Fourier transform of function on locally compact Abelian groups taking value in Banach spaces *

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Abstract

We consider Fourier transform of vector-valued functions on a locally compact group $G$, which take value in a Banach space $X$, and are square-integrable in Bochner sense. If $G$ is a finite group then Fourier transform is a bounded operator. If $G$ is an infinite group then Fourier transform $F: L_2(G, X) \to L_2(\hat{G}, X)$ is a bounded operator if and only if Banach space $X$ is isomorphic to a Hilbert one.

1 Fourier transform over groups $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{T}$

In the paper [1] J. Peetre proved an extension of Hausdorff–Young’s theorem describing image of $L_q(\mathbb{R})$ under Fourier transform. He considered vector-valued $x \in L_q(\mathbb{R}, X), 1 \leq q \leq 2$ on the real axis taking value in Banach space $X$, and integrable in Bochner sense, i.e. weakly measurable with finite norm

$$||x||_{L_q(\mathbb{R}, X)} = \left( \int_\mathbb{R} ||x(t)||_X^q dt \right)^{1/q}.$$  

J. Peetre noted, that for $q = 2$ in all known to him cases Fourier transform

$$\mathcal{F}: L_2(\mathbb{R}, X) \to L_2(\mathbb{R}, X), \quad (\mathcal{F}x)(s) = \int_\mathbb{R} x(t)e^{-2\pi ist} dt.$$  

was bounded only if $X$ was isomorphic to a Hilbert space. In [3] Polish mathematician S. Kwapien in fact proved the following

**Theorem 1** Statements below are equivalent:
1) Banach space $X$ is isomorphic to a Hilbert one.
2) There exists $C > 0$ such that for any positive integer $n$ and $x_0, x_1, x_{-1}, \ldots, x_n, x_{-n} \in X$

$$\int_0^1 \left\| \sum_{k=-n}^{n} e^{2\pi ikt} \cdot x_k \right\|^2 dt \leq C \sum_{k=-n}^{n} \|x_k\|^2.$$  

3) There exists $C > 0$ such that for any positive integer $n$ and $x_0, x_1, x_{-1}, \ldots, x_n, x_{-n} \in X$

$$C^{-1} \sum_{k=-n}^{n} \|x_k\|^2 \leq \int_0^1 \left\| \sum_{k=-n}^{n} e^{2\pi ikt} \cdot x_k \right\|^2 dt.$$  

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4) Fourier transform $\mathcal{F}$ initially defined on a dense subspace of simple functions $D_{\mathcal{F}} \subset L_2(\mathbb{R}, X)$,

$$D_{\mathcal{F}} = \left\{ x(t) = \sum_{k=1}^{n} I_{A_k}(t) \cdot x_k \right\},$$

is a bounded operator. Here $A_k$ are disjoint subset of finite measure in $\mathbb{R}$, $I_{A_k}$ are indicators (i.e. functions equal to 1 on $A_k$ and to 0 elsewhere), $x_k$ are vectors in $X$.

Let us define Fourier transform of a vector-valued function over integers $\mathbb{Z}$ by

$$\mathcal{F}_{\mathbb{Z}} : L_2(\mathbb{Z}, X) \to L_2(\mathbb{T}, X) : (x_k) \mapsto \sum_{k \in \mathbb{Z}} e^{-2\pi i k t} \cdot x_k,$$

Here $\mathbb{T}$ denotes a one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, which is isomorphic to $[0, 1]$ with the length in the category of spaces with a measure.

Statement 2) in Theorem means that $\mathcal{F}_{\mathbb{Z}} I_{\mathbb{Z}}$ is a bounded operator (here $I_{\mathbb{Z}}$ denotes the operator of changing variable $I_{\mathbb{Z}} : (x_k) \mapsto (x_{-k})$, which is an isometry). That is why $\mathcal{F}_{\mathbb{Z}}$ is bounded on a dense subspace of $L_2(\mathbb{Z}, X)$ consisting of compactly-supported functions, and can be continued to the whole $L_2(\mathbb{Z}, X)$.

Statement 3) in Theorem means that inverse Fourier transform

$$\mathcal{F}_{\mathbb{Z}}^{-1} : L_2(\mathbb{T}, X) \to L_2(\mathbb{Z}, X)$$

is a bounded operator.

## 2 Generalization and necessary facts

Fourier transform of Banach space-valued functions on a group different from $\mathbb{R}, \mathbb{Z}, \mathbb{T}$ (namely on additive group of $p$-adic field $\mathbb{Q}_p$) was considered in [4].

Now it’s natural to look at the general case of arbitrary locally compact group $G$. We consider functions on $G$ taking value in Banach space $X$, that are square-integrable in Bochner sense, and Fourier transform

$$\mathcal{F} \equiv \mathcal{F}_G : L_2(G, X) \to L_2(\hat{G}, X), \quad (\mathcal{F}x)(\xi) = \int_G \langle \xi, t \rangle_G x(t) d\mu_G(t).$$

Here $\hat{G}$ is Pontryagin dual to $G$ (group of characters), $\langle \xi, t \rangle_G$ is the canonical pairing between $\hat{G}$ and $G$, $\mu_G$ is Haar measure.

First we recall some necessary results. We refer to [5] for results in harmonic analysis, and to [6] for structure theory of locally compact groups, Bruhat–Schwartz theory is exposed in [7].

Fix a dual Haar measure $\mu_{\hat{G}}$ on $\hat{G}$ such that scalar Steklov–Parseval’s equality holds

$$||\varphi||^2_{L_2(\hat{G})} = \int_G |\varphi|^2 d\mu_G = \int_{\hat{G}} |\mathcal{F}\varphi|^2 d\mu_{\hat{G}} = ||\mathcal{F}\varphi||_{L_2(\hat{G})}.$$ 

Let $\mathcal{S}(G)$ denote Bruhat–Schwartz space of “smooth fastly decreasing” functions on $G$. It’s useful to take a dense subspace

$$D_{\mathcal{F}} = L_2(G) \otimes X \subset L_2(G, X),$$
as initial domain of $\mathcal{F}$, where it acts by

$$\mathcal{F}\left(\sum_{k=1}^{n} \varphi_k(t) \cdot x_k\right) = \sum_{k=1}^{n} ((\mathcal{F}\varphi_k)(\xi) \cdot x_k).$$

To show denseness of $D_{\mathcal{F}}$ and denseness of $\mathcal{S}(G) \otimes X \subset L_2(G) \otimes X$ consider indicator $I_A$ of arbitrary measurable subset of finite measure (i.e. functions equal to 1 on $A$ and to 0 elsewhere). Clearly, $I_A \in L_2(G)$, and it can be approximated by elements of $\mathcal{S}(G)$ using convolution with any $\delta$-net consisting of Bruhat–Schwartz functions. By definition of $L_2(G, X)$ finite linear combinations

$$\sum_{k=1}^{n} I_{A_k}(t) \cdot x_k \in L_2(G) \otimes X \equiv D_{\mathcal{F}}$$

are dense in $L_2(G, X)$. Here $A_k$ are measurable disjoint subsets of finite measure in $G$, $I_{A_k}$ are indicators, and $x_k \in X$.

Due to the fact that scalar Fourier transform is a bijection from $L_2(G)$ into $L_2(\hat{G})$, and is a bijection from $\mathcal{S}(G)$ into $\mathcal{S}(\hat{G})$, the restriction of vector-valued Fourier transform onto $L_2(G) \otimes X$ is a bijection into $L_2(\hat{G}) \otimes X$, and its restriction onto $\mathcal{S}(G) \otimes X$ is a bijection to $\mathcal{S}(\hat{G}) \otimes X$.

To handle the case of infinite group $G$ we need a theorem describing structure of locally compact Abelian groups $[6]$.

**Theorem 2** Let $G$ be a locally compact Abelian group. Then $G$ is a union of open compactly generated subgroups $H$. Topology of $G$ is the topology of inductive limit. In turn, each compactly-generated subgroup $H \subset G$ is a projective limit of elementary factor-groups $H/K$, where $K \subset H$ are compact. “Elementary” here means that $H/K$ is isomorphic to cartesian product

$$H/K \cong R^{a_{H,K}} \times T^{b_{H,K}} \times Z^{c_{H,K}} \times F_{H,K},$$

where $a_{H,K} \geq 0$, $b_{H,K} \geq 0$, $c_{H,K} \geq 0$, and $F_{H,K}$ is a finite group.

**Definition 1** We say that $G$ contains an $R$-component if for some $H, K$ in Theorem 2 number $a_{H,K}$ is positive in elementary factor-group $R^{a_{H,K}} \times T^{b_{H,K}} \times Z^{c_{H,K}} \times F_{H,K}$.

In the same way we use phrases “group $G$ contains $Z$-component”, “group $G$ contains $T$-component”.

Now recall some properties of Pontryagin duality.

Consider an exact sequence of homomorphisms (i.e. image of each homomorphism coincides the kernel of the following one)

$$0 \to K \to G \to G/K \to 0, \quad 0 \to H \to G \to G/H \to 0,$$

where $K$ is compact, and $H$ is open. Dual groups $\hat{G}/K$, $\hat{G}/H$ can be identified with annihilators

$$K_{\hat{G}} = \{ \chi \in \hat{G} : \forall g \in K, \langle \chi, g \rangle = 1 \}, \quad H_{\hat{G}} = \{ \chi \in \hat{G} : \forall g \in H, \langle \chi, g \rangle = 1 \}.$$

Moreover, $K_{\hat{G}}$ is an open subgroup, and $H_{\hat{G}}$ is compact. One has dual oppositely-directed exact sequences

$$0 \leftarrow \hat{G}/K_{\hat{G}} \leftarrow \hat{G} \leftarrow K_{\hat{G}} \leftarrow 0, \quad 0 \leftarrow \hat{G}/H_{\hat{G}} \leftarrow \hat{G} \leftarrow H_{\hat{G}} \leftarrow 0.$$

If $K \subset H$, then $K_{\hat{G}} \supset H_{\hat{G}}$.

It’s not hard to see that Fourier transform of a function on $G$, that is supported in open subgroup $H$ and is constant on cosets of compact subgroup $K \subset H$, is a function on $\hat{G}$ supported in $K_{\hat{G}}$ and constant on cosets of $H_{\hat{G}}$. 

3
3 The case of an arbitrary local compact group

We will prove some necessary lemmas before formulating the main result.

**Lemma 1** Assume that Banach space $X$ is isomorphic to a Hilbert one, i.e. there exists inner product $(\cdot,\cdot)_{X}$ on $X$ such that for some $C > 0$ the following inequality is true

$$C^{-1}(x,x)^{1/2} \leq \|x\|_{X} \leq C(x,x)^{1/2}.$$ 

Then Fourier transform $\mathcal{F}: L_{2}(G,X) \rightarrow L_{2}(\hat{G},X)$ is bounded.

**Proof.** Consider vector-valued Steklov–Parseval’s equality

$$(\mathcal{F}\varphi,\mathcal{F}\varphi)_{L_{2}(\hat{G},X)} = \int_{G} \langle \mathcal{F}\varphi(\xi),\mathcal{F}\varphi(\xi) \rangle d\mu_{\hat{G}}(\xi) = \int_{G} \langle \varphi(t),\varphi(t) \rangle d\mu_{G}(t) = (\varphi,\varphi)_{L_{2}(G,X)},$$

which can be easily checked for $\varphi \in L_{2}(G) \otimes X$ by means of axioms for inner product, scalar Steklov–Parseval’s equality and cross-norm’s property

$$(\varphi_{1} \otimes x_{1},\varphi_{2} \otimes x_{2})_{L_{2}(G,X)} = (\varphi_{1},\varphi_{2})_{L_{2}(G)} \cdot (x_{1},x_{2})_{X}.$$ 

Then we have on a dense subspace

$$\|\mathcal{F}\varphi\|_{L_{2}(G,X)} \leq C(\mathcal{F}\varphi,\mathcal{F}\varphi)_{L_{2}(\hat{G},X)} = C(\varphi,\varphi)_{L_{2}(G,X)} \leq C^{2}\|\varphi\|_{L_{2}(G,X)},$$

so we can extend $\mathcal{F}$ by continuity onto the whole $L_{2}(G,X)$.

If $G$ is a finite group, then space $L_{2}(G,X)$ is isomorphic to the finite Cartesian product $X^{G}$. Pontryagin’s dual group $\hat{G}$ is isomorphic to $G$ itself. Self-dual Haar measure possesses the property $\mu_{G}(G) = \sqrt{|G|}$. Fourier transform, also known as discrete Fourier transform, becomes

$$(\mathcal{F}\varphi)(\xi) = \frac{1}{\sqrt{|G|}} \sum_{t \in G} \langle \xi,t \rangle \varphi(t).$$

**Theorem 3** If $G$ is a finite group, then Fourier transform $\mathcal{F}: L_{2}(G,X) \rightarrow L_{2}(\hat{G},X)$ is bounded for any Banach space $X$.

**Proof.** It follows from inequality

$$\|\mathcal{F}\varphi\|_{L_{2}(\hat{G},X)}^{2} = \sum_{\xi \in \hat{G}} \left(\frac{1}{\sqrt{|G|}} \sum_{t \in G} \langle \xi,t \rangle \varphi(t)\right)^{2} \leq \frac{1}{|G|} \sum_{\xi \in \hat{G}} \left(\sum_{t \in G} \|\langle \xi,t \rangle \varphi(t)\|_{X}\right)^{2} = \frac{1}{|G|} \sum_{\xi \in \hat{G}} \left(\sum_{t \in G} \|\varphi(t)\|_{X}\right)^{2} \leq |G| \sum_{t \in G} \|\varphi(t)\|_{X}^{2} = |G| \cdot \|\varphi\|_{L_{2}(G,X)}^{2}.$$ 

Now pass to infinite groups.

**Lemma 2** Let group $G$ contain $\mathbf{R}$-component, $\mathbf{T}$-component or $\mathbf{Z}$-component. Then boundedness of Fourier transform

$$\mathcal{F}: L_{2}(G,X) \rightarrow L_{2}(\hat{G},X)$$

implies isomorphism of Banach space $X$ to a Hilbert one.
**Proof.** Consider the case, when group $G$ contains $\mathbf{R}$-component. There are open compactly generated subgroup $H$ in $G$ and compact subgroup $K \subset H$ such that $H/K \cong \mathbf{R}^a \times \mathbf{T}^b \times \mathbf{Z}^c \times \mathbf{F}$, where $a \geq 1$.

Let $\tau_1 : H \to H/K$ be a canonical projection, $\tau_2 : H/K \to \mathbf{R}$ be the projection on the first coordinate of $\mathbf{R}^a$, $\tau = \tau_2 \circ \tau_1$.

Consider helper functions $\psi_{\mathbf{R},i} \in L_2(\mathbf{R})$, $2 \leq i \leq a$, $\psi_{\mathbf{T},j} \in L_2(\mathbf{T})$, $1 \leq j \leq b$, $\psi_{\mathbf{Z},k} \in L_2(\mathbf{Z})$, $1 \leq k \leq c$, $\psi_F \in L_2(\mathbf{F})$, each of them having norm equal to 1 in corresponding space. Consider injection $J : L_2(\mathbf{R},X) \to L_2(H,X)$,

$$J : \varphi \mapsto \left( (\varphi \circ \tau) \otimes \bigotimes_{i=2}^{a} \psi_{\mathbf{R},i} \otimes \bigotimes_{j=1}^{b} \psi_{\mathbf{T},j} \otimes \bigotimes_{k=1}^{c} \psi_{\mathbf{Z},k} \otimes \psi_F \right).$$

It is easy to see that the injection $J$ is isometric. There exists a unique injection $\widehat{J} : L_2(\mathbf{R},X) \to L_2(\widehat{H},X)$, which is also an isometry, and for which the following diagram is commutative

$$\begin{array}{ccc}
L_2(\mathbf{R},X) & \xrightarrow{\mathcal{F}_R} & L_2(\mathbf{R},X) \\
J \downarrow & & \downarrow \widehat{J} \\
L_2(H,X) & \xrightarrow{\mathcal{F}_H} & L_2(\widehat{H},X).
\end{array}$$

Space $L_2(H,X)$ can be identified with a closed subspace of $L_2(G,X)$ consisting of functions, that are 0 almost everywhere outside $H$. Space $L_2(\widehat{H},X)$ can be identified with a closed subspace of $L_2(\widehat{G},X)$ consisting of functions, which are constant on cosets of $H^\perp$ (recall, that $\widehat{H} \cong \widehat{G}/H_G^\perp$).

Fourier transform $\mathcal{F}_H$ is the restriction of $\mathcal{F}_G$ and thus, bounded. Fourier transform $\mathcal{F}_R = (J)^{-1} \mathcal{F}_H J$ is continuous as a composition of continuous maps. By Theorem [11] statement 4) space $X$ is isomorphic to a Hilbert one.

Cases when group $G$ contains $\mathbf{T}$-component or $\mathbf{Z}$-component are considered similarly. ▮

Consider the case, when group $G$ does not contain $\mathbf{R}$-, $\mathbf{Z}$- or $\mathbf{T}$-elements. Then all compactly generated subgroups $H \subset G$ are projective limits of finite subgroups with discrete topology, i.e. they are profinite groups.

Profinite groups are characterized by the following lemma [9].

**Lemma 3** Topological group $H$ is a profinite one if and only if it

- a) possesses Hausdorff’s property;
- b) is compact;
- c) is totally disconnected, i.e. for any two points $x, y \in H$ there exists subset $U \subset H$ that is both open and closed, such that $x \in U$ and $y \notin U$.

Any profinite group $H$ is either discrete (and finite by virtue of compactness) or non-discrete (and therefore infinite).

Consider non-discrete profinite group $H$. We normalize Haar measure on $H$ with $\mu_H(H) = 1$.

Group $H$ is a Lebesgue space, i.e. it is isomorphic as a space with measure to segment $[0,1]$ with length $[5,8]$. This fact can be proved selecting sequence of compact subgroups $K_n \subset H$ such that $K_1 \subset K_2 \subset \ldots$ and cardinality of quotient groups $M_n := \|H/K_n\|$ tends to $+\infty$. If one numbers cosets of $K_n$ properly, he becomes able to identify them with the intervals of length $1/M_n$ in $[0,1]$ up to a subsets of zero measure. By $\tau$ denote this isomorphism of spaces with measure.

5
A system of functions \((r_i)_{i=1,2,...}\), similar to the Rademacher’s system on \([0, 1]\) can be constructed on group \(H\). This is an orthogonal system of functions taking values \(\{+1, -1\}\) on subsets of measure \(\frac{1}{2}\). Saying in probability-theoretical language functions \(r_i\) are realizations of independent random variables taking values \(\{+1, -1\}\) with probability \(\frac{1}{2}\). One can simply assume \(r_i = r_i^\infty \circ \tau\), where \(r_i^\infty(t) = \sin 2^i \pi t\) on \([0, 1]\) is the usual Rademacher’s functions.

We need a criterion, which is proved in [3].

**Theorem 4** The following statements are equivalent:

1) Banach space \(X\) is isomorphic to a Hilbert one.

2) There exists constant \(C > 0\) such that for any finite collection of vectors \(x_1, x_2, ..., x_n \in X\) two-sided Khinchin’s type inequality holds

\[
C^{-1} \sum_{i=1}^{n} \|x_i\|^2 \leq \mathbb{E} \left[ \sum_{i=1}^{n} r_i x_i \right]^2 = \int_{H} \left[ \sum_{i=1}^{n} r_i(t) x_i \right]^2 dt \leq C \sum_{i=1}^{n} \|x_i\|^2, 
\]

where \(r_i\) are independent random variables taking values \(\{+1, -1\}\) with probability \(\frac{1}{2}\), and symbol \(\mathbb{E}\) denotes expectation.

As in [3] we formulate a Lemma, which shows importance of the system \((r_i)\) on \(H\) and allows us to consider arbitrary basis in \(L_2(H)\) instead of \((r_i)\). By \(dt\) denote Haar measure on \(H\).

**Lemma 4** Let \(X\) be a Banach space, \((f_i)\) be an orthonormal complete system in \(L_2(H)\). Assume that for some \(C > 0\) and for any collection \(x_1, x_2, ..., x_n \in X\) there is inequality

\[
\int_{H} \left[ \sum_{i=1}^{n} f_i(t) x_i \right]^2 dt \leq C \sum_{i=1}^{n} \|x_i\|^2 \quad \text{resp.,} \quad C^{-1} \sum_{i=1}^{n} \|x_i\|^2 \leq \int_{H} \left[ \sum_{i=1}^{n} f_i(t) x_i \right]^2 dt.
\]

Then for the same constant \(C > 0\) and for any collection \(x_1, x_2, ..., x_n \in X\) one also has

\[
\int_{H} \left[ \sum_{i=1}^{n} r_i(t) x_i \right]^2 dt \leq C \sum_{i=1}^{n} \|x_i\|^2 \quad \text{resp.,} \quad C^{-1} \sum_{i=1}^{n} \|x_i\|^2 \leq \int_{H} \left[ \sum_{i=1}^{n} f_i(t) x_i \right]^2 dt.
\]

**Proof.** As Rademacher’s system \((r_i)\) is orthonormal and \((f_k)\) are complete, we can find for a given \(\varepsilon > 0\) an increasing sequences of indices \((k_j)\), \((m_j)\) and orthonormal sequence \((h_j)\) such that

\[
h_j = \sum_{k=k_j}^{k_j+1} (h_j, f_k) \cdot f_k, \quad \int_{H} |h_j(t) - r_{m_j}(t)|^2 dt < \frac{\varepsilon}{2^j}.
\]

For a fixed \(n\) and fixed \(x_1, x_2, ..., x_n \in X\) we have

\[
\int_{H} \left[ \sum_{i=1}^{n} r_i(t) \cdot x_i \right]^2 dt = \int_{H} \left[ \sum_{i=1}^{n} r_{m_j}(t) \cdot x_i \right]^2 dt.
\]

By the triangle inequality

\[
\left( \int_{H} \left[ \sum_{i=1}^{n} r_{m_j}(t) \cdot x_i \right]^2 dt \right)^{1/2} \leq \left( \int_{H} \left[ \sum_{j=1}^{n} (r_{m_j}(t) - h_j(t)) \cdot x_j \right]^2 dt \right)^{1/2} + \left( \int_{H} \left[ \sum_{j=1}^{n} h_j(t) \cdot x_j \right]^2 dt \right)^{1/2},
\]

\[
+ \left( \int_{H} \left[ \sum_{j=1}^{n} h_j(t) \cdot x_j \right]^2 dt \right)^{1/2} \leq \sqrt{\varepsilon} \left( \sum_{j=1}^{n} \|x_j\|^2 \right)^{1/2} + \left( \int_{H} \left[ \sum_{j=1}^{n} h_j(t) \cdot x_j \right]^2 dt \right)^{1/2}.
\]
As $1 = \|h_j\|^2 = \sum_{k=k_j}^{k_j+1} |(h_j, f_k)|^2$, we get

$$\int_{\hat{H}} \left\| \sum_{j=1}^{n} h_j(t) \cdot x_j \right\|^2 dt = \int_{\hat{H}} \left\| \sum_{j=1}^{n} \sum_{k=k_j}^{k_j+1} (h_j, f_k) \cdot f_k \right\| x_j \right\|^2 dt \leq C \sum_{j=1}^{n} \sum_{k=k_j}^{k_j+1} |(h_j, f_k)|^2 \|x_j\|^2 = C \sum_{j=1}^{n} \|x_j\|^2.$$

Thus,

$$\int_{\hat{H}} \left\| \sum_{i=1}^{n} r_i(t) \cdot x_i \right\|^2 dt \leq (\sqrt{\varepsilon} + \sqrt{C})^2 \sum_{i=1}^{n} \|x_i\|^2.$$

As $\varepsilon$ is arbitrary, we obtain the desired inequality. Proof in the case of reverse type inequality is analogous.

**Corollary 4.1** Let $X$ be a Banach space, and $(f_i)$ be a complete orthonormal system in $L_2(H)$. Space $X$ is linearly isomorphic to a Hilbert one if and only if there exists constant $C > 0$ such that for any set of vectors $x_1, x_2, \ldots, x_n \in X$ one has

$$C^{-1} \sum_{i=1}^{n} \|x_i\|^2 \leq \int_{\hat{H}} \left\| \sum_{i=1}^{n} f_i(t) \cdot x_i \right\|^2 dt \leq C \sum_{i=1}^{n} \|x_i\|^2.$$

In Corollary 4.1, isomorphism of $X$ to a Hilbert space follows from two-sided inequality. Knowledge of profinite groups’ structure allows us to switch from lower estimate to upper one and vice versa as shown below.

First recall that Bruhat–Schwartz space on a profinite group $H$ and on dual discrete $\hat{H}$ consists of locally constant functions with compact support. Spaces $S(H)$ and $S(\hat{H})$ are inductive limit of finite-dimensional spaces and carry the strongest locally convex topology $\mathfrak{T}$.

Now we are going to study the case of vector-valued Fourier transform on a profinite non-discrete group $H$. As $H$ is a compact infinite group, its dual $\hat{H}$ is a discrete infinite group.

**Lemma 5** Let $X$ be a Banach space, and $H$ be a profinite non-discrete group. The following statements are equivalent:

1) $X$ is linearly isomorphic to a Hilbert space.

2) There exists constant $C > 0$ such that for any set of vectors $x_1, \ldots, x_n \in X$ and characters $\xi_1, \ldots, \xi_n \in \hat{H}$ one has

$$\int_{\hat{H}} \left\| \sum_{k=1}^{n} \langle \xi_k, t \rangle x_k \right\|^2 dt \leq C \sum_{k=1}^{n} \|x_k\|^2.$$

2) Fourier transform $\mathcal{F}_{\hat{H}} : L_2(\hat{H}, X) \to L_2(H, X)$ and inverse Fourier transform $F_{H}^{-1} = \mathcal{I}_H \mathcal{F}_{\hat{H}}$ are bounded. Here $\mathcal{I}_H$ is an isometrical operator of changing variable $x(t) \mapsto x(-t)$.

3) There exists constant $C > 0$ such that for any set of vectors $x_1, \ldots, x_n \in X$ and characters $\xi_1, \ldots, \xi_n \in \hat{H}$ one has

$$C^{-1} \sum_{k=1}^{n} \|x_k\|^2 \leq \int_{\hat{H}} \left\| \sum_{k=1}^{n} \langle \xi_k, t \rangle x_k \right\|^2 dt.$$
Inverse Fourier transform $\mathcal{F}_H^{-1} : L_2(H, X) \rightarrow L_2(\hat{H}, X)$ and Fourier transform $\mathcal{F}_H = \mathcal{I}_H \mathcal{F}_H^{-1}$ are bounded.

**Proof.** Lemma 1.4 yields implications 1) $\Rightarrow$ 2'), 1) $\Rightarrow$ 3').

To prove equivalences 2') $\iff$ 2), 3') $\Rightarrow$ 3) it's enough to consider

$$h = \sum_{k=1}^{n} I_{\{\xi_k\}} \cdot x_k \in \mathcal{S}(\hat{H}) \otimes X,$$

where $\xi_k \in \hat{H}, x_k \in X, n \in \mathbb{N}$. By definition of Fourier transform $\mathcal{F} : L_2(\hat{H}, X) \rightarrow L_2(H, X)$ and by chosen normalization of Haar measure on $H$ we get

$$\|h\|^2_{L_2(H, X)} = \sum_{k=1}^{n} \|x_k\|^2, \quad \|\mathcal{F}h\|^2_{L_2(H, X)} = \int_{H} \left\| \sum_{k=1}^{n} \langle \xi_k, t \rangle x_k \right\|^2 dt.$$

Equivalence follows from the density of $\mathcal{S}(\hat{H}) \otimes X$ in $L_2(\hat{H}, X)$, density of $\mathcal{S}(H) \otimes X$ in $L_2(H, X)$, and bijectivity of $\mathcal{F}_H, \mathcal{F}_H$ in the corresponding spaces.

By Corollary 4.1 one also has 2) & 3) $\Rightarrow$ 1).

Now assume boundedness condition 2'). Let's look at Fourier transform $\mathcal{F}_H$ on a subspace $\mathcal{S}(H) \otimes X$. For this consider arbitrary compact subgroup $K \subset H$, for which $|H/K| < +\infty$. We identify functions, which are constant on $K$, with elements of $\mathcal{S}(H/K) \otimes X$.

By finiteness of $H/K$ there exist an isomorphism $\alpha : \hat{H}/K \rightarrow H/K$. Adjoint isomorphism $\alpha^* : \hat{H}/K \rightarrow H/K$ is defined by

$$\langle \xi_1, \alpha(\xi_2) \rangle_H = \langle \alpha^*(\xi_1), \xi_2 \rangle_{\hat{H}}.$$

Consider operator

$$R_\alpha : \mathcal{S}(H/K) \otimes X \rightarrow \mathcal{S}(\hat{H}/K) \otimes X : (R_\alpha \psi)(\xi') = \psi(\alpha(\xi')) \cdot |H/K|^ {-\frac{1}{2}}.$$

It’s an isometry, because

$$\|R_\alpha \psi\|^2 = \sum_{\xi' \in \hat{H}/K} ||R_\alpha \psi(\xi')||^2 \mu_{\hat{H}/K}(\xi') = \sum_{\xi' \in \hat{H}/K} ||\psi(\alpha(\xi'))|^2 \cdot |H/K|^ {-\frac{1}{2}} =$$

$$= [t := \alpha(\xi')] = \sum_{t \in H/K} ||\psi(t)||^2 \cdot |H/K|^{-1} = \sum_{t \in H/K} ||\psi(t)||^2 \mu_{H/K}(t) = ||\psi||^2.$$

Now one has

$$\langle \mathcal{F}_H R_\alpha \psi(t'), t' \rangle = \sum_{\xi' \in \hat{H}/K} \langle \xi', \xi' \rangle_{\hat{H}/K} \psi(\alpha(\xi')) |H/K|^{-\frac{1}{2}} =$$

$$= \langle \mathcal{F}_H R_\alpha \psi, \alpha^* \rangle H/K \psi(t) |H/K|^{-\frac{1}{2}} = \sum_{t \in H/K} \langle \alpha^*(\xi), \alpha^{-1}(t) \rangle_H \psi(t) \mu_{H/K}(t) =$$

$$= \sum_{t \in H/K} \langle \xi, t \rangle_H \psi(t) \mu_{H/K}(t) = \langle \mathcal{F}_H \psi, t \rangle_H.$$
Thus, restriction of \( \mathcal{F}_H \) onto \( S(H/K) \otimes X \) has the same norm as \( \mathcal{F}_{H/K} \) does. As \( K \) is arbitrary, it implies continuity of \( \mathcal{F}_H \) on \( L_2(H, X) \), and implication \( 2) \Rightarrow 3) \) is true. Implication \( 3) \Rightarrow 2) \) can be proved in the same way.

Now one has enough implications to see the equivalence of all statements.

Let’s pass to the general case of Fourier transform on a locally compact Abelian group \( G \). This is our main result.

**Theorem 5** Let \( X \) be a Banach space, let \( G \) be a locally compact Abelian group. Space \( X \) is linearly isomorphic to a Hilbert one if and only if Fourier transform

\[
\mathcal{F} : L_2(G, X) \to L_2(\hat{G}, X)
\]

is bounded.

**Proof.** Existence of isomorphism is a sufficient condition for boundedness of \( \mathcal{F} \) as shown in Lemma 1.

Now assume that Fourier transform is bounded.

If \( G \) contains \( \mathbb{R} \)-, \( \mathbb{T} \)- or \( \mathbb{Z} \)-component, then isomorphism of \( X \) to a Hilbert space follows from Lemma 2 and we are done.

Otherwise, group \( G \) does not contain \( \mathbb{R} \), \( \mathbb{T} \) or \( \mathbb{Z} \)-components. In this case all compactly generated open subgroups \( H \subset G \) are profinite.

If some of these \( H \) is non-discrete, then we consider space \( L_2(H, X) \) as a closed subspace in \( L_2(G, X) \) (one can simply assume that function from \( L_2(H, X) \) are zero outside \( H \)). We identify \( L_2(\hat{H}, X) \) with a subspace of \( L_2(\hat{G}, X) \) consisting of functions constant on cosets of annihilator \( H^\perp \subset \hat{G} \). Fourier transform on \( L_2(H, X) \) is the restriction of Fourier transform from \( L_2(G, X) \) and is bounded. Isomorphism of \( X \) to a Hilbert space follows from Lemma 5, statement 3)′. And we are done.

If all subgroups \( H \subset G \) considered are discrete (their compactness implies finiteness), then by Theorem 2 group \( G \) is an inductive limit of discrete subgroups. By properties of Pontryagin duality (in the language of Cathegory theory one can say that passing to a dual group is an exact functor) dual group \( \hat{G} \) is a projective limit of \( \hat{H} \). Groups \( \hat{H} \) are dual to finite discrete \( H \). Thus, \( \hat{H} \) are isomorphic \( H \), and are finite discrete themselves. Group \( \hat{G} \) is profinite. As \( G \) is infinite, \( \hat{G} \) is non-discrete.

Boundedness of Fourier transform \( \mathcal{F} : L_2(G, X) \to L_2(\hat{G}, X) \) is equivalent to boundedness of inverse Fourier transform \( \mathcal{F}^{-1}_{\hat{G}} \) on profinite non-discrete \( \hat{G} \). Isomorphism of \( X \) to a Hilbert space follows from Lemma 5 statement 2)′.

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