K-NN active learning under local smoothness assumption

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Abstract

There is a large body of work on convergence rates either in passive or active learning. Here we first outline some of the main results that have been obtained, more specifically in a nonparametric setting under assumptions about the smoothness of the regression function (or the boundary between classes) and the margin noise. We discuss the relative merits of these underlying assumptions by putting active learning in perspective with recent work on passive learning. We design an active learning algorithm with a rate of convergence better than in passive learning, using a particular smoothness assumption customized for k-nearest neighbors. Unlike previous active learning algorithms, we use a smoothness assumption that provides a dependence on the marginal distribution of the instance space. Additionally, our algorithm avoids the strong density assumption that supposes the existence of the density function of the marginal distribution of the instance space and is therefore more generally applicable.

Keywords: Nonparametric learning, active learning, nearest-neighbors, smoothness condition.

1 Introduction

Active learning is a machine learning approach for reducing the data labeling effort. Given an instance space \(X\) or a pool of unlabeled data \(\{X_1, \ldots, X_w\}\) provided by a distribution \(P_X\), the learner focuses its labeling effort only on the most "informative" points so that a model built from them can achieve the best possible guarantees \([6]\). Such guarantees are particularly interesting when they are significantly better than those obtained in passive learning \([10]\). In the context of this work, we consider binary classification (where the label \(Y\) of \(X\) takes its value in \(\{0, 1\}\)) in a nonparametric setting. Extensions to multiclass classification and adaptive algorithms are discussed in the last section.

The nonparametric setting has the advantage of providing guarantees with many informations such as the dependence on the dimensional and distributional parameters by using some hypotheses on the regularity of the decision boundary \([4]\), on the regression function \([20, 15]\), and on the geometry of instance space (called strong density assumption) \([1, 15, 20]\). One of the initial works on nonparametric active learning \([4]\) assumed that the decision boundary is the graph of a smooth function, that a margin assumption very similar to Tsybakov’s noise assumption \([18]\) holds, and that distribution \(P_X\) is uniform. This led to a better guarantee than in passive learning. Instead of the assumption on the decision boundary, other works \([20, 15]\) supposed rather that the regression function is smooth (in some sense). This assumption, along with Tsybakov’s noise assumption and strong density assumption also gave a better guarantee than in passive learning. Moreover, unlike in \([4]\), they provided algorithms that are adaptive with respect to the margin’s noise and to the smoothness parameters.

However, recent work \([5]\) pointed out some disadvantages of the preceding smoothness assumption, and extended it in the context of passive learning with \(k\)-nearest neighbors (\(k\-nn\)) by using another smoothness

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assumption that is able to sharply characterize the rate of convergence for all probability distributions that satisfy it.

In this paper, we thus extend the work of [3] to the active learning setting, and provide a novel algorithm that outputs a classifier with the same rate of convergence as other recent algorithms that were using more restrictive hypotheses, as for example [20][15].

Section 2 introduces general definitions, Section 3 presents previous work on convergence rates in active and passive non-parametric learning, with a special emphasis on the assumptions related to our work. Section 4 describes our algorithm, Section 5 provides the theoretical motivations behind our algorithm, Section 6 concludes this paper with a discussion of possible extensions of this work, and in Appendix A we provide the proofs of the main results.

2 Preliminaries

We begin with some general definitions and notations about active learning in binary classification, then summarize the main assumptions that are typically used to study the rate of convergence of active learning algorithms in the framework of statistical learning theory.

2.1 Active learning setting

Let \((X, \rho)\) a metric space. In this paper we set \(X = \mathbb{R}^d\) and refer to it as the instance space, and take \(\rho\) the Euclidean metric. Let \(Y = \{0, 1\}\) the label space. We assume that the couples \((X, Y)\) are random variables distributed according to an unknown probability \(P\) over \(X \times Y\). Let us denote \(P_X\) the marginal distribution of \(P\) over \(X\).

Given \(w \in \mathbb{N}\) and an i.i.d sample \((X_1, Y_1), \ldots, (X_w, Y_w)\) drawn according to probability \(P\), the learning problem consists in minimizing the risk \(\mathcal{R}(f) = P(Y \neq f(X))\) over all measurable functions, called classifiers \(f : X \rightarrow Y\).

In active learning, the labels are not available from the beginning but we can request iteratively at a certain cost (to a so-called oracle) a given number \(n\) of samples, called the budget (\(n \leq w\)). In passive learning, all labels are available from the beginning, and \(n = w\). At any time, we choose to request the label of a point \(X\) according to the previous observations. The point \(X\) is chosen to be most “informative”, which amounts to belonging to a region where classification is difficult and requires more labeled data to be collected. Therefore, the goal of active learning is to design a sampling strategy that outputs a classifier \(\hat{f}_n\) whose excess risk is as small as possible with high probability over the requested samples, as reviewed in [6][10][7].

Given \(x\) in \(X\), let us introduce \(\eta(x) = E(Y|X = x) = P(Y = 1|X = x)\) the regression function. As done in [17], it is easy to show that the function \(f^*(x) = 1_{\eta(x) \geq 1/2}\) achieves the minimum risk and that \(\mathcal{R}(f^*) = E_X(\min(\eta(X), 1 - \eta(X)))\). Because \(P\) is unknown, the function \(f^*\) is unreachable and thus the aim of a learning algorithm is to return a classifier \(\hat{f}_n\) with minimum excess risk \(\mathcal{R}(\hat{f}_n) - \mathcal{R}(f^*)\) with high probability over the sample \((X_1, Y_1), \ldots, (X_n, Y_n)\).

2.2 k-nearest neighbors (k-nn) classifier

Given two integers \(k, n\) such that \(k < n\), and a test point \(X \in X\), the k-nn classifier predicts the label of \(X\) by giving the majority vote of its \(k\) nearest neighbors amongst the sample \(X_1, \ldots, X_n\). For \(k = 1\), the k-nn classifier returns the label of the nearest neighbor of \(X\) amongst the sample \(X_1, \ldots, X_n\). Often \(k\) grows with \(n\), in which case the method is called \(k_n\)-nn. For a complete discussion of nearest neighbors classification, see for example [3][5].

2.3 Regularity, noise and strong density assumptions

Let \(B(x, r) = \{x' \in X, \; \rho(x, x') < r\}\) and \(\bar{B}(x, r) = \{x' \in X, \; \rho(x, x') \leq r\}\) the open and closed balls (with respect to the Euclidean metric \(\rho\)), respectively, centered at \(x \in X\) with radius \(r > 0\). Let \(\text{supp}(P_X) = \{x \in X, \; \forall r > 0, \; P_X(B(x, r)) > 0\}\) the support of the marginal distribution \(P_X\).
H2. Then there is a constant

\[ \lambda > 0 \]

Theorem 1. Suppose that \( X \subset \mathbb{R}^d \) where \( d \) is the dimension of the instance space.

\[ |\eta(x) - \eta(x')| \leq L \rho(x, x')^\alpha. \] (H1)

The notion of Hölder continuity ensures that the proximity between two closest (according to the metric \( \rho \)) points is reflected in a similar value for the conditional probability \( \eta(x) \).

This definition remains true for a general metric space, but for the case where \( \rho \) is the Euclidean metric, we should always have \( 0 < \alpha \leq 1 \), otherwise \( \eta \) becomes constant.

Definition 2 (Strong density).
Let \( P \) the distribution probability defined over \( \mathcal{X} \times \mathcal{Y} \) and \( P_X \) the marginal distribution of \( P \) over \( \mathcal{X} \). We say that \( P \) satisfies the **strong density** assumption if there exists some constants \( r_0 > 0, c_0 > 0, p_{\text{min}} > 0 \) such that for all \( x \in \text{supp}(P_X) \):

\[ \lambda(B(x, r) \cap \text{supp}(P_X)) \geq c_0 \lambda(B(x, r)), \ \forall r \leq r_0 \]

and \( p_X(x) > p_{\text{min}} \).

(Strong density)

where \( p_X \) is the density function of the marginal distribution \( P_X \) and \( \lambda \) is the Lebesgue measure.

The strong density assumption ensures that, given a realisation \( X = x \) according to \( P_X \), there exists an infinite number of realisations \( X_1 = x_1, \ldots, X_m = x_m, \ldots \) in a neighborhood of \( x \).

Definition 3 (Margin noise).
We say that \( P \) satisfies **margin noise** or **Tsybakov’s noise** assumption with parameter \( \beta \geq 0 \) if for all \( 0 < \epsilon \leq 1 \)

\[ P_X(x \in \mathcal{X}, |\eta(x) - 1/2| < \epsilon) < C \epsilon^\beta, \] (H3)

for \( C := C(\beta) \in [1, +\infty] \).

The margin noise assumption gives a bound on the probability that the label of the training points in the neighborhood of a test point \( x \) differs from the label of \( x \) given by the conditional probability \( \eta(x) \). It also describes the behavior of the regression function in the vicinity of the decision boundary \( \eta(x) = \frac{1}{2} \).

When \( \beta \) goes to infinity, we observe a ”jump” of \( \eta \) around to the decision boundary, and then we obtain Massart’s noise condition \cite{19}. Small values of \( \beta \) allow for \( \eta \) to ”cuddle” \( \frac{1}{2} \) when we approach the decision boundary.

Definition 4 ((\( \alpha, L \))-smooth).
Let \( 0 < \alpha \leq 1 \) and \( L > 1 \). The regression function is **(\( \alpha, L \))-smooth** if for all \( x, z \in \text{supp}(P_X) \) we have:

\[ |\eta(x) - \eta(z)| \leq L P_X(B(x, \rho(x, z)))^{\alpha/d}, \] (H4)

where \( d \) is the dimension of the instance space.

Theorem 1 \cite{1} states that the \( (\alpha, L) \)-smooth assumption \cite{14} is more general than the Hölder continuity assumption \cite{11}.

**Theorem 1.** \cite{3}
Suppose that \( \mathcal{X} \subset \mathbb{R}^d \), that the regression function \( \eta \) is \( (\alpha_h, L_h) \)-Hölder continuous, and that \( P_X \) satisfies \( \text{H2} \). Then there is a constant \( L > 1 \) such that for any \( x, z \in \text{supp}(P_X) \), we have:

\[ |\eta(x) - \eta(z)| \leq L P_X(B(x, \rho(x, z)))^{\alpha_h/d}. \]
3 Convergence rates in nonparametric active learning

3.1 Previous works

Active learning theory has been mostly studied during the last decades in a parametric setting, see for example [2, 11, 7] and references therein. One of the pioneering works studying the achievable limits in active learning in a nonparametric setting [4] required that the decision boundary is the graph of a Hölder continuous function with parameter $\alpha$ (H1). Using a notion of margin noise (with parameter $\beta$) very similar to (H3), the following minimax rate was obtained:

$$O \left( n^{-\frac{\beta}{2\beta+\gamma-2}} \right),$$

where $\gamma = \frac{d-1}{\alpha}$ and $d$ is the dimension of instance space ($\mathcal{X} = \mathbb{R}^d$).

Note that this result assumes the knowledge of smoothness and margin noise parameters, whereas an algorithm that achieves the same rate, but that adapts to these parameters was proposed recently in [16].

In this paper, we consider the case where the smoothness assumption refers to the regression function both in passive and in active learning.

In passive learning, by assuming that the regression function is Hölder continuous (H1), along with (H3) and (H2), the minimax rate was established by [1]:

$$O \left( n^{-\frac{\alpha(\beta+1)}{2\alpha+d}} \right).$$

In active learning, using the same assumptions (H1), (H3) and (H2), with the additional condition $\alpha \beta < d$, the following minimax rate was obtained [15]:

$$\tilde{O} \left( n^{-\frac{\alpha(\beta+1)}{2\alpha+d-\alpha\beta}} \right),$$

where $\tilde{O}$ indicates that there may be additional logarithmic factors. This active learning rate given by [3] thus represents an improvement over the passive learning rate (2) that uses the same hypotheses.

With another assumption on the regression function relating the $L_2$ and $L_\infty$ approximation losses of certain piecewise constant or polynomial approximations of $\eta$ in the vicinity of the decision boundary, the same rate (3) was also obtained by [20].

3.2 Link with $k$-nn classifiers

For practicals applications, an interesting question is if $k$-nn classifiers attain the rate given by [2] in passive learning and by [3] in active learning.

In passive learning, under assumptions (H1), (H3) and (H2), and for suitable $k_n$, it was shown in [5] that $k_n$-nn indeed achieves the rate (2).

In active learning a pool-based algorithm that outputs a $k$-nn classifier has been proposed in [14], but its assumptions differ from ours in terms of smoothness and noise, and the number of queries is constant. Similarly, the algorithm proposed in [9] outputs a 1-nn classifier based on a subsample of the initial pool, such that the label of each instance of this subsample is determined with high probability by the labels of its neighbors. The number of neighbors is adaptively chosen for each instance in the subsample, leading to the minimax rate (3) under the same assumptions as in [15].

To obtain more general results on the rate of convergence for $k$-nn classifiers in metric spaces under minimal assumptions, the more general smoothness assumption given by (H4) was used in [5]. By using a $k$-nn algorithm, and under assumptions (H3) and (H4), the rate of convergence obtained in [5] is also on the order of (2). Additionally, using assumption (H4) instead of (H1) removes the need for the strong density assumption (H2), which therefore allows more classes of probability.
3.3 Contributions of the current work

In this paper, we provide an active learning algorithm under the assumptions (H1) and (H3), that were used in passive learning in [5]. The $\alpha$-smooth assumption (H4) involves a dependence on the marginal distribution $P_X$, and holds for any pair of distributions $P_X$ and $\eta$, which allows the use of discrete probability. However in active learning the Hölder continuity assumption (H1) is typically used, along with the strong density assumption (H2), which assumes the existence of the density $p_X$ of the marginal probability $P_X$. By using assumption (H4) instead of (H1) and thereby avoiding (H2), our algorithm removes unnecessary restrictions on the distribution that would exclude important densities (e.g., Gaussian) as noticed in [8].

In the following, we will show that the rate of convergence of our algorithm remains the same as [3], despite the use of more general hypotheses.

4 KALLS algorithm

4.1 Setting

As explained in Section 2.1, we consider an active learning setting with a pool of i.i.d unlabeled examples $\mathcal{K} = \{X_1, X_2, \ldots, X_w\}$. Let $n \leq w$ the budget, that is the maximum number of points whose label we are allowed to query to the oracle. Recall that in a passive learning setting, we would have $n = w$. The objective of the algorithm is to build a subsample $\{X_{t_i}, i \geq 1\}$ whose labels are considered most “informative”, and which we call the active set. More precisely, a point $X_{t_i}$ is considered “informative” if its label cannot be inferred from the previous observations $X_{t_j}$ (with $t_j < t_i$). The sequence $(t_i)_{i \geq 1}$ of indices is an increasing sequence of integers, starting arbitrarily with $X_{t_1} = X_1$ and stopping when the budget $n$ is attained or when $X_{t_i} = X_w$ for some $t_i$.

When a point $\{X_{t_i}\}$ is considered informative, instead of requesting its label, we request the labels of its nearest neighbors, as was done in [9]. This is reasonable for practical situations where the uncertainty about the label of $X_{t_i}$ has to be overcome, and it is related to the $(\alpha, L)$-smooth assumption (H4). Note that it differs from the setting of [16], where the label of $X_{t_i}$ is requested several times. The number of neighbors $k_{t_i}$ is adaptively determined such that, while respecting the budget, we can predict with high confidence the true label as $f^*(X_{t_i})$ of $X_{t_i}$ by empirical mean of the labels of its $k_{t_i}$ nearest neighbors.

The final active set output by the algorithm will thus be $\widehat{S} = \{(X_{t_1}, \widehat{Y}_{t_1}), \ldots, (X_{t_l}, \widehat{Y}_{t_l})\}$ with $t_1 \leq \cdots \leq t_l \leq n$. This set $\widehat{S}$ is the set of points considered to be informative, and is obtained by removing the points that are too noisy and thus that require too many labels. We show that the active set $\widehat{S}$ is sufficient to predict the label of any new point with a 1-nn classification rule $\widehat{f}_{n,w}$.

4.2 Algorithm

Before beginning the description of our algorithm, let us introduce some variables and notations. The precise form of the expressions below will be justified in Section 5.

For $\epsilon, \delta \in (0, 1)$, $k \geq 1$, set:

$$b_{\delta,k} = \sqrt{\frac{2}{k} \left( \log \left( \frac{1}{\delta^2} \right) + \log \log \left( \frac{1}{\delta} \right) + \log \log(ek) \right)}.$$  

(4)

$$k(\epsilon, \delta) = \frac{c}{\Delta^2} \left[ \log(\frac{1}{\delta}) + \log \log \left( \frac{1}{\delta} \right) + \log \log \left( \frac{512\sqrt{c}}{\Delta} \right) \right]$$

(5)

where

$$\Delta = \max\left( \frac{\epsilon}{2}, \left( \frac{\epsilon}{2C} \right)^{\frac{1}{\eta+1}} \right), \quad c \geq 7.10^6.$$  

(6)

Let

$$\phi_n = \sqrt{\frac{1}{n} \left( \log \left( \frac{1}{\delta} \right) + \log \log \left( \frac{1}{\delta} \right) \right)}.$$  

(7)
For \( X_s \in \mathcal{K} = \{X_1, \ldots, X_w\} \), we denote henceforth by \( X_s^{(k)} \) its \( k \)-th nearest neighbor in \( \mathcal{K} \), and \( Y_s^{(k)} \) the corresponding label.

For an integer \( k \geq 1 \), let

\[
\tilde{\eta}_k(X_s) = \frac{1}{k} \sum_{i=1}^{k} Y_s^{(i)}, \quad \tilde{\eta}_k(X_s) = \frac{1}{k} \sum_{i=1}^{k} \eta(X_s^{(i)}).
\] (8)

Below we provide a description of the KALLS algorithm (Algorithm 1), that aims at determining the active set defined in Section 4.1 and the related 1-nn classifier \( \hat{f}_{n,w} \) under the assumptions \( \text{(H4)} \) and \( \text{(H3)} \). The complete proofs of the convergence of the algorithm are in Section 5 and Appendix A.

For the KALLS algorithm, the inputs are a pool \( \mathcal{K} \) of unlabelled data of size \( w \), the budget \( n \), the smoothness parameters \( (\alpha, L) \) from \( \text{(H4)} \), the margin noise parameters \( (\beta, C) \) from \( \text{(H3)} \), a confidence parameter \( \delta \in (0, 1) \) and an accuracy parameter \( \epsilon \in (0, 1) \). For the moment, these parameters are fixed from the beginning. Adaptive algorithms such as \([15]\) could be exploited, in particular for the \( \alpha \) and \( \beta \) parameters.

The final active set is obtained such that, with high confidence, the 1-nn classifier \( \hat{f}_{n,w} \) based on it agrees with the Bayes classifier at points that lie beyond some margin \( \Delta_o > 0 \) of the decision boundary.

Formally, given \( x \in \mathcal{X} \) such that \( |\eta(x) - 1/2| > \Delta_o \), we have \( \hat{f}_{n,w}(x) = 1_{\eta(x) \geq 1/2} \) with high confidence. We will show that, with a suitable choice of \( \Delta_o \), the hypothesis \( \text{(H3)} \) leads to the desired rate of convergence \( \text{(3)} \).

**Algorithm 1: k-nn Active Learning under Local Smoothness (KALLS)**

**Input:** a pool \( \mathcal{K} = \{X_1, \ldots, X_w\} \), label budget \( n \), smoothness parameters \( (\alpha, L) \), margin noise parameters \( (\beta, C) \), confidence parameter \( \delta \), accuracy parameter \( \epsilon \).

**Output:** 1-nn classifier \( \hat{f}_{n,w} \)

1. \( s = 1 \) \( \triangleright \) index of point currently examined
2. \( \hat{S} = \emptyset \) \( \triangleright \) current active set
3. \( t = n \) \( \triangleright \) current label budget
4. \( I = \emptyset \) \( \triangleright \) set of "informative points"; used for providing the label complexity
5. **while** \( t > 0 \) and \( s < w \) **do**
6. \[ \text{Let } \delta_s = \frac{\delta}{2s} \]
7. \[ T = \text{Reliable}(X_s, \delta_s, \alpha, L, I) \]
8. **if** \( T = \text{True} \) **then**
9. \[ s = s + 1 \]
10. **else**
11. \[ (\hat{Y}, Q_s) = \text{ConfidentLabel}(X_s, k(\epsilon, \delta_s), t, \delta) \]
12. \[ \hat{L}B_s = \left\lfloor \frac{1}{|Q_s|} \sum_{(X,Y) \in Q_s} Y - \frac{1}{2} \right\rfloor - b_{\delta_s,|Q_s|} \] \( \triangleright \) Lower bound guarantee on \( |\eta(X_s) - 1/2| \)
13. \[ I = I \cup \{(X_s, \hat{L}B_s, |Q_s|)\} \]
14. \[ t = t - |Q_s| \]
15. **if** \( \hat{L}B_s \geq 0.1b_{\delta_s,|Q_s|} \) **then**
16. \[ \hat{S} = \hat{S} \cup \{(X_s, \hat{Y})\} \]

17. \( \hat{f}_{n,w} \leftarrow \text{Learn} \left( \hat{S} \right) \)

KALLS (Algorithm 1) uses two main subroutines: Reliable and ConfidentLabel which are detailed below in Sections 4.3 and 4.4, respectively. It also uses a small subroutine called Learn to output the final 1-nn classifier \( \hat{f}_{n,w} \) (Section 4.5).
4.3 Reliable subroutine

The **Reliable** subroutine is a binary test that checks if the label of the current point $X_s$ can be inferred with high confidence using the labels of the points currently in the active set. If it is the case, the point $X_s$ is not considered to be informative, its label is not requested and it is not added to the active set.

When a point $X_s' \in \mathcal{S}$ is relatively far away from the decision boundary, the subroutine **ConfidentLabel** provides a lower confidence bound $O(\mathbf{LB}_{s'}) \leq |\eta(X_s') - \frac{1}{2}|$. For a new point $X_s$, we have a low degree of uncertainty (in which case, $X_s$ is considered to be uninformative) if $|\eta(X_s) - \frac{1}{2}|$ entails the same confidence lower bound $O(\mathbf{LB}_{s'})$ as for some previous informative point $X_s'$. We can see that by smoothness assumption, it suffices to have $P_X(B(X_s, \rho(X_s', X_s))) \leq O((\hat{\mathbf{LB}}_{s'})^{d/\alpha})$. Because $P_X$ is unknown, we use the subroutine **BerEst** to adaptively estimate with high probability (over the data) $P_X(B(X_s, \rho(X_s', X_s)))$ up to $O((\hat{\mathbf{LB}}_{s'})^{d/\alpha})$.

**Algorithm 2: Reliable subroutine**

**Input:** an instance $X$, a confidence parameter $\delta$, smoothness parameters $\alpha$, $L$, a set $I \subset \mathcal{X} \times \mathbb{R} \times \mathbb{N}$

**Output:** $T_1$

1. for $(X', c, k) \in I$ do
2.  if $c \geq 0$ then
3.    $\hat{p}_{X'} = \operatorname{Estprob}(\rho(X, X'), \left(\frac{c}{64L}\right)^{d/\alpha}, 50, \delta)$
4.  if $\exists (X', c, k) \in I$ such that $\hat{p}_{X'} \leq \frac{75}{94} \left(\frac{c}{64L}\right)^{d/\alpha}$ then
5.    $T = True$
6.  else
7.    $T = False$

The **Reliable** subroutine uses $\operatorname{EstProb}(r, \epsilon_o, 50, \delta)$ as follows:

1. Call the subroutine $\operatorname{BerEst}(\epsilon_o, \delta, 50)$.

2. To draw a single $p_i$ in $\operatorname{BerEst}(\epsilon_o, \delta, 50)$, sample randomly an example $X_i$ from $K$, and set $p_i = \mathbb{1}_{X_i \in B(X, r)}$.

**Algorithm 3: BerEst subroutine (Bernoulli Estimation)**

**Input:** accuracy parameter $\epsilon_o$, confidence parameter $\delta'$, budget parameter $u$. $\triangleright$ $u$ does not depend on the label budget $n$

**Output:** $\hat{p}$

1. Sample $p_1, \ldots, p_4$
2. $S = \{p_1, \ldots, p_4\}$
3. $K = \frac{3u}{\epsilon_o} \log\left(\frac{8u}{\delta' \epsilon_o}\right)$
4. for $i = 3 : \log_2(u \log(2K/\delta')/\epsilon_o)$ do
5.  $m = 2^i$
6.  $S = S \cup \{p_m/2+1, \ldots, p_m\}$
7.  $\hat{p} = \frac{1}{m} \sum_{j=1}^{m} p_j$
8.  if $\hat{p} > u \log(2m/\delta')/m$ then
9.    Break
10. Output $\hat{p}$
4.4 ConfidentLabel subroutine

If the point $X_s$ is considered informative, the **ConfidentLabel** subroutine is used to determine with a given level of confidence, the label of the current point $X_s$. This is done by using the labels of its $k(\epsilon, \delta_s)$ nearest neighbors, where $k(\epsilon, \delta_s)$ is chosen such that, with high probability, the empirical majority of $k(\epsilon, \delta_s)$ labels differs from the majority in expectation by less than some margin, and all the $k(\epsilon, \delta_s)$ nearest neighbors are at most at some distance from $X_s$.

![Algorithm 4: ConfidentLabel subroutine](image)

4.5 Learn subroutine

The **Learn** subroutine takes as input the set of points that were considered informative and relatively less noisy, we apply passive learning on this subset by using the 1-nn classifier.

![Algorithm 5: Learn subroutine](image)
5 Theoretical motivations

This Section provides the main results and theoretical motivations behind the \textsc{Kalls} algorithm. Before that, let us recall $\mathcal{K} = \{X_1, \ldots, X_w\}$ is the pool of unlabeled data and $n$ is the budget. Let us denote by $\mathcal{A}_{a,w}$ the set of active learning algorithms on $\mathcal{K}$, and $\mathcal{P}(\alpha, \beta) := \text{the set of probabilities that satisfy the hypotheses (H4) and (H3), where } \alpha \text{ is the parameter in (H4) and } \beta \text{ in (H3).}$ For $A \in \mathcal{A}_{a,w}$, we denote by $\hat{f}_{A,n,w} := f_{n,w}$ the classifier that is provided by $A$.

Theorem 2 is the main result of this paper, which provides bounds on the excess risk for the \textsc{Kalls} algorithm. The main idea of its proof is sketched in Section 5.2, while a detailed proof is in Appendix [A].

5.1 Main results

**Theorem 2** (Excess risk for the \textsc{Kalls} algorithm.)

Let the set $\mathcal{P}(\alpha, \beta)$ such that $\alpha \beta < d$ where $d$ is the dimension of the input space $X = \mathbb{R}^d$. Then, we have:

$$\inf_{A \in \mathcal{A}_{a,w}} \sup_{P \in \mathcal{P}(\alpha, \beta)} \mathbb{E}_n \left[ R(\hat{f}_{n,w}) - R(f^*) \right] \leq \tilde{O} \left( n^{\frac{\alpha(\beta+1)}{\alpha d + \alpha - \alpha \beta}} \right),$$

where $\mathbb{E}_n$ is with respect to the randomness of the algorithm $A \in \mathcal{A}_{a,w}$.

The result (9) is also be stated below (Theorem 3) in a more practical form using label complexity. This latter form (10) will be used in the proof.

**Theorem 3** (Label complexity for the \textsc{Kalls} algorithm.)

Let the set $\mathcal{P}(\alpha, \beta)$ such that $\alpha \beta < d$. Let $\epsilon, \delta \in (0, 1)$. For all $n, w \in \mathbb{N}$ such that:

- if
  $$n \geq \tilde{O} \left( \left( \frac{1}{\epsilon} \right)^{\frac{2 \alpha + d - \alpha \beta}{\alpha(\beta+1)}} \right),$$
  $$w \geq \tilde{O} \left( \left( \frac{1}{\epsilon} \right)^{\frac{2 \alpha + d}{\alpha(\beta+1)}} \right)$$

and
  $$w \geq \frac{400 \log \left( \frac{12800\alpha^2}{\epsilon L \tilde{c} \phi_n^d / \alpha} \right)}{(1 - \delta)^{d/\alpha}},$$

where $L$ is defined in (H4), $\bar{c} = 0.1$ and $\phi_n$ is defined by (7),

then with probability at least $1 - \delta$ we have:

$$\inf_{A \in \mathcal{A}_{a,w}} \sup_{P \in \mathcal{P}(\alpha, \beta)} \left[ R(\hat{f}_{n,w}) - R(f^*) \right] \leq \epsilon.$$ 

5.2 Proof sketch

For a classifier $\hat{f}_{n,w}$, it is well known\cite{17} that the excess of risk is:

$$R(\hat{f}_{n,w}) - R(f^*) = \int_{\{x, \hat{f}_{n,w}(x) \neq f^*(x)\}} |2\eta(x) - 1|dP_X(x).$$

We thus aim to proof that (10) is a sufficient condition to guarantee (with probability $\geq 1 - \delta$), that $\hat{f}_{n,w}$ agrees with $f^*$ on the set $\{x, |\eta(x) - 1/2| > \Delta_o\}$, for $\Delta_o > 0$.

Introducing $\Delta_o$ in (14) leads to:

$$R(\hat{f}_{n,w}) - R(f^*) \leq 2\Delta_o P_X(|\eta(x) - 1/2| < \Delta_o).$$

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Therefore, if $\Delta_o \leq \frac{\epsilon}{2}$ then, $R(\hat{f}_{n,w}) - R(f^*) \leq \epsilon$. Otherwise, if $\Delta_o > \frac{\epsilon}{2}$, by hypothesis (H3), we have $R(\hat{f}_{n,w}) - R(f^*) \leq 2\Delta_o^{\beta+1}$. In the latter case, setting $\Delta_o = \left(\frac{\epsilon}{2e}\right)^{\frac{1}{\beta+1}}$ guarantees $R(\hat{f}_{n,w}) - R(f^*) \leq \epsilon$. Altogether, using for $\Delta_o$ the value $\Delta = \max(\frac{\epsilon}{2}, \left(\frac{\epsilon}{2e}\right)^{\frac{1}{\beta+1}})$ guarantees $R(\hat{f}_{n,w}) - R(f^*) \leq \epsilon$. This explains the expression (6).

We organize the proof of Theorem 2 in three main steps:

1. **Adaptive label requests on informative points:**
   We design two events $A_1, A_2$ with $P(A_1 \cap A_2) \geq 1 - \frac{3\delta}{16}$ such that:
   - Given an informative point $X_s$, the following relations hold on $A_1 \cap A_2$ for all $k \geq 1$:
     \[
     |\hat{\eta}_k(X_s) - \eta_k(X_s)| \leq \sqrt{\frac{2\log\left(\frac{32s^2}{\delta}\right)}{k}} \quad \text{(see Lemma 4 (Hoeffding’s inequality))} \quad (16)
     \]
     and
     \[
     |\hat{\eta}_k(X_s) - \eta_k(X_s)| \leq b_{\delta,k} \quad \text{(see Lemma 5).} \quad (17)
     \]
   In addition, if $|\eta(X_s) - \frac{1}{2}| \geq \frac{1}{2}\Delta$ and if the budget permits ($n = w = +\infty$), by using (16), the cut-off condition used in Algorithm 4
   \[
   \left| \frac{1}{k} \sum_{i=1}^{k} Y^{(i)} - \frac{1}{2} \right| \leq 2b_{\delta,s,k}
   \]
   will be violated and we can predict (by using (17)) the correct label of $X_s$ after at most $k(\epsilon, \delta_s)$ requests, with $k(\epsilon, \delta_s) \leq k(\epsilon, \delta_s)$.
   The intuition behind this is to adapt (with respect to the noise) the number of labels requested; i.e., fewer label requests on a less-noisy point (i.e., $|\eta(X_s) - \frac{1}{2}| \geq \frac{1}{2}\Delta$), and more label requests on a noisy point. This provides significant savings in the number of requests needed to predict with high probability the correct label.
   - In the event $A_1 \cap A_2$, any informative point $X_s$ falls in a high density region such that all the $k(\epsilon, \delta_s)$ nearest neighbors of $X_s$ are within at most some distance to $X_s$, and the condition (H1) is sufficient to have $k(\epsilon, \delta_s) \leq w$.

2. **Condition to be an informative point**
   Let $I = \{(X_{s'}, LB_{s'}, |Q_{s'}|), s' \leq w\}$ the set defined in KALLS, where
   \[
   \widehat{LB}_{s'} = \left| \frac{1}{|Q_{s'}|} \sum_{(X,Y) \in Q_{s'}} Y - \frac{1}{2} \right| - b_{s',|Q_{s'}|}
   \]
   and $Q_{s'}$ is defined in subroutine ConfidentLabel (Algorithm 4). We design an event $A_3$ with $P(A_3) \geq 1 - \delta/16$ and such that:
   - Given $X_s$ a current point that we want to know if it is informative or not, On the event $A_3$, if there exists $s' < s$ such that $(X_{s'}, LB_{s'}, |Q_{s'}|) \in I$, and
     \[
     \widehat{LB}_{s'} \geq \bar{c}b_{s',|Q_{s'}|} \quad \text{and} \quad \hat{p}_{X_{s'}} \leq \frac{75}{94} \left( \frac{1}{64L\widehat{LB}_{s'}} \right)^{d/\alpha} \quad \text{(with } \bar{c} = 0.1) \quad (18)
     \]
   (where $\hat{p}_{X_{s'}} := \text{Estprob}(\rho(X_s, X_{s'}), \left( \frac{1}{64L\widehat{LB}_{s'}} \right)^{d/\alpha}, 50, \delta_s)$) then
   \[
   P_X (B(X_s, \rho(X_s, X_{s'}))) \leq \left( \frac{1}{64L\widehat{LB}_{s'}} \right)^{d/\alpha}. \quad (19)
   \]
In this case, let $X_{s'}$ be such a point that satisfies (18) and (19), we can easily prove that when $X_{s'}$ is relatively far from the boundary i.e., $|\eta(X_{s'}) - \frac{1}{2}\Delta| \geq \frac{32}{63} \hat{L}B_{s'}$ (20) and easily deduce by using the smoothness assumption, (19) and (20), that the points $X_s$ and $X_{s'}$ have the same label, then we do not need to use $X_s$ in the subroutine ConfidentLabel (Algorithm 4). In addition, (12) is a sufficient condition such that the number of points used in Estprob($\rho(X_s, X_{s'})$, ($\frac{1}{64\hat{L}B_{s'}}$)$^{d/\alpha}$, $50, \delta_s$) is lower than $w$.

3. Label the instance space and label complexity

Let $I = \{(X_{s'}, \hat{L}B_{s'}, |Q_{s'}|), s' \leq w\}$ the set defined in KALLS, where

$$\hat{L}B_{s'} = \frac{1}{|Q_{s'}|} \sum_{(X, Y) \in Q_{s'}} Y - \frac{1}{2} - b_{s', |Q_{s'}|}$$

and $Q_{s'}$ is defined in subroutine ConfidentLabel (Algorithm 4). Let $s_I = \max\{s, (X_{s'}, \hat{L}B_{s'}, |Q_{s'}|) \in I\}$ and $\hat{S}$ the final active set in KALLS. We design two events $A_4$ and $A_5$ with $P(A_4 \cap A_5) \geq 1 - \delta/4$ such that:

- For all $x \in X$ with $|\eta(x) - \frac{1}{2}| > \Delta$, on $A_1 \cap A_2 \cap A_3 \cap A_4$, if

$$s_I \geq T_{\epsilon, \delta} \quad (\text{where } T_{\epsilon, \delta} = \frac{1}{p_{\epsilon}} \ln\left(\frac{8}{\delta}\right), \text{ and } \hat{p}_{\epsilon} = \left(\frac{\Delta}{128L}\right)^{d/\alpha})$$

(21) and if equations (12) and (11) hold, we prove that

$$|\eta(X_{x,1}^{(1)}) - \frac{1}{2}| \geq \frac{1}{2}\Delta \quad \text{and} \quad \hat{f}_{n,w}(x) = f^*(x) = f^*(X_{x,1}^{(1)}),$$

(22) where $X_{x,1}^{(1)}$ is the nearest neighbor of $x$ in the final active set $\hat{S}$, and $\hat{f}_{n,w}$ the 1-nn classifier on $\hat{S}$. Additionally, on $A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5$ we prove that (10) is sufficient to obtain (21).

Finally, if (11), (11) and (12) hold simultaneously, the final classifier $\hat{f}_{n,w}$ agrees with the Bayes classifier $f^*$ on $\{x, |\eta(x) - 1/2| > \Delta\}$. Thus, (13) holds with probability at least $1 - (\frac{\delta}{16} + \frac{\delta}{8} + \frac{\delta}{16} + \frac{\delta}{8} + \frac{\delta}{8}) = 1 - \delta/2 > 1 - \delta$.

6 Conclusion

In this paper we have reviewed the main results for convergence rates in a nonparametric setting for active learning, with a special emphasis on the relative merits of the assumptions about the smoothness and the margin noise. By putting active learning in perspective with recent work on passive learning that used a particular smoothness assumption customized for $k$-nn, we provided a novel active learning algorithm with a rate of convergence comparable to state-of-the art active learning algorithms, but with less restrictive assumptions. Interesting future directions include an extension to multi-class instead of binary classification. For example, in passive learning setting, (22) provides a step in this direction, since it extends the work of [5] to the context of multiclass. Adaptive algorithms, i.e. where the parameters $\alpha, \beta$ describing the smoothness and margin noise are unknown should also be explored in our setting. Previous work in this direction was done in [15, 20].

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A Detailed proof of Theorem 3

This Appendix is organized as follows: in Section A.1, we introduce some notations, in Section A.2, we adaptively determine the number of label requests needed to accurately predict the label of a point that is relatively far from the boundary decision. In Section A.4, we provide some lemmas that illustrate a sufficient condition for a point to be informative, in Section A.5, we give theorems that allow us to classify each instance relatively far from the decision boundary. Finally, in Section A.6, we provide the label complexity and establish Theorem 3.

A.1 Notations

Some notations that will be used throughout the proofs are listed here for convenience.

As defined in Section 2.3, let $B(x, r) = \{x' \in X, \rho(x, x') \leq r\}$ and $\overline{B}(x, r) = \{x' \in X, \rho(x, x') < r\}$ the open and closed balls (with respect to the Euclidean metric $\rho$), respectively, centered at $x \in X$ with radius $r > 0$. Let $\text{supp}(P_X) = \{x \in X, \forall r > 0, P_X(B(x, r)) > 0\}$ the support of the marginal distribution $P_X$.

For $p \in (0, 1]$, and $x \in \text{supp}(P_X)$, let us define

$$r_p(x) = \inf\{r > 0, P_X(B(x, r)) \geq p\}. \quad (23)$$

Let us recall for $X_s \in \mathcal{K} = \{X_1, \ldots, X_w\}$, we denote by $X_s^{(k)}$ its $k$-th nearest neighbor in $\mathcal{K}$, and $Y_s^{(k)}$ the corresponding label.

For an integer $k \geq 1$, let

$$\hat{\eta}_k(X_s) = \frac{1}{k} \sum_{i=1}^{k} Y_s^{(i)}, \quad \eta_k(X_s) = \frac{1}{k} \sum_{i=1}^{k} \eta(X_s^{(i)}). \quad (24)$$

A.2 Adaptive label requests on informative points

Lemma 1 (Chernoff [21]).
Suppose $X_1, \ldots, X_m$ are independent random variables taking value in $\{0, 1\}$. Let $X$ denote their sum and $\mu = E(X)$ its expected value. Then, for any $\delta > 0$,

$$P_m(X \leq (1 - \delta)\mu) \leq \exp(-\delta^2 \mu/2),$$

where $P_m$ is the probability with respect to the sample $X_1, \ldots, X_m$.

Lemma 2 (Logarithmic relationship, [23]).
Suppose $a, b, c > 0$, $abc^{c/a} > 4 \log_2(e)$, and $u \geq 1$. Then:

$$u \geq 2c + 2a \log(ab) \Rightarrow u > c + a \log(bu).$$

Lemma 3 (Chaudhuri and Dasgupta, [2]).
For $p \in (0, 1]$, and $x \in \text{supp}(P_X)$, let us define $r_p(x) = \inf\{r > 0, P_X(B(x, r)) \geq p\}$. For all $p \in (0, 1]$, and $x \in \text{supp}(P_X)$, we have:

$$P_X(B(x, r_p(x)) \geq p).$$

Theorem 4.
Let $\epsilon, \delta \in (0, 1)$. Set $\Delta = \max(\epsilon, (\frac{\epsilon}{2C})^{\frac{1}{d+1}})$, and $p_\epsilon = (\frac{31\Delta}{1024L})^{d/\alpha}$, where $\alpha, L, \beta, C$ are parameters used in [113] and [114].
For $p \in (0, 1]$, and $x \in \text{supp}(P_X)$, let us introduce $r_p(x) = \inf\{r > 0, P_X(B(x, r)) \geq p\}$ and $k_s := k(\epsilon, \delta_s)$ defined in (4) (where $\delta_s = \frac{\delta}{2^{2s+1}}$).
For $k, s \geq 1$, set $\tau_{k_s} = \sqrt{\frac{2}{L}} \log\left(\frac{\delta_s \epsilon^2}{\Delta}\right)$. There exists an event $A_1$ with probability at least $1 - \frac{\delta}{10}$, such that on $A_1$, for all $1 \leq s \leq w$, if

$$k_s \leq (1 - \tau_{k_s})p_\epsilon(w - 1) \quad (25)$$
then the $k_s$ nearest neighbors of $X_s$ (in the pool $\mathcal{K}$) belong to the ball $B(X_s, r_{p_\epsilon}(X_s))$. Additionally, the condition

$$w \geq \hat{O} \left( \left( \frac{1}{\epsilon} \right)^{\frac{2n+m}{m+1}} \right)$$

is sufficient to have (25).

Proof.

Fix $x \in \text{supp}(P_X)$. For $k \in \mathbb{N}$, let us denote $X_x^{(k)}$, the $k^{th}$ nearest neighbor of $x$ in the pool. We have,

$$P(\rho(x, X_x^{(k+1)}) > r_{p_\epsilon}(x)) \leq P(\sum_{i=1}^{w} \mathbb{1}_{X_i \in B(x, r_{p_\epsilon}(x))} \leq k_s).$$

Then, by using Lemma 1 and Lemma 3 and if $k_s$ satisfies (25), we have:

$$P(\rho(x, X_x^{(k+1)}) > r_{p_\epsilon}(x)) \leq P\left(\sum_{i=1}^{w} \mathbb{1}_{X_i \in B(x, r_{p_\epsilon}(x))} \leq (1 - \tau_{k_s, s})P_\epsilon(w - 1)\right)$$

$$\leq \exp(-\tau_{k_s, s}^2(w - 1)P_\epsilon(B(x, r_{p_\epsilon}(x))/2)$$

$$\leq \exp(-\tau_{k_s, s}^2(w - 1)/2)$$

$$\leq \exp(-\log(32s^2/\delta))$$

$$= \frac{\delta}{32s^2}.$$

Fix $x = X_s$. Given $X_s$, there exists an event $A_{1,s}$, such that $P(A_{1,s}) \geq 1 - \delta/(32s^2)$, and on $A_{1,s}$, if

$$k_s \leq (1 - \tau_{k_s, s})p_\epsilon(w - 1),$$

we have $B(X_s, r_{p_\epsilon}(X_s)) \cap \{X_1, \ldots, X_w\} \geq k_s$. By setting $A_1 = \bigcap_{s \geq 1} A_{1,s}$, we have $P(A_1) \geq 1 - \delta/16$, and on $A_1$, for all $1 \leq s \leq w$, if $k_s \leq (1 - \tau_{k_s, s})p_\epsilon(w - 1)$, then $B(X_s, r_{p_\epsilon}(X_s)) \cap \{X_1, \ldots, X_w\} \geq k_s$.

Now, let us proof that the condition (26) is sufficient to guarantee (25): the relation (25) implies

$$w \geq \frac{k_s}{(1 - \tau_{k_s, s})p_\epsilon} + 1. \tag{27}$$

We can see by a bit calculus, that $\tau_{k_s, s} \leq \frac{1}{2}$, and then

$$\frac{k_s}{(1 - \tau_{k_s, s})p_\epsilon} + 1 \leq \frac{2k_s}{p_\epsilon} + 1$$

$$\leq \frac{4c}{p_\epsilon \Delta^2} \left( \text{because } \frac{k_s}{p_\epsilon} \geq 1 \right)$$

$$= \frac{4c}{p_\epsilon \Delta^2} \left[ \log\left( \frac{32s^2}{\delta} \right) + \log\log\left( \frac{32s^2}{\delta} \right) + \log\log\left( \frac{512\sqrt{\epsilon}}{\Delta} \right) \right]$$

$$= \frac{b}{\Delta^2 + \frac{4}{n}} \left[ \log\left( \frac{32s^2}{\delta} \right) + \log\log\left( \frac{32s^2}{\delta} \right) + \log\log\left( \frac{512\sqrt{\epsilon}}{\Delta} \right) \right]$$
Where \( b = 4c \left( \frac{1024L}{31} \right)^{d/\alpha} \).

\[
\frac{\kappa_s}{(1 - \tau_{k_s,s}^s)p_{k_s}} + 1 \leq \tilde{C} \left( \frac{1}{\epsilon} \right)^{2\alpha + d \alpha (d + 1)} \left[ \log \left( \frac{32s^2}{\delta} \right) + \log \log \left( \frac{32s^2}{\delta} \right) + \log \log \left( \frac{512\sqrt{\epsilon}}{\Delta} \right) \right]
\]

as \( \Delta = \max (\epsilon, \left( \frac{\epsilon}{2C} \right)^{\frac{1}{\alpha + 1}}) \), where \( \tilde{C} = b \left( 2C \right)^{2\alpha + d \alpha (d + 1)} \).

\[
\leq \tilde{C} \left( \frac{1}{\epsilon} \right)^{2\alpha + d \alpha (d + 1)} \left[ 2 \log \left( \frac{32s^2}{\delta} \right) + \log \left( \frac{512\sqrt{\epsilon}}{\epsilon} \right) \right]
\]

as \( \log(x) \leq x \), and \( \Delta \geq \epsilon \).

\[
\leq 2\tilde{C} \left( \frac{1}{\epsilon} \right)^{2\alpha + d \alpha (d + 1)} \left[ \log(s^2) + \log \left( \frac{16384\sqrt{\epsilon}}{\delta \epsilon} \right) \right]
\]

\[
\leq 4\tilde{C} \left( \frac{1}{\epsilon} \right)^{2\alpha + d \alpha (d + 1)} \left[ \log(s) + \log \left( \frac{16384\sqrt{\epsilon}}{\delta \epsilon} \right) \right]
\]

\[
\leq 4\tilde{C} \left( \frac{1}{\epsilon} \right)^{2\alpha + d \alpha (d + 1)} \left[ \log(w) + \log \left( \frac{16384\sqrt{\epsilon}}{\delta \epsilon} \right) \right]
\]

(28)

Now, we are going to apply the Lemma 2. If we set in Lemma 2

\[
a = 4\tilde{C} \left( \frac{1}{\epsilon} \right)^{2\alpha + d \alpha (d + 1)}, \quad c = 4\tilde{C} \left( \frac{1}{\epsilon} \right)^{2\alpha + d \alpha (d + 1)} \log \left( \frac{16384\sqrt{\epsilon}}{\delta \epsilon} \right), \quad b = 1
\]

we can easily see that \( c \geq a, \ a \geq 4 \) and then

\[
abc^{c/a} \geq 4e > \log_2(\epsilon).
\]

then, the relation

\[
w \geq 4\tilde{C} \left( \frac{1}{\epsilon} \right)^{2\alpha + d \alpha (d + 1)} \left( \log \left( \frac{16384\sqrt{\epsilon}}{\delta \epsilon} \right) + \log \left( 4\tilde{C} \left( \frac{1}{\epsilon} \right)^{2\alpha + d \alpha (d + 1)} \right) \right)
\]

is sufficient to guarantee (28).

Let us note that the guarantee obtained in the preceding theorem corresponds to that obtained in passive setting \( (w = n) \).

A.3 Motivation for choosing \( k_s \) for \( X_s \)

Lemma 4 (Hoeffding). [12]

- **First version:**
  Let \( X \) be a random variable with \( E(X) = 0, \ a \leq X \leq b \), then for \( v > 0 \),

  \[
  E(e^{vX}) \leq e^{v^2(b-a)^2/8}.
  \]

- **Second version:**
  Let \( X_1, \ldots, X_m \) be independent random variables such that \(-1 \leq X_i \leq 1, \ (i = 0, \ldots, m) \). We define the empirical mean of these variables by

  \[
  \bar{X} = \frac{1}{m} \sum_{i=1}^{m} X_i.
  \]

Then we have:

\[
P(|X - E(\bar{X})| \geq t) \leq \exp(-mt^2/2)
\]
Lemma 5 (Kaufmann et al. [13]). Let \( \zeta(u) = \sum_{k \geq 1} k^{-1} \). Let \( X_1, X_2, \ldots \) be independent random variables, identically distributed, such that, for all \( v > 0 \), \( E(e^{vX_1}) \leq e^{v^2 \sigma^2 / 2} \). For every positive integer \( t \), let \( S_t = X_1 + \ldots + X_t \). Then, for all \( \gamma > 1 \) and \( r \geq \frac{8}{(e - 1)^2} \):
\[
P \left( \bigcup_{t \in \mathbb{N}^*} \left\{ |S_t| > \sqrt{2\sigma^2 t(r + \gamma \log \log(elt))} \right\} \right) \leq \sqrt{e} \zeta(1 - \frac{1}{2r})(\frac{\sqrt{r}}{2\sqrt{2}} + 1)^\gamma \exp(-r).
\]

Lemma 6.
Let \( m \geq 1 \) and \( u \geq 20 \). Then we have:
\[
m \geq 2u \log(\log(u)) \implies m \geq u \log(\log(m)).
\]

Proof.
Define \( \phi(m) = m - u \log(\log(m)) \), and let \( m_0 = 2u \log(\log(u)) \). We have:
\[
\phi(m_0) = 2u \log(\log(u)) - u(\log(2u \log(\log(u)))) = 2u \log(\log(u)) - u(\log(2u) + \log(\log(u)))
\]
It can be shown numerically that \( \phi(m_0) \geq 0 \) for \( u \geq 20 \).

Also, we have: \( \phi'(m) = \frac{m \log(m) - u m \log(m)}{m \log(m)} \geq 0 \) for all \( m \geq m_0 \) (notice that \( m_0 \geq u \) for \( u \geq 20 \)). Then it is easy to see that \( \phi(m) \geq \phi(m_0) \) for all \( m \geq m_0 \). This establishes the lemma.

Theorem 5.
Let \( \delta \in (0, 1) \), and \( \epsilon \in (0, 1) \). Let us assume that \( w \) satisfies \( \text{(1)} \). For \( X_s \), set \( \tilde{k}(\epsilon, \delta_s) \) (with \( \delta_s = \frac{\delta}{32s^2} \)) as
\[
\tilde{k}(\epsilon, \delta_s) = \frac{c}{4|\eta(X_s) - \frac{1}{2}|^2} \left[ \log(\frac{32\sigma^2}{\delta}) + \log \log(\frac{32\sigma^2}{\delta}) + \log \log \left( \frac{256\sqrt{e}}{|\eta(X_s) - \frac{1}{2}|} \right) \right],
\]
where \( c \geq 7.10^6 \). For \( k \geq 1 \), \( s \leq w \), let \( \Delta = \max(\frac{c}{4}, (\frac{c}{2r})^{\frac{1}{2+r}}) \) and \( b_{\delta, s, k} \) defined in \( \text{(1)} \).

Then, there exists an event \( A_2 \), such that \( P(A_2) \geq 1 - \delta/8 \), and on \( A_1 \cap A_2 \), we have:

1. For \( k \geq 1 \), \( \tilde{\eta}_k(X_s) \) and \( \bar{\eta}_k(X_s) \) defined in \( \text{(21)} \), for all \( s \in \{1, \ldots, w\} \),
\[
|\tilde{\eta}_k(X_s) - \bar{\eta}_k(X_s)| \leq b_{\delta, s, k}.
\]

2. For all \( s \leq w \), if \( |\eta(X_s) - \frac{1}{2}| \geq \frac{1}{2} \Delta \), then, \( \tilde{k}(\epsilon, \delta_s) \leq k(\epsilon, \delta_s) \), and the subroutine \( \text{ConfidentLabel}(X_s) := \text{ConfidentLabel}(X_s, k(\epsilon, \delta_s), t = \infty, \delta_s) \) uses at most \( \tilde{k}(\epsilon, \delta_s) \) label requests.

We also have
\[
\left| \frac{1}{k_s} \sum_{i=1}^{k_s} Y^{(i)}_s - \frac{1}{2} \right| \geq 2b_{\delta, \bar{k}, s}
\]
and
\[
f^*(X_s) = \mathbb{I}_{\tilde{\eta}_{\bar{k}_s}(X_s) \geq \frac{1}{2}}.
\]
Where \( \bar{k}_s \) is the number of requests made in \( \text{ConfidentLabel}(X_s) \).

Proof.

1. Let us begin with the proof of the first part of Theorem 5.

Here, we follow the proof of Theorem 8 in [13], with few additional modifications.

Let \( s \in \{1, \ldots, w\} \). Set \( S_k = \sum_{i=1}^{k_s} \left( Y^{(i)}_s - \eta(X^{(i)}_s) \right) \). Given \( \{X_1, \ldots, X_w\} \), \( E(Y^{(k)}_s - \eta(X^{(k)}_s)) = 0 \),
and the random variables \( \{Y_s^{(i)} - \eta(X_s^{(i)}), i = 1, \ldots, k\} \) are independent. Then by Lemma 41 given \( \{X_1, \ldots, X_w\} \), as \( Y_s^{(1)} - \eta(X_s^{(1)}) \) takes values in \([-1, 1]\), we have \( E(e^{v(Y_s^{(1)} - \eta(X_s^{(1)}))}) \leq e^{v^2/2} \) for all \( v > 0 \). Furthermore, set \( z = \log(32^2/\delta) \), and \( r = z + 3 \log(z) \). We have \( r \geq \frac{8}{(e-1)^2} \), and by Lemma 5 with \( \gamma = 3/2 \), we have:

\[
P \left( \bigcup_{k \in \mathbb{N}^*} \left\{ |S_k| > \sqrt{2k(r + \gamma \log \log(ek))} \right\} \right) \leq \sqrt{e} \zeta(3/2(1 - \frac{1}{2r})) \left( \frac{r}{2\sqrt{2}} + 1 \right)^{3/2} \exp(-r)
\]

\[
= \frac{\sqrt{e}}{8} \zeta \left( \frac{3}{2} - \frac{3}{4(z + 3 \log(z))} \right) \left( \frac{\sqrt{z + 3 \log(z) + \sqrt{8}})^{3/2}}{z^3} \frac{\delta}{32^2} \right)
\]

It can be shown numerically that for \( z \geq 2.03 \), which holds for all \( \delta \in (0, 1) \), \( s \geq 1 \),

\[
\frac{\sqrt{e}}{8} \zeta \left( \frac{3}{2} - \frac{3}{4(z + 3 \log(z))} \right) \left( \frac{\sqrt{z + 3 \log(z) + \sqrt{8}})^{3/2}}{z^3} \frac{\delta}{32^2} \right) \leq 1.
\]

Then, we have, given \( s \in \{1, \ldots, w\} \), there exists an event \( A'_{2,s} \) such that \( P(A'_{2,s}) \geq 1 - \delta/32s^2 \), and simultaneously for all \( k \geq 1 \), we have:

\[
|S_k| \leq \sqrt{2k \left( \log \left( \frac{32^2}{\delta} \right) + \log \left( \frac{32^2}{\delta} \right) + \log \log(ek) \right)}.
\]

By setting \( A'_{2} = \bigcap_{s \geq 1} A'_{2,s} \), we have \( P(A'_{2}) \geq 1 - \delta/16 \), and on \( A'_{2} \), we have for all \( s \in \{1, \ldots, w\} \), for all \( k \geq 1 \),

\[
|\bar{\eta}_k(X_s) - \bar{\eta}_k(X_s)| \leq b_{\delta,s,k}.
\]

2. For the proof of the second part of Theorem 5, we are going to show that there exists an event \( A''_2 \) such that (30) and (31) hold on \( A'_2 \cap A''_2 \cap A_1 \).

Given \( \{X_1, \ldots, X_w\} \), and \( X_s \in \{X_1, \ldots, X_w\} \), by Lemma 4 there exists an event \( A''_{2,s} \), with \( P(A''_{2,s}) \geq 1 - \delta/32s^2 \), and on \( A''_{2,s} \), we have:

\[
|\bar{\eta}_k(X_s) - \bar{\eta}_k(X_s)| \leq \sqrt{\frac{2 \log(32^2/\delta)}{k}}.
\]

This implies that:

\[
|\bar{\eta}_k(X_s) - \frac{1}{2}| \geq |\bar{\eta}_k(X_s) - \frac{1}{2}| - \sqrt{\frac{2 \log(32^2/\delta)}{k}}.
\]

On the event \( A_1 \), we have, for all \( k \leq k_s \), by the \( \alpha \)-smoothness assumption (44),

\[
|\eta(X_s) - \eta(X_s^{(k)})| \leq \frac{31}{1024} \Delta.
\]

And then, if \( |\eta(X_s) - \frac{1}{2}| \geq \frac{1}{2} \Delta \), then \( |\eta(X_s) - \frac{1}{2}| \geq \frac{1}{32} \Delta \). The relation (33) becomes

\[
|\eta(X_s^{(k)}) - \frac{1}{2}| \geq \frac{1}{1024} |\eta(X_s) - \frac{1}{2}|.
\]

Then (32) becomes:

\[
|\bar{\eta}_k(X_s) - \frac{1}{2}| \geq \frac{1}{1024} |\eta(X_s) - \frac{1}{2}| - \sqrt{\frac{2 \log(32^2/\delta)}{k}}.
\]

A sufficient condition for \( k \) to satisfy (33), is

\[
\frac{1}{1024} |\eta(X_s) - \frac{1}{2}| - \sqrt{\frac{2 \log(32^2/\delta)}{k}} \geq 2b_{\delta,s,k}.
\]
and then:
\[
\frac{1}{1024} |\eta(X_s) - \frac{1}{2}| - \sqrt{2 \log\left(\frac{32s^2}{k}\right)} \geq 2 \sqrt{2 \left(\log\left(\frac{32s^2}{\delta}\right) + \log\log\left(\frac{32s^2}{\delta}\right) + \log\log(ek)\right)}
\]
this implies:
\[
k \geq \frac{1024}{|\eta(X_s) - \frac{1}{2}|^2} \left(\sqrt{2 \log\left(\frac{32s^2}{\delta}\right)} + 2 \sqrt{2 \left(\log\left(\frac{32s^2}{\delta}\right) + \log\log\left(\frac{32s^2}{\delta}\right) + \log\log(ek)\right)}\right)^2.
\] (35)

On the other hand, the right-hand side is smaller than:
\[
\frac{1024}{|\eta(X_s) - \frac{1}{2}|^2} \left(\sqrt{2 \log\left(\frac{32s^2}{\delta}\right)} + 2 \sqrt{2 \log\left(\frac{32s^2}{\delta}\right) + 2 \sqrt{2 \log\log\left(\frac{32s^2}{\delta}\right) + 2 \sqrt{2 \log\log(ek)}}\right)^2.
\]
To deduce (35), it suffices to have the expression into brackets lower than:
\[
\frac{\sqrt{k}}{32} |\eta(X_s) - \frac{1}{2}|
\]
Then, it suffices to have simultaneously:
\[
\sqrt{2 \log\left(\frac{32s^2}{\delta}\right)} \leq \frac{\sqrt{k}}{9 \, 32} |\eta(X_s) - \frac{1}{2}|
\]
\[
\sqrt{2 \log\log\left(\frac{32s^2}{\delta}\right)} \leq \frac{\sqrt{k}}{6 \, 32} |\eta(X_s) - \frac{1}{2}|
\]
\[
\sqrt{2 \log\log(ek)} \leq \frac{\sqrt{k}}{6 \, 32} |\eta(X_s) - \frac{1}{2}|
\]
Equivalently, we have:
\[
k \geq \frac{1024}{|\eta(X_s) - \frac{1}{2}|^2} 162 \log\left(\frac{32s^2}{\delta}\right)
\] (36)
\[
k \geq \frac{1024}{|\eta(X_s) - \frac{1}{2}|^2} 72 \log\log\left(\frac{32s^2}{\delta}\right)
\] (37)
\[
k \geq \frac{1024}{|\eta(X_s) - \frac{1}{2}|^2} 72 \log\log(ek)
\] (38)
We can apply the Lemma 6 in (38) by taking: \( m = ek \) and \( u = \frac{73728e}{|\eta(X_s) - \frac{1}{2}|} \). We have \( m \geq 1 \) and \( u \geq 20 \) and then, a sufficient condition to have (38) is:
\[
k \geq 2 \frac{73728e}{|\eta(X_s) - \frac{1}{2}|^2} \log\log\left(\frac{73728e}{|\eta(X_s) - \frac{1}{2}|^2}\right)
\]
or
\[
k \geq 4 \frac{73728e}{|\eta(X_s) - \frac{1}{2}|^2} \log\log\left(\frac{\sqrt{73728e}}{|\eta(X_s) - \frac{1}{2}|}\right)
\] (39)
We can easily see that \( \tilde{k}_s := \tilde{k}(\epsilon, \delta_s) \) satisfies (36), (37), (39). Then
\[
\frac{1}{k_s} \sum_{i=1}^{k_s} Y_s^{(i)} - \frac{1}{2} \geq 2b_{\delta_s, \tilde{k}_s}.
\] (40)
As $|\eta(X_s) - \frac{1}{2}| \geq \frac{1}{2}\Delta$, we can easily see that $\hat{k}(\epsilon, \delta_s) \leq k(\epsilon, \delta_s)$. By taking the minimum value $\bar{k}_s = \hat{k}(\epsilon, \delta_s)$ that satisfies (40), we can see that when the budget allows us, the subroutine ConfidentLabel requests $\bar{k}_s$ labels, and we have:

$$\left| \frac{1}{\bar{k}_s} \sum_{i=1}^{\bar{k}_s} Y'_s - \frac{1}{2} \right| \geq 2b_{\delta_s, \bar{k}_s}. \quad (41)$$

By setting $A'_2 = \cap_{s \geq 1} A''_{2, s}$, we have $P(A'_2) \geq 1 - \delta/16$, and we can deduce (30).

We have on $A'_2$, for all $s \leq w$, $k \leq k(\epsilon, \delta_s)$,

$$|\hat{\eta}(X_s) - \bar{\eta}_k(X_s)| \leq b_{\delta_s, k}.$$ 

And then, on $A_1 \cap A'_2$, we have for all $s \leq w$, $k \leq k(\epsilon, \delta_s)$:

$$|\eta(X_s) - \bar{\eta}_k(X_s)| \leq |\eta(X_s) - \bar{\eta}_k(X_s)| + |\bar{\eta}(X_s) - \bar{\eta}_k(X_s)|$$

$$\leq \frac{31}{1024} \Delta + b_{\delta_s, k}. \quad (42)$$

Assume without loss of generality that $\eta(X_s) \geq \frac{1}{2}$, which leads to:

$$\frac{\hat{\eta}_{k_s}(X_s)}{2} - \frac{1}{2} = \hat{\eta}_{k_s}(X_s) - \eta(X_s) + \eta(X_s) - \frac{1}{2}$$

$$\geq -|\hat{\eta}_{k_s}(X_s) - \eta(X_s)| + \eta(X_s) - \frac{1}{2}. \quad (43)$$

If $\eta(X_s) - \frac{1}{2} \geq \frac{1}{2}\Delta$, with (42), the expression (43) becomes:

$$\frac{\hat{\eta}_{k_s}(X_s)}{2} - \frac{1}{2} \geq -\frac{31}{1024} \Delta - b_{\delta_s, \bar{k}_s} + \frac{1}{2}\Delta$$

$$= \frac{481}{1024} \Delta - b_{\delta_s, \bar{k}_s}$$

$$\geq -b_{\delta_s, \bar{k}_s}. \quad (44)$$

On the other hand, we have by (30),

$$|\hat{\eta}_{k_s}(X_s) - \frac{1}{2}| \geq 2b_{\delta_s, \bar{k}_s},$$

that is to say:

$$\frac{\hat{\eta}_{k_s}(X_s)}{2} - \frac{1}{2} \geq 2b_{\delta_s, \bar{k}_s} \text{ or } \frac{\hat{\eta}_{k_s}(X_s)}{2} - \frac{1}{2} \leq -2b_{\delta_s, \bar{k}_s}.$$

By (44), we have necessarily $\frac{\hat{\eta}_{k_s}(X_s)}{2} - \frac{1}{2} \geq 2b_{\delta_s, \bar{k}_s}$, and then:

$$\frac{\hat{\eta}_{k_s}}{2} - \frac{1}{2} \geq \max(-b_{\delta_s, \bar{k}_s}, 2b_{\delta_s, \bar{k}_s}) = 2b_{\delta_s, \bar{k}_s} \geq 0,$$

Thus we can easily deduce (31). By setting $A_2 = A'_2 \cap A''_2$, we have $P(A_2) \geq 1 - \delta/8$ and on $A_1 \cap A_2$, the item 1 and item 2 hold simultaneously.
A.4 Sufficient condition to be an informative point

As noticed in Section 4.3 a sufficient condition for a point $X_t$ to be considered as not informative is:

$$P_X(B(X_t, \rho(X_t, X_s))) \leq O((\hat{L}B_s)^{d/\alpha}).$$

(45)

for some previous informative point $X_s$ (with $\hat{L}B_s > 0$ defined in Lemma 8). Because $P_X$ is unknown, we provide a computational scheme sufficient to obtain (45).

Firstly we follow the general procedure used in [14] to estimate adaptively the expectation of a Bernoulli random variable. And secondly, we apply it to the Bernoulli variable $\mathbb{1}_A$ where $A = \{x, x \in B(X_t, r)\}$.

Lemma 7 (Kontorovich et al.). [14]

Let $\delta' \in (0, 1)$, $\epsilon_o > 0$, $t \geq 7$ and set $g(t) = 1 + \frac{\sqrt{2}}{t}$. Let $p_1, p_2, \ldots \in \{0, 1\}$ be i.i.d Bernoulli random variables with expectation $p$. Let $\hat{p}$ be the output of $\text{BerEst}(\epsilon_o, \delta', t)$. There exists an event $A'$, such that $P(A') \geq 1 - \delta'$, and on $A'$, we have:

1. If $\hat{p} \leq \frac{\epsilon_o}{g(t)}$ then $p \leq \epsilon_o$.

2. The number of random draws in the $\text{BerEst}$ subroutine (Algorithm 3) is at most $\frac{8t \log(\frac{d}{\alpha})}{\epsilon_o}$, where $\psi := \max(\epsilon_o, \frac{p}{g(t)})$.

Lemma 8.

Let $\epsilon, \delta \in (0, 1)$, $r > 0$. As in Section 4.2, for $k \geq 1$, let us define

$$b_{\delta, k} = \sqrt{\frac{2}{k} \left( \log \left( \frac{1}{\delta} \right) + \log \log \left( \frac{1}{\delta} \right) + \log \log(ek) \right)}.$$

For $(X_s, \hat{L}B_s, |Q_s|) \in I$, (where $I$ is the set defined in KALLS), where

$$\hat{L}B_s = \left\lfloor \frac{1}{|Q_s|} \sum_{(X, Y) \in Q_s} Y - \frac{1}{2} \right\rfloor \hat{b}_{\delta, |Q_s|}$$

and $Q_s$ is defined in subroutine $\text{ConfidentLabel}$ (Algorithm 4). Let us assume that $w$ satisfies \[12\]. There exists an event $A_3$, such that $P(A_3) \geq 1 - \delta/16$, we have, on $A_3$, for all $s \leq w$:

If there exists $1 \leq s' \leq s$ and $(X_{s'}, \hat{L}B_{s'}, |Q_{s'}|) \in I$, such that:

$$\hat{L}B_{s'} \geq \tilde{c} b_{\delta, |Q_{s'}|} \text{ and } \hat{p}_{X_{s'}} \leq \frac{75}{94} \left( \frac{1}{64L} \hat{L}B_{s'} \right)^{d/\alpha} \quad \text{(with } \tilde{c} = 0.1)$$

(46)

(where $\hat{p}_{X_{s'}} := \text{Estprob}(\rho(X_s, X_{s'}), \left( \frac{1}{64L} \hat{L}B_{s'} \right)^{d/\alpha}, 50, \delta_s$), then

$$P_X(B(X_s, \rho(X_s, X_{s'}))) \leq \left( \frac{1}{64L} \hat{L}B_{s'} \right)^{d/\alpha}.$$

Proof.

By following the scheme of subroutine Estprob, this Lemma is a direct application of Lemma 7 by taking for all $s \leq w$, $t = 50$, $\epsilon_o = \left( \frac{1}{64L} \hat{L}B_s \right)^{d/\alpha}$, $\delta' = \delta_s$, $r = \rho(X_s, X_{s'})$, $A_3, s := A'$. And then, if we set $A_3 = \cap_{s \geq 1} A_{3, s}$, we have $P(A_3) \geq 1 - \delta/16$, and (46) follows immediately.

On the other hand, for all $s \leq w$, the number of draws in Estprob($\rho(X_s, X_{s'}), \left( \frac{1}{64L} \hat{L}B_{s'} \right)^{d/\alpha}, 50, \delta_s$) is always lower than $w$. Indeed, by Lemma 7 the number of draws is at most:

$$N := \frac{400 \log(\frac{12800w^2}{\epsilon_o})}{\psi} \quad \text{where } \psi := \max(\frac{1}{64L} \hat{L}B_{s'}^{d/\alpha}, \frac{75}{94} P_X(B(X_s, \rho(X_s, X_{s'})))).$$
Then we have:

\[
N \leq \frac{400 \log \left( \frac{12800 s^2}{\delta (\frac{1}{4L} LB_{s'})^{d/\alpha}} \right)}{\left( \frac{1}{4L} \bar{c} \delta_{x',|Q_{s'}|} \right)^{d/\alpha}} \quad (\text{as } LB_{s'} \geq \bar{c} b_{x',|Q_{s'}|})
\]

\[
\leq \frac{400 \log \left( \frac{12800 s^2}{\delta (\frac{1}{4L} \hat{c} b_{x',|Q_{s'}|})^{d/\alpha}} \right)}{\left( \frac{1}{4L} \bar{c} \delta_{x',|Q_{s'}|} \right)^{d/\alpha}} \quad (\text{we can easily see that } b_{x',|Q_{s'}|} \geq \phi_{n})
\]

\[
\leq w \quad (\text{by } 47).
\]

\[\square\]

A.5 Label the instance space

Theorem 6. Let \( \epsilon, \delta \in (0, 1) \). Let \( T_{\epsilon,\delta} = \frac{1}{p_{c}} \ln \left( \frac{\tilde{p}_{c}}{\delta} \right) \), and \( \tilde{p}_{c} = \left( \frac{\Delta}{128L} \right)^{d/\alpha}, \) with \( \Delta = \max \left( \frac{1}{2}, \left( \frac{\epsilon}{2c} \right)^{\frac{1}{p_{c}}} \right) \). Let \( I \subset X \times \mathbb{R} \times \mathbb{N} \) the set used in KALLS.

Set \( s_I = \max I' \) with \( I' = \{ s, (X_s, \hat{L}B_{s'}, |Q_s|) \} \) (index of the last informative point). There exists an event \( A_4 \) such that \( P(A_4) \geq 1 - \delta/8 \), and on \( A_1 \cap A_2 \cap A_3 \cap A_4 \), we have

1. \[ \sup_{x \in \text{supp}(P_X)} \min_{\bar{X} \in \{ X_1, \ldots, X_{T_{\epsilon,\delta}} \}} P_X(B(x, \rho(\bar{X}, x))) \leq \tilde{p}_{c}. \] (47)

2. If \( w \) satisfies (11) and (12) and the following condition holds

\[
s_I \geq T_{\epsilon,\delta},
\]

then, for all \( x \in \text{supp}(P_X) \) such that \( |\eta(x) - \frac{1}{2}| > \Delta \), there exists \( s := s(x) \in I' \) such that:

\[
|\eta(X_s) - \frac{1}{2}| \geq \frac{1}{2} \Delta
\]

and

\[
f^*(x) = f^*(X_s)
\]

Proof.

1. Let us begin by proving the first part of Theorem 6

As in Section A.1, for \( x \in \text{supp}(P_X) \), let us introduce

\[
r_{\tilde{p}_{c}}(x) = \inf \{ r > 0, P_X(B(x, r)) \geq \tilde{p}_{c} \}.
\]

By Lemma 8, we have \( P_X(B(x, r_{\tilde{p}_{c}}(x)) \geq \tilde{p}_{c} \). Then each \( \bar{X} \in \{ X_1, \ldots, X_{T_{\epsilon,\delta}} \} \) belongs to \( B(x, r_{\tilde{p}_{c}}(x)) \)
with probability at least $\tilde{p}_\epsilon$. If we denote $\hat{P}$ the probability over the data, we have:

$$
\hat{P}(\exists \tilde{X} \in \{X_1, \ldots, X_{T,\delta}\}, P_X(B(x, \rho(x, \tilde{X})) \leq \tilde{p}_\epsilon) = 1 - \hat{P}(\forall \tilde{X} \in \{X_1, \ldots, X_{T,\delta}\}, P_X(B(x, \rho(x, \tilde{X})) > \tilde{p}_\epsilon)
$$

$$
= 1 - \prod_{i=1}^{T,\delta} \hat{P}(P_X(B(x, \rho(x, X_i)) > \tilde{p}_\epsilon)
$$

$$
\geq 1 - \prod_{i=1}^{T,\delta} \hat{P}(\rho(x, X_i) > r_{\tilde{p}_\epsilon}(x))
$$

$$
= 1 - \prod_{i=1}^{T,\delta} (1 - \hat{P}(\rho(x, X_i) \leq r_{\tilde{p}_\epsilon}(x)))
$$

$$
\geq 1 - (1 - \tilde{p}_\epsilon)^{T,\delta}
$$

$$
\geq 1 - \exp(-T,\delta \tilde{p}_\epsilon)
$$

$$
= 1 - \delta/8.
$$

Then, there exists an event $A_4$, such that $P(A_4) \geq 1 - \delta/8$ and (47) holds. And then, we can easily conclude the first part.

2. For the second part of Theorem 6, let $x \in \text{supp}(P_X)$. By (47), on $A_4$ there exists $X_x \in \{X_1, \ldots, X_{T,\delta}\}$ such that:

$$
P_X(B(x, \rho(x, X_x))) \leq \tilde{p}_\epsilon.
$$

(51)

By (H4), we have:

$$
|\eta(x) - \eta(X_x)| \leq \frac{1}{128} \Delta < \frac{1}{32} \Delta.
$$

(52)

Then if $|\eta(x) - \frac{1}{2}| > \Delta$, we have:

$$
(1 - \frac{1}{32})\Delta < |\eta(x) - \frac{1}{2}| < (1 + \frac{1}{32})\Delta.
$$

(53)

As $s_I \geq T,\delta$, then there exists $s'$ such that $X_x := X_{s'}$ and $X_{s'}$ passes through the subroutine Reliable.

We have two cases:

a) $X_{s'}$ is uninformative. Then there exists $s < s'$, such that

$$
\tilde{L}B_s \geq 0.1b_{\delta_s, |Q_s|} \text{ and } \hat{p}_{X_s} \leq \frac{75}{94} \left( \frac{1}{64L} \tilde{L}B_s \right)^{d/\alpha}
$$

(where $\hat{p}_{X_s} := \text{Estprob}(\rho(X_s, X_{s'}), \left( \frac{1}{64L} \tilde{L}B_s \right)^{d/\alpha}, 50, \delta_s$), then

$$
P_X(B(X_{s'}, \rho(X_s, X_{s'}))) \leq \left( \frac{1}{64L} \tilde{L}B_s \right)^{d/\alpha}.
$$

(54)

Necessary, we have $|\eta(X_s) - \frac{1}{2}| \geq \frac{1}{32} \Delta$. Indeed, if $|\eta(X_s) - \frac{1}{2}| < \frac{1}{32} \Delta$, then on $A_1 \cap A_2$, by denoting $k_s$ the number of request labels in $\text{ConfidentLabel}(X_s) := \text{ConfidentLabel}(X_s, k(\epsilon, \delta_s), t, \delta_s)$,
(where \( t = n - \sum_{s_i \in I', s_i < s} |Q_{s_i}| \)) we have:

\[
\overline{LB}_s = |\bar{\eta}_{k_s}(X_s) - \frac{1}{2}| - b_{\delta_s, k_s}
\]

\[
\leq |\bar{\eta}_{k_s}(X_s) - \bar{\eta}_{k_s}(X_s)| + |\bar{\eta}_{k_s}(X_s) - \frac{1}{2}| - b_{\delta_s, k_s}
\]

\[
\leq |\bar{\eta}_{k_s}(X_s) - \frac{1}{2}| \quad \text{(by (29))}
\]

\[
\leq |\eta(X_s) - \frac{1}{2}| + \frac{1}{32}(1 - \frac{1}{32}) \Delta \quad \text{(by \[\text{H4}\] and Theorem \[\text{H4}\])} \quad (55)
\]

\[
< \frac{1}{32} \Delta + \frac{1}{32}(1 - \frac{1}{32}) \Delta
\]

\[
= \frac{63}{1024} \Delta \quad (56)
\]

By \[\text{H4}\] and (51), we have:

\[
|\eta(X_{s'}) - \frac{1}{2}| \leq |\eta(X_s) - \frac{1}{2}| + \frac{1}{64} \overline{LB}_s
\]

\[
< \frac{1}{32} \Delta + \frac{1}{64} \frac{63}{1024} \Delta \quad \text{(by (56))}
\]

\[
= (\frac{1}{32} + \frac{1}{64} \cdot \frac{63}{1024}) \Delta
\]

\[
\leq (1 - \frac{1}{32}) \Delta
\]

that contradicts (53), then we have \( |\eta(X_s) - \frac{1}{2}| \geq \frac{1}{32} \Delta \). Therefore, by (54), (55), we have:

\[
P_X(B(X_{s'}, \rho(X_s, X_{s'}))) \leq \left( \frac{1}{64L \overline{LB}_s} \right)^{d/\alpha}
\]

\[
\leq \left( \frac{1}{64L} \left( |\eta(X_s) - \frac{1}{2}| + \frac{1}{32}(1 - \frac{1}{32}) \Delta \right) \right)^{d/\alpha}
\]

\[
\leq \left( \frac{1}{64L} \left( |\eta(X_s) - \frac{1}{2}| + (1 - \frac{1}{32}) |\eta(X_s) - \frac{1}{2}| \right) \right)^{d/\alpha}
\]

\[
= \left( \frac{1}{64L} \left( 2 - \frac{1}{32} \right) |\eta(X_s) - \frac{1}{2}| \right)^{d/\alpha}
\]

\[
= \left( \frac{63}{2048L} |\eta(X_s) - \frac{1}{2}| \right)^{d/\alpha} \quad . \quad (57)
\]

On the other hand, by (51), we have:

\[
P_X(B(x, \rho(X_{s'}, x))) \leq \tilde{p}_\epsilon
\]

\[
= \left( \frac{1}{128L} \Delta \right)^{d/\alpha}
\]

\[
\leq \left( \frac{1}{128L} |\eta(x) - \frac{1}{2}| \right)^{d/\alpha} \quad . \quad (58)
\]
We have:

\[
|\eta(x) - \eta(X_s)| \leq |\eta(x) - \eta(X_{s'})| + |\eta(X_{s'}) - \eta(X_s)|
\]
\[
\leq L.P_X(B(x, \rho(X_{s'}, x)))^{\alpha/d} + L.P_X(B(X_{s'}, \rho(X_{s'}, X_s)))^{\alpha/d} \quad \text{(by (14))}
\]
\[
\leq \frac{1}{128}|\eta(x) - \frac{1}{2}| + \frac{63}{2048}|\eta(X_s) - \frac{1}{2}| \quad \text{(by (57) and (58))}
\]
\[
\leq \frac{1}{128}|\eta(x) - \frac{1}{2}| + \frac{63}{2048}| \eta(X_{s'}) - \frac{1}{2}| \quad \text{(by (14) and (57))}
\]
\[
\leq \frac{1}{128}|\eta(x) - \frac{1}{2}| + \frac{63}{1985}(1 + \frac{1}{128})|\eta(x) - \frac{1}{2}| \quad \text{(by (52))}
\]
\[
= \frac{79}{1985}|\eta(x) - \frac{1}{2}|
\quad \text{(60)}
\]

b) \textit{X}_{s'} \textit{ is informative}. In this case, \( s = s' \) and then we always obtains the equation (60), which becomes

\[
|\eta(X_s) - \frac{1}{2}| \geq \left(1 - \frac{79}{1985}\right)|\eta(x) - \frac{1}{2}|
\]
\[
\geq \left(1 - \frac{79}{1985}\right)\Delta
\]
\[
\geq \frac{1}{2}\Delta
\quad \text{(62)}
\]

Then

\[
|\eta(X_s) - \frac{1}{2}| \geq \frac{1}{2}\Delta
\quad \text{(63)}
\]

On \( A_1 \cap A_2 \), by Theorem 5, the subroutine \textit{ConfidentLabel}(X_s) uses at most \( \tilde{k}(\epsilon, \delta) \) request labels, and returns the correct label (with respect to the Bayes classifier) of \( X_s \).

Let us proof that \( f^*(x) = f^*(X_s) \). Let us assume without loss of generality that \( \eta(X_s) - \frac{1}{2} \geq 0 \). We will show that \( \eta(x) - \frac{1}{2} \geq 0 \). We have:

\[
\eta(x) - \frac{1}{2} = \eta(x) - \eta(X_s) + \eta(X_s) - \frac{1}{2}
\]
\[
\geq \eta(X_s) - \frac{1}{2} - \frac{79}{1985}|\eta(x) - \frac{1}{2}| \quad \text{(by (60))}
\]
\[
\geq (1 - \frac{79}{1985})|\eta(x) - \frac{1}{2}| - \frac{79}{1985}|\eta(x) - \frac{1}{2}| \quad \text{(by (60))}
\]
\[
= \frac{1827}{1985}|\eta(x) - \frac{1}{2}|
\]
\[
\geq 0
\]

Then \( f^*(x) = f^*(X_s) \).

\[\square\]

**Theorem 7.**

Let \( x \in \text{supp}(P_X) \) such that \( |\eta(x) - \frac{1}{2}| > \tilde{\Delta} \). Let \( I \subset X \times \mathbb{R} \times \mathbb{N} \) the set used in \textit{KALLS}.

Set \( s_I = \max I' \), with \( I' = \{s, (X_s, \text{LB}_{s}, \{Q_s\}) \in I\} \) (\( s_I \) is the index of the last informative point). Let us assume that \( s_I \geq T_{c, \delta} \). Let \( \hat{S} \) the final active set use in subroutine \textit{Learn} (1). Let \( \hat{f}_{n,w} \) the output of the subroutine \textit{Learn}. Let us assume that \( w \) satisfies (11) and (12). We have on \( A_1 \cap A_2 \cap A_3 \cap A_4 \)

\[
\hat{f}_{n,w}(x) = f^*(x).
\]
A.6 Label complexity

Lemma 9.
Let us assume that \( w \) satisfies (11), (12), and \( \psi \geq A_\epsilon, \delta \). Then, there exists an event \( A_5 \) such that \( P(A_5) \geq 1 - \delta/8 \), and on \( A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \). The condition (10) is sufficient to guarantee (18).

Proof.
Let \( I \subset \mathcal{X} \times \mathbb{R} \times \mathbb{N} \) the set used in KALLS.
Set \( s_I = \max I' \), with \( I' = \{ s, (X, \hat{L}B_s, |Q_s|) \in I \} \) (index of the last informative point). We consider two cases:

1. **First case:** \( s_I = w \): we can easily see that (18) is satisfied, and we have trivially that the condition (10) is sufficient to guarantee (18).

2. **Second case:** \( s_I < w \): then the total number of label requests up to \( s_I \) is:

\[
\sum_{s \in I'} |Q_s| \quad (64)
\]

where \( Q_s \) is the output in the subroutine \( \text{ConfidentLabel} \). Let \( s \in I' \). For brevity, let us denote \( \text{ConfidentLabel}(X, t) := \text{ConfidentLabel}(X, k(\epsilon, \delta_s) t, \delta_s) \), where \( t = n - \sum_{s_i \in I', s_i < s} |Q_{s_i}| \) (the budget parameter ). If \( s \neq s_I \), the subroutine \( \text{ConfidentLabel}(t, X) \) implicitly assumes that the process of label request do not takes into account the constraint related to the budget \( n \) (very large budget with respect to \( k(\epsilon, \delta_s) \)) such that \( \text{ConfidentLabel}(X, t) = \text{ConfidentLabel}(X, t = \infty) \). Then we have:

\[
n > \sum_{s \in I'} |Q_s| 
\]

On the other hand, we want to guarantee the condition (18). For this, necessary for all \( s \in I' \), such that \( s \leq T_{\epsilon, \delta} \), and \( s < s_I \), at the end of the subroutine \( \text{ConfidentLabel}(X, t) \), the budget \( n \) is not yet reached and then we can replace the relation (65) by

\[
n > \sum_{s \in I'} |Q_s| 
\]

(66)

Then, necessarily, (18) holds when (66) holds.

Also, for \( s \in I' \), by theorem (13), if we assume that \( |\eta(X_s) - \frac{1}{2}| \geq \frac{1}{2} \Delta \), we have that \( |Q_s| \leq \hat{k}(\epsilon, \delta_s) \), and the subroutine \( \text{ConfidentLabel}(X, t) \), (with \( t = n - \sum_{s_i \in I', s_i < s} |Q_{s_i}| \)) terminates when the cut-off condition (30) is satisfied. The left hand side of (66) by is equal to:

\[
\sum_{s \in I'} |Q_s| + \sum_{s \in I'} \sum_{s < s_I} |Q_s| \quad (67)
\]

Firstly, let us consider the first term in (67) and denote it by \( T_1 \). Let us denote by \( B_s \) the event:

\[
B_s = \{|\eta(X_s) - \frac{1}{2}| \geq \frac{1}{2} \Delta\}.
\]

We have

\[
1_{B_s} = \sum_{j=1}^{m_s} 1_{B_{s,j}}
\]

(68)

where

\[
B_{s,j} = \{2^{-j-1} \Delta \leq |\eta(X_s) - \frac{1}{2}| \leq 2^j \Delta \} \quad \text{and} \quad m_s = \max \left( 0, \left \lceil \log_2 \left( \frac{1}{2} \Delta \right) \right \rceil \right).
\]
Then,

\[ T_1 \leq \sum_{s \in I'} \tilde{k}(\epsilon, \delta_s) \quad \text{by Theorem 5} \]

\[ = \sum_{s \in I'} \sum_{j=1}^{m_s} \tilde{k}(\epsilon, \delta_s) \mathbb{1}_{B_{s,j}} \quad (69) \]

On \( B_{s,j} \),

\[ \tilde{k}(\epsilon, \delta_s) \leq \frac{c}{2^{2j} \Delta^2} \left[ \log(\frac{32s^2}{\delta}) + \log \log(\frac{32s^2}{\delta}) + \log \left( \frac{512\sqrt{\epsilon}}{2\gamma \Delta} \right) \right] \]

\[ \leq \frac{c}{2^{2j} \Delta^2} \left[ 2\log(\frac{32s^2}{\delta}) + \log \left( \frac{512\sqrt{\epsilon}}{\Delta} \right) \right] \quad (70) \]

Then (69) becomes:

\[ T_1 \leq \frac{c}{\Delta^2} \left[ 2 \log(\frac{32T_{e,\delta}^2}{\delta}) + \log \left( \frac{512\sqrt{\epsilon}}{\Delta} \right) \right] \sum_{j=1}^{m_s} 2^{-2j} \sum_{s \in I'} \sum_{s \leq T_{e,\delta}} \mathbb{1}_{B_{s,j}} \]

\[ \leq \frac{c}{\Delta^2} \left[ 2 \log(\frac{32T_{e,\delta}^2}{\delta}) + \log \left( \frac{512\sqrt{\epsilon}}{\Delta} \right) \right] \sum_{j=1}^{m_s} 2^{-2j} \sum_{s \leq T_{e,\delta}} \mathbb{1}_{B_{s,j}} \quad (71) \]

By Lemma 4 there exists an event \( A_5 \) such that \( P(A_5) \geq 1 - \delta/8 \), and on \( A_5 \), we have for all \( j \leq m_e \),

\[ \sum_{s \leq T_{e,\delta}} \mathbb{1}_{B_{s,j}} \leq T_{e,\delta} \mathbb{P}_X(x, |\eta(x) - \frac{1}{2}| \leq 2^j \frac{1}{2} \Delta) + T_{e,\delta} \sqrt{\frac{1}{2T_{e,\delta}}} \log \left( \frac{8}{\delta} \right) \]

\[ \leq \frac{c}{\Delta^2} \left( T_{e,\delta} \mathbb{P}_X(x, |\eta(x) - \frac{1}{2}| \leq 2^j \frac{1}{2} \Delta) + T_{e,\delta} \sqrt{\frac{1}{2T_{e,\delta}}} \log \left( \frac{8}{\delta} \right) \right) \]

\[ \leq \frac{c}{\Delta^2} \left( T_{e,\delta} 2^{\beta j} \frac{1}{2^\beta} C \Delta^\beta + \frac{1}{\sqrt{p_e}} \log \left( \frac{8}{\delta} \right) \right) \quad \text{by (H3)} \]

\[ = 2^\beta (j-1) O \left( \left( \frac{1}{\epsilon} \right)^{\frac{2\alpha + d - \alpha \beta}{\alpha(\beta + 1)}} \right) + O \left( \left( \frac{1}{\epsilon} \right)^{\frac{4\alpha + d}{2\alpha(\beta + 1)}} \right) \]

Then, (71) becomes:

\[ T_1 \leq \tilde{O} \left( \left( \frac{1}{\epsilon} \right)^{\frac{2\alpha + d - \alpha \beta}{\alpha(\beta + 1)}} \right) \sum_{j=1}^{m_s} 2^{(\beta - 2)j} + \tilde{O} \left( \left( \frac{1}{\epsilon} \right)^{\frac{4\alpha + d}{2\alpha(\beta + 1)}} \right) \sum_{j=1}^{m_s} 2^{-2j} \]

\[ \leq \max \left( \tilde{O} \left( \left( \frac{1}{\epsilon} \right)^{\frac{2\alpha + d - \alpha \beta}{\alpha(\beta + 1)}} \right), \tilde{O} \left( \left( \frac{1}{\epsilon} \right)^{\frac{4\alpha + d}{2\alpha(\beta + 1)}} \right) \right), \quad (72) \]
where $\tilde{O}$ includes the logarithmic terms.
Secondly, by using the same argument as with the term $T_1$, the second term $T_2$ in (67) also satisfies the same relation (72). Then the term in (67) is less than:

$$\max \left( \tilde{O} \left( \left( \frac{1}{\epsilon} \right)^{\frac{2\alpha + d - \alpha \beta}{\alpha(\beta + 1)}} \right), \tilde{O} \left( \left( \frac{1}{\epsilon} \right)^{\frac{4\alpha + d}{2\alpha(\beta + 1)}} \right) \right)$$

(73)

Then if

$$n \geq \max \left( \tilde{O} \left( \left( \frac{1}{\epsilon} \right)^{\frac{2\alpha + d - \alpha \beta}{\alpha(\beta + 1)}} \right), \tilde{O} \left( \left( \frac{1}{\epsilon} \right)^{\frac{4\alpha + d}{2\alpha(\beta + 1)}} \right) \right)$$

we have that $n$ satisfies (66), and (48) is necessary satisfied.