The Hawking Energy on the Past Lightcone in Cosmology

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This work studies the Hawking energy in a cosmological context. The past lightcone of a point in spacetime is the natural geometric structure closely linked to cosmological observations. By slicing the past lightcone into a 1-parameter family of spacelike 2-surfaces, the evolution of the Hawking energy down the lightcone is studied. Strong gravitational fields may generate lightcone self-intersections and wave front singularities. We show that in the presence of swallow-tail type singularities, the Hawking energy and its variation along the null generators of the lightcone remains well-defined and subsequently discuss its positivity and monotonicity.

I. INTRODUCTION

A general and consistent notion of energy or mass poses a difficulty in the context of general relativity. Due to the weak equivalence principle, the energy momentum distribution of the gravitational field is locally vanishing for a freely-falling observer moving along a geodesic. Because of these local considerations, quasilocal constructions were brought forward, see for instance [1] and references therein. Amongst the candidates is a construction by Hawking [2], which will be referred to as Hawking energy in the following. Based on a closed spacelike 2-surface $S$ in spacetime, it phenomenologically aims to relate the energy/matter content enclosed by $S$ to the amount of light bending on $S$.

A sensible energy definition should match previously established concepts of energy in highly symmetric or asymptotic settings, such as the ADM- or Bondi-mass, but still be general enough to also apply to more general set-ups. A central challenge for many quasilocal energy definitions is to provide a sensible meaning to the concepts of positivity and monotonicity in a physically realistic and general enough context. For asymptotically flat spacetimes filled with matter obeying the dominant energy condition (DEC), several positive mass theorems for the ADM-mass [3–5] as well as for the Bondi-mass [6–8] were established. Later, positivity of mass was extended also to asymptotically AdS-spacetimes and to Einstein-Maxwell theory [9]. Closely linked to the question of positivity is the Penrose conjecture [10, 11], relating the total mass of a spacetime to the area of the outermost apparent horizon. The corresponding Riemannian version, the Riemann-Penrose inequality, was proven by Huisken & Ilmanen [12] using the observation by Geroch that the Hawking energy behaves monotonously under the inverse mean curvature flow [13]; it was also independently proven by Bray [14].

This work is structured as follows. The Hawking energy and its main properties are discussed in section II, before the cosmological set-up together with the slicing construction of the lightcone is explained in section III. The weak lensing regime in absence of self-intersections is studied in section IV. The effect of self-intersections on the lightcone geometry is addressed in section V. Section VI establishes the well-definedness of the Hawking energy and its derivative in the presence of swallow-tail type singularities. The rescaling freedom of the null generators of the lightcone is discussed in section VII, before moving to a discussion on monotonicity and possible extensions in sections VIII & IX. We conclude in section X.

II. HAWKING ENERGY

Unless stated otherwise, we assume a globally hyperbolic Lorentzian spacetime $(M,g)$ satisfying the Einstein field equations (EFEs):

$$R_{ab} - \frac{1}{2}R g_{ab} = 8\pi T_{ab} ,$$

with Ricci tensor $R_{ab}$, Ricci scalar $R$, and the energy-momentum tensor $T_{ab}$ satisfying the DEC. A potential
cosmological constant can be accommodated in $T_{ab}$ in the following discussions. The signature convention is $(- + + +)$ and we use units in which $c = G = 1$.

A spacelike 2-surface $S$ in $M$ uniquely defines two distinct orthogonal null congruences, both of which are either future or past directed, represented by two null vector fields $l^a$ and $n^a$. These congruences are often referred to as outgoing and ingoing, their expansion scalars are denoted by $\theta_+ = \nabla_a l^a$ and $\theta_- = \nabla_a n^a$ respectively, where $\nabla_a$ is the covariant derivative associated with the spacetime metric $g$. Hawking's original definition [2] of the energy $E(S)$ associated with a spacelike surface $S$ of spherical topology reads:

$$E(S) := \frac{\sqrt{A(S)}}{(4\pi)^{3/2}} \left( 2\pi + \frac{1}{4} \int_S \theta_+ \theta_- dS \right), \quad (2)$$

where $A(S) = \int_S dS$ denotes the area of the surface $S$ given in terms of the pullback $dS$ of the canonical spacetime volume form onto $S$. This definition satisfies several important limits briefly reviewed here, see also [1, 20]:

(i) The Hawking energy of any point in spacetime should vanish, hence $E(S) \to 0$ for $S$ degenerating to a point.

(ii) For a small sphere of (area) radius $r \to 0$ about point $p$, one finds for the leading order in $r$ [20]:

$$E(S) \sim r^5 B_{abcd} t^a t^b t^c t^d \geq 0 \quad \text{in vacuum} \quad (3)$$

$$E(S) \sim r^3 T_{ab} t^a t^b \quad \text{in non-vacuum} \quad (4)$$

with the Bel-Robinson tensor $B_{abcd}$, $t^a \in T_p M$ a unit timelike vector orthogonal to $S$, and the energy-momentum tensor $T_{ab}$. If the DEC holds, then $E(S) \geq 0$ also in the non-vacuum case.

(iii) For large spheres near null infinity $\mathcal{I}^\pm$, the Bondi-Sachs energy is recovered [2]: $E(S) \to E_{\text{Bondi-Sachs}}$.

(iv) For large spheres near spatial infinity $\mathcal{I}^0$, the ADM-mass is recovered [1]: $E(S) \to E_{\text{ADM}}$.

(v) In a spherically symmetric spacetime, the Hawking energy coincides with the Misner-Sharp energy, e.g. [27].

(vi) If $S$ is a metric sphere in Minkowski spacetime: $E(S) = 0$ [1].

(vii) Given a null hypersurface with $\theta_+ = 0$, for instance a non-expanding horizon or a Killing horizon. For any spacelike spherical cross section $S$, one finds:

$$E(S) = \sqrt{\frac{A(S)}{16\pi}}. \quad (5)$$

In particular, for a cross section of the event horizon of a Kerr-Newman black hole, the irreducible mass $M_{\text{irr}}$ is recovered, see e.g. [20].

Two other properties one would expect from an energy definition are positivity and monotonicity. However, it appears that this is not given in the general case. Concerning positivity, it is worth pointing out that (vi) only holds for metric spheres and not for arbitrary topological spheres on Minkowski spacetime. In fact, the Hawking energy might become negative for suitably shaped spheres$^2$. In order to maintain a vanishing energy for any spacelike topological sphere in Minkowski space, Hayward proposed a modification by including shear and twist terms [28]. However, it is negative for small spheres in vacuum [29]. A general positivity result for maximal slices was obtained in [19]. Furthermore, one would naturally expect the energy to increase if the domain, i.e. the surface $S$, is enlarged. Since in general there are many ways to enlarge $S$, one would have to specify a particular construction to give a more precise meaning to the statement. Eardley was able to construct a special family of surfaces along which the Hawking energy increases monotonously [20]. This result is essential in order to establish monotonicity in the weak lensing case and will be discussed in greater detail in section IV.

In the light of these results, a natural question is whether positivity and monotonicity of the Hawking energy can be established in particular, physically relevant set-ups, such as the past lightcone of an observer in cosmology.

### III. COSMOLOGICAL SET-UP

The cosmological context in which we aim to answer this question is provided by the observational approach by Ellis and others [16, 17]. Based solely on data on the past lightcone of an observer, it aims at deducing the spacetime geometry in the vicinity of the lightcone without further model assumptions. Mathematically, it constitutes a characteristic final value problem, see e.g. [30, 31] and references therein, with final data given on the past lightcone and a solution in the chronological past of the event is constructed by propagating the data on the lightcone into its interior via the EFEs. The observer is assumed to be a point $p$ in spacetime $M$ and a future-pointing normalised timelike vector $u^a \in T_p M$. This is a good approximation as long as the duration of observation is negligible compared to the dynamical timescale of the universe. Almost all cosmologically relevant information, such as light and gravitational waves, travels with the speed of light, hence the central geometric object of

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1 The Bel-Robinson tensor is defined as $B_{abcd} := C_{acdf} C_{bdf} - \frac{3}{2} g_{[a[c} p_{d]f]} C_{[c}^{jk} f_{d]}$.

2 In general, the Hawking energy turns negative if according to (21) the mean curvature $H$ of $S$ within the spacelike hypersurface $\Sigma$ is large enough compared to the mean curvature $\tau$ of $\Sigma$ in $M$. 

interest is the past lightcone $C^-(p)$ of the observer at $p \in M$. It is a null hypersurface and can be uniquely constructed once the point $p \in M$ is specified. In Minkowski spacetime, it is an undistorted cone with topology $\mathbb{R} \times S^2$. However, the presence of matter or other inhomogeneities will in general deform the lightcone. Two regimes can be distinguished:

- **Weak Lensing Regime**: the lightcone remains an embedded surface, but is weakly deformed, preserving the $\mathbb{R} \times S^2$ topology. Hence, no multiple images of the same source appear.

- **Strong Lensing Regime**: the lightcone is strongly deformed and intersects itself. Changes in topology cause multiple imaging.

More formally, the past lightcone $C^-(p)$ of a cosmological observer $(p, u^\alpha)$ in a globally hyperbolic spacetime $M$ is the image of the exponential map $\exp_p$ along past-pointing null vectors $\in T_p M$ on its maximal domain of definition. Sufficiently close to $p$, the exponential map is always injective. At self-intersections, the exponential map fails to be injective, i.e. points may be reached along multiple null geodesics starting at $p$. Another crucial observation is that past null geodesics issued at $p$ are initially part of the boundary $\tilde{I}^-(p)$ of the chronological past $I^-(p)$ of $p$, but might leave the boundary into the interior. Thus, they are not exclusively confined to $\tilde{I}^-(p)$ but rather to $\tilde{I}^-(p) \cup I^-(p)$. The last point along a null generator $\gamma(\tau)$ in $\tilde{I}^-(p)$ is called cut point of $\gamma$. The union of all cut points of all past-pointing null generators is then referred to as cut locus $L^-(p)$ of the past lightcone $C^-(p)$. Any point of a generator beyond the cut point lies in the chronological past of $p$ and therefore can also be reached along a timelike curve from $p$. At a cut point, multiple null generators intersect, either infinitesimally close generators resulting in a conjugate point, or globally different generators, see Fig. 1. Furthermore, since $\tilde{I}^-(p)$ is an achronal boundary and therefore a Lipschitz continuous submanifold [32], the same holds for the part of the lightcone contained in the boundary, $C^-(p) \cap \tilde{I}^-(p)$. Additionally, the cut locus has measure zero in $\tilde{I}^-(p)$, thus, $C^-(p) \cap I^-(p)$ is differentiable everywhere except at $p$ and the cut locus [25].

In this cosmological set-up, we study the properties of the Hawking energy on the past lightcone $C^-(p)$ of a cosmological observer. In particular, we are interested in monotonicity properties of $E$ along a family of two dimensional slices ($S_t$) down the lightcone. Before turning to more formal and rigorous statements in sections IV & VIII, we first provide an intuitive argument in favour of monotonicity.

The past lightcone in Fig. 1 can be sliced into two dimensional spacelike surfaces, for instance by a one-parameter family of (partial) Cauchy surfaces $\Sigma_t$. The part of such a lightcone slice contained in the past causal boundary $\tilde{I}^-(p)$ is denoted by $S_t$: $S_t := C^-(p) \cap \tilde{I}^-(p) \cap \Sigma_t$. Since $\tilde{I}^-(p)$ is the past causal boundary of $I^-(p)$, any matter respecting the DEC can only leave $I^-(p)$ to the future, in particular, nothing can enter $I^-(p)$ from outside. Therefore, taking two different slices $S_t$ and $S_{t'}$ with $t < t'$ as depicted in Fig. 2, matter may only leave $I^-(p)$ between $t$ and $t'$. Turning the argument around, the surfaces $S_t$ should enclose more and more matter towards the past. Each $S_t$ is typically a closed, spacelike surface and thus has an associated Hawking energy $E(S_t)$. By the above argument, the Hawking energy should then be monotonously increasing along the family ($S_t$) down the lightcone. Though, this naive argument only holds for $\theta_+ > 0$ everywhere on $C^-(p)$ as we shall see later.

It is crucial to note that this argument only holds for surfaces which are part of the causal boundary. As mentioned above, the lightcone generators leave the boundary after self-intersections and the interior parts of $C^-(p)$, that is the part contained in the chronological past, can be penetrated by timelike curves. Therefore, we have to exclude the interior parts of $C^-(p)$ and restrict our monotonicity discussion to the part of the lightcone contained in the causal boundary $C^-(p) \cap \tilde{I}^-(p)$. Also from a geometric point of view, $C^-(p) \cap \tilde{I}^-(p)$ is a much

![FIG. 1. Lightcone $C^-(p)$ of point $p$ going through a gravitational lensing event causing $C^-(p)$ to intersect itself at the cut locus $L^-(p)$ (blue), which is part of the exterior $C^-(p) \cap \tilde{I}^-(p)$. Two different null generators (red) intersect at the cut locus, after which they turn into the interior $I^-(p)$ (dashed). The conjugate point $q$, where infinitesimally close generators intersect, is of swallow-tail type. Two cusp ridges originating from $q$ remain in $I^-(p)$.](image)
Thus, if $E$ the next sections. The following results can be understood
ing caustics, a discussion including these can be found in
the interior of $I^−(p)$. better-behaved hypersurface than $C^−(p)$ since it is a
Lipschitz manifold, whereas $C^−(p)$ might in general fail
to be a manifold due to complicated self-intersections
in the past of $I^−(p)$. In the subsequent sections, we do not use Cauchy sur-
faces, but rather adopt the following construction in order
to generate the 1-parameter family of lightcone slices. We
start with an initial lightcone cut $S$ sufficiently close to $p$,
guaranteeing that $S$ is a topological sphere and that
the Hawking energy is positive due to the small sphere
limit [26]. The part of the lightcone in the past of $S$ is
generated by the past-pointing null geodesics associ-
ated with the null generators $l^a$. The lightcone can then
be sliced into constant affine parameter distance slices
$S_\lambda$, with $\lambda \geq 0$ and $S_{\lambda=0} = S$. However, since $l^a$ is
a null vector field, we have a pointwise rescaling freedom
$l^a \rightarrow \alpha l^a$, with $\alpha > 0$ a function on $S$. After fixing the
rescaling freedom in a suitable manner, as is done in the
subsequent sections, we walk down a unit distance along
the generators and arrive at a new spacelike cut, where
the rescaling procedure is repeated etc. The function $\alpha$
extends to a function on $C^−(p) \cap I^−(p)$ and encodes the
particular choice of the lightcone foliation $(S_\lambda)$. Changing
from one foliation with corresponding affine parameter $\lambda$
to another one with $\lambda$ relates the change of $E$ for each
foliation via
\begin{equation}
\frac{\partial E}{\partial \lambda} = \frac{\partial E}{\partial \lambda} \frac{\partial \lambda}{\partial \lambda} .
\end{equation}
Thus, if $E$ is monotonously increasing along the $\lambda$-foliation, it increases along the new foliation if $\frac{\partial \lambda}{\partial \lambda} > 0$, that is, if $\lambda$ is an increasing function of $\lambda$.

IV. WEAK LENSING REGIME

This section addresses monotonicity in scenarios excluding
causics, a discussion including these can be found in
the next sections. The following results can be understood
as an application of Eardley’s findings [20] to lightcones.
Each lightcone slice on the boundary, $S_\lambda \subset C^−(p) \cap I^−(p)$,
comes with an associated Hawking energy $E(S_\lambda)$. In the
following, it is assumed that each slice $S_\lambda$ is topologically
a sphere, a brief discussion of different topologies can be
found in section IX. The change of the Hawking energy
(2) assigned to $S_\lambda$ along the outgoing null direction $l^a$
is given by $\partial_\lambda E(S_\lambda) \equiv \dot{E}(S_\lambda)$:
\begin{equation}
\dot{E}(S_\lambda) = \frac{E(S_\lambda)}{2A(S_\lambda)} \int_{S_\lambda} \theta_+ dS_\lambda +
\frac{\sqrt{A(S_\lambda)}}{(4\pi)^{3/2}} \int_{S_\lambda} \left\{ \theta_+ \dot{\theta}_- + \theta_- \dot{\theta}_+ + \theta_+ \dot{\theta}_- \right\} dS_\lambda ,
\end{equation}
where we used $A(S) = \int_S dS$ and $(dS) = \theta_+ dS$. At the same
time, the vector field $l^a$ will be taken to be identical
to the null generators of the past lightcone $C^−(p)$. Next,
we will make use of the Sachs equation for the evolution of $\theta_+$ (see e.g. [32]):
\begin{equation}
\dot{\theta}_+ = -\frac{1}{2} \theta_+^2 - \sigma_{ab} \sigma^{ab} - R_{ab} l^a l^b .
\end{equation}
Since $l^a$ generates a null hypersurface, the vorticity term
in the general Sachs equation is vanishing and thus absent
in (8). $\sigma_{ab}$ denotes the shear tensor of the congruence $l^a$.
Using the EFEs, the right hand side of (8) is non-positive
if the DEC holds. The evolution equation of $\theta_-$ along $l^a$
can be derived from [33]:
\begin{equation}
\dot{\theta}_- = D_\alpha \Omega^\alpha + \Omega_\alpha \Omega^\alpha - \frac{1}{2} R l^a l^b R_{ab} - \theta_+ \theta_- .
\end{equation}
It describes the change of the expansion of the ingoing
null congruence $n^a$ along the outgoing one. $\Omega_\alpha = \nabla_\alpha n^a$
denotes the change of the ingoing null vector along the
outgoing one. $h_{ab}$ denotes the two dimensional Riemann-
ian metric on $S_\lambda$ defined by the pullback of the spacetime
metric $g_{ab}$ onto $S_\lambda$. They are related via
\begin{equation}
h_{ab} = g_{ab} + l_a n_b + n_a l_b .
\end{equation}
$D_\alpha$ is the covariant derivative on $S_\lambda$ compatible with its
induced metric $h_{ab}$ and related to the spacetime covariant
derivative $\nabla$ via the projection operator onto $TS_\lambda$, $\Pi^b_a =
\delta^b_a + l_a n_b + n_a l_b$. For instance, $D_a X^b = \Pi^b_a \Pi^d_c \nabla_c X^d$ for
any $X^a \in TS_\lambda$. The two dimensional Ricci scalar of $S_\lambda$
is denoted by $2R$. Using the EFE, we find
\begin{equation}
h_{ab} R_{ab} = R + 2 R_{ab} l^a n^b = 16 \pi T_{ab} l^a n^b \geq 0 ,
\end{equation}
if the DEC is satisfied. Inserting (8), (9), and (11) into
(7) yields:
\begin{equation}
\dot{E}(S_\lambda) = \frac{E(S_\lambda)}{2A(S_\lambda)} \int_{S_\lambda} \theta_+ dS_\lambda +
\frac{\sqrt{A(S_\lambda)}}{(4\pi)^{3/2}} \int_{S_\lambda} \left\{ - \theta_+ \left( \frac{1}{2} \theta_+^2 + \sigma_{ab} \sigma^{ab} + R_{ab} l^a l^b \right) + \theta_+ \left( D_\alpha \Omega^\alpha + \Omega_\alpha \Omega^\alpha - \frac{1}{2} R + 8 \pi T_{ab} l^a n^b \right) \right\} dS_\lambda .
\end{equation}
Eardley [20] established a monotonicity results for a particular family of surfaces (\(S_r\)). Starting off with a surface \(S\) with \(\theta_+ > 0 \) and \(\theta_- \leq 0\) almost everywhere. One can define a constant \(r\) on \(S\) by \(A(S) := 4\pi r^2\). Although \(n^a\) is normalised such that \(n^a l_a = -1\), there is still a pointwise rescaling freedom of \(l^a\) left: \(l^a \rightarrow \alpha n^a\) with \(\alpha > 0\). It is used to rescale \(l^a\) such that \(\theta_+ = \frac{2}{r}\). Since \(\partial_{\lambda} r = 1\), \(r\) is also a parameter along the congruence. In fact, \(r\) corresponds to a luminosity distance function for a source of luminosity \(L\) and flux \(F\):

\[
r = \sqrt{\frac{L}{4\pi F}} = \sqrt{\frac{AF}{4\pi F}} = \sqrt{\frac{A}{4\pi}}.
\]

Starting with the initial surface \(S\) being a lightcone section arbitrarily close to the tip \(p\), the remaining lightcone is foliated by level surfaces \(S_r\) of constant \(r\). Along this special family of surfaces \(S_r\), (12) can be further simplified by inserting the explicit expressions for \(A\) and \(\theta_+\):

\[
\hat{E}(S_r) = \frac{1}{4\pi} \int_{S_r} \left\{ -\frac{r}{4} \theta_+ (\sigma_{ab}\sigma^{ab} + R_{ab}l^a l^b) + \frac{1}{2} \left( \Omega_\alpha \Omega^\alpha + \frac{1}{2} R + R_{ab} l^a l^b \right) \right\} dS_r,
\]

where the Gauss-Bonnet theorem for a sphere \(\int S^2 R dS = 8\pi\) was used as well as \(\int S^2 \sigma_{ab} \sigma^{ab} dS = 0\), because \(S\) is a closed surface. Extending the assumption \(\theta_+ > 0\) and \(\theta_- \leq 0\) to all \(S_r\), and further assuming the DEC, we find the right hand side of (14) to be non-negative, because \(\sigma_{ab}\sigma^{ab} \geq 0\) and \(\Omega_\alpha \Omega^\alpha \geq 0\), which can be verified by direct calculation. Using the EFEs, the curvature terms are simplified. Yet, it was shown that the multitude of these self-intersections can be divided into stable and unstable ones in the following sense. The set of points in spacetime \(M\) that can be reached by the outgoing, respectively ingoing, null geodesic congruence emanating from an orientable, spacelike, smooth surface \(S\) in \(M\) is called wavefront, see e.g. [25]. The caustic of a wavefront is defined to be the set of points where the wavefront fails to be an immersed submanifold of \(M\). In particular, the past lightcone \(C^-(p)\) of \(p\) is a wavefront if \(S\) is chosen suitably close to \(p\). Stability refers to arbitrarily small perturbations of the initial surface \(S\), see e.g. [23] for more details. A classification of stable caustics of wavefronts was established by [22, 23, 35, 36], using Arnol’d’s singularity theory of Lagrangian and Legendrian maps [24, 37]. Of particular relevance for the present work, Low showed [35, 36] that only two types of stable caustics appear in the intersection of a lightcone with a spacelike hypersurface, referred to as cusp and swallow-tail singularities.

Before discussing more general set-ups, we study the simple lensing configuration in Fig.1, where both cusp and swallow-tail singularities are present in \(C^-(p) \cap \Sigma_t\). The presence of self-intersections renders \(C^-(p)\) Lipschitz continuous on the measure zero set of self-intersections, in other words, the light cone remains smooth almost everywhere. In the following discussion, we assume that the cut locus \(L^-\) also has measure zero in each slice \(S_t = C^-(p) \cap \Sigma_t\), i.e. \(S_t\) is smooth almost everywhere, which is certainly satisfied for the configuration in Fig.1.

Thus in the most general case discussed here, the surface \(S\) may contain a measure zero set of isolated singular points. Since singular points are conjugate points along the null generators with respect to \(p\), the expansion of the lightcone generators \(\theta_+ = -\infty\) at a singular point. Hence, \(\theta_+\) is a smooth function almost everywhere on \(S\), apart from the singular points. Large regions of \(S\) will display a positive \(\theta_+\), and by continuity, any singular point is surrounded by a neighbourhood with negative \(\theta_+\).
Also, at least for a spherically symmetric lens (cf. Fig.1) and related configurations, only swallow-tail singular points can be found in $C^-(p) \cap \hat{I}^-(p)$, cusp singular points appear only at self-intersections in the interior $I^-(p)$. Hence, it suffices to take only swallow-tail singular points on $S$ into account, additionally, one could conjecture that of all stable singular points according to Arnol’d, only swallow-tail ones appear in $C^-(p) \cap \hat{I}^-(p) \cap \Sigma_i$.

VI. HAWKING ENERGY IN THE PRESENCE OF SINGULARITIES

Given the above set-up of a light cone including caustics, it is a natural question whether or not the Hawking energy for a surface containing these types of singularities is well-defined. In particular, since at singular points $\theta_+ = -\infty$, one might wonder whether the integral $\int_S \theta_+ \theta_- \, dS$ in (2) is well-defined, i.e. finite, for $S$ containing singular points. In the following, we show that this is indeed the case for swallow-tail singularities only, whereas the integral is divergent if $S$ contains cusp points.

The past-pointing null vectors $l^a$ and $n^a$ orthogonal to the spacelike codimension-2 surface $S$ can be decomposed into a timelike (future-pointing) unit normal $t^a$ as well as a spacelike unit normal $v^a$ via

$$l^a = \frac{1}{\sqrt{2}} (-t^a + v^a) \quad \& \quad n^a = \frac{1}{\sqrt{2}} (-t^a - v^a) . \quad (15)$$

Following [1], the tangent bundle $TM$ of the spacetime $M$ can be decomposed into the sum of the tangent bundle $TS$ of $S$ and the normal bundle $NS$ of $S$ by using the corresponding projectors

$$\Pi^a_b := \delta^a_b + t^a t_b - v^a v_b = \delta^a_b + l^a_n b + n^a l_b \quad \text{and} \quad O^a_b := \delta^a_b - \Pi^a_b \quad (16)$$

respectively. For example, the spacetime metric $g$ can be decomposed into the intrinsic metric $h$ on $S$ and an orthogonal part, cf. (10):

$$g_{ab} = h_{ab} - t_a t_b + r_a r_b = h_{ab} - l_a n_b - n_a l_b \quad . \quad (17)$$

Corresponding to each normal, there is an associated extrinsic curvature $\tau_{ab}$ and $H_{ab}$:

$$\tau_{ab} = \Pi_a_b \nabla_c l^c \quad \& \quad H_{ab} = \Pi_a_b \nabla_c v^c \quad . \quad (18)$$

$\tau_{ab}$ is the extrinsic curvature (or second fundamental form) of the spacelike hypersurface $\Sigma$ with timelike normal $t^a$ embedded in spacetime. $\Sigma$ creates the lightcone slices: $S = C^-(p) \cap \hat{I}^-(p) \cap \Sigma$. $H_{ab}$ is the extrinsic curvature of the spacelike 2-surface $S$ with spacelike normal $v^a$ within the spacelike hypersurface $\Sigma$. Taking the trace results in the mean curvatures $\tau$ and $H$ of $S$ in each direction. Using the definitions $\theta_+ = \nabla_a l^a$ and $\theta_- = \nabla_a n^a$, together with (15) & (18), yields the following relation between the null expansions and mean curvatures:

$$\theta_\pm = \frac{1}{\sqrt{2}} (-\tau \pm H) . \quad (19)$$

Because $\theta_+ = -\infty$ at singular points, (19) implies that one of the mean curvatures has to diverge. Since $\tau$ is the mean curvature of the smooth spacelike hypersurface $\Sigma$, it is finite, and hence, $H \to -\infty$ at singular points. The product of expansions can be written as the norm of the main curvature vector $Q^a$ of $S$:

$$Q^a := -\theta_- l^a - \theta_+ n^a = \tau t^a - H v^a \quad \text{thus} \quad -Q^a Q_a = 2 \theta_+ \theta_- = \tau^2 - H^2 \quad . \quad (20)$$

In the generic case where $\theta_+ > 0$ and $\theta_- < 0$, $Q^a$ is spacelike, it is null if one of the expansions is zero, for example on horizons, and becomes timelike if the expansions have the same sign, for instance for trapped surfaces. These results imply that every singular point on $S$ is surrounded by a "trapped ring", where $\theta_+ \theta_- > 0$ (see Fig.3).

Using (20), the integral expression appearing in (2) becomes

$$E(S) = \frac{\sqrt{A(S)}}{(4 \pi)^{3/2}} \left( 2 \pi + \frac{1}{8} \int_S \left( \tau^2 - H^2 \right) \, dS \right) . \quad (21)$$

The fact that $H$ diverges at singular points was also more formally established in [38] (c.f. corollary 3.5), where the authors proved that the mean curvature of a hypersurface in a Riemannian manifold diverges at swallow-tail or cusp singular points. Although $H$ is divergent, one might still hope that $\int_S H^2 \, dS$ in (21) is finite. In the following, we show that this is the case only for swallow-tail singularities, whereas the integral diverges for cusp singularities. Therefore, the Hawking energy is well-defined, i.e. finite-valued, for Lipschitz surfaces only containing swallow-tail singularities.

It suffices to study the integral in a neighbourhood $Q \subset S$ of a singular point, because the mean curvature is always finite-valued at non-singular points of $S$. Below, we arrive at explicit expressions for the mean curvature near

![Fig. 3. A singular point $q$ on $S$ is surrounded by a trapped, ring-like region (grey), where the product of the null expansions $\theta_+ \theta_-$ is positive.](image)
a cusp and swallow-tail singular point, if $S$ is embedded in Euclidean space, and find that $\int Q H^2 dS$ diverges for a cusp, but is finite for a swallow-tail point. This result can be immediately replaced to the Riemannian case by replacing the Euclidean metric $g$ in the calculation below with its Riemannian counterpart, altering the result only by finite factors.

The following calculation and notation follows [38]. Given a smooth map $f : M \to N$ from an oriented 2-manifold $M$ into an oriented Riemannian 3-manifold $N$ with metric $g$. $f$ is called an instantaneous wavefront if there exists a unit vector field $\nu^a \in N$ along $f$ such that $g(f_n, \nu) = 0 \ \forall X \in TM$. $\nu^a$ is called the normal vector of the instantaneous wavefront $f$. An instantaneous wavefront is the intersection of a wavefront with a spacelike hypersurface [25]. $q \in M$ is called a singular point of the front $f$, if $f$ is not an immersion at $q$. A singular point is called cusp point or swallow-tail point respectively, if it is locally diffeomorphic to

$$f_C(u,v) := (u^2, u^3, v) \text{ or } f_S(u,v) := (3u^4 + u^2v, 4u^3 + 2uv, v)$$

at $(u,v) = (0,0)$. The mean curvature $H$ of the front $f$ with normal vector $\nu^a$ is

$$H := \frac{EN - 2FM + GL}{4\lambda^2} ,$$

with $f_u = \partial_u f$, $f_v = \partial_v f$, $E = g(f_u, f_u)$, $F = g(f_u, f_v)$, $G = g(f_v, f_v)$, $|\lambda| = \sqrt{EG - F^2}$, $L = -g(f_u, \nu_u)$, $M = -g(f_v, \nu_v) = -g(f_u, \nu_v)$, $N = -g(f_u, \nu_v)$. Computing the mean curvature for cusp and swallow-tail point singularities near the singular point $(0,0)$ yields:

$$H_C = -\frac{3}{2u(9u^2 + 4)^{3/2}} \text{ and }$$

$$H_S = \frac{u^4 + 4u^2 + 1}{8(6u^2 + v)(u^4 + u^2 + 1)^{3/2}} .$$

Inserting these into the integral expression in (21) using $dS = |\lambda| du dv$ and setting the integration range

$$Q_C = \{ v \in [b_1, b_2], u \in [-a, a] \}$$

yields

$$\int_{Q_C} H_C^2 dS = \frac{9}{4} \frac{v_{b_2}b_2}{a} \int_{-a}^{a} \frac{du}{|u|(9u^2 + 4)^{5/2}}$$

$$\approx 0 \int_{-\infty}^{0} \frac{9}{128} v_{b_1}b_1 \int_{-b_1}^{0} \frac{du}{|u|} = 9 \frac{a}{128}(b_2 - b_1) \cdot 2 \ln(|u|)_{0}^{a} = +\infty$$

(26)

for the cusp case. In the case of the swallow-tail, we must be careful to only integrate over the outer part of the surface, i.e. the part contained in $C^{-}(p) \cap \Gamma(p)$, see Fig.4. This is done by restricting the integration range to $v \geq -2a^2$ in the above parametrization.

Next, we address the derivative of $E$ along the null generators (12), which can be further simplified by making use of the contracted Gauss equation:

$$2R = h^{ac}h^{bd}R_{abcd} - \theta_+ \theta_- + 2\sigma^+_a \sigma^a_- .$$

(28)

Applying the Ricci decomposition of the Riemann tensor (see also [39]) and using the metric decomposition (17) together with the EFE yields:

$$h^{ac}h^{bd}R_{abcd} = h^{ac}h^{bd}C_{abcd} + 16\pi T_{ab}n^a n^b \text{ with }$$

$$h^{ac}h^{bd}C_{abcd} = 2C_{\text{int}} ,$$

(29)

after using the symmetries of the Weyl tensor. The Weyl tensor term vanishes if $l^a$ belongs to a null geodesic congruence. This can be seen by taking the shear evolution equation along the null congruence, contracting it with $n^a n^b$ and noting that $\sigma^+_a n^a = 0$:

$$C_{\text{int}} = -n^a n^b \nabla_l \sigma^a_+ - \theta_+ \sigma^+_a n^a n^b = 0 .$$

(30)

Summarising, we find

$$2R = 16\pi T_{\text{int}} - \theta_+ \theta_- + 2\sigma^+_a \sigma^a_- .$$

(31)

Next, the term $\Omega_0 \Omega^a$ appearing in (12) can be expressed in terms of the energy momentum tensor.

FIG. 4. Front containing a swallow-tail singularity. The outer part $\subset \Gamma^-(p)$ is coloured in red and satisfies $v \geq -2a^2$. It is a zoom-in of Fig.1 near the swallow-tail point $q$. $Q_S = \{ v \in [-2a^2, a], u \in [-b, b] \}$, yielding

$$\int_{Q_S} H_S^2 dS = \int_{-b}^{b} du \int_{-2u^2}^{a} dv \frac{(u^4 + 4u^2 + 1)^2}{32(6u^2 + v)(u^4 + u^2 + 1)^{3/2}}$$

$$= \int_{-b}^{b} du \frac{(u^4 + 4u^2 + 1)^2}{32(6u^2 + v)(u^4 + u^2 + 1)^{3/2}} \ln \left( \frac{3 + a}{4u^2} \right)$$

$$\approx 1 \frac{1}{32} \int_{-b}^{b} du \ln \left( \frac{a}{4u^2} \right)_{0}^{b} < \infty .$$

(27)

Summarising, we showed that the integrals in (21) are finite, thus the Hawking energy for a topological sphere $S$ containing swallow-tail singularities is well-defined.
Recalling that \( \nabla \cdot n_a = 0 = \partial_a n^a \), we are left with
\[
\nabla_l (n^a \nabla_l n_a) = \nabla_l \left( \frac{1}{2} \nabla_l (n_a n^a) \right) = 0 \quad \Leftrightarrow \quad \Omega_a \Omega^a = -n^a \nabla_l \Omega_a . \quad (32)
\]
Using the evolution equation for \( \Omega_a \) along the null generators (cf. [33]),
\[
\nabla_l \Omega_a = -\Theta^a_\nu \Omega_\nu - \Theta_\nu \Omega_\nu + 8\pi T_{\mu \nu} l^\mu + \frac{1}{2} D_a \theta_+ - D_a \sigma^a_+ \eta_+ ,
\]
and contracting it with \( n^a \), we end up with
\[
\Omega_a \Omega^a = -8\pi T_{l n} . \quad (34)
\]
Thus, inserting (31) and (34) into (12), we find
\[
\dot{E} = \frac{E(S)}{A(S)} \int_S \theta_+ dS + \frac{\sqrt{A(S)}}{(4\pi)^{3/2}} \int_S \left[ -\left( \theta_- \sigma^+_{ab} \sigma^{ab} + \theta_+ \sigma^+_{ab} \sigma^{ab} \right) - 8\pi \left( \theta_- T_{l l} + \theta_+ T_{l n} \right) + \theta_+ D_a \Omega^a \right] dS . \quad (35)
\]

This expression describes how the Hawking energy changes along constant affine parameter slices of spherical topology of the past lightcone \( C^- (p) \). The shear and matter effects separate into two different contributions.

Before discussing the terms and addressing monotonicity, we first comment on whether or not the first derivative \( \dot{E} \), in addition to the energy itself, is well-defined. One can check with (25) that \( \int_S H_S dS \) as well as \( \int_S H_S^2 dS \) are finite, however, \( \int_S H_S^3 dS \) diverges. Since the first as well as the terms involving the energy momentum tensor are proportional to \( H \), they are finite. Because \( S \) is a manifold without boundary, \( \int_S \theta_+ D_a \Omega^a = -\int_S \Omega^a D_a \theta_+ \), and \( D_a \theta_+ \) is proportional to \( \partial_a H \) and \( \partial_a H \). Again, one can check explicitly with the help of (25) that these derivatives are finite. Turning to the shear terms, we note that the shear tensors \( \sigma^\pm_{ab} \) are also diverging at singular points, in particular in the same way as \( \theta_\pm \) for non-degenerate singular points such as cusp or swallow-tail [40]. Therefore, \( \partial_+ \sigma^\pm_{ab} \) and \( \partial_+ \sigma^\pm_{ab} \) are of order \( H^3 \), hence their integrals over \( S \) diverge. However, close to a singular point, both terms have opposite signs and cancel each other. This can be seen by rewriting the expression with the help of \( Q^a_{c c} := h^a_n h^c_n \nabla_d h^c_n \) and noting that \( \Theta_{ab} = -l_c Q^c_{a b}, \; \Xi_{ab} = -n_c Q^c_{a b} \).
\[
\theta_- \sigma^+_{ab} + \theta_+ \sigma^+_{ab} = \sigma^+_{a c} Q^c_{a b} . \quad (36)
\]

Next, expressing \( Q^a \) and \( Q^c_{a c} \) in terms of the timelike and spacelike unit normals \( l^a \) and \( n^a \) yields:
\[
\sigma^+_{a b} Q^c_{a c} = \frac{1}{\sqrt{2}} \tau_{a b} (\tau^{a b} + H^{a b}) + \frac{1}{2} \tau^2 \theta_+ - \frac{1}{\sqrt{2}} H H_{a b} \tau^{a b} + \frac{1}{\sqrt{2}} H H_{a b} H^{a b} - \frac{1}{2} \theta_+ H^2 . \quad (37)
\]

All terms apart from the last two are at most of the order \( H^2 \) and thus integrable. The last two terms are diverging as \( H^3 \) near a singular point, but being of opposite sign, they precisely cancel each other. Hence, the integral of the shear terms is also finite.

Summarising, we found that the Hawking energy as well as its first derivative along the null generators of the past lightcone are well-defined even for surfaces including swallow-tail type singularities. Knowing that (35) is a well-defined quantity, we now address the rescaling freedom of \( l^a \) before studying the monotonicity of (35).

\[\text{VII. CHOICE OF RESCALING}\]

As mentioned earlier, once a scaling function \( \alpha \) is chosen, the Hawking energy will monotonously increase and be positive along the family of constant (affine) parameter surfaces \( (S_\lambda) \) associated with this rescaling, if and only if (35) is positive. Hayward [41] pointed out that the sign of \( \int_S \theta_\pm dS \) is not an invariant under rescaling \( l^a \). In particular, if \( \theta_\pm \) changes its sign on \( S \), \( \int_S \theta_\pm dS \) can take any sign and value by constructing an appropriate rescaling function \( \alpha \) on \( S \). In fact, since all terms appearing in (35) are not invariant under rescaling, one can use the rescaling freedom to simplify its right-hand-side. As in the weak lensing case, the term \( \int_S \theta_+ D_a \Omega^a \) can be eliminated even if \( \theta_+ \) is not strictly positive anymore. Under rescaling \( l^a \to \alpha l^a, \; \alpha > 0 \), \( \Omega_a \) transforms as
\[
\Omega_a \to \Omega + D_a \ln \alpha \quad \Rightarrow \quad D_a \Omega^a \to D_a \Omega^a + D_a D^a \ln \alpha . \quad (38)
\]
leading to the following Poisson equation for \( \alpha \) on \( S \), if the rescaling is used to eliminate \( D_a \Omega^a \):
\[
D_a D^a \ln \alpha = -D_a \Omega^a . \quad (39)
\]

We have two cases to consider depending on whether \( S \) is smooth or Lipschitz.

\[\text{A. Poisson equation on smooth Riemannian manifold}\]

For smooth \( S \), we have the following existence theorem for the Poisson equation:
Theorem: On a closed Riemannian manifold $M$, if $\rho$ is a smooth function satisfying $\int_M \rho = 0$, there exists a smooth solution to $\Delta \Phi = \rho$, unique up to the addition of a constant.

Since $S$ is a closed manifold, $\int_S D_\alpha \Omega^\alpha = 0$ and therefore we can find an $\alpha$ such that $D_\alpha \Omega^\alpha = 0$ after rescaling.

B. Poisson equation on a Lipschitz manifold

If $S$ is only Lipschitz continuous, we would still like to eliminate $\int_S \theta_+ D_\alpha \Omega^\alpha$. Because the Poisson equation \((39)\) contains second derivatives, it is ill-defined on a Lipschitz manifold. However, one can adapt a weak (i.e. distributional) formulation in the following way. Recall that a Riemannian Lipschitz manifold $(M, q)$ is a manifold $M$ equipped with a positive-definite metric $q$, for which all transition maps are locally Lipschitz functions. By Rademacher’s theorem, a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable almost everywhere (w.r.t to the n-dim. Lebesque measure). We denote the linear space of all Lipschitz continuous functions $\phi : M \to \mathbb{R}$ for which the norm

$$||\phi||^2 := \int_M (\phi^2 + |\nabla \phi|^2) d\mu < \infty \quad (40)$$

is finite by Lip$^{1,2}(M)$. This norm is well-defined because Rademacher’s theorem ensures the existence of the gradient almost everywhere. We then define the Sobolev space $W^{1,2}(M)$ as the Cauchy completion of Lip$^{1,2}(M)$ with respect to the above norm $|| \cdot ||$. If $M$ is a compact, connected, oriented, Lipschitz manifold without boundary, the weak version of the Poisson equation reads

$$- \int_M \langle \nabla \phi, \nabla f \rangle d\mu = \int_M g \phi d\mu \quad , \quad (41)$$

$\forall \phi \in W^{1,2}(M)$, given that $f \in W^{1,2}(M)$ and $g \in L^2(M)$ (see theorem 1.3 in \[42\]). Thus, provided that $\theta_+ \in W^{1,2}(M)$ and $D_\alpha \Omega^\alpha \in L^2(M)$, we can find a function $\alpha$ such that

$$\int_S \theta_+ (D_\alpha \Omega^\alpha + D_\alpha D^\alpha \ln \alpha) dS = 0 \quad . \quad (42)$$

Hence, one way to use the rescaling freedom is to eliminate the term $\int_S \theta_+ D_\alpha \Omega^\alpha dS$ in \((35)\), provided that $\theta_+ \in W^{1,2}(S)$. $D_\alpha \Omega^\alpha$ is in $L^2(M)$ because of $\int_S D_\alpha \Omega^\alpha dS = 0$. If $\theta_+ \notin W^{1,2}(S)$, we can use the rescaling of $l^\alpha$ to achieve $\int_S \theta_+ dS > 0$, but then assumptions on $D_\alpha \Omega^\alpha$ have to be made.

VIII. MONOTONICITY

The crucial difference to the weak lensing case is that there now exists a region of negative $\theta_+$ on $S$ connected to the singular point. This implies that locally, the area decreases along $l^\alpha$ although the total area of $S$ can still increase if $A(S) = \int_S \theta_+ dS > 0$. It is precisely this region in which energy can now be injected into the interior of $l^-(p)$ from the exterior along causal curves. Hence, the naive monotonicity argument related to Fig. 2 holds only for regions with positive $\theta_+$ and fails in regions of negative $\theta_+$. In general, two different effects concerning monotonicity have to be taken into account:

(i) A variation in the area $A$ leads to a change in the energy, because the amount of matter enclosed by $S$ changes. This effect is manifested in the first term in \((35)\), describing nothing other than the change of $A$ along the null generators $l^\alpha$.

(ii) Energy may leave $l^-(p)$ only in regions with $\theta_+ > 0$, and enter $l^-(p)$ only where $\theta_+ < 0$. This is accounted for by the second integral in \((35)\), stating two contributions: shear and matter. The first corresponds to energy transported by the pure gravitational field in the form of gravitational waves, and is even present in vacuum. The latter contribution is due to matter encoded in the energy momentum tensor satisfying the DEC. A potential cosmological constant can be accommodated in the energy momentum tensor.

Assume in the following that $A$ is increasing along the family of surfaces, i.e. $\int_S \theta_+ dS > 0$. In the case of a vacuum spacetime, the matter terms vanish and one only has to deal with the net flux of in- and outgoing shear contributions. Furthermore, by the Goldberg-Sachs theorem \[43\], the geodesic congruence $l^\alpha$ in a vacuum spacetime $M$ is shear free, i.e. $\sigma^+_{ab} = 0$, if and only if $M$ is algebraically special, that is $l^\alpha$ is a repeated principle null direction, see also \[44\] and references therein for generalisations. Demanding vanishing shear within the class of non-vacuum spacetimes imposes a strong constraint, see \[44, 45\].

So far, the studied configuration contained only one strong gravitational lens, see Fig. 1. Nevertheless, the obtained results can easily be generalised to configurations with multiple isolated strong lensing events taking place, that is, the swallow-tail singular points on $S$ have to be isolated. The more lensing events happen, the larger the fraction of $S$ with negative $\theta_+$. Having more and more lensing events present will ultimately turn $\int_S \theta_+ dS$ negative and therefore the whole lightcone will refocus. This indicates that enough energy is concentrated in the interior to cause the shrinking of $S$.

IX. EXTENSIONS

Until now, the discussion was restricted to a family of topological 2-spheres. In the following, we briefly review how a change of topology affects the results. One could imagine that more complicated lensing configurations
may cause $C^{-}(p) \cap \dot{I}^{-}(p)$ to still consist of one connected component, but to be topologically different from $\mathbb{R} \times S^2$. The Hawking energy can be generalised to arbitrary closed orientable surfaces $\dot{S}$, characterised by their genus $g$, by using the Gauss Bonnet theorem $\int_{\dot{S}} R \, d\dot{S} = 8\pi(1 - g)$, see for instance [28]. Then, (2) reads instead

$$E(\dot{S}) := \sqrt{A(\dot{S})} \left( 8\pi(1 - g(\dot{S})) + \int_{\dot{S}} \theta_{+} \, d\dot{S} \right).$$

(43)

However, if $g(\dot{S}) \geq 1$, and the surface is non-trapped on average in the sense of [41], i.e. $\int_{\dot{S}} \theta_{+} \, d\dot{S} < 0$, the Hawking energy is negative.

In principle, it is also possible that $C^{-}(p) \cap \dot{I}^{-}(p)$ splits into $n$ disconnected components of genus $g_i$:

$$C^{-}(p) \cap \dot{I}^{-}(p) = \mathbb{R} \times S_{g_1} \times \cdots \times S_{g_n}.$$  

(44)

This can happen for instance in situations topologically similar to a Schwarzschild black hole (see e.g. [25]). Hayward [28] observed that the Hawking energy for $n$ disconnected surfaces is superadditive. If we denote $S_1 \cup \cdots \cup S_n = S_{\cup}$, then

$$E_{\cup} = \sqrt{\frac{A_1}{A_1}} E_1 + \cdots + \sqrt{\frac{A_n}{A_n}} E_n > E_1 + \cdots + E_n.$$  

(45)

This property is in contrast to the expected subadditivity of gravitational systems.

\section{X. Conclusions}

The Hawking energy provides a reasonable definition of energy in the setting of cosmology. The past lightcone of a point $p$ in spacetime is closely linked to cosmological observations and therefore provides the ideal geometric structure to study the properties of the Hawking energy in a physical set-up. The part of it within the causal boundary, $C^{-}(p) \cap \dot{I}^{-}(p)$, provides the mathematical arena in which the Hawking energy is studied. It is a Lipschitz continuous manifold, potentially containing points where $\theta_+$ is singular, and is assumed to have spherical topology: $C^{-}(p) \cap \dot{I}^{-}(p) \simeq \mathbb{R} \times S^2$. This seems to be the natural case, but it would be interesting to get a better understanding whether, and under what circumstances other topologies may arise.

Assuming that the universe is described by a globally hyperbolic spacetime in which all matter obeys the DEC, strong gravitational fields may cause the lightcone to intersect itself locally at singular points, or globally. Since these singular points are conjugate to $p$, the presence of singularities indicates the existence of regions on $S$ where the expansion $\theta_+$ of the null generators is negative. The only two stable types of singularities appearing in lightcone slices are cusp and swallow-tail singularities [35]. Further restricting such a slice to the causal boundary seems to suggest that only swallow-tail singularities appear in $S_{\lambda} = C^{-}(p) \cap \dot{I}^{-}(p) \cap \Sigma_{\lambda}$. Therefore, two natural regimes arise. The weak lensing regime, in which self-intersections are absent, exhibits a positive expansion parameter $\theta_+ > 0$ everywhere on smooth surfaces $S$. In contrast, in the strong lensing regime, $S$ is only Lipschitz and contains swallow-tail singularities, leading to regions where $\theta_+ < 0$.

In the weak lensing regime, the Hawking energy is positive and monotonously increases along the null generators of the past lightcone, following directly from Eardley’s results [20] and the small sphere limit [26]. Although swallow-tail singularities are present in the strong lensing set-up, the energy and its derivative remain well-defined, however, monotonicity (35) depends upon two effects. Firstly, the area of $S$ changes along the null generators. Secondly, and in contrast to the weak lensing case, matter may enter the interior of $\dot{I}^{-}(p) \cap C^{-}(p)$ through regions where $\theta_+ < 0$. Hence in general, the Hawking energy is not monotinous along the past lightcone anymore and monotonicity depends on the balance of in- and outgoing energy flux.

Of course, the above results equally apply to the future lightcone. Furthermore, since in the above construction the lightcone is distinguished from other null hypersurfaces only by the small sphere limit in the vicinity of the tip of the cone, the above equations also apply to more general null hypersurfaces of the same topology.

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[1] L. B. Szabados, “Quasi-Local Energy-Momentum and Angular Momentum in GR: A Review Article,” Living Rev. Rel. 7, 4 (2004).

[2] S. Hawking, “Gravitational radiation in an expanding universe,” J. Math. Phys. 9, 598–604 (1968).

[3] R. Schoen and S.-T. Yau, “On the Proof of the positive mass conjecture in general relativity,” Commun. Math. Phys. 65, 45–76 (1979).

[4] R. Schoen and S. T. Yau, “Proof of the positive mass theorem. ii,” Comm. Math. Phys. 79, 231–260 (1981).
