Resonant scattering effect in spectroscopies of interacting atomic gases

M J Leskinen, J Kajala and J J Kinnunen
Department of Applied Physics, 02015 Aalto University, PO Box 5100, Finland
E-mail: jami.kinnunen@tkk.fi

New Journal of Physics 12 (2010) 083041 (24pp)
Received 4 March 2010
Published 17 August 2010
Online at http://www.njp.org/
doi:10.1088/1367-2630/12/8/083041

Abstract. We consider spectroscopies of strongly interacting atomic gases, and we propose a model for describing the coupling between quasiparticles and gapless phonon-like modes. Our model explains features in a wide range of different experiments in both fermionic and bosonic atom gases in various spectroscopic methods.

1 Author to whom any correspondence should be addressed.
1. Introduction

Experimental setups in ultracold atomic gases, allowing easy optical access to the sample and the well-established imaging techniques, have resulted in a wide range of precise and well understood spectroscopies. Radio-frequency (RF) [1]–[8], Raman [9, 10], Bragg [11]–[14] and lattice modulation spectroscopies [15]–[17] have plenty in common, and they can all be described effectively as an absorption of a single photon of momentum $k$ and frequency $\omega$.

The agreement between spectroscopic experiments and theories in ultracold gases has often been at most qualitative, despite the various theoretical approaches, such as mean field [18]–[26], $t$-matrix [27] or other diagrammatic approaches [28]–[31]. Strange peaks and features (such as anomalously broad spectral peaks) in the experiments have been attributed to the lack of coherence [14], defects in a lattice [15] or center-of-mass motion of pairs [8], suggesting that those features are caused by imperfections in the experimental setups. These anomalies have been all but neglected in theoretical descriptions and, instead, a lot of effort has been put into improving the underlying many-body theories by including thermal and quantum fluctuations [32]–[34] in order to obtain quantitative agreement with the well-understood features in the spectra. Here we show that those neglected anomalies seen in a wide range of different experiments have a common source.

While the breakdown of a mean-field theory in strongly interacting systems is fully anticipated, the treatment of beyond mean-field effects is exceedingly difficult. As a result, most...
dependent Schrödinger equation, approximate wavefunction in the second-order perturbation theory, purposes, it is enough to consider the first three terms of the above expansion, yielding the series expansion of the S-matrix yields perturbation theories of different orders. For our time-dependent Hamiltonian $H(t) = H_0 + V(t)$, and the system evolves according to the time-dependent Schrödinger equation,

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle.$$ 

This can be solved to yield the state at time $t$ as

$$|\psi(t)\rangle = e^{-iH_0t/\hbar} S(t) |\psi(0)\rangle,$$

where the scattering S-matrix is

$$S(t) = 1 + \frac{1}{i\hbar} \int_0^t dt' V_1(t') + \left(\frac{1}{i\hbar}\right)^2 \int_0^t dt' \int_0^t dt'' V_1(t') V_1(t'') + \cdots,$$

where $V_1(t) = e^{iH_0t/\hbar} V(t) e^{-iH_0t/\hbar}$ is the perturbation expressed in the interaction picture. This formulation of S-matrix readily offers easy access to the perturbative description. Cutting the series expansion of the S-matrix yields perturbation theories of different orders. For our purposes, it is enough to consider the first three terms of the above expansion, yielding the approximate wavefunction in the second-order perturbation theory,

$$|\psi(t)\rangle \approx e^{-iH_0t/\hbar} \left[ 1 + \frac{1}{i\hbar} \int_0^t dt' V_1(t') + \left(\frac{1}{i\hbar}\right)^2 \int_0^t dt' \int_0^t dt'' V_1(t') V_1(t'') \right] |\psi(0)\rangle. \quad (1)$$

Ultimately, we are interested in the expectation value of some observable $\mathcal{O}$. Keeping terms only up to the second order in the perturbation $V$, we obtain

$$\langle \mathcal{O}(t) \rangle = \langle \psi(t)|\mathcal{O}|\psi(t)\rangle = \langle \mathcal{O} \rangle + \frac{1}{\hbar^2} \int_0^t dt' \int_0^t dt'' \langle V_1(t') \mathcal{O}_1(t) V_1(t'') \rangle$$

$$- \frac{1}{\hbar^2} \int_0^t dt' \int_0^{t'} dt'' \left[ \langle \mathcal{O}_1(t) V_1(t') V_1(t'') \rangle + \langle V_1(t'') V_1(t') \mathcal{O}_1(t) \rangle \right], \quad (2)$$

where the expectation values on the right-hand side are calculated for the initial state $|\psi(0)\rangle$. Should we be interested in the rate of change of $\langle \mathcal{O}(t) \rangle$ (as is the case when measuring current), one can differentiate equation (2) with respect to time $t$ and obtain

$$\frac{d}{dt} \langle \mathcal{O}(t) \rangle = 2 \frac{1}{\hbar^2} \int_0^t dt' \Re \left[ \langle V_1(t') \mathcal{O}_1(t) V_1(t') \rangle - \langle \mathcal{O}_1(t) V_1(t') V_1(t') \rangle \right]. \quad (3)$$

New Journal of Physics 12 (2010) 083041 (http://www.njp.org/)

Theoretical studies have relied on the mean-field approximation, partly because of the apparently good agreement between theory and experiment. However, as we will soon point out, various experiments performed with strongly interacting systems have features that cannot be explained by mean-field theories. Here we propose an efficient diagrammatic expansion for the mean-field theories to describe an important class of coherent processes.

2. Linear response spectroscopy

We begin by deriving the linear response theory for a weak perturbation $V$. While the calculation is a simple textbook derivation, it will help to illuminate the problem encountered with the mean-field linear response theory.

Let us assume that the system is initially (at time $t = 0$) in a ground state of a (many-body) Hamiltonian $H_0$, denoted by $|\psi(0)\rangle$. Switching on perturbation $V(t)$ yields the full time-dependent Hamiltonian $H(t) = H_0 + V(t)$, and the system evolves according to the time-dependent Schrödinger equation,

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle.$$ 

This can be solved to yield the state at time $t$ as

$$|\psi(t)\rangle = e^{-iH_0t/\hbar} S(t) |\psi(0)\rangle,$$

where the scattering S-matrix is

$$S(t) = 1 + \frac{1}{i\hbar} \int_0^t dt' V_1(t') + \left(\frac{1}{i\hbar}\right)^2 \int_0^t dt' \int_0^t dt'' V_1(t') V_1(t'') + \cdots,$$

where $V_1(t) = e^{iH_0t/\hbar} V(t) e^{-iH_0t/\hbar}$ is the perturbation expressed in the interaction picture. This formulation of S-matrix readily offers easy access to the perturbative description. Cutting the series expansion of the S-matrix yields perturbation theories of different orders. For our purposes, it is enough to consider the first three terms of the above expansion, yielding the approximate wavefunction in the second-order perturbation theory,

$$|\psi(t)\rangle \approx e^{-iH_0t/\hbar} \left[ 1 + \frac{1}{i\hbar} \int_0^t dt' V_1(t') + \left(\frac{1}{i\hbar}\right)^2 \int_0^t dt' \int_0^t dt'' V_1(t') V_1(t'') \right] |\psi(0)\rangle. \quad (1)$$

Ultimately, we are interested in the expectation value of some observable $\mathcal{O}$. Keeping terms only up to the second order in the perturbation $V$, we obtain

$$\langle \mathcal{O}(t) \rangle = \langle \psi(t)|\mathcal{O}|\psi(t)\rangle = \langle \mathcal{O} \rangle + \frac{1}{\hbar^2} \int_0^t dt' \int_0^t dt'' \langle V_1(t') \mathcal{O}_1(t) V_1(t'') \rangle$$

$$- \frac{1}{\hbar^2} \int_0^t dt' \int_0^{t'} dt'' \left[ \langle \mathcal{O}_1(t) V_1(t') V_1(t'') \rangle + \langle V_1(t'') V_1(t') \mathcal{O}_1(t) \rangle \right], \quad (2)$$

where the expectation values on the right-hand side are calculated for the initial state $|\psi(0)\rangle$. Should we be interested in the rate of change of $\langle \mathcal{O}(t) \rangle$ (as is the case when measuring current), one can differentiate equation (2) with respect to time $t$ and obtain

$$\frac{d}{dt} \langle \mathcal{O}(t) \rangle = 2 \frac{1}{\hbar^2} \int_0^t dt' \Re \left[ \langle V_1(t') \mathcal{O}_1(t) V_1(t') \rangle - \langle \mathcal{O}_1(t) V_1(t') V_1(t') \rangle \right]. \quad (3)$$

New Journal of Physics 12 (2010) 083041 (http://www.njp.org/)
where $\Re$ is the real part, and we have assumed here that $V(t)$ and $\mathcal{O}$ are Hermitian operators. In order to continue this discussion on linear response, we need to specify the perturbation $V$ and the observable $\mathcal{O}$.

2.1. Radio-frequency (RF) spectroscopy

We will first consider the case of RF spectroscopy. A radio-frequency field of frequency $\omega$ and amplitude $\Omega$ drives atoms from hyperfine state $|2\rangle$ to \textit{initially unoccupied} hyperfine state $|3\rangle$. As the wavelength of the RF field is much longer than any relevant length scale in the system, the RF photons do not affect the momentum of the atoms but only couple to the hyperfine state degrees of freedom. The perturbation in the rotating wave approximation is then given by

$$
\mathcal{O} = N_3 = \sum_k \epsilon_k \sigma_k.
$$

Since the hyperfine state $|3\rangle$ is initially empty, $N_3|\psi(0)\rangle = 0$, the formula for the current simplifies into the form

$$
\frac{d}{dt} \langle N_3(t) \rangle = \frac{2|\Omega|^2}{\hbar^2} \int_0^t \Re \langle e^{iH_0 t'}/\hbar} V(t) e^{-iH_0(t-t')/\hbar} N_3 e^{-iH_0(t-t')/\hbar} V(t') e^{-iH_0 t'/\hbar} \rangle,
$$

where we have written the operators in the Schrödinger picture. Operating by $N_3$ within the expectation value on the right-hand side of equation (4) will necessarily give 1, since the single perturbation $V(t')$ has transferred only one atom into the hyperfine state $|3\rangle$ and we assume that $N_3$ commutes with $H_0$. Thus, we have

$$
\frac{d}{dt} \langle N_3(t) \rangle = \frac{2|\Omega|^2}{\hbar^2} \sum_{k_p} \Re \int_0^t \langle e^{-i\delta (t-t')/\hbar} (\epsilon_k e^{iH_0 t'/\hbar} c_{3k} \sigma_k e^{-iH_0 t'/\hbar} c_{3k}^\dagger) \rangle,
$$

where we have assumed that the initial state $|\psi(0)\rangle$ is an eigenstate of $H_0$ with eigenenergy equal to zero (a nonzero energy would provide only phase factor). This can be further simplified to yield

$$
\frac{d}{dt} \langle N_3(t) \rangle = \frac{2|\Omega|^2}{\hbar^2} \sum_{k_p} \Re \int_0^t \langle e^{-i\delta' t'/\hbar} G_2(k, -t') G_3(k, t') \rangle.
$$

The expectation value on the right-hand side is in principle a two-particle (or particle–hole) Green’s function propagating within a many-body sea of fermions. However, in the special case where the hyperfine state $|3\rangle$ is noninteracting, the two-particle propagator separates into a product of one-particle and one-hole propagators,

$$
\frac{d}{dt} \langle N_3(t) \rangle = \frac{2|\Omega|^2}{\hbar^2} \sum_k \Re \int_0^t \langle e^{-i\delta' t'/\hbar} G_2(k, -t') \rangle \langle G_3(k, t') \rangle,
$$

where $-iG_2(k, -t') = (\epsilon_k e^{-iH_0 t'/\hbar} c_{2k})$ is the propagator of a hole in single-particle state $|2, k\rangle$ and $iG_3(k, t') = (\epsilon_k e^{-iH_0 t'/\hbar} c_{3k}^\dagger)$ is the propagator of an atom in state $|3, k\rangle$. Being a noninteracting Green’s function (and since hyperfine state $|3\rangle$ is initially empty), the particle propagator $G_3(k, t')$ is simply $-ie^{-i\epsilon_k t'/\hbar}$, where $\epsilon_k$ is the energy of an atom with momentum $k$, and we end up with

$$
\frac{d}{dt} \langle N_3(t) \rangle = \frac{2|\Omega|^2}{\hbar^2} \sum_k \Re \int_0^t \langle e^{-i\epsilon_k t'/\hbar} iG_2(k, -t') \rangle.
$$
In the limit of a very long pulse $t \to \infty$ (but keeping the perturbation weak $t\Omega/\hbar \ll 1$), the integral yields the Fourier transform of the real-time hole propagator,

$$\frac{d}{dt} \langle N_3(t) \rangle = \frac{2|\Omega|^2}{\hbar^2} \sum_k \Im G_2(k, \epsilon_k + \delta),$$

where $\Im$ is the imaginary part. Thus the RF spectrum probes the spectral function of the hole $A_2(k, \epsilon) = 2\Im G_2(k, \epsilon)$ which is ultimately determined by the operator $e^{-iH_0 t/\hbar}$.

While the simple Bardeen–Cooper–Schrieffer (BCS) mean-field treatment has proven to be able to describe spectroscopies of even strongly interacting gases rather well in many cases, quantitative agreement between theory and the experiments has turned out to be very hard to achieve. Therefore, it is worth looking more closely at what information has been lost in the BCS mean-field approximation. In particular, we are looking for non-BCS processes that are important for the hole propagator in order to obtain quantitative agreement with the observed spectra.

2.2. Bardeen–Cooper–Schrieffer (BCS) mean-field approximation

We want to describe RF spectroscopy of a two-component Fermi gas. Since the RF-field couples one of the two hyperfine states into a third excited state, the full system is described by a three-component Hamiltonian, but one in which the third state $|3\rangle$ is initially empty. The Hamiltonian of the three-component system is

$$H_0 = \sum_{k\sigma} \epsilon_k c_{k\sigma}^\dagger c_{k\sigma} + \frac{1}{2} \sum_{kpq\sigma\sigma'} g_{\sigma\sigma'} c_{k,\sigma}^\dagger c_{-k+q,\sigma'} c_{-p+q,\sigma'} c_{p,\sigma},$$

where $\epsilon_k$ is the single-particle kinetic energy, $\sigma \in \{1, 2, 3\}$ and $g_{\sigma\sigma'}$ is the interaction strength between the atoms in hyperfine states $|\sigma\rangle$ and $|\sigma'\rangle$ in contact interaction potential approximation. The standard approach uses the two-body scattering T-matrix with $g_{\sigma\sigma'} = \frac{4\pi \hbar^2 |a_{\sigma\sigma'}|}{V}$, where $a_{\sigma\sigma'}$ is the scattering length and $V$ is the volume. Initially, we will assume that only $g_{12}$ is nonzero, but later on we will consider also the general case in which all interactions can be nonzero.

The BCS mean-field approximation has proven to be very simple but still very effective in describing two-component Fermi gases, even in the very strongly interacting regime. The approximation amounts to first considering only opposite momentum scatterings $q = 0$ in the Hamiltonian $H_0$ and then replacing the quartic interaction in $H_0$ by quadratic mean-field interaction, resulting in a BCS mean-field Hamiltonian,

$$H_0^{\text{BCS}} = \sum_{k\sigma} \epsilon_k c_{k\sigma}^\dagger c_{k\sigma} + \Delta_{12} \sum_k c_{k,1}^\dagger c_{-k,2} + \text{h.c.},$$

where the pairing field $\Delta_{12} = g_{12} \sum_p \langle c_{-p,2} c_{p,1} \rangle$. Here we have neglected the Hartree energy shift, but it needs to be included when studying, for example, gases in optical lattices in order to obtain correct bound state energies.

The BCS mean-field theory can be solved exactly by diagonalizing the BCS Hamiltonian using the Bogoliubov transformation. This yields the Hamiltonian in the quasiparticle basis,

$$H_0^{\text{BCS}} = \sum_{k\sigma} E_k \gamma_{k,\sigma}^\dagger \gamma_{k,\sigma} + C,$$
where $C$ is a constant and the operators $\gamma_{k,\sigma} (\gamma_{k,\sigma}^\dagger)$ annihilate (create) quasiparticle excitation of momentum $k$ and energy $E_k = \sqrt{(\epsilon_k - \mu)^2 + \Delta_{k2}^2}$, with $\mu$ being the chemical potential of the atoms in hyperfine states $|1\rangle$ and $|2\rangle$, and $\epsilon_k$ is the single-particle excitation energy. The quasiparticle ‘spin’ $\sigma \in \{ \uparrow, \downarrow \}$ does not refer to the hyperfine level of an atom but simply to the fact that there are two branches of quasiparticle excitations. The hole spectral function can be shown to be

$$G_2(k, \epsilon) = \frac{u_k^2 n_F(E_k)}{\epsilon - E_k + i\eta} + \frac{v_k^2 n_F(-E_k)}{\epsilon + E_k + i\eta},$$

where the Bogoliubov coefficients $u_k^2 = \frac{1}{2} (1 + \frac{\epsilon_k - \mu}{\Delta_{k2}})$ and $v_k^2 = 1 - u_k^2$. We have assumed that the system is in equilibrium and the quasiparticle distributions are given by the Fermi–Dirac distribution $n_F(\epsilon) = 1/(1 + e^{\beta \epsilon})$, $\beta = 1/(k_B T)$, $k_B$ is the Boltzmann constant and $T$ is the temperature. We consider only zero temperature $T = 0$ results in this paper.

At zero temperature, the BCS hole spectral function (6) consists of a single sharp peak (the second peak appears at finite temperatures), describing a quasiparticle excitation with infinite lifetime. This property of infinite lifetime can easily be seen from the form of the BCS Hamiltonian $H_0^{BCS}$. Creating a hole in state $|k, 2\rangle$ will necessarily render any atom in state $| -k, 1\rangle$ effectively noninteracting, since the Hamiltonian $H_0^{BCS}$ describes only scatterings between atoms with opposite momenta.

In order to consider quasiparticle lifetime effects, one would need to go beyond the BCS approximation and include residual interactions between the quasiparticles,

$$H_0^{res} = g_{12} \sum_{k,p,q \neq 0} c_{k,p}^\dagger c_{-k+q,2}^\dagger c_{-p+q,2} c_{p,1}^\dagger.$$

This approach has been studied in the context of nuclear physics [35] and high-temperature superconductors [36]. It has also been discussed in the present context in [37] using a fully self-consistent calculation of the many-body T-matrix. The self-consistent nature allowed the inclusion of Anderson–Bogoliubov (AB) phonon effects, and these were shown to result in the broadening of spectral features. The theory was, however, highly involved, and the present theory can be seen as an attempt to include important quasiparticle lifetime effects in a simpler way, namely using first-order perturbation theory.

While equation (7) for residual interactions is enough for our purposes, it is very instructive to briefly consider its presentation in the BCS quasiparticle basis. Writing the residual interaction $H_0^{res} = H_{40} + H_{31} + H_{22}$ in the quasiparticle basis yields [36, 37]

$$H_{40} = g_{12} \sum_{k,p,q \neq 0} v_{k+p} v_{k} v_{p} u_{p-q} \gamma_{k,\uparrow} \gamma_{-k-q,\downarrow} \gamma_{-p,\downarrow} \gamma_{p+q,\uparrow} + h.c.,$$

$$H_{31} = g_{12} \sum_{k,p,q \neq 0,\sigma} (v_{p} v_{p+q} v_{k} u_{k-q} - u_{p} u_{p+q} u_{k} v_{k-q}) \gamma_{k,\uparrow} \gamma_{k-q,\sigma} \gamma_{-p,\downarrow} \gamma_{p+q,\uparrow} + h.c. ,$$

and

$$H_{22} = g_{12} \sum_{k,p,q \neq 0} \left[ (u_{q-k} u_{k} u_{p-q} - v_{q-k} v_{k} v_{p-q}) \gamma_{q-k,\uparrow} \gamma_{q-k,\downarrow} \gamma_{p,\downarrow} \gamma_{p,\uparrow} + (u_{k} u_{p-q} v_{p} v_{k-q} + v_{k} v_{p-q} u_{p} u_{k-q}) \gamma_{k,\uparrow} \gamma_{p,\downarrow} \gamma_{q-k,\uparrow} \gamma_{q-k,\downarrow} + u_{k+q} u_{p+q} v_{p} v_{k+q} \gamma_{p+q,\uparrow} \gamma_{p+q,\downarrow} \gamma_{k,\uparrow} + v_{k+q} v_{p+q} u_{p} u_{k+q} \gamma_{p+q,\uparrow} \gamma_{p+q,\downarrow} \gamma_{k,\uparrow} \right].$$

New Journal of Physics 12 (2010) 083041 (http://www.njp.org/)
Clearly, the number of quasiparticles does not need to be conserved as a single excitation can be split into several excitations through scattering and vice versa. Moreover, since quasiparticles have minimum energy cost of $\Delta$, transitions to states with different numbers of quasiparticles are gapped.

However, these quasiparticle interactions also provide coupling to the collective AB mode, which is a phonon-like excitation and thus gapless. The description of this mode requires nonperturbative treatment, such as random phase approximation \[38\], and while the AB phonon arises already from the mean-field theory, the actual coupling between the quasiparticle excitations and these collective modes is provided by the residual interactions $H_{\text{res}}^0$ [37]. What our theory does is to include approximatively the coupling between the quasiparticle excitation and the phonon modes. We will later also consider settings in which the couplings to different numbers of quasiparticle excitations will be important and such processes may actually arise naturally from the perturbative treatment. Furthermore, we will also consider systems at temperatures above the critical temperature where the superfluid order parameter vanishes. In such a case, the BCS model yields an ideal finite temperature Fermi gas without pairing correlations, binding energies or AB phonons. In that limit, the particle and the quasiparticle become the same.

3. Beyond BCS mean-field approximation

Our goal is an effective theory which can include the effect of interactions between quasiparticles approximatively and can easily be incorporated into the standard mean-field BCS theory. As our starting point, we take the residual interactions neglected in the BCS mean-field theory $H_{\text{res}}^0$ and treat these as a perturbation in the hole propagator. Writing the time-evolution operator $e^{-iHt/\hbar}$ in equation (1) in terms of the scattering S-matrix as

$$e^{-iH_{\text{res}}^0 t/\hbar} = e^{-iH_{\text{BCS}}^0 t/\hbar} S_{\text{res}}(t)$$

and expanding the S-matrix $S_{\text{res}}(t) = 1 + \frac{1}{\hbar} \int_0^t dt' H_{\text{res}}^0(t') + \cdots$, where $H_{\text{res}}^0(t') = e^{iH_{\text{BCS}}^0 t'} \times H_{\text{res}}^0 e^{-iH_{\text{BCS}}^0 t'}$, we obtain the first-order correction (in terms of $H_{\text{res}}^0$),

$$|\psi(t)\rangle_1 = \left(\frac{1}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt'' e^{iH_{\text{res}}^0 (t-t')/\hbar} \times [H_{\text{res}}^0 e^{-iH_{\text{BCS}}^0 (t'-t'')/\hbar} V(t'') + V(t') e^{-iH_{\text{BCS}}^0 (t'-t'')/\hbar} H_{\text{res}}^0 e^{-iH_{\text{BCS}}^0 t''/\hbar}] |\psi(0)\rangle.$$  \hspace{1cm} (11)

Here, we have only included the first-order term (with respect to RF-coupling $V$) from equation (1) since the zero- and second-order terms do not contribute to the current in equation (4). The above equation is a result of applying perturbation theory to both residual interaction $H_{\text{res}}^0$ and the RF-field $V(t)$.

At this stage, we notice the similarity to two-photon processes in quantum optics, such as the two-photon Bragg scattering or two-photon Raman transitions that also incorporate correlated couplings with two different fields. In the present case, the two fields are the RF-field and the atoms in the hyperfine state $|1\rangle$. The main difference, however, is that in our model the atom–atom interaction has many different scattering channels, including very long wavelength (low momentum exchange $q$) transitions. Nonetheless, the similarity is encouraging—we are here dealing with physics that has manifested in other systems as well.
Figure 1. The lowest-order photon absorption $\hbar \omega$, $q_L$ and quasiparticle scattering $g_{12}$ process. The absorption of the photon by an atom in the ground state of $|2\rangle$ produces an atom in hyperfine state $|3\rangle$ and one hole (backward propagator) excitation in $|2\rangle$. The hole excitation in $|2, k\rangle$ then scatters with a hole in $|1, p\rangle$, resulting in holes at states $|2, k+q\rangle$ and $|1, p-q\rangle$. As is clear from this description, the photon absorption and the atom–atom scattering processes are connected through the hole degrees of freedom. If the photon has finite momentum $q_L$, such as in Bragg spectroscopy, the atom will also feel a momentum change worth $q_L$. We approximate that $|1\rangle$-atoms are described by bare propagators, and hence the atom–atom scattering process does not produce additional excitations experiencing mean-field energy shifts (such as BCS pairing gap $\Delta$). The process qualitatively describes the coupling of the hole excitation to a gapless phonon mode.

The first term in equation (11) can be depicted pictorially as in figure 1 (the second term corresponds to figure 3, as will be discussed later). The diagram has been drawn for a more general spectroscopy than RF-spectroscopy by including also the photon momentum $q_L$. Below we will assume that $q_L = 0$ for RF-spectroscopy, but later on in the discussion regarding Bragg spectroscopy the momentum of the photon can be significant. The diagram describes a path for the wavefunction and one will need to calculate the squared norm of the amplitude in order to find expectation values, such as a Green’s function. As in the discussion regarding linear response above, a first-order perturbation for the wavefunction corresponds to a second-order perturbation for the propagator.

In the case of RF-spectroscopy, the diagram describes a correlated process in which a RF-photon creates an atom in state $|3, k\rangle$ and a hole in $|2, k\rangle$. The hole then scatters with a hole in state $|1, p\rangle$ into a new set of holes $|2, k+q\rangle$ and $|1, p-q\rangle$. Another way to describe the diagram is that an atom in state $|2, k\rangle$ absorbs the photon and is transferred into state $|3, k\rangle$. After the RF-photon absorption, another pair of atoms, originally in states $|2, k+q\rangle$ and $|1, p-q\rangle$, scatter into states $|2, k\rangle$, $|1, p\rangle$, filling the hole created by the RF-transition. This description can be better illustrated with figure 2. While the latter description sounds like the processes (RF-photon absorption and atom–atom scattering) are disconnected, the first interpretation shows that they are indeed connected through hole degrees of freedom. However, below we will use the latter description as it generalizes more easily. The corresponding diagrams for holes can easily be formulated as the two representations are completely interchangeable.

In the following, we will assume that the propagator lines for atoms in hyperfine state $|1\rangle$ are described by bare Green’s functions and only the atoms (or holes) in hyperfine state $|2\rangle$ are described by dressed propagators. If we look at the residual interactions $H_{res}^0$ in the quasiparticle basis, it is clear that in the first order of scattering the single quasiparticle created by the RF-field
Figure 2. An alternative way of seeing the process in figure 1. The absorption of the photon by an atom in the ground state of $|2\rangle$ produces a particle in hyperfine state $|3\rangle$. After the photon absorption, another pair of atoms in states $|2, k + q\rangle$ and $|1, p - q\rangle$ scatter into states $|2, k\rangle$, $|1, p\rangle$, filling the hole created by the photon absorption.

couples only to a state with three quasiparticles (through equation (9)) and not to the gapless AB phonon with a single quasiparticle. This problem is generic to a perturbative approach, as the description of the collective AB phonon would require self-consistent, or nonperturbative, treatment. The assumption invoked here allows us to capture elements from the AB phonon since the scattering does not introduce extra quasiparticles. Indeed, in the zero-momentum scattering asymptote $q \rightarrow 0$, the energy shift due to the scattering becomes $\hbar c q$, where $c = \frac{\hbar p}{m}$. Close to the Fermi surface, $p \approx k_F$, this yields the speed of sound as expected for the AB phonon.

Thus, by using bare propagators for atoms in state $|1\rangle$, figure 2 yields an approximative description of the coupling to the AB mode. To elaborate, by allowing the single quasiparticle excitation to scatter into lower-energy states, the hole wavefunction will spread over several momentum states leading into the decay of the excitation. However, since the process is correlated with the RF-photon absorption, it is possible for the quasiparticle excitation to scatter also to a higher-energy state if the RF photon has excess energy. Simply put, this means that the excess energy of the RF photon is transferred into extra kinetic energy of atoms. However, since the RF transition conserves momentum, this change in kinetic energy is provided by the atom–atom scattering. Similarly, given extra energy from the probing photon, a quasiparticle at the bottom of the quasiparticle energy band (corresponding to a hole at the Fermi surface) can decay by scattering to higher-energy states, acquiring a finite lifetime. This might be connected to the observation of a finite linewidth of such quasiparticles in [37].

Figure 2 can be added to the standard linear response calculation of the RF spectroscopy, thus generalizing the linear response calculation to include couplings with gapless phonon modes. As already discussed in [37], such coupling broadens the sharp quasiparticle peak in the hole spectral function. We will discuss the implications later on in section 5.

Including figure 2 yields the RF transition (for zero photon momentum $q_L = 0$)

$$
\int_0^\infty dt \int_0^t dt' e^{i(\delta E_1 + \delta E_2)t} \left( g_{12} c_{1, p}^+ c_{2, k}^+ c_{2, k+q} c_{1, p-q} \right) e^{-i\delta E_1(t-t')} \left( \Omega c_{3, k}^+ c_{2, k} \right) |\psi(0)\rangle e^{-q t'},
$$

where $\delta E_1 = \epsilon^3_k - \mu + E^2_k - \delta$ is the energy change due to RF-photon absorption (describing the energy of a single particle in state $|k, 3\rangle$ and hole in $|2, k\rangle$), $\delta E_2 = \epsilon^1_p - \epsilon^1_{p-q} + E^2_{k+q} - E^2_k$ is the energy change due to the scattering (corresponding to the kinetic energy difference of the
atom in hyperfine state $|1\rangle$ and the change in the hole excitation energy due to the scattering of the hole from state $|2, k\rangle$ to state $|2, k + q\rangle$, and the factor $e^{-i\eta'}$ has been added to guarantee convergence. The factor $\eta$ plays the role of linewidth of the RF field and it can be interpreted as a switching off of the RF field. Here we have included the hyperfine state indices in the single-particle and quasiparticle energies for clarity. The time integrals will yield the energy conservation but, in order to calculate the actual transition probability amplitude, one will also need to calculate the norm of the state,

$$|\psi\rangle = c_{1,p}^\dagger c_{2,k}^\dagger c_{2,k+q}c_{1,p-q}c_{3,k}^\dagger c_{2,k}|\psi(0)\rangle,$$

which depends on the details of the initial state $|\psi(0)\rangle$. The transition probability can be obtained by calculating the norm of this transition. However, as will be seen later, there are also other diagrams describing the same transition, and these will need to be summed coherently before calculating the probabilities.

The time integrals can be easily evaluated, yielding

$$P_{\text{hole}}^{k,p,q}|\psi\rangle = \frac{\Omega}{\delta E_1 + i\eta'} \frac{g_{12}c_{1,p}^\dagger c_{2,k}^\dagger c_{2,k+q}c_{1,p-q}c_{3,k}^\dagger c_{2,k}|\psi(0)\rangle}{\delta E_1 + \delta E_2 + i\eta}$$

$$= \frac{\Omega}{\epsilon_k^3 - \mu + E_k^2 - \delta + i\eta'} \frac{g_{12}c_{1,p}^\dagger c_{2,k}^\dagger c_{2,k+q}c_{1,p-q}c_{3,k}^\dagger c_{2,k}|\psi(0)\rangle}{\epsilon_p^3 + \epsilon_q^3 + E_{k+q} - \mu - \epsilon_{p-q}^1 - \delta + i\eta},$$

where we have added a lifetime broadening $\eta'$ to the (virtual) intermediate state. We use the lifetime broadening $\eta' = |g_{12}|N_1$, where $N_1$ is the number of particles in state $|1\rangle$, because the two-body scatterings are the dominant decay channel in a dilute gas.

For narrow linewidth $\eta' \ll \eta'$, the transition probability amplitude has a single peak centered on $\delta = (\epsilon_k^3 + \epsilon_p^3) - (\mu - E_{k+q}^2 + \epsilon_{p-q}^1)$. This corresponds to the energy change of a process in which one transfers one dressed ground state particle from state $|2, k+q\rangle$ and one bare particle from state $|1, p - q\rangle$ into two bare particles in states $|3, k\rangle$ and $|1, p\rangle$. Note that in the limit $q \rightarrow 0$, the peak approaches the standard linear response result $\delta = \epsilon_k^3 - \mu + E_k^2$, the asymptotic difference being linear in $q$, as already discussed above.

### 4. Sum rule considerations

Based on very general arguments, the atom–atom interaction effects should vanish from the RF spectra when the interaction strengths between atoms in hyperfine states $|1\rangle$ and $|2\rangle$ ($g_{12}$) and between states $|1\rangle$ and $|3\rangle$ ($g_{13}$) are equal [22, 40]. Thus, we expect that figure 2 needs to be combined with other diagrams of the same order in photon coupling and atom–atom interactions. The other first-order diagrams are shown in figures 3 and 4. The contributions from these are

$$P_{12}^{k,p,q}|\psi\rangle = \frac{\Omega c_{1,p}^\dagger c_{2,k}^\dagger c_{1,p-q}^\dagger c_{3,k}^\dagger c_{2,k+q}c_{1,p-q}c_{3,k}^\dagger c_{2,k}|\psi(0)\rangle}{E_k^2 + \epsilon_p^1 + E_{k+q}^2 - \epsilon_{p-q}^1 + \epsilon_k^3 + E_{k+q}^2 - \mu - \epsilon_{p-q}^1 - \delta + i\eta}$$

and

$$P_{13}^{k,p,q}|\psi\rangle = \frac{\Omega}{\epsilon_k^3 - \mu + E_{k+q}^2 - \delta + i\eta'} \frac{g_{13}c_{1,p}^\dagger c_{3,k+q}c_{1,p-q}c_{3,k+q}c_{2,k+q}}{\epsilon_p^3 + \epsilon_k^3 + E_{k+q}^2 - \mu - \epsilon_{p-q}^1 - \delta + i\eta}.$$
Figure 3. The other first-order diagram for the photon absorption $\hbar \omega$, $q_L$ and atom–atom scattering $g_{12}$ process. Here the atom in hyperfine state $|2\rangle$ scatters, producing one-particle excitation and one-hole excitation. The particle excitation is then transferred to the hyperfine state $|3\rangle$ by the photon absorption.

Figure 4. The first-order diagram for the RF-photon absorption $\hbar \omega$, $q_L$ and for the atom–atom scattering $g_{13}$ process. Here again the RF-photon creates a particle in hyperfine state $|3\rangle$ and a hole excitation in $|2\rangle$. However, now the scattering takes place in the 13-channel. Assuming that initially the hyperfine state $|3\rangle$ is empty, this is the only first-order diagram contributing in this interaction channel.

We refer to figures 2–4 collectively as term resonant scattering and absorption (RSA) diagrams.

The quasiparticle line entering the scattering vertex in figure 3 describes a (mean-field) ground state pair (energy $-E_{k+q}^2 < 0$) and the line leaving the scattering vertex is excited ($E_{k+q}^2 > 0$). The role of the photon is now to transfer the particle excitation to the state $|3\rangle$. Since the quasiparticle has already been excited in the atom–atom scattering process, the photon absorption itself does not create further excitations. This also follows from the observation that the total energy difference for all three diagrams is the same as they describe the same transition. Figures 2 and 4 do not create more than one quasiparticle excitation, and hence the process in figure 3 cannot result in more than one excitation.

Figure 3 warrants an even closer examination. It describes a process in which the atoms scatter before the photon is absorbed. The first part of the diagram (the scattering) can be understood as a first-order correction to the mean-field ground state. Indeed, neglecting the photon line for a second, the state obtained from the first-order time-dependent perturbation theory, when the perturbation (in this case the atom–atom interaction) is switched on adiabatically, is the state one would have obtained from the first-order time-independent perturbation theory. The perturbative correction to the state can be understood as a fluctuation around the unperturbed state, and the photon in figure 3 couples to these fluctuations. While figure 3 could be removed by including the first-order time-independent perturbative correction to the initial state $|\psi(0)\rangle$ (that is, by including the fluctuations in the state $|\psi(0)\rangle$), figure 2
cannot. If we interpret figure 3 as a process in which fluctuations absorb the photon, figure 2 corresponds to a process in which the fluctuations are enhanced by the absorption of the photon. Clearly, the latter process cannot be included in the initial state $|\psi(0)\rangle$ as the photon field is absent at time $t = 0$. The present formulation as time-dependent perturbation theory allows an easy and consistent (in the sense of the sum rules discussed below) way to include the effect of fluctuations in the spectroscopic probes.

4.1. Asymptotic high-energy scaling

The form of equation (13) has interesting implications for the asymptotic tail of the RF spectrum. In a uniform three-dimensional (3D) system, the single-particle energy dispersion is $\epsilon_k = \frac{\hbar^2 k^2}{2m}$. In the high-energy asymptote of the spectrum $\delta \gg \epsilon_F$, pairing effects described by the gap $\Delta$ have little effect (allowing the approximation $E_\delta^2 = \sqrt{(\epsilon_k - \mu)^2 + \Delta^2} \approx \epsilon_k$) and resonant processes are dominated by scatterings to high-momentum states $p, k \gg k_F$, where $k_F$ is the Fermi momentum. Assuming that only low momentum states are initially occupied, one has $k + q \approx p - q \approx 0$ (hence $k \approx -p$ and $q$ is large). Now equation (13) yields

$$P_{q,k,p}^{12-\text{asymptote}} |\psi\rangle = \frac{g_{12}}{2^{\frac{3}{2}} \hbar^2 m + i\eta} \frac{\Omega c^\dagger_{3,k} c^\dagger_{2,k} c^\dagger_{1,p} c^\dagger_{2,k+q} c_{1,p-q} |\psi(0)\rangle}{2^{\frac{3}{2}} \hbar^2 m - \delta + i\eta}. \quad (15)$$

In the large $k$ limit, the probabilities of the states $|2, k\rangle$ and $|1, p = -k\rangle$ being empty approach unity, and likewise the occupation probabilities for states $|1, p - q \approx 0\rangle$ and $|2, k + q \approx 0\rangle$ approach unity. Calculating the transition probability $|P_{q,k,p}^{12-\text{asymptote}} |\psi\rangle|^2$ and integrating over $k$, one obtains the transition probability for a single particle in state $|2, k + q\rangle$ to be transferred to hyperfine state $|3, k\rangle$,

$$P_{q,k,p}^{12-\text{asymptote}} \sim \int_0^\infty d\epsilon \sqrt{\epsilon} \left| \frac{g_{12}}{2^{\frac{3}{2}} \hbar^2 m + i\eta} \frac{\Omega c^\dagger_{3,k} c^\dagger_{2,k} c^\dagger_{1,p} c^\dagger_{2,k+q} c_{1,p-q} |\psi(0)\rangle}{2^{\frac{3}{2}} \hbar^2 m - \delta + i\eta} \right|^2, \quad (16)$$

where the $\sqrt{\epsilon}$ factor comes from the density of states in a 3D uniform system. For narrow linewidth $\eta \ll \delta$, only energies $\epsilon$ close to $\delta/2$ contribute to the integral and it can be approximated as

$$P_{q,k,p}^{12-\text{asymptote}} \sim \int_0^\infty d\epsilon \sqrt{\delta/2} \left| \frac{g_{12}}{\delta + i\eta'} \frac{\Omega}{2\epsilon - \delta + i\eta} \right|^2 = \sqrt{\delta/2} (\frac{g_{12}\Omega}{\delta^2 + \eta'^2})^2 \frac{\pi}{\eta}. \quad (17)$$

Assuming $\delta \gg \eta'$, we obtain the asymptotic high RF detuning tail $P_{q,k,p}^{\text{asymptote}} \sim \delta^{-1.5}$. It is important to note that this result is independent of a possible ‘quasiparticle tail’ in the occupation numbers $v_{2,k}$. Figures 2 and 4 also produce high-energy tails for the spectra. The asymptotic form of the latter diagram is

$$P_{q,k,p}^{13-\text{asymptote}} \sim \int_0^\infty d\epsilon \sqrt{\epsilon} \left| \frac{\Omega}{-\delta + i\eta'} \frac{g_{13}}{2\epsilon - \delta + i\eta} \right|^2, \quad (18)$$

which yields the same asymptotic behavior as equation (17) but is proportional to $g_{13}$ instead of $g_{12}$. Unlike the above two asymptotic forms, the high-energy asymptote of figure 2 requires occupation of high-momentum states and it is given by

$$P_{q,k,p}^{\text{hole-asymptote}} \sim \int_0^\infty d\epsilon \sqrt{\epsilon} \left| \frac{\Omega}{2\epsilon - \delta + i\eta} \frac{g_{12} v_{2,\epsilon}}{2\epsilon - \delta + i\eta} \right|^2, \quad (19)$$

New Journal of Physics 12 (2010) 083041 (http://www.njp.org/)
where $v_{2,\epsilon}$ is the probability amplitude that there is an atom in hyperfine state 2 with energy $\epsilon$. For a BCS-like state, this is given by the Bogoliubov coefficient $v_{2,\epsilon}^2 \approx \frac{1}{2} (1 - \epsilon/\sqrt{\epsilon^2 + \Delta^2})$ yielding the same asymptotic high-energy tail $\delta^{-1.5}$. Here we have neglected terms of the order $v_{4,k}$, which is justified assuming that the occupation probabilities do not have too fat a tail. The most important point to note, however, is that only equation (19) depends on high-energy excitations in the initial state $|\psi(0)\rangle$.

The $\delta^{-1.5}$ scaling law has been suggested already in various studies [41]–[44], where the scaling law comes from the high-energy tail of the occupied atom states (the $v_{2,k}$ or $v_{2,\epsilon}$ tail). In the present theory, no such many-body correlations are needed in the initial state—the power law arises directly from the RSA process. In particular, the RSA scaling law applies in the limit $\Delta \to 0$ and also to all momentum $k$ modes separately, resulting in a $\delta^{-1.5}$ tail in the spectral function of every $k$ mode. This will have implications in particular for the momentum resolved spectroscopy [8], which we will consider in section 5.1.

All the RSA diagrams lead into the same final state $|\psi\rangle$ and thus they interfere coherently. A closer examination of the orderings of the creation and annihilation operators shows that, because of fermionic anticommutation rules, equation (14) has opposite sign to equations (12) and (13), and hence the three contributions interfere destructively. This is easy to see by operating on equation (14) with the identity operator $1 = c_{2,k}^{\dagger} c_{2,k} + c_{2,k} c_{2,k}^{\dagger}$. In particular, for identical interactions ($g_{12} = g_{13}$) and for equal dispersions ($\epsilon_{2,k} = \epsilon_{2,k}',$ for all $k$), the three RSA diagrams combine into a single sequential process. Such a process does not allow energy exchange between the quasiparticle and the emitted phonon. Thus, in the symmetric case, our model does not provide any energy shift to the Rf spectrum. This is in agreement with sum rules.

The sum rule argument must hold for all orders of atom–atom scatterings, and hence it will hold also (or especially) in the nonperturbative treatment. Thus, the picture from our model, in which the RF-photon absorption and the excitation of the phonon are correlated, should also hold for a real system in which the phonon is the AB phonon.

If the final state $|3\rangle$ is interacting, also the BCS mean-field response changes, and one needs to include the Aslamazov–Larkin (AL) and Maki–Thompson (MT) contributions. These resemble our beyond mean-field diagrams. However, these are strictly mean-field effects and do not contribute to the quasiparticle excitation lifetimes. The effect of these diagrams have been studied in the context of RF spectroscopy of atomic gases in [31, 45].

5. Comparison with experimental results: fermionic gases

All spectroscopies used in the context of ultracold atom gases can be understood as an effective photon absorption, and hence we expect diagrams similar to figure 2 to play a role. The main difference in various spectroscopic methods is the atom degree of freedom that is affected: hyperfine state (RF and Raman spectroscopies), momentum state (Bragg and Raman spectroscopies) or lattice band (lattice modulation and Bragg spectroscopies). Furthermore, the diagrams (with slight modifications) can also be applied to bosonic atoms. In the following, we will analyze seven different experiments that we believe show the effect of the RSA process.

5.1. Radio-frequency spectroscopy of a Fermi gas

An experiment performed in JILA in 2008 [8, 26] applied momentum resolved RF spectroscopy to probe the single-particle excitation spectrum of a two-component Fermi gas. The spectrum
in figure 3(c) of [8] was shown to be broader than expected, and the effect was assumed to follow from the center-of-mass motion of the pairs in the Bose–Einstein condensate (BEC) side of the Feshbach resonance. However, in a 3D Fermi gas, also the RSA process produces the broadening effect. Furthermore, figure 4 in [8] shows asymmetric spectra for atoms of a given momentum. For a fixed momentum \( k \), the linear response theory predicts a symmetric spectrum centered at the single-particle resonance at \( \xi_k - \mu + E_k + U_{12} n_1 \). The RSA diagram allows the transition with more energetic RF photons, producing a tail on the negative side of the single-particle peak.

The present theory can easily be adapted for describing this experiment. Due to high temperatures, we assume that the initial state can be described by a normal finite temperature Fermi gas, implying that the quasiparticle energies in equations (12) and (13) can be replaced by single-particle dispersion \( E^2_k = |\epsilon_k - \mu| \). The contributions from the two equations can be combined, and we obtain in the limit of narrow linewidth (\( \eta \to 0 \)) the transition probability

\[
|P_{k,p,q}|^2 = \frac{\Omega^2 g_{12} n_F(\epsilon_{p-q}) n_F(\epsilon_{k+q}) n_F(-\epsilon_p)}{(\delta)^2 + \eta^2} \delta(\epsilon_{k+q} + \epsilon_p - \epsilon_{p-q} - \delta),
\]

where \( \delta(x) \) is the Dirac delta function and \( n_F(x) = 1/(1 + e^{x/k_B\beta}) \) is the Fermi–Dirac distribution at temperature \( T = 1/k_B\beta \), where \( k_B \) is the Boltzmann constant. The lifetime broadening \( \eta \) should in principle depend on the energy of the excitation, but here we will use the average lifetime obtained from the Galitskii equation [46],

\[
\eta = \frac{2 (k_Fa)^2}{\pi} \frac{d^3 p}{d^3 p} \left( \frac{1 - |p|}{k_F} \right)^2 \theta(k_F - |p|) = \frac{4}{5\pi}.
\]

where we have replaced \( k_Fa = -2 \). This value is motivated by a two-body scattering calculation, which yields a \( k \)-dependent scattering amplitude [46]. It should be noted that the qualitative results do not depend sensitively on the choice of this lifetime broadening. The momentum-resolved spectra can be calculated by summing over momenta \( p \) and \( q \) in equation (20) and plotting the transition probabilities as functions of momentum \( k \) and energy \( \epsilon_k - \delta \). Doing this, we obtain the spectra shown in figure 5, which can be directly compared with momentum resolved spectra in [8, 47]. Notice, however, that the latter experiment is performed at temperatures below the critical temperature, and it is somewhat surprising that, in this case, we obtain a good match with the experimental spectra, having assumed a gapless state. This warrants a more detailed analysis, which will be a topic of further work. Despite the simplicity of the current model, the qualitative agreement is excellent. Clearly, normal-state correlations are sufficient for producing the back-bending \( -\epsilon_k \) dispersion, as also seen in [44], where a more complete T-matrix calculation was carried out for \( k \gg k_F \). The present model, however, can also describe the low-momentum regime where the hole degrees of freedom become important.

It is worth pointing out that possibly the most controversial assumption made in our theory, namely assuming that the atoms in hyperfine state \( |1\rangle \) are described by bare propagators, was not needed in the above calculation of the momentum-resolved spectrum for a normal state since the dressed and bare propagators are identical.

In a recent experiment by the same group [48], the high-momentum asymptote was studied also by measuring the expansion of the atom gas after abruptly switching off the interactions. This asymptote was found to match very well with the high-momentum tail obtained from the RF spectrum, in agreement with the theory of S. Tan [41, 42]. It is worth noting the similarity

\[\text{New Journal of Physics 12 (2010) 083041 (http://www.njp.org/)}\]
Figure 5. The momentum resolved RF-spectra for the gapless system at zero temperature (left) and for $T = 0.2 T_F$ (right). For low momenta $k \ll k_F$, the spectrum follows the free particle dispersion. For high momenta $k \gg k_F$, the spectrum follows a negative $-\frac{\hbar^2 k^2}{2m}$ dispersion. The transition region around $k_F$ shifts towards higher momenta for increasing temperature in qualitative agreement with the experimental results. The white curves indicate the centers of Gaussian fits to the individual spectra for different momenta $k$. These are to be compared with the experimental data in [8, 47].

between transferring a particle into a noninteracting hyperfine state $|3\rangle$ and switching off the interactions. While in the first the moment of the transition is not well defined (reflected as a time integral in the response), in the latter the moment when the interactions are switched off is determined by the external magnetic field. Despite this important difference, one can apply the RSA model also to the latter setting. The only contributing diagram is figure 3, as the scattering must take place before the interactions are switched off. Now the corresponding integral formula is

$$\frac{i}{\hbar} \int_0^T dt \ e^{i\left(E_k^2 t + E_{k+q}^2 t - E_{p-q}^2 t\right)/\hbar} \left(g_{12} c_{1,p}^\dagger c_{2,k}^\dagger c_{2,k+q} c_{1,p-q}\right) |\psi(0)\rangle e^{-\eta t}. \quad (22)$$

Evaluating this equation, and taking the limit $T \to \infty$ (one could also keep the time $T$ finite, but we want to compare the result with RF spectra for which we considered the limit of infinitely long pulse), yields

$$\frac{g_{12}}{E_k^2 + \epsilon_p^1 + E_{k+q}^2 - \epsilon_{p-q}^1 + i\eta} c_{1,p}^\dagger c_{2,k+q} c_{1,p-q} |\psi(0)\rangle. \quad (23)$$

Now the probability of finding an atom in hyperfine state $|2\rangle$ with momentum $k$ is

$$\left| \sum_{p,q} \frac{g_{12}}{E_k^2 + \epsilon_p^1 + E_{k+q}^2 - \epsilon_{p-q}^1 + i\eta} c_{1,p}^\dagger c_{2,k+q} c_{1,p-q} |\psi(0)\rangle \right|^2. \quad (24)$$

As above, in the discussion of the high-energy tail of the RF-spectrum, we obtain constraints $p - q \approx k + q \approx 0$, and the energy change due to the scattering becomes $E_k^2 + \epsilon_p^1 + E_{k+q}^2 - \epsilon_{p-q}^1 \approx 2\epsilon_k$, yielding the perceived occupation probability for momentum state $k$ in the high-momentum $k \to \infty$ limit,

$$\frac{g_{12}^2}{2\epsilon_k + i\eta} \sim \frac{1}{k^4}. \quad (25)$$
This $k^{-4}$ occupation probability produces the $\delta^{-3/2}$ asymptote in the RF spectrum in linear response theory, and this asymptotic behaviour is also exactly the same as obtained for the RF spectrum in the RSA model. Thus, the tail observed in [48] is also in agreement with the RSA model. However, we have made no assumptions regarding the occupation numbers for the initial state $|\psi(0)\rangle$.

5.2. Bragg spectroscopy of a Fermi gas

Another experiment used Bragg spectroscopy to study a strongly interacting Fermi gas [12]. The experiment fitted very well with the theoretical mean-field predictions, including both the heights and the widths of the spectral peaks. In the case of Bragg spectroscopy, the hyperfine state of the atom is not changed in the photon absorption but only the momentum state. Therefore, the hyperfine state $|3\rangle$ in the RSA diagrams corresponds to hyperfine state $|2\rangle$. Thus, the final-state interactions discussed in the context of RF spectroscopy match the initial interactions $g_{13} = g_{12} = g$, and the diagrams interfere destructively. Thus one does not expect any contribution from the RSA effect to the Bragg spectroscopy in a Fermi gas. This is in good agreement with experiment. This observation shows that the Bragg spectroscopy could be a good way of measuring single-particle properties in a Fermi gas since the RSA-type beyond mean-field effects are suppressed.

6. Comparison with experimental results: bosonic gases

The RSA effect can also be applied to bosonic atoms. However, especially in the presence of a BEC, the effect of different diagrams is changed, because atoms within the condensate do not feel the exchange interaction channel [49]. In the following, we will first consider a superfluid Bose gas and experiments performed on such systems, then later on discuss the Mott insulator state and the corresponding experiments.

6.1. Resonant scattering and absorption (RSA) effect in Bose–Einstein condensate (BEC)

Let us consider a single-component Bose gas and let us assume, for the sake of simplicity, that all atoms are initially Bose–Einstein condensed into a zero-momentum state. Now the relevant RSA diagrams for Bragg and lattice modulation spectroscopies are figures 3 and 4, but, since there is only one hyperfine component present, the hyperfine states $|1\rangle$, $|2\rangle$ and $|3\rangle$ are the same. Figure 2 does not contribute in the case of an ideal BEC, because the diagram would require atoms initially in two different momentum states $k$ and $k+q$, and hence it requires uncondensed atoms. Of course, such a process would be allowed for nonideal BECs and may be important in the case of strongly interacting BECs where quantum depletion is important or at high temperatures.

As in the case of fermions, these bosonic RSA diagrams interfere coherently, but the only contribution comes from momenta $k+q = p - q = 0$. However, figure 4 also experiences the exchange interaction channel as the atom–atom scattering does not take place between two condensed atoms but between one condensed atom and one atom with momentum $q_L$. Hence, the two RSA diagrams do not fully cancel each other. This partial cancelation of the RSA contribution produces interesting effects in the case of a superfluid BEC in an optical lattice.
6.2. Lattice modulation spectroscopy of a Bose gas

A superfluid Bose gas in a 1D lattice was studied using lattice modulation spectroscopy [15] and Bragg spectroscopy [14, 50]. In the experiments, the atoms were transferred within the first lattice band [15, 50], as well as between the first lattice band and the second and third bands [14], while changing the atom momentum due to the momentum carried by the photons. The transition to the second band was as predicted by the standard Bogoliubov (mean-field) theory, but the spectral peak of the transition to the third band and transitions within the first band were anomalously broad (see figure 3 in [14], figure 3(a) in [50] and figure 1 in [15]). These effects were attributed to a loss of coherence [14], quantum depletion of the condensate [15], and beyond mean-field correlation and thermal effects in a 1D system [50].

The present theory suggests an alternative explanation. Since atoms in different lattice bands have reduced overlap, the interband interactions are reduced. In the harmonic potential approximation, atoms in the second band have an extra factor 0.5 in the interactions with the first band (and in particular with the condensate), and the atoms in the third band have a factor 0.375. Including these factors, the bosonic RSA diagrams will cancel each other for the transition to the second band (as the contribution from figure 4 is reduced by factor 0.5, canceling the exchange interaction channel effect) but will yield only a factor $1 - 2 \times 0.375 = 0.25$ (contribution from figure 4 is multiplied by 0.375) for the transition to the third band and a factor 1 for the transitions within the first band. Thus, the bosonic RSA effect does not affect the transitions to the second lattice band, but it is present for the transitions to the first and third bands. Similarly to the fermionic gases, the bosonic RSA effect will spread the spectral peaks. This is in qualitative agreement with the experimental observations. Furthermore, the width of the transition to the third band approaches the width of the third band, and the width of the transitions within the first band is approximately twice the first band width, in agreement with present theory. Figure 6(a) shows the schematic energy-level diagram for the transition to the third band.

6.3. Bragg spectroscopy of a Bose gas

The Bragg spectroscopy of a Bose gas has been studied also in a harmonic trap. The experiment in JILA in 2008 applied Bragg spectroscopy to study a strongly interacting BEC [13, 25, 51]. The width of the spectral peak showed anomalous broadening in the strongly interacting regime (see figure 3 in [13]). The bosonic RSA process explains the broadening by allowing transitions at higher energies, as in the above discussion regarding the 1D BEC experiments [14, 15, 50] and the momentum-resolved RF spectroscopy [8]. Moreover, the bosonic RSA process provides transitions also at lower energies, since noncondensed (at finite momentum $k$) atoms can be scattered into the condensate before absorbing the Bragg photon (in which case the atom–atom scattering before the photon absorption feels the exchange interaction channel but the opposite process does not). This reduces the kinetic energy of the atoms, allowing transitions at lower frequencies and thus further contributing to the width of the spectrum. These atom–atom scatterings down into the condensate are enhanced by the coherent state, and thus the RSA processes, scatterings up from the condensate and scatterings down into the condensate, are equally likely, even when the quantum depletion is weak. Since the first process produces a high-energy tail due to transitions to higher kinetic-energy states, the latter process produces a similar low-energy tail. Hence, the total spectral lineshape remains symmetric, as also observed in the experiment [13].

New Journal of Physics 12 (2010) 083041 (http://www.njp.org/)
Figure 6. Bosonic RSA energy-level scheme relevant for Bragg scattering and lattice modulation experiments [14], [50], and [15]. (a) The atom–atom interaction $U$ excites two particles (two dashed arrows) from the zero-momentum condensate. One of the atoms is then scattered by the photon $\delta$ into the third band (the single dot-dashed line). (b) In the Mott state, the atom–atom interaction $U$ will yield a large energy shift due to the Mott gap, leading into a two-peak structure in the spectrum. The Bragg photon $\delta$ provides an additional momentum kick (dot-dashed line) and needs to provide the energy for creation of the two Mott excitations.

Note that a similar process as described by the RSA effect was crucial for the experiment, as the actual measured response was the total momentum of the atom gas [13]. The single particle excitations created by the Bragg scattering decayed due to scatterings with other atoms, and the excess momentum yielded by the Bragg photon was thus transferred into center-of-mass momentum of the atom cloud. Thus, the single-particle excitation was turned into a collective motion of the cloud. What our RSA effect now suggests is that these atom–atom scatterings and the initial Bragg scattering are coherent in the sense that they are able to exchange energy as well as momentum.
6.4. Resonant scattering and absorption (RSA) effect in bosonic Mott insulator

In the case of a Mott insulator state in a Bose gas in an optical lattice, figure 2 contributes as the condensate fraction vanishes and higher momentum states become populated in the ground state. Since there is no condensate present, all interactions experience the exchange interaction channel. However, the biggest difference in the interpretation of these diagrams as compared to superfluid gases is that the excitations created by the photon absorption do not couple to gapless excitations, as the gapless excitations in the Mott state are spin-flip excitations and these are not induced by atom–atom scatterings. Instead, the scattering of two atoms in the Mott insulator will necessarily create two single-particle excitations that are gapped by the Mott gap $U$, where $U$ is the on-site interaction energy. The energy-level scheme of the bosonic RSA process in a Mott insulator is shown in figure 6(b), demonstrating how the process produces two Mott excitations.

The 1D Bose gases in the Mott insulator regime were studied using Bragg spectroscopy [50] and lattice modulation spectroscopy [15]. The measured spectra, such as in figures 1(f)–(h) in [50] and figure 1 in [15], show two distinct peaks, the second resonance at twice the energy of the first resonance. The second peak was attributed to defects in the lattice, with atoms being transferred to doubly occupied sites [15]. These observations fit in the picture drawn above as the additional peaks observed in [50] and [15], and they can be understood as a creation of two Mott excitations with energy cost equal to twice the Mott gap $g$. The process is made resonant by sufficiently energetic photon absorption, reflected as the second peak in the spectra. Notice that in the second order of atom–atom scatterings, one can also create three Mott excitations. Such processes should result in spectral peaks at three times the Mott gap.

6.5. Summary

To summarize, we have analyzed seven experiments in the light of our theory, and found that there is a qualitative match. However, for the case of momentum–resolved RF spectroscopy, we performed quantitative analysis, obtaining good agreement between experiment and theory. As in the case of momentum–resolved spectroscopy in Fermi gases, calculating quantitative predictions for the other experiments using the RSA theory can be done. For example, in the case of spectroscopies of superfluid BECs in [13]–[15], [50], the starting point could be an ideal BEC and the interactions would then be considered only in the first-order perturbation theory by the present diagrammatic method. The approximation can also be easily improved by using the Bogoliubov theory, and the subsequent ground state, as the starting point. This is, indeed, an interesting topic for further work.

7. Single-band Fermi–Hubbard model

Besides experiments, also exact numerical results provide a good platform for testing the theory. In this section, we will study a Fermi gas in a 1D optical lattice described by the single-band Fermi–Hubbard model. We will compare the predictions from the RSA theory to the exact RF-spectra obtained using the time-evolved block decimation (TEBD) method.

In the language used in these systems, the RF photon can be understood as creating a spin excitation, and the RSA process describes the decay of this excitation by scattering into lower-energy spin excitation and creating a gapless charge excitation. The charge excitation thus plays the role of the AB phonon. We apply the TEBD method as it allows us to access the exact RF
spectrum of the system. The time evolution of the system is determined by the Fermi–Hubbard Hamiltonian,

$$H(t) = -J \sum_{i, \sigma \in \{1, 2, 3\}} c_{i\sigma}^\dagger c_{i+1\sigma} + \text{h.c.} + U \sum_i n_{i2} n_{i1} + \Omega e^{2it/\hbar} \sum_i c_{i3}^\dagger c_{i2} + \text{h.c.},$$  \hspace{1cm} (26)$$

where $U < 0$ is the attractive on-site interaction strength between atoms in states $|1\rangle$ and $|2\rangle$ and the density operator $n_{i\sigma} = c_{i\sigma}^\dagger c_{i\sigma}$, where $c_{i\sigma}^{(\dagger)}$ destroys (creates) a particle of hyperfine state $|\sigma\rangle$ in lattice site $i$. The state $|3\rangle$ is assumed to be noninteracting. Below, we use units that have the hopping strength $J = 1$.

We use the TEBD algorithm for solving the ground state of the Fermi–Hubbard Hamiltonian in the absence of the RF field $\Omega = 0$ and for the empty hyperfine state $|3\rangle$. Starting from this ground state, we switch on the RF field and calculate the time evolution. Figure 7 shows the fraction of atoms in state $|3\rangle$ after the RF pulse as a function of RF-detuning $\delta$.

The BCS theory is known to fail in 1D lattices. Even though the mean-field model reproduces exact ground state energies faithfully, the behavior of the pairing gap is phenomenally bad. In order to fix this, we do not solve the pairing gap equation but rather derive them from the TEBD ground state following the procedure in [52]. We use these values $\Delta_{\text{TEBD}}$ and $\mu_{\text{TEBD}}$ in the BCS-type quasiparticle dispersion energies $E_k^{\mu} = \sqrt{(\epsilon_k - \mu_{\text{TEBD}})^2 + \Delta_{\text{TEBD}}^2}$, where the lattice dispersion $\epsilon_k = -2J \cos k$ for $k \in [-\pi, \pi]$, and in the Bogoliubov coefficients for the occupation numbers $\nu_k^2 = \frac{1}{2}(1 - \Delta_k^2/\epsilon_k)$. The linear response RF spectra, obtained using these TEBD derived parameters in the quasiparticle dispersion $E_k^{\mu}$, agree very well with parts of the observed TEBD spectra. However, the high-energy tails of the spectra are missing from the linear response spectra, as the spectra are consistently too narrow. This shows that the BCS-type hole spectral function does not capture all of the relevant physics of the RF-photon absorption. The question is now whether this is a problem with the BCS-like description of the ground state (that is, how well does the BCS-like quasiparticle dispersion work even when used with the TEBD-derived parameters?) or with the dynamics described by the linear response theory. With this caveat in mind, let us consider how the RSA process would affect the picture.

Figure 7 also shows the RSA spectra from equations (12) and (13). This contribution has been separated into two parts, one corresponding to atom–atom scatterings to higher kinetic-energy states $\Delta K_{kpq} = E_{k+q}^1 + \epsilon_p^1 - E_k^2 + \epsilon_{p-q}^1 > 0$ and one for scatterings to lower kinetic-energy states $\Delta K_{kpq} < 0$. While the former process can be understood as a phonon excitation, the latter process may be unphysical and possibly an artifact from our approximative approach. We will neglect these scatterings to lower-energy states for a moment and discuss them later on.

The combination of the standard linear response spectrum and the RSA spectrum seems to describe rather well all qualitative features of the exact TEBD spectrum: equation (7) reproduces the main peak of the exact spectrum and equation (12) provides the correct width for the spectrum (neglecting the $\Delta K < 0$ contribution). Indeed, the $8J$ width of the spectrum can be observed in all TEBD spectra at different filling fractions and different interaction strengths (although resolution becomes difficult for weak interactions $-U < 4J$). This width in the 1D lattice is also a general property of equations (12) and (13). The low-energy end of the spectrum is provided by low momentum $|2, k = 0\rangle$-atoms that scatter from $|1, p = k_F\rangle$-atoms at the Fermi surface, but the change in momentum is small ($q = 0$). Effectively, it corresponds to no scattering at all, and the resonant energy for the transition is $\delta_{\text{low}} = \epsilon_0^3 - \mu + E_0^2$. The high-energy end of the spectrum is provided by $|2, k = \pi\rangle$-atoms at the edge of the lattice band.
Figure 7. Exact RF spectra in 1D lattice. The TEBD method is used for calculating the exact RF spectra for filling fractions (per spin state) (a) 0.3 \((1.1 J, -1.45 J)\), (b) 0.5 \((1.1 J, 0.0 J)\) and (c) 0.7 \((1.1 J, 1.45 J)\), with corresponding BCS fitting parameters shown as \((\Delta, \mu)\), respectively. The on-site interaction strength is \(U = -5 J\), RF-coupling strength \(\Omega = 0.025 J\) and the length of the pulse is \(T = 10 \hbar J^{-1}\). Shown are also linear response (mean-field) spectra and spectra from the RSA model for both scatterings to higher and lower kinetic-energy states (see the text for discussion). The RSA transition probabilities are not properly normalized and hence the RSA spectra have been divided by 100 in order to better fit them in the picture. We have used 40 lattice sites in these simulations.

scattering from \(|1, p = 0\rangle\)-atoms in the bottom of the Fermi sea with momentum exchange \(q = -\pi\), yielding resonant energy \(\delta_{\text{high}} = \epsilon_3^3 + E_0^2 + \epsilon_1^1 - \epsilon_0^0 - \mu\). Thus, the total width of the spectrum equals \(\delta_{\text{high}} - \delta_{\text{low}} = \epsilon_3^3 + \epsilon_1^1 - \epsilon_0^0 - \epsilon_3^0 = 8 J\), independent of the chemical potential \(\mu\) (and hence filling fraction) and the excitation gap \(\Delta\). This is in excellent, quantitative agreement...
with the exact TEBD spectra. Notice also that this implies that the width of the spectrum is actually insensitive to the way in which $\Delta$ and $\mu$ are determined, and even the values from the BCS theory would yield the same total width; only the positions of the peaks would be affected. Finally, the quasiparticle dispersions vanish from the calculation of the total width, and hence, even if one relaxed the assumption of the BCS-type quasiparticle dispersion $E^\text{fit}_k$ somewhat, the result would remain the same.

At high filling fractions, our model predicts a low-energy tail extending beyond the $8J$ width in disagreement with the TEBD spectra. The tail is caused by scatterings to lower-energy states $\Delta K_{kpq} < 0$. Since the $|1\rangle$-atoms are forbidden from scattering to lower kinetic-energy states due to the presence of a well-formed Fermi sphere, the lowering of the energy is possible only if the increase in the kinetic energy of the $|1\rangle$-atom is lower than the decrease in the quasiparticle energy of the $|2\rangle$-atom. Thus, we have condition

$$0 > \Delta K_{kpq} = (E_{k+q} - E_k) + (\epsilon_p - q - \epsilon_p) + (\epsilon_p - q - \epsilon_p).$$

(27)

In the long scattering wavelength limit $q \to 0$, this yields

$$\frac{dE_k}{dk} < -\frac{d\epsilon_p}{dp} \bigg|_{k=k_F},$$

(28)

where the right-hand side must be evaluated close to the Fermi surface because of the well-formed Fermi sphere for $|1\rangle$-atoms. The left-hand side yields $\frac{\epsilon_k - \mu}{E_k} \frac{d\epsilon_p}{dk}$, where the prefactor $(\epsilon_k - \mu)/E_k$ has a magnitude of less than 1. However, since the quasiparticles can populate any momentum state $k$, it is possible that equation (28) is satisfied for some $k$, as long as the single-particle dispersion is not too rapidly increasing function at the Fermi momentum $k_F$. In particular, at high fillings, the single-particle dispersion $\epsilon_k$ becomes concave at the Fermi surface and the scatterings to lower kinetic-energy states $\Delta K < 0$ become possible. At half filling, the single-particle dispersion has the highest slope and no $\Delta K < 0$ scatterings are possible. At low fillings, there are some scattering channels as well, but the contribution is very small as compared to the scatterings to higher-energy states $\Delta K > 0$ (typically at least one order of magnitude lower).

8. Discussion

In conclusion, we have formulated a first-order scattering theory that adds to the understanding of interdisciplinary spectroscopies. Despite being of the first order in the photon coupling, the theory goes beyond standard mean-field linear response theories by incorporating an approximative coupling to phonon-like modes. Similar phenomena have been extensively studied in nuclear physics since the 1960s [35], and it would be interesting to compare the present approach with the models used in nuclear physics. The resonant scattering process in ultracold atom gases becomes increasingly important with stronger interactions, and its effect can be seen in a wide range of experiments—here we have analyzed seven of them.

Acknowledgments

This work was supported by the National Graduate School in Materials Physics and Academy of Finland (project no. 217043). We acknowledge inspiring discussions with F Massel and A-P Jauho, and the use of CSC—IT Center for Science Ltd computing resources.
References

[1] Gupta S, Hadzibabic Z, Wierlein M W Z, Stan C A, Dieckmann K, Schunck C H, van Kempen E G M, Verhaar B J and Ketterle W 2003 Radio-frequency spectroscopy of ultracold fermions Science 300 1723

[2] Regal C A, Ticknor C, Bohn J L and Jin D S 2003 Creation of ultracold molecules from a Fermi gas of atoms Nature 424 47

[3] Chin C, Bartenstein M, Altmeyer A, Riedl S, Jochim S, Denschlag J H and Grimm R 2004 Observation of the pairing gap in a strongly interacting Fermi gas Science 305 1128

[4] Moritz H, Stöferle T, Günter K, Köhl M and Esslinger T 2005 Confinement induced molecules in a 1D Fermi gas Phys. Rev. Lett. 94 210401

[5] Bartenstein M et al 2005 Precise determination of $^6$Li cold collision parameters by radio-frequency spectroscopy on weakly bound molecules Phys. Rev. Lett. 94 103201

[6] Schunck C H, Shin Y, Schirotzek A, Wierlein M W Z and Ketterle W 2007 Pairing without superfluidity: the ground state of an imbalanced Fermi mixture Science 316 867

[7] Schunck C H, Shin Y, Schirotzek A and Ketterle W 2008 Determination of the fermion pair size in a resonantly interacting superfluid Nature 454 739

[8] Stewart J T, Gaebler J P and Jin D S 2008 Using photoemission spectroscopy to probe a strongly interacting Fermi gas Nature 454 07172

[9] Blakie P B 2006 Raman spectroscopy of Mott insulator states in optical lattices New J. Phys. 8 157

[10] Dao T-L, Georges A, Dalibard J, Salomon C and Carusotto I 2007 Measuring the one-particle excitations of ultracold fermionic atoms by stimulated Raman spectroscopy Phys. Rev. Lett. 98 240402

[11] Stenger J, Inouye S, Chikkatur A P, Stamper-Kurn D M, Pritchard D E and Ketterle W 1999 Bragg spectroscopy of a Bose–Einstein condensate Phys. Rev. Lett. 82 4569

[12] Veeravalli G, Kuhnle E, Dyke P and Vale C J 2008 Bragg spectroscopy of a strongly interacting Fermi gas Phys. Rev. Lett. 101 250403

[13] Papp S B, Pino J M, Wild R J, Ronen S, Wieman C E, Jin D S and Cornell E A 2008 Bragg spectroscopy of a strongly interacting $^{85}$Rb Bose–Einstein condensate Phys. Rev. Lett. 101 135301

[14] Fabbri N, Clément D, Fallani L, Fort C, Modugno M, van der Stam K M R and Inguscio M 2009 Excitations of Bose–Einstein condensates in a one-dimensional periodic potential Phys. Rev. A 79 043623

[15] Stöferle T, Moritz H, Schori C, Köhl M and Esslinger T 2004 Transition from a strongly interacting 1D superfluid to a Mott insulator Phys. Rev. Lett. 92 130403

[16] Schori C, Stöferle T, Moritz H, Köhl M and Esslinger T 2004 Excitations of a superfluid in a three-dimensional optical lattice Phys. Rev. Lett. 93 240402

[17] Jördens R, Strohmaier N, Günter K, Moritz H and Esslinger T 2008 A Mott insulator of fermionic atoms in an optical lattice Nature 455 07244

[18] Törmä P and Zoller P 2000 Laser probing of Cooper pairs Phys. Rev. Lett. 85 487

[19] Kinnunen J, Rodriguez M and Törmä P 2004 Pairing gap and in-gap excitations in trapped fermionic superfluids Science 305 1131

[20] Ohashi Y and Griffin A 2005 Single-particle excitations in a trapped gas of Fermi atoms in the BCS–BEC crossover region Phys. Rev. A 72 013601

[21] He Y, Chen Q and Levin K 2005 Radio-frequency spectroscopy and the pairing gap in trapped Fermi gases Phys. Rev. A 72 011602

[22] Yu Z and Baym G 2006 Spin-correlation functions in ultracold paired atomic-fermion systems: sum rules, self-consistent approximations, and mean fields Phys. Rev. A 73 063601

[23] Basu S and Mueller E J 2008 Final-state effects in the radio frequency spectrum of strongly interacting fermions Phys. Rev. Lett. 101 060405

[24] Massignan P, Bruun G M and Stoof H T C 2008 Twin peaks in RF spectra of Fermi gases at unitary in the normal phase Phys. Rev. A 77 031601

[25] Kinnunen J J and Holland M 2009 Bragg spectroscopy of a strongly interacting Bose–Einstein condensate New J. Phys. 11 013030
[26] Chen Q and Levin K 2009 Momentum resolved radio frequency spectroscopy in trapped Fermi gases Phys. Rev. Lett. 102 190402

[27] Punk M and Zwerger W 2007 Theory of RF-spectroscopy of strongly interacting fermions Phys. Rev. Lett. 99 170404

[28] Kinnunen J and Törmä P 2006 Beyond linear response spectroscopy of ultracold Fermi gases Phys. Rev. Lett. 96 070402

[29] Perali A, Pieri P and Strinati G C 2008 Competition between final-state and pairing-gap effects in the radio-frequency spectra of ultracold Fermi atoms Phys. Rev. Lett. 100 010402

[30] He Y, Chien C-C, Chen Q and Levin K 2009 Temperature and final state effects in radio frequency spectroscopy experiments on atomic Fermi gases Phys. Rev. Lett. 102 020402

[31] Pieri P, Perali A and Strinati G C 2009 Enhanced paraconductivity-like fluctuations in the radiofrequency spectra of ultracold Fermi atoms Nat. Phys. 5 736

[32] Haussmann R 1994 Properties of a Fermi liquid at the superfluid transition in the crossover region between BCS superconductivity and Bose–Einstein condensation Phys. Rev. B 49 12975

[33] Chen Q, Stajic J, Tan S and Levin K 2005 BCS–BEC crossover: from high temperature superconductors to ultracold superfluids Phys. Rep. 412 1

[34] Pieral A, Pieri P, Strinati G C and Castellani C 2002 Pseudogap and spectral function from superconducting fluctuations to the bosonic limit Phys. Rev. B 66 024510

[35] Soloviev V G 1992 Theory of Atomic Nuclei: Quasiparticles and Phonons (London: Taylor and Francis)

[36] Coiffer D 1990 Effect of the operator nature of the superconducting order parameter in the presence of an added quasiparticle Phys. Rev. B 42 6040

[37] Haussmann R, Punk M and Zwerger W 2009 Spectral functions and rf response of ultracold fermionic atoms Phys. Rev. A 80 063612

[38] Anderson P W 1958 Random-phase approximation in the theory of superconductivity Phys. Rev. 112 1900

[39] Minguzzi A, Ferrari G and Castin Y 2001 Dynamic structure factor of a superfluid Fermi gas Eur. Phys. J. D 17 49

[40] Baym G, Pethick C J, Yu Z and Zwierlein M W 2007 Coherence and clock shifts in ultracold Fermi gases with resonant interactions Phys. Rev. Lett. 99 190407

[41] Tan S 2008 Energetics of a strongly correlated Fermi gas Ann. Phys. 323 2952

[42] Tan S 2008 Large momentum part of a strongly correlated Fermi gas Ann. Phys. 323 2971

[43] Braaten E and Platter L 2008 Exact relations for a strongly interacting Fermi gas from the operator product expansion Phys. Rev. Lett. 100 205301

[44] Schneider W and Randeria M 2010 Universal short-distance structure of the single-particle spectral function of dilute Fermi gases Phys. Rev. A 81 021601

[45] He Y et al 2009 Temperature and final state effects in radio frequency spectroscopy experiments on atomic Fermi gases Phys. Rev. Lett. 102 020402

[46] Fetter A L and Walecka J D 2003 Quantum Theory of Many-Particle Systems (New York: Dover)

[47] Gaebler J P, Stewart J T, Drake T E, Jin D S, Perali A, Pieri P and Strinati G C 2010 Observation of pseudogap behavior in a strongly interacting Fermi gas Nat. Phys. 6 569

[48] Stewart J T, Gaebler J P, Drake T E and Jin D S 2010 Verification of universal relations in a strongly interacting Fermi gas Phys. Rev. Lett. 104 235301

[49] Pethick C J and Smith H 2001 Bose–Einstein Condensation in Dilute Gases (Cambridge: Cambridge University Press)

[50] Clément D, Fabbri N, Fallani L, Fort C and Inguscio M 2009 Exploring correlated 1D Bose gases from the superfluid to the Mott-insulator state by inelastic light scattering Phys. Rev. Lett. 102 155301

[51] Ronen S 2009 The dispersion relation of a Bose gas in the intermediate- and high-momentum regimes J. Phys. B: At. Mol. Opt. Phys. 42 055301

[52] Marsiglio F 1997 Evaluation of the BCS approximation for the attractive Hubbard model in one dimension Phys. Rev. B 55 575