Timelike surfaces in Minkowski space with a canonical null direction

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Abstract

Given a constant vector field $Z$ in Minkowski space, a timelike surface is said to have a canonical null direction with respect to $Z$ if the projection of $Z$ on the tangent space of the surface gives a lightlike vector field. In this paper we describe these surfaces in the ruled case. For example when the Minkowski space has three dimensions then a surface with a canonical null direction is minimal and flat. On the other hand, we describe several properties in the non ruled case and we partially describe these surfaces in four-dimensional Minkowski space. We give different ways for building these surfaces in four-dimensional Minkowski space and we finally use the Gauss map for describe another properties of these surfaces.

Keywords: Timelike surfaces; canonical null direction; principal direction.

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Introduction

We consider $\mathbb{R}^{n,1}$ the $(n + 1)$–dimensional Minkowski space defined by $\mathbb{R}^{n+1}$ endowed with the metric of signature $(n, 1)$

$$\langle \cdot, \cdot \rangle = -dx_1^2 + dx_2^2 + \ldots + dx_{n+1}^2.$$ 

A surface $M$ in $\mathbb{R}^{n,1}$ is said to be timelike if the metric $\langle \cdot, \cdot \rangle$ induces a Lorentzian metric, i.e. a metric of signature $(1, 1)$, on $M$.  

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Definition 1. We say that a timelike surface $M$ in $\mathbb{R}^{n,1}$ has a canonical null direction with respect to a constant vector field $Z$ in $\mathbb{R}^{n,1}$ if the tangent part $Z^\top$ of $Z$ is a lightlike vector field along $M$, i.e. $Z^\top$ is nonzero and $\langle Z^\top, Z^\top \rangle = 0$. We will say that $Z$ defines a null direction on the surface.

In this paper, we are interested in the description of timelike surfaces with a canonical null direction in Minkowski space. We will begin by describing the compatibility equations which determine a canonical null direction on a surface and we will see that there exists two different cases for consider: the ruled and the non ruled case. We give a complete description of these surfaces in the ruled case (Theorem 2.2). On the other hand, we give several properties in the non ruled case and we partially describe these surfaces in four-dimensional Minkowski space (Proposition 3.7 and Theorem 3.10). We also give different ways for building these surfaces in four-dimensional Minkowski space and we finally use the Gauss map for describe another properties of these surfaces.

The notion of a canonical null direction only makes sense for timelike submanifolds in the $n + 1$-dimensional Minkowski space and it is inspired in the concept of surfaces with canonical principal direction with respect to a parallel vector field defined by F. Dillen and his collaborators in [4] and [5]. The second author together with E. García and O. Palmas in [6] investigated the case of hypersurfaces with a canonical principal direction with respect to a closed conformal vector field.

The paper is organized as follows. In Section 1 we describe the compatibility equations which determine a canonical null direction on a timelike surface and we give some properties about their geometry. In Corollary 1.8 we proved that if a surface in $\mathbb{R}^{n,1}$ has parallel mean curvature then it is minimal. In Section 2 we give a classification of these surfaces in Minkowski space in the ruled case. In Section 3 we study the non ruled case: we give some properties and we partially describe these surfaces in four-dimensional Minkowski space.

1 The compatibility equations

We consider a timelike surface $M$ in $\mathbb{R}^{n,1}$ with a canonical null direction $Z$. We can assume that $Z$ is a unit spacelike vector field; therefore, using the natural decomposition $Z = Z^\top + Z^\perp$ and since $\langle Z^\top, Z^\top \rangle = 0$ we have that $\langle Z^\perp, Z^\perp \rangle = 1$. Here and below we denote by $\langle \cdot, \cdot \rangle$ the metric on the Minkowski space, on $TM$ and on the normal bundle $NM$.

We will denote by $II : TM \times TM \to NM$ the second fundamental form of the immersion $M \subset \mathbb{R}^{n,1}$ given by

$$II(X,Y) = \nabla_X Y - \nabla_X Y,$$

where $\nabla$ and $\nabla$ are the Levi Civita connections of $\mathbb{R}^{n,1}$ and $M$, respectively. As usual, if $\nu \in NM$, $A_\nu : TM \to TM$ stands for the symmetric operator such
that

\[ \langle A_{\nu}(X), Y \rangle = \langle II(X, Y), \nu \rangle, \]

for all \( X, Y \in TM \). Finally, we denote by \( \nabla^\perp \) the Levi Civita connection of the normal bundle \( NM \). The following lemma is fundamental.

**Lemma 1.1.** We have

\[ \nabla_X Z^\top = A_{Z^\top}(X) \quad \text{and} \quad \nabla_X Z^\perp = -II(Z^\top, X), \]

(1)

for all \( X \in TM \).

**Proof.** Using the Gauss and Weingarten equations, we obtain that

\[ 0 = \nabla_X Z = \nabla_X Z^\top + \nabla_X Z^\perp \]

\[ = \nabla_X Z^\top - A_{Z^\perp}(X) + II(Z^\top, X) + \nabla_X Z^\perp; \]

the result follows by taking tangent and normal parts. \( \square \)

**Lemma 1.2.** We have

\[ A_{Z^\perp}(Z^\top) = 0 \quad \text{and} \quad \nabla_Z Z^\top = 0. \]

In particular, \( Z^\top \) is a canonical principal direction on the surface.

**Proof.** Using (1) we get

\[ \langle A_{Z^\perp}(Z^\top), X \rangle = \langle II(Z^\top, X), Z^\perp \rangle = -\langle \nabla_X Z^\perp, Z^\perp \rangle = -\frac{1}{2}X \langle Z^\perp, Z^\perp \rangle = 0, \]

for all \( X \in TM \). Finally, \( \nabla_Z Z^\top = A_{Z^\perp}(Z^\top) = 0. \) \( \square \)

Let us consider \( W \) a lightlike vector field tangent to \( M \) (i.e. \( W \) is nonzero and \( \langle W, W \rangle = 0 \)) such that \( \langle Z^\top, W \rangle = -1 \).

**Remark 1.3.** If we consider the frame \( (Z^\top, W) \) of lightlike vector fields on \( TM \) (with \( \langle Z^\top, W \rangle = -1 \)), the mean curvature vector of the immersion is given by

\[ \vec{H} := \frac{1}{2}tr(II) II = -II(Z^\top, W). \]

We define the function \( a := \langle II(W, W), Z^\perp \rangle. \)

**Lemma 1.4.** The Levi-Civita connection of \( M \) satisfies the following relations:

\[ \nabla_{Z^\top} Z^\top = 0 = \nabla_{Z^\top} W, \quad \nabla_W Z^\top = -aZ^\top \quad \text{and} \quad \nabla_W W = aW. \]

In particular, \( [Z^\top, W] = aZ^\top. \)
Proof. The first equality was given in Lemma 1.2. Now, \( \langle W, W \rangle = 0 \) implies that \( \langle \nabla_{Z^T} W, W \rangle = 0 \); and \( \langle Z^T, W \rangle = -1 \) implies \( 0 = \langle \nabla_{Z^T} Z^T, W \rangle + \langle Z^T, \nabla_{Z^T} W \rangle = \langle Z^T, \nabla_{Z^T} W \rangle \); therefore,
\[
\nabla_{Z^T} W = -\langle \nabla_{Z^T} W, W \rangle Z^T - \langle \nabla_{Z^T} W, Z^T \rangle W = 0.
\]
In a similar way, using \( \langle Z^T, Z^T \rangle = 0 \) we deduce that \( \langle \nabla_W Z^T, Z^T \rangle = 0 \); using (1) we get \( \langle \nabla_W Z^T, W \rangle = \langle A_{Z^T} (W), W \rangle = \langle II(W, W), Z^T \rangle = a \); thus,
\[
\nabla_W Z^T = -\langle \nabla_W Z^T, W \rangle Z^T - \langle \nabla_W Z^T, Z^T \rangle W = -a Z^T.
\]
On the other hand, since \( \langle \nabla_W W, W \rangle = 0 \), and \( \langle \nabla_W W, Z^T \rangle = -\langle W, \nabla_W Z^T \rangle = \langle W, a Z^T \rangle = -a \), we deduce that,
\[
\nabla_W W = -\langle \nabla_W W, W \rangle Z^T - \langle \nabla_W W, Z^T \rangle W = a W.
\]
Finally, \( [Z^T, W] = \nabla_{Z^T} W - \nabla_W Z^T = a Z^T \), because \( \nabla_{Z^T} W = 0 \). \( \square \)

We have the following relations for the curvature tensors of \( M \).

**Proposition 1.5.** The curvature tensor \( R \) and the normal curvature tensor \( R^\perp \) of \( M \) in \( \mathbb{R}^{n,1} \) are given by
\[
R(Z^T, W)Z^T = Z^T (a) Z^T \quad \text{and} \quad R^\perp(Z^T, W)Z^\perp = a II(Z^T, Z^T).
\]

Proof. Using the equalities of Lemma 1.4, we get
\[
R(Z^T, W)Z^T = \nabla_W \nabla_{Z^T} Z^T - \nabla_{Z^T} \nabla_W Z^T + \nabla_{[Z^T,W]} Z^T
= -\nabla_{Z^T} (aZ^T) + \nabla_{(a Z^T)} Z^T
= Z^T (a) Z^T.
\]
On other hand, by (1) we have
\[
R^\perp(Z^T, W)Z^\perp = \nabla_W \nabla_{Z^T} Z^\perp - \nabla_{Z^T} \nabla_W Z^\perp + \nabla_{[Z^T,W]} Z^\perp
= -\nabla_W (II(Z^T, Z^T)) + \nabla_{Z^T} (II(W, Z^T)) - a II(Z^T, Z^T);
\]
by Codazzi equation and the equalities of Lemma 1.4, we obtain that
\[
-\nabla_W (II(Z^T, Z^T)) + \nabla_{Z^T} (II(W, Z^T))
= - \left( \nabla_W II \right) (Z^T, Z^T) - II(W, Z^T, Z^T) - II(Z^T, W, Z^T)
+ \left( \nabla_{Z^T} II \right) (W, Z^T) + II(Z^T, W, Z^T) + II(W, Z^T, Z^T)
= 2a II(Z^T, Z^T),
\]
this finish the proof. \( \square \)
Corollary 1.6. The Gaussian curvature of $M$ is given by
\[ K = \frac{\langle R(Z^T, W)Z^T, W \rangle}{|Z^T|^2 - \langle Z^T, W \rangle^2} = Z^T(a). \]

Using the formula above for the Gauss curvature $K$, we will find a relation between the norm of the mean curvature vector and the Gaussian curvature.

Proposition 1.7. The mean curvature vector and its derivative satisfies the following relations:
\[ \nabla_{\overset{\perp}{W}} \overset{\perp}{H} = -\nabla_{\overset{\perp}{W}} (II(W, W)) \quad \text{and} \quad |\overset{\perp}{H}|^2 = -\langle \nabla_{\overset{\perp}{W}} \overset{\perp}{H}, Z^\perp \rangle. \tag{2} \]
Moreover, we have
\[ K = |\overset{\perp}{H}|^2 - \langle II(W, W), II(Z^T, Z^T) \rangle. \]

Proof. By Codazzi equation and the formulae of Lemma 1.4, we have
\[
\nabla_{Z^T}(II(W, W)) = \left( \nabla_{Z^T}II \right)(W, W) + 2II(\nabla_{Z^T}W, W) \\
= \left( \nabla_{Z^T}II \right)(Z^T, W) \\
= \nabla_{Z^T}II(Z^T, W) - II(\nabla_{Z^T}Z^T, W) - II(Z^T, \nabla_{Z^T}W) \\
= -\nabla_{Z^T}H + aII(Z^T, W) - aII(Z^T, W) \\
= -\nabla_{Z^T}H.
\]

On other hand, since $\langle \overset{\perp}{H}, Z^\perp \rangle = -\langle II(Z^T, W), Z^\perp \rangle = 0$ (see Lemma 1.2), from (1) we get
\[
0 = W(\overset{\perp}{H}, Z^\perp) = \langle \nabla_{\overset{\perp}{W}} \overset{\perp}{H}, Z^\perp \rangle + \langle \overset{\perp}{H}, \nabla_{\overset{\perp}{W}} Z^\perp \rangle = \langle \nabla_{\overset{\perp}{W}} \overset{\perp}{H}, Z^\perp \rangle - \langle \overset{\perp}{H}, II(Z^T, W) \rangle = \langle \nabla_{\overset{\perp}{W}} \overset{\perp}{H}, Z^\perp \rangle + |\overset{\perp}{H}|^2.
\]
Therefore, by Corollary 1.6 and the equalities in (1)-(2) we obtain
\[
K = |\overset{\perp}{H}|^2 - \langle II(W, W), II(Z^T, Z^T) \rangle
\]
which proves the assertion.

Corollary 1.8. If the mean curvature vector $\overset{\perp}{H}$ is parallel then the surface $M$ is minimal, i.e. $\overset{\perp}{H} = 0$.

Proof. This is a consequence of the second equality in Proposition 1.7.

The normal curvature tensor $R^\perp$ is determined by the vector $II(Z^T, Z^T)$, which is orthogonal to $Z^\perp$ (see the proof of Lemma 1.2: $\langle II(Z^T, Z^T), Z^\perp \rangle = 0$). Therefore, we can consider two cases: when $II(Z^T, Z^T) = 0$ (the ruled case) and when $II(Z^T, Z^T) \neq 0$ (the non ruled case).
2 The ruled case

In this section we study the case of a timelike surface $M$ in $\mathbb{R}^{n,1}$ with a canonical null direction $Z$ such that $II(Z^\top, Z^\top) = 0$. By Remark 1.3 and Proposition 1.5, the Gauss curvature and the normal curvature tensor satisfy the following relations:

$$|\vec{H}|^2 - K = 0 \quad \text{and} \quad R^\perp = 0.$$  \hspace{1cm} (3)

The timelike surfaces in four-dimensional pseudo Euclidean space for which (3) is valid are called umbilic (if $II - \langle \cdot, \cdot \rangle \vec{H} = 0$) or quasi-umbilic (if $II - \langle \cdot, \cdot \rangle \vec{H} \neq 0$). See e.g. [1, 2].

The surfaces in $\mathbb{R}^{2,1}$ such that $|\vec{H}|^2 - K = 0$ were classified in [3].

**Remark 2.1.** The normal vector field $Z^\perp$ is parallel if and only if $M$ is minimal. Let us verify this fact. By (1), we have

$$\nabla_{Z^\top} Z^\perp = -II(Z^\top, Z^\top) = 0 \quad \text{and} \quad \nabla_W Z^\perp = -II(Z^\top, W) = \vec{H},$$

which proves the assertion.

The next result gives a local description of a timelike surface $M$ in $\mathbb{R}^{n,1}$ with a canonical null direction $Z$ such that $II(Z^\top, Z^\top) = 0$. We moreover assume that $Z$ is not orthogonal to the surface; otherwise, $M$ should be any surface in a hyperplane orthogonal to $Z$.

**Theorem 2.2.** A timelike surface $M$ in $\mathbb{R}^{n,1}$ has a canonical null direction with respect to $Z$ and satisfies the condition $II(Z^\top, Z^\top) = 0$ if and only if $M$ can be locally parametrized by

$$\psi(x, y) = \alpha(x) + y \ Z^\top(x),$$

where $\alpha(x)$ is a lightlike curve in $\mathbb{R}^{n,1}$, $Z^\top(x)$ is the restriction of the null vector field $Z^\top$ along $\alpha$ and where the following conditions holds

- $Z$ is not orthogonal to $\alpha'(x)$ for every $x$,
- the vectors $\alpha'(x)$ and $Z^\top(x)$ are linearly independent for every $x$,
- the position vectors $Z^\top(x)$ gives a curve in a timelike hyperplane.

**Proof.** Let us consider a coordinate system $(x, y) \mapsto \psi(x, y)$ of $M$ such the metric of $M$ is given by

$$\langle \cdot, \cdot \rangle = -2\lambda(x, y)dx dy,$$

where $\lambda$ is some positive function; we moreover assume that $Z^\top = \frac{\partial \psi}{\partial y}$ satisfies $II(Z^\top, Z^\top) = 0$. By calculating the Christoffel symbols of the metric we get that

$$\nabla_{Z^\top} Z^\top = \frac{1}{\lambda} \frac{\partial \lambda}{\partial y} Z^\top.$$
Thus, $T := \frac{1}{\lambda}Z^\top$ satisfies that $\nabla_{Z^\top}T = 0$ and $\text{II}(T, Z^\top) = 0$. Since $\nabla_{Z^\top}T = \nabla_{Z^\top}T + \text{II}(T, Z^\top) = 0$, we have that

$$T(\psi(x,y)) = T(\psi(x,0)) + \int_0^y \frac{\partial}{\partial u}(T(\psi(x,u)))du = T(\psi(x,0)).$$

Then, $Z^\top(\psi(x,y)) = \frac{\lambda(x,y)}{\lambda(x,0)}Z^\top(\psi(x,0))$, and therefore,

$$\psi(x,y) = \psi(x,0) + \int_0^y \frac{\partial \psi}{\partial u}(x,u)du$$

$$= \psi(x,0) + \left(\int_0^y \frac{\lambda(x,u)}{\lambda(x,0)}du\right)Z^\top(\psi(x,0)).$$

So, $\psi$ can be written as

$$\psi(x,y) = \alpha(x) + f(x,y) Z^\top(x),$$

where $\alpha(x) := \psi(x,0)$ is a lightlike curve in $\mathbb{R}^{n,1}$ and $Z^\top(x) := Z^\top(\psi(x,0))$ is a lightlike vector field along $\alpha$. Since $\psi(x,0) = \alpha(x)$, we have that $f(x,0) = 0$; moreover

$$\frac{\partial f}{\partial y} = \frac{\lambda(x,y)}{\lambda(x,0)} > 0.$$

So, the formulae $x' = x$ and $y' = f(x,y)$, define local coordinates such that (4) is valid. Moreover, since $Z = Z^\top + Z^\perp$, we have that $Z$ is a spacelike constant vector with $\langle Z, Z^\top \rangle = 0$, in particular $\langle Z, Z^\perp(x) \rangle = 0$ for all $x$; thus the positions vectors $Z^\perp(x)$ are orthogonal to $Z$ and so they are contained in the timelike hyperplane orthogonal to $Z$.

Reciprocally, suppose that $M$ is parametrized as in (4). Since the positions vectors $Z^\top(x)$ lives in a timelike hyperplane, we can choose a constant spacelike vector in the spacelike line orthogonal to the hyperplane. So, $\langle Z, Z^\top(x) \rangle = 0$ for all $x$. This implies that the tangent part of $Z$ is $\frac{\partial u}{\partial y} = Z^\top(x)$ because $Z$ is not orthogonal to $\alpha'(x)$. Finally, since $M$ is a ruled surface with rules in the direction $Z^\top(x)$, we deduce that $\text{II}(Z^\top, Z^\top) = 0$. \qed

### 2.1 Timelike surfaces in $\mathbb{R}^{2,1}$

In this case, the normal vector $Z^\perp$ is parallel; by (1) we have that $\text{II}(Z^\top, X) = 0$, for all $X \in TM$; see Remark 2.1. Using moreover (3) we get:

**Corollary 2.3.** A timelike surface $M$ in $\mathbb{R}^{2,1}$ with a canonical null direction $Z$ is flat and minimal.

**Theorem 2.4.** A timelike surface $M$ in $\mathbb{R}^{2,1}$ with a canonical null direction $Z$ can be locally parametrized by

$$\psi(x,y) = \alpha(x) + y T_0,$$

where $\alpha(x)$ is a lightlike curve in $\mathbb{R}^{2,1}$, $T_0$ is some constant lightlike vector along $\alpha$, and the vectors $\alpha'(x)$ and $T_0$ are linearly independent for every $x$. 7
Proof. Let us observe that in this case we have that \( II(Z^\top, Z^\top) = 0 \). We can adapt the proof of Proposition 2.2 to obtain that \( M \) can be locally parametrized as in (4). The second fundamental form in the coordinates \((x, y)\) is given by \( II = (d^2\psi)^N \). The mean curvature vector \( \vec{H} = \frac{1}{2}g^{ij}II_{ij} = \frac{1}{\langle \alpha', Z^\top \rangle} \{ (Z^\top)' \}' \) satisfies the relation
\[
K = |\vec{H}|^2 = \frac{|(Z^\top)'|^2}{\langle \alpha', Z^\top \rangle^2}.
\]
Therefore, the condition \( K = |\vec{H}|^2 = 0 \), is equivalent to \( |(Z^\top)'|^2 = 0 \). Since \( |Z^\top|^2 = 0 \) and \( \langle (Z^\top)', Z^\top \rangle = 0 \), we have the relation \( (Z^\top)'(x) = h(x)Z^\top(x) \). Thus, by integration we get \( Z^\top(x) = H(x)Z^\top(0) \) where \( H \) is a smooth function such that \( H(0) = 1 \). Using the change of variable \( x' = x, y' = yH(x) \), and writing \( T_0 := Z^\top(0) \), we find that \( M \) is parametrized by \( \alpha(x') + y'T_0 \), for small values of \( x' \) and \( y' \).

Reciprocally, suppose that \( M \) is parametrized as in (5). Thus, a spacelike constant vector \( Z \) in \( \mathbb{R}^{2,1} \) such that \( \langle Z, T_0 \rangle = 0 \), defines a canonical null direction on \( M \). Moreover, \( Z^\top(x) = H(x)T_0 \), for some smooth function \( H(x) \). \( \Box \)

### 3 The non ruled case

In this section we study the case of a timelike surface \( M \) in \( \mathbb{R}^{n,1} \) with a canonical null direction \( Z \) such that \( II(Z^\top, Z^\top) \neq 0 \). We note that, as a consequence of Proposition 1.5 and Corollary 1.6, we have the following:

**Corollary 3.1.** Let us assume that the surface \( M \) has normal curvature tensor \( R^\perp \) identically zero (i.e. the function \( a \) is identically zero). Then the Gauss curvature \( K \) is also constant zero.

We note that, if we assume that \( \nabla a \) is a multiple of \( Z^\top \) we get that the Gauss curvature \( K = Z^\top(a) = \langle \nabla a, Z^\top \rangle \) (Corollary 1.6) is zero. We will describe the converse statement. We need some lemmas.

**Lemma 3.2.** There is a local smooth function \( f : M \to \mathbb{R} \) such that \( \nabla f = Z^\top \). Moreover, \( f \) is a harmonic function, i.e. \( \Delta f = 0 \).

**Proof.** We consider the 1–form \( \theta(X) = \langle X, Z^\top \rangle \), for all \( X \in TM \). Using the equalities of Lemma 1.4, we get \( \theta \) is a closed 1–form, i.e. \( d\theta = 0 \); thus, there exists a function \( f : M \to \mathbb{R} \) such that \( df = \theta \), that is \( \nabla f = Z^\top \).

We compute the laplacian of the function \( f \). In the orthonormal frame \( \left( \frac{Z^\top + W}{\sqrt{2}}, \frac{Z^\top - W}{\sqrt{2}} \right) \) on \( TM \), we get
\[
\Delta f = -2\text{Hess}f(Z^\top, W) = -2\langle \nabla_{Z^\top} \nabla f, W \rangle = -2\langle \nabla_{Z^\top} Z^\top, W \rangle = 0,
\]
because \( \nabla_{Z^\top} Z^\top = 0 \) (see Lemma 1.2). \( \Box \)
Lemma 3.3. The laplacian of the function \( a = \langle II(W, W), Z^\perp \rangle \) is given by
\[
\Delta a = -2Ka - 2W(K),
\]
where \( K \) is the Gauss curvature of the surface. In particular, if the Gauss curvature is zero, \( a \) is a harmonic function.

Proof. In the same frame, as in the proof of Lemma 3.2, we get
\[
\Delta a = -2\text{Hess } f(W, Z^\top) = -2\langle \nabla W \nabla a, Z^\top \rangle.
\]
On the other hand, since \( K = Z^\top(a) = \langle \nabla a, Z^\top \rangle \) (Corollary 1.6), using Lemma 1.4 we obtain
\[
W(K) = W(\nabla a, Z^\top) = \langle \nabla W \nabla a, Z^\top \rangle + \langle \nabla a, \nabla W Z^\top \rangle = -\frac{1}{2} \Delta a - a \langle \nabla a, Z^\top \rangle,
\]
which is the equality of the lemma.

Proposition 3.4. The Gauss curvature \( K \) is zero if and only if there exists a harmonic function \( a_1 : M \to \mathbb{R} \) such that \( \nabla a = a_1 Z^\top \).

Proof. We assume that the Gauss curvature \( K \) is zero: since \( K = Z^\top(a) = \langle \nabla a, Z^\top \rangle \) (Corollary 1.6), there exists a smooth function \( a_1 : M \to \mathbb{R} \) such that \( \nabla a = a_1 Z^\top \) because \( Z^\top \) is a null vector field. Using Lemmas 3.2 and 3.3, we obtain
\[
0 = \Delta a = \text{div}(\nabla a) = \text{div}(a_1 \nabla f) = \langle \nabla a_1, \nabla f \rangle + a_1 \Delta f = \langle \nabla a_1, Z^\top \rangle,
\]
thus, there exists a smooth function \( a_2 : M \to \mathbb{R} \) such that \( \nabla a_1 = a_2 Z^\top \). The laplacian of the function \( a_1 \) is given by
\[
\Delta a_1 = -2\langle \nabla W \nabla a_1, Z^\top \rangle = -2 \langle W(a_2) Z^\top + a_2 \nabla W Z^\top, Z^\top \rangle = 0.
\]
Note that, we can continue with this procedure.

3.1 Timelike surfaces in \( \mathbb{R}^{3,1} \)

In this case, we consider the normalized vector field
\[
\nu := \frac{II(Z^\top, Z^\top)}{|II(Z^\top, Z^\top)|} \in NM.
\]
Note that \( \nu \) is orthogonal to \( Z^\perp \) (see Lemma 1.2). We recall that \( Z^\perp \) is a space-like vector field with \( \langle Z^\perp, Z^\perp \rangle = 1 \). So, \( (Z^\perp, \nu) \) defines an oriented orthonormal frame of the normal bundle \( NM \) along \( M \).

Corollary 3.5. The normal curvature of the surface \( M \) in \( \mathbb{R}^{3,1} \) is given by
\[
K_N = a |II(Z^\top, Z^\top)|.
\]
Proof. Using the Ricci equation, in the orthonormal frame \((\frac{Z^T}{\sqrt{2}}, \frac{W}{\sqrt{2}})\) on \(TM\), we obtain

\[
K_N = \left\langle (A_{Z^\perp} \circ A_\nu - A_\nu \circ A_{Z^\perp}) \left( \frac{Z^T + W}{\sqrt{2}} \right), \frac{Z^T - W}{\sqrt{2}} \right\rangle
\]

\[
= -\langle (A_{Z^\perp} \circ A_\nu - A_\nu \circ A_{Z^\perp})(Z^T), W \rangle
\]

\[
= \langle R^\perp (Z^T, W)Z^\perp, \nu \rangle;
\]

we get the result by replacing the second equality given in Proposition 1.5.

Now, we will give a relation between the Gauss curvature, the normal curvature and the mean curvature vector of \(M\) in \(\mathbb{R}^{3,1}\).

**Lemma 3.6.** In the orthonormal frame \((Z^\perp, \nu)\) orthogonal to \(M\), we have the following relation

\[
II(W, W) = \frac{K_N}{|II(Z^T, Z^T)|} Z^\perp + \frac{\|\tilde{H}\|^2 - K}{|II(Z^T, Z^T)|} \nu.
\]

In particular, \(|II(W, W)|^2|II(Z^T, Z^T)|^2 = (\|\tilde{H}\|^2 - K)^2 + K_N^2\).

**Proof.** We have

\[
II(W, W) = \langle II(W, W), Z^\perp \rangle Z^\perp + \langle II(W, W), \nu \rangle \nu
\]

\[
= aZ^\perp + \frac{\langle II(W, W), II(Z^T, Z^T) \rangle}{|II(Z^T, Z^T)|} \nu,
\]

we get the result by using Corollary 3.5 and Proposition 1.7.

Using the lemma above we have the following description in a simple case:

**Proposition 3.7.** Consider a timelike surface \(M\) in \(\mathbb{R}^{3,1}\) with a canonical null direction \(Z\) such that \(II(Z^T, Z^T) \neq 0\). If \(M\) is minimal and has flat normal bundle (i.e. \(K_N = 0\)) then it can be parametrized as

\[
\psi(x, y) = \alpha(x) + y W_0,
\]

where \(\alpha'(x) = Z^T(x)\), \(\alpha''(x) = II(Z^T, Z^T)\) (\(\alpha\) is a geodesic of \(M\)), \(W_0\) is some constant lightlike tangent vector along \(\alpha\) and the vectors \(\alpha'(x)\) and \(W_0\) are linearly independent for every \(x\).

**Proof.** Since \(a = 0\) (i.e. \(K_N = 0\)), by Lemma 1.4 we have that \(Z^T\) and \(W\) are parallel vector fields and \([Z^T, W] = 0\). So, there exists a coordinate system \((x, y) \mapsto \psi(x, y)\) of \(M\) such that

\[
\frac{\partial \psi}{\partial x}(x, y) = Z^T(\psi(x, y)) \quad \text{and} \quad \frac{\partial \psi}{\partial y}(x, y) = W(\psi(x, y)).
\]
We have that $\Pi(W, \cdot) = 0$: indeed, $\Pi(W, Z^\top) = -\tilde{H} = 0$ and $\Pi(W, W) = 0$ because $K = K_N = |\tilde{H}|^2 = 0$ in Lemma 3.6. Since $\nabla W = 0$, we get that $\nabla W = 0$; thus,

$$W(\psi(x, y)) = W(\psi(x, 0)) + \int_0^y \frac{\partial}{\partial u} W(\psi(x, u)) du = W(\psi(x, 0)), \quad (6)$$

this implies that,

$$\psi(x, y) = \psi(x, 0) + \int_0^y \frac{\partial \psi}{\partial u}(x, u) du = \psi(x, 0) + \int_0^y W(\psi(x, u)) du = \psi(x, 0) + y W(\psi(x, 0)).$$

In the same way, let us observe that

$$W(\psi(x, y)) = W(\psi(0, y)) + \int_0^x \frac{\partial}{\partial r} W(\psi(r, y)) dr = W(\psi(0, y)). \quad (7)$$

The equalities (6)-(7) imply that $W(\psi(x, y)) =: W_0$ is constant, i.e.

$$\psi(x, y) = \alpha(x) + y W_0,$$

where $\alpha(x) := \psi(x, 0)$ is a lightlike curve such that $\alpha'(x) = Z^\top(\psi(x, 0))$. □

The following example describes a timelike surface in $\mathbb{R}^{3,1}$ with a canonical null direction $Z$ such that $\Pi(Z^\top, Z^\top) \neq 0$ which is minimal but has normal curvature not zero. Here and below, we denote by $\{e_1, e_2, e_3, e_4\}$ the canonical basis of the four-dimensional Minkowski space; of course $e_1$ is a timelike vector.

**Example 3.8.** Let us consider the surface $M$ in $\mathbb{R}^{3,1}$ parametrized as

$$\psi(x, y) = \alpha(x) + \beta(y),$$

where $\alpha$ and $\beta$ are two lightlike curves contained in the timelike hyperplanes orthogonal to $e_4$ and $e_3$, respectively, and satisfy the following conditions

- $\langle \alpha'(x), \beta''(y) \rangle \neq 0$ for every $(x, y)$,
- $\langle e_3, \alpha'(x) \rangle \neq 0$ for every $x$,
- $\beta''(y)$ (resp. $\alpha''(x)$) is not lightlike: in other case, $\beta''(y)$ would be linearly dependent to $\beta'(y)$ and thus $\beta$ would be a lightlike line in $\mathbb{R}^{3,1}$.

We have that $M$ is a minimal timelike surface in $\mathbb{R}^{3,1}$ with normal curvature not zero and has a canonical null direction with respect to $e_3$ with $\Pi(e_3^\top, e_3^\top) \neq 0$.

Indeed, note that $M$ is a timelike surface because its tangent plane is generated by the linearly independent lightlike tangent vectors $\psi_x = \alpha'(x)$ and
\( \psi_y = \beta'(y) \). On the other hand, since the curve \( \beta \) is orthogonal to \( e_3 \), the tangent part of \( e_3 \) is given by

\[
e_3^\top = \frac{\langle e_3, \beta'(y) \rangle}{\langle \alpha'(x), \beta'(y) \rangle} \alpha'(x) + \frac{\langle e_3, \alpha'(x) \rangle}{\langle \alpha'(x), \beta'(y) \rangle} \beta'(y) = \frac{\langle e_3, \alpha'(x) \rangle}{\langle \alpha'(x), \beta'(y) \rangle} \beta'(y);
\]

this proves that \( e_3^\top = \lambda(x,y) \beta'(y) \), where \( \lambda(x,y) := \frac{\langle e_3, \alpha'(x) \rangle}{\langle \alpha'(x), \beta'(y) \rangle} \) is not zero, is a lightlike direction on the surface \( M \). Since \( \nabla_{e_3^\top} e_3^\top = 0 \) (Lemma 1.2), we obtain

\[
\nabla_{\beta'(y)} \beta'(y) = -\frac{1}{\lambda(x,y)} \frac{\partial \lambda}{\partial y} \beta'(y).
\]

Therefore,

\[
\beta''(y) = \nabla_{\beta'(y)} \beta'(y) + II(\beta'(y), \beta'(y)) = -\frac{1}{\lambda(x,y)} \frac{\partial \lambda}{\partial y} \beta'(y) + II(\beta'(y), \beta'(y));
\]

since \( \beta'(y) \) and \( \beta''(y) \) are linearly independent, we get that \( II(\beta'(y), \beta'(y)) \neq 0 \), that is

\[II(e_3^\top, e_3^\top) = \lambda(x,y)^2 II(\beta'(y), \beta'(y)) \neq 0.\]

Now, since \( \psi_x = 0 \) we get that \( \nabla_{\psi_x} \psi_y = 0 \) and \( II(\psi_x, \psi_y) = 0 \), in particular, \( M \) is minimal. We finally prove that \( M \) has normal curvature not zero: we consider the lightlike tangent vector

\[W := -\frac{1}{\lambda(x,y) \langle \alpha'(x), \beta'(y) \rangle} \alpha'(x) = -\frac{1}{\langle e_3, \alpha'(x) \rangle} \alpha'(x),\]

which is such that \( \langle e_3^\top, W \rangle = -1 \); since \( \nabla_W e_3^\top = -a e_3^\top \) (Lemma 1.4) we obtain

\[a = \frac{1}{\lambda(x,y) \langle e_3, \alpha'(x) \rangle} \frac{\partial \lambda}{\partial x};\]

according to Corollary 3.5, \( K_N = 0 \) if and only if \( a = 0 \), that is, if and only if \( \frac{\partial \lambda}{\partial x} = 0 \), the last equality is equivalent to

\[
\lambda = \frac{\langle e_3, \alpha''(x) \rangle}{\langle \alpha''(x), \beta'(y) \rangle}
\]

which is valid when \( \alpha''(x) \) is linearly dependent to \( \alpha'(x) \). We finally give an explicit numerical example of this situation: consider

\[\alpha(x) = (\cosh x, \sinh x, x, 0) \quad \text{and} \quad \beta(y) = (\cosh y, y, 0, \sinh y),\]

defined on a domain for \( (x, y) \) where \( \langle \alpha'(x), \beta'(y) \rangle \neq 0 \).
3.2 Timelike surfaces in $\mathbb{R}^{3,1}$ as a graph of a function

In this section, we will study the situation when a surface is given as the graph of a smooth function.

Let $f, g : U \subset \mathbb{R}^2 \to \mathbb{R}$ be two smooth functions and consider the surface

$$M := \{(f(x, y), g(x, y), x, y) \in \mathbb{R}^{3,1} \mid x, y \in U\} \subset \mathbb{R}^{3,1}$$

given as a graph of the function $(x, y) \to (f(x, y), g(x, y))$. A global parametrization of this surface is given by

$$\psi : U \subset \mathbb{R}^2 \to \mathbb{R}^{3,1}, \quad \psi(x, y) = (f(x, y), g(x, y), x, y).$$

The tangent vectors to the surface are $\psi_x = (f_x, g_x, 1, 0)$ and $\psi_y = (f_y, g_y, 0, 1)$, and the components of the induced metric $\langle \cdot, \cdot \rangle$ in $M$ are given by

$$E := \langle \psi_x, \psi_x \rangle = 1 - f_x^2 + g_y^2, \quad F := \langle \psi_x, \psi_y \rangle = -f_x f_y + g_x g_y$$

and

$$G := \langle \psi_y, \psi_y \rangle = 1 - f_y^2 + g_y^2.$$

The determinant of this metric is

$$\det \langle \cdot, \cdot \rangle = EG - F^2 = 1 - |\nabla f|^2 + |\nabla g|^2 - \langle \nabla f, \nabla g \rangle^2,$$

where the right hand side is calculated on $\mathbb{R}^2$ with its standard Riemannian flat metric; in particular, $M$ is a timelike surface if and only if $\det \langle \cdot, \cdot \rangle < 0$.

**Proposition 3.9.** Let $M$ be a timelike surface in $\mathbb{R}^{3,1}$ given as in (8). Then $M$ has a canonical null direction with respect to $e_4$ (resp. $e_3$) if and only if $\psi_x$ (resp. $\psi_y$) is a lightlike vector field along $M$. In that situation we have

$$e_4^\top = \frac{1}{F} \psi_x \quad \text{(resp. } \quad e_3^\top = \frac{1}{F} \psi_y \text{).}$$

**Proof.** We have to calculate the tangent part of $e_4$ along $M$ (the case for the vector $e_3$ is similar therefore it will be omitted): writing

$$e_4^\top = a \psi_x + b \psi_y,$$

we get

$$\langle e_4^\top, \psi_x \rangle = aE + bF \quad \text{and} \quad \langle e_4^\top, \psi_y \rangle = aF + bG;$$

therefore

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \langle e_4, \psi_x \rangle \\ \langle e_4, \psi_y \rangle \end{pmatrix} = \begin{pmatrix} \langle e_4^\top, \psi_x \rangle \\ \langle e_4^\top, \psi_y \rangle \end{pmatrix} = \begin{pmatrix} E \\ F \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

and thus

$$a = \frac{-F}{EG - F^2} \quad \text{and} \quad b = \frac{E}{EG - F^2}. $$
Finally, we get
\[
\langle e_4^\top, e_4^\top \rangle = a^2 E + b^2 G + 2abF = \frac{E}{EG - F^2}
\]
that is, \(e_4^\top\) is a lightlike vector field along \(M\) if and only if
\[
E = \langle \psi_x, \psi_x \rangle = 1 - f_x^2 + g_x^2 = 0,
\]
i.e. if and only if \(\psi_x\) is a lightlike vector field.

**Theorem 3.10.** Let \(M\) be a timelike surface in \(\mathbb{R}^{3,1}\) given as in (8). Assume that \(M\) has a canonical null direction with respect to \(e_3\) and \(e_4\). Then \(M\) is minimal if and only if \(M\) can be locally parametrized by
\[
\psi(x, y) = \alpha(x) + \beta(y)
\]
where \(\alpha\) and \(\beta\) are two lightlike curves contained in the timelike hyperplanes orthogonal to \(e_4\) and \(e_3\), respectively.

**Proof.** A global basis for the normal bundle \(NM\) is given by the vector fields
\[
\xi_1 := (1, 0, f_x, f_y) \quad \text{and} \quad \xi_2 := (0, -1, g_x, g_y).
\]
The components of the induced metric in \(NM\) are given by
\[
L := \langle \xi_1, \xi_1 \rangle = |\nabla f|^2 - 1, \quad M := \langle \xi_1, \xi_2 \rangle = \langle \nabla f, \nabla g \rangle, \quad N := \langle \xi_2, \xi_2 \rangle = |\nabla g|^2 + 1,
\]
and satisfies \(LN - M^2 > 0\). We are going to calculate the condition for \(M\) to be minimal. Using Proposition 3.9, we have that the tangent vectors \(\psi_x\) and \(\psi_y\) of \(M\) are lightlike; therefore, \(M\) is minimal if and only if \(II(\psi_x, \psi_y) = 0\). In general, for \(i, j \in \{x, y\}\), we have
\[
II(\psi_i, \psi_j) = \langle \nabla_{\psi_i} \psi_j \rangle^\perp = a \xi_1 + b \xi_2,
\]
where \(\nabla\) is the Levi Civita connection of \(\mathbb{R}^{3,1}\) and
\[
\left( \begin{array}{c}
\alpha \\
\beta
\end{array} \right) = \frac{1}{LN - M^2} \left( \begin{array}{cc}
N & -M \\
-M & L
\end{array} \right) \left( \begin{array}{c}
\langle \nabla_{\psi_i} \psi_j, \xi_1 \rangle \\
\langle \nabla_{\psi_i} \psi_j, \xi_2 \rangle
\end{array} \right).
\]
Since \(\nabla_{\psi_i} \psi_j = (f_{ij}, g_{ij}, 0, 0)\), we get
\[
\langle \nabla_{\psi_i} \psi_j, \xi_1 \rangle = -f_{ij} \quad \text{and} \quad \langle \nabla_{\psi_i} \psi_j, \xi_2 \rangle = -g_{ij}.
\]
We deduce that,
\[
II(\psi_x, \psi_x) = -\frac{f_{xx}N + g_{xx}M}{LN - M^2} \xi_1 + \frac{f_{xx}M - g_{xx}L}{LN - M^2} \xi_2,
\]
\[
II(\psi_y, \psi_y) = -\frac{f_{yy}N + g_{yy}M}{LN - M^2} \xi_1 + \frac{f_{yy}M - g_{yy}L}{LN - M^2} \xi_2,
\]
\[
II(\psi_x, \psi_y) = -\frac{f_{xy}N + g_{xy}M}{LN - M^2} \xi_1 + \frac{f_{xy}M - g_{xy}L}{LN - M^2} \xi_2.
\]
Therefore, $M$ is minimal if and only if
\[
\begin{pmatrix}
-N & M \\
M & -L
\end{pmatrix}
\begin{pmatrix}
f_{xy} \\
g_{xy}
\end{pmatrix}
= \begin{pmatrix}
-f_{xy}N + g_{xy}M \\
f_{xy}M - g_{xy}L
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix};
\]
since $LN - M^2 > 0$, we obtain that $f_{xy} = 0 = g_{xy}$. Thus, by integration we get
\[
f(x, y) = \alpha_1(x) + \beta_1(y) \quad \text{and} \quad g(x, y) = \alpha_2(x) + \beta_2(y)
\]
This implies that $\psi$ can be written as in (9) with $\alpha(x) = (\alpha_1(x), \alpha_2(x), x, 0)$ (orthogonal to $e_4$) and $\beta(y) = (\beta_1(y), \beta_2(y), 0, y)$ (orthogonal to $e_3$). Let us observe that in this case $\psi_x$ and $\psi_y$ are lightlike vectors if and only if $\alpha$ and $\beta$ are lightlike curves.

We consider the isometric embedding of $\mathbb{R}^{2,1}$ in $\mathbb{R}^{3,1}$ given by
\[
\mathbb{R}^{2,1} := (e_4)^{\perp},
\]
where $e_4$ is the fourth vector of the canonical basis of $\mathbb{R}^{3,1}$.

**Proposition 3.11.** Let $M_0$ be a Lorentzian surface in $\mathbb{R}^{2,1}$, $f : M_0 \to \mathbb{R}$ be a given smooth function. Let us consider the surface obtained as the graph of $f$, i.e.
\[
M := \{(p, f(p))| p \in M_0 \} \subset \mathbb{R}^{3,1},
\]
with the induced metric. Then $M$ has a canonical null direction with respect to $e_4$ if and only if $\nabla f$ is a lightlike vector field on $M_0$.

**Proof.** The surface $M$ is parametrized by the immersion
\[
\psi : M_0 \to \mathbb{R}^{3,1}, \quad \psi(p) = (p, f(p)).
\]
We consider a local orthonormal frame $(X_1, X_2)$ of $TM_0$ with $e_j = \langle X_j, X_j \rangle$ and such that $\epsilon_1 \epsilon_2 = -1$ ($M_0$ is Lorentzian). Moreover, $X_1$ and $X_2$ are orthogonal to $e_4$. Using the immersion $\psi$, we can get the induced local frame on $TM$,
\[
Y_j := d\psi(X_j) = X_j + df(X_j)e_4, \quad j = 1, 2.
\]
So, the induced metric $\langle \cdot, \cdot \rangle$ on $M$ is given by the matrix
\[
\begin{pmatrix}
\langle Y_1, Y_1 \rangle & \langle Y_1, Y_2 \rangle \\
\langle Y_2, Y_1 \rangle & \langle Y_2, Y_2 \rangle
\end{pmatrix}
= \begin{pmatrix}
\epsilon_1 + (\nabla f, X_1)^2 & \langle \nabla f, X_1 \rangle \langle \nabla f, X_2 \rangle \\
\langle \nabla f, X_1 \rangle \langle \nabla f, X_2 \rangle & \epsilon_2 + (\nabla f, X_2)^2
\end{pmatrix},
\]
and its determinant is
\[
det(\langle \cdot, \cdot \rangle) = -1 + \epsilon_2 (\nabla f, X_1)^2 + \epsilon_1 (\nabla f, X_2)^2 = -(1 + \epsilon_1 (\nabla f, X_1)^2 + \epsilon_2 (\nabla f, X_2)^2)
\]
\[= -(1 + \langle \nabla f, \nabla f \rangle)
\]
(since $\nabla f = \epsilon_1 (\nabla f, X_1) X_1 + \epsilon_2 (\nabla f, X_2) X_2$); therefore, $M$ is a timelike surface if and only if $\langle \nabla f, \nabla f \rangle > -1$. On the other hand, we have $e_4^\top = a Y_1 + b Y_2$ where
\[
\begin{pmatrix}
a \\
b
\end{pmatrix}
= \frac{1}{\det \langle \cdot, \cdot \rangle} \begin{pmatrix}
\epsilon_2 + \langle \nabla f, X_2 \rangle^2 & -\langle \nabla f, X_1 \rangle \langle \nabla f, X_2 \rangle \\
-\langle \nabla f, X_1 \rangle \langle \nabla f, X_2 \rangle & \epsilon_1 + \langle \nabla f, X_1 \rangle^2
\end{pmatrix}
\begin{pmatrix}
\langle e_4, Y_1 \rangle \\
\langle e_4, Y_2 \rangle
\end{pmatrix};
\]
since, $\langle e_4, Y_1 \rangle = \langle \nabla f, X_1 \rangle$ and $\langle e_4, Y_2 \rangle = \langle \nabla f, X_2 \rangle$ we obtain that
\[
\begin{pmatrix}
a \\
b
\end{pmatrix}
= \frac{1}{\det \langle \cdot, \cdot \rangle} \begin{pmatrix}
\epsilon_2 \langle \nabla f, X_1 \rangle \\
\epsilon_1 \langle \nabla f, X_2 \rangle
\end{pmatrix}
= \frac{1}{\det \langle \cdot, \cdot \rangle} \begin{pmatrix}
\epsilon_1 \langle \nabla f, X_1 \rangle \\
\epsilon_2 \langle \nabla f, X_2 \rangle
\end{pmatrix},
\]
therefore,
\[
\langle e_4^\top, e_4^\top \rangle = \frac{1}{\det \langle \cdot, \cdot \rangle} (\epsilon_1 \langle \nabla f, X_1 \rangle Y_1 + \epsilon_2 \langle \nabla f, X_2 \rangle Y_2).
\]
Thus
\[
\langle e_4^\top, e_4^\top \rangle = \frac{1}{\det \langle \cdot, \cdot \rangle} (\epsilon_1 \langle \nabla f, X_1 \rangle^2 + \epsilon_2 \langle \nabla f, X_2 \rangle^2)
- 2 \langle \nabla f, X_1 \rangle \langle \nabla f, X_2 \rangle
= \frac{1}{\det \langle \cdot, \cdot \rangle} (\langle \nabla f, \nabla f \rangle + \langle \nabla f, \nabla f \rangle^2).
\]
Now, it is clear that $e_4^\top$ is a lightlike vector field on $M$ if and only if $\langle \nabla f, \nabla f \rangle$ is either 0 or $-1$; but the case $\langle \nabla f, \nabla f \rangle = -1$ is not possible because we would have that $\det \langle \cdot, \cdot \rangle = 0$.

The following proposition generalize Lemma 3.2.

**Proposition 3.12.** Let $M$ be a Lorentzian surface, $f : M \to \mathbb{R}$ be a given smooth function. If the gradient $\nabla f$ is a lightlike vector field then the integral curves of $\nabla f$ are geodesics and $f$ is a harmonic function, i.e. $\Delta f = 0$.

**Proof.** Note that $\nabla f \neq 0$ because it is lightlike vector field, in particular $f$ is not a constant function. By a direct computation we get
\[
0 = X \langle \nabla f, \nabla f \rangle = 2 \langle \nabla_X \nabla f, \nabla f \rangle = 2 \text{Hess } f(\nabla f, X),
\]
for all $X \in TM$; in particular, $\nabla_{\nabla f} \nabla f = 0$ because $\langle \nabla_X \nabla f, \nabla f \rangle = \langle \nabla_{\nabla f} \nabla f, X \rangle$. On the other hand, let $W$ be another lightlike vector field defined locally on $M$ such that $\langle \nabla f, W \rangle = -1$. We compute the laplacian of the function $f$ : in the orthonormal frame \( \left( \frac{\nabla f + W}{\sqrt{2}}, \frac{\nabla f - W}{\sqrt{2}} \right) \) on $TM$, we get
\[
\Delta f = -2 \text{Hess } f(\nabla f, W) = -2 \langle \nabla_{\nabla f} \nabla f, W \rangle = 0,
\]
because $\nabla_{\nabla f} \nabla f = 0$. \qed
Example 3.13. Let us consider the timelike surface

\[ M_0 := \{(x, \exp iy) \mid x \in \mathbb{R}, \ y \in (0, 2\pi)\} \subset \mathbb{R}^{2,1}, \]

and the function \( f : M_0 \to \mathbb{R} \) given by \( f(x, \exp iy) = y - x \). The level curve \( \gamma(y) = (y - c, \exp iy) \) is a lightlike geodesic in \( M_0 \) for all constant \( c \in \mathbb{R} \). Indeed, we only have to remark that \( \gamma'(y) = (1, i \exp iy) = \partial_x + \partial_y \) is a lightlike vector field. On the other hand, we compute the gradient of the function \( f \) since \( M_0 \) is a Lorentzian product, we have that \( \partial_x = e_1 \) and \( \partial_y = -\sin y \ e_2 + \cos y \ e_3 \) are an orthonormal frame along \( M_0 \), therefore,

\[ \nabla f = -(\partial_x f)\partial_x + (\partial_y f)\partial_y = \partial_x + \partial_y = \gamma'(y), \]

which is a lightlike vector field on \( M_0 \); from Proposition 3.12 we obtain that \( \gamma \) is a geodesic. Finally, by Proposition 3.11, the timelike surface

\[ M := \{(x, \exp iy, y - x) \mid x \in \mathbb{R}, \ y \in (0, 2\pi)\} \subset \mathbb{R}^{3,1} \]

has a canonical null direction with respect to \( e_4 \).

3.3 Another properties using the Gauss map

We consider \( \Lambda^2 \mathbb{R}^{3,1} \), the vector space of bivectors of \( \mathbb{R}^{3,1} \) endowed with its natural metric \( \langle \cdot, \cdot \rangle \) of signature \((3,3)\). The Grassmannian of the oriented timelike 2–planes in \( \mathbb{R}^{3,1} \) identifies with the submanifold of unit and simple bivectors

\[ \mathcal{Q} := \{ \eta \in \Lambda^2 \mathbb{R}^{3,1} \mid \langle \eta, \eta \rangle = -1, \ \eta \wedge \eta = 0 \}, \]

and the oriented Gauss map of a timelike surface in \( \mathbb{R}^{3,1} \) with the map

\[ G : M \to \mathcal{Q}, \quad p \mapsto G(p) = u_1 \wedge u_2, \]

where \((u_1, u_2)\) is a positively oriented orthonormal basis of \( T_p M \). The Hodge * operator \( \Lambda^2 \mathbb{R}^{3,1} \to \Lambda^2 \mathbb{R}^{3,1} \) is defined by the relation

\[ \langle * \eta, \eta' \rangle = \eta \wedge \eta', \]

for all \( \eta, \eta' \in \Lambda^2 \mathbb{R}^{3,1} \), where we identify \( \Lambda^4 \mathbb{R}^{3,1} \) to \( \mathbb{R} \) using the canonical volume element \( e_1 \wedge e_2 \wedge e_3 \wedge e_4 \) of \( \mathbb{R}^{3,1} \). It satisfies \( *^2 = -id_{\Lambda^2 \mathbb{R}^{3,1}} \) and thus \( i := -* \) defines a complex structure on \( \Lambda^2 \mathbb{R}^{3,1} \). We also define the map \( H : \Lambda^2 \mathbb{R}^{3,1} \times \Lambda^2 \mathbb{R}^{3,1} \to \mathbb{C} \) by

\[ H(\eta, \eta') = \langle \eta, \eta' \rangle + i \ \eta \wedge \eta', \]

for all \( \eta, \eta' \in \Lambda^2 \mathbb{R}^{3,1} \). This is a \( \mathbb{C} \)-bilinear map on \( \Lambda^2 \mathbb{R}^{3,1} \), and we have

\[ \mathcal{Q} = \{ \eta \in \Lambda^2 \mathbb{R}^{3,1} \mid H(\eta, \eta) = -1 \}. \]

The bivectors

\[ \{ e_1 \wedge e_2, \ e_2 \wedge e_3, \ e_3 \wedge e_1 \} \]

form an orthonormal basis (with respect to the norm \( H \)) of \( \Lambda^2 \mathbb{R}^{3,1} \) as a complex space of signature \((-,-,+,-)\). Using this basis of \( \Lambda^2 \mathbb{R}^{3,1} \), the Grassmannian \( \mathcal{Q} \) is identified with a complex hyperboloid of one sheet

\[ \mathcal{Q} \simeq \{ (z_1, z_2, z_3) \in \mathbb{C}^3 \mid -z_1^2 + z_2^2 - z_3^2 = -1 \}. \]

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3.3.1 Timelike surfaces with a canonical null direction

We consider an oriented timelike surface \( M \) in \( \mathbb{R}^{3,1} \) with a canonical null direction \( Z \) (with \( \langle Z, Z \rangle = 1 \) and) such that \( II(Z^\top, Z^\top) \neq 0 \). We recall that \( W \) is a lightlike vector field tangent to \( M \) such that \( \langle Z^\top, W \rangle = -1 \), and that \( Z^\perp \) is a unit vector field normal to \( M \). As before, we consider the unit vector field normal to the surface

\[
\nu := \frac{II(Z^\top, Z^\top)}{|II(Z^\top, Z^\top)|} \in NM;
\]

recall that \( \nu \) is orthogonal to \( Z^\perp \) (see the proof of Lemma 1.2). We moreover suppose that

\[
e_1 := \frac{Z^\top + W}{\sqrt{2}}, \quad e_2 := \frac{Z^\top - W}{\sqrt{2}}, \quad e_3 := Z^\perp \text{ and } e_4 := \nu,
\]

is an oriented and orthonormal basis of \( \mathbb{R}^{3,1} \), and define the orthonormal basis (11) of \( \Lambda^2 \mathbb{R}^{3,1} \), with respect to the form \( H \).

**Lemma 3.14.** The Gauss map of \( M \) is given by \( G = W \wedge Z^\top \), and satisfies

\[
dG(Z^\top) = -\bar{H} \wedge Z^\top + W \wedge II(Z^\top, Z^\top) \text{ and } dG(W) = \bar{H} \wedge W - Z^\top \wedge II(W, W).
\]

**Proof.** We only need to compute

\[
G = e_1 \wedge e_2 = \frac{1}{2}(Z^\top + W) \wedge (Z^\top - W) = W \wedge Z^\top.
\]

The differential of the expression above is given by

\[
dG(u) = (\nabla_u W + II(W, u)) \wedge Z^\top + W \wedge (\nabla_u Z^\top + II(Z^\top, u))
\]

for all \( u \in T_p M \); using the identities of Lemma 1.4 we conclude the result. \( \square \)

We define the bivectors

\[
N_1 := \frac{1}{\sqrt{2}}(e_2 \wedge e_3 + e_3 \wedge e_1) \quad \text{and} \quad N_2 := \frac{1}{\sqrt{2}}(-e_2 \wedge e_3 + e_3 \wedge e_1);
\]

\( N_1 \) and \( N_2 \) are linearly independent, they satisfy \( H(N_1, N_1) = H(N_2, N_2) = 0 \) and \( H(N_1, N_2) = -1 \). Explicitly, \( N_1 \) and \( N_2 \) are given by

\[
N_1 = Z^\perp \wedge W \quad \text{and} \quad N_2 = Z^\perp \wedge Z^\top;
\]

moreover, with respect to the complex structure \( i = -\ast \) defined on \( \Lambda^2 \mathbb{R}^{3,1} \), of a direct computation we get

\[
iN_1 = W \wedge \nu \quad \text{and} \quad iN_2 = -Z^\top \wedge \nu,
\]

and the volume element is given by \( -iN_1 \wedge N_2 \).
Lemma 3.15. We have the following identities:

- \( dG(Z^\top) = i|II(Z^\top, Z^\top)|N_1 - i\langle \vec{H}, \nu \rangle N_2, \)
- \( dG(W) = -i\langle \vec{H}, \nu \rangle N_1 + \left( \frac{K_N}{|II(Z^\top, Z^\top)|} + i\frac{|\vec{H}|^2 - K}{|II(Z^\top, Z^\top)|} \right) N_2. \)

Proof. Since 0 = \( \langle II(Z^\top, W), Z_\perp \rangle = -\langle \vec{H}, Z_\perp \rangle \), we have
\[
\vec{H} = \langle \vec{H}, Z_\perp \rangle Z_\perp + \langle \vec{H}, \nu \rangle \nu = \langle \vec{H}, \nu \rangle \nu, \tag{13}\]
replacing this, and the relation given in Proposition 3.6 in the identities of Lemma 3.14, we easily get the result.

The pull-back of the form \( H \) by the Gauss map. The pull-back by the Gauss map \( G : M \rightarrow Q \subset \Lambda^2 \mathbb{R}^3 \) of the form \( H \) (defined in (10)) permits to define, for all \( p \in M \), the complex quadratic form on the tangent space \( T_p M \)
\[
G^* H_p : T_p M \rightarrow \mathbb{C}, \quad u \mapsto H(dG_p(u), dG_p(u)).
\]
This form is analogous to the third fundamental form of the classical theory of surfaces in Euclidean space. We will describe some properties of this quadratic form for a timelike surface with a canonical null direction.

Lemma 3.16. We have the following identities

- \( H(dG(Z^\top), dG(Z^\top)) = -2|II(Z^\top, Z^\top)|\langle \vec{H}, \nu \rangle, \)
- \( H(dG(W), dG(W)) = -2\frac{1}{|II(Z^\top, Z^\top)|}\langle \vec{H}, \nu \rangle \left( |\vec{H}|^2 - K - iK_N \right), \)
- \( H(dG(Z^\top), dG(W)) = \left( 2|\vec{H}|^2 - K \right) - iK_N. \)

Proof. The proof of these equalities is obtained by a direct computation using the expressions of \( dG(Z^\top) \) and \( dG(W) \) given in Lemma 3.15.

Proposition 3.17. The discriminant of the complex quadratic form \( G^* H \) satisfies
\[
\text{disc } G^* H := -\det G^* H = -\left( K + iK_N \right)^2,
\]
where \( K \) and \( K_N \) are the Gauss and normal curvatures of the surface \( M \).

Proof. Using the identities of Lemma 3.16, by a direct computation we get
\[
\det G^* H = \left( 2|\vec{H}|^2 - K - iK_N \right)^2 - 4|\vec{H}|^2 \left( |\vec{H}|^2 - K - iK_N \right) = K^2 + 2iKK_N - K_N^2,
\]
which implies the result.
Proposition 3.18. The complex quadratic form $G^*H$ is zero at every point of $M$ if and only if $M$ is minimal and has flat normal bundle.

Proof. We recall that $M$ is minimal if and only if $\langle \tilde{H}, \nu \rangle = 0$ (identity (13)), and that normal curvature zero implies Gauss curvature zero (see Corollary 3.1). Using the identities of Lemma 3.16, since $H(Z^\top, Z^\top) \neq 0$, we easily get the result.

The interpretation of the condition $G^*H \equiv 0$ is the following: for all $p$ in $M$, the space $dG_p(T_pM)$ belongs to $G(p) + \{ \xi \in \Lambda^2_{\mathbb{R}^{3,1}} | H(G(p), \xi) = 0 = H(\xi, \xi) \} \subset T_{G(p)} Q$; this set is the union of two complex lines through $G(p)$ in the Grassmannian $Q$ of the oriented and timelike planes of $\mathbb{R}^{3,1}$; explicitly, these complex lines are given by $G(p) + \mathbb{C}N_1$ and $G(p) + \mathbb{C}N_2$. In particular, the first normal space in $p$ is 1-dimensional, i.e. the osculator space of the surface is degenerate at every point $p$ of $M$.

Asymptotic directions on the surface. For all $p \in M$, we consider the real quadratic form

$$\delta : T_p M \to \mathbb{R}, \quad u \mapsto dG_p(u) \wedge dG_p(u),$$

where $\Lambda^4_{\mathbb{R}^{3,1}}$ is identified with $\mathbb{R}$ by means of the volume element $-iN_1 \wedge N_2 \simeq 1$. A non-zero vector $u \in T_p M$ defines an asymptotic direction at $p$ if $\delta(u) = 0$. The opposite of the determinant of $\delta$, with respect to the metric on $M$,

$$\Delta := -\det \delta,$$

is a second order invariant of the surface; $\Delta \leq 0$ if and only if there exists asymptotic directions; $\Delta$ is negative if and only if the surface admits two distinct asymptotic directions at every point. We refer to [1, Section 4] (see also [2]) for a complete description of the asymptotic directions of a timelike surface in $\mathbb{R}^{3,1}$.

We will compute the invariant $\Delta$ and describe the asymptotic directions of a timelike surface with a canonical null direction.

Proposition 3.19. At every point of $M$ we have:

$$\Delta = -K_N^2$$

where $K_N$ is the normal curvature of $M$. In particular, there exists asymptotic directions at every point of $M$. 

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Proof. Since $\delta$ is the imaginary part of the quadratic form $G^*H$, we have $\delta(Z^\top) = 0$ (first equality of Lemma 3.16). Using the identities of Lemma 3.15, by a direct computation we get

$$\Delta = - \left[dG(Z^\top) \wedge dG(W)\right]^2 + \delta(Z^\top)\delta(W) = -K_N^2$$

since $dG(Z^\top) \wedge dG(W) = -K_N iN_1 \wedge N_2 \simeq K_N$.

**Proposition 3.20.** At every point of $M$, $Z^\top$ is an asymptotic direction; moreover, $W$ is an asymptotic direction if and only if $M$ is minimal or has flat normal bundle.

Proof. Since $\delta$ is the imaginary part of $G^*H$, using the identities of Lemma 3.16 we have

$$\delta(Z^\top) = 0 \quad \text{and} \quad \delta(W) = -2\frac{1}{\|H(Z^\top, Z^\top)\|} \langle \vec{H}, \nu \rangle K_N$$

which implies the results.

According to Proposition 3.19, if the normal curvature $K_N$ is not zero, there exists two distinct asymptotic directions at every point of the surface. From Proposition 3.20, $Z^\top$ is an asymptotic direction; by a direct computation, we describe the other asymptotic direction.

**Proposition 3.21.** If the surface $M$ has not zero normal curvature, there exists two different asymptotic directions given by

$$Z^\top \quad \text{and} \quad \frac{\langle \vec{H}, \nu \rangle}{\|H(Z^\top, Z^\top)\|} Z^\top + W$$

at every point of the surface.

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