Constructing Perfect Steganographic Systems *

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Abstract

We propose steganographic systems for the case when covertex
(tcontainers) are generated by an i.i.d. or a finite-memory distribution, with
known or unknown statistics. The probability distributions
of covertexts with and without hidden information are the same; this
means that the proposed stegosystems are perfectly secure, i.e. an
observer cannot determine whether hidden information is being trans-
mittted. In contrast, existing results only include methods for which
the distributions of covertexts with and without hidden text are close
but not equal.

The speed of transmission of hidden information can be made
arbitrary close to the theoretical limit — the Shannon entropy of the
source of covertexts. All the proposed algorithms are polynomial-time
in all arguments. An interesting feature of our stegosystems is that
they do not require any (secret or public) key. In other words, shared
secret is not required to obtain perfect steganographic security.

Keywords: Steganography, Information Hiding, Information Theory,
Shannon entropy.

1 Introduction

The goal of steganography can be described as follows. Alice and Bob can
exchange messages of a certain kind (called covertexts) over a public channel.
The covertexs can be, for example, a sequence of photographic images,
videos, text emails and so on. Alice wants to pass some secret information
to Bob so that Eve, the observer, cannot notice that any hidden information
is being passed. Thus, Alice should use the covertexs to hide the secret
text. It may be assumed that Alice and Bob share a secret key. A classical

*Some of the results were reported at ISIT’07 [14]
illustration from [16] states the problem in terms of communication in a prison: Alice and Bob are prisoners who want to concoct an escape plan passing each other messages which can be read by a ward.

Perhaps the first information–theoretic approach to steganography was taken by Cachin [1]. In this work the sequence of covertext is modelled by a memoryless finite-alphabet distribution. Besides laying out basic definitions of steganographic protocols and their security, Cachin has constructed a steganographic protocol, which, relying on the fact that the probability distribution of covertexts is known, assures that the distributions of covertexts with and without hidden information are statistically close (but, in general, are not equal). For the case of an unknown distribution, a universal steganographic system was proposed, in which this property holds only asymptotically with the size of the hidden message going to infinity. Distribution-free stegosystems are of particular practical importance, since in reality covertexts can be a sequence of graphical images, instant or email messages, that is, sources for which the distribution is not only unknown but perhaps cannot be reasonably approximated. Cachin has also defined perfectly secure steganographic systems as those for which the probability distribution of covertexts with and without hidden information are the same. It is worth noting that, since C. Shannon’s celebrated paper “Communication theory of secrecy systems” [15], the information–theoretic approach was efficiently applied to many problems of secrecy systems, see e.g. [8] and references therein.

We follow the information–theoretic approach to steganography of [1] and propose constructions of perfectly secure stegosystems. This means that in these stegosystems covertexts with and without hidden information are statistically indistinguishable. In other words, Eve does not get any information on whether hidden text is being transmitted within the covertext sequence. This is an improvement over the universal stegosystem of Cachin in that we replace $\varepsilon$-security in asymptotic by perfect security for any message length. Moreover, we relax theoretical assumptions of [1] by allowing the source of covertexts to have an infinite alphabet and to have a finite memory.

For any stegosystem the next property after its security that is of interest is its capacity. The capacity of a stegosystem can be defined as the number of hidden bits transmitted per letter of covertext. For the case when the covertexts are drawn from a finite alphabet we show that our stegosystems have maximal possible capacity: the number of hidden bits per covertext approaches (with the length of the block growing) the Shannon entropy of the source of covertexts. On the other hand, if the size of the alphabet of
the covertext source and its minentropy tend to infinity then, in the case of a memoryless source of covertexts, the number of bits of hidden text per letter of covertext tends to \( \log(n!)/n \) where \( n \) is the (fixed) size of blocks used for hidden text encoding.

Another feature of our stegosystems is that they do not require a secret key. Thus, the constructions presented demonstrate that in order to achieve perfect steganographic security no secret has to be shared between the communicating parties. Clearly, in this case Eve (the observer) can retrieve the secret message being transmitted; however, she will not be able to say whether it is a secret message or a random noise. This property of our stegosystems (as indeed their secrecy) relies heavily on the fact that the secret message transmitted is indistinguishable from a Bernoulli i.i.d. sequence of bits (random noise). This is a standard assumption that can be easily fulfilled if Alice uses the Vernam cipher (a one-time pad) to encode the secret before transmitting. For this, obviously, a secret key is required. In other words, a secret key can be used to obtain cryptographic security, but it is not required to obtain steganographic security, as long as the hidden information is already indistinguishable from random noise. This also means that the proposed stegosystems can be directly applied for covert open-key cryptographic communication.

The main idea behind the stegosystems we propose is the following. Suppose that for a sequence of covertexts generated by a source, we can find a set of covertexts that had the same probability of being generated as the given one. Moreover, assume that each of these other sequences defines this set uniquely. Then instead of transmitting the sequence actually generated, we can transmit one of the sequences in the set, whose number corresponds to the secret text we want to pass. This does not change the probabilistic characteristics of the source, provided the hidden text consists of i.i.d. equiprobable bits. Therefore, an observer cannot tell whether secret information is being passed. Consider a simple example. Suppose that Alice wants to pass a single bit, and assume that the source of covertexts is i.i.d., but its distribution is unknown. Alice reads two symbols from the source, say \( ab \). She knows that (since the source is i.i.d.) the probability of \( ba \) was the same. So if Alice’s secret bit to pass is 0 she transmits \( ab \) and if she needs to pass 1 then she transmits \( ba \). However, if the source has generated \( aa \) then Alice cannot pass the secret bit, but she has to transmit \( aa \) anyway, to preserve the probabilistic characteristics of the source. (This example is considered in more details in Section 3) The same idea was used by von Neumann [10] in his method of extracting random equiprobable bits from a source of i.i.d. (but not necessarily equiprobable) symbols. Von
Neumann’s method is not optimal in the sense that it does not extract all randomness from the source; in particular, pairs of the type \(aa\) are not used. The steganographic method that we have outlined has the same disadvantage, namely the rate of transmission of hidden symbols is far from maximal. To construct a stegosystem that has a better rate of transmission, one can use the same ideas that were used to generalize von Neumann’s randomness extractor. One way is to iterate von Neumann’s procedure (e.g. use pairs of pairs \(aa...bb\) in the same way as pairs were used). This idea is due to Peres [11]; it can also be used to construct a stegosystem, but we do not consider it here. The stegosystems we propose are constructed based on the ideas similar to those used by Elias [4] for generalizing von Neumann’s randomness extractor. The idea is as follows. For a sequence of symbols of length \(n\) output by an i.i.d. source (with unknown characteristics), all its permutations have the same probability. To pass secret information, Alice transmits the permuted sequence whose number (in the set of all permutations) encodes her message. A stegosystem based on this principle achieves (asymptotically with the block length \(n\) growing) maximal possible rate of transmission of hidden text: the Shannon entropy of the source of covertexts. Moreover, an advantage of this idea is that it can be used beyond i.i.d. sources of covertexts. In particular, we propose a stegosystem for \(k\)-order Markovian sources (for any given \(k\)), also by considering sequences whose probability is the same as that of the given one.

The fact that such a stegosystem has maximal possible rate of transmission is based on results of the so-called Theory of Types [2], widely used in Information Theory. Two sequences are said to be of the same type if the have the same probability of being output (by a given source). The result that is of importance for our construction is the relation of the size of the set of sequences of the same type as a given one to their empirical entropy, and thus to the entropy of the source. It is interesting to observe that a distribution-free stegosystem of Cachin [1] is also based on ideas of the Theory of Types (but this stegosystem is not perfectly secure).

The rest of the paper is organized as follows. In the next section we present the basic definitions. Section 3 provides an example of a simple perfectly secure stegosystem. This stegosystem does not have the maximal capacity, but it demonstrates the ideas used in the stegosystems that we present in the subsequent sections. In Section 4 we present a (perfectly secure) stegosystem for the case of a memoryless source of covertexts, which has the mentioned asymptotic properties of the rates of hidden text transmission, and in Section 5 we briefly describe how it can be algorithmically realized in practice. Section 6 contains an extension of the stegosystem of
Section 4 to the case when the source of covertexts has a (finite) memory. Finally, Section 7 contains a discussion.

2 Notations and definitions

We use the following model for steganography, mainly following [1]. It is assumed that Alice has an access to an oracle which generates covertexts according to some fixed but unknown distribution of covertexts $\mu$. Covertexts belong to some (possibly infinite) alphabet $A$. Alice wants to use this source for transmitting hidden messages. It is assumed that Alice does not know the distribution of covertexts generated by the oracle, but this distribution is either memoryless or has a finite memory; moreover, a bound on the memory of the source of covertexts is known to all the parties (and is used in the stegosystems as a parameter).

A hidden message is a sequence of letters from $B = \{0, 1\}$ generated independently with equal probabilities of 0 and 1. We denote the source of hidden messages by $\omega$. This is a commonly used model for the source of secret messages, since it is assumed that secret messages are encrypted by Alice using a key shared only with Bob. If Alice uses the Vernam cipher (a one-time pad) then the encrypted messages are indeed generated according to the Bernoulli $1/2$ distribution, whereas if Alice uses modern block or stream ciphers the encrypted sequence “looks like” a sequence of random Bernoulli $1/2$ trials. (Here “looks like” means indistinguishable in polynomial time, or that the likeness is confirmed experimentally by statistical data, see, e.g. [9, 13].) The third party, Eve is a passive adversary: Eve is reading all messages passed from Alice to Bob and is trying to determine whether secret messages are being passed in the covertexts or not. Clearly, if covertexts with and without hidden information have the same probability distribution ($\mu$) then it is impossible to distinguish them. Finite groups of (covertext, hidden, secret) letters are sometimes called (covertext, hidden, secret) words. Elements of $A$ ($B$) are usually denoted by $x$ ($y$).

The steganographic protocol can be summarized in the following definition.

**Definition 1** (steganographic protocol). Alice draws a sequence of covertexts $x^* = x_1, x_2, \ldots$ generated by a source of covertexts $\mu$, where $x_i$, $i \in \mathbb{N}$ belong to some (finite or infinite) alphabet $A$.

Alice has a sequence $y^* = y_1, y_2, \ldots$ of secret text generated by a source $\omega$ i.i.d. equiprobable bits $y_i$: $\omega(y_i = 0) = \omega(y_i = 1) = 1/2$, independently for all $i \in \mathbb{N}$. Alice also has access to a private random sequence $\Delta =$
δ₁, δ₂, . . . of i.i.d. equiprobable bits. The sources µ, ω, and ∆ are assumed independent.

A stegosystem St is a pair of functions: the encoder, that maps \( A^n \times \{0, 1\}^\infty \times \{0, 1\}^\infty \) (a block of covertexts, a secret sequence and randomness) to \( A^n \), where \( n \in \mathbb{N} \) is an optional parameter (the block length), whose value is known to all parties. The decoder is a function from \( A^n \) to \( \{0, 1\}^* \).

From \( x^* \), \( y^* \) and \( \Delta \) Alice using a stegosystem St obtains a steganographic sequence \( X = X_1, X_2, \ldots \) that is transmitted over a public channel to Bob.

Bob (and any possible Eve) receives \( X \) and obtains using the decoder \( St^{-1}\(X\) the resulting sequence \( y^* \).

For convenience of notation, the definition is presented in terms of an infinite sequence of secret text (and random source). It means that a stegosystem can use as many or as few bits of the hidden text for transmission in a given block as is needed. In practice, of course, Alice has only a finite sequence to pass, which may result in that she will run out of secret bits when transmitting the last block of covertexts. In this case we assume that the end of each message can always be determined (e.g. there is always an encrypted “end of message” sign in the end), so that Alice can fill up the remainder by random noise.

Observe that we require by definition of a steganographic system that the decoding is always correct. Moreover, we do not consider noisy channels or active adversaries, so that Bob always receives what Alice has transmitted.

Note also that there is no secret key in the protocol. A secret key may or may not be used before entering into steganographic communication in order to obtain the hidden sequence \( x^* \); however, this is out of scope of the protocol.

**Definition 2** (perfect security). A steganographic system is called (perfectly) secure if the sequence of covertexts \( x^* \) and the steganographic sequence \( X \) have the same distribution: \( Pr(x_1, \ldots, x_n \in C) = Pr(X_1, \ldots, X_n \in C) \)

for any (measurable) \( C \subset A^n \) and any \( n \in \mathbb{N} \), where the probability is taken with respect to all distributions involved: \( \mu, \omega \) and \( \Delta \).

### 3 Example: a simple perfectly secure stegosystem

Next we present a simple stegosystem that demonstrates the main ideas used in the general stegosystem, which we develop in the next section. The stegosystem described in this section does not use randomization.
Consider a situation in which not only the secret letters are drawn (using \( \omega \)) from a binary alphabet, but also the source of covertexts \( \mu \) generates i.i.d. symbols from the alphabet \( A = \{a, b\} \) (not necessarily with equal probabilities). Suppose that Alice has to transmit the sequence \( y^* = y_1 y_2 \ldots \) generated according to \( \omega \) and let there be given a covertext sequence \( x^* = x_1 x_2 \ldots \) generated by \( \mu \). For example, let

\[
y^* = 01100 \ldots, \quad x^* = aababbaaabaaaabb \ldots.
\]

The sequences \( x^* \) and \( y^* \) are encoded in a new sequence \( X \) (to be transmitted to Bob) such that \( y^* \) is uniquely determined by \( X \) and the distribution of \( X \) is the same as the distribution of \( x^* \) (that is, \( \mu \); in other words, \( X \) and \( x^* \) are statistically indistinguishable).

The encoding is carried out in two steps. First let us group all symbols of \( x^* \) into pairs, and denote

\[
aa = u, \quad bb = u, \quad ab = v_0, \quad ba = v_1.
\]

In our example, the sequence \( x^* \) is represented as

\[
x^* = aababbaaabaaaabb \ldots = u v_1 v_1 v_0 v_1 u u u \ldots
\]

Then \( X \) is acquired from \( x^* \) as follows: all pairs corresponding to \( u \) are left unchanged, while all pairs corresponding to \( v_k \) are transformed to pairs corresponding to \( v_{y_1} v_{y_2} v_{y_3} \ldots \); in our example

\[
X = aa abba aa ba ab aa aa bb \ldots
\]

Decoding is obvious: Bob groups the symbols of \( X \) into pairs, ignores all occurrences of \( aa \) and \( bb \) and changes \( ab \) to 0 and \( ba \) to 1.

The properties of the described stegosystem, which we call \( St_0^2 \), are summarized in the following (nearly obvious) statement.

**Proposition 1.** Suppose that a source \( \mu \) generates i.i.d. random variables taking values in \( A = \{a, b\} \) and let this source be used for encoding secret messages consisting of a sequence of i.i.d. equiprobable binary symbols using the method \( St_0^2 \). Then the sequence of symbols output by the stegosystem obeys the same distribution \( \mu \) as the input sequence.

The proof of this statement is simple, and we omit it since in the next section a stegosystem is presented that has stronger properties. It is also easy to see that the same method can be used when the alphabet \( A \) is not binary. Indeed, everything that we need to construct \( St_0^2(A) \) is that there is some (partial) ordering on the set \( A \). Then Alice can use each consecutive pair \( a_1 a_2 \) such that either \( a_1 < a_2 \) or \( a_2 < a_1 \) to transmit one bit of the secret text.
4 General construction of a universal stegosystem for i.i.d. sources

In this section we consider the general construction of universal stegosystem which has the desired asymptotic properties. As before, Alice needs to transmit a sequence $y^* = y_1 y_2 \ldots$ of secret binary messages drawn by an i.i.d. source $\omega$ with equal probabilities of 0 and 1, and let there be given a sequence of covertexts $x^* = x_1 x_2 \ldots$ drawn i.i.d. by a source $\mu$ from an alphabet $A$. First we break the sequence $x^*$ into blocks of $n$ symbols each, where $n > 1$ is a parameter. Each block will be used to transmit several symbols from $y^*$ (for example, in the previously constructed stegosystem $St_0^2(A)$ each block of length 2 was used to transmit 1 or 0 symbols). However, in the general case a problem arises which was not present in the construction of $St_0^2(A)$. Namely, we have to align the lengths of the blocks of symbols from $x^*$ and from $y^*$, and for this we will need randomization. The problem is that the probabilities of blocks from $y^*$ are divisible by powers of 2, which is not necessarily the case with blocks from $x^*$.

We now present a formal description. Let $u$ denote the first $n$ symbols of $x^*$: $u = x_1 \ldots x_n$, and let $\nu_u(a)$ be the number of occurrences of the symbol $a$ in $u$. Define the set $S_u$ as consisting of all words of length $n$ in which the frequency of each letter $a \in A$ is the same as in $u$: $S_u = \{v \in A^n : \forall a \in A \nu_v(a) = \nu_u(a)\}$.

Observe that the $\mu$-probabilities of all members of $S_u$ are equal. Let there be given some ordering on the set $S_u$ (for example, lexicographical) which is known to both Alice and Bob (and to anyone else) and let $S_u = \{s_0, s_1, \ldots, s_{|S_u| - 1}\}$ with this ordering.

Denote $m = \lfloor \log_2 |S_u| \rfloor$, where $|y|$ stands for the largest integer not greater than $y$. Consider the binary expansion of $|S_u|$:

$$|S_u| = (\alpha_m, \alpha_{m-1}, \ldots, \alpha_0),$$

where $\alpha_m = 1$, $\alpha_j \in \{0, 1\}$, $m > j \geq 0$. In other words,

$$|S_u| = 2^m + \alpha_{m-1}2^{m-1} + \alpha_{m-2}2^{m-2} + \ldots + \alpha_0.$$  

Define a random variable $\Delta$ as taking each value $i \in \{0, 1, \ldots, m\}$ with probability $\alpha_i 2^i / |S_u| :$

$$p(\Delta = i) = \alpha_i 2^i / |S_u|.$$  

Alice, having read $u$, generates a value of the random variable $\Delta$, say $d$, and then reads $d$ symbols from $y^*$. Consider the word $r^*$ represented by these
symbols as an integer which we denote by \( r \). Then we encode the word \( r^* \) (that is, \( d \) bits of \( y^* \)) by the word \( s_\tau \) from the set \( S_u \), where

\[
\tau = \sum_{l=d+1}^{m} \alpha_l 2^l + r. \tag{4}
\]

(In other words, the word \( s_\tau \) is being output by the coder.)

Then Alice reads the next \( n \)-bit word, and so on. Denote the constructed stegosystem by \( St_0^n(A) \).

To decode the received sequence Bob breaks it into blocks of length \( n \) and repeats all the steps in the reversed order: by the current word \( u \) he obtains \( S_u \) and \( \tau \). To obtain \( d \) Bob finds the largest number \( d' \in \{0, \ldots, m\} \) such that \( \alpha_{d'} \neq 0 \) and \( \tau < \sum_{l=d}^{m} \alpha_l 2^l \). Then he proceeds to finding \( r \) and \( r^* \); that is, he finds \( |r^*| \) next symbols of the secret sequence \( y^* \).

Consider an example which illustrates all the steps of the calculation. Let \( A = \{a, b, c\} \), \( n = 3 \), \( u = bac \). Then \( S_u = \{abc, acb, bac, bca, cab, cba\} \), \( |S_u| = 6 \), \( m = 2 \), \( \alpha_2 = 1 \), \( \alpha_1 = 1 \), \( \alpha_0 = 0 \). Let the sequence of secret messages be \( y^* = 0110 \ldots \). Suppose the value of \( \Delta \) generated by Alice is 1. Then she reads one symbol of \( y^* \) (in this case 0) and calculates \( r = 0 \), \( r^* = 0 \), \( \tau = 2^2 + 0 = 4 \) and finds the codeblock \( s_4 = cab \). To decode the message, Bob from the block \( cab \) calculates \( \tau = 4 \), \( r = 0 \), \( r^* = 0 \) and finds the next symbol of the secret sequence — 0.

It is clear from the construction that encoding and decoding is unambiguous and the decoding is always correct. The following theorem establishes perfect security of the obtained stegosystem, and its rate of transmission of hidden text.

**Theorem 1.** Suppose that an unknown source \( \mu \) generates i.i.d. random variables taking values in some alphabet \( A \). Let this source be used for encoding secret messages consisting of a sequence of i.i.d. equiprobable binary symbols using the described method \( ST_0^n(A) \) with \( n > 1 \). Then

(i) \( ST_0^n(A) \) is perfectly secure: the sequence of symbols output by the stegosystem obeys the same distribution \( \mu \) as the input sequence,

(ii) the average number of secret symbols per covertext \( (L_n) \) satisfies the following inequality

\[
L_n \geq \frac{1}{n} \left( \sum_{u \in A^n} \mu(u) \log \frac{n!}{\prod_{a \in A} \nu_u(a)!} - 3 \right), \tag{5}
\]

where \( \mu(u) \) is the \( \mu \)-probability of the word \( u \) and \( \nu_u(a) \) is the number of occurrences of the letter \( a \) in the word \( u \).
Proof. For for any covertext word $u$ we have $\mu(u) = 1/|S_u|$. Therefore, to prove the first statement it is sufficient to show that the probability of occurrence of each $u$ in the output sequence is also $1/|S_u|$. Consider the first block to be transmitted. Let $y_1, y_2, \ldots$ be the hidden text to transmit, and let $r_d$ be the integer represented by $y_1, \ldots, y_d$, that is $r_d = \sum_{k=d}^{1} y_k 2^k$. Using the notation $S_u = \{s_\tau : \tau = 1, \ldots, m\}$ we find that for each $\tau$ the probability of $s_\tau$ in the transmitted steganographic sequence is

$$P(\tau) = \sum_{k=0}^{m} P(\tau|\Delta = k)P(\Delta = k) = \sum_{k=0}^{m} P(\tau|\Delta = k) \frac{\alpha_k 2^k}{|S_u|} = \sum_{k=0}^{m} P(\tau - \sum_{l=k+1}^{m} \alpha_l 2^l = r_k|\Delta = k) \frac{\alpha_k 2^k}{|S_u|} = \sum_{k=0}^{m} 2^{-k} \frac{\alpha_k 2^k}{|S_u|} = \frac{1}{|S_u|}.$$ 

where the first and the last equalities are trivial, the second follows from the definition of $\Delta$ (3), the third from (4) and the fourth from the fact that $y_i$ are independent and equiprobable random bits.

The second statement can be obtained by direct calculation of the average number of symbols from $y^*$ encoded by one block. Indeed, from (3) we find that for each covertext word $u$ the expected number of transmitted symbols $L_n$ is

$$L_n = \sum_{k=1}^{m} \alpha_k \frac{k}{|S_u|} 2^k \geq \frac{1}{|S_u|} \left( m|S_u| - \sum_{k=1}^{m} |S_u|k2^{-k} \right).$$ 

Having taken into account the identity $\sum_{k=0}^{\infty} k/2^k = 2$ and the definition $m = \lfloor \log|S_u| \rfloor$ we get $L_n \geq m - 2 \geq \log |S_u| - 3$. It remains to notice that $|S_u| = \prod_{a \in A} \nu_a(a)!$ for each word $u$. 

Let us now consider the asymptotic behaviour of $L_n$ when $n \to \infty$. The following statement establishes (in asymptotic) the maximum possible rate of transmission of hidden text of the stegosystem $S_{l_n}(A)$.

**Corollary 1.** If the alphabet $A$ is finite then the average number of hidden symbols per letter $L_n$ goes to the Shannon entropy $h(\mu)$ of the source $\mu$ as the block length $n$ goes to infinity; here by definition $h(\mu) = -\sum_{a \in A} \mu(a) \log \mu(a)$. 

Proof. The statement follows from (5) and the law of large numbers. Indeed, it is a well-known fact of Information Theory (e.g. [5]) that as $n \to \infty$ with probability 1 we have $\log |S_u|/n \to h(\mu)$.
In many real stegosystems the alphabet $A$ is huge (it can consist, for example, of all possible digital photographs of given file format, or of all possible e-mail messages). In such a case it is interesting to consider the asymptotic behaviour of $L_n$ with fixed $n$ when the alphabet size $|A|$ goes to infinity. For this we need to define the so-called min-entropy of the source $\mu$:

$$H_\infty(\mu) = \min_{a \in A} \{- \log \mu(a)\}.$$  

(6)

Corollary 2. Assume the conditions of Theorem 1 and fix the block length $n > 1$. If $|A| \rightarrow \infty$ and $H_\infty(\mu) \rightarrow \infty$ then $L_n$ tends to $(\log(n!) - O(1))/n$.

Proof. From $H_\infty(\mu) \rightarrow \infty$ it follows that $\max_{a \in A} \mu(a) \rightarrow 0$. Therefore the probability that all letters in a block are different goes to 1, so that the bound in (5) approaches $(\log(n!) - 3)/n$.

5 Complexity of encoding and decoding

Consider the resource complexity of the stegosystem $St_n^0(A)$. The only algorithmically non-trivial part of this stegosystem is in finding the rank of a given block $u$ in the set $S_u$ of all its permutations, and, vice versa, finding a block given its rank. (It is clear that all other operations can be performed in linear time.)

Consider this computational problem in some detail. To store all possible words from the set $S_u$ would require memory of order $|A'|^n n \log |A'|$ bits, (where $A' \subset A$ is the set of all symbols that occur in $u$ and $n = |u|$; without loss of generality in the sequel we assume $A = A'$), which is practically unacceptable for large $n$. However, there are algorithms for solving this problem with polynomial resource complexity. The first such algorithm, that uses polynomial memory with the time of calculation $cn^2$, $c > 0$, per letter, was proposed in [7] (see also [3]). The time of calculation of the fastest known algorithm is $O(\log^3 n)$, see [12].

Next we briefly present the ideas behind the algorithm from [7]. Assume the alphabet $A$ is binary. Let $S$ be a set of $n$-length binary words with $w$ 1’s.

The main observation is the following formula which gives a lexicographical number of any word $v = x_1 \ldots x_n \in S$:

$$\text{rank}(x_1 \ldots x_n) = \sum_{k=1}^{n} x_k \left( \frac{n-k}{w - \sum_{i=1}^{k-1} x_i} \right),$$

(7)

where $\binom{t}{m} = t!/(m!(t-m)!)$, $0! = 1$, $\binom{t}{m} = 0$, if $t < m$. The proof of this well-known equality can be found, for example, in [6] [12]. As an
example, for $n = 4, w = 2, v = 1010$ we have

$$\text{rank}(1010) = \binom{3}{2} + \binom{1}{1} = 4.$$  

The computation by (7) can be performed step by step based on the following obvious identities:

$$\binom{t}{p} = \binom{t-1}{p-1} \cdot \frac{t}{p}, \quad \binom{t}{p} = \binom{t-1}{p} \cdot \frac{t}{t-p}.$$  

A direct estimation of the number of multiplications and divisions gives a polynomial time of calculations by (7). The method of finding the word $v$ based on its rank and generalization for non-binary alphabet are based on the same equality (7); a detailed analysis can be found in [6, 12].

6 A stegosystem for $k$-order Markov sources of covertexts

In this section we describe a stegosystem, which is an extension of the stegosystem described in the Section 4 to the case of $k$-order Markov sources, where $k > 0$. The main idea is the same; first, the given sequence of covertexts is divided into blocks, say, of length $n > 2k$. For each block $x = (x_1, \ldots, x_n)$, Alice finds all sequences of covertexts of lengths $n$ that have the same probability as $x$ and also have the same $k$ leading and $k$ trailing symbols (the latter has to be done so that the probability of the sequence of blocks as a whole is intact). Then Alice enumerates all these sequences, and transmits the one whose number codes her hidden text. As before, to find the sequences that have the same probability as the given one, this probability itself does not have to be known. In fact, words that have the same number of occurrence of all subwords of length $k+1$ have the same probability, for any $k$-order Markov distribution.

We now present a formal description.

Definition 3. A source (of covertexts) $\mu$ is called (stationary) $k$-order Markov, if

$$\mu(x_{n+1} = a|x_n = a_n, x_{n-1} = a_{n-1}, \ldots, x_1 = a_1) = \mu(x_{k+1} = a|x_k = a_n, x_{n-1} = a_{n-1}, \ldots, x_1 = a_{n-k+1})$$

for all $n \in \mathbb{N}$ and all $a, a_1, a_2, \ldots, a_n \in A$.  

Alice is given a sequence of covertexts \(x^* = x_1, x_2, \ldots\) from an alphabet \(A\) generated by a \(k\)-Markov source \(\mu\), where \(k \geq 0\) is given. As in the i.i.d. case, the stegosystem depends on a parameter \(n\) — the block length, which we require to be greater than \(2k\).

Let \(u\) denote the first \(n\) symbols of \(x^*: u = x_1 \ldots x_n\) (the first block), let \(\nu_u(a_1, \ldots, a_{k+1})\) be the number of occurrences of the subword \(a_1, \ldots, a_{k+1}\) in \(u\). Define the set \(S_u\) as consisting of all words of length \(n\) in which the frequency of each subword of length \(k+1\) is the same as in \(u\), and for whose first and last \(k\) symbols are the same as in \(u\):

\[
S_u = \left\{ v \in A^n : \forall s \in A^{k+1} \nu_v(s) = \nu_u(s); \forall t \in \{1, \ldots, k, n - k + 1, \ldots, n\} v_t = u_t \right\}. \tag{8}
\]

With the set \(S_u\) so defined, the rest of the definition is the same as the definition of the stegosystem \(St_0^n(A)\): in order to encode her secret text, Alice enumerates \(S_u\), and finds the element whose number corresponds to the secret text; the length of the block of hidden text is defined using an auxiliary random variable \(\Delta\), just as before. For the second and subsequent blocks the encoding is analogous. The decoding is also analogous. Denote the described stegosystem \(St_k^n(A)\).

The \(k\)-order (conditional) Shannon entropy \(h_m(\mu)\) of a source \(\mu\) is defined as follows:

\[
h_m(\mu) = \sum_{v \in A^m} \mu(v) \sum_{a \in A} \mu(a|v) \log \mu(a|v). \tag{9}
\]

**Theorem 2.** Suppose that an unknown \(k\)-order Markov source \(\mu\) generates a sequence of covertext taking values in some alphabet \(A\), where \(k \geq 0\) is known. Let this source be used for encoding secret messages consisting of a sequence of i.i.d. equiprobable binary symbols using the described method \(St_k^n(A)\) with \(n > 1\). Then

(i) the sequence of symbols output by the stegosystem obeys the same distribution \(\mu\) as the input sequence,

(ii) If the alphabet \(A\) is finite then the average number of hidden symbols per letter \(L_n\) goes to the \(k\)-order Shannon entropy \(h_k(\mu)\) of the source \(\mu\) as \(n\) goes to infinity.

**Proof.** To prove (i) observe that if, as before, \(x_1^*, x_2^* \ldots\) denotes the sequence generated by the source of covertexts, and \(X_1, X_2, \ldots\) the transmitted sequence, then by construction (cf. the proof of Theorem [1]) we have
\[ P(X_1, \ldots, X_n) = P(x_1^*, \ldots, x_n^*) \] where \( n \) is the length of the block. For the second block we have

\[
P(X_{n+1}, \ldots, X_{2n} | X_1, \ldots, X_n) = P(X_{n+1}, \ldots, X_{2n} | X_{n-k+1}, \ldots, X_n) = P(x_{n+1}^*, \ldots, x_{2n}^* | x_{n-k+1}^*, \ldots, x_n^*),
\]

where the first equality follows from the \( k \)-Markov property, the second is by construction (the last \( k \) symbols of each block are kept intact), and the last one (as before) holds because the hidden texts are equiprobable, as are the elements of \( S_u \). The same holds for all the following blocks, thereby establishing the equality of distributions (i).

Let \( S'_u \) be the set of all strings of length \( n = |u| \) that have the same \( k \)-type as \( u \), that is, the same frequencies of subwords of length \( k \): \( S'_u = \{ v \in A^n : \forall s \in A^{k+1} \nu_v(s) = \nu_u(s) \} \). In other words, \( S'_u \) is the same as \( S_u \) except the \( k \) first and last symbols are not fixed. Using a result of the theory of types \([2]\), for any \( u \) for the size of the set \( S'_u \) we have

\[
\log |S'_u| = nh_k(P_u) + o(n),
\]

where \( h_k(P_u) \) is the \( k \)-th order entropy of the \( k \)-order Markov distribution \( P_u \) defined by the empirical frequencies of the word \( u \). Since the set \( S_u \) is not more than a constant times smaller than \( S'_u \) we also have \( \log |S_u| = nh_k(P_u) + o(n) \). Moreover, the law of large numbers implies that \( h_k(u) \rightarrow h_k(\mu) \) for \( \mu \)-almost every sequence \( u \) as its size \( n \) goes to infinity. Therefore, \( \log |S_u| = nh_k(\mu) + o(n) \) with \( \mu \)-probability 1. It remains to observe that the expected number of hidden letters per block is lower bounded by \( \log |S_u| - 3 \).

As was the case with \( Str_n^0(A) \), the only algorithmically nontrivial part of the stegosystem \( Str_n^k(A) \) is enumeration of all the sequences of the same type as a given one. Although it is clear that a polynomial-time algorithm for this problem can be found, based on the same ideas as outlined in Section 5, we cannot point to any source describing such an algorithm.

7 Discussion

We have proposed two stegosystems (with and without randomization) for which the output sequence of covertexes with hidden information is statistically indistinguishable from a sequence of covertexes without hidden information. The proposed stegosystems are based on the assumption that the source of covertexes has either zero (Sections 3 and 4) or finite (Section 6) memory. Even the former assumption is reasonable if we want to embed a secret message into a sequence of, say, graphical images, videos or texts of...
a certain kind. If, for example, we want to use just one image to transmit (a large portion of) a secret text then our covertexts are parts of the image, which are clearly not i.i.d. How to extend the ideas developed in this work to the case of non-i.i.d. covertexts is perhaps the main open question.

However, as it was mentioned in the introduction, the main ideas behind our stegosystems are not limited to the case of finite-memory sources. What we required is that for any given block of covertexts one can find a set of other blocks of the same probability, and such that each of these other blocks determines this set uniquely. Sequences of the same probability are called sequences of the same type. Sets of sequences of the same type can be identified for some distributions other than those with finite memory, for example for renewal processes (see [2] and references therein). Moreover, if there is one such set whose probability is close to 1 (as is the case, for a large block size, for finite-memory sources, or for aforementioned renewal processes) then the rate of transmission of hidden text will be close to the entropy of the source. It is challenging to find (high-probability) sets of equiprobable covertexts that would be relevant for those steganographic applications in which only a single object (such as a video or an image) is available for embedding the secret message.

Another matter of interest from the algorithmic point of view is randomization. The stegosystems proposed in Sections 4 and 6 use randomization. The first observation here is that if, instead of using random numbers, Alice uses pseudorandom numbers which are indistinguishable from truly random in polynomial time, then it is easy to see that the same hardness property carries over to the security of the stegosystem: the distribution of covertexts with and without secret information will be indistinguishable in polynomial. In other words, the security of the stegosystem will be at least as good as that of the random number generator involved. Besides, the same concerns the assumption on the secret text. Namely, we have assumed that it is a sequence of independent observations of tosses of an unbiased coin. Without this assumption, we can say that, in general, it will be as hard to distinguish the sequence of covertexts with hidden information from that without, as it is to distinguish the secret text from i.i.d. unbiased coin tosses. Another observation is that perfectly secure stegosystems that do not use randomization can be constructed. For example, the stegosystem $St_2(A)$ of Section 3 does not use randomization. The stegosystems of Sections 4 and 6 can be made non-randomized, sacrificing (at most) half of the rate of secret text transmission. Indeed, the need for randomization in these stegosystems arises from the fact that the size of the sets $S_u$ may not be a power of 2. A non-randomized version of the stegosystem can be obtained as follows. Alice
uses only a subset of $|S_u|$ that has as its size the largest power of 2 smaller than $|S_u|$, to encode the secret message; this is done exactly as in $St_n^k$ for the case $\Delta = m$. In case the actual block of covertexts observed does not belong to this set, she transmits this block unchanged, without coding any secret message. The rate of transmission of secret text for this non-randomized stegosystem is asymptotically lower bounded by $h(\mu)/2$ (instead of $h(\mu)$). However, such a stegosystem does not use random numbers.

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