DEGENERATION OF HODGE STRUCTURES ON I-SURFACES

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Abstract. Work of Green, Griffiths, Laza, and Robles suggests that the moduli space of (smoothable) stable surfaces should admit a natural stratification defined via Hodge theoretic data. In the case of stable surfaces with $K^2_X = 1$ and $\chi(X) = 3$ we compute the Hodge type of all examples known to us and show that all predicted degenerations are geometrically realised.

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1. Introduction

The study of surfaces of general type and their moduli spaces constructed by Gieseker is a classical topic of investigation. In recent years, additional interest has arisen from the construction of a modular compactification, the moduli space of stable surfaces.

Already the Gieseker moduli space $\mathcal{M}_{K^2,\chi}$ can be arbitrarily singular and, in general, the compactification $\overline{\mathcal{M}}_{K^2,\chi}$ also does not have good geometric properties. However, in their investigation of compactifications of period domains, Green, Griffiths, Laza, and Robles suggested the existence of a stratification of the closure of the Gieseker moduli space determined by Hodge-theoretic data. In its coarsest form, the Hodge type reflects the weights with respect to the mixed Hodge structure of the cohomology classes contributing to $p_g$. Several new features (and difficulties) occur for $p_g \geq 2$. We will explain some of this in Section 3, but want to mention here that the final aim would be to use even more information from the Hodge structure to study the geometry of the moduli space, compare [GGR21].

One of the easiest test cases for these ideas are minimal surfaces with $K^2_X = 1$ and $\chi(X) = 3$, baptised I-surfaces by Green et al. These had been studied by the three last named authors in [FPR17] as part of their investigation of Gorenstein
stable surfaces with $K_X^2 = 1$. The interaction of the Hodge theoretic and geometric approaches, also manifest at the 2021 Meeting “Moduli and Hodge Theory, IMSA Miami” has kept our interest in this class of surfaces alive and led to the papers [FPRR22] and [CFP+21].

In the present paper, our purpose is modest: we want to exhibit a zoo of examples of I-surfaces, mostly non-normal, and show how relatively easy geometric constructions realise all degenerations of the Hodge type predicted by general theory (cf. [Rob17]). As our understanding of the moduli space of I-surfaces is still limited, there is no way we can claim to give a complete discussion in any sense.

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2. Stable surfaces and Kollár’s glueing

The (semi-log)-canonical reference for the basic definitions and properties of stable surfaces is [Kol13].

We simply recall that a stable surface $X$ has ample $\mathbb{Q}$-Cartier canonical divisor $K_X$ and semi-log-canonical singularities.

If $X$ is a stable normal surface and $(X, x)$ is an isolated surface singularity, then $(X, x)$ is either a rational or elliptic singularity. In the second case, considering the minimal resolution $\sigma: (\bar{Y}, E) \to (X, x)$, $x$ is said to be simple elliptic if $E$ is smooth of genus 1, and is said to be a cusp if $p_a(E) = 1$ and $E$ is a cycle of rational curves. The degree of the singularity is $-E^2$.

If $X$ is a non-normal stable surface we consider its normalisation $\pi: \bar{X} \to X$ and we recall that the non-normal locus $D \subset X$ and its preimage $\bar{D} \subset \bar{X}$ are pure of codimension 1, i.e., they are curves. Since $X$ has ordinary double points at the generic points of $D$, the map on normalisations $\bar{D}^\nu \to \bar{D}^\nu$ is the quotient by an involution $\tau$. Kollár’s glueing principle says that $X$ can be uniquely reconstructed from $(\bar{X}, \bar{D}, \tau: \bar{D}^\nu \to \bar{D}^\nu)$ via the following two push-out squares:

\[
\begin{array}{ccc}
\bar{X} & \xleftarrow{\iota} & \bar{D} \\
\downarrow{\pi} & & \downarrow{\pi} \\
X & \xleftarrow{\iota} & D
\end{array}
\]

(2.1)

More precisely, we have the following.

**Theorem 2.2** ([Kol13, Thm. 5.13]) — **Associating to a stable surface $X$ the triple $(\bar{X}, \bar{D}, \tau: \bar{D}^\nu \to \bar{D}^\nu)$ induces a one-to-one correspondence**

\[
\left\{ \text{stable surfaces} \right\} \leftrightarrow \left\{ (\bar{X}, \bar{D}, \tau) \mid \begin{array}{l}
(\bar{X}, \bar{D}) \text{ log-canonical pair with } K_{\bar{X}} + \bar{D} \text{ ample,} \\
\tau: \bar{D}^\nu \to \bar{D}^\nu \text{ involution s.th. } \text{Diff}_{\bar{D}^\nu}(0) \text{ is } \tau\text{-invariant.}
\end{array} \right\}
\]

where $\text{Diff}_{\bar{D}^\nu}(0)$ is the different (see [Kol13, 5.11] for the definition).
2.A. I-surfaces. An I-surface $X$ is a stable surface with $K_X^2 = 1$ and $\chi(X) = 3$. If $X$ is smooth this is equivalent to saying that $X$ is a minimal surface of general type with $K_X^2 = 1$, $p_g(X) = 2$ and $q(X) = 0$. 

By Theorem 2.2 a triple $(\bar{X}, \bar{D}, \tau)$ corresponds to an I-surface if and only if the following four conditions are satisfied:

**Stable pair condition:** $(\bar{X}, \bar{D})$ is an lc pair such that $K_{\bar{X}} + \bar{D}$ is an ample $\mathbb{Q}$-Cartier divisor.

$K_X^2$-condition: $(K_{\bar{X}} + D)^2 = 1$.

**Glueing condition:** $\tau: \bar{D}^\nu \to \bar{D}^\nu$ is an involution such that $\text{Diff}_{\bar{D}^\nu}(0)$ is $\tau$-invariant.

$\chi$-condition: The holomorphic Euler-characteristic of the non-normal locus $D$ is $\chi(D) = 3 - \chi(\bar{X}) + \chi(\bar{D})$.

In [FPR17] it was shown that the classical description of smooth surfaces of general type with $K_X^2 = 1$ and $\chi(X) = 3$ extends to the Gorenstein case, i.e.:

- a Gorenstein I-surface $X$ is canonically embedded as a hypersurface of degree 10 in (the smooth locus of) $\mathbb{P}(1, 1, 2, 5)$;
- the moduli space $\mathbb{M}^{(\text{Gor})}_{13}$ of Gorenstein stable surfaces with $K^2 = 1$ and $\chi = 3$ is irreducible and rational of dimension 28.
- for a Gorenstein I-surface $X$, the bicanonical map is a degree 2 morphism $\varphi_2: X \to \mathbb{Q}_2 \subset \mathbb{P}^3$, where $\mathbb{Q}_2$ is the quadric cone, branched on the vertex $o$ and on a quintic section $B$ of $\mathbb{Q}_2$ not containing $o$.
- conversely, if $B$ is a quintic section of $\mathbb{Q}_2$ such that $(\mathbb{Q}_2, \frac{1}{2}B)$ is a log-canonical pair, then the double cover of $\mathbb{Q}_2$ branched on $B$ and $o$ is a stable I-surface; it is Gorenstein if $B$ does not contain $o$, else it is 2-Gorenstein.

3. Stratification of $\mathcal{M}$ from Hodge structures

3.A. The mixed Hodge structure on $H^2(X)$. Here we follow roughly the exposition of [Ant20].

The stratification we will define and study later is motivated by work (partially in progress) of Green, Griffiths, Kerr, Laza and Robles [GGR21, KPR19, Rob16, Rob17] about degenerations of Hodge structures. Roughly, there should be a stratification of the moduli space of the surfaces under investigation, according to the type of polarised mixed Hodge structure on $H^2(X)$. It should be noted that the details about this Hodge-theoretic stratification are not fully understood yet. Therefore, we can only give an informal description. We refer to Robles’ exposition [Rob17] and the references therein for more details; for the basic theory of mixed Hodge structures see Durfee’s short introduction [Dur83] and the comprehensive account by Peters and Steenbrink [PS08].

Given a flat degeneration $\mathfrak{X} \to \Delta$, where $\Delta \subset \mathbb{C}$ is the unit disc and where all fibres $\mathfrak{X}_s$, $s \in \Delta^* = \Delta - 0$ are smooth and projective, we can can associate a limiting polarised mixed Hodge structure (LMHS for short) with the family of Hodge structures $H^2(\mathfrak{X}_s; \mathbb{C})$ (see [GGR21]). Moreover, we can compare this LMHS with Deligne’s natural mixed Hodge structure on the cohomology of the special fibre $H^2(\mathfrak{X}_0; \mathbb{C})$ via the specialisation map. In general, the two mixed Hodge structures are related by the Clemens–Schmid exact sequence (cf. [Cle77]) and its generalizations (cf. [KLS21, KL20]).

For our application we consider the case where $X = \mathfrak{X}_0$ is a stable surface. In this case we consider Deligne’s mixed Hodge structure on $X$, since $X$ is cohomologically insignificant. That is, for every $n$, the Deligne mixed Hodge structure on $H^n(\mathfrak{X}_0; \mathbb{C})$
and the limiting mixed Hodge structure on $H^n(X_s; \mathbb{C})$ agree on $(p, q)$-components where $pq = 0$. This follows since a projective variety with at worst du Bois singularities is cohomologically insignificant by a result of Steenbrink [St81, Theorem 2], and semi-log-canonical surfaces are Du Bois (cf. Kollár [Kol13, Corollary 6.32] or Kovács, Schwede, Smith [KSS10, Theorem 4.16]). See also [KLS21, Thm 9.3 and Cor. 9.9] for a generalisation of the Clemens–Schmid exact sequence and applications to slc varieties.

Now we focus on $I$-surfaces (see §2.A). Assume that $X$ is smoothable. Since $h^{2,0}(X) = h^{0,2}(X) = p_g(X) = 2$ for a smooth minimal surface $X$ of general type satisfying $K^2_X = 1$ and $\chi(X) = 3$, our case of interest corresponds to the Hodge numbers $h^{1,1} = 2$. That is, the Deligne splitting $H^2(X; \mathbb{C}) = \bigoplus_{p+q=2} H^2(X; \mathbb{C})^{(p,q)}$ of the limiting mixed Hodge structure has a Hodge diamond (indicating $\dim H^2(X; \mathbb{C})^{(p,q)}$)

\[
\begin{array}{c}
\hat{1}_1 \\
\hat{0}_0 \quad \hat{0}_1 \\
\hat{1}_2 \\
\hat{0}_2
\end{array}
\]

Figure 1. Degeneration diagram for the Hodge types.

Definition 3.1 — Let $X$ be an $I$-surface (see §2.A). Then $X$ is said to be of $Hodge type \hat{r}_s$ if $r = \dim H^2(X; \mathbb{C})^{(0,0)}$ and $s = \dim H^2(X; \mathbb{C})^{(1,0)}$, where $H^2(X; \mathbb{C})^{(p,q)}$ is the $(p,q)$-component of Deligne’s mixed Hodge structure on $H^2(X; \mathbb{C})$.

Among these Hodge types there is the so-called polarised relation, defined by $\hat{r}_s \leq \hat{t}_u$ if and only if $r \leq t$ and $r + s \leq t + u$. This relation comes from representation theory and has a geometric meaning which follows by the analysis of the extension of period domain and the period map (cf. Robles [Rob17]). Roughly speaking $\hat{r}_s \leq \hat{t}_u$ means that the Hodge type $\hat{t}_u$ is “more degenerate” than $\hat{r}_s$.

In our case the polarised relations among Hodge types for an $I$-surface are illustrated in the Degeneration Diagram in Figure 1 (cf. [Rob17, Example 4.10] for a detailed explanation).

Smooth varieties have Hodge structures of pure weight, that is, they are of type $\hat{0}_0$. Our purpose is to show that we can realise in several different ways all the polarised relations illustrated in Figure 1 by considering the stratification given by the singularities of $I$-surfaces. We will ignore $h^{1,1}$ just as we will ignore canonical surface singularities.
3.B. The Mayer–Vietoris exact sequence. Let us provide some information on how to compute Deligne’s mixed Hodge structure on the second cohomology and more specifically the Hodge type in practice.

**Proposition 3.2** — There are Mayer–Vietoris type exact sequences of mixed Hodge structure in the following situations:

(i) Let $X$ be a demi-normal surface, $\pi: \breve{X} \to X$ its normalisation, $D \subset X$ the non-normal locus and $\breve{D} \subset \breve{X}$ its preimage (see §2). Then the commutative diagram associated to the normalisation map

\[
\begin{array}{ccc}
\breve{X} & \xleftarrow{\iota} & \breve{D} \\
\downarrow{\pi} & & \downarrow{\pi|_{\breve{D}}} \\
X & \xleftarrow{\iota} & D
\end{array}
\]

induces an exact sequences of mixed Hodge structures

\[
\begin{align*}
H^1(X, \mathbb{C}) & \xrightarrow{(\pi^{*}, \iota^{*})} H^1(\breve{X}, \mathbb{C}) \oplus H^1(D, \mathbb{C}) \xrightarrow{\iota^{*}-\pi|_{D}^{*}} H^1(\breve{D}, \mathbb{C}) \\
\to H^2(X, \mathbb{C}) & \xrightarrow{(\pi^{*}, \iota^{*})} H^2(\breve{X}, \mathbb{C}) \oplus H^2(D, \mathbb{C}) \xrightarrow{\iota^{*}-\pi|_{D}^{*}} H^2(\breve{D}, \mathbb{C}) \to \cdots
\end{align*}
\]

(ii) Let $\breve{X}$ be a normal surface and $\sigma: \breve{Y} \to \breve{X}$ a resolution. Let $S = \{p_1, \ldots, p_k\}$ be the singular points of $X$ and $E_i$ the exceptional curve over $p_i$. Then there is an exact sequence

\[
\begin{align*}
H^1(\breve{X}, \mathbb{C}) & \xrightarrow{\sigma^{*}} H^1(\breve{Y}, \mathbb{C}) \to \bigoplus \, H^1(E_i, \mathbb{C}) \\
\to H^2(\breve{X}, \mathbb{C}) & \xrightarrow{\sigma^{*}} H^2(\breve{Y}, \mathbb{C}) \to \bigoplus \, H^2(E_i, \mathbb{C})
\end{align*}
\]

(iii) Let $C$ be a curve with singular points $p_1, \ldots, p_r$ and $\nu: C^{\nu} = \bigsqcup \, C_i^{\nu} \to C$ the normalisation. Then there is an exact sequence

\[
\begin{align*}
0 & \to H^0(C, \mathbb{C}) \to H^0(C^{\nu}, \mathbb{C}) \oplus \bigoplus \, H^0(p_i, \mathbb{C}) \to \bigoplus \, H^0(\nu^{-1}(p_i), \mathbb{C}) \\
\to H^1(C, \mathbb{C}) & \xrightarrow{\nu^{*}} \bigoplus \, H^1(C_i^{\nu}, \mathbb{C}) \to 0
\end{align*}
\]

**Proof.** The Proposition follows from [PS08, Corollary-Definition 5.37]. Concerning (iii), a detailed exposition in the case where $C$ has only nodes is given in [Dur83, §4].

In particular for a stable surface $X$, if $(X, p)$ is a simple elliptic singularity with minimal resolution $\sigma: (\breve{Y}, E) \to (X, p)$, then $E$ is a smooth elliptic curve, hence $H^1(E, \mathbb{C})$ carries a pure Hodge structure of weight 1. If $p$ is a cusp then $E$ is a cycle of rational curves, hence $H^1(E, \mathbb{C})$ carries a mixed Hodge structure concentrated in weight 0. The restriction maps in the Mayer–Vietoris exact sequences described above can be used to compute the mixed Hodge structure on $X$ (cf. [Ant20, §2]).

**Remark 3.6** — We collect some slogans about what determines the mixed Hodge type on the second cohomology, that follow from the above exact sequences:
Smooth varieties have Hodge structures of pure weight.
Rational singularities do not affect the Hodge type of a stable surface. In particular, a surface $X$ with only rational singularities has Hodge structure of pure weight.
Isolated irrational singularities may introduce the HS of the exceptional curve.
Non-normal glueings may introduce the HS of the non-normal locus.
On a curve, the weight filtration on the first cohomology captures the genus of the normalisation.

Remark 3.7 — We want to emphasise again that the Hodge type $\phi_{r,s}$ is a numerical datum, which forgets a lot of information contained in the mixed Hodge structure. In particular, both the integral structure as well as the extension data of the graded pieces should capture much more of the geometry of the surface, including information about the rational singularities. We refer to [GGR21, §5.2] for the period mapping and a detailed description of the extension data for I-surfaces.

4. Examples of degenerations in I-surfaces

4.A. Double covers of the cone. Smooth I-surfaces can be realised as double covers of the quadric cone $Q_2 \subset \mathbb{P}^3$, branched over on the vertex $o$ and on a smooth quintic section $B$ of $Q_2$ not containing $o$. Therefore we can obtain degenerations of a smooth I-surface by letting $B$ acquire suitable singularities (see §2.A).

Since the quadric cone can be seen as the image of the second Veronese map of $\mathbb{P}(1,1,2)$, we identify $Q_2$ with $\mathbb{P}(1,1,2)$ and we describe I-surfaces that arise as double covers of the quadric cone in $\mathbb{P}^3$ as a hypersurfaces $X$ of degree ten in $\mathbb{P}(1,1,2,5)$ given by an equation
$$z^2 - f_{10}(x_0, x_1, y) = 0.$$ Denote by $B = \{ f_{10} = 0 \} \subset \mathbb{P}(1,1,2)$ the branch locus of the double cover. Then the following hold:
- $X$ has slc singularities if and only if $(\mathbb{P}(1,1,2), \frac{1}{2}B)$ is an lc pair.
- $X$ is non-normal if and only if $f$ has a multiple factor.
- $X$ is Gorenstein if and only if the branch curve $B$ does not pass through the vertex, that is, the singular point $(0:0:1)$ of $\mathbb{P}(1,1,2)$.

Let us quickly review what kind of singularities we can allow on the branch curve $B$ in a smooth point of $\mathbb{P}(1,1,2)$ so that $X$ is slc.
If $p \in B$ is an isolated singularity of $B$, then one of the following occurs:
- $p$ is a negligible singularity: $p$ is a double point or a triple point such that every point infinitely near to $p$ is at most double for the strict transform of $B$. The preimage of $P$ in $X$ is a canonical singularity.
- $p$ is a quadruple point such that for every point $q$ infinitely near to $p$ the local intersection number at $q$ of the strict transform of $B$ and the exceptional curve is at most 2. The preimage of $p$ in $X$ is an elliptic Gorenstein singularity of degree 2. The exceptional curve in the minimal resolution is a smooth elliptic curve if and only if the quadruple point is ordinary, else it is a cycle of rational curves.
- $p$ is a $[3,3]$-point, namely $p$ is a triple point with an infinitely near triple point $p_1$ such that for every point $q$ infinitely near to $p_1$ the local intersection number at $q$ of the strict transform of $B$ and the exceptional curve is at most 2. The preimage of $p$ in $X$ is an elliptic Gorenstein singularity of degree 1.
The exceptional curve in the minimal resolution is a smooth elliptic curve if and only if the infinitely near triple point is ordinary, else it is a cycle of rational curves.

4.A.1. Normal double covers of the cone. Clearly, a smooth I-surface has Hodge type $\hat{0},0$ and we want to show quickly that all possible degenerations of the Hodge type already occur for normal surfaces. A different stratification by number and degree of elliptic points was given in [FPR17]. But since cusps were not considered in loc. cit., we cannot cite all desired degenerations directly from there. For the reader’s convenience, we reproduce the resulting stratification together with the corresponding Hodge type in Figure 2. Some information on the general element in each stratum is given in Table 1, where

$$\mathcal{M}_{d_1, \ldots, d_k} = \left\{ X \in \mathcal{M}_{1,3}^{(Gor)} \mid X \text{ is normal and has exactly } k \text{ elliptic singularites of degree } d_1 \leq \cdots \leq d_k \right\}.$$ 

Table 1. Irreducible strata of normal surfaces in $\mathcal{M}_{1,3}^{(Gor)}$

| stratum | dimension | minimal resolution $\hat{X}$ | $\kappa(\hat{X})$ |
|---------|-----------|-------------------------------|------------------|
| $\mathcal{M}_{\emptyset}$ = $\mathcal{M}_{1,3}$ | 28 | general type | 2 |
| $\mathcal{M}_2$ | 20 | blow up of a K3-surface | 0 |
| $\mathcal{M}_1$ | 19 | minimal elliptic surface with $\chi(\hat{X}) = 2$ | 1 |
| $\mathcal{M}_{2,2}$ | 12 | rational surface | $-\infty$ |
| $\mathcal{M}_{1,2}$ | 11 | rational surface | $-\infty$ |
| $\mathcal{M}_{1}^{R}$ | 10 | rational surface | $-\infty$ |
| $\mathcal{M}_{1,1}^{E}$ | 10 | blow up of an Enriques surface | 0 |
| $\mathcal{M}_{1,1,2}$ | 2 | ruled surface with $\chi(\hat{X}) = 0$ | $-\infty$ |
| $\mathcal{M}_{1,1,1}$ | 1 | ruled surface with $\chi(\hat{X}) = 0$ | $-\infty$ |

Now we explain one possibility to realise everything in Figure 1: fix two general points $p_1, p_2$ on quadric the cone and take three general planes $H_3, H_4, H_5$ containing both points. Now consider the family of all pairs of planes $H_1, H_2$ such that if one takes $B = \sum_i H_i$ as branch divisor then the resulting surface is a normal stable surface $X$. 
Then the following happens:

(i) if \( H_1, H_2 \) are general, then \( X \) has canonical singularities;
(ii) if \( H_1 \) is general and \( H_2 \) passes through \( p_2 \) in a general way, then \( p_2 \) is an ordinary quadruple point for \( B \) and \( X \) has one simple elliptic singularity of degree 2;
(iii) if \( H_1 \) is general and \( H_2 \) is tangent to \( H_3 \) in \( p_2 \), then \( p_2 \) is a degenerate quadruple point and \( X \) has one cusp singularity of degree 2;
(iv) if \( H_1 \) passes through \( p_1 \) in a general direction, then then \( p_1 \) and \( p_2 \) are ordinary quadruple points and \( X \) has two elliptic singularities of degree 2;
(v) if \( H_1 \) passes through \( p_1 \) in a general direction and \( H_2 \) is tangent to \( H_3 \) in \( p_2 \), then \( X \) has one elliptic and one cusp singularity of degree 2;
(vi) if \( H_1 \) is tangent to \( H_3 \) at \( p_1 \), then \( X \) has two cusp singularities of degree 2.

According to general principles ((3.4)), this should realise all degenerations in Figure 1, if we check that the exceptional curves in the resolution give rise to independent classes in \( H^2(X) \). This can be easily checked using the classification of cases given in [FPR17], that we have summarized in Table 1.

In case (ii) the minimal resolution \( \tilde{Y} \) of \( X \) is the blow-up of a K3 surface. Denoting by \( E_2 \) the elliptic curve coming from the resolution of the singular point over \( P_2 \), equation (3.4) reads as follows

\[
0 = H^1(\tilde{Y}, \mathbb{C}) \longrightarrow H^1(E_2, \mathbb{C}) \longrightarrow H^2(X, \mathbb{C}) \longrightarrow \sigma^* H^2(\tilde{Y}, \mathbb{C}) \longrightarrow 0
\]

In this case \( H^1(E_2, \mathbb{C}) \) carries a pure Hodge structure of weight 1 with \( h^{1,0}(E_2) = 1 \). Therefore the mixed Hodge structure of \( X \) is of type \( \hat{0}_{0,1} \), that is, we have realised the degeneration of Hodge types \( \hat{0}_{0,0} \to \hat{0}_{0,1} \).

If we are in case (iv) then our surface \( X \) has two simple elliptic singularities and the minimal resolution \( \tilde{Y} \) of \( X \) is a rational surface. Arguing as above, equation (3.4) shows that the classes of the exceptional divisors \( E_1 \) and \( E_2 \) give independent classes in \( H^2(X) \), hence we obtain a further degeneration of Hodge types \( \hat{0}_{0,1} \to \hat{0}_{0,2} \).

If we are in case (iii) then \( X \) has one cusp singularity of degree 2 over \( P_2 \) and its minimal resolution is again the blow-up of a K3 surface. This is in fact a degeneration of case (ii). Denoting by \( E_2 = \sum_{i=1}^r E_{2,i} \) the cycle of rational curve coming from the resolution of the singular point over \( p_2 \), and by \( \{ q_1, \ldots, q_r \} \) the set of singular points of \( E_2 \), equation (3.4) and equation (3.5) show that we get a mixed Hodge structure of type \( \hat{0}_{1,0} \). Indeed we get

\[
0 = H^1(\tilde{Y}, \mathbb{C}) \longrightarrow H^1(E_2, \mathbb{C}) \longrightarrow H^2(X, \mathbb{C}) \longrightarrow \sigma^* H^2(\tilde{Y}, \mathbb{C}) \longrightarrow 0
\]

To study \( H^1(E_2, \mathbb{C}) \) we consider the exact sequence (3.5)

\[
0 \longrightarrow \mathbb{C} = H^0(E_2, \mathbb{C}) \longrightarrow \bigoplus_i H^0(E_{2,i}, \mathbb{C}) \oplus \bigoplus_i H^0(q_i, \mathbb{C}) \longrightarrow \bigoplus_i H^0(\nu^{-1}(q_i), \mathbb{C}) \longrightarrow H^1(E_2, \mathbb{C}) \longrightarrow \bigoplus_i H^1(E_{2,i}, \mathbb{C})
\]

Now it is \( \bigoplus_i H^0(E_{2,i}, \mathbb{C}) \cong \mathbb{C}^r \cong \bigoplus_i H^0(q_i, \mathbb{C}), \bigoplus_i H^0(\nu^{-1}(q_i), \mathbb{C}) \cong \mathbb{C}^{2r} \) and \( \bigoplus_i H^1(E_{2,i}, \mathbb{C}) = 0 \). Moreover \((H^1(\tilde{Y}))^{0,0} = (H^2(\tilde{Y}))^{0,0} = 0\), hence \((H^2(X))^{0,0}\) is isomorphic to the \((0,0)\) components of \( H^1(E_2) \). Summing up, we have obtained the degeneration \( \hat{0}_{0,0} \to \hat{0}_{0,1} \to \hat{0}_{1,0} \).
In case (v) and case (vi) we repeat the above arguments for the two singular points, obtaining the degeneration $\mathcal{O}_{0,0} \to \mathcal{O}_{0,1} \to \mathcal{O}_{1,0} \to \mathcal{O}_{1,1}$ and $\mathcal{O}_{0,0} \to \mathcal{O}_{0,1} \to \mathcal{O}_{1,0} \to \mathcal{O}_{2,0}$, respectively.

**Remark 4.1** — In Table 1 we find degenerations $X$ with three elliptic points and minimal resolution a (possibly non-minimal) surface $\bar{X}$ ruled over an elliptic curve. Therefore, the only possible exceptional elliptic cycles are smooth elliptic curves embedded as multi-sections in $\bar{X}$. Thus the relevant part of (3.4) becomes

$$0 \longrightarrow H^1(\bar{X}, \mathbb{C}) \longrightarrow \oplus_{i=1}^2 H^1(E_i, \mathbb{C}) \longrightarrow H^2(X, \mathbb{C}) \longrightarrow H^2(\bar{X}, \mathbb{C})$$

because each multisection carries, via its map to the base curve of the ruled surface the first rational cohomology. Thus the Hodge type turns out to be $\mathcal{O}_{0,2}$. Note that none of the elliptic points can degenerate to a cusp because of the geometry of the minimal resolution. If we want to degenerate the Hodge type further we need to degenerate $X$ to a non-normal surface.

**4.A.2. Non-normal double covers of the cone.** If we want to consider non-normal double covers of the cone then we have a non-reduced branch divisor, that is, $f = gd^2$ and $B = B_0 + 2D$,

and the normalisation diagram (2.1) becomes

\[
\begin{array}{ccc}
\bar{X} & \xleftarrow{i} & \bar{D} \\
\downarrow{\pi} & & \downarrow{\pi} \\
X & \xleftarrow{i} & D \\
\text{branched over } B_0 \text{ and vertex} & & \text{branched over } B_0 \cap D \text{ and vertex} \\
\mathbb{P}(1,1,2)
\end{array}
\]

If both $B_0$ and $D$ are general then the normalisation is smooth and so are the conductor curves. The possibilities are listed in Table 2. The first two cases yield Gorenstein non-normal I-surfaces and are described in [FPR17]. The other three cases are new and yield 2-Gorenstein I-surfaces, which have singular normalisation.

The list is obtained by considering the possible degrees of $B_0$ and $D$. The genus of $D$ follows by using the adjunction formula for a complete intersection curve in $\mathbb{P}^3$. $K_X$ and $K_D$ can be computed via the Riemann-Hurwitz formula looking at the double covers $X \to \mathbb{P}(1,1,2)$ branched over $B_0$ and the vertex, respectively, $D \to D$ branched over $B_0 \cap D$ and the vertex.

Note that a smooth curve in $\mathbb{Q}_2$ passes through the vertex if its degree is odd. If such a curve is a component of $B$ then $X$ is not Gorenstein.

The curve $D$ can have at most nodes as singularities. If $p$ is a node of $D$ then $p \notin B_0$ and in the double cover we get a degenerate cusp locally given by the equation $z^2 - u^2 v^2 = 0$. If $p$ is a smooth point of $D$, then one of the following occurs:

- $p \notin B_0$ and the preimage in $X$ is a normal crossing point.
- $D$ and $B_0$ intersect transversely at $p$ and the preimage in $X$ is a pinch point.
- The local intersection multiplicity $(D, B_0)_p = 2$ and $B_0$ is either smooth or has an $A_n$ singularity at $p$. The preimage is a degenerate cusp, usually denoted with $T_{2,q,\infty}$ for $q \geq 3$ and locally described by the equation $z^2 - u^2(v^2 - u^{q-2})$. 
This can be checked by blowing up and analysing the cases similarly to what is done in [LR12] or from the point of view of log-canonical threshold in [KSC04, 6.5]. Equipped with this knowledge of possible singularities we will use the ideas from Remark 3.6 to show that in each of the cases in Table 2, an appropriately chosen degeneration of the branch data will allow for all possible degenerations of the Hodge type depicted in Figure 1.

One could view $T_{2,3,\infty}$ as a non-normal version of the elliptic or cusp singularity coming from a $[3,3]$ point, while the others are non-normal versions of the quadruple point.

Table 2. Non-normal double covers of $\mathbb{P}(1,1,2)$ for general $D$ and $B_0$

| Hodge-Type | $\deg d$ | $\deg g$ | $\bar{X}$ | $g(D)$ | $g(\bar{D})$ |
|------------|----------|----------|-----------|--------|-------------|
| $\hat{0}_{0,2}$ | 2 | 6 | del Pezzo of degree 1 | 2 | 0 |
| $\hat{0}_{0,2}$ | 4 | 2 | $\mathbb{P}^2$ | 3 | 1 |
| $\hat{0}_{0,1}$ | 1 | 8 | K3 | 1 | 0 |
| $\hat{0}_{0,2}$ | 3 | 4 | del Pezzo of degree 4 | 2 | 0 |
| $\hat{0}_{0,2}$ | 5 | 0 | two copies of $\mathbb{P}(1,1,2)$ | 2 + 2 | 2 |

$(\deg d, \deg g) = (2,6)$: In this case (3.3) gives an exact sequence

$$0 \to H^1(\bar{D}, \mathbb{C}) \to H^2(X, \mathbb{C}) \to H^2(\bar{X}, \mathbb{C}) \to 0$$

where $\bar{X}$ is a del Pezzo surface of degree one, $D$ a rational curve and $\bar{D}$ is a genus 2 curve (see [FPR17, Proposition 4.4]); the Hodge type is determined by the curve only, because the del Pezzo surface $\bar{X}$ has $p_g(\bar{X}) = 0$.

Thus we can degenerate further by choosing $B_0$ and $D$ such that they are tangent in one or two points. Each tangency will introduce a node in $\bar{D}$ and by (3.5) introduce a one-dimensional piece of weight zero in the Hodge structure. This gives the degeneration of Hodge types $\hat{0}_{0,2} \to \hat{0}_{1,1} \to \hat{0}_{2,0}$.

To obtain the entire string $\hat{0}_{0,0} \to \hat{0}_{0,1} \to \hat{0}_{0,2} \to \hat{0}_{1,1} \to \hat{0}_{2,0}$, we consider a degeneration of the branch divisor $B$. Take $B = B_0 + D_1 + D_2$, with $D_1$ and $D_2$ linearly equivalent but distinct. For $D_1$ and $D_2$ general we have negligible singularities, i.e. Hodge type $\hat{0}_{0,0}$. Letting $D_1$ and $D_2$ pass through a point $p_0 \in B_0$, with suitable tangent conditions, we obtain a [3,3]-point, hence a degeneration of type $\hat{0}_{0,1}$. Letting $D_1 = D_2$, we get our surface $X$.

$(\deg d, \deg g) = (4,2)$: In this case $\bar{X} = \mathbb{P}^2$ and (3.3) gives an exact sequence

$$0 \to H^1(D, \mathbb{C}) \to H^1(\bar{D}, \mathbb{C}) \to H^2(X, \mathbb{C}) \to H^2(\mathbb{P}^2, \mathbb{C}) \to 0.$$
through a point \( p_0 \in B_0 \), so that we obtain a \([3,3]\)-point, and then letting \( D_1 = D_2 \), we get our surface \( X \).

\((\deg d, \deg g) = (1,8)\): \( X \) is a singular K3 surface (it has two singularities of type \( A_1 \) over the vertex of the cone), \( D \) is rational curve and \( \bar{D} \) is a genus one curve. In this case \( X \) is not Gorenstein. Identifying \( \mathbb{P}(1,1,2) \) with the quadric cone \( Q_2 \subset \mathbb{P}^3 \), we can obtain such degeneration choosing four suitable plane sections through a point.

This can be achieved by first letting \( B_0 \) acquire an ordinary quadruple point and then degenerate quadruple point where two local branches are tangent. Identifying \( \mathbb{P}(1,1,2) \) with the quadric cone \( Q_2 \subset \mathbb{P}^3 \), we can obtain degeneration choosing four suitable plane sections through a point.

This gives the degeneration \( \diamondsuit_{0,1} \to \diamondsuit_{0,2} \to \diamondsuit_{1,1} \) if \( D \) remains general. If in addition we allow \( D \) to be tangent to \( B_0 \) then \( \bar{D} \) will be a nodal curve of genus one, giving the missing degenerations from Figure 1.

\((\deg d, \deg g) = (3,4)\): In this case \( X \) is a singular del Pezzo surface of degree four, \( D \) is rational curve and \( \bar{D} \) is a genus two curve. Again the Hodge type is purely determined by \( D \) and can degenerate \( \diamondsuit_{0,2} \to \diamondsuit_{1,1} \to \diamondsuit_{2,0} \) by letting \( D \) and \( B_0 \)

acquire one or two tangencies.

As in the previous cases we obtain the entire string of Hodge type degenerations, considering \( B = B_0 + D_1 + D_2 \), with \( D_1 \) and \( D_2 \) linearly equivalent passing through a point \( p_0 \in B_0 \), and then letting \( D_1 = D_2 \).

\((\deg d, \deg g) = (5,0)\): This is the first case we encounter where \( X \) is reducible, with \( \bar{D} = D \cup D \) and (3.3) becomes

\[
0 \to H^1(D, \mathbb{C}) \to H^1(D, \mathbb{C}) \oplus^2 \to H^2(X, \mathbb{C}) \to
\]

\[
\to H^2(Q_2, \mathbb{C}) \oplus H^2(D, \mathbb{C}) \to H^2(D, \mathbb{C}) \oplus^2 \to 0.
\]

We see that the Hodge type is purely determined by \( D \) and can produce the degeneration \( \diamondsuit_{0,2} \to \diamondsuit_{1,1} \to \diamondsuit_{2,0} \) by letting \( D \) acquire one or two nodes.

4.B. Two K3 surfaces. It was observed in [FPR17, Example 4.7] that one can obtain examples of non-normal I-surfaces as follows:

Consider in \( \mathbb{P}^3 \) two planes \( P_1 \) and \( P_2 \) and let \( B = B_1 + B_2 \) be a quintic section of \( P_1 \cup P_2 \). Let \( L \) be the intersection line and \( L_i \subset P_i \) its preimages. Then the double cover

\[
\tilde{X} = X_1 \cup X_2 \quad \xrightarrow{\pi} \quad X
\]

branched over \( B = L_i \)

\[
P_1 \cup P_2 \quad \xrightarrow{} \quad P_1 \cup P_2
\]

defines, for a general choice of quintic section a 2-Gorenstein I-surface.

For a general quintic sections the curves \( B_i \) are smooth and intersect the lines \( L_i \) transversely, so \( X_i \) is a singular K3 surface with five nodes over \( L_i \cap B_i \). It is at these points that \( X \) is not Gorenstein.

To compute the mixed Hodge structure we use (3.3) again and get

\[
0 = (H^1(L_1, \mathbb{C}) \oplus H^1(L_2, \mathbb{C})) \to H^2(X, \mathbb{C}) \to H^2(X_1, \mathbb{C}) \oplus H^2(X_2, \mathbb{C}) \oplus H^2(L, \mathbb{C})
\]
so that $X$ has Hodge type $\hat{\phi}_{0,0}$ if the branch curve is general. \footnote{One might want to compare it with the fact that if we take two elliptic curves meeting in a node, then the Hodge structure on the first cohomology is still pure, despite the curve being singular.}

We can get all possible degenerations from Figure 1 by letting $B_1$ and then $B_2$ independently acquire first an ordinary and then a degenerate quadruple point.

Explicitly, pick a point $p_i \in P_i$ and consider three general planes through both points. The resulting ordinary triple point does not affect the Hodge type, since it gives rise to a canonical singularity. Now it is easy to choose different quadrics, which have the required intersection or tangency at the $p_i$.

### 4.C. Double covers of a different cone.

As proved in [FPRR22] an I-surface with a unique $T$-singularity of type $\frac{1}{18}(1, 5)$ is a complete intersection in $\mathbb{P}(1, 1, 2, 3, 5)$ given by equations

$$x_1^3 - yx_2 = z^2 - f_{10}(x_1, x_2, y, u) = 0$$

(where $\deg x_i = 1, \deg y = 2, \deg u = 3, \deg z = 5$) for a sufficiently general polynomial $f_{10}$ and we can thus consider $X$ as a double cover over a rational surface

$$X \to S = \text{Proj} \mathbb{C}[x_1, x_2, y, u]/(x_1^3 - x_2y)$$

branched over $B = \{ f_{10} = 0 \}$.

For a general branch divisor $B$ the surface $X$ has a unique singularity, which is rational, and thus has Hodge type $\hat{\phi}_{0,0}$. We want to explore some degenerations to non-normal surfaces by degenerating the branch locus but have to be more careful than in Section 4.A to remain s.l.c.

Thus for simplicity we will only consider the case where the non reduced part of the branch divisor avoids the point $(0 : 0 : 0 : 1) \in \mathbb{P}(1, 1, 2, 3)$, that is, we consider equations of the form

$$f_{10} = g_4(x_1, x_2, y, u) \cdot (u - h_3(x_1, x_2, y))^2.$$  

Then the homogenous coordinate ring of the non-reduced part of the branch divisor $D$ is

$$\mathbb{C}[x_1, x_2, y, u]/(x_1^3 - x_2y, u - h_3) \cong \mathbb{C}[x_1, x_2, y]/(x_1^3 - x_2y),$$

which describes a smooth rational curve and does not depend on $h_3$ at all.

The normalisation $\bar{X}$ is given by adjoining the ratio $w := z/(u - h_3)$ of weight 2 to the coordinate ring of $X$. Thus we may eliminate $z$ using $w$ and $\bar{X}$ is

$$\{ x_1^3 - x_2y = w^2 - g_4 = 0 \} \subset \mathbb{P}(1, 1, 2, 3, 2).$$

This is a rational surface with ample anti-canonical sheaf and for general $g_4$, $\bar{X}$ has two $A_1$ singularities occurring at the intersection points of $\bar{X}$ and $\mathbb{P}(2) \subset \mathbb{P}(1, 1, 2, 3, 2)$ and one $\frac{1}{18}(1, 5)$ singularity, occurring at $(0 : 0 : 0 : 1 : 0)$.

In total, by (3.3) the Hodge type is completely controlled by $\bar{D}$, which is the induced double cover of $D$ described by using $h_3$ to eliminate $u$ from the equations for $\bar{X}$:

$$\{ x_1^3 - x_2y = w^2 - g_4(x_1, x_2, y, h_3) = 0 \} \subset \mathbb{P}(1, 1, 2, 2)$$

For suitable choices of $g_4$ and $h_3$ we can arrange for $\bar{D}$ to be either smooth of genus 2 or to have one or two nodes. These give rise to degenerate cusps on X thus giving the degenerations $\hat{\phi}_{0,2} \to \hat{\phi}_{1,1} \to \hat{\phi}_{2,0}$.
4.C.1. A K3 surface and a rational surface. We consider now a non-normal degeneration of Example 4.C. Let $X$ be the complete intersection in $\mathbb{P}(1,1,2,3,5)$ given by equations

$$x_2y = z^2 - f_{10}(x_1, x_2, y, u) = 0$$

for sufficiently general $f_{10}$. As in section 4.C, $X$ is a double cover of $S$: $(x_2y = 0)$ in $\mathbb{P}(1,1,2,3)$. This time, $S$ has two components $S_1: (y = 0)$ and $S_2: (x_2 = 0)$ which are the weighted projective planes $S_1 \cong \mathbb{P}(1,1,3)$ and $S_2 \cong \mathbb{P}(1,2,3)$. They are glued along a common weighted projective line $L := \mathbb{P}(1,3) \cong \mathbb{P}^1$. The double cover $X_1$ of $S_1$ is a singular del Pezzo surface with one $\frac{1}{3}(1,1)$ singularity over the point $(0:0:1)$ in $\mathbb{P}(1,1,3)$ (see [RS03] for many more details). The double cover $X_2$ of $S_2$ is a K3 surface with an $A_2$ singularity over the point $(0:0:1)$ in $\mathbb{P}(1,2,3)$. The double curve of $X$ is an elliptic curve $D$ which is the preimage of $L$. The two singular points are identified to give a singularity on $X$ which is analytically isomorphic to the quotient of $\{ x_2y = 0 \} \subset \mathbb{A}^3$ by the order three action $(x_2, y, z) \mapsto (\omega x_2, \omega^2 y, \omega^2 z)$ where $\omega$ is a primitive third root of unity.

Using (3.3) again, we compute the mixed Hodge structure:

$$0 \to H^1(D, \mathbb{C}) \to H^1(D, \mathbb{C}) \overset{\oplus}{\to} H^2(X, \mathbb{C}) \to H^2(X_1, \mathbb{C}) \oplus H^2(X_2, \mathbb{C}) \oplus H^2(D, \mathbb{C}),$$

so $X$ has Hodge type $\phi_{0,1}$. If we allow $D$ to get a node (i.e. $f_{10}$ to become tangent to $L$) then we get Hodge type $\phi_{1,0}$ and so we have recovered all the types in Figure 1.

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