DERESONATING A TATE PERIOD

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Abstract. We introduce a technique to compute monodromy periods in certain families of algebraic varieties by perturbing ("deresonating") the fiberwise Betti to de Rham comparison off the motivic setting. As an application, we find Apery periods of Grassmannians \(G(2, N)\) and identify the Apery numbers for the equations D3 of the Mukai threefolds with certain \(L\)-values. We show that the argument of the \(L\)-function is 3 for the rational and 2 for the non-rational Mukai threefolds.

1. Apery limits for equations D3 and deresonating

How can one read the topology or geometry of a Fano variety off its Landau–Ginzburg model? Van Enckevort and van Straten were able \([vEvS06]\), given the LGs of certain four–dimensional Fanos \(F\), to recover the Chern classes of the restriction of the tangent bundle to the Calabi–Yau anticanonical section in \(F\). L. Katzarkov has suggested recently to study the birational type of \(F\) by looking at its Landau–Ginzburg model. We study this question for the class of Fano 3–folds considered by Sh. Mukai, namely, the complete intersections in the Grassmannians of simple algebraic groups. We link the rational type of the generic variety in a family with a certain period of Tate(–Artin) type, the so called Apery constant.

1.1. Apery limits. We shall say that a linear homogeneous recurrence \(R\) with polynomial coefficients is a recurrence of the Apery type, if there is a Dirichlet character with \(L\)-function \(L(s)\), an argument \(s_0 \in \mathbb{Z}, s_0 > 1\) and two solutions of \(R\)

\[ u(n) = a_n, b_n \in \mathbb{Q} \]

such that

\[ \lim_{n \to +\infty} \frac{b_n}{a_n} = c L(s_0), \quad c \in \mathbb{Q}^*. \]

We shall say that the limit above is an Apery limit of the recurrence \(R\).

The solution spaces of the recurrences of the arithmetic type we consider are typically endowed with two filtrations. One is the Dwork type filtration by the rate of growth of the denominators. The other indicates how many of the leading terms of the solution annihilate. One can specify what \(a_n\) and \(b_n\) are in the presence of the two filtrations and speak of the Apery constants in such cases. In our situation, \(a_n\) is characterized simply as the unique integral solution normalized by \(a_0 = 1\), and \(b_n\), as the unique solution with \(b_0 = 0, b_1 = 1\). We refer the reader to \([AvSZ08]\) for intriguing numerical findings on the Apery constants for recurrences that arise from certain differential equations of order 4 and 5.
In this language, our result says that the quantum recurrences (sec. 2) of the Mukai threefolds $V_{10}, V_{12}, V_{14}, V_{16}, V_{18}$ are recurrences of the Apery type, the respective Apery constants being $\frac{1}{10}\zeta(2), \frac{1}{6}\zeta(3), \frac{1}{7}\zeta(2), \frac{7}{32}\zeta(3), \frac{1}{3}L(\chi_3, 3)$. We see, in particular, that the argument of the $L$–function is 3 for the rational and 2 for the non–rational Mukai threefolds.

For the rational varieties, one can compute the Apery constant by the method of Beukers that uses modularity of the regularized quantum differential equation (sec. 3). For the non-rational varieties (sec. 4) we introduce a new method, deresonating.

1.2. Deresonating. The formula

$$\log \Gamma(1-t) = \gamma t + \sum_{i=2}^{\infty} \zeta(i)\frac{t^i}{i}$$

suggests treating a Tate motive as if it were a resonance limit of a non–motivic entity in a wider hypergeometric realm. Put $D = t\frac{d}{dt}$, and let $\mathcal{D} = \mathbb{C}[t, t^{-1}, D]$ denote the ring of differential operators on the torus. A hypergeometric $D$–module

$$\mathcal{H}(\alpha_i, \beta_j) = \mathcal{D}/\mathcal{D} \left( \prod_{i=1}^{n} (D-\alpha_i) - t \prod_{j=1}^{n} (D-\beta_j) \right)$$

with rational indices $\alpha_i, \beta_j$ is motivic, i. e. is a subquotient of a variation of periods in a pencil of varieties over $\mathbb{G}_m$ defined over $\overline{\mathbb{Q}}$. The local system of its solutions is endowed with both Betti and de Rham structures.

No rational structure can exist in the case of irrational exponents, and yet, according to Dwork, a motivic quantity in (a pullback of) a hypergeometric family should be extended along the space of hypergeometric indices. A gamma structure on a hypergeometric $D$–module manifests itself as a rational structure in the case of rational exponents and gives rise to an extension of the Betti to de Rham comparison along the space of exponents, i.e. in the non-motivic direction (hypergeometric non-periods).

One might attempt to use l’Hospital’s rule to extract certain Tate–type periods out of expressions in gamma values as follows:

1) realize a Tate motive in a degenerate limiting fiber in a family of hypergeometric pure motives;
2) perturb the hypergeometric exponents to a non-resonant set;
3) pass to the degenerate nonresonant non-period matrix; compute it;
4) let the perturbation parameters tend back to 0.

We call the process of so perturbing a Tate type period to an expression in gamma–values deresonating.

We will change slightly the proposed setup, deresonating the Apery constants, which are frequently periods in families related to hypergeometric families. The Apery constants are monodromy periods as opposed to fiberwise periods. The role of the Apery constants in topology can be explained in short as follows. Quantum topology (Dub98 4.2.1], Iri07, KKP08) has discerned in the Todd genus (the topological embodiment of what seems to be an atomic thing, the logarithm of the
multiplicative group law) the couple of \textit{gamma genera}:

\[
\frac{t}{1 - \exp(-t)} = \exp\left(\frac{1}{2}t\right)\Gamma(1 + \frac{t}{2\pi i})\Gamma(1 - \frac{t}{2\pi i}).
\]

so that, in particular, Riemann–Roch–Hirzebruch reads

\[
\chi(A, B) = \int \text{ch} A^* \gamma(X)^* \text{ch} B \gamma(X)
\]

for a Calabi–Yau \(X\). The individual gamma halves are invisible to the classical Riemann–Roch–Hirzebruch but are seen by its quantum counterpart, the \textit{Dubrovin conjecture} on the monodromy of the regularized quantum DE of a Fano variety. Each Lefschetz submodule in the cohomology of a Fano variety corresponds to a resonant summand of the local quantum DE at infinity. Finally, the monodromy of a DE is related to the recurrence on the expansion coefficients of its solutions via the Apery limits by a classical argument of Beukers. Thus, in Galkin’s recent formulation \url{http://www.mi.ras.ru/~galkin/work/zetagrass.pdf}, a system of Apery constants corresponds naturally to the Lefschetz decomposition of the cohomology of a Fano, which leads to a definition of the \textit{Apery class}. A conjecture put forward by Galkin and Iritani relates the Apery class to the gamma class for such Fano varieties as Grassmannians.

By deresonating the simplest Apery constant in the simplest family of Grassmannians, \(G(2, N)’s\), we find (Theorem 1.1) its value to be \(\frac{6}{N^2(N + 1)}\zeta(2)\). Formula \(4.3.43\) shows how the perturbed Apery constant, as an expression in hypergeometric exponents, is assembled from the matrix of the base change between the Frobenius basis and the gamma basis (which is a finite expression in gamma values) and the limit ratio which is an algebraic expression in the hypergeometric exponents (the \textit{sine formula}). The sine formula is the simplest particular case of a more general Vandermonde determinant formula; it corresponds to the choice of the second wedge as the polynomial functor. It would be interesting to compare this method with the methods of \cite{Bro06, Car02, GM04} where applicable.

The very first Landau–Ginzburg models had been studied by Beukers and Peters \cite{BP84} and Beukers and Stienstra \cite{SB85} long before they were introduced in the context of mirror symmetry. Namely, they showed that the recurrence that Apery had used to prove irrationality of \(\zeta(3)\) (resp. to find a measure of irrationality of \(\zeta(2)\)) translated into Picard–Fuchs equation in a family of \(K3\) surfaces (resp. elliptic curves). We have identified \cite{Gold07} these families with the Landau–Ginzburg models of the Fano threefold \(V_{12}\) (resp. del Pezzo surface of degree 5):

\textbf{1.3. Apery’s recurrence for \(\zeta(3)\).} Apery proved irrationality of \(\zeta(3)\) in 1979 by considering the recurrence

\[
n^3u_n - (34n^3 - 51n^2 + 27n - 5)u_{n-1} + (n - 1)^3u_{n-2} = 0.
\]

Denote by \(a_n\) he solution of the recurrence with \(a_0 = 1, a_1 = 5\) and by \(b_n\) the solution that satisfies \(b_0 = 0, b_1 = 1\). Then \cite{MP93}

\[
(i) \quad |\zeta(3) - \frac{6b_n}{a_n}| = \sum_{k=n+1}^{\infty} \frac{6}{k^3a_ka_{k-1}} = o(a_n^{-2});
\]
(ii) All $a_n$'s are integral; the denominator of $b_n$ divides $12 \text{LCM}(1, 2, \ldots, n)^3$;

(iii) $a_n = O(\alpha^n)$ where $\alpha$ is the root of the characteristic polynomial $x^2 - 34x + 1$ that is greater in absolute value;

Put $\frac{6b_n}{a_n} = \frac{p_n}{q_n}$ with coprime integral $p_n, q_n$. Then it follows from $\text{LCM}(1, 2, \ldots, n) \leq (1 + \epsilon)^n$ that $|\zeta(3) - \frac{p_n}{q_n}| = o(q_n)^{(-1+\delta)}$ with some $\delta > 0$.

The key assertion here is (ii), which follows from the fact that the solutions $a_n$ $b_n$ are iterated binomial sums:

\[
a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left( \sum_{m=1}^{k} \frac{1}{m^2} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m} \binom{n+m}{m}} \right).
\]

Put $D = \frac{\partial}{\partial t}$. Denote $A(t) = \sum a_n t^n$, $B(t) = \sum b_n t^n$. Put $L = D^3 - t(2D + 1)(17D^2 + 17D + 5) + t^2(D + 1)^3$. Then $LA = 0$ and $(D - 1)LB = 0$.

1.4. **Theorem of Beukers and Peters** [BP84]. Assume that $t \neq 0, 1, (\sqrt{2} \pm 1)^4, \infty$. Then:

(i) The surface $S_t^1 : 1 - (1 - XY)Z - tXYZ(1 - X)(1 - Y)(1 - Z) = 0$ is birationally equivalent to a K3 surface;

(ii) The form\[\omega_t = \frac{dX \wedge dZ}{XZ(1 - t(1 - X)(1 - Y)(1 - Z))} \bigg|_{S_t^1}\]
is the unique holomorphic 2–form on $X_T$;

(iii) $\text{rk Pic} X_t \geq 19$ (and is 19 for generic $t$);

(iv) The periods $y$ of the form $\omega_t$ satisfy the differential equation $Ly = 0$.

We denote by $i$ the square root of $-1$.

2. **Constructing quantum DEs and recurrences**

Let $X$ be a Picard rank one Fano threefold. Denote by $-K$ the anticanonical class of $X$. Consider a one–dimensional torus $G_m = \text{Spec} \mathbb{C}[t, t^{-1}]$.

2.1. The following is the standard procedure to obtain quantum differential equations and recurrences (cf e.g. [Gol07]).

**Step 1.** Define a trilinear functional $\langle \alpha, \beta, \gamma \rangle$ on $H^*(X)$ setting

\[
\langle \alpha, \beta, \gamma \rangle = \sum t^d \cdot \text{[number of maps $\mathbb{P}^1 \to X$ of degree $d$ with respect to $-K$ such that 0 maps into a representative of $\alpha$, 1 maps into a representative of $\beta$, $\infty$ maps into a representative of $\gamma$]}
\]

\[1\text{One can choose } \delta = \frac{\log \alpha - 3}{\log \alpha + 3}. \]
One has:
\[ \langle \cdot, \cdot \rangle : (H^*(X) \otimes \mathbb{C}[t, t^{-1}])^\otimes 3 \to \mathbb{C}[t, t^{-1}] \].

**Step 2.** Extend the Poincare pairing \( \langle \cdot, \cdot \rangle \) to the trivial vector bundle \( \mathcal{H} = H^*(X) \otimes \mathbb{C}[t, t^{-1}] \) horizontally.

**Step 3.** Turn the trilinear form into a multiplication law:
\[ (\alpha \cdot \beta, \gamma) = \langle \alpha, \beta, \gamma \rangle. \]

**Step 4.** Introduce a connection (= a D–module structure) in \( \mathcal{H} \): for \( h \in H^*(X) = H^*(X) \otimes \mathbb{C}[t, t^{-1}] \) one sets
\[ Dh = -K \cdot h \]
(here \( h \) is understood to be \( h \otimes 1 \)).

**Step 5.** “Convolute the system” into a single scalar equation using \( 1 \otimes 1 \) for the cyclic vector:
\[ \sum b_{ij} t^j D^i (1 \otimes 1) = 0. \]

**Step 6.** Translate this into a recurrence in \( u \), the expansion coefficients of the solutions,
\[ R^{irreg} : \sum b_{ij} u(n-j)(n-i) = 0 \]

**Step 7.** Pass to the equation \( R^{reg} \) whose solution is \( u(n)n! \):
\[ \sum_i u(n-i) \sum_j b_{ij} (n-i)^j \to \sum_i u(n-i) \sum_j b_{ij} (n-i)^j \frac{n!}{(n-i)!}. \]

2.2. We define the Mukai threefolds to be those Fano threefolds with Picard rank 1 that are complete intersections in the Grassmannians of simple Lie groups other than projective spaces. They were considered by Sh. Mukai in [Muk92].

\[ \begin{align*}
V_{10} & \quad \text{a section of the Grassmannian } G(2, 5) \text{ by a quadric and a codimension 2 plane} \\
V_{12} & \quad \text{a section of the orthogonal Grassmannian } O(5, 10) \text{ by a codimension 7 plane} \\
V_{14} & \quad \text{a section of the Grassmannian } G(2, 6) \text{ by a codimension 5 plane} \\
V_{16} & \quad \text{a section of the lagrangian Grassmannian } L(3, 6) \text{ by a codimension 3 plane} \\
V_{18} & \quad \text{a section of } G_2/P \text{ by a codimension 2 plane}
\end{align*} \]

2.3. The corresponding differential operators [Gol07, Prz07].

\[ \begin{align*}
V_{10} : & \quad D^4 - 2t(1 + 2D)(11D^2 + 11D + 3) - 4t^2(D + 1)(2D + 3)(1 + 2D) \\
V_{12} : & \quad D^4 - t(1 + 2D)(17D^2 + 17D + 5) + t^2(D + 1)^3 \\
V_{14} : & \quad D^3 - 4t(1 + 2D)(13D^2 + 13D + 4) - 3t(D + 1)(3D + 4)(3D + 2) \\
V_{16} : & \quad D^3 - 4t(1 + 2D)(3D^2 + 3D + 1) + 16t^2(D + 1)^3 \\
V_{18} : & \quad D^3 - 4t(1 + 2D)(3D^2 + 3D + 1) - 27t^2(D + 1)^3
\end{align*} \]

2.4. **Theorem.** The quantum recurrences of the Mukai threefolds \( V_{10}, V_{12}, V_{14}, V_{16}, V_{18} \) are recurrences of the Apery type. The respective Apery constants are:

\[ \begin{align*}
V_{10} : & \quad \frac{1}{10} \zeta(2) \\
V_{12} : & \quad \frac{1}{6} \zeta(3) \\
V_{14} : & \quad \frac{1}{7} \zeta(2) \\
V_{16} : & \quad \frac{7}{32} \zeta(3) \\
V_{18} : & \quad \frac{1}{3} L(\chi_3, 3)
\end{align*} \]
3. $V_{12}, V_{16}, V_{18}$: RATIONAL CASES

3.1. Theorem of Beukers. [Beu87, 1.2] Let $F(\tau) = \sum_{n=1}^{\infty} c_n q^n$, $q = e^{2\pi i \tau}$ be a modular form of weight 4 and conductor $N$ which is Atkin–Lehner odd, i.e. satisfies

$$F(-1/N \tau) = -(\tau \sqrt{N})^4 F(\tau).$$

Put

$$f(\tau) = \sum_{n=1}^{\infty} \frac{c_n}{n^3} q^n.$$

Denote $L(F, s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}$. Put $h(\tau) = f(\tau) - L(F, 3)$. Then

$$h(\tau) = -(\tau \sqrt{N})^2 h(-1/N \tau).$$

3.2. “Eisenstein harmonics”. We will need finite linear combination of “elementary Eisenstein series”

$$E_{2,i}(Q) = -\frac{1}{24} i (1 - 24 \sum_{n=1}^{\infty} \sigma(n) Q^n)$$

and

$$E_{4,i}(Q) = \frac{1}{240} i^2 (1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) Q^n).$$

An implication of 3.1 is the following proposition worked out by Beukers in the case $N = 6$.

3.3. Proposition. For the equations that correspond to the cases $V_{2N}$ with $N = 6, 8, 9$ the following hold:

(i) The function $\sum a_n t^n = \Phi(q(t))$ is an Atkin–Lehner odd weight 2 modular form of level $N$, i.e. $\Phi(-1/N \tau) = -N \tau^2 \Phi(\tau)$;

(ii) $\sum b_n (t(q))^n = \sum c_i \frac{t^i}{i^3} q^i$, with the coefficients $c_i$ coming from a weight 4 modular form $F$, as in Beukers’s theorem;

(iii) $\Phi(-1/N \tau)(f(-1/N \tau) - L(F, 3)) = \Phi(\tau)(f(\tau) - L(F, 3))$;

(iv) the solution

$$\sum (b_n - L(F, 3) a_n) t^n = \Phi(\tau) h(\tau)$$

extends analytically beyond the singularity $i/\sqrt{N}$.

Proof.

(i) This was established in [Gol07].
(ii) Straightforward. The expressions of $F$ in terms of the “Eisenstein harmonics” and the shape of $L$–function is the table below.

| Variety | $\Phi$ |
|---------|--------|
| $V_{12}$ | $5E_{2,1} - E_{2,2} + E_{2,3} - 5E_{2,6}$ |
| $V_{16}$ | $4E_{2,1} - 2E_{2,2} + 2E_{2,4} - 4E_{2,8}$ |
| $V_{18}$ | $3E_{2,1} - 3E_{2,9}$ |

| Variety | $F$ | $L(s)$ |
|---------|-----|--------|
| $V_{12}$ | $E_4 - 7E_{4,2} + 7E_{4,3} - E_{4,6}$ | $(1 - 7 \cdot 2^{2-s} + 7 \cdot 3^{2-s} - 6^{2-s})\zeta(s)\zeta(s - 3)$ |
| $V_{16}$ | $E_4 - 21/4E_{4,2} + 21/4E_{4,4} - E_{4,8}$ | $(1 - 21/4 \cdot 2^{2-s} + 21/4 \cdot 4^{2-s} - 8^{2-s})\zeta(s)\zeta(s - 3)$ |
| $V_{18}$ | $\sum_{n=1}^{\infty} (\frac{n}{3})\sigma_3(n)q^n$ | $\prod_p(1 - (\frac{p}{3})p^{3-s})^{-1}(1 - (\frac{p}{3})p^{-s})^{-1}$ |

(iii) Follows from (iii).
(iv) Follows from (iii).

3.4. Corollary. One has for the varieties $V_{12}, V_{16}, V_{18}$

$$\lim_{n \to +\infty} \frac{b_n}{a_n} = L(F, 3).$$

**Proof.** The assertions that $\lim_{n \to +\infty} \frac{b_n}{a_n} = x$ and that the solution $\sum b_n t^n - x \sum a_n t^n$ extends beyond the radius of convergence of $\sum a_n t^n$ and $\sum b_n t^n$ are equivalent. One has only to note that the singularity with the smaller absolute value is uniformized by the point $\tau = i/\sqrt{N}$ in each of these cases.

4. $V_{10}, V_{14}$: THE NON–RATIONAL CASES

4.1. **Theorem.** Let $N$ be an integer $\geq 5$. Then the Apery constant of the Grassmannian $G(2, N)$ is $\frac{6}{N^2(N + 1)}\zeta(2)$: there are two solutions of the regularized
quantum recurrence for $G(2, N)$

$$a_{G(2,N)}^n, b_{G(2,N)}^n,$$

such that $a_n \in \mathbb{Z}$, $a_0 = 1$, $b_0 = 0$, $b_N = 1$

$$\lim_{n \to +\infty} \frac{b_{G(2,N)}^n}{a_{G(2,N)}^n} = \frac{6}{N^2(N + 1)} \zeta(2).$$

### 4.2. Corollary.

One has:

(i) the Apery constant for $V_{10}$ is $\frac{1}{10} \zeta(2)$;

(ii) the Apery constant for $V_{14}$ is $\frac{1}{7} \zeta(2)$;

**Proof of the corollary.** i) The quantum Lefschetz theorem (cf e.g. [Gat03]) implies that for the variety $V_{10}$ one has

$$a_{V_{10}}^n = \frac{5}{2} a_{G(2,5)}^n \frac{(n!)^3(2n)!}{5n!}, b_{V_{10}}^n = b_{G(2,5)}^n \frac{(n!)^3(2n)!}{5n!}.$$

ii) Quantum Lefschetz says that for $V_{14}$ one has

$$a_{V_{14}}^n = 6 a_{G(2,6)}^n \frac{(n!)^6}{6n!}, b_{V_{14}}^n = b_{G(2,6)}^n \frac{(n!)^6}{6n!}.$$

### 4.3. Proof of Theorem 4.1.

(i) The quantum differential operator for $G(2, N)$ is the second wedge of the $N$–Kummer pullback of the differential operator $(D - 1/2)^N + t$ (which is, up to a convention, the second wedge of the q.d.o for the projective space $G(1, N)$). The second wedge generates the ideal in the ring of the differential operators $\mathcal{C}[t, D]$, that annihilates all $2 \times 2$ minors of the fundamental matrix of the q.d.o of the projective space). This is a theorem of Bertram–Ciocan–Fontanine–Kim–Sabbah, [BCFK05], [KS08].

(ii) Deresonate the hypergeometric D–module as follows. Introduce the operator

$$L_{dr} = (D - 1/2 - u)(D - 1/2 + u)(D - 1/2 - e)(D - 1/2 + e)(D - 1/2)^N + t.$$

Let $S_u$ (resp. $S_e$) be the solution whose expansion starts with $t^{1/2+u}$ (resp.$t^{1/2+e}$), and let $S_{-u}$ (resp. $S_{-e}$) be the solution whose expansion starts with $t^{1/2-u}$ (resp. $t^{1/2-e}$); put

$$\Gamma(n) = \frac{(-1)^n}{\Gamma(1/2 + e + n)\Gamma(1/2 - e + n)\Gamma(1/2 + u + n)\Gamma(1/2 - u + n)\Gamma(1/2 + n)\Gamma(1/2 - n)^{N-4}};$$

then

$$S_{-e} = \sum_{n=0}^{\infty} \frac{\Gamma(1/2 - e + n)t^{1/2-e+n}}{\Gamma(1/2 + n)\Gamma(1/2 - n)^{N-4}}.$$
Let
\[ R_u = S_u S'_u - S'_u S_u \]
\[ R_e = S_e S'_e - S'_e S_e \]
be the two solutions of \( L u^2 R = 0 \), and put \( R_u = \sum r_u^{(n)} t^n \), \( R_e = \sum r_e^{(n)} t^n \). Then, as we shall see in 4.4, the sine formula holds
\[ \lim_{n \to +\infty} \frac{r_u^{(n)}}{r_u^{(0)}} = \frac{\sin(2\pi u)}{\sin(2\pi e)} \]

(iii) Put
\[ A^{dr}(t) = \frac{R_e}{r_e^{(0)}} \]
\[ B^{dr}(t) = \frac{r_u^{(0)} R_u - r_u^{(0)} R_e}{r_u^{(0)} r_u^{(1)} - r_e^{(0)} r_e^{(1)}} \]
Then \( A^{dr}(t) \) is a deresonation of \( A(t) = \sum a_N^{(2,N)} t^n (N n)!, \) and \( B^{dr}(t) \) is a deresonation of \( B(t) = N \sum b_N^{(2,N)} t^n (N n)!, \) and the limit of the ratio of \( n \)–th respective coefficients is a perturbation of the Apery constant of the Grassmannian:
\[ \lim_{n \to +\infty} \frac{r_e^{(n)} r_u^{(0)} - r_u^{(0)} r_e^{(n)}}{r_u^{(0)} r_u^{(1)} - r_e^{(0)} r_e^{(1)}} \]}

(iv) Combining it with the sine formula, we arrive at the following expression for the perturbed Apery constant:
\[ \text{p. A. c.} = \frac{1}{N} \frac{\sin(2\pi u)/\sin(2\pi e) r_e^{(0)} r_u^{(0)2} - r_u^{(0)} r_e^{(0)}}{r_u^{(1)} r_u^{(0)} - r_e^{(1)} r_u^{(0)}} \]

It is a routine check that the limit of the p. A. c. as \( u, e \to 0 \) is \( \frac{\pi^2}{N^2(N + 1)} \).
It only remains to check the sine formula.

4.4. Proof of the sine formula. It looks probable that the sine formula holds for any nonnegative \( N \), not necessarily integral. Let us give a sketch of a proof for even \( N \). We will use Dubrovin’s extension of the Thom–Sebastiani formula [Dub99], [Dub04], that expresses the monodromy of the so called second structural connection of a product of two Frobenius manifolds in terms of the monodromy data of the second structural connections of the factors. This approach

\[ \text{Don Zagier has suggested an approach to the sine formula which is based on the Poisson summation and does not require integrality of } N. \]
will work in a much wider framework: one can apply arbitrary polynomial functors to Kloosterman type objects and compute the Apery limits for the resulting DE’s.

Identify $\mathbb{C}^{\text{BV}}$ with the real space $\mathbb{C}^{\text{CA}}$ and fix a linear form $h: \mathbb{C}^{\text{CA}} \to \mathbb{C}$ in general position. Consider the following objects of linear algebra: a vector space endowed with a non–symmetric bilinear form $\langle \cdot, \cdot \rangle$ and a choice of a semiorthogonal basis $\langle v_1, \ldots, v_n \rangle$ of $V$ compatible with a marking $\mu = \mu_V: [1, \ldots, n] \to \mathbb{C}$:

\[ [v_i, v_j] = 0 \text{ when } h(\mu(i)) > h(\mu(j)), \]
\[ [v_i, v_i] = 1 \text{ when } 1 \leq i \leq n. \]

These may be used to produce polarized local systems on $\mathbb{C} \setminus \{\mu(i)\}$ by identifying the fiber with $V$, endowing it with the form $\langle \cdot, \cdot \rangle$ (resp. $(\cdot, \cdot)$) — the (skew)symmetrization of the form $\langle \cdot, \cdot \rangle$, choosing infinity for the base point, joining it with the points $\mu_i = \mu(i)$ with the level rays $h(x) = h(\mu_i)$ as paths and requiring that the turn around $\mu_i$ act in the monodromy representation by the reflection w.r. to $v_i$. Vice versa, if there is such a local system and the paths are given, one can consider the fiber with the basis of vanishing cycles and pass from the (skew)symmetrized form to the non–symmetric one according to the order given by the values of $h(\mu_i)$.

These objects may be “tensorized”: consider $V \otimes W, \langle \cdot, \cdot \rangle, \langle v_1 \otimes w_1, \ldots, v_n \otimes w_n \rangle, \mu$

so that

\[ [v_i \otimes w_j, v_k \otimes w_l] = [v_i, v_k][w_j, w_l] \]

and

\[ \mu(i, j)_{v \otimes w} = \mu_V(i) + \mu_W(j), \]

then passing to the (skew)symmetrized local system.

For simplicity, perturb our hypergeometric differential operator $L^{dr}$ further to a non–resonance one

\[ \prod_{i=1}^{N}(D - \alpha_i) + t; \]

assume also that every index $\alpha_i$ occurs along with $1 - \alpha_i$.

(i) Put $t = -w^{-N}$, so that the coordinate $w$ is the Kummer pullback of the coordinate $t$. Clearly, wedging commutes with the Kummer pullback. We use the minus sign for simplicity so as to deal with the roots of unity and not $-1$. The final result is not affected, nor is it affected by the convention to expand the local solutions around infinity and not zero.

(ii) The monodromy of the regularized differential operator

\[ \prod_{i=1}^{N}(D + i) - w^{N} \prod_{i=1}^{N}(D + N \alpha_i) \]

can be described according to [GM09] (cf also [Gol01, 1.2]) as follows. Put

\[ H(y) = \frac{1 - y^N}{\prod(1 - y \exp(2\pi i \alpha_i))}, \]

and expand $H(y) = 1 + \sum_{i=1}^{\infty} c_i y^i$. Consider the $N$–dimensional $\mathbb{C}$–vector space $V$ with the basis $v_i$, endowed with the symmetric bilinear form given
by
\[(v_i, v_i) = 2 \quad 1 \leq i \leq N,\]
\[(v_i, v_j) = c_{|i-j|} \quad i \neq j.\]

Set \(U = \mathbb{P}^1 \setminus \{N\text{-th roots of unity } , \infty\}\). Interpret the monodromy representation as acting in the space \(V, (\cdot, \cdot)\) so that the turn around \(\exp(2\pi i(j - 1)/N)\) acts by the reflection with respect to \(v_j\). Linear algebra shows that the expansion of the eigenvectors of the local monodromy around \(\infty\) with respect to the dual basis \(\hat{v}_k\) is given by the Vandermonde matrix ([GM09]):
\[e_i = \sum_j \exp(2\pi i\alpha_j) \hat{v}_j.\]

Choose the eigenvectors \(e_1, e_2, e_3, e_4\) to correspond to the solutions \(S_{-e}, S_{e}, S_{-u}, S_u\).

(iii) According to Dubrovin [Dub99, ch. 4, 5], the monodromy of the regularized second wedge of the operator
\[1 - w^N \prod_{i=1}^N (D + N\alpha_i)\]
can be described as follows. Let \(V_{ij} = v_i \wedge v_j\) be the elements of the basis of the \(\frac{N(N-1)}{2}\) dimensional \(\mathbb{C}\)-vector space \(\Lambda^2 V\). Introduce the semiorthonormal form
\[[V_{ij}, V_{kl}] = [v_i, v_k][v_j, v_l] - [v_i, v_l][v_j, v_k],\]
and denote by \([\cdot, \cdot]\) its skewsymmetrization. The set of singularities of our DE is
\[U = \mathbb{C} \setminus \{\text{all sums of pairs of distinct roots of unity}\},\]
and the turn around \(\exp(2\pi i(k - 1)/N) + \exp(2\pi i(l - 1)/N)\) acts by the reflection with respect to \(V_{kl}\). This assertion requires specifying the loops explicitly. In our situation, a loop around \(\mu_{12} = \exp(2\pi i \cdot 0/N) + \exp(2\pi i \cdot 1/N)\) can be chosen to be of the simplest shape: a ray from infinity to \(\mu_{12}\) — a turn along a small circle around \(\mu_{12}\) — the way back along the same ray.

The eigenvectors of the monodromy around \(\infty\) have the form [Dub99, ch. 5]
\[E_{jk} = e_j \wedge e_k.\]

It is also clear that the solutions \(R^e, R^u\) correspond to \(E_{12}, E_{44}\) in the adopted notation. Therefore the expansion coefficient at \(V_{12}\) in the expansion of \(E_{12}\) with respect to the dual basis \(\hat{V}_{ij}\) equals \(\sin(2\pi e)\), and the coefficient at \(V_{12}\) in the expansion of \(E_{34}\) in \(\hat{V}_{ij}\)’s equals \(\sin(2\pi u)\).

(iv) One notes now that the sine formula
\[\lim_{n \to +\infty} \frac{r^{(n)}_e}{r^{(n)}_u} = \frac{\sin(2\pi e)}{\sin(2\pi u)}\]
means that the solution \(\sin(2\pi u)R^e - \sin(2\pi e)R^u\) can be extended analytically beyond the radius of convergence of \(R^e\) and \(R^u\). This is equivalent to
Figure 1. It is clear how to use the monodromy formula by shifting a little the singular points. The parallel lines are the level lines of $h$.

the coefficient at $\hat{V}_{12}$ in the expansion of $\sin(2\pi u)R^e - \sin(2\pi e)R^u$ being 0, which has been proven above. This finishes the proof of the sine formula and Theorem 4.1.

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