ANOTHER APPROACH TO WEIGHTED INEQUALITIES
FOR A SUPERPOSITION OF COPSON AND HARDY OPERATORS

RZA MUSTAFAYEV AND MERVE YILMAZ

ABSTRACT. In this paper, we present a solution to the inequality

\[ \left( \int_0^\infty \left( \int_0^\infty \left( \int_0^\infty h(t) dt \right)^q u(x) dx \right)^{\frac{1}{q}} ds \right) \leq C \left( \int_0^\infty h^p v \right)^{\frac{1}{p}}, \quad h \in \mathbb{M}^+(0,\infty), \]

using a combination of reduction techniques and discretization. Here \( 1 \leq p < \infty, 0 < q, r < \infty \) and \( u, v, w \) are weight functions on \( (0,\infty) \).

1. Introduction

Throughout this paper by \( \mathbb{M}^+(0,\infty) \) we denote the set of all non-negative measurable functions on \( (0,\infty) \). A weight is a function \( v \in \mathbb{M}^+(0,\infty) \) such that

\[ 0 < \int_0^x v(t) dt < \infty \quad \text{for all} \quad x \in (0,\infty). \]

The family of all weight functions (also called just weights) on \( (0,\infty) \) is given by \( \mathcal{W}(0,\infty) \). In the following, assume that \( u, v, w \in \mathcal{W}(0,\infty) \).

The investigation of weighted iterated Hardy-type inequalities started with the study of the inequality

\[ \left( \int_0^\infty \left( \int_0^\infty h(t) dt \right)^{\frac{1}{q}} u(x) dx \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty h^p v \right)^{\frac{1}{p}}, \quad h \in \mathbb{M}^+(0,\infty). \]

Inequality (1.1) have been considered in the case \( q = 1 \) in [2] (see also [3]), where the result was presented without proof, and in the case \( p = 1 \) in [4] and [25], where the special type of weight function \( v \) was considered. Recall that the inequality has been completely characterized in [7] and [8] in the case \( 0 < q < \infty, 0 < r \leq \infty, 1 \leq p < \infty \) by using discretization and anti-discretization methods; but, the obtained results were restricted to non-degenerate weights. Another approach to get the characterization of inequality (1.1) was presented in [24]. However, this characterization involves auxiliary functions, which make conditions more complicated.

As it was mentioned in [9] the characterization of "dual" inequality

\[ \left( \int_0^\infty \left( \int_0^\infty h(t) dt \right)^{\frac{1}{q}} u(x) dx \right) \leq C \left( \int_0^\infty h^p v \right)^{\frac{1}{p}}, \quad h \in \mathbb{M}^+(0,\infty) \]

can be easily obtained from the solutions of inequality (1.1), which was presented in [5].

Another pair of "dual" weighted iterated Hardy-type inequalities are

\[ \left( \int_0^\infty \left( \int_0^\infty h(t) dt \right)^{\frac{1}{q}} u(x) dx \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty h^p v \right)^{\frac{1}{p}}, \quad h \in \mathbb{M}^+(0,\infty) \]

and

\[ \left( \int_0^\infty \left( \int_0^\infty h(t) dt \right)^{\frac{1}{q}} u(x) dx \right)^{\frac{1}{p}} \leq C \left( \int_0^\infty h^p v \right)^{\frac{1}{p}}, \quad h \in \mathbb{M}^+(0,\infty). \]

The characterization of all four inequalities can be reduced to the characterization of the weighted Hardy-type inequalities on the cones of non-increasing or non-decreasing functions, when \( 1 < p < \infty \). This approach provides solution of iterated inequalities by so-called "flipped" conditions (see, [9] and [10]). In the case when \( p = 1 \), [9]...
contains solutions of inequalities (1.1) - (1.4) with weight functions \( \int_0^x v \) and \( (\int_x^\infty v)^{-1} \) on the right-hand side, as well.

Different approach to solve (1.3) has been given in [22] when \( p = 1 \) using a combination of reduction techniques and discretization. The "classical" conditions ensuring the validity of (1.3) was recently presented in [19]. Inequalities (1.2) and (1.4) were recently characterized by using discretization techniques in [6] and [17], respectively.

In the present paper we solve inequality (1.2) using a combination of reduction techniques and discretization, which essentially shortens the proof. Similar approach was applied to inequality (1.3) in [22] when \( p = 1 \). Our approach consists of the following steps: We prove that

\[
\left( \int_0^\infty \left( \int_x^\infty \left( \int_0^x h(q,s)ds \right)^q w(t)dt \right)^{1\over q} u(x)dx \right)^{1\over p} \leq \left\| 2^{1\over q} \left( \int_x^\infty \left( \int_s^\infty h(q,t)dt \right)^q w(s)ds \right)^{1\over q} \right\|_{L^p(\mathbb{Z})} + \left( \int_0^\infty u(x) \left( \sup_{v \in \{x, x+1\}} \left( \int_v^\infty w(t)dt \right)^{1\over p} \right) \right)^{1\over q} \left( \int_0^x h(t)dt \right)^{1\over q} dx \right)^{1\over p}
\]

with constants independent of \( h \in \mathfrak{H}^+(0, \infty) \), where \( \{x_k\}_{k \in \mathbb{Z}} \) is a covering sequence mentioned in Definition 3.1 (see Theorem 3.3). Consequently, inequality (1.2) holds if and only if both inequalities

\[
\left( 2^{1\over q} \left( \int_x^\infty \left( \int_s^\infty h(q,t)dt \right)^q w(s)ds \right)^{1\over q} \right) \leq C \left( \int_{x_n}^{x_{n+1}} h^{q\ell}(t)dt \right)^{1\over p}, \quad h \in \mathfrak{H}^+(0, \infty)
\]

and

\[
\left( \int_0^\infty u(x) \left( \sup_{v \in \{x, x+1\}} \left( \int_v^\infty w(t)dt \right)^{1\over p} \right) \right)^{1\over q} \left( \int_0^x h(t)dt \right)^{1\over q} dx \right)^{1\over p}
\]

hold (see Theorem 3.4). Recall that the solution of inequality (1.6) is known (see Theorem 4.1). Discrete characterization of inequality (1.5) is presented in Section 5 (see Lemma 5.1). Continuous sufficient conditions for inequality (1.5) are studied in Section 6 (see Lemma 6.1). In Section 7, we show that these conditions are necessary for inequality (1.2) (see Lemma 7.1). The proof of the main statement is given in Section 8.

We pronounce that the characterizations of inequalities (1.1) - (1.4) are important because many inequalities for classical operators can be reduced to them. These inequalities play an important role in the theory of weighted Morrey-type spaces and Cesàro function spaces (see [5], [11], [12] and [13]). Note that using characterizations of weighted Hardy inequalities it is easy to obtain the characterization of the boundedness of bilinear Hardy-type inequalities (see, for instance, [18] and [1]).

For a given weight function \( v, 0 \leq x < y \leq \infty \) and \( 1 \leq p < \infty \), set

\[
\sigma_p(x,y) := \left( \int_x^y v(t)^{1-p} dt \right)^{1\over p} \quad \text{ess sup}_{x < t < y} v(t)^{1-p} \quad \text{when} \quad 1 < p < \infty,
\]

\[
\sigma_p(x,y) := \left( \int_x^y v(t)^{-1} dt \right)^{1\over p} \quad \text{when} \quad p = 1.
\]

Our main result, which coincides with [6, Theorem A], reads as follows:

**Theorem 1.1.** Let \( 1 \leq p < \infty, 0 < q, r < \infty, u \in \mathcal{W}(0, \infty) \cap C(0, \infty) \) and \( v, w \in \mathcal{W}(0, \infty) \).

(a) Let \( p \leq \min\{q, r\} \). Then inequality (1.2) holds if and only if \( F_1 < \infty \) and \( F_2 < \infty \), where

\[
F_1 := \sup_{x \in (0, \infty)} \left( \int_0^x u(t)^{1-q} dt \right)^{1\over q} \left( \int_x^\infty w(t)^{1-r} dt \right)^{1\over r} \sigma_p(0, x) < \infty,
\]

and

\[
F_2 := \sup_{x \in (0, \infty)} \left( \int_x^\infty \left( \int_t^\infty w(t)^{1-r} dt \right)^{1\over r} u(t)dt \right)^{1\over q} \sigma_p(0, x) < \infty.
\]

Moreover, if \( C \) is the best constant in (1.2), then \( C \approx F_1 + F_2 \).

(b) Let \( r < p < q \). Then inequality (1.2) holds if and only if \( F_3 < \infty \) and \( F_4 < \infty \), where

\[
F_3 := \left( \int_0^\infty \left( \sup_{x \in [x, \infty)} \left( \int_t^\infty w(t)^{1-r} dt \right)^{1\over r} \sigma_p(0, x) \right) \left( \int_0^x u(t)^{1-q} dt \right)^{1\over q} \left( \int_0^x u(t)^{-1} dt \right)^{1\over p} \right)^{1\over p} < \infty,
\]
and

\[ F_4 := \left( \int_0^\infty \left( \int_x^\infty \left( \int_y^\infty w(t) \, dt \right) \frac{r}{p} \left( \int_x^\infty w(z) \left( \sigma_{p}(0,x) \right)^{\frac{p}{p^*}} u(x) \, dx \right)^{\frac{r}{p}} \right)^{\frac{1}{r}} \right)^{\frac{1}{q}} < \infty. \]

Moreover, if \( C \) is the best constant in (1.2), then \( C \approx F_3 + F_4 \).

(c) Let \( q \leq p \leq r \). Then inequality (1.2) holds if and only if \( F_2 < \infty \) and \( F_5 < \infty \), where

\[ F_5 := \sup_{t \in (0,\infty)} \left( \int_0^t \left( \int_x^\infty \left( \int_y^\infty w(x) \left( \sigma_{p}(0,x) \right)^{\frac{p}{p^*}} \, dx \right)^{\frac{r}{p}} \right)^{\frac{1}{q}} \right)^{\frac{1}{r}} < \infty. \]

Moreover, if \( C \) is the best constant in (1.2), then \( C \approx F_2 + F_5 \).

(d) Let \( \max(q, r) < p \). Then inequality (1.2) holds if and only if \( F_4 < \infty \) and \( F_6 < \infty \), where

\[ F_6 := \left( \int_0^\infty \left( \int_0^t \left( \int_x^\infty \left( \int_y^\infty w(x) \left( \sigma_{p}(0,x) \right)^{\frac{p}{p^*}} \, dx \right)^{\frac{r}{p}} \right)^{\frac{1}{q}} \right)^{\frac{1}{r}} \right)^{\frac{1}{q}} < \infty. \]

Moreover, if \( C \) is the best constant in (1.2), then \( C \approx F_4 + F_6 \).

2. Notations and Preliminaries

Throughout the paper, we always denote by \( C \) a positive constant, which is independent of main parameters but it may vary from line to line. However a constant with subscript or superscript such as \( C_1 \) does not change in different occurrences. By \( a \leq b \), \( (b \geq a) \) we mean that \( a \leq \lambda b \), where \( \lambda > 0 \) depends on inessential parameters. If \( a \leq b \) and \( b \leq a \), we write \( a \approx b \) and say that \( a \) and \( b \) are equivalent. Unless a special remark is made, the differential element \( dx \) is omitted when the integrals under consideration are the Lebesgue integrals. We put \( 0 \cdot \infty = 0, \infty / \infty = 0 \) and \( 0/0 = 0 \).

Let \( 0 \neq \mathcal{Z} \subseteq \mathbb{Z} := \mathbb{Z} \cup \{-\infty, +\infty\}, 0 < q \leq +\infty \) and \( \{w_k\} = \{w_k\}_{k \in \mathcal{Z}} \) be a sequence of positive numbers. We denote by \( \ell^q((w_k), \mathcal{Z}) \) the following discrete analogue of a weighted Lebesgue space: if \( 0 < q < +\infty \), then

\[ \ell^q((w_k), \mathcal{Z}) = \left\{ \{a_k\}_{k \in \mathcal{Z}} : \|\{a_k\}\|_{\ell^q((w_k), \mathcal{Z})} := \left( \sum_{k \in \mathcal{Z}} |a_k w_k|^q \right)^{\frac{1}{q}} < +\infty \right\}, \]

and

\[ \ell^\infty((w_k), \mathcal{Z}) = \left\{ \{a_k\}_{k \in \mathcal{Z}} : \|\{a_k\}\|_{\ell^\infty((w_k), \mathcal{Z})} := \sup_{k \in \mathcal{Z}} |a_k w_k| < +\infty \right\}. \]

If \( w_k = 1 \) for all \( k \in \mathcal{Z} \), we write simply \( \ell^q(\mathcal{Z}) \) instead of \( \ell^q((w_k), \mathcal{Z}) \).

The following inequality is a straightforward consequence of the discrete Hölder inequality:

\[ \|a_k b_k\|_{\ell^1(\mathcal{Z})} \leq \|a_k\|_{\ell^\infty(\mathcal{Z})} \|b_k\|_{\ell^1(\mathcal{Z})}. \]

**Definition 2.1.** Let \( N, M \in \mathbb{Z}, N < M \). A positive sequence \( \{\tau_k\}_{k=N}^M \) is called geometrically increasing if there is \( \alpha \in (1, +\infty) \) such that

\[ \tau_k \geq \alpha \tau_{k-1} \quad \text{for all} \quad k \in \{N + 1, \ldots, M\}. \]

Proofs of the following statement can be found in [20] and [21].

**Lemma 2.2.** Let \( q \in (0, +\infty) \), \( N, M \in \mathbb{Z}, N \leq M \), \( \mathcal{Z} = \{N, N + 1, \ldots, M - 1, M\} \) and let \( \{\tau_k\}_{k=N}^M \) be a geometrically increasing sequence. Then

\[ \left\| \left\{ \tau_k \sum_{m=k}^M d_m \right\} \right\|_{\ell^q(\mathcal{Z})} \approx \|\tau_k a_k\|_{\ell^q(\mathcal{Z})} \]

and

\[ \left\| \left\{ \tau_k \sup_{k \leq m \leq M} a_m \right\} \right\|_{\ell^q(\mathcal{Z})} \approx \|\tau_k a_k\|_{\ell^q(\mathcal{Z})} \]

for all non-negative sequences \( \{a_k\}_{k=N}^M \).
Given two (quasi-) Banach spaces $X$ and $Y$, we write $X \hookrightarrow Y$ if $X \subset Y$ and if the natural embedding of $X$ in $Y$ is continuous.

The discrete version of the classical Landau resonance theorems is given in the following proposition. Proofs can be found, for example, in [15].

**Proposition 2.3.** ([15, Proposition 4.1]) Let $0 < p, r < +\infty, 0 \neq \mathcal{Z} \subseteq \overline{\mathcal{Z}}$ and let $\{v_k\}_{k \in \mathbb{Z}}$ and $\{w_k\}_{k \in \mathbb{Z}}$ be two sequences of nonnegative numbers. Assume that
\[
\ell^p(\{v_k\}, \mathcal{Z}) \hookrightarrow \ell^r(\{w_k\}, \mathcal{Z}).
\]
Then
\[
\| (w_k v_k^{-1}) \|_{\ell^r(\mathcal{Z})} \leq C,
\]
where $1/r := (1/r - 1/p)_+$ and $C$ stands for the norm of embedding (2.2).

We will use the following well-known characterizations of weights for which the weighted Hardy-type inequality holds (see, for instance, [25]).

**Theorem 2.4.** Let $1 \leq p < \infty, 0 < q < \infty$ and $v, w \in \mathcal{W}(0, \infty)$.

(i) Let $p \leq q$. Then
\[
\sup_{f \in \mathcal{W}^+(0, \infty)} \left( \int_0^\infty \left( \int_0^x f^q w(x) \, dx \right)^{\frac{p}{q}} \right)^{\frac{q}{p}} \leq \sup_{x \in (0, \infty)} \left( \int_x^\infty w \right)^{\frac{q}{p}} \sigma_p(0, x).
\]

(ii) Let $q < p$. Then
\[
\sup_{f \in \mathcal{W}^+(0, \infty)} \left( \int_0^\infty \left( \int_0^x f w(x) \, dx \right)^{\frac{p}{q}} \right)^{\frac{q}{p}} \leq \left( \int_0^\infty \left( \int_x^\infty w \right)^{\frac{q}{p}} \sigma_p(0, x) \, dx \right)^{\frac{p}{q}}.
\]

We next quote the following result concerning characterization of inequality involving supremum operator.

**Theorem 2.5.** ([14, Theorems 4.1 and 4.4]) Let $1 \leq p < \infty, 0 < r < \infty$. Assume that $u \in \mathcal{W}(0, \infty) \cap C(0, \infty)$ and $v, w \in \mathcal{W}(0, \infty)$.

(a) Let $p \leq r$. Then the inequality
\[
\left( \int_0^\infty \sup_{y \in [0, \infty]} u(y) \int_0^y g(s) \, ds \right)^{\frac{1}{p}} w(x) \, dx \right)^{\frac{1}{r}} \leq C \left( \int_0^\infty g(x)^p v(x) \, dx \right)^{\frac{1}{p}} w(x) \, dx
\]
holds for all $g \in \mathcal{W}^+(0, \infty)$ if and only if
\[
C_1 := \sup_{x \in (0, \infty)} \left( \int_0^x w \right)^{\frac{1}{p}} \left( \sup_{t \in [0, \infty]} u(t) \right) \sigma_p(0, x) < \infty
\]
and
\[
C_2 := \sup_{x \in (0, \infty)} \left( \int_x^\infty \left( \sup_{t \in [r, \infty)} u(t) \right) \sigma_p(0, x) < \infty.
\]
Moreover, the least constant $C$ such that (2.3) holds for all $g \in \mathcal{W}^+$ satisfies $C \approx C_1 + C_2$.

(b) Let $r < p$. Then inequality (2.3) holds for all $g \in \mathcal{W}^+(0, \infty)$ if and only if
\[
C_3 := \left( \int_0^\infty \sup_{y \in [0, \infty]} \sup_{\tau \in [0, \infty]} u(y) \sigma_p(0, \tau) \right)^{\frac{p}{r}} \left( \int_0^\infty w \right)^{\frac{p}{r}} \sigma_p(0, x) < \infty
\]
and
\[
C_4 := \left( \int_0^\infty \left( \int_x^\infty w \right)^{\frac{p}{r}} \sigma_p(0, x) \right)^{\frac{p}{r}} < \infty.
\]
Moreover, the least constant $C$ such that (2.3) holds for all $g \in \mathcal{W}^+$ satisfies $C \approx C_3 + C_4$.

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\(^{1}\)For any $a \in \mathbb{R}$ denote by $a_+ = a$ when $a > 0$ and $a_+ = 0$ when $a \leq 0$. 

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3. Equivalence and reduction theorems

In this section we prove the equivalence and reduction theorems.

**Definition 3.1.** Assume that $u$ is a weight function on $(0, +\infty)$. If $\int_0^\infty u(t) dt = +\infty$, let $\{x_k\}_{k=0}^{\infty} \subset (0, \infty)$ be a strictly increasing sequence such that $\int_0^{x_k} u(t) dt = 2^k$, $k \in \mathbb{Z}$. Denote $M := +\infty$ and $\mathcal{Z} = \mathbb{Z}$, when $\int_0^\infty u(t) dt = \infty$. If $\int_0^\infty u(t) dt < +\infty$, define a strictly increasing sequence $\{x_k\}_{k=0}^{M}$ such that $\int_0^{x_k} u(t) dt = 2^k$, $-\infty < k \leq M$, where $M$ satisfies the inequality $2^M \leq \int_0^\infty u(t) dt < 2^{M+1}$. Denote $x_{M+1} := \infty$ and $\mathcal{Z} := \{k \in \mathbb{Z} : k \leq M\}$, when $\int_0^\infty u(t) dt < \infty$. Obviously, $\cup_{k \in \mathcal{Z}} \{x_k, x_{k+1}\} = (0, \infty)$ in both cases. The sequence $\{x_k\}_{k \in \mathcal{Z}}$ is called a covering sequence.

**Remark 3.2.** We shall use the following equivalences without mentioning anytime we need them.

Assume that $\{x_k\}_{k \in \mathcal{Z}}$ is a covering sequence. Clearly,

$$\int_{x_{k-1}}^{x_k} u \approx 2^k, \quad k \in \mathcal{Z}.$$  

Moreover,

$$\int_{x_{k-1}}^{x_k} \left( \int_{x_{k-1}}^{x_k} u \right)^{\frac{r}{q}} (\int_0 u) dt \approx \left( \int_{x_{k-1}}^{x_k} \left( \int_0 u \right)^{\frac{r}{q}} \right) dt \approx 2^k \frac{p}{q}$$

when $0 < r < p < \infty$.

Our equivalency statement reads as follows:

**Theorem 3.3.** Let $0 < q, r < \infty$ and $u, v, w \in \mathcal{W}(0, \infty)$. Assume that $\{x_k\}_{k \in \mathcal{Z}}$ is a covering sequence. Then

$$\left( \int_0^\infty \left( \int_x^\infty \left( \int_0^h \right)^q w(t) dt \right)^{\frac{r}{q}} u(x) dx \right)^{\frac{1}{r}} \approx \left\| \left\{ 2^k \left( \int_{x_k}^{x_{k+1}} \left( \int_0^h \right)^q w(t) dt \right)^{\frac{1}{q}} \right\} \right\|_{l_1(\mathcal{Z})}$$

with constants independent of $h \in \mathcal{W}^+(0, \infty)$.

**Proof.** Since

$$\left( \int_0^\infty \left( \int_x^\infty \left( \int_0^h \right)^q w(t) dt \right)^{\frac{r}{q}} u(x) dx \right)^{\frac{1}{r}} = \left\| \left\{ \left( \int_{x_k}^{x_{k+1}} \left( \int_x^\infty \left( \int_0^h \right)^q w(t) dt \right)^{\frac{1}{q}} u(x) dx \right) \right\} \right\|_{l_1(\mathcal{Z})}$$

$$\leq \left\| \left\{ \left( \int_{x_{k+1}}^{x_k} \left( \int_x^\infty \left( \int_0^h \right)^q w(t) dt \right)^{\frac{1}{q}} u(x) dx \right) \right\} \right\|_{l_1(\mathcal{Z})}$$

$$= \left\| \left\{ 2^k \left( \int_{x_k}^{x_{k+1}} \left( \int_0^h \right)^q w(t) dt \right)^{\frac{1}{q}} \right\} \right\|_{l_1(\mathcal{Z})},$$

and

$$\left( \int_0^\infty \left( \int_x^\infty \left( \int_0^h \right)^q w(t) dt \right)^{\frac{r}{q}} u(x) dx \right)^{\frac{1}{r}} \approx \left\| \left\{ \left( \int_{x_{k-1}}^{x_k} \left( \int_x^\infty \left( \int_0^h \right)^q w(t) dt \right)^{\frac{1}{q}} u(x) dx \right) \right\} \right\|_{l_1(\mathcal{Z})}$$

$$\approx \left\| \left\{ \left( \int_{x_{k-1}}^{x_k} \left( \int_0^h \right)^q w(t) dt \right)^{\frac{1}{q}} \right\} \right\|_{l_1(\mathcal{Z})},$$
then
\[ \left( \int_0^\infty \left( \int_x^\infty \left( \int_0^t h^q w(t) dt \right)^{\frac{q}{q-1}} u(x) dx \right)^{\frac{q}{q-1}} \right)^{\frac{q}{q}} \]
\[ \approx \left\| \left\{ 2^\frac{q}{q+1} \left( \int_{x_k}^{x_{k+1}} \left( \int_0^t h^q w(t) dt \right)^{\frac{q}{q-1}} \right)^{\frac{q}{q}} \right\}_{k \in \mathbb{Z}} \right\|_{L^q(\mathbb{Z})} . \]

By Lemma 2.2, we get that
\[ \left( \int_0^\infty \left( \int_x^\infty \left( \int_0^t h^q w(t) dt \right)^{\frac{q}{q-1}} u(x) dx \right)^{\frac{q}{q-1}} \right)^{\frac{q}{q}} \]
\[ \approx \left\| \left\{ 2^\frac{q}{q+1} \left( \int_{x_k}^{x_{k+1}} \left( \int_0^t h^q w(t) dt \right)^{\frac{q}{q-1}} \right)^{\frac{q}{q}} \right\}_{k \in \mathbb{Z}} \right\|_{L^q(\mathbb{Z})} \]
\[ + \left\| \left\{ 2^\frac{q}{q+1} \left( \int_{x_k}^{x_{k+1}} w \left( \int_0^t h \right)^{\frac{q}{q-1}} \right)^{\frac{q}{q}} \right\}_{k \in \mathbb{Z}} \right\|_{L^q(\mathbb{Z})} . \]

(3.1)

Since
\[ \left\| \left\{ 2^\frac{k}{k+1} \left( \int_{x_k}^{x_{k+1}} w \right)^{\frac{q}{q-1}} \left( \int_0^{x_k} h \right) \right\}_{k \in \mathbb{Z}} \right\|_{L^q(\mathbb{Z})} \]
\[ \leq \left\| \left\{ \left( \int_{x_k}^{x_{k+1}} u(x) dx \right) \left[ \sup_{s \in [x_k, x_{k+1}]} \left( \int_s^{x_{k+1}} w \right)^{\frac{q}{q-1}} \left( \int_0^s h \right) \right] \right\}_{k \in \mathbb{Z}} \right\|_{L^q(\mathbb{Z})} \]
\[ \leq \left\| \left\{ \left( \int_{x_k}^{x_{k+1}} u(x) \left[ \sup_{s \in [x_k, x_{k+1}]} \left( \int_s^{x_{k+1}} w \right)^{\frac{q}{q-1}} \left( \int_0^s h \right) \right]^{\frac{q}{q-1}} \right) dx \right\}_{k \in \mathbb{Z}} \right\|_{L^q(\mathbb{Z})} \]
\[ \leq \left\| \left\{ \left( \int_{x_k}^{x_{k+1}} u(x) \left[ \sup_{s \in [x_k, x_{k+1}]} \left( \int_s^{x_{k+1}} w \right)^{\frac{q}{q-1}} \left( \int_0^s h \right) \right]^{\frac{q}{q-1}} \right) dx \right\}_{k \in \mathbb{Z}} \right\|_{L^q(\mathbb{Z})} \]
\[ \leq \left( \int_0^\infty u(x) \left[ \sup_{s \in [x, x_{k+1}]} \left( \int_s^\infty w \right)^{\frac{q}{q-1}} \left( \int_0^s h \right) \right]^{\frac{q}{q-1}} dx \right)^{\frac{q}{q}} , \]
on using (3.1), we arrive at
\[ \left( \int_0^\infty \left( \int_x^\infty \left( \int_0^t h^q w(t) dt \right)^{\frac{q}{q-1}} u(x) dx \right)^{\frac{q}{q-1}} \right)^{\frac{q}{q}} \]
\[ \leq \left\| \left\{ 2^\frac{q}{q+1} \left( \int_{x_k}^{x_{k+1}} \left( \int_0^t h^q w(t) dt \right)^{\frac{q}{q-1}} \right)^{\frac{q}{q}} \right\}_{k \in \mathbb{Z}} \right\|_{L^q(\mathbb{Z})} \]
\[ + \left( \int_0^\infty u(x) \left[ \sup_{s \in [x, x_{k+1}]} \left( \int_s^\infty w \right)^{\frac{q}{q-1}} \left( \int_0^s h \right) \right] \right)^{\frac{q}{q}} \]

For the converse, note that, by (3.1),
\[ \left\| \left\{ 2^\frac{q}{q+1} \left( \int_{x_k}^{x_{k+1}} \left( \int_0^t h^q w(t) dt \right)^{\frac{q}{q-1}} \right)^{\frac{q}{q}} \right\}_{k \in \mathbb{Z}} \right\|_{L^q(\mathbb{Z})} \]
\[ \leq \left( \int_0^\infty \left( \int_x^\infty \left( \int_0^t h^q w(t) dt \right)^{\frac{q}{q-1}} u(x) dx \right)^{\frac{q}{q}} \right)^{\frac{q}{q}} \].

On the other hand,
\[ \left( \int_0^\infty u(x) \left[ \sup_{s \in [x, x_{k+1}]} \left( \int_s^\infty w \right)^{\frac{q}{q-1}} \left( \int_0^s h \right) \right] \right)^{\frac{q}{q}} \]
Moreover, the least constant C such that
\[
\left\| 2^\frac{1}{\tau} \left( \int_{x_k}^{x_{k+1}} \left( \int_{x_k}^{s} h(t) \, dt \right)^\tau \right)^{\frac{1}{\tau}} \right\|_{\ell^r(\mathbb{Z})}
\]
\[\leq \left( \int_0^{\infty} u(x) \left( \sup_{x \in [x,\infty)} \left( \int_s^\infty h(t) \, dt \right)^{\frac{q}{p'}} \right) \, dx \right)^{\frac{1}{r}}.
\]
We arrive at
\[
\left\| 2^\frac{1}{\tau} \left( \int_{x_k}^{x_{k+1}} \left( \int_{x_k}^{s} h(t) \, dt \right)^\tau \right)^{\frac{1}{\tau}} \right\|_{\ell^r(\mathbb{Z})}
\]
\[+ \left( \int_0^{\infty} u(x) \left( \sup_{x \in [x,\infty)} \left( \int_s^\infty w(t) \, dt \right)^{\frac{1}{p'}} \right) \, dx \right)^{\frac{1}{r}}
\]
\[\leq \left( \int_0^{\infty} u(x) \left( \int_s^\infty h(t) \, dt \right)^{\frac{q}{p'}} \, dx \right)^{\frac{1}{r}}
\]
by combining the previous two inequalities.

The proof is completed. □

So, we are able to formulate our reduction statement.

**Theorem 3.4.** Let \(0 < q, r < \infty\) and \(u, v, w \in \mathcal{W}(0, \infty)\). Assume that \(\{x_k\}_{k \in \mathbb{Z}}\) is a covering sequence. Then inequality (1.2) holds if and only if both of inequalities (1.5) and (1.6) hold.

**Proof.** The proof of the statement immediately follows from Theorem 3.3. □

4. **Solution of inequality (1.6)**

In this section we present the solution of inequality (1.6).

**Theorem 4.1.** Let \(1 \leq p < \infty, 0 < r < \infty\). Assume that \(u, v, w \in \mathcal{W}(0, \infty)\).

(a) Let \(p \leq r\). Then inequality (1.6) holds for all \(g \in \mathfrak{M}^+(0, \infty)\) if and only if
\[
D_1 := \sup_{x \in (0,\infty)} \left( \int_0^x u \left( \int_0^\infty w \right)^\frac{1}{r} \sigma_p(0, x) \right) < \infty
\]
and
\[
D_2 := \sup_{x \in (0,\infty)} \left( \int_0^\infty \left( \int_0^\infty w \right)^\frac{1}{r} u(0, t) \right) \left( \int_0^x u \left( \int_0^\infty w \right)^\frac{1}{r} \sigma_p(0, x) \right) < \infty.
\]
Moreover, the least constant \(C\) such that (1.6) holds for all \(g \in \mathfrak{M}^+(0, \infty)\) if and only if
\[D_1 + D_2 < \infty.
\]

(b) Let \(r < p\). Then inequality (1.6) holds for all \(h \in \mathfrak{M}^+(0, \infty)\) if and only if
\[
D_3 := \left( \int_0^\infty \left( \sup_{x \in [0,\infty)} \left( \int_x^\infty \int_0^\infty w \right)^\frac{1}{p} \sigma_p(0, \tau) \right)^{\frac{p}{p'}} \left( \int_0^x u \left( \int_0^\infty w \right)^\frac{1}{r} \sigma_p(0, x) \right) \, dx \right)^{\frac{r}{p'}} < \infty
\]
and
\[
D_4 := \left( \int_0^\infty \left( \int_0^\infty \left( \int_y^\infty w \right)^\frac{1}{p} \sigma_p(0, \tau) \right)^{\frac{p}{p'}} \left( \int_0^\infty \left( \int_x^\infty w \right)^\frac{1}{r} \sigma_p(0, x) \right) \, dy \right)^{\frac{r}{p'}} < \infty.
\]
Moreover, the least constant \(C\) such that (1.6) holds for all \(g \in \mathfrak{M}^+(0, \infty)\) if and only if
\[D_3 + D_4 < \infty.
\]

**Proof.** The statement directly follows from Theorem 2.5. □

5. **Discrete solution of inequality (1.5).**

Now we give a discrete characterization of inequality (1.5).

**Lemma 5.1.** Let \(1 \leq p < \infty, 0 < q, r < \infty\) and \(u, v, w \in \mathcal{W}(0, \infty)\). Assume that \(\{x_k\}_{k \in \mathbb{Z}}\) is a covering sequence.

(i) If \(p \leq q\), then inequality (1.5) holds with constant independent of \(h \in \mathfrak{M}^+(0, \infty)\) if and only if \(A_1 < \infty\), where
\[
A_1 := \left\| 2^\frac{1}{\tau} \left( \sup_{x \in [x_k, x_{k+1})} \left( \int_x^{x_{k+1}} w \right)^\frac{1}{r} \sigma_p(0, x) \right) \right\|_{\ell^r(\mathbb{Z})}.
\]
Moreover, if $C$ is the best constant in (1.5), then $C \approx A_1$.

(ii) If $q < p$, then inequality (1.5) holds with constant independent of $h \in \mathfrak{M}^+(0, \infty)$ if and only if $A_2 < \infty$, where

$$A_2 := \left\| 2^{\frac{1}{2}} \left( \int_{x_k}^{x_{k+1}} \left( \int_x^{x_k} w(s) ds \right)^{q \rho} w(x)[\sigma_p(x, x)]^{\frac{pq}{p-q}} dx \right) \right\|_{L^q(Z)}.$$

Moreover, if $C$ is the best constant in (1.5), then $C \approx A_2$.

**Proof.** We give the proof of the second case. The first one can be proved similarly, so its proof is omitted.

**Necessity:** Suppose that inequality (1.5) holds with constant $C$ independent of $h \in \mathfrak{M}^+(0, \infty)$.

Since, by Theorem 2.4, (ii), for any $k \in \mathbb{Z}$

\[
\sup \left\{ \left( \int_{x_k}^{x_{k+1}} h^q w(x) dx \right)^{\frac{1}{q}} : \int_{x_k}^{x_{k+1}} h^p v = 1 \right\} = \left( \int_{x_k}^{x_{k+1}} \left( \int_x^{x_k} w(s) ds \right)^{q \rho} w(x)[\sigma_p(x, x)]^{\frac{pq}{p-q}} dx \right)^{\frac{p-q}{pq}},
\]
then there exists $h_k \in \mathfrak{M}^+(0, \infty)$ such that \text{supp} $h_k \subset (x_k, x_{k+1})$, \text{supp} $h_k \subset (x_k, x_{k+1})$, and $\int_{x_k}^{x_{k+1}} h_k^p v = 1$ and

\[
\left( \int_{x_k}^{x_{k+1}} \left( \int_{j_k}^{x_{k+1}} h_k^q w(x) dx \right)^{\frac{1}{q}} \right) \geq \frac{1}{2} \left( \int_{x_k}^{x_{k+1}} \left( \int_x^{x_k} w(x)[\sigma_p(x, x)]^{\frac{pq}{p-q}} dx \right)^{\frac{p-q}{pq}} dx \right)^{\frac{pq}{p-q}}.
\]

Define

\[
h = \sum_{m \in \mathbb{Z}} a_m h_m,
\]
where $\{a_k\}_{k \in \mathbb{Z}}$ is any sequence of nonnegative numbers.

Since $h \in \mathfrak{M}^+(0, \infty)$, then inequality (1.5) holds for function (5.1) as well. Thus the inequality

\[
\left\| \left\{ a_k 2^{\frac{1}{2}} \left( \int_{x_k}^{x_{k+1}} \left( \int_x^{x_k} w(s) ds \right)^{\frac{q \rho}{p-q}} w(x)[\sigma_p(x, x)]^{\frac{pq}{p-q}} dx \right) \right\} \right\|_{L^q(Z)} \leq C \| (a_k) \|_{L^q(Z)}
\]
holds for all sequences $\{a_k\}_{k \in \mathbb{Z}}$ of nonnegative numbers.

By Proposition 2.3, we arrive at

\[
A_2 = \left\| \left\{ 2^{\frac{1}{2}} \left( \int_{x_k}^{x_{k+1}} \left( \int_x^{x_k} w(s) ds \right)^{\frac{q \rho}{p-q}} w(x)[\sigma_p(x, x)]^{\frac{pq}{p-q}} dx \right) \right\} \right\|_{L^q(Z)} \leq C.
\]

**Sufficiency:** Assume that $A_2 < \infty$. By Theorem 2.4, (ii), applying inequality (2.1), we have that

\[
\left\| \left\{ 2^{\frac{1}{2}} \left( \int_{x_k}^{x_{k+1}} h^q w(s) ds \right)^{\frac{1}{q}} \right\} \right\|_{L^q(Z)} \leq \left\| \left\{ 2^{\frac{1}{2}} \left( \int_{x_k}^{x_{k+1}} h^q w(s) ds \right)^{\frac{1}{q}} \right\} \right\|_{L^q(Z)} \leq A_2 \left\| \left\{ h^p v \right\} \right\|_{L^q(Z)}.
\]

Thus, inequality (1.5) holds, and, if $C$ is the best constant in (1.5), then

\[
C \leq A_2.
\]

The proof is completed. \qed
6. Continuous sufficient conditions for inequality (1.5)

Lemma 6.1. Let $1 \leq p < \infty$, $0 < q, r < \infty$ and $u, v, w \in W(0, \infty)$.

(i) Let $p \leq \min\{q, r\}$. If

$$B_1 := \sup_{t \in (0, \infty)} \left( \int_0^t u \right)^{\frac{1}{q}} \left( \int_t^\infty w \right)^{\frac{1}{r}} \sigma_p(0, t) < \infty,$$

then inequality (1.5) holds with constant independent of $h \in \mathcal{W}(0, \infty)$. Moreover, if $C$ is the best constant in (1.5), then $C \leq B_1$.

(ii) Let $r < p \leq q$. If

$$B_2 := \left( \int_0^\infty \left( \int_0^t u \right)^{\frac{1}{q}} u(t) \left( \esssup_{x \in (t, \infty)} \left( \int_x^\infty w \right)^{\frac{1}{r}} \sigma_p(0, x) \right)^{\frac{p}{pr}} dt \right)^{\frac{p}{pr}} < \infty,$$

then inequality (1.5) holds with constant independent of $h \in \mathcal{W}(0, \infty)$. Moreover, if $C$ is the best constant in (1.5), then $C \leq B_2$.

(iii) Let $q < p \leq r$. If

$$B_3 := \sup_{t \in (0, \infty)} \left( \int_0^t u \right)^{\frac{1}{q}} \left( \int_t^\infty w \right)^{\frac{1}{r}} \sigma_p(0, x) \left( \esssup_{x \in (0, x)} \left( \int_x^\infty w \right)^{\frac{1}{q}} w(x) \right)^{\frac{p}{pq}} dx < \infty,$$

then inequality (1.5) holds with constant independent of $h \in \mathcal{W}(0, \infty)$. Moreover, if $C$ is the best constant in (1.5), then $C \leq B_3$.

(iv) Let $\max\{q, r\} < p$. If

$$B_4 := \left( \int_0^\infty \left( \int_0^t u \right)^{\frac{1}{q}} u(t) \left( \int_t^\infty w \right)^{\frac{1}{r}} w(x) \left[ \sigma_p(0, x) \right]^{\frac{p}{pq}} dx \right)^{\frac{pq}{pr}} < \infty,$$

then inequality (1.5) holds with constant independent of $h \in \mathcal{W}(0, \infty)$. Moreover, if $C$ is the best constant in (1.5), then $C \leq B_4$.

Proof. (i) Let $p \leq \min\{q, r\}$. Assume that $B_1 < \infty$.

Recall that, if $F$ is a non-negative non-decreasing function on $(0, \infty)$, then

$$\esssup_{t \in (0, \infty)} F(t) G(t) = \esssup_{t \in (0, \infty)} F(t) \esssup_{\tau \in (t, \infty)} G(\tau)$$

(see, for instance, [16]).

On using (6.1), we get that

$$A_1 = \sup_{k \in \mathbb{Z}} 2^k \left( \esssup_{\tau \in (x_k, x_{k+1})} \left( \int_{x_k}^{x_{k+1}} w \right)^{\frac{1}{r}} \sigma_p(x_k, x) \right).$$

$$\leq \sup_{k \in \mathbb{Z}} \left( \int_{x_k}^{x_{k+1}} w \right)^{\frac{1}{r}} \esssup_{x \in (x_k, x_{k+1})} \left( \int_x^\infty w \right)^{\frac{1}{r}} \sigma_p(0, x) \right)$$

$$\leq \sup_{t \in (0, \infty)} \left( \int_0^t u \right)^{\frac{1}{q}} \esssup_{x \in (0, x)} \left( \int_x^\infty w \right)^{\frac{1}{r}} \sigma_p(0, x) \right)$$

$$= \sup_{t \in (0, \infty)} \left( \int_0^t u \right)^{\frac{1}{q}} \left( \int_t^\infty w \right)^{\frac{1}{r}} \sigma_p(0, t) = B_1.$$

Thus $A_1 < \infty$, and the statement follows from Lemma 5.1, (i).

(ii) Let $r < p \leq q$. Assume that $B_2 < \infty$. Clearly,

$$A_1 = \left( \sum_{k \in \mathbb{Z}} 2^k \left( \esssup_{\tau \in (x_k, x_{k+1})} \left( \int_{x_k}^{x_{k+1}} w \right)^{\frac{1}{r}} \sigma_p(x_k, x) \right)^{\frac{pq}{pr}} \right)^{\frac{pr}{pq}}$$

$$\leq \left( \sum_{k \in \mathbb{Z}} \int_{x_k}^{x_{k+1}} \left( \int_0^t u \right)^{\frac{1}{q}} u(t) dt \cdot \esssup_{\tau \in (x_k, x_{k+1})} \left( \int_x^\infty w \right)^{\frac{1}{r}} \sigma_p(0, x) \right)^{\frac{pq}{pr}}.$$
\[
\leq \left( \sum_{k \in \mathbb{Z}} \int_{x_k}^{x_{k+1}} \left( \int_0^t u(t) \left( \frac{1}{w(x)} \sigma_p(0, x) \right)^{\frac{1}{p-q}} dt \right) \right)^{\frac{p-r}{p}}.
\]

Thus $A_1 < \infty$, and the statement follows from Lemma 5.1, (i).

(iii) Let $q < p \leq r$. Assume that $B_3 < \infty$. We have that

\[
A_2 = \sup_{k \in \mathbb{Z}} 2^{\frac{k}{r}} \left( \int_{x_k}^{x_{k+1}} \left( \int_x^{x_{k+1}} w \right)^{\frac{1}{p-r}} w(x) \left[ \sigma_p(x_k, x) \right]^{\frac{p}{p-r}} dx \right)^{\frac{r}{p}}
\]

\[
\leq \sup_{k \in \mathbb{Z}} \left( \int_{x_k}^{x_{k+1}} \left( \int_0^t u(t) \left( \int_x^{x_{k+1}} w \right)^{\frac{1}{p-r}} w(x) \left[ \sigma_p(x_k, x) \right]^{\frac{p}{p-r}} dx \right)^{\frac{r}{p}} dt \right)
\]

\[
\leq \sup_{t \in (0, \infty)} \left( \int_0^t u(t) \left( \int_t^{x_{k+1}} \left( \int_x^{x_{k+1}} w \right)^{\frac{1}{p-r}} w(x) \left[ \sigma_p(x_k, x) \right]^{\frac{p}{p-r}} dx \right)^{\frac{r}{p}} dt \right)
\]

\[
\leq \left( \int_0^t u(t) \left( \int_t^{x_{k+1}} \left( \int_x^{x_{k+1}} w \right)^{\frac{1}{p-r}} w(x) \left[ \sigma_p(x_k, x) \right]^{\frac{p}{p-r}} dx \right)^{\frac{r}{p}} dt \right)
\]

Thus $A_2 < \infty$, and the statement follows from Lemma 5.1, (ii).

(iv) Let $\max(q, r) < p$. Assume that $B_4 < \infty$. We have that

\[
A_2 = \left( \sum_{k \in \mathbb{Z}} 2^{\frac{k}{r}} \left( \int_{x_k}^{x_{k+1}} \left( \int_x^{x_{k+1}} w \right)^{\frac{1}{p-r}} w(x) \left[ \sigma_p(x_k, x) \right]^{\frac{p}{p-r}} dx \right)^{\frac{r}{p}} \right)
\]

\[
\leq \left( \sum_{k \in \mathbb{Z}} \int_{x_k}^{x_{k+1}} \left( \int_0^t u(t) \left( \int_x^{x_{k+1}} w \right)^{\frac{1}{p-r}} w(x) \left[ \sigma_p(x_k, x) \right]^{\frac{p}{p-r}} dx \right)^{\frac{r}{p}} dt \right)
\]

\[
\leq \left( \sum_{k \in \mathbb{Z}} \int_{x_k}^{x_{k+1}} \left( \int_0^t u(t) \left( \int_t^{x_{k+1}} \left( \int_x^{x_{k+1}} w \right)^{\frac{1}{p-r}} w(x) \left[ \sigma_p(x_k, x) \right]^{\frac{p}{p-r}} dx \right)^{\frac{r}{p}} dt \right)
\]

Thus $A_2 < \infty$, and the statement follows from Lemma 5.1, (ii).

The proof is completed.

7. Necessity of conditions $B_i$, $i = 1, 4$ for inequality (1.2)

In this section, we show that conditions obtained in previous section are necessary for inequality (1.2).

**Lemma 7.1.** Let $1 \leq p < \infty$, $0 < q$, $r < \infty$ and $u, v, w \in W(0, \infty)$. Assume that inequality (1.2) holds.

(i) If $p \leq \min(q, r)$, then $B_1 < \infty$. Moreover, if $C$ is the best constant in (1.2), then $B_1 \leq C$.

(ii) If $r \leq \min(q, r)$, then $B_2 < \infty$. Moreover, if $C$ is the best constant in (1.2), then $B_2 \leq C$.

(iii) If $q < p \leq r$, then $B_3 < \infty$. Moreover, if $C$ is the best constant in (1.2), then $B_3 \leq C$.

(iv) If $\max(q, r) < p$, then $B_4 < \infty$. Moreover, if $C$ is the best constant in (1.2), then $B_4 \leq C$.

**Proof.** Let $1 \leq p < \infty$, $0 < q$, $r < \infty$. Assume that inequality (1.2) holds. Suppose that $\{x_k\}_{k \in \mathbb{Z}}$ is a covering sequence.

(i) Let $p \leq \min(q, r)$. Since $B_1 = D_1$, then the statement follows by Theorem 3.4 and Theorem 4.1.

(ii) Let $r < p \leq q$. We get, by applying Lemma 2.2, that

\[
B_2 \leq \left( \sum_{k \in \mathbb{Z}} 2^{k \frac{p}{p-r}} \left( \sup_{x \in [x_k, x_{k+1}]} \left( \int_x^{x_{k+1}} w \right)^{\frac{1}{q}} \sigma_p(0, x)^{\frac{p}{p-r}} \right)^{\frac{r}{p}} \right).
\]

It is easy to see that for any $k \in \mathbb{Z}$

\[
\sup_{x \in [x_k, x_{k+1}]} \left( \int_x^{x_{k+1}} w \right)^{\frac{1}{q}} \sigma_p(0, x) \leq \sup_{x \in [x_k, x_{k+1}]} \left( \int_x^{x_{k+1}} w \right)^{\frac{1}{q}} \sigma_p(x_k, x) + \sup_{r \in (0, \infty)} \left( \int_0^{\infty} w \right)^{\frac{1}{q}} \sigma_p(0, r)
\]

Thus $A_1 < \infty$, and the statement follows from Lemma 5.1, (i).

(iii) Let $q < p \leq r$. Assume that $B_3 < \infty$. We have that

\[
A_2 = \sup_{k \in \mathbb{Z}} 2^{k \frac{p}{p-r}} \left( \int_{x_k}^{x_{k+1}} \left( \int_x^{x_{k+1}} w \right)^{\frac{1}{q}} \sigma_p(x_k, x)^{\frac{p}{p-r}} dx \right)^{\frac{r}{p}}
\]

\[
\leq \sup_{k \in \mathbb{Z}} \left( \int_{x_k}^{x_{k+1}} \left( \int_0^t u(t) \left( \int_x^{x_{k+1}} w \right)^{\frac{1}{q}} \sigma_p(x_k, x)^{\frac{p}{p-r}} dx \right)^{\frac{r}{p}} dt \right)
\]

\[
\leq \sup_{t \in (0, \infty)} \left( \int_0^t u(t) \left( \int_t^{x_{k+1}} \left( \int_x^{x_{k+1}} w \right)^{\frac{1}{q}} \sigma_p(x_k, x)^{\frac{p}{p-r}} dx \right)^{\frac{r}{p}} dt \right)
\]

Thus $A_2 < \infty$, and the statement follows from Lemma 5.1, (ii).

The proof is completed.
Thus
\[ B_2 \leq \left( \sum_{k \in \mathbb{Z}} 2^{k \frac{p}{p-r}} \left( \sup_{u \in [x_k, x_{k+1}]} \left( \int_x^{x_{k+1}} w \sigma_p(x_k, x) \right)^{\frac{p}{p-r}} \right) \right)^{\frac{p-r}{p}} \]
\[ + \left( \sum_{k \in \mathbb{Z}} 2^{k \frac{p}{p-r}} \left( \sup_{\tau \in \mathbb{R}} \left( \int_{\tau}^{\infty} w \sigma_p(0, \tau) \right)^{\frac{p}{p-r}} \right) \right)^{\frac{p-r}{p}} \]
\[ = A_1 + \left( \sum_{k \in \mathbb{Z}} 2^{k \frac{p}{p-r}} \left( \sup_{\tau \in \mathbb{R}} \left( \int_{\tau}^{\infty} w \sigma_p(0, \tau) \right)^{\frac{p}{p-r}} \right) \right)^{\frac{p-r}{p}}. \]

Denote by
\[ E_1 := \left( \sum_{k \in \mathbb{Z}} 2^{k \frac{p}{p-r}} \left( \sup_{\tau \in \mathbb{R}} \left( \int_{\tau}^{\infty} w \sigma_p(0, \tau) \right)^{\frac{p}{p-r}} \right) \right)^{\frac{p-r}{p}}. \]

Clearly,
\[ E_1 = \left( \sum_{k \in \mathbb{Z}} \left( \int_{x_{k-1}}^{x_k} \left( \int_{x_{k-1}}^{x_k} u \sigma_p(0, u) dt \right) \left( \sup_{\tau \in \mathbb{R}} \left( \int_{\tau}^{\infty} w \sigma_p(0, \tau) \right)^{\frac{p}{p-r}} \right) \right)^{\frac{p-r}{p}} \right)^{\frac{p-r}{p}} \]
\[ \leq \left( \sum_{k \in \mathbb{Z}} \left( \int_{x_{k-1}}^{x_k} \left( \sup_{\tau \in \mathbb{R}} \left( \int_{\tau}^{\infty} w \sigma_p(0, \tau) \right)^{\frac{p}{p-r}} \right)^{\frac{p-r}{p}} \right)^{\frac{p-r}{p}} \right)^{\frac{p-r}{p}} \]
\[ \leq \left( \int_{\mathbb{R}} \left( \sup_{\tau \in \mathbb{R}} \left( \int_{\tau}^{\infty} w \sigma_p(0, \tau) \right)^{\frac{p}{p-r}} \right)^{\frac{p-r}{p}} \right)^{\frac{p-r}{p}} = D_3. \]

Combining yields
\[ B_2 \leq A_1 + D_3, \]
and the statement follows by Theorem 3.4, Lemma 5.1 and Theorem 4.1.

(iii) Let \( q < p \leq r \). Applying Lemma 2.2, we get that
\[ B_3 \leq \sup_{k \in \mathbb{Z}} 2^{\frac{k}{q}} \left( \int_{x_k}^{x_{k+1}} \left( \int_{x_k}^{x_{k+1}} w \sigma_p(0, x) \right)^{\frac{p}{q}} dx \right)^{\frac{p}{p-q}}. \]

Integrating by parts, for any \( k \in \mathbb{Z} \), it is easy to see that
\[ \left( \int_{x_k}^{x_{k+1}} \left( \int_{x_k}^{x_{k+1}} w \sigma_p(0, x) \right)^{\frac{p}{q}} dx \right)^{\frac{p}{p-q}} \]
\[ \leq \left( \int_{x_k}^{x_{k+1}} \left( \int_{x_k}^{x_{k+1}} w \sigma_p(x_k, x) \right)^{\frac{p}{q}} dx \right)^{\frac{p}{p-q}} + \sup_{\tau \in \mathbb{R}} \left( \int_{\tau}^{\infty} w \sigma_p(0, t) \right)^{\frac{p}{q}}. \]

Thus,
\[ B_3 \leq \sup_{k \in \mathbb{Z}} 2^{\frac{k}{q}} \left( \int_{x_k}^{x_{k+1}} \left( \int_{x_k}^{x_{k+1}} w \sigma_p(x_k, x) \right)^{\frac{p}{q}} dx \right)^{\frac{p}{p-q}} \]
\[ + \sup_{k \in \mathbb{Z}} 2^{\frac{k}{q}} \sup_{\tau \in \mathbb{R}} \left( \int_{\tau}^{\infty} w \sigma_p(0, t) \right)^{\frac{p}{q}} \]
\[ = A_2 + \sup_{k \in \mathbb{Z}} 2^{\frac{k}{q}} \sup_{\tau \in \mathbb{R}} \left( \int_{\tau}^{\infty} w \sigma_p(0, t) \right)^{\frac{p}{q}}. \]

Denote by
\[ E_2 := \sup_{k \in \mathbb{Z}} 2^{\frac{k}{q}} \sup_{\tau \in \mathbb{R}} \left( \int_{\tau}^{\infty} w \sigma_p(0, t) \right)^{\frac{p}{q}}. \]

Since
\[ E_2 = \sup_{k \in \mathbb{Z}} \left( \int_{0}^{x_k} u \sigma_p(0, t) \right)^{\frac{p}{q}} \]
we arrive at

\[ B_3 \leq A_2 + D_1 \]

and the statement follows by Theorem 3.4, Lemma 5.1 and Theorem 4.1.

(iv) Let \( \max\{q, r\} < p \). Applying Lemma 2.2, we get that

\[ B_4 \leq \left( \sum_{k \in \mathbb{Z}} 2^{k \frac{p}{p-r}} \left( \int_{x_k}^{x_{k+1}} \left( \int_y^{\infty} w(x) \left[ \sigma_{p}(0, x) \right]^{\frac{p}{p-r}} dx \right)^{\frac{p-r}{p}} \right)^{\frac{p}{p-r}} \right)^{\frac{p-r}{p}}. \]

On using (7.1), we have that

\[ B_4 \leq \left( \sum_{k \in \mathbb{Z}} 2^{k \frac{p}{p-r}} \left( \int_{x_k}^{x_{k+1}} \left( \int_y^{\infty} w(x) \left[ \sigma_{p}(x_k, x) \right]^{\frac{p}{p-r}} dx \right)^{\frac{p-r}{p}} \right)^{\frac{p}{p-r}} \right)^{\frac{p-r}{p}} + \left( \sum_{k \in \mathbb{Z}} 2^{k \frac{p}{p-r}} \sup_{x \in [x_k, x_{k+1})} \left( \int_y^{\infty} \left[ \sigma_{p}(0, t) \right]^{\frac{p}{p-r}} dy \right)^{\frac{p-r}{p}} \right)^{\frac{p-r}{p}} = A_2 + E_1 \leq A_2 + D_3, \]

and the statement follows by Theorem 3.4, Lemma 5.1 and Theorem 4.1.

The proof is completed.

\[ \square \]

8. Proof of the main statement.

We are now in position to prove our main result.

**Proof of Theorem 1.1.**

Note that \( F_1 = D_1, F_2 = D_2, F_3 = D_3 \) and \( F_4 = D_4 \).

Obviously, for any \( t \in (0, \infty) \)

\[
\left( \int_t^{\infty} w(x) \right)^{\frac{1}{q}} \sigma_{p}(0, t) \leq \left( \int_t^{\infty} \left( \int_y^{\infty} w(y) dy \right)^{\frac{p}{p-q}} w(y) dy \right)^{\frac{p-q}{p}} \sigma_{p}(0, t)
\]

(8.1)

Thus

\[ F_1 = B_1 = \sup_{x \in (0, \infty)} \left( \int_0^x u \right)^{\frac{1}{q}} \left( \int_x^{\infty} w(x) \right)^{\frac{1}{q}} \sigma_{p}(0, x) \leq \sup_{x \in (0, \infty)} \left( \int_0^x u \right)^{\frac{1}{q}} \left( \int_x^{\infty} \left( \int_y^{\infty} w(y) \right)^{\frac{p}{p-q}} w(y) dy \right)^{\frac{p-q}{p}} \right)^{\frac{p-q}{p}} = B_3 = F_5. \]

By inequality (8.1), we get that

\[
\sup_{x \in (0, \infty)} \left( \int_t^{\infty} w(x) \right)^{\frac{1}{q}} \sigma_{p}(0, t) \leq \sup_{x \in (0, \infty)} \left( \int_0^x \left( \int_y^{\infty} w(y) \right)^{\frac{p}{p-q}} w(y) \left[ \sigma_{p}(0, y) \right]^{\frac{p}{p-q}} dy \right)^{\frac{p-q}{p}} \leq \left( \int_0^x \left( \int_y^{\infty} w(y) \left[ \sigma_{p}(0, y) \right]^{\frac{p}{p-q}} dy \right)^{\frac{p-q}{p}} \right)^{\frac{p-q}{p}} = B_3 = F_5.
\]

Thus

\[ F_3 = B_2 = \left( \int_0^x \sup_{x \in [x_k, x_{k+1})} \left( \int_t^{\infty} w(x) \right)^{\frac{1}{q}} \sigma_{p}(0, t) \right)^{\frac{p}{p-q}} \left( \int_0^x u \right)^{\frac{p}{p-q}} u(x) dx \right)^{\frac{p}{p-q}} \leq \left( \int_0^x \left( \int_0^x u \right)^{\frac{p}{p-q}} u(x) \left( \int_y^{\infty} w(y) \left[ \sigma_{p}(0, y) \right]^{\frac{p}{p-q}} dy \right)^{\frac{p-q}{p}} dx \right)^{\frac{p}{p-q}} = B_3 = F_6. \]

So, the proof of the statement immediately follows from Theorem 3.4 and 4.1, Lemmas 6.1 and 7.1. \( \square \)
