A KIRBY COLOR FOR KHOVANOV HOMOLOGY

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ABSTRACT. We construct a Kirby color in the setting of Khovanov homology: an ind-object of the annular Bar-Natan category that is equipped with a natural handle slide isomorphism. Using functoriality and cabling properties of Khovanov homology, we define a Kirby-colored Khovanov homology that is invariant under the handle slide Kirby move, up to isomorphism. Via the Manolescu–Neithalath 2-handle formula, Kirby-colored Khovanov homology agrees with the $\mathfrak{gl}_2$ skein lasagna module, hence is an invariant of 4-dimensional 2-handlebodies.

1. Introduction

In the context of quantum link invariants (such as the Jones polynomial) a Kirby color\(^1\) is a certain linear combination of cabling patterns yielding a framed link invariant that is invariant under the second (handle slide) Kirby move [Kir78]. The resulting “Kirby-colored” quantum invariant of a link $\mathcal{L} \subset S^3$ can then be regarded as an invariant of the 4-manifold obtained by attaching 2-handles to the 4-ball $B^4$ along the link $\mathcal{L} \subset \partial B^4$ [Lic92]. Perhaps the most famous examples of this process are the Witten–Reshetikhin–Turaev invariants [Wit89, RT91], though we note these are special in that they admit a further renormalization that is invariant under the first (blow-up) Kirby move\(^2\).

Khovanov homology [Kho00] is a bigraded homology theory for links $\mathcal{L} \subset S^3$, which categorifies the Jones polynomial. One of the longest-standing problems regarding Khovanov homology is whether (and if so, how) it extends to an invariant of 3- or 4-manifolds. The goal of this paper is to propose a solution to this problem that proceeds by developing a Kirby color for Khovanov homology.

1.1. The Kirby color. To describe our work with slightly more precision, recall that in the (pre-categorified) context of framed link invariants associated with a linear ribbon category, the set of cabling patterns forms an algebra. For the Jones polynomial, this is the Kauffman bracket skein algebra of the thickened annulus. Analogously, in the categorified context the cabling patterns form a monoidal

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\(^1\)This name seems to have come into use sometime before 2001 [Bla03], although the concept is older. Another common name in the tensor category literature is “(virtual) regular object”, see e.g. [EGNO15, p.270].

\(^2\)Being invariant under both Kirby moves, the Witten–Reshetikhin–Turaev invariants are thus 3-manifold invariants, depending only on the boundary of the aforementioned 4-dimensional 2-handlebody.
category: for Khovanov homology this role is played by the Bar-Natan category of the thickened annulus, which hereafter will be denoted $\mathcal{ABN}$. For this introduction, it suffices to note that objects of $\mathcal{ABN}$ are embedded 1-manifolds in the annulus and morphisms are formal linear combinations of (dotted) cobordisms embedded in the thickened annulus, modulo local relations (from [BN05, §11.2]). Moreover, $\mathcal{ABN}$ is $\mathbb{Z}$-graded and linear, with monoidal structure given by inserting one annulus into the interior of another. See §3.1 for the precise definitions.

We let $c$ denote the object of $\mathcal{ABN}$ which is a single essential circle, so $c^n$ denotes $n$ concentric essential circles. The symmetric group $S_n$ acts on $c^n$, thus the symmetric power $\text{Sym}^n(c)$ exists in the Karoubi completion $\text{Kar}(\mathcal{ABN})$ as the image of the symmetrizing idempotent.

**Definition 1.1.** Let $\omega := \omega_0 + \omega_1$, where the summands are the colimits

\[
\omega_0 := \text{colim} \left( \text{Sym}^0(c) \to q^{-2}\text{Sym}^2(c) \to q^{-4}\text{Sym}^4(c) \to \cdots \right)
\]

\[
\omega_1 := \text{colim} \left( q^{-1}\text{Sym}^1(c) \to q^{-3}\text{Sym}^3(c) \to q^{-5}\text{Sym}^5(c) \to \cdots \right),
\]

regarded as objects of an appropriate completion $\overline{\mathcal{ABN}}$ (see Convention 4.1). Here, the maps are given by dotted annulus cobordisms, and the grading shifts $q^{-n}$ ensure that these maps are degree zero.

The object $\omega \in \overline{\mathcal{ABN}}$ is the titular Kirby color for Khovanov homology. In contrast to the precategorified situation, it is an object of a monoidal category rather than an element of an algebra, and it manifestly does not require working at a root of unity. In particular, the Kirby color $\omega$ for Khovanov homology does not categorify any Kirby element for the $\mathfrak{sl}_2$ Witten–Reshetikhin–Turaev invariants!

Nevertheless, the Kirby color $\omega$ behaves like a categorified Kirby element [Vir06] in the sense that it leads to link invariants that are invariant under handle slide. In the categorical setting, handle slide invariance is an additional structure, rather than a property. This structure is best phrased in the annular setting, namely using the relative Bar-Natan category of the annulus $\mathcal{PBN}$, wherein objects are annular curves with boundary (see §4.2). This category is a module category for $\mathcal{ABN}$ and contains two special objects

\[ L := \begin{array}{c} \bullet \\ \end{array}, \quad R := \begin{array}{c} \bullet \\ \end{array}. \]

**Theorem A** (Handle slide). The Kirby color $\omega$ is equipped with the following handle slide structure:

- (Lemma 4.13) There exists a distinguished isomorphism $\omega \cdot L \cong \omega \cdot R$ in an appropriate completion $\overline{\mathcal{PBN}}$, that we call the elementary handle slide. Graphically, this may be depicted as in the first isomorphism of (1).

\[ \begin{array}{c} \bullet \omega \\ \end{array} \cong \begin{array}{c} \bullet \omega \\ \end{array}, \quad \begin{array}{c} \bullet \omega \\ \end{array} \cong \begin{array}{c} \bullet \omega \\ \end{array}. \]

- (Lemma 4.14) Compositions of elementary handle slides assemble into a collection of handle slide isomorphisms, which are natural with respect to cobordisms involving the “sliding strands” (illustrated as $D$ in the second isomorphism of (1)).

In Theorem 4.15, we further show that the Kirby color $\omega$, together with its handle slide isomorphisms, constitutes an object of the Drinfeld center of the completed relative Bar-Natan category $\overline{\mathcal{PBN}}$ of

\[ \text{See Remark 6.6 for a more sophisticated choice, which will not be necessary for the purpose of this paper.} \]
the annulus, considered as a bimodule for the monoidal Bar-Natan bicategory \( \mathcal{BN} \) associated to the rectangle. The proof of Theorem A uses a generators and relations presentation of a certain subcategory of \( \mathcal{PBN} \) that we provide in Theorem 4.12.

1.2. Manifold invariants and TQFT context. In order to port our results on handle slide invariance from the annular setting to links in \( S^3 \), we need a well-defined notion of cabling in Khovanov homology. The following is a straightforward consequence of the functoriality of Khovanov homology:

**Theorem B** (Cabling in Khovanov homology). Let \( \mathcal{L} = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_r \) be an \( r \)-component framed oriented link in \( S^3 \). There is a functor

\[
\text{K}h_{\mathcal{L}} : \mathcal{ABN}^{\times r} \to \text{Vect}^{Z \times Z}
\]

sending \((c_1, \ldots, c_r)\) to the Khovanov homology of the cable \( \mathcal{L}_c := \mathcal{K}_1^{c_1} \cup \cdots \cup \mathcal{K}_r^{c_r} \).

Here, \( \mathcal{K}_i^{c_i} \) denotes the \( c_i \)-fold parallel cable of the component \( \mathcal{K}_i \) and \( \text{Vect}^{Z \times Z} \) denotes the category of bigraded vector spaces. The functor in Theorem B requires certain choices that fix the (original) well-known sign ambiguity in the functoriality of Khovanov homology; this is discussed further in §6.1 where we re-state and prove this result as Theorem 6.1. Since \( \text{Vect}^{Z \times Z} \) is closed under all of the relevant operations used to define \( \mathcal{ABN} \) (grading shifts, direct sums and summands, filtered colimits), cabling extends to a functor

\[
\text{K}h_{\mathcal{L}} : (\mathcal{ABN})^{\times r} \to \text{Vect}^{Z \times Z}.
\]

**Definition 1.2** (Kirby-colored Khovanov homology). Let \( \mathcal{L} \) be a framed oriented link in \( S^3 \) with a decomposition into sublinks \( \mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \). Set

\[
\text{K}h(\mathcal{L}_1 \cup \mathcal{L}_2^\omega) := \text{K}h_{\mathcal{L}_1 \cup \mathcal{L}_2}(c, \ldots, c, \omega, \ldots, \omega)
\]

in which all the components of \( \mathcal{L}_1 \) carry the label \( c \), and all the components of \( \mathcal{L}_2 \) carry the label \( \omega \).

**Remark 1.3.** In [GLW18, §7.5] Grigsby–Licata–Wehrli consider a different colimit of Khovanov homologies of cables of a knot \( \mathcal{K} \), which appears to be unrelated to our Kirby colored Khovanov homology.

Theorems A and B suggest that \( \text{K}h(\mathcal{L}_1 \cup \mathcal{L}_2^\omega) \) may be a diffeomorphism invariant of the pair \((B^4(\mathcal{L}_2), \mathcal{L}_1)\), where \( B^4(\mathcal{L}_2) \) is the 4-dimensional 2-handlebody obtained by attaching 2-handles to \( B^4 \) along the components of \( \mathcal{L}_2 \), and \( \mathcal{L}_1 \) is regarded as a link in the boundary 3-manifold \( S^3(\mathcal{L}_2) := \partial B^4(\mathcal{L}_2) \). To establish this directly, one would further need to incorporate 1- and 3-handles in a way that they satisfy an appropriate cancellation property with \( \omega \).

The resulting 4-manifold invariant would a priori be valued in isomorphism classes of bigraded vector spaces, rather than valued in bigraded vector spaces. An upgrade to the latter would in particular require not only isomorphisms associated to handle slide Kirby moves, but a coherent family thereof. For this, one would need a classification of movie moves for Kirby moves, in analogy with the Carter–Saito movie moves for isotopic link cobordisms [CS98]. We are not aware of such a classification.

In the absence of both of the above, we instead establish \( \text{K}h(\mathcal{L}_1 \cup \mathcal{L}_2^\omega) \) as a bigraded vector space-valued invariant of \((B^4(\mathcal{L}_2), \mathcal{L}_1)\) by comparison with the \( gl_N \) skein lasagna module 4-manifold invariants from [MWW19], whose invariance is manifest.

**Theorem C.** Let \( \mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \) be a framed oriented link in \( S^3 \). Decorate all the components of \( \mathcal{L}_1 \) with \( c \), and all the components of \( \mathcal{L}_2 \) with \( \omega \). The following bigraded vector spaces are isomorphic:

1. the Kirby-colored Khovanov homology \( \text{K}h(\mathcal{L}_1 \cup \mathcal{L}_2^\omega) \),
2. the cabled \( N = 2 \) Khovanov–Rozansky homology of \( \mathcal{L}_1 \cup \mathcal{L}_2^\omega \) as defined by Manolescu–Neithalath [MN22] for split unions \( \mathcal{L}_1 \sqcup \mathcal{L}_2^\omega \) and extended to the general case in [MWW22], and
3. the \( N = 2 \) skein lasagna module (degree zero blob homology) of \((B^4(\mathcal{L}_2), \mathcal{L}_1)\) from [MWW19].

As a consequence, the Kirby-colored Khovanov homology \( \text{K}h(\mathcal{L}_1 \cup \mathcal{L}_2^\omega) \) is an invariant of the pair \((B^4(\mathcal{L}_2), \mathcal{L}_1)\), valued in bigraded vector spaces.
The isomorphisms (2) \( \cong (3) \) have been established in [MN22, MWW22]. In §6, we verify (1) \( \cong (2) \) by showing that cabling with the Kirby color \( \omega \) implements the Manolescu–Neithalath 2-handle formula from [MN22] that defines (2).

**Remark 1.4.** The \( \mathfrak{sl}_2 \)-version of the skein lasagna module of \((B^4(L_2), L_1)\) is graded by the relative second homology group \(H_2(B^4(L_2), L_1; \mathbb{Z}/2)\). The isomorphism from Theorem C identifies this grading with the direct sum decomposition of \( \text{Kh}_{L_1 \cup L_2}(c, \ldots, c; \omega, \ldots, \omega) \) inherited from \( \omega = \omega_0 \oplus \omega_1 \).

### 1.3. Diagrammatics.

A theorem of Russell [Rus09] implies that the monoidal category \( \mathcal{ABN} \) admits a diagrammatic presentation in terms of the **dotted Temperley–Lieb category** \( \mathcal{dTL} \) (see our §3.1 for a definition). The latter is a non-semisimple \( \mathbb{Z} \)-graded monoidal category that contains the familiar Temperley–Lieb category as its degree zero subcategory. A further goal in the present paper is to extend the graphical calculus for \( \mathcal{dTL} \) to give a presentation of the monoidal category obtained from \( \mathcal{ABN} \) by adjoining the Kirby objects \( \omega_0 \) and \( \omega_1 \). This is accomplished in §5.

One interesting feature of this extended calculus is the necessity of infinite sums of diagrams. The following gives the flavor of the sort of relations one encounters:

\[
\begin{align*}
&\begin{array}{c}
\begin{array}{c}
\left[0\right]
\end{array}
\end{array} \\
&\begin{array}{c}
\begin{array}{c}
\left[0\right]
\end{array}
\end{array}
\end{align*}
= \sum_{n \geq 0} \frac{(-1)^{\frac{n(n-1)}{2}}}{n!} \begin{array}{c}
\begin{array}{c}
\left[n\right]
\end{array}
\end{array} \\
&\begin{array}{c}
\begin{array}{c}
\left[0\right]
\end{array}
\end{array} \\
&\begin{array}{c}
\begin{array}{c}
\left[0\right]
\end{array}
\end{array} \\
&\begin{array}{c}
\begin{array}{c}
\left[0\right]
\end{array}
\end{array} \\
&\begin{array}{c}
\begin{array}{c}
\left[0\right]
\end{array}
\end{array} \\
&\begin{array}{c}
\begin{array}{c}
\left[0\right]
\end{array}
\end{array} \\
&\begin{array}{c}
\begin{array}{c}
\left[0\right]
\end{array}
\end{array}
\end{align*}
\]

Here, the \([0]\)-label indicates the Kirby object \( \omega_0 \), and this equation gives a decomposition of the identity morphism of \( \omega_0 \otimes \omega_0 \) into mutually orthogonal idempotents. This relation reflects a certain **quasi-idempotence** property of the Kirby object, namely that \( \omega_0 \otimes \omega_0 \cong \bigoplus_{n \geq 0} q^{-2n}\omega_0 \) (with similar statements for \( \omega_i \otimes \omega_j \)). See Corollary 5.25 for details.

### 1.4. The Kirby color as representing planar evaluation.

For the remainder of this introduction, we phrase all of our results in terms of the diagrammatic category \( \mathcal{dTL} \), instead of \( \mathcal{ABN} \). The Kirby color as representing planar evaluation.

**Remark 1.5.** We would like to emphasize that the surprising feature in representing the functor \( \text{Pol}^* \) is not that a representing object exists in some appropriate completion of \( \mathcal{dTL} \). Rather, it is that the representing object \( \omega \) can be described very explicitly as the, arguably, simplest non-trivial directed system (i.e. ind-object) over \( \mathcal{dTL} \). Furthermore, the explicit description is essential for computations of Kirby-colored Khovanov homology (Definition 1.2) and for the diagrammatics described in §1.3.
The functor $\text{Pol}: \text{dTL} \to \text{Vect}^\mathbb{Z}$ also has an algebraic description via $\mathfrak{sl}_2$ representation theory. As mentioned above, the usual (undotted) Temperley-Lieb category $\text{TL}$ (at circle-value 2) can be regarded as the subcategory of degree zero morphisms in $\text{dTL}$. As is well-known, there is a fully faithful monoidal functor $\text{TL} \to \text{Rep}(\mathfrak{sl}_2)$ that sends $c \mapsto V$, the defining 2-dimensional representation of $\mathfrak{sl}_2$. If we forget the action of the Chevalley generators $E, F \in \mathfrak{sl}_2$ but remember the weight grading, we obtain a functor $\text{TL} \to \text{Vect}^\mathbb{Z}$ that coincides with $\text{Pol}|_{\text{TL}}$. Now $\text{Pol}: \text{dTL} \to \text{Vect}^\mathbb{Z}$ may be thought of as the extension of this functor to $\text{dTL}$, defined by sending the \textquotedblleft dot\textquotedblright{} endomorphism of $c$ to the action of $E \in \mathfrak{sl}_2$ on $V$.

**Remark 1.6.** Since the Karoubi (idempotent) completion of $\text{TL}$ is semisimple, it is not hard to see that the restricted functor $\text{Pol}^*|_{\text{TL}}: \text{TL}^{op} \to \text{Vect}^\mathbb{Z}$ is representable by the object

$$\bigoplus_{n \geq 0} \text{Pol}(\text{Sym}^n(c))^* \otimes \text{Sym}^n(c)$$

where here $\text{Sym}^n(c)$ is the object in Kar($\text{TL}$) corresponding to the simple finite-dimensional $\mathfrak{sl}_2$-module $\text{Sym}^n(V)$. Since $\text{Pol}(\text{Sym}^n(c))$ has graded dimension equal to the quantum integer $[n+1]$, this formula is reminiscent of the familiar formula for the Kirby element in $\mathfrak{sl}_2$ Witten–Reshetikhin–Turaev theory.

However, it is the category $\text{dTL}$ (non-semisimple even after Karoubi completing) that naturally arises in Khovanov homology, and the representing object for $\text{Pol}^*: \text{dTL} \to \text{Vect}^\mathbb{Z}$ need not be (and indeed is not) isomorphic to the representing object for its restriction to $\text{TL}$. This elucidates why our Kirby color does not categorify the familiar Kirby element from $\mathfrak{sl}_2$ Witten–Reshetikhin–Turaev theory.

**Conventions.** All results in this paper hold over any field $\mathbb{K}$ of characteristic zero. We let $\mathbb{k}$ denote an arbitrary field. Knots and links are always framed and oriented.

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## 2. Categorical Background

In this section we recall category-theoretic constructions that will be used throughout. In particular, we review graded, linear categories and discuss (co)limits and various completions in this setting.

### 2.1. Graded linear categories.

We begin with some basic notions, mostly for the purpose of establishing notation. Let $\Gamma$ be an abelian group and $\mathbb{k}$ a field. A $\Gamma$-graded vector space is a $\Gamma$-indexed collection of $\mathbb{k}$-vector spaces $(V_i)_{i \in \Gamma}$. Given a pair of $\Gamma$-graded vector spaces $V, W$, we let $\text{HOM}(V, W)$ denote the $\Gamma$-graded vector space which is given in degree $i \in \Gamma$ by

$$\text{HOM}^i(V, W) := \prod_{j \in \Gamma} \text{Hom}_\mathbb{k}(V_j, W_{i+j}).$$

We let $\text{Vect}_0^\Gamma$ denote the category with objects $\Gamma$-graded vector spaces and with morphisms

$$\text{Hom}_{\text{Vect}_0^\Gamma}(V, W) := \text{HOM}^0(V, W).$$

The category $\text{Vect}_0^\Gamma$ is symmetric monoidal, with tensor product given by

$$((V_i)_{i \in \Gamma} \otimes (W_j)_{j \in \Gamma})_k := \bigoplus_{i+j=k} V_i \otimes W_j.$$
Further, the tensor-hom adjunction
\[ \text{Hom}_{\text{Vect}_0^\Gamma}(U \otimes V, W) \cong \text{Hom}_{\text{Vect}_0^\Gamma}(U, \text{HOM}(V, W)) \]
holds in $\text{Vect}_0^\Gamma$. In other words, $\text{Vect}_0^\Gamma$ is closed with internal hom given by HOM. Since $\text{Vect}_0^\Gamma$ is further (co)complete (i.e. has all (co)limits, see below), it is natural to consider categories enriched thereover.

**Definition 2.1.** A $\Gamma$-graded $k$-linear category is a category $\mathcal{C}$ that is enriched over $\text{Vect}_0^\Gamma$. If $\mathcal{C}$ is a $\Gamma$-graded $k$-linear category, then we let $\mathcal{C}_0$ (its degree zero subcategory) denote the non-full subcategory with the same objects as $\mathcal{C}$ and with morphisms the degree zero morphisms in $\mathcal{C}$.

**Remark 2.2.** An isomorphism in a $\Gamma$-graded $k$-linear category $\mathcal{C}$ will always mean an element of $\text{Hom}_{\mathcal{C}_0}(X, Y)$ with a two-sided inverse (necessarily also degree zero).

The prototypical example of a $\Gamma$-graded $k$-linear category is the category $\text{Vect}^\Gamma$ with objects $\Gamma$-graded vector spaces and with morphisms
\[ \text{Hom}_{\text{Vect}^\Gamma}(V, W) := \text{HOM}(V, W) \]
The degree zero subcategory of $\text{Vect}^\Gamma$ is simply $\text{Vect}_0^\Gamma$.

We will need to adjoin the images of idempotents to $\Gamma$-graded $k$-linear categories. This is accomplished with the following operation.

**Definition 2.3** (Karoubi completion). Let $\mathcal{C}$ be a $\Gamma$-graded $k$-linear category. The (graded) Karoubi envelope $\text{Kar}(\mathcal{C})$ is the category whose objects are pairs $(X, e)$ in which $X \in \mathcal{C}$ and $e$ is a degree zero idempotent endomorphism of $X$, i.e. $e \in \text{End}^0_{\mathcal{C}}(X)$ and $e^2 = e$. Morphisms in $\text{Kar}(\mathcal{C})$ are defined by
\[ \text{Hom}_{\text{Kar}(\mathcal{C})}((X, e), (X', e')) := \{ f \in \text{Hom}_{\mathcal{C}}(X, X') \mid f = e' \circ f \circ e \} \]
In other words, the graded Karoubi envelope of $\mathcal{C}$ has the same objects as the usual Karoubi envelope of $\mathcal{C}_0$. Often, we will use abbreviated notation for objects in $\text{Kar}(\mathcal{C})$, denoting the object $(X, e)$ simply by the idempotent $e$ itself.

### 2.2. Shifts and biproducts.
We now discuss (in turn) shifts and biproducts in the graded linear setting. Our eventual aim is Definition 2.10, which gives the completion of a $\Gamma$-graded $k$-linear categories with respect to these notions.

Given $j \in \Gamma$, let $q^j$ denote the endofunctor of $\text{Vect}_0^\Gamma$ (or $\text{Vect}^\Gamma$) that shifts grading up by $j$. In other words, if $V = (V_i)_{i \in \Gamma}$ is a graded vector space, then $q^jV$ is the graded vector space with $(q^jV)_i = V_{i-j}$. The following extends this notion to graded linear categories.

**Definition 2.4.** Let $X, Y \in \mathcal{C}$ be objects of a $\Gamma$-graded $k$-linear category and let $k \in \Gamma$. We denote $Y \cong q^k X$ if $Y$ is equipped with an invertible morphism $\sigma \in \text{Hom}_{\mathcal{C}}(X, Y)$. Such objects are called shifts of $X$.

Note that the morphism $\sigma$ in Definition 2.4 is (typically) not an isomorphism in $\mathcal{C}$ (see Remark 2.2). Shifts of $X$ are characterized by the following universal property, whose proof is a straightforward exercise.

**Lemma 2.5.** The following are equivalent.

1. $Y \cong q^k X$.
2. $\text{Hom}_{\mathcal{C}}(Y, -) \cong q^{-k} \text{Hom}_{\mathcal{C}}(X, -)$ as functors $\mathcal{C} \to \text{Vect}^\Gamma$.
3. $\text{Hom}_{\mathcal{C}}(-, Y) \cong q^k \text{Hom}_{\mathcal{C}}(-, X)$ as functors $\mathcal{C} \to \text{Vect}^\Gamma$.

We next discuss biproducts. We emphasize to the reader that we will use the standard notation $\coprod$ and $\prod$ for general category-theoretic coproducts and products (reviewed in the graded linear setting below), and reserve the symbol $\bigoplus$ for biproducts, which we now recall. Note that a $\Gamma$-graded $k$-linear
category can be viewed as enriched over pointed sets: in each Hom-space there exists a zero-morphism $0_{Y,X} \in \text{Hom}_C(X, Y)$ satisfying

- $f \circ 0_{Y,X} = 0_{Z,X}$ for all $f \in \text{Hom}_C(Y, Z)$, and
- $0_{Y,X} \circ g = 0_{Y,Z}$ for all $g \in \text{Hom}_C(Z, X)$.

**Definition 2.6.** Let $C$ be a $\Gamma$-graded $k$-linear category. Given objects $Y, X_i \in C$ where $i$ ranges over a (possibly infinite) set $I$, we say that $Y$ is the biproduct of the $X_i$, and denote this $Y \cong \bigoplus_i X_i$, provided $Y$ is equipped with morphisms $\sigma_i \in \text{Hom}_C(X_i, Y)$ and $\pi_i \in \text{Hom}_C(Y, X_i)$ such that

- (i) $\pi_i \circ \sigma_j = \delta_{ij} \text{id}_{X_i}$ if $i = j$ and $0_{Y,X}$ if $i \neq j$,
- (ii) the collection $\{\sigma_i\}_{i \in I}$ gives $Y$ the structure of the coproduct $\bigsqcup_{i \in I} X_i$, and
- (iii) the collection $\{\pi_i\}_{i \in I}$ gives $Y$ the structure of the product $\prod_{i \in I} X_i$.

**Remark 2.7.** More generally, the above definition could be used in any category enriched over pointed sets. Our condition (i) is readily seen to be equivalent to the condition sometimes seen in the literature, namely that the “canonical comparison map” $\prod_{i \in I} X_i \to \bigsqcup_{i \in I} X_i$ is an isomorphism.

In the (graded) linear setting, finite biproducts are easily recognized, as they are characterized “equationally.” The following is standard, see e.g. [ML98, §VIII.2, Theorem 2].

**Lemma 2.8** (Finite biproduct recognition). Let $C$ be a $\Gamma$-graded $k$-linear category and let $\sigma_i \in \text{Hom}_C(X_i, Y)$ and $\pi_i \in \text{Hom}_C(Y, X_i)$ for $i = 1, \ldots, k$. The maps $\{\pi_i\}_{i = 1}^k$ and $\{\sigma_i\}_{i = 1}^k$ exhibit $Y$ as the biproduct of the $X_i$ if and only if:

- $\pi_i \circ \sigma_j = \delta_{ij} \text{id}_{X_i}$, and
- $\text{id}_Y = \sum_{i = 1}^k \sigma_i \circ \pi_i$. \hfill \Box

**Remark 2.9.** An immediate consequence of Lemma 2.8 is that any linear functor $C \to D$ sends finite biproducts to finite biproducts (i.e. biproducts are absolute colimits in categories enriched in abelian groups). Shifts are similarly preserved by all graded linear functors.

In §2.4 below we give an extension of Lemma 2.8 that gives a similar “equational” characterization of certain infinite biproducts. Note that infinite biproducts do sometimes exist “in nature”, e.g. the direct sum of countably many graded vector spaces is a biproduct provided the direct sum is locally finite.

It is possible to formally adjoin both grading shifts and finite biproducts to a given $\Gamma$-graded $k$-linear category that may lack them.

**Definition 2.10.** If $C$ is a $\Gamma$-graded $k$-linear category, then the $\Gamma$-additive completion of $C$ is the category $\text{Mat}(C)$ wherein objects are formal expressions $\bigoplus_{i \in I} q^{k_i} X_i$ where $I$ is a finite set, $k_i \in \Gamma$ and $X_i \in C$. Morphisms are given by matrices, i.e.

$$\text{Hom}_{\text{Mat}(C)} \left( \bigoplus_{i \in I} q^{k_i} X_i, \bigoplus_{j \in J} q^{l_j} Y_j \right) := \bigoplus_{(j,i) \in J \times I} q^{l_j - k_i} \text{Hom}_C(X_i, Y_j).$$

**Remark 2.11.** If $C$ already admits grading shifts and finite biproducts, then the canonical inclusion $C \hookrightarrow \text{Mat}(C)$ is an equivalence.

We conclude this section with a discussion of copowers, which are a common generalization of both shifts and certain finite biproducts.

**Definition 2.12.** Let $C$ be a $\Gamma$-graded $k$-linear category. Given $X \in C$ and $V \in \text{Vect}^\Gamma$, the copower $V \otimes X$ is the object (unique up to canonical isomorphism when it exists) characterized by isomorphisms

$$\text{Hom}_C(V \otimes X, Y) \cong \text{Hom}_{\text{Vect}^\Gamma}(V, \text{Hom}_C(X, Y))$$

natural in $Y$. 

Example 2.13. If $V = q^k k$, then the copower $V \otimes X$ coincides with the shift $q^k X$. If $V = \prod_i q^k k$ (not necessarily finite), then the copower $V \otimes X$ satisfies

$$V \otimes X \cong \prod_i q^k X.$$  

2.3. Colimits and ind-completion. We assume the reader is familiar with the standard notions of limits and colimits in ordinary category theory. Although there is a rich theory of (co)limits in the setting of enriched categories, our purposes require only slight elaboration on the usual definitions. (Essentially, we require that all diagrams and structure maps have degree zero.) We now recall the necessary background, restricting to the case of colimits in $\text{Vect}_\Gamma$-enriched categories for efficiency of exposition.

Definition 2.14. Let $\mathcal{C}$ be a $\Gamma$-graded $k$-linear category. A diagram in $\mathcal{C}$ is a small category $\mathcal{I}$ and a functor $\alpha: \mathcal{I} \to \mathcal{C}$ that factors through the canonical inclusion $\mathcal{C}_0 \to \mathcal{C}$. The colimit of $\alpha$ in $\mathcal{C}$ is the object $\text{colim}_{\alpha} \alpha(i)$ (unique up to canonical isomorphism if it exists) characterized by

$$\text{Hom}_{\mathcal{C}}(\text{colim}_{\alpha} \alpha(i), -) \cong \lim_{i \in \mathcal{I}} \text{Hom}_{\mathcal{C}}(\alpha(i), -)$$

as functors $\mathcal{C} \to \text{Vect}^\Gamma$.

We will refer to a diagram $\alpha: \mathcal{I} \to \mathcal{C}$ as a $\mathcal{I}$-indexed and will call $\mathcal{I}$ the indexing category.

Remark 2.15. We draw the reader’s attention to some details in Definition 2.14. For fixed $Y \in \mathcal{C}$, the limit $\lim_{i \in \mathcal{I}} \text{Hom}_{\mathcal{C}}(\alpha(i), Y)$ can be calculated in $\text{Vect}_0^\Gamma$ since the structure maps have degree zero by hypothesis on $\alpha$. This limit can be described as an explicit graded subspace of $\prod_{i \in \mathcal{I}} \text{Hom}_{\mathcal{C}}(\alpha(i), Y)$. Letting $Y$ vary yields the functor $\mathcal{C} \to \text{Vect}^\Gamma$ appearing in the right-hand side of (2). By Remark 2.2, the structure maps in $\text{Hom}_{\mathcal{C}}(\alpha(i), \text{colim}_{\alpha} \alpha(i))$ (obtained by plugging the identity morphism of $\text{colim}_{\alpha} \alpha(i)$ into the left-hand side of (2)) are necessarily degree zero.

Remark 2.16. Colimits as in Definition 2.14 are called conical colimits in the language of enriched category theory, to distinguish them from the more general notion of weighted colimits. All explicit references to colimits in this paper are conical, so we drop the adjective. The notion of (conical) limit is dual.

Example 2.17. Let $\mathcal{I}$ be a (small) discrete category, i.e. a category wherein the only morphisms are the identity morphisms. An $\mathcal{I}$-indexed diagram in $\mathcal{C}$ then consists of a set of objects $\{X_i\}_{i \in \mathcal{I}}$ (given by $X_i = \alpha(i)$) and we have $\text{colim}_{i \in \mathcal{I}} \alpha(i) = \prod_{i \in \mathcal{I}} X_i$ and $\lim_{i \in \mathcal{I}} \alpha(i) = \prod_{i \in \mathcal{I}} X_i$. As noted in Remark 2.15, the inclusion $\rho_i: X_i \to \prod_{i \in \mathcal{I}} X_i$ and projection $\pi_i: \prod_{i \in \mathcal{I}} X_i \to X_i$ are degree zero.

We will make particular use of filtered colimits, and now review all we will need concerning them.

Definition 2.18. A small category $\mathcal{I}$ is filtered provided the following conditions hold:

- given objects $i, j \in \mathcal{I}$, there exists an object $k$ and morphisms $i \to k$ and $j \to k$, and
- given parallel morphisms $f: i \to j$ and $g: i \to j$ in $\mathcal{I}$, there exists a morphism $h: j \to k$ so that $hf = hg$.

If $\mathcal{C}$ is a $\Gamma$-graded $k$-linear category, then a directed system in $\mathcal{C}$ is an $\mathcal{I}$-indexed diagram in $\mathcal{C}$ where $\mathcal{I}$ is a (small) filtered category; a filtered colimit in $\mathcal{C}$ is the colimit of such a directed system.

It is often useful to compute (filtered) colimits by restricting to certain subcategories of the indexing category. More generally, suppose that $F: \mathcal{I} \to \mathcal{J}$ is a functor and that $\alpha: \mathcal{J} \to \mathcal{C}$ is a $\mathcal{J}$-indexed diagram in $\mathcal{C}$. The structure maps $\alpha(F(i)) \to \text{colim}_{j \in \mathcal{J}} \alpha(j)$ then induce a canonical comparison map

$$\text{colim}_{i \in \mathcal{I}} \alpha(F(i)) \to \text{colim}_{j \in \mathcal{J}} \alpha(j).$$
Definition 2.19. Let $F: \mathcal{J} \to \mathcal{B}$ be a functor between (indexing) categories. The functor $F$ is called \textbf{final} if the canonical comparison map \eqref{eq:comparison-map} is an isomorphism for all diagrams $\alpha: \mathcal{J} \to \mathcal{C}$ such that both colimits involved exist.

We will also refer to a full subcategory $\mathcal{I} \subset \mathcal{J}$ as \textbf{final} if the inclusion functor is final. The following is a standard criterion for establishing that a functor is final (it is a special case of the equivalent characterization of final from \cite{ML98, §IX.3}).

Lemma 2.20. If $F: \mathcal{J} \to \mathcal{B}$ is a functor such that:

- for each $j \in \mathcal{J}$ there is a morphism $f_j : F(i) \to F(j)$ for some $i \in \mathcal{I}$, and
- given $i \in \mathcal{I}$ and parallel morphisms $f_1, f_2 : j \to F(i)$ in $\mathcal{B}$ there exists a morphism $g : i \to i'$ in $\mathcal{J}$ such that $F(g) \circ f_1 = F(g) \circ f_2$,

then $F$ is final. \hfill $\square$

Example 2.21. The poset $(\mathbb{Z}, \leq)$ is a small filtered category. A $\mathbb{Z}$-indexed diagram in a $\Gamma$-graded $k$-linear category $\mathcal{C}$ can be understood as a sequence of objects and degree zero morphisms

$$\cdots \xrightarrow{f_{-1}} A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots$$

in $\mathcal{C}$. The subcategory $(\mathbb{N}, \leq) \subset (\mathbb{Z}, \leq)$ is final.

Recall that a category $\mathcal{C}$ is called \textbf{cocomplete} when colimits of all diagrams in $\mathcal{C}$ exists therein. When this is not the case, it is always possible to embed $\mathcal{C}$ into its \textbf{presheaf category} $\mathcal{C}^{\text{op}}$ (of contravariant functors to $\text{Vect}^\Gamma$, when $\mathcal{C}$ is $\Gamma$-graded $k$-linear) which is cocomplete; however, the latter is often intractable. If one is primarily interested in filtered colimits, the following provides an intermediary means to formally adjoin such colimits to a $\Gamma$-graded $k$-linear category.

Definition 2.22. Let $\mathcal{C}$ be a $\Gamma$-graded $k$-linear category. The \textbf{ind-completion} of $\mathcal{C}$ is the category $\text{Ind}(\mathcal{C})$ with objects the directed systems $\alpha: \mathcal{J} \to \mathcal{C}$. Given directed systems $\alpha: \mathcal{J} \to \mathcal{C}$ and $\beta: \mathcal{J} \to \mathcal{C}$, the morphism space is given by

$$\text{Hom}_{\text{Ind}(\mathcal{C})}(\alpha, \beta) := \lim_{i \in \mathcal{J}} \colim_{j \in \mathcal{J}} \text{Hom}_{\mathcal{C}}(\alpha(i), \beta(j))$$

where on the right-hand side the colimit and limit are taken in $\text{Vect}^\Gamma$ (equivalently here, in $\text{Vect}^\Gamma_{k}$).

An object of $\text{Ind}(\mathcal{C})$ will be informally referred to as an \textbf{ind-object} of $\mathcal{C}$. We will denote ind-objects either by explicitly displaying the directed system, e.g.

$$\alpha(0) \to \alpha(1) \to \cdots$$

or simply by writing\footnote{Since the objects of $\text{Ind}(\mathcal{C})$ are “formal” colimits, it is common to denote them by “$\colim_{i \in \mathcal{I}} \alpha(i)$”; however, we drop the quotation marks since it will be clear from context whether the colimit is an object of $\text{Ind}(\mathcal{C})$ or an object in a category admitting filtered colimits.} $\colim_{i \in \mathcal{J}} \alpha(i)$. In the following, we collect a number of useful facts/observations that will facilitate working in $\text{Ind}(\mathcal{C})$; almost all can be found in \cite{KS06}.

Remark 2.23. Let $\mathcal{J}$ be a (small) category and let $\mathcal{C}$ be $\Gamma$-graded $k$-linear. The collection of $\mathcal{J}$-indexed diagrams in $\mathcal{C}$ forms a $\Gamma$-graded $k$-linear category $\mathcal{C}^{\mathcal{J}}$ wherein morphisms are homogeneous natural transformations. If $\mathcal{J}$ is filtered, then there is a functor $\mathcal{C}^{\mathcal{J}} \to \text{Ind}(\mathcal{C})$, since natural transformations of diagrams induce maps of colimits. In fact, such functors describe all morphisms in $\text{Ind}(\mathcal{C})$. To wit, suppose we are given a morphism $f \in \text{Hom}_{\text{Ind}(\mathcal{C})}(\alpha, \beta)$ from an ind-object $\alpha: \mathcal{J} \to \mathcal{C}$ to $\beta: \mathcal{J} \to \mathcal{C}$. Then, there exists a filtered category $\mathcal{K}$ and final functors $F_1: \mathcal{K} \to \mathcal{J}$ and $F_2: \mathcal{K} \to \mathcal{J}$ and a natural transformation $\bar{f}: \alpha \circ F_1 \Rightarrow \beta \circ F_2$ so that the induced morphism

$$\colim_{i \in \mathcal{J}} \alpha(i) \cong \colim_{k \in \mathcal{K}} \alpha(F_1(k)) \xrightarrow{\bar{f}} \colim_{k \in \mathcal{K}} \beta(F_2(k)) \cong \colim_{j \in \mathcal{J}} \beta(j)$$
agrees with \( f \). (Here, we also denote the map induced on colimits by \( \hat{f} \) by \( \hat{f} \) as well.) This description e.g. elucidates composition of morphisms in \( \text{Ind}(\mathcal{C}) \).

**Remark 2.24.** Suppose that \( \mathcal{D} \) is a \( (\Gamma\text{-graded } k\text{-linear}) \) category that is closed under all (small) filtered colimits. Recall that an object \( K \in \mathcal{D} \) is called *compact* provided for all filtered colimits \( \operatorname{colim}_{i \in I} \alpha(i) \) in \( \mathcal{D} \) we have that

\[
\operatorname{Hom}_\mathcal{D}(K, \operatorname{colim}_{i \in I} \alpha(i)) = \operatorname{colim}_{i \in I} \operatorname{Hom}_\mathcal{D}(K, \alpha(i)).
\]

If \( \mathcal{C} \) is a \( \Gamma\text{-graded } k\text{-linear} \) category, then we may regard it as a full subcategory of \( \text{Ind}(\mathcal{C}) \) via the fully faithful functor \( \mathcal{C} \hookrightarrow \text{Ind}(\mathcal{C}) \) that sends an object \( X \) in \( \mathcal{C} \) to the directed system that is constant at \( X \) and indexed by the one element poset. Under this inclusion, \( \mathcal{C} \) is equivalent to the full subcategory of compact objects in \( \text{Ind}(\mathcal{C}) \). This implies that \( \text{Ind}(\mathcal{C}) \) is compactly generated, i.e. all objects are filtered colimits of compact objects.

Further, \( \text{Ind}(\mathcal{C}) \) admits all filtered colimits and the inclusion \( \mathcal{C} \hookrightarrow \text{Ind}(\mathcal{C}) \) is initial among all functors from \( \mathcal{C} \) to categories that admit all filtered colimits. Explicitly, if \( \mathcal{D} \) admits all filtered colimits, then we can extend any functor \( F: \mathcal{C} \to \mathcal{D} \) to \( \text{Ind}(\mathcal{C}) \) by declaring that \( F(\text{colim}_{i \in I} \alpha(i)) = \text{colim}_{i \in I} F(\alpha(i)) \).

**Remark 2.25.** Remark 2.24 implies that the Yoneda embedding \( \mathcal{C} \hookrightarrow \mathcal{C}^\wedge \) factors through the inclusion \( \mathcal{C} \hookrightarrow \text{Ind}(\mathcal{C}) \). Indeed, \( \text{Ind}(\mathcal{C}) \) admits an alternative definition as the full subcategory of \( \mathcal{C}^\wedge \) of objects that are filtered colimits of representable functors.

Now suppose that \( \mathcal{C} \) is a *monoidal* \( \Gamma\text{-graded } k\text{-linear} \) category. The presheaf category \( \mathcal{C}^\wedge \) is then monoidal under Day convolution \([\text{Day}70]\). This restricts to a monoidal structure on \( \text{Ind}(\mathcal{C}) \) that commutes with filtered colimits in each factor, i.e.

\[
\text{colim}_{i \in I} \alpha(i) \otimes \text{colim}_{j \in J} \beta(j) \cong \text{colim}_{i,j \in I \times J} \alpha(i) \otimes \beta(j).
\]

Note that the inclusion \( \mathcal{C} \hookrightarrow \text{Ind}(\mathcal{C}) \) is monoidal.

### 2.4. Recognizing infinite biproducts

We conclude our categorical background with the following technical result, which provides an analogue of Lemma 2.8 for certain infinite biproducts.

**Lemma 2.26 (Biprodut Recognition).** Let \( \mathcal{D} \) be a compactly generated \( \Gamma\text{-graded } k\text{-linear} \) category, and let morphisms \( \pi_i \in \operatorname{Hom}_{\mathcal{D}_k}(Y_i, Y_i) \) and \( \sigma_i \in \operatorname{Hom}_{\mathcal{D}_k}(Y_i, Y) \) be given for \( i \in I \) (an arbitrary indexing set). The following conditions suffice for the maps \( \{\pi_i\}_{i \in I} \) and \( \{\sigma_i\}_{i \in I} \) to exhibit \( Y \) as the biproduct of the \( Y_i \):

(i) \( \pi_i \circ \sigma_j = \delta_{i,j} \text{id}_{Y_i} \),

(ii) for each compact \( K \in \mathcal{D} \) and \( k \in \Gamma \), we have that \( \operatorname{Hom}_\mathcal{D}^k(K, Y_i) = 0 \) for all but finitely many \( i \in I \), and

(iii) for each compact \( K \in \mathcal{D} \) and \( f \in \operatorname{Hom}_\mathcal{D}(K, Y) \), we have that \( f = \sum_{i \in I} \sigma_i \circ \pi_i \circ f \).

(\text{Note that the sum in (iii) is finite by (ii).})

**Proof.** We first show that the maps \( \{\pi_i\}_{i \in I} \) exhibit \( Y \) as a product \( Y \cong \prod_{i \in I} Y_i \). For this, we must construct a two-sided inverse to the assignment:

\[
\Phi: \operatorname{Hom}_\mathcal{D}(X, Y) \to \prod_{i \in I} \operatorname{Hom}_\mathcal{D}(X, Y_i), \quad F \mapsto (\pi_i \circ F)_{i \in I}
\]

where \( X \in \mathcal{C} \) is arbitrary.

To construct the inverse \( \Phi' \), we approximate \( X \) by compact objects, i.e. we write \( X = \text{colim}_\alpha K_\alpha \) with \( K_\alpha \) compact. Now let \( (f_i)_{i \in I} \in \prod_{i \in I} \operatorname{Hom}_\mathcal{C}(X, Y_i) \) be given. For each \( \alpha \), set

\[
F'_\alpha := \sum_{i \in I} \sigma_i \circ f_i \circ \iota_\alpha \in \operatorname{Hom}_\mathcal{D}(K_\alpha, Y),
\]
where \( \iota_\alpha \in \text{Hom}_D(K_\alpha, X) \) are the structure maps of the colimit. Since \( K_\alpha \) is compact and \( f_i \circ \iota_\alpha \in \text{Hom}_D(K_\alpha, Y_i) \), this sum is finite (in each homogeneous degree), hence gives a well-defined morphism. Furthermore, the collection of maps \( F'_\alpha \in \text{Hom}_D(K_\alpha, Y) \) is compatible with the morphisms \( \gamma_{\beta, \alpha} : K_\alpha \to K_\beta \) in the directed system (whose colimit is \( X \)) in the following sense:

\[
F'_\beta \circ \gamma_{\beta, \alpha} = \sum_{i \in I} \sigma_i \circ f_i \circ \iota_\beta \circ \gamma_{\beta, \alpha} = \sum_{i \in I} \sigma_i \circ f_i \circ \iota_\alpha = F'_\alpha.
\]

We hence let

\[
\Psi' : \prod_{i \in I} \text{Hom}_D(X, Y_i) \to \text{Hom}_D(X, Y), \quad \Psi'((f_i)_{i \in I}) := \text{colim}_{\alpha} F'_\alpha.
\]

Since \( X = \text{colim}_\alpha K_\alpha \), we have that \( \text{Hom}_D(X, Y) \cong \text{lim}_\alpha \text{Hom}_D(K_\alpha, Y) \). Thus, we may check that \( \Psi' \circ (\Phi(F)) = F \) by showing that this holds upon precomposing with each \( \iota_\alpha \). We thus compute:

\[
\Phi(F) \circ \iota_\alpha = \Phi\left(\sum_{i \in I} \sigma_i \circ f_i \circ \iota_\alpha\right) = \sum_{i \in I} \sigma_i \circ \pi_i \circ (F \circ \iota_\alpha) = F \circ \iota_\alpha
\]

as desired. To see that \( \Phi(\Psi'(f_i)_{i \in I}) = (f_i)_{i \in I} \), we must check that

\[
\pi_i \circ \Phi'(f_i) = f_i
\]

as morphisms \( X \to Y_i \) for all \( i \in I \). Again, it suffices to prove equality holds upon precomposing with each \( \iota_\alpha \), so we compute

\[
\pi_i \circ \Phi'(f_i) = \pi_i \circ \sum_{j \in I} \sigma_j \circ f_j \circ \iota_\alpha = \sum_{j \in I} \pi_i \circ \sigma_j \circ f_j \circ \iota_\alpha = f_i \circ \iota_\alpha
\]

as desired.

It remains to show that the maps \( \{\sigma_i\}_{i \in I} \) exhibit \( Y \) as a coproduct \( Y \cong \coprod_{i \in I} Y_i \). To see this, we must construct a two-sided inverse to

\[
\Psi : \text{Hom}_D(Y, Z) \to \prod_{i \in I} \text{Hom}_D(Y_i, Z), \quad G \mapsto (G \circ \sigma_i)_{i \in I}
\]

where \( Z \in D \) is arbitrary. We begin by approximating \( Y \) by compact objects, i.e. we write \( Y = \text{colim}_\beta C_\beta \) with \( C_\beta \in D \) compact. Our candidate two-sided inverse to \( \Psi \) is defined by:

\[
(4) \quad \Psi' : \prod_{i \in I} \text{Hom}_D(Y_i, Z) \to \text{Hom}_D(Y, Z), \quad \Psi'(g_i)_{i \in I} \circ \iota_\beta = \sum_{i \in I} g_i \circ \pi_i \circ \iota_\beta.
\]

Here, \( \iota_\beta \in \text{Hom}_D(C_\beta, Y) \) are the structure maps of the colimit, and we again use that \( \text{Hom}_D(Y, Z) \cong \text{lim}_\beta \text{Hom}_D(C_\beta, Z) \) to see that the formulae specify \( \Psi'(g_i)_{i \in I} \in \text{Hom}_D(Y, Z) \). For each \( \beta \), the sum in (4) is finite (thus well-defined) by condition (ii) applied to the maps \( \pi_i \circ \iota_\beta \in \text{Hom}_D(C_\beta, Y_i) \).

We compute that:

\[
\Psi'(\Psi(G)) \circ \iota_\beta = \sum_{i \in I} (G \circ \sigma_i) \circ \pi_i \circ \iota_\beta = G \circ \sum_{i \in I} \sigma_i \circ \pi_i \circ \iota_\beta \quad \text{for each } \beta,
\]

for each \( \beta \), thus \( \Psi'(\Psi(G)) = G \). To see that \( \Psi'(\Psi'(g_i)_{i \in I}) = (g_i)_{i \in I} \), we must show that \( \Psi'(g_i)_{i \in I} \circ \sigma_i = g_i \) for all \( i \). Since \( \text{id}_{Y_i} = \pi_i \circ \sigma_i \), it suffices to show that, for all \( i \in I \), \( \Psi'(g_i)_{i \in I} \circ \pi_i = g_i \circ \pi_i \) as maps \( Y \to Z \). Yet again, since \( Y = \text{colim}_\beta C_\beta \), it suffices to show that this identity holds after precomposing with \( \iota_\beta' \) for all \( \beta' \). In other words, we have reduced the problem to establishing the identity

\[
\text{colim}_{\beta} \left( \sum_{j \in I} g_j \circ \pi_j \circ \iota_\beta \right) \circ \sigma_i \circ \pi_i \circ \iota_{\beta'} = g_i \circ \pi_i \circ \iota_{\beta'}
\]

for all \( i \) and all \( \beta' \). For this, we will use compactness of \( C_\beta \) in an essential way.
To wit, since $Y = \text{colim}_\beta C_\beta$ is a filtered colimit and $C_{\beta'}$ is compact, we have that
$$\text{Hom}_D(C_{\beta'}, Y) = \text{Hom}_D(C_{\beta'}, \text{colim}_\beta C_\beta) = \text{colim}_\beta \text{Hom}_D(C_{\beta'}, C_\beta).$$
Further, since the colimit is taken over a directed system, every element of the latter lies in the image of the map
$$\text{Hom}_D(C_{\beta'}, C_{\beta_0}) \xrightarrow{\iota_{\beta_0} \circ -} \text{Hom}_D(C_{\beta'}, Y)$$
for some $\beta_0$. It follows that we may write
$$\sigma_i \circ \pi_i \circ \iota_{\beta'} = \iota_{\beta_0} \circ \nu$$
for some $\nu \in \text{Hom}_D(C_{\beta'}, C_{\beta_0})$. We then compute
$$\text{colim}_\beta \left( \sum_{j \in I} g_j \circ \pi_j \circ \iota_\beta \right) \circ \sigma_i \circ \pi_i \circ \iota_{\beta'} = \text{colim}_\beta \left( \sum_{j \in I} g_j \circ \pi_j \circ \iota_\beta \right) \circ \iota_{\beta_0} \circ \nu$$
$$= \sum_{j \in I} g_j \circ \pi_j \circ \iota_{\beta_0} \circ \nu$$
$$= \sum_{j \in I} g_j \circ \pi_j \circ \sigma_i \circ \pi_i \circ \iota_{\beta'}$$
$$= g_i \circ \pi_i \circ \iota_{\beta'}$$
as desired. □

3. DOTTED TEMPERLEY–LIEB AND ANNULAR BAR-NATAN CATEGORIES

In this section, we recall the dotted Temperley–Lieb category and a theorem of Russell [Rus09] showing that it gives an explicit monoidal presentation for the Bar-Natan skein module of the annulus. We further discuss a variation on the latter in which curves (and cobordisms) are permitted to meet the boundary of the (thickened) annulus. Such categories are module categories for the Bar-Natan skein module of the annulus (without boundary), and we give a presentation for the 2-boundary version and its action by the dotted Temperley–Lieb category.

From now on, we work with a field $\mathbb{K}$ of characteristic zero.

3.1. The categories $dTL$ and $ABN$.

**Definition 3.1.** The dotted Temperley–Lieb category $dTL$ is the $\mathbb{Z}$-graded $\mathbb{K}$-linear pivotal category freely generated by a single (symmetrically) self-dual object $c$ and an endomorphism $x \in \text{End}^2_{dTL}(c)$, modulo a certain monoidal ideal generated by three relations. Using the standard graphical language for pivotal categories and denoting the endomorphism $x$ as a dot, the relations are as follows:

$$\bigcirc = 2, \quad \updownarrow^2 := 0, \quad \downarrow + \downarrow = \bigcup + \bigcup.$$

These relations, together with the relation that a dotted circle equals zero (which is implied by (5) since $\text{char}(\mathbb{K}) \neq 2$), are called the SBN relations in [AF07].

**Remark 3.2.** Unpacking Definition 3.1, we see that objects in $dTL$ are the tensor powers $c^n := c^{\otimes n}$ of the generating object $c$ for $n \in \mathbb{N}$. The morphism space $\text{Hom}_{dTL}(c^n, c^m)$ is the $\mathbb{Z}$-graded $\mathbb{K}$-vector space spanned by dotted $(m, n)$-planar tangles (with $m$ points at the bottom and $n$ points at the top), modulo the relations in (5) and planar isotopy (which is implicit in the word “pivotal” above). The degree of such a dotted tangle equals $2(\# \text{ of dots})$. Note that the relations in (5) imply that

$$\updownarrow + \updownarrow = \bigcup.$$
Remark 3.3. The endomorphism algebras of \(d\text{TL}\) are also called “nil-blob algebras”. In \([LPRH21]\), it is shown that they arise naturally as diagrammatically defined endomorphism subalgebras of type \(\tilde{A}_1\) Soergel bimodules and as singular weight idempotent truncations of suitable KLR algebras.

Remark 3.4. Let \(V = \text{span}_\mathbb{Q}(v_+, v_-)\) denote the graded \(\mathbb{Q}\)-vector space with basis elements of degree \(\deg(v_{\pm}) = \pm 1\). The dotted Temperley–Lieb category \(d\text{TL}\) over \(K = \mathbb{Q}\) acts on tensor powers of \(V\) as follows:

\[
\bigcup : V \otimes V \to \mathbb{Q}, \quad \begin{cases}
v_+ \otimes v_+ &\mapsto 0 \\
v_+ \otimes v_- &\mapsto 1 \\
v_- \otimes v_+ &\mapsto 1 \\
v_- \otimes v_- &\mapsto 0
\end{cases},
\bigcap : V \to V, \quad \begin{cases}v_+ &\mapsto 0 \\
v_- &\mapsto v_+
\end{cases},
\]

The cup and the cap maps can be interpreted as the \(q = 1\) reduction of morphisms between tensor powers of type II vector representations of the quantum group \(U_q(\mathfrak{sl}_2)\). The dot corresponds to the action of the quantum group Chevalley generator \(E\). The action is faithful, as we will see (in different language) in §3.5.

Remark 3.5. Restricting to the subcategory of degree zero morphisms in \(d\text{TL}\) recovers the circle-value 2 specialization of the Temperley–Lieb category. Recall that this category is equivalent to the full subcategory of \(\mathfrak{sl}_2\) representations tensor generated by the defining representation. In the following, the Temperley–Lieb category will always refer to this specialization, which we denote by TL.

We next discuss the Bar-Natan category \([BN05]\) of an orientable surfaces \((\Sigma, p)\) with marked points on its boundary, which is the natural setting for Khovanov homology (of tangles in thickened surfaces). When \(\Sigma = \mathbb{D}\) and \(|p| = m + n\), this categorifies the Hom-space \(\text{Hom}_{\text{TL}}(c^m, c^n)\). Surprisingly (at least to those unfamiliar with the theory of trace decategorification), we’ll see below that in the case when \(\Sigma = \mathcal{A}\) is the annulus and \(p = \emptyset\), this category agrees with \(d\text{TL}\).

Definition 3.6. Let \(\Sigma\) be an orientable surface with (possibly empty) boundary and let \(p \subset \partial \Sigma\) be finite. The Bar-Natan category is \(\text{BN}(\Sigma; p) := \text{Mat}(\text{BN}(\Sigma; p))\) where \(\text{BN}(\Sigma; p)\) is the \(\mathbb{Z}\)-graded \(K\)-linear category defined as follows. Objects in \(\text{BN}(\Sigma; p)\) are smoothly embedded 1-manifolds \(C \subset \Sigma\) with boundary \(\partial C = p\) meeting \(\partial \Sigma\) transversely. Given objects \(C_1, C_2\), \(\text{Hom}_{\text{BN}}(C_1, C_2)\) is the \(\mathbb{Z}\)-graded \(K\)-module spanned by embedded orientable cobordisms \(W \subset \Sigma \times [0, 1]\) with corners (when \(p \neq \emptyset\)) from \(C_1\) to \(C_2\), modulo the following local relations:

\[
(7) \quad \begin{array}{c}
\text{cylinder} = \text{disk} + \text{disk} \\
\text{cylinder with dot} = 0, \\
\text{disk with dot} = 1, \\
\text{dot} = 0.
\end{array}
\]

The degree of a cobordism with corners \(W : C_1 \to C_2\) is given by \(\deg(W) = \frac{1}{2}|p| - \chi(W)\), and a dot on a surface is used as shorthand for taking connect sum with a torus at that point and multiplying by \(\frac{1}{2}\). For example,

\[
(8) \quad \text{dot} := \frac{1}{2} \text{torus}.
\]
Remark 3.7. It is possible to define the Bar-Natan category integrally, if we work with “formally dotted” cobordisms. In this case, the first relation in (7) (the so-called neck-cutting relation) implies the characterization of a dot given in (8), after clearing denominators.

Remark 3.8. The first three relations in (7) imply that the neck-cutting relation gives an idempotent decomposition for the identity morphism of any null-homotopic circle in Σ, i.e. we have the “circle removal” isomorphism

\[ \bigcirc \cong q^{-1} \emptyset \oplus q \emptyset \]

in BN(Σ; p) for such circles.

Definition 3.9. Let ABN := BN(S1 × [0, 1]; ∅) denote the Bar-Natan category associated to the annulus with no points on the boundary. We refer to ABN as the annular Bar-Natan category.

Remark 3.10. ABN is a monoidal category, with tensor product given by glueing one (thickened) annulus inside the other, so that A ⊗ B is “A inside B”.

The following is essentially a re-packaging of a theorem of Russell [Rus09].

Proposition 3.11. There is a fully faithful monoidal functor Φ: dTL↪→ABN defined by “rotating dotted Temperley–Lieb diagrams” around the annular core, i.e.

\[ \bullet \mapsto \bullet \quad \bullet \mapsto \bullet \quad \bullet \mapsto \bullet \quad \bullet \mapsto \bullet \]

The induced functor Mat(dTL) → ABN is an equivalence of categories.

Proof. Fullness and essential surjectivity (after passing to Mat(dTL)) follow as in [QR18, Propositions 3.8 and 4.2], which establish the analogous result in the setting of gl2 foams. Faithfulness follows from [Rus09, Theorem 2.1]. □

Remark 3.12. We could have worked in greater generality throughout §3.1 and in the following. Generalizing Definition 3.1, for graded commutative ring R and fixed elements h ∈ R2, t ∈ R4, one can also consider the equivariant R-linear dotted Temperley–Lieb category dTLR, defined analogously as above, but subject to the relations:

\[ \bigcirc = 2, \quad \bullet^2 := h + t \quad \bullet^+ \quad \bullet^- h \quad \bigcup \bigcup - h \bigcup \bigcup \]

If 2 is invertible, the third relation implies that the dotted circle evaluates to h. All of the above results carry over to the equivariant setting, provided we work with analogously generalized categories BN^R(Σ; p).

3.2. Basic structure of dTL. Since dTL is a Z-graded category, its center Z(dTL) (i.e. the endo-natural transformations of the identity functor) is a Z-graded algebra.

Lemma 3.13. There is a map of graded algebras \( \mathbb{K}[s, z]/(s^2 = 1) \rightarrow Z(dTL) \) defined by

\[ z_{|c} := \sum (-1)^n (\cdots (-1)^{n-1} \cdots) \]

and \( s_{|c} = (-1)^n \text{id}_{c} \).

Proof. Straightforward. □

In Corollary 3.23 we prove that this algebra map is an isomorphism.
Remark 3.14. We will use the abbreviation $z_n := z_{|c_n}$. We have $z_n^{n+1} = 0$ because each of the $n$ strands in $\text{id}_c$ can carry at most one dot.

Remark 3.15. The involution $d\text{TL} \to d\text{TL}$ given by reflecting all diagrams across a vertical axis sends $z_n \mapsto (-1)^{n-1}z_n$, i.e. $z \mapsto -sz$.

The object $c_n \in d\text{TL}$ carries an action of the symmetric group $\mathfrak{S}_n$, by defining

\[ (11) \]

However, this does not quite make $d\text{TL}$ into a symmetric monoidal category, because dots slide only up to a sign:

\[ \cdot = -\cdot, \quad \cdot = -\cdot \]

It is sometimes convenient to normalize away this sign.

Convention 3.16. Let $x_i \in \text{End}_{d\text{TL}}(c_n)$ denote a dot on the $i$-th strand and set $\bar{x}_i := (-1)^{i-1}x_i$, i.e.

\[ \bar{x}_i = (-1)^{i-1}x_i = (-1)^{i-1} \]

where

\[ \bar{x} := \cdots \]

We will sometimes also denote $f \in \mathbb{K}[x_1, \ldots, x_n]/(x_i^2) = \mathbb{K}[\bar{x}_1, \ldots, \bar{x}_n]/(\bar{x}_i^2) \subset \text{End}_{d\text{TL}}(c^n)$ by

\[ \frac{\bar{x} \cdots \bar{x}}{f} \]

Note that $w(f(\bar{x}_1, \ldots, \bar{x}_n))w^{-1} = f(\bar{x}_w(1), \ldots, \bar{x}_w(n))$ for all $w \in \mathfrak{S}_n$ and all $f \in \mathbb{K}[\bar{x}_1, \ldots, \bar{x}_n]/(\bar{x}_i^2)$, and $z_n = \bar{x}_1 + \cdots + \bar{x}_n$.

3.3. The symmetrizing idempotent.

Definition 3.17. Let $P_n \in \text{End}_{d\text{TL}}(c^n)$ denote the symmetrizing idempotent, also known as the $n$th Jones–Wenzl projector (at circle-value 2), defined by

\[ P_n := \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} := \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \begin{array}{c} \vdots w \\ \vdots \\ \vdots \\ \vdots \end{array} \]

The Jones–Wenzl projector $P_n$ is uniquely characterized by the following properties:

- $P_n^2 = P_n$,
- $P_n w = P_n = w P_n$ for all $w \in \mathfrak{S}_n$ (equivalently, $P_n$ “kills turnbacks”), and
- $P_n \neq 0$

and admits an inductive description via

\[ (12) \]

For more details, see e.g. [KL94].
Lemma 3.18. The following relations hold in dTL:

\[(13)\quad P_n x_i P_n = -P_n x_{i+1} P_n \quad \text{for } 1 \leq i \leq n - 1\]

\[(14)\quad P_n x_1 \cdots x_k P_n = \frac{(-1)^{\binom{k}{2}} (n-k)!}{n!} z_n^k P_n\]

\[(15)\quad \frac{n-1}{P_n} \quad = \quad \frac{n-1}{P_n} \quad \text{for } 0 \leq k \leq n - 3 \quad \text{(and its vertical reflection)}\]

Proof. Throughout the proof we use the auxiliary variables \(\bar{x}_i = (-1)^{i-1} x_i\) from Convention 3.16, as well as their straightforward interaction with permutations \(w \in \mathfrak{S}_n\). In particular, we will make repeated use of the relation

\[(17)\quad P_n f(\bar{x}_1, \ldots, \bar{x}_n) P_n = P_n w f(\bar{x}_1, \ldots, \bar{x}_n) w^{-1} P_n = P_n f(\bar{x}_{w(1)}, \ldots, \bar{x}_{w(n)}) P_n\]

which holds since \(P_n\) absorbs permutations. Equation (13) is now immediate.

For (14), recall that \(z_n = \bar{x}_1 + \cdots + \bar{x}_n\), so \(z_n^k = k! \sum_{i_1 < \cdots < i_k} \bar{x}_{i_1} \cdots \bar{x}_{i_k}\). Now, compute:

\[z_n^k P_n = P_n z_n^k P_n = k! \sum_{i_1 < \cdots < i_k} P_n \bar{x}_{i_1} \cdots \bar{x}_{i_k} P_n = k! \binom{n}{k} P_n \bar{x}_1 \cdots \bar{x}_k P_n.\]

In the first equality we use that \(z_n\) is central and that \(P_n^2 = P_n\), and in the last equality we use (17). Accounting for the signs gives (14), since \(\bar{x}_1 \cdots \bar{x}_k = (-1)^{\binom{k}{2}} x_1 \cdots x_k\).

For (15), it suffices to consider the \(n = 3\) case, by the \(\ell = 3\) case of the “absorption” relation:

\[(18)\quad (\text{id}_{i-1} \otimes P_\ell \otimes \text{id}_{n-i-\ell+1}) P_n = P_n = P_n (\text{id}_{i-1} \otimes P_\ell \otimes \text{id}_{n-i-\ell+1}).\]

We compute using the last relation in (5) that

\[\frac{n-1}{P_n} = - \frac{n-1}{P_n} + \frac{n-1}{P_n} + \frac{n-1}{P_n} = \frac{n-1}{P_n}.\]

Lastly,

\[\frac{n-1}{P_n} \quad = \quad \frac{n-1}{P_n} \quad - \quad \frac{n-1}{P_n} \quad \text{which gives (16) since the relations in (5) imply that a dotted circle is zero.}\]
3.4. The Karoubi envelope of dTL. In this section, we give an explicit presentation of Kar(dTL) in terms of the objects $P_n$. Recall that the Karoubi envelope of the usual Temperley-Lieb category TL is generated by the (images of the) Jones-Wenzl idempotents $P_n$. Further, these objects are simple in Kar(TL), with one-dimensional endomorphism algebras, and no nonzero morphisms $P_n \to P_m$ for $n \neq m$. The situation for dTL is quite different: there are many interesting morphisms in Kar(dTL) between $P_n$ for various $n$.

Motivated by Lemma 3.18, we now define certain morphisms in Kar(dTL).

**Definition 3.19.** For each $n \geq 0$, let $U_n : P_n \to P_{n+2}$ and $D_{n+2} : P_{n+2} \to P_n$ be the following morphisms in Kar(dTL):

\[
U_n := \begin{array}{ccc}
\ldots & P_{n+2} & \\
\ldots & \vdots & \\
P_n & \cdot & P_n
\end{array},
\quad D_{n+2} := (n+2)(n+1) \begin{array}{ccc}
\ldots & P_{n+2} & \\
\ldots & \vdots & \\
P_n & \cdot & P_n
\end{array}.
\]

Let $z_n : P_n \to P_n$ denote (by slight abuse of notation) the element of End$_{dTL}(P_n)$ given by the central element $z$ from Lemma 3.13.

**Lemma 3.20.** The morphisms in Definition 3.19 satisfy the following relations:

\[
z_n^{n+1} = 0, \quad U_{n-2}D_n = -z_n^2 = D_n U_{n-2}, \quad z_n U_{n-2} = U_{n-2} z_{n-2}, \quad z_n D_{n+2} = D_{n+2} z_{n+2}
\]

**Proof.** The first relation is immediate (see Remark 3.14). Next, we have

\[
U_{n-2}D_n = n(n-1) \begin{array}{ccc}
\ldots & P_n & \\
\ldots & \vdots & \\
P_n & \cdot & P_n
\end{array} = (6) \begin{array}{ccc}
\ldots & P_n & \\
\ldots & \vdots & \\
P_n & \cdot & P_n
\end{array} = n(n-1) \quad (16) \begin{array}{ccc}
\ldots & P_n & \\
\ldots & \vdots & \\
P_n & \cdot & P_n
\end{array} = -z_n^2.
\]

Next, we compute

\[
D_{n} U_{n-2} = n(n-1) \begin{array}{ccc}
\ldots & P_n & \\
\ldots & \vdots & \\
P_n & \cdot & P_n
\end{array} = (12) \begin{array}{ccc}
\ldots & P_n & \\
\ldots & \vdots & \\
P_n & \cdot & P_n
\end{array} = n(n-1) \quad - (n-1)^2 \begin{array}{ccc}
\ldots & P_{n-1} & \\
\ldots & \vdots & \\
P_{n-1} & \cdot & P_{n-1}
\end{array} = (16) \begin{array}{ccc}
\ldots & P_{n-1} & \\
\ldots & \vdots & \\
P_{n-1} & \cdot & P_{n-1}
\end{array} = -z_{n-2}^2.
\]

Lastly, the final two relations follow from centrality of $z$. □

Using Lemma 3.20 (and Proposition 3.26 below), we obtain a generators-and-relations presentation for Mat(Kar(dTL)) as a $\mathbb{Z}$-additive $\mathbb{K}$-linear category.

**Theorem 3.21.** The category Mat(Kar(dTL)) is equivalent to the category of finitely-generated graded projective modules for the path algebra of the quiver (with two connected components)

\[
\begin{array}{c}
P_0 \\
\downarrow D_2
\end{array} \overset{u_0}{\leftarrow} \begin{array}{c}
P_2 \\
\downarrow U_2
\end{array} \overset{z_2}{\leftarrow} \begin{array}{c}
P_4 \\
\downarrow U_4
\end{array} \overset{z_4}{\leftarrow} \cdots, \quad
\begin{array}{c}
P_1 \\
\downarrow D_3
\end{array} \overset{u_1}{\leftarrow} \begin{array}{c}
P_3 \\
\downarrow U_3
\end{array} \overset{z_3}{\leftarrow} \begin{array}{c}
P_5 \\
\downarrow U_5
\end{array} \overset{z_5}{\leftarrow} \cdots
\end{array}
\]

subject to the relations in Lemma 3.20.
Similar quiver-descriptions exist for other (non-semisimple) variations of Temperley–Lieb categories, e.g. in positive characteristic [TW21] and at roots of unity [STWZ21]. However, such quiver descriptions do not capture the monoidal structure that is present in all these cases. For background on locally unital algebras, such as path algebras of infinite quivers, and their modules, see [BS18, §2.2].

**Proof.** Let \( \mathcal{C} \) denote the category of finitely-generated graded projective modules for the quiver algebra described in the statement. The indecomposable objects in \( \mathcal{C} \) are precisely the projective modules corresponding to the vertices \( P_n \), which we will denote by the same symbol. The assignments

\[
P_n \mapsto P_n, \quad z_n \mapsto z_n, \quad U_n \mapsto U_n, \quad D_n \mapsto D_n
\]

then determine a functor \( \mathcal{C} \to \text{Mat}(\text{Kar}(\text{dTL})) \), and it remains to show that this establishes an equivalence of categories.

The subcategory of \( \text{dTL} \) consisting of morphisms of degree zero is precisely the Temperley–Lieb category \( \text{TL} \). By Definition 2.3, the indecomposable objects in \( \text{Mat}(\text{Kar}(\text{dTL})) \) are thus the same as in \( \text{Mat}(\text{Kar}(\text{TL})) \), i.e. they are the (images of the) Jones–Wenzl projectors and their shifts. Hence, the functor \( \mathcal{C} \to \text{Mat}(\text{Kar}(\text{dTL})) \) is essentially surjective.

It remains to see that it is fully faithful. For this, we assume that \( m \leq n \) and give an explicit spanning set for \( \text{Hom}_{\text{Mat}(\text{Kar}(\text{dTL}))}(P_m, P_n) \); the \( m > n \) case is analogous (but also follows from this case via duality). By definition, this Hom-space is spanned by elements \( P_n X P_m \) where \( X \) is an \((m,n)\)-planar tangle, with each component possibly carrying a single dot. If \( X \) has any un-dotted caps at its bottom or un-dotted cups at its top, then \( P_n X P_m = 0 \). Otherwise, we can use (15) to express

\[
P_n X P_m = \pm \frac{(k - \ell)!}{m!} U_{n-2} \cdots U_m z_m^{n+\ell-k}.
\]

Here, \( k \) is the number of “through strands” in \( X \), \( \ell \) is the number of dots on those strands, and \( \pm \) indicates some signs, not necessarily equal.

Thus, \( \text{Hom}_{\text{Kar}(\text{dTL})}(P_m, P_n) \) is spanned by the elements \( \{ U_{n-2} \cdots U_m z_m^k \}_{k=0}^m \) when \( m \leq n \) and by the elements \( \{ z_m^k D_{n+2} \cdots D_m \}_{k=0}^n \) when \( m > n \). Hence, the functor \( \mathcal{C} \to \text{Mat}(\text{Kar}(\text{dTL})) \) is full. In order to see that it is faithful, it suffices to show that the above spanning sets are the images of spanning sets of the corresponding morphism spaces in \( \mathcal{C} \) (an easy check using the relations on the quiver underlying \( \mathcal{C} \)) and are linearly independent. The latter follows by showing that the spanning morphisms act linearly independently on the polynomial representation of \( \text{dTL} \) considered in §3.5, and we delay this check until that section. \( \square \)

**Corollary 3.22.** Let \( m - n \in 2\mathbb{Z} \). There is an isomorphism of graded vector spaces

\[
\text{Hom}_{\text{Kar}(\text{dTL})}(P_m, P_n) \cong q^{m-n}[K[z]/z^{1+\min(m,n)}]
\]

induced by composition with the maps \( D_r \) and \( U_r \), which identifies \( z \) with the endomorphism \( z_{\min(m,n)} \in \text{End}_{\text{Kar}(\text{dTL})}^2(P_{\min(m,n)}) \).

If \( m - n \) is odd, then clearly \( \text{Hom}_{\text{Kar}(\text{dTL})}(P_m, P_n) = 0 \) for parity reasons.

**Corollary 3.23.** The map \( K[s,z]/(s^2 - 1) \to Z(\text{dTL}) \) from Lemma 3.13 is an isomorphism of graded algebras.
Proof. We will construct the inverse isomorphism \( Z(\text{dTL}) \to \mathbb{K}[s, z]/(s^2 = 1) \) explicitly. Suppose we are given \( f \in Z(\text{dTL}) \). For each \( n \), let \( f_n \) denote the restriction of \( f \) to \( P_n \). By Corollary 3.22, post-composing with the dotted cup \( U_n \) gives an isomorphism of graded vector spaces \[
abla \text{dTL}(P_n) \cong q^{-2} \text{Hom} \text{dTL}(P_n, P_{n+2}) \, ,
abla
\]
and pre-composing with \( U_n \) induces a map \[
abla \text{dTL}(P_{n+2}) \to q^{-2} \text{Hom} \text{dTL}(P_n, P_{n+2}) \cong \text{dTL}(P_n) \, .
\]
By centrality, \( f_n \) and \( f_{n+2} \) must have the same image in \( \text{dTL}(P_n) \), so \( (f_n)_{n \in \mathbb{N}} \) defines an element of the limit \[
abla \lim_{n \in I} \text{End} \text{dTL}(P_n) \cong \lim_{n \in I} \mathbb{K}[z_n]/(z_n^n) \]
where \( I \) is the poset \( 2\mathbb{N} \bigcup (2\mathbb{N}+1) \) (disconnected since the maps go in steps of 2). This limit is isomorphic to \( \mathbb{K}[z] \otimes \mathbb{K}[z] \), which in turn is isomorphic to \( \mathbb{K}[s, z]/(s^2 = 1) \) via the map sending \[
f \otimes g \mapsto \frac{1+s}{2} f + \frac{1-s}{2} g.
\]
It is straightforward to check that this is an inverse to the algebra map from Lemma 3.13. \( \square \)

Remark 3.24. In the proof above, we have used that the limit of the inverse (cofiltered) system \[
\cdots \leftarrow \mathbb{K}[z]/(z^n) \leftarrow \mathbb{K}[z]/(z^{n+2}) \leftarrow \cdots
\]
(in which all maps are canonical projections) is \( \mathbb{K}[z] \). We obtain the polynomial ring \( \mathbb{K}[z] \) and not the power series ring \( \mathbb{K}[[z]] \), since the limit is computed in the category of \( \mathbb{Z} \)-graded vector spaces.

3.5. The polynomial representation of \( \text{dTL} \). In §1.4 and Remark 3.4 we have seen that the fiber functor from the Temperley–Lieb category to vector spaces (namely representations of \( \mathfrak{sl}_2 \)) admits an extension to the dotted Temperley–Lieb category, valued in graded vector spaces, in which the dot acts as the Chevalley generator \( E \). In the present section we will give an alternative description of this action, the polynomial representation, and prove that it is faithful.

Definition 3.25. Let \( \{x_i\}_{i=1}^{\infty} \) be variables of degree 2. The polynomial representation of \( \text{dTL} \) is the \( \mathbb{K} \)-linear monoidal functor \( \text{Pol} : \text{dTL} \to \text{Vect}^Z \) given on objects by \( \text{Pol}(c^n) = q^{-n} \mathbb{K}[x_1, \ldots, x_n]/(x_i^2) \). This functor is given on generating morphisms by:

- The dot \( x \in \text{End} \text{dTL}(c^1) \) maps to multiplication by \( x \) on \( \text{Pol}(c) = q^{-1} \mathbb{K}[x]/(x^2) \).
- The cap morphism \( \bigcup : c^2 \to c^0 \) maps to the degree zero linear map \( q^{-2} \mathbb{K}[x_1, x_2]/(x_i^2) \to \mathbb{K} \) sending \( x_1 \mapsto 1, x_2 \mapsto 0, \) and \( x_1 x_2 \mapsto 0 \).
- The cup morphism \( \bigcap : c^0 \to c^2 \) maps to the degree zero linear map \( \mathbb{K} \to q^{-2} \mathbb{K}[x_1, x_2]/(x_i^2) \) sending \( 1 \mapsto x_1 + x_2 \).

It is straightforward to check that the relations in \( \text{dTL} \) are satisfied by the above assignments, hence \( \text{Pol} \) is well-defined.

Proposition 3.26. The polynomial representation \( \text{Pol} : \text{dTL} \to \text{Vect}^Z \) is faithful (but not full).

For the reader’s convenience we include a detailed proof, although analogous results are well-known in the context of (nil) blob algebras.

Proof. We abbreviate \( \text{Pol}(n) := \text{Pol}(c^n) \) for \( n \in \mathbb{N} \). We have a commutative diagram \[
\begin{array}{ccc}
\text{Hom} \text{dTL}(c^m, c^n) & \to & \text{Hom} \text{dTL}(c^{m+n}, c^{2n}) \\
\downarrow & & \downarrow \\
\text{Hom}_{\mathbb{K}} \text{Pol}(m, \text{Pol}(n)) & \to & \text{Hom}_{\mathbb{K}} \text{Pol}(m + n, \text{Pol}(2n))
\end{array}
\]

\( \text{Pol}(\text{cap}) \) : \text{Pol}(c^0) \to \text{Pol}(c^2) \).
of \(\mathbb{Z}\)-graded \(K\)-vector spaces, where \(\text{cap} : c^{2n} \rightarrow c^0\) denotes the nested cap morphism in \(d\text{TL}\), the vertical maps are applications of \(\text{Pol}\), and we are abbreviating by writing \(\text{Hom}_{\text{Vect}}^{c^2} = \text{Hom}_K\). In the bottom row we are using the fact that \(\text{Pol}\) is monoidal, so \(\text{Pol}(m + n) \cong \text{Pol}(m) \otimes \text{Pol}(n)\) for all \(m\).

The composition along each row is an isomorphism, so injectivity of the leftmost vertical arrow will follow from the injectivity of the rightmost. To show this, consider the spanning set \(\mathcal{B} = \{d_i\}_{i \in I}\) for \(\text{Hom}_{\text{dTL}}(c^{m+n}, c^0)\), consisting of dotted Temperley–Lieb diagrams with the property that all of their dots are adjacent to the unbounded region above the diagram. The last relation in (5) indeed shows that such diagrams span, since any dotted Temperley–Lieb diagram not satisfying this property can be rewritten as a linear combination of diagrams where each dot has either the same number or fewer strands separating it from the unbounded region, and at least one has strictly fewer.

We will find a set \(\mathcal{B}^* = \{\delta_i^*\}_{i \in I} \subset \text{Pol}(m + n)\) of monic monomials and a total order on \(I\), such that

\[
d_i(\delta_j^*) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i > j \end{cases}
\]

This means that the pairing matrix \((d_i(\delta_j^*)))_{i,j \in I}\) is unitriangular with respect to the order on \(I\). As a consequence, these elements act linearly independently on \(\text{Pol}(m + n)\), i.e. are linearly independent in \(\text{Hom}_{\text{Vect}}^{c^2}\)(\(\text{Pol}(m + n), \text{Pol}(0)\)). (This further shows that \(\mathcal{B}\) is in fact a basis for \(\text{Hom}_{\text{dTL}}(c^{m+n}, c^0)\).)

Given \(d_i \in \mathcal{B}\), define \(\delta_i^*\) as an ordered monomial \(x_{i_1} \cdots x_{i_k}\) with \(i_1 < \cdots < i_k\) by the following rule: for every cap in \(d_i\) record the indices \(a < b\) of its boundary points. Now include \(x_a\) (but not \(x_b\)) in the monomial if there is no dot on the cap and include neither variable if there is a dot on the cap. By construction, we have that \(d_i(\delta_i^*) = 1\).

The total order on \(I\) giving (19) corresponds to the following total order on such monomials: we declare \(i > j\) if the total degree \(\delta_i^*\) is greater than that of \(\delta_j^*\). We compare monomials of equal degree using an appropriate lexicographic order determined by \(x_1 > x_2 > \cdots\). Precisely, \(x_{i_1} \cdots x_{i_k} > x_{j_1} \cdots x_{j_k}\) if there is an index \(l \geq 1\) for which \(i_p = j_p\) for \(1 \leq p \leq l - 1\) (regarded as vacuously true if \(l = 1\)), and \(i_l < j_l\). For example, on degree three monomials we have

\[
x_1x_2x_3 > x_1x_2x_4 > x_1x_3x_4 > x_2x_3x_4 > \cdots.
\]

Given a monomial \(\delta_j^* = x_{j_1} \cdots x_{j_k} \in \mathcal{B}_i^*\) with \(k\) variables, we can (re)construct the corresponding dotted Temperley–Lieb diagram \(d_j\) as follows. In order to evaluate non-trivially on \(\delta_j^*\), the diagram \(d_j\) needs precisely \(k\) undotted caps and \((m + n)/2 - k\) dotted caps. The positions of the undotted caps are uniquely determined by \(\delta_j^*\) via the following greedy algorithm. We start with the variable \(x_{j_k}\) of largest index, then there must be an undotted cap connecting the boundary points \(j_k\) and \(j_k + 1\). (The sought-after second endpoint must be \(j_{k+1}\), but, if it were \(j_{k+1}\), it would require a dotted cap nested within, which is impossible in \(\mathcal{B}\).) Now remove \(x_{j_{k+1}}\) from the monomial and iterate until the monomial is 1. Finally, connect all of the remaining points with dotted caps, with none nested in one another.

Finally, suppose that \(d_i \in \mathcal{B}\) is another diagram such that \(d_i(\delta_j^*) \neq 0\) (and so \(d_i\) has exactly \(k\) undotted caps), then we claim that \(i \leq j\). Indeed, \(d_i(\delta_j^*) \neq 0\) if and only if every undotted cap in \(d_i\) has one (but not both) of the variables corresponding to its boundary points present in \(\delta_j^*\). If all the variables correspond to the left boundary point, then \(i = j\). Otherwise there exists a smallest index \(0 \leq l \leq k\), such that the variable \(x_{i_l}\) corresponds to the right endpoint of a cap in \(d_i\); denote the variable for the left endpoint by \(j_l\). Then, \(\delta_i^* = x_{j_1} \cdots x_{j_{l-1}} x_{i_l} \cdots\) and \(\delta_j^* = x_{j_1} \cdots x_{j_{l-1}} x_{j_l} \cdots\) with \(i_l > j_l\), so \(i < j\). \(\square\)

**Remark 3.27.** Proposition 3.26 shows that \(\text{End}_{d\text{TL}}(c^n)\) can be viewed as the subalgebra of

\[
\text{End}_K \left( \mathbb{K}[x_1, \ldots, x_n]/(x_i^2) \right)
\]
generated by multiplication by \(x_i\) and the image of \(\text{End}_{\text{TL}}(c^n)\). In this way, one can arrive at a definition of \(\text{dTL}\) using only (undotted) \(\text{TL}\) and its polynomial representation.

**Proposition 3.28.** The polynomial representation extends to a functor \(\text{Pol}: \text{Kar}(\text{dTL}) \to \text{Vect}^\mathbb{Z}\) (denoted by the same symbol), and we have

\[
\text{Pol}(P_n) \cong q^{-n}K[z]/(z^{n+1}) \subset q^{-n}K[x_1, \ldots, x_n]/(x_i^2).
\]

where \(z = x_1 - x_2 + \cdots + (-1)^{n-1}x_n\).

**Proof.** The extension to \(\text{Kar}(\text{dTL})\) is immediate, since \(\text{Vect}^\mathbb{Z}\) is idempotent complete.

It remains to compute \(\text{Pol}(P_n)\). Recall from (11) that there is an algebra map \(K[\mathfrak{S}_n] \to \text{End}_{\text{dTL}}(c^n)\) and that the Jones-Wenzl idempotent \(P_n\) is the image of the symmetrizing idempotent \(\frac{1}{n!} \sum_{w \in \mathfrak{S}_n} w\).

On the level of the polynomial representation, the action of \(\mathfrak{S}_n\) is given by

\[
\text{Pol}(w) = \varepsilon \circ w \circ \varepsilon,
\]

where \(w\) is the standard action on \(K[x_1, \ldots, x_n]/(x_i^2)\) by permuting variables, and \(\varepsilon\) is the algebra automorphism of \(K[x_1, \ldots, x_n]/(x_i^2)\) sending \(x_i \mapsto (-1)^{i-1}x_i = \bar{x}_i\). In other words, the action of \(\text{Pol}(w)\) is by permuting the variables \(\{x_1, -x_2, \ldots, \pm x_n\} = \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}\).

It follows that the image of \(P_n\) is the \(\mathfrak{S}_n\)-invariant subspace \((K[x_1, -x_2, \ldots, \pm x_n]/(x_i^2))^{\oplus n}\). This is isomorphic to the subalgebra generated by the first power sum symmetric polynomial \(z = x_1 - x_2 + \cdots + x_n\), since all higher-degree power sum symmetric polynomials are zero. The relation \(z^{n+1} = 0\) follows from Remark 3.14.

We finish this section by giving two alternative interpretations of the polynomial representation. A third will appear in Example 6.3.

**Remark 3.29.** The polynomial representation admits the following diagrammatic presentation, which we will revisit and extend in §5. For each \(n \in \mathbb{N}\), let \(\text{Pol}'(n)\) denote the \(K\)-vector space formally spanned by diagrams of the form

\[
\begin{array}{c}
\vdots \\
D \\
\vdots
\end{array}
\]

where \(D\) is a diagram in \(\text{Hom}_{\text{dTL}}(c^m, c^n)\) for some \(m\), modulo the \(\text{dTL}\) relations among black strands, together with the local relations:

\[
\begin{array}{c}
\bigcup \quad = \quad \big| \\
\bigcup \quad = \quad \big| + \\
\bigcup \quad = \quad \big| = \text{zero} \\
\bigcup \quad = \quad \big|
\end{array}
\]

We make \(\text{Pol}'(n)\) into a \(\mathbb{Z}\)-graded vector space by declaring that the degree of a diagram is \(2 \{\#\text{dots}\} - n\). The assignment \(n \mapsto \text{Pol}'(n)\) extends to a functor \(\text{dTL} \to \text{Vect}^\mathbb{Z}\), with the action of morphisms in \(\text{dTL}\) given by vertical stacking. It is not hard to see that \(\text{Pol} \cong \text{Pol}'\) as functors \(\text{dTL} \to \text{Vect}^\mathbb{Z}\).

**Remark 3.30.** As shown in [QR18, §3] in the analogous setting of \(\mathfrak{gl}_2\) foams, there is an equivalence of categories \(\mathcal{ABN} \cong \text{hTr}(\mathcal{BN}) \cong \text{vTr}(\mathcal{BN})\) where \(\text{hTr}\) and \(\text{vTr}\) denote horizontal and (an appropriate notion of) vertical trace of a bicategory, respectively. As discussed in [BHLW17, §9], this implies that \(\text{dTL} \cong \mathcal{ABN}\) acts naturally on the center of objects of the bicategory \(\mathcal{BN}\), i.e. on the \(\mathbb{Z}\)-graded vector space

\[
\bigoplus_{n \in \mathbb{N}} \text{End}_{\mathcal{BN}}(\text{id}_n).
\]
For example, to the cap morphism $\bigcap: c^2 \to c^0$ in $\text{dTL}$ we can associate the following surface with boundary:

(a sphere with two disks removed), and we can “plug in” any endomorphism of the identity 1-morphism $\text{id}_2$ to obtain a closed (dotted) cobordism, i.e. an endomorphism of the identity 1-morphism $\text{id}_0$. Using the identification

$$\text{End}_{\mathcal{BN}}(\text{id}_n) \cong (\mathbb{K}[x]/(x^2))^\otimes n \cong \mathbb{K}[x_1, \ldots, x_n]/(x_i)^2$$

one recovers the polynomial representation (up to shift).

4. Kirby color and handle slides

In this section, we define our Kirby color and establish its handle slide invariance.

**Convention 4.1.** To simplify notation in the remainder of the paper, we will abbreviate by writing $\text{dTL} := \text{Ind}(\text{Mat}(\text{Kar}(\text{dTL})))$ and $\mathcal{ABN} := \text{Ind}(\text{Mat}(\text{Kar}(\mathcal{ABN})))$.

4.1. The Khovanov Kirby color. Recall from §2.3 that objects of $\text{dTL}$ are given by directed systems of objects in $\text{Mat}(\text{Kar}(\text{dTL}))$. We now introduce our Kirby objects as such.

**Definition 4.2.** For $k \geq 0$, the Kirby object of winding number $k$ is the object $\omega_k \in \text{dTL}$ given by the following directed system in $\text{Kar}(\text{dTL})$:

$$(20) \quad \omega_k := (q^{-k} P_k U_k \to q^{-k-2} P_{k+2} U_{k+2} \to q^{-k-4} P_{k+4} \to \cdots)$$

where $U_{k+2i}$ is as in Definition 3.19. For $k < 0$, we set $\omega_k := \omega_{k|}$. We immediately observe that the object $\omega_k$ only depends on the parity of $k$.

**Lemma 4.3.** For $k \in \mathbb{Z}$ there are canonical isomorphisms $\omega_k \cong \omega_{k+2}$ in $\text{dTL}$.

**Proof.** Without loss of generality, we may assume that $k \geq 0$. By Lemma 2.20, the inclusion of $(k+2) + 2\mathbb{N} \hookrightarrow k + 2\mathbb{N}$ of filtered categories is final, thus it induces an isomorphism $\omega_{k+2} \cong \omega_k$ in $\text{dTL}$. □

The reference to winding numbers in Definition 4.2 may seem obscure, given that Lemma 4.3 shows that only the parity of $k$ is relevant. The appropriate context for this terminology is a conjectural $\mathfrak{gl}_N$-analogue of (20) that we discuss in Section 7.1. In the present context, however, the following construction combines the distinct Kirby objects.

**Definition 4.4.** The total Kirby color is $\omega := \omega_0 \oplus \omega_1$.

**Remark 4.5.** There are analogous definitions for Kirby colors defined as objects of a full subcategory of the pro-completion of $\text{Mat}(\text{Kar}(\text{dTL}))$. Our definition is chosen to be compatible with [MN22].

In §4.4 below (after reviewing further background), we establish the handle slide isomorphism for the total Kirby color. Before doing so, we pause to establish an algebraic result that will be used later. Note that $\text{Vect}^\mathbb{Z}$ has grading shifts, direct sums and summands, and filtered colimits, so Remark 2.24 implies that the polynomial representation from §3.5 extends to a functor:

$$(21) \quad \text{Pol}: \text{dTL} \to \text{Vect}^\mathbb{Z}$$

\footnote{In the definition of $\mathcal{ABN}$ the operation Mat can be omitted since $\mathcal{ABN}$ has Mat already built in and $\text{Kar}(\mathcal{ABN})$ still has finite biproducts and grading shifts.}
(denoted by the same symbol as before). We thus evaluate this functor on the Kirby objects \(\omega_k\).

**Proposition 4.6.** There is an isomorphism

\[
\text{Pol}(\omega_k) \cong \bigoplus_{i=0}^{\infty} q^{-2i}K = K \oplus q^{-2}K \oplus q^{-4}K \oplus \cdots
\]

of \(\mathbb{Z}\)-graded \(K\)-vector spaces. With respect to this isomorphism, the value of \(\text{Pol}\) on the canonical map \(q^{-k}P_k \rightarrow \omega_k\) corresponds to the inclusion \(\bigoplus_{i=0}^{k} q^{-2i}K \hookrightarrow \bigoplus_{i=0}^{\infty} q^{-2i}K\).

**Proof.** Proposition 3.28 gives that

\[
\text{Pol}(P_k) \cong q^{-k}K[z]/(z^{k+1})
\]

which has graded dimension \(\dim_q(\text{Pol}(P_k)) = [k+1] = q^k + q^{k-2} + \cdots + q^{-k}\). It is straightforward to verify that

\[
q^{-k}\text{Pol}(P_k) \xrightarrow{\text{Pol}(U_k)} q^{-k-2}\text{Pol}(P_{k+2})
\]

is full rank, hence the colimit of the system \(\cdots \rightarrow q^{-k}\text{Pol}(P_k) \rightarrow q^{-k-2}\text{Pol}(P_{k+2}) \rightarrow \cdots\) is the union of these graded vector spaces.

**Remark 4.7.** Proposition 4.6 suggests a representation-theoretic interpretation of the Kirby objects: for \(k \geq 0\) the graded dimension of \(q^k\text{Pol}(\omega_k)\) coincides with the character of the (dual) Verma module for \(\mathfrak{sl}_2\) of highest weight \(k\). Since there is a degree zero morphism \(P_k \rightarrow q^k\omega_k\) induced by the inclusion of the former in the directed system defining the latter, we see that \(q^k\text{Pol}(\omega_k)\) is most-naturally viewed in this regard as the dual Verma module \(\nabla(k)\), which has the irreducible \(\mathfrak{sl}_2\) representation of highest weight \(k\) as a submodule. In Corollary 5.14 below, we show that \(\text{End}_{\text{TL}}(\omega_k) \cong K[z]\) for a degree two element \(z\). As in Remark 3.4, the action of this generator on \(q^k\text{Pol}(\omega_k)\) again corresponds to the action of the Chevalley generator \(E\) on \(\nabla(k)\). (Note that \(E\) does not act nilpotently on \(\nabla(k)\).)

### 4.2. The punctured Bar-Natan categories.

In order to state (and prove) the handle slide invariance of \(\omega\), we now further discuss the (annular) Bar-Natan categories \(\mathcal{BN}(S^1 \times [0,1]; p)\) from Definition 3.6 in the case when \(p \neq \emptyset\). To facilitate the description of additional algebraic structure on these categories, we will work with a punctured square \([0,1]^2 \setminus \{(\frac{1}{2}, \frac{1}{2})\}\) instead of the annulus.

**Definition 4.8.** For \(n \in \mathbb{N}\) let \(p_n \subset (0,1)\) denote a chosen set of \(n\) distinct points. Let

\[
\mathcal{BN}_m = \mathcal{BN}([0,1]^2; p_m \times \{0\} \cup p_n \times \{1\})
\]

and

\[
\mathcal{PBN}_m = \mathcal{BN}([0,1]^2 \setminus \{(\frac{1}{2}, \frac{1}{2})\}; p_m \times \{0\} \cup p_n \times \{1\})
\]

be the Bar-Natan categories of the square and punctured-square with marked points at the bottom and top.

**Remark 4.9.** In §4.3 below we will continue to draw curves in \([0,1]^2 \setminus \{(\frac{1}{2}, \frac{1}{2})\}\) as curves in \(S^1 \times [0,1]\). The arcs of the outer boundary \(S^1 \times \{1\}\) corresponding to the top and bottom of the square \([0,1]^2\) will be clear from context (or entirely irrelevant).

We have the following structures on these categories, which we describe via glueing operations on the products of (punctured) squares with the interval \([0,1]\) (i.e. the space within which Bar-Natan cobordisms are embedded):

1. A composition functor

\[
*: \mathcal{BN}_m \times \mathcal{BN}_k \rightarrow \mathcal{BN}_k
\]
(2) An external tensor product functor
\[
\boxtimes : \mathcal{B}_m^{n_1} \times \mathcal{B}_m^{n_2} \to \mathcal{B}_m^{n_1 + n_2},
\]
(3) A top and bottom action of $\mathcal{B}_n$ on $\mathcal{P}_n$, which is to say a functor
\[
\mathcal{B}_m^{n_1} \times \mathcal{P}_m^{n_1} \times \mathcal{B}_m^{n_2} \to \mathcal{P}_m^{n_2},
\]
denoted $(X, M, Y) \mapsto X \star M \star Y$.

(4) A left and right action of $\mathcal{B}_n$ on $\mathcal{P}_n$, which is to say a functor
\[
\mathcal{B}_m^{n_1} \times \mathcal{P}_m^{n_2} \times \mathcal{B}_m^{n_3} \to \mathcal{P}_m^{n_1 + n_2 + n_3},
\]
denoted $(X, M, Y) \mapsto X \boxtimes M \boxtimes Y$.

(5) An action of $\mathcal{A}_n$ on $\mathcal{P}_n$, which is to say a functor
\[
\mathcal{A}_m^n \times \mathcal{P}_m^n \to \mathcal{P}_m^n,
\]
denoted $(C, M) \mapsto C \bullet M$, defined by inserting $C$ in a neighborhood of the puncture.

The operations (1) and (2) make $\mathcal{B}_n$ into a monoidal bicategory with 1-morphism categories $\mathcal{B}_n^m$.

The operations (3) and (4) make $\mathcal{P}_n$ into a bimodule over $\mathcal{B}_n$ in two ways, corresponding to the two binary operations $\star$ and $\boxtimes$ on $\mathcal{B}_n$. The operation (5) makes $\mathcal{P}_n$ into a left module over $\mathcal{A}_n$ in a way which commutes with all the aforementioned structures; if we identify $\mathcal{P}_n^0 \cong \mathcal{A}_n$, then this recovers the monoidal structure $\otimes$ on $\mathcal{A}_n$ from Remark 3.10. There are various relationships between these actions, all of which are apparent from the above descriptions, e.g.
\[
(X_1 \boxtimes Y_1 \boxtimes Z_1) \star (X_2 \boxtimes M \boxtimes Z_2) \star (X_3 \boxtimes Y_3 \boxtimes Z_3) \equiv (X_1 \star X_2 \star X_3) \boxtimes (Y_1 \star M \star Y_3) \boxtimes (Z_1 \star Z_2 \star Z_3).
\]

Remark 4.10. All of the above structure persists upon proceeding to the completion $\mathcal{P}_n^\infty := \text{Ind(Mat(Kar(\mathcal{P}_n)))}$.

4.3. The 2-point category. We now extend the Temperley-Lieb type presentation of $\mathcal{A}_n$ to a similar presentation of its module category $\mathcal{P}_n^1$.

Definition 4.11. Let $\text{MdTL}$ be the $\mathbb{Z}$-graded $\mathbb{K}$-linear category with objects pairs $(n, Z) \in \mathbb{N} \times \{L, R\}$ and generated as a module category over $\text{dTL}$ (acting on the left) by morphisms
\[
Z \times Z \in \text{End}^2((0, Z)), \quad Z \times Y \in \text{Hom}^1((1, Y), (0, Z))
\]
for $Y, Z \in \{L, R\}$ with $Y \neq Z$, modulo local relations. Setting

$$Z Y := Z Y$$

the relations are

$$Z Z = Z \cdot Z + Z Z \cdot 0, \quad Y \cdot Z = 0 \cdot Y, \quad Z Z = Y \cdot Z, \quad Z^2 = 0.$$ 

The module structure is given on objects by $e^m \otimes (n, Z) = (m + n, Z)$ and given on morphisms by placing dotted Temperley–Lieb diagrams on the left.

**Theorem 4.12.** The assignments

\[
L \mapsto \begin{array}{c}
\cdot
\end{array}, \quad R \mapsto \begin{array}{c}
\cdot
\end{array}
\]

and

\[
L \mapsto \begin{array}{c}
\begin{array}{c}
D
\end{array}
\end{array}, \quad R \mapsto \begin{array}{c}
\begin{array}{c}
D
\end{array}
\end{array}
\]

determine a fully faithful functor $\Phi: \text{MdTL} \rightarrow \mathcal{PBN}_1$ of module categories, i.e. such that the diagram

$$\text{dTL} \otimes \text{MdTL} \rightarrow \text{MdTL}$$

commutes. The induced functor $\text{Mat}(\text{MdTL}) \rightarrow \mathcal{PBN}_1$ is an equivalence of categories.

In the dimensionally-reduced calculus of $\text{MdTL}$, the left side of diagrams corresponds to the inside of the annulus (close to the puncture) and the right side to the outside boundary.

**Proof.** It is straightforward to check that the images of the relations in (22) are satisfied in $\mathcal{PBN}_1$, e.g. the image of the first relation holds by neck cutting. Hence, they determine a functor that satisfies (23) by definition.

In order to show that this functor is fully faithful, we will establish isomorphisms between Hom-spaces in $\text{MdTL}$ and $\mathcal{PBN}_1$ and certain Hom-spaces in $\text{dTL}$ and $\mathcal{ABN}$. First, note that the relations in (22) imply that spanning sets for $\text{Hom}_{\text{MdTL}}((m, Z), (n, Z))$ and $\text{Hom}_{\text{MdTL}}((m, Y), (n, Z))$ for $Y \neq Z$ are given by

\[
\left\{ \begin{array}{c}
\begin{array}{c}
D
\end{array}
\end{array} \right| D \in \text{Hom}_{\text{dTL}}(e^m, e^n), \quad i = 0, 1 \right\} \quad \text{and} \quad \left\{ \begin{array}{c}
\begin{array}{c}
D
\end{array}
\end{array} \right| D \in \text{Hom}_{\text{dTL}}(e^m, e^{n+1}) \right\}.
\]
Thus,
\begin{equation}
\dim \left( \text{Hom}_{\text{MdTL}} \left( (m, Z), (n, Z) \right) \right) \leq 2 \dim \left( \text{Hom}_{\text{dTL}}(c^m, c^n) \right)
\end{equation}
and
\begin{equation}
\dim \left( \text{Hom}_{\text{MdTL}} \left( (m, Y), (n, Z) \right) \right) \leq \dim \left( \text{Hom}_{\text{dTL}}(c^m, c^{n+1}) \right).
\end{equation}

We now restrict to the case when $Z = L$, since the $Z = R$ case is analogous. Consider the composition of homogeneous linear maps
\begin{equation}
\text{Hom}_{\text{MdTL}} \left( (m, L), (n, L) \right) \rightarrow \text{Hom}_{\text{PBN}} \left( \bullet \left( \begin{array}{c} m \\ n \end{array} \right) \right) \cong q \text{Hom}_{\text{ABN}} \left( \bullet \left( \begin{array}{c} m \\ n \end{array} \right) \right) \cong \text{Hom}_{\text{dTL}}(c^m, c^{n+1}).
\end{equation}

Here, the first isomorphism exists because both Hom-spaces describe the “Bar-Natan skein module” spanned by (dotted) surfaces in the solid torus with boundary consisting of $m + n$ longitudes and one null-homotopic circle. (Informally, the isomorphism can be implemented by “sliding” the union of the strand at the bottom (i.e. source side) of the cobordism that meets the boundary and the vertical segments to the top (i.e. target side) of the cobordism.) Since the composition of these maps sends
\[
\begin{pmatrix} L \\ D \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ D \end{pmatrix}, \quad \begin{pmatrix} D \\ L \end{pmatrix} \mapsto \begin{pmatrix} D \\ 0 \end{pmatrix}
\]

the first map in (26) is surjective, thus an isomorphism by (24).

Similarly, the composition
\begin{equation}
\text{Hom}_{\text{MdTL}} \left( (m, R), (n, L) \right) \rightarrow \text{Hom}_{\text{PBN}} \left( \bullet \left( \begin{array}{c} m \\ n \end{array} \right) \right) \cong q \text{Hom}_{\text{ABN}} \left( \bullet \left( \begin{array}{c} m \\ n \end{array} \right) \right) \cong \text{Hom}_{\text{dTL}}(c^m, c^{n+1})
\end{equation}

(wherein the first isomorphism can be seen as before by “sliding” the bottom and vertical segments in a cobordism to the top) sends
\[
\begin{pmatrix} \cdots \\ D \\ \cdots \end{pmatrix} \mapsto \begin{pmatrix} D \end{pmatrix}
\]

Therefore, the first map in (25) is surjective, hence an isomorphism by (25). Thus, $\Phi: \text{MdTL} \hookrightarrow \text{PBN}_1$ is fully faithful.

Finally, the induced functor $\Phi: \text{Mat}(\text{MdTL}) \hookrightarrow \text{PBN}_1$ is essentially surjective by the circle removal isomorphism (9).
4.4. Invariance under handle slide. We are now ready to establish an isomorphism

\[ \omega_k \cdot Z \cong \omega_{k+1} \cdot \tau(Z) \]

of directed systems in \( \overline{PBN}_1 \), which we describe in the equivalent category

\[ \text{MdTL} := \text{Ind} (\text{Mat}(\text{Kar}(\text{MdTL}))) \].

For notational convenience, we will use the involution \( \tau : \text{MdTL} \to \text{MdTL} \) sending \( L \mapsto R \) and \( \tau(X \bullet Z) = X \bullet \tau(Z) \) for all \( X \in \text{dTL} \).

**Lemma 4.13** (Elementary Handle Slide). Let \( Z \in \{ L, R \} \) and \( k \in \mathbb{Z} \). There is an isomorphism

\[ \omega_k \bullet Z \cong \omega_{k+1} \bullet \tau(Z) \].

**Proof.** We may assume that \( k \geq 0 \). Consider the following diagram:

\[ \cdots \to q^{-n-1}P_{n+1} \bullet \tau(Z) \to q^{-n-3}P_{n+3} \bullet \tau(Z) \to \cdots \]

\[ \cdots \to q^{-n}P_n \bullet Z \to q^{-n-2}P_{n+2} \bullet Z \to \cdots \]

which commutes by (15). This constitutes a natural transformation of directed systems and hence determines a morphism \( \omega_k \bullet Z \to \omega_{k+1} \bullet \tau(Z) \) in \( \text{MdTL} \); see Remark 2.23. The composition of two such maps is computed component-wise as follows:

\[ \begin{array}{ccc}
\cdots & \to & q^{-n}P_n \bullet Z \\
\tau(Z) & \downarrow & \tau(Z) \\
\cdots & \to & q^{-n-2}P_{n+2} \bullet Z \\
\end{array} \]

Thus, the components of the composition \( \omega_k \bullet Z \to \omega_{k+1} \bullet \tau(Z) \to \omega_{k+2} \bullet Z \) are the transition maps in the directed system \( \omega_k \bullet Z \). This is precisely the inverse to the isomorphism \( (\omega_{k+2} \cong \omega_k) \bullet \text{id}_Z \) induced from Lemma 4.3. \( \square \)

For the following, we consider the (completed) 4-point category \( \overline{PBN}_2 \) and its objects:

\[ LL := \begin{array}{c}
\bullet
\end{array}, \quad CC := \begin{array}{c}
\bullet
\end{array}, \quad RR := \begin{array}{c}
\bullet
\end{array}, \quad LR := \begin{array}{c}
\bullet
\end{array}. \]
Lemma 4.14. There are commutative diagrams

\[
\begin{align*}
\omega_k \cdot LL & \cong \omega_{k+1} \cdot LR \cong \omega_{k+2} \cdot RR, \\
\omega_k \cdot CC & \cong \omega_{k+2} \cdot CC, \\
\omega_k \cdot L & \cong \omega_{k+1} \cdot R
\end{align*}
\]

wherein all horizontal maps are elementary handle slide isomorphisms, \(s\) denotes saddle cobordisms, and \(x_L\) and \(x_R\) are the dot endomorphisms of \(L\) and \(R\) illustrated in Theorem 4.12.

In other words, the handle slide isomorphisms from Lemma 4.13 are natural with respect to saddles and dot morphisms in \(\mathcal{PBN}\).

Proof. Commutativity of the second diagram is clear from the second relation in (22). Commutativity of the first is best seen when expressed in terms of cobordisms as in Theorem 4.12, where it corresponds to changing the order in which distant saddle cobordisms are applied. \(\square\)

Recall that, for an \(\mathcal{M}\)-bimodule category \(\mathcal{B}\) over a monoidal category \(\mathcal{M}\), the Drinfeld center of \(\mathcal{B}\) is the category, whose objects are pairs \((\mathcal{B}, \tau_B)\) consisting of an object \(B \in \mathcal{B}\) and a natural isomorphism \(\tau_B : B \boxtimes - \Rightarrow - \boxtimes B\) satisfying the usual coherence conditions of a half-braiding. The following theorem uses the analogous (via categorification) concept of the Drinfeld center of a bimodule over a monoidal bicategory. As we will not use this formulation in the rest of the paper, we omit a detailed description and study of such Drinfeld centers.

Theorem 4.15 (Handle Slide). The Kirby color \(\omega\), viewed as an object of \(\mathcal{PBN}\), together with half-braidings assembled from the isomorphisms in Lemma 4.13, defines an object of the Drinfeld center of the \(\mathcal{BN}\)-bimodule \(\mathcal{PBN}\).

Here we view \(\mathcal{PBN}\) as a \(\mathcal{BN}\)-bimodule under the operation \(\boxtimes\) described in item (4) of §4.2.

Proof sketch. The objects of \(\mathcal{BN}\) are generated (as 1-morphisms in a monoidal bicategory, and under direct sums and shifts) by cup, cap, and identity tangles. The half-braidings can be assembled accordingly, and their building blocks are only interesting for identity tangles, where we use the elementary handle slides from Lemma 4.13 (and their inverses). Lemma 4.14 implies that these candidate half-braidings satisfy the requisite coherence conditions. \(\square\)

5. Diagrammatic presentation

In this section, we study the monoidal category obtained from \(\mathcal{dTL}\) by adjoining the objects \(\omega_0\) and \(\omega_1\). Precisely, in Theorem 5.22 below we give an explicit diagrammatic description of the full monoidal subcategory of \(\mathcal{dTTL}\) generated by \(\mathcal{dTL}\) and the Kirby objects \(\omega_0, \omega_1\).

5.1. Kirby colored diagrammatics: first approximation. To begin, we define a strict monoidal category \(\mathcal{KdTL}'\) via generators and relations and a monoidal functor \(\mathcal{KdTL}' \to \mathcal{dTTL}\). This will give a useful framework for studying the Kirby objects in \(\mathcal{dTTL}\).

Definition 5.1. Let \(\mathcal{KdTL}'\) be the \(\mathbb{Z}\)-graded \(\mathbb{K}\)-linear monoidal category given by adjoining to \(\mathcal{dTTL}\) two new generating objects \(\omega_i'\) for \(i \in \mathbb{Z}/2\mathbb{Z}\) (abbreviated in the diagrams below simply as \([i]\)) and morphisms

\[
\begin{align*}
\iota_n := & \in \text{Hom}^-\left([n] \to [n]\right), \\
\iota_{n+j} := & \in \text{Hom}^0\left([n] \to [n+j]\right), \\
\iota_{[i]+[j]} := & \in \text{End}^2\left([i] \to [i+j]\right).
\end{align*}
\]
for $n \geq 0$ and $i, j \in \mathbb{Z}/2\mathbb{Z}$, modulo the defining relations

$$(28a)$$

$[i][i] = [i][i]$, $[i][i+k] = [i][i+k]$, $[i+j][i+j] = [i+j][i+j]$.

$$(28b)$$

$$[i][i+j] = [i][i+j] + (-1)^i [i][i+j], \quad [i][n] = [i][n].$$

$$(28c)$$

$$[i+j][i+j] = 0,$$

for all integers $i, j, n \geq 0$. Here (for $\iota_0$) and below (for $\iota_1$), we use the notation

$\iota_0 := [0], \quad \iota_1 := [0].$

Since all of the new generating morphisms have codomain $\omega'_i$ and all new relations involve such morphisms, it follows that there is a fully faithful inclusion $dTL \rightarrow KdTL$. An object of $\text{Kar}(KdTL)$ will be called finite if it is in the image of $\text{Kar}(dTL)$ and will be called infinite otherwise. In other words, an infinite object has at least one tensor factor of the form $\omega'_i$.

Remark 5.2. Relations $(28a)$ say that $\omega'_0 \oplus \omega'_1$ has the structure of a $\mathbb{Z}/2\mathbb{Z}$-graded algebra object in $\text{Mat}(\text{Kar}(KdTL'))$. Further, note that the first relation in $(28c)$ pairs with $(12)$ to show that

$$[n] = [n] \in \mathbb{N}.$$

Remark 5.3. The category $KdTL'$ admits a $\otimes$-contravariant involution defined on diagrams by reflecting in a vertical line and multiplying every dot by the sign $(-1)^{i-1}$ where $i$ is the parity of the strand: 1 for black strands and $i$ for a pink strand with label $[i]$.

The following makes the connection between the formally defined category $KdTL'$ and $\overline{dTL}$.

Theorem 5.4. There exists a unique monoidal functor $\varphi : KdTL' \rightarrow \overline{dTL}$ such that:

- $\varphi$ extends the embedding $dTL \rightarrow \overline{dTL}$
- $\varphi(\omega'_i) = \omega_i$.
- $\varphi$ is given on the generating morphisms of $KdTL'$ as follows:

$$\varphi \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right) = \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right),$$

$$\varphi \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right) = \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right),$$

$$\varphi \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right) = \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right),$$

$$\varphi \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right) = \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right).$$
Proof. Uniqueness is clear, since any such \( \varphi \) extends the embedding of \( dTL \) and the images of the additional generators of \( KdTL' \) are explicitly specified. It thus suffices to check that these assignments \( \varphi \) actually define a functor \( KdTL' \to dTL \), i.e. that \( \varphi \) respects all defining relations from Definition 5.1. This follows easily, once we unpack the notation for the images of the generating morphisms.

For the first, note that

\[
\text{Hom}_{dTL}(c^m, \omega_m) = \text{Hom}_{dTL}(c^m, \colim_{k \in \mathbb{N}} q^{-m-2k}P_{m+2k}) = \colim_{k \in \mathbb{N}} \text{Hom}_{dTL}(c^m, q^{-m-2k}P_{m+2k})
\]

so \([P_m]\) simply denotes the equivalence class in the colimit of the map \( P_m \in \text{Hom}_{dTL}(c^m, q^{-m}P_m) \). For the second, it follows from Remark 2.25 and Lemma 2.20 that

\[
\omega_i \otimes \omega_j \cong \colim_{(k,\ell) \in \mathbb{N}^2} q^{-i-2k}P_{i+2k} \otimes q^{-j-2\ell}P_{j+2\ell} \cong \colim_{n \in \mathbb{N}} q^{-i-j-4n}P_{i+2n} \otimes P_{j+2n}
\]

so

\[
\text{Hom}_{dTL}(\omega_i \otimes \omega_j, \omega_{i+j}) \cong \lim_{n \in \mathbb{N}} \text{Hom}_{dTL}(q^{-i-j-4n}P_{i+2n} \otimes P_{j+2n}, q^{-i-j-4k}P_{i+j+4k}).
\]

(Here, we additionally use that \( m + 4N \mapsto m + 2N \) is final.) Elements in this limit are given by a stable family (indexed by \( n \in \mathbb{N} \)) of morphisms in

\[
\colim_{k \in \mathbb{N}} \text{Hom}_{dTL}(q^{-i-j-4n}P_{i+2n} \otimes P_{j+2n}, q^{-i-j-4k}P_{i+j+4k}),
\]

and morphisms in the latter can be exhibited as the equivalence class of a single morphism in

\[
\text{Hom}_{dTL}(q^{-i-j-4n}P_{i+2n} \otimes P_{j+2n}, q^{-i-j-4k}P_{i+j+4n}).
\]

The notation above denotes such a stable family of equivalence classes. The image of the third generator is depicted analogously. \( \square \)

The diagrammatic category \( KdTL' \) is insufficient to describe \( dTL \). Indeed, it is a consequence of Corollary 5.25 below that the functor \( \varphi: KdTL' \to dTL \) is not full. For example, that corollary shows that \( \text{Hom}_{dTL}(\omega_0, \omega_0 \otimes \omega_0) \) is uncountably infinite-dimensional, while it is straightforward to see that \( \text{Hom}_{KdTL'}(\omega_0, \omega_0 \otimes \omega_0) \) is countably infinite dimensional. This issue will be remedied in §5.4, where we extend \( KdTL' \) to allow certain infinite sums of diagrams.

5.2. More diagrammatic relations. Introduce the following shorthand:

\[
\begin{array}{c}
\text{[i+1]} \\
\text{[i+1]} \\
\text{[i+1]} \\
\end{array}
\begin{array}{c}
= \\
\cdot \\
\cdot \\
\end{array}
\begin{array}{c}
\text{[i+1]} \\
\text{[i+1]} \\
\text{[i+1]} \\
\end{array}
\]

Lemma 5.5. The following relations, as well as the images of these under the involution from Remark 5.3, hold in \( KdTL' \):

\[(29a)\]

\[
\begin{array}{c}
\text{[i+j+1]} \\
\text{[i+j+1]} \\
\text{[i+j+1]} \\
\text{[i+j+1]} \\
\text{[i+j+1]} \\
\end{array}
\begin{array}{c}
= \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\end{array}
\begin{array}{c}
\text{[i+j+1]} \\
\text{[i+j+1]} \\
\text{[i+j+1]} \\
\text{[i+j+1]} \\
\text{[i+j+1]} \\
\end{array}
\]

\[(29b)\]

\[
\begin{array}{c}
\text{[i+1]} \\
\text{[i+1]} \\
\text{[i+1]} \\
\end{array}
\begin{array}{c}
= \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\end{array}
\begin{array}{c}
\text{[i+1]} \\
\text{[i+1]} \\
\text{[i+1]} \\
\end{array}
\]
\[(29c)\]

\[
\begin{array}{c}
\begin{array}{c}
\text{(29d)}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{(29e)}
\end{array}
\end{array}
\]

Observe that the only relation that picks up a sign under the left-right reflection is \((29b)\), whose image yields the relation

\[
\begin{array}{c}
\begin{array}{c}
\text{Proof. Relations (29a), (29b), and (29c) are easy to check, thus we leave this as an exercise. Relation (29d) follows from}
\end{array}
\end{array}
\]

together with cup annihilation relation in (29c). The first relation in (29c) follows from

\[
\begin{array}{c}
\begin{array}{c}
\text{followed by dotted cup absorption (29c). The second relation in (29e) follows from (29d) together with the undotted cup annihilation (29c).} \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{Definition 5.6. For } n \in \mathbb{N} \text{ set:}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{Informally, the } n \text{-labeled hollow circle can be interpreted as indicating } n \text{ dots missing. We record some relations involving these morphisms.}
\end{array}
\end{array}
\]

\[
\text{Lemma 5.7.}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{where in both instances of the second formula there are } n \text{ undotted black strands running between the pink strands.}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{Informally, the } n \text{-labeled hollow circle can be interpreted as indicating } n \text{ dots missing. We record some relations involving these morphisms.}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{Lemma 5.7.}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{where in both instances of the second formula there are } n \text{ undotted black strands running between the pink strands.}
\end{array}
\end{array}
\]
In Equation (30a), it is understood that the expression in the middle is zero for $n = 0$. 

Proof. Relation (30a) is an exercise (note one equation implies the other via the involution from Remark 5.3). Relation (30b) is immediate if $n = 0$ or $m = 0$. Otherwise we use (30a) to decrease both $m$ and $n$ and the result follows by induction. Relation (30c) can be proved similarly by iterating (30a). □

5.3. The polynomial representation, revisited. The diagrammatic category $\mathbb{K}dTL'$ allows for straightforward proofs of isomorphisms in $\overline{dTL}$. For instance, in light of relation (29d), we may view the graphical calculus in $\mathbb{K}dTL'$ as an extension of the graphical calculus for the polynomial representation from Remark 3.29. The following is a key consequence. Recall that we denote $\omega = \omega_0 \oplus \omega_1 \in dTL$.

Theorem 5.8. There is an isomorphism $\omega \otimes - \cong \text{Pol}(-) \otimes \omega$ of functors $dTL \rightarrow \overline{dTL}$.

Proof. Remarks 2.24 and 2.25 show that both $\omega \otimes -$ and $\text{Pol}(-) \otimes \omega$ preserve filtered colimits, so Remark 2.9 implies that it suffices to give an isomorphism of restricted functors $$(\omega \otimes -)|_{dTL} \cong (\text{Pol}(-) \otimes \omega)|_{dTL} : dTL \rightarrow \overline{dTL}.$$ Since functors preserve isomorphisms and using Theorem 5.4, it further suffices to show that there is an isomorphism $\omega' \otimes - \cong \text{Pol}(-) \otimes \omega'$ of functors $dTL \rightarrow \text{Mat}(\text{Kar}(\mathbb{K}dTL'))$, where $\omega' := \omega_0' \oplus \omega_1'$. Since $\text{Pol}$ is monoidal, we need only establish the isomorphism on generating objects/morphisms. Equation (29d) gives that 

$$(31)$$

and (29c) implies that the summands on the right-hand side are orthogonal idempotents. Hence, we have the following direct sum decomposition in $\text{Mat}(\text{Kar}(\mathbb{K}dTL'))$: $$\omega' \otimes c \cong q^{-1}\omega' \oplus q\omega' \cong q^{-1}\mathbb{K}[x]/(x^2) \otimes \omega' = \text{Pol}(c) \otimes \omega'.$$

It is straightforward to check that the induced isomorphisms $\omega' \otimes c^n \cong \text{Pol}(c) \otimes \omega'$ are natural with respect to the generating morphisms in $dTL$, essentially by Remark 3.29. For example, the components of the isomorphism $\text{Pol}(c) \otimes \omega' \cong c \otimes \omega'$ corresponding to the subspaces $\mathbb{K} \cdot 1$ and $\mathbb{K} \cdot x$ in $\text{Pol}(c)$ are

and it follows from (31) that the dot generator in $dTL$ maps the first component to the second and the second to zero, as desired. □

Remark 5.9. Continuing along the lines of Remark 3.4, Theorem 5.8 shows that in the presence of the Kirby color, the generating object $c$ of $dTL$ acts as the multiplicity space $\text{Pol}(c) = V$, which we can
identify with the vector representation of \( \mathfrak{sl}_2 \). In particular, any linear endomorphism \( \varphi \in \text{End}_K(V) \) can be expressed diagrammatically as an element of \( \text{End}(\omega_k \otimes c) \), e.g. (31) reflects the identity
\[
\text{id}_V = F \circ E + E \circ F \in \text{End}_K(V).
\]

The direct sum decomposition from the proof of Theorem 5.8 generalizes as follows.

**Proposition 5.10.** The identity morphism of \( \omega \otimes P_n \) has the following decomposition into orthogonal idempotents in \( KdTL' \):

\[
\begin{array}{ccc}
\cdots & \pmb{P}_n & \cdots \\
\ldots & & \ldots \\
\end{array}
\quad = \quad \frac{(-1)}{n!} \sum_{k+l=n} \sum_{k \leq l} \begin{array}{c}
\cdots \\
\pm \frac{k}{n}
\end{array}
\]

**Proof.** Iterating (31) gives the relation

\[
\begin{array}{ccc}
\cdots & \pmb{P}_n & \cdots \\
\ldots & & \ldots \\
\end{array}
\quad = \quad \sum_{f} \begin{array}{c}
\cdots \\
\pm \hat{f}
\end{array}
\]

where the sum ranges over all square-free monic monomials \( f \) in the variables \( \{x_1, \ldots, x_n\} \) and \( \hat{f} \) denotes the square-free monic monomial complementary to \( f \) (uniquely characterized by \( f \cdot \hat{f} = x_1 \cdots x_n \)).

Placing a pair of symmetrizers on the top and bottom of (33) yields

\[
\begin{array}{ccc}
\cdots & \pmb{P}_n & \cdots \\
\ldots & & \ldots \\
\end{array}
\quad = \quad \frac{1}{n!} \sum_{k=0}^{n} \begin{array}{c}
\cdots \\
\pm \frac{k}{n}
\end{array}
\]

Indeed, the first equality holds since we can simultaneously sort all the dotted strands to the left on the bottom and right on the top. All of the signs involved in this operation cancel: for every dotted strand slid left on the bottom, there is a complementary undotted strand which is slid to the left in the top half of the diagram. For the second equality, we first use that \( P_n x_{k+1} \cdots x_n P_n = (-1)^k(n-k) P_n x_1 \cdots x_{n-k} P_n \), and then apply (14). The result then follows from the identity \( (\frac{k}{2}) + (\frac{n-k}{2}) + k(n-k) = \binom{n}{2} \).

Theorem 5.8 allows for the computation of \( \text{Hom}_{\overline{dTL}}(X, \omega) \) for any compact object \( X \in \overline{dTL} \). To begin, we have.

**Lemma 5.11.** \( \text{Hom}_{\overline{dTL}}(P_0, \omega) \cong \mathbb{K} \).
Proof. First, note that $\text{Hom}_{\overline{dTL}}(P_0, \omega_1) = 0$ for parity reasons, so
\[ \text{Hom}_{\overline{dTL}}(P_0, \omega_0 + \omega_1) \cong \text{Hom}_{\overline{dTL}}(P_0, \omega_0) \oplus \text{Hom}_{\overline{dTL}}(P_0, \omega_1) = \text{Hom}_{\overline{dTL}}(P_0, \omega_0). \]
Using 3.22, we then compute
\[ \text{Hom}_{\overline{dTL}}(P_0, \omega_0) = \text{colim} \left( \text{Hom}_{\text{Kar}(dTL)}(P_0, P_0) \xrightarrow{\text{Hom}(P_0, U_0)} \text{Hom}_{\text{Kar}(dTL)}(P_0, q^{-2}P_2) \xrightarrow{\text{Hom}(P_0, U_2)} \cdots \right) \]
\[ \cong \text{colim} \left( \mathbb{K} \xrightarrow{id} \mathbb{K} \xrightarrow{id} \cdots \right) \cong \mathbb{K}. \]

Proposition 5.12. There is an isomorphism of functors $\overline{dTL}^\text{op} \to \text{Vect}^{\mathbb{Z}}$:
\[ (34) \quad \text{Pol}^* \cong \text{Hom}_{\overline{dTL}}(-, \omega), \]
where $\text{Pol}^*$ is $\text{Pol}$ followed by the contravariant graded dual functor $\text{HOM}(-, \mathbb{K}) : \text{Vect}^{\mathbb{Z}} \to \text{Vect}^{\mathbb{Z}}$.

Proof. We first restrict the domains of the left and right-hand sides of (34) to $dTL$. If $X \in dTL$, then using duality in $dTL$ and Theorem 5.8, we compute
\[ \text{Hom}_{\overline{dTL}}(X, \omega) \cong \text{Hom}_{\overline{dTL}}(P_0, \omega \otimes X^\vee) \cong \text{Hom}_{\overline{dTL}}(c^0, \text{Pol}(X^\vee) \otimes \omega) \]
\[ \cong \text{Pol}(X^\vee) \otimes \text{Hom}_{\overline{dTL}}(c^0, \omega) \cong \text{Pol}(X^\vee) \cong \text{Pol}(X)^* . \]
This last isomorphism holds as $\text{Pol}$ commutes with duals since it is monoidal. Completing the domain with respect to direct sums, direct summands, and grading shifts then gives an isomorphism of functors
\[ (35) \quad \text{Pol}^*[\text{Mat}(\text{Kar}(dTL))] \cong \text{Hom}_{\overline{dTL}}(-, \omega)|_{\text{Mat}(\text{Kar}(dTL))}. \]
Both the left and right-hand sides of (34) send filtered colimits in $\overline{dTL}$ to cofiltered limits in $\text{Vect}^{\mathbb{Z}}$. Since every object of $\overline{dTL}$ is by definition a filtered colimit in $\text{Mat}(\text{Kar}(dTL))$, the isomorphism of restrictions (35) induces the isomorphism (34). \qed

Remark 5.13. Proposition 5.12 implies that the Kirby color $\omega$ is a representing object for the functor $\text{Pol}^* : \overline{dTL}^\text{op} \to \text{Vect}^{\mathbb{Z}}$, i.e.
\[ \text{Pol}^* \cong \text{Hom}_{\overline{dTL}^\text{op}}(\omega, -). \]
Equivalently, using the above isomorphism, one can show
\[ \text{Hom}_{\text{Vect}^{\mathbb{Z}}}(\text{Pol}(-), V) \cong \text{Hom}_{\overline{dTL}}(-, V \otimes \omega) \]
for any object $V \in \text{Vect}^{\mathbb{Z}}$. This shows that $V \mapsto V \otimes \omega$ is the right-adjoint to $\text{Pol}$, and $\omega$ is its value on the 1-dimensional vector space $\mathbb{K}$ concentrated in degree zero.

Using Proposition 5.12, we can compute morphisms between Kirby objects.

Corollary 5.14. $\text{End}_{\overline{dTL}}(\omega_i) \cong \mathbb{K}[z]$ where $z$ is an element of degree 2.

Proof. If $i \neq j$, then $\text{Hom}_{\overline{dTL}}(\omega_i, \omega_j) = 0$ for parity reasons. Thus, we have the isomorphisms
\[ \text{End}_{\overline{dTL}}(\omega_i) \cong \text{Hom}_{\overline{dTL}}(\omega_i, \omega_i) \cong \text{Pol}(\omega_i)^* \cong \bigoplus_{n=0}^{\infty} q^{2n} \mathbb{K} \cong \mathbb{K}[z] \]
of graded vectors spaces. Here, we have used Proposition 4.6 for the third isomorphism. For the algebraic structure, we note that all of the above isomorphisms commute with the action of the center $Z(dTL) \cong \mathbb{K}[s, z]/(s^2 = 1)$, provided we let $s$ acts by the constant $(-1)^i$ on the right-hand side. \qed

Finally, we compute the morphisms from a Kirby object to any compact object.

Lemma 5.15. Let $X \in \text{Mat}(\text{Kar}(dTL))$, then $\text{Hom}_{\overline{dTL}}(\omega_1, X) = 0$. 

Proof. It suffices to consider \( X = P_m \) for some \( m \). We compute \( \text{Hom}_{\text{KdTL}}(\omega_i, P_m) \) as the limit of the inverse system
\[
\text{Hom}_{\text{KdTL}}(q^{-i}P_i, P_m) \leftarrow \cdots \leftarrow \text{Hom}_{\text{KdTL}}(q^{-2n}P_{i+2n}, P_m) \leftarrow \text{Hom}_{\text{KdTL}}(q^{-2n-2}P_{i+2n+2}, P_m) \leftarrow \cdots
\]
of \( \mathbb{Z} \)-graded vector spaces. By Corollary 3.22, each term is given by
\[
\text{Hom}(q^{-i-2n}P_{i+2n}, P_m) \cong q^{i+2n}q^{i+2n-m}|\mathbb{K}[z]/(z)_{1+\min(m,i+2n)}.
\]
when \( i + 2n - m \) is even and is zero otherwise. In the non-zero case, the \( n^{th} \) term is supported in degrees \( > i + 2n \), hence the limit vanishes. \( \square \)

Remark 5.16. The vanishing in Lemma 5.15 further supports our assertion from Remark 4.7 that \( q^i\text{Pol}(\omega_k) \) should be interpreted as the dual Verma module \( \nabla(k) \), rather than the Verma module \( \Delta(k) \).

5.4 Kirby colored diagrammatic completion. As discussed at the end of \( \S \) 5.1, the category \( \text{KdTL}' \) is not sufficient to describe all the structure we have observed for the Kirby objects \( \omega_i \) in \( \text{dTL} \).

For example, it is an easy consequence of Theorem 5.8 (and Remark 2.25) that \( \omega \otimes \omega \cong \bigoplus_{n=0}^{\infty} q^{-2n}\omega \)
in \( \text{dTL} \), and in Corollary 5.25 below we establish the stronger result that \( \omega \otimes \omega \cong \bigoplus_{n=0}^{\infty} q^{-2n}\omega \).

However, this isomorphism cannot be described using \( \text{KdTL}' \). Indeed, this would require specifying a countably-infinite number of inclusion and projection morphisms giving the isomorphism, which would then express the identity morphism of \( \omega' \otimes \omega' \) as an infinite sum of idempotents projecting onto various shifts of \( \omega' \). This is impossible in \( \text{KdTL}' \) for two reasons:

- The morphisms in \( \text{KdTL}' \) are finite linear combinations of diagrams.
- None of the relations (28a)–(29c) changes the topology of the pink part of the diagram.

To accommodate this inadequacy, we will adjoin certain infinite sums of morphisms to \( \text{KdTL}' \). Informally, the allowed infinite sums are exactly those that get truncated to finite sums when precomposing with a morphism out of a finite object, and two infinite sums are considered equivalent if all of their finite truncations agree. We now work towards making this precise.

Definition 5.17. Let \( \text{Incl} \) denote the smallest set of morphisms in \( \text{Kar}(\text{KdTL}') \) that is closed under tensor product and contains \( \text{id}_{P_n} = P_n \in \text{End}_{\text{Kar}(\text{KdTL}')}(P_n) \) and \( i_n \in \text{Hom}_{\text{Kar}(\text{KdTL}')}(P_n, \omega'_n) \) for all \( n \in \mathbb{N} \). Further, given an object \( T \in \text{KdTL}' \), let the set \( \text{Incl}(T) \) of canonical inclusions into \( T \) be the set of all morphisms \( i \in \text{Incl} \) with codomain \( T \).

Before we proceed, we establish some useful facts about canonical inclusions. First, note that by definition the domain of any canonical inclusion is a finite object in \( \text{Kar}(\text{KdTL}') \). There is a transitive, reflexive relation on \( \text{Incl}(T) \) given by declaring that \( i \leq i' \) provided there exists a morphism \( \vartheta \) (necessarily in \( \text{Kar(dTL)} \)) such that \( i = i' \circ \vartheta \).

Lemma 5.18. Given canonical inclusions \( i: X \to T \) and \( i': X' \to T \) with \( i \leq i' \), then there is a unique morphism \( \vartheta \in \text{Hom}_{\text{KdTL}'}(X, X') \) with \( i = i' \circ \vartheta \). Consequently, for all objects \( T \in \text{KdTL}' \), the relation \( \leq \) defines a partial order on \( \text{Incl}(T) \).

Proof. If \( T \) is a finite object, then \( i = i' = \text{id}_T \), since the only canonical inclusions into a finite object are identity maps, by definition. If \( T \) is infinite, then the morphism \( \vartheta \in \text{Hom}_{\text{KdTL}'}(X, X') \) in the equation \( i = i' \circ \vartheta \) must be a composition of dotted cup maps between the appropriate tensor product of objects \( P_n \). Any two such compositions with fixed domain and codomain are equal, by (6). \( \square \)

Example 5.19. We have \( \text{Incl}(\omega'_i) = \{ i_m \in \text{Hom}_{\text{Kar}(\text{KdTL}')}(P_m, \omega'_i) \mid m \in \mathbb{N}, m \equiv i \mod 2 \} \cong \mathbb{N} \)
as partially ordered sets. Similarly,
\[
\text{Incl}(\omega'_i \otimes \omega'_j) = \{ i_m \otimes i_n \in \text{Hom}_{\text{Kar}(\text{KdTL}')}(P_m \otimes P_n, \omega'_i \otimes \omega'_j) \mid m, n \in \mathbb{N}, (m, n) \equiv (i, j) \mod 2 \} \cong \mathbb{N}^2
\]
where we use the product partial order on the latter.

Next, we observe that morphisms with domain a finite object admit factorizations as follows.

**Lemma 5.20.** Any morphism in $\text{Kar}(\text{KdTL}')$ with domain a finite object can be expressed in the form $\iota \circ D$ where $D$ is a morphism in $\text{Kar}(\text{dTL})$ and $\iota$ is a canonical inclusion.

**Proof.** Since identity morphisms in $\text{dTL}$ are canonical inclusions, it suffices to show that if $D'$ is a morphism in $\text{Kar}(\text{KdTL}')$ and $\iota'$ is a canonical inclusion, then $D' \circ \iota'$ can be expressed in the form $\iota \circ D$ where $D$ is a morphism in $\text{Kar}(\text{dTL})$ and $\iota$ is a canonical inclusion. However, this can simply be checked on each generating morphism of $\text{KdTL}'$. □

**Definition 5.21.** Let $\text{KdTL}$ denote the category with the same objects as $\text{KdTL}'$ and morphisms defined as follows. Given objects $S, T \in \text{KdTL}$, let $\text{Hom}^k_{\text{KdTL}}(S, T)$ to be the set of equivalence classes of formal expressions $\sum_{i \in I} D_i$, wherein $I$ is some (potentially infinite) indexing set, $D_i \in \text{Hom}^k_{\text{KdTL}'}(S, T)$, and such that for each $i \in \text{Incl}(S)$ we have $D_i \circ \iota = 0$ for all but finitely many $i$.

Two such infinite sums $\sum_{i \in I} D_i$ and $\sum_{j \in J} E_j$ are declared equivalent provided that for all canonical inclusions $\iota \in \text{Incl}(S)$ the finite sums $\sum_{i \in I} (D_i \circ \iota)$ and $\sum_{j \in J} (E_j \circ \iota)$ are equal as morphisms in $\text{Kar}(\text{KdTL}')$.

It is straightforward to check that $\text{KdTL}$ inherits composition and tensor product from $\text{KdTL}'$, and thus naturally carries the structure of a $\mathbb{Z}$-graded $K$-linear monoidal category. For example, composition of morphisms in $\text{KdTL}$ is given formally:

$$
\left( \sum_{j \in J} E_j \right) \circ \left( \sum_{i \in I} D_i \right) := \sum_{(j, i) \in J \times I} (E_j \circ D_i).
$$

That this is again a morphism in $\text{KdTL}$ follows from Lemma 5.20. Equation (36) further implies that the infinite sums $\sum_{i \in I} D_i$ satisfy the appropriate notion of distributivity with respect to finite sums in $\text{KdTL}'$.

Our main goal in this section is to establish the following result.

**Theorem 5.22.** The functor $\varphi: \text{KdTL}' \to \text{dTL}$ from Theorem 5.4 extends to a fully faithful functor $\text{KdTL} \to \text{dTL}$ that we also denote by $\varphi$:

$$
\begin{array}{ccc}
\text{KdTL} & \longrightarrow & \text{KdTL}' \\
\downarrow & & \downarrow \\
\text{dTL} & \longrightarrow & \text{dTL}
\end{array}
$$

We will first check the existence of the extension. The proof of fullness and faithfulness requires additional preparation.

**Proof (existence of extension).** If $X$ is a finite object, then we have $\text{Hom}^k_{\text{KdTL}}(X, T) = \text{Hom}^k_{\text{dTL}'}(X, T)$ since the only canonical inclusions into finite objects are identity morphisms. This determines the extension on such Hom-spaces.

If $f \in \text{Hom}^k_{\text{KdTL}}(S, T)$ for $S$ infinite, then we obtain a family of morphisms $f \circ \iota \in \text{Hom}^k_{\text{KdTL}'}(S, T)$ indexed by $\iota \in \text{Incl}(S)$. Note that each finite object $S_i$ is obtained from $S$ by simply replacing each instance of $\omega_i$ with some $P_n$ for $n \equiv i \mod 2$. As such, we have a directed system of dotted cup morphisms relating the objects $S_i$ (the unique morphisms given in Lemma 5.18). Precomposition with these morphisms makes the collection $\{\text{Hom}^k_{\text{KdTL}'}(S_i, T) \mid \iota \in \text{Incl}(S)\}$ into an inverse system, and the limit of this system is $\text{Hom}^k_{\text{KdTL}'}(S, T)$. The collection of maps $\{f \circ \iota\}_{\iota \in \text{Incl}(S)}$ is stable by (28c), hence it gives an element of the limit. The extension is thus given by $f \mapsto \{f \circ \iota\}_{\iota \in \text{Incl}(S)} \in \text{Hom}^k_{\text{dTL}'}(S, T)$. It is
immediate from Definition 5.21 that this assignment is independent of the representative of equivalence class.

In order to establish fully faithfulness in Theorem 5.22, we first prove some relations in the completed category $KdTL$ which will facilitate the proof.

**Lemma 5.23.** The following relations hold in $KdTL$:

\[
\left(37a\right) \quad \frac{(-1)^{\binom{n}{2}}}{n!} \sum_{n \geq 0} \delta_{n,m} = \sum_{n \geq 0} \frac{(-1)^{\binom{n}{2} + n(n-j)}}{n!} \delta_{n,m}.
\]

**Proof.** Using Proposition 5.10 and equation (30c), we compute:

Thus, the left- and right-hand sides of the first equation in (37a) are equal in $\text{Kar}(KdTL')$ upon precomposing with all $\iota \in \text{Incl}(\omega_i \otimes \omega_j \otimes \omega_k)$. Hence, they are equal in $KdTL$. The second equation in (37a) follows from the first by application of the left-right reflection symmetry from Remark 5.3, noting that each of the dots on the pink strand gives a sign $(-1)^{j-1}$.

Precomposing (37a) with the appropriate pink unit maps gives the following relations.

**Corollary 5.24.** The following identities hold in $KdTL$:

\[
\left(37b\right) \quad \frac{(-1)^{\binom{n}{2}}}{n!} \sum_{n \geq 0} \frac{(-1)^{\binom{n}{2} + n(j-1)}}{n!} \delta_{n,m}.
\]

**Corollary 5.25.** We have that $\omega_i \otimes \omega_j \cong \bigoplus_{k \geq 0} q^{-2k} \omega_{i+j}$ in $dTL$.

**Proof.** Relation (30b) and the associativity relation in (28a) imply that

\[
\left(38\right) \quad \delta_{n,m} = \frac{(-1)^{\binom{n}{2}}}{n!} \delta_{n,m}.
\]
Thus, equation (37b) expresses $\text{id}_{\omega_i} \otimes \text{id}_{\omega_j}$ as a sum of orthogonal idempotents. The result then follows by applying the functor $\text{KdTL} \to \text{dTL}$ to this decomposition of identity and then using the Biproduct Recognition Lemma 2.26. Note that condition (ii) therein holds by Proposition 5.12 and condition (iii) holds by Definition 5.21.

The analogous result is true in the purely diagrammatic category $\text{KdTL}$, but, since we have not yet shown that $\text{KdTL}$ is compactly generated, we cannot directly use the Biproduct Recognition Lemma 2.26. Nonetheless, we can repeat the argument in the proof thereof, using the following technical fact.

**Lemma 5.26.** Let $X$ be a finite object in $\text{KdTL}$ and let $T \in \text{KdTL}$ be arbitrary. Then, $\text{Hom}_{\text{KdTL}}(X,T)$ is bounded above in $q$-degree.

**Proof.** The object $X$ has a dual, so we have $\text{Hom}_{\text{KdTL}}(X,T) \cong \text{Hom}_{\text{KdTL}}(c^0,X^\vee \otimes T)$. Thus, without loss of generality we may assume $X = c^0$. Further, using Theorem 5.8 we may assume that $T$ is a tensor product of Kirby objects $T = \omega_i' \otimes \cdots \otimes \omega_r'$. By Lemma 5.20, $\text{Hom}_{\text{KdTL}}(c^0,\omega_i' \otimes \cdots \otimes \omega_r')$ is spanned by morphisms of form $\iota \circ D$ where $D \in \text{Hom}_{\text{KdTL}}(c^0,c^2N)$ and $\iota = \iota_{n_1} \otimes \cdots \otimes \iota_{n_r}$ with $n_1 + \cdots + n_r = 2N$. Since $\iota$ therefore has degree $-2N$ and $D$ has degree at most $2N$, $\text{Hom}_{\text{KdTL}}(c^0,T)$ is supported in non-positive degrees. 

**Proposition 5.27.** We have that $\omega_i' \otimes \omega_j' \cong \bigoplus_{k \geq 0} q^{-2k} \omega_{i+j}'$ in $\text{KdTL}$.

**Proof.** The proof parallels that of the Biproduct Recognition Lemma 2.26. Let $Y := \omega_i' \otimes \omega_j'$ and $N_n = q^{-i-j-2n} \omega_{i+j}'$ and consider the projection and inclusion morphisms:

$$
\sigma_n := {\begin{array}{c} [i] \otimes \cdots \otimes [i] \\
[j] \rightarrow [i+j]
\end{array}} : Y_n \rightarrow Y, \quad \pi_n := \frac{(-1)^{\binom{j}{2}}}{n!} {\begin{array}{c} [i] \\
[j] \rightarrow [i+j]
\end{array}} : Y \rightarrow Y_n.
$$

We have

$$
\pi_n \circ \sigma_m = \delta_{n,m} \text{id}_{\omega_{i+j}'} , \quad \text{id}_{\omega_i'} \otimes \text{id}_{\omega_j'} = \sum_{n=0}^{\infty} \sigma_n \circ \pi_n
$$

by (38) and (37b), respectively.

We first claim that the collection of maps $\{\pi_n\}_{n \in \mathbb{N}}$ exhibits $Y$ as a product $Y \cong \prod_{n \in \mathbb{N}} Y_n$. For this, we must construct a two-sided inverse to the assignment:

$$
\Phi : \text{Hom}_{\text{KdTL}}(T,Y) \rightarrow \prod_{n \in \mathbb{N}} \text{Hom}_{\text{KdTL}}(T,Y_n), \quad F \mapsto (\pi_n \circ F)_{n \in \mathbb{N}}.
$$

where $T \in \text{KdTL}$ is an arbitrary object. Consider the map

$$
\Phi' : \prod_{n \in \mathbb{N}} \text{Hom}_{\text{KdTL}}(T,Y_n) \rightarrow \text{Hom}_{\text{KdTL}}(T,Y), \quad (f_n)_{n \in \mathbb{N}} \mapsto \sum_{n=0}^{\infty} \sigma_n \circ f_n.
$$

To see that this is well-defined, it suffices to consider the case when each component $f_n \in \text{Hom}(T,Y_n)$ is homogeneous of degree zero. Since $Y_n = q^{-i-j-2n} \omega_{i+j}'$, this means that $f_n$ corresponds to a degree $i + j + 2n$ element of $\text{Hom}(T,\omega_{i+j}')$. Fix a canonical inclusion $\iota \in \text{Incl}(T)$ with domain $T_i$, then in $\text{Hom}_{\text{KdTL}}(T,\omega_{i+j}')$ we have that

$$
\deg(f_n \circ \iota) = i + j + 2n + \deg(\iota).
$$

Lemma 5.26 then implies that $f_n \circ \iota = 0$ for all but finitely many $n$. It is straightforward to check that $\Phi$ and $\Phi'$ are two-sided inverses.
Next, we must show that the collection of maps \( \{ \sigma_n \}_{n \in \mathbb{N}} \) exhibits \( Y \) as a coproduct \( Y \cong \coprod_{n \in \mathbb{N}} Y_n \), i.e. that, for all objects \( T \in KdTL \), the assignment

\[
\Psi: \text{Hom}_{KdTL}(Y,T) \to \prod_{n \in \mathbb{N}} \text{Hom}_{KdTL}(Y_n,T), \quad G \mapsto \{ G \circ \sigma_n \}_n
\]

has a 2-sided inverse. Consider

\[
\Psi': \prod_{n \in \mathbb{N}} \text{Hom}_{KdTL}(Y_n,T) \to \text{Hom}_{KdTL}(Y,T), \quad \{ g_n \}_n \mapsto \sum_{n=0}^{\infty} g_n \circ \pi_n .
\]

Given any canonical inclusion \( \iota \in \iota(\omega'_i) \), the first relation in Lemma 3.20 implies that \( \pi_n \circ \iota = 0 \) for all but finitely many \( n \). Hence, this assignment is well-defined, and we leave it as an exercise to show that \( \Psi \) and \( \Psi' \) are two-sided inverses.

Lastly, the condition (i) in Definition 2.6 holds by (39).

At last, we establish the following.

Proof (fully faithfulness in Theorem 5.22). Given two objects \( S, T \) in \( KdTL \), we need to show that

\[
\varphi: \text{Hom}_{KdTL}(S,T) \to \text{Hom}_{dTL}(\varphi(S),\varphi(T))
\]

is an isomorphism. If \( T \) is finite, then these Hom-spaces are zero unless \( S \) is finite as well. In that case, both Hom-spaces are simply the corresponding Hom-space in \( \text{Kar}(dTL) \) so there is nothing to show.

Thus, suppose that \( T \) is infinite. Using (the proof of) Theorem 5.8, Corollary 5.25 and Proposition 5.27, we immediately reduce to the case where \( T = \omega'_i \). We now compute all of the relevant Hom-spaces case-by-case.

1. If \( S = X \) is finite, then without loss of generality we may assume \( X = e^n \) and \( i \equiv n \mod 2 \), since otherwise \( \text{Hom}_{KdTL}(X,Y) = 0 \) and \( \text{Hom}_{dTL}(\varphi(X),\varphi(Y)) = 0 \). Proposition 5.12 gives that

\[
\text{Hom}_{dTL}(\varphi(X),\varphi(T)) = \text{Hom}_{dTL}(e^n,\omega_i) \cong \text{Pol}^*(e^n)
\]

while Definition 5.21 and Lemma 5.20 imply that

\[
\text{Hom}_{KdTL}(e^n,\omega'_i) = \text{Hom}_{KdTL}(e^n,\omega'_i) = \text{span} \left\{ \left[ \begin{array}{c} n \\ f \end{array} \right] \right\}
\]

where \( f \) ranges over all square free monic monomials in \( \mathbb{K}[x_1, \ldots, x_n] \). The isomorphism from Theorem 5.8, in the explicit form provided by (33), identifies these Hom-spaces.

2. If \( S \) is infinite, then as above we immediately reduce to the case of \( S = \omega'_j \). Without loss of generality we may assume \( i \equiv j \mod 2 \), since otherwise both Hom-spaces are zero. Definition 5.21 gives that

\[
\text{Hom}_{KdTL}(\omega'_j,\omega'_i) = \text{Hom}_{KdTL}(\omega'_j,\omega'_i) = \text{span} \left\{ \left[ \begin{array}{c} i \\ f \end{array} \right], \left[ \begin{array}{c} j \\ n \end{array} \right] \right\}_{n \in \mathbb{N}}
\]

and the result follows from Corollary 5.14. \(\square\)
6. Kirby-colored Khovanov homology

In this section, we review cabling properties of Khovanov homology and use them to define the Kirby-colored Khovanov homology of a link. Using results of Manolescu–Neithalath [MN22], we show that Kirby-colored Khovanov homology recovers the skein lasagna modules from [MWW19] evaluated on 4-dimensional 2-handlebodies, and thus gives a 4-manifold invariant.

6.1. Cabling and Khovanov homology. As in the rest of the paper, we work over a field \( \mathbb{K} \) of characteristic zero. The following construction depends on a fully functorial version of Khovanov homology. While we prefer Blanchet’s oriented model [Bla10] because of its compatibility with the \( \mathfrak{gl}_N \) version developed in [QR16, ETW18], the precise fix to functoriality will not play a role here. See e.g. the introduction of [BHKW19] for a discussion of several of the means of fixing the functoriality of Khovanov homology.

**Theorem 6.1** (Cabling in Khovanov homology). Let \( \mathcal{L} = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_r \) be an \( r \)-component framed oriented link in \( S^3 \). There is a functor

\[
\text{Kh}_\mathcal{L} : \mathcal{ABN}^{\times r} \to \text{Vect}^{\mathbb{Z} \times \mathbb{Z}}
\]

sending \( (c_{m_1}, \ldots, c_{m_r}) \) to the Khovanov homology of the cable \( \mathcal{K}_1^{m_1} \cup \cdots \cup \mathcal{K}_r^{m_r} \) of \( \mathcal{L} \) obtained by replacing the component \( \mathcal{K}_j \) by its \( m_j \)-fold parallel cable, with parallelism determined by the framing of \( \mathcal{K}_j \), and with alternating orientations\(^6\).

*Proof.* Let \( \mathcal{C}_A \) denote the category whose objects are oriented, embedded 1-manifolds in \( A := S^1 \times [0,1] \) and whose morphisms are formal \( \mathbb{K} \)-linear combinations of oriented link cobordisms properly embedded in \( A \times [0,1] \). Similarly, let \( \mathcal{C}_{S^3} \) denote the category whose objects are oriented links in \( S^3 \) and whose morphisms are formal \( \mathbb{K} \)-linear combinations of link cobordisms in \( S^3 \times [0,1] \). A framed link \( \mathcal{L} \) determines an embedding \( \mathcal{A}^{\times r} \to S^3 \), which gives a functor \( \Phi_\mathcal{L} : \mathcal{C}_A^{\times r} \to \mathcal{C}_{S^3} \).

Meanwhile, Khovanov homology (with a chosen fix of its functoriality) gives a functor \( \text{Kh} : \mathcal{C}_{S^3} \to \text{Vect}^{\mathbb{Z} \times \mathbb{Z}} \). Since \( \text{Vect}^{\mathbb{Z} \times \mathbb{Z}} \) is closed under direct sums and grading shifts, the composite \( \text{Kh} \circ \Phi_\mathcal{L} : \mathcal{C}_A^{\times r} \to \text{Vect}^{\mathbb{Z} \times \mathbb{Z}} \) extends to \( \text{Mat}(\mathcal{C}_A)^{\times r} \). By definition, \( \mathcal{ABN} \) is a quotient of \( \text{Mat}(\mathcal{C}_A) \) and the functor \( \text{Kh} \circ \Phi_\mathcal{L} \) descends to this quotient, since the local relations that define \( \mathcal{ABN} \) are satisfied by the Khovanov invariant. The resulting functor \( \mathcal{ABN}^{\times r} \to \text{Vect}^{\mathbb{Z} \times \mathbb{Z}} \) is \( \text{Kh}_\mathcal{L} \). \( \square \)

**Remark 6.2.** We work with alternating orientations in the components \( \mathcal{K}_n \) so that the annulus cobordisms \( \mathcal{K}_n \leftrightarrow \mathcal{K}_n^{n+2} \) (the images under \( \Phi_\mathcal{L} \) of cap and cup morphisms in \( \text{dTL} \subset \mathcal{ABN} \)) are oriented.

We will also use \( \text{Kh}_\mathcal{L} \) to denote the corresponding functors \( \text{dTL}^{\times r} \to \text{Vect}^{\mathbb{Z} \times \mathbb{Z}} \) and \( \text{dTL}^{\times r} \to \text{Vect}^{\mathbb{Z} \times \mathbb{Z}} \). Note that the latter exists since \( \text{Vect}^{\mathbb{Z} \times \mathbb{Z}} \) is further closed under direct summands and (filtered) colimits.

**Example 6.3.** If \( \mathcal{U} \) is the 0-framed unknot, then \( \text{Kh}_\mathcal{U} \) coincides with the polynomial representation (after forgetting about the homological grading, which is trivial on the image of \( \text{Kh}_\mathcal{U} \)). On the level of objects, \( \text{Kh}_\mathcal{U}(e^n) \) is the Khovanov homology of an \( n \)-component unlink, which is isomorphic to \( \text{Pol}(n) \) (with graded dimension \( (q + q^{-1})^n \)). The verification that cups, caps, and dots act in the same way under \( \text{Kh}_\mathcal{U} \) and \( \text{Pol} \) is a straightforward exercise.

For the purposes of this paper, we take the following definition of a colored link. (Recall Theorem \( B \) and Definition 1.2 from the introduction.)

\(^6\)If \( m_j \) is even then the orientations are balanced; if \( m_j \) is odd there is one more component oriented parallel to \( \mathcal{K}_j \) than antiparallel.
Definition 6.4. A colored link is a framed oriented link \( L \subset S^3 \) with an ordering of its components, \( L = K_1 \cup \cdots \cup K_r \) together with a choice of objects \( X_1, \ldots, X_r \in \text{dTL} \). We write the data of a colored link as \( \mathcal{L} := \mathcal{L}^X := K_1^X \cup \cdots \cup K_r^X \), where \( X := (X_1, \ldots, X_r) \). Lastly, the Khovanov homology of a colored link is denoted \( \text{Kh}(\mathcal{L}^X) := \text{Kh}_L(X_1, \ldots, X_r) \).

Further, we abbreviate \( K_1^{m_1} \cup \cdots \cup K_r^{m_r} \) by \( K_t^{m_t} \cup \cdots \cup K_r^{m_r} \), which is compatible with the usage of the latter in Theorem 6.1 under the functor \( \Phi_L \) appearing in its proof.

The \( n \)-strand braid group \( B_n \) acts on \( \text{Kh}(K^n) \) by cobordisms that braid parallel components of the cable. Strictly speaking, the action of a braid also permutes the chosen orientations of the components of \( K^n \); however, these orientations only affect the Khovanov invariant up to an overall shift, so there is no harm in regarding braids as acting by automorphisms on \( \text{Kh}(K^n) \). Work of Grigsby-Licata-Wehrli [GLW18] shows that this braid group action factors through the symmetric group \( \mathfrak{S}_n \), and, moreover, that this action coincides with the one induced by the \( \mathfrak{S}_n \) action on \( e^n \in \text{dTL} \) from (11).

Before discussing Kirby-colored Khovanov homology, we take the opportunity to prove a folklore theorem that identifies the invariant subspace \( \text{Kh}(K^n)^{\mathfrak{S}_n} \) under this action with both our invariant \( \text{Kh}(K^{P_n}) \) and another invariant appearing in the literature: Khovanov’s categorification of the \( n \)-colored Jones polynomial from [Kho05]. We denote this latter invariant by \( \text{Kh}(K; n) \).

Proposition 6.5. If \( K \) is a framed knot and \( n \in \mathbb{N} \), then \( \text{Kh}(K^n)^{\mathfrak{S}_n} \cong \text{Kh}(K^{P_n}) \cong \text{Kh}(K; n) \).

Proof. Recall that \( V \) denotes the vector representation of \( \mathfrak{sl}_2 \). One of the central ideas in [Kho05] is the construction of an explicit complex of \( \mathfrak{sl}_2 \)-representations which gives a resolution (in the category of finite-dimensional \( \mathfrak{sl}_2 \) representations) of \( \text{Sym}^n(V) \) by tensor powers of \( V \). Equivalently, Khovanov provides a complex \( C_n \in K^b(\text{Kar}(\text{TL})) \) that gives a resolution of \( P_n \in K^b(\text{Kar}(\text{TL})) \).

In the language of the present paper, Khovanov’s colored invariant \( \text{Kh}(K; n) \) is defined by extending \( \text{Kh}_K : \text{dTL} \to \text{Vect}^{Z \times Z} \) to a functor between homotopy categories of complexes

\[
\text{Kh}_K : K^b(\text{dTL}) \to K^b(\text{Vect}^{Z \times Z})
\]

and setting \( \text{Kh}(K; n) := H^\bullet(\text{Kh}_K(C_n)) \), where here \( H^\bullet(-) \) denotes taking homology. Since \( P_n \simeq C_n \) in \( K^b(\text{dTL}) \) and \( \text{Kh}_K \) is additive, it is then immediate that

\[
\text{Kh}(K; n) = H^\bullet(\text{Kh}_K(C_n)) \cong H^\bullet(\text{Kh}_K(P_n)) = \text{Kh}_K(P_n) = \text{Kh}(K^{P_n}).
\]

Lastly, note that \( \text{Kh}(K^n)^{\mathfrak{S}_n} \) can be identified with the object \( (\text{Kh}(K^n), Q_n) \in \text{Kar}(\text{Vect}^{Z \times Z}) \cong \text{Vect}^{Z \times Z} \), where \( Q_n = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} w \) is the symmetrizing idempotent for the action of \( \mathfrak{S}_n \) on \( \text{Kh}(K^n) \) by braiding cobordisms. The aforementioned results in [GLW18] imply\(^7\) that \( Q_n = \text{Kh}_K(P_n) \), thus

\[
\text{Kh}(K^n)^{\mathfrak{S}_n} = (\text{Kh}(K^n), Q_n) = (\text{Kh}_K(e^n)), \text{Kh}_K(P_n)) = \text{Kh}_K((e^n, P_n)) = \text{Kh}(K^{P_n}). \]

Remark 6.6. Note that the cabling operation from (40) in the proof of Proposition 6.5 assembles Khovanov homologies into a new complex. Although not needed here, a more sophisticated cabling operation on the level of Khovanov chain complexes can be realized in the form of a dg (or \( A_\infty \)) functor from the “derived horizontal trace” [GHW21] of the Bar-Natan bicategory to complexes of \( Z \)-graded vector spaces. We expect that all the constructions in this paper should have analogues in the setting of derived traces, but we do not investigate them here.

For brevity, we will often denote the colored knot \( K^{P_n} \) by \( K^{(n)} \).

Example 6.7. The invariant \( \text{Kh}(K^{(n)}) \) of a knot colored by a Kirby object is computed as

\[
\text{Kh}(K^{(n)}) = \underset{n \in \mathbb{N}}{\text{colim}} \, q^{-n-2n} \text{Kh}(K^{(n+2n)})
\]

\(^7\)Their results are stated in the setting of annular Khovanov homology, but their proof holds in the non-annular setting. See [MN22, Proposition 3.6] and also the discussion in [GW19, §6.1].
where the maps in the colimit are induced by dotted annulus cobordisms. As in Example 6.3, we can identify \( \text{K}(U^{m_1}) \) with \( \text{Pol}(\omega_1) \). This agrees with the \( N = 2 \) case of [MN22, Theorem 1.2], which computes Manolescu–Neithalath’s cabled Khovanov–Rozansky homology of the 0-framed unknot (in any given homology class). A generalization is given below in Theorem 6.8.

If instead we color by the total Kirby color \( \omega = \omega_0 \oplus \omega_1 \), then \( \text{K}(\mathcal{K}^{\omega}) \) is computed as

\[
\text{K}(\mathcal{K}^{\omega}) = \text{colim}_{m \in \mathbb{N}} q^{-m} \text{K}(\mathcal{K}^{(m)})
\]

where for the purposes of this colimit we must use the nonstandard partial order on \( \mathbb{N} \) given by \( m \leq m' \) iff \( m \leq m' \) and \( m \equiv m' \mod 2 \). (This colimit isn’t filtered, but the result is a direct sum of two filtered colimits.)

### 6.2. Kirby colored Khovanov homology and skein modules

To prepare for our main theorem on Kirby colored Khovanov homology, let \( \mathcal{L} \subset S^3 \) be a framed oriented link with a decomposition into sublinks \( \mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \). Denote by \( B^4(\mathcal{L}_2) \) the 4-manifold obtained by attaching 2-handles to the 4-ball \( B^4 \) along \( \mathcal{L}_2 \subset S^3 = \partial B^4 \). We regard \( \mathcal{L}_1 \) as a link in the boundary 3-manifold \( S^3(\mathcal{L}_2) := \partial B^4(\mathcal{L}_2) \).

Write \( \mathcal{L}_2 \) in terms of its components as \( \mathcal{L}_2 = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_r \). For \( \mathcal{L} = (i_1, \ldots, i_r) \in \{0, 1\}^r \), we let \( \mathcal{L}_1 \cup \mathcal{L}_2^{(i)} \) denote the colored link wherein the components of \( \mathcal{L}_1 \) are viewed as colored by the generating object \( c \) of \( \text{dTL} \) and \( \mathcal{L}_2^{(i)} = \mathcal{K}_1^{i_1} \cup \cdots \cup \mathcal{K}_r^{i_r} \).

**Theorem 6.8.** There is an isomorphism

\[
\text{K}(\mathcal{L}_1 \cup \mathcal{L}_2^{(i)}) \cong S^0(\mathcal{L}_2; \mathcal{L}_1, i, \mathbb{K})
\]

of \( \mathbb{Z} \times \mathbb{Z} \)-graded vector spaces, where the right-hand side denotes the \( \mathfrak{gl}_2 \) skein lasagna module (i.e. degree zero blob homology) as defined in [MWW19], and \( i \) is interpreted as a class in the second relative homology group \( H_2(\mathcal{L}_2, \mathcal{L}_1; \mathbb{Z}) \).

**Proof (sketch).** The result is a straightforward consequence of the Manolescu–Neithalath 2-handle formula [MN22] for the \( \mathfrak{gl}_N \) skein lasagna module \( S^0 \) in its slightly generalized form from [MWW22], together with some optimizations that are currently only possible in the \( N = 2 \) case.

Observe that the Kirby-colored Khovanov homology \( \text{K}(\mathcal{L}_1 \cup \mathcal{L}_2^{i}) \) can be computed as in Example 6.7:

\[
(41) \quad \text{K}(\mathcal{L}_1 \cup \mathcal{L}_2^{i}) \cong \text{colim}_{\mathbb{Z} \in \mathbb{N}^r} \text{K}(\mathcal{L}_1 \cup \mathcal{K}_1^{i_1} \cup \cdots \cup \mathcal{K}_r^{i_r})
\]

Next, for the purposes of stating the 2-handle formula, let us temporarily use the notation \( \mathcal{K}^{m+n} \) to denote the \((m+n)\)-cable of a framed oriented knot in which \( m \) strands have the orientation inherited from \( \mathcal{K} \) and \( n \) have the opposite orientation. The product of braid groups \( \text{Br}_m \times \text{Br}_n \) acts on \( \mathcal{K}^{m+n} \) by cobordisms which braid similarly-oriented components.

The 2-handle formula expresses the skein lasagna module \( S^0(\mathcal{L}_2; \mathcal{L}_1, 1, \mathbb{K}) \) as

\[
(42) \quad \left( \bigoplus_{\mathbb{Z} \in \mathbb{N}^r} q^{-|\mathbb{Z}|+2|\mathbb{Z}|} \text{K}(\mathcal{L}_1 \cup \mathcal{K}_1^{i_1} \cup \cdots \cup \mathcal{K}_r^{i_r}) \right) \sim
\]

where \( |\mathbb{Z}| = i_1 + \cdots + i_r \) and \( |\mathbb{Z}| = n_1 + \cdots + n_r \). The relations imposed on this direct sum are as follows:

1. Partially symmetrize, i.e. take coinvariants with respect to the parabolic subgroup \( \mathfrak{S}_{i_1+n_j} \times \mathfrak{S}_{n_j} \) acting via cobordisms braiding the similarly-oriented components of \( \mathcal{K}_j^{i_1+n_j+\partial\mathcal{K}} \).
2. Kill the action of the annulus cobordisms.
3. Identify along the images of dotted annulus maps.

Taken together, (1) and (2) accomplish the full symmetrization on each component with respect to \( \mathfrak{S}_{i_1+2n_j} \). The direct sum with these relations is hence precisely the colimit calculated in (41).
Remark 6.9. Both of the invariants \( \text{Kh}(\mathcal{L}_1 \cup \mathcal{L}_2^m) \) and \( S^2_\omega(B^4(\mathcal{L}_2); \mathcal{L}_1, i, \mathbb{K}) \) are defined for arbitrary \( i \in \mathbb{Z}^r \). The 2-periodicity of Kirby objects \( \omega_i \cong \omega_{i+2} \) ensures that the former depends only on the parity of the components of \( i \). On the other hand, the same is only true of the skein lasagna invariants \( S^2_\omega(B^4(\mathcal{L}_2); \mathcal{L}_1, i, \mathbb{K}) \) up to overall grading shift, because the components of \( K^{i+n+\pi} \) can be given an alternating orientation only if \( i \in \{0, 1\} \). This disagreement between Kirby-colored Khovanov homology and skein lasagna modules would disappear had we worked with \( \mathfrak{gl}_2 \) foams instead of the Bar-Natan category.

7. Future directions

To conclude, we list and comment on questions related to the Kirby color for Khovanov homology that we find interesting, but which exceed the scope of this paper.

7.1. Kirby color in other constructions. Khovanov homology admits many different constructions using technology from different areas of mathematics, including combinatorics, geometric representation theory, and symplectic geometry. We expect that the Kirby color \( \omega \) has an avatar in each of these constructions. It is an interesting problem to identify the Kirby color intrinsically in each of these settings.

Similarly, we expect that Kirby colors also exist for other link homology theories, provided they extend to functorial tangle invariants. Key examples are the \( \mathfrak{gl}_N \) link homology theories originally constructed by Khovanov–Rozansky [KR08]. In this setting, the role of the dotted Temperley–Lieb category is played by the dotted \( \mathfrak{gl}_N \) web category \( \text{dWeb}_N \). Isomorphism classes of objects in this category are indexed by pairs of natural numbers \((k, \ell) \in \mathbb{Z}^2_{\geq 0}\), which are represented by \( k \) upward-oriented boundary points next to \( \ell \) downward-oriented boundary points. Morphisms in \( \text{dWeb}_N \) are \( \mathbb{K} \)-linear combinations of dotted trivalent graphs, and all preserve the difference \( k - \ell \). The relation \( x^2 = 0 \) in \( \text{dTL} \) generalizes to the relation \( x^N = 0 \) in \( \text{dWeb}_N \).

In the same way as \( \text{dTL} \) describes the annular Bar-Natan category \( \mathcal{ABN} \), the dotted web category \( \text{dWeb}_N \) describes the category of annular \( \mathfrak{gl}_N \) foams (after appropriate completion). A version of the latter (its positive half) appears in [QR18], and it is possible to adapt the arguments therein to the entire category of annular \( \mathfrak{gl}_N \) foams, defined by feeding the canopolis of \( \mathfrak{gl}_N \) foams from [ETW18] into the machinery from [QW21, §3.1]. This yields the equivalence between \( \text{dWeb}_N \) and annular \( \mathfrak{gl}_N \) foams. Under this equivalence, the image of the standard object \((k, \ell) \in \text{dWeb}_N \) is a collection of concentric essential circles in the annulus of which \( k \) have the standard orientation and \( \ell \) have the opposite orientation. Therefore, the total winding number of this object around the core of the annulus is \( k - \ell \).

The analogue of Jones-Wenzl projectors in \( \text{dWeb}_N \) are idempotent morphisms \( P_{k, \ell} \), one for each object \((k, \ell) \). For any winding number \( m \in \mathbb{Z} \), we expect that one can assemble the morphisms \( P_{m+n, n} \) for \( n \geq \max(0, -m) \) into directed systems, with transition maps involving \((N - 1)\)-fold dotted cup morphisms. We conjecture that the ind-objects \( \omega_m \) giving the colimits of these directed systems play an analogous role to the Kirby colors developed in the present paper for \( \mathfrak{gl}_N \) link homology. In the \( \mathfrak{gl}_N \) setting, directed systems of different winding numbers are not isomorphic: there are nonzero morphisms between them.

7.2. Representing other modules for the annular category. As discussed in Remark 5.13, the Kirby color is a representing object for the (dual of) \( \text{Pol} \), thus in light of Example 6.3 also for planar evaluation of \( \mathcal{ABN} \). A natural follow-up is to determine whether other evaluation functors \( \rho: \mathcal{ABN} \to \text{Vect}^r \) can be represented by objects in \( \mathcal{ABN} \). Interesting examples of such \( \rho \)s include the following:

- Annular Khovanov homology [APS04] factors through a functor \( \mathcal{ABN} \to \text{Vect}^{\mathbb{Z} \times 2} \), which involves an additional annular grading. In this case, one can ask for an object representing each annular degree.
By Theorem 6.1, any framed knot $K \subset S^3$ determines a functor $\text{Kh}_K : ABV \to \text{Vect}^{\mathbb{Z} \times \mathbb{Z}}$. If it exists, a representing object for $\omega_K$ in $ABV$ would capture the data of all colored Khovanov homologies of $K$, the Khovanov homologies of all parallel cables of $K$, as well as all linear maps associated by Khovanov homology to a certain class of link cobordisms which are supported over a tubular neighborhood of $K$. If $\omega_K$ could be expressed in terms of $\omega =: \omega_U$ and compact objects of $ABV$, this would amount to a quantification of the growth behavior [Wed19] of the colored Khovanov homologies of $K$ relative to those for the unknot $U$. We speculate that this may provide an approach to formalizing a notion of $q$-holonomicity of colored Khovanov homology, in categorical analogy with the $q$-holonomicity of the colored Jones polynomial [GL05].

7.3. 2-handles in the Asaeda–Frohman TQFT. Our primary interest in the Kirby color $\omega$ is due to its relevance for computing 4-manifold invariants associated with Khovanov homology. However, $\omega$ may also play a role in the $(3 + \epsilon)$-dimensional TQFT associated with the Bar-Natan monoidal bicategory $BN$, whose associated 3-manifold invariants are the Bar-Natan skein modules from [AF07, Rus09, Kai09, Fad16]. Specifically, we expect that $\omega \in ABV$ models 3-dimensional 2-handle attachments in this theory. This parallels the fact that the Kirby color for 2-handle attachments in the 4-dimensional Crane–Yetter-type TQFT based on a braided fusion category $C$ also serves as the Kirby color in the 3-dimensional Turaev–Viro-type TQFT associated with the fusion category $C$ (with the braiding forgotten).

Our results may therefore be of particular interest in the program to categorify the Kauffman bracket skein modules of 3-manifolds presented via Heegaard decompositions. This approach first seeks to categorify the Kauffman bracket skein algebra of the Heegaard surface in the form of a monoidal category, and then to take a categorified version of a 2-sided quotient by 2-handle relations. This program was initiated in [QW21] with the construction of such a categorified skein algebra which was partially based on conjectural functoriality properties of Khovanov homology under foams. The requisite functoriality properties have recently been addressed in [Que22]. It is unclear whether the resulting (candidate) categorified skein algebra is compatible with the Kirby color as a model for 2-handles, since the construction in [QW21] requires a certain truncation by higher degree morphisms, including the dotted annulus maps. The latter, of course, are essential in the definition of the Kirby color, so it is an interesting problem to reconcile this divide.

7.4. Kirby color via coends: In the context of 3-manifold invariants associated with (not necessarily semisimple) ribbon categories $\mathcal{C}$, a useful method for encoding cabling procedures is in terms of the coend of the functor $\mathcal{C}^\text{op} \otimes \mathcal{C} \to \mathcal{C}$, $(X,Y) \mapsto X^* \otimes Y$. If it exists, the coend $A \in \mathcal{C}$ represents the relative skein module of $\mathcal{C}$-labelled ribbon graphs in the thickened annulus with boundary points in a specified disk on the boundary, viewed as a $\mathcal{C}$-module via the action at this boundary disk. A Kirby element can then be defined [Vir06] as an element of $\text{Hom}_\mathcal{C}(1,A)$ such that the associated cabled link invariants are constant under the second Kirby move. Since morphisms are taken here from the monoidal unit $1 \in \mathcal{C}$, this models the non-relative skein of the thickened annulus. We speculate that an analogous framework can be provided in the categorified case by considering a suitable completion of the Bar-Natan monoidal bicategory $BN$.

7.5. Kirby color in positive characteristic. Throughout the present paper, we have worked in characteristic zero. In characteristic $p > 0$, the description of symmetric objects $\text{Sym}^k(c)$ as images of symmetrizers on $c^k$ as in Definition 3.17 no longer holds when $k \geq p$. Thus, it is unclear if a directed system like $\omega$ can be constructed in this case. However, the category of tilting modules for $\text{SL}(2)$, which is modeled by the Temperley–Lieb category, admits interesting tensor ideals in the modular case, reminiscent of (but richer than) quantum groups at a root of unity. It is an interesting question
whether Khovanov homology in positive characteristic admits a “smaller” Kirby object, roughly in parallel with the Kirby element for the Jones polynomial at a root of unity being a finite sum.

7.6. Connections with Category $\mathcal{O}$. As discussed in Remarks 4.7 and 5.16, for $k \geq 0$ there are strong parallels between the shifted Kirby objects $q^k \omega_k$ and the $\mathfrak{sl}_2$ dual Verma modules $\nabla(k)$ of highest weight $k$. To recap:

- $\text{Pol}(q^k \omega_k)$ and $\nabla(k)$ have the same graded dimension.
- There is a degree zero monic morphism $P_k \hookrightarrow q^k \omega_k$ in $\text{dTL}$ akin to the inclusion of $\mathfrak{sl}_2$-modules $\text{Sym}^k(V) \hookrightarrow \nabla(k)$.
- $\text{Hom}_{\text{dTL}}(q^k \omega_k, X) = 0$ for any compact $X \in \text{dTL}$ and $\text{Hom}_{\text{Rep}(\mathfrak{sl}_2)}(\nabla(k), W) = 0$ for any finite-dimensional $W \in \text{Rep}(\mathfrak{sl}_2)$.

It would be interesting to understand the precise relation between $\text{dTL}$ and the BGG category $\mathcal{O}(\mathfrak{sl}_2)$, and to extend this relation to the $\mathfrak{gl}_N$ setting discussed in §7.1.

7.7. A homotopy colimit. Instead of modelling the Kirby color as a filtered colimit, we can also build a similar object as a homotopy colimit of the directed system (20). For $k \geq 0$ this is the 2-term complex $\Omega_k$:

$$
\begin{align*}
&\begin{array}{c}
q^{-k} P_k \\
U_k
\end{array} \longrightarrow \\
&\begin{array}{c}
q^{-k} P_k \\
U_k
\end{array}
\end{align*}
$$

obtained by taking coproducts in the columns, with the right-hand column in homological degree zero. The complex $\Omega_k$ can be considered as an object of the dg category $\text{Ch}^b(\text{Kar}(\text{dTL})^\oplus)$ of bounded chain complexes over $\text{Kar}(\text{dTL})^\oplus$, where the superscript $\oplus$ refers to completion with respect to countable coproducts. Gaussian elimination immediately implies $\Omega_k \simeq \Omega_{k+2}$, the analogue of Lemma 4.3. We expect that $\Omega_k$ and $\omega_k$ are quasi-isomorphic when considered as objects of a suitably defined derived category of $\text{dTL}$.

One advantage of $\Omega_k$ is that for $k \in \{0, 1\}$ it is manifestly filtered, with subquotients of the form

$$C_l := \left( q^{-l+2} P_{l-2} \overset{U_l}{\longrightarrow} q^{-l} P_l \right)$$

for $l \geq 0$, where we declare $P_l = 0$ for $l < 0$. Following an observation of Elijah Bodish, these complexes have the property that all morphisms $C_l \rightarrow C_{l'}$ are null-homotopic when $l \leq l'$, and thus may be part of a highest weight structure. We plan to investigate these structures in future work.

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