Semiclassical expansion of the Bethe scalar products in the XXZ spin chain

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ABSTRACT: In this note we adress the problem of performing the semiclassical expansion of the scalar product of Bethe states in the case of the XXZ spin chain. Our approach closely follows the one developed in [1]: after expressing the scalar products in terms of a central quantity - the $B$ functional, we reinterpret it as a grand partition function of a gas of particles with pairwise interaction potential. This work has been done as part of the author’s master thesis under the supervision of Ivan Kostov and Didina Serban at Institut de Physique Théorique (CEA Saclay) in September 2015 - February 2016, and present a concise summary of the results obtained at that time.

KEYWORDS: XXZ spin chain, semiclassical expansion
1 Position of the problem

In this note we are concerned with some aspects of the scalar products of the XXZ spin chain. Spin chains provide one of the most famous example of integrable systems, namely systems which can be exactly solved and for which the quantities of physical interest can be computed analytically. In the case of spin chains, the procedure that extracts the physical information bears the name of the Algebraic Bethe Ansatz and yields the so called Bethe vectors, which are the eigenvectors of the hamiltonian of the spin chain. Here we propose to study the scalar products between such vectors, in a special limit. In addition to be of intrinsic interest for the study of integrable systems, this problem (and more broadly the understanding of the behaviour of spin chains) is also motivated by its applications to the study of the AdS/CFT correspondence, where it has been shown since a decade that integrability plays a major role.

First we define what we mean by semiclassical expansion of Bethe states. The system under study is the XXZ spin chain of length \( L \) whose \( R \) matrix is given by:

\[
R(u) = \begin{pmatrix}
\text{sh}(u + i\eta) & 0 & 0 & 0 \\
0 & \text{sh}(u) & \text{sh}(i\eta) & 0 \\
0 & \text{sh}(i\eta) & \text{sh}(u) & 0 \\
0 & 0 & 0 & \text{sh}(u + i\eta)
\end{pmatrix}
\]

(1.1)

From the \( R \) matrix we define the transfer matrix \( T \) written in compact form as:

\[
T(u) = \begin{pmatrix}
A(u) & B(u) \\
C(u) & D(u)
\end{pmatrix}
\]

(1.2)

The Bethe vectors are then defined as multiple actions of the operators \( B \) and \( C \) on the vacuum of the system \( |\Omega\rangle \) (bold letters will denote ordered sets of complex numbers : \( a = (a_1, ..., a_N) \)):

\[
|u\rangle := B(u_1)...B(u_M)|\Omega\rangle \quad \langle v| := \langle \Omega| C(v_1)...C(v_M)
\]

(1.3)

where we assumed that the rapidity sets \( u \) and \( v \) satisfy the Bethe equations (\( \theta_m \) represents the inhomogeneity located at the site \( m \) of the chain):

\[
\prod_{k=1}^{M} \frac{\text{sh}(u_j - u_k + i\eta)}{\text{sh}(u_j - u_k - i\eta)} = -\prod_{m=1}^{L} \frac{\text{sh}(u_j - \theta_m + i\eta)}{\text{sh}(u_j - \theta_m)}
\]

(1.4)

The scalar product of Bethe vectors is then given by:

\[
\langle v|u\rangle = \langle \Omega| \prod_{k=1}^{M} C(v_k) \prod_{k=1}^{M} B(u_k)|\Omega\rangle
\]

(1.5)

These scalar products will be our main object of study. We aim to find the asymptotical expansion of (1.5) in the semiclassical limit, ie in the limit \( M \rightarrow \infty \), and our reasoning will closely follow the papers [1] and [2], where the same problem has been resolved in the case of a XXX spin chain. The main results of this work are given by (2.18), which expresses (1.5) in terms of a specific quantity \( B \), which should admit a particularly simple semiclassical expansion and (3.14)/(3.15), which are the first steps towards this expansion. We first start by proving the formula (2.18).

2 The B functional

We now consider two Bethe vectors : \( |v\rangle \) and \( |u\rangle \). As it is explained in [2], it is possible to bring the scalar product to the form:

\[
\langle v|u\rangle = \prod_{j=1}^{M} A(v_j)D(u_j)S_{u,v}
\]

(2.1)

where the functions

\[
A(u) = \prod_{m=1}^{L} \text{sh}(u - \theta_m + i\eta) \quad D(u) = \prod_{m=1}^{L} \text{sh}(u - \theta_m)
\]

(2.2)
and some condensed notations for the different quantities that will appear frequently in the following:

In order to make contact with the more familiar XXX case, we define the exponential variables:

$$t(X) = \frac{\text{sh}(in)}{\text{sh}(X) \text{sh}(X + in)}$$

$$e^{2ip_{a}(v)} = \frac{1}{\prod_{m=1}^{L} \text{sh}(X - u_{k} + in)} \prod_{k=1}^{M} \text{sh}(X - u_{k} - in)$$

These results are simple generalizations of the XXX case obtained in [2]; the quantity $$\Omega$$ is the so called Slavnov kernel [4], and $$p_{a}$$ is called the pseudo-momentum. It is related to the Bethe equations by:

$$e^{2ip_{a}(u)} = -1 \quad \forall u \in \mathbf{u} \text{ satisfying Bethe equations}$$

In order to make contact with the more familiar XXX case, we define the exponential variables:

$$q = e^{-2\eta} \quad z = e^{2u} \quad w = e^{2v} \quad \xi = e^{2\theta}$$

and some condensed notations for the different quantities that will appear frequently in the following:

$$Q_{a}(X) = \prod_{u \in \mathbf{a}} (X - a) \quad Q_{a}^{\pm}(X) = \prod_{u \in \mathbf{a}} (X - q^{\pm 1}a) \quad E_{a}^{\pm}(X) = \frac{Q_{a}^{\pm}(X)}{Q_{a}(X)}$$

Then it is possible to express the Bethe equations and the pseudo-momentum in terms of $$E_{a}^{\pm}$$. We write the function $$t$$ in terms of exponential variables and factorize the Slavnov kernel in the following manner (the action of the operator $$q^{z\partial_{z}}$$ on a function results in the multiplication of its argument by $$q$$):

$$\Omega(z, w) = (1 - e^{2ip_{a}(w)q^{-w0_{w}}})(q^{z\partial_{z}} - 1) \frac{2z}{z - qw}$$

We can use the factorization of the Slavnov kernel to expand the factor $$S_{uv}$$ in equation (2.1):

$$S_{uv} = \frac{\det_{jk} \Omega(u_{j}, v_{k})}{\det_{jk} \Omega(u_{j}, v_{j})} \prod_{u \in \mathbf{u}} \prod_{w \in \mathbf{w}} (1 - e^{2ip_{a}(w)q^{-w0_{w}}}) \prod_{u \in \mathbf{u}} (q^{z\partial_{z}} - 1) \det_{jk} \left( \frac{1}{z_{j} - qw_{k}} \right)$$

where

$$K = q^{M/2} \prod_{j=1}^{M} \frac{z_{j}}{\prod_{j \neq j} w_{j}}$$

It appears that $$S_{uv}$$ is mainly the action of some differential operator on the Cauchy determinant $$\det_{jk} \left( \frac{1}{z_{j} - qw_{k}} \right)$$. It is known that a Cauchy determinant can be split as a product of VanderMonde determinants, divided by some polynomial. Using this fact here, and setting $$V^{a} = \prod_{1 \leq j < k \leq N} (a_{j} - a_{k})$$ we get

$$S_{uv} = (-1)^{M} K \prod_{u \in \mathbf{u}} \prod_{w \in \mathbf{w}} \left( 1 - q^{\frac{1}{2}} \frac{E_{a}^{+}(w)}{E_{a}^{-}(w)} \right) V(w) \times \frac{1}{V(z)} \prod_{z \in \mathbf{a}} \left( 1 - \frac{E_{a}^{+}(z)}{q^{M-1} - E_{a}^{-}(z)} \right) V(z)$$

Looking carefully at the previous expression of $$S_{uv}$$, we see that a single relevant quantity appears twice. This motivates the definition of the $$B$$ functional

$$B_{a}^{\pm}[f] = \frac{1}{V(z)} \prod_{z \in \mathbf{a}} \left( 1 - f(z)q^{z\partial_{z}} \right) V(z)$$

Clearly, the $$B$$ functional is the XXZ analogue of the A functional defined in [2]. We have to find functional equations satisfied by the $$B$$ functional to combine the two $$B$$ factors in the expression (2.11) in a single $$B$$ functional defined on the set $$\mathbf{z} \cup \mathbf{w}$$. We introduce the following generic notation for partitions: $$z = z' \cup z''$$, and perform an expansion on partitions:

$$B_{a}^{\pm}[f] = \sum_{z' \cup z''} \prod_{z' \in z'} \left( -f(z'')q^{z'/(M-1}) \right) \prod_{z'' \in z''} E_{a}^{\pm}(z'') q^{-\frac{1}{2}z''^{2}}$$
It is now possible to obtain the two functional equations on $B$:

$$B^+_z[f] = B^-_z[q^{2(M-1)}\left(1 - \frac{q}{1 - q^v}E^+_z\right) f]$$
$$B^-_z[f] = B^+_z[q^{-2(M-1)}\left(1 - \frac{q}{1 - q^v}E^-_z\right)^{-1} f]$$

(2.14)

Using the $B$ functional and the associated differential operator $\hat{B}$, the quantity $S_{z,w}$ is given by:

$$S_{z,w} = (-1)^MK\hat{B}q^{L/2}\left(\frac{E^+_z}{E^+_\xi}\right)\hat{B}\left(\frac{1}{q^{M-1}}E^-_w\right)$$

(2.15)

To conclude, we need to use the functional identities derived above, in the previous expression of $S_{z,w}$, as well as Bethe equations for the set $z$. This gives a more symmetric expression for $S_{z,w}$:

$$S_{z,w} = (-1)^MK\hat{B}q^{L/2}\left(\frac{E^+_z}{E^+_\xi}\right)\hat{B}\left(\frac{q^{L/2}}{E^-_w}E^-_w\right)$$

(2.16)

The symmetry of the previous formula allows to obtain $S_{z,w}$ as a single $B$ functional on the set $z \cup w$:

$$S_{z,w} = (-1)^MKB^-_{z\cup w}\left[\frac{q^{L/2}}{E^-_w}E^-_w\right]$$

(2.17)

The final result for the scalar product is then:

$$\langle \psi|u \rangle = \prod_{j=1}^{M} A(v_j)D(u_j)(-1)^MKB^-_{z\cup w}\left[\frac{q^{L/2}}{E^-_w}E^-_w\right]$$

(2.18)

where both usual and exponential variables have been used. The result is similar to the expression for the XXX case, derived in [2]. The next section outline the investigation of the $B$ functional in the semiclassical limit, done in [1] for the XXX case.

3 Semi-classical expansion of the scalar products

We aim now to study the situation where the number of rapidities is taken to be infinitely large. We restrict ourselves to the study of the quantity $B^-_y[f]$, where $y$ is a set of complex numbers that do not necessarily satisfy the Bethe equations, and $f$ is a rational function depending on the inhomogeneities. We can also suppose that the elements of $y$ are located on a macroscopical arc in the complex plane, and their number is finite. Following the steps in [1], we aim to express $B^-_y[f]$ as an integral. The representation of the Izergin-Korepin determinant (which is an analytical result for the Bethe scalar product (1.5)) by shift operators (of type $q^{\alpha \alpha'}$) was done in [5]. Here we obtain:

$$B^-_y[f] = \sum_\alpha (-1)^\alpha \det \left( \frac{E_j y_i(1 - q^{-1})}{y_j - q^{-1}y_i} \right)$$

(3.1)

where we have set : $E_j = f(y_j)\prod_{k \neq j} \frac{y_k - y^{-1}y_j}{y_k - y_j}$. This can be combined to:

$$B^-_y[f] = \det(I - K)$$

(3.2)

with $K$ the $N \times N$ matrix:

$$K_{jk} = \frac{E_j y_i(1 - q^{-1})}{y_j - q^{-1}y_i}$$

(3.3)

Then representing the determinant with fermions [1] (normalised by $\langle \psi(x)\psi^*(y) \rangle = \frac{1}{x-y}$):

$$B^-_y[f] = \langle 0| \exp \left( \sum_{i=1}^{N} E_j y_i(1 - q^{-1})\psi^*(y_j)\psi(q^{-1}y_j) \right)|0 \rangle$$

(3.4)

Which can be rewritten as:

$$B^-_y[f] = \langle 0| \exp \left( - \int_{C_y} \frac{dx}{2\pi i} H_0(x)\psi^*(x)\psi(q^{-1}x) \right)|0 \rangle$$

(3.5)
where the contour $C_Y$ encircles the complex numbers $y$ and

$$H_q(X) := f(X) \prod_{i=1}^N \frac{q^{-1}X - y_k}{X - y_k} \quad (3.6)$$

Representing fermion fields as vertex operators ($\psi(x) = e^{\phi(x)} \psi^+(x) = e^{-\phi(x)}$) and calculating their correlation functions, we get

$$B_q[f] = \sum_{n=0}^{\infty} \sum_{l=1}^n \frac{(-1)^n}{n!} \prod_{j=1}^n \int_{C_Y} \frac{dx_j}{2\pi i} \left( \frac{H_q(x_j)}{-\left(1 - q^{-1}\right)x_j} \right) \prod_{k>j} \frac{(x_j - x_k)^2}{(x_j - qx_k)(x_j - q^{-1}x_k)} \quad (3.7)$$

Indeed, the factors in the second product are the correlation functions of two vertex operators $V_q(x)$:

$$V_q(x) = e^{\phi(x)} - \phi(x) \quad (3.8)$$

which are given by:

$$\langle 0 | V_q(x) V_q(y) | 0 \rangle = \frac{(x - y)^2}{(x - qy)(x - q^{-1}y)} \quad (3.9)$$

This integral formula for $B_q[f]$ is reminiscent of the grand canonical partition function for a gas of particles in a pairwise interaction potential. The previous expression of $B_q[f]$ is therefore well adapted to perform a cluster expansion [6]. Define:

$$1 + f_{jk} = \frac{(x_j - x_k)^2}{(x_j - qx_k)(x_j - q^{-1}x_k)} \quad (3.10)$$

We set $Z_n$ to be the canonical partition function of the gas of particles:

$$Z_n = \frac{(-1)^n}{n!} \prod_{j=1}^n \int_{C_Y} \frac{dx_j}{2\pi i} \left( \frac{H_q(x_j)}{-\left(1 - q^{-1}\right)x_j} \right) \prod_{j<k} (1 + f_{jk}) \quad (3.11)$$

The idea of the cluster expansion is the following: we expand the product $\prod_{j<k}(1 + f_{jk})$ as a sum, whose terms are represented by graphs. To each graph we associate a weight (the value of the integral of the corresponding term). This gives:

$$Z_n = \frac{(-1)^n}{n!} \sum_G W[G] \quad (3.12)$$

where the summation is performed on all possible graphs $G$ with weights $W[G]$. We now apply the cluster expansion of [6]. Defining $U_l$ to be:

$$U_l = \int_{C_Y} \prod_{i=1}^l \frac{dx}{2\pi i} \left( \frac{H_q(x_j)}{-\left(1 - q^{-1}\right)x_j} \right) \sum_{G \in l\text{-cluster}} W[G] \quad (3.13)$$

The cluster expansion yields:

$$Z_n = (-1)^n \sum_{l} U_l^{m_l} \prod_{(m)} U_l^{m_l} \quad (3.14)$$

where the sum is calculated with the constraint $\sum_{l=1}^n m_l l = n$. The final result for $B_q[f]$ is therefore:

$$B_q[f] = \sum_{n=0}^{\infty} Z_n \quad (3.15)$$

We would like to compute this sum in the limit $N \rightarrow \infty$, ie when we can relax the constraint $\sum_{l=1}^n m_l l = n$. In this case, $\sum_{n=0}^{\infty} Z_n$ can be written in an elegant way as an exponential of a certain power series [6]. This is what we would like to do, but we have to cope with a major obstruction: the term $U_l$ depends on the number $N$ through the function $H_q$. Hence, it appears to be difficult to find the limit $N \rightarrow \infty$ of $B_q[f]$ from the relation (3.15), without assuming precise constraints on the set of rapidities $y$. Nevertheless it is possible to formulate a conjecture on what the leading term should be: since in the XXX case the leading
term of the scalar product was essentially given by the exponential of the dilogarithm function [1], we can expect for the XXZ case the same kind of identity by involving this time the quantum dilogarithm:

\[
\log(v|u) \sim \int_{C_{u\cup v}} \frac{dx}{2\pi} \log \Psi(e^{ip_u(x)}+ip_v(x))
\]

(3.16)

where the contour \(C_{u\cup v}\) encircles the rapidities \(u\) and \(v\) and quantum dilogarithm is defined by:

\[
\Psi(x) = \prod_{n=0}^{\infty} \left(1-xq^n\right) \quad |q| < 1
\]

(3.17)

The motivation for this conjecture is the following : it is possible to recover the XXX spin chain from the XXZ spin chain by sending the parameter \(\eta\) to 0. On the other hand, if we set \(q = e^\eta\) and set \(\eta \to 0\) we have :

\[
\Psi(x) = \frac{1}{\sqrt{1-x}} e^{i \frac{\pi}{2} \text{Li}_2(x)} (1 + \mathcal{O}(\eta))
\]

(3.18)

which involves the same expression that was found for the leading term in the semiclassical expansion of the XXX case in [1].

4 Conclusion

In this note, our main result (2.18) allows to write a XXZ Bethe scalar product in terms of a specific quantity: the \(B\) functional, which is well designed for the semiclassical expansion. The natural way to continue our investigation of the XXZ inner product is to understand the structure of the Bethe roots in this case. This could give a hint on which exact hypothesis we should fix on the set of rapidities \(y\) in order to obtain a manageable mathematical calculation for the semiclassical expansion.

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Note added

After this research project was completed, the paper [3] by Y. Jiang and J. Brunekreef appeared where the same problem has been adressed.

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