Droplet-Edge Operators in Nonrelativistic Conformal Field Theories

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Abstract: We consider the large-charge expansion of the charged ground state of a Schrödinger-invariant, nonrelativistic conformal field theory in a harmonic trap, in general dimension $d$. In the existing literature, the energy in the trap has been computed to next-to-leading order (NLO) at large charge $Q$, which comes from the classical contribution of two higher-derivative terms in the effective field theory. In this note, we explain the structure of operators localized at the edge of the droplet, where the density drops to zero. We list all operators contributing to the ground-state energy with nonnegative powers of $Q$ in the large-$Q$ expansion. As a test, we use dimensional regularization to reproduce the calculation of the NLO ground state energy by Kravec and Pal [7], and we recover the same universal coefficient for the logarithmic term as in that work. We refine the derivation by presenting a systematic operator analysis of the possible edge counterterms, showing that different choices of cutoff procedures must yield the same renormalized result up to an enumerable list of Wilson coefficients for conformally invariant local counterterms at the droplet edge. We also demonstrate the existence of a previously unnoticed edge contribution to the ground-state operator dimension of order $Q^{\frac{d}{2} - \frac{1}{2}}$ in $d$ spatial dimensions. Finally, we show there is no bulk or edge counterterm scaling as $Q^0$ in two spatial dimensions, which establishes the universality of the order $Q^0$ term in large-$Q$ expansion of the lowest charged operator dimension in $d = 2$. 
Contents

1 Introduction and Summary 1

2 The interacting unitary fermion NRCFT and its large-charge universality class 4
  2.1 NRCFT: General structure 4
  2.2 EFT description of the large-charge sector 5
  2.3 Relation to the relativistic state-operator correspondence 5
  2.4 Leading-order terms in the bulk effective action 6
  2.5 Leading-order quantities in infinite volume 7
  2.6 Leading-order quantities in a harmonic trapping potential 8
  2.7 Next-to-leading order (NLO) terms in the bulk effective action 9
  2.8 Criteria for admissible terms 10
    2.8.1 Gauge invariance 10
    2.8.2 Diffeomorphism invariance 11
    2.8.3 Conformal invariance 11
  2.9 The dressing rule for bulk operators 12
    2.9.1 General comments on dressing rules 12
    2.9.2 Dressing rule in the Son-Wingate EFT 13
    2.9.3 Bipartite decomposition and $\mu$-scaling of bulk operators 13
  2.10 Equations of motion 14
    2.10.1 NLO corrections to the dispersion relation 14
    2.10.2 First-excited spectrum in the harmonic potential 15
  2.11 Comments on bulk terms beyond NLO 15
    2.11.1 $\mu$-scalings of higher-derivative terms in the harmonic potential 16

3 Structure of local operators at the droplet edge 18
  3.1 General comments 18
  3.2 The $Y$-dressing rule for edge operators 19
  3.3 The tripartite structure of droplet-edge operators 20
    3.3.1 The $\delta(X)$ factor 20
    3.3.2 The $Y$-dressing 21
  3.4 Example: A simple droplet-edge operator and its $\mu$-scaling 22
  3.5 Expansion in phonons 22
  3.6 Classical $\mu$-scaling and $Q$-scaling of the droplet-edge operators 24
    3.6.1 $\mu$-scalings of (unintegrated) dressed edge operators 24
    3.6.2 $\mu$-scalings of integrated edge terms 24
    3.6.3 Classification of Lagrangian perturbations by their classical $\mu$-scaling 26
  3.7 A sufficient condition for allowed edge operators 27
4 Energies in the harmonic trap at NLO
4.1 Energy shift from the Lagrangian perturbation
4.2 Conformal sharp cutoff in $d = 2$
4.3 Evaluation of the bare energy at order $c_1^1$ in dimensional regularization, near $d = 2$
4.4 Identification and coefficient of the boundary counterterm in dimensional regularization, at $d = 2$
4.5 Energy at order $c_1^1$ in $d = 2$, with the counterterm included
4.6 Contributions of other subleading operators with positive $Q$-scaling in $d = 2$
4.6.1 The bulk term $L_{c_2}$
4.6.2 The edge term $O_{b_2} \equiv [m^{-1}Z]_{\text{edge}}$
4.7 Upper bound on quantum $\mu$-scalings of contributions of edge operators to the ground state energy
4.7.1 UV cutoff and scaling of the propagator
4.7.2 A crude bound on the quantum $\mu$-scaling of general edge operators
4.7.3 No quantum corrections to the vacuum energy with nonnegative $\mu$-scaling in $d = 2$

5 Conclusions

A Conventions
A.1 Summary of relationships among the variously-defined $\xi$-coefficients and the $c_0$ coefficient in the recent literature
A.2 Conventions for NLO coefficients
A.3 Comparison of names for coefficients and local quantities, making explicit the factors of $m$ and $\hbar$
A.4 Convention-sensitivity of scheme-dependent constants in minimal subtraction

1 Introduction and Summary

Various zero-temperature phase transitions are purported to be described by “quantum critical behavior,” defined by a nonrelativistic conformal field theory. In this paper, we deal with the NRCFT studied by Son and Wingate [1] that describes interacting fermions at zero temperature when a parameter of the Hamiltonian has been tuned (in all existing cases, by dialing the strength of a background magnetic field near a Feshbach resonance; see, e.g., [2, 3] for early experimental realizations) so that the fermionic scattering length becomes infinite. This notional fixed point, originally posed by Bertsch [4, 5], is referred to in the literature as a unitary fermi gas.

In the case of a trapping potential where the charge is supported in a finite region, the precision of the computation of the ground-state energy is limited by one’s understanding...
of the nature of the Hamiltonian density near the edge of the atomic droplet, where the charge density goes to zero.

To make this slightly more explicit, let us briefly introduce some basics of the Son-Wingate NRCFT. The symmetries of the theory include temporal- and spatial-translation symmetry, atom-number (mass) conservation, Galilean relativistic invariance, and scale invariance, which is enhanced to a nonrelativistic conformal symmetry. The full group generated by these symmetries is known as the Schrödinger group. This symmetry is respected by free, nonrelativistic fermions in any spatial dimension $d$ (but there are believed to be non-free examples realized in nature). The condensate droplet effective theory descends from a general treatment of massive Schrödinger particles $\psi$, coupled to an external potential consistent with the symmetries of the system. Invariance under a phase rotation of $\psi$ is a global $U(1)$ symmetry. At infinite scattering length, the remaining physical degree of freedom in the theory is the phase of the condensate; for bosonic particles this phase is that of the field $\psi$ in the usual way, while for fermions one can identify it with (half) the phase of a Cooper pair. The global $U(1)$ symmetry is broken by the choice of ground state, which fixes the charge $Q$, and fluctuations around this ground state are (conformal) Goldstone fields $\chi$; the classical superfluid ground state with chemical potential $\mu$ is at $\chi = \mu t$. The expansion of the EFT in the region of finite density is an expansion in derivatives of $\chi$, where the conformal dimension is adjusted to marginality by fractional negative powers of

$$X \equiv \chi - A_0 - \frac{1}{2m} (\bar{\partial} \chi - \bar{A})^2,$$

(1.1)

and where the background potential has $\bar{A} = 0$ for the case of the harmonic trap. The density $\rho$ goes as $(m X)^{\frac{d}{2}}$ in spatial dimension $d$, so the expansion is in powers of $\rho^{-\frac{1}{2}} \bar{\partial}$, which breaks down at the droplet edge where the density falls to zero.

The present state of understanding was advanced by [6, 7], wherein the large-charge expansion of the NRCFT was presented as a way to perturbatively suppress quantum effects below leading-order contributions in the Lagrangian. The authors of [7], in particular, studied the system in a harmonic potential trap, though the quantum effects of the theory remain uncontrolled in the absence of a detailed and complete treatment of the structure of counterterms beyond those supported in the bulk of the density distribution.

More specifically, the breakdown of the derivative expansion in the interior is associated with unknown contributions to the operator dimension, scaling with positive powers of the charge $Q$. These unknown contributions are parametrically larger than contributions from quantum-mechanical fluctuations of the $\chi$ field, which start with the Casimir energy at order $Q^0$. Without the ability to test these quantum-mechanical contributions against experiment, or against other methods of calculation, it remains open whether the Lagrangian of [1] is a true effective theory of the large-charge sector, or whether there are other light degrees of freedom at large $Q$ in addition to the conformal Goldstone field $\chi$. It is thus crucial to understand the contributions to the energy from singularities near the droplet edge, where the density falls to zero, if we are to probe the quantum-mechanical completeness of the $\chi$ theory as a description of the low-energy states of the large-charge Hilbert space. In this paper we aim to do precisely this. We will explain the general
structure of the droplet-edge terms and demonstrate their role in the renormalization of the Hamiltonian at the classical and quantum level.

In spite of the singularity of the classical solution of the EFT near the droplet edge, the edge effects are in fact under control within the EFT itself (as we will show), with the singular behavior being parametrized by new Wilson coefficients for additional effective terms localized at the droplet edge. To put the analysis in context, a similar situation occurs in the case of the large-spin expansion of the relativistic effective string with freely moving endpoints \[9\]. Here, the length of the string goes as \(R_{\text{phys}} = E_{\text{IR}}^{-1} = \sqrt{J/\alpha'}\), and the local strong-coupling scale in the interior of the worldsheet is the total energy of the string \(E_{\text{UV}}^{(\text{interior})} = \sqrt{J/\alpha'}\), while the effective loop-counting parameter is \(E_{\text{IR}}/E_{\text{UV}} = 1/J\).

The classical solution controlling leading-order observables is singular at the boundary of the worldsheet, and the large-spin expansion appears to break down entirely. On closer examination \[10\], however, one finds this is not so. At the boundary of the worldsheet, the local strong coupling scale drops to \(E_{\text{UV}}^{(\text{boundary})} = J^{-\frac{1}{4}}\alpha'^{-\frac{3}{2}}\). Quantum effects near the boundary are not as strongly suppressed as in the interior, but they are still suppressed: The effective loop-counting parameter near the boundary is \(J^{-\frac{1}{4}}\) rather than \(J^{-1}\).

The breakdown of the bulk derivative expansion near the droplet edge of the NRCFT is resolved in precisely the same way as in the case of the effective string. At the droplet edge, the field appearing in fractional or negative powers in the derivative expansion (which we refer to as the “dressing field,” by analogy with the case of the effective string) is

\[
Y/m \equiv \langle \hat{\partial}^2 X \rangle /m, \tag{1.2}
\]

instead of \(X\). The infrared energy scale in the harmonic trap is simply the trapping frequency \(\omega\), and the local strong-coupling energy scale, by way of the chemical potential \(\mu\), is \(\langle (Y/m) \rangle \propto (\omega^2 \mu)^{\frac{1}{4}} \sim Q^{\frac{1}{4}} \mu \), instead of \(\mu \sim Q^{\frac{1}{2}} \omega\). (Here we use the proportionality \(Q \propto \mu^d\) between the chemical potential and total charge in the harmonic potential, as reviewed in Sec. 2.6.) It follows that higher derivative terms at the edge are suppressed by powers of \(E_{\text{UV}}^{(\text{edge})}/E_{\text{IR}} \sim (\mu/\omega)^{\frac{1}{2}} \sim Q^{\frac{1}{4}}\), rather than powers of the hierarchy \(E_{\text{UV}}^{(\text{interior})}/E_{\text{IR}} \sim Q^{\frac{1}{2}}\), which suppresses quantum effects and higher-derivative terms in the bulk.

The plan of the paper is as follows:

- In Section 2 we review the basic setup of the theory, including leading-order and next-to-leading-order (NLO) effects in homogeneous ground states in the harmonic trap in \(d\) dimensions, as calculated previously in, e.g., \[1, 7\].

- In Section 3 we systematize the construction of edge counterterms and give rules for counting the \(\mu\)-scaling of their contribution to the ground state energy in the harmonic potential.

- In Section 4 we calculate the contributions from NLO operators in the bulk using dimensional regularization, reproducing the results of \[7\] up to nonlogarithmic contributions at order \(Q^{\frac{1}{2}}\), corresponding to the (now elucidated) conformally invariant counterterms at the droplet edge.

- In Section 5 we discuss the results and summarize our conclusions.
2 The interacting unitary fermion NRCFT and its large-charge universality class

Let us begin by reviewing the structure of the effective Lagrangian of [1]. We will keep our conventions as close as possible to those of Son and Wingate, deviating only when necessary to extend those conventions in a natural way to arbitrary complex spatial dimension \( d \), or to make contact with alternate conventions used in recent literature, such as [7].

2.1 NRCFT: General structure

The best-known interacting NRCFT is the theory of interacting fermions at the so-called unitary limit, where the scattering length becomes infinite. Femions at unitarity, unlike free fermions, are an example of a system with no additional internal symmetries. Collections of various types of fermionic atoms [11–16] are all believed to flow to the same critical point at zero temperature.

The interacting critical point can be distinguished easily from the free critical point, for instance, by the ground state energy density for a homogeneous state of fermion density \( \rho \). In general complex spatial dimension \( d \), however, there is no natural choice for the number of spin states, and we need to exercise caution in expressing the free energy density if we wish to make contact with the conventions of [17], for instance. In [17], the authors normalize the ground state energy of the system relative to two species of scalar fermions in spatial dimension \( d \), rather than a spinorial \( SO(d) \) multiplet of spinning fermions.\(^1\) The ratio of the ground state energy density of the interacting theory (which we shall refer to as the Son-Wingate (SW) theory) to that of the corresponding free theory with two scalar particle species at fixed \( \rho \) is thus explicitly stated:

\[
\xi \equiv \frac{\mathcal{H}_{\text{interacting NRCFT}}}{n_{\text{free fermion; 2 scalar species}}} \bigg|_{\text{same } \rho} . \tag{2.1}
\]

Known as the Bertsch parameter, \( \xi \) constitutes a characteristic dimensionless number of the critical point. Straightforwardly, if \( \xi \) is not equal to 1, the critical point does not describe the free fermion system. The parameter is particular to the NRCFT in question, analogous to the \( c_{3/2} \) parameter of the large-charge relativistic conformal EFT in \( 2 + 1 \) dimensions [18–22], or its higher-dimensional [21, 23–26] and nonabelian-symmetric [21, 27–29] counterparts. The value of \( \xi \) is approximately the same for all experimental realizations of the critical theory referred to above (and different from unity, of course); it can also be extracted numerically via Monte Carlo simulation [30, 31], placing its estimated value in three spatial dimensions around

\[
\xi_{d=3} \simeq 0.4 . \tag{2.2}
\]

\(^1\)For more detail, see the formula given below eqn. (18) of [17] for the Fermi momentum \( k_F \) in terms of the density of the free fermion system in \( d \) dimensions: The authors write \( k_F = \left[2^{d-1} \pi^{d/2} \Gamma\left(\frac{d}{2} + 1\right) n\right]^{1/2} \), with \( n \) denoting the ground-state fermion density (we use \( \rho \)). Formula (18) of [17] is correct for a system of free fermions with exactly two fermion states per momentum level, independent of the dimension \( d \). For a different number \( a_s \) of fermion states at a given momentum, the relationship would be \( k_F = \left[a_s^{-1} 2^{d-1} \pi^{d/2} \Gamma\left(\frac{d}{2} + 1\right) n\right]^{1/2} \). We discuss this further in Sec. A.1.
2.2 EFT description of the large-charge sector

As it stands, the definition of this NRCFT is unclear, a priori. As with the Wilson-Fisher $O(2)$ model [32], interacting NRCFTs are generically strongly coupled with non-infinitesimal anomalous dimensions, and any Lagrangian description would necessarily be of Wilsonian type, with an infinite number of higher-derivative terms for whatever field content, all of them comparable in size (in units of the cutoff). Outside a perturbative treatment of the theory near a weak-coupling region (such as an $\epsilon$-expansion, as in [32], or the large-$N$ expansion), there is currently no known way to verify or exclude the existence of a renormalization-group fixed point with a given set symmetries and degrees of freedom.

Even assuming the existence of a Wilsonian description, such a treatment is of limited direct utility for the computation of many observables of interest. For certain limits in observable-space, however, the Wilsonian description of the fixed point becomes effectively weakly coupled. In the case of generic relativistic CFT, the properties of the ground state at charge $Q$ can be computed in an asymptotic expansion in inverse powers of $Q$ [9, 10, 18–20, 28, 29, 33–42]. In examples with $U(1)$ symmetry, this large-charge sector is described by a conformally invariant effective Lagrangian for a single Goldstone boson $\chi$; in nonabelian and supersymmetric examples, the effective theories are described by generalizations that extend to the minimal field content as dictated by the symmetries.

The proposal in [1] comprises an analogous conformally invariant Lagrangian in the nonrelativistic case, describing the quantum critical points in interacting fermion systems at unitarity. The theory is purported to control the dynamics of the large-charge limit of the NRCFT and, as in the aforementioned cases in relativistic CFTs, the large-charge sector admits an EFT for a single conformal goldstone mode $\chi$, with a controlled derivative expansion wherein higher-order terms make parametrically suppressed contributions at large $Q$. Various leading-order and next-to-leading-order quantities have been computed for homogeneous ground states in vanishing background fields in infinite volume [1, 43], in finite volume in flat space [6], and in a harmonic trapping potential [1, 7]. The energy in the latter case is of particular importance, due to the nature of the state-operator correspondence (more on this below).

Note that this EFT may actually describe the large-charge dynamics of more than one critical point with the same symmetries. There may be other NRCFTs described by the same EFT with different values of $\xi$, as well as other subleading Wilson coefficients, but the particular critical point describing the unitary Fermi gas stands as a useful testbed as a highly generic, nonempty nonrelativistic CFT.

2.3 Relation to the relativistic state-operator correspondence

As noted, there is an analogous large-charge EFT for the Wilson-Fisher $O(2)$ model [18] [19], and in fact the Son-Wingate EFT works out very similarly. The difference most relevant in the present context is that the state-operator correspondence in NRCFT is not between arbitrary local operators and states on the sphere, but between positively-charged local operators only, and states in flat space in a harmonic potential. States of zero or
negative particle number do not correspond to any quantum state under the NRCFT state-operator correspondence.

Specifically, for positively charged operators of charge $Q$, the corresponding state is a state of charge $Q$ in infinite volume, with a nontrivial background potential

$$A_0(\vec{x}) = \frac{m_0^2}{2} \vec{x}^2 .$$  \hfill (2.3)

The scaling dimensions of charged operators in the NRCFT are equal to $\omega^{-1}$ times the energies of the corresponding states in the harmonic potential.

This is conceptually similar the $O(2)$ model, but the salient difference is that, while the charged ground state of the $O(2)$ model on the sphere is spatially homogeneous [18], the charged ground state in the harmonic trap is inhomogeneous, with the density falling to zero at some radius defining the finite extent of the droplet. The singularity at the edge of the droplet is resolved by unknown short-distance physics. For purposes of low-energy, long-wavelength observables, the effects of these unknown dynamics can be absorbed into effective terms in the Hamiltonian that are localized at the surface of the droplet. Unlike the case in a translationally invariant ground state, however, the singularity contributes at the classical level; the Hamiltonian requires regularization and renormalization even for tree-level consistency. The analysis of the classical UV singularity and the edge counterterms that cancel the classical divergence is the central subject of this paper.

2.4 Leading-order terms in the bulk effective action

The leading-order bulk effective Lagrangian for the Son-Wingate [1] boson $\chi$ is

$$\mathcal{L}_{SW} \ni c_0 m_0^{d} \mathbf{X}^{1+d} ,$$  \hfill (2.4)

where $d$ is the spatial dimension, and $c_0$ is a parameter. The charge density and Hamiltonian density appear as

$$\rho = \frac{\delta \mathcal{L}}{\delta \dot{\chi}} = \frac{\delta \mathcal{L}}{\delta \mathbf{X}} = \left( 1 + \frac{d}{2} \right) c_0 m_0^{d} \mathbf{X}^{d} ,$$

$$\mathcal{H} = \dot{\chi} \rho - \mathcal{L} = \frac{d}{2} \mathcal{L} + A_0 \rho .$$  \hfill (2.5)

In nonrelativistic CFT, the parameter $m$ is dimensionless, i.e., inert under rescaling. It can consistently be set equal to 1 (as some authors [7] choose to do) without breaking conformal invariance, though we leave $m$ indicated explicitly throughout the present paper. In NRCFT with Schrödinger symmetry, energy scales with twice the conformal dimension of spatial momentum, and we will adopt the convention that spatial momentum has scaling dimension 1 while energy has scaling dimension 2, which leaves the spatial coordinates $\vec{x}$ and temporal coordinate $t$ to scale with dimensions $-1$ and $-2$, respectively.

The dynamics of the $\chi$ theory is under perturbative control in a homogeneous ground state in infinite volume. The scattering of goldstone excitations above such a ground state can be computed reliably in a perturbative expansion, including quantum effects. To capture quantum effects consistently, subleading terms in the derivative expansion [1] must
be included at the appropriate order, while the symmetries of the Schrödinger group must be preserved by the process of regularization and renormalization. The renormalization of the theory is itself under perturbative control so long as the ultraviolet cutoff $\Lambda$ is set at a scale parametrically higher than the infrared energy scale $E_{IR}$, and set lower than the bulk strong-coupling scale $E_{UV} \sim \mu \propto (X) \propto \langle \rho \rangle^{\frac{d}{2}}$:

$$E_{IR} \ll \Lambda \ll E_{UV} .$$  \hspace{1cm} (2.6)

In the limit $E_{IR} \ll E_{UV}$, this prescription (2.6) can be imposed consistently. Observables, including quantum effects, can be expressed as a series in $E_{IR}/E_{UV}$, with only a finite number of higher-derivative effective terms or loops in Feynman diagrams contributing at a given order in $E_{IR}/E_{UV}$. In particular, no higher-derivative term or quantum correction can contribute to the energy of the homogeneous ground state with constant charge density in infinite volume.

2.5 Leading-order quantities in infinite volume

On the grounds of dimensional analysis, the energy density of the charged homogeneous ground state in infinite volume is goes as

$$H \propto m^{-1} \rho^{\frac{d+2}{d}} \propto m^{\frac{d}{2}} \mu^{1+\frac{d}{2}} .$$  \hspace{1cm} (2.7)

The proportionality relations follow strictly from dimensional analysis, scale invariance, and the existence of a thermodynamic limit at finite chemical potential $\mu$ and zero temperature. The coefficients of proportionality may depend on the theory however, and are expressed in terms of the dimensionless, theory-dependent constant $\xi$ (2.1).

Under the conventions of [17], with two scalar species of fermions in dimension $d$ (see the discussion above (2.1)), the energy density of the charged homogeneous ground state of the unitary theory in infinite volume, in terms of the charge density $\rho$ and the Bertsch parameter $\xi$, is

$$H\left[ \text{charge density } \rho, \text{ interacting NRCFT} \right] = \frac{d}{\pi^2} \xi \epsilon_{FF} \rho .$$  \hspace{1cm} (2.8)

The (free) Fermi energy $\epsilon_{FF}$ in dimension $d$, via the $d$-dimensional (free) Fermi momentum $k_F$, is

$$\epsilon_{FF} = \frac{k_F^2}{2m} = \frac{1}{2m} \left( 2^{d-1} \pi^{d/2} \Gamma \left( \frac{d}{2} + 1 \right) \right)^{2/d} .$$  \hspace{1cm} (2.9)

The parameter $\xi$ can, in turn, be expressed in terms of the chemical potential

$$\mu = \frac{d+2}{d} \frac{d}{\rho^2} H\left[ \text{charge density } \rho, \text{ interacting NRCFT} \right]$$  \hspace{1cm} (2.10)

as

$$\xi = \frac{\mu}{\epsilon_{FF}} .$$  \hspace{1cm} (2.11)

In terms of $c_0$,

$$\xi = (2\pi)^{-1} \left( \frac{1}{d} \Gamma \left( \frac{d}{2} + 2 \right) c_0 \right)^{-\frac{2}{d}} ,$$  \hspace{1cm} (2.12)
\[c_0 = \frac{2}{\Gamma\left(\frac{d}{2}+2\right)} \left(2\pi\right)^{-\frac{d}{4}} \xi^{-\frac{d}{2}}.\]  

(2.13)

In \(d = 3\) spatial dimensions, this reduces to

\[
\xi \bigg|_{d=3} = 2^{(2/15)^{2/3}} -3_{0}^{2/3\pi^{4/3}}, \quad c_0 \bigg|_{d=3} = \frac{2^{5/2}}{15^{2\xi^{3/2}}}. 
\]

(2.14)

as in [1].

### 2.6 Leading-order quantities in a harmonic trapping potential

Let us now introduce the harmonic trap. The classical solution in the potential (2.3) is just

\[\chi = \mu t, \quad X = \mu - \frac{m\omega^2}{2}x^2,\]

(2.15)

defined within the radius of nonvanishing charge density,

\[X(\vec{x}) > 0, \quad |\vec{x}| < R \equiv \frac{\sqrt{\mu}}{\omega \sqrt{m}}.\]

(2.16)

Before working out the explicit formulae for the energies in the harmonic potential in \(d\) dimensions, we can start by deriving the scalings from dimensional analysis. The Lagrangian and Hamiltonian densities scale as \(m^{2/d} \mu^{1+2/d}\), and the size of the droplet goes as \(R \sim m^{-1/2} \omega^{-1} \mu^{1/2}\). So the total energy at leading order goes as \(E \sim m^{2/d} \mu^{1+2/d} R^{d} \sim \omega^{-d} \mu^{d+1}\).

Then the charge scales as \(Q \sim \frac{dE}{d\mu} \sim m^{-2/d} (\mu/\omega)^d\), so the chemical potential in terms of the charge is \(\mu \sim \omega Q^{d/2}\).

With these general scalings in place, we write the explicit leading-order formulae for the total Lagrangian, potential, energy, charge, etc., by integrating the leading-order Lagrangian density (2.4) over the region \(|\vec{x}| < R\). Using (2.15), (2.16), and the formula for the area of the unit \((d-1)\)-sphere,

\[A_{(d-1)} = \frac{2 \pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)},\]

we have

\[L = c_0 \left(2\pi\right)^{d/2} \left(\frac{\mu}{\omega}\right)^{d+1} \omega = \frac{2}{\Gamma\left(\frac{d}{2}+2\right)} \xi^{-\frac{d}{2}} \left(\frac{\mu}{\omega}\right)^{d+1} \omega,\]

(2.18)

at leading order, where we have used the identity (2.13) relating \(c_0\) to \(\xi\).

Differentiating with respect to \(\mu\), the leading-order relationship between the chemical potential \(\mu\) and charge \(Q\) in the isotropic harmonic trap with frequency \(\omega\), is

\[Q = c_0 \left(2\pi\right)^{d/2} \left(\frac{\mu}{\omega}\right)^{d} \left(\frac{\mu}{\omega}\right)^{d} = \frac{2}{\Gamma\left(\frac{d+1}{2}\right)} \xi^{-\frac{d}{2}} \left(\frac{\mu}{\omega}\right)^{d}.\]

(2.19)

For what follows, it is convenient to retain the inverse expressions for the chemical potential:

\[\mu = \omega c_0^{-1} \left(2\pi\right)^{-\frac{1}{2}} \left[\Gamma\left(d+1\right)\right]^{-\frac{1}{2}} Q^{\frac{1}{2}} = \omega \xi^{\frac{1}{2}} \left[\Gamma\left(d+1\right)\right]^{-\frac{1}{2}} Q^{\frac{1}{2}}.\]

(2.20)
Since $L$ scales as $\mu^{d+1}$, we have
\[
\mu Q = (d+1) L , \quad H = dL = \frac{d}{d+1} \mu Q , \quad L = \frac{1}{d+1} \mu Q = \frac{1}{d} H . \tag{2.21}
\]
Thus, the leading-order expression for operator dimensions in terms of $Q$ and either $c_0$ or the Bertsch parameter $\xi$ is easily obtained:
\[
\Delta_{\text{leading order}} (Q) = \frac{1}{\omega} E_{\text{leading order}} (Q) = \frac{d}{\omega} L_{\text{leading order}} (Q) = c_0 - \frac{d}{d+1} (2\pi)^{-\frac{1}{2}} \left[ \frac{\Gamma(d+1)}{\Gamma(\frac{d}{2}+2)} \right]^{\frac{1}{2}} Q^{\frac{d+1}{d}} = \xi^{\frac{d}{2}} - \frac{d}{d+1} \left[ \frac{\Gamma(d+1)}{2} \right]^{\frac{1}{2}} Q^{\frac{d+1}{d}} . \tag{2.22}
\]

2.7 Next-to-leading order (NLO) terms in the bulk effective action

Some higher-derivative effective terms contributing beyond leading order in the expansion in $E_{IR}/E_{UV}$ have been worked out in [1]. The two terms allowed by diffeomorphism invariance and conformal invariance, which contribute at NLO, are
\[
\mathcal{L}_{\text{NLO}} = \mathcal{L}_{c_1} + \mathcal{L}_{c_2} + (\text{other}) . \tag{2.23}
\]
Defining
\[
Y \equiv (\tilde{\partial}X)^2 , \tag{2.24}
\]
\[
Z \equiv [\tilde{\partial}^2 A_0 - \frac{1}{\ell^2 m}(\tilde{\partial}^2 \chi)^2] , \tag{2.25}
\]
we have
\[
\mathcal{L}_{c_1} \equiv c_1 m^{\frac{1}{2}(d-2)} X^{\frac{d}{2}-2} Y , \tag{2.26}
\]
\[
\mathcal{L}_{c_2} \equiv -c_2 d^2 m^{\frac{1}{2}(d-2)} X^{\frac{d}{2}-1} Z , \tag{2.27}
\]
while the “other” terms are further suppressed in the large-charge expansion. For the sake of clarity in the context of the existing literature, it is worth noting that this convention (namely, the assignment of $c_n$ coefficients to certain terms in the theory) aligns with that of [1], among others (albeit in general dimension $d$). Other authors [7] make different choices while retaining the $c_n$ notation, so some caution is in order.

Since the higher-order terms come with two additional spatial derivatives relative to the leading term, they are dressed to conformality, and so are of relative order $1/(R^2 X) = O(\omega^2/\mu^2)$ compared to the leading term. In terms of $Q$, using eqn. (2.20), they are of relative order $O(Q^{-\frac{1}{d}})$ compared to the leading term. As for edge operators, we shall see in Sec. 3.4 that the leading operator has a $\mu$-scaling $\mu^{\frac{d-1}{d+1}}$ after integration, which is the same size as the integrated subleading bulk operator in $d = 2$, and is strictly smaller in
In all, the relative size of corrections from subleading operators is always of relative order \( \omega^2/\mu^2 = O(Q^{-\frac{d}{2}}) \).

The authors of [1] justify the restriction to these terms by the symmetries of the theory. In addition to the obvious translational, scale, and galilean symmetries, [1] demands a conformally invariant and nonrelativistically-"generally-covariant" coupling to background fields, such as the gauge field and metric. In this paper we are going to focus primarily on the first-order contributions of the \( c_1 \) term to illustrate the universality of renormalization of NLO interactions in \( d = 2 \), though in section (4.6.1) we briefly consider the first-order contribution of the \( c_2 \) term as well.

2.8 Criteria for admissible terms

At this point, it is sensible to enumerate, at the broadest level, the criteria for allowed higher-derivative terms in the effective action. This will help to establish structural boundaries for the subsequent organization of allowed operators when we eventually address the space of viable counterterms at the droplet edge. Let us emphasize that, for the sake of completeness, we are including some criteria here that ultimately do not play such a central role in the classification of possible terms at the droplet edge. In constraining available edge terms, for instance, we need not appeal heavily to diffeomorphism invariance, or conformal invariance beyond rigid scale invariance.\(^2\) These considerations generally enter in very generic ways (like specifying that \( Z \) and \( Y \) are conformal primaries, say, or appealing to the fact that the dressing rule on the edge requires delta-function support expressed as \( \delta(X) \), rather than \( \delta(r-R) \), which is implicitly a consequence of diffeomorphism invariance). Even so, these criteria play a role, and it is useful to elucidate the extent of their influence.

2.8.1 Gauge invariance

The internal symmetry \( \chi \to \chi + \text{(const.)} \) of the \( \text{sw} \) theory can be understood as a particle number symmetry, which enforces mass conservation in a nonrelativistic system. This symmetry has a special role, appearing as a commutator in the algebra of generators of the Schrödinger symmetry [43]. Like any internal symmetry of a quantum field theory, one can couple it to a background gauge connection, even though it is a global symmetry. When coupled to a background gauge field, the global symmetry can be promoted to a ‘local gauge symmetry’ in the appropriate sense, so long as the conservation of the original current is exact. To be careful, by local gauge symmetry we do not mean that the action is invariant under a local symmetry transformation of the dynamical fields alone. Rather, the system has a local gauge symmetry in the ‘spurionic’ sense, in which the transformation acts both on the dynamical fields and the background gauge field.

The \( \text{sw} \) theory can be thought of as an effective theory of the phase variable \( \chi \) of the complex fermion field,

\[
\psi = e^{-i\chi} \sqrt{\psi^\dagger \psi}.
\] (2.28)

\(^2\)To be sure, in the bulk theory at NLO we need conformal invariance explicitly to eliminate one otherwise-admissible term.
In this role, $\chi$ transforms uniquely under a gauge transformation $\alpha(x, t)$:

$$\chi \rightarrow \chi + \alpha .$$

(2.29)

To build a covariant Lagrangian for $\chi$ under this local symmetry, we must promote partial derivatives to covariant derivatives,

$$\partial_\mu \chi \rightarrow \nabla_\mu \chi \equiv \partial_\mu \chi - A_\mu ,$$

(2.30)

with the nondynamical background field $A_\mu$ transforming as

$$A_\mu \rightarrow A_\mu + \partial_\mu \alpha .$$

(2.31)

### 2.8.2 Diffeomorphism invariance

Other than topological field theories, quantum field theories are never diffeomorphism invariant in the fully dynamical sense. That is, non-topological QFTs are never invariant under a general diffeomorphism transformation acting on the dynamical fields alone. Rather, in non-topological QFT, diffeomorphism invariance is meant in the same ‘spurionic’ sense discussed above, in which theories with global symmetries can be made covariant under local gauge transformations. I.e., the action for the dynamical fields and (non-dynamical) background metric is invariant under a combined transformation of both, rather than a transformation of the background metric alone. For relativistic quantum field theories, invariance in this sense under diffeomorphisms connected to the identity is equivalent to the conservation of the stress tensor.

Nonrelativistic theories are clearly not fully diffeomorphism-invariant, even in this spurionic sense; the formulation of a nonrelativistic theory intrinsically involves singling out a particular timelike direction. However, it was shown \[1\] that an even weaker version of the spurionic diffeomorphism symmetry usefully constrains the interactions of the system. By introducing a metric $g_{ab}$ on vectors pointing in a purely spacelike direction, one can covariantize the system in the canonical way under time-independent diffeomorphisms $\delta x^a = \xi^a(x)$ of the spatial coordinates $x^a$. By introducing additional terms, proportional to $\dot{\xi}^a$, in the transformations of the metric and background gauge connection $A_\mu$, one may covariantize the system further under spatial diffeomorphisms depending on time, $\delta x^a = \xi^a(x, t)$. (For details of the covariantization and useful elements of nonrelativistically-diffeomorphic tensor calculus, see \[1\].) It is in this sense that ‘diffeomorphism invariance’ is imposed as a symmetry of the large-charge effective action for $\chi$ in \[1\], which functions as a nontrivial constraint on terms in the EFT, beyond the constraints imposed by galilean symmetry and scale invariance.

### 2.8.3 Conformal invariance

Beyond spatial diffeomorphism invariance, one can also consider reparametrizations of the time coordinate, together with a time-dependent rescaling of the metric and gauge field,

$$\delta t = -t , \quad \delta g_{ab} = - g_{ab} ,$$

(2.32)
\( \delta A_i = 0 \), \( \delta A_0 = +A_0 \). \hfill (2.32)

The symmetry algebra of NRCFT contains rigid scale transformations and directly generalizes relativistic conformal symmetry to the nonrelativistic case. This nonrelativistic conformal group preserves the leading-order action (2.4) of the SW theory [1], and constrains its possible higher-derivative corrections as well.

2.9 The dressing rule for bulk operators

2.9.1 General comments on dressing rules

Implicit in the construction of the effective theory is a dressing rule, controlling which singular local functionals of the fields are allowed to appear as effective terms in the action. More precisely, this is a rule specifying which composite or composites of dynamical fields can appear as denominators. A dressing rule of some kind is always a logically necessary ingredient in the construction of any conformally invariant effective Lagrangian. Since conformal invariance is never broken directly by the dynamics, there is no external dimensional parameter that can cancel the conformal dimensions of numerators of higher-derivative terms in the effective action, so the denominators must be some sort of dynamical field or composite thereof.

Given the central role of the principle of the dressing rule in our subsequent analysis, it is helpful to mention a few contextual examples. One familiar case is that of the dressing rule in the effective actions for superconformal theories with vacuum manifolds when the conformal symmetry is spontaneously broken by an expectation value of the vacuum moduli. In simple cases, such as a vacuum manifold of complex dimension 1 in \( \mathcal{N} \geq 2 \) superconformal symmetry in \( D \geq 3 \), or \( \mathcal{N} \geq 1 \) superconformal symmetry in \( D \geq 4 \), the dressing rule is always uniquely determined. In such theories, numerators of vanishing \( R \)-charge and scaling dimension \( \Delta \) are always dressed with the factor

\[
(\phi^* \phi)^{-\frac{\Delta - D}{2\Delta \phi}}, \hfill (2.33)
\]

where \( \Delta \phi \) is the conformal dimension of the modulus \( \phi \). Similar rules hold in superconformal theories where the vacuum manifold is of the minimal dimension allowed by the unbroken and spontaneously broken symmetries of the system, such as superconformal gauge theories with rank-one gauge group [33–35, 44].

In simple nonsupersymmetric CFT with global symmetries, the dressing rule for the large-charge EFT is also frequently uniquely determined. For instance, in the large-charge EFT of the Wilson-Fisher critical \( O(2) \) model, the dressing field is \( |\partial \chi| \), where \( \chi \) is again the phase of the complex field; the same dressing rule holds for other large-charge limits in the same universality-class, such as the \( \mathbb{C}P(n) \) models at large topological charge [45]. Analogous dressing rules hold for the higher \( O(2N) \) models at large Noether charge [29], and for the worldsheet CFT of effective strings [10, 36, 46–48].

The general prescription for a dressing rule in a large-charge EFT is as follows: Consider all scalar conformal primaries \( \{X_i\} \), with conformal dimensions \( \Delta_i \), having nonzero expectation values in the charged ground state. To each such operator, assign a \( Q \)-scaling
exponent $\beta_i$, meaning that the vev of each $X_i$ scales as $Q^{\beta_i}$ in the ground state with large charge density $Q$. The fields $X_i$ that maximize the ratio $\beta_i/\Delta_i$ are the ones that participate in the dressing rule. If there is only one such field that is algebraically independent in the low-energy Hilbert space, then that field is the unique dressing field.

Ultimately, the rationale for the dressing rule is the principle of naturalness. The dressing field, raised to the power $1/\Delta$, gives the scaling of the energy $m_{\text{heavy}}$ of the heavy excitations above the large-charge ground state (which are integrated out). The mass formula must be invariant under all symmetries of the system and, at large charge, will be dominated by the invariant with the largest ratio $\beta_i/\Delta_i$.

To be sure, there are known examples in which the dressing rule is not unique, including $\mathcal{N} = 2$, $D = 4$ theories with gauge group of rank greater than 1, and the large-charge EFT of nonsupersymmetric theories with Abelian global symmetry of rank greater than 1 [19]. In such theories, the dressing field can involve an unknown function of dimensionless ratios of dressing fields. Such EFTs, though still having some predictive power, are more weakly constrained relative to theories with a unique dressing rule. For further general comments on dressing rules, see the related discussions in [10, 34, 47].

### 2.9.2 Dressing rule in the Son-Wingate EFT

In the nonrelativistic case the criterion is the same: Candidate dressing fields are all fields with a vev, with the maximum ratio of $Q$-scaling exponent to conformal dimension. For the Son-Wingate EFT [1], the field $X$ has conformal dimension $d+2$ and $\mu$-scaling exponent $\mu^1$, which works out to $Q^{2/d}$ in the homogenous ground state, or $Q^{1/d}$ in the harmonic trap. All other invariants have a lower ratio of $\mu$-scaling exponent to conformal dimension, making $X$ the unique dressing field (where it is nonvanishing). Thus, the dressing rule in the bulk states that the only singular functionals allowed to appear are singular powers of $X$ itself, dressing nonsingular monomials in $\chi$ and its appropriately covariantized derivatives.

This structure must change at the droplet edge, where $X$ vanishes. As we will show in Sec. 3, the operator structure of the Hamiltonian is reorganized on the droplet edge under a different dressing rule, with $(m^2 Y)^{1/3}$ playing the role of the dressing field for edge operators that $X$ plays for bulk operators.

### 2.9.3 Bipartite decomposition and $\mu$-scaling of bulk operators

The $X$-dressing rule implies a bipartite decomposition for operators in the EFT:

$$\mathcal{O} = X^{-\mu} \mathcal{O}_{\text{undressed}},$$

(2.34)

where $\mathcal{O}_{\text{undressed}}$ is a polynomial in $\chi$ (and its derivatives), plus terms involving couplings to gauge and metric background fields:

$$\mathcal{O}_{\text{undressed}} = \sum \prod A [\partial_{x}^{m_A} \partial_\mu^{n_A} \chi] + \text{(background couplings)}$$

(2.35)

where $\partial_\mu^\nu$ is short for a linear combination of operators of the form $\partial_{x_{i_1}} \partial_{x_{i_2}} \cdots \partial_{x_{i_m}}$. The dimension of the operator above is

$$\Delta_{\text{und}} \equiv \Delta(\mathcal{O}_{\text{undressed}}) = \sum A (m_A + 2n_A),$$

(2.36)
and the dimension of the corresponding dressed operator is
\[ \Delta(\mathcal{O}) = \Delta_{\text{undressed}} - 2p. \]  
(2.37)

An operator that can appear in the conformal EFT lagrangian density must be gauge invariant and must also be a conformal scalar primary of weight \( d + 2 \). This criterion determines the exponent \( p \) of the \( X \)-dressing:
\[ p = \frac{1}{2} \Delta_{\text{undressed}} - \left( 1 + \frac{d}{2} \right). \]  
(2.38)

### 2.10 Equations of motion

Varying the leading order action with respect to \( \chi \), we find
\[ 0 = -\partial_t (X \frac{\delta}{\partial t}) + \frac{1}{m} \partial_i \left[ (\partial^i \chi) X \frac{\delta}{\partial t} \right]. \]  
(2.39)

Note that the leading-order EOM allows us to eliminate temporal derivatives of \( X \) in favor of terms with just spatial derivatives. This is useful when one wants to count independent operators in the action beyond leading order. Linearizing the EOM in a trivial background, \( \chi = \mu t + \tilde{\chi} \), we have
\[ \ddot{\tilde{\chi}} = \frac{2\mu}{d m} (\nabla^2 \tilde{\chi}) + O(\tilde{\chi}^2), \]  
(2.40)

so the leading-order dispersion relation is
\[ \omega^2 = c_s^2 p^2, \quad c_s^2 = \frac{2\mu}{d m}. \]  
(2.41)

#### 2.10.1 NLO corrections to the dispersion relation

Note that the dispersion relation is not fixed by conformal symmetry. Higher-derivative corrections at NLO and beyond that are consistent with conformal symmetry affect the dispersion relation by terms proportional to the coefficients \( c_{1,2} \). For instance, the leading-order dispersion relation (2.41) is corrected \([1, 6]\) as
\[ \omega(p) = c_s |p| + (\delta \omega(p))_{\text{NLO}}, \]  
(2.42)

\[ (\delta \omega(p))_{\text{NLO}} \equiv -\frac{(2\pi)^{d/2} \Gamma(\frac{d}{2})}{\sqrt{2^d \xi^d}} \frac{d}{m^{p/2} \mu^{1/2}} \left( c_1 + \frac{d}{2} c_2 \right) \frac{\bar{p}^3}{m^{3/2} \mu^{1/2}}, \]  
(2.43)

at leading order in \( c_{1,2} \). Note that our conventions for the normalizations of \( c_{0,1,2} \) and \( \xi \) agree with those of \([1, 6]\)\(^3\), and we have also written the quantity \( d_0^2 \), defined in \([6]\), in terms of \( \xi \) rather than \( c_0 \) (see eq. (A.17) in the Appendix).

\(^3\)We, like \([1]\), set \( \hbar \to 1 \), while \([6]\) leave \( \hbar \) dimensionful.
2.10.2 First-excited spectrum in the harmonic potential

Now consider the EOM in the harmonic potential (2.3), which describes nonnegatively charged local operators in radial quantization in NRCFT \[43\]. In the harmonic potential \(A_0 = \frac{1}{2} \omega^2 \bar{x}^2\), the linearized EOM reads:

\[
0 = \ddot{\chi} - \frac{2}{md} (\mu - A_0) \nabla^2 \chi + \frac{1}{m} \omega^2 \bar{x} \cdot \vec{\partial} \dot{\chi} \tag{2.44}
\]

Using the dispersion relation (2.44) in the harmonic potential, the linearized excitation spectrum about the ground state was computed in \[7\]. The result is a first-excited spectrum labelled by integers \(n, \ell \geq 0\), with energies

\[
\epsilon(n, \ell) \equiv \omega \left( \frac{4}{d} n^2 + 4n + \frac{4}{d} \ell n - \frac{4}{d} n + \ell \right)^{\frac{1}{2}}. \tag{2.45}
\]

For \(\ell = 1\) and \(n = 0\), this has energy \(+\omega\), and for \(\ell = 0\) and \(n = 1\) it has energy \(+2\omega\), so these are precisely the conformal raising operators corresponding to \(\vec{\partial}_x\) and \(\partial_t\), respectively, under the NRCFT state-operator correspondence reviewed in sec. 2.3.

This spectrum gives the leading-order spectrum of primaries with dimension \(O(1)\) above the ground state. The sum over these linearized frequencies, times a factor of \(\frac{1}{2}\), also gives an \(O(Q^0)\) contribution to the ground state energy via the Coleman-Weinberg formula. Sometimes, as in \(d = 2\), there are no bulk or boundary counterterms of order \(Q^0\), and this contribution is universal after renormalization of bulk and boundary divergences.

The details of the renormalization of loop divergences is beyond the level of detail we will attempt in the present article, and we defer it to future work \[49\]. However, we will give a crude bound on the size of quantum effects of edge operators in sec. (4.7), and we will briefly discuss the implications of that bound in sec. 5.

2.11 Comments on bulk terms beyond NLO

We can use the \(X\)-dressing rule (2.34) for bulk operators to estimate the effect of higher-derivative terms in the action on observables at a given length, time, momentum, or energy scale. For a process characterized by a momentum scale \(p\), a NLO term makes a subleading contribution suppressed by \(p^2/\mu\) for every two additional spatial derivatives in the term,\(^4\) which come along with an additional \(X^{-1} \sim \mu^{-1}\). For a process characterized by a length scale \(L\), we have a factor of \((L \sqrt{m\mu})^{-1}\) for each additional derivative. So, for instance, an operator in the Lagrangian of the form

\[
L \ni \mathcal{O} \equiv \kappa (\partial_x X)^{2k}/X^{3k-\frac{d}{2}-1}
\]

contributes to correlators as

\[
\langle \cdots \rangle_{\mathcal{O}(\kappa^l)} = \langle \cdots \rangle_{\mathcal{O}(\kappa^0)} \times (L^2 m\mu)^{-k}, \tag{2.47}
\]

for a correlation function \(\langle \cdots \rangle\) characterized by the distance scale \(L\). We denote this property by saying that the operator \(\mathcal{O} \equiv (\partial_x X)^{2k}/X^{3k-\frac{d}{2}-1}\) has \(\mu\)-scaling \(\mu^{1+\frac{d}{2}-k}\), to be

\(^4\)Note again that one loses no generality by restricting to spatial derivatives; we can always eliminate time derivatives by the leading-order EOM.
understood as an abstract property of a term in the EFT action. Implicitly, the dimension will be compensated by powers of the infrared scale, which will be taken to lie parametrically below the UV scale set by $\mu$.

To further simplify the discussion, we can also leave out the factor $\mu^{1+\frac{d}{2}}$ of the $\mu$-scaling of the leading order term $X^{1+\frac{d}{2}}$, and speak of terms as having a relative $\mu$-scaling against leading-order quantities ($\mu^{-k}$ in the example at hand). The relative $\mu$-scaling gives us a simple way to further distinguish between NLO terms in the action in order of importance. Thus, both terms (2.24), (2.25) in the NLO action (2.23) have relative $\mu$-scaling $\mu^{-1}$, and should contribute to any (renormalized) observable with an additional power of $p^2/\mu$ or $1/(\mu L^2)$ for each insertion of a power of the NLO term.

This assignment of $\mu$-scalings to operators treats the IR scale as being independent of $\mu$. This is useful for infinite-volume computations and the local study of correlation functions, so we might refer to it as the “infinite volume” $\mu$-scaling if we want to distinguish such $\mu$-scalings from cases in which the natural infrared scale also has a nontrivial scaling with $\mu$ (as with global observables in the harmonic potential). Also note that the $\mu$-scalings we have discussed here apply only to renormalized observables. For bare observables, the $\mu$-scaling is the same but the momentum $p$ may be replaced with a power of the momentum cutoff or inverse distance cutoff. In EFT, this is normally relevant only for quantum processes, but for backgrounds with classical singularities, such as the edge of the density distribution in a harmonic potential, a cutoff may appear in the contributions of NLO terms to classical processes as well. It is this sort of UV-divergent contribution that is of particular importance for our present goals.

### 2.11.1 $\mu$-scalings of higher-derivative terms in the harmonic potential

From the discussion above, it is clear that the bulk expansion is an expansion in $1/(m\mu L^2)$, where $L$ is the characteristic distance scale of a process. We now use this general analysis to estimate the effect of higher-derivative terms on the ground state energy in the harmonic potential. In this background, the only available distance scale $L$ is the size of the droplet $R = \sqrt{2\mu}/(\omega\sqrt{m})$ on which the charge density is supported (2.16). So, in the harmonic potential, it is useful to assign a given NLO term a $\mu$-scaling that includes the $\mu$-scaling of the IR distance scale $R$ as well. Per the note of caution above, this is a different $\mu$-scaling than the “infinite volume” $\mu$-scaling we have already discussed, which applies to observables in which the relevant infrared scale is $\mu$-independent.

In the infinite-volume $\mu$-scaling, the leading-order action scales as $\mu^{1+\frac{d}{2}}$. In the harmonic-potential $\mu$-scaling, one also has to include the volume of the spatial integration region, which goes as $L^d = R^d \propto \mu^{\frac{d}{2}}$, so the harmonic-potential $\mu$-scaling of the leading-order action is $\mu^{d+1}$.

Next, we can consider subleading terms in the action. For every two additional spatial derivatives in a NLO term in the action, there is an additional factor of $1/(m\mu L^2) \propto \omega^2/\mu^2$ in its contribution to an observable, relative to the leading-order contribution. So the $\mu$-scaling of an integrated NLO term in the harmonic potential is two powers of $\mu$ less for every additional two spatial derivatives in the numerator.
Denoting the bulk harmonic-potential $\mu$-scaling of an (unintegrated) operator by $\alpha(O)$, we have
\[ \alpha(X) = 1 , \quad \alpha(Y) = 1 , \quad \alpha(Z) = 0 . \] (2.48)

Note that these $\mu$-scalings can be read off directly from the behavior of the classical solution as we take $\mu$ large while keeping the location $x$ of the operator fixed inside the droplet at $|x| < R$.

For conformally dressed operators, we have
\[ \alpha(X^{d+1}) = \frac{d}{2} + 1 , \quad \alpha(Y X^{d-2}) = \frac{d}{2} - 1 , \quad \alpha(Z X^{d-1}) = \frac{d}{2} - 1 . \] (2.49)

For integrated conformally dressed operators, we simply include the additional factor of the size $R^d \propto \mu^{\frac{d}{2}}$ of the droplet, to get
\[ \alpha_{\text{integrated}}(X^{d+1}) = d + 1 , \]
\[ \alpha_{\text{integrated}}(Y X^{d-2}) = d - 1 , \]
\[ \alpha_{\text{integrated}}(Z X^{d-1}) = d - 1 . \] (2.50)

In general, every pair of spatial derivatives in a conformal bulk term must come with an additional $(\omega^2/\mu^2)$ in the harmonic potential, so for conformal terms with no time derivatives we have
\[ \alpha_{\text{integrated}}(O) = d + 1 - \#(\partial_x) , \]
\[ \int d^d x \ O \sim [(\mu/\omega)^{d+1-\#(\partial_x)}] \times \omega . \] (2.51)

In $d = 2$, we see that for terms without spatial derivatives the only potential contributions larger than order $\mu^0 \leftrightarrow Q^0$ come from terms with two derivatives, which have already been examined in [1], and written here in (2.23),(2.24),(2.25). For $d = 3$, there are four-derivative terms that can contribute classically at order $Q^0$, the same order as the first quantum correction. The same is always true in any odd spatial dimension: there are always $(d+1)$-derivative terms that contribute at the same order as the one-loop vacuum energy, order $Q^0$.

In even spatial dimensions there are no bulk terms contributing at order $Q^0$ to the renormalized vacuum energy; in some cases, such as $d = 2$, we will see further that there are no edge terms that ever contribute at order $Q^0$ either, indicating that the order $Q^0$ term is universal and calculable in $d = 2$. 

- 17 -
3 Structure of local operators at the droplet edge

3.1 General comments

As noted earlier, the phenomenon of localized singularities in classical solutions describing large-quantum-number states is ubiquitous in the subject of large-quantum-number, or LQN, expansions. The primary difference between the droplet edge and a fixed-position boundary is that the position of a true boundary is just that: it is fixed. Indeed, the dynamical fluctuations of the droplet edge are incorporated in the fluctuations of the $\chi$ field, within the regime of validity of the large-charge EFT.

In the case of the open effective string with Neumann boundary conditions, it was understood [10] that the proper way to interpret these singularities is not as a signal that the EFT breaks down altogether, but that the organization of operators changes, wherein a different dressing rule applies for the denominators of operators in the singular region, relative to that in the bulk.5 Qualitatively, this can be understood in terms of the parametric scaling of energies of heavy excitations at large charge, which are integrated out. In the singular region, these heavy modes are parametrically lighter than the corresponding heavy modes in the bulk, but they still have energies scaling with a positive power of the charge or chemical potential. For the CFT to break down, the energies of the heavy modes would have to go to zero, but they do not. In the singular region, the dressing field is simply proportional to the local expression describing the parametric energy scale of the heavy modes. In practice, this means the dressing field is always a power of the EFT operator with the largest $Q$-scaling (or $\mu$-scaling) per conformal dimension, both in the bulk and in the singular region.6

Despite the dynamical nature of the droplet edge, as opposed to a fixed boundary, we can still treat the droplet edge in EFT because the fluctuations in its position admit an energy cost that can be estimated according to the phonon energy spectrum (2.45). These energies are $O(\omega^1\mu^0)$ for phonons whose angular momenta $\ell$ and radial wavenumbers $n$ are $O(1)$. The low-energy Hilbert space can accommodate only some fixed number of such excitations (depending on where one places the cutoff $\Lambda$), so the fluctuations of the coordinate position of the boundary are actually rather small. Thus, we are safe treating the position of the boundary as a semiclassically fixed entity, so long as we incorporate the effects of the fluctuations of its position systematically in perturbation theory, while maintaining all the symmetries of the system.

The formal method for doing this generalizes the way we treat boundary or defect operators. For a boundary operator, we write $\Delta \mathcal{L} = \delta(\sigma_1 = \sigma_1^{(0)}) \mathcal{O}(\sigma)$, if $\sigma$ are the coordinates and the boundary is normal to the $\sigma_1$ direction. For a droplet-edge operator, we accommodate the fluctuations of the droplet edge by letting the argument of the $\delta$-

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5For a related discussion in the case of fold singularities in effective strings, see [48].

6When such an operator is unique, as in [9, 10, 18, 19, 28, 29, 33–41, 44], there is a unique choice of dressing field for the large-charge EFT. The dressing operator is not always unique, however, as in the case of a generic CFT with $U(1)^2$ symmetry [19]. In these cases the EFT still has predictive power, but somewhat less so absent additional information.
function be the dynamical field $X$ rather than a coordinate:

$$\Delta L_{\text{edge}} = \delta(X) O(x) \ .$$  

(3.1)

The large-$Q$ expansion is of course only an asymptotic expansion, but in this expansion we can understand $\delta(X)$ concretely as a boundary operator at the classical droplet edge $R = \sqrt{2\mu/(\omega \sqrt{m})}$, plus a series of higher-derivative boundary operators that are parametrically suppressed at large $\mu$. While expressions like $\delta(X)$ appear to be singular, we proceed with the understanding that the EFT is regularized and renormalized, and all local operators are defined at an energy scale $\Lambda$ that is parametrically below the scale set by the chemical potential. We give a concrete expansion of $\delta(X)$ in fluctuations in sec. 3.5.

### 3.2 The $Y$-dressing rule for edge operators

In sec. 2.9 we discussed the bulk dressing rule. Namely, the only field allowed to appear in the (bulk) effective action raised to negative or fractional powers is the composite $X$ itself. The appearance of such functions of $X$ are innocuous so long as one does not try to use the effective theory in a regime where $X$ is small compared to the relevant infrared energy scale. Of course, this is precisely the problem for describing the complete system with finite droplet extent, since $X$ vanishes at the edge of the trapped droplet. Thus, we need to adopt an applicable dressing rule at the edge. The dressing operator for near-edge operators should be the operator with the lowest $\mu$-scaling per unit conformal dimension that is nonvanishing at the edge. This is the operator $Y$, defined in eq. (2.24):

$$Y \equiv (\partial X)^2 \ .$$  

(3.2)

The $Y$-dressing rule at the edge thus allows terms such as $O/Y^p$, where $O$ is a polynomial in dynamical and background fields and their derivatives, but not terms such as $O/Z^q$, with $Z$ defined in eqn. (2.25) above.

To understand why the former ($O/Y^p$) is allowed for a nonsingular droplet edge but not the latter ($O/Z^q$), the reasoning is similar to the reasoning for dressing rules in other cases, including the $X$-dressing rule for bulk operators. As with the bulk theory, the principle can be understood as a consequence of naturalness: The denominators of allowed terms should be powers of the energies $E$ of unknown integrated-out excitations, which should be presumed to have the most generic possible energy formula allowed by the symmetries. The formula for $E = E(X, Y, Z, \cdots)$ can depend on the local fields in a general way, consistent with conformal invariance, which forces $E$ to be a scalar primary of weight 2, but with no other constraints. At a point in the bulk, where $X$ is nonvanishing, a generic energy formula $E = E(X, Y, Z, \cdots)$ will be dominated by the term of largest possible $\mu$-scaling per unit conformal dimension, which is $X$. Only when $X$ is vanishing should the formula for $E$ be dominated by the invariant with the next largest $\mu$-scaling per unit conformal dimension, i.e., $Y$.

As we shall see, the $Y$-dressing rule at the edge has consequences for the $\mu$-scaling laws associated with edge operators. Namely, operators such as $Y^pZ^q$ are not allowed at the edge unless $q$ is a nonnegative integer. This can be understood easily on physical grounds:
Consider the ground state of the system in a constant electric field $E$ in the $x_1$-direction. For such a system the ground state solution has $X = E(x_1 - x_1^{(0)})$. The physics of the infinite, flat droplet edge at $x_1 = x_1^{(0)}$ is locally identical to the physics of the circular edge in the harmonic potential, but the $Z$-invariant vanishes identically everywhere in the ground state of the constant electric field. The properly chosen dressing rule must give an organization of operators that makes sense in the neighborhood of any edge that is locally of a generic type with well-defined unit normal vector. An organization of operators dressed with singular powers of $Z$ is ill-defined at the edge of the charge distribution in a constant electric field, so we conclude that operators dressed with fractional or negative powers of $Z$ are not allowed. Similar considerations forbid any dressing field other than $Y$ for a droplet edge of nondegenerate type with well-defined normal vector at the edge.\footnote{Of course, one can always consider more general singular configurations in which the background electric potential is organized so that the edge has a cusp or corner, or where an open set of the locus $X = 0$ has a double zero, or otherwise where there exists an ill-defined normal vector at the edge. In such geometries, the $Y$ field vanishes at the edge and the organization of operators would be different again. In the most generic such situations we would expect a $Z$-dressing rule to apply. However interesting to contemplate, we do not consider such situations in the present article.}

### 3.3 The tripartite structure of droplet-edge operators

The discussion above thus gives a general recipe for constructing edge operators:

- Start with a nonsingular bulk operator with no undifferentiated $X$ fields. For lack of a better name, let us say that this plays the role of an undressed numerator.
- Append a factor of $\delta(X)$, so that the term is supported only at the edge of the droplet.
- Further append powers of $Y$ to make the operator a conformal primary of marginal conformal weight.

That is, edge operators can be thought of as being generated by terms with a simple tripartite structure:

$$
O_{\text{edge}} = \delta(X) Y^{-p} O_{\text{numerator}}.
$$

Let us now discuss each sector of this structure. First, $O_{\text{numerator}}$ is an arbitrary monomial in $\chi$ and its derivatives, modulo positive powers of $X$ and terms that vanish by the EOM. In fact, the properties of $O_{\text{numerator}}$ are just inherited from the analogous requirements pertaining to the operator dressing rule in the bulk theory (see Sec. 2.9), so we need not revisit that discussion in full detail. The other two parts of the dressing structure are unique to the edge theory, and they deserve further commentary.

#### 3.3.1 The $\delta(X)$ factor

It is convenient to consider the $\delta$-function as an explicit part of the dressing rule, since all edge terms must appear with precisely one power of $\delta(X)$. Formally, the operator $\delta(X)$ is a conformal primary of nonrelativistic scaling dimension $-2$, where the spatial partial
derivative is counted with weight 1 and the time derivative is counted with weight 2. The \(\delta\)-function of an object always carries the negative of the conformal dimension of that object, in any kind of semiclassical expansion. Of course, going to higher-order in insertions of the edge-term itself will force us to regularize the \(\delta\)-function, but at first order in the insertion of the edge term the regularization is irrelevant.

The \(\delta(X)\) factor is more intuitive if one uses the inverse chain rule for \(\delta\)-functions
definition to rewrite it as a \(\delta\)-function in \(X\)-space, multiplied by an operator-valued measure factor. That is, the \(\delta\)-function of an operator \(\hat{O}\) can be understood as

\[
\delta(\hat{O}) = \delta(r - \hat{r}_0) \left| \frac{\partial}{\partial r} \hat{O} \right|^{-1}
\]

where \(r\) is simply a coordinate label for local operators, and \(\hat{r}_0 = \hat{r}_0(\theta, t)\) is an operator denoting the radial position of the vanishing locus of \(\hat{O}\) at a particular angular direction \(\theta\) and time \(t\). For all purposes in the present paper, we will be considering vacuum expectation values of the edge operators. In the classical approximation in the vacuum, \(\hat{r}_0(\theta, t)\) is a \(c\)-number, independent of \(\theta\) and \(t\), equal to the classical radius of the droplet:

\[
\hat{r}_0(\theta, t) \simeq R = \frac{\sqrt{2\mu}}{m\omega}.
\]

Thus, in this approximation, we retrieve via (2.24):

\[
\delta(X) \simeq \delta(X_{\text{classical}}) = \left| \frac{\partial}{\partial x} X_{\text{classical}} \right|^{-1} \delta(|x| - R)
\]

\[
= Y^{-\frac{1}{2}} \delta(|x| - R)
\]

\[
= \frac{1}{m\omega^2 R} \delta(|x| - R) .
\]

In section 3.5, we will see how to go beyond this approximation to incorporate quantum fluctuations as operator contributions to \(\delta(X)\).

### 3.3.2 The Y-dressing

Finally, as discussed above, naturalness (among other considerations) dictates that droplet-edge terms be dressed to conformality with powers of \(Y\), which itself has conformal dimension 6. For an edge operator \(O_{\text{edge}}\) appearing in the conformal EFT Lagrangian density, with the \(Y\)-factor appearing as \(Y^{-p}\), conformal invariance of the perturbing Hamiltonian will thus force \(p = -\frac{1}{6}(d + 4 - \Delta_{\text{und}})\), where \(\Delta_{\text{und}}\) is the dimension of the undressed numerator factor \(O_{\text{numerator}}\). This exponent is chosen so that the total (momentum) scaling dimension of \(O_{\text{edge}}\) is equal to \(d + 2\), with the scaling dimension of \(\delta(X)\) counted as \(-2\). So, altogether, the droplet-edge terms are of the form

\[
O_{\text{edge}} = \delta(X) \left( m^2 Y \right)^{\frac{1}{6}(d + 4 - \Delta_{\text{und}})} \hat{O}_{\text{numerator}} .
\]

In the following sections we will calculate the power law with which an edge operator of the form (3.7) contributes. I.e., we want to characterize the exponent \(\gamma\) such that

\[
\langle \int d^d x \hat{O}_{\text{edge}} \rangle \lesssim \mu^\gamma
\]
at large charge.
3.4 Example: A simple droplet-edge operator and its $\mu$-scaling

The simplest droplet-edge operator is the case where the numerator is simply the identity. To make the operator conformally invariant, we need to dress it with $Y$ so that the total weight is marginal:

$$O_{\text{edge}} = \delta(X) Y^{\frac{d+1}{4}} . \quad (3.8)$$

At the classical level, we can simply replace $X$ with its classical profile. Per the discussion above, translating the $\delta$-function of $X$ into a $\delta$-function of position in the radial direction, we obtain

$$\delta(X) = |x|^{-\frac{1}{2}} \delta(|x| - R).$$

The classical ground-state value of $Y$ evaluated at $r = R$ is

$$Y_{\text{classical}} |_{|x|=R} = m^2 \omega^4 R^2 = 2 m \omega^4 \mu . \quad (3.9)$$

The $\mu$-scaling of $Y$ is $\sim \mu^{+1}$, and the $\mu$-scaling of $\delta(X)$ is $Y^{-\frac{1}{2}} \sim \mu^{-\frac{1}{2}}$, so, in total, the $\mu$-scaling of $O_{\text{edge}}$ is $\mu^{\frac{d+1}{6}}$. Since the size of the integration region is $R^{d-1} \propto \mu^{\frac{d-1}{2}}$, the $\mu$-scaling of the integrated operator is $\mu^{\frac{d-1}{3}}$.

3.5 Expansion in phonons

To go beyond the classical level, we need to understand the meaning of the expression $\delta(X)$ as interpreted in EFT. Nonperturbatively, there may or may not be some ambiguity in the definition of $\delta(O)$, but it can always be given a meaningful definition in an effective theory with a specified regularization procedure. For instance, if $O$ vanishes at more than one value of $r$ at a given time and angular direction, $\hat{r}_0(\Omega, t)$ can be defined to be the largest value of $r$ at which $\hat{O}(r, \Omega, t) = 0$, or some variation on that rule. In semiclassical perturbation theory around a classical ground state for which $\hat{O}$ vanishes exactly once, at a distance $r = r_0^{[O]}$ from the origin, the operator $\hat{r}_0^{[O]}$ can be represented as

$$\hat{r}_0^{[O]} = r_0^{[O]} + \hat{\delta} r_0^{[O]} , \quad (3.10)$$

where $\hat{\delta} r_0^{[O]}$ has a vanishing expectation value in the vacuum, and its fluctuations are suppressed in the semiclassical expansion.

This situation describes exactly the case $\hat{O} = X$ in the harmonic trap, where the classical vanishing radius $r_0^{[X]} = \langle \hat{r}_0^{[X]} \rangle$ is just the droplet radius $R = \sqrt{2\mu}/(\omega \sqrt{m})$, and the quantum fluctuations of $\hat{r}_0^{[X]}$ in the ground state are suppressed by powers of the total fermion charge $Q$. In perturbation theory about the ground state of the harmonic trap at large $Q$, the fluctuations around the classical configuration of $X$ are small, so the fluctuations $\hat{\delta} r_0^{[X]}$ of $\hat{r}_0^{[X]}$ about the vev $r_0^{[X]} = \langle \hat{r}_0^{[X]} \rangle = R$ are also small. We can use the suppression of fluctuations $\hat{\delta} r_0^{[X]}$ to represent the droplet-edge operators as a series of operators supported exactly at the classical vanishing radius.

To make this concrete, we can break up the operator $\hat{X}$ into vev and fluctuations,

$$\hat{X} = X_{\text{cl}} + \delta \hat{X} , \quad (3.11)$$
where $X_{\text{cl}}$ is the classical vev of $X$ at some given chemical potential in the harmonic trap. We can then write the $\delta$-function as

$$
\delta(X) = \delta(X_{\text{cl}} + \tilde{X}) \\
= \delta(X_{\text{cl}}) + \delta'(X_{\text{cl}}) \tilde{X} + \frac{1}{2} \delta''(X_{\text{cl}}) (\tilde{X})^2 + O\left( (\tilde{X})^3 \delta''(X_{\text{cl}}) \right).
$$

(3.12)

First, the zero-fluctuation term is simply

$$
\delta(X_{\text{cl}}) = |\partial_r X_{\text{cl}}|^{-1} \delta(|x| - R).
$$

(3.13)

Now, to understand the localized distribution appearing in the one- and two-fluctuation terms, we can use various $\delta$-function identities and the form of the classical solution

$$
X_{\text{cl}}(r) = \left< X(r) \right> = \mu - \frac{m^2}{\omega} r^2,
$$

(3.14)

to derive the expansion of the $\delta(X)$ operator in fluctuations:

$$
\delta(X) = \delta(|x| - R) \left[ \frac{1}{m \omega^2 R} - \frac{1}{m^2 \omega^4 R^3} \tilde{X} + \frac{1}{m^2 \omega^4 R^2} \tilde{X}_r + \frac{3}{2 m^3 \omega^6 R^5} \tilde{X}^2 - \frac{3}{2 m^3 \omega^6 R^4} \partial_r \left[ \tilde{X}^2 \right] + \frac{1}{2 m^3 \omega^6 R^4} \partial^2_r \left[ \tilde{X}^2 \right] \right]_{r=R}

+ \delta'(|x| - R) \left[ - \frac{1}{m \omega^2 R^2} \tilde{X} + \frac{3}{2 m \omega^4 R^3} \tilde{X}^2 - \frac{1}{m^2 \omega^6 R^5} \partial_r \left[ \tilde{X}^2 \right] \right]_{r=R}

+ \delta''(|x| - R) \frac{1}{2 m^3 \omega^6 R^5} \left[ \tilde{X}^2 \right]_{r=R} + O\left( \tilde{X}^3 \right).
$$

(3.15)

So, we have traded the abstract expression $\delta(X)$ for an explicit expression in terms of dynamical quantum fields evaluated at $r = R$, times purely $c$-number distributions in the radial direction, with calculable coefficients.

To be sure, we are concerned in this paper with simply analyzing the energy of the classical ground state in the trap, with all bulk and edge operators allowed in the action that are consistent with the symmetries. However, a few comments beyond the present application are useful for context:

- While the above is all at the classical level, this representation of $\delta(X)$ is well-defined at the quantum level without further information, as long as we are working to first-order in insertions of the droplet-edge fields.

- Even at classical level, working beyond first order in the coefficients of the edge-operators requires we resolve the singularity of the distributions $\delta(|x| - R), \delta'(|x| - R), \cdots$ by giving the $\delta$-functions a finite width $\sim (m \Lambda)^{-\frac{1}{2}}$, where $\Lambda$ is the Wilsonian energy cutoff satisfying our double hierarchy

$$
E_{IR} \ll \Lambda \ll \mu = E_{UV}.
$$

(3.16)
• We can also use this Wilsonian type of analysis to determine the parametrization of the boundary conditions for $\chi$ at the droplet edge. Even at semiclassical level, this is useful for a systematic analysis of the phonon energy spectrum in the $1/Q$ expansion. Beyond leading order in $Q$, the parametrization of boundary conditions also has a hierarchical expansion in negative powers of $Q$; the parameters in the space of boundary conditions are related to the parameters in the space of allowed droplet-edge operators.

3.6 Classical $\mu$-scaling and $Q$-scaling of the droplet-edge operators

We have now seen that the edge operators have a well-defined operational meaning beyond the purely classical analysis of the ground state. In particular, with our understanding of fluctuations under control, we can proceed to compute the leading-order effects of various boundary terms in the $1/Q$ expansion of the energy of the ground state in the harmonic potential, in analogy with the bulk $\mu$-scalings $\alpha(O)$ we defined for bulk operators in the harmonic potential.

3.6.1 $\mu$-scalings of (unintegrated) dressed edge operators

Using our Jacobian (3.4),(3.6), we have

$$\delta(X) = \left| \frac{dX}{dr} \right|^{-1} \delta(|x| - R) = Y^{-\frac{1}{2}} \delta(|x| - R) ,$$

plus fluctuation terms, which we ignore when computing the classical ground state energy.

To be clear, the scaling behavior of edge operators is distinct from bulk operators, insofar as we are defining the $\mu$-scaling of the former by their behavior as we scale up $\mu$ and evaluate the operator $O(x)$ at a point on the edge $|x| = R = \sqrt{2\mu}/(\omega\sqrt{m})$, which itself grows with $\mu$. That is, we can assign operators a boundary $\mu$-scaling exponent $\beta(O)$ that can be different from its bulk $\mu$-scaling exponent. For instance, the operator $Y$ in the trap background is proportional to $m^2\omega^4r^2$, so at fixed $r$ it scales as $\mu^0$. However, the bulk $\mu$-scaling exponent involves evaluating $Y$ at $|x| = R = \sqrt{2\mu}/(\omega\sqrt{m})$, which means the boundary $\mu$-scaling exponent of $Y$ is $+1$,

$$\beta(Y) = +1 .$$

(3.18)

As for $Z$, its classical value is simply proportional to $m\omega^2$, independent of position, so its boundary $\mu$-scaling exponent vanishes (as does its bulk $\mu$-scaling exponent):

$$\beta(Z) = 0 .$$

(3.19)

3.6.2 $\mu$-scalings of integrated edge terms

When integrating an edge operator, we must first take care to estimate the $\mu$-scaling of the $\delta$-function contribution $\delta(X)$. As noted above in eqn. (3.17), the translation of $\delta(X)$ into a $\delta$-function of radial position comes with an extra measure factor of $Y^{-\frac{1}{2}}$,

$$\delta(X) = Y^{-\frac{1}{2}} \delta(|x| - R) ,$$

(3.20)
\[ \beta[\delta(X)] = -\frac{1}{2} \beta(Y) = -\frac{1}{2} . \] (3.21)

Generally, edge operators are integrated over a \((d-1)\)-dimensional sphere of radius \(R\), and \(R\) itself scales as \(\mu^{\frac{1}{2}}\) (2.16). If we introduce the operation \(\mathcal{I}\), which promotes an undressed operator to its dressed counterpart and performs the appropriate integration, we can compactly encode the sequence of manipulations needed to extract the \(\mu\)-scalings of interest for edge operators. Namely, following the edge-operator decomposition in eqn. (3.7), we obtain

\[ \mathcal{I}[O_{\text{undressed}}] \equiv \int d^d x O_{\text{edge}} = \int d^d x \delta(X) (m^2 Y)^{\frac{d+1-\Delta_{\text{und}}}{6}} O_{\text{undressed}} , \] (3.22)

which has the classical value

\[ \langle \mathcal{I}[O_{\text{undressed}}] \rangle = A_{d-1} R^{d-1} (m^2 Y)^{\frac{d+1-\Delta_{\text{und}}}{6}} \langle O_{\text{undressed}} \rangle \]

\[ \propto \mu^{\frac{2d-1}{3} - \frac{\Delta_{\text{und}}}{6}} \langle O_{\text{undressed}} \rangle . \] (3.23)

The brackets in the equation above denote the classical value of the undressed operator evaluated at the boundary of the droplet edge in the classical ground state solution in the harmonic trap. If the \(\mu\)-scaling of the undressed operator is

\[ \langle O_{\text{undressed}} \rangle \sim \mu^{\beta_{\text{und}}} , \] (3.24)

then the integrated and dressed operator scales as

\[ \langle \mathcal{I}[O_{\text{undressed}}] \rangle \sim \mu^{\gamma_{\text{total}} \equiv \mu^{\frac{2d-1}{3} + \gamma_{\text{term}}}} , \] (3.25)

where we define the exponents to be

\[ \gamma_{\text{term}} \equiv \beta_{\text{und}} - \frac{1}{6} \Delta_{\text{und}} , \]

\[ \gamma_{\text{total}} \equiv \frac{2d-1}{3} + \sum_a \gamma_{\text{term}} a . \] (3.26)

The exponent \(\gamma_{\text{term}}\) can be thought of as the effective \(\mu\)-scaling exponent of the dressed operator \((m^2 Y)^{\frac{d+1-\Delta_{\text{und}}}{6}} O_{\text{undressed}}\), which is just the undressed operator with a power of the boundary dressing appended to bring the dressed operator to zero scaling dimension.

This recipe for constructing conformal edge operators and computing their scaling at large quantum number is essentially identical to that of the analysis for bulk [47] and boundary [10] operators in effective string theory. As discussed above, and now made explicitly clear, the one difference is that the position of the edge is operator-valued rather than fixed, and so instead of a delta function of a fixed coordinate position, as in [47], we have an operator-valued \(\delta\)-function \(\delta(X)\) to restrict the support of the operator in the effective theory of the finite droplet in NRCFT.
3.6.3 Classification of Lagrangian perturbations by their classical $\mu$-scaling

To classify droplet-edge terms in the action by their $Q$-scaling (or equivalently their $\mu$-scaling), we would like to show that there are only a finite number of undressed operators $O_{\text{und}}$ with $\beta_{\text{und}} - \frac{\Delta_{\text{und}}}{6}$ greater than any given number, where $\Delta_{\text{und}}$ and $\beta_{\text{und}}$ are the conformal dimension of $O_{\text{und}}$ and its boundary $\mu$-scaling, respectively. Let us start by considering what type of undressed $O_{\text{und}}$ we can make from the derivatives of $X$. To every undressed operator $O_{\text{und}}$, we assign the combination $\gamma \equiv \beta_{\text{und}} - \frac{1}{6} \Delta_{\text{und}}$.

| $O_{\text{undressed}}$ | $\beta_{\text{und}}$ | $\Delta_{\text{und}}$ | $\gamma_{\text{term}} \equiv \beta_{\text{und}} - \frac{1}{6} \Delta_{\text{und}}$ |
|-------------------------|-----------------------|-----------------------|-------------------------------------------------|
| $1$                     | $0$                   | $0$                   | $0$                                             |
| $\partial_x X$          | $+\frac{1}{2}$        | $+3$                  | $0$                                             |
| $(\partial_x)^2 X$      | $0$                   | $+4$                  | $-\frac{2}{3}$                                 |

It helps to start with a table of some simple objects (see Tab. 1). In this table we have included only objects with nonzero vevs in the classical vacuum solution, so that their classical $\mu$-scaling exponent $\beta_{\text{und}}$ is well-defined. Note that nothing contributes with positive $\gamma_{\text{term}}$.

It is especially worth noting what does not appear in Table 1. We have omitted, for instance, $\partial_t X$ and $\partial_x^\ell X$ for $\ell \geq 3$, because these vanish classically. We have also omitted background terms proportional to the spatial magnetic field $F_{ij}$ and its derivatives, because the magnetic field is zero in the background defining the harmonic trap. Background terms proportional to the electric field and its derivatives can also be left off. The electric field $E_i$ is nonvanishing in the harmonic trap, but the spatial gradient $X_{,i}$ of the classical solution $X$ is equal to $E_i$, so we have the relation $E_i = X_{,i} + (\text{vevless operators})$. Finally, we also omit spatial gradients of $\chi$. The term $\chi_{,i}$ itself is not gauge-invariant, and its gauge-invariant completion $\nabla_i \chi = F_{ij}$ vanishes identically in this background. We summarize these omissions in Table 2.

| $O$                                           | Classical vacuum solution of the harmonic trap:                  |
|-----------------------------------------------|-----------------------------------------------------------------|
| $\partial_t X$                               | $X$ is constant in time                                           |
| $\partial_x^\ell X, \ell \geq 3$             | $X$ is quadratic in $x$                                           |
| $F_{ij}$                                      | vanishes in this background                                      |
| $E_i$                                         | $E_i = X_{,i} + (\text{vevless operators})$                      |
| $\nabla_i \cdots \nabla_{ik} \chi$           | zero because $\nabla_i \chi = F_{ij} = 0$                      |

So, put simply, the total $\mu$-scaling exponent of dressed edge operators is

$$\gamma_{\text{total}} \equiv \frac{2d-1}{3} + \sum_a \gamma_a = \frac{2d-1}{3} - \frac{2}{3} \#(X_{,ij}) = \frac{1}{6} \left( 2[d - \#(X_{,ij})] - 1 \right). \quad (3.27)$$
Of course, we can translate this into a $Q$-scaling, since $\mu \propto Q^{\hat{d}}$ (by eqn. (2.20)), so the $Q$-scaling of integrated edge terms is

$$\mathcal{I} \left[ \prod_{\alpha} C^{(\alpha)}_{\text{und}} \right] \propto Q^{\gamma_{\text{total}}}.$$  \hfill (3.28)

Per eqn. (3.27), the important arithmetic is contained in the number of $X_{ij}$ in the operator, and, in particular, the only droplet-edge operators with nonnegative $\mu$-scaling at the classical level are those with

$$\#(X_{ij}) < d.$$  \hfill (3.29)

Note that only a finite number of operators exist with scaling exponent $\gamma_{\text{total}}$ greater than a specified floor $\gamma_{\text{min}}$. In particular, the classification above reduces the set of independent Wilson coefficients contributing classically at a given order to a finite basis. Any operator with a nonzero classical value can be built out of the three operators in Table 1, modulo operators that vanish classically. The identity is trivial algebraically, so we only have to count powers of $X_i$ and $X_{ij}$. However, any pair of $X_i$ in which the vector indices are contracted with each other gives a factor of $Y$, which is effectively trivial because it is absorbed by the $Y$-dressing. The only way to make algebraically independent boundary operators with nonzero classical values in the ground state configuration is then to include powers of $X_{ij}$, contracting them either with the metric, with $X_i$, or with each other. Each power of $X_{ij}$ contained in the numerator, has a $\gamma$-contribution of $\gamma = -\frac{d}{3}$. So there are only a finite number of terms that can be constructed with the total $\gamma$ exponent greater than any fixed amount: Any lower limit on $\gamma_{\text{total}}$ sets an upper limit on the number of $X_{ij}$ that can be included in a term. With any fixed number of $X_{ij}$, there are only a finite number of invariant terms that can be constructed.

### 3.7 A sufficient condition for allowed edge operators

So far in this section we have performed a coarse analysis of scaling laws for edge operators, with the aim of bounding the number of operators that can appear at a given order. We have not so far given any sort of lower bound for the number of independent operators with a given $\mu$-scaling at any order, and we have not explicitly constructed any specific operators that are allowed, beyond the simplest example $m^{-2} \delta(X)(m^2 Y)^{\frac{2}{d-2}}$. It is of course important to be sure that one can reliably construct all edge operators at a given order; the derivation of sum rules and other relations among terms in the large-$Q$ expansion of observables, depends on knowing the number of independent Wilson coefficients at a given order, including edge Wilson coefficients.

We do not attempt it here because of certain subtleties involved in the criteria for gauge invariance and conformal invariance at the edge. The naive criteria for invariance are plausibly too strict in the presence of the $\delta$-function that appears in edge terms. For instance, there may be local operators on which a conformal lowering operator produces a term proportional to an undifferentiated $X$. Such a term would not be primary in the bulk, but is primary when dressed with a $\delta(X)$ and turned into an edge operator. Due to
these and related subtleties, we defer a fuller analysis of sufficient conditions for allowed edge operators to future work.

However there is one recipe for constructing invariant edge operators that already generates examples contributing certain \( \mu \)–scalings to the energy, that appear to have gone unnoticed in the literature so far. Since bulk operators are already gauge-invariant and conformally invariant, we can always start with a bulk operator and construct an edge operator by the following recipe:

- Start with a bulk term in bipartite form (2.34), \( O_{\text{bulk}} = O_{\text{undressed}} X^{1+\frac{d}{2} - \frac{1}{2} \Delta_{\text{und}}} \). With no loss of generality we can assume \( O_{\text{undressed}} \) contains no undifferentiated powers of \( X \).

- Strip off the \( X \)-dressing, to isolate the numerator \( O_{\text{undressed}} \). The key point is that \( O_{\text{undressed}} \) must be a gauge-invariant conformal primary, since \( X \) is a conformal primary, and terms in the Lagrangian density must be gauge-invariant and primary as well.

- Then, re-dress the numerator \( O_{\text{undressed}} \) as an edge term by appending the edge dressing \( \delta(X)(Y/m)^{\frac{1}{2}(d+4-\Delta_{\text{und}})} \), to obtain a new edge operator

  \[
  O_{\text{edge}} = \delta(X)(Y/m)^{\frac{1}{2}(d+4-\Delta_{\text{und}})} O_{\text{undressed}}. \tag{3.30}
  \]

- The dressing factor \( (Y/m)^{\frac{1}{2}(d+4-\Delta_{\text{und}})} \) is a conformal primary of weight \( d + 4 - \Delta_{\text{und}} \) and the \( \delta \)-function factor \( \delta(X) \) is a conformal primary of weight \(-2\), so the new edge operator (3.30) is a conformal primary of weight \( 2 + d \), and is an allowed term in the action.

Again we emphasize this construction may not necessarily produce all possible edge terms, but it is a concrete construction that produces some examples of edge terms that might not be noticed otherwise, and indeed do not seem to have shown up in the literature so far.

In particular we can apply this construction to the bulk term \( L_{C_2} \propto m_{\frac{1}{2}(d-2)} X^{\frac{d}{2} - 1} Z \) showing up as a term (2.27) in the bulk action. Here we have \( O_{\text{undressed}} = m_{\frac{1}{2}(d-2)} Z \) so and \( \Delta_{\text{und}} = 4 \), so the corresponding edge operator is

\[
O_{\text{edge}} = m^{\frac{d+3}{2}} \delta(X) (Y/m)^{\frac{d}{2}} Z = m^{-1} \delta(X) (m^2 Y)^{\frac{d}{2}} Z \tag{3.31}
\]

The undressed term \( Z \) has \( \mu \)-scaling exponent \( \beta_{\text{und}} = 0 \) and a nonzero expectation value in the classical ground-state solution; so by formula (3.26) the integral of the dressed edge \( O_{\text{edge}} \) has \( \mu \)-scaling \( \gamma_{\text{total}} = \frac{2d-3}{2d-3} \). We compute its contribution in sec. 4.6.2.

In this section we have analyzed the \( \mu \)-scalings of edge operators at the classical level in the EFT, reducing to a finite problem the enumeration of edge operators that contribute classically with \( \mu \)-scaling above any given exponent in any given dimension \( d \). This analysis is incomplete without an analysis of the quantum contributions of operators which vanish classically in the large-charge vacuum, both tree-level contributions at second and higher order in perturbation theory, and loop contributions. We will give a loose upper bound on the quantum scalings of vevless operators in the next section, that will suffice to make the counting of terms that may contribute quantum mechanically, into a finite problem.
4 Energies in the harmonic trap at NLO

4.1 Energy shift from the Lagrangian perturbation

To simplify the overall treatment, we focus here (mostly) on the \( c_1 \) term in the NLO bulk Lagrangian, which is sufficient to illustrate the issues of principle involved in the renormalization of near-edge singularities with boundary counterterms. The integrated \( c_2 \) term in the Lagrangian is convergent in \( d = 2 \), and for completeness we discuss it briefly in section 4.6.1.

The \( c_1 \) term in the NLO bulk lagrangian, given in eqs. (2.23), (2.24), and (2.26), is

\[
\mathcal{L} \ni m \frac{d^2}{\pi^2} c_1 X^{\frac{d-2}{2}} (\partial_i X \partial^i X) .
\] (4.1)

To evaluate its effect on the energy of the charged ground state in the trapping potential, we now recall a simple lemma from classical mechanics: For a charged ground state of a classical system with a global symmetry, the first-order perturbation of the energy is just the negative of the perturbing Lagrangian, evaluated in the (unperturbed) ground state. This lemma generalizes the usual relationship between the first-order perturbing Lagrangian and first-order perturbing Hamiltonian for the overall ground state. In fact, we can reduce the more general case to the case of the overall ground state by adding an explicit chemical potential to the action.

For the case at hand, if \( c_1 \) is a coupling constant multiplying a small term in the Lagrangian, then

\[
E(Q) \big|_{O(c_1)} = -L(\mu) \big|_{O(c_1), \mu \to \mu_0(Q)} ,
\] (4.2)

where \( \mu_0(Q) \) is the expression for the chemical potential evaluated at charge \( Q \) in the unperturbed action at \( c_1 = 0 \). According to the lemma above, we have

\[
(\Delta E)_{[c_1]} = H_{[c_1]} = -L_{[c_1]} = - \int d^d x \mathcal{L}_{[c_1]}
\]
\[
= -m \frac{2\pi^2}{\Gamma(d)} c_1 \int d^d x X^{\frac{d-2}{2}} (\partial_i X \partial^i X) ,
\] (4.3)

where \( X \) is evaluated in the unperturbed classical solution (i.e., the \( c_1 = 0 \) classical solution). The energy shift thus appears as

\[
(\Delta E)_{[c_1]} = -m \frac{2\pi^2}{\Gamma(d)} c_1 \omega^4 \int d^d x r^2 \left( \mu - \frac{m_\omega^2}{2} r^2 \right)^{\frac{d-2}{2}} .
\] (4.4)

Evaluating the integral in angular variables, using formula (2.17) for the area of the unit \( d-1 \) sphere, and formula (2.16) for the classical size \( R \) of the droplet, we can straightforwardly evaluate the integral in \( d \) spatial dimensions:

\[
(\Delta E)_{[c_1]} = -4\pi \mu c_1 \left( \sqrt{2\pi} \frac{\mu}{\omega} \right)^{d-2} \left( \frac{\Gamma(\frac{d+1}{2}) \Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2}) \Gamma(d)} \right) .
\] (4.5)

We would also like to express this in terms of the total charge \( Q \) in the trap. Using the relationship (2.20), we have

\[
\mu^{d-1} = \xi \frac{d-1}{2} \left( \frac{1}{2} \right)^{\frac{d-1}{2}} \omega^{d-1} Q^{\frac{d-1}{2}} + O(c_1 Q^{\frac{d-3}{2}}) ,
\] (4.6)
so, in terms of $Q$,

$$\left(\Delta E\right)_{[c^1, \text{ fixed } Q]} = -2 c_1 \omega (2\pi)^{\frac{d}{2}} \xi \frac{d+1}{2} \left[ \frac{\Gamma(d+1)}{\Gamma(d)} \right]^{\frac{d+1}{2}} \frac{\Gamma}\left(\frac{d-1}{2}\right)^{\frac{d-1}{2}} \left[ Q \right]^{\frac{d-1}{2}} \right.$$

$$\left. \quad \times \frac{\Gamma(d+1)\Gamma(d-1)}{\Gamma(d)} Q^{\frac{d-1}{2}} \right). \quad (4.7)$$

Alternatively, in terms of $c_0$,

$$\left(\Delta E\right)_{[c^1, \text{ fixed } Q]} = -2^{\frac{d}{2} - 3} \frac{(2\pi)^{\frac{d}{2}} c_1^2 \omega}{c_0^{d+1} d} d^\frac{1}{2} \left( d + 2 \right)^{\frac{d-1}{2}} \frac{\Gamma(d+1)\Gamma(d-1)}{\Gamma(d)\Gamma(d+1)} Q^{\frac{d-1}{2}} \right). \quad (4.8)$$

This calculation is convergent in dimension $d \geq 3$. For $d = 3$ in particular, we have

$$\left(\Delta E\right)_{[c^1]} \bigg|_{d=3} = -4\pi c_1 \sqrt{2\pi} \frac{\mu^2}{\omega} \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{4}\right)}{\Gamma\left(\frac{d}{4}\right)\Gamma\left(\frac{d}{4}\right)} = -3\sqrt{2} \pi^2 c_1^2 \mu^2 \omega. \quad (4.9)$$

Or, in terms of $Q$,

$$\mu \bigg|_{d=3, \text{ leading order}} = \xi \frac{1}{2} \left( 3 Q \right)^{\frac{1}{2}} \omega, \quad (4.10)$$

so

$$\left(\Delta E\right)_{[c^1]} \bigg|_{d=3} = -\frac{3\pi^2}{2} \pi^2 c_1 \xi \omega Q^{\frac{2}{3}} \bigg. \quad (4.11)$$

This agrees with the $c^1_1$ term in eqn. (9) of [1] (with $\omega_1 = \omega_2 = \omega_3 = \omega$).

In $d = 2$ the expression (4.8) is divergent, so we have to regulate and renormalize it. In the following several sub-sections we will deal with this computation. In Sec. 4.2 we do so with a sharp cutoff in $d = 2$. Then, in 4.3, 4.4, and 4.5, we will employ dimensional regularization to cross-check the result. Namely, we derive the universal coefficient of the $\mu \log(\mu/\omega)$ term in the energy, which in terms of the particle-number charge $Q$ corresponds to a term of order $Q^{\frac{d}{2}} \log(Q)$.

### 4.2 Conformal sharp cutoff in $d = 2$

Let us first calculate the energy in $d = 2$ by cutting off the integral explicitly near the droplet edge. It will turn out that the sharp conformal cutoff is equivalent to the “cloud radius” cutoff used in [1, 7], but organized in a more manifestly conformal way.

The simplest conformally invariant way to regulate the integral is to specify the cutoff $X = \epsilon (Y/m)^{\frac{1}{3}}$. With this definition of the cutoff, the parameter $\epsilon$ is dimensionless in $h = 1, m \neq 0$ dimensional analysis. Since both $X$ and $Y$ are primary fields, the cutoff parameter $\epsilon$ is actually conformally invariant.

Note that the integral diverges if taken in the limit of fixed $\mu$ and $\epsilon \rightarrow 0$. This is not how the limit should ever be taken, though; one should always regulate and renormalize at fixed $\epsilon$ and take $\mu$ large. At any fixed $\epsilon$, the loop corrections and higher-derivative corrections are suppressed by powers of $\mu$, even near the edge. This is similar to the situation in effective string theory, in which the derivative expansion of the worldsheet EFT is reorganized at the boundary of an open string with freely-moving endpoints [10]. In these cases, the EFT itself does not actually break down, but some other invariant takes over as the dressing operator appearing in denominators of effective terms in the singular region: The dressing
rule at the boundary (or defect, or droplet edge) is not the same as the dressing rule in the bulk, but there is still a well-defined dressing rule and a well-defined derivative expansion that generates a perturbative expansion at large quantum number.

In the case of droplet-edge operators, the dressing field at the edge is $Y$. So it is only the expansion in $\partial/X^{1/2}$ that breaks down near the droplet edge, not the derivative expansion of the EFT altogether. The low-energy expansion is reorganized into a derivative expansion in $\partial/Y^{1/6}$. This fact is important when we come to the point of fully classifying operators at the droplet edge.

In the classical solution, the formula for the cutoff point, is

$$R_\epsilon = R - \delta_\epsilon , \quad (4.12)$$

$$\delta_\epsilon \simeq 2^{-1/2} m^{-\frac{1}{2}} \omega^{-\frac{1}{4}} \mu^{-\frac{1}{2}} \epsilon , \quad (4.13)$$

where the error is of $O(\epsilon^2 m^{-\frac{1}{2}} \omega^{-\frac{1}{2}} \mu^{-\frac{1}{2}} \epsilon)$.

Cutting off the integral at the point $r = R - \delta_\epsilon$ (and working in $d = 2$), we get

$$L_{[c_1]} \to \int_{r < R - \delta_\epsilon} d^d x \mathcal{L}_{c_1, \mu} = A_{(d-1)} \int_0^{R-\delta_\epsilon} dr r^{d-1} \mathcal{L}_{c_1, \mu}$$

$$= 2\pi \int_0^{R-\delta_\epsilon} dr r \mathcal{L}_{c_1, \mu}$$

$$= -4\pi c_1 \mu \log(\delta/R) + O(\mu^1)$$

$$= \frac{8\pi c_1 \mu}{3} \log \left( \frac{\mu}{\omega \epsilon^2} \right) + O(\mu^1) . \quad (4.14)$$

The nonlogarithmic term is scheme-dependent and not calculable within the EFT. Rather, its coefficient can be absorbed into a local edge counterterm, as we shall see in sec. 4.4.

The leading-order relationship between chemical potential $\mu$ and charge $Q$ in the isotropic harmonic trap with frequency $\omega$, is

$$\left( \frac{\mu}{\omega} \right) = \xi^{\frac{1}{2}} Q^{\frac{1}{2}}, \quad (4.15)$$

or equivalently

$$Q = \xi^{-1} \left( \frac{\mu}{\omega} \right)^2 , \quad (4.16)$$

which are taken from (2.19), (2.20), and evaluated in $d = 2$. Substituting in these leading-order relations, we have

$$\left( \Delta E \right)_{c_1, Q} = -\left( \Delta L \right)_{c_1, \mu} \bigg|_{\mu \to Q_0(\mu)}$$

$$= -\frac{8\pi c_1}{3} \xi^{\frac{1}{2}} Q^{\frac{1}{2}} \frac{1}{\omega} \log \left( \frac{\xi^{\frac{1}{2}} Q^{\frac{1}{2}}}{\epsilon^2} \right) + O(Q^{\frac{1}{2}} \omega) \quad (4.17)$$
Note that we employ here the mechanics lemma introduced above.

Now we will cancel the divergence with a counterterm to derive the renormalized energy. This discussion is essentially equivalent to the derivation in [7], the only difference being an emphasis on doing the regularization and renormalization with manifest conformal covariance.

In $d = 2$, the operator $\delta(X)Y$ is dimension 4 and its integral scales as $\mu^1$, according to formulas (3.25), (3.26), in the case where the undressed operator is the identity. Adding this term with a coefficient proportional to $c_1 \log(\epsilon^{-\frac{1}{2}})$ cancels the $\epsilon$-dependence of the bare term (4.17), leaving a cutoff-independent result: (4.14):

$$(\Delta E)_{c_1, Q, \text{renormalized}} = -\frac{4\pi c_1}{3} \xi^\frac{1}{2} Q^\frac{1}{d} \omega \log(\xi Q) + O(Q^\frac{1}{d} \omega) .$$  (4.18)

We will now go on to recalculate this answer in dimensional regularization, checking that we get the same result for the coefficient of $Q^\frac{1}{d} \log(Q)$ in the operator dimension in $d = 2$. Note that there can be no conformal boundary counterterm with a logarithmic dependence on $\mu$ (since the argument of the logarithm must be dimensionless, and there is no dimensionful parameter available), so we expect the coefficient of the $Q \log(Q)$ term to be universal and scheme-independent, given the value of the bulk coefficient $c_1$. We shall now check this expectation by calculating the same contribution to the energy in dimensional regularization.

4.3 Evaluation of the bare energy at order $c_1^1$ in dimensional regularization, near $d = 2$

Let us return to eq. (4.5), which we recap here for convenient reference,

$$(\Delta E)_{\epsilon^1} = -4\pi \mu c_1 \left(\sqrt{2\pi \mu} \right)^{d-2} \left(\frac{\Gamma\left(\frac{d}{2}+1\right)\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(d)}\right) ,$$  (4.19)

and expand the expression near $d = 2$. Namely, we find

$$(\Delta E)_{\epsilon^1} = -\frac{8\pi c_1 \mu}{d-2} - 8\pi c_1 \mu \log \left(\frac{d}{2}\right) + O \left[\mu^1 (d-2)^0\right] + O \left[\mu^1 (d-2)^1\right] .$$  (4.20)

There is a divergence proportional $1/(d-2)$ with coefficient of order $\mu^1$. We will now see that this divergence is an ultraviolet divergence corresponding to a local boundary counterterm.

4.4 Identification and coefficient of the boundary counterterm in dimensional regularization, at $d = 2$

To understand the form of the counterterm, we refer to Sec. 3 and consult the results of the boundary operator analysis therein. In eqns. (3.27) and (3.29), we classified all possible boundary operators that could contribute classically at order $\mu^0$ or larger, which of course should include any possible counterterm to cancel the divergence in expression (4.20). By formula (3.27), an edge operator scaling as $\mu^1$ in $d = 2$ must have no $X_{ij}$ appearing within. The only available scalar operator is then the dressed identity, since additional powers of $(\partial X)^2 = Y$ are, by definition, cancelled by the $Y$-dressing to adjust...
the conformal dimension to marginality. So the only available counterterm is a multiple of \( \delta(X) Y^+ \):

\[
\Delta H_{\text{edge}} \ni \kappa_d \mathcal{I}[1],
\]

with

\[
\mathcal{I}[1] \equiv m^{-2} \int d^d x \delta(X) (m^2 Y)^{\frac{d+4}{d-4}}
\]

Now we can evaluate this integral, again using dimensional regularization.

Recalling from above that

\[
\delta(X) = |\partial_r X|^{-1} \delta(|x|-R) = Y^{-\frac{d}{2}} \delta(|x|-R),
\]

the integrated edge operator \( \mathcal{I}[1] \) is thus

\[
\mathcal{I}[1] = m^{\frac{d-2}{2}} R^{d-1} A_{(d-1)} Y^{d+1} \bigg|_{r=R},
\]

where \( Y \) is evaluated at \( r = R = \sqrt{2\mu/(\omega \sqrt{m})} \). Then, using the classical value (3.9) of the \( Y \) invariant at \( r = R \), we have

\[
\mathcal{I}[1] = 2\mu A_{(d-1)} \left( \frac{2\mu}{\omega} \right)^{\frac{2}{d-2}},
\]

\[
\Delta H_{\text{edge}} \ni 2\kappa_d \mu A_{(d-1)} \left( \frac{2\mu}{\omega} \right)^{\frac{2}{d-2}}.
\]

We have shown the integrated term \( \mathcal{I}[1] \) is the only available counterterm scaling as large as \( \mu^{+1} \) in \( d = 2 \). All other local edge terms of that size are ruled out by a combination of conformal invariance and the \( Y \)-dressing rule. We conclude that the counterterm must be proportional to \( \mathcal{I} \), with a possibly \( d \)-dependent numerical coefficient.

We emphasize that the coefficient \( \kappa_d \) must really be “numerical”, rather than a ratio of scales \( (\mu/\omega)^{d-2} \)-dependent exponent, since a counterterm must be constructed out of local observables and background couplings. The only way one could possibly get such a ratio as a local term would be to realize it as a term of the form \( Y^\alpha / Z^\beta \). But edge operators containing fractional powers of \( Z \) are excluded by the dressing rule (Sec. 3.2).

Indeed, the coefficient \( \kappa_d \) is fixed by the necessity of cancelling the \( (d-2)^{-1} \) term in the energy shift \( E_{[c_1]} \) as calculated in (4.20):

\[
(\Delta E)_{[c_1]}^{(\text{bulk})} = -\frac{8\pi c_1 \mu}{d-2} - 8\pi c_1 \mu \log \left( \frac{2\mu}{\omega} \right) + O \left( \mu^1 (d-2)^0 \right) + O \left( (d-2)^1 \right),
\]

coming from the divergent integral of the \( c_1 \) term in the bulk action for \( d \leq 2 \). In the usual way, to cancel the divergent term we must add \( \mathcal{I}[1] \) with a coefficient \( \kappa_d \equiv \kappa / (d-2) \), determined by the condition that the divergence cancel. Using formula (4.25) for the
evaluation of the droplet-edge term, we see that we need
\[
H_{\text{edge}} \ni \kappa d \mathcal{I}[1] \bigg|_{d \to 2} = \frac{\kappa}{d-2} \times \mathcal{I}[1] \bigg|_{d \to 2}
\]
\[
= \frac{\kappa}{d-2} \times (2\mu) \times A_{(d-1)} \big|_{d \to 2}
\]
\[
= \frac{8\pi c_1 \mu}{d-2} ,
\] (4.27)
so we take \( \kappa = 2c_1 \), which means
\[
\kappa_d \equiv \frac{2c_1}{d-2} .
\] (4.28)
Thus, our edge-Hamiltonian counterterm, in conformal-edge minimal-subtraction, is
\[
(\Delta H)_{\text{edge}} = \frac{2c_1}{d-2} \mathcal{I}[1] ,
\] (4.29)
or, in terms of the Lagrangian,
\[
(\Delta L)_{\text{edge}} = -\frac{2c_1}{d-2} \mathcal{I}[1] .
\] (4.30)
Written out explicitly, we have
\[
(\Delta L)_{\text{edge}} = -\frac{2c_1}{d-2} m^{-2} \int d^d x \delta(X) (m^2 Y) \frac{4\pi}{\omega} .
\] (4.31)

### 4.5 Energy at order \( c_1 \) in \( d = 2 \), with the counterterm included

To calculate the contribution of the edge Lagrangian near \( d = 2 \), including the finite term, we have
\[
(\Delta E)_{\text{edge}} = -(\Delta L)_{\text{edge}} = \frac{2c_1}{d-2} A_{(d-1)} m^{d-2} R^{d-1} Y \frac{4\pi}{\omega} \bigg|_{r=R} .
\] (4.32)
Using eqns. (2.16), (2.17), and (3.9), and expanding near \( d = 2 \), we get
\[
(\Delta E)_{\text{edge}} \to \frac{8\pi c_1 \mu}{d-2} + \frac{16\pi c_1}{3} \mu \log \left( \frac{\mu}{\omega} \right) + 4\pi c_1 \gamma_E .
\]
Ignoring the nonlogarithmic finite piece and restoring notation to reflect that we are working with the \( O(c_1) \) sector of the theory, we have
\[
(\Delta E)_{|c_1|}^{(\text{edge})} = \frac{8\pi c_1 \mu}{d-2} + \frac{16\pi c_1}{3} \mu \log \left( \frac{\mu}{\omega} \right) + O \left[ \mu^1 (d-2)^0 \right] .
\] (4.33)
Adding the counterterm contribution to the bulk contribution (4.20), the total is
\[
(\Delta E)_{|c_1|}^{(\text{total})} = (\Delta E)_{|c_1|}^{(\text{bulk})} + (\Delta E)_{|c_1|}^{(\text{edge})} = -\frac{8\pi c_1}{3} \mu \log \left( \frac{\mu}{\omega} \right) + O \left[ \mu^1 (d-2)^0 \right] ,
\] (4.34)
where the latter part is finite, scheme-dependent, and nonlogarithmic. The \( \mu^1 \) term can be absorbed into the finite part of the coefficient of the counterterm \( \delta(X)Y \). Note, of course, that the coefficient of the \( \mu \log(\mu/\omega) \) term agrees with the value (4.18) computed in a conformally-invariant sharp-cutoff regulator in Sec. 4.2, which also agrees with the value
computed with essentially the same type of regulator in [7]. Written in terms of $Q$, using the relations (4.15), (4.16), we have

$$ (\Delta E)^{(\text{total})}_{[c_1]} = -\frac{4\pi c_1}{\xi} \xi^\frac{1}{2} Q^\frac{1}{2} \omega \log (\xi Q) + O \left( \omega \xi^\frac{1}{2} Q^\frac{1}{2} \right). $$

There is some ambiguity in the extension of the definition of $\xi$ in ref [1] to general dimension $d$, that translates into an order $(d-2)^0 \mu^1$ term in the energy, in $d = 2$. This term can be re-absorbed into the $(d-2)^0$ piece of the coefficient of the counterterm. We give the details of the scheme-dependence in section A.1 of the Appendix, and discuss its consequences for the scheme-dependence of the finite piece of the coefficient of the $\mu + 1$ edge counterterm in $d = 2$, in Sec. A.4.

4.6 Contributions of other subleading operators with positive $Q$-scaling in $d = 2$

So far we have analyzed in detail the effect of a bulk subleading operator (2.26), (2.24) and a boundary counterterm (3.8) on the ground state energy in $d$ dimensions, particularly in the case $d = 2$, in which the bulk contribution is logarithmically divergent. We did this to motivate a systematic analysis of boundary counterterms and their classical contributions to the vacuum energy. We have not yet analyzed the effects of other subleading operators that contribute to the ground state energy with nonnegative powers of $Q$, because the analysis of those terms proceeds very similarly to the cases we have already considered. For completeness, we can briefly present the results.

4.6.1 The bulk term $\mathcal{L}_{c_2}$

First we consider the term $\mathcal{L}_{c_2}$ written in (2.27), which we recap here:

$$ \mathcal{L}_{c_2} \equiv -c_2 d^2 m^\frac{1}{2}(d-2) X^\frac{d}{2} - Z, $$

$$ Z \equiv \partial^2 A_0 - \frac{1}{\sigma m} (\partial^2 \chi)^2. $$

In contrast to the contribution of the $c_1$ term, the contribution of the $c_2$ term is convergent in $d = 2$.\(^8\) The classical value of $Z$ in the harmonic potential (2.3) is

$$ Z_{\text{classical}} = m \omega^2 d, $$

and so in $d = 2$ we have

$$ Z = 2m \omega^2. $$

In $d = 2$ there is no $X$-dressing at all of the $\mathcal{L}_{c_2}$ term, and so

$$ \mathcal{L}_{c_2} = -4c_2 Z = -8c_2 m \omega^2. $$

Integrated over the extent of the droplet, this is just the droplet area times $-2c_2 m \omega^2$:

$$ (\Delta E)_{[c_2]} = -(\Delta L)_{[c_2]} = 8\pi c_2 m \omega^2 R^2 = 16\pi c_2 \mu. $$

This term is obviously finite, and contributes parametrically at the same scale as the boundary counterterm $\delta(X)Y$ in two dimensions.

---

\(^8\) The $c_2$ contribution to the energy in the harmonic potential is also convergent in $d = 3$ as well; the result is given in Eqn. (9) of [1].
4.6.2 The edge term $\mathcal{O}_{b_2} \equiv [m^{-1}Z]_{\text{edge}}$

The operator $Z$ also participates in the edge term

$$\mathcal{O}_{b_2} \equiv [m^{-1}Z]_{\text{edge}} \equiv m^{-1} \delta(X)(m^2 Y)^{\frac{1}{3}} Z,$$  \hspace{1cm} (4.42)

which is nonzero classically, and whose integrated contribution scales as $\mu^{\frac{1}{3}}$ in $d = 2$, by formulas (3.25), (3.26). Concretely, by (4.39), (3.9), and (2.16), we have the classical value

$$\langle \mathcal{O}_{b_1} \rangle = \langle [m^{-1}Z]_{\text{edge}} \rangle = m^{-\frac{1}{3}} \langle Z Y^{-\frac{1}{6}} \rangle \delta(|x| - R),$$

$$\langle I |m^{-1}Z| \rangle \equiv \int d^2 x \langle \mathcal{O}_{b_1} \rangle = 2\pi R m^{-\frac{1}{3}} \langle Z Y^{-\frac{1}{6}} \rangle = 4\pi (2\omega^2 \mu)^{\frac{1}{3}}.$$  \hspace{1cm} (4.43)

This term does not arise as a UV-divergent counterterm in either of the conformal regulators we have considered in $d = 2$ (either the conformal sharp cutoff or conformal dimensional regularization). It could arise in principle as a divergent counterterm in some other conformal cutoff, but we do not know of one.

In any complete NRCFT realizing the conformal EFT, including the unitary fermi gas, one expects all possible effective terms to arise, and so one expects the edge operator $[m^{-1}Z]_{\text{edge}}$ to appear with a finite coefficient. It would be interesting to learn the coefficient of this edge operator in the unitary fermi gas, by any experimental, numerical, or theoretical methods available. Possible theoretical tools might include a nonrelativistic analog of the large-charge conformal bootstrap \cite{50}, or a nonrelativistic analog of the powerful large-charge double-scaling techniques recently invented for the study of nonsupersymmetric \cite{51–55} and supersymmetric \cite{56–58} relativistic conformal field theories.

4.7 Upper bound on quantum $\mu$-scalings of contributions of edge operators to the ground state energy

We conclude this section with a note on quantum corrections, which requires giving an estimate of the quantum mechanical $\mu$-scalings of operators with vanishing expectation value in the classical solution. The estimate entails an analysis of the regularization and renormalization of the quantum fluctuations of the $\chi$ field in the harmonic trap. We defer a detailed analysis to later work \cite{49}, but in this section we will give a loose upper bound that is sufficient to ensure that only a finite number of operators can contribute quantum mechanically to an observable at or above a given order in $\mu$, in any given spatial dimension $d$.

As a particular application, we will show that in $d = 2$ there are no tree or loop graphs contributing at order $Q^0$, other than the one-loop vacuum bubble representing the Casimir energy. This implies that the order $Q^0$ term in the dimension of the lowest operator with particle number $Q$ in two spatial dimensions is universal and calculable.

\footnote{See e.g. \cite{8} for comments on the subject}
4.7.1 UV cutoff and scaling of the propagator

We now briefly consider the quantum effects of operators with vanishing expectation value at the classical level in the ground-state solution in the harmonic potential. The bare quantum expectation value of an operator is cutoff-dependent, and this cutoff-dependence must be considered carefully for these operators, since the UV-divergent quantum contribution is the leading effect controlling the $\mu$-scaling.

In considering these contributions it is important to recall the hierarchical separation (2.6) between the Wilsonian cutoff and the UV scale. In the limit $Q \to \infty$, the cutoff on energy and/or momentum is taken to be parametrically lower than the UV scale set by $\mu$. So when counting the quantum contributions of operators, time and/or spatial derivatives never contribute positive powers of $\mu$, even at the quantum level in UV-divergent diagrams.

To obtain an estimate, we must bound the scaling of the $\chi$ propagator. The Gaussian terms in the Lagrangian density go as $\mu^d \chi^2$ and $\mu^d (\partial_x \chi)^2$. At energies of order $\omega$ and momenta of order $R^{-1} \sim \omega^2/\mu$, the $\chi$ propagator then goes as

$$\langle \hat{\chi}(p,E) \hat{\chi}(p,-E) \rangle \sim \mu^{1-\frac{d}{2}} \omega^{-2},$$

(4.44)

while $E \sim \omega$ and $p \sim R^{-1} \sim \omega \sqrt{m/\mu}$. The anisotropy of the propagator and the singularity near the boundary complicate the analysis, and we postpone a detailed treatment to later work [49]. For now, we will compute our estimate by imposing an energy cutoff $\Lambda$ that is independent of $\mu$, and a momentum cutoff $p_{\text{max}} = m^{\frac{d}{2}} \Lambda/\sqrt{\mu}$ that goes to zero as the inverse square root of the chemical potential. This cutoff suffices for an analysis of the quantum contributions to the vacuum energy in the harmonic potential, since the infrared momentum scale is $R^{-1} \sim m^{\frac{d}{2}} \omega/\sqrt{\mu}$, so we still have $\sqrt{m\mu} \gg p_{\text{max}} \gg R^{-1}$, so long as we take $\mu \gg \Lambda \gg \omega$:

$$p < p_{\text{max}} \equiv \Lambda \sqrt{m/\mu}, \quad E < \Lambda, \quad \Lambda \ll \mu.$$  \hspace{1cm} (4.45)

With this cutoff, the quantum scaling of a fluctuation $\hat{\chi} \equiv \chi - \langle \chi \rangle$ is given by the square root of the propagator,

$$\hat{\chi} \sim \mu^{\frac{2-d}{4}}.$$  \hspace{1cm} (4.46)

4.7.2 A crude bound on the quantum $\mu$-scaling of general edge operators

Thus, multi-derivatives of $\hat{\chi}$ scale as

$$O_{n_x,n_t} \equiv \partial_x^{n_x} \partial_t^{n_t} \hat{\chi} \sim \mu^{-\frac{d-2}{4} - \frac{n_x}{2}}.$$  \hspace{1cm} (4.47)

We then have that

$$\beta(O_{n_x,n_t}) = -\frac{d-2}{4} - \frac{n_x}{2},$$

$$\Delta(O_{n_x,n_t}) = n_x + 2n_t,$$

$$\gamma_{\text{term}}(O_{n_x,n_t}) = -\frac{d-2}{4} - \frac{n_x}{2} - \frac{n_x}{6} - \frac{n_t}{3} = -\frac{3d-6+8n_x+4n_t}{12},$$  \hspace{1cm} (4.48)

- 37 -
for each multiderivative of $\chi$, and

$$\gamma_{\text{term}}(O_{\text{undressed}}) = -\frac{1}{12} \sum_{a} 3d - 6 + 8n^{(a)}_x + 4n^{(a)}_t , \quad (4.49)$$

for a general monomial in differentiated fields.

We can improve the bound further with some simple considerations. For each $\chi$, there must be at least one $\partial_x$ or one $\partial_t$, by global charge conservation. First, consider the case $n_x = 0$ and $n_t = 1$, that is, the term $\dot{\chi} \equiv \dot{\chi} - \mu$. The term $\dot{\chi}$ must be completed to $X$ by gauge invariance and conformal invariance, so we must consider the quantum $\mu$-scaling of $\dot{X} \equiv X - \langle X \rangle$. But $X$ vanishes as an operator\footnote{It vanishes identically, not just as an expectation value.} at the edge. Multiplying by the factor $\delta(X)$ in the dressing kills the operator: $X \delta(X) = 0$. So there is no term with $n_t = 1$ and $n_x = 0$.

Therefore, we can assume that $2n_x + n_t \geq 2$ for a nonvanishing, globally symmetric edge term. This means

$$3d - 6 + 8n^{(a)}_x + 4n^{(a)}_t \geq 3d + 2 , \quad (4.50)$$

and so

$$\gamma_{\text{term}}(O_{\text{undressed}}) \leq -\frac{3d+2}{12} n_\chi , \quad (4.51)$$

giving an upper bound on the total $\mu$-scaling of such a term after dressing and integration:

$$\gamma_{\text{total}} \leq \frac{2d-1}{3} - \frac{3d+2}{12} n_\chi = \frac{1}{12} [8d - 4 - (3d + 2)n_\chi] . \quad (4.52)$$

The estimate (4.52) is weaker than one might like: It allows a growing number of operators contributing with positive $Q$-scaling to the large-charge vacuum energy in the harmonic potential. The maximum number of $\chi$-fluctuations in an operator with positive $\gamma_{\text{total}}$ grows with spatial dimension as $n^{(\text{max})}_\chi \sim \frac{8}{3}(d - 2)$. With some further refinement, the bound can be strengthened and the number of contributing operators reduced [49], but the simple bound (4.52) suffices to make the classification of contributing edge operators tractable in low spatial dimension. In particular, for $d = 2$ we can show that there are no edge terms making quantum contributions to the vacuum energy at order $Q^0$ or greater.

**4.7.3 No quantum corrections to the vacuum energy with nonnegative $\mu$-scaling in $d = 2$**

For $d = 2$, the only edge terms that could possibly contribute with a positive $\mu$-scaling are $\dot{X}$ and $\nabla_x \chi$, both of which have integrated dressed $\mu$-scaling $\leq \frac{1}{3}$, which is order $\mu^{-\frac{1}{4}}$ relative to the leading term $\mu^3$. Both of these undressed operators vanish in the classical ground state solution, and can contribute to the vacuum energy in the harmonic potential only through their quantum effects. Since the one-point functions of these terms vanish at tree level, the only possible contributions can come from tree-level contributions with more than one insertion, or from loop corrections.
Connected contributions with more than one insertion are too small to contribute with positive $\mu$-scaling. Each insertion of a (dressed and integrated) $\dot{X}$ term or $\nabla_x \chi$ term in the path integral would lower the $\mu$-scaling by $\frac{2}{3}$ relative to the leading-order energy $\mu^3$. At least two insertions would needed to get a nonvanishing contribution, which would bring down the total $\mu$-scaling of the two-insertion effect to $-\frac{7}{3}$. Thus, in $d = 2$ we conclude that tree graphs involving vevless operators can only contribute terms with negative $\mu$-scaling to the vacuum energy.

Loop corrections to the contribution with one insertion are also too small to contribute. At tree level, the largest edge operator is the dressed identity, whose integral $I[1]$ enters at order $\mu^{2d-1}$. The action in $d$ dimensions scales as $\mu^{1+d}$, so the loop-counting parameter is $\mu^{-(d+1)}$. Thus, a one-loop correction to the contribution of $I[1]$ can be no larger than $\mu^{-\frac{3}{2}d+\frac{3}{2}}$. We conclude that contributions with even one loop and one edge operator all have negative $Q$-scaling in any dimension, including $d = 2$.

5 Conclusions

In this paper we have done the following:

- We have given a concrete representation for operators localized at the edge of the region of support of the particle droplet in a nonrelativistic CFT, where the density falls to zero.
- We have bounded the classical and quantum contributions of higher-derivative operators at the droplet edge, reducing the set of operators contributing at any given order in the large-$Q$ expansion to a finite set.
- We have used our classification of edge operators to demonstrate the scheme-independence of the $Q^\frac{1}{2} \log(Q)$ term in the operator dimension in two spatial dimensions.
- We have uncovered a second edge operator contributing to the lowest charged operator dimension with a positive $Q$-scaling; this operator goes as $\mu^\frac{1}{3}$ in $d = 2$ and as $\mu^{+1}$ in $d = 3$. This coefficient of this edge operator is scheme-independent in the sense that it does not renormalize a bulk UV divergence. The appearance of edge terms with third-integer powers of the chemical potential is an interesting and unexpected outcome of the operator analysis, reminiscent of the appearance of quarter-integer powers of the angular momentum for effective string theory at large spin [10] [59].

While a detailed analysis of the quantum effects of conformal edge operators is postponed to future work [49], a coarse bound on the size of quantum effects shows that no term contributes classically or quantum-mechanically at order $Q^0$ or larger in $d = 2$. This allows a straightforward calculation of the one-loop vacuum energy in the harmonic trap in $d = 2$, with the result:

$$E_{\mathcal{D}(Q^0)} \bigg|_{Q^0} = -0.29416 \omega .$$

(5.1)
The result (5.1) is a renormalized Casimir-type energy obtained by a renormalized Coleman-Weinberg sum over the energy spectrum (2.45). Due to the absence of an order $Q^0$ counterterm in $d = 2$, the renormalized value of this term is universal and unambiguous.

More broadly, we have derived an asymptotic expansion of the ground state energy of the $Q$-particle state in a harmonic potential with trapping frequency $\omega$ in $d = 2$, with the structure

$$E = E_{\frac{3}{2}} + E_{\frac{1}{2}} + E_{\frac{1}{6}} + E_0 + \text{(negative $Q-$scalings)},$$

$$E_{\frac{3}{2}} \equiv \frac{2}{3} \xi^{\frac{1}{2}} Q^{\frac{3}{2}} \omega,$$

$$E_{\frac{1}{2}} \equiv -\frac{4\pi c_1}{3} \xi^{\frac{1}{2}} Q^{\frac{1}{2}} \log (\xi Q) \omega + \alpha_{\frac{1}{2}} Q^{\frac{1}{2}} \omega,$$

$$E_{\frac{1}{6}} \equiv \alpha_{\frac{1}{6}} Q^{\frac{1}{6}} \omega,$$

$$E_0 \equiv -0.29416 \omega.$$

The leading term $E_{\frac{3}{2}}$ is determined solely by the leading term $c_0 m X^2$ in the action. The logarithmic part of the $E_{\frac{1}{2}}$ term comes solely from the $L_{c_1}$ term in the EFT, while the nonlogarithmic term is scheme-dependent, depending on the details of the cutoff procedure, on the coefficients of the $\delta(X)Y$ edge term, and also on the coefficient of the $Z$ bulk term. The $E_{\frac{1}{6}}$ term is scheme-independent and depends only on the (finite) coefficient of the edge term $\delta(X)Y^{-\frac{1}{3}}Z$. This term is gauge-invariant, conformally invariant, and by naturalness presumably appears with a nonzero coefficient in the edge Hamiltonian of any UV completion of the Son-Wingate EFT, including the interacting NRCFT describing the unitary fermi gas at quantum criticality. To our knowledge, this term and its higher-dimensional analogs have not been discussed in any analysis of the unitary fermi gas, and it would be interesting to constrain its coefficient by any available means.

The $E_0$ coefficient, the explicit details of whose calculation we defer to future work [49], is universal, not just among all UV completions of the critical unitary fermi gas, but among any other $2 + 1$-dimensional nonrelativistic CFT that lies in the same “large-charge universality class” as that of the critical unitary fermi gas. Such distinct NRCFTs would have other values for the coefficients $\xi, c_1, \alpha_{\frac{1}{2}}, \alpha_{\frac{1}{6}}$, and so on, but should exhibit the same structure of the asymptotic expansion, the same leading-order excitation spectrum, and the same value of the universal coefficient of the $E_0$ term. Though we are not aware of any specific examples, one possible application of the present work may be to suggest possible NRCFTs in the same large-charge universality class, which could be constructed using the large-charge expansion as a clue to the structure of the full theory.

Of course, by the NRCFT state-operator correspondence, the terms above correspond to the asymptotic expansion of the scaling dimensions of the lowest charged operators in

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11See for example Sec. 7 of [17] and references therein for a discussion of other NRCFTs, some of which may lie in the same (or a similar) universality class as the unitary fermi gas at large charge.
the NRCFT with particle number $Q$:

$$\Delta(Q) = \Delta_{3/2} + \Delta_{1/2} + \Delta_{1} + \Delta_{0} + \text{(negative $Q$ - scalings)},$$

$$\Delta_{3/2} = \frac{2}{3} \xi^{3/2} Q^{3},$$

$$\Delta_{1/2} = -\frac{4\pi c_1}{3} \xi^{1/2} Q^{3/2} \log(\xi Q) + \alpha_1 Q^{1},$$

$$\Delta_{1} = \alpha_1 Q^{1},$$

$$\Delta_{0} = -0.29416 . \hspace{1cm} (5.3)$$

As a final advertisement, we will analyze the quantum effects of edge counterterms and give details of the computation of the universal $Q^0$ term (5.1), in the future [49].

**Note**

While this paper was in preparation we learned of another forthcoming work with related results [60].

**A Conventions**

For convenience, we include here a dictionary of translations among conventions in the recent literature.

**A.1 Summary of relationships among the variously-defined $\xi$-coefficients and the $c_0$ coefficient in the recent literature**

We define $\xi$ in this paper the same way as defined in [17]. There, $\xi$ is defined as the ratio of the unitary Fermi gas ground state energy density at a given fermion density, to the energy density of the free Fermi gas at the same fermion density:

$$\xi = \frac{\mathcal{H}_{\text{interacting}}(\rho)}{\mathcal{H}_{\text{free}}(\rho)}, \hspace{1cm} (A.1)$$

where $\mathcal{H}$ is the ground-state energy density in the infinite-volume limit at fermion density $\rho$, with vanishing background potential, in $d$ dimensions. However, the denominator in expression (A.1) is not well-defined as an arbitrary function of spatial dimension $d$ without further information. Specifically, as noted in the text above, the energy density at a given fermion density depends on the number $a_s$ of spin-and-species states. Ref. [17]'s convention is to define *the free Fermi gas* as a system of fermions with two identical spinless fermion species, for all spatial dimensions $d$. This convention is not stated explicitly, but can be inferred from the formula for the (free) Fermi momentum $k_F$ in terms of the free fermion
density, which [17] denotes by $n$, equivalent to our $\rho$. The formula, given below eqn. (18) of [17], states

$$k_F = \left[ 2^{d-1} \pi^{d/2} \Gamma\left(\frac{d}{2} + 1\right) n \right]^{\frac{1}{d}}$$

(A.2)
in general spatial dimension $d$. The relationship between the ground state energy density and the Fermi momentum depends of course on the number $a_s$ of free fermion states per momentum level; in terms of $a_s$ the general relationship is

$$k_F = \left[ a_s^{-1} 2^{d} \pi^{d/2} \Gamma\left(\frac{d}{2} + 1\right) \rho \right]^{\frac{1}{d}}$$

(A.3)

So, the more precise characterization of $\xi$ with [17]’s implicit definition is to express

$$\xi \equiv \frac{\mathcal{H}_{\text{interacting}}(\rho)}{\mathcal{H}_{\text{free}}(\rho)} \bigg|_{a_s = 2}$$

(A.4)

Given this definition of $\xi$, the relationship to the coefficient $c_0$ in the effective Lagrangian is

$$\xi = (2\pi)^{-\frac{1}{d}} \left( \frac{1}{2} \Gamma \left(\frac{d}{2} + 2\right) c_0 \right)^{-\frac{2}{d}}$$

(A.5)

$$c_0 = \frac{2}{\Gamma\left(\frac{d}{2} + 2\right)} (2\pi)^{-\frac{d}{2}} \xi^{\frac{d}{2}}$$

(A.6)

The convention for the definition of the Bertsch coefficient $\xi$ in [7] is slightly different. Denoting the coefficient appearing there by $\xi_{[KP]}$, we have

$$\xi_{[KP]} = \left( \frac{1}{2} \Gamma(d + 1) \right)^{\frac{1}{d}} \xi^{\frac{d}{2}}$$

(A.7)

$$\xi = \left( \frac{2}{\Gamma\left(d + 1\right)} \right)^{\frac{2}{d}} \xi_{[KP]}^{\frac{d}{2}}$$

(A.8)

so that the relationship between the $c_0$ coefficient and the coefficient $\xi_{[KP]}$ of [7] is given by

$$\xi_{[KP]} = \left[ c_0 \left( \frac{2\pi}{\Gamma\left(\frac{1}{2}\right)} \right)^{\frac{d}{2}} \times \Gamma\left(\frac{d}{2} + 1\right) \right]^{-\frac{1}{d}}$$

(A.9)

$$c_0 = \left[ \frac{\left(2\pi\right)^{\frac{d}{2}} \times \Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(d+1\right)} \right] \left( \xi_{[KP]} \right)^{-\frac{d}{2}}$$

(A.10)

A.2 Conventions for NLO coefficients

In eqns. (2.23), (2.26), and (2.27), the NLO bulk Lagrangian is given by

$$\mathcal{L} = m_1^{\frac{1}{2}(d-2)} c_1 X^{-\frac{d-2}{2}} Y - d^2 m_1^{\frac{1}{2}(d-2)} c_2 X^{-\frac{d}{2}} Z$$

(A.11)

with the terms normalized in (2.24), (2.25) as

$$Y \equiv \left(\bar{\partial} X\right)^2$$

(A.12)
\[ Z \equiv \left[ \partial^2 A_0 - \frac{1}{\ell^2 m} (\partial^2 \chi)^2 \right] . \]  
(A.13)

These are the same conventions used in [1] (SW), albeit restricted to \( d = 3 \):

\[ c_1, \text{SW} = c_1, \text{here}_{|d=3} , \quad c_2, \text{SW} = c_2, \text{here}_{|d=3} . \]  
(A.14)

Our conventions are also the same as those of [6], except that ref. [6] sets the background gauge potential to zero:

\[ \mathcal{L}_1, \text{Favrod et al.} = \mathcal{L}_1, \text{here} , \quad \mathcal{L}_2, \text{Favrod et al.} = \mathcal{L}_2, \text{here}_{|A_0 \to 0} , \]  
(A.15)

with

\[ c_1, \text{Favrod et al.} = c_1, \text{here} , \quad c_2, \text{Favrod et al.} = c_2, \text{here} . \]  
(A.16)

Also note that the derived quantity \( d_0^2 \), defined below eqn. (3.5) of [6], is given by

\[ d_0^2 = \frac{1}{4} \times \Gamma \left( \frac{d}{2} \right) \left( 2\pi \right)^{\frac{d}{2}} \xi^\frac{d}{2} \]  
(A.17)

when written in terms of \( \xi \).

A.3 Comparison of names for coefficients and local quantities, making explicit the factors of \( m \) and \( h \)

To proceed further, we note some differences in conventions for dimensional analysis, among various relevant works in the literature:

- Ref. [7] sets \( m \to 1 \) but [1, 6] do not, and neither do we;
- For better or worse, we, [1], and [7] all set \( h \to 1 \), though [6] does not.

With these in mind, the reader may apply:

\[ \theta_{\text{Son-Wingate}} = \theta_{\text{Favrod-Orlando-Reffert}} = \chi_{\text{Kravec-Pal}} = \chi_{\text{here}} , \]  
(A.18)

\[ X_{\text{Son-Wingate}} = X_{\text{Kravec-Pal}} = U_{\text{Favrod-Orlando-Reffert}} = X_{\text{here}} , \]  
(A.19)

\[ n_{\text{Son-Wingate}} = n_{\text{Kravec-Pal}} = \rho_{\text{here}} = \frac{\delta \mathcal{L}}{\delta \chi} = \hbar \rho_{\text{Favrod-Orlando-Reffert}} , \]  
(A.20)

\[ m_{\text{Son-Wingate}} = m_{\text{here}} = m_{\text{Favrod-Orlando-Reffert}} = 1_{\text{Kravec-Pal}} . \]  
(A.21)
At the level of the leading-order Lagrangian, and in utterly explicit detail:

\[
P(X)_{\text{Son-Wingate}} = \mathcal{L}_{\text{Son-Wingate}} \quad \text{(by eqn. (62) of Son-Wingate)}
\]

\[
= \mathcal{L}_{\text{[here]}} \quad \text{(because the action is the action)}
\]

\[
= \mathcal{L}_{\text{[Kravec-Pal]}}
\]

\[
= \frac{1}{\hbar} \mathcal{L}_{\text{[Favrod-Orlando-Reffert]}} \quad \text{(because Favrod et al. keep \(\hbar\))}
\]

\[
= \left( c_0 X^{1 + \frac{d}{2}} \right)_{\text{[Kravec-Pal]}} \quad \text{(they set \(m \to 1\)}
\]

\[
= \left( m^{\frac{d}{2}} c_0 X^{1 + \frac{d}{2}} \right)_{\text{[Son-Wingate/Nishida-Son]}}
\]

\[
= \left( h^{-\frac{d}{2}} m^{\frac{d}{2}} \right) \left( c_0 U^{1 + \frac{d}{2}} \right)_{\text{[Favrod-Orlando-Reffert]}} \quad \text{(they keep \(\hbar\) explicit)}
\]

\[
= \left( m^{\frac{d}{2}} c_0 X^{1 + \frac{d}{2}} \right)_{\text{[here]}} \quad \text{(we do not keep \(\hbar\) explicit).} \quad (A.22)
\]

So the dictionary of \(c_0\)-coefficients is

\[
c_0_{\text{[Kravec-Pal]}} = m^{\frac{d}{2}} c_0_{\text{[Son-Wingate]}} = m^{\frac{d}{2}} c_0_{\text{[here]}} = h^{-\frac{d}{2}} m^{\frac{d}{2}} c_0_{\text{[Favrod-Orlando-Reffert]}} \quad \text{(A.23)}
\]

\[
c_0_{\text{[here]}} = c_0_{\text{[Son-Wingate]}} = h^{-\frac{d}{2}} \left( c_0_{\text{[Favrod-Orlando-Reffert]}} \right) = m^{\frac{d}{2}} c_0_{\text{[Kravec-Pal]}} \quad \text{(A.24)}
\]

\[
c_0_{\text{[Favrod-Orlando-Reffert]}} = h^{\frac{d}{2}} c_0_{\text{[here]}} = h^{\frac{d}{2}} c_0_{\text{[Son-Wingate]}} = m^{-\frac{d}{2}} h^{\frac{d}{2}} c_0_{\text{[Kravec-Pal]}} \quad \text{(A.25)}
\]

### A.4 Convention-sensitivity of scheme-dependent constants in minimal subtraction

If we redefine \(c_0\) in a dimension-dependent way,

\[
c_0 \to \tilde{c}_0 \equiv f(d) \times c_0 \quad \text{(A.26)}
\]

then even if we take \(f(d = 2) = 1\) and recalculate near \(d = 2\) with the same “minimal” subtraction (that is, subtracting just the coefficient of the pole \(1/d^2\)), we get a different finite part of the operator dimension by terms proportional to \(f'(d = 2)\). The \(d\)-dependent redefinition (A.26) of \(c_0\) is equivalent to redefining \(\xi\) by

\[
\xi \to \tilde{\xi} = f(d)^{-\frac{d}{2}} \xi \quad \text{(A.27)}
\]
where we have used the relationship (A.6) between $\xi$ and $c_0$. So the result (4.35) for the $O(c_1)$ piece of $\Delta Q$, which goes as $\xi^{d-2}$, transforms as

$$\Delta Q \bigg|_{O(c_1)} \rightarrow \left[ f(d) \right]^{-\frac{d}{2}} \Delta Q \bigg|_{O(c_1)} .$$  \hspace{1cm} (A.28)

If we parametrize the bare $\Delta Q \bigg|_{O(c_1)}$ near $d = 2$ by

$$\Delta Q \bigg|_{O(c_1)} = \frac{b_{-1}}{d-2} + b_0 (d-2)^0 + O((d-2)) ,$$  \hspace{1cm} (A.29)

then the coefficients $b_{-1,0}$ transform as

$$b_{-1} \rightarrow \left[ f(2) \right]^{-\frac{1}{2}} b_{-1} ,$$

$$b_0 \rightarrow \left[ f(2) \right]^{-\frac{1}{2}} \left[ b_0 - \frac{1}{2} f''(2)b_{-1} \right] .$$  \hspace{1cm} (A.30)

In other words, the term $b_0$ does not transform covariantly under a change in the regularization procedure even within dimensional regularization, transforming with an additive shift proportional to $f''(2)b_0$. So, even within dimensional regularization with minimal subtraction, there is an ambiguity that affects the renormalized answer.

Of course, there is nothing metaphysical about scheme ambiguities: In local quantum field theories, they always correspond to coefficients of local terms in the Hamiltonian. The particular scheme-dependence corresponding to the ambiguity (A.30) corresponds to a counterterm localized at the edge of the atom droplet in the harmonic trap, specifically the edge counterterm $\delta(X)Y^{+1}$. In the present article we have taken care to remove this ambiguity by defining our Lagrangian couplings in all dimensions $d$; see the discussion in Sec. A.1.

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