The limit shape of the height function in the six-vertex model with domain-wall boundary conditions

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Abstract. The height function of the six-vertex model with the domain-wall boundary conditions in the free fermion point is computed by the Monte Carlo algorithm. The numerical results are in good agreement with the analytical expression for the limit shape height function. This paper is a "warm up" for the forthcoming one, where the two-point correlation function for the height function is calculated.

1. Introduction
The six-vertex is the integrable lattice model of statistical physics [1, 2, 3]. It was initially introduced as the model of the two-dimensional ice on a square lattice to study melting and crystallization [4]. The states in this model are configurations of arrows on edges which satisfy the ice rule: each edge of the lattice obtains an orientation, an arrow, in such a way that for each vertex there are exactly two incoming and two outgoing edges. The six types of possible vertices are presented in figure 1. The partition function of the six-vertex model in square lattice is given by a sum of configurations

$$Z = \sum_{\text{conf}} \prod_{i=1}^{N} \prod_{j=1}^{N} w_{\text{conf}}^{(i,j)},$$

where the Boltzmann weight $w_{\text{conf}}^{(i,j)}$ for each vertex takes a predefined value (see figure 1). These weights are the matrix elements of the $R$-matrix. They satisfy the Yang-Baxter equation [2, 5]. The model has exact solutions in some special cases [6, 7] and is under active theoretical [8-23] and numerical investigation [24-27]. The six-vertex model on $N \times N$ square lattice with domain-wall boundary conditions [6] is known to develop a limit shape when $N \to \infty$ [28, 29]. States of the model can be regarded as configurations of lattice paths [25]. The lattice paths can be regarded as the level curves of a step function defined on faces. This step function is, in turn, called the height function. The limit shape phenomenon means that in the thermodynamic limit the normalized random height function converges to a continuous function $h(x, y)$. This function is linear (“frozen”) outside of a curve which is usually called “the Arctic curve” [30]. Its shape depends on the parameters of the model and boundary conditions. Inside this curve, the height...
Figure 1. Local configurations and weights of the six-vertex model.

function is smooth and strictly convex function \( h_0(x,y) \). We will focus on the case of symmetric model with weights \( a_1 = a_2 = a, b_1 = b_2 = b, c_1 = c_2 = c \). The parameter
\[
\Delta = \frac{a^2 + b^2 - c^2}{2ab}
\]
plays an important role in the model. It characterizes the phase the model is in on a torus. We study the case \( \Delta = 0 \) when fluctuations around the limit shape are described by the Gaussian free field [31, 19]. In this case, the model is mapped to a dimer model [29] and the partition function can be computed as the determinant of the Kasteleyn matrix [32].

In the current report, we compare the analytical height function obtained based on the results of A. G. Pronko [33] and the numerical one for the case \( \Delta = 0 \). The numerical height function is computed by the Markov chain Monte Carlo algorithm developed in [25]. The numerical results are obtained by averaging over random states of the Markov process. A good agreement of the analytical and numerical results is observed. This report is a “warm up” for the forthcoming paper with N. Yu. Reshetikhin, where the two-point correlation function for the height function is calculated.

2. The limit shape height function
In this section, we describe the limit shape height function \( h_0(x,y) \) for the six-vertex model with the domain-wall boundary conditions. We use the result of F. Colomo, V. Noferini and A. G. Pronko [33] where the function
\[
D(x,y) = \alpha (1-\alpha) \left[ \frac{(y-x)^2}{\alpha} + \frac{(1-x-y)^2}{1-\alpha} - 1 \right]
\]
was derived. The Arctic curve for the square domain \( 0 \leq x \leq 1, 0 \leq y \leq 1 \) is defined by the ellipse \( D(x,y) = 0 \). Here, the parameter \( \alpha \) is determined by values of Boltzmann weights of the model as \( \alpha = b/a \). It corresponds to the Baxter’s parametrization \( \alpha = \tan (u) \) [2].

The following result has been obtained by A. G. Pronko:
\[
\frac{\partial y}{\partial y} h_0(x,y) = \begin{cases} 
\frac{1}{\pi} \arccot \left( \frac{-n y}{\sqrt{-D(x,y)}} \right), & (x,y) \in D(x,y) < 0 \\
0, & (x,y) \in A \cup B \\
1, & (x,y) \in C \cup D
\end{cases}
\]

Here \( n_y = (1-\alpha)(y-x) + \alpha(1-x-y) = x + (2\alpha - 1)y - \alpha \). Regions \( A, B, C, D \) are shown in figure 2. In figure 3, left plot, we show the function (1) for \( \alpha = 9/25 \). Inside the elliptic arctic curve, it is given by the nontrivial part of Eq. (1) and outside that one it equals to zero or one.
**Figure 2.** The ellipse $D(x, y) = 0$ is the boundary of the limit shape. The height function is smooth inside the ellipse and linear outside. Here we also show the lines $n_x = 0$, $n_y = 0$ and areas $A, B, C, D$.

Integrating Eq. (1), we obtain the following analytical expression for the limit shape height function:

$$h_0(x, y) = \begin{cases} 0 & (x, y) \in D(x, y) < 0 \& x < 1 - \alpha \\ \frac{1}{2} \mathcal{H}(x, y) + \frac{x}{2} - \frac{1}{2} & (x, y) \in D(x, y) < 0 \& x > 1 - \alpha \\ x & (x, y) \in A \\ y & (x, y) \in B \\ x + y - 1 & (x, y) \in C \\ y & (x, y) \in D \end{cases}$$

(2)

where

$$\mathcal{H}(x, y) = y \arccot \left[ \frac{-n_y}{\sqrt{-D(x, y)}} \right] - \frac{1}{2} \arctan \left[ \frac{-x^2 + (y - \alpha)(\alpha - 1) + x(1 + y - 2y\alpha)}{(1 - \alpha - x)\sqrt{-D(x, y)}} \right] + \left( \frac{1}{2} - x \right) \arctan \left[ \frac{-n_x}{\sqrt{-D(x, y)}} \right].$$

Here $n_x = (1 - \alpha)(y - x) + \alpha(x + y - 1) = y + (2\alpha - 1)x - \alpha$. The height function $h_0(x, y)$ for $\alpha = 9/25$ is shown in figure 3, right plot.

**3. Numerical calculation of the height function**

For calculation of the height function, we use the Markov chain Monte Carlo algorithm to generate a sequence of the random configurations of the six-vertex model developed in [25, 27]. The algorithm is based on the special choice of the transition probabilities, see [25]. It is also known as the Metropolis algorithm [34, 35]. This algorithm and similar ones have already been used in a number of related studies [26, 37-39].

The idea of the method is to create a random process that will follow most likely configurations in the model. Transition probabilities should satisfy the detailed balance condition [35]. This condition (plus some minor technical assumptions such as nondegeneracy of the largest eigenvalue)
Figure 3. Left plot: the density of horizontal edges occupied by paths, $\partial_y h_0(x, y)$, for $\alpha = 9/25$. Right plot: the limit shape height function $h_0(x, y)$ [Eq. (2)] of the six-vertex model inside the arctic curve. The parameter $\alpha = 9/25$ guarantees that the process converges to the Boltzmann distribution. The average of an observable along the states generated by this process tends to the expectation value as a number of iterations of the process grows.

This procedure is especially effective when the Boltzmann distribution is concentrated in a small vicinity of the most likely configuration (the limit shape). In dimer models it was proven [39] that there exists such a state. The probability for any other state to be “macroscopically distant” from it is exponentially suppressed. The six-vertex model in the free fermionic point ($\Delta = 0$) is equivalent to the dimer model. Therefore, in this case we have the same structure of probability distribution. The concentration of random configurations near the limit shape makes the numerical computation of observables easy when the Markov process is thermalized. To do it, we observe fluctuations of the normalized volume under the height function:

$$\frac{1}{N^3} \sum_{(n,m) \in D_N} h(n, m).$$

The normalized volume “drifts”, when the process is not yet thermalized, then it starts to fluctuate around the normalized volume under the limit shape $h_0$. The numerical experiments show that for the lattice of size $60 \times 60$ and $\Delta = 0$ we can start averaging after about $10^6$ elementary random changes of the paths.

Once the thermalization is achieved, we compute an observable by averaging:

$$\langle O \rangle = \frac{O(s_1) + \cdots + O(s_K)}{K}.$$ 

Here $s_i$ is a random configuration, $K$ is the total number of configurations. The right side is a random variable, but as $K \to \infty$ it converges to the Boltzmann expectation value.

From Monte Carlo simulations we obtained the numerical height function for each configuration for $\Delta = 0$ and different lattice sizes. An example of such a numerical result is shown in figure 4, left plot, for the lattice size $40 \times 40$. We see that the numerical height function fluctuates inside the Arctic curve. Outside the Arctic curve the numerical data is fixed, so it looks like linear. The fluctuations inside the Arctic curve can be avoided and the
Figure 4. Left plot: the numerical height function of a single configuration for $\Delta = 0$ and for the lattice size of $40 \times 40$. Central plot: the numerical height function as a result of averaging over $10^4$ random configurations. Right plot: the difference between the theoretical height function and the one obtained from averaging.

The central plot of figure 4 shows the result of averaging over $10^4$ configurations. Now, the numerical height function is almost identical to the function $h_0(x, y)$ [compare with figure 3, right plot]. Nevertheless, there is a difference between the averaged height function and $h_0(x, y)$ calculated by Eq. (2). This difference is presented in figure 4, right plot. One can see that it reveals the Airy asymptotics near the boundary of the limit shape [40]. The calculations for other lattice sizes demonstrate that this difference vanishes as $N \to \infty$.

4. Conclusion
We presented the numerical computation of the height function in the six-vertex model with domain-wall boundary conditions in the free fermion point ($\Delta = 0$). The results are in good agreement with the ones derived analytically. They are the stepping stone for numerical computation of the correlation functions which will be done in the near future.

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