Model with Strong $\gamma_4 T$-violation*

R. Friedberg$^1$ and T. D. Lee$^{1, 2}$

1. Physics Department, Columbia University
   New York, NY 10027, U.S.A.
2. China Center of Advanced Science and Technology (CCAST/World Lab.)
   P.O. Box 8730, Beijing 100080, China

Abstract

We extend the $T$ violating model of the paper on "Hidden symmetry of the CKM and neutrino-mapping matrices" by assuming its $T$-violating phases $\chi_\uparrow$ and $\chi_\downarrow$ to be large and the same, with $\chi = \chi_\uparrow = \chi_\downarrow$. In this case, the model has 9 real parameters: $\alpha_\uparrow$, $\beta_\uparrow$, $\xi_\uparrow$, $\eta_\uparrow$ for the $\uparrow$-quark sector, $\alpha_\downarrow$, $\beta_\downarrow$, $\xi_\downarrow$, $\eta_\downarrow$ for the $\downarrow$ sector and a common $\chi$. We examine whether these nine parameters are compatible with ten observables: the six quark masses and the four real parameters that characterize the CKM matrix (i.e., the Jarlskog invariant $J$ and three Eulerian angles). We find that this is possible only if the $T$-violating phase $\chi$ is large, between $-120^0$ to $-135^0$. In this strong $T$ violating model, the smallness of the Jarlskog invariant $J \approx 3 \times 10^{-5}$ is mainly accounted for by the large heavy quark masses, with $m_c < m_t \approx m_b \approx .02$, as well as the near complete overlap of $t$ and $b$ quark, with $(c|b) = -.04$.

PACS: 12.15.Ff, 11.30.Er

Key words: Jarlskog invariant, CKM matrix, strong $\gamma_4 T$-violation

*This research was supported in part by the U.S. Department of Energy (grant no. DE-FG02-92- ER40699)
1. Introduction

In a previous paper on the "Hidden symmetry of the CKM and neutrino-mapping matrices" [1], we have posited a mass-generating Hamiltonian $H_\uparrow + H_\downarrow$ where

$$
H_\uparrow = \alpha_\uparrow |q_3^\uparrow - \xi_\uparrow q_2^\uparrow|^2 + \beta_\uparrow |q_2^\uparrow - \eta_\uparrow q_1^\uparrow|^2 + \beta_\uparrow |q_3^\uparrow - \xi_\uparrow \eta_\uparrow q_1^\uparrow|^2 \\
H_\downarrow = \alpha_\downarrow |q_3^\downarrow - \xi_\downarrow q_2^\downarrow|^2 + \beta_\downarrow |q_2^\downarrow - \eta_\downarrow q_1^\downarrow|^2 + \beta_\downarrow |q_3^\downarrow - \xi_\downarrow \eta_\downarrow q_1^\downarrow|^2
$$

(1.1)

with $\alpha$, $\beta$, $\xi$, $\eta$ real. This conserves $T$ and leads to zero masses for the light quarks $u$ and $d$. We then modified (1.1) by replacing $\xi$, $\eta$ with the corresponding $T$ violating factors $\xi_\uparrow e^{i\chi_\uparrow}$ and $\xi_\downarrow e^{i\chi_\downarrow}$. To first order in $\chi_\uparrow$ and $\chi_\downarrow$ we found a relation of proportionality between $\mathcal{J}$, the Jarlskog invariant measuring $T$-violation, and a linear combination of square roots of the light masses. The ratio agreed roughly with known values. We shall call this the "weak $\gamma_4$-model" because to make the calculation we assumed $\chi_\uparrow$, $\chi_\downarrow$ to be small.

There were two reasons for dissatisfaction with this model. First, why not introduce the phase factor into $\eta$ or $\xi \eta$, yielding different physics? And second, when we estimated not only $\mathcal{J}$ but the individual matrix elements of $U_{CKM}$, we found that the data required $\chi_\uparrow$ and $\chi_\downarrow$ to be large angles, not small.

We now present a new model, the "strong $\gamma_4$-model". In this model we introduce phase factors independently into all three terms, but require them to have the same values in $H_\uparrow$ and $H_\downarrow$. Thus we take the mass-generating Hamiltonian to be $H_\uparrow + H_\downarrow$ where

$$
H_\uparrow = \alpha_\uparrow |q_3^\uparrow - \xi_\uparrow e^{i\theta} q_2^\uparrow|^2 + \beta_\uparrow |q_2^\uparrow - \eta_\uparrow e^{i\omega} q_1^\uparrow|^2 + \beta_\uparrow |q_3^\uparrow - \xi_\uparrow \eta_\uparrow e^{-i\tau} q_1^\uparrow|^2 \\
H_\downarrow = \alpha_\downarrow |q_3^\downarrow - \xi_\downarrow e^{i\theta} q_2^\downarrow|^2 + \beta_\downarrow |q_2^\downarrow - \eta_\downarrow e^{i\omega} q_1^\downarrow|^2 + \beta_\downarrow |q_3^\downarrow - \xi_\downarrow \eta_\downarrow e^{-i\tau} q_1^\downarrow|^2
$$

(1.2)

It is now easily seen that the masses and CKM matrix depend on the phases only through the sum $\chi = \rho + \omega + \tau$. Accordingly, without loss of generality, we set $\rho = \omega = 0$, $\tau = \chi$. The mass-generating Hamiltonian can then be written as

$$
\left(\overline{q}_1^\uparrow, \overline{q}_2^\uparrow, \overline{q}_3^\uparrow\right) M_\uparrow \left(\begin{array}{c} q_1^\uparrow \\ q_2^\uparrow \\ q_3^\uparrow \end{array}\right) + \left(\overline{q}_1^\downarrow, \overline{q}_2^\downarrow, \overline{q}_3^\downarrow\right) M_\downarrow \left(\begin{array}{c} q_1^\downarrow \\ q_2^\downarrow \\ q_3^\downarrow \end{array}\right)
$$
where \( q_i^\uparrow, q_i^\downarrow \) and \( \bar{q}_i^\uparrow, \bar{q}_i^\downarrow \) are related to the corresponding Dirac field operators \( \psi(q_i(\uparrow)), \psi(q_i(\downarrow)) \) and their hermitian conjugate \( \psi^\dagger(q_i(\uparrow)), \psi^\dagger(q_i(\downarrow)) \) by

\[
q_i^{\dagger/\downarrow} = \psi(q_i(\uparrow/\downarrow)) \quad \text{and} \quad \bar{q}_i^{\dagger/\downarrow} = \psi^\dagger(q_i(\uparrow/\downarrow)) \gamma_4,
\]

(1.3)

\[
M_{1/\downarrow} = \begin{pmatrix}
\beta \eta^2 (1 + \xi^2) & -\beta \eta & -\beta \xi \eta e^{i\chi} \\
-\beta \eta & \beta + \alpha \xi^2 & -\alpha \xi \\
-\beta \xi \eta e^{-i\chi} & -\alpha \xi & \alpha + \beta
\end{pmatrix}_{\uparrow/\downarrow},
\]

(1.4)

with the arrow-subscripts \( \uparrow, \downarrow \) referring to \( \alpha, \beta, \xi, \eta \), but not to \( \chi \).

In diagonalizing (1.4) we do not assume, as in the weak \( \gamma_4 \)-model, that \( \chi \) is small. We find that the smallness of \( \mathcal{J} \) is mainly accounted for by the large heavy masses with

\[
\frac{m_c}{m_t} < \frac{m_s}{m_b} \approx .02
\]

(1.5)

and by the nearly complete overlap of the statevectors for \( t \) and \( b \) since

\[
|(u|b)| < |(c|b)| \approx 0.04.
\]

(1.6)

We have been able to carry out complete calculations in which the only approximations are based on the smallness of \( \frac{m_s}{m_b}, \frac{m_t}{m_c} \) and \( (c|b) \). These calculations are described in Sections 2 and 3; we give here a brief outline.

We diagonalize \( M_{\uparrow} \) and \( M_{\downarrow} \) with the aid of parameters \( r_{\uparrow,\downarrow}, B_{\uparrow,\downarrow}, \Phi_{\uparrow,\downarrow}, S, L \) to be defined in the next two sections. They are shown there to satisfy the following ten equations (to first order in small quantities):

\[
\frac{1 - r_{\uparrow}^2}{r_{\uparrow}^2} \sin^2 B_{\uparrow} = \frac{4m_u m_c}{(m_c - m_u)^2},
\]

(1.7)

\[
\frac{1 - r_{\downarrow}^2}{r_{\downarrow}^2} \sin^2 B_{\downarrow} = \frac{4m_d m_s}{(m_s - m_d)^2},
\]

(1.8)

\[
\sin^2 \frac{1}{2} \chi = \frac{1 - r_{\uparrow}^2}{\sin^2 2\Phi_{\uparrow}} = \frac{1 - r_{\downarrow}^2}{\sin^2 2\Phi_{\downarrow}},
\]

(1.9)

\[
L = \sqrt{\frac{m_s m_d}{m_b}} - \sqrt{\frac{m_c m_u}{m_t}},
\]

(1.10)

\[
S = \sin(\Phi_{\uparrow} - \Phi_{\downarrow}) = (c|b),
\]

(1.11)
\[(u|b) + S \sin \frac{1}{2} B_\uparrow |^2 = L^2 \cos^2 \frac{1}{2} B_\uparrow, \quad (1.12)\]

\[Im(u|b) = -L \frac{\cos \frac{1}{2} B_\uparrow \cos \frac{1}{2} \chi}{r_\uparrow} \quad (1.13)\]

and

\[(u|s) = \sin \frac{1}{2} (B_\downarrow - B_\uparrow). \quad (1.14)\]

Our strategy of solution is as follows. We take \(m_s, m_c, m_b, m_t,\) as well as \((u|s), (u|b)\) and \((c|b),\) to be given from data (see table 1). Then we have eleven unknowns \(r_\uparrow, B_\downarrow, \Phi_\uparrow, S, L, \chi, m_d, m_u\) constrained by ten independent equations given above (with (1.9)and (1.11), each counted as two equations). Taking a trial value of \(\sin \frac{1}{2} B_\uparrow,\) we are able to solve numerically for the other ten unknowns by a self-correcting double iteration that converges to 4 decimal stability after \(36 = 6 \times 6\) passes. We find that \(m_u\) is particularly sensitive to variations in \(\sin \frac{1}{2} B_\uparrow;\) a variation of 30\% in the latter carries \(m_u\) through the whole of its experimental range from 1.5 to 3.0\,MeV/c^2. Meanwhile \(m_d\) varies by only 25\%, from 5.2 to 6.5\,MeV/c^2, well within the experimental range, 3.0 to 8.0\,MeV/c^2. The value of \(\chi\) must be taken as negative and is in the neighborhood of \(-125^0\), between \(-120^0\) and \(-135^0\). We have also tried deviations in \(m_s, m_b, (c|b), Re(u|b)\) and \(Im(u|b).\) Only in the case of \(m_s\) does it appear that a maximal deviation \((-25\%)\) from the "best value" might push \(m_d\) outside the range given by data. (See Tables 1 and 2, and Fig. 1).

The next two sections are devoted to defining the parameters that appear in (1.7)-(1.14) and proving that these equations are satisfied. In Section 2, we discuss the separate diagonalization of \(M_\uparrow\) and \(M_\downarrow,\) and in Section 3, we examine the CKM matrix.

In Section 4, we discuss briefly a third model[2], which we may call a \(i\gamma_5\) model, because its Hamiltonian contains a term in \(i\gamma_4\gamma_5\) as well as the usual one in \(\gamma_4\).
2. Diagonalization of $M_{\uparrow}$ and $M_{\downarrow}$

In this section, we shall drop the arrow-subscripts and write (1.4) as

$$M = \begin{pmatrix} T^2\beta & -T\beta \cos\Phi & -T\beta \sin\Phi e^{ix} \\ -T\beta \cos\Phi & \alpha \tan^2\Phi + \beta & -\alpha \tan\Phi \\ -T\beta \sin\Phi e^{-ix} & -\alpha \tan\Phi & \alpha + \beta \end{pmatrix}, \quad (2.1)$$

where

$$\Phi = \tan^{-1}\xi \quad (2.2)$$

$$T = \eta\sqrt{1 + \xi^2} \quad (2.3)$$

so that $T^2\beta = \beta\eta^2(1 + \xi^2)$, $\sin\Phi = \xi/\sqrt{1 + \xi^2}$, $\cos\Phi = 1/\sqrt{1 + \xi^2}$ and (2.1)=(1.4). We denote the eigenvalues of $M$ by $m_l$, $m_m$, $m_h$ (light, medium, heavy), and seek a unitary matrix $W$ (with $WW^\dagger = 1$) such that

$$M = W \begin{pmatrix} m_l & 0 & 0 \\ 0 & m_m & 0 \\ 0 & 0 & m_h \end{pmatrix} W^\dagger. \quad (2.4)$$

The $W$ matrix will be built up in stages, as we shall discuss. First we isolate the heavy mass by writing

$$M = \Omega \begin{pmatrix} (n) & L \\ L^* & \mu_h \end{pmatrix} \Omega^\dagger \quad (2.5)$$

where

$$\Omega^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\Phi} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.6)$$

$$\mu_h = \alpha \sec^2\Phi + \beta \quad (2.7)$$

$$L = T\beta \cos\Phi \sin\Phi(1 - e^{ix}) \quad (2.8)$$

and

$$(n) = \beta \begin{pmatrix} T^2 & -T(\cos^2\Phi + \sin^2\Phi e^{ix}) \\ -T(\cos^2\Phi + \sin^2\Phi e^{-ix}) & 1 \end{pmatrix}. \quad (2.9)$$
Thus, (2.1) can be obtained by a simple substitution of (2.6)-(2.9) into (2.5). Next, we diagonalize the $2 \times 2$ matrix ($n$) of (2.9) by setting

$$\cos^2 \Phi + \sin^2 \Phi e^{i\chi} = re^{iA}$$

(2.10)

with $r, A$ both real. Then

$$(n) = \beta \begin{pmatrix} \frac{T^2}{2} & -Tr e^{iA} \\ -Tr e^{-iA} & 1 \end{pmatrix}$$

$$= e^{\frac{1}{2}i\tau_z A} e^{-\frac{1}{2}i\tau_y B} \begin{pmatrix} \mu_l & 0 \\ 0 & \mu_m \end{pmatrix} e^{\frac{1}{2}i\tau_y B} e^{-\frac{1}{2}i\tau_z A}$$

(2.11)

provided that

$$\mu_m + \mu_l = \beta(1 + T^2)$$

$$(\mu_m - \mu_l) \cos B = \beta(1 - T^2)$$

(2.12)

$$(\mu_m - \mu_l) \sin B = 2\beta Tr.$$ By quadratic combination of (2.12) we obtain

$$\mu_m \mu_l = \beta^2 T^2 (1 - r^2);$$

(2.13)

then, by dividing the above equation by the square of the last line of (2.12), we have

$$\frac{4\mu_m \mu_l}{(\mu_m - \mu_l)^2} = \frac{1 - r^2}{r^2} \sin^2 B$$

(2.14)

which leads to (1.7) and (1.8).

Also, by applying the Law of Sines to the complex triangle described by (2.10), followed by trigonometric identities, we find

$$\cos \left(\frac{1}{2} \chi - A\right) = \frac{\cos \frac{1}{2} \chi}{r},$$

(2.15)

a relation that will be useful later.

Applying (2.11) to (2.5), we now have

$$M = \Omega V \begin{pmatrix} \mu_l & 0 & L \Delta^* \cos \frac{1}{2} B \\ 0 & \mu_m & -L \Delta^* \sin \frac{1}{2} B \\ L^* \Delta \cos \frac{1}{2} B & -L^* \Delta \sin \frac{1}{2} B & \mu_h \end{pmatrix} V^\dagger \Omega^\dagger,$$

(2.16)
where
\[ \Delta = e^{\frac{1}{2}iA} \]  
and
\[ V^\dagger = \begin{pmatrix} e^{\frac{1}{2}i\tau_yB} e^{-\frac{1}{2}i\tau_zA} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]  
Thus \( M \) is almost diagonalized. Let us study the magnitude of \( L \). From (2.13) and (2.10) we find
\[ \mu_m\mu_l = \beta^2 T^2 (1 - r^2) = 2\beta^2 T^2 (1 - \cos \chi) \cos^2 \Phi \sin^2 \Phi \]  
and comparing this with (2.8) we have
\[ |L| = 2|T\beta \cos \Phi \sin \Phi \sin \frac{1}{2}\chi| = \sqrt{\mu_m\mu_l}. \]  
Hence, if we write
\[ \begin{pmatrix} \mu_l & 0 & L\Delta^* \cos \frac{1}{2}B \\ 0 & \mu_m & -L\Delta^* \sin \frac{1}{2}B \\ L^* \Delta \cos \frac{1}{2}B & -L^* \Delta \sin \frac{1}{2}B & \mu_h \end{pmatrix} = P \begin{pmatrix} m_l & 0 & 0 \\ 0 & m_m & 0 \\ 0 & 0 & m_h \end{pmatrix} P^\dagger \]  
the elements of \( P \) will differ from those of the unit matrix by \( O\left[ \frac{\sqrt{m_m m_l}}{m_h} \right] << 1 \). 
A careful examination shows that all the \( m \)'s may be approximated by \( \mu \)'s; in particular, we also have \( |\frac{\mu_m}{m_l} - 1| \sim O[\frac{m_m}{m_h}] \). Therefore (2.14) becomes
\[ \frac{4m_m m_l}{(m_m - m_l)^2} = \frac{1 - r^2}{r^2} \sin^2 B \]  
and (1.7) and (1.8) are established.

Also, (1.9) is a direct consequence of (2.13) and (2.20). We may take (1.10) as the definition of \( L \), and from (2.20) we may write it as
\[ L = \frac{|L_\uparrow|}{m_b} - \frac{|L_\downarrow|}{m_t}. \]  
The first equality of (1.11) is the definition of \( S \). Thus what remains is to establish the second part of (1.11), and (1.12)-(1.14). This requires studying the CKM matrix which relates "up" to "down" eigenstates, as we shall see.
3. The CKM Matrix

In this section we restore the arrow subscripts ↑, ↓. On account of (2.16) and (2.21), the matrix $W$ defined in (2.4) is given by

$$W_{↑,↓}^\dagger = P_{↑,↓}^\dagger V_{↑,↓}^\dagger \Omega_{↑,↓}^\dagger.$$  \hspace{1cm} (3.1)

If we define

$$U = W_{↑}^\dagger W_{↓} = P_{↑}^\dagger U_0 P_{↓}$$  \hspace{1cm} (3.2)

where

$$U_0 = V_{↑}^\dagger \Omega_{↑}^\dagger \Omega_{↓} V_{↓}$$

$$= \begin{pmatrix}
(e^{\frac{1}{2}i\tau_y B_{↑}} e^{-\frac{1}{2}i\tau_z A_{↑}}) & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & e^{i(\Phi_{↑} - \Phi_{↓})\tau_y} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$ \hspace{1cm} (3.3)

then $U$ transforms eigenstates of $M_{↓}$ into eigenstates of $M_{↑}$, provided that the phases of the eigenstates are suitably chosen. To obtain the CKM matrix $U_{CKM}$, which relates eigenstates whose phases follow a standard convention, we shall need an additional transformation

$$U_{CKM} = Q_{↑}^\dagger U Q_{↓}$$  \hspace{1cm} (3.4)

where $Q_{↑,↓}$ are diagonal unitary matrices to be chosen presently.

In evaluating (3.3) it is convenient to introduce new symbols:

$$\delta = \Delta_{↑} \Delta_{↓}^* = e^{\frac{1}{2}i(A_{↑} - A_{↓})},$$  \hspace{1cm} (3.5)

$$\Gamma = \cos \frac{1}{2} B_{↑}, \quad \gamma = \cos \frac{1}{2} B_{↓},$$  \hspace{1cm} (3.6)

$$\Sigma = \sin \frac{1}{2} B_{↑}, \quad \sigma = \sin \frac{1}{2} B_{↓},$$  \hspace{1cm} (3.7)

$$S = \sin(\Phi_{↑} - \Phi_{↓}) \text{ and } C = \cos(\Phi_{↑} - \Phi_{↓}).$$  \hspace{1cm} (3.8)

We note that the first equation in (3.8) is the same in (1.11). By using (3.5)-(3.8), we find $U_0$ of (3.3) can be written as

$$U_0 = \begin{pmatrix}
\delta^* \Gamma \gamma + C \delta \Sigma \sigma & -\delta^* \Gamma \sigma + C \delta \Sigma \gamma & S \Delta_{↑} \Sigma \\
-\delta^* \Sigma \gamma + C \delta \Gamma \sigma & \delta^* \Sigma \sigma + C \delta \Gamma \gamma & S \Delta_{↑} \Gamma \\
-S \Delta_{↓} \sigma & -S \Delta_{↓} \gamma & C
\end{pmatrix}.$$ \hspace{1cm} (3.9)
The next step is to prepare for a perturbative treatment of (3.2) by writing

$$P_{\uparrow, \downarrow} \approx I + p_{\uparrow, \downarrow} \quad (3.10)$$

where (in arrowless notation)

$$p^\dagger = \frac{1}{m_h} \begin{pmatrix} 0 & 0 & -\Delta^* L \cos \frac{1}{2} B \\ 0 & 0 & \Delta^* L \sin \frac{1}{2} B \\ \Delta L^* \cos \frac{1}{2} B & -\Delta L^* \sin \frac{1}{2} B & 0 \end{pmatrix} \quad (3.11)$$

We note that by putting (3.11) into (3.10), we can satisfy (2.21) to first order in $L$.

Thus we have

$$U \approx U_0 + U' \quad (3.12)$$

where

$$U' = p^\dagger U_0 + U_0 p_{\downarrow} \quad (3.13)$$

Let us carefully evaluate the lower left element of $p^\dagger U_0$:

$$(p^\dagger U_0)_{31} = \frac{1}{m_t} (L^*_\uparrow \Delta^* \cos \frac{1}{2} B_{\uparrow}) (\delta^* \Gamma \gamma + C \delta \Sigma \sigma)$$

$$+ \frac{1}{m_t} (-L^*_\uparrow \Delta^* \sin \frac{1}{2} B_{\uparrow}) (-\delta^* \Sigma \gamma + C \delta \Gamma \sigma)$$

$$= \frac{L^*_\uparrow \Delta^* \gamma}{m_t} [\Gamma (\delta^* \Gamma \gamma + C \delta \Sigma \sigma) + \Sigma (\delta^* \Sigma \gamma - C \delta \Gamma \sigma)]$$

$$= \frac{L^*_\uparrow \Delta^* \gamma}{m_t} (\Gamma^2 + \Sigma^2) \gamma = \frac{L^*_\uparrow \Delta^* \gamma}{m_t} \quad (3.14)$$

(Note how the calculation converts $\Delta_{\uparrow}$ to $\Delta_{\downarrow}$ and $\Gamma$ to $\gamma$.) The corresponding element of $U_0 p_{\downarrow}$ is trivial:

$$(U_0 p_{\downarrow})_{31} = C \left( \frac{1}{m_b} \Delta^*_\downarrow L_{\downarrow} \cos \frac{1}{2} B_{\downarrow} \right)^* = -\frac{L^*_\downarrow}{m_b} \Delta_{\downarrow} \gamma C \quad (3.15)$$

Anticipating that $B_{\uparrow}$ will turn out fairly small, $\sim 0.2$, we now observe that the matrix element $U_{23}$ is going to be dominated by $(U_0)_{23} = S \Delta_{\uparrow} \Gamma \sim S \Delta_{\uparrow}$. Therefore, $S$ must have magnitude $\sim .04$. It follows that $C \sim 1 - \frac{1}{2} S^2$ can
be replaced by 1, and that all elements of \( U' \) other than \((U')_{13,23,31,32}\) being of order \( S \cdot \sqrt{m_b m_t} \), can be neglected.

Thus, by repeating for \((U')_{13,23,32}\) the calculations leading to (3.14) and (3.15), we have

\[
U' \simeq \begin{pmatrix}
0 & 0 & +\left( \frac{L_1}{m_b} - \frac{L_{13}}{m_t} \right) \Delta_1 \Gamma \\
0 & 0 & -\left( \frac{L_1}{m_b} - \frac{L_{13}}{m_t} \right) \Delta_1 \Sigma \\
-\left( \frac{L_1}{m_b} - \frac{L_{13}}{m_t} \right) \Delta_1 \gamma & +\left( \frac{L_{13}^*}{m_b} - \frac{L_{13}}{m_t} \right) \Delta_1 \sigma & 0
\end{pmatrix}
\] (3.16)

But from (2.8), taking \( T, \beta, \cos \Phi, \sin \Phi \) positive, we find

\[
\frac{L_\downarrow}{|L_\downarrow|} = L_\uparrow = \frac{1 - e^{i\chi}}{|1 - e^{i\chi}|}
\] (3.17)

and so

\[
\frac{L_\downarrow}{m_b} - \frac{L_\uparrow}{m_t} = \frac{1 - e^{i\chi}}{|1 - e^{i\chi}|} \mathcal{L}
\] (3.18)

by (2.23). We now anticipate that \( \chi \) will have to be negative in order to make everything come out right. Hence,

\[
\frac{1 - e^{i\chi}}{|1 - e^{i\chi}|} = \frac{1}{2 \sin \frac{1}{2} \chi} e^{ \frac{1}{2} i \chi} (-2i \sin \frac{1}{2} \chi) = +ie^{ \frac{1}{2} i \chi}
\] (3.19)

and (3.16) leads to

\[
U' \simeq \begin{pmatrix}
0 & 0 & +ie^{ \frac{1}{2} i \chi} \mathcal{L} \Delta_1^* \Gamma \\
0 & 0 & -ie^{ \frac{1}{2} i \chi} \mathcal{L} \Delta_1^* \Sigma \\
+ie^{ \frac{1}{2} i \chi} \mathcal{L} \Delta_1 \gamma & -ie^{ \frac{1}{2} i \chi} \mathcal{L} \Delta_1 \sigma & 0
\end{pmatrix}
\] (3.20)

For reasons shortly to be evident, let us now introduce the phase factors

\[
\varepsilon_{\uparrow, \downarrow} = -ie^{ \frac{1}{2} i \chi} (\Delta_{\uparrow, \downarrow}^*)^2 = e^{ \frac{1}{2} i \chi} e^{i(\frac{1}{2} \chi - A_{\uparrow, \downarrow})}
\] (3.21)

Then we have

\[
U' = \begin{pmatrix}
0 & 0 & -\varepsilon_\uparrow \mathcal{L} \Delta_\uparrow \Gamma \\
0 & 0 & +\varepsilon_\uparrow \mathcal{L} \Delta_\uparrow \Sigma \\
+\varepsilon_\downarrow^* \mathcal{L} \Delta_\downarrow^* \gamma & -\varepsilon_\downarrow^* \mathcal{L} \Delta_\downarrow^* \sigma & 0
\end{pmatrix}
\] (3.22)

10
In treating (3.9), let us note that since $\Phi_\uparrow - \Phi_\downarrow \approx \sin^{-1} S$ is small, $A_\uparrow - A_\downarrow$ is also small by (2.10). Hence $|\text{Im } \delta|$ is small (see (3.5)) and $1 - \text{Re } \delta$ is second order. So $\text{Re } \delta$ can be taken $= 1$, and the imaginary parts of $(U_0)_{11,12,21,22}$ can be adjusted by small adjustments in $Q_\uparrow, Q_\downarrow$. We shall treat such adjustments imprecisely and simply neglect these imaginary parts. By taking $C' \to 1$ and using (3.6)-(3.7), we find

$$
\left( \begin{array}{cc}
(U_0)_{11} & (U_0)_{12} \\
(U_0)_{21} & (U_0)_{22}
\end{array} \right) = \left( \begin{array}{cc}
\Gamma \gamma + \Sigma \sigma & -\Gamma \sigma + \Sigma \gamma \\
-\Sigma \gamma + \Gamma \sigma & \Sigma \sigma + \Gamma \gamma
\end{array} \right)
$$

$$
= \left( \begin{array}{cc}
\cos \frac{1}{2}(B_\downarrow - B_\uparrow) & -\sin \frac{1}{2}(B_\downarrow - B_\uparrow) \\
\sin \frac{1}{2}(B_\downarrow - B_\uparrow) & \cos \frac{1}{2}(B_\downarrow - B_\uparrow)
\end{array} \right). \quad (3.23)
$$

Now $B_\downarrow - B_\uparrow$ must be positive to fit $U_{13}$ and $U_{31}$, and so $U_{12}$ is negative, whereas the standard presentation gives $(U_{CKM})_{12}$ positive. Therefore, we shall use the $Q$-transformation to change the sign of the first row and column, and also to remove the factors $\Delta_\uparrow, \Delta_\downarrow^*$ now appearing in the third row and column. Thus

$$
Q_\uparrow = \left( \begin{array}{ccc}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \Delta_\downarrow
\end{array} \right), \quad Q_\downarrow = \left( \begin{array}{ccc}
+1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \Delta_\downarrow^*
\end{array} \right) \quad (3.24)
$$

and

$$
U_{CKM} = Q_\uparrow^* U_0 Q_\downarrow + Q_\uparrow^* U' Q_\downarrow
$$

$$
= \left( \begin{array}{ccc}
\cos \frac{1}{2}(B_\downarrow - B_\uparrow) & \sin \frac{1}{2}(B_\downarrow - B_\uparrow) & -\Sigma \Sigma + \varepsilon_\uparrow \varepsilon_L \Gamma \\
-\sin \frac{1}{2}(B_\downarrow - B_\uparrow) & \cos \frac{1}{2}(B_\downarrow - B_\uparrow) & \Sigma \Gamma + \varepsilon_\downarrow \varepsilon_L \Sigma \\
\Sigma \sigma - \varepsilon_\uparrow \varepsilon_L \gamma & -\Sigma \gamma - \varepsilon_\uparrow \varepsilon_L \sigma & 1
\end{array} \right). \quad (3.25)
$$

where we have again allowed a slight imprecision of phase in the $(3, 3)$ element.

Comparing (3.25) with the array

$$
U_{CKM} = \left( \begin{array}{ccc}
(u|d) & (u|s) & (u|b) \\
(c|d) & (c|s) & (c|b) \\
(t|d) & (t|s) & (t|b)
\end{array} \right), \quad (3.26)
$$

we obtain the second half of (1.11) and (1.12)-(1.14).
Note: there is an ambiguity, $\Phi^\dagger >\quad$ or $<\frac{\pi}{4}$. We take both $\Phi'$s $>\frac{\pi}{4}$, so that $|A| > |\chi - A|$ and hence $|A| > |\frac{1}{2}\chi|$. Since $\chi$ and $A$ are negative, $\frac{1}{2}\chi - A > 0$ and hence $Re\, \varepsilon^\dagger > 0$, as required in $(u|b)$ and $(t|d)$. Because $Im\, \varepsilon^\dagger = -\cos(\frac{1}{2}\chi - A)$, we can then derive (1.13) by using (2.15).

4. The "Timeon" Model

The merit of the "strong $\gamma_4$ $T$-violation model" examined in this paper suggests that there may be large $T$-violation somewhere in physics although its manifestation in the quark mass sector is small. In the "strong $\gamma_4$ $T$-violation model" the $T$-violating effects are produced by the phase $\chi$ which enters non-linearly into the Hamiltonian. This non-linear interaction makes it difficult to construct a renormalizable quantum field theory that can be extended beyond the mass matrix. For this and other reasons, we have considered a different model[3] in which the $T$-violating effect enters linearly; therefore, the model can lead to a renormalizable field theory, called "timeon".

In the timeon theory, the mass-generating Hamiltonian can be written by replacing $M_{1/4}$ in (1.4) by

$$G_{1/4} + i\gamma_5 F_{1/4},$$

(4.1)

where $G_{1/4}$ and $F_{1/4}$ are real symmetric matrices, and the $F_{1/4}$ term in $i\gamma_4$ arises from coupling to the vacuum expectation value of a new $T$-negative and $P$-negative field $\tau(x)$, the timeon field. Thus, the whole field theory conserves $T$, but $T$-violation arises from the spontaneous symmetry breaking that makes the vacuum expectation value

$$\tau_0 = <\tau(x)>_{\text{vac}} \neq 0.$$  

(4.2)

The timeon field $\tau(x)$ is real, so that there is no Goldstone boson[4]. However, the oscillation of $\tau(x)$ around its vacuum expectation value $\tau_0$ gives rise to a new particle, called "timeon", whose production can lead to large $T$-violating effects. In Ref. 3, it is shown that the parameters determining $G_{1/4}$ and $F_{1/4}$ can be adjusted to simulate an arbitrary complex $\gamma_4$ model, as far as the quark masses are concerned, but not the CKM matrix. Thus, for example,
in the timeon $\gamma_5$-model the light quark masses in the small mass limit turn out to be proportional to $J$, whereas in the $\gamma_4$-model, they are proportional to $J^2$.

References

[1] R. Friedberg and T. D. Lee, Ann. Phys. 323 (2008) 1087

[2] Particle Data Group, J. Phys. G33 (2006) 1

[3] R. Friedberg and T. D. Lee, arXiv:0809.3633

[4] J. Goldstone, Nuovo Cimento 9 (1961) 154

Table 1

| Parameter | "Best" value |
|-----------|--------------|
| $m_s$     | 95 MeV       |
| $m_b$     | 4.5 GeV      |
| $(c|b)$    | 0.04         |
| $Re(u|b)$ | 0.002        |
| $Im(u|b)$ | −0.003       |

These values are used to obtain the top two rows in Table 2.
Table 2

Values of $m_u$, $m_d$ and $\chi$ calculated from the strong $\gamma_4$-model

| Input parameters | $m_u$(MeV) | $m_d$(MeV) | $\cos \frac{1}{2} \chi$ |
|------------------|------------|------------|-------------------------|
| As in Table 1    | 1.45       | 5.18       | .487                    |
|                  | 3.16       | 6.50       | .428                    |
| Table 1 except $m_s = 85$MeV | 1.39  | 5.43       | .479                    |
|                  | 3.29       | 6.86       | .418                    |
| Table 1 except $m_s = 105$MeV | 1.52  | 5.00       | .490                    |
|                  | 3.09       | 6.22       | .433                    |
| Table 1 except $m_b = 4.2$GeV | 1.63  | 4.83       | .483                    |
|                  | 3.33       | 6.02       | .427                    |
| Table 1 except $m_b = 4.7$GeV | 1.61  | 5.68       | .476                    |
|                  | 3.53       | 7.14       | .417                    |
| Table 1 except $(c|b) = 0.035$ | 1.40  | 4.86       | .507                    |
|                  | 2.98       | 5.96       | .454                    |
| Table 1 except $(c|b) = 0.045$ | 1.51  | 5.52       | .468                    |
|                  | 3.36       | 7.07       | .405                    |
| Table 1 except $Re(u|b) = 0.0015$ | 1.63  | 4.74       | .525                    |
|                  | 3.33       | 5.96       | .463                    |
| Table 1 except $Re(u|b) = 0.0025$ | 1.72  | 6.09       | .432                    |
|                  | 2.96       | 7.06       | .397                    |
| Table 1 except $Im(u|b) = -0.0025$ | 1.64  | 4.93       | .428                    |
|                  | 2.75       | 5.81       | .389                    |
| Table 1 except $Im(u|b) = -0.0035$ | 1.73  | 5.96       | .510                    |
|                  | 2.93       | 6.83       | .473                    |

Table 2 (footnotes)

The values of five input parameters are taken as in Table 1, except for single departures as shown in the left-hand column here. For each setting of the input parameters, there is a one-parameter family of solutions of Eqs. (1.7)-(1.14). We show two members of each family, chosen roughly to span the experimental range of $m_u$ from 1.5 to 3.0 MeV. The corresponding values of $m_d$ stay within its experimental range from 3 to 8 MeV, and $\chi$ remains large from $-120^0$ to $-135^0$.  

14
Figure 1. $m_d$ versus $m_u$ for $m_s = 95\text{MeV}$, $(c|b) = 0.04$, $(u|b) = 0.002 - 0.003i$ and $m_b = 4.2\text{GeV}$, $4.5\text{GeV}$ and $4.7\text{GeV}$.