On Streaming Algorithms for the Steiner Cycle and Path Cover Problem on Interval Graphs and Falling Platforms in Video Games

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Abstract
We introduce a simplified model for platform game levels with falling platforms based on interval graphs and show that solvability of such levels corresponds to finding Steiner cycles or Steiner paths in the corresponding graphs. Linear time algorithms are obtained for both of these problems. We also study these algorithms as streaming algorithms and analyze the necessary memory with respect to the maximum number of intervals contained in another interval. This corresponds to understanding which parts of a level have to be visible at each point to allow the player to make optimal deterministic decisions.

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1 Introduction

In 2D platform games it is a common game mechanism to include platforms that fall or break after the player visits them once. Additionally, it is often the case that the player has to collect certain items (coins, stars, ...) that are placed on some of these platforms and afterwards get back to the start or reach the exit of the level. Popular examples of video games that are (partially) based on these principles include Super Mario Bros., Donkey Kong Country and Super Mario Land1 (see Figure 1). We study solvability of levels based on these principles by introducing a toy model of such video games, in which all platforms (except for the target/starting point) have this falling property. The reachability between two platforms is modeled via an interval graph, which in many cases is a reasonable simplification. Then, the solvability of a level boils down to either finding a Steiner cycle or a Steiner path in the corresponding interval graph. To our knowledge, these problems have not been studied for

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1 Super Mario Bros., Donkey Kong Country and Super Mario Land are trademarks of Nintendo. Sprites are used here under Fair Use for educational purposes.
this specific graph class. The Hamiltonian cycle and Hamiltonian path problem, which are special cases of the Steiner variants, are extensively studied for interval graphs and can be solved in linear time, if the intervals are given as a right endpoint sorted list \([5, 1, 6]\).

In this work we generalize the algorithms of Manacher et al. \([6]\) to the Steiner setting and obtain first linear time algorithms for the Steiner path cover and Steiner cycle problem on interval graphs. A second important aspect when considering 2D game levels is the fact that the screen size is limited, so the whole level is not visible to the player at once. By studying our algorithms as single pass streaming algorithms we state precisely which parts of a level have to be visible to the player to deterministically decide how to play at each time. Alternatively, this can be interpreted as a memory bound for the streaming algorithms in terms of a natural graph parameter for interval graphs.

For a more general model for platform game levels based on intersection graphs of two dimensional boxes these problems are known to be NP-hard. Such graphs are generalizations of grid graphs for which already the Hamiltonian path problem is known to be NP-hard \([3]\).

## 2 Definitions and Preliminary Results

Given an interval \(i = [x, y]\) we denote the starting point \(x\) by \(l(i) = x\) and the endpoint \(y\) by \(r(i) = y\). Let \(I = (i_1, i_2, \ldots, i_n)\) be a list or set of intervals. We denote by \(G(I)\) the interval graph of \(I\). The vertices of this graph correspond to the intervals of \(I\). Two intervals \(i, i' \in I\) are connected by an edge in \(G(I)\) if \(i \cap i' \neq \emptyset\).

For an arbitrary graph \(G = (V, E)\) a list of vertices \(P = (i_1, i_2, \ldots, i_l)\) is a (simple) path if those vertices are pairwise distinct and for each \(j = 1, 2, \ldots, l - 1\) it holds that \(\{i_j, i_{j+1}\} \in E\). The start of \(P\) is denoted by \(\text{start}(P) = i_1\) and the end of \(P\) is denoted by \(\text{end}(P) = i_l\). We define \(\text{rev}(P)\) as the reverse path \((i_l, i_{l-1}, \ldots, i_1)\) of \(P\). If in addition \(\{i_l, i_1\} \in E\) we call \(P\)
a (simple) cycle. For ease of writing we sometimes abuse notation and consider $P$ as a set instead of a list, to allow for the use of set operations. Given two paths $P$ and $Q$ and a vertex $i$ we also write $(P, Q)$ for the concatenation of $P$ and $Q$ and $(P, i)$ for the concatenation of $P$ and $i$. Given a set $S \subseteq V$, a Steiner cycle is a cycle $C$ in $G$ such that $S \subseteq C$. A Steiner path cover of $G$ is a set $\{P_1, P_2, \ldots, P_k\}$ of paths in $G$ such that $S \subseteq \bigcup_{j=1}^{k} P_j$. The Steiner path cover number $\pi_S(G)$ is the the minimum cardinality of a Steiner path cover. If $\pi_S(G) = 1$ we say that $G$ has a Steiner path. A set $C \subseteq V$ is called a cutset of $G$ if $G - C$ is disconnected. A set of vertices $T \subseteq V$ is called an island with respect to $C$, if $T$ is not adjacent to any vertex in $V \setminus (C \cup T)$. $T$ is called an $S$-island with respect to $C$, if $T$ is an island with respect to $C$ and $S \cap T \neq \emptyset$.

The following two results are generalizations of two observations by Hung and Chung [3], easily verified by the pigeonhole principle.

- **Proposition 1.** Let $C$ be a cutset of $G$ and $g_S$ the number of connected components $K$ in $G - C$ such that $K \cap S \neq \emptyset$. Then, $\pi_S(G) \geq g_S - |C|$.

- **Proposition 2.** Let $C$ be a cutset of $G$ and $g_S$ the number of connected components $K$ in $G - C$ such that $K \cap S \neq \emptyset$. If $g_S > |C|$, then $G$ has no Steiner cycle.

We use these results to solve the Steiner path cover problem (see Section 3) and the Steiner cycle problem (see Section 4) on interval graphs efficiently. In the whole paper we assume that $|S|$ is known to the algorithms and queries $i \in S$ can be performed in $O(1)$ time.

### 3 The Steiner Path Cover Problem

We show that the basic greedy principle, that is the core of efficient algorithms for the path cover problem on interval graphs, can be generalized by the introduction of neglectable intervals. The basic greedy principle to find paths in interval graphs was introduced independently by Manacher et. al [6] and Arikati et al. [1].

Given a right endpoint sorted list of interval $i_1, i_2, \ldots, i_n$ the algorithm iteratively constructs a path $P$. It starts with the path $P := (i_1)$ containing only the first interval. Then it repetitively extends $P$ by the neighbor of end($P$) not in $P$ with minimum right endpoint. If no such extension is possible the algorithm terminates with the current path $P$ as an output. We denote this algorithm by $\text{GP}$ and the path $P$ obtained by this algorithm by $\text{GP}(I)$.

For a path $P = \text{GP}(I) = (i_1, i_2, \ldots, i_l)$ obtained by the algorithm if executed on an interval graph $G(I)$, we define $L(P)$, the set of intervals that exceed beyond the right endpoint of the end of $P$, i.e. $L(P) = \{i \in P : r(i) > r(\text{end}(P))\}$. Based on this we recursively define $C(P)$, the set of covers of the path $P$. If $L(P) = \emptyset$, we also set $C(P) = \emptyset$. Otherwise, let $j$ be the maximum index such that $i_j \in L(P)$. We set $C(P) = \{i_j\} \cup C(P')$ for $P' = (i_1, i_2, \ldots, i_{j-1})$.

For $C(P) = \{c_1, c_2, \ldots, c_k\}$ and $P = (P_0, c_1, P_1, c_2, \ldots, c_k, P_k)$ Manacher et al. [6] proved that for each $j = 0, 1, \ldots, k$ it holds that $P_j$ is an island with respect to $C(P)$ and if $I \setminus P = \emptyset$ also $I \setminus P$ is an island with respect to $C(P)$. We call such a decomposition of $P$ a decomposition into covers and islands.

Manacher et al. [6] also observed the following important properties of a decomposition into covers and islands.

- **Proposition 3.** Let $P = \text{GP}(I) = (i_1, i_2, \ldots, i_l)$.
  1. If $i_j \in C(P)$ it holds that $r(i_j) > r(i_{j+1})$.
  2. If $P = (P_0, c_1, P_1, c_2, \ldots, c_k, P_k)$ is a decomposition into covers and islands it holds that $L(P_j) = \emptyset$ for each $j = 0, 1, \ldots, k$. 


To illustrate the notions introduced above, consider the intervals in Figure 2 given as a right endpoint-sorted list \( I = (i_1, i_2, \ldots, i_{12}) \). Algorithm \( \mathbf{GP} \) starts by setting \( P = (i_1) \).

![Figure 2 An interval model I of twelve endpoint-sorted intervals.](image)

Neighbors of \( i_1 \) are \( \{i_2, i_4, i_6\} \), and since \( r(i_2) < \min\{r(i_4), r(i_6)\} \) we extend \( P \) by \( i_2 \), i.e. \( P = (i_1, i_2) \). Among neighbors of \( i_2 \) that are not already in \( P \), \( i_3 \) has the smallest right endpoint, so \( P \) is extended to \( P = (i_1, i_2, i_3) \). Next candidates for the extension are \( \{i_4, i_6\} \) among which we chose \( i_4 \), i.e. \( P = (i_1, i_2, i_3, i_4) \). Next, the only possible extension is by \( i_6 \), hence \( P = (i_1, i_2, i_3, i_4, i_6) \). Among the next candidates for extension \( \{i_5, i_{10}\} \), interval \( i_5 \) is chosen. At this point the algorithm terminates and outputs \( P = (i_1, i_2, i_3, i_4, i_6, i_5) \), since there is no neighbor of \( i_5 \) that is not already in \( P \).

Now we find a decomposition into covers and islands of \( P \). Since \( r(i_6) > r(\text{end}(P) = i_5) \), we have that \( L(P) = \{i_6\} \), and \( C(P) = \{i_6\} \cup C(P^r = (i_1, i_2, i_3, i_4)) \). \( L(P^r) \) is the empty set, so the decomposition process is over and we have that the decomposition into covers and islands of \( P \) is given by \( C(P) = \{i_6\} \) and \( \mathbf{GP} = (P_0, i_6, P_1) \), where \( P_0 = (i_1, i_2, i_3, i_4) \) and \( P_1 = (i_6) \). Note that \( P_0, P_1 \) and \( I \setminus P \) are islands with respect to \( C(P) = \{i_6\} \). Furthermore, note that our decomposition satisfies the properties in Proposition 4.

Given the fact that in the Steiner variant of the problem only the intervals in \( S \) have to be visited, we introduce neglectable intervals. Let \( P \) be the current path at any point of the algorithm and \( i' \) be the next extension. We call \( i' \) neglectable with respect to \( \text{end}(P) \), if \( i' \notin S \) and \( r(i') < r(\text{end}(P)) \), i.e. \( \text{end}(P) \in L((P, i')) \). We modify the algorithm \( \mathbf{GP} \) such that it skips neglectable intervals with respect to the end of the current path. Analogously to \( \mathbf{GP} \) this modification is denoted by \( \mathbf{GP}_S \). We define the set \( N_i \) of intervals that are not contained in \( \mathbf{GP}_S(I) \) since they are neglectable with respect to \( i \) for some path \( P \) during the execution of \( \mathbf{GP}_S \), where \( i = \text{end}(P) \). We denote by \( N \) the set of all such neglectable intervals obtained during the entire run of \( \mathbf{GP}_S \).

**Proposition 4**. Let \( P = \mathbf{GP}_S(I) \) be the path obtained by \( \mathbf{GP}_S \) for a given list of intervals \( I \) and \( P = (P_0, c_1, P_1, c_2, \ldots, P_{k-1}, c_k, P_k) \) its decomposition into covers and islands in \( G(I \setminus N) \). Let \( C(P) = \{c_1, c_2, \ldots, c_k\} \), then it holds for all \( j = 0, 1, \ldots, k \) that \( P_j \cap S \neq \emptyset \), i.e. \( P_j \) is an \( S \)-island with respect to \( C(P) \) in \( G(I \setminus N) \). It even holds that \( P_j \cup N_{c_j} \) contains at least one \( S \)-island with respect to \( C(P) \) in \( G(I) \).

**Proof**. It is easy to see that this decomposition into covers and islands exists, since if \( P = \mathbf{GP}_S(I) \) it follows by construction that \( P = \mathbf{GP}(I \setminus N) \).

The fact that \( P_j \) is an \( S \)-island with respect to \( C(P) \) in \( G(I \setminus N) \) is a trivial consequence of the decomposition into covers and islands. Since \( c_j \) is used before every interval in \( N_{c_j} \) we have that the left endpoint of every interval in \( N_{c_j} \) is larger than the left endpoint of \( c_j \). The right endpoints of each of those intervals is smaller than the right endpoint of \( c_j \) by definition of neglected intervals. But this directly implies that \( C(P) \) separates also \( N_{c_j} \) from the rest of \( G(I) \), except for possibly \( P_j \).
Based on this we can obtain an easy procedure to solve the Steiner path cover problem on interval graphs. We start with \( P = \emptyset \) and apply the algorithm \( \text{GP}_S \). After termination let \( P = \text{GP}_S(I) \). We add \( P \) to our partial solution \( \mathcal{P} \) and find the smallest index \( j \) such that \( i_j \in S \) and \( i_j \) is not in any path currently contained in \( \mathcal{P} \). Then we apply \( \text{GP}_S \) again to the list of intervals \( i_j, i_{j+1}, \ldots, i_n \), until all intervals in \( S \) are covered by one of the paths in \( \mathcal{P} \). The algorithm terminates with the Steiner path cover \( \mathcal{P} \) as its output.

| Theorem 5. | The Steiner path cover obtained by iterated application of \( \text{GP}_S \) is optimal. |

**Proof.** Let \( P_1, P_2, \ldots, P_l \) be the paths obtained by the given algorithm and \( C' = \bigcup_{j=1}^{l} C(P_j) \) be the union of all the covers in the decomposition into covers and islands of each path. Then, by repeated application of Lemma 4 we obtain that there are \( l + |C'| \) S-islands with respect to \( C' \) in \( G(I) \). By Proposition 1 we then know that \( \pi_S(G(I)) = l \), so our solution is an optimal Steiner path cover.

To illustrate our algorithm for the Steiner path cover problem we again consider the example in Figure 2. In the case when \( S = I \), i.e., all intervals need to be covered, our algorithm runs \( \text{GP}_S(I) \) which outputs \( P^* = (i_1, i_2, i_3, i_6, i_5) \), and then it runs \( \text{GP}_S(I \setminus P^*) \) which outputs \( P'' = (i_7, i_8, i_9, i_{10}, i_{12}, i_{11}) \), and the algorithm terminates. Therefore, for \( S = I \) we have that \( \pi_S(I) = 2 \). Now let's say that \( S = \{i_2, i_4, i_6, i_8, i_{10}, i_{12}\} \). \( \text{GP}_S(I) \) starts with the element of \( S \) with the smallest right endpoint which is \( i_2 \). Then it extends the path with \( i_2, i_4 \) and then \( i_6 \). After that, the algorithm neglects \( i_5 \) since \( r(i_5) < r(i_6) \) and \( i_5 \notin S \). Next, the path is extended by \( i_{10} \), then \( i_7 \) is neglected, but \( i_8 \) is added to the path (since \( i_8 \in S \)). Then the path is extended by \( i_9 \) and finally by \( i_{12} \). Interval \( i_{11} \) is neglected. The output of the algorithm is the path \( P = (i_2, i_3, i_4, i_6, i_{10}, i_8, i_9, i_{12}) \), so \( \pi_S(I) = 1 \). Note that the key factor that allowed us to cover the set \( S \) with only one path is the fact that we could neglect \( i_5 \).

By using the Deferred-queue approach by Chang et al. [2] this algorithm can be implemented in \( O(n) \) time.

4 The Steiner Cycle Problem

To solve the Steiner cycle problem we first run our algorithm for the Steiner cover problem (see Section 3). If \( \pi_S > 1 \) we know that there cannot exist a Steiner cycle. Otherwise, let \( P = (i_1, i_2, \ldots, i_l) \) be the obtained Steiner path in \( G(I) \).

Based on \( P \) we construct two paths \( Q \) and \( R \). We start by setting \( R = (i_1) \) and \( Q = (i_2) \). Then, we iteratively process the intervals \( i_3 \) to \( i_n \). If in the step of processing interval \( i_j \) we have that \( \text{end}(Q) = i_{j-1} \), we consider the following two cases. If \( i_j \cap \text{end}(R) \neq \emptyset \), we extend \( R \) by \( i_j \), i.e. \( R = (R, i_j) \). Otherwise, we extend \( Q \) by \( i_j \), i.e. \( Q = (Q, i_j) \). If on the other hand in this step we have that \( \text{end}(R) = i_{j-1} \) we check symmetrically if \( i_j \cap \text{end}(Q) \neq \emptyset \). If this is the case we extend \( Q \) by \( i_j \) and if not we extend \( R \) by \( i_j \).

If in the end of this process \( \text{end}(Q) = i_1 \) and \( \text{end}(R) = i_l \), or vice versa, we try to connect \( Q \) and \( \text{rev}(R) \) to a Steiner cycle. To achieve this we check if \( \text{end}(Q) \) and \( \text{end}(R) \) are directly connected, i.e. \( \text{end}(Q) \cap \text{end}(R) \neq \emptyset \), or if there is an interval \( i' \) among the intervals \( I' \subseteq I \), whose right endpoints \( r(i') > i_j \) for all \( j = 1, 2, \ldots, l \) such that both \( \text{end}(Q) \cap i' \neq \emptyset \) and \( \text{end}(R) \cap i' \neq \emptyset \). In any of those two cases we can connect \( Q \) and \( \text{rev}(R) \) to a Steiner cycle. Otherwise, the algorithm returns that no Steiner cycle exists.

| Theorem 6. | The given algorithm correctly decides the existence of a Steiner cycle in \( G(I) \) and obtains such a cycle if possible. |
Proof. If the algorithm finds a Steiner cycle this is obviously true. Also, by correctness of the algorithm for the Steiner path cover (Theorem 5), if no Steiner path is found we correctly determine that no Steiner cycle can exist.

Otherwise, let us assume that the algorithm did not find a Steiner cycle. Without loss of generality, let \( \text{end}(R) = i_h \) with \( h < l - 1 \) and consider the path \( P' = (i_1, i_2, \ldots, i_h) \) and its decomposition into covers and islands. Since \( R \) was not extended by any of the intervals \( i_{h+2}, i_{h+3}, \ldots, i_n \), we have that \( C(P') \cup \{i_{h+1}\} \) separates the islands of \( P' \) from \( \{i_{h+2}, i_{h+3}, \ldots, i_l\} \). In addition since \( \text{end}(R) \) and \( \text{end}(Q) \) could not be connected with any interval in \( I' \) it holds for all interval \( i' \in I' \) that \( 1(i') > r(i_h) \). Combining this with point 2 of Proposition 3 we observe that \( \{i_{h+2}, i_{h+3}, \ldots, i_l\} \cup I' \) is non-empty and an \( S \)-island with respect to \( C(P') \cup \{i_{h+1}\} \).

By Lemma 4 there are at least \( |C(P')| + 1 \) \( S \)-island with respect to \( C \cup \{i_{h+1}\} \). So, by Proposition 2 there does not exist a Steiner cycle in \( G(I) \).

Given a Steiner path \( P \), the paths \( Q \) and \( R \) can be easily constructed in \( O(n) \) time. This gives a linear time algorithm for the Steiner cycle problem in interval graphs.

Now we illustrate our algorithm for the Steiner cycle problem on interval graphs with the example given in Figure 3. The given instance has 10 intervals \( I = \{i_1, i_2, \ldots, i_{10}\} \) and \( S = \{i_2, i_5, i_8\} \). Intervals in \( S \) are represented with the red color. First we run \( \text{GP}_S(I) \).

\[ \hspace{2cm} \text{Figure 3 An instance of the Steiner cycle problem on an interval graph with } S = \{i_2, i_5, i_8\}. \]

It starts the path with \( i_2 \) and then extends it with \( i_3 \) and \( i_5 \) before neglecting \( i_4 \). Then it proceeds by extending the path with \( i_6 \), \( i_7 \), finishing with \( i_8 \). Hence it obtains the Steiner path \( P = (i_2, i_3, i_5, i_6, i_7, i_8) \). In an attempt to create a Steiner cycle, we partition \( P \) into two paths \( R \) and \( Q \). We initialize them with the first two intervals in \( P \), that is, \( R = (i_2) \) and \( Q = (i_3) \). Now we consider \( Q \) to be the current path, and \( R \) to be the previous path. In each step we consider the next interval of \( P \), and in the case that it intersect the end of the previous path, we extend the previous path and make it the current path. Otherwise we add the interval to the current path. So, interval \( i_5 \) is the next interval in \( P \), and it does not intersect \( \text{end}(R) = i_2 \), hence we add it to \( Q \), making it \( Q = (i_3, i_5) \). The next interval is \( i_6 \), and it intersects \( \text{end}(R) = i_2 \), hence we extend \( R \) and make it the current path, so \( R = (i_2, i_6) \). Next interval \( i_7 \) does not intersect \( \text{end}(Q) = i_5 \) so we extend \( R \), making it \( R = (i_2, i_6, i_7) \). Finally, interval \( i_8 \) does not intersect \( \text{end}(Q) = i_5 \) so we extend \( R \), making it \( R = (i_2, i_6, i_7, i_8) \). This ends our partition of \( P \) with the resulting subpaths \( R = (i_2, i_6, i_7, i_8) \) and \( Q = (i_3, i_5) \). Since \( \text{end}(R) = i_8 \) and \( \text{end}(Q) = i_5 \) do not intersect, we cannot connect them into a cycle. The only remaining chance to do so is using an interval from \( I' = \{i \in I' \cap P : r(i) > r(\text{end}(P))\} = \{i_9, i_{10}\} \). Luckily, \( i_9 \) intercept both \( \text{end}(R) = i_8 \) and \( \text{end}(Q) = i_5 \), and can be used to connect \( R \) and \( Q \) into a cycle. The Steiner cycle is then given by \( (R, i_9, \text{rev}(Q)) = (i_2, i_6, i_7, i_8, i_9, i_5, i_3) \).

Now let us consider a modified instance of Figure 3 where \( i_4 \) is also an element of \( S \). Then \( \text{GP}_S(I) \) would output the path \( P = (i_2, i_3, i_5, i_4, i_6, i_7, i_8) \), and the subsequent partition of
$P$ would give $R = (i_2, i_6, i_7, i_8)$ and $Q = (i_3, i_5, i_4)$. But now there is no interval in $I'$ that connects $\text{end}(R) = i_8$ and $\text{end}(Q) = i_4$, so our algorithm outputs that there is no Steiner cycle. In order to verify that there is no Steiner cycle we can follow the arguments in the proof of Theorem 6, which gives us a cutset $C = \{i_5, i_6\}$ that separates $I$ into three $S$-islands, and hence, by Proposition 2 guarantees that there is no Steiner cycle.

5 Streaming Algorithms – The Problem of Limited Screen Size

An important question when considering solvability of game levels is which parts of a level have to be visible to the user at any time for them to deterministically know how to play correctly. To answer this question for our toy model, we study the algorithms from Section 3 and 4 as streaming algorithms. We assume that the input stream is presented as a sequence of right endpoint sorted intervals which can only be examined in one pass. As its output the streaming algorithm has to write the list of intervals giving the paths or cycle.

First, consider the algorithm $\text{GP}_S$. In each step this algorithm needs access to the next interval on the stream that is connected with the current path $\text{end}(P)$. If the next interval $i$ on the stream is not connected to $\text{end}(P)$ there can be two reasons. This interval could either be in a new different connected component than $P$, or it could be connected to $P$ via another interval $i'$ with $r(i') > r(i)$. Intervals of this kind are all completely contained in $i'$. After processing and storing all such intervals we clearly know whether the graph is disconnected or the path $P$ can be extended and we can further process the stored intervals. This motivates the introduction of the parameter $\kappa(I)$, the maximum number of intervals contained in another. Based on this parameter we observe that $\text{GP}_S$ can be implemented as a single pass streaming algorithm with $O(\kappa(I))$ additional storage. Based on this we obtain the following result.

Theorem 7. Given $\kappa(I)$ the Steiner path cover problem on interval graphs can be solved by a single pass streaming algorithm in $O(n)$ time with $O(\kappa(I))$ additional storage.

Remark. If $\kappa(I)$ is not known to the algorithm the same result only holds assuming $G(I)$ is connected. Otherwise in the case of a disconnected interval graph the algorithm can not decide after $O(\kappa(I))$ steps that the graph is disconnected. It has to continue to store the intervals from the stream till the end, because there is no way of knowing if a future interval will be connected to $\text{end}(P)$ for the current path $P$.

On the other hand if $\kappa(I)$ is known we can stop this process after storing $\kappa(I)$ intervals since we know that no more of them can be contained in another interval and terminate with the current path $P$.

To solve the Steiner cycle problem, a single pass streaming algorithm can no longer first run $\text{GP}_S$ and then construct the two paths $Q$ and $R$, since this would need two passes. Also the output of the cycle is only possible in a single pass, without a large amount of additional memory, if the two paths $Q$ and $R$ are accepted as an output instead of the list for the Steiner cycle. In the application to platform games this is not a problem since here a player actually is doing first a pass from the left to the right and then another pass from the right to the left. So correct construction of $Q$ during the first pass is enough to guarantee the possibility of getting back to the exit later. This path for the way back can then be easily found doing a simple greedy approach (see the description in the end of the current section). The construction of $Q$ and $R$ can be incorporated into the streaming variant of $\text{GP}_S$ described above without the need for additional memory. In addition to $\text{end}(P)$ we also store $\text{end}(Q)$ and $\text{end}(R)$. This way in each step of the algorithm we can decide whether the next
interval extending $P$ should be appended to $Q$ or $R$, by the same method as explained in Section 4. This only needs additional memory for storing both $\text{end}(Q)$ and $\text{end}(R)$ compared to just executing $\text{GP}_S$.

**Theorem 8.** Given $\kappa(I)$ the Steiner cycle problem on interval graphs can be solved by a single pass streaming algorithm in $O(n)$ time with $O(\kappa(I))$ additional storage.

It is important to note that from the view of a player the additional storage in the streaming algorithms does not correspond to storage needed to decide the next step of the game but to the range of the level that has to be visible to the player. It covers the fact that the player has to be able to see at least the next two intervals reachable from its current position and all the intervals before that in a right endpoint sorted order. The two things a player needs to remember at each point of the game are $\text{end}(Q)$ the platform it is currently on and $\text{end}(R)$. The algorithm can also be simplified in the following way.

Assume the player is currently located on the interval $\text{end}(Q)$. There are two possible cases. In the first case the last step was jumping onto $\text{end}(Q)$. Let $i$ be the interval reachable from $\text{end}(Q)$ with $r(i)$ minimum, such that $i$ is not neglectable with respect to $\text{end}(Q)$. If $i \cap \text{end}(R) \neq \emptyset$ we extend $R$, so the player remembers $\text{end}(R) = i$. Otherwise the player jumps to $i$, so $\text{end}(Q) = i$. If neither is possible the current level is unsolvable. In the second case the last step was an extension of $R$, so $\text{end}(R)$ was updated. Let $i$ be the interval reachable from end($R$) with $r(i)$ minimum, such that $i$ is not neglectable with respect to $\text{end}(R)$. If $i \cap \text{end}(Q) \neq \emptyset$ the player jumps to $i$, so $\text{end}(Q) = i$. Otherwise we extend $R$ so the player remembers that $\text{end}(R) = i$. If neither is possible the current level is also unsolvable. If the last interval in $S$ is either visited by the player, i.e. is equal to $\text{end}(Q)$ or reached by $R$, i.e. is equal to $\text{end}(R)$ the player tries to reach $\text{end}(R)$ from $\text{end}(Q)$ by jumping there directly or using an interval $i'$ with $r(i') > \max\{r(\text{end}(Q)), r(\text{end}(R))\}$. If this is not possible the player determines that the current level is unsolvable. Otherwise it can easily get back to the exit visiting all the unvisited intervals in $S$ by reconstructing a maybe permuted version of the path rev($R$). Let $i$ be the interval the player is currently on. In each step it can greedily jump to the reachable interval $i'$ with maximum left endpoint $l(i')$, such that $i'$ is not neglectable with respect to $i$ in the reverse sense. This means we can neglect jumping to $i'$ if $l(i) < l(i')$ and $i' \notin S$. This is just an application of $\text{GP}_S$ in reverse direction. Since the path $R$ exists, by the optimality of $\text{GP}_S$ for the path cover problem, using this strategy the player finds a path $R'$ covering all intervals in $S$ and returning to the start of the level.

## 6 Conclusion

We obtained linear time algorithms for both the Steiner path cover problem and the Steiner cycle problem, assuming the intervals are given as a right endpoint sorted list. We also analyzed those algorithms as single pass streaming algorithms to study solvability of a simplified model for platform game levels.

Our simplification reduced those levels to a one-dimensional interval graph model. The hamiltonian cycle and path problems for two-dimensional generalizations of interval graphs are known to be NP-hard. It would be of interest to study special cases of these problems inspired from game levels. Furthermore the analysis of streaming algorithms for interval graphs is a natural extension to classic algorithms for interval graphs. Understanding other efficient algorithms for different problems on interval graphs in this model is a very interesting area for further research.
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