Littlewood-Paley Characterization for Musielak-Orlicz-Hardy Spaces Associated with Operators

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Abstract Let $X$ be a space of homogeneous type. Assume that $L$ is an non-negative second-order self-adjoint operator on $L^2(X)$ with (heart) kernel associated to the semigroup $e^{-tL}$ that satisfies the Gaussian upper bound. In this paper, the authors introduce a new characterization of the Musielak-Orlicz-Hardy Space $H_{\varphi,L}(X)$ associated with $L$ in terms of the Lusin area function where $\varphi$ is a growth function. Further, the authors prove that the Musielak-Orlicz-Hardy Space $H_{L,G,\varphi}(X)$ associated with $L$ in terms of the Littlewood-Paley function is coincide with $H_{\varphi,L}(X)$ and their norms are equivalent.

Keywords Musielak-Orlicz Hardy spaces · Heat semigroup · Gaussian estimate · Nonnegative self-adjoint operator · Area and Littlewood-Paley functions

Mathematics Subject Classification 42B20 · 42B25 · 42B30 · 42B35 · 46E30 · 47B38

1. Introduction

Recently, the study of the Hardy spaces associated with operators has been in the spotlight. This topic was initiated by Auscher et al. [2], who studied the Hardy space $H^1_1(\mathbb{R}^n)$ associated with the operator $L$ whose heat kernel satisfies the pointwise Poisson upper bounded condition. Later on, the adapted BMO theory has been presented by Duong and Yan [4, 5], under the assumption that the heat kernel associated to $L$ satisfies the pointwise Gaussian estimate. The theory of the Hardy space $H^p(\mathbb{R}^n)$ for $0 < p < 1$ associated with the operator $L$ satisfying the Davies-Gaffney estimates was established by Yan [13]. It is then quite natural to consider the weighted Hardy spaces $H^p_{L,\omega}(\mathbb{R}^n)$ associated with an operator $L$ and a weight function $\omega$. Song and Yan [12] first introduced the weighted Hardy space $H^1_{L,\omega}(\mathbb{R}^n)$ associated with the Schrödinger operator $L$ for $\omega \in A_\infty(\mathbb{R}^n)$. Recently, Duong et al. [6] considered two kinds of weighted Hardy spaces on the homogeneous spaces $X$ associated with an operator whose kernel satisfying the Gaussian upper bound. For $0 < p \leq 1$ and $\omega \in A_\infty$, they first studied the weighted Hardy space $H^p_{L,S,\omega}(X)$ which defined in terms of the Lusin area function, and secondly turned to consider the weighted Hardy space $H^p_{L,G,\omega}(X)$ which defined in terms of the Littlewood-Paley function. Finally, they obtained the equivalence between the two kinds of weighted Hardy spaces by adding the Moser...
Definition 1.1. Let $L$ satisfies (H1) and (H2), and $\varphi$ be a growth function. A function $f \in H^2(X)$ is said to be in $\tilde{H}_{\varphi,L}(X)$ if $S_L(f) \in L^2(X)$. Moreover, define

$$||f||_{\tilde{H}_{\varphi,L}(X)} := ||S_L(f)||_{L^2(X)} := \inf \left\{ \lambda \in (0, \infty) : \int_X \varphi \left( x, \frac{S_L(f)(x)}{\lambda} \right) d\mu(x) \leq 1 \right\}. $$

The Musielak-Orlicz-Hardy space $H_{\varphi,L}(X)$ is defined to be the completion of $\tilde{H}_{\varphi,L}(X)$ with the quasi-norm $||.||_{\tilde{H}_{\varphi,L}(X)}$. 

On the other hand, Ky [11] presented a new Musielak-Orlicz-Hardy space, $H_{\varphi}(\mathbb{R}^n)$, defined via a growth function $\varphi$ (see Sect.2 below for the definition of growth function). As an natural generalization, the Musielak-Orlicz-Hardy space $H_{\varphi,L}$ defined via the Lusin area function associated with an operator $L$ that satisfies the Davies-Gaffney estimate, which contains the weighted Hardy space $H^p_{L,S,w}(X)$ in [6], had been introduced and systematically studied by Yang et al. in [14] later on. Characterizations of $H_{\varphi,L}$, including the atom, the molecule, etc. was obtained in [14]. However, to characterize $H_{\varphi,L}$, Yang et al. needed to impose an extra assumption that the growth function $\varphi$ satisfies the uniformly reverse Hölder condition.

Throughout this article (X, $d$, $\mu$) is a metric measure space endowed with a distance $d$ and a non-negative Borel doubling measure $\mu$. And we assume that $L$ is a densely defined operator on $L^2(X)$ and satisfies the following two conditions in different sections of this paper.

(H1) $L$ is a second-order non-negative self-adjoint operator on $L^2(X)$;

(H2) The kernel of $e^{-\lambda t}L$, denote by $p_t(x,y)$, is a measurable function on $X \times X$ and satisfies the Gaussian estimates, namely, there exist positive constants $C_1$ and $C_2$ such that, for all $t > 0$, and $x,y \in X$,

$$|p_t(x,y)| \leq \frac{C_1}{V(x, \sqrt{t})} \exp \left( -\frac{d(x,y)^2}{C_2 t} \right),$$

where $V(x, \sqrt{t}) = \mu(B(x, \sqrt{t}))$.

Given an operator $L$ that satisfying (H1) and (H2) and a function $f \in L^2(X)$, we consider the following Littlewood-Paley function $G_L(f)$ and Lusin area function $S_L(f)$ associated with the heat semigroup generated by $L$

$$G_L(f)(x) := \left( \int_0^\infty |t^2 L e^{-t^2} L f(x)|^2 dt \right)^{1/2}$$

and

$$S_L(f)(x) := \left( \int_0^\infty \int_{d(x,y)<t} |t^2 L e^{-t^2} L f(y)|^2 \frac{d\mu(y)}{\mu(B(x,t))} dt \right)^{1/2}. $$

In this paper, Musielak-Orlicz Hardy spaces $H_{\varphi,L}$ and $H_{L,G,\varphi}$ will be concerned. Their definitions are as follows.

Definition 1.1. Let $L$ satisfies (H1) and (H2), and $\varphi$ be a growth function. A function $f \in H^2(X)$ is said to be in $\tilde{H}_{\varphi,L}(X)$ if $S_L(f) \in L^2(X)$. Moreover, define
**Definition 1.2.** Let $L$ satisfy (H1) and (H2), and $\varphi$ be a growth function. A function $f \in H^2(X)$ is said to be in $\tilde{H}_{L, G, \varphi}(X)$ if $G_L(f) \in L^2(X)$. Moreover, define
\[
\|f\|_{\tilde{H}_{L, G, \varphi}(X)} := \|G_L(f)\|_{L^2(X)} := \inf \left\{ \lambda \in (0, \infty) ; \int_X \varphi \left( x, \frac{G_L(f)(x)}{\lambda} \right) \, d\mu(x) \leq 1 \right\}.
\]
The Musielak-Orlicz-Hardy space $H_{L, G, \varphi}(X)$ is defined to be the completion of $\tilde{H}_{L, G, \varphi}(X)$ with the quasi-norm $\|\|_{H_{L, G, \varphi}(X)}$.

What deserves to be mentioned the most is that the Musielak-Orlicz Hardy space $H_{\varphi,L}$ introduced in [14] is associated with $L$ satisfying the Davies-Gaffney estimates, while the operator $L$ in Definition 1.1 and Definition 1.2 satisfies the stronger Gaussian estimates.

Motivated by the work of [6, 14, 10], the first contribution of this paper is to establish a discrete characterization for the two kinds of Musielak-Orlicz-Hardy spaces $H_{\varphi,L}(X)$ and $H_{L,G,\varphi}(X)$ defined above. This generalizes the results presented in [6, 10] since $H_{\varphi,L}(X)$ contains $H_{L,S,w}^p(X)$ and $H_{L,G,\varphi}(X)$ contains $H_{L,G,w}^p(X)$. Also, by removing the uniformly reverse Hölder condition on growth function $\varphi$, our work improves a part of results of Yang and Yang [14]. The second goal of this article is to prove that $H_{\varphi,L}(X)$ and $H_{L,G,\varphi}(X)$ are equivalent, which improves the result about the behavior of Littlewood-Paley $g$-function $G_L$ on $H_{\varphi,L}$ proved in [14, Theorem 6.3].

Our main approach is inspired by the results in [8, 6]. The layout of this article is as follows. We first recall some basic facts and known results in Sect. 2. In Sect. 3, we first establish discrete characterizations for $H_{\varphi,L}(X)$ and $H_{L,G,\varphi}(X)$ and then obtain the consistency between $H_{\varphi,L}(X)$ and $H_{L,G,\varphi}(X)$ in the sense of norm as a corollary.

Throughout this paper, we mean by writing $a \equiv b$ that variables $a$ and $b$ are equivalent, namely, there exist positive constants $C_1$ and $C_2$ independent of $a$ and $b$ such that $C_1b \leq a \leq C_2b$.

**2. Preliminaries**

**2.1. Metric Measure Spaces**

Let $(X, d, \mu)$ be a metric measure space, namely, $d$ is a metric and $\mu$ a nonnegative Borel regular measure on $X$. Throughout out this paper, for any fixed $x \in X$ and $r \in (0, \infty)$, we denote the open ball centered at $x$ with radius $r$ by
\[
B(x, r) := \{ y \in X ; d(x, y) < r \},
\]
and we set $V(x, r) := \mu(B(x, r))$. Moreover, we assume that $X$ is of homogeneous type, that is, there exists a constant $C_D \in [1, \infty)$ such that, for all $x \in X$ and $r \in (0, \infty)$,
\[
V(x, 2r) \leq C_D V(x, r) < \infty.
\]

Condition (4) is also called the doubling condition which implies that the following strong homogeneity property that, for some positive constants $C$ and $n$,
\[
V(x, \lambda r) \leq C \lambda^n V(x, r)
\]
uniformly for all $\lambda \in [1, \infty)$, $x \in X$, and $r \in (0, \infty)$. And as is shown by Grigor’yan et al. [9], let $C_D$ be as in (4) and $m = \log_2 C_D$, then for all $x, y \in X$ and $0 < r \leq R < \infty$ we have
\[
V(x, R) \leq C_D \left[ \frac{R + d(x, y)}{r} \right]^m V(y, r).
\]
Using the doubling condition \([4]\), it is trivial to show that for any \(N > n\), there exists a constant \(C_N\) such that for all \(x \in X\) and \(t > 0\),

\[
\int_X \left(1 + r^{-1}d(x,y)\right)^-N d\mu(y) \leq C_N V(x,t). \tag{7}
\]

We further have the following dyadic cubes decomposition on spaces of homogeneous type constructed by Christ [3].

**Lemma 2.1.** Let \((X, d, \mu)\) be a space of homogeneous type. Then, there exist a collection \(\{Q^k_\alpha \subset X : k \in \mathbb{Z}, \alpha \in I_k\}\) of open subsets, where \(I_k\) is some index set, and constants \(\delta \in (0, 1)\), and \(C_1, C_2 > 0\), such that

(i) \(\mu\left(X \setminus \bigcup \alpha Q^k_\alpha\right) = 0\), for each fixed \(k\) and \(Q^k_\alpha \cap Q^k_\beta = \emptyset\) if \(\alpha \neq \beta\);

(ii) for any \(\alpha, \beta, k, l\) with \(k \leq l\), either \(Q^k_\beta \subset Q^k_\alpha\) or \(Q^l_\alpha \cap Q^k_\beta = \emptyset\);

(iii) for each \((k, \alpha)\) and each \(l < k\), there exists a unique \(\beta \in I_l\) such that \(Q^l_\alpha \subset Q^k_\beta\);

(iv) \(\text{diam}\left(Q^k_\alpha\right) \leq C_1 \delta^k\);

(v) each \(Q^k_\alpha\) contains some ball \(B(z^k_\alpha, C_2 \delta^k)\), where \(z^k_\alpha \in X\).

We can think of \(Q^k_\alpha\) as being a dyadic cube with diameter roughly \(\delta^k\) centered at \(y_{Q^k_\alpha}\), and we then set \(\ell(Q^k_\alpha) = C_1 \delta^k\). The precise value \(C_1\) is nonessential, and as was proved by Christ [3], in what follows, we without loss of generality assume \(C_1 = \delta^{-1}\).

### 2.2. Growth Functions and Their Properties

Recall from [14] that a nonnegative nondecreasing function \(\Phi\) defined on \([0, +\infty)\) is said to be an **Orlicz function** if \(\Phi(0) = 0\), \(\Phi(t) > 0\) for all \(t \in (0, \infty)\) and \(\lim_{t \to \infty} \Phi(t) = \infty\). The function \(\Phi\) is said to be of **upper type \(p\)** (resp., **lower type \(p\)**) for some \(p \in [0, \infty)\), if there exists a positive constant \(C\) such that, for all \(t \in [1, \infty)\) (resp., \(t \in [0, 1]\)) and \(s \in [0, \infty)\), \(\Phi(st) \leq Ct^p \Phi(s)\). And it is trivial that an Orlicz function is of upper type 1, if it is of upper type \(p \in (0, 1)\).

Let \(\varphi : X \times [0, +\infty) \to [0, +\infty)\) be a function, such that, for any \(x \in X\), \(\varphi(x, \cdot)\) is an Orlicz function. We say that \(\varphi\) is of **uniformly upper type \(p\)** (resp., **uniformly lower type \(p\)**) for some \(p \in [0, \infty)\), if there exists a positive constant \(C\) such that, for all \(x \in X\), \(t \in [1, \infty)\) (resp., \(t \in [0, 1]\)) and \(s \in [0, \infty)\),

\[
\varphi(x, st) \leq Ct^p \varphi(x, s). \tag{8}
\]

As in Ky [11], a function \(\varphi : X \times [0, +\infty) \to [0, +\infty)\) is said to be **uniformly locally integrable**, if for all \(t \in [0, \infty)\), \(x \mapsto \varphi(x, t)\) is measurable and for all bounded subsets \(K \subset X\),

\[
\sup_{t \in [0, \infty)} \left\{ \varphi(x, t) \int_K \varphi(y,t) d\mu(y) \right\} d\mu(x) < \infty.
\]

Following [14, 15], we next recall the definition of the Uniform Muchenhoupt Class and its properties.
Definition 2.1. Let \( \varphi : X \times [0, +\infty) \to [0, +\infty) \) be uniformly locally integrable. The function \( \varphi(\cdot, t) \) is said to satisfy the uniformly Muckenhoupt condition for some \( q \in [1, \infty) \), denoted by \( \varphi \in \mathcal{A}_q(X) \), if, when \( q \in (1, \infty) \),

\[
\mathcal{A}_q(\varphi) := \sup_{t \in (0, \infty)} \sup_{B \subset X} \left\{ \frac{1}{\mu(B)} \int_B \varphi(x, t) \, d\mu(x) \right\} \\
\leq \left\{ \frac{1}{\mu(B)} \int_B \varphi(y, t)^{-q'/q} \, d\mu(y) \right\}^{q'/q} < \infty,
\]

where \( 1/q + 1/q' = 1 \), or

\[
\mathcal{A}_1(\varphi) := \sup_{t \in (0, \infty)} \sup_{B \subset X} \frac{1}{\mu(B)} \int_B \varphi(x, t) \, d\mu(x) \left\{ \text{esssup}_{y \in B} [\varphi(y, t)]^{-1} \right\} < \infty.
\]

We further define \( \mathcal{A}_\infty(X) := \bigcup_{q \in [1, \infty)} \mathcal{A}_q(X) \) and let

\[
q(\varphi) := \inf \{ q \in [1, \infty) : \varphi \in \mathcal{A}_q(X) \}
\]

to be the critical indices of \( \varphi \).

The following properties of \( \mathcal{A}_\infty(X) \) and their proofs are similar to those in Yang et al. [15], and we omit the details. In what follows, we use the notation

\[
\varphi(E, t) := \int_E \varphi(x, t) \, d\mu(x)
\]

for any measurable subset \( E \) of \( X \) and \( t \in [0, \infty) \). And \( \mathcal{M} \) denotes the Hardy-Littlewood maximal function on \( X \), i.e., for all \( x \in X \),

\[
\mathcal{M}(f)(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y),
\]

where the supremum is taken over all balls \( B \ni x \).

Lemma 2.2.

1. \( \mathcal{A}_1(X) \subset \mathcal{A}_p(X) \subset \mathcal{A}_q(X) \), for \( 1 \leq p \leq q < \infty \).
2. If \( \varphi \in \mathcal{A}_p(X) \) with \( p \in (1, \infty) \), then there exists some \( q \in (1, p) \) such that \( \varphi \in \mathcal{A}_q(X) \).
3. If \( \varphi \in \mathcal{A}_p(X) \) with \( p \in (1, \infty) \), then there exists a positive constant \( C \) such that, for all measurable functions \( f \) on \( X \) and \( t \in [0, \infty) \),

\[
\int_X [\mathcal{M}(f)(x)]^p \varphi(x, t) \, d\mu(x) \leq C \int_X |f(x)|^p \varphi(x, t) \, d\mu(x).
\]

4. If \( \varphi \in \mathcal{A}_p(X) \) with \( p \in [1, \infty) \), then there exists a positive constant \( C \) such that, for all balls \( B \subset X \) and measurable subset \( E \subset B \) and \( t \in [0, \infty) \),

\[
\frac{\varphi(B,t)}{\varphi(E,t)} \leq C \left[ \frac{\mu(B)}{\mu(E)} \right]^p.
\]

We now introduce the growth functions and their properties which can be found in [11, 14].

Definition 2.2. A function \( \varphi : X \times [0, +\infty) \to [0, +\infty) \) is called a growth function if the following remain true:

(i) \( \varphi \) is a Musielak-Orlicz function, namely,

\[
\varphi(x) \text{ is a Musielak-Orlicz function, namely,}
\]

(i) \( \varphi(x) \) is a Musielak-Orlicz function, namely,
Proof For any fixed \( r \in (0, p_1/q(\varphi)) \), let \( \tilde{\varphi}(x,t) = \varphi(x, t^{1/r}) \). We claim that \( \tilde{\varphi} \) is of uniformly lower type \( \frac{p_2}{r} \) and upper type \( \frac{p_1}{r} \). By assumption, there exists a constant \( C_1 \), such that

\[
\tilde{\varphi}(x, st) = \varphi(x, s^{1/r} t^{1/r}) \leq C_1 s^{p_1/r} \varphi(x, s^{1/r}) = C_1 t^{p_1/r} \varphi(x, s)
\]

for all \( t \in [0,1], x \in X \) and \( s \in [0,\infty) \). In the mean time, there exists another constant \( C_2 \), such that

\[
\tilde{\varphi}(x, st) = \varphi(x, s^{1/r} t^{1/r}) \leq C_2 t^{p_2/r} \varphi(x, s^{1/r}) = C_2 t^{p_2/r} \varphi(x, s)
\]

for all \( t \in [1,\infty), x \in X \) and \( s \in [0,\infty) \).

2.3. Musielak-Orlicz Space

In this subsection we recall the Musielak-Orlicz Space and obtain a vector-valued inequality.

The Musielak-Orlicz space \( L^\varphi(X) \) contains all measurable functions \( f \) which satisfy \( \int_X \varphi(x, |f(x)|) d\mu(x) < \infty \) with Luxembourg norm

\[
\|f\|_{L^\varphi(X)} := \inf \left\{ \lambda \in (0,\infty) : \int_X \varphi\left(x, \frac{|f(x)|}{\lambda}\right) d\mu(x) \leq 1 \right\}.
\]

The following Lemma of Musielak-Orlicz Fefferman-Stein vector-valued inequality is obtained by Obtained by Yiyu et al. \([16]\). In what follows, the space \( L^\varphi(\ell^p, X) \) is defined to be the set of all \( \{f_j\}_{j \in \mathbb{Z}} \) satisfying \( \left[ \sum_j |f_j|^{p_2} \right]^{1/p_2} \in L^\varphi(X) \) and we let

\[
\left\|\{f_j\}_{j \in \mathbb{Z}}\right\|_{L^\varphi(\ell^p, X)} := \left\| \left[ \sum_j |f_j|^{p_2} \right]^{1/p_2} \right\|_{L^\varphi(X)}.
\]

Lemma 2.4. Let \( p \in (1,\infty) \), \( \varphi \) be a Musielak-Orlicz function with uniformly lower type \( p_1 \) and upper type \( p_2 \), and \( \varphi \in \mathcal{A}_q(X) \) for \( q \in (1,\infty) \). If \( q(\varphi) < p_1 < p_2 < \infty \), then there exists a positive constant \( C \) such that, for all \( \{f_j\}_{j \in \mathbb{Z}} \in L^\varphi(\ell^p, X) \),

\[
\int_X \varphi\left(x, \left[ \sum_j M(f_j)(x)^p \right]^{1/p} \right) d\mu(x) \leq C \int_X \varphi\left(x, \left[ \sum_j |f_j(x)|^{p_1} \right]^{1/p_1} \right) d\mu(x).
\]

Corollary 2.1. Let \( p \) and \( \varphi \) be as in Lemma 2.4, then for all \( r \in (0, p_1/q(\varphi)) \), we have

\[
\int_X \varphi\left(x, \left[ \sum_j M(f_j)(x)^p \right]^{1/r} \right) d\mu(x) \leq C \int_X \varphi\left(x, \left[ \sum_j |f_j(x)|^p \right]^{1/r} \right) d\mu(x).
\]
Since \( q(\tilde{\varphi}) = q(\varphi) \), for arbitrary \( r \in (0, p_1/q(\varphi)) \), it is trivial that

\[
q(\tilde{\varphi}) = q(\varphi) < \frac{p_1}{r} \leq \frac{p_2}{r} < \infty.
\]

By employing Lemma 2.4, we obtain

\[
\int_X \varphi \left( x, \left| \sum_j M(f_j)(x)^p \right|^\frac{1}{p} \right)d\mu(x) = \int_X \tilde{\varphi} \left( x, \left| \sum_j M(f_j)(x)^p \right|^\frac{1}{p} \right)d\mu(x)
\leq C \int_X \varphi \left( x, \left| \sum_j f_j(x)^p \right|^\frac{1}{p} \right)d\mu(x)
= C \int_X \varphi \left( x, \left| \sum_j f_j(x)^p \right|^\frac{1}{p} \right)d\mu(x).
\]

\[\square\]

2.4. \( AT_{L,M} \)-Family Associated with Operator \( L \)

Recalling that \( X \) is a space that satisfies the strong homogeneity property (5) with homogeneous dimension \( n \). In the view of Lemma 2.1, there exists a collection \( \{Q^k_a \subset X; k \in \mathbb{Z}, \alpha \in I_k\} \) of open subsets, where \( I_k \) is the index set, such that for every \( k \in \mathbb{Z} \),

\[
X = \bigcup_{\alpha \in I_k} Q^k_a
\]

with properties of \( Q^k_a \) as in Lemma 2.1. In what follows, such open subsets \( \{Q^k_a \subset X; k \in \mathbb{Z}, \alpha \in I_k\} \) is said to be a family of dyadic cubes of \( X \). And we now turn to introduce the \( AT_{L,M} \)-family associated with an operator \( L \) whose definition can also be found in [6].

**Definition 2.3.** Suppose that an operator \( L \) satisfies (H1) and (H2) and \( M \in \mathbb{N} \). A collection of functions \( \{a_Q\}_{Q: Dyadic} \) is said to be an \( AT_{L,M} \)-family associated with \( L \), if for each dyadic cube \( Q \), there exists a function \( b_Q \in D(L^{2M}) \) such that

1. \( a_Q = L^M(b_Q); \)
2. \( \text{supp}(L^k(b_Q)) \subset 3Q, k = 0, 1, \cdots, 2M; \)
3. \( \ell(Q)^2L^k(b_Q) \leq \ell(Q)^{2M}\mu(Q)^{-1/2}, k = 0, 1, \cdots, 2M. \)

Here, \( D(L) \) denotes the domain of operator \( L \), and by \( L^k \) the \( k \)-fold composition of \( L \) with itself.

With this definition, we can decompose an \( L^2 \) function into \( AT_{L,M} \)-family. Given a function \( f \in L^2(X) \), we say that \( f \) has an \( AT_{L,M} \)-expansion, if there exists sequence \( s = \{s_Q\}_{Q: Dyadic}, 0 \leq s_Q < \infty \) and an \( AT_{L,M} \)-family \( \{a_Q\}_{Q: Dyadic} \) in \( L^2(X) \) such that

\[
f = \sum_{Q: Dyadic} s_Q a_Q. \quad (9)
\]

We then define by \( W_f(x) \) the function related to the sequence \( s = \{s_Q\}_{Q: Dyadic}, 0 \leq s_Q < \infty \) as

\[
W_f(x) := \left( \sum_{Q: Dyadic} \mu(Q)^{-1/2} |s_Q|^2 \chi_Q(x) \right)^{1/2}. \quad (10)
\]
Proposition 2.1. Given an operator \( L \) that satisfies (H1)-(H2) and \( f \in L^2(X) \). Then for all \( M \in \mathbb{N} \), \( f \) has an \( AT_{L,M} \)-expansion. Moreover, let \( Q_x^k \) and \( \delta \) be as in Lemma 2.1 we have

\[
s_{q_x^k} = \left( \int_{\delta^{k+1}} \int_{Q_x^k} |r^2 Le^{-r^2} f(y)|^2 d\mu(y) \frac{dy}{t} \right)^{1/2}.
\]

The proof of Proposition 2.1 can be found in [6, Proof of Theorem 3.2]. We omit the details.

3. Musielar-Orlicz Hardy Space \( H_{\varphi,L} \) and its Equivalent Characterization

In this section, we begin to study the Musielar-Orlicz-Hardy space, and in what follows, we always assume the operator \( L \) satisfies (H1) and (H2), and \( \varphi \) is a growth function which is defined in Definition 2.2. With some basic notations set forth in Sect. 2, we first establish the following characterization for the Hardy space \( H_{\varphi,L} \).

Theorem 3.1. Suppose \( L \) is an operator that satisfies (H1) and (H2). Let \( \varphi \) be a growth function with uniformly lower type \( p_1 \) and \( f \in H_{\varphi,L}(X) \cap L^2(X) \), then for all natural number \( M > nq(\varphi)/(2p_1) \), \( f \) has an \( AT_{L,M} \)-expansion such that

\[
\|f\|_{H_{\varphi,L}(X)} \approx \|W_f\|_{L^2(X)}.
\]

Before we prove Theorem 3.1, we need to introduce some notions and establish some results as follows.

For any \( \nu \in (0, \infty) \) and \( x \in X \), let \( \Gamma_\nu(x) := \{ (y, t) \in X \times (0, \infty) : d(x, y) < \nu t \} \) be the cone of aperture \( \nu \) with vertex \( x \in X \). For any closed subset \( F \) of \( X \), denote by \( \mathcal{R}_\nu(F) \) the union of all cones with vertices in \( F \), i.e., \( \mathcal{R}_\nu(F) = \bigcup_{x \in F} \Gamma_\nu(x) \). In what follows, we denote \( \Gamma_1(x) \) and \( \mathcal{R}_1(F) \) simply by \( \Gamma(x) \) and \( \mathcal{R}(F) \), respectively. For any open subset \( O \) of \( X \), we establish the following geometric property of \( \mathcal{R}(O^c) \) which generalizes a similar result obtained by Aguilera and Segovia [1, Lemma 1] in the case of Euclidean space.

Lemma 3.1. Suppose that \((X, d, \mu)\) is a space of homogeneous type with constant \( C_D > 1 \) such that (4) holds. Let \( O \) be an open subset of \( X \), \( F = O^c \) and \( X_D \) its characteristic function. If for \( \nu > 1 \), we define \( O^* \) as

\[
O^* := \{ x \in X ; M(X_D)(x) > (4\nu)^{-2\log_2 C_D} \}
\]

and let \( F^* = (O^*)^c \), then we have

(i) \( \mathcal{R}_\nu(F^c) \) is contained in \( \mathcal{R}(F) \).
(ii) If \((z, t) \in \mathcal{R}_\nu(F^c) \), then there exists some constant \( C_\nu \) such that

\[
V(z, t) < \frac{1}{C_\nu} \mu(B(z, t) \cap F).
\]

Proof The lemma is trivial if \( \mathcal{R}_\nu(F^c) = \emptyset \). We then with no loss of generality assume that \( \mathcal{R}_\nu(F^c) \neq \emptyset \), which implies that \( O \neq X \). We then first prove (i). If \((z, t) \in \mathcal{R}_\nu(F^c) \), then either \( z \in F \) or \( z \in O \). In the first case it is apparent that \((z, t) \in \mathcal{R}(F) \), since \( d(z, z) = 0 < t \).

If we are in the second case, i.e., \( z \in O \), let \( \delta \) be the distance from \( z \) to the closed and non-empty set \( F \). This number \( \delta \) is positive and finite, and \( B(z, \delta) \) is contained in \( O \). The assumption
that \((z, t) \in \mathcal{R}_r(F^*)\) implies that there is \(y \in F^*\) with \(d(z, y) < vt\). Thus, writing \(r = \delta + d(z, y)\), we get \(B(z, \delta) \subset B(y, r)\) and also

\[
B(z, \delta) \subset B(z, \delta) \cap O \subset B(y, r) \cap O,
\]

which together with the definition of \(O^*\), implies that

\[
V(z, \delta) \leq \mu(B(y, r) \cap O) \leq (4\nu)^{-2\log_2 C_D} V(y, r)
\]

since \(y \in F^*\).

By using (6) twice, we also have

\[
V(y, r) \leq C_D (r\delta^{-1})^{\log_2 C_D} V(y, \delta)
\]

\[
\leq C_D^2 (r\delta^{-1})^{\log_2 C_D} (1 + \delta^{-1} d(y, z))^{\log_2 C_D} V(z, \delta)
\]

\[
= (2r\delta^{-1})^{2\log_2 C_D} V(z, \delta).
\]

From these inequalities, we get that

\[
\delta \leq \frac{r}{2\nu}.
\]

Recalling that \(r = \delta + d(z, y)\) and \(d(z, y) < vt\), we obtain

\[
\delta \leq \frac{\delta + d(z, y)}{2\nu} < \frac{\delta + vt}{2\nu}
\]

and since \(\nu > 1\), it follows that \(\delta < t\). Then by the very definition of \(\delta\), there exists an \(x \in F\), satisfying \(d(x, z) < t\), which means that \((z, t) \in \mathcal{R}(F)\). This proves (i).

We then turn our attention to (ii). If \((z, t) \in \mathcal{R}_r(F^*)\), there is \(y \in F^*\) such that \(d(z, y) < vt\). Then \(B(z, t) \subset B(y, (1 + \nu) t)\) and since \(y \in F^*\), we get

\[
\mu(B(z, t) \cap O) \leq \mu(B(y, (1 + \nu) t) \cap O) \leq (4\nu)^{-2\log_2 C_D} V(y, (1 + \nu) t),
\]

and therefore

\[
\mu(B(z, t) \cap O) \leq (4\nu)^{-2\log_2 C_D} C_D (1 + \nu)^{\log_2 C_D} V(y, t)
\]

\[
\leq C_D^2 (4\nu)^{-2\log_2 C_D} (1 + \nu)^{\log_2 C_D} (1 + t^{-1} d(y, z))^{\log_2 C_D} V(z, t)
\]

\[
< \left(\frac{1 + \nu}{2\nu}\right)^{2\log_2 C_D} V(z, t).
\]

Now from \(V(z, t) = \mu(B(z, t) \cap O) + \mu(B(z, t) \cap F)\), we obtain

\[
\left[1 - \left(\frac{1 + \nu}{2\nu}\right)^{2\log_2 C_D}\right] V(z, t) < \mu(B(z, t) \cap F),
\]

which implies (ii). \(\square\)

Next we introduce the following variant of Lusin-area function associated with \(L\). For all \(\nu \in (0, \infty), f \in L^2(X)\) and \(x \in X\), let

\[
S_{L, \nu}(f)(x) := \left(\int_0^{\infty} \int_{d(x, y) < \nu} |y^2 L e^{-2L}(f)(y)|^2 \frac{d\mu(y)}{V(x, t)} \frac{dt}{t}\right)^{1/2}.
\]

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We also have the following two Lemmas for the variant of Lusin-area function that associated with $L$ which generalize the results of Aguilera and Segovia [1, Lemma 2] and Yang et al. [15, Lemma 3.3.5].

**Lemma 3.2.** Assume that $L$ satisfies (H1) and (H2). Let $\varphi \in A_p(X)$, $1 \leq p < \infty$, and $O$ be an open subset of $X$. If $O^*$ is the set associated to $O$ as in Lemma 3.1 with some $\nu > 1$, then there exists a finite constant $C$, which is independent of $O$, such that for all $\lambda \in (0, \infty)$ and $f \in L^2(X)$,

$$\int_{F^*} |S_{L^*}(f)(x)|^2 \varphi(x, \lambda) \, d\mu(x) \leq C \int_F |S_L(f)(x)|^2 \varphi(x, \lambda) \, d\mu(x),$$

where $F^* = (O^*)^c$ and $F = O^c$.

**Proof** For any $x \in F^*$, $(y,t) \in \Gamma_\nu(x)$, we observe that $d(x,y) < nt$, and hence by (6),

$$V(x,t)^{-1} \leq C_D \left(1 + t^{-1}d(x,y)\right)^{\log C_D} V(y,t)^{-1} < C_D(1 + \nu)^{\log C_D} V(y,t)^{-1}.$$

It follows that

$$\int_{F^*} |S_{L^*}(f)(x)|^2 \varphi(x, \lambda) \, d\mu(x)$$

$$\leq C_D(1 + \nu)^{\log C_D} \int_{F^*} \left(\int_{\Gamma_\nu(x)} |t^2 L e^{-t^2 L} (f)(x)|^2 \frac{d\mu(y)}{V(y,t)} \right) \varphi(x, \lambda) \, d\mu(x)$$

$$= C_{\nu,D} \int_{R^*_{t,F^*}} |t^2 L e^{-t^2 L} (f)(y)|^2 V(y,t)^{-1} \varphi(B(y,nt) \cap F^* \setminus \lambda) \frac{d\mu(y)}{t} dt. \tag{11}$$

We then employ Lemma 2.2 (iv) to the set $E = B(y,t)$ and $B = B(y,nt)$, to get

$$\varphi(B(y,nt) \setminus \lambda) \leq C(2\nu)^{\log C_D} \varphi(B(y,t) \setminus \lambda). \tag{12}$$

Applying Lemma 2.2 (iv) once again to $E = B(y,t) \cap F$ and $B = B(y,t)$, we get

$$\varphi(B(y,t) \setminus \lambda) \leq C \left(\frac{V(y,t)}{\mu(B(y,t) \cap F)}\right)^p \varphi(B(y,t) \cap F \setminus \lambda). \tag{13}$$

Therefore, from (12) and (13), plus part (ii) of Lemma 3.1 we have

$$\varphi(B(y,nt) \setminus \lambda) \leq C \varphi(B(y,t) \cap F \setminus \lambda).$$

From this estimate it follows that the last integral in (11) is bounded by

$$C \int_{R^*_{t,F^*}} |t^2 L e^{-t^2 L} (f)(y)|^2 V(y,t)^{-1} \varphi(B(y,t) \cap F \setminus \lambda) \frac{d\mu(y)}{t} dt.$$

Finally, in the view of Lemma 3.1 (i), we observe that $R_\nu(F^*) \subset R(F)$, it follows immediately that
the last integral above is bounded by
\[ C \int_{R(F)} |t^2Le^{-2L}(f)(y)|^2 V(y, t)^{-1} \varphi(B(y, t) \cap F, \lambda) \frac{d\mu(y)}{t} dt \]
\[ = C \int_{F} \left( \int_{\Gamma_{y}(x)} |t^2Le^{-2L}(f)(y)|^2 \frac{d\mu(y)}{V(y, t)} dt \right) \varphi(x, \lambda) d\mu(x) \]
\[ \leq C \int_{F} |S_L(f)(x)|^2 \varphi(x, \lambda) d\mu(x), \]
where the last inequality follows from the fact that
\[ V(y, t)^{-1} \leq C_D\left(1 + t^{-1}d(x, y)\right)^{\log C_D} V(x, t)^{-1} < C_D^2 V(x, t)^{-1} \]
for \((y, t) \in \Gamma_{\nu}(x)\). This proves the lemma.

\[ \square \]

**Lemma 3.3.** Assume that \( L \) satisfies \((H1)\) and \((H2)\). Let \( q \in (1, \infty) \), \( \varphi \) be as in Definition 2.2 and \( \varphi \in A^q(X) \). Then for all \( \nu \in (0, \infty) \) there exists a positive constant \( C_{\nu} \), such that, for all measurable functions \( f \),
\[ \int_X \varphi(x, S_{L,\nu}(f)(x)) d\mu(x) \leq C_{\nu} \int_X \varphi(x, S_L(f)(x)) d\mu(x). \]

**Proof** If \( \nu \in (0, 1] \), the conclusion is trivial. We further suppose that \( \nu \in (1, \infty) \). For all \( \lambda \in (0, \infty) \), let
\[ O_\lambda := \{ x \in X; S_L(f)(x) > \lambda \} \]
and
\[ O^*_\lambda := \{ x \in X; M(X_{O_\lambda})(x) > (4\nu)^{-\log C_D} \}. \]
where \( M \) is the Hardy-Littlewood maximal function. Since \( \varphi \in A^q(X) \), it follows from Lemma 2.2(iii),
\[ \varphi(O^*_\lambda, \lambda) = \varphi\left( \{ x \in X; M(X_{O_\lambda})(x) > (4\nu)^{-\log C_D} \}, \lambda \right) \]
\[ \leq \int_X (4\nu)^{\rho\log C_D} (M(X_{O_\lambda})(x))^q \varphi(x, \lambda) d\mu(x) \]
\[ \leq C \varphi(O_\lambda, \lambda). \quad (14) \]

Let \( F_\lambda := O^*_\lambda \), \( F^*_\lambda := (O_\lambda)^C \) and apply Lemma 2.2 to get
\[ \int_{F^*_\lambda} |S_{L,\nu}(f)(x)|^2 \varphi(x, \lambda) d\mu(x) \leq C \int_{F^*_\lambda} |S_L(f)(x)|^2 \varphi(x, \lambda) d\mu(x). \quad (15) \]
Thus, from (14) and (15), it follows that

$$\varphi\left(\{x \in X; S_{L^p}(f)(x) > \lambda\}, \lambda\right) \leq \varphi\left(O^{*}_\lambda, \lambda\right) + \lambda^2 \int_{F_{\lambda}^*} |S_{L^p}(f)(x)|^2 \varphi(x, \lambda) d\mu(x)$$

$$\leq C \varphi\left(O^{*}_\lambda, \lambda\right) + \lambda^2 \int_{F_{\lambda}^*} |S_{L^p}(f)(x)|^2 \varphi(x, \lambda) d\mu(x)$$

$$\leq C \left[ \varphi\left(O^{*}_\lambda, \lambda\right) + \lambda^{-2} \int_{F_{\lambda}^*} |S_{L^p}(f)(x)|^2 \varphi(x, \lambda) d\mu(x) \right]$$

which, together with the assumption $\nu \in (1, \infty)$, Lemma 2.3 and the uniformly upper type 1 of $\varphi$, we further get that

$$\int_X \varphi(x, S_{L^p}(f)(x)) d\mu(x) \leq C \int_X \varphi\left(O^{*}_\lambda, \lambda\right) d\mu(x) + \int_0^\infty \varphi\left(\{x \in X; S_{L^p}(f)(x) > \lambda\}, \lambda\right) d\lambda$$

This finishes the proof of Lemma 3.3.

Moreover, we also need the following Lemma, whose standard proof can be found in [7], we omit the details. And in what follows, we recall that the Hardy-Littlewood maximal operator $M$ on $(X, \mu, d)$ is defined by

$$M(f)(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y).$$

where the supremum is taken over all balls $B \ni x$. 
Lemma 3.4. Suppose $0 < q \leq 1$ and $N > n/q$, where $n$ is the doubling dimension of the space in (5). Fix an integer $k$, and let $\{s_{Q^k}\}_{x \in I_k}$ be as in Proposition 2.1 then for any subsequence $I_k' \subset I_k$ and each $x \in X$,

$$\sum_{\alpha \in I_k'} \frac{|s_{Q^k}|}{\left[ 1 + \ell(Q^k_*^{-1} d(x, y^k_\alpha)) \right]^N} \leq C \left[ M \left( \sum_{\alpha \in I_k'} |s_{Q^k_*}|^{q} X(\cdot) \right) \right]^{1/q}(x).$$

where $y^k_\alpha$ denotes the center of $Q^k_\alpha$.

Proof of theorem 3.7. For any fixed $f \in H_{\varphi,L}(X) \cap L^2(X)$, we let $\lambda_0 = \|f\|_{L^1(X)}$ and $\lambda_1 = \|W_f\|_{L^\infty(X)}$. It suffices to show that for all $\lambda \in (0, \infty)$,

$$\int_X \varphi \left( x, \frac{S_L(f)(x)}{\lambda} \right) d\mu(x) \leq \int_X \varphi \left( x, \frac{|W_f(x)|}{\lambda_1} \right) d\mu(x).$$

(16)

In fact, if (16) holds for all $\lambda \in (0, \infty)$, then there exists a constant $C_0$ such that

$$\int_X \varphi \left( x, \frac{S_L(f)(x)}{\lambda_1} \right) d\mu(x) \leq C_0 \int_X \varphi \left( x, \frac{|W_f(x)|}{\lambda_1} \right) d\mu(x) \leq C_0,$$

which, together with (8), implies that

$$\int_X \varphi \left( x, \frac{G_L(f)(x)}{C_1 \lambda_1} \right) d\mu(x) \leq 1$$

for some constants $C_1$, and hence we have $\lambda_0 \leq C_1 \lambda_1$. In a similar fashion, one can prove $\lambda_1 \leq C_2 \lambda_0$ for some constants $C_2$, and get the desired property.

Let $f$ be a function in $H_{\varphi,L}(X) \cap L^2(X)$. In the view of Lemma 2.1 for any fixed $(x, k) \in X \times Z$ there exists a unique $\alpha \in I_k$, such that $x \in Q^k_\alpha$. Let $Q^k_\alpha$ denote the such $Q^k_\alpha$, and we write

$$W_f(x) = \left\{ \sum_{k \in Z} \sum_{\alpha \in I_k} \mu(Q^k_\alpha)^{-1/2} \left| s_{Q^k_\alpha} \right| X_{Q^k_\alpha}(x) \right\}^{1/2} = \left\{ \sum_{k \in Z} \mu(Q^k_\alpha)^{-1} \left| s_{Q^k_\alpha} \right|^2 \right\}^{1/2} = \left\{ \sum_{k \in Z} \sum_{\delta \in \mathbb{Z}} \mu(Q^k_\alpha)^{-1} \int_{\partial Q^k_\alpha} |r^2 L e^{-r^2 L} f(y)|^2 d\mu(y) \frac{dt}{t} \right\}^{1/2},$$

(17)

where $\delta \in (0, 1)$ is a constant as in Lemma 2.1 and the last quantity follows from Proposition 2.1.

Moreover, by (iv) and (v) of Lemma 2.1 we know that for any fixed $(x, k) \in X \times Z$ there exists $z^k \in Q^k_\alpha$ and constants $C_1 \in (0, 1)$, $C_2 = \delta^{-1}$ such that $\text{diam} \left( Q^k_\alpha \right) \leq C_2 \delta^{-}\ell \left( Q^k_\alpha \right)$ and

$$B \left( z^k, C_1 \delta^k \right) \subset Q^k_\alpha \subset B \left( x, C_2 \delta^k \right) \subset B \left( x, C_2 \delta^{-1}t \right)$$

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for all $t \in (\delta^{k+1}, \delta^k)$. It follows immediately from (6) that
\[
\mu(Q^k) \leq V(x, C_1 \delta^k)^{-1}
\]
\[
\leq C \left( 1 + \frac{d(x, z^k)}{C_1 \delta^k} \right)^m V(x, C_1 \delta^k)^{-1}
\]
\[
\leq CV(x, C_1 \delta^k)^{-1}
\]
\[
\leq CV(x, \delta^k)^{-1}
\]
\[
\leq CV(x, t)^{-1},
\]
where the last but one inequality follows from the fact that $V(x, \delta^k) \leq C D C_1^{-m} V(x, C_1 \delta^k)$ with $C_1 \in (0, 1)$. Hence, by this estimate and (17), we have
\[
W_f(x) \leq C \left( \sum_{k \in \mathbb{Z}} \sum_{a \in I_k} \int_{z^k}^{z^{k+1}} V(x, t)^{-1} \int_{d(x, z^k)}^{d(x, z^{k+1})} |t^2 L e^{-t^2 L} f(y)|^2 \frac{d \mu(y)}{V(x, t)} \frac{dt}{t} \right)^{1/2}
\]
\[
= C \left( \int_0^\infty \int_{d(x, y) < \delta^j} |t^2 L e^{-t^2 L} f(y)|^2 \frac{d \mu(y)}{V(x, t)} \frac{dt}{t} \right)^{1/2}
\]
\[
= CS_L \delta^{-2} (f)(x),
\]
which, together with Lemma [5.3], we deduce the $\geq$ inequality of (16).

It remains to establish the reverse inequality. In the view of Proposition [2.1] we write $f = \sum_{k \in \mathbb{Z}} \sum_{a \in I_k} s_{Q_{a}^k} a_{Q_{a}^k}$. Let $\delta$ be as in Lemma [2.1] we get
\[
S_L(f)(x)
\]
\[
= \left( \int_0^\infty \int_{d(x, y) < \delta} |t^2 L e^{-t^2 L} (f)(y)|^2 \frac{d \mu(y)}{V(x, t)} \frac{dt}{t} \right)^{1/2}
\]
\[
= \left( \int_0^\infty \int_{d(x, y) < \delta} |t^2 L e^{-t^2 L} (\sum_{k \in \mathbb{Z}} \sum_{a \in I_k} s_{Q_{a}^k} a_{Q_{a}^k})(y)|^2 \frac{d \mu(y)}{V(x, t)} \frac{dt}{t} \right)^{1/2}
\]
\[
= \left( \sum_{j \in \mathbb{Z}} \int_{d(x, y) < \delta^j} \int_{d(x, y) < \delta} |t^2 L e^{-t^2 L} (\sum_{k \in \mathbb{Z}} \sum_{a \in I_k} s_{Q_{a}^k} a_{Q_{a}^k})(y)|^2 \frac{d \mu(y)}{V(x, t)} \frac{dt}{t} \right)^{1/2}
\]
\[
\leq \left( \sum_{j \in \mathbb{Z}} \int_{d(x, y) < \delta^j} \int_{d(x, y) < \delta} |t^2 L e^{-t^2 L} (\sum_{k \leq j} \sum_{a \in I_k} s_{Q_{a}^k} a_{Q_{a}^k})(y)|^2 \frac{d \mu(y)}{V(x, t)} \frac{dt}{t} \right)^{1/2}
\]
\[
+ \left( \sum_{j \in \mathbb{Z}} \int_{d(x, y) < \delta^j} \int_{d(x, y) < \delta} |t^2 L e^{-t^2 L} (\sum_{j < k \leq \infty} \sum_{a \in I_k} s_{Q_{a}^k} a_{Q_{a}^k})(y)|^2 \frac{d \mu(y)}{V(x, t)} \frac{dt}{t} \right)^{1/2}.
\]

We now estimate the first part of (18). For any $k > j$ and $a \in I_k$, noting that $a_{Q_{a}^k}^j = L^{M} b_{Q_{a}^k}$, we write
\[
|t^2 L e^{-t^2 L} (a_{Q_{a}^k})(y)| = |t^2 L^{M+1} e^{-t^2 L} (b_{Q_{a}^k})(y)| = r^{-2M} \left( t^2 L \right)^{M+1} e^{-t^2 L} (b_{Q_{a}^k})(y).
\]
Let $n$ be as in (5), since $M > nq(\varphi)/(2p_1)$, we can choose some $q = r$ with $r$ be as in Corollary 2.1 such that $2M > n/q$. We then let $N$ be some positive number such that $2M > N > n/q$. Then by Definition 2.3 the upper bound of the kernel of $(t^2L)^{M+1}e^{-tL}$ and (7), we get

$$\left|t^2Le^{-tL}(a_{Q^k}) (y)\right| \leq \frac{C_s}{V(y,t)} t^{-2M} \ell(Q^k)^{2M} \mu(Q^k)^{-1/2} \int_{3Q^k_a} \exp \left(- \frac{d(y,z)^2}{C_s \ell^2} \right) d\mu(z)$$

$$\leq Ct^{-2M} \ell(Q^k)^{2M} \mu(Q^k)^{-1/2} \left(\frac{1}{t+d(y,z^k_a)}\right)^N,$$

where we denote by $z^k_a$ the center of $Q^k_a$. By the fact that $d(x,y) < t$, we further obtain

$$\left(\int_{d(x,y) < t} |t^2Le^{-tL}(a_{Q^k}) (y)|^2 \frac{d\mu(y)}{V(x,t)}\right)^{1/2} \leq Ct^{-2M} \ell(Q^k)^{2M} \mu(Q^k)^{-1/2} \left(\frac{1}{t+d(x,y)}\right)^{2N} \frac{d\mu(y)}{V(x,t)} \right)^{1/2}$$

$$\leq Ct^{-2M} \ell(Q^k)^{2M} \mu(Q^k)^{-1/2} \left(1 + t^{-1}d(x,z^k_a)\right)^{-N}.$$

Hence, we have

$$\left(\int_{d(x,y) < t} |t^2Le^{-tL}(\sum_{k<j} \sum_{a \in I_k} s_{Q^k_a} a_{Q^k_a}) (y)|^2 \frac{d\mu(y)}{V(x,t)}\right)^{1/2}$$

$$\leq C \sum_{k<j} \sum_{a \in I_k} t^{-2M} \ell(Q^k)^{2M} \mu(Q^k)^{-1/2} \left|s_{Q^k_a}\right| \left(1 + t^{-1}d(x,z^k_a)\right)^{-N}$$

$$\leq C \sum_{k<j} t^{(2M-N)(k-j)} \left|s_{Q^k_a}\right| \left[1 + \ell(Q^k_a)^{-1}d(x,z^k_a)\right]^N$$

$$\leq C \sum_{k<j} t^{(2M-N)(k-j)} \left|s_{Q^k_a}\right| \mu(Q^k_a)^{-q^2} \left[X_{Q^k_a} (\cdot)\right](x) \right]^{1/q},$$

where the last inequality follows from Lemma 3.4.

Estimate of the second part of (13). For any $k \leq j$ and $a \in I_k$, we write

$$\left|t^2Le^{-tL}(a_{Q^k}) (y)\right| = t^2 \left|e^{-tL}(L(a_{Q^k})) (y)\right|.$$

Then by Definition 2.3 the Gaussian estimate (11) and inequality (7), we get

$$\left|t^2Le^{-tL}(a_{Q^k}) (y)\right| \leq \frac{C_s}{V(y,t)} t^2 \ell(Q^k)^2 \mu(Q^k)^{-1/2} \int_{3Q^k_a} \exp \left(- \frac{d(y,z)^2}{C_s \ell^2} \right) d\mu(z)$$

$$\leq Ct^2 \ell(Q^k)^2 \mu(Q^k)^{-1/2} \left(1 + t^{-1}d(y,z^k_a)\right)^{-N},$$

and (7), we get
which together with the fact that \( d(x, y) \leq t \) further implies that
\[
\left( \int_{d(x, y) < t} |r^2 Le^{-r^2} (a \tilde{G}_d) (y)|^2 \frac{d\mu (y)}{V (x, t)} \right)^{1/2} \\
\leq C r^2 \ell (Q_a^k)^{-\frac{1}{2}} \mu (Q_a^k)^{-\frac{1}{2}} \left( \int_{d(x, y) < t} \left( \ell (Q_a^k) + d(x, y) \right)^{2N} \frac{d\mu (y)}{V (x, t)} \right)^{1/2} \\
\leq C r^2 \ell (Q_a^k)^{-\frac{1}{2}} \mu (Q_a^k)^{-\frac{1}{2}} \left( 1 + \ell (Q_a^k)^{-1} d(x, z_a^k) \right)^{\frac{N}{2}}.
\]
Therefore, by employing Lemma 3.4 once again, we get
\[
\left( \int_{d(x, y) < t} |r^2 Le^{-r^2} \left( \sum_{k \leq j, a \in A_k} s \tilde{G}_d a \tilde{G}_d \right) (y)|^2 \frac{d\mu (y)}{V (x, t)} \right)^{1/2} \\
\leq C \sum_{k \leq j} \sum_{a \in A_k} r^2 \ell (Q_a^k)^{-\frac{1}{2}} \mu (Q_a^k)^{-\frac{1}{2}} |s \tilde{G}_d| \left( 1 + \ell (Q_a^k)^{-1} d(x, z_a^k) \right)^{\frac{N}{2}} \\
\leq C \sum_{k \leq j} \delta^{2(k-j)} \left( \sum_{a \in A_k} |s \tilde{G}_d|^2 \mu (Q_a^k)^{-q/2} X_{Q_a^k} (\cdot) (x) \right)^{1/2} \\
\leq C \sum_{k \leq j} \delta^{2(k-j)} \left( \sum_{a \in A_k} |s \tilde{G}_d|^2 \mu (Q_a^k)^{-q/2} X_{Q_a^k} (\cdot) (x) \right)^{1/2}.
\tag{20}
\]
Set \( F_k (x) := M \left( \sum_{a \in A_k} |s \tilde{G}_d|^2 \mu (Q_a^k)^{-q/2} X_{Q_a^k} (\cdot) \right) (x) \). For any fix \( j \in \mathbb{Z} \), we let \( \beta > 0 \), \( \tau = 1 \) if \( k > j \) and \( \tau = -1 \) if \( k \leq j \). We now turn to estimate \( \left( \sum_k \delta^{\beta(k-j)} F_k (x) \right)^{1/2} \) for the case either \( k > j \) or \( k \leq j \). Since
\[
\delta^{\beta(k-j)} = \frac{\beta \delta^\beta}{1 - \delta^\beta} \int_{\delta^{\beta(k-j)-1}}^{\delta^{\beta(k-j)-1}} s^{\beta-1} ds,
\]
we let \( E_k := \left[ \delta^{\beta(k-j)-1}, \delta^{\beta(k-j)-1} \right] \) and it follows that
\[
\left( \sum_k \delta^{\beta(k-j)} F_k (x) \right)^{1/2} = C_\beta \left( \sum_k \int_{\delta^{\beta(k-j)-1}}^{\delta^{\beta(k-j)+1}} F_k (x) s^{\beta-1} ds \right)^{1/2} \\
= C_\beta \left( \int_0^1 \sum_k \chi_{E_k} (s) F_k (x) s^{1/2} ds \right)^{1/2} \\
\leq C_\beta \left( \int_0^1 s^{\beta-1} ds \right) \left( \int_0^1 \left( \sum_k \chi_{E_k} (s) F_k (x) \right)^{1/2} s^{\beta-1} ds \right) \\
\leq C_\beta \int_0^1 \left( \sum_k \chi_{E_k} (s) F_k (x) \right)^{1/2} s^{\beta-1} ds \\
= C_\beta \sum_k \int_{\delta^{\beta(k-j)-1}}^{\delta^{\beta(k-j)+1}} F_k (x) s^{2/2} s^{\beta-1} ds.
\]

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Combining now (18)-(21), we get the following estimate for $S_L(f)$,

$$S_L(f)(x) \leq \left( \sum_{j \in \mathbb{Z}} \int_{|d(x,y)| < \delta_j} \int_{d(x,y) < \delta_j} |r^2 Le^{-r^2L} (\sum_{k \geq j} \sum_{\alpha \in h_k} s_{Q^{k}_{\alpha}} a_{Q^{k}_{\alpha}}) (x) |^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2}$$

$$+ \left( \sum_{j \in \mathbb{Z}} \int_{|d(x,y)| < \delta_j} \int_{d(x,y) < \delta_j} |r^2 Le^{-r^2L} (\sum_{k \leq j} \sum_{\alpha \in h_k} s_{Q^{k}_{\alpha}} a_{Q^{k}_{\alpha}}) (x) |^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2}$$

$$\leq C \left( \sum_{j \in \mathbb{Z}} \int_{|d(x,y)| < \delta_j} \left( \sum_{k \geq j} \delta^{2(M-N)(k-j)} F_k(x) \right)^{1/2} \frac{dt}{t} \right)$$

$$+ \left( \sum_{j \in \mathbb{Z}} \int_{|d(x,y)| < \delta_j} \left( \sum_{k \leq j} \delta^{2(j-k)} F_k(x) \right)^{1/2} \frac{dt}{t} \right)$$

$$= C \left( \sum_{k \in \mathbb{Z}} F_k(x)^{2/q} \left( \sum_{j \geq k} \delta^{2(M-N)(k-j)} + \sum_{j < k} \delta^{2(j-k)} \right) \right)^{1/2}$$

$$\leq C \left( \sum_{k \in \mathbb{Z}} F_k(x)^{2/q} \right)^{1/2} .$$

where we used (21) with $\beta = 2M - N$, $\tau = 1$ in the first sum and (21) with $\beta = 2$, $\tau = -1$ in the second sum. Using this bound, we apply Corollary 2.1 to get,

$$\int_X \varphi (x, S_L(f)(x)/\lambda) \, d\mu(x)$$

$$\leq C \int_X \varphi \left( x, \lambda^{-1} \left( \sum_{k \in \mathbb{Z}} F_k(x)^{2/q} \right)^{1/2} \right) \, d\mu(x)$$

$$\leq C \int_X \varphi \left( x, \lambda^{-1} \left( \sum_{k \in \mathbb{Z}} \left( \sum_{\alpha \in h_k} |s_{Q^{k}_{\alpha}}| \mu(Q^{k}_{\alpha}) \right)^{1/2} \right)^{2/q} \right) \, d\mu(x)$$

$$= C \int_X \varphi \left( x, \lambda^{-1} \left( \sum_{k \in \mathbb{Z}} \sum_{\alpha \in h_k} |s_{Q^{k}_{\alpha}}|^2 \mu(Q^{k}_{\alpha})^{-1} X_{Q^{k}_{\alpha}}(x) \right)^{1/2} \right) \, d\mu(x)$$

$$= C \int_X \varphi \left( x, W_f(x)/\lambda \right) \, d\mu(x) ,$$

which proves the $\leq$ inequality in (16) and completes the proof of the theorem. 

□
We now turn to characterize the Musielak-Orlicz Hardy space $H_{L,G,p}$ and have the following result.

**Theorem 3.2.** Suppose $L$ is an operator that satisfies (H1) and (H2). Let $\varphi$ be a growth function with uniformly lower type $p_1$ and $f \in H_{L,G,p}(X) \cap L^2(X)$, then for all natural number $M > nq(\varphi)/(2p_1)$, $f$ has an $AT_{L,M}^*$-expansion such that

$$
\|f\|_{H_{L,G,p}(X)} \equiv \|W_f\|_{L^2(X)}.
$$

We prove Theorem 3.2 by borrowing some ideas from Duong et al. [6, Proof of Theorem 3.2].

To this end, we start with listing some known facts as follows.

Moreover, we also need the following Lemma, whose proof is standard, we omit the details.

**Lemma 3.6.** Let $n$ and $m$ be as in (5) and (6), and suppose that $N > n + m$. Then there exists a constant $C > 0$ such that for all measurable functions $f$ on $(X, \mu, d)$, $t > 0$ and each $y \in X$,

$$
\int_X \frac{|f(x)|}{V(x,t)[1 + t^{-1}d(x,y)]^N} d\mu(x) \leq CM(f)(y).
$$

**Proof of theorem 3.2** For any fixed $f \in H_{L,G,p}(X) \cap L^2(X)$, we let $\lambda_0 = \|f\|_{H_{L,G,p}(X)}$ and $\lambda_1 = \|W_f\|_{L^2(X)}$. It suffices to show that for all $\lambda \in (0, \infty)$,

$$
\int_X \varphi\left(x, \frac{G_L(f)(x)}{\lambda}\right) d\mu(x) \equiv \int_X \varphi\left(x, \frac{W_f(x)}{\lambda}\right) d\mu(x).
$$

(22)

In fact, if (22) holds for all $\lambda \in (0, \infty)$, then there exists a constant $C_0$ such that

$$
\int_X \varphi\left(x, \frac{G_L(f)(x)}{\lambda_1}\right) d\mu(x) \leq C_0 \int_X \varphi\left(x, \frac{W_f(x)}{\lambda_1}\right) d\mu(x) \leq C_0,
$$

where $\varphi$ is the growth function.
which, together with (8), implies that

\[ \int_X \varphi \left( x, \frac{G_L(f)(x)}{C_1A_1} \right) d\mu(x) \leq 1 \]

for some constants \( C_1 \), and hence we have \( \lambda_0 \leq C_1A_1 \). In a similar fashion, one can prove \( \lambda_1 \leq C_2A_0 \) for some constants \( C_2 \), and get the desired property.

Now we fix arbitrary \( \lambda \in (0, \infty) \) and prove (22). In the view of Lemma 2.1 for any fixed \( (x,k) \in X \times \mathbb{Z} \) there exists a unique \( \alpha \in I_k \), such that \( x \in Q^k_\alpha \). Let \( Q^k_\alpha \) denote the such \( Q^k_\alpha \) and we write

\[
W_f(x) = \left\{ \sum_{\alpha \in I_k} \sum_{k \in \mathbb{Z}} \left\{ \mu(Q^k_\alpha)^{-1} \left| s_{Q^k_\alpha} X_{Q^k_\alpha} (x) \right|^2 \right\}^{1/2} \right\}
\]

\[
= \left\{ \sum_{k \in \mathbb{Z}} \mu(Q^k_\alpha)^{-1} \left| s_{Q^k_\alpha} \right|^2 \right\}^{1/2}
\]

\[
= \left\{ \sum_{k \in \mathbb{Z}} \int_{\delta^{k+1}}^{\delta^k} \mu(Q^k_\alpha)^{-1} \left| \int_{Q^k_\alpha} \left| t^2 L e^{-t^2 L} f(y) \right|^2 d\mu(y) \frac{dt}{t} \right| \right\}^{1/2}, \quad (23)
\]

where \( \delta \in (0,1) \) is a constant as in Lemma 2.1 and the last quantity follows from Proposition 2.1. Moreover, by (iv) and (v) of Lemma 2.1, we know that for any fixed \( (x,k) \in X \times \mathbb{Z} \) there exists \( z^k_\delta \in Q^k_\alpha \) and constants \( C_3 \in (0,1), C_4 > 0 \) such that \( \text{diam} \left( Q^k_\alpha \right) \leq C_4 \delta^k \) and

\[ B\left( z^k_\delta, C_3 \delta^k \right) \subset Q^k_\alpha \subset B\left( x, C_4 \delta^k \right) \subset B\left( x, C_4 \delta^{-1} \right) \]

for all \( t \in (\delta^{k+1}, \delta^k) \). Then for each \( k \in \mathbb{Z} \) we compute by (5) and (6),

\[
\int_{\delta^{k+1}}^{\delta^k} \mu(Q^k_\alpha)^{-1} \left| \int_{Q^k_\alpha} t^2 L e^{-t^2 L} f(y) \right|^2 d\mu(y) \frac{dt}{t}
\]

\[
\leq \int_{\delta^{k+1}}^{\delta^k} \mu \left( B \left( x, C_3 \delta^k \right) \right)^{-1} \left| \int_{B(x,C_4 \delta^k)} t^2 L e^{-t^2 L} f(y) \right|^2 d\mu(y) \frac{dt}{t}
\]

\[
\leq C \int_{\delta^{k+1}}^{\delta^k} \mu \left( B \left( x, C_4 \delta^k \right) \right)^{-1} \mu \left( B \left( x, C_3 \delta^k \right) \right)^{-1} \left| \int_{B(x,C_3 \delta^k)} t^2 L e^{-t^2 L} f(y) \right|^2 d\mu(y) \frac{dt}{t}
\]

\[
\leq C \int_{\delta^{k+1}}^{\delta^k} \text{esssup}_{y \in B(x,C_3 \delta^k)} t^2 L e^{-t^2 L} f(y) \frac{dt}{t}
\]

\[
\leq C \int_{\delta^{k+1}}^{\delta^k} \left| M^*_a \left( f, t \right) \right|^2 \frac{dt}{t} \quad (24)
\]

for some appropriate constant \( C \), where \( M^*_a \left( f, t \right) \) is the Fefferman-Stein-type maximal function, with some large enough constants \( a \) to be selected. And the last inequality follows from
\[
\text{esssup}_{y \in B_{r}} |f^{2} L e^{-t^{2} L} f(y)| = \text{esssup}_{y \in B_{r}} \frac{|t^{2} L e^{-t^{2} L} f(y)|^{2}}{[1 + t^{-1} d(x,y)]^{2a}} \leq (1 + C_{1} \delta^{-1}) \left[ M_{a,L}^{*}(f)(x,t) \right]^{2a}.
\]

Combining now (23) and (24), we have the following estimate of \( W_{f}(x) \),

\[
W_{f}(x) \leq C \left\{ \int_{0}^{\infty} \left[ M_{a,L}^{*}(f)(x,t) \right]^{2} \frac{dt}{t} \right\}^{1/2} = C \left\{ \sum_{k \in \mathbb{Z}} \int_{2^{k}}^{2^{k+1}} \left[ M_{a,L}^{*}(f)(x,t) \right]^{2} \frac{dt}{t} \right\}^{1/2} = C \left\{ \sum_{k \in \mathbb{Z}} \int_{1}^{2} \left[ M_{a,L}^{*}(f)(x,2^{-k}t) \right]^{2} \frac{dt}{t} \right\}^{1/2}.
\]

In the view of Lemma 3.5, we see that for any \( \beta > 0, r > 0 \) and \( a > m/2 \), there exists a constant \( C \) such that for all \( f \in L^{2}(X), k \in \mathbb{Z}, x \in X \) and \( t \in [1, 2) \),

\[
\left| M_{a,L}(f)(x,2^{-k}t) \right| \leq C \sum_{j=k}^{\infty} 2^{(j-k)p} \int_{X} \frac{\left| (2^{-j}t)^{2} L e^{-2^{-j}t L} f(z) \right|^{r}}{V(z,2^{-k})|1 + 2^{k}d(x,z)|^{ar}} d\mu(z).
\]

Let \( r \in (0, 1) \) be as in Corollary 2.1 with \( p = 2/r > 1 \). Fix some \( \beta > 0 \) and choose \( a > m/2 \) such that \( ar > m + n \). We then take the norm \( \left[ \int_{1}^{2} \left| \frac{f(t)}{t} \right|^{2} dt \right]^{1/2} \) on the both sides of (25), and employ the Minkowski's inequality to get

\[
\left[ \int_{1}^{2} \left| M_{a,L}^{*}(f)(x,2^{-k}t) \right|^{2} \frac{dt}{t} \right]^{1/2} \leq C \left\{ \sum_{j=k}^{\infty} 2^{(j-k)p} \left\{ \int_{X} \frac{\left| (2^{-j}t)^{2} L e^{-2^{-j}t L} f(z) \right|^{r}}{V(z,2^{-k})|1 + 2^{k}d(x,z)|^{ar}} d\mu(z) \right\}^{2/r} \frac{dt}{t} \right\}^{1/2} \leq C \sum_{j=k}^{\infty} 2^{(j-k)p} \left[ \int_{X} \frac{\left| (2^{-j}t)^{2} L e^{-2^{-j}t L} f(z) \right|^{2} dt/t}{V(z,2^{-k})|1 + 2^{k}d(x,z)|^{ar}} d\mu(z) \right]^{1/2} \leq C \sum_{j=k}^{\infty} 2^{(j-k)p} \left[ \int_{X} \frac{\left| (2^{-j}t)^{2} L e^{-2^{-j}t L} f(z) \right|^{2} dt/t}{V(z,2^{-k})|1 + 2^{k}d(x,z)|^{ar}} d\mu(z) \right]^{1/2} \leq CM \left\{ \sum_{j=k}^{\infty} 2^{(j-k)p} \left[ \int_{X} \frac{\left| (2^{-j}t)^{2} L e^{-2^{-j}t L} f(z) \right|^{2} dt/t}{V(z,2^{-k})|1 + 2^{k}d(x,z)|^{ar}} d\mu(z) \right]^{1/2} \right\} (x) := CM(F_{k})(x),
\]
where \( F_k(x) := \sum_{j=k}^{\infty} 2^{-(j-k)\beta r} \left[ \int_1^2 |(2^{-j}t)^2 L e^{-(2^{-j}i)^2 L f(x)}|^2 dt/t \right]^{r/2} \), and the last inequality follows from Lemma 3.6. It follows that

\[
\int_X \varphi \left( x, \frac{|W_j|}{\lambda} \right) d\mu(x) 
\leq \int_X \varphi \left( x, C \left( \sum_{k \in \mathbb{Z}} \left[ \int_1^2 \left[ M_{a,L}^*(f) \left( x, 2^{-k}i \right) \right]^2 dt/t \right]^{r/2} \right)^{1/2} \right) d\mu(x) 
= \int_X \varphi \left( x, C \left( \sum_{k \in \mathbb{Z}} \left[ \left( \int_1^2 \left[ M_{a,L}^*(f) \left( x, 2^{-k}i \right) \right]^2 dt/t \right]^{r/2} \right)^{1/2} \right) \right) d\mu(x) 
\leq \int_X \varphi \left( x, C \left( \sum_{k \in \mathbb{Z}} \left[ \mathcal{M}(F_k(x)) \right]^{2/r} \right)^{1/2} \right) d\mu(x) 
\leq C \int_X \varphi \left( x, \left( \sum_{k \in \mathbb{Z}} F_k(x)^{2/r} \right) \right)^{1/2} d\mu(x),
\]

where we used the Corollary 2.1 in the last inequality.

We now turn to estimate \( F_k(x)^{2/r} \). For any \( k \in \mathbb{Z} \), we recall

\[
F_k(x) = \sum_{j=k}^{\infty} 2^{-(j-k)\beta r} \left[ \int_1^2 |(2^{-j}t)^2 L e^{-(2^{-j}i)^2 L f(x)}|^2 dt/t \right]^{1/2}.
\]

Since

\[
2^{-(j-k)\beta r} = \frac{\beta r}{1 - 2^{-\beta r}} \int_{2^{-j-k}}^{2^{-j-k+1}} s^{-\beta r - 1} ds,
\]

we let \( E_j := [2^{-j-k}, 2^{-j-k+1}] \) and it follows that

\[
F_k(x) = C \sum_{j=k}^{\infty} \int_{2^{-j-k}}^{2^{-j-k+1}} \left[ \int_1^2 \left| (2^{-j}t)^2 L e^{-(2^{-j}i)^2 L f(x)} \right|^2 \frac{dt}{t} \right]^{r/2} \frac{ds}{s^{\beta r + 1}}
= C \sum_{j=k}^{\infty} \int_1^\infty \left[ \int_1^2 \left| (2^{-j}t)^2 L e^{-(2^{-j}i)^2 L f(x)} \right|^2 \frac{dt}{t} \right]^{r/2} \chi_{E_j}(s) \frac{ds}{s^{\beta r + 1}}
= C \int_1^\infty \sum_{j=k}^{\infty} \left[ \int_1^2 \left| (2^{-j}t)^2 L e^{-(2^{-j}i)^2 L f(x)} \right|^2 \frac{dt}{t} \right]^{r/2} \chi_{E_j}(s) \frac{ds}{s^{\beta r + 1}}.
\]
We then apply the Hölder’s inequality to obtain
\[
F_k(x)^{2r} = \left[ C \int_1^\infty \sum_{j=k}^{\infty} \left( \int_1^\infty |(2^{-j}t)^2 L e^{-(2^{-j}t)^2 L} f(x)|^2 \frac{dt}{t} \right)^{r/2} \right]^{2/r} \leq C \left( \int_1^\infty s^{-br-1} ds \right)^{2/r} \times \int_1^\infty \left( \sum_{j=k}^{\infty} \left( \int_1^\infty |(2^{-j}t)^2 L e^{-(2^{-j}t)^2 L} f(x)|^2 \frac{dt}{t} \right)^{r/2} \right) \frac{ds}{s^{b+1}} \left( \sum_{j=k}^{\infty} \left( \int_1^\infty |(2^{-j}t)^2 L e^{-(2^{-j}t)^2 L} f(x)|^2 \frac{dt}{t} \right) \right)^{2r} \frac{ds}{s^{b+1}} = C \left( \sum_{j=k}^{\infty} \left( \int_1^\infty |(2^{-j}t)^2 L e^{-(2^{-j}t)^2 L} f(x)|^2 \frac{dt}{t} \right) \right)^{2r} \frac{ds}{s^{b+1}} = C \left( \sum_{j=k}^{\infty} \left( \int_1^\infty |(2^{-j}t)^2 L e^{-(2^{-j}t)^2 L} f(x)|^2 \frac{dt}{t} \right) \right)^{2r} \frac{ds}{s^{b+1}}.
\]  
(27)

Summation by all \( k \in \mathbb{Z} \), we have
\[
\sum_{k \in \mathbb{Z}} F_k(x)^{2r} \leq C \sum_{k \in \mathbb{Z}} \sum_{j \geq k} \left( 2^{-(j-k)b} \int_1^\infty |(2^{-j}t)^2 L e^{-(2^{-j}t)^2 L} f(x)|^2 \frac{dt}{t} \right) = C \sum_{j \in \mathbb{Z}} \sum_{k \leq j} \left( 2^{-(j-k)b} \int_1^\infty |(2^{-j}t)^2 L e^{-(2^{-j}t)^2 L} f(x)|^2 \frac{dt}{t} \right) = C (1 - 2^{-br}) \sum_{j \in \mathbb{Z}} \int_1^\infty |(2^{-j}t)^2 L e^{-(2^{-j}t)^2 L} f(x)|^2 \frac{dt}{t} = C \sum_{j \in \mathbb{Z}} \int_{2^{-j}}^{2^{-j+1}} |l^2 L e^{-t^2 L} f(x)|^2 \frac{dt}{t} = C \int_0^\infty |l^2 L e^{-t^2 L} f(x)|^2 \frac{dt}{t} = CG_L(f)(x)^2,
\]

which, together with (26) and (5), yields the \( \geq \) inequality in (22).

It remains to establish the reverse inequality. In the view of Proposition 2.1, we write \( f =
\[ \sum_{k \in \mathbb{Z}} \sum_{\alpha \in I_k} s_{Q_k} a_{Q_k}. \] Let \( \delta \) be as in Lemma 2.1 we get

\[ G_L(f)(x) = \left( \int_0^\infty |r^2Le^{-r^2L}(f)(x)|^2 \frac{dt}{t} \right)^{1/2} \]

\[ = \left( \int_0^\infty |r^2Le^{-r^2L} \left( \sum_{k \in \mathbb{Z}} \sum_{\alpha \in I_k} s_{Q_k} a_{Q_k} \right)(x) |^2 \frac{dt}{t} \right)^{1/2} \]

\[ = \left( \sum_{j \in \mathbb{Z}} \int_{\delta^j} |r^2Le^{-r^2L} \left( \sum_{k \in \mathbb{Z}} \sum_{\alpha \in I_k} s_{Q_k} a_{Q_k} \right)(x) |^2 \frac{dt}{t} \right)^{1/2} \]

\[ \leq \left( \sum_{j \in \mathbb{Z}} \int_{\delta^j} |r^2Le^{-r^2L} \left( \sum_{k \in \mathbb{Z}} \sum_{\alpha \in I_k} s_{Q_k} a_{Q_k} \right)(x) |^2 \frac{dt}{t} \right)^{1/2} \]

\[ + \left( \sum_{j \in \mathbb{Z}} \int_{\delta^j} |r^2Le^{-r^2L} \left( \sum_{k \in \mathbb{Z}} \sum_{\alpha \in I_k} s_{Q_k} a_{Q_k} \right)(x) |^2 \frac{dt}{t} \right)^{1/2} . \tag{28} \]

We now estimate the first part of (28). For any \( k > j \) and \( \alpha \in I_k \), noting that \( a_{Q_k} = L^M b_{Q_k} \), we write

\[ |r^2Le^{-r^2L}(a_{Q_k})(x)| = |r^2Le^{-r^2L} \left( b_{Q_k} \right)(x)| = r^{-2M} \left| \left( r^2L \right)^{M+1} e^{-r^2L} \left( b_{Q_k} \right)(x) \right| . \]

Let \( n \) be as in (5), since \( M > nq(\varphi)/(2p_1) \), we can choose some \( q = r \) with \( r \) be as in Corollary 2.1 such that \( 2M > n/q \). We then let \( N \) be some positive number such that \( 2M > N > n/q \). Then by Definition 2.3, the upper bound of the kernel of \( \left( r^2L \right)^{M+1} e^{-r^2L} \) and (7), we get

\[ |r^2Le^{-r^2L}(a_{Q_k})(x)| \]

\[ \leq \frac{C_5}{V(x,t)} r^{-2M} \ell(Q_k) 2^M \mu(Q_k)^{-1/2} \int_{3Q_k} \exp \left( - \frac{d(x,y)^2}{C_6 t^2} \right) d\mu(y) \]

\[ \leq Cr^{-2M} \ell(Q_k) 2^M \mu(Q_k)^{-1/2} \left( \frac{t}{t + d(x,y_k)} \right)^N , \]

where we denote by \( y_k \) the center of \( Q_k \). Hence,

\[ |r^2Le^{-r^2L} \left( \sum_{k \in \mathbb{Z}} \sum_{\alpha \in I_k} s_{Q_k} a_{Q_k} \right)(x) | \]

\[ \leq C \sum_{k > j} \sum_{\alpha \in I_k} r^{-2M} \ell(Q_k) 2^M \mu(Q_k)^{-1/2} \left| \frac{s_{Q_k}}{[1 + r^{-1}d(x,y_k)]^N} \right| \]

\[ \leq C \sum_{k > j} \delta^{(2M-N)(k-j)} \sum_{\alpha \in I_k} \mu(Q_k)^{-1/2} \left| \frac{s_{Q_k}}{[1 + \ell(Q_k)^{-1}d(x,y_k)]^N} \right| \]

\[ \leq C \sum_{k > j} \delta^{(2M-N)(k-j)} \left[ M \left( \sum_{\alpha \in I_k} |s_{Q_k}| \mu(Q_k)^{-q/2} X_{Q_k}(\cdot)(x) \right)^{1/q} \right], \tag{29} \]
where the last inequality follows from Lemma 3.4.

Estimate of the second part of (28). For any \( k \leq j \) and \( \alpha \in I_k \), we write

\[
|t^2 \mathcal{L}^{-2} \left( a_{Q_k^a} \right)(x)| = t^2 \left| \mathcal{L}^{-2} \left( L \left( a_{Q_k^a} \right) \right)(x) \right|.
\]

Then by Definition 2.3 the Gaussian estimate (11) and inequality (7), we get

\[
|t^2 \mathcal{L}^{-2} \left( a_{Q_k^a} \right)(x)| \leq \frac{C_s}{V(x,t)} t^2 \ell \left( Q_k^a \right)^{-2} \mu \left( Q_k^a \right)^{-1/2} \int_{Q_k^a} \exp \left( -\frac{d(x,y)^2}{C_3 t^2} \right) d\mu(y).
\]

which implies that

\[
|t^2 \mathcal{L}^{-2} \left( \sum_{k \leq j} \sum_{a \in \mathbb{L}} s_{Q_k^a} a_{Q_k^a} \right)(x)| \leq C \sum_{k \leq j} \sum_{a \in \mathbb{L}} t^2 \ell \left( Q_k^a \right)^{-2} \mu \left( Q_k^a \right)^{-1/2} \left| s_{Q_k^a} \right| \left[ 1 + \ell \left( Q_k^a \right)^{-1} d(x,y_k^a) \right]^{-N/2},
\]

By the same technique we used in (27), we combine now (28)-(30), and get the following estimate of \( G_L(f) \),

\[
G_L(f)(x) \leq \left( \sum_{j \in \mathbb{Z}} \int_{2^{j-1}}^{2^j} \left| t^2 \mathcal{L}^{-2} \left( \sum_{k \leq j} \sum_{a \in \mathbb{L}} s_{Q_k^a} a_{Q_k^a} \right)(x) \right|^2 \frac{dt}{t} \right)^{1/2} + \left( \sum_{j \in \mathbb{Z}} \int_{2^{j-1}}^{2^j} \left| t^2 \mathcal{L}^{-2} \left( \sum_{k \leq j} \sum_{a \in \mathbb{L}} s_{Q_k^a} a_{Q_k^a} \right)(x) \right|^2 \frac{dt}{t} \right)^{1/2}
\]

\[
\leq C \left( \sum_{j \in \mathbb{Z}} \int_{2^{j-1}}^{2^j} \left| \sum_{k \geq j} \sum_{a \in \mathbb{L}} s_{Q_k^a} a_{Q_k^a} \right|^2 \left| G_k(x) \right|^{1/4} \frac{dt}{t} \right)^{1/2} + \sum_{j \in \mathbb{Z}} \int_{2^{j-1}}^{2^j} \left| \sum_{k \leq j} \sum_{a \in \mathbb{L}} s_{Q_k^a} a_{Q_k^a} \right|^2 \left| G_k(x) \right|^{1/4} \frac{dt}{t} \right)^{1/2}
\]

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\[ \int_X \varphi(x, G_L f(x)/\lambda) d\mu(x) \leq C \int_X \varphi \left( x, \lambda^{-1} \left( \sum_{k \in \mathbb{Z}} G_k(x)^{2q} \right)^{1/2} \right) d\mu(x) \]
\[ \leq C \int_X \varphi \left( x, \lambda^{-1} \left( \sum_{k \in \mathbb{Z}} \left| \delta_x \right|^q \mu(Q_k^{-1})^{-q/2} X_{\varphi}(x) \right)^{2/q} \right) d\mu(x) \]
\[ = C \int_X \varphi \left( x, \lambda^{-1} \left( \sum_{k \in \mathbb{Z}} \sum_{\alpha \in I_k} |s_{\alpha}^e|^{q/2} \mu(Q_k^{-1})^{-q/2} X_{\varphi}(x) \right)^{1/2} \right) d\mu(x) \]
\[ = C \int_X \varphi \left( x, W_f(x)/\lambda \right) d\mu(x), \]

which proves the reverse inequality in (22) and completes the proof of the theorem.

Using Theorem 3.1 and Theorem 3.2, we immediately obtain the following result.

**Theorem 3.3.** Suppose \( L \) is an operator that satisfies (H1) and (H2). Let \( \varphi \) be a growth function with uniformly lower type \( \rho_1 \). Then the spaces \( H_{\varphi,L}(X) \) and \( H_{L,G,\varphi}(X) \) coincide and their norms are equivalent.

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