q-ANALOG SINGULAR HOMOLOGY OF CONVEX SPACES

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Dedicated to Professor A. Reyes, in the occasion of his 76th birthday.

Abstract. In this article we study some interesting properties of the q-Analog singular homology, which is a generalization of the usual singular homology, suitably adapted to the context of N-complex and amplitude homology [7]. We calculate the q-Analog singular homology of a convex space. Although it is a local matter; this is an important step in order to understand the presheaf of q-chains and its algebraic properties. Our result is consistent with those of Dubois-Violette & Henneaux [4]. Some of these results were presented for the XVIII Congreso Colombiano de Matemáticas in Bucaramanga, 2011.

Introduction

The fact that singular homology satisfies the homotopy axiom is a well known result of topological algebra. It can be understood in several ways. From the topological scope it asserts that any topological space that is homotopic to a single point, must have no topological holes. More than this, homotopic spaces have the same singular homology and homotopic maps induce the same maps between the respective homology groups. A customary proof can be carried out by means of these mathematical facts,

(1) The cone construction [2, p.33].
(2) A Leibnitz rule for the convex product of singular chains [1, p.220].
(3) The double composition of border map ∂ vanishes, i.e. ∂² = 0, which means that singular chains constitute a usual chain complex.

On the other hand, the theory of N-complexes has raised in the last years as a new homology theory with a broad field of applications in quantum physics [4]. Let N ≥ 3 be a prime integer. A N-complex is a graded module whose border map ∂ vanishes in the N-th composition, i.e. ∂^N = 0. The m-amplitude homologies are defined for 1 ≤ m ≤ N − 1; see [3, 6]. For instance, take a complex N-th root of the identity, q ∈ C; i.e. q^N = 1. Then there can be defined q-simplicial chains, as singular chains that are linear combinations of singular simplexes where the constants are taken on the ring Z[q] and the border map is adequately adapted. Several examples will be treated here below.

The main result in this article is that any convex Euclidean space has the same q-Analog singular homology of a singleton. This is a consequence of the algebraic structure induced by the border map and the combinatorial properties of q-numbers. In order to prove this,

(1) We use the fact that ∂^N = 0, i.e. q-Analog singular chains are a graded N-differential module.
(2) We extend the cone construction to a convex product for the q-Analog singular homology.
(3) We obtain a q-Leibnitz rule for the convex product and a formula for the Newton’s polynomials.
(4) We construct a geometric N-homotopy operator by means of the convex product.

An open question we hope to answer in the future is to demonstrate that q-Analog singular chains satisfy the Mayer-Vietoris property.

The article has been organized as follows. In the sections §1, §2 we summarize some usual facts of q-numbers and N-complexes. Section §3 is devoted to q-singular chains and more examples. In section §4 we define the
convex product and show the Leibnitz rule. The last section is devoted to prove the homotopy axiom for $q$-Analog singular homology, which is our main result.

1. $q$-NUMBERS

Recall the definition of $q$-numbers and some of their properties [3].

1.1. $q$-numbers. Let $q \in \mathbb{C}$ be a complex non trivial $N$-th root of the identity i.e. $q^N = 1$ and $q \neq 1$. In the classical literature $N$ is assumed to be a prime integer and $q = \exp(2\pi i/N)$, see [4]. The basic $q$-numbers are

$$[k]_q = \frac{1 - q^k}{1 - q} = 1 + q + \cdots + q^{k-1} \quad \forall k \in \mathbb{N}$$

Notice that $[N]_q = 0$. The $q$-factorial numbers are

$$[k]_q! = [1]_q \cdot [2]_q \cdots [k]_q \quad 0 \leq k \leq N - 1$$

Finally, the $q$-combinatorial numbers are

$$\binom{k}{l}_q = \frac{[k]_q!}{[l]_q! \cdot [(k-l)]_q!} \quad \forall 0 \leq l \leq k \leq N - 1$$

Since $N \geq 2$ the polynomial $q^N - 1 = 0$ is irreducible in $\mathbb{R}$, so $\mathbb{R}[q] = \mathbb{C}$ is the field of complex numbers. In particular, $q$-numbers $[k]_q \neq 0$ have multiplicative inverse in $\mathbb{R}[q]$. The following properties follow from the definition of $q$-numbers, we leave the details to the reader.

Lemma 1.1.1. Let $1 \leq k \leq n, m \leq N - 1$. Then,

1. $[m + n]_q = [m]_q + q^m [n]_q$.
2. If $m$ is prime relative to $N$, then $[m]_q$ is a unit in $\mathbb{Z}[q]$; and its multiplicative inverse is $[a]_q^{-1}$ where $an + bn = 1$ for some integers $a, b$.
3. $\binom{n}{k}_q + q^k \binom{n}{k+1}_q = \binom{n+1}{k+1}_q + q^{n-k} \binom{n}{k}_q$.
4. $[n]_q! = \sum_{\sigma \in S_n} q^{a_{\sigma(n)}}$ where $S_n$ is the $n$th symmetric group and $\sigma$ runs over all permutations of $n$ elements.

2. $N$-COMPLEXES

Let us fix a positive integer $N \geq 2$ and a principal ideal domain $(R, +, \cdot, 1)$ as the underlying ring of constants (usually we will take $R = \mathbb{Z}[q]$). A $N$-complex is a generalization of usual chain complexes, and presents a similar behavior taking into account the integer $N$, which is called the amplitude of the complex [3] [4].

2.1. $N$-complexes. A $N$-complex is a pair $(M, \partial)$ such that $M$ is a module and $\partial : M \rightarrow M$ is a linear endomorphism such that the $N$-th composition $\partial^N = 0$ vanishes. We call $\partial$ the border map. For any integer $1 \leq m \leq N - 1$, we consider the submodules

$$M \xrightarrow{\partial^{m(N-m)}} M \xrightarrow{\partial^m} M \quad \text{and} \quad B_m(M) = \text{Im} \left( \partial^{N-m} \right) \subset \ker \left( \partial^m \right) = Z_m(M)$$

An element of $Z_m(M)$ (resp. $B_m(M)$) is a $m$-amplitude cycle (resp. border). The homology of $M$ with amplitude $m$ is the quotient module

$$H_m(M) = \frac{Z_m(M)}{B_m(M)}$$

The total homology of $M$ is the graded module

$$H(M) = \{ H_m(M) : 1 \leq m \leq N - 1 \}$$
A morphism of $N$-complexes $\xymatrix{ (M, \partial) \ar[r]^f & (M', \partial') }$ is a linear morphism $f$ such that $f \partial = \partial' f$. The induced arrow is well defined on each amplitude homology $H_m (M) \xymatrix{ \ar[r]^f & H_m (M') }$, and passes to the total homology $H (M) \xymatrix{ \ar[r]^f & H (M') }$.

For any short exact sequence of $N$-complexes

\[
0 \xymatrix{ \ar[r] & M \ar[r]^\alpha & M' \ar[r]^\beta & M'' \ar[r] & 0 }
\]

there is a version of the snake lemma, and a connecting morphism $H_m (M) \xymatrix{ \ar[r]^\alpha & H_{N-m} (M) }$ from which arises an exact hexagon,

\[
\begin{array}{ccc}
H_m (M) & \xymatrix{ \ar[r]^\beta & H_m (M') } & \xymatrix{ \ar[l]_\alpha & H_{N-m} (M) } \\
\xymatrix{ \ar[u]_\partial & H_m (M'') } & & \xymatrix{ \ar[l]_\beta & H_{N-m} (M'') } \xymatrix{ \ar[u]_\alpha & } \\
& \xymatrix{ \ar[r]_\partial & H_{N-m} (M''') } & \xymatrix{ \ar[l]^\beta & H_{N-m} (M''') } \xymatrix{ \ar[u]_\alpha & } \end{array}
\]

2.2. Graded $N$-complexes. A graded $N$-differential module is a pair $(M_\ast, \partial)$ such that $M_\ast = \{ M_k : k \in \mathbb{Z} \}$ is a graded module and $\partial$ is a $(-1)$-graded linear endomorphism $M_k \xymatrix{ \ar[r]^\partial & M_{k-1} }$ such that $\partial^N = 0$. The properties of $N$-complexes can be extended to the graded case. The amplitude homology is now a bigraded module $H (M) = \{ H_{m,k} (M) : 1 \leq m \leq N - 1, k \in \mathbb{Z} \}$ depending on the amplitude $m$ and the degree $k$. The inclusion $i$ and the border map $\partial$ induce, respectively, well defined maps in the bigraded homology.

2.3. Examples.

(1) Any finite sequence of modules and morphisms

\[
0 \xymatrix{ \ar[r] & M_1 \ar[r]^\partial_1 & M_2 \ar[r]^\partial_2 & \cdots \ar[r]^\partial_{N-2} & M_{N-1} \ar[r]^\partial_{N-1} & M_N \ar[r] & 0 }
\]

is a graded $N$-complex.

(2) With a little abuse of notation let us write

\[
\mathbb{Z}[q] \xymatrix{ \ar[r]^q & \mathbb{Z}[q] }
\]

for the linear function that maps any element $\alpha \in \mathbb{Z}[q]$ to $[n]_q \cdot \alpha$. According to (1.1.1) (2), since $N$ is prime, $[n]_q \neq 0$ has a multiplicative inverse in $\mathbb{Z}[q]$ for $1 \leq n \leq N - 1$. The above map is a module isomorphism between free $\mathbb{Z}[q]$-modules.

\[
0 \xymatrix{ \ar[r] & \mathbb{Z}[q] \ar[r]^q & \mathbb{Z}[q] \ar[r]^q & \cdots \ar[r]^q & \mathbb{Z}[q] \ar[r]^q & 0 \ar[r] & 0 \ar[r] & \cdots }
\]

is a $N$-complex; we use to denote it by $\left( \mathbb{Z}[q], \left[ * \right]_q \right)$. A straightforward calculation shows that

\[
H_{m,n} \left( \mathbb{Z}[q], \left[ * \right]_q \right) = \begin{cases} 
\mathbb{Z}[q] & 1 \leq n = m \leq N - 2 \\
0 & \text{else}
\end{cases}
\]
One can construct $N$-differential modules with smooth differential forms on $\mathbb{R}^n$; see [3, 6]. There are also $N$-complexes with geometric singular chains on any topological space. For more details see the next sections.

2.4. Homotopy of $N$-complexes. Given any two morphisms of $N$-differential modules $M \xrightarrow{f,g} M'$, we say that they are homotopic and write $f \sim g$ iff there is a sequence of morphisms of modules $M \xrightarrow{K_m} M'$, for $0 \leq m \leq N - 1$, satisfying

$$\sum_{m=0}^{N-1} \left(\partial'\right)^m K_m \partial^{N-m-1} = (f - g)$$

The sequence of morphisms $K = \{K_m\}_m$ is a homotopy from $g$ to $f$. The existence of homotopies is an equivalence relation between morphisms of $N$-complexes; homotopic morphisms induce the same maps in the amplitude homologies. An alternative way to see that this is suitable definition of homotopy between morphisms of differential $N$-modules is to follow [7][p.4-5]. Consider, for any pair of $N$-differential graded modules $(M, \partial)$ and $(N, \delta)$, the graded module $\text{Hom}(M, N)$ with the $N$-differential operator given by

$$D(f) = \sum_{i=0}^{N-1} q^{\deg(f) + 1} \delta f \partial^{N-i-1}$$

A morphism $M \xrightarrow{f} N$ is compatible with the differentials iff it is a $D$-cycle, and then it induces a well defined morphism on the $k$-amplitude homologies $H_k(M) \xrightarrow{f} H_k(N)$ for $1 \leq k \leq N - 1$. Then, two differential morphisms $f, g$ (with $\deg(f) = \deg(g) = 0$ as above) are homotopic iff their difference $f - g$ is a $D$-border in $\text{Hom}(M, N)$. This happens iff there exists a morphism $M \xrightarrow{K} N$ such that $\deg(K) = (N - 1)$ and $(f - g) = D(k)$. Notice that the morphism $K$ has degree $\deg(K) = N - 1$.

3. $q$-Chains

3.1. The $N$-complex of $q$-chains on a simplicial set. Recall the construction of simplicial $q$-chains [3, 6]. A simplicial set is a family of non-empty sets and maps

$$X_{n+1} \xrightarrow{\partial_i} X_n \quad 0 \leq i \leq n, \ n \in \mathbb{N}$$

such that their compositions (1) satisfy

$$\partial_i \partial_j = \partial_j \partial_{i+1} \quad \forall j \leq i$$

An element of $X_n$ is a basic chain of dimension $n$. Let $N$ and $q$ be as in [11,1]. Take the polynomial extension $\mathbb{Z}[q]$ as the ring of constants. The $(N, q)$-complex generated by $X$ is the graded free $\mathbb{Z}[q]$-module that on each degree $n$ is spanned by $X_n$ as a linear basis.

$$qC_n(X) = \mathbb{Z}[q] \langle X_n \rangle = \bigoplus_{x \in X_n} \mathbb{Z}[q] \cdot x \quad n \in \mathbb{N}$$

As usual we assume the convention $qC_n(X) = 0$ for $n < 0$. The border map is the graded linear morphism

$$qC_n(X) \xrightarrow{\partial} qC_{n-1}(X) \quad \partial = \sum_{i=0}^{n} q^i \partial_i$$

We must check that our definition makes sense.

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1We write $fg$ for the composition $f(g(x))$ on each $x$ where it makes sense.
Lemma 3.1.1. [Iteration rule for the border map] The following equality holds

\[ \partial^k = [k]_q ! \cdot \sum_{i_1 \leq \cdots \leq i_k} q^{i_1+\cdots+i_k} \partial_{i_k} \cdots \partial_{i_1}; \quad 0 \leq k \leq N \]

Therefore, \((\mathcal{C}_q(X), \partial)\) is a graded \(N\)-complex.

[Proof] Apply the definition of the border map \(\partial\) and property [1.1.1](4). See [6]. \(\square\)

In particular, since \([N]_q = 0\) we get \(\partial^N = 0\), so \(\mathcal{C}_q(X)\) is a \(N\)-complex.

3.2. Singular \(q\)-chains. A geometric realization is given by the \(N\)-complex of Singular \(q\)-chains. For each integer \(n \in \mathbb{N}\) we write \(\Delta^n \) for the standard \(n\)-simplex, i.e. the convex hull generated on \(\mathbb{R}^{n+1}\) with the standard basis \(\{e_0, \ldots, e_n\}\). A linear map \(\Delta^n \xymatrix{\ar[r]^L & \Delta^m}\) is determined by its values on \(e_0, \ldots, e_n\); we write \(L = \langle x_0, \ldots, x_n \rangle\) to mean that \(x_i = L(e_i)\) for \(i = 0, \ldots, n\). Take

\[ \Delta^n \xymatrix{\ar[r]^\lambda_j & \Delta^{n+1}} \]

\[ \lambda_j = \{e_0, \ldots, e_j, \ldots, e_{n+1}\} \quad j = 0, \ldots, n \]

where \(\hat{e}_j\) means to omit the element \(e_j\). Given a topological space \(X \neq \emptyset\) we define \(X_n\) as the set of all continuous maps \(\Delta^n \xymatrix{\ar[r]^\sigma & X}\). An element of \(X_n\) is a simplex on \(X\). For each \(0 \leq j \leq n\) the \(j\)th-face map \(X_n \xymatrix{\ar[r]^\partial_j & X_{n-1}}\) is given by the composition \(\partial_j(\sigma) = \sigma \lambda_j\). This family is a simplicial set in our previous sense. The \(N\)-complex of singular \(q\)-chains on a topological space \(X\)

\[ q\mathcal{C}_n(X) = \mathcal{C}_n \left( \left\{ X_n : n \in \mathbb{N} \right\} \cup \left\{ \partial_j : X_n \xymatrix{\ar[r] & X_{n-1} : 0 \leq j \leq n, n \in \mathbb{N} \right\} \right) \]

is the \((N,q)\)-complex generated by the singular \(q\)-simplexes and face maps. An element \(\xi \in q\mathcal{C}_n(X)\) is a singular \(q\)-chain of dimension \(n\); it can be written a linear combination \(\xi = a_1 \sigma_1 + \cdots + a_r \sigma_r\) where each \(a \in \mathbb{Z}[q]\) is a polynomial and each \(\sigma \in X_n\) is a simplex of dimension \(n\) on \(X\). We also write \(n = \dim(\xi)\). The standard singular \((N,q)\)-homology of \(X\) is the homology of this \(N\)-complex

\[ qH_{m,n}(X) = qH_m \left( q\mathcal{C}_n(X) \right) \quad 1 \leq m \leq n - 1, n \in \mathbb{N} \]

3.3. Example: \(q\)-homology of a point. If \(P = \{p\}\) is a single point; then \(P_n = \{\sigma_n\}\) where \(\Delta^n \xymatrix{\ar[r]^\sigma & P}\) is the constant map. The module

\[ q\mathcal{C}_n(P) = \mathbb{Z}[q] \cdot \sigma_n \cong \mathbb{Z}[q] \]

is isomorphic to the ring of constants \(\mathbb{Z}[q]\) through the change of basis \(\sigma_n \mapsto 1\). All face maps \(\partial_0 = \cdots = \partial_n\) coincide. The border operator \(\partial q\mathcal{C}_n(P) \xymatrix{\ar[r]^\partial & q\mathcal{C}_{n-1}(P)}\) is the zero map for \(n = 0\). For \(n \geq 1\)

\[ \partial(a\sigma_n) = \left( \partial_0 + q \partial_1 + q^2 \partial_2 + \cdots + q^n \partial_n \right) (a\sigma_n) = \left( 1 + \cdots + q^n \right) \partial_0 (a\sigma_n) = [n+1]_q a\sigma_{n-1} \]

can be seen as the multiplication by the element \([n+1]_q a\);
It vanishes when \( n + 1 \) a positive multiple of \( N \). In any other case \([n + 1]_q \neq 0\) is a unit in \( \mathbb{Z}[q] \); see \([1.1.1](2)\), so \( \partial \) is a module isomorphism (though not a ring isomorphism). Therefore,

\[
qH_{m,n}(P) = \begin{cases} 
\mathbb{Z}[q] & 0 \leq n = m - 1 \leq N - 2 \\
0 & \text{else}
\end{cases}
\]

coincides with the amplitude homology of the \( N \)-complex given in the first examples \([2.3](2)\).

### 3.4. Exact sequence of a pair.

Given a topological space \( X \) and a subspace \( A \subset X \); we consider as usual the short exact sequence

\[
\begin{array}{cccccccc}
0 & \rightarrow & qSC_n(A) & \rightarrow & qSC_n(X) & \rightarrow & qSC_n(X, A) & \rightarrow & 0
\end{array}
\]

The exact hexagon of \([2.1](6)\) splits to a long exact sequence

\[
\begin{array}{cccccccccccccccc}
\ldots & \rightarrow & qH_{m,n}(A) & \rightarrow & \cdots & \rightarrow & qH_{m,n}(X) & \rightarrow & qH_{m,n}(X, A) & \rightarrow & qH_{N-m,n-m}(A) & \rightarrow & \cdots
\end{array}
\]

\[
\begin{array}{cccccccccccccccc}
\frac{\partial}{\mapsto} & \rightarrow & qH_{N-m,n-m}(X, A) & \rightarrow & \cdots & \rightarrow & \cdots & \rightarrow & 0
\end{array}
\]

In the sequel, given an exact sequence from a splitted hexagon as above we will just write

\[
\begin{array}{cccccccccccccccc}
\ldots & \rightarrow & qH_{m,n}(A) & \rightarrow & qH_{m,n}(X) & \rightarrow & qH_{m,n}(X, A) & \rightarrow & qH_{N-m,n-m}(A) & \rightarrow & \cdots
\end{array}
\]

for short. In particular, this one is the \((N,q)\)-homology sequence of the pair \((X,A)\). There is also a \((N,q)\)-homology sequence of a triple \((X,A,B)\)

\[
\begin{array}{cccccccccccccccc}
\ldots & \rightarrow & qH_{m,n}(A,B) & \rightarrow & qH_{m,n}(X,B) & \rightarrow & qH_{m,n}(X, A) & \rightarrow & qH_{N-m,n-m}(A,B) & \rightarrow & \cdots
\end{array}
\]

As usual, the connecting morphism is obtained by chasing in the diagram.

### 4. Convex product

Now we extend the usual cone construction \([2\text{ p. 38}]\) to a convex product, this will be the operation between \(q\)-chains in order to have a geometric \(N\)-homotopy. Our goal is to construct a homotopy operator \(K\) as in \([2.4]\) from the index map to the identity map in \(\mathbb{R}^{N-1}\). Since the cone constructions increases the dimension in 1, a first attempt should be to iterate the conification from \(N-2\) different affinely independent chosen points. An easier way is to take a convex combination between any two different singular simplexes, we develop this idea.

### 4.1. Convex product.

Suppose that \(X \subset \mathbb{R}^d\) is a convex subspace. Given two simplexes \(\Delta^m \rightarrow X \rightarrow \Delta^n\) and a point

\[
(\alpha; \beta) = (\alpha_0, \ldots, \alpha_m; \beta_0, \ldots, \beta_n) \in \Delta^{m+n+1}
\]
write $|\alpha| = \alpha_0 + \cdots + \alpha_m$ and $|\beta| = \beta_0 + \cdots + \beta_n$; so $|\alpha| + |\beta| = 1$. Consider

$$\tau \ast \sigma : \Delta^{m+n+1} \longrightarrow X$$

$$\tau \ast \sigma(\alpha; \beta) = \begin{cases} \tau(\alpha) & |\beta| = 0 \\ \sigma(\beta) & |\alpha| = 0 \\ |\alpha| \cdot \tau\left(\frac{\alpha}{|\alpha|}\right) + |\beta| \cdot \sigma\left(\frac{\beta}{|\beta|}\right) & \text{else} \end{cases}$$

The simplex $\tau \ast \sigma$ above is unique for each pair $(\tau, \sigma)$ so the map can be extended to a bilinear operation

$$\varphi(\text{SC}_m(X) \times \varphi(\text{SC}_n(X)) \longrightarrow \varphi(\text{SC}_{m+n}(X))$$

For $m = 0$, $\tau(\mathbf{e}_n) = P$ is a single point and $\tau \ast \sigma = P(\sigma)$ is the conification of $\sigma$ to the vertex $P$. In general, $\tau \ast \sigma$ can be thought as a convex combination of $\tau$ and $\sigma$. The convex product satisfies nice properties with respect to the border map.

**Lemma 4.1.1. [Leibnitz rule]** Let $\tau \in \varphi(\text{SC}_m(X)$ and $\sigma \in \varphi(\text{SC}_n(X)$. If $mn > 0$ then

$$\partial(\tau \ast \sigma) = \partial(\tau) \ast \sigma + q^{m+1} \tau \ast \partial(\sigma)$$

**Proof** By the bilinearity of the border map we can suppose that $\tau, \sigma$ are singular simplexes. Apply the definition of the border map, see [3,1]. The face maps behave as follows,

$$\partial_i(\tau \ast \sigma) = \begin{cases} (\partial_i \tau) \ast \sigma & 0 \leq i \leq m \\ \tau \ast (\partial_{i-m-1} \sigma) & m + 1 \leq i \end{cases}$$

\[\square\]

4.2. **Newton’s terms.** Our main goal on this § is to prove a general formula for $\partial^k (\tau \ast \sigma)$. If $\tau, \sigma$ are 0-dimensional singular simplexes then, by definition of the border map; the border of the 1-simplex $\tau \ast \sigma = [\tau(\mathbf{e}_n), \sigma(\mathbf{e}_n)]$ is

$$\partial(\tau \ast \sigma) = \sigma(\mathbf{e}_n) + q\tau(\mathbf{e}_n) = \sigma + q\tau$$

This is the simplest counter-example of the Leibnitz rule since, at [4,1.1] the right side of the equation vanishes. Also, if $m = \dim(\tau) = 0$ and $n = \dim(\sigma) > 0$, applying the definition of the border map we get

$$\partial(\tau \ast \sigma) = \sigma + q\tau \ast \partial(\sigma)$$

This is exactly what happens in the usual case for $N = 2$ and $q = -1$, see [2] p.35 eq.(4.9)]; we will use this in the sequel. Broadly speaking, since $\dim(\tau \ast \sigma) = m + n + 1$, for $k \geq m + n + 2$ all terms in a Newton’s polynomial should vanish. One can conjecture that, for $\min\{m, n\} \leq k \leq m + n + 1$ some of the terms vanish and others perhaps not. Given a singular simplex $\Delta^m \longrightarrow X$, the Newton’s terms of $\tau$ is

$$\mathcal{N}^i(\tau) = \begin{cases} \partial^i(\tau) & 0 \leq i \leq m \\ [m + 1]_q! & i = m + 1 \\ 0 & i \geq m + 2 \end{cases}$$

We will show the following statement,

**Proposition 4.2.1. [Newton’s polynomial]** Let $\tau \in \varphi(\text{SC}_m(X)$ and $\sigma \in \varphi(\text{SC}_n(X)$. Then

$$\partial^k (\tau \ast \sigma) = \sum_{i=0}^{k} q^{(m+1-k-i)} \cdot \left[\begin{array}{c} k \\ i \end{array}\right] \mathcal{N}^{k-i}(\tau) \ast \mathcal{N}^i(\sigma) \quad \forall \ k \geq 0$$

In order to do so, our plan is to check the stationary behavior which begins as soon as $k > \min\{m, n\}$; this will be listed in a sort of lemmas called the tail formulae. We now carry out the plan.
Lemma 4.2.2. [Tail formula #1] Given a simplex $\Delta^m \xrightarrow{\tau} X$ let $T_j = \tau(e_j)$ for $j = 0, \ldots, m$. Then

$$\partial^m(\tau) = [m]_q! \cdot \sum_{j=0}^{m} q^j T_{m-j}$$

[Proof] Let us apply twice the iteration rule of the border map [3.1.1] and reorder the indexes. We get

$$\partial^2(\tau) = \sum_{i=0}^{m-1} \sum_{j=0}^{m} q^{i+j} \partial_i \partial_j(\tau) = (1 + q) \sum_{0 \leq i \leq j \leq m-1} q^{i+j} \partial_i \partial_j(\tau)$$

Notice that the indexes $i, j$ run over $0 \leq i \leq j \leq m - 1$. For $\partial^i(\tau)$ the indexes $i_1, \ldots, i_k$ run over $0 \leq i_1 \leq \cdots \leq i_k \leq m - k + 1$. Finally, for $\partial^m(\tau)$ the indexes $i_1, \ldots, i_m$ run over $0 \leq i_1 \leq \cdots \leq i_m = m - m + 1 = 1$. Therefore,

$$\partial^m(\tau) = [m]_q! \cdot \sum_{0 \leq i_1 \leq \cdots \leq i_m \leq 1} q^{i_1 + \cdots + i_m} \partial_{i_1} \cdots \partial_{i_m}(\tau)$$

$$= [m]_q! \left( \frac{q}{0^+ + 0 + m} \partial_0(\tau) + q \frac{q}{0^+ + 0 + 1} \partial_1 \partial_0(\tau) + \cdots + q \frac{q}{0^+ + \cdots + m} \partial_m(\tau) \right)$$

$$= [m]_q! \left( q \partial_0(\tau) + q \partial_1 \partial_0(\tau) + \cdots + q \partial_m(\tau) \right)$$

$$= [m]_q! \cdot \sum_{j=0}^{m} q^j T_{m-j}$$

□

Lemma 4.2.3. [Tail formula #2] Given two chains $\tau \in \_SC_m(X)$ and $\sigma \in \_SC_n(X)$,

$$\partial^m(\tau) \ast \partial^n(\sigma) = [m + 1]_q! \partial^m(\tau) \ast q [n + 1]_q! \partial^n(\sigma)$$

[Proof] By the bilinearity of the convex product and the linearity of the border map, we can assume that $\Delta^m \xrightarrow{\tau} X \leftarrow \Delta^n$ are two simplexes. By lemma [4.2.2] let us write

$$\partial^m(\tau) = [m]_q! \sum_{j=0}^{m} q^j T_{m-j} \hspace{1cm} \partial^n(\sigma) = [n]_q! \sum_{i=0}^{n} q^i S_{n-i}$$

in a suitable form. Then

$$\partial \left( \partial^m(\tau) \ast \partial^n(\sigma) \right) = \partial \left( [m]_q! \sum_{j=0}^{m} q^j T_{m-j} \ast [n]_q! \sum_{i=0}^{n} q^i S_{n-i} \right)$$

$$= [m]_q! [n]_q! \sum_{j=0}^{m} \sum_{i=0}^{n} q^j q^i \partial (T_{m-j} \ast S_{n-i})$$

$$= [m]_q! [n]_q! \sum_{j=0}^{m} \sum_{i=0}^{n} q^j q^i (S_{n-i} + q \cdot T_{m-j})$$

$$= [m]_q! \left( \sum_{j=0}^{m} q^j \right) \left( [n]_q! \sum_{i=0}^{n} q^i S_{n-i} \right) + q [n]_q! \left( \sum_{i=0}^{n} q^i \right) \left( [m]_q! \sum_{j=0}^{m} q^j T_{m-j} \right)$$

$$= [m + 1]_q! \partial^m(\tau) + q [n + 1]_q! \partial^n(\sigma)$$

as desired. □
Lemma 4.2.4. [Tail formulæ #3] Let $\tau \in qSC_m(X)$ and $\sigma \in qSC_n(X)$. If $mn > 0$ then

\[
\partial^k (\partial^m(\tau) \ast \sigma) = \begin{cases} 
[m+1]_q \cdot \left( \sum_{j=0}^{m} q^j T_{m-j} \right) \ast \sigma & 1 \leq k \leq n \\
[m+1]_q \cdot \left( \sum_{j=0}^{m} q^j \partial (T_{m-j} \ast \sigma) \right) & k = n + 1 \\
0 & \text{else}
\end{cases}
\]

[Proof] By the bilinearity of the convex product and linearity of the border map, it is enough to show it on the generators. Assume that $\sigma, \tau$ are simplexes. Let $T_j = \tau(e_j)$ for $j = 0, \ldots, m$. By (4.2.2) on $\partial^m(\tau)$, equation (11) at (4.2.1) on $T_j \ast \sigma$ for each $j$, the linearity of $\partial$ and the bilinearity of the cone-product;

\[
\partial (\partial^m(\tau) \ast \sigma) = \partial \left( [m]_q \cdot \left( \sum_{j=0}^{m} q^j T_{m-j} \right) \ast \sigma \right) = [m]_q \cdot \sum_{j=0}^{m} q^j \partial (T_{m-j} \ast \sigma) = [m]_q \cdot \sum_{j=0}^{m} q^j (\sigma + q T_{m-j}) = [m]_q \cdot \sum_{j=0}^{m} q^j \partial (\tau) \ast \sigma = [m+1]_q \cdot \sigma + q \cdot \partial^m(\tau) \ast \partial (\sigma)
\]

This proves the equality for $k = 1$; for $2 \leq k \leq n$ apply this rule and use induction on $k$. For $k = n + 1$, by direct calculations

\[
\partial^{n+1} (\partial^m(\tau) \ast \sigma) = \partial \left( \partial^m(\tau) \ast \sigma \right) = \partial \left( [m+1]_q \cdot \left( \sum_{j=0}^{m} q^j T_{m-j} \right) \ast \sigma \right) = [m+1]_q \cdot \left( \sum_{j=0}^{m} q^j \partial (T_{m-j} \ast \sigma) \right) = [m+1]_q \cdot \left( \sum_{j=0}^{m} q^j \partial (\tau) \ast \sigma + q \cdot \partial^{n+1} (\tau) \ast \partial (\sigma) \right)
\]

For the last term we now apply lemma (4.2.3). Then,

\[
\partial^{n+1} (\partial^m(\tau) \ast \sigma) = [m+1]_q \cdot \left( \sum_{j=0}^{m} q^j \partial (\tau) \ast \sigma + q \cdot \partial^{n+1} (\tau) \ast \partial (\sigma) \right)
\]

as desired. \qed

A similar expression can be obtained for $\partial^k (\tau \ast \partial^n (\sigma))$, though we will not need it here.

4.3 Proof of Proposition (4.2.1) We will proceed by double induction on $n + m$ and $k$. For $n + m = 0$ we have $n = m = 0$. Consider the following cases: $k = 0$ which is trivial, $k = 1$ which gives equation (10) at (4.2.1) for $k \geq 2$ we get $\partial^k (\tau \ast \sigma) = 0$ by a dimension argument. This proves (4.2.1) for $n + m = 0$ and $k \geq 0$. For $m + n > 0$ fix some simplexes $\tau, \sigma$ with respective dimensions $m, n$. Let us assume the inductive hypotheses, i.e. that (4.2.3) holds for any pair of simplexes $\tau', \sigma'$ with respective dimensions $m', n'$ such that $m' + n' < m + n$. For $k = 0$ there is nothing to prove. For $k \geq \dim (\tau \ast \sigma) + 1 = m + n + 2$, by a dimension argument, the left side of the Newton’s polynomial at (4.2.1) vanishes. Also, all the terms $N^{k-1}(\tau) \ast N^n(\sigma)$ in the right side vanish since, for any $i \leq k$, we have $i \leq n \Rightarrow k - i > m$ and $k - i \leq m \Rightarrow i > n$, so the statement holds. Hence we only have to check (4.2.1) for $1 \leq k \leq m + n + 1$. 


For $k = 1$ the statement of (4.2.1) is the Leibnitz rule (4.1.1). Notice that $m = \deg(\tau)$ so the power $q^{m+1}$ in the statement of (4.1.1) depends on $\tau$; i.e. $\partial(\tau \ast \sigma) = \partial(\tau) \ast \sigma + q^{\deg(\tau)+1} \tau \ast \partial(\sigma)$. Assume the inductive hypothesis for $k \leq \min\{m,n\} - 1$. Then, by the linearity of the border map,

$$\partial^{k+1}(\tau \ast \sigma) = \partial \left( \partial^k(\tau \ast \sigma) \right) = \partial \left[ \sum_{i=0}^k q^{i(m+1-k+i)} \cdot \left[ \begin{array}{c} k \\ i \end{array} \right] \partial^{k-i}(\tau) \ast \partial^i(\sigma) \right] = \sum_{i=0}^k q^{i(m+1-k+i)} \cdot \left[ \begin{array}{c} k \\ i \end{array} \right] \partial^{k-i}(\tau) \ast \partial^i(\sigma)$$

Since $k \leq \min\{m,n\} - 1$, all the terms $\partial \left( \partial^{k-i}(\tau) \ast \partial^i(\sigma) \right)$ in the last sum satisfy the hypothesis of (4.1.1). Apply the Leibnitz rule to each of them. We get

$$\partial^{k+1}(\tau \ast \sigma) = \partial^{k+1}(\tau) \ast \sigma + \sum_{i=1}^k q^{i(1+i-m-k+i+1)} \cdot \left( \left[ \begin{array}{c} k \\ i \end{array} \right] \cdot q^{i+1} \cdot \left[ \begin{array}{c} k \\ i+1 \end{array} \right] \right) \partial^{k-i}(\tau) \ast \partial^{i+1}(\sigma) + \tau \ast \partial^{k+1}(\sigma)$$

By property (4.1.1) (3) the sum of q-combinatorial numbers can be arranged, so

$$\partial^{k+1}(\tau \ast \sigma) = \sum_{i=0}^{k+1} q^{i(1+i-m-k+i+1)} \cdot \left( \left[ \begin{array}{c} k+1 \\ i+1 \end{array} \right] q^{i+1} \right) \partial^{k-i}(\tau) \ast \partial^{i+1}(\sigma)$$

as desired. We have proved (4.2.1) for $0 \leq k \leq \min\{m,n\}$.

For $\min\{m,n\} + 1 \leq k \leq m + n + 1$ consider the following cases.

- **$m < n$:** We check directly (4.2.1) for $k = m + 1 \leq n$. Notice that

$$\partial^{m+1}(\tau \ast \sigma) = \partial \left( \partial^m(\tau \ast \sigma) \right) = \partial \left( \sum_{i=1}^m q^{i(i+1)} \cdot \left[ \begin{array}{c} m \\ i \end{array} \right] \partial^{m-i}(\tau) \ast \partial^i(\sigma) \right)$$

For $k = m$,

$$= \partial \left( \partial^m(\tau) \ast \sigma \right) + \sum_{i=1}^m q^{i(i+1)} \cdot \left[ \begin{array}{c} m \\ i \end{array} \right] \partial^{m-i}(\tau) \ast \partial^i(\sigma)$$

linearity of $\partial$

$$= \left[ m+1 \right] q^{1} \ast \sigma + q \partial^m(\tau) \ast \partial(\sigma) + \sum_{i=1}^m q^{i(i+1)} \cdot \sum_{i=1}^m \left[ \begin{array}{c} m \\ i \end{array} \right] \partial^{m-i}(\tau) \ast \partial^{i+1}(\sigma) + q^{1} \partial^1(\tau) \ast \partial(\sigma) \ast \partial^1(\sigma)$$

Group similar terms

$$= \left[ m+1 \right] q^{1} \ast \sigma + \sum_{i=1}^m q^{i(i+2)} \cdot \left( \left[ \begin{array}{c} m \\ i+1 \end{array} \right] + q \cdot \left[ \begin{array}{c} m+1 \\ i \end{array} \right] \right) \partial^{m-i}(\tau) \ast \partial^{i+1}(\sigma)$$

This proves (4.2.1) for $k = m + 1$. Let us assume again, by induction on $k$, that we have proved it for any integer from 0 to some $m + 1 \leq k \leq n$. Then, by linearity of the border map and the inductive hypothesis,

$$\partial^{k+1}(\tau \ast \sigma) = \partial \left( \partial^k(\tau \ast \sigma) \right) = \partial \left( \sum_{i=0}^k q^{i(m+1-k+i)} \cdot \left[ \begin{array}{c} k \\ i \end{array} \right] N^{k-i}(\tau) \ast N^i(\sigma) \right) = \sum_{i=0}^k q^{i(m-k+1+i)} \cdot \left[ \begin{array}{c} k \\ i \end{array} \right] \partial \left( N^{k-i}(\tau) \ast N^i(\sigma) \right)$$

By definition of the Newton’s terms at (12) $N^{k-i}(\tau)$ vanishes for $k - i \geq m + 2$. Take only take the terms satisfying $0 \leq k - i \leq m + 1$; i.e. $k - m - 1 \leq i \leq k$. We get,

$$\partial^{k+1}(\tau \ast \sigma) = \sum_{i=k-m-1}^k q^{i(m-k+1+i)} \cdot \left[ \begin{array}{c} k \\ i \end{array} \right] \partial \left( N^{k-i}(\tau) \ast N^i(\sigma) \right)$$

$$= \left[ \begin{array}{c} k \\ k - m - 1 \end{array} \right] q \partial^{m+1}(\tau) \ast \partial^{m-1}(\sigma) + \sum_{i=k-m}^k q^{i(m-k+1+i)} \cdot \left[ \begin{array}{c} k \\ i \end{array} \right] \partial \left( \partial^{k-i}(\tau) \ast \partial^i(\sigma) \right)$$

$$= \left[ m+1 \right] q \left[ \begin{array}{c} k \\ k - m - 1 \end{array} \right] q^{m}(\tau) \ast \partial^{m}(\sigma) + \left[ \begin{array}{c} k \\ k - m \end{array} \right] \partial^{m}(\tau) \ast \partial^{m}(\sigma)$$

$$+ \sum_{i=k-m}^k q^{i(m-k+1+i)} \cdot \left[ \begin{array}{c} k \\ i \end{array} \right] \partial \left( \partial^{k-i}(\tau) \ast \partial^i(\sigma) \right)$$
In the last expression, apply the tail formula (4.2.3) to the second term, and the Leibniz rule (4.1.7) to the terms in the last sum.

\[
\partial^{k+1} (\tau \ast \sigma) = [m + 1]_q \left[ \frac{k + 1}{k - m} \right] q \partial^{m} (\sigma) + \sum_{i=k-m+1}^{k} \left[ \frac{k}{i-1} \right] q \partial^{i} (\tau) + \sum_{i=k-m+1}^{k} \left[ \frac{k}{i} \right] q \partial^{k+1} (\sigma) \partial^{i} (\tau)
\]

Regroup similar terms. Apply property (4.1.11) (3) on the \( q \)-combinatorial numbers;

\[
\partial^{k+1} (\tau \ast \sigma) = [m + 1]_q \left[ \frac{k + 1}{k - m} \right] q \partial^{m} (\sigma) + \sum_{i=k-m+1}^{k} \left[ \frac{k}{i-1} \right] q \partial^{i} (\tau) + \sum_{i=k-m+1}^{k} \left[ \frac{k}{i} \right] q \partial^{k+1} (\sigma) \partial^{i} (\tau)
\]

Include the vanishing terms of the form \( N^{k-i+1} (\tau) \ast N^{i} (\sigma) \) for \( 0 \leq i \leq k - m - 1 \). We obtain

\[
\partial^{k+1} (\tau \ast \sigma) = \sum_{i=0}^{k+1} q^{i(m-k+i)} \left[ \frac{k+1}{i} \right] q N^{k-i+1} (\tau) \ast N^{i} (\sigma)
\]

This is the complete expression of the right term in (4.2.4) for \( k+1 \). Thus we have proved the statement for \( 0 \leq k \leq n + 1 \). Finally, for \( n + 2 \leq k \leq m + n + 1 \) a similar argumentation can be carried out. The tail formulæ must be used in both extremes of the sum.

* \( m \geq n \): We leave the details to the reader.

\[ \square \]

### 4.3.1. Zeroth \( q \)-homology group, augmentation

Since \( \Delta^0 = \{e_0\} \) is a singleton, each 0-dimensional simplex \( \sigma \) in \( X \) can be identified to its image point \( x = \sigma(e_0) \in X \). The 0-th module of \( q \) chains is then

\[
qSC_0 (X) = \bigoplus_{\sigma \in X_0} \mathbb{Z} [q] \cdot \sigma \cong \bigoplus_{x \in X} \mathbb{Z} [q] \cdot x
\]

Consider the morphism

\[
qSC_0 (X) \overset{\epsilon}{\longrightarrow} \mathbb{Z} [q] \quad \sum_i \alpha_i x_i \mapsto \sum_i \alpha_i
\]

Given a \( m \)-simplex \( \Delta^m \overset{\tau}{\longrightarrow} X \) the element \( \partial^m (\tau) \) is a 0-dimensional chain. Let us write \( P_j = \tau(e_j) \) for \( j = 0, \ldots, m \). Applying (4.1.2) we get

\[
\epsilon (\partial^m (\tau)) = [m]_q ! \cdot \sum_{j=0}^{m} q^j = [m + 1]_q !
\]
In particular, for $m = N - 1$ we get $\epsilon \left( \partial^{N-1}(\tau) \right) = 0$ and

\[(12) \quad \mathcal{H}_{1,0}(X) \xrightarrow{\gamma} \mathbb{Z}[q] \quad [\tau] \mapsto \epsilon(\tau)\]

is a well defined linear surjective morphism.

The constant map $X \xrightarrow{\text{id}} P$ induces a morphism of $N$-complexes

\[\varphi SC_n(X) \xrightarrow{\gamma} \varphi SC_n(P)\]

called the **augmentation**. The reduced $q$-homology

\[\tilde{q}H_{m,n}(X) = \ker \left\{ \varphi H_{m,n}(X) \xrightarrow{\gamma} \varphi H_{m,n}(P) \right\}\]

is the kernel of the corresponding homology morphism. By equation (6) at §3.3,

\[qH_{m,n}(X) = \begin{cases} \mathbb{Z}[q] \oplus \tilde{q}H_{m,n}(X) & 1 \leq n = m \leq N - 2 \\ \tilde{q}H_{m,n}(X) & \text{else} \end{cases}\]

A reduced $q$-homology sequence of the pair

\[\cdots \xrightarrow{\tilde{q}H_{m,n}(A)} \tilde{q}H_{m,n}(X) \xrightarrow{qH_{m,n}(A)} \varphi H_{m,n}(X, A) \xrightarrow{\partial} \tilde{q}H_{-m,n-m}(A) \xrightarrow{\cdots}\]

can also be deduced.

5. **$q$-Analog Singular Homology of Convex Spaces**

We arrive to the main result of this article.

5.1. **The index map.** In complete analogy with the usual case ($N = 2, q = -1$), the index map is, in general, the morphism

\[(\varphi SC_*(X), \partial) \xrightarrow{\gamma} (\mathbb{Z}[q], [\ast]_q)\]

that sends each $n$-simplex to $1 \in \mathbb{Z}[q]$ in the corresponding degree, for $0 \leq n \leq N - 2$; and vanishes for $n \geq N - 1$.

**Theorem 5.1.1.** Let $X \subset \mathbb{R}^{N-1}$ be a convex space. Then the index map \(\varphi SC_*(X) \xrightarrow{\eta} \mathbb{Z}[q]\) induces an isomorphism in $N$-homology.

**[Proof]** We follow essentially the same argumentation of [2, p.38]. We will define a map

\[\mathbb{Z}[q] \xrightarrow{\hat{\rho}} \varphi SC_*(X)\]

The composition $\eta \hat{P} = id$ must be the identity map on the $(N - 1)$-complex $(\mathbb{Z}[q], [\ast])$; so $\hat{P}(1) = \nu_n$ will be a single singular $n$-simplex for $0 \leq n \leq N - 2$ and it will vanish for $n \geq N - 1$. The other composition $\hat{P}\eta$ will be $N$-homotopic to the identity map $id$ on $\varphi SC_*(X)$ in the sense of [2, 3]. In order to explain better how we will pick the $\nu_n$’s we will construct a homotopy operator

\[\varphi SC_n(X) \xrightarrow{\kappa} \varphi SC_{n-N+1}(X)\]

and show how it works. We proceed by steps.
• Definition of $K$: Fix some singular $N - 2$-dimensional simplex $\Delta^{N-2} \rightarrow X$. Since $N$ is a prime integer, $[k]_q$ is a unit in $\mathbb{Z}[q]$ for $1 \leq k \leq N - 1$ and therefore $[N - 1]_q!$ is also a unit; see §1.1.1(2). We define

$$K(\sigma) = \frac{1}{[N - 1]_q!} \cdot (i \ast \sigma)$$

(13)

Up to the correction by the constant, $K$ is essentially the convex product of $i$ and $\sigma$; and it can be uniquely extended to $qSC_\ast \left( \mathbb{R}^{N-1} \right)$ by linearity.

• $K$ is a $N$-homotopy: We verify that $K$ satisfies [2.4] Fix a singular simplex $\sigma \in qSC_n(X)$. By §4.2.1 we have

$$\partial_k K \partial^N_{N-1-k-1}(i) = 1 \cdot \sum_{i=0}^{k} \left( \frac{k}{i} \right)_q \cdot N^{k-1}(i) \ast N^j \left( \partial^N_{N-1-k-1}(\sigma) \right)$$

Although $\partial^j(i)$ is a chain and not a simplex, since $\partial^j \left( \partial^i(\sigma) \right) = \partial^{i+j}(\sigma)$ we will assume the following convention,

$$\mathcal{N}^j \left( \partial^i(\sigma) \right) = \left\{ \begin{array}{ll} \partial^{i+j}(\sigma) & j \leq n - i \\ [n + 1]_q! & j = n - i + 1 \\ 0 & \text{else} \end{array} \right.$$ 

Therefore

$$\partial_k K \partial^N_{N-k-1}(i) = 1 \cdot \sum_{i=0}^{k} \left( \frac{k}{i} \right)_q \cdot N^{k-1}(i) \ast N^{N-k-1+i}(\sigma)$$

Taking sums in both sides,

$$\sum_{k=0}^{N-1} \partial_k K \partial^N_{N-k-1}(i) = 1 \cdot \sum_{k=0}^{N-1} \sum_{i=0}^{k} \left( \frac{k}{i} \right)_q \cdot N^{k-1}(i) \ast N^{N-k-1+i}(\sigma)$$

Let us reorder and group all similar terms taking $l = k - i$. We arrive to the following expression

$$\sum_{k=0}^{N-1} \partial_k K \partial^N_{N-k-1}(i) = \frac{1}{[N - 1]_q!} \cdot \sum_{i=0}^{N-1} \alpha_i \mathcal{N}^j(i) \ast \mathcal{N}^{N-1-i}(\sigma)$$

(14)
Let us look for instance the following array of the coefficients $\alpha_{k,l}$ for $N = 7$. The vertical sums of the entries in the table correspond to the values of $\alpha_{l,i}$.

![Table of the coefficients $\alpha_{k,l}$ for $N = 7$. Each horizontal row corresponds to some $0 \leq k \leq 6$ and each vertical column corresponds to a fixed $l = (k - i)$. The powers of $q$ have been simplified with the identity $q^7 = 1$.](image)

These coefficients can be simplified by using the properties of $q$-numbers. A simple inspection suggests that $\alpha_{l} = 0$ for $0 \leq l \leq N - 2$. This is, indeed, the case. Let us write

$$\alpha_l = \sum_{i=k-1}^{l} \alpha_{k,i} = \sum_{i=k-1}^{l} q^{i(N-1-k+i)} \left\{ \begin{array}{c} k \\ i \end{array} \right\}_q$$

$$= \sum_{i=0}^{N-l-1} q^{i(N-1-i)} \left\{ \begin{array}{c} l+i \\ i \end{array} \right\}_q$$

$$= \sum_{i=0}^{s} q^{i} \cdot \left[ \begin{array}{c} N-1-s+i \\ i \end{array} \right]_q$$

$$= \sum_{i=0}^{s} q^{i} \cdot \left[ \begin{array}{c} N-1-s+i \\ N-1-s \end{array} \right]_q = \beta_s$$

symmetry of combinatorials

We check that $\beta_s = \alpha_{N-1-s} = 0$ for $1 \leq s \leq N - 1$. For $s = 1$,

$$\beta_1 = q^0 + q^1 \left[ \begin{array}{c} N-1 \\ N-2 \end{array} \right]_q = 1 + q [N-1]_q = [N]_q = 0 = \alpha_{N-2}$$
Assume that $\beta_s = 0$ for some $s \leq N - 2$. Then,

$$\beta_{s+1} = \sum_{i=0}^{s+1} q^{(s+1)\iota} \left[ \begin{array}{c} N - 2 - s + i \\ N - 2 - s \end{array} \right] = \sum_{i=0}^{s+1} q^{i\iota} \left[ \begin{array}{c} N - 2 - s + i \\ N - 2 - s \end{array} \right]$$

by definition

$$= 1 + \sum_{i=1}^{s+1} q^{i\iota} \left[ \begin{array}{c} N - 1 - s + i \\ N - 1 - s \end{array} \right] - \left[ \begin{array}{c} N - 2 - s + i \\ N - 1 - s \end{array} \right]$$

by (11.1) (3)

$$= 1 + \sum_{i=1}^{s+1} q^{i\iota} \left[ \begin{array}{c} N - 1 - s + i \\ N - 1 - s \end{array} \right] - \sum_{i=1}^{s+1} q^{i\iota} \left[ \begin{array}{c} N - 2 - s + i \\ N - 1 - s \end{array} \right]$$

$$= 1 + \left( (-1)^j + \sum_{i=0}^s q^{i\iota} \left[ \begin{array}{c} N - 1 - s + i \\ N - 1 - s \end{array} \right] + q^{(s+1)\iota} \left[ \begin{array}{c} N \\ N - 1 - s \end{array} \right] \right)$$

$$- \sum_{i=1}^{s+1} q^{i\iota} \left[ \begin{array}{c} N - 2 - s + i \\ N - 1 - s \end{array} \right]$$

split the first sum

$$= \sum_{i=0}^s q^{i\iota} \left[ \begin{array}{c} N - 1 - s + i \\ N - 1 - s \end{array} \right] - \sum_{i=0}^s q^{(s+1)\iota} \left[ \begin{array}{c} N - 1 - s + j \\ N - 1 - s \end{array} \right]$$

$[N]_q = 0, i = j + 1$(2nd sum)

$$= (1 - q^{\iota})\beta_s = 0$$

by definition

By equation (14), the definition of the Newton’s terms and a dimension argument on $\sigma$, we deduce that

$$\sum_{k=0}^{N-1} \partial^k K \partial^{N-k-1} (\sigma) = \frac{\alpha_{N-1}}{[N-1]_q} N^{N-1} (i) * X^\sigma (\sigma) = \sigma$$

whenever $n = \dim(\sigma) \geq N - 1$, and the whole sum in the left term vanishes when $n < N - 1$. In other words,

$$\sum_{k=0}^{N-1} \partial^k K \partial^{N-k-1} (\sigma) = \begin{cases} \sigma & \text{dim}(\sigma) \geq N - 1 \\ 0 & \text{else} \end{cases}$$

\[\square\]

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