On the dispersion theory of $\pi\pi$ scattering

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Abstract. Recent developments in low energy pion physics are reviewed, emphasizing the strength of dispersion theory in this context. As an illustration of the method, I discuss some consequences of the forward dispersion relation obeyed by the isoscalar component of the scattering amplitude.

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Pions play a crucial role whenever the strong interaction is involved at low energies – the Standard Model prediction for the muon magnetic moment provides a good illustration. My talk dealt with the remarkable theoretical progress made in low energy pion physics in recent years.

In the first part, I discussed the current theoretical and experimental knowledge of the $S$-wave $\pi\pi$ scattering lengths ($a_0^0$, $a_0^2$) in some detail, because these play a central role. Simulations of QCD on a lattice now reach sufficiently small quark masses for a meaningful extrapolation to the values of physical interest to become possible [1, 2]. Chiral perturbation theory not only describes the dependence on the quark masses, but also allows one to calculate the leading finite volume effects [3]. The values obtained for the coupling constants $\ell_3$ and $\ell_4$ are consistent with the estimates given in [4, 5]. Since these control the leading corrections to the low energy theorems for $a_0^0$ and $a_0^2$, the lattice results at the same time provide a rough check of the remarkably precise predictions for these quantities obtained in [5] (in the following, this paper is referred to as CGL). The exotic scattering length $a_0^2$ can also be extracted directly from the volume dependence of the energy levels occurring on the lattice. The result obtained in [6] is in good agreement with the prediction as well. The current state of our knowledge concerning $a_0^0$ and $a_0^2$ is briefly summarized in [7].

The second part of the talk covered recent results established with dispersive methods. The upshot of this development is that, in the threshold region, the $\pi\pi$ scattering amplitude is now known to an amazing degree of accuracy [5]. In particular, we know how to calculate mass and width of the lowest resonance of QCD [8]. The actual uncertainty in the pole position is smaller than the estimate given in the 2006 edition of the Review of Particle Physics [9], by more than an order of magnitude. The progress made in this field heavily relies on the fact that the dispersion theory of $\pi\pi$ scattering is particularly simple: the $s$-, $t$- and $u$-channels represent the same physical process. As a consequence, the scattering amplitude can be represented as a dispersion integral over the imaginary part and the integral exclusively extends over the physical region [10]. The projection of
the amplitude on the partial waves leads to a dispersive representation for these, the Roy
equations. For a detailed discussion, I refer to [11].

In the present article, I wish to explain the essence of the dispersive approach, avoiding
technical machinery as much as possible. Although the Roy equations represent an
optimal and comprehensive framework for the low energy analysis of the $\pi\pi$ scattering
amplitude, the main points can be seen in a simpler context: forward dispersion relations
[13]. More specifically, I consider the component of the scattering amplitude with
$s$-channel isospin $I = 0$, which I denote by $T^0(s,t)$. It satisfies a twice subtracted fixed-$t$
dispersion relation in the variable $s$. In the forward direction, $t = 0$, this relation reads

$$\operatorname{Re} T^0(s,0) = c_0 + c_1 s + \frac{s(s - 4M_{\pi}^2)}{\pi} P \int_{4M_{\pi}^2}^{\infty} \frac{dx \operatorname{Im} T^0(x,0)}{x(x - 4M_{\pi}^2)(x - s)} +$$

$$+ \frac{s(s - 4M_{\pi}^2)}{\pi} \int_{4M_{\pi}^2}^{\infty} dx \left\{ \operatorname{Im} T^0(x,0) - 3 \operatorname{Im} T^1(x,0) + 5 \operatorname{Im} T^2(x,0) \right\} / 3 x(x - 4M_{\pi}^2)(x + s - 4M_{\pi}^2).$$

The symbol $P$ indicates that the principal value must be taken. The first integral accounts
for the discontinuity across the right hand cut, while the second represents the analogous
contribution from the left hand cut, where the components of the scattering amplitude with
$I = 1, 2$ also show up. According to the optical theorem, the imaginary part of
the forward scattering amplitude represents the total cross section: in the normalization
of [11], we have $\operatorname{Im} T^I(s,0) = \sqrt{s(s - 4M_{\pi}^2)} \sigma^I_{tot}(s)$. The subtraction constants are also
determined by physical quantities – the $S$-wave scattering lengths:

$$c_0 + c_1 s = 32 \pi \left\{ a_0^0 + (2a_0^0 - 5a_0^2) \frac{s - 4M_{\pi}^2}{12M_{\pi}^2} \right\}.$$  (2)

A dispersion relation of the above type also holds for other processes. What is special
about $\pi\pi$ is that the contribution from the crossed channels can be expressed in terms of
observable quantities – total cross sections in the case of forward scattering.

The right hand side of equation (1) can be evaluated with the available representations of
the scattering amplitude. The contribution from the left hand cut is dominated by the
$\rho$-meson, which generates a pronounced peak in the total cross section with $I = 1$.
This contribution is known very accurately from $e^+e^- \rightarrow \pi^+\pi^-$ and $\tau^\pm \rightarrow \nu\pi^0\pi^0$.
Since the channel with $I = 2$ is exotic, it does not contain any resonances and – at low
energies – only generates a minor correction. In the physical region, $s > 4M_{\pi}^2$, the entire
contribution from the crossed channels is a smooth function that varies only slowly with
the energy. Note, however, that this contribution is by no means small.

The angular momentum barrier suppresses the higher partial waves: at low energies,
the first term in the partial wave decomposition

$$\operatorname{Re} T^0(s,0)/(32\pi) = \operatorname{Re} t^0(s) = 5 \operatorname{Re} t^1(s) + \ldots ,$$  (3)

represents the most important contribution. In the vicinity of the threshold, where the
contribution from the $D$-wave is small, the dispersion relation (1) thus essentially
determines the real part of the isoscalar $S$-wave. For brevity, I refer to this wave as $S^0$.

Figure 1a is based on the representation of the scattering amplitude in CGL, where
the low energy behaviour of the $S$- and $P$-wave phase shifts was determined by solving
the Roy equations below 800 MeV. That calculation required input for (a) the imaginary parts of the higher partial waves, (b) the imaginary parts of the $S$- and $P$-waves above 800 MeV and (c) the $S$-wave scattering lengths. For (a) and (b), we relied on the literature [12], while for (c), we used the low energy theorems of chiral perturbation theory. The Roy equations then yield an approximate representation for the real parts of all partial waves, throughout their region of validity. Together with the input used for the imaginary parts, this also fixes the phase shifts and elasticities in that region. The dashed lines in figures 1a, 2 and 3b are calculated in this way. I emphasize that, above 800 MeV, these curves amount to an extrapolation. The uncertainties in the representation of the scattering amplitude are discussed in detail in CGL, but are not shown in the figures, which are calculated with the central values. In the threshold region, the uncertainties are tiny, but they grow with the energy. In particular, Figure 3b shows that the extrapolation overestimates the inelasticity in the region between 800 MeV and $2M_K$ – in reality, a significant amount of inelasticity only arises when the $K\bar{K}$ channel opens.

In order to demonstrate that the representation of the scattering amplitude in CGL is consistent with equation (1), I evaluate the function $\text{Re} T^0(s,0)$ with it and remove the $D^0$-wave, setting $\text{Re} t^0_0(s)|_{\text{f.d.r.}} = \text{Re} T^0(s,0)/(32\pi) - 5 \text{Re} t^0_0(s)$ and thereby ignoring partial waves with $\ell \geq 4$. Figure 1a shows that, below 800 MeV, the f.d.r. is indeed obeyed very well. No wonder: if the partial waves satisfy the Roy equations, then the sum over all of these automatically obeys the f.d.r. The difference between the two curves arises from the neglected higher partial waves, which become more important if the energy is increased.

Figure 1b shows the result obtained if the representation in CGL is replaced by the one in [13] (in the following, this paper is referred to as KPY). Visibly, the forward dispersion relation is not obeyed well, but this is to be expected: that parametrization does not rely on dispersion theory, but represents a phenomenological fit to data sets that are subject to large errors. Note also that the figure does not show the uncertainties of the partial wave analysis in KPY, which are considerable: the difference between the two curves is covered by these.

As discussed in [13, 14], the forward dispersion relations can be used to improve the partial wave representation. In the following, I apply the method of [11], which is very suitable for the purpose: it suffices to replace the Roy equation for $\text{Re} t^0_0$ by the f.d.r. for $\text{Re} T^0$. If the subtraction constants and all partial waves except $t^0_0$ are treated as known,
the problem may be given the following mathematical form:

Choose a “matching point” $s_m$ and prescribe the function $\text{Im}t_0^0(s)$ above $s_m$ as well as the elasticity $\eta_0^0(s)$ below $s_m$. Find solutions of equation (1) for $s < s_m$ that respect the unitarity relation between $\text{Re}t_0^0$, $\text{Im}t_0^0$ and $\eta_0^0$.

In the framework of the Roy equations, this problem was discussed in detail in [11]. If the matching point is taken below the energy where the phase goes through $90^\circ$, then there is exactly one solution. This is the situation considered in CGL, where the matching point and the central value of the phase at that point were set equal to 800 MeV and $82.3^\circ$, respectively. The dashed lines in figures 1a, 2 and 3b represent the resulting unique solution.

Next, I observe that, near 970 MeV, the representations given for $S^0$ in KPY and CGL are very similar – in either case, the real part reaches the lower unitarity limit in the immediate vicinity of that energy. For this reason, I now discuss the situation for the case where the matching point is taken at $\sqrt{s_m} = 970$ MeV (full vertical line) and where the input for the imaginary parts is taken from KPY: Above 1.42 GeV, I use the Regge parametrization of the forward scattering amplitudes given in that reference. At lower energies, the partial wave decomposition is used. With the exception of $S^0$ below 970 MeV, all of the partial waves are taken from KPY (central values of the parameters, throughout). Finally, below 970 MeV, the elasticity of $S^0$ is set equal to 1, while the phase is left open – the f.d.r. is used to determine it. In order to be able to compare the result with the one obtained with CGL, I keep the subtraction constants fixed at the

FIGURE 2. Phase of the isoscalar $S$-wave.
FIGURE 3. Violation of f.d.r. for \( T^0 \) and elasticity of the isoscalar \( S \)-wave.

central values in CGL. The range used for \( a_0^0, a_0^2 \) in KPY is consistent with that.

With this input, the value of the phase at the matching point is 142°. As discussed in detail in [11], the mathematical problem posed above leads to a curiosity if the phase at the matching point exceeds 90°: the solution of the forward dispersion relation then fails to be unique – there is an entire family of solutions. In the present case, there is a one-parameter family, which may be labeled with the value of the phase somewhere below the matching point, at 800 MeV, for example. In particular, we may select the solution for which the phase at 800 MeV agrees with the central parametrization in KPY, \( \delta_0^0(800\text{MeV}) = 92° \).

This particular solution is physically unacceptable, however, because it contains a strong cusp at the matching point. An enlargement of the region around this point is shown in figure 3a, where the difference \( \Delta \) between the left and right hand sides of equation (1) is plotted as a function of the energy. By construction, the difference vanishes below the matching point – within the accuracy to which the solutions are worked out. The cusp manifests itself as a spike in the vicinity of the matching point. The occurrence of a cusp is a generic feature of the manifold of solutions to the mathematical problem specified above. The amplitude of the cusp decreases if the phase at 800 MeV is lowered. There is a unique solution for which a cusp does not occur, in the sense that the first derivative is continuous at the matching point. This solution is reached if the phase at 800 MeV is lowered to about 81°. Figure 3a shows that the violation of the f.d.r. is then postponed to \( K\bar{K} \) threshold, but it persists: with the input specified above, equation (1) does not have a physically acceptable solution.

The problem originates in \( D^0 \), the isoscalar \( D \)-wave – indeed, in KPY, the parametrization used for the elasticity of this wave is mentioned as a potential culprit. Around \( K\bar{K} \) threshold, the real part of \( D^0 \) is by no means negligible and it is essential that the representation used for it is consistent with analyticity. In this regard, there is a difference between the mathematical problem specified above and the Roy equations, where the input of the calculation exclusively involves imaginary parts. As was noted already in the pioneering work of Basdevant, Froggatt and Petersen [15], dispersion theory im-
poses very strong constraints on the low energy behaviour of the higher partial waves. The representation for \( D^0 \) in KPY is not consistent with these constraints. In particular, phase space strongly suppresses the inelasticity generated by the \( K\bar{K} \) states: \( 1 - \eta_0^2 \) only grows with the fifth power of the kaon momentum.

The problem encountered with the behaviour of the solutions around \( K\bar{K} \) threshold disappears if the representation for \( D^0 \) is taken from the solution of the Roy equation for \( \text{Re} t_0^0 \), retaining the parametrization of KPY only for the other waves. Alternatively, the parametrization of \( D^0 \) given in [16] can be used – this is less accurate, but barely makes any difference around \( K\bar{K} \) threshold. The full lines in figures 3a and 3b show that equation (1) does now admit physically acceptable solutions. As before, there is a one-parameter family of solutions, which differ in the strength of the cusp at the matching point. The particular solution shown is obtained by minimizing the difference between the right and left hand sides of equation (1) on the entire interval from \( 2M_\pi \) to \( 2M_K \). For this solution, the cusp is too small to be visible in the figures.

As was to be expected, the solution represents a compromise between the two curves in figure 1b. Below \( 2M_K \), the difference between the left and right hand sides of equation (1) is less than \( 10^{-3} \), but above 1050 MeV, the real part of the solution departs from the real part of the parametrization in KPY, which is indicated as a dash-dotted line (shown only above 970 MeV, where the imaginary part of that parametrization is used as an input). The real part and the phase of the solution are displayed as full lines in figures 1c and 2. At low energies, the solution of the forward dispersion relation runs within the shaded region, which represents the uncertainty band in CGL. The value of the phase at the upper end is \( \delta_0^0(800 \text{ MeV}) = 80^\circ \). This confirms one of the main results in CGL: below 800 MeV, the solution is not sensitive to the behaviour at high energies. On the other hand, above the matching point, the phase very closely follows the parametrization of the \( S^0 \)-wave used in the input. As discussed in detail in [11], in the context of the Roy equations, this happens whenever the input stems from an analytic parametrization of the partial wave in question. As can be seen in figure 3b, shifting the matching point up has the effect of removing the unitarity violation in the elastic region – the output for \( \eta_0^0 \) now differs appreciably from KPY only above 1050 MeV.

I conclude that – in the specific framework used here, where all partial waves except the isoscalar \( S \)-wave are treated as known – the constraint imposed on the scattering amplitude by the forward dispersion relation for \( T^0(s,t) \) is essentially equivalent to the Roy equation for \( \text{Re} t_0^0(s) \). Likewise, the Roy equations for \( \text{Re} t_1^1(s) \) and \( \text{Re} t_0^2(s) \) can be replaced by the forward dispersion relations for \( T^1(s,t) \) and \( T^2(s,t) \). In all three cases, the subtraction constants are determined by the \( S \)-wave scattering lengths. The forward dispersion relations do not provide a handle on the higher partial waves, but the Roy equations do. As we have pushed the matching point up, the dispersive framework now correlates the value of the phase at 800 MeV with the behaviour of the scattering amplitude above \( K\bar{K} \) threshold. In particular, if the representation in KPY represents a good approximation above \( 2M_K \) and if the theoretical predictions for \( a_0^0 \) and \( a_0^2 \) are correct, then the phase at 800 MeV must be in the vicinity of \( 80^\circ \).

In the above, I did not discuss the experimental information at all. Instead, I merely showed that – in view of the sharp theoretical predictions for the subtraction constants – the properties of the amplitude above \( K\bar{K} \) threshold very strongly constrain the behaviour
at lower energies. A more systematic investigation is under way, which aims at extending the work described in CGL to somewhat higher energies. The matching point can be pushed up all the way to the limit of validity of the Roy equations. The price to pay is that the contributions from the high energy region then become more important. We are using a Regge representation for the asymptotic domain and invoke experimental information as well as sum rules to pin down the residue functions. A short account of this work is given in [17] and more should soon be ready for publication, in particular also the application to the electromagnetic form factor of the pion, for which an accurate representation is needed in connection with the Standard Model prediction for the magnetic moment of the muon.

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