Recognizing generalized Petersen graphs in linear time

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Abstract
By identifying a local property which structurally classifies any edge, we show that the family of generalized Petersen graphs can be recognized in linear time.

Keywords: graph recognition, generalized Petersen graphs, linear algorithms

The generalized Petersen graphs, introduced by Coxeter [6] and named by Watkins [13], are cubic graphs formed by connecting the vertices of a regular polygon to the corresponding vertices of a star polygon. Various aspects of their structure have been extensively studied. Examples include identifying generalized Petersen graphs that are Hamiltonian [1, 2, 4], hypo-Hamiltonian [3], Cayley [10, 12], or partial cubes [9], and finding their automorphism group [7] or determining isomorphic members of the family [11]. Additional aspects of the mentioned family are well surveyed in [5, 8].

Figure 1: The generalized Petersen graph $G(10,4)$, also known as the Desargues graph.

In this paper we give a linear-time recognition algorithm for the family of generalized Petersen graphs. In particular, we identify a local property which structurally classifies...
any edge, whenever our graph is a generalized Petersen graph. We start by providing the necessary definitions and the analysis of 8-cycles; after the preliminaries in Section 1, we describe 8-cycles in Section 2. In Section 3 we introduce our main lemma, and describe and analyze the recognition procedure. At the end we mention how our Algorithm 2 behaves when the input graph is small.

1. Preliminaries

We follow the notations of Watkins [13] for the family GP of generalized Petersen graphs. For given integers \( n \) and \( k < n/2 \), we define the generalized Petersen graph \( G(n,k) \) as the graph with vertex-set \( \{u_0, u_1, \ldots, u_{n-1}, w_0, w_1, \ldots, w_{n-1}\} \) and edge-set \( \{u_iu_{i+1}\} \cup \{u_iv_i\} \cup \{v_iv_{i+k}\} \), for \( i = 0, 1, \ldots, i - 1 \), where all subscripts are taken modulo \( n \). We partition the edges of \( G(n,k) \) as

- the edges \( E_O(G) \) from the outer rim (of type \( u_iu_{i+1} \)) inducing a cycle of length \( n \);
- the edges \( E_I(G) \) from the inner rims (of type \( v_iv_{i+k} \)) inducing \( \gcd(n,k) \) cycles of length \( \frac{n}{\gcd(n,k)} \);
- the spokes \( E_S(G) \) (of type \( u_iv_i \)) inducing a perfect matching in \( G(n,k) \).

For each edge \( e \in E(G) \), let \( \sigma_G(e) \) be the number of 8-cycles containing \( e \), and let \( \mathcal{P}_G \) be a partition of the edge-set of \( G \), corresponding to the values of \( \sigma_G(e) \), for \( e \in E(G) \). The mapping \( \sigma \) plays a crucial role in the structural classification of edges in Section 2; for example, any spoke \( e \) of the Petersen graph \( G(5,2) \) has the value \( \sigma_{G(5,2)}(e) = 4 \) (see Figure 2).

![Figure 2: Any spoke \( e \) from the Petersen graph \( G(5,2) \) has the value \( \sigma_{G(5,2)}(e) = 4 \).](image)

By symmetry, the edges within any of the sets \( \{E_O(G), E_S(G), E_I(G)\} \) has the same value of \( \sigma_G \). This implies the following remark:

**Remark.** For a generalized Petersen graph \( G \), the partition \( \{E_O(G), E_S(G), E_I(G)\} \) is a refinement of \( \mathcal{P}_G \).

For any \( G \in \text{GP} \), we define \( \sigma_o \) to be \( \sigma_G(e) \), where \( e \) is a member of \( E_O(G) \); \( \sigma_s \) and \( \sigma_i \) are defined similarly.

Given a graph \( G \), we use the standard neighborhood notations for vertices \( N_k(v) = \{w \in V(G): d_G(v, w) = k\} \) or edges \( N_k(e) = \{e' \in E(G): d_{L(G)}(e, e') = k\} \), where \( L(G) \) is the line graph of \( G \) and \( d_G(v, w) \) is the distance between \( v \) and \( w \) in \( G \). The notation \( G[S] \) corresponds to the subgraph of \( G \) induced by \( S \), where \( S \subseteq V(G) \) or \( S \subseteq E(G) \). Graphs
of order at most 80 are considered to be small graphs, and can clearly be recognized in constant time, so in what follows we assume that our graphs are not small – that is, are large. In this paper we study the large members of GP with respect to their values of $\sigma_o, \sigma_s, \sigma_i$, and in particular determine that such a triple can correspond to only one of nine possibilities, described below in (1). We proceed with the description of the possible 8-cycles in $G(n,k)$.

2. A description of 8-cycles

Given a graph $G \simeq G(n,k)$, fix a partitioning of $E(G)$ into $E_O(G), E_S(G), \text{and } E_I(G)$, as above. For $i \in [0, n - 1]$, define the rotation $\rho$ on $V(G)$ by $\rho(v_i) = v_{i+1}$ and $\rho(w_i) = w_{i+1}$. Note that $\rho \in \text{Aut}(G)$.

All 8-cycles of $G$ may be partitioned into equivalence classes with respect to $\rho$. For a fixed 8-cycle $C$, we say that $C$ has $c$-fold rotational symmetry if its $\rho$-orbit is of order $c$. Then, by considering the number of its edges in $E_O(G), E_S(G), E_I(G)$ and the order of its $\rho$-orbit, we can explicitly describe its contribution to the global values $\sigma_o, \sigma_s, \text{and } \sigma_i$. These contributions uniquely describe the type of cycle $C$, and are denoted by $\delta_o, \delta_s, \text{and } \delta_i$, respectively. Whenever $\delta_o, \delta_s, \text{and } \delta_i$ are used in this paper, the type of corresponding cycle(s) is always explicitly determined, or is clear from the context.

If $C$ uses $s$ edges from $E_S(G)$, we say that $s$ is the spoke value of $C$. Note that the spoke value is always 0, 2, or 4, and that the spoke value coincides with $\delta_s$ only when the rotational symmetry of $C$ is $n$-fold. For the eight specific triples of $\delta_o, \delta_s, \text{and } \delta_i$ we name the corresponding eight types of such cycles by $\{C_i\}_{i=0}^{7}$, as described in Table 1. We will see later that no other types are possible for the generalized Petersen graphs.

| label | $C_0$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ | $C_7$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\delta_o$ | 0     | 1     | 2     | 3     | 4     | 5     | 1     | 2     |
| $\delta_s$ | 0     | 2     | 2     | 2     | 2     | 2     | 2     | 4     |
| $\delta_i$ | 1     | 5     | 4     | 3     | 2     | 1     | 1     | 2     |

Table 1: The labeling of all 8 possible types of 8-cycles in $G(n,k)$ is uniquely defined by the corresponding values of $(\delta_o, \delta_s, \delta_i)$.

We proceed by listing each of the possible 8-cycles by its spoke values $s$, describing its rotational symmetries, the existence conditions, and the corresponding values of $\delta_o, \delta_s, \text{and } \delta_i$. We summarize these results in Table 2. Watkins [13] has made a list of 8-cycles for generalized Petersen graphs, similar to our Table 2, but did not explain why his list is complete.
We next focus on the values of $s$ in turn.

8-cycles with $s = 0$.

We first note that, since $G$ is large, the cycle in the outer rim cannot be of length 8, and so the edge-set of any such cycle is a subset of $E_I (G)$. Observe that all the inner cycles in $E_I (G)$ are (by definition) of length $n / \gcd (n, k)$, and have $n / 8$-fold rotational symmetry, so $(\delta_o, \delta_s, \delta_i) = (0, 0, 1)$. Furthermore, it is easy to see that either there are no 8-cycles with $i + j = 8$, or there are $n / 8$ such cycles, with $k = n / 8$ or $3n / 8$. We label these cycles $C_0$.

8-cycles with $s = 2$.

We next focus on the 8-cycles that contain two edges from $E_S (G)$. Any such 8-cycle contains $o$ edges in $E_O (G)$ and $6 - o$ edges in $E_I (G)$, and may be rotated by the $n$-fold rotational symmetry of $G$. We further observe that $1 \leq o \leq 5$, and label these 8-cycles $C_o$. It follows that $(\delta_o, \delta_s, \delta_i) = (o, 2, 6 - o)$. We now consider all possibilities with respect to their value $o$.

$C_1$ When $o = 1$, any corresponding cycle is of type $\{u_0, v_0, v_k, v_{2k}, v_{3k}, v_{4k}, v_{5k}, u_{5k}\}$ if and only if $5k \pm 1 \equiv 0 \pmod n$.

$C_2$ Similarly, if such an 8-cycle exists, then $4k \equiv \pm 2 \pmod n$, and since $k < n / 2$, it follows that $k = (n \pm 2) / 4$ or $k = (2n - 2) / 4$. In the former case $n \equiv 2 \pmod 4$, and in the latter case $n$ is odd.

$C_3$ If $o = 3$, then $3k \equiv n \pm 3 \pmod n$. So 3 divides $n$, and $k = 1, (n / 3) - 1$, and $(n / 3) - 1$.

$C_4$ If $o = 4$, then $2k \equiv \pm 4 \pmod n$. Since $k < n / 2$, there are again two possibilities: $k = 2$ or $k = (n - 4) / 2$.

$C_5$ If $o = 5$, then $k \equiv 5 \pmod n$. 

| label | a representative of an 8-cycle | existence conditions |
|-------|--------------------------------|----------------------|
| $C_0$ | $[v_0, v_k, \ldots, v_{7k}]$ | $n$ or $3n = 8k$ |
| $C_1$ | $[u_0, u_0, v_0, v_k, v_{2k}, v_{3k}, v_{4k}, v_{5k}, u_{5k}]$ | $n$ or $2n = 5k \pm 1$ |
| $C_2$ | $[u_0, u_1, u_2, v_2, v_{2+k}, v_{2+2k}, v_{2+3k}, v_{2+4k}]$ | $n$ or $2n = 4k + 2$ |
| $C_3$ | $[u_0, u_1, u_2, v_3, v_{3}, v_{n(3)+2}, v_{(2n/3)+1}, v_0]$ | $k = (n/3) - 1$ |
| $C_4$ | $[u_0, u_1, \ldots, u_4, v_4, v_{(n+4)/2}, v_0]$ | $k = (n/2) - 2$ |
| $C_5$ | $[u_0, u_1, \ldots, u_4, v_4, v_2, v_0]$ | $k = 2$ |
| $C_6$ | $[u_0, u_1, v_1, v_{n/2}, u_{n/2}, u_{(n+2)/2}, v_{(n+2)/2}, u_0]$ | $k = (n/2) - 1$ |
| $C_7$ | $[u_0, u_1, v_1, v_{k+1}, u_{k+1}, u_k, v_k, v_0]$ | $n \geq 4$ |

Table 2: Characterization of all 8-cycles from $G(n,k)$.
8-cycles with $s = 4$.

First, we consider a special type of 8-cycle of the form

$$\{u_0, u_1, v_1, v_{n/2}, u_{n/2}, u_{(n+2)/2}, v_{(n+2)/2}, v_0\}$$

whenever $n = 2k + 2$. These 8-cycles have only $(n/2)$-fold rotational symmetry, so $\delta_o = 1$, $\delta_s = 2$, and $\delta_i = 1$. We label these cycles $C_6$.

Finally, consider the 8-cycles with four edges from $E_S(G)$ of type

$$\{v_iu_iu_{i+k}v_{i+k}v_{i+k+1}u_{i+k+1}u_{i+1}v_{i+1}\},$$

where $0 \leq i \leq n - 1$. Since $G$ is assumed to be large, these cycles are always present, and are denoted by $C_7$. Since they have $n$-fold symmetry, $\delta_o = 2$, $\delta_s = 4$, and $\delta_i = 2$.

3. Recognizing generalized Petersen graphs

Using the structure of 8-cycles in $G$, we have the following property.

**Lemma 1.** Let $G(n, k) \in \text{GP}$ be a large graph, and let $P$ be a partition of its edge-set, corresponding to the values of $\sigma(G(n, k)).$ Then $P$ contains a part of size $n$.

**Proof.** Let $G = G(n, k)$. By the above remark, it is enough to prove that $|P| > 1$. In addition to type 7, more than one additional type of 8-cycle may coexist in $G$.

We first show that this can never happen in a large graph. Indeed, assume that two distinct types of 8-cycles exist, and notice that, depending on their type, there are up to six possible pairs of $(n, k)$-equations, always yielding a constant bound on $n$ that does not exceed 40. In particular, the highest bound on $n$ is attained by testing the coexistence of 8-cycles labeled $C_0$ and $C_5$, where two values of $n$ are not feasible, while the remaining two are 8 and 40 — that is, the graph $G$ is small.

But then, there are at most eight distinct possibilities for the corresponding values of $(\sigma_o, \sigma_s, \sigma_i)$. In particular,

$$(\sigma_o, \sigma_s, \sigma_i) \in \{(2, 4, 3), (3, 6, 7), (4, 6, 6), (5, 6, 5), (6, 6, 4), (7, 6, 3), (3, 6, 3), (2, 4, 2)\}. \quad (1)$$

where the values above are obtained by adding $(2, 4, 2)$ to the values $(\delta_o, \delta_s, \delta_i)$ for a $C_t$-labeled cycle — see Table [I]. To prove this, it is enough to observe that in the eight possible triples from [I], the values never coincide. \hfill \qed

3.1. A recognition algorithm

In this section we describe a simple procedure Recognize$(G)$, which runs in time $O(n)$ and determines whether the input graph $G$ is a member of GP. It is described in Algorithm [2] and uses an additional procedure Extend$(G, U)$ that is given in Algorithm [I].

The tasks of these Algorithms [I] and [2] basically correspond to identifying the vertices of the outer rim, and checking whether this outer rim can be extended to a proper member of GP. Regarding Algorithm [I] for a connected cubic input graph $G$ of order $2n$, and given the $n$-set of vertices $U$ that induces either $E_O(G)$ or $E_I(G)$, a procedure Extend$(G, U)$ runs in $O(n)$ and decides whether $G \in \text{GP}$. Indeed, once one determines $E_O(G)$ from $U$ (up to
Algorithm 1 Extend \((G, O)\)

**Require:** a cubic graph \(G\) on \(2n\) vertices, and a set \(U \subseteq V(G)\)

1. if \(G[O] \not\cong C_n\) then
2. \quad if \(|O| \neq n\) or \(G[V(G) \setminus O] \not\cong C_n\) then
3. \quad \quad return False
4. \quad else
5. \quad \quad \(O \leftarrow V(G) \setminus O\)
6. \quad relabel \(O = \{v_0, v_1, \ldots, v_{n-1}\}\) cyclically w.r.t. \(C_n\)
7. for \(0 \leq i < n\) do
8. \quad relabel the unique node from \(N_G(v_i) \setminus \{v_i+1, v_{i-1}\}\) by \(w_i\)
9. \quad \(k \leftarrow\) smallest number with \(w_k \in N(w_0)\)
10. for \(1 \leq i < n\) do
11. \quad if \(v_i v_{i+k} \notin E(G)\) then
12. \quad \quad return False
13. \quad return True

line 6), a bijection \(V(G) \to \{v_0, v_1, \ldots, v_{n-1}, w_0, w_1, \ldots, w_{n-1}\}\) is established and \(k\) is easily determined (see line 9). So it is enough to check that the edges of \(G\) indeed map to the edges of \(G(n, k)\).

Algorithm 2 basically consists of:

i. determining the values of \(\sigma_G(e)\) for each edge \(e \in E(G)\);

ii. determining an \(n\)-subset of \(E(G)\) which is also a member of \(\{E_O(G), E_S(G), E_I(G)\}\), whenever \(G \in GP\);

iii. identifying a vertex-set \(U\) which is one of \(\{u_i\}_{i=0}^{n-1}\) or \(\{v_i\}_{i=0}^{n-1}\), and running \(\text{Extend}(G, U)\) accordingly.

We comment on the above three statements in turn.

i. For any \(e \in E(G)\), all 8-cycles that contain \(e\) lie within its 4-neighborhood. If \(G' = G[\bigcup_{i=0}^{4} N_i(e)]\), then \(\sigma_G(e) = \sigma_{G'}(e)\). But since \(G\) is cubic, \(G'\) has constant order, bounded above by 61, so calculating \(\sigma_G(e)\) takes \(O(1)\) time, and the whole loop at line 2 altogether takes at most \(O(|E(G)|) = O(n)\) time. In line 5 according to its value of \(\sigma_G\), each edge is classified to the corresponding part from an edge partition \(P\).

ii. While line 6 is trivially of constant time-complexity, its correctness is provided by Lemma 1. Note that if \(\min_{U \in P} |U| \neq n\), then the same lemma allows us to simply return \(\text{False}\).

iii. If \(G\) is a generalized Petersen graph, and if the set \(U\) selected in line 6 corresponds to either \(E_O(G)\) or \(E_I(G)\) (we can verify this by checking whether \(U\) is a 2-factor), then we need only run a sub-procedure \(\text{Extend}(G, U)\). If \(U\) is not a 2-factor, then \(G \setminus U\) clearly is one. Since its biggest component \(C_{\text{max}}\) must correspond to an outer-rim \(E_O(G)\), it is again enough to call a sub-procedure \(\text{Extend}(G, C_{\text{max}})\).
Algorithm 2 Recognize\((G)\)

**Require:** a cubic graph \(G\) on \(2n\) vertices

1. \(\mathcal{P} \leftarrow\) an (initially) empty edge-dictionary of \(G\), labeled by integers
2. **for** \(e \in E(G)\) **do**
3. \(G' \leftarrow G \cup_{i=0}^{4} N_i(v)\)
4. \(s_e \leftarrow \sigma_{G'}(e)\)
5. \(\mathcal{P}(s_e) \leftarrow \mathcal{P}(s_e) \cup \{e\}\)
6. \(U \leftarrow\) the smallest member from the partition \(\mathcal{P}\)
7. **if** \(|U| \neq n\) **then**
8. return False
9. **if** \(G[U]\) admits a leaf **then**
10. \(U \leftarrow\) the largest component of \(G - U\)
11. **if** \(G[U]\) is a 2-factor **then**
12. return \(\text{extend}(G, U)\)
13. **else**
14. return False

### 3.2. Recognizing small graphs

It is clear that, for graphs on 80 or less vertices, the membership of \(GP\) may theoretically be determined in a constant time. However, the task may not be easy in practice. We now argue that the Algorithm 2 fails for precisely ten members of \(GP\), and that it can safely be used on any graphs of order not equal to 6, 8, 10, 16, 20, 24, 26, 48, or 52.

It is clear that Lemma 1 cannot hold for all small graphs. Indeed, by Frucht et al. [7], we know that there exist precisely seven pairs \((n,k)\) for which \(G(n,k)\) is edge-transitive: in particular, \((n,k) = (4,1), (5,2), (8,3), (10,2), (10,3), (12,5)\) or \((24,5)\). In these cases all edges have the same value for \(\sigma\).

Using a computer one can easily calculate the \(\sigma\)-partitions for the remaining 373 \((n,k)\)-pairs which correspond to small generalized Petersen graphs. We have checked whether the corresponding \(\sigma\)-partition of edges consists of more than one part and have determined that, in addition to the above seven pairs, there are three additional members of \(GP\) for which \(\sigma_{G(n,k)}(e_1) = \sigma_{G(n,k)}(e_2)\) for any pair of edges \(e_1, e_2 \in G(n,k)\). These three additional cases are \(G(3,1), G(13,5),\) and \(G(26,5)\). In the case \(G(3,1)\), we trivially have \(\sigma(e) = 0\) for each edge \(e\). The remaining two cases contain 8-cycles of types \(C_1, C_5,\) and \(C_7\), and in these cases we have

\[
(\sigma_o, \sigma_s, \sigma_t) = (1, 2, 5) + (5, 2, 1) + (2, 4, 2) = (8, 8, 8),
\]

so \(\sigma(e) = 8\) for each edge \(e\).

**Corollary 1.** Let \(G\) be a graph on \(n\) vertices, where \(n \notin \{6, 8, 10, 16, 20, 24, 26, 48, 52\}\). Then, Algorithm 2 runs in \(O(n)\) time and decides whether \(G \in GP\).

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