Oscillations in the solar atmosphere: some diagnostic aspects

G. Worrall*†
Department of Physics and Astronomy, The Open University, Walton Hall, Milton Keynes MK7 6AA

ABSTRACT
Oscillations in velocity and intensity are detectable over a range of height in the solar atmosphere. Properties of the atmosphere can be deduced from the stratification revealed by the corresponding power and phase-difference spectra, potentially in a manner independent of optical spectroscopy. In this paper, published acoustic power spectra and phase-difference spectra, mainly derived from observations in the NaD₂ line, are interpreted through a simple model of the oscillations in an exploratory investigation along these lines. The difference between the ‘heights of formation’ at two positions in the line profile is deduced from the ratio of oscillatory power. The corresponding phase difference between oscillations in velocity at the two heights is compared with the phase spectrum derived from observations. Deduced acoustically, the ‘height of formation’ of the core of the D₂ line is about 700 km above that of the pressure-broadened wing.

The indicated acoustic cut-off frequency is about 4.5 mHz, a value significantly lower than that usually quoted. A variety of observational evidence is consistent with this value. The low value is probably a consequence of radiative damping, with a relaxation time of about 55 s.

The phase relations between the vertical and horizontal components of motion, the temperature and the pressure are considered. At low frequencies the velocity/intensity phase difference, for oscillations of moderate to high degree, is significantly affected by the horizontal motion, even if observations are restricted to a relatively small window at disc centre.

There is evidence that at the edge of the Doppler cores of strong absorption lines, fluctuations in intensity are in anti-phase with fluctuations in temperature.

Key words: Sun: atmosphere – Sun: oscillations.

1 INTRODUCTION
Since the confirmation by Deubner (1975) that the oscillations with periods around 5 min in the solar atmosphere are the fringes of global modes, the oscillations have been exploited as a probe of the interior of the Sun. The speed of sound, as a function of radial distance, is deduced through very precise determinations of the frequencies of modes identified by radial order, \( n \), and spherical harmonic degree, \( l \). Observations of the oscillations can also tell us something about the atmosphere, although in this context the modal frequencies in themselves carry no information. However, because the oscillations can be observed at different heights in the atmosphere (by simultaneous measurements in different spectral lines or at different positions in the same spectral line), there is the possibility of tracing the vertical profile of the waves of a particular frequency and horizontal scale. Ruiz-Cobo, Rodriguez-Hidalgo & Collados (1997) have explored some aspects of the stratification with optical depth. Their investigation is based on a time series of recorded profiles of the K i 7699-Å line. The parameters of a model atmosphere were adjusted to achieve a fit to each individual profile. In this way the amplitude of the oscillatory velocity was deduced as a function of continuum optical depth, the independent variable of the model atmospheres. The introduction to their paper has extensive references to earlier work on the use of oscillations in atmospheric diagnostics. Other techniques include wave travel-time measurements (Duvall et al. 1993; Jefferys et al. 1994) and the related time–distance analysis (Jefferys et al. 1997).

The differential equation governing steady-state oscillations in a model of the atmosphere has oscillations which vary with height, \( z \), essentially through the dependence of the speed of sound, \( c \), on \( z \). The solution of this equation at a particular frequency, the wavefunction, determines the ratio of the amplitudes of, and the phase difference between, oscillations at any two different heights. Each of these quantities can be determined from observations at two positions in the profile of an absorption line, and to be acceptable the wavefunction must be compatible with both. In this way a postulated \( c(z) \) could be tested and refined using power spectra and phase-difference spectra determined from observations at

*Present address: Birdswood, Eardisley, Hereford HR3 6NJ.
†E-mail: gw52@tutor.open.ac.uk

© 2002 RAS
Oscillations in the solar atmosphere

2 PROPERTIES OF THE ATMOSPHERE FROM POWER AND PHASE SPECTRA

Suppose the space and time dependence of oscillating quantities to be given by $exp[i(\omega t - k_x x - K z)]$, where $x$ is a horizontal coordinate in the direction of the horizontal component, $k_x$, of the wave vector, $z$ is the vertical coordinate taken positive upwards, $K$ is the vertical component of the wave vector, and $\omega$ is the angular frequency, taken to be real, as is appropriate with the interpretation of power and phase spectra in mind. The linearized equations of motion (see, for example, Gough 1991), within a plane-parallel atmosphere, lead to the dispersion equation:

$$\omega^4 - \omega^2 c^2 (k_x^2 + K^2) + (\gamma - 1) g^2 k_x^2 + i y g \omega^2 K = 0,$$

where $c$ is the speed of sound, $g$ the acceleration due to gravity and $\gamma$ the adiabatic exponent. At a given frequency the solution for $K$ is

$$K = i \frac{\omega_a}{c} \pm \frac{1}{c} \left[ \omega^2 - \omega_a^2 + \frac{N^2 \omega_a^2}{\omega^2} - \omega_a^2 \right]^{1/2},$$

where $\omega_a(=y g/2c)$ is the acoustic cut-off frequency, $N = (\gamma - 1)^{1/2} g/c$ is the buoyancy frequency and $\omega_b(=ck_b)$ is the Lamb frequency. The critical frequencies, between which $K$ has no real part, are given by

$$\omega_b^2 = \frac{1}{2} (\omega_a^2 + \omega_b^2) \pm \frac{1}{2} \left[ (\omega_a^2 + \omega_b^2)^2 - 4N^2 \omega_b^2 \right]^{1/2}. \quad (2)$$

Separating $K$ into its real and imaginary parts, $K = k_x + i \kappa$, the solution corresponding to upward propagation of energy is

$$k_x(\omega) = \pm \frac{1}{c} \left[ \omega^2 - \omega_a^2 + (\gamma - 1) \frac{g^2 k_x^2}{\omega^2} - c^2 k_x^2 \right]^{1/2}, \quad \kappa = \frac{\omega_a}{c}, \quad (3)$$

the positive value of $k_x$ being taken for $\omega > \omega_a$, the negative value for $\omega < \omega_a$.

For $\omega_a < \omega < \omega_b$, the evanescent region, where oscillatory power is greatest, the solution with oscillatory energy decreasing upward is

$$k_x = 0, \quad \kappa(\omega) = \frac{\omega_a}{c} - \frac{1}{c} \left[ \omega_a^2 - \omega^2 - (\gamma - 1) \frac{g^2 k_x^2}{\omega^2} + c^2 k_x^2 \right]^{1/2}. \quad (4)$$

Suppose that velocity power spectra, and $(V - V)$ phase spectra, are available from observations made at wavelength shifts $\Delta \lambda_i$ and

© 2002 RAS, MNRAS 335, 628–640
Δλ, from the centre of a spectral line (with Δλ > Δλ). According to this simple model, the phase difference between oscillations in velocity at these two positions, at an angular frequency ω0 (ω0 > ω), is Δφ = kωΔz, where Δz is the difference, z2 − z1, between the ‘heights of formation’ for the two line profile positions. If, at frequency ω0 (ω0 < ω), P1 is the oscillatory power in velocity at Δλ1, and P2 that at Δλ2, then, for the same model, ln(P2/P1) = 2kω8Δz. The ratio, ln(P2/P1)/Δφ = 2kω8/8(ω1P18), is β, say, of these two quantities is independent of Δz, and depends on γ and on the parameter characterizing the isothermal atmosphere, namely the scaleheight, H = P0/P0(γ), where P0 and P0 are, respectively, the pressure and density in the undisturbed atmosphere.

For oscillation modes of low angular degree (k, ≃ 0) the ratio β, considered as given by observations, determines, through equations (3) and (4), the cut-off frequency ω0, = −1/2γg/2γ(P0/P0). Otherwise the ratio β depends explicitly on the sound speed, and more sensitively the larger the value of k. Since c2 = (γP0/P0) we see that for these modes the ratio β depends on the product of γ and P0/P0, whereas for k, = 0 modes it depends on the ratio of these quantities. In principle, then, power and phase spectra for both kinds of mode, deduced from observations at the same two positions in the line profile, allow both γ and P0/P0 to be determined. The largest value of k, for which oscillation modes have frequencies well below the cut-off frequency is about 1.5 Mm (≃ 1000). Assuming the power ratio to have been determined at 3.5 mHz and the phase difference at 8 MHz, we find, using approximately known values, that δc/c ≃ −2βδ/β; thus the expression for β, considered as an equation for c, is not hopeless ill-conditioned. Of course, for a particular value of k, we would try to match Δφ as a function of frequency, and power and phase spectra for a range of values of k, are potentially available. Supposing the speed of sound to have been determined we can find Δz, in geometric units, from the same observations.

Power and phase spectra deduced from observations in the NaD2 line have been published by Deubner, Waldschik & Stevens (1996, hereafter DWS). The recorded traces of the line profile were analysed by the ‘lamdiameter’ method at 16 positions in the profile shown in fig. 1 of their paper. These positions range from the pressure-broadened wing of the line to the line centre, and so the observations are responding to atmospheric fluctuations from the photosphere to beyond the temperature minimum. Individual p modes cannot be picked out in these spectra (DWS, fig. 4), deduced from a 4.2-h time series, as the spectra are plotted on a very tight frequency scale. However, the ‘5-min peak’ is prominent and it is relatively simple to read off the power at this maximum. We find

\[ \log P_1 = -2.10 \pm 0.02 \text{ at the w15 position (far wing)}, \]
\[ \log P_2 = -1.26 \pm 0.02 \text{ at line centre}, \]
and so

\[ \ln(P_2/P_1) = 1.93 \pm 0.09 \text{ at } 3.2 \text{ mHz } (\omega_0/2\pi). \]  

(5)

From the corresponding (V − V) phase spectrum (DWS, fig. 6) we find

\[ \Delta \phi = (1.75 \pm 0.05) \pi \text{ at } 9 \text{ mHz } (\omega_0/2\pi). \]  

(6)

Comparison of the traces of (V − I) spectra with the grey-scale, two-dimensional spectra (DWS, figs 4 and 5) indicates that the traces are sections at small k, Since k2 (equation 4) and k3 (equation 3) vary very slowly with k, where k, is small, it is appropriate to set k, = 0 to obtain
Analyses. The mal atmosphere, that is $H/\Delta z$ or $\gamma$, is taken to be 5. The NaD$_2$ line, from Deubner et al. (1996, g. 4). The hatched region represents the measured ($V - V$) phase difference between line centre and the pressure-broadened wing in the NaD$_2$ line (Deubner et al., fig. 6).

This corresponds to $T_{5000} = 0.32$, using the relation between $T_{5000}$ and $T_{5000}$ in table 1 of Worrall (1973), and this occurs at $h \approx 60$ km in the HSRA. Accordingly, the ‘height of formation’ of the D$_2$ line core is $775 \pm 40$ km, a result which is consistent with spectroscopic analyses.

If $\gamma$ is taken to be 5/3, with the same specification of the isothermal atmosphere, that is $H = 110$ km, we find the speed of sound to be 7.07 km s$^{-1}$ and the acoustic cut-off at 5.14 MHz. The value of $\kappa$, from (4), is then $9.93 \times 10^{-4}$ km$^{-1}$, and the corresponding difference in height from the w15 position to the line centre is $\Delta z = 975 \pm 50$ km, making the ‘height of formation’ of the D$_2$ line core 1035 $\pm 50$ km. It would be very difficult to reconcile this height with spectroscopic analyses.

Fig. 1 shows the phase difference, $k_1(\omega)\Delta z$, with $k_1$ from equation (3), for $H = 110$ km, and for the two cases $\gamma = 1.28$ and 5/3, compared with the observed ($V - V$) phase difference between oscillations in the damping wing (w15 position) and the line centre, taken from fig. 6 of DWS. In each case the height difference, $\Delta z$, is that calculated from the ratio of velocity power. As far as I know, previous computations of ($V - V$) phase spectra have relied upon height differences deduced from traditional spectroscopic analysis. The straight lines through the origin are the corresponding high-frequency asymptotes. The calculated phase difference matches the trend of the observations quite well at higher frequencies where the real situation, that the solar atmosphere is not isothermal, is having a lesser effect.

### 3 VELOCITY/INTENSITY PHASE RELATIONS

As well as power spectra, fig. 4 of DWS presents ($V - I$) phase spectra for each line profile position. At the positions near line centre these spectra display jumps in phase of almost 180° at about 8 mHz, and it is with the interpretation of these spectacular features that the authors are mainly concerned. The spectrum for the w15 position has no such feature and it is here, in the pressure-broadened wing, that the link between temperature and intensity is probably simplest of all line profile positions. The phase difference between the vertical component of velocity and temperature has been computed by Marmolino & Severino (1991) for an isothermal atmosphere with uniform radiative relaxation time, and for the adiabatic case. The ($V - I$) phase spectrum from observations at the w15 position resembles the calculated adiabatic spectrum in that at high frequency the phase difference is small, decreasing with decreasing frequency, approaching $-90^\circ$ at about 4.5 mHz, where there is an abrupt change in slope. This again indicates that the cut-off frequency is about 4.5 mHz. At lower frequencies the ($V - I$) phase difference increases towards zero, whereas it would be expected to remain at $-90^\circ$. Subsequent observations indicate that it is probably the background spectrum, that is, the signal between the oscillation modes, which is the cause of this feature. The authors’ conclusion that the behaviour is close to adiabatic. The analysis covers the frequency range 2 to 4.7 mHz, so the expected change in slope towards higher frequencies is not apparent.

Power spectra and velocity/intensity phase spectra extending to higher frequencies, and for high-degree modes, have been generated using observations made with the MDI on SOHO by Straus et al. (1999). A detailed spectral trace is presented for $l \approx 300$ (Straus et al., fig. 4). As in the spectrum for the w15 position for NaD$_2$ observations, the ($V - I$) phase difference is small at high frequency, decreasing to a minimum around $-95^\circ$ at about 4.5 mHz. It is clear that at frequencies below 4 mHz the phase difference is much more negative at the p modes than between the modes: for instance, it is about $-35^\circ$ at the p$_3$ mode whereas between the modes it is about $+55^\circ$. Velocity/intensity spectra at even higher resolution have been obtained from observations with the Global Oscillations Network Group (GONG) network at moderate $l$-values (Oliviero et al. 1999). At $l = 200$ the ($V - I$) phase difference at the p$_3$ mode is about $-75^\circ$, while that between the modes is about $+75^\circ$.

Further prominent features of the calculated ($T - V$) phase spectra (Marmolino & Severino 1991) for modes of high degree are the 180° jumps in phase at the f mode and at the Lamb mode. Although there are no global modes at frequencies below that of the f mode, oscillations may yet be present, and if the frequency of the Lamb mode could be inferred from observed velocity/intensity phase spectra then the speed of sound would follow directly. Traces of ($V - I$) phase spectra at $l = 800$ and 1000 (Jefferys, private communication) show that the phase difference decreases from positive values at frequencies above the approximate frequency of the Lamb mode to negative values below it, and is zero at 1.1 mHz for $l = 800$, and at 1.4 mHz for $l = 1000$. If these frequencies are identified as those of the Lamb mode, the corresponding speed of sound is 6.0 km s$^{-1}$ ($l = 800$) and 6.1 km s$^{-1}$ ($l = 1000$), similar to the value deduced in the previous section. The spectral feature referred to here can be
seen as the narrow white band between the regions labelled 2 and 3
in the colour-coded, two-dimensional (v, l) phase spectrum shown
as the upper left panel of fig. 3 of Straus et al. (1999). However,
at low frequency, horizontal motion dominates; this is taken into
account in the next section.

3.1 Phase relations: vertical and horizontal motion
In this section the equations of motion, for the adiabatic case, are
used to link the amplitudes and phases of vertical and horizontal-
components of motion, and the temperature, to oscillations in
pressure.

In terms of the small perturbations in density, \( \rho' = \rho - \rho_0 \), and
pressure, \( P' = P - P_0 \), at a given position, the linearized equations
of motion for a plane-parallel atmosphere can be written

\[
\frac{\partial \rho'}{\partial t} + v_z \frac{\partial \rho_0}{\partial z} + \rho_0 \nabla \cdot \mathbf{v} = 0,
\]

(7)

\[
\frac{\partial \rho}{\partial t} + \nabla P' - \mathbf{g} \rho' = 0,
\]

(8)

\[
\frac{\partial P'}{\partial t} + v_x \frac{\partial P_0}{\partial z} + \gamma P_0 \nabla \cdot \mathbf{v} = 0,
\]

(9)

where \( \mathbf{g} = -g \mathbf{\hat{z}} \) is the acceleration due to gravity, and \( \mathbf{v} \) is the velocity,
with horizontal and vertical components \( v_x \) and \( v_z \), respectively.

Writing

\[
\frac{\mathbf{v}_x}{\mathbf{V}_0} = \frac{\mathbf{v}_z}{\mathbf{V}_0} = \frac{P'}{P_0 A_F} = \exp(i(\omega t - k_x x - k_z z)),
\]

(10)

the equations of motion allow the velocity amplitudes \( \mathbf{V}_0 \) and \( \mathbf{V}_0 \)
to be expressed in terms of the amplitude, \( A_F \), in relative pressure.

The \( x \)-component of equation (8) leads to

\[
\mathbf{V}_0 = \mathbf{V}_0 = \frac{P_0 k_x}{\rho_0 c} A_F,
\]

(11)

and equation (9) gives

\[
A_F = -i \frac{\gamma g}{\omega c^2} \mathbf{V}_0 + \frac{(\omega^2 - k_x^2)}{\gamma (k_x^2 + k_z^2)} A_F.
\]

Taking \( \mathbf{V}_0 \) from equation (11),

\[
\mathbf{V}_0 = \frac{(\omega^2 - k_x^2)}{\gamma \omega (k_x^2 + k_z^2)} A_F.
\]

Writing \( K = k_x + i \kappa \),

\[
\mathbf{V}_0 = \frac{(\omega^2 - k_x^2)}{\gamma \omega [k_x^2 + (\kappa/c^2 - \kappa^2)]} A_F = V_0 \exp(i\varphi) A_F,
\]

(12)

where

\[
\varphi = \arctan \left[ \frac{(\kappa/c^2 - \kappa)}{k_x} \right]
\]

(13)

is the phase lead of the vertical component of velocity over the
pressure.

As far as solar observations are concerned, the velocity phase
depends on the position on the disc at which oscillations are followed.
Consider the line of sight at angle \( \theta \) to the vertical in the atmosphere.
The resultant velocity in this direction is

\[
v(\theta) = v_z \sin \theta + v_x \cos \theta,
\]

which, on taking the real parts of equations (10), can be written

\[
v(\theta) = V \cos(\omega t - k_x x - k_z z + \delta),
\]

with \( \delta \) given by

\[
\cot \delta = \cot \varphi + \frac{(V_0/\rho_0)^2}{\sin \varphi \tan \theta}.
\]

(14)

The result for \( \delta \) has been written this way so as to separate the term
dependent on direction.

In the high-frequency limit each component of velocity is in phase
with the pressure, and \( \delta \approx \varphi = 0 \). At lower frequencies, but still well
above that of the fundamental mode, \( V_0/\rho_0 \) is small, and \( \delta \approx \varphi \)
(the vertical component dominating, excepting, of course, the case
in which observations are made close to the solar limb). In the
frequency range of evanescent waves \( \varphi = 90^\circ \), and \( \delta \) is significantly
dependent on direction if \( V_0/\rho_0 \) is not small: that is, at frequencies
near and below the fundamental mode. At frequencies below \( \omega_c \), in
the regime of internal gravity waves, \( V_0/\rho_0 \) is greater than unity,
increasing rapidly with decreasing frequency, and the velocity phase
is strongly dependent on direction.

3.2 Velocity–temperature phase relations
We have

\[
\frac{T'}{T_0} = -(\gamma - 1) \nabla \cdot \mathbf{\xi} = -(\gamma - 1) \left( \frac{\partial \xi_x}{\partial x} + \frac{\partial \xi_z}{\partial z} \right),
\]

where \( T' = T - T_0 \). Using \( \mathbf{v} = \partial \mathbf{\xi}/\partial t \), equations (10) lead to

\[
\frac{T'}{T_0} = \frac{(\gamma - 1)}{\omega} (k_x \mathbf{V}_0 + K \mathbf{V}_0) \exp[i(\omega t - k_x x - k_z z)].
\]

The relative phase is fixed by the amplitude factor:

\[
A_T = \frac{(\gamma - 1)}{\omega} (k_x \mathbf{V}_0 + K \mathbf{V}_0).
\]

Taking \( \mathbf{V}_0 \) from equation (11) and \( \mathbf{V}_0 \) from equation (12), and
writing \( K = k_x + i \kappa \), we find \( A_T = A_T \exp(i\eta) \), where

\[
A_T = \frac{\gamma - 1}{\omega} \left( \left[ \omega^2 k_x^2 + (\kappa/c^2 - \kappa) \right]^2 \right)^{1/2} A_F
\]

(15)

and

\[
\eta = \arctan \left[ \frac{g k_x (\omega^2 - \kappa)}{g k_x k_x^2 - (\kappa/c^2 - \kappa)} \right]
\]

(16)

is the phase lead of temperature oscillations over pressure.

The velocity leads the pressure by \( \delta \), given by equation (14), so
the temperature leads the velocity by \( (\eta - \delta) \). For vertical oscillations
(\( k_z = 0 \)) \( \delta = \varphi \), and we find that the temperature leads the velocity
by \( \arctan(k_x/k_z) \).

Fig. 2 shows velocity–temperature phase relations and relative
amplitudes for the case \( l = 800 \), calculated from these results, for
the model \( H = 110 \text{ km} \), \( \gamma = 1.28 \) adopted in Section 2. The \( V-T \)
phase spectrum for vertical motion corresponds to the adiabatic case
(\( l \approx 700 \)) in fig. 3 of Marmolino & Severino (1991). In the region
between the upper cut-off frequency and the Lamb mode (most of the
evanescence region) the horizontal motion is \( 90^\circ \) behind the vertical.
The \( 180^\circ \) jump at the f mode occurs in the phase of the temperature
relative to the pressure; that at the Lamb mode occurs in the vertical
velocity.

In the published MDI observations (Straus et al. 1999), the
intensity \( I \), from which the \( I - V \) phase spectra are generated,
Oscillations in the solar atmosphere

Figure 2. The phase difference, velocity – temperature (upper panel), and relative amplitudes (lower panel) as a function of frequency for oscillations of degree l = 800 in an isothermal atmosphere ($H = 110$ km, $g = 274$ m s$^{-2}$) with $\gamma = 1.28$. The symbol 'Vtl' labels the vertical component; 'Hzl' labels the horizontal component. In the upper panel solid lines indicate the $(V - T)$ phase difference for oscillations at the same height; dashed lines the $(V - T)$ phase difference for oscillations 70 km higher than $V$. The dots indicate the mean phase difference between line-of-sight velocity and temperature within a 400 × 400 arcsec$^2$ window at disc centre, weighted by fractional projected area and limb-darkening. The lower panel shows velocity amplitudes (km s$^{-1}$, left-hand scale) and the amplitude of the fractional temperature change (right-hand scale) as a multiple of the amplitude, $A_p$, of the oscillation in fractional pressure.

is measured at line centre, and so will be responding to fluctuations in temperature somewhat higher in the atmosphere than the velocity fluctuations, deduced from Doppler shifts at the sides of the line profile. This has an effect on the $(V - T)$ phase in the regions of propagating waves. The height difference, 70 km, was chosen to match, approximately, the observed phase difference at higher frequency. The effect is severe at very low frequencies since $k_z \rightarrow -\infty$ as $\nu \rightarrow 0$. Unless the observations are representing the velocity and intensity fluctuations at exactly the same height, there will be very rapid variations in the phase difference as $\nu \rightarrow 0$.

The field of view of the detector is restricted to 400 × 400 arcsec$^2$. The phase lead, $\delta$, of the line-of-sight velocity over the pressure depends upon position in the aperture. A mean value, weighted by fractional projected area and limb-darkening, was calculated as

$$\bar{\delta} = \int_0^{\theta_c} \delta(\theta) P(\theta) L(\theta) \, d\theta,$$

where $\theta_c$ is the angle to the local vertical in the atmosphere at the corners of the square aperture, and $\delta(\theta)$ is taken from equation (14). $P(\theta) \delta \theta$ is the fractional projected area between angles $\theta$ and $\theta + \delta \theta$ within the aperture, given by

$$P(\theta) = \frac{(\pi - 4\alpha) \sin 2\theta}{4 \sin^2 \theta},$$

where $\theta_c$ is the value of $\theta$ at the middle of the sides of the square, with

$\alpha = 0$ for $0 < \theta < \theta_c$

and

$\alpha = \arccos(\sin \theta_c / \sin \theta)$ for $\theta_c < \theta < \theta_c$.

The limb-darkening factor was taken to be $L(\theta) = A + B \cos \theta + C \cos^2 \theta$, with the coefficients $A$, $B$ and $C$ from table III of Pierce & Slaughter (1977) at wavelength 6791.4 Å, considered to be sufficiently close to the wavelength of the line.
For the 400 × 400 arcsec$^2$ window at disc centre, taking the angular radius of the Sun to be 960 arcsec, $\theta_0$ is only 12.0 ($\theta_0 \cong 17^\circ$); nevertheless, as can be seen in Fig. 2, the horizontal motion has a significant effect on the weighted mean ($V - T$) phase difference at lower frequencies. In the region of maximum power, between the upper cut-off frequency and the f mode, the ($V - T$) phase difference is more negative by a few degrees. Between the f mode and the Lamb mode, the ‘plateau’ (Deubner et al. 1990) appears as a concave slope declining towards lower frequencies, rather as the observations show. Radiative damping leads to a similar effect (Marmolino & Severino 1991, fig. 2d) but it is reinforced by the horizontal motion. At the Lamb mode the vertical component of velocity is zero, the horizontal component and the temperature are each in phase with the pressure, and so the ($V - T$) phase difference is zero, supporting the suggestion, just before Section 3.1, that this position of zero phase difference in observed spectra can be identified as the Lamb mode.

### 3.3 The f mode

At the frequency of the f mode, $\omega_f = \sqrt{(g\kappa)}$, equation (4) gives

$$\kappa_l = \kappa_{-l} = \frac{g}{2c^2} - \frac{1}{c} \left( \frac{g^2}{4c^2} - \frac{g\kappa}{c^2} - k_z^2 \right)^{1/2}$$

This means, from equation (15), that the temperature oscillations vanish, and, from equations (11) and (12), that the amplitudes of the vertical and horizontal motion are the same. In the power spectra from MDI data for $l \simeq 300$ (Straus et al. 1999, fig. 4), and from the GONG data for $l = 200$ (Oliviero et al. 1999, fig. 1), it is noticeable that while the f mode is apparent in the velocity power spectra, it is absent from the intensity power spectra, in accord with this result. However, the f mode is apparent in both intensity and velocity power spectra for modes of high degree ($l \gtrsim 650$), and the MDI data indicate that for these modes the ($V - I$) phase difference is well defined at about $-50^\circ$ (Straus et al. 1999, fig. 2). Although, in this simple model, the amplitude of the temperature oscillation is zero, the phase of both the velocity and the temperature relative to the pressure is defined. From equation (16) the temperature leads the pressure by

$$\eta_l = \arctan \left[ \frac{k_z (g/c^2 - k_z)}{k_z^2 - (g/c^2 - \kappa)(\kappa - k_z)} \right] = \arctan \left( \frac{g/c^2 - \kappa}{k_z} \right) = \phi_l,$$

where $\phi_l$ is the phase lead of the vertical velocity over the pressure (equation 13); that is, the temperature is in phase with the vertical component of the velocity. Since $k_z$ is zero, $\eta_l = \phi_l = 90^\circ$.

From equation (14) the resultant velocity at angle $\theta$ to the local vertical leads the pressure by $\delta_l$, where $\cos \delta_l = \tan \theta$. Thus $\delta_l = (90^\circ - \theta)$ and the velocity at $\theta$ leads the temperature by $\delta_l - \eta_l = (90^\circ - \theta) - \eta_l = -\theta$. Within the 400-arcsec square aperture the mean value of $\theta$ weighted by projected area and limb-darkening is $9.1^\circ$, so in this simple model the ($V - T$) phase difference $-9.1^\circ$, quite different from the measured $-50^\circ$. In the caption in their fig. 2, Straus et al. (1999) apparently imply that the expected adiabatic value, for vertical oscillations, is $-90^\circ$.

### 4 OTHER EVIDENCE FOR A LOW SOLAR CUT-OFF FREQUENCY

#### 4.1 Variation with frequency of ratio of oscillatory power at two different heights

In an isothermal atmosphere the amplitude of oscillation varies as $\exp(\kappa z)$, so that the ratio of oscillatory power at heights separated by $\Delta z$ is $\exp(2\kappa \Delta z)$. In the evanescent region the factor $\kappa$ increases very rapidly with frequency as the cut-off is approached from below, and is constant above (see Section 2, equation 4). Therefore it should be possible to locate the cut-off from the observed ratio of oscillatory power as a function of frequency.

There are hardly any published power spectra that allow this to be done. The velocity power spectra from the observations in NaD$_2$ in fig. 4 of DWS are on so tight a scale that the technique is hardly feasible. However, fig. 4 of Straus et al. (1999) shows power spectra, deduced from MDI observations, for $l \simeq 300$, on a sufficiently large scale. There are two intensity power spectra: one at line centre ($I$); the other in the local continuum ($C$). The fluctuations at line centre will be responding to oscillations higher in the atmosphere than those in the continuum, and a variation in the ratio of oscillatory power such as described in the previous paragraph is apparent at a glance.

If we express the amplitude of temperature oscillations as $|T'|/|T| = A \exp(\kappa z)$ then the amplitude of oscillations can be written as $|\Delta T'|/|T| = A \exp(\kappa z_0)$, where $R$ is the ratio of intensity amplitude to temperature amplitude, and $z_0$ is the height in the atmosphere corresponding to the wavelength of observation (for observations made at disc centre, $z_0$ is approximately the height at which the optical depth at the wavelength of observation is unity). Thus the ratio of intensity power at two different wavelengths is

$$|\Delta I'_2|/|\Delta I'_1|^2 = (R_2/R_1)^2 \exp(2\kappa \Delta z_0),$$

from which

$$\Delta \log P = 2 \log (R_2/R_1) + 2 \log(e) \kappa \Delta z_0.$$  

Although the ratio $R$ varies very strongly with wavelength shift from spectral line centre, it varies only weakly with acoustic wave frequency, so that the frequency dependence of $\Delta \log P$ follows that of the amplitude growth factor $\kappa$. For frequencies below the cut-off $\kappa$ is given by equation (4) in Section 2, but, at $l \simeq 300$, $\omega_1$ is very small compared with $\omega_2$ and $N^2$, so that

$$\kappa = \kappa_{-l} = \frac{\omega_1}{c} \left( \frac{\omega_1^2 - \omega_2^2}{\omega_1^2 - \omega_2^2} \right)^{1/2},$$

from which

$$\frac{d\kappa}{\omega} = 2\pi \frac{d\omega}{d\omega} \simeq 2\pi \frac{\omega}{\omega} \frac{\omega_1}{\omega_2} \frac{\omega_2}{\omega_1} \hbar^2,$$

where

$$\hbar^2 = \frac{\omega_2 - \omega_1}{\omega_1^2} (\omega_1^2 - \omega_2^2)^{1/2}.$$  

We see that $\kappa$ increases ever more rapidly as the frequency approaches the cut-off, with $d\kappa/d\omega \to \infty$ as $\omega \to \omega_1$.

Fig. 3 shows $(\log P_1 - \log P_C)$ plotted against frequency, from the power spectra of fig. 4 of Straus et al. (1999) ($\log P$ is plotted in these spectra). The values were read at the frequencies of the p modes and pseudo-modes, as indicated by the dotted lines in that figure. If one imagines trying to fit a curve to these points, which matches the frequency dependence of $\kappa$ as described above, it is apparent that the cut-off frequency, indicated by the position of a vertical tangent, is at about 4.5 mHz.

Velocity power spectra are more appropriate for this purpose because the extra complication introduced by the link between temperature and intensity fluctuations, leading to the factor $R$, is avoided.
Velocity power spectra from lambdameter observations in the Mgb line were kindly made available by Sebastian Steffens. The observations were made at the Observatorio del Teide, Tenerife, and the power spectra are based on 196 min scanning 10 positions separated by 7.5 arcsec with a slit 1.56 arcsec wide. The power derived from the light curves at all solar positions was averaged to produce the spectra made available. The power was read off at intervals of 0.5 mHz, from 1 to 7 mHz, from the power spectrum at the line centre (P1) and that in the wing at 0.260 Å from line centre (P2), with the result shown in Fig. 4. Since the power was plotted on a linear scale, it is natural to plot ln(power ratio). In this case, ln(P2/P1) approaches zero at zero frequency, as expected for modes of low angular degree, in the absence of any factors R. The curve is 2κ Δz0, with κ calculated for γ = 1.28 and H = 110 km as in Section 2 (ωκ = 4.5 mHz), and Δz0 = 400 km. This height difference is not an optimum-fit value, but is that which follows from the measured power ratio where the power is greatest, at about 3.5 mHz. Above the cut-off frequency the model gives 3.65 for ln(P2/P1), which is significantly greater than the measured values. This is as expected in the presence of radiative damping. In this simple model, at frequencies above the cut-off, κ has the value 1/2H, and so is independent of the value of γ, whereas, as is apparent from fig. 1 of Souffrin (1972), a radiative relaxation time of about 55 s corresponds to a value of κ, above the cut-off frequency, approximately 0.78 times the adiabatic value, leading to ln(P2/P1) ≈ 2.85.

4.2 Time–distance analysis and chromospheric modes

Time–distance analysis (Duvall et al. 1993) has been carried out on solar acoustic waves at frequencies above the cut-off (Jefferies et al. 1997). The results show satellite ridges in t–Δ plots indicating that partial reflection is occurring at a level high in the atmosphere as well as at the photosphere. The difference in time between satellite feature and main feature, at a particular angular separation, Δ, corresponds to the round-trip time for wave energy between the photosphere and upper reflecting layer. In this analysis high-frequency waves were selected by a Gaussian filter centred on ν = 6.75 mHz, l ≈ 125, with σ1 = 0.75 mHz and σ2 = 33. For l ≈ 125 (κσ ≈ 0.18 Mm–1) the Lamb frequency, about 0.18 mHz, is so far below the acoustic cut-off that the dispersion equation for vertical oscillations is a good approximation: that is, we can take kσ = (ω2 − 2ω2)1/2/c in evaluating the group velocity, vg = ∂ω/∂kσ, to obtain

\[ v_g = c\left(\omega^2 - 2\omega^2\right)^{1/2}/\omega. \]

As the frequency approaches the cut-off, vg tends to zero with dv/vdω tending to infinity: that is, wave propagation is very dispersive at frequencies just above the cut-off, with the round-trip time for wave energy between the two partially reflecting layers approaching infinity at the cut-off frequency. Thus, if waves of frequency just above the cut-off frequency contribute significantly to the time–distance analysis, the satellite feature will be very asymmetric, with a long tail on the side of greater time. In the analysis referred to, the central frequency, 6.75 mHz, is only 2γ above what is generally supposed to be the cut-off frequency and, since the oscillatory power decreases with increasing frequency, waves just above the cut-off would be making a significant contribution. From an examination of the profiles of the satellite features, at fixed Δ, it cannot be stated that these profiles are definitely asymmetric, although quite often there are hints of long tails (Jefferies, private communication). It could be argued that this in itself is evidence for an acoustic cut-off at a significantly lower frequency than is generally supposed. For a given spread of frequencies around the central sampling frequency, the dispersion of wave traveltimes is very much

Figure 3. Difference in log of oscillatory power (intensity) between line centre and continuum for the Ni i 6768 Å line as a function of wave frequency. Values read from Straus et al. (1999, fig. 4).

Figure 4. Log of ratio of oscillatory power (velocity) between line centre and 0.260 Å from line centre, for the Mgb line (from power spectra obtained by S. Steffens). The line shows 2κ(ω)Δz, with Δz = 400 km, for an isothermal atmosphere (H = 110 km, g = 274 m s–2) with γ = 1.28.
reduced if the cut-off is at about 4.5 mHz, which is 3σ below 6.75 mHz.

Partial reflection at two levels in the solar atmosphere means that standing wave modes should exist between those two levels. Partial reflection occurs, for waves at frequencies above the cut-off, at the edges of the acoustic barrier (Worrall 1991) which is about 2000 km wide. Chromospheric modes occur at those frequencies, νc, at which the barrier width, L, is a whole number of half-wavelengths: that is, for kc = nπ/L, where n is an integer. Since kc = (ω2 − ωc2)1/2/c, this leads to

\[ νc^2 = νa^2 + n^2(c/2L)^2. \]  

(18)

Evidence for chromospheric modes at about 4.7 and 5.8 mHz appeared in the early observations (Ca II 3933 Å, Doppler) of solar oscillations (see Noyes 1967). In the following decades a number of observers have found these modes elusive. Harvey et al. (1993) have shown that the chromospheric modes are detectable, in high-resolution power spectra, in the background, that is between the p mode, or pseudo-mode, peaks. From a 1988 time series (17 d) of measures of intensity in the core of Ca II 3933 Å made at the South Pole, they found a mode at about 4.8 mHz. The mode at about 5.8 mHz, of the early observations, was not detected; this may be because observations in the core of Ca II 3933 Å are responding to fluctuations close to the temperature node at height L/2 in this mode. (The Doppler observations have been made further from the absorption line centre, and so were responding to fluctuations deeper in the atmosphere.) However, a mode at about 5.5 mHz was detected in intensity observations in one wing of Hz. From an analysis of a 1994 time series (19 d) of intensity measurements in the core of Ca II 3933 Å, Harvey et al. (1996) found modes at about 5 and 7 mHz. These authors also detected the lowest frequency mode, at about 4.6 mHz, in an analysis of GONG data, covering 36 d.

The lowest frequency mode has also been found in observations of a pure rotation line of OH (Deming et al. 1988). The authors quote 4.3 mHz for the frequency, although the reproduced power spectra seem to indicate about 4.5 mHz. Radio noise power spectra at 3.3- and 3.5-mm wavelength, arising from the low chromosphere, indicate a component at 5.6 mHz (Simon & Shimabukuro 1971).

All in all, these observations suggest that we might make the following identifications: ν1 = 4.8 ± 0.2 mHz, ν2 = 5.6 ± 0.2 mHz, ν3 = 7.0 ± 0.2 mHz. The degree to which these values conform to the simple model of equation (18) is shown in Fig. 5. The lines indicated lead to ν1 = 4.4 ± 0.3 mHz and (c/2L)2 = (3.28 ± 0.5) × 10−6 s−2. Again we find a low cut-off frequency consistent with the previous results. The linear fit of Fig. 5 suggests that ν4 ≈ 8.5 mHz. There is evidence for a chromospheric mode at about this frequency in the intensity power spectra from the NaD2 observations (DWS) at the w1 to w4 lamblambdaer positions. The velocity–power ratio (5-min peak to line wing) discussed in Section 2 indicates that this region of the line profile is responding to fluctuations at about 750 km above the photosphere, which is just where the n = 4 mode would have a temperature antinode. The lower antinode, at about 250 km above the photosphere, would be just below the region sampled by the w14 position, but the power at this lower level would be much less than that at the higher antinode.

The slope of the linear fit to the data of Fig. 5 allows the round-trip traveltime, Δt, between the two partially reflecting levels to be estimated. This time is given by

\[ Δt = \frac{2L}{νc} = \frac{2Lω}{c(ω^2 − ω_c^2)^{1/2}} = \frac{ω}{(ω^2 − ω_c^2)^{1/2}} \frac{1}{√S}. \]

Figure 5. Linear fit to squared frequencies of chromospheric modes. See equation (18). The three lines are the best-fitting line and extreme lines drawn by eye.

where S is the slope of the linear fit. Taking ν = 6.75 mHz and, from Fig. 5, ν4 = 4.4 ± 0.3 mHz, S = (3.28 ± 0.5) × 10−6 s−2, we find Δt = 730 ± 100 s.

From the separation of satellite and main features in the t–Δ plots, Jefferies et al. (1997) find [by their method (a)] Δt ≈ 11 min. [Method (b) gives a much shorter time but assumes that the atmosphere is non-dispersive.]

4.3 Variation of p-mode linewidths with frequency

The width of the peak corresponding to a p mode in oscillation power spectra is a measure of the rate of energy loss from the oscillation mode. As the frequency increases towards the cut-off frequency at the top of the acoustic barrier, the progressively greater leakage of wave energy into the corona must make a major contribution to the linewidth; p-mode linewidths do increase very steeply with frequency from about 3 to 5.5 mHz (Duvall et al. 1991). It might be expected that there is a clue to the cut-off frequency in this variation.

In a paper concerned with the effect of the position of the acoustic source on the transition from p modes to pseudo-modes, or high-frequency peaks, Vorontsov et al. (1998) represent the acoustic barrier as parabolic (in ω2 as a function of height) so as to exploit the simple result for the reflection coefficient available from the equivalent potential-barrier problem in quantum mechanics. It is interesting to consider the implications of the same model for p-mode linewidths.

Suppose that the amplitude reflection coefficient of the acoustic barrier is r, and the transmission coefficient is (1 − r), and consider a wave of angular frequency ω and amplitude A incident on the barrier from beneath. After multiple transits of the sub-photospheric cavity, of acoustic depth T, the resultant displacement in the region of observation can be written

\[ ψ = A(1 − r)[1 + r \exp(−i2ωT) + r^2 \exp(−i4ωT) + \cdots] \exp(iωt) \]

\[ = \frac{A(1 − r)}{1 − r \exp(−i2ωT)} \exp(iωt), \]

\[ \]
and the power
\[ P = \psi \psi^* = \frac{A^2(1 - r)^2}{1 + r^2 - 2r \cos(2\omega T)}. \] (19)

An oscillation mode exists at those frequencies, \( \omega_p \), such that \( \cos(2\omega T) = 1 \), and \( P \) has its maximum value \( A^2 \); these frequencies are \( \omega_p = n\pi/T \), \( \nu_p = n/2T \), where \( n \) is an integer. The modes are separated in frequency by \( 1/2T \). This is what Vorontsov et al. (1998) refer to as the one-way spectrum; no account has been taken of the wave initially travelling downwards from the source which is responsible for the high-frequency peaks, or pseudo-modes, in this model.

According to equation (19) the power falls to half of its maximum at \( \omega = \omega_b \), such that
\[ \frac{(1 - r)^2}{1 + r^2 - 2r \cos(2\omega_bT)} = \frac{1}{2}, \]
from which
\[ \cos(2\omega_bT) = 1 - \frac{(1 - r)^2}{2r}. \]

Putting \( \omega_b = \omega_p \pm \delta \omega/2 \), so that \( \delta \omega \) is the full width at half-maximum power, we have
\[ \cos(2\omega_bT) = \cos[2(\omega_p \pm \delta \omega/2)/T] = \cos(\delta \omega T), \]

since \( 2\omega_b T = 2\pi \). Thus
\[ \cos(\delta \omega T) = 1 - \frac{(1 - r)^2}{2r}, \]
or
\[ 2 \sin^2 \left( \frac{\delta \omega T}{2} \right) = \frac{(1 - r)^2}{2r}, \]

from which the frequency linewidth is
\[ \Gamma = \frac{\delta \omega}{2\pi} = \frac{1}{\pi T} \arcsin \left( \frac{1 - r}{2\sqrt{r}} \right). \]

In this way, for this simple model, we can relate the mode linewidth to the frequency separation of the modes \((1/2T)\) and the barrier reflection coefficient \( r \).

For the parabolic barrier, previously referred to, we have
\[ r^2 = \left[ 1 + \exp \left( \frac{\omega^2 - \omega_0^2}{\omega_0^2} \right) \right]^{-1} \]
[Vorontsov et al. 1998, equation (14)], but note that this result, taken from Landau & Lifshitz (1977), gives the flux reflection coefficient, \( r^2 \), rather than \( r \). Here \( \omega_b \) is the frequency at the top of the parabolic barrier and \( \omega_0 \) is a parametric value which fixes the width of the barrier; \( r^2 = 1/2 \) at \( \omega_0 \), and rises to \( e/(e - 1) \) at \( \omega = (\omega_0^2 - \omega_0^2)^{1/2} \).

Fig. 6 shows calculated linewidths with the values of \( \omega_m \) and \( \omega_0 \) chosen to give a reasonable fit to the measured linewidths at the lower frequencies of this range. The measurements are corrected average widths for \( 100 \leq \ell \leq 250 \) from observations of intensity fluctuations in the core of Ca II 3933 Å made at the South Pole in 1988 (kindly made available by Stuart Jeffries). For this range of \( \ell \)-values the mean frequency separation of p modes is about 0.32 mHz, corresponding to an acoustic depth, \( T \), of about 1600 s. The indicated frequency at the top of the acoustic barrier is once again well below the generally assumed cut-off.

5 DISCUSSION AND CONCLUSIONS

We have seen that there is a variety of observational evidence for the existence of an acoustic cut-off frequency in the solar atmosphere.

![Figure 6](https://example.com/figure6.png)

**Figure 6.** p-mode linewidths, \( \Gamma \), as a function of frequency, corresponding to leakage of wave energy at a parabolic, acoustic barrier (see text). Solid lines are calculated for \( \nu_m = 4.35 \) mHz, \( \nu_0 = 2.2 \) mHz; \( T \) is the acoustic depth of the sub-photospheric cavity. Dashed lines are calculated for \( \nu_m = 5.0 \) mHz, \( \nu_0 = 2.2 \) mHz (the steeper of the two) and \( \nu_m = 5.0 \) mHz, \( \nu_0 = 3.0 \) mHz (the values adopted by Vorontsov et al. 1998). In each case \( T = 1600 \) s. The points are observationally determined, corrected linewidths for modes with \( 100 < \ell < 250 \) (S.M. Jeffries, private communication).
Such evidence is required if an isothermal representation of the atmosphere is to be meaningful at all. The indicated cut-off frequency is about 4.5 mHz, significantly lower than the value corresponding to $\gamma = 5/3$. It seems likely that the lower effective value of $\gamma(z) < 1.28$ is due to radiative damping with a relaxation time of about 55 s. The speed of sound, deduced from the apparent Lamb frequency, is an accord with this value. In the exploratory investigation of this paper, adiabatic oscillations, with an effective value of $\gamma$, in an isothermal model atmosphere were considered so as to make use of the simple analytical results then available.

An analysis of some aspects of published, power and phase-difference spectra deduced from observations in the NaD$_2$ line is described in Section 2. An isothermal model of the atmosphere allows a fairly self-consistent interpretation of the ratio of oscillatory power at the 5-min peak and the $(V - V)$ phase difference between the core and the wing of the line. It is possible to deduce the difference in height corresponding to these two positions in the spectral line profile. Acoustic spectra can be a powerful tool not only in probing the structure of the atmosphere of the quiet Sun, but also in providing a method of linking physical depth with optical depth within the profile of an absorption line, as an alternative to classical, spectroscopic analysis. The difference in physical depth obtained here, about 700 km, is reliable only in so far as the adopted cut-off frequency certainly varies with height. Reference was made in Section 3.1 to the analysis of a time-series of profiles of the K $\lambda 7699$-Å line by Ruiz-Cobo et al. (1997). This yielded a sequence of velocity power spectra labelled by continuum optical depth (Ruiz-Cobo et al. 1997, fig. 7). The ratio of power (at 3.5 mHz) at log $\tau = -4.2$ to that at log $\tau = 0$ is approximately 8.0, corresponding to a difference in height of about 600 km for the isothermal model adopted here. Since log $\tau = -4.2$ is reached just above the temperature minimum, this value is not unreasonable.

The analysis could be founded on a more realistic model of the atmosphere leading to the sound speed as a function of height. It would then be necessary, in general, to obtain the corresponding wavefunctions numerically, and the problem of specifying the boundary condition in the outer atmosphere arises. The analysis is based on the phase difference between oscillations at two different heights at higher frequencies (well above the cut-off), and on the ratio of oscillatory power at the same two heights at lower frequencies (well below the cut-off). The former depends on conditions only between the two heights, and so is insensitive to the outer boundary condition. The corresponding ratio of oscillatory power does depend on the profile of the wave which, strictly speaking, depends on conditions high in the atmosphere, in the corona, where the structure is neither uniform nor constant. However, for the frequencies well below the cut-off, where the oscillatory power is greatest, the energy density diminishes very rapidly with height.

For oscillations of 5-min period the energy density at the transition to the corona is less than $10^{-5}$ that at the photosphere. [It is consistent that observations from instruments on board SOHO of chromospheric and coronal lines in the ultraviolet indicate that the power spectrum peaks arising from p modes are not detected above the transition region to the corona (Deubner & Steffens 1999).] Thus the calculated ratio of oscillatory power at the two positions in the line profile is probably not very sensitive to the outer boundary condition. For instance, the power ratio from the JWKH solution depends only on conditions at and between $z_1$ and $z_2$, and is dominated, as long as $z_2$ is substantially greater than $z_1$, by the factor

$$\exp \left[ \int_{z_1}^{z_2} \kappa(z) \, dz \right],$$

where $\kappa(z)$ is the local value of the imaginary part of the vertical component of the wave vector from the appropriate local dispersion equation (as in Gough 1991, section 5.4). Moreover, it is possible to argue that if the profile of these low-frequency waves, in the region of the photosphere, and hence in deeper regions too, were sensitive to the variable conditions in the corona then there would be no long-lived oscillation modes.

Observationally, a time-series of spectral line profiles is required. Useful analyses can result from power and phase-difference spectra with the frequency resolution afforded by a time-series a few hours long. Since the case of the isothermal model indicates that it is the waves of large horizontal scale (small values of $k_x$) that determine the ratio of $\gamma$ to $P_0/\rho_0$, while those with large $k_x$ give a handle on the product, we would seek to make use of observations over a wide range of values of $k_x$, resolved according to $k_x$.

Velocity/intensity phase-difference spectra are also widely used as a diagnostic tool. In Section 3 the relative amplitudes and phases of vertical and horizontal components of motion, the temperature and the pressure were considered. A conspicuous feature of the calculated $(V - T)$ phase spectra, Fig. 2, is the sharp change in gradient at about $-90^\circ$ at the cut-off frequency. There is no hint of such a feature in the observed velocity/intensity spectra at what is the generally supposed cut-off, somewhat above 5 mHz, corresponding to $\gamma = 5/3$. The spectra deduced from MDI observations (Straus et al. 1999) have a minimum in the $(V - I)$ phase difference of about $-95^\circ$ at about 4.5 mHz for $l = 200$, the frequency at which this minimum occurs increasing slightly as the degree $l$ increases. The position of the minimum matches quite closely that of the sharp change in gradient in the calculated $(V - T)$ phase-difference spectra at the cut-off.

At low frequencies the phase of the resultant, line-of-sight velocity is significantly influenced by the horizontal motion, even if observations are restricted to a relatively small window at the centre of the solar disc. In any interpretation of velocity/intensity phase differences at low frequencies, the size and shape of the window of observation should be taken into account.

ACKNOWLEDGMENTS

I would like to thank S. M. Jeffries for data and preprints, for pointing out a number of relevant papers, and for several discussions. I have also benefited from discussions with D. O. Gough, J. W. Harvey and F. -L. Deubner and the Wurzburg group; I am grateful to S. Steffens for data and to F. Schmitz for reprints. Ian van Breda has always been ready to lend an ear, and I am indebted to him for much technical help and advice in preparing this paper.

REFERENCES

Curtis G. W., Jeffries J. T., 1967, ApJ, 150, 1061
Deming D., Glencar D. A., Kaufl H.-U., Espenak F., 1988, in Christensen-Dalsgaard J., Frandsen S., eds, Proc. IAU Symp. 123, Advances in Helio- and Asteroseismology. Reidel, Dordrecht, p. 425
Deubner F.-L., 1975, A&A, 44, 371
Deubner F.-L., Steffens S., 1999, in Wilson A., ed., Proc. 9th European Meeting on Solar Physics, ESA SP-448, ESA, Noordwijk, p. 149
Deubner F.-L., Fleck B., Marmolino C., Severino G., 1990, A&A, 236, 509
Deubner F.-L., Weidenschlag Th., Steffens S., 1996, A&A, 307, 936 (DWS)

© 2002 RAS, MNRAS 335, 628–640
Oscillations in the solar atmosphere

APPENDIX A

1 Temperature variations and intensity variations

As the temperature fluctuates, the observed intensity will be influenced by the consequent variations in the source function, \( S(\tau) \), and in the absorption coefficient \( \alpha \); here \( \tau \) is the optical depth at radiation frequency \( \nu \). A positive fluctuation in temperature will lift the source function, but if the absorption coefficient also increases we ‘see’ to a lesser depth, where the source function is lower (in the usual case in which the source function is increasing with depth); the effects on \( S \) and \( \alpha \) are in competition. Frandsen (1988) pointed out the possibility of the phase difference between velocity and intensity being distorted by this effect. There is evidence in published power spectra and velocity/intensity phase difference spectra that the correlation between temperature and intensity is in fact reversed at the edges of the Doppler cores of strong absorption lines. First a simple, approximate argument is given to show that such an anticorrelation is not unreasonable. The argument exploits the Eddington–Barbier relation, which has often been used in exploratory investigations, as a simple link between the source function and the observed intensity; see Jeffries (1968) for several examples.

According to the Eddington–Barbier relation, for observations made at the centre of the solar disc, \( I \sim S(\tau=1) \). Thus, if the temperature at \( \tau = 1 \) increases by \( \delta T \), we have

\[
\delta I \sim (\delta S)_{\tau=1} = \left( \frac{\partial S}{\partial T} \right)_{\tau=1} \delta T = \left( \frac{\partial S}{\partial \tau} \right)_{\tau=1} \delta \tau
\]

where \( \delta \tau \) is the corresponding increase in optical depth. (Here subscripts \( \nu \) have been omitted from the quantities \( \tau \), \( S \) and \( I \).) The second term has a minus sign since, if \( \delta \tau \) is positive, unit optical depth now occurs where the unperturbed optical depth is smaller by \( \delta \tau \). Taking \( D \) to be geometric depth into the atmosphere,

\[
\delta \tau = \int_0^{\tau_1} \frac{d\tau}{\tau} \delta \alpha = \int_0^{\tau_1} \frac{\delta \alpha}{\alpha} d\tau
\]

where \( \tau_1 \) is the depth at which \( \tau = 1 \), and \( \delta \alpha/\alpha \) is non-zero only where \( \delta \tau \) is non-zero. Making the rather rough approximation of a uniform temperature perturbation over the region where \( \alpha \) is significant,

\[
\delta \tau \simeq \frac{\delta \alpha}{\alpha} \int_0^{\tau_1} \alpha d\tau = \frac{\delta \alpha}{\alpha} \tau = \frac{\delta \alpha}{\alpha} \quad \text{at} \quad \tau = 1
\]

Thus

\[
\delta I \simeq \frac{\delta S}{S} \left( \frac{\partial S}{\partial \tau} \right)_{\tau=1} \left( \frac{1}{D} \frac{\delta \alpha}{\alpha} \right) \tau = \frac{\delta \alpha}{\alpha}
\]

Where local thermodynamic equilibrium (LTE) can be assumed to prevail, that is for continuum radiation, in the far wings of strong absorption lines, and, approximately, for absorption lines that are not too strong, the source function can be taken to be the Planck function \( B_{\nu}(T) = B \), say.

Then

\[
\frac{\partial S}{\partial T} \delta T = \frac{\partial B}{\partial T} \delta T = FB \frac{\delta T}{T}
\]

where

\[
F = \frac{h v}{k T} \exp(hv/kT) - 1
\]

and so

\[
\frac{\delta I}{I} \simeq \frac{\delta T}{T} - \left( \frac{\partial B}{B} \frac{\delta T}{T} \right) \frac{\delta \alpha}{\alpha}
\]

Taking \( \lambda = 5890 \, \text{Å} \) and \( T = 5000 \, \text{K} \), with solar observations in the region of the NaD lines in mind, we find \( F \approx 4.9 \).

From the HSRA (Gingerich et al. 1971) we find, for the continuum at 5000 Å, that \( (\partial B/\partial \tau)/B \approx 0.54 \). If the dependence of the continuum absorption coefficient on temperature is as \( \alpha \approx T^n \), this simple analysis is consistent with a positive correlation of intensity with temperature in the continuum, if \( n \) is not greater than 9.

Where LTE is not an adequate approximation, for example in the Doppler cores of strong lines such as the NaD lines, only a complete non-LTE computation can lead to definite results. However, it is possible to make some estimates. The source function is the line source function, which, in the relevant region of the atmosphere, is not close-coupled to the local kinetic temperature, and falls far below the Planck function. It may well be less sensitive to fluctuations in the local kinetic temperature for the same reason.

Turning to the term representing the effect on the absorption coefficient, an analysis of the D-line cores (Curtis & Jeffries 1967; Worrall 1971) suggests that \( \partial S/\partial \tau \) is about unity at 0.05 Å
from line centre (compared with about 0.5 for the continuum). In the Doppler core the continuum absorption coefficient is negligibly small compared with the line absorption coefficient, which is given by

$$\alpha(\Delta\lambda) = \frac{An}{\Delta\lambda_0} \exp\left[-\frac{\left(\frac{\Delta\lambda}{\Delta\lambda_0}\right)^2}{2}\right],$$

where $\Delta\lambda_0 = (\lambda/c)\sqrt{2kT/m}$. $A$ is a constant and $n$ is the number density of absorbing atoms; the line absorption coefficient is strongly dependent on temperature through the Doppler width $\Delta\lambda_0$. (Here $c$ is the speed of light, $k$ is the Boltzmann constant and $m$ is the mass of an atom.) Considering just the temperature dependence,

$$\frac{\delta\alpha}{\alpha} = \frac{1}{2}\frac{\partial\alpha}{\partial T} \delta T = \left[\left(\frac{\Delta\lambda}{\Delta\lambda_0}\right)^2 - 1\right] \frac{\delta T}{T} \approx 7.5 \frac{\delta T}{T}$$

at the edge of the Doppler core where $\Delta\lambda \cong 0.1 \AA$.

It seems possible, therefore, that changes in the absorption coefficient could outweigh the changes in the source function, resulting in an inverse correlation between temperature fluctuations and intensity fluctuations at the edge of the Doppler core of strong absorption lines. There is evidence that this is so in the analysis of the lambdameter observations of the solar NaD$_2$ line (DWS). Fig. 4 of DWS displays $(V - I)$ phase spectra out to 20 mHz at 16 different positions in the line profile, from the core to $\Delta\lambda = 0.419 \AA$ in the damping wing. In this last spectrum, labelled w15, the $(V - I)$ phase difference approaches zero at high frequency, as expected, as it does at those positions in the deep core of the line. However, for the positions labelled w5 to w13 inclusive, the $(V - I)$ phase difference approaches 180° at high frequency, as it would if the temperature/intensity correlation were reversed. These positions are at the edge of the Doppler core of the line profile, just where such an inverse correlation is most likely, as we have seen. If this interpretation is correct, there should be positions in the line profile, between those at which the $(V - I)$ phase difference at high frequency reverses, at which the fluctuations in intensity vanish; the fluctuations in the source function being just balanced by the effect on the absorption coefficient. The same fig. 4 (of DWS) presents velocity and intensity power spectra at each position in the line profile. Values read from these power spectra, converted to a linear scale, are plotted against wavelength shift from line centre in Fig. A1 (plotted as crosses), and in velocity, $P_v$ (plotted as circles), as a function of wavelength shift from the line centre of the NaD$_2$ line, taken from Deubner et al. (1996, fig. 4). Upper panel: power at the local maximum, 3.2 mHz. Lower panel: power at 20 mHz.

Fig. A1 at this transition in phase, because there is only one position available in the profile beyond the w14 position, and that is far out in the damping wing at $\Delta\lambda = 0.419 \AA$, while w14 is at $\Delta\lambda = 0.226 \AA$.

Other relevant $(V - I)$ phase spectra have been published by Lites & Chipman (1979) and Lites, Chipman & White (1982). These are from solar observations made at the edges of the Doppler cores of the lines Fe i 5576 Å, Mg i 5173 Å, Ca ii 8498 Å and Ca ii 8542 Å. For the moderately strong, photospheric Fe i line the $(V - I)$ phase difference tends to zero at high frequency, as expected, but for the very strong Mg i and Ca ii 8498-Å lines the $(V - I)$ phase difference tends to 180° at high frequency, again suggesting that the intensity fluctuations at the edges of the Doppler cores of these strong lines are in antiphase with the temperature. The $(V - I)$ phase difference from observations in the very strong Ca ii 8542-Å line is distinctly different: about −45° over the observed range from about 2.5 to 8 mHz. It is notable that the 8542-Å line has much more extensive damping wings than the other strong lines involved here.

![Figure 1. Oscillatory power in intensity, $P_I$ (plotted as crosses), and in velocity, $P_v$ (plotted as circles), as a function of wavelength shift from the line centre of the NaD$_2$ line, taken from Deubner et al. (1996, fig. 4). Upper panel: power at the local maximum, 3.2 mHz. Lower panel: power at 20 mHz.](https://academic.oup.com/mnras/article-abstract/335/3/628/1014232)