continuous image of $K$ contains no nonmetrizable zero-dimensional compact subspaces. Koszmider [17] presented several consistent constructions of such spaces $K$ discussing some additional properties they may have. Building on Koszmider’s ideas, in Sect. 7 we make our modest contribution to the subject showing that, under various set-theoretic assumptions, these examples can be weakly Radon-Nikodým.

## 2 Orthogonal classes of compact spaces

All topological spaces considered in this paper are assumed to be Hausdorff. For a class $C$ of compact spaces we introduce two orthogonal classes as follows.

**Definition 4** (a) $C^\perp$ is the class of those compact spaces $K$ such that every continuous image of $K$ that belongs to $C$ is metrizable;

(b) $C^{(\perp)}$ is the class of those compact spaces $K$ such that every continuous image of any closed subspace of $K$ that belongs to $C$ is metrizable.
Obviously, $C^{(\perp)} \subseteq C^{\perp}$ for every class of compact spaces $C$. Note that if $C \subseteq D$ then $D^{\perp} \subseteq C^{\perp}$. We also record the following general facts.

**Lemma 5** Let $C$ be a class of compact spaces that is closed under continuous images. Then

(i) $C \subseteq C^{\perp\perp}$;

(ii) $C^{\perp} = C^{\perp\perp\perp}$.

**Proof** If $K \notin C^{\perp\perp}$ then $K$ has a nonmetrizable continuous image $L \in C^{\perp}$. Then, in particular, $L \notin C$ so $K \notin C$; this proves (i).

Note that $C^{\perp}$ is automatically closed under continuous images. Hence, $C^{\perp} \subseteq C^{\perp\perp\perp}$ by (i). For the reverse inclusion, apply the previously mentioned fact that $C \subseteq D$ implies $D^{\perp} \subseteq C^{\perp}$ to (i). $\square$

Although $C^{\perp}$ is always closed under continuous images, it might not be closed under subspaces. Nevertheless, $C^{(\perp)}$ is always closed under subspaces and continuous images by the very definition and the following fact, which is analogous to [17, Lemma 4.1].

**Lemma 6** For any class $C$ of compact spaces, $C^{(\perp)}$ coincides with the class of those compact spaces $K$ such that every subspace of every continuous image of $K$ is metrizable whenever it is in $C$.

**Proof** It is immediate that every subspace of any continuous image of $K$ is a continuous image of a subspace of $K$. On the other hand, if $K_1$ is a subspace of $K$ and $f : K_1 \to L$ is a continuous surjection then, assuming that $L$ is embedded in $[0, 1]^\kappa$ for some $\kappa$, by the Tietze extension theorem, there is a continuous extension $\hat{f} : K \to [0, 1]^\kappa$. Consequently, $L$ is a subspace of a continuous image of $K$. $\square$

We finish this section with the following elementary fact.

**Lemma 7** For any class $C$ of compact spaces, $C^{(\perp)}$ is the class of hereditarily $C^{\perp}$ compact spaces, i.e. the class of those compact spaces $K$ such that every closed subspace of $K$ belongs to $C^{\perp}$.

**Proof** If $K \in C^{(\perp)}$ and $L \subseteq K$ is a subspace, then every continuous image of $L$ belonging to $C$ is metrizable, so $L \in C^{\perp}$.

On the other hand, if $K$ is hereditarily $C^{\perp}$ and $L \subseteq K$ is a subspace, then $L \in C^{\perp}$ and therefore any continuous image of $L$ belonging to $C$ is metrizable, so $K \in C^{(\perp)}$. $\square$

In the next sections we study the classes $C^{\perp}$ and $C^{(\perp)}$ for several well known classes of compact spaces $C$. Most often, we focus on the class $C^{\perp}$ and we do not pay too much attention to the class $C^{(\perp)}$. This is due to the fact that, once we have a characterization for the class $C^{\perp}$, the best characterization for the class $C^{(\perp)}$ that we are able to obtain is the one given by a direct application of Lemma 7. Nevertheless, sometimes this characterization for the class $C^{(\perp)}$ is obscure and even not useful enough to easily determine whether this class is nontrivial. This will be the case for $C$ the class of zero-dimensional compacta (see Sect. 7)

## 3 Orthogonal classes and ccc compacta

Rosenthal [27] proved that a compact space $K$ is ccc if and only if every weakly compact subset of the Banach space $C(K)$ is separable. This classical result may be translated to
saying that Eberlein ccc compact spaces are metrizable. Recall that an Eberlein compactum (resp. uniformly Eberlein compactum) is a compact space homeomorphic to a weak compact subset of a Banach space (a Hilbert space, respectively). For all undefined classes of compacta that are mentioned below we refer the reader to Negrepontis [25] or Fabian [7].

We denote classes of compacta by their names written in small capitals so, for example, EBERLEIN (UNIEBERLEIN) is the class of Eberlein compacta (uniform Eberlein compacta, respectively) while CCC stands for all compact ccc spaces. Moreover, we denote by UE(1) the class of ‘singleton-supported compact spaces’, i.e. those compact spaces which embed into \( \{ x \in \mathbb{R}^\kappa : x(\gamma) \neq 0 \text{ for at most one } \gamma < \kappa \} \) for some \( \kappa \). Clearly, every compact space in UE(1) is Uniformly Eberlein (recall Farmaki’s characterization of uniformly Eberlein compacta in [8, Theorem 2.10]).

**Remark 8** The following chain of implications holds

\[
\text{UE}(1) \subset \text{UNIEBERLEIN} \subset \text{EBERLEIN} \subset \text{TALAGRAND} \subset \text{GUL’KO}.
\]

Indeed, Talagrand [30] proved that every Eberlein compact space is Talagrand compact and, motivated by Rosenthal’s Theorem mentioned above, he asked whether every Talagrand or Gul’ko ccc compact space is metrizable (see [31, Problème 5] and [32, Problème 7.9]). These questions were solved positively by Argyros and Negrepontis in [3]. The following may be seen as a summary of these results.

**Proposition 9** Let \( C \) be any of the classes in Remark 8. Then \( C^\perp = \text{CCC} \). As a consequence, \( C^{(\perp)} \) coincides with the class of hereditarily ccc compact spaces.

**Proof** As \( \text{UE}(1) \subset \text{GUL’KO} \), we have \( \text{GUL’KO}^\perp \subseteq \text{UE}(1)^\perp \). Clearly, the class CCC is closed under continuous images. So if a continuous image \( L \) of \( K \in \text{CCC} \) is Gul’ko then \( L \) is metrizable by the result from [3]. Thus, \( \text{CCC} \subseteq \text{GUL’KO}^\perp \).

On the other hand, if \( K \) is not ccc and \( \{ W_\alpha : \alpha < \omega_1 \} \) are pairwise disjoint nonempty open sets, one can take for every \( \alpha \) a nonzero continuous function \( f_\alpha : K \to [0, 1] \) such that \( f_\alpha | K \setminus W_\alpha = 0 \). Then the diagonal mapping maps \( K \) onto a nonmetrizable subset of

\[
\{ x \in \mathbb{R}^{\omega_1} : x(\gamma) \neq 0 \text{ for at most one } \gamma < \omega_1 \},
\]

so \( K \notin \text{UE}(1)^\perp \). This proves that \( \text{CCC} = \text{UE}(1)^\perp = \text{GUL’KO}^\perp \) and the first part of the proposition follows from Remark 8.

The second assertion follows from Lemma 7.

**Corollary 10** \( \text{CCC} = \text{CCC}^{\perp\perp} \).

**Proof** Applying Lemma 5 and Proposition 9

\[
\text{CCC} = \text{EBERLEIN}^\perp = \text{EBERLEIN}^{\perp\perp\perp} = \text{CCC}^{\perp\perp},
\]

and we are done.

A natural class generalizing all the classes mentioned in Remark 8 is the class of Corson compacta, i.e. those compact spaces \( K \) that are homeomorphic to a subspace of a \( \Sigma \)-product

\[
\Sigma(\mathbb{R}^\kappa) := \{ x \in \mathbb{R}^\kappa : x \text{ has countable support} \},
\]

for some cardinal \( \kappa \). As we will recall below, it is consistent that there are ccc Corson compact spaces which are not metrizable; in such a case CORSON\(^\perp\) is a proper subclass of CCC.
In the spirit of Talagrand’s question mentioned above, one may wonder what is the largest class whose orthogonal is $\ccc$. Note that Corollary 10 provides some answer—the largest class whose orthogonal is $\ccc$ is just $\ccc^\perp$. Nevertheless, no handy characterization of $\ccc^\perp$ in terms of popular classes of compact spaces seems to be available.

**Proposition 11** Under Martin’s Axiom $\MA_{\omega_1}$, $\ccc^\perp = \SEPARABLE^\perp$.

**Proof** Clearly, $\ccc^\perp \subseteq \SEPARABLE^\perp$. If $K \not\in \ccc^\perp$ then $K$ has a nonmetrizable $\ccc$ continuous image $L$. It is easy to construct a continuous image $L_1$ of $L$ of weight $\omega_1$. Then $L_1$ is also $\ccc$ and the standard application of Martin’s Axiom implies that $L_1$ is separable. Consequently, $K \not\in \SEPARABLE^\perp$.

Recall that $\omega_1$ is a precaliber of a topological space $X$ if every uncountable family of nonempty open subsets of $X$ contains an uncountable centered subfamily. If we take a compact space $K$ then the notion of a precaliber coincides with that of a caliber: $\omega_1$ is a caliber of $K$ if for every uncountable family $\mathcal{G}$ of nonempty open subsets of $K$, $|\{G \in \mathcal{G} : x \in G\}| \geq \omega_1$ for some $x \in K$ (cf. [5]). Write $\cal{c}(\omega_1)$ for the class of compact spaces having caliber $\omega_1$.

**Remark 12** $\cal{c}(\omega_1) = \ccc$ is equivalent to $\MA_{\omega_1}$. Indeed, the condition $K \in \cal{c}(\omega_1)$ is equivalent to saying that every point-countable family of nonempty open subsets of $K$ is countable; the latter is called Shanin’s condition, see Todorcevic [35].

**Proposition 13** $\CORSON^\perp = \cal{c}(\omega_1)$. Thus, $\CORSON^\perp = \ccc$ is equivalent to $\MA_{\omega_1}$.

**Proof** Note first that if $L$ is nonmetrizable Corson compact then $\omega_1$ is not a caliber of $L$. Indeed, suppose that $L$ is embedded into the $\Sigma$-product $\Sigma(\mathbb{R}^\kappa)$ for some $\kappa$. Then the set $V_\alpha = \{x \in K : x(\alpha) \neq 0\}$ is nonempty for uncountably many $\alpha$’s. On the other hand, no uncountable subfamily of $V_\alpha$’s can have nonempty intersection by the very definition of a $\Sigma$-product.

If $\omega_1$ is a caliber of a compact space $K$ then it is also a caliber of every continuous image $L$ of $K$. Hence such $L$ is either metrizable or not Corson by above.

If $\omega_1$ is not a caliber of $K$ then take a family $\{G_\alpha : \alpha < \omega_1\}$ witnessing that fact. We can of course assume that $G_\alpha \neq K$ for every $\alpha$. If $g_\alpha : K \rightarrow [0, 1]$ is a nonzero continuous function vanishing outside $G_\alpha$ then the diagonal mapping $\Delta_\alpha g_\alpha : K \rightarrow [0, 1]^\omega_1$ maps $K$ onto a nonmetrizable Corson compactum.

**Remark 14** From Proposition 13 it follows that the failure of $\MA_{\omega_1}$ is equivalent to the existence of a nonmetrizable $\ccc$ Corson compact space $K$. It is indeed equivalent to the existence of a zero-dimensional nonmetrizable $\ccc$ Corson compact space $L$. It is a well known elementary fact that any Corson compact space is a continuous image of a zero-dimensional Corson compact space $L$ (cf. for instance [1, Corollary 24]). By restriction to a minimal closed subspace where surjectivity is preserved the surjection $f : L \rightarrow K$ can be supposed to be irreducible: nonempty open sets are mapped onto sets with nonempty interior. From irreducibility it follows that if $K$ is $\ccc$, then $L$ is also $\ccc$. Summarizing, if $K$ is a nonmetrizable $\ccc$ Corson compact space $K$, then $L$ shares the same properties and it is moreover zero-dimensional.

We recall below that Proposition 11 does not hold in ZFC. We have three natural classes of compacta

$$\SEPARABLE \subseteq \SPM \subseteq \CCC,$$
where SPM is the class of compact spaces carrying a strictly positive probability Borel measure. Therefore,

\[ \text{CCC}^\perp \subseteq \text{SPM}^\perp \subseteq \text{SEPARABLE}^\perp. \]

Obviously, every $\aleph_0$-monolithic compact space (i.e. a compactum in which all separable subspaces are metrizable), in particular every Corson compactum, is in $\text{SEPARABLE}^\perp$, since the class of $\aleph_0$-monolithic compacta is stable under continuous images. There is a large number of examples, starting from those constructed by Haydon, Kunen and Talagrand under CH and described in [25], of a nonseparable Corson compact space $K \in \text{spm}$. They indicate that the class $\text{spm} \cap \text{SEPARABLE}^\perp$ is nontrivial (i.e. contains nonmetrizable spaces) under CH. Kunen and van Mill [19] showed that, in this context, CH may be relaxed to saying that the restriction of MA$_{\omega_1}$ to measure algebras fails.

Another result showing that Proposition 11 may be violated is based on classical Gaifman’s example of a ccc compact space admitting no strictly positive measure. The proof below follows closely the description of the Gaifman space given by Comfort and Negrepontis [5,Theorem 6.23] and its modification from [2].

By a Lusin set we mean an uncountable $Z \subset \mathbb{R}$ such that $Z \cap N$ is countable for every nowhere dense $N \subset \mathbb{R}$. Recall that a family $\mathcal{A}$ of subsets of some set $X$ is said to be adequate if $B \subset A \in \mathcal{A}$ always implies $B \in \mathcal{A}$ and $A \in \mathcal{A}$ for every $A \subset X$ satisfying $|A|^{<\omega} \subset \mathcal{A}$. Those two properties imply that an adequate family $\mathcal{A}$ defines a compact subset of $2^X$ in a natural way.

Given a compact space $K$, we write $P(K)$ for the set of regular probability Borel measures on $K$. In section 6, we consider $P(K)$ as a topological space. Recall that $P(K)$ is compact in the weak* topology inherited from the dual Banach space $C(K)^*$.

Recall that $K \in \text{spm}$ is equivalent to saying that $K$ is the support of some regular measure $\mu \in P(K)$ (see [5]). Recall also that if $g : K \to L$ is a continuous surjection of compacta and $\nu \in P(L)$ then there is $\mu \in P(K)$ such that $g[\mu] = \nu$ (meaning that $\nu(B) = \mu(g^{-1}[B])$ for every Borel set $B \subset L$).

**Proposition 15** If there is a Lusin set then there is a nonseparable Corson compact space in $\text{ccc} \cap \text{SPM}^\perp$.

**Proof** Fix a Lusin set $Z$. Let $\{T_n : n \geq 2\}$ be an enumeration of intervals in $\mathbb{R}$ with rational endpoints. For every $n$ we pick a pairwise disjoint family $\{T_{n,k} : 1 \leq k \leq n^2\}$ of nonempty subintervals of $T_n$.

We define an adequate family $\mathcal{A}$ of subsets of $Z$ as follows:

\[ A \in \mathcal{A} \iff (\forall n) \left( |\{k \leq n^2 : A \cap T_{n,k} \neq \emptyset\} | \leq n \right). \]

Then we consider a compact space $K \subset 2^Z$ defined by $\mathcal{A}$, that is

\[ \chi_A \in K \iff A \in \mathcal{A}. \]

The space $K$ is ccc; in fact it is proved in [5] that the family of all nonempty open subsets of $K$ may be written as $\bigcup_n \mathcal{G}_n$ where every $\mathcal{G}_n$ contains no pairwise disjoint subfamily of size $n$.

The space $K$ is Corson compact since $\mathcal{A}$ contains no uncountable set — note that if $X \subset Z$ is uncountable then $X$ is not meager in $\mathbb{R}$ so it is dense in some $T_n$ and intersects every $T_{n,k}$; therefore $X \notin \mathcal{A}$.

**Claim.** Every $\mu \in P(K)$ has a metrizable support.
Indeed, fix such a measure $\mu$ and write $V_z = \{x \in K : x(z) = 1\}$. If we suppose that $\mu$ is not concentrated on a metrizable subspace of $K$ then $\mu(V_z) > 0$ for uncountably many $z \in Z$. Then there is $\varepsilon > 0$ and an uncountable $Z_0 \subset Z$ such that $\mu(V_z) \geq \varepsilon$ for $z \in Z_0$. By the Lusin property $Z_0$ is not meager in $\mathbb{R}$ so its closure contains $T_n$ for infinitely many $n$; fix such $n$ with $n > 1/\varepsilon$. Pick $z_k \in T_{n,k} \cap Z_0$ for every $k \leq n^2$ and consider the family of sets $V_{z_k}, k \leq n^2$ each of measure $\geq \varepsilon$. It follows that there is $I \subset \{1, 2, \ldots, n^2\}$ with $|I| \geq \varepsilon \cdot n^2$ and such that $W = \cap_{i \in I} V_{z_i} \neq \emptyset$. To see this note that, otherwise, we would have

$$\sum_{i=1}^{n^2} \chi_{V_{z_i}} < \varepsilon \cdot n^2,$$

and $\varepsilon \cdot n^2 \leq \int K \sum_{i=1}^{n^2} \chi_{V_{z_i}} \, d\mu < \varepsilon \cdot n^2.$$

On the other hand $W = \emptyset$ by the very definition of $A$, as $\varepsilon \cdot n^2 > n$, which is a contradiction.

Using Claim we can check that $K \in \text{SPM}$: let $g : K \to L$ be a continuous surjection. If $L$ is nonmetrizable and $v \in P(L)$ then there is $\mu \in P(K)$ such that $g[\mu] = v$. By Claim, $\mu(K_0) = 1$ for some metrizable $K_0 \subset K$; consequently, $v$ lives on a metrizable subspace $g[K_0]$ of $L$ and hence $v$ cannot be strictly positive.

In order to provide more examples of spaces in $\text{ccc}^\perp$, we use the following simple lemma.

**Lemma 16** Let $C_1$, $C_2$ and $C_3$ be three classes of compact spaces with $C_2$ stable under continuous images and such that $C_1 \cap C_2 \subseteq C_3$. Then, $C_3^\perp \cap C_2 \subseteq C_1^\perp$.

**Proof** Suppose that $K \in C_3^\perp \cap C_2$ and take a continuous image $L$ of $K$ which is in $C_1$. Then, since $C_3^\perp$ and $C_2$ are stable under continuous images, we have $L \in C_3^\perp \cap C_2 \cap C_1 \subseteq C_3^\perp \cap C_3$, so $L$ is metrizable.

Let $C_{dm}$ be the class of compacta $K$ having the property that every continuous image of $K$ has a dense metrizable subspace. Notice that this class is stable under continuous images. It contains the class of fragmentable compact spaces and the (consistently) more general class of Stegall compact spaces (see, c.f., [7, Theorems 3.1.5, 3.1.6 and 5.1.11]).

**Lemma 17** Any ccc compact space in $C_{dm}$ is separable.

**Proof** Take $K \in \text{ccc} \cap C_{dm}$; since $K$ is a continuous image of itself, there is a metrizable dense subset $D \subseteq K$. As dense subspaces of ccc spaces are ccc, we conclude that $D$ is a ccc metrizable topological space, so it is separable and, therefore, $K$ is also separable.

**Proposition 18** $\text{SEPARABLE}^\perp \cap C_{dm} \subseteq \text{ccc}^\perp$. In particular, any $\aleph_0$-monolithic fragmentable compactum $K$ is in $\text{ccc}^\perp$.

**Proof** Lemma 17 says that we can apply Lemma 16 to $C_1 = \text{ccc}$, $C_2 = C_{dm}$ and $C_3 = \text{SEPARABLE}$ and this yields $\text{SEPARABLE}^\perp \cap C_{dm} \subseteq \text{ccc}^\perp$.

The last part of the proposition follows from the fact that any $\aleph_0$-monolithic compact space is in $\text{SEPARABLE}^\perp$.

Ordinal interval spaces $[0, \alpha]$ serve as examples of compact spaces $K$ as above; they are scattered and therefore fragmentable.

We finish this section with a remark: Every nonmetrizable continuous image of a compact space $K$ has a further continuous image of weight $\omega_1$. Therefore, if a class $C$ is stable under continuous images, then a compact space $K$ belongs to $C^\perp$ if and only if no continuous...
image of $K$ of weight $\omega_1$ belongs to $\mathcal{C}$. In this sense, there is some similarity between the questions we are considering here and reflection problems of the sort studied by Tkachuk [33] and Tkachenko and Tkachuk [34]: Given a class of compacta $\mathcal{C}$, can we say that $K \in \mathcal{C}$ once we know that every continuous image of $K$ of weight $\leq \omega_1$ is in $\mathcal{C}$? Answering two questions from [34], Magidor and Plebanek [20] gave a consistent example of a scattered non Corson compact space of weight $\omega_2$ all of whose continuous images of weight $\omega_1$ are uniform Eberlein. Such a space is, in a sense, an extreme example in the class $\text{ccc}^\perp$: it is not in $\text{UniEberlein}^\perp$ because uniform Eberlein compacta are $\text{ccc}$ only when they are metrizable.

4 The orthogonal class of (weakly) Radon-Nikodým compacta

A compact space is Radon-Nikodým if it is homeomorphic to a weak* compact subset of $X^*$ for some Banach space $X$ which is Asplund, that is every separable subspace $Y$ of $X$ has a separable dual. Note that if $X$ is Asplund then it does not contain an isomorphic copy of $\ell_1$; hence $\text{RN} \subset \text{WRN}$. The reverse inclusion does not hold in the following strong sense.

**Lemma 19** Every hereditarily Lindelöf compact space is in $\text{RN}^\perp$. In particular, the class $\text{WRN} \cap \text{RN}^\perp$ is nontrivial, i.e. it contains nonmetrizable compacta.

**Proof** Let $K$ be an hereditarily Lindelöf compact space and suppose that $L$ is a Radon-Nikodým continuous image of $K$. Since the Lindelöf property is stable under continuous images, $L$ is hereditarily Lindelöf and therefore metrizable by [24,Theorem 5.8]. Thus, $K$ belongs to $\text{RN}^\perp$.

The second statement of the lemma follows from the fact that there exist nonmetrizable hereditarily Lindelöf compact spaces which are $\text{WRN}$. One example is given by the **double arrow space**, also known as the **split interval** (see [24,Example 5.9] and [12,Corollary 8.8] or [13,Subsect. 4.2]).

Since $\text{Eberlein} \subset \text{RN} \subset \text{WRN}$, we have

$$\text{WRN}^\perp \subset \text{RN}^\perp \subset \text{Eberlein}^\perp = \text{ccc}.$$  

By Lemma 19, the set $\text{WRN} \cap \text{RN}^\perp$ is nontrivial. Let us observe that $\text{WRN}^\perp \neq \text{ccc}$: for instance, $\beta \mathbb{N} \in \text{ccc} \setminus \text{WRN}^\perp$ since $\beta \mathbb{N}$ can be continuously mapped onto any separable compact space.

Dyadic spaces provide natural examples of compact spaces that are in $\text{WRN}^\perp$. Recall that a compact space is said to be dyadic if it is a continuous image of the Cantor cube $2^\kappa$ for some $\kappa$.

**Proposition 20** Every dyadic compactum belongs to $\text{WRN}^\perp$.

**Proof** If $K$ is dyadic then, by a result due to Gerlits [14], every nonmetrizable continuous image $L$ of $K$ contains a copy of $2^{\omega_1}$, so $L$ is not in $\text{WRN}$ by [12,Remark 10.7].

As mentioned in the introduction, following [4] we say that a Boolean algebras $\mathcal{A}$ is **weakly Radon-Nikodým** (denoted $\mathcal{A} \in \text{WRN}(\mathcal{B})$) if its Stone space is in the class $\text{WRN}$. Theorem 2 gives an internal characterizations of such algebras. It is convenient to rephrase that purely Boolean condition in the following form (see [4,Proposition 3.2] for the proof).

**Theorem 21** For a Boolean algebra $\mathcal{B}$, $\mathcal{B} \in \text{WRN}(\mathcal{B})$ if and only if $\mathcal{B}$ can be decomposed into countably many parts, none of which contains an infinite independent sequence.
Recall that, for a class $C$ of Boolean algebras, we denote by $C^\perp$ the class of Boolean algebras containing no uncountable Boolean subalgebra in $C$. Theorem 21 yields the following characterization of $\text{WRN}(B)^\perp$.

**Proposition 22**  Given a Boolean algebra $\mathcal{B}$, $\mathcal{B} \in \text{WRN}(B)^\perp$ if and only if every uncountable subset of $\mathcal{B}$ contains an infinite independent subset.

**Proof**  Consider an uncountable set $\Gamma \subset \mathcal{B}$ where $\mathcal{B} \in \text{WRN}(B)^\perp$. Let $L$ be the Stone space of the Boolean subalgebra $\mathfrak{A}$ of $\mathcal{B}$ generated by $\Gamma$; then $L$ is a continuous image of $\text{ult}(\mathcal{B})$. Since $\Gamma$ is uncountable, $L$ is not metrizable and so it is not weakly Radon-Nikodým. In particular, this and Theorem 21 immediately imply that $\Gamma$ contains an infinite independent sequence.

For the reverse implication suppose that $\mathfrak{A}$ is a subalgebra of $\mathcal{B}$ such that $\mathfrak{A} \in \text{WRN}(B)$. Then $\mathfrak{A}$ is generated by some $\mathfrak{G} = \bigcup_n \mathfrak{G}_n$, where no $\mathfrak{G}_n$ contains an infinite independent sequence. It follows that every $\mathfrak{G}_n$ is countable as well.

**Remark 23**  We can compare the above characterization of Boolean algebras in $\text{WRN}(B)^\perp$ with the following: $\text{ult}(\mathcal{B})$ is scattered if and only if every uncountable subalgebra of $\mathcal{B}$ contains an infinite independent sequence. This follows from the fact that, given $\mathfrak{A} \subset \mathcal{B}$, $\text{ult}(\mathfrak{A})$ is not scattered if and only if $\text{ult}(\mathfrak{A})$ maps continuously onto the Cantor set $2^\omega$.

## 5 Corson compacta and $\text{WRN}(B)$

As we have seen, all dyadic compacta are in the class $\text{WRN}^\perp$. To find other examples of nonmetrizable spaces from $\text{WRN}^\perp$ we need some auxiliary results.

Consider any subspace $X$ of some cube $2^\kappa$. We say that some $C \subset X$ is determined by (coordinates in) $J \subset \kappa$ if

$$
\pi_J^{-1} \pi_f(C) \cap X = C.
$$

In other words: if $x \in C$, $y \in X$, $x|J = y|J$ then $y \in C$. Notice that the ambient space $X$ is relevant in this definition, even if we do not mention it for economy of language. Recall that if $K \subset 2^\kappa$ is compact then, by the Stone-Weierstrass theorem, every clopen subset of $K$ is determined by a finite number of coordinates.

Let us first fix $n = \{0, 1, \ldots, n - 1\}$ and some $X \subset 2^n$; for any $s \subset n$ and $\varphi : s \to 2$ write

$$
C(\varphi) = \{x \in X : x|s = \varphi\}.
$$

**Lemma 24**  Let $X \subset 2^n$. Suppose that $J \subset n$ and $C \subset X$ is a subset that is not determined by coordinates in $J$ (this, in particular, implies that $C$ is a proper subset of $X$). Then there are $s \subset n$, $k \in s \setminus J$ and $\varphi, \psi : s \to 2$ such that

(i) $\varphi|(s \setminus \{k\}) = \psi|(s \setminus \{k\})$;

(ii) $\emptyset \neq C(\varphi) \subset C$;

(iii) $\emptyset \neq C(\psi) \subset X \setminus C$.

**Proof**  Since $C$ is not determined by $J$, there are $x \in C$ and $y \in X \setminus C$ such that $x|J = y|J$. Choose such a pair $x$, $y$ that the set $\Delta(x, y) = \{i < n : x(i) \neq y(i)\}$ has the minimal possible size. As $x$, $y$ are different, the set $\Delta(x, y)$ is not empty; choose any $k \in \Delta(x, y)$; note that $k \notin J$. Put $s = (n \setminus \Delta(x, y)) \cup \{k\}$ and define $\varphi, \psi : s \to 2$ so that $\phi|s = x|s$ and...
ψ|s = y|s. Then (i) is granted and we have \( C(φ), C(ψ) \neq \emptyset \), so it remains to verify the inclusions in (ii) and (iii).

Suppose that \( z ∈ C(φ) \) but \( z ∈ X \setminus C \). Then \( Δ(x, z) ⊆ Δ(x, y) \setminus \{k\} \), a contradiction with the minimality of \( Δ(x, y) \).

We verify (iii) in a similar manner: Suppose that \( z ∈ C(ψ) \) but \( z ∈ C \). Then again \( Δ(z, y) \) is a proper subset of \( Δ(x, y) \).

Consider now a compact space \( K ⊆ 2^κ \) for some \( κ \). As before we write

\[
C(φ) = \{ x ∈ K : x|s = φ \},
\]

for any finite \( s ⊆ κ \) and \( φ : s → 2 \).

**Corollary 25** Suppose that \( C \) is an uncountable family of clopen subsets of \( K \). Then there are families \( \{C_α : α < ω_1\} ⊆ C \), \( \{s_α : α < ω_1\} ⊆ [κ]^{<ω} \), a one-to-one function \( ξ : ω_1 → κ \) and \( φ_α, ψ_α : s_α → 2 \) such that for every \( α < ω_1 \)

(i) \( φ_α|(s_α \setminus ξ(α)) = ψ_α|(s_α \setminus ξ(α)) \); 
(ii) \( \emptyset \neq C(φ_α) ⊆ C_α \); 
(iii) \( \emptyset \neq C(ψ_α) ⊆ K \setminus C_α \); 
(iv) The sets \( \{s_α : α < ω_1\} \) form a Δ-system with root \( R \); 
(v) The restrictions \( φ_α|_R = ψ_α|_R \) are the same for all \( α \).

**Proof** Note that for every countable set \( J ⊆ κ \) there are only countably many clopens that are determined by coordinates in \( J \). Therefore, a simple induction using Lemma 24 gives \( s_α \)'s and \( ξ \) satisfying (i)–(iii) and such that \( ξ(α) \notin s_β \) whenever \( β < α \). We get (iv) using the Δ-system lemma. The elements \( ξ(α) \) after the first one cannot be in the root since they do not belong to previous \( s_β \) and this implies that \( φ_α|_R = ψ_α|_R \). There are only finitely many possibilities for these restrictions to \( R \), so passing to a further uncountable subfamily we get (v).

Motivated by the notion of an adequate family described in Sect. 3, we shall now consider any ccc compact space \( K ⊆ 2^κ \) and its ‘adequate closure’ \( \widehat{K} \), where

\[
\widehat{K} = \{ x ∈ 2^κ : \text{there is } y ∈ K \text{ such that } x ≤ y \}.
\]

In what follows, we write \( f ≤ g \) for functions defined on possibly different subsets of \( κ \) if \( f(α) ≤ g(α) \) whenever \( α \) belongs simultaneously to the domains of \( f \) and \( g \). Recall that we say that a zero-dimensional compact space belongs to \( \text{WRN}(B) \) if its clopen algebra belongs to \( \text{WRN}(B)^⊥ \). As we shall see in the next section, compact spaces in \( \text{WRN}(B)^⊥ \) might not belong to \( \text{WRN}^⊥ \).

**Theorem 26** If a compact space \( K ⊆ 2^κ \) is ccc then its adequate closure \( \widehat{K} \) belongs to \( \text{WRN}(B)^⊥ \).

**Proof** In view of Proposition 22, we are going to show that every uncountable subfamily of \( \text{Clop}(\widehat{K}) \) contains an infinite independent sequence.

Using Corollary 25 it is sufficient to consider \( \{C_α : α < ω_1\} ⊆ \text{Clop}(\widehat{K}) \) together with \( \{s_α : α < ω_1\} ⊆ [κ]^{<ω} \), a one-to-one function \( ξ : ω_1 → κ \) and \( φ_α, ψ_α : s_α → 2 \) such that for every \( α < ω_1 \)

(i) \( φ_α|(s_α \setminus ξ(α)) = ψ_α|(s_α \setminus ξ(α)) \); 
(ii) \( \emptyset \neq C(φ_α) ⊆ C_α \);
(iii) \( \emptyset \neq C(\psi_\alpha) \subset \tilde{K} \setminus C_\alpha \);
(iv) The sets \( \{ s_\alpha : \alpha < \omega_1 \} \) form a \( \Delta \)-system with root \( R \);
(v) The restrictions \( \phi_\alpha|_R = \psi_\alpha|_R \) are the same for all \( \alpha \).

Now we show that there is an infinite independent set of pairs of the form \( (C(\psi_\alpha), C(\psi_\alpha)) \). Consider \( h_\alpha = \max \{ \phi_\alpha, \psi_\alpha \} \in \{ \phi_\alpha, \psi_\alpha \} \). By definition of \( \tilde{K} \), for every \( x \in C(h_\alpha) \) there is \( y_x \in K \) such that \( x \leq y_x \). Thus, we can find \( g_\alpha : s_\alpha \to 2 \) such that \( g_\alpha \geq h_\alpha \) and \( C(g_\alpha) \cap K \neq \emptyset \), for every \( \alpha < \omega_1 \).

We apply ccc to the family \( \{ C(g_\alpha) \cap K : \alpha < \omega_1 \} \) of clopens to get \( y \in \tilde{K} \) belonging to \( C(g_\alpha) \) for \( \alpha \) in an infinite set \( A \). In order to prove independence we need to find \( x \in \tilde{K} \) in any intersection of the form

\[
C(\phi_{\alpha_1}) \cap \cdots \cap C(\phi_{\alpha_n}) \cap C(\psi_{\beta_1}) \cap \cdots \cap C(\psi_{\beta_m})
\]

with the \( \alpha_i \) and \( \beta_i \) distinct elements of \( A \). The desired \( x \leq y \) is obtained by taking the value at each \( s_{\alpha_i} \setminus R \) to coincide with \( \phi_{\alpha_i} \), the value at each \( s_{\beta_i} \setminus R \) to coincide with \( \psi_{\beta_i} \), the value at \( R \) to coincide with common value of all the \( \phi_\alpha \) and \( \psi_\alpha \), and the value at the rest of \( \omega_1 \) to be zero.

\[ \square \]

**Corollary 27** If \( MA_{\omega_1} \) does not hold then there is a nonmetrizable zero-dimensional compact space in \( \text{CORSON} \cap \text{WRN}(B)^\perp \).

**Proof** It is a well known fact that any zero-dimensional Corson compact space can be embedded into the intersection \( 2^\kappa \cap \Sigma(2^\kappa) \) for some cardinal \( \kappa \). Thus, the conclusion follows from an application of Theorem 26 to a zero-dimensional nonmetrizable ccc Corson compactum \( K \subset 2^\kappa \cap \Sigma(2^\kappa) \) (see Remark 14). Notice that \( \tilde{K} \) is also Corson.

\[ \square \]

A Boolean algebra \( \mathcal{B} \) has a precaliber \( (\kappa, \lambda) \) if every family in \( \mathcal{B} \) of size \( \kappa \) contains a centered subfamily of size \( \lambda \). Mimicking this definition (it may refer either to Boolean algebras or topological spaces) we can form the following. Say that a Boolean algebra \( \mathcal{B} \) has an independence-precaliber \( (\kappa, \lambda) \) if every subfamily of \( \mathcal{B} \) of size \( \kappa \) contains an independent subfamily of size \( \lambda \). Such a notion was already considered in the context of measure algebras, see [6] and also [10]. With this terminology, Proposition 22 says that \( \mathcal{B} \in \text{WRN}(B)^\perp \) if and only if \( \mathcal{B} \) has an independence-precaliber \( (\omega_1, \omega) \). Arguing as in Corollary 27 we conclude the following.

**Corollary 28** Suppose that \( MA_{\omega_1} \) does not hold. Then there is an uncountable Boolean algebra \( \mathcal{B} \) such that

(i) \( \mathcal{B} \) has an independence-precaliber \( (\omega_1, \omega) \), i.e. \( \mathcal{B} \in \text{WRN}(B)^\perp \);

(ii) \( \mathcal{B} \) contains no uncountable independent family and therefore does not have an independence-precaliber \( (\omega_1, \omega_1) \).

**Remark 29** We enclose two comments on Theorem 26.

(1) The process of forming the adequate closure of a compact space \( K \subset 2^\kappa \) may be described in terms of the underlying Boolean algebras. The algebra \( \mathfrak{A} = \text{Clop}(K) \) has generators \( a_i = \{ x \in K : x_i = 1 \} \), the algebra \( \tilde{\mathfrak{A}} = \text{Clop}(\tilde{K}) \) has generators \( \tilde{a}_i = \{ x \in \tilde{K} : x_i = 1 \} \), and the algebra \( \mathfrak{F} = \text{Clop}(2^\kappa) \) is the free Boolean algebra with generators \( e_i = \{ x \in 2^\kappa : x_i = 1 \} \). Consider the continuous surjection \( f : K \times 2^\kappa \to \tilde{K} \) given by \( f(x, y) = (\min(x_i, y_i))_{i < \kappa} \), and the dual homomorphism \( f^* : \tilde{\mathfrak{A}} \to \text{Clop}(K \times 2^\kappa) = \mathfrak{A} \otimes \mathfrak{F} \) given by \( f^*(a) = f^{-1}(a) \). Notice that \( f^*(\tilde{a}_i) = a_i \otimes e_i \) and \( f^* \) is an isomorphism onto
its range. Thus, \( \mathfrak{A} \) can be described as the subalgebra of the free product \( \mathfrak{A} \otimes \mathfrak{F} \) generated by the elements \( a_i \otimes e_j \). Any set of generators in a Boolean algebra can be represented as the 1-coordinate sets of an embedding into \( 2^\kappa \), so an equivalent formulation of Theorem 26 is that if \( \{ a_i : i < \kappa \} \) is a set of generators of a ccc Boolean algebra \( \mathfrak{A} \) then the algebra generated by \( \{ a_i \otimes e_i : i < \kappa \} \) inside \( \mathfrak{A} \otimes \mathfrak{F} \) belongs to \( \text{WRN}(B)^\perp \).

(2) The proof of Theorem 26 also shows that other chain conditions on \( K \) give analogous independence conditions on \( \bar{K} \). For example, if \( K \) has a precaliber \( \omega_1 \) (that is, \( K \in \text{CORSON}^{\perp} \)), then \( \text{Clop}(\bar{K}) \) has an independence-precaliber \( (\omega_1, \omega_1) \).

6 Between \( \text{WRNB} \) and \( \text{WRN}(B) \)

The main result from the previous section leads us to the following question.

**Problem 30** Is there a (consistent) example of a nonmetrizable Corson compact space in \( \text{WRN}^{\perp} \)?

In particular, we do not know if the space \( \bar{K} \) discussed in Theorem 26 is orthogonal to all (not necessarily zero-dimensional) weakly Radon-Nikodým compacta.

In connection to Problem 30 we shall now present another construction showing that a compact space in \( \text{WRN}(B)^\perp \) might not belong to \( \text{WRN}^{\perp} \).

**Theorem 31** Let \( L \) be a ccc compact and convex subspace of \( \mathbb{R}^\kappa \) (for some \( \kappa \)). Then \( L \) is a continuous image of a space \( K \) belonging to \( \text{WRN}(B)^\perp \).

**Proof** With \( L \subset \mathbb{R}^\kappa \) given, we first fix some notation. For any \( a \in \mathbb{R} \) and \( \xi < \kappa \) write

\[
V^{0}_\xi(a) = \{ x \in L : x(\xi) < a \}, \quad V^{1}_\xi(a) = \{ x \in L : x(\xi) > a \}.
\]

Moreover, we denote \( \Delta = \{ (p, q) \in \mathbb{Q}^2 : p < q \} \).

We shall consider functions \( f : \kappa \times \Delta \to 2 \). For such \( f \) and a finite set \( s \subset \kappa \times \Delta \) we set

\[
\text{core}_s(f) = \{ x \in L : \forall (\xi, p, q) \in s \ (f(\xi, p, q) = 0 \Rightarrow x_\xi < q), \ (f(\xi, p, q) = 1 \Rightarrow x_\xi > p) \}
\]  

\[
= \bigcap \left\{ V^{0}_\xi(q) : \exists p, \ (\xi, p, q) \in s \text{ and } f(\xi, p, q) = 0 \right\}
\]  

\[
\cap \bigcap \left\{ V^{1}_\xi(p) : \exists q, \ (\xi, p, q) \in s \text{ and } f(\xi, p, q) = 1 \right\}.
\]

Note that \( \text{core}_s(f) \) is in fact determined by \( f | s \) so below we also consider \( \text{core}_s(\varphi) \) whenever some \( \varphi : s \to 2 \) is given.

We define the space \( K \) as follows

\[
K = \{ f : \text{core}_s(f) \neq \emptyset \text{ for every finite } s \subset \kappa \times \Delta \},
\]

and check that \( L \) is a continuous image of \( K \) and that \( K \in \text{WRN}(B)^\perp \). By the very definition, \( K \) is a compact subspace of \( 2^{\kappa \times \Delta} \).

**Claim 1.** There is a continuous surjection \( \theta : K \to L \).

Define \( \theta(f) \) to be the unique point in \( \bigcap_s \text{core}_s(f) \) (where the intersection is taken over all finite \( s \subset \kappa \times \Delta \)). To see that the definition is correct note that such an intersection is nonempty by compactness of \( L \). Moreover, \( \bigcap_s \text{core}_s(f) \) cannot contain two distinct points \( x, x' \in L \) for, otherwise, we have \( x(\xi) \neq x'(\xi) \) for some \( \xi < \kappa \); say that \( x(\xi) < x'(\xi) \) and
we can pick rational numbers \( p, q \) so that \( x(\xi) < p < q < x'(\xi) \). Then examine the value of \( f(\xi, p, q) \) to get a contradiction. Moreover, if \( x \in L \), then it is immediate that the function \( f : \kappa \times \Delta \to 2 \) given by \( f(\xi, p, q) = 1 \) if \( x(\xi) > p \) and zero otherwise belongs to \( K \) and satisfies \( \theta(f) = x \), so \( \theta \) is surjective.

To verify the continuity of \( \theta \) note that sets of the form \( V^1_\xi(p) \), \( V^0_\xi(q) \) form a subbase of the topology on \( L \). The following equalities show that the preimages of these sets under \( \theta \) are open:

\[
\theta^{-1}(V^1_\xi(p)) = \bigcup_{p', q \in \mathbb{Q}, p < p' < q} \{ f \in K : f(\xi, p', q) = 1 \},
\]

\[
\theta^{-1}(V^0_\xi(q)) = \bigcup_{p, q' \in \mathbb{Q}, p < q' < q} \{ f \in K : f(\xi, p, q') = 0 \}.
\]

We carefully check the first one, the second one being analogous. If \( f(\xi, p', q) = 1 \) for some \( q > p' > p \) then \( \text{core}_s(f) \subset V^1_\xi(p') \) for any \( s \) that contains \((\xi, p', q)\). So

\[
\theta(f) \in \bigcap_s \text{core}_s(f) \subset V^1_\xi(p') \subset V^1_\xi(p).
\]

For the reverse inclusion, suppose that \( f(\xi, p', q) = 0 \) whenever \( p < p' < q \). Then \( \text{core}_s(f) \subset V^0_\xi(q) \) whenever \((\xi, p', q)\) \( \in \) \( s \). Therefore

\[
\theta(f) \in \bigcap_s \text{core}_s(f) \subset \bigcap_{q > p} V^0_\xi(q).
\]

It follows that \( \theta(f) \xi \leq p \), so that \( \theta(f) \notin V^1_\xi(p) \).

It remains to prove that \( K \in \text{WRN}(B)^\perp \). Consider first a finite set \( s \subset \kappa \times \Delta \) and two functions \( \varphi, \psi : s \to 2 \) that differ only at \( (\xi_0, p_0, q_0) \in s \); say that \( \varphi(\xi_0, p_0, q_0) = 0 \) and \( \psi(\xi_0, p_0, q_0) = 1 \). Assume that the clopens \( C(\varphi) \) and \( C(\psi) \) are nonempty.

Set \( s' = s \setminus \{ (\xi_0, p_0, q_0) \} \); note that \( \text{core}_{s'}(\varphi) = \text{core}_{s'}(\psi) \) and

\[
\text{core}_{s'}(\varphi) = \text{core}_{s'}(\varphi) \cap V^0_{\xi_0}(q_0), \quad \text{core}_{s'}(\psi) = \text{core}_{s'}(\psi) \cap V^1_{\xi_0}(p_0).
\]

**Claim 2.** The set \( U = \text{core}_{s'}(\varphi) \cap V^0_{\xi_0}(q_0) \cap V^1_{\xi_0}(p_0) = \text{core}_{s'}(\varphi) \cap \text{core}_{s'}(\psi) \) is not empty.

This follows from the convexity of \( L \): take \( x \in \text{core}_{s'}(\varphi) \) and \( y \in \text{core}_{s'}(\psi) \). Then \( x(\xi_0) < q_0 \) and \( y(\xi_0) > p_0 \) so there is \( z \) lying on the segment joining \( x \) and \( y \) such that \( z(\xi_0) \in (p_0, q_0) \). Hence \( z \in U \) (note that every core is a convex subset of \( L \)).

In order to prove that \( K \in \text{WRN}(B)^\perp \) we check the criterion of Proposition 22 as we did before: if \( C \subset \text{Clop}(K) \) is an uncountable family then, as in Theorem 26, we find uncountably many \( C_\alpha \in C \) for which there are nonempty sets \( C(\varphi_\alpha) \subset C_\alpha \) and \( C(\psi_\alpha) \subset K \setminus C_\alpha \), where \( \varphi_\alpha \) and \( \psi_\alpha \) have the same finite domain \( s_\alpha = s'_\alpha \cup \{ \xi_\alpha \} \) and differ at exactly one point \( \xi_\alpha \). As stated in Corollary 25, the assignment \( \alpha \mapsto \xi_\alpha \) can be taken one-to-one. We can also suppose that the sets \( s'_\alpha \) form a \( \Delta \)-system with root \( R \), that \( \xi_\alpha \notin s'_\beta \) and \( \varphi_\alpha \restriction R = \varphi_\beta \restriction R \) for all \( \alpha, \beta \). For every \( \alpha \) consider the open set

\[
U_\alpha = \text{core}_{s'_\alpha}(\varphi_\alpha) \cap \text{core}_{s'_\alpha}(\psi_\alpha),
\]

as in Claim 2. Since \( L \) is \( ccc \), it follows from Claim 2 that we can find an infinite sequence of indices \( \alpha_1, \alpha_2, \ldots \) such that \( U_{\alpha_1} \cap U_{\alpha_2} \cap \cdots \cap U_{\alpha_n} \neq \emptyset \) for every \( n \). We claim that the sequence of pairs \( (C(\varphi_{\alpha_n}), C(\psi_{\alpha_n})) \) is independent, as desired. For this, we must find
$f \in C(\gamma_1) \cap C(\gamma_2) \cap \cdots \cap C(\gamma_n)$ for any choice of $\gamma_i \in \{\varphi_i, \psi_i\}$. Pick $x \in U_{a_1} \cap \cdots \cap U_{a_n}$ and then define $f$ so that $f|_{s_{a_i}}$ agrees with $\gamma_{a_i}$ for all $i \leq n$, while for $t = (\xi, p, q) \notin \bigcup_{i=1}^n s_{a_i}$, we declare $f(t) = 1$ when $x(\xi) > p$ and $f(t) = 0$ otherwise. Notice that there are no conflicts in this definition because of all the previous refinements on the family. We have that $f \in K$ because $x \in \bigcap_i \text{core}_s(f)$, and $f \in C(\gamma_1) \cap C(\gamma_2) \cap \cdots \cap C(\gamma_n)$. \hfill \qed

Take any nonmetrizable WRN separable compact space $L_0$ (e.g. the split interval). Then, the space $L = P(L_0)$ of all regular probability measures on $L_0$ is a separable (so is $ccc$) nonmetrizable convex WRN compact space.\footnote{The fact that $P(K)$ is WRN whenever $K$ is WRN is a consequence of the characterization of WRN compacta as those compact spaces $K$ for which $C(K)$ is weakly precompactly generated, but also as those compact spaces which can be weak*-embedded into the dual ball of a weakly precompactly generated Banach space; see [23, Theorem 2.1.4 and 2.1.5].} We can apply Theorem 31 to get the following, somewhat surprising, result.

**Corollary 32** The class $\text{WRN}(B)^{\perp} \setminus \text{WRN}^{\perp}$ contains zero-dimensional (necessarily nonmetrizable) compact spaces.

### 7 The orthogonal class of zero-dimensional compacta

In this section we denote by $\text{ZERO- DIMENSIONAL}^{\perp}$ the class of zero-dimensional compact spaces. The orthogonal class $\text{ZERO- DIMENSIONAL}^{\perp}$ can be easily characterized as follows:

**Lemma 33** A compact space $K$ belongs to $\text{ZERO- DIMENSIONAL}^{\perp}$ if and only if it contains at most countably many different clopens.

**Proof** Notice that $K$ has countably many different clopens if and only if its clopen algebra is countable, which in turn is equivalent to the fact that the Stone space $\text{ult}(\text{Clop}(K))$ of its clopen algebra is metrizable. Bearing in mind that $K$ always maps continuously onto $\text{ult}(\text{Clop}(K))$, we obtain that if $K \in \text{ZERO- DIMENSIONAL}^{\perp}$ then it contains at most countably many different clopens. On the other hand, suppose that $f : K \to L$ is a continuous map onto a zero-dimensional compact space $L$. If $L$ were nonmetrizable, then it would contain uncountably many clopens and, since the preimage of a clopen is a clopen, $K$ would have the same property. Thus, if $K$ contains at most countably many different clopens then it belongs to $\text{ZERO- DIMENSIONAL}^{\perp}$. \hfill \qed

From the previous lemma and Lemma 7 one could give a characterization for the class $\text{ZERO- DIMENSIONAL}^{\perp}$ which turns out to be unsatisfactory in the sense that it does not seem to be useful to determine whether this class contains nonmetrizable compact spaces. Let us recall that even a simpler question, whether there are nonmetrizable compact spaces containing no zero-dimensional nonmetrizable closed subspaces, is somewhat delicate: Koszmider [17] gave the first ZFC example of such a space; Marciszewski [21] gave a consistent example which is Eberlein compact.

The class $\text{ZERO- DIMENSIONAL}^{\perp}$ has been studied in the literature; see, e.g., [17, Question 1.1(2)], [15, Question 12 (374)] and [16, Question 4 (1176)]. It seems to be an open problem whether it is consistent that $\text{ZERO- DIMENSIONAL}^{\perp}$ is trivial (i.e. it consists solely of metrizable compacta). Nevertheless, it may happen (in some models of set theory) that $\text{ZERO- DIMENSIONAL}^{\perp}$ contains some nonmetrizable Corson compact spaces and compact
spaces that are not hereditarily separable; on the other hand, this orthogonal class can contain neither nonmetrizable Eberlein compacta nor Rosenthal compacta (see [17,Proposition 4.2]). Moreover, CORSON ∩ ZERO-DIMENSIONAL is trivial under MAω1. Namely, by [17,Proposition 4.2(1)] every compact space \( K \) in ZERO-DIMENSIONAL has no uncountable discrete subspace, in particular, \( K \) is ccc. If \( K \) is also Corson, then under MAω1, it must be metrizable. Another recent result shows that a connected version of Kunen’s \( L \)-space constructed under CH is in CORSON ∩ ZERO-DIMENSIONAL [26,Theorem 3.4].

Let us note that any Souslin line is a WRN compact space since it is a linearly ordered compact space (see [12,Theorem 8.7] or [13,Theorem 4.5]). Moreover, any such line belongs to ZERO-DIMENSIONAL, which can be demonstrated using the argument from [17,Proposition 4.2(5)]). We show below, using the so called split compact spaces introduced by Koszmider (see [17,Definition 2.1]), that nonmetrizable WRN compact spaces in ZERO-DIMENSIONAL can be constructed under Martin’s Axiom and the negation of CH.

**Definition 34** Let \( M \) be a metric compact space, \( L \) a compact space, \( \kappa \) an ordinal, \( \{ r_\xi : \xi < \kappa \} \) a set of distinct points of \( L \) and \( f_\xi : L \setminus \{ r_\xi \} \to M \) a continuous function for every \( \xi < \kappa \). The split \( L \) induced by \( \{ f_\xi : \xi < \kappa \} \) is the subspace \( K \) of \( L^{[\kappa]} \times M^\kappa \) consisting of points of the form

\[
\{ x_\xi, t : \xi < \kappa, \ t \in M \} \cup \{ x_r : r \in L \setminus \{ r_\xi : \xi < \kappa \} \},
\]

where

- \( x_\xi, t(*) = r_\xi, x_\xi, t(\xi) = t \) and \( x_\xi, t(\eta) = f_\eta(r_\xi) \) if \( \eta \neq \xi \).
- \( x_r(*) = r \) and \( x_r(\xi) = f_\xi(r) \) for all \( r \in L \setminus \{ r_\xi : \xi < \kappa \} \) and every \( \xi < \kappa \).

The classical split interval is an example of a split compact space of this form. We provide in Theorem 37 a sufficient condition for a split compact space to be WRN. For that purpose, we need to extend the concept of independent functions to functions taking values in any compact space.

**Definition 35** Let \( K \) and \( M \) be compact spaces. A sequence of functions \( f_n : K \to M \) is said to be \( M \)-independent if there exist closed disjoint sets \( C, C' \) in \( M \) such that \( (f_n^{-1}(C), f_n^{-1}(C'))_{n \in \mathbb{N}} \) is independent.

We say that the sequence of functions \( f_{\xi,n} : L \setminus \{ r_{\xi,n} \} \to M \) is \( M \)-independent if there exist extensions (possibly not continuous) \( g_{\xi,n} : L \to M \) of \( f_{\xi,n} \) for each \( n \in \mathbb{N} \) such that the sequence \( g_{\xi,n} \) is \( M \)-independent.

Notice that \( f_{\xi,n} \) is \( M \)-independent if and only if every extension provides an \( M \)-independent sequence, i.e. if \( g_n \) and \( h_n \) are different extensions of \( f_{\xi,n} \) then the sequence \( g_n \) is \( M \)-independent if and only if \( h_n \) is \( M \)-independent. Namely, if \( g_n \) is not \( M \)-independent then for every closed disjoint sets \( C, C' \) of \( M \) there are disjoint finite subsets \( S_1, S_2 \) of \( \mathbb{N} \) such that

\[
\left( \bigcap_{k \in S_1} g_k^{-1}(C) \right) \cap \left( \bigcap_{k' \in S_2} g_k^{-1}(C') \right) = \emptyset.
\]

But then

\[
\left( \bigcap_{k \in S_1} h_k^{-1}(C) \right) \cap \left( \bigcap_{k' \in S_2} h_k^{-1}(C') \right) \subset \{ r_{\xi,n} : n \in S_1 \cup S_2 \}
\]
is a finite set. Now, a suitable choice of finite sets $S_1' \supseteq S_1$ and $S_2' \supseteq S_2$ shows that $h_n$ is not $M$-independent.

The following lemma is a simple extension of the well-known Rosenthal Theorem which states that every sequence of functions $f_n : S \rightarrow [0, 1]$ defined on a set $S$ contains a pointwise convergent subsequence or a $[0, 1]$-independent subsequence [28].

**Lemma 36** Let $S$ be a set, $M$ a metric compact space and $f_n : S \rightarrow M$ a sequence of functions. Then $f_n$ has a pointwise convergent subsequence or an $M$-independent subsequence. Moreover, if $S$ is compact, the functions $f_n$ are continuous and the sequence $f_n$ is $M$-independent, then it does not have pointwise convergent subsequences.

**Proof** Take $q : M \rightarrow [0, 1]^\mathbb{N}$ an embedding from $M$ into the Hilbert cube and denote by $q_n$ the $n$th-coordinate function of $q$. Suppose $f_n$ does not have an $M$-independent subsequence. Then, $q_1 \circ f_n$ does not have a $[0, 1]$-independent subsequence. By Rosenthal’s Theorem, there exists a convergent subsequence of $q_1 \circ f_n$. A standard diagonal argument provides a subsequence $f_{n_k}$ such that $(q_m \circ f_{n_k})_{k \in \mathbb{N}}$ converges for every $m \in \mathbb{N}$. Thus, $f_{n_k}$ is a convergent subsequence of $f_n$.

For the last part of the lemma, take $C$ and $C'$ closed disjoint sets witnessing the $M$-independence of the sequence $f_n : S \rightarrow M$. Let $(f_{n_k})_k$ be any subsequence. Since $S$ is compact and the functions $f_n$ are continuous, we have that $(f_{n_k}^{-1}(C), f_{n_k}^{-1}(C'))_{n \in \mathbb{N}}$ is an independent sequence consisting of compact subsets of $S$. Thus, we can take

$$t \in \bigcap_{k \in \mathbb{N}} \left( f_{n_k}^{-1}(C) \cap f_{n_k}^{-1}(C') \right).$$

The sequence $f_{n_k}(t)$ cannot be convergent since $C$ and $C'$ are disjoint closed sets. \qed

**Theorem 37** Let $K$ be the split $L$ induced by $\{ f_\xi : \xi < \kappa \}$, where $L$ and $\{ f_\xi : \xi < \kappa \}$ are as in Definition 34. If $\{ f_\xi : \xi < \kappa \}$ does not contain $M$-independent sequences and $L$ is WRN, then $K$ is WRN.

**Proof** Denote by $\pi_\ast : K \rightarrow L$ the projection onto the first coordinate (i.e. $\pi_\ast(x) = x(\ast)$ for every $x \in K \subseteq L^{(\ast)} \times M^{\ast}$) and by $\pi_\xi : K \rightarrow M$ the projection onto the coordinate $\xi$, i.e. $\pi_\xi(x) = x(\xi)$ for every $x \in K$, $\xi < \kappa$. We claim that $\{ \pi_\xi : \xi < \kappa \}$ does not contain $M$-independent sequences. Suppose that a sequence $\pi_{\xi_n}$ is $M$-independent. Since $\{ f_\xi : \xi < \kappa \}$ does not contain $M$-independent sequences, by Lemma 36 we may suppose, passing to a further subsequence if necessary, that $f_{\xi_n}$ is pointwise convergent, in the sense that $(f_{\xi_n}(x))_{n \in \mathbb{N}, \xi_n \neq x}$ converges for every $x \in L$. Notice that for every $x \in K$, $\pi_{\xi_n}(x) = f_{\xi_n}(x(\ast))$ for all except at most one $n \in \mathbb{N}$. Thus, the sequence $\pi_{\xi_n}$ is a pointwise convergent sequence of continuous functions over a compact space and therefore it does not contain $M$-independent subsequences due to Lemma 36, which yields a contradiction. Hence $\{ \pi_\xi : \xi < \kappa \}$ does not contain $M$-independent sequences. Since $L$ is WRN, there exists a family $\mathcal{F}$ of continuous functions from $L$ to $[0, 1]$ separating points and with no independent sequences (Theorem 1). Notice that the family of functions

$$\{ \pi_\xi : \xi < \kappa \} \cup \{ f \circ \pi_\ast : f \in \mathcal{F} \}$$

separates the points of $K$.

Now take $q : M \rightarrow [0, 1]^{\mathbb{N}}$ an embedding from $M$ into the Hilbert cube, with $q_n$ the coordinate functions of $q$. Set $\mathcal{F}_n = \{ q_n \circ \pi_{\xi} \mid \xi < \kappa \}$ and $\mathcal{F}' = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$. Then, $\mathcal{F}'$ does not contain independent sequences of functions. It follows that $\mathcal{F}' \cup \{ f \circ \pi_\ast : f \in \mathcal{F} \}$ is a family of continuous functions which separates the points of $K$ and with no independent subsequences. Therefore, $K$ is WRN again by Theorem 1. \qed
Example 38  Set $L = [0, 1]^2$, $M = S$, where $S$ is the unit circumference in $\mathbb{R}^2$ with the Euclidean metric, \{r_\xi : \xi < \kappa\} \subset L$ and $f_\xi : L \setminus \{r_\xi\} \to M$ defined as $f_\xi(x) = \frac{x - r_\xi}{d(x,r_\xi)}$ for every $\xi < \kappa$, where $d$ is the Euclidean distance in $[0,1]^2$. Let $K$ be the split $L$ induced by \{r_\xi : \xi < \kappa\}. The compact space $K$ is said to be a Filippov space (see, e.g., [9] or [18]). We claim that $K$ is WRN. By Theorem 37, it is sufficient to check that every sequence $f_{\xi_n}$ does not contain an $M$-independent subsequence or, equivalently, that it contains a convergent subsequence. However, since $r_{\xi_n}$ is a sequence in $[0,1]^2$, we may suppose without loss of generality that $r_{\xi_n}$ converges to some $r \in [0,1]^2$. But then notice that $f_{r_{\xi_n}}(x) = \frac{x - r_{\xi_n}}{d(x,r_{\xi_n})}$ converges to $\frac{x - r}{d(x,r)}$ for every $x \neq r$. Passing to a subsequence if necessary, we may suppose that the sequence $f_{r_{\xi_n}}(r)$ is also convergent. Thus, $f_{r_{\xi_n}}$ does not contain $M$-independent subsequences and we conclude that $K$ is WRN.

The original construction of a Filippov space, where the set \{r_\xi : \xi < \kappa\} is a Luzin set (so it works under CH), gives a nonmetrizable hereditarily Lindelöf space, which by Example 38 and [17, Proposition 4.4] is also in ZERO- DIMENSIONAL. Furthermore, even under Martin’s Axiom and the negation of CH this class of spaces can be used to produce relevant examples of nonmetrizable compact spaces in WRN ∩ ZERO- DIMENSIONAL:

Corollary 39  It is consistent with Martin’s Axiom and the negation of CH that there is a WRN nonmetrizable compact space in ZERO- DIMENSIONAL.

Proof  It is a consequence of Example 38 and [17, Theorem 4.5], where it is proved that it is consistent with Martin’s Axiom and the negation of CH that there is a nonmetrizable Filippov space in ZERO- DIMENSIONAL. □

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