Generalized quantum current algebras

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Abstract

Two general families of new quantum deformed current algebras are proposed and identified both as infinite Hopf family of algebras, a structure which enable one to define “tensor products” of these algebras. The standard quantum affine algebras turn out to be a very special case of both algebra families, in which case the infinite Hopf family structure degenerates into standard Hopf algebras. The relationship between the two algebra families as well as their various special examples are discussed, and the free boson representation is also considered.

Quantum groups, since proposed by Drinfeld [1, 2], have attracted the attentions of both mathematicians and theoretical physicists for more than ten years. And even after so many years’ extensive studies, the interest in quantum groups and their various extensions–quantum affine algebras [3, 19], Yangian doubles [17] and so on–is still not faded.

Theoretical physicists are fascinated about quantum groups and their extensions because these algebraic objects are the right candidates for describing the dynamical symmetries in certain models in integrable quantum field theories and/or exactly solvable statistical physics. Mathematicians are interested in quantum groups and their extensions because these are the first known nontrivial Hopf algebras–non-commutative non-co-commutative associative algebras expected for quite some years but not realized until the discovery of Drinfeld.

Recently, accompanying the search and investigation for elliptic quantum groups [7, 8, 9], it is realized that there are more general algebraic structures which are of interests to both mathematicians and physicists. One such example is the elliptic quantum groups proposed by Felder et al [7, 8, 9] which belong to the class of quasi-triangular quasi-Hopf algebras [5]–certain twists [15] of the standard Hopf algebra structures. From a pure mathematical point of view, the most important significance for the discovery of elliptic quantum groups might be that it reveals the possibility for the existence of non-co-associative and non-commutative non-co-commutative associative algebras. Such algebras also happen to describe the dynamical symmetry of some statistical models and hence greatly attracted the attention of theoretical physicists.

Further investigations along this line showed that there are several kinds of elliptic quantum groups [7, 8, 9, 10, 11, 12, 14, 13, 16, 18, 21], some are recognized as quasi-triangular quasi-Hopf algebras and some are still not. Due to the lack of co-algebraic structures for some of the elliptic

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quantum groups, it is nature to ask if the structure of quasi-triangular quasi-Hopf algebras is still adequate for describing the co-algebraic structures for these algebras. The answer to this question remains open and it seems very hard to get it solved. Alternatively, it may be reasonable to try some other formulations to describe the co-algebraic structures for some of the new elliptic quantum groups, even if the answer to the above question finally proves to be a positive “yes”. The infinite Hopf family of algebras described in \[13, 14\] is just such a candidate.

For standard quantum affine algebras and Yangian doubles, three major realizations are proved to exists, i.e. the Yang-Baxter type realization (also known as Reshetikhin-Semenov-Tian-Shansky realization \[15\]), Drinfeld realization \[3, 4\] and Drinfeld current realization (or simply current realization) \[5\]. The first and last realizations are usually used to describe the co-algebraic structures and for the investigation of infinite dimensional representations respectively. The standard Hopf algebra structure cannot be written closely using the current realization only. However, there exists an alternative Hopf algebra structure for quantum affine algebras and Yangian doubles which closes over the currents only. Such a structure is first discovered by Drinfeld in the case of quantum affine algebras and is called Drinfeld’s Hopf structures \[1\]. It is generally believed that the three realizations exist for all quantum deformations of the universal enveloping algebras of the classical affine Lie algebras, although only one or two realizations for some recently discovered quantum algebras are known to exist.

In this paper, we shall study two general families of quantum current algebras which contain many well-known quantum current algebras as special cases. We shall show that both families of current algebras have the structure of infinite Hopf family of algebras, which ensure a proper definition of fused representations (i.e. tensor product representations via comultiplications).

Our investigation will be restricted in the current realization only. Strictly speaking, the algebras we shall consider are not algebras but “functional algebras” \[2, 20\] generalizing the concept of a usual algebra. However we shall abuse the language in the context and call the object under investigation “algebras”.

Throughout the following, \(g\) will be a classical simply-laced Lie algebra with Cartan matrix \((A_{ij})\).

1

**Definition 1.1** Let \(q, p\) be generic parameters. We choose the set of analytic functions \(\Psi_{ij}(z|q)\) \((i, j = 1, ..., \text{rank}(g))\) such that they depend on the suffices \(i, j\) only through the Cartan matrix elements \(A_{ij}\) of \(g\) and that

\[
\Psi_{ij}(z|q) = \Psi_{ji}(z^{-1}|q)^{-1}.
\]

The functional current algebra \(A^{(1)}(\{\Psi\}, z)|q,p,c\) is defined as the associative algebra generated by the currents \(E_i(z), F_i(z), \text{invertible } H_i^{\pm}(z)\) with \(i = 1, 2, ..., \text{rank}(g)\), central element \(c\) and
the unit 1 with relations

\begin{align*}
H^\pm_i(z)H^\pm_j(w) &= \frac{\Psi_{ij}(\frac{z}{w}|q)}{\Psi_{ij}(\frac{z}{w}|\tilde{q})} H^\pm_j(w)H^\pm_i(z), \\
H^\pm_i(z)H^-_j(w) &= \frac{\Psi_{ij}(\frac{z}{w}p^c/2|q)}{\Psi_{ij}(\frac{z}{w}p^{-c/2}|\tilde{q})} H^-_j(w)H^+_i(z), \\
H^\pm_i(z)E_j(w) &= \Psi_{ij}(\frac{z}{w}p^{c/4}|\tilde{q}) E_j(w)H^\pm_i(z), \\
E_i(z)E_j(w) &= \Psi_{ij}(\frac{z}{w}q) E_j(w)E_i(z), \\
F_i(z)F_j(w) &= \Psi_{ij}(\frac{z}{w}|\tilde{q})^{-1} F_j(w)F_i(z), \\
[E_i(z), F_j(w)] &= \frac{\delta_{ij}}{(p^c - p^{-c})} \left[ \delta \left( \frac{z}{w}p^{-c/4} \right) H^+_i(wp^{c/4}) - \delta \left( \frac{w}{z}p^{-c/2} \right) H^+_i(zp^{c/4}) \right],
\end{align*}

where

\begin{align*}
\tilde{q} &= qp^c, \\
f^+_i(z_1/w, z_2/w|q) &= \left( \Psi_{ii}(\frac{z_2}{z_1}|q) + 1 \right) \left( \Psi_{ij}(\frac{w}{z_2}|q) \Psi_{ij}(\frac{w}{z_1}|q) + 1 \right), \\
f^-_i(z_1/w, z_2/w|\tilde{q}) &= \left( \Psi_{ii}(\frac{\tilde{q}}{z_1})^{-1} + 1 \right) \left( \Psi_{ij}(\frac{w}{z_1}|\tilde{q})^{-1} \Psi_{ij}(\frac{w}{z_2}|\tilde{q})^{-1} + 1 \right).
\end{align*}

Let \( \{A_n, n \in \mathbb{Z}\} \) be a family of associative algebras over \( C \) with unit. Let \( \{v_i^{(n)}\}, i = 1, ..., \dim(A_n) \) be a basis of \( A_n \). The maps

\[ \tau^\pm_n : A_n \to A_{n\pm 1} \]

\[ v_i^{(n)} \mapsto v_i^{(n\pm 1)} \]

are morphisms from \( A_n \) to \( A_{n\pm 1} \). For any two integers \( n, m \) with \( n < m \), we can specify a pair of morphisms

\[ \tau^{(m,n)} = Mor(A_m, A_n) \equiv \tau^+_m \cdots \tau^+_n \tau^+_n : A_n \to A_m, \]
\[ \tau(n, m) = Mor(A_n, A_m) \equiv \tau_{n+1}^\cdots \tau_{m-1}^- \tau_m^- : A_m \rightarrow A_n \]  

with \( \tau(m, n)\tau(n, m) = id_m, \ \tau(m, n)\tau(n, m) = id_n \). Clearly the morphisms \( \tau(m, n), \ n, m \in Z \) satisfy the associativity condition \( \tau(m, n)\tau(p, n) = \tau(m, n) \) and thus make the family of algebras \( \{ A_n, n \in Z \} \) into a category.

**Definition 1.2** The category of algebras \( \{ A_n, \ \{ \tau(n, m) \}, \ n, m \in Z \} \) is called an infinite Hopf family of algebras if on each object \( A_n \) of the category one can define the morphisms \( \Delta_n^+ : A_n \rightarrow A_n \otimes A_{n+1}, \ \epsilon_n : A_n \rightarrow C \) and antimorphisms \( S_n^- : A_n \rightarrow A_{n+1} \) such that the following axioms hold,

1. \((\epsilon_n \otimes id_{n+1}) \circ \Delta_n^+ = \tau_n^+, \ (id_{n-1} \otimes \epsilon_n) \circ \Delta_n^- = \tau_n^- \) \hspace{1cm} (a1)
2. \( m_{n+1} \circ (S_n^+ \otimes id_{n+1}) \circ \Delta_n^+ = \epsilon_{n+1} \circ \tau_n^+, \ m_{n-1} \circ (id_{n-1} \otimes S_n^-) \circ \Delta_n^- = \epsilon_{n-1} \circ \tau_n^- \) \hspace{1cm} (a2)
3. \((\Delta_n^- \otimes id_{n+1}) \circ \Delta_n^+ = (id_{n-1} \otimes \Delta_n^+) \circ \Delta_n^- \) \hspace{1cm} (a3)

in which \( m_n \) is the algebra multiplication for \( A_n \).

Now let us consider the structure of infinite Hopf family of algebras for our algebra \( A^{(1)}(\{ \Psi \}, \tilde{g})_{q,p,c} \).

This algebra is determined uniquely by the defining relations provided the following data are fixed: \( g, q, p, c \).

In general, given a series of \( c_n, n \in Z \), we can define the series of parameters \( \{ q_n, n \in Z \} \) by the relations

\[ q_{n+1}/q_n = p^{c_n}, \]

starting from the data \( q_1 = q, c_1 = c \). It is obvious that \( \tilde{q} = q_2 \). In the same fashion we can define \( \tilde{q}_n = q_{n+1} \) and write the algebras \( A_n = A^{(1)}(\{ \Psi \}, \tilde{g})_{q_n,p,c_n} \). The generating currents \( H^+_i(z), E_i(z) \) and \( F_i(z) \) for the algebra \( A_n \) are denoted as \( H^+_i(z|n), E_i(z|n) \) and \( F_i(z|n) \) respectively.

We collect the families of algebras \( A = \{ A_n, n \in Z \} \). This family of algebras can be turned into a category if we introduce the morphisms \( \tau_n^\pm \)

\[ \tau_n^\pm : A_n \rightarrow A_{n\pm 1} \]
\[ H^+_i(z|n) \rightarrow H^+_i(z|n \pm 1) \]
\[ E_i(z|n) \rightarrow E_i(z|n \pm 1) \]
\[ F_i(z|n) \rightarrow F_i(z|n \pm 1) \]
\[ c_n \rightarrow c_{n\pm 1} \]

and defining the compositions \( \tau(n, m) \) as did in [11].

The following proposition say that \( A \) is an infinite Hopf family of algebras.
Proposition 1.3 The category $A$ of algebras $\{A_n, n \in \mathbb{Z}\}$ form an infinite Hopf family of algebras with the Hopf family structures given as follows:

- the comultiplications $\Delta_n^\pm$:

\[
\Delta_n^+ c_n = c_n + c_{n+1},
\]
\[
\Delta_n^+ H_i^+(z|n) = H_i^+(zp^{c_{n+1}/4}|n) \otimes H_i^+(zp^{-c_n/4}|n+1),
\]
\[
\Delta_n^+ H_i^-(z|n) = H_i^-(zp^{-c_{n+1}/4}|n) \otimes H_i^-(zp^{c_n/4}|n+1),
\]
\[
\Delta_n^+ E_i(z|n) = E_i(z|n) \otimes 1 + H_i^-(zp^{c_n/4}|n) \otimes E_i(zp^{c_n/2}|n+1),
\]
\[
\Delta_n^+ F_i(z|n) = 1 \otimes F_i(z|n) + F_i(zp^{c_n+1/2}|n) \otimes H_i^+(zp^{c_n/4}|n+1),
\]
\[
\Delta_n^- c_n = c_{n-1} + c_n,
\]
\[
\Delta_n^- H_i^+(z|n) = H_i^+(zp^{c_n/4}|n-1) \otimes H_i^+(zp^{-c_{n-1}/4}|n),
\]
\[
\Delta_n^- H_i^-(z|n) = H_i^-(zp^{-c_{n+1}/4}|n-1) \otimes H_i^-(zp^{c_n-1/4}|n),
\]
\[
\Delta_n^- E_i(z|n) = E_i(z|n-1) \otimes 1 + H_i^-(zp^{c_n/4}|n-1) \otimes E_i(zp^{c_n-1/2}|n),
\]
\[
\Delta_n^- F_i(z|n) = 1 \otimes F_i(z|n) + F_i(zp^{c_n/2}|n-1) \otimes H_i^+(zp^{-c_n/4}|n);
\]

- the counits $\epsilon_n$:

\[
\epsilon_n(c_n) = 0,
\]
\[
\epsilon_n(1) = 1,
\]
\[
\epsilon_n(H_i^+(z|n)) = 1,
\]
\[
\epsilon_n(E_i(z|n)) = 0,
\]
\[
\epsilon_n(F_i(z|n)) = 0;
\]

- the antipodes $S_n^\pm$:

\[
S_n^+ c_n = -c_{n\pm 1},
\]
\[
S_n^+ H_i^+(z|n) = [H_i^+(z|n \pm 1)]^{-1},
\]
\[
S_n^+ H_i^-(z|n) = [H_i^-(z|n \pm 1)]^{-1},
\]
\[
S_n^+ E_i(z|n) = -H_i^-(zp^{-c_n\pm 1/4}|n \pm 1) \otimes E_i(zp^{-c_n\pm 1/2}|n \pm 1),
\]
\[
S_n^+ F_i(z|n) = -F_i(zp^{-c_n\pm 1/2}|n \pm 1) H_i^+(zp^{-c_n\pm 1/4}|n \pm 1)^{-1}.
\]

Despite these unusual co-algebraic structures, we can still define tensor product homomorphisms for both algebra families. We have

Proposition 1.4 The comultiplication $\Delta_n^+$ induces an algebra homomorphism

\[
\rho: \mathcal{A}^{(1)}(\{\Psi\}, \tilde{g})_{q_n, p, c_n + c_{n+1}} \rightarrow \mathcal{A}^{(1)}(\{\Psi\}, \tilde{g})_{q_n, p, c_n} \otimes \mathcal{A}^{(1)}(\{\Psi\}, \tilde{g})_{q_{n+1}, p, c_{n+1}}
\]
\[
X \mapsto \Delta_n^+ X,
\]
where \( X \in \mathcal{A}^{(1)}(\{\Psi\}, \overline{g})_{q_n, p, c_n + c_{n+1}}, \) \( \bar{X} \in \mathcal{A}^{(1)}(\{\Psi\}, \overline{g})_{q_n, p, c_n} \) and

\[
\bar{X} = \begin{cases}
\frac{c_n}{H^+(z|n)} & \text{if } X = \begin{cases}
c_n + c_{n+1} & \\
H^+(z|n) & \\
E_i(z|n) & \\
E_i(z|n+1) & \\
F_i(z|n) & \\
F_i(z|n+1) & 
\end{cases}.
\end{cases}
\]

Likewise, the comultiplication \( \Delta^n_- \) induces an algebra homomorphism

\[
\bar{\rho} : \mathcal{A}^{(1)}(\{\Psi\}, \overline{g})_{q_{n-1}, p, c_{n-1} + c_n} \to \mathcal{A}^{(1)}(\{\Psi\}, \overline{g})_{q_{n-1}, p, c_{n-1}} \otimes \mathcal{A}^{(1)}(\{\Psi\}, \overline{g})_{q_n, p, c_n}
\]

where \( X \in \mathcal{A}^{(1)}(\{\Psi\}, \overline{g})_{q_{n-1}, p, c_{n-1} + c_n}, \) \( \bar{X} \in \mathcal{A}^{(1)}(\{\Psi\}, \overline{g})_{q_{n-1}, p, c_{n-1}} \) and

\[
\bar{X} = \begin{cases}
c_n & \text{if } X = \begin{cases}
c_n - c_{n-1} & \\
H^+(z|n) & \\
E_i(z|n) & \\
E_i(z|n+1) & \\
F_i(z|n) & \\
F_i(z|n+1) & 
\end{cases}.
\end{cases}
\]

**Corollary 1.5** Let \( m \) be a positive integer. The iterated comultiplication \( \Delta^{(m)}_n = (id_n \otimes id_{n+1} \otimes \ldots \otimes id_{n+m-2} \otimes \Delta^{+}_{n+m-1} \circ (id_n \otimes id_{n+1} \otimes \ldots \otimes id_{n+m-3} \otimes \Delta^{+}_{n+m-2}) \ldots \circ (id_n \otimes \Delta^{+}_{n+1}) \circ \Delta^{+}_n \) induces an algebra homomorphism \( \rho^{(m)} \)

\[
\rho^{(m)} : \mathcal{A}^{(1)}(\{\Psi\}, \overline{g})_{q^{(n)}, p, c_n + c_{n+1} + \ldots + c_{n+m}} \to \mathcal{A}^{(1)}(\{\Psi\}, \overline{g})_{q^{(n)}, p, c_n} \otimes \mathcal{A}^{(1)}(\{\Psi\}, \overline{g})_{q^{(n+1)}, c_{n+1} + \ldots} \otimes \mathcal{A}^{(1)}(\{\Psi\}, \overline{g})_{q^{(n+m)}, p, c_{n+m}}
\]

in the spirit of Proposition 1.4.

**Remark 1.6** We stress here that the maps \( \rho, \bar{\rho} \) and \( \rho^{(m)} \) are algebra homomorphisms, whilst \( \tau^+_n, \tau^{(n,m)} \) and \( \Delta^n_- \) etc are only algebra morphisms. The difference between algebra morphisms and algebra homomorphisms lies in that the latter preserves the structure functions whilst the former does not.

**2**

**Definition 2.1** Let \( q, \bar{q} \) be generic parameters. We choose the set of analytic functions \( \Psi_{ij}(z|q) \) (\( i, j = 1, \ldots, \text{rank}(g) \)) such that they obey the condition \([4]\). The functional current algebra \( \mathcal{A}^{(2)}(\{\Psi\}, \overline{g})_{q, \bar{q}, \beta, \gamma} \) is defined as the associative algebra generated by the currents \( E_i(z), F_i(z), \) invertible \( H^\pm_i(z) \) with \( i = 1, 2, \ldots, \text{rank}(g) \), central elements \( \beta, \gamma \), and the unit 1 with relations
\[ H^\pm_i(z) H_j^\mp(w) = \frac{\Psi_{ij}(\frac{z}{w}, q)}{\Psi_{ij}(\frac{\bar{z}}{\bar{w}}, \bar{q})} H_j^\pm(w) H_i^\mp(z), \] (12)

\[ H_i^+(z) H_j^-(w) = \frac{\Psi_{ij}(\frac{z}{w}, (\beta - 1)q^{1/2}|\bar{q})}{\Psi_{ij}(\frac{\bar{z}}{\bar{w}}, (\beta - 1)^{-1/2}|\bar{q})} H_j^-(w) H_i^+(z), \] (13)

\[ H_i^+(z) E_j(w) = \Psi_{ij}(\frac{z}{w}, \beta - 1/2|q) E_j(w) H_i^+(z), \] (14)

\[ H_i^-(z) E_j(w) = \Psi_{ij}(\frac{z}{w}, \gamma - 1/2|q) E_j(w) H_i^-(z), \] (15)

\[ H_i^+(z) F_j(w) = \Psi_{ij}(\frac{z}{w}, \beta^2/\bar{q})^{-1} F_j(w) H_i^+(z), \] (16)

\[ H_i^-(z) F_j(w) = \Psi_{ij}(\frac{z}{w}, \gamma^2/\bar{q})^{-1} F_j(w) H_i^-(z), \] (17)

\[ E_i(z) E_j(w) = \Psi_{ij}(\frac{z}{w}, q) E_j(w) E_i(z), \] (18)

\[ F_i(z) F_j(w) = \Psi_{ij}(\frac{z}{w}, \bar{q})^{-1} F_j(w) F_i(z), \] (19)

\[ [E_i(z), F_j(w)] = \frac{\delta_{ij}}{(q/\bar{q} - 1)} \left[ \delta \left(\frac{z}{w}\right) H_i^+(w) (\beta - 1/2) - \delta \left(\frac{w}{z}\gamma^{-1}\right) H_i^-(w) (\gamma - 1/2) \right], \] (20)

\[ E_i(z_1) E_i(z_2) E_j(w) = f_{ij}^+(z_1/w, z_2/w|q) E_i(z_1) E_j(w) E_i(z_2) + E_j(w) E_i(z_1) E_i(z_2) \]

\[ \quad + (\text{replacement } z_1 \leftrightarrow z_2) = 0, \quad A_{ij} = -1, \] (21)

\[ F_i(z_1) F_i(z_2) F_j(w) = f_{ij}^-(z_1/w, z_2/w|\bar{q}) F_i(z_1) F_j(w) F_i(z_2) + F_j(w) F_i(z_1) F_i(z_2) \]

\[ \quad + (\text{replacement } z_1 \leftrightarrow z_2) = 0, \quad A_{ij} = -1, \] (22)

where

\[ f_{ij}^+(z_1/w, z_2/w|q) = \frac{\left(\Psi_{ij}(\frac{z_1}{z_2}, q) + 1\right) \left(\Psi_{ij}(\frac{w}{z_1}, q) \Psi_{ij}(\frac{w}{z_2}, q) + 1\right)}{\Psi_{ij}(\frac{w}{z_1}, q) + \Psi_{ij}(\frac{z_1}{z_2}, q) \Psi_{ij}(\frac{w}{z_2}, q)}, \]

\[ f_{ij}^-(z_1/w, z_2/w|\bar{q}) = \frac{\left(\Psi_{ij}(\frac{w}{z_1}, \bar{q})^{-1} + 1\right) \left(\Psi_{ij}(\frac{w}{z_2}, \bar{q})^{-1} \Psi_{ij}(\frac{w}{z_1}, \bar{q})^{-1} + 1\right)}{\Psi_{ij}(\frac{w}{z_2}, \bar{q})^{-1} + \Psi_{ij}(\frac{w}{z_1}, \bar{q})^{-1} \Psi_{ij}(\frac{w}{z_2}, \bar{q})^{-1}}. \]

This algebra is determined uniquely by the defining relations provided the following data are fixed: \( q, \bar{q}, \tilde{q} \).

Now let us choose an arbitrary set of parameters \( \{q(n), n \in Z\} \) and define \( \tilde{q}(n) = q(n+1) \). We collect the family of algebras \( B = \{B_n, n \in Z\} \) where \( B_n \equiv A((\Psi), \tilde{q}) q(n), q(n), \beta_n, \gamma_n \). The generating currents \( H_i^\pm(z) \), \( E_i(z) \) and \( F_i(z) \) for the algebra \( B_n \) are denoted as \( H_i^\pm(z|n) \), \( E_i(z|n) \) and \( F_i(z|n) \) respectively.

The algebra family \( B \) can also be turned into a category in the same way as we did for the family \( A \), provided the basic morphisms \( \tau_n^\pm \) are given as follows,

\[ \tau_n^\pm : B_n \rightarrow B_{n\pm 1} \]
Proposition 2.2. The category $B$ of algebras $\{B_n, n \in Z\}$ form an infinite Hopf family of algebras with the Hopf family structures given as follows:

- **the comultiplications $\Delta_n^\pm$:**
  \[ \Delta_n^+ \beta_n = \beta_n \beta_{n+1}, \]
  \[ \Delta_n^+ \gamma_n = \gamma_n \gamma_{n+1}, \]
  \[ \Delta_n^+ H_i^+(z|n) = H_i^+(z(\beta_{n+1})^{1/2}|n) \otimes H_i^+(z(\beta_n)^{1/2}|n + 1), \]
  \[ \Delta_n^+ H_i^-(z|n) = H_i^-(z(\gamma_{n+1})^{1/2}|n) \otimes H_i^-(z(\gamma_n)^{1/2}|n + 1), \]
  \[ \Delta_n^+ E_i(z|n) = E_i(z|n) \otimes 1 + H_i^-(z(\gamma_n)^{1/2}|n) \otimes E_i(z \gamma_n|n + 1), \]
  \[ \Delta_n^+ F_i(z|n) = 1 \otimes F_i(z|n + 1) + F_i(\beta_{n+1}^{-1}|n) \otimes H_i^+(z(\beta_{n+1})^{-1/2}|n + 1), \]

- **the counits $\epsilon_n$:**
  \[ \epsilon_n(\beta_n) = 1, \]
  \[ \epsilon_n(\gamma_n) = 1, \]
  \[ \epsilon_n(1_n) = 1, \]
  \[ \epsilon_n(H_i^+(z|n)) = 1, \]
  \[ \epsilon_n(E_i(z|n)) = 0, \]
  \[ \epsilon_n(F_i(z|n)) = 0; \]

- **the antipodes $S_n^\pm$:**
\[ S^+_n \beta_n = (\beta_n^{\pm 1})^{-1}, \]
\[ S^+_n \gamma_n = (\gamma_n^{\pm 1})^{-1}, \]
\[ S^+_n H^+_i(z|n) = [H^+_i(z|n \pm 1)]^{-1}, \]
\[ S^+_n H^-_i(z|n) = [H^-_i(z|n \pm 1)]^{-1}, \]
\[ S^+_n E_i(z|n) = -H^-_i(z(\gamma_n^{\pm 1})^{-1/2}|n \pm 1)^{-1}E_i(z(\gamma_n^{\pm 1})^{-1}|n \pm 1), \]
\[ S^+_n F_i(z|n) = -F_i(z(\beta_n^{\pm 1})|n \pm 1)H^+_i(z(\beta_n^{\pm 1})^{1/2}|n \pm 1)^{-1}. \]

The co-algebraic structure for the family \( B \) is also rather unusual, and we can understand such a structure more deeply by considering the tensor product homomorphisms for such algebra families. We have

**Proposition 2.3** The comultiplication \( \Delta^+_n \) induces an algebra homomorphism

\[ \rho : \mathcal{A}^{(2)}(\{\Psi\}, \bar{g})_{q(n), q(n+2)}^{\beta_n, \beta_{n+1}, \gamma_n, \gamma_{n+1}} \rightarrow \]
\[ \mathcal{A}^{(2)}(\{\Psi\}, \bar{g})_{q(n), q(n+1)}^{\beta_n, \beta_{n+1}, \gamma_n} \otimes \mathcal{A}^{(2)}(\{\Psi\}, \bar{g})_{q(n), q(n+2)}^{\beta_{n+1}, \gamma_n, \gamma_{n+1}} \]
\[ X \mapsto \Delta^+_n \bar{X}, \]

where \( X \in \mathcal{A}^{(2)}(\{\Psi\}, \bar{g})_{q(n), q(n+2)}^{\beta_n, \beta_{n+1}, \gamma_n, \gamma_{n+1}}, \bar{X} \in \mathcal{A}^{(2)}(\{\Psi\}, \bar{g})_{q(n), q(n+1)}^{\beta_n, \beta_{n+1}, \gamma_n} \) and

\[ \bar{X} = \begin{cases} 
\beta_n \\
\gamma_n \\
H^+_n(z|n) \\
E_i(z|n) \\
F_i(z|n)
\end{cases} \quad \text{if} \quad X = \begin{cases} 
\beta_n \beta_{n+1} \\
\gamma_n \gamma_{n+1} \\
H^+_n(z|n) \\
E_i(z|n) \\
F_i(z|n)
\end{cases}. \]

Likewise, the comultiplication \( \Delta^-_n \) induces an algebra homomorphism

\[ \bar{\rho} : \mathcal{A}^{(2)}(\{\Psi\}, \bar{g})_{q(n-1), q(n+1)}^{\beta_{n-1}, \beta_n, \gamma_{n-1}, \gamma_n} \rightarrow \]
\[ \mathcal{A}^{(2)}(\{\Psi\}, \bar{g})_{q(n-1), q(n)}^{\beta_{n-1}, \beta_n, \gamma_{n-1}} \otimes \mathcal{A}^{(2)}(\{\Psi\}, \bar{g})_{q(n), q(n+1)}^{\beta_n, \gamma_n, \gamma_{n+1}} \]
\[ X \mapsto \Delta^-_n \bar{X}, \]

where \( X \in \mathcal{A}^{(2)}(\{\Psi\}, \bar{g})_{q(n-1), q(n+1)}^{\beta_{n-1}, \beta_n, \gamma_{n-1}, \gamma_n}, \bar{X} \in \mathcal{A}^{(2)}(\{\Psi\}, \bar{g})_{q(n), q(n+1)}^{\beta_{n-1}, \beta_n, \gamma_{n-1}} \) and

\[ \bar{X} = \begin{cases} 
\beta_n \\
\gamma_n \\
H^+_n(z|n) \\
E_i(z|n) \\
F_i(z|n)
\end{cases} \quad \text{if} \quad X = \begin{cases} 
\beta_{n-1} \beta_n \\
\gamma_{n-1} \gamma_n \\
H^+_n(z|n-1) \\
E_i(z|n-1) \\
F_i(z|n-1)
\end{cases}. \]
Corollary 2.4 Let $m$ be a positive integer. The iterated comultiplication $\Delta_n^{(m)+} = (id_n \otimes id_{n+1} \otimes \ldots \otimes id_{n+m-2} \otimes \Delta_{n+m-1}^{+}) \circ (id_n \otimes id_{n+1} \otimes \ldots \otimes id_{n+m-3} \otimes \Delta_{n+m-2}^{+}) \ldots \circ (id_n \otimes \Delta_{n+1}^{+}) \circ \Delta_n^{+}$ induces an algebra homomorphism $\rho^{(m)}$

\[
\rho^{(m)} : A^{(2)}(\{\Psi\}, \hat{g})_{q(m), q(n+1), \beta_n, \beta_{n+1}, \ldots, \beta_{n+m}, \gamma_{n+1}, \ldots, \gamma_{n+m}} \rightarrow A^{(2)}(\{\Psi\}, \hat{g})_{q(n+1), \beta_n, \gamma_n} \otimes A^{(2)}(\{\Psi\}, \hat{g})_{q(n+1), q(n+2), \beta_{n+1}, \gamma_{n+1}} \otimes \ldots \otimes A^{(2)}(\{\Psi\}, \hat{g})_{q(n+m), q(n+m+1), \beta_{n+m}, \gamma_{n+m}}
\]

in the spirit of Proposition 2.3.

Remark 2.5 From the point of view of quantized affine algebras, the family $B$ of algebras seems more natural and symmetric: with the quantized (or deformed) Cartan part of currents splitted into positive and negative halves, why should the central element remain as a whole?

3

Now it is the point to consider the relationship between the algebras in the families $A$ and $B$. We note that the algebra $A^{(2)}(\{\Psi\}, \hat{g})_{q, \tilde{q}, \beta, \gamma}$ in the family $B$ has one more generator than the algebra $A^{(1)}(\{\Psi\}, \hat{g})_{q, p, c}$ in the family $A$ and hence both algebras cannot be identical in general. However, it is possible to introduce certain restrictions to the algebras in the family $B$ so that the algebras in the family $B$ can be related to the one in the family $A$. That means, the family $A$ is some special case of the family $B$.

Now we illustrate some examples of such restrictions in due course.

First let us recall that, in the algebra $A^{(2)}(\{\Psi\}, \hat{g})_{q, \tilde{q}, \beta, \gamma}$, the parameters $q, \tilde{q}$ are generic and their ratio is not assumed to be related to the central elements $\beta, \gamma$. This is in contrast to the case of $A^{(1)}(\{\Psi\}, \hat{g})_{q, p, c}$ in which the ratio of $q$ and $\tilde{q}$ is related to $p^c$. Now if we can consider the algebra $A^{(2)}(\{\Psi\}, \hat{g})_{q, \tilde{q}, \beta, \gamma}$ with the restrictions $\beta = \gamma^{-1} = p^{-c/2}, \tilde{q} = qp^c$ where $p$ is some constant, then the algebra $A^{(2)}(\{\Psi\}, \hat{g})_{q, \tilde{q}, \beta, \gamma}$ will become $A^{(1)}(\{\Psi\}, \hat{g})_{q, p, c}$, with corresponding generating currents identified. In this special case, the co-algebraic structures of both algebras also coincide.

Another special case is given as follows. In the algebra $A^{(2)}(\{\Psi\}, \hat{g})_{q, \tilde{q}, \beta, \gamma}$, let $\tilde{q}/q = p^c$, $p$ being some constant. Then the map

\[
\mu : A^{(1)}(\{\Psi\}, \hat{g})_{q, p, c} \rightarrow A^{(2)}(\{\Psi\}, \hat{g})_{q, \tilde{q}, \beta, \gamma}
\]

\[
E_i(z) \rightarrow E_i(z\gamma^{-1/2})
\]

\[
F_i(z) \rightarrow F_i(z\beta^{1/2})
\]

\[
H_i^{\pm}(z) \rightarrow H_i^{\pm}(z(\beta^{-1}\gamma)^{-1/4})
\]

\[
p^c \rightarrow \beta^{-1}\gamma
\]
gives a homomorphism from $A^{(1)}(\{\Psi\}, \hat{g})_{q,p,c}$ to $A^{(2)}(\{\Psi\}, \hat{g})_{q,\tilde{q},\beta,\gamma}$ as associative algebras. However, under this case, the co-algebraic structures for the algebra $A^{(1)}(\{\Psi\}, \hat{g})_{q,p,c}$ are not mapped into those for the algebra $A^{(2)}(\{\Psi\}, \hat{g})_{q,\tilde{q},\beta,\gamma}$.

It is interesting to mention that, the homomorphism $\mu$, though cannot map the co-algebraic structures correctly, can provide a way of obtaining certain bosonic realizations of the algebra $A^{(2)}(\{\Psi\}, \hat{g})_{q,\tilde{q},\beta,\gamma}$ from that of the algebra $A^{(1)}(\{\Psi\}, \hat{g})_{q,p,c}$, and vice versa. The simplest starting point will be the $c = 1$ bosonic realization for the algebra $A^{(1)}(\{\Psi\}, \hat{g})_{q,p,c}$, which would lead to a bosonic realization for the algebra $A^{(2)}(\{\Psi\}, \hat{g})_{q,\tilde{q},\beta,\gamma}$ at $\beta = q, \gamma = \tilde{q} = qp$. We shall come back to this point in Section 5. Before going to the bosonic representations, we would like to present some concrete examples for the structure functions $\Psi_{ij}(z|q)$ to show how general our algebras are.

4

In this section we present some examples for the structure functions $\Psi_{ij}(z|q)$ of the algebras $A^{(1)}(\{\Psi\}, \hat{g})_{q,p,c}$ and $A^{(2)}(\{\Psi\}, \hat{g})_{q,\tilde{q},\beta,\gamma}$.

First comes a special case in which $\Psi_{ij}(z|q)$ are generic analytic functions of $z$ satisfying the condition (1) but are independent of $q$. Then the parameters $q_n$ will not appear at all in both the family A and B, and the whole family A will degenerate into a single standard Hopf algebra which is nothing but the generalized quantum current algebra given by Ding and Iohara in [2] (we should change the notation $p \rightarrow q^2$ to compare with [2]). The family B will also degenerate into a single Hopf algebra which is, to our knowledge, not considered elsewhere earlier.

Let $\psi(z)$ be an analytical function of $z$ such that

$$\psi(z) = -z\psi(z^{-1}),$$

whose definition may depends on the parameters $(q, p)$ or $(q, \tilde{q})$. We define

$$\Psi_{ij}(z|q) = (-1)^{A_{ij}x^{-A_{ij}}} \frac{\psi(zx^{A_{ij}})}{\psi(zx^{-A_{ij}})},$$

where $x$ is an arbitrary function of the deformation parameters, $x = x(q,p)$ for the case of $A^{(1)}(\{\Psi\}, \hat{g})_{q,p,c}$ and $x = x(q,\tilde{q})$ for the case of $A^{(2)}(\{\Psi\}, \hat{g})_{q,\tilde{q},\beta,\gamma}$. The functions $\Psi_{ij}(z|q)$ given in (23) fulfill the condition (1) and hence can be used to give examples for either the algebra $A^{(1)}(\{\Psi\}, \hat{g})_{q,p,c}$ or the algebra $A^{(2)}(\{\Psi\}, \hat{g})_{q,\tilde{q},\beta,\gamma}$.

We note that the signature factor $(-1)^{A_{ij}}$ does not affect the condition (1) and thus can be omitted from eq. (23), and that would of course lead to a slightly different definition of the algebras.
\[
\Psi_{ij}(z|q) = x^{-A_{ij}} \frac{\psi(zx^{A_{ij}})}{\psi(zx^{-A_{ij}})}.
\]

Let us present some more special cases for the function \(\psi(z)\).

The rational function

\[
\psi(z) = 1 - z
\]

is our first choice. If we further choose \(x = p^{1/2}\) in this case, the corresponding family \(A\) will be identical to \(U_q(\hat{g})\) with \(q = p^{1/2}\).

Let \(\theta_q(z)\) be the elliptic function

\[
\theta_q(z) = (z|q)_\infty (qz^{-1}|q)_\infty (q|q)_\infty,
\]

\[
(z|q_1, ..., q_m)_\infty = \prod_{i_1, i_2, ..., i_m = 0}^\infty (1 - z q_1^{i_1} q_2^{i_2} ... q_m^{i_m}).
\]

We can choose

\[
\psi(z) = \theta_q(z)
\]

with \(y = y(q, p)\) for the case of \(A^{(1)}(\{\Psi\}, \hat{g})_{q, p, c}\) and \(y = y(q, \tilde{q})\) for the case of \(A^{(2)}(\{\Psi\}, \hat{g})_{q, \tilde{q}, \beta, \gamma}\).

If \(y = \text{const.}\) and \(x = p^{1/2}\), the algebra \(A^{(1)}(\{\Psi\}, \hat{g})_{q, p, c}\) will become the elliptic quantum group mentioned in [8, 9, 10].

It is interesting to mention that the algebra \(A^{(1)}(\{\Psi\}, \hat{g})_{q, p, c}\) with \(\psi(z)\) chosen as \(\theta_q(z)\) and \(x\) chosen as \(p^{1/2}\) form an elliptic generalization of the algebra \(A_{h, z}(\hat{g})\), which is the representative of the first known infinite Hopf family of algebras. In the scaling limit, the elliptic algebra family \(A\) will become the algebra family containing \(A_{h, z}(\hat{g})\). With the same choices of \(\psi(z)\) and \(x\), the algebra \(A^{(2)}(\{\Psi\}, \hat{g})_{q, \tilde{q}, \beta, \gamma}\) will become a new type of elliptic quantum group, which, in the case of \(\beta = q, \gamma = \tilde{q}\), will tend to the algebra of modifies screening currents with identification of parameters \(p = \tilde{q}/q\).

We emphasis that our construction enable us to introduce more free parameters and obtain multi-parameter quantum current algebras from a very general setting.

Last we should mention that this manuscript considers only those algebras with multiplicative spectral parameters. We could as well consider the cases with additive spectral parameters, and in those cases the structure functions (denoted as \(\Psi_{ij}(u|\eta)\)) should behave as

\[
\Psi_{ij}(-u|\eta) = \Psi_{ji}(u|\eta)^{-1}.
\]
The algebra \( A_{\hbar, \eta}(\hat{g}) \) is actually a concrete example for the algebra family \( A \) with structure functions given by

\[
\Psi_{ij}(u|\eta) = \frac{\text{sh} \pi \eta(u - i\hbar A_{ij}/2)}{\text{sh} \pi \eta(u + i\hbar A_{ij}/2)}.
\]

5

Having established the infinite Hopf family of algebras structure of the algebra families \( A \) and \( B \), we now turn to consider their simplest infinite dimensional representation, i.e. the free boson realization.

The purpose of this section is to conduct a general method for obtaining a particular (lowest level) free boson realization for a given quantum current algebra. We shall show that the free boson realization for the algebra \( A^{(1)}(\{\Psi\}, \hat{g})_{q,p,c} \) at \( c = 1 \) and that for the algebra \( A^{(2)}(\{\Psi\}, \hat{g})_{q, \tilde{q}, \beta, \gamma} \) at \( \beta = q, \gamma = \tilde{q} \equiv qp \) can actually be obtained from the same set of bosonic fields.

Let us start our construction by considering the generating relations for \( E_i(z)E_j(w), F_i(z)F_j(w) \) and the commutator relations \([E_i(z), F_j(w)]\) respectively. We notice that the \( E_i(z)E_j(w), F_i(z)F_j(w) \) relations are the same for both algebras under consideration. This is a very important feature for our consideration. In order to obtain a free boson realization, we need to introduce some Riemann decomposition for the structure functions \( \Psi_{ij}(z|q) \). Suppose this decomposition is given by

\[
\Psi_{ij}(z|q) = \frac{\Phi_{ij}(z|q)}{\Phi_{ji}(z^{-1}|q)},
\]

then we can rewrite the \( E_i(z)E_j(w), F_i(z)F_j(w) \) relations in the form

\[
\Phi_{ji} \left( \frac{w}{z} | q \right) E_i(z)E_j(w) = \Phi_{ij} \left( \frac{z}{w} | q \right) E_i(w)E_j(z),
\]

\[
\Phi_{ij} \left( \frac{z}{w} | \tilde{q} \right) F_i(z)F_j(w) = \Phi_{ji} \left( \frac{w}{z} | \tilde{q} \right) F_i(w)F_j(z).
\]

In order to obtain a bosonic realization for the above relations, it is enough to write down some bosonic expressions also denoted \( E_i(z) \) and \( F_i(z) \) such that they satisfy the relations

\[
\Phi_{ji} \left( \frac{w}{z} | q \right) E_i(z)E_j(w) =: E_i(z)E_j(w) :,
\]

\[
\Phi_{ij} \left( \frac{z}{w} | \tilde{q} \right) F_i(z)F_j(w) =: F_i(z)F_j(w) :,
\]

where :: means the standard normal ordering of bosonic expressions. In the meantime, we set

\[
^2\text{Actually, the standard definition of the } q^-\text{affine current algebra was given with the Riemann decomposition explicitly introduced into the } E_i(z)E_j(w), F_i(z)F_j(w) \text{ relations like in (24) and (25).}
\]
\begin{equation}
E_i(z)F_j(w) = \Upsilon_{ij}(z, w) : E_i(z)F_j(w) :,
\tag{29}
\end{equation}
\begin{equation}
F_j(w)E_i(z) = \bar{\Upsilon}_{ji}(z, w) : E_i(z)F_j(w) :,
\tag{30}
\end{equation}

where, in order to yield the \(\delta\)-function terms in the relations (8) and (20), we need the functions \(\Upsilon_{ij}(z, w)\) and \(\bar{\Upsilon}_{ji}(z, w)\) to satisfy the relation

\begin{equation}
\Upsilon_{ij}(z, w) - \bar{\Upsilon}_{ji}(z, w) = \delta_{ij} \left( \left( \frac{w}{z} p^{-1/2} \right) g^{(1)+}(wp^{1/4}) - \left( \frac{w}{z} p^{-1/2} \right) g^{(1)-}(zp^{1/4}) \right)
\tag{31}
\end{equation}

for the case of the algebra \(\mathcal{A}^{(1)}(\{\Psi\}, \hat{g})_{q,p,c}\) at \(c = 1\) or

\begin{equation}
\Upsilon_{ij}(z, w) - \bar{\Upsilon}_{ji}(z, w) = \frac{\delta_{ij}}{(q/q - 1)} \left[ \delta \left( \frac{z}{w} q \right) g^{(2)+}(wq^{-1/2}) - \delta \left( \frac{w}{z} q^{-1} \right) g^{(2)-}(zq^{1/2}) \right]
\tag{32}
\end{equation}

for the case of the algebra \(\mathcal{A}^{(2)}(\{\Psi\}, \hat{g})_{q,q',\beta,\gamma}\) at \(\beta = q, \gamma = q\), where \(g^{(l)\pm}(z)\), \(l = 1, 2\) are some power functions of the arguments which could be absorbed into the definition of \(H^{\pm}_i(z)\) as normalization factors. Since the above two algebras at the given values of central elements are homomorphic as mentioned in the end of Section 3, we shall proceed with only the case of \(\mathcal{A}^{(1)}(\{\Psi\}, \hat{g})_{q,p,c}\) at \(c = 1\) and obtain the case of \(\mathcal{A}^{(2)}(\{\Psi\}, \hat{g})_{q,q',\beta,\gamma}\) at \(\beta = q, \gamma = q\) as a trivial result of the homomorphism.

From experiences in the study of bosonic realizations for standard \(q\)-affine algebras, we know that the choices

\begin{align*}
\Upsilon_{ij}(z, w) &= \begin{cases} 
\frac{1}{z^2(1-\hat{p}q^{1/2})(1-\hat{p}q^{-1/2})} & \text{for } A_{ij} = 2, \\
\sim \text{some regular expressions} & \text{for } A_{ij} = -1, 0
\end{cases} \\
\bar{\Upsilon}_{ji}(z, w) &= \Upsilon_{ij}(w, z)
\end{align*}

fulfills the condition (31), with \(g^{(1)+}(z) = g^{(1)-}(z) = z^{-2}\).

According to the above analysis, we now introduce the ansatz for the bosonic expressions \(E_i(z)\) and \(F_i(z)\)

\begin{align*}
E_i(z) &= \exp \varphi_i(z), \\
F_i(z) &= \exp \psi_i(z),
\tag{32}
\end{align*}

where

\begin{align*}
\varphi_i(z) &= Q_i + \log(Az) P_i + \sum_{n \neq 0} u[n] a_i[n] z^{-n}, \\
\psi_i(z) &= -Q_i - \log(Bz) P_i - \sum_{n \neq 0} v[n] a_i[n] z^{-n},
\tag{33}
\end{align*}
A and B are some constants to be related to the deformation parameters, \( u[n] \) and \( v[n] \) are all functions of the integer \( n \) which are independent of \( z \), and \( Q_i, P_i \) and \( a_i[n] \) are bosonic operators whose commutation relations are to be determined. The operators \( Q_i \) and \( P_i \) here play the role of zero mode generators for the bosonic fields \( \varphi_i(z) \) and \( \psi_i(z) \).

Following from the above ansatz, we can write immediately

\[
E_i(z) E_j(w) = \exp(\langle \varphi_i(z) \varphi_j(w) \rangle) : E_i(z) E_j(w) :, \tag{36}
\]

\[
F_i(z) F_j(w) = \exp(\langle \psi_i(z) \psi_j(w) \rangle) : F_i(z) F_j(w) :, \tag{37}
\]

\[
E_i(z) F_j(w) = \exp(\langle \varphi_i(z) \psi_j(w) \rangle) : E_i(z) F_j(w) :, \tag{38}
\]

which, compared to eqs. (27-30), yield

\[
\langle \varphi_i(z) \varphi_j(w) \rangle = -\log \left[ \Phi_{ji}(w|z) \right], \tag{39}
\]

\[
\langle \psi_i(z) \psi_j(w) \rangle = -\log \left[ \Phi_{ij}(z|w) \right], \tag{40}
\]

\[
\langle \varphi_i(z) \psi_j(w) \rangle = \log [\Upsilon_{ij}(z,w)]. \tag{41}
\]

For any concrete set of functions \( \Phi_{ij}(z|q) \), these last equations serve as a good starting point to determine the unknown coefficients \( A, B, u[n], v[n] \) as well as the unknown commutation relations for \( Q_i, P_i \) and \( a_i[n] \) respectively. Of course the solution need not to be unique.

In the concrete case when \( \Psi_{ij}(z|q) \) is given by eq. (23) with \( \psi(z) = \theta_q(z) \) and \( x = p^{1/2} \), i.e.

\[
\Psi_{ij}(z|q) = (-1)^{A_{ij}} p^{-A_{ij}/2} \frac{\theta_q(zp^{A_{ij}/2})}{\theta_q(zp^{-A_{ij}/2})}, \tag{42}
\]

we have the following explicit result.

First the Heisenberg algebra \( H_{q,p}(g) \) with generators \( a_i[n], P_i, Q_i, i = 1, \ldots, \text{rank}(g) \), \( n \in \mathbb{Z} \setminus \{0\} \) can be introduced by writing down the generating relations

\[
[a_i[n], a_j[m]] = \frac{1}{n} \frac{(1 - q^{-n})(p^{nA_{ij}/2} - p^{-nA_{ij}/2})(1 - (pq)^n)}{1 - p^n} \delta_{n,m},
\]

\[
[P_i, Q_j] = A_{ij},
\]

where \( (A_{ij}) \) is the Cartan matrix for the Lie algebra \( g \). Let

\[
A = B = 1,
\]

\[
u[n] = \frac{(pq)^{-1/2}}{q^n - 1}, \quad v[n] = \frac{q^{-1/2}}{(pq)^{-n} - 1}\]

then we have
Proposition 5.1 The following bosonic expressions give a level $c = 1$ realization for the algebra $\mathcal{A}^{(1)}(\{\Psi\},\hat{g})_{q,p,c}$ with $\Psi_{ij}(z|q)$ chosen as in (32), on the Fock space of the Heisenberg algebra $\mathcal{H}_{q,p}(g)$.

$$E_i(z) = \exp[\varphi_i(z)];$$
$$F_i(z) = \exp[\psi_i(z)];$$
$$H^+_i(z) = z^{-2} : E_i(zp^{1/4})F_i(zp^{-1/4}) :,$$
$$H^-_i(z) = z^{-2} : E_i(zp^{-1/4})F_i(zp^{1/4}) :.$$ 

Of course, this result can be readily mapped into a representation of $\mathcal{A}^{(2)}(\{\Psi\},\hat{g})_{q,\tilde{q},\beta,\gamma}$ at $\tilde{q} = qp, \beta = q, \gamma = qp$ using the homomorphism $\mu$.

So far we obtained two families of quantum current algebras $\{\mathcal{A}^{(1)}(\{\Psi\},\hat{g})_{q_n,p_n,c_n}, n \in \mathbb{Z}\}$ and $\{\mathcal{A}^{(2)}(\{\Psi\},\hat{g})_{q_n,\tilde{q}(n),\beta_n,\gamma_n}, n \in \mathbb{Z}\}$ and established their structures as infinite Hopf family of algebras. The generality of the defining relations for these two family of algebras indicates that the infinite Hopf family of algebras exists much broader than the standard Hopf algebras. Actually, taken from the point of view of defining tensor product representations, the standard Hopf algebra structure is by no means superior to the infinite Hopf family of algebras, because both kinds of structures allow one to obtain fused representations from the tensor category of the set of seed algebras.

It is interesting to mention that the comultiplications appearing in such co-structures are all of the Drinfeld type, which closes over the currents themselves and does not require the resolution to the inverse problem (Riemann problem) of the Ding-Frenkel homomorphism [16]. Recall that two kinds of comultiplications (and thus two kinds of Hopf algebra structures) are known for the standard $q$ affine algebras. While the $q$ affine algebras are considered as the most trivial cases of infinite Hopf family of algebras, only the Drinfeld type co-structures find their place in the generalized co-structure, whilst the standard Hopf algebra structure find no counterpart in our present study. This is because we studied here only the current algebra formulation. In order to have a complete generalization of both Hopf algebra structures of the $q$ affine algebras, it seems that we have to go to the Yang-Baxter realization as well, and it is highly probable that in that realization, the quasi-triangular quasi-Hopf algebra structure may take some place. We leave this problem to future study.

We should emphasis that this work is only a preliminary study for the new quantum current algebras. Besides the definition and infinite Hopf family structure, we know very little about these algebras, especially their detailed representation theory, vertex operators, Yang-Baxter type realizations etc. The physical applications should also be considered.
On the other hand, the structures of infinite Hopf family of algebras is still poorly understood yet. We do not know whether there exists a quantum double construction over the infinite Hopf family of algebras and, if not, what kind of new structure will take the place of the standard quantum doubles. Also, the classical counterpart of the infinite Hopf family of algebras is unknown and it seems that all these problems deserve further investigations.

Finally, from experiences of studying various (deformed) affine algebras, we know that given a (quantum deformed) affine algebra there must exist an accompanied (deformed) Virasoro algebra, and the latter is highly expected to have important applications in physics of 1+1 dimensions. Therefore, given the two new family of quantum current algebras, it seems very interesting to find/construct the corresponding deformed Virasoro algebras.

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