Relative symplectic Steinberg group

Andrei Lavrenov*

Abstract

We give two definitions of relative symplectic Steinberg group and show that they coincide.

Introduction

The main result of the present paper is a “relative version” of the another presentation for symplectic Steinberg groups obtained in [2].

First, we give a definition of a relative symplectic group StSp_{2l}(R, I, \Gamma) for any form ideal (I, \Gamma) in R. For maximal form parameter \Gamma = I it has already appeared in [3] and is shown to have good homological properties. We also consider case of a maximal form parameter and establish another “coordinate-free” presentation for StSp_{2l}(R, I, \Gamma_{\text{max}}) = StSp_{2l}(R, I). The last result is a relativisation of the [2], but it is not a generalisation, since we need more relations for the presentation. Moreover, in our proofs we use the results of [2].

Namely, the main theorem of [2] states the following.

Theorem. Let l \geq 3, then the symplectic Steinberg group StSp(2l, R) can be defined by the set of generators

\[ \{ X(u, v, a) \mid u, v \in V, \text{u is a column of} \ a \text{symplectic elementary matrix,} \langle u, v \rangle = 0, \ a \in R \} \]

*The author acknowledges support of the RSCF project 14-11-00297 “Decomposition of unipotents in reductive groups”.

\[ \text{arXiv:1412.2421v1 [math.KT]} \ 8 \text{Dec} 2014. \]
and relations

\[ X(u, v, a)X(u, w, b) = X(u, v + w, a + b + \langle v, w \rangle), \quad (P1) \]
\[ X(u, va, 0) = X(v, ua, 0) \quad \text{where } v \text{ is also a column of a symplectic elementary matrix}, \quad (P2) \]

\[ X(u', v', b)X(u, v, a)X(u', v', b)^{-1} = X(T(u', v', b)u, T(u', v', b)v, a), \quad (P3) \]

where \( T(u, v, a) \) is an ESD-transformation

\[ w \mapsto w + u(\langle v, w \rangle + a\langle u, w \rangle) + \langle u, w \rangle. \]

For usual generators of the symplectic Steinberg group the following identities hold

\[ X_{ij}(a) = X(e_i, e_{-j}a\varepsilon_{-j}, 0) \quad \text{for } j \neq -i, \quad X_{i, -i}(a) = X(e_i, 0, a). \]

We prove the following theorem in the case of maximal form parameter.

**Theorem 1.** Let \( l \geq 3 \), then the relative symplectic Steinberg group \( \text{StSp}_{2l}(R, I) \) can be defined by the set of generators

\[ \{(u, v, a, b) \mid u, v \in V, \; u \text{ is a column of a symplectic elementary matrix}, \; \langle u, v \rangle = 0, \; a, b \in I\} \]

and relations

\[ (u, vr, a, b) = (u, v, ar, b) \quad \text{for any } r \in R, \quad (T1) \]
\[ (u, v, a, b)(u, w, a, c) = (u, v + w, a, b + c + a^2\langle v, w \rangle), \quad (T2) \]
\[ (u, v, a, 0)(u, v, b, 0) = (u, v, a + b, 0), \quad (T3) \]
\[ (u, v, a, 0) = (v, u, a, 0) \quad \text{for } v \text{ a column of a symplectic elementary matrix}, \quad (T4) \]

\[ (u', v', a', b')(u, v, a, b)(u', v', a', b')^{-1} = (T(u', v'a', b')u, T(u', v'a', b')v, a, b), \quad (T5) \]
\[ (u, v, a, 0) = (u, 0, 0, 2a), \quad (T6) \]
\[ (u + vr, 0, 0, a) = (u, 0, 0, a)(v, 0, 0, ar^2)(v, u, ar, 0) \quad \text{for } v, \; u + vr \; \text{also columns of symplectic elementary matrices}. \quad (T7) \]

and for usual relative generators the following identities hold

\[ Y_{ij}(a) = (e_i, e_{-j}, a\varepsilon_{-j}, 0) \quad \text{for } j \neq -i, \quad Y_{i, -i}(a) = (e_i, 0, 0, a). \]

The author plans to generalise the above result for the case of arbitrary form parameter.
1 Relative symplectic Steinberg group

In the sequel \( R \) denotes an arbitrary associative commutative unital ring, \( V = R^{2l} \) denotes a free right \( R \)-module with basis numbered \( e_{-l}, \ldots, e_{-1}, e_1, \ldots, e_l \), \( l \geq 3 \). For the vector \( v \in V \) its \( i \)-th coordinate will be denoted by \( v_i \), i.e. \( v = \sum_{i=-l}^{l} e_i v_i \). By \( \langle \ , \rangle \) we denote the standard symplectic form on \( V \), i.e \( \langle e_i, e_j \rangle = \text{sgn}(i)\delta_{i,-j} \). We will usually write \( \varepsilon_i \) instead of \( \text{sgn}(i) \).

Observe that \( \langle u, u \rangle = 0 \) for any \( u \in V \).

**Definition.** Define the symplectic group \( \text{Sp}(V) = \text{Sp}_{2l}(R) \) as the group of automorphisms of \( V \) preserving the symplectic form \( \langle \ , \rangle \),

\[
\text{Sp}(V) = \{ f \in \text{GL}(V) \mid \langle f(u), f(v) \rangle = \langle u, v \rangle \ \forall \ u, v \in V \}.
\]

**Definition (Eichler–Siegel–Dickson transformations).** For \( a \in R \) and \( u, v \in V \), \( \langle u, v \rangle = 0 \), denote by \( T(u, v, a) \) the automorphism of \( V \) s.t. for \( w \in V \) one has

\[
T(u, v, a) : w \mapsto w + u(\langle v, w \rangle + a\langle u, w \rangle) + v\langle u, w \rangle.
\]

We refer to the elements \( T(u, v, a) \) as the (symplectic) ESD-transformations.

**Lemma 1.** Let \( u, v, w \in V \) be three vectors such that \( \langle u, v \rangle = 0, \langle u, w \rangle = 0 \), and let \( a, b \in R \). Then

a) \( T(u, v, a) \in \text{Sp}(V) \),

b) \( T(u, v, a)T(u, w, b) = T(u, v+w, a+b+\langle v, w \rangle) \),

c) \( T(u, va, 0) = T(v, ua, 0) \),

d) \( gT(u, v, a)g^{-1} = T(gu, gv, a) \ \forall \ g \in \text{Sp}(V) \).

**Remark.** Observe that \( T(u, 0, 0) = 1 \) and \( T(u, v, a)^{-1} = T(u, -v, -a) \).

In the present paper all commutators are left-normed, \( [x, y] = xyx^{-1}y^{-1} \), we denote \( xyx^{-1}y^{-1} \) by \( xy \).

**Lemma 2.** For \( u, v \in V \) such that \( u_i = u_{-i} = v_i = v_{-i} = 0, \langle u, v \rangle = 0 \), and \( a \in R \) one has

\[
[T(e_i, u, 0), T(e_{-i}, v, a)] = T(u, v\varepsilon_i, a)T(e_{-i}, -ua\varepsilon_{-i}, 0).
\]

**Definition.** A form ideal \( (I, \Gamma) \) is a pair of an ideal \( I \trianglelefteq R \) and a relative form parameter \( \Gamma \) of level \( I \), i.e., an additive subgroup of \( I \) such that
a) ∀ \( a \in I \) holds \( 2a \in \Gamma \),

b) ∀ \( r \in R, \forall a \in I \) holds \( ra^2 \in \Gamma \),

c) ∀ \( \alpha \in \Gamma, \forall r \in R \) holds \( \alpha r^2 \in \Gamma \).

**Remark.** If \( 2 \in R^\times \) the only possible choice for a relative form parameter is \( \Gamma = I \).

**Definition.** Define \( T_{ij}(a) = T(e_i, e_j a \varepsilon_j, 0) \) and \( T_{i,-i}(a) = T(e_i, 0, a) \), where \( a \in R, i, j \in \{-l, \ldots, -1, 1, \ldots, l\}, i \not\in \{\pm j\} \). We refer to these elements as the *elementary symplectic transvections*. A normal subgroup of \( \text{Sp}(V) \)

\[ \text{Ep}_2(l, I, \Gamma) = \text{Sp}(V) \langle T_{ij}(a), T_{i,-i}(\alpha) \mid i \not\in \{\pm j\}, a \in I, \alpha \in \Gamma \rangle \]

is called the *relative elementary symplectic group* corresponding to the form ideal \((I, \Gamma)\).

**Definition.** The *symplectic Steinberg group* \( \text{StSp}_2(l, R) \) is the group generated by the formal symbols \( X_{ij}(r), i \neq j, r \in R \) subject to the Steinberg relations

\[ X_{ij}(r) = X_{-j,-i}(-r \varepsilon_i \varepsilon_j), \quad (S0) \]
\[ X_{ij}(r)X_{ij}(s) = X_{ij}(r + s), \quad (S1) \]
\[ [X_{ij}(r), X_{jk}(s)] = 1, \text{ for } h \not\in \{j,-i\}, k \not\in \{i,-j\}, \quad (S2) \]
\[ [X_{ij}(r), X_{jk}(s)] = X_{ik}(rs), \text{ for } i \not\in \{-j,-k\}, j \neq -k, \quad (S3) \]
\[ [X_{i,-i}(r), X_{-i,j}(s)] = X_{ij}(rs \varepsilon_i)X_{-j,j}(-rs^2), \quad (S4) \]
\[ [X_{ij}(r), X_{j,-i}(s)] = X_{i,-i}(2rs \varepsilon_i). \quad (S5) \]

The next lemma is a straightforward consequence of Lemmas 1 and 2.

**Lemma 3.** There is a natural epimorphism

\[ \phi: \text{StSp}_2(l, R) \to \text{Ep}_2(l, R) = \text{Ep}_2(R, R, R) \]

sending the generators \( X_{ij}(a) \) to the corresponding elementary transvections \( T_{ij}(a) \). In other words, the Steinberg relations hold for the elementary transvections.

If group \( G \) acts on group \( H \) from the left, we will denote the image of \( h \in H \) under the homomorphism corresponding to the element \( g \in G \) by \( ^g h \), the element \( ^g h \cdot h^{-1} \) by \([g, h]\) and the element \( h \cdot ^g h^{-1} \) by \([h, g]\).
Definition. Define the \textit{relative symplectic Steinberg group} \(\text{StSp}_2l(R, I, \Gamma)\) corresponding to the form ideal \((I, \Gamma)\) as a formal group with the action of the \textit{absolute Steinberg group} \(\text{StSp}_2l(R)\) defined by the set of (relative) generators \(\{Y_{ij}(a) \mid i \notin \{\pm j\}, a \in I\} \cup \{Y_{i,-i}(\alpha) \mid \alpha \in \Gamma\}\) subject to the following relations

\[
Y_{ij}(a) = Y_{-j,-i}(-a\varepsilon_i\varepsilon_j), \quad (KL0)
\]
\[
Y_{ij}(a)Y_{ij}(b) = Y_{ij}(a+b), \quad (KL1)
\]
\[
[X_{ij}(r), Y_{hk}(a)] = 1, \text{ for } h \notin \{j, -i\}, k \notin \{i, -j\}, \quad (KL2)
\]
\[
[X_{ij}(r), Y_{jk}(a)] = Y_{ik}(ra), \text{ for } i \notin \{-j, -k\}, j \neq -k, \quad (KL3)
\]
\[
[X_{i,-i}(r), Y_{i,-j}(a)] = Y_{ij}(ra\varepsilon_i)Y_{-j,-i}(-ra^2), \quad (KL4)
\]
\[
[Y_{i,-i}(\alpha), X_{-i,j}(r)]=Y_{ij}(ar\varepsilon_i)Y_{-j,-i}(-ar^2), \quad (KL5)
\]
\[
[X_{ij}(r), Y_{j,-i}(a)] = X_{i,-i}(2ra\varepsilon_i), \quad (KL6)
\]
\[
X_{ij}(a)Y_{hk}(b) = Y_{ij}(a)Y_{hk}(b). \quad (KL7)
\]

In other words we consider a free group generated by symbols \((g, x) = g^x\) where \(g\) is from the absolute Steinberg group and \(x\) is from the set of relative generators, \(\text{StSp}_2l(R)\) naturally acts on this free group via \(f(g, x) = (fg, x)\) and then we define a relative symplectic Steinberg group as a factor of the described free group modulo normal subgroup generated by KL0–KL7.

Definition. Obviously, there is a natural map

\[\varphi : \text{StSp}_2l(R, I, \Gamma) \to \text{Sp}_2l(R).\]

Then its kernel is denoted by \(K_2\text{Sp}_2l(R, I, \Gamma)\).

2 Auxiliary constructions

Definition. Define the \textit{relative Steinberg unipotent radical}

\[U_1 = \langle Y_{ij}(a), Y_{i,-i}(\alpha) \mid i \notin \{\pm j\}, a \in I, \alpha \in \Gamma\rangle \leq \text{StSp}_2l(R, I, \Gamma)\]

and the (absolute) Steinberg parabolic subgroup

\[P_1 = \langle X_{kh}(a) \mid i \notin \{h, -k\}, a \in R\rangle \leq \text{StSp}_2l(R)\]

Lemma 4 (Levi decomposition). For \(g \in P_1, u \in U_1\) one has

\[g^u \in U_1.\]
Lemma 5. One has

\[ [i]U_1, \langle i \rangle U_1 \leq \langle Y_{i-1}(\alpha) \rangle, \quad [i]U_1, \{ Y_{i-1}(\alpha) \} = 1. \]

Corollary. Every element of \( [i]U_1 \) can be expressed in the form

\[ Y_{i-1}(\alpha)Y_{i-1}(a_{i-1}) \cdots Y_{i-1}(a_1)Y_{i-1}(a_1) \cdots Y_{i-1}(a_i). \]

Lemma 6. The restriction of the natural projection \( \varphi: StSp_{2i}(R, I, \Gamma) \rightarrow Sp_{2i}(R) \) to \( [i]U_1 \) is injective

\[ [i]U_1 \cong \varphi([i]U_1). \]

Proof. Take an element \( x \in [i]U_1 \). Using the above corollary, decompose \( x \) as

\[ x = Y_{i-1}(a_{i-1})Y_{i-1}(a_{i-1}) \cdots Y_{i-1}(a_1)Y_{i-1}(a_1) \cdots Y_{i-1}(a_i). \]

Then \( \varphi(x) = 1 \) implies that \( a_i = 0 \) for all \( i \). \( \square \)

Lemma 7. Take \( v \in I^{2i} \) such that \( v_{-i} = 0 \) and \( \alpha \in \Gamma \). Denote

\[ v_- = \sum_{k < 0} e_kv_k \quad \text{and} \quad v_+ = \sum_{k > 0} e_kv_k. \]

Then

\[ T(e_i, v, \langle v_-, v_+ \rangle + \alpha) = T_{i-1}(\alpha + 2v_i) \cdot T_{-i-1}(v_{-i}) \cdots T_{-i-1}(v_{-i})T_{1-1}(v_{1,i}) \cdots T_{i-1}(v_{i,i}). \]

Proof. Assume that \( i > 0 \), for \( i < 0 \) the proof looks exactly the same. Since \( T_{j-1}(v_{j,i}) = T(e_i, e_jv_j, 0) \) for \( j \neq -i \) and

\[ T_{i-1}(2v_i) = T(e_i, 0, 2v_i) = T(e_i, e_iv_i, 0), \]

one has

\[ T_{-i-1}(v_{-i}) \cdots T_{-i-1}(v_{-i}) = T(e_i, e_{-1}v_{-i}, 0) \cdots T(e_i, e_{-1}v_{-1}, 0) = T(e_i, v_{-i}, 0), \]

and

\[ T_{i-1}(2v_i)T_{1-1}(v_{1,i}) \cdots T_{i-1}(v_{i,i}) = T(e_i, e_{i}v_{i}, 0)T(e_i, e_{1}v_{1}, 0) \cdots T(e_i, e_{i}v_{i}, 0) = T(e_i, v_{+}, 0), \]

6
so that the right hand side of the desired equality is in fact equal to

\[
T_{i,-i}(\alpha) T(e_i, v_-, 0) T(e_i, v_+, 0) = \\
= T(e_i, 0, \alpha) T(e_i, v, \langle v_-, v_+ \rangle) = T(e_i, v, \langle v_-, v_+ \rangle + \alpha).
\]

\[\square\]

**Definition.** For \( v \in I^{2l} \) with \( v_- = 0 \) and \( a \in R \) such that \( a - \langle v_-, v_+ \rangle \in \Gamma \), define

\[
Y(e_i, v, a) = (\phi_{|0U_1})^{-1}(T(e_i, v, a)).
\]

**Remark.** By Lemma 7, \( T(e_i, v, a) \) indeed lies in \( \phi(i)U_1 \). Moreover, the same lemma provides the following decomposition.

**Lemma 8.** For \( v \in I^{2l} \) such that \( v_- = 0 \), \( a \in R \) such that \( a - \langle v_-, v_+ \rangle \in \Gamma \), one has

\[
Y(e_i, v, a) = Y_{i,-i}(a + 2v_i - \langle v_-, v_+ \rangle) \\
\cdot Y_{i,-i}(v_1 \varepsilon_i) \cdots Y_{i,-i}(v_1 \varepsilon_i) Y_{i,-i}(v_i \varepsilon_i) \cdots Y_{i,-i}(v_1 \varepsilon_i).
\]

**Corollary.** In particular, \( Y(e_{-j}, -e_j a \varepsilon, 0) = Y_{ij}(a) \) for \( i \notin \{ \pm j \} \) and \( Y(e_i, 0, \alpha) = Y_{i,-i}(\alpha) \).

**Lemma 9.** For \( v, w \in I^{2l} \) such that \( v_- = w_- = 0 \) and \( a, b \in R \) such that \( a - \langle v_-, v_+ \rangle, b - \langle w_-, w_+ \rangle \in \Gamma \), one has

\[
Y(e_i, v, a) Y(e_i, w, b) = Y(e_i, v + w, a + b + \langle v, w \rangle).
\]

**Proof.** Obviously, \( (v + w)_- = 0 \). Moreover,

\[
a + b + \langle v, w \rangle - ((v + w)_-, (v + w)_+) = a - \langle v_-, v_+ \rangle + b - \langle w_-, w_+ \rangle \in \Gamma,
\]

so that the right hand side of this equality is well-defined. Now, it remains to observe that the images of the elements on both sides under \( \varphi \) coincide. \[\square\]

**Corollary.** One has \( Y(e_i, 0, 0) = 1 \) and \( Y(e_i, v, a)^{-1} = Y(e_i, -v, -a) \).

**Lemma 10.** For \( f \in E_{2l}(R), v \in I^{2l} \) one has

\[
\langle (fv)_-, (fv)_+ \rangle - \langle v_-, v_+ \rangle \in \Gamma.
\]
Proof. We may assume that \( f \) is an elementary transvection. For short root transvection one has

\[
\langle (T_{ij}(r)v)_-, (T_{ij}(r)v)_+ \rangle - \langle v_-, v_+ \rangle =
- (v_i + v_j r)v_i - v_j (v_{-j} - v_{-i} r \varepsilon_i \varepsilon_j) + v_i v_{-i} + v_j v_{-j} =
- v_i v_j r (\varepsilon_i \varepsilon_j - 1) \in 2I \subseteq \Gamma.
\]

For long root transvection one has

\[
\langle (T_{i,-i}(r)v)_-, (T_{i,-i}(r)v)_+ \rangle - \langle v_-, v_+ \rangle =
- (v_i + v_{-i} r \varepsilon_i)v_i - v_i v_{-i} = - \varepsilon_i r v_i^2 \in \Gamma.
\]

\[\square\]

Lemma 11. For \( g \in (i)P_1, v \in I^{2l} \) such that \( v_{-i} = 0, a \in R \) such that \( a - \langle v_-, v_+ \rangle \in \Gamma \), one has

\[gY(e_i, v, a) = Y(e_i, \phi(g)v, a).\]

Proof. First, observe that since \( T_{kh}(a)e_i = e_i \) for \( i \notin \{h, -k\} \) one has \( \phi(g)e_i = e_i \). Thus,

\[\langle e_i, \phi(g)v \rangle = \langle \phi(g)e_i, \phi(g)v \rangle = \langle e_i, v \rangle = 0,
\]

i.e., \( (\phi(g)v)_- = 0 \). By Lemma 10, \( a - \langle (\phi(g)v)_-, (\phi(g)v)_+ \rangle \in \Gamma \), so the right hand side of the desired equation is well-defined. Finally, observe that the images of both sides under \( \varphi \) coincide. \( \square \)

Lemma 12. For \( j \neq -i, a \in I \), one has \( Y(e_i, e_j a, 0) = Y(e_j, e_i a, 0) \).

Proof. For \( i = j \) the claim is obvious. Let \( i \neq j \), then

\[Y(e_i, e_j a, 0) = Y_{-j,i}(a \varepsilon_i) = Y_{-i,j}(-a \varepsilon_j) = Y(e_j, e_i a, 0).
\]

\[\square\]

Remark. In the absolute situation \((I, \Gamma) = (R, R)\), we will write \( X(e_i, v, a) \) instead of \( Y(e_i, v, a) \). These are exactly the elements, which appear in the another presentation.

Lemma 13. For \( v \in V \) such that \( v_{-j} = v_k = v_{-k} = 0, k \notin \{\pm j\}, a \in R \), \( b \in I \) one has

\[\left[ Y(e_k, e_j b, 0), X(e_{-k}, v, a) \right] = Y(e_j, vb \varepsilon_k, ab^2)Y(e_{-k}, -e_j ab \varepsilon_{-k}, 0).
\]
Similarly, we have 

\[ Y(h, v) = Y(e, e, b, 0, X(e, v, a))Y(e, e, a, 0). \]

**Corollary.** For \( v \in V \) such that \( v_j = v_k = v_k = 0, k \not\in \{\pm j\}, a \in R, \]
\( b \in I \) such that any \( Y \)

\[ \begin{align*}
\text{For} \quad Y \in I^2 & \quad \text{ such that } v_j = v_k = 0, r \in R, a \quad \text{such that } a - \langle v_-, v_+ \rangle \in \Gamma, \ 
onumber
\text{one has the following decomposition} \\
Y(e, b, a) = Y(e, e, b, 0, X(e, v, a))Y(e, a, b, 0). 
\end{align*} \]

**Proof.** Denote \( x = X_{j-k}(e, r) = X(e, e, r, 0), w = Y_{-k}(a - \langle v_-, v_+ \rangle), \)
\( y = Y_{-l}(v, e) \quad \text{such that } \)
\( \text{one has} \\
\begin{align*}
[X(e, e, r, 0), Y(e, v, a)] = Y(e, v, a)Y(e, v, a).
\end{align*} \]

Assume that \( j < 0 \), for \( j > 0 \) the proof looks exactly the same. For \( h \not\in \{\pm j, \pm k\} \) one has

\[ [X_{j-k}(e), Y_{h,k}(v)] = [X_{j-k}(e), Y_{-k}(v)] = Y_{j-k}(v). \]

For \( h = j \) one has

\[ [X_{j-k}(e), Y_{j,k}(v)] = [X_{j-k}(e), Y_{j-k}(v)] = Y_{j,k}(2v). \]

Since any \( Y_{h,j}(\hat{a}) \) commutes with any \( Y_{h,j}(\hat{b}) \) for \( h, t \not\in \{\pm k, -j\} \), \( h, t < 0 \), we have

\[ [x, y] = \prod_{h < 0} [X_{j-k}(e), Y_{h,k}(v)] = \]
\[ = Y_{j-k}(2v) \prod_{h < 0} Y_{h,j}(v) = Y(e, v, 0). \]

Similarly,

\[ [x, z] = \prod_{h > 0} Y_{h,j}(v) = Y(e, v, 0). \]
Then,

\[ y[[x, z]] = Y(e_{-k}, v, 0)Y(e_j, v + r\varepsilon_k, 0) = \]
\[ = Y(e_j, v + r\varepsilon_k, 0)[Y(e_j, -v + r\varepsilon_k, 0), Y(e_{-k}, v, 0)] = \]
\[ = Y(e_j, v + r\varepsilon_k, 0)[X(e_j, -v + r\varepsilon_k, 0), Y(e_{-k}, v, 0)] = \]
\[ = Y(e_j, v + r\varepsilon_k, 0)Y(e_{-k}, e_j\langle v_-, v_+\rangle r\varepsilon_k, 0). \]

Next,

\[ [x, w] = [X_{j,-k}(r\varepsilon_k), Y_{-k,k}(a - \langle v_-, v_+\rangle)] = \]
\[ = \left( [Y_{-k,k}(a - \langle v_-, v_+\rangle), X_{k,-j}(r\varepsilon_j)] \right)^{-1} = \]
\[ = \left( Y_{-k,-j}(a - \langle v_-, v_+\rangle)Y_{j,-j}(a - \langle v_-, v_+\rangle)^{-1} = \right) \]
\[ = Y_{j,-j}(ar^2 - \langle v_- r\varepsilon_k, v_+ r\varepsilon_k \rangle)Y(e_j, e_{-k}(a - \langle v_-, v_+\rangle) r\varepsilon_k, 0). \]

Obviously, \([x, w]\) commutes with \(y\) and \(z\), thus, finally,

\[ [x, yzw] = Y_{j,-j}(ar^2 - \langle v_- r\varepsilon_k, v_+ r\varepsilon_k \rangle)Y_{j,-j}(2rv_j\varepsilon_k) \]
\[ \cdot \prod_{h \neq j \atop h < 0} Y_{h,-j}(rv_h\varepsilon_k\varepsilon_j) \prod_{h > 0} Y_{h,-j}(rv_h\varepsilon_k\varepsilon_j) Y(e_{-k}, e_j\langle v_-, v_+\rangle r\varepsilon_k, 0) \]
\[ \cdot Y(e_{-k}, e_j(a - \langle v_-, v_+\rangle) r\varepsilon_k, 0) = Y(e_j, v r\varepsilon_k, a r^2) Y(e_{-k}, e_j a r\varepsilon_k, 0). \]

\[ \square \]

**Corollary.** For \( j \notin \{ \pm k \}, v \in I^{2l} \) such that \( v_- = v_k = v_{-k} = 0, r \in R, a \) such that \( a - \langle v_-, v_+\rangle \in \Gamma \), one has

\[ Y(e_j, v r, a r^2) = [X(e_k, e_j r, 0), Y(e_{-k}, v r\varepsilon_k, a)] Y(e_{-k}, e_j a r\varepsilon_{-k}, 0). \]

**Definition.** For \( u \in V, v \in I^{2l} \) such that \( \langle u, v \rangle = 0, u_i = u_{-i} = v_i = v_{-i} = 0, a \in R \) such that \( a - \langle v_-, v_+\rangle \in \Gamma \) denote

\[ Y_{(i)}(u, v, a) = [X(e_i, u, 0), Y(e_{-i}, u a \varepsilon_{-i}, a)] Y(e_{-i}, u a \varepsilon_{-i}, 0). \]

**Remark.** Due to Lemma 2 one has \( \varphi(Y_{(i)}(u, v, a)) = T(u, v, a). \)

**Remark.** For \( v \in I^{2l} \) such that \( v_- = v_i = v_{-i} = 0, a \in R \) such that \( a - \langle v_-, v_+\rangle \in \Gamma \), one has

\[ Y_{(i)}(e_j, v, a) = Y(e_j, v, a) \]

by Lemma 14. One can also obtain the following result.
Lemma 15. For \( v \in V \) with \( v_j = v_i = v_{-i} = 0 \), \( b \in I \) one has
\[
Y_{(i)}(v, e_j b, 0) = Y(e_j, vb, 0).
\]

Proof. The proof of Lemma 15 from the Another presentation paper can be repeated verbatim. \( \square \)

Lemma 16. Consider \( u \in V \), \( v \in I^2 \) such that \( \langle u, v \rangle = 0 \), \( u_i = u_{-i} = u_j = u_{-j} = 0 \), \( v_i = v_{-i} = v_j = v_{-j} = 0 \), \( r \in R \) and \( a \in R \) such that \( a - \langle v_-, v_+ \rangle \in \Gamma \). Then one has
\[
Y_{(i)}(u, vr, ar^2) = Y_{(j)}(ur, v, a).
\]

Proof. Denote \( x = X(e_i, u, 0) \), \( q = Y(e_{-i}, vr \varepsilon_i, ar^2) \), \( c = Y(e_{-i}, uar \varepsilon_{-i}, 0) \), then by the very definition
\[
Y_{(i)}(u, vr, ar^2) = [x, q]c.
\]
Denote \( y = X(e_j, e_{-i}r \varepsilon_{-i}, 0) \), \( z = Y(e_{-j}, v \varepsilon_j, a) \), \( w = Y(e_{-j}, e_{-i}ar \varepsilon \varepsilon_{-j}, 0) \), then by Lemma 14
\[
q = [y^{-1}, z]w
\]
and
\[
[x, q] = [x, [y^{-1}, z]w] = [x, [y^{-1}, z]] . [y^{-1}, z][x, w].
\]
Using Hall–Witt identity,
\[
[x, [y^{-1}, z]] = y^{-1}x[[x^{-1}, y], z] . y^{-1}z[[z^{-1}, x], y].
\]
One can check using Lemma 11 that \( x \) acts trivially on \( z \), i.e., \( [z^{-1}, x] = 1 \), thus,
\[
[x, [y^{-1}, z]] = y^{-1}x[[x^{-1}, y], z].
\]
Next,
\[
[x^{-1}, y] = [X(e_i, -u, 0), X(e_j, -e_{-i}r \varepsilon_i, 0)] = X(-u, -e_j r, 0) = X(e_j, ur, 0).
\]
Denote \( d = [x^{-1}, y] = X(e_j, ur, 0) \), one can show that \( [x, d] = 1 \), thus,
\[
[x, [y^{-1}, z]] = y^{-1}x[d, z].
\]
Observe that \( y \) acts trivially on \( q \), thus, \( y[x, q] = [yx, q] \). Since \( yx = xy[y^{-1}, x^{-1}] \) and
\[
[y^{-1}, x^{-1}] = [X(e_{-i}, e_j r \varepsilon_i, 0), X(e_i, -u, 0)] = X(e_j, ur, 0)
\]
also acts trivially on $q$, one has $y[x, q] = [x, q]$, thus,

$$[x, q] = y[x, q] = y[x, [y^{-1}, z]] \cdot y[y^{-1}, z][x, w] = [d, z] \cdot y[y^{-1}, z][x, w].$$

Denote

$$h = [x, w] = [X(e_i, u, 0), Y(e_{-i}, e_{-j} a r \varepsilon_i \varepsilon_j, 0)] = Y(e_{-j}, u a r \varepsilon_{-j}, 0),$$

then $[w, h] = 1$, thus, $[y^{-1}, z] h = [y^{-1}, z] w h = g h = h$, so,

$$[x, q] = [d, z] \cdot y h = [d, z] \cdot h^{-1} \cdot y.$$  

Since

$$[y, h^{-1}] = [X(e_j, -e_{-i} r \varepsilon_i, 0), Y(e_{-j}, u a r \varepsilon_j, 0)] = Y(e_{-i}, u a r \varepsilon_{-i}, 0) = c,$$

one has

$$[x, q] c = [d, z] h$$

or,

$$Y_{(j)}(u, v r, a r^2) = Y_{(j)}(u r, v, a).$$

\[\square\]

**Definition.** Define the (absolute) Levi subgroup $(i)L_1 = (i)P_1 \cap (-i)P_1$.

**Remark.** Observe that for $g \in (i)L_1$ one has $\phi(g) e_i = e_i$ and $\phi(g) e_{-i} = e_{-i}$. Indeed, the first equality holds for $g = X_{k h}(a)$ with $\{k, h\} \neq i$ and the second one for $g = X_{k h}(a)$ with $\{k, h\} \neq -i$.

**Lemma 17.** For $u \in V$, $v \in I^2l$ such that $\langle u, v \rangle = 0$, $u_i = u_{-i} = v_i = v_{-i} = 0$, $a \in R$ such that $a - \langle v_{-i}, v_i \rangle \in \Gamma$, $g \in (i)L_1$, one has

$$g Y_{(i)}(u, v, a) g^{-1} = Y_{(i)}(\phi(g) u, \phi(g) v, a).$$

**Remark.** Since $\langle \phi(g) u, e_i \rangle = \langle \phi(g) u, \phi(g) e_i \rangle = \langle u, e_i \rangle = 0$, one can see that $(\phi(g) u)_{-i} = 0$ and similarly $(\phi(g) u)_{-i} = (\phi(g) v)_{-i} = (\phi(g) v)_{i} = 0$. Using Lemma 10 one obtains, that $a - \langle (\phi(g) u)_{-i}, (\phi(g) v)_{i} \rangle \in \Gamma$. Thus, $Y_{(i)}(\phi(g) u, \phi(g) v, a)$ is well-defined.

**Proof.** The proof of Lemma 17 from the Another presentation paper works for this situation as well. \[\square\]

**Remark.** Lemma 7 implies that for $v$ such that $v_{-i} = v_j = v_{-j} = 0$, $j \notin \{ \pm i \}$ one has $Y(e_i, v, a) \in (i)L_1$. 

12
Remark. For \( w \) orthogonal to both \( u \) and \( v \), \( \langle u, w \rangle = \langle v, w \rangle = 0 \), one has \( T(u, v, a)w = w \). Below, in the computations this fact is frequently used without any special reference.

**Lemma 18.** For \( j \not\in \{\pm 1\} \) and \( u \in V \) such that \( u_i = u_{-i} = u_j = u_{-j} = 0 \), \( v \in I^2 \) such that \( v_i = v_{-i} = 0 \) and \( \langle u, v \rangle = 0 \), and for \( a, b \in I \) such that \( a - \langle v_-, v_+ \rangle \in \Gamma \), one has

\[
Y_{(i)}(u, v, a)Y_{(i)}(u, e_j b, 0) = Y_{(i)}(u, v + e_j b, a + v_{-j} b \varepsilon_{-j}).
\]

**Proof.** Start with the right-hand side (one can check that it is well-defined)

\[
Y_{(i)}(u, v + e_j b, a + v_{-j} b \varepsilon_{j}) = [X(e_i, u, 0), Y(e_{-i}, (v+e_j b) \varepsilon_i, a+v_{-j} b \varepsilon_j)]Y(e_{-i}, u(a+v_{-j} b \varepsilon_{-j}) \varepsilon_{-i}, 0).
\]

Decompose \( Y(e_{-i}, (v+e_j b) \varepsilon_i, a + v_{-j} b \varepsilon_j) \) inside the commutator and use the familiar identity \([a, bc] = [a, b] \cdot b^i [a, c]\) to obtain

\[
[X(e_i, u, 0), Y(e_{-i}, (v+e_j b) \varepsilon_i, a + v_{-j} b \varepsilon_j)] = [X(e_i, u, 0), Y(e_{-i}, v \varepsilon_i, a)] \cdot Y(e_{-i}, v \varepsilon_i, a) Y_{(i)}(u, e_j b, 0) = [X(e_i, u, 0), Y(e_{-i}, v \varepsilon_i, a)] \cdot Y(e_{-i}, v \varepsilon_i, a) Y_{(i)}(u, v + e_j b, a + v_{-j} b \varepsilon_{-j}).
\]

Observe that in general \( X(e_{-i}, v \varepsilon_i, a) \) does not lie in \( (j)P_1 \), but \( X(e_j, u b, 0) \) always lies in \( (j)P_1 \). So that we can compute the conjugate as follows

\[
Y(e_{-i}, v \varepsilon_i, a) Y_{(j)}(e_j, u b, 0) = Y(e_j, u b, 0) Y(e_{-i}, v \varepsilon_i, a) Y(e_{-i}, v \varepsilon_i, a) = Y(e_j, u b, 0) Y(e_{-i}, v \varepsilon_i, a) Y(e_{-i}, v \varepsilon_i, a) Y(e_{-i}, v \varepsilon_i, a) = Y(e_j, u b, 0) Y(e_{-i}, v \varepsilon_i, a).
\]

Thus,

\[
Y_{(i)}(u, v + e_j b, a + v_{-j} b \varepsilon_j) = [X(e_i, u, 0), Y(e_{-i}, v \varepsilon_i, a)] Y(e_j, u b, 0) \cdot Y(e_{-i}, v \varepsilon_i, a) Y(e_{-i}, v \varepsilon_i, a) Y(e_{-i}, v \varepsilon_i, a) Y(e_{-i}, v \varepsilon_i, a) Y(e_{-i}, v \varepsilon_i, a).
\]
Finally, it remains to observe that $X(e_j, ub, 0) \in (-i)P_1$ and thus one has $[Y(e_j, ub, 0), Y(e_{-i}, uae_{-i}, 0)] = 1$ and

$$Y_{(i)}(u, v+e_jb, a+v_jb\varepsilon_j) = [X(e_i, u, 0), Y(e_{-i}, v\varepsilon_i, a)]Y(e_{-i}, uae_{-i}, 0) \cdot Y(e_j, ub, 0) = Y_{(i)}(u, v, a)Y_{(i)}(u, e_jb, 0).$$

\[ \square \]

**Definition.** For $u \in V$ such that $u_i = u_{-i} = 0$, $v \in I^{2l}$ such that $\langle u, v \rangle = 0$, and $a \in R$ such that $a - \langle v_{-i}, v_{+i} \rangle \in \Gamma$ define

$$Y_{(i)}(u, v, a) = Y_{(i)}(u, v - e_i v_i - e_{-i}v_{-i}, a - v_i v_{-i} \varepsilon_i) Y(e_i, uv_i, 0) Y(e_{-i}, uv_{-i}, 0).$$

**Remark.** Observe that the above definition coincides with the old one for $v$ with $v_i = v_{-i} = 0$, so that we can use the same notation for the generator. Observe also that the right-hand side is well-defined.

**Lemma 19.** For $g \in (i)L_1$, $u \in V$ such that $u_i = u_{-i} = 0$, $v \in I^{2l}$ such that $\langle u, v \rangle = 0$, and $a \in R$ such that $a - \langle v_{-i}, v_{+i} \rangle \in \Gamma$ one has

$$g Y_{(i)}(u, v, a) g^{-1} = Y_{(i)}(\phi(g)u, \phi(g)v, a).$$

**Proof.** Since $g \in (i)L_1$ one gets

$$(\phi(g)v)_i = \langle \phi(g)v, e_{-i}\varepsilon_i \rangle = \langle \phi(g)v, \phi(g)e_{-i}\varepsilon_i \rangle = \langle v, e_{-i}\varepsilon_i \rangle = v_i$$

and similarly $(\phi(g)v)_{-i} = v_{-i}$. Then

$$g Y_{(i)}(u, v, a) = g Y_{(i)}(u, v - e_i v_i - e_{-i}v_{-i}, a - v_i v_{-i} \varepsilon_i) \cdot g Y(e_i, uv_i, 0) \cdot g Y(e_{-i}, uv_{-i}, 0) =$$

$$= Y_{(i)}(\phi(g)u, \phi(g)v - \phi(g)e_i v_i - \phi(g)e_{-i}v_{-i}, a - v_i v_{-i} \varepsilon_i) \cdot Y(e_i, \phi(g)uv_i, 0) Y(e_{-i}, \phi(g)uv_{-i}, 0) =$$

$$= Y_{(i)}(\phi(g)u, \phi(g)v - e_i (\phi(g)v)_i - e_{-i}(\phi(g)v)_{-i}, a - (\phi(g)v)_i (\phi(g)v)_{-i} \varepsilon_i) \cdot Y(e_i, \phi(g)u(\phi(g)v)_i, 0) Y(e_{-i}, \phi(g)u(\phi(g)v)_{-i}, 0) =$$

$$= Y_{(i)}(\phi(g)u, \phi(g)v, a).$$

\[ \square \]
Remark. Obviously, in the absolute case \((I, \Gamma) = (R, R)\) for \(u, v \in V\) such that \(u_i = u_{-i} = u_j = u_{-j} = 0\) and \(v_j = v_{-j} = 0, j \not\in \{\pm i\}, a \in R\), one has that such an element \(X_{(j)}(u, v, a)\) lies in \(\hat{(j)}L_1\). Indeed, it is a product of elements from \(\hat{(j)}L_1\) by definition.

Lemma 20. For \(j \not\in \{\pm i\}, u \in V\) such that \(u_i = u_{-i} = u_j = u_{-j} = 0, v \in I^j\) such that \([u, v] = 0\), and \(a \in R\) such that \(a - [v_-, v_+] \in \Gamma\) one has

\[
Y_{(j)}(u, v, a) = Y_{(j)}(u, v, a).
\]

Proof. Denote \(\tilde{v} = v - e_i v_i - e_{-i} v_{-i}, \tilde{a} = a - v_i v_{-i} \varepsilon_i\). Then by Lemma 18 one gets

\[
Y_{(i)}(u, \tilde{v}, \tilde{a}) = Y_{(i)}(u, \tilde{v} - e_{-j} v_{-j}, \tilde{a} - v_j v_{-j} \varepsilon_j) Y_{(i)}(u, e_{-j} v_{-j}, 0) =
\]

\[
= Y_{(i)}(u, \tilde{v} - e_{-j} v_{-j} - e_j v_j, \tilde{a} - v_j v_{-j} \varepsilon_j) Y_{(i)}(u, e_j v_j, 0) Y_{(i)}(u, e_{-j} v_{-j}, 0). 
\]

Further, denote \(\tilde{v} = e_j v_j - e_{-j} v_{-j}\) and \(\tilde{a} = a - v_j v_{-j} \varepsilon_j\). Then one has

\[
Y_{(i)}(u, v, a) = Y_{(i)}(u, \tilde{v}, \tilde{a}) Y(e_j, uv_j, 0) Y(e_{-j}, uv_{-j}, 0) 
\]

\[
\cdot Y(e_i, uv_i, 0) Y(e_{-i}, uv_{-i}, 0).
\]

Changing roles of \(i\) and \(j\) one gets

\[
Y_{(j)}(u, v, a) = Y_{(j)}(u, \tilde{v}, \tilde{a}) Y(e_i, uv_i, 0) Y(e_{-i}, uv_{-i}, 0). 
\]

\[
\cdot Y(e_j, uv_j, 0) Y(e_{-j}, uv_{-j}, 0).
\]

But \(Y_{(i)}(u, \tilde{v}, \tilde{a}) = Y_{(j)}(u, \tilde{v}, \tilde{a})\) by Lemma 16. Finally, it remains to observe that \(Y(e_{-i}, uv_{-i}, 0)\) and \(Y(e_i, uv_i, 0)\) commute with both \(Y(e_j, uv_j, 0)\) and \(Y(e_{-j}, uv_{-j}, 0)\). This is obvious from the fact that \(X(e_{-i}, uv_{-i}, 0)\) and \(X(e_i, uv_i, 0)\) lie in \(\hat{(j)}L_1\). \(\square\)

Remark. For \(u\) equal to the base vector \(e_j\) using Lemma 13 one gets

\[
Y_{(i)}(e_j, v, a) =
\]

\[
= Y_{(i)}(e_j, v - e_i v_i - e_{-i} v_{-i}, a - v_i v_{-i} \varepsilon_i) Y(e_i, e_j v_i, 0) Y(e_{-i}, e_j v_{-i}, 0) =
\]

\[
= Y(e_j, v - e_i v_i - e_{-i} v_{-i}, a - v_i v_{-i} \varepsilon_i) Y(e_j, e_i v_i, 0) Y(e_{-j}, e_{-i} v_{-i}, 0) =
\]

\[
= Y(e_j, v, a).
\]

Definition. For \(u\) having at least two pairs of zeros the element \(Y_{(i)}(u, v, a)\) does not depend on the choice of \(i\) by Lemma 20. In this situation we will often omit the index in the notation,

\[
Y(u, v, a) = Y_{(j)}(u, v, a).
\]

15
Definition. For $u \in V$, $w \in I^{2n}$, such that $\langle u, w \rangle = 0$, $w_i = w_{-i} = 0$ and $a \in \langle w_-, w_+ \rangle + \Gamma$ define

$$Z_{(i)}(u, w, a) = Y_{(i)}(u - e_i u_i - e_{-i} u_{-i}, w, a) Y(e_i u_i + e_{-i} u_{-i}, w, a).$$

Our next objective is to show that $Z_{(i)}(u, 0, a)$ does not depend on the choice of $i$.

Lemma 21. For $u \in V$ such that $u_i = u_{-i} = u_j = u_{-j} = 0$, $v \in I^2$ such that $v_i = v_{-i} = 0$, $\langle u, v \rangle = 0$, $a \in R$ such that $a - \langle v_-, v_+ \rangle \in \Gamma$, $b \in \Gamma$ one has

$$Y(u, v, a + b) = Y(u, v, a)Y(u, 0, b).$$

Proof. Decompose $Y(u, v, a + b)$ as follows

$$Y(u, v, a + b) = Y_{(i)}(u, v, a + b) = [X(e_i, u, 0), Y(e_{-i}, v_{\varepsilon_i}, a + b)] Y(e_{-i}, u(a + b) \varepsilon_{-i}, 0).$$

Using $[a, bc] = [a, b] \cdot b[a, c]$ we obtain

$$[X(e_i, u, 0), Y(e_{-i}, v_{\varepsilon_i}, a + b)] = [X(e_i, u, 0), Y(e_{-i}, v_{\varepsilon_i}, a)] Y(e_{-i}, 0, b) = [X(e_i, u, 0), Y(e_{-i}, v_{\varepsilon_i}, a)] \cdot Y(e_{-i}, v_{\varepsilon_i}, a) [X(e_i, u, 0), Y(e_{-i}, 0, b)].$$

Since $\langle u, v \rangle = 0$ one has

$$X(e_{-i}, v_{\varepsilon_i}, a) Y(e_{-i}, ub_{\varepsilon_{-i}}, 0) = Y(e_{-i}, ub_{\varepsilon_{-i}}, 0),$$

and thus

$$Y(u, v, a + b) = [X(e_i, u, 0), Y(e_{-i}, v_{\varepsilon_i}, a + b)] Y(e_{-i}, ub_{\varepsilon_{-i}}, 0) Y(e_{-i}, 0, b) Y(e_{-i}, 0, b) = [X(e_i, u, 0), Y(e_{-i}, v_{\varepsilon_i}, a)] \cdot Y(e_{-i}, v_{\varepsilon_i}, a) [X(e_i, u, 0), Y(e_{-i}, 0, b)].$$

Recall that $X_{ij}(u, 0, b) \in (i) L_1$ acts trivially on both $Y(e_{-i}, v_{\varepsilon_i}, a)$ and $Y(e_{-i}, ub_{\varepsilon_{-i}}, 0)$, so that

$$Y(u, v, a + b) = [Y(e_i, u, 0), Y(e_{-i}, v_{\varepsilon_i}, a)] Y(e_{-i}, 0, 0) Y(u, 0, b) = Y_{(i)}(u, v, a) Y(u, 0, b).$$

$\square$
**Remark.** For $u \in V$ having at least two pairs of zeros one has $Y(u, 0, 0) = 1$ and $Y(u, 0, a)^{-1} = Y(u, 0, -a)$.

**Lemma 22.** For $u \in V$ such that $u_i = u_{-i} = u_j = u_{-j} = 0$, $v \in I^{2l}$ such that $\langle u, v \rangle = 0$ and $v_i v_{-i} \in \Gamma$, $a \in R$ such that $a - \langle v_-, v_+ \rangle \in \Gamma$ one has

$$Y(u, v, a) = Y(u, v - e_i v_i - e_{-i} v_{-i}, a)Y(u, e_i v_i + e_{-i} v_{-i}, 0).$$

**Proof.** By definition

$$Y(u, v, a) = Y_{(i)}(u, v, a) =$$

$$= Y_{(i)}(u, v - e_i v_i - e_{-i} v_{-i}, a - v_i v_{-i} \varepsilon_i)Y(e_i, u v_i, 0)Y(e_{-i}, u v_{-i}, 0).$$

Denote $\tilde{v} = v - e_i v_i - e_{-i} v_{-i}$, then by the previous lemma

$$Y(u, \tilde{v}, a - v_i v_{-i} \varepsilon_i) = Y(u, \tilde{v}, a)Y(u, 0, -v_i v_{-i} \varepsilon_i),$$

and thus

$$Y(u, v, a) = Y(u, \tilde{v}, a)Y(u, 0, -v_i v_{-i} \varepsilon_i)Y(e_i, u v_i, 0)Y(e_{-i}, u v_{-i}, 0) =$$

$$= Y(u, \tilde{v}, a)Y_{(i)}(u, e_i v_i + e_{-i} v_{-i}, 0).$$

**Corollary.** Consider $u \in V$, $w \in I^{2n}$, such that $\langle u, w \rangle = 0$, $w_i = w_{-i} = 0$, $a \in \langle w_-, w_+ \rangle + \Gamma$, $j \notin \{ \pm i \}$ and denote $v = e_i u_i + e_{-i} u_{-i}$, $v' = e_j u_j + e_{-j} u_{-j}$, $	ilde{u} = u - v$, $\tilde{v} = \tilde{u} - v'$. Then one has

$$Z_{(i)}(u, w, a) = Y_{(i)}(\tilde{u}, w, a)Y(v, w, a)Y(v, \tilde{u} a, 0) =$$

$$= Y_{(i)}(\tilde{u}, w, a)Y(v, w, a)Y(v, \tilde{u} a, 0)Y(v, v' a, 0).$$

**Proof.** Since $l \geq 3$ the vector $v$ has at least two pairs of zeros, so that one can apply the previous lemma. \hfill $\Box$

**Lemma 23.** For $j, k \notin \{ \pm i \}$, $u, v \in V$ such that $u_i = u_{-i} = u_j = u_{-j} = 0$ and $v_i = v_{-i} = v_k = v_{-k} = 0$, $\langle u, v \rangle = 0$, $w \in I^{2l}$ such that $w_i = w_{-i} = 0$, $\langle u, w \rangle = \langle v, w \rangle = 0$ and $a \in \langle w_-, w_+ \rangle + \Gamma$ holds

$$Y_{(i)}(u + v, w, a) = Y(u, w, a) \cdot Y(v, w, a) \cdot Y(v, u a, 0).$$

17
Proof. Decomposing $X(e_i, u+v, 0) = X(e_i, v, 0)X(e_i, u, 0)$ inside the commutator and using $[ab, c] = a[b, c] - [a, c]$ we get

\[
Y(e_i, u+v, w, a) = [X(e_i, u+v, 0), Y(e_{-i}, w, a)]Y(e_{-i}, u+v, a)e_{-i}, 0) = [X(e_i, v, 0)X(e_i, u, 0), Y(e_{-i}, va_{-i}, 0)Y(e_{-i}, va_{-i}, 0) = X(e_i, u, 0), Y(e_{-i}, w, a)] \cdot [X(e_i, v, 0), Y(e_{-i}, w, a)].
\]

\[
Y(e_{-i}, va_{-i}, 0)Y(e_{-i}, ma_{-i}, 0) = X(e_i, u, 0), Y(e_{-i}, w, a)].
\]

\[
Y_{(i)}(u, w, a) = Y_{(i)}(v, w, a) = Y_{(i)}(v, w, a).
\]

Observe that $Y_{(k)}(v, w, a) \in \mathbb{L}_1$ commutes with $Y(e_{-i}, -ua_{-i}, 0)$, moreover, $X(e_i, v, 0) \in \mathbb{L}_1$ and acts trivially on $Y_{(k)}(v, w, a)$, thus $Y_{(k)}(v, w, a)$ commutes with $X(e_i, v, 0)Y(e_{-i}, -ua_{-i}, 0)$. Thus, we get

\[
X(e_i, v, 0)[X(e_i, u, 0), Y(e_{-i}, w, a)] \cdot Y(v, w, a) \cdot Y(e_{-i}, ua_{-i}, 0) = X(e_i, u, 0), Y(e_{-i}, w, a)] \cdot X(e_i, v, 0)Y(e_{-i}, ua_{-i}, 0).
\]

Finally, notice that $X_{(k)}(v_{-i}, u, 0) \in \mathbb{L}_1$ commutes with $Y(e_i, w_{-i}, a)$.
3 Case of maximal form parameter

In this section we assume that $\Gamma = I$.

**Lemma 24.** For $k, j \not\in \{\pm i\}, u, v \in V$ such that $u_i = u_{-i} = u_j = u_{-j} = 0$ and $v_i = v_{-i} = v_k = v_{-k} = 0$, $\langle u, v \rangle = 0$, and $a \in I$, one has

$$Y(u, va, 0) = Y(v, ua, 0).$$

**Proof.** By Lemma 23 one has

$$Y(v, ua, 0) = Y(v, 0, -a)Y(u, 0, -a)Y_{(i)}(u + v, 0, a)$$

and similarly

$$Y(u, va, 0) = Y(u, 0, -a)Y(v, 0, -a)Y_{(i)}(v + u, 0, a).$$

But $X_{(j)}(u, 0, -a) \in \langle j \rangle L_1$ acts trivially on $Y_{(i)}(v, 0, -a)$. \hfill $\square$

**Lemma 25.** For $u \in V, a \in I$ and $j \not\in \{\pm i\}$

$$Z_{(i)}(u, 0, a) = Z_{(j)}(u, 0, a).$$

**Proof.** Set

$$v = e_iz_i + e_{-i}z_{-i}, \quad v' = e_ju_j + e_{-j}u_{-j}, \quad \tilde{u} = u - v, \quad \tilde{u}' = \tilde{u} - v'.$$

Lemmas 22 and 23 imply that

$$Z_{(i)}(u, 0, a) = Y_{(i)}(\tilde{u}, 0, a)Y(v, 0, a)Y(v, \tilde{u}a, 0) =$$

$$= Y_{(i)}(\tilde{u}, 0, a)Y(v, 0, a)Y(v, \tilde{u}a, 0)Y(v, v'a, 0) =$$

$$= Y(\tilde{u}, 0, a)Y(v', 0, a)Y(v', \tilde{u}a, 0)Y(v, 0, a)Y(v, \tilde{u}a, 0)Y(v, v'a, 0).$$

Interchanging roles of $i$ and $j$ one has

$$Z_{(j)}(u, 0, a) =$$

$$= Y(\tilde{u}, 0, a)Y(v, 0, a)Y(v, \tilde{u}a, 0)Y(v', 0, a)Y(v', \tilde{u}a, 0)Y(v', va, 0).$$

By Lemma 24 one has $Y(v, v'a, 0) = Y(v', va, 0)$. Now we have to show that $Y(v, 0, a)Y(v, \tilde{u}a, 0)$ commutes with $Y(v', 0, a)Y(v', \tilde{u}a, 0)$, and with this $Y(v, \tilde{u}a, 0)$ commutes with $Y_{(j)}(v', \tilde{u}a, 0)$ by Lemma 19. Next, we will show that

$$[Y(v, \tilde{u}a, 0), Y(v', \tilde{u}a, 0)] = 1.$$
By Lemma 24 one has
\[ Y(v, \tilde{u}a, 0) = Y(\tilde{u}, va, 0) \]
and
\[ Y(v', \tilde{u}a, 0) = Y(\tilde{u}, v'a, 0). \]

Now, Lemma 22 implies that
\[ Y(v, \tilde{u}a, 0) Y(v', \tilde{u}a, 0) = Y(\tilde{u}, va, 0) \]
\[ Y(v', \tilde{u}a, 0) Y(v, \tilde{u}a, 0) = Y(v', \tilde{u}a, 0) Y(v, \tilde{u}a, 0). \]

Finally, it remains to observe that \( Y(j)(v, 0, a) \) commutes with both \( Y(k)(v', 0, a) \) and \( Y(k)(v', \tilde{u}a, 0) \) by Lemma 19.

Remark. Since \( Z(i)(u, 0, a) \) does not depend on the choice of \( i \) we will often omit the index in the notation
\[ Z(u, 0, a) = Z(i)(u, 0, a). \]

Our next objective is to prove the following formula describing the action of \( \text{StSp}_{2d}(R) \) on the long-root type generators, namely
\[ g Z(u, 0, a) = Z(\phi(g)u, 0, a). \]

Lemma 26. For any \( u \in V, \) any \( a \in \Gamma, \) \( b \in R \) and any index \( i, \) one has
\[ X_{i, -i}(b) Z(u, 0, a) = Z(T_{i, -i}(b)u, 0, a). \]

Proof. The proof of Lemma 26 from the Another Presentation paper can be repeated verbatim.

Lemma 27. For \( u \in V, \) \( v, w \in I^{2d} \) and \( j \notin \{\pm i\} \) such that
\[ u_i = u_{-i} = u_j = u_{-j} = 0, \quad v_j = v_{-j} = 0, \quad w_j = w_{-j} = 0, \]
and \( \langle u, v \rangle = 0, \) \( \langle u, w \rangle = 0, \) \( \langle v, w \rangle = 0, \) one has
\[ Y(u, v, 0) Y(u, w, 0) = Y(u, v + w, 0). \]

Proof. The proof of Lemma 27 from the Another Presentation paper can be repeated verbatim.
Lemma 28. Let $\text{Card}\{\pm i, \pm j, \pm k\} = 6$ and let $v \in I^{2l}$, $v' \in V$ be vectors having only $\pm i$-th and $\pm j$-th non-zero coordinates respectively; consider also $v'' \in V$ such that $(v'')_i = (v'')_{-i} = (v'')_j = (v'')_{-j} = 0$. Set $w = v' + v''$. Then
\[
Y(v'', v, 0)Y(v', v, 0) = Y_{(i)}(w, v, 0).
\]

Proof. The proof of Lemma 28 from the Another Presentation paper can be repeated verbatim.

Lemma 29. For any $j \notin \{\pm k\}$, any $u \in V$ and any $a \in I$, $b \in R$, one has
\[
X_{jk}(b)Z(u, 0, a) = Z(T_{jk}(b)u, 0, a).
\]

Proof. The proof of Lemma 29 from the Another presentation paper can be repeated almost verbatim. The only difference is that one should use another approach to show that $X_{jk}(b)$ acts trivially on $Y(v, 0, a)$ and $Y(v, \tilde{u}a, 0)$. Namely, one should observe first that $X_{jk}(b) \in (i)L_1$ acts trivially on $Y(e_j(v, 0, a)) = Y(e_j, v, 0)$. Afterwards, decompose
\[
Y(v, 0, a) = [X(e_{-j}, v, 0), Y(e_j, 0, a)]Y(e_j, va\varepsilon_j, 0),
\]
thus,
\[
X_{jk}(b)Y(v, 0, a) = [X(T_{jk}(b)e_{-j}, v, 0), Y(e_j, 0, a)]Y(e_j, va\varepsilon_j, 0).
\]
Since
\[
X(T_{jk}(b)e_{-j}, v, 0) = X(e_{-j} - e_{-k}a\varepsilon_k\varepsilon_j, v, 0) = X(e_{-j}, v, 0)X(-e_{-k}a\varepsilon_k\varepsilon_j, v, 0) = X(e_{-j}, v, 0)X(e_{-k}, -va\varepsilon_k\varepsilon_j, 0),
\]
we obtain that
\[
X_{jk}(b)Y(v, 0, a) = [X(e_{-j}, v, 0)X(e_{-k}, -va\varepsilon_k\varepsilon_j, 0), Y(e_j, 0, a)]Y(e_j, va\varepsilon_j, 0) = X(e_{-j}, v, 0)[X(e_{-k}, -va\varepsilon_k\varepsilon_j, 0), Y(e_j, 0, a)]Y(v, 0, a).
\]
Finally, it remains to observe that $X(e_{-j}, v, 0) \in (j)L_1$ acts trivially on $Y(e_j, 0, a)$. \qed
**Corollary.** Lemmas 26 and 29 imply that for any $g \in \text{StSp}(2l, R)$, one has

$$g \cdot Z(u, 0, a) = Z(\phi(g)u, 0, a).$$

**Lemma 30.** The set of elements $\{Z(u, 0, a) \mid u \in V, a \in I\}$ generates $\text{StSp}_{2l}(R, I)$ as a group.

**Proof.** Firstly, choosing some $i$ and $j$ such that $\text{Card}\{\pm i, \pm j\} = 4$ one has

$$Y(j)(e_i, 0, a) = Y(e_i, 0, a)Y(0, 0, a)Y(0, e_i a, 0) = Y(e_i, 0, a) = Z_{i, -i}(a).$$

Now, choosing $k \notin \{\pm i, \pm j\}$, taking $u = e_{-k}$, $v = -e_j \varepsilon_k$ and any $a \in I$, and using Lemma 23 we obtain

$$X_{jk}(a) = Y(u, va, 0) = Y(-e_j \varepsilon_k, 0, -a)Y(e_{-k}, 0, -a)Y(e_{-k} - e_j \varepsilon_k, 0, a) = Z_{i}(e_{-k} - e_j \varepsilon_k, 0, a).$$

**Corollary.** Clearly, $\text{Ker} \phi$ acts trivially on $\text{StSp}_{2l}(R, I)$.

**Lemma 31.** For $u \in V, a, b \in I$ one has

$$Z(u, 0, a)Z(u, 0, b) = Z(u, 0, a + b).$$

**Proof.** The proof of Lemma 31 from the Another Presentation paper can be repeated verbatim.

**Lemma 32.** For $j \notin \{\pm i\}$, $u \in V$ such that $u_i = u_{-i} = u_j = u_{-j} = 0$ and $a \in I, b \in R$, one has

$$Y(ub, 0, a) = Y(u, 0, ab^2).$$

**Proof.** The claim follows directly from Lemma 16.

**Lemma 33.** For $u \in V, a \in I, b \in R$, one has

$$Z(ub, 0, a) = Z(u, 0, ab^2).$$

**Proof.** The proof of Lemma 33 from the Another Presentation paper can be repeated verbatim.
Definition. For \( u, v \in V \) such that \( \langle u, v \rangle = 0 \), \( a \in I \) set
\[
Z(v, u, a, 0) = Z(v, 0, -a)Z(u, 0, -a)Z(u + v, 0, a)Z(u, 0, -a).
\]

Lemma 34. For \( g \in \text{StSp}(2l, \mathbb{R}) \) and \( u, v \in V \) such that \( \langle u, v \rangle = 0 \), \( a \in I \), \( b \in R \), one has
\[
a) \ Z(v, u, a, 0) = Z(u, v, a, 0); \quad b) \ Z^{g}(u, v, a, 0) = Z(\phi(g)u, \phi(g)v, a, 0); \quad c) \ Z(u, ub, a, 0) = Z(u, 0, 2ab).
\]

Proof. Since \( a) \) and \( b) \) are obvious, it remains only to check \( c) \). By the very definition we have
\[
Z(u, ub, a, 0) = Z(u, 0, -a)Z(ub, 0, -a)Z(ub + u, 0, a).
\]
Then, using Lemma 33 and then Lemma 31, we get
\[
Z(u, 0, -a)Z(ub, 0, -a)Z(ub + u, 0, a) = Z(u, 0, -a)Z(u, 0, -ab^2)Z(u, 0, a(b + 1)^2) = Z(u, 0, 2ab).
\]

Lemma 35. Consider \( u, w \in V \) such that
\[
\langle u, w \rangle = 0, \quad w_i = w_{-i} = w_j = w_{-j} = 0,
\]
where \( i \notin \{ \pm j \} \). Then
\[
Z(u + w, 0, a) = Z(u, 0, a)Z(w, 0, a)Y(w, u, a, 0).
\]

Proof. The proof of Lemma 35 from the Another Presentation paper can be repeated verbatim. \( \square \)

Lemma 36. For \( v \in V \) such that \( v_{-i} = 0 \), \( a \in I \), one has
\[
Z(e_i, v, a, 0) = Y(e_i, va, 0).
\]

Proof. In the statement of Lemma 35 take \( u = v, w = e_i \). \( \square \)

Corollary. Let \( u \) be a column of a symplectic elementary matrix, \( v, w \in V \) such that \( \langle u, v \rangle = \langle u, w \rangle = 0 \), \( a, b \in I \) such that \( va = wb \). Then one has
\[
Z(u, v, a, 0) = Z(u, w, b, 0).
\]
Proof. Take \( g \in \text{StSp}_{2l}(R) \) such that \( \phi(g)u = e_i \). Then one has

\[
Z(u, v, a, 0) = g Z(e_i, \phi(g)^{-1}v, a, 0) = g Y(e_i, \phi(g)^{-1}va, 0) = g Z(e_i, \phi(g)^{-1}w, b, 0) = Z(u, w, b, 0).
\]

\[\square\]

Lemma 37. Consider \( a \in I, r \in R \) and \( u, v \in V \) such that \( \langle u, v \rangle = 0 \) and assume also that \( v \) is a column of a symplectic elementary matrix. Then

\[
Z(u + vr, 0, a) = Z(u, 0, a)Z(v, 0, a^2)Z(v, u, ar, 0).
\]

Proof. Take \( g \in \text{StSp}_{2l}(R) \) such that \( \phi(g)v = e_i \). Then,

\[
g Z(u + vr, 0, a) = Z(\phi(g)u + e_ir, 0, a).
\]

Now, Lemma 35 (and Lemma 36) imply that

\[
Z(\phi(g)u + e_ir, 0, a) = Z(\phi(g)u, 0, a)Z(e_i r, 0, a)Y(e_i r, \phi(g)ua, 0) =
\]

\[
= g \left( Z(u, 0, a)Z(v, 0, a^2)Z(v, u, ar, 0) \right).
\]

\[\square\]

Lemma 38. Let \( u \in V \) be column of a symplectic elementary matrix and let \( v, w \in V \) be arbitrary columns such that \( \langle u, v \rangle = \langle u, w \rangle = 0 \), \( a, b \in I \). Then

a) \( Z(u, v, a, 0)Z(u, w, a, 0) = Z(u, v + w, a, 0)Z(u, 0, a^2 \langle v, w \rangle) \),

b) \( Z(u, v, a, 0)Z(u, v, b, 0) = Z(u, v, a + b, 0) \).

Proof. Use the same trick as in Lemma 37. \[\square\]

Definition. For \( u, v \in V \) such that \( \langle u, v \rangle = 0 \), \( a, b \in I \), set

\[
Z(u, v, a, b) = Z(u, v, a, 0)Z(u, 0, b).
\]

Lemma 39. Assume that \( u, u' \) are columns of symplectic elementary matrices, and let \( v, v', w \in V \) be arbitrary columns such that \( \langle u, v \rangle = \langle u, w \rangle = 0 \),
and $\langle u', v' \rangle = 0$. Then for any $a, a', b, b' \in I, r \in R$ one has

\begin{enumerate}
\item $Z(u, vr, a, b) = Z(u, v, ar, b),$  
\item $Z(u, v, a, b)Z(u, w, a, c) = Z(u, v + w, a, b + c + a^2\langle v, w \rangle),$  
\item $Z(u, v, a, 0)Z(u, v, b, 0) = Z(u, v, a + b, 0),$  
\item $Z(u, v, a, 0) = Z(v, u, a, 0),$  
\item $Z(u', v', a', b')Z(u, v, a, b)Z(u', v', a', b')^{-1} = Z(T(u', v' a', b')u, T(u', v' a', b')v, a, b),$  
\item $Z(u, v, a, 0) = Z(u, 0, 0, 2a),$  
\item $Z(v + ur, 0, 0, a) = Z(v, 0, 0, ar^2)Z(u, v, 0, 0)Z(u, v, 0, ar, 0).$
\end{enumerate}

**Proof.** For b) use Lemmas 38 and 31, the rest was already checked. \qed

**Definition.** Let the relative symplectic van der Kallen group $StSp_{2l}^\ast(R, I)$ be the group defined by the set of generators

$$\{(u, v, a, b) \in V \times V \times I \times I \mid u \text{ is a column of a symplectic elementary matrix, } \langle u, v \rangle = 0\}$$

and relations

\begin{align*}
(u, vr, a, b) &= (u, v, ar, b) \text{ for any } r \in R, \quad (T1) \\
(u, v, a, b)(u, w, a, c) &= (u, v + w, a, b + c + a^2\langle v, w \rangle), \quad (T2) \\
(u, v, a, 0)(u, v, b, 0) &= (u, v, a + b, 0), \quad (T3) \\
(u, v, a, 0) &= (v, u, a, 0) \text{ for } v \text{ a column of a symplectic elementary matrix, } (T4) \\
(u', v', a', b')(u, v, a, b)(u', v', a', b')^{-1} &= (T(u', v' a', b')u, T(u', v' a', b')v, a, b), \quad (T5) \\
(u, u, a, 0) &= (u, 0, 0, 2a), \quad (T6) \\
(u + vr, 0, 0, a) &= (u, 0, 0, a)(v, 0, 0, ar^2)(v, u, ar, 0) \text{ for } u + vr \text{ also columns of symplectic elementary matrices. (T7)}
\end{align*}

**Remark.** Clearly, Lemma 39 amounts to the existence of a homomorphism

$$\varpi: StSp_{2l}^\ast(R, I) \to StSp_{2l}(R, I),$$

sending $(u, v, a, b)$ to $Z(u, v, a, b)$.  

25
Lemma 40. Any triple \((u, v, a) \in V \times V \times R\) defines a homomorphism
\[ \alpha_{u,v,a} : \text{StSp}_{2l}^*(R, I) \to \text{StSp}_{2l}^*(R, I) \]
sending generators \((u', v', a', b')\) to \((T(u, v, a)u', T(u, v, a)v', a', b')\).

Proof. To show that \(\alpha_{u,v,a}\) is well-defined we have to check that T1–T7 hold
for the images of the generators. All of them are obvious, except T5, which
is checked below.

\[(T(u, v, a)u', T(u, v, a)v', a', b')(T(u, v, a)u'', T(u, v, a)v'', a'', b'').\]
\[\cdot (T(u, v, a)u', T(u, v, a)v', a', b')^{-1} =\]
\[(T(T(u, v, a)u', T(u, v, a)v'a', b')T(u, v, a)u'', T(u, v, a)v'', a'', b'') =\]
\[(T(u, v, a)T(u', v', a')u'', T(u, v, a)T(u', v', a')v'', a'', b'').\]

\[\square\]

Lemma 41. There exists a well-defined homomorphism
\[ \text{StSp}_{2l}(R) \to \text{Aut}(\text{StSp}_{2l}^*(R, I)) \]
sending \(X(u, v, a)\) to \(\alpha_{u,v,a}\), i.e., absolute Steinberg group acts on relative
van der Kallen group.

Proof. We need to verify that P1–P3 hold for \(\alpha_{u,v,a}\). We check P1 below,
P2 and P3 are left to the reader.

\[\alpha_{u,v,a}\alpha_{u,w,b}(u', v', a', b') = \alpha_{u,v,a}(T(u, w, b)u', T(u, w, b)v', a', b') =\]
\[= (T(u, v, a)T(u, w, b)u', T(u, v, a)T(u, w, b)v', a', b') =\]
\[= (T(u, v + w, a + b + (v, w))u', T(u, v + w, a + b + (v, w))v', a', b') =\]
\[= \alpha_{u,v+w,a+b+(v,w)}(u', v', a', b').\]

\[\square\]

Remark. Notice that \(\varpi\) preserves the action of \(\text{StSp}_{2l}(R)\).

Definition. Set
\[ i_{j}(a) = (e_{-j}, e_{i}, a_{e_{-j}}, 0) \text{ for } i \notin \{\pm j\}, \]
\[ i_{-j}(a) = (e_{i}, 0, 0, a). \]
Lemma 42. Steinberg relations KL0–KL7 hold for \(i_j(a)\) and \(i_{-i}(a)\).

Proof. To check KL0 use T4, for KL1 use T3 and T2. KL2 follows from the definition of the action. Below we verify the rest.

(KL3) \([X_{ij}(r), j_k(b)] = (e_{-k}, T(e_i, e_{-j}r\varepsilon_{-j}, 0)e_j, b\varepsilon_{-k}, 0)(e_{-k}, e_j, b\varepsilon_{-k}, 0)^{-1} = (e_{-k}, e_j + e_ir, b\varepsilon_{-k}, 0)(e_{-k}, -e_j, b\varepsilon_{-k}, 0) = (e_{-k}, e_ir, b\varepsilon_{-k}, 0) = (e_i, e_{-k}, r\varepsilon_{-k}, 0) = i_k(rb);\)

(KL4) \([X_{i_{-i}}, (-i_j)(b)] = (e_{-j}, T(e_i, 0, r)e_{-i}, b\varepsilon_{-j}, 0) \cdot (-i_j(-r\varepsilon_{-j})) = (e_{-j}, e_{-i} + e_ir\varepsilon_{i}, b\varepsilon_{-j}, 0)(e_{-j}, -e_{-i}, b\varepsilon_{-j}, 0) = (e_{-j}, e_ir\varepsilon_{i}, b\varepsilon_{-j}, -r\varepsilon_{-j}) = (e_{-j}, e_ir\varepsilon_{i}, b\varepsilon_{-j}, 0)(e_{-j}, 0, b\varepsilon_{-j}, -r\varepsilon_{-j}) = (e_{-j}, e_i, r\varepsilon_{i}\varepsilon_{-j}, 0)(e_{-j}, 0, -r\varepsilon_{-j}) = i_j(r\varepsilon_{i}) \cdot (-j_j(-r\varepsilon_{-j}));\)

(KL5) \([i_{-i}(a), X_{-i_j}(s)] = (e_i, 0, 0, a)(T(e_{-i}, e_{-j}s\varepsilon_{-j}, 0)e_i, 0, 0, -a) = (e_i, 0, 0, a)(e_{-j}, e_{-i} + e_{-j}s\varepsilon_{-j}\varepsilon_{-i}, 0, 0, -a) = (e_i, 0, 0, a)(e_{-j}, 0, -a)(e_{-j}, 0, 0, -as^2)(e_{-j}, e_i, -as\varepsilon_{i}\varepsilon_{j}, 0) = (e_{-j}, e_i, as\varepsilon_{i}\varepsilon_{-j}, 0)(e_{-j}, 0, 0, -as^2) = i_j(as\varepsilon_{i}) \cdot (-j_j(-as^2));\)

(KL6) \([X_{ij}(r), j_{-i}(b)] = (e_i, T(e_i, e_{-j}r\varepsilon_{-j}, 0)e_j, b\varepsilon_{i}, 0)(e_i, e_j, b\varepsilon_{i}, 0) = (e_i, e_{-j}r, b\varepsilon_{i}, 0)(e_i, -e_j, b\varepsilon_{i}, 0) = (e_i, e_{-j}, b\varepsilon_{i}, 0) = (e_i, e_{-j}, r\varepsilon_{i}, 0) = (e_i, 0, 0, 2r\varepsilon_{i}) = i_{-i}(2r\varepsilon_{i}).\)

The definition of the action together with T5 imply KL7.

Corollary. There is a homomorphism

\[\varphi: \text{StSp}_{2l}(R, I) \rightarrow \text{StSp}_{2l}^*(R, I)\]

sending \(Y_{ij}(a)\) to \(ij(a)\) and preserving the action of \(\text{StSp}_{2l}(R)\). Obviously, \(\varpi \varphi = 1\), i.e. \(\varphi\) is a splitting for \(\varpi\).
Lemma 43. For $v \in V$ such that $v_{-i} = 0$, set

$$
\tilde{v}_- = \sum_{i \in \{\pm j\} \atop i < 0} e_i v_i \quad \text{and similarly} \quad \tilde{v}_+ = \sum_{i \in \{\pm j\} \atop i > 0} e_i v_i.
$$

Then

$$(e_i, v, a, b) =
= i_{-i}(b + 2av_i - a^2(\tilde{v}_-, \tilde{v}_+)) \prod_{i < 0} j_{-i}(av_j \varepsilon_i) \prod_{i > 0} j_{-i}(av_j \varepsilon_i).$$

Proof.

$$(e_i, v, a, b) = (e_i, 0, a, b)(e_i, v, a, 0) =
= (e_i, 0, 0, b)(e_i, e_i v_i, a, 0)(e_i, \tilde{v}_- + \tilde{v}_+, a, 0) =
= (e_i, 0, 0, b)(e_i, e_i, av_i, 0)(e_i, 0, a, -a^2(\tilde{v}_- , \tilde{v}_+) ) \cdot (e_i, \tilde{v}_- + \tilde{v}_+, a, a^2(\tilde{v}_- , \tilde{v}_+) ) =
= (e_i, 0, 0, b)(e_i, 0, 0, 2av_i)(e_i, 0, 0, -a^2(\tilde{v}_- , \tilde{v}_+) ) \cdot (e_i, \tilde{v}_- , a, 0)(e_i, \tilde{v}_+, a, 0) =
\cdot \prod_{i \in \{\pm j\} \atop i < 0} (e_i, e_j v_j, a, 0) \prod_{i \in \{\pm j\} \atop i > 0} (e_i, e_j v_j, a, 0) =
= i_{-i}(b + 2av_i - a^2(\tilde{v}_-, \tilde{v}_+)) \prod_{i < 0} j_{-i}(av_j \varepsilon_i) \prod_{i > 0} j_{-i}(av_j \varepsilon_i).$$

Lemma 44.

$$\varpi \varpi = 1$$

Proof. Obviously, $\varpi \varpi(\iota_j(a)) = \iota_j(a)$, then $\varpi \varpi((e_i, v, a, b)) = (e_i, v, a, b)$ by the previous lemma. For arbitrary column of symplectic elementary matrix $u$ take $g \in \text{StSp}_{2\ell}(R)$ such that $\phi(g)e_i = u$ and use that $g$ and $\varpi$ preserve action.

$$\varpi \varpi((u, v, a, b)) = g \varpi(g^{-1}(\varphi(g)^{-1}v, a, b)) =
= g \varpi((e_i, \varphi(g)^{-1}v, a, b)) = \varphi(e_i, \varphi(g)^{-1}v, a, b) = (u, v, a, b).$$
References

[1] van der Kallen W., “Another presentation for Steinberg groups”, *Indag. Math.*, **39**:4 (1977), 304–312.

[2] Lavrenov A., “Another presentation for symplectic Steinberg groups”, arXiv.

[3] Sinchuk S., *Parabolic factorizations of reductive groups*, Ph. D. Thesis, Saint-Petersburg State University, 2013.