Decomposition theorems for a generalization of the holonomy Lie algebra of an arrangement

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Abstract

In [7] it is proved that the holonomy Lie algebra of an arrangement of hyperplanes through origo decomposes as a direct product of Lie algebras in degree at least two if and only if a certain (computable) condition is fulfilled. We prove similar results for a class of Lie algebras which is a generalization of the holonomy Lie algebras. The proof methods are the same as in [7].

1 Introduction

The holonomy Lie algebra of a projective hyperplane arrangement in $\mathbb{C}^l$ is defined by generators and relations as follows (see [2]). Let $x_1, \ldots, x_n$ be the names of $n$ hyperplanes which serve as generators of a free (ordinary, i.e. non-super) Lie algebra over the complex numbers. For all maximal subsets $\{y_1, \ldots, y_k\}$ of the hyperplanes whose intersection has codimension two, there are relations $[y_i, \sum_{j=1}^k y_j]$ for $i = 1, \ldots, k$. The Koszul dual (see [3]) of the enveloping algebra of the holonomy Lie algebra has the following description. It is the exterior algebra on $e_1, \ldots, e_n$ dual to the generators $x_1, \ldots, x_n$, modulo the following relations. For each triple $a, b, c$ of the generators, corresponding to a triple of hyperplanes whose intersection has codimension two, there is the relation $ab - ac + bc$. The Orlik-Solomon algebra $\mathcal{O}$, i.e., the cohomology of the complement of the union of the hyperplanes, is a quotient of this algebra with relations of higher degrees obtained in a similar way (see [2]). The enveloping algebra of the holonomy Lie algebra is the subalgebra generated by the elements of degree one in the Yoneda algebra $\text{Ext}_\mathcal{O}(\mathbb{C}, \mathbb{C})$, (see [5]). The holonomy Lie algebra may be defined over any coefficient ring. There is a result by Kohno ([4]) that the holonomy
Lie algebra over the rationals is equal to the graded associated Lie algebra of the fundamental group of the arrangement tensored with the rationals.

The holonomy Lie algebra is composed of “localized” Lie algebras obtained by restricting to the hyperplanes in a maximal set of hyperplanes whose intersection has codimension two. These Lie algebras are free, in degree at least two, on \( k - 1 \) generators, where \( k \) is the size of the maximal set. In [7] it is studied under what circumstances the holonomy Lie algebra decomposes as a direct product of these localized Lie algebras in degree at least two.

We will generalize this by studying any set of “localized” Lie algebras (not even graded) with the only and important property that two different sets of generators have at most one generator in common, which obviously is the case for holonomy Lie algebras. We will prove a decomposition theorem for “closed subarrangements” introduced in [7] and we also get decomposition results in the case when the Lie algebra is not fully decomposable. Finally, we give an application with an example of the Lie structure of a non-decomposable holonomy algebra.

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2 General theory

Let \( C \) be a commutative ring with unit.

Definition 2.1 A set-arrangement on \( X = \{x_1, \ldots, x_n\} \), where \( n \geq 2 \), is a subset \( \mathcal{A} \) of all subsets of \( X \) such that each set in \( \mathcal{A} \) has at least two elements, two different sets in \( \mathcal{A} \) has at most one element in common, and the union of all subsets in \( \mathcal{A} \) is equal to \( X \).

Definition 2.2 A Lie algebra \( \mathcal{L} \) of a set-arrangement \( \mathcal{A} \) on \( X = \{x_1, \ldots, x_n\} \) is a free ordinary (i.e., non-super) Lie algebra, \( \mathcal{F}(X) \), with coefficients in \( C \), modulo all relations \([x_i, x_j]\) where \( \{x_i, x_j\} \) is not a subset of any \( A \in \mathcal{A} \) together with a set of relations \( R' \), \( R = \cup_{A \in \mathcal{A}} R_A \), where \( R_A \in \mathcal{F}(A)_{\geq 2} \) and \( \mathcal{F}(A)_i \) refers to the submodule generated by all Lie monomials of word length \( i \). The Lie algebra \( \mathcal{F}(A)/\langle R_A \rangle \) is called the localized Lie algebra of \( \mathcal{L} \) at \( A \) and denoted \( \mathcal{L}_A \).
Remark 2.3 The set-arrangement of a holonomy Lie algebra may be defined to consist of all maximal sets of hyperplanes whose intersection have codimension two. This may include some two-elements sets, whose generators commute by definition. We will instead define the set-arrangement of a holonomy Lie algebra as before but excluding the two-elements sets; the commutation of generators from two-elements sets will follow from the definition of a Lie algebra of a set-arrangement. Here is an example of a Lie algebra of a set-arrangement with a two-element set whose generators do not commute and it is not naturally graded (it is however graded letting $x, y, z, u, v$ have the degrees $2, 2, 2, 1, 1$). We will see in the next section that it is decomposable; in fact, the derived Lie algebra is a direct product of the derived localized Lie algebras (which can be proved to be free Lie algebras on two generators).

Example 2.4

Let $\mathcal{L}$ be a Lie algebra of a set-arrangement $\mathcal{A}$. There is a natural Lie algebra morphism $s_A : \mathcal{L}_A \to \mathcal{L}$ for each $A \in \mathcal{A}$. Also, we may define a Lie algebra morphism $\pi_A : \mathcal{L} \to \mathcal{L}_A$ by sending all variables not in $A$ to zero (and a variable in $A$ to itself). This is well-defined, since if $r \in R_B$ where $B \neq A$, then each Lie monomial in $r$ contains at least one variable which is not in $A$, since $|A \cap B| \leq 1$ and $[x, x] = 0$; also if for all $B \in \mathcal{A}$, \{a, b\} $\nsubseteq B$, then $a \notin A$ or $b \notin A$ and hence $\pi_A([a, b]) = 0$.

We have $\pi_A \circ s_A = \text{id}_{\mathcal{L}_A}$ for each $A \in \mathcal{A}$. Now consider the derived functor $\mathcal{L}' = [\mathcal{L}, \mathcal{L}]$. We have for each $A \in \mathcal{A}$, $s'_A : \mathcal{L}'_A \to \mathcal{L}'$, $\pi'_A : \mathcal{L}' \to \mathcal{L}'_A$ and $\pi'_A \circ s'_A = \text{id}_{\mathcal{L}'_A}$. Define $C$-module morphisms $s'$ and $\pi'$ as

$$s' = \sum_{A \in \mathcal{A}} s'_A : \bigoplus_{A \in \mathcal{A}} \mathcal{L}'_A \to \mathcal{L}'$$

$$\pi' = \bigoplus_{A \in \mathcal{A}} \pi'_A : \mathcal{L}' \to \bigoplus_{A \in \mathcal{A}} \mathcal{L}'_A$$

Proposition 2.5 We have $\pi'_B \circ s'_A = 0$ if $B \neq A$, $\pi' \circ s' = \text{id}_{\mathcal{L}'_A}$, $\pi'$ is surjective, $s'$ is injective, and the $C$-submodule, $\text{im}(s') = \sum \text{im}(s'_A)$, of $\mathcal{L}'$ is a direct sum.
Proof. It is enough to prove the first assertion. This follows in the same way as above. Indeed, an element in \( \text{im}(s'_A) \) is represented by an element in \( \mathcal{F}(A)_{\geq 2} \), in which each Lie monomial contains at least one generator which is not in \( B \). \( \square \)

**Theorem 2.6** The following are equivalent for a Lie algebra \( \mathcal{L} \) of a set-arrangement \( \mathcal{A} \) on \( X \).

1) \([x, \text{im}(s'_A)] = 0 \) for all \( A \in \mathcal{A} \) and all \( x \in X \setminus A \)
2) \( \text{im}(s'_A) \) is an ideal in \( \mathcal{L} \) for all \( A \in \mathcal{A} \)
3) \( \mathcal{L}' = \sum_{a \in \mathcal{A}} \text{im}(s'_A) \)
4) \( s' \) is surjective
5) \( \pi' \) is injective

If the conditions above are satisfied, then

\[ \mathcal{L}' = \bigoplus_{A \in \mathcal{A}} \text{im}(s'_A) \cong \bigoplus_{A \in \mathcal{A}} \mathcal{L}'_A \]

as Lie algebras.

**Proof.** 1) \( \Rightarrow \) 2): obvious, since \([x, \text{im}(s'_A)] \subseteq \text{im}(s'_A) \) if \( x \in A \).

2) \( \Rightarrow \) 3): By the properties of the defining ideal for \( \mathcal{L} \), it follows that all elements in \( \mathcal{L}' \) which are represented by \([x, y] \), where \( x, y \in X \), belong to \( \sum \text{im}(s'_A) \). But as an ideal in \( \mathcal{L} \), \( \mathcal{L}' \) is generated by such elements and hence 3) follows from 2).

3) \( \Rightarrow \) 4): obvious

4) \( \Rightarrow \) 5): \( s' \) is an isomorphism by 4) and Proposition 2.5. Since \( \pi' \circ s' = \text{id}_{\oplus \mathcal{L}'_A} \) it follows that \( \pi' \) is the inverse of \( s' \).

5) \( \Rightarrow \) 1): We have \( \pi'_A([x, \text{im}(s'_A)]) = 0 \) if \( x \notin A \). Also, by Proposition 2.5, \( \pi'_B([x, \text{im}(s'_A)]) = 0 \) if \( B \neq A \). Hence \( \pi'([x, \text{im}(s'_A)]) = 0 \) if \( x \notin A \) and hence 1) follows from 5).

The last statement follows, since by 1) and 2) \( [\text{im}(s'_A), \text{im}(s'_B)] = 0 \) if \( A \neq B \). \( \square \)

**Example 2.7** (continuation from Example 2.4) Observe that we have the relations \([x, v] = [u, v] = [y, u] = 0 \) and that we have a symmetry by \( x \leftrightarrow y \).
and \( u \leftrightarrow v \). From the relations follows that \( \text{im}(s'_B) \) is contained in the subalgebra generated by \( x, u \) and also contained in the subalgebra generated by \( z, u \). By symmetry, \( \text{im}(s'_C) \) is contained in the subalgebra generated by \( y, v \) and also contained in the subalgebra generated by \( z, v \). From this it follows that \( \text{im}(s'_B) \) is contained in the subalgebra generated by \( x, u \) and also contained in the subalgebra generated by \( z, u \). By symmetry, \( \text{im}(s'_C) \) is contained in the subalgebra generated by \( y, v \) and also contained in the subalgebra generated by \( z, v \). From this it follows that \( [v, \text{im}(s'_B)] = [u, \text{im}(s'_C)] = 0 \). The induction start follows, since we just proved that \( [y, [z, u]] = 0 \). Suppose \( [y, r] = 0 \), where \( r \in \text{im}(s'_B) \). We must prove that \( [y, [z, r]] = [y, [u, r]] = 0 \). By Jacobi and induction we have \( [y, [z, r]] = [[y, z], r] + [z, [y, r]] = [[v, [v, y]], r] + 0 = 0 \), where the last equality follows by induction and the fact from above that \( [v, \text{im}(s'_B)] = 0 \). Since \( [y, u] = 0 \) we also get by induction that \( [y, [u, r]] = 0 \). Hence we have proved that \( [a, \text{im}(s'_B)] = 0 \) if \( a = y, v \). By symmetry we also get that \( [a, \text{im}(s'_C)] = 0 \) if \( a = x, u \). In order to be able to use Theorem 2.6 it remains to prove \( [a, \text{im}(s'_A)] = 0 \) if \( a = z, u, v \). This is proved by induction in the same manner as above.

Hence we get by the theorem that the derived Lie algebra is isomorphic to the direct product of the derived Lie algebras \( L'_A, L'_B, L'_C \).

Looking more carefully at the above example, one would like to find conditions such that the induction step follows directly when proving 1) in Theorem 2.6 by induction. Here is one such condition which is true for holonomy Lie algebras (but not for the Lie algebra in Example 2.3, the generators in a two-set must commute).

**Definition 2.8** A Lie algebra \( L \) of a set-arrangement \( A \) is said to satisfy the replacement condition if for all \( A, B \in A, A \neq B \) and all \( x \in A \cap B \), \( L'_A \) is contained in the Lie subalgebra of \( L_A \) generated by \( A \setminus \{x\} \).

A holonomy Lie algebra satisfies the replacement condition since if \( A = \{y_1, \ldots, y_k\} \) and \( x = y_j \) then for all \( i \neq j \) we have the relation \([x, y_i] = \sum_{r \neq j} [y_i, y_r]\).

**Theorem 2.9** Let \( L \) be a Lie algebra of a set-arrangement \( A \) on \( X \) and suppose \( L \) satisfies the replacement condition. Then the conditions 1) to 5) in Theorem 2.6 are equivalent to

6) \([x, [y, z]] = 0 \) for all \( A \in A \) and all \( y, z \in A \) and \( x \in X \setminus A \)


Proof. Of course, we only have to prove 6) implies 1). Suppose 6) is true. We prove by induction over the word length of $r$ that $[x, r] = 0$ if $x \in X \setminus A$ and $r$ is a Lie monomial, of word length at least two, built from generators in $A$. This is true when the word length of $r$ is two by 6). Suppose now $r$ is a Lie monomial such that $[x, r] = 0$ for all $x \in X \setminus A$. Suppose $a \in A$. We want to prove that $[x, [a, r]] = 0$ for all $x \in X \setminus A$. By Jacobi and induction, it is enough to prove that $[[x, a], r] = 0$. If $[x, a] \neq 0$, there is a $B \in \mathcal{A}$, $B \neq A$ such that $x, a \in B$. By replacement, there is an $s$ belonging to the Lie subalgebra of $\mathcal{L}_B$ generated by $B \setminus \{a\}$ such that $[x, a] = s$. Since $B \cap A = \{a\}$, the variables occurring in $s$ do not belong to $A$. Hence by Jacobi and induction $[s, r] = 0$. \qed

3 Closed sub-arrangements

Definition 3.1 If $\mathcal{A}$ is a set-arrangement and $B \subseteq \mathcal{A}$, supp$(B)$ is defined as $\cup_{B \in \mathcal{B}} B$. Then $B$ is a set-arrangement on supp$(B)$.

Following [7] we make the definition of a “closed” sub-arrangement.

Definition 3.2 If $\mathcal{A}$ is a set-arrangement and $B \subseteq \mathcal{A}$, $B$ is called a closed sub-arrangement of $\mathcal{A}$ if for all $A \in \mathcal{A} \setminus B$, $|A \cap \text{supp}(B)| \leq 1$.

Given a closed sub-arrangement $B$ of $\mathcal{A}$ and a Lie algebra $\mathcal{L}$ of $\mathcal{A}$, we define the localized Lie algebra at $B$ as $\mathcal{L}_B = \mathcal{F} \left( \text{supp}(B) \right) / \langle \mathcal{R}_B \rangle$, where

$$\mathcal{R}_B = \cup_{B \in \mathcal{B}} \mathcal{R}_B \cup \{ [x, y]; \ x, y \in \text{supp}(B) \text{ and } \forall B \in \mathcal{B} \ {x, y} \nsubseteq B \}$$

This extends the old definition of a localized Lie algebra since $\{A\}$ is a closed sub-arrangement of $\mathcal{A}$ for all $A \in \mathcal{A}$ and $\mathcal{R}_{\{A\}} = \mathcal{R}_A$.

By the “closed” condition, there is a Lie algebra map

$$s_B : \mathcal{L}_B \to \mathcal{L}$$

A map $\mathcal{F} \left( \text{supp}(A) \right) \to \mathcal{L}_B$ is defined by sending the generators $x \notin \text{supp}(B)$ to zero (and not changing the other generators). We want to prove that the map factors through $\mathcal{L}$. If for all $A \in \mathcal{A}$, $\{x, y\} \nsubseteq A$, then $[x, y] = 0$ in $\mathcal{L}_B$. Moreover, suppose $r \in \mathcal{R}_A$, $A \notin \mathcal{B}$, then any non-zero Lie monomial
in $r$ must contain at least one generator not in $\text{supp}(B)$, since $B$ is closed. Hence $r$ is mapped to zero and we have defined a map

$$\pi_B : \mathcal{L} \to \mathcal{L}_B$$

We have $\pi_B \circ s_B = \text{id}_{\mathcal{L}_B}$. Hence, $s_B$ is injective and $\pi_B$ is surjective and $\mathcal{L}$ is a semidirect product of $\text{im}(s_B)$ and $\text{ker}(\pi_B) = \text{the ideal generated by supp}(A) \setminus \text{supp}(B)$.

**Proposition 3.3** Let $B$ be a closed sub-arrangement of $A$ and let $\mathcal{L}$ be a Lie algebra of $A$. Then $\mathcal{L}'$ is the direct product of $\text{im}(s'_B)$ and $\text{ker}(\pi'_B)$ if

$$[x, \text{im}(s'_B)] = 0 \quad \text{for all } x \in \text{supp}(A) \setminus \text{supp}(B)$$

**Proof.** Obvious, since by Jacobi and induction we get, $[\text{ker}(\pi'_B), \text{im}(s'_B)] = 0$. □

Suppose now we have pairwise disjoint closed sub-arrangements $B_i$, $i = 1, \ldots, k$ such that $\cup B_i = A$. As in the previous section we may define $C$-module morphisms $s'$ and $\pi'$ as

$$s' = \sum_{i=1}^{k} s'_{B_i} : \bigoplus_{i=1}^{k} \mathcal{L}'_{B_i} \to \mathcal{L}'$$

$$\pi' = \bigoplus_{i=1}^{k} \pi'_{B_i} : \mathcal{L}' \to \bigoplus_{i=1}^{k} \mathcal{L}'_{B_i}$$

**Proposition 3.4** We have $\pi'_{B_i} \circ s'_{B_j} = 0$ for $i \neq j$, $\pi' \circ s' = \text{id}_{\mathcal{L}'_{B_i}}$, $\pi'$ is surjective, $s'$ is injective and the $C$-submodule, $\text{im}(s') = \sum \text{im}(s'_{B_i})$, of $\mathcal{L}'$ is a direct sum.

**Proof.** We only need to prove the first assertion. It may happen that $\text{supp}(B_i)$ and $\text{supp}(B_j)$ have several elements in common, but these elements commute, since if $x, y \in \text{supp}(B_i) \cap \text{supp}(B_j)$ and $[x, y] \neq 0$, then there is $A \in A$ such that $x, y \in A$. By the closed condition, it follows that $A \in B_i \cap B_j$ which is a contradiction if $i \neq j$. Now we can argue in the same way as in the proof of 2.5. Indeed, in any non-zero monomial built up by elements in $\text{supp}(B_j)$ there is at least one element which does not belong to $\text{supp}(B_i)$ if $i \neq j$. □
Theorem 3.5 Let $B_i$, $i=1,\ldots,k$ be pairwise disjoint closed sub-arrangements of a set-arrangement $A$ on $X$, such that $\cup B_i = A$. The following are equivalent for a Lie algebra $L$ of $A$.

1) $[x,\im(s'_{B_i})] = 0$ for all $i=1,\ldots,k$ and all $x \in X \setminus \supp(B_i)$
2) $\im(s'_{B_i})$ is an ideal in $L$ for all $i=1,\ldots,k$
3) $L' = \sum_{i=1}^{k} \im(s'_{B_i})$
4) $s'$ is surjective
5) $\pi'$ is injective

If the conditions above are satisfied, then

$$L' = \bigoplus_{i=1}^{k} \im(s'_{B_i}) \cong \bigoplus_{i=1}^{k} L'_{B_i}$$

as Lie algebras.

Proof. Very much the same as the proof of 2.6. In the proof of 5) $\Rightarrow$ 1) we use Proposition 3.4. $\square$

Theorem 3.6 Suppose the premises of theorem 3.5 are true and suppose that $L$ satisfies the replacement condition (see Definition 2.8), then the conditions 1) to 5) in Theorem 3.5 are equivalent to

6) $[x,[y,z]] = 0$ for all $y,z \in \supp(B_i)$, $x \in X \setminus \supp(B_i)$, $i=1,\ldots,k$

Proof. This is almost identical to the proof of Theorem 2.9. One only has to replace $A$ by $\supp(B_i)$, replace $B \neq A$ by $B \notin B_i$, and use that $B_i$ is closed. $\square$

We end this section with a computable sufficient condition which gives partial decomposition in the case the Lie algebra satisfies the replacement condition.

Proposition 3.7 Let $B$ be a closed sub-arrangement of $A$ and let $L$ be a Lie algebra of $A$ which satisfies the replacement condition. Then $L'$ is the direct product of $\im(s'_{B})$ and $\ker(\pi'_{B})$ if

$[x,[y,z]] = 0$ for all $x \in \supp(A) \setminus \supp(B)$ and all $y,z \in \supp(B)$

Moreover, $\ker(\pi_B)$ is the Lie subalgebra generated by $\supp(A) \setminus \supp(B)$. 8
The first claim follows from Proposition 3.3 and the proof of Theorem 3.6 since the proof there is valid for each \( i \) separately. The last claim is proved in the same way as Theorem 3.6. \( \square \)

4 An example

As an application of Theorem 3.6, we will study the holonomy Lie algebra of the hyperplane arrangement studied in [1] with notation \((10_3)_3\) (this example was shown to me by Jan-Erik Roos). It consists of 10 planes through origo in \( \mathbb{C}^3 \). There are 10 combinations of three planes whose intersection has codimension 2 (and no larger sets). We let \( 6, 7, 8, 1, 9, 4, 5, 2, 3, 10 \) be the names of the planes in the order given in [1]. We get the following set-arrangement on \{1, 2, \ldots , 10\}.

\[
\mathcal{A} = \{\{1, 2, 3\}, \{1, 4, 5\}, \{2, 4, 6\}, \{3, 5, 6\}, \\
\{1, 7, 8\}, \{2, 7, 10\}, \{3, 7, 9\}, \{4, 8, 10\}, \{5, 9, 10\}, \{6, 8, 9\}\}
\]

Put \( \mathcal{B}_1 = \{\{1, 2, 3\}, \{1, 4, 5\}, \{2, 4, 6\}, \{3, 5, 6\}\} \). It is easy to see that \( \mathcal{B}_1 \) is a closed sub-arrangement of \( \mathcal{A} \). Its Lie algebra is the Lie algebra of the graphic arrangement \( K_4 \), the complete graph with four vertices, see [7]. Put \( \mathcal{B}_i = \{A_i\}, \ i = 2, \ldots , 7 \) where \( A_2, \ldots , A_7 \) are the remaining sets in \( \mathcal{A} \).

The holonomy Lie algebra, \( \mathcal{L} \), has for each set \( \{a, b, c\} \) in \( \mathcal{A} \) the relations \( [a, b] = [b, c] = [c, a] \) together with all relations \( [x, y] \) such that \( \{x, y\} \) is not a subset of any set in \( \mathcal{A} \).

We have that \( \mathcal{L} \) is graded by letting all generators have degree one. In degree two we have the 10 generators \( [1, 2], [1, 4], \ldots , [6, 8] \). To be able to use Theorem 3.6 we have to check that \( [1, 2], [1, 4], [2, 4], [3, 5] \) are annihilated by \( 7, 8, 9, 10 \) and that \( [1, 7] \) is annihilated by \( 2, 3, 4, 5, 6, 9, 10 \) and so on. In total there are 58 triple products that must be proved to be zero. To do this one can use the program \texttt{liedim} [6]; it can compute over any prime field (but not over the integers). Over the rationals the program shows that all the 58 products are zero and hence the Lie algebra \( \mathcal{L}' \) is the direct sum of seven localized Lie algebras.

To analyze \( \mathcal{L}'_{\mathcal{B}_1} \), put \( A = \{1, 2, 3\} \). By Proposition 3.7, \( \mathcal{L}'_{\mathcal{B}_1} \) is a semidirect product of \( \mathcal{L}'_A \) by the Lie subalgebra generated by \( 4, 5, 6 \) (in degree \( \geq 2 \)). The Lie algebra \( \mathcal{L}_A \) is also a semidirect product of the free Lie algebra on \( 3 \) by the free Lie algebra on \( 1, 2 \), where \( 3.1 = [1, 2], \ 3.2 = [2, 1] \).
Hence we get that $\mathcal{L}_{B_1}$ is $\mathcal{F}(4, 5, 6)/I \rtimes (\mathcal{F}(1, 2) \rtimes \mathcal{F}(3))$ for some ideal $I$. On the other hand, by defining the operations of 1, 2, 3 on $\mathcal{F}(4, 5, 6)$ and checking that the relations $[1, 2] - [2, 3], [1, 2] - [3, 1]$ operate as zero one can deduce that also $\mathcal{F}(4, 5, 6) \rtimes (\mathcal{F}(1, 2) \rtimes \mathcal{F}(3))$ is a quotient of $\mathcal{L}_{B_1}$ and hence we get

$$\mathcal{L}_{B_1} \cong \mathcal{F}(4, 5, 6) \rtimes (\mathcal{F}(1, 2) \rtimes \mathcal{F}(3))$$

Finally, we get the following decomposition of $\mathcal{L}$.

$$\mathcal{L} \cong (\mathcal{F}(4, 5, 6) \rtimes (\mathcal{F}(1, 2) \rtimes \mathcal{F}(3))) \oplus \bigoplus_{i=2}^{7} \mathcal{L}_i$$

where $\mathcal{L}_i \cong \mathcal{F}(1, 2) \rtimes \mathcal{F}(3), i = 2, \ldots, 7$.

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