Evidence for Bohmian velocities from conditional Schrödinger equation

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It is often argued that measurable predictions of Bohmian mechanics cannot be distinguished from those of a theory with arbitrarily modified particle velocities satisfying the same equivariance equation. By considering the wave function of a closed system in a state with definite total energy, we argue that a distinction in measurable predictions is possible. Even though such a wave function is time-independent, the conditional wave function for a subsystem depends on time through the time-dependent particle trajectories not belonging to the subsystem. If these trajectories can be approximated by classical trajectories, then the conditional wave function can be approximated by a wave function which satisfies Schrödinger equation in a classical time-dependent potential, which is in good agreement with observations. However, such an approximation cannot be justified for particle velocities significantly deviating from the Bohmian ones, implying that Bohmian velocities are observationally preferred.

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I. INTRODUCTION

The Bohmian formulation of quantum mechanics (QM) in terms of deterministic particle trajectories \[1\], originally developed to offer a possible resolution of the interpretative difficulties with QM (see also \[2,3\]), today offers also a practical tool in various applications \[4,5\]. A recent work has demonstrated that Bohmian particle trajectories can even be weakly measured \[6,7\], so in a sense they are no longer “hidden variables”. Namely, these weak measurements demonstrate that Bohmian trajectories are not more hidden than the wave function, i.e., that both can be measured as properties of large statistical ensembles, but not as properties of individual systems. Whether the Bohmian trajectories, or wave functions, or both, have reality at the level of individual systems is observationally still unclear.

A frequent argument (see, e.g., \[9,10\]) against reality of Bohmian trajectories is the claim that their measurable predictions cannot be distinguished from those of a theory with arbitrarily modified particle velocities satisfying the same equivariance equation

\[
\partial_t |\psi|^2 + \sum_{a=1}^N \nabla_a (|\psi|^2 u_a) = 0, \tag{1}
\]

where \(\Psi(x_1, \ldots, x_N, t)\) is the wave function for \(N\) particle positions \(x_a\), and \(u_a(x_1, \ldots, x_N, t)\) is the velocity of particle \(a\) as a function of time \(t\) and all particle positions \(x_1, \ldots, x_N\). Eq. (1) is satisfied for any velocity of the form

\[
u_a = v_a + \frac{j_a}{|\psi|^2}, \tag{2}
\]

where \(j_a(x_1, \ldots, x_N, t)\) is an arbitrary vector field with the property \(\sum_{a=1}^N \nabla_a j_a = 0\), \(v_a\) is the Bohmian velocity \(v_a = m_a^{-1} \nabla_a S\), \(S(x_1, \ldots, x_N, t)\) is defined by the wave function in the polar form \(\Psi = Re^{iS/h}\), and \(m_a\) is the mass of particle \(a\). The arguments for preferring Bohmian velocity \(u_a = v_a\) (i.e., \(j_a = 0\)) include the analogy with classical Hamilton-Jacobi mechanics \[11,12\], Galilean invariance \[13\], and the relation with weak measurement of velocity \[9,10\]. Nevertheless, from an observational point of view neither of these arguments seems conclusive.

In this paper we present a new observationally supported argument for preferring Bohmian velocity \(u_a = v_a\). For that purpose, suppose that a closed system (e.g., the whole Universe) is in a state with definite total energy \(E\). The Schrödinger equation implies that the wave function has the form \(\Psi(x_1, \ldots, x_N, t) = \psi(x_1, \ldots, x_N)e^{-iEt/h}\), implying that the wave function is essentially time-independent, in the sense that \(|\psi|^2 = |\psi|^2\) does not depend on time. On the other hand, the observed world clearly depends on time. How the observed time dependence can be reconciled with a time-independent wave function of the Universe \(\psi\)?

An obvious answer is that our Universe is simply not in a state with definite total energy, but in a superposition of many different Hamiltonian eigenstates. However, such an answer does not seem satisfying \[13\] because such a state of matter in a superposition of different energies would imply that the gravitational field, determined by energy of matter, is in a superposition of states corresponding to macroscopically different gravitational fields, which would contradict the classical appearance of observed gravitational fields. Just as decoherence theory applied to coupling between charge and electromagnetic field explains the charge superselection rule \[14,15\], a similar superselection rule preventing superpositions of different matter energies is expected from decoherence theory applied to coupling between matter and gravity. This argument against time dependence in QM is also closely related to the problem of time in canonical quan-
tum gravity [16, 17], where the total energy of matter and gravity must be exactly zero at any point in space, thus preventing any time dependence of the wave function.

A key to the resolution of the problem of time-dependence lies in the observation that most interpretations of QM introduce some additional time dependence in the system (e.g., through wave-function collapse, classical macro-world, free human interventions, or time-dependent hidden variables), implying that time evolution of the Universe is not described completely by the Schrödinger equation alone. In Appendix A we briefly review how various interpretations deal with it. The main purpose of the present paper is to explain in detail how the Bohmian formulation of QM explains the origin of time dependence in closed quantum systems with definite energy, and how that provides an observational evidence for preferring Bohmian velocities over other velocities in (2) consistent with (1).

The basic idea is not difficult to understand. Suppose that one is interested in a subsystem consisting of \( n \leq N \) particles. For that purpose one can consider the conditional wave function

\[
\psi_c(x_1, \ldots, x_n, t) = \psi(x_1, \ldots, x_n, X_{n+1}(t), \ldots, X_N(t)),
\]

where \( X_{n+1}(t), \ldots, X_N(t) \) are the actual particle trajectories of all other particles, with velocities \( dX_a(t)/dt = u_a \). The conditional wave function has a well-understood role in explaining the illusion of wave-function collapse \[18\]. In this paper we study a different role of conditional wave function, as a tool to understand the origin of time dependence in classical mechanics, with emphasis on the concepts of conditional Hamiltonian and conditional Hamilton-Jacobi equation. Then in the central section, Sec. III, we generalize these results in classical mechanics to those in quantum mechanics, by presenting a detailed derivation of the conditional Schrödinger equation. In Sec. IV we discuss the physical relevance of the conditional Schrödinger equation, where we explain in more detail how conditional Schrödinger equation explains the origin of time evolution in closed quantum systems with definite energy, and how that prefers Bohmian velocity over other velocities consistent with (1). Finally, the conclusions are drawn in Sec. V.

II. CONDITIONAL CLASSICAL MECHANICS

A. Conditional Hamiltonian

Consider a closed classical system of two particles with positions \( x_1 \) and \( x_2 \), described by the Hamiltonian

\[
H(x_1, p_1, x_2, p_2) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(x_1, x_2).
\]

(We consider only two particles moving in only one dimension just to simplify the notation. The generalization of our results to arbitrary number of particles in three dimensions is trivial.) We distinguish the abstract time-independent canonical variables \( x_a, p_a, a = 1, 2 \), from the physical time-dependent particle trajectories \( X_a = X_a(t), P_a = P_a(t) \). The latter are given by the Hamilton equations of motion

\[
\frac{dX_a}{dt} = \frac{\partial H}{\partial P_a}, \quad \frac{dP_a}{dt} = -\frac{\partial H}{\partial X_a}.
\]

Now assume that trajectory \( X_2(t), P_2(t) \) of the second particle is known. The dynamics of the first particle can be described by the conditional Hamiltonian

\[
H_c(x_1, p_1, t) = H(x_1, p_1, X_2(t), P_2(t)).
\]

Eq. (8) for \( a = 1 \) is equivalent to

\[
\frac{dX_1}{dt} = \frac{\partial H_c}{\partial P_1}, \quad \frac{dP_1}{dt} = -\frac{\partial H_c}{\partial X_1}.
\]
which are nothing but Hamilton equations of motion for the system described by the conditional Hamiltonian. Using (7), we see that (9) can be written in a more explicit form

\[ H_c(x_1, p_1, t) = \frac{p_1^2}{2m_1} + \frac{P_2^2(t)}{2m_2} + V_c(x_1, t), \]  

(11)

where

\[ V_c(x_1, t) = V(x_1, X_2(t)). \]  

(12)

The term \( P_2^2(t)/2m_2 \) in (11) depends only on time, not on \( x_1 \) or \( p_1 \), so this term is irrelevant for the Hamilton equations of motion (10). Consequently, (10) is valid also if \( H_c \) given by (11) is replaced by a simpler conditional Hamiltonian

\[ \hat{H}_c(x_1, p_1, t) = \frac{p_1^2}{2m_1} + V_c(x_1, t). \]  

(13)

Unlike (7), the conditional Hamiltonian (13) has an explicit dependence on time and the corresponding energy is not conserved. This is because only the first particle is a dynamical variable in the conditional Hamiltonian, and consequently the system described by the conditional Hamiltonian is not closed.

### B. Conditional Hamilton-Jacobi equation

The conditional Hamiltonian can be used to introduce the conditional Hamilton-Jacobi equation. We start from the total principal function \( S_{tot}(x_1, x_2, t) \) satisfying the Hamilton-Jacobi equation for the whole system

\[ H(x_1, \partial_1 S_{tot}, x_2, \partial_2 S_{tot}) = -\partial_t S_{tot}, \]  

(14)

where \( \partial_a \equiv \partial/\partial x_a, \partial_t \equiv \partial/\partial t. \) Since the total energy \( E \) is conserved, the solution of (14) has the form

\[ S_{tot}(x_1, x_2, t) = S(x_1, x_2) - Et, \]  

(15)

so (14) reduces to the time-independent Hamilton-Jacobi equation

\[ H(x_1, \partial_1 S, x_2, \partial_2 S) = E. \]  

(16)

To simplify further mathematical expressions, we introduce a short-hand notation

\[ F(x_1, x_2) \equiv F, \]
\[ F(x_1, X_2(t)) \equiv F_c, \]
\[ F(X_1(t), x_2(t)) \equiv F_t, \]  

(17)

for any function \( F. \) With this notation, the particle trajectories \( S \) with Hamiltonian (11) are equivalent to

\[ \frac{dX_a(t)}{dt} = \frac{1}{m_a}(\partial_a S)_t, \]  

(18)

where

\[ m_a \frac{dX_a(t)}{dt} = P_a(t). \]  

(19)

Now assume that \( X_2(t) \) is known. Clearly, (18) implies that the remaining trajectory \( X_1(t) \) can be calculated as

\[ \frac{dX_1(t)}{dt} = \frac{1}{m_1}(\partial_1 S_c)_t, \]  

(20)

where

\[ S_c(x_1, t) = S(x_1, X_2(t)) \]  

(21)

is the conditional principal function.

On the other hand, we have seen in Sec. IIA that dynamics of the remaining particle can equivalently be described by the simplified conditional Hamiltonian (13). Since Hamilton formalism is equivalent to Hamilton-Jacobi formalism, it implies that the remaining particle can be described by the conditional Hamilton-Jacobi equation

\[ \hat{H}_c(x_1, \partial_1 \hat{S}_c, t) = -\partial_t \hat{S}_c, \]  

(22)

the solution of which determines the particle trajectory

\[ \frac{dX_1(t)}{dt} = \frac{1}{m_1}(\partial_1 \hat{S}_c)_t. \]  

(23)

Here \( \hat{S}_c(x_1, t) \) is a solution of (22), which gives the same velocity \( dX_1/dt \) at any given \( x_1 \) as \( S_c(x_1, t) \) given by (21), so

\[ \partial_t \hat{S}_c = \partial_t S_c \]  

(24)

for any \( x_1. \) Consequently, \( \hat{S}_c \) and \( S_c \) are related as

\[ \hat{S}_c(x_1, t) = S_c(x_1, t) + f(t), \]  

(25)

where \( f(t) \) is some function which depends only on \( t. \)

### III. CONDITIONAL QUANTUM MECHANICS

#### A. Conditional wave function in the polar form

Now we are ready to generalize the results of Sec. II to quantum mechanics. The total system is described by the total wave function \( \psi_{tot}(x_1, x_2, t) \) satisfying the Schrödinger equation

\[ H(x_1, \hat{p_1}, x_2, \hat{p_2})\psi_{tot} = i\hbar \partial_t \psi_{tot}, \]  

(26)

where

\[ \hat{p_a} = -i\hbar \partial_a. \]  

(27)

We are interested in a solution with a definite total energy \( E, \) so \( \psi_{tot}(x_1, x_2, t) = \psi(x_1, x_2)e^{-iEt/\hbar} \) and (26) reduces to

\[ H(x_1, \hat{p_1}, x_2, \hat{p_2})\psi = E\psi. \]  

(28)
The solution can be written in the polar form
\[ \psi(x_1, x_2) = R(x_1, x_2) e^{iS(x_1, x_2)/\hbar}, \] (29)
so complex equation (28) is equivalent to a set of two real equations
\[ H_Q(x_1, \partial_1 S, x_2, \partial_2 S) = E, \] (30)
\[ \partial_1 (R^2 v_1) + \partial_2 (R^2 v_2) = 0, \] (31)
where
\[ H_Q(x_1, \partial_1 S, x_2, \partial_2 S) = H(x_1, \partial_1 S, x_2, \partial_2 S) + Q_1(x_1, x_2) + Q_2(x_1, x_2), \] (32)
\[ Q_a = -\frac{\hbar^2}{2m_a} \frac{\partial^2 R}{R}, \] (33)
\[ v_a = \frac{1}{m_a} \partial_a S, \] (34)
and \( H(x_1, \partial_1 S, x_2, \partial_2 S) \) is the classical Hamiltonian. We see that (30) has the same form as the classical time-independent Hamilton-Jacobi equation (16), with the replacement
\[ V(x_1, x_2) \to V_Q(x_1, x_2) = V(x_1, x_2) + Q(x_1, x_2), \] (35)
where \( Q = Q_1 + Q_2 \).

In the Bohmian interpretation of QM, quantum particles have definite trajectories analogous to classical trajectories (13)
\[ \frac{dX_a(t)}{dt} = \frac{1}{m_a} (\partial_a S)_t, \] (36)
with the only difference that now \( S \) satisfies the quantum Hamilton-Jacobi equation (30), instead of the classical Hamilton-Jacobi equation (16). So, if \( X_2(t) \) is known, then the motion of the remaining particle can be calculated from
\[ \frac{dX_1(t)}{dt} = \frac{1}{m_1} (\partial_1 S_c)_t, \] (37)
where \( S_c(x_1, t) = S(x_1, X_2(t)) \). Equivalently, it can also be calculated from
\[ \frac{dX_1(t)}{dt} = \frac{1}{m_1} (\partial_1 \tilde{S}_c)_t, \] (38)
where \( \tilde{S}_c(x_1, t) \) satisfies the quantum analogue of (22)
\[ \dot{H}_{Q_c}(x_1, \partial_1 \tilde{S}_c, t) = -\partial_t \tilde{S}_c, \] (39)
with
\[ \dot{H}_{Q_c}(x_1, \partial_1 \tilde{S}_c, t) = \frac{(\partial_1 \tilde{S}_c)^2}{2m_1} + V_c(x_1, t) + Q_1c(x_1, t) + Q_2c(x_1, t). \] (40)

Analogously to (25), we see that
\[ \tilde{S}_c(x_1, t) = S_c(x_1, t) + f(t), \] (41)
where \( f(t) \) is some function which depends only on \( t \).

Now consider (31) along the trajectory \( x_2 = X_2(t) \). Clearly, (31) for \( x_2 = X_2(t) \) can be written as
\[ \partial_1 (R_c^2 v_{1c}) + (\partial_2 R^2_c) v_{2c} + R_c^2 (\partial_2 v_{2c}) = 0. \] (42)

On the other hand, we have
\[ \partial_t R_c^2 = \frac{\partial (R_c^2)}{\partial X_2} \frac{dX_2(t)}{dt} = (\partial_2 R^2_c) v_{2t}. \] (43)

This can be used to eliminate \((\partial_2 R^2_c)\) from (42), which leads to
\[ \frac{v_{2c}}{v_{2t}} \partial_t R_c^2 + \partial_1 (R_c^2 v_{1c}) + R_c^2 (\partial_2 v_{2c}) = 0. \] (44)

### B. Conditional Schrödinger equation

Eqs. (39), (40), (31), and (44) are exact equations. In Appendix B we show that they can be combined into an exact pseudo-Schrödinger equation. Here, however, we are more interested in the case when the trajectory \( X_2(t) \) can be approximated by a classical trajectory. This happens when \( R(x_1, x_2) \) does not depend much on \( x_2 \). (For a more detailed discussion of the classical limit in Bohmian mechanics, see also [19].) More precisely, if \( L_2 \) is a typical length scale within which \( R \) changes significantly by a change of \( x_2 \), then \( X_2(t) \) behaves classically if
\[ L_2 \gg \hbar/P_2, \] (45)
where \( P_2 = m_2 \dot{X}_2 \) is the momentum. In this approximation we can take \( \partial_2 R \approx 0 \), so (33) implies that
\[ Q_2 \approx 0. \] (46)

Similarly, we have \( \partial_2 Q_1 \approx 0, \partial_2 Q_2 \approx 0 \), implying that the quantum force \( -\partial_2 (Q_1 + Q_2) \) on the second particle is negligible. Therefore the motion of the second particle is approximately governed by local classical forces, implying that velocity of the second particle does not depend much on the position of the first particle at the same time, so
\[ v_{2c}(x_1, t) \approx v_{2t}(t). \] (47)

Similarly, the quantity \((\partial_2 v_{2c})\) does not depend much on \( x_1 \), so we can write
\[ (\partial_2 v_{2c}) \approx -\Gamma(t), \] (48)
where \( \Gamma(t) \) is a function which depends only on \( t \). Inserting all these approximations into equations (39), (40),
these equations reduce to a set of two independent approximate equations
\[
\frac{(\partial_t S_c)^2}{2m_1} + V_c + Q_{1c} + \dot{f}(t) = -\partial_t S_c,
\]
where \( f(t) = df(t)/dt \) and all functions with the label \( c \) are functions of \( x_1 \) and \( t \).

We see that the left-hand side of (50) looks like an equivariance equation, but the right-hand side does not due to the \( \Gamma \)-term. To find a true equivariance equation, we define a new quantity
\[
\rho_c(x_1, t) = \frac{R_c^2(x_1, t)}{N(t)},
\]
where \( N(t) \) will be fixed from the requirement that \( \rho_c(x_1, t) \) should satisfy an equivariance equation. Inserting (51) into (50), we get
\[
N(t)[\partial_t \rho_c + \partial_t (\rho_c v_{1c})] + \rho_c[N(t) + (\Gamma(t)N(t)) = 0.
\]
The requirement of equivariance is the vanishing of the first square bracket in (52)
\[
\partial_t \rho_c + \partial_t (\rho_c v_{1c}) = 0,
\]
which is satisfied if the second square bracket in (52) vanishes as well
\[
N(t) + \Gamma(t)N(t) = 0.
\]
Eq. (54) is a differential equation with the solution
\[
N(t) = N(0)e^{-\int_0^t dt \Gamma(t)}.
\]
Note that \( \Gamma(t) \) is not a positive-definite function, so the exponent in (55) may be either positive or negative for various values of \( t \). On the other hand, we choose \( N(0) \) to be positive, implying that \( N(t) \) is positive for all \( t \).

Next we introduce the wave function
\[
\tilde{\psi}_c(x_1, t) = \tilde{R}_c(x_1, t)e^{i\tilde{S}_c(x_1, t)/\hbar},
\]
where
\[
\tilde{R}_c(x_1, t) = \frac{R_c(x_1, t)}{\sqrt{N(t)}},
\]
and \( \tilde{S}_c(x_1, t) \) is given by [41]. Now (51) can be written as \( \rho_c = \tilde{R}_c^2 \), so (52) can be written as
\[
\partial_t \tilde{R}_c^2 + \partial_t (\tilde{R}_c^2 \tilde{v}_{1c}) = 0,
\]
where, due to (54) and (41),
\[
\tilde{v}_{1c} = \frac{1}{m_1} \partial_t \tilde{S}_c = \frac{1}{m_1} \partial_t S_c.
\]
Similarly, (49) can be written as
\[
\frac{(\partial_t \tilde{S}_c)^2}{2m_1} + V_c + \tilde{Q}_{1c} = -\partial_t \tilde{S}_c,
\]
where, due to (33) and (57),
\[
\tilde{Q}_{1c} = -\frac{\hbar^2}{2m_1} \frac{\partial_t^2 \tilde{R}}{\tilde{R}} = -\frac{\hbar^2}{2m_1} \frac{\partial_t^2 \tilde{R}}{\tilde{R}}.
\]
Eqs. (58) and (60), together with (56), are easily recognized to be equivalent to the conditional Schrödinger equation
\[
\left[ -\frac{\hbar^2}{2m_1} \frac{\partial_t^2}{\partial_t} + V_c(x_1, t) \right] \tilde{\psi}_c(x_1, t) = i\hbar \partial_t \tilde{\psi}_c(x_1, t).
\]
Eq. (62) is our final result, which represents a quantum analogue of the classical Hamilton-Jacobi equation (22) with (13). Note, however, that (22) with (13) is an exact result, while (62) is only an approximation. Eq. (62) can also be obtained in a different way, by using a WKB-like expansion in powers of \( 1/\sqrt{m_1} \). A similar result has also been obtained under more restrictive assumptions in [21].

In practical descriptions of open physical systems, effective Schrödinger equations of the form of (62) are often taken as a starting point, with heuristic justifications for the use of classical time-dependent potentials in quantum mechanics. (Recall that, in general, a classical time-dependent potential is defined as the conditional potential (6) in which \( X_{n+1}(t), \ldots, X_{N}(t) \) are classical trajectories.) Our detailed analysis above shows how such an effective description can be obtained from first principles, by using conditional wave functions.

IV. PHYSICAL RELEVANCE OF THE CONDITIONAL SCHRODINGER EQUATION

After the technical developments in Secs. III and IV, now we want to understand the physical relevance of the conditional Schrödinger equation.

A. When different velocities cannot be distinguished

In modeling quantum measurements, one often argues (either in Bohmian [1–4] or non-Bohmian [12, 22] context) that the ordinary Schrödinger equation itself, without particle trajectories, provides a non-trivial time evolution \( \Psi(x_1, x_2, t) \) of the form
\[
\psi(x_1)\phi_0(x_2) \rightarrow \sum_p c_p \psi_b(x_1)\phi_p(x_2),
\]
where \( \psi(x_1) = \sum_p c_p \psi_b(x_1) \) is the initial wave function of the measured system, \( \phi_0(x_2) \) is the initial wave function
of the measuring apparatus, \( \psi_s(x_1) \) are normalized eigenstates of the measured observable, and \( \phi_b(x_2) \) are normalized macroscopically distinct pointer states well localized in the \( x_2 \)-space. Then particle trajectories \( X_a(t) \) with any velocities

\[
\frac{dX_a(t)}{dt} = u_a(X_1(t), X_2(t), t)
\]  
(64)

satisfying the equivariance equation

\[
\partial_t \lvert \Psi \rvert^2 + \partial_1 (\lvert \Psi \rvert^2 u_1) + \partial_2 (\lvert \Psi \rvert^2 u_2) = 0
\]  
(65)

will imply that \( X_2 \) will enter the channel \( \phi_b(x_2) \) with probability \( \lvert c_b \rvert^2 \), in agreement with observations. For that purpose, the Bohmian velocity \( u_a = v_0 \) is not better than any other velocity \( u_a \) satisfying (65). The localized wave packets \( \phi_b(x_2, t) \) themselves move approximately classically owing to the Ehrenfest theorem, even if the trajectory \( X_2(t) \) is highly non-classical at lengths shorter than the width of the wave packet. So if the only role of particle trajectories is to pick up some particular channel \( \phi_b(x_2) \) in (63), then observations cannot distinguish between different velocities \( u_a(x_1, x_2, t) \) satisfying (65). Some even argue that it makes the very existence of such trajectories superfluous [23, 24].

**B. When different velocities can be distinguished**

In this paper, however, we point out that picking up some particular channel in (63) is not the only role of particle trajectories. In modeling time evolutions of the form (63), one usually describes it with a Schrödinger equation in which the Hamiltonian itself has an explicit dependence on time. (For instance, one often uses a coupling between the measured system and the apparatus that starts at some particular time \( t_0 \) and ends at another particular time \( t_1 \).) But fundamental Hamiltonians do not have an explicit dependence on time, so such a description is at best an effective one. One might argue that the time dependence comes from a classical time-dependent environment, but if the macroscopic apparatus is to be ultimately described by QM as in (63), then, at the fundamental level, the environment should also be quantum. On the other hand, if the closed system as a whole is in a state with definite total energy, then no time evolution at all, including the time evolution in (63), can be described by the Schrödinger equation alone. Some additional time-dependence is needed.

In this paper we propose that the time evolution of the wave function in (63) is fundamentally a consequence of time-dependent particle trajectories. More precisely, we propose that the time-dependent wave function \( \Psi(x_1, x_2, t) \) in (63) is actually a conditional wave function

\[
\Psi(x_1, x_2, t) = \Psi(x_1, x_2, X_3(t)),
\]  
(66)

where \( X_3(t) \) represents trajectories of the particles constituting the environment. (Here by “environment” we mean the controllable environment degrees of freedom which are responsible for preparations of quantum experiments. They should not be mixed with uncontrollable environment degrees of freedom responsible for decoherence [13, 22].)

It is known from experience that classical models of environment provide an approximation that agrees well with observations. Essentially, a classical model of a time-dependent environment can be reduced to a Hamiltonian with a time-dependent potential of the form

\[
V(x_1, x_2, t) = V(x_1, x_2, X_3(t)),
\]  
(67)

where \( X_3(t) \) is a classical trajectory. But from the results of Sec. 1113 we know that conditional wave function approximately satisfies a Schrödinger equation with such a classically time-dependent potential, provided that \( X_3(t) \) can actually be approximated by a classical trajectory. We have seen that such an approximation is justified if the velocities \( u_a \) are the Bohmian velocities \( v_a \).

On the other hand, with an arbitrary modification of velocities as in Eq. (2), such an approximation does not work. (Even though it may work for some sufficiently small modifications.) The reason is the fact that \( V(x_1, x_2, X_3(t)) \) would not have approximately classical dependence on time if \( X_3(t) \) did not have approximately classical dependence on time. As a consequence, the conditional Schrödinger equation (62) with a classical time-dependence of the potential could not be obtained. Thus we conclude that observations prefer Bohmian velocities \( u_a = v_a \) over other arbitrary modifications of the form of (2). With arbitrary modifications, or more precisely with velocities significantly deviating from the Bohmian ones, it would be difficult to explain why quantum systems described by the time-dependent potentials of the form (67) with classical time-dependence \( X_3(t) \) are in agreement with observations.

It is also useful to illustrate the meaning of “significant” deviations by more specific examples. If a vector field \( f_a(x_1, \ldots, x_N, t) \) has the property \( \sum_a \nabla_a f_a = 0 \), then so does the vector field \( \lambda f_a \) with an arbitrary constant \( \lambda \). Therefore the velocity of the form

\[
u_a = v_a + \lambda \frac{f_a}{|\Psi|^2},
\]  
(68)

satisfies the equivariance equation (11) for any \( \lambda \). Clearly, the Bohmian velocity \( v_a \) cannot be experimentally distinguished from (68) with a sufficiently small \( \lambda \). In this case, the deviation of (68) from \( v_a \) is not significant. By contrast, a significant deviation of (68) from \( v_a \) corresponds to the case in which \( \lambda \) is sufficiently large. Even if the quantum potential is exactly zero, in which case \( v_a \) coincides with classical velocity, the velocity (68) will be very different from the classical velocity if \( \lambda \) is sufficiently large.

An even more specific example is provided by the case in which \( |\Psi|^2 \) in (11) does not depend on time. In this case one can choose \( f_a = |\Psi|^2 v_a \), so (68) becomes

\[
u_a = (1 + \lambda) v_a.
\]  
(69)
Thus the actual velocities are proportional to the Bohmian ones, but not equal to them. An extreme example is provided by $\lambda = -1$, in which case particles do not move at all. It should be obvious that particles which do not move cannot explain why the Universe (with a time-independent total wave function) depends on time, despite the fact that (1) is satisfied.

In addition, note that the condition (45), needed for validity of the classical approximation, is not satisfied for well-localized wave packets. Just the opposite, this condition is best satisfied for very delocalized wave functions, such as a plane wave $e^{ik_2 x_2}$. When this condition is met, then the wave function does not look like a classical particle with a time-dependent trajectory. For that reason, the role of actual Bohmian particle trajectories is essential.

C. Is conditional wave function essential?

One might argue that our argument for preferring Bohmian velocities over more general velocities can be reduced to the explanation of the classical limit in QM, and that conditional wave functions are a red herring which are not essential for the argument.

To see if this is true, consider the following question: If nature is quantum, then why some objects seem to have classical trajectories? The standard (non-Bohmian) answer, presented in a somewhat over-simplified form, is this: Some objects have classical trajectories because localized wave packets have classical trajectories, owing to the Ehrenfest theorem. However, there is a problem with that answer. If the wave function of the closed system has definite total energy, then the wave function does not depend on time, and therefore wave packets do not depend on time, and therefore wave packets do not move at all. So again, if wave packets do not move, then why do we see that some objects move with classical trajectories?

The Bohmian formulation of QM provides a very clear answer. The objects which move are not wave packets, but the particles. But if Bohmian velocities are replaced with more general velocities, then, in general, this motion cannot be approximately classical. Therefore observations prefer Bohmian velocities over more general velocities.

In this way one concludes that Bohmian velocities are preferred without invoking conditional wave functions. There is, however, a problem with the argument above. This argument uses the assertion that “the objects which move are not wave packets, but the particles”. This assertion cannot be completely true because there is experimental evidence that, in many cases, wave packets do move. But then again, how can moving wave packets be compatible with the total wave function which does not depend on time? The Bohmian answer is: because these time-dependent wave functions are conditional wave functions. So this is why it is important to analyze the behavior of conditional wave functions, and to show that they are compatible with approximations which we already know to be in agreement with observations.

V. CONCLUSION

It is well known that QM has both deterministic aspects (the time-evolution of the wave function) and probabilistic aspects (the probabilistic interpretation of the wave function). For a given deterministically-evolving wave function of a subsystem, our results do not change the well-established result that the probabilistic measurable predictions of Bohmian mechanics do not depend on the choice of $J_0$ in (2). What our results demonstrate is that there are deterministic measurable predictions, regarding the evolution of wave function itself, which are sensible to different choices of $J_0$.

More specifically, in this paper we have presented a detailed derivation of the conditional Schrödinger equation satisfied by the conditional wave function, in which the time dependence of the wave function comes from the time dependence of known particle trajectories. If the known particle trajectories can be approximated by classical trajectories, then the conditional Schrödinger equation turns out to coincide with a Schrödinger equation in a classical time-dependent potential, which is known to be a good description in agreement with observations. The approximation by classical trajectories is justified if the particle velocities are the Bohmian velocities, but is not justified if the particle velocities are arbitrarily modified in a way that does not ruin validity of the quantum equivariance equation. We have used these results to propose that Bohmian particle trajectories are the fundamental source of time dependence in closed systems with definite energy, and to argue that observations prefer Bohmian velocities over other velocities compatible with the quantum equivariance equation.

Our main line of reasoning can be summarized in a simple form as follows:

- In practical applications of QM, one often uses an approximation (which agrees with observations) according to which some particles are described by a Schrödinger equation in a classical time-dependent potential.

- Bohmian mechanics needs to explain why that approximation is justified.

- The conditional wave function of these particles approximately obeys the Schrödinger equation in a classical time-dependent potential only if other environment particles, which are responsible for the time-dependence of the potential, follow approximately classical trajectories.

- Given that the total wave function of a closed system is time independent, the environment particles
can follow approximately classical trajectories if the Bohmian velocity formula is assumed, but not if the general velocity formula (significantly deviating from the Bohmian one) is assumed.

- Therefore, Bohmian mechanics justifies the approximation in the first item above if the Bohmian velocity formula is assumed, but not if the general velocity formula (significantly deviating from the Bohmian one) is assumed.

Another, even more concise way to summarize our results, is to point out that we have found a loophole in the usual proof that all velocities satisfying the quantum equivariance equation give the same measurable predictions. The usual proof takes for granted that the evolution (63) does not depend on the existence of particle trajectories, which, indeed, represents the core of the proof. However, taking it for granted overlooks the idea that the wave function of a closed system should not depend on time. When this overlooked idea is taken into account, one finds that the evolution (63) does depend on particle trajectories, because the wave function evolving with time is the conditional wave function.

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Appendix A: Time evolution in various interpretations of QM

Consider a closed system in a state with definite energy. How time evolution is possible in such a system? Unfortunately, an interpretation-independent answer to this question does not exist. In this Appendix we briefly review different answers provided by different interpretations.

“Copenhagen”-collapse interpretation. According to this interpretation proposed by von Neumann [23], everything, including the observer, is described by the wave function. However, the time-evolution of the wave function is not always governed by the Schrödinger equation. Instead, the act of observation is associated with a wave-function collapse. The collapse introduces an additional time-dependence in the system, present in the evolution by the Schrödinger equation. In this interpretation the act of observation plays a fundamental role, but the concept of observation itself is not described by physics.

“Copenhagen” interpretation with classical macro-world. According to this interpretation, usually attributed to Bohr, quantum mechanics can be applied only to the micro-world, not to the macro-world. The macro-world is described by classical mechanics, so the time evolution in the macro-world is not governed by a Schrödinger equation. In a closed system a quantum micro-subsystem interacts with a classical macro-subsystem, so that the time-dependence of the latter induces a time dependence of the former.

Modern instrumental “Copenhagen” interpretation. This is a widely-used practically oriented interpretation of QM (see, e.g., [20]), in which QM is nothing but a tool used to predict the probabilities of measurement outcomes for given measurement preparations. The measurement preparations are freely chosen by experimentalists. The experimentalists themselves are not described by QM. The free manipulations by experimentalists introduce additional time-dependence in the system not described by the Schrödinger equation. Within such an interpretation, the concept of wave function of the whole Universe does not make sense.

Objective collapse. In this interpretation the Schrödinger equation is modified by adding a stochastic term due to which the wave function collapses independently on any observers. The best known example of such a modification is the GRW theory [27].

Hidden variables. In this class of interpretations, the physical objects observed in experiments are not the wave functions, but some other time-dependent variables $\lambda(t)$. Even if the wave function governed by the Schrödinger equation is time-independent, the “hidden” variable $\lambda(t)$ may depend on time. The best known and most successful model of such variables is given by the Bohmian interpretation [1], some aspects of which are studied in more detail in the present paper.

Statistical ensemble. According to this interpretation, the wave function is only a property of a statistical ensemble of similarly prepared systems and tells nothing about properties of individual physical systems [28]. So if a wave function is time-independent, it does not mean that individual systems do not depend on time. This interpretation can be thought of as an agnostic variant of the hidden-variable interpretation, in the sense that the existence of hidden variables is compatible (and perhaps even natural) with the statistical-ensemble interpretation, but the statistical-ensemble interpretation refrains from saying anything more specific about them.

Consistent histories. In this interpretation [29], the wave function is a tool to assign a probability to a given time-dependent history of the physical system. In this sense, it is similar to hidden-variable interpretations. However, to avoid non-localities typically associated with normal hidden-variable theories, the consistent-histories interpretation replaces the classical propositional logic with a different kind of logic [30].

Many worlds. According to the many-world interpretation, the Universe as a whole is nothing but a wave function evolving according to the Schrödinger equation [31, 52]. So, if wave function of the Universe is a state with definite total energy, at first sight it seems impossi-
Appendix B: Pseudo-Schrödinger equation for conditional wave function

Consider a wave function \( \psi(x, t) \) satisfying equation of the form

\[
\left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + U(x, t) + h(t) + i\hbar \frac{\Gamma(x, t)}{2} \right] \psi = i\hbar \partial_t \psi,
\]

where \( U(x, t), h(t), \Gamma(x, t) \) are some real, but otherwise unspecified functions. Eq. (B17) looks similar to a Schrödinger equation, but the \( \Gamma \)-term makes the operator in the square brackets non-hermitian, implying that time-evolution of \( \psi \) is not unitary. Besides, the functions \( U(x, t), h(t), \Gamma(x, t) \) may in fact be functionals of \( \psi(x, t) \), in which case (B17) is not a linear equation. For those reasons, we refer to (B17) as a pseudo-Schrödinger equation.

Now let us write \( \psi \) in the polar form

\[
\psi(x, t) = R(x, t)e^{iS(x, t)/\hbar}.
\]

Using the identities

\[
-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = \left[ \frac{(\partial_x S)^2}{2m} + Q - \frac{i\hbar}{2R^2} \partial_x (R^2 v) \right] \psi,
\]

\[
i\hbar \partial_t \psi = \left[ \frac{i\hbar}{2R^2} (\partial_x R^2) - (\partial_x S) \right] \psi,
\]

where

\[
Q = \frac{\hbar^2}{2m} \frac{\partial^2 R}{R}, \quad v = \frac{1}{m} \partial_x S,
\]

and multiplying (B11) with \( \psi^* \) from the left, one finds that the complex equation (B11) is equivalent to a set of two real equations

\[
\frac{(\partial_x S)^2}{2m} + U + Q + \hbar = -\partial_t S, \tag{B6}
\]

\[
\frac{2i\hbar}{\partial_t R^2 + \partial_x (R^2 v)} = R^2 \Gamma. \tag{B7}
\]

Next we want to show that the exact results of Sec. [11A] can be written in the form of (B6) and (B7). For that purpose we first observe that (39), (40), and (41) can be written as

\[
\frac{(\partial_t S_c)^2}{2m_1} + V_c + Q_{2c} + Q_{1c} + \dot{f} = -\partial_t S_c. \tag{B8}
\]

Second, we write (41) as

\[
\partial_t R_c^2 + \partial_t (R_c^2 v_{1c}) = R_c^2 \left[ \frac{v_{2c}}{v_{2t}^2} - 1 \right] \frac{\partial_t R_c^2}{R_c^2} - (\partial_2 v_{2c})_c(x_1, t). \tag{B9}
\]

We see that (B8) and (B9) have the same form as (B6) and (B7), respectively, with the identifications \( m = m_1, x = x_1, R = R_c, v = v_{1c}, Q = Q_{1c} \), provided that we take

\[
U(x_1, t) = V_c(x_1, t) + Q_{2c}(x_1, t), \tag{B10}
\]

\[
h(t) = \dot{f}(t), \tag{B11}
\]

\[
\Gamma(x_1, t) = \left( \frac{v_{2c}(x_1, t)}{v_{2t}(t)} - 1 \right) \frac{\partial_t R_c^2(x_1, t)}{R_c(x_1, t)} - (\partial_2 v_{2c})_c(x_1, t). \tag{B12}
\]

In other words, the conditional wave function \( \psi_c(x_1, t) \) satisfies the pseudo-Schrödinger equation (B11)

\[
\left[ -\frac{\hbar^2}{2m_1} \frac{\partial^2 \psi_c}{\partial x_1^2} + U(x_1, t) + \dot{f}(t) + i\hbar \frac{\Gamma(x_1, t)}{2} \right] \psi_c = i\hbar \partial_t \psi_c,
\]

where \( U(x_1, t) \) and \( \Gamma(x_1, t) \) are given by (B10) and (B12), respectively. We see that (B10) and (B12) are functionals of \( \psi_c(x_1, t) \), so (B11) is a non-linear equation.

Finally, the \( \dot{f} \)-term in (B11) can be eliminated by defining a new wave function

\[
\tilde{\psi}_c(x_1, t) = \psi_c(x_1, t) e^{i\frac{f(t)}{\hbar}}. \tag{B14}
\]

The new wave function satisfies

\[
\left[ -\frac{\hbar^2}{2m_1} \frac{\partial^2 \tilde{\psi}_c}{\partial x_1^2} + U(x_1, t) + i\hbar \frac{\Gamma(x_1, t)}{2} \right] \tilde{\psi}_c = i\hbar \partial_t \tilde{\psi}_c. \tag{B15}
\]

A similar pseudo-Schrödinger equation has also been obtained in [7] and [33].

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