QUANTUM FIELD THEORY AT FINITE TEMPERATURE: AN INTRODUCTION

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ABSTRACT

In these notes we review some properties of Statistical Quantum Field Theory at equilibrium, i.e Quantum Field Theory at finite temperature. We explain the relation between finite temperature quantum field theory in $(d,1)$ dimensions and statistical classical field theory in $d+1$ dimensions. This identification allows to analyze the finite temperature QFT in terms of the renormalization group and the theory of finite size effects of the classical theory. We discuss in particular the limit of high temperature (HT) or the situation of finite temperature phase transitions. There the concept of dimensional reduction plays an essential role. Dimensional reduction in some sense reflects the known property that quantum effects are not important at high temperature.

We illustrate these ideas with several standard examples, $\phi^4$ field theory, the non-linear $\sigma$ model and the Gross–Neveu model, gauge theories. We construct the corresponding effective reduced theories at one-loop order, using the technique of mode expansion of fields in the imaginary time variable. In models where the field is a vector with $N$ components, the large $N$ expansion provides another specially convenient tool to study dimensional reduction.

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1 Finite (and high) temperature field theory: General remarks

Study of quantum field theory (QFT) at finite temperature was initially motivated by cosmological problems [1] and more recently has gained additional attention in connection with high energy heavy ion collisions and speculations about possible phase transitions. References to initial papers like [2,3] can be found in Gross et al [4]. More recently several new review papers and textbooks have been published [5]. Since we are interested here only in equilibrium physics the imaginary time formalism will be used throughout these notes. Non-equilibrium processes can be described either in the same formalism after analytic continuation to real time or alternatively by Schwinger’s Closed Time Path formalism in the more convenient path integral formulation [6].

In these notes we specially want to emphasize that additional physical intuition about QFT at finite temperature at equilibrium can be gained by realizing that it can be also considered as an example of classical statistical field theory in systems with finite sizes [7].

Quantum field theory at finite temperature is the relativistic generalization of finite temperature non-relativistic quantum statistical mechanics. There it is known that quantum effects are only important at low temperature. More precisely the important parameter is the ratio of the thermal wave-length \(\hbar/\sqrt{mT}\) and the length scale which characterizes the variation of the potential (for smooth potentials). Only when this ratio is large are quantum effects important. Increasing the temperature is at leading order equivalent to decrease \(\hbar\). Note that, from the point of view of the path integral representation of quantum mechanics, the transition from quantum to classical statistical mechanics appears as a kind of dimensional reduction: in the classical limit path (one dimensional system) integrals reduce to ordinary (zero dimensional points) integrals.

We discuss here quantum field theory at finite temperature in \((d,1)\) dimensions, at equilibrium. We want to explore its properties and in particular study the relevance of quantum effects at high temperature. Note that high temperature now means either that the theory contains massless particles or that one is working in the ultra-relativistic limit where the temperature, in energy units, is much larger than the rest energy of massive particles. In particular we want to understand the conditions under which statistical properties of finite temperature QFT in \((d,1)\) dimension can be described by an effective classical statistical field theory in \(d\) dimension.

1.1 Finite temperature quantum field theory

The static properties of finite temperature QFT can be derived from the partition function \(Z = \text{tr} e^{-H/T}\), where \(H\) is the hamiltonian of the quantum field theory and \(T\) the temperature. For a simple theory with boson fields \(\phi\) and
euclidean action $S(\phi)$, the partition function is given by the functional integral

$$Z = \int [d\phi] \exp \left[ -S(\phi) \right],$$  \hspace{1cm} (1.1)

where $S(\phi)$ is the integral of the lagrangian density $\mathcal{L}(\phi)$

$$S(\phi) = \int_0^{1/T} d\tau \int d^d x \mathcal{L}(\phi),$$

and the field $\phi$ satisfies periodic boundary conditions in the (imaginary) time direction

$$\phi(\tau = 0, x) = \phi(\tau = 1/T, x).$$

The quantum field theory may also involve fermions. Fermion fields $\psi(\tau, x)$ instead satisfy anti-periodic boundary conditions

$$\psi(\tau = 0, x) = -\psi(\tau = 1/T, x).$$

*Classical statistical field theory and renormalization group.* The quantum partition function (1.1) has also the interpretation of the partition function of a classical statistical field theory in $d+1$ dimension. The zero temperature limit of the quantum theory corresponds to the usual infinite volume classical partition function. Correlation functions thus satisfy the renormalization group (RG) equations of the corresponding $d+1$ dimensional theory.

In this interpretation finite temperature for the quantum partition function (1.1) corresponds to a finite size $L = 1/T$ in one direction for the classical partition function. General results obtained in the study of finite size effects \[8\] also apply here \[9\]. RG equations are only sensitive to short distance singularities, and therefore finite size effects do not modify RG equations \[9,10\]. Finite size effects affect only the solution of the RG equations, because a new dimensionless, RG invariant, variable appears which can be written as the product $Lm_L$, where the correlation length $\xi_L = 1/m_L$ characterizes the decay of correlation functions in space directions.

For $L$ finite (non-vanishing temperature), we expect a cross-over from a $d+1$-dimensional behaviour when the correlation length $\xi_L$ is small compared to $L$, to the $d$-dimensional behaviour when $\xi_L$ is large compared to $L$. This regime can be described by an effective $d$-dimensional theory. Note that in quantum field theory the initial microscopic scale $\Lambda^{-1}$, where $\Lambda$ is the QFT cut-off, always appears. Therefore, even at high temperature $L \to 0$, the product $L\Lambda$ remains large.

*Mode expansion.* As a consequence of periodicity, fields can be expanded in eigenmodes in the time direction and the corresponding frequencies are quantized. For boson fields

$$\phi(x, t) = \sum_{\omega_n = 2n\pi/L} e^{i\omega_n t} \phi_n(x).$$  \hspace{1cm} (1.2)
In the case of fermions anti-periodic conditions lead to the expansion
\[
\psi(x, t) = \sum_{\omega_n=(2n+1)\pi/L} e^{i\omega_n t} \psi_n(x). \tag{1.3}
\]

When \( T = L^{-1} \gg m \), where \( m \) is the zero-temperature physical mass of boson fields, a situation realized at high temperature in the QFT sense, or when the mass vanishes, a non-trivial physics exists for momenta much smaller than the temperature \( T \) or distances much larger than \( L \). In this limit one expects to be able to treat all non-zero modes perturbatively: the perturbative integration over the non-zero modes leads to an effective field theory for the zero-mode, with a \( d \)-dimensional action \( S_L \).

Fermions instead, due to anti-periodic conditions, have no zero modes. In the same limit fermions can be completely integrated out.

Apart from high temperature there is another situation where we expect this mode integration to be useful, in the case of a finite temperature second order phase transition. Then it is the finite temperature correlation \( \xi_L \) which diverges, and this induces a non-trivial long distance physics.

Remarks.

(i) The mode expansion (1.2, 1.3) is well-suited to simple situations where the field belongs to a linear space. In the case of non-linear \( \sigma \) models or gauge theories the separation of the zero-mode will be a more complicated issue.

(ii) More precisely the zero-mode has to be treated differently from other modes when the correlation length \( \xi_L = 1/m_L \) in the space directions is large compared to \( L \), i.e. \( m_L \ll T \). This condition is equivalent to \( m \ll T \) only at leading order in perturbation theory.

1.2 Dimensional reduction and effective field theory

To construct the effective \( d \)-dimensional theory, we thus keep the zero mode and integrate perturbatively over all other modes. It is convenient to introduce some notation
\[
\phi(x, t) = \varphi(x) + \chi(x, t), \tag{1.4}
\]
where \( \varphi \) is the zero mode and \( \chi \) the sum of all other modes (equation (1.2))
\[
\chi(x, t) = \sum_{n \neq 0} e^{i\omega_n t} \phi_n(x), \quad \omega_n = 2n\pi/L. \tag{1.5}
\]

The action \( S_L \) of the reduced theory is defined by
\[
e^{-S_L(\varphi)} = \int [d\chi] \exp[-\mathcal{S}(\varphi + \chi)]. \tag{1.6}
\]
At leading order in perturbation theory one simply finds

\[ S_L(\phi) = L \int d^d x \mathcal{L}(\phi). \] (1.7)

We note that \( L \) plays, in this leading approximation, the formal role of \( 1/\hbar \), and the large \( L \) expansion corresponds to a loopwise expansion. The length \( L \) is large with respect to \( \Lambda^{-1} \). If \( LA \) is the relevant expansion parameter, which means that the perturbative expansion is dominated by large momentum (UV) contributions, then the effective \( d \) dimensional theory can still be studied with perturbation theory. This is expected for large number of space dimensions where theories are non renormalizable. However, another dimensionless combination can be found, \( mL \), which at high temperature is small. This may be the relevant expansion parameter for theories which are dominated by small momentum (IR) contributions, a problem which arises at low dimension \( d \). Then perturbation theory is no longer possible or useful.

An important parameter in the full effective theory is really \( LmL \). Therefore an important question is whether the integration over non-zero modes, beyond leading order, generates a mass for the zero mode.

**Loop corrections to the effective action.** After integration over non-zero modes the effective action contains all possible interactions. In the high temperature limit one can perform a *local expansion* of the effective action. One expects, but this has to be checked carefully, that in general higher order corrections coming from the mode integration will generate terms which renormalize the terms already present at leading order, and additional interactions suppressed by powers of \( 1/L \). Exceptions are provided by gauge theories where new low dimensional interactions are generated by the breaking of \( O(d, 1) \) invariance.

**Renormalization.** If the initial \( d, 1 \) dimensional theory has been renormalized, the complete theory is finite in the formal infinite cut-off limit. However, as a consequence of the zero-mode subtraction, cut-off dependent terms may remain in the reduced \( d \)-dimensional action. These terms provide the necessary counter terms which render the perturbative expansion of the effective field theory finite. The effective can thus be written

\[ S_L(\phi) = S_L^{(0)}(\phi) + \text{counter-terms}. \]

Correlation functions have finite expressions in terms of the parameters of the effective action, in which the counter-terms have been omitted. The first part \( S_L^{(0)}(\phi) \) thus satisfies the RG equations of the \( d + 1 \) theory [9,12].

Finally the local expansion breaks down at momenta of order \( L^{-1} \). Actually the temperature \( L^{-1} \) plays the role of an intermediate cut-off. Determining the finite parts may involve some careful calculations.

**The finite temperature correlation length.** As already stressed, a first and important problem is to understand the behaviour of the effective mass of the
zero-mode generated by integrating out the non-zero modes. If this mass $m_L$ is of order of the QFT temperature $T = L^{-1}$, the zero-mode is no longer different from other modes. The IR problem disappears and one expects to again be able to use perturbation theory. Actually one should be able to rearrange the $d + 1$ perturbation theory to treat all modes in the same way.

Conversely if the mass of the zero-mode remains much smaller than the temperature, then perturbation theory may be invalidated by IR contributions. However then one can use the local expansion of the effective action to study the non-trivial IR properties.

At high temperature the QFT remains with only one explicit length scale $L$. The quantity $L m_L$, where $m_L$ is the physical mass of the complete theory, then only depends on dimensionless ratios. If $L m_L$ is of order one, the final zero mode acquires a mass comparable to the other modes. Note that this is what happens in theories with non-trivial IR fixed points.

2 The example of the $\phi^4_{d,1}$ quantum field theory

We first study the example of a simple scalar field theory. The scalar field $\phi$ is a $N$-component vector and the hamiltonian $\mathcal{H}(\Pi, \phi)$ is $O(N)$ symmetric

$$\mathcal{H}(\Pi, \phi) = \frac{1}{2} \int d^d x \Pi^2(x) + \Sigma(\phi), \quad (2.1)$$

with

$$\Sigma(\phi) = \int d^d x \left\{ \frac{1}{2} [\nabla \phi(x)]^2 + \frac{1}{2}(r_c + r)\phi^2(x) + \frac{1}{4!} u(\phi^2(x))^2 \right\}. \quad (2.2)$$

A cut-off $\Lambda$ as usual is implied, to render the field theory UV finite. The quantity $r_c(u)$ has the form of a mass renormalization. It is defined by the condition that at zero temperature, $T = 0$, when $r$ vanishes the physical mass $m$ of the field $\phi$ vanishes. At $r = 0$ a transition occurs between a symmetric phase, $r > 0$, and a broken phase, $r < 0$. We recall that the field theory is meaningful only if the physical mass $m$ is much smaller than the cut-off $\Lambda$. This implies either (the famous fine tuning problem) $|r| \ll \Lambda^2$ or, for $N \neq 1$, $r < 0$ which corresponds to a spontaneously broken symmetry with massless Goldstone modes. This latter situation will be examined in section 4 within the more suitable formalism of the non-linear $\sigma$-model.

Note that we sometimes will set

$$u = \Lambda^{3-d} g, \quad (2.3)$$

where $g$ is dimensionless.
The finite temperature quantum partition function reads

\[ Z = \int [d\phi] \exp[-S(\phi)], \quad (2.4) \]

with periodic boundary conditions in the time direction, and

\[ S(\phi) = \int_0^L dt \left[ \int d^d x \frac{1}{2} (d_t \phi)^2 + \Sigma(\phi) \right], \quad (2.5) \]

where \( T \equiv 1/L \) is the temperature.

We now construct the effective \( d \)-dimensional theory and discuss its validity. Note, however, that this construction is useful only if the IR divergences are strong enough to invalidate perturbation theory. Therefore we do not expect the construction to be very useful if the initial theory has a dimension \( d + 1 > 4 \), because the reduced \( d \)-dimensional theory has a finite perturbation expansion even in the massless limit. This is a property we will check by discussing the dimension \( d = 4 \).

2.1 Renormalization group at finite temperature

As already explained, some useful information can be obtained from renormalization group analysis. One important quantity is the product \( Lm_L \), where \( \xi_L = m_L^{-1} \) is the finite temperature (finite size) correlation length, and \( m_L \) therefore the mass of the zero-mode in the effective theory.

The zero temperature theory satisfies the RG equations of a \( d + 1 \) dimensional field theory in infinite volume. The dimension \( d = 3 \) is special, since then the \( \phi^4_{d+1} \) theory is just renormalizable.

Dimensions \( d > 3 \). For \( d > 3 \) the theory is non-renormalizable, which means that the gaussian fixed point \( u = 0 \) is stable. The coupling constant \( u = g \Lambda^{3-d} \) is small in the physical range, and perturbation theory is applicable. At zero temperature the physical mass in the symmetric phase scales like in the free theory

\[ m \propto r^{1/2}. \]

The leading corrections to the two-point function due to finite temperature effects are of order \( u \). Therefore in the symmetric phase, for dimensional reasons,

\[ m_L = 1/\xi_L \propto (r + \text{const.} \ g \Lambda^{3-d} L^{1-d})^{1/2}. \]

This expression has several consequences.

If at zero temperature the symmetry is broken, then a phase transition occurs at a temperature \( T_c \) which scales like

\[ T_c = 1/L_c \propto \Lambda \left( -r/\Lambda^2 \right)^{1/(d-1)}. \]
which means high temperature, since the physical mass scale is \((-r)^{1/2}\). In particular in the case \(N = 1\), the critical temperature is large with respect to the initial physical mass \(m \propto (-r)^{1/2}\):

\[ T_c \propto m^{2/(d-1)} \Lambda^{(d-3)/(d-1)} \gg m. \]

At high temperature or in the massless theory \((r = 0)\) the effective mass \(m_L\) increases like

\[ m_L \propto (\Lambda)^{(3-d)/2} \ll 1. \]

The behaviour \(L m_L \ll 1\) implies the validity of the mode expansion.

**Dimension \(d = 3\).** The theory is just renormalizable and logarithmic deviations from naive scaling appear. RG equations take the form

\[
\left[ \Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} - \frac{n}{2} \eta(g) - \eta_2(g) r \frac{\partial}{\partial r} \right] \Gamma^{(n)}(p_i; r, g, \Lambda) = 0. \tag{2.6}
\]

The product \(L m_L = F(\Lambda L, g, r L^2)\) is a dimensionless RG invariant, thus it satisfies

\[
\left[ \Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} - \eta_2(g) r \frac{\partial}{\partial r} \right] F = 0.
\]

The solution can be written

\[ L m_L = F(\Lambda L, L^2 r, g) = F(\Lambda L, g(\lambda), L^2 r(\lambda)), \tag{2.7} \]

where \(\lambda\) is a scale parameter, and \(g(\lambda), r(\lambda)\) the corresponding running parameters (or effective parameters at scale \(\lambda\))

\[ \lambda \frac{dg(\lambda)}{d\lambda} = \beta(g(\lambda)), \quad \lambda \frac{dr(\lambda)}{d\lambda} = -r(\lambda) \eta_2(g(\lambda)). \]

The form of the RG \(\beta\)-function

\[ \beta(g) = \frac{(N + 8)}{48\pi^2} g^2 + O(g^3), \tag{2.8} \]

implies that the theory is IR free, i.e. that \(g(\lambda) \to 0\) for \(\lambda \to 0\). The effective coupling constant at the physical scale is logarithmically small. For example to reach the scale \(T = 1/L\) we have to choose \(\lambda = 1/\Lambda L \ll 1\), and thus

\[ g(1/\Lambda L) \sim \frac{48\pi^2}{(N + 8) \ln(\Lambda L)}. \tag{2.9} \]

From

\[ \eta_2(g) = - \frac{N + 2}{48\pi^2} g + O(g^2), \]
one also finds
\[ r(1/\Lambda L) \propto \frac{r}{(\ln \Lambda L)^{(N+2)/(N+8)}}. \] (2.10)

Therefore RG improved perturbation theory can be used, and we expect results
to be qualitatively similar to those of \( d > 3 \), except that powers of \( \Lambda \) are replaced
by powers of \( \ln \Lambda \).

**Dimensions \( d = 2 \).** The three-dimensional classical theory has an IR fixed
point \( g^* \). Then finite size scaling (equation (2.7)) predicts, in the symmetric
phase,
\[ Lm_L = f(rL^{1/\nu}), \]
where is \( \nu \) the exponent of the three-dimensional system. Therefore \( m_L \) remains
of order \( T \) and the zero-mode is special only if the function \( f(x) \) is small (com-
pared to 1).

This happens at a phase transition, but in an effective two-dimensional theory
a phase transition is possible only for \( N = 1 \). Then if \( f(x) \) vanishes at \( x = x_0 \),
for \( |x - x_0| \ll 1 \), i.e. when the temperature \( T \) is close to the critical temperature
\( T_c \),
\[ T_c = x_0^{-\nu}(-r)^{\nu} \propto m, \quad |T - T_c| \ll (-r)^{\nu} \propto m, \]
the IR properties are described by an effective two-dimensional theory. If the
system has a finite temperature phase transition, it is at zero temperature in a
broken phase.

Again the situation of a broken phase at zero temperature for \( N \neq 1 \) will be
examined separately.

### 2.2 One-loop effective action

**Mode expansion and effective action at leading order.** To construct the effec-
tive field theory in \( d \) dimensions one expands the field in eigenmodes in the time
direction (equation (1.2)). One then calculates the effective action (1.6) by inte-
grating perturbatively over all non-zero modes. In the notation (2.2) the result
at leading order simply is
\[ S_L(\varphi) = L \Sigma(\varphi). \] (2.11)

Note that if we rescale \( \varphi \) into \( \varphi L^{-1/2} \) the coupling constant is changed into \( u/L \)
\[ S_L(\varphi) = \int d^d x \left\{ \frac{1}{2} [\nabla \varphi(x)]^2 + \frac{1}{2} r \varphi^2(x) + \frac{1}{4!} (u/L) (\varphi^2(x))^2 \right\}. \] (2.12)

At this order \( r_c \) vanishes and therefore has been omitted. The action (2.12) gen-
erates a perturbation theory. In terms of a dimensionless coupling \( g = u \Lambda^{d-3} \),
the expansion parameter is \( (\Lambda/m_L)^{4-d} g/\Lambda L \). For \( d \geq 4 \) it is always small because
\( \Lambda L \) is large. For \( d = 3 \) the expansion parameter is of order \( g/Lm_L \). Since dimen-
sional reduction is useful only for \( Lm_L \) small, the situation is subtle because the
running coupling constant $g(1/\Lambda L)$ at scale $L^{-1} \ll \Lambda$, renormalized by higher modes, is also small: $g(1/\Lambda L) = O(1/\ln(\Lambda L))$. A more detailed discussion of the situation requires a one-loop calculation.

For $d < 3$ IR singularities are always present both in the initial and the reduced theory, and the small coupling regime can never be reached for interesting situations. The $\varepsilon = 3 - d$ expansion can be useful in some limits, otherwise the problem has to be studied by non-perturbative methods.

One-loop calculation. The one-loop contribution takes the form ($\ln \det = \text{tr} \ln$)

$$
\delta S_L = \frac{1}{2} \text{tr} \ln \left[ (-d^2 - \Delta_d + r + \frac{1}{6} u \varphi^2)\delta_{ij} + \frac{1}{3} u \varphi_i \varphi_j \right] - (\varphi = 0).
$$

The situation of interest here is when the correlation length $\xi = 1/m$ of zero-temperature or the infinite volume $d + 1$ system is at least of the order of the size $L$. In this situation we expect to be able to make a local expansion in $\varphi$.

The leading order in the derivative expansion can be obtained by treating $\varphi(x)$ as a constant. To calculate the one-loop contribution to the reduced $\delta S_L$ it is then convenient to use the identity

$$
\text{tr} \ln A - \text{tr} \ln B = \text{tr} \int_0^\infty \frac{ds}{s} \left( e^{-Bs} - e^{-As} \right).
$$

This leads to Schwinger’s representation of the one-loop contribution

$$
\delta S_L = -\frac{1}{2} \int d^dx \int_0^\infty \frac{ds}{s} \int \frac{dp}{(2\pi)^d} \sum_{n \neq 0} [(N - 1) e^{-s(p^2 + \omega_n^2 + r + u \varphi^2(x)/6)}
$$

$$
+ e^{-s(p^2 + \omega_n^2 + r + u \varphi^2(x)/2)} - (\varphi \equiv 0)].
$$

This expression has UV divergences which appear as divergences at $s = 0$. A possible regularization, which we will adopt here, consists in cutting the $s$ integration at $s = 1/\Lambda^2$. The integral over momenta is simple. The sum over the frequencies $\omega_n$ can be expressed in terms of the function $\vartheta_0(s)$ (equation $[A2.8]$) which satisfies

$$
\vartheta_0(s) = s^{-1/2} \vartheta_0(1/s).
$$

One obtains

$$
\delta S_L(\varphi) = -\frac{1}{2} \frac{1}{(4\pi)^{d/2}} \int d^d x \int_1^{\infty} \frac{ds}{s^{1+d/2}} e^{-sr} \left( \vartheta_0(4\pi s/L^2) - 1 \right)
$$

$$
\times \left[ (N - 1) e^{-us\varphi^2/6} + e^{-us\varphi^2/2} - N \right].
$$

(2.13)

After the change of variables $4\pi s/L^2 \leftrightarrow s$ the expression becomes

$$
\delta S_L(\varphi) = -\frac{1}{2} L^{-d} \int d^d x \int_{s_0}^{\infty} \frac{ds}{s^{1+d/2}} e^{-sL^2r/4\pi} \left( \vartheta_0(s) - 1 \right)
$$

$$
\times \left[ (N - 1) e^{-usL^2\varphi^2/24\pi} + e^{-usL^2\varphi^2/8\pi} - N \right].
$$

(2.14)
with $s_0 = 4\pi/L^2 \Lambda^2$. We now perform a small $\varphi$ expansion. At this order this expansion makes sense only if $L^2 r > -4\pi^2$, a condition which more generally involves the dimensionless scale invariant ratio $L^2 m^2$, and implies being not too far in the ordered region. This is not surprising since an expansion around $\varphi = 0$ makes sense only if the field expectation value is small.

Order $\varphi^2$. Note that for the quadratic term the local approximation was not needed because the corresponding one-loop diagram is a constant

$$
\delta S^{(2)}_L = \frac{1}{12} (N + 2) u G_2 \int d^d x \varphi^2(x),
$$

where the constant $G_2$ is given by

$$
G_2(r, L) = \frac{L^{2-d}}{4\pi} \int_{s_0}^\infty ds \frac{e^{-rL^2 s/4\pi}}{s^{d/2}} (\vartheta_0(s) - 1). \tag{2.16}
$$

Order $\varphi^4$. The quartic term is proportional to the initial interaction

$$
\delta S^{(4)}_L = -\frac{1}{144} (N + 8) u^2 G_4 \int d^d x (\varphi^2(x))^2 \tag{2.17}
$$

with

$$
G_4 = \frac{L^{4-d}}{(4\pi)^2} \int_{s_0}^\infty ds \frac{e^{-rL^2 s/4\pi}}{s^{d/2-1}} (\vartheta_0(s) - 1) = -\frac{\partial}{\partial r} G_2. \tag{2.18}
$$

The one-loop reduced action. We first keep only the terms already present in the tree approximation. The value of $r_c$ correspond to the mass renormalization which renders the $T = 0$ theory massless at one-loop order. Thus $G_2$ has to be replaced by $[G_2]_r$. Using the same regularization one obtains $(d > 1)$

$$
[G_2]_r = G_2 - \frac{L}{(2\pi)^{d+1}} \int \frac{d^{d+1}k}{k^2} = G_2 - \frac{2L\Lambda^{d-1}}{(d-1)(4\pi)^{(d+1)/2}}.
$$

After the rescaling $\varphi \mapsto \varphi L^{-1/2}$ the effective action can be written

$$
S_L(\varphi) = \int d^d x \left\{ \frac{1}{2} [\nabla \varphi(x)]^2 + \frac{1}{2} \sigma_2 \varphi^2(x) + \frac{1}{4!} \sigma_4 (\varphi^2(x))^2 \right\}, \tag{2.19}
$$

with

$$
\sigma_2 = r + \frac{1}{6} (N + 2) u [G_2]_r/L, \quad \sigma_4 = u/L - \frac{1}{6} (N + 8) u^2 G_4/L^2.
$$
Other interactions. For space dimensions $d < 5$ the coefficients of the other interaction terms are no longer UV divergent. Since the zero-mode contribution has been subtracted no IR divergence is generated even in the massless limit. In this limit the coefficients are thus proportional to powers of $L$ obtained by dimensional analysis (in the normalization (2.12))

$$ \delta S_L^{(2n)} \propto g^n (\Lambda L)^{-n(d-3)} L^{n(d-2)-d} \int d^d x (\varphi^2(x))^n, \quad (2.20)$$

and therefore increasingly negligible at high temperature at least for $d \geq 3$.

The local expansion of the one-loop determinant also generates monomials with derivatives. No term proportional to $(\partial_\mu \varphi)^2$ is generated at one-loop order. All other terms with derivative are finite for $d < 5$, and thus the structure of the coefficients again is given by dimensional analysis. To $2k$ derivatives corresponds an additional factor $L^{2k}$.

Finally for $r \neq 0$ but $rL^2 \ll 1$, we can expand in powers of $r$ and the previous arguments immediately generalize.

3 High temperature and critical limits

We now examine two interesting situations. First we discuss $r \to 0$ which corresponds to a massless theory at zero temperature in the QFT context (and to the critical temperature of the $d+1$ dimensional statistical field theory). This will give the leading contributions in the high temperature limit. It will prove useful to also keep terms linear in $r\varphi^2$.

Then we consider another situation, which, from the classical statistical point of view, corresponds to choose the parameter $r$ as a function of $L$ to remain at the critical point, and from the QFT point of view to choose $T = 1/L$, the temperature, to take a critical value $T_c$, if possible. Then the correlation length diverges and the effective field theory for the zero-mode is indeed a $d$-dimensional field theory for a massless scalar field. We again find a situation of dimensional reduction.

The massless limit. The constants $[G_2]_r$ and $G_4$ for $r = 0$ become

$$[G_2]_r = \frac{1}{2\pi^{(d+1)/2}} \Gamma((d-1)/2) \zeta(d-1)L^{2-d} - \frac{2\Lambda^{d-2}}{(d-2)(4\pi)^{d/2}},$$

$$G_4 = \frac{\pi^{d/2-4}}{8} \Gamma(2-d/2)\zeta(4-d)L^{4-d} + \frac{2\Lambda^{d-4}}{(d-4)(4\pi)^{d/2}} + \frac{2L\Lambda^{d-3}}{(d-3)(4\pi)^{(d+1)/2}}, \quad (3.1)$$

where the results of appendices A1.2 have been used. The expression for $[G_2]_r$ is the sum of two terms, a renormalized mass term for the zero-mode, and the one-loop counterterm which renders the two-point function one-loop finite in the
reduced theory. The expression for $G_4$ contains a finite temperature contribution, a zero-temperature renormalization for $d \geq 3$ and a counter-term for the reduced theory for $d \geq 4$.

Finally from the value of $G_4$ and the relation (2.18) we immediately obtain the term linear in $r$ in $G_2$

$$G_2(r, L) = G_2(0, L) - rG_4(0, L) + O(r^2). \quad (3.3)$$

### 3.1 Dimension $d = 4$

For $d > 3$ the coupling constant $u$ which is of order $\Lambda^{3-d}$ is very small. The renormalized mass generated for the zero-mode is of order $L^{-1}(L\Lambda)^{(3-d)/2}$ and thus small in the temperature scale, justifying a mode expansion.

Let us examine more precisely the $d = 4$ case, keeping the contribution of order $r$ in $G_2$. Then

$$[G_2]_r = \frac{1}{4\pi^2 L^2} \zeta(3) - \frac{\Lambda^2}{16\pi^2} + r \left[ -\frac{\gamma}{16\pi^2} - \frac{2}{(4\pi)^{5/2}} \Lambda L + \frac{1}{8\pi^2} \ln(\Lambda L) \right] + O(r^2)$$

$$G_4 = \frac{\gamma}{16\pi^2} + \frac{2}{(4\pi)^{5/2}} \Lambda L - \frac{1}{8\pi^2} \ln(\Lambda L),$$

where $\gamma$ is Euler’s constant, $\gamma = -\psi(1)$. The infinite volume terms proportional to $\Lambda L$ which induces a finite renormalization $g \mapsto g_r$ of the dimensionless $\phi^4$ coupling $g$, and $r \mapsto r_r$

$$u = g/\Lambda, \quad g_r = g - \frac{1}{(4\pi)^{5/2}} \frac{N + 8}{3} g^2, \quad r_r = r - \frac{1}{(4\pi)^{5/2}} \frac{N + 2}{3} g r.$$

The remaining cut-off dependent terms of $[G_2]_r$ and $G_4$ will render the effective $d = 4$ theory one-loop finite. Using expression (2.19) and introducing the small dimensionless (effective) coupling constant $\lambda_L$

$$\lambda_L = g_r/(\Lambda L),$$

we can write the effective action at one-loop order

$$S_L(\varphi) = \int d^4 x \left\{ \frac{1}{2} [\nabla \varphi(x)]^2 + \frac{1}{2} r_L \varphi^2(x) + \frac{1}{4!} g_L (\varphi^2(x))^2 \right\} + \delta S_{L,\Lambda}(\varphi), \quad (3.4)$$

where $\delta S_{L,\Lambda}$ is the sum of one loop counter-terms

$$\delta S_{L,\Lambda}(\varphi) = \int d^4 x \left[ \frac{1}{2} \delta r_L \varphi^2(x) + \frac{1}{4!} \delta g_L (\varphi^2(x))^2 \right]$$
and the various coefficients are
\[ r_L = r_\tau + \frac{1}{6}(N + 2) \frac{\zeta(3)}{4\pi^2} L^{-2} \lambda_L, \quad \delta r_L = \frac{N + 2}{96\pi^2} (-\Lambda^2 + 2r_\tau \ln(\Lambda L) - \gamma r_\tau) \lambda_L, \]
\[ g_L = \lambda_L, \quad \delta g_L = \frac{N + 8}{96\pi^2} (2 \ln(\Lambda L) - \gamma) \lambda_L^2. \]

Other local interactions. In the same normalization an interaction term with 2n fields and 2k derivatives is proportional to \( g_L^n L^{2n-4+2k} \) and thus negligible in the situations under study for \( n > 2 \) or \( n = 2, k > 0 \).

One-loop calculation with reduced action. To check that indeed one-loop counter-terms have been provided, we calculate the one-loop contributions to the two-point function \( \Gamma^{(2)} \) and the four-point function \( \Gamma^{(4)} \) at zero momentum.

For the one-loop contribution \( r_L = r_\tau = r \) because the differences are of order \( \lambda_L \). Taking into account the counter-terms one finds
\[
\Gamma^{(2)}_{\text{one loop}} = \frac{N + 2}{6} \frac{\lambda_L}{(2\pi)^4} \int^\Lambda \frac{d^4k}{k^2 + r} - \frac{N + 2}{96\pi^2} \Lambda^2 \lambda_L + \frac{N + 2}{96\pi^2} (2 \ln(\Lambda L) - \gamma) r \lambda_L
\]
\[ = \frac{N + 2}{96\pi^2} (\ln(r L^2) - 1) \lambda_L r. \]

Therefore the complete two-point function at one-loop order is
\[
\Gamma^{(2)}(p) = p^2 + r_\tau + \frac{1}{6}(N + 2) \frac{\zeta(3)}{4\pi^2} L^{-2} \lambda_L + \frac{N + 2}{96\pi^2} (\ln(r_\tau L^2) - 1) \lambda_L r_\tau. \quad (3.5)
\]

For the four-point function at zero momentum we find
\[
\Gamma^{(4)}_{\text{one loop}} = -\frac{N + 8}{6} \frac{1}{(2\pi)^4} \int^\Lambda \frac{d^4k}{(k^2 + r)^2} \lambda_L^2 + \frac{N + 8}{96\pi^2} (2 \ln(\Lambda L) - \gamma) \lambda_L^2
\]
\[ = \frac{N + 8}{96\pi^2} \ln(r_\tau L^2) \lambda_L^2. \]

Thus the complete four-point function reads
\[
\Gamma^{(4)}(p_i = 0) = \lambda_L + \frac{N + 8}{96\pi^2} \ln(r_\tau L^2) \lambda_L^2 + O(\lambda_L^3).
\]

We note that \( L^{-1} \) plays the role of the cut-off in the reduced theory. We also find large logarithms which can be summed by the RG of the four-dimensional reduced theory. However, because the initial coupling constant \( \lambda_L \) is very small, it has no time to run.

The massless theory. At leading order in the massless theory \( r = 0 \) we find an effective mass
\[
Lm_L = \left[ \frac{1}{24\pi^2} (N + 2) \zeta(3) \lambda_L \right]^{1/2} \ll 1.
\]
Although the induced mass remains small, because the effective four-dimensional theory has at most logarithmic IR singularities, and the effective coupling is of order $1/\Lambda L$, the reduced theory can still be solved by perturbation theory.

For $r \neq 0$ but still such that $rL^2$ is small (high temperature) the product $Lm_L$ remains small, the term $rL^2 = L^2m^2$ becoming dominant when $mL \gg (\Lambda L)^{-1/2}$.

The critical temperature. We now calculate the critical temperature. Note that for dimensions $d \geq 3$ we can study the effective theory by perturbation theory and renormalization group. If we start from four or lower dimensions perturbative methods are no longer applicable, because the massless theory is IR divergent.

Using the expression (3.3) one finds at leading order

$$r + \frac{(N+2)\zeta(3)}{24\pi^2L^2} \lambda_L = 0.$$  

This equation justifies a small $r$ expansion, and shows in particular that a phase transition is possible only if at $T = 0$ (zero QFT temperature) the system is in the ordered phase. The critical temperature $T_c$ has the form

$$T_c \sim ((N+2)\zeta(3))^{-1/3}(2\pi)^{2/3}\langle \phi \rangle |^{2/3} \gg m,$$  

where $|\langle \phi \rangle|$ is the zero temperature field expectation value and $m \propto \sqrt{-r}$ the physical mass-scale. This result confirms that the critical temperature is in the high temperature region.

3.2 Dimension $d \leq 3$

Dimension $d = 3$. For $d = 3$ and at order $r$ we now obtain

$$[G_2]_r = \frac{1}{12L} - \frac{2\Lambda}{(4\pi)^{3/2}} - \frac{L}{8\pi^2} \ln(\Lambda L) - \frac{1}{2}\gamma - \ln(4\pi) \right) r.$$  

$$G_4 = \frac{L}{8\pi^2} \ln(\Lambda L) - \frac{1}{2}\gamma - \ln(4\pi).$$

The coupling constant $u$ is dimensionless $u \equiv g$. Then $G_4$ just yields the one-loop contribution to the perturbative expansion of the running coupling $g(1/\Lambda L)$,

$$g(1/\Lambda L) = g - \frac{N + 8}{48\pi^2} \ln(\Lambda L) - \frac{1}{2}\gamma - \ln(4\pi) \right) g^2.$$  

In fact we know from RG arguments that all quantities will only depend on the running coupling constant.

In the same way $G_2$ contains a one-loop contribution to perturbative expansion of the running $\phi^2$ coefficient $r(1/\Lambda L)$

$$r(1/\Lambda L)/r = 1 - \frac{N + 2}{48\pi^2} \ln(\Lambda L) - \frac{1}{2}\gamma - \ln(4\pi) \right) g.$$  

$$G_4 = \frac{L}{8\pi^2} \ln(\Lambda L) - \frac{1}{2}\gamma - \ln(4\pi).$$
The $d = 3$ effective theory is super-renormalizable, and thus requires only a mass renormalization. In $G_2$ we find two terms, one cut-off dependent which is a one-loop counter-term, and the second which gives a mass to the zero-mode. The one-loop effective action takes the form

$$S_L(\varphi) = \int d^3x \left\{ \frac{1}{2} |\nabla \varphi(x)|^2 + \frac{1}{2} r_L \varphi^2(x) + \frac{1}{4!} g_L (\varphi^2(x))^2 \right\} + \text{one loop counter-terms},$$

with

$$r_L = r(1/\Lambda L) + \frac{N + 2}{72} \frac{g(1/\Lambda L)}{L^2}, \quad g_L = \frac{g(1/\Lambda L)}{L}.$$

Expanding in powers of $g$ one can check explicitly that, as in the case $d = 4$, the term proportional to $\Lambda$ in $[G_2]_r$ is indeed the one-loop counter-term and one is left with a one-loop contribution to $\Gamma^{(2)}$:

$$-\frac{N + 2}{48\pi} g \frac{\sqrt{r}}{L}.$$

Again we note that $L^{-1}$ now plays the role of the cut-off.

The massless theory. For $r = 0$

$$(Lm_L)^2 = \frac{N + 2}{72} g(L\Lambda).$$

The solution to the RG equation tells us that when $L \Lambda \gg 1$ $g(L\Lambda)$ goes to zero as $1/\ln(L\Lambda)$ (equation (2.9)). Therefore

$$(Lm_L)^2 \sim \frac{2\pi^2(N + 2)}{3(N + 8)} \frac{1}{\ln(\Lambda L)}.$$

The mass of the zero-mode is smaller, though only logarithmically smaller, than the other modes, justifying the mode expansion. Moreover the perturbative expansion of the three dimensional effective theory is, for small momenta, an expansion in $g(\Lambda L)/L$ divided by the mass which is of order $\sqrt{g(\Lambda L)/L}$. The expansion parameter thus is $\sqrt{g(\Lambda L)}$ which is small, due to the IR freedom of the four-dimensional theory. Higher order calculations have been performed [13,14]. The convergence, however, is expected to be extremely slow and therefore summation techniques have been proposed [15]. General summation methods, which have been used in the calculation of 3D critical exponents, should be useful here also [16,17].

The situation $r = 0$ is representative of a regime, in which $|L^2 r(1/\Lambda L)| = L^2 m^2$ is small compared to, or of the order of $g(L\Lambda)$, where the physical mass scale $m$ is for $r > 0$ also proportional to the physical mass.
The critical temperature. If at zero temperature the system is in an ordered phase \((r < 0)\), at higher temperature a phase transition occurs at a critical temperature \(T_c = 1/L\), which at leading order is solution of the equation

\[
r_L = r(1/\Lambda L) + \frac{N + 2 \, g(1/\Lambda L)}{72 \, L^2} = 0.
\]

This relation can be rewritten in various forms, for example

\[
\sqrt{(N + 2)/12} \, T_c \sim |\langle \phi \rangle| \propto m \sqrt{\ln(\Lambda/m)} \propto (-r)^{1/2} (\ln(-r))^{3/(N+8)}, \tag{3.11}
\]

where \(m\) and \(\langle \phi \rangle\) are the physical mass scale and field expectation value resp. at zero temperature. We note that the critical temperature is large compared to the mass scale \(m\) and thus belongs to the high temperature regime. The critical theory, which can no longer be studied by perturbative methods, is the theory relevant to a large class of phase transitions in statistical physics. It has been studied by a number of different methods.

Other local interactions. In the same normalization an interaction term with \(2n\) fields and \(2k\) derivatives is proportional to \(g^n L^{n-3+2k}\) and thus negligible in the situations under study for \(n > 2\) or \(n = 2, k > 0\), because even the zero-mode mass is large.

Dimension \(d < 3\). Renormalization group arguments imply that the finite \(L\) correlation length \(\xi_L\) is proportional to \(L\) for \(d < 3\). Since \(L m_L\) is of order unity, there is no justification anymore to treat the zero mode separately. To calculate correlation functions at momenta small compared to the temperature, and for small field expectation value, a local expansion of the type of chiral perturbation theory still makes physical sense, but it is necessary to modify the perturbative expansion. For example in the \(d+1\) field theory one can add and subtract a mass of order \(1/L\) a mass term for the zero mode. This temperature dependent mass renormalization modifies the propagator and introduces an IR cut-off. One then determines the mass term by demanding cancellation of the one-loop correction to the mass.

An alternative strategy is to work in \(d = 3 - \varepsilon\) dimension and use the \(\varepsilon\)-expansion. Then the zero-mode effective mass is formally small of order \(\sqrt{\varepsilon}/L\), and the expansion parameter is \(\sqrt{\varepsilon}\).

The critical temperature. For \(d = 2\), a problem which arises only for \(N = 1\), RG equations lead to the scaling relation

\[
T_c \propto |\langle \phi \rangle|^2/(1+\eta) \propto m,
\]

where \(\eta\) is the 3D Ising model exponent \(\eta \approx 0.03\).
4 The non-linear $\sigma$ model in the large $N$ limit

We now discuss another, related, example: the non-linear $\sigma$ model because the presence of Goldstone modes introduces some new aspects in the analysis. Moreover, due the non-linear character of the group representation, one is confronted with difficulties which also appear in non-abelian gauge theories. Actually the non-linear $\sigma$ model and non-abelian theories share another property: both are asymptotically free in the dimensions in which they are renormalizable [18,19].

Before dealing with the non-linear $\sigma$ in the perturbative framework, we discuss the finite temperature properties in the large $N$ limit. Large $N$ methods are particularly well suited to study finite temperature QFT because one is confronted with a problem of crossover between different dimensions. We recall that it has been proven within the framework of the $1/N$ expansion that the non-linear $\sigma$ model is equivalent to the $((\phi^2)^2$ field theory, both quantum field theories generating two different perturbative expansions of the same physical model [19].

The non-linear $\sigma$ model. The non-linear $\sigma$ model is an $O(N)$ symmetric quantum field theory, with an $N$-component scalar field $S(x)$ which belongs to a sphere, i.e. satisfies the constraint $S^2(x) = 1$. To study the model in the large $N$ limit, it is convenient to enforce the constraint by a $\delta$-function in its Fourier representation. We thus write the partition function of the non-linear $\sigma$ model:

$$Z = \int [dS(x)d\lambda(x)] \exp [−S(S, \lambda)], \quad (4.1)$$

with:

$$S(S, \lambda) = \frac{1}{2t} \int d^{d+1}x \left[ (\partial_\mu S(x))^2 + \lambda(x) (S^2(x) - 1) \right], \quad (4.2)$$

where the $\lambda$ integration runs along the imaginary axis. The parameter $t$ is the coupling constant of the quantum model as well as the temperature of the corresponding classical theory in $d+1$ dimensions.

Note that to compare the expectation value of $S$ with the expectation of the field $\phi$ of the $\phi^4$ field theory one must set $S = t^{1/2} \phi$.

4.1 The large $N$ limit at zero temperature

We briefly recall the solution of the $\sigma$-model in the large $N$ limit at zero temperature. Integrating over $N-1$ components of $S$ and calling $\sigma$ the remaining component, we obtain [18,20]:

$$Z = \int [d\sigma(x)d\lambda(x)] \exp [−S_N(\sigma, \lambda)], \quad (4.3)$$
with:

\[ S_N (\sigma, \lambda) = \frac{1}{2t} \int \left[ (\partial_\mu \sigma)^2 + (\sigma^2(x) - 1) \lambda(x) \right] d^{d+1}x + \frac{1}{2} (N-1) \text{tr} \ln [-\Delta + \lambda(\cdot)] \]  

(4.4)

The large $N$ limit is here taken at $tN$ fixed. The functional integral can then be calculated by steepest descent. At leading order we replace $N-1$ by $N$. The saddle point equations are:

\[ m^2 \sigma = 0, \]  

(4.5a)

\[ \sigma^2 = 1 - \frac{Nt}{(2\pi)^{d+1}} \int^\Lambda \frac{d^{d+1}k}{k^2 + m^2}, \]  

(4.5b)

where we have set $\langle \lambda(x) \rangle = m^2$. For $t$ small the field expectation value $\sigma$ is different from zero, the $O(N)$ symmetry is broken and thus $m$, which is the mass of the $\pi$-field, vanishes. Equation (4.5b) gives the field expectation value:

\[ \sigma^2 = 1 - \frac{t}{t_c}, \]  

(4.6)

where we have introduced $t_c$, the critical coupling constant where $\sigma$ vanishes:

\[ \frac{1}{t_c} = \frac{N}{(2\pi)^{d+1}} \int^\Lambda \frac{d^{d+1}k}{k^2}. \]  

(4.7)

Above $t_c$, $\sigma$ instead vanishes, the symmetry is unbroken, and $m$ which is now the common mass of the $\pi$- and $\sigma$-field is given by the gap equation

\[ \frac{1}{(2\pi)^{d+1}} \int^\Lambda \frac{d^{d+1}k}{k^2 + m^2} = \frac{1}{Nt}. \]

Depending on space dimension we thus find:

(i) For $d > 3$:

\[ m \propto \sqrt{t - t_c}. \]  

(4.8)

(iii) For $d = 3$

\[ \frac{1}{t_c} - \frac{1}{t} \sim \frac{N}{8\pi^2} m^2 \ln(\Lambda/m). \]  

(4.9)

(iv) For $d = 2$

\[ m = \frac{4\pi}{N} \left( \frac{1}{t_c} - \frac{1}{t} \right). \]  

(4.10)

(v) For $d = 1$

\[ m \propto \Lambda e^{-2\pi/NI}. \]  

(4.11)

The physical domain then corresponds to $t$ small, $t = O(1/\ln(\Lambda/m))$. 
4.2 Finite temperature

As we have already explained, finite temperature $T$ corresponds to finite size $L = 1/T$ in the corresponding $d + 1$ dimensional classical theory.

The saddle point or gap equation (4.5b), in the symmetric phase $\sigma = 0$, becomes [21]

\[
1 = N t \frac{1}{(2\pi)^d L} \int d^d k \sum_n \frac{1}{m_n^2 + k^2 + (2\pi n/L)^2},
\]

\[
= N t L^{1-d} \frac{1}{4\pi} \int_{s_0}^\infty \frac{ds}{s^{d/2}} e^{-m_n^2 L^2 s/4\pi} \vartheta_0(s),
\]

with $s_0 = 4\pi/L^2 \Lambda^2$.

Here $\xi_L = m_L^{-1}$ has the meaning of a correlation length in the space directions.

A phase transition is possible only if the integral is finite for $m_L = 0$. IR divergences can come only from the contribution of the zero-mode: since the integral is $d$-dimensional, a phase transition is possible only for $d > 2$. This is already an example of dimensional reduction $d + 1 \rightarrow d$.

We have seen that from the point of view of perturbation theory a crossover between different dimensions is a source of technical difficulties because IR divergences are more severe in lower dimensions. Instead the large $N$ expansion is particularly well suited to the study of this problem because it exists for any dimension.

**Dimension $d = 1$.** Let us first examine the case $d = 1$. This corresponds to a situation where even at zero temperature, $L = \infty$, the phase always is symmetric and the mass is given by equation (4.11). Using the gap equation for zero temperature with the same cut-off, we write the finite temperature gap equation

\[
\int_0^\infty ds s^{-1/2} e^{-m_n^2 L^2 s/4\pi} \left[ \vartheta_0(s) - s^{-1/2} \right] + \int_{s_0}^{s_1} \frac{ds}{s} e^{-z^2 s/4\pi} = 0,
\]

with $s_1 = s_0 m^2/m_L^2$. It follows

\[
\ln(m_L/m) = \frac{1}{2} \int_0^\infty ds s^{-1/2} e^{-m_n^2 L^2 s/4\pi} \left[ \vartheta_0(s) - s^{-1/2} \right].
\]

High temperature corresponds to $m \ll 1/L$ and thus we also expect to $m_L \gg m$. The integral diverges only for $m_L L \rightarrow 0$, and it is then dominated by the contribution of the zero-mode,

\[
\frac{1}{4\pi} \int_0^\infty ds s^{-1/2} e^{-z^2 s/4\pi} \left[ \vartheta_0(s) - s^{-1/2} \right] = \frac{1}{2z} + \frac{1}{2\pi} (\ln z + \gamma - \ln(4\pi)) + O(z),
\]
and therefore
\[ L m_L = -\pi / \ln(m_L/m) \sim -\pi / \ln(mL). \]  
(4.14)

As we will see later, the logarithmic decrease at high temperature of the product \( L m_L \) corresponds to the UV asymptotic freedom of the classical non-linear \( \sigma \) model in two dimensions.

**Dimensions \( d > 1 \).** In higher dimensions the system can be in either phase at zero temperature depending on the value of the coupling constant \( t \). Introducing the critical coupling constant \( t_c \) we can then rewrite the gap equation
\[ G\Lambda(Lm_L) = L^{d-1} \left( \frac{1}{Nt} - \frac{1}{Nt_c} \right), \]  
(4.15)
\[ G\Lambda(z) \equiv \frac{1}{4\pi} \int_{s_0}^{\infty} ds \ s^{-d/2} \left[ e^{-z^2s/4\pi} \vartheta_0(s) - s^{-1/2} \right]. \]  
(4.16)

**Dimension \( d = 2 \).** At finite temperature the phase is always symmetric because no phase transition is possible in two dimensions. The function \( G\Lambda \) has a limit for large cut-off \( G_\infty \) and the gap equation thus has a scaling form for \( d = 2 \) as predicted by finite size RG arguments. For \( t > t_c \) (and \( t - t_c \) small) the r.h.s. involves the product of the mass \( m \) at zero temperature (equation (4.10)) by \( L \)
\[ L m = 4\pi G_\infty(Lm_L). \]

with
\[ G_\infty(z) = -\frac{1}{2\pi} \ln(2 \sinh(z/2)), \]  
(4.17)

For \( t = t_c \), a situation relevant to high temperature QFT, we find the equation \( G_\infty(z) = 0 \). The solution is
\[ m_L L = 2 \ln((1 + \sqrt{5})/2), \]

and therefore the mass of the zero-mode is proportional to the mass of the other modes.

The zero-mode instead dominates if \( L m_L \) is small and this can arise only in the situation \( t < t_c \), i.e. when the symmetry is broken at zero temperature. We then have to examine the behaviour of \( G_\infty(z) \) for \( z \) small. From the explicit expression (4.17) we obtain
\[ G_\infty(z) \xrightarrow{z \to 0} -\frac{1}{2\pi} \ln z + O(z^2) \quad \Rightarrow \quad L m_L = \exp \left[ -\frac{2\pi L}{N} \left( \frac{1}{t} - \frac{1}{t_c} \right) \right]. \]  
(4.18)

Dimensional reduction makes sense only for \( L m_L \) small. On the other hand the physical scale in the broken phase is \( m \propto 1/t - 1/t_c \). Therefore \( L m_L \) is small only for \( L m \) large, i.e. at low but non-zero temperature, a somewhat surprising
situation, and a precursor of the zero temperature phase transition. Another possibility corresponds to \( t < t_c \) fixed and thus a physical scale of order \( \Lambda \): this is the situation of chiral perturbation theory, and corresponds to the deep IR (perturbative) region where only Goldstone particles propagate. Then \( Lm_L \) is small even at high temperature. Note that the mass \( m_L \) has, when the coupling constant \( t \) goes to zero or \( L \to \infty \), the exponential behaviour characteristic of the dimension two.

For \( t < t_c \) the equation can also be written

\[
Lm_L = e^{-2\pi L(\langle \phi \rangle)^2/N},
\]

where \( \langle \phi \rangle \) is the field expectation value in the normalization of the \( \phi^4 \) field theory.

**Dimension \( d = 3 \).** For \( d = 3 \) the situation is different because a phase transition is possible in a three-dimensional classical theory. This is consistent with the existence of the quantity \( G_\infty(0) > 0 \) which appears in the relation between coupling constant and temperature at the critical point \( m_L = 0 \):

\[
\frac{1}{t} - \frac{1}{t_c} = \frac{N}{12L^2}.
\] (4.19)

For a coupling constant \( t \) which corresponds to a phase of broken symmetry at zero temperature \( (t < t_c) \), one now finds a transition temperature

\[
T_c = L_c^{-1} \sim (12/N)^{1/2} |\langle \phi \rangle|,
\]

a result consistent with equation (3.11).

Studying more generally the saddle point equations one can derive all other properties of this system. Another limit of interest is the high temperature QFT. For \( z \neq 0 \) the coefficient of \( z^2 \) in expression (4.15) has a cut-off dependence

\[
G_\Lambda(z) = \frac{1}{12} - \frac{z}{4\pi} + \frac{z^2}{16\pi} [-2 \ln(\Lambda) - \gamma + 2 \ln(4\pi)] + O(z^3).
\]

At \( t = t_c \) we find that \( (m_L L)^2 \) is of order \( 1/\ln(\Lambda L) \). Thus at leading order

\[
(m_L L)^2 = \frac{2\pi^2}{3 \ln(\Lambda L)},
\]

in agreement with equation (3.10).

**Dimension \( d = 4 \).** From \( G_\infty(0) = \zeta(3)/4\pi^2 \) and the simple relation

\[
\frac{\partial G_\Lambda(d = 4, z)}{\partial z^2} = -\frac{1}{4\pi} G_\infty(d = 2, z) - \frac{L\Lambda}{16\pi^{5/2}},
\]

we find:
(i) The critical temperature $T_c$ for $d = 4$

$$T_c = L_c^{-1} \sim (2\pi)^{2/3} (N\zeta(3))^{-1/3} |\langle \phi \rangle|^{2/3},$$

again consistent with equation (3.6).

(ii) In the massless limit $G(z) = 0$ leads to

$$L^2 m_L^2 \sim \pi^{-1/2} \zeta(3)/L\Lambda,$$

in agreement with the behaviour found in section 3.1.

The $((\phi)^2)^2$ field theory at large $N$. To compare with the situation in the $\phi^4$ theory of section 3.2 it is interesting to also write the corresponding gap equation for $d = 4$ in the large $N$ limit. One finds

$$L^2 m_L^2 = \frac{Ng}{6\Lambda L} G_\Lambda(Lm_L).$$

For $Lm_L$ small one expands

$$L^2 m_L^2 = \frac{Ng}{6\Lambda L} \left[ \zeta(3) + \frac{L\Lambda}{4\pi^{3/2}} m_L^2 L^2 + \frac{1}{8\pi^2} m_L^2 L^2 \ln(Lm_L) + O(m_L^2 L^2) \right].$$

At leading order one finds

$$L^2 m_L^2 = \frac{Ng}{6\pi^2} g_L, \quad g_L = \frac{1}{\Lambda L} \frac{g}{1 + Ng/(34\pi^{5/2})}.$$

This behaviour is consistent with the behaviour found in section 3.1 and the behaviour (4.20).

5 The non-linear $\sigma$ model: Dimensional reduction

We want now to derive the reduced effective action for the non-linear $\sigma$ model. Because the space of fields is a sphere, a simple mode expansion destroys the geometry of the model. Several strategies then are available. We explore here two of them, and mention a third one.

A first possibility, which we will not discuss because we can do better, is based on solving the constraint $S^2(x) = 1$ by parametrizing the field $S(x)$

$$S(x) = \{\sigma(x), \pi(x)\},$$

and eliminating locally the field $\sigma(x)$ by:

$$\sigma(x) = (1 - \pi^2(x))^{1/2}.$$
One then performs a mode expansion on $\pi(x)$, and integrates perturbatively over the non-zero-modes. This mode expansion somewhat butchers the geometry and this is the main source of complications. Otherwise, provided one uses dimensional regularization (or lattice regularization, but the calculations are much more difficult) to deal with the functional measure, this strategy is possible.

However one can find other methods in which the geometric properties are obvious. One convenient method involves parametrizing the zero-mode in terms of a time-dependent rotation matrix which rotates the field zero-mode to a standard direction in the spirit of section 36.5 of [7]. Here instead we describe a method based on the introduction of an auxiliary field. This will allow us to use a more physical cut-off regularization of Pauli–Villars type.

5.1 Linearized formalism. Renormalization group

We again start from the action in the form (4.2).

$$S(S, \lambda) = \frac{\Lambda^{d-1}}{2t} \int d^{d+1}x \left[ (\partial_\mu S(x))^2 + \lambda(x) (S^2(x) - 1) \right],$$  \hspace{1cm} (5.1)

but we have rescaled the coupling constant in such a way that $t$ now is dimensionless. The correlation functions of the $S$ field satisfy RG equations

$$\left[ \Lambda \frac{\partial}{\partial \Lambda} + \beta(t) \frac{\partial}{\partial t} - \frac{n}{2} \zeta(t) \right] \Gamma^{(n)}(p_i; t, L, \Lambda) = 0,$$ \hspace{1cm} (5.2)

with

$$\beta(t) = (d - 1)t + O(t^2).$$

The solution can be written

$$\Gamma^{(n)}(p_i; t, L, \Lambda) = m^{d+1}(t) M_0^{-n}(t) F^{(n)}(p_i/m(t), Lm(t)).$$ \hspace{1cm} (5.3)

with

$$M_0(t) = \exp \left[ -\frac{1}{2} \int_0^t \frac{\zeta(t')}{\beta(t')} dt' \right],$$ \hspace{1cm} (5.4)

$$m(t) = \frac{1}{\xi(t)} = \Lambda t^{-1/(d-1)} \exp \left[ -\int_0^t \left( \frac{1}{\beta(t')} - \frac{1}{(d-1)t'} \right) dt' \right].$$ \hspace{1cm} (5.5)

The RG functions are related to properties of the zero temperature theory. The function $m(t)$ has the nature of a physical mass. In the broken phase it is a crossover scale between the large momentum critical behaviour and the small momentum perturbative behaviour. The function $M_0(t)$ is proportional to the field expectation value.
For $d = 1$

$$\beta(t) = -\frac{N - 2}{2\pi} t^2 + O(t^3), \quad \zeta(t) = \frac{N - 1}{2\pi} t + O(t^2),$$  (5.6)

and the definition (5.3) has to be modified

$$m(t) \propto \Lambda \exp \left[ - \int^t \frac{dt'}{\beta(t')} \right] \Rightarrow \ln(m/\Lambda) = -\frac{2\pi}{(N - 2)t} + O(\ln t).$$  (5.7)

Another way to express the solution of RG equations at finite temperature is to introduce the coupling $t_L$ at scale $L$

$$\ln(\Lambda L) = \int_{t_L}^t \frac{dt'}{\beta(t')};$$  (5.8)

where $t_L$ is a function of $t$ and $L$ only through the combination $Lm(t)$

$$\ln Lm(t) = -\int_{t_L}^{t_L} \frac{dt'}{\beta(t')}.$$  (5.9)

For $d > 1$ and $t < t_c$ fixed, the equation (5.8) implies that $t_L$ approaches the IR fixed point $t = 0$ at fixed temperature

$$t_L \sim 1/(Lm(t))^{d-1}.$$  (5.10)

In the mass scale $m(t)$ which is of order $\Lambda$, this is a low temperature regime, where finite temperature effects can be calculated from perturbation theory and renormalization group.

At $t_c$, and more generally in the critical domain, techniques based on an $\varepsilon = d - 1$ expansion can be used. Since $t_c$ is a RG fixed point $t_L(t_c) = t_c$.

Finally in two dimensions ($d = 1$) we see from equation (5.9) that $t_L$ goes to zero for $Lm(t)$ small, i.e. at high temperature, because $t = 0$ then is a UV fixed point,

$$t_L \sim \frac{2\pi}{(N - 2) \ln(m(t)L)},$$  (5.11)

and this is the limit in which the two-dimensional perturbation theory is useful.

5.2 Dimensional reduction

We expand the fields in eigenmodes in the time dimension, and keep the tree and one loop contributions. We call $\phi, \rho$ the zero momentum modes and $S_L(\phi, \rho)$ the reduced $d$ dimensional action. At leading order we find

$$S_L(\phi, \rho) = LS(\phi, \rho).$$  (5.12)
The one-loop contribution now is
\[ \delta S_L = \frac{1}{2} N \text{tr} \ln(-\Delta + \rho) + \frac{1}{2} \text{tr} \ln \left[ \varphi(-\Delta + \rho)^{-1} \varphi \right] \]
\[ = \frac{1}{2} (N - 1) \text{tr} \ln(-\Delta + \rho) + \frac{1}{2} \text{tr} \ln \left[ \varphi(-\Delta + \rho)^{-1} \varphi(-\Delta + \rho) \right]. \quad (5.13) \]

The form of the last term may surprise, until one remembers that the perturbative expansion is performed around a non-vanishing value of \( \varphi \).

We use the identity, obtained after some commutations,
\[ \varphi(-\Delta + \rho)^{-1} \varphi(-\Delta + \rho) = \varphi \cdot \varphi + \varphi(-\Delta + \rho)^{-1} [(\Delta \varphi) + 2 \partial_\mu \varphi \partial_\mu]. \]

At this order \( \varphi \cdot \varphi = 1 \) and we expect that \( \rho \) can be neglected because it yields an interaction of higher dimension, which is therefore negligible in the long distance limit. If we then expand \( \text{tr} \ln \) we see that the first term yields a term with two derivatives and higher orders yield additional derivatives which also are sub-leading in the long distance limit. The first term yields
\[ \text{tr} \varphi(-\Delta + \rho)^{-1} [(\Delta \varphi) + 2 \partial_\mu \varphi \partial_\mu] \sim \text{tr}(\partial_\mu \varphi)^2 (-\Delta)^{-1}, \]
where the relations
\[ \varphi \cdot \partial_\mu \varphi = 0, \quad (\partial_\mu \varphi)^2 + \varphi \cdot \Delta \varphi = 0, \]
valid at leading order, have been used.

In the same way we expand the first term in (5.13) in powers of the field \( \rho \). At leading order only one term is relevant, and we thus obtain
\[ \delta S_L = \frac{1}{2} G_2 \int d^d x \left[ (\partial_\mu \varphi(x))^2 + (N - 1) \rho(x) \right], \]
where the constant \( G_2 \) defined in (2.16) has to taken at \( r = 0 \). One finds (appendix A1.2)
\[ G_2 = \frac{1}{2 \pi^{(d+1)/2} \Gamma((d-1)/2)} \zeta(d-1) L^{2-d} - \frac{2 \Lambda^{d-2}}{(d-2)(4\pi)^{d/2}} + \frac{2 L \Lambda^{d-1}}{(d-1)(4\pi)^{(d+1)/2)}, \quad (5.14) \]

We conclude that at one-loop order
\[ S_L(\varphi, \rho) = \frac{L \Lambda^{d-1}}{2t} \int d^d x \left[ (Z_\varphi/Z_t)(\partial_\mu \varphi(x))^2 + \rho(x) (\varphi^2(x) - Z_\varphi^{-1}) \right], \]
with
\[
Z_t = 1 + (N - 2)\Lambda^{1-d}L^{-1}G_2t + O(t^2)
\]
\[
Z_\varphi = 1 + (N - 1)\Lambda^{1-d}L^{-1}G_2t + O(t^2).
\]

**Dimension** $d = 2$ [22,23]. For $d = 2$ the constant $G_2$ in equation (5.14) has a UV contribution which is three dimensional of order $\Lambda$, and a two-dimensional contribution of order $\ln(\Lambda L)$, corresponding to the omitted zero-mode
\[
G_2 = \frac{1}{2\pi} (\ln(\Lambda L) - \frac{1}{2} \gamma) + \frac{2}{(4\pi)^{3/2}} \Lambda L.
\]

The term proportional to $\Lambda L$ generates a finite renormalization of $t$
\[
t_\tau = t + 2(4\pi)^{-3/2}(N - 2)t^2,
\]
and of the field $\varphi$
\[
\varphi = \left[1 - (4\pi)^{-3/2}(N - 1)t\right] \varphi_\tau.
\]

We now introduce the effective coupling constant $g$
\[
g_L = t_\tau / (\Lambda L).
\]

The effective action becomes
\[
S_L(\varphi_\tau, \rho_\tau) = \frac{1}{2g_L} \int d^2x \left[ (\tilde{Z}_\varphi/Z_g) (\partial_\mu \varphi_\tau(x))^2 + \rho_\tau(x) \left( \varphi_\tau^2(x) - \tilde{Z}_\varphi^{-1} \right) \right].
\]

We verify that the remaining factors $Z_g, \tilde{Z}_\varphi$ render the reduced theory one-loop finite
\[
Z_g = 1 - \frac{N - 2}{2\pi} (\ln(\Lambda L) - \frac{1}{2} \gamma) g_L + O(g_L^2)
\]
\[
\tilde{Z}_\varphi = 1 - \frac{N - 1}{2\pi} (\ln(\Lambda L) - \frac{1}{2} \gamma) g_L + O(g_L^2).
\]

The solution of the two-dimensional non-linear $\sigma$ model then requires non-perturbative techniques, but the two-dimensional RG tells us
\[
\ln(m_L L) \propto - \frac{2\pi}{(N - 2)g_L} = - \frac{2\pi \Lambda}{(N - 2)t} = - \frac{2\pi}{N - 2} Lm(t),
\]
where the last equation involves the three-dimensional RG. The result is consistent with equation (4.18).

**Dimension** $d = 1$. Then
\[
G_2 = \frac{L}{2\pi} \left[ \frac{1}{2} \gamma - \ln(\pi) + \ln(\Lambda L) \right].
\]
The reduced one-dimensional theory is of course finite. Therefore $Z_\varphi$ and $Z_t$ are the renormalization factors which are associated with the change from the scale $\Lambda$ to the temperature scale $1/L$. We set

$$\varphi_t = Z_\varphi^{1/2} \varphi = [1 + (N - 1)G_2 t/2L] \varphi \quad (5.15a)$$

$$\frac{1}{g} = \frac{1}{tZ_t} = \frac{1}{t} - (N - 2)G_2/L + O(t). \quad (5.15b)$$

Both quantities $Z_\varphi$ and $g$ satisfy the RG equations of the zero temperature field theory

$$\Lambda \frac{\partial Z_\varphi}{\partial \Lambda} + \beta(t) \frac{\partial Z_\varphi}{\partial t} - \zeta(t)Z_\varphi = 0, \quad \Lambda \frac{\partial g}{\partial \Lambda} + \beta(t) \frac{\partial g}{\partial t} = 0,$$

where the RG functions at this order are given in (5.6).

The one-dimensional non-linear $\sigma$ model again cannot be solved by perturbation theory, but since it corresponds to a simple angular momentum squared hamiltonian, it can be solved exactly. The difference between the energies of the ground state energy and first excited state is

$$m_L = \frac{1}{2} \left( N - 1 \right) g/L.$$

Expressing $t$ in terms of the mass scale (5.7), which is proportional to the physical mass, we obtain

$$\frac{1}{m_L L} \sim - \frac{1}{\pi} \frac{N - 2}{N - 1} \ln(mL), \quad (5.16)$$

a result consistent with equation (4.14). The result reflects the UV asymptotic freedom of the non-linear $\sigma$ model in two dimensions; the effective coupling constant decreases at high temperature where $mL \to 0$.

### 5.3 Matching conditions

If the explicit form of the reduced theory can be guessed, another strategy is available, based on matching conditions. The idea is to calculate some physical observables in $d + 1$ dimensions and to expand them for high temperature thus $L$ small. One then calculates the same quantities in the guessed reduced theory in $d$ dimensions. Identifying the two set of results, one obtains the relations between the parameters of the initial and reduced action [23,14,24]. One advantage of the method is the possibility to check the ansatz of dimensional reduction by calculating more quantities than needed, and requiring consistency. In addition one has a better control of the correspondence for what concerns large momentum effects. The main drawback is that one is often led to calculate detailed expressions, here the two-point correlation function in an external field, where the most part is not useful (related to IR properties). Contributions of the zero-mode have to be separated for each diagram.
To guess the reduced theory the main guiding principles are power counting and symmetries, as usual for effective low energy field theories.

In what follows dimensional regularization is used to avoid the functional measure problem: in the absence of a Lagrange multiplier, a Pauli–Villars cut-off does not regularize the $O(N)$ invariant measure. The more “physical” lattice regularization is also available, but explicit calculations are more difficult.

Finally, in the dimensions of interest, it is necessary to add an explicit symmetry breaking linear in the field, to avoid IR divergences. Once the correspondence between the parameters of the finite temperature and the reduced theories has been determined one can take the symmetric limit.

We consider the finite temperature $d + 1$ dimensional theory

\[
S(S) = \frac{\Lambda^{d-1} Z_S}{2t Z_t} \int d^{d+1}x (\partial_\mu S(x))^2 - \frac{\Lambda^{d-1}}{t} \int d^{d+1}x \mathbf{h} \cdot S(x),
\]

where the \text{MS} scheme is used to define renormalization constants, and

\[
S^2(x) = Z_S^{-1}.
\]

Therefore $t$ is the effective coupling constant at scale $\Lambda$.

RG equations in an external field $\mathbf{h}$ take the form

\[
\left[ \Lambda \frac{\partial}{\partial \Lambda} + \beta(t) \frac{\partial}{\partial t} - \frac{n}{2} \zeta(t) + \rho(t) \mathbf{h} \cdot \frac{\partial}{\partial \mathbf{h}} \right] \Gamma^{(n)}(p_i; t, \mathbf{h}, L, \Lambda) = 0,
\]

where the new RG function is not independent:

\[
\rho(t) = 1 - d + \frac{1}{2} \zeta(t) + \beta(t)/t.
\]

**Dimensional reduction.** We compare it with the zero temperature $d$ dimensional theory

\[
S(\varphi) = \frac{L^{2-d} Z_\varphi}{2g Z_g} \int d^d x (\partial_\mu \varphi(x))^2 - \frac{L^{2-d}}{g} \int d^d x \mathbf{h} \cdot \varphi(x),
\]

where the \text{MS} scheme is used to define renormalization constants, and

\[
\varphi^2(x) = Z_\varphi^{-1}.
\]

The coupling constant $g$ instead is the effective coupling at the temperature scale $L^{-1}$. 
We expect that between the two fields $\mathbf{S}$ and $\varphi$ some renormalization will be required.

The one-loop diagrams are listed in figure 1. In the reduced model, at one-loop order the two-point function is

$$
\Gamma^{(2)}_d(p) = \frac{L^{2-d}}{g} \left( p^2 Z_\varphi / Z_g + h Z_\varphi^{1/2} \right) + \left[ p^2 + \frac{1}{2} (N - 1) h \right] I(h) + O(g),
$$

where $h = |h|$ and

$$
I(h) = \frac{1}{(2\pi)^d} \int \frac{d^d p}{p^2 + h}.
$$

At finite temperature, in the $d + 1$ theory, one finds instead

$$
\Gamma^{(2)}_{d+1}(p_0 = 0, p) = \frac{\Lambda^{d-1} L}{t} \left( p^2 Z_\mathbf{S} / Z_t + h Z_\mathbf{S}^{1/2} \right) + \left[ p^2 + \frac{1}{2} (N - 1) h \right] G_2(h, L) + I(h) + O(t),
$$

where the contribution $I(h)$ of the zero-mode has been separated explicitly and the function $G_2(r = h, L)$ is defined in (2.11). In the limit $h = 0$

$$
G_2(0, L) = N_d \frac{\pi}{\sin(\pi d/2)} (2\pi)^{d-2} \zeta(2 - d) L^{2-d}.
$$

where $N_d$ is the usual loop factor

$$
N_d = \frac{2}{(4\pi)^{d/2} \Gamma(d/2)} = \frac{1}{2\pi} + O(d - 2).
$$

Dimension $d = 2$. For $d \to 2$ the renormalization constants at one-loop in the $\overline{\text{MS}}$ scheme are

$$
Z_g = 1 + (N - 2) \frac{N_d}{d - 2} g, \quad Z_\varphi = 1 + (N - 1) \frac{N_d}{d - 2} g.
$$
In particular
\[ Z/Z_g = 1 + \frac{N_d}{d - 2} g_t. \]

Therefore the renormalized \(d\)-dimensional two-point function reads
\[ \Gamma^{(2)}_d(p) = \frac{1}{g}(p^2 + h) + \left[p^2 + \frac{1}{2}(N - 1)h\right] I_r(h), \]
with
\[ I_r(h) = \lim_{d \to 2} I(h) + \frac{N_d}{d - 2} L^{2-d} = \frac{-1}{4\pi} \ln(hL^2). \] (5.28)

In the finite temperature theory no renormalization is required because the theory is non-renormalizable, and dimensional regularization cancels all power divergences. Thus
\[ \Gamma^{(2)}_{d+1}(p) = \frac{\Lambda L}{t} \left(p^2 + h\right) + \left[p^2 + \frac{1}{2}(N - 1)h\right] \left(G_2(h, L) + I(h)\right) + O(t), \] (5.29)
with
\[ G_2(0, L) = \frac{N_d}{d - 2} - \frac{1}{2\pi} \ln L + O(d - 2). \]

We note that at this order no field renormalization is required to compare the two functions and then
\[ \frac{1}{g} = \frac{\Lambda L}{t} + O(t). \]

Dimension \(d = 1\). In \(d = 1\) dimension the reduced theory has no divergences and the one-loop expression reads
\[ \Gamma^{(2)}_d(p) = \frac{L}{g}(p^2 + h) + \left[p^2 + \frac{1}{2}(N - 1)h\right] I(h) + O(g). \]

We compare this expression with the finite temperature two-point function, calculated in the \(\overline{\text{MS}}\) scheme (with renormalization scale \(\Lambda\)). For this purpose we have to subtract to expression (5.25) the \(\overline{\text{MS}}\) counterterm. For \(d \to 1\) we find
\[ [G_2]_t(0, L) = \frac{L}{2\pi} \left(\ln(\Lambda L) + \gamma - \ln(4\pi)\right), \] (5.30)
and therefore
\[ \Gamma^{(2)}_{d+1}(p) = \frac{L}{t} \left(p^2 + h\right) + \left[p^2 + \frac{1}{2}(N - 1)h\right] \left[I(h) + \frac{L}{2\pi} \left(\ln(\Lambda L) + \gamma - \ln(4\pi)\right)\right] + O(t). \] (5.31)
This time we have also to take into account the field renormalization. We set
\[ \varphi(x) = S(x) \sqrt{Z_{\varphi}}, \quad Z_{S\varphi} = 1 + (N - 1) (\ln(\Lambda L) + \gamma - \ln(4\pi)) \frac{t}{2\pi}, \]
and
\[ \frac{1}{t} = \frac{1}{gZ_{tg}}, \quad Z_{gt} = 1 - (\ln(\Lambda L) + \gamma - \ln(4\pi)) \frac{t}{2\pi}, \]
or inverting the relation
\[ \frac{1}{g} = \frac{1}{t} - \frac{(N - 2)}{2\pi} (\ln(\Lambda L) + \gamma - \ln(4\pi)) + O(t), \]
a result which can also be obtained by the method of section 36.5 of [7]. The results for \( Z_{gt} \) and \( g \) are consistent with the equations (5.15).

6 The Gross–Neveu in the large \( N \) expansion

To gain some intuition about the role of fermions at finite temperature we now examine a simple model of self-interacting fermions, the Gross–Neveu (GN) model. The GN model is described in terms of a \( U(N) \) symmetric action for a set of \( N \) massless Dirac fermions \( \{\psi^i, \bar{\psi}^\dagger_i\} \):
\[
S(\bar{\psi}, \psi) = - \int d^{d+1}x \left[ \bar{\psi} \cdot \partial \psi + \frac{1}{2} G \left( \bar{\psi} \cdot \psi \right)^2 \right]. \tag{6.1}
\]
The GN model has in all dimensions a discrete symmetry
\[
x = \{x_1, x_2, \ldots, x_d\} \mapsto \tilde{x} = \{-x_1, x_2, \ldots, x_d\}, \quad \left\{ \begin{array}{l}
\psi(x) \mapsto \gamma_1 \psi(\tilde{x}), \\
\bar{\psi}(x) \mapsto -\bar{\psi}(\tilde{x}) \gamma_1,
\end{array} \right.
\]
which prevents the addition of a mass term. In even dimensions it implies a discrete chiral symmetry, and in odd dimensions it corresponds to space reflection. Below, to simplify, we will speak about chiral symmetry, irrespective of dimensions.

The GN model is renormalizable in \( d = 1 \) dimension, where it is asymptotically free and the chiral symmetry is always broken at zero temperature.

We recall that within the \( 1/N \) expansion it can be proven that the GN model is equivalent to the GNY (Y for Yukawa) model, a model with the same symmetry, but with an elementary scalar particle coupled to fermions through a Yukawa-like interaction, which is renormalizable in four dimensions [24]. This equivalence provides a simple interpretation to some of the results that follow.

Since fermions at finite temperature have no zero modes, limited insight about the physics of the model can be gained from perturbation theory; all fermions are simply integrated out. Therefore we study here the GN model within the framework of the \( 1/N \) expansion.
6.1 The GN model at zero temperature, in the large $N$ limit

We first recall the properties of the GN model at zero temperature \cite{24,25} in the large $N$ limit. To generate the large $N$ expansion one introduces an auxiliary field $\sigma$, replaces the action (6.1) by an equivalent action:

$$S((\bar{\psi}, \psi, \sigma)) = \int d^{d+1}x \left[ -\bar{\psi} \cdot (\partial + \sigma) \psi + \frac{1}{2G} \sigma^2 \right],$$

and integrates over $N - 1$ fermions. One finds ($\psi \equiv \psi_1$)

$$S_N(\bar{\psi}, \psi, \sigma) = \int d^{d+1}x \left[ -\bar{\psi}(\partial + \sigma)\psi + \frac{1}{2G} \sigma^2 \right] - (N - 1) \text{tr} \ln (\partial + \sigma). \quad (6.2)$$

For $G = O(1/N)$ the corresponding partition function can be calculated by the steepest descent method. The saddle point (or gap) equation obtained by differentiating with respect to $\sigma$ has the trivial solution $\sigma = 0$ and (at leading order for $N \to \infty$ we can replace $N - 1$ by $N$)

$$\frac{1}{G} = \frac{N'}{(2\pi)^{d+1}} \int_{\Lambda} \frac{d^{d+1}k}{k^2 + \sigma^2}, \quad (6.3)$$

where $N' = N \text{tr} 1$ is the total number of fermions.

Note that at leading order for $N$ large the scalar field expectation value $\sigma = \langle \sigma \rangle$ is also the fermion mass $m_\psi = \langle \sigma \rangle$.

For $d = 1$ the chiral symmetry is spontaneously broken for all $G > 0$, and one finds

$$\sigma = m_\psi \propto \Lambda e^{-\pi/N}.$$

For $d > 1$ a phase transition occurs at a value $G_c$ such that

$$\frac{1}{G_c} = \frac{N'}{(2\pi)^{d+1}} \int_{\Lambda} \frac{d^{d+1}k}{k^2}.$$

For $G < G_c$ the saddle point is $\sigma = 0$ and the chiral symmetry is preserved. For $G > G_c$ the chiral symmetry is broken, and for $d < 3$

$$\sigma \propto (G - G_c)^{1/(d-1)},$$

which implies that the physical region corresponds to $|G - G_c|$ small.

For $d = 3$ logarithmic corrections appear and one finds instead

$$\sigma^2 \ln(\Lambda/\sigma) \sim \frac{8\pi^2}{N'} \left( \frac{1}{G_c} - \frac{1}{G} \right),$$

a reflection of the IR triviality of the effective renormalizable GNY model.
In higher dimensions the model is equivalent to a weakly interacting GNY model, with an IR stable gaussian fixed point and

\[ \sigma \propto (G - G_c)^{1/2}. \]

In the broken phase the \( \sigma \)-propagator is given by

\[ \Delta^{-1}_\sigma(p) = \frac{N'}{2(2\pi)^{d+1}} \left( p^2 + 4\sigma^2 \right) \int^A \frac{d^{d+1}k}{(k^2 + \sigma^2) \left[ (p + k)^2 + \sigma^2 \right]}, \quad (6.4) \]

where the saddle point equation has been used. The mass of the scalar field \( m_\sigma = 2 \langle \sigma \rangle \), is such at leading order the \( \sigma \) particle is a fermion bound state at threshold \( (m_\sigma = 2m_\psi) \).

In the chiral symmetric phase \( G < G_c \) instead one finds

\[ \Delta^{-1}_\sigma(p) = \frac{1}{G} - \frac{1}{G_c} + \frac{N'}{2(2\pi)^{d+1}} p^2 \int^A \frac{d^{d+1}k}{k^2 (p + k)^2}, \quad (6.5) \]

a reflection of the property that the \( \sigma \) particle now is a resonance which can decay into a fermion pair.

6.2 The GN model at finite temperature

Due to the anti-periodic boundary conditions fermions have no zero modes, and at high temperature can be integrated out, yielding an effective action for the periodic scalar field \( \sigma \). In the situations in which the \( \sigma \) mass is small compared with the temperature, one can perform a mode expansion of the \( \sigma \) field, integrate over the non-zero modes and obtain a local \( d \)-dimensional action for the zero-mode. It is important to realize that, since the reduced action is local and symmetric in \( \sigma \mapsto -\sigma \), it describes the physics of the Ising transition with short range interactions (unlike what happens at zero temperature). The question which then arises is the possibility of a breaking of this remaining symmetry of Ising type. If a transition exists and is continuous, the \( \sigma \)-mass vanishes at the transition and a potential IR problem appears.

Additional effects due to the addition of a chemical potential will not be considered in these notes [27].

After integration over all fermions we obtain a non-local action \( S_N \) for the field \( \sigma \),

\[ S_N(\sigma) = \frac{1}{2G} \int_0^L d\tau \int d^d x \, \sigma^2 - N \text{tr} \ln (\delta + \sigma), \quad (6.6) \]

where \( L \) is the inverse temperature \( T = 1/L \), and the \( \sigma \) field satisfies periodic boundary conditions in the time direction. As we have seen a non-trivial perturbation theory is obtained by expanding for large \( N \).
The gap equation. The gap equation at finite temperature again splits into two equations \( \sigma = 0 \) and

\[
\frac{L}{G} = N'G_2(\sigma, L), \tag{6.7}
\]

\[
G_2(\sigma, L) = \frac{1}{(2\pi)^d} \sum_n \int_\Lambda \frac{d^dk}{k^2 + \omega_n^2 + \sigma^2}, \quad \omega_n = (2n + 1)\pi/L.
\]

Using Schwinger’s representation, and the corresponding regularization, the function \( G_2 \) can be expressed in terms of another function \( \vartheta_1(s) \), of elliptic type (equation (A2.15)),

\[
G_2(\sigma, L) = \frac{L^{2-d}}{4\pi} \int_{s_0}^{s} \frac{ds}{s^{d/2}} e^{-s L^2 \sigma^2 / (4\pi)} \vartheta_1(s), \tag{6.8}
\]

with \( s_0 = 4\pi/(AL)^2 \). From equation (A2.16) we learn that \( \vartheta_1(s) = 1/\sqrt{s} \) for \( s \to 0 \), up to exponentially small corrections. The function \( G_2(\sigma, L) \) has a regular small \( \sigma \) expansion. The two first terms are

\[
G_2(\sigma, L) = G_2(0, L) + \sigma^2 G_4 + O(\sigma^4),
\]

with for \( d < 5 \) (equations (A1.6,A1.5))

\[
G_2(0, L) = \frac{4L^{2-d}}{(4\pi)^{(d+1)/2}}(1 - 2^{d-2})\Gamma((d-1)/2)\zeta(d-1) + \frac{1}{d-1} \frac{2LA^{d-1}}{(4\pi)^{(d+1)/2}}, \tag{6.9}
\]

\[
G_4 = L^{4-d} \frac{\Gamma(2-d/2)}{8\pi^{d/2}} (2^{4-d} - 1) \zeta(4-d) + \frac{1}{d-3} \frac{2LA^{d-3}}{(4\pi)^{(d+1)/2}}. \tag{6.10}
\]

Finally the propagator \( \Delta_\sigma(p) \equiv \Delta_\sigma(\omega = 0, p) \) of the \( \sigma \) zero-mode, in the broken phase, is given by (after use of the gap equation (6.7))

\[
\Delta_\sigma^{-1}(p) = \frac{N'}{2(2\pi)^dL} (p^2 + 4\sigma^2) \sum_n \int_\Lambda \frac{d^dk}{(k^2 + \omega_n^2 + \sigma^2) [(p + k)^2 + \omega_n^2 + \sigma^2]}.
\]

We again find that the \( \sigma \)-mass is \( 2\sigma \). In the symmetric phase instead the propagator of the zero mode is given by

\[
\frac{1}{N'\Delta_\sigma(p)} = \frac{1}{N'G} - \frac{G_2(0, L)}{L} + \frac{p^2}{2(2\pi)^dL} \sum_n \int_\Lambda \frac{d^dk}{(k^2 + \omega_n^2)((p + k)^2 + \omega_n^2)}. \tag{6.12}
\]
6.3 Phase structure for $d > 1$

For $d > 1$ we introduce the critical value $G_c$ where $\sigma$ vanishes at zero temperature ($L = \infty$),

$$[G_2]_r(\sigma, L) = G_2 - \frac{L}{N'G_c} = \frac{L^{2-d}}{4\pi} \int_{s_0}^L \frac{ds}{s^{d/2}} \left[ e^{-sL^2\sigma^2/(4\pi)} \vartheta_1(s) - s^{-1/2} \right], \quad (6.13)$$

The function $[G_2]_r(\sigma, L)$ is a decreasing function of $\sigma$, thus

$$[G_2]_r(\sigma, L) \leq [G_2]_r(0, L) = L^{2-d} \mathcal{I}_2(d), \quad (6.14)$$

with

$$\mathcal{I}_2(d) = \frac{4}{(4\pi)^{(d+1)/2}} \left( 1 - 2^{d-2} \right) \Gamma((d-1)/2) \zeta(d-1). \quad (6.15)$$

The integral is always negative and therefore the gap equation (6.7) has a solution only for $G > G_c$, i.e. when at zero temperature chiral symmetry is broken. For $d < 3$ the integral (6.13) converges at $s = 0$ and the gap equation (6.7) takes a scaling form. For $d = 2$ it can be expressed in terms of the fermion mass at zero temperature $m_\psi$

$$Lm_\psi = - \int_0^L \frac{ds}{s} \left( e^{-sL^2\sigma^2/(4\pi)} \vartheta_1(s) - s^{-1/2} \right).$$

The phase transition. A phase transition between the two Ising phases takes place at a temperature $T_c = L_c^{-1}$ where $\sigma$ solution to the equation (6.7) vanishes:

$$L_c^{d-1} \left( \frac{1}{G} - \frac{1}{G_c} \right) = N'\mathcal{I}_2(d).$$

Since the r.s.h. of the gap equation (6.7) is a decreasing function of $\sigma$, the $\sigma \to -\sigma$ Ising symmetry is broken for $T < T_c$ and restored for $T > T_c$.

It is interesting to express the critical temperature in terms of the fermion mass $m_\psi$. For $d > 3$ one finds

$$T_c = (L_c)^{-1} \propto m_\psi (\Lambda/m_\psi)^{(d-3)/(d-1)} \gg m_\psi.$$

Therefore the critical temperature is a high temperature in the scale of the particle masses.

For $d = 3$ the critical temperature is given by

$$L_c^2 \left( \frac{1}{G_c} - \frac{1}{G} \right) = \frac{N'}{48}.$$
Therefore
\[ T_c \sim \frac{\sqrt{6}}{\pi} m_\psi \sqrt{\ln(\Lambda/m_\psi)} \propto \sqrt{G - G_c}, \]
which again corresponds to a high temperature regime.

Finally for \( d = 2 \) the critical temperature is proportional to the fermion mass:
\[ T_c = \frac{1}{2 \ln 2} m_\psi. \]

**Local expansion.** When the \( \sigma \) mass or expectation value are small compared to \( 1/L \) we can perform a local expansion of the action (6.6), and study it to all orders in the \( 1/N \) expansion. Consistency requires that one also performs a mode expansion of the field \( \sigma \) and retains only the zero mode. In the reduced theory \( 1/L \) plays the role of a large momentum cut-off.

The first terms of the effective \( d \) dimensional action are
\[ S_d(\sigma) = \int d^d x \left[ \frac{1}{2} Z_\sigma (\partial_\mu \sigma)^2 + \frac{1}{2} r \sigma^2 + \frac{1}{4!} u \sigma^4 \right], \quad (6.16) \]
where terms of order \( \sigma^6 \) and \( \partial^2 \sigma^4 \) and higher have been neglected, and the three parameters are given by
\[ Z_\sigma = \frac{1}{2} N' G_4, \quad r = \frac{L}{G} - N' G_2(0, L), \quad u = 6 N' G_4, \]
where \( G_4 = L^{4-d} \mathcal{I}_4(d) \). For \( d > 1 \) after the shift of the coupling constant one finds
\[ r = \frac{L}{G} - \frac{L}{G_c} - N' L^{2-d} \mathcal{I}_2(d) = N' \mathcal{I}_2(d) L \left( L^{-d} - L^{1-d} \right). \quad (6.17) \]

Though, after rescaling of the field we observe that the effective \( \sigma^4 \) coupling is logarithmically small, close enough to the critical temperature the effective theory cannot be solved by perturbative methods.

For \( d = 2 \) the integrals are UV finite after the shift of \( G \). The effective theory describes the physics of the two-dimensional Ising model.

**Dimensional reduction and \( \sigma \) mass.** For \( d > 3 \), in the symmetric phase \( L < L_c \), the \( \sigma \) mass behaves like
\[ m_\sigma^2 \propto L^{-2} (\Lambda L)^{3-d} \left[ 1 - (L/L_c)^{d-1} \right], \]
and thus is small with respect to \( L \), justifying dimensional reduction. Moreover, after rescaling of the field \( \sigma Z_\sigma^{1/2} \mapsto \sigma \) one sees that the effective \( \sigma^4 \) coupling is of order
\[ u/Z_\sigma^2 \propto L^{d-4} (\Lambda L)^{3-d}. \]
For \( d \geq 4 \) the coupling is small, the physics perturbative, and no additional analysis is required.

For the mathematical case \( 3 < d < 4 \), the situation is more subtle. For dimensional reasons the true expansion parameter is

\[
m_{\sigma}^{d-4} u/Z_{\sigma}^2 \propto (m_{\sigma} L)^{d-4} (\Lambda L)^{3-d}.
\]

At high temperature, i.e. for \( 1 - L/L_c \) positive and finite, \( Lm_{\sigma} \propto (\Lambda L)^{(3-d)/2} \) and one finds

\[
m_{\sigma}^{d-4} u/Z_{\sigma}^2 \propto (\Lambda L)^{(3-d)(d-2)/2},
\]

which is small. On the other hand for \( |T-T_c| \propto L - L_c \) small enough perturbation theory is no longer useful.

For \( d = 3 \) dimensional reduction is justified near the critical temperature, but the reduced model is non-perturbative. Since the coefficient \( G_4 \) has still a logarithmic UV contribution, in the symmetric phase at high temperature one finds

\[
Lm_{\sigma} \propto 1/\sqrt{\ln(\Lambda L)},
\]

which also justifies a local expansion. The effective coupling

\[
\frac{u}{m_{\sigma} Z_{\sigma}^2} \propto \frac{1}{\sqrt{\ln(\Lambda L)}},
\]

and thus the reduced model can be solved using perturbation theory. This is a situation we have already met in the example of the \( \phi^4 \) field theory, and which reflects the IR triviality of the GNY model.

In dimension \( d = 2 \), near the critical temperature the \( \sigma \) mass is small and local and mode expansions can be performed. The reduced theory cannot be solved by perturbation theory.

In general the model obeys simple scaling relations (a reflection of the existence of a non-trivial IR fixed point in the GNY model). At high temperature \( \sigma = 0 \), and the \( \sigma \) mass if of order \( 1/L \). No local expansion is justified and necessary.

### 6.4 Dimension \( d = 1 \)

The situation \( d = 1 \) is doubly special, since at zero temperature chiral symmetry is always broken and at finite temperature the Ising symmetry is never broken. The GN model is renormalizable and UV free

\[
\beta(G) = -\frac{N' - 2}{2\pi} G^2 + O(G^3).
\]

The RG invariant mass scale \( \Lambda(G) \), to which all masses at zero temperature of the rich GN spectrum are proportional, has the form

\[
\Lambda(G) \propto \Lambda \exp \left[ -\int_0^G \frac{dG'}{\beta(G')} \right].
\]
Physical masses are small with respect to $\Lambda$ when $G$ is small

$$\ln(\Lambda/m_\psi) = \frac{2\pi}{(N'-2)G} + O(\ln G).$$

At finite temperature all masses, in the sense of inverse of the correlation length in the space direction, have a scaling property. For example the $\sigma$ mass has the form

$$Lm_\sigma = f(Lm_\psi).$$

One can also express the scaling properties by introducing a temperature dependent coupling constant $G_L$ defined by

$$\int_G^{G_L} \frac{dG'}{\beta(G')} = -\ln(L\Lambda).$$

At high temperature $G_L$ decreases

$$G_L \sim \frac{2\pi}{(N'-2)\ln(Lm_\psi)}.$$

We therefore expect a trivial high temperature physics with weakly interacting fermions.

At finite temperature the large $N$ gap equation becomes

$$\frac{2\pi}{N'G} = \ln(\Lambda L) + \frac{1}{2} \int_0^{\infty} \frac{ds}{s^{1/2}} \left[ e^{-L^2\sigma^2 s/(4\pi)} \vartheta_1(s) - s^{-1/2} \theta(1-s) \right],$$

which, in terms of the zero-temperature mass scale

$$m_\psi = \Lambda e^{-2\pi/N'G},$$

can be rewritten:

$$\ln(m_\psi L) = F(\sigma L),$$

$$F(z) = -\frac{1}{2} \int_0^{\infty} \frac{ds}{s^{1/2}} \left[ e^{-z^2 s/(4\pi)} \vartheta_1(s) - s^{-1/2} \theta(1-s) \right].$$

The function $F(z)$ is an increasing function, with a finite limit for $z \to 0$ and which behaves like $\ln z$ for $z \to \infty$, in such a way that at low temperature one recovers $m_\sigma = 2m_\psi$. Again we find a phase transition at a temperature $L_c^{-1} \propto m_\psi$. This mean-field-like prediction of a phase transition contradicts the well-known property of the absence of phase transitions in an Ising-type model in $d = 1$ dimension. In the high temperature phase $\sigma = 0$ and one finds

$$\frac{1}{N'\Delta_\sigma(p)} = \frac{1}{2\pi} \left[ -\ln(Lm_\psi) - \gamma + \frac{1}{2} \ln \pi \right] + \sum_{n \geq 0} \frac{p^2}{L \omega_n (p^2 + 4\omega_n^2)}.$$
from which one derives a scaling relation

\[ L m_\sigma = f(L m_\psi). \]

If we assume that dimensional reduction is justified we can return to the action (6.16). At leading order we find a simple model in quantum mechanics: the quartic anharmonic oscillator. Straightforward considerations show that the correlation length, inverse of the \( \sigma \) mass parameter, becomes small only when the coefficient of \( \sigma^2 \) is large and negative. This happens only at low temperature, where the two lowest eigenvalues of the hamiltonian corresponding to space direction are almost degenerate, a precursor of the zero temperature phase transition. One then finds

\[ L m_\sigma \propto (\ln m_\psi L)^{5/4} e^{-\text{const.}(\ln m_\psi L)^{3/2}}. \]

7 Abelian gauge theories

We first discuss the abelian case which is much simpler, because the mode decomposition is consistent with the gauge structure. Some additional problems arising in non-abelian gauge theories will be considered in next section. Because the gauge field has a number of components which depends on the number of space dimensions, the mode expansion have some new properties and affects gauge transformations. The simplest non-trivial example of a gauge theory is QED, a theory which is IR free in four dimensions, and therefore from the RG point of view has properties similar to the scalar \( \phi^4 \) field theory. Another example is provided by the abelian Higgs model but since it has a first order phase transition, it has a more limited validity.

7.1 Mode expansion and gauge transformations

We decompose a general gauge field \( A_\mu(t, x) \) into the sum of a zero mode \( B_\mu(x) \) and the sum of all non-zero modes \( Q_\mu(t, x) \)

\[ A_\mu(t, x) = B_\mu(x) + Q_\mu(t, x). \]

At finite temperature \( T = L^{-1} \), \( Q_\mu(t, x) \) thus satisfies

\[ \int_0^L dt Q_\mu(t, x) = 0. \tag{7.1} \]

With this decomposition gauge transformations

\[ \delta A_\mu(t, x) = \partial_\mu \varphi(t, x), \]

become

\[ \delta B_\mu(x) = \partial_\mu \varphi_0(t, x), \quad \delta Q_\mu(t, x) = \partial_\mu \varphi_1(t, x), \quad \varphi = \varphi_0 + \varphi_1. \tag{7.2} \]
Since $\delta B_\mu$ does not depend on $t$ we conclude that $\varphi_0(t, x)$ must have the special form
\[ \varphi_0(t, x) = F(x) + \Omega t, \]  
where $\Omega$ is a constant. The space components $B_i$ transform as the components of a $d$-dimensional gauge field; the time component $B_0$ is a $d$-dimensional scalar field which is translated by a constant
\[ \delta B_0(x) = \Omega. \]

Symmetry which respect to the translation (7.4) implies that the scalar field $B_0$ is massless.

The condition (7.1) then implies
\[ \partial_i \int_0^L dt \varphi_1(t, x) = 0, \quad \int_0^L dt \partial_i \varphi_1(t, x) = 0 \Rightarrow \varphi_1(0, x) = \varphi_1(L, x). \]

The transformations of the gauge field $Q_\mu$ are thus specified by periodic functions $\varphi_1(t, x)$ with a constant zero-mode, which can be set to zero.

Finally we verify that the function $\varphi = \varphi_1 + \varphi_0$ is such that $\partial_\mu \varphi$ is periodic as it should, since $A_\mu$ is periodic.

*Matter fields.* We now couple the gauge field to matter, for instance charged fermions $\psi(t, x), \bar{\psi}(t, x)$. At finite temperature fermion fields satisfy anti-periodic boundary conditions. To the gauge transformation (7.2) corresponds for the fermions
\[ \psi(t, x) = e^{i\varphi(t, x)} \psi'(t, x), \quad \bar{\psi}(t, x) = e^{-i\varphi(t, x)} \bar{\psi}'(t, x). \]

Anti-periodicity implies that
\[ \varphi(L, x) = \varphi(0, x) \pmod{2\pi}. \]

Since $\varphi_1$ is periodic, the condition implies for the constant $\Omega$ in (7.3)
\[ \Omega = 2n\pi/L. \]  

This restriction on the transformation (7.4) of the scalar component $B_0$ has important consequences. As a result of quantum corrections generated by the interactions with charged matter, the scalar field $B_0$ does not remain massless. Instead the thermodynamic potential is a periodic function of $B_0$ with period $2\pi/L$. 
7.2 Gauge field coupled to fermions: quantization

We now consider a gauge field coupled to an $N$-component massless charged fermion

$$S(\bar{\psi}, \psi, A_\mu) = \int d^{d+1}x \left[ \frac{1}{4e^2} F_{\mu\nu}^2(x) - \bar{\psi}(x) \cdot (\partial + iA) \psi(x) \right]. \quad (7.6)$$

The theory has RG properties which bear some similarities with the $\phi^4$ theory; it is renormalizable for $d = 3$ and IR free (trivial). It can be solved in the large $N$ limit. Finally in dimension $d = 1$ it reduces to the massless Schwinger model which can be solved exactly even at finite temperature \cite{28}, because bosonization methods still work.

The temporal gauge. To calculate the partition function we first quantize in the temporal gauge $A_0(x,t) = 0$ because the corresponding hamiltonian formalism is simple. The action becomes

$$S(\bar{\psi}, \psi, A_\mu) = \int d^dxdt \left[ \frac{1}{4e^2} \left( 2A_0^2 + F_{ij}^2(t,x) \right) - \bar{\psi}(t,x) \cdot (\partial + iA) \psi(t,x) \right]. \quad (7.7)$$

In calculating $\text{tr} e^{-LH}$ we have to remember that Gauss’s law has still to be imposed. This means that the trace has to be taken in the subspace of wave functionals invariant under space-dependent gauge transformations. To project onto this subspace we impose periodic conditions in the time direction up to a gauge transformation:

$$A_i(L, x) = A_i(0, x) - L \partial_i \varphi(x),$$
$$\psi(L, x) = e^{iL\varphi(x)} \psi(0, x),$$

and integrate over the gauge transformation $\varphi(x)$. We then set

$$A_i(t, x) = A_i'(t, x) - t \partial_i \varphi(x),$$
$$\psi(t, x) = e^{it\varphi(x)} \psi'(t, x).$$

where the fields $A_i', \psi', \bar{\psi}'$ now are periodic and anti-periodic resp.. This induces two modifications in the action

$$\int dxdt (\partial_t A_i)^2 \mapsto \int dxdt (\partial_t A_i)^2 + L \int dx (\partial_i \varphi(x))^2$$
$$\int dxdt \bar{\psi}(t, x) \gamma_0 \partial_t \psi(t, x) \mapsto \int dxdt \bar{\psi}(t, x) \gamma_0 (\partial_t + i\varphi(x)) \psi(t, x).$$

Therefore $\varphi(x)$ is simply the residual zero mode of the $A_0$ component

$$\varphi(x) \equiv B_0(x).$$
Its presence is a direct consequence of enforcing Gauss’s law.

The field theory has a $d$-dimensional gauge invariance with the zero mode $B_i(x)$ of $A_i(t, x)$ as gauge field. In addition it contains $d$ families of neutral vector fields with masses $2\pi n/L$, $n \neq 0$, quantized in a unitary, and thus non-renormalizable gauge.

Note that from the technical point of view the usual difficulties which appear in perturbation calculations with the temporal gauge (the gauge field propagator is singular) reduce to the need for quantizing the remaining zero-mode, and to the non explicit renormalizability. The latter problem can be solved with the help of dimensional regularization for example (for gauge invariant observables). An alternative possibility of course is to introduce a renormalizable gauge. The change of gauges can be performed by the standard zero-temperature method [7].

**Covariant gauge.** To change to the covariant gauge we introduce a time component $A_0$ for the gauge field (periodic in time) and multiply the functional measure by the corresponding $\delta$-function

$$1 = \int [dQ_0(t, x)] \prod_{t, x} \delta(Q_0).$$

The action can then be written in a gauge invariant form. We then introduce a second identity in the functional integral

$$1 = \det(-\partial^2) \int [d\varphi_1] \delta(\partial_\mu Q_\mu + \partial^2 \varphi_1 - h(t, x)), \quad (7.8)$$

where $\varphi_1$ and $h(t, x)$ are two periodic functions without zero-mode. We perform the gauge transformation

$$Q_\mu + \partial_\mu \varphi_1 \mapsto Q_\mu.$$ 

The $\varphi_1$ dependence remains only in $\delta(Q_0 - \partial_t \varphi_1)$, and the integration over $\varphi_1$ yields a constant. Integration over $h(t, x)$ with a gaussian weight yields the standard covariant gauge

$$S_{\text{gauge}} = \frac{L}{2\xi} \int dx (\partial_i B_i(x))^2 + \frac{1}{2\xi} \int dt dx (\partial_\mu Q_\mu)^2 \equiv \frac{1}{2\xi} \int dt dx (\partial_\mu A_\mu)^2.$$ 

since

$$\partial_\mu A_\mu = \partial_t Q_0(t, x) + \partial_i B_i(x) + \partial_i Q_i(t, x).$$ 

We conclude that the gauge fixing term is just obtained by substituting the mode decomposition into the gauge fixing term of the zero-temperature action. From the point of view of the $B$ gauge field this corresponds to a quantization in the covariant gauge in $d$ dimensions.

Note that the transformation from the temporal gauge to the covariant gauge generates a determinant (equation (7.8)) which is field-independent, but contributes to the free energy.
7.3 Dimensional reduction

At finite temperature, to generate the effective action for the gauge field zero-modes, we have to integrate over all fermion modes (anti-periodic boundary conditions) and over the non-zero modes $Q_\mu(t,x)$ of the gauge field. At leading order one finds a free theory containing a gauge field $B_i$ and a massless scalar $B_0$. At one-loop order only fermion modes contribute. Replacing the gauge field $A_\mu$ by its zero mode $B_\mu$ and performing the fermion integration explicitly we find the effective action

$$S_L = L \int d^d x \left[ \frac{1}{2e^2} (\partial_i B_0)^2 + \frac{1}{4e^2} F_{ij}^2(B) \right] - N \text{tr} \ln(\partial + iB),$$  \hspace{1cm} (7.9)

where latin indices mean space indices.

An important issue is the behaviour of induced mass of the time component $B_0 = \varphi$ of the gauge field. We thus first calculate the effective potential for constant $\varphi$.

The effective potential. The effective potential for constant field $\varphi$ is then given by

$$V(\varphi) = -\frac{1}{2} N' \sum_n \frac{1}{(2\pi)^d} \int d^d k \ln \left[ k^2 + (\varphi + (2n + 1)\pi/L)^2 \right],$$

where $N' = N \text{tr} 1$ is the total number of fermion degrees of freedom.

Its evaluation involves the new function

$$\sum_n e^{-t[(2n+1)\pi/L+\varphi]^2} \equiv \vartheta_2(4\pi t/L^2; \nu, 0) = \frac{L}{2\sqrt{\pi t}} \sum_n (-1)^n \cos(nL\varphi) e^{-n^2 L^2/4t},$$

where $\nu = \frac{1}{2} + L\varphi/2\pi$, and the general Poisson’s formula \ref{A2.13} has been used. Then

$$V(\varphi) = \frac{1}{2} N' \frac{1}{(4\pi)^{d/2}} \int \frac{dt}{t^{1+d/2}} \left[ \vartheta_2(4\pi t/L^2; \nu, 0) - \vartheta_2(4\pi t/L^2; 1/2, 0) \right],$$  \hspace{1cm} (7.10)

where $V(\varphi)$ has been shifted so as to vanish for $\varphi = 0$. After the usual change of variables $t = L^2 s/4\pi$ we find

$$V(\varphi) = \frac{1}{2} N' L^{-d} \int_{s_0}^{\infty} \frac{ds}{s^{1+d/2}} \left[ \vartheta_2(s; \nu, 0) - \vartheta_2(s; 1/2, 0) \right],$$  \hspace{1cm} (7.11)

We note

$$\vartheta_2(s; \nu, 0) - \vartheta_2(s; 1/2, 0) = s^{-1/2} \sum_n (-1)^{n+1} [1 - \cos(nL\varphi)] e^{-n^2 \pi^2/s},$$
which shows that the $n = 0$ term cancels and thus the potential is UV finite. This is not too surprising since in the zero temperature, large $L$, limit no gauge field mass or quartic potential are generated.

The potential has an extremum at $\varphi = 0$. The coefficient of $\varphi^2$ is:

$$V(\varphi) = \frac{1}{2} L^{2-d} \varphi^2 K_2(d) + O(\varphi^4)$$

$$K_2(d) = N' \frac{8}{(4\pi)^{d/2+1/2}} \Gamma((d+1)/2) (2^{d-2} - 1) \zeta(d-1).$$

The constant $K_2(d)$ is positive showing that $\varphi = 0$ is a minimum. More generally the potential is

$$V(\varphi) = K(d) L^{-d} \mathcal{V}(L \varphi), \quad K(d) = N' \frac{1}{\pi^{d/2+1/2}} \Gamma((d+1)/2),$$

where the function $\mathcal{V}(z)$ is given by the sum

$$\mathcal{V}(z) = \sum_{n=1}^\infty (-1)^{n+1} \frac{(1 - \cos(nz))}{n^{d+1}} = \frac{1 - \cos z}{2\Gamma(d+1)} \int_0^\infty d\tau \frac{\tau^d \tanh(\tau/2)}{\cosh \tau + \cos z}.$$

Its derivative has the sign of $\sin z$, it is negative for $-\pi < z < 0$ and positive for $0 < z < \pi$. Therefore $z = 0$ is the unique minimum in the interval $-\pi < z < \pi$.

A special case is $d = 1$ for which one finds

$$\mathcal{V}'(z) = \frac{1}{2} z \quad \text{for } |z| < \pi \quad \text{and thus } \mathcal{V}(z) = \frac{1}{4} z^2.$$
generated for instance by a Pauli–Villars regulator field, a fermion of large mass \( \Lambda \),
\[
\frac{1}{3(4\pi)^{(d+1)/2}} \Gamma((3 - d)/2)(\Lambda L)^{d-3}.
\]
These contributions generate in dimensions \( d \geq 3 \) a renormalization \( e \mapsto e_r \) of the coupling constant.

Discussion. We thus obtain a mass term which is proportional to \( e_r L^{(1-d)/2} \). If \( e \) is generic, i.e. of order 1 at the microscopic scale \( 1/\Lambda \), then \( e \propto \Lambda^{(3-d)/2} \) and the scalar mass is proportional to \( (\Lambda L)^{(3-d)/2}/L \). It is thus large with respect to the vector masses for \( d < 3 \) and small for \( d > 3 \). We conclude that for \( d < 3 \) the scalar field can be completely integrated out, but for \( d > 3 \) it survives. Note that for \( d > 3 \) additional corrections are even smaller because UV divergences cannot compensate the small coupling constant. For \( d = 3 \) QED is IR free,
\[
\beta_e = \frac{N}{6\pi^2} e^4 + O(e^6),
\]
e\(_r\) has to be replaced by the effective coupling constant \( e(1/\Lambda L) \) with is logarithmically small
\[
e^2(1/\Lambda L) \sim \frac{6\pi^2}{N \ln(\Lambda L)},
\]
and the scalar mass thus is still small
\[
m_\varphi^2 \propto \frac{1}{L^2 \ln(\Lambda L)}.
\]
The separation between zero and non-zero modes remains justified. The situation is completely analogous to high temperature \( \phi^4 \) field theory, and perturbation for the same reason remains applicable.

Finally if one is interested in IR physics only, one can in a second step integrate over the massive scalar field \( \varphi \).

For more details and more systematic QED calculations see \([29,30,31]\).

7.4 The abelian Higgs model

The abelian gauge fields interacting with charged scalar fields has also been investigated \([32,33,30]\) as a toy model to study properties of the electro-weak phase transition at finite temperature. The gauge action reads
\[
S(A_\mu, \phi) = \int dt d^d x \left[ \frac{1}{4e^2} F_{\mu\nu}^2 + |D_\mu \phi|^2 + U(|\phi|^2) \right],
\]
where the quartic potential \( U(|\phi|^2) \)
\[
U(z) = rz + \frac{1}{6} gz^2,
\]
is such that the $U(1)$ symmetry is broken at zero temperature.

The model can directly be quantized in the unitary (non-renormalizable) gauge and calculations of gauge-independent observables can be performed with dimensional regularization. Below we use instead the temporal gauge because the unitary gauge becomes singular near the phase transition.

One limitation of the model is that RG shows that in $3 + 1$ dimensions the hypothesis of second order phase transition is inconsistent, and therefore the transition is most likely first order. Indeed, in a more general model with $N$ charged scalars for $d = 3$ the RG $\beta$-functions are

\[ \beta_g = \frac{1}{24\pi^2} [(N + 4)g^2 - 18ge^2 + 54e^4] \]
\[ \beta_{e^2} = \frac{1}{24\pi^2} N e^4. \]

The origin $e^2 = g = 0$ is a stable IR fixed point only for $N \geq 183$. For $N$ small, the continuum model remains meaningful if initially the coupling constants are small enough, in such a way that by the time the running coupling constants reach the physical scale, they have not yet reached the region of instability. The transition then is weak first order.

Presumably the same result applies for small values of $N$ to the three-dimensional classical statistical field theory, which is also the Landau–Ginzburg model of superconductivity.

Neglecting all non-zero modes we obtain one massive vector field degenerated in mass with a scalar field, and the Higgs field. We expect the degeneracy between vector and scalar masses to be lifted by the integration over non-zero modes. To construct the reduced action we quantize in the temporal gauge $A_0 = B_0$ and we set

\[ A_i = B_i + Q_i, \quad \phi = \varphi + \chi. \]

The reduced action at leading order is simply

\[ S_L(B_{\mu}, \varphi) = L \int d^d x \left[ \frac{1}{2e^2} (\partial_\mu B_0)^2 + \frac{1}{4e^2} F_{ij}^2(B) + |D_i \varphi|^2 + |\varphi|^2 B_0^2 + U(|\varphi|^2) \right], \]

where the covariant derivative now refers to the gauge field $B_i$

\[ D_i = \partial_i + i B_i. \]

At one-loop order we need the terms quadratic in $Q_{\mu}, \chi$. In the gaussian integration over $Q_{\mu}, \chi$ at high temperature the leading effects come from the shift in the masses. We can therefore take $B_0$ and $\varphi$ constant. The quadratic terms in the action relevant for the $\varphi$ mass shift are

\[ S_2 = \int dt d^d x \left[ \frac{1}{2e^2} (\partial_i Q_i)^2 + \frac{1}{4e^2} F_{ij}^2(Q) + |\partial_i \chi|^2 + |\partial_i \chi + i Q_i \varphi|^2 + \frac{2}{3} g |\varphi|^2 |\chi|^2 \right], \]
where we have omitted the term proportional to \( r|\chi|^2 \), a high temperature approximation. The integration over \( \chi \) yields a term

\[
e^2 |\varphi|^2 \sum_{\omega} \int d^d k Q_i(\omega, k) \left( \delta_{ij} - \frac{k_i k_j}{k^2 + \omega^2} \right) Q_j(-\omega, -k),
\]

and thus finally a contribution \( \delta r \) to the coefficient of \( |\varphi|^2 \)

\[
\delta r = G_2(0, L)(de^2 + 2g/3)/L,
\]

where \( G_2 \) is defined by equation (2.16). For \( d = 3 \) we have found \( G_2 = 1/12L \) (section 3.2) and therefore

\[
\delta r = \frac{1}{36L^2}(9e^2 + 2g).
\]

For the contribution to the \( B_0 \) mass the relevant quadratic action is

\[
S_2 = \int dt d^d x |D_\mu \chi|^2.
\]

In the limit of constant \( B_\mu \), the space components \( B_i \) can be eliminated by a gauge transformation. The remaining \( B_0 \) component cannot be eliminated because \( \chi \) satisfies periodic boundary conditions in the time direction. Instead, as we have already discussed, the mode integration generates a periodic potential for \( B_0 \)

\[
\int d^d x \sum_{\omega} \int d^d k \ln \left[ \left( \omega + B_0(x) \right)^2 + k^2 \right].
\]

Expanding to order \( B_0^2 \) we obtain the mass term

\[
(d-1)G_2 \int d^d x B_0^2(x),
\]

where the identity, valid in dimensional regularization,

\[
\frac{1}{(2\pi)^d} \sum_{\omega} \int \frac{d^d k \omega^2}{(k^2 + \omega^2)^2} = (1 - d/2)G_2,
\]

has been used. Thus for \( d = 3 \)

\[
m_{B_0}^2 = \frac{e^2}{3L^2}.
\]

As we have discussed several times, in three dimensions additional UV contributions transform the parameters \( e^2, g, r \) into the one-loop expansion of the running
parameters at scale $1/\Lambda L$. Finally, for completeness, let us point out that the coefficient of $|\varphi|^2 B_0^2$ gets renormalized. At one-loop one finds

$$1 \mapsto 1 + \frac{e^2 + g}{12\pi^2}.$$ \hspace{1cm} (7.13)

Discussion. We assume that at zero temperature the $U(1)$ symmetry is broken which implies that the coefficient $r$ in the potential $U$ (equation (7.13)) is sufficiently negative. When the temperature increases, the coefficient of $|\varphi|^2$ increases until a critical temperature is reached where the $U(1)$ symmetry is restored. Near the transition the scalar field $B_0$ is massive and therefore the effective theory relevant for the phase transition is simply the classical $U(1)$ gauge model

$$\tilde{S}_L(B_i, \varphi) = \int d^d x \left[ \frac{1}{4\tilde{e}^2} F_{ij}^2 + |D_i \varphi|^2 + \tilde{U}(|\varphi|^2) \right], \hspace{1cm} (7.15)$$

where the parameters $\tilde{e}^2$ and in $\tilde{U}$ can be obtained by integrating the reduced action also over the heavy field $B_0$.

8 Non-abelian gauge theories

Pure non-abelian gauge theories, as well as non-abelian gauge coupled to a small number of fermions, are UV asymptotically free in four dimensions. From the RG point of view, we expect some similarities with the non-linear $\sigma$ model (in different dimensions). In particular in $d = 3$ dimensions the effective coupling constant $g(L)$ decreases at high temperature, $g(L) \propto 1/\ln(mL)$, where $m$ is the RG invariant mass scale of the gauge theory.

Notation. We consider fields $A_\mu$ written as matrices in the adjoint representation of some compact group $G$. Gauge transformations take the form

$$A'_\mu(x) = g(x)A_\mu(x)g^{-1}(x) + g(x)\partial_\mu g^{-1}(x), \hspace{1cm} (8.1)$$

where $g$ is a group element.

Covariant derivatives $D_\mu$ acting on fields $\varphi$ belonging the adjoint representation are

$$D_\mu \varphi = \partial_\mu \varphi + [A_\mu, \varphi]. \hspace{1cm} (8.2)$$

The corresponding curvature $F_{\mu\nu}(x)$ tensor is

$$F_{\mu\nu}(x) = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \hspace{1cm} (8.3)$$

and the pure gauge action

$$S(A) = -\frac{1}{4g^2} \text{tr} \int d^{d+1}x \mathbf{F}^2_{\mu\nu}. \hspace{1cm} (8.4)$$
Temporal gauge. The situation is here more complicated because the mode decomposition is not gauge invariant. Thus we first quantize, choosing the temporal gauge. Then the space components $A_i$ must be periodic up a gauge transformation which again enforces Gauss’s law:

$$A_i(L, x) = g(x)A_i(0, x)g^{-1}(x) + g(x)\partial_i g^{-1}(x),$$

We parametrize the group element $g$ in terms of an element $\varphi$ of the Lie algebra:

$$g(x) = e^{L\varphi(x)},$$

and introduce

$$g(t, x) = e^{t\varphi(x)},$$

After the gauge transformation

$$A_i(t, x) = g(t, x)A_i'(t, x)g^{-1}(t, x) + g(t, x)\partial_i g^{-1}(t, x),$$

the new field $A_i'$ becomes strictly periodic. The gauge component $A_0$ which vanishes before the transformation becomes

$$0 = g(t, x)A_0'(t, x)g^{-1}(t, x) + g(t, x)\partial_i g^{-1}(t, x),$$

and therefore

$$A_0'(t, x) = \varphi(x).$$

Again the temporal gauge reduces the time-component $A_0$ to its zero-mode, the field $\varphi(x)$.

Let us express the pure gauge action (8.4) in terms of the new fields

$$S(A, \varphi) = -\frac{1}{2g^2} \text{tr} \int dt \int d^d x \left( \partial_i \varphi(x) - \partial_i A_i + [A_i, \varphi] \right)^2 - \frac{1}{4g^2} \text{tr} \int dt \int d^d x \mathbf{F}^2_{ij}$$

$$= -\frac{1}{2g^2} \text{tr} \int dt \int d^d x \left( D_i \varphi - \partial_i A_i \right)^2 - \frac{1}{4g^2} \text{tr} \int dt \int d^d x \mathbf{F}^2_{ij}. $$

Dimensional reduction. We now separate the zero-modes of the space components of the gauge field

$$A_i(t, x) = B_i(x) + Q_i(t, x),$$

with

$$Q_i(t, x) = \sum_{n \neq 0} e^{2i\pi nt/L} Q_{n,i}(x).$$

Then

$$D_i(A)\varphi = D_i(B)\varphi + [Q_i, \varphi].$$
In the same way

\[ F_{ij}(A) = F_{ij}(B) + D_i Q_j - D_j Q_i + [Q_i, Q_j], \]

where the covariant derivative now refers to the gauge field \( B \).

We see that the resulting action is gauge invariant with respect to time-independent gauge transformations with gauge field \( B \). The gauge field is coupled to a massless scalar \( \varphi \) and massive vector fields with masses \( 4\pi^2 n^2 / L^2 \), all belonging to the adjoint representation.

The problem of quantization then reduces to the quantization of the field \( B \). If we choose a covariant gauge then, not only is the theory quantized, but the vector fields do not have the kind of singular propagator typical of the temporal gauge. A question remains, massive vector fields lead to non-renormalizable theories. A way to solve this problem is to go over to a covariant gauge. We introduce a time component \( A_0 \) for the gauge field (periodic in time) and multiply the functional measure by the corresponding \( \delta \)-function. The action is a function only of the sum \( A_0(t, x) + \varphi(x) \). Let us thus temporarily call \( \tilde{A}_\mu \) the field

\[ \tilde{A}_i = A_i, \quad \tilde{A}_0 = A_0(t, x) + \varphi(x). \]

The \( \delta \)-function then becomes \( \delta(\tilde{A}_0 - \varphi) \). We then perform the standard manipulations to pass to the covariant gauge with gauge function \( \partial_\mu \tilde{A}_\mu \). Eventually we shall get the gauge average of the constraint \( \delta(\tilde{A}_0 - \varphi) \). This gives a determinant which only depends on \( \varphi \), while \( \varphi \) appears nowhere else. The integral over \( \varphi \) factorizes and gives a constant factor. Of course in the process we have introduced ghost fields which satisfy periodic boundary conditions, unlike ordinary fermions.

**One-loop calculation of the effective \( \varphi \) potential.** We expect that quantum corrections generated by the integration over non-zero modes give a mass to the scalar field \( \varphi \), as in the abelian example.

For \( \varphi \) constant, and omitting the massless gauge field we find a simplified action. It is convenient here to introduce the generators of the Lie algebra of the compact group \( G \), in the form of hermitian matrices \( \tau^\alpha \)

\[ \text{tr} \tau^\alpha \tau^\beta = \delta_{\alpha\beta}, \quad [\tau^\alpha, \tau^\beta] = if_{\alpha\beta\gamma} \tau^\gamma, \]

where the structure constants \( f_{\alpha\beta\gamma} \) are chosen antisymmetric. We then set

\[ Q_i = iQ_i^\alpha \tau^\alpha, \quad \varphi = i\varphi^\alpha \tau^\alpha. \]

Then the relevant \( Q \) action is

\[ S_2(Q) = \frac{1}{2g^2} \int dt d^d x \left[ \frac{1}{2} \left( \partial_t Q_j^\alpha - \partial_j Q_i^\alpha \right)^2 + \left( \partial_t Q_i^\alpha + f_{\alpha\beta\gamma} Q_i^\beta \varphi^\gamma \right)^2 \right]. \]
The integration yields a determinant which generates an additive contribution to the effective action

\[ V(\varphi) = \frac{1}{2} \sum_{n \neq 0} \text{tr} \ln \left[ \left( k^2 \delta_{ij} - k_i k_j + \omega_n^2 \delta_{ij} \right) \delta_{\alpha\beta} + \delta_{ij} \left( 2i \omega_n f_{\alpha\beta\gamma} \varphi^\gamma + f_{\alpha\gamma\delta} f_{\beta\epsilon\delta} \varphi^\gamma \varphi^\epsilon \right) \right] - (\varphi = 0), \]

(8.5)

with \( \omega_n = 2\pi n/L \). The result can be written as the sum of two contributions, along \( k \) and transverse. The first contribution is

\[ V_1(\varphi) = \frac{1}{2} \sum_x \sum_{n \neq 0} \text{tr} \ln \left[ \omega_n^2 \delta_{\alpha\beta} + (2i \omega_n f_{\alpha\beta\gamma} \varphi^\gamma + f_{\alpha\gamma\delta} f_{\beta\epsilon\delta} \varphi^\gamma \varphi^\epsilon) \right] - (\varphi = 0) \]

\[ = \sum_x \sum_{n \neq 0} \text{tr} \ln \left( \delta_{\alpha\beta} + i f_{\alpha\beta\gamma} \varphi^\gamma / \omega_n \right). \]

\[ = \ln \det \left[ \prod_x 2\Phi^{-1} L^{-1} \sinh(L\Phi/2) \right], \]

(8.6)

where we have introduced the matrix \( \Phi \) with elements

\[ \Phi^{\alpha\beta} = f_{\alpha\beta\gamma} \varphi^\gamma. \]

This term contributes to the \( \varphi(x) \) integration measure and yields a factor at each point \( x \)

\[ \prod_x d\varphi(x) \det \frac{L\Phi(x)/2}{\sinh(L\Phi(x)/2)}, \]

which cancels the invariant group measure in the \( \varphi \) parametrization (see Appendix A3).

The second term, after division by the space volume, then reads

\[ V_2(\varphi) = \frac{1}{2} (d - 1) \sum_{n \neq 0} \frac{1}{(2\pi)^d} \int d^d k \ln \det \left[ k^2 \delta_{\alpha\beta} + (\omega_n + i\Phi)^2_{\alpha\beta} \right] - (\varphi = 0). \]

Using Schwinger’s parametrization we obtain

\[ V_2(\varphi) = \frac{1}{2} (d - 1) \frac{1}{(4\pi)^{d/2}} \sum_{n \neq 0} \int_0^\infty \frac{dt}{t^{1+d/2}} \text{tr} \left[ e^{-\omega_n^2 t} - e^{(\omega_n + i\Phi)^2 t} \right], \]

and therefore

\[ V_2(\varphi) = -\frac{1}{2} (d - 1) \frac{1}{(4\pi)^{d/2}} \int_0^\infty \frac{dt}{t^{1+d/2}} \text{tr} \left[ \varphi_2(4\pi t/L^2; \nu, 0) - \varphi_2(4\pi t/L^2; 0, 0) - e^{t\Phi^2} + 1 \right], \]

(8.7)
where \( \nu \) now is a matrix \( \nu = iL\Phi/2\pi \). We change variables \( 4\pi t/L^2 = s \) and find

\[
V_2(\varphi) = -\frac{1}{2}(d-1)\Gamma((d+1)/2)\int_0^{\infty} \frac{ds}{s^{3/2+d/2}} \sum_{n=1}^{\infty} e^{-n^2/s} [1 - \cos(nL\varphi)] + \frac{(d-1)\Gamma(-d/2)}{(4\pi)^{d/2}} \varphi^d
\]

The end of the calculation is somewhat similar to the abelian case. We complete the calculation for the \( SU(2) \) group and obtain

\[
V_2(\varphi) = -(d-1)\Gamma((d+1)/2)\int_0^{\infty} \frac{ds}{s^{3/2+d/2}} \sum_{n=1}^{\infty} e^{-n^2/s} [1 - \cos(nL\varphi)] + \frac{(d-1)\Gamma(-d/2)}{(4\pi)^{d/2}} \varphi^d,
\]

where \( \varphi \) now is a three component vector and \( \nu = L\varphi/2\pi \). Using the Poisson formula (A2.16),

\[
\vartheta_2(s; \nu, 0) - \vartheta_2(s; 0, 0) = \frac{1}{\sqrt{s}} \sum_n (e^{2i\pi \nu n} - 1) e^{-n^2/s} = -\frac{1}{\sqrt{s}} \sum_n (1 - \cos(2\pi \nu)) e^{-n^2/s},
\]

we can rewrite the expression

\[
V_2(\varphi) = (d-1)\Gamma((d+1)/2)\int_0^{\infty} \frac{ds}{s^{3/2+d/2}} \sum_{n=1}^{\infty} e^{-n^2/s} [1 - \cos(nL\varphi)] + \frac{(d-1)\Gamma(-d/2)}{(4\pi)^{d/2}} \varphi^d.
\]

where dimensional regularization has been used. The last term, which comes from the zero mode, is there to provide a counter-term to the \( \varphi \) four-point function for \( d = 4 \). The sum can be replaced by another integral

\[
\sum_{n=1}^{\infty} \frac{1 - \cos(nL\varphi)}{n^{d+1}} = \frac{1}{2\Gamma(d+1)} \int_0^{\infty} \frac{ds}{s^{d}} \left( \frac{2}{e^s - 1} - \frac{1}{e^{-s+iL\varphi} - 1} - \frac{1}{e^{-s-iL\varphi} - 1} \right)
\]

\[
= \frac{1 - \cos(L\varphi)}{2\Gamma(d+1)} \int_0^{\infty} \frac{ds}{s^{d}} \left( \frac{1}{\tanh(s/2)} [\cosh s - \cos(L\varphi)] \right).
\]

As in the QED case \( \varphi = 0 \) is the minimum of the potential (which is also periodic in \( \varphi \)), and the generated mass is UV finite

\[
V_2(\varphi) = (d-1)\Gamma((d+1)/2)\zeta(d-1)L^2 \varphi^2 + O(\varphi^4),
\]

because gauge invariance ensures the absence of mass terms for gauge fields.
For $d > 3$ again the mass is small. For $d < 3$ it is large and the scalar field can be integrated out. For $d = 3$ the situation is different from the QED case because the theory is UV asymptotically free. We expect a situation similar to the non-linear $\sigma$ model in two dimensions: the effective coupling constant at high temperature is logarithmically small, $g(L) \propto 1/\ln(mL)$, $m$ being the RG invariant mass scale of the gauge theory (related to the $\beta$-function). Thus we can trust the effective reduced field theory. However the effective theory most likely cannot be solved by perturbative methods.

Remarks. Detailed calculations have been performed for models of physical interest like QCD [29,34] with the problem of the quark-gluon plasma phase and the Higgs sector of the Standard Model with the question of the $SU(2) \times U(1)$ symmetry restoration [37]. In QCD the problem of slow convergence also arises and various summation schemes have been proposed [30].

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APPENDICES
FEYNMAN DIAGRAMS AT FINITE TEMPERATURE

In this appendix we summarize a few definitions and identities useful for general one-loop calculations. But first a short section is devoted to a few reminders on group measures, useful for gauge theories.

A1 One-loop calculations

We give here some technical details about explicit calculations of one-loop diagrams.

A1.1 One-loop diagrams: general remarks

Let us add a few remarks concerning the calculation of Feynman diagrams in finite temperature field theory. In the review we have used techniques which have been developed for the more general finite size problems. It is based on the introduction of Schwinger’s parameters and write the momentum space propagator $\Delta(p)$,

$$\Delta(p) = \frac{1}{p^2 + \mu^2} = \int_0^\infty dt \ e^{-t(p^2 + \mu^2)},$$

a method also used at zero temperature. After this transformation some zero temperature gaussian integrals over momenta are replaced by discrete sum over integers which can no longer be calculated exactly. However, dimensional continuation can still be defined. In the review we have mostly considered simple one-loop diagrams of the form $D_\gamma$ which can be written

$$D_\gamma \equiv \frac{1}{(2\pi)^dL} \sum_n \int \frac{d^dp}{(p^2 + \omega_n^2 + \mu^2)^\sigma}$$

$$= \frac{1}{(2\pi)^d\Gamma(\sigma)} \frac{1}{L} \int_0^\infty dt t^{\sigma-1} \sum_n \int d^d p \ e^{-t(p^2 + \omega_n^2 + \mu^2)},$$

with $\omega_n = 2\pi n/L$.

In terms of the function $\vartheta_0(s)$ defined by (A2.8), the sums can be written

$$D_\gamma = \frac{1}{(4\pi)^{d/2}\Gamma(\sigma)} \frac{1}{L} \int_0^\infty dt t^{\sigma-d/2-1} e^{-t\mu^2} \vartheta_0(4t\pi/L^2)$$

$$= \frac{1}{(4\pi)^{\sigma}\Gamma(\sigma)} L^{2\sigma-d-1} \int_0^\infty ds s^{\sigma-d/2-1} e^{-s\mu^2 L^2/4\pi} \vartheta_0(s).$$

The identity (equation (A2.10))

$$\vartheta_0(s) = (1/s)^{1/2} \vartheta_0(1/s)$$
shows, in particular, that the zero temperature limit is approached exponentially when the mass $\mu$ is finite. Indeed from

$$\vartheta_0(s) - \frac{1}{2\pi} \int d\omega \, e^{-\pi s\omega^2} \sim 2s^{-1/2} e^{-\pi/s}$$

one concludes

$$D_\gamma(L) - D_\gamma(L = \infty) \sim \frac{L^{2\sigma-d-1}}{(4\pi)^{\sigma-1}} e^{-\mu L} \frac{L^{\mu}}{L^{\mu}}.$$

Other analytic techniques. We just mention here the more traditional and more specific techniques also available in finite temperature quantum field theory. The idea is the following: in the mixed $d$-momentum, time representation the propagator is the two-point function $\Delta(t, p)$ of the harmonic oscillator with frequency $\omega(p) = \sqrt{p^2 + m^2}$ and time interval $L$:

$$\frac{1}{p^2 + \omega^2 + m^2} \mapsto \Delta(t, p) = \frac{1}{2\omega(p)} \cosh \left(\frac{(L/2 - |t|)\omega(p)}{2}\right).$$

Summing over all frequencies is equivalent to set $t = 0$. For the simple one-loop diagram one finds

$$\frac{1}{2\pi} \int \frac{d\omega}{\omega^2 + p^2 + m^2} \mapsto \frac{1}{2\omega(p)} \cosh \left(\frac{\omega(p)L/2}{2}\right) \frac{1}{\sinh(\omega(p)L/2)}.$$

This again allows a simple separation into IR and UV contributions:

$$\frac{1}{2\omega(p) \tanh(L\omega(p)/2)} = \frac{1}{2\omega(p)} + \frac{1}{\omega(p)(1 - e^{-L\omega(p)})},$$

where the first term is the zero-temperature result, and the second term, which involves the relativistic Bose statistical factor, decreases exponentially at large momentum.

Finally we calculate the massless propagator, with excluded zero-mode.

The massless propagator. The Fourier representation

$$\Delta(t, x) = \frac{1}{(2\pi)^d L} \sum_{n \neq 0} \int \frac{d^d p e^{-ipx - i\omega_n t}}{p^2 + \omega_n^2}, \quad (A1.1)$$

with $\omega_n = 2\pi n/L$ can be transformed into an infinite sum

$$\Delta(t, x) = \frac{\Gamma((d-1)/2)}{4\pi^{(d+1)/2}} \sum_{n} \left[x^2 + (t + nL)^2\right]^{(1-d)/2} - \frac{\Gamma(d/2 - 1)}{L^d \pi^{d/2}} x^{2-d}. \quad (A1.2)$$
A1.2 Some one-loop calculations

Many one-loop results which have been used can be derived from the simple integral

\[
G_2(r, L) = \frac{L^{2-d}}{4\pi} \int_{s_0}^{\infty} \frac{ds}{s^{d/2}} e^{-rL^2s/4\pi} \left( \vartheta_0(s) - 1 \right),
\]

(A1.3)

with \( s_0 = 4\pi/L^2\Lambda^2 \). We first note that the function for dimension \( d \) is related to the function for the dimension \( d - 2 \).

\[
\frac{\partial G_2(d)}{\partial r} = -G_4(d) = -\frac{L^2}{4\pi} G_2(d - 2).
\]

The expansion of the function for \( r \) small has been several times needed. We first calculate the value at \( r = 0 \)

\[
G_2(0, L) = \frac{L^{2-d}}{4\pi} \int_{s_0}^{\infty} \frac{ds}{s^{d/2}} \left( \vartheta_0(s) - 1 \right).
\]

(A1.4)

A method which can be used quite often in these one-loop calculations is the following: one calculates explicitly the integral in dimensions \( d \) in which it converges at \( s = 0 \) and subtracts the integral on \([0, s_0]\). One then proceeds by analytic continuation

\[
G_2 = \frac{L^{2-d}}{4\pi} \int_0^{\infty} \frac{ds}{s^{d/2}} \left( \vartheta_0(s) - 1 \right) - \frac{L^{2-d}}{4\pi} \int_{s_0}^{\infty} \frac{ds}{s^{d/2}} \left( \vartheta_0(s) - 1 \right)
\]

\[
= \frac{1}{2} \pi^{d/2-2} L^{2-d} \Gamma(1 - d/2) \zeta(2 - d) + \frac{2L\Lambda^{d-1}}{(d-1)(4\pi)^{d+1}/2} - \frac{2\Lambda^{d-2}}{(d-2)(4\pi)^{d/2}}.
\]

where the first integral is given by equation (A2.11) and in the second, for small \( s \), \( \vartheta_0(s) \) has been replaced by \( 1/\sqrt{s} \). Note that from the identity (A2.4) the constant can also be written

\[
\frac{1}{2} \pi^{d/2-2} \Gamma(1 - d/2) \zeta(2 - d) = \frac{1}{2} \pi^{-(d+1)/2} \Gamma((d-1)/2) \zeta(d-1) = N_d \Gamma(d-1) \zeta(d-1).
\]

Additional terms in the small \( r \) expansion can be obtained by Mellin transform

\[
M(\lambda) = \int_0^{\infty} dr \ r^{\lambda-1} G_2(r, L)
\]

\[
= \frac{L^{2-d}}{4\pi} \int_0^{\infty} \frac{ds}{s^{d/2}} \left( \vartheta_0(s) - 1 \right) \int_0^{\infty} dr \ r^{\lambda-1} e^{-rL^2s/4\pi}
\]

\[
= (4\pi)^{\lambda-1} L^{2-d-2\lambda} \Gamma(\lambda) \int_{s_0}^{\infty} \frac{ds}{s^{d/2+\lambda}} \left( \vartheta_0(s) - 1 \right).
\]
The last integral can then be obtained from $G_2(r,0)$ by replacing $d$ by $d + 2\lambda$.

$$M(\lambda)/\Gamma(\lambda) = \frac{2^{2\lambda-1}}{\pi^{(d+1)/2}} \Gamma(\lambda + (d - 1)/2) \zeta(d - 1 + 2\lambda) L^{2-d-2\lambda} + \frac{2LA^{d-1+2\lambda}}{(2\lambda + d - 1)(4\pi)^{(d+1)/2}} - \frac{2A^{d-2+2\lambda}}{(2\lambda + d - 2)(4\pi)^{d/2}}.$$

The residues of the pole in $\lambda$ of the Mellin transform yield the terms of the small $r$ expansion

$$Ar^\beta \mapsto \frac{A}{(\lambda + \beta)}.$$  

Also

$$Ar^\beta \ln r \mapsto -\frac{A}{(\lambda + \beta)^2}.$$  

**Fermions.** In the case of the GN model we need two integrals. With the help of the relations (A2.1, A2.17) one finds

$$G_4 = \frac{L^{4-d}}{(4\pi)^2} \int_{s_0} ds \, s^{1-d/2} \vartheta_1(s)$$

$$= L^{4-d} \Gamma(2-d/2) \frac{8\pi^{4-d/2}}{(4\pi)^{(d+1)/2}} (2^{4-d} - 1) \zeta(4 - d) + \frac{1}{d - 3} \frac{2LA^{d-3}}{(4\pi)^{(d+1)/2}}.$$  

The second integral is

$$G_2(0, L) = \frac{L^{2-d}}{4\pi} \int_{s_0} ds \, s^{-d/2} \vartheta_1(s) = \frac{4L^{2-d}}{\Gamma((d - 1)/2)(4\pi)^{(d+1)/2}} \Gamma((d - 1)/2) \zeta(d - 1) + \frac{1}{d - 1} \frac{2LA^{d-1}}{(4\pi)^{(d+1)/2}}.$$  

### A2 \(\Gamma, \psi, \zeta, \theta\)-functions: a few useful identities

**\(\Gamma, \psi, \zeta\)-functions.** Two useful identities on the $\Gamma$ function are

$$\sqrt{\pi} \Gamma(2z) = 2^{2z-1} \Gamma(z) \Gamma(z + 1/2), \quad \Gamma(z) \Gamma(1 - z) \sin(\pi z) = \pi.$$  

They translate into relations for the $\psi(z)$ function, $\psi(z) = \Gamma'(z)/\Gamma(z)$

$$2\psi(2z) = 2 \ln 2 + \psi(z) + \psi(z + 1/2), \quad \psi(z) - \psi(1 - z) + \pi / \tan(\pi z) = 0.$$  

We also need Riemann’s $\zeta$ function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$
It is useful, for what follows, to remember the reflection formula

$$\zeta(s)\Gamma(s/2) = \pi^{s-1/2}\Gamma((1-s)/2)\zeta(1-s), \quad (A2.4)$$

which can be written in different forms using \(\Gamma\)-function relations. Moreover

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} = (2^{1-s} - 1)\zeta(s), \quad (A2.5)$$

Finally (\(\gamma = -\psi(1)\))

$$\zeta(1 + \varepsilon) = \frac{1}{\varepsilon} + \gamma, \quad (A2.6)$$

$$\zeta(\varepsilon) = -\frac{1}{2}(2\pi)^\varepsilon + O(\varepsilon^2). \quad (A2.7)$$

\textit{Jacobi’s \(\theta\)-functions.} We define the function \(\vartheta_0(s)\) (related to Jacobi’s elliptic functions, see below)

$$\vartheta_0(s) = \sum_{n=-\infty}^{+\infty} e^{-\pi sn^2}. \quad (A2.8)$$

Poisson’s summation formula is useful in this context. Let \(f(x)\) be a function which has a Fourier transform

$$\tilde{f}(k) = \int dx f(x) e^{i2\pi kx}.$$ 

Then from

$$\sum_{k=-\infty}^{+\infty} e^{i2\pi kx} = \sum_{l=-\infty}^{+\infty} \delta(x-l),$$

follows Poisson’s formula

$$\sum_{k=-\infty}^{+\infty} \tilde{f}(k) = \sum_{l=-\infty}^{+\infty} f(l). \quad (A2.9)$$

Applying this relation to the function \(e^{-\pi sx^2}\) one finds the identity:

$$\vartheta_0(s) = (1/s)^{1/2}\vartheta_0(1/s). \quad (A2.10)$$

The two integrals, which are related by the change of variables \(s \mapsto 1/s\), are needed

$$\int_0^{\infty} ds \; s^{\alpha/2-1} \left[ \vartheta_0(s) - 1 \right] = 2\pi^{-\alpha/2}\Gamma(\alpha/2)\zeta(\alpha), \quad (A2.11)$$

$$\int_0^{\infty} ds \; s^{\alpha/2-1} \left[ \vartheta_0(s) - 1/\sqrt{s} \right] = 2\pi^{\alpha/2-1/2}\Gamma[(1-\alpha)/2]\zeta(1-\alpha), \quad (A2.12)$$
where $\zeta(s)$ is Riemann’s $\zeta$-function.
For fermion and gauge theories calculations we need the more general function
\[
\vartheta_2(s; \nu, \lambda) = e^{-\pi s \nu^2} \theta_3(\lambda + i \nu s, e^{-\pi s}) = \sum_n e^{-\pi s (n+\nu)^2 + 2i \pi n \lambda}, \tag{A2.13}
\]
where $\theta_3$ is an elliptic Jacobi’s function, which, applying Poisson’s formula, can be shown to satisfy
\[
\vartheta_2(s; \nu, \lambda) = \vartheta_2(s; -\nu, -\lambda) = s^{-1/2} \vartheta_2(1/s; \lambda, -\nu). \tag{A2.14}
\]
For fermions the relevant function is $\vartheta_1(s)$
\[
\vartheta_1(s) \equiv \vartheta_2(s; 1/2, 0) = \sum_n e^{-(n+1/2)^2 \pi s}. \tag{A2.15}
\]
From (A2.14) we obtain
\[
\vartheta_1(s) = \frac{1}{\sqrt{s}} \sum_n (-1)^n e^{-\pi n^2/s}. \tag{A2.16}
\]
Finally ($\alpha > 1$)
\[
\int_0^\infty ds \, s^{\alpha/2-1} \vartheta_1(s) = 2(2^\alpha - 1)\pi^{-\alpha/2} \Gamma(\alpha/2) \zeta(\alpha) \tag{A2.17}
\]

**A3 Group measure**

For the example of non-abelian gauge theories we derive the form of the group measure in the representation of group elements as exponentials of elements of the Lie algebra. The notation and conventions are the same as in section 8.

We set
\[
g = e^\xi,
\]
and we will determine the invariant measure in terms of the components $\xi^\alpha$
\[
\xi = i \tau^\alpha \xi^\alpha.
\]
We thus introduce a time dependent group element $g$ as
\[
g(t) = e^{t \xi}, \quad g(1) = g.
\]
We also need the element of the Lie algebra $L^\alpha(t)$
\[
L^\alpha(t) = \frac{\partial g(t)}{\partial \xi^\alpha} g^{-1}(t).
\]
It satisfies the differential equation
\[
\frac{d}{dt} L^\alpha = i\tau^\alpha + [\xi, L^\alpha], \quad L^\alpha(0) = 0. \tag{A3.1}
\]
This equation can also be written in component form, setting
\[
L^\alpha = iL^{\alpha\beta}\tau^\beta.
\]
Then
\[
\frac{d}{dt} L^{\alpha\beta} = \delta_{\alpha\beta} + f_{\gamma\beta\delta}\xi^\gamma L^{\alpha\delta}. \tag{A3.2}
\]
We call \(\Lambda\) the matrix of elements \(L^{\alpha\beta}\) and introduce the antisymmetric matrix \(X\) of elements
\[
X^{\alpha\beta} = f_{\alpha\beta\gamma}\xi^\gamma.
\]
The solution of equation (A3.2) then can be written
\[
\Lambda(t) = \int_0^t dt' e^{X(t'-t)} = X^{-1}(1 - e^{-tX}).
\]
The metric tensor corresponding to the group is \(g_{\alpha\beta}\)
\[
g_{\alpha\beta} = -\text{tr} L^\alpha(1)L^\beta(1) = L^{\alpha\gamma}(1)L^\beta\gamma(1) = (\Lambda\Lambda^T)_{\alpha\beta}, \tag{A3.3}
\]
and the group invariant measure is
\[
dg \equiv (\det g_{\alpha\beta})^{1/2} \prod_\alpha d\xi^\alpha = (\det \Lambda\Lambda^T)^{1/2} \prod_\alpha d\xi^\alpha. \tag{A3.4}
\]
Then
\[
\Lambda\Lambda^T = -X^{-2} (1 - e^{-X}) (1 - e^X) = -4X^{-2}\sinh^2(X/2)
\]
\[
= -\prod_{n \neq 0} (1 + X^2/(2\pi n)^2),
\]
where we recognize an expression which appears in equation (8.6).
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