PLANAR GRAPHS WITHOUT CYCLES OF LENGTHS 4 AND 5 AND CLOSE TRIANGLES ARE DP-3-COLORABLE

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Abstract. Montassier, Raspaud, and Wang (2006) asked to find the smallest positive integers \(d_0\) and \(d_1\) such that planar graphs without \(\{4, 5\}\)-cycles and \(d_\Delta \geq d_0\) are 3-choosable and planar graphs without \(\{4, 5, 6\}\)-cycles and \(d_\Delta \geq d_1\) are 3-choosable, where \(d_\Delta\) is the smallest distance between triangles. They showed that \(2 \leq d_0 \leq 4\) and \(d_1 \leq 3\). In this paper, we show that the following planar graphs are DP-3-colorable: (1) planar graphs without \(\{4, 5\}\)-cycles and \(d_\Delta \geq 3\) are DP-3-colorable, and (2) planar graphs without \(\{4, 5, 6\}\)-cycles and \(d_\Delta \geq 2\) are DP-3-colorable. DP-coloring is a generalization of list-coloring, thus as a corollary, \(d_0 \leq 3\) and \(d_1 \leq 2\). We actually prove stronger statements that each pre-coloring on some cycles can be extended to the whole graph.

1. Introduction

Coloring of planar graphs has a long history. The famous Four Color Theorem states that every planar graph is properly 4-colorable, where a graph is properly \(k\)-colorable if there is a function \(c\) that assigns an element \(c(v) \in [k] := \{1, 2, \ldots, k\}\) to each \(v \in V(G)\) so that adjacent vertices receive distinct colors.

Grötzsch [17] showed every planar graph without 3-cycles is 3-colorable. But it is NP-complete to decide whether a planar graph is 3-colorable. There were heavy research on sufficient conditions for a planar graph to be 3-colorable. Three typical conditions are the following:

- One is in the spirit of the Steinberg’s conjecture (recently disproved) or Erdős’s problem that forbids cycles of certain lengths. Borodin, Glebov, Raspaud, and Salavatipour [11] showed that planar graphs without \(\{4, 5, 6, 7\}\)-cycles are 3-colorable, and it remains open to know if one can allow 7-cycle.
- Havel [16] proposed to make \(d_\Delta\) large enough, where \(d_\Delta\) is the smallest distance between triangles. Dvořák, Kral, and Thomas [14] showed that \(d_\Delta \geq 10^{100}\) suffices.
• The Bordeaux approach \cite{12} combines the two kinds of conditions. Borodin and Glebov \cite{10} showed that planar graphs without 5-cycles and \(d^\Delta \geq 2\) are 3-colorable. It is conjectured \cite{12} that \(d^\Delta \geq 1\) suffices.

Vizing \cite{27}, and independently Erdős, Rubin, and Taylor \cite{15} introduced list coloring as a generalization of proper coloring. A list assignment \(L\) gives each vertex \(v\) a list \(L(v)\) of available colors. A graph \(G\) is \(L\)-colorable if there is a proper coloring \(c\) of \(V(G)\) such that \(c(v) \in L(v)\) for each \(v \in V(G)\). A graph \(G\) is \(k\)-choosable if \(G\) is \(L\)-colorable for each \(L\) with \(|L(v)| \geq k\). Clearly, a proper \(k\)-coloring is an \(L\)-coloring when \(L(v) = \{k\}\) for all \(v \in V(G)\).

While list coloring provides a powerful tool to study coloring problems, some important techniques used in coloring (for example, identification of vertices) are not feasible in list coloring. Therefore, it is often the case that a condition that suffices for coloring is not enough for the corresponding list-coloring. Thomassen \cite{25, 26} showed that every planar graph is 5-choosable and every planar graph without \(\{3, 4\}\)-cycles is 3-choosable, but Voigt \cite{28, 29} gave non-4-choosable planar graphs and non-3-choosable triangle-free planar graphs.

Sometimes we do not know if a stronger condition would help. For example, Borodin (\cite{8}, 1996) conjectured that planar graphs without cycles of lengths from 4 to 8 are 3-choosable.

In the spirit of Bordeaux conditions, Montassier, Raspaud, and Wang \cite{24} gave the following conditions for a planar graph to be 3-choosable:

**Theorem 1.1** (Montassier, Raspaud, and Wang \cite{24}). A planar graph \(G\) is 3-choosable if

- \(G\) contains no cycles of lengths 4 and 5 and \(d^\Delta \geq 4\), or
- \(G\) contains no cycles of lengths from 4 to 6 and \(d^\Delta \geq 3\).

There exist planar graphs without 4-, 5-cycles and \(d^\Delta = 1\) that are not 3-choosable.

They asked for the optimal conditions on \(d^\Delta\) for the same conclusions.

Very recently, Dvořák and Postle \cite{13} introduced DP-coloring (under the name correspondence coloring), which helped them confirm the conjecture by Borodin mentioned above. DP-coloring is a generalization of list-coloring, but it allows identification of vertices in some situations.

**Definition 1.1.** Let \(G\) be a simple graph with \(n\) vertices, and \(L\) be a list assignment of \(V(G)\). For each vertex \(u \in V(G)\), let \(L_u = \{u\} \times L(u)\). For each edge \(uv \in E(G)\), let \(M_{uv}\) be a matching (maybe empty) between the sets \(L_u\) and \(L_v\) and let \(\mathcal{M}_L = \{M_{uv} : uv \in E(G)\}\), called the matching assignment. Let \(G_L\) be the graph that satisfies the following conditions

- \(V(G_L) = \bigcup_{u \in V(G)} L_u\).
- for all \(u \in V(G)\), the set \(L_u\) forms a clique.
- if \(uv \in E(G)\), then the edges between \(L_u\) and \(L_v\) are those of \(M_{uv}\)
- if \(uv \notin E(G)\), then there are no edges between \(L_u\) and \(L_v\).

If \(G_L\) contains an independent set of size \(n\), then \(G\) has an \(\mathcal{M}_L\)-coloring. The graph \(G\) is DP-\(k\)-colorable if, for each \(k\)-list assignment \(L\) and each matching assignment \(\mathcal{M}_L\) over \(L\), it has an \(\mathcal{M}_L\)-coloring. The minimum \(k\) such that \(G\) is DP-\(k\)-colorable is the DP-chromatic number of \(G\), denoted by \(\chi_{DP}(G)\).
As in list coloring, we refer to the elements of $L(v)$ as colors and call the element $i \in L(v)$ chosen in the independent set of an $\mathcal{M}_L$-coloring as the color of $v$.

We should note that DP-coloring and list coloring can be quite different. For example, Bernshteyn [2] showed that the DP-chromatic number of every graph $G$ with average degree $d$ is $\Omega(d/\log d)$, while Alon [1] proved that $\chi_l(G) = \Omega(\log d)$ and the bound is sharp.

Much attention was drawn on this new coloring, see for example, [2, 3, 4, 5, 6, 7, 18, 19, 20, 23, 22]. We are interested in DP-coloring of planar graphs. Dvořák and Postle [13] noted that Thomassen’s proofs [25] for choosability can be used to show $\chi_{DP}(G) \leq 5$ if $G$ is a planar graph, and $\chi_{DP}(G) \leq 3$ if $G$ is a planar graph with no 3-cycles and 4-cycles. Some sufficient conditions were given in [18, 19, 23] for a planar graph to be DP-4-colorable. Sufficient conditions for a planar graph to be DP-3-colorable are obtained in [21] and [22]. In particular,

**Theorem 1.2.** (21, 22) A planar graph is DP-3-colorable if it has no cycles of length $\{4, 9, a, b\}$, where $(a, b) \in \{(5, 6), (5, 7), (6, 7), (6, 8), (7, 8)\}$.

In this paper, we use DP-coloring to improve the results in Theorem 1.1. To state our results, we have to introduce extendability. Let $G$ be a graph and $C$ be a subgraph of $G$. Then $(G, C)$ is DP-3-colorable if every DP-3-coloring of $C$ can be extended to $G$.

A 9-cycle $C$ is bad if it is the outer 9-cycle in a subgraph isomorphic to the graphs in Figure 1. A 9-cycle is good if it is not a bad 9-cycle.

**Theorem 1.3.** Let $G$ be a planar graph that contains no $\{4, 5\}$-cycles and $d^\Delta \geq 3$. Let $C_0$ be a 3-, 6-, 7-, or 8-cycle or a good 9-cycle in $G$. Then each DP-3-coloring of $C_0$ can be extended to $G$.

**Theorem 1.4.** Let $G$ be a planar graph that contains no $\{4, 5, 6\}$-cycles and $d^\Delta \geq 2$. Let $C_0$ be a cycle of length 7, 8, 9 or 10 in $G$. Then each DP-3-coloring of $C_0$ can be extended to $G$.

The proofs of Theorem 1.3 and 1.4 use identification of vertices. We shall note that the planar graphs in the following corollary was not known to be 3-choosable.

**Corollary 1.5.** The following planar graphs are DP-3-colorable (thus also 3-choosable):

- no $\{4, 5\}$-cycles and $d^\Delta \geq 3$, or
- no $\{4, 5, 6\}$-cycles and $d^\Delta \geq 2$. 

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**Figure 1.** bad 9-cycles.
Proof. Let $G$ be a planar graph under consideration. Note that $G$ is DP-3-colorable if $G$ contains no 3-cycle. So we may assume that $G$ contains a 3-cycle. Then by Theorem 1.4, $G$ is DP-3-colorable when $d^1 \geq 3$. So we let $d^1 \geq 2$ and assume that $G$ contains no $\{4,5,6\}$-cycles. By Theorem 1.2 we may assume that $G$ contains a cycle of length in $\{7,8,9\}$. Now by Theorem 1.4 $G$ is DP-3-colorable.

We use discharging method to prove the results. One part of the proofs is to show some structures to be reducible, that is, a coloring outside of the structure can be extended to the whole graph. The following lemma from [21] provides a powerful tool to prove the reducibility.

**Lemma 1.6. [21]** Let $k \geq 3$ and $H$ be a subgraph of $G$. If the vertices of $H$ can be ordered as $v_1, v_2, \ldots, v_\ell$ such that the following hold

1. $v_iv_\ell \in E(G)$, and $v_1$ has no neighbor outside of $H$,
2. $d(v_\ell) \leq k$ and $v_\ell$ has at least one neighbor in $G - H$,
3. for each $2 \leq i \leq \ell - 1$, $v_i$ has at most $k - 1$ neighbors in $G[\{v_1, \ldots, v_{i-1}\}] \cup (G - H)$,

then a DP-$k$-coloring of $G - H$ can be extended to a DP-$k$-coloring of $G$.

We end the introduction with some notations used in the paper. All graphs mentioned in this paper are simple. A $k$-vertex ($k^+$-vertex, $k^-$-vertex) is a vertex of degree $k$ (at least $k$, at most $k$). The same notation will be applied to faces and cycles. We use $V(G)$ and $F(G)$ to denote the set of vertices and faces in $G$, respectively. An $(\ell_1, \ell_2)$-edge is an edge $e = v_1v_2$ with $d(v_i) = \ell_i$. An $(\ell_1, \ell_2, \ldots, \ell_k)$-face is a $k$-face $f = [v_1v_2\ldots v_k]$ with $d(v_i) = \ell_i$, respectively. Recall that two faces are adjacent if they share a common edge, and are intersecting if they share a common vertex. A vertex is incident to a face if it is on the face, and is adjacent to a face if it is not on the face but adjacent to a vertex on the face. A vertex in $G$ is light if it is incident to a 3-face. If $C$ is a cycle in an embedding of $G$, we use $int(C)$ and $ext(C)$ to denote the sets of vertices located inside and outside a cycle $C$, respectively. The cycle $C$ is called a separating cycle if $int(C) \neq \emptyset \neq ext(C)$. An edge $uv \in E(G)$ is straight if every $(u,c_1)(v,c_2) \in M_{uv}$ satisfies $c_1 = c_2$. We note that if all edges in a subgraph are straight, then a DP-3-coloring on the subgraph is the same as a proper 3-coloring.

**2. Proof of Theorem 1.3**

Let $(G, C_0)$ be a counterexample to Theorem 1.3 with minimum number of vertices, where $C_0$ is a 3-, 6-, 7-, 8-cycle or a good 9-cycle. Below we let $G$ be a plane graph. The following was shown in [21] for every non-DP-3-colorable graphs.

**Lemma 2.1.** For each $v \in G - C_0$, $d(v) \geq 3$ and for each $v \in C_0$, $d(v) \geq 2$.

**Lemma 2.2.** There exist no separating $\{3,6,7,8\}$-cycles or good 9-cycle.

Proof. First of all, we note that $C_0$ cannot be a separating cycle. For otherwise, we may extend the coloring of $C_0$ to both inside $C_0$ and outside $C_0$, respectively, then combine them to get a coloring of $G$. So we may assume that $C_0$ is the outer face of the embedding of $G$. 

Let $C \neq C_0$ be a separating $\{3,6,7,8\}$-cycle or good 9-cycle in $G$. By the minimality of $G$, the coloring of $C_0$ can be extended to $G - \text{int}(C)$. Now that $C$ is colored, thus by the minimality of $G$ again, the coloring of $C$ can be extended to $\text{int}(C)$. Combine inside and outside of $C$, we have a coloring of $G$, which is extended from the coloring of $C_0$, a contradiction. \hfill \Box

By Lemma 2.2, if $C$ is a bad 9-cycle, then the subgraph in Figure 1 that contains $C$ must be induced. From now on, we will let $C_0$ be the outer face of $G$. Likewise, if $C_0$ contains a chord, then by Lemma 2.2, $G$ contains no other vertices, so the coloring on $C_0$ is also a coloring of $G$. Therefore, we may assume that $C_0$ is chordless as well. A vertex is internal if it is not on $C_0$ and a face is internal if it contains no vertex of $C_0$.

For convenience, a $6^+$-face $f$ in $G$ is bad if $d(f) = 6$ and $f$ is adjacent to a 3-face, otherwise, it is good. Let $f$ be a $(3,3,3,3,3,3)$-face adjacent to a 3-face $f'$. We call the vertex $v$ on $f'$ but not on $f$ the roof of $f$, and $f$ the base of $v$.

**Lemma 2.3.** Let $f$ be an internal 6-face in $G$ and $f_1$ be an internal $(3,3,4)$-face adjacent to $f$. Then each of the followings holds:

(a) $f$ cannot contain vertices of another 3-face;

(b) If $f$ is a $(3,3,3,3,4)$-face such that $f$ and $f_1$ share a common $(3,4)$-edge, then the other $(3,4)$-edge of $f_1$ cannot be on another internal $(3,3,3,3,3,3,4)$-face.

(c) If $f$ is a $(3,3,3,3,3)$-face, then $f_1$ cannot be adjacent to an internal $(3,3,3,3,3,3,4)$-face. This means a 4-vertex on an internal $(3,3,3,3,3,3,4)$-face cannot be a roof.

*Proof.* (a) follows from the condition on the distance of triangles. To show (b) and (c), let $f_1 = [xyz]$ so that $xy$ is the common edge of $f_1$ and $f = [xyu_1u_2u_3u_4]$ and $d(x) \leq d(y)$. Let $f_2 = [zv_1v_2v_3v_4y]$ be the (3,3,3,3,4)-face adjacent to $f_1$.

(b) We have $d(y) = 4$ and $d(x) = d(z) = d(u_i) = d(v_i) = 3$ for $i \in [4]$. Order the vertices on $f$ and $f_2$ as

$$y, v_4, v_3, v_2, v_1, z, x, u_4, u_3, u_2, u_1.$$  

Let $S$ be the set of vertices in the list. By Lemma 1.6, a DP-3-coloring of $(G - S, C_0)$ can be extended to $(G, C_0)$, a contradiction.

(c) We have $d(z) = 4$ and $d(x) = d(y) = d(u_i) = d(v_i) = 3$ for $i \in [4]$, and $u_1 = v_4$. Order the vertices on $f$ and $f_2$ as

$$x, z, v_1, v_2, v_3, y, u_1, u_2, u_3, u_4.$$  

Let $S$ be the set of vertices in the list. By Lemma 1.6, a DP-3-coloring of $(G - S, C_0)$ can be extended to $G$, a contradiction. \hfill \Box

**Lemma 2.4.** Let $f = [v_1v_2v_3v_4v_5v_6]$ be an internal 6-face that is adjacent to an internal $(3,3,3)$-face $f_1 = [v_1v_2v_{12}]$, then $d(v_3) \geq 4$ or $d(v_6) \geq 4$.

*Proof.* We assume that $d(v_3) = d(v_6) = 3$, and use $v$ to denote the neighbor of $v_{12}$ other than $v_1, v_2$. First we may rename the lists of vertices in $\{v, v_{12}, v_2, v_3, v_4\}$ so that each edge in $\{v_1v_2, vv_{12}, v_{12}v_2, v_2v_3, v_3v_4\}$ is straight.
Consider the graph $G'$ obtained from $G = \{v_{12}, v_1, v_2, v_3, v_6\}$ by identifying $v_4$ and $v$. We claim that no new cycles of length from 3 to 5 or multiple edges are created, for otherwise, there is a path of length 2, 3, 4 or 5 from $v$ to $v_4$ in $G = \{v_{12}, v_1, v_2, v_3, v_6\}$, which together with $v_{12}, v_1, v_2, v_3$ forms a separating \{678\}-cycle or good 9-cycle, a contradiction to Lemma 2.2. Clearly, $\Delta(G') \geq 3$. Finally, we claim that no new chord in $C_0$ is formed in $G'$, for otherwise, $v \in C_0$ and $v_4$ is adjacent to a vertex on $C_0$, then there is a path between $v_4$ and $v$ on $C_0$ with length at most four, which with $v_3v_2v_{12}$ forms a separating \{6, 7, 8\}-cycle or good 9-cycle.

By minimality of $(G, C_0)$, the $DP$-3-coloring of $C_0$ can be extended to a $DP$-3-coloring of $G'$. Now keep the colors of all vertices in $G'$ and color $v_4$ and $v$ with the color of the identified vertex. Now properly color $v_3$, and then color $v_{12}$ with the color of $v_3$, which we can do because the edges $vv_{12}, v_1v_2v_3, v_3v_4$ are straight and the color of $v_3$ is different from the one on $v_4$ and $v$. Now color $v_6, v_1, v_2$ properly in the order, we obtain a coloring of $G$, a contradiction.

\[\Box\]

**Lemma 2.5.** Let $P = xu_1u_2yyv_1v_2z$ be a path in $int(C_0)$ and $f = [x'y'z']$ be an internal $(3, 3, 3)$-face. Consider the graph $G'$ obtained from $G = \{x, u_1, u_2, y, y', x', z'\}$ by identifying $z$ and $y''$. Since $\Delta(G) \geq 3$, $v_1$ and $v_2$ cannot be on triangles. We claim that no new cycles of length from 1 to 5 are created, for otherwise, there is a path of length 2, 3, 4 or 5 from $y''$ to $z$ in $G = \{x, u_1, u_2, y, y', x', z'\}$, which together with $y, y', z'$ forms a separating \{6, 7, 8\}-cycle or good 9-cycle, a contradiction to Lemma 2.2. Clearly, $\Delta(G') \geq 3$. Finally, we claim that no new chord in $C_0$ is formed in $G'$, for otherwise, $y'' \in C_0$ and $z$ is adjacent to a vertex on $C_0$, then there is a path between $y''$ and $z$ on $C_0$ with length at most four, which again forms a good separating cycle with $yy'x'$ of forbidden length.

By minimality of $(G, C_0)$, the $DP$-3-coloring of $C_0$ can be extended to a $DP$-3-coloring of $G'$. Now keep the colors of all vertices in $G'$ and color $y''$ and $z$ with the color of the identified vertex. Now properly coloring $y$ and coloring $z'$ with the color of $y$, and coloring $u_2, u_1, x, x', y'$ in order, we obtain a coloring of $G$, a contradiction.

\[\Box\]

We use $\mu(x)$ to denote the initial charge of a vertex or face $x$ in $G$ and $\mu^*(x)$ to denote the final charge after the discharging procedure. We use $\mu(v) = 2d(v) - 6$ for each vertex $v$, $\mu(f) = d(f) - 6$ for each face $f \neq C_0$, and $\mu(C_0) = d(C_0) + 6$. Then by Euler formula, $\sum_{x \in V(G) \cup F(G)} \mu(x) = 0$. To lead to a contradiction, we shall prove that $\mu^*(x) \geq 0$ for all $x \in V \cup F$ and $\mu^*(C_0)$ is positive. For shortness, let $F_k = \{f : f$ is a $k$-face and $V(f) \cap C_0 \neq \emptyset\}$. 


We use the following discharging rules:

(R1) Each internal 4\(^+\)-vertex gives \(\frac{3}{2}\) to its incident 3-face, and \(\frac{1}{2}\) to its base or incident (3, 3, 3, 3, 3, 4)-face. Each internal 4-vertex gives 1 to its adjacent (3, 3, 3)-face and \(\frac{1}{2}\) to its incident 6-faces that are not adjacent to its adjacent 3-face, and each internal 5\(^+\)-vertex gives 2 to its adjacent (3, 3, 3)-face and \(\frac{1}{2}\) to its incident 6-faces that are not adjacent to its adjacent 3-face.

(R2) Each 7\(^+\)-face or non-internal 6-face other than \(C_0\) gives 1 to each of its adjacent internal 3-faces and the rest to the outer face. Each internal 6-face gives \(\frac{1}{2}\) to its adjacent internal 3-face when it shares an (3, 4\(^+\))-edge with the 3-face, or contains a 4\(^+\)-vertex that is not adjacent to a (3, 3, 3)-face, or it is a (3, 3, 3, 3, 3, 3)-face.

(R3) The outer face \(C_0\) gets \(\mu(v)\) from each \(v \in C_0\), gives 3 to each intersecting 3-face and 1 to each adjacent bad 6-face with an internal 3-face.

We first check the final charge of vertices in \(G\). By (R3), each vertex on \(C_0\) has final charge 0. So let \(v\) be an internal vertex of \(G\). Then by Lemma 2.1, \(d(v) \geq 3\). Note the \(\mu^*(v) = 0\) if \(d(v) = 3\).

Let \(d(v) = k \geq 5\). If \(v\) is on a 3-face, then it is not adjacent to other 3-faces, so by (R1), it gives \(\frac{3}{2}\) to the 3-face, \(\frac{1}{2}\) to each other incident face and possibly \(\frac{1}{2}\) to its base (at most one by definition), so \(\mu^*(v) \geq 2k - 6 - \frac{3}{2} - \frac{1}{2} \cdot k = \frac{3}{2}(k - 5) \geq 0\). If \(v\) is adjacent to a 3-face, then it is not on or adjacent to other 3-faces, so by (R1), it gives at most 2 to the 3-face, and \(\frac{1}{2}\) to each other incident 6-faces that are not adjacent to the 3-face, hence \(\mu^*(f) \geq 2k - 6 - 2 - \frac{1}{2}(k - 2) > 0\). If \(f\) is not on or adjacent to 3-faces, then by (R1), its final charge is \(\mu^*(f) \geq 2k - 6 - \frac{1}{2}k > 0\).

Let \(d(v) = 4\). Let \(f_i\) for \(1 \leq i \leq 4\) be the incident face of \(v\) in clockwise order. First assume that \(v\) is on a 3-face. By Lemma 2.3 (b) and (c), \(v\) cannot be a roof and on a (3, 3, 3, 3, 3, 4)-face at the same time, so by (R1), \(v\) gives out at most \(\frac{1}{2}\) to 6-faces and \(\frac{3}{2}\) to the 3-face, thus \(\mu^*(v) \geq 0\). Now assume that \(v\) is adjacent to a 3-face. Then \(v\) cannot be adjacent other 3-faces. By (R1), \(v\) gives at most 1 to the 3-face and \(\frac{1}{2}\) to each of the other 6-faces that are not adjacent to the 3-face, and \(\mu^*(v) \geq 2 - 1 - \frac{1}{2} \cdot 2 = 0\). Finally assume that \(v\) is not on or adjacent to any 3-face. Then by (R1), \(\mu^*(v) \geq 2 - \frac{1}{2} \cdot 4 = 0\).

Now we check the final charge of faces. Let \(d(f) = 3\). If \(f\) contains vertices of \(C_0\), then by (R3), \(\mu^*(f) = 0\). So we assume that \(f\) is internal. If \(f\) is incident with at least two \(4^+\)-vertices, then \(f\) gets \(\frac{3}{2}\) from each of the incident \(4^+\)-vertices by (R1), thus \(\mu^*(f) \geq -3 + \frac{3}{2} \cdot 2 = 0\). If \(f\) is incident with exactly one \(4^+\)-vertex, then \(f\) gets \(\frac{3}{2}\) from the incident \(4^+\)-vertex by (R1) and gets \(\frac{1}{2}\) from each of the incident \(6^+\)-face by (R2).

Now we assume that \(f = [x'y'z']\) is an internal (3, 3, 3)-face. Let \(xx', yy', zz' \in E(G)\) and let \(f_1, f_2, f_3\) be the three adjacent faces of \(f\) so that \(f_1\) contains \(x, x', y', y\) and \(f_2\) contains \(y, y', z', z\). If \(f\) is adjacent to three \(7^+\)- or non-internal 6-faces, then it gets 1 from each by (R2) and its final charge is at least 0. So we may assume that it is adjacent to an internal 6-face, say \(f_1\). By Lemma 2.4 \(f\) is adjacent to at least one internal \(4^+\)-vertex (say \(y\)) which is on \(f_1\). If \(f\) is adjacent to three internal 6-faces, then by Lemma 2.4, one of \(x\) and \(z\) is a \(4^+\)-vertex, and by Lemma 2.5 either one of \(x, y, z\) is a \(5^+\)-vertex, in which case by (R1),
\( \mu^*(f) \geq -3+2+1 = 0 \), or they are all 4-vertices, in which case by (R1), \( \mu^*(f) \geq -3+1\cdot 3 = 0 \), or one of them (say \( x \)) is a 3-vertex and other two are 4-vertices, in which case by Lemma 2.5, \( f_1 \) and \( f_3 \) both contain 4\(^+\)-vertices that are not adjacent to \( f \) so by (R1) and (R2), \( f \) gets 1 + 1 from the two 4-vertices and \( \frac{1}{2} \cdot 2 \) from \( f_1 \) and \( f_3 \). Likewise, if \( f_2 \) and \( f_3 \) are both 7\(^-\) or non-internal 6-faces, then by (R1) and (R2), \( \mu^*(f) \geq -3 + 1 + 1 \cdot 2 = 0 \). So we may assume that one of \( f_2 \) or \( f_3 \) is an internal 6-face and the other is a 7\(^-\) or non-internal 6-face. If \( f_3 \) is an internal 6-face, then by Lemma 2.4, \( x \) or \( z \) is a 4\(^+\)-vertex, thus by (R1) \( f \) gets 1 \cdot 2 from the two adjacent 4\(^+\)-vertices and by (R2) \( f \) gets 1 from \( f_2 \). So we may assume that \( f_2 \) is an internal 6-face and \( f_3 \) is a 7\(^-\) or non-internal 6-face, and furthermore assume that \( x, z \) are 3-vertices and \( d(y) = 4 \). Now by Lemma 2.5, \( f_1 \) and \( f_2 \) both contain 4\(^+\)-vertices that are not adjacent to \( f \), so by (R2), \( f \) gets \( \frac{1}{2} \cdot 2 \) from \( f_1 \) and \( f_2 \), 1 from \( f_3 \), and by (R1), 1 from \( y \), and we have \( \mu^*(f) \geq -3 + 3 = 0 \).

Since \( G \) contains no 4- or 5-cycles, we only need to check the 6\(^+\)-faces. If \( d(f) \geq 7 \), then \( f \) is adjacent to at most \( \lfloor \frac{d(f)}{4} \rfloor \) 3-faces, so after (R1), \( \mu^*(f) \geq d(f) - 6 - \lfloor \frac{d(f)}{4} \rfloor \geq 0 \).

Let \( d(f) = 6 \). If \( f \) is good or \( f \) contains vertices of \( C_0 \), then \( \mu^*(f) = 0 \). Now we assume that \( f \) is an internal bad 6-face that is adjacent to an internal 3-face \( f' = [xyz] \) on edge \( xy \) with \( d(x) \leq d(y) \).

- If \( d(x), d(y) \geq 4 \), then \( f \) gives nothing to \( f' \). So \( \mu^*(f) = \mu(f) = 0 \).
- If \( d(x) = 3 \) and \( d(y) \geq 5 \), then \( f \) gets \( \frac{1}{2} \) from \( y \) and gives \( \frac{1}{2} \) to \( f' \). Thus \( \mu^*(f) \geq 6 - 6 + \frac{1}{2} - \frac{1}{2} = 0 \).
- If \( d(x) = d(y) = 3 \), then by (R2), \( f \) gives \( \frac{1}{2} \) to \( f' \) only when \( f \) contains a 4\(^+\)-vertex that is not adjacent to the 3-face, in which case, \( f \) gets \( \frac{1}{2} \) from the 4\(^+\)-vertex by (R1). So we always have \( \mu^*(f) \geq 0 \).
- Let \( d(x) = 3 \) and \( d(y) = 4 \). If \( f \) is an internal \((3,3,3,3,3,4)\)-face, then it gets \( \frac{1}{2} \) from \( y \), or else \( f \) contains another 4\(^+\)-vertex, from which \( f \) gets \( \frac{1}{2} \). Thus \( \mu^*(f) \geq 6 - 6 + \frac{1}{2} - \frac{1}{2} = 0 \).

We call a bad 6-face \( f \) in \( F_6 \) \textit{special} if \( f \) is adjacent to one internal 3-face.

\textbf{Lemma 2.6.} The final charge of \( C_0 \) is positive.

\textit{Proof.} Assume that \( \mu^*(C_0) \leq 0 \). Let \( E(C_0, G - C_0) \) be the set of edges between \( C_0 \) and \( G - C_0 \). Let \( e' \) be the number of edges in \( E(C_0, G - C_0) \) that is not on a 3-face and \( x \) be the number of charges \( C_0 \) receives by (R3). Let \( f_3 = |F_3| \) and \( f_6 \) be the number of special 6-faces. By (R3) and (R4), the final charge of \( C_0 \) is

\[
\mu^*(C_0) = d(C_0) + 6 + \sum_{v \in C_0} (2d(v) - 6) - 3f_3 - f_6 + x
\]

\[
= d(C_0) + 6 + \sum_{v \in C_0} 2(d(v) - 2) - 2d(C_0) - 3f_3 - f_6 + x
\]

\[
= 6 - d(C_0) + 2|E(C_0, G - C_0)| - 3f_3 - f_6 + x
\]

\[
\geq 6 - d(C_0) + f_3 + 2e' - f_6 + x,
\]

where the last equality follows from that each 3-face in \( F_3 \) contains two edges in \( E(C_0, G - C_0) \).
Note that for each special 6-face $f$, no edge in $E(C_0, G - C_0) \cap E(f)$ is on 3-faces. Then $e' \geq f_6$. When $e' = f_6$, $C_0$ is adjacent to at least three 6-faces, so $e' = f_6 \geq 3$, and it follows that $d(C_0) = 9$ and $x = f_3 = 0$ and $e' = f_6 = 3$, in which case, we have a bad 9-cycle as in the second graph in Figure 1. So we may assume that $e' \geq f_6 + 1$. Thus

$$\mu^*(C_0) \geq 6 - d(C_0) + f_3 + 2e' - f_6 + x \geq 6 - d(C_0) + f_3 + x + f_6 + 2.$$  

Since $\mu^*(C_0) \leq 0$, $d(C_0) \geq 8$. So if $f_6 = 1$, then $d(C_0) = 9$ and $(f_3, x, e') = (0, 0, 2)$. Now that the 6-face shares at most one edge with $C_0$, $C_0$ is adjacent to a 10-face $f$ that contains at least five consecutive 2-vertices on $C_0$, thus by (R3), $x \geq d(f) - 6 - \left\lceil \frac{d(f)-7}{4} \right\rceil \geq 0$, a contradiction.

Therefore, we may assume that $f_6 = 0$, and $f_3 + 2e' + x \leq d(C_0) - 6 \leq 3$. So $e' \leq 1$.

Let $e' = 1$. It follows that $f_3 \leq 1$.

- Let $f_3 = 1$. Then $d(C_0) = 9$ and $x = 0$. Since $C_0$ is not a bad 9-cycle, $C_0$ is adjacent to a 7-face $f$ and $f$ is adjacent to the 3-face, so by (R3), $f$ gives at least 1 to $C_0$, that is, $x \geq 1$, a contradiction.

- Let $f_3 = 0$. Then $d(C_0) \geq 8$ and $x \leq 1$. Note that $C_0$ is adjacent to a 9-face $f$ that contains at least $d(C_0) - 1$ consecutive 2-vertices, thus by (R3), $f$ gives at least $d(f) - 6 - \left\lceil \frac{d(f)-7}{4} \right\rceil \geq 2$ to $C_0$, a contradiction to $x \leq 1$.

Finally let $e' = 0$. Then $f_3 + x \leq d(C_0) - 6$, and each edge in $E(C_0, G - C_0)$ is on a 3-face. Note that we may assume that $f_3 > 0$, for otherwise $G = C_0$. Now follow the boundaries of the 7-faces adjacent to $C_0$, each of the $f_3$ triangles is encountered twice, thus the 7-faces do not give charge to at least 2$f_3$ triangles, so $x \geq 2f_3$. It follows $f_3 = 1$ and $d(C_0) = 9$. In this case, $C_0$ is adjacent to a 10-face $f$ that contains at least 7 consecutive 2-vertices on $C_0$. Then by (R3), $f$ gives at least $d(f) - 6 - \left\lceil \frac{d(f)-9}{4} \right\rceil \geq 3$ to $C_0$, a contradiction to $x = 2$. □

3. Proof of Theorem 1.4

Let $(G, C_0)$ be a counterexample to Theorem 1.4 with minimum number of vertices, where $C_0$ is a 7-, 8-, 9- or 10-cycle. Let $G$ be a plane graph.

**Lemma 3.1.** For each $v \notin C_0$, $d(v) \geq 3$.

**Proof.** Let $v \notin C_0$ be a vertex with $d(v) \leq 2$. Any $\mathcal{M}_L$-coloring of $G - v$ can be extended to $G$ since $v$ has at most $d(v)$ elements of $L(v)$ forbidden by the colors selected for the neighbors of $v$, while $|L(v)| = 3$. □

**Lemma 3.2.** The graph $G$ has no separating cycles of length 7, 8, 9 or 10.

**Proof.** First of all, we note that $C_0$ cannot be a separating cycle. For otherwise, we may extend the coloring of $C_0$ to both inside $C_0$ and outside $C_0$, respectively, then combine them to get a coloring of $G$. So we may assume that $C_0$ is the outer face of the embedding of $G$.

Let $C \neq C_0$ be a separating cycle of length 7, 8, 9 or 10 in $G$. By the minimality of $G$, the coloring of $C_0$ can be extended to $G - \text{int}(C)$. Now that $C$ is colored, thus by the minimality of $G$ again, the coloring of $C$ can be extended to $\text{int}(C)$. Combine inside and outside of $C$, we have a coloring of $G$, which is extended from the coloring of $C_0$, a contradiction. □
So we may assume that $C_0$ is the outer face of the embedding of $G$ in the rest of this paper. Like in the previous section, we may assume that $C_0$ is chordless. A face is \textit{internal} if none of its vertices is on $C_0$, and a vertex is \textit{internal} if it is not on $C_0$.

\textbf{Lemma 3.3.} Let $f$ be an internal 7-face that is adjacent to an internal (3, 3, 3)-face and is incident with at least six 3-vertices. Then none of the followings occur

(a) $f$ contains a (3, 4)-edge that is on an internal (3, 3, 4)-face.

(b) $f$ contains seven 3-vertices and is adjacent to an internal (3, 3, 4\textsuperscript{+})-face.

(c) $f$ is adjacent to another internal (3, 3, 3)-face.

\textit{Proof.} Let $f = [v_1 v_2 \cdots v_7]$, and $v_1 v_2$ be the (3, 3)-edge that is on an internal (3, 3, 3)-face $[v_1 v_2 v_{12}]$. Since $d^\Delta(G) \geq 2$, by symmetry we may assume that $v_4 v_5$ is on a 3-face $[v_4 v_5 v_{45}]$.

(a) or (b): If $d(v_4) \leq 4$ and $d(v_5) = 3$, then let $S$ be the set of vertices listed as:

$$v_2, v_3, v_4, v_{45}, v_5, v_6, v_7, v_1, v_{12}.$$  

If $d(v_5) = 4$, then let $S$ be the set of vertices listed as:

$$v_1, v_7, v_6, v_5, v_{45}, v_4, v_3, v_2, v_{12}.$$  

By Lemma 1.6, a DP-3-coloring of $G - S$ can be extended to $G$, a contradiction.

(c) Suppose otherwise that the 3-face $[v_4 v_5 v_{45}]$ is an internal (3, 3, 3)-face. Let $v$ be the neighbor of $v_{45}$ not on $f$. Since $f$ is incident with at least six 3-vertices, by symmetry we may assume that $d(v_6) = 3$. We can rename the lists of vertices in $\{v, v_{45}, v_4, v_5, v_6, v_7\}$ so that each edge in $\{v_7 v_6, v_6 v_5, v_5 v_4, v_3 v_4, v_{45} v, v_{45} v\}$ is straight.

Consider the graph $G'$ obtained from $G - \{v_6, v_5, v_4, v_{45}\}$ by identifying $v_7$ and $v$. We claim that no new cycles of length from 3 to 6 are created, for otherwise, there is a path of length 3, 4, 5 or 6 from $v$ to $v_7$ in $G - \{v_6, v_5, v_4, v_{45}\}$, which together with $v_6, v_5, v_{45}$ forms a separating cycle of length 7, 8, 9 or 10, a contradiction to Lemma 3.2. Since none of $v_7$ and $v$ is on a triangle, $d^\Delta(G') \geq 2$. Finally, we claim that no new chord in $C_0$ is formed in $G'$, for otherwise, $v \in C_0$ and $v_7$ is adjacent to a vertex on $C_0$, then there is a path between $v_7$ and $v$ on $C_0$ with length at most four, which again forms a separating cycle with $v_6 v_5 v_{45}$ of forbidden length.

By minimality of $(G, C_0)$, the DP-3-coloring of $C_0$ can be extended to a DP-3-coloring $\phi$ of $G'$. Now keep the colors of all other vertices in $G'$ and color $v_7$ and $v$ with the color of the identifying vertex. For $x \in \{v_4, v_5, v_6, v_{45}\}$, let $L^*(x) = L(x) \setminus \cup_{ux \in E(G)}\{c' \in L(u) : (u, c)(x, c') \in C_{ux} \text{ and } (u, c) \in \phi\}$. Then $|L^*(v_4)| = |L^*(v_5)| = 3$ and $|L^*(v_6)| \geq 1$. So we can extend $\phi$ to a DP-3-coloring of $G$ by color $v_6$ and $v_{45}$ with the same color and then color $v_4, v_5$ in order, a contradiction. \hfill \Box

We use $\mu(x)$ to denote the initial charge of a vertex or face $x$ in $G$ and $\mu^*(x)$ to denote the final charge after the discharging procedure. We use $\mu(v) = 2d(v) - 6$ for each vertex $v$, $\mu(f) = d(f) - 6$ for each face $f \neq C_0$, and $\mu(C_0) = d(C_0) + 6$. Then by Euler formula, $\sum_{x \in V(G) \cup F(G)} \mu(x) = 0$. To lead to a contradiction, we shall prove that $\mu^*(x) \geq 0$ for all $x \in V \cup F$ and $\mu^*(C_0)$ is positive.
For shortness, let $F_k = \{ f : f$ is a $k$-face and $V(f) \cap C_0 \neq \emptyset \}$. We call a 7-face $f$ in $F_7$ special if $f$ is adjacent to two internal 3-faces. We call a 4-vertex $v$ on a 7+-face $f$ rich to $f$ if $v$ is not on a 3-face adjacent to $f$.

We have the following discharging rules:

(R1) Each internal 3-face gets $\frac{1}{2}$ from each incident 4+-vertex and then gets its needed charge evenly from adjacent faces.

(R2) Each internal 7-face gets $\frac{1}{2}$ from each incident rich 4-vertex or 5+-vertex.

(R3) After (R1) and (R2), each 7+-face gives all its remaining charges to $C_0$.

(R4) The outer face $C$ of $G$ gives all its remaining charges to $C_0$.

Lemma 3.4. Every vertex $v$ and every face other than $C_0$ in $G$ has nonnegative final charge.

Proof. We first check the final charges of vertices in $G$. Let $v$ be a vertex in $G$. If $v \in C_0$, then by (R4) $\mu^*(v) = 0$. If $v \notin C_0$, then by Lemma 3.1 $d(v) \geq 3$. If $d(v) = 3$, then $\mu^*(v) = \mu(v) = 0$. Note that each vertex can be incident to at most one 3-face since $d^\Delta(G) \geq 2$. Let $d(v) = 4$. If $v$ is light, then $v$ gives $\frac{3}{2}$ to the adjacent 3-face and $\frac{1}{2}$ to the incident 7-face to which $v$ is rich by (R1) and (R2). If $v$ is not light, then $v$ gives at most $\frac{1}{2}$ to each incident face by (R2). In either case, $\mu^*(v) \geq 2 \cdot 4 - 6 - \max\{\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 4\} \geq 0$. If $d(v) \geq 5$, then $v$ gives $\frac{3}{2}$ to at most one incident 3-face and at most $\frac{1}{2}$ to each other incident face. So $\mu^*(v) \geq 2d(v) - 6 - \frac{3}{2} - \frac{1}{2} \cdot (d(v) - 1) > 0$.

Now we check the final charges of faces other than $C_0$ in $G$. Since $G$ contains no 4, 5, 6-cycles, a 3-face in $G$ is adjacent to three 7+-faces. Thus, by (R1) and (R4) each 3-face has nonnegative final charge. Let $f$ be a 7+-face in $G$. By (R1) $f$ only needs to give 1 to each adjacent internal (3, 3, 3)-face and $\frac{1}{2}$ to each adjacent internal (3, 3, 4+)-face. Since $d^\Delta(G) \geq 2$, $f$ is adjacent to at most $\lfloor \frac{d(f)}{3} \rfloor$ 3-faces. If $d(f) \geq 6$, then $\mu^*(f) \geq d(f) - 6 - 1 \cdot \lfloor \frac{d(f)}{3} \rfloor \geq 0$. Let $d(f) = 7$. Note that $f$ gives at most 1 to each adjacent 3-face by (R1). If $f$ is in $F_7$ or adjacent to at most one internal 3-face, then by (R1) and (R4), $\mu^*(f) \geq 7 - 6 - \max\{1, 1 \cdot 2 - 1\} = 0$.

Therefore, we may assume that $f$ is an internal 7-face and adjacent to two internal 3-faces. If none of the 3-faces is a (3, 3, 3)-face, or one of the two 3-faces contains more than one 4+-vertex, then $f$ gives out at most 1 to the 3-faces, so $\mu^*(f) \geq 0$. Thus, we may assume that $f$ is adjacent to a (3, 3, 3)-face $f_1$ and a (3, 3, 4+)-face $f_2$. If $f_2$ shares a (3, 4+)-edge with $f$, then by Lemma 3.3 (a) $f$ contains a rich 4-vertex or 5+-vertex, which gives $\frac{1}{2}$ to $f$ by (R2). So $\mu^*(f) \geq 7 - 6 - 1 - \frac{1}{2} + \frac{1}{2} = 0$. If $f_2$ shares a (3, 3)-edge with $f$, then by Lemma 3.3 (b) and (c), $f$ contains at least one 4+-vertex if $f_2$ is a (3, 3, 4+)-face and at least two 4+-vertices if $f_2$ is a (3, 3, 3)-face, respectively. By (R2) $f$ gets $\frac{1}{2}$ from each incident rich 4-vertex or 5+-vertex. So $\mu^*(f) \geq 7 - 6 - \max\{1 + \frac{1}{2} - \frac{1}{2}, 1 \cdot 2 - \frac{1}{2} \cdot 2\} = 0$. □

Lemma 3.5. The final charge of $C_0$ is positive.

Proof. Let $E(C_0, G - C_0)$ be the set of edges between $C_0$ and $G - C_0$. Let $\epsilon'$ be the number of edges in $E(C_0, G - C_0)$ that is not on a 3-face and $x$ be the number of charges $C_0$ receives by (R3). Let $f_3 = |F_3|$ and $f_7$ be the number of special 7-faces. By (R3) and (R4), the final
charge of $C_0$ is at least
\[
\mu^*(C_0) = d(C_0) + 6 + \sum_{v \in C_0} (2d(v) - 6) - 3f_3 - f_7 + x
\]
\[
\geq d(C_0) + 6 + \sum_{v \in C_0} 2d(v) - 2d(C_0) - 3f_3 - f_7 + x
\]
\[
\geq 6 - d(C_0) + 2|E(C_0, G - C_0)| - 3f_3 - f_7 + x
\]
\[
= 6 - d(C_0) + f_3 + 2e' - f_7 + x,
\]
where the last equality follows from that each 3-face in $F_3$ contains two edges in $E(C_0, G - C_0)$.

Let $f$ be a 7*-face adjacent to $C_0$. A path on $f$ is charge-friendly if no vertex on it is on a triangle that needs charge from $f$ (which means triangles on the paths are in $F_3$). Let $P$ be a charge-friendly path on $f$. Then $f$ gives at least $d(f) - 6 - \left\lfloor \frac{d(f) + 1 - |V(P)|}{3} \right\rfloor$ to $C_0$, and thus
\begin{equation}
(1) \quad x \geq d(f) - 6 - \left\lfloor \frac{d(f) + 1 - |V(P)|}{3} \right\rfloor \geq \frac{2}{3}d(f) - 9 + \frac{|V(P)| - 1}{3}.
\end{equation}

Since $d^\Delta(G) \geq 2$, each special 7-face must share exactly one edge or one vertex with $C_0$ and each edge in $E(C_0, G - C_0) \cap E(f)$ is not on 3-faces. Then $e' \geq f_7$, with equality only if $e' = f_7 = d(C_0)$ and $f_3 = 0$, in which case, $\mu^*(C_0) \geq 6 - d(C_0) + d(C_0) > 0$. So we may assume that $e' \geq f_7 + 1$. Then
\[
f_7 = 0 \text{ when } d(C_0) \leq 8, \quad f_7 \leq 1 \text{ when } d(C_0) = 9, \text{ and } f_7 \leq 2 \text{ when } d(C_0) = 10,
\]
for otherwise, $\mu^*(C_0) \geq 6 - d(C_0) + f_3 + 2e' - f_7 + x \geq 6 - d(C_0) + f_3 + x + f_7 + 2 > 0$. Now assume that $\mu^*(C_0) \leq 0$. We consider a few cases.

**Case 1.** $f_7 = 0$. From $\mu^*(C_0) \geq 6 - d(C_0) + f_3 + 2e' - f_7 + x = 6 - d(C_0) + f_3 + 2e' + x$, we have $f_3 + 2e' + x \leq d(C_0) - 6 \leq 10 - 6 = 4$. So $e' \leq 2$.

Let $e' = 2$. Then $d(C_0) = 10$ and $f_3 = x = 0$. It follows that $G$ is adjacent to a 7*-face $f$ that contains at least four consecutive 2-vertices, thus $f$ contains a charge-friendly path $P$ with $|V(P)| \geq 6$, so by (1) $x \geq \frac{2}{3}(7 - 9) + \frac{6 - 1}{3} > 0$, a contradiction.

Let $e' = 1$. It follows that $f_3 \leq 2$.

- If $f_3 = 2$, then $d(C_0) = 10$ and $x = 0$. Now $C_0$ is adjacent to a 7*-face that contains a path with a triangle at one end and having at least two consecutive 2-vertices, thus, $f$ contains a charge-friendly path $P$ with $|V(P)| \geq 6$, so by (1) $x \geq \frac{2}{3}(7 - 9) + \frac{6 - 1}{3} > 0$, a contradiction.

- If $f_3 = 1$, then $d(C_0) \geq 9$. Note that $C_0$ contains at most three 3*-vertices. If $d(C_0) = 9$, then $x = 0$ and $C_0$ is adjacent to a 7*-face that contains a path with a triangle at one end and having at least three consecutive 2-vertices. Thus, $f$ contains a charge-friendly path $P$ with $|V(P)| \geq 7$, so by (1) $x \geq \frac{2}{3}(7 - 9) + \frac{7 - 1}{3} > 0$, a contradiction. If $d(C_0) = 10$, then $x \leq 1$ and $C_0$ is adjacent to a 8*-face that contains a path with a triangle at one end and having at least four consecutive 2-vertices. Thus, $f$ contains a charge-friendly path $P$ with $|V(P)| \geq 8$, so by (1) $x \geq \frac{2}{3}(8 - 9) + \frac{8 - 1}{3} > 1$, a contradiction.
• If \( f_3 = 0 \), then \( d(C_0) \geq 8 \) and \( x \leq 2 \). Note that \( C_0 \) is adjacent to a \( 9^+ \)-face that contains at least \( d(C_0) - 1 \) consecutive 2-vertices, thus \( f \) contains a charge-friendly path of at least \( d(C_0) + 1 \) vertices, so \( x \geq \frac{2}{3}(9 - 9) + \frac{d(C_0)+1-1}{3} > 2 \) by (1), a contradiction.

Finally let \( e' = 0 \). Then \( f_3 + x \leq d(C_0) - 6 \), and each edge in \( E(C_0, G - C_0) \) is on a 3-face. Note that we may assume that \( f_3 > 0 \), for otherwise \( G = C_0 \). Now follow the boundaries of the \( 7^+ \)-faces adjacent to \( C_0 \), each of the \( f_3 \) triangles is encountered twice, thus the \( 7^+ \)-faces do not give charge to at least \( 2f_3 \) triangles. So \( x \geq 2f_3 \). It follows \( f_3 = 1 \) and \( d(C_0) \geq 9 \). In this case, \( C_0 \) is adjacent to a \( 9^+ \)-face \( f \) that contains at least \( d(C_0) - 2 \) consecutive 2-vertices and a triangle at one end, thus \( f \) contains a charge-friendly path of at least \( d(C_0) + 2 \) vertices, so by (1), \( x \geq \frac{2}{3}(9 - 9) + \frac{d(C_0)+2-1}{2} > 3 \), a contradiction.

Case 2. \( f_3 = 1 \). As \( \mu^+(C_0) \geq 6 - d(C_0) + f_3 + 2e' - f_7 + x \geq 6 - d(C_0) + f_3 + x + f_7 + 2 \), either \( d(C_0) = 9 \) and \( (f_3, x, e') = (0, 0, 2) \), or \( d(C_0) = 10 \) and \( f_3 + x + 2e' \leq 5 \). In the former case, \( C_0 \) is adjacent to a \( 9^+ \)-face that contains seven 2-vertices, thus by (1), \( x \geq \frac{2}{3}(9 - 9) + \frac{7-1}{3} > 0 \), a contradiction. Consider the latter case. It follows that \( e' = 2 \) and \( f_3 + x \leq 1 \). So if \( f_3 = 0 \), then \( C_0 \) is adjacent to a \( 9^+ \)-face that contains eight consecutive 2-vertices, thus \( x \geq \frac{2}{3}(9 - 9) + \frac{8-1}{3} > 2 \) by (1), a contradiction; if \( f_3 = 1 \), then \( C_0 \) is adjacent to a \( 7^+ \)-face \( f \) that contains at least three consecutive 2-vertices and a triangle at one end, thus \( f \) contains a charge-friendly path of at least 6 vertices, so thus \( x \geq \frac{2}{3}(7 - 9) + \frac{6-1}{3} > 0 \) by (1), a contradiction again.

Case 3. \( f_7 = 2 \). Then \( \mu^+(C_0) \geq 6 - 10 + f_3 + 2e' - f_7 + x \geq -4 + f_3 + x + f_7 + 2 \), we have \( f_3 = x = 0 \) and \( e' = 3 \). Thus, \( C_0 \) is a 10-face and the two \( 7 \)-faces must share an edge in \( E_0 \). Then \( C_0 \) is adjacent to a \( 8^+ \)-face \( f' \) that contains seven consecutive 2-vertices. Thus \( x \geq \frac{2}{3}(8 - 9) + \frac{7-1}{3} > 0 \) by (1), a contradiction.

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