A NOTE ON ANDREWS INEQUALITIES

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Abstract. In this note, we prove some extensions of Andrews inequality.

1. Introduction

In his work, Ben Andrews has proved the following geometric inequality for closed Riemannian manifolds with positive Ricci curvature:

**Theorem 1.** Let \((M, g)\) be an \(n\) dimensional closed Riemannian manifold with positive Ricci curvature, \(\text{Ric}_g\). Then for any smooth function \(\phi \in C^\infty(M, g)\),

\[
\frac{n}{n-1} \int_M \phi^2 \, dv_g \leq \int_M \text{Ric}^{-1}(\nabla \phi, \nabla \phi) \, dv_g,
\]

where the identity holds if and only if \((M, g)\) is conformal to the \(n\)-sphere \((S^n, g_{S^n})\) with its standard metric. In addition, \(g\) is invariant under an \(O(n-1)\) isometry group.

Inequality (1.1) has seen applications in the study of Ricci flow, see [BLN]. More recently, Gursky and Streets [GS1, GS2] have used (1.1) in a different area of geometric analysis, namely, conformal geometry of 4-manifolds. \(\sigma_k\) Yamabe problem is a recent focus in the study of conformal geometry. It is well known that in dimension 4, the positivity of \(\sigma_1\) and \(\sigma_2\) curvatures implies the positivity of Ricci curvature. Thus, (1.1) is applicable to such manifolds. Motivated by a similar construction in Kähler geometry, Gursky-Streets [GS1] have defined a formal metric structure on all conformal metrics over a 4-manifolds \(M\) with positive \(\sigma_2\) curvature. They then applied (1.1) to establish the geodesic convexity of a crucial functional defined by Chang-Gursky-Yang [CGY1]. Through further careful analysis, Gursky and Streets are able to prove the uniqueness of the \(\sigma_2\) Yamabe problem solution. This is a surprising result in contrast with the general dimension situation for the standard Yamabe problem.

In this note, we are prove some extensions of Andrews inequality (1.1) and explore further applications in conformal geometry.

First we discuss the case of conic manifolds.

**Definition 2.** Let \((M^n, g_0)\) be a compact smooth Riemannian \(n\)-manifold. For some \(k \in \mathbb{N}\), assume that \(p_i \in M\) and \(0 > \beta_i > -1\). Define a conformal divisor

\[
D = \sum_{i=1}^{k} p_i \beta_i,
\]

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Let $\gamma(x)$ be a function in $C^\infty(M^n \setminus \{p_i\})$ such that near $p_i$, $\gamma(x) - \beta_i \log r_i$ is locally smooth, where $r_i = \text{dist}_{g_0}(x, p_i)$ is the distance to $p_i$. Let
\begin{equation}
(1.2) \quad g_D = e^{2\gamma} g_0.
\end{equation}
g_D is a metric conformal to $g_0$ on $M^n \setminus \{p_i\}$ which has a conical singularity at each $p_i$.

We define the conformal class of $g_D$
\[ [g_D] := \{ g_w = e^{2w} g_D : w \in C^\infty(M, g_0) \cap C^\infty_{\text{loc}}(M - \{p_i\}) \}, \text{ for some } \beta \in (0, 1). \]
Note $[g_D]$ depends only on $(M^n, g_0)$ and $D$. Let $g_1 \in [g_D]$. We call a 4-tuple $(M^n, g_0, D, g_1)$ a conic $n$-manifold, where $g_0$ is the background metric and $g_1$ is the conic metric.

We remark here that our definition of conic singularity is in the sense of Cheeger-Colding [CC1, CC2], which is different from the similar notation used in studies of Kähler geometry. For discussion of related conformal geometry problems on conic manifolds, see for example [FW1, FW2, FW3] and [FM]. We also note that for analytical consideration, we may relax the regularity of the conformal factor function.

We mention a particular type of conic manifolds that are first discussed in Chang-Han-Yang [CHY].

**Definition 3.** $(S^n, g_{S^n}, D, g)$ is called a conic $n$-sphere of football type if $D = \beta p + \beta q$ where $\beta \in (-1, 0)$ and $p, q$ are two distinct points on $S^n$.

**Remark 4.** For a conic $n$-sphere of football type, by a simple conformal transform, we may assume that $p, q$ are antipodal points on $S^n$. It is also clear that a conic sphere of football type is conformally flat.

Our first main result is the following

**Theorem 5.** (Conic Andrews) Let $(M, g_0, D, g)$ be a compact $n$-dimensional conic manifold with positive Ricci curvature and let $\alpha > 0$ such that $\alpha + 2 \min(\beta_i) > 0$. Then for $\phi \in C^\alpha(M, g_0) \cap C^\infty(M \setminus D, g_0)$ with $\int_M \phi dv_g = 0$,
\begin{equation}
(1.3) \quad \frac{n}{n-1} \int_{M \setminus D} \phi^2 dv_g \leq \int_{M \setminus D} \text{Ric}^{-1}(\nabla \phi, \nabla \phi) dv_g.
\end{equation}
In particular, the identity holds if and only if $(M, g_0, D, g)$ is conformal to a conic sphere of football type with an $O(n-1)$ isometry group.

The second part of our note is on manifolds with boundary. Our result is the following

**Theorem 6.** Let $(M, g)$ be a connected $n$-dimensional compact manifold with boundary $\partial M$. Suppose that $M$ has positive Ricci curvature. Let $\Pi$ be the second fundamental form near $\partial M$, and assume that $\Pi \geq 0$ or equivalently, $\partial M$ is weakly convex. Then, for any $\phi$ such that $\int_M \phi dv_g = 0$, we have
\begin{equation}
(1.4) \quad \frac{n}{n-1} \int_M \phi^2 dv_g \leq \int_M \text{Ric}^{-1}(\nabla \phi, \nabla \phi) dv_g,
\end{equation}
where the equality holds if and only if $(M, g)$ is conformal to an $n$-dimensional hemisphere $S^n_+$ with its standard metric; furthermore, $g$ admits an $O(n-1)$ isometry group.
Remark 7. In [Es], Escobar has studied the Laplacian eigenvalues with Neumann boundary condition for manifolds with $Ric \geq (n-1)$ and convex boundary. He proved that the first eigenvalue $\lambda_1 \geq n$ with the identity holds if and only if the manifold is isometric to the standard hemisphere. In our case, $M$ is less restricted when the equality in (1.4) holds.

In summary, Theorems 5 and Theorem 6 are some natural extensions of the original Andrews inequality, Theorem 1. We note that for various Andrews inequalities we have obtained, the equality cases all imply the conformal-flat-ness and the existence of a relatively large isometry group.

We would like to make some further comments. First, we raise a general question which may be of interest. For manifolds with positive Ricci curvature, we may define for the following functional for non-zero square integrable functions:

$$F(\phi) = \frac{\int_M Ric^{-1}(\nabla \phi, \nabla \phi) \, dv_g}{\int_M \phi^2 \, dv_g}$$

and (1.1) obviously gives an lower bound estimate. The corresponding Euler equation is thus

$$\delta (Ric^{-1} d\phi) = \lambda \phi,$$

which can be viewed as an eigenvalue problem of an elliptic operator.

Obviously, when the manifold is Einstein, this is just the well-known eigenvalue problem for the Laplacian operator. However, there are known examples of non-Einstein manifolds with positive Ricci curvature. For instance, if $k \geq 4$, the connected sum $M = \#_k \mathbb{CP}^2$ admits no Einstein metric [Be], while there always exists a metric on $M$ with positive Ricci curvature due to Perelman [P]. Sha-Yang [SY1, SY2] has constructed a class of Ricci positive manifolds which admit arbitrarily large Betti numbers. It may be of interest to study sharp Andrews type inequality in these special cases.

Another direction of interest is to study the almost sharp Andrews inequality. We may expect some rigidity results. For instance, when the first eigenvalue in (1.5) is almost $\frac{n}{n-1}$, one may naively wish the manifold to be a sphere. However, Anderson [An] constructs a metric on $\mathbb{CP}^n$ with $Ric \geq n-1$ such that the Laplacian first eigenvalue is arbitrarily close to $n+1$. We easily check that the first eigenvalue of (1.5) in Anderson’s example can be arbitrarily close to $\frac{n}{n-1}$. For further discussions on eigenvalue problems and rigidity results, see [Cg, Co, Pe2].

The rest of the paper is organized as follows. In Section 2, we discuss the warped product structure which is crucial in our study of sharp Andrews inequality. In Section 3, we prove Theorem 5. In Section 4, we prove Theorem 6.

2. WARPED PRODUCT

In this section, we discuss the geometry of warped products. Most of the results in this section are well known and can be found in [Pa, IT, Ta, CCT, WY, CSZ, CMM]. We state them for future use.

Definition 8. Let $(M^n, g)$ be a $n$-dimensional Riemannian manifold. We call $M$ is a warped product if $M \simeq [a, b] \times N^{n-1}$ and $g$ can be written as $g = dr^2 + f(r)^2 g_N$.
for some \( n - 1 \)-dimensional Riemannian manifold \((N^{n-1}, g_N)\) and a positive function \(f : [a, b] \to \mathbb{R}\). We use the following notation to denote a warped product:

\[ M = [a, b] \times_f N^{n-1}. \]

Let \( \phi_r : N^{n-1} \hookrightarrow M^n \) be the embedding of \( N \) at \( r \in (a, b) \). Let \( \{x^i\}_{i=1,\ldots,n-1} \) be a local coordinate on \( N^{n-1} \). Then, \( (r, x^i) \) gives a coordinate on \( M^n \). Let \( g, \nabla, R_{ijkl}, \text{Ric} \) be the metric, connection, curvature tensor and Ricci curvature tensor on \( M^n \) respectively and \( \bar{g} = g_N, \bar{\nabla}, \bar{R}_{ijkl}, \bar{\text{Ric}} \) be the corresponding metric, connection, curvature tensor and Ricci curvature tensor on \( N \), respectively. By simple calculation, we have:

**Lemma 9.** Using notations as above, we have

\[
\nabla_{\partial_i} \partial_k = (\ln f(r))' \partial_i, \quad \nabla_{\partial_i} \partial_r = (\ln f(r))' \partial_i, \\
\nabla_{\partial_i} \partial_j = -f(r) f'(r) \bar{g}_{ij} \partial_r + \nabla_{\partial_i} \partial_j.
\]

The second fundamental form \( \Pi = (h_{ij}) \) of the embedding \( \phi_r \) is given by

\[
\Pi = -f(r)^{-1} f'(r) \bar{g}_{ij}.
\]

By Gauss-Codazzi equation, we have

\[
R_{ijkl} - f(r)^2 \bar{R}_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}, \quad R_{rrij} = \left((\ln f)' + (\ln f')^2\right) g_{ij}.
\]

For Ricci curvature tensor, we have

\[
\text{Ric}_{ij} = \left(-\frac{f''}{f} - (n - 2) \left(\frac{f'}{f}\right)^2\right) g_{ij} + \bar{\text{Ric}}_{ij},
\]

\[
\text{Ric}_{rr} = -(n - 1) \left(\frac{f''}{f}\right), \quad \text{Ric}_{ir} = 0.
\]

The next proposition characterizes the warped product structure, see [Pe].

**Proposition 10.** If there exists a function \( u \) on \( M \) such that

\[
\nabla^2 u - \frac{\Delta u}{n} g = 0,
\]

and \( du \neq 0 \), then \( M \) is locally a warped product. If \( du(p) = 0 \) at \( p \in M \) and \( \Delta u(p) \neq 0 \), then \( g = dr^2 + f(r)^2 ds_{n-1}^2 \), where \( ds_{n-1}^2 \) is the standard metric on \( n - 1 \) sphere.

**Proof.** We sketch the proof here. Let \( \alpha = \frac{du}{|du|} \), where \( |du| \) is computed using \( g^{T^*M} \), a metric on \( T^*M \) that is canonically induced from \( g \). By (2.2) and a direct computation using (2.2), we get that \( d\alpha = 0 \). Thus, we may define a local function \( r \) such that \( dr = \alpha \). Thus, we have

\[
dr = \frac{du}{|du|}.
\]

Let \( H = \{x \in M, r = r_0\} \) be a level set of \( r \). Thus, for \( x \in M \) near \( H \), since \( |dr| = 1 \), the distance from \( x \) to \( H \) can be computed as \( |r - r_0| \). By (2.3), for any vector field \( V \) tangent to \( TH \), we have \( du(V) = 0 \). Thus, \( u \) is a function of \( r \). Let \( f(r) = |\nabla u(r)| \). The metric \( g \) on \( M \) can then be locally written as \( g = dr^2 + f(r)^2 g_{H,r} \) in an open domain \((r_0 - \epsilon, r_0 + \epsilon) \times H \) in \( M \). Here \( g_{H,0} \) is a metric on \( H \). A direct computation shows that \( g_{H,r} \) is invariant along the \( \partial_r \) direction. Hence \( g_{H,r} = g_H \). For a local critical point \( p \in M \) of \( u \), it is easy to see that \( H \) has to be \( S^{n-1} \). \( \square \)
Proposition 11. We use notations as given in Definition 8 and Proposition 10. If Ricci curvature is positive, then $u$ is a Morse function, i.e. critical points of $u$ are non-degenerate. Moreover, critical points of $u$ are local maximums or minimums.

Proof. Since Ricci curvature is positive, by (2.1), $f''/f < 0$. Hence, $f(r)$ is strictly concave. Therefore, along each geodesic generated by $\partial r$, $u(r)$ has at most one maximum and one minimum. Recall $f(r(p)) = |\nabla u(p)|$. If $p$ is a critical point of $u$, then $f(r(p)) = 0$ and $\nabla^2 u = f'(r)g$.

Thus $p$ can be a local maximum, minimum, or a degenerate critical point. Now, suppose that $p$ is a degenerate critical point of $u$. Let $r_1 = r(p)$. Then, $f(r_1) = 0$ and $f'(r_1) = 0$, hence $f(r) < 0$ in a neighborhood of $r_1$ which contradicts with the fact that $f$ is non-negative. Thus, $p$ is non-degenerate. □

Corollary 12. Let $(M, g)$ be a connected compact manifold with positive Ricci curvature. If there is a non-vanishing function $u$ satisfying (2.2), then $M$ is diffeomorphic to a sphere with warped product structure

$$ g = dr^2 + f(r)^2 g_{S^{n-1}}, $$

where $g_{S^{n-1}}$ is the standard metric on $S^{n-1}$.

Remark 13. By Morse theory, if $M$ only has maximum and minimum as its critical points, then $M$ is homeomorphic to a sphere. The diffeomorphism follows from the warped product structure.

A similar argument is used to derive the following result for conic manifolds.

Theorem 14. If a conic manifold $(M, g_0, D, g)$ with positive Ricci curvature admits a function $u \in C^{2,\alpha+2\beta}(B_\delta(p_i), g_0) \cap C^\infty(M \setminus D, g_0)$ such that

$$ \text{Hess}_u = \frac{\Delta u}{n} g, \quad M \setminus D, $$

then $|\nabla u|$ is constant on the level set of $u$. Let $f(r) = |\nabla u|$. Then $(M, g)$ has to be one of following:

$A$: $g = dr^2 + f(r)^2 g_{S^{n-1}}$ on $M \setminus \{q\} = S^{n-1} \times (0, \infty)$ with $\lim_{r \to 0} f(r)/r = 1 + \beta$, here $\beta \in (-1, 0)$ is the conic coefficient at conic point $q \in D$. 

\[\text{Figure 2.1. } f(r) \text{ is concave}\]
$B: g = dr^2 + f(r)^2gs_{n-1}$ on $M \setminus \{q, p\} = S^{n-1} \times (a, b)$ with $\lim_{r \to b} \frac{f(r)}{r} = \beta_b$, here $\beta_b \in (-1, 0)$ is the conic coefficient at conic point $p \in D$, and $\lim_{r \to a} \frac{f(r)}{r} = 1$.

$C: g = dr^2 + f(r)^2gs_{n-1}$ on $M \setminus \{q, p\} = S^{n-1} \times (a, b)$ with $\lim_{r \to b} \frac{f(r)}{r} = \beta_b$ and $\lim_{r \to a} \frac{f(r)}{r} = 1 + \beta_a$, here $\beta_a, \beta_b \in (-1, 0)$ are conic coefficients at $q, p \in D$ respectively.

Proof. We follow the proof of Proposition 10 to define $\alpha, r, r_0, H$ and $f(r)$. Hence, $H$ is a hypersurface in $M$. We consider $\gamma(r)$, a maximally extended flow line of $\frac{\partial}{\partial r} = \nabla u/|\nabla u|$. Thus, $\gamma(r)$ is a geodesic and $|\gamma'(r)| = 1$. Since $M$ is compact, $\gamma$ is of finite length. Furthermore, $u|_\gamma$ is increasing with respect to $r$. Also, recall $f(r) = |\nabla u(r)|$.

Let $p_1, p_2 \in \partial \{\gamma(r)\}$. Then, we have $f(r(p_i)) = 0, i = 1, 2$. Therefore, $p_i$ is either a critical point of $u$, or a conic point. We claim that there are finitely many choices of such $p_i$ for all possible choices of $\gamma$.

Fix $i \in \{1, 2\}$. If $p_i$ is one of the conic points, then, by definition they are isolated. We claim that, near $p_i$, the metric $g$ can be written as $g = dr^2 + f(r)^2gs_{n-1}$, where $f(r) \geq 0, f(0) = 0$.

Actually near conic point $p_i$, let $g = e^{2\theta}d(x, p_i)^{2\beta}g_0$, where $\theta$ is $C^\alpha(M, g_0) \cap C^\infty(M \setminus D, g_0)$. Locally, there exists a function $v_2$ which is smooth such that $v(x) = v(d_g(p_i, x), \theta) = v(0) + C't^\alpha + v_2 = v_1 + v_2$, where $t = d_g(p_i, x)$ and $v_1 = v(0) + C't^\alpha$. Let $dr = e^{v_i(t)}t^{\beta}dt$. Then, locally $g_0 = dt^2 + g_1$, where $g_1$ is a metric on $S^{n-1}$. In particular, since $g_0$ is a smooth metric, we have $\lim_{t \to 0} g_0/t^2 = \lim_{t \to 0} gs_{n-1}$ and $\lim_{t \to 0}(\frac{\partial g_0}{\partial t} - \frac{2}{3}g_0) = 0$. For metric $g$, locally $g = e^{2\theta}d(x, p_i)^{2\beta}g_0 = e^{2\theta_1}t^{2\beta}dt^2 + g_1 = dr^2 + f(r)^2gs_{n-1},$

Moreover, $f'(r) \neq 0$ for $r \in (0, \epsilon)$, since $\lim_{r \to 0} f'(r) = 1 + \beta_i$.

If $p_i$ is a regular point of $M$, then it is a critical point of $u$. On a maximally extended interval $(a, b)$ of $\gamma(r)$, $g = dr^2 + f(r)^2g_H$. By the concavity of $f$, the critical point is non-degenerate, hence they are isolated. In summary, we show that all possible choices of $p_i$ are isolated. Hence, there exist finitely many $p_i$.

Next, we consider the general geodesic flow in $\frac{\partial}{\partial r}$ direction. Since $dr$ is closed, in a neighborhood of $H$, $dr$ is integrable. Adjusting by a constant, we may assume that $r|_H = 0$. Then for $\epsilon$ small, $r^{-1}(-\epsilon, \epsilon)$ has a warped product structure. There exists a $a > 0$ such that $\lim_{r \to a} f(r) = 0$. Since all critical points are isolated, we know that for geodesic flow $\gamma(x, r)$:

$$\lim_{r \to a} \gamma(x, r) = p, \forall x \in H,$$

which means that all nearby points flow to one particular choice of $p_i$.

Similarly, there exists some $b > 0$, such that $\lim_{r \to b} f(r) = 0$. Note that we may have at most two critical points for warped product metric.
We have the diffeomorphism
\[ F : (b, a) \times H \to M\setminus\{p, q\} \]
and on \( M\setminus\{p, q\} \), by the above argument
\[ g = dr^2 + f(r)^2 g_H. \]
As \( g \) is conic metric at \( p \), \( f(r)^2 g_H = f(r)^2 g_{S^{n-1}} \) with \( \lim_{r \to 0} f(r)/r = 1 + \beta \), here \( \beta \) is the conic coefficient for \( p \).

We summarize our result. For a conic manifold, there exists at least one conic point and we have three cases as below.

Case A: only one conic point \( q \) and without loss of generality \( F : (0, +\infty) \times H \to M\setminus\{q\} \).

We have
\[ g = dr^2 + f(r)^2 g_{S^{n-1}}, \]
on \( M\setminus\{q\} \) with \( \lim_{r \to 0} f(r)/r = 1 + \beta \), here \( \beta \) is the conic coefficient of point \( q \).

Case B: two critical points \( p, q \) including only one conic point \( p \) and \( F : (b, a) \times H \to M\setminus\{p, q\} \).

We have that \( g = dr^2 + f(r)^2 g_{S^{n-1}} \) on \( M\setminus\{q, p\} \) with \( \lim_{r \to b} |\frac{f(r)}{r}| = 1 + \beta_b \), where \( \beta_b \) is the the conic coefficient of point \( p \), and \( \lim_{r \to a} |\frac{f(r)}{r}| = 1 \). In this case, due to the existence of regular critical point \( p \), we see that \( H \) is topological \( S^{n-1} \).

Case C: two conic points \( p, q \) and \( F : (b, a) \times H \to M\setminus\{p, q\} \).

We have that \( g = dr^2 + f(r)^2 g_{S^{n-1}} \) on \( M\setminus\{q, p\} \) with \( \lim_{r \to b} |\frac{f(r)}{r}| = 1 + \beta_b \), where \( \beta_b \) is the conic coefficient of point \( p \), and \( \lim_{r \to a} |\frac{f(r)}{r}| = 1 + \beta_a \), where \( \beta_a \) is the conic coefficient of point \( q \).

In particular, if \( g \) is smooth, then \((M, g)\) is conformal to warped product \( N \times (-\infty, +\infty) \), or Euclidean space or a sphere. \( \square \)

3. Conic Andrews inequality

In this section, we deal with conic manifolds. We first prove an existence result for Laplacian type equations, which is known to experts. We provide a proof here just for completeness. We then prove Theorem 5.

We begin with a weighted Sobolev inequality from [ChC1].

**Lemma 15.** For \( \alpha > \tau + 2 > 0 \), and \( \varphi \in W^{1, 2}_0(B^n, |x|^\gamma) \), there exists a positive constant \( K_q \) such that
\[ K_q \int_B |\varphi|^q |x|^\alpha dx \leq \int_B |\nabla \varphi|^2 |x|^\gamma dx, \]
where \( \frac{(n+\alpha)}{p} + 1 = \frac{n+\beta}{2} \) and \( 2 \leq q < p \) and especially, \( K_2 = \frac{4}{(n+\tau-2)^2} \).

**Lemma 16.** For a conic manifold \((M, g_0, D, g)\), given \( \phi \in C^\alpha(M, g_0) \cap C^\infty(M\setminus D, g_0) \) satisfying \( \int_M \phi dv_g = 0 \) for \( 0 < \alpha < 1 \), there exists a unique solution \( u \) such that \( \Delta_g u = \phi \) and \( u \in C^{2, \alpha+2\beta_1}(B_b(p_1), g_0) \cap C^\infty(M\setminus D, g_0) \) for \( \alpha + 2 \min \{\beta_i\} > 0 \).

**Proof.** This should be already known somewhere and we give a sketch of standard proof for completeness.

Step 1: we prove that \( W^{1, 2}(dv_g) \) is compact in \( L^2(dv_g) \).

Let \( \{u_k\} \) be a bounded sequence in \( W^{1, 2}(dv_g) \). Denote \( \{\chi_i\} \) as a partition of unity.

\[ ||u_k - u||^2_{L^2(dv_g)} \leq \sum ||\chi_i(u_k - u)||^2_{L^2(dv_g)}. \]
On compact set \(\text{supp} \chi_i\), \(x_i u_k\) is compact in \(L^2(dv_g)\) from Lemma \(\text{[15]}\). By a diagonal process, we have a Cauchy subsequence \(\{u_k\}\) in \(L^2(dv_g)\).

Step 2: we claim that for given \((M^n, g_0, D, g)\), there exists a positive constant \(C_1 > 0\) such that \(\int_{M^n} |\nabla_g v|^2 \geq C_1 (\int_{M^n} v^2 dv_g)\) for \(\int_{M^n} v dv_g = 0\).

Otherwise, there exists a sequence \(v_i\) such that \(i \int_{M^n} |\nabla_g v_i|^2 \leq (\int_{M^n} v_i^2 dv_g)\) and \(\int_{M^n} v_i dv_g = 0\).

Let \(\bar{v}_i = \frac{v_i}{||v_i||_{L^2(dv_g)}}\). We have \(||\bar{v}_i||_{L^2(dv_g)} = 1\) and \(\int_{M^n} |\nabla_g \bar{v}_i|^2 \leq \frac{1}{i}\). So \(\bar{v}_i\) is bounded in \(W^{1,2}(dv_g)\).

By compactness, there exists a subsequence \(\bar{v}_i\) and \(v \in L^2(dv_g)\) such that \(||\bar{v}_i - v||_{L^2(dv_g)} \to 0\) as \(i \to \infty\). We have \(\nabla v = 0\) a.e. With \(\int_{M^n} v dv_g = 0\), \(v = 0\), which is contradicted to \(||\bar{v}_i||_{L^2(dv_g)} = 1\). The claim is proved.

Step 3: We define \(I(v) = \frac{1}{2} \int_{M^n} |\nabla_g v|^2 dv_g - \int_{M^n} \phi dv_g\) and \(\mu = \inf \{I(v) | v \in W^{1,2}(M^n, g, D), \int_{M^n} v dv_g = 0\}\).

\[
I(v) \geq \frac{1}{2} \int_{M^n} |\nabla_g v|^2 dv_g - (\int_{M^n} v^2 dv_g)^{1/2} (\int_{M^n} \phi dv_g)^{1/2} \\
\geq \frac{1}{2} \int_{M^n} |\nabla_g v|^2 dv_g - C_1 (\int_{M^n} |\nabla_g v|^2 dv_g)^{1/2} (\int_{M^n} \phi^2 dv_g)^{1/2} \\
\geq \left(\frac{1}{2} - \epsilon\right) \int_{M^n} |\nabla_g v|^2 dv_g - C_2 \int_{M^n} \phi^2 dv_g
\]

Thus, \(\inf I(v)\) exists for small \(\epsilon\). We can take a minimizing sequence \(v_j, j \in \mathbb{N}\). By the compactness, there exists a function \(u\) attaining the minimum.

There exists a solution \(u \in W^{1,2}(M^n, g, D)\) such that \(\Delta_g u = \phi\) with \(\phi \in C^\alpha(M, g_0) \cap C^\infty(M\setminus D, g_0)\). Near singularity \(p_i\), \(\Delta_g u = d(x, p_i)^{2\beta_i}\phi\) and near \(p_i\), \(u \in C^{2,\alpha+2\beta_i}(B_{\delta_i}(p_i), g_0) \cap C^\infty(M\setminus D, g_0)\) by classical Schauder theory. \(\square\)

Now, we may give the proof of Theorem \(\text{[5]}\)

Proof of Theorem \(\text{[5]}\). For \(\int_{M^n} \phi dv_g = 0\), there exists a solution \(u \in C^{2,\alpha+2\beta_i}(B_{\delta_i}(p_i), g_0) \cap C^\infty(M\setminus D, g_0)\) such that \(\Delta_g u = \phi\).

Near \(p_i \in D, r = d_{g_0}(x, p_i)\),

\[
\Gamma^k_{ij}(x) \approx r^{-2\beta_i} r^{2\beta_i - 1} = \frac{1}{r}, \quad \Gamma^k_{ij} u_k \approx r^{-1+1+\alpha+2\beta_i} = r^{\alpha+2\beta_i},
\]

\[
\nabla^j u^{ij} \approx r^{-4\beta_i} r^{\alpha+2\beta_i + 1}, \quad \nabla_{ij} u \approx r^{\alpha+2\beta_i}, \quad dv_g \approx r^{n\beta_i + n-1} dr ds_{n-1}.
\]

We obtain \(|\nabla^2 u - \frac{\Delta u}{n} g^{ij}_{g_0} dv_g| \approx C r^{2\alpha+n(1+\beta_i) - 1} dr \text{ and } \int_{B_{\delta_i}(p_i) \setminus \{p_i\}} |\nabla^2 u - \frac{\Delta u}{n} g^{ij}_{g_0} dv_g|
\] is then well defined. Moreover, \(\lim_{\delta_i \to 0} \int_{B_{\delta_i}(p_i) \setminus \{p_i\}} |\nabla^2 u - \frac{\Delta u}{n} g^{ij}_{g_0} dv_g = 0\). The proof is similar to Andrews’s original proof and the only difference is to check the following
\[
\int_{M^n \setminus \partial B} |\nabla^2 u - \frac{\Delta u}{n} g_j^2| dv_g
\]
\[
= \int_{M^n \setminus \partial B} \nabla_i \nabla_j u \nabla^i \nabla^j u - \frac{\Delta u}{n} g_j^2 dv_g
\]
\[
= \int_{M^n \setminus \partial B} \nabla_i \nabla_j u \nabla^i \nabla^j u + \int_{M^n \setminus \partial B} \frac{\Delta u}{n} g_{ij} \frac{\Delta u}{n} g^{ij} dv_g - 2 \int_{M^n \setminus \partial B} \nabla_i \nabla_j u \frac{\Delta u}{n} g^{ij} dv_g
\]
\[
= \lim_{\varepsilon \to 0} \left( - \int_{M^n \setminus \cup_{l=1}^k B_{\varepsilon}(p_l)} \nabla^i \nabla_j u \nabla^i \nabla^j u dv_g + \sum_{l=1}^k \int_{\partial B_{\varepsilon}(p_l)} \nabla_i \nabla_j u \nabla^i \nabla^j u dv_g \right)
\]
\[
+ \int_{M^n \setminus \partial B} \frac{\Delta u}{n}^2 dv_g - \int_{M^n \setminus \partial B} \frac{2(\Delta u)^2}{n} dv_g
\]
\[
= - \int_{M^n \setminus \partial B} \frac{(\Delta u)^2}{n} dv_g - \int_{M^n \setminus \partial B} \nabla^i \nabla_j u \nabla^i u dv_g + \lim_{\varepsilon \to 0} \sum_{l=1}^k \int_{\partial B_{\varepsilon}(p_l)} \nabla_i \nabla_j u \nabla^i u dv_g.
\]

Let us see
\[
\lim_{\varepsilon \to 0} \int_{\partial B_{\varepsilon}(p_l)} \nabla_i \nabla_j u \nabla^i u dv_g \equiv \lim_{\varepsilon \to 0} \varepsilon^{n+2\beta_i} \varepsilon^{n+2\beta_i} = 0
\]
if \(n + 2\alpha + 4\beta_l > 0\).

\[
- \int_{M^n \setminus \partial B} \nabla^i \nabla_i \nabla_j u \nabla^j u dv_g
\]
\[
= - \int_{M^n \setminus \partial B} R_{ij} u^i u^j dv_g - \int_{M^n \setminus \partial B} \nabla_j \Delta u \nabla^j u dv_g
\]
\[
= - \int_{M^n \setminus \partial B} R_{ij} u^i u^j dv_g + \lim_{\varepsilon \to 0} \int_{M^n \setminus \cup_{l=1}^k B_{\varepsilon}(p_l)} (\Delta u)^2 dv_g - \lim_{\varepsilon \to 0} \sum_{l=1}^k \int_{\partial B_{\varepsilon}(p_l)} \Delta u \nabla^i u dv_g
\]
\[
= - \int_{M^n \setminus \partial B} R_{ij} u^i u^j dv_g + \int_{M^n \setminus \partial B} (\Delta u)^2 dv_g.
\]

Therefore
\[
\int_{M^n \setminus \partial B} |\nabla^2 u - \frac{\Delta u}{n} g_j^2| dv_g = - \int_{M^n \setminus \partial B} R_{ij} u^i u^j dv_g + \frac{n-1}{n} \int_{M^n \setminus \partial B} (\Delta u)^2 dv_g.
\]

We also get
\[
\int_{M^n \setminus \partial B} \frac{1}{n} (\nabla^i \phi + b R_{ij} u^k) (\nabla^j \phi + b R_{ij} u^l) dv_g
\]
\[
= \int_{M^n \setminus \partial B} \frac{1}{n} (\nabla^i \phi \phi + b R_{ij} u^k \phi) (\nabla^j \phi \phi + b R_{ij} u^l \phi) dv_g
\]
\[
= \int_{M^n \setminus \partial B} \frac{1}{n} (\nabla^i \phi \phi + b R_{ij} u^k \phi) (\nabla^j \phi \phi + b R_{ij} u^l \phi) dv_g + \int_{M^n \setminus \partial B} b^2 R_{ij} u^i u^j dv_g
\]
\[
= \int_{M^n \setminus \partial B} \frac{1}{n} (\nabla^i \phi \phi + b R_{ij} u^k \phi) (\nabla^j \phi \phi + b R_{ij} u^l \phi) dv_g + \int_{M^n \setminus \partial B} b^2 R_{ij} u^i u^j dv_g - 2 \int_{M^n \setminus \partial B} b \Delta u \phi dv_g + 2 \lim_{\varepsilon \to 0} \sum_{l=1}^k \int_{\partial B_{\varepsilon}(p_l)} b^k \phi \phi ds_g
\]
\[
= \int_{M^n \setminus \partial B} \frac{1}{n} (\nabla^i \phi \phi + b R_{ij} u^k \phi) (\nabla^j \phi \phi + b R_{ij} u^l \phi) dv_g - 2 \int_{M^n \setminus \partial B} b^2 R_{ij} u^i u^j dv_g - 2 \int_{M^n \setminus \partial B} b \phi^2 dv_g.
Taking $a = 1/b$, we obtain

\[ 0 \leq \int_{M^n \setminus D} |\nabla^2 u - \frac{\Delta u}{n} g|^2 dv_g + a^2 \int_{M^n \setminus D} (Rc^{-1})^{ij} (\nabla_i \phi + bR_{ik} u^k)(\nabla_j \phi + bR_{jl} u^l) \]

\[ = -\int_{M^n \setminus D} R_{ij} u^i u^j dv_g + \frac{n-1}{n} \int_{M^n \setminus D} (\Delta u)^2 dv_g + a^2 \left( \int_{M^n \setminus D} (Rc^{-1})^{ij} \phi_i \phi_j dv_g \right) \]

\[ + \frac{n-1}{n} \int_{M^n \setminus D} (\Delta u)^2 dv_g - 2a \int_{M^n \setminus D} \phi^2 dv_g + a^2 \int_{M^n \setminus D} (Rc^{-1})^{ij} \phi_i \phi_j dv_g. \]

So

\[ (-\frac{n-1}{n} + 2a) \frac{1}{a^2} \int_{M^n \setminus D} \phi^2 dv_g \leq \int_{M^n \setminus D} (Rc^{-1})^{ij} \phi_i \phi_j dv_g. \]

Taking $a = \frac{n-1}{n}$, we get the inequality.

When the Andrews inequality (1.3) is an equality, we have for $\Delta u = \phi$,

\[ \nabla^2 u - \frac{\Delta u}{n} g = 0 \quad \text{on} \quad M \setminus D \]

and

\[ (\nabla_i \phi + \frac{n}{n-1} R_{ik} u^k = 0 \quad \text{on} \quad M \setminus D. \]

By Theorem 14, the metric $g$ can be written as

\[ g = f(r)^2 dS_{n-1} + dr^2. \]

where $f = u'$. Since $\int_M \Delta u dv_g = \int_M \phi dv_g = 0$, we have $f(0) = f(a)$. Thus, by Case C of Theorem 14 the conic coefficients of $p$ and $q$ are identical. A direct computation from (3.2) shows that the metric is conformally flat. \(\square\)

4. Andrews Inequality for Manifolds with Boundary

In this section, we prove Theorem 1.4. We first prove that the inequality (1.4) and then prove the rigidity result when the equality holds.

**Proof of Inequality (1.4)**. Given $\phi \in C^\infty(M)$, we consider the following equation with Neumann boundary condition:

\[ \begin{cases} \Delta u = \phi, & x \in M, \\ u_\nu = 0, & x \in \partial M. \end{cases} \tag{4.1} \]

Here (4.1) is clearly solvable when $\int_M \phi = 0$. As in the conic case, we consider the following inequality

\[ 0 \leq \int_M \left| \nabla^2 u - \frac{\Delta u}{n} g \right|^2 dv_g + a^2 \int_M (Ric^{-1})^{ij} (\nabla_i \phi + \frac{1}{a} Ric_{jk} u^k)(\nabla_j \phi + \frac{1}{a} Ric_{jk} u_k)dv_g. \tag{4.2} \]

Note

\[ \int_M \left| \nabla^2 u - \frac{\Delta u}{n} g \right|^2 dv_g = \int_M |\nabla^2 u|^2 dv_g - \frac{1}{n} \int_M |\Delta u|^2 dv_g. \tag{4.3} \]
By Bochner formula:

(4.4) \[ |\nabla^2 u|^2 = \Delta \left( \frac{1}{2} |\nabla u|^2 \right) - \nabla u \cdot \nabla \Delta u - Ric(\nabla u, \nabla u). \]

Combine (4.3) and (4.4), we have

(4.5) \[ \int_M |\nabla^2 u|^2 = \int_{\partial M} \left( \frac{1}{2} \partial_{\nu} |\nabla u|^2 - \Delta u \partial_{\nu} u \right) d\sigma + \int_M |\Delta u|^2 dv_g - \int_M Ric(\nabla u, \nabla u) dv_g. \]

Here, \( \nu \) is the outer normal vector. In the second term in (4.2), we obtain

(4.6) \[ a^2 \int_M (Ric^{-1})^{ij}(\nabla_i \phi + \frac{1}{a} Ric^k_i u_k)(\nabla_j \phi + \frac{1}{a} Ric^k_j u_k) dv_g \]

Note

(4.7) \[ \int_M \nabla \cdot \nabla udv_g = \int_{\partial M} \phi \partial_{\nu} ud\sigma - \int_M \phi^2 dv_g = \int_{\partial M} \Delta u \partial_{\nu} u d\sigma - \int_M \phi^2 dv_g. \]

Combining (4.2), (4.5), (4.6), and (4.7), we get

(4.8) \[ 0 \leq \left( \frac{n-1}{n} - 2a \right) \int_M \phi^2 + a^2 \int_M (Ric^{-1})^{ij} \nabla_i \phi \nabla_j \phi \]

Next, we compute boundary terms. With \( u_{\nu} = 0 \) on \( \partial M \), we write

(4.9) \[ \nabla u|_{\partial M} = u_i e_i, \]

where \( e_1, e_2 \cdots e_{n-1} \in TM \) form a local orthonormal frame tangential to the boundary. Then

(4.10) \[ \nu(|\nabla u|^2) = 2(\nabla_u \nabla u, \nabla u) = 2\text{Hess}_u(\nabla u, \nu). \]

Recalling

\[ \Pi(\alpha, \beta) = \langle \nabla_{\alpha} \beta, -\nu \rangle|_{\partial M} \]

is the second fundamental form of the boundary \( \partial M \) with respect to the inner normal vector field \(-\nu\). Since \( u_{\nu} = 0 \), we obtain

(4.11) \[ 0 = e_k(\nabla_u \nu) = \langle \nabla_{e_k} \nabla_u \nu \rangle + \langle \nabla_u \nabla_{e_k} \nu \rangle = \text{Hess}_u(e_i, \nu) + \Pi(e_k, \nabla u). \]

Thus, by (4.9) and (4.11)

\[ 2\text{Hess}_u(\nabla u, \nu) = 2u_i \text{Hess}_u(e_i, \nu) = -2u_i \Pi(e_i, \nabla u) = -2\Pi(\nabla u, \nabla u). \]

Notice that the boundary term in (4.8) is

(4.12) \[ \int_{\partial M} \left( \frac{1}{2} \partial_{\nu} |\nabla u|^2 + (2a - 1) \Delta u \partial_{\nu} u \right) d\sigma = -\int_{\partial M} \Pi(\nabla u, \nabla u) d\sigma, \]
since \( u_\nu = 0 \). By (4.8) and the assumption \( II \geq 0 \), we have
\[
0 \leq \int_{\partial M} II(\nabla u, \nabla u) \leq \left( \frac{n-1}{n} - 2a \right) \int_M \phi^2 + a^2 \int_M (\text{Ric}^{-1})^{ij} \nabla_i \phi \nabla_j \phi.
\]
Therefore,
\[
- \left( \frac{n-1}{n} - 2a \right) \int_M \phi^2 \leq a^2 \int_M (\text{Ric}^{-1})^{ij} \nabla_i \phi \nabla_j \phi.
\]
Then the boundary Andrews inequality (1.4) follows by choosing \( a = \frac{n-1}{n} \). \( \square \)

Next, we discuss the equality case of (1.4). An immediate consequence of the proof of (1.4) is the following corollary:

**Corollary 17.** The equality in Andrews inequality (1.4) holds if and only if there is a function \( u \) such that

\[
\begin{align*}
(4.13) \\
\begin{cases}
\nabla^2 u - \frac{\Delta u}{n} g = 0, \\
II(\nabla u, \nabla u)|_{\partial M} = 0,
\end{cases}
\end{align*}
\]

where \( \phi = \Delta u \).

**Proof.** The first restriction \( \nabla^2 u - \frac{\Delta u}{n} g = 0 \) implies that \( M \) is locally a warped product. In fact, locally we have the warped product structure,
\[
g_M = dr^2 + f(r)^2 g_N,
\]
where \( dr = \frac{du}{|du|} \), and \( f = |du| \), see Proposition 10. Then, \( \phi = f'(r) \), and by (2.1), we see that
\[
\nabla \phi + \frac{n}{n-1} \text{Ric}(\nabla u) = 0,
\]
holds automatically. \( \square \)

Since \( u \) restricted on \( \partial M \) also yields a locally warped product structure, we now state a rigidity result for \( \partial M \).

**Corollary 18.** If the equality in Andrews inequality (1.4) holds, then \( (\partial M, g|_{\partial M}) \) is conformal to a \( n-1 \) sphere with the standard structure.

**Proof.** Let \( \bar{u} = u|_{\partial M} \) be the restriction of \( u \) on the boundary. Let \( \bar{g}, \bar{\nabla} \) be the induced metric and connection on \( \partial M \). Since \( u_\nu = 0 \), we have that \( \nabla \bar{u} = \nabla u|_{\partial M} \).

Let \( e_1 \cdots e_{n-1} \) be a local orthonormal frame on \( \partial M \). We compute the Hessian of \( \bar{u} \):
\[
\text{Hess}_{\bar{u}}(e_i, e_j) = \langle \nabla_{e_i} \nabla \bar{u}, e_j \rangle_{\bar{g}}
\]
\[
= \langle \nabla_{e_i} \nabla u - \langle \nabla_{e_i} \nabla u, \nu \rangle \nu, e_j \rangle_{g}
\]
\[
= \langle \nabla_{e_i} \nabla u, e_j \rangle_{g}
\]
\[
= \text{Hess}_{\bar{u}}(e_i, e_j)|_{\partial M}.
\]
Therefore, by (4.13),
\[
\text{Hess}_{\bar{u}} = \frac{\Delta u}{n} g|_{\partial M} = \frac{\Delta u}{n} \bar{g} = \frac{\Delta \bar{u}}{n-1} \bar{g}.
\]
Then, using Corollary 12 for \( \partial M \) and \( \bar{u}, \partial M \) is a warped sphere. We finish the proof. \( \square \)

If \( \partial M \) is connected, by Theorem 1, \( \partial M \) is conformal to the standard \( n-1 \) sphere. In order to finish our argument for \( M \), we need to establish the connect-ness of \( \partial M \).
Proposition 19. If $M$ is connected, then $u$ can only have one maximum and one minimum. Furthermore, $\partial M$ is connected and $M$ has to be a hemisphere.

Proof. This is a straightforward application of Morse theory. We first observe that by Proposition 11, there are only maximums and minimums on $M$. Suppose that $\partial M$ has at least two connected components. Then $u$ has at least one maximum on each boundary component which also happens to be the local maximum of $u$ in $M$. Let $p_1, \cdots, p_k$ be maximum points of $u$. Consider the flow given by

$$\frac{d}{dt} \phi_t(x) = -\nabla u(\phi_t(x)).$$

Note the flow $\phi$ is compatible with the boundary since $\nabla u$ is tangent to the boundary. Let

$$\mathcal{D}(p_i) = \{x \in M : \lim_{t \to -\infty} \phi_t(x) = p_i\}.$$ 

be the unstable manifold of $p_i$. For two different local maximums $p_1$ and $p_2$, if critical point $q \in \mathcal{D}(p_1) \cap \mathcal{D}(p_2)$, then $q$ has to be a critical point of $u$ with non-trivial index which however does not exist. Therefore, $\mathcal{D}(p_1) \cap \mathcal{D}(p_2) = \emptyset$, which implies that $M$ is not connected. This is a contradiction. We have thus proved that $\partial M$ is connected.

Proof of Theorem 2. We prove the rigidity result when the equality holds in (1.4). By Proposition 19, there is only one maximum $p$ and one minimum $q$ on $M$ which also lie on the boundary. Since $p$ is non-degenerate, near $p$, level sets of $u$ near local maximum are diffeomorphic to a $(n-1)$ dimensional hemisphere by Morse Lemma. Take $\epsilon$ small and let $L$ be the level set $u^{-1}(u(p) - \epsilon)$. For generic choice of $\epsilon$, $L$ is a smooth hypersurface with boundary. Since $\nabla u = \nabla u|_{\partial M}$, we see that $M$ has a warped product metric given by

$$g = dr^2 + f(r)^2 g_L,$$

where $r$ and $f = |\nabla u|$ are globally defined since $M - \{p,q\}$ is contractible. Note $f(r(p)) = 0$ if $p \in \partial M$ is a critical point of $u$. Without loss of generality, we assume that $r(p) = 0$. By restricting the warped product structure to $\partial M$, we see that

$$\lim_{r \to 0} (f(r)^2/r^2) g_{\partial L} = g_{S^{n-2}}.$$

In fact, by choosing a geodesic polar coordinate at $p$ on $\partial M$ and $\nabla u = \nabla u|_{\partial M}$, we see (4.14) easily. On the other hand, $p$ is also a maximum point on $M$. Since $p \in \partial M$, by the same argument,

$$\lim_{r \to 0} (\frac{f(r)}{r})^2 g_L = g_{S^{n-1}_+},$$

where $g_{S^{n-1}_+}$ is the standard metric on unit hemisphere. Therefore, $g_L = c^2 \cdot g_{S^{n-1}_+}$ for some constant $c$ and the metric on $M$ is given by

$$g = dr^2 + \tilde{f}(r)^2 g_{S^{n-1}_+},$$

for some $\tilde{f}(r) = \frac{1}{2} f(r)$. Now, this warped product structure can be extended to the whole manifold except the critical points of $u$. Same argument works for the global minimum point $q$. Thus, the warped product structure can be extended to $q$. Finally, a direct computation shows that the second fundamental form of $\partial M$ vanishes identically, which implies that $\partial M$ is totally geodesic. We have finished the proof. \qed
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