Critical and Maximum Independent Sets of a Graph

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Abstract

Let $G$ be a simple graph with vertex set $V(G)$. A set $S \subseteq V(G)$ is independent if no two vertices from $S$ are adjacent. By $\text{Ind}(G)$ we mean the family of all independent sets of $G$, while $\text{core}(G)$ and $\text{corona}(G)$ denote the intersection and the union of all maximum independent sets, respectively.

The number $d(X) = |X| - |N(X)|$ is the difference of $X \subseteq V(G)$, and a set $A \in \text{Ind}(G)$ is critical if $d(A) = \max\{d(I) : I \in \text{Ind}(G)\}$ [23].

Let $\text{ker}(G)$ and $\text{diadem}(G)$ be the intersection and union, respectively, of all critical independent sets of $G$ [13].

In this paper, we present various connections between critical unions and intersections of maximum independent sets of a graph. These relations give birth to new characterizations of König-Egerváry graphs, some of them involving $\text{ker}(G)$, $\text{core}(G)$, corona $(G)$, and $\text{diadem}(G)$.

Keywords: maximum independent set, maximum critical set, ker, core, corona, diadem, maximum matching, König-Egerváry graph.

1 Introduction

Throughout this paper $G$ is a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. If $X \subseteq V(G)$, then $G[X]$ is the subgraph of $G$ induced by $X$. By $G - W$ we mean either the subgraph $G[V(G) - W]$, if $W \subseteq V(G)$, or the subgraph obtained by deleting the edge set $W$, for $W \subseteq E(G)$. In either case, we use $G - w$, whenever $W = \{w\}$. If $A, B \subseteq V(G)$, then $(A, B)$ stands for the set $\{ab : a \in A, b \in B, ab \in E(G)\}$. 

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The neighborhood \( N(v) \) of \( v \in V(G) \) is the set \( \{ w : w \in V(G) \text{ and } vw \in E(G) \} \). In order to avoid ambiguity, we use also \( N_G(v) \) instead of \( N(v) \). The neighborhood \( N(A) \) of \( A \subseteq V(G) \) is \( \{ v \in V(G) : N(v) \cap A \neq \emptyset \} \), and \( N[A] = N(A) \cup A \).

A set \( S \subseteq V(G) \) is independent if no two vertices from \( S \) are adjacent, and by \( \text{Ind}(G) \) we mean the family of all the independent sets of \( G \). An independent set of maximum size is a maximum independent set of \( G \), and \( \alpha(G) = \max\{|S| : S \in \text{Ind}(G)\} \).

**Theorem 1.1** \([4]\) An independent set \( X \) is maximum if and only if every independent set \( S \) disjoint from \( X \) can be matched into \( X \).

For a graph \( G \), let \( \Omega(G) \) denote the family of all its maximum independent sets,

\[
\text{core}(G) = \bigcap\{S : S \in \Omega(G)\} \quad \text{and} \quad \text{corona}(G) = \bigcup\{S : S \in \Omega(G)\}.
\]

It is clear that \( N(\text{core}(G)) \subseteq V(G) - \text{corona}(G) \), and there exist graphs satisfying \( N(\text{core}(G)) \neq V(G) - \text{corona}(G) \) (for some examples, see the graphs from Figure 11 where \( \text{core}(G_1) = \{a, b\} \) and \( \text{core}(G_2) = \{x, y, z\} \)).

The problem of whether \( \text{core}(G) \neq \emptyset \) is \( \text{NP} \)-hard \([3]\).

![Figure 1: \( V(G_1) = \text{corona}(G_1) \cup N(\text{core}(G_1)) \cup \{d\}, V(G_2) = \text{corona}(G_2) \cup N(\text{core}(G_2)) \).](image)

A matching is a set \( M \) of pairwise non-incident edges of \( G \). If \( A \subseteq V(G) \), then \( M(A) \) is the set of all the vertices matched by \( M \) with vertices belonging to \( A \). A matching of maximum cardinality, denoted \( \mu(G) \), is a maximum matching.

**Lemma 1.2 (Matching Lemma)** \([3]\) If \( A \in \text{Ind}(G), \Lambda \subseteq \Omega(G), \) and \( |\Lambda| \geq 1 \), then there exists a matching from \( A - \bigcap \Lambda \to \bigcup \Lambda - A \).

For \( X \subseteq V(G) \), the number \( |X| - |N(X)| \) is the difference of \( X \), denoted \( d(X) \). The critical difference of \( G \) is \( \max\{d(X) : X \subseteq V(G)\} \). The number \( \max\{|I| : I \in \text{Ind}(G)\} \) is the critical independence difference of \( G \), denoted \( \text{id}(G) \). Clearly, \( d(G) \geq \text{id}(G) \). It was shown in \([23]\) that \( d(G) = \text{id}(G) \) holds for every graph \( G \). If \( A \) is an independent set in \( G \) with \( d(X) = \text{id}(G) \), then \( A \) is a critical independent set \([23]\).

For example, consider the graph \( G \) of Figure 2 where \( X = \{v_1, v_2, v_3, v_4\} \) is a critical set, while \( I = \{v_1, v_2, v_3, v_6, v_7\} \) is a critical independent set. Other critical sets are \( \{v_1, v_2\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_3, v_4, v_6, v_7\} \).

It is known that finding a maximum independent set is an \( \text{NP} \)-hard problem \([6]\). Zhang proved that a critical independent set can be found in polynomial time \([23]\).

**Theorem 1.3** \([4]\) Each critical independent set is included in a maximum independent set.
Figure 2: core($G$) = \{v_1, v_2, v_6, v_{10}\} is a critical set, since $d(\text{core}(G)) = 1 = d(G)$.

Theorem 1.3 leads to an efficient way of approximating $\alpha(G)$ [22]. Moreover, every critical independent set is contained in a maximum critical independent set, and such a maximum critical independent set can be found in polynomial time [9].

Theorem 1.4 [9] There is a matching from $N(S)$ into $S$ for every critical independent set $S$.

It is well-known that $\alpha(G) + \mu(G) \leq |V(G)|$ holds for every graph $G$. Recall that if $\alpha(G) + \mu(G) = |V(G)|$, then $G$ is a König-Egerváry graph [5, 21]. For example, each bipartite graph is a König-Egerváry graph as well. Various properties of König-Egerváry graphs can be found in [8, 12, 18]. It turns out that König-Egerváry graphs are exactly the graphs having a critical maximum independent set [10]. In [14] it was shown the following.

Lemma 1.5 [14] $d(G) = \alpha(G) - \mu(G)$ holds for each König-Egerváry graph $G$.

Using this finding, we have strengthened the characterization from [10].

Theorem 1.6 [14] For a graph $G$, the following assertions are equivalent:
(i) $G$ is a König-Egerváry graph;
(ii) there exists some maximum independent set which is critical;
(iii) each of its maximum independent sets is critical.

For a graph $G$, let ker($G$) be the intersection of all its critical independent sets [13], and diadem($G$) = \bigcup\{S : S \text{ is a critical independent set}\}.

In this paper we present several properties of critical unions and intersections of maximum independent sets leading to new characterizations of König-Egerváry graphs, in terms of core($G$), corona($G$), and diadem($G$).

2 Preliminaries

Let $G$ be the graph from Figure 2 the sets $X = \{v_1, v_2, v_3\}$, $Y = \{v_1, v_2, v_4\}$ are critical independent, and the sets $X \cap Y$, $X \cup Y$ are also critical, but only $X \cap Y$ is also independent. In addition, one can easily see that ker($G$) = \{v_1, v_2\} \subseteq \text{core}(G)$, and ker($G$) is a minimal critical independent set of $G$.

Theorem 2.1 [13] For a graph $G$, the following assertions are true:
(i) ker($G$) \subseteq \text{core}(G);
(ii) if $A$ and $B$ are critical in $G$, then $A \cup B$ and $A \cap B$ are critical as well;
(iii) $G$ has a unique minimal independent critical set, namely, ker($G$).
Various properties of \( \ker(G) \) and \( \core(G) \) can be found in [15, 17, 20].

As an immediate consequence of Theorem 2.1 we have the following.

**Corollary 2.2** For every graph \( G \), diadem\((G)\) is a critical set.

For instance, the graph \( G \) from Figure 2 has diadem\((G)\) = \( \{v_1, v_2, v_3, v_4, v_6, v_7, v_{10}\} \), which is critical, but not independent.

The graph \( G_1 \) from Figure 1 has \( d(G_1) \) = 1 and \( d(\text{corona}(G_1)) \) = 0, which means that corona\((G_1)\) is not a critical set. Notice that \( G_1 \) is not a König-Egerváry graph.

Combining Theorems 1.6 and 2.1 (ii), we deduce the following.

**Corollary 2.3** If \( G \) is a König-Egerváry graph, then both \( \core(G) \) and corona\((G)\) are critical sets. Moreover, corona\((G)\) = \( \bigcup \{A : A \text{ is a maximum critical independent set}\} \).

The converse of Corollary 2.3 is not necessarily true; e.g., the graph \( G_2 \) in Figure 1 is not a König-Egerváry graph, while \( \core(G_2) \) and corona\((G_2)\) are critical.

### 3 Unions and intersections of maximum independent sets

**Theorem 3.1** Let \( \Lambda \subseteq \Omega(G) \), and \( |\Lambda| \geq 1 \). Then

\[
d\left(\bigcup \Lambda\right) = |\bigcap \Lambda| + |\bigcup \Lambda| - |V(G)| \geq \max_{S \in \Omega} d(S).
\]

In particular,

\[
d(\text{corona}(G)) = |\text{corona}(G)| + |\core(G)| - |V(G)| \geq 2\alpha(G) - |V(G)| = \max_{S \in \Omega(G)} d(S).
\]

**Proof.** Every vertex in \( \bigcup \Lambda - \bigcap \Lambda \) has a neighbor in \( \bigcup \Lambda - \bigcap \Lambda \), since \( \Lambda \subseteq \Omega(G) \). Therefore, \( N\left(\bigcup \Lambda\right) = \left(\bigcup \Lambda - \bigcap \Lambda\right) \cup \left(V(G) - \bigcup \Lambda\right) \), which implies

\[
d\left(\bigcup \Lambda\right) = |\bigcup \Lambda| - |N\left(\bigcup \Lambda\right)| = |\bigcup \Lambda| - \left|\left(\bigcup \Lambda - \bigcap \Lambda\right) \cup \left(V(G) - \bigcup \Lambda\right)\right| =
\]

\[
|\bigcap \Lambda| - \left(|V(G)| - |\bigcup \Lambda|\right) = |\bigcap \Lambda| + |\bigcup \Lambda| - |V(G)|.
\]

On the other hand, for every \( S \in \Omega(G) \) we have

\[
d(S) = \alpha(G) - (|V(G)| - \alpha(G)) = 2\alpha(G) - |V(G)|.
\]

Since \( |\bigcap \Lambda| + |\bigcup \Lambda| \geq 2\alpha(G) \), we obtain

\[
d\left(\bigcup \Lambda\right) = |\bigcap \Lambda| + |\bigcup \Lambda| - |V(G)| \geq 2\alpha(G) - |V(G)| = d(S),
\]

as required.
In particular, if $\Lambda = \Omega(G)$, then $\bigcup \Lambda = \text{corona}(G)$, $\bigcap \Lambda = \text{core}(G)$, and the conclusion follows. $\blacksquare$

Notice that if $A$ is a critical independent set in a graph $G$ having $d(G) > 0$, then $A \cap S \neq \emptyset$ holds for every $S \in \Omega(G)$, because $\emptyset \neq \ker(G) \subseteq A \cap \ker(G) \subseteq A \cap S$, according to Theorem 3.1(i).

**Proposition 3.2** Let $A$ be a critical independent set of a graph $G$ with $\ker(G) = \emptyset$, and $\Lambda = \{S \in \Omega(G) : A \cap S = \emptyset\}$. Then $|\bigcap \Lambda| \geq |A|$.

**Proof.** Let $S \in \Lambda$. Since $A$ is critical and $d(G) = 0$, it follows that $|A| = |\alpha(G)|$. By Theorem 1.1 there is a matching from $A$ into $S$, because $A$ is independent and disjoint from $S$. Consequently, we infer that $\alpha(G) \subseteq S$. Hence, we obtain $|\bigcap \Lambda| \geq |\alpha(G)| = |A|$, as required. $\blacksquare$

**Theorem 3.3** Let $\Lambda \subseteq \Omega(G)$, and $|\Lambda| \geq 1$.

(i) If $\bigcup \Lambda$ is critical, then $|N(\bigcap \Lambda)| + |\bigcup \Lambda| = |V(G)|$, and $\bigcap \Lambda$ is critical.

(ii) If $\bigcap \Lambda$ is critical, then

$$|N(\bigcap \Lambda)| + |\bigcup \Lambda| \leq |V(G)|,$$

and $d\left(\bigcap \Lambda\right) \geq 2\alpha(G) - |V(G)|$.

**Proof.** (i) By definition of $d(G)$ and Theorem 3.1 we get

$$d(G) = d\left(\bigcup \Lambda\right) = |\bigcap \Lambda| + |\bigcup \Lambda| - |V(G)| \geq d\left(\bigcap \Lambda\right) = |\bigcap \Lambda| - |N(\bigcap \Lambda)|.$$

Hence we infer that $|N(\bigcap \Lambda)| + |\bigcup \Lambda| \geq |V(G)|$. Thus, $|N(\bigcap \Lambda)| + |\bigcup \Lambda| = |V(G)|$.

Moreover, we deduce that

$$d\left(\bigcup \Lambda\right) = |\bigcap \Lambda| + |\bigcup \Lambda| - |V(G)| = |\bigcap \Lambda| - |N(\bigcap \Lambda)| = d\left(\bigcap \Lambda\right),$$

i.e., $\bigcap \Lambda$ is a critical set.

(ii) By definition of $d(G)$ and Theorem 3.1 we have

$$d(G) = d\left(\bigcap \Lambda\right) = |\bigcap \Lambda| - |N(\bigcap \Lambda)| \geq$$

$$\geq d\left(\bigcup \Lambda\right) = |\bigcap \Lambda| + |\bigcup \Lambda| - |V(G)| \geq 2\alpha(G) - |V(G)|,$$

which completes the proof. $\blacksquare$

In particular, taking $\Lambda = \Omega(G)$ in Theorem 3.3 we obtain the following.

**Corollary 3.4** If $\text{corona}(G)$ is a critical set, then $|\text{corona}(G)| + |N(\text{core}(G))| = |V(G)|$ and $\text{core}(G)$ is critical.
Notice that if core($G$) is critical, then corona($G$) is not necessarily critical. For example, the graph $G_1$ from Figure 11 has $d(G_1) = d(\text{core}(G_1)) = 1$, while corona($G_1$) is not a critical set.

**Theorem 3.5** If $G$ is a Kőnig-Egerváry graph, then

(i) $|\bigcap \Lambda| + |\bigcup \Lambda| = 2\alpha(G)$ holds for every family $\Lambda \subseteq \Omega(G)$, $|\Lambda| \geq 1$;

(ii) $|\text{corona}(G)| + |\text{core}(G)| = 2\alpha(G)$.

**Proof.** (i) By Theorems 1.6 and 2.1(ii), both $\bigcup \Lambda$ and $\bigcap \Lambda$ are critical sets. According to Lemma 1.5, we have

$$d\left(\bigcup \Lambda\right) = \left|\bigcup \Lambda\right| - |N\left(\bigcup \Lambda\right)| = \alpha(G) - \mu(G),$$

$$d\left(\bigcap \Lambda\right) = \left|\bigcap \Lambda\right| - |N\left(\bigcap \Lambda\right)| = \alpha(G) - \mu(G).$$

Hence, $|\bigcap \Lambda| + |\bigcup \Lambda| = 2\alpha(G) - 2\mu(G) + |N\left(\bigcup \Lambda\right)| + |N\left(\bigcap \Lambda\right)|$.

By Theorem 3.3(i), we infer that

$$|N\left(\bigcup \Lambda\right)| + |N\left(\bigcap \Lambda\right)| = |N\left(\bigcup \Lambda\right)| + |V(G)| - |\bigcup \Lambda| =$$

$$= |N\left(\bigcap \Lambda\right)| + \alpha(G) + \mu(G) - |\bigcup \Lambda| =$$

$$= \alpha(G) + \mu(G) - d\left(\bigcup \Lambda\right) = 2\mu(G).$$

Consequently, we obtain $|\bigcap \Lambda| + |\bigcup \Lambda| = 2\alpha(G)$, as claimed.

(ii) It follows from Part (i), by taking $\Lambda \subseteq \Omega(G)$. ■

The graph $G_2$ from Figure 11 has $|\text{corona}(G_2)| + |\text{core}(G_2)| = 13 > 12 = 2\alpha(G_2)$. On the other hand, there is a non-Kőnig-Egerváry graph, namely $G_1$ in Figure 11, that satisfies $|\text{corona}(G_1)| + |\text{core}(G_1)| = 10 = 2\alpha(G_1)$.

If $\bigcap \Lambda$ is a critical set, then $\bigcup \Lambda$ is not necessarily critical. For instance, consider the graph $G$ from Figure 3 and $\Lambda = \{S_1, S_2\}$, where $S_1 = \{x, y, u\}$ and $S_2 = \{x, y, w\}$. Clearly, $\bigcap \Lambda = \{x, y\} = \text{core}(G)$ is critical, while $\bigcup \Lambda = \{x, y, u, w\}$ is not a critical set.

![Figure 3: core($G$) = ker($G$) = \{x, y\}](image)

**Theorem 3.6** Let $\Lambda \subseteq \Omega(G)$, and $|\Lambda| \geq 1$. Then $G$ is a Kőnig-Egerváry graph if and only if $\bigcup \Lambda$ is critical and $|\bigcap \Lambda| + |\bigcup \Lambda| = 2\alpha(G)$. 

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Proof. Combining Theorems 1.6 and 2.1(ii), we infer that $\bigcup \Lambda$ is critical. The equality $|\bigcap \Lambda| + |\bigcup \Lambda| = 2 \alpha(G)$ holds by Theorem 3.5(i).

Conversely, according to Theorem 3.3(i), the set $\bigcap \Lambda$ is critical. Hence, by Theorem 1.4, there is a matching, say $M$, from $\bigcup \Lambda - S$ to $\bigcap \Lambda$. Theorem 3.3(i) ensures that $|N(\bigcap \Lambda)| + |\bigcup \Lambda| = |V(G)|$, which means that $\bigcup \Lambda \cup N(\bigcap \Lambda) = V(G)$. To complete the proof that $G$ is a König-Egerváry graph, one has to find a matching from $\bigcup \Lambda - S$ into $S - \bigcap \Lambda$ for some maximum independent set $S \in \Lambda$. Actually, in accordance with Lemma 1.2, there is a matching, say $M$, from $S - \bigcap \Lambda$ to $\bigcup \Lambda - S$. Since $|\bigcap \Lambda| + |\bigcup \Lambda| = 2 \alpha(G)$ and $|S| = \alpha(G)$, we infer that $|S - \bigcap \Lambda| = |\bigcup \Lambda - S|$. Consequently, $M$ is a perfect matching, and this shows that $M$ is also a matching from $\bigcup \Lambda - S$ into $S - \bigcap \Lambda$, as required. ■

Remark 3.7 If $\bigcap \Lambda$ is critical and $|\bigcap \Lambda| + |\bigcup \Lambda| = 2 \alpha(G)$, then $G$ is not necessarily a König-Egerváry graph. For example, let $G$ be the graph from Figure 3, and $\Lambda = \{S_1, S_2\}$, where $S_1 = \{x, y, u\}$, $S_2 = \{x, y, w\}$. Hence, $\bigcap \Lambda = \{x, y\} = \ker(G)$ is a critical set, $|\bigcap \Lambda| + |\bigcup \Lambda| = 6 = 2 \alpha(G)$, while $G$ is not a König-Egerváry graph. Clearly, the set $\bigcup \Lambda = \{x, y, u, w\}$ is not critical.

Figure 4: $G$ is a non-König-Egerváry graph with $\ker(G) = \{x, y\}$.

If $A_1, A_2$ are independent sets such that $A_1 \cup A_2$ is critical, then $A_1$ and $A_2$ are not necessarily critical. For instance, consider the graph $G$ from Figure 3, where $A_1 \cup A_2 = \{u, v, x, y\}$ is a critical set, while none of $A_1 = \{u, x\}$ and $A_2 = \{v, y\}$ is critical. The case is different whenever the two independent sets are also maximum.

Corollary 3.8 The following assertions are equivalent:
(i) $G$ is a König-Egerváry graph;
(ii) for every $S_1, S_2 \in \Omega(G)$, the set $S_1 \cup S_2$ is critical;
(iii) there exist $S_1, S_2 \in \Omega(G)$, such that $S_1 \cup S_2$ is critical.

Proof. (i) $\Rightarrow$ (ii) It follows combining Theorem 1.6(iii) and Theorem 2.1(ii).
(ii) $\Rightarrow$ (iii) Clear.
(iii) $\Rightarrow$ (i) It is true according to Theorem 3.6 because $|\bigcap \Lambda| + |\bigcup \Lambda| = |S_1 \cup S_2| + |S_1 \cap S_2| = |S_1| + |S_2| = 2 \alpha(G)$ is automatically valid for every family $\Lambda \subseteq \Omega(G)$ with $|\Lambda| = 2$. ■
Remark 3.9 If \( G \) is a König-Egerváry graph, then \( S \cup A \) is not necessarily critical for every \( S \in \Omega(G) \) and \( A \in \text{Ind}(G) \). For instance, consider the graph \( G \) in Figure 5 and \( S = \{a, b, c, d\} \in \Omega(G) \). The sets \( A_1 = \{v\} \) and \( A_2 = \{w\} \) are independent, \( S \cup A_1 \) is critical (because \( N(S \cup A_1) = \{u, v, w, a\} \)), while \( S \cup A_2 \) is not critical (as \( N(S \cup A_2) = \{u, v, w, c, d\} \)).

Corollary 3.10 The following assertions are equivalent:

(i) \( G \) is a König-Egerváry graph;
(ii) for every \( S \in \Omega(G) \) there exists \( A \in \text{Ind}(G) \), such that the set \( S \cup A \) is critical;
(iii) there are \( S \in \Omega(G) \) and \( A \in \text{Ind}(G) \), such that the set \( S \cup A \) is critical.

Proof. (i) \( \Rightarrow \) (ii) By Theorem 1.6(iii), we know that every \( S \in \Omega(G) \) is critical. Hence, \( S \cup A \) is critical for any \( A \subseteq S \).

(ii) \( \Rightarrow \) (iii) Clear.

(iii) \( \Rightarrow \) (i) If \( A \subseteq S \), the result follows by Theorem 1.6. Otherwise, we can suppose that \( S \cap A = \emptyset \). By Theorem 1.1, we know that \( |N(A) \cap S| \geq |A| \). Since

\[
|N(S \cup A)| \geq |N(S) \cup S| + |N(S)| \geq |A| + |N(S)|,
\]

we obtain

\[
d(G) = d(S \cup A) = |S \cup A| - |N(S \cup A)| \\
\leq (|S| + |A|) - (|A| + |N(S)|) = d(S).
\]

Therefore, \( d(S) = d(G) \), i.e., \( S \) is a critical set. According to Theorem 1.6, \( G \) is a König-Egerváry graph.

4 \( \ker(G) \) and \( \text{diadem}(G) \) in König-Egerváry graphs

Theorem 4.1 If \( G \) is a König-Egerváry graph, then

(i) \( \text{diadem}(G) = \text{corona}(G) \), while \( \text{diadem}(G) \subseteq \text{corona}(G) \) is true for every graph;
(ii) \( |\ker(G)| + |\text{diadem}(G)| \leq 2\alpha(G) \).

Proof. (i) Every \( S \in \Omega(G) \) is a critical set, by Theorem 1.6. Hence we deduce that \( \text{corona}(G) \subseteq \text{diadem}(G) \). On the other hand, for every graph each critical independent set is included in a maximum independent set, according to Theorem 1.3. Thus, we infer that \( \text{diadem}(G) \subseteq \text{corona}(G) \). Consequently, the equality \( \text{diadem}(G) = \text{corona}(G) \) holds.

(ii) It follows by combining Part (i), Theorem 3.5(ii) and Theorem 2.1(i).
The König-Egerváry graphs from Figure 6 satisfy \(|\ker(G)| + |\text{diadem}(G)| < 2\alpha(G)|

The graph \(G_1\) from Figure 7 is a non-bipartite König-Egerváry graph, such that \(\ker(G_1) = \text{core}(G_1)\) and \(\text{diadem}(G_1) = \text{corona}(G_1)\). The combination of \(\text{diadem}(G) \subset \text{corona}(G)\) and \(\ker(G) = \text{core}(G)\) is realized by the non-König-Egerváry graph \(G_2\) from Figure 7 because \(\ker(G_2) = \text{core}(G_2)\) and \(\text{diadem}(G_2) \cup \{z, t, v, w\} = \text{corona}(G_2)\).

\[
\begin{align*}
\text{Figure 6: } & G_1 \text{ and } G_2 \text{ are König-Egerváry graphs. } \ker(G_1) = \{x, y\} \text{ and } \ker(G_2) = \emptyset. \\
\end{align*}
\]

The König-Egerváry graphs from Figure 6 satisfy \(|\ker(G)| + |\text{diadem}(G)| < 2\alpha(G)|.

\[
\begin{align*}
\text{Figure 7: } & \text{core}(G_1) = \{a, b\} \text{ and } \text{core}(G_2) = \{x, y\}. \\
\end{align*}
\]

Corollary 4.2 The following assertions are equivalent:

(i) \(G\) is a König-Egerváry graph;

(ii) \(\text{diadem}(G) = \text{corona}(G)\) and \(\text{corona}(G) + |\text{core}(G)| + |\text{corona}(G)| = 2\alpha(G)\);

(iii) \(\text{corona}(G)\) is a critical set and \(|\text{core}(G)| + |\text{corona}(G)| = 2\alpha(G)|

Proof. (i) \(\Rightarrow\) (ii) It is true, by applying Theorem 4.1(i) and Theorem 3.5(ii).

(ii) \(\Rightarrow\) (iii) It follows from Corollary 2.2.

(iii) \(\Rightarrow\) (i) Take \(\Lambda = \Omega(G)\) and use Theorem 3.6.

\[
\begin{align*}
\text{Figure 7: } & \text{core}(G_1) = \{a, b\} \text{ and } \text{core}(G_2) = \{x, y\}. \\
\end{align*}
\]

Notice that the graph \(G_1\) from Figure 7 satisfies \(|\text{core}(G_1)| + |\text{corona}(G_1)| = 2\alpha(G_1)|

while \(d(\text{corona}(G_1)) = 0 < d(G_1) = 1\), i.e., \(\text{corona}(G_1)\) is not a critical set, because \(\text{corona}(G_1) = V(G_1) - \{c, d\}\) and \(N(\text{corona}(G_1)) = V(G_1) - \{a, b\}\). On the other hand, the graph \(G_2\) from Figure 7 satisfies \(|\text{core}(G_2)| + |\text{corona}(G_2)| = 13 > 12 = 2\alpha(G_2)|

while \(\text{corona}(G_2)\) is a critical set.

5 Conclusions

In this paper we focus on interconnections between critical unions and intersections of maximum independent sets, with emphasis on König-Egerváry graphs. In [19] we showed that \(2\alpha(G) \leq |\text{core}(G)| + |\text{corona}(G)|\) is true for every graph, while the equality \(\text{diadem}(G) = \text{corona}(G)\) holds for each König-Egerváry graph \(G\), by Theorem 4.1(i).
According to Theorem 2.1(i), ker($G$) ⊆ core($G$) for every graph. On the other hand, Theorem 1.3 implies the inclusion diadem($G$) ⊆ corona($G$). Hence

$$|\ker(G)| + |\text{diadem}(G)| \leq |\text{core}(G)| + |\text{corona}(G)|$$

for each graph $G$. These remarks together with Theorem 4.1(ii) motivate the following.

**Conjecture 5.1** $|\ker(G)| + |\text{diadem}(G)| \leq 2\alpha(G)$ is true for every graph $G$.

When it is proved one can conclude that the following inequalities:

$$|\ker(G)| + |\text{diadem}(G)| \leq 2\alpha(G) \leq |\text{core}(G)| + |\text{corona}(G)|$$

hold for every graph $G$.

Theorem 4.1 claims that $\text{diadem}(G) = \text{corona}(G)$ is a necessary condition for $G$ to be a König-Egerváry graph, while Corollary 4.2 shows that, apparently, this equality is not enough. These facts motivate the following.

**Conjecture 5.2** If $\text{diadem}(G) = \text{corona}(G)$, then $G$ is a König-Egerváry graph.

The graphs in Figure 8 are non-König-Egerváry graphs; core($G_1$) = \{a, b, c, d\} and it is a critical set, while core($G_2$) = \{x, y, z, w\} and it is not critical.

![Figure 8](image)

Figure 8: Both $G_1$ and $G_2$ are non-König-Egerváry graphs.

By Corollary 2.3, core($G$) is a critical set for every König-Egerváry graph. It justifies the following.

**Problem 5.3** Characterize graphs such that core($G$) is a critical set.

It is known that the sets ker($G$) and core($G$) coincide for bipartite graphs [16]. Notice that there are non-bipartite graphs enjoying the equality ker($G$) = core($G$); e.g., the graphs from Figure 9 where only $G_1$ is a König-Egerváry graph.

![Figure 9](image)

Figure 9: core($G_1$) = ker($G_1$) = \{x, y\} and core($G_2$) = ker($G_2$) = \{a, b\}.

There is a non-bipartite König-Egerváry graph $G$, such that ker($G$) ≠ core($G$). For instance, the graph $G_1$ from Figure 6 has ker($G_1$) = \{x, y\}, while core($G_1$) = \{x, y, u, v\}. The graph $G_2$ from Figure 6 has ker($G_2$) = \emptyset, while core($G_2$) = \{w\}. We propose the following.

**Problem 5.4** Characterize (König-Egerváry) graphs satisfying ker($G$) = core($G$).
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