The branching problem for generalized Verma modules, with application to the pair

\((so(7), \text{Lie } G_2)\)

Extended version with tables

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Abstract

We discuss the branching problem for generalized Verma modules \(M_\lambda(g, p)\) applied to couples of reductive Lie algebras \(\bar{g} \hookrightarrow g\). The analysis is based on projecting character formulas to quantify the branching, and on the action of the center of \(U(\bar{g})\) to explicitly construct singular vectors realizing part of the branching. We demonstrate the results on the pair \(Lie \ G_2 \hookrightarrow so(7)\) for both strongly and weakly compatible with \(i(Lie \ G_2)\) parabolic subalgebras and a large class of inducing representations.

Key words: Generalized Verma modules, branching problems, branching rules, character formulas, singular vectors, non-symmetric pairs, \((so(7), \text{Lie } G_2)\).

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1 Introduction

For an embedding of reductive complex algebraic Lie groups, \(\bar{G} \hookrightarrow G\) with Lie algebras \(\bar{g}\) and \(g\), the branching rules of generalized Verma \(g\)-modules over \(\bar{g}\) are a central problem of representation theory, harmonic analysis and geometry.

To our best knowledge, the most general treatment of the problem is given in [11], [12]. The article [11] restricts to a couple of reductive Lie algebras \((g, i(\bar{g}))\) and a parabolic subalgebra \(p \subset g\) compatible with \(i(\bar{g})\), and the question of discrete decomposability of an element in the Bernstein-Gelfand-Gelfand parabolic
category $\mathcal{O}^\mathfrak{p}$ over $\mathfrak{g}$ is achieved by employing the geometrical properties of the double coset $N_G(i(\mathfrak{g}))\backslash G/P$. Here $N_G(i(\mathfrak{g}))$ denotes the normalizer of $i(\mathfrak{g})$ in $G$.

We assume that $\mathfrak{g}$ is a semisimple Lie algebra, $i(\mathfrak{g})$ is reductive in $\mathfrak{g}$ and $i(\mathfrak{b}) \subset \mathfrak{b} \subset \mathfrak{p}$, where $\mathfrak{b}$ and $\mathfrak{b}$ are Borel subalgebras of respectively $\mathfrak{g}$ and $\mathfrak{g}$. Let $M_\lambda(\mathfrak{g}, \mathfrak{p})$ be the generalized Verma $\mathfrak{g}$-module induced from the irreducible finite-dimensional $\mathfrak{p}$-module with highest weight $\lambda$. We define the branching problem of $M_\lambda(\mathfrak{g}, \mathfrak{p})$ over $\mathfrak{g}$ to be the problem of finding all $\mathfrak{b}$-singular vectors in $M_\lambda(\mathfrak{g}, \mathfrak{p})$, that is, the set of all vectors annihilated by image of the nilradical of $\mathfrak{b}$ on which the image of the Cartan subalgebra of $\mathfrak{b}$ has diagonal action.

The branching problem can be split into two closely related sub-problems. First, prove that $M_\lambda(\mathfrak{g}, \mathfrak{p})$ has (finite or infinite) Jordan-Hölder series over $\mathfrak{g}$, enumerate the $\mathfrak{b}$-highest weights $\mu$ appearing in the series, and give their multiplicity as a function of $\mu$ and $\lambda$. Second, for each $\mu$, compute explicitly a basis for the $\mathfrak{b}$-singular vectors of weight $\mu$.

In the present article, under certain technical assumptions, we reduce the first step of the branching problem to the problem of understanding a single central character block in Category $\mathcal{O}^\mathfrak{p}$. More precisely, we express the character of $M_\lambda(\mathfrak{g}, \mathfrak{p})$ over $\mathfrak{g}$ as a sum of characters of $M_\mu(\mathfrak{g}, \mathfrak{p})$ over $\mathfrak{g}$ and so reduce the branching rules of $M_\lambda(\mathfrak{g}, \mathfrak{p})$ over $\mathfrak{g}$ to those of $M_\mu(\mathfrak{g}, \mathfrak{p})$ over $\mathfrak{g}$. We also conjecture that our technical assumptions are not necessary, and conjecturally our result on multiplicities holds in full generality.

For the second step of the branching problem, with additional assumptions, we show how to find explicitly the highest weight $\mathfrak{b}$-singular vectors at a certain “top level” $\mathfrak{g}$-submodule of $M_\lambda(\mathfrak{g}, \mathfrak{p})$, and give sufficient conditions for this “top level” to equal the entire $M_\lambda(\mathfrak{g}, \mathfrak{p})$.

We illustrate our results on the pilot example Lie $G_2 \hookrightarrow so(7)$. Finally, we present tables with $\mathfrak{b}$-singular vectors and multiplicities for a large set of inducing representations for the pair Lie $G_2 \hookrightarrow so(7)$.

We recall that for an arbitrary $\mathfrak{g}$-module $M$, the Fernando-Kac subalgebra of $\mathfrak{g}$ associated to $M$ is the Lie subalgebra of elements that act locally finitely on every vector $v \in M$. As the Fernando-Kac subalgebra associated to $M_\lambda(\mathfrak{g}, \mathfrak{p})$ is $\mathfrak{p}$, it follows that the Fernando-Kac subalgebra of $\mathfrak{g}$ associated to $M_\lambda(\mathfrak{g}, \mathfrak{p})$ equals $i^{-1}(i(\mathfrak{g}) \cap \mathfrak{p})$. Then our requirement that $\mathfrak{p}$ contains the image of a Borel subalgebra of $\mathfrak{g}$ implies the discrete decomposability of $M_\lambda(\mathfrak{g}, \mathfrak{p})$ over $i(\mathfrak{g})$ (Lemma 3.2).

Our viewpoint of the branching problem differs from that of [11] in that we assume that $i(\mathfrak{b}) \subset \mathfrak{p}$. If we drop the requirement $i(\mathfrak{b}) \subset \mathfrak{p}$, it appears that there is no good understanding of the simple $\mathfrak{g}$-modules with Fernando-Kac subalgebras of the form $i^{-1}(i(\mathfrak{g}) \cap \mathfrak{p})$. Even more, there appears to be no complete understanding of the structure of the Lie algebra $i^{-1}(i(\mathfrak{g}) \cap \mathfrak{p})$. We note that if $\mathfrak{p}$ does not contain an image of the Borel subalgebra of $\mathfrak{g}$, we can restrict
our attention to a maximal reductive in \( \mathfrak{g} \) subalgebra \( \mathfrak{g}_1 \) with the property that it has a Borel subalgebra whose image is contained in \( \mathfrak{p} \). If \( \mathfrak{g}_1 \neq \{0\} \), the branching problem of \( M_\lambda(\mathfrak{g}, \mathfrak{p}) \) over \( \mathfrak{g}_1 \) is well-posed, and the results of our paper apply to \( \mathfrak{g}_1 \).

Let \( \mathfrak{p} \supset \mathfrak{b} \) be a parabolic subalgebra of \( \mathfrak{g} \) and define the parabolic subalgebra \( \mathfrak{p} \subset \mathfrak{g} \) by \( i(\mathfrak{p}) = i(\mathfrak{g}) \cap \mathfrak{p} \). Let \( M_\lambda(\mathfrak{g}, \mathfrak{p}) \) be the generalized Verma \( \mathfrak{g} \)-module equipped with \( \mathfrak{g} \)-module structure induced by \( i \). Then one can write \( M_\lambda(\mathfrak{g}, \mathfrak{p}) \) as a direct limit of certain \( \mathfrak{g} \)-submodules \( M_n \) so that the \( \mathfrak{h} \)-character \( \text{ch} M_n \) of \( M_n \) decomposes as a sum of the form

\[
\text{ch} M_n = \sum m_n(\mu, \lambda) \text{ch}(M_{\mu}(\mathfrak{g}, \mathfrak{p})).
\]

Under the technical Condition A given in Definition 3.5 below, we prove that the coefficients \( m_n(\mu, \lambda) \) in the above expression have limits \( m(\mu, \lambda) \) (allowing \( m(\mu, \lambda) = +\infty \)). We do not know of an example where Condition A fails and we may conjecture that it is a consequence of our assumption \( \mathfrak{p} \supset \mathfrak{b} \supset i(\mathfrak{b}) \).

If Condition A holds, in Theorem 3.10 we prove that either all non-zero \( m(\mu, \lambda) \) are simultaneously equal to \( +\infty \), or they are all simultaneously finite, and in the latter case we give a formula computing them. For our pilot example \( \text{Lie} \, G_2 \hookrightarrow so(7) \), Theorem 3.10 implies that \( \mathfrak{p}(1,0,0) \) is the only proper parabolic subalgebra of \( so(7) \) with finite branching over \( \text{Lie} \, G_2 \), i.e., branching for which \( m(\mu, \lambda) \neq 0 \) for only finitely many \( \mu \). We note that this gives an example in which an infinite dimensional \( \mathfrak{g} \)-module has finite branching over a subalgebra \( i(\mathfrak{g}) \) of rank strictly smaller than the rank of \( \mathfrak{g} \).

On Condition A, in Theorem 3.12, we explain how to compute \( m(\mu, \lambda) \) as a piecewise quasi-polynomial in the coordinates of \( \mu \) and \( \lambda \) and give an upper bound for the degrees of the piecewise quasi-polynomials. In our pilot example \( \text{Lie} \, G_2 \hookrightarrow so(7) \), the degree in question is 1.

We prove that Condition A holds for all parabolic subalgebras \( \mathfrak{p} \) weakly compatible with \( i(\mathfrak{g}) \). In particular, Condition A holds for a parabolic subalgebra \( \mathfrak{p} \) compatible with \( i(\mathfrak{g}) \) in the sense of \( [11, \text{Section 3}] \). As we have assumed that \( i(\mathfrak{b}) \subset \mathfrak{b} \subset \mathfrak{p} \), it follows that all parabolic subalgebras \( \mathfrak{p} \) of \( so(7) \) are weakly compatible with \( i(\text{Lie} \, G_2) \) (of them only 4 are strongly compatible, Corollary 5.3), and therefore Condition A holds for all parabolic subalgebras relative to the pair \( \text{Lie} \, G_2 \hookrightarrow so(7) \).

In Section 4 we discuss the problem of explicitly constructing \( \mathfrak{b} \)-singular vectors in \( M_\lambda(\mathfrak{g}, \mathfrak{p}) \). Under additional assumptions given in Theorem 4.5 we use the Harish-Chandra isomorphism theorem and the corresponding elements in the center of \( U(\mathfrak{g}) \) to find a certain a set of singular vectors that realize the “top level” (see (21)) of the branching problem. For a fixed dimension of the inducing finite dimensional \( \mathfrak{p} \)-module, our assumptions on \( \lambda \) exclude a certain Zariski-closed subset of \( \mathfrak{h}^* \), however allowing the Zariski-closed subset to be the
entire $\mathfrak{h}^*$. Corollary 4.6 gives a sufficient criteria for $M_\lambda(\mathfrak{g}, \mathfrak{p})$ to decompose as a direct sum of (in general, reducible) generalized Verma $\mathfrak{g}$-modules.

In our pilot example Lie $G_2 \rightarrow so(7)$, in the case of the parabolic subalgebra $\mathfrak{p}_{(1,0,0)}$, the technical assumptions on $\lambda$ exclude finitely many values for a fixed dimension of the inducing $\mathfrak{p}_{(1,0,0)}$-module. Except for these values, in Theorem 5.5 we give explicit bases of the $\mathfrak{b}$-singular vectors of $M_\lambda(so(7), \mathfrak{p}_{(1,0,0)})$ for which $m(\mu, \lambda) \neq 0$ for the 6 one-parameter families $\lambda = x_1 \omega_1, x_1 \omega_1 + \omega_2, x_1 \omega_1 + \omega_3, x_1 \omega_1 + 2\omega_2, x_1 \omega_1 + \omega_1 + \omega_2, x_1 \omega_1 + 2\omega_3$. In Corollary 5.7 we prove that, except for the (explicitly computed) exceptional values for $x_1$ from the preceding Theorem, at least 5 out of the 6 families of $\mathfrak{b}$-singular vectors give a decomposition of $M_\lambda(so(7), \mathfrak{p}_{(1,0,0)})$ as a direct sum of generalized Verma modules $M_\mu(\text{Lie } G_2, \mathfrak{p}(1,0))$.

In Tables 8-13, for $\mathfrak{p} \neq \mathfrak{p}_{(1,0,0)}$, we tabulate the “top level” sets of $\mathfrak{b}$-singular vectors in $M_\lambda(so(7), \mathfrak{p})$ for certain one or two-parameter families of the highest weights $\lambda$. For the finite dimensional branching $\mathfrak{p} \simeq so(7)$, Table 8 gives the vectors realizing the branching of all irreducible finite dimensional $so(7)$-modules over Lie $G_2$ for highest weight with fundamental coordinate sum less than or equal to 2.

A geometric motivation of the present study may be given as follows. Let $G, \tilde{G}$ be the connected and simply connected Lie groups with Lie algebras $\mathfrak{g}, \tilde{\mathfrak{g}}$. Let $P$ be the parabolic subgroup of $G$ with Lie algebra $\mathfrak{p}$. Then there is a well-known equivalence between invariant differential operators acting on induced representations and homomorphisms of generalized Verma modules, realized by the natural pairing

$$\text{Ind}^G_P(V_\lambda(L)^*) \times M_\lambda(\mathfrak{g}, \mathfrak{p}) \rightarrow \mathbb{C},$$

where $V_\lambda(L)$ denotes the finite-dimensional irreducible $L$-module obtained by exponentiating $V_\lambda(1)$, $V_\lambda(L)^*$ is its dual, and $\text{Ind}^G_P$ denotes induction. As a consequence, the singular vectors constructed in the article induce invariant differential operators acting between induced representations of $j(\tilde{G})$. It is quite interesting to construct these invariant differential operators, in particular their curved extensions as lifts to homomorphisms of corresponding semiholonomic generalized Verma modules. More information on the relationship between conformal geometry in dimension 5 and filtered structure on the tangent space (i.e., a distribution) related to exceptional Lie group $G_2$, can be found in [6] and references therein.

We end this section by proposing a series of (open to our best knowledge) sub-problems that could eventually lead to a solution of the branching problem. The central $\tilde{\mathfrak{g}}$-characters (the constants by which the center of $U(\tilde{\mathfrak{g}})$ acts on cyclic $\tilde{\mathfrak{g}}$-modules) that appear in $\tilde{\mathfrak{g}}$-submodules of $M_\lambda(\mathfrak{g}, \mathfrak{p})$ that afford a central character are exhausted by the central $\tilde{\mathfrak{g}}$-characters of $M_\mu(\tilde{\mathfrak{g}}, \tilde{\mathfrak{p}})$ for which $m(\mu, \lambda) > 0$. Therefore, one needs to first find all $\mathfrak{b}$-singular vectors $v_\mu$ of $\mathfrak{h}$-weight $\mu$ in
$M_\lambda(g, p)$, for which $m(\mu, \lambda) > 0$. In other words, one needs to extend the results of the current paper beyond the “top-level” branching given in Theorem 4.5, and in addition explore what happens when the Zariski-open Condition B fails. To achieve this task, it appears most natural to use iterated translation functors. Here, by translation functors we broadly mean the decomposition into indecomposables of the $\mathfrak{g}$-module $M_\mu(\mathfrak{g}, \mathfrak{p}) \otimes \mathfrak{g}/\mathfrak{g}$ as a function of the coordinates of the highest weight $\mu$.

Next, one must solve a problem involving a single central character block in the Bernstein-Gelfand-Gelfand category $\mathcal{O}^{\mathfrak{p}}$ - namely, find all $\mathfrak{b}$-singular vectors of $M_\mu(\mathfrak{g}, \mathfrak{p})$. In general, $M_\mu(\mathfrak{g}, \mathfrak{p})$ has $\mathfrak{b}$-singular vectors other than the highest one, whose weight is of the form $w(\mu + \rho_\mathfrak{g}) - \rho_\mathfrak{g}$, where $w$ is in the Weyl group of $\mathfrak{g}$. In the case of $\mathfrak{p} = \mathfrak{b}$, the multiplicities of these vectors are 0 or 1 by [5, Chapter 7]; for a general $\mathfrak{p}$ one could use Kazhdan-Lusztig polynomials to compute the multiplicities in question. Next, one should compute the $\mathfrak{b}$-singular vectors in $M_\mu(\mathfrak{g}, \mathfrak{p})$ explicitly. For $\mathfrak{p} = \mathfrak{b}$, an algorithm for this has been described in [9, Chapter 4]. For a general $\mathfrak{p}$, one can use the approach of analysis of generalized Verma modules developed in [12], based on distribution Fourier transform. Here the analysis should be a lot more involved as one must account both standard and non-standard generalized Verma module homomorphisms (see [13]). Non-standard homomorphisms have been classified only for parabolic subalgebras with commutative nilradicals ([3]) and maximal parabolic subalgebras ([15]), both cases just for scalar generalized Verma modules.

In addition to the branching of the generalized Verma module $M_\mu(\mathfrak{g}, \mathfrak{p})$ over $\mathfrak{g}$, one has to solve one final problem arising from the embedding $\mathfrak{g} \hookrightarrow \mathfrak{g}$. More precisely, given a $\mathfrak{b}$-singular vector $v_\mu$ of $\mathfrak{b}$-weight $\mu$ in $M_\lambda(g, p)$, one needs to decide whether $v_\mu$ generates a $\mathfrak{g}$-submodule isomorphic to $M_\mu(\mathfrak{g}, \mathfrak{p})$, or a proper quotient module (if the latter is the case, one also needs tools to understand which quotients are generated).

2 Notation and preliminaries

We fix the base field to be $\mathbb{C}$. For an arbitrary Lie algebra $t$ we denote by $U(t)$ its universal enveloping algebra and by $ad$ we denote the adjoint action in $t$ and $U(t)$. Given a $t$-module $M$, we say that a vector $v \in M$ is $t$-singular if the one-dimensional vector space $\mathbb{C}v$ is preserved by $t$ (i.e., if $\mathbb{C}v$ is a one-dimensional $t$-submodule of $M$).

For a semisimple Lie algebra $\mathfrak{g}$, the quadratic Casimir element $c_1 \in U(\mathfrak{g})$ is defined as $\sum_i e_i f_i$, where $\{e_i\}, \{f_i\}$ are two bases of $\mathfrak{g}$, dual with respect to the Killing form $K(e, f) := \text{Tr}(ade \circ adf)$.

Let $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ be reductive Lie algebras. We recall that a subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ is called reductive in $\mathfrak{g}$ if any irreducible finite-dimensional $\mathfrak{g}$-module is a semisimple $\mathfrak{g}'$-module (i.e., decomposes as a direct sum of irreducible $\mathfrak{g}'$-modules). Let
i be an embedding \( \bar{\mathfrak{g}} \hookrightarrow \mathfrak{g} \) such that \( i(\bar{\mathfrak{g}}) \) is reductive in \( \mathfrak{g} \). Let \( \bar{\mathfrak{h}} \) be a Cartan subalgebra of \( \bar{\mathfrak{g}} \). Since the base field is \( \mathbb{C} \) and \( i(\bar{\mathfrak{g}}) \) is reductive in \( \mathfrak{g} \), we can extend \( i(\bar{\mathfrak{h}}) \) to a Cartan subalgebra \( \mathfrak{h} \supset i(\bar{\mathfrak{h}}) \) of \( \mathfrak{g} \).

We denote by \( \Delta(\mathfrak{g}) \) the root system of \( \mathfrak{g} \) with respect to \( \mathfrak{h} \) and by \( \Delta(\bar{\mathfrak{g}}) \) the root system of \( \bar{\mathfrak{g}} \) with respect to \( \bar{\mathfrak{h}} \). Then \( \mathfrak{g} \) and \( \bar{\mathfrak{g}} \) have the vector space decompositions

\[
\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\beta \in \Delta(\mathfrak{g})} \mathfrak{g}_\beta, \quad \bar{\mathfrak{g}} = \bar{\mathfrak{h}} \oplus \bigoplus_{\alpha \in \Delta(\bar{\mathfrak{g}})} \bar{\mathfrak{g}}_\alpha,
\]

where \( \mathfrak{g}_\beta \) (respectively, \( \bar{\mathfrak{g}}_\alpha \)) are the \( \Delta(\mathfrak{g}) \)- (respectively, \( \Delta(\bar{\mathfrak{g}}) \)-) root spaces. The Killing form on \( \mathfrak{g} \) (respectively \( \bar{\mathfrak{g}} \)) induces a non-degenerate symmetric bilinear form on \( \mathfrak{h}^* \) (respectively \( \bar{\mathfrak{h}}^* \)); we rescale this form on each simple component of \( \mathfrak{g} \) (respectively, \( \bar{\mathfrak{g}} \)) so that the long roots have length \( \sqrt{2} \) for types \( A_n, B_n, C_n, D_n, E_6, E_7, E_8 \), length 2 for types \( F_4 \), and length \( \sqrt{5} \) for type \( G_2 \). We denote the so obtained bilinear form on \( \mathfrak{h}^* \) (respectively, \( \bar{\mathfrak{h}}^* \)) by \( (\cdot, \cdot)_\mathfrak{g} \) (respectively, \( (\cdot, \cdot)_{\bar{\mathfrak{g}}} \)). For any weight \( \beta \in \mathfrak{h}^* \) (respectively \( \alpha \in \bar{\mathfrak{h}}^* \)) we define \( h_\beta \in \mathfrak{h} \) (respectively \( h_\alpha \in \bar{\mathfrak{h}} \)) to be the dual element of \( \beta \), i.e., we require that \( [h_\beta, g_\gamma] = (\beta, \gamma)_\mathfrak{g} g_\gamma \) (respectively \( [h_\alpha, \bar{g}_\gamma] = (\alpha, \gamma)_{\bar{\mathfrak{g}}} \bar{g}_\gamma \)) for all \( g_\gamma \in \mathfrak{g}_\gamma \) (respectively \( \bar{g}_\gamma \in \bar{\mathfrak{g}}_\gamma \)).

We define an embedding

\[
\iota : \bar{\mathfrak{h}}^* \hookrightarrow \mathfrak{h}^*
\]

by requiring

\[
[i(\bar{h}_\gamma), g_\beta] = (\iota(\gamma), \beta)_\mathfrak{g} g_\beta
\]

for all \( \beta \in \Delta(\mathfrak{g}) \) and all \( \gamma \in \bar{\mathfrak{h}}^* \). There exists a positive integer \( D \) such that \( D(\alpha_1, \alpha_2)_\mathfrak{g} = (\iota(\alpha_1), \iota(\alpha_2))_\mathfrak{g} \) for all \( \alpha_1, \alpha_2 \in \mathfrak{h}^* \); the integer \( D \) is called Dynkin index of the embedding \( \bar{\mathfrak{g}} \hookrightarrow \mathfrak{g} \). [?] Let

\[
\text{pr} : \mathfrak{h}^* \rightarrow \bar{\mathfrak{h}}^*
\]

denote the canonical projection map naturally induced by \( \iota \).

We say that a \( \mathfrak{g} \)-module \( M \) is a \( \mathfrak{h} \)-weight module if it has a vector space decomposition into countably many one-dimensional \( \mathfrak{h} \)-submodules. All \( \mathfrak{g} \)-modules discussed in the current paper are \( \mathfrak{h} \)-weight modules, in particular so is every finite dimensional completely reducible \( \mathfrak{g} \)-module as well as every generalized Verma \( \mathfrak{g} \)-module. Given a \( \mathfrak{h} \)-weight \( \mathfrak{g} \)-module \( M \), denote by \( \text{Weights}_\mathfrak{h}(M) \) the ordered \( (\dim M) \)-tuple of elements of \( \mathfrak{h}^* \) consisting of the \( \mathfrak{h} \)-weights of \( M \) with multiplicities. In the case that \( \dim M < \infty \), we order weights in increasing graded lexicographic order with respect to coordinates given in a simple basis of the ambient Lie algebra. Define \( \text{Supp}_\mathfrak{h} M \) to be the set of \( \mathfrak{h} \)-weights of \( M \).

Given a \( \mathfrak{g} \)-module \( M \) with action \( \cdot \), we equip \( M \) with a \( \bar{\mathfrak{g}} \)-module structure by requesting that an element \( \bar{g} \in \bar{\mathfrak{g}} \) act on an element \( m \in M \) by \( i(\bar{g}) \cdot m \). In
particular, \( \mathfrak{g} \) is a \( \tilde{\mathfrak{g}} \) module under the \( \text{ad} \circ i \) action of \( \tilde{\mathfrak{g}} \). Given a \( \mathfrak{g} \)-module \( M \) which as a \( \tilde{\mathfrak{g}} \)-module is a \( \mathfrak{h} \)-weight module, the definitions of \( \mathfrak{h} \) and \( \mathfrak{b} \) imply the following relation:

\[
\beta \in \text{Weights}_\mathfrak{h} M \Rightarrow \text{pr} \beta \in \text{Weights}_\mathfrak{g} M.
\]

For each \( \alpha \in \Delta(\mathfrak{g}) \) and \( \beta \in \Delta(\tilde{\mathfrak{g}}) \), we fix a Chevalley-Weyl basis \( \tilde{\alpha} \in \tilde{\mathfrak{g}}_\alpha \) and \( \tilde{\beta} \in \tilde{\mathfrak{g}}_\beta \). In a Chevalley-Weyl basis we have that there exist integer structure constants \( \bar{n}_{\alpha_1,\alpha_2} \in \mathbb{Z} \), \( n_{\beta_1,\beta_2} \in \mathbb{Z} \), for which

\[
\begin{align*}
[g_{\alpha_1},g_{\alpha_2}] & = \bar{n}_{\alpha_1,\alpha_2}g_{\alpha_1+\alpha_2}, \\
[g_{\alpha_1},g_{-\alpha_1}] & = \frac{1}{2} \bar{h}_{\alpha_1}, \\
[g_{\beta_1},g_{-\beta_1}] & = n_{\beta_1,\beta_2}g_{\beta_1+\beta_2},
\end{align*}
\]

(2)

where we define \( \bar{g}_\mu := 0 \), \( g_\mu := 0 \) whenever \( \mu \) is not a root of the respective root system. We note that the relations (2) imply that the elements \( [g_\alpha,g_{-\alpha}],g_\alpha,g_{-\alpha} \) form a standard \( h,e,f \)-triple parametrization of the \( \mathfrak{sl}(2) \)-subalgebra they generate.

Let \( \mathfrak{b} \supset \tilde{\mathfrak{b}} \) be a Borel subalgebra of \( \tilde{\mathfrak{g}} \). As the base field is \( \mathbb{C} \), \( i(\tilde{\mathfrak{b}}) \) can be extended to a Borel subalgebra \( \mathfrak{b} \supset \mathfrak{h} \) of \( \mathfrak{g} \). In the present paper we assume one such choice of Borel subalgebras \( \mathfrak{b} \supset \tilde{\mathfrak{b}}, \mathfrak{b} \supset \mathfrak{h} \supset i(\tilde{\mathfrak{h}}), i(\tilde{\mathfrak{b}}) \subset \mathfrak{b} \) to be fixed. All parabolic subalgebras \( \mathfrak{p} \) of \( \mathfrak{g} \) are assumed to contain \( \mathfrak{b} \), and similarly, all parabolic subalgebras \( \tilde{\mathfrak{p}} \) of \( \tilde{\mathfrak{g}} \) are assumed to contain \( \tilde{\mathfrak{b}} \).

We set \( \mathfrak{n}_- \) to be opposite nilradical of \( \mathfrak{n} \), i.e., the Lie subalgebra generated by the root spaces opposite to the root spaces lying in \( \mathfrak{n} \). Set \( \tilde{\mathfrak{p}} \) to be the preimage of the intersection of \( \mathfrak{p} \) with \( i(\tilde{\mathfrak{g}}) \), i.e., set

\[
\tilde{\mathfrak{p}} := i^{-1}(i(\tilde{\mathfrak{g}}) \cap \mathfrak{p})
\]

As \( \tilde{\mathfrak{p}} \) contains \( \tilde{\mathfrak{b}} \), it is a parabolic subalgebra of \( \tilde{\mathfrak{g}} \). Set \( \bar{\mathfrak{n}} \) to be the nilradical of \( \tilde{\mathfrak{p}} \) and \( \bar{\mathfrak{n}}_- \) to be its opposite nilradical.

If \( \mathfrak{g} \) is of rank \( n \), we parametrize the subsets of its positive simple roots by \( n \)-tuples of 0’s and 1’s. A parabolic subalgebra \( \mathfrak{p} \) containing \( \mathfrak{b} \) is parametrized by indicating which positive simple roots of \( \mathfrak{g} \) are weights of \( \mathfrak{n} \) (those roots are also called “crossed roots”). For example, the parabolic subalgebra \( \mathfrak{P}_{(1,0,0)} \) of \( \mathfrak{so}(7) \) stands for the parabolic subalgebra in which the first simple root is crossed out.

Define \( \mathfrak{l} \) to be the reductive Levi part of \( \mathfrak{p} \) that contains \( \mathfrak{h} \) and similarly define \( \bar{\mathfrak{l}} \) to be the reductive part of \( \tilde{\mathfrak{p}} \) that contains \( \tilde{\mathfrak{h}} \). The fact that \( i(\tilde{\mathfrak{h}}) \subset \mathfrak{h} \) implies that \( i(\mathfrak{l}) = \mathfrak{l} \cap i(\tilde{\mathfrak{g}}) \). As \( i(\mathfrak{l}) \subset \mathfrak{l} \), it follows that \( i(\tilde{\mathfrak{g}}) \cap \mathfrak{n} \) is a \( \mathfrak{l} \)-submodule. We claim that \( \mathfrak{n}_- \) splits as the direct sum of the \( \mathfrak{l} \)-module \( \mathfrak{n}_- \cap i(\tilde{\mathfrak{g}}) \) and its complement submodule isomorphic to \( \mathfrak{n}_-/(\mathfrak{n}_- \cap i(\tilde{\mathfrak{g}})) \) - indeed, this follows as \( \bar{\mathfrak{l}} \) is reductive in \( \tilde{\mathfrak{g}} \), which is by definition reductive in \( \mathfrak{g} \).

We set \( \mathfrak{s} := \{ [\mathfrak{l},\mathfrak{l}] \} \). We say that a weight \( \lambda \in \mathfrak{h}^* \) is integral with respect to \( \mathfrak{s} \) and dominant with respect to \( \mathfrak{s} \cap \mathfrak{b} \) if \( (\lambda,\beta)_{\mathfrak{g}} \) is a positive integer for all
positive roots $\beta$ of $\mathfrak{t}$. Given a weight $\lambda \in \mathfrak{h}^*$ that is integral with respect to $\mathfrak{s}$ and dominant with respect to $\mathfrak{s} \cap \mathfrak{b}$, we denote by $V_\lambda(\mathfrak{l})$ the irreducible finite dimensional $\mathfrak{l}$-module of highest weight $\lambda$. $V_\lambda(\mathfrak{l})$ can be regarded as a $\mathfrak{p}$-module with trivial action of $\mathfrak{n}$. We note that as a partial case of our notation, for $\mathfrak{p} \simeq \mathfrak{g}$, $V_\lambda(\mathfrak{g})$ denotes the irreducible finite dimensional $\mathfrak{g}$-module of highest weight $\lambda$. In a similar fashion we define $\bar{\mathfrak{s}} := [\mathfrak{i}, \mathfrak{i}]$, the irreducible $\mathfrak{p}$-module $V_\mu(\bar{\mathfrak{g}})$ and the irreducible $\bar{\mathfrak{g}}$-module $V_\mu(\bar{\mathfrak{g}})$.

In order to compute in $V_\lambda(\mathfrak{g})$, for an arbitrary reductive Lie algebra $\mathfrak{g}$, we need to realize the following steps.

1. Produce a set of words $u_1, \ldots, u_k \in U(\mathfrak{g})$ such that $u_1 \cdot v_\lambda, \ldots, u_k \cdot v_\lambda$ give a basis of $V_\lambda(\mathfrak{g})$. Let $m_i := u_i \cdot v_\lambda$.

2. For each simple generator $g_\alpha \in \mathfrak{g}$, compute the matrix of the action of $g_\alpha$ on the $m_i$’s, i.e., compute the numbers $b_{is}$ for which $g_\alpha \cdot m_i = \sum_s b_{is} m_s$.

In order to deal with 1), we use a non-commutative monomial basis of $V_\lambda(\mathfrak{g})$ as described in [14], §1. More precisely, the elements $u_i$ are chosen to be non-commutative (non-PBW ordered) monomials in the simple Chevalley-Weyl generators, whose exponents are given by the set of adapted strings $S^\lambda_\mathfrak{w}$ described in the Definition after [14], Lemma 1.3. In order to solve 2), we use the Shapovalov form, [21], [10]. The algorithm we used for solving 1) and 2) is described in [10]. The explicit computations in Section 5 were carried out with the help of a C++ program written for the purpose, [13], and we omit all computation details.

For $\lambda \in \mathfrak{h}^*$ integral with respect to $\mathfrak{s}$ and dominant with respect to $\mathfrak{s} \cap \mathfrak{b}$, define

$$M_\lambda(\mathfrak{g}, \mathfrak{p}) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V_\lambda(\mathfrak{l})$$

to be the generalized Verma $\mathfrak{g}$-module induced from $V_\lambda(\mathfrak{l})$. Similarly define $M_\mu(\bar{\mathfrak{g}}, \bar{\mathfrak{p}}) := U(\bar{\mathfrak{g}}) \otimes_{U(\bar{\mathfrak{p}})} V_\mu(\bar{\mathfrak{l}})$ to be the generalized Verma $\bar{\mathfrak{g}}$-module induced from $V_\lambda(\bar{\mathfrak{l}})$. We have the vector space isomorphism

$$M_\lambda(\mathfrak{g}, \mathfrak{p}) \simeq U(\mathfrak{n}_-) \otimes V_\lambda(\mathfrak{l}) \quad (3)$$

Given a vector space $V$, by $S^*(V) := \bigoplus_{i=0}^\infty S^i(V)$ we denote the symmetric tensor algebra of $V$, where $S^i(V)$ is generated over $\mathbb{C}$ by monomials of the form $\sum_{\sigma \in S_i} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)}$.

We now fix notation for the Weyl character formula. We denote the Weyl group of a reductive Lie algebra $\mathfrak{t}$ by $W(\mathfrak{t})$. To each element $\beta \in \mathfrak{h}^*$ we assign the formal exponent $x^\beta$ with multiplication $x^\beta_1 \cdot x^\beta_2 = x^\beta_1 + \beta_2$ and denote by $\mathbb{Q}[x^\gamma]_{\gamma \in \mathfrak{h}^*}$ the commutative associative $\mathbb{Q}$-algebra generated by all such elements with no further relations. We define $\mathbb{Q}[x^\gamma]_{\gamma \in \mathfrak{h}^*}$ as the vector space generated by formal infinite $\mathbb{Q}$-linear combinations of the monomials $x^\alpha$. Similarly, to each element $\alpha \in \mathfrak{h}^*$ we assign the formal exponent $y^\alpha$ with multiplication
\[ y^{\alpha_1} y^{\alpha_2} = y^{\alpha_1 + \alpha_2} \]

and denote by \( \mathbb{Q}[y^\gamma]_{\gamma \in \mathfrak{h}^*} \) the commutative associative \( \mathbb{Q} \)-algebra generated by all such elements with no further relations.

We define the \( \mathbb{Q} \)-algebra homomorphism

\[
\Pr : \mathbb{Q}[x^\gamma]_{\gamma \in \mathfrak{h}^*} \to \mathbb{Q}[y^\gamma]_{\gamma \in \mathfrak{h}^*}, \quad x^\gamma \mapsto y^{\text{pr}(\gamma)}. \tag{4}
\]

Given a weight \( \mathfrak{g} \)-module \( M \) with finite-dimensional \( \mathfrak{h} \)-weight spaces \( M(\beta) \), the character of \( \text{ch} M \in \mathbb{Q}[x^\gamma]_{\gamma \in \mathfrak{h}^*} \) is defined via

\[
\text{ch} M := \sum_{\beta \in \text{Weights}_\mathfrak{h} M} x^\beta = \sum_{\beta \in \text{Supp}_\mathfrak{h} M} \dim M(\beta) x^\beta.
\]

Similarly, for a \( \mathfrak{g} \)-module \( N \) we define \( \overline{\text{ch}} \) to be the character with respect to \( \bar{\mathfrak{h}} \),

\[
\overline{\text{ch}} N := \sum_{\alpha \in \text{Weights}_\mathfrak{h} N} y^\alpha.
\]

**Definition 2.1** Let \( a := \sum_{\gamma \in \mathfrak{h}^*} a(\gamma) x^\gamma \in \mathbb{Q}[x^\gamma]_{\gamma \in \mathfrak{h}^*} \). We say that \( a \) is pr-finite if for any weight \( \delta \in \mathfrak{h}^* \) the set \( \{ \gamma \in \mathfrak{h}^* | \text{pr}(\gamma) = \delta \text{ and } a(\gamma) \neq 0 \} \) is finite for every \( \delta \).

The set of pr-finite elements \( \text{PrFin} \) of \( \mathbb{Q}[x^\gamma]_{\gamma \in \mathfrak{h}^*} \) is a vector subspace of \( \mathbb{Q}[x^\gamma]_{\gamma \in \mathfrak{h}^*} \), containing \( \mathbb{Q}[x^\gamma]_{\gamma \in \mathfrak{h}^*} \). We can extend the map \( \Pr : \mathbb{Q}[x^\gamma]_{\gamma \in \mathfrak{h}^*} \to \mathbb{Q}[y^\gamma]_{\gamma \in \mathfrak{h}^*} \) to a map \( \Pr : \text{PrFin} \to \mathbb{Q}[y^\gamma]_{\gamma \in \mathfrak{h}^*} \) via

\[
\Pr(a) := \sum_{\delta \in \mathfrak{h}} \left( \sum_{\text{pr}(\gamma) = \delta} a(\gamma) \right) x^\delta. \tag{5}
\]

We note that if \( \text{ch}(M) \) is pr-finite, then we have the equality

\[
\Pr(\text{ch}(M)) = \overline{\text{ch}}(M). \tag{6}
\]

### 3 Character formulas and branching laws for generalized Verma modules

We recall that a finitely generated \( \mathfrak{g} \)-module is of finite length if it has finite Jordan-Hölder composition series over \( \mathfrak{g} \). We recall the following definition, [11].

**Definition 3.1** A \( \mathfrak{g} \)-module \( M \) is discretely decomposable if there is an increasing filtration \( M_n \) by \( \mathfrak{g} \)-submodules of finite length such that \( \bigcup_n M_n = M \). Let \( \mathfrak{p} \) be a parabolic subalgebra of \( \mathfrak{g} \). We say that \( M \) is discretely decomposable in Bernstein-Gelfand-Gelfand parabolic category \( \mathcal{O}^\mathfrak{p} \) if in addition all modules \( M_n \) from the filtration can be chosen to lie in \( \mathcal{O}^\mathfrak{p} \).
Definition 3.1 implies the following.

**Lemma 3.2** The generalized Verma module $M_\lambda(g,p)$ is discretely decomposable as a $\bar{g}$-module in the category $\mathcal{O}_\bar{p}$.

**Proof.** Let $v$ be the $b$-highest weight vector of $M_\lambda(g,p)$. Let $g \simeq U_1 \subset U_2 \subset \cdots \subset U_n \subset \cdots$ be the standard filtration of $U(g)$ by degree. Let $M_n$ be the $\bar{g}$-module generated by the vector space $U_n \cdot v$. It is a straightforward check that $M_n$ belongs to category $\mathcal{O}_\bar{p}$. The so constructed modules $M_n$ provide the filtration required by Definition 3.1. □

In [11, Section 3], the branching laws of $M_\lambda(g,p)$ over $\bar{g}$ are explored under the additional assumption that $p$ is compatible with $i(\bar{g})$, i.e., that $p$ can be defined using a hyperbolic grading element that lies in $i(\bar{h})$. If $p$ is a parabolic subalgebra compatible with $i(\bar{g})$ we have that $i(\bar{n}) \subset \bar{n}$. Without the assumption that $p$ is $i(\bar{g})$-compatible this no longer has to hold, see Lemma 5.2.3.(b),(c). Nevertheless, the following Lemma holds.

**Lemma 3.3**

1. The map $pr$ maps $Supp_\mathfrak{h} n$ surjectively onto $Supp_\mathfrak{h} \bar{n}$.

2. $n$ has an $\bar{l}$-submodule isomorphic to $\bar{n}$.

Both claims of the Lemma remain true if we interchange $n$ with $n_-$ and $\bar{n}$ with $\bar{n}_-$.

**Proof.** 1. Consider a $\mathfrak{h}$-weight $\gamma$ of $\bar{n}$. Then $i(\bar{g}_\gamma)$ is a $\mathbb{C}$-linear combination of elements of the weight spaces $g_\beta$ for which $pr(\beta) = \gamma$. Suppose all $pr$-preimages of $\gamma$ in $\Delta(g)$ are roots of $l$. This implies that $i(\bar{g}_{-\gamma})$ is a subspace of $l$, which implies that $\bar{g}_\gamma \subset l$. Contradiction. Therefore there exists a weight $\beta \in Weights_\mathfrak{h} n$ satisfying $pr(\beta) = \gamma$. The first part of the Lemma now follows from the fact that the $\mathfrak{h}$-weights of $\bar{n}$ have multiplicity one (as the root spaces of $\bar{g}$ have no multiplicities).

2. Let $\mu \in \mathfrak{h}^*$ be a $\mathfrak{b} \cap \mathfrak{l}$-maximal weight of $\bar{n}$ and let $w$ be a $\mathfrak{b} \cap \mathfrak{l}$-singular vector of $\bar{n}$ of weight $\mu$ under the adjoint action of $\mathfrak{l}$ on $\bar{n}$. Then there exist complex numbers $a_\beta$, for $\beta \in \Delta(g)$ and $pr(\beta) = \mu$, such that

$$i(w) = \sum_{\beta \in \Delta(g), pr(\beta) = \mu} a_\beta g_\beta \quad .$$

(7)

For a fixed $\mu$, let

$$I_1 := \{ \beta \in Supp_\mathfrak{b}(n) | pr(\beta) = \mu \} \quad ,$$

$$I_2 := \{ \beta \in Supp_\mathfrak{b}(l) | pr(\beta) = \mu \} \quad .$$

(8)
The disjoint union $I_1 \sqcup I_2$ equals $\{ \beta \in \Delta(g) | \text{pr}(\beta) = \mu \}$. Consider the element

$$v := \sum_{\beta \in I_1} a_\beta s_\beta .$$

We claim that $v$ is non-zero. Indeed, otherwise we would have that $i(w) \in I$ and consequently $i(w) \in i(I)$, which is impossible.

We claim next that $v$ is $b \cap L$-singular. Indeed, let $\tilde{m}$ be the nilradical of $\tilde{b}$. As $n$ is a $L$-module under the $\text{ad} \circ i$ action (as it is a $L$-module), we have that $[i(\tilde{m} \cap I), v] \subset n$. On the other hand, $[i(\tilde{m} \cap I), i(w)] = 0$ and therefore $[i(\tilde{m} \cap I), v] = [i(\tilde{m} \cap I), v - i(w) + i(w)] = [i(\tilde{m} \cap I), v - i(w)] \subset L$, where the latter follows as $v - i(w) \in I$. Thus $[i(\tilde{m} \cap I), v] \in n \cap I = \{0\}$. The latter implies that $v$ is $b \cap L$-singular as $i(\tilde{h}) \subset \tilde{h}$.

For each $b \cap L$-maximal weight of $\tilde{n}$, the preceding construction yields a $b \cap L$-singular non-zero vector of the same $h$-weight. As $\tilde{n}$ is multiplicity free as a $L$-module (as it is multiplicity free as a $h$-module), the construction yields one set of $b \cap L$-singular vectors under the $\text{ad} \circ i$ action generating an $L$-submodule of $n$ isomorphic to $\tilde{n}$. □

Remark. The complex numbers $a_\beta$ in (7) depend non-trivially on the embedding map $i$. It appears that this dependence is not yet fully understood. A discussion of the subject can be found in [11, §3.3].

In view of Lemma 3.3, the following definition of the numbers $m(\mu, \lambda)$ is a straightforward generalization of the numbers denoted with the same letter in [11, §3.4, (3.3)].

**Definition 3.4** Let $\lambda \in h^*$ be integral with respect to $s$ and dominant with respect to $s \cap b$. Let $N$ be any $L$-submodule of $n_-$ isomorphic to $\tilde{n}_-$ (under the $\text{ad} \circ i$-action). We define

$$m(\mu, \lambda) := \dim \text{Hom}_L(V_\mu(\tilde{I}), V_\lambda(\tilde{L}) \otimes S^*(n_-/N)) .$$

As $i(\tilde{I})$ is reductive in $\tilde{g}$, the $L$-module $N$ has a complement $L$-submodule $Q \simeq n_-/N$. As $S^n(n_-)$ is an $L$-module, so is $S^n(Q)$, and we can identify the $L$-module $S^n(n_-/N)$ with $S^n(Q)$.

### 3.1 The cones $C, C'$

For an ordered tuple of weights $X \subset h^*$, let $\text{Cone}_{\mathbb{Z}_{>0}}(X)$ denote the set of the possible finite $\mathbb{Z}_{>0}$-linear combinations of the elements of $X$.

The following Condition A will play a crucial role in our ability to write closed form character formulas for $M_\lambda(g, p)$.

**Definition 3.5** [Condition A] We recall that $i(\tilde{b}) \subset b \subset p$, $\bar{p} := i^{-1}(p)$ and that $n_-$ is the nilradical opposite to the nilradical of $p$. Let $N$ be as in Definition 3.4.
Define the cones $C \subset C'$ by

\[
C := \text{Cone}_{\mathbb{Z}_{\geq 0}}(\text{Weights}_h(n_-/N)),
\]
\[
C' := \text{Cone}_{\mathbb{Z}_{\geq 0}}(\text{Weights}_h(n_-)) .
\]

We say that $p$ satisfies Condition A if exactly one of the two holds.

1. $0 \in C$.

2. $0 \notin C'$.

The failure of Condition A is equivalent to $0 \in C'$ and $0 \notin C$. We immediately note that we don’t have an example for which Condition A fails, and we may conjecture that it always holds. The $I$-module $N$ is not uniquely determined, but its $\mathfrak{h}$-weights are (they equal the $\mathfrak{h}$-weights of $\hat{n}_-$). Consequently the weights of $n_-/N$ are obtained by removing the $\mathfrak{h}$-weights of $\hat{n}_-$ from the $\mathfrak{h}$-weights of $n_-$ accounting for multiplicities; thus the cone $C$ does not depend on the choice of $N$.

Example. As an example of $0 \notin C'$, we may pick $p \simeq g$ to be the full parabolic subalgebra and $\hat{g}$ to be any reductive subalgebra (then $0 \notin C' = \emptyset$). As an example of $0 \in C'$, we can choose subalgebra $i(\hat{g})$ such that a root $\beta$ of $g$ projects to the zero $\mathfrak{h}$-weight, and chose the parabolic subalgebra $p$ such that $\beta$ is not a root of its Levi part. An example for which $0 \in C'$ but none of the roots of $g$ projects to zero $\mathfrak{h}$-weight can be given as follows. Pick $g \simeq so(2n+2)$; its root system can be given in $\varepsilon$-notation as $\Delta(so(2n+1)) := \{ \pm \varepsilon_i \pm \varepsilon_j | 1 \leq i < j \leq n + 1 \}$. Declare the roots $\{ \varepsilon_i \pm \varepsilon_j | i < j \}$ to be positive, and define $b$ accordingly. Pick $i(\hat{g})$ to be the subalgebra generated by the root spaces $\{ g_{\pm \varepsilon_i \pm \varepsilon_j} | 2 \leq i < j \leq n + 1 \}$; this subalgebra is isomorphic to $so(2n)$. Pick $p \supset b$ to be any parabolic subalgebra for which the root $\varepsilon_1 - \varepsilon_2$ is “crossed out”, i.e., $\varepsilon_1 - \varepsilon_2$ is not a root of $p$. Then the root spaces $g_{-\varepsilon_1 - \varepsilon_2}$ and $g_{-\varepsilon_1 + \varepsilon_2}$ belong to $n_-$. Therefore the weight $0 = \text{pr}(-2\varepsilon_1) = \text{pr}(-\varepsilon_1 - \varepsilon_2 + (-\varepsilon_1 + \varepsilon_2))$ belongs to $C'$, however none of the roots of $so(2n+2)$ projects to zero. Another example of $0 \in C'$ can be given by $g \simeq sl(3)$ and $\hat{g} \simeq sl(2)$, with $i(sl(2))$ equal to the regular $sl(2)$-subalgebra of $sl(3)$ and $p \simeq b$. However, if we pick $i(sl(2))$ as the principal subalgebra of $sl(3)$, then our prerequisite requirement that $i(\mathfrak{h}) \subset \mathfrak{h} \subset b$ implies $0 \notin C'$ no matter which of the 4 parabolic subalgebras $p \supset b$ we choose.

Definition 3.6 We define $p$ to be weakly compatible with $i(\hat{g})$ if there exists an element $\hat{h} \in \hat{h}$ such that $\gamma(i(\hat{h})) \in \mathbb{Q}$ for all $\gamma \in \Delta(\mathfrak{g})$ and

1. $\alpha(\hat{h}) > 0$ for all $\alpha \in \text{Weights}_\mathfrak{h} \hat{n}$,

2. $\beta(i(\hat{h})) \geq 0$ for all $\beta \in \text{Weights}_\mathfrak{h} n$.

Lemma 3.7 Suppose $p$ is weakly compatible with $i(\hat{g})$ (Definition 3.6). Then Condition A holds, i.e.,

$0 \notin C \iff 0 \notin C'$.
Proof. \((\Leftarrow)\) is trivial; we are proving the other implication. Weights\(_h\) \(n = -\) Weights\(_h\) \(n\) and therefore there exists an element \(h \in \mathfrak{h}\) for which \(\alpha(h) < 0\) for all \(\alpha \in \text{Weights}_h\) \(n\) and for which pr\((\beta)(h)\) \(\leq 0\) for all \(\beta \in \text{Weights}_h\) \(n\).

Suppose

\[
0 = \sum_{\beta \in \text{Weights}_h\ n} n_\beta \text{pr}(\beta)
\]

for some non-negative numbers \(n_\beta > 0\). Evaluate both sides of the above expression on \(h\) to obtain that \(0 = \sum_{\beta \in \text{Weights}_h\ n} n_\beta \text{pr}(\beta)(h)\), where all summands are non-positive and therefore must be zero. Let \(\gamma\) be a weight with pr\(\gamma \in \text{Weights}_h\) \(n\). Then pr\(\gamma(h) < 0\) and therefore \(n_\gamma = 0\). Therefore 0 \(\in C'\) implies that 0 \(\in C\), which proves the Lemma. \(\square\)

The notion of a parabolic subalgebra weakly compatible with \(i(\mathfrak{g})\) is a generalization of the notion of a compatible parabolic subalgebra, as the following Proposition shows.

**Proposition 3.8** Let \(p\) be a parabolic subalgebra of \(\mathfrak{g}\) compatible with \(i(\mathfrak{g})\) in the sense of [11, Section 3]. Then \(p\) is weakly compatible with \(i(\mathfrak{g})\) and Condition A holds.

**Proof.** By the definition of a parabolic subalgebra compatible with \(i(\mathfrak{g})\) there exists an element \(h \in \mathfrak{h}\) for which \(\beta(i(h)) > 0\) for all \(\beta \in \text{Weights}_h\) \(n\). Let \(\alpha \in \text{Weights}_h\) \(n\). By Lemma 5.3 there exists \(\beta \in \text{Weights}_h\) \(n\) with pr\(\beta) = \alpha\). Therefore \(\alpha(h) = \text{pr}(\beta)(h) = \beta(i(h)) > 0\). \(\square\)

We recall that \(\mathbb{Q}[\gamma]_{\gamma \in \mathfrak{h}^*}\) is not a ring, as products are not necessarily well defined: for example, the product of \(\sum_{n=0}^{\infty} y^{n\alpha}\) and \(\sum_{n=0}^{\infty} y^{-n\alpha}\). However, if 0 \(\notin C'\), the vector subspace \(\mathbb{Q}[\gamma]_{\gamma \in \mathfrak{h}^*} C'\) given by

\[
\mathbb{Q}[\gamma]_{\gamma \in \mathfrak{h}^*} C' := \sum_{\mu \in \mathfrak{h}^*} \sum_{\gamma \in \mu + C'} a(\gamma) y^\gamma | a(\gamma) \in \mathbb{Q}\]

(9)

is a ring, where the outer \(\sum\) sign stands vector space sum, i.e., the vector space generated by finite linear combinations of the elements of the vector spaces under the \(\sum\) sign.

Therefore if 0 \(\notin C'\) we can write

\[
\overline{\text{ch}} S^*(n_-/N) = \prod_{\alpha \in \text{Weights}_h(n_-/N)} \frac{1}{(1 - y^\alpha)}
\]

\[
\overline{\text{ch}} S^*(n_-) = \prod_{\alpha \in \text{Weights}_h(n_-)} \frac{1}{(1 - y^\alpha)},
\]

(10)

where \(\frac{1}{(1 - y^\alpha)}\) denotes the multiplicative inverse of \((1 - y^\alpha)\) in \(\mathbb{Q}[\gamma]_{\gamma \in \mathfrak{h}^*} C'\), given by the geometric series sum formula, and the products are taken in the ring \(\mathbb{Q}[\gamma]_{\gamma \in \mathfrak{h}^*} C'\).
Let $\lambda \in \mathfrak{h}^*$ be dominant with respect to $\mathfrak{b} \cap \mathfrak{s}$ and integral with respect to $\mathfrak{s} = [\mathfrak{l}, \mathfrak{l}]$, where we recall $\mathfrak{l}$ is the reductive Levi part of $\mathfrak{g}$. As $\bar{\mathfrak{l}}$ is reductive in $\mathfrak{g}$, the $\bar{\mathfrak{l}}$-module $V_\lambda(\bar{\mathfrak{l}})$ decomposes as a direct sum of irreducible $\bar{\mathfrak{l}}$-modules.

Therefore for $\mu \in \bar{\mathfrak{h}}^*$ we can define the numbers $n(\mu, \lambda)$ as the multiplicity of the module $V_\mu(\bar{\mathfrak{l}})$ in $V_\lambda(\bar{\mathfrak{l}})$. The numbers $n(\mu, \lambda)$ are then computed via the finite sum

$$\bar{\text{ch}}V_\lambda(\bar{\mathfrak{l}}) = \sum_{\mu \in \bar{\mathfrak{h}}^*} n(\mu, \lambda) \bar{\text{ch}}V_\mu(\bar{\mathfrak{l}}).$$

(11)

For an arbitrary sequence of elements $P_1, \ldots, P_n, \ldots \in \mathbb{Q}[[y^\gamma]]_{\gamma \in \bar{\mathfrak{h}}^*}$, let the functions $p_n : \bar{\mathfrak{h}}^* \to \mathbb{Q}$ be defined via $P_n := \sum_{\gamma \in \bar{\mathfrak{h}}^*} p_n(\gamma) y^\gamma$. We say that the sequence $P_1, \ldots, P_n, \ldots \in \mathbb{Q}[[y^\gamma]]_{\gamma \in \bar{\mathfrak{h}}^*}$ has a limit if for all $\gamma \in \bar{\mathfrak{h}}^*$ the limit $\lim_{n \to \infty} P_n(\gamma)$ exists. If so, we define $\lim_{n \to \infty} P_n$ by

$$\lim_{n \to \infty} P_n := \sum_{\gamma \in \bar{\mathfrak{h}}^*} \left( \lim_{n \to \infty} p_n(\gamma) \right) y^\gamma.$$

The following Lemma follows from the definition of $\bar{\text{ch}}$ and the geometric series sum formula.

**Lemma 3.9**

1. $\sum_{\mu \in \bar{\mathfrak{h}}^*} m(\mu, \lambda) \bar{\text{ch}}V_\lambda(\bar{\mathfrak{l}}) = \lim_{n \to \infty} \left( \bar{\text{ch}}V_\lambda(\bar{\mathfrak{l}}) \prod_{i=0}^{n} \text{ch}(S^i(n_-/N)) \right)$

$$= \lim_{n \to \infty} \left( \left. \bar{\text{ch}}V_\lambda(\bar{\mathfrak{l}}) \prod_{\alpha \in \text{Weights}_{\bar{\mathfrak{h}}}(n_-/N)} \frac{1}{1 - y^{\alpha}} \right|_{\alpha = 0} := n. \right),$$

(12)

where the division and multiplication operations are carried out in the ring $\mathbb{Q}[y^\gamma]_{\gamma \in \bar{\mathfrak{h}}^*}$, we allow $m(\mu, \lambda) = +\infty$, and for $\alpha = 0$ we have defined $\left. \frac{1}{1 - y^{\alpha}} \right|_{\alpha = 0} := n$.

2. Suppose $0 \notin C$. Then

$$\sum_{\mu \in \bar{\mathfrak{h}}^*} m(\mu, \lambda) \bar{\text{ch}}V_\lambda(\bar{\mathfrak{l}}) = \bar{\text{ch}}V_\lambda(\bar{\mathfrak{l}}) \prod_{\alpha \in \text{Weights}_{\bar{\mathfrak{h}}}(n_-/N)} \frac{1}{1 - y^{\alpha}} \right),$$

(13)

where the inverse and multiplication operations are taken in the ring $\mathbb{Q}[[y^\gamma]]_C$, defined by (9).

### 3.2 Character formulas and multiplicity functions $m(\mu, \lambda)$

In the following Theorem we explore what happens when we apply the map $\text{Pr}$ to the character $\text{ch}M_\lambda(\mathfrak{g}, \mathfrak{p})$. The observation of the Theorem is that we can separate the $\text{Pr}$-images of the denominators $(1 - x^\beta)$ in the generating function
of the character formula according to the (possibly zero) multiplicity with which \( \text{pr}(\beta) \) belongs to \( \text{Weights}_\beta \bar{n}_- \) and \( \text{Weights}_\beta \bar{n}_- / N \).

**Theorem 3.10** Let \( \lambda \in \mathfrak{h}^* \) be a weight dominant with respect to \( \mathfrak{h} \cap \mathfrak{s} \) and integral with respect to \( \mathfrak{s} \).

1. Suppose \( 0 \notin C' \). Then we have that \( \text{ch} M_\lambda(g, p) \) is pr-finite and

\[
\text{ch} M_\lambda(g, p) = \text{Pr}(\text{ch} M_\lambda(g, p)) = \sum_{\mu \in \mathfrak{h}^*} m(\mu, \lambda) \text{ch} M_\mu(\bar{g}, \bar{p}).
\]

In particular \( m(\mu, \lambda) \in \mathbb{Z}_{\geq 0} \).

In the language of [11], in the Grothendieck group of \( \mathcal{O}_\bar{p} \) of the Bernstein-Gelfand-Gelfand category \( K(\mathcal{O}_\bar{p}) \) there is equality

\[
\text{ch} M_\lambda(g, p) = \bigoplus_{\mu \in \mathfrak{h}^*} m(\mu, \lambda) \text{ch} M_\mu(\bar{g}, \bar{p}).
\]

2. If \( 0 \in C \), we have that if \( m(\mu, \lambda) \neq 0 \) then \( m(\mu, \lambda) = +\infty \).

**Proof.** 1. By (3) we have that

\[
\text{ch} M_\lambda(g, p) = \left( \prod_{\beta \in \text{Weights}_\beta(n_-)} \frac{1}{1 - x^\beta} \right) \text{ch} V_\lambda(l),
\]

and similarly

\[
\text{ch} M_\mu(\bar{g}, \bar{p}) = \left( \prod_{\alpha \in \text{Weights}_\alpha(\bar{n}_-)} \frac{1}{1 - y^\alpha} \right) \text{ch} V_\lambda(l).
\]

By (15) and (10) \( \text{ch} M_\lambda(g, p) \) is pr-finite. Therefore we can compute

\[
\text{Pr} \left( \prod_{\beta \in \text{Weights}_\beta(n_-)} \frac{1}{1 - x^\beta} \right) = \prod_{\beta \in \text{Weights}_\beta(n_-)} \frac{1}{1 - y^{\text{pr}(\beta)}} = \prod_{\alpha \in \text{Weights}_\alpha(n_- / N)} \frac{1}{1 - y^\alpha},
\]

where the multiplication is taken in the ring \( \mathbb{Q}[[y^\gamma]]_{C'} \). The computation

\[
\text{Pr} \text{ch} M_\lambda(g, p) = \text{\text{ch} V}_\lambda(l) \prod_{\alpha \in \text{Weights}_\alpha(n_- / N)} \frac{1}{1 - y^\alpha} \prod_{\alpha \in \text{Weights}_\alpha(\bar{n}_-)} \frac{1}{1 - y^\alpha}
\]

\[
= \sum_{\mu} m(\mu, \lambda) \text{ch} V_\mu(l) \prod_{\alpha \in \text{Weights}_\alpha(\bar{n}_-)} \frac{1}{1 - y^\alpha}
\]

\[
= \sum_{\mu} m(\mu, \lambda) \text{ch} M_\mu(\bar{g}, \bar{p})
\]
completes the proof of 1).

2. By (12), for any positive integer \( n \), we have that \( m(\mu, \lambda) \) is greater than or equal to the coefficient of \( y^\mu \) in the expression

\[
\overline{ch}V_\lambda(l) \prod_{\alpha \in \text{Weights}_g(n_-/N)} (1 + y^\alpha + \cdots + y^{n\alpha}).
\]

Consider the expression

\[
A(n) := \prod_{\alpha \in \text{Weights}_g(n_-/N)} (1 + y^\alpha + \cdots + y^{n\alpha}).
\]

The condition \( 0 \in C \) implies there exist non-negative integers \( t_\alpha \), not simultaneously vanishing, with \( 0 = \sum_{\alpha \in \text{Weights}_g(n_-/N)} t_\alpha \alpha \). If \( \mu = \sum_{\alpha \in \text{Weights}_g(n_-/N)} m_\alpha \alpha \) for some integers \( m_\alpha \geq 0 \), the coefficient in front of \( y^\mu \) in \( A(n) \) is greater than or equal to \((n - \max_\alpha m_\alpha)/(\max_\alpha t_\alpha)\). Therefore, as \( n \to \infty \), the coefficient in front of \( y^\mu \) in \( A(n) \) tends to \( \infty \). As the character \( \overline{ch}V_\lambda(l) \) is polynomial (in the variables \( y^{\psi_j} \) for \( \psi_j \) - the fundamental \( \mathfrak{h} \)-weights of \( \mathfrak{g} \)) with positive coefficients, the statement follows. \( \square \)

For an ordered tuple \( X \subset \mathfrak{h}^* \), denote by \( P_X : \mathfrak{h}^* \to \mathbb{Z}_{\geq 0} \cup \{\infty\} \) the vector (Kostant) partition function with respect to \( X \). More precisely, \( P_X(\alpha) \) is defined to be the number of ways to write \( \alpha \) as a non-negative integral combination of the elements of \( X \). For overview of the theory of vector partition functions we direct the reader to [1] and the references therein.

We conclude this section by expressing the multiplicities \( m(\mu, \lambda) \) via vector partition functions. Vector partition functions can be written in closed form as piecewise quasi-polynomials, for example using the Szenes-Vergne formula, [17], and therefore so can the function \( m(\mu, \lambda) \). Here, closed form formula means a formula which can be evaluated with a constant number of arithmetic operations in the coordinates of \( \mu \) and \( \lambda \). Although we do not use it in the remainder of the paper, we find that Theorem 3.12 is important on its own. As its proof is constructive, it will allow the use of computers to compute the piecewise quasi-polynomial functions \( m(\mu, \lambda) \). In addition, Theorem 3.12 gives an upper bound on the growth of \( m(\mu, \lambda) \) as a function of \( \mu \) and \( \lambda \), as the example at the end of the section shows.

We give first a definition of piecewise quasi-polynomials.

**Definition 3.11** We say that a function \( p : \mathbb{C}^n \to \mathbb{Q} \) is an elementary piecewise quasi-polynomial if there exist

1. A finite set of non-strict linear inequalities with rational coefficients cutting off a (not necessarily bounded) set \( C \subset \mathbb{R}^n \subset \mathbb{C}^n \),

2. A rational lattice \( \Lambda \) (i.e., a \( \mathbb{Z} \)-span of a set of vectors with rational coordinates),
3. A rational polynomial \( p_{C, \Lambda}(y_1, \ldots, y_n) \), such that
\[
p(y_1, \ldots, y_n) = \begin{cases} 
p_{C, \Lambda}(y_1, \ldots, y_n) & \text{if } (y_1, \ldots, y_n) \in \Lambda \cap C \\
0 & \text{otherwise}
\end{cases}
\]
We say that the degree of \( p_{C, \Lambda} \) is the degree of the elementary piecewise quasi-polynomial \( p \).

We say that a function \( p \) is piecewise quasi-polynomial if it can be written as a finite sum of elementary piecewise quasi-polynomials. We say that \( p \) is of degree \( d \) if \( p \) can be written as a linear combination of elementary piecewise quasi-polynomials, each of degree less than or equal to \( d \), and \( d \) is the smallest integer with this property.

**Theorem 3.12** Assume the notation of Theorem 3.10 and \( 0 \notin C \). Then the multiplicity function \( m(\mu, \lambda) \) is piecewise quasi-polynomial (in the coordinates of \( \mu \) and \( \lambda \), i.e., a quasi-polynomial of \( \dim \mathfrak{h} + \dim \mathfrak{h} \) variables). The piecewise quasi-polynomial is of degree not exceeding the integer \( \frac{1}{2} \left( \dim \mathfrak{g} - \dim \mathfrak{g} - \dim \mathfrak{h} - \dim \mathfrak{h} \right) \).

**Proof.** Let \( \lambda \in \mathfrak{h}^* \) be integral with respect to \( \mathfrak{s} = [l, l] \) and dominant with respect to \( \mathfrak{b} \cap \mathfrak{s} \), where we recall that \( l \) is the reductive Levi part of \( \mathfrak{p} \). The Weyl character formula asserts that
\[
\text{ch} V_{\lambda}(l) = \left( \prod_{-\alpha \in \text{Weights}_{\mathfrak{h}}(l \cap \mathfrak{h})} \frac{1}{1 - x^\alpha} \right) \left( \sum_{w \in W(l)} \text{sign}(w) x^{w(\lambda + \rho_l) - \rho_l} \right),
\]
where \( \rho_l := \frac{1}{2} \sum_{\beta \in \text{Weights}_{\mathfrak{s}}(l \cap \mathfrak{s})} \beta \). By (3) and the Weyl character formula we have that
\[
\text{ch} M_{\lambda}(\mathfrak{g}, \mathfrak{p}) = \left( \prod_{\alpha \in \text{Weights}_{\mathfrak{s}}(\mathfrak{m} \cap \mathfrak{b})} \frac{1}{1 - x^\alpha} \right) \text{ch} V_{\lambda}(l)
\]
\[
= \left( \prod_{-\alpha \in \text{Weights}_{\mathfrak{s}}(\mathfrak{b})} \frac{1}{1 - x^\alpha} \right) \sum_{w \in W(l)} \text{sign}(w) x^{w(\lambda + \rho_l) - \rho_l} \quad (18)
\]
Similarly, we get that
\[
\overline{\text{ch}} M_{\mu}(\mathfrak{g}, \mathfrak{p}) = \left( \prod_{-\alpha \in \text{Weights}_{\mathfrak{s}}(\mathfrak{b})} \frac{1}{1 - y^\alpha} \right) \sum_{w \in W(l)} \text{sign}(w) y^{w(\mu + \rho_l) - \rho_l} \quad (19)
\]
where \( \rho_l := \frac{1}{2} \sum_{\alpha \in \Delta^+(l)} \alpha \). Let \( \mathfrak{m} \) and \( \mathfrak{m} \) be the nilradicals opposite to the nilradicals of \( \mathfrak{b} \) and \( \mathfrak{b} \) and let \( M \) be the \( \mathfrak{h} \)-module \( M := \mathfrak{m} / i(\mathfrak{m}) \) under the
ad ◦ i-action. We substitute (19) in (14) from Theorem 3.10, and cancel the common denominator \( \prod_{\alpha \in \text{Weights}_\mathfrak{h}} \frac{1}{1-y^\alpha} \) from both sides to obtain

\[
\left( \prod_{\alpha \in \text{Weights}_\mathfrak{h}} \frac{1}{1-y^\alpha} \right) \sum_{w \in \mathcal{W}(l)} \text{sign}(w)y^{\text{pr}(w(\lambda + \rho_l)-\rho_l)} = \sum_{\mu} \sum_{w \in \mathcal{W}(l)} \text{sign}(w)y^{\mu(\rho_l)-\rho_l}, \tag{20}
\]

where \( \mu \) runs over the weights integral with respect to \( \tilde{\mathfrak{b}} = [\tilde{\mathfrak{l}}, \tilde{\mathfrak{f}}] \) and dominant with respect to \( \tilde{\mathfrak{b}} \cap \tilde{\mathfrak{s}} \).

Set \( X := \text{Weights}_\mathfrak{h} M \) (with multiplicities) and let \( P_X \) be the vector partition function with respect to \( X \).

Fix a weight \( \mu \) dominant with respect to \( \tilde{\mathfrak{b}} \cap \tilde{\mathfrak{s}} \). In the right hand side of (20), among all monomials of the form \( m(\mu, \lambda) \text{sign}(w)y^{\mu(\rho_l)-\rho_l} \), the only monomial whose exponent is dominant with respect to \( \tilde{\mathfrak{b}} \cap \tilde{\mathfrak{s}} \) is \( m(\mu, \lambda) \), obtained for \( w = \text{id} \). Therefore we can compare the coefficients in front of all \( y^\gamma \) for which \( \gamma \) is \( \tilde{\mathfrak{b}} \cap \tilde{\mathfrak{l}} \)-dominant to get that

\[
m(\mu, \lambda) = \sum_{\mu \text{ is } \tilde{\mathfrak{b}} \cap \tilde{\mathfrak{l}} \text{-dominant}} \text{sign}(w)P_X (\mu - \text{pr}(w(\lambda + \rho_l)-\rho_l))
\]

\[
= \sum_{\mu \text{ is } \tilde{\mathfrak{b}} \cap \tilde{\mathfrak{l}} \text{-dominant}} \text{sign}(w)P_X (u_w(\mu, \lambda) + \tau_w),
\]

where we have set

\[
\tau_w := - \text{pr}(w(\rho_l)-\rho_l), \quad u_w := - \text{pr} \circ w, \quad u_w(\mu, \lambda) := \mu + u_w(\lambda).
\]

By standard results on vector partition functions (see [?] and the references therein), \( P_X \) is a piecewise quasi-polynomial of degree less than or equal to the number of elements of \( X \) minus the dimension of the ambient vector space, i.e.,

\[
\begin{align*}
\dim M - \dim \tilde{\mathfrak{g}} &= \dim \tilde{\mathfrak{m}}_+ - \dim \tilde{\mathfrak{m}}_- - \dim \tilde{\mathfrak{h}} = \frac{1}{2}(\dim \mathfrak{g} - \dim \tilde{\mathfrak{h}}) - \frac{1}{2}(\dim \mathfrak{g} - \dim \tilde{\mathfrak{h}}) - \frac{1}{2}(\dim \mathfrak{g} - \dim \tilde{\mathfrak{h}}) = \frac{1}{2}(\dim \mathfrak{g} - \dim \tilde{\mathfrak{g}} - \dim \mathfrak{h} - \dim \tilde{\mathfrak{h}}).
\end{align*}
\]

\[\square\]

**Example.** For Lie \( G_2 \hookrightarrow so(7) \), we have that \( \frac{1}{2}(\dim \mathfrak{g} - \dim \tilde{\mathfrak{g}} - \dim \mathfrak{h} - \dim \tilde{\mathfrak{h}}) = 1 \) and the piecewise quasi-polynomial \( m(\mu, \lambda) \) is linear. For \( \mathbb{C} \oplus so(2n + 1) \hookrightarrow so(2n + 2) \), \( so(2n) \hookrightarrow so(2n + 1) \) and \( gl(n) \hookrightarrow sl(n + 1) \) we have that \( \frac{1}{2}(\dim \mathfrak{g} - \dim \tilde{\mathfrak{g}} - \dim \mathfrak{h} - \dim \tilde{\mathfrak{h}}) = 0 \), and the piecewise quasi-polynomial \( m(\mu, \lambda) \) is bounded by a constant; in those cases, for finite dimensional representations, we know from the classical branching rules that the constant is 1 ("multiplicity free branching").
4 Constructing $\mathfrak{b}$-singular vectors in $M_{\lambda}(\mathfrak{g}, \mathfrak{p})$

We recall that $V_{\lambda}(\mathfrak{l})$ denotes the finite dimensional $\mathfrak{p}$-module inducing $M_{\lambda}(\mathfrak{g}, \mathfrak{p})$. By Weyl’s semisimplicity theorem $V_{\lambda}(\mathfrak{l})$ decomposes as a direct sum of irreducible $\mathfrak{l}$-modules (we recall that $\mathfrak{l} = i^{-1}(i(\mathfrak{g}) \cap \mathfrak{l})$). By Theorem 3.10, the component $V_{\mu}(\mathfrak{l})$ of highest weight $\mu$ in the $\mathfrak{l}$-decomposition of $V_{\lambda}(\mathfrak{l})$ appears with multiplicity $n(\mu, \lambda)$. By Theorem 3.10, the multiplicity $m(\mu, \lambda)$ of $M_{\mu}(\mathfrak{g}, \mathfrak{p})$ in $M_{\lambda}(\mathfrak{g}, \mathfrak{p})$ is equal to the $\mathfrak{l}$-multiplicity of $V_{\mu}(\mathfrak{l})$ in $S^{*}(\mathfrak{n}_{-}/\mathfrak{N}) \otimes V_{\lambda}(\mathfrak{l})$. Therefore $m(\mu, \lambda) \geq n(\mu, \lambda)$.

With some assumptions, in the present Section we assign to each $\mathfrak{b} \cap \mathfrak{l}$-singular vector $v$ in $V_{\lambda}(\mathfrak{l})$ a $\mathfrak{b}$-singular vector $v'$ in $M_{\lambda}(\mathfrak{g}, \mathfrak{p})$. More precisely, assuming that $\lambda$ satisfies Condition B (Definition 4.1 below), given a weight $\mu \in \mathfrak{h}^{*}$ we show a method for constructing $n(\mu, \lambda)$ linearly independent $\mathfrak{b}$-singular vectors in $M_{\lambda}(\mathfrak{g}, \mathfrak{p})$. In the particular case that $\mathfrak{p}$ has a finite branching problem over $i(\mathfrak{g})$ (Definition 4.1 below), it follows that $n(\mu, \lambda) = m(\mu, \lambda)$ and the construction exhausts all $\mathfrak{b}$-singular vectors of weight $\mu$ for which $m(\mu, \lambda) \neq 0$.

**Definition 4.1** We say that the parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ has a finite branching problem over $i(\mathfrak{g})$ if for $\lambda \in \mathfrak{h}^{*}$ there are only finitely many $\mu \in \mathfrak{h}^{*}$ with $m(\mu, \lambda) \neq 0$.

The above definition is equivalent to requesting that $M_{\lambda}(\mathfrak{g}, \mathfrak{p})$ has finite Jordan-Hölder series as a $\mathfrak{g}$-module. In view of Theorem 3.11 it is also equivalent to requesting that $\dim \mathfrak{n} = \dim \mathfrak{\check{n}}$. Theorem 3.11 furthermore implies that the definition does not depend on the choice of inducing representation. A non-trivial example of Definition 4.1 is given by Lie $G_{2} \rightarrow i(\mathfrak{g})$ for $\mathfrak{p} \cong \mathfrak{p}_{(1,0,0)}$.

Our construction is carried out in two steps. First, we decompose the inducing finite dimensional representation $V_{\lambda}(\mathfrak{l})$ over $\mathfrak{l}$ by producing a spanning set for the $\mathfrak{b} \cap \mathfrak{l}$ singular vectors in $V_{\lambda}(\mathfrak{l})$. Second, we project the obtained $\mathfrak{b} \cap \mathfrak{l}$ singular vectors to $\mathfrak{b}$-singular, using an element in the center of $U(\mathfrak{g})$.

We need the following definition.

**Definition 4.2** Let $P \in \mathbb{Q}[[[y^{\gamma}]]_{\gamma \in \mathfrak{h}^{*}}$. We say that $\mu \in \mathfrak{h}^{*}$ is $\mathfrak{b}$-maximal in $P$ if $y^{\mu}$ has non-zero coefficient in $P$ and we have that $y^{\mu + \alpha}$ has coefficient zero in $P$ for all positive roots $\alpha$ of $\mathfrak{g}$.

We describe a procedure for computing the decomposition $\mathfrak{b}$-$\mathfrak{l}$. Consider the $\mathfrak{p}$-module $V_{\lambda}(\mathfrak{l})$ inducing $M_{\lambda}(\mathfrak{g}, \mathfrak{p})$. We compute $\mathfrak{ch}V_{\lambda}(\mathfrak{l})$ using the Freudenthal formula with respect to $\mathfrak{l}$ (see e.g., [14] §22.3, 22.4). Then we compute the projection $\Pr(\mathfrak{ch}V_{\lambda}(\mathfrak{l})) = \mathfrak{ch}V_{\lambda}(\mathfrak{l})$. Next we find a $\mathfrak{b}$-maximal weight $\mu_{1}$ in $\Pr(\mathfrak{ch}V_{\lambda}(\mathfrak{l}))$. Then $V_{\mu_{1}}(\mathfrak{l})$ appears as a summand in the $\mathfrak{l}$-decomposition of $V_{\lambda}(\mathfrak{l})$. Next we compute $\mathfrak{ch}V_{\mu_{1}}(\mathfrak{l})$ using the Freudenthal formula with respect to $\mathfrak{l}$ and subtract $\mathfrak{ch}V_{\mu_{1}}(\mathfrak{l})$ from $\mathfrak{ch}V_{\lambda}(\mathfrak{l})$. If we obtain zero, we are done, otherwise we find another $\mathfrak{b}$-maximal weight $\mu_{2}$ in the remaining character $\mathfrak{ch}V_{\lambda}(\mathfrak{l}) - \mathfrak{ch}V_{\mu_{1}}(\mathfrak{l})$
Proposition 4.3 Suppose the parabolic subalgebra $p \subset g$ has finite branching problem over $i(\mathfrak{g})$. Let $\lambda \in \mathfrak{h}^*$ be dominant with respect to $\mathfrak{b} \cap \mathfrak{s}$ and integral with respect to $\mathfrak{s}$. Then we have the $\mathfrak{g}$-module isomorphism

$$Q_\lambda \simeq M_\lambda(\mathfrak{g}, p) \ .$$

Proof. By Lemma 3.3 the map $pr$ maps $\text{Weights}_\mathfrak{h} \, n_-$ bijectively onto $\text{Weights}_\mathfrak{n} \, \tilde{n}_-$. Let $\beta_1, \ldots, \beta_l$ be elements of the ordered tuple $\text{Weights}_\mathfrak{h} \, n_-$, and let $\alpha_1, \ldots, \alpha_l$

and subtract $\mathfrak{ch} V_{\mu_2}(l)$, and so on. On each step of the algorithm, the remaining character is a finite sum of monomials with non-negative integer coefficients. Furthermore, on each step the sum of the coefficients of all monomials decreases strictly, and therefore the algorithm terminates after a finite number of steps. By recording the intermediate summands $\mathfrak{ch} V_{\mu}(l)$ appearing in the algorithm, we obtain the decomposition $\mathfrak{ch} V(l)$.

We are now in a position to compute the $\mathfrak{b} \cap \mathfrak{l}$-singular vectors of $V_\lambda(l)$ using linear algebra and the remarks on finite dimensional representations in Section 2. Indeed, for each weight $\mu \in \mathfrak{h}^*$ with $n(\mu, \lambda) > 0$, we compute the intersection of the eigenspaces of the linear operators given by the action of $i(\tilde{\mathfrak{g}})$, $i(\tilde{\mathfrak{h}}) - \mu(\tilde{\mathfrak{h}})$ on $V_\lambda(l)$, where $\alpha$ runs over the simple positive roots of $\mathfrak{l}$.

We discuss shortly the center of $U(\mathfrak{g})$. For an arbitrary semisimple Lie algebra $\mathfrak{g}$, the center of $U(\mathfrak{g})$ is a polynomial algebra (of rank equal to $\dim \mathfrak{h}$), as follows from the Harish-Chandra homomorphism Theorem ([7] §23.3) and a Theorem of Chevalley ([8, Chapter 3]). Let $\tilde{c}_1, \ldots, \tilde{c}_k$ with $k = \dim \mathfrak{h}$ be a basis of the center of $U(\mathfrak{g})$ consisting of homogeneous elements, where $\tilde{c}_1$ is chosen to be the quadratic Casimir element. The elements $\tilde{c}_j$ are known, a list of their degrees for each type can be found in [20, page 260]. A discussion of how to construct a basis $\tilde{c}_1, \ldots, \tilde{c}_k$ can be found in [2] and the references therein.

For an arbitrary $\mu \in \mathfrak{h}^*$, the elements of the center of $U(\mathfrak{g})$ act by scalars on the Verma module $M_\mu(\mathfrak{g}, \mathfrak{b})$. Let the constant by which $\tilde{c}_j$ acts on $M_\mu(\mathfrak{g}, \mathfrak{b})$ be $p_j(\mu)$, i.e.,

$$p_j(\mu)v = \tilde{c}_j \cdot v \quad \text{for all } v \in M_\mu(\mathfrak{g}, \mathfrak{b}) \ .$$

We say that two weights $\mu$ and $\nu$ in $\mathfrak{h}^*$ are linked if there exists an element $w$ of the Weyl group of $\mathfrak{g}$ such that $w(\mu + \rho_\mathfrak{g}) = \nu$, where $\rho_\mathfrak{g}$ is the half-sum of the positive roots of $\mathfrak{g}$. By the Harish-Chandra homomorphism Theorem, $\mu$ and $\nu$ are linked if and only if $p_j(\mu) = p_j(\nu)$ for $j = 1, \ldots, \dim \mathfrak{h}$.

Let $Q_\lambda$ denote the “top level” $\mathfrak{g}$-submodule of $M_\lambda(\mathfrak{g}, \mathfrak{b})$ given by

$$Q_\lambda := i(U(\tilde{\mathfrak{g}})) \otimes_{i(U(\tilde{\mathfrak{h}}))} V_\lambda(l) \ ,$$

where $V_\lambda(l)$ is equipped with the $\mathfrak{p}$ action induced from the embedding $i$. We note that $Q_\lambda$ lies in the BGG category $\mathcal{O}^p$.

Before we proceed with the construction of $\mathfrak{b}$-singular vectors in $Q_\lambda$, we note the following.
be the elements of \( \text{Weights}_\mathfrak{n}_- \), such that \( \text{pr}(\beta_i) = \alpha_i \). Let \( v_1, \ldots, v_k \) be a basis of \( V_\lambda(l) \). By \( \{ \} \) the set
\[
A := \{ g_{\beta_1}^{n_1} \cdots g_{\beta_l}^{n_l} \cdot v_s | n_i \in \mathbb{Z}_{\geq 0}, s = 1, \ldots, k \}
\]
is a vector space basis of \( M_\lambda(\mathfrak{g}, \mathfrak{p}) \). Define a partial order \( \succ \) among the monomials \( g_{\beta_1}^{n_1} \cdots g_{\beta_l}^{n_l} \cdot v_s \) by requesting that \( g_{\beta_1}^{n_1} \cdots g_{\beta_l}^{n_l} \cdot v_s \succ g_{\beta_1}^{m_1} \cdots g_{\beta_l}^{m_l} \cdot v_t \) if and only if \( \sum_{i=1}^{l} n_i > \sum_{i=1}^{l} m_i \). The partial order does not depend on \( s \) and \( t \). Consider the element
\[
m(n_1, \ldots, n_l, s) := i(g_{\alpha_1}^{n_1} \cdots g_{\alpha_l}^{n_l}) \cdot v_s \quad .
\]

Lemma \[ \text{3.3} \] and the finiteness of the branching problem imply the vector space isomorphism \( i(\mathfrak{n}) \simeq \mathfrak{n} / \mathfrak{l} \). Therefore by the proof of Lemma \[ \text{3.3} \] for each index \( j \) there exists a non-zero complex number \( a_j \) and an element \( u_j \in \mathfrak{l} \) such that \( i(g_{\alpha_j}) \in a_j g_{\beta_j} + u_j \) and so
\[
m(n_1, \ldots, n_l, s) = (a_1 g_{\beta_1} + u_1)^{n_1} \cdots (a_l g_{\beta_l} + u_l)^{n_l} \cdot v_s \quad . \tag{22}
\]

Rewrite \( m(n_1, \ldots, n_l, s) \) in the monomial basis \( A \). As \( \mathfrak{n}_- \) is an ideal in \( \mathfrak{l} \oplus \mathfrak{n}_- \), \[ \text{22} \] and the PBW theorem imply that \( m(n_1, \ldots, n_l, s) \) has unique \( \succ \)-maximal monomial in the basis \( A \), proportional to \( g_{\beta_1}^{n_1} \cdots g_{\beta_l}^{n_l} \cdot v_s \). Therefore a straightforward filtration argument shows that, for fixed \( P \in \mathbb{Z}_{\geq 0} \), the linear span of
\[
\{ m(n_1, \ldots, n_l, s) \mid \sum_{i=1}^{l} n_i \leq P, s = 1, \ldots, k \}
\]
equals the linear span of
\[
\{ g_{\beta_1}^{n_1} \cdots g_{\beta_l}^{n_l} \cdot v_s | \sum_{i=1}^{l} n_i \leq P, s = 1, \ldots, k \}
\]
which proves the statement. \( \square \)

In order to associate \( \mathfrak{b} \cap \mathfrak{l} \)-singular vectors of \( V_\lambda(l) \) to \( \mathfrak{b} \)-singular vectors in \( M_\lambda(\mathfrak{g}, \mathfrak{p}) \), we need the following condition on \( \lambda \).

**Definition 4.4 (Condition B)** Let \( \lambda \in \mathfrak{h}^* \) be integral with respect to \( \mathfrak{s} = [\mathfrak{l}, \mathfrak{l}] \) and dominant with respect to \( \mathfrak{b} \cap \mathfrak{s} \). We say that \( \lambda \) satisfies Condition B if for each pair of \( \mathfrak{h} \)-weights \( \mu \neq \nu \) for which \( n(\mu, \lambda) > 0 \) and \( n(\nu, \lambda) > 0 \), we have that \( \nu \) and \( \mu \) are not linked.

We say that \( \lambda \) satisfies the strong Condition B if for all pairs \( \mu \neq \nu \) with \( n(\mu, \lambda) > 0 \) and \( n(\nu, \lambda) > 0 \), we have that \( p_1(\mu) \neq p_1(\nu) \), where \( p_1(\mu), p_1(\nu) \) are the constants given by the action of the quadratic Casimir element \( \tilde{e}_1 \) of \( \tilde{\mathfrak{g}} \) on \( M_\mu(\tilde{\mathfrak{g}}, \tilde{\mathfrak{b}}) \), \( M_\nu(\tilde{\mathfrak{g}}, \tilde{\mathfrak{b}}) \).
We note that, for the two extreme cases, the full parabolic subalgebra \( \tilde{p} \simeq \tilde{g} \), and \( \tilde{p} \simeq \tilde{b} \), every possible weight \( \lambda \) satisfies Condition B. Indeed, for the full parabolic \( \tilde{p} = \tilde{g} \), all \( \mu \) for which \( n(\mu, \lambda) > 0 \) are dominant with respect to \( \tilde{b} \) and integral with respect to \( \tilde{g} \) and therefore cannot be pairwise linked. For the minimal parabolic \( \tilde{p} \simeq \tilde{b} \), the inducing module is one-dimensional and the condition is trivially satisfied.

**Theorem 4.5** Let \( \lambda \in \mathfrak{h}^* \) be a integral with respect to \( s = [l, l] \) and dominant with respect to \( \mathfrak{b} \cap s \). Suppose \( \lambda \) satisfies Condition B (Definition 4.4). Let \( \mu \in \mathfrak{h}^* \) such that \( n(\mu, \lambda) > 0 \). Then there exists an element \( \tilde{d} \) in the center of \( U(\tilde{g}) \), such that \( i(\tilde{d}) \cdot (1 \otimes_U \tilde{p}) v_{\mu} \) is a non-zero \( \tilde{b} \)-singular vector in \( M_\lambda(\mathfrak{g}, \mathfrak{p}) \) for any \( \tilde{b} \cap l \)-singular vector \( v_{\mu} \) in \( V_\lambda(l) \) of \( \tilde{h} \)-weight \( \mu \).

**Proof.** Let \( \tilde{h} \) be a grading element that defines \( \tilde{p} \), i.e., an element such that \( \alpha(\tilde{h}) > 0 \) for \( \alpha \in \text{Weights}_{\tilde{g}} \mathfrak{n} \) and \( \alpha(\tilde{h}) = 0 \) for \( \alpha \in \text{Weights}_{\tilde{h}} \mathfrak{l} \). Let

\[
a := \max_{\alpha(\nu, \lambda) > 0} \nu(\tilde{h}) .
\]

For each \( s \in \mathbb{Z} \), let \( W_s \) be the sum of all \( \mathfrak{h} \)-weight spaces of \( V_\lambda(l) \) of weight \( \nu \) for which \( \nu(\tilde{h}) \geq a - s \). Then \( W_s \) is a \( \tilde{p} \)-module. Set

\[
Q_s := i(U(\tilde{g})) \otimes i(U(\tilde{p})) W_s , \quad R_s := Q_s/Q_{s-1} ,
\]

where \( Q_{-1} := \{0\} \), where \( Q_s \) are realized as \( \tilde{g} \)-submodules of \( M_\lambda(\mathfrak{g}, \mathfrak{p}) \).

Let

\[
\mu(\tilde{h}) = a - m .
\]

We will prove by induction on \( s \) that for each \( s < m \) there exists an element \( \tilde{d}_s \) in the center of \( U(\tilde{g}) \) such that \( i(\tilde{d}_s) \cdot Q_s = 0 \) and

\[
i(\tilde{d}_s) \cdot (1 \otimes_U \tilde{p}) v_{\mu} = x(1 \otimes_U \tilde{p}) v_{\mu} \mod Q_{m-1} \quad (23)
\]

for some non-zero complex number \( x \), where the latter computation is carried out in \( R_m \).

Our induction hypothesis holds trivially for the base case \( s = -1 \) with \( d_{-1} := 1 \). Suppose that there exists an element \( \tilde{d}_{s-1} \) that satisfies the induction hypothesis. Let \( v_{\nu} \in V_\lambda(l) \) be a \( \mathfrak{b} \cap l \)-singular vector of \( \mathfrak{h} \)-weight \( \nu \) for which \( \nu(\tilde{h}) = a - s \). As Condition B holds, there exists an element \( \tilde{d}_{\nu} \) such that \( \tilde{d}_{\nu} \cdot M_\nu(\tilde{g}, \tilde{b}) = \{0\} \) and \( \tilde{d}_{\nu} \cdot M_\mu(\tilde{g}, \tilde{b}) \neq \{0\} \).

Let \( \mathfrak{m} \) be the nilradical of \( \tilde{b} \) and \( \bar{\mathfrak{m}} \) its opposite nilradical. By the PBW theorem we have the vector space isomorphism \( U(\tilde{g}) \simeq U(\bar{\mathfrak{m}}) \otimes U(\tilde{h}) \otimes U(\mathfrak{m}) \) and we can write

\[
\tilde{d}_{\nu} = \sum_k \tilde{m}_k^- \tilde{h}_k \tilde{m}_k^+ \quad (24)
\]
for some monomials \( \bar{m}_{k}^{-} \in U(\bar{m}_{-}) \), \( \bar{h}_{k} \in U(\bar{h}) \), \( \bar{m}_{k}^{+} \in U(\bar{m}) \), where the summands are assumed linearly independent. Consider the action of a summand \( \bar{m}_{k}^{-} \bar{h}_{k} \bar{m}_{k}^{+} \) in \( \mathfrak{g} \) on \( 1 \otimes_{i(U(\mathfrak{p}))} v_{\nu} \). As \( i(\bar{m} \cap \mathfrak{u}) \cdot v_{\nu} = 0 \), it follows that whenever the monomial \( m_{k}^{+} \) is not constant it maps \( 1 \otimes_{i(U(\mathfrak{p}))} v_{\nu} \) to \( Q_{s-1} \). If \( m_{k}^{-} \) is a non-constant monomial, \( m_{k}^{+} \) is a non-constant monomial too (as elements of the center of \( U(\mathfrak{g}) \) have zero weight and the summands in \( 24 \) are linearly independent). Let \( u \) be the sum of the summands in \( 24 \) that lie in \( U(\bar{h}) \). As we have chosen \( \bar{d}_{\nu} \) to annihilate \( \nu_{\mathfrak{p}}(\bar{g}, \bar{b}) \), we have that \( u \cdot v_{\nu} = 0 \). The preceding discussion now implies that \( i(\bar{d}_{\nu}) \cdot (1 \otimes_{i(U(\mathfrak{p}))} v_{\nu}) \in Q_{s-1} \) and therefore the product \( \bar{d}_{s-1} \bar{d}_{\nu} \) annihilates \( 1 \otimes_{i(U(\mathfrak{p}))} v_{\nu} \). In addition, the above discussion shows

\[
i(\bar{d}_{\nu}) \cdot (1 \otimes_{i(U(\mathfrak{p}))} v_{\nu}) = (1 \otimes_{i(U(\mathfrak{p}))} u \cdot v_{\mu}) \mod Q_{m-1} \tag{25}
\]

and \( (1 \otimes_{i(U(\mathfrak{p}))} u \cdot v_{\mu}) \mod Q_{m-1} \) is a non-zero multiple of \( (1 \otimes_{i(U(\mathfrak{p}))} v_{\nu}) \mod Q_{m-1} \) as \( \bar{d}_{\nu} \cdot \nu_{\mathfrak{p}}(\bar{g}, \bar{b}) \neq \{0\} \). Set \( \bar{d}_{s} \) to be the product

\[
\bar{d}_{s} := \bar{d}_{s-1} \prod_{n(\nu, \lambda) > 0, \nu(h) = a - s} \bar{d}_{\nu} .
\]

Then \( \bar{d}_{s} \) is an element with the desired properties, which completes our induction argument.

We claim \( \bar{d} := \bar{d}_{m-1} \) satisfies the requirements of the theorem. Indeed, \( i(\bar{d}) \cdot (1 \otimes U(\mathfrak{p})) v_{\mu} \neq 0 \) by the construction of \( \bar{d} \) and if \( \bar{m} \in \bar{m} \), then \( i(\bar{m}) \cdot (i(\bar{d}) \cdot (1 \otimes U(\mathfrak{p})) v_{\mu}) \neq i(\bar{d}) \cdot i(\bar{m}) \cdot (1 \otimes U(\mathfrak{p})) v_{\mu} \) = 0. □

The proof of Theorem 4.3 and Proposition 4.3 imply the following.

**Corollary 4.6** Suppose the parabolic subalgebra \( \mathfrak{p} \subset \mathfrak{g} \) has finite branching problem over \( i(\bar{g}) \). Let \( \lambda \in \mathfrak{h}^{*} \) be dominant with respect to \( \mathfrak{b} \cap \mathfrak{s} \) and integral with respect to \( \mathfrak{s} \). Suppose \( \lambda \) satisfies Condition B. Suppose in addition \( n(\mu, \lambda) \leq 1 \) for all \( \mu \in \mathfrak{h}^{*} \). Then we have the \( \bar{g} \)-module isomorphism

\[
\nu_{\mathfrak{p}}_{\mathfrak{m}}(\bar{g}, \bar{p}) \simeq \bigoplus_{n(\mu, \lambda) = 1} \nu_{\mathfrak{p}}_{\mathfrak{m}}(\bar{g}, \bar{p}) .
\]

**Proof.** By Condition B and the fact that \( n(\mu, \lambda) = 1 \), all summands on the right hand side have pairwise zero intersections. By Proposition 4.3 and Theorem 4.3 each of the modules on the right hand side is contained as a subset in the left hand side. By Proposition 4.3 the right hand side contains a vector space basis of \( \nu_{\mathfrak{p}}_{\mathfrak{m}}(\bar{g}, \bar{p}) \). This proves the statement. □

We conclude this section with a discussion on Condition B. Let \( \mathbf{x} := (x_{1}, \ldots, x_{\dim \mathfrak{h}}) \) be indeterminate variables and let \( \mathbf{z} := (z_{1}, \ldots, z_{\dim \mathfrak{h}}) \), \( z_{i} \in \mathbb{Z}_{\geq 0} \) be non-negative integers. Let \( j \) be the set of simple roots of \( \bar{g} \). Let the fundamental weight that corresponds to the simple root \( \eta_{j} \) be denoted by \( \omega_{j} \).
Let $I_1$ be the subset the elements of $I$ that are roots of $l$ and define

$$
a(x) := \sum_{\alpha_i \in I_1} x_i \omega_i^i \\
b(z) := \sum_{\alpha_i \in I_2} z_i \omega_i^i \\
\lambda(x, z) := a(x) + b(z)
$$

(26)

For a fixed value of $z$, we claim that the set of the $\lambda$ satisfying Condition B is a Zariski open set in the variables $x_1, \ldots, x_{\dim \mathfrak{h}}$. Indeed, as a $\mathfrak{g}$-module, $V_{\lambda(\mathfrak{h}, \mathfrak{g})}(l)$ is isomorphic to $V_{\lambda(\mathfrak{h}, \mathfrak{g})}(s)$, and therefore the number of values of $a$ for which $n(pr(\lambda(x, z))) = \alpha, \lambda(x, z)) \neq 0$ is a function independent of $x_1, \ldots, x_{\dim \mathfrak{h}}$. For each $\mu \neq \nu \in \mathfrak{h}^*$ with $n(\mu, \lambda(x, z)) \neq 0$, the condition that $\mu$ and $\nu$ are not linked is a finite (as $\mathcal{W}(\mathfrak{g})$ is finite) set of linear $\neq$-inequalities in the coordinates of $\mu$ and $\nu$, which in turn are linear functions in the $x_i$’s. Therefore for a fixed value of $z$, Condition B is determined by a finite set of linear $\neq$-inequalities in the variables $x_1, \ldots, x_{\dim \mathfrak{h}}$ and is therefore Zariski open.

We note that it is possible the Zariski open set on the variables $x_1, \ldots, x_{\dim \mathfrak{h}}$ is the empty set: this could happen if one of the linear $\neq$-inequalities is of the form $a \neq 0$ for a non-zero constant $a$.

For a fixed value of $z$, we claim that the set of $\lambda$ satisfying the strong Condition B is also a Zariski open set in the variables $x_1, \ldots, x_{\dim \mathfrak{h}}$ (which may, again, be the empty set). Indeed, let $\alpha_i$ be the simple roots of $\mathfrak{g}$, and define $q_i(x, z)$ via $pr(\lambda(x, z)) =: \sum q_i(x, z) \alpha_i$. The elements $q_i(x, z)$ are linear functions of the $x_i$’s. By the preceding discussion, a weight $\mu \in \mathfrak{h}^*$ with $n(\mu, pr(\lambda(x, z))) > 0$ is of the form $pr(\lambda(x, z)) - \alpha_i$ where $\alpha_i \in \mathfrak{h}^*$ runs over some finite set of weights with coordinates that do not depend on the $x_i$. Therefore for $n(\mu, pr(\lambda(x, z))) > 0$, the function $p_1(\mu)$ is a quadratic polynomial in the variables $x_1, \ldots, x_{\dim \mathfrak{h}}$ with quadratic term depending only on $\lambda(x, z)$. Therefore the function $p_1(\mu) - p_1(\nu)$ is linear function in the variables $x_1, \ldots, x_{\dim \mathfrak{h}}$, which proves our claim.

5 The pair Lie $G_2 \hookrightarrow so(7)$

In this section we study the branching problem for the pair Lie $G_2 \hookrightarrow so(7)$.

Our original motivation for studying the example of Lie $G_2 \hookrightarrow so(7)$ comes from a natural conformal geometry problem in dimension 5 (notice that $so(7)$ is the complexification of the conformal Lie algebra in dimension 5). There is a remarkable connection between the geometry of generic 2-plane fields $D$ on a manifold $M$ of dimension 5 and pseudo-Riemannian metrics of signature $(3, 4)$ in dimension 7 whose holonomy is the split real form of $G_2$. The identification proceeds in two stages. The first step, due to Cartan, is the equivalence of such fields $D$ to a conformal class of metrics of signature $(2, 3)$ on $M$. The second
step canonically associates to a conformal structure of signature \((2,3)\) on \(M\) a Ricci-flat metric \(\tilde{g}\) of signature \((3,4)\) on an open subset of \((\mathbb{R}_+ \times M \times \mathbb{R})\) containing \((\mathbb{R}_+ \times M \times \{0\})\). Furthermore, the metrics constructed in this way satisfy \(\text{Hol}(\tilde{g}) \subset \text{Lie} G_2\), where \(\text{Hol}(\tilde{g})\) is the holonomy of the metric \(\tilde{g}\). We note that metrics with holonomy reduced to \(\text{Lie} G_2\) in dimension 7 are not easy to construct.

On the other hand, a geometrical characterization of the reduction of the structure group \(\text{SO}(7)\) down to \(G_2\) for a given inducing representation \(V_\lambda\) is given by invariant differential operators acting on sections of the associated vector bundles, intertwined by actions of \(\text{so}(7)\) and \(\text{Lie} G_2\).

Finally, by \([\text{1}]\), we can transform the problem of finding differential invariants for \((\text{so}(7), \text{Lie} G_2), V_\lambda\) into an algebraic question on homomorphisms between generalized Verma modules, which correspond to solutions of the branching problem. For the homomorphisms constructed in the present Section, we need to further construct lifts of homomorphisms to the category of semiholonomic generalized Verma modules. We hope to address this geometrically interesting topic in a future work.

The example \(\text{Lie} G_2 \hookrightarrow \text{so}(7)\) is of special algebraic interest as well. In the branching problem examples in which the subalgebra is of rank 1 (such as, for example, the principal \(\text{sl}(2)\)-subalgebra of \(\text{sl}(3)\)) the quasipolynomials that govern the branching via Theorem \([3.12]\) are in one variable. In our point of view these examples are not sufficiently “generic” - here, we note that there are fundamental differences in the vector partition function theory in one and two or more variables. These differences stem from the fact that polynomials in one variable are a principal ideal domain, which in turn implies “non-generic” properties such as uniqueness of the partial fraction decompositions used to compute the vector partition functions.

The pairs of type \(B_2 \simeq C_2 \hookrightarrow A_3 \simeq D_3, A_3 \hookrightarrow B_3, A_3 \hookrightarrow C_3\) are, by Theorem \([3.12]\) of quasipolynomial degree \(0 = \frac{1}{2}(\dim g - \dim \tilde{g} - \dim h - \dim \tilde{h})\), and we do not consider them sufficiently “generic”. Of the remaining low-dimensional examples there is only one in which the subalgebra is simple, namely that of \(\text{Lie} G_2 \hookrightarrow \text{so}(7)\).

We note that the finite-dimensional branching laws for the pair \((\text{so}(7), \text{Lie} G_2)\) were studied in \([16]\). However, \([16]\) does not address the task of finding explicit formulas for the \(b\)-singular vectors realizing the branching, which is ultimately needed for the geometric applications described in the preceding paragraph. It is important to note that formulas for the finite dimensional branching can be extracted using the infinite-dimensional branching problem. We make use of this idea in Corollary \([5.8]\) which states that the \(b\)-singular vector formulas derived in an infinite-dimensional setting in Theorem \([5.5]\) remain valid for finite dimensional representations as well.
5.1 Structure of the embedding $\text{Lie } G_2 \xrightarrow{i} so(7)$ and the corresponding parabolic subalgebras

We describe first the structure of $so(7)$ as a Lie $G_2$-module and the parabolic subalgebras $\mathfrak{p}$ of $so(7)$ relative to the parabolic subalgebras $i(\mathfrak{p})$ of $i(\text{Lie } G_2)$.

We start by fixing a Chevalley-Weyl basis of the Lie algebra $so(2n + 1)$. Let the defining vector space $V$ of $so(2n + 1)$ have a basis $e_1, \ldots, e_n, e_0, e_{-1}, \ldots e_{-n}$. Let the defining symmetric bilinear form $B$ of $so(2n + 1)$ be given by the matrix

$$B := \begin{pmatrix}
0_{n\times n} & 0_{n\times 1} & 1_{n\times n} \\
0_{1\times n} & 1 & 0_{1\times n} \\
1_{n\times 1} & 0_{n\times 1} & 0_{n\times n}
\end{pmatrix},$$

alternatively defined by $B(e_i, e_j) := 0, i \neq -j$, $B(e_i, e_{-i}) = 1$, $B(e_i, e_0) := 0$, $B(e_0, e_0) := 1$, or alternatively defined as an element of $S^2(V^*)$,

$$B := \sum_{i=-n}^{n} e_i^* \otimes e_{-i}^* = (e_0)^2 + 2 \sum_{i=1}^{n} e_i^* e_{-i}^*, \quad (27)$$

under the identification $v^*w^* := \frac{1}{2n} (v^* \otimes w^* + w^* \otimes v^*)$.

In the basis $e_1, \ldots, e_n, e_0, e_{-1}, \ldots e_{-n}$, the matrices of the elements of $so(2n + 1)$ are of the form

$$\begin{pmatrix}
A & v_1 & \vdots \\
& \ddots & \vdots \\
v_n & & -v_1 & \ldots & -v_n \\
-w_1 & \vdots & & -A^T \\
-w_n \\
D = -D^T
\end{pmatrix},$$

i.e., all matrices $C$ such that $C^T B + BC = 0$. We fix $e_1^*, \ldots, e_n^*, e_0^*, e_{-1}^*, \ldots e_{-n}^*$ to be basis of $V^*$ dual to $e_1, \ldots, e_n, e_0, e_{-1}, \ldots e_{-n}$. We identify elements of $\text{End}(V)$ with elements of $V \otimes V^*$. In turn, we identify elements of $\text{End}(V)$ with their matrices in the basis $e_1, \ldots, e_n, e_0, e_{-1}, \ldots e_{-n}$.

Fix the Cartan subalgebra $\mathfrak{h}$ of $so(2n + 1)$ to be the subalgebra of diagonal matrices, i.e., the subalgebra spanned by the vectors $e_i \otimes e_i^* - e_{-i} \otimes e_{-i}^*$. Then the basis vectors $e_1, \ldots, e_n, e_0, e_{-1}, \ldots e_{-n}$ are a basis for the $\mathfrak{h}$-weight vector decomposition of $V$. Let the $\mathfrak{h}$-weight of $e_i, i > 0$, be $\varepsilon_i$. Then the $\mathfrak{h}$-weight of $e_{-i}, i > 0$ is $-\varepsilon_i$, and an $\mathfrak{h}$-weight decomposition of $so(2n + 1)$ is given by the elements $g_{\varepsilon_i - \varepsilon_j} := e_i \otimes e_j^* - e_j \otimes e_i^*, g_{\pm (\varepsilon_i + \varepsilon_j)} := e_{\pm i} \otimes e_{\mp j}^* - e_{\pm j} \otimes e_{\mp i}^*$ and $g_{\pm \varepsilon_i} := \sqrt{2} (e_{\pm i} \otimes e_0^* - e_0 \otimes e_{\mp i}^*)$, where $i, j > 0$.

Define the symmetric bilinear form $\langle \cdot, \cdot \rangle_\mathfrak{h}$ on $\mathfrak{h}^*$ by $\langle \varepsilon_i, \varepsilon_j \rangle_\mathfrak{h} = 1$ if $i = j$ and zero otherwise.
The root system of $so(2n+1)$ with respect to $h$ is given by $\Delta(g) := \Delta^+(g) \cup \Delta^-(g)$, where we define

$$\Delta^+(g) := \{\varepsilon_i \pm \varepsilon_j | 1 \leq i < j \leq n\} \cup \{\varepsilon_i | 1 \leq i \leq n\}$$

and $\Delta^-(g) := -\Delta^+(g)$. We fix the Borel subalgebra $b$ of $so(2n+1)$ to be the subalgebra spanned by $h$ and the elements $g_{\alpha}, \alpha \in \Delta^+(g)$. The simple positive roots corresponding to $b$ are then given by

$$\eta_1 := \varepsilon_1 - \varepsilon_2, \ldots, \eta_{n-1} := \varepsilon_{n-1} - \varepsilon_n, \eta_n := \varepsilon_n.$$ 

We recall that the $i^{th}$ fundamental weight $\omega_i \in b^*$ is defined by $\langle \omega_i, \eta_j \rangle_b := \delta_{ij}$. The fundamental weights of $so(2n+1)$ are then given by

$$\omega_1 := \varepsilon_1$$
$$\omega_2 := \varepsilon_1 + \varepsilon_2$$
$$\vdots$$
$$\omega_{n-1} := \varepsilon_1 + \cdots + \varepsilon_{n-1}$$
$$\omega_n := 1/2(\varepsilon_1 + \cdots + \varepsilon_n).$$

For the remainder of this Section we fix the odd orthogonal Lie algebra to be $so(7)$.

We abbreviate the Chevalley-Weyl generator $g_{\alpha} \in so(7)$ as $g_i$ according to the Table below. The rows are sorted by the graded lexicographic order on the simple coordinates of the roots. We furthermore set $h_1 := [g_1, g_{-1}], h_2 := [g_2, g_{-2}], h_3 := 1/2[g_3, g_{-3}]$.

| generator | root | simple coord. | root $\varepsilon$-notation |
|-----------|------|---------------|------------------------------|
| $g_{-9}$ | $(-1, -2, -2)$ | $-\varepsilon_1 - \varepsilon_2$ |
| $g_{-8}$ | $(-1, -1, -2)$ | $-\varepsilon_1 - \varepsilon_3$ |
| $g_{-7}$ | $(0, -1, -2)$ | $-\varepsilon_2 - \varepsilon_3$ |
| $g_{-6}$ | $(-1, -1, -1)$ | $-\varepsilon_1$ |
| $g_{-5}$ | $(0, -1, -1)$ | $-\varepsilon_2$ |
| $g_{-4}$ | $(-1, -1, 0)$ | $-\varepsilon_1 + \varepsilon_3$ |
| $g_{-3}$ | $(0, 0, -1)$ | $-\varepsilon_3$ |
| $g_{-2}$ | $(0, -1, 0)$ | $-\varepsilon_2 + \varepsilon_3$ |
| $g_{-1}$ | $(-1, 0, 0)$ | $-\varepsilon_1 + \varepsilon_2$ |
| $h_1$ | $(0, 0, 0)$ | $0$ |
| $h_2$ | $(0, 0, 0)$ | $0$ |
| $h_3$ | $(0, 0, 0)$ | $0$ |
| $g_1$ | $(1, 0, 0)$ | $\varepsilon_1 - \varepsilon_2$ |
| $g_2$ | $(0, 1, 0)$ | $\varepsilon_2 - \varepsilon_3$ |
Table 1: Generators of so(7)

| generator | root simple coord. | root ε-notation |
|-----------|--------------------|-----------------|
| $g_3$     | $(0, 0, 1)$        | $\varepsilon_3$ |
| $g_4$     | $(1, 1, 0)$        | $\varepsilon_1 - \varepsilon_3$ |
| $g_5$     | $(0, 1, 1)$        | $\varepsilon_2$ |
| $g_6$     | $(1, 1, 1)$        | $\varepsilon_1$ |
| $g_7$     | $(0, 1, 2)$        | $\varepsilon_2 + \varepsilon_3$ |
| $g_8$     | $(1, 1, 2)$        | $\varepsilon_1 + \varepsilon_3$ |
| $g_9$     | $(1, 2, 2)$        | $\varepsilon_1 + \varepsilon_2$ |

Let now $\bar{g} = \text{Lie } G_2$. One way of defining the positive root system of Lie $G_2$ is by setting it to be the set of vectors

$$\Delta(\bar{g}) := \{\pm (1, 0), \pm (0, 1), \pm (1, 1), \pm (2, 1), \pm (3, 1), \pm (3, 2)\}.$$  \hspace{1cm} (29)

We set $\alpha_1 := (1, 0)$ and $\alpha_2 := (0, 1)$. We fix a bilinear form $\langle \cdot, \cdot \rangle_{\bar{g}}$ on $\bar{h}$, proportional to the one induced by Killing form by setting

$$\left( \begin{array}{c}
\langle \alpha_1, \alpha_1 \rangle_{\bar{g}} & \langle \alpha_1, \alpha_2 \rangle_{\bar{g}} \\
\langle \alpha_2, \alpha_1 \rangle_{\bar{g}} & \langle \alpha_2, \alpha_2 \rangle_{\bar{g}}
\end{array} \right) :=
\left( \begin{array}{cc}
2 & -3 \\
-3 & 6
\end{array} \right).$$

In an $\langle \cdot, \cdot \rangle_{\bar{g}}$-orthogonal basis the root system of Lie $G_2$ is often drawn as $\star \rightarrow \star \rightarrow \rightarrow \rightarrow$. The fundamental weights of Lie $G_2$ relative to the simple basis $\{\alpha_1, \alpha_2\}$ are then given by $\psi_1 := 2\alpha_1 + \alpha_2$, $\psi_2 := 3\alpha_1 + 2\alpha_2$. We fix a basis for the Lie algebra Lie $G_2$ by giving a set of Chevalley-Weyl generators $\bar{g}_i$, $i \in \{\pm 1, \cdots, \pm 6\}$, and by setting $\bar{h}_1 := [\bar{g}_1, \bar{g}_{-1}]$, $\bar{h}_2 := 3[\bar{g}_2, \bar{g}_{-2}]$ according to the following table.

Table 2: Generators of Lie $G_2$

| generator | root simple coord. |
|-----------|--------------------|
| $\bar{g}_{-6}$ | $(-3, -2)$ |
| $\bar{g}_{-5}$ | $(-3, -1)$ |
| $\bar{g}_{-4}$ | $(-2, -1)$ |
| $\bar{g}_{-3}$ | $(-1, -1)$ |
| $\bar{g}_{-2}$ | $(0, -1)$ |
| $\bar{g}_{-1}$ | $(-1, 0)$ |
| $\bar{h}_1$ | $(0, 0)$ |
| $\bar{h}_2$ | $(0, 0)$ |
| $\bar{g}_1$ | $(1, 0)$ |

\footnote{For the bilinear form induced by the Killing form of Lie $G_2$, the long root has squared length 36, so the coefficient of proportionality is 6}
Table 2: Generators of Lie $G_2$

| generator | root simple coord. |
|-----------|--------------------|
| $\bar{g}_2$ | (0, 1) |
| $\bar{g}_3$ | (1, 1) |
| $\bar{g}_4$ | (2, 1) |
| $\bar{g}_5$ | (3, 1) |
| $\bar{g}_6$ | (3, 2) |

All embeddings $\text{Lie } G_2 \xrightarrow{i} \text{so}(7)$ are conjugate over $\mathbb{C}$. Following [16], one such embedding is given via

\[ i(\bar{g}_{\pm 2}) := g_{\pm 2}, \quad i(\bar{g}_{\pm 1}) := g_{\pm 1} + g_{\pm 3} \, . \]  

(30)

As $\bar{g}_{\pm 1}, \bar{g}_{\pm 2}$ generate the Lie algebra $\text{Lie } G_2$, the preceding data determines the map $i$. The structure constants of $\text{Lie } G_2$ are readily available from a number of computer algebra systems, including ours, and one can directly check that the map $i$ is a Lie algebra homomorphism. Alternatively, we can use $i(\bar{g}_{\pm 1}), i(\bar{g}_{\pm 2})$ to generate a Lie subalgebra of $\text{so}(7)$, verify that this subalgebra is indeed 14-dimensional and simple, and finally use this 14-dimensional image to compute the structure constants of $\text{Lie } G_2$.

Let $\text{pr} : \mathfrak{h}^* \to \mathfrak{h}^*$ be the map naturally induced by $i$. Then

\[ \text{pr}(\varepsilon_1 - \varepsilon_2) = \text{pr}(\varepsilon_3) = \alpha_1, \quad \text{pr}(\varepsilon_2 - \varepsilon_3) = \alpha_2 \, , \]  

or equivalently

\[ \text{pr}(\omega_1) = \text{pr}(\omega_3) = \psi_1, \quad \text{pr}(\omega_2) = \psi_2 \, . \]  

Conversely, $\iota : \tilde{\mathfrak{h}}^* \to \mathfrak{h}^*$ (see Section [2]) is the map

\[ \iota(\alpha_2) = 3\eta_2 - 3\varepsilon_2 - 3\varepsilon_3, \quad \iota(\alpha_1) = \eta_1 + 2\eta_3 = \varepsilon_1 - \varepsilon_2 + 2\varepsilon_3 \, . \]

The following Lemma gives the pairwise inclusions between the parabolic subalgebras of $\text{so}(7)$ and the embeddings of the parabolic subalgebras of Lie $G_2$.

**Lemma 5.1** For the pair $G_2 \xrightarrow{i} \text{so}(7)$, let $\mathfrak{h}, \mathfrak{b}, \mathfrak{p}, \tilde{\mathfrak{h}}, \tilde{\mathfrak{b}}, \tilde{\mathfrak{p}}$ denote respectively Cartan, Borel and parabolic subalgebras with the assumptions that $i(\mathfrak{h}) \subset \mathfrak{h} \subset \mathfrak{b}$, $i(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{b}} \subset \tilde{\mathfrak{b}}$, $i(\tilde{\mathfrak{b}}) \subset \mathfrak{b} \subset \mathfrak{p}$, $\tilde{\mathfrak{b}} \subset \mathfrak{p}$. Then we have the following inclusion diagram for all
possible values of $p, \bar{p}$.

If a path of arrows exists from one node of the diagram to the other, then the corresponding parabolic subalgebras lie inside one another. If in the diagram a direct arrow exists from a parabolic subalgebra $\bar{p}$ of Lie $G_2$ to a parabolic subalgebra $p$ of $so(7)$, then $\bar{p} = i^{-1}(i(\bar{g}) \cap p)$.

Proof. Since a parabolic subalgebra of a reductive Lie algebra is a direct sum of root spaces, (30) implies that $i(\bar{g}_{\pm 1})$ belongs to $p$ if and only if both $g_{\pm 1}$ and $g_{\pm 2}$ belong to $p$. Similarly we have that $i(\bar{g}_{\pm 2})$ belongs to $p$ if and only if $g_{\pm 2}$ belongs to $p$. This proves the Lemma. $\square$

The following Lemma describes the structure of $so(7)$ as a module over the Levi part of a parabolic subalgebra of Lie $G_2$. The proof of the Lemma is a straightforward computation. Recall from Section 2 that $\mathfrak{h}$ stands for Cartan subalgebra, $p$ stands for parabolic subalgebra, $\mathfrak{b}$ stands for Borel subalgebra, $\bar{p}$ stands for parabolic subalgebra, $\mathfrak{l}$ stands for the reductive Levi part of $p$, $\mathfrak{s} := [\mathfrak{l}, \mathfrak{l}]$ and $\mathfrak{n}_{-}$ stands for the nilradical opposite to the nilradical of $p$.

Lemma 5.2 Let $\mathfrak{h}, \mathfrak{b}, \mathfrak{s}, \mathfrak{l}, \mathfrak{p}, \mathfrak{n}_{-} \subset$ Lie $G_2$ depend on the subalgebras $\mathfrak{h}, \mathfrak{n}_{-}, \mathfrak{b}, \mathfrak{s}, \mathfrak{l}, \mathfrak{p} \subset$ $so(7)$ as in Section 2. We recall the key requirements that $\bar{p} = i^{-1}(i(\bar{g}) \cap p)$ and $i(\bar{b}) \subset \mathfrak{b}$.

1. Suppose $\bar{p} \simeq \bar{p}(0,0) \simeq$ Lie $G_2$. Then $so(7)$ decomposes over Lie $G_2$ as the direct sum of Lie $G_2$-modules $V_{\psi_2}(\text{Lie } G_2) \oplus V_{\psi_1}(\text{Lie } G_2)$, where $V_{\psi_1}(\text{Lie } G_2)$ is the 7-dimensional simple Lie $G_2$-module and $V_{\psi_2}(\text{Lie } G_2) \simeq$ Lie $G_2$ is the adjoint 14-dimensional Lie $G_2$-module. A weight vector basis of $V_{\psi_1}(\text{Lie } G_2)$ is given by the following table.
2. Suppose \( \ddot{p} \simeq \ddot{p}_{(0,1)} \). Then \( \ddot{i} \simeq sl(2) + \ddot{h} \) and \( \ddot{s} = [\ddot{i}, \ddot{i}] \simeq sl(2) \). The \( \ddot{b} \cap \ddot{s} \)-positive root of \( \ddot{s} \) can be identified with \( \alpha_1 \) and so(7) decomposes over \( \ddot{s} \) as
\[
2V_{\frac{3}{2} \alpha_1} (\ddot{s}) \oplus 2V_{\alpha_1} (\ddot{s}) \oplus 2V_{\frac{1}{2} \alpha_1} (\ddot{s}) \oplus V_0 (\ddot{s}) \quad (\text{the summands are of respective dimensions } 2 \times 4 + 2 \times 3 + 2 \times 2 + 3 \times 1).
\]
By Lemma 5.1, 7 \( p \simeq p_{(0,1,0)} \). Therefore \( \dim n_- = 7 \), \( i(n_-) \subset n_- \), and \( n_- \) equals the maximal \( \ddot{i} \)-stable subspace of the nilradical of the opposite Borel subalgebra of so(7). In addition, \( n_- \) is isomorphic as an \( \ddot{s} \)-module to \( V_{\frac{3}{2} \alpha_1} (\ddot{s}) \oplus V_{\alpha_1} (\ddot{s}) \oplus V_0 (\ddot{s}) \) and is spanned respectively by the \( \ddot{h} \)-weight vectors
\[
\{6g_8, -2g_6 - 2g_7 - g_4 + g_5, g_2\}, \{g_6 - 2g_7, 2g_4 + g_5\}
\]
and \( \{g_9\} \). The coefficients of the preceding elements are chosen so that \( \text{ad}(i(\ddot{g}_-)) \) sends the highest weight vectors to the lower ones.

The set \( \text{Weights}_5 (n_-/N) \) equals \( \{-\psi_1, \psi_1 - \psi_2\} \).

3. Suppose \( \ddot{p} \simeq p_{(1,0)} \). Then \( \ddot{i} \simeq sl(2) + \ddot{h} \) and \( \ddot{s} = [\ddot{i}, \ddot{i}] \simeq sl(2) \). The \( \ddot{b} \cap \ddot{s} \)-positive root of \( \ddot{s} \) can be identified with \( \alpha_2 \) and so(7) decomposes over \( \ddot{s} \) as
\[
2V_{3 \alpha_2} (\ddot{s}) \oplus 6V_{2 \alpha_2} (\ddot{s}) \oplus 6V_0 (\ddot{s}) \quad (\text{the summands are of respective dimensions } 1 \times 3 + 6 \times 2 + 6 \times 1 \text{). The nilradical of the opposite Borel subalgebra of so(7) has a maximal \( \ddot{i} \)-stable subspace isomorphic as a \( \ddot{s} \)-module to } 3V_{\alpha_2} (\ddot{s}) \oplus 2V_0 (\ddot{s}), \text{ spanned respectively by the } \ddot{h} \text{-weight vectors } \{-g_9, g_8\},
\]
\[
\{g_4 - g_5, g_1 + g_3\}, \{2g_4 - g_5, 2g_4 - g_3\}, \{g_6 + g_7\}, \{g_6 + 2g_7\}
\]
and \( \{2g_7\} \). The coefficients of the preceding elements are chosen so that \( \text{ad}(i(\ddot{g}_-)) \) sends the highest weight vectors to the lower ones.

By Lemma 5.1, 7 \( p \simeq p_{(1,0,1)}, p_{(0,0,1)} \) or \( p_{(1,0,0)} \).
Corollary 5.3 All 8 parabolic subalgebras \( p \supset b \supset h \) of \( so(7) \) are weakly compatible with \( i(Lie \ G_2) \). Furthermore, \( p_{(1,0,1)}, p_{(0,1,0)}, p_{(1,1,1)} \) and \( p_{(0,0,0)} \) are compatible with \( i(Lie \ G_2) \) in the sense of [17, Section 3] and the remaining 4 parabolic subalgebras of \( so(7) \) are not.

5.2 Constructing \( b \)-singular vectors in \( M_\lambda(so(7), p) \) for \( Lie \ G_2 \hookrightarrow so(7) \)

The remarks after Definition 4.1 and Lemma 5.2 imply the following Corollary.

Corollary 5.4 The parabolic subalgebra \( p \subset so(7) \) has a finite branching problem over \( i(Lie \ G_2) \) if and only if \( p \simeq p_{(1,0,0)} \) or \( p \simeq so(7) = p_{(0,0,0)} \).

In the present Section we apply Section 4 to the pair \( Lie \ G_2 \hookrightarrow so(7) \). Let \( \bar{c}_1 \) be the quadratic Casimir element of \( U(Lie \ G_2) \). A formula for a degree 6 homogeneous element, linearly (and algebraically) independent on \( \bar{c}_1^3 \) can be found in [2].

In order to simplify computations, we only use the quadratic Casimir operator and the strong Condition B (Definition 4.4). As it turns out in all of our examples, for a fixed value of \( z \), the set of weights \( \lambda(x, z) \) satisfying the strong Condition B is non-empty and Zariski open, and therefore includes almost all values of \( x_1, x_2, x_3 \).
Theorem 5.5

Let $\mathfrak{m}$ be the highest weight vector of the generalized Verma module $M(\mathfrak{so}(7), p_{(1,0,0)})$. We recall $\omega_1 = \varepsilon_1$, $\omega_2 = \varepsilon_1 + \varepsilon_2$, $\omega_3 = 1/2(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$ are the fundamental weights of $\mathfrak{so}(7)$. Recall $\mathfrak{b}$ is the Borel subalgebra of $\mathfrak{lie} G_2$.

1. Suppose $\lambda = x_1 \omega_1 + \omega_2$. Then

\[
\begin{align*}
\nu_{\lambda,0} & : = \nu_{\lambda} \\
\nu_{\lambda,1} & : = ((-x_1 - 2)g_{-3}g_{-2} - 4g_{-4} + 2g_{-2}g_{-1}) \cdot \nu_{\lambda} \\
\nu_{\lambda,2} & : = ((x_1^2 + 2x_1)g_{-3}g_{-2} + (x_1 - 1)g_{-6} \\
& \quad + (-2x_1 - 1)g_{-1}g_{-3}g_{-2} - 2g_{-4}g_{-1} + 2g_{-1}g_{-2}) \cdot \nu_{\lambda}
\end{align*}
\]

are linearly independent $\mathfrak{b}$-singular vectors. Moreover, for $x_1 \notin \{-1, -7/2, -6\}$, the Lie $G_2$-submodules generated by $\nu_{\lambda,i}$ have pairwise empty intersections.

2. Suppose $\lambda = x_1 \omega_1 + \omega_3$. Then

\[
\begin{align*}
\nu_{\lambda,0} & : = \nu_{\lambda} \\
\nu_{\lambda,1} & : = (-x_1g_{-3} + g_{-1}) \cdot \nu_{\lambda} \\
\nu_{\lambda,2} & : = ((2x_1 + 5)g_{-3}g_{-2}g_{-3} - g_{-6} \\
& \quad + 2g_{-4}g_{-3} - 2g_{-1}g_{-2}g_{-3}) \cdot \nu_{\lambda}
\end{align*}
\]

are linearly independent $\mathfrak{b}$-singular vectors. Moreover, for $x_1 \notin \{-5, -3, -1\}$, the Lie $G_2$-submodules generated by $\nu_{\lambda,i}$ have pairwise empty intersections.
3. Suppose $\lambda = x_1\omega_1 + 2\omega_2$. Then

\[
v_{\lambda,0} := v_\lambda
\]
\[
v_{\lambda,1} := (-4g_{-4} + (-x_1 - 3)g_{-3}g_{-2} + 2g_{-1}g_{-2}) \cdot v_\lambda
\]
\[
v_{\lambda,2} := ((4x_1 - 4)g_{-6} - 8g_{-4}g_{-1} + (2x_1^2 + 6x_1)g_{-2}^2g_{-2} + (-4x_1 - 4)g_{-1}g_{-3}g_{-2} + 4g_{-2}g_{-2}) \cdot v_\lambda
\]
\[
v_{\lambda,3} := ((-10x_1 - 20)g_{-4}g_{-3}g_{-2} + 20g_{-4}g_{-1}g_{-2} - 20g_{-2}^2 + (-x_1^2 - 5x_1 - 6)g_{-2}g_{-2}^2 + (x_1^2 + 5x_1 + 6)g_{-2}g_{-2}^2g_{-2} - 5x_1 + 10)g_{-1}g_{-3}g_{-2} - 10g_{-2}g_{-2} \cdot v_\lambda
\]
\[
v_{\lambda,4} := ((12x_1^2 + 48x_1 + 48)g_{-9} + (6x_1^2 + 24x_1 + 24)g_{-8}g_{-2} + (12x_1^2 + 60x_1 - 168)g_{-6}g_{-4} + (3x_1^3 + 24x_1^2 - 72)g_{-6}g_{-3}g_{-2} + (-6x_1^2 - 30x_1 + 84)g_{-6}g_{-1}g_{-2} + (6x_1^3 + 51x_1^2 + 72x_1 - 12)g_{-4}g_{-2}^2g_{-2}g_{-2} + (-18x_1^2 - 138x_1 - 84)g_{-4}g_{-1}g_{-3}g_{-2} + (24x_1 + 168)g_{-4}g_{-2}^2g_{-2}g_{-2} + (-24x_1 - 168)g_{-2}g_{-1}g_{-4}g_{-2} + (1/2x_1^2 + 6x_1^3 + 41/2x_1^2 + 21x_1)g_{-3}g_{-3}g_{-2} + (-3x_1^3 - 57/2x_1^2 - 54x_1 - 18)g_{-1}g_{-2}g_{-2}^2 + (3x_1^2 + 18x_1 + 24)g_{-1}g_{-2}g_{-3}g_{-2} + (-12x_1 - 84)g_{-1}g_{-2}^2g_{-2} + (9x_1^2 + 69x_1 + 42)g_{-2}^2g_{-2}g_{-2}^2 \cdot v_\lambda
\]
\[
v_{\lambda,5} := ((8x_1^2 + 24x_1 + 16)g_{-9}g_{-1} + (-24x_1^2 - 72x_1 + 112)g_{-8}g_{-4} + (-8x_1^3 - 36x_1^2 - 12x_1 + 56)g_{-8}g_{-3}g_{-2} + (16x_1^2 + 48x_1 - 48)g_{-8}g_{-1}g_{-2} + (16x_1^2 + 8x_1 - 168)g_{-6}g_{-4}g_{-1} + (-4x_1^4 - 14x_1^3 + 12x_1^2 + 34x_1 - 28)g_{-6}g_{-2}^2g_{-2} + (8x_1^2 + 12x_1^2 - 60x_1 - 24)g_{-6}g_{-1}g_{-3}g_{-2} + (-8x_1^3 - 4x_1 + 84)g_{-6}g_{-1}g_{-2} + (-4x_1^3 + 28x_1 + 56)g_{-2}g_{-2}^2 + (8x_1^3 + 28x_1^2 - 12x_1 - 32)g_{-2}g_{-1}g_{-3}g_{-2} + (16x_1 + 56)g_{-4}g_{-3}g_{-1}g_{-2} + (-16x_1^2 - 48x_1 + 28)g_{-4}g_{-2}^2g_{-1}g_{-2} + (-16x_1 - 56)g_{-2}^2g_{-2}^2 + (-1/3x_1^3 - 5/2x_1^4 - 5x_1^3 + 5/6x_1^2 + 7x_1)g_{-3}g_{-2}^2 + (4/3x_1^2 + 20/3x_1^2 + 5x_1^2 - 25/3x_1 - 14/3)g_{-3}g_{-2}^2g_{-2} + (-8x_1 - 28)g_{-1}g_{-2}^2g_{-2} + (8x_1^2 + 24x_1 - 14)g_{-3}g_{-1}g_{-2}^2g_{-2} + (-4x_1^4 - 14x_1^3 + 2x_1 + 8)g_{-1}g_{-2}g_{-3}g_{-2} + (4x_1 + 8)g_{-1}g_{-2}^2g_{-2}g_{-2} \cdot v_\lambda
\]

are linearly independent $\tilde{b}$-singular vectors.

Moreover, for $x_1 \notin \{0, -1, -4, -3, -9/2, -2, -8, -6, -7, -5, -9\}$, the Lie $G_2$-submodules generated by $v_{\lambda,i}$ have pairwise empty intersections.
4. Suppose $\lambda = x_1 \omega_1 + \omega_2 + \omega_3$. Then

\[
v_{\lambda,0} := v_{\lambda}
v_{\lambda,1} := (-x_1 g_{-3} + g_{-1}) \cdot v_{\lambda}
v_{\lambda,2} := (5g_{-1}g_{-2} + (-2x_1 - 4)g_{-3}g_{-2} + (x_1 + 2)g_{-2}g_{-3} - 5g_{-4}) \cdot v_{\lambda}
v_{\lambda,3} := ((1/6)x_1^3 + 5/3x_1^2 + 8/3x_1)g_{-3}^2 g_{-2}
+ (-1/2x_1^2 - x_1)g_{-3}g_{-2}g_{-3}
+ (1/6x_1^2 + 4/3x_1 - 3)g_{-6} + (2/3x_1^2 + 10/3x_1 - 1)g_{-4}g_{-3}
+ (-2/3x_1^2 - 16/3x_1 - 3)g_{-1}g_{-3}g_{-2} + (-x_1 - 7)g_{-1}g_{-4}
+ (x_1 + 7)g_{-1}^2 g_{-2} + (x_1 + 2)g_{-1}g_{-2}g_{-3}) \cdot v_{\lambda}
v_{\lambda,4} := ((1/6)x_1^3 + 5/3x_1^2 + 17/3x_1 + 6)g_{-3}g_{-2}g_{-3}
+ (-1/8x_1^2 - 5/4x_1 - 2)g_{-3}^2 g_{-2}
+ (-1/12x_1^2 - 2/3x_1 - 9/4)g_{-6} + (1/2x_1^2 + 35/12x_1 + 31/12)g_{-4}g_{-3}
+ (-1/6x_1^2 + 11/12x_1 - 7/6)g_{-3}g_{-2}g_{-3}
+ (-1/6x_1^2 - 13/12x_1 - 1/4)g_{-1}g_{-3}g_{-2}
+ (-1/3x_1 - 23/12)g_{-4}g_{-3} + (1/3x_1 + 23/12)g_{-1}g_{-2}g_{-3}) \cdot v_{\lambda}
v_{\lambda,5} := ((-x_1^3 - 9/2x_1^2 - 3/2x_1^2 + 7x_1)g_{-3}^2 g_{-2} + (-3x_1^2 - 3x_1)
+ 18x_1 - 12)g_{-6}g_{-3} + (3x_1^2 + 21/2x_1^2 - 9/2x_1 - 9)g_{-1}g_{-2}^2 g_{-3}
+ (-3x_1^2 - 15x_1 + 18)g_{-8}g_{-5} + (3x_1^2 + 3x_1 - 30)g_{-6}g_{-1}
+ (6x_1^2 + 12x_1 - 24)g_{-4}g_{-1}g_{-3}
+ (-6x_1^2 - 18x_1 + 12)g_{-2}g_{-1}^2 g_{-2} + (-3x_1^2 - 3x_1 + 6)g_{-1}g_{-3}g_{-2}g_{-3}
+ (-6x_1 - 21)g_{-5}g_{-1}^2 + (6x_1 + 21)g_{-1}^2 g_{-2} + (3x_1 + 6)g_{-1}^2 g_{-2}g_{-3}) \cdot v_{\lambda}
v_{\lambda,6} := ((1/2)x_1^3 + 29/4x_1^2 + 153/4x_1^2 + 173/2x_1 + 70)g_{-3}^2 g_{-2}g_{-3}
+ (1/2x_1^2 + 11/2x_1^2 + 19x_1 + 20)g_{-6}g_{-2}g_{-3}
+ (-x_1^3 - 11/2x_1^2 - 38x_1 - 40)g_{-6}g_{-3}g_{-2}
+ (-1/6x_1^3 - 29/12x_1^2 - 51/4x_1^2 - 173/6x_1 - 70/3)g_{-2}^3 g_{-2}g_{-3}
+ (4x_1^2 + 93/2x_1^2 + 349/2x_1 + 210)g_{-4}g_{-3}g_{-2}g_{-3}
+ (-1/2x_1^3 - 27/4x_1^2 - 121/4x_1 - 45)g_{-4}g_{-2}g_{-3}
+ (-2x_1^2 - 93/4x_1^2 - 349/4x_1 - 105)g_{-1}g_{-3}g_{-2}g_{-3}
+ (-5/2x_1^2 - 45/2x_1 - 50)g_{-9}
+ (-1/2x_1^2 + 27/4x_1 + 45x_1 + 100)g_{-6}g_{-2}g_{-3}
+ (5/2x_1^2 + 45/2x_1 - 50)g_{-6}g_{-4}
+ (5/2x_1^2 + 45/2x_1 + 50)g_{-6}g_{-1}g_{-2}
+ (-5x_1^2 - 45x_1 - 100)g_{-4}g_{-1}g_{-2}g_{-3}
+ (1/2)x_1^2 + 27/4x_1^2 + 121/4x_1 + 45)g_{-4}g_{-2}g_{-3}g_{-2}
+ (5/2x_1^2 + 45/2x_1 + 50)g_{-2}^2 g_{-2}g_{-3}) \cdot v_{\lambda}
\[ v_{\lambda, 7} := (-1/24x_1^6 - 37/48x_1^5 - 265/48x_1^4 - 115/6x_1^3 
- 129/4x_1^3 - 21/x_1)g_3^2g_2^2g_3 - 3 
+ (-1/4x_1^5 - 15/4x_1^4 - 177/4x_1^3 - 147/4x_1^2 - 6x_1 + 36)g_{-6g_{-3g_{-2g_{-3}}}g_{-2g_{-3}}} 
+ (1/8x_1^2 + 15/8x_1^4 + 10x_1^3 + 87/4x_1^2 + 51/4x_1 - 9)g_{-6g_{-3g_{-2}}} 
+ (-1/12x_1^5 - 7/6x_1^4 - 139/24x_1^3 - 287/24x_1^2 - 17/2x_1)g_{-4g_2^3g_{-2g_{-2}}} 
+ (1/8x_1^2 + 2x_1^4 + 187/16x_1^3 + 487/16x_1^2 + 33x_1 + 9)g_{-1g_2^2g_2^2g_{-2g_{-3}}} 
+ (-1/4x_1^4 - 9/4x_1^3 - 8x_1^2 - 33/2x_1 + 18)g_{-9g_{-3g_{-3}}} 
+ (-1/4x_1^4 - 3x_1^3 - 53/4x_1^2 - 51/2x_1 - 18)g_{-8g_{-2g_{-3}}} 
+ (1/4x_1^4 + 15/4x_1^3 + 37/2x_1 + 69/2x_1 + 18)g_{-8g_{-3g_{-2}}} 
+ (1/8x_1^2 + 3/2x_1^3 + 41/8x_1^2 + 9/4x_1 - 9)g_2^2g_6 
+ (-1/4x_1^4 - 11/4x_1^3 - 7x_1^2 + 9x_1 + 36)g_{-6g_{-4g_{-3}}} 
+ (1/4x_1^3 + 3x_1^3 + 21/2x_1^2 + 25/4x_1^2 - 15)g_{-6g_{-1g_{-2g_{-3}}}g_{-2g_{-3}}} 
+ (-1/4x_1^4 - 13/4x_1^3 - 14x_1^2 - 43/2x_1 - 6)g_{-6g_{-1g_{-3g_{-2}}}g_{-2g_{-3}}} 
+ (-1/4x_1^4 - 29/8x_1^3 - 145/8x_1^2 - 73/2x_1 - 24)g_2^2g_2^2g_{-2g_{-3}} 
+ (1/2x_1^2 + 29/4x_1^3 + 145/4x_1^2 + 73x_1 + 48)g_{-4g_{-1g_{-3g_{-2}}}g_{-2g_{-3}}} 
+ (-1/12x_1^4 - x_1^3 - 25/6x_1^2 - 27/4x_1 - 3)g_{-1g_{-2g_{-3}}}g_{-9g_{-3g_{-2}}} 
+ (-1/2x_1^4 - 31/4x_1^3 - 143/4x_1 + 51)g_{-8g_{-1g_{-2}}} 
+ (1/4x_1^2 + 23/8x_1^2 + 87/8x_1 + 27/2)g_2^2g_{-2g_{-3}}g_{-2g_{-2}} 
+ (1/4x_1^3 + 13/4x_1^2 + 27/2x_1 + 18)g_2^2g_{-2g_{-3}}g_{-3} 
+ (1/2x_1^2 + 13/2x_1^2 + 27x_1 + 36)g_{-8g_{-4}} 
+ (-1/4x_1^4 - 23/8x_1^2 - 87/8x_1 - 27/2)g_{-4g_{-1g_{-3g_{-2}}}g_{-2g_{-2}}} 
+ (1/4x_1^2 + 13/4x_1^2 + 27/2x_1 + 18)g_{-6g_{-2g_{-3}}}g_{-2g_{-2}} 
+ (1/2x_1^2 + 13/2x_1^2 + 27x_1 + 36)g_{-4g_{-1g_{-2}}}g_{-3} 
+ (-1/2x_1^2 - 13/2x_1^2 - 27x_1 - 36)g_{-4g_{-1g_{-2}}}g_{-2g_{-3}} 
+ (-1/4x_1^3 - 13/4x_1^2 - 27/2x_1 - 18)g_{-6g_{-4g_{-2}}}g_{-1g_{-1}} 
+ (-5/4x_1^2 - 35/4x_1 - 15)g_{-9g_{-1}} \cdot v_\lambda \]

are linearly independent 8-singular vectors.
5. Suppose $\lambda = x_1 \omega_1 + 2 \omega_3$. Then

$$v_{\lambda, 0} := v_\lambda$$

$$v_{\lambda, 1} := ((-x_1) g_{-3} + 2 g_{-1}) \cdot v_\lambda$$

$$v_{\lambda, 2} := ((2x_1^2 - 2x_1) g_{-3} + (-4x_1 + 4) g_{-1} g_{-3} + 4g_{-2}^2) \cdot v_\lambda$$

$$v_{\lambda, 3} := ((-2x_1 + 8) g_{-6} + (2x_1 + 8) g_{-4} g_{-3} + (2x_1^2 + 14x_1 + 24) g_{-3} g_{-2} g_{-3} + (-2x_1 - 8) g_{-1} g_{-2} g_{-3}) \cdot v_\lambda$$

$$v_{\lambda, 4} := ((-x_1^2 - 10x_1 - 21) g_{-8} + (-1/2x_1^3 - 7/2x_1^2 - 15/2x_1 - 9/2) g_{-6} g_{-3} + (x_1^2 + 7x_1 + 12) g_{-6} g_{-1} + (x_1^3 + 13x_1^2 + 10x_1 - 3/2) g_{-4} g_{-3} + (-x_1^2 - 7x_1 - 12) g_{-4} g_{-1} g_{-3} + (1/2x_1^4 + 5x_1^2) + 33/2x_1^2 + 18x_1) g_{-3} g_{-2} g_{-3} + (-x_1^3 - 19/2x_1^2 - 28x_1 - 51/2) g_{-1} g_{-3} g_{-2} g_{-3} + (3/2x_1^2 + 9x_1 + 27/2) g_{-1} g_{-2} g_{-3}^2 + (x_1^2 + 7x_1 + 12) g_{-2} g_{-1} g_{-2} g_{-3}) \cdot v_\lambda$$

$$v_{\lambda, 5} := ((4/3x_1^3 + 38/3x_1^2 + 118/3x_1 + 40) g_{-9} g_{-3} + (-4/3x_1^3) - 28/3x_1 - 16) g_{-9} g_{-1} + (-4/3x_1^3 - 28/3x_1 - 16) g_{-8} g_{-4} + (+4/3x_1^3 + 38/3x_1^2 + 118/3x_1 + 40) g_{-8} g_{-2} g_{-3} + (+4/3x_1^3 + 14x_1^2) + 146/3x_1 + 56) g_{-6} g_{-4} g_{-3} + (+4/3x_1^3 + 18x_1^2 + 90x_1^2 + 592/3x_1 + 160) g_{-6} g_{-3} g_{-2} g_{-3} + (-2/3x_1^2 - 9x_1^1 - 45x_1^2 - 296/3x_1 - 80) g_{-6} g_{-2} g_{-2} g_{-3} + (-4/3x_1^3 + 14x_1^2 - 146/3x_1 + 56) g_{-6} g_{-1} g_{-2} g_{-3} + (-2/3x_1^2 + 22/3x_1^2 - 80/3x_1 - 32) g_{-2}^2 g_{-6} + (+4/3x_1^3 - 52/3x_1^3 - 251/3x_1^2 - 533/3x_1 - 140) g_{-4}^2 g_{-3} g_{-2} g_{-3} + (+4/3x_1^3 + 14x_1^2 + 146/3x_1 + 56) g_{-4} g_{-1} g_{-2} g_{-2} g_{-3} + (-4/3x_1^3 - 14x_1^2 - 146/3x_1 + 56) g_{-2}^2 g_{-4}^2 g_{-3} + (+1/3x_1^3 - 11/2x_1^4 - 36x_1^3 - 701/6x_1^2 - 188x_1 - 120) g_{-4}^2 g_{-3} g_{-2} g_{-3} + (+2/3x_1^2 + 26/3x_1^3 + 251/6x_1^2 + 533/6x_1 + 70) g_{-1} g_{-3} g_{-2}^2 g_{-3} + (-2/3x_1^3 - 7x_1^2 - 73/3x_1 - 28) g_{-1} g_{-2} g_{-2}^2 g_{-3}) \cdot v_\lambda$$

are linearly independent $\delta$-singular vectors. Moreover, for $x_1 \notin \{ -5, -3, -6, -4, -7/2, -1, -7, 0, -2 \}$, the Lie $G_2$-submodules generated by $v_{\lambda, i}$ have pairwise empty intersections.

**Proof.** Fix the base field to be the field $\mathbb{C}(x_1)$ of rational functions in the variable $x_1$. 

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We recall $l$ denotes the reductive Levi part of $\mathfrak{p}_{(1,0,0)}$. Then $l \simeq \mathfrak{ch}_l \oplus \mathfrak{so}(5)$, where the $\mathfrak{so}(5)$-part is generated by the simple Lie algebra generators $g_2$, $g_{-2}$, $g_3$ and $g_{-3}$. The finite dimensional $l$-module $V_\lambda(l)$, inducing the generalized Verma module $M_\lambda(g,p)$, has a basis of the form $u_i \cdot v_\lambda$, where each element $u_i$ is a product of elements of the form $g_{s_i} g_{t_j}$, where $s_i, t_j \in \mathbb{Z}_{\geq 0}$. Such monomial bases as well as the actions of $g_2$ and $g_3$ on them are given in the following table.

Table 4: Monomial bases of the inducing modules of certain generalized Verma modules for $\mathfrak{p}_{(1,0,0)}$. The weights are sorted in increasing $gr$-lex-order with respect to their coordinates in simple basis (weights that are lower in the table have $gr$-lex-higher weight).

| Highest weight $\lambda = x_1 \omega_1 + \omega_2$ (restricts to natural $\mathfrak{so}(5)$-module) |
| --- |
| Element | monomial expression | action of $g_2$ | action of $g_3$ |
| $m_1$ | $(x_1 + 2) \omega_1 - \omega_2$ | $g_{-2} g^3_5 g_{-2} v_\lambda$ | $m_2$ | 0 |
| $m_2$ | $(x_1 + 1) \omega_1 + \omega_2 - 2 \omega_3$ | $g_{-3} g_{-2} v_\lambda$ | 0 | $2m_3$ |
| $m_3$ | $(x_1 + 1) \omega_1 + \omega_2 - 2 \omega_3$ | $g_{-3} g_{-2} v_\lambda$ | 0 | $2m_4$ |
| $m_4$ | $(x_1 + 1) \omega_1 - \omega_2 + 2 \omega_3$ | $g_{-2} v_\lambda$ | $m_5$ | 0 |
| $m_5$ | $x_1 \omega_1 + \omega_2$ | $v_\lambda$ | 0 | 0 |

| Highest weight $\lambda = x_1 \omega_1 + \omega_2$ (restricts to spinor $\mathfrak{so}(5)$-module) |
| --- |
| Element | monomial expression | action of $g_2$ | action of $g_3$ |
| $m_1$ | $(x_1 + 1) \omega_1 - \omega_2$ | $g_{-3} g_{-2} g_{-3} v_\lambda$ | 0 | $m_2$ |
| $m_2$ | $(x_1 + 1) \omega_1 - \omega_2 + \omega_3$ | $g_{-2} g_{-3} v_\lambda$ | $m_3$ | 0 |
| $m_3$ | $x_1 \omega_1 + 2 \omega_2 - \omega_3$ | $g_{-3} v_\lambda$ | 0 | $m_4$ |
| $m_4$ | $x_1 \omega_1 + \omega_3$ | $v_\lambda$ | 0 | 0 |

| Highest weight $\lambda = x_1 \omega_1 + 2 \omega_2$ |
| --- |
| Element | monomial expression | action of $g_2$ | action of $g_3$ |
| $m_1$ | $(x_1 + 4) \omega_1 - 2 \omega_2$ | $g_{-3} g_{-2} g_{-3} v_\lambda$ | $2m_2$ | 0 |
| $m_2$ | $(x_1 + 3) \omega_1 - 2 \omega_3$ | $g_{-3} g_{-2} g_{-3} v_\lambda$ | $2m_3$ | $4m_4$ |
| $m_3$ | $(x_1 + 2) \omega_1 + 2 \omega_2 - 4 \omega_3$ | $g_{-3} g_{-2} g_{-3} v_\lambda$ | 0 | $4m_5$ |
| $m_4$ | $(x_1 + 3) \omega_1 - \omega_2$ | $g_{-3} g_{-2} g_{-3} v_\lambda$ | $m_5$ | $6m_6$ |
| $m_5$ | $(x_1 + 2) \omega_1 + 2 \omega_2 - 2 \omega_3$ | $g_{-3} g_{-2} g_{-3} v_\lambda$ | 0 | $6m_8$ |
| $m_6$ | $(x_1 + 1) \omega_1 + 2 \omega_2$ | $g_{-3} g_{-2} g_{-3} v_\lambda$ | $2m_7$ | 0 |
| $m_7$ | $(x_1 + 2) \omega_1 + 2 \omega_2 - 2 \omega_3$ | $g_{-3} g_{-2} g_{-3} v_\lambda$ | $2m_9$ | $m_{10}$ |
| $m_8$ | $(x_1 + 2) \omega_1 + 2 \omega_2 - 2 \omega_3$ | $g_{-3} g_{-2} g_{-3} v_\lambda$ | $2m_9$ | $6m_{10}$ |
| $m_9$ | $(x_1 + 1) \omega_1 + 2 \omega_2 - 2 \omega_3$ | $g_{-3} g_{-2} g_{-3} v_\lambda$ | 0 | $2m_{11}$ |
| $m_{10}$ | $(x_1 + 2) \omega_1 + 2 \omega_2 - 2 \omega_3$ | $g_{-3} g_{-2} g_{-3} v_\lambda$ | $2m_{11}$ | $4m_{12}$ |
| $m_{11}$ | $(x_1 + 1) \omega_1 + 2 \omega_2 - 2 \omega_3$ | $g_{-3} g_{-2} g_{-3} v_\lambda$ | 0 | $2m_{13}$ |
| $m_{12}$ | $(x_1 + 2) \omega_1 + 2 \omega_2 - 2 \omega_3$ | $g_{-3} g_{-2} g_{-3} v_\lambda$ | $2m_{13}$ | 0 |
| $m_{13}$ | $(x_1 + 1) \omega_1 + 2 \omega_2 - 2 \omega_3$ | $g_{-3} g_{-2} g_{-3} v_\lambda$ | $2m_{14}$ | 0 |
| $m_{14}$ | $x_1 \omega_1 + 2 \omega_2$ | $v_\lambda$ | 0 | 0 |

| Highest weight $\lambda = x_1 \omega_1 + 2 \omega_2 + \omega_3$ |
| --- |
| Element | monomial expression | action of $g_2$ | action of $g_3$ |
| $m_1$ | $(x_1 + 3) \omega_1 - \omega_2 - \omega_3$ | $g_{-3} g_{-2} g_{-3} g_{-2} v_\lambda$ | $m_2$ | $m_3$ |
| $m_2$ | $(x_1 + 2) \omega_1 + \omega_2 - 2 \omega_3$ | $g_{-3} g_{-2} g_{-3} g_{-2} v_\lambda$ | 0 | $3m_5$ |
| $m_3$ | $(x_1 + 3) \omega_1 - 2 \omega_2 + \omega_3$ | $g_{-3} g_{-2} g_{-3} g_{-2} v_\lambda$ | $2m_4$ | 0 |
| $m_4$ | $(x_1 + 2) \omega_1 - \omega_3$ | $g_{-3} g_{-2} g_{-3} g_{-2} v_\lambda$ | $2m_6$ | $3m_7$ |
| $m_5$ | $(x_1 + 2) \omega_1 - \omega_3$ | $g_{-3} g_{-2} g_{-3} g_{-2} v_\lambda$ | $2/3m_6$ | $4m_8$ |
| $m_6$ | $(x_1 + 3) \omega_1 + 2 \omega_2 - 3 \omega_3$ | $g_{-3} g_{-2} g_{-3} g_{-2} v_\lambda$ | 0 | $3m_{10}$ |
| $m_7$ | $(x_1 + 2) \omega_1 - \omega_3$ | $g_{-3} g_{-2} g_{-3} g_{-2} v_\lambda$ | $m_{10}$ | $2m_{11}$ |
| $m_8$ | $(x_1 + 2) \omega_1 - \omega_3$ | $g_{-3} g_{-2} g_{-3} g_{-2} v_\lambda$ | $2m_9$ | $3m_{11}$ |
| $m_9$ | $(x_1 + 1) \omega_1 + 2 \omega_2 + \omega_3$ | $g_{-3} g_{-2} g_{-3} g_{-2} v_\lambda$ | 0 | $m_{12} + m_{13}$ |
| $m_{10}$ | $(x_1 + 1) \omega_1 + \omega_2 - \omega_3$ | $g_{-3} g_{-2} g_{-3} g_{-2} v_\lambda$ | 0 | $4m_{13}$ |
Table 4. The resulting \( \mathbb{C} \)-computed as the eigenspace of the action of operator \( \mathbb{C} \) over \( h \) in \( \mathfrak{so}(7) \) generated by \( g_2 \) and \( g_3 \).

By Lemma 5.1 \( \bar{p} = \bar{p}_{(1,0)} \subset \text{Lie } G_2 \), and \( i(\bar{\mathfrak{s}}) = i(\bar{[\mathfrak{h}, \mathfrak{i}]}) \) equals the \( sl(2) \)-subalgebra of \( \mathfrak{so}(7) \) generated by \( g_2 \) and \( g_3 \).

We note that \( C_{h_2} \oplus C_{g_3} \) is a Borel subalgebra of \( \mathfrak{s} = [\mathfrak{h}, \mathfrak{i}] \). We can decompose \( V_\lambda(i) \) over \( h + i(\mathfrak{i}) \) to obtain the decomposition indicated in the following table.

**Table 5:** Decomposition of \( V_\lambda(i) \) over \( h + i(\mathfrak{i}) \). For \( \nu \in \mathfrak{h}^* \), we abbreviate \( V_\nu(h + i(\mathfrak{i})) \) as \( V_\nu \).

| Element | Monomial expression | Action of \( g_2 \) | Action of \( g_3 \) |
|---------|---------------------|----------------------|----------------------|
| \( m_1 \) | \( (x_1 + 2)\omega_1 - 2\omega_2 + 3\omega_3 \) | \( g_2^2g_3^{-3}v_\lambda \) | 0 |
| \( m_2 \) | \( (x_1 + 2)\omega_1 - 2\omega_2 \) | \( g_2^2g_3^{-3}v_\lambda \) | 2m_3 |
| \( m_3 \) | \( (x_1 + 1)\omega_1 + \omega_2 - 2\omega_3 \) | \( g_2^2g_3^{-3}v_\lambda \) | 0 |
| \( m_4 \) | \( (x_1 + 2)\omega_1 - 2\omega_2 + 2\omega_3 \) | \( g_2^2g_3^{-3}v_\lambda \) | 2m_6 |
| \( m_5 \) | \( (x_1 + 1)\omega_1 \) | \( g_2^{-3}g_3^{-3}v_\lambda \) | 0 |
| \( m_6 \) | \( (x_1 + 1)\omega_1 \) | \( g_2^{-3}g_3^{-3}v_\lambda \) | m_7 |
| \( m_7 \) | \( x_1\omega_1 + 2\omega_2 - 2\omega_3 \) | \( g_2^0g_3^{-2}v_\lambda \) | 0 |
| \( m_8 \) | \( (x_1 + 1)\omega_1 - 2\omega_2 + 2\omega_3 \) | \( g_2^{-3}g_3^{-3}v_\lambda \) | 0 |
| \( m_9 \) | \( x_1\omega_1 + \omega_2 \) | \( g_2^0g_3^{-2}v_\lambda \) | 0 |
| \( m_{10} \) | \( x_1\omega_1 + 2\omega_3 \) | \( v_\lambda \) | 0 |

The \( C_{h_2} \oplus C_{g_2} \)-singular vectors realizing the above decomposition are computed as the eigenspace of the action of operator \( g_2 \) using the fourth column of Table 4. The resulting \( C_{h_2} \oplus C_{g_2} \)-singular vectors are given in the following table.
Let \( \bar{\lambda} := \text{pr}(\lambda) \) be the projection of \( \lambda \) onto \( \check{h}^* \). By Theorem 5.1 (c) and Lemma 5.1 (c) we have that \( P_{1,0,0} \) has a finite branching problem over \( i(\text{Lie } G_2) \) and therefore the coefficient of \( V_\nu \) in Table 6 equals \( m(\text{pr}(\nu), \text{pr}(\lambda)) \). This correspondence is one to one as no two different weights \( \nu \) appearing in Table 6 project to the same weight in \( \check{h}^* \) by Lemma 5.1 (c) as one sees in the third column of Table 6.

The quadratic Casimir element \( \check{c}_1 \) of \( \text{Lie } G_2 \) is given by

\[
36 \check{c}_1 = \check{h}_1^2 + 3 \check{h}_1 \check{h}_2 + 3 \check{h}_2^2 + 15 \check{h}_2 + 9 \check{g} - 6 \check{g}_6 \\
+ 9 \check{h}_1 + 9 \check{g} - 5 \check{g}_5 + 3 \check{g} - 4 \check{g}_4 + 3 \check{g} - 3 \check{g}_3 + 3 \check{g} - 2 \check{g}_2 + 9 \check{g} - 1 \check{g}_1 
\]
and its embedding \( i(\bar{c}_1) \) is given by
\[
12(\bar{c}_1) = 3h_2^2 + 3h_1h_2 + 6h_2h_3 + h_1^2 + 4h_1h_3 + 4h_3^2 + 10h_3 + 3g_{-9}g_9 + 5h_1 + 9h_2 \\
+ 3g_{-8}g_8 + g_{-7}g_7 + g_{-6}g_6 + g_{-5}g_5 + g_{-4}g_4 + g_{-3}g_3 + g_{-2}g_2 \\
+ g_{-1}g_1 + g_1 + g_2 + g_3 + g_4 + g_5 + g_6 + g_7 + g_8 + g_9 \\
= \psi_1 + \psi_2.
\]

Let \( v_\mu \) be the highest weight vector of a Lie \( G_2 \)-highest weight module of highest weight \( \mu = y_1\psi_1 + y_2\psi_2 \), (we recall \( \psi_1 = \alpha_1 + 2\alpha_2 \) and \( \psi_2 = 2\alpha_1 + 3\alpha_2 \) are the fundamental weights of Lie \( G_2 \)). Then \( \bar{c}_1 \) acts on \( v_\mu \) by a constant, i.e., \( \bar{c}_1 \cdot v_\mu = p_1(\mu)v_\mu \) (see Section 4). To compute the coefficient \( p_1(\mu) \), in the expression for \( \bar{c}_1 \) we carry out the substitutions \( h_2 \mapsto 3y_1, h_1 \mapsto y_2 \), and set all terms involving the generators \( \bar{y}_i \) to zero:
\[
p_1(\mu) := 1/12y_2^2 + 1/4y_1y_2 + 5/12y_2 + 1/4y_1^2 + 3/4y_1.
\]

Therefore we can compute the action of \( i(\bar{c}_1) \) on each component of a decomposition of \( M_\lambda(g, p) \) into \( \mathfrak{b} \)-highest modules in the last column of Table 6.

Consider a \( \mathbb{C}h_2 \oplus \mathbb{C}g_2 \)-singular vector \( m \) of \( \mathfrak{b} \)-weight \( \mu \) given in the second column of Table 6. As the base field is \( \mathbb{C}(x_1) \), the strong Condition B (Definition 4.3) is trivially satisfied: indeed, for pairwise different values of \( \mu \), the values \( p_1(\mu) \) in the fourth column of Table 6 are pairwise different elements of \( \mathbb{C}(x_1) \). Therefore by Theorem 5.3, the vectors of the form \( \left( \prod_{\nu \in \mu}(i(\bar{c}_1) - p_1(\nu)) \right) \cdot m \) are \( \mathfrak{b} \)-singular. The vectors \( v_{\lambda,k} \), given in the statement of the current theorem, are indeed up to a scalar in \( \mathbb{C}(x_1) \) equal to \( \left( \prod_{\nu \in \mu}(i(\bar{c}_1) - y(\nu)) \right) \cdot m \). Since the monomials used in the expressions for \( v_{\lambda,i} \) are linearly independent and the coefficients do not have a common zero, substitutions in the variable \( x_1 \) yield non-zero vectors defined over the field \( \mathbb{C} \). As the structure constants of \( \mathfrak{so}(7) \) are identical over all fields of characteristic 0, substitutions of the variable \( x_1 \) give the desired \( \mathfrak{b} \)-singular vectors.

For \( x_1 \) not belonging to the sets listed in the statement of the theorem, the values of \( p_1(\mu) \) are pairwise different. Therefore for those values the Lie \( G_2 \)-modules generated by the vectors \( v_{i,\lambda} \) have pairwise empty intersections, which completes the proof of the Theorem. \( \square \)

Except for finitely many values of \( x_1 \), the formulas for \( \mathfrak{b} \)-singular vectors given in Theorem 5.5 hold in arbitrary highest weight modules of highest weight \( \lambda \). In fact, the following observation holds.

**Corollary 5.6** Let \( \lambda \in \mathfrak{h}^* \) be one of the weights given in Theorem 5.5. Let \( M \) be a \( \mathfrak{so}(7) \)-module that has a \( \mathfrak{b} \)-singular vector \( v_\lambda \) of \( \mathfrak{b} \)-weight \( \lambda \). Let the vectors \( v_{\lambda,i} \) be defined by the same formulas as in Theorem 5.5. For a fixed \( v_{\lambda,i} \), let \( x_1 \) be a number different from the numbers indicated in the right column of Table 6 below. Recall \( \mathfrak{b} \) is the Borel subalgebra of Lie \( G_2 \).

Then \( v_{\lambda,i} \) is non-zero and therefore a \( \mathfrak{b} \)-singular vector under the action on \( M \) induced by the embedding Lie \( G_2 \rightarrow \mathfrak{so}(7) \).

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| \( v_{\lambda,i} \) | \( \lambda = x_1 \omega_1 + \omega_3 \) | \( x_1 \notin \text{roots of } S_{\lambda,i} \) |
|-----------------|------------------|--------------------------|
| \( v_{\lambda,2} \) | \( 2x_1^3 + 27x_1^2 + 133x_1 + 285x_1 + 225 \) | -3, -5, -5/2 |
| \( v_{\lambda,1} \) | \( x_1^2 + 1 \) | 0, -1 |

| \( v_{\lambda,i} \) | \( \lambda = x_1 \omega_1 + \omega_2 \) | \( x_1 \notin \text{roots of } S_{\lambda,i} \) |
|-----------------|------------------|--------------------------|
| \( v_{\lambda,2} \) | \( 2x_1^4 + 13x_1^3 + 25x_1^2 + 14x_1 \) | 0, -1, -2, -7/2 |
| \( v_{\lambda,1} \) | \( x_1^2 + 8x_1 + 12 \) | -2, -6 |

| \( \lambda = x_1 \omega_1 + 2 \omega_3 \) | \( x_1 \notin \text{roots of } S_{\lambda,i} \) |
|-----------------|------------------|--------------------------|
| \( v_{\lambda,5} \) | \( 4x_1^{10} + 152x_1^9 + 2579x_1^8 + 25730x_1^7 \) | -7/2, -5/2, -3, -5, -4, -6 |
| \( v_{\lambda,4} \) | \( +167312x_1^6 + 740582x_1^5 + 2260753x_1^4 \) | 0, -1, -3, -4, -7, -7/2 |
| \( v_{\lambda,3} \) | \( +4700490x_1^3 + 6371352x_1^2 \) | 0, 1, -1 |
| \( v_{\lambda,2} \) | \( +5084640x_1 + 1814400 \) | 0, -2 |
| \( v_{\lambda,1} \) | \( 2x_1^4 + 49x_1^3 + 495x_1^2 + 2678x_1^1 \) | 0, 1, -1 |
| \( v_{\lambda,7} \) | \( x_1^4 + 17x_1^3 + 28x_1^2 + 288x_1 + 288 \) | -3, -4, -6 |
| \( v_{\lambda,6} \) | \( x_1^4 - x_1^2 \) | 0, 1, -1 |
| \( v_{\lambda,5} \) | \( x_1^2 + 2x_1 \) | 0, -2 |

| \( \lambda = x_1 \omega_1 + 2 \omega_2 \) |
|-----------------|------------------|--------------------------|
| \( v_{\lambda,2} \) | \( 2x_1^3 + 25x_1^2 + 113x_1 + 205x_1 + 225 \) | 0, 1, -1, -2, -3, -4, -7/2 |
| \( v_{\lambda,1} \) | \( 53x_1^6 - 230x_1^5 - 168x_1^4 \) | 0, -1, -2, -3, -4, -7/2 |
| \( v_{\lambda,4} \) | \( 2x_1^7 + 47x_1^6 + 431x_1^5 + 1978x_1^4 \) | 0, -1, -2, -3, -4, -7/2 |
| \( v_{\lambda,3} \) | \( +4804x_1^2 + 5896x_1 + 2880 \) | 0, -1, -2, -3, -4, -7/2 |
| \( v_{\lambda,2} \) | \( +3004x_1^2 + 2016x_1 \) | 0, -1, -2, -3, -4, -7/2 |
| \( v_{\lambda,1} \) | \( x_1^2 + 9x_1 + 14 \) | -2, -7 |

\( \omega \in \mathbb{R} \) for each \( v_{\lambda,i} \)
Table 7: Values of $x_1$ for each $v_{\lambda,i}$

| Vector $v_{\lambda,i}$ | $x_1$ $\notin$ roots of $Sh_{\lambda,i} =$ |
|------------------------|-----------------------------------------------|
| $v_{\lambda,5}$        | $4x_1^9 + 80x_1^8 + 655x_1^7 + 2780x_1^6$   |
|                        | $-4536x_1^5$                                 |
| $v_{\lambda,4}$        | $2x_1^8 + 65x_1^7 + 870x_1^6 + 6193x_1^5 + 25234x_1^4$ |
|                        | $+58716x_1^3 + 72216x_1^2 + 36288x_1$         |
| $v_{\lambda,3}$        | $x_1^4 + 20x_1^3 + 137x_1^2 + 370x_1 + 336$ |
| $v_{\lambda,2}$        | $x_1^2 + 9x_1 + 23x_1^2 + 15x_1 + 27$         |
| $v_{\lambda,1}$        | $x_1 + 12 + 27$                               |

Proof. Let $\tau$ denote the transpose anti-automorphism of $U(\mathfrak{g})$, i.e., the linear map of $U(\mathfrak{g})$ defined by

$$
\tau(g_{-\beta}) := g_{\beta}, \quad \beta \in \Delta(so(7))
$$

$$
\tau(g_{\beta_1} \cdots g_{\beta_k}) := \tau(g_{\beta_k}) \cdots \tau(g_{\beta_1}), \quad \beta_j \in \Delta(so(7))
$$

$$
\tau(h) := h, \quad h \in \mathfrak{h}.
$$

Let $u_{\lambda,i} \in U(so(7))$ be an element for which $v_{\lambda,i} = u_{\lambda,i} \cdot v_\lambda$ (one such element is given in Theorem 5.5). Define $Sh_{\lambda,i} := \tau(u_{\lambda,i})u_{\lambda,i} \cdot v_\lambda$. Although we will not use this, we note that $Sh_{\lambda,i}$ does not depend on the choice of $u_{\lambda,i}$ (see, e.g., [10]).

A short consideration shows that $Sh_{\lambda,i}$ is a multiple of $v_\lambda$. Up to a rational scalar this multiple is indicated in the second column of Table 7. The scalar is chosen so that all polynomials have integral relatively prime coefficients and the leading coefficient is positive. The roots of the polynomials $Sh_{\lambda,i}$ are all rational and are indicated in the last column of Table 7 (some of the roots have multiplicity higher than 1). If $Sh_{\lambda,i}$ does not equal to zero for a given value of $x_1$, then $v_{\lambda,i}$ cannot vanish for that value of $x_1$. This proves the statement.

Corollary 4.6 implies the following. The case of $\lambda = x_1\omega_1 + \omega_2 + \omega_3$ is excluded as $m(x_1\psi_1 + \psi_2, \lambda) = 2$.

Corollary 5.7 We have the following $\mathfrak{g}$-module isomorphisms.

1. $M_{x_1\omega_1}(so(7), p(1,0,0)) \simeq M_{x_1\psi_1}(Lie G_2, p(1,0))$.

2. Suppose $x_1 \notin \{-1, -7/2, -6\}$. Then

$$
M_{x_1\omega_1 + \omega_2}(so(7), p(1,0,0)) \simeq M_{x_1\psi_1 + \psi_2}(Lie G_2, p(1,0))
$$

$$
\oplus M_{(x_1+1)\psi_1}(Lie G_2, p(1,0))
$$

$$
\oplus M_{(x_1-1)\psi_1 + \psi_2}(Lie G_2, p(1,0))
$$

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3. Suppose \( x_1 \notin \{-5, -3, -1\} \). Then

\[
M_{x_1\omega_1 + \omega_3} (so(7), p_{(1,0,0)}) \cong M_{(x_1+1)\psi_1} (\text{Lie } G_2, p_{(1,0)}) \\
\oplus M_{(x_1-1)\psi_1 + \psi_2} (\text{Lie } G_2, p_{(1,0)}) \\
\oplus M_{x_1\psi_1} (\text{Lie } G_2, p_{(1,0)}).
\]

4. Suppose \( x_1 \notin \{0, -1, -4, -3, -9/2, -2, -8, -6, -7, -5, -9\} \). Then

\[
M_{x_1\omega_1 + 2\omega_2} (so(7), p_{(1,0,0)}) \cong M_{x_1\psi_1 + 2\psi_2} (\text{Lie } G_2, p_{(1,0)}) \\
\oplus M_{(x_1+1)\psi_1 + \psi_2} (\text{Lie } G_2, p_{(1,0)}) \\
\oplus M_{(x_1+2)\psi_1} (\text{Lie } G_2, p_{(1,0)}) \\
\oplus M_{(x_1-1)\psi_1 + 2\psi_2} (\text{Lie } G_2, p_{(1,0)}) \\
\oplus M_{x_1\psi_1 + \psi_2} (\text{Lie } G_2, p_{(1,0)}) \\
\oplus M_{(x_1-2)\psi_1 + 2\psi_2} (\text{Lie } G_2, p_{(1,0)}).
\]

5. Suppose \( x_1 \notin \{-5, -3, -6, -4, -7/2, -1, -7, 0, -2\} \). Then

\[
M_{x_1\omega_1 + 2\omega_3} (so(7), p_{(1,0,0)}) \cong M_{(x_1+1)\psi_1} (\text{Lie } G_2, p_{(1,0)}) \\
\oplus M_{x_1\psi_1 + \psi_2} (\text{Lie } G_2, p_{(1,0)}) \\
\oplus M_{(x_1+1)\psi_1} (\text{Lie } G_2, p_{(1,0)}) \\
\oplus M_{(x_1-2)\psi_1 + 2\psi_2} (\text{Lie } G_2, p_{(1,0)}) \\
\oplus M_{(x_1-1)\psi_1 + \psi_2} (\text{Lie } G_2, p_{(1,0)}) \\
\oplus M_{x_1\psi_1} (\text{Lie } G_2, p_{(1,0)}).
\]

5.4 Parabolic subalgebras \( p \subset so(7) \) with non-finite branching problem over \( i(\text{Lie } G_2) \) and the case \( p \cong so(7) \)

For a parabolic subalgebra \( p \subset so(7) \), let \( \lambda(x, z) \) be as in Section 4. In the current Section, for each parabolic subalgebra \( p \subset so(7) \) other than \( p_{(1,0,0)} \), for \( z_1 + z_2 + z_3 \leq 5, \ z_i \in \mathbb{Z}_{\geq 0} \), we compute the strong Condition B (Definition 4.4). For \( z_1 + z_2 + z_3 \leq 2 \), for each \( \mu \in \mathfrak{h}^* \) with \( n(\mu, \lambda(x, z)) > 0 \) we construct \( n(\mu, \lambda(x, z)) \) \( \mathfrak{b} \)-singular vectors in \( M_{\lambda(x, z)} (so(7), p) \) of weight \( \mu \).

As it turns out, the constructed \( \mathfrak{b} \)-singular vectors remain \( \mathfrak{b} \)-singular independent of the strong Condition B, however they may fail to generate the “top level” Lie \( G_2 \)-module \( Q_{\lambda(x, z)} \) defined by (21).

By direct observation, for any parabolic subalgebra of \( so(7) \) and \( \lambda(x, z) \) with \( z_1 + z_2 + z_3 \leq 2 \), the set of weights \( \lambda(x, z) \) satisfying the strong Condition B (Definition 4.4) is non-empty. Therefore, if we change the base field to \( \mathbb{C}(x_1, x_2, x_3) \), the strong Condition B never fails, as in the field \( \mathbb{C}(x_1, x_2, x_3) \) a non-trivial \( \mathbb{Q} \)-linear combination of the \( x_i \) is non-zero by definition. Therefore
for \( z_1 + z_2 + z_3 \leq 2 \), Theorem 4.5 applies over \( \mathbb{C}(x_1, x_2, x_3) \), and specializations of the variable \( x_i \) need to only exclude a proper Zariski-open set for the \( x_i \)’s. We present the computation of the \( \mathfrak{b} \)-singular vectors in \( M_{\lambda(x, z)}(so(7), p) \), \( z_1 + z_2 + z_3 \leq 2 \) over the field \( \mathbb{C}(x_1, x_2, x_3) \) in Tables 10-13 (for \( p \simeq p_{(1,0,0)} \) the result is already given in Theorem 5.5). The computations use Sections 3 and 4 in a similar fashion to Theorem 5.5 and we do not present details. We note that by direct observation, the \( \mathfrak{b} \)-singular vectors in the tables remain linearly independent under substitutions of the variables \( x_1, x_2, x_3 \) with constants.

The linear \( \neq \)-inequalities that determine the strong Condition B (over the field \( \mathbb{C} \)) are computed explicitly in Tables 14-19 (including the case \( p \simeq p_{(1,0,0)} \)).

In addition we compute the numbers \( n(\mu, \lambda(x, z)) \) for \( z_1 + z_2 + z_3 \leq 5 \) with coefficients in \( \mathbb{C}(x_1, x_2, x_3) \), as described in Section 4.

We supplement the infinite dimensional branching by Table 8, where we list the \( \mathfrak{b} \)-singular vectors that decompose the finite dimensional \( so(7) \)-modules \( z_1\omega_1 + z_2\omega_2 + z_3\omega_3 \) with \( z_1 + z_2 + z_3 \leq 2 \) over \( \text{Lie } G_2 \). The numbers \( n(\lambda, \mu) \) for the finite dimensional branching can be computed using [16] and we omit the corresponding table.

### 5.4.1 Tables of \( \mathfrak{b} \)-singular vectors induced from \( V_\lambda(I) \)

In the first and second columns we write the module \( V_\lambda(I) \), abbreviated as \( V_\lambda \), and its dimension. The weight \( \lambda \in h^* \) is taken with coefficients in \( \mathbb{C}(x_1, x_2, x_3) \). In the third and fourth columns we list the \( I \)-summands from the decomposition of \( V_\lambda(I) \) over \( I \) and their dimensions. We abbreviate \( V_\mu(I) \) as \( V_\mu \). In the fifth column we give an \( \mathfrak{b} \cap I \)-singular vector corresponding to each summand \( V_\lambda \). In the sixth column we give the Casimir projector whose existence is given by Theorem 4.5 and in the last column we give the corresponding \( \mathfrak{b} \)-singular vector as given by Theorem 4.5. We note that for the case of the full parabolic subalgebra \( p \simeq so(7) \) the last two columns are not applicable. We also note that the vectors listed in the last column remain \( \mathfrak{b} \)-singular independent of the strong Condition B.

| so(7)  | dim. | Lie \( G_2 \) dim. | b-singular vectors |
|--------|------|---------------------|--------------------|
| \( V_0 \) | 1 | \( V_0 \) 1 | \( v_\lambda \) |
| \( V_{\omega_3} \) | 8 | \( V_0 \) 1 | \( g^{-1}g^{-2g-3} \cdot v_\lambda \) |
| & | \( V_{\omega_1} \) 7 | \( v_\lambda \) |
| \( V_{\omega_2} \) | 21 | \( V_{\psi_1} \) 7 | \( g^{-1}g^{-2} \cdot v_\lambda \) |
| & | \( V_{\omega_2} \) 14 | \( v_\lambda \) |
| \( V_{\omega_3} \) | 7 | \( V_{\psi_1} \) 7 | \( v_\lambda \) |
Table 8: Decompositions of finite dimensional $so(7)$-modules over Lie $G_2$

| so(7) | dim. | Lie $G_2$ | dim. | b-singular vectors |
|-------|------|------------|------|-------------------|
| $V_{2\omega_3}$ | 35   | $V_0$       | 1    | $g^2/1g^2/2g^2/3 \cdot v_\lambda$
|       |      |             |      | $-g^2/1g^2/3 \cdot v_\lambda$
|       |      |             |      | $+2g^2/1g^2/2g^2/3 \cdot v_\lambda$
|       |      |             |      | $-2g^2/1g^2/3_1g^2/2g^2/3 \cdot v_\lambda$
|       |      |             |      | $+2g^2/1g^2/3_1g^2/2g^2/3 \cdot v_\lambda$
| $V_{\psi_1}$ | 7    | $g^2/1g^2/2g^2/3 \cdot v_\lambda$
|       |      |             |      | $-g^2/1g^2/3 \cdot v_\lambda$
|       |      |             |      | $+1/2g^2/1g^2/2g^2/3 \cdot v_\lambda$
|       |      |             |      | $-g^2/1g^2/3_1g^2/2g^2/3 \cdot v_\lambda$
| $V_{\psi_1}$ | 27   | $v_\lambda$
| $V_{\omega_2+\omega_3}$ | 112  | $V_{\psi_1}$ | 7    | $g^2/1g^2/2g^2/3 \cdot v_\lambda$
|       |      |             |      | $-4/5g^2/1g^2/3g^2/3 \cdot v_\lambda$
|       |      |             |      | $+2/5g^2/1g^2/2g^2/3 \cdot v_\lambda$
|       |      |             |      | $-4/5g^2/1g^2/3_1g^2/2g^2/3 \cdot v_\lambda$
|       |      |             |      | $+2/5g^2/1g^2/3_1g^2/2g^2/3 \cdot v_\lambda$
|       |      |             |      | $-2/15g^2/1g^2/3_1g^2/2g^2/3 \cdot v_\lambda$
| $V_{\psi_2}$ | 14   | $g^2/1g^2/2g^2/3 \cdot v_\lambda$
|       |      |             |      | $-g^2/1g^2/3 \cdot v_\lambda$
|       |      |             |      | $-g^2/1g^2/3_1g^2/2g^2/3 \cdot v_\lambda$
|       |      |             |      | $+3g^2/1g^2/2g^2/3 \cdot v_\lambda$
| $V_{2\psi_1}$ | 27   | $g^2/1g^2/2g^2/3 \cdot v_\lambda$
|       |      |             |      | $+1/2g^2/1g^2/2g^2/3 \cdot v_\lambda$
| $V_{\psi_1+\psi_2}$ | 64   | $v_\lambda$
| $V_{2\omega_2}$ | 168  | $V_{2\psi_1}$ | 27   | $g^2/1g^2/2g^2/3 \cdot v_\lambda$
|       |      |             |      | $-2/3g^2/1g^2/3g^2/3 \cdot v_\lambda$
|       |      |             |      | $+2/3g^2/1g^2/2g^2/3 \cdot v_\lambda$
|       |      |             |      | $+1/2g^2/1g^2/2g^2/3 \cdot v_\lambda$
|       |      |             |      | $-2/15g^2/1g^2/3_1g^2/2g^2/3 \cdot v_\lambda$
| $V_{\psi_1+\psi_2}$ | 64   | $g^2/1g^2/2g^2/3 \cdot v_\lambda$
|       |      |             |      | $+1/2g^2/1g^2/2g^2/3 \cdot v_\lambda$
| $V_{2\psi_2}$ | 77   | $v_\lambda$
| $V_{\omega_1+\omega_3}$ | 48   | $V_{\psi_1}$ | 7    | $g^2/1g^2/2g^2/3 \cdot v_\lambda$
|       |      |             |      | $-g^2/1g^2/3g^2/3 \cdot v_\lambda$
|       |      |             |      | $+5g^2/1g^2/2g^2/3 \cdot v_\lambda$
|       |      |             |      | $-7g^2/1g^2/3g^2/3 \cdot v_\lambda$
| $V_{\psi_2}$ | 14   | $g^2/1g^2/2g^2/3 \cdot v_\lambda$
|       |      |             |      | $-g^2/1g^2/3g^2/3 \cdot v_\lambda$
| $V_{2\psi_3}$ | 27   | $v_\lambda$
| $V_{\omega_1+\omega_2}$ | 105  | $V_{\psi_1}$ | 7    | $g^2/1g^2/2g^2/3 \cdot v_\lambda$
|       |      |             |      | $-g^2/1g^2/3g^2/3 \cdot v_\lambda$
|       |      |             |      | $+g^2/1g^2/2g^2/3 \cdot v_\lambda$
Table 8: Decompositions of finite dimensional $so(7)$-modules over Lie $G_2$

| $so(7)$ | dim. | Lie $G_2$ | dim. | b-singular vectors |
|---------|------|-----------|------|--------------------|
| $V_{2\psi_1}$ | 27 | | | $-g_{-2}g_{-1} \cdot v_\lambda$
| | | | | $-2g_{-1}g_{-2} \cdot v_\lambda$
| | | | | $+3/2g_{-3}g_{-2} \cdot v_\lambda$
| $V_{\psi_1+\psi_2}$ | 64 | | | $v_\lambda$
| $V_{2\omega_1}$ | 27 | $V_{2\psi_1}$ | 27 | $v_\lambda$ |
Table 9: \(\mathfrak{b}\)-singular vectors in \(M(\mathfrak{so}(7), \mathfrak{p}(0,1,0))\), corresponding to \(V_\lambda(l)\), where \(l\) is the reductive Levi part of \(\mathfrak{p}(0,1,0)\). \(\mathfrak{I}\) denotes the reductive Levi part of \(\mathfrak{p}(0,1) \subset \text{Lie } G_2\).

| \(V_\lambda(l)\) | dim. l-decomp. dim. | \(b \cap l\)-singular vectors | Casimir projector | Corresp. \(\mathfrak{b}\)-singular vectors |
|-----------------|-----------------|----------------------------|-----------------|-----------------|
| \(V_{x_2\omega_2}\) | 1 | \(V_{x_2\psi_2}\) | 1 | \(-v_\lambda\) | \(id\) | \(-v_\lambda\) |
| \(V_{x_2\omega_2 + \omega_3}\) | 2 | \(V_{\psi_1 + x_2\psi_2}\) | 2 | \(-v_\lambda\) | \(id\) | \(-v_\lambda\) |
| \(V_{x_1 + x_2\omega_2}\) | 2 | \(V_{\psi_1 + x_2\psi_2}\) | 2 | \(-v_\lambda\) | \(id\) | \(-v_\lambda\) |
| \(V_{x_2\omega_2 + 2\omega_3}\) | 3 | \(V_{\psi_1 + x_2\psi_2}\) | 3 | \(-v_\lambda\) | \(id\) | \(-v_\lambda\) |
| \(V_{\omega_1 + x_2\omega_2 + \omega_3}\) | 4 | \(V_{(x_2+1)\psi_2}\) | 1 | \(-g_{-1} \cdot v_\lambda\) | \(12(\iota(\bar{c}_1) - (1/4x_2^2 + 5/4x_2 - 2g_{-1} \cdot v_\lambda + 7/6))\) | \(-2g_{-1} \cdot v_\lambda\) |
| | | \(V_{\psi_1 + x_2\psi_2}\) | 3 | \(-v_\lambda\) | \(id\) | \(-v_\lambda\) |
| \(V_{2\omega_1 + x_2\omega_2}\) | 3 | \(V_{2\psi_1 + x_2\psi_2}\) | 3 | \(-v_\lambda\) | \(id\) | \(-v_\lambda\) |

Table 10: \(\mathfrak{b}\)-singular vectors in \(M(\mathfrak{so}(7), \mathfrak{p}(0,0,1))\), corresponding to \(V_\lambda(l)\), where \(l\) is the reductive Levi part of \(\mathfrak{p}(0,0,1)\). \(\mathfrak{I}\) denotes the reductive Levi part of \(\mathfrak{p}(1,0) \subset \text{Lie } G_2\).

| \(V_\lambda(l)\) | dim. l-decomp. dim. | \(b \cap l\)-singular vectors | Casimir projector | Corresp. \(\mathfrak{b}\)-singular vectors |
|-----------------|-----------------|----------------------------|-----------------|-----------------|
| \(V_{x_3\omega_3}\) | 1 | \(V_{x_3\psi_1}\) | 1 | \(-v_\lambda\) | \(id\) | \(-v_\lambda\) |
| \(V_{x_2 + x_3\omega_3}\) | 3 | \(V_{(x_3+1)\psi_1}\) | 1 | \(-g_{-1}g_{-2} \cdot v_\lambda\) | \(12(\iota(\bar{c}_1) - (1/12x_3^2 + 2/3x_3 + 1))\) | \(-g_{-5}v_\lambda - g_{-3}g_{-2} \cdot v_\lambda\) |
| | | \(V_{x_3\psi_1 + \psi_2}\) | 2 | \(-v_\lambda\) | \(id\) | \(-v_\lambda\) |
| \(V_{x_1 + x_3\omega_3}\) | 3 | \(V_{(x_3-1)\psi_1 + \psi_2}\) | 2 | \(-g_{-1} \cdot v_\lambda\) | \(12(\iota(\bar{c}_1) - (1/12x_3^2 + 7/12x_3 + 1/2))\) | \(-g_{-3}v_\lambda + x_3g_{-1} \cdot v_\lambda\) |
| | | \(V_{(x_3+1)\psi_1}\) | 1 | \(-v_\lambda\) | \(id\) | \(-v_\lambda\) |
Table 10: $\mathfrak{b}$-singular vectors in $M_{\lambda}(so(7), p_{(0,0,1)})$, corresponding to $V_{\lambda}(l)$, where $l$ is the reductive Levi part of $p_{(0,0,1)}$. $\bar{l}$ denotes the reductive Levi part of $\bar{p}_{(1,0)} \subset \text{Lie } G_2$.

| $V_{\lambda}(l)$ | dim. | $l$-decomp. | dim. | $\mathfrak{b} \cap l$-singular vectors | Casimir projector | Corresp. $\mathfrak{b}$-singular vectors |
|------------------|------|-------------|------|----------------------------------------|------------------|---------------------------------------|
| $V_{2\omega_2 + x_3 \omega_3}$ | 6    | $V_{(x_3+2)\psi_1}$ | 1    | $-g_{-2}^2 g_{-2} \cdot v_{\lambda}$ | $12(i(\bar{c}_1) - (1/12 x_3^2 + 5/6 x_3 + 7/4))$ | $-8g_{-25} v_{\lambda}$ |
|                  |      | $V_{(x_3+1)\psi_1 + \psi_2}$ | 2    | $-g_{-1} g_{-2} \cdot v_{\lambda}$ | $12(i(\bar{c}_1) - (1/12 x_3^2 + 11/12 x_3 + 5/2))$ | $-2g_{-5} v_{\lambda} - g_{-3} g_{-2} \cdot v_{\lambda}$ |
|                  |      | $V_{x_3 \psi_1 + 2\psi_2}$ | 3    | $-v_{\lambda}$ | $id$ | $-v_{\lambda}$ |
| $V_{\omega_1 + \omega_2 + x_3 \omega_3}$ | 8    | $V_{x_3 \psi_1 + \psi_2}$ | 2    | $-g_{-2}^2 g_{-2} \cdot v_{\lambda}$ | $12(i(\bar{c}_1) - (1/12 x_3^2 + 3/4 x_3 + 7/6))$ | $(4x_3 + 26)g_{-5} v_{\lambda}$ |
|                  |      | $V_{(x_3+2)\psi_1}$ | 1    | $-g_{-2} g_{-1} \cdot v_{\lambda}$ | $12(i(\bar{c}_1) - (1/12 x_3^2 + 3/4 x_3 + 5/3))$ | $-18g_{-6} v_{\lambda} + (-4x_3 - 44)g_{-7} v_{\lambda}$ |
|                  |      | $V_{(x_3+1)\psi_1 + 2\psi_2}$ | 3    | $-g_{-1} \cdot v_{\lambda}$ | $12(i(\bar{c}_1) - (1/12 x_3^2 + 5/6 x_3 + 7/4))$ | $-18g_{-5} v_{\lambda} - 18g_{-25} v_{\lambda}$ |
|                  |      | $V_{x_3 \psi_1 + 2\psi_2}$ | 3    | $-v_{\lambda}$ | $id$ | $-v_{\lambda}$ |
Table 10: $\mathfrak{b}$-singular vectors in $M_\lambda(s\mathfrak{o}(7), p_{(0, 0, 1)})$, corresponding to $V_\lambda(\mathfrak{l})$, where $\mathfrak{l}$ is the reductive Levi part of $p_{(0, 0, 1)}$. $\bar{\mathfrak{l}}$ denotes the reductive Levi part of $\bar{p}_{(1, 0)} \subset \text{Lie } G_2$.

| $V_\lambda(\mathfrak{l})$ | dim. | $\mathfrak{t}$-decomp. | dim. | $\mathfrak{b} \cap \mathfrak{t}$-singular vectors | Casimir projector | Corresp. $\mathfrak{b}$-singular vectors |
|--------------------------|------|------------------------|------|---------------------------------|------------------|---------------------------------|
| $V_{(x_3+1)\psi_1+\psi_2}$ | 2 | $-v_\lambda$ | | | $-v_\lambda$ | |
| $V_{(x_3-2)\psi_1+2\psi_2}$ | 3 | $-g_{-1}^2 \cdot v_\lambda$ | 12$(i(\hat{c}_1) - (1/12x_3^2 + 2/3x_3 + 1))12(i(\hat{c}_1) - (1/12x_3^2 + 3/4x_3 + 7/6))$ | $12(i(\hat{c}_1) - (1/12x_3^2 + 2/3x_3 + 1))12(i(\hat{c}_1) - (1/12x_3^2 + 3/4x_3 + 7/6))$ | $-4g_{-3}^2 v_\lambda$ | $(4x_3 - 4)g_{-3}g_{-1} \cdot v_\lambda$ | $(2x_3^2 + 2x_3)g_{-1}^2 \cdot v_\lambda$ |
| $V_{x_3\psi_1+\psi_2}$ | 2 | $-g_{-1} \cdot v_\lambda$ | | | | |
| $V_{(x_3+2)\psi_1}$ | 1 | $-v_\lambda$ | | | $-v_\lambda$ | |

Table 11: $\mathfrak{b}$-singular vectors in $M_\lambda(s\mathfrak{o}(7), p_{(1, 1, 0)})$, corresponding to $V_\lambda(\mathfrak{l})$, where $\mathfrak{l}$ is the reductive Levi part of $p_{(1, 1, 0)}$. $\bar{\mathfrak{l}}$ denotes the reductive Levi part of $\bar{p}_{(1, 1)} \subset \text{Lie } G_2$.

| $V_\lambda(\mathfrak{l})$ | dim. | $\mathfrak{t}$-decomp. | dim. | $\mathfrak{b} \cap \mathfrak{t}$-singular vectors | Casimir projector | Corresp. $\mathfrak{b}$-singular vectors |
|--------------------------|------|------------------------|------|---------------------------------|------------------|---------------------------------|
| $V_{x_1\omega_1+x_2\omega_2}$ | 1 | $V_{x_1\psi_1+x_2\psi_2}$ | 1 | $-v_\lambda$ | | $-v_\lambda$ |
| $V_{x_1\omega_1+x_2\omega_2+x_3\omega_3}$ | 2 | $V_{(x_1-1)\psi_1+(x_2+1)\psi_2}$ | 1 | $-g_{-3} \cdot v_\lambda$ | $12(i(\hat{c}_1) - (1/4x_2^2 + 1/4x_1x_2 + x_2 + 1/12x_2^2 + 7/12x_1 + 1/2))$ | $-g_{-1}v_\lambda + x_1g_{-3} \cdot v_\lambda$ |
| $V_{(x_1+1)\psi_1+x_2\psi_2}$ | 1 | $-v_\lambda$ | | | $-v_\lambda$ | |
Table 11: $\bar{b}$-singular vectors in $M_\lambda(\mathfrak{so}(7),\mathfrak{p}_{(1,1,0)})$, corresponding to $V_\lambda(l)$, where $l$ is the reductive Levi part of $\mathfrak{p}_{(1,1,0)}$. $\bar{l}$ denotes the reductive Levi part of $\bar{\mathfrak{p}}_{(1,1)}^{(0,1)} \subset \text{Lie } G_2$.

| $V_\lambda(l)$ | dim. | 1-decomp. | dim. $\mathfrak{b} \cap 1$-singular vectors | Casimir projector | Corresp. $\mathfrak{b}$-singular vectors |
|----------------|------|-----------|---------------------------------------------|------------------|---------------------------------------|
| $V_{x_1 \psi_1 + x_2 \psi_2}$ 3 | $V_{(x_1-2)\psi_1 + (x_2+2)\psi_2}$ 1 | $-g^2_{-3} \cdot v_\lambda$ | $12((\bar{c}_1) - (1/4x_1^2 + 1/4x_1x_2)$ | $-4g^2_{-3}v_\lambda$ |
| | | | $+5/4x_2 + 1/12x_1^2$ | $+(4x_1 - 4)g_{-1}g_{-3} \cdot v_\lambda$ |
| | | | $+2/3x_1 + 1))$ | $+(−2x_1^2 + 2x_1)g^2_{-3} \cdot v_\lambda$ |

Table 12: $\mathfrak{b}$-singular vectors in $M_\lambda(\mathfrak{so}(7),\mathfrak{p}_{(1,0,1)})$, corresponding to $V_\lambda(l)$, where $l$ is the reductive Levi part of $\mathfrak{p}_{(1,0,1)}$. $\bar{l}$ denotes the reductive Levi part of $\bar{\mathfrak{p}}_{(0,1)}^{(1,0)} \subset \text{Lie } G_2$.

| $V_\lambda(l)$ | dim. | 1-decomp. | dim. $\mathfrak{b} \cap 1$-singular vectors | Casimir projector | Corresp. $\mathfrak{b}$-singular vectors |
|----------------|------|-----------|---------------------------------------------|------------------|---------------------------------------|
| $V_{x_1 \psi_1 + x_2 \psi_2}$ 1 | $V_{(x_1+1)\psi_1}$ 1 | $-v_\lambda$ | $id$ | $-v_\lambda$ |
| $V_{x_1 \omega_1 + x_2 \omega_2 + x_3 \omega_3}$ 2 | $V_{(x_1+1)\psi_1 + \omega_2}$ 2 | $-v_\lambda$ | $id$ | $-v_\lambda$ |
| $V_{x_1 \omega_1 + 2x_2 \omega_2 + x_3 \omega_3}$ 3 | $V_{(x_1+1)\psi_1 + 2\psi_2}$ 3 | $-v_\lambda$ | $id$ | $-v_\lambda$ |
Table 13: $\mathfrak{b}$-singular vectors in $M_\lambda(\mathfrak{so}(7),\mathfrak{p}_{(0,1,1)})$, corresponding to $V_\lambda(\mathfrak{l})$, where $\mathfrak{l}$ is the reductive Levi part of $\mathfrak{p}_{(0,1,1)}$. $\mathfrak{l}$ denotes the reductive Levi part of $\mathfrak{p}_{(1,1)} \subset \text{Lie } G_2$.

| $V_\lambda(\mathfrak{l})$ | dim. | $\mathfrak{l}$-decomp. dim. | $\mathfrak{b} \cap \mathfrak{l}$-singular vectors | Casimir projector | Corresp. $\mathfrak{b}$-singular vectors |
|--------------------------|------|-----------------|---------------------------------|-----------------|---------------------------------|
| $V_{x_2\omega_2 + x_3\omega_3}$ | 1 | $V_{x_2\psi_1 + x_2\psi_2}$ | 1 | $-v_\lambda$ | 12$(\mathfrak{i}(\mathfrak{c}_1) - (1/12x_3^2 + 1/4x_2x_3)$ +7/12$x_3 + 1/4x_2^2$ +x_2 + 1/2$)$ | $-g_3v_\lambda + x_3g_{-1} \cdot v_\lambda$ |
| $V_{\omega_1 + x_2\omega_2 + x_3\omega_3}$ | 2 | $V_{(x_3-1)\psi_1 + (x_2+1)\psi_2}$ | 1 | $-g_{-1} \cdot v_\lambda$ | 12$(\mathfrak{i}(\mathfrak{c}_1) - (1/12x_3^2 + 1/4x_2x_3)$ +2/3$x_3 + 1/4x_2^2$ +5/4$x_2 + 1)$ | $-4g_{-1}^2v_\lambda$ |
| | | $V_{(x_3+1)\psi_1 + x_2\psi_2}$ | 1 | $-v_\lambda$ | $id$ | $-v_\lambda$ |
| $V_{2(\omega_1 + x_2\omega_2 + x_3\omega_3)}$ | 3 | $V_{(x_3-2)\psi_1 + (x_2+2)\psi_2}$ | 1 | $-g_{-1}^2 \cdot v_\lambda$ | 12$(\mathfrak{i}(\mathfrak{c}_1) - (1/12x_3^2 + 1/4x_2x_3)$ +3/4$x_3 + 1/4x_2^2$ +5/4$x_2 + 7/6)$ | $+(4x_3 - 4)g_{-3}g_{-1} \cdot v_\lambda$ |
| | | $V_{x_3\psi_1 + (x_2+1)\psi_2}$ | 1 | $-g_{-1} \cdot v_\lambda$ | 12$(\mathfrak{i}(\mathfrak{c}_1) - (1/12x_3^2 + 1/4x_2x_3)$ +3/4$x_3 + 1/4x_2^2$ +5/4$x_2 + 7/6)$ | $-2g_{-3}v_\lambda + x_3g_{-1} \cdot v_\lambda$ |
| | | $V_{(x_3+2)\psi_1 + x_2\psi_2}$ | 1 | $-v_\lambda$ | $id$ | $-v_\lambda$ |
5.4.2 Tables of $\mathfrak{n}(\mu, \lambda)$

In the left column we write the module $V_\lambda(I)$, abbreviated as $V_\lambda$. The weight $\lambda \in \mathfrak{h}^*$ is taken with coefficients in $\mathbb{C}(x_1, x_2, x_3)$. In the right column we write the decomposition of $V_\lambda(I)$ as a direct sum of $V_\mu(I)$-modules, $\mu \in \mathfrak{h}^*$. For $\mu \in \mathfrak{h}^*$ we abbreviate $V_\mu(I)$ by $V_\mu$. In the row below each pair of columns, we indicate, for each $\lambda \in \mathfrak{h}^*$ with $\mathbb{C}(x_1, x_2, x_3)$, the Zariski open conditions on the $x_i$’s which imply the strong Condition B. We recall that $\psi_1, \psi_2$ stand for the fundamental weights of Lie $G_2$ and $\omega_1, \omega_2, \omega_3$ stand for the fundamental weights of $so(7)$.

Table 14: Decompositions of inducing $\mathfrak{p}(1,0,0)$-modules over $I$, where $I$ is the reductive Levi part of $\mathfrak{p}(1,0,0)$ and $I$ is the reductive Levi part of $\mathfrak{p}(1,0)$.

| $V_\lambda(I)$ | Decomposition over $I$ |
|----------------|------------------------|
| $V_{x_1 \omega_1 + \omega_3}$ | $V_{x_1 \psi_1}$ + $V_{(x_1+1)\psi_1}$ + $V_{(x_1-1)\psi_1 + \psi_2}$ + $V_{x_1 \psi_1}$ |
| $V_{x_2 \omega_1 + \omega_2}$ | Strong Condition B: $(-x_1 - 5) \neq 0, (-x_1 - 3) \neq 0, (-x_1 - 1) \neq 0$ |
| $V_{x_3 \omega_1 + 2\omega_3}$ | $V_{x_1 \psi_1 + \psi_2}$ + $V_{(x_1+1)\psi_1}$ + $V_{(x_1+1)\psi_1}$ + $V_{(x_1-1)\psi_1 + \psi_2}$ + $V_{x_1 \psi_1}$ |
| $V_{x_4 \omega_1 + \omega_2 + \omega_3}$ | Strong Condition B: $(-x_1 - 6) \neq 0, (-x_1 - 1) \neq 0, (-x_1 - 1) \neq 0$ |
| $V_{x_5 \omega_1 + 2\omega_3}$ | $V_{x_1 \psi_1 + \psi_2} + 2V_{x_1 \psi_1 + \psi_2} + V_{(x_1+1)\psi_1}$ + $V_{(x_1+1)\psi_1}$ + $V_{(x_1-1)\psi_1 + \psi_2}$ + $V_{x_1 \psi_1}$ |
| $V_{x_6 \omega_1 + 3\omega_3}$ | Strong Condition B: $(-x_1 - 9) \neq 0, (-x_1 - 4) \neq 0, (-x_1 - 3) \neq 0, (-x_1 - 1) \neq 0, (-x_1 - 1) \neq 0$ |
| $V_{x_7 \omega_1 + 2\omega_3}$ | $V_{x_1 \psi_1 + \psi_2} + V_{x_1 \psi_1 + \psi_2} + V_{(x_1+1)\psi_1}$ + $V_{(x_1+1)\psi_1}$ + $V_{(x_1+1)\psi_1}$ + $V_{x_1 \psi_1}$ |
| $V_{x_8 \omega_1 + 3\omega_3}$ | Strong Condition B: $(-x_1 - 9) \neq 0, (-x_1 - 4) \neq 0, (-x_1 - 3) \neq 0, (-x_1 - 1) \neq 0, (-x_1 - 1) \neq 0$ |
| $V_{x_9 \omega_1 + \omega_2}$ | $V_{x_1 \psi_1 + \psi_2}$ + $V_{x_1 \psi_1 + \psi_2}$ + $2V_{x_1 \psi_1 + \psi_2}$ + $2V_{x_1 \psi_1 + \psi_2}$ + $2V_{x_1 \psi_1 + \psi_2}$ + $2V_{x_1 \psi_1 + \psi_2}$ |
| $V_{x_{10} \omega_1 + 2\omega_3}$ | Strong Condition B: $(-x_1 - 9) \neq 0, (-x_1 - 4) \neq 0, (-x_1 - 3) \neq 0, (-x_1 - 1) \neq 0, (-x_1 - 1) \neq 0$ |

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Table 14: Decompositions of inducing $\mathfrak{p}_{(1,0,0)}$-modules over $\overline{I}$, where $I$ is the reductive Levi part of $\mathfrak{p}_{(1,0,0)}$ and $\overline{I}$ is the reductive Levi part of $\mathfrak{p}_{(1,0)}$.

| $V_{\lambda}(I)$ | Decomposition over $I$ |
|------------------|-------------------------|
| Strong Condition B: $(r_1 - 10) \neq 0$, $(r_1 - 5) \neq 0$, $(r_1 - 2) \neq 0$, $(r_1 - 4) \neq 0$, $(r_1 - 1) \neq 0$, $(-r_1 - 9) \neq 0$, $(r_1 - 3) \neq 0$, $(-r_2 - 11) \neq 0$, $(r_1 - 7) \neq 0$, $(r_1 - 9) \neq 0$, $(r_1 - 15) \neq 0$, $(-r_5 - 26) \neq 0$, $(-r_4 - 25) \neq 0$, $(-r_4 - 19) \neq 0$, $(-r_4 - 7) \neq 0$, $(-r_4 - 11) \neq 0$, $(-r_5 - 19) \neq 0$, $(-r_5 - 10) \neq 0$, $(r_1 - 11) \neq 0$, $(r_1 - 6) \neq 0$, $(r_1 - 8) \neq 0$, $(-r_2 - 3) \neq 0$, $(-r_4 - 12) \neq 0$, $(-r_4 - 7) \neq 0$, $(-r_4 - 11) \neq 0$, $(r_1 - 17) \neq 0$, $(-r_5 - 23) \neq 0$, $(-r_5 - 29) \neq 0$, $(-r_5 - 33) \neq 0$, $(r_1 - 4) \neq 0$, $(r_1 - 5) \neq 0$, $(-r_2 - 13) \neq 0$, $(r_1 - 10) \neq 0$, $(r_1 - 7) \neq 0$, $(-r_5 - 34) \neq 0$, $(r_1 - 2) \neq 0$, $(r_1 - 9) \neq 0$, $(r_1 - 3) \neq 0$, $(-r_2 - 9) \neq 0$, $(r_1 + 1) \neq 0$, $(-r_5 - 11) \neq 0$, $(-r_4 - 33) \neq 0$, $(r_1 - 2) \neq 0$, $(r_1 + 2) \neq 0$, $(r_1 + 3) \neq 0$, $(-r_2 - 11) \neq 0$, $(-r_4 - 10) \neq 0$, $(r_1 - 15) \neq 0$, $(r_1 - 8) \neq 0$, $(r_1 - 7) \neq 0$, $(-r_5 - 26) \neq 0$, $(-r_4 - 25) \neq 0$, $(-r_5 - 19) \neq 0$, $(-r_4 - 7) \neq 0$, $(-r_4 - 31) \neq 0$, $(r_1 - 20) \neq 0$, $(-r_5 - 24) \neq 0$, $(-r_4 - 11) \neq 0$, $(-r_5 - 21) \neq 0$, $(-r_4 - 32) \neq 0$, $(-r_1 - 16) \neq 0$ |

| $V_{\lambda}(I)$ | Decomposition over $\overline{I}$ |
|------------------|-------------------------|
| $V_{x_1 \omega_1 + 2 \omega_2 + \omega_3}$ | $2V_{x_1 \psi_1} + 2V_{x_1 \psi_1 + \psi_2} + V_{x_1 \psi_1 + \psi_2} + V_{x_1 \psi_1 + 2 \psi_2}$ |
| Strong Condition B: $(r_1 - 10) \neq 0$, $(r_1 - 5) \neq 0$, $(r_1 - 2) \neq 0$, $(r_1 - 4) \neq 0$, $(r_1 - 1) \neq 0$, $(-r_1 - 9) \neq 0$, $(r_1 - 3) \neq 0$, $(-r_2 - 11) \neq 0$, $(r_1 - 7) \neq 0$, $(r_1 - 9) \neq 0$, $(r_1 - 15) \neq 0$, $(-r_5 - 26) \neq 0$, $(-r_4 - 25) \neq 0$, $(-r_4 - 19) \neq 0$, $(-r_4 - 7) \neq 0$, $(-r_4 - 11) \neq 0$, $(r_1 - 11) \neq 0$, $(r_1 - 6) \neq 0$, $(r_1 - 8) \neq 0$, $(-r_2 - 3) \neq 0$, $(-r_4 - 12) \neq 0$, $(-r_4 - 7) \neq 0$, $(-r_4 - 11) \neq 0$, $(r_1 - 17) \neq 0$, $(-r_5 - 23) \neq 0$, $(-r_5 - 29) \neq 0$, $(-r_5 - 33) \neq 0$, $(r_1 - 2) \neq 0$, $(r_1 - 9) \neq 0$, $(r_1 - 3) \neq 0$, $(-r_2 - 9) \neq 0$, $(r_1 + 1) \neq 0$, $(-r_5 - 11) \neq 0$, $(-r_4 - 33) \neq 0$, $(r_1 - 2) \neq 0$, $(r_1 + 2) \neq 0$, $(r_1 + 3) \neq 0$, $(-r_2 - 11) \neq 0$, $(-r_4 - 10) \neq 0$, $(r_1 - 15) \neq 0$, $(r_1 - 8) \neq 0$, $(r_1 - 7) \neq 0$, $(-r_5 - 26) \neq 0$, $(-r_4 - 25) \neq 0$, $(-r_5 - 19) \neq 0$, $(-r_4 - 7) \neq 0$, $(-r_4 - 31) \neq 0$, $(r_1 - 20) \neq 0$, $(-r_5 - 24) \neq 0$, $(-r_4 - 11) \neq 0$, $(-r_5 - 21) \neq 0$, $(-r_4 - 32) \neq 0$, $(-r_1 - 16) \neq 0$ |
| Table 14: Decompositions of inducing $p_{(1,0)}$-modules over $\overline{l}$, where $l$ is the reductive Levi part of $p_{(1,0)}$ and $\overline{l}$ is the reductive Levi part of $p_{(1,0)}$. |
|-----------------|-----------------|
| $V_{\lambda}(l)$ | Decomposition over $l$ |
| Strong Condition B: ($x_1 + \frac{1}{2}) \neq 0$, ($x_1 - \frac{1}{2}) \neq 0$, ($x_1 - \frac{1}{4}) \neq 0$, ($x_1 - \frac{3}{4}) \neq 0$, ($x_1 - \frac{5}{4}) \neq 0$, ($x_1 - \frac{7}{4}) \neq 0$, ($x_1 - \frac{9}{4}) \neq 0$, ($x_1 - \frac{11}{4}) \neq 0$, ($x_1 - \frac{13}{4}) \neq 0$, ($x_1 - \frac{15}{4}) \neq 0$, ($x_1 - \frac{17}{4}) \neq 0$ |
| $V_{x_1\omega_1+2\omega_2+2\omega_3}$ | $V_{(x_1-3)\psi_1+4\psi_2} \oplus V_{x_1\psi_1+3\psi_2} \oplus 3V_{x_1\psi_1+2\psi_2}$ |
| Strong Condition B: ($x_1 + \frac{1}{2}) \neq 0$, ($x_1 - \frac{1}{2}) \neq 0$, ($x_1 - \frac{1}{4}) \neq 0$, ($x_1 - \frac{3}{4}) \neq 0$, ($x_1 - \frac{5}{4}) \neq 0$, ($x_1 - \frac{7}{4}) \neq 0$, ($x_1 - \frac{9}{4}) \neq 0$, ($x_1 - \frac{11}{4}) \neq 0$, ($x_1 - \frac{13}{4}) \neq 0$, ($x_1 - \frac{15}{4}) \neq 0$, ($x_1 - \frac{17}{4}) \neq 0$ |
| $V_{x_1\omega_1+3\omega_2+2\omega_3}$ | $V_{(x_1-3)\psi_1+4\psi_2} \oplus 2V_{x_1\psi_1+3\psi_2} \oplus 2V_{x_1\psi_1+2\psi_2}$ |
| Strong Condition B: ($x_1 + \frac{1}{2}) \neq 0$, ($x_1 - \frac{1}{2}) \neq 0$, ($x_1 - \frac{1}{4}) \neq 0$, ($x_1 - \frac{3}{4}) \neq 0$, ($x_1 - \frac{5}{4}) \neq 0$, ($x_1 - \frac{7}{4}) \neq 0$, ($x_1 - \frac{9}{4}) \neq 0$, ($x_1 - \frac{11}{4}) \neq 0$, ($x_1 - \frac{13}{4}) \neq 0$, ($x_1 - \frac{15}{4}) \neq 0$, ($x_1 - \frac{17}{4}) \neq 0$ |
| $V_{x_1\omega_1+4\omega_2}$ | $V_{(x_1-3)\psi_1+4\psi_2} \oplus 3V_{x_1\psi_1+3\psi_2}$ |

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Table 14: Decompositions of inducing $p_{(1,0,0)}$-modules over $\Lambda$, where $\Lambda$ is the reductive Levi part of $p_{(1,0,0)}$ and $I$ is the reductive Levi part of $p_{(1,0)}$.

| $V_\Lambda(I)$ | Decomposition over $I$ |
|----------------|-------------------------|
| Strong Condition B: $-x_1 \neq 0$, $-x_1 - 7 \neq 0$, $-x_1 - 5 \neq 0$, $-2x_1 - 15 \neq 0$, $-x_1 - 11 \neq 0$, $-x_1 - 13 \neq 0$, $-2x_1 - 13 \neq 0$, $-x_1 - 9 \neq 0$, $-x_1 - 2 \neq 0$, $-5x_1 - 43 \neq 0$, $-x_1 - 14 \neq 0$, $-x_1 - 16 \neq 0$, $-x_1 - 6 \neq 0$, $-x_1 - 1 \neq 0$, $-x_1 - 4 \neq 0$, $-5x_1 - 27 \neq 0$, $-2x_1 - 7 \neq 0$, $-1 \neq 0$, $-x_1 - 12 \neq 0$, $-x_1 - 3 \neq 0$, $-2x_1 - 21 \neq 0$, $-x_1 - 8 \neq 0$, $-x_1 - 11 \neq 0$, $-2x_1 - 19 \neq 0$, $-5x_1 - 39 \neq 0$, $-2x_1 - 11 \neq 0$, $-5x_1 - 38 \neq 0$, $-4x_1 - 15 \neq 0$, $-7x_1 - 54 \neq 0$, $-5x_1 - 26 \neq 0$, $-x_1 - 23 \neq 0$, $-x_1 - 15 \neq 0$, $-x_1 - 18 \neq 0$, $-x_1 - 16 \neq 0$, $-5x_1 - 22 \neq 0$, $-2x_1 - 5 \neq 0$, $-x_1 + 2 \neq 0$, $-5x_1 - 13 \neq 0$, $-7x_1 - 37 \neq 0$, $-x_1 - 22 \neq 0$, $-4x_1 - 37 \neq 0$, $-5x_1 - 52 \neq 0 |
| $V_{x_1 \omega_1 + 5 \omega_3}$ | $V_{(x_1-3)\psi_1+4\psi_2} \oplus V_{(x_1)\psi_1+2\psi_2} \oplus V_{(x_1+3)\psi_1+\psi_2}$ $V_{x_1\psi_1+\psi_2} \oplus V_{(x_1+5)\psi_1} \oplus V_{(x_2-1)\psi_1+3\psi_2} \oplus V_{(x_2-2)\psi_1+3\psi_2}$ $V_{(x_1+1)\psi_1+2\psi_2} \oplus V_{(x_1-1)\psi_1+2\psi_2} \oplus V_{(x_2+2)\psi_1+\psi_2}$ $V_{(x_1+1)\psi_1+\psi_2} \oplus V_{(x_1-1)\psi_1} \oplus V_{(x_1+1)\psi_1} \oplus V_{(x_2+4)\psi_1}$ $V_{(x_1+2)\psi_1} \oplus V_{(x_2-5)\psi_1+5\psi_2} \oplus V_{(x_2-4)\psi_1+4\psi_2}$ $V_{(x_2-3)\psi_1+3\psi_2} \oplus V_{(x_2-2)\psi_1+2\psi_2} \oplus V_{(x_2-1)\psi_1+\psi_2}$ $V_{x_1\psi_1}$ |
| Strong Condition B: $-x_1 - 13 \neq 0$, $-x_1 - 5 \neq 0$, $-x_1 + 1 \neq 0$, $-x_1 - 3 \neq 0$, $-x_1 + 2 \neq 0$, $-2x_1 - 5 \neq 0$, $-x_1 \neq 0$, $-x_1 - 10 \neq 0$, $-x_1 - 4 \neq 0$, $-x_1 - 11 \neq 0$, $-x_1 - 12 \neq 0$, $-x_1 - 7 \neq 0$, $-2x_1 - 9 \neq 0$, $-5x_1 - 21 \neq 0$, $-4x_1 - 11 \neq 0$, $-x_1 - 4 \neq 0$, $-7x_1 - 29 \neq 0$, $-1 \neq 0$, $-x_1 - 25 \neq 0$, $-x_1 - 10 \neq 0$, $-2x_1 - 11 \neq 0$, $-x_1 - 6 \neq 0$, $-2x_1 - 15 \neq 0$, $-x_1 - 9 \neq 0$, $-5x_1 - 29 \neq 0$, $-7x_1 - 33 \neq 0$, $-4x_1 - 19 \neq 0$, $-4x_1 - 15 \neq 0$, $-x_1 - 15 \neq 0$, $-x_1 - 8 \neq 0$, $-5x_1 - 26 \neq 0$, $-7x_1 - 39 \neq 0$, $-x_1 - 20 \neq 0$, $-2x_1 - 33 \neq 0$, $-4x_1 - 29 \neq 0$, $-5x_1 - 24 \neq 0$, $-7x_1 - 38 \neq 0$, $-x_1 - 13 \neq 0$, $-5x_1 - 37 \neq 0$, $-4x_1 - 25 \neq 0$, $-7x_1 - 37 \neq 0$, $-2x_1 - 7 \neq 0$, $-7x_1 - 31 \neq 0$, $-x_1 - 19 \neq 0$, $-x_1 - 31 \neq 0$, $-x_1 - 14 \neq 0$, $-x_1 - 16 \neq 0$, $-5x_1 - 31 \neq 0$, $-x_1 - 17 \neq 0$, $-7x_1 - 41 \neq 0$, $-4x_1 - 21 \neq 0$, $-7x_1 - 32 \neq 0$, $-x_1 - 21 \neq 0$ |
| $V_{x_1 \omega_1 + 2 \omega_2}$ | $2V_{(x_1+3)\psi_1+4\psi_2} \oplus V_{x_1\psi_1+3\psi_2} \oplus 2V_{x_1\psi_1+2\psi_2}$ $V_{x_1\psi_1+3\psi_2} \oplus V_{x_2\psi_1+\psi_2} \oplus V_{(x_1+2)\psi_1+\psi_2} \oplus V_{(x_2-4)\psi_1+5\psi_2}$ $V_{x_1\psi_1+4\psi_2} \oplus 2V_{x_1\psi_1+2\psi_2} \oplus 2V_{x_2\psi_1+3\psi_2} \oplus 2V_{x_2\psi_1+5\psi_2}$ $2V_{(x_1+1)\psi_1+2\psi_2} \oplus 2V_{(x_1-1)\psi_1+2\psi_2} \oplus V_{(x_2-2)\psi_1+2\psi_2}$ $2V_{(x_1+1)\psi_1+\psi_2} \oplus 2V_{(x_1+3)\psi_1+\psi_2} \oplus 2V_{(x_1+2)\psi_1+\psi_2}$ $V_{x_1\psi_1} \oplus V_{(x_1+5)\psi_1} \oplus V_{(x_1+1)\psi_1}$ $V_{x_1-5}\psi_1+5\psi_2 \oplus V_{x_1-4}\psi_1+4\psi_2 \oplus V_{x_1-3}\psi_1+3\psi_2 \oplus V_{x_1-2}\psi_1+2\psi_2 \oplus V_{(x_1-1)\psi_1}$ |
Table 14: Decompositions of inducing $p_{(1,0,0)}$-modules over $\tilde{I}$, where $I$ is the reductive Levi part of $p_{(1,0,0)}$ and $\tilde{I}$ is the reductive Levi part of $p_{(1,0)}$.

| $V_{\lambda}(I)$ | Decomposition over $\tilde{I}$ |
|------------------|----------------------------------|
| $V_{\lambda}(I)$ | $V_{\lambda}(I)$                  |
Table 14: Decompositions of inducing $\mathfrak{p}_{(1,0,0)}$-modules over $\tilde{I}$, where $I$ is the reductive Levi part of $\mathfrak{p}_{(1,0,0)}$ and $\tilde{I}$ is the reductive Levi part of $\mathfrak{p}_{(1,0)}$.

| $V_{A}(I)$ | Decomposition over $I$ |
|------------|-------------------------|
| $V_{x_{1}^3 + 3x_{2} + 2x_{3}}$ | $V_{(x_{1} - 3)}\psi_{1} + 5\psi_{2} \oplus 2V_{(x_{1} - 3)}\psi_{1} + 4\psi_{2} \oplus V_{x_{1}\psi_{1} + 4\psi_{2}}$ |
| $V_{x_{1}^3 \omega_{1} + 3\omega_{2} + 2\omega_{3}}$ | $3V_{x_{1}\psi_{1} + 3\psi_{2}} \oplus 2V_{x_{1}\psi_{1} + 2\psi_{2}} \oplus V_{(x_{1} - 4)}\psi_{1} + 5\psi_{2}$ |
| $V_{x_{1} \omega_{1} + 3\omega_{2} + 2\omega_{3}}$ | $2V_{(x_{1} - 2)}\psi_{1} + 5\psi_{2} \oplus 2V_{(x_{1} - 2)}\psi_{1} + 4\psi_{2} \oplus 2V_{(x_{1} - 1)}\psi_{1} + 4\psi_{2}$ |
| $V_{x_{1} \omega_{1} + 3\omega_{2} + 2\omega_{3}}$ | $2V_{(x_{1} - 2)}\psi_{1} + 3\psi_{2} \oplus V_{(x_{1} - 2)}\psi_{1} + 3\psi_{2} \oplus 3V_{(x_{1} - 1)}\psi_{1} + 3\psi_{2}$ |
| $V_{x_{1} \omega_{1} + 3\omega_{2} + 2\omega_{3}}$ | $2V_{(x_{1} + 1)}\psi_{1} + 5\psi_{2} \oplus V_{(x_{1} + 3)}\psi_{1} + 2\psi_{2} \oplus 3V_{(x_{1} - 1)}\psi_{1} + 2\psi_{2}$ |
| $V_{x_{1} \omega_{1} + 3\omega_{2} + 2\omega_{3}}$ | $2V_{(x_{1} + 1)}\psi_{1} + 2\psi_{2} \oplus 2V_{(x_{1} + 2)}\psi_{1} + 2\psi_{2}$ |
| $V_{x_{1} \omega_{1} + 3\omega_{2} + 2\omega_{3}}$ | $V_{(x_{1} + 1)}\psi_{1} + 2\psi_{2} \oplus 2V_{(x_{1} + 2)}\psi_{1} + 2\psi_{2}$ |
| $V_{x_{1} \omega_{1} + 3\omega_{2} + 2\omega_{3}}$ | $2V_{(x_{1} + 3)}\psi_{1} + 6\psi_{2} \oplus V_{(x_{1} + 1)}\psi_{1} + 6\psi_{2}$ |
| $V_{x_{1} \omega_{1} + 3\omega_{2} + 2\omega_{3}}$ | $V_{(x_{1} + 1)}\psi_{1} + 5\psi_{2} \oplus V_{(x_{1} + 3)}\psi_{1} + 6\psi_{2}$ |
| $V_{x_{1} \omega_{1} + 3\omega_{2} + 2\omega_{3}}$ | $V_{(x_{1} + 1)}\psi_{1} + 5\psi_{2} \oplus V_{(x_{1} + 3)}\psi_{1} + 6\psi_{2}$ |
| $V_{x_{1} \omega_{1} + 3\omega_{2} + 2\omega_{3}}$ | $V_{(x_{1} + 1)}\psi_{1} + 5\psi_{2} \oplus V_{(x_{1} + 3)}\psi_{1} + 5\psi_{2}$ |
| $V_{x_{1} \omega_{1} + 3\omega_{2} + 2\omega_{3}}$ | $V_{(x_{1} + 1)}\psi_{1} + 5\psi_{2} \oplus V_{(x_{1} + 3)}\psi_{1} + 5\psi_{2}$ |

| $V_{A}(\tilde{I})$ | Decomposition over $\tilde{I}$ |
|------------|-------------------------|
| $V_{x_{1}^3 + 3x_{2} + 2x_{3}}$ | $V_{(x_{1} - 3)}\psi_{1} + 5\psi_{2} \oplus 2V_{(x_{1} - 3)}\psi_{1} + 4\psi_{2} \oplus 2V_{x_{1}\psi_{1} + 4\psi_{2}}$ |
| $V_{x_{1}^3 \omega_{1} + 3\omega_{2} + 2\omega_{3}}$ | $3V_{x_{1}\psi_{1} + 3\psi_{2}} \oplus 2V_{x_{1}\psi_{1} + 2\psi_{2}} \oplus V_{(x_{1} - 4)}\psi_{1} + 5\psi_{2}$ |
| $V_{x_{1} \omega_{1} + 3\omega_{2} + 2\omega_{3}}$ | $2V_{(x_{1} - 2)}\psi_{1} + 5\psi_{2} \oplus 2V_{(x_{1} - 2)}\psi_{1} + 4\psi_{2} \oplus 2V_{(x_{1} - 1)}\psi_{1} + 4\psi_{2}$ |
| $V_{x_{1} \omega_{1} + 3\omega_{2} + 2\omega_{3}}$ | $2V_{(x_{1} - 2)}\psi_{1} + 3\psi_{2} \oplus V_{(x_{1} - 2)}\psi_{1} + 3\psi_{2} \oplus 3V_{(x_{1} - 1)}\psi_{1} + 3\psi_{2}$ |
| $V_{x_{1} \omega_{1} + 3\omega_{2} + 2\omega_{3}}$ | $2V_{(x_{1} + 1)}\psi_{1} + 5\psi_{2} \oplus V_{(x_{1} + 3)}\psi_{1} + 2\psi_{2} \oplus 3V_{(x_{1} - 1)}\psi_{1} + 2\psi_{2}$ |
| $V_{x_{1} \omega_{1} + 3\omega_{2} + 2\omega_{3}}$ | $2V_{(x_{1} + 1)}\psi_{1} + 2\psi_{2} \oplus 2V_{(x_{1} + 2)}\psi_{1} + 2\psi_{2}$ |
| $V_{x_{1} \omega_{1} + 3\omega_{2} + 2\omega_{3}}$ | $V_{(x_{1} + 1)}\psi_{1} + 2\psi_{2} \oplus 2V_{(x_{1} + 2)}\psi_{1} + 2\psi_{2}$ |
| $V_{x_{1} \omega_{1} + 3\omega_{2} + 2\omega_{3}}$ | $2V_{(x_{1} + 3)}\psi_{1} + 6\psi_{2} \oplus V_{(x_{1} + 1)}\psi_{1} + 6\psi_{2}$ |
| $V_{x_{1} \omega_{1} + 3\omega_{2} + 2\omega_{3}}$ | $V_{(x_{1} + 1)}\psi_{1} + 6\psi_{2} \oplus V_{(x_{1} + 3)}\psi_{1} + 6\psi_{2}$ |
| $V_{x_{1} \omega_{1} + 3\omega_{2} + 2\omega_{3}}$ | $V_{(x_{1} + 1)}\psi_{1} + 6\psi_{2} \oplus V_{(x_{1} + 3)}\psi_{1} + 5\psi_{2}$ |
| $V_{x_{1} \omega_{1} + 3\omega_{2} + 2\omega_{3}}$ | $V_{(x_{1} + 1)}\psi_{1} + 6\psi_{2} \oplus V_{(x_{1} + 3)}\psi_{1} + 5\psi_{2}$ |
| $V_{x_{1} \omega_{1} + 3\omega_{2} + 2\omega_{3}}$ | $V_{(x_{1} + 1)}\psi_{1} + 6\psi_{2} \oplus V_{(x_{1} + 3)}\psi_{1} + 5\psi_{2}$ |
Table 14: Decompositions of inducing $\mathfrak{p}_{(1,0,0)}$-modules over $\overline{I}$, where $I$ is the reductive Levi part of $\mathfrak{p}_{(1,0,0)}$ and $\overline{I}$ is the reductive Levi part of $\mathfrak{p}_{(1)}$.

| $V_{\lambda}(I)$ | Decomposition over $\overline{I}$ |
|------------------|----------------------------------|
| **Strong Condition B:** $(-x_1 - 15) \neq 0$, $(-x_1 - 7) \neq 0$, $(-x_1 - 1) \neq 0$, $(-x_1 - 10) \neq 0$, $(-x_1 - 2) \neq 0$, $(-x_1 - 8) \neq 0$, $(-5x_1 - 1) \neq 0$, $(-x_2 - 13) \neq 0$, $(-x_3 - 33) \neq 0$, $(-4x_1 - 17) \neq 0$, $(-x_1 - 16) \neq 0$, $(-x_1 - 11) \neq 0$, $(-2x_1 - 17) \neq 0$, $(-x_1 - 6) \neq 0$, $(-x_1 + 2) \neq 0$, $(-2x_1 - 13) \neq 0$, $(-x_1 - 4) \neq 0$, $(-5x_1 - 28) \neq 0$, $(-4x_1 - 13) \neq 0$, $(-5x_1 - 16) \neq 0$, $(-x_1 - 46) \neq 0$, $(-x_1 - 27) \neq 0$, $(-x_1 - 13) \neq 0$, $(-x_1 - 14) \neq 0$, $(-x_1 - 12) \neq 0$, $(-2x_1 - 21) \neq 0$, $(-x_1 - 9) \neq 0$, $(-5x_1 - 59) \neq 0$, $(-x_1 - 3) \neq 0$, $(-5x_1 - 44) \neq 0$, $(-x_1 - 28) \neq 0$, $(-2x_1 - 23) \neq 0$, $(-2x_1 - 7) \neq 0$, $(-5x_1 - 42) \neq 0$, $(-5x_1 - 27) \neq 0$, $(-7x_1 - 59) \neq 0$, $(-5x_1 - 18) \neq 0$, $(-7x_1 - 44) \neq 0$, $(-5x_1 - 39) \neq 0$, $(-x_2 - 71) \neq 0$, $(-x_1 - 29) \neq 0$, $(-x_1 - 33) \neq 0$, $(-x_1 - 18) \neq 0$, $(-x_1 - 34) \neq 0$, $(-x_1 - 19) \neq 0$, $(-2x_1 - 11) \neq 0$, $(-5x_1 - 37) \neq 0$, $(-2x_1 - 5) \neq 0$, $(-7x_1 - 52) \neq 0$, $(-5x_1 - 22) \neq 0$, $(-7x_1 - 37) \neq 0$, $(-3x_1 - 23) \neq 0$, $(-5x_1 - 13) \neq 0$, $(-5x_1 - 23) \neq 0$, $(-5x_1 - 11) \neq 0$, $(-x_1 + 3) \neq 0$, $(-4x_1 - 9) \neq 0$, $(-7x_1 - 39) \neq 0$, $(-7x_1 - 27) \neq 0$, $(-3x_1 - 19) \neq 0$, $(-8x_1 - 41) \neq 0$, $(-4x_1 - 43) \neq 0$, $(-5x_1 - 43) \neq 0$, $(-5x_1 - 31) \neq 0$, $(-4x_1 - 19) \neq 0$, $(-7x_1 - 61) \neq 0$, $(-4x_1 - 39) \neq 0$, $(-5x_1 - 38) \neq 0$, $(-5x_1 - 26) \neq 0$, $(-7x_1 - 54) \neq 0$, $(-8x_1 - 71) \neq 0$, $(-4x_1 - 15) \neq 0$, $(-x_1 - 23) \neq 0$, $(-2x_1 - 39) \neq 0$, $(-5x_1 - 57) \neq 0$, $(-4x_1 - 41) \neq 0$, $(-x_1 - 22) \neq 0$, $(-2x_1 - 37) \neq 0$, $(-5x_1 - 52) \neq 0$, $(-4x_1 - 37) \neq 0$, $(-x_1 - 24) \neq 0$, $(-2x_1 - 41) \neq 0$ |

| $V_{\lambda}(I)$ | Decomposition over $\overline{I}$ |
|------------------|----------------------------------|
| $V_{\lambda}(I)$ | $V_{(x_1-3)}\psi_3^1+5\psi_2^1 + V_{(x_1+3)^1}\psi_1^1+4\psi_2^1 + V_{(x_1-4)^1}\psi_1^1+4\psi_2^1 + V_{(x_1-5)^1}\psi_1^1+4\psi_2^1$ |

| $V_{x_1\omega_1+5\omega_2}$ | $V_{x_1\omega_1+5\omega_2}$ |
|------------------|----------------------------------|
| $V_{x_1\omega_1+5\omega_2}$ | $V_{x_1\omega_1+5\omega_2}$ |

| **Strong Condition B:** $(-x_1 - 16) \neq 0$, $(-x_1 - 8) \neq 0$, $(-x_1 - 11) \neq 0$, $(-x_1 - 2) \neq 0$, $(-2x_1 - 17) \neq 0$, $(-5x_1 - 52) \neq 0$, $(-x_1 - 6) \neq 0$, $(-x_1 + 1) \neq 0$, $(-2x_1 - 15) \neq 0$, $(-x_2 - 13) \neq 0$, $(-x_3 - 33) \neq 0$, $(-4x_1 - 17) \neq 0$, $(-x_1 - 9) \neq 0$, $(-x_1 - 7) \neq 0$, $(-x_1 - 15) \neq 0$, $(-x_1 - 10) \neq 0$, $(-5x_1 - 47) \neq 0$, $(-2x_1 - 9) \neq 0$, $(-7x_1 - 66) \neq 0$, $(-5x_1 - 32) \neq 0$, $(-1) \neq 0$, $(-x_1 - 28) \neq 0$, $(-x_1 - 14) \neq 0$, $(-x_1 - 13) \neq 0$, $(-2x_1 - 23) \neq 0$, $(-x_1 - 4) \neq 0$, $(-5x_1 - 64) \neq 0$, $(-x_1 - 17) \neq 0$, $(-x_1 - 12) \neq 0$, $(-x_1 - 27) \neq 0$, $(-2x_1 - 21) \neq 0$, $(-5x_1 - 59) \neq 0$, $(-7x_1 - 78) \neq 0$, $(-x_1 - 3) \neq 0$, $(-5x_1 - 44) \neq 0$, $(-2x_1 - 13) \neq 0$, $(-x_1 + 2) \neq 0$, $(-5x_1 - 28) \neq 0$, $(-4x_1 - 13) \neq 0$, $(-5x_1 - 16) \neq 0$, $(-7x_1 - 46) \neq 0$, $(-4x_1 - 47) \neq 0$, $(-5x_1 - 48) \neq 0$, $(-4x_1 - 51) \neq 0$, $(-2x_1 - 19) \neq 0$, $(-5x_1 - 42) \neq 0$, $(-2x_1 - 7) \neq 0$, $(-5x_1 - 27) \neq 0$, $(-2x_1 - 59) \neq 0$, $(-x_1 - 18) \neq 0$, $(-3x_1 - 26) \neq 0$, $(-7x_1 - 44) \neq 0$, $(-4x_1 - 43) \neq 0$, $(-5x_1 - 43) \neq 0$, $(-5x_1 - 31) \neq 0$, $(-7x_1 - 61) \neq 0$, $(-4x_1 - 19) \neq 0$, $(-8x_1 - 79) \neq 0$, $(-x_1 - 34) \neq 0$, $(-x_1 - 19) \neq 0$, $(-x_2 - 29) \neq 0$, $(-x_3 - 18) \neq 0$, $(-x_1 - 24) \neq 0$, $(-2x_1 - 41) \neq 0$, $(-5x_1 - 23) \neq 0$, $(-5x_1 - 11) \neq 0$, $(-2x_1 - 39) \neq 0$, $(-7x_1 - 39) \neq 0$, $(-x_1 - 12) \neq 0$, $(-2x_1 - 37) \neq 0$, $(-5x_1 - 52) \neq 0$, $(-4x_1 - 37) \neq 0$, $(-x_1 - 24) \neq 0$, $(-2x_1 - 41) \neq 0$ |

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Table 15: Decompositions of inducing $p_{(0,1,0)}$-modules over $\bar{I}$, where $I$ is the reductive Levi part of $p_{(0,1,0)}$ and $\bar{I}$ is the reductive Levi part of $p_{(0,1)}$.

| $V_{\lambda}(I)$ | Decomposition over $I$ |
|-------------------|-------------------------|
| $V_{x_2\omega_2}$ | $V_{x_2\psi_1}$ |
| $V_{x_2\omega_2 + \omega_1}$ | $V_{\psi_1 + x_2\psi_2}$ |
| $V_{\omega_1 + x_2\omega_2}$ | $V_{\psi_1 + x_2\psi_2}$ |
| $V_{x_2\omega_2 + 2\omega_1}$ | $V_{2\psi_1 + x_2\psi_2}$ |
| $V_{\omega_1 + x_2\omega_2 + \omega_3}$ | $V_{(x_2 + 1)\psi_2} \oplus V_{2\psi_1 + x_2\psi_2}$ |

Strong Condition B: $-1 \neq 0$

| $V_{2\omega_1 + x_2\omega_2}$ | $V_{2\psi_1 + x_2\psi_2}$ |
| $V_{x_2\omega_2 + 3\omega_1}$ | $V_{3\psi_1 + x_2\psi_2}$ |
| $V_{\omega_1 + 2x_2\omega_2 + 2\omega_3}$ | $V_{\psi_1 + (x_2 + 1)\psi_2} \oplus V_{3\psi_1 + x_2\psi_2}$ |

Strong Condition B: $-1 \neq 0$

| $V_{2\omega_1 + x_2\omega_2 + \omega_3}$ | $V_{\psi_1 + (x_2 + 1)\psi_2} \oplus V_{3\psi_1 + x_2\psi_2}$ |

Strong Condition B: $-1 \neq 0$

| $V_{4\omega_1 + x_2\omega_3}$ | $V_{3\psi_1 + x_2\psi_2}$ |
| $V_{x_2\omega_2 + 4\omega_1}$ | $V_{4\psi_1 + x_2\psi_2}$ |
| $V_{\omega_1 + x_2\omega_2 + 3\omega_3}$ | $V_{2\psi_1 + (x_2 + 1)\psi_2} \oplus V_{4\psi_1 + x_2\psi_2}$ |

Strong Condition B: $-1 \neq 0$

| $V_{2\omega_1 + x_2\omega_2 + 2\omega_3}$ | $V_{(x_2 + 2)\psi_2} \oplus V_{2\psi_1 + (x_2 + 1)\psi_2} \oplus V_{4\psi_1 + x_2\psi_2}$ |

Strong Condition B: $-1 \neq 0$

| $V_{3\omega_1 + x_2\omega_3 + \omega_3}$ | $V_{2\psi_1 + (x_2 + 1)\psi_2} \oplus V_{4\psi_1 + x_2\psi_2}$ |

Strong Condition B: $-1 \neq 0$

| $V_{4\omega_1 + x_2\omega_2}$ | $V_{4\psi_1 + x_2\psi_2}$ |
| $V_{x_2\omega_2 + 5\omega_3}$ | $V_{5\psi_1 + x_2\psi_2}$ |
| $V_{\omega_1 + x_2\omega_2 + 4\omega_3}$ | $V_{3\psi_1 + (x_2 + 1)\psi_2} \oplus V_{5\psi_1 + x_2\psi_2}$ |

Strong Condition B: $-1 \neq 0$

| $V_{2\omega_1 + x_2\omega_2 + 3\omega_3}$ | $V_{3\psi_1 + (x_2 + 1)\psi_2} \oplus V_{\psi_1 + (x_2 + 2)\psi_2} \oplus V_{5\psi_1 + x_2\psi_2}$ |

Strong Condition B: $-1 \neq 0$

| $V_{3\omega_1 + x_2\omega_2 + 2\omega_3}$ | $V_{3\psi_1 + (x_2 + 1)\psi_2} \oplus V_{\psi_1 + (x_2 + 2)\psi_2} \oplus V_{5\psi_1 + x_2\psi_2}$ |

Strong Condition B: $-1 \neq 0$

| $V_{4\omega_1 + x_2\omega_2 + \omega_3}$ | $V_{3\psi_1 + (x_2 + 1)\psi_2} \oplus V_{5\psi_1 + x_2\psi_2}$ |

Strong Condition B: $-1 \neq 0$

| $V_{5\omega_1 + x_2\omega_2}$ | $V_{5\psi_1 + x_2\psi_2}$ |

Table 16: Decompositions of inducing $p_{(0,0,1)}$-modules over $\bar{I}$, where $I$ is the reductive Levi part of $p_{(0,0,1)}$ and $\bar{I}$ is the reductive Levi part of $p_{(1,0)}$.

| $V_{\lambda}(I)$ | Decomposition over $I$ |
|-------------------|-------------------------|
| $V_{x_3\omega_3}$ | $V_{x_3\psi_1}$ |
| $V_{\omega_1 + x_3\omega_3}$ | $V_{x_3\psi_1 + \psi_2} \oplus V_{(x_3 + 1)\psi_1}$ |
Table 16: Decompositions of inducing $\mathfrak{p}_{(0,0,1)}$-modules over $\overline{I}$, where $I$ is the reductive Levi part of $\mathfrak{p}_{(0,0,1)}$ and $\overline{I}$ is the reductive Levi part of $\mathfrak{p}_{(1,0)}$.

| $V_\lambda(\ell)$ | Decomposition over $I$ |
|-------------------|------------------------|
| $V_{\omega_1+3}\omega_3$ | $V(x_3+1)\psi_1 \oplus V(x_3-1)\psi_1+\psi_2$ |
| Strong Condition B: $(-x_3 - 6) \neq 0$ |
| $V_{2\omega_2+x_3\omega_3}$ | $V_{x_3} \psi_1+2\psi_2 \oplus V(x_3+1)\psi_1+\psi_2 \oplus V(x_3+2)\psi_1$ |
| Strong Condition B: $(-x_3 - 1) \neq 0$ |
| $V_{\omega_1+\omega_2+x_3\omega_3}$ | $V_{x_3} \psi_1+2\psi_2 \oplus V(x_3+1)\psi_1+\psi_2 \oplus V(x_3+2)\psi_1+\psi_2 \oplus V(x_3+3)\psi_1$ |
| Strong Condition B: $(-x_3 - 9) \neq 0$, $(-x_3 - 7) \neq 0$, $(-x_3 - 8) \neq 0$ |
| $V_{2\omega_1+x_3\omega_3}$ | $V_{x_3} \psi_1+\psi_2 \oplus V(x_3+1)\psi_1+2\psi_2 \oplus V(x_3+2)\psi_1+\psi_2$ |
| Strong Condition B: $(-x_3 - 12) \neq 0$, $(-x_3 - 10) \neq 0$, $(-x_3 - 11) \neq 0$, $(-x_3 - 9) \neq 0$, $(-x_3 - 8) \neq 0$ |
| $V_{3\omega_2+x_3\omega_3}$ | $V_{x_3} \psi_1+3\psi_2 \oplus V(x_3+1)\psi_1+2\psi_2 \oplus V(x_3+2)\psi_1+2\psi_2 \oplus V(x_3+3)\psi_1$ |
| Strong Condition B: $(-x_3 - 12) \neq 0$, $(-x_3 - 10) \neq 0$, $(-x_3 - 11) \neq 0$, $(-x_3 - 9) \neq 0$, $(-x_3 - 8) \neq 0$ |

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Table 16: Decompositions of inducing $\mathfrak{p}(0,0,1)$-modules over $\overline{I}$, where $I$ is the reductive Levi part of $\mathfrak{p}(0,0,1)$ and $\overline{I}$ is the reductive Levi part of $\mathfrak{p}(1,0)$.

| V_{A(I)} | Decomposition over $\overline{I}$ |
|----------|--------------------------------|
| Strong Condition B: $-x_5 \neq 0$, $(-x_5 - 1) \neq 0$, $(-x_3 - 13) \neq 0$, $(-x_3 - 5) \neq 0$, $(-x_3 - 2) \neq 0$, $-1 \neq 0$, $(-x_3 - 12) \neq 0$, $(-x_3 - 7) \neq 0$, $(-x_3 - 3) \neq 0$, $(-x_3 - 22) \neq 0$, $(-x_3 - 10) \neq 0$, $(-x_3 - 11) \neq 0$, $(-x_3 - 8) \neq 0$, $(-x_3 - 4) \neq 0$, $(-x_3 - 6) \neq 0$, $(-x_3 - 18) \neq 0$, $(-x_3 - 9) \neq 0$ |

$V_{3\omega_1+\omega_2+3\omega_3}$

$V_{(x_3-3)}\psi_1 + 3\psi_2 + V_{(x_3+1)}\psi_1 + 3\psi_2 + V_{(x_3+2)}\psi_1 + 3\psi_2$

Strong Condition B: $(-x_3 - 1) \neq 0$, $(-x_3 - 1) \neq 0$, $(-x_3 - 3) \neq 0$, $(-x_3 - 5) \neq 0$, $(-x_3 - 3) \neq 0$, $(-x_3 - 7) \neq 0$, $(-x_3 - 3) \neq 0$, $(-x_3 - 12) \neq 0$, $(-x_3 - 11) \neq 0$, $(-x_3 - 8) \neq 0$, $(-x_3 - 18) \neq 0$, $(-x_3 - 9) \neq 0$

$V_{3\omega_1+3\omega_3}$

$V_{(x_3-3)}\psi_1 + 3\psi_2 + V_{(x_3+1)}\psi_1 + 3\psi_2 + V_{(x_3+2)}\psi_1 + 3\psi_2$

Strong Condition B: $(-x_3 - 1) \neq 0$, $(-x_3 - 1) \neq 0$, $(-x_3 - 3) \neq 0$, $(-x_3 - 5) \neq 0$, $(-x_3 - 3) \neq 0$, $(-x_3 - 7) \neq 0$, $(-x_3 - 3) \neq 0$, $(-x_3 - 12) \neq 0$, $(-x_3 - 11) \neq 0$, $(-x_3 - 8) \neq 0$, $(-x_3 - 18) \neq 0$, $(-x_3 - 9) \neq 0$
### Table 16: Decompositions of inducing $p(0,0,1)$-modules over $\bar{I}$, where $I$ is the reductive Levi part of $p(0,0,1)$ and $\bar{I}$ is the reductive Levi part of $p(1,0)$.

| $V_A(I)$ | Decomposition over $\bar{I}$ |
|----------|--------------------------------|
| $V_{\lambda}(I)$ | Strong Condition B: $(-x_3 + 1) \neq 0$, $(-x_3 - 1) \neq 0$, $(-x_3 - 2) \neq 0$, $(-x_3 - 3) \neq 0$, $(-x_3 - 4) \neq 0$, $(-x_3 - 5) \neq 0$ |
| $V_{4\omega_1 + \omega_2 + x_3 \omega_3}$ | $V_{(x_3-3)\psi_1 + 4\psi_2} \oplus V_{(x_3-2)\psi_1 + 4\psi_2} \oplus V_{(x_3-1)\psi_1 + 3\psi_2} \oplus V_{(x_3+2)\psi_1 + 2\psi_2} \oplus V_{(x_3+1)\psi_1 + 3\psi_2} \oplus V_{(x_3+3)\psi_1 + 2\psi_2} \oplus V_{(x_3+2)\psi_1 + \psi_2} \oplus V_{(x_3+1)\psi_1 + \psi_2} \oplus V_{(x_3+4)\psi_1 + \psi_2} \oplus V_{(x_3+5)\psi_1}$ |

### Table 17: Decompositions of inducing $p(1,1,0)$-modules over $\bar{I}$, where $I$ is the reductive Levi part of $p(1,1,0)$ and $\bar{I}$ is the reductive Levi part of $p(1,1)$.

| $V_A(I)$ | Decomposition over $\bar{I}$ |
|----------|--------------------------------|
| $V_{\lambda}(I)$ | Strong Condition B: $(-x_1 - 1) \neq 0$, $(-x_1 - 2) \neq 0$, $(-x_1 - 3) \neq 0$, $(-x_1 - 4) \neq 0$, $(-x_1 - 5) \neq 0$ |
| $V_{x_1 \omega_1 + x_2 \omega_2}$ | $V_{x_1 \psi_1 + x_2 \psi_2}$ |
| $V_{x_1 \omega_1 + x_2 \omega_2 + \omega_3}$ | $V_{(x_1+1)\psi_1 + x_2 \psi_2} \oplus V_{(x_1-1)\psi_1 + (x_2+1)\psi_2}$ |
| $V_{x_1 \omega_1 + x_2 \omega_2 + 2\omega_3}$ | $V_{x_1 \psi_1 + (x_2+1)\psi_2} \oplus V_{(x_1+2)\psi_1 + x_2 \psi_2} \oplus V_{(x_1-2)\psi_1 + (x_2+2)\psi_2}$ |
| $V_{x_1 \omega_1 + x_2 \omega_2 + 3\omega_3}$ | $V_{(x_1-1)\psi_1 + (x_2+2)\psi_2} \oplus V_{(x_1+1)\psi_1 + (x_2+1)\psi_2} \oplus V_{(x_1+3)\psi_1 + x_2 \psi_2} \oplus V_{(x_1-3)\psi_1 + (x_2+3)\psi_2}$ |
| $V_{x_1 \omega_1 + x_2 \omega_2 + 4\omega_3}$ | $V_{(x_1+2)\psi_1 + (x_2+1)\psi_2} \oplus V_{(x_1-2)\psi_1 + (x_2+3)\psi_2} \oplus V_{x_1 \psi_1 + (x_2+2)\psi_2} \oplus V_{(x_1-4)\psi_1 + (x_2+4)\psi_2}$ |
| $V_{x_1 \omega_1 + x_2 \omega_2 + 5\omega_3}$ | $V_{(x_1-1)\psi_1 + (x_2+3)\psi_2} \oplus V_{(x_1+3)\psi_1 + (x_2+1)\psi_2} \oplus V_{(x_1+1)\psi_1 + (x_2+2)\psi_2} \oplus V_{(x_1-3)\psi_1 + (x_2+4)\psi_2} \oplus V_{(x_1+5)\psi_1 + x_2 \psi_2} \oplus V_{(x_1-5)\psi_1 + (x_2+5)\psi_2}$ |
Table 18: Decompositions of inducing $p_{(1,0,1)}$-modules over $\bar{l}$, where $l$ is the reductive Levi part of $p_{(1,0,1)}$ and $\bar{l}$ is the reductive Levi part of $p_{(1,0)}$.

| $V_{\Lambda}(l)$ | Decomposition over $l$ |
|-------------------|------------------------|
| $V_{x_1\omega_1+x_3\omega_3}$ | $V_{(x_3+x_1)}\psi_1$ |
| $V_{x_1\omega_1+x_2\omega_2+x_3\omega_3}$ | $V_{(x_3+x_1)}\psi_1+\psi_2$ |
| $V_{x_1\omega_1+2\omega_2+x_3\omega_3}$ | $V_{(x_3+x_1)}\psi_1+2\psi_2$ |
| $V_{x_1\omega_1+3\omega_2+x_3\omega_3}$ | $V_{(x_3+x_1)}\psi_1+3\psi_2$ |
| $V_{x_1\omega_1+4\omega_2+x_3\omega_3}$ | $V_{(x_3+x_1)}\psi_1+4\psi_2$ |
| $V_{x_1\omega_1+5\omega_2+x_3\omega_3}$ | $V_{(x_3+x_1)}\psi_1+5\psi_2$ |

Table 19: Decompositions of inducing $p_{(0,1,1)}$-modules over $\bar{l}$, where $l$ is the reductive Levi part of $p_{(0,1,1)}$ and $\bar{l}$ is the reductive Levi part of $p_{(1,1)}$.

| $V_{\Lambda}(l)$ | Decomposition over $l$ |
|-------------------|------------------------|
| $V_{x_2\omega_2+x_3\omega_3}$ | $V_{x_3}\psi_1+\psi_2$ |
| $V_{\omega_1+x_2\omega_2+x_3\omega_3}$ | $V_{(x_3+1)}\psi_1+\psi_2 \oplus V_{(x_3)}\psi_1+(x_2+1)\psi_2$ |
| $V_{2\omega_1+x_2\omega_2+x_3\omega_3}$ | $V_{x_3}\psi_1+(x_2+1)\psi_2 \oplus V_{(x_3+2)}\psi_1+\psi_2 \oplus V_{(x_3-2)}\psi_1+(x_2+2)\psi_2$ |
| $V_{3\omega_1+x_2\omega_2+x_3\omega_3}$ | $V_{(x_3-1)}\psi_1+(x_2+2)\psi_2 \oplus V_{(x_3+1)}\psi_1+(x_2+1)\psi_2 \oplus V_{(x_3+3)}\psi_1+x_2\psi_2$ |
| $V_{4\omega_1+x_2\omega_2+x_3\omega_3}$ | $V_{x_3}\psi_1+(x_2+2)\psi_2 \oplus V_{(x_3+2)}\psi_1+(x_2+1)\psi_2 \oplus V_{(x_3-2)}\psi_1+(x_2+3)\psi_2$ |
| $V_{5\omega_1+x_2\omega_2+x_3\omega_3}$ | $V_{(x_3-1)}\psi_1+(x_2+3)\psi_2 \oplus V_{(x_3+3)}\psi_1+(x_3+1)\psi_2 \oplus V_{(x_3+1)}\psi_1+(x_2+2)\psi_2 \oplus V_{(x_3-3)}\psi_1+(x_2+4)\psi_2 \oplus V_{(x_3+5)}\psi_1+x_2\psi_2 \oplus V_{(x_3-5)}\psi_1+(x_2+5)\psi_2$ |

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