OPTIMAL INVESTMENT-REINSURANCE POLICY WITH REGIME SWITCHING AND VALUE-AT-RISK CONSTRAINT

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Abstract. This paper studies an optimal investment-reinsurance problem for an insurance company which is subject to a dynamic Value-at-Risk (VaR) constraint in a Markovian regime-switching environment. Our goal is to minimize its ruin probability and control its market risk simultaneously. We formulate the problem as an infinite horizontal stochastic control problem with the constrained strategies. The dynamic programming technique is applied to derive the coupled Hamilton-Jacobi-Bellman (HJB) equations and the Lagrange multiplier method is used to tackle the dynamic VaR constraint. Furthermore, we propose an efficient numerical method to solve those HJB equations. Finally, we employ a practical example from the Korean market to verify the numerical method and analyze the optimal strategies under different VaR constraints.

1. Introduction. This paper is concerned with the classical optimization problem for an insurance company, i.e., minimizing its ruin probability. To accomplish this, the insurance company can enter reinsurance contracts to transfer parts of its risks to the reinsurance companies. At the same time, it has the opportunities to invest in the stock markets to diversify its wealth and risk. The study of optimal investment and reinsurance strategies to minimize the ruin probability has attracted a great deal of interest. See, for example, [3], [2], [18], [4], and the references therein.

It has been widely recognized that the regime switching models can reflect the structural changes of the economic environment and capture some important features of the financial markets such as time-varying volatility. Triggered by the recent economic recession, the dynamics of the finance market fluctuate drastically nowadays. Furthermore, the insurance companies as the major institutions in the finance market could be significantly affected by economic cycles. The switching behaviors in the finance and the insurance markets have been verified empirically in the literature, such as [1], [11], [5], and [8]. Recent work on the risk models for the insurance company with regime-switching can be also found in [23], [6], [14], and [13]. Specially, Jang and Kim [13] investigate the optimal investment-reinsurance
problem under a two-state regime-switching economy. Besides, they offer a practical example with carefully estimated parameters from the Korean market and a numerical scheme to derive the optimal policy.

Due to the violent fluctuation in the market, it is important for every company to control its risk at a reasonable level. This raises the issues of risk measurement and management. The concept of Value-at-Risk (VaR) is defined as the maximum loss of a portfolio over a given horizon, at a given confidence level. For a risk measurement, it has gained an increasing popularity in recent years. Recently, there have been a lot of works about the portfolio selection problem with the VaR constraint, such as [9], [21], [12], and [22]. We should mention that Yiu et al. in [22] introduce the concept of maximum Value-at-Risk (MVaR) when they deal with the portfolio selection problem with regime-switching. They define the MVaR constraint as the maximum value of the VaRs over different regimes. In addition, quite a few papers focus on the optimal strategies for the insurance company with the VaR constraint, such as [15], [16], and [7]. In [7], Chen et al. investigate the optimal investment-reinsurance policy for an insurance company with a dynamic VaR constraint. Their aim is also to minimize the ruin probability of the company. However, they have not considered the regime-switching phenomenon in the market.

Inspired by [13], [22], and [7], we shall incorporate the regime-switching and the MVaR constraint into the optimal investment-reinsurance problem in this paper. Our objective is to minimize the ruin probability and fulfill the regulators’ requirement on the market risk at the same time, when the key parameters, such as interest rates, stock returns and volatility, the drift and volatility of insurance claims, can vary according to the two regimes. We should mention the related work [17], in which the authors also study an optimal investment-reinsurance problem for an insurer who faces dynamic risk constraint in a regime-switching environment. Their goal is to maximize the expected utility of terminal wealth. Comparing with it, we focus on an more conservative strategy, since we try to avoid various risks for an insurer, such as the bankrupt and the market fluctuation. Therefore, we can provide a useful alternative choice in the bad economic environment. Technically, we use the dynamic programming principle and the Lagrange multiplier method to derive the system of coupled HJB equations. We also provide a practical example in which the ruin probabilities and the optimal strategies are obtained by solving the HJB equations numerically. By means of the theoretical analysis and the numerical experiment, we shall find the impacts of the MVaR constraint on the ruin probabilities and the optimal strategies in both the economic regimes.

The rest of the paper are organized as follows. In Section 2, our new model is introduced. First, we describe the dynamics of the insurance company’s surplus, in which the parameters switch between two states over time according to a continuous-time Markovian chain; then develop the MVaR constraint; and finally present the mathematical formulation of the optimal problem. In Section 3, we shall derive two coupled HJB equations with the control constraints using the dynamic programming principle and the Lagrange multiplier method. In Section 4, we shall introduce the numerical method to solve the HJB equations, and provide a numerical example to analyze the optimal strategies. In Section 5, we shall conclude the paper.

2. The model. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, where $\mathcal{F}_t = \sigma(Y(t), W_s(t), W_c(t))$ is the natural filtration induced by $Y(t), W_s(t)$ and $W_c(t)$. Processes $\{Y(t)\}_{t \geq 0}$, $\{W_s(t)\}_{t \geq 0}$ and $\{W_c(t)\}_{t \geq 0}$ are defined below.
2.1. The surplus process with regime-switching. Here, we consider an insurance company that simultaneously attempts to optimally allocate its financial assets and to develop a strategic plan to reinsure some of its claims. The financial market consists of two kinds of assets, which are a risk-free asset $B$ and a risky asset $S$. In addition, the insurer’s claim process is denoted by $C(t)$, and its surplus process without the investment-reinsurance strategies is denoted by $R(t)$.

Set $Y(t)$ as a continuous-time two-state Markovian chain with two states: Regime 1 and Regime 2. They are interpreted as different states of an economy. Following the work [10], we adopt the canonical representation of these states. Assume that $Y(t) \in \{e_1, e_2\}$, where $e_1 = (1,0)'$ and $e_2 = (0,1)'$. The dynamic of $Y(t)$ has the semi-martingale decomposition as:

$$Y(t) = Y(0) + \int_0^t QY(s)ds + M(t),$$

where $M(t)$ is two-valued martingale with respect to the filtration generated by $Y(s), s \leq t$. The constant $2 \times 2$ matrix $Q = (q_{ij})_{2\times2}$ is called the rate matrix, and satisfies $q_{12} = -q_{11} = \lambda_1$ and $q_{21} = -q_{22} = \lambda_2$. The set-up implies that Regime $i$ changes to Regime $j$ ($j \neq i$) with intensity $\lambda_i$, namely,

$$P(Y(t + dt) = j \mid Y(t) = i) = \lambda_i dt,$$

for an infinitesimal length of time $dt$. See [22] for more details.

Suppose that the interest rate $r(t)$ of the risk-free asset $B$ is

$$r(t) = r(t, Y(t)) = \langle r, Y(t) \rangle,$$  \hspace{1cm} (2.1)

where $\langle \cdot, \cdot \rangle$ denotes an inner production and $r := (r_1, r_2)$. Suppose that $\mu(t)$ and $\sigma(t)$ denote the return rate and the volatility of the risky asset $S$, respectively. We have that

$$\mu(t) = \mu(t, Y(t)) = \langle \mu, Y(t) \rangle, \quad \sigma(t) = \sigma(t, Y(t)) = \langle \sigma, Y(t) \rangle,$$  \hspace{1cm} (2.2)

where $\mu := (\mu_1, \mu_2)$ and $\sigma := (\sigma_1, \sigma_2)$ with $\mu_1 > r_1$ and $\sigma_i > 0$ for each $i = 1, 2$.

Let $W_s(t)$ denote the standard Brownian motion. The dynamics of those assets are

$$dB(t) = r(t)B(t)dt,$$  \hspace{1cm} (2.3)

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW_s(t).$$  \hspace{1cm} (2.4)

In this paper, we adopt a commonly accepted idea that the surplus of an insurance company is relatively large compared to the size of the individual claims. Therefore, the surplus process the insurer can be well approximated by a diffusion form. Let $W_c(t)$ denote the standard Brownian motion, independent of $W_s(t)$. The aggregate insurance claim process $C(t)$, which represents all the money paid to the insured at the time $t$, follows a diffusion approximation process:

$$dC(t) = \alpha(t)dt - \beta(t)dW_c(t),$$  \hspace{1cm} (2.5)

with $\alpha(t)$ as the rate and $\beta(t)$ as the volatility. Similarly, we suppose that the rate and volatility also change according to the economic state. Therefore, we assume that

$$\alpha(t) = \alpha(t, Y(t)) = \langle \alpha, Y(t) \rangle, \quad \beta(t) = \beta(t, Y(t)) = \langle \beta, Y(t) \rangle,$$  \hspace{1cm} (2.6)

where $\alpha := (\alpha_1, \alpha_2)$ and $\beta := (\beta_1, \beta_2)$ with $\alpha_i > 0, \beta_i > 0$ for each $i = 1, 2$.

We assume that the premium is paid to the insurer continuously over time and that the premium rate is proportional to the expected payout. The premium rate is supposed to be evaluated using the expected value principle with risk loading.
In particular, the premium rate is \((1 + \eta)\alpha(t)\), where \(\eta\) is the safety loading of the insurance. In the absence of control, the surplus process is governed by

\[
dR(t) = (1 + \eta)\alpha(t)dt - dC(t) = \alpha(t)\eta dt + \beta(t)dW_{\alpha}(t). \quad (2.7)
\]

Furthermore, we assume that at each moment, the insurer has an option to purchase a proportional reinsurance protection or to acquire new business. The function \(u(t) : R^+ \to [0, +\infty)\) stands for the value of risk exposure which means that the insurance company will cover the 100\(u(t)%\) of the claims. A proportional reinsurance premium is paid continuously at the constant rate \((1 + \theta)(1 - u(t))\alpha(t)\), where \(\theta\) is the safety loading of the proportional reinsurance and satisfies \(\theta \geq \eta\). In this situation, the surplus process becomes

\[
dR(t) = (1 + \eta)\alpha(t)dt - u(t)dC(t) - (1 + \theta)(1 - u(t))\alpha(t)dt = (\eta - \theta + \theta u(t))\alpha(t)dt + u(t)\beta(t)dW_{\alpha}(t). \quad (2.8)
\]

The insurer has an opportunity of investing in the stock markets, which means that he/she can invest in the risk-free asset and the risky asset. The amount invested in the risk-free asset is set as \(X(t) - \pi(t)\) and the risky asset is set as \(\pi(t)\). The surplus process with the investment–reinsurance strategies is denoted by the wealth process \(X(t)\). The insurer’s wealth process \(X(t)\) is given by

\[
dX(t) = (r(t)X(t) + (\mu(t) - r(t))\pi(t) + [\eta - \theta + \theta u(t)]\alpha(t))dt + \sigma(t)\pi(t)dW_{\pi}(t) + \beta(t)u(t)dW_{\alpha}(t).
\]

To obtain an easier expression, we denote the reinsurance–investment strategies as \(\phi(t) = (u(t), \pi(t))^T\). Further, we introduce the notations \(\hat{\theta}(t) = \theta\alpha(t)\),

\[
\hat{\mu}(t) = \begin{pmatrix} \hat{\theta}(t) \\ \mu(t) - r(t) \end{pmatrix}, \quad \hat{\sigma}(t) = \begin{pmatrix} \beta(t) & 0 \\ 0 & \sigma(t) \end{pmatrix}, \quad W(t) = \begin{pmatrix} W_{\pi}(t) \\ W_{\alpha}(t) \end{pmatrix},
\]

and \(\hat{\eta}(t) = \eta\alpha(t)\).

With the initial surplus \(x\) and the policy \(\phi(t)\), the wealth process of the insurance company satisfies the following stochastic differential equation (SDE):

\[
dX(t) = rX(t)dt + [\phi^T(t)\hat{\mu} + (\hat{\eta}(t) - \hat{\theta}(t))]dt + \phi^T(t)\hat{\sigma}(t)dW(t), \quad (2.9)
\]

\[
X(0) = x. \quad (2.10)
\]

2.2. The MVaR constraint. In the sequel, we introduce the MVaR constraint into our model. First of all, we should mention that the constraint discussed below is not strictly VaR constraints in two regimes, but rather the VaR constraints in the approximate from. Our motivation of approximation is to obtain the explicit form of the constraint, while capturing the main features of the VaR condition in the same time. Now we focus on the strategies over the time interval \([t, t + h]\), in which \(h > 0\) is small enough and \(t \geq 0\). We will offer two assumptions about the approximation form below. Firstly, we assume that asset allocation and reinsurance strategies do not change over this short time period. This means that there is no trading between two constraint re-evaluations, and the risk exposure remains roughly constant in the given time period. That is, \(\phi(l) = \phi(t)\) for \(l \in [t, t + h]\). Secondly, we assume that there is no regime switching in the small time interval. For instance, the market state keeps in Regime \(i\) for \(l \in [t, t + h]\). Those settings are reasonable based on the fact that \(h\) is small enough. On one hand, the insurer can only adjust its reinsurance/new business and investment policy at discrete time in real life. On the other hand, the regime changes from Regime \(i\) to Regime \(j\) over the time interval.
\[ [t, t + h] \] with probability \( \lambda_i h \). It shows that the probability of staying in the same state \((1 - \lambda_i h)\) approximately equals 1 if \( h \) is small enough. Similar setting is also adopted by [22]. Under such assumptions, we have the following result according to Ito’s formula:

\[
X(t + h) - e^{r_i h} X(t) = \frac{e^{r_i h} - 1}{r_i} \left( \phi^T(t) \bar{\mu}_i + (\bar{\eta}_i - \bar{\theta}_i) \right) + \phi^T(t) \tilde{\sigma}_i \int_t^{t + h} e^{r_i (t + h - l)} dW(l)
\]  

(2.11)

Define the discounted net loss over the interval \([t, t + h]\) by

\[
\Delta_i X(t, h) = e^{r_i h - 1} X(t) - X(t + h).
\]  

(2.12)

We figure out from (2.11) that \( \Delta_i X(t, h) \) can be regarded as the normal distribution in the approximation form. Moreover, the mean and variance of \( \Delta_i X(t + h) \) are

\[
E(\Delta_i X(t, h)) = -\frac{e^{r_i h} - 1}{r_i} (\phi^T(t) \bar{\mu}_i + (\bar{\eta}_i - \bar{\theta}_i))
\]

and

\[
Var(\Delta_i X(t, h)) = \frac{e^{2r_i h} - 1}{2r_i} \phi^T(t) \tilde{\sigma}_i \tilde{\sigma}_i^T \phi(t),
\]

respectively.

Then, the VaR\((t, h, i, \beta)\) with the probability level \( \beta \) over the time interval \([t, t + h]\) is defined as

\[
P(\Delta_i X(t, h) \geq \text{VaR}(t, h, i, \beta)) = \beta.
\]  

(2.13)

After simple computation, we have that

\[
\text{VaR}(t, h, i, \beta) = -\frac{e^{r_i h} - 1}{r_i} (\phi^T(t) \bar{\mu}_i + (\bar{\eta}_i - \bar{\theta}_i)) - \Psi^{-1}(\beta) \sqrt{\frac{e^{2r_i h} - 1}{2r_i} \phi^T(t) \tilde{\sigma}_i \tilde{\sigma}_i^T \phi(t)},
\]

where \( \Psi^{-1}(\cdot) \) is the inverse function of the standard normal distribution function.

We define the MVaR as the maximum value of the VaRs over two states of the chain, expressed as

\[
\text{MVaR}(t, h, \beta) = \max_{i=1,2} \text{VaR}(t, h, i, \beta).
\]  

(2.14)

Then, the constraint of restricting the MVaR at the level \( R \) is

\[
\text{MVaR}(t, h, \beta) \leq R,
\]  

(2.15)

which is equivalent to the following two constraints:

\[
\text{VaR}(t, h, i, \beta) \leq R, \quad i = 1, 2.
\]  

(2.16)

That is,

\[
-\frac{e^{r_i h} - 1}{r} (\phi^T(t) \bar{\mu}_i + (\bar{\eta}_i - \bar{\theta}_i)) - \Psi^{-1}(\beta) \sqrt{\frac{e^{2r_i h} - 1}{2r_i} \phi^T(t) \tilde{\sigma}_i \tilde{\sigma}_i^T \phi(t)} \leq R, \quad i = 1, 2.
\]  

(2.17)

After some manipulation, inequality (2.17) can be rewritten as

\[
\hat{k}_i \sqrt{\phi^T(t) \tilde{\sigma}_i \tilde{\sigma}_i^T \phi(t)} \leq \phi^T(t) \bar{\mu}_i + c_i, \quad i = 1, 2,
\]  

(2.18)
where
\[
\tilde{k}_i = -\Psi^{-1}(\beta) \sqrt{\frac{r_i e^{2r_i h} - 1}{2(e^{r_i h} - 1)^2}}, \quad c_i = \frac{r_i}{e^{r_i h} - 1} R + (\tilde{\eta}_i - \tilde{\theta}_i). \quad (2.19)
\]

We call the sets of the possible strategies satisfying (2.18) as the control spaces of Regime \(i, i = 1, 2\).

**Remark 2.1.** Here, we restrict that \(c_i > 0\), i.e.,
\[
\frac{r_i}{e^{r_i h} - 1} R + (\tilde{\eta}_i - \tilde{\theta}_i) > 0, \quad i = 1, 2. \quad (2.20)
\]
Note that with \(h\) small enough the left hand side of (2.20) can be relatively bigger than zero. Hence, the restriction of (2.20) is mild.

**Remark 2.2.** Set
\[
\Lambda_i = \sqrt{\tilde{\mu}_i^T (\tilde{\sigma}_i \tilde{\sigma}_i^T)^{-1} \tilde{\mu}_i}. \quad (2.21)
\]
A formal calculation suggests that if \(\tilde{k}_i > \Lambda_i\), the control space of Regime \(i\) will be the first quadrant of an ellipse; otherwise, it will be the first quadrant of a parabolic. We refer to [7] for details.

To sum up, we have the definition about the set of the possible policies.

**Definition 2.3.** Denote by \(\Gamma\) the set of reinsurance-investment policies that satisfy MVaR constraint, i.e.,
\[
\Gamma = \left\{ \phi(t) : \tilde{k} \sqrt{\phi^T(t)\tilde{\sigma}(t)\tilde{\sigma}(t)^T} \phi(t) \leq \phi^T(t)\tilde{\mu}(t) + c \right\}. \quad (2.22)
\]
We also call \(\Gamma\) the control space.

### 2.3. The problem formulation.
To provide rigorous mathematical formulation of our problem, we start with the definition of the set of admissible policies \(\Phi\).

**Definition 2.4.** A policy \(\phi(t)\) is said to be admissible if
(1) \(\phi(t)\) is an adapted process on \((\Omega, \mathcal{F}_t, P)\);
(2) \(\phi(t)\) satisfies the integrability condition: \(\int_0^t |\phi(s)|^2 ds < \infty\), almost surely, for all \(t \geq 0\). Here, \(|\cdot|\) denotes the Euclidean norm;
(3) the SDE (2.9) has a unique solution corresponding to \(\phi(t)\).

Besides, we assume that the policy \(\phi(t)\) is Markovian, which implies that \(\phi(t) = \phi(X(t), Y(t))\) for some function \(\phi : [0, +\infty) \times \{1, 2\} \rightarrow [0, +\infty) \times [0, +\infty)\). With slight abuse of notation, we do not distinguish \(\phi\) and \(\tilde{\phi}\). For the simplicity of notation, we can use the expression \(\phi_i(x) = (u_i(x), \pi_i(x))^T\) to symbolize \(\phi(x, i)\) if there is no confusion. Now we can set the admissible policies \(\Phi\) as
\[
\Phi = \{ \phi \in \Gamma : \phi \text{ is admissible Markovian control} \}. \quad (2.22)
\]
Once a policy \(\phi\) is given, the corresponding wealth process can be expressed as \(X^\phi\) and follows the dynamics
\[
dX^\phi(t) = rX^\phi(t)dt + [\phi^T(t)\tilde{\mu}(t) + (\tilde{\eta}(t) - \tilde{\theta}(t))]dt + \phi^T(t)\tilde{\sigma}(t)dW(t).
\]
Furthermore, we define the ruin time
\[
\tau^\phi = \inf\{t > 0 : X^\phi(t) \leq 0\}. \quad (2.23)
\]
The performance index of $\phi$ is the ruin probability under that policy:

$$ J(x, i; \phi) = P\{\tau^\phi < \infty | X^\phi(0) = x, Y(0) = i \}. $$

(2.24)

Here, $x \geq 0$ is the initial surplus and $i \in \{1, 2\}$ is the initial state.

Now, our problem can be formulated as

$$ \min_{\phi \in \Phi} J(x, i; \phi) $$

subject to

$$ dX^\phi(t) = rX^\phi(t)dt + [\phi^T(t)\tilde{\mu}(t) + (\tilde{\eta}(t) - \tilde{\theta}(t))]dt + \phi^T(t)\tilde{\sigma}(t)dW(t). $$

(2.26)

3. Hamilton-Jacobi-Bellman equations and the optimality conditions.

3.1. Hamilton-Jacobi-Bellman equations. The problem (2.25)-(2.26) is an infinite horizon control problem. To solve this problem, we define the value functions $V_t(x, i)$, $i = 1, 2$, as

$$ V_t(x, i) = \min_{\phi \in \Phi} P\{\tau^\phi < \infty | X^\phi(t) = x, Y(t) = i \}. $$

(3.27)

Since the problem has the infinite time horizon, we can use $V_i(x)$ to symbolize $V_t(x, i)$ for $i = 1, 2$.

Using the dynamic programming principle, we induce a system of two HJB equations for the value functions:

$$ \min_{\phi_i} \left\{ V_i'(x)[\phi_i^T(x)\tilde{\mu}_i + r_i x + (\tilde{\eta}_i - \tilde{\theta}_i)] + \frac{1}{2} \phi_i^T(x)\tilde{\sigma}_i \tilde{\sigma}_i^T \phi_i V_i''(x) + \lambda_i(V_j(x) - V_i(x)) \right\} = 0, \quad 0 < x < \infty, \quad j \neq i \in \{1, 2\}, $$

(3.28)

subject to the MVaR constraint

$$ \tilde{k}_i \sqrt{\phi_i^T(x)\tilde{\sigma}_i \tilde{\sigma}_i^T \phi_i(x)} \leq \phi_i^T(x)\tilde{\mu}_i + c_i, \quad i = 1, 2. $$

(3.29)

Once we solve (3.28)-(3.29), we can derive the minimal ruin probability and the corresponding optimal strategies of the problem (2.25)-(2.26).

Next, we will state some properties of value functions, which are essential for solving those HJB equations. Now will discuss the boundary conditions.

**Lemma 3.1.** Set

$$ x^* = \max_{i=1,2} \left\{ \frac{\tilde{\theta}_i - \tilde{\eta}_i}{r_i} \right\}. $$

(3.30)

Then, we have $V_i(x) \equiv 0$ for $x \geq x^*$ and $i = 1, 2$.

**Proof.** Consider the possible policy $\phi_i = \phi^0 \equiv (0, 0)^T$, $i = 1, 2$, which means that the company takes the full reinsurance and there is no risky asset in both regimes. It can be verified that these strategies satisfy the MVaR constraint (3.29), since $c_i > 0$ for $i = 1, 2$. In this paper, we assume that the regime does not change in a small time interval $[t, t + h]$, for any $t \geq 0$. We let $X(t) = x \geq x^*$ and the market
state be Regime $i$ in time interval $[t, t + \Delta t]$, for any $\Delta t \leq h$. According to the formula (2.11), we have that

$$X(t + \Delta t) - X(t) = (e^{r_i \Delta t} - 1) \left[ X(t) - \frac{\beta_i - \eta_i}{r_i} \right] \geq 0,$$

which implies that $X(t + \Delta t) \geq x^*$. Repeating the procedure, we can obtain that $X(s) \geq x^*$ for any $s \geq t$. Therefore, we have $\tau^\phi = \infty$ and

$$0 \leq V_i(x) \leq J(x, i; \phi^0) = P(\tau^\phi < \infty | X(t) = x, Y(t) = i) = 0, \quad i = 1, 2.$$ 

This completes the proof. 

Based on the above lemma, we can provide the following proposition about the boundary conditions.

**Proposition 3.2.** The value functions $V_i(x)$ ($i = 1, 2$) satisfy that

$$V_i(0) = 1 \quad \text{and} \quad V_i(x^*) = 0, \quad i = 1, 2. \quad (3.31)$$

The first condition implies that the ruin probability should be one if the insurance company has no wealth at the beginning. The second condition is associated with a so-called safe level, at which the insurance company can get rid of default risks.

In the next proposition, we will focus on the discussion of the value functions for $x \in [0, x^*)$, which will be automatically extended to the entire domain.

**Proposition 3.3.** The value functions $V_i(x)$, $i = 1, 2$ are non-increasing functions when $x \in [0, x^*)$. Furthermore, there exists $x^c \in [0, x^*)$ to satisfy that

$$V_i(x) > 0, \quad V_i'(x) < 0 \quad \text{for} \quad x \in [0, x^c),$$

and $V_i(x) \equiv 0$ for $x \in [x^c, x^*)$.

**Proof.** Since $V_i(x)$, $i = 1, 2$, are the minimal ruin probabilities, it is easy to verify that they are non-increasing functions, and see [19] and [20] for the details. Now, we utilize the method in [7] to prove that $V_i(x)$ ($i = 1, 2$) are strictly decreasing in the region where $V_i(x) > 0$.

For any $0 \leq x < y < \infty$, let $\phi_{x,i} \in \Phi$ be an admissible policy and denote by $\tau_{x,i}$ the ruin time for the initial surplus $x$ and the initial Regime $i$ under policy $\phi_{x,i}$. Choose Markovian control

$$\phi_{y,i} = \begin{cases} \phi_{x,i} & \text{for} \quad 0 \leq t \leq \tau_{x,i}, \\ \phi_1 & \text{for} \quad t > \tau_{x,i}. \end{cases}$$

Here $\phi_1 = (\epsilon, 0)$ is constant, which satisfies the inequality (2.18) and

$$\theta \epsilon - (\theta - \eta) > 0.$$ 

Denote that $\bar{\epsilon} = (\theta \epsilon - (\theta - \eta))$. Let $\tau_{y,i}$ be the ruin time with the initial surplus $y$, the initial Regime $i$ under policy $\phi_{y,i}$. Obviously, we have $\tau_{y,i} \geq \tau_{x,i}$ a.e. for $i = 1, 2$. By the Markov property, the probability of ruin now becomes

$$J(y, i; \phi_{y,i}) = E[1_{\tau_{x,i} < \infty} E[1_{\tau_{y,i}} | F_{x,i}]].$$

$$= E[1_{\tau_{x,i} < \infty} P_{t \geq 0} \{ X^{\phi_{x,i}}(t) \leq 0 \} | F_{x,i}]$$

$$= E[1_{\tau_{x,i} < \infty} P_{t \geq \tau_{x,i}} \inf_{s \geq 0} \{ X^{\phi_{x,i}}(s + \tau_{x,i}) \leq 0 \} | F_{x,i}]$$

$$= E[1_{\tau_{x,i} < \infty} P_{s \geq 0} \{ X^{\phi_{x,i}}(s) \leq 0 \} | F_{x,i}].$$
Note that $X^{\phi_{x,i}}(\tau_{x,i}) = y - x$. However, we are not sure in which regime the insurer is at the ruin time $\tau_{x,i}$. Therefore, there exists a constant $p \in [0,1]$ to satisfy that

$$P_{s\geq 0}[X^{\phi_{x,i}}(s + \tau_{x,i}) \leq 0|\mathcal{F}_{\tau_{x,i}}] = p J(y - x, i; \phi_1) + (1 - p) J(y - x, j; \phi_1),$$

for $j \in \{1, 2\}$ and $j \neq i$. This implies that

$$J(y, i; \phi_{y,i}) = J(x, i; \phi_{x,i})(p J(y - x, i; \phi_1) + (1 - p) J(y - x, j; \phi_1)).$$

In order to discuss $J(y - x, 1; \phi_1)$ or $J(y - x, 2; \phi_1)$, we focus on the wealth process with the initial surplus $y - x$ under the policy $\phi_1$. The surplus can be expressed as

$$X^{\phi_i}(t) = y - x + \int_0^t \{r(s) X^{\phi_i}(s) + \theta(s) \epsilon - (\theta(s) - \eta(s))\} ds + \int_0^t \epsilon \sigma(s) dW_s$$

$$\geq y - x + \epsilon t + \int_0^t \epsilon \tilde{\sigma}(s) dW_s,$$

Let $\sigma_{\text{max}} = \max\{\sigma_1, \sigma_2\}$. Then, we have from $y - x + \epsilon t > 0$ that for $i = 1, 2$

$$J(y - x, i; \phi_1) = 1 - P(X^{\phi_i}(t) > 0 \text{ for all } t \geq 0)$$

$$\leq 1 - P(y - x + \epsilon t + \epsilon \int_0^t \sigma(s) dW_s \text{ for } t \geq 0)$$

$$\leq e^{-\frac{2(x-y)}{\epsilon \sigma_{\text{max}}}}.$$

Taking the infimum over all strategies $\phi_{x,i}$ yields

$$V_i(y) \leq e^{-\frac{2(x-y)}{\epsilon \sigma_{\text{max}}}} V_i(x) < V_i(x).$$

To sum up, there exist $x_i \in [0, x^*)$ for $i = 1, 2$ satisfying $V_i(x) i = 1, 2$ are strictly decreasing for $x \in (0, x_i)$ for $i = 1, 2$.

Now, we prove $x_1 = x_2$. If $x_1 > x_2$, the HJB equation (3.28) can not hold in the interval $(x_2, x_1)$ for $i = 2$, since the left-hand side of this equation is equal to $\lambda_2 V_1 > 0$. Similarly, the equation can not hold in the interval $(x_1, x_2)$ for $i = 1$, if $x_2 > x_1$. Therefore, we have $x_1 = x_2$ and set $x^c = x_1 = x_2$. This completes the proof.

3.2. The optimality conditions. Now we study the optimality conditions for $x \in [0, x^c)$, where the value functions $V_i(x), i = 1, 2$, are strictly positive. The basic thought comes from [7], which uses the Lagrange functions to tackle the optimality conditions. As in [7], we assume that $V_i(x) i = 1, 2$ are strictly convex functions where $x \in [0, x^c)$. Denote the Lagrange functions by

$$L_i(\pi, L_i) = V_i'(x)[\phi_i^T(x) \mu_i + r_i x + (\eta_i - \tilde{\eta}_i)] + \frac{1}{2} \phi_i^T(x) \tilde{\sigma}_i \tilde{\sigma}_i^T \phi_i V_i''(x)$$

$$+ \lambda_i(V_j(x) - V_i(x)) + L_i \left[ \phi_i^T(x) \mu_i + c_i - \tilde{k}_i \sqrt{\phi_i^T(x) \tilde{\sigma}_i \tilde{\sigma}_i^T \phi_i(x)} \right],$$

where $L_i \geq 0, i = 1, 2$, are the Lagrange multipliers. Denote the optimal policy in Regime $i$ with the MVaR constraint by $\phi_i^*$. Now the first order necessary conditions
for \( i = 1, 2 \) are given by

\[
L_i \geq 0; \tag{3.32}
\]

\[
(\phi_i^*)^T \mu_i + c_i - \bar{k}_i \sqrt{(\phi_i^*)^T \tilde{\sigma}_i \tilde{\sigma}_i^T \phi_i^*} \geq 0; \tag{3.33}
\]

\[
L_i \left[ (\phi_i^*)^T \mu_i + c_i - \bar{k}_i \sqrt{(\phi_i^*)^T \tilde{\sigma}_i \tilde{\sigma}_i^T \phi_i^*} \right] = 0; \tag{3.34}
\]

\[
\tilde{\mu}_i V_i' + \tilde{\sigma}_i \tilde{\sigma}_i^T \phi_i'' V_i'' + L_i \left[ \tilde{\mu}_i - \bar{k}_i \frac{\tilde{\sigma}_i \tilde{\sigma}_i^T \phi_i^*}{\sqrt{(\phi_i^*)^T \tilde{\sigma}_i \tilde{\sigma}_i^T \phi_i^*}} \right] = 0. \tag{3.35}
\]

If \( \phi_i^* \) takes values in the interior of control space, we have that

\[
\tilde{\mu}_i V_i' + \tilde{\sigma}_i \tilde{\sigma}_i^T \phi_i'' V_i'' = 0,
\]

which implies that

\[
\phi_i^* = -\frac{V_i''}{V_i'} (\tilde{\sigma}_i \tilde{\sigma}_i^T)^{-1} \tilde{\mu}_i. \tag{3.36}
\]

In this situation, the inequality (3.33) becomes

\[
(\bar{k}_i \Lambda_i - \Lambda_i^2) \left( -\frac{V_i''}{V_i'} \right) - c_i \leq 0. \tag{3.37}
\]

If the above inequality holds, then the optimal policy \( \phi_i^* \) can take values inside the control space, which means that the equation (3.36) holds. Otherwise, the optimal policy \( \phi_i^* \) must be chosen on the boundary of the control space, which means that

\[
(\phi_i^*)^T \mu_i + c_i - \bar{k}_i \sqrt{(\phi_i^*)^T \tilde{\sigma}_i \tilde{\sigma}_i^T \phi_i^*} = 0.
\]

After some computations, we derive that

\[
\phi_i^* = m_i (\tilde{\sigma}_i \tilde{\sigma}_i^T)^{-1} \tilde{\mu}_i, \quad \text{where} \quad m_i = \frac{c_i}{\Lambda_i (\bar{k}_i - \Lambda_i)}, \quad i = 1, 2. \tag{3.38}
\]

Now the key to derive the optimal strategy is to find when the inequality (3.37) holds. If \( \bar{k}_i \leq \Lambda_i \), the inequality (3.37) holds automatically, since \( V_i(x) \) is the non-increasing and convex function in this paper. That is to say, the conclusion that the optimal policy is inside the control space follows. If \( \bar{k}_i > \Lambda_i \), (3.37) holds when

\[
-\frac{V_i''}{V_i'} \leq m_i, \quad \text{which implies that the optimal policy takes values inside the control space. Otherwise, (3.38) holds and the optimal policy is chosen on the boundary of the control space.}
\]

To sum up, we arrive at the next lemma.

**Lemma 3.4.** Assume that the value functions \( V_i(x) \) and the optimal policy \( \phi_i^*(x) \), \( i = 1, 2 \), are defined in (3.28)-(3.29). Then, they satisfy the following condition in the interval \([0, x^c]\) for \( i = 1, 2 \):

\[
\phi_i^*(x) = \begin{cases} 
\min \left[ \frac{V_i'(x)}{V_i''(x)}, m_i \right] (\tilde{\sigma}_i \tilde{\sigma}_i^T)^{-1} \tilde{\mu}_i, & \text{if} \quad \bar{k}_i > \Lambda_i, \\
-\frac{V_i'(x)}{V_i''(x)} (\tilde{\sigma}_i \tilde{\sigma}_i^T)^{-1} \tilde{\mu}_i, & \text{if} \quad \bar{k}_i \leq \Lambda_i.
\end{cases} \tag{3.39}
\]

Now we are in the position to provide the main result of this section. The next theorem is the consequence of Proposition 3.3 and Lemma 3.4.
Theorem 3.5. The value functions $V_i(x)$, $i = 1, 2$, defined in (3.28)-(3.29) satisfy the following system of coupled Hamilton-Jacobi-Bellman equations in the region $[0, x^c)$:

$$V'_i(x) \left[ (\phi^*_i(x))^T \hat{\mu}_i + r_i x + (\tilde{\eta}_i - \tilde{\theta}_i) \right] + \lambda_i (V_j(x) - V_i(x)) + \frac{1}{2} (\phi^*_i(x))^T \hat{\sigma}_i \hat{\sigma}_i^T \phi^*_i(x) V''_i(x) = 0,$$

with the boundary conditions $V_i(0) = 1$ and $V_i(x^c) = 0$ for $i = 1, 2$. The corresponding optimal policy $\phi^*_i(x)$ satisfies (3.39).

Remark 3.6. From the above theorem, we can see that for each $i = 1, 2$, the Hamilton-Jacobi-Bellman equation (3.40) can be transformed to two kinds of nonlinear partial differential equations as follows:

1. If $k_i \leq \Lambda_i$ or $-\frac{V'_i}{V''} \leq m_i$, it is equal to

$$\left( r_i x + (\tilde{\eta}_i - \tilde{\theta}_i) \right) V'_i(x) + \lambda_i (V_j(x) - V_i(x)) - \frac{1}{2} \Lambda_i^2 \frac{(V'_i(x))^2}{V''_i(x)} = 0; \tag{3.41}$$

2. Otherwise, it is equal to

$$(\Lambda_i^2 m_i + r_i x + (\tilde{\eta}_i - \tilde{\theta}_i)) V'_i(x) + \lambda_i (V_j(x) - V_i(x)) + \frac{1}{2} \Lambda_i^2 m_i^2 V''_i(x) = 0. \tag{3.42}$$

Before we provide the numerical method to solve the above equations (3.40), we discuss how the MVaR constraint affects the optimal policy.

First, we discuss the conditions $k_i \leq \Lambda_i$, $i = 1, 2$. When this condition holds for Regime $i$, it shows in (3.39) that the MVaR constraint takes no effect on the policy in Regime $i$ directly. However, we should mention that this constraint may affect the policy in Regime $j$, and therefore may affect the policy in Regime $i$ indirectly. The reason is that the optimal policies in the two regimes are related based on the coupled HJB equations (3.40). Furthermore, if the conditions hold for the both regimes, then the constraint has no effect on the optimal policy at all.

Then, we focus on the important values $m_i$, $i = 1, 2$, shown in (3.39). When $k_i > \Lambda_i$, the MVaR constraint will bring some upper bounds for the optimal policy in Regime $i$, expressed by

$$u^*_i \leq m_i \frac{\mu_i - r_i}{\sigma_i^2}, \quad \pi^*_i \leq m_i \frac{\theta_i \alpha_i}{\beta_i^2}. \tag{3.43}$$

Whether or not the the optimal policy achieves the upper bounds depends on the comparison between $m_i$ and $-\frac{V'_i}{V''}$. If $-\frac{V'_i}{V''} \leq m_i$ for both $i = 1, 2$, then the MVaR constraint takes no effect at all.

Finally, we discuss how the parameters about the MVaR constraint affect the optimal policy. There are three parameters involved in the constraint, which are the time period $h$, the probability level $\beta$ and the MVaR level $R$. Since the time period $h$ is fixed for a certain problem, we only talk about $\beta$ and $R$ in the following discussions. From (2.20), (2.21) and (3.38), we can conclude that $k_i$ is a decreasing function of $\beta$ and $m_i$ is an increasing function of $\beta$ and $R$. Therefore, the MVaR constraint will be more strict with the smaller probability level $\beta$ and the smaller MVaR level $R$. 
4. The numerical experiments.

4.1. The numerical method. In this subsection, we shall present an iterative algorithm in order to solve our problem. We adopt the method in [13] and revise it to deal with the MVar constraint. Since we cannot find the accurate value of \( x^e \) before the whole problem is solved, we compute the problem with the wealth interval \([0, x^+]\). We divide the interval into \( N \) subintervals with the same length \( \Delta x = \frac{x^+}{N} \). We set the node points as \( x_n = n \Delta x, \ n = 0, 1, 2, \cdots, N \), and compute the values of the ruin probabilities \( V_i(x) \), \( i = 1, 2 \), on these points. The value of \( x^e \) will be derived simultaneously in this procedure. To obtain a sufficiently precise numerical solution, we will choose a sufficiently large \( N \).

In the iterative algorithm, we use the solution of the single-regime case as the initial guess. Its analytical form can be found in Theorem 4.1 of [7]. The ruin probabilities after the \( k \)th iteration are denoted as \( V_i^k(x) \), \( k = 1, 2, \cdots, K \). To adopt the MVaR constraint, we introduce the regions \( I_i \subset [0, x^+] \), \( i = 1, 2 \), to satisfy that the optimal strategies \( \phi_i^+(x) \) are inside the control space when \( x \in I_i \), and on the boundary of the control space when \( x \not\in I_i \). Based on (3.41)-(3.42), we have the following modified finite difference relationships between the \( k \)th and the \((k-1)\)th round numerical solutions:

\[
(r_i x_n + (\bar{\mu}_i - \bar{\theta}_i)) \frac{V_i^{k-1}(x_{n+1}) - V_i^{k-1}(x_{n-1})}{2 \Delta x} + \lambda_i (V_i^k(x_n) - V_i^k(x_{n-1})) - \frac{\Lambda_i^2}{8} \frac{(V_i^{k-1}(x_{n+1}) - V_i^{k-1}(x_{n-1}))^2}{V_i^{k-1}(x_{n+1}) - 2V_i^k(x_n) + V_i^{k-1}(x_{n-1})} = 0, \quad x_n \in I_i, \quad (4.44)
\]

\[
(\Lambda_i^2 m_i + r_i x_n + (\bar{\mu}_i - \bar{\theta}_i)) \frac{V_i^{k-1}(x_{n+1}) - V_i^{k-1}(x_{n-1})}{2 \Delta x} + \lambda_i (V_i^k(x_n) - V_i^k(x_{n-1})) + \frac{1}{2} m_i^2 \Lambda_i^2 \frac{V_i^{k-1}(x_{n+1}) - 2V_i^k(x_n) + V_i^{k-1}(x_{n-1})}{(\Delta x)^2} = 0, \quad x_n \not\in I_i, \quad (4.45)
\]

where \( n = 1, 2, \cdots, N - 1 \) and \( i = 1, 2 \). In fact, the numerical scheme (4.44) has the similar form to [13], while the scheme (4.45) is derived by using the condition \( \phi_i^+(x) = m_i (\bar{\sigma}_i \bar{\sigma}_i^T)^{-1} \bar{\mu}_i \). We let the iterative number \( K \) be big enough to make sure that the numerical solution is stabilized when \( k \) closes to \( K \).

Now we provide a numerical procedure to find the effective control constraint regions \( \tilde{I}_i, \ i = 1, 2 \). First, we use the regions in the single-regime case as the initial guesses. We implement the above iterative method with these constraint regions until the solutions are stabilized. Then, we update the control constraint regions, using the stabilized solutions \( V_i^K(x) \), \( i = 1, 2 \). Define \( N_c \) to satisfy that \( V_i^K(x_n) \equiv 0 \) for \( n > N_c \) and \( V_i^K(x_n) > 0 \) for \( n \leq N_c \). For \( n = 1, 2, \cdots, N_c \), define

\[
\psi_i(x_n) = -\frac{(V_i^K(x_{n+1}) - V_i^K(x_n)) \Delta x}{V_i^K(x_{n+1}) - 2V_i^K(x_n) + V_i^K(x_{n-1})}, \quad i = 1, 2, \quad (4.46)
\]

The new regions \( \tilde{I}_i, \ i = 1, 2 \), are set to satisfy the following condition:

\[
\left[ x_n - \frac{\Delta x}{2}, x_n + \frac{\Delta x}{2} \right] \subset \tilde{I}_i, \quad \text{if} \quad \psi_i(x_n) > m_i \quad \text{and} \quad \kappa_i > \Lambda_i; \quad (4.47)
\]

\[
\left[ x_n - \frac{\Delta x}{2}, x_n + \frac{\Delta x}{2} \right] \subset [0, x^+] \setminus \tilde{I}_i, \quad \text{if} \quad \psi_i(x_n) \leq m_i \quad \text{or} \quad \kappa_i \leq \Lambda_i, \quad (4.48)
\]

To sum up, the iteration algorithm can be implemented by following procedure.
Step 1. Based on the single-regime case, set 0-th numerical ruin probabilities $V_i^0$ and the initial control constraint regions $I_i$, $i = 1, 2$.

Step 2. Calculate the ruin probabilities by the iteration method:

Step 2-1. Set the boundary values to be $V_i^k(x_0) = 1$ and $V_i^k(x_N) = 0$ for $k = 1, 2, \cdots, K$.

Step 2-2. For $k = 1, 2, \cdots, K$, calculate $V_i^k(x_n)$ for $i = 1, 2, \cdots, N-1$ by solving the system of equations (4.44) and (4.45) simultaneously.

Step 3. Compute $\psi_i$ by (4.46) and the new constraint regions $\tilde{I}_i$ by (4.47)-(4.48). If $\tilde{I}_i$ is close to $I_i$ enough, stop. Otherwise, set $I_i = \tilde{I}_i$ and $V_i^0 = V_i^K$ for $i = 1, 2$. Go to Step 2.

4.2. Numerical examples. Now we present a numerical example, in which the basic parameters come from [13]. In that paper, most of the parameters are estimated by using the monthly data of Korea property and casualty insurance claims as well as the Korea Composite Stock Price Index from 1997 to 2012. Specifically, we use $\mu_1 = \mu_2 = 0.0684$, $\sigma_1 = 2.3283$, $\sigma_2 = 0.5003$, $r_1 = 0.0117$ and $r_2 = 0.0282$ as the parameters for the stock, and $\alpha_1 = \alpha_2 = 1.7136$, $\beta_1 = 0.0745$, $\beta_2 = 0.5829$, $\theta = 0.115$ and $\eta = 0.1$ as the parameters for the insurance claims. It can be seen that the volatility is much higher during Regime 2 than during Regime 1. The insurer is likely to suffer from a high volatility of the stock and insurance claims during Regime 2. Therefore, we call Regime 1 as the good economic state and Regime 2 the bad economic state from now. In addition, the intensities of the regime-switching are estimated as $\lambda_1 = 0.1063$ and $\lambda_2 = 0.7682$. In our paper, we assume that the correlation parameters between stock prices and insurance claims are zero. This is a widely accepted assumption in the existing literature. Besides, the estimated correlation parameters are very small in [13], which also verifies the rationality of our choice.

We now provide the parameters involved in the MVaR constraint. We choose the time interval $h = 0.0022$ (less than 1 day) to make sure that the asset allocation and risk exposure do not change over this time period. The confidence level is chosen as $\beta = 0.01$ and the MVaR level $R$ is chosen to vary in the region $[0.003, 0.01]$. We can verify that the conditions (2.20) are satisfied for $R \in [0.003, 0.01]$ in both regimes, which means that $c_i > 0$, $i = 1, 2$. Furthermore, we exploit $N = 1000$ node points in the interval $[0, x^*]$, in which $x^*$ can be obtained as 2.1969. The iteration number is chosen as $K = 20000$ to obtain a sufficiently precise numerical solution.

After some simple computations, we can derive that $\Lambda_1 = 2.6563$, $\Lambda_2 = 0.3474$, $\tilde{k}_i = 49.4971$, $i = 1, 2$. This implies that $\tilde{k}_i >> \Lambda_i$, $i = 1, 2$. Therefore, the MVaR constraint could be effective for the both regimes, depending on the values of $m_i$, $i = 1, 2$. After deriving the values of $m_i$, we can gain the upper bounds for $u_i^*$ and $\pi_i^*$ based on (3.43).

Figures 1 and 2 draw the reinsurance/new business strategies $u_i^*$ ($i = 1, 2$) when MVaR levels $R = 0.005$, $R = 0.006$ and $R = 0.01$. It shows that when $R = 0.01$, the strategies in the two regimes are inside the control space. This result is very similar to [13]. The poor insurers in Regime 1 keep more polices when the wealth level increases, while the ones with sufficient wealth keep fewer polices with the increasing of wealth. The authors in [13] have explained the reason. They point out that the poor insurers have a tendency to behave myopically with the fear of the regime shift, whereas the ones with sufficient wealth make reinsurance plans
with a long-term view. The insurers in Regime 2 keep fewer policies than the ones in Regime 2 due to the bad economic state. They will reduce their retained policies when the wealth level increases. Besides, we can derive that the upper bounds of $u^*_i$, $i = 1, 2$, are 1.2915 and 0.1536 in this case, which are much higher than actual values of $u^*_i$. Therefore, the MVaR constraint takes no effect on $u^*_i$, $i = 1, 2$ for $R = 0.01$.

The cases when $R = 0.005$ and $R = 0.006$ indicate that the insurers in Regime 1 will take less risk exposure when the MVaR constraint is required. Specially, the retained proportion of the polices has the upper bounds 0.7720 for $R = 0.005$ and 0.6421 for $R = 0.006$. The insurers in Regime 2 have not been affected by the MVaR constraint. In fact, the upper bounds of $u^*_2$ are 0.09187 for $R = 0.005$, and 0.07641 for $R = 0.006$, which are much higher than values of $u^*_2$ in those cases. The explanation is that the insurers in the bad economic state is already very prudent, hence the MVaR constraint has less effect on them than the ones in Regime 1.

Figures 3 and 4 draw the amounts invested in the risky assets $\pi^*_i$, $i = 1, 2$. They can show that the tendency of $\pi^*_i$ is as the same as the one of the reinsurance policies $u^*_i$. This numerical result coincides with our analytical result.
Figures 3 and 4 draw the value functions $V_i(x)$ with different MVaR levels. We find that the ruin probabilities under the optimal strategies in the two regimes are strictly decreasing functions of the wealth level. Furthermore, the ruin probabilities when the MVaR constraint takes effect (the case $R = 0.003$) are higher than the cases when the MVaR constraint takes no effect (the case $R = 0.01$). However, the differences between the two value functions are very small in the both regimes. This result is similar to the conclusion in [7].

5. Conclusion. In this paper, we consider the optimal reinsurance-investment problem with regime switching and value-at-risk constraint. Our objective is to minimize the ruin probability and to control the market risk at the same time. We assume that the key parameters about the insurance and finance market vary according to two regimes. The MVaR constraint is introduced to measure and control the market risk. We formulate our problem as the stochastic optimal control problem, and use the dynamic programming principle and Lagrange multiplier method to derive the constrained coupled Hamilton-Jacobi-Bellman equations. Then, we briefly analyze how the MVaR constraint effects the optimal strategies based on the forms of those equations. Moreover, we propose an effective iterative numerical
scheme to solve HJB equations and obtain the optimal strategies. Finally, we offer some practical example. Through the theoretical analysis and numerical calculation, we prove that the MVaR constraint would bring the constant upper bounds for the optimal reinsurance-investment strategies if it is strict enough. Furthermore, the insurers in the good economic state tend to be affected by the MVaR constraint more than the ones in the bad state which are already prudent. In the mean time, the regime-switching also affects the optimal strategies, and the tendency of the strategies coincides with the the numerical result in [13], in which the MVaR constraint is not considered.

REFERENCES

[1] A. Ang and G. Bekaert, International asset allocation with regime shifts, Review of Financial Studies, 15 (2002), 1137–1187.
[2] S. Browne, Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin, Mathematics of Operations Research, 20 (1995), 937–958.
[3] H. Bühlmann, Mathematical Methods in Risk Theory, Springer, Berlin, 1970.
[4] Y. Cao and X. Zeng, Optimal proportional reinsurance and investment with minimum probability of ruin, *J. Nanjing Norm. Univ. Nat. Sci. Ed.*, **36** (2013), 1–9.

[5] R. Chen, K. A. Wong and H. C. Lee, Underwriting cycles in Asia, *Journal of Risk and Insurance*, **66** (1999), 29–47.

[6] P. Chen and S. C. P. Yam, Optimal proportional reinsurance and investment with regime-switching for mean-variance insurers, *Insurance: Mathematics & Economics*, **53** (2013), 871–883.

[7] S. Chen, Z. Li and K. Li, Optimal investment-reinsurance policy for an insurance company with VaR constraint, *Insurance: Mathematics & Economics*, **47** (2010), 144–153.

[8] S. Choi and P. D. Thistle, The property/liability insurance cycle: A comparison of alternative models, *Southern Economic Journal*, **68** (2002), 530–548.

[9] D. Cuoco, H. He and S. Isaenko, Optimal dynamic trading strategies with risk limits, *Operations Research*, **56** (2001), 358–368.

[10] R. J. Elliott, L. Aggoun and J. B. Moore, *Hidden Markov Models: Estimation and Control*, Springer, 1995.

[11] H. G. Fung and R. C. Witt, Underwriting cycles in property and liability insurance: An empirical analysis of industry and byline data, *Journal of Risk and Insurance*, **65** (1998), 539–561.

[12] A. Gundel and S. Weber, Utility maximization under a shortfall risk constraint, *Journal of Mathematical Economics*, **44** (2008), 1126–1151.

[13] B. G. Jang and K. T. Kim, Optimal reinsurance and asset allocation under regime switching, *Journal of Banking and Finance*, **56** (2015), 37–47.

[14] Z. Jin, G. Yin and F. Wu, Optimal reinsurance strategies in regime-switching jump diffusion models: Stochastic differential game formulation and numerical methods, *Insurance: Mathematics & Economics*, **53** (2013), 733–746.

[15] Z. Liang and J. Guo, Optimal proportional reinsurance under two criteria: Maximizing the expected utility and minimizing the value at risk, *Anziam Journal*, **51** (2010), 449–463.

[16] J. Liu, K. F. C. Yiu, R. C. Loxton, K. L. Teo, Optimal investment and proportional reinsurance with risk constraint, *Journal of Mathematical Finance*, **3** (4) (2013) 437–447.

[17] J. Liu, K. F. C. Yiu, T. K. Siu and W. K. Ching, Optimal investment-reinsurance with dynamic risk constraint and regime switching, *Scandinavian Actuarial Journal*, **3** (2013), Article ID:38147, 11 pages.

[18] H. Schmidli, *Stochastic Control in Insurance*, Springer, London, 2008.

[19] H. Schmidli, Optimal proportional reinsurance policies in a dynamic setting, *Scandinavian Actuarial Journal*, **1** (2001), 55–68.

[20] M. I. Taksar and C. Markussen, Optimal dynamic reinsurance policies for large insurance portfolios, *Finance and Stochastics*, **7** (2003), 97–121.

[21] K. F. C. Yiu, Optimal portfolios under a value-at-risk constraint, *Journal of Economic Dynamics & Control*, **28** (2004), 1317–1334.

[22] K. F. C. Yiu, J. Liu, T. K. Siu and W. K. Ching, Optimal portfolios with regime switching and value-at-risk constraint, *Automatica*, **46** (2010), 979–989.

[23] C. Zhu, Optimal control of the risk process in a regime-switching environment, *Automatica*, **47** (2011), 1570–1579.

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