Global existence of solutions of the Boltzmann equation for Bose-Einstein particles with anisotropic initial data

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Abstract In this paper we prove the global in time existence and uniqueness of solutions of the spatially homogeneous Boltzmann equation for Bose-Einstein particles for the hard sphere model for bounded anisotropic initial data. The main idea of our proof is as follows: we first establish an intermediate equation which is closely related to the original equation and is relatively easily proven to have global in time and unique solutions, then we use the multi-step iterations of the collision gain operator to obtain a desired uniform $L^\infty$-bound for the solution of the intermediate equation so that it becomes the solution of the original equation.

Keywords Boltzmann equation; Bose-Einstein particles; Global existence; Iteration of collision operator; Hard sphere model

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1 Introduction

The Boltzmann equation for Bose-Einstein particles describes time evolution of dilute Bose gases. Derivations of such quantum Boltzmann equations can be found for instance in [17], [19], [4], [2], [5]. In this paper we consider the case of spatially homogeneous equation and study the global in time existence and uniqueness of solutions for anisotropic initial data. The spatially homogeneous Boltzmann equation for
According to (1.6), the precise meaning of (1.8) is that the function \( B(v-v_*,\sigma) \) for quantum Boltzmann equation, takes the form

\[
\frac{\partial f}{\partial t} = \int_{\mathbb{R}_3 \times S^2} B(v-v_*,\sigma) \left[ f'(f'_*) - f_{v_*}(f + f_{v_*}) \right] d\sigma dv_*
\]  

(1.1)

with \((t,v) \in [0,\infty) \times \mathbb{R}_3\), where \( h > 0 \) is the Planck’s constant, \( f = f(t,v) \) is the density of the number of particles at time \( t \in [0,\infty) \) with the velocity \( v \in \mathbb{R}_3 \), \( f' = f(t,v') \), \( f_{v_*} = f(t,v_*) \), and \( v,v_* \) are velocities of two particles before and after their collision, where

\[ v' = \frac{v + v_*}{2} + \frac{|v-v_*|}{2\sigma}, \quad v'_* = \frac{v + v_*}{2} - \frac{|v-v_*|}{2\sigma}. \]  

(1.2)

Recall that the equation (1.1) can also be written in \( \omega \)-representation by

\[
\frac{\partial f}{\partial t} = \int_{\mathbb{R}_3 \times S^2} \tilde{B}(v-v_*,\omega) \left[ f'(f'_*) - f_{v_*}(f + f_{v_*}) \right] d\omega dv_*
\]

(1.3)

where

\[
v' = v - \langle v-v_*,\omega \rangle \omega, \quad v'_* = v + \langle v-v_*,\omega \rangle \omega
\]

(1.4)

\[ \sigma = n - 2\langle n,\omega \rangle \omega, \quad \tilde{B}(v-v_*,\omega) = 2\langle n,\omega \rangle B(v-v_*,\sigma) \]

(1.5)

\[ n = \frac{v-v_*}{|v-v_*|}, \quad \text{and } B(v-v_*,\sigma) \text{ is given in the equation (1.1)} \text{ (see e.g. Chapter 2.4 in [20]).} \]

In the rest part of this paper we will always use the \( \sigma \)-representation of the equation (1.1).

The function \( B(v-v_*,\sigma) \) in the equation (1.1) is called the collision kernel which, according to [1, 2, 3] for quantum Boltzmann equation, takes the form

\[
B(v-v_*,\sigma) = \frac{|v-v_*|}{h^4} \left[ \hat{\Phi} \left( \frac{|v-v'_*|}{\hbar} \right) + \hat{\Phi} \left( \frac{|v-v'_*|}{\hbar} \right) \right]^2
\]

(1.6)

where the radially symmetric function \( \hat{\Phi}(|\xi|) := \Phi(|\xi|) = \int_{\mathbb{R}} \Phi(x)e^{-i\xi x} dx \) is the Fourier transform of an interacting potential \( \Phi(x) = \Phi(|x|) \) which is assumed (as in many cases) to be radially symmetric. From (1.6) and (1.2) one sees that \( B(v-v_*,\sigma) \) is a function of \(|v-v_*|, |\langle v-v_*,\sigma \rangle|\), i.e.

\[
B(v-v_*,\sigma) = B_1(|v-v_*|, |\langle v-v_*,\sigma \rangle|), \quad (v,v_*,\sigma) \in \mathbb{R}_3 \times \mathbb{R}_3 \times S^2
\]

(1.7)

with a nonnegative measurable function \( B_1 \) on \( \mathbb{R}_0 \times \mathbb{R}_0 \) determined by \( \hat{\Phi}(|\cdot|) \).

Investigation experience shows that the most possible case for obtaining a global solution for anisotropic initial data is the hard sphere model or the asymptotically hard sphere model:

\[
B(v-v_*,\sigma) \sim |v-v_*| \quad \text{when } |v-v_*| \gg 1.
\]

(1.8)

According to (1.8), the precise meaning of (1.8) is that the function \( \hat{\Phi}(|\xi|) \) satisfies the following condition:

\[
a_0 \frac{|v-v_*|^\beta}{1 + |v-v_*|^\beta} \leq \left( \hat{\Phi}(|v-v'|) + \hat{\Phi}(|v-v'_|) \right)^2 \leq b_0
\]

(1.9)

where \( \beta \geq 3 \) and \( 0 < a_0 \leq b_0 \leq \infty \) are constants independent of \( h \). This includes a physical case where the interaction potential \( \Phi \) is equal to a Dirac \( \delta \)-function \( \delta(x) \) plus a small attractive force’s potential \( -U(x) \):

\[
\Phi(x) = \delta(x) - U(x), \quad x \in \mathbb{R}^3.
\]

(1.10)
For instance one may take

\[ U(x) = U(|x|) = \frac{1}{4\pi|x|} \exp(-|x|), \quad x \in \mathbb{R}^3 \]

which is Yukawa potential in \( \mathbb{R}^3 \); its Fourier transform is

\[ \hat{U}(\xi) = \hat{U}(|\xi|) = \frac{1}{1 + |\xi|^2}, \quad \xi \in \mathbb{R}^3. \]

From (1.10) we have \( \hat{\Phi}(|\xi|) = 1 - \hat{U}(\xi) = 1 - \frac{1}{1 + |\xi|^2} \) so that

\[ \frac{1}{8} \frac{|v - v_\star|^4}{1 + |v - v_\star|^2} \leq (\hat{\Phi}(|v - v'|) + \hat{\Phi}(|v - v_\star|))^2 \leq 4, \]

which satisfies (1.9) with \( \beta = 4 \), \( a_0 = \frac{1}{8} \) and \( b_0 = 4 \). Of course if \( U(x) = 0 \), i.e. if \( \Phi(x) = \delta(x) \), then \( B(v - v_\star, \sigma) = \frac{1}{\pi} |v - v_\star| \) which corresponds to the hard sphere model.

To simplify notations we may set \( \hbar = 1 \). That is, in the rest of this paper we need only to consider the equation (1.1) for the scale normalized case \( \hbar = 1 \). In fact, by the scaling transforms

\[ \tilde{f}(t, v) = h^3 f(h^3 t, v), \quad f(t, v) = h^{-3} \tilde{f}(h^{-3} t, v), \quad \Psi(|x|) = h \Phi(|hx|) \] (1.11)

we have \( \hat{\Psi}(|\xi|) = \hat{\Psi}(\xi) = \frac{1}{h^3} \hat{\Phi}(\frac{\xi}{h}) = \frac{1}{h^3} \hat{\Phi}(\frac{|\xi|}{h}) \) so that it is easily checked that \( f \) is a solution of the equation (1.1) if and only if \( \tilde{f} \) is a solution of the scale normalized equation:

\[ \frac{\partial f}{\partial t} = \iint_{\mathbb{R}^3 \times S^2} B(v - v_\star, \sigma) [f f_s'(1 + f + f_s) - f f_s(1 + f' + f_s')] d\sigma d\nu \] (1.12)

with

\[ B(v - v_\star, \sigma) = |v - v_\star| (\hat{\Phi}(|v - v'|) + \hat{\Phi}(|v - v_\star|))^2. \] (1.13)

Correspondingly, the condition (1.9) is rewritten in terms of \( \hat{\Psi} \) as

\[ \frac{a_0}{h^4} \frac{|v - v_\star|^\beta}{h^2 + |v - v_\star|^2} \leq (\hat{\Psi}(|v - v'|) + \hat{\Psi}(|v - v_\star|))^2 \leq \frac{b_0}{h^4}. \] (1.14)

Let us denote

\[ a = \frac{a_0}{h^4}, \quad b = \frac{b_0}{h^4}, \quad \text{and assume that } 0 < h \leq 1. \] (1.15)

Then from (1.14) and (1.13) we have

\[ a \frac{|v - v_\star|^\beta}{1 + |v - v_\star|^2} \leq (\hat{\Psi}(|v - v'|) + \hat{\Psi}(|v - v_\star|))^2 \leq b, \]

\[ a \frac{|v - v_\star|^\beta}{1 + |v - v_\star|^2} |v - v_\star| \leq B(v - v_\star, \sigma) \leq b |v - v_\star|. \] (1.16)

One of the main difficulties in proving the global in time existence of solutions of Eq. (1.12) with general \( L^1 \) initial data is the divergence of the cubic term (see e.g. [10]):

\[ \sup_{f \geq 0, \|f\|_{L^1} \leq 1} \iint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} B(v - v_\star, \sigma) f(v) f(v_s) f(v_s') d\sigma d\nu_s dv = \infty \] (1.17)
Another main difficulty comes from the low temperature effect which yields the Bose-Einstein condensation and that the regular part of the equilibrium (i.e. the Bose-Einstein distribution) is unbounded near the origin; these together with the result of convergence to equilibrium (see [11]) imply that for low temperature there must be no such a global solution that is bounded on $[0, \infty) \times \mathbb{R}^3$.

So far, basic results for Eq. (1.12) on global in time existences, singular behavior, long time behavior, kinetics of Bose-Einstein condensation, etc. are all concerned with solutions for isotropic initial data (which implies that the solutions are also isotropic), see for instance [6],[7],[8],[11],[14],[18]. In a recent work [3], Briant and Einav considered a wide class of bounded solutions of the equation (1.12) and proved the local in time existence, uniqueness, stability and moment production of solutions of Eq. (1.12) without the isotropic assumption on the initial data.

Our aim of this paper is to prove the global in time existence and uniqueness of solutions of Eq. (1.12) for bounded anisotropic initial data $f_0$ satisfying (1.22) (see Theorem 1.2 below). As mentioned above, this must belong to the case of high temperature.

In order to state our main result we need to introduce the definition of solutions of Eq. (1.12) and some related notations.

1.1 Definition of solutions

Our working function spaces are the usual weighted Lebesgue $L^p$ spaces $L^p_s(\mathbb{R}^3)$ defined by

$$f \in L^p_s(\mathbb{R}^3) \iff \|f\|_{L^p_s} := \left( \int_{\mathbb{R}^3} \langle v \rangle^s |f(v)|^p dv \right)^{1/p} < \infty$$

where $1 \leq p < \infty$, $0 \leq s < \infty$ and $\langle v \rangle^s = (1 + |v|^2)^{s/2}$. For $p = \infty$, $0 \leq s < \infty$ we define

$$\|f\|_{L^\infty_s} = \text{ess sup}_{v \in \mathbb{R}^3} \langle v \rangle^s |f(v)| \quad \text{and} \quad f \in L^\infty_s(\mathbb{R}^3) \iff \|f\|_{L^\infty_s} < \infty.$$ 

As usual let $Q(f)$ denote the collision integral in the right hand side of Eq. (1.12):

$$Q(f)(v) = \int_{\mathbb{R}^3 \times S^2} B(v - v_*, \sigma) [f' f_*' (1 + f + f_*) - f f_*(1 + f' + f_*')] d\sigma dv_*, \quad v \in \mathbb{R}^3. \quad (1.18)$$

And we denote $Q(f)(t, v) := Q(f(t, \cdot))(v)$ for the case $f = f(t, v)$.

**Definition 1.1.** Given $0 \leq f_0 \in L^1_2(\mathbb{R}^3) \cap C(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3; L^1_2(\mathbb{R}^3))$. We say that a nonnegative function $f \in C([0, \infty) \times \mathbb{R}^3) \cap L^\infty([0, \infty); L^1_2(\mathbb{R}^3))$ is a solution to Eq. (1.12) with the initial datum $f(0, \cdot) = f_0$, if $f$ satisfies the following (i),(ii):

(i) for any $v \in \mathbb{R}^3$, the function $t \mapsto f(t, v)$ belongs to $C^1([0, \infty)),

(ii) $f$ satisfies Eq. (1.12) on $[0, \infty) \times \mathbb{R}^3$, i.e.

$$\frac{\partial}{\partial t} f(t, v) = Q(f)(t, v) \quad \forall (t, v) \in [0, \infty) \times \mathbb{R}^3. \quad (1.19)$$
Furthermore, if $f$ conserves the mass, momentum and energy, i.e. if
\[ \int_{\mathbb{R}^3} (1, v, |v|^2) f(t, v) dv = \int_{\mathbb{R}^3} (1, v, |v|^2) f_0(v) dv \quad \forall \, t \in [0, \infty) \] (1.20)
then $f$ is called a conservative solution.

1.2 Moments and kinetic temperature
Moments $M_k(f)$ of order $k \in [0, s]$ for $0 \leq f \in L_1^1(\mathbb{R}^3)$ are defined by
\[ M_k(f) = \int_{\mathbb{R}^3} |v|^k f(v) dv. \]
For $0 \leq f \in L_1^1(\mathbb{R}^3)$, $mM_0(f)$ and $\frac{m}{T}M_2(f)$ are the mass and kinetic energy of a particle system per unit space volume, where $m$ is the mass of one particle. Without confusion we also call $M_0(f)$ and $M_2(f)$ the mass and energy. In this paper we always assume that initial data $0 \leq f_0 \in L_1^1(\mathbb{R}^3)$ satisfy $M_0(f_0) > 0$. Denote $M_0 = M_0(f_0)$, $M_2 = M_2(f_0)$. By the conservation of mass and energy, the kinetic temperature $T$ of the particle system is defined by (see e.g. Chapter 2 in [3])
\[ T = \frac{1}{3k_B} \cdot \frac{mM_2}{M_0}, \]
and the critical temperature $T_c$ corresponding to Eq. (1.12) is given by (see e.g. [11] and references therein)
\[ T_c = \frac{m\zeta(5/2)}{2\pi k_B \zeta(3/2)} \left( \frac{M_0}{\zeta(3/2)} \right)^{2/3} \]
where $k_B$ is the Boltzmann constant, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ ($s > 1$) is the Riemann-Zeta function. By calculation we have
\[ \frac{T}{T_c} = \frac{2\pi[\zeta(3/2)]^{5/3}}{3\zeta(5/2)} \cdot \frac{M_2}{M_0^{5/3}}. \] (1.21)
As mentioned above, if $T/T_c \leq 1$ (i.e. the case of low temperature), there is no bounded solution on $[0, \infty) \times \mathbb{R}^3$. The case of $T/T_c > 1$ but not too large is difficult. While the case $T/T_c >> 1$ is easily proved to be necessary for having global bounded solutions for small initial data as concerned in this paper, see Remark 1.4 below.

1.3 Main result
Our main result of the paper is the following

**Theorem 1.2.** Suppose the collision kernel $B$ in [1,13] satisfies (1.16) with $\beta \geq 3$. Let $0 \leq f_0 \in L_1^1(\mathbb{R}^3) \cap L_1^\infty(\mathbb{R}^3) \cap C(\mathbb{R}^3)$ satisfies $\int_{\mathbb{R}^3} v f_0(v) dv = 0$, $M_2(f_0) = \int_{\mathbb{R}^3} |v|^2 f_0(v) dv > 0$, and
\[ \left( \|f_0\|_{L^1} + \|f_0\|_{L^\infty} \right) \left( \frac{\|f_0\|_{L^1}}{\min\{\|f_0\|_{L^1}, M_2(f_0)\}} \right)^{4\beta} \leq \frac{1}{2^{3\beta+11}} \left( \frac{4\beta + 2}{4\beta + 4} \right)^{2\beta+2} \left( \frac{\alpha}{\beta} \right)^{2(\beta+1)} \] (1.22)
Theorem 1.2 for the original equation (1.1) is equivalent to velocity the initial data satisfying the condition (1.22) exist extensively. Bes id es, the assumption of zero mean-

Remark 1.3. From (1.9) and (1.15) (i.e. Remark 1.6. also holds true for mild solutions, where the definition of mild solutions is given in Definition 2.3 below. In order to prove the global in time existence of solutions of Eq.(1.12), we consider two types of approx-

Remark 1.4. In the proof of Theorem 1.2 we will prove that this smallness of \( \| f_0 \|_{L^\infty} \) does not lose generality. In fact, for the case \( v_0 := \frac{1}{M_0} \int_{\mathbb{R}^3} \nu f_0(v) dv \neq 0 (M_0 = \int_{\mathbb{R}^3} f_0(v) dv) \), it is easily seen that for the solution \( \tilde{f}(t,v) \) of Eq.(1.12) with the initial datum \( \tilde{f}_0(v) := f_0(v + v_0) \) (which has zero mean-velocity), the \( \nu \)-translation \((t,v) \mapsto f(t,v) := \tilde{f}(t,v - v_0)\) is the solution of Eq.1.12 with the initial datum \( f_0(v) \).

Remark 1.5. From our proof of Theorem 1.2 it is easily seen that if we do not assume that the initial data \( f_0 \) are continuous, i.e. if we only assume that \( 0 \leq f_0 \in L^1_\beta(\mathbb{R}^3) \cap L^\infty_\beta(\mathbb{R}^3) \), then the above main result also holds true for mild solutions, where the definition of mild solutions is given in Definition 2.3 below.

Remark 1.6. From (1.9) and (1.15) (i.e. \( a/b = a_0/b_0 \)) we see that our initial assumption (1.22) in Theorem 1.2 for the original equation (1.1) is equivalent to

\[
(\| f_0 \|_{L^1} + \| f_0 \|_{L^\infty}) \left( \frac{\| f_0 \|_{L^1_\beta}}{\min\{\| f_0 \|_{L^1}, M_2(f_0)\}} \right)^{4\beta} \leq \frac{1}{2^{3\beta-11}} \frac{(4\beta + 2)^{2\beta+1}}{(4\beta + 4)^{2\beta+2}} \left( \frac{a}{b} \right)^{2(\beta+1)} \frac{1}{h^3}
\]

which is easily satisfied for any given initial datum \( f_0 \) if \( h \) is small enough.

1.4 Strategy and organization of the paper

In order to prove the global in time existence of solutions of Eq.1.12, we consider two types of approximate equations: cutoff equations and the intermediate equation. In cutoff equations the collision kernel \( B \) is cut off as \( B_n = B \wedge n \) and the solutions \( f \) in the collision integrals are cut off as \( f \wedge n, f \wedge K \) (with constants \( n > 0, K > 0 \)). Here

\[
x \wedge y := \min\{x,y\}, \quad x, y \in \mathbb{R}.
\]
Then with $K$ fixed we use the $L^1$ relative compactness of mild solutions $\{f^n\}_{n=1}^\infty$ of cutoff equations to obtain a mild solution $f$ of the intermediate equation. In the intermediate equation the collision kernel $B$ is the original kernel, but the mild solution $f$ in the collision integrals are partly modified as $f \wedge K$.

We finally use carefully multi-step iterations of the collision gain operator $Q^+ (\cdot, \cdot)$ with a suitable choice of $K$ to obtain the uniform $L^\infty$ estimate $f \leq K$ for the solution $f$ of the intermediate equation so that $f$ is the solution of Eq. (1.12).

The paper is organized as follows: In Section 2 we give some basic definitions and then present some technical lemmas and propositions which will be used in the subsequent sections. Section 3 is the proof of existence and $L^1$ relative compactness of mild solutions $f^n$ of cutoff equations. In Section 4 we prove the existence of a bounded mild solution $f$ of the intermediate equation. The proof of Theorem 1.2 is concluded in Section 5.

2 Some properties of collision operations

This section is a preparation for proving our main result. We begin by recalling a few elementary properties of collision integrals which are used in deriving basic equalities and estimates. From the $\omega$-representation (1.4) and the identity $|v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2$ one sees that for any $\omega \in S^2$, $(v, v_*) \mapsto (v', v'_*)$ is an orthogonal linear transformation on $\mathbb{R}^6$. It is this property that makes the proof of some elementary properties of collision integrals relatively easy. On the other hand, the $\sigma$-representation (1.2) has the advantage that gives a nice structure for the collision integrals. The two representations however are equivalent in representing collision integrals. In fact it is not difficult to prove the following identity (for all $F$ which are nonnegative measurable or satisfy required integrability):

$$\int_{S^2} B(v - v_*, \sigma) F(v', v'_*) d\sigma = \int_{S^2} B(v - v_*, \omega) F(v', v'_*) d\omega, \quad v, v_* \in \mathbb{R}^3$$

(2.1)

where $(v', v'_*)$ in the left hand side and in the right hand side are given by $\sigma$-representation (1.2) and $\omega$-representation (1.4) respectively. This property allows us to translate some elementary properties of collision integrals with the $\sigma$-representation into those with the $\omega$-representation so that they can be proven rigorously (see also Chapter 2.4 in [20]). For instance, applying (2.1) and the $\omega$-representation one deduces the following general identity with the $\sigma$-representation (which is often used in deriving fundamental properties of collision integrals):

$$\int_{S^2} \int_{S^2} \int_{S^2} \int_{S^2} B(v - v_*, \sigma) F(v', v'_*, v, v_* d\sigma d\nu_* d\nu d\nu_*$

$$= \int_{S^2} \int_{S^2} \int_{S^2} \int_{S^2} B(v - v_*, \sigma) F(v, v_*, v', v'_*) d\sigma d\nu_* d\nu d\nu_*$

(2.2)

where $(v', v'_*)$ is given by the $\sigma$-representation (1.2) and $F$ are nonnegative Lebesgue measurable functions on $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ or satisfy required integrability.
2.1 Collision integral operators and approximate equations

In this subsection, we introduce definitions of collision integral operators, two types of approximate equations and mild solutions.

**Definition 2.1.** Fix $K > 0, n \in \mathbb{N}$. Suppose the collision kernel $B$ in (1.13) satisfies (1.10). For any nonnegative measurable functions $f, g$ on $\mathbb{R}^3$, we define for any $v \in \mathbb{R}^3$

$$Q^+(f, g)(v) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} B(v - v_*, \sigma) f' g'_* d\sigma dv_*$$

which is called the (bilinear) collision gain operator, and

$$Q^+(f)(v) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} B(v - v_*, \sigma) f' f'_* (1 + f + f_*) d\sigma dv_*,$$

$$Q^-(f)(v) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} B(v - v_*, \sigma) f f_* (1 + f' + f'_*) d\sigma dv_*,$$

$$Q^+_K(f)(v) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} B(v - v_*, \sigma) f' f'_* (1 + f \wedge K + f_* \wedge K) d\sigma dv_*,$$

$$L^+_K(f)(v) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} B(v - v_*, \sigma) f_* (1 + f' \wedge K + f'_* \wedge K) d\sigma dv_*,$$

$$Q^-_K(f)(v) = f(v) L^-_K(f)(v) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} B(v - v_*, \sigma) f f_* (1 + f' \wedge K + f'_* \wedge K) d\sigma dv_*,$$

$$Q_0(f)(v) = Q^+_K(f)(v) - Q^-_K(f)(v)$$

for $f \in L^1_1(\mathbb{R}^3)$,

$$Q^+_{n,K}(f)(v) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} B_n(v - v_*, \sigma) (f' \wedge n)(f'_* \wedge n)(1 + f \wedge K + f_* \wedge K) d\sigma dv_*,$$

$$Q^-_{n,K}(f)(v) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} B_n(v - v_*, \sigma) (f \wedge n)(f'_* \wedge n)(1 + f' \wedge K + f'_* \wedge K) d\sigma dv_*,$$

$$Q_{n,K}(f)(v) = Q^+_{n,K}(f)(v) - Q^-_{n,K}(f)(v)$$

where $B_n(v - v_*, \sigma) = B(v - v_*, \sigma) \wedge n$.

**Definition 2.2.** Given any $K > 0, n \in \mathbb{N}$. Suppose the collision kernel $B$ in (1.13) satisfies (1.10). Our cutoff equation of Eq. (1.12) mentioned above is defined by

$$\frac{\partial}{\partial t} f(t, v) = Q_{n,K}(f)(t, v), \quad (t, v) \in [0, \infty) \times \mathbb{R}^3,$$

and the intermediate equation of Eq. (1.12) is defined by

$$\frac{\partial}{\partial t} f(t, v) = Q_K(f)(t, v), \quad (t, v) \in [0, \infty) \times \mathbb{R}^3.$$

Here $Q_{n,K}(f)(t, v) = Q_{n,K}(f(t, \cdot))(v)$, $Q_K(f)(t, v) = Q_K(f(t, \cdot))(v)$.

**Definition 2.3.** Let $n \in \mathbb{N}$, $B_n = B \wedge n$ with $B$ the collision kernel satisfying (1.10). Let $Q(\cdot), Q^\pm(\cdot)$, $Q_K(\cdot), Q^\pm_K(\cdot)$, $Q_{n,K}(\cdot)$, $Q^\pm_{n,K}(\cdot)$ be the collision operators defined in (1.13), (2.4) – (2.12) respectively. Let $Q_{\ast}(\cdot), Q_{\ast}^\pm(\cdot)$ be one of the three couples $Q(\cdot), Q^\pm(\cdot)$; $Q_K(\cdot), Q^\pm_K(\cdot)$; $Q_{n,K}(\cdot), Q^\pm_{n,K}(\cdot)$. Given any
$0 \leq f_0 \in L^1_2(\mathbb{R}^3)$, we say that a nonnegative measurable function $(t,v) \mapsto f(t,v)$ on $[0,\infty) \times \mathbb{R}^3$ is a mild solution of the equation

$$\frac{\partial}{\partial t} f(t,v) = Q_*(f)(t,v), \quad (t,v) \in [0,\infty) \times \mathbb{R}^3$$

with the initial datum $f(0,\cdot) = f_0$, if $f$ satisfies $f \in L^\infty([0,\infty); L^1_2(\mathbb{R}^3))$ and there is a null set $Z \subset \mathbb{R}^3$, which is independent of $t$, such that for all $t \in [0,\infty)$ and all $v \in \mathbb{R}^3 \setminus Z$,

$$\int_0^t Q_\pm(f)(\tau,v) \, d\tau < \infty,$$

$$f(t,v) = f_0(v) + \int_0^t Q_*(f)(\tau,v) \, d\tau.$$

Furthermore if $f$ also conserves the mass, momentum and energy, i.e. if $f$ satisfies (1.20), then $f$ is called a conservative mild solution. For any given $0 < T < \infty$, by replacing $[0,\infty)$ with $[0,T]$ we also define (conservative) mild solutions on $[0,T] \times \mathbb{R}^3$.

**Remark 2.4.** For the cutoff case $Q(\cdot) = Q_{n,K}, Q_\pm(\cdot) = Q_{n,K}^\pm(\cdot)$, the null sets $Z$ in the definition of mild solutions can be chosen as an empty set. See Proposition 3.1 below.

### 2.2 Some lemmas and propositions

This subsection is a collection of technical lemmas and propositions. We begin with the proofs of two important lemmas. These two lemmas together with their corollary will help us to obtain the moment estimates of mild solutions of approximate equations (2.13) and (2.14).

**Lemma 2.5.** Let $k > 1, x \geq 0, y \geq 0$. Then we have

$$(k - 1) \min\{x^k, y^k\} \leq (x + y)^k - x^k - y^k \leq (2^k - 2) \max\{x^{k-\lambda}y^\lambda, y^{k-\lambda}x^\lambda\}$$

where $0 \leq \lambda \leq \min\{1, k/2\}$.

**Proof.** The second inequality in (2.15) relies on the following elementary inequality:

$$(1 + X)^k \leq 1 + X^k + (2^k - 2)X^\lambda, \quad X \in [0,1].$$

To prove (2.16) we first assume that $k \geq 2$. Let

$$\varphi(X) = (1 + X)^k - 1 - X^k - (2^k - 2)X, \quad X \in [0,1].$$

Since $\varphi$ is convex on $[0,1]$, it follows that $\varphi(X) \leq \max\{\varphi(0), \varphi(1)\} = 0$ for all $X \in [0,1]$. Thus

$$(1 + X)^k \leq 1 + X^k + (2^k - 2)X \leq 1 + X^k + (2^k - 2)X^\lambda \quad \forall X \in [0,1].$$

Here in the last inequality we used the assumption $0 \leq \lambda \leq \min\{1, k/2\} = 1$. Next assume that $1 < k \leq 2$. We consider

$$\psi(X) = (1 + X)^k X^{-\lambda} - X^{-\lambda} - X^{k-\lambda}, \quad X \in (0,1].$$
Using convexity of the derivative $\psi'(X)$ it is not difficult to prove that the function $\psi(X)$ is increasing on $(0, 1]$. Then we have

$$(1 + X)^k X^{-\lambda} - X^{-\lambda} - X^k - \lambda = \psi(X) \leq \psi(1) = 2^k - 2 \quad \forall X \in (0, 1].$$

This gives

$$(1 + X)^k - 1 - X^k \leq (2^k - 2)X^\lambda \quad \forall X \in [0, 1].$$

combining the above we have (2.10).

Now let $x \geq 0, y \geq 0$. Without loss of generality we may assume $x \leq y$ and $y > 0$. It follows from (2.10) that

$$(x + y)^k - x^k - y^k = y^k \left(1 + \frac{x}{y}\right)^k - 1 - \left(\frac{x}{y}\right)^k$$

$$\leq y^k (2^k - 2) \left(\frac{x}{y}\right)^\lambda = (2^k - 2) \max\{x^{k-\lambda} y^{\lambda}, y^{k-\lambda} x^{\lambda}\}, \quad k > 1.$$

This proves the second inequality in (2.15). Next using Bernoulli’s inequality we have

$$(x + y)^k = y^k \left(1 + \frac{x}{y}\right)^k \geq y^k \left(1 + k \frac{x}{y}\right) = y^k + kxy^{k-1} \geq y^k + kx^k, \quad k > 1.$$

Then

$$(x + y)^k - x^k - y^k \geq (k-1)x^k = (k-1)\min\{x^k, y^k\}, \quad k > 1.$$

This ends up the proof. \(\Box\)

**Lemma 2.6.** Let $c \geq 0$, $\varrho(v) = c + |v|^2$, $v \in \mathbb{R}^3$, and let $k > 1$, $0 \leq \lambda \leq \min\{1, k/2\}$. Then for all $(v, v_*, \sigma) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2$

$$[\varrho(v')]^k + [\varrho(v_*')]^k - [\varrho(v)]^k - [\varrho(v_*)]^k$$

$$\leq 2(2^k - 2)\{[\varrho(v)]^{k-\lambda} [\varrho(v_*)]^{\lambda} + [\varrho(v)]^{\lambda} [\varrho(v_*)]^{k-\lambda}\} - 4^{-k}(k-1)[\kappa(\theta)]^k [\varrho(v)]^k \quad (2.17)$$

where $v', v_*$ are given by (1.2), and

$$\kappa(\theta) = \min\{(1 - \sin(\theta/2))^2, (1 - \cos(\theta/2))^2\}, \quad \theta = \arccos(\langle \mathbf{n}, \sigma \rangle) \in [0, \pi] \quad (2.18)$$

$$\mathbf{n} = \frac{v - v_*}{|v - v_*|} \quad \text{if} \quad v \neq v_*; \quad \mathbf{n} = \mathbf{e}_1 \quad \text{if} \quad v = v_* \quad (2.19)$$

**Proof.** Given any $(v, v_*, \sigma) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2$, denote

$$\varrho = \varrho(v), \quad \varrho_* = \varrho(v_*), \quad \varrho' = \varrho(v'), \quad \varrho_*' = \varrho(v_*'),$$

$$D_k = (\varrho')^k + (\varrho_*')^k - (\varrho)^k - (\varrho_*)^k.$$

Then it follows from the identity $\varrho' + \varrho_*' = \varrho + \varrho_*$ that

$$D_k = (\varrho + \varrho_*)^k - (\varrho)^k - (\varrho_*)^k = [(\varrho' + \varrho_*')^k - (\varrho')^k - (\varrho_*')^k].$$
By Lemma 2.3 we conclude
\[(\varrho + \varrho_*)^k - (\varrho)^k - (\varrho_*)^k \leq (2^k - 2) \max\{(\varrho)^{k-\lambda} \varrho_\lambda, \varrho_\lambda^{\lambda}(\varrho_*)^{k-\lambda}\}\]
and
\[(\varrho' + \varrho'_*)^k - (\varrho')^k - (\varrho'_*)^k \geq (k - 1) \min\{(\varrho')^k, (\varrho'_*)^k\}.
\]
Thus
\[D_k \leq (2^k - 2) \max\{(\varrho)^{k-\lambda} \varrho_\lambda, \varrho_\lambda^{\lambda}(\varrho_*)^{k-\lambda}\} - (k - 1) \min\{(\varrho')^k, (\varrho'_*)^k\}. \tag{2.20}\]
This implies first that (2.17) holds for \(\kappa(\theta) = 0\). Now suppose that \(\kappa(\theta) > 0\), i.e. \(0 < \cos(\theta/2) < 1\). By definition of \(\nu', \nu_*'\) we have
\[|\nu' - \nu| = |\nu - \nu_*| \sin(\theta/2), \quad |\nu_*' - \nu| = |\nu - \nu_*| \cos(\theta/2).
\]
From this we have
\[|\nu'| = |\nu' - \nu + \nu| \geq |\nu| - |\nu' - \nu| = |\nu| - |\nu - \nu_*| \sin(\theta/2),
\]
\[|\nu_*'| = |\nu_*' - \nu + \nu| \geq |\nu| - |\nu_*' - \nu| = |\nu| - |\nu - \nu_*| \cos(\theta/2).
\]
Let
\[M(\theta) = \max\left\{\frac{\sin(\theta/2)}{1 - \sin(\theta/2)}, \frac{\cos(\theta/2)}{1 - \cos(\theta/2)}\right\}.
\]
Then it follows from \(\max\{\sin(\theta/2), \cos(\theta/2)\} \geq 1/2\) that \(M(\theta) \geq 1\). At this stage we will look at two possibilities.

**Case 1:** \(|\nu| \geq 2M(\theta)|\nu_*|\). By definitions of \(\nu', \nu_*'\) and \(\cos(\theta)\), we have
\[|\nu'| \geq |\nu| - |\nu - \nu_*| \sin(\theta/2) \geq |\nu| - |\nu| \sin(\theta/2) - |\nu_*| \sin(\theta/2)
\]
\[= |\nu|(1 - \sin(\theta/2)) - |\nu_*| \sin(\theta/2) \geq \frac{1 - \sin(\theta/2)}{2} |\nu|,
\]
\[|\nu_*'| \geq |\nu| - |\nu - \nu_*| \cos(\theta/2) \geq |\nu| - |\nu| \cos(\theta/2) - |\nu_*| \cos(\theta/2)
\]
\[= |\nu|(1 - \cos(\theta/2)) - |\nu_*| \cos(\theta/2) \geq \frac{1 - \cos(\theta/2)}{2} |\nu|.
\]
These imply
\[\varrho' = c + |\nu'|^2 \geq c + \frac{1}{4} (1 - \sin(\theta/2))^2 |\nu|^2 \geq \frac{1}{4} (1 - \sin(\theta/2))^2 (c + |\nu|^2) = \frac{1}{4} (1 - \sin(\theta/2))^2 \varrho \geq \frac{1}{4} \kappa(\theta) \varrho,
\]
\[\varrho_*' = c + |\nu_*'|^2 \geq c + \frac{1}{4} (1 - \cos(\theta/2))^2 |\nu|^2 \geq \frac{1}{4} (1 - \cos(\theta/2))^2 (c + |\nu|^2) = \frac{1}{4} (1 - \cos(\theta/2))^2 \varrho \geq \frac{1}{4} \kappa(\theta) \varrho,
\]
and so
\[(k - 1) \min\{(\varrho')^k, (\varrho_*')^k\} \geq (k - 1) \frac{1}{4^k}[\kappa(\theta)]^k(\varrho)^k,
\]
hence, by (2.20), we have
\[D_k \leq (2^k - 2) \max\{(\varrho)^{k-\lambda} \varrho_\lambda, \varrho_\lambda^{\lambda}(\varrho_*)^{k-\lambda}\} - 4^{-k}(k - 1)[\kappa(\theta)]^k(\varrho)^k. \tag{2.21}\]
Case 2: $|v| \leq 2M(\theta)|v_*|$. Then (notice that $M(\theta) \geq 1$)

$$
\theta = c + |v|^2 \leq c + [2M(\theta)]^2|v_*|^2 \leq [2M(\theta)]^2(c + |v|^2) = [2M(\theta)]^2\varrho_*.
$$

From the inequality

$$
\min\{(1 - x)^2, (1 - y)^2\} \max\left\{\frac{x}{1 - x}, \frac{y}{1 - y}\right\} \leq \frac{1}{4} \quad \forall 0 < x, y < 1
$$

we have

$$
\kappa(\theta)M(\theta) = \min\{(1 - \sin(\theta/2))^2, (1 - \cos(\theta/2))^2\} \max\left\{\frac{\sin(\theta/2)}{1 - \sin(\theta/2)}, \frac{\cos(\theta/2)}{1 - \cos(\theta/2)}\right\} \leq \frac{1}{4}.
$$

This together with $\lambda \leq \frac{k}{4}$ and $k - 1 \leq 2^k - 2$ ($k > 1$) gives

$$
4^{-k}(k - 1)[\kappa(\theta)]^k(\varrho_*)^k \leq 4^{-k}(2^k - 2)(1/2)^k(\varrho_*)^k\gamma^k.
$$

Therefore, using (2.20) (omitting the negative term) we have

$$
D_k \leq (2^k - 2)(1 + (1/2)^k)\max\{(\varrho)^{k-\lambda}(\varrho_*)^{k-\gamma}, (\varrho)^{\gamma}(\varrho_*)^{\lambda}\} - 4^{-k}(k - 1)[\kappa(\theta)]^k(\varrho_*)^k.
$$

Combining (2.21) and (2.22) gives (2.17).

Corollary 2.7. Let $s > 2$. Then for all $(v, v_*, \sigma) \in \mathbb{R}^3 \times \mathbb{R}^3 \times S^2$ and $0 \leq \gamma \leq \min\{2, s/2\}$ we have

$$
\langle v'\rangle^s + \langle v_*'\rangle^s - \langle v\rangle^s - \langle v_*\rangle^s \leq 2(2^{s/2} - 2)\langle v\rangle^{s-\gamma}\langle v_\gamma\rangle + \langle v\rangle\gamma\langle v_*\rangle^{s-\gamma} - 2^{-s}(s/2 - 1)[\kappa(\theta)]^{s/2}\langle v\rangle^s,
$$

$$
|v'|^s + |v_*'|^s - |v|^s - |v_*|^s \leq 2(2^{s/2} - 2)(|v|^{s-\gamma}|v_*|^\gamma + |v|^\gamma|v_*|^{s-\gamma}) - 2^{-s}(s/2 - 1)[\kappa(\theta)]^{s/2}|v|^s
$$

where $v', v_*'$ are given in (1.2) and $\kappa(\theta), \theta$ are given in (2.18) and (2.19) respectively.

Proof. In Lemma 2.6, we take $k = s/2 > 1, \gamma = 2\lambda$ and take $c = 1, 0$ respectively, and then we deduce the above inequalities.

The lemma below will be frequently used when we deal with the estimates of some kinds of collision integral operators with cutoff such as $Q_K$ and $Q_{n,K}$.

Lemma 2.8. Let $x, y, z \geq 0$. Then

$$
|x \wedge z - y \wedge z| \leq |x - y|, \quad x \leq y \implies x \wedge z \leq y \wedge z, \quad (x + y) \wedge z \leq x \wedge z + y \wedge z.
$$

Proof. Fix $z \geq 0$ and let $f(X) = X \wedge z = \frac{1}{2}(X + z - |X - z|), X \geq 0$. The properties in this lemma follow easily from the fact that $f$ is concave and non-decreasing on $[0, \infty)$.

The next lemma deals with the completeness of some function spaces (e.g. $L^\infty([0, \infty); L^1(\mathbb{R}^3))$). This completeness is of course important when using for instance the fixed point theorem of contractive mappings to prove the existence of solutions of some integral equations. A proof of such a lemma should be able to find from some textbooks. For convenience of the reader we would like to present here a proof.
Lemma 2.9. Let \( \Omega \subset \mathbb{R}^N \) (with \( N \in \mathbb{N} \)) be a Lebesgue measurable set, let \( I = [0, \infty) \) or \( I = [0, T] \) with \( 0 < T < \infty \). Define

\[
L^\infty(I; L^1(\Omega)) = \left\{ f : I \times \Omega \to [-\infty, \infty] \text{ is measurable on } I \times \Omega \left| \sup_{t \in I} \int_{\Omega} |f(t,v)| dv < \infty \right\} \tag{2.23}
\]

with the norm

\[
\|f\| := \sup_{t \in I} \|f(t)\|_{L^1(\Omega)} = \sup_{t \in I} \|f(t, \cdot)\|_{L^1(\Omega)} = \sup_{t \in I} \int_{\Omega} |f(t,v)| dv, \quad f \in L^\infty(I; L^1(\Omega)). \tag{2.24}
\]

Then \( L^\infty(I; L^1(\Omega)) \) with the norm \( \| \cdot \| \) is a Banach space.

Furthermore, let \( \{f^n\}_{n=1}^\infty \) be a bounded sequence in \( L^\infty(I; L^1(\Omega)) \) satisfying that for any \( t \in I \), \( \{f^n(t, \cdot)\}_{n=1}^\infty \) is a Cauchy sequence in \( L^1(\Omega) \). Then there is a function \( f \in L^\infty(I; L^1(\Omega)) \) such that \( \|f^n(t, \cdot) - f(t, \cdot)\|_{L^1(\Omega)} (n \to \infty) \) for all \( t \in I \). Besides if assume in addition that all \( f^n \) are nonnegative on \( I \times \Omega \), then \( f \) is also nonnegative on \( I \times \Omega \).

**Proof.** It is obvious that \( L^\infty(I; L^1(\Omega)) \) is a real normed linear space. We first prove the second part in **Step 1** since the first part (to be proved in **Step 2**) is relatively easy.

**Step 1.** By the assumption and the compactness of \( L^1(\Omega) \) we know that for any \( t \in I \) there is a function \( g(t, \cdot) \in L^1(\Omega) \) such that

\[
\lim_{n \to \infty} \|f^n(t) - g(t)\|_{L^1(\Omega)} = 0, \quad \sup_{t \in I} \|g(t)\|_{L^1(\Omega)} \leq \sup_{n \geq 1} \|f^n\| < \infty. \tag{2.25}
\]

In order to prove measurability in full variables we consider an integrable weight function:

\[
\rho(t) = e^{-t} \quad \text{if} \quad I = [0, \infty); \quad \rho(t) = 1 \quad \text{if} \quad I = [0, T].
\]

We need to show that \( \{f^n\}_{n=1}^\infty \) is a Cauchy sequence in \( L^1(I \times \Omega, \rho(t)dt dv) \). In fact we have

\[
\sup_{m \geq n} \int_I \int_{\Omega} \rho(t)|f^m(t,v) - f^n(t,v)| dv dt = \sup_{m \geq n} \int_I \rho(t)||f^m(t) - f^n(t)||_{L^1(\Omega)} dt \\
\leq \int_I \rho(t) \sup_{m \geq n} ||f^m(t) - f^n(t)||_{L^1(\Omega)} dt = \int_I \rho(t) \omega_n(t) dt
\]

where \( \omega_n(t) = \sup_{m \geq n} ||f^m(t) - f^n(t)||_{L^1(\Omega)} \). Next, from \( (2.24) \) we have

\[
\lim_{n \to \infty} \omega_n(t) = 0 \quad \forall t \in I; \quad \sup_{n \in \mathbb{N}, t \in I} \omega_n(t) \leq 2 \sup_{n \geq 1} \|f^n\| < \infty.
\]

This together with Lebesgue’s dominated convergence implies that \( \int_I \rho(t) \omega_n(t) dt \to 0 \) \((n \to \infty)\). Thus \( \{f^n\}_{n=1}^\infty \) is a Cauchy sequence in \( L^1(I \times \Omega, \rho(t)dt dv) \). Since \( L^1(I \times \Omega, \rho(t)dt dv) \) is complete, there is a function \( h \in L^1(I \times \Omega, \rho(t)dt dv) \) such that

\[
\int_I \rho(t)||f^n(t) - h(t)||_{L^1(\Omega)} dt = \int_I \rho(t)||f^n(t,v) - h(t,v)||_{L^1(\Omega)} dv dt \to 0 \quad \text{as} \quad n \to \infty. \tag{2.26}
\]

Now consider two sets

\[
Z_1 = \{ t \in I \mid h(t, \cdot) \notin L^1(\Omega) \}, \\
Z_2 = \{ t \in I \setminus Z_1 \mid \|f^n(t) - h(t)\|_{L^1(\Omega)} \neq 0 \}.
\]
We prove that $Z_1, Z_2$ are null sets. From Fubini’s theorem we know that $\text{mes}(Z_1) = 0$. To prove that $\text{mes}(Z_2) = 0$, we first use Fatou’s lemma to obtain
\[
\int_I \rho(t) \liminf_{n \to \infty} \|f^n(t) - h(t)\|_{L^1(\Omega)} \, dt \leq \liminf_{n \to \infty} \int_I \rho(t) \|f^n(t) - h(t)\|_{L^1(\Omega)} \, dt = 0
\]
which implies that
\[
Z_3 := \{ t \in I \setminus Z_1 \mid \liminf_{n \to \infty} \|f^n(t) - h(t)\|_{L^1(\Omega)} \neq 0 \}
\]
has measure zero. Next take any $t \in I \setminus (Z_1 \cup Z_3)$. We have $h(t) \in L^1(\Omega)$ and
\[
\liminf_{n \to \infty} \|f^n(t) - h(t)\|_{L^1(\Omega)} = 0
\]
which implies that there exists a subsequence $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$ (depending on $t$) such that
\[
\lim_{k \to \infty} \|f^{n_k}(t) - h(t)\|_{L^1(\Omega)} = 0.
\]
Comparing this with (2.25) we conclude that $h(t, v) = g(t, v)$ a.e. $v \in \mathbb{R}^3$. Thus using (2.25) again we obtain
\[
\|f^n(t) - h(t)\|_{L^1(\Omega)} = \|f^n(t) - g(t)\|_{L^1(\Omega)} \to 0 \quad (n \to \infty)
\]
which implies that $t \in I \setminus Z_2$. This proves that $I \setminus (Z_1 \cup Z_3) \subset I \setminus Z_2$, i.e. $Z_2 \subset Z_1 \cup Z_3$ and so $\text{mes}(Z_2) = 0$. Let $Z = Z_1 \cup Z_2$ and define
\[
f(t, v) = \begin{cases} g(t, v), & (t, v) \in Z \times \Omega; \\ h(t, v), & t \in (I \setminus Z) \times \Omega. \end{cases}
\]
Then $f$ is measurable on $I \times \Omega$ (since $\text{mes}(Z) = 0$), and for any $t \in I$, the function $v \mapsto f(t, v)$ belongs to $L^1(\Omega)$. Also it is easily checked that
\[
\lim_{n \to \infty} \|f^n(t) - f(t)\|_{L^1(\Omega)} = 0 \quad \text{for every } t \in I. \tag{2.28}
\]
From this and the boundedness of $\{f^n\}_{n=1}^\infty$ in $L^\infty(I; L^1(\Omega))$ we have $\sup_{t \in I} \|f(t)\|_{L^1(\Omega)} \leq \sup_{n \geq 1} \|f^n\| < \infty$. Thus $f \in L^\infty(I; L^1(\Omega))$.

Finally suppose in addition that all $f^n$ are nonnegative on $I \times \Omega$, then from (2.25) and (2.20) we see that $g, h$ can be chosen as nonnegative functions on $I \times \Omega$. It follows from the definition (2.27) that $f$ is also nonnegative on $I \times \Omega$. Thus we have finished the proof of the second part of this lemma.

**Step 2.** We now prove the first part of this lemma. Let $\{f^n\}_{n=1}^\infty$ be a Cauchy sequence in $L^\infty(I; L^1(\Omega))$. Then $\{f^n\}_{n=1}^\infty$ is bounded in $L^\infty(I; L^1(\Omega))$ and recalling the definition of $\| \cdot \|$ we see from **Step 1** that there exists a function $f \in L^\infty(I; L^1(\Omega))$ such that (2.25) holds true. From this we have $\|f^n(t) - f(t)\|_{L^1(\Omega)} = \lim_{m \to \infty} \|f^n(t) - f^m(t)\|_{L^1(\Omega)}$ and so
\[
\|f^n - f\| = \sup_{t \in I} \|f^n(t) - f(t)\|_{L^1(\Omega)} \leq \sup_{m \geq n} \|f^n - f^m\| \to 0 \quad (n \to \infty).
\]
This proves that $\{f^n\}_{n=1}^{\infty}$ converges in $L^\infty(I;L^1(\Omega))$ and so $L^\infty(I;L^1(\Omega))$ is a Banach space. \hfill \Box

The lemma below will help us to prove a property that if $f$ is a mild solution of Eq.\,(2.14) and satisfies $f(t,v) \leq K$ for all $(t,v) \in [0,\infty) \times (\mathbb{R}^3 \setminus Z)$ with a null set $Z$ independent of $t$, then $f$ is a mild solution of Eq.\,(1.22).

**Lemma 2.10.** Let $I \subset \mathbb{R}$ be an interval, $\Omega \subset \mathbb{R}^N$ an Lebesgue measurable set. Let $f, g : I \times \Omega \to [-\infty, \infty]$ be Lebesgue measurable functions satisfying

(i) for almost every $v \in \Omega$, $t \mapsto f(t,v)$, $t \mapsto g(t,v)$ are continuous on $I$, 

(ii) $f(t,v) = g(t,v)$ for almost every $(t,v) \in I \times \Omega$.

Then there is a common null set $Z \subset \Omega$, such that $f(t,v) = g(t,v)$ for all $(t,v) \in I \times (\Omega \setminus Z)$.

**Proof.** By assumption (i), there is a null set $Z_1 \subset \Omega$ such that for every $v \in \Omega \setminus Z_1$, $t \mapsto f(t,v)$, $t \mapsto g(t,v)$ are continuous on $I$. Let $S = \{(t,v) \in I \times \Omega \mid f(t,v) \neq g(t,v)\}$, $S_v = \{t \in I \mid (t,v) \in S\}, v \in \Omega$. By assumption (ii) we have $\text{mes}(S) = 0$ and so by Fubini’s theorem, the set $Z_2 = \{v \in \Omega \mid S_v \text{ is not a null set}\}$ has measure zero. Let $Z = Z_1 \cup Z_2$. Then $\text{mes}(Z) = 0$ and for any $(t,v) \in I \times (\Omega \setminus Z)$, if $t \in I \setminus S_v$, then $f(t,v) = g(t,v)$; if $t \in S_v$, then since $S_v$ is a null set which implies that $I \setminus S_v$ is dense in $I$, there is a sequence $\{t_n\}_{n=1}^{\infty} \subset I \setminus S_v$ such that $t_n \to t (n \to \infty)$ and so, by continuity, $f(t,v) = \lim_{n \to \infty} f(t_n,v) = \lim_{n \to \infty} g(t_n,v) = g(t,v)$. \hfill \Box

**Lemma 2.11.** If $f(t)$ is absolutely continuous on $[0,T]$, $G(y)$ is Lipschitz continuous on $[a,b]$ and $f([0,T]) \subset [a,b]$. Then

$$G(f(t)) = G(f(0)) + \int_0^t G_1(f(\tau)) f'(\tau) d\tau, \quad t \in [0,T]$$

where $G_1(y) = \frac{dy}{dy} G(y)$ a.e. on $[a,b]$.

**Proof.** See [16], p.223 Theorem 4.3, p.263 Theorem 4.9, and note that by assumption of the lemma the function $t \mapsto G(f(t))$ is also absolutely continuous on $[0,T]$. \hfill \Box

Our next lemma will be used to prove that condition (1.22) can imply the very high temperature condition (1.24).

**Lemma 2.12.** Given constants $0 < p < q < \infty$. Let $\phi$ be measurable on $[0,\infty)$ with $0 \leq \phi \leq 1$ and $0 < \int_0^\infty r^{q-1} \phi(r) dr < \infty$. Then

$$\left(p \int_0^\infty r^{p-1} \phi(r) dr\right)^{\frac{1}{p}} \leq \left(q \int_0^\infty r^{q-1} \phi(r) dr\right)^{\frac{1}{q}},$$

and the equality sign holds if and only if there is a constant $0 < R < \infty$ such that $\phi = 1_{[0,R]}$ a.e. on $[0,\infty)$.

**Proof.** See [9], p.382 Lemma 4. \hfill \Box
The next two propositions are concerned with computation formula and basic estimates for certain general collision integrals. They are very useful in obtaining the pointwise estimates of some kinds of collision integral operators.

**Proposition 2.13.** Let \( W(z, \sigma, v) = W_1(|z|, (\frac{\sigma}{|z|}, \sigma), v) \) where \( W_1(r, s, v) \) is a nonnegative Lebesgue measurable function on \([0, \infty) \times [-1, 1] \times \mathbb{R}^3\). Then

\[
\int_{\mathbb{R}^3} \int_{S^2} W(v - v_*, \sigma, v') d\sigma dv_* = \int_{\mathbb{R}^3} \int_{S^2} \frac{1}{\sin^3(\theta/2)} W(\frac{v - v_*}{\sin(\theta/2)}, \sigma, v_*) d\sigma dv_* ,
\]

\[
\int_{\mathbb{R}^3} \int_{S^2} W(v - v_*, \sigma, v') d\sigma dv_* = \int_{\mathbb{R}^3} \int_{S^2} \frac{1}{\cos^3(\theta/2)} W(\frac{v - v_*}{\cos(\theta/2)}, \sigma, v_*) d\sigma dv_* ,
\]

where \( v', v_*' \) are given in (1.2), \( \theta \) is given in (2.10), \( \sin(\theta/2) = \sqrt{\frac{1 - (n, \sigma)}{2}} \), \( \cos(\theta/2) = \sqrt{\frac{1 + (n, \sigma)}{2}} \).

**Proof.** See [12], p.1715 Proposition 2.1.

**Proposition 2.14.** Let \( p, q, \gamma \geq 0, 0 \leq g \in L^1_{p+\gamma}(\mathbb{R}^3) \cap L^\infty_p(\mathbb{R}^3), 0 \leq f \in L^1_{q+\gamma}(\mathbb{R}^3) \cap L^\infty_q(\mathbb{R}^3) \). Then

\[
\int_{\mathbb{R}^3} \|v - v_*\|^\gamma (v')^p (v_*')^q g(v') f(v'_*) d\sigma dv_* \leq 2^{(3+\gamma)/2} |S^2| \left( \|f\|_{L^\infty_p} \|g\|_{L^1_{p+\gamma}} + \|g\|_{L^\infty_q} \|f\|_{L^1_{q+\gamma}} \right) (v)^\gamma
\]

for all \( v \in \mathbb{R}^3 \), where \( v', v_*' \) are given in (1.2).

**Proof.** From Proposition 2.13 and the inequality \(|v - v_*| \leq (v) (v_*)\) we have

\[
\int_{\mathbb{R}^3} \int_{S^2} \frac{|v - v_*|^\gamma (v')^p (v_*')^q g(v') f(v'_*)}{\sin^3(\theta/2)} d\sigma dv_*
\]

\[
= \int_{\mathbb{R}^3} \int_{S^2} \frac{|v - v_*|^\gamma (v')^p (v_*')^q g(v') f(v'_*)}{\sin^3(\theta/2)} 1_{\{ \frac{|v - v_*|}{\sin(\theta/2)} \leq 0 \}} d\sigma dv_*
\]

\[
+ \int_{\mathbb{R}^3} \int_{S^2} \frac{|v - v_*|^\gamma (v')^p (v_*')^q g(v') f(v'_*)}{\sin^3(\theta/2)} 1_{\{ \frac{|v - v_*|}{\sin(\theta/2)} > 0 \}} d\sigma dv_*
\]

\[
\leq \|f\|_{L^\infty_p} \int_{\mathbb{R}^3} \int_{S^2} \frac{|v - v_*|^\gamma}{\sin^3(\theta/2)} (v)^p (v_*)^q g(v_*) \frac{1}{\sin^3(\theta/2)} 1_{\{ \frac{|v - v_*|}{\sin(\theta/2)} \geq \sqrt{\frac{1 - (n, \sigma)}{2}} \}} d\sigma dv_*
\]

\[
+ \|g\|_{L^\infty_q} \int_{\mathbb{R}^3} \int_{S^2} \frac{|v - v_*|^\gamma}{\cos^3(\theta/2)} (v)^q f(v_*) \frac{1}{\cos^3(\theta/2)} 1_{\{ \frac{|v - v_*|}{\cos(\theta/2)} \geq \sqrt{\frac{1 + (n, \sigma)}{2}} \}} d\sigma dv_*
\]

\[
\leq 2^{(3+\gamma)/2} \|f\|_{L^\infty_p} \int_{\mathbb{R}^3} \int_{S^2} |v - v_*|^\gamma (v)^p (v_*)^q g(v_*) d\sigma dv_* + 2^{(3+\gamma)/2} \|g\|_{L^\infty_q} \int_{\mathbb{R}^3} \int_{S^2} |v - v_*|^\gamma (v)^q f(v_*) d\sigma dv_*
\]

\[
\leq 2^{(3+\gamma)/2} |S^2| \left( \|f\|_{L^\infty_p} \|g\|_{L^1_{p+\gamma}} + \|g\|_{L^\infty_q} \|f\|_{L^1_{q+\gamma}} \right) (v)^\gamma, \quad v \in \mathbb{R}^3.
\]

\[\square\]

The proposition below will be used to prove the existence of mild solutions of Eq (2.13) in Section 3.

**Proposition 2.15.** For any \( n \in \mathbb{N} \), let \( B_n = B \wedge n \) with \( B \) the collision kernel satisfying (1.10). For any \( 0 \leq f, g \in L^1(\mathbb{R}^3) \) define

\[
I_n(f, g) = \int_{\mathbb{R}^3} \int_{S^2} B_n(v - v_*, \sigma)(f' \wedge n)(f'_* \wedge n)(1 + f \wedge K + f_* \wedge K)
\]

\[
- (g' \wedge n)(g'_* \wedge n)(1 + g \wedge K + g_* \wedge K) d\sigma dv_* dv.
\]

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Then
\[ I_n(f, g) \leq (1 + 2K)|S|^2|n||f||L^1 + ||g||L^1)\|f - g\|_{L^1} + 2^{7/2}|S|^2|n||g \wedge n||L^\infty ||g\|_{L^1} \|f - g\|_{L^1}. \] (2.30)

**Proof.** Compute
\[ I_n(f, g) = \int_{\mathbb{R}^3} B_n(v - v_*, \sigma)\langle f' \wedge n \rangle\langle f \wedge n \rangle(1 + f + K + f_\wedge K) \]
\[- (g' \wedge n)(g_\wedge n)(1 + g \wedge K + g_\wedge K)\|d\sigma dv, dv \]
\[ \leq (1 + 2K)\int_{\mathbb{R}^3} B_n(v - v_*, \sigma)\langle f' - g' \rangle\langle f \wedge n \rangle \]
\[ + (g' \wedge n)(f_\wedge n - g_\wedge n) + (g' \wedge n)(g_\wedge n)(|f - g| + |f_\wedge - g_\wedge|)\|d\sigma dv, dv \]
\[ \leq (1 + 2K)|S|^2|n\int_{\mathbb{R}^3} |f - g|d\sigma dv \quad (1 + 2K)|S|^2|n\int_{\mathbb{R}^3} g|f_\wedge - g_\wedge|d\sigma dv \]
\[ + 2n\int_{\mathbb{R}^3} |f - g|\int_{\mathbb{R}^3} (g' \wedge n)(g_\wedge n)d\sigma dv, dv \]
where we used (2.2). According to Proposition 2.14 (with \( p = q = \gamma = 0 \)) we have for any \( v \in \mathbb{R}^3 \)
\[ \int_{\mathbb{R}^3} (g' \wedge n)(g_\wedge n)d\sigma dv \leq 2^{5/2}|S|^2||g \wedge n||L^\infty \|g \wedge n||L^1. \]
Combining the above gives (2.30). \( \square \)

The following proposition is concerned with estimates of collision gain operator \( Q^+(\cdot, \cdot) \).

**Proposition 2.16.** Let \( Q^+(\cdot, \cdot) \) be defined by (2.3) with the collision kernel \( B \) satisfying (1.16) and let \( 0 \leq f, g \in L^1_1(\mathbb{R}^3) \). Then
\[ Q^+(f, g)(v) = Q^+(g, f)(v) \quad \forall v \in \mathbb{R}^3. \] (2.31)

Furthermore if \( 0 \leq f \in L^1_1(\mathbb{R}^3) \cap L^\infty (\mathbb{R}^3) \), then
\[ Q^+(f, f)(v) \leq 2^{5/2}\pi b\|f\|_{L^\infty} \|f\|_{L^1_1}(v) \quad \forall v \in \mathbb{R}^3 \] (2.32)
\[ \|Q^+(f, f)\|_{L^1} \leq 4\pi b\|f\|_{L^1_1}^2 \] (2.33)
\[ \|Q^+(f, f)\|_{L^2} \leq 2^{3/2+\pi}b\|f\|_{L^1_1}^2 \|f\|_{L^2_1}. \] (2.34)

**Proof.** Recalling (1.2) and (1.7), (2.31) can be easily proved by using the change of variables \((v_*, \sigma) \to (v_*, -\sigma)\). Next, it follows from (1.16) and Proposition 2.14 (with \( p = q = 0, \gamma = 1 \)) that
\[ Q^+(f, f)(v) \leq b\int_{\mathbb{R}^3} |v - v_*|f'f_\wedge d\sigma dv, dv \leq 8b|S|^2||f||_{L^\infty} \|f\|_{L^1_1}(v) \quad \forall v \in \mathbb{R}^3 \]
which proves (2.32). To prove (2.33), we have
\[ \|Q^+(f, f)\|_{L^1} \leq b\int_{\mathbb{R}^3} \langle v \rangle |f|f_\wedge d\sigma dv, dv \leq b|S|^2||f||_{L^1_1}^2 \quad \forall t \geq 0. \]
Finally, using Hölder’s inequality, Proposition 2.14 (with $p = q = \gamma = 0$) and the fact that $|v - v_*| \leq \langle v \rangle \langle v_* \rangle$ we have

$$
\|Q^+(f, f)\|_{L^2}^2 \leq \int_{\mathbb{R}^3} \left( b \int_{\mathbb{R}^3 \times S^2} |v - v_*| f' f'_* d\sigma d\nu_* \right)^2 dv
$$

$$
\leq b^2 \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3 \times S^2} |v - v_*|^2 f' f'_* d\sigma d\nu_* \right) \left( \int_{\mathbb{R}^3 \times S^2} f' f'_* d\sigma d\nu_* \right) dv
$$

$$
\leq 4 \sqrt{2} b^2 \|S^2\| (\|f\|_{L^2}^2) \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} (\langle v \rangle \langle v_* \rangle)^2 f f_* d\sigma d\nu_* dv
$$

$$
= 4 \sqrt{2} b^2 \|S^2\|^2 (\|f\|_{L^2}^2) \|f\|_{L^2}^2 \|f\|_{L^2}^2.
$$

This gives (2.34). □

The last proposition below gives us very important tools which make the multi-step iterations of the collision gain operator $Q^+ (\cdot, \cdot)$ work well. It should be noted that in this proposition the constants given in the explicit version are helpful in applications as will be seen in the proof of Theorem 1.2.

**Proposition 2.17.** Let $Q^+ (\cdot, \cdot)$ be defined by (2.3) with the collision kernel $B$ satisfying (1.10). Then

(a) If $0 \leq f \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, $0 \leq g, h \in L^1(\mathbb{R}^3)$, then

$$
Q^+ (f, Q^+(g, h))(v) \leq 2^{5 + \frac{2}{p}} \pi^{\frac{1}{p}} b^2 \|f\|_{L^2}^\frac{1}{p} \|f\|_{L^1}^\frac{1}{p} \|g\|_{L^1} \|h\|_{L^1} \text{ for all } v \in \mathbb{R}^3. \tag{2.35}
$$

(b) If $1 \leq p \leq 2$ and $0 \leq f \in L^1_{\frac{1}{p}}(\mathbb{R}^3)$, $0 \leq g, h \in L^1_{\frac{1}{p}}(\mathbb{R}^3)$, then

$$
\|Q^+ (f, Q^+(g, h))\|_{L^p} \leq 2^{4 + \frac{2}{p}} \pi^{1 + \frac{1}{p}} b^2 \|f\|_{L^1_{\frac{1}{p}}} \|f\|_{L^1_{\frac{1}{p}}} \|g\|_{L^1_{\frac{1}{p}}} \|h\|_{L^1_{\frac{1}{p}}} \tag{2.36}
$$

and consequently (with $p = 1, 2$)

$$
Q^+ (Q^+(f, Q^+(g, g)), Q^+(h, h))(v) \leq 2^{11} \pi^3 b^4 \|f\|_{L^1}^\frac{1}{2} \|f\|_{L^1}^\frac{1}{2} \|g\|_{L^1}^\frac{1}{2} \|g\|_{L^1}^\frac{1}{2} \|h\|_{L^1}^2 \tag{2.37}
$$

for all $v \in \mathbb{R}^3$.

**Proof.** From Lemma 2.1 in [15] we have

$$
Q^+ (f, Q^+(g, h))(v) = \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} K_B(v, v_*, w, w_*) f(v_*) g(w) d\nu_* d\sigma d\nu_* \tag{2.38}
$$

where $K_B : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ is defined by

$$
K_B(v, v_*, w, w_*) = \begin{cases} 
\frac{8}{|v - v_*||w - w_*|} \zeta \left( \left( \frac{2v - (w + w_*)}{|w - w_*|} \right) \right) & \frac{B(w - w_*, \sigma) B(w' - v_*, \sigma')}{|w' - v_*|} \, d\nu \, d\sigma \\
0 & \text{if } |v - v_*||w - w_*| \neq 0 \tag{2.39}
\end{cases}
$$

where

$$
\zeta(t) := (1 - t^2)^\frac{3}{2} \mathbb{1}_{(-1,1)}(t), \quad t \in \mathbb{R}, \quad \mathbb{S}^1(n) = \{ \omega \in \mathbb{S}^2 | \omega \perp n \}.
$$

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and $d^1\omega$ denotes the sphere measure element of $S^1(\mathbf{n})$. If $|v - v_*| |w - w_*| \neq 0$, then we compute

$$K_B(v, v_*, w, w_*) = \frac{8}{|v - v_*||w - w_*|} \zeta \left( \left\langle \mathbf{n}, \frac{2v - (w + w_*)}{|w - w_*|} \right\rangle \right) \int_{S^1(\mathbf{n})} B(w - w_*, \sigma) B(w' - v_*, \sigma) d^1\omega$$

$$\leq b^2 \frac{8}{|v - v_*||w - w_*|} \zeta \left( \left\langle \mathbf{n}, \frac{2v - (w + w_*)}{|w - w_*|} \right\rangle \right) \int_{S^1(\mathbf{n})} \frac{|w - w_*||w' - v_*|}{|w' - v_*|} d^1\omega$$

$$= 8b^2 |S^1| \frac{1}{|v - v_*|} \zeta \left( \left\langle \mathbf{n}, \frac{2v - (w + w_*)}{|w - w_*|} \right\rangle \right) \leq 8b^2 |S^1| \frac{1}{|v - v_*|}. \quad (2.40)$$

Thus from (2.38), (2.40), and Lemma 2.5 in [15] we have for any $v \in \mathbb{R}^3$

$$Q^+(f, Q^+(g, h))(v) \leq \iint_{\mathbb{R}^3 \times \mathbb{R}^3} 8b^2 |S^1| \frac{1}{|v - v_*|} f(v_*) g(w) h(w_*) dv_* dw_*$$

$$\leq 8b^2 |S^1| ||g||_L^1 ||h||_L^1 \int_{\mathbb{R}^3} \frac{1}{|v - v_*|} f(v_*) dv_* \leq 16b^2 |S^1| ||S^2|^{\frac{1}{2}} ||f||_{L^2} \frac{4}{3} ||g||_L^1 ||h||_L^1$$

which gives (2.39).

Define

$$T_{w, w_*}(f)(v) := \int_{\mathbb{R}^3} K_B(v, v_*, w, w_*) f(v_*) dv_* \quad \text{for all } v, w, w_* \in \mathbb{R}^3.$$

Then from (2.38) we have for any $1 \leq p \leq 2$

$$||Q^+(f, Q^+(g, h))||_{L^p} \leq \iint_{\mathbb{R}^3 \times \mathbb{R}^3} ||T_{w, w_*}(f)||_{L^p} g(w) h(w_*) dw dw_* \quad (2.41)$$

Next we define for any $1 \leq \alpha \leq 2$

$$J_\alpha(w, w_*) := \int_{\mathbb{R}^3} \frac{1}{|v - v_*|^{\alpha}} 1_{(-1,1)} \left( \left\langle \mathbf{n}, \frac{2v - (w + w_*)}{|w - w_*|} \right\rangle \right) dv, \quad w, w_* \in \mathbb{R}^3.$$

Making the change of variable

$$v = v_* + \frac{|w - w_*|}{2} r \sigma, \quad dv = \frac{|w - w_*|}{2} r^2 dr d\sigma$$

we have

$$J_\alpha(w, w_*) = \left| \frac{w - w_*}{2} \right|^{3-\alpha} \int_{S^2} I(\langle u, \sigma \rangle) d\sigma$$

where

$$I(\langle u, \sigma \rangle) = \int_0^\infty r^{2-\alpha} 1_{\{r + \langle u, \sigma \rangle < 1\}} dr, \quad u = \frac{2v_* - (w + w_*)}{|w - w_*|}.$$

If $\langle u, \sigma \rangle \geq 1$, then $I(\langle u, \sigma \rangle) = 0$; if $\langle u, \sigma \rangle < 1$, then

$$I(\langle u, \sigma \rangle) \leq 2(1 + |u|)^{2-\alpha} \leq 2 \left( \frac{|w - w_*|}{2} + \frac{|w + w_*|}{2} + |v_*| \right)^{2-\alpha} \leq \frac{2(|v_*| (w) (w_*)^{2-\alpha}}{|w - w_*|^{2-\alpha}}$$

where we used

$$|w - w_*| \leq \langle w \rangle \langle w_* \rangle,$$

$$|w' - v_*| = |w' - v + v - v_*| = \sqrt{|v - v_*|^2 + |v - w'|^2},$$

$$|v - v_*| \leq |w' - v_*| \leq \frac{|w + w_*|}{2} + \frac{|w - w_*|}{2} + |v_*| \leq \langle w \rangle \langle w_* \rangle.$$
It follows that for any $w, w_\ast \in \mathbb{R}^3$

$$J_n(w, w_\ast) = \left| \frac{w - w_\ast}{2} \right|^{3-\alpha} \int_{S^2} I((u, \sigma))\,d\sigma \leq |S^2||w - w_\ast|\langle \langle w_\ast \rangle \langle w_\ast \rangle \rangle^{2-\alpha}. \quad (2.42)$$

Combining this with (2.40) we deduce

$$\left( \int_{\mathbb{R}^3} |K_B(v, u, w, w_\ast)|^p\,dv \right)^{\frac{1}{p}} \leq 8b^2|S^1|(|J_p(w, w_\ast)|)^{\frac{1}{p}}$$

$$\leq 8b^2|S^1||S^2|^{\frac{1}{p}}|w - w_\ast|^{\frac{1}{p}}\langle \langle w_\ast \rangle \langle w_\ast \rangle \rangle^{\frac{2-\alpha}{p}}$$

$$\leq 8b^2|S^1||S^2|\langle \langle w_\ast \rangle \rangle^{\frac{2-\alpha}{p}}|\langle \langle w_\ast \rangle \rangle^{\frac{2-\alpha}{p}}\ , \quad 1 \leq p \leq 2.$$ 

Therefore using Lemma 2.4 in [15] and the assumption on $f$ we conclude that for any $1 \leq p \leq 2$

$$\|T_{w, w_\ast}(f)\|_{L^p} \leq 8b^2|S^1||S^2|\langle \langle w_\ast \rangle \rangle^{\frac{2-\alpha}{p}}\|f\|_{L^1_{\alpha}} \quad \forall w, w_\ast \in \mathbb{R}^3 \quad (2.43)$$

which together with (2.33) proves (2.36).

Finally we prove (2.37). According to (2.35) and (2.36) (with $p = 1, 2$) we have for all $v \in \mathbb{R}^3$

$$Q^+(Q^+(f, Q^+(g, g)), Q^+(h, h))(v)$$

$$\leq 16b^2|S^1||S^2|^{\frac{3}{2}}\|h\|_{L^3}^2\|Q^+(f, Q^+(g, g))\|_{L^3}^{\frac{1}{2}}\|Q^+(f, Q^+(g, g))\|_{L^3}^{\frac{1}{2}}$$

$$\leq 16b^2|S^1||S^2|^{\frac{3}{2}}\|h\|_{L^3}^2(8b^2|S^1||S^2|\|f\|_{L^1_{\alpha}}\|g\|_{L^1_{\alpha}}^2)^{\frac{1}{2}}(8b^2|S^1||S^2|\|f\|_{L^1_{\alpha}}\|g\|_{L^1_{\alpha}}^2)^{\frac{1}{2}}$$

$$= 2^{11}n^3b^4\|f\|_{L^1_{\alpha}}^\frac{3}{2}\|f\|_{L^1_{\alpha}}^\frac{1}{2}\|g\|_{L^1_{\alpha}}^\frac{3}{2}\|g\|_{L^1_{\alpha}}^\frac{1}{2}\|h\|_{L^3}^2.$$ 

\[
\square
\]

3 Mild solutions of cutoff equations

This section is a further preparation for obtaining mild solutions of Eq.(2.14). We will first prove the existence of mild solutions $\{f^n\}_{n=1}^{\infty}$ of Eq.(2.13). Then we will show that $\{f^n(t)\}_{n=1}^{\infty}$ satisfy the $L^1$ relative compactness conditions, etc. Throughout this section, the constant $K > 0$ in Eq.(2.13) is fixed.

3.1 Existence of mild solutions of cutoff equations

In order to prove the global in time existence of mild solutions of Eq.(2.13), we first use fixed point theorem of contractive self-mappings to prove the existence of mild solutions of Eq.(2.13) on $[0, T] \times \mathbb{R}^3$ for some $0 < T < \infty$, then we will prove the conservation of the mass of $f$ and extend $f$ to $[0, \infty) \times \mathbb{R}^3$ so that the extended $f$ is a mild solution of Eq.(2.13) on $[0, \infty) \times \mathbb{R}^3$.

**Proposition 3.1.** For any $0 \leq f_0 \in L^1_1(\mathbb{R}^3)$ and any $n \in \mathbb{N}$, there exists a conservative mild solution $f^n$ of Eq.(2.13) on $[0, \infty) \times \mathbb{R}^3$ with the initial datum $f_0$ such that (see Remark 2.4)

$$f^n(t, v) = f_0(v) + \int_0^t Q_{n,K}(f^n)(\tau, v)\,d\tau \quad \forall (t, v) \in [0, \infty) \times \mathbb{R}^3$$

where $Q_{n,K}(\cdot)$ is defined in (2.12) with collision kernel $B$ satisfying (1.16).
**Proof. Step 1.** For any $T \in (0, \infty)$ recall the definition of $L^\infty([0,T]; L^1(\mathbb{R}^3))$ in (2.29) and define

$$\mathcal{A}_T = \{ f \in L^\infty([0,T]; L^1(\mathbb{R}^3)) : \|f\| := \sup_{t \in [0,T]} \|f(t)\|_{L^1} \leq 2\|f_0\|_{L^1} \}$$

with the distance

$$\|f - g\| = \sup_{t \in [0,T]} \|f(t) - g(t)\|_{L^1}, \quad f, g \in \mathcal{A}_T.$$ 

From Lemma 2.39 it is obvious that $\mathcal{A}_T$ is a closed subset of the Banach space $(L^\infty([0,T]; L^1(\mathbb{R}^3)))$ and so $\mathcal{A}_T$ is complete. Fix $n \in \mathbb{N}$. Let

$$J_n(f)(t, v) = f_0(v) + \int_0^t Q_{n,K}(|f|)(\tau, v) d\tau, \quad (t, v) \in [0, \infty) \times \mathbb{R}^3, \quad f \in \mathcal{A}_T.$$ 

To obtain contractiveness of $J_n$ we choose

$$T = T_n = \frac{1}{16(1 + 2K + 2^{3/2})|S^2|n\|f_0\|_{L^1}}.$$ 

Now we need to prove that $J_n : \mathcal{A}_{T_n} \to \mathcal{A}_{T_n}$ and

$$\sup_{t \in [0, T_n]} \|J_n(f)(t) - J_n(g)(t)\|_{L^1} \leq \frac{1}{2} \sup_{t \in [0, T_n]} \|f(t) - g(t)\|_{L^1}, \quad f, g \in \mathcal{A}_{T_n}. \quad (3.1)$$

In fact for any $f \in \mathcal{A}_{T_n}$ and any $t \in [0, T_n]$ we have

$$\begin{align*}
\int_{\mathbb{R}^3} |Q_{n,K}(|f|)(t, v)| dv & \leq \iint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} B_n(v - v_\ast, \sigma)(|f| \wedge n)(|f_\ast| \wedge n)(1 + |f| \wedge K + |f_\ast| \wedge K) d\sigma dv_\ast dv \\
& + \iint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} B_n(v - v_\ast, \sigma)(|f| \wedge n)(|f_\ast| \wedge n)(1 + |f'| \wedge K + |f_\ast'| \wedge K) d\sigma dv_\ast dv \\
& = 2L_n(|f(t)|, 0) \leq 2(1 + 2K)|S^2|n\|f(t)\|_{L^1}^2 \leq 8(1 + 2K)|S^2|n\|f_0\|_{L^1}^2 < \infty
\end{align*}$$

where we used (2.2) and Proposition 2.15. This implies that $(t, v) \mapsto J_n(f)(t, v)$ is measurable on $[0, T_n] \times \mathbb{R}^3$ and

$$\begin{align*}
\int_{\mathbb{R}^3} |J_n(f)(t, v)| dv & \leq \|f_0\|_{L^1} + \int_0^t d\tau \int_{\mathbb{R}^3} |Q_{n,K}(|f|)(\tau, v)| dv \\
& \leq \|f_0\|_{L^1} + 8(1 + 2K)|S^2|n\|f_0\|_{L^1} T_n \|f_0\|_{L^1} \leq 2\|f_0\|_{L^1} \quad \forall t \in [0, T_n]. \quad (3.2)
\end{align*}$$

Thus $\sup_{t \in [0, T_n]} \|J_n(f)(t)\|_{L^1} \leq 2\|f_0\|_{L^1}$ and so $J_n(f) \in \mathcal{A}_{T_n}$. Next for any $f, g \in \mathcal{A}_{T_n}$ and any $t \in [0, T_n]$ we have (by using Proposition 2.15)

$$\begin{align*}
\|J_n(f)(t) - J_n(g)(t)\|_{L^1} & \leq \int_0^t d\tau \int_{\mathbb{R}^3} |Q_{n,K}(|f|)(\tau, v) - Q_{n,K}(|g|)(\tau, v)| dv \\
& = 2 \int_0^t \mathcal{I}_{n,K}(|f(\tau)|, |g(\tau)|) d\tau \leq 2 \int_0^t (1 + 2K)|S^2|n\|f(\tau)\|_{L^1} + \|g(\tau)\|_{L^1}) \|f(\tau) - g(\tau)\|_{L^1} \\
& + 2n 2^{5/2}|S^2|n\|f(\tau)\|_{L^1} \|g(\tau)\|_{L^1} \|f(\tau) - g(\tau)\|_{L^1} d\tau \\
& \leq 2 \int_0^t (1 + 2K)|S^2|n\|f_0\|_{L^1} \|f(\tau) - g(\tau)\|_{L^1} + 2n 2^{5/2}|S^2|n\|f_0\|_{L^1} \|f(\tau) - g(\tau)\|_{L^1} d\tau \\
& \leq 8(1 + 2K + 2^{3/2})|S^2|n\|f_0\|_{L^1} T_n \sup_{t \in [0, T_n]} \|f(t) - g(t)\|_{L^1} \leq \frac{1}{2} \sup_{t \in [0, T_n]} \|f(t) - g(t)\|_{L^1}. \quad (3.3)
\end{align*}$$
This proves (3.1). Thus there exists a unique $f \in \mathcal{A}_{T_n}$ such that $J_n(f) = f$.

**Step 2.** Given any $n \in \mathbb{N}$. Let $T = T_n > 0$ be defined in **Step 1** and let $f^n := f$ obtained in **Step 1** be the unique fixed point of $J_n : \mathcal{A}_{T_n} \to \mathcal{A}_{T_n}$. From Proposition 2.14 and (3.2) we have for any $(t, v) \in [0, T_n] \times \mathbb{R}^3$

$$|Q_{n,K}^+(f^n)(t, v)| \leq \int_{\mathbb{R}^3 \times S^2} B_n(v - v_*, \sigma)(|f^n| \wedge n)(|f^n| \wedge n)(1 + |f^n| \wedge K + |f^n| \wedge K) d\sigma dv_*$

$$\leq (1 + 2K)n \int_{\mathbb{R}^3 \times S^2} (|f^n| \wedge n)(|f^n| \wedge n) d\sigma dv_* \leq 2\mathbb{E} (1 + 2K)^2 n^2 |S^2| ||f(t)||_{L^1} \leq 2\mathbb{E} (1 + 2K)^2 n^2 |S^2| ||f_0||_{L^1}.$$

$|Q_{n,K}^-(f^n)(t, v)| \leq n^2(1 + 2K) \int_{\mathbb{R}^3 \times S^2} |f^n| d\sigma dv_* \leq 2(1 + 2K)n^2 |S^2| ||f_0||_{L^1}.$

Fix any $v \in \mathbb{R}^3$. The function $t \mapsto f^n(t, v)$ is absolutely continues on $[0, T_n]$. Since $(\cdot)^+$ is Lipschitz continues, it follows from Lemma 2.11 that $t \mapsto (-f^n(t, v))^+$ is absolutely continues about $t \in [0, T_n]$. Since $f^n(0, v) = f_0(v) \geq 0$, it follows that for all $(t, v) \in [0, T_n] \times \mathbb{R}^3$

$$(-f^n(t, v))^+ = \int_0^t \left(-\frac{d}{dt}f^n(\tau, v)\mathbf{1}_{\{f^n(\tau, v) \leq 0\}}\right)d\tau = \int_0^t \left(-Q_{n,K}(f^n)(\tau, v)\mathbf{1}_{\{f^n(\tau, v) \leq 0\}}\right)d\tau$$

$$= \int_0^t \left(\int_{\mathbb{R}^3} |f^n(\tau, v_*)|d\sigma_*\right)f^n(\tau, v_*)\mathbf{1}_{\{f^n(\tau, v) \leq 0\}}d\tau$$

$$\leq (1 + 2K)n|S^2| \int_0^t \left(\int_{\mathbb{R}^3} |f^n(\tau, v_*)|d\sigma_*\right)f^n(\tau, v_*)\mathbf{1}_{\{f^n(\tau, v) \leq 0\}}d\tau$$

$$\leq 2(1 + 2K)n|S^2|||f_0||_{L^1}.$$

By Gronwall’s lemma we conclude that $(-f^n(t, v))^+ = 0$ for all $(t, v) \in [0, T_n] \times \mathbb{R}^3$. Thus we have proved $f^n(t, v) \geq 0$ on $[0, T_n] \times \mathbb{R}^3$ and so $f^n$ satisfies

$$f^n(t, v) = f_0(v) + \int_0^t Q_{n,K}(f^n)(\tau, v)d\tau \quad \forall (t, v) \in [0, T_n] \times \mathbb{R}^3.$$

Next we prove that $f^n(t)$ conserves the mass on $[0, T_n]$. For any $\varphi \in L^\infty(\mathbb{R}^3)$ and any $0 \leq t \leq T_n$, we have by using (3.2) that

$$\int_{\mathbb{R}^3} \varphi(v)(Q_{n,K}^+(f^n)(t, v) + Q_{n,K}^-(f^n)(t, v))dv$$

$$\leq (1 + 2K)n\|\varphi\|_{L^\infty} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} f^n(t, v')f^n(t, v_*) + f^n(t, v)f^n(t, v_*) d\sigma dv_* dv$$

$$\leq 2(1 + 2K)n\|\varphi\|_{L^\infty}|S^2|(\sup_{0 \leq t \leq T_n}||f^n(t)||_{L^1})^2 \leq 2(1 + 2K)n\|\varphi\|_{L^\infty}|S^2|(2||f_0||_{L^1})^2.$$

This integrability allows us to compute (as usual)

$$\int_{\mathbb{R}^3} \varphi(v)Q_{n,K}(f^n)(t, v)dv$$

$$= \int_{\mathbb{R}^3} \varphi(v) + \varphi(v_*) \int_{\mathbb{R}^3 \times S^2} B_n(v - v_*, \sigma)(|f^n| \wedge n)(|f^n| \wedge n)(1 + |f^n| \wedge K + |f^n| \wedge K)$$

$$- (f^n \wedge n)(f^n \wedge n)(1 + f^n \wedge K + f^n \wedge K)d\sigma dv_*.$$
According to Fubini’s theorem it follows that
\[
\int_{\mathbb{R}^3} \phi(v) f^n(t, v) dv = \int_{\mathbb{R}^3} \phi(v) f_0(v) dv + \frac{1}{2} \int_0^t d\tau \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B_n(v - v_*, \sigma)(f^n \wedge n)(f^n \wedge n) \times (1 + f^n \wedge K + f^n \wedge K)(\phi(v') + \phi(v') - \phi(v) - \phi(v_*)) d\sigma dv_* dv,
\]
\(t \in [0, T_n].\) \hspace{1cm} (3.4)

Taking \(\phi(v) \equiv 1\) gives the conservation of mass of \(f^n:\)
\[
\int_{\mathbb{R}^3} f^n(t, v) dv = \int_{\mathbb{R}^3} f_0(v) dv = \|f_0\|_{L^1} < \infty, \quad \forall t \in [0, T_n].
\]

Since \(f^n\) is nonnegative and conserves the mass, it follows from the choice of \(T_n\) that with the same number \(T_n > 0,\) the function \(f^n\) can be extended from the time interval \([0, T_n]\) to \([T_n, 2T_n], [2T_n, 3T_n], \ldots.\)
Therefore we obtain a function \(0 \leq f^n \in L^\infty([0, \infty); L^1(\mathbb{R}^3))\) which conserves the mass and satisfies
\[
f^n(t, v) = f_0(v) + \int_0^t Q_{n,K}(f^n)(\tau, v) d\tau \quad \forall (t, v) \in [0, \infty) \times \mathbb{R}^3.
\]

To prove that \(f^n\) also conserves momentum and energy we consider truncation \(\phi(v) = \langle v \rangle^s \wedge R\) with \(s \geq 1\) and \(0 < R < \infty.\) From Lemma 2.8 we have
\[
\phi(v') + \phi(v') - \phi(v) - \phi(v_*) \leq \phi(v') + \phi(v') \leq 2^s(\langle v \rangle^s \wedge R) + 2^s(\langle v_* \rangle^s \wedge R). \hspace{1cm} (3.5)
\]

Suppose \(f_0 \in L^1_1(\mathbb{R}^3).\) Then, since \(\phi \in L^\infty(\mathbb{R}^3)\) and \(f^n\) conserves the mass, it follows from [3.4] and [3.5] that
\[
\int_{\mathbb{R}^3} (\langle v \rangle^s \wedge R) f^n(t, v) dv = \int_{\mathbb{R}^3} (\langle v \rangle^s \wedge R) f_0(v) dv + \frac{1}{2} \int_0^t d\tau \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B_n(v - v_*, \sigma)(f^n \wedge n)(f^n \wedge n) \times (1 + f^n \wedge K + f^n \wedge K)(\phi(v') + \phi(v') - \phi(v) - \phi(v_*)) d\sigma dv_* dv
\]
\[
\leq \|f_0\|_{L^1} + 2^s(1 + 2K)n|S^2| \int_0^t d\tau \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} (\langle v \rangle^s \wedge R)(f^n \wedge n)(f^n \wedge n) d\sigma dv_* dv
\]
\[
\leq \|f_0\|_{L^1} + 2^s(1 + 2K)n|S^2|\|f_0\|_{L^1} \int_0^t d\tau \int_{\mathbb{R}^3} (\langle v \rangle^s \wedge R) f^n(\tau, v) dv, \quad t \geq 0.
\]

By Gronwall’s lemma this gives
\[
\int_{\mathbb{R}^3} (\langle v \rangle^s \wedge R) f^n(t, v) dv \leq \|f_0\|_{L^1} \exp \left(2^s(1 + 2K)n|S^2|\|f_0\|_{L^1} t\right), \quad t \geq 0.
\]

Letting \(R \to \infty\) we conclude from Fatou’s lemma that \(f^n(t, \cdot) \in L^1_1(\mathbb{R})\) and \(f^n(t) = f_0(t)\) almost everywhere.

Next for any \(s \geq 1\) and any \(0 < T < \infty\) we have by using [2.2] that
\[
\sup_{t \in [0, T]} \int_{\mathbb{R}^3} (\langle v \rangle^s \wedge R)(Q_{n,K}(f^n)(t, v) + Q_{n,K}(f^n)(t, v)) dv
\]
\[
\leq 2n(1 + 2K) \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} 2^s(\langle v \rangle^s \wedge R)(f^n(t, v)) dv dv
\]
\[
\leq 2n(1 + 2K)^2|S^2| \left(\|f_0\|_{L^1} \exp \left(2^s(1 + 2K)n|S^2|\|f_0\|_{L^1} T\right) \right)^2 < \infty.
\]
This means that the test function can be chosen as \( \varphi(v) = \langle v \rangle^s \). Thus (again recall definition of \( \| \cdot \|_{L^1} \))

\[
\|f^n(t)\|_{L^1} = \|f^n_0\|_{L^1} + \frac{1}{2} \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B_n(v - v_* \sigma)(f^n \wedge n)(f^n_* \wedge n) \times (1 + f^n' \wedge K + f^n'' \wedge K)(\langle v' \rangle^s + \langle v'_* \rangle^s - \langle v \rangle^s - \langle v_* \rangle^s) d\sigma dv_* dv, \quad t \in [0, \infty).
\]

3.2 Moment estimates

In order to prove the \( L^1 \) relative compactness of solutions \( \{f^n\}_{n=1}^\infty \) of Eq. \((2.13)\), we need to establish uniform moment estimates of \( \{f^n\}_{n=1}^\infty \).

**Proposition 3.2.** For any \( n \in \mathbb{N} \), let \( B_n = B \wedge n \) with \( B \) the collision kernel satisfying \((1.10)\) and let \( Q_{n,K}(\cdot) \) be the collision operator defined in \((2.12)\). For any \( 0 \leq f_0 \in L^1_s(\mathbb{R}^3) \) with \( 3 \leq s < \infty \), let \( f^n \) be a conservative mild solution of Eq. \((2.13)\) corresponding to the kernel \( B_n \) with the initial datum \( f_0 \). Then

\[
\sup_{n \geq 1} \|f^n(t)\|_{L^1} \leq \left( \|f_0\|_{L^1}^{\frac{1}{s-1}} + (1 + 2K)^{2\frac{2}{s-2}}|S|^2 b(\|f_0\|_{L^1}) \frac{t^{\frac{s-1}{s-2}}}{t} \right)^{s-2}, \quad t \in [0, \infty).
\]

**Proof.** From \((3.10)\) and Corollary \((2.21)\) we have for a.e. \( t \in [0, T] \)

\[
\frac{d}{dt} \|f^n(t)\|_{L^1} = \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B_n(v - v_* \sigma)(f^n \wedge n)(f^n_* \wedge n)(1 + f^n' \wedge K + f^n'' \wedge K) \times (\langle v' \rangle^s + \langle v'_*_s \rangle^s - \langle v \rangle^s - \langle v_* \rangle^s) d\sigma dv_* dv
\]

\[
\leq \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B_n(v - v_* \sigma)(f^n \wedge n)(f^n_* \wedge n)(1 + f^n' \wedge K + f^n'' \wedge K) \times 2^{s/2} (\langle v \rangle^{s-2} (v_*)^2 + \langle v \rangle^{s-2} (v_*^s)^2) d\sigma dv_* dv
\]

\[
\leq (1 + 2K)^{2\frac{s}{s-2}}|S|^2 b \int_{\mathbb{R}^3 \times \mathbb{R}^3} (\langle v \rangle + \langle v_* \rangle)(\langle v \rangle^{s-2} (v_*)^2 + \langle v \rangle^2 (v_*^s)^2) f^n f^n_* dv_* dv
\]

\[
= (1 + 2K)^{2\frac{s}{s-2}}|S|^2 b \left( \|f_0\|_{L^1} \|f^n(t)\|_{L^1} + \|f^n(t)\|_{L^1}^2 \right).
\]

Recall that for any \( 2 < p \leq s \), by writing \( p = 2 + \frac{2}{s-2} + \frac{s-2}{s-2} \) and using Hölder’s inequality we have

\[
\|f\|_{L^p} \leq \left( \|f\|_{L^1} \right)^{s-2} \left( \|f\|_{L^1} \right)^{s-2}, \quad f \in L^1_s(\mathbb{R}^3).
\]

It follows that

\[
\frac{d}{dt} \|f^n(t)\|_{L^1} \leq (1 + 2K)^{2\frac{s}{s-2}}|S|^2 b \left( \|f_0\|_{L^1} \right)^{s-2} \left( \|f^n(t)\|_{L^1} \right)^{s-2} := A \|f^n(t)\|_{L^1}^{\frac{s-2}{s-2}} \quad \text{for a.e. } t \in [0, T].
\]
Thus denoting $\alpha = \frac{1}{2}$ we have with $u(t) := \|f^n(t)\|_{L^1}$ that
\[
(u(t))^\alpha - (u(0))^\alpha \leq \alpha [u(t)]^{\alpha-1} u'(t) \leq \alpha [u(t)]^{\alpha-1} A[u(t)]^{1-\alpha} = A\alpha \quad \text{a.e.} \quad t \in [0,T].
\]
Here we note that by conservation of mass we have $u(t) = \|f^n(t)\|_{L^1} \geq \|f_0\|_{L^1} > 0$ and from (3.6) we know that $u(t)$ is absolutely continuous on any bounded interval. It follows that $t \mapsto [u(t)]^\alpha$ is also absolutely continuous on any bounded interval. Thus
\[
[u(t)]^\alpha \leq [u(0)]^\alpha + A\alpha t, \quad u(t) \leq ([u(0)]^\alpha + A\alpha t)^{\frac{1}{\alpha}} \quad \forall t \in [0,T]
\]
and we conclude
\[
\sup_{n \geq 1} \|f^n(t)\|_{L^1} \leq \left( \|f_0\|_{L^1} + (1 + 2K)2^{\frac{n}{2}} b \|S^2\|_b \|f_0\|_{L^1}^2 t \right)^{\frac{1}{s-2}}, \quad t \in [0,T].
\]

\[\Box\]

### 3.3 $Q^+$ iteration and $L^1$ compactness

Our first use of multi-step iterations of $Q^+$ is in the proof of the following proposition which gives $L^1_t$-stability estimates for mild solutions of Eq. (2.13), Eq. (2.14), and Eq. (1.12). One will see that it is this multi-step iterations of $Q^+$ that enable us to obtain useful $L^\infty_t$ estimates of solutions.

**Proposition 3.3.** Let $B$ be the collision kernel satisfying (1.10). For any fixed $n \in \mathbb{N}$, let $B_n = B \cap n$ and let $Q(\cdot), Q_K(\cdot), Q_{\alpha,K}(\cdot)$ be the collision operators defined in (1.18), (2.9), (2.12) respectively. With $0 \leq f_0, g_0 \in L^1_3(\mathbb{R}^3) \cap L^\infty_3(\mathbb{R}^3)$ being initial data, let $f, g$ be the conservative mild solutions of Eq. (2.14) and $f = f^n, g = g^n$ the conservative mild solutions of Eq. (2.13), respectively. Also assume in both cases that $f, g$ satisfy the estimate $\text{(3.7)}$ with $s = 3$, i.e.

\[
\begin{align*}
\|f(t)\|_{L^1_3} &\leq \|f_0\|_{L^1_3} + (1 + 2K)2^{\frac{n}{2}} b \|S^2\|_b \|f_0\|_2^2 t, \quad t \in [0,\infty) \\
\|g(t)\|_{L^1_3} &\leq \|g_0\|_{L^1_3} + (1 + 2K)2^{\frac{n}{2}} b \|S^2\|_b \|g_0\|_2^2 t, \quad t \in [0,\infty). \quad (3.8)
\end{align*}
\]

Then
\[
\sup_{t \in [0,T]} \|f(t) - g(t)\|_{L^1_3} \leq C_{T,K} \|f_0 - g_0\|_{L^1_3} \quad \forall 0 < T < \infty \quad (3.9)
\]

where $C_{T,K} = C_{T,K}(\|f_0\|_{L^\infty_3}, \|f_0\|_{L^1_3}, \|g_0\|_{L^\infty_3}, \|g_0\|_{L^1_3}) < \infty$ is independent of $n$, $[0,\infty)^4 \ni (y_1, y_2, y_3, y_4) \mapsto C_{T,K}(y_1, y_2, y_3, y_4)$ is a continuous function and is monotone non-decreasing with respect to each $y_1, y_2, y_3, y_4 \in [0,\infty)$.

Furthermore, for any $0 \leq f_0, g_0 \in L^1_3(\mathbb{R}^3) \cap L^\infty_3(\mathbb{R}^3)$, let $0 \leq f, g \in L^\infty_{loc}\left((0,\infty); L^1_3(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)\right)$ be mild solutions of Eq. (1.12) corresponding to the kernel $B$ with the initial data $f_0, g_0$ respectively. Then
\[
\sup_{t \in [0,T]} \|f(t) - g(t)\|_{L^1_3} \leq C_T \|f_0 - g_0\|_{L^1_3} \quad \forall 0 < T < \infty \quad (3.10)
\]

where $C_T < \infty$ depends only on $T$ and $\|f_0\|_{L^\infty_3}$, $\sup_{t \in [0,T]} \|f(t)\|_{L^\infty_3}$, $\sup_{t \in [0,T]} \|f(t)\|_{L^\infty}$, $\sup_{t \in [0,T]} \|g(t)\|_{L^1_3}$, $\sup_{t \in [0,T]} \|g(t)\|_{L^\infty}$.

\[\text{sup}

Proof. The proof is divided into two steps.

Step 1. Let \( Q^+ (\cdot, \cdot) \) be defined in (2.3) and fix any \( 0 < T < \infty \), and let \( Z \subset \mathbb{R}^3 \) is a null set appeared in Definition 2.24. Notice first that under the assumption in the proposition we have

\[
f(t, v) \leq f_0(v) + (1 + 2K) \int_0^T Q^+(f_{\tau_1}, f_{\tau_1})(v) d\tau_1, \quad (t, v) \in [0, T] \times (\mathbb{R}^3 \setminus Z) \tag{3.11}
\]

where to shorten notation we denote the solution \( f_t(v) \equiv f(t, v) \). Using the inequality (3.11) to \( f(\tau_1, v) \) and substituting the right hand side of the inequality into the two arguments of \( Q^+(f_{\tau_1}, f_{\tau_1}) \), and then making further iteration, we compute for any \( (t, v) \in [0, T] \times (\mathbb{R}^3 \setminus Z) \) that

\[
f(t, v) \leq f_0(v) + (1 + 2K) \int_0^T Q^+(f_0, f_0)(v) d\tau_1 + 2(1 + 2K)^2 \int_0^T \int_0^T Q^+(Q^+(f_{\tau_2}, f_{\tau_2}), f_0)(v) d\tau_3 d\tau_1 \]

\[
+ (1 + 2K)^3 \int_0^T \int_0^T \int_0^T Q^+(Q^+(f_{\tau_2}, f_{\tau_2}), Q^+(f_0, f_0))(v) d\tau_3 d\tau_2 d\tau_1
\]

\[
+ (1 + 2K)^4 \int_0^T \int_0^T \int_0^T \int_0^T Q^+(Q^+(f_{\tau_2}, f_{\tau_2}), Q^+(f_0, Q^+(f_{\tau_4}, f_{\tau_4}))(v) d\tau_4 d\tau_3 d\tau_2 d\tau_1
\]

(3.12)

Next, using Proposition 2.14 and \( |v - v_*| = |v' - v'_*| \leq \langle v' \rangle \langle v'_* \rangle \) we have

\[
Q^+(f_0, f_0)(v) \leq b \int_{\mathbb{R}^3 \times S^2} \langle v' \rangle \langle v'_* \rangle f_0(v') f_0(v'_*) d\sigma dv_* \leq 2 \hat{b} \delta \| f_0 \|_{L^\infty} \| f_0 \|_{L^1} , \quad v \in \mathbb{R}^3 \tag{3.13}
\]

In order to use Proposition 2.17 to obtain uniform estimates of \( \| f(t) \|_{L^\infty} \) and \( \| g(t) \|_{L^2} \) let us define

\[
(f)_s(v) := \langle v \rangle^s f(v) \quad \text{with} \quad s \geq 0.
\]

It is easily seen that for any \( 0 \leq f, g, h \in L^1_s(\mathbb{R}^3) \) and any \( v \in \mathbb{R}^3 \) we have

\[
(Q^+(f, g))_s(v) \leq Q^+(Q^+(f, (g)_s))(v)
\]

\[
(Q^+(f, Q^+(g, h)))_s(v) \leq Q^+(Q^+(f)_s, Q^+(g)_s)(v)
\]

where we used the fact that \( \langle v \rangle \leq \langle v' \rangle \langle v'_* \rangle \). From these we see that the inequality (3.11) hence all inequalities in (3.14) hold also for the function \( (f)_s(t, v) = \langle v' \rangle^s f(t, v) \). In other words, this means that if \( f \) is a solution of the inequality (3.11), so is \( (f)_s \), for any \( s \geq 0 \). Take \( s = 2 \). Then combining this with (2.31), (2.33 - 2.37), (3.7), (3.8), (3.12) and (3.13) we obtain a uniform estimate:

\[
\sup_{0 \leq t \leq T} (\| t \|_{L^\infty} + \| f(t) \|_{L^1} + \| g(t) \|_{L^2} + \| g(t) \|_{L^1}) \leq \tilde{C}_{T,K} (\| f_0 \|_{L^\infty} + \| f_0 \|_{L^1} + \| g_0 \|_{L^\infty} + \| g_0 \|_{L^1}) =: \tilde{C}_{T,K} < \infty \tag{3.14}
\]

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where and below $\tilde{C}_{T,K}(y_1, y_2, y_3, y_4)$ denotes a continuous function on $[0, \infty)^4$ which is monotone non-decreasing with respect to each variable $y_1, y_2, y_3, y_4 \in [0, \infty)$.

In order for our proof of stability estimates to cover both Eq.(2.13) and Eq.(2.14), we denote $Q_* = Q_K, Q_*^\pm = Q_K^\pm$ and $Q_* = Q_{n_K}, Q_*^\pm = Q_{n_K}^\pm$ respectively, and let

$$
\psi(t, v) = \text{sign}(f(t, v) - g(t, v)).
$$

Then we have

$$
|f(t, v) - g(t, v)| = |f_0(v) - g_0(v)| + \int_0^t (Q_*(f)(\tau, v) - Q_*(g)(\tau, v))\psi(\tau, v)d\tau.
$$

For further estimates we need to show that $Q_+^\pm(f), Q_+^\pm(g)$ belong to $L^\infty([0, T]; L_1^2(\mathbb{R}^3))$ so that there is no problem of integrability. In fact we have, for instance for $Q_+^\pm(f)$,

$$
\int_{\mathbb{R}^3} (v)^2(Q^+_\pm(f)(t, v) + Q^-_\pm(f)(t, v))dv \\
\leq (1 + 2K) \int_{\mathbb{R}^3} (v)^2 B(v - v_*) |f'_\pm + f_\pm|dv_* \ dv \\
\leq (1 + 2K)b \int_{\mathbb{R}^3} (1 + |v'|^2 + |v'_*|^2)|v - v_*| |f'_\pm + f_\pm|dv_* \ dv \\
\leq 2(1 + 2K)b \int_{\mathbb{R}^3} (v)^2|f_\pm|dv_* \ dv = 2(1 + 2K)b |S|^2 \|f(t)\|_{L_1^2}^2 \tag{3.15}
$$

where we used (2.2). From this integrability we have

$$
\|f(t) - g(t)\|_{L_1^2} = \|f_0 - g_0\|_{L_1^2} + \int_0^t \int_{\mathbb{R}^3} (Q_*(f)(\tau, v) - Q_*(g)(\tau, v))\psi(\tau, v)dv,
$$

$$
\int_{\mathbb{R}^3} (v)^2(Q_*(f)(\tau, v) - Q_*(g)(\tau, v))\psi(\tau, v)dv \\
\leq \int_{\mathbb{R}^3} b|v - v_*|((1 + 2K)(f + g)|f_\pm - g_\pm| + (f_\pm + g_\pm)(|f'_\pm - g'_\pm| + |f'_\pm - g'_\pm|))dv_* \ dv \\
\leq b(1 + 2K)|S|^2 \int_{\mathbb{R}^3} (v)^2|v - v_*| |f_\pm - g_\pm|dv_* \ dv \\
+ 2b \int_{\mathbb{R}^3} |f - g| \int_{\mathbb{R}^3} (v)^2|v - v_*| |f'_\pm| |f'_\pm + g'_\pm|dv_* \ dv + 2b \int_{\mathbb{R}^3} |f - g| \int_{\mathbb{R}^3} (v)^2|v - v_*| |g'_\pm| |f'_\pm + g'_\pm|dv_* \ dv.
$$

Further estimates: From $|v - v_*| \leq (v)\langle v_* \rangle$ we have

$$
\int_{\mathbb{R}^3} (v)^2|v - v_*| |f_\pm - g_\pm|dv_* \ dv \leq \|f + g\|_{L_1^2} \|f - g\|_{L_1^2}.
$$

Combining this with Proposition 2.13 ($p = 2, q = 0, \gamma = 1$) gives

$$
\int_{\mathbb{R}^3} (v)^2(Q_*(f)(\tau, v) - Q_*(g)(\tau, v))\psi(\tau, v)dv \\
\leq b(1 + 2K)|S|^2 \|f + g\|_{L_1^2} \|f - g\|_{L_1^2} \\
+ 8b|S|^2 \||f| \|_{L_1^2} \|f\|_{L_1^2} + \|f\|_{L_1^2} \|g\|_{L_1^2} + \|g\|_{L_1^2} \|f\|_{L_1^2} + \|g\|_{L_1^2} \|g\|_{L_1^2} \|f - g\|_{L_1^2} \\
\leq (b(1 + 2K)|S|^2 C_{T,K} + 8b|S|^2 C_{T,K}^2) \|f(\tau) - g(\tau)\|_{L_1^2}, \quad \tau \in [0, T]
$$

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where we used (3.14). Thus we obtain
\[ \|f(t) - g(t)\|_{L^2} \leq \|f_0 - g_0\|_{L^2} + (b(1 + 2K)|S^2|\tilde{C}_{T,K} + 8b|S^2|\tilde{C}_{T,K}^2) \int_0^t \|f(\tau) - g(\tau)\|_{L^2} \, d\tau \]
for all \( t \in [0, T] \). We then conclude from Gronwall’s lemma that
\[ \|f(t) - g(t)\|_{L^2} \leq C_{T,K} \|f_0 - g_0\|_{L^2} \quad \forall t \in [0, T] \]
where \( C_{T,K} = C_{T,K} (\|f_0\|_{L^\infty}, \|f_0\|_{L^1}, \|g_0\|_{L^\infty}, \|g_0\|_{L^1}) = \exp \left[ (b(1 + 2K)|S^2|\tilde{C}_{K,T} + 8b|S^2|\tilde{C}_{K,T}^2)T \right] \). From (3.14) we see that the function \((y_1, y_2, y_3, y_4) \mapsto C_{T,K}(y_1, y_2, y_3, y_4)\) is independent of \( n \), continuous on \([0, \infty)^4\), and monotone non-decreasing with respect to each variable \( y_1, y_2, y_3, y_4 \in [0, \infty) \). This proves (3.9).

**Step 2.** We now prove (3.10). Fix any \( 0 < T < \infty \) and let \( \overline{K} = \max \left\{ \sup_{0 \leq t \leq T} \|f(t)\|_{L^\infty}, \sup_{0 \leq t \leq T} \|g(t)\|_{L^\infty} \right\} \). Following a similar argument in Step 1 we see that \( f \) satisfies (3.11), (3.12) with \( K = \overline{K} \). Then it follows from the proof of (3.14) that
\[
\sup_{0 \leq t \leq T} \left( \|f(t)\|_{L^\infty} + \|f(t)\|_{L^1} + \|g(t)\|_{L^\infty} + \|g(t)\|_{L^1} \right)
\leq \overline{C}_{T,\overline{K}} (\|f_0\|_{L^\infty}, \sup_{0 \leq t \leq T} \|f(t)\|_{L^1}, \sup_{0 \leq t \leq T} \|f(t)\|_{L^\infty}, \sup_{0 \leq t \leq T} \|g(t)\|_{L^1}, \sup_{0 \leq t \leq T} \|g(t)\|_{L^\infty}) =: \overline{C}_{T,\overline{K}}
\]
where \( \overline{C}_{T,\overline{K}} \) depends only on \( T \) and \( \|f_0\|_{L^\infty}, \sup_{0 \leq t \leq T} \|f(t)\|_{L^1}, \sup_{0 \leq t \leq T} \|f(t)\|_{L^\infty}, \sup_{0 \leq t \leq T} \|g(t)\|_{L^1}, \sup_{0 \leq t \leq T} \|g(t)\|_{L^\infty} \). Thus, (3.10) can be proved with the same argument in the rest part of the proof in Step 1 by replacing \( K \), \( \tilde{C}_{T,K} \) with \( \overline{K}, \overline{C}_{T,\overline{K}} \) respectively. \( \square \)

As an immediate application of this proposition we obtain the following continuity estimates for mild solutions of Eq. (2.13) and Eq. (2.14).

**Proposition 3.4.** Let \( 0 \leq f_0 \in L^3_1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \) and let \( Q_n, K(\cdot) \) be defined in (2.12) with \( B_n = B \cap n \) and \( B \) satisfying (1.10). With the same initial datum \( f_0 \), let \( f^n \) and \( f \) be mild solutions of Eq. (2.13) and Eq. (2.14) respectively. Then
\[
\sup_{n \geq 1, t \in [0, T]} \|f^n(t, \cdot + h) - f^n(t)\|_{L^2} \leq C_{T,K} \|f_0(\cdot + h) - f_0\|_{L^2} \quad \forall |h| \leq 1, \forall T \in (0, \infty) \quad (3.16)
\]
\[
\sup_{n \geq 1} \|f^n(t_1) - f^n(t_2)\|_{L^1} \leq 2b(1 + 2K)|S^2|\|f_0\|_{L^1} \|f_0\|_{L^2} |t_1 - t_2| \quad \forall t_1, t_2 \in [0, \infty) \quad (3.17)
\]
\[
\sup_{n \geq 1} \|f(t, \cdot + h) - f(t)\|_{L^2} \leq C_{T,K} \|f_0(\cdot + h) - f_0\|_{L^2} \quad \forall |h| \leq 1, \forall T \in (0, \infty) \quad (3.18)
\]
\[
\|f(t_1) - f(t_2)\|_{L^1} \leq 2b(1 + 2K)|S^2|\|f_0\|_{L^1} \|f_0\|_{L^2} |t_1 - t_2| \quad \forall t_1, t_2 \in [0, \infty) \quad (3.19)
\]
where \( 0 < C_{T,K} < \infty \) depends only on \( T, K, \|f_0\|_{L^\infty}, \|f_0\|_{L^2} \). Consequently the sequence \( \{f^n(t, \cdot)\}_{n=1}^\infty \) is both equicontinuous in \( L^1_2(\mathbb{R}^3) \) uniformly in local time and equicontinuous in \( C([0, \infty); L^1(\mathbb{R}^3)) \).

**Proof.** We need only to prove the estimates for \( f^n \) since the proof for \( f \) is completely the same. From the structure of the collision \( B \) (see (1.7)), it is easily seen that the velocity translation \( g^n(t, v) := \)
$f^n(t,v+h)$ is still a mild solution to Eq.\eqref{2.13} with the initial datum $g_0(v) := f_0(v+h)$. Since for any $|h| \leq 1$ and any $s \in \{2,3\}$

$$
\|f_0(\cdot + h)\|_{L^1} \leq 3^{s/2}\|f_0\|_{L^1}, \quad \|f_0(\cdot + h)\|_{L^\infty} \leq 3^{s/2}\|f_0\|_{L^\infty}
$$

it follows from \eqref{3.9} that \eqref{3.10} holds true. Next for any $h \in \mathbb{R}^3$ with $|h| \leq 1$ we have $|\langle v \rangle^2 - \langle v-h \rangle^2| f_0(v) \leq 3|h|\langle v \rangle f_0(v)$ so that

$$
\|f_0(\cdot + h) - f_0\|_{L^1} \leq 3|h|\|f_0\|_{L^1} + \|\bar{f}_0(\cdot + h) - \bar{f}_0\|_{L^1} \to 0 \quad \text{as} \quad h \to 0 \quad (3.20)
$$

where $\bar{f}_0(v) = \langle v \rangle^2 f_0(v)$. This together with \eqref{3.10} proves the uniform local in time $L^1(\mathbb{R}^3)$-equicontinuity of the sequence $\{f^n(t,\cdot)\}_{n=1}^\infty$.

Finally for any $t_1, t_2 \geq 0$, using Cauchy-Schwarz inequality and the conservation of mass and energy of $f^n$ we have

$$
\int_{\mathbb{R}^3} |f^n(t_1,v) - f^n(t_2,v)| dv \
\leq \int_{t_1 \wedge t_2} \int\int\int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} B_n(v-v_*,\sigma)(f^n(\cdot \wedge n)(f^n(\cdot \wedge n))(1 + f^n \wedge K + f^n \wedge K) d\sigma dv dv \
+ \int_{t_1 \wedge t_2} \int\int\int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} B_n(v-v_*,\sigma)(f^n(\cdot \wedge n)(f^n(\cdot \wedge n))(1 + f^n \wedge K + f^n \wedge K) d\sigma dv dv \
\leq 2b(1+2K)|S^2| \int_{t_1 \wedge t_2} \left( \int_{\mathbb{R}^3} \langle v \rangle f^n(\tau,v) dv \right)^2 d\tau \leq 2b(1+2K)|S^2| \|f_0\|_{L^1} \|f_0\|_{L^2} |t_1 - t_2|.
$$

This proves \eqref{3.17}. $\square$

4 Mild solutions of the intermediate equation

In this section we first prove the existence of conservative mild solutions of Eq.\eqref{2.14}. Then we will use multi-step iterations of the collision gain operator $Q^+ (\cdot , \cdot)$ to prove further estimates for these mild solutions which are used in proving Theorem 1.2. Throughout this section, the same constant $K > 0$ in Eq.\eqref{2.13} and Eq.\eqref{2.14} is fixed.

4.1 Existence of mild solutions of the intermediate equation

**Proposition 4.1.** Let $Q_K(\cdot )$ be defined in \eqref{2.9} with $B$ satisfying \eqref{1.10}, and let $0 \leq f_0 \in L^1_3(\mathbb{R}^3) \cap L^\infty_3(\mathbb{R}^3)$. Then Eq.\eqref{2.14} has a conservative mild solution $f \in L^\infty([0,\infty); L^1_3(\mathbb{R}^3))$ with the initial datum $f_0$, i.e.

$$
f(t,v) = f_0(v) + \int_0^t Q_K(f)(\tau,v) d\tau, \quad (t,v) \in [0,\infty) \times (\mathbb{R}^3 \setminus Z).\quad (4.1)
$$

Here $Z$ is a null set independent of $t$.

**Proof.** Our proof consists of two steps. In the first step we shall use mild solutions of Eq.\eqref{2.13}.
Step 1. Let \( \{f^n\}_{n=1}^{\infty} \) be a sequence of mild solutions of Eq. (2.13) obtained in Proposition 3.1 with the same initial data \( f_0^n = f_0, n = 1, 2, 3, \ldots \). By conservation of mass and energy of \( \{f^n\}_{n=1}^{\infty} \) we have

\[
\sup_{n \geq 1, t \geq 0} \|f^n(t)\|_{L^2} = \|f_0\|_{L^2} < \infty.
\]

Next for any fixed \( t \geq 0 \) we deduce from (3.13) that \( \{f^n(t, \cdot)\}_{n=1}^{\infty} \) is equicontinuous in \( L^1(\mathbb{R}^3) \). These imply that \( \{f^n(t, \cdot)\}_{n=1}^{\infty} \) is a relatively compact set in \( L^1(\mathbb{R}^3) \) (for any fixed \( t > 0 \)). Using diagonal argument we can find a common subsequence \( \{n_k\}_{k=1}^{\infty} \subset \mathbb{N} \) such that for any \( r \in \mathbb{Q} \cap [0, \infty), \{f^{n_k}(r, \cdot)\}_{k=1}^{\infty} \) is a Cauchy sequence in \( L^1(\mathbb{R}^3) \). Then for any fixed \( t \geq 0 \), let us consider \( \{f^{n_k}(t, \cdot)\}_{k=1}^{\infty} \). It follows from (3.17) that for any \( \varepsilon > 0 \), there exists an \( r \in \mathbb{Q} \cap [0, \infty) \), such that

\[
\|f^{n_k}(t) - f^{n_k}(r)\|_{L^1} \leq \frac{\varepsilon}{3} \text{ for all } k \in \mathbb{N}, \text{ and thus}
\]

\[
\|f^{n_k}(t) - f^{n_j}(t)\|_{L^1} \leq \|f^{n_k}(t) - f^{n_k}(r)\|_{L^1} + \|f^{n_j}(t) - f^{n_j}(r)\|_{L^1} + \|f^{n_k}(r) - f^{n_j}(r)\|_{L^1} \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \|f^{n_k}(r) - f^{n_j}(r)\|_{L^1} \quad \forall k, j \in \mathbb{N}.
\]

Since \( \{f^{n_k}(r, \cdot)\}_{k=1}^{\infty} \) is a Cauchy sequence in \( L^1(\mathbb{R}^3) \), it follows from the arbitrariness of \( \varepsilon \) that \( \{f^{n_k}(t, \cdot)\}_{k=1}^{\infty} \) is a Cauchy sequence in \( L^1(\mathbb{R}^3) \) (for any fixed \( t \geq 0 \)), i.e.

\[
\lim_{k \to \infty} \|f^{n_k}(t) - f^{n_k}(t)\|_{L^1} = 0 \quad \forall t \in [0, \infty).
\]

Step 2. We prove that the above function \( \tilde{f} \), after a modification on a null set, is a conservative mild solution of Eq. (2.13). Let \( 0 < R < \infty \) and let \( 0 < T < \infty \). Define \( \varphi(v) = \langle v \rangle^3 \wedge R \) for any \( v \in \mathbb{R}^3 \). Since \( \varphi \in L^\infty(\mathbb{R}^3) \) and \( f_0 \in L^1_3(\mathbb{R}^3) \), it follows from Proposition 3.2 and (4.3) that for any \( t \in [0, T] \)

\[
\int_{\mathbb{R}^3} \langle (v)^3 \wedge R \rangle \tilde{f}(t, v) dv = \lim_{k \to \infty} \int_{\mathbb{R}^3} \langle (v)^3 \wedge R \rangle f^{n_k}(t, v) dv
\]

\[
\leq \lim_{k \to \infty} \int_{\mathbb{R}^3} (v)^3 f^{n_k}(t, v) dv \leq \|f_0\|_{L^1_3} + (1 + 2K)2^2 b|S^2|\|f_0\|_{L^1_3}^2 T := C_{f_0,T} < \infty
\]

and so, by Fatou’s lemma,

\[
\|\tilde{f}(t, v)\|_{L^1_3} \leq C_{f_0,T}.
\]

Next from (3.15) we have for any \( t \in [0, T] \)

\[
\int_{\mathbb{R}^3} (v)^2 (Q^+_K(\tilde{f})(t, v) + Q^-_K(\tilde{f})(t, v)) dv \leq 2(1 + 2K)b|S^2|\|\tilde{f}\|_{L^1_3}^2 \leq 2(1 + 2K)b|S^2|C_{f_0,T}^2 < \infty.
\]

This together with the fact that \( 0 \leq Q_{n,K}^\pm(\tilde{f}) \leq Q_K^\pm(\tilde{f}) \) and Fatou’s lemma implies that

\[
\int_0^T dt \int_{\mathbb{R}^3} |\tilde{f}(t, v) - f_0(v)| \, dv \leq \int_0^T dt \lim_{k \to \infty} \int_0^t d\tau \left( \int_{\mathbb{R}^3} |Q_{n,K}(\tilde{f})(\tau, v) - Q_K(\tilde{f})(\tau, v)| \, dv \right)
\]

\[
+ \int_{\mathbb{R}^3} |Q_{n,K}(f^{n_k})(\tau, v) - Q_{n,K}(\tilde{f})(\tau, v)| \, dv.
\]
To estimate the first term in the right hand side of the above inequality we compute
\[\int_{\mathbb{R}^3} |Q_{n_k,K}(\tilde{f})(\tau,v) - Q_K(\tilde{f})(\tau,v)|\,dv\]
\[\leq 2(1 + 2K) \iint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^2} \left| B_{n_k}(v - v_*) \sigma(\tilde{f} \wedge n_k)(\tilde{f}_* \wedge n_k) - B_{n_k}(v - v_*) \sigma \tilde{f}_* \right| \,d\sigma dv_* dv\
\[+ |B(v - v_*) \sigma(\tilde{f} \wedge n_k)(\tilde{f}_* \wedge n_k) - B(v - v_*) \sigma \tilde{f}_*| \right) \,d\sigma dv_* dv\
\leq 8b(1 + 2K) \iint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^2} \langle v \rangle \langle v_* \rangle |(\tilde{f} \wedge n_k)(\tilde{f}_* \wedge n_k) - \tilde{f}_*| \,d\sigma dv_* dv \\
\leq 16b(1 + 2K) C^2_{f_0,T} \quad \text{(uniformly in } \tau \in [0,T]).\] (4.8)

Since \(\langle v \rangle \langle v_* \rangle |(\tilde{f} \wedge n_k)(\tilde{f}_* \wedge n_k) - \tilde{f}_*| \leq 2 \langle v \rangle \langle v_* \rangle \tilde{f}_*,\) it follows from Lebesgue’s dominated convergence that
\[\iint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^2} \langle v \rangle \langle v_* \rangle |(\tilde{f} \wedge n_k)(\tilde{f}_* \wedge n_k) - \tilde{f}_*| \,d\sigma dv_* dv \to 0 \quad \text{as } k \to \infty.\]

This together with (4.8) implies that
\[\int_{\mathbb{R}^3} |Q_{n_k,K}(\tilde{f})(\tau,v) - Q_K(\tilde{f})(\tau,v)|\,dv \to 0 \quad \text{as } k \to \infty. \tag{4.9}\]

Then from (4.8) and Lebesgue’s dominated convergence we obtain
\[\lim_{k \to \infty} \int_0^t \int_{\mathbb{R}^3} |Q_{n_k,K}(\tilde{f})(\tau,v) - Q_K(\tilde{f})(\tau,v)|\,dv = 0 \quad \forall \, t \in [0,\infty). \tag{4.10}\]

Following a similar method we have
\[\lim_{k \to \infty} \int_0^t \int_{\mathbb{R}^3} |Q_{n_k,K}(f^{n_k})(\tau,v) - Q_{n_k,K}(\tilde{f})(\tau,v)|\,dv = 0 \quad \forall \, t \in [0,\infty). \tag{4.11}\]

Combining these with (4.7) we conclude
\[\int_{\mathbb{R}^3} dv \int_0^T |\tilde{f}(t,v) - f_0(v) - \int_0^t Q_K(\tilde{f})(\tau,v)\,d\tau|\,dt = 0.\]

Since \(0 < T < \infty\) is arbitrary, it follows from Fatou’s lemma that
\[\int_{\mathbb{R}^3} dv \int_0^\infty |\tilde{f}(t,v) - f_0(v) - \int_0^t Q_K(\tilde{f})(\tau,v)\,d\tau|\,dt = 0. \tag{4.12}\]

Let \(f(t,v) := |f_0(v) + \int_0^t Q_K(\tilde{f})(\tau,v)\,d\tau|\), \((t,v) \in [0,\infty) \times \mathbb{R}^3.\) We see that \(f\) is nonnegative and there is a null set \(Z_1 \subset \mathbb{R}^3\) (independent of \(t\)) such that for any fixed \(v \in \mathbb{R}^3 \setminus Z_1\) the function \(t \mapsto f(t,v)\) is continuous on \([0,\infty).\) This is because the functions \((\tau,v) \mapsto Q_K^\pm(\tilde{f})(\tau,v)\) belong to \(L^1([0,T] \times \mathbb{R}^3)\) for all \(0 < T < \infty,\) and so by Fubini’s theorem there is a null set \(Z_1 \subset \mathbb{R}^3\) such that \(\int_0^T |Q_K(\tilde{f})(\tau,v)|\,d\tau < \infty\) for all \(0 < T < \infty\) and all \(v \in \mathbb{R}^3 \setminus Z_1.\) From (4.12) and the nonnegativity of \(\tilde{f}\) we see that
\[f(t,v) = \tilde{f}(t,v) \quad \text{for a.e. } (t,v) \in [0,\infty) \times \mathbb{R}^3. \tag{4.13}\]

Thus
\[\int_0^\infty dt \int_{\mathbb{R}^3} |f(t,v) - \tilde{f}(t,v)|\,dv = 0.\]
This implies, using Fubini’s theorem, that there exists a null $\tilde{Z} \subset [0, \infty)$ such that
\[
\int_{\mathbb{R}^3} |f(t, v) - \tilde{f}(t, v)|dv = 0 \quad \forall t \in [0, \infty) \setminus \tilde{Z}.
\]
This together with (4.13) implies that for any $0 < T < \infty$ and any $t \in [0, T] \setminus \tilde{Z}$, $\|f(t)\|_{L^1} = \|\tilde{f}(t)\|_{L^1} \leq C_{f_0, T} < \infty$. Since $t \mapsto f(t, v)$ is continuous on $[0, \infty)$ for any fixed $v \in \mathbb{R}^3 \setminus Z$, it follows from Fatou’s lemma and (4.14) that
\[
\|f(t)\|_{L^1} \leq C_{f_0, T} \quad \forall t \in [0, T], \forall 0 < T < \infty.
\]
Thus, from (3.15) we have
\[
\sup_{0 \leq t \leq T} \|Q_K^\pm(f)(t, \cdot)\|_{L^1} < \infty \quad \text{for any } 0 < T < \infty.
\]
This together with (4.13), the boundedness of the mapping $\varphi \mapsto \varphi \land K$, the formula (2.22) of change variables, and the arbitrariness of $T$ imply that
\[
Q_K(f)(t, v) = Q_K(f)(t, v) \quad \text{a.e.} \quad (t, v) \in [0, \infty) \times \mathbb{R}^3.
\]
Combining this with (4.12) and (4.13) leads to
\[
f(t, v) = f_0(v) + \int_0^t Q_K(f)(\tau, v)d\tau \quad \text{a.e.} \quad (t, v) \in [0, \infty) \times \mathbb{R}^3.
\]
Since by definition of $f$, $f(t, v)$ is fully measurable on $[0, \infty) \times \mathbb{R}^3$ and the function $t \mapsto f(t, v)$ is continuous on $[0, \infty)$ for almost every $v \in \mathbb{R}^3$, while using Fubini’s theorem it is easily seen that the function $f_0(v) + \int_0^t Q_K(f)(\tau, v)d\tau$ is fully measurable on $[0, \infty) \times \mathbb{R}^3$ and $t \mapsto f_0(v) + \int_0^t Q_K(f)(\tau, v)d\tau$ is continuous on $t \in [0, \infty)$ for almost every $v \in \mathbb{R}^3$, it follows from Lemma 2.4 that there is a null set $Z \subset \mathbb{R}^3$ such that
\[
f(t, v) = f_0(v) + \int_0^t Q_K(f)(\tau, v)d\tau \quad \forall (t, v) \in [0, \infty) \times (\mathbb{R}^3 \setminus Z).
\]
To prove that $f$ is a conservative mild solution of Eq. (2.14), we now need only to prove the conservation law of $f$. Let $\varphi \in C(\mathbb{R}^3)$ satisfy $|\varphi(v)| \leq C|\varphi|^2$ for constant $0 < C < \infty$. From (4.15) and (4.16) we have
\[
\int_{\mathbb{R}^3} \varphi(v)f(t, v)dv = \int_{\mathbb{R}^3} \varphi(v)f_0(v)dv + \int_0^t \int_{\mathbb{R}^3} \varphi(v)^2Q_K(f)(\tau, v)dv, \quad t \in [0, \infty)
\]
and so for a.e. $t \in [0, \infty)$
\[
\frac{d}{dt} \int_{\mathbb{R}^3} \varphi(v)f(t, v)dv = \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^2} B(v - v_*, \sigma)f_{\varphi}(1 + f' \land K + f_*' \land K)
\times (\varphi(v') + \varphi(v'_*) - \varphi(v) - \varphi(v_*))d\sigma dv_* dv.
\]
Taking $\varphi(v) = 1$ implies that $f$ conserves the mass. While taking $\varphi(v) = |v - a|^2$ with any constant vector $a \in \mathbb{R}^3$ we also have $\varphi(v') + \varphi(v'_*) - \varphi(v) - \varphi(v_*) \equiv 0$ and thus
\[
\int_{\mathbb{R}^3} |v - a|^2 f(t, v)dv = \int_{\mathbb{R}^3} |v - a|^2 f_0(v)dv \quad \forall t \in [0, \infty).
\]
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By choosing $a = (0, (1, 0, 0), (0, 1, 0), (0, 0, 1))$ respectively and using the conservation of mass of $f$ we conclude that $f$ conserves energy and momentum:

$$
\int_{\mathbb{R}^3} |v|^2 f(t, v) dv = \int_{\mathbb{R}^3} |v|^2 f_0(v) dv, \quad \int_{\mathbb{R}^3} v f(t, v) dv = \int_{\mathbb{R}^3} v f_0(v) dv \quad \forall t \in [0, \infty).
$$

\[\Box\]

### 4.2 Some estimates for collision integral of mild solutions of the intermediate equation

Recall that in order to use mild solutions $f$ of Eq. (2.14) to prove the global in time existence of solutions of Eq. (1.12), one needs only to prove that $f \leq K$ for a suitable $0 < K < \infty$. To do this we will establish the next two Propositions and use Duhamel’s formula to deduce certain pointwise estimates for $f$ in Proposition 4.4. Then we will give further propositions which are based on multi-step iterations of the collision gain operator $Q^+ (\cdot, \cdot)$ and the estimates obtained in Proposition 4.4. For notational convenience we will use in this subsection the notation $f_1 (v) = f(t, v)$.

**Proposition 4.2.** Let $Q_K (\cdot)$ be defined in (2.9) with $B$ satisfying (1.16). For any $0 \leq f_0 \in L^1_1 (\mathbb{R}^3) \cap L^\infty_3 (\mathbb{R}^3)$, let $f$ be a mild solution of Eq. (2.14) obtained in Proposition 4.1 corresponding to the kernel $B$ with the initial datum $f_0$. Then

$$
\|f(t)\|_{L^3} \leq \max \{1, C_1\} \|f_0\|_{L^3_1} \quad \forall t \geq 0
$$

where

$$
C_1 = \frac{(128(\sqrt{2} - 1)(1 + 2K)b + 2(\frac{22}{7}\sqrt{2} - \frac{31}{5})a)\|f_0\|^2_{L^3_1}}{\omega(\frac{22}{7}\sqrt{2} - \frac{31}{5})\|f_0\|_{L^1_1} \|f_0\|_{L^3_1}}.
$$

**Proof.** We first prove that (4.17) and (4.18) hold true for the case $f_0 \in \bigcap_{s \geq 3} L^1_3 (\mathbb{R}^3)$. For any $s \geq 3$ and $R > 0$, define $\varphi (v) = \langle v \rangle^s \wedge R$. Since $\varphi \in L^\infty (\mathbb{R}^3)$, it follows from Proposition 3.2 and the same proof of (4.14) that

$$
\int_{\mathbb{R}^3} ((\langle v \rangle^s \wedge R) f(t, v) dv \leq \int_{\mathbb{R}^3} (\langle v \rangle^s \wedge R) \tilde{f}(t, v) dv = \lim_{k \to \infty} \int_{\mathbb{R}^3} (\langle v \rangle^s \wedge R) f^{n_k}(t, v) dv

\leq \left(\|f_0\|_{L^3_1}^{\frac{1}{s-2}} + (1 + 2K)^{\frac{2s}{s+2}}|S|^2 |b| (\|f_0\|_{L^3_1})^{\frac{s-1}{s-2}} \frac{t}{s-2}\right)^{s-2} \quad \forall t \geq 0
$$

where $\tilde{f}$ and $f^{n_k}$ are defined in Proposition 4.1. Letting $R \to \infty$ we obtain by Fatou's lemma that

$$
\|f(t)\|_{L^3_1} \leq \left(\|f_0\|_{L^3_1}^{\frac{1}{s-2}} + (1 + 2K)^{\frac{2s}{s+2}}|S|^2 |b| (\|f_0\|_{L^3_1})^{\frac{s-1}{s-2}} \frac{t}{s-2}\right)^{s-2}.
$$

This implies that $\sup_{0 \leq t \leq T} \|f(t)\|_{L^3_1} < \infty$ for all $s \geq 3$ and all $0 < T < \infty$. Since $|v - v_s| \leq \langle v \rangle \langle v_s \rangle$ and $\langle v' \rangle^* \leq \langle v \rangle^*(\langle v_s \rangle^*)$ it follows that for any $t \in [0, \infty)$

$$
\|Q^+_K (f)(t, \cdot)\|_{L^3_1} \leq (1 + 2K)b|S|^2 \|f(t)\|^2_{L^3_1} + \|Q^+_K (f) (t, \cdot)\|_{L^3_1} \leq (1 + 2K)b|S|^2 \|f(t)\|_{L^3_1} \|f(t)\|_{L^3_1}.
$$

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Thus for any \( s \geq 3 \) and any \( 0 < T < \infty \) we have \( \sup_{0 \leq t \leq T} \|Q^K(f)(t)\|_{L^1} < \infty \). Then it follows from Fubini’s theorem that

\[
\|f(t)\|_{L^1} = \|f_0\|_{L^1} + \int_0^t d\tau \int_{\mathbb{R}^3} \langle v \rangle^s Q^K(f)(\tau, v)dv \quad \forall \ t \in [0, \infty).
\]

From (4.20) with \( s = 3 \) and Corollary 2.7 we have for a.e. \( t \in [0, \infty) \),

\[
\frac{d}{dt} \|f(t)\|_{L^1} = \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \langle \langle \rangle \rangle^3 - \langle \rangle^3 B(v - v_*, \sigma)f_f_{\cdot}(1 + f' \wedge K + f'' \wedge K)d\sigma dv_*dv
\]

\[
\leq 4(\sqrt{2} - 1)(1 + 2K) \iint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B(v - v_*, \sigma)\langle \langle \rangle \rangle^2 f_f_{\cdot}dv_*dv_*dv
\]

\[
= 4(\sqrt{2} - 1)(1 + 2K) I_1 - \frac{1}{32} I_2
\]

where \( \kappa(\theta) = \min\{(1 - \cos^2(\theta/2)), (1 - \sin^2(\theta/2))\} \). For \( I_1 \) and \( I_2 \) we compute

\[
I_1 \leq b \iint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \langle \langle \rangle \rangle^3 f_f_{\cdot}d\sigma dv_*dv = b|S^2|\|f_0\|_{L^1}\|f(t)\|_{L^1},
\]

\[
I_2 \geq a \int_{\mathbb{S}^2} \kappa(\theta)^{\frac{3}{2}}d\sigma \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|v - v_*|^{\beta + 1}}{1 + |v - v_*|}\langle \langle \rangle \rangle f_f_{\cdot}dv_*dv
\]

\[
\geq a C_0 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (|v - v_*| - \frac{|v - v_*|}{1 + |v - v_*|}) f_f_{\cdot}dv_*dv
\]

\[
\geq a C_0 (\|f_0\|_{L^1}\|f(t)\|_{L^1} - \|f(t)\|_{L^1}\|f(t)\|_{L^1} - \|f_0\|_{L^1}\|f(t)\|_{L^1})
\]

\[
\geq a C_0 (\|f_0\|_{L^1} \frac{\|f(t)\|_{L^1}^2}{\|f_0\|_{L^1}^2} - 2\|f_0\|_{L^1}^2 f(t)\|_{L^1})
\]

where we used the conservation of mass and energy of \( f \) and \( C_0 = \int_{S^2}|\kappa(\theta)|^{\frac{3}{2}}d\sigma = \pi \frac{2\sqrt{2}^3 - 31}{4} \pi \). Combining (4.21), (4.22) and (4.23) we obtain for a.e. \( t \in [0, \infty) \),

\[
\frac{d}{dt} \|f(t)\|_{L^1} \leq -c_1\|f(t)\|_{L^1}^2 + c_2\|f(t)\|_{L^1}
\]

where

\[
c_1 = \frac{(2^2\sqrt{2} - 31)p\pi|f_0|_{L^1}^2}{32|f_0|_{L^1}^2}, \quad c_2 = (4(\sqrt{2} - 1)(1 + 2K)b|S^2| + \frac{1}{16}(\frac{22}{5}\sqrt{2} - \frac{31}{5})\pi a)|f_0|_{L^1}.
\]

Here we note that by conservation of mass we have \( \|f(t)\|_{L^1} \geq \|f_0\|_{L^1} > 0 \) and from (4.19) we know that \( \frac{1}{\|f(t)\|_{L^1}^2} \) is absolutely continuous on any bounded interval. It follows that \( t \mapsto \frac{\exp(c_2 t)}{\|f(t)\|_{L^1}^2} \) is also absolutely continuous on any bounded interval. Thus

\[
\frac{d}{dt} \left( \frac{\exp(c_2 t)}{\|f(t)\|_{L^1}^2} \right) \geq c_1 \exp(c_2 t) \quad \text{a.e.} \quad t \in [0, \infty).
\]
Proposition 4.3. Let \( L_K(\cdot), Q_K(\cdot) \) be defined in (2.74), (2.75) respectively with \( B \) satisfying (1.10). For any \( 0 \leq f_0 \in L^1_3(\mathbb{R}^3) \cap L^\infty_3(\mathbb{R}^3) \) satisfying \( \int_{\mathbb{R}^3} v f_0(v) dv = 0 \), let \( f_t(v) = f(t, v) \) with the initial datum \( f_0 \) be a mild solution of Eq. (2.14) obtained in Proposition 4.1 corresponding to the kernel \( B \). Then we have

\[
L_K(f_t)(v) \geq a \frac{\min\{M_0, M_2\}}{2^\beta \|f_t\|_{L^1_3}^\beta} (v) \quad \forall (t, v) \in [0, \infty) \times \mathbb{R}^3
\]  

(4.24)

where \( M_0 = \int_{\mathbb{R}^3} f_0(v) dv, M_2 = \int_{\mathbb{R}^3} |v|^2 f_0(v) dv \).

**Proof.** Using Hölder’s inequality we have for any \( v \in \mathbb{R}^3 \)

\[
|v|^2 M_0 + M_2 = \int_{\mathbb{R}^3} |v - v_*|^2 f_t(v_*) dv_* \\
\leq \left( \int_{\mathbb{R}^3} \frac{|v - v_*|}{1 + |v - v_*|^{-\beta}} f_t(v_*) dv_* \right)^{\frac{2}{\beta+1}} \left( \int_{\mathbb{R}^3} |v - v_*|^{\frac{2\beta}{\beta+1}} (1 + |v - v_*|^{-\beta}) \frac{2\beta}{\beta+1} f_t(v_*) dv_* \right)^{\frac{\beta+1}{\beta+2}} \\
\leq \left( \int_{\mathbb{R}^3} \frac{|v - v_*|^{\beta+1}}{1 + |v - v_*|^2} f_t(v_*) dv_* \right)^{\frac{1}{\beta}} \left( \int_{\mathbb{R}^3} (1 + |v - v_*|^{\frac{2\beta}{\beta+1}}) f_t(v_*) dv_* \right)^{\frac{\beta+1}{\beta}}.
\]

From this inequality and the definition of \( L_K(f_t)(v) \) we deduce that for any \( (t, v) \in [0, \infty) \times \mathbb{R}^3 \)

\[
L_K(f_t)(v) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} B(v - v_*, \sigma) f_t(v_*) (1 + f_t(v') \land K + f_t(v') \land K) d\sigma dv_* \geq a \left( \int_{\mathbb{R}^3} |v - v_*|^{\beta+1} f_t(v_*) dv_* \right)^{\frac{1}{\beta+1}} \left( \int_{\mathbb{R}^3} (1 + |v - v_*|^{\frac{2\beta}{\beta+1}}) f_t(v_*) dv_* \right)^{\frac{\beta+1}{\beta}} \\
\geq a \frac{\min\{M_0, M_2\} (|v|^2 + 1)^{\frac{\beta+1}{2\beta}}}{2^\beta \|f_t\|_{L^1_3}^{\beta-1} \langle v \rangle^\beta} \geq a \frac{\min\{M_0, M_2\}}{2^\beta \|f_t\|_{L^1_3}^{\beta-1} \langle v \rangle^\beta}.
\]
where we used the assumption $\beta \geq 3$ and the conservation of mass, momentum ($\int_{\mathbb{R}^3} vf_0(v)dv = 0$) and energy of $f$.

\begin{proposition}
Let $Q^+(\cdot, \cdot)$, $Q_K(\cdot)$ be defined in \ref{eq:4.1.14}, \ref{eq:4.1.15} respectively with $B$ satisfying \ref{eq:4.1.14}. For any $0 \leq f_0 \in L^3_3(\mathbb{R}^3) \cap L^\infty_3(\mathbb{R}^3)$ satisfying $\int_{\mathbb{R}^3} vf_0(v)dv = 0$, let $f_t(v) \equiv f(t, v)$ be a mild solution of Eq.\ref{eq:2.1.14} obtained in Proposition \ref{prop:4.1} corresponding to the kernel $B$ with the initial datum $f_0$. Let

$$E'_t(v) = e^{-\int_0^t L_K(f_{\tau})(v)dv}, \quad 0 \leq \tau < t < \infty, \quad v \in \mathbb{R}^3,$$

$$\tilde{E}'_t(v) = \exp \left( -a \frac{\min\{M_0, M_2\}}{2^3 C_2^{2/3}} (t-\tau)(v) \right), \quad 0 \leq \tau \leq t < \infty, \quad v \in \mathbb{R}^3,$$

$$E'_\tau = \exp \left( -a \frac{\min\{M_0, M_2\}}{2^3 C_2^{2/3}} (t-\tau) \right), \quad 0 \leq \tau \leq t < \infty, \quad v \in \mathbb{R}^3$$

where $M_0 = \int_{\mathbb{R}^3} f_0(v)dv$, $M_2 = \int_{\mathbb{R}^3} |v|^2 f_0(v)dv$, and $C_2 = \max\{1, C_1\}\|f_0\|_{L^3_3}$ is the right hand side of \ref{eq:4.1.17} in Proposition \ref{prop:4.2}. Then for all $(t, v) \in [0, \infty) \times (\mathbb{R}^3 \setminus Z)$ (where $Z \subset \mathbb{R}^3$ is a null set as mentioned in Proposition \ref{prop:4.4}) we have

$$f_t(v) \leq E'_0(v)f_0(v) + (1 + 2K) \int_0^t \tilde{E}'_{\tau}(v)Q^+(f_{\tau}, f_{\tau})(v)d\tau \leq E'_0(v)f_0(v) + (1 + 2K) \int_0^t \tilde{E}'_{\tau}Q^+(f_{\tau}, f_{\tau})(v)d\tau,$$

(4.25)

and

$$E'_t(v) \leq \tilde{E}'_t(v) \leq E'_\tau, \quad 0 \leq \tau \leq t < \infty, \quad v \in \mathbb{R}^3.$$

(4.27)

In particular

$$E'_0(v) \leq \tilde{E}'_0 = \exp \left( -a \frac{\min\{M_0, M_2\}}{2^3 C_2^{2/3}} t \right), \quad (t, v) \in [0, \infty) \times \mathbb{R}^3.$$

(4.28)

\begin{proof}
Inequalities \ref{eq:4.2.14} and \ref{eq:4.2.15} follow from Propositions \ref{prop:4.1} and \ref{prop:4.2}. For every $v \in \mathbb{R}^3 \setminus Z$, the function $t \mapsto L_K(f_t)(v)$ is bounded on $[0, \infty)$ so that $t \mapsto \int_0^t L_K(f_{\tau})(v)d\tau$ is Lipschitz continuous hence $t \mapsto e^{L_0} L_K(f_t)(v)d\tau$ is Lipschitz on every bounded interval, it follows that the function $t \mapsto e^{L_0} L_K(f_t)(v)d\tau f(t, v)$ is absolutely continuous on every bounded interval. Thus it holds Duhamel's formula:

$$f_t(v) e^{L_0} L_K(f_t)(v)d\tau = f_0(v) + \int_0^t Q^+_K(f_{\tau})(v)e^{L_0} L_K(f_{\tau})(v)d\tau, \quad (t, v) \in [0, \infty) \times (\mathbb{R}^3 \setminus Z)$$

which is rewritten with $E'_0(v), E'_t(v)$ as

$$f_t(v) = E'_0(v)f_0(v) + \int_0^t E'_\tau(v)Q^+_K(f_{\tau})(v)d\tau, \quad (t, v) \in [0, \infty) \times (\mathbb{R}^3 \setminus Z).$$

(4.29)

Then \ref{eq:4.2.15}, \ref{eq:4.2.16} follow from \ref{eq:4.2.10} and \ref{eq:4.2.11}.

\end{proof}

From \ref{eq:4.2.15} we see that a uniform pointwise estimate for $f$ can be obtained from an appropriate pointwise estimate for $Q^+(f, f)$. We will establish such an estimate for $Q^+(f, f)$ in Proposition \ref{prop:4.1}, It
will be seen that a successful method to do this is to use multi-step iterations of the collision gain operator $Q^+(\cdot, \cdot)$, which we put into Propositions 4.5 - 4.7.

**Proposition 4.5.** Let $Q^+(\cdot, \cdot), Q_K(\cdot)$ be defined in (2.26), (2.29) respectively with $B$ satisfying (1.10). For any $0 \leq f_0 \in L^1_0(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ satisfying $\int_{\mathbb{R}^3} v f_0(v) dv = 0$, let $f_t(v) \equiv f(t, v)$ be the conservative mild solution of Eq. (2.11) obtained in Proposition 4.4 corresponding to the kernel $B$ with the initial datum $f_0$. Then for all $t \in [0, \infty), v \in \mathbb{R}^3 \setminus Z$

\[
Q^+(f_t, f_0)(v) \leq 2^8 b \pi \|f_0\|_{L^\infty} \|f_0\|_{L^1_2}(v) \\
+ (1 + 2K) \cdot 2^5 + \frac{a}{2} b \pi \|f_0\|_{L^1_2} \|f_0\|_{L^1_1} \frac{2^8 C_2}{2 a (\min\{M_0, M_2\})^{\frac{3}{2}}} (4.30)
\]

where $Z \subset \mathbb{R}^3$ is a null set given in Proposition 4.7, $M_0 = \int_{\mathbb{R}^3} f_0(v) dv$, $M_2 = \int_{\mathbb{R}^3} |v|^2 f_0(v) dv$, and $C_2 = \max\{1, C_1\} \|f_0\|_{L^1_1}$ is the right hand side of (4.17) in Proposition 4.2.

**Proof.** From (4.26), (2.31), (2.32) and (2.35) we have for all $(t, v) \in [0, \infty) \times (\mathbb{R}^3 \setminus Z)$

\[
Q^+(f_t, f_0)(v) \leq Q^+(f_0, f_0)(v) + (1 + 2K) \int_0^t \mathcal{E}_r Q^+(f_{\tau}, f_\tau, f_0)(v) d\tau \\
\leq 8 b \pi \|f_0\|_{L^\infty} \|f_0\|_{L^1_2}(v) + (1 + 2K) \int_0^t \mathcal{E}_{\tau} 16 b^2 \pi \|f_0\|_{L^1_2} \|f_0\|_{L^1_1} \frac{2^8}{2 a (\min\{M_0, M_2\})^{\frac{3}{2}}} d\tau \\
\leq 8 b \pi \|f_0\|_{L^\infty} \|f_0\|_{L^1_2}(v) + (1 + 2K) 16 b^2 \pi \|f_0\|_{L^1_2} \|f_0\|_{L^1_1} \frac{2^8}{2 a (\min\{M_0, M_2\})^{\frac{3}{2}}} \int_0^t \mathcal{E}_r d\tau \\
\leq 8 b \pi \|f_0\|_{L^\infty} \|f_0\|_{L^1_2}(v) + (1 + 2K) 16 b^2 \pi \|f_0\|_{L^1_2} \|f_0\|_{L^1_1} \frac{2^8}{2 a (\min\{M_0, M_2\})^{\frac{3}{2}}} (4.31)
\]

where we used the conservation of mass and energy of $f$ and $\|f_0\|_{L^2} \leq \|f_0\|_{L^\infty} \|f_0\|_{L^1_1}$.

**Proposition 4.6.** Let $Q^+(\cdot, \cdot), Q_K(\cdot)$ be defined in (2.26), (2.29) respectively with $B$ satisfying (1.11). For any $0 \leq f_0 \in L^1_0(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ satisfying $\int_{\mathbb{R}^3} v f_0(v) dv = 0$, let $f_t(v) \equiv f(t, v)$ be a conservative mild solution of Eq. (2.11) corresponding to the kernel $B$ with the initial datum $f_0$. Then for all $t, \tau \in [0, \infty)$ and $v \in \mathbb{R}^3 \setminus Z$

\[
Q^+(Q^+(f_t, f_0), Q^+(f_\tau, f_\tau))(v) \leq 2^8 b \pi \|f_0\|_{L^1_2} \|f_0\|_{L^1_1} \frac{2^8}{2 a (\min\{M_0, M_2\})^{\frac{3}{2}}} (4.31)
\]

where $Z \subset \mathbb{R}^3$ is a null set given in Proposition 4.7, $M_0 = \int_{\mathbb{R}^3} f_0(v) dv$, $M_2 = \int_{\mathbb{R}^3} |v|^2 f_0(v) dv$, and $C_2 = \max\{1, C_1\} \|f_0\|_{L^1_1}$ is the right hand side of (4.17) in Proposition 4.2.

**Proof.** According to Proposition 4.4 we have for any $t, \tau \in [0, \infty)$ and any $v \in \mathbb{R}^3 \setminus Z$

\[
Q^+(Q^+(f_t, f_0), Q^+(f_\tau, f_\tau))(v) \leq Q^+(Q^+(f_t, f_0), Q^+(f_\tau, f_\tau))(v) \\
+ (1 + 2K) \int_0^t \mathcal{E}_r Q^+(Q^+(f_t, Q^+(f_\tau, f_\tau)), Q^+(f_\tau, f_\tau))(v) d\tau (4.32)
\]

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First we estimate the second term in the right hand side of (4.32). Due to (2.37), Hölder’s inequality and the conservation of mass and energy of \( f \) we have

\[
\int_0^t \mathbb{F}_{\tau_1} Q^+(Q^+(f_t, Q^+(f_{\tau_1}, f_{\tau_1})), Q^+(f_{\tau_1}, f_{\tau_1}))(v) d\tau_1
\leq 2^{7}b^4 |S|^1 |S|^2 ||f_0||^{\frac{3}{2}}_{L^1} ||f_0||^{\frac{1}{2}}_{L^2} \int_0^t \mathbb{F}_{\tau_1} d\tau_1 \\
\leq 2^{7}b^4 |S|^1 |S|^2 ||f_0||^{\frac{3}{2}}_{L^1} ||f_0||^{\frac{1}{2}}_{L^2} \cdot \frac{2^{3}C_2^{\frac{2-1}{3}}} {a'(\min\{M_0, M_2\})^{\frac{3}{2}}}
\]

(4.33)

For the first term in the right hand side of (4.32) we deduce from (2.31), (2.33) and (2.37) that

\[
Q^+(Q^+(f_t, f_0), Q^+(f_{\tau_1}, f_{\tau_1}))(v)
\leq Q^+(Q^+(f_0, f_0), Q^+(f_{\tau_1}, f_{\tau_1}))(v)
\]

\[
+(1 + 2K) \int_0^t \mathbb{F}_{\tau_1} Q^+(Q^+(f_0, Q^+(f_{\tau_1}, f_{\tau_1})), Q^+(f_{\tau_1}, f_{\tau_1}))(v) d\tau_1 \\
\leq 16b^2 |S|^1 |S|^2 \|f_0\|_{L^1} \|f_0\|_{L^2} (b|S|^2 \|f_0\|_{L^1} \|f_0\|_{L^2})^{\frac{1}{2}} (2^{4}b^4 |S|^2 \|f_0\|_{L^1} \|f_0\|_{L^2}) \frac{2^{3}C_2^{\frac{2-1}{4}}} {a'(\min\{M_0, M_2\})^{\frac{4}{2}}}
\]

(4.34)

where we used the conservation of mass and energy of \( f \). Combining (4.32), (4.33) and (4.34) gives (4.31).

\(\Box\)

**Proposition 4.7.** Let \( Q^+(-, -) \), \( Q_K(-) \) be defined in (2.3), (2.9) respectively with \( B \) satisfying (1.10).

For any \( 0 \leq f_0 \in L^1_{\mathbb{R}^3} \cap L^\infty_{\mathbb{R}^3} \) satisfying \( \int_{\mathbb{R}^3} v f_0(v) dv = 0 \), let \( f_t(v) \equiv f(t, v) \) be a mild solution of Eq.(4.17) obtained in Proposition 4.7 corresponding to the kernel \( B \) with the initial datum \( f_0 \). Then for any \( t, \tau \in [0, \infty) \) and \( v \in \mathbb{R}^3 \setminus Z \)

\[
Q^+(f_t, Q^+(f_{\tau_1}, f_{\tau_1}))(v)
\leq 2^{5+2} b^2 \pi \|f_0\|^{\frac{3}{2}}_{L^1} \|f_0\|^{\frac{1}{2}}_{L^2} + (1 + 2K) \frac{2^{3}C_2^{\frac{2-1}{3}}} {a'(\min\{M_0, M_2\})^{\frac{3}{2}}} \left( 2^{8+4} b^3 \pi \|f_0\|^{\frac{3}{2}}_{L^1} \|f_0\|^{\frac{1}{2}}_{L^2} \|f_0\|^{\frac{1}{2}}_{L^\infty} \\
\right)
\]

(4.35)

where \( Z \subset \mathbb{R}^3 \) is a null set given in Proposition 4.7, \( M_0 = \int_{\mathbb{R}^3} f_0(v) dv, M_2 = \int_{\mathbb{R}^3} |v|^2 f_0(v) dv \), and \( C_2 = \max\{1, C_1\} \|f_0\|_{L^1} \) is the right hand side of (4.17) in Proposition 4.2.

**Proof.** From (2.31), (2.35), (4.29) and (4.31) we have for any \( t, \tau \in [0, \infty) \) and any \( v \in \mathbb{R}^3 \setminus Z \)

\[
Q^+(f_t, Q^+(f_{\tau_1}, f_{\tau_1}))(v)
\]
\[ \leq Q^+(E_0 f_0 + (1 + 2K) \int_0^t \mathcal{E}_r Q^+(f_r, f_r) \, dr, Q^+(f_r, f_r))(v) \]
\[ \leq Q^+(f_0, Q^+(f_r, f_r))(v) + (1 + 2K) \int_0^t \mathcal{E}_r Q^+(Q^+(f_r, f_r), Q^+(f_r, f_r)) \, dr \]
\[ \leq 16b^2|S|^1|S|^2^{\frac{1}{2}}f_0 \|f_0\|_{L^2}\|f_0\|_{L^\infty} + (1 + 2K) \frac{2^8 b^4 |S|^1|S|^2^{\frac{1}{2}}\|f_0\|_{L^2}\|f_0\|_{L^2}}{a(\min\{M_0, M_2\})^{\frac{1}{2}}} \]

where we used the conservation of mass and energy of \( f \). This gives (4.35).

**Proposition 4.8.** Let \( Q^+(\cdot, \cdot) \), \( Q_K(\cdot) \) be defined in (2.9), (2.10) respectively with \( B \) satisfying (1.16).

For any \( 0 \leq f_0 \in L_1^4(\mathbb{R}^3) \cap L_\infty^3(\mathbb{R}^3) \) satisfying \( \int_{\mathbb{R}^3} v f_0(v) \, dv = 0 \), let \( f_t(v) \equiv f(t, v) \) be a mild solution of Eq. (4.7) obtained in Proposition 4.7 corresponding to the kernel \( B \) with the initial datum \( f_0 \). Then for any \( (t, v) \in [0, \infty) \times (\mathbb{R}^3 \setminus Z) \) we have

\[ Q^+(f_t, f_t)(v) \leq A(f_0)(v) + P(f_0) \quad (4.36) \]

where

\[
A(f_0) = 2^5 b\pi \|f_0\|_{L^\infty} \|f_0\|_{L^1},
\]

\[
P(f_0) = (1 + 2K) 2^{6+\frac{1}{2}} 2^{12} \pi^{\frac{1}{2}} \|f_0\|_{L^2} \|f_0\|_{L^\infty} 2^\frac{b}{2} C_2 \frac{2^\frac{b}{2} - 1}{a(\min\{M_0, M_2\})^{\frac{1}{2}}} \\
+ \left( 1 + 2K \right) 2^{6+\frac{1}{2}} 2^{12} \pi^{\frac{1}{2}} \|f_0\|_{L^2} \|f_0\|_{L^\infty} + (1 + 2K) 2^\frac{b}{2} C_2 \frac{2^\frac{b}{2} - 1}{a(\min\{M_0, M_2\})^{\frac{1}{2}}} \\
\times \left( \left( 1 + 2K \right) 2^{6+\frac{1}{2}} 2^{12} \pi^{\frac{1}{2}} \|f_0\|_{L^2} \|f_0\|_{L^\infty} + (1 + 2K) 2^\frac{b}{2} C_2 \frac{2^\frac{b}{2} - 1}{a(\min\{M_0, M_2\})^{\frac{1}{2}}} \right) \]

where \( Z \subset \mathbb{R}^3 \) is a null set given in Proposition 4.7, \( M_0 = \int_{\mathbb{R}^3} f_0(v) \, dv \), \( M_2 = \int_{\mathbb{R}^3} |v|^2 f_0(v) \, dv \), and \( C_2 = \max\{1, C_1\} \|f_0\|_{L^1} \) is the right hand side of (4.17) in Proposition 4.2.

**Proof.** This is just a calculation: according to Propositions 4.4–4.7, we have for all \( (t, v) \in [0, \infty) \times (\mathbb{R}^3 \setminus Z) \)

\[
Q^+(f_t, f_t)(v) \leq Q^+(E_0 f_0 + (1 + 2K) \int_0^t \mathcal{E}_r Q^+(f_r, f_r) \, dr, f_t)(v) \]
\[ \leq Q^+(f_0, f_t)(v) + (1 + 2K) \int_0^t \mathcal{E}_r Q^+(Q^+(f_r, f_r), f_t) \, dr \]
\[ \leq 8b|S|^1 \|f_0\|_{L^\infty} \|f_0\|_{L^1} + (1 + 2K) 16b^2 |S|^1 |S|^2^{\frac{1}{2}} \|f_0\|_{L^2} \|f_0\|_{L^\infty} \frac{2^\frac{b}{2} C_2 \frac{2^\frac{b}{2} - 1}{a(\min\{M_0, M_2\})^{\frac{1}{2}}}}{a(\min\{M_0, M_2\})^{\frac{1}{2}}} \\
+ \int_0^t \mathcal{E}_r \, dr \left( (1 + 2K) 16b^2 |S|^1 |S|^2^{\frac{1}{2}} \|f_0\|_{L^2} \|f_0\|_{L^\infty} + (1 + 2K)^2 \frac{2^\frac{b}{2} C_2 \frac{2^\frac{b}{2} - 1}{a(\min\{M_0, M_2\})^{\frac{1}{2}}}}{a(\min\{M_0, M_2\})^{\frac{1}{2}}} \right) \\
\times \left[ (1 + 2K) 2^\frac{b}{2} |S|^1 |S|^2^{\frac{1}{2}} \|f_0\|_{L^2} \|f_0\|_{L^\infty} + (1 + 2K) 2^\frac{b}{2} C_2 \frac{2^\frac{b}{2} - 1}{a(\min\{M_0, M_2\})^{\frac{1}{2}}} \right] \]
From (5.1) we see that in order to prove

Having made sufficient preparations in previous sections we can now turn to the

5 Proof of Theorem 1.2

To be clear we divide the proof into four steps.

Proof of Theorem 1.2. To be clear we divide the proof into four steps.

Step 1. We prove that the mild solution \( f \) of Eq. (2.14) obtained in Proposition 4.1 is a mild solution of Eq. (1.12) by proving \( f \leq K \) with a suitable constant \( K > \|f_0\|_{L^\infty} \). Let \( Z \subset \mathbb{R}^3 \) be the null set appeared in Proposition 4.1 and let

\[
C_0 = \left( \frac{22}{5} \sqrt{2} - \frac{31}{5} \right) \pi, \quad C_1 = \{1, C_1\} \|f_0\|_{L^0}, \quad C_3 = \frac{2^\beta C_2^{\frac{\beta-1}{\beta}}}{a(\min\{M_0, M_2\})^{\frac{\beta-1}{\beta}}}
\]

where \( C_1 \) is given in (4.18) in Proposition 4.1. Using Propositions 4.3 and 4.8 we compute for all \((t, v) \in [0, \infty) \times (\mathbb{R}^3 \setminus Z)\)

\[
f(t, v) \leq f_0(v) + (1 + 2K) \int_0^t \tilde{E}_1(v) A(f_0)(v) + \tilde{E}_2 P(f_0) d\tau
\]

\[
\leq f_0(v) + (1 + 2K) C_3 \left\{ 8b|S|^2 \|f_0\|_{L^\infty} \|f_0\|_{L^1} + (1 + 2K) 16b^2|S|^1 |S^2|^\frac{5}{4} \|f_0\|_{L^1} \|f_0\|_{L^\infty} \right\}
\]

\[
+ C_3 \left[ (1 + 2K) 16b^2|S|^1 |S^2|^\frac{1}{4} \|f_0\|_{L^1} \|f_0\|_{L^\infty} \right] + (1 + 2K)^2 C_3
\]

\[
\times \left\{ 2^{1+5} b^3|S|^1 |S^2|^\frac{5}{4} \|f_0\|_{L^1} \|f_0\|_{L^2} \|f_0\|_{L^\infty} + (1 + 2K)^2 b^4 |S|^1 |S^2|^2 \|f_0\|_{L^1} \|f_0\|_{L^2} \right\}
\]

\[
\leq f_0(v) + 2^8 |S|^1 |S^2| (1 + 2K)^4 C_4
\]

where

\[
C_4 = b C_3 (M_0 + M_2) \|f_0\|_{L^\infty} + (b C_3)^2 M_0 ^{\frac{5}{4}} \|f_0\|_{L^\infty} + (b C_3)^3 M_0 ^{\frac{7}{4}} (M_0 + M_2) ^{\frac{5}{4}} \|f_0\|_{L^\infty}
\]

\[
+ (b C_3)^4 M_0 ^{\frac{9}{4}} (M_0 + M_2) ^{\frac{7}{4}}.
\]

From (5.1) we see that in order to prove \( f(t, v) \leq K \) for all \((t, v) \in [0, \infty) \times (\mathbb{R}^3 \setminus Z)\), it suffices to prove that \( \|f_0\|_{L^\infty} + 2^8 |S|^1 |S^2| (1 + 2K)^4 C_4 \leq K \), i.e. to prove that

\[
C_4 \leq \frac{K - \|f_0\|_{L^\infty}}{2^8 |S|^1 |S^2| (1 + 2K)^4}
\]

\[
(5.3)
\]

\[
(5.1)
\]
Denoting \( \rho_0 := M_2/M_0 \), we will prove (5.3) by discussing the following four cases.

**Case 1:** \( C_1 \geq 1 \) and \( \rho_0 \geq 1 \). In this case we have

\[
bC_3 \leq C(a, b) \rho_0^{\beta} \cdot \frac{1}{M_2}
\]

where

\[
C(a, b) = \frac{2^{2\beta-1}(2^{6}(1 + 2K)|S^2|)^{\frac{2\beta+1}{\beta}}}{\rho_0^{\beta}} \left( \frac{b}{a} \right)^{\frac{2\beta+1}{\beta}}
\]

and we have used \( 0 < C_0 < 1, 0 < a \leq b < \infty \). By definition of \( C_4 \) in (5.2), \( \rho_0 \geq 1 \) and \( C(a, b) > 1 \), this gives

\[
C_4 + \frac{\|f_0\|_{L^\infty}}{2^{8}|S^1|^2|S^2|(1 + 2K)\frac{1}{\beta}} \leq C(a, b) \rho_0^{\beta-1}(1 + \rho_0)\|f_0\|_{L^\infty} + C(a, b)^3 \rho_0^{\beta-1}M_0^{\frac{2}{\beta}} \|f_0\|_{L^\infty} + C(a, b)^5(1 + \rho_0)M_0 + \|f_0\|_{L^\infty}
\]

where we used the inequality \( x^p y^{1-p} \leq px + (1-p)y \) for all \( x, y > 0 \) and \( 0 \leq p \leq 1 \). Combining this with (5.4) we see that in order to prove (5.3) it suffices to prove that

\[
\rho_0^{4\beta-3}(M_0 + \|f_0\|_{L^\infty}) \leq \frac{C_0^{2(\beta-1)K}}{2^{20\beta-6}|S^1|^2|S^2|^2}(1 + 2K)^{2(\beta+1)} \left( \frac{a}{b} \right)^{2\beta+2}. \tag{5.5}
\]

**Case 2:** \( C_1 \geq 1 \) and \( \rho_0 < 1 \). For this case we have

\[
bC_3 \leq C(a, b) \rho_0^{\beta} \cdot \frac{1}{M_0}.
\]

An argument similar to the one used in **Case 1** shows that

\[
C_4 + \frac{\|f_0\|_{L^\infty}}{2^{8}|S^1|^2|S^2|(1 + 2K)\frac{1}{\beta}} \leq 4C(a, b)^4 \rho_0^{\beta-1}(M_0 + \|f_0\|_{L^\infty}) \tag{5.6}
\]

Following the same argument as in **Case 1** we see that in this case in order to prove (5.3) it suffices to prove that

\[
\frac{1}{\rho_0^{\beta}}(M_0 + \|f_0\|_{L^\infty}) \leq \frac{C_0^{2(\beta-1)K}}{2^{20\beta-6}|S^1|^2|S^2|^2}(1 + 2K)^{2(\beta+1)} \left( \frac{a}{b} \right)^{2\beta+2}. \tag{5.7}
\]

**Case 3:** \( C_1 < 1 \) and \( \rho_0 \geq 1 \). In this case we have

\[
bC_3 \leq 2^\beta \left( \frac{\|f_0\|_{L^1}}{M_0} \right)^{\frac{2\beta+1}{\beta}} \frac{b}{a} \cdot \frac{1}{M_0} := \widetilde{C}(a, b, M_0, \|f_0\|_{L^1}) \frac{1}{M_0}.
\]

Inserting this into the expression of \( C_4 \) and noticing that \( \widetilde{C}(a, b, M_0, \|f_0\|_{L^1}) \geq 1 \) we compute

\[
C_4 + \frac{\|f_0\|_{L^\infty}}{2^{8}|S^1|^2|S^2|(1 + 2K)\frac{1}{\beta}} \leq 4\rho_0\widetilde{C}(a, b, M_0, \|f_0\|_{L^1})^4(M_0 + \|f_0\|_{L^\infty}). \tag{5.8}
\]

Thus, to prove (5.3), it suffices to prove that

\[
M_0 + \|f_0\|_{L^\infty} \leq \frac{K}{2^{10}|S^1|^2|S^2|\rho_0\widetilde{C}(a, b, M_0, \|f_0\|_{L^1})^4(1 + 2K)^4}. \tag{5.9}
\]
While from the definition of $\bar{C}(a, b, M_0, \|f_0\|_{L^1})$ we see that in order to prove (5.9) it suffices to prove that

$$(M_0 + \|f_0\|_{L^\infty}) \left( \frac{\|f_0\|_{L^1}}{M_0} \right)^{2\beta-1} \leq \frac{K}{2^{4\beta+10}\|S^1\|^2\|S^2\|^2(1 + 2K)^4} \left( \frac{a}{b} \right)^4.$$  (5.10)

**Case 4:** If $\beta_1 < 1$ and $\rho_0 < 1$. For this case we have

$$bC_3 \leq \frac{3}{2} \left( \frac{\|f_0\|_{L^1}}{M_2} \right)^{2\beta-1} \frac{1}{M_2} =: \bar{C}(a, b, M_2, \|f_0\|_{L^1}) \frac{1}{M_2}.$$  

An argument similar to the one used in **Case 3** shows that

$$C_4 + \frac{\|f_0\|_{L^\infty}}{2^{\|S^1\|^2\|S^2\|^2}(1 + 2K)^4} \leq \frac{1}{\rho^4} \bar{C}(a, b, M_2, \|f_0\|_{L^1})^4(M_0 + \|f_0\|_{L^\infty})^4.$$  (5.11)

Following the same argument as in **Case 3**, we see that in order to prove (5.3) it suffices to prove

$$(M_0 + \|f_0\|_{L^\infty}) \left( \frac{\|f_0\|_{L^1}}{M_2} \right)^{2\beta+1} \leq \frac{K}{2^{4\beta+10}\|S^1\|^2\|S^2\|^2(1 + 2K)^4} \left( \frac{a}{b} \right)^{2\beta+2}.$$  (5.12)

Now we summarize these four cases. From (5.3), (5.7), (5.9) and (5.11) we see that in order to prove (5.5) it suffices to prove

$$(\|f_0\|_{L^1} + \|f_0\|_{L^\infty}) \left( \frac{\|f_0\|_{L^1}}{\min\{M_0, M_2\}} \right)^{2\beta} \leq \frac{C_0^{2(\beta-1)} K}{2^{3\beta-6}\|S^1\|^2\|S^2\|^2(1 + 2K)^{2\beta+1}} \left( \frac{a}{b} \right)^{2\beta+2}.$$  (5.13)

Maximizing the right hand side of it with respect to $K$ gives that a good choice of $K$ is $K = \frac{1}{4\beta+2}$. Thus to prove proved (5.5) with $K = \frac{1}{4\beta+2}$, we need only to prove that

$$(\|f_0\|_{L^1} + \|f_0\|_{L^\infty}) \left( \frac{\|f_0\|_{L^1}}{\min\{M_0, M_2\}} \right)^{2\beta} \leq \frac{1}{2^{3\beta-6}\|S^1\|^2\|S^2\|^2(1 + 2K)^{2\beta+1}} \left( \frac{a}{b} \right)^{2\beta+2},$$

but this is just the given condition (1.72). Thus we have proved that $f(t, v) \leq K = \frac{1}{4\beta+2}$ for all $(t, v) \in [0, \infty) \times (\mathbb{R}^3 \setminus Z)$. Then by elementary calculation we have $Q_K(f)(t, v) = Q(f)(t, v)$ for a.e. $(t, v) \in [0, \infty) \times \mathbb{R}^3$. This implies that $f(t, v) = f_0(v) + \int_0^t Q(f)(\tau, v)d\tau$ for a.e. $(t, v) \in [0, \infty) \times \mathbb{R}^3$. Since for almost every $v \in \Omega$, $t \mapsto f(t, v)$, $t \mapsto \int_0^t Q(f)(\tau, v)d\tau$ are continuous on $[0, \infty)$, it follows from Lemma 2.10 that there exists a null set (still denote it as $Z$) $Z \subset \mathbb{R}^3$, such that for all $(t, v) \in [0, \infty) \times (\mathbb{R}^3 \setminus Z)$, $f(t, v) = f_0(v) + \int_0^t Q(f)(\tau, v)d\tau$, i.e.

$$f(t, v) = f_0(v) + \int_0^t d\tau \int_{\mathbb{R}^3 \times S^2} B(v - v_*, \sigma) [f' f_*(1 + f + f_*) - f f_*(1 + f' + f'_*)] d\sigma dv_*.$$  (5.14)

This proves that $f$ is a mild solution of Eq. (1.12). Moreover combining (5.4), (5.6), (5.8) and (5.11) leads to

$$f(t, v) \leq 2^{3\beta} \left( \frac{b}{a} \right)^{2\beta+1} \frac{(2\beta + 2)^{2\beta-2}}{(2\beta + 3)^{2\beta-2}} \left( \frac{\|f_0\|_{L^1}}{\min\{M_0, M_2\}} \right)^{2\beta} \left( \|f_0\|_{L^1} + \|f_0\|_{L^\infty} \right) =: C_{L^\infty}(f_0)$$

for all $(t, v) \in [0, \infty) \times (\mathbb{R}^3 \setminus Z)$. 42
Step 2. We will use Lemma 2.14 to prove the high temperature condition (1.24). Define \( \phi(r) := \frac{1}{4\pi \| f_0 \|_{L^\infty}} \int_{S^2} f_0(r\sigma) d\sigma \) for \( r \in [0, \infty) \). Then \( 0 \leq \phi(r) \leq 1 \) for all \( r \in [0, \infty) \) and

\[
\frac{M_0}{4\pi \| f_0 \|_{L^\infty}} = \int_0^\infty r^2 \phi(r) dr, \quad \frac{M_2}{4\pi \| f_0 \|_{L^\infty}} = \int_0^\infty r^4 \phi(r) dr.
\]

According to the inequality (2.24), it follows from (1.21) and (1.22) that

\[
\frac{T}{T_\infty} \geq \frac{2\pi \left(3/2\right) ^2}{3\zeta(5/2)} \frac{3^2}{5(4\pi) ^2 \| f_0 \|_{L^\infty} ^2} \geq \frac{2 \pi \left(3/2\right) ^2}{3\zeta(5/2)} \frac{3^2}{5(4\pi) ^2} \frac{b}{a} ^{\frac{4\beta + 4}{3\beta} + 2} \geq 1,
\]

which proves (1.24).

Step 3. We prove that, after a modification on a \( \nu \)-null set, the bounded mild solution \( f \) of Eq.(2.13) obtained in Step 1 is a solution of Eq.(1.12). To do this we first derive a similar version of Duhamel’s formula: for every \( v \in \mathbb{R}^3 \setminus Z \) (with \( \text{mes}(Z) = 0 \)) and every \( t \geq 0 \)

\[
f(t,v) = f_0(v) + \int_0^t dr \int_{\mathbb{R}^3 \times S^2} B(v - v_*,\sigma)[f'f'_*(1 + f_*) + f'f'_* - ff_*(1 + f_*)]d\sigma dv_*,
\]

\[
= f_0(v) + \int_0^t \tilde{Q}^+(f)(\tau,v)d\tau - \int_0^t f(\tau,v)\tilde{L}(f)(\tau,v)d\tau
\]

where

\[
\tilde{Q}^+(f)(t,v) = \int_{\mathbb{R}^3 \times S^2} B(v - v_*,\sigma)f'f'_*(1 + f_*)d\sigma dv_*, \quad (5.15)
\]

\[
\tilde{L}(f)(t,v) = \int_{\mathbb{R}^3 \times S^2} B(v - v_*,\sigma)[f_*(1 + f_*) - f'f'_*]d\sigma dv_*, \quad (5.16)
\]

Thus for every \( v \in \mathbb{R}^3 \setminus Z \) and for almost every \( t \in [0, \infty) \)

\[
\frac{\partial f(t,v)}{\partial t} = \tilde{Q}^+(f)(t,v) - f(t,v)\tilde{L}(f)(t,v).
\]

As before, for every \( v \in \mathbb{R}^3 \setminus Z \) the function \( t \mapsto \tilde{L}(f)(t,v) \) is bounded and the function \( t \mapsto \int_0^t \tilde{L}(f)(\tau,v)d\tau \) is Lipschitz continuous hence \( t \mapsto e^{\int_0^t \tilde{L}(f)(\tau,v)d\tau} \) is Lipschitz on every bounded interval. It follows that the function \( t \mapsto e^{\int_0^t \tilde{L}(f)(\tau,v)d\tau} f(t,v) \) is absolutely continuous on every bounded interval and thus it holds Duhamel’s formula:

\[
f(t,v) = f_0(v)e^{-\int_0^t \tilde{L}(f)(\tau,v)d\tau} + \int_0^t e^{-\int_0^\tau \tilde{L}(f)(\tau',v)d\tau'} \tilde{Q}^+(f)(\tau,v)d\tau \quad (5.17)
\]

for all \((t,v) \in [0, \infty) \times (\mathbb{R}^3 \setminus Z)\).

To prove the continuity of the solution \( f \) we also need the following proposition concerning the continuity of the collision integrals (5.15), (5.16):

**Proposition 5.1.** Let \( B \) be the collision kernel satisfying (1.10), \( Q \) the collision operators defined in (1.18). Let \( 0 \leq f_0 \in L^1_3(\mathbb{R}^3) \cap L^\infty_3(\mathbb{R}^3) \cap C(\mathbb{R}^3) \) satisfy \( \int_{\mathbb{R}^3} v f_0(v) dv = 0 \) and (1.22), and let \( f \) with the initial datum \( f_0 \) be a mild solution of Eq.(1.12) obtained in Step 1 of the proof of Theorem 1.1. Then \((t,v) \mapsto \tilde{Q}^+(f)(t,v), (t,v) \mapsto \tilde{L}(f)(t,v) \) are both continuous on \([0, \infty) \times \mathbb{R}^3\).
A proof of this proposition will be given later. Right now we continue the proof of Theorem 1.2. Let us define
\[
g(t, v) = f_0(v)e^{-\int_0^t \tilde{L}(f)(\tau, v)d\tau} + \int_0^t e^{-\int_0^\tau \tilde{L}(f)(\sigma, v)d\sigma} \tilde{Q}^+(f)(\tau, v)d\tau, \quad (t, v) \in [0, \infty) \times \mathbb{R}^3. \tag{5.18}
\]
We see from Proposition 5.1 that \(0 \leq g \in C([0, \infty) \times \mathbb{R}^3)\). Comparing (5.18) with (5.17), we have
\[
g(t, v) = f(t, v) \quad \text{for any } (t, v) \in [0, \infty) \times (\mathbb{R}^3 \setminus Z) \quad \text{and} \quad \sup_{0 \leq t \leq T} \| Q^\pm(g)(t) \|_{L^1} < \infty, \quad \sup_{0 \leq t \leq T} \| Q^\pm(f)(t) \|_{L^1} < \infty
\]
for any \(T > 0\). Then we conclude with simple calculation that \(Q(g)(t, v) = Q(f)(t, v)\) for a.e. \((t, v) \in [0, \infty) \times \mathbb{R}^3\). This implies that \(g(t, v) = f_0(v) + \int_0^t Q(g)(\tau, v)d\tau\) for a.e. \((t, v) \in [0, \infty) \times \mathbb{R}^3\). Since
\[
\sup_{0 \leq t \leq T} |Q^\pm(g)(t, v)| < \infty \quad \text{for any } T \in [0, \infty) \text{ and any } v \in \mathbb{R}^3,
\]
it follows that \(t \mapsto \int_0^t Q(g)(\tau, v)d\tau\) is continuous on \([0, \infty)\). Combining this with \(g \in C([0, \infty) \times \mathbb{R}^3)\) and Lemma 2.10 implies that there exists a null set \(Z_0 \subset \mathbb{R}^3\) such that for all \((t, v) \in [0, \infty) \times (\mathbb{R}^3 \setminus Z_0)\), \(g(t, v) = f_0(v) + \int_0^t Q(g)(\tau, v)d\tau\). From Proposition 5.1 and \(g \in C([0, \infty) \times \mathbb{R}^3)\) we see that \(Q(g) = \tilde{Q}^+(g) = \tilde{L}(g) \in C([0, \infty) \times \mathbb{R}^3)\). This together with \(g \in C([0, \infty) \times \mathbb{R}^3)\) and the fact that \(\sup_{0 \leq t \leq T} |Q^\pm(g)(t, v)| < \infty \quad \text{for any } T \in [0, \infty)\) imply that
\[
g(t, v) = f_0(v) + \int_0^t Q(g)(\tau, v)d\tau \quad \forall (t, v) \in [0, \infty) \times \mathbb{R}^3.
\]
Next from (5.14) and \(\sup_{t \in [0, \infty)} \| g(t) \|_{L^1_3} = \sup_{t \in [0, \infty)} \| f(t) \|_{L^1_3} < \infty\), we can make a similar calculation as done in the proof of Proposition 5.1 to obtain that for every fixed \(v \in \mathbb{R}^3\), the function \(t \mapsto Q(g)(t, v)\) is continuous on \([0, \infty)\). It follows that
\[
\frac{\partial}{\partial t} g(t, v) = Q(g)(t, v) \quad \forall (t, v) \in [0, \infty) \times \mathbb{R}^3.
\]
Then by the continuity of \(g\) and (5.14), it is easily seen that \(g\) satisfies the \(L^\infty\) estimate (1.23). Thus from Definition 1.14 we see that \(g\) is a solution of Eq. (1.12) with initial datum \(0 \leq g(0, \cdot) = f_0 \in L^1_3(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \cap C(\mathbb{R}^3)\). To keep the same notation of solution as in the theorem, we now rewrite \(g\) as \(f\) and thus we have proved the global in time existence of a classical solution \(0 \leq f \in L^\infty([0, \infty); L^1_3(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \cap C(\mathbb{R}^3))\) of Eq. (1.12) with the initial datum \(0 \leq f_0 \in L^1_3(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \cap C(\mathbb{R}^3)\).

**Step 4.** We prove that the classical solution \(f\) obtained in **Step 3** is unique in \(L^\infty([0, \infty); L^1_3(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \cap C(\mathbb{R}^3))\). But this follows easily from (5.10).

To finish the proof of Theorem 1.2, we now need only to finish the

**Proof of Proposition 5.1.** In the following we denote \(C_{*, \ast, \ldots, \ast} \) to be any finite and positive constants that depend only on their arguments \(*, \ast, \ldots, \ast\), and they may have different values in different lines. We divide the proof into four steps.
Step 1. Let $C_{L^\infty} = C_{L^\infty}(f_0)$ be defined in (5.14). Then, \( \sup_{t \leq 0} \|f(t)\|_{L^\infty} \leq C_{L^\infty} \). Therefore

\[
\begin{align*}
\|f(s) - f(t)\|_{L^1} &\leq \int_s^t d\tau \int_{\mathbb{R}^3} |Q(f)(\tau, v)| |(1 + |v|^2)dv \\
&\leq 2(1 + 2C_{L^\infty}) \int_s^t d\tau \int_{\mathbb{R}^3} |v - v_*| f(\tau, v)f(\tau, v_*) (1 + |v|^2 + |v_*|^2) dv dv_* \\
&\leq 2(1 + 2C_{L^\infty}) \int_s^t \|f(\tau)\|_{L^1}^2 d\tau \leq C_{f_0} |t - s| \quad \forall 0 \leq s < t < \infty \quad (5.19)
\end{align*}
\]

where we used \( \sup_{t \geq 0} \|f(t)\|_{L^1} \leq \max \{1, C_1\} \|f_0\|_{L^1} \) (see Proposition 4.2).

Step 2. Fix any \( 1 \leq R < \infty \). Let \( v \in \mathbb{R}^3 \) satisfy \( |v| < R \). For any \( 0 < \delta < \frac{1}{2} \) and any \( t, s \in [0, \infty) \), we compute

\[
\begin{align*}
&|\tilde{Q}^+(f)(t, v) - \tilde{Q}^+(f)(s, v)| \\
&\leq (1 + C_{L^\infty}) \int_{\mathbb{R}^3 \times \mathbb{S}^2} B(v - v_*, \sigma) \left( f(s, v') |f(s, v')| - f(t, v') |f(t, v')| + f(t, v') |f(t, v') - f(s, v')| \\
&+ f(t, v') |f(t, v') - f(s, v')| \right) dv dv_* \\
&\leq b(1 + C_{L^\infty}) \int_{\mathbb{R}^3 \times \mathbb{S}^2} |v - v_*| \left( f(s, v') |f(s, v')| - f(t, v') |f(t, v')| + f(t, v') |f(t, v') - f(s, v')| \\
&+ f(t, v') |f(t, v') - f(s, v')| \right) dv dv_* \\
&\leq b(1 + C_{L^\infty}) \left( 2C_{L^\infty} \delta^4 \frac{4\pi(v)}{f(s)} |f(s) - f(t)|_{L^1} + 2C_0 C_{L^\infty} \langle v \rangle (\|f_0\|_{L^1} \|f_0\|_{L^1}^{1/2}) \frac{1}{\delta} \\
&+ C_2^4 \frac{4\pi(v)}{f(s)} |f(s) - f(t)|_{L^1} \right) \\
&\leq C_{f_0} R \left( \delta^4 |f(s) - f(t)|_{L^1} + \delta + |f(s) - f(t)|_{L^1} \right) \quad (5.20)
\end{align*}
\]

where we have used Proposition 2.13 to obtain

\[
\begin{align*}
\int_{\mathbb{R}^3 \times \mathbb{S}^2} |v - v_*| f(s, v') |f(s, v')| - f(t, v') |f(t, v')| dv dv_* &= \int_{\mathbb{R}^3 \times \mathbb{S}^2} |v - v_*| f(s, v') |f(s, v')| - f(t, v') |f(t, v')| dv dv_* \\
&= \int_{\mathbb{R}^3 \times \mathbb{S}^2} 1_{\{|v| \geq \delta\}} |v - v_*| f(s, v') |f(s, v')| - f(t, v') |f(t, v')| dv dv_* \\
&+ \int_{\mathbb{R}^3 \times \mathbb{S}^2} 1_{\{|v| < \delta\}} |v - v_*| f(s, v') |f(s, v')| - f(t, v') |f(t, v')| dv dv_* \\
&\leq C_{L^\infty} \int_{\mathbb{R}^3 \times \mathbb{S}^2} 1_{\{|v| \geq \delta\}} |v - v_*| f(s, v') |f(s, v')| - f(t, v') |f(t, v')| dv dv_* \\
&+ C_{L^\infty} \int_{\mathbb{R}^3 \times \mathbb{S}^2} 1_{\{|v| < \delta\}} |v - v_*| f(s, v') |f(s, v')| - f(t, v') |f(t, v')| dv dv_* \\
&= C_{L^\infty} \int_{\mathbb{R}^3 \times \mathbb{S}^2} \frac{1}{\sin^2(\theta/2)} 1_{\{|v| \geq \delta\}} |v - v_*| f(s, v_*) - f(t, v_*) dv dv_* \\
&+ C_{L^\infty} \int_{\mathbb{R}^3 \times \mathbb{S}^2} \frac{1}{\cos^2(\theta/2)} 1_{\{|v| < \delta\}} |v - v_*| f(s, v_*) - f(t, v_*) dv dv_* \\
&= C_{L^\infty} \delta^4 \int_{\mathbb{R}^3 \times \mathbb{S}^2} |v - v_*| f(s, v_*) - f(t, v_*) dv dv_* + C_0 C_{L^\infty} \int_{\mathbb{R}^3 \times \mathbb{S}^2} 1_{\{|v| < \delta\}} |v - v_*| f(s, v_*) dv dv_* \\
&\leq C_{L^\infty} \delta^4 4\pi(v) |f(s) - f(t)|_{L^1} + C_0 C_{L^\infty} \langle v \rangle (\|f_0\|_{L^1} \|f_0\|_{L^1}^{1/2}) \frac{1}{\delta}
\end{align*}
\]
and
\[
\int_{\mathbb{R}^3 \times S^2} |v - u_s| f(t, v') f(t, v_s') |f(s, v_s) - f(t, v_s)| |\sigma dv_s \leq C_L^2 \approx 4\pi \langle v \rangle \|f(s) - f(t)\|_{L^1_2}.
\]
Minimizing the right-hand side of (5.20) with respect to \( \delta > 0 \) gives
\[
\sup_{|v| < R} | \tilde{Q}^+(f)(t, v) - \tilde{Q}^+(f)(s, v) | \leq C_{f_0, R}(\|f(s) - f(t)\|_{L^1_2})^{1/5} \quad \forall s, t \in [0, \infty).
\]
Following the similar argument we also have
\[
\sup_{|v| < R} | \tilde{L}(f)(t, v) - \tilde{L}(f)(s, v) | \leq C_{f_0, R}(\|f(s) - f(t)\|_{L^1_2})^{1/5} \quad \forall s, t \in [0, \infty).
\]
From (5.19), (5.20) and (5.21) we can see that for any \( u \in \mathbb{R}^3 \) with \( \|v\| < R \), the functions \( t \mapsto \tilde{Q}^+(f)(t, v) \) and \( t \mapsto \tilde{L}(f)(t, v) \) are both uniformly continuous on \( [0, \infty) \).

**Step 3.** Fix any \( 0 < T < \infty, t \in [0, T] \). For any \( h \in \mathbb{R}^3 \) with \( \|h\| < 1 \), since \( B \) satisfies (1.7), it is immediately verified that the velocity-translation \( g(t, v) := f(t, v + h) \) is a mild solution to Eq. (1.12) with the initial datum \( g_0(v) = f_0(v + h) \). For \( 0 < \delta < \frac{1}{2} \) and \( |v| < R \), using Propositions 2.13 and 3.10 and doing a similar calculation as in **Step 2** we have
\[
\sup_{t \in [0, T], |v| < R} | \tilde{Q}^+(f)(t, v + h) - \tilde{Q}^+(f)(t, v) | = \sup_{t \in [0, T], |v| < R} | \tilde{Q}^+(g)(t, v) - \tilde{Q}^+(f)(t, v) |
\leq b(1 + C_{L, \infty}) (C_{f_0, R,T} \delta^{-4}) \|f_0(\cdot + h) - f_0\|_{L^1_2} + C_{f_0, R}\delta + C_{f_0, R,T} \|f_0(\cdot + h) - f_0\|_{L^1_2}.
\]
Minimizing the right-hand side with respect to \( \delta > 0 \) gives
\[
\sup_{t \in [0, T], |v| < R} | \tilde{Q}^+(f)(t, v + h) - \tilde{Q}^+(f)(t, v) | \leq C_{f_0, R,T} (\|f_0(\cdot + h) - f_0\|_{L^1_2})^{1/5} \quad \forall |h| < 1.
\]
Following a similar argument we also have
\[
\sup_{t \in [0, T], |v| < R} | \tilde{L}(f)(t, v + h) - \tilde{L}(f)(t, v) | \leq C_{f_0, R,T} (\|f_0(\cdot + h) - f_0\|_{L^1_2})^{1/5} \quad \forall |h| < 1.
\]
From (5.21), (5.22) and (5.23) we see that for any fixed \( T \in (0, \infty) \) and any \( t \in [0, T] \), the functions \( v \mapsto \tilde{Q}^+(f)(t, v) \) and \( v \mapsto \tilde{L}(f)(t, v) \) are both uniformly continuous in \( |v| < R \).

**Step 4.** Finally we prove that \( (t, v) \mapsto \tilde{Q}^+(f)(t, v) \) is continuous on \( [0, \infty) \times \mathbb{R}^3 \). Fix any \( (t, v) \in [0, \infty) \times \mathbb{R}^3 \). We have
\[
|Q^+(f)(s, u) - Q^+(f)(t, v)| \leq |Q^+(f)(s, u) - Q^+(f)(s, v)| + |Q^+(f)(s, v) - Q^+(f)(t, v)|.
\]
From **Step 2** and **Step 3** (taking for instance \( R = 1 + |v| \)) we see that for \( \forall \varepsilon > 0 \) there exists \( 0 < \delta = \delta_{t, v} < 1 \) such that if \( |s - t| < \delta \) (with \( s \geq 0 \)) and \( |u - v| < \delta \), then \( |Q^+(f)(s, v) - Q^+(f)(t, v)| < \varepsilon/2 \) and \( |Q^+(f)(s, u) - Q^+(f)(s, v)| < \varepsilon/2 \). From (5.24) we conclude that \( Q^+(f)(\cdot, \cdot) \) is continuous at \( (t, v) \).

This finishes the proof of Proposition 5.1 and thus the proof of Theorem 1.2 is completed.

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