Decidability results for ATL with imperfect information and perfect recall∗

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Abstract

Alternating-time Temporal Logic (ATL∗) is a central logic for multiagent systems. It has been extended in various ways, notably with imperfect information (ATL∗i). Since the model-checking problem against ATL∗i for agents with perfect recall is undecidable, studies have mostly focused either on agents without memory, or on alternative semantics to retrieve decidability. In this work, we establish new, strong decidability results for agents with perfect recall. We first prove a meta-theorem that allows the transfer of decidability results for classes of multiplayer games with imperfect information, such as games with hierarchical observation, to the model-checking problem for ATL∗i. We also establish that model checking ATL∗ with strategy context and imperfect information for hierarchical instances is decidable.

1 Introduction

In formal system verification, model checking is a well-established method to automatically check the correctness of a system [8, 31, 9]. It consists in modelling the system as a mathematical structure, expressing its desired behaviour as a formula from some suitable logic, and checking whether the model satisfies the formula. In the nineties, interest has arisen in the verification of multiagent systems (MAS), in which various entities (the agents) interact and can form coalitions to attain certain objectives. This led to the development of logics that allow reasoning about strategic abilities in MAS [2, 27, 19, 35, 1, 7].

Alternating-time Temporal Logic (ATL∗), introduced by Alur, Henzinger, and Kupferman [2], plays a central role in this line of work. This logic, interpreted on concurrent game structures, extends CTL∞ with strategic modalities. These modalities allow one to reason about the existence of strategies for coalitions of agents to force the system’s behaviour to satisfy certain temporal properties. ATL∗ has been extended in many ways, and among these extensions an important one is ATL∗ with strategy context [6, 25]. In ATL∗, strategies of all agents are forgotten at each new strategic modality. In ATL∗ with strategy context (ATL∗sc) instead they are stored in a strategy context, and are forgotten only when replaced by a new strategy or when the formula explicitly unbinds the agent from her strategy. Thanks to this additional expressive power, ATL∗sc can express important game theoretic concepts such as the existence of Nash Equilibria [25].

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In many real-life scenarios, such as poker, agents do not always know precisely what is the current state of the system. Instead, they have a partial view, or observation, of the state. This fundamental feature of MAS is called imperfect information, and it is known to quickly bring about undecidability when involved in strategic problems, especially when agents have perfect recall of the past, which is a usual and important assumption in games with imperfect information and epistemic temporal logics [13]. For instance, solving multiplayer games with imperfect information and perfect recall, i.e., deciding the existence of a distributed winning strategy in such games, is already undecidable for reachability objective, as proven by Peterson, Reif and Azhar [28]. Since such games are easily captured by ATL∗ with imperfect information (ATL∗ i), model checking ATL∗ i with perfect recall is also undecidable [2].

However, it is known that restricting attention to cases where some sort of hierarchy exists on the different agents’ information yields decidability for several problems related to the existence of strategies. Synthesis of distributed systems, which implicitly uses perfect recall and is undecidable in general [30], is decidable for hierarchical architectures [23]. Actually, for branching-time specifications, distributed synthesis is decidable exactly on architectures free from information forks, for which the problem can be reduced to the hierarchical case [14]. For richer specifications from alternating-time logics, being free of information forks is no longer sufficient, but distributed synthesis is decidable precisely on hierarchical architectures [32]. Similarly, solving multiplayer games with imperfect information and perfect recall, i.e., checking for the existence of winning distributed strategies, is decidable for ω-regular winning conditions when there is a hierarchy among players, each one observing more than those below [29, 23]. Recently, it has been proven that this assumption can be relaxed while maintaining decidability: the problem remains decidable if the hierarchy can change along a play, or even if transient phases without such a hierarchy are allowed [5].

Our contribution. In this work we establish several decidability results for model checking ATL∗ i with perfect recall, with and without strategy context, all related to notions of hierarchy. Our first result is a theorem that allows the transfer of decidability results for classes of multiplayer games with imperfect information, such as those mentioned above, to the model-checking problem for ATL∗ i. This theorem essentially states that if solving multiplayer games with imperfect information, perfect recall and omega-regular objectives is decidable on some class of concurrent game structures, then model checking ATL∗ i with perfect recall is also decidable on this class of models (a simple bottom-up algorithm that evaluates innermost strategic modalities in every state of the model suffices). As a direct consequence we easily obtain new decidability results for the model checking of ATL∗ i on several classes of concurrent game structures.

Our second contribution considers ATL∗ with imperfect information and strategy context (ATL∗ sc,i). Because there are in general infinitely many possible strategy contexts, the bottom-up approach used for ATL∗ i cannot be used here. Instead we build upon the proof presented in [25] to establish the decidability of model checking ATL∗ sc,i, by reduction to the model-checking problem for Quantified CTL∗ (QCTL∗). The latter extends CTL∗ with second-order quantification on atomic propositions, and it has been studied in a number of works [33, 21, 22, 15, 24]. QCTL∗ i, an imperfect-information extension of QCTL∗, has recently been introduced, and its model-checking problem was proven decidable for the class of hierarchical formulas [4]. In this paper, we define a notion of hierarchical instances for the ATL∗ sc,i model-checking problem: informally, an ATL∗ sc,i formula ϕ together with a concurrent game structure G is a hierarchical instance if outermost strategic modalities in ϕ concern agents who observe less in G. We adapt the proof from [25] and reduce the model-checking problem for ATL∗ sc,i on hierarchical instances to the model-checking problem for hierarchical
QCTL$_i^*$ formulas. We obtain that model checking hierarchical instances of ATL$_{sc,i}^*$ with perfect recall is decidable.

**Related work.** The model-checking problem for ATL$_i^*$ is known to be decidable when agents have no memory [33], and the case of agents with bounded memory reduces to that of no memory. Another way to retrieve decidability is to assume that all agents in a coalition have the same information, either because their observations of the system are the same, or because they can communicate and share their observations [16, 11, 17, 20]. This idea was also used recently to establish a decidability result for ATL$_{sc,i}^*$ [26] when all agents have the same observation of the game.

The results we establish here thus strictly extend previously known results on the decidability of model checking ATL$_i^*$ and ATL$_{sc,i}^*$ with perfect recall and standard semantics, and they hold for vast, natural classes of instances, that all rely on notions of hierarchy, which seems to be inherent to all decidable cases of strategic problems for multiple entities with imperfect information and perfect recall.

**Outline.** After setting some basic definitions in Section 2, we present our transfer theorem and its various corollaries concerning the model checking problem for ATL$_i^*$ in Section 3. In Section 4 we prove that when restricted to hierarchical instances, model checking ATL$_{sc,i}^*$ is decidable, and we conclude in Section 5.

2 Preleminaries

Let $\Sigma$ be an alphabet. A finite (resp. infinite) word over $\Sigma$ is an element of $\Sigma^*$ (resp. $\Sigma^\omega$). The empty word is noted $\epsilon$, and $\Sigma^+ = \Sigma^* \setminus \{\epsilon\}$. The length of a word is $|w| := 0$ if $w$ is the empty word $\epsilon$, if $w = w_0w_1 \ldots w_n$ is a finite nonempty word then $|w| := n + 1$, and for an infinite word $w$ we let $|w| := \omega$. Given a word $w$ and $0 \leq i, j \leq |w| - 1$, we let $w_i$ be the letter at position $i$ in $w$ and $w[i, j]$ be the subword of $w$ that starts at position $i$ and ends at position $j$. For $n \in \mathbb{N}$ we let $[n] := \{1, \ldots, n\}$. Finally, let us fix a countably infinite set of atomic propositions $AP$ and let $AP \subset AP$ be some finite subset of atomic propositions.

2.1 Kripke structures

A Kripke structure over $AP$ is a tuple $S = (S, R, \ell)$ where $S$ is a set of states, $R \subseteq S \times S$ is a left-total transition relation and $\ell : S \rightarrow 2^{AP}$ is a labelling function.

A pointed Kripke structure is a pair $(S, s)$ where $s \in S$. A path in a structure $S = (S, R, \ell)$ is an infinite word $\lambda$ over $S$ such that for all $i \in \mathbb{N}$, $(\lambda_i, \lambda_{i+1}) \in R$. For $s \in S$, Paths(s) is the set of all paths that start in $s$.

2.2 Infinite trees

Let $X$ be a finite set. An $X$-tree $\tau$ is a nonempty set of words $\tau \subseteq X^+$ such that

= there exists $r \in X$, called the root of $\tau$, such that each $u \in \tau$ starts with $r$;
= if $u \cdot x \in \tau$ and $u \neq \epsilon$, then $u \in \tau$, and
= if $u \in \tau$ then there exists $x \in X$ such that $u \cdot x \in \tau$.

The elements of a tree $\tau$ are called nodes. If $u \cdot x \in \tau$, we say that $u \cdot x$ is a child of $u$. Similarly to Kripke structures, a path is an infinite sequence of nodes $\lambda = u_0u_1 \ldots$ such that for all $i$, $u_{i+1}$ is a child of $u_i$, and Paths(u) is the set of paths that start in node $u$. An

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1 i.e., for all $s \in S$, there exists $s'$ such that $(s, s') \in R$. 

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AP-labelled X-tree, or (AP, X)-tree for short, is a pair $t = (\tau, \ell)$, where $\tau$ is an X-tree called the domain of $t$ and $\ell : \tau \to 2^{AP}$ is a labelling.

Definition 1 (Tree unfoldings). Let $S = (S, R, \ell)$ be a Kripke structure over AP, and let $s \in S$. The tree-unfolding of $S$ from $s$ is $(AP, S)$-tree $t_S(s) = (\tau, \ell')$, where $\tau$ is the set of all finite paths that start in $s$, and for every $u \in \tau$, $\ell'(u) = \ell(\text{last}(u))$.

3 ATL* with imperfect information

In this section we recall the syntax and semantics of ATL* with imperfect information and synchronous perfect-recall semantics, or ATL$_i^*$ for short, and establish a meta-theorem on the decidability of its model-checking problem.

3.1 Definitions

We first introduce the models of the logics we study. For the rest of the paper, let us fix a way:

This assumption, as well as the choice of a unique set of moves for all agents, is made to ease presentation.

Definition 1 (Tree unfoldings). Let $S = (S, R, \ell)$ be a Kripke structure over AP, and let $s \in S$. The tree-unfolding of $S$ from $s$ is $(AP, S)$-tree $t_S(s) = (\tau, \ell')$, where $\tau$ is the set of all finite paths that start in $s$, and for every $u \in \tau$, $\ell'(u) = \ell(\text{last}(u))$.

In this work we consider agents with synchronous perfect recall, meaning that the

Definition 2. A concurrent game structure with imperfect information (or CGS$_i$ for short) over AP is a tuple $G = (V, E, \ell, \sim_a)_{a \in \text{Ag}}$ where $V$ is a non-empty finite set of positions, $E : V \times M^{\text{Ag}} \to V$ is a transition function, $\ell : V \to 2^{AP}$ is a labelling function and for each agent $a \in \text{Ag}$, $\sim_a$ is an equivalence relation.

In a position $v \in V$, each agent $a$ chooses a move $m_a \in M_a$, and the game proceeds to position $E(v, m)$, where $m \in M^{\text{Ag}}$ stands for the joint move $(m_a)_{a \in \text{Ag}}$ (note that we assume $E(v, m)$ to be defined for all $v$ and $m$). For each position $v \in V$, $\ell(v)$ is the finite set of atomic propositions that hold in $v$, and $\sim_a$ represents the observation of agent $a$: for two positions $v, v' \in V$, $v \sim_a v'$ means that agent $a$ cannot tell the difference between $v$ and $v'$. We may write $v \in G$ for $v \in V$. A pointed CGS$_i$ $(G, v)$ is a CGS$_i$ $G$ together with a position $v \in G$.

In Section 3.2 we also use nondeterministic CGS$_i$, which are as in Definition 2 except that they have a transition relation $E \subseteq V \times M^{\text{Ag}} \times V$ instead of a transition function. In a position $v$, after every agent has chosen a move, forming a joint move $m \in M^{\text{Ag}}$, a special agent called Nature (not in Ag) chooses a next position $v'$ such that $(v, m, v') \in E$ (see [5] for detail). In the following, unless explicitly specified, CGS$_i$ always refers to deterministic CGS$_i$. The following definitions also concern deterministic CGS$_i$, but they can be adapted to nondeterministic ones in an obvious way.

A finite (resp. infinite) play is a finite (resp. infinite) word $\rho = v_0 \ldots v_n$ (resp. $\pi = v_0v_1\ldots$) such that for all $i$ with $0 \leq i < |\rho| - 1$ (resp. $i \geq 0$), there exists a joint move $m$ such that $E(v_i, m) = v_{i+1}$. A finite (resp. infinite) play $\rho$ (resp. $\pi$) starts in a position $v$ if $\rho_0 = v$ (resp. $\pi_0 = v$). We let $\text{Plays}(G, v)$ be the set of plays, either finite or infinite, that start in $v$.

In this work we consider agents with synchronous perfect recall, meaning that the observational equivalence relation for each agent $a$ is extended to finite plays the following way: $\rho \sim_a \rho'$ if $|\rho| = |\rho'|$ and $\rho_i \sim_a \rho_i'$ for every $i \in \{0, \ldots, |\rho| - 1\}$. A strategy for agent $a$ is a function $\sigma : V^+ \to M$ such that $\sigma(\rho) = \sigma(\rho')$ whenever $\rho \sim_a \rho'$. The latter constraint captures the essence of imperfect information, which is that agents can base their strategic decisions on the information they observe, rather than on the exact sequence of moves. This approach simplifies the analysis of complex systems and provides a solid foundation for the development of algorithms and tools for verifying properties of such systems.
choices only on the information available to them, and removing this constraint yields the semantics of classic ATL with perfect information.

A strategy profile for a coalition \( A \subseteq A_{\text{g}} \) is a mapping \( \sigma_A \) that assigns a strategy to each agent \( a \in A \); for \( a \in A \), we may write \( \sigma_a \) instead of \( \sigma_A(a) \). An infinite play \( \pi \) follows a strategy profile \( \sigma_A \) for a coalition \( A \) if for all \( i \geq 0 \), there exists a joint move \( m \) such that \( E(\pi_i, m) = \pi_{i+1} \) and for each \( a \in A, m_a = \sigma_a(\pi[0, i]) \). For a strategy profile \( \sigma_A \) and a position \( v \in V \), we define the outcome \( \text{Out}(v, \sigma_A) \) of \( \sigma_A \) in \( v \) as the set of infinite plays that start in \( v \) and follow \( \sigma_A \).

The syntax of \( \text{ATL}_i^* \) is the same as that of \( \text{ATL}^* \), and is given by the following grammar:

\[
\varphi ::= p \mid \neg \varphi \mid \varphi \lor \varphi \mid \langle A \rangle \varphi \mid X \varphi \mid \varphi U \varphi,
\]

where \( p \in AP \) and \( A \subseteq A_{\text{g}} \).

\( X \) and \( U \) are the classic next and until operators, respectively, while the strategic operator \( \langle A \rangle \) quantifies on strategy profiles for coalition \( A \).

The semantics of \( \text{ATL}_i^* \) is defined with regards to a CGS, \( G = (V, E, \ell, \{\sim_a\}_{a \in A_{\text{g}}}) \), an infinite play \( \pi \) and a position \( i \geq 0 \) along this play, by induction on formulas:

\[
\begin{align*}
G, \pi, i &\models p \quad \text{if } p \in \ell(\pi_i) \\
G, \pi, i &\models \neg \varphi \quad \text{if } G, \pi, i \not\models \varphi \\
G, \pi, i &\models \varphi \lor \varphi' \quad \text{if } G, \pi, i \models \varphi \text{ or } G, \pi, i \models \varphi' \\
G, \pi, i &\models \langle A \rangle \varphi \quad \text{if there exists a strategy profile } \sigma_A \text{ s.t.} \\
&\quad \text{for all } \pi' \in \text{Out}(\pi_i, \sigma_A), G, \pi', 0 \models \varphi \\
G, \pi, i &\models X \varphi \quad \text{if } G, \pi, i + 1 \models \varphi \\
G, \pi, i &\models \varphi U \varphi' \quad \text{if there exists } j \geq i \text{ s.t. } G, \pi, j \models \varphi' \text{ and,} \\
&\quad \text{for all } k \text{ s.t. } i \leq k < j, G, \pi, k \models \varphi.
\end{align*}
\]

An \( \text{ATL}_i^* \) formula \( \varphi \) is closed if every temporal operator (\( X \) or \( U \)) in \( \varphi \) is in the scope of a strategic operator \( \langle A \rangle \). Since the semantics of a closed formula \( \varphi \) does not depend on the future, we may write \( G, v \models \varphi \) if \( G, \pi, 0 \models \varphi \) for any infinite play \( \pi \) that starts in \( v \).

The model-checking problem for \( \text{ATL}_i^* \) consists in deciding, given a closed \( \text{ATL}_i^* \) formula \( \varphi \) and a finite pointed CGS \( \langle G, v \rangle \), whether \( G, v \models \varphi \).

## 3.2 Model checking \( \text{ATL}_i^* \)

It is well known that the model-checking problem for \( \text{ATL}_i^* \) is undecidable for agents with perfect recall [2], as it can easily express the existence of distributed winning strategies for multiplayer reachability games with imperfect information and perfect recall, which was proved undecidable by Peterson, Reif and Azhar [28]. A direct proof of this undecidability result for \( \text{ATL}_i^* \) is also presented in [12]. However, there are classes of multiplayer games with imperfect information that are decidable. For many years, the only known decidable case was that of hierarchical games, in which there is a total preorder among players, each player observing at least as much as those below her in this preorder [29][23]. Recently, this result has been extended by relaxing the assumption of hierarchical observation. In particular, it has been shown that the problem remains decidable if the hierarchy can change along a play, or if transient phases without such a hierarchy are allowed [5]. We establish that these results transfer to the model-checking problem for \( \text{ATL}_i^* \).

We remind that a concurrent game with imperfect information is a pair \( ((G, v), W) \) where \( (G, v) \) is a pointed nondeterministic CGS, and \( W \) is a property of infinite plays called the winning condition. The strategy problem is, given such a game, to decide whether there exists
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a strategy profile for the grand coalition $Ag$ to enforce the winning condition against Nature (for more details see, e.g., [3]).

Before stating our transfer theorem we need to introduce a couple of additional notions. First we introduce a notion of abstraction over a group of agents. Informally, abstracting a CGS, $G$ over an agent consists in erasing her from the group of agents and letting Nature play for her in $G$.

**Definition 3** (Abstraction). Let $A \subseteq Ag$ be a group of agents and let $G = (V, E, \ell, \{\sim_a\}_{a \in Ag})$ be a CGS. The abstraction of $G$ from $A$ is the nondeterministic CGS $G\rangle_A$ over set of agents $Ag \setminus A$ defined as $G\rangle_A := (V, E', \ell, \{\sim_a\}_{a \in Ag \setminus A})$, where for every $v \in V$ and $m \in M^{Ag \setminus A}$,

$$(v, m, v') \in E'$$ if $$\exists m' \in M^A$$ s.t. $E(v, (m, m')) = v'$.

Thanks to this notion we can define the following problem:

**Definition 4** (A-strategy problem). The A-strategy problem takes as input a pointed CGS $(G, v)$, a set $A \subseteq Ag$ of agents and a winning condition $W$, and returns the answer to the strategy problem for the game $((G\rangle_A, v), W)$. The A-strategy problem for $(G, v)$ with winning condition $W$ thus consists in deciding whether there is a strategy profile for agents in $A$ to enforce $W$ against everybody else.

Finally we introduce the following notion, which simply captures the change of initial position in a game from a position $v$ to another position $v'$ reachable from $v$:

**Definition 5** (Initial shifting). Let $G$ be a CGS and let $v, v' \in G$. The pointed CGS $(G, v')$ is an initial shifting of $(G, v)$ if $v'$ is reachable from $v$ in $G$.

We are now ready to state our first result.

**Theorem 6.** If $C$ is a class of pointed CGS closed under initial shifting and such that the A-strategy problem with $\omega$-regular objective is decidable on $C$, then model checking $ATL^*_A$ is decidable on $C$.

**Proof.** Let $C$ be such a class of pointed CGS, and let $(\varphi, (G, v))$ be an instance of the model-checking problem for $ATL^*_A$ on $C$. A bottom-up algorithm consists in evaluating each innermost subformula of $\varphi$ of the form $(A)\varphi'$, where $\varphi'$ is thus an LTL formula, on each position $v'$ of $G$ reachable from $v$. Evaluating $(A)\varphi'$ on $v'$ amounts to solving an instance of the A-strategy problem with $\omega$-regular objective (recall that LTL properties are $\omega$-regular). By assumption $(G, v) \in C$, and because $C$ is closed by initial shifting and $v'$ is reachable from $v$, we have that $(G, v') \in C$. Also by assumption, the A-strategy problem for $\omega$-regular winning conditions is decidable on $C$. We thus have an algorithm to evaluate each $(A)\varphi'$ on each $v'$. One can then mark positions of the game with fresh atomic propositions indicating these formulas hold, and repeat the procedure until all strategic operators have been eliminated. It then remains to evaluate a boolean formula in the initial position $v$. □

Let us recall for which classes of nondeterministic CGS, the strategy problem is known to be decidable. A (nondeterministic or deterministic) CGS $G$ has hierarchical observation if there exists a total preorder $\preceq$ over $Ag$ such that if $a \preceq b$ and $v \sim_a v'$, then $v \sim_b v'$.

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3 Observe that if $A = Ag$ then $G\rangle_{Ag \setminus A} = G$, and Nature thus does not do anything. This is coherent with the fact that for agents with perfect recall $(Ag)\varphi \equiv E\varphi$, where $E$ is the CTL path quantifier, even for imperfect information.
This notion was refined in [5] to take into account the agents’ memory, using the notion of information set: for a finite play $\rho \in \text{Plays}(G, v)$ and an agent $a$, the information set of agent $a$ after $\rho$ is $I^a(\rho) := \{ \rho' \in \text{Plays}(G, v) \mid \rho \sim_a \rho' \}$. A finite play $\rho$ yields hierarchical information if there is a total preorder $\preceq$ over $Ag$ such that if $a \preceq b$, then $I^a(\rho) \subseteq I^b(\rho)$. If all finite plays in $\text{Plays}(G, v)$ yield hierarchical information for the same preorder over agents, $(G, v)$ yields static hierarchical information. If this preorder can vary depending on the play, $(G, v)$ yields dynamic hierarchical information. The last generalisation consists in allowing for transient phases without hierarchy: if every infinite play in $\text{Plays}(G, v)$ has infinitely many prefixes that yield hierarchical information, $(G, v)$ yields recurring hierarchical information.

▶ Proposition 7. Hierarchical observation as well as static, dynamic and recurring hierarchical information are preserved by abstraction.

Proof. Abstraction removes agents without affecting observations of remaining ones. The result thus follows from the respective definitions of hierarchical observation and of static, dynamic and recurring hierarchical information. ▶

▶ Proposition 8. Hierarchical observation as well as static, dynamic and recurring hierarchical information are preserved by initial shifting.

This is obvious for hierarchical observation. For the other cases we establish Lemma 9 below. It is then easy to check that Proposition 8 holds.

▶ Lemma 9. If a finite play $v \cdot \rho \cdot v' \cdot \rho'$ yields hierarchical information in $(G, v)$, so does $v' \cdot \rho'$ in $(G, v')$, with the same preorder among agents.

Proof. Assume that $v \cdot \rho \cdot v' \cdot \rho'$ yields hierarchical information in $(G, v)$ with preorder $\preceq$ over $Ag$. Suppose towards a contradiction that there are agents $a, b \in Ag$ such that $a \preceq b$ but $I^a(v' \cdot \rho') \not\subseteq I^b(v' \cdot \rho')$. This means that there is $v' \cdot \rho'' \in \text{Plays}(G, v')$ such that $v' \cdot \rho' \sim_a v'' \cdot \rho''$ but $v' \cdot \rho' \not\sim_b v' \cdot \rho''$. By definition of synchronous perfect recall relations we then have that $v \cdot \rho \cdot v' \cdot \rho' \sim_a v \cdot v' \cdot \rho''$ and $v \cdot \rho \cdot v' \cdot \rho' \not\sim_b v \cdot v' \cdot \rho''$. This implies that $I^a(v \cdot v' \cdot \rho' \cdot \rho') \not\subseteq I^b(v \cdot v' \cdot \rho' \cdot \rho')$, which contradicts the fact that $a \preceq b$. Therefore for all agents $a, b$ such that $a \preceq b$ we have $I^a(v' \cdot \rho') \subseteq I^b(v' \cdot \rho')$, and thus $v' \cdot \rho'$ yields hierarchical information with preorder $\preceq$. ▶

Let $C_{\text{obs}}$ (resp. $C_{\text{stat}}, C_{\text{dyn}}, C_{\text{rec}}$) be the class of pointed CGS$_i$ with hierarchical observation (resp. static, dynamic, recurring hierarchical information). We instantiate Theorem 6 to obtain three decidability results for ATL$^*_i$.

▶ Theorem 10. Model checking ATL$^*_i$ is decidable on the class of CGS$_i$ with hierarchical observation.

Proof. By Proposition 8 $C_{\text{obs}}$ is closed under initial shifting. It is proven in [23] that the strategy problem is decidable for games with hierarchical observation and $\omega$-regular objectives. Since, by Proposition 4 all pointed nondeterministic CGS$_i$ obtained by abstracting agents from CGS$_i$ in $C_{\text{obs}}$ also yield hierarchical observation, we get that the $A$-strategy problem with $\omega$-regular objectives is decidable on $C_{\text{obs}}$. We can therefore apply Theorem 6 on $C_{\text{obs}}$. ▶

It is proven in [5] that the strategy problem with $\omega$-regular objectives is also decidable for games with static hierarchical information and for games with dynamic hierarchical information. Since Proposition 7 and Proposition 8 also hold for $C_{\text{stat}}$ and $C_{\text{dyn}}$, with the same argument as in the proof of Theorem 10 we obtain the following results as consequences of Theorem 6.
Theorem 11. Model checking $\text{ATL}^*_i$ is decidable on the class of $\text{CGS}_i$ with static hierarchical information.

Theorem 12. Model checking $\text{ATL}^*_i$ is decidable on the class of $\text{CGS}_i$ with dynamic hierarchical information.

Note that in fact, since $\mathcal{C}_{\text{obs}} \subset \mathcal{C}_{\text{stat}} \subset \mathcal{C}_{\text{dyn}}$, Theorem 10 and Theorem 11 are also obtained as corollaries of Theorem 12, but we wanted to illustrate how Theorem 6 can be applied to obtain decidability results for different classes of $\text{CGS}_i$.

Remark. The last result in [5] establishes that the strategy problem is decidable for games with recurring hierarchical information, but only for observable $\omega$-regular winning conditions, i.e., when all agents can tell whether a play is winning or not. Now considering $\text{ATL}^*_i$ on $\mathcal{C}_{\text{dyn}}$ we could require atomic propositions to be observable for all agents; in that case we could evaluate the inner-most strategy quantifiers using the above-mentioned result. But then the fresh atomic propositions that mark positions where these subformulas hold (see the proof of Theorem 6) would not, in general, be observable by all agents. So on $\mathcal{C}_{\text{rec}}$ we could obtain a decision procedure for the fragment of $\text{ATL}^*_i$ without nested non-trivial strategy quantifiers, where non-trivial means for coalitions other than the empty coalition or the one made of all agents (which, we recall, are simply the $\text{CTL}$ path quantifiers). We do not state it explicitly because it does not seem of much interest.

Concerning complexity, the strategy problem for games with imperfect information and hierarchical observation is already nonelementary [30, 28], hence the following result:

Corollary 13. Model checking $\text{ATL}^*_i$ is nonelementary on games with hierarchical observation, hence also for games with static or dynamic hierarchical information.

We now turn to $\text{ATL}_i$ with imperfect information and strategy context, and study its model-checking problem.

4 $\text{ATL}_i$ with strategy context

While in $\text{ATL}$ strategies for all agents are forgotten each time a new strategy quantifier is met, in $\text{ATL}$ with strategy context ($\text{ATL}_{\text{sc}}$) [6, 10, 25] agents keep using the same strategy as long as the formula does not say otherwise. In this section we consider $\text{ATL}_{\text{sc}}$ with imperfect information ($\text{ATL}_{\text{sc},i}$). As far as we know, the only existing work on this logic is [26], which proved its model-checking problem to be decidable in the case where all agents have the same observation of the game. We extend significantly this result by establishing that the model-checking problem is decidable as long as strategy quantification is hierarchical, in the sense that if there is a strategy quantification for agent $a$ nested in a strategy quantification for agent $b$, then $b$ should observe no more than $a$. In other terms, innermost strategic quantifications should concern agents who observe more.

4.1 Syntax and semantics

The models are still $\text{CGS}_i$. To remember which agents are currently bound to a strategy, and what these strategies are, the semantics uses strategy contexts. Formally, a strategy context for a set of agents $B \subseteq \text{Ag}$ is a strategy profile $\sigma_B$. We define the composition of strategy contexts as follows. If $\sigma_B$ is a strategy context for $B$ and $\sigma_A$ is a new strategy profile for coalition $A$, we let $\sigma_A \circ \sigma_B$ be the strategy context for $A \cup B$ defined as $\sigma_{A \cup B} : a \mapsto \begin{cases} \sigma_A(a) & \text{if } a \in A, \\ \sigma_B(a) & \text{otherwise}. \end{cases}$
So if \( a \) is assigned a strategy by \( \sigma_A \), her strategy in \( \sigma_A \circ \sigma_B \) is \( \sigma_A(a) \). If she is not assigned a strategy by \( \sigma_A \), her strategy remains the one given by \( \sigma_B \), if any.

Also, given a strategy context \( \sigma_B \) and a set of agents \( A \subseteq Ag \), we let \( (\sigma_B)_{\setminus A} \) be the strategy context obtained by restricting \( \sigma_B \) to the domain \( B \setminus A \).

Finally, because agents who do not change their strategy keep playing the one they were assigned, if any, we cannot forget the past at each strategy quantifier, as in the semantics of \( ATL^* \) (see Section 3.1). We thus define the outcome of a strategy profile \( \sigma_A \) after a finite play \( \rho \), written \( \text{Out}(\rho, \sigma_A) \), as the set of infinite plays \( \pi \) that start with \( \rho \) and then follow \( \sigma_A \): \( \pi \in \text{Out}(\rho, \sigma_A) \) if \( \pi = \rho \cdot \pi' \) for some \( \pi' \), and for all \( i \geq |\rho| - 1 \), there exists a joint move \( m \in M_A^\pi \) such that \( E(\pi_i, m) = \pi_{i+1} \) and for each \( a \in A \), \( m_a = \sigma_a(\pi[0, i]) \).

To differentiate from \( ATL^* \), in \( ATL^*_{sc,i} \) the strategy quantifier for a coalition \( A \) is written \( \langle A \rangle \) instead of \( (A) \). \( ATL^*_{sc,i} \) also has an additional operator, \( [A] \), that releases agents in \( A \) from their current strategy, if they have one. The syntax of \( ATL^*_{sc,i} \) is the same as that of \( ATL^*_{sc} \) and is thus given by the following grammar:

\[ \varphi ::= p \mid \neg \varphi \mid \varphi \lor \varphi \mid \langle A \rangle \varphi \mid [A] \varphi \mid X \varphi \mid X \lor \varphi, \]

where \( p \in AP \) and \( A \subseteq Ag \).

\[ \text{Remark. In } [23] \text{ the syntax of } ATL^*_{sc,i} \text{ contains in addition operators } \langle \neg \rangle \text{ and } \langle \neg \rangle \text{ for complement coalitions. While they add expressivity when the set of agents is not fixed, and are thus of interest when considering expressivity or satisfiability, they are redundant if we consider model checking, which is our case in this work. To simplify presentation we thus choose not to consider them here.} \]

The semantics of \( ATL^*_{sc,i} \) is defined with regards to a \( CGS_i \), \( G = (V, E, \ell, \{\sim_a\}_{a \in Ag}) \), an infinite play \( \pi \), a position \( i \in \mathbb{N} \) along this play, and a strategy context \( \sigma_B \). The semantics is defined by induction on formulas:

\[ \begin{align*}
G, \pi, i &\models_{\sim a} p \quad \text{if } p \in \ell(\pi_i) \\
G, \pi, i &\models_{\sim a} \neg \varphi \quad \text{if } G, \pi, i \not\models_{\sim a} \varphi \\
G, \pi, i &\models_{\sim a} \varphi \lor \varphi' \quad \text{if } G, \pi, i \models_{\sim a} \varphi \text{ or } G, \pi, i \models_{\sim a} \varphi' \\
G, \pi, i &\models_{\sim a} \langle A \rangle \varphi \quad \text{if there exists a strategy profile } \sigma_A \text{ s.t. for all } \pi' \in \text{Out}(\pi[0, i], \sigma_A \circ \sigma_B), G, \pi', i \models_{\sim a \circ \sigma_B} \varphi \\
G, \pi, i &\models_{\sim a} [A] \varphi \quad \text{if } G, \pi, i \models_{\sim a \circ \sigma_B} \varphi \text{ and, for all } k \text{ such that } i \leq k < j, G, \pi, k \models_{\sim a} \varphi. \\
\end{align*} \]

The notion of closed formula is as defined in Section 3.1 and once more, the semantics of a closed formula \( \varphi \) being independent from the future, we may write \( G, v \models_{\sim a} \varphi \) instead of \( G, \pi, 0 \models_{\sim a} \varphi \) for any infinite play \( \pi \) that starts in position \( v \). We also write \( G, v \models \varphi \) if \( G, v \models_{\sim a} \varphi \), that is if \( \varphi \) holds in \( v \) with the empty strategy context.

The model-checking problem for \( ATL^*_{sc,i} \) consists in deciding, given a closed \( ATL^*_{sc,i} \) formula \( \varphi \) and a finite pointed \( CGS_i \) \( (G, v) \), whether \( G, v \models \varphi \).

We now present \( QCTL^* \) with imperfect information, or \( QCTL^*_i \) for short, before proving our main result on the model-checking problem for \( ATL^*_{sc,i} \) by reducing it to the model-checking problem for a decidable fragment of \( QCTL^*_i \).

### 4.2 \( QCTL^* \) with imperfect information

Quantified \( CTL^* \), or \( QCTL^* \) for short, is an extension of \( CTL^* \) with second-order quantifiers on atomic propositions that has been well studied [31, 21, 22, 24]. It has recently been
further extended to take into account imperfect information, resulting in the logic called QCTL\(^*\) with imperfect information, or QCTL\(^*_i\) [3, 4]. We briefly present this logic, as well as a decidability result on its model-checking problem proved in [3, 4] and that we rely on to establish our result on the model checking of ATL\(^*_i\).

Imperfect information is incorporated into QCTL\(^*\) by considering Kripke models with internal structure in the form of local states, like in distributed systems (see for instance [15]), and then parameterising quantifiers on atomic propositions with observations that define what portions of the states a quantifier can “observe”. The semantics is then adapted to capture the idea of quantification on atomic propositions being made with partial observation.

Let us fix a collection \(\{L_i\}_{i \in [n]}\) of \(n\) disjoint finite sets of local states. We also let \(X_n = L_1 \times \ldots \times L_n\).

\[\textbf{Definition 14.} A compound Kripke structure (CKS) over } AP \text{ is a Kripke structure } S = (S, R, \ell) \text{ such that } S \subseteq X_n.\]

The syntax of QCTL\(^*_i\) is that of QCTL\(^*\), except that quantifiers over atomic propositions are parameterised by a set of indices that defines what local states the quantifier can “observe”. It is thus defined by the following grammar:

\[
\varphi ::= p \mid \neg \varphi \mid \varphi \lor \varphi \mid E\varphi \mid \exists p. \varphi \mid X\varphi \mid \varphi U\varphi
\]

where \(p \in AP\) and \(o \subseteq \mathbb{N}\) is a finite set of indices. We use standard abbreviations: \(\top := p \lor \neg p\), \(\bot := \neg \top\), \(F \varphi := \top U \varphi\), \(G \varphi := \neg F \neg \varphi\) and \(A \varphi := \neg E \neg \varphi\).

A finite set \(o \subseteq \mathbb{N}\) is called an observation, and two states \(s = (l_1, \ldots, l_n)\) and \(s' = (l'_1, \ldots, l'_n)\) are \(o\)-indistinguishable, written \(s \equiv_o s'\), if for all \(i \in [n] \cap o\), it holds that \(l_i = l'_i\).

The intuition is that a quantifier with observation \(o\) must choose the valuation of atomic propositions uniformly with respect to \(o\). Note that in [3], two semantics are considered for QCTL\(^*_i\), just like in [24] for QCTL\(^*\): the structure semantics and the tree semantics. In the former, formulas are evaluated directly on the structure, while in the latter the structure is first unfolded into an infinite tree. Here we only present the tree semantics, as it is this one that allows us to capture agents with perfect recall. But we first need a few more definitions.

For \(p \in AP\), two labelled trees \(t = (\tau, \ell)\) and \(t' = (\tau', \ell')\) are equivalent modulo \(p\), written \(t \equiv_p t'\), if \(\tau = \tau'\) and for each node \(u \in \tau\), \(\ell(u) \setminus \{p\} = \ell'(u) \setminus \{p\}\). So \(t \equiv_p t'\) if they are the same trees, except for the labelling of proposition \(p\).

This notion of equivalence modulo \(p\) is the one used to define quantification on atomic propositions in QCTL\(^*\): intuitively, an existential quantification over \(p\) chooses a new labelling for valuation \(p\), all else remaining the same, and the evaluation of the formula continues from the current node with the new labelling. For imperfect information we need to express the fact that this new labelling for a proposition is done uniformly with regards to the quantifier’s observation.

First, we define the notion of indistinguishability between two nodes in the unfolding of a CKS. Let \(o\) be an observation, let \(\tau\) be an \(X_n\)-tree (which may be obtained by unfolding some pointed CKS), and let \(u = s_0 \ldots s_i\) and \(u' = s'_0 \ldots s'_i\) be two nodes in \(\tau\). The nodes \(u\) and \(u'\) are \(o\)-indistinguishable, written \(u \equiv_o u'\), if \(i = j\) and for all \(k \in [0, \ldots, i]\), we have \(s_k \approx_o s'_k\). Observe that this definition corresponds to the notion of synchronous perfect recall in CGS\(_i\), (see Section 3.1). We now define what it means for the labelling of an atomic proposition to be uniform with regards to an observation.

\[\textbf{Definition 15.} Let } t = (\tau, \ell) \text{ be a labelled } X_n\text{-tree, let } p \in AP \text{ be an atomic proposition and } o \subseteq \mathbb{N} \text{ an observation. Tree } t \text{ is } o\text{-uniform in } p \text{ if for every pair of nodes } u, u' \in \tau \text{ such that } u \equiv_o u', \text{ we have } p \in \ell(u) \iff p \in \ell(u').\]
The satisfaction relation $|=_{\ell} (t$ is for tree semantics) is now defined as follows, where $t = (\tau, \ell)$ is a labelled $X_{\ell}$-tree, $\lambda$ is a path in $\tau$ and $i \in \mathbb{N}$ a position along that branch:

$$
t, \lambda, i \models p \quad \text{if } p \in \ell(\lambda_i)
$$

$$
t, \lambda, i \models \neg \varphi \quad \text{if } t, \lambda, i \not\models \varphi
$$

$$
t, \lambda, i \models \varphi \lor \varphi' \quad \text{if } t, \lambda, i \models \varphi \text{ or } t, \lambda, i \models \varphi'
$$

$$
t, \lambda, i \models \mathcal{E} \varphi \quad \text{if there exists } \lambda' \in \text{Paths}(\lambda_i) \text{ such that } t, \lambda', 0 \models \varphi
$$

$$
t, \lambda, i \models \exists \varphi \quad \text{if there exists } t' \equiv_p t \text{ such that } t' \text{ is } o \text{-uniform in } p \text{ and } t', \lambda, i \models \varphi
$$

$$
t, \lambda, i \models X \varphi \quad \text{if } t, \lambda, i + 1 \models \varphi
$$

$$
t, \lambda, i \models \varphi \mathcal{U} \varphi' \quad \text{if there exists } j \geq i \text{ such that } t, \lambda, j \models \varphi' \text{ and for } i \leq k < j, t, \lambda, j \models \varphi
$$

Similarly to $\text{ATL}_{sc,i}^+$ we say that a $\text{QCTL}^*_i$ formula is closed if all temporal operators are in the scope of a path quantifier. The semantics of such formulas depending only on the current node, for a closed formula $\varphi$ we may write $t \models \varphi$ for $t, \models \varphi$, where $r$ is the root of $t$, and given a pointed CKS $(S, s)$ and a $\text{QCTL}^*_i$ formula $\varphi$, we write $S, s \models \varphi$ if $t_S(s) \models \varphi$.

**Remark.** In [3] the syntax is presented with path formulas distinguished from state formulas, and the semantics is defined accordingly. To make the presentation more uniform with that of $\text{ATL}_{sc,i}^+$ we chose here a different, but equivalent, presentation.

**Remark.** Note that when $n$ is fixed, the propositional quantifier with perfect information from $\text{QCTL}^*_i$ is equivalent to the $\text{QCTL}^*_i$ quantifier that observes all the components, i.e., the quantifier parameterised with observation $[n]$.

The model-checking problem for $\text{QCTL}^*_i$ is the following: given a closed $\text{QCTL}^*_i$ formula $\varphi$ and a finite pointed CKS $(S, s)$, decide whether $S, s \models \varphi$.

We now define the class of $\text{QCTL}^*_i$ formulas for which the model-checking problem is known to be decidable with the tree semantics.

**Definition 16.** A $\text{QCTL}^*_i$ formula $\varphi$ is hierarchical if for all subformulas $\varphi_1, \varphi_2$ of the form $\varphi_1 = \exists \varphi_1 p_1$, $\varphi_1'$ and $\varphi_2 = \exists \varphi_2 p_2$, $\varphi_2'$ where $\varphi_2$ is a subformula of $\varphi_1'$, we have $o_1 \subseteq o_2$.

The following result is proved in [3], where $\text{QCTL}_{sc,i}^+$ is the set of hierarchical $\text{QCTL}^*_i$ formulas:

**Theorem 17 ([3]).** Model checking $\text{QCTL}_{sc,i}^+$ with tree semantics is decidable.

4.3 Model checking $\text{ATL}_{sc,i}^+$

We establish that model checking $\text{ATL}_{sc,i}^+$ is decidable on a class of instances whose definition relies on the notion of hierarchical observation.

**Definition 18.** Let $\mathcal{G} = (V, E, \ell, \{\sim_a\}_{a \in Ag})$ be a CGS, and let $a, b \in Ag$ be two agents. Agent $a$ observes no more than agent $b$ in $\mathcal{G}$, written $a \lesssim_b \text{ b}$, if for every pair of positions $v, v' \in V$, $v \sim_a v'$ implies $v \sim_a v'$. We say that $A \subseteq Ag$ is hierarchical in $\mathcal{G}$ if $\lesssim_A$ is a total preorder on $A$.

If a set of agents $A$ is hierarchical in a CGS, $\mathcal{G}$, we thus may talk about maximal and minimal agents in $A$, referring to maximal and minimal elements of $A$ for the relation $\lesssim_A$.

The essence of the requirement that makes the problem decidable is the same as for the decidability result on $\text{QCTL}^*_i$ (Theorem [7]): nesting of quantifiers (here, strategy quantifiers) should be hierarchical, with those observing more inside those observing less. However, unlike in $\text{QCTL}^*_i$, in $\text{ATL}_{sc,i}^+$ observations are not part of formulas, but rather they are given by the models. We thus define the notion of hierarchical $\text{ATL}_{sc,i}^+$ formula with respect to a CGS.
Decidability results for ATL with imperfect information and perfect recall

**Definition 19.** Let $\Phi$ be an $\text{ATL}^*_{sc,i}$ formula and let $\mathcal{G}$ be a CGS$_i$. We say that $\Phi$ is *hierarchical in* $\mathcal{G}$ if:

- for every subformula $\varphi$ of the form $\varphi = \langle A \rangle \varphi'$, $A$ is hierarchical in $\mathcal{G}$, and
- for all subformulas $\varphi_1, \varphi_2$ of the form $\varphi_1 = \langle A_1 \rangle \varphi'_1$ and $\varphi_2 = \langle A_2 \rangle \varphi'_2$ where $\varphi_2$ is a subformula of $\varphi'_1$, maximal agents of $A_1$ observe no more than minimal agents of $A_2$.

An instance $(\Phi, (\mathcal{G}, v))$ of the model-checking problem for $\text{ATL}^*_{sc,i}$ is **hierarchical** if $\Phi$ is hierarchical in $\mathcal{G}$.

In the rest of the section we establish the following:

**Theorem 20.** Model checking $\text{ATL}^*_{sc,i}$ is decidable on the class of hierarchical instances.

We build upon the proof in [25] that establishes the decidability of the model-checking problem for $\text{ATL}^*_{sc,i}$ by reduction to the model-checking problem for $\text{QCTL}^*$. The main difference is that we reduce to the model-checking problem for $\text{QCTL}^*_{i,c}$ instead, using quantifiers parameterised with observations corresponding to agents’ observations. We also need to make a couple of adjustments to obtain formulas in the decidable fragment $\text{QCTL}^*_{i,c}$.

Let $(\Phi, (\mathcal{G}, v_i))$ be a hierarchical instance of the $\text{ATL}^*_{sc,i}$ model-checking problem, where $\mathcal{G} = (V, E, \ell, \{\sim_a\}_{a \in \text{Ag}})$ is a CGS$_i$ over $\text{AP}$. In the reduction we will transform $\Phi$ into an equivalent $\text{QCTL}^*_{i,c}$ formula $\Phi'$ in which we need to refer to the current position in the model $\mathcal{G}$, and also to talk about moves taken by agents. To do so, we consider the additional sets of atomic propositions $\text{AP}_v := \{p_v \mid v \in V\}$ and $\text{AP}_m := \{p^a_m \mid a \in \text{Ag} \text{ and } m \in M\}$, that we take disjoint from $\text{AP}$.

First we define the CKS $\mathcal{S}_{\mathcal{G}}$ on which $\Phi'$ will be evaluated. Since the models of the two logics use different ways to represent imperfect information (equivalence relations on positions for CGS$_i$ and local states for CKS) this requires a bit of work. First, for each $v \in V$ and $a \in \text{Ag}$, let us define $[v]_a$ as the equivalence class of $v$ for relation $\sim_a$. Now, noting $\text{Ag} = \{a_1, \ldots, a_n\}$, we define for each $i \in [n]$ the set $L_i := \{[v]_{a_i} \mid v \in V\}$ of local states for agent $a_i$. Since we need to know the actual position of the CGS$_i$ to define the dynamics, we also let $L_{n+1} := V$. States of $\mathcal{S}_{\mathcal{G}}$ will thus be tuples in $L_1 \times \ldots \times L_n \times L_{n+1}$. For each $v \in \mathcal{G}$, let $s_v := ([v]_{a_1}, \ldots, [v]_{a_n}, v)$ be its corresponding state in $\mathcal{S}_{\mathcal{G}}$.

We can now define $\mathcal{S}_{\mathcal{G}} := (S, R, \ell')$, where

- $S := \{s_v \mid v \in V\}$,
- $R := \{(s_v, s_{v'}) \mid \exists m \in M^{\text{Ag}} \text{ s.t. } E(v, m) = v'\}$, and
- $\ell'(s_v) := \ell(v) \cup \{p_u\}$.

To make the connection between finite plays in $\mathcal{G}$ and nodes in tree unfoldings of $\mathcal{S}_{\mathcal{G}}$, let us define, for every finite play $\rho = v_0 \ldots v_k$, the node $u_\rho := s_{v_0} \ldots s_{v_k}$ in $t_{\mathcal{S}_{\mathcal{G}}}(s_{v_0})$ (which exists, by definition of $\mathcal{S}_{\mathcal{G}}$ and of tree unfoldings). Observe that the mapping $\rho \mapsto u_\rho$ is in fact a bijection between the set of finite plays starting in a given position $v$ and the set of nodes in $t_{\mathcal{S}_{\mathcal{G}}}(s_v)$.

Now it should be clear that giving to a propositional quantifier in $\text{QCTL}^*_{i,c}$ observation $s_v := \{i\}$, for $i \in [n]$, amounts to giving him the same observation as agent $a_i$. Formally, one can prove the following lemma, simply by applying the definitions of observational equivalence in the two frameworks:

**Lemma 21.** For all finite plays $\rho, \rho'$ starting in $v$, $\rho \sim_{a_i} \rho'$ iff $u_\rho \approx_{o_i} u_{\rho'}$ in $t_{\mathcal{S}_{\mathcal{G}}}(s_v)$. 
We now describe the translation from \( \text{ATL}_{sc,i} \) formulas to \( \text{QCTL}^i_\ast \) formulas. First we recall the translation from \cite{25} for the perfect-information case.

The translation from \( \text{ATL}_{sc} \) to \( \text{QCTL}^\ast \) is parameterised by a coalition \( B \subset A^g \), that conveys the set of agents who are currently bound to a strategy. It is defined by induction on \( \Phi \) as follows:

\[
\begin{align*}
\varphi^B & := \varphi \\
\varphi \lor \varphi^B & := \varphi^B \lor \varphi^B \\
\mathbf{X} \varphi^B & := \mathbf{X} \varphi^B \\
\neg \varphi^B & := \neg \varphi^B \\
\langle A \rangle \varphi^B & := \mathbf{A} \varphi^B, \\
\langle \Phi_{\text{strat}}(A) \rangle \varphi^B \land \langle \Phi_{\text{out}}(A \cup B) \rangle & := (\Phi_{\text{strat}}(A) \land \Phi_{\text{out}}(A \cup B)) \land \mathbf{A}(\text{G}_{\text{out}} \rightarrow \text{p}_{\text{out}}^A),
\end{align*}
\]

The only non-trivial case is for formulas of the form \( \langle A \rangle \varphi \). For the rest of the section, we let \( M = \{ m_1, \ldots, m_l \} \). Now, if \( A = \{ a_{i_1}, \ldots, a_{i_k} \} \), we define

\[
\langle A \rangle \varphi^B := \exists m_1^{a_{i_1}} \ldots m_l^{a_{i_k}} \ldots m_{i_k}^{a_{i_k}}.p_{\text{out}}. \\
(\Phi_{\text{strat}}(A) \land \Phi_{\text{out}}(A \cup B) \land \mathbf{A}(\text{G}_{\text{out}} \rightarrow \text{p}_{\text{out}}^A)),
\]

where

\[
\Phi_{\text{strat}}(A) := \bigwedge_{a \in A} \mathbf{A} G \bigwedge_{m \in M} (m^a \land \bigwedge_{m' \neq m} \neg m'^a)
\]

and

\[
\Phi_{\text{out}}(A) := p_{\text{out}} \land \mathbf{A} G [\neg p_{\text{out}} \rightarrow \mathbf{A} X \neg p_{\text{out}}] \land \mathbf{A} G \left[ p_{\text{out}} \rightarrow \bigvee_{v \in V} \bigvee_{m \in M^A} \left( p_v \land p_m \land \mathbf{A} X \left( \bigvee_{v' \in E(v, m)} p_{v'} \leftrightarrow p_{\text{out}} \right) \right) \right].
\]

In \( \Phi_{\text{out}}(A) \), for \( m = (m_a)_{a \in A} \in M^A \), notation \( p_m \) stands for the propositional formula \( \bigwedge_{a \in A} m^a \) which characterizes the joint move \( m \) that agents in \( A \) play in \( m \). Also, \( E(v, m) \) is the set of possible next positions when the current one is \( v \) and agents in \( A \) play \( m \), and it is defined as \( E(v, m) := \{ E(v, (m_a, m')) | m' \in M^{A \setminus A} \} \).

The idea of this translation is the following: first, for each agent \( a \in A \) and each possible move \( m \in M \), an existential quantification on the atomic proposition \( m^a \) “chooses” for each finite play \( \rho \) of \( (G, v) \) (or, equivalently, for each node \( u_{\rho} \) of the \( t \mathcal{S}_G(s_{\rho}) \)) whether agent \( a \) plays move \( m \) in \( \rho \) or not. Formula \( \Phi_{\text{strat}}(A) \) ensures that each agent \( a \) chooses exactly one move in each finite play, and thus that atomic propositions \( m^a \) characterise a strategy for her. An atomic proposition \( p_{\text{out}} \) is then used to mark the paths that follow the currently fixed strategies: formula \( \Phi_{\text{out}}(A \cup B) \) states that \( p_{\text{out}} \) marks exactly the outcome of strategies just chosen for agents in \( A \), as well as those of agents in \( B \), that were chosen previously by a strategy quantifier “higher” in \( \Phi \).

Note that we simplified slightly \( \Phi_{\text{strat}}(A) \) and \( \Phi_{\text{out}}(A) \), using the fact that unlike in \cite{25}, we have assumed in our definition of \( \text{CGS}_i \) that the set of available moves is the same for all agents in all positions (see Footnote 2).

It is proven in \cite{23} that this translation is correct, in the sense that for every \( \text{ATL}_{sc} \) closed formula \( \varphi \) and pointed perfect-information concurrent game structure \( (G, v) \), letting \( \mathcal{S}_G \) be

\footnote{Here we abuse language: the construction depends on the model \( \mathcal{G} \) and is therefore not a translation in the usual sense.}
as described above but removing the local states for all agents and keeping only the $L_{n+1}$ component, we have:

$$\mathcal{G}, v \models \varphi \iff t_{\mathcal{G}_v}(s_v) \models \mathbf{F}^\emptyset.$$  

We now explain how to adapt this translation to the case of imperfect information. Observe that the only difference between $\text{ATL}^*_c$ and $\text{ATL}^*_c,i$ is that in the latter, strategies must be defined uniformly over indistinguishable finite plays, i.e., a strategy $\sigma$ for an agent $a$ must be such that if $\rho \sim_a \rho'$, then $\sigma(\rho) = \sigma(\rho')$. To enforce that the strategies coded by atomic propositions $m^a$ in $\langle A \rangle \varphi$ are uniform, we use the propositional quantifiers with partial observation of $\text{QCTL}^*_c$. Formally, we define a translation $\sim_B$ from $\text{ATL}^*_c,i$ to $\text{QCTL}^*_c$. It is defined exactly as the one from $\text{ATL}^*_c$ to $\text{QCTL}^*$, except for the following inductive case.

If $A = \{a_1, \ldots, a_k\}$ we let

$$\langle A \rangle \varphi^B := \exists a_1 a_1^1 \ldots m_{k}^a \ldots \exists a_k a_{ik}^1 \ldots m_{k}^a \exists_{\text{out}}.$$

$$ \big( \Phi_{\text{strat}}(A) \land \Phi_{\text{out}}(A \cup B) \land (A(\text{G}_{\text{out}} \rightarrow \mathbf{F}^A_{\text{out}})) \big),$$

where $\Phi_{\text{strat}}(A)$ and $\Phi_{\text{out}}(A)$ are defined as before, and $\exists_{\text{out}}$ is a macro for $\exists^{[1, \ldots, n+1]}_{\text{out}}$ (see Remark 4.2).

So the only difference from the previous translation is that now, the labelling of each atomic proposition $m^a_i$ must be $a_i$-uniform. This means that if two nodes $u$ and $u'$ in $t_{\mathcal{G}_v}(s_v)$ are $a_i$-indistinguishable, then $u$ is labelled with $m^a_i$ if and only if $u'$ also is. In other words, in the strategy coded by atomic propositions $m^a_i$, agent $a_i$ plays $m$ in $u$ if and only if she also plays it in $u'$, and thus this strategy is uniform (recall that, by Lemma 21, observation $o_i$ correctly reflects agent $a_i$’s observation in $t_{\mathcal{G}_v}(s_v)$). It is then clear that this translation is correct:

$$\mathcal{G}, v_i \models \Phi \iff t_{\mathcal{G}_v}(s_v) \models \mathbf{F}^\emptyset.$$  \hspace{1cm} (1)

However, even if we have taken $(\Phi, (\mathcal{G}, v_i))$ to be a hierarchical instance, $\mathbf{F}^\emptyset$ is not in the decidable fragment $\text{QCTL}^*_c$. Indeed, with the current definition of observations $\{o_i\}_{i \in [n]}$, hierarchical observation in $\mathcal{G}$ does not imply hierarchical observation in $\mathcal{G}_v$: since $a_i = \{i\}$, for $i \neq j$ it is never the case that $a_i \subseteq a_j$. Still, we note that if agent $a_j$ observes no more than agent $a_i$, then letting $a_i$ see also what agent $a_j$ sees does not increase her knowledge of the situation:

\textbf{Lemma 22.} If $a_j \not\approx_G a_i$, then for all finite plays $\rho, \rho'$ that start in the same position, $u_\rho \approx_{a_i} u_{\rho'}$ if and only if $u_\rho \approx_{a_i \cup a_j} u_{\rho'}$.

\textbf{Proof.} Assume that $a_j \not\approx_G a_i$. It is enough to see that for every pair of states $s_v, s_{v'}$ in $\mathcal{S}_G$, we have $s_v \approx_{a_i} s_{v'}$ if and only if $s_v \approx_{a_i \cup a_j} s_{v'}$. The right-to-left implication is obvious: if two states have the same $i$-th and $j$-th components, in particular they have the same $i$-th component. For the other direction, assume that $s_v \approx_{a_i} s_{v'}$. This means that $[v]_{a_i} = [v']_{a_i}$, and thus that $v \approx_{a_j} v'$. Since $a_j \not\approx_G a_i$, we also have that $v \approx_{a_j} v'$, and thus that $[v]_{a_j} = [v']_{a_j}$, and it follows that $s_v \approx_{a_i \cup a_j} s_{v'}$. \hfill \blacktriangleleft

In the light of this Lemma 22 we can safely redefine observations as follows: for each $i \in [n]$, we let

$$a_i' := \bigcup_{j \neq a_j} a_j.$$
Observe that in fact \( o'_i = \{ j : a_j \preceq_G a_i \} \). Informally, a quantifier with observation \( o'_i \) sees what agent \( a_i \) observes (note that \( \preceq_G \) is reflexive), as well as what agents that see no more than \( a_i \) observe.

Let us define a new version of the translation \( \sim^B \). First, \( \Phi \) being hierarchical in \( G \), for each subformula of \( \Phi \) of the form \( \langle A \rangle \varphi \) we have that \( A \) is hierarchical in \( G \). It is thus possible to choose for agents in \( A \) an indexing \( A = \{ a_{i_1}, \ldots, a_{i_k} \} \) such that for all \( 1 \leq c < d \leq k \), we have \( a_{i_c} \preceq_G a_{i_d} \).

Now the translation remains the same as before except for the following inductive case:

If \( A = \{ a_{i_1}, \ldots, a_{i_k} \} \), where for all \( 1 \leq c < d \leq k \), we have \( a_{i_c} \preceq_G a_{i_d} \), we let

\[
\langle A \rangle \varphi := \exists^{a'_{i_1}} m_{i_1}^{a_{i_1}} \ldots m_{i_k}^{a_{i_k}} \exists^{a'_{i_k}} m_{i_1}^{a_{i_k}} \ldots m_{i_k}^{a_{i_k}} \exists \varphi_{\text{out}},
\]

where \( \Phi_{\text{strat}}(A) \) and \( \varphi_{\text{out}}(A) \) are defined as before.

From Lemma \[22\] we have that this new translation is still correct in the sense of Equation \[1\]. In addition, for all \( 1 \leq c < d \leq k \) we have \( o'_{i_c} \subseteq o'_{i_d} \).

Now consider formula \( \tilde{\Phi}^\emptyset \). Because \( \Phi \) is hierarchical in \( G \), for every pair of subformulas \( \varphi_1, \varphi_2 \) of the form \( \varphi_1 = \langle A_1 \rangle \varphi'_1 \) and \( \varphi_2 = \langle A_2 \rangle \varphi'_2 \) where \( \varphi_2 \) is a subformula of \( \varphi'_1 \), maximal agents of \( A_1 \) observe no more than minimal agents of \( A_2 \). It is then easy to see that \( \tilde{\Phi}^\emptyset \) would be hierarchical if there were not the perfect-information quantifications on atomic proposition \( p_{\text{out}} \) that break the monotony of observations along subformulas when there are nested strategic quantifiers. We explain how to remedy this last problem.

We remove altogether proposition \( p_{\text{out}} \), and we use instead the formula \( \psi_{\text{out}}(A) \) defined below to characterise which paths are in the outcome of the currently-fixed strategies:

\[
\psi_{\text{out}}(A) := G \left( \bigwedge_{v \in V} \bigwedge_{m \in M^A} p_v \wedge p_m \Rightarrow X \bigvee_{v' \in B(v, m)} p_{v'} \right).
\]

Clearly, this formula holds in a path \( \lambda \) of \( t_{G}(s_0) \) marked with propositions \( m^a \) characterising strategies for agents in \( A \), if at each point along \( \lambda \) corresponding to some position \( v \), the next point in \( \lambda \) corresponds to a position \( v' \) that can be attained from \( v \) when agents in \( A \) each play the move prescribed by their current strategy. The last modification to \( \sim^B \) is thus the following:

If \( A = \{ a_{i_1}, \ldots, a_{i_k} \} \), where for all \( 1 \leq c < d \leq k \), we have \( a_{i_c} \preceq_G a_{i_d} \), we let

\[
\langle A \rangle \varphi := \exists^{a'_{i_1}} m_{i_1}^{a_{i_1}} \ldots m_{i_k}^{a_{i_k}} \exists^{a'_{i_k}} m_{i_1}^{a_{i_k}} \ldots m_{i_k}^{a_{i_k}} \Phi_{\text{strat}}(A) \wedge A \left( \psi_{\text{out}}(A \cup B) \Rightarrow \tilde{\varphi}^{A \cup B} \right),
\]

where \( \Phi_{\text{strat}}(A) \) is defined as before.

It follows from the above considerations that this translation is still correct in the sense of Equation \[1\], and one can check that \( \tilde{\Phi}^\emptyset \) is a hierarchical \( \text{QCTL}^1 \) formula. We conclude the proof by recalling that by Theorem \[17\] model checking \( \text{QCTL}^1_{\preceq, \subset} \) is decidable.

Concerning complexity, model checking \( \text{ATL}_{\text{sc}, \subset} \) being already nonelementary \[25\], so is it for \( \text{ATL}_{\text{sc}, \subset} \).

5 Conclusion

In this work we established new decidability results for the model-checking problem of \( \text{ATL}^* \) with imperfect information and perfect recall as well as its extension with strategy context.
Should new decidable classes of multiplayer games with imperfect information be discovered, and assuming the reasonable property of closure under initial shifting, our transfer theorem (Theorem 6) would entail new decidability results also for ATL∗. As for ATL∗sc,i, it would be interesting to investigate whether a meaningful notion of hierarchical instances based on, e.g., dynamic or recurring hierarchical information instead of hierarchical observation as here, could lead to stronger decidability results.

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