Operads, deformation theory and $F$-manifolds

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§0. Introduction

0.1. Little disks operad and Hertling-Manin’s $F$-manifolds. Frobenius manifolds created by Dubrovin in 1991 from rich theoretical physics material have been found since in many different fragments of mathematics — quantum cohomology and mirror symmetry, complex geometry, symplectic geometry, singularity theory, integrable systems — raising hopes for unifying them into one picture. It also became clear that the notion of Frobenius manifold is not broad enough to cover all objects of the associated working categories; say, on the $B$-side of the mirror symmetry it applies only to extended moduli spaces of Calabi-Yau manifolds, the latter forming a rather small subcategory of the category of complex manifolds. In 1998 Hertling and Manin [HeMa] introduced a weaker notion of $F$-manifold which is, by definition, a pair $(M,\mu_2)$ consisting of a smooth supermanifold $M$ and a smooth $\mathcal{O}_M$-linear associative graded commutative multiplication on the tangent sheaf, $\mu_2 : \otimes^2 \mathcal{O}_M \mathcal{T}_M \to \mathcal{T}_M$, satisfying the integrability condition,

$$[\mu_2,\mu_2] = 0,$$

where $[\mu_2,\mu_2] : \otimes^4 \mathcal{O}_M \mathcal{T}_M \to \mathcal{T}_M$ is given explicitly by

$$[[\mu_2,\mu_2](X,Y,Z,W) := \allowbreak [\mu_2(X,Y),\mu_2(Z,W)] - \mu_2([\mu_2(X,Y),Z],W) - (1)^{(|X|+|Y|+|Z|)} [\mu_2(Z,[\mu_2(X,Y),W])] - \mu_2(X,[Y,\mu_2(Z,W)]) - (1)^{|Y|(|Z|+|W|)} [\mu_2(X,\mu_2(Z,W)),Y] + (1)^{|Y|(|Z|)} [\mu_2(X,\mu_2(Z,[Y,W])) + \mu_2(X,\mu_2([Y,Z],W))] + (1)^{|Y|(|Z|)} [\mu_2([X,Z],\mu_2(Y,W))] + (1)^{|W|(|Y|+|Z|)} [\mu_2([X,W],\mu_2(Y,Z))].$$

A non-trivial part of the above definition is an implicit assertion that $[\mu_2,\mu_2]$ is a tensor, i.e. $\mathcal{O}_M$-polylinear in all four inputs. It is here where the assumption that $\mu_2$ is both graded commutative and associative plays a key role.

Any Frobenius manifold is an $F$-manifold. Any $F$-manifold with semi-simple product $\mu_2$ can be made into a Frobenius manifold [HeMa]. Hertling in his book [He] explained in detail how $F$-manifolds turn up in the singularity theory.

In this paper we show that (cohomology/strong homotopy, see below) $F$-manifolds arise naturally in every mathematical structure which admits an action of the chain operad of the little disks operad (or its more compact version, $G_{\infty}$-operad [GetJo]). In particular, we prove
Theorem A. Let $A$ be either a complex or symplectic structure on a compact manifold. Then the smooth part, $M_{reg}$, of the extended moduli space $M$ of deformations of $A$ is canonically an $F$-manifold.

0.2. Cohomology $F$-manifolds. It is often not a pleasure to work with objects like $M_{reg} \subset M$ in Theorem A; moreover, their existence is not guaranteed for many reasonable deformation problems.

The germ, $(M, *)$, of, in general, singular moduli space $M$ (if it exists at all) at the distinguished point $*$ always admits a smooth dg resolution, $(M, *, \partial)$ [Me2]. The latter consists of a germ of a smooth graded pointed manifold, $(M, *)$, and a germ of a smooth degree 1 vector field, $\partial$, satisfying two conditions,

$$[\partial, \partial] = 0, \quad \text{and} \quad \partial I \subset I^2,$$

where $I$ is the ideal of the distinguished point. The relation between $(M, *)$ (which may not exist as an analytic space) and $(M, *, \partial)$ (which always exists for any deformation problem!) is given by the well known formula of “nonlinear cohomology”,

$$M \simeq \frac{\text{Zeros}(\partial)}{\text{Im} \partial},$$

representing $M$ as the quotient of the zero set of the vector field $\partial$ by the integrable distribution

$$\text{Im} \partial := \{X \in T_M : X = [\partial, Y] \text{ for some } Y \in T_M\},$$

which, as it is easy to check, is tangent to $\text{Zeros}(\partial)$.

With any dg manifold $(M, \partial)$ one can associate two cohomology sheaves: the cohomology structure sheaf,

$$H(\mathcal{O}_M) := \frac{\text{Ker} \partial : \mathcal{O}_M \to \mathcal{O}_M}{\text{Im} \partial : \mathcal{O}_M \to \mathcal{O}_M},$$

and the cohomology tangent sheaf,

$$\mathcal{H}T_M := \frac{\text{Ker} \text{Lie}_\partial : T_M \to T_M}{\text{Im} \text{Lie}_\partial : T_M \to T_M},$$

which is a sheaf of $H(\mathcal{O}_M)$-modules (in fact, a sheaf of Lie $H(\mathcal{O}_M)$-algebras).

We define a cohomology $F$-manifold to be a dg manifold $(M, \partial)$ together with a graded commutative associative $H(\mathcal{O}_M)$-polylinear product $\mu_2 : \mathcal{H}T_M \times \mathcal{H}T_M \to \mathcal{H}T_M$, such that the integrability condition, $[\mu_2, \mu_2] = 0$, holds. This notion also makes sense in the category of formal dg manifolds.

Theorem B. If the operad $G_\infty$ acts on a dg vector space $(V, d)$, then the formal graded manifold associated with the cohomology vector space $H(V, d)$ is canonically a cohomology $F$-manifold.

Let us emphasize again that the notion of (cohomology) $F$-manifold is diffeomorphism invariant. Though the input in Theorem B belongs to the category of vector spaces which one can geometrically interpret as pointed affine (=flat) manifolds, the output lies in the category of
general smooth graded manifolds with morphisms being arbitrary (not necessary, linear) smooth maps. Thus the output of Theorem B belongs to the realm of differential geometry.

Recent proofs of Deligne’s conjecture [Ko2, KoSo1, McSm, Ta, Vo] together with Theorem B imply

**Corollary C.** (i) Let $A$ be an associative $k$-algebra. The formal manifold associated with the Hochschild cohomology $H^\bullet(A, A)$ is naturally a cohomology $F$-manifold.

(ii) Let $X$ be a compact topological space. The formal manifold associated with its singular cohomology $H^\bullet(X, k)$ is naturally a cohomology $F$-manifold.

**0.3. $F_\infty$-manifolds.** Instead of passing to cohomology sheaves as above, one can adopt the notion of $F$-manifold to the category of dg manifolds by constructing its strong homotopy version. We do it in this paper with the help of the $\mathcal{G}_\infty$-operad (cf. Theorem B).

Let $(\mathcal{M}, \mathfrak{g}, \ast)$ be a formal dg manifold and let

$$\mu_\bullet = \{\mu_n\}_{n \geq 1} : \otimes_{\mathcal{O}_\mathcal{M}}^* T\mathcal{M} \to T\mathcal{M}$$

be a structure of $C_\infty$-algebra on the tangent sheaf. We call it geometric if $\mu_1 = \text{Lie}_\mathfrak{g}$ and $\mu_{\bullet \geq 2}$ are morphisms of $\mathcal{O}_\mathcal{M}$-modules, i.e. are tensors. If all $\mu_n$ except $\mu_2$ vanish, this structure reduces to the structure of graded commutative associative product as in Sect. 0.1. Note that $(\otimes_{\mathcal{O}_\mathcal{M}}^* T\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} T\mathcal{M}, \text{Lie}_\mathfrak{g})$ is a complex of (sheaves of) $\mathcal{O}_\mathcal{M}$-modules. Its cohomology is denoted by $H(\otimes_{\mathcal{O}_\mathcal{M}}^* T\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} T\mathcal{M})$.

Choosing a torsion-free affine connection $\nabla$ on $\mathcal{M}$, one can construct an extension of the Hertling-Manin’s “bracket” $[\mu_2, \mu_2]$ to geometric $C_\infty$-structures,

$$[\mu_\bullet, \mu_\bullet]^{\nabla}(X_1, \ldots, X_\bullet, Y_1, \ldots, Y_\bullet) := [\mu_\bullet(X_1, \ldots, X_\bullet), \mu_\bullet(Y_1, \ldots, Y_\bullet)] + \text{correction terms},$$

producing thereby a collection of tensors, $[\mu_\bullet, \mu_\bullet]^{\nabla} : \otimes_{\mathcal{O}_\mathcal{M}}^{*,*} T\mathcal{M} \to T\mathcal{M}$, satisfying the following two conditions,

- $\text{Lie}_\mathfrak{g}[\mu_\bullet, \mu_\bullet]^{\nabla} = 0$,
- the cohomology class,

$$[[\mu_\bullet, \mu_\bullet]] \in H(\otimes_{\mathcal{O}_\mathcal{M}}^* T\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} T\mathcal{M}),$$

produced by $[\mu_\bullet, \mu_\bullet]^{\nabla}$ does not depend on the choice of the connection $\nabla$ and hence gives a well-defined invariant of the geometric $C_\infty$-structure. Moreover, this invariant depends only on the homotopy class of that structure.

The correction terms to $[[\mu_\bullet, \mu_\bullet]]$ can, in principle, be read off from the structural equations of the $\mathcal{G}_\infty$-operad, as explained in Sect. 4. However, all the basic properties of the bracket $[\mu_\bullet, \mu_\bullet]^{\nabla}$, such as its existence, $\mathcal{O}_\mathcal{M}$-linearity, $\text{Lie}_\mathfrak{g}$-closedness etc., can be proved without doing this sort of explicit calculations.

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1There is no way to remember the original flat structure of the input unless the combination $(G_\infty \text{action}, V, d)$ is formal as a $L_\infty$-algebra, and one makes a particular choice of a homotopy class of formality maps.

2Here and everywhere in this paper $k$ stands for a field of characteristic 0. Every vector space is implicitly assumed to be over $k$.

3The assumption that $\mu_\bullet$ is a $C_\infty$-structure is important. The construction does not work for geometric $A_\infty$-structures.
Definition D. An $F_\infty$-manifold is a dg manifold, $(\mathcal{M}, \partial, *)$, together with a homotopy class of geometric $C_\infty$-structures, $\{\mu_\bullet : \bigotimes_{\mathcal{M}} \mathcal{T}_\mathcal{M} \to \mathcal{T}_\mathcal{M}\}$, satisfying the integrability condition,

$$[[\mu_\bullet, \mu_\bullet]] = 0.$$ 

Clearly, any $F_\infty$-manifold gives naturally rise to a cohomology $F$-manifold. In fact, the cohomology $F$-manifolds discussed above in Sect. 0.2 are precisely of this type:

**Theorem E.** All statements of Theorem B and Corollary C remain true if one replaces

$$\text{cohomology } F\text{-manifold } \longrightarrow F_\infty\text{-manifold}.$$ 

0.4. Content. In §1 and §2 we remind basic notions and notations of the (homotopy) theory of operads and discuss in detail some particular examples. The main result in §3 is an explicit graphical description, Proposition 3.6.1, of the cobar construction for the operad of non-commutative Gerstenhaber algebras and a surprisingly nice geometric interpretation, Theorem 3.9.2, of the derived category of algebras over that operad. In §4 we outline an operadic guide to the extended deformation theory (as a more informative alternative to the classical idea of deformation functor) and, in that context, prove all the claims made in the Introduction.

§1. Operads and their algebras

1.1. Operads. By an operad in this paper we always understand what is usually called a *nonunital or pseudo-operad* [Mar1], that is, a pair of collections,

$$\mathcal{O} = \left(\{\mathcal{O}(n)\}_{n \geq 1}, \{\circ_{i}^{n,n'}\}_{n,n' \geq 1; 1 \leq i \leq n} \right),$$

where each $\mathcal{O}(n)$ is a $\mathbb{Z}$-graded vector space equipped with a linear action of the permutation group $\mathbb{S}_n$ (the collection $\{\mathcal{O}(n)\}_{n \geq 1}$ will sometimes be called an $\mathbb{S}$-*module*), and each $\circ_{i}^{n,n'}$ is a linear equivariant map,

$$\circ_{i}^{n,n'} : \mathcal{O}(n) \otimes \mathcal{O}(n') \to \mathcal{O}(n + n' - 1),$$

such that, for any $f \in \mathcal{O}(n)$, $f' \in \mathcal{O}(n')$ and $f'' \in \mathcal{O}(n'')$, one has

$$\left(f \circ_{i}^{n,n'} f'\right) \circ_{j+n'-1}^{n+n'-1,n''} f'' = (-1)^{|f'||f''|} \left(f \circ_{j}^{n,n'} f''\right) \circ_{i}^{n+n'-1,n''} f', \quad \forall 1 \leq i < j \leq n,$

and

$$f \circ_{i}^{n,n'+n''-1} \left(f' \circ_{j}^{n',n''} f''\right) = \left(f \circ_{i}^{n,n'} f'\right) \circ_{i+j-1}^{n+n'-1,n''} f'', \quad \forall 1 \leq i \leq n, 1 \leq j \leq n'.$$

Equivariance of $\circ_{i}^{n,n'}$ above means that for any $\sigma \in \mathbb{S}_n$ and $\sigma' \in \mathbb{S}_{n'}$ one has

$$(\sigma f) \circ_{i}^{n,n'} (\sigma' f') = (\sigma f) \circ_{i}^{n,n'} (\sigma' f')$$

where $(\sigma' \sigma) \in \mathbb{S}_{n+n'-1}$ is given by inserting the permutation $\sigma'$ into the $i$th place of $\sigma$. 


An ideal in an operad \( O \) is a collection \( I \) of \( S_n \)-invariant subspaces \( \{ I(n) \subset O(n) \}_{n \geq 1} \) such that \( f \circ_i^{n,n'} f' \in I(n + n' - 1) \) whenever \( f \in I(n) \) or \( f' \in I(n') \); in particular, \( I \) is a suboperad of \( O \). It is clear that the quotient \( S \)-module \( \{ O(n)/I(n) \}_{n \geq 1} \) has a naturally induced structure of an operad called a quotient operad.

An operad \( O \) with \( O(1) = 0 \) is called simply connected.

### 1.2. Free operads and trees.

A morphism of operads, \( f : O \to O' \), is, by definition, a morphism of the associated \( S \)-modules, \( \{ f(n) : O(n) \to O'(n) \}_{n \geq 2} \), which commutes in the obvious way with all the operations \( \circ_i^{n,n'} \). Operads form a category.

The forgetful functor

\[
\text{Category of operads} \quad \{ \{ O(n) \}_{n \geq 1} , \{ \circ_i^{n,n'} \}_{n,n' \geq 1} \}_{1 \leq i \leq n} \quad \longrightarrow \quad \text{Category of } S\text{-modules} \quad \{ \{ O(n) \}_{n \geq 1} \},
\]

has a left adjoint functor, \( \text{Free} \), which associates to an arbitrary collection, \( \mathcal{E} = \{ \mathcal{E}(n) \}_{n \geq 1} \), of graded vector \( S_n \)-spaces the free operad, \( \text{Free}(\mathcal{E}) \). It is best described in terms of trees as follows (see [GiKa, GetJo, KoSo1] for more details).

An \([n]\)-tree \( T \) is, by definition, the data \((V_T, N_T, \phi_T)\) consisting of

- a stratified finite set \( V_T = V_T^i \sqcup V_T^i \) whose elements are called vertices; elements of the subset \( V_T^i \) (resp. \( V_T^j \)) are called internal (resp. tail) vertices;
- a bijection \( \phi : V_T^i \to \{ 1, 2, \ldots, n \} =: [n] \);
- a map \( N_T : V_T \to V_T \) satisfying the conditions: (i) \( N_T \) has only one fixed point root \( T \) which lies in \( V_T^i \) and is called the root vertex, (ii) \( N_T^k(v) = \text{root}_T, \forall v \in V_T \) and \( k > 1 \), (iii) for all \( v \in V_T^i \), the cardinality, \( \#v \), of the set \( N_T^{-1}(v) \) is greater than or equal to 1, while for all \( v \in V_T^j \) one has \( \#v = 0 \).

The number \( \#v \) is often called the valency of the vertex \( v \); the pairs \((v, N_T(v))\) are called edges.

Given an \( S \)-module \( \mathcal{E} = \{ \mathcal{E}(n) \}_{n \geq 1} \), we can associate to an \([n]\)-tree \( T \) the vector space

\[
\mathcal{E}(T) := \bigotimes_{v \in V_T^i} \mathcal{E}(\#v).
\]

Its elements are interpreted as \([n]\)-trees whose internal vertices are decorated with elements of \( \mathcal{E} \). The permutation group \( S_n \) then acts on this space via relabelling the tail vertices (i.e changing \( \phi_T \) to \( \sigma \circ \phi_T, \sigma \in S_n \)).

Now, as an \( S \)-module the free operad \( \text{Free}(\mathcal{E}) \) is defined as

\[
\text{Free}(\mathcal{E})(n) = \bigoplus_{[\text{[n]-trees } T]} \mathcal{E}(T),
\]

where the summation goes over all isomorphism classes of \([n]\)-trees. The composition, say \( f \circ_i^{n,n'} f' \), is given by gluing the root vertex of the decorated \([n]\)-tree \( f' \in \text{Free}(\mathcal{E})(n') \) with the \( i \)-labelled tail vertex of the decorated \([n]\)-tree \( f \). The new numeration, \( \phi : V_T^i \to [n + n' - 1] \), of tails is clear.
Any free operad is naturally graded, $\text{Free}(\mathcal{E}) = \bigoplus_{p=1}^{\infty} \text{Free}^p(\mathcal{E})$, where $\text{Free}^p(\mathcal{E})$ is the $S$-submodule of $\text{Free}(\mathcal{E})$ spanned by all possible isomorphism classes of $\mathcal{E}$-decorated trees with precisely $p$ internal vertices.

1.3. Example. Let $V$ be a $\mathbb{Z}$-graded vector space. The associated $S$-module,

$$\mathcal{E}_V = \{\mathcal{E}_V(n) := \text{Hom}(V^\otimes n, V)\},$$

has a natural structure of operad with compositions, $f \circ_i^{n,n'} f'$, given by inserting the output of $f' \in \text{Hom}(V^{\otimes n'}, V)$ into the $i$-th input of $f \in \text{Hom}(V^{\otimes n}, V)$.

An algebra over an operad $\mathcal{O}$ is, by definition, a $\mathbb{Z}$-graded vector space $V$ together with a morphism of operads $\mathcal{O} \rightarrow \mathcal{E}_V$.

1.4. Example. Let $A$ be an $S$-module given by

$$A(n) := \begin{cases} k[S_2][0] & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases}$$

where here and below the symbol $k[S_n][p]$ stands for the graded vector space whose only non-vanishing homogeneous component lies in degree $-p$ and equals the regular representation $k[S_n]$ of the permutation group $S_n$. If we identify the natural basis, $id$ and $(12)$, of $k[S_2]$ with planar $[2]$-corollas,

\[ \begin{array}{c} \bullet \\ \bullet \end{array} \quad \text{and} \quad \begin{array}{c} \bullet \\ \bullet \end{array} \]

then the associated free operad $\{\text{Free}(A)(n), \circ_i^{n,n'}\}$ can be represented as a linear span of all possible (isomorphism classes of) binary planar $[n]$-trees, e.g.

\[ \begin{array}{c} \bullet \\ \bullet \end{array} \quad \bullet \quad \begin{array}{c} \bullet \\ \bullet \end{array} \quad \bullet \quad \begin{array}{c} \bullet \\ \bullet \end{array} \]

with the compositions $\circ_i^{n,n'}$ given simply by gluing the root vertex of a planar $[n']$-tree to the $i$th tail vertex of an $[n]$-tree (the new numeration of tails is clear). Indeed an isomorphism class of an $\{id, (12)\}$-decorated abstract (=space) binary tree of Subsect. 1.2 has a natural representative which lies in a fixed plane in the space and which is consistent with the interpretation of the labelling set $\{id, (12)\}$ as the set of planar $[2]$-corollas; more importantly, the resulting correspondence

\[ \left\{ \text{isomorphism classes of } \{id, (12)\}-\text{decorated abstract binary trees} \right\} \rightarrow \left\{ \text{isomorphism classes of planar binary numbered trees} \right\} \]

is one-to-one.

\[ ^4 \text{More generally, for a } \mathbb{Z}\text{-graded vector space } V = \oplus_{i \in \mathbb{Z}} V^i, \text{ the symbol } V[p] \text{ stands for the } \mathbb{Z}\text{-graded vector space with } V[p]^i := v^{i+p}. \]
Algebras over $Free(A)$ are not that interesting objects — they are just graded vector spaces $V$ together with a fixed element of $\Hom(V^\otimes 2, V)$ which can be arbitrary.

Let $I_A$ be the ideal in $Free(A)$ generated by $3!$ vectors of the form

\[
\begin{array}{c}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]

The quotient operad $Free(A)/I_A$ is denoted by $Ass$ for the obvious reason — its algebras are nothing but the usual graded associative algebras. As an $S$-module, $Ass(n) \simeq k[S_n]$.

1.5. Example. Consider an $S$-module,

\[ Comm(n) := 1_n[0] \]

where $1_n$ stands for the trivial representation of the permutation group $S_n$. This $S$-module can be made into an operad $Comm$ by defining the compositions $c_{i,n,n'}^i$ to be the identity maps. It is not hard to check that $Comm$-algebras are graded commutative associative algebras in the usual sense.

For later reference it will be convenient to represent the operad $Comm$ as a quotient of a free operad. For this purpose we first consider an $S$-module $C$,

\[ C(n) := \begin{cases} 1_2[0] & \text{if } n = 2 \\ 0 & \text{otherwise.} \end{cases} \]

If we identify a basis vector of $1_n[0]$ with the unique (up to an isomorphism) space corolla (i.e. the one embedded in $\mathbb{R}^3$)

\[
\begin{array}{c}
1 & 2 \\
\end{array}
\]

then the associated free operad \{Free($C$)(n)\} can be represented as a linear span of all possible isomorphism classes of binary space $[n]$-trees; for example, Free($C$)(3) is a 3-dimensional vector space spanned by the following space $[3]$-trees

\[
\begin{array}{c}
\begin{array}{c}
1 & 2 & 3 \\
\end{array}
\end{array}
\]

The composition in $Free(C)$ is given by gluing the root vertex of one space tree to a tail vertex of another one. The new numeration of tail vertices is clear.

Let $I_C$ be the ideal in $Free(C)$ generated by 2 vectors of the form

\[
\begin{array}{c}
\begin{array}{c}
1 & 2 & 3 \\
\end{array}
\end{array}
\]

and

\[
\begin{array}{c}
\begin{array}{c}
1 & 3 & 2 \\
\end{array}
\end{array}
\]
The quotient operad $Free(C)/I_C$ is clearly isomorphic to $Comm$.

1.6. Example. Let $C$ be an $S$-module given by

$$
\mathcal{C}(n) := \begin{cases} 
1_2[-1] & \text{if } n = 2, \\
0 & \text{otherwise}.
\end{cases}
$$

If we identify, as in Example 1.4, a basis vector of the one dimensional vector space $1_2[1]$ with the unique (up to isomorphism) space $[2]$-corolla

then the associated free operad $\{Free(\mathcal{C}(n))\}$ can be represented as a linear span of all possible (isomorphism classes of) binary space $[n]$-trees with the composition given by gluing the root vertex of one space tree to a tail vertex of another one.

Let $I_C$ be the ideal in $Free(\mathcal{C})$ generated by the following $S_3$-invariant vector,

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \ \ \ \ 2 \\
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
3 \ \ \ \ 1 \\
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
2 \ \ \ \ 3 \\
\end{array}
\end{array}
\end{array}.
\end{array}
\]

Algebras over the associated quotient operad, $Lie := Free(\mathcal{C})/I_C$, are graded vector spaces $V$ equipped with a degree $-1$ element $\nu \in Hom(\otimes^2 V, V)$ satisfying the Jacobi condition,

$$
\nu(\nu(v_1, v_2), v_3) + (-1)^{|v_3|(|v_1|+|v_2|)}\nu(\nu(v_3, v_1), v_2) + (-1)^{|v_1|(|v_2|+|v_3|)}\nu(\nu(v_2, v_3), v_1) = 0.
$$

Setting

$$
[v_1 \bullet v_2] := (-1)^{|v_1|}\nu(v_1, v_2)
$$

we recover the notion of (odd) Lie algebra $[Ma]$. It is, of course, the same thing as the usual graded Lie algebra structure on the shifted graded vector space $V[1]$ but for our purposes it is more suitable not to make this shift; thus in the present paper by a graded Lie algebra we always understand an algebra over the operad $Lie$, i.e. a pair $(V, [\bullet \bullet])$ with $[\bullet \bullet] : \otimes^2 V \to V$ having degree $-1$ and satisfying (odd) Jacobi identity.

1.7. Example. Let $\mathcal{A}$ be an $S$-module given by $\mathcal{A}(n) := \mathcal{A}(n) \oplus \mathcal{C}(n)$. Its only non-vanishing component $\mathcal{A}(2)$ is a 3-dimensional vector space spanned by two planar corollas in degree 0 and one space corolla in degree $-1$,

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \ \ \ \ 2 \\
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
2 \ \ \ \ 1 \\
\end{array}
\end{array}
\end{array}
\end{array}\]

and

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \ \ \ \ 2 \\
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
2 \ \ \ \ 1 \\
\end{array}
\end{array}
\end{array}.
\end{array}
\]

The associated free operad $Free(\mathcal{A})$ can be represented as a linear span of all possible isomorphism classes of binary $[n]$-trees in the 3-space $\mathbb{R}^3$ with the condition that all “planar” corollas are perpendicular to a fixed line in $\mathbb{R}^3$. The composition in $Free(\mathcal{A})$ is given again by gluing the root vertex of one such partially planar/ partially space tree to a tail vertex of another one.
Let $\mathcal{I}_{A\mathcal{C}}$ be the ideal in $\text{Free}(\mathcal{A}\mathcal{C})$ generated by the following $3!$ vectors,

\[
\begin{array}{ccc}
  i_1 & i_2 & i_3 \\
  \downarrow & \downarrow & \downarrow \\
  i_1 & i_2 & i_3 \\
\end{array}
\]

Algebras over the quotient operad,

\[
\mathcal{G}_{\text{erst}} := \text{Free}(\mathcal{L})/ < \mathcal{I}_A, \mathcal{I}_L, \mathcal{I}_{A\mathcal{C}} >,
\]

are called (non-commutative) Gerstenhaber algebras. These are triples, $(V, \circ, \{,\})$, consisting of a graded vector space $V$, a degree 0 associative product, $\circ : V \otimes V \to V$ and a degree $-1$ Lie bracket, $\{,\} : \circ^2 V \to V$ which satisfy the following compatibility condition,

\[
[a \circ (b \circ c)] = [a \circ b] \circ c + (-1)^{(\tilde{a}+1)\tilde{b}} \circ [a \circ c],
\]

for all homogeneous $a, b, c \in V$.

1.8. Example. A Gerstenhaber algebra $(V, \circ, \{,\})$ is called graded commutative if such is the product $\circ$. Let us denote by $\mathcal{G}$ the operad which governs graded commutative Gerstenhaber algebras.

1.9. Remark. There is a canonical map of operads,

\[
\mathcal{G} \longrightarrow \mathcal{G}_{\text{erst}},
\]

corresponding to the obvious functor

\[
\left\{ \begin{array}{c}
\text{A category of} \\
\mathcal{G}\text{-algebras}
\end{array} \right\} \longrightarrow \left\{ \begin{array}{c}
\text{A category of} \\
\mathcal{G}_{\text{erst}}\text{-algebras}
\end{array} \right\}
\]

which simply forgets graded commutativity of the associated product.

Both operads $\mathcal{G}$ and $\mathcal{G}_{\text{erst}}$ are composed from a pair of simpler operads, $(\text{Comm}, \text{Lie})$ and, respectively, $(\text{Ass}, \text{Lie})$. The difference, however, is that the composition of $(\text{Comm}, \text{Lie})$ into $\mathcal{G}$ satisfies the distributive law [Mar2], while the composition of $(\text{Ass}, \text{Lie})$ into $\mathcal{G}_{\text{erst}}$ does not. Indeed, “opening” the expression

\[
[(a_1 \circ a_2) \circ (a_3 \circ a_4)] \simeq
\]

in two possible ways,

\[
[(a_1 \circ a_2) \circ (a_3 \circ a_4)] = a_1 \circ [a_2 \circ (a_3 \circ a_4)] + (-1)^{|a_2|(|a_3|+|a_4|+1)} a_1 \circ (a_3 \circ a_4) \circ a_2
\]

\[
= a_1 \circ [a_2 \circ a_3] \circ a_4 + (-1)^{|a_3|(|a_2|+1)} a_1 \circ a_3 \circ [a_2 \circ a_4]
\]

\[
+ (-1)^{|a_2|(|a_3|+|a_4|+1)} \left( [a_1 \circ a_3] \circ a_4 \circ a_2 + (-1)^{|a_3|(|a_1|+1)} a_3 \circ [a_1 \circ a_4] \circ a_2 \right),
\]
\[(a_1 \circ a_2) \bullet (a_3 \circ a_4) = [(a_1 \circ a_2) \bullet a_3] \circ a_4 + (-1)^{|a_2|(|a_1|+|a_2|+1)} a_3 \circ [(a_1 \circ a_2) \bullet a_4]
\]
\[= a_1 \circ [a_2 \bullet a_3] \circ a_4 + (-1)^{|a_2|(|a_3|+1)} [a_1 \bullet a_3] \circ a_2 \circ a_4
\]
\[+(-1)^{|a_3|(|a_1|+|a_2|+1)} (a_3 \circ a_1 \circ [a_2 \bullet a_4] + (-1)^{|a_2|(|a_4|+1)} a_3 \circ [a_1 \bullet a_4] \circ a_2)
\]

and then decomposing the associated relation in \(Gerst(4)\) into irreducibles, one gets

\[
\begin{array}{c}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\bullet & \bullet & \bullet & \bullet
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{cccc}
2 & 1 & 3 & 4 \\
\bullet & \bullet & \bullet & \bullet
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\bullet & \bullet & \bullet & \bullet
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{cccc}
2 & 1 & 4 & 3 \\
\bullet & \bullet & \bullet & \bullet
\end{array}
\end{array}
\]

The resulting relations in \(Gerst(4)\) are non-trivial unless the product \(\circ\) is graded commutative.

1.10. Example. Let \(AC\) be an S-module given by \(AC(n) := A(n)[-1] \oplus Comm(n)\). Its only non-vanishing component \(AC(2)\) is a 3-dimensional vector space spanned by two planar corollas of degree 1 and one space corolla of degree 0,

\[
\begin{array}{c}
\begin{array}{cccc}
1 & 2 \\
\bullet & \bullet
\end{array}
\end{array}
, \quad
\begin{array}{c}
\begin{array}{cccc}
2 & 1 \\
\bullet & \bullet
\end{array}
\end{array}
, \quad
\begin{array}{c}
\begin{array}{cccc}
1 & 2 \\
\bullet & \bullet
\end{array}
\end{array}
, \quad
\begin{array}{c}
\begin{array}{cccc}
1 & 2 \\
\bullet & \bullet
\end{array}
\end{array}
\]

As in example 1.7, the associated free operad \(Free(\mathcal{AC})\) can be represented as a linear span of all possible isomorphism classes of binary \([n]\)-trees in the 3-space \(\mathbb{R}^3\) with the condition that all planar corollas are perpendicular to a fixed line in \(\mathbb{R}^3\). The composition in \(Free(\mathcal{AC})\) is given by gluing the root vertex of one such partially planar/ partially space tree to a tail vertex of another one.

Let \(I_{AC}\) be the ideal in \(Free(\mathcal{AC})\) generated by the following 3+3=3! vectors,

\[
\begin{array}{c}
\begin{array}{cccc}
i_1 & i_2 & i_3 \\
\bullet & \bullet & \bullet
\end{array}
\end{array}
, \quad
\begin{array}{c}
\begin{array}{cccc}
i_1 & i_2 & i_3 \\
\bullet & \bullet & \bullet
\end{array}
\end{array}
, \quad
\begin{array}{c}
\begin{array}{cccc}
i_1 & i_2 & i_3 \\
\bullet & \bullet & \bullet
\end{array}
\end{array}
, \quad
\begin{array}{c}
\begin{array}{cccc}
i_1 & i_2 & i_3 \\
\bullet & \bullet & \bullet
\end{array}
\end{array}
\]

and

\[
\begin{array}{c}
\begin{array}{cccc}
i_1 & i_2 & i_3 \\
\bullet & \bullet & \bullet
\end{array}
\end{array}
, \quad
\begin{array}{c}
\begin{array}{cccc}
i_1 & i_2 & i_3 \\
\bullet & \bullet & \bullet
\end{array}
\end{array}
, \quad
\begin{array}{c}
\begin{array}{cccc}
i_1 & i_2 & i_3 \\
\bullet & \bullet & \bullet
\end{array}
\end{array}
, \quad
\begin{array}{c}
\begin{array}{cccc}
i_1 & i_2 & i_3 \\
\bullet & \bullet & \bullet
\end{array}
\end{array}
\]

Algebras over the quotient operad,

\[
\mathcal{H} := Free(\mathcal{AC})/ < I_{A[-1]} , I_C , I_{AC} >
\]
are triples, \((V, \circ, \bullet)\), consisting of a graded vector space \(V\), a degree 0 associative graded commutative product, \(\circ : V \odot V \to V\) and a degree 1 associative product \(\bullet : V \otimes V \to V\) which satisfy the following compatibility conditions,

\[
a \bullet (b \circ c) = (a \bullet b) \circ c + (-1)^{(\tilde{a}+1)\tilde{b}} b \circ (a \bullet c),
\]

\[
(a \circ b) \bullet c = a \circ (b \bullet c) + (-1)^{(\tilde{c}+1)\tilde{b}} (a \bullet c) \circ b,
\]

for all homogeneous \(a, b, c \in V\).

§2. Strongly homotopy algebras

2.1. Dg operads. A differential graded (shortly, dg) operad is an operad,

\[
\mathcal{O} = \left( \{\mathcal{O}(n)\}_{n \geq 1}, \{\mathcal{O}_i^{n,n'}\}_{n,n' \geq 1}^{1 \leq i \leq n} \right),
\]

in the sense of §1 together with a degree 1 equivariant linear map \(d : \mathcal{O}(n) \to \mathcal{O}(n), \forall n\), satisfying the conditions,

\[
d^2 = 0,
\]

\[
d \left( f \mathcal{O}_i^{n,n'} f' \right) = (df) \mathcal{O}_i^{n,n'} f' + (-1)^{|f|} f \mathcal{O}_i^{n,n'} df', \quad \forall f \in \mathcal{O}(n), f' \in \mathcal{O}(n').
\]

The associated cohomology \(\mathbb{S}\)-module \(H(\mathcal{O}) := \{H^*(\mathcal{O}(n))\}_{n \geq 1}\) has an induced operad structure.

A morphism, \(f : (\mathcal{O}, d) \to (\mathcal{O}', d')\), of dg operads is, by definition, a morphism of operads \(f : \mathcal{O} \to \mathcal{O}'\) which commutes in the obvious sense with the differentials. A morphism, \(f : (\mathcal{O}, d) \to (\mathcal{O}', d')\) is called a quasi-isomorphism if the induced morphism of the cohomology operads, \([f] : H(\mathcal{O}) \to H(\mathcal{O}')\), is an isomorphism.

If \((V, d)\) is a dg vector space, then \((\mathcal{E}_V, d_{\text{ind}})\) is naturally a dg operad where \(d_{\text{ind}} : \text{Hom}(\otimes^* V, V) \to \text{Hom}(\otimes^* V, V)\) is the differential which is naturally induced by \(d\) and which we denote from now on by the same symbol \(d\).

An algebra over a dg operad \((\mathcal{O}, d)\) is a dg vector space \((V, d)\) together with a morphism of dg operads \((\mathcal{O}, d) \to (\mathcal{E}_V, d)\).

It was shown in [Mar1] that for any simply connected dg operad \((\mathcal{O}, d)\) there exists a unique (up to an isomorphism) triple \(\mathcal{O}_\infty := (\text{Free}(\mathcal{E}), d, f)\) where

(i) \(\text{Free}(\mathcal{E})\) is the free operad generated by an \(\mathbb{S}\)-module \(\mathcal{E} = \{\mathcal{E}(n)\}_{n \geq 2}\);

(ii) \(d\) is the differential in \(\text{Free}(\mathcal{E})\) which is decomposable in the sense that \(df \in \text{Free}^{\geq 2}(\mathcal{E})(n)\) for any \(f \in \mathcal{E}(n), n \geq 2\).

(iii) \(f : (\text{Free}(\mathcal{E}), d) \to (\mathcal{O}, d)\) is a quasi-isomorphism of dg operads.

This operad \(\mathcal{O}_\infty\) is called the minimal resolution\(^5\) of the operad \(\mathcal{O}\). Such minimal resolutions play a very important role in the homotopy theory of operadic algebras which we discuss below after considering a few examples.

\(^5\)An operad \(\mathcal{O}_\infty := (\text{Free}(\mathcal{E}), d)\) satisfying relations (i) and (ii) is often called minimal.
2.2. Remark. It is clear that to define a particular dg operad, \((O,d)\), is the same thing as to define its algebras, i.e. the image of the map \((O,d) \to \mathcal{E}_V\) for some “variable” graded vector space \(V\). Moreover, for this purely descriptive purpose it is enough to assume that \(\dim V < \infty\). We always make such an assumption when applying this method to concrete examples; in particular, there is never a problem with replacing \(V\) by its dual, \(V^* = \text{Hom}(V,k)\).

2.3. Example: operad \(A_\infty\). Let \(\hat{A}\) be an \(S\)-module given by,

\[
A(n) := \begin{cases} 
0 & \text{if } n = 1 \\
k[S_n][n - 2] & \text{if } n \geq 2.
\end{cases}
\]

If we identify the natural basis of \(k[S_n][n - 2]\) with \textit{planar} \([n]\)-corollas,

\[
\begin{array}{c}
\vdots \\
i_1 & \cdots & i_n \\
\vdots
\end{array}
\]

then the associated free operad \(A_\infty := \{Free(\hat{A})(n), o_i^{n,n'}\}\) can be represented as a linear span of all possible (isomorphism classes of) \textit{planar} \([n]\)-trees with the compositions \(o_i^{n,n'}\) given simply by gluing the root vertex of a planar \([n']\)-tree to the \(i\)th tail vertex of an \([n]\)-tree.

One can make \(A_\infty\) into a dg operad with the differential \(d\) given on generators by

\[
d = \sum_{l+p=n+1 \atop l,p \geq 1} \sum_{s=0}^{p-1} (-1)^{l+s(l+1)} 
\]

The associated cohomology operad, \(H(A_\infty,d)\), is in fact isomorphic to \(\text{Ass} [GiKa]\). Hence, the natural morphism of dg operads,

\[
f : (A_\infty,d) \rightarrow (\text{Ass},0),
\]

defined to be identity on \([2]\)-corollas and zero on \([n \geq 3]\)-corollas, is a quasi-isomorphism. Thus \((A_\infty,d)\) is the minimal resolution of \(\text{Ass}\).

An algebra over the dg operad \(A_\infty\) is called a \textit{strongly homotopy associative algebra} or, shortly, an \(A_\infty\)-\textit{algebra}. This is a dg vector space \((V,d)\) equipped with degree \(2-n\) multilinear operations \(\mu_n : \otimes^n V \to V, n \geq 2\), such that for any \(N \geq 1\) and any \(v_1, \ldots, v_N \in V\),

\[
\sum_{l+p=N+1 \atop l,p \geq 1} (-1)^r \mu_p(v_1, \ldots, v_s, \mu_l(v_{s+1}, \ldots, v_{s+l}), v_{s+l+1}, \ldots, v_N) = 0,
\]

where \(\mu_1 = d\) and \(r = l + s(l + 1) + p(|v_1| + \ldots + |v_s|)\). If all \(\mu_n\) except \(n = 2\) vanish, the above equation translates into the associative condition for the product \(v_1 \circ v_2 := \mu_2(v_1, v_2)\). Strongly homotopy associative algebras have been invented by Stasheff [St] in his study of spaces homotopy equivalent to loop spaces.

2.4. Example: operad \(C_\infty\). The dg operad \(A_\infty\) has a commutative analog, \(C_\infty\), which provides us with the minimal resolution of the operad \(\text{Comm}\) from Example 1.5. Using remark 2.2
one can describe \( C_\infty \) as follows: a \( C_\infty \)-algebra is, by definition, an \( A_\infty \)-algebra \((V, \{\mu_n\}_{n \geq 1})\) such that every multilinear operation \( \mu_n : \otimes^n V \to V \) is a Harrison cochain, that is, vanishes on every shuffle product which is given on generators by the formula

\[
(v_1 \otimes \ldots \otimes v_i)\mathfrak{m}(v_{i+1} \otimes \ldots \otimes v_n) = \sum_{\sigma \in \text{Sh}(i,n)} (-1)^\sigma v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}.
\]

Here \((-1)^\sigma\) is the standard Koszul sign and \( \text{Sh}(i,n) \) stands for a subset of \( \mathbb{S}_n \) consisting of all permutations satisfying \( \sigma^{-1}(1) < \ldots < \sigma^{-1}(i), \sigma^{-1}(i+1) < \ldots < \sigma^{-1}(n) \). In particular, \( \mu_2 \) must be graded commutative, \( \mu_2(v_1, v_2) = (-1)^{|v_1||v_2|} \mu_2(v_2, v_1) \).

### 2.5. Example: operad \( L_\infty \)

Let \( \hat{L} \) be an \( \mathbb{S} \)-module given by,

\[
\hat{L}(n) := \begin{cases} 0 & \text{if } n = 1 \\ 1_n[2n - 3] & \text{if } n \geq 2. \end{cases}
\]

If we identify a basis vector of the one dimensional vector space \( 1_n[2n - 3] \) with the (unique, up to an isomorphism) space \([n]-\text{corolla},\]

then the associated free operad \( L_\infty := (\{\text{Free}(\hat{L})(n), \circ_i^{n,n'}\}) \) can be represented as a linear span of all possible (isomorphism classes of) space \([n]-\text{trees} with the compositions } \circ_i^{n,n'} \text{ given by gluing the root vertex of a planar } [n'] \text{-tree to the } i \text{th tail vertex of an } [n]-\text{tree. The point is that } L_\infty \text{ can be naturally made into a dg operad with the differential } d \text{ given on the generators by}.

\[
d = \sum_{I_1 \cup I_2 = (1, \ldots, n)} \sum_{\#I_1 \geq 2, \#I_2 \geq 2} (-1)^\sigma \mu_{\#I_2 + 1}(\mu_{\#I_1}(v_{I_1}), v_{I_2}).
\]

The associated cohomology operad, \( H(L_\infty, d) \), is isomorphic to \( \text{Lie} \) [GiKa]. Hence, the natural morphism of dg operads,

\[
f : (L_\infty, d) \longrightarrow (\text{Lie}, 0),
\]

defined to be identity on \([2]-\text{corollas} \) and zero on \([n \geq 3]-\text{corollas}, \) is a quasi-isomorphism. Thus \((L_\infty, d)\) is the minimal resolution of the operad of Lie algebras.

An algebra over the dg operad \( L_\infty \) is called a strongly homotopy Lie algebra or, shortly, a \( L_\infty \)-algebra. This is a dg vector space \((V, d)\) equipped with degree \( 3 - 2n \) multilinear operations \( \nu_n : \otimes^n V \to V, n \geq 2, \) such that for any \( N \geq 1 \) and any \( v_1, \ldots, v_N \in V, \)

\[
\sum_{I_1 \cup I_2 = (1, \ldots, n)} \sum_{\#I_1 \geq 1, \#I_2 \geq 0} (-1)^\sigma \mu_{\#I_2 + 1}(\mu_{\#I_1}(v_{I_1}), v_{I_2}) = 0,
\]
where $\mu_1 := d$, $(-1)\sigma$ is the standard Koszul sign associated with a permutation of the elements $v_1, \ldots, v_N$, and

$$v_I := v_{i_1} \otimes \ldots \otimes v_{i_l}, \quad \text{for } I = (i_1, \ldots, i_l) \subset (1, \ldots, n).$$

### 2.6. Example: operad $\mathcal{G}_\infty$.

Using remark 2.2 one can describe the minimal resolution, $\mathcal{G}_\infty$, of the operad, $\mathcal{G}$, of graded commutative Gerstenhaber algebras as follows [GetJo] \textsuperscript{6}.

Let $V$ be a finite dimensional graded vector space and

$$\text{Lie}(V^*[−2]) = \bigoplus_{k=1}^{\infty} \text{Lie}^k(V^*[−2])$$

the free graded Lie algebra generated by the shifted dual vector space, i.e.

$$\text{Lie}^1(V^*[−2]) := V^*[−2], \quad \text{Lie}^k(V^*[−2]) := \left[V^*[−2] \otimes \text{Lie}^{k−1}(V^*[−2])\right].$$

The Lie bracket on $\text{Lie}(V^*[−2])$ extends in the usual way to the completed (with respect to the natural filtration, $\text{Lie}^{≥k}(V^*[−2])$) graded commutative associative algebra

$$\hat{\circ}^*\text{Lie}(V^*[−2]) = \prod_{k=0}^{\infty} \circ^k\text{Lie}(V^*[−2]),$$

making the latter into a graded commutative Gerstenhaber algebra.

### 2.6.1. Proposition [GetJo] (see also [Ta, TaTs]).

A $G_\infty$-algebra structure on a finite-dimensional vector space $V$ is, by definition, a differential, of the free $\mathcal{G}$-algebra $\hat{\circ}^*\text{Lie}(V^*[−2])$.

There is a one-to-one correspondence between arbitrary derivations, $D$, of $\hat{\circ}^*\text{Lie}(V^*[−2])$ and arbitrary collections of linear maps,

$$m_{k_1, \ldots, k_n} : V^*[−2] \to \text{Lie}^{k_1}(V^*[−2]) \otimes \ldots \otimes \text{Lie}^{k_n}(V^*[−2]).$$

Upon dualization the latter go into linear homogeneous maps,

$$m_{k_1, \ldots, k_n} : \frac{V^* \otimes V^* \cdots V^*}{\text{shuffle products}} \to V,$$

of degree $3−n−k_1−\ldots−k_n$. The condition $D^2 = 0$ translates into a well-defined set of quadratic equations for $m_{k_1, \ldots, k_n}$ which say, in particular, that $m_1$ is a differential on $V$ and that the product, $v_1 \cdot v_2 := (−1)^{i_1}m_2(v_1, v_2)$, together with the Lie bracket, $[v_1 \cdot v_2] := −(−1)^{i_1}m_{1,1}(v_1, v_2)$, satisfy the Poisson identity up to a homotopy given by $m_2, 1$. Hence the associated cohomology space $H(V, \mu_1)$ is a graded commutative Gerstenhaber algebra with respect to the binary operations induced by $m_2$ and $m_{1,1}$.

### 2.7. A tower of approximations to the $G_\infty$ operad.

Let $V$ be a (finite-dimensional, see Remark 2.2) vector space and $\hat{\circ}^*\left(\text{Lie}V^*[−2]\right)$ the associated free $\mathcal{G}$-algebra. It is easy to see that the multiplicative ideal, $I := <\text{Lie}^{≥2}V^*[−2]>$, generated by the commutant of $\text{Lie}V^*[−2]$, as well as its multiplicative power $I^n, n \geq 2$, are also $\text{Lie}$ ideals of $\hat{\circ}^*\left(\text{Lie}V^*[−2]\right)$. Hence the

\textsuperscript{6} Actually, in [GetJo] this operad was called Fulton-MacPherson operad. We, however, follow in this paper the terminology and notations used in [TaTs]
quotient, $\odot^\bullet (\text{Lie} V^*[-2]) / I^n$, $n \geq 1$, has a canonical $\mathcal{G}$-algebra structure (note, however, that in the case $n = 1$ the induced Lie brackets vanish).

2.7.1. Definition of operad $\mathcal{G}_\infty^{(n)}$: A $\mathcal{G}_\infty^{(n)}$-algebra structure on a finite-dimensional vector space $V$ is a differential of the quotient $\mathcal{G}$-algebra $\odot^\bullet (\text{Lie} V^*[-2]) / I^n$, $n \geq 1$.

The following statements are obvious.

2.7.2. Lemma. $\mathcal{G}_\infty^{(1)} = \mathcal{L}_\infty$.

2.7.3. Lemma. For any $n \geq 2$, there is a commutative diagram,

$$
\begin{array}{ccc}
\mathcal{G}_\infty^{(n)} & \longrightarrow & \mathcal{G}_\infty \\
\mathcal{L}_\infty & \longrightarrow & L_\infty
\end{array}
$$

with all arrows being canonical cofibrations\(^7\) of operads.

We shall give below in this paper a fairly explicit description of the next two floors, $\mathcal{G}_\infty^{(2)}$ and $\mathcal{G}_\infty^{(3)}$, of the tower,

$$
\mathcal{L}_\infty = \mathcal{G}_\infty^{(1)} \longrightarrow \mathcal{G}_\infty^{(2)} \longrightarrow \mathcal{G}_\infty^{(3)} \longrightarrow \ldots \longrightarrow \mathcal{G}_\infty^{(n)} \longrightarrow \ldots ,
$$

of cofibrant approximations to the operad $\mathcal{G}_\infty = \text{colim} \mathcal{G}_\infty^{(n)}$. Interestingly enough, the operad $\mathcal{G}_\infty^{(2)}$ is closely related (through the cobar construction) to the operad $\mathcal{Gerst}$ and governs Frobenius$_{\infty}$ manifolds introduced in [Me2], while the operad $\mathcal{G}_\infty^{(3)}$ governs strong homotopy generalizations of Hertling-Manin’s $F$-manifolds.

2.8. Homotopy theory. The categories of operads and of their algebras belong to a class of so called closed model categories [Qu] which have a particularly nice homotopy theory. Here is a brief outline of all the relevant notions and facts we use in the paper (see, e.g., [DS, GeMa] for more details and proofs).

2.8.1. Definition. A closed model category is, by definition, a category $\text{Cat}$ with three distinguished classes of morphisms — (i) weak equivalences, $\mathcal{E}$, (ii) fibrations, $\mathcal{F}$, and (iii) cofibrations, $\mathcal{F}^\circ$, — which are closed under composition and contain all identity maps. The following axioms must be satisfied:

CMC1: Finite limits and colimits exist in $\text{Cat}$.
CMC2: If $f$ and $g$ are morphisms in $\text{Cat}$ such that their composition $fg$ is defined, then if any two of the three maps $f, g, fg$ are weak equivalences then so is the third morphism.
CMC3: Given any commutative diagram of the form,

$$
\begin{array}{ccc}
A & \overset{i}{\longrightarrow} & B \\
\downarrow{f} & & \downarrow{g} \\
A' & \overset{i'}{\longrightarrow} & B'
\end{array}
\quad \begin{array}{c} \text{and} \end{array}
\begin{array}{ccc}
B & \overset{p}{\longrightarrow} & A \\
\downarrow{f} & & \downarrow{f} \\
B' & \overset{p'}{\longrightarrow} & A'
\end{array}
$$

\(^7\)The notion of cofibration is explained in Sect. 2.8.
with \( pi \) and \( p'i' \) being the identity maps. If \( g \) is in \( \mathcal{E}, \mathcal{F} \) or \( \mathcal{F}^0 \), then so is \( f \).

**CMC4:** Given any commutative solid arrow diagram of the form,

\[
\begin{array}{c}
A \\
\downarrow f
\end{array} \quad \begin{array}{c}
\bullet
\end{array} \quad \begin{array}{c}
\downarrow g
\end{array} \quad \begin{array}{c}
B
\end{array}
\]

A dotted arrow \( h \) commuting with all other maps exists in either of the following two situations:

(i) \( f \in \mathcal{F}^0 \cap \mathcal{E} \) and \( g \in \mathcal{F} \), or (ii) \( f \in \mathcal{F}^0 \) and \( g \in \mathcal{F} \cap \mathcal{E} \).

**CMC5** Any morphism can be factored in two ways: (i) \( F = pi \) with \( i \in \mathcal{F} \) and \( p \in \mathcal{F} \cap \mathcal{E} \), and (ii) \( f = pi i \in \mathcal{F} \cap \mathcal{E} \) and \( p \in \mathcal{F} \).

### 2.8.2. Definitions.

(i) If a pair of morphisms, \( f : A \to A' \) and \( g : B \to B' \), satisfies the condition CMC4, then we say that \( f \) has the \textit{left lifting property} (LLP) with respect to \( g \) or that \( g \) has the \textit{right lifting property} (RLP) with respect to \( f \).

(ii) The morphisms in \( \mathcal{F} \cap \mathcal{E} \) are called \textit{acyclic fibrations}. The morphisms in \( \mathcal{F}^0 \cap \mathcal{E} \) are called \textit{acyclic cofibrations}.

(iii) The axiom CMC1 implies, in particular, that every closed model category \( \mathcal{C} \) at has both an initial object \( \emptyset \) and a terminal object \( * \). An object \( A \) of \( \mathcal{C} \) is called \textit{fibrant} (resp., \textit{cofibrant}) if \( A \to * \) is a fibration (resp., \( \emptyset \to A \) is a cofibration).

### 2.8.3. Facts

[GetJo, Hi]. (i) The category of dg operads is a closed model category with

- weak equivalences \( \mathcal{E} = \{ \) the morphisms, \( f : (\mathcal{O},d) \to (\mathcal{O}',d') \), of dg operads which induce isomorphism, \( [f] : H(\mathcal{O}) \to H(\mathcal{O}') \), in cohomology\( \} \);
- fibrations \( \mathcal{F} = \{ \) surjective morphisms of dg operads \( \};
- cofibrations \( \mathcal{F}^0 = \{ \) the morphisms which have LLP with respect to all acyclic fibrations \( \} \).

(ii) Given a dg operad \( \mathcal{O} \), the associated category of \( \mathcal{O} \)-algebras is a closed model category with the classes of morphism \( \mathcal{E}, \mathcal{F} \) and \( \mathcal{F}^0 \) defined in close analogy to (i).

(iii) Every object in the closed model categories (i) and (ii) is obviously fibrant.

### 2.8.4. Homotopy and derived categories of a closed model category.

From now on \( \mathcal{C} \) at stands for a closed model category. Moreover, we assume for simplicity that every object in \( \mathcal{C} \) at is fibrant (as in the two examples above).

Two morphisms, \( f, g : A \to B \), are called (right) \textit{homotopic} if there exists a path object, for \( B \) (that is an object \( B^I \) together with a diagram

\[
B \xrightarrow{i} B^I \xrightarrow{p} B \times B, \quad i \in \mathcal{E},
\]

which factors the diagonal map \( B \xrightarrow{(id,id)} B \times B \) such that the product map \( (f,g) : A \to B \times B \) lifts to a map \( H : A \to B^I \). Such a map \( H \) is called a (right) homotopy from \( f \) to \( g \). This, in fact, defines an equivalence relation \( \sim \) in \( \text{Hom}_{\mathcal{C} \at}(A,B) \) for any objects \( A, B \). However, the associated homotopy classes of maps,

\[
\pi(A,B) := \frac{\text{Hom}_{\mathcal{C} \at}(A,B)}{\sim},
\]

16
do not necessarily compose, \( \pi(A, B) \times \pi(B, C) \to \pi(A, C) \), unless the objects involved are cofibrant.

By \textbf{CMC5}(i), the map \( \emptyset \to A \) can be factored, \( \emptyset \to QA \xrightarrow{p_A} A \), with \( QA \) being cofibrant and \( p_A \) a weak equivalence. Such an object \( QA \) is called a \textit{cofibrant resolution} of \( A \). Usually, cofibrant resolutions are constructed by the method of “adding a new variable and killing a cycle”. A nice illustration of the method at work is, for example, Markl’s [Mar1] original construction of the minimal resolution \( O_\infty \) of a simply connected dg operad \( O \); Markl’s minimal resolutions give an important class of cofibrant objects in the category of dg operads.

Another remarkable fact is that not only every object, but also every morphism, \( f : A \to B \), has a “cofibrant resolution” \( Qf \) making the following diagram commutative,

\[
\begin{array}{ccc}
QA & \xrightarrow{Qf} & QB \\
\downarrow^{p_A} & & \downarrow^{p_B} \\
A & \xrightarrow{f} & B
\end{array}
\]

Moreover, the homotopy class of such maps \([Qf]\) is defined uniquely by the homotopy class \([f]\).

The \textit{homotopy category}, \( \text{Ho}(\text{Cat}) \), is the category with the same objects as \( \text{Cat} \) and with morphisms given by

\[
\text{Hom}_{\text{Ho}(\text{Cat})}(A, B) := \pi(QA, QB).
\]

Clearly, there is a canonical functor \( \alpha : \text{Cat} \to \text{Ho}(\text{Cat}) \), which is the identity on objects and sends morphisms \( f \) to \([Qf]\).

The derived category, \( \text{D}(\text{Cat}) \), is the category obtained from \( \text{Cat} \) by localization with respect to weak equivalences; put another way, this is a category together with a functor \( F : \text{Cat} \to \text{D}(\text{Cat}) \) satisfying two conditions,

- \( F(f) \) is an isomorphism for each weak equivalence \( f \);
- every functor \( G : \text{Cat} \to \text{Cat}' \) sending weak equivalences into isomorphisms factors uniquely through \((\text{D}(\text{Cat}), F)\),

\[
G : \text{Cat} \xrightarrow{F} \text{D}(\text{Cat}) \xrightarrow{G'} \text{Cat}',
\]

for some functor \( G' \).

Note that the definition of \( \text{D}(\text{Cat}) \) involves only one class, \( \mathcal{E} \), of the three classes which define the closed model structure in \( \text{Cat} \). Nevertheless, one of the central results in Quillen’s [Qu] theory of closed model categories asserts the equivalence,

\[
\text{Ho}(\text{Cat}) \simeq \text{D}(\text{Cat}),
\]

of the two categories associated to \( \text{Cat} \).

\textbf{2.8.5. Transfer Theorem.} \textit{Let} \( \mathcal{P} \) \textit{be a cofibrant dg operad and} \( f : V \to V' \) \textit{a quasi-isomorphism of dg vector spaces. For any} \( \mathcal{P} \)-algebra structure on \( V \) (resp. \( V' \)), \textit{there exists a} \( \mathcal{P} \)-algebra structure on \( V' \) (resp. \( V \)) \textit{so that} \( V \) \textit{and} \( V' \) \textit{are equivalent as} \( \mathcal{P} \)-algebras (i.e. there exists a} \( \mathcal{P} \)-algebra \( V'' \) \textit{and a pair of quasi-isomorphisms of} \( \mathcal{P} \)-algebras, \( V \leftarrow V'' \rightarrow V' \)).

This is a well known fact. We show the proof only for completeness (cf. [BeMo]).
Proof. The dg operads $\mathcal{E}_V$ and $\mathcal{E}_{V'}$, if viewed only as dg $\mathbb{S}$-modules, have two natural maps,

$$\mathcal{E}_V \xrightarrow{f^*} \mathcal{E}_{V,V'}, \quad \mathcal{E}_{V'} \xrightarrow{f^*} \mathcal{E}_{V,V'},$$

to the dg $\mathbb{S}$-module $\mathcal{E}_{V,V'} := \{ \text{Hom}(V^\otimes n, V') \}, \quad n \geq 1$. Define a dg $\mathbb{S}$-module, $\mathcal{E}_f = \{ \mathcal{E}_f(n) \}, \quad n \geq 1$, by setting

$$\mathcal{E}_f(n) := \{ (v, v') \in \mathcal{E}_V(n) \times \mathcal{E}_{V'}(n) \mid f v = v' f^\otimes n \},$$

or, alternatively, by the pullback diagram of dg $\mathbb{S}$-modules,

$$\begin{array}{ccc}
\mathcal{E}_f & \xrightarrow{i_1} & \mathcal{E}_V \\
\downarrow & & \downarrow \\
\mathcal{E}_{V'} & \xrightarrow{f^*} & \mathcal{E}_{V,V'}.
\end{array}$$

It is easy to check that $\mathcal{E}_f$ inherits from the $\mathcal{E}_V$ and $\mathcal{E}_{V'}$ not only the $\mathbb{S}$-module structure, but also the compositions $\circ_i^{n,n'}$. Thus $\mathcal{E}_f$ is a dg operad with maps $i_1$ and $i_2$ above being morphisms of dg operads. Moreover, $i_1$ and $i_2$ are weak equivalences (recall that we are working over a field of characteristic zero). Hence, as objects in $D(\text{Oper})$, the dg operads $\mathcal{E}_V$, $\mathcal{E}_f$ and $\mathcal{E}_{V'}$ are isomorphic. Since the derived category of dg operads is equivalent to the homotopy category, we get isomorphisms of sets,

$$\pi(\mathcal{P}, \mathcal{E}_V) \simeq \pi(\mathcal{P}, \mathcal{E}_f) \simeq \pi(\mathcal{P}, \mathcal{E}_{V'}).$$

If, say, $V$ is a $\mathcal{P}$-algebra, the homotopy equivalence class of the structure map $\phi : \mathcal{P} \to \mathcal{E}_V$ gives rise to an element $[\phi_f]$ in $\pi(\mathcal{P}, \mathcal{E}_f)$. As $\mathcal{P}$ is cofibrant, the latter has a representative, $\phi_f \in \text{Hom}_{\text{Oper}}(\mathcal{P}, Q\mathcal{E}_f)$, where $p_f : Q\mathcal{E}_f \to \mathcal{E}_f$ is some cofibrant resolution of $\mathcal{E}_f$. By construction, the composition $i_1 \circ p_f \circ \phi_f : \mathcal{P} \to \mathcal{E}_V$ is homotopy equivalent to the original structure map $\phi : \mathcal{P} \to \mathcal{E}_V$. Finally, another composition, $i_1 \circ \partial_f \circ \phi_f : \mathcal{P} \to \mathcal{E}_V$, makes $V'$ into a $\mathcal{P}$ algebra which, in the derived category of $\mathcal{P}$-algebras, is obviously isomorphic to $\phi$. Analogously one proves the dual statement.

\[ \square \]

2.8.6. Sh algebras. An algebra over a cofibrant operad is called a strongly homotopy (or, shortly, sh) algebra. By the Theorem above, sh algebraic structures can be transferred by quasi-isomorphisms of complexes.

2.9. Markl’s theory of sh maps. The beauty of sh algebras, the transfer property 2.8.5, is spoiled by the fact that to compare such structures on quasi-isomorphic dg spaces $V$ and $V'$ (we refer to 2.8.5 again) one has to resort to a chain of strict $\mathcal{P}$-algebra morphisms, $V \leftarrow V'' \to V'$, involving a third party which is often hard to construct explicitly. One may try to overcome this deficiency by appropriately extending the notion of map between sh algebras.

Markl made in [Mar3, Mar4] an interesting suggestion which, in the setting of the proof of the Transfer Theorem, can be illustrated as follows. First one observes that the operad $\mathcal{E}_f$ is in fact a two coloured operad with one colour associated to $V$ and another one to $V'$. Next one constructs a two coloured cofibrant resolution, $Q\mathcal{E}_f$, and then defines the set of sh maps between the $\mathcal{P}$-algebras $V$ and $V'$ as the set of all algebras over the dg operad $Q\mathcal{E}_f$. In this way Markl was able to prove stronger versions of the Transfer Theorem [Mar3]. The problem, however, with this approach is that it is not yet clear whether or not such sh maps can be composed making the pair (sh algebras, sh maps) into a genuine category. At present, this is known to be true.
only for a class of sh algebras associated with Koszul operads. In particular, it is true for $A_\infty$, $C_\infty$- and $L_\infty$-algebras reproducing thereby the well established theory of sh maps of these three classes of sh algebras. For later reference we review below a few basic facts (see, e.g., [Ko1, Pr]).

2.10. Sh maps of $A_\infty$- and $C_\infty$-algebras. An $A_\infty$-structure on a vector space $V$ can be suitably represented as a codifferential, $\mu : (T^*V[1], \Delta) \to (T^*V[1], \Delta)$, of the free tensor coalgebra cogenerated by $V[1]$. A sh map, $f : (V, \mu) \to (\hat{V}, \hat{\mu})$, of $A_\infty$-algebras is, by definition, a morphism of the associated differential coalgebras, $f : (T^*V[1], \Delta, \mu) \to (T^*\hat{V}[1], \Delta, \hat{\mu})$. Such a map is equivalent to a set of linear maps $\{ f_n : V \otimes^n \to \hat{V}, \ n \geq 1 \}$ of degree $1 - n$ which satisfy the equations,

$$
\sum_{1 \leq k_1 < k_2 < \ldots < k_i = n} (-1)^{i+r} \hat{\mu}_i (f_{k_1}(v_1, \ldots, v_{k_1}), f_{k_2-k_1}(v_{k_1+1}, \ldots, v_{k_2}), \ldots, f_{n-k_{i-1}}(v_{k_{i-1}+1}, \ldots, v_{n}))
$$

$$
= \sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^{(l(v_1+\ldots+v_j+n)+j(l-1)} f_k(v_1, \ldots, v_j, \mu(v_{j+1}, \ldots, v_{j+l}), v_{j+l+1}, \ldots, v_n),
$$

for arbitrary $v_i \in V$.

The pair $(Ob = A_\infty$-algebras, $Mor = sh$ maps) forms a category called the category of $A_\infty$-algebras.

A sh map $f = \{ f_n \} : (V, \mu) \to (\hat{V}, \hat{\mu})$ is called a quasi-isomorphism if the associated map of dg vector spaces, $f_1 : (V, \mu_1) \to (\hat{V}, \hat{\mu}_1)$, induces an isomorphism in cohomology.

Two sh maps, $f, g : (T^*V[1], \Delta, \mu) \to T^*\hat{V}[1], \Delta, \hat{\mu}$, are said to be homotopic if there is a homogeneous map, $h : T^*V[1] \to T^*\hat{V}[1]$, of degree $-1$ such that

$$
\Delta h = (f \otimes h + h \otimes g)\Delta, \quad f - g = \hat{\mu} \circ h + \mu \circ h.
$$

Remarkably enough, homotopy induces an equivalence relation in the set of sh maps $(V, \mu) \to (\hat{V}, \hat{\mu})$ [Pr]. Moreover, a sh map $f = \{ f_n \} : (V, \mu) \to (\hat{V}, \hat{\mu})$ is a quasi-isomorphism if and only if it is a homotopy equivalence. Thus the derived category of $A_\infty$-algebras is simply the quotient of the category $A_\infty$-algebras by the above homotopy relation!

For $C_\infty$-algebras one has a similar list of definitions and results.

2.11. Sh maps of $L_\infty$-algebras. A $L_\infty$-algebra structure, $\nu = \{ \nu_n : \odot^n V \to V, |\nu_n| = 3 - 2n \}$, on a vector space $V$ can be compactly described as a codifferential, $\nu : (\odot^*V[2], \Delta) \to (\odot^*V[2], \Delta)$, of the free cocommutative tensor coalgebra cogenerated by $V[2]$. A sh map, $f : (V, \nu) \to (\hat{V}, \hat{\nu})$, of $L_\infty$-algebras is, by definition, a morphism of the associated differential cocommutative coalgebras, $f : (\odot^*V[2], \Delta, \nu) \to (\odot^*V[2], \Delta, \hat{\nu})$. Such a map is equivalent to a set of linear maps $\{ f_n : \odot^n V \to \hat{V}, \ n \geq 1 \}$ of degree $2 - 2n$ which satisfy the equations similar to the ones in Subsect. 2.10. The notions of quasi-isomorphism and homotopy are similar as well.

The pair $(Ob = L_\infty$-algebras, $Mor = sh$ maps) forms a category called the category of $L_\infty$-algebras.

Dualizing the above formulae for a finite-dimensional vector space $V$ one arrives at a beautiful geometric formulation of $L_\infty$-algebras and their sh maps [Ko1]:
• A $L_\infty$-algebra structure on $V$ can be identified with a smooth degree 1 vector field $\vec{\nu}$ on the pointed flat graded manifold $(V[2], 0)$ which satisfies the conditions $[\vec{\nu}, \vec{\nu}] = 0$ and $\vec{\nu}|_0 = 0$. Explicitly, the identification, $$\vec{\nu} \leftrightarrow \{ \nu_n : \otimes^n V \to V \},$$ is given by the formula, $$\vec{\nu} = \sum_{n=1}^{\infty} \sum_{\alpha, \beta_1, \ldots, \beta_n} \frac{(-1)^r}{n!} t^{\beta_1} \ldots t^{\beta_n} \mu_{\beta_1, \ldots, \beta_n} \frac{\partial}{\partial t^{\alpha}},$$ where $\{t^\alpha, \alpha = 1, \ldots, \dim V\}$ is the basis of $V^*[−2]$ associated to a basis, $\{e_\alpha\}$, of $V$ (so that $|t^\alpha| = −|e_\alpha| + 2$), $$r = (2n - 3)(|e_\alpha_1| + \ldots + |e_\alpha_n|) + \sum_{k=2}^{n} |e_\alpha_k|(|e_\alpha_1| + \ldots + |e_\alpha_{k-1}|),$$ and $\mu_{\beta_1, \ldots, \beta_n} \in k$ are given by $$\mu_n(e_{\beta_1}, \ldots, e_{\beta_n}) = \sum_{\alpha} \mu_{\beta_1, \ldots, \beta_n} e_\alpha.$$

• A sh map of $f : (V, \nu_*) \to (\hat{V}, \hat{\nu}_*)$, of $L_\infty$-algebras is a smooth map of pointed graded manifolds, $f : (V^*[−2], 0) \to (\hat{V}^*[−2], 0)$ such that $f_*(\vec{\nu})$ is well defined and coincides with $\vec{\nu}$. Put another way, a sh map of $L_\infty$-algebras is just a morphism of the associated pointed dg manifolds.

A $L_\infty$-algebra $(V, \{\nu_n\}_{n \geq 1})$ with $\nu_1 = 0$ is called minimal (equivalently, the homological vector field $\vec{\nu}$ has zero at the distinguished point of order $\geq 2$).

2.11.1. Facts [Ko1]. (i) Every $L_\infty$-algebra is quasi-isomorphic to a minimal one.
(ii) There is a one-to-one correspondence between quasi-isomorphisms of $L_\infty$-algebras and diffeomorphisms of the associated dg manifolds.

2.11.2. Fact [Me2]. The canonical functor
$$\left\{ \begin{array}{c}
\text{The category of } L_\infty\text{-algebras} \\
\text{of } L_\infty\text{-algebras}
\end{array} \right\} \longrightarrow \left\{ \begin{array}{c}
\text{The derived category of } L_\infty\text{-algebras} \\
\text{of } L_\infty\text{-algebras}
\end{array} \right\},$$
when restricted to minimal $L_\infty$-algebras, becomes simply a forgetful functor,
$$\left( M, *, \text{flat structure, } \vec{\nu} \right) \longrightarrow \left( M, *, \vec{\nu} \right),$$
which forgets the flat (=affine) structure on $(M, *) = (V^*[2], 0)$.

Thus a homotopy class of minimal $L_\infty$-algebras is nothing but a pointed formal dg manifold, $(M, *, \vec{\nu})$, with no preferred choice of local coordinates. Moreover, the derived (=homotopy) category of $L_\infty$-algebras is equivalent to the purely geometric category of formal dg manifolds.
§3. Cobar construction for \emph{Gerst}

3.1. Cobar construction. For an \(S\)-module \(O = \{0(n)\}_{n \geq 1}\) we set \(O(m)\) to be an \(S\)-module given by the tensor product,

\[ O(m)(n) := \mathcal{O}(n) \otimes_k \Lambda_n^{\otimes m}[m(n - 1)], \]

where \(\Lambda_n\) is the sign representation of the permutation group \(S_n\). If \(O\) is a dg operad, then \(O(m)\) is naturally a dg operad as well: a structure of \(O(m)\)-algebra on a dg vector space \(V\) is the same as a structure of \(O\)-algebra on the shifted dg vector space \(V[m]\).

Let \(O = \{\mathcal{O}(n), c_i^{n,n'}, d\}\) be a simply connected dg operad and let \(\mathcal{O}^*(-1)\) stand for the \(S\)-module \(\{\mathcal{O}(n)^*[-1]\}\). It was shown in [GiKa] that the free operad associated to the \(S\)-module \(\text{Free}(\mathcal{O}[n][-1])\) can be naturally made into a \emph{differential} operad, \(\mathcal{D}(\mathcal{O}) = (\text{Free}(\mathcal{O}[n][-1])[-1]), \delta\), with the differential \(\delta\) defined by that in \(O\) and the compositions \(c_i^{n,n'}\). This construction gives rise to a functor, \(\mathcal{D} : \text{Oper}_1 \to \text{Oper}_1\), on the category of simply connected dg operads with the property that \(\mathcal{D}(\mathcal{D}(\mathcal{O}))\) is canonically quasi-isomorphic to the original operad \(O\). This functor is called a \emph{cobar construction}.

3.2. Koszul duals. An operad \(O\) is called \emph{quadratic} if it can be represented as a quotient,

\[ O = \frac{\text{Free}(\mathcal{E})}{<R>}, \]

of the free operad generated by an \(S\)-module \(\mathcal{E}\) with \(\mathcal{E}(n) = 0\) for \(n \neq 2\) by an ideal generated by an \(S_3\)-invariant subspace \(R\) in \(\text{Free}(\mathcal{E})(3)\). For example, operads \emph{Comm}, \emph{Ass} and \emph{Lie} are quadratic.

The \emph{Koszul dual} of a quadratic operad \(O = \text{Free}(\mathcal{E})/ <R>\) is, by definition, the quadratic operad \(O' = \text{Free}(\mathcal{E})/ <R^\perp>\) where \(\mathcal{E}\) is the \(S\)-module whose only non-vanishing component is \(\mathcal{E}(2) = \mathcal{E}(2)^* \otimes \Lambda_2\) and \(R^\perp\) is the annihilator of \(R\), i.e. the kernel of the natural map \(\text{Free}(\mathcal{E})(3) \to R^*\).

Applying cobar construction to the Koszul dual of a quadratic operad \(O\) one gets a cofibrant dg operad \(\mathcal{D}(O')\) together with a canonical map of dg operads [GiKa],

\[ (\mathcal{D}(O'), \delta) \to (O, 0). \]

Whatever the operad \(O\) is, the associated \(\mathcal{D}(O')\)-algebras are strong homotopy ones.

3.3. Koszul operads. A quadratic operad \(O\) is called \emph{Koszul} if the canonical map \((\mathcal{D}(O'), \delta) \to (O, 0)\) is a quasi-isomorphism. In such a case the cobar construction applied to the Koszul dual operad \(O'\) provides us with the minimal resolution, \(O_\infty = \mathcal{D}(O')\), of the operad \(O\) (see Sect. 2.1).

3.4. Examples [GetJo, GiKa]. The operads \emph{Ass}, \emph{Comm}, \emph{Lie} and \(\mathcal{G}\) are Koszul with

\[ \text{Ass}' = \text{Ass}, \quad \text{Comm}' = \text{Lie}, \quad \text{Lie}' = \text{Comm}, \quad \mathcal{G}' = \mathcal{G}\{1\}. \]

Thus the operads \(A_\infty := \mathcal{D}(\text{Ass}), C_\infty := \mathcal{D}(\text{Lie}), L_\infty := \mathcal{D}(\text{Comm})\) and \(\mathcal{G}_\infty := \mathcal{D}(\mathcal{G}\{1\})\) are minimal resolutions of the operads, \emph{Ass}, \emph{Comm}, \emph{Lie} and \(\mathcal{G}\) respectively. This explains all the claims made in Examples 2.3—2.6.
3.5. Proposition. $\text{Gerst}^1 = \mathcal{H}\{1\}$ and $\mathcal{H}^1 = \text{Gerst}\{1\}$, where $\mathcal{H}$ is the operad defined in Example 1.10.

Proof is straightforward.

3.6. Cobar construction for $\text{Gerst}^1$. Surprisingly enough, the minimal operad $\text{Gerst}_\infty := \mathcal{D}(\text{Gerst}^1)$ turns out to be a much more elementary object than its “graded commutative” analogue $\mathcal{G}_\infty = \mathcal{D}(\mathcal{G}^1)$. In this section we present an explicit and simple description of the cobar construction $\text{Gerst}_\infty$ in terms of partially planar/partially space trees (reflecting its nature as a composition of the operads $A_\infty$ and $L_\infty$, see below), and in the next section we show that strong homotopy $\text{Gerst}_\infty$-algebras admit a nice geometric interpretation.

The main reason behind that acclaimed simplicity of $\mathcal{D}(\text{Gerst}^1) = \mathcal{D}(\mathcal{H}\{1\})$ is the non-distributive nature of the operad $\mathcal{H}$ (cf. Remark 1.9): “opening” the left- and right hand sides of the following two equalities in the operad $\mathcal{H}$,

$$
\bullet \bullet \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow 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then $Gerst_\infty$, as an $\mathbb{S}$-module, equals the linear span of all possible (isomorphism classes of) partially plane partially space trees formed by these corollas, while the compositions $o_{i}^{n,n'}$ are given simply by gluing the root vertex of an $[n']$-tree to the $i$th tail vertex of an $[n]$-tree. To complete the description of the operad $Gerst_\infty$ we need only to compute Ginzburg-Kapranov’s cobar differential $d$.

3.6.1. Proposition. The cobar differential in $Gerst_\infty$ is given on generators by

$$d = \sum_{I_1 \cup I_2 = (1, \ldots, n) \atop \#I_1 \geq 2, \#I_2 \geq 2} I_1 \cdot I_2 ,$$

$$d = \sum_{I_1 \cup I_2 = (p+1, \ldots, n) \atop \#I_1 \geq 5, \#I_2 \geq 1} I_1 \cdot I_2 ,$$

$$+ \sum_{I_1 \cup I_2 = (p+1, \ldots, n) \atop \#I_1 \geq 1, \#I_2 \geq 0} \sum_{s=1}^{p} I_1 \cdot I_2 ,$$

$$+ \sum_{I_1 \cup I_2 = (p+1, \ldots, n) \atop \#I_1 \geq 2, \#I_2 \geq 2} \sum_{s=0}^{m-1} (-1)^{l+s(l+1)} I_1 \cdot I_2 .$$

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Proof is straightforward though very tedious.

**Corollary 3.6.2.** There is a canonical cofibration of operads, \( \mathcal{L}_\infty \to \mathcal{G}erst_\infty \).

**Corollary 3.6.3.** A \( \mathcal{G}erst_\infty \)-algebra is the data,

\[
\left( V, \{ \nu_n \}_{n \geq 1}, \{ \mu_{p,n} \}_{p, n \geq 2} \right),
\]

consisting of a graded vector space \( V \) and two collections of homogeneous linear maps,

- \( \nu_n : \otimes^n V \to V \) of degree \( 3 - 2n \), \( n = 1, 2, 3, \ldots \), and
- \( \mu_{n,p} : (\otimes^n V) \otimes (\otimes^p V) \to V \) of degree \( 2 - n - 2p \), \( k = 2, 3, 4, \ldots \), \( n = 0, 1, 2, \ldots \),

which satisfy the equations, for any \( a_1, \ldots, a_n, b_1, \ldots, b_N \in V \),

\[
\sum_{S_1 \sqcup S_2 = (1, \ldots, N)} (-1)^{|S_1|+1} \nu_{|S_1|+1}(\nu_{|S_1|}(b_{S_1}), b_{S_2}) = 0, \quad N \geq 1,
\]

and, for \( n \geq 2, N \geq 0 \),

\[
\sum_{S_1 \sqcup S_2 = (1, \ldots, N)} (-1)^{\sigma} \left\{ \nu_{|S_2|+1}(\mu_{n,|S_1|}(a_1, \ldots, a_n; b_{S_1}), b_{S_2}) \right\}
\]

\[
- (-1)^{\tilde{a}_1 + \ldots + \tilde{a}_n - n} \mu_{n,|S_2|+1}(a_1, \ldots, a_n; \nu_{S_1}(b_{S_1}), b_{S_2})
\]

\[
- \sum_{j=1}^n (-1)^{\tilde{a}_1 + \ldots + \tilde{a}_{j-1} + \tilde{a}_{j+1} + \ldots + \tilde{a}_n} b_{S_1} - n \mu_{n,|S_2|}(a_1, \ldots, a_{j-1}, \nu_{S_1}(b_{S_1}), a_{j+1}, \ldots, a_n; b_{S_2})
\]

\[
= \sum_{k+l=n+1} \sum_{j=0}^{k} \sum_{k, l \geq 2} (-1)^r \mu_{k,|S_2|}(a_1, \ldots, a_j, \mu_{l,|S_1|}(a_{j+1}, \ldots, a_{j+l}; b_{S_1}), a_{j+l+1}, \ldots, a_n; b_{S_2}),
\]

where \( (-1)^\sigma \) is the standard Koszul sign of the shuffle permutation \( b_1 \otimes \ldots \otimes b_N \to b_{S_1} \otimes b_{S_2} \), and \( r = j + l(n - j - l) + l(\tilde{a}_1 + \ldots + \tilde{a}_j) + (\tilde{a}_{j+l+1} + \ldots + \tilde{a}_n) \).

Thus the operations \( \nu_\bullet \) define on \( V \) the structure of \( L_\infty \) algebra, while the operations \( \mu_{\bullet,0} \) define on \( V \) the structure of \( A_\infty \) algebra; the remaining operations \( \mu_{\bullet,1} \geq 1 \) are homotopies, and homotopies of homotopies, and \ldots, which make these two basic structures Poisson-type consistent. In the special case when all operations but \( \nu_2 \) and \( \mu_{2,0} \) vanish the \( \mathcal{G}erst_\infty \)-algebra \( V \) becomes nothing but a \( \mathcal{G}erst \)-algebra, i.e. a (non-commutative, in general) Gerstenhaber algebra.

### 3.7. Coalgebra interpretation

The notion of \( \mathcal{H} \)-algebra (which is a graded vector space \( V \) equipped with two consistent associative multiplications, \( : V \otimes V \to V \) and \( \bullet : V \otimes V \to V \), of degrees 0 and 1 respectively, see Sect. 1.10) can be naturally dualized leading to the notion of \( \mathcal{H} \)-coalgebra.

Let \( V \) be a graded vector space and \( (\bar{BV} := \otimes_{\geq 1} V[1], \Delta_\bullet) \) the (reduced, as \( \otimes^0 V[1] \) is omitted) free tensor coalgebra generated by \( V[1] \). Let \( (\otimes \bullet (\bar{BV}[1]), \Delta_\circ) \) the completed (with respect to the natural filtration \( \mathcal{F}^{2r} := \otimes_{\geq 1} V[1] \)) free graded cocommutative coalgebra generated
by $BV[1]$ (the latter is $BV$ with shifted grading). Let $I \subset \hat{\otimes}^\bullet(BV[1])$ be the coideal generated by $\otimes^{\geq 2}V[1]$. The standard coassociative coproduct $\Delta_\ast$ in $BV$ extends naturally to a degree -1 coproduct in the quotient $\hat{\otimes}^\bullet(BV[1])/I^2$ making the latter into an $\mathcal{H}$-coalgebra. (Here $I^2$ is the square of the coideal, i.e. the set of elements $x \in I$ such that $\Delta_\ast x \in I \otimes I$.)

3.7.1. Proposition. There is a one-to-one correspondence,

$$\left\{ \begin{array}{c}
\text{Gerst}_\infty\text{-algebra structures in } V \\
\text{codifferentials in the cofree } H\text{-coalgebra } (\hat{\otimes}^\bullet(BV[1])/I^2, \Delta_\ast, \Delta_\ast)
\end{array} \right\} \leftrightarrow \left\{ \begin{array}{c}
\text{geometric } K\text{-structure in } V
\end{array} \right\}.$$  

Proof. A coderivation $D$ of the coalgebra $\hat{\otimes}^\bullet(BV[1])/I^2$ is equivalent to a degree 1 linear map, $\hat{\otimes}^\bullet(BV[1])/I^2 \rightarrow V[2]$, vanishing on $k$, the direct summand of constants. As $\hat{\otimes}^\bullet(BV[1])/I^2 = \hat{\otimes}^\bullet(V[2]) \oplus (\otimes^{\geq 2}V[1])[1] \otimes \hat{\otimes}^\bullet(V[2])$, this is the same as two collections of homogeneous linear maps, $\{\nu_n : \otimes^n V \rightarrow V\}$ of degree $3 - 2n \}_{n \geq 1}$ and $\{\mu_{n,p} : (\otimes^n V) \otimes (\otimes^p V) \rightarrow V\}$ of degree $2 - n - 2p \}_{n \geq 2, p \geq 0}$. It is straightforward to check that the condition $D^2 = 0$ translates precisely into the equations of Corollary 3.6.3. \hfill $\square$

3.8. Geometric interpretation. $\mathcal{L}_\infty$-structure on a finite-dimensional vector space $V$ has a beautiful geometric interpretation [Ko1] as a smooth degree 1 vector field $\vec{v}$ on the pointed affine graded formal manifold $(V[2], 0)$ which satisfies the conditions $[\vec{v}, \vec{v}] = 0$ and $\vec{v}|_0 = 0$.

Surprisingly enough, $\text{Gerst}_\infty$-structure is also of purely geometric nature.

3.8.1. Definitions. (i) A geometric $A_\infty$-structure (respectively, geometric $C_\infty$-structure) on a graded manifold $\mathcal{M}$ is the data $(\vec{\nu}, \mu_\ast)$ consisting of

(a) a smooth homological vector field $\vec{\nu}$ making $\mathcal{M}$ into a dg manifold;

(b) a collection of maps, $\mu_\ast = \{\mu_n\}_{n \geq 1} : \otimes^n_{\mathcal{O}_\mathcal{M}} T\mathcal{M} \rightarrow T\mathcal{M}$, with $\mu_1 = \text{Lie}_{\vec{\nu}}$, making the tangent sheaf $T\mathcal{M}$ into a sheaf of $A_\infty$-algebras (respectively, $C_\infty$-algebras).

(ii) A geometric $A_\infty$-structure/$C_\infty$-structure on a pointed graded manifold $(\mathcal{M}, \ast)$ is called minimal if $\vec{\nu} I \subset I^2$ where $I$ is the ideal of the distinguished point $\ast \in \mathcal{M}$.

3.8.2. Theorem. There is a one-to-one correspondence

$$\left\{ \begin{array}{c}
\text{Gerst}_\infty\text{-algebra structures in } V \\
\text{geometric } A_\infty\text{-structures on the pointed affine formal manifold } \mathcal{M} = (V[2], 0)
\end{array} \right\} \leftrightarrow \left\{ \begin{array}{c}
\text{affine formal manifold } \mathcal{M}
\end{array} \right\}.$$  

Proof. Let $\{e_\alpha, \alpha = 1, \ldots, \dim V\}$ be basis of $V$ and $\{t^\alpha\}$ the associated dual basis of $V^*[-2]$ which we identify with coordinate functions on $V[2]$. Set $t := \sum \alpha t^\alpha e_\alpha$, and let $\tau$ be the isomorphism of $\mathcal{O}_\mathcal{M}$-modules

$$\tau : T\mathcal{M} \rightarrow \mathcal{O}_\mathcal{M} \otimes V,$$

$$X = \sum \alpha X^\alpha(t) \partial/\partial t^\alpha \rightarrow \tau(X) := \sum \alpha X^\alpha(t)e_\alpha.$$
Let \( \{\nu_n : \odot^n V \to V\}_{n \geq 1} \) and \( \{\mu_{n,p} : (\odot^n V) \otimes (\odot^p V) \to V\}_{n \geq 2, p \geq 0} \) be a \( \text{Gerst}_{\infty}\)-algebra structure on \( V \). Define the vector field,
\[
\vec{\nu} := \sum_{n=1}^{\infty} \frac{1}{n!} \tau^{-1} \circ \nu_n(t, \ldots, t),
\]
and the tensors, for \( n \geq 2 \),
\[
\mu_n : \bigotimes_{\mathcal{O}_M} T_M \to T_M,
\]
\[
X_1 \otimes \ldots \otimes X_n \to \sum_{p=1}^{\infty} \frac{1}{p!} \tau^{-1} \circ \mu_{n,p}(\tau(X_1), \ldots, \tau(X_n); t, \ldots, t)
\]
on \( \mathcal{M} \). Then the equations of Corollary 3.6.3 translate precisely into the statement that \((\vec{\nu}, \mu_\bullet)\) is a geometric \( A_{\infty}\)-structure on \( M \).

This argument also works in the opposite direction through the Taylor decomposition of all the tensors at the distinguished point. \( \Box \)

### 3.9. \( \text{Gerst}_{\infty}\)-manifolds.

Since the operad \( \text{Gerst}_{\infty}\) is minimal, its algebras are strong homotopy ones, i.e. can be transferred via quasi-isomorphisms. In this subsection we essentially give a purely geometric description of the derived(=homotopy) category of \( \text{Gerst}_{\infty}\)-algebras.

**3.9.1. Definition.** A \( \text{Gerst}_{\infty}\)-manifold is a smooth manifold \( M \) together with a homotopy class (in the sense of 2.10), \((\bar{\nu}, [\mu_\bullet])\), of minimal geometric \( A_{\infty}\)-algebra structures on the tangent sheaf.

Note that we do not assume in the above definition that \( M \) is affine: the notion of \( \text{Gerst}_{\infty}\)-manifold is built on a collection of tensors satisfying a system of diffeomorphism covariant differential equations. The following tautologically formulated theorem is one of the main results of this section; it essentially describes a functor from the category of \( \text{Gerst}_{\infty}\)-algebras to a subcategory of the category of formal dg manifolds.

**3.9.2. Theorem.** If the operad \( \text{Gerst}_{\infty}\) acts on a dg vector space \((V, d)\), then the formal graded manifold associated with the cohomology vector space \( H(V, d) \) is canonically a \( \text{Gerst}_{\infty}\)-manifold.

**Proof.** Since we work over a field in this paper, there always exists a quasi-isomorphism of complexes \((V, d) \to (H(V, d), 0)\) and hence an induced structure of \( \text{Gerst}_{\infty}\)-algebra on \( H(V, d) \). This structure, however, is not canonical. What we have to show is that, first reinterpreting this induced structure as a geometric \( A_{\infty}\)-structure \((\bar{\nu}, \mu_\bullet)\) on the pointed flat formal manifold \((\mathcal{M}, *) = (H(V, d)[2], 0)\), then passing to the associated homotopy class, \((\bar{\nu}, [\mu_\bullet])\), just in the sense of \( A_{\infty}\)-algebras, and finally forgetting the flat structure one gets at the end the structure on \( M \) which does not depend on any choices made. It is precisely this structure which was termed in 3.9.1 an \( \text{Gerst}_{\infty}\)-manifold.

By Theorem 3.8.1, our input is a formal affine dg manifold,
\[
(M \simeq V[2], \vec{\nu} = \{\nu_\bullet\} \text{ with } \nu_1 = d, \text{ } * = 0),
\]
together with a geometric \( A_{\infty}\)-structure \( \mu_\bullet \). Let us choose a cohomological splitting of the complex \((V, d)\), i.e. a decomposition of the \( \mathbb{Z} \)-graded vector space \( V \) into a direct sum,
\[
V = H(V, d) \oplus B \oplus B[-1],
\]
in such a way that the differential vanishes when restricted to the summands \( H(V, D) \oplus B[-1] \) while on the remaining summand it equals the shifted by \([1]\) identity map \( B \to B[-1] \). According to Kontsevich [Ko1], such a splitting can be lifted to an isomorphism of formal affine dg manifolds,

\[
(M, \bar{\nu}, *) \simeq (M, \bar{\partial}, *) \times (B, \bar{d}_{DR}, *),
\]

where

- \((M, \bar{\partial}, *)\) is the formal affine minimal dg manifold whose tangent space at \(*\) is \( H(V, d)[2] \),
- \((B, \bar{d}_{DR}, *)\) is the formal affine dg manifold whose tangent space at \(*\) is \( B[2] \oplus B[1] \), homological vector field \( \bar{d}_{DR} \) is linear and coincides precisely with the usual De Rham differential when one identifies the structure sheaf, \( \mathcal{O}_{B,*} \), with the De Rham algebra of smooth formal differential forms on the vector space \( B \). In particular, both the cohomology groups, \( H(\mathcal{O}_{B,*}, \bar{d}_{DR}) \) and \( H(\mathcal{T}_{B,*}, Lie_{\bar{d}_{DR}}) \) are trivial.

Let \( \pi_1 : M \to M \) and \( \pi_2 : M \to B \) be the projections associated with the chosen above cohomological splitting. There is an associated decomposition of complexes of vector spaces (note that differentials are not \( B \)-linear),

\[
(\mathcal{T}_{M,*}, Lie_\bar{\nu}) = (\pi_1^* \mathcal{T}_{M,*}, Lie_\bar{\nu}) \oplus (\pi_2^* \mathcal{T}_{B,*}, Lie_{\bar{d}_{DR}}).
\]

The tangent vector space at \(* \in B\) can be identified with \( B[2] \oplus B[1] \). Let \( H : B[2] \oplus B[1] \to B[2] \oplus B[1] \) be a degree \(-1\) linear map which is equal to zero on the summand \( B[2] \) and is equal to the shifted by \([-1]\) identity map \( B[1] \to B[2] \) on the remaining summand. Denote by the same letter \( H \) its natural \( \mathcal{O}_{B,*}\)-linear extension to \( \mathcal{T}_{B,*} \). It is an easy calculation to check that the identity automorphism of the tangent sheaf to \( M \) decomposes as follows,

\[
Id = pr \oplus Lie_\bar{\nu} \circ \pi_2^*(H) \oplus \pi_2^*(H) \circ Lie_\bar{\nu},
\]

where \( \pi_2^*(H) \) is assumed to act as zero on the summand \( \pi_1^* \mathcal{T}_{M,*} \), and \( pr \) stands for the canonical projection \( \mathcal{T}_{M,*} \to \pi_1^* \mathcal{T}_{M,*} \). Thus we have constructed an \( \mathcal{O}_M\)-linear homotopy, \( \pi_2^*(H) : \mathcal{T}_{M,*} \to \mathcal{T}_{M,*} \), associated with quasi-isomorphic complexes of \( k \)-linear vector spaces \( (\mathcal{T}_{M,*}, Lie_\bar{\nu}) \) and \( (\pi_1^* \mathcal{T}_{M,*}, Lie_\bar{\nu}) \).

Next step is to employ, say, the explicit formulae of [Me1, KoSo2], to construct an \( \mathcal{A}_\infty \)-algebra structure,

\[
(\hat{\mu}_{n\geq 2} : \otimes_{\mathcal{O}_{M}}^n \pi_1^* \mathcal{T}_M \to \pi_1^* \mathcal{T}_M; \mu_1 = Lie_\bar{\nu}),
\]

on the lifted tangent sheaf \( \pi_1^* \mathcal{T}_M \). The key fact that the resulting \( \hat{\mu}_{n\geq 2} \) are tensors is ensured by \( \mathcal{O}_M\)-linearity of the constructed homotopy.

Finally one repeats the above procedure using the standard contraction homotopy (which is \( \pi^{-1} \mathcal{O}_M\)-linear) of the cohomologically trivial De Rham complex \( (\mathcal{O}_{B,*}, \bar{d}_{DR}) \) to induce a geometric \( \mathcal{A}_\infty \)-algebra structure, \((\bar{\partial}, \hat{\mu}_*)\), on \( M \). It is easy to check that the associated homotopy class (in the sense of 2.11 for \( \bar{\partial} \), and 2.10 for \( \hat{\mu}_* \)) does not depend on the choice of a particular factorization of the dg manifold \( (M, \bar{\nu}, 0) \) into a direct product of a minimal dg manifold and a linearly contractible one.

\[\square\]

3.9.3. Remark. The special case of the above Theorem when \((V, d, \text{Gerst}_\infty\text{-action})\) is just a (non-commutative) Gerstenhaber algebra was proved in [Me2] by explicit perturbative calculations.
3.10. Homotopy commutative sibling of $\mathcal{Gerst}_\infty$. Theorem 3.9.2 motivates the following definition.

3.10.1. Definition. $\mathcal{Gerst}_\infty$ is the operad whose algebras are given by

$$\begin{align*}
\{ \text{A $\mathcal{Gerst}_\infty$-algebra structure in } V \} := \{ \text{A geometric $C_\infty$-structure on the pointed affine formal manifold } (V[2],0) \}.
\end{align*}$$

Almost repeating 3.9.1 and 3.9.2 one obtains the notion of $\mathcal{Gerst}_\infty$-manifold (which was in fact introduced earlier [Me2] under the name Frobenius$_\infty$ manifold) and the statement:

3.10.2. Theorem. If the operad $\mathcal{Gerst}_\infty$ acts on a dg vector space $(V,d)$, then the formal graded manifold associated with the cohomology vector space $H(V,d)$ is canonically a $\mathcal{Gerst}_\infty$-manifold.

§4 Deformation theory

4.1. Deformation functor. The traditional approach to the deformation theory of a mathematical structure $A$ is based on the idea of deformation functor, $\text{Def}_g$, on Artinian rings.

Initially that idea was applied to Artinian rings concentrated in degree 0 so that the tangent space, $\text{Def}_g(k[\varepsilon]/\varepsilon^2)$, to the deformation functor (which is the same as the Zariski tangent space, $T_AM$, to the moduli space at the distinguished point) equals some particular homogeneous bit, $H^i(g,d)$, of the $\mathbb{Z}$-graded cohomology group, $H^*(g,d) = \oplus_{i \in \mathbb{Z}} H^i(g,d)$, of the dg Lie algebra, $(g,d)$, controlling the deformations of $A$. The next homogeneous bit, $H^{i+1}(g,d)$, absorbs the obstructions to exponentiating infinitesimal deformations from $H^i(g,d)$ to genuine ones.

Recent studies in mirror symmetry led to an extension [BaKo, Ma, Me2] of the deformation functor first to $\mathbb{Z}$-graded and then differential $\mathbb{Z}$-graded Artinian rings. The dg extension of $\text{Def}_g$ always produces smooth formal dg moduli spaces $(M,\partial,*)$ with Zariski tangent space, $T_AM$, isomorphic to the full cohomology group $H^*(g,d)$ and with obstructions encoded into a homological vector field on $M$ [Ko1, Me2]. The table below compares the two deformation functors in three important examples:

| $A$                        | $T_AM$ in classical $\text{Def}$ | $T_AM$ in extended $\text{Def}$ |
|----------------------------|----------------------------------|----------------------------------|
| Complex manifold $(M,J)$   | $H^1(M,T_M)$                     | $H^*(M,\wedge^\bullet T_M)$     |
| Symplectic manifold $(M,\omega)$ | $H^2(M,\mathbb{R})$             | $H^*(M,\mathbb{R})$             |
| Associative algebra $(A,\circ)$ | $\text{Hoch}^2(A,A)$           | $\text{Hoch}^*(A,A)$            |

These developments lead naturally to a question: What happens to $A$ when it is deformed in the generic direction in $H^*(g,d)$ (rather than in $H^i(g,d)$)? Or, equivalently, what is the universal structure $\mathcal{A}$ over the extended moduli space $\mathcal{M}$? Thanks to Stasheff [St], we know the answer to this question in the case $A = \text{Associative algebra}$: deforming any given associative algebra $A$ along a generic tangent vector in $T_AM = \text{Hoch}^*(A,A)$ one obtains, if all obstructions vanish, an $A_\infty$-algebra. To author’s knowledge, infinity versions of such notions as complex and
symplectic structure are still a mystery, and the deformation theory in the form of deformation functor gives no clue to its solution. Moreover, the dg extension of Def makes it evident that one does not really need Artinian rings to do the deformation theory — the dg versal moduli space, $(M, \partial, \ast)$, representing the functor $\text{Def}_g$ on dg Artinian rings is nothing but the image of $(g, d)$ under the canonical functor

$$\left\{ \begin{array}{c} \text{the category of} \\ \text{dg Lie algebras} \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{the derived category of} \\ \text{dg Lie algebras} \end{array} \right\}.$$ 

Thus the Artinian functor approach to deformation theory is tantamount to a perturbative computation of the minimal $\mathcal{L}_\infty$-model for the controlling dg Lie algebra.

### 4.2. An operadic guide to deformation theory

We lose too much information about the deformed mathematical structure $\mathcal{A}$ if we naively understand the extended deformation theory as outlined above in Sect. 4.1, i.e., as the deformation functor $\text{Def}_g$ extended to (differential) $\mathbb{Z}$-graded Artinian rings. Here is a suggestion on what one should do instead.

**Step 1:** Associate to the mathematical structure $\mathcal{A}$ we wish to deform a “controlling” Deformation Algebra, $(g, [\cdot, \cdot], d, \text{ADD})$, consisting of a dg Lie algebra $^9$ $(g = \bigoplus_{i \in \mathbb{Z}} g^i, [\cdot, \cdot], d)$, and a collection of some additional algebraic operations, ADD, on $g$. For example,

- if $\mathcal{A}$ is a symplectic or complex structure (see Examples 4.4 and 4.5 below), then Deformation Algebra is a graded commutative Gerstenhaber algebra, and ADD is just a graded commutative product consistent with the dg Lie algebra structure via Poisson type identities.

- if $\mathcal{A}$ is an associative algebra structure (see Example 4.6 below), then Deformation Algebra is what is called in [GeVo] a homotopy Gerstenhaber algebra, and ADD is an infinite series of operations called braces.

**Step 2:** Find a cofibrant resolution, $\mathcal{D}A_\infty$, of the operad $\mathcal{D}A$ describing species Deformation Algebra obtained in the previous step. In many important cases there exists the minimal cofibrant model $\mathcal{D}A_\infty$ of $\mathcal{D}A$ whose differential is decomposable.

**Step 3:** As the operad $\mathcal{D}A_\infty$ is cofibrant, its algebras are strong homotopy ones, that is, $\mathcal{P}$-algebra structures can be transferred by quasi-isomorphisms of complexes (see Theorem 2.8.5). Then choosing a cohomological splitting of $(g, d)$, one induces on the cohomology space $H(g, d)$ a canonical homotopy class of $\mathcal{D}A_\infty$-algebras (which is independent of the splitting used.)

**Step 4:** Try to find a geometric interpretation (called extended moduli space) of the homotopy class of minimal $\mathcal{D}A_\infty$-algebra structures canonically induced on $H(g, d)$ in the previous step, that is, try to interpret the latter in terms of sections of some natural vector bundles (equipped with $\partial$-connections, tensorial algebraic structures, . . . ) over a formal pointed dg manifold $(M, \partial, \ast)$ whose tangent space at the unique geometric point is precisely $H(g, d)$.

### 4.3. Motivation and evidence

All the steps above are functorial with respect to the choice of input, Deformation Algebra. Making a particular choice means essentially a choice of

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$^9$More generally, one can replace the dg Lie algebra structure on $g$ with a $\mathcal{L}_\infty$-algebra structure.

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precision with which we want to do the deformation theory. The most crude one is to apply first the obvious forgetful functor,

$$F : \{\text{Deformation Algebra}\} \longrightarrow \{\text{dg Lie Algebra}\}$$

and then apply the operadic algorithm. All the four steps can be easily fulfilled (see Sect. 2.11) with the final outcome at Step 4 being a smooth formal minimal dg manifold $$(\mathcal{M}, \bar{\partial}, \ast).$$ Note that this outcome is well defined only up to an action of the formal diffeomorphism group $Diff(\mathcal{M}, \ast),$

$$(\mathcal{M}, \bar{\partial}, \ast) \sim (\mathcal{M}, f_{\ast} \bar{\partial}, \ast), \quad \forall f \in Diff(\mathcal{M}, \ast).$$

Thus, whatever the choice of Deformation Algebra in Step 1, the outcome of Step 4 will always be at least a smooth dg manifold with its automorphism group $Diff(\mathcal{M}, \ast).$ As we expect ADD in the input of the deformation theory to be consistent, in some or other sense, with the underlying dg Lie algebra structure, it is natural to expect that what we get from ADD in Step 4 will also be consistent with the homological vector field $\bar{\partial}$ and behave reasonably well under the action of $Diff(\mathcal{M}, \ast).$ The main results of this paper (see the Introduction for the list) show that these expectations are met in such important cases as, e.g., extended deformations of $T^n$ where

$$\ast = R^n,$$

in Step 4 will also be consistent with the homological vector field $\bar{\partial}$ and behave reasonably well under the action of $Diff(\mathcal{M}, \ast).$ The main results of this paper (see the Introduction for the list) show that these expectations are met in such important cases as, e.g., extended deformations of complex and symplectic structures. This is why we believe that Step 4, the most questionable one in the above programme, makes sense.

### 4.4. Example (deformations of complex structures).

The dg commutative Gerstenhaber algebra controlling extended deformations of a given complex structure on a real $2n$-dimensional manifold $X$ is given by

$$\mathfrak{g} = \left( \bigoplus_{i=0}^{2n} \mathfrak{g}^i, \mathfrak{g}^i = \bigoplus_{p+q=i} \Gamma(X, \wedge^p \mathcal{T}_X \otimes \Omega^q_X), \ [\bullet, \bar{\partial}, \circ] \right)$$

where $\mathcal{T}_X$ stands for the sheaf of holomorphic vector fields, $\Omega^q_X$ for the sheaf of smooth differential forms of type $(s, q),$ and $[\bullet] = \text{Schouten brackets} \otimes \text{wedge product of forms},$ and $\circ = \text{wedge product of polyvector fields} \otimes \text{wedge product of forms}.$

### 4.5. Example (deformations of Poisson and symplectic structures).

The dg commutative Gerstenhaber algebra controlling deformations of a given Poisson structure, $\nu \in \Gamma(X, \wedge^2 \mathcal{T}_X),$ on a real smooth manifold $X$ is given by

$$\left( \bigoplus_{i=0}^{\dim X} \Gamma(X, \wedge^i \mathcal{T}_X), [\bullet] = \text{Schouten brackets}, d = [\nu \bullet \ldots], \text{wedge product of polyvector fields} \right),$$

where $\mathcal{T}_X$ stands for the sheaf of real tangent vectors.

If $\nu$ is non-degenerate, that is, $\nu = \omega^{-1}$ for some symplectic form $\omega$ on $X,$ then the natural “lowering of indices map” $\omega^{\wedge i} : \wedge^i \mathcal{T}_X \to \Omega_X^i$ sends $[\nu \bullet \ldots]$ into the usual de Rham differential. The image of the Schouten brackets under this isomorphism we denote by $[\bullet]_\omega.$ In this way we make the de Rham complex of $X$ into a dg commutative Gerstenhaber algebra,

$$\mathfrak{g} = \left( \bigoplus_{i=0}^{\dim X} \Gamma(X, \Omega_X^i), [\bullet]_\omega, d = \text{de Rham differential, wedge product of forms} \right),$$

which controls the extended deformations of the symplectic structure $\omega.$ More explicitly,

$$[\kappa_1 \bullet \kappa_2]_\omega := (-1)^{\kappa_1} [i_\nu, d](\kappa_1 \wedge \kappa_2) - (-1)^{\kappa_1} ([i_\nu, d] \kappa_1) \wedge \kappa_2 - \kappa_1 \wedge [i_\nu, d] \kappa_2, \quad \forall \kappa_1, \kappa_2 \in \wedge^* \mathcal{T}_X,$$
with \( i_\nu : \Omega^\bullet_X \to \Omega^{\bullet - 2}_X \) being the natural contraction with the 2-vector \( \nu \).

### 4.6. Example (deformations of holomorphic vector bundles)

Let \( E \to X \) be a holomorphic vector bundle on a complex manifold \( X \). The standard Lie algebra structure in the endomorphism sheaf \( \text{End}(E) \) extends naturally to \( \wedge^\bullet \mathcal{O}_X \text{End}(E) \cong \oplus \mathcal{O}_X[\text{End}(E)[1]] \) in such a way that \( (\wedge^\bullet \mathcal{O}_X \text{End}(E), \wedge, \cdot) \) becomes a sheaf of graded commutative Gerstenhaber algebras.

The dg commutative Gerstenhaber algebra controlling extended deformations of a given holomorphic structure in \( E \to X \) is given by

\[
g = \left( \Gamma(X, \wedge^\bullet \mathcal{O}_X \text{End}(E) \otimes \Omega^0_X \cdot), \lbrack \cdot \rbrack, \bar{\partial}, \circ \right)
\]

where \( \lbrack \cdot \rbrack = \lbrack \cdot \rbrack_E \otimes \text{wedge product of forms} \), and \( \circ = \text{wedge product of endomorphisms} \otimes \text{wedge product of forms} \).

### 4.7. Homotopy Gerstenhaber algebras

Let \( V \) be a graded vector space and \( (BV := \otimes^{\bullet \geq 0} V[1], \Delta) \) the free tensor coalgebra cogenerated by \( V[1] \).

#### 4.7.1. Definitions

(i) A \( B_\infty \)-algebra structure on a graded vector space \( V \) is the structure of dg bialgebra,

\[
(BV, \Delta, \circ : BV \otimes BV \to BV, d : BV \to BV),
\]
on the tensor coalgebra \( (BV, \Delta) \) such that the element \( 1 \in k = \otimes^0 V[1] \) is the identity element.

(ii) A homotopy Gerstenhaber algebra structure on a graded vector space \( V \) is a structure of \( B_\infty \)-algebra such that multiplication \( \circ \) preserves the filtration \( F_r := \otimes^{\bullet \leq r} V[1] \). We denote by \( h\mathcal{G} \) the operad whose algebras are homotopy Gerstenhaber algebras.

The \( h\mathcal{G} \)-algebra structure on \( V \) can be described by a collection of homogeneous maps,

\[
M_k : \otimes^k V \to V,
M_{1;k} : \otimes^{k+1} V \to V,
\]
satisfying a system of quadratic equations written explicitly in [Vo]. In particular, the operations \( M_2 \) and \( M_{1;1} \) induce on \( H(V, M_1) \) the structure of graded commutative Gerstenhaber algebra. Thus there is a canonical map of operads, \( p : h\mathcal{G} \to \mathcal{G} \).

#### 4.7.2. Example (higher order Steenrod operations)

Let \( S_\bullet X \) be the singular chain complex, of a topological space \( X \). Elements of \( S_n X \) are formal linear combinations of continuous maps, \( \sigma : \Delta[n] \to X \), from the standard \( n \)-simplex \( \Delta[n] \) to \( X \). For such a map \( \sigma \in S^n X \) and a \( k \)-face, \( f : \Delta[k] \to \Delta[n] \), of the standard simplex spanned by vertices \( n_0 = f(0), \ldots, n_k = f(k) \) (\( f \) being injective and monotone), denote by \( \sigma[n_0, \ldots, n_k] \in S_k X \) the associated composition,

\[
\sigma[n_0, \ldots, n_k] : \Delta[k] \xrightarrow{f} \Delta[n] \xrightarrow{\sigma} X.
\]

Let \( S^\bullet X := \text{Hom}_k(S_\bullet, k) \) be the associated cochain complex of \( X \). It was noted by Ger-
stenhaver and Voronov in [GeVo] using earlier results of Baues that the data,

\[ M_1(\phi)(\sigma) := \sum_{k=0}^{n+1} (-1)^k \phi([0, 1, \ldots, \hat{k}, \ldots, n]), \]

\[ M_2(\phi, \psi)(\sigma) := \sum_{k=0}^{n} \phi([0, \ldots, k])\psi([k, \ldots, n]), \quad \forall \phi, \psi \in S^*X, \]

\[ M_k := 0 \text{ for } k \geq 3, \]

\[ M_{1,k}(\phi_0; \phi_1, \ldots, \phi_n)(\sigma) := \phi_0(\sigma[0, n_1, n_1 + n_2, \ldots, n_1 + \ldots + n_k]) \]

\[ + \phi_1(\sigma[0, 1, \ldots, n_1, n_1 + n_2, \ldots, n_1 + \ldots + n_k]) \]

\[ \vdots \]

\[ + \phi_k(\sigma[n_1, \ldots, n_k]) \quad \forall \phi_0 \in S^kX, \phi_1 \in S^{n_1}X, \ldots, \phi_k \in S^{n_k}X, \]

make \( S^*X \) into an \( hG \)-algebra. The operation \( M_{1,1} \) is nothing but the Steenrod operation \( \cup_1 \).

**4.7.3. Example (deformations of associative algebras).** Let \( A \) be an associative algebra, and \( C^\bullet(A, A) := \text{Hom}_k(A^\bullet, A) \) its Hochschild complex with the differential,

\[ (d\phi)(a_1, \ldots, a_{n+1}) := a_1\phi(a_2, \ldots, a_{n+1}) \]

\[ + \sum_{k=1}^{n} (-1)^k \phi(a_1, \ldots, a_{k-1}, a_k, a_{k+1}, \ldots, a_{n+1}) \]

\[ \quad + (-1)^{k+1} \phi(a_1, a_2, \ldots, a_n)a_{n+1}. \]

It was shown in [GeVo] that the data,

\[ M_1 := d, \]

\[ M_2(\phi, \psi)(a_1, \ldots, a_{k+l}) := \phi(a_1, \ldots, a_k)\psi(a_{k+1}, \ldots, a_{k+l}) \quad \forall \phi \in C^k(A, A), \psi \in C^l(A, A) \]

\[ M_k := 0 \text{ for } k \geq 3, \]

and, for any \( \phi_0, \phi_1, \ldots, \phi_k \in C^\bullet(A, A), \)

\[ M_{1,k}(\phi_0; \phi_1, \ldots, \phi_n)(a_1, \ldots, a_m) := \]

\[ \sum_{\sigma} (-1)^{\sum_{p=1}^{n}(|\phi_\sigma|-1)i_p} \phi_0(a_1, \ldots, a_{i_1}, \phi_1(a_{i_1}, \ldots), \ldots, a_{i_n}, \phi_n(a_{i_n}, \ldots), \ldots, a_m) \]

where the summation runs over all possible ordered substitutions of \( \phi_1, \ldots, \phi_n \) into \( \phi_0 \), makes the Hochschild complex into an \( hG \)-algebra.

This \( hG \)-algebra controls deformations of the associative algebra structure in \( A \).

**4.8. Deligne’s conjecture.** The operations \( M_2 \) and \( M_{1,1} \) induce on the Hochschild cohomology, \( H^\bullet(A, A) \), the structure of \( G \)-algebra. Deligne conjectured that this action of the operad \( G \) on \( H^\bullet(A, A) \) can be lifted to the action of \( G_\infty \) on \( C^\bullet(A, A) \). This conjecture, which was recently proved in [Ko2, KoSo1, McSm, Ta, Vo], is essentially the same as the following statement.

**4.8.1. Theorem [Ta, TaTs, Vo].** There is a natural morphism of operads, \( f : G_\infty \to hG \), such that the diagram,

\[ \begin{array}{ccc}
G_\infty & \xrightarrow{f} & hG \\
q.-iso. \downarrow & & \downarrow p \\
\hat{G} & \end{array} \]

is commutative.
commutes.

4.9. Approximations to deformation theory. The cofibrant resolution, $DA_\infty$, of the operad $DA$ describing species Deformation Algebra could be so complicated that it would be unrealistic to ask for an immediate geometric interpretation of the homotopy class of minimal $DA_\infty$-algebra structures as in Step 4.

In Steps 3 and 4 one can therefore replace $DA_\infty$ by its “approximation”, a cofibrant operad $\overline{DA}_\infty$ fitting the commutative diagram

$$
\begin{array}{ccc}
\overline{DA}_\infty & \xrightarrow{i} & DA_\infty \\
\downarrow j & & \downarrow q-iso. \\
\mathcal{G} & \xleftarrow{\text{G-iso.}} & \end{array}
$$

for some natural morphisms of operads $i$ and $j$, $i$ being preferably a cofibration.

4.10. Approximations to $G_\infty$ deformation theory. As discussed in Sections 4.4-4.6, extended deformations of basic geometric structures are described by the operad $G_\infty$. In view of theorem 4.8.1, the same operad can be applied to the deformation theory of associative algebras.

At present we have no complete picture of the geometric object behind a homotopy class of minimal $G_\infty$-algebras. We appeal instead to the infinite tower of cofibrant approximations to $G_\infty$ introduced in Sect. 2.7,

$$
\mathcal{L}_\infty = G^{(1)}_\infty \to G^{(2)}_\infty \to G^{(3)}_\infty \to \ldots \to G^{(n)}_\infty \to \ldots \to G_\infty,
$$

and, in the rest of this paper, attempt to give such a picture for the first three floors of this tower using all the previous results. The ground floor, $G^{(1)}_\infty$, corresponds, in view of 2.7.2, to the forgetful functor 4.3 so that $G^{(1)}_\infty$-approximation to the deformation theory of examples 4.4-4.6 and 4.7.3 simply says that the extended moduli space is a formal minimal dg manifold $(M, \bar{\partial}, \ast, \bar{\partial})$. If its non-linear cohomology (see Introduction),

$$
M \simeq \frac{\text{Zeros}(\bar{\partial})}{\text{Im} \bar{\partial}},
$$

makes sense in some geometric category, it is precisely the versal moduli space associated with the $\mathbb{Z}$-graded extension of the classical deformation functor $\text{Def}_g$.

4.11. Theorem. There is a canonical isomorphism of operads,

$$
G^{(2)}_\infty = \text{Gerst}_\infty^c,
$$

where $\text{Gerst}_\infty^c$ is defined in Sect. 3.10.1.

Proof. The statement follows immediately from the definition 3.10.1 and the proof of Theorem 3.4.2 in [Me2] (see also Sect. 4.12 for a reconstruction of that argument). □

4.11.1. Corollary. If the operad $G_\infty$ acts on a dg vector space $(V,d)$, then the formal graded manifold associated with the cohomology vector space $H(V,d)$ is canonically a $\text{Gerst}_\infty^c$-manifold.

Proof. The statement follows from 4.11 and 3.9.2. □
4.11.2. Corollary [Me2]. Let $A$ be one of the structures 4.4-4.6 or 4.7.3. The extended moduli space of deformations of $A$ is naturally a $\text{Gerst}_{\infty}$-manifold, i.e. a dg manifold equipped with a homotopy class of geometric $C_{\infty}$-structures on the tangent bundle.

4.11.3. Notation. For a graded module $V$ over a graded ring $\mathcal{O}$ we set

$$\bigotimes_{\mathcal{O}} V := (\bigotimes_{\mathcal{O}} V[1]) [-1]$$

and

$$\bigotimes_{\text{shuffle products}} V := \left( \bigotimes_{\text{shuffle products}} V[1] \right) [-1].$$

4.12. Proof of Theorems A, B and E. The $G_{\infty}$-algebra structure on $V$ induces canonically a homotopy class of minimal $G_{\infty}$-algebra structures on its cohomology, $H(V,d)$. Let $\Theta$ be any representative of this homotopy class.

Let $\{\partial_{\alpha}, \alpha = 1, \ldots, \dim H(V,d)\}$ be a homogeneous basis of $H(V,d)[2]$ and $\{t^\alpha\}$ the associated dual basis of $H(V,d)^*[2]$. We identify the latter with coordinate functions on the formal manifold $(\mathcal{M}, \ast)$ associated with $H(V,d)[2]$, and the former with basis vector fields $\partial/\partial t^\alpha$. We can assume without loss of generality\(^{10}\) that degrees of all $\partial_{\alpha}$ vanish mod $2\mathbb{Z}$.

Let $t^{B_i}, i \geq 2$, stand for the Lie polynomial,

$$t^{B_i} := \left[ \ldots \left[ t^{\beta_1} \cdot t^{\beta_2} \cdot t^{\beta_3} \right] \ldots \cdot t^{\beta_i} \right],$$

and

$$\partial_{B_i} \equiv (\partial_{\beta_1} | \partial_{\beta_2}) \ldots | \partial_{\beta_i})$$

for the image of the tensor product,

$$\partial_{\beta_1} \otimes \partial_{\beta_2} \otimes \ldots \otimes \partial_{\beta_i},$$

under the degree $n - 1$ composition

$$\otimes^n H(V,d)[2] \to \otimes^n H(V,d)[2].$$

Note that under our assumption the parities of $t^{B_i}$ and $\partial_{B_i}$ are both equal to $(i - 1) \mod 2\mathbb{Z}$. Here and below we use $B_i$ to denote the multi-index $\beta_1 \beta_2 \ldots \beta_i$.

By Proposition 2.6.1, the $G_{\infty}$-algebra structure $\Theta$ is the same as the differential, $\delta$, of the Gerstenhaber algebra $\hat{\text{Lie}}(H(V,d)^*[2])$. The latter is uniquely determined on the generators,

$$\delta t^\alpha = \sum_{\substack{k \geq 0, n \geq 0 \\beta_1 \ldots \beta_k \beta_1 \ldots \beta_n \in k}} \frac{1}{n!} \Theta_{B_1 \ldots B_k \beta_1 \ldots \beta_n} \cdot t^{B_1} \cdots t^{B_k} t^{\beta_1} \cdots t^{\beta_n},$$

for some homogeneous constants $\Theta_{B_1 \ldots B_k \beta_1 \ldots \beta_n} \in k$ (see footnote 9). Here and below juxtaposition of $t^\ast$s means their symmetric product $\circledast$.

\(^{10}\) For example, we could opt to work in the category of graded manifolds over graded base spaces, “sources of $\mathbb{Z}$-graded constants”; then all the signs lost under our assumption in “natural over the base” calculations can be easily restored through the condition that the expression under study is functorial with respect to the base space change.
We re-arrange the above data into a smooth “vector field”,

\[ \partial := \sum_{n \geq 0} \frac{1}{n!} \Theta^\alpha_{0, \beta_1 \ldots \beta_n} t^{\beta_1} \ldots t^{\beta_n} \frac{\partial}{\partial t}, \]

and a collection of maps, for \( k \geq 2 \),

\[ \Theta^{[k]} : \odot_k^c (\otimes^c T_M) \rightarrow T_M \]

\[ \partial B_1 \odot \ldots \odot \partial B_k \rightarrow \sum_{n \geq 0} \frac{1}{n!} \Theta^\alpha_{B_1 \ldots B_k; \beta_1 \ldots \beta_n} t^{\beta_1} \ldots t^{\beta_n} \frac{\partial}{\partial t}. \]

Thus the defining equation for the differential \( \delta \) takes the form,

\[ \sum_\alpha (\delta t^\alpha) \partial_\alpha = \partial + \sum_{k \geq 1} \Theta^{[k]} (\partial B_1, \ldots, \partial B_k) t^{B_1} \ldots t^{B_k}. \]

The ideal \( I := \langle \operatorname{Lie}^2 V^*[-2] \rangle \) is generated by \( t^{B_i} \).

Using the identities,

\[ \delta f(t) = \sum_\alpha (\delta t^\alpha) \frac{\partial f}{\partial t_\alpha}, \]

and

\[ [t^\alpha \bullet f(t)] = \sum_\beta [t^\alpha \bullet t^\beta] \frac{\partial f}{\partial t^\beta}, \]

where \( f(t) \) is an arbitrary smooth function on \((M, *)\), it is not hard to study the transformation properties of the defined above fields \( \partial \) and \( \Theta^{[k]} \) under an arbitrary formal (non-linear) change of coordinates,

\[ t^\alpha \rightarrow \tilde{t}^\alpha = f^\alpha(t), \]

and conclude that

(i) \( \partial \) is indeed a vector field on \( M \);

(ii) the map \( \Theta^{[1]} \) factors through the composition

\[ \Theta^{[1]} : \otimes^c T_M \rightarrow \otimes^c T_M \rightarrow \otimes^c T_M, \]

i.e. represents a family of tensors, \( \mu_\bullet : \otimes^c T_M \rightarrow \otimes^c T_M \), vanishing on shuffle products;

(iii) the maps \( \Theta^{[k]} \) are sections of certain jet bundles (of order \( k - 1 \)) on \( M \).

A similar calculation shows that

(iv) the equation \( \sum_\alpha (\delta^2 t^\alpha) \partial_\alpha \equiv 0 \) mod \( I \) implies \( [\partial, \partial] = 0 \); this is essentially Lemma 2.7.2;

(v) the equation \( \sum_\alpha (\delta^2 t^\alpha) \partial_\alpha \equiv 0 \) mod \( I^2 \) implies that the data \( \Theta^{[1]} \simeq \mu_\bullet \) is nothing but a geometric \( C^\infty \)-structure on the dg manifold \((M, *, \partial)\); this is essentially Theorem 4.11.
To prove the statements A, B and E we have to move one level up and study the equation

\[ \sum_{\alpha} (\delta^2 t^\alpha) \partial_\alpha = 0 \mod I^3, \]

which now involves the non-tensorial object \( \Theta[2] \), a section of the bundle \( J^1(\otimes O_M T_M) \otimes O_M (\otimes O_M T_M) \otimes O_M T_M \). It is convenient for our purposes to choose a splitting of this bundle, say, an affine torsion-free affine connection \( \nabla \) on \( M \). Then we can replace non-tensorial objects

\[ t^B_n = \left[ \ldots [t^{\beta_1} \cdot t^{\beta_2} \cdot t^{\beta_3}] \ldots \cdot t^{\beta_n} \right], \]

by tensorial ones,

\[ \tilde{t}^B_n = \left[ \ldots \left( [t^{\beta_1} \cdot t^{\beta_2}]' \cdot t^{\beta_3} \right)' \ldots \cdot t^{\beta_n} \right]', \]

which are given recursively by

\[ [t^{\beta_1} \cdot t^{\beta_2}]' = [t^{\beta_1} \cdot t^{\beta_2}] \]

\[ [t^{\beta_1} \cdot t^{\beta_2}]' \cdot t^{\beta_3} = \left( [t^{\beta_1} \cdot t^{\beta_2}]' \cdot t^{\beta_3} + \Gamma^{\beta_3}_{\mu \nu} \circ [t^{\mu} \cdot t^{\beta_2}]' \circ [t^{\nu} \cdot t^{\beta_3}]' \right) \]

\[ + \Gamma_{\mu \nu}^{\beta_2} \circ [t^{\mu} \cdot t^{\beta_1}]' \circ [t^{\nu} \cdot t^{\beta_3}]' \]

\[ \ldots = \ldots \]

\[ \tilde{t}^B_n = \left[ \tilde{t}^{B_{n-1}} \cdot t^{\beta_n} + \sum_{i=1}^{n-1} \Gamma_{\mu \nu}^{\beta_i} \circ \tilde{t}^{B_{n-1}(\beta_i, \mu \nu)} \circ [t^{\mu} \cdot t^{\beta_n}]' \right], \]

where \( B_{n-1}(\beta_i, \mu) \) stands for the multi-index \( \beta_1 \ldots \beta_i \mu \beta_{i+1} \ldots \beta_{n-1} \) and \( \Gamma_{\mu \nu}^{\beta_i} \) are the Christoffel symbols of the connection \( \nabla \) in the coordinate system \( \{ t^\alpha \} \). Thus the defining equation for the differential \( \delta \) can be written in the form,

\[ \sum_{\alpha} (\delta t^\alpha) \partial_\alpha = \partial + \sum_{k \geq 1} \tilde{\Theta}^{[k]}(\partial_{B_1}, \ldots, \partial_{B_k}) \circ \tilde{t}^{B_1} \circ \ldots \circ \tilde{t}^{B_k}, \]

where all the coefficients, \( \Theta^{[k]} \), are tensors rather than sections of the jet bundles.

Now, to prove Theorems A and B it is enough to understand the first non-trivial component,

\[ [t^{\beta_1} \cdot t^{\beta_2}] \circ [t^{\gamma_1} \cdot t^{\gamma_2}], \]

of the formal power series \( \sum_{\alpha} (\delta^2 t^\alpha) \partial_\alpha \mod I^3 \). The only terms of \( \sum_{\alpha} (\delta t^\alpha) \partial_\alpha \mod I^3 \) which contribute to that component are written explicitly below,

\[ \sum_{\alpha} (\delta t^\alpha) \partial_\alpha = \partial + \sum (\tilde{\Theta}^{[1]}(\partial_{\beta_1} | \partial_{\beta_2}) \circ [t^{\beta_1} \cdot t^{\beta_2}] + \sum (\tilde{\Theta}^{[1]}(\partial_{\beta_1} | \partial_{\beta_2} | \partial_{\beta_3}) \circ [t^{\beta_1} \cdot t^{\beta_2} \cdot t^{\beta_3}]' \]

\[ + \sum (\tilde{\Theta}^{[2]}((\partial_{\beta_1} | \partial_{\beta_2}), (\partial_{\gamma_1} | \partial_{\gamma_2})) \circ [t^{\beta_1} \cdot t^{\beta_2}] \circ [t^{\gamma_1} \cdot t^{\gamma_2}] + \ldots \]

Applying to the shown terms the differential \( \delta \) and ignoring all components of the equation \( \sum_{\alpha} (\delta^2 t^\alpha) \partial_\alpha = 0 \mod I^3 \) except the chosen one, one gets an equation,

\[ [\mu_2, \mu_2] = Lie_\delta \mu_{2,2}, \quad (*) \]
where
\[ [\mu_2, \mu_2]^\nabla (X, Y, Z, W) = [\mu_2, \mu_2]_{HM} (X, Y, Z, W) \]
\[ - (\text{Lie}_S \mu_2)(W, \nabla X Z + \nabla_X Z Y) - (\text{Lie}_S \mu_3)(W, \nabla W X + \nabla_X W Y) \]
\[ - (\text{Lie}_S \mu_3)(W, \nabla Y Z + \nabla_Y Z X) - (\text{Lie}_S \mu_3)(Z, \nabla W Y + \nabla_Y W X) \]
\[ + \mu_3(W, (\text{Lie}_S \nabla)_Z X + (\text{Lie}_S \nabla)_X Z, Y) \]
\[ + \mu_3(W, (\text{Lie}_S \nabla)_Z X + (\text{Lie}_S \nabla)_X Z, Y) \]
\[ + \mu_3(W, (\text{Lie}_S \nabla)_Z Y + (\text{Lie}_S \nabla)_Y Z, X) \]
\[ + \mu_3(W, (\text{Lie}_S \nabla)_Z Y + (\text{Lie}_S \nabla)_Y Z, X) \],

\( \mu_2, \mu_3 \) and \( \mu_{2,2} \) are, respectively, the compositions,
\[
\mu_2 : \odot^2_{\nabla_M} T_M \xrightarrow{\sim} \odot^2_{\nabla_M} T_M \xrightarrow{\text{shuffle products}} \odot^2_{\nabla_M} T_M \xrightarrow{\text{shuffle products}} \Theta^3 T_M, \\
\mu_3 : \odot^3_{\nabla_M} T_M \xrightarrow{\sim} \odot^3_{\nabla_M} T_M \xrightarrow{\text{shuffle products}} \odot^3_{\nabla_M} T_M \xrightarrow{\text{shuffle products}} \Theta^3 \nabla T_M \\
\mu_{2,2} : \odot^2_{\nabla_M}(\odot^2_{\nabla_M} T_M) \xrightarrow{\sim} \odot^2_{\nabla_M} \odot^2_{\nabla_M} T_M \xrightarrow{\text{shuffle products}} \odot^2_{\nabla_M} T_M \xrightarrow{\text{shuffle products}} \Theta^3 T_M, \\
\]
\( [\mu_2, \mu_2]_{HM} \) is the Hertling-Manin bracket (see Sect. 0.1), and \( X, Y, Z, W \) are arbitrary smooth even (for simplicity of presentation) vector fields on \( M \). As the r.h.s. of the equation (\( \ast \)) is obviously a tensor, the l.h.s. must represent a tensor as well. (The original Hertling-Manin bracket can not possibly be a tensor without correction terms as the product \( \mu_2 \) is associative only up to homotopy.) Changing the affine connection,
\[ \nabla \rightarrow \nabla' = \nabla + S, \]
for some symmetric tensor \( S : \odot^2_{\nabla_M} T_M \rightarrow T_M \), we get
\[ [\mu_2, \mu_2]^\nabla' = [\mu_2, \mu_2]^\nabla - 2\text{Lie}_S \nu, \]
where
\[ \nu(X, Y, Z, W) = \mu_3(W, S(Z, X, Y)) + \mu_3(Z, S(W, X, Y)) + \mu_3(W, S(Z, Y, X)) + \mu_3(Z, S(W, Y, X)). \]
Thus the cohomology class, \( [\mu_2, \mu_2] \in H(\odot^2_{\nabla_M}(\odot^2_{\nabla_M} T_M)) \), associated with \( [\mu_2, \mu_2]^\nabla \) does not depend on the choice of \( \nabla \). The equation (\( \ast \)) implies Theorems A and B almost immediately.

To prove Theorem E one needs to understand a general structure of the components,
\[ \iota^{B_k} \odot \iota^{B_l}, \]
of the formal power series \( \sum_{\alpha} (\delta^2 \iota^\alpha) \partial_\alpha \mod I^3 \). Here are the terms of \( \sum_\alpha (\delta \iota^\alpha) \partial_\alpha \mod I^3 \) contributing to that component,
\[ \sum_{\alpha} (\delta \iota^\alpha) \partial_\alpha = \partial + \sum_{k \geq 2, B_k} \Theta^1(\partial_{B_k}) \odot \iota^{B_k} + \sum_{k, l \geq 2, B_k, B_l} \Theta^2(\partial_{B_k}, \partial_{B_l}) \odot \iota^{B_k} \odot \iota^{B_l} + \ldots \].

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It is not hard to see that the projection of the equation \( \sum_{\alpha} (\delta^2 t^\alpha) \partial_\alpha = 0 \mod I^3 \) to that component gives the equation of the form,

\[
[\mu_k(X_1, \ldots, X_k), \mu_l(Y_1, \ldots, Y_k)] + \text{correction terms} = (\text{Lie}_3 \mu_{k,l})(X_1, \ldots, X_k, Y_1, \ldots, Y_k),
\]

where the brackets stand for the standard commutator of vector fields, and \( \mu_k, \mu_l \) (\( \mu_{k,l} \)) are homogeneous components of \( \Theta^{[1]} \) (respectively, \( \Theta^{[2]} \)). Again, as the r.h.s. of the above equation is obviously a tensor, the l.h.s. must be a tensor as well. This tensor, \( [\mu_k, \mu_l] \), is \( \text{Lie}_3 \)-closed and, moreover, defines a vanishing cohomology class \( [[\mu_k, \mu_l]] \) of the complex of \( \mathcal{O}_M \)-modules, \( \otimes \mathcal{O}_M \mathcal{T}_M^* \otimes \mathcal{T}_M, \text{Lie}_3 \).

\[\square\]

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