Cooperation of anti-aligning and aligning shell-model forces for \( N = Z \)

K. Neergård

Fjordtoften 17, 4700 Næstved, Denmark

For two neutrons and two protons or two neutron holes and two proton holes in a single \( j \)-shell, the ground state with only a pairing force and the lowest angular momentum zero state with only an attractive force acting solely on pairs of a quasineutron and a quasiproton with maximally aligned angular momenta overlap considerably for all realistic \( j \), and the state produced with one of these forces has a large content of the pair angular momentum formally favored by the other force. Therefore these two strongly attractive components of a realistic effective two-quasinucleon interaction cooperate to produce a ground state which is essentially a linear combination of both states with comparable coefficients. In the \( 1f_{7/2} \) and \( 1g_{9/2} \) shells, the state produced with only a pairing force makes up about 80 \% of the ground state. The overlaps of the latter with both states decrease with increasing \( j \).

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In a recent article \[1\], Qi et al. discuss the ground-state structure of nuclei with two neutrons and two protons or two neutron holes and two proton holes in spherical shells such as \( 1f_{7/2}, 1g_{9/2} \), or \( 1h_{11/2} \) with a high single-nucleon angular momentum \( j \). The authors compare contributions of the configurations

\[
((j_1,j_2)J_e(j_3,j_4)J_e)0
\]

with \( J_e = 0 \) (where ‘e’ stands for ‘equal’) and

\[
((j_1,j_3)J_d(j_2,j_4)J_d)0
\]

with \( J_d = 2j \) (where ‘d’ stands for ‘different’) in a vector-coupling notation with the total magnetic quantum number suppressed. The 1st and 2nd quasinucleons are the neutrons or neutron holes and the 3rd and 4th quasinucleons the protons or proton holes, and all \( j_i \) are equal to \( j \). In single-\( j \)-shell calculations with interactions taken from experimental data and the classic analysis by Schiff and True \[2\], they find that \( J_e = 0 \) contributes 51-62 \% whereas \( J_d = 2j \) contributes 92-95 \%. From these and other results of calculations they infer that in heavy nuclei with \( N = Z \), a spin-aligned isoscalar pair mode replaces as the dominant coupling scheme the isovector pairing mode prevalent in the bulk of the chart of nuclides.

For a deeper analysis of these observations the Pauli principle and isospin conservation should be taken into account. The Pauli principle requires that \( J_e \) is even, so this is understood throughout in the following. The states \( |J_e \rangle \) then span the angular momentum \( I = 0 \) space. While isospin is conserved in the model, and the calculated ground states have isospin \( T = 0 \), the individual \( |J_e \rangle \) are not eigenstates of \( T \) for \( j \geq 3/2 \). The \( I = 0 \) space has a \([2j/3]\)-dimensional subspace with \( T = 0 \). If finite-dimensional, that is, for \( j \geq 3/2 \), the orthogonal subspace has \( T = 2 \) \[3\]. It can be shown by Flowers’s method \[4\] that in each such finite-dimensional maximal eigenspace of \( T \) within the \( I = 0 \) space, a one-dimensional subspace carries the seniority \( s = 0 \) representation of the symplectic group \( Sp(2j + 1) \) while the orthogonal subspace, if finite-dimensional, belongs to the representation \( s = 4 \), \( t = T \), where \( t \) is the reduced isospin. Since \( |J_e = 0 \rangle \) is symplectically invariant, it belongs to the \( s = 0 \) subspace (two-dimensional for \( j \geq 3/2 \)) of the \( I = 0 \) space. Therefore, in each finite-dimensional maximal eigenspace of \( T \) within the \( I = 0 \) space the state \( |s = 0 \rangle \) is obtained by projecting \( |J_e = 0 \rangle \) onto that space and thus has the maximal content of \( J_e = 0 \). Fig. \[1\] shows for \( j = 9/2 \) the distribution of \( J_e \) in the \( I = s = T = 0 \) state \( |\phi \rangle \). It is seen that \( J_e = 0 \) makes up only 73 \% of it. Therefore 51-62 \% of \( J_e = 0 \) in the calculated ground states is equivalent to \( |\phi \rangle \) contributing 70-85 \%. A very similar picture emerges for other \( j \).

Frauendorf and Sheikh have pointed out \[5\] that the product of a neutron and a proton BCS state commonly employed to model isovector pairing may be viewed as a state deformed in isospace, which may rotate in this space giving rise to isorotational bands. A microscopic model of such a superfluid isorotation was examined by me \[6\]. Since \( |J_e = 0 \rangle \) is composed of one quasineutron and one quasiproton Cooper pair, it may be seen analogously as a state deformed in isospace, and the projection onto \( T = 0 \) seen as the construction of an isorotational band head out of this intrinsic state. Naturally, with only four quasinucleons this analogy must not be pushed too far because the isorotational band terminates already at \( T = 2 \).

FIG. 1: \( J_e \)-distribution in \( |\phi \rangle \) for \( j = 9/2 \).

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*Electronic address: kai@kaineergard.dk*
FIG. 2: $n(J,|\phi\rangle)$ as a function of $J$ for $j = 9/2$. The sum is six.

For any $I = 0$ state $|\psi\rangle$, I define the average number

$$n(J,|\psi\rangle) = \langle \psi | \sum_{i<k} P_{J_{ik}=J} |\psi\rangle$$

of pairs with $J_{ik} = J$, where $J_{ik}$ is the combined angular momentum of the $i$th and $k$th quasinucleons, and $P_{J_{ik}=J}$ the projection onto the $J_{ik} = J$ space. I also introduce the general symbol

$$V_J = c_J \sum_{i<k} P_{J_{ik}=J}$$

for a two-nucleon interaction acting solely on quasineutron pairs with combined angular momentum $J$. A $V_0$ is symplectically invariant and thus commutes with $s$ and $t$. It follows from expressions given by Edmonds and Flowers [2] that it has eigenvalue zero for $s = 4$, $t = T$. Within the $I = T = 0$ space, it is therefore proportional to the projection onto the one-dimensional $s = 0$ subspace, and $|\phi\rangle$ is the ground state when a pairing force, that is, an attractive $V_0$, is the only two-quasineutron interaction.

Fig. 2 shows the distribution of $n(J,|\phi\rangle)$ for $j = 9/2$. It is seen that in spite of $|\phi\rangle$ being the ground state when an attractive $V_0$ is the only two-quasineutron interaction, no more than about one third of the pairs have $J = 0$. The next most frequent $J$, possessed by about one sixth of the pairs, is $2j$. This demonstrates that a force which formally favors a certain pair angular momentum does not prevent the majority of pairs from having very different angular momenta. It can be inferred, moreover, that an additional attractive $V_{2j}$ will help to stabilize $|\phi\rangle$. I digress with the remark that the broad distribution found here seems to lend some doubt to the reliability of the distribution of the pair angular momentum as an indicator of the pairing structure of a multi-nucleon state.

The state $|J_d = 2j\rangle$ is not antisymmetric in the quasineutrons or the quasiprotons and thus violates the Pauli principle. For $j = 9/2$ only 50% of it belongs to the even-$J_e$ space. This percentage is typical of all $j$ and $J_d$. The projection of $|J_d = 2j\rangle$ onto the even-$J_e$ space is an exact $T = 0$ state. I shall prove that this is true, in fact, of the projection of any state $|\psi\rangle$ which is symmetric in the 1st and 3rd quasinucleons and in the 2nd and 4th quasinucleons onto the space of states that are antisymmetric in the quasineutrons and in the quasiprotons. (Almost the same proof applies to states which are symmetric in the 1st and 4th quasinucleons and in the 2nd and 3rd quasinucleons.)

To this end, notice that, denoting by $n$, $T$, and $t_i$ the number of quasinucleons, the total isospin, and the isospin of the $i$th quasinucleon, we have

$$T^2 = n \times \frac{3}{4} + 2 \sum_{i<k} t_i \cdot t_k$$

$$= n \times \frac{3}{4} + 2 \sum_{i<k} \left( \frac{1}{2}(\delta_{ik})_t - \frac{1}{4} \right)$$

$$= n \times \frac{3}{4} + 2 \frac{n(n-1)}{2} \times \left( -\frac{1}{4} \right) + \sum_{i<k} (\delta_{ik})_t$$

$$= n - \frac{n^2}{4} + \sum_{i<k} (\delta_{ik})_t,$$

where $(\delta_{ik})_t$ is the transposition of the isospins of the $i$th and $k$th quasinucleons with state vectors that are antisymmetric in all quantum numbers including those of isospin. Since with such state vectors we have

$$(\delta_{ik})_t = -(\delta_{ik}),$$

where $(\delta_{ik})$ is the transposition of the quantum numbers other than those of isospin, Eq. (1) can be written

$$T^2 = n - \frac{n^2}{4} - \sum_{i<k} (\delta_{ik}). \tag{2}$$

This may be used to express the isospin of state vectors like those considered here with a specified type of each quasinucleon. For $n = 4$ the first two terms in the expression (2) cancel, so the $T = 0$ space is the kernel of the sum of transpositions. (This could also be inferred, more abstractly, from the fact that its Young frame [22] is self-conjugate.) Since the sum of transpositions commutes with the considered projection

$$P = \frac{1}{4} (1 - (12))(1 - (34)),$$

and by assumption

$$P(12) = P(34) = P, \quad (13)|\psi\rangle = (24)|\psi\rangle = |\psi\rangle,$$

$$P(1 - (12))(14) = (14)(1 - (24)),$$

$$P(1 - (12))(23) = (23)(1 - (13)),$$

we get

$$\left( \sum_{i<k} (\delta_{ik}) \right) P|\psi\rangle = P \left( \sum_{i<k} (\delta_{ik}) \right) |\psi\rangle = 0.$$  

This completes the proof.

As discovered by Moya de Guerra et al. [8] and discussed by Zamick and Escuderos [9], the interactions $V_J$ are separable in the $l = 0$ space for odd $J$:

$$\langle J_e | \sum_{i<k} P_{J_{ik}=J} |J'_e\rangle$$

$$= \sum_{i<k} \langle J_e | J_{ik} = J_{lm} = J |J_{ik} = J_{lm} = J|J'_e\rangle$$

$$= 4 \langle J_e | J_d = J \rangle \langle J_d = J |J'_e\rangle,$$  

[Here, the subscripts $e$ and $d$ refer to even and odd quasinucleons, respectively.]
The state reported by Qi et al. to make up 92-95 % of the calculated ground states is $|\chi\rangle = NP|J_4 = 2J\rangle$ with a normalization factor $N$ such that $\langle \chi|\chi\rangle = 1$, and thus the lowest $I = 0$ state when an attractive $V_2$ is the only two-quasineutron interaction. (Zamick and Escuderos have shown [11] that it is not the ground state for $j = 9/2$.) The $J_2$-distribution in this state is shown for $j = 9/2$ in Fig. 3. Like that of $|\phi\rangle$ it is dominated by low $J_2$ while the high $J_2$ are virtually absent. A difference is that in $|\chi\rangle$ the components with $J_2 = 0$ and 2 contribute almost equally with some predominance of the latter. A similar picture emerges for all $j$ with $J_2 = 2$ contributing more than $J_2 = 0$ for $j \geq 7/2$.

Fig. 4 shows the distribution of $n(J,|\chi\rangle)$ for $j = 9/2$. Almost exactly one third of the pairs have $J = 2j$ while the rest have predominantly low $J$. This distribution is thus quite similar to that of $|\phi\rangle$ with the low and high $J$ interchanged, and one can infer that an attractive $V_0$ helps to stabilize $|\chi\rangle$.

The overlap $|\langle \phi|\chi\rangle|^2$ is shown in Fig. 5 as a function of $j$. For $j \leq 3/2$ this overlap is one because there is only one $I = T = 0$ state. For higher $j$ it decreases with $j$, but far from becoming orthogonal, $|\phi\rangle$ and $|\chi\rangle$ maintain a considerable overlap for all realistic $j$. For $j = 9/2$ it is 52 %.

I turn to calculations for the $1f_{7/2}$ and $1g_{9/2}$ shells with realistic interactions $\sum J V_J$. In both shells the $I = T = 0$ space is three-dimensional, so I introduce a third basic vector $|\xi\rangle$ orthogonal to both $|\phi\rangle$ and $|\chi\rangle$. The states $|\phi\rangle$, $|\chi\rangle$, and $|\xi\rangle$ span the $I = T = 0$ space, but as already noted, $|\phi\rangle$ and $|\chi\rangle$ are not orthogonal. The results of calculations are shown in Table I with the ground state denoted by $|\psi\rangle$. Those for an attractive interaction $V_0$ or $V_2$ can be inferred from the preceding discussion but are repeated for reference. Otherwise the interactions ‘SchTr’ are taken from the appendix of the aforesaid study by Schiffer and True with ‘emp.’ referring to the empirical matrix elements and ‘fit’ to those derived from a universal interaction fitted to the data. The matrix elements of the interaction ‘EZB’ are taken from Page 4 of a recent report by Escuderos, Zamick and Bayman [10]. They were derived from the spectrum of $^{42}$Sc. Finally, the interaction ‘Qi’ was kindly suggested to me by Chong Qi as suitable for single-$j$-shell calculations in the $1g_{9/2}$ shell. Its matrix elements are given in Table I. The interactions ‘EZB’ and ‘Qi’ have $c_0 = 0$ by definition. Evidently, adding a constant to all $c_J$ does not change the calculated states.

By construction, an attractive $V_0$ gives $|\langle \phi|\psi\rangle|^2 = 1$ and $|\langle \chi|\psi\rangle|^2 = |\langle \phi|\chi\rangle|^2$, and oppositely for an attractive $V_2$. For the realistic interactions, $|\langle \chi|\psi\rangle|^2$ is, indeed, consistently larger than $|\langle \phi|\psi\rangle|^2$. However, both are close to one and considerably larger than $|\langle \phi|\chi\rangle|^2$. The expectation of a cooperation of the two most attractive terms in these interaction, $V_0$ and $V_2$, is borne out.
TABLE II: Matrix elements in MeV of the interaction ‘Qi’.

| $J$ | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $c_J$ | 1.220 | 1.458 | 1.592 | 2.283 | 1.882 | 2.549 | 1.930 | 2.688 | 0.626 |

remarkably. The overlaps shown in the table result from $|\psi\rangle \propto |\phi\rangle + \alpha|\chi\rangle + \beta|\xi\rangle$ with $\beta \approx 0$ and $\alpha = 1.65$-1.97 except for the empirical 1f$_{7/2}$ interaction of Schiffer and True, which gives $\alpha = 2.85$. This deviation from what is found otherwise may be due to shortcoming of the data whence these empirical matrix elements were extracted.

The entire interaction including the terms with $1 \leq J \leq 2j - 1$ is involved in producing the almost constant $\alpha \approx 1.8$. Thus $\alpha$ is different when the interactions are modified by setting $c_J = c$ for $1 \leq J \leq 2j - 1$ with $c = 0$ for the interactions ‘SchTr’ and $c = 2$-3 MeV for the interactions ‘EZB’ and ‘Qi’. Then $\beta = 0$ and $\alpha = \sqrt{r^2 + a^2} + a$ with $r = (c-c_2)n(2j,|\chi\rangle)/\langle c-c_0)n(0,|\phi\rangle)$ and $a = (r-1/2)|\langle \phi|\chi\rangle|$, which gives $\alpha = 0.57$-1.37.

Most notably both $|\langle \phi|\psi\rangle|^2$ and $|\langle \chi|\psi\rangle|^2$ are less in the 1g$_{9/2}$ than in the 1f$_{7/2}$ shell. This can be understood as due to the increasing angle $\cos^{-1}|\langle \phi|\chi\rangle| = 38^\circ$ and 44$^\circ$ for $j = 7/2$ and 9/2, respectively. With this increasing angle the norms of the perpendicular projections onto both basic states of a unit state vector proportional to $|\phi\rangle + \alpha|\chi\rangle$ with constant $\alpha$ decrease.

The overlaps $|\langle \xi|\psi\rangle|^2$ are extremely small due to a sub- tle cancellation in the calculations among the individual $\langle \phi|V_j|\xi\rangle$ and $\langle \chi|V_j|\xi\rangle$ with $1 \leq J \leq 2j - 1$.

To summarize and conclude, I studied the ground states of nuclei with two neutrons and two protons or two neutron holes and two proton holes in a single $j$-shell. I found that the ground state $|\phi\rangle$ produced with only a pairing force $V_0$ and the lowest $I = 0$ state $|\chi\rangle$ produced with only an attractive force $V_2$ acting solely on pairs of a quasineutron and a quasiproton with maximally aligned angular momenta have a considerable overlap $|\langle \phi|\chi\rangle|^2$ for all realistic $j$. Moreover, in these states no more than about one third of the quasinucleon pairs have the angular momentum, 0 or 2$j$, formally favored by the force, and about one sixth of the pairs have the angular momentum formally favored by the other force. Therefore, these two strongly attractive terms in the effective two-quasinucleon interaction cooperate to produce a ground state $|\psi\rangle$ found in calculations to be roughly proportional to $|\phi\rangle + 1.8|\chi\rangle$. Both $|\langle \phi|\psi\rangle|^2$ and $|\langle \chi|\psi\rangle|^2$ decrease with increasing $j$ due to decreasing $|\langle \phi|\chi\rangle|^2$. In the 1f$_{7/2}$ and 1g$_{9/2}$ shells, $|\langle \phi|\psi\rangle|^2$ is about 80%, so if $|\phi\rangle$, which is the sole state with $I = s = T = 0$, where $I$ is the angular momentum, $s$ the seniority, and $T$ the isospin, is seen as a manifestation of conventional isovector pairing, the latter is far from absent from these ground states.

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[1] C. Qi, J. Blomqvist, T. Bäck, B. Cederwall, A. Johnson, R. J. Liotta, and R. Wyss, Phys. Rev. C 84, 021301 (2011).
[2] J. P. Schiffer and W. W. True, Rev. Mod. Phys. 48, 191 (1976).
[3] C. Qi, Phys. Rev. C 81, 034318 (2010).
[4] B. H. Flowers, Proc. R. Soc. A 212, 248 (1952).
[5] S. G. Fraudenfor and J. A. Sheikh, Nucl. Phys. A645, 509 (1999); Phys. Scr. T88, 162 (2000).
[6] K. Neergård, Phys. Rev. C 80, 044313 (2009).
[7] A. R. Edmonds and B. H. Flowers, Proc. R. Soc. A 214, 515 (1952).
[8] E. Moya de Guerra, A. A. Raduta, L. Zamick, and P. Sarriugueri, Nucl. Phys. A727, 3 (2003).
[9] L. Zamick and A. Escuderos, Phys. Rev. C 87, 044302 (2013).
[10] A. Escuderos, L. Zamick, and B. F. Bayman (2005), arXiv:nucl-th/0506050.