ON DECOMPOSABILITY OF POSITIVE MAPS BETWEEN $M_2(\mathbb{C})$ AND $M_n(\mathbb{C})$

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Abstract. A map $\varphi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ is decomposable if it is of the form $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1$ is a CP map while $\varphi_2$ is a co-CP map. A partial characterization of decomposability for maps $\varphi : M_2(\mathbb{C}) \to M_3(\mathbb{C})$ is given.

1. Introduction

Let $\varphi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map. We say that $\varphi$ is positive if $\varphi(A)$ is a positive element in $M_n(\mathbb{C})$ for every positive matrix from $M_m(\mathbb{C})$. If $k \in \mathbb{N}$, then $\varphi$ is said to be $k$-positive (respectively $k$-copositive) whenever $[\varphi(A_{ij})]_{i,j=1}^k$ (respectively $[\varphi(A_{ij})]_{i,j=1}^k$) is positive in $M_k(M_n(\mathbb{C}))$ for every positive element $[A_{ij}]_{i,j=1}^k$ of $M_k(M_n(\mathbb{C}))$. If $\varphi$ is $k$-positive (respectively $k$-copositive) for every $k \in \mathbb{N}$ then we say that $\varphi$ is completely positive (respectively completely copositive). Finally, we say that the map $\varphi$ is decomposable if it has the form $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1$ is a completely positive map while $\varphi_2$ is a completely copositive one.

By $\mathcal{P}(m,n)$ we denote the set of all positive maps acting between $M_m(\mathbb{C})$ and $M_n(\mathbb{C})$ and by $\mathcal{P}_1(m,n)$ – the subset of $\mathcal{P}(m,n)$ composed of all positive unital maps (i.e. such that $\varphi(\mathbb{I}) = \mathbb{I}$). Recall that $\mathcal{P}(m,n)$ has the structure of a convex cone while $\mathcal{P}_1(m,n)$ is its convex subset.

In the sequel we will use the notion of a face of a convex cone.

Definition 1.1. Let $C$ be a convex cone. We say that a convex subcone $F \subset C$ is a face of $C$ if for every $c_1, c_2 \in F$ the condition $c_1 + c_2 \in F$ implies $c_1, c_2 \in F$.

A face $F$ is said to be a maximal face if $F$ is a proper subcone of $C$ and for every face $G$ such that $F \subseteq G$ we have $G = F$ or $G = C$.

The following theorem of Kye gives a nice characterization of maximal faces in $\mathcal{P}(m,n)$.

Theorem 1.2 ([10]). A convex subset $F \subset \mathcal{P}(m,n)$ is a maximal face of $\mathcal{P}(m,n)$ if and only if there are vectors $\xi \in \mathbb{C}^m$ and $\eta \in \mathbb{C}^n$ such that $F = F_{\xi, \eta}$ where

$$F_{\xi, \eta} = \{ \varphi \in \mathcal{P}(m,n) : \varphi(P_{\xi})\eta = 0 \}$$

and $P_{\xi}$ denotes the one-dimensional orthogonal projection in $M_m(\mathbb{C})$ onto the subspace generated by the vector $\xi$.

The aim of this paper is to discuss the problem whether it is possible to find concrete examples of the decomposition $\varphi = \varphi_1 + \varphi_2$ onto completely positive and completely copositive parts where $\varphi \in \mathcal{P}(m,n)$. It is well known (see [11] [14]) that every elements of $\mathcal{P}(2,2)$, $\mathcal{P}(2,3)$ and $\mathcal{P}(3,2)$ are decomposable. In [10] we proved that if $\varphi$ is extremal element of $\mathcal{P}(1,2)$ then its decomposition is unique. Moreover, we provided a full description of this decomposition. In the case $m = 2$ or $n > 2$ the problem of finding decomposition is still unsolved. In this paper we consider the next step for solving this problem, namely for the case $m = 2$ and $n = 3$. Our approach will be based on the method of the so called Choi matrix.
Recall (see [3, 10] for details) that if $\varphi : M_m \to M_n$ is a linear map and $\{E_{ij}\}_{i,j=1}^m$ is a system of matrix units in $M_m(\mathbb{C})$, then the matrix

\begin{equation}
H_\varphi = [\varphi(E_{ij})]_{i,j=1}^m \in M_m(M_n(\mathbb{C})),
\end{equation}

is called the Choi matrix of $\varphi$ with respect to the system $\{E_{ij}\}$. Complete positivity of $\varphi$ is equivalent to positivity of $H_\varphi$ while positivity of $\varphi$ is equivalent to block-positivity of $H_\varphi$ (see [3, 10]). Recall (see Lemma 2.3 in [10]) that in the case $m = n = 2$ the general form of the Choi matrix of a positive map $\varphi$ is the following

\begin{equation}
H_\varphi = \begin{bmatrix}
    a & c & 0 & y \\
    \bar{c} & b & \bar{t} & z \\
    0 & \bar{z} & 0 & 0 \\
    \bar{y} & 0 & \bar{u} & 0
\end{bmatrix}
\end{equation}

where $a, b, u \geq 0$, $c, y, z, t \in \mathbb{C}$ and the following inequalities are satisfied:

1. $|c|^2 \leq ab$,
2. $|t|^2 \leq bu$,
3. $|y| + |z| \leq (au)^{1/2}$.

It will turn out that in the case $n = 3$ the Choi matrix has the form which similar to (1.3) but some of the coefficients have to be matrices. The main result of our paper is the generalization of Lemma 2.3 from [10] in the language of some matrix inequalities. It is worth to pointing out that technical lemmas leading to this generalization are formulated and proved for more general case, i.e. for $\varphi : M_2(\mathbb{C}) \to M_{n+1}(\mathbb{C})$ where $n \geq 2$.

### 2. Main results

In this section we will make one step further in the analysis of positive maps and we will examine maps $\varphi$ in $P_1(2, n+1)$ where $n \geq 1$. Let $\{e_1, e_2\}$ and $\{f_1, f_2, \ldots, f_{n+1}\}$ denote the standard orthonormal bases of the spaces $\mathbb{C}^2$ and $\mathbb{C}^n$ respectively, and let $\{E_{ij}\}_{i,j=1}^m$ and $\{F_{kl}\}_{k,l=1}^{n+1}$ be systems of matrix matrix units in $M_2(\mathbb{C})$ and $M_{n+1}(\mathbb{C})$ associated with these bases. We assume that $\varphi \in F_{\xi, \eta}$ for some $\xi \in \mathbb{C}^2$ and $\eta \in \mathbb{C}^{n+1}$. By taking the map $A \mapsto V^* \varphi(WAW^*)V$ for suitable $W \in U(2)$ and $V \in U(n+1)$ we can assume without loss of generality that $\xi = e_2$ and $\eta = f_1$. Then the Choi matrix of $\varphi$ has the form

\begin{equation}
H = \begin{bmatrix}
    a & c_1 & c_2 & \ldots & c_n \\
    \bar{c}_1 & \bar{b}_{11} & \bar{b}_{12} & \ldots & \bar{b}_{1n} \\
    \bar{c}_2 & \bar{b}_{21} & \bar{b}_{22} & \ldots & \bar{b}_{2n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    \bar{c}_n & \bar{b}_{n1} & \bar{b}_{n2} & \ldots & \bar{b}_{nn} \\
    \bar{y}_1 & \bar{t}_{11} & \bar{t}_{12} & \ldots & \bar{t}_{1n} \\
    \bar{y}_2 & \bar{t}_{21} & \bar{t}_{22} & \ldots & \bar{t}_{2n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    \bar{y}_n & \bar{t}_{n1} & \bar{t}_{n2} & \ldots & \bar{t}_{nn} \\
\end{bmatrix}
\end{equation}

We introduce the following notations:

- $C = [\begin{array}{cccc}
    c_1 & c_2 & \ldots & c_n
\end{array}]$, $Y = [\begin{array}{cccc}
    y_1 & y_2 & \ldots & y_n
\end{array}]$, $Z = [\begin{array}{cccc}
    z_1 & z_2 & \ldots & z_n
\end{array}]$,

- $B = [\begin{array}{cccc}
    b_{11} & b_{12} & \ldots & b_{1n} \\
    b_{21} & b_{22} & \ldots & b_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    b_{n1} & b_{n2} & \ldots & b_{nn}
\end{array}]$, $T = [\begin{array}{cccc}
    t_{11} & t_{12} & \ldots & t_{1n} \\
    t_{21} & t_{22} & \ldots & t_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    t_{n1} & t_{n2} & \ldots & t_{nn}
\end{array}]$, $U = [\begin{array}{cccc}
    u_{11} & u_{12} & \ldots & u_{1n} \\
    u_{21} & u_{22} & \ldots & u_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    u_{n1} & u_{n2} & \ldots & u_{nn}
\end{array}]$. 


The matrix (2.1) can be rewritten in the following form

\[
H = \begin{bmatrix}
    a & C & x & Y \\
    C^* & B & Z^* & T \\
    \bar{x} & \bar{Z} & 0 & 0 \\
    Y^* & T^* & 0 & U
\end{bmatrix}.
\]

The symbol 0 in the right-bottom block has three different meanings. It denotes \([0 \ 0 \ \ldots \ 0]\)
or\([0 \ 0 \ \ldots \ 0]\) respectively. We have the following

Proposition 2.1. Let \(\varphi : M_2(\mathbb{C}) \to M_{n+1}(\mathbb{C})\) be a positive map with the Choi matrix of the form (2.2). Then the following relations hold:

1. \(a \geq 0\) and \(B, U\) are positive matrices,
2. if \(a = 0\) then \(C = 0\), and if \(a > 0\) then \(C^*C \leq aB\),
3. \(x = 0\),
4. the matrix \([B \ T^* U] \in M_2(M_n(\mathbb{C}))\) is block-positive.

Proof. It follows from positivity of \(\varphi\) that blocks on main diagonal, i.e. \(\varphi(E_{11}) = \begin{bmatrix} a & C \\ C^* & B \end{bmatrix}\) and \(\varphi(E_{22}) = \begin{bmatrix} 0 & 0 \\ 0 & U \end{bmatrix}\), must be positive matrices. This immediately implies (1) and (2) (cf. [14]).

From block-positivity of \(H\) we conclude that the matrix

\[
\begin{bmatrix}
    a & x \\
    \bar{x} & 0
\end{bmatrix} = \begin{bmatrix}
    \langle f_1, \varphi(E_{11})f_1 \rangle & \langle f_1, \varphi(E_{12})f_1 \rangle \\
    \langle f_1, \varphi(E_{21})f_1 \rangle & \langle f_1, \varphi(E_{22})f_1 \rangle
\end{bmatrix}
\]

is a positive element of \(M_2(\mathbb{C})\). So \(x = 0\), and (3) is proved. The statement of point (4) is an obvious consequence of block-positivity of \(H\). \(\square\)

For \(X = [x_1 \ x_2 \ \ldots \ x_n] \in M_{1,n}(\mathbb{C})\) we define \(\|X\| = (\sum_{i=1}^{n}|x_i|^2)^{1/2}\). By \(|X|\) we denote the square \((n \times n)\)-matrix \((X^*X)^{1/2}\). Let us observe that for any \(X \in M_{1,n}(\mathbb{C})\) we have

\(|X| = \|X\|P_\xi\)

where \(\xi_X = \|X\|^{-1}X^*\) and \(P_\xi\) denotes the orthogonal projection onto the one-dimensional subspace in \(\mathbb{C}^n\) generated by a vector \(\xi \in \mathbb{C}^n\) (we identify elements of \(M_{n,1}(\mathbb{C})\) with vectors from \(\mathbb{C}^n\)).

We have the following

Lemma 2.2. A map \(\varphi\) with the Choi matrix of the form

\[
H = \begin{bmatrix}
    a & C & 0 & Y \\
    C^* & B & Z^* & T \\
    0 & Z & 0 & 0 \\
    Y^* & T^* & 0 & U
\end{bmatrix}
\]

is positive if and only if the inequality

\[
|\langle Y^*, \Gamma^* \rangle + \langle Z^*, \Gamma^* \rangle + \text{Tr} (A^*T) |^2 \leq [aa + \text{Tr} (A^*B) + 2\Re (C^*, \Gamma^*)] \text{Tr} (A^*U)
\]

holds for every \(\alpha \in \mathbb{C}\), matrices \(\Gamma = [\gamma_1 \ \gamma_2 \ \ldots \ \gamma_n]\) and \(\Lambda = [\lambda_{11} \ \lambda_{12} \ \ldots \ \lambda_{1n} \ \lambda_{21} \ \lambda_{22} \ \ldots \ \lambda_{2n} \ \vdots \ \vdots \ \lambda_{n1} \ \lambda_{n2} \ \ldots \ \lambda_{nn}]\),

\(\gamma_i \in \mathbb{C}, \lambda_{ij} \in \mathbb{C}\) for \(i, j = 1, 2, \ldots, n\), such that

1. \(\alpha \geq 0\) and \(\Lambda \geq 0\),
2. \(\Gamma^*\Gamma \leq \alpha \Lambda\).
The superscript \( \tau \) denotes the transposition of matrices.

**Proof.** Obviously, the map \( \varphi \) is positive if and only if \( \omega \circ \varphi \) is a positive functional on \( M_2(\mathbb{C}) \) for every positive functional \( \omega \) on \( M_{n+1}(\mathbb{C}) \).

Let \( \omega \) be a linear functional on \( M_{n+1}(\mathbb{C}) \). Recall that positivity of \( \omega \) is equivalent to its complete positivity. Hence, the Choi matrix \( H_{ij} = \{ \omega(F_{ij}) \}_{(i,j=1)}^{n+1} \) of \( \omega \) is a positive element of \( M_{n+1}(\mathbb{C}) \). Let us denote \( \alpha = \omega(F_{11}), \gamma_j = \omega(F_{1,j+1}) \) for \( j = 1, 2, \ldots, n \), \( \lambda_{ij} = \omega(F_{i+1,j+1}) \) for \( i, j = 1, 2, \ldots, n \) and \( \Gamma \) and \( A \) are defined as in the statement of the lemma. Then, we came to the conclusion that \( \omega \) is a positive functional if and only if the matrix

\[
\begin{bmatrix}
\alpha & \Gamma \\
\Gamma^* & A
\end{bmatrix}
\]

is a positive element of \( M_{n+1}(\mathbb{C}) \). This is equivalent to the conditions (1) and (2) from the statement of the lemma.

Similarly, if \( \omega' \) is a linear functional on \( M_2(\mathbb{C}) \), then its positivity is equivalent to positivity of the matrix \( \{ \omega'(E_{ij}) \}_{(i,j=1)}^{n+1} \). Consequently, \( \omega' \) is positive if and only if \( \omega'(E_{ii}) \geq 0 \) for \( i = 1, 2 \), \( \omega'(E_{21}) = \omega'(E_{12}) \) and

\[
|\omega'(E_{12})|^2 \leq \omega'(E_{11})\omega'(E_{22})
\]

(cf. Corollary 8.4 in [11]).

Now, assume that \( \alpha, \Gamma \) and \( A \) are given and the conditions (1) and (2) are fulfilled. In described above way it corresponds to some positive functional \( \omega \) on \( M_{n+1}(\mathbb{C}) \). Let \( \omega' = \omega \circ \varphi \). Then

\[
\begin{align*}
\omega'(E_{11}) &= a a + \sum_{i,j=1}^{n} \lambda_{ij} b_{ij} + 2 \Re \left( \sum_{i=1}^{n} \gamma_i c_i \right) = a a + \text{Tr}(A^*B) + 2 \Re(C^*, \Gamma^*) \\
\omega'(E_{22}) &= \sum_{i,j=1}^{n} \lambda_{ij} u_{ij} = \text{Tr}(A^*U) \\
\omega'(E_{12}) &= \sum_{i=1}^{n} \gamma_i y_i + \sum_{i=1}^{n} \bar{\gamma_i} z_i + \sum_{i,j=1}^{n} \lambda_{ij} v_{ij} = (Y^*, \Gamma^*) + (Z^*, \Gamma^*) + \text{Tr}(A^*T).
\end{align*}
\]

It follows from the above equalities that (2.6) is equivalent to the inequality (2.4). Hence, the statement of the lemma follows from the remark contained in the first paragraph of the proof. \( \square \)

**Proposition 2.3.** If the assumptions of Propositions 2.1 are fulfilled, then

\[
|Y| + |Z| \leq a^{1/2} U^{1/2}.
\]

**Proof.** Firstly, let us observe that the inequality (2.4) can be written in the form

\[
\begin{align*}
&\left| \langle Y^*, \Gamma^* \rangle + \langle Z^*, \Gamma^* \rangle \right|^2 + \left| \text{Tr}(A^*T) \right|^2 + \\
&+ 2 \Re \left[ \left( \langle Y^*, \Gamma^* \rangle + \langle Z^*, \Gamma^* \rangle \right) \text{Tr}(A^*T) \right] \leq [a a + \text{Tr}(A^*B) + 2 \Re(C^*, \Gamma^*)] \text{Tr}(A^*U) \\
&\text{Putting } - \Gamma \text{ instead of } \Gamma \text{ we preserve the positivity of the matrix (2.5) and the above inequality takes the form}
\end{align*}
\]

\[
\begin{align*}
&\left| \langle Y^*, \Gamma^* \rangle + \langle Z^*, \Gamma^* \rangle \right|^2 + \left| \text{Tr}(A^*T) \right|^2 + \\
&- 2 \Re \left[ \left( \langle Y^*, \Gamma^* \rangle + \langle Z^*, \Gamma^* \rangle \right) \text{Tr}(A^*T) \right] \leq [a a + \text{Tr}(A^*B) - 2 \Re(C^*, \Gamma^*)] \text{Tr}(A^*U)
\end{align*}
\]

If we add both above inequalities and divide the result by 2, then we get

\[
\left| \langle Y^*, \Gamma^* \rangle \right|^2 + \left| \langle Z^*, \Gamma^* \rangle \right|^2 + 2 \Re \langle Y^*, \Gamma^* \rangle \langle Z^*, \Gamma^* \rangle \leq [a a \text{Tr}(A^*U) + \text{Tr}(A^*B) \text{Tr}(A^*U) - \left| \text{Tr}(A^*T) \right|^2
\]

and consequently

\[
\left| \langle Y^*, \Gamma^* \rangle \right|^2 + \left| \langle Z^*, \Gamma^* \rangle \right|^2 + 2 \Re \langle Y^*, \Gamma^* \rangle \langle Z^*, \Gamma^* \rangle \leq a a \text{Tr}(A^*U) + \text{Tr}(A^*B) \text{Tr}(A^*U) - \left| \text{Tr}(A^*T) \right|^2
\]

Now, let \( \eta \) be an arbitrary unit vector from \( \mathbb{C}^n \). Then
\[
(2.12) \quad \|(|Y| + |Z|)\eta\| \leq \|Y\| + \|Z\| = \|P_y\eta\| + \|P_z\eta\|
\]
\[
= \|Y\| \left|\langle \xi_y, \eta \rangle \right| + \|Z\| \left|\langle \xi_z, \eta \rangle \right| = \left|\langle Y^*, \eta \rangle \right| + \left|\langle Z^*, \eta \rangle \right|
\]

Let \( \varepsilon > 0 \). Put \( \alpha = \varepsilon^{-1} \) and let \( \Gamma = e^{i\theta} \eta^* \), where \( \theta \) is such that
\[
e^{2i\theta} \langle Y^*, \eta \rangle \langle Z^*, \eta \rangle = \left|\langle Y^*, \eta \rangle\right| \left|\langle Z^*, \eta \rangle\right|
\]
hence
\[
\langle Y^*, \Gamma \rangle \langle Z^*, \Gamma \rangle = \left|\langle Y^*, \Gamma \rangle\right| \left|\langle Z^*, \Gamma \rangle\right|.
\]
Put also \( \Lambda = \varepsilon \Gamma^* \Gamma \). Then from (2.11) and (2.12) we have
\[
\|(|Y| + |Z|)\eta\|^2 \leq a \text{Tr}((\Gamma^* \Gamma)^7 U) + \varepsilon^2 \left( \text{Tr}(A^* B) \text{Tr}(A^* U) - \text{Tr}(A^* U)^2 \right)
\]
Since \( \varepsilon \) is arbitrary then we have
\[
\|(|Y| + |Z|)\eta\|^2 \leq a \text{Tr}((\Gamma^* \Gamma)^7 U) = a \text{Tr}(P_y U) = a \langle \eta, U \eta \rangle = \|a^{1/2} U^{1/2} \eta\|^2
\]
and the proof is finished since \( \|Y \|^2 \leq a U \) implies \( |Y| + |Z| \leq a^{1/2} U^{1/2} \). \( \Box \)

**Lemma 2.4.** Let \( \varphi : M_2(\mathbb{C}) \to M_{n+1}(\mathbb{C}) \) be a linear map with the Choi matrix of the form (2.3). Then

1. the map \( \varphi \) is completely positive if and only if the following conditions hold:
   - (A1) \( Z = 0 \),
   - (A2) the matrix
     \[
     \begin{bmatrix}
     a & C & Y \\
     C^* & B & T \\
     Y^* & T^* & U
     \end{bmatrix}
     \]
     is a positive element of \( M_{2n+1}(\mathbb{C}) \).

   In particular, the condition (A2) implies:
   - (A3) if \( B \) is an invertible matrix, then \( T^* B^{-1} T \leq U \),
   - (A4) \( C^* C \leq a B \),
   - (A5) \( Y^* Y \leq a U \),

2. the map \( \varphi \) is completely co-positive if and only if
   - (B1) \( Y = 0 \),
   - (B2) the matrix
     \[
     \begin{bmatrix}
     a & C & Z \\
     C^* & B & T^* \\
     Z^* & T & U
     \end{bmatrix}
     \]
     is a positive element of \( M_{2n+1}(\mathbb{C}) \).

   In particular, the condition (B2) implies:
   - (B3) if \( B \) is invertible, then \( T B^{-1} T^* \leq U \),
   - (B4) \( C^* C \leq a B \),
   - (B5) \( Z^* Z \leq a U \),

**Proof.** It is rather obvious that the conditions (A1) and (A2) imply positivity of the matrix (2.3), and consequently the complete positivity of the map \( \varphi \). On the other hand it is easy to see that (A2) is a necessary condition for positivity of the matrix (2.3). In order to finish the proof of the first part of point (1) one should show that positivity of (2.3) implies \( Z = 0 \).

Let \( L_1 \) be a linear subspace generated by the vector \( f_1 \) and let \( L_2 \) be a subspace spanned by \( f_2, f_3, \ldots, f_{n+1} \), so \( \mathbb{C}^{n+1} = L_1 + L_2 \). Any vector \( v \in \mathbb{C}^{n+1} \) can be uniquely decomposed onto the sum \( v = v^{(1)} + v^{(2)} \), where \( v^{(i)} \in L_i \), \( i = 1, 2 \). Blocks of the matrix (2.3) are interpreted as operators. Namely: \( B, T, U : L_2 \to L_2, C, Y, Z : L_2 \to L_1, \) and \( a : L_1 \to L_1 \). Recall (cf. [13]), that the positivity of the matrix (2.3) is equivalent to the following inequality
\[
\left\langle v_1, \begin{bmatrix}
    a & C & Z \\
    C^* & B & T^* \\
    Z^* & T & U
    \end{bmatrix} v_1 \right\rangle + \left\langle v_2, \begin{bmatrix}
    0 & Y & Z \\
    Y^* & T & T^* \\
    Z^* & T & U
    \end{bmatrix} v_2 \right\rangle \geq 0
\]
for any \( v_1, v_2 \in \mathbb{C}^{n+1} \). This is equivalent to
\[
(2.13) \quad \left\langle v_1^{(1)}, a v_1^{(1)} \right\rangle + \left\langle v_1^{(2)}, B v_1^{(2)} \right\rangle + \left\langle v_2^{(2)}, U v_2^{(2)} \right\rangle + 2 \Re \left\langle v_1^{(1)}, C v_1^{(2)} \right\rangle + 2 \Re \left\langle v_1^{(1)}, Y v_2^{(2)} \right\rangle + 2 \Re \left\langle v_2^{(1)}, Z v_1^{(2)} \right\rangle + 2 \Re \left\langle v_2^{(2)}, T v_2^{(2)} \right\rangle \geq 0
\]
where \( v_j = v_j^{(1)} + v_j^{(2)} \) for \( j = 1, 2 \), and \( v_1^{(1)}, v_2^{(1)} \in L_1 \) and \( v_1^{(2)}, v_2^{(2)} \in L_2 \). Assume that \( v_1^{(1)} = 0 \), \( v_2^{(2)} = 0 \), \( v_1^{(2)} \) is an arbitrary element of \( L_2 \), and \( v_2^{(1)} = -rv_1^{(2)} \) for some \( r > 0 \). Then (2.13) reduces to

\[
\langle v_1^{(2)}, Bv_1^{(2)} \rangle - r \left\| Zv_1^{(2)} \right\|^2 \geq 0.
\]

This inequality holds for any \( v_1^{(2)} \in L_2 \) and \( r > 0 \). It is possible only for \( Z = 0 \).

To show the second part of the point (1) one needs to notice that positivity of the matrix

\[
\begin{bmatrix}
\alpha & C & Y \\
C^* & B & T \\
Y^* & T^* & U
\end{bmatrix}
\]

implies the positivity of the matrices \( \begin{bmatrix} B & T & U \end{bmatrix} \), \( \begin{bmatrix} a & C & \phi \\
C^* & B & \chi \\
Y^* & T^* & \mu
\end{bmatrix} \) and \( \begin{bmatrix} a & Y & \rho \\
C & \phi & \chi \\
Y^* & U & \mu
\end{bmatrix} \).

Recall that the map is completely copositive if and only if the partial transposition of the matrix (2.15) is positive. The partial transposition (cf. [10]) of (2.16) is equal to

\[
\begin{bmatrix}
a & C & 0 & Z \\
C^* & B & Y^* & T^* \\
0 & Y & 0 & 0 \\
Z^* & T^* & 0 & U
\end{bmatrix}.
\]

So, to prove the point (2) we can use the arguments as in proof of the point (1).

Now, assume that \( \varphi : M_2(\mathbb{C}) \to M_3(\mathbb{C}) \) is a unital positive map, and \( \varphi \in F_{e_2,f_1} \). Hence its Choi matrix has the form

\[
\begin{bmatrix}
1 & 0 & 0 & Y \\
0 & B & Z^* & T \\
0 & Z & 0 & 0 \\
Y^* & T^* & 0 & U
\end{bmatrix},
\]

where \( B \) and \( U \) are positive matrices such that \( B + U = 1 \) and conditions listed in Propositions 2.31 and 2.32 are satisfied. From the theorem of Woronowicz (cf. [14]) it follows that there are maps \( \varphi_1, \varphi_2 : M_2(\mathbb{C}) \to M_3(\mathbb{C}) \) such that \( \varphi = \varphi_1 + \varphi_2 \), and \( \varphi_1 \) is a completely positive map while \( \varphi_2 \) is a completely copositive one. From Definition 2.1 we conclude that both \( \varphi_1 \) and \( \varphi_2 \) are contained in the face \( F_{e_2,f_1} \). So, from Proposition 2.4 it follows that their Choi matrices \( H_1 \) and \( H_2 \) are of the form

\[
H_1 = \begin{bmatrix}
a_1 & C & 0 & Y \\
B_1 & T_1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
Y^* & T^*_1 & 0 & U_1
\end{bmatrix}, \quad H_2 = \begin{bmatrix}
a_2 & -C & 0 & 0 \\
-B_2 & Z & 0 & 0 \\
0 & T_2 & 0 & U_2
\end{bmatrix},
\]

where \( a_i, b_i, u_i \geq 0 \) for \( i = 1, 2 \), and the following equalities hold: \( a_1 + a_2 = a, T_1 + T_2 = T, B_1 + B_2 = B \) and \( U_1 + U_2 = U \). In [14] we proved that if \( \varphi : M_2(\mathbb{C}) \to M_3(\mathbb{C}) \) is from a large class of extremal positive unital maps, then the maps \( \varphi_1 \) and \( \varphi_2 \) are uniquely determined (cf. Theorem 2.7 in [14]). Motivated by this type of decomposition and the results given in this section (we ‘quantized’ the relations (1)-(3) given at the end of Section 1) we wish to formulate the following conjecture:

Assume that \( \varphi : M_2(\mathbb{C}) \to M_3(\mathbb{C}) \) is a positive unital map such that \( U \neq 0, Y \neq 0, Z \neq 0 \) and \( |Y| + |Z| = U^{1/2} \). Then the decomposition \( \varphi = \varphi_1 + \varphi_2 \) onto completely positive and completely copositive parts is uniquely determined.

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