A Simple Bound for Resilient Submodular Maximization with Curvature

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1 Introduction

Resilient submodular maximization refers to the combinatorial problems studied by Nemhauser et al. [5] and Fisher et al. [3] and asks how to maximize an objective given a number of adversarial removals. For example, one application of this problem is multi-robot sensor planning with adversarial attacks [9, 6]. However, more general applications of submodular maximization are also relevant. Tzoumas et al. [8] obtain near-optimal solutions to this problem by taking advantage of a property called curvature to produce a mechanism which makes certain bait elements interchangeable with other elements of the solution that are produced via typical greedy means [3, 5]. This document demonstrates that—at least in theory—applying the method for selection of bait elements to the entire solution can improve that guarantee on solution quality.

Specifically, this document provides a brief description and proof for an algorithm that improves upon bounds for resilient submodular maximization with curvature by Tzoumas et al. [8, (6), (7)]. The approach builds on work by Zhou et al. [11] and takes aspects of that analysis to a natural conclusion.1

The proposed algorithm maximizes objective values for individual solution elements on their own i.e. \( f(x) \) for \( x \in \Omega \) subject to some constraint where \( f \) is a submodular, monotonic, normalized set function and \( \Omega \) is the ground set.2 This contrasts with typical methods which maximize marginal gains of the form \( f(x) - f(X) \) where \( X \subseteq \Omega \) is typically a partial solution. Intuitively, the approach that this document describes will perform well because the curvature of \( f \) prevents objective values for sets from deviating far from the sum of values for solution elements considered individually.

Curvature is defined as follows:

Definition 1 (Curvature). Consider a monotonic and submodular set function \( f \). The curvature of \( f \) is

\[

\nu = 1 - \min_{x \in \Omega} \frac{f(\Omega) - f(\Omega \setminus x)}{f(x)}

\]

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2This document arises, in part, from discussions with Vasileios Tzoumas and Lifeng Zhou. Additionally, another version of this analysis appears in [10].

For brevity we omit some definitions. Please refer to existing works on submodular maximization [5, 3] for more detailed discussion.
Additionally, the resilient submodular maximization problem seeks to maximize $f$ given a number $\alpha$ of adversarial removals.\footnote{In this document, the removals are subject only to a cardinality constraint (forming a uniform matroid). Tzoumas et al. \cite{8} also discuss a limited class of problems where the removals are subject to a partition matroid constraint. Although we do not present results for this class of problems directly, similar results can be obtained and produce same bound.}

**Problem 1** (Resilient submodular maximization). Consider a submodular, monotonic, normalized set function $f : 2^\Omega \rightarrow \mathbb{R}$ where $\Omega$ is the ground set and $(\mathcal{I}, \Omega)$ is a matroid. The goal is to maximize $f$ subject to $\alpha$ worst-case removals. That is,

$$\max_{A \in \mathcal{I}} \min_{B \subseteq A, |B| \leq \alpha} f(A \setminus B). \tag{2}$$

Selecting solution elements for value on their own $f(x)$ will admit exchange of individual solution elements (which solve maximization subproblems exactly) for elements of the optimal solution to obtain an approximation guarantee of $1 - \nu$. However, applying methods for greedy submodular maximization \cite{8,41–43} incurs a penalty in the form of the suboptimality of the greedy procedure for an approximation factor of $\frac{1}{1+\nu}$.

In the following, we assume familiarity with background material (matroids, set functions, monotonicity, and submodularity) although readers may refer to related works for background material \cite{8}.

2 Approach

Algorithm 1 produces feasible, suboptimal solutions $A^{\text{sol}}$ to Prob. 1 by repeatedly maximizing objective values for individual solution elements until it obtains a complete solution.

Additionally, if $(\Omega, \mathcal{I})$ is a partition matroid, maximization steps can run over individual blocks of the partition as in local greedy methods \cite{3,4,7} and in parallel as well. Likewise, maximization steps can be executed in a distributed manner such as for multi-robot applications \cite{11}.

\begin{algorithm}[h]
\begin{algorithmic}[1]
\State $A^{\text{sol}} \leftarrow \emptyset$
\While{$|A^{\text{sol}}| < \text{rank}(\mathcal{I})$}
\State $a \leftarrow \arg \max_{a \in \Omega \setminus A^{\text{sol}}, A^{\text{sol}} \cup \{a\} \in \mathcal{I}} f(\{a\})$
\State $A^{\text{sol}} \leftarrow A^{\text{sol}} \cup \{a\}$
\EndWhile
\State \textbf{return} $A^{\text{sol}}$
\end{algorithmic}
\end{algorithm}

3 Analysis

Let us begin with a few definitions. Let $A^*$ be an optimal solution to Prob. 1, and let $B^*(A)$ be an optimal removal so that for any $A \in \mathcal{I}$, then

$$B^*(A) \in \arg \min_{B \subseteq A, |B| \leq \alpha} f(A \setminus B). \tag{3}$$
For brevity, the remaining solution elements after removing $B^*(A)$ are denoted $R(A) = A \setminus B^*(A)$.

We can now state the main result:

**Theorem 1** (Approximation performance). Algorithm 1 outputs feasible solutions to Prob. 1 that satisfy

\[
f(R(A_{sol})) \geq (1 - \nu)f(R(A^*)),
\]

observing that $f(R(A^*))$ is the optimal objective value for (2).

**Proof.** Due to curvature (1), submodularity, and monotonicity, objective values over remaining elements are lower bounded in terms of sums over individual elements as

\[
f(R(A_{sol})) \geq (1 - \nu) \sum_{a \in R(A_{sol})} f(a),
\]

and this statement follows from [8, Lemma 2].

Now, to account for the partition matroid, [1, Theorem 1] implies existence of a bijection $\pi : A_{sol} \to A^*$ between $A_{sol}$ and $A^*$ (which are each bases of $\mathcal{I}$) that enables exchange of elements between the two solutions so that for any $a \in A_{sol}$ then $A_{sol} \setminus \{a\} \cup \{\pi(a)\} \in \mathcal{I}$. Further, $\pi$ may be defined as the identity on the intersection $A_{sol} \cap A^*$.

That is, either $a = \pi(a)$ (the given solution element is also part of the optimal solution) or $\pi(a) \notin A_{sol}$, and $\pi(a)$ was also a feasible solution to the maximization step at which $a$ was selected (Alg. 1, line 3). Then, following from these two cases of the identity and feasibility during the maximization step,

\[
f(a) \geq f(\pi(a)), \quad \text{for all } a \in A_{sol}.
\]

Substituting (6) into (5), we get

\[
f(R(A_{sol})) \geq (1 - \nu) \sum_{a \in R(A_{sol})} f(\pi(a)).
\]

Then, define $R_{a^*} = \{\pi(a) \mid a \in R(A_{sol})\}$ as the subset of the optimal solution ($R_{a^*} \subseteq A^*$) with elements corresponding to the remaining elements of the our suboptimal solution. Applying submodularity to (7) and writing the result in terms of $R_{a^*}$,

\[
f(R(A_{sol})) \geq (1 - \nu)f(R_{a^*}).
\]

Given that $|A^* \setminus R_{a^*}| = \alpha$, equation (3) implies that $f(R_{a^*}) \geq f(R(A^*))$. Then, substituting back into (8),

\[
f(R(A_{sol})) \geq (1 - \nu)f(R(A^*)).
\]

This completes the proof.

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\footnote{For an example, please refer to discussion by Filmus and Ward [2].}
4 Conclusion

The algorithm which this document describes is trivial and likely not of much interest for any applications. Yet, the approximation guarantee improves on results by Tzoumas et al. [8] which have since been applied in several recent works [6, 9, 11]. Moreover, application, in part, of a greedy algorithm [8] is desirable and may produce superior performance in practice if the number of removals is uncertain or if the removals are not actually adversarial. Instead, these results may encourage further improvements to suboptimality guarantees for Prob. 1 or study of more specialized resilient submodular maximization problems that may exhibit different behaviors.

Acknowledgments

I would like to thank Vasileios Tzoumas and Lifeng Zhou for looking over and discussing this document.

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