TOPOLOGICALLY ISOTOPIC AND SMOOTHLY INEQUIVALENT 2-SPHERES IN SIMPLY CONNECTED 4-MANIFOLDS WHOSE COMPLEMENT HAS A PRESCRIBED FUNDAMENTAL GROUP.

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Abstract: We describe a procedure to construct infinite sets of pairwise smoothly inequivalent 2-spheres in simply connected 4-manifolds, which are topologically isotopic and whose complement has a prescribed fundamental group that satisfies some conditions. This class of groups include cyclic groups and the binary icosahedral group. These are the first known examples of such exotic embeddings of 2-spheres in 4-manifolds. Examples of locally flat embedded 2-spheres in a non-smoothable 4-manifold whose complements are homotopy equivalent to smoothly embedded ones are also given.

1. Main results

The first main result of this note is the following theorem.

Theorem A. Fix $p \geq 2$. There is an infinite set

\[ \{S_{n,p} : n \in \mathbb{Z} \} \]

of smoothly embedded 2-spheres in $2\mathbb{C}P^2 \# 4\mathbb{C}P^2$ that satisfies the following properties.

- Any two elements are topologically isotopic.
- There is a diffeomorphism of pairs
  \[ (2\mathbb{C}P^2 \# 4\mathbb{C}P^2, S_{n_1,p}) \to (2\mathbb{C}P^2 \# 4\mathbb{C}P^2, S_{n_2,p}) \]
  if and only if $n_1 = n_2$.
- The fundamental group of the complement is
  \[ \pi_1(2\mathbb{C}P^2 \# 4\mathbb{C}P^2 \setminus \nu(S_{n,p})) = \mathbb{Z}/p \]
  for every $n \in \mathbb{Z}$.
- $[S_{n,p}] \neq 0 \in H_2(2\mathbb{C}P^2 \# 4\mathbb{C}P^2; \mathbb{Z})$ for every $n \in \mathbb{Z}$.
- Surgery along each of these 2-spheres yields an infinite set of pairwise homeomorphic and pairwise non-diffeomorphic closed smooth 4-manifolds with fundamental group $\mathbb{Z}/p$.

Theorem A provides the first known example of an infinite set of 2-spheres smoothly embedded in a simply connected 4-manifold that are pairwise topologically isotopic, pairwise smoothly inequivalent and having a complement with finite cyclic fundamental group. Schwartz [28, Theorem 2] pointed out the existence of closed simply connected 4-manifolds containing pairs of smoothly embedded 2-spheres that are both smoothly equivalent and topologically isotopic, but not

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smoothly isotopic. Examples of these exotic embeddings of 2-spheres in closed 4-manifolds with simply connected complement have been constructed by Akbulut [3] and Auckly-Kim-Melvin-Ruberman [6]. Exotic embeddings of surfaces with positive genus in simply connected 4-manifolds and complement having non-trivial fundamental group are found in Kim [19] and Kim-Ruberman [20]. An ingredient in the proof of Theorem A is of independent interest: we point out in Theorem 1 that constructions of inequivalent smooth structures on simply connected 4-manifolds of Fintushel-Stern [11, 12] can be extended to produce such structures on 4-manifolds with non-trivial fundamental group too.

The second main result provides a construction procedure of topologically equivalent yet smoothly inequivalent homologically essential 2-spheres whose complement can be chosen to have the same fundamental group as a wide range of $Q$-homology 4-spheres. We work with the modified Seiberg-Witten $SW'_X$ invariant of a closed 4-manifold $X$ as defined, for example, in [10, Section 2], and denote by $B_X$ the set of basic classes.

**Theorem B.** Let \( \{Z_n : n \in \mathbb{Z}\} \) be an infinite set of closed smooth simply connected 4-manifolds with pairwise different integer invariants
\[
S_n = \max \{|SW'_{Z_n}(kZ_n)| : kZ_n \in B_{Z_n}\},
\]
which are pairwise homeomorphic to a given closed 4-manifold $Z$ and such that the connected sum $Z_n \# S^2 \times S^2$ is diffeomorphic to $Z \# S^2 \times S^2$ for every $n \in \mathbb{Z}$. Let $M$ be a closed smooth 4-manifold with $H_*(M; \mathbb{Q}) \cong H_*(S^4; \mathbb{Q})$ and set $\pi := \pi_1 M$. Suppose that there is a loop $\alpha \subset M$ and a choice of framing such that
\[
S^2 \times S^2 = M \setminus \nu(\alpha) \cup D^2 \times S^2.
\]
There is an infinite set
\[
\{S_{n,\pi} : n \in \mathbb{Z}\}
\]
of smoothly embedded 2-spheres in $Z \# S^2 \times S^2$ that satisfies the following properties.

- There is a homeomorphism of pairs
  \[
  (Z \# S^2 \times S^2, S_{n_1,\pi}) \to (Z \# S^2 \times S^2, S_{n_2,\pi})
  \]
  for every $n_i \in \mathbb{Z}$.
- There is a diffeomorphism of pairs
  \[
  (Z \# S^2 \times S^2, S_{n_1,\pi}) \to (Z \# S^2 \times S^2, S_{n_2,\pi})
  \]
  if and only if $n_1 = n_2$.
- The fundamental group of the complement is
  \[
  \pi_1(Z \# S^2 \times S^2 \setminus \nu(S_{n,\pi})) = \pi
  \]
  and its homology class satisfies
  \[
  [S_{n,\pi}] \neq 0 \in H_2(Z \# S^2 \times S^2; \mathbb{Z})
  \]
  for every $n \in \mathbb{Z}$.
- Surgery along each of these 2-spheres yields an infinite set $\{Z_n \# M : n \in \mathbb{Z}\}$ of pairwise non-diffeomorphic closed smooth 4-manifolds with fundamental group $\pi$, and that are pairwise homeomorphic to the connected sum $Z \# M$.

Please see [10, Proof of Theorem 1] for details on the definition of the invariant (1.1). Fintushel-Stern constructed infinite sets as in the hypothesis of Theorem B for $Z = \mathbb{C}P^2 \# k \mathbb{C}P^2$ for $2 \leq k \leq 7$ in [11, 12]. Baykur-Sunukjian [9] showed that Fintushel-Stern’s examples become diffeomorphic after a connected sum with...
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A single copy of \( S^2 \times S^2 \). Examples of \( \mathbb{Q} \)-homology 4-spheres \( M \) that satisfy the hypothesis are spun 4-manifolds with the fundamental group of any lens space and the Poincaré homology 3-sphere. A similar result holds if \( S^2 \times \tilde{S}^2 \) is substituted for the non-trivial bundle \( S^2 \times S^2 \). It is possible to strengthen the conclusion of Theorem B to topologically isotopic 2-spheres, although we do not pursue this endeavor here; see Sunukjian [29].

A contribution of this note is to point out the simplicity of the proofs of Theorem A and Theorem B. The reader will notice that the 4-manifolds in the last clause of Theorem A are smoothly reducible (cf. [15, Definition 10.1.17]), while those in the last clause of Theorem B are not. We explain in Remark 2 how an instance of Theorem B implies the claims on the existence of the homeomorphism of pairs and the non-existence of the diffeomorphism of pairs of Theorem A. An independent proof of Theorem A is given in Section 2.7 as well. The following consequence of Theorem B is another contribution.

**Corollary C.** Let \( G \) be a finite cyclic group or the icosahedral group
\[
\langle g_1, g_2 : g_1^5 = (g_1 g_2)^2 = g_2^3 \rangle.
\]
There is an infinite set of smoothly embedded 2-spheres in \( 2\mathbb{C}P^2 \# 4\mathbb{C}P^2 \) that are pairwise topologically equivalent, yet pairwise smoothly inequivalent, and the fundamental group of the complement is \( G \).

These are the first examples of exotic embeddings of 2-spheres in simply connected 4-manifolds whose complement has a fundamental group isomorphic to the binary icosahedral group among several other choices of groups. We exhibit interesting smooth embeddings of nullhomotopic 2-spheres in the fourth main result of this note.

**Theorem D.** There is an infinite set
\[
\{S_n : n \in \mathbb{Z}\}
\]
of 2-spheres smoothly embedded in \( 2\mathbb{C}P^2 \# 4\mathbb{C}P^2 \) that satisfies the following properties.

- The fundamental group of the complement of an element in (1.3) is

\[
\pi_1(2\mathbb{C}P^2 \# 4\mathbb{C}P^2 \setminus \nu(S_n)) = \mathbb{Z}
\]

and \([S_n] = 0 \in \pi_2(2\mathbb{C}P^2 \# 4\mathbb{C}P^2)\) for every \( n \in \mathbb{Z} \).

- There is a diffeomorphism of pairs

\[
(2\mathbb{C}P^2 \# 4\mathbb{C}P^2, S_{n_1}) \rightarrow (2\mathbb{C}P^2 \# 4\mathbb{C}P^2, S_{n_2})
\]

if and only if \( n_1 = n_2 \).

Notice that elements in (1.3) do not bound a smoothly embedded 3-ball in \( 2\mathbb{C}P^2 \# 4\mathbb{C}P^2 \). The smoothly inequivalent embeddings of homotopically trivial 2-spheres of Theorem D are related to a construction of an infinite set of closed smooth 4-manifolds with infinite cyclic fundamental group and the homology of the connected sum \( 2\mathbb{C}P^2 \# 4\mathbb{C}P^2 \# S^1 \times S^3 \), which is given in Theorem 2.

While any 2-sphere in a closed simply connected 4-manifold can be assumed to be regularly immersed, Hambleton-Kreck [16] and Lee-Wilczyński [22, 23] completely characterized when a homology class of non-zero divisibility can be represented by a locally flat embedded 2-sphere. The fifth and last result to be mentioned in
this introduction records the existence of a myriad of explicit examples of locally flat embedded 2-spheres in closed simply connected 4-manifolds whose exteriors are homotopy equivalent but not homeomorphic.

**Theorem E.** For every \( p \geq 2 \), there is a locally flat embedded 2-sphere

\[
S_p \subset \# \mathbb{CP}^2 \mathbb{CP}^2
\]

whose complement has finite cyclic group \( \mathbb{Z}/p \), and it is homotopy equivalent to the complement of a smoothly embedded 2-sphere

\[
S_p' \subset \mathbb{CP}^2 \mathbb{CP}^2.
\]

Theorem E is essentially derived from an existence result of non-smoothable \( \mathbb{Q} \)-homology 4-spheres due to Hambleton-Kreck [17]. Other interesting examples are found in Kasprowski-Lambert-Cole-Land-Lecuona [18].

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2. **Proofs**

2.1. **Infinitely many inequivalent smooth structures.** Fintushel-Stern [12, Theorem 1] showed that there is a nullhomologous 2-torus \( T \) smoothly embedded in \( \mathbb{CP}^2 \mathbb{CP}^2 \) such that performing surgeries on \( T \) results in infinitely many inequivalent smooth structures on \( \mathbb{CP}^2 \mathbb{CP}^2 \). We point out that changing the coefficients of the torus surgery on \( T \) introduces homotopically non-trivial loops to the resulting 4-manifold, and their procedure also yields infinitely many smooth structures on 4-manifolds with prescribed cyclic fundamental group. The latter will serve as raw material to construct the knotted 2-spheres.

We introduce terminology to state the result and follow the notation in [12, Section 3]. Let \( T \subset X \) be a smoothly embedded 2-torus with trivial tubular neighborhood \( \nu(T) = T^2 \times D^2 \). Let \( \{ a, b \} \) be loops in \( T \) that form a symplectic basis of \( \pi_1(T) = \mathbb{Z}^2 \), and let \( \{ S_a^1, S_b^1 \} \) be loops in \( \partial \nu(T) = T^2 \times \partial D^2 = T^2 \times S^1 \) that are homologous to \( a \) and \( b \), respectively. The meridian of \( T \) is denoted by \( \mu_T \) and it is any curve in the same isotopy class of the curve \( \{ x \} \times \partial D^2 \subset \partial \nu(T) \). The smooth 4-manifold

\[
X_{T, S_b^1}(p/n) := (X \setminus \nu(T)) \cup_{\varphi} (T^2 \times D^2)
\]

where the gluing diffeomorphism satisfies \( \varphi_\ast([\partial D^2]) = n[S_b^1] + p[\mu_T] \) is said to be obtained by performing a \( p/n \)-torus surgery to \( X \) on \( T \) along the curve \( b \).

We first consider the case of finite cyclic fundamental group and postpone the infinite cyclic case to the end of the section.

**Theorem 1.** Fix \( p \geq 2 \). There is a smoothly embedded nullhomologous 2-torus \( T \subset \mathbb{CP}^2 \mathbb{CP}^2 \) and a nullhomologous curve in its complement \( S_b^1 \subset \mathbb{CP}^2 \mathbb{CP}^2 \setminus \nu(T) \) such that performing a \( (p/n) \)-torus surgery to \( \mathbb{CP}^2 \mathbb{CP}^2 \) on \( T \) along \( S_b^1 \) yields an infinite set

\[
\{ X_{T, S_b^1}(p/n) : n \in \mathbb{Z} \}
\]
of pairwise non-diffeomorphic 4-manifolds such that every element is homeomorphic to the connected sum
\[(2.3) \quad \mathbb{CP}^2 \# 3\mathbb{CP}^2 \# \Sigma_p,\]
where \(\Sigma_p\) is a \(\mathbb{Q}\)-homology 4-sphere with fundamental group \(\pi_1\Sigma_p = \mathbb{Z}/p\).

**Proof.** The only contribution in this note to the work of Fintushel-Stern [11, 12] that provides a proof of Theorem 1 is the change in a coefficient of the torus surgery. We then employ a homeomorphism criteria of Hambleton-Kreck to pin down the existence of a nullhomologous torus \(T \subset X\) and the curve \(b \subset T\) with framing \(S_b \subset X \setminus \nu(T)\) as in the statement of Theorem 1. Build \(X_{T,S_b}(p/n)\) as in (2.1). Since \(\{T\} = 0 \in H_2(X; \mathbb{Z})\), we have that \(H_1(X_{T,S_b}(p/n); \mathbb{Z}) = \mathbb{Z}/p\) for every \(n \in \mathbb{Z}\); in this notation, \(p = 0\) corresponds to \(\mathbb{Z}\). We now fix \(p \geq 2\).

To see that the fundamental group of \(X_{T,S_b}(p/n)\) is \(\mathbb{Z}/p\), we take a closer look at the constructions of Fintushel-Stern in [11, 12], where six torus surgeries along six nullhomologous 2-tori \(\{T_{1,i}, T_{2,i} : i = 1, 2, 3\}\) are performed to \(X\) to produce a symplectic 4-manifold \(Q\) with fundamental group \(\pi_1(Q) = \mathbb{Z}/p\) and that contains six Lagrangian 2-tori \(\{L_{1,i}, L_{2,i} : i = 1, 2, 3\}\) [11, Proposition 7], [12, p. 77]. The complements of these 2-tori are the same, i.e.,
\[(2.4) \quad X \setminus \bigcup_{i=1}^{3}(\nu(T_{1,i}) \cup \nu(T_{2,i})) = Q \setminus \bigcup_{i=1}^{3}(\nu(L_{1,i}) \cup \nu(L_{2,i})).\]

By applying six surgeries to the symplectic 4-manifold \(Q\) along the Lagrangian 2-tori with a given choice of surgery curves [11, Theorem 2], one obtains an infinite set of inequivalent smooth structures on \(X\). The first five surgeries are \([1/1]\)-torus surgeries, while the last one is a \(1/n\)-torus surgery [11, p. 1685]. In particular, this infinite set can be obtained by performing torus surgeries to \(X\) on six nullhomologous 2-tori. For our purposes, we perform the first five surgeries verbatim as in the proof of [11, Theorem 2], change the surgery coefficients of the sixth surgery in order to obtain an infinite set
\[(2.5) \quad \{X_{T,S_b}(p/n) : n \in \mathbb{Z}\}\]
for a fixed \(p \geq 2\). It follows from the Seifert-van Kampen theorem that the fundamental group is \(\pi_1(X_{T,S_b}(p/n)) = \mathbb{Z}/p\) [8, p. 321] for every \(n \in \mathbb{Z}\): a detailed account on the computation of the fundamental group of the 4-manifolds obtained with such a change in the surgery coefficient can be found in several places in the literature, for example, [8, Section 5], [5, p. 595]. We have explained so far that six surgeries on six nullhomologous 2-tori in \(X\) as in [11, Theorem 2] produce an infinite set \(\{X_{T,S_b}(p/n) : n \in \mathbb{Z}\}\) of 4-manifolds with fundamental group \(\mathbb{Z}/p\).

We now appeal to the main result in [12, Section 8]. Fintushel-Stern showed that the first five surgeries on \(X\) do not change the diffeomorphism type of \(X\) and, thus, that there is a single nullhomologous 2-torus \(T \subset X\) along with a nullhomologous curve \(S_b \subset X \setminus \nu(T)\) such that an \(1/n\)-torus surgery produce an infinite set of smooth structures on \(X\), as we had mentioned before [12, Theorem 1.1]. Thus, we conclude that each element in the set \(\{X_{T,S_b}(p/n) : n \in \mathbb{Z}\}\) is obtained by performing a \(p/n\)-torus surgery on \(T \subset X\) along \(S_b\).
We now argue that these 4-manifolds are homeomorphic to \((2.3)\). An inclusion-exclusion argument indicates that the Euler characteristic is unchanged under torus surgeries, i.e.,

\[
\chi(X_{T,S^1}(p/n)) = \chi(X) = 6.
\]

Novikov additivity \cite{15} Remark 9.1.7 implies

\[
\sigma(X_{T,S^1}(p/n)) = \sigma(X) = -2,
\]

and we conclude that the second Stiefel-Whitney class of \(X_{T,S^1}(p/n)\) does not vanish employing a result of Rohklin cf. \cite{15} Theorem 1.2.29. A classification result of Hambleton-Kreck \cite{17} Theorem C allows us to conclude that the 4-manifold \(X_{T,S^1}(p/n)\) is homeomorphic to \(\mathbb{C}P^2 \#_3 \mathbb{C}P^2 \# \Sigma_p\), where \(\Sigma_p\) is a closed smooth 4-manifold with Euler characteristic two and signature zero for every \(n \in \mathbb{Z}\) and \(p \geq 2\).

To argue that we have constructed infinitely many 4-manifolds that are pairwise non-diffeomorphic, we compute their Seiberg-Witten invariants using an argument well-documented in the literature \cite{4,8,10,11,12}. We reproduce the argument here for the sake of completeness, which requires us to describe the relation between the Seiberg-Witten invariants of the 4-manifolds \(X_{T,S^1}(p/n)\), \(X\) and \(X_{T,S^1}(1/0)\). Given a characteristic element \(k_0 \in H_2(X_{T,S^1}(0/1); \mathbb{Z})\), there are unique characteristic elements \(k_X \in H_2(X_{T,S^1}(0/1); \mathbb{Z}) = H_2(X; \mathbb{Z})\) and \(k_{p/n} \in H_2(X_{T,S^1}(p/n); \mathbb{Z})\) \cite{4} Remark 4, \cite{11} p. 64. The 4-manifolds \(X_{T,S^1}(1/0)\) and \(X_{T,S^1}(0/1)\) will have at most one basic class up to sign in our setting; cf. \cite{4,12,11}. As described in \cite{12} Section 3, the 4-manifold \(X_{T,S^1}(0/1)\) has infinite cyclic fundamental group and it admits a symplectic structure \cite{12} Section 4 (cf. \cite{10} Section 3). A result of Taubes \cite{31} says that the canonical class \(k_0 = -c_1(X_{T,S^1}(0/1))\) is a basic class of \(X_{T,S^1}(0/1)\) and \(\text{SW}_{X_{T,S^1}(0/1)}(\pm k_0) = \pm 1\). Moreover, the adjunction inequality (see \cite{4} Section 2.1) implies that \(k_0 \in \mathcal{B}\) is the only basic class up to sign.

It follows that there is a unique \(k_{p/n} \in \mathcal{B}_{X_{T,S^1}(p/n)}\) up to sign for every \(n \geq 1\), and the product formula of Morgan-Mrowka-Szabo \cite{24} Theorem 1.1] yields

\[
\text{SW}_{X_{T,S^1}(p/n)}(k_{p/n}) = p \cdot \text{SW}_X(k_X) + n \cdot \sum_i \text{SW}_{X_{T,S^1}(0/1)}(k_0 + i[T_0]).
\]

There is a 2-torus \(T_0 \subset X_{T,S^1}(0/1)\) that is geometrically dual to the core 2-torus \(T_0 \subset X_{T,S^1}(0/1)\) of the surgery. Along with this fact, an adjunction inequality argument implies that the sum on the right handside of \((2.8)\) contains at most one non-vanishing term; see \cite{4] Section 4.1] for the argument. We have the equality

\[
\text{SW}_{X_{T,S^1}(p/n)}(k_{p/n}) = p \cdot \text{SW}_X(k_X) + n \cdot \text{SW}_{X_{T,S^1}(0/1)}(k_0)
\]

and we conclude that there is an infinite set of pairwise non-diffeomorphic closed 4-manifolds \((2.2)\). This concludes the proof of Theorem \cite{1}.

What is obtained when we set \(p = 0\) in the statement of Theorem \cite{1} and the previous proof, is an infinite set \(\{X_{T,S^1}(0/n) : n \in \mathbb{Z} - \{0\}\}\) of pairwise non-diffeomorphic closed 4-manifolds with infinite cyclic fundamental group and the same homology of the connected sum \(2\mathbb{C}P^2 \#_4 \mathbb{C}P^2 \# S^1 \times S^3\); cf. Fintushel-Stern \cite{12} Theorem 1.1.
Theorem 2. There is a smoothly embedded nullhomologous 2-torus \( T \subset \mathbb{CP}^2 \# 3 \overline{\mathbb{CP}^2} \) and a nullhomologous curve in its complement \( S^1_b \subset \mathbb{CP}^2 \# 3 \overline{\mathbb{CP}^2} \) such that performing a \((0/n)\)-torus surgery to \( \mathbb{CP}^2 \# 3 \overline{\mathbb{CP}^2} \) on \( T \) along \( S^1_b \) yields an infinite set

\[
\{ X_{T,S^1_b}(0/n) : n \in \mathbb{Z} - \{0\} \}
\]

of pairwise non-diffeomorphic 4-manifolds with infinite cyclic fundamental group and such that every element has the homology of the connected sum

\[
2\mathbb{CP}^2 \# 4 \overline{\mathbb{CP}^2} \# S^1 \times S^3.
\]

Similar statements to Theorem 1 and Theorem 2 for further choices of homeomorphism types of 4-manifolds with cyclic fundamental group are produced by employing other results of Fintushel-Stern in [12].

2.2. 2-spheres whose complement has a prescribed fundamental group.
Let \( X \) be a closed smooth 4-manifold whose fundamental group has a presentation

\[
\pi_1 X = \langle g_1, \ldots, g_j : r_1, \ldots, r_k \rangle
\]

such that adding the relation \( g_1 = 1 \) to it for a given generator \( g_1 \), one obtains the trivial group. Cyclic groups and the group \( \langle g_1, g_2 : g_1^5 = (g_1g_2)^2 = g_2^3 \rangle \) are examples of such groups.

Let \( \alpha_1 \subset X \) be a based loop whose homotopy class is \([\alpha_1]\) = \( g_1 \in \pi_1 X \). Build the closed simply connected 4-manifold

\[
Z := X \setminus \nu(\alpha_1) \cup (D^2 \times S^2)
\]

and consider the belt 2-sphere

\[
S := \{0\} \times S^2 \subset D^2 \times S^2 \subset Z.
\]

Further information is needed on the framing of the loop \( \alpha_1 \subset X \) to pin down the diffeomorphism or homeomorphism type of \( Z \). Once this is taken care of, this process provides a 2-sphere (2.14) smoothly embedded in \( Z \) and whose complement has fundamental group \( G \). A topological construction of locally flat 2-surfaces in topological 4-manifolds is obtained by using locally flat embedded submanifolds in the surgery (2.13); see [13, Section 9.3] for existence and uniqueness results on tubular neighborhoods of locally flat embedded submanifolds.

We set some notation and construct the 2-spheres of Theorem A using this procedure in the following example. It includes the choice of framing on the loop whose homotopy class generates the fundamental group.

Example 1. Fix \( p \geq 2 \) and an integer \( n \in \mathbb{Z} \). Consider the 4-manifold \( X_{T,S^1_b}(p/n) \) in the set (2.2) and let \( \tilde{T} \subset X_{T,S^1_b}(p/n) \) be the core 2-torus of the surgery. Let \( \alpha \subset X_{T,S^1_b}(p/n) \) be a loop such that the 4-manifold

\[
Z_{n,p} := (X_{T,S^1_b}(p/n) \setminus \nu(\alpha)) \cup (D^2 \times S^2)
\]

is simply connected and consider the belt 2-sphere

\[
S_{n,p} := \{0\} \times S^2 \subset D^2 \times S^2 \subset Z_{n,p}.
\]
Notice that the loop $\alpha$ lies on the boundary of $\partial \nu(\hat{T})$. The framing on the loop $\alpha$ is induced by the product framing of core torus of the $(p/n)$-torus surgery. The complement of the 2-sphere (2.10) has fundamental group

$$\pi_1(Z_{n,p} \setminus \nu(S_n)) = \mathbb{Z}/p,$$

and homology class of (2.10) satisfies $[S_{n,p}] \neq 0 \in H_2(Z_{n,p}; \mathbb{Z})$. Moreover, the 4-manifold $X_{n,p}$ is recovered by applying surgery to $Z_{n,p}$ along $S_{n,p}$.

2.3. The ambient 4-manifold of Theorem A and Theorem D. We prove in this section that the 2-spheres (2.16) of Example 1 are all smoothly embedded in $2\mathbb{C}P^2 \# 4\mathbb{C}P^2$, and postpone to Section 2.7 the proof that they are pairwise smoothly inequivalent.

Proposition 1. The 4-manifold $Z_{n,p}$ defined in (2.15) is diffeomorphic to the connected sum $2\mathbb{C}P^2 \# 4\mathbb{C}P^2$ for every $n \in \mathbb{Z}$ and a fixed $p \geq 2$. In particular, there is an infinite set

$$(2.18) \{S_{n,p} : n \in \mathbb{Z}\}$$

of 2-spheres smoothly embedded in $Z = 2\mathbb{C}P^2 \# 4\mathbb{C}P^2$ such that the complement $Z \setminus \nu(S_{n,p})$ has fundamental group $\mathbb{Z}/p$ for every $n \in \mathbb{Z}$.

Proof. We use an argument due to Moishezon [25, Lemma 13] (see Gompf’s description too [14, Lemma 3]) and work of Baykur-Sunukjian [9] to establish the diffeomorphism type of our 4-manifolds. We follow the notation in [14, Lemma 3], fix an $n \in \mathbb{Z}$ and a $p \geq 2$, and consider the 4-manifold $X_{T,b}(p/n)$ in (2.2) that is constructed from $\mathbb{C}P^2 \# 3\mathbb{C}P^2$ using torus surgeries and the 4-manifold $Z_{n,p}$ built in (2.15). Perform a torus surgery to $X_{T,b}(p/n)$ which identifies the loop that generates its fundamental group with the normal disk to the 2-torus to obtain a simply connected 4-manifold $\hat{N}$; this gluing map is described on [14, page 101]. The latter 4-manifold can also be obtained by applying a torus surgery to $\mathbb{C}P^2 \# 3\mathbb{C}P^2$. Moishezon’s argument implies that $Z_{n,p}$ is obtained from $\hat{N}$ by doing surgery along a loop [15, §5.2], i.e., $Z_{n,p} = N^* = \hat{N} \# S^2 \times S^2$ [15, Proposition 5.2.3, Proposition 5.2.4]. Results of Baykur-Sunukjian [9, Section 3] imply that $\hat{N} \# S^2 \times S^2$ is diffeomorphic to $\mathbb{C}P^2 \# 3\mathbb{C}P^2 \# S^2 \times S^2 = 2\mathbb{C}P^2 \# 4\mathbb{C}P^2$. Since the choice of $n$ and $p$ was arbitrary, we conclude that $Z_{n,p}$ is diffeomorphic to $2\mathbb{C}P^2 \# 4\mathbb{C}P^2$ for every $n \in \mathbb{Z}$ and $p \geq 2$. \qed

A tweak to the proof of Proposition 1 pins down the diffeomorphism type of the 4-manifolds constructed in the proof of Theorem D.

2.4. Topological isotopy. Locally-flat embeddings of 2-spheres in 4-manifolds whose complement has finite cyclic fundamental group have been studied by Lee-Wilczynski [22] and Hambleton-Kreck [16, Theorem 4.5]. The next result from their work is of particular importance for our purposes.

Theorem 3. (Lee-Wilczynski, Hambleton-Kreck). Let $X$ be a closed simply connected topological 4-manifold such that $b_3(X) > |\sigma(X)| + 2$ and let $h \in H_2(X; \mathbb{Z})$ be a homology class of non-zero divisibility $p \neq 0$. Let $S_1, S_2 \subset X$ be locally flat embedded 2-spheres with homology classes $[S_1] = [S_2] = h \in H_2(X; \mathbb{Z})$, and whose
complement has fundamental group \( \pi_1(X \setminus \nu(S_1)) = \mathbb{Z}/p = \pi_1(X \setminus \nu(S_2)) \) for \( p \geq 2 \). If
\[
(2.19) \quad b_2(X) > \max_{0 \leq j < p} |\sigma(X) - 2j(p - j)(1/p^2)h \cdot h|,
\]
then there is a topological isotopy between \( S_1 \) and \( S_2 \).

Notice that our ambient 4-manifold \( 2\mathbb{C}P^2 \# 4\overline{\mathbb{C}P^2} \) is within the range of the hypothesis of Theorem 3. Moreover, the homology class of the belt 2-sphere (2.16) of Example 1 has non-zero divisibility and self-intersection zero by construction. We conclude that the 2-spheres that were constructed in the previous sections are all topologically isotopic to each other by Theorem 3.

**Corollary 1.** The infinite set \( \{S_{n,p} : n \in \mathbb{Z} \} \) of Proposition 1 is made of smoothly embedded 2-spheres in \( 2\mathbb{C}P^2 \# 4\overline{\mathbb{C}P^2} \) that are pairwise topologically isotopic.

### 2.5. Some examples of \( \mathbb{F} \)-homology 4-spheres

Constructions of 4-manifolds that have the same \( \mathbb{F} \)-homology as \( S^4 \) are not scarce in the literature. For example, a surgery theory construction of \( \mathbb{Q} \)-homology 4-spheres with finite cyclic fundamental group is given by Hambleton-Kreck in [17, Proposition 4.1]. Their examples include 4-manifolds with non-vanishing Kirby-Siebermann invariant and they admit no smooth structure. We describe two constructions of such objects in this section.

The first involves doing surgery on the product of a 3-manifold with a circle. Spun closed smooth 4-manifolds form a classical set of examples of 4-manifolds that share the homology of \( S^4 \) with \( \mathbb{F} \)-coefficients and whose fundamental group is a 3-manifold group. We briefly recall their construction and suppose that \( N \) is a closed orientable 3-manifold. A homology 4-sphere \( \Sigma_N \) with fundamental group \( \pi_1 \Sigma_N = \pi_1 N \) is constructed as
\[
(2.20) \quad \Sigma_N := (N \times S^1) \setminus \nu(\{pt\} \times S^1) \cup_{\text{id}} (D^2 \times S^2),
\]
where we use the identity map to identify the common boundary. There is another choice of framing, yet results of Plotnick [26] state that there is a unique diffeomorphism class of (2.20) for the 3-manifolds employed in this paper. If \( N \) is an \( \mathbb{F} \)-homology 3-sphere, then \( \Sigma_N \) is an \( \mathbb{F} \)-homology 4-sphere. There are two principal choices of 3-manifold used in the proofs of our results.

- For \( N = L(p, 1) \), we obtain a \( \mathbb{Q} \)-homology 4-sphere \( \Sigma_{L(p, 1)} \) with fundamental group \( \pi_1 \Sigma_{L(p, 1)} = \mathbb{Z}/p \).
- For \( N = \Sigma(2, 3, 5) \), we obtain a \( \mathbb{Z} \)-homology 4-sphere \( \Sigma_{\Sigma(2,3,5)} \) with fundamental group \( \pi_1 \Sigma_{\Sigma(2,3,5)} = \langle a, b : a^5 = (ab)^2 = b^3 \rangle \).

A second construction of smooth \( \mathbb{Q} \)-homology 4-spheres with finite cyclic fundamental group is through handlebodies. Gompf-Stipsicz’s depiction of a pair of orientable \( S^2 \)-bundles over \( \mathbb{R}P^2 \) in [15, Figure 5.46] describes a handlebody of a pair of \( \mathbb{Q} \)-homology 4-spheres with fundamental group of order two whose second Stiefel-Whitney class can be chosen to vanish or not depending on the \( n \)-framing of one of the two 2-handles. Handlebodies of pairs of \( \mathbb{Q} \)-homology 4-spheres \( \{\Sigma_{p,n} : n \in \{0, 1\} \} \) with fundamental group
\[
\pi_1 \Sigma_{p,n} = \mathbb{Z}/p
\]
for every \( p \geq 2 \) and second Stiefel-Whitney class
\[
w_2 \Sigma_{p,n} = n
\]
consisting of one 0-handle, one 1-handle, one 0-framed 2-handle, one \(n\)-framed 2-handle, one 3-handle, and one 4-handle are drawn as a straight-forward extension of the \(p = 2\) case [15, Figure 5.46].

2.6. 2-spheres in simply connected 4-manifolds via \(\mathbb{F}\)-homology 4-spheres.

The 4-manifolds of the previous section and the procedure of Section 2.2 yields knotted 2-spheres smoothly embedded in the total space of an \(S^2\)-bundle over \(S^2\). The case of most interest for us is summarized in the following lemma.

**Lemma 1.** (Sato [27, §3]). There is a smoothly embedded 2-sphere \(S_p \hookrightarrow S^2 \times S^2\) whose complement has fundamental group \(\mathbb{Z}/p\) for every \(p \geq 2\).

There is a smoothly embedded 2-sphere \(S_G \hookrightarrow S^2 \times S^2\) whose complement has fundamental group \(G = \langle a, b : a^5 = (ab)^2 = b^3 \rangle\) or \(\mathbb{Z}/p\).

A variation of the proof of Proposition [1] yields a proof of Lemma [1] by using Moishezon's argument [25], a lemma of Gompf [14, Lemma 1.6] and a result of Akbulut [1, Theorem]; cf. Tange [30]. Another proof of Lemma [1] is obtained by using handlebodies [1, 2, 15].

2.7. Proof of Theorem [A]. We collect the results of previous sections into a proof of the following theorem, which is equivalent to Theorem [A].

**Theorem 4.** Fix \(p \geq 2\). There is an infinite set

\[
\{ S_{n,p} \subset 2\mathbb{C}P^2 \# 4\mathbb{C}P^2 : n \in \mathbb{Z} \}
\]

made of topologically isotopic 2-spheres whose complement has fundamental group \(\mathbb{Z}/p\), and for which doing surgery on each element yields the infinite set (2.2) of pairwise non-diffeomorphic smooth 4-manifolds in the homeomorphism class of \(\mathbb{C}P^2 \# 3\mathbb{C}P^2 \# \Sigma_p\).

In particular, there is a diffeomorphism of pairs

\[
(2\mathbb{C}P^2 \# 4\mathbb{C}P^2, S_{n_1,p}) \rightarrow (2\mathbb{C}P^2 \# 4\mathbb{C}P^2, S_{n_2,p})
\]

if and only if \(n_1 = n_2\), and the infinite set (2.21) consists of pairwise smoothly inequivalent 2-spheres.

**Proof.** The infinite set (2.21) was constructed in Section 2.2. The fundamental group of the complement of any 2-sphere is a prescribed finite cyclic group; see Example 1. Corollary 1 says that elements in (2.21) are pairwise topologically isotopic. As indicated in Example 1 the 4-manifold \(X_{T, S^2} (p/n)\) in the infinite set (2.2) is obtained by carving out a tubular neighborhood \(\nu(S_{n,p})\) of a 2-sphere in (2.21) from \(2\mathbb{C}P^2 \# 4\mathbb{C}P^2\), and capping off the boundary with \(S^1 \times D^3\). Given that the infinite set (2.2) is made of pairwise non-diffeomorphic 4-manifolds, we conclude that the infinite set (2.21) is made of pairwise smoothly inequivalent 2-spheres. □

**Remark 1.** A minor modification to the previous argument yields a proof of Theorem [B].

2.8. Proof of Theorem [B]. Let \(\{ Z_n : n \in \mathbb{Z} \}\) be an infinite set of pairwise non-diffeomorphic 4-manifolds in the homeomorphism class of \(Z\). Taking a connected sum with any \(\mathbb{Q}\)-homology 4-sphere \(M\) yields an infinite set

\[
\{ Z_n \# M : n \in \mathbb{Z} \}
\]
of reducible pairwise non-diffeomorphic 4-manifolds that are pairwise homeomorphic to \( Z \# M \). The smooth structures are distinguished with the Seiberg-Witten invariant of the connected sums using the fact that \( b_1(M) = 0 = b_1^+(M) \) and results of Kotschick-Morgan-Taubes [21]. By hypothesis, there is a Spin\(^c\)-structure on \( Z_n \) for which the Seiberg-Witten invariant \( SW_{Z_n} \) is non-zero. As explained in [21] Proof of Proposition 2], the Spin\(^c\)-structure can be extended to the connected sum \( Z_n \# M \) and conclude that there is a Spin\(^c\)-structure for which \( SW_{Z_n \# M} = SW_{Z_n} \). This implies that the infinite set \( \{ Z_n \# M : n \in \mathbb{Z} \} \) consists of pairwise non-diffeomorphic 4-manifolds that are pairwise homeomorphic to \( Z \# M \).

We do surgery along the loop \( \alpha \subset Z_n \# M \) as in the hypothesis of Theorem [13] verbatim to the procedure described in Example [1] to construct an infinite set
\[
(2.24) \quad \{ S_{n, \pi} : n \in \mathbb{Z}, \pi = \pi_1 M \}
\]
that are smoothly embedded in \( Z \# S^2 \times S^2 \) and whose complement has fundamental group \( \pi = \pi_1 M \). By construction we obtain a homeomorphism of pairs between \( (Z \# S^2 \times S^2, S_{n_1, \pi}) \) and \( (Z \# S^2 \times S^2, S_{n_2, \pi}) \) for every \( n_1, n_2 \in \mathbb{Z} \). Surgery on the belt 2-sphere \( S_{n, \pi} \subset Z \# S^2 \times S^2 \) gives us \( Z_n \# M \) back. Since the infinite set (2.23) is made of pairwise non-diffeomorphic 4-manifolds, we conclude that there is no diffeomorphism of pairs
\[
(2.25) \quad (Z \# S^2 \times S^2, S_{n_1, \pi}) \to (Z \# S^2 \times S^2, S_{n_2, \pi})
\]
if \( n_1 \neq n_2 \).

\[\Box\]

Remark 2. We elaborate on an argument to prove Theorem [4] by using the construction procedure of Theorem [12]. The ingredients that satisfy the hypothesis of the latter are the following. Take the infinite set \( \{ Z_n : n \in \mathbb{Z} \} \) of pairwise non-diffeomorphic 4-manifolds that are homeomorphic to \( \mathbb{C}P^2 \# 3\mathbb{C}P^2 \) that was constructed by Fintushel-Stern in [12]. These 4-manifolds have different Seiberg-Witten invariant. A result of Baykur-Sunukjian [9] implies that \( Z_n \# S^2 \times S^2 \) is diffeomorphic to \( 2\mathbb{C}P^2 \# 4\mathbb{C}P^2 \) for every \( n \in \mathbb{Z} \). As the 4-manifold \( M \) in the statement of Theorem [12] use the \( Q \)-homology 4-sphere \( \Sigma_{p,0} \) that was discussed in Section 2.6 with \( \pi_1 \Sigma_{p,0} = \mathbb{Z}/p \). Build the infinite set
\[
(2.26) \quad \{ Z_n \# \Sigma_{p,0} : n \in \mathbb{Z} \}
\]
of closed reducible 4-manifolds that are homeomorphic to \( \mathbb{C}P^2 \# 3\mathbb{C}P^2 \# \Sigma_{p,0} \). The set (2.20) consists of pairwise non-diffeomorphic 4-manifolds, where the diffeomorphism classes are distinguished by their Seiberg-Witten invariants [21] Proposition 2]. Proceed as in the proof of Theorem [12] and build an infinite set (2.24) of pairwise smoothly inequivalent 2-spheres. These submanifolds have the required properties by construction and they are pairwise topologically isotopic by Theorem [4].

2.9. Proof of Corollary [4] We check that the hypothesis of Theorem [13] are met in these cases. As the infinite set \( \{ Z_n : n \in \mathbb{Z} \} \) we can take the infinite inequivalent smooth structures on \( \mathbb{C}P^2 \# 3\mathbb{C}P^2 \) constructed by Fintushel-Stern [12]. Work of Baykur-Sunukjian [9] Theorem 1 implies that \( Z_n \# S^2 \times S^2 = 2\mathbb{C}P^2 \# 4\mathbb{C}P^2 \) for every \( n \in \mathbb{Z} \); this connected sum is the simply connected 4-manifold in the statement of Corollary [4]. The \( Q \)-homology 4-spheres with the desired fundamental group were constructed in Section 2.6; see Lemma [1].
2.10. **Proof of Theorem E**. Hambleton-Kreck [17, Proposition 4.1] used surgery to prove the existence of a $Q$-homology 4-sphere $M_p$ with non-zero second Stiefel-Whitney class $w_2(M_p) \neq 0$, non-vanishing Kirby-Siebenmann invariant $KS(M_p) \neq 0$, and fundamental group $\pi_1 M_p = \mathbb{Z}/p$ for every $p \geq 2$. Carve out the loop in $M_p$ whose homotopy class generates the group $\pi_1 M_p = \mathbb{Z}/p$, and glue back a locally flat copy of $D^2 \times S^2$ to obtain a simply connected 4-manifold $\hat{M}$ with Euler characteristic $\chi(\hat{M}) = \chi(M_p) + 2 = 4$, signature $\sigma(\hat{M}) = \sigma(M_p)$, second Stiefel-Whitney class $w_2(\hat{M}) \neq 0$ and Kirby-Siebenmann invariant $KS(\hat{M}) \neq 0$. A result of Freedman-Quinn [13, Section 10.1] states that $\hat{M}$ is homeomorphic to the connected sum $\# \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ of the Chern manifold and the complex projective space with the opposite orientation for every $p \geq 2$ (cf. [15, Theorem 1.2.27]). The fundamental group of the complement of the belt 2-sphere $S_p$ of the surgery is isomorphic to $\pi_1 M_p = \mathbb{Z}/p$.

To produce the smoothly embedded 2-sphere $S'_p \subset \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ as in (1.5) and prove the last clause of Theorem E perform surgery to the smooth $Q$-homology 4-sphere $\Sigma_{p,1}$ described in Section 2.6.

**References**

[1] S. Akbulut, *Scharlemann's manifold is standard*, Ann. of Math. 149 (1999), 497 - 510.
[2] S. Akbulut, *4-manifolds*, Oxford Graduate Texts in Mathematics, 25. Oxford University Press, 2016, vii + 263 pp.
[3] S. Akbulut, *Isotoping 2-spheres in 4-manifolds*, in Proceedings of the 21st. Gökova Geometry-Topology Conference 2014, 264 - 266, GGT, Gökova, 2014.
[4] A. Akhmedov, R. I. Baykur and B. D. Park, *Constructing infinitely many smooth structures on small 4-manifolds*, J. Topol. 1 (2008), 409 - 428.
[5] A. Akhmedov and B. D. Park, *Exotic smooth structures on small 4-manifolds with odd signatures*, Invent. Math. 181 (2010), 577 - 603.
[6] D. Auckly, H. J. Kim, P. Melvin and D. Ruberman, *Stable isotopy in four dimensions*, Jour. London Math. Soc. 91 (2015), 439 - 463.
[7] D. Auckly, H. J. Kim, P. Melvin, D. Ruberman and H. Schwartz, *Isotopy of surfaces in 4-manifolds after a single stabilization* Adv. in Math. 341 (2018), 609 - 615.
[8] S. Baldwin and R. Kirby, *Construction of small symplectic 4-manifolds using Luttinger surgery*, J. Diff. Geom. 82 (2008), 919 - 940.
[9] R. I. Baykur and N. Sunukjian, *Round handles, logarithmic transforms and smooth 4-manifolds*, J. Topol. 6 (2013), 49 - 63.
[10] R. A. Fintushel, B. D. Park, and R. J. Stern, *Reverse engineering small 4-manifolds*, Algeb. Geom. Topol. 7 (2007), 2103 - 2116.
[11] R. A. Fintushel and R. J. Stern, *Pinwheels and nullhomologous surgery on 4-manifolds with $b^+ = 1$*, Alg. Geom. Topol. 11 (2011), 1649 - 1699.
[12] R. A. Fintushel and R. J. Stern, *Surgery on nullhomologous tori*, Geom. Topol. Monographs 18 (2012), 61 - 81.
[13] M. H. Freedman and F. Quinn, *Topology of 4-manifolds*, Princeton Mathematical Series 39, Princeton University Press, Princeton, NJ, 1990, viii + 259 pp.
[14] R. E. Gompf, *Sums of elliptic surfaces*, J. Differential Geom. 34 (1991), 93 - 114.
[15] R. E. Gompf and A. I. Stipsicz, *4-Manifolds and Kirby Calculus*, Graduate Studies in Mathematics, 20. Amer. Math. Soc., Providence, RI, 1999. xv + 557 pp.
[16] I. Hambleton and M. Kreck, *Cancellation of hyperbolic forms and topological four-manifolds*, J. Reine Angew. Math. 443 (1993), 21 - 47.
I. Hambleton and M. Kreck, *Cancellation, elliptic surfaces and the topology of certain four-manifolds*, J. Reine Angew. Math. 444 (1993), 79 -100.

D. Kasprzowski, P. Lambert-Cole, M. Land and A. G. Lecuona, *Topologically flat embedded 2-spheres in specific simply connected 4-manifolds*, MATRIX Annals, (2019), 111 - 116.

H. J. Kim, *Modifying surfaces in 4-manifolds by twist spanning*, Geom. Topol. 10 (2006), 27 - 56.

D. Kasprowski, P. Lambert-Cole, M. Land and A. G. Lecuona, *Topologically flat embedded 2-spheres in specific simply connected 4-manifolds*, MATRIX Annals, (2019), 111 - 116.

H. J. Kim, *Modifying surfaces in 4-manifolds by twist spanning*, Geom. Topol. 10 (2006), 27 - 56.

R. Lee and D. M. Wilczyński, *Originally flat 2-spheres in simply connected 4-manifolds*, Comment. Math. Helv. 6 (1990), 388 - 412.

R. Lee and D. M. Wilczyński, *Representing homology classes by locally flat surfaces of minimum genus*, Amer. J. Math. 119 (1997), 1119 - 1137.

J. W. Morgan, T. S. Mrowka and Z. Szabó, *Product formulas along $T^3$ for Seiberg-Witten invariants*, Math. Res. Lett. 4 (1997), 915 - 929.

B. Moishezon, *Complex surfaces and connected sums of complex projective planes*, Lect. Notes in Math., 603, Springer, Berlin, 1977.

S. P. Plotnick, *Equivariant intersection forms, knots in $S^4$, and rotations in 2-spheres*, Trans. Amer. Math. Soc. 296 (1986), 543 - 575.

Y. Sato, *Locally flat 2-knots in $S^2 \times S^2$ with the same fundamental group*, Trans. Amer. Math. Soc. 323 (1991), 911 - 920.

H. Schwartz, *Equivalent non-isotopic spheres in 4-manifolds*, J. Topol. 12 (2019), 1396 - 1412.

N. S. Sunukjian, *Surfaces in 4-manifolds: concordance, isotopy, and surgery*, Int. Math. Res. Not. 17 (2015), 7950 - 7978.

M. Tange, *The link surgery of $S^2 \times S^2$ and Scharlemann’s manifolds*, Hiroshima Math. J. 44 (2014), 35 - 62.

C. H. Taubes, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Lett. 1 (1994), 809 - 822.