Calculation of the dispersion curves of a functionally graded hollow cylinder

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Abstract. The method based on a Peano expansion of the matricant is proposed to calculate the dispersion curves of the radially graded hollow cylinder. The state vector and corresponding system matrix are found for isotropic and radially graded hollow cylinders. The effectiveness of the inhomogeneous model and matricant method is demonstrated on the dispersion spectra for the homogeneous hollow cylinder and rod. High sensitivity of SAW dispersion curves to various power law gradient profiles are obtained on numerical results for both hard and soft inhomogeneous coatings.

1. Introduction

In a radially graded cylinder, the composition change continuously along the radial direction, resulting in corresponding changes of material properties. Radial inhomogeneity may occur naturally as in wood, or else due to damage (chemically attacked concrete pile) or manufacturing techniques (heat treating, welding, wintering) which create continuous transition zone. Mainly two numerical methods have been introduced to predict the dispersion behaviors: a hybrid method of combining finite element and Fourier transformation methods [1] and a Legendre polynomial expansion method [2]. The two approaches lead to seeking an exact solution of an actually modified problem and add some questions of accuracy and validity of the results in numerical computing. The Peano expansion method [3] of keeping the continuity and property variation of the authentic problem has been demonstrated to be an exact solution for a graded plate [4, 5]. The objective of this paper is to apply this method on functionally graded cylindrical structures. Numerical results on dispersion curves of the SAW are presented to show the influence of various gradient profiles of hard and soft inhomogeneous coatings.

2. Governing equations

Consider the acoustic wave propagation in an inhomogeneous and isotropic hollow cylinder of infinite length, with external and internal radii $a$ and $b$. The density $\rho$ and two Lamé coefficients $\lambda$ and $\mu$ vary along the radial direction. The displacement $U$ and the stress vector $\sigma$ are sought in the form:

$$U(r, \theta, t) = u(r) \exp [j(\omega t - \nu \theta)],$$

$$\Sigma(r, \theta, t) = \sigma(r) \exp [j(\omega t - \nu \theta)].$$

(1)

$\nu (\nu=k_0a)$ being the horizontal wave number, $\omega$ the angular frequency. Generally no explicit analytical

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solution exists for the corresponding wave equation of a second order differential equation with varying coefficients. The state vector approach [3] is applied to obtain the matrix differential equation:

\[ d\mathbf{X}(r)/dr = \mathbf{A}(r)\mathbf{X}(r) \]  

(2)

where state vector \( \mathbf{X} = (u_r, u_\theta, \sigma_{rr}, \sigma_{r\theta})^T \) (\(i=r, \theta, z\)) is the displacement–traction and \( \mathbf{A} \) is the system matrix depending on the elastic properties of the inhomogeneous medium, and on \( v \) and \( \omega \). For an isotropic and radially graded cylinder, the state vector and the system matrix for P-SV coupled waves are:

\[ \mathbf{X}(r) = (u_r, u_\theta, \sigma_{rr}, \sigma_{r\theta})^T \]  

(3)

and

\[ \mathbf{A}(r) = \begin{pmatrix}
-r^{-1}\lambda(r)\left[\lambda(r) + 2\mu(r)\right]^{-1} & jvr^{-1}\lambda(r)\left[\lambda(r) + 2\mu(r)\right]^{-1} & \lambda(r) + 2\mu(r)\left[\lambda(r) + 2\mu(r)\right]^{-1} & 0 \\
-j\mu(r)\omega^2 & -jv\xi(r)/r^2 & -r^{-1}\lambda(r)\left[\lambda(r) + 2\mu(r)\right]^{-1} & jvr/r \\
-jv\xi(r)/r^2 & v^2\xi(r)/r^2 - \rho(r)\omega^2 & jvr^{-1}\lambda(r)\left[\lambda(r) + 2\mu(r)\right]^{-1} & -2/r
\end{pmatrix} \]  

(4)

with

\[ \xi(r) = 4\mu(r)\lambda(r) + \mu(r) \]  

(5)

3. Solution and dispersion equation

The matricant \( \mathbf{M}(r, r_0) \) is the fundamental solution of Eq. (2) with the form:

\[ \mathbf{X}(r) = \mathbf{M}(r, r_0)\mathbf{X}(r_0) \]  

(6)

where \( r_0 \) is a reference point on the \( r \)-axis. The matricant can be calculated by the Peano expansion:

\[ \mathbf{M}(r, r_0) = \mathbf{I} + \int_{r_0}^r \mathbf{A}(\zeta_1) d\zeta_1 + \int_{r_0}^r \mathbf{A}(\zeta_1) \mathbf{A}(\zeta_2) d\zeta_2 d\zeta_1 + ... \]  

(7)

The explicit form of the system matrix \( \mathbf{A} \) allows a convenient factorization by \( \omega \). In Eq. (4), \( \mathbf{A} \) can be rewritten as a series in \( \omega \) with coefficients depending only on \( r \) (through the upper integration limit) and \( v \). This factorization can lead to a matrix polynomial form of the matricant.

Let the cylindrical faces be subjected to the traction-free boundary condition:

\[ \sigma_{rr}|_{r=b} = \sigma_{r\theta}|_{r=a} = 0 \quad (i=r, \theta) \]  

(8)

Combining Eq. (8) with Eq. (6) leads to the P-SV wave dispersion equation in the form:

\[ \det \mathbf{M}_3(a, b) = 0 \]  

(9)

where \( \mathbf{M}_3 \) is the left off-diagonal block of \( \mathbf{M} \). The left hand side of Eq (9) can be arranged as a polynomial, whose zeros are the eigenfrequencies for a prescribed value of \( v \). Thus the obtained set of pairs \( (\omega, v) \) describe the dispersion curves. The dispersion for a graded coating on a rod can be further deduced by putting into Eq. (6) the analytical state vector solution for the inner rod of radius \( b \). A desirable accuracy can be reached by choosing an appropriate way of calculating the Peano expansion.

4. Numerical results

To demonstrate the effectiveness of the Peano expansion method, the dispersion spectrum has been calculated for aluminum hollow cylinders. The density, and the longitudinal and shear wave velocities, are chosen to be \( \rho = 2700 \text{ kg/m}^3 \), \( V_L = 6400 \text{ m/s} \) and \( V_S = 3110 \text{ m/s} \) through this paper.

The dispersion curves of various P-SV wave modes are clearly observed in figures 1(a) and 1(b). More modes appear in the same frequency and wave number window when the thickness increases, i.e. \( b/a \) the ratio of inner radius \( b \) over outer radius \( a \) decreases. The first mode \( (A0) \) behaves exactly like the first anti-symmetric mode of a plate when the outer radius \( a \) turns to infinite. Notations \( S0, A1, S1, ... \) for a homogeneous plate are used to denote the high order P-SV waves. But they behave differently to those for a homogeneous plate due to the presence of the cylindrical curvature. These dispersion curves could be also obtained on a single homogeneous layered model [6].
The dispersion spectrum is further calculated for an aluminum rod. The cylinder is composed of two parts: a graded tube of 1mm thickness and a homogeneous rod of 8 mm diameter. The theoretical solution of the inner rod is found on authors’ previous model [7]. The spectrum is then compared to the dispersion curves by taking the two parts as a single homogeneous rod of 10 mm diameter. The dispersion behaviors of various P-SV wave modes are clearly observable in figure 2. The maximum in the spectrum is in good coincidence to the corresponding dashed and white lines of the dispersion curves calculated by the analytical model on a single homogeneous rod. They are Rayleigh (R) [8] and various orders of Whishpering Gallery (W1, W2,….) waves [9]. The agreement implies the capability of both models on predicting the dispersive behavior.

To effectively demonstrate the inhomogeneous model and its theoretical solution, the influence of various gradient profiles on the surface acoustic wave (SAW) behavior has been evaluated. Figure 3 shows various elastic property profiles for a 1mm thick coating on an 8mm-diameter aluminum rod. The gradient profile of the coating is assumed to have the power law profile as

\[ \Delta E / E = \pm 0.2 \left[ \left( r - b \right) / \left( a - b \right) \right]^n \]  

(10)

Here \( n=0 \) stands for a homogeneous coating, and \( n>0 \) for a graded coating. 20% increase or decrease of the elastic modulus are supposed for hard coating and soft coating respectively. For the comparison, the SAW dispersion curve is also obtained for a 10mm diameter aluminum rod without...
any coating. The corresponding profile is a straight line with 0% variation of elastic modulus.

Figure 4 displays the SAW dispersion curves for several coated cylinders with profiles displayed in figure 3. Shown in figure 4(a) for hard coatings, there is a clear downward shift to the homogeneous rod without any coating for frequency greater than 1 MHz as the power law exponent \( n \) increases. This is partly caused by the decrease of the mean stiffness of the coating. The large shift in higher frequency between the homogeneous coating \((n=0)\) and no coating is due to the penetration depth of SAW in the order of its wave length. The reverse order of the shift in the low frequency range (<1 MHz) is possibly due to the curvature of the cylindrical surface [8]. On the other hand, there is a clear upward shift in figure 4(b) for soft coatings instead of a downward shift in figure 4(a) for hard coatings as number \( n \) increases. This is also due to the same reasons for the hard coating. The almost opposite effect further emphasizes the effectiveness of the theoretical solution for the graded cylinder.

5. Conclusion

The Peano expansion method is applied on graded cylinders. The corresponding system matrix and the matricant solution are found for predicting the dispersion behaviour. Dispersion spectra are calculated for two aluminium hollow cylinders with different thickness, and high sensitivity to ratio of inner over outer radii is obtained. Dispersion spectrum for a homogeneous aluminium rod is calculated in agreement with that of authors’ previous model. Numerical SAW dispersion curves are observed with high sensitivity to various gradient profiles of hard and soft inhomogeneous coatings.

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