An (inverse) Pieri formula for Macdonald polynomials of type $C$

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Mathematics Subject Classifications: 33D52

Abstract

We give an explicit Pieri formula for Macdonald polynomials attached to the root system $C_n$ (with equal multiplicities). By inversion we obtain an explicit expansion for two-row Macdonald polynomials of type $C$.

1 Introduction

In the eighties, I. G. Macdonald introduced a new family of orthogonal polynomials, which are Laurent polynomials in several variables, and generalize the Weyl characters of compact simple Lie groups [16, 17]. In the most simple situation, a family $P^{(R)}_\lambda(q,t)$ of such polynomials, depending rationally on two parameters $q,t$, is attached to any reduced irreducible root system $R$.

These orthogonal polynomials are elements of the group algebra of the weight lattice of $R$, invariant under the action of the Weyl group. They are indexed by the set $P^+$ of dominant weights. For $t = q$, they correspond to the Weyl characters $\chi^{(R)}_\lambda$ of compact simple Lie groups.

When $R$ is of type $A$, the orthogonal polynomials $P^{(A)}_\lambda(q,t)$ correspond to the symmetric functions $P_\lambda(q,t)$ previously studied in [14, 15].

Let $\lambda$ be a dominant weight of $R$, $r$ some positive integer and $\omega_s$ some fundamental weight. By analogy with the type $A$ case, the product decomposition

$$P^{(R)}_{r\omega_s} P^{(R)}_\lambda = \sum_{\tau \in \Sigma} C_{\tau} P^{(R)}_{\lambda + \tau}$$

is called a “Pieri formula” for the Macdonald polynomials of type $R$. 

1
When $R = A_n$ the range $\Sigma$ and the coefficients $C_\tau$ are explicitly known for $r = 1, s$ arbitrary and $s = 1, r$ arbitrary \cite{13} (6.24), p. 340. Moreover there is a duality property connecting these two cases.

The situation is very different when $R$ is not of type $A$. General results \cite{18} entail that $\Sigma$ is formed by the “integral points” of the convex hull of the Weyl group orbit of $r \omega_s$. But no formula is known for the coefficients $C_\tau$, except when $\tau$ belongs to the boundary of this convex hull. Finding a general explicit expression of $C_\tau$ is still a very difficult open problem.

In this paper we perform some computation of $C_\tau$ when $R = C_n$. For this root system we give the explicit decomposition of the product $P_{r \omega_1} P_{s \omega_1}$. Remarkably the coefficients $C_\tau$ appear to be fully factorized (which is not a general property). Moreover this Pieri formula is easily inverted, which yields the explicit expansion of any $P_{\lambda_1 \omega_1 + \lambda_2 \omega_2}$ in terms of products $P_{r \omega_j} P_{s \omega_k}$.

For $n = 2$ this expansion completely determines the Macdonald polynomials attached to $C_2$ and its dual root system $B_2$.

However our results are obtained under the technical assumption that the same parameter $t$ is associated with short and long roots. The case of distinct parameters is yet unknown and appears to be much more intricate.

The paper is organized as follows. Sections 2 and 3 are devoted to general facts about Macdonald polynomials, including the Pieri formula for a (quasi) minuscule weight. In Section 4 these results are specified for the root system $C_n$. Our Pieri formula is presented in Section 5, proved in Section 6 and inverted in Section 7. Technical material is given in Sections 8-10, including $\lambda$-ring calculus and a very remarkable rational identity.

This multivariate identity presents an interest by itself. Section 12 is devoted to some basic $q$-hypergeometric identities obtained by its “(multiple) principal specialization”, and outlines its links with previous results.

Acknowledgements. Thanks are due to the anonymous referees for helpful comments. I am much indebted to Michael Schlosser for advice and discussions, in particular for making me aware of \cite{21}.

## 2 Macdonald polynomials

In this section we introduce our notations, and recall some general facts about Macdonald polynomials. For more details the reader is referred to \cite{16, 17, 18}.

The most general class of Macdonald polynomials is associated with a pair of root systems $(R, S)$, spanning the same vector space and having the same Weyl group, with $S$ reduced. Throughout this paper, we shall only consider the case $S = R$.

Let $V$ be a finite-dimensional real vector space endowed with a positive definite symmetric bilinear form $\langle u, v \rangle$. For all $v \in V$, we write $|v| = \langle v, v \rangle^{1/2}$ and $v^\vee = 2v/|v|^2$.

Let $R \subset V$ be a reduced irreducible root system, $W$ the Weyl group of $R$, $R^+$ the set of positive roots, $\{\alpha_1, \ldots, \alpha_n\}$ the basis of simple roots, and $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$ the dual root system.
The fundamental weights $\omega_i$ are defined by $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. Let

$$Q = \sum_{i=1}^{n} \mathbb{Z} \alpha_i, \quad Q^+ = \sum_{i=1}^{n} \mathbb{N} \alpha_i$$

be the root lattice of $R$ and its positive octant, and

$$P = \sum_{i=1}^{n} \mathbb{Z} \omega_i, \quad P^+ = \sum_{i=1}^{n} \mathbb{N} \omega_i$$

the weight lattice of $R$ and the cone of dominant weights. A partial order is defined on $P$ by $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q^+$. Let $A$ denote the group algebra over $\mathbb{R}$ of the free Abelian group $P$. For each $\lambda \in P$ let $e^\lambda$ denote the corresponding element of $A$, subject to the multiplication rule $e^\lambda e^\mu = e^{\lambda+\mu}$. The set $\{ e^\lambda, \lambda \in P \}$ is an $\mathbb{R}$-basis of $A$.

The Weyl group $W$ acts on $P$, hence on $A$ by $w(e^\lambda) = e^{w\lambda}$. Let $A^W$ denote the subspace of $W$-invariants in $A$. Such elements are called “symmetric polynomials”. There are two important examples of a basis of $A^W$, both indexed by dominant weights $\lambda \in P^+$.

The first one is given by the orbit-sums

$$m_\lambda = \sum_{\mu \in W\lambda} e^\mu.$$ 

The second one is provided by the Weyl characters defined as follows. Let

$$\delta = \prod_{\alpha \in R^+} (e^{\alpha/2} - e^{-\alpha/2}) = e^{-\rho} \prod_{\alpha \in R^+} (e^\alpha - 1),$$

with $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \in P^+$. Then $w\delta = \varepsilon(w)\delta$ for any $w \in W$, where $\varepsilon(w) = \det(w) = \pm 1$. For all $\lambda \in P$ the element

$$\chi_\lambda = \delta^{-1} \sum_{w \in W} \varepsilon(w) e^{w(\lambda + \rho)}$$

is in $A^W$, and the set $\{ \chi_\lambda, \lambda \in P^+ \}$ forms an $\mathbb{R}$-basis of $A^W$.

Let $0 < q < 1$. For any real number $k \in \mathbb{R}$, the classical $q$-shifted factorial $(u; q)_k$ is defined by

$$(u; q)_\infty = \prod_{j \geq 0} (1 - uq^j), \quad (u; q)_k = (u; q)_\infty / (uq^k; q)_\infty.$$ 

For each $\alpha \in R$ let $t_\alpha = q^{k_\alpha}$ be a positive real number such that $t_\alpha = t_\beta$ if $|\alpha| = |\beta|$. There are at most two different values for the $t_\alpha$’s, depending on whether $\alpha$ is a short or a long root. We define

$$\rho_k = \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha \alpha, \quad \rho_k^\vee = \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha \alpha^\vee.$$ 

Observe that $\rho_k^\vee$ is not $2\rho_k/|\rho_k|^2$. 

3
If \( f = \sum_{\lambda \in P} a_\lambda e^\lambda \in A \), let \( \bar{f} = \sum_{\lambda \in P} a_\lambda e^{-\lambda} \) and \([f]_1\), its constant term \( a_0 \). The inner product defined on \( A \) by

\[
\langle f, g \rangle_{q,t} = \frac{1}{|W|} [f \bar{g} \Delta],
\]

with \(|W|\) the order of \( W \) and

\[
\Delta = \prod_{\alpha \in R} \frac{(e^\alpha ; q)_{\infty}}{(t_\alpha e^\alpha ; q)_{\infty}} = \prod_{\alpha \in R} (e^\alpha ; q)_{k_\alpha}
\]
is non degenerate and \( W \)-invariant.

There exists a unique basis \( \{ P_\lambda, \lambda \in P^+ \} \) of \( A^W \), called Macdonald polynomials, such that

(i) \( P_\lambda = m_\lambda + \sum_{\mu \in P^+, \mu < \lambda} a_{\lambda \mu}(q,t) m_\mu \),

where the coefficients \( a_{\lambda \mu}(q,t) \) are rational functions of \( q \) and the \( t_\alpha \)'s,

(ii) \( \langle P_\lambda, P_\mu \rangle_{q,t} = 0 \) if \( \lambda \neq \mu \).

It is clear that the \( P_\lambda \)'s, if they exist, are unique. Their existence is proved as eigenvectors of an operator \( A^W \to A^W \), self-adjoint with respect to \( \langle \cdot, \cdot \rangle_{q,t} \) and having its eigenvalues all distinct. This operator may be constructed as follows \[16, 17\].

A weight \( \pi \) of \( R^\vee \) is called minuscule if \( \langle \pi, \alpha \rangle \in \{0, 1\} \) for \( \alpha \in R^+ \), and quasi-minuscule if \( \langle \pi, \alpha \rangle \in \{0, 1, 2\} \) for \( \alpha \in R^+ \). Let \( T_\pi \) be the translation operator defined on \( A \) by \( T_\pi(e^\lambda) = q^{\langle \pi, \lambda \rangle} e^\lambda \) for any \( \lambda \in P \), and

\[
\Phi_\pi = T_\pi(\Delta_+) / \Delta_+ \quad \text{with} \quad \Delta_+ = \prod_{\alpha \in R^+} (e^\alpha ; q)_{k_\alpha}.
\]

Two situations may be considered.

First case : \( \pi \) is a minuscule weight of \( R^\vee \).

Such a weight only exists when \( R \) is of type \( B, C, D \) or \( E_6, E_7 \). It is necessarily a fundamental weight of \( R^\vee \). Let \( E_\pi \) be the self-adjoint operator defined by

\[
E_\pi f = \sum_{w \in W} w(\Phi_\pi T_\pi f),
\]

where \( \Phi_\pi \) is now given by

\[
\Phi_\pi = \prod_{\alpha \in R^+} \frac{1 - t_\alpha e^\alpha}{1 - e^\alpha} = \prod_{\alpha \in R^+} \frac{1 - t_\alpha^{\langle \pi, \alpha \rangle} e^\alpha}{1 - e^\alpha}.
\]

Macdonald polynomials \( P_\lambda \) are defined as eigenvectors of \( E_\pi \), namely

\[
E_\pi P_\lambda = c_\lambda P_\lambda \quad \text{with} \quad c_\lambda = q^{\langle \pi, \rho_k \rangle} \sum_{w \in W} q^{\langle w\pi, \lambda + \rho_k \rangle}.
\]
Second case: \( \pi \) is a quasi-minuscule weight of \( R^\vee \).

When \( R \) is \( E_8 \), \( F_4 \) or \( G_2 \), \( R^\vee \) has no minuscule weight. However a quasi-minuscule weight always exists, given by \( \pi = \varphi^\vee \), where \( \varphi \) is the highest root of \( R \). For the types just mentioned, it is the only one.

In other words, the family \( \{ \alpha \in R^+ : \langle \pi, \alpha \rangle = 2 \} \) is either empty (if \( \pi \) is minuscule) or reduced to the single element \( \varphi \) (if \( \pi = \varphi^\vee \)).

For \( \pi \) a quasi-minuscule weight, let \( R^\vee \) have no minuscule weight. However a quasi-minuscule weight always exists, given by \( \pi = \varphi^\vee \), where \( \varphi \) is the highest root of \( R \). For the types just mentioned, it is the only one.

In other words, the family \( \{ \alpha \in R^+ : \langle \pi, \alpha \rangle = 2 \} \) is either empty (if \( \pi \) is minuscule) or reduced to the single element \( \varphi \) (if \( \pi = \varphi^\vee \)).

For \( \pi \) a quasi-minuscule weight, let \( F^\pi \) be the self-adjoint operator defined by

\[
F^\pi f = \sum_{w \in W} w (\Phi^\pi (T^\pi - 1) f),
\]

where \( \Phi^\pi \) is now given by

\[
\Phi^\pi = \prod_{\alpha \in R^+} \frac{1 - t_\alpha e^\alpha}{1 - e^{\langle \pi, \alpha \rangle}} \prod_{\alpha \in R^+} \frac{1 - t_\alpha e^\alpha 1 - qt_\alpha e^\alpha}{1 - q e^\alpha}.
\]

Then Macdonald polynomials \( P_\lambda \) are eigenvectors of \( F^\pi \). We have

\[
F^\pi P_\lambda = c'_\lambda P_\lambda \quad \text{with} \quad c'_\lambda = q^{\langle \pi, \rho_k \rangle} \sum_{w \in W} (q^{\langle w^\pi, \lambda + \rho_k \rangle} - q^{\langle w^\pi, \rho_k \rangle}).
\]

If \( \pi \) is minuscule, the definitions of \( E^\pi \) and \( F^\pi \) are equivalent because we have

\[
\sum_{w \in W} w \Phi^\pi = \sum_{w \in W} \prod_{\alpha \in R^+} \frac{1 - t_\alpha e^{w_\alpha}}{1 - e^{w_\alpha}} = q^{\langle \pi, \rho_k \rangle} \sum_{w \in W} q^{\langle w^\pi, \rho_k \rangle},
\]

which is a consequence of the Macdonald identity

\[
\sum_{w \in W} \prod_{\alpha \in R^+} \frac{1 - u_\alpha e^{-w_\alpha}}{1 - e^{-w_\alpha}} = \sum_{w \in W} \prod_{\alpha \in R^+} u_\alpha,
\]

proved in [19, Theorem 2.8] for any family of indeterminates \( \{ u_\alpha, \alpha \in R^+ \} \).

We may regard any \( f = \sum_{\lambda \in P} f_\lambda e^\lambda \in A \), with only finitely many nonzero coefficients, as a function on \( V \) by putting for any \( x \in V \),

\[
f(x) = \sum_{\lambda \in P} f_\lambda q^{\langle \lambda, x \rangle}.
\]

With this convention we obviously have

\[
(T^\tau f)(x) = f(x + \tau).
\]

Then Macdonald polynomials satisfy the two following properties proved by Cherednik [2]:
(i) Specialization. For any \( \lambda \in P^+ \) we have
\[
P_\lambda(\rho_k^\vee) = q^{-(\lambda,\rho_k^\vee)} \prod_{\alpha \in R^+} (q(\rho_k,\alpha^\vee)t_\alpha; q)_{\langle \lambda,\alpha^\vee \rangle} / (q^{(\rho_k,\alpha^\vee)}; q)_{\langle \lambda,\alpha^\vee \rangle}.
\]

(ii) Symmetry. Let \( P^\vee \) be the weight lattice of \( R^\vee \), and for any \( \mu \in (P^\vee)^+ \) let \( P_\mu \) be the associated Macdonald polynomial. Define
\[
\tilde{P}_\lambda = P_\lambda / P_\lambda(\rho_k^\vee), \quad \tilde{P}_\mu = P_\mu / P_\mu(\rho_k).
\]
Then we have
\[
\tilde{P}_\lambda(\mu + \rho_k^\vee) = \tilde{P}_\mu(\lambda + \rho_k).
\]

3 The Pieri formula for a (quasi) minuscule weight

For any vector \( \tau \in V \) define
\[
\Sigma(\tau) = C(\tau) \cap (\tau + Q) = \bigcap_{w \in W} w(\tau - Q^+)
\]
with \( C(\tau) \) the convex hull of the Weyl group orbit \( W_\tau \).

Let \( \lambda \in P^+ \) and \( \omega \) be a fundamental weight. We consider the Pieri formula
\[
P_\omega P_\lambda = \sum_{\tau \in \Sigma} C_\tau P_{\lambda + \tau},
\]
with \( C_\tau \) for \( \tau \in P^+ \) yet unknown.

By general results [13] (5.3.8), p. 104, it is known that the range \( \Sigma \) on the right-hand side is equal to \( \Sigma(\omega) \). But an explicit formula for the coefficients \( C_\tau \) is yet unknown.

However when \( \omega \) is a minuscule or quasi-minuscule weight of \( R \), an explicit expression for \( C_\tau \) can be derived from the definition of Macdonald polynomials attached to the dual root system \( R^\vee \). Although this duality method is known to experts (see [3] Appendix or [4] Section 4), we think useful to enter into details.

Observe that if \( \omega \) is a minuscule weight of \( R \) we have \( \Sigma(\omega) = W \omega \) and \( P_\omega = m_\omega \).

**Theorem 1.** Let \( \omega \in P^+ \) be a minuscule (fundamental) weight of \( R \). Then we have
\[
P_\omega P_\lambda = \sum_{\tau \in W_\omega} C_\tau P_{\lambda + \tau},
\]
with
\[
C_\tau = \prod_{\alpha \in R^+} \frac{1 - q^{(\lambda+\rho_k,\alpha^\vee)}t_\alpha^{-1}}{1 - q^{(\lambda+\rho_k,\alpha^\vee)} - 1} \frac{1 - q^{(\lambda+\rho_k,\alpha^\vee)-1}t_\alpha}{1 - q^{(\lambda+\rho_k,\alpha^\vee)-1}}.
\]
Equivalently

\[ P_\omega \tilde{P}_\lambda = \sum_{\tau \in W_\omega} C_\tau \tilde{P}_{\lambda + \tau}, \]

with

\[ C_\tau = q^{-\langle \tau, \rho_k' \rangle} \prod_{\alpha \in R^+} \frac{1 - t_\alpha^{\langle \tau, \alpha \rangle} q^{\langle \lambda + \rho_k, \alpha \rangle}}{1 - q^{\langle \lambda + \rho_k, \alpha \rangle}}. \]

**Proof.** The equivalence of both formulations is a consequence of the specialization formula, together with the fact that for \( \alpha \in R^+ \) we have \( \langle \tau, \alpha \rangle \in \{-1, 0, 1\} \), since \( \langle \omega, \alpha \rangle \in \{0, 1\} \) and \( W \) permutes roots.

It is equivalent to prove the second formulation for the dual root system \( R^\vee \). Then it writes as

\[ m_\pi \tilde{P}_\mu = \sum_{\tau \in W_\pi} C_\tau \tilde{P}_{\mu + \tau}, \]

with \( \pi \) a minuscule weight of \( R^\vee \), \( \mu \in (P^\vee)^+ \) and

\[ C_\tau = q^{-\langle \tau, \rho_k \rangle} \prod_{\alpha \in R^+} \frac{1 - t_\alpha^{\langle \tau, \rho_k \rangle} q^{\langle \mu + \rho_k', \alpha \rangle}}{1 - q^{\langle \mu + \rho_k', \alpha \rangle}}. \]

Let \( \kappa \in P^+ \) arbitrary. The associated Macdonald polynomial may be defined by

\[ E_\pi P_\kappa = c_\kappa P_\kappa \]

with

\[ c_\kappa = q^{\langle \pi, \rho_k \rangle} \sum_{w \in W} q^{\langle w\pi, \kappa + \rho_k \rangle} = q^{\langle \pi, \rho_k \rangle} |W_\pi| m_\pi (\kappa + \rho_k), \]

and \( W_\pi \) the stabilizer of \( \pi \) in \( W \). On the other hand, \( E_\pi \) is given by

\[ E_\pi = \sum_{w \in W} w (\Phi_\pi T_\pi) = \sum_{w \in W} \prod_{\alpha \in R^+} \frac{1 - t_\alpha^{\langle \pi, \alpha \rangle} e^{w\alpha}}{1 - e^{w\alpha}} T_{w\pi}. \]

This can be written as

\[ E_\pi = q^{\langle \pi, \rho_k \rangle} |W_\pi| \sum_{\tau \in W_\pi} q^{-\langle \tau, \rho_k \rangle} \prod_{\alpha \in R^+} \frac{1 - t_\alpha^{\langle \tau, \alpha \rangle} e^{\alpha}}{1 - e^{\alpha}} T_{\tau}, \]

because for any \( w \in W \) we have

\[ \prod_{\alpha \in R^+} \frac{1 - t_\alpha^{\langle \pi, \alpha \rangle} e^{w\alpha}}{1 - e^{w\alpha}} = \prod_{\alpha \in R^+ \cap wR^+} \frac{1 - t_\alpha^{\langle w\pi, \alpha \rangle} e^{\alpha}}{1 - e^{\alpha}} \prod_{\alpha \in R^+ \cap wR^+} t_\alpha^{-\langle w\pi, \alpha \rangle} \frac{1 - t_\alpha^{\langle \pi, \alpha \rangle} e^{\alpha}}{1 - e^{\alpha}} \]

and

\[ q^{\langle \pi - w\pi, \rho_k \rangle} = \prod_{\alpha \in R^+ \cap -wR^+} t_\alpha^{-\langle w\pi, \alpha \rangle}. \]
Thus for \( x \in V \) the definition of \( P_\kappa \) writes as
\[
m_\kappa(\kappa + \rho_k) P_\kappa(x) = \sum_{\tau \in \mathcal{W}_\kappa} q^{-\langle \tau, \rho_k \rangle} \prod_{\alpha \in R^+} \frac{1 - t_\alpha^{\langle \tau, \alpha \rangle}}{1 - q^{\langle \alpha, x \rangle}} P_\kappa(x + \tau).
\]
Choosing \( x = \mu + \rho_k^\vee \), dividing both sides by \( P_\kappa(\rho_k^\vee) \) and making use of the symmetry property, we can conclude. \( \square \)

There is an analogous result when \( \omega \) is a quasi-minuscule weight. Then we have \( \Sigma(\omega) = \mathcal{W}_\omega \cup \{0\} \) and \( P_\omega = m_\omega + \text{constant}. \)

**Theorem 2.** Let \( \omega \in P^+ \) be a quasi-minuscule weight of \( R \). Then we have
\[
(P_\omega - P_\omega(\rho_k^\vee)) P_\lambda = \sum_{\tau \in \mathcal{W}_\omega} \left( C_\tau P_{\lambda + \tau} - D_\tau P_\lambda \right),
\]
with
\[
C_\tau = \prod_{\alpha \in R^+} \frac{1 - q^{\langle \lambda + \rho_k, \alpha \rangle} t_\alpha^{-1}}{1 - q^{\langle \lambda + \rho_k, \alpha \rangle}} \prod_{\alpha \in R^+} \frac{1 - q^{\langle \lambda + \rho_k, \alpha \rangle} t_\alpha^{-1}}{1 - q^{\langle \lambda + \rho_k, \alpha \rangle}}
\]

\[
D_\tau = q^{-\langle \tau, \rho_k^\vee \rangle} \prod_{\alpha \in R^+} \frac{1 - q^{\langle \lambda + \rho_k, \alpha \rangle} t_\alpha^{1}}{1 - q^{\langle \lambda + \rho_k, \alpha \rangle}} \prod_{\alpha \in R^+} \frac{1 - q^{\langle \lambda + \rho_k, \alpha \rangle} t_\alpha^{1}}{1 - q^{\langle \lambda + \rho_k, \alpha \rangle}}.
\]

Equivalently
\[
(P_\omega - P_\omega(\rho_k^\vee)) \tilde{P}_\lambda = \sum_{\tau \in \mathcal{W}_\omega} q^{-\langle \tau, \rho_k^\vee \rangle} \prod_{\alpha \in R^+} \frac{q^{\langle \lambda + \rho_k, \alpha \rangle} t_\alpha^{1}}{q^{\langle \lambda + \rho_k, \alpha \rangle}} \frac{q^{\langle \alpha, \tau \rangle}}{q^{\langle \alpha, \tau \rangle}} \tilde{P}_{\lambda + \tau} - \tilde{P}_\lambda,
\]
with \( \epsilon_\alpha \) the sign of \( \langle \tau, \alpha \rangle \).

**Proof.** The equivalence of both formulations results from the specialization formula
\[
q^{-\langle \tau, \rho_k^\vee \rangle} \frac{P_\lambda(\rho_k^\vee)}{P_{\lambda + \tau}(\rho_k^\vee)} = \prod_{\alpha \in R^+} \frac{q^{\langle \lambda + \rho_k, \alpha \rangle} t_\alpha^{1}}{q^{\langle \lambda + \rho_k, \alpha \rangle} t_\alpha^{1}},
\]


together with the fact that for \( \alpha \in R^+ \) we have \( \langle \tau, \alpha \rangle \in \{-2, -1, 0, 1, 2\} \), since \( W \) permutes roots.

It is enough to prove the second formulation for the dual root system \( R^\vee \). Let \( \pi \) a quasi-minuscule weight of \( R^\vee \) and \( \kappa \in P^+ \) arbitrary. The Macdonald polynomial \( P_\kappa \) may be defined by \( F_\pi P_\kappa = c'_\kappa P_\kappa \) with
\[
c'_\kappa = q^{\langle \pi, \rho_k \rangle} \sum_{w \in W} (q^{\langle w \pi, \kappa + \rho_k \rangle} - q^{\langle w \pi, \rho_k \rangle}) = q^{\langle \pi, \rho_k \rangle} |W_\pi| (m_\pi(\kappa + \rho_k) - m_\pi(\rho_k)).
\]
But $F_\pi$ is given by

$$F_\pi = \sum_{w \in W} w(\Phi_\pi (T_\pi - 1)) = \sum_{w \in W} \prod_{\alpha \in R^+} \frac{1 - t_\alpha^{(\pi,\alpha)} e^{w_\alpha}}{1 - e^{w_\alpha}} \prod_{\alpha \in R^+} \frac{1 - t_\alpha e^{w_\alpha}}{1 - q t_\alpha e^{w_\alpha}} (T_{w_\pi} - 1).$$

As in the proof of Theorem 1, the first product at the right-hand side is

$$\prod_{\alpha \in R^+} \frac{1 - t_\alpha^{(\pi,\alpha)} e^{w_\alpha}}{1 - e^{w_\alpha}} = q^{\langle \pi, - w_\pi, \rho_k \rangle} \prod_{\alpha \in R^+} \frac{1 - t_\alpha^{(w_\pi, \alpha)} e^{\alpha}}{1 - e^{\alpha}}.$$ 

The second product may be written

$$\prod_{\alpha \in R^+} \frac{1 - t_\alpha e^{w_\alpha}}{1 - t_\alpha^2 e^{w_\alpha}} \frac{1 - q t_\alpha e^{w_\alpha}}{1 - q e^{w_\alpha}} = \prod_{\gamma \in R^+} \frac{1 - t_\gamma^{\pm 1} e^\gamma}{1 - t_\gamma^{\pm 2} e^\gamma} \frac{1 - q^{\pm 1} t_\gamma^{\pm 1} e^\gamma}{1 - q^{\pm 1} t_\gamma^{\pm 2} e^\gamma}.$$ 

Finally, writing $\epsilon_\alpha$ for the sign of $\langle \tau, \alpha \rangle$, we have

$$F_\pi = q^{\langle \pi, \rho_k \rangle} |W_\pi| \sum_{\tau \in W_\pi} q^{-\langle \tau, \rho_k \rangle} \prod_{\alpha \in R^+} \frac{(t_\alpha^{\epsilon_\alpha}, q^{\epsilon_\alpha})_{\langle \pi, \alpha \rangle}}{(e^{\alpha}, q^{\epsilon_\alpha})_{\langle \pi, \alpha \rangle}} (T_{\tau} - 1).$$

The definition of $P_\kappa$ yields

$$(m_\pi(\kappa + \rho_k) - m_\pi(\rho_k)) P_\kappa(x) = \sum_{\tau \in W_\pi} q^{-\langle \tau, \rho_k \rangle} \prod_{\alpha \in R^+} \frac{(q^{\langle \alpha, x \rangle} t_\alpha^{\epsilon_\alpha}, q^{\epsilon_\alpha})_{\langle \pi, \alpha \rangle}}{(q^{\langle \alpha, x \rangle}, q^{\epsilon_\alpha})_{\langle \pi, \alpha \rangle}} (P_\kappa(x + \tau) - P_\kappa(x)).$$

Choosing $x = \mu + \rho_k^\vee$, with $\mu \in (P^\vee)^+$, dividing both sides by $P_\kappa(\rho_k^\vee)$ and making use of the symmetry property, we get

$$(m_\pi - m_\pi(\rho_k)) P_\mu = \sum_{\tau \in W_\pi} q^{-\langle \tau, \rho_k \rangle} \prod_{\alpha \in R^+} \frac{(q^{\langle \alpha, \mu + \rho_k^\vee \rangle} t_\alpha^{\epsilon_\alpha}, q^{\epsilon_\alpha})_{\langle \pi, \alpha \rangle}}{(q^{\langle \alpha, \mu + \rho_k^\vee \rangle}, q^{\epsilon_\alpha})_{\langle \pi, \alpha \rangle}} \left( \frac{P_\mu(\rho_k)}{P_{\mu + \tau}(\rho_k)} P_{\mu + \tau} - P_\mu \right).$$

We conclude by dividing both sides by $P_\mu(\rho_k)$. \square

4. The root system $C_\ell$

From now on we assume that $R = C_\ell$. We identify $V$ with $\mathbb{R}^n$ with the standard basis $\varepsilon_1, \ldots, \varepsilon_n$. Defining $x_i = e^{\varepsilon_i}$, $1 \leq i \leq n$, we regard Macdonald polynomials as Laurent polynomials of $n$ variables $x_1, \ldots, x_n$.

The set of positive roots is the union of short roots $R_1 = \{ \varepsilon_i \pm \varepsilon_j, 1 \leq i < j \leq n \}$ and long roots $R_2 = \{ 2 \varepsilon_i, 1 \leq i \leq n \}$. Throughout this paper we shall assume that the same parameter $t = q^k$ is associated with short and long roots. We have $\rho_k = k \sum_{i=1}^n (n-i+1) \varepsilon_i$ and $\rho_k^\vee = k \sum_{i=1}^n (n-i+\frac{1}{2}) \varepsilon_i$. 

9
The Weyl group $W$ is the semi-direct product of the permutation group $S_n$ by $(\mathbb{Z}/2\mathbb{Z})^n$. It acts on $V$ by signed permutation of components. The fundamental weights are given by $\omega_i = \sum_{j=1}^{n} \varepsilon_j$, $1 \leq i \leq n$. The dominant weights $\lambda \in P^+$ can be identified with vectors $\lambda = \sum_{i=1}^{n} \lambda_i \varepsilon_i$ such that $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a partition.

The dual root system $B_n$ has one minuscule weight $\pi = \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_n)$. Its $W$-orbit is formed by vectors $\frac{1}{2}(\sigma_1 \varepsilon_1 + \ldots + \sigma_n \varepsilon_n)$ with $\sigma \in (-1, +1)^n$. We have

$$\Phi_\pi = \prod_{i=1}^{n} \frac{1 - tx_i^2}{1 - x_i^2} \prod_{1 \leq i < j \leq n} \frac{1 - tx_i x_j}{1 - x_i x_j},$$

and the translation operator $T_\pi$ acts on $A$ by

$$T_\pi f(x_1, \ldots, x_n) = f(q^{\frac{1}{2}} x_1, \ldots, q^{\frac{1}{2}} x_n).$$

The Macdonald operator $E_\pi$ can be written as

$$E_\pi f = \sum_{\sigma \in (-1, +1)^n} \prod_{i=1}^{n} \frac{1 - tx_i^{2\sigma_i}}{1 - x_i^{2\sigma_i}} \prod_{1 \leq i < j \leq n} \frac{1 - tx_i^{\sigma_i} x_j^{\sigma_j}}{1 - x_i^{\sigma_i} x_j^{\sigma_j}} f(q^{\sigma_1/2} x_1, \ldots, q^{\sigma_n/2} x_n).$$

For any dominant weight $\lambda = \sum_{i=1}^{n} \lambda_i \varepsilon_i$, equivalently for any partition $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, the Macdonald polynomial $P_\lambda$ is defined, up to a constant, by

$$E_\pi P_\lambda = e_\lambda P_\lambda \quad \text{with} \quad e_\lambda = \prod_{i=1}^{n} (q^{\lambda_i/2} t^{n_i+1} + q^{-\lambda_i/2}).$$

The root system $C_n$ has one minuscule weight $\omega_1 = \varepsilon_1$ and one quasi-minuscule weight $\omega_2 = \varepsilon_1 + \varepsilon_2$.

For any partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ and any integer $1 \leq i \leq n$, we denote by $\lambda^{(i)}$ (resp. $\lambda^{(i)}$) the partition $\mu$ (if it exists) such that $\mu_j = \lambda_j$ for $j \neq i$ and $\mu_i = \lambda_i + 1$ (resp. $\mu_i = \lambda_i - 1$). By Theorem 1 the Pieri formula for $\omega_1$ writes as

$$P_{\varepsilon_1} P_\lambda = \sum_{k=1}^{n} (a_k^+ P_{\lambda^{(k)}} + a_k^- P_{\lambda^{(k)}}),$$

with

$$a_k^+ = \prod_{i=1}^{k-1} \frac{1 - q^{\lambda_i - \lambda_k} t^{k-i-1}}{1 - q^{\lambda_i - \lambda_k} t^{k-i}} \frac{1 - q^{\lambda_i - \lambda_k - 1} t^{k-i+1}}{1 - q^{\lambda_i - \lambda_k - 1} t^{k-i}},$$

$$a_k^- = \frac{1 - q^{\lambda_k} t^{n-k}}{1 - q^{\lambda_k} t^{n-k+1}} \frac{1 - q^{\lambda_k - 1} t^{n-k+2}}{1 - q^{\lambda_k - 1} t^{n-k+3}},$$

$$\times \prod_{\substack{i=1 \atop i \neq k}}^{n} \frac{1 - q^{\lambda_i + \lambda_k} t^{2n-i-k+1}}{1 - q^{\lambda_i + \lambda_k} t^{2n-i-k+2}} \frac{1 - q^{\lambda_i - \lambda_k - 1} t^{2n-i-k+3}}{1 - q^{\lambda_i - \lambda_k - 1} t^{2n-i-k+2}},$$

$$\times \prod_{i=k+1}^{n} \frac{1 - q^{\lambda_k - \lambda_i} t^{i-k-1}}{1 - q^{\lambda_k - \lambda_i} t^{i-k}} \frac{1 - q^{\lambda_k - \lambda_i - 1} t^{i-k+1}}{1 - q^{\lambda_k - \lambda_i - 1} t^{i-k}}.$$
5 A Pieri formula

In this paper we shall not study the general Pieri formula for $C_n$, which gives the explicit decomposition of the product $P_{rs}$, $P_\lambda$ with $r$ some positive integer, $1 \leq s \leq n$ and $\lambda$ any dominant weight.

We shall only consider the particular case where $s = 1$ and $\lambda$ is a multiple of $\omega_1$. Nevertheless this will produce a deep result.

For any dominant weight $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i$ we normalize Macdonald polynomials by

$$Q_\lambda = \prod_{1 \leq i \leq j \leq n} \frac{(q^{\lambda_i-\lambda_j} t^{j-i+1}; q)_{\lambda_j-\lambda_{j+1}}}{(q^{\lambda_i-\lambda_j+1} t^{-i}; q)_{\lambda_j-\lambda_{j+1}}} P_\lambda.$$  \hspace{1cm} (5.1)

Observe that the normalization factor is identical with the non combinatorial expression of $b_\lambda$, the factor appearing in the classical normalization $Q_\lambda = b_\lambda P_\lambda$ for Macdonald polynomials of type $A$ ( [15] pp. 338-339] and e.g. [24 Equ. 2.6]). For instance we have

$$Q_{\lambda_1 \varepsilon_1} = \frac{(t; q)_{\lambda_1}}{(q; q)_{\lambda_1}} P_{\lambda_1 \varepsilon_1}, \quad Q_{\lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2} = \frac{(t; q)_{\lambda_1-\lambda_2}}{(q; q)_{\lambda_1-\lambda_2}} \frac{(t; q)_{\lambda_1}}{(q; q)_{\lambda_1}} \frac{(q^{\lambda_1-\lambda_2+2} t^2; q)_{\lambda_2}}{(q^{\lambda_1-\lambda_2} t^2; q)_{\lambda_2}} P_{\lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2}.$$

We shall prove the following remarkable Pieri formula.

**Theorem 3.** For any partition $\lambda = (\lambda_1, \lambda_2)$ we have

$$Q_{\lambda_1 \varepsilon_1} Q_{\lambda_2 \varepsilon_1} = \sum_{(i,j) \in \mathbb{N}^2} c_{ij}(\lambda_1, \lambda_2) Q_{(\lambda_1+i-j) \varepsilon_1 + (\lambda_2-i-j) \varepsilon_2},$$

with

$$c_{ij}(\lambda_1, \lambda_2) = \frac{(t; q)_i}{(q; q)_i} \frac{(t; q)_j}{(q; q)_j} \frac{(q^{\lambda_1-\lambda_2+i+j+1}; q)_i}{(q^{\lambda_1-\lambda_2+i+j}; q)_i} \frac{(q^{\lambda_1-\lambda_2-i-j+1}; q)_j}{(q^{\lambda_1-\lambda_2-i-j}; q)_j}.$$

Here two remarks are needed. Firstly, in [20] and [11, Theorem 4] it was shown independently that the generating function of the polynomials $Q_{r \varepsilon_1}$ is given by

$$\sum_{r \in \mathbb{N}} u^r Q_{r \varepsilon_1} = \prod_{i=1}^n \frac{(tux_i; q)_\infty}{(ux_i; q)_\infty} \frac{(tu/x_i; q)_\infty}{(u/x_i; q)_\infty}.$$  \hspace{1cm} (5.2)

By an obvious two-fold product, Theorem 3 amounts to give the explicit Macdonald development of

$$\prod_{i=1}^n \prod_{j=1}^2 \frac{(tju_i; q)_\infty}{(uj_i; q)_\infty} \frac{(tu_j/x_i; q)_\infty}{(u_j/x_i; q)_\infty},$$

which is reminiscent of the celebrated Cauchy formula [15] (4.13), p. 324] for Macdonald polynomials of type $A$. 

11
Secondly, the Pieri coefficients \( c_{ij}(\lambda_1, \lambda_2) \) appear to be fully factorized. One may wonder whether this remarkable property keeps verified in the general case

\[
P_{r_1} P_\lambda = \sum_{\tau \in \Sigma(r_1)} C_\tau P_{\lambda+\tau},
\]

with \( \lambda \) arbitrary. However this is not true.

Actually computer calculations, performed for \( n = 2 \), show that for \( r = 2 \) all coefficients \( C_\tau \) factorize but \( C_0 \). For \( r = 3 \) all \( C_\tau \) factorize but those with \( \tau \in \{ \pm \varepsilon_1, \pm \varepsilon_2 \} \). For \( r = 4 \) all coefficients factorize but those with \( \tau \in \{ \pm 2\varepsilon_1, \pm 2\varepsilon_2, \pm \varepsilon_1 \pm \varepsilon_2 \} \).

In general when \( \tau \) belongs to the boundary of the convex hull of the Weyl group orbit of \( r\varepsilon_1 \), \( C_\tau \) is fully factorized. For values of \( \tau \) inside this convex hull, we expect \( C_\tau \) to write as a sum of factorized terms, the number of which increases according to the distance of \( \tau \) to the boundary.

Of course the above remarks are only valid if the dominant weight \( \lambda \) keeps generic.

For specific values of \( \lambda \), the Pieri coefficients may all factorize. Theorem 3 shows that it is the case for \( \lambda = s\omega_1 \), but we also conjecture the property for \( \lambda = s\omega_2 \).

6 Proof of Theorem 3

The proof relies on the following property giving the action of the Macdonald operator \( E_\pi \) on a product \( Q_{\lambda_1 \varepsilon_1} Q_{\lambda_2 \varepsilon_1} \). This result will be proved in Section 10. Obviously for \( \lambda_2 = 0 \) we recover (4.2).

**Theorem 4.** For any partition \( \lambda = (\lambda_1, \lambda_2) \) we have

\[
E_\pi(Q_{(\lambda_1)}Q_{(\lambda_2)}) = e_\lambda Q_{(\lambda_1)}Q_{(\lambda_2)} + (1 - t) \prod_{i=1}^{n-2} (t^i + 1)
\times \left( q^{-\frac{1}{2}(\lambda_1-\lambda_2)} t^{n-1} \sum_{k=1}^{\lambda_2} q^{-k} (1 - q^{2k+\lambda_1-\lambda_2}) Q_{(\lambda_1+k)}Q_{(\lambda_2-k)} - q^{-\frac{1}{2}(\lambda_1+\lambda_2)} \sum_{k=1}^{\lambda_2} t^{k-1} (1 - q^{2k+\lambda_1+\lambda_2 t^{2n}}) Q_{(\lambda_1-k)}Q_{(\lambda_2-k)} \right).
\]

**Proof of Theorem 3.** It is done by induction on \( \lambda_2 \). Firstly we consider the case \( \lambda_2 = 1 \). Writing (4.3) for \( \lambda = (r) \), we obtain

\[
Q_{(1)}Q_{(r)} = Q_{(r,1)} + \frac{1 - t}{1 - q} \frac{t^{r+1}}{1 - q^r t} Q_{(r+1)} + \frac{1 - t}{1 - q^r t^{2n}} \frac{1 - q}{1 - q^r t^{2n-1}} Q_{(r-1)},
\]

which proves Theorem 3 for \( \lambda_2 = 1 \).
In a second step, assuming the property true for $\lambda_2 \leq s - 1$ we prove it for $\lambda_2 = s$. For this purpose we write the Macdonald development of $Q(r)Q(s)$ as

$$Q(r)Q(s) = \sum_{\mu \in P^+} c_\mu(r, s) Q_\mu.$$ 

Applying $E_\pi$, Theorem 4 yields

$$\sum_{\mu \in P^+} c_\mu(r, s) (e_\mu - e_{(r,s)}) Q_\mu = (1 - t) \prod_{i=1}^{n-2} (t^i + 1)$$

$$\times \left( q^{-\frac{1}{2}(r-s)} t^{n-1} \sum_{k=1}^{s} q^{-k} (1 - q^{2k+r-s}) Q_{(r+k)}Q_{(s-k)} ight)$$

$$- q^{-\frac{1}{2}(r+s)} \sum_{k=1}^{s} t^{k-1} (1 - q^{2k+r+s} t^{2n}) Q_{(r-k)}Q_{(s-k)}.$$

Then we apply the inductive hypothesis to the products of Macdonald operators on the right-hand side. We obtain at once that the partitions $\mu$ are of the form $(r+i-j, s-i-j)$ with $i+j \leq s$.

Moreover by identification of coefficients we have

$$c_{i,j}(r, s) (e_{(r+i-j,s-i-j)} - e_{(r,s)}) = (1 - t) \prod_{i=1}^{n-2} (t^i + 1)$$

$$\times \left( q^{-\frac{1}{2}(r-s)} t^{n-1} \sum_{k=1}^{i} q^{-k} (1 - q^{2k+r-s}) c_{i-k,j}(r+k, s-k) ight)$$

$$- q^{-\frac{1}{2}(r+s)} \sum_{k=1}^{j} t^{k-1} (1 - q^{2k+r+s} t^{2n}) c_{i,j-k}(r-k, s-k).$$

Therefore putting $a = q^{r-s}$ and $b = q^{r+s} t^{2n}$, we have only to prove the identity

$$\frac{(t; q)_i (t; q)_j (aq^{i+1}; q)_i (bq^{-j-1}; 1/q)_j}{(q; q)_i (q; q)_j (aq^i t; q)_i (bq^{-j}/t; 1/q)_j}$$

$$\left( b^{1/2} q^{-j}/t + a^{1/2} q^i + a^{-1/2} q^{-i}/t + b^{-1/2} q^j - b^{1/2}/t - a^{1/2} - a^{-1/2}/t - b^{-1/2} \right) =$$

$$(1 - t)/t \left( a^{-1/2} \sum_{k=1}^{i} q^{-k} (1 - aq^{2k}) \frac{(t; q)_i (t; q)_j (aq^{i+k+1}; q)_i (bq^{-j-1}; 1/q)_j}{(q; q)_i (q; q)_j (aq^{i+k} t; q)_i (bq^{-j}/t; 1/q)_j} ight.$$

$$- b^{-1/2} \sum_{k=1}^{j} t^{k} (1 - bq^{-2k}) \frac{(t; q)_i (t; q)_j (aq^{i+1}; q)_i (bq^{-j-k-1}; 1/q)_j}{(q; q)_i (q; q)_j (aq^{i} t; q)_i (bq^{-j-k}/t; 1/q)_j}.$$
The latter is a consequence of the stronger result

\[
\frac{(t; q)_i (aq^{i+1}; q)_i}{(q; q)_i (aq^i t; q)_i} \left( a^{1/2}q^i + a^{-1/2}q^{-i}/t - a^{1/2} - a^{-1/2}/t \right) = (1 - t)a^{-1/2}/t \sum_{k=1}^{i} q^{-k}(1 - aq^{2k}) \frac{(t; q)_{i-k} (aq^{i+k+1}; q)_{i-k}}{(q; q)_{i-k} (aq^{i+k} t; q)_{i-k}},
\]

and its analog obtained by substituting \((b, j, 1/q, 1/t)\) to \((a, i, q, t)\).

This identity is easily proved, because it can be transformed into

\[
\sum_{k=1}^{i} q^{i-k}(1 - aq^{2k}) \frac{(q^{i-k+1}; q)_k}{(q^{i-kt}; q)_k} \frac{(aq^i t; q)_k}{(aq^{i+1}; q)_k} = \frac{1 - q^i}{1 - t} (1 - aq^i t),
\]

which is the classical summation formula

\[
\sum_{k=1}^{i} q^{i-k}(1 - aq^{2k}) \frac{(q^{i-k+1}; q)_k}{(q^{i-kt}; q)_k} \frac{(aq^i t; q)_k}{(aq^{i+1}; q)_k} = \frac{1 - q^i}{1 - a} \frac{1 - q^{-i}/t}{1 - 1/t},
\]

for a terminating very-well-poised \(_6\phi_5\) basic hypergeometric series \([5, (2.4.2)]\).

\[\square\]

7 An inverse Pieri formula

In view of (5.2), the following result completely determines the “two-row” Macdonald polynomials of type \(C_n\) (hence all Macdonald polynomials for \(C_2\)).

**Theorem 5.** For any partition \(\lambda = (\lambda_1, \lambda_2)\) we have

\[
Q_{\lambda \in \lambda_1 + \lambda_2} = \sum_{(i, j) \in \mathbb{N}^2 \atop 0 \leq i + j \leq \lambda_2} C_{ij}(\lambda_1, \lambda_2) Q_{(\lambda_1 + i - j) \in \lambda_1} Q_{(\lambda_2 - i - j) \in \lambda_1},
\]

with

\[
C_{ij}(\lambda_1, \lambda_2) = t^{i+j} \frac{(1/t; q)_i (1/t; q)_j}{(q; q)_i (q; q)_j} \frac{(q^{\lambda_1 - \lambda_2 + 1}; q)_i}{(q^{\lambda_1 - \lambda_2 + 1} t; q)_i} \frac{1 - q^{\lambda_1 - \lambda_2 + 2i}}{1 - q^{\lambda_1 - \lambda_2 + i}} \times \frac{(q^{\lambda_1 + \lambda_2 - 2j}; 1/q)_j}{(q^{\lambda_1 + \lambda_2 - 2j 2^n}; 1/q)_j} \frac{1 - q^{\lambda_1 + \lambda_2 - 2j 2^n}}{1 - q^{\lambda_1 + \lambda_2 - 2j}}.
\]

Starting from Theorem 3, the proof is obtained by inverting infinite multi-dimensional matrices.

An infinite one-dimensional matrix \((f_{ij})_{i, j \in \mathbb{Z}}\) is said to be lower-triangular if \(f_{ij} = 0\) unless \(i \geq j\). Two infinite lower-triangular matrixes \((f_{ij})_{i, j \in \mathbb{Z}}\) and \((g_{kl})_{k, \ell \in \mathbb{Z}}\) are said to be mutually inverse if \(\sum_{i \geq j \geq k} f_{ij} g_{jk} = \delta_{ik}\).
The simplest case of such a pair is given by Bressoud’s matrix inverse [1], which states
that, defining

\[ A_{ij}(u, v) = (u/v)^i \frac{(u/v; q)_{i-j}(u; q)_{i+j}}{(q; q)_{i-j}(vq; q)_{i+j}} \frac{1-vq^{2j}}{1-v}, \]

the matrices \( A(u, v) \) and \( A(v, u) \) are mutually inverse. We refer to [7] for a generalization
of [1] and to [11] for some applications.

An equivalent formulation of Bressoud’s matrix inverse is obtained by considering the
matrices

\[ f_{ij} = (u/v)^i \frac{(u; q)_{2j}}{(vq; q)_{2j}} \frac{1-vq^{2i}}{1-v} A_{ij}(v, u), \]

\[ g_{kl} = (v/u)^l \frac{(vq; q)_{2k}}{(u; q)_{2k}} \frac{1-vq^{2l}}{1-v} A_{kl}(u, v), \]

which are mutually inverse because we have

\[ f_{ij} g_{kl} = \delta_{jl}, \]

for all \( j, l \in \mathbb{Z}^2 \), where \( \delta_{jl} \) is the usual Kronecker symbol.

Starting from the one-dimensional Bressoud’s pair with \( u/v = t \), and taking a two-fold
product, we easily obtain the following pair of mutually inverse two-dimensional matrices

\[ f_{jk} = t^{j_1-k_1} \frac{(1/t; q)_{j_1-k_1}(u_1q^{2k_1}; q)_{j_1-k_1}}{(q; q)_{j_1-k_1}(tu_1q^{2k_1+1}; q)_{j_1-k_1}} \frac{1-u_1q^{2j_1}}{1-u_1q^{2k_1}} \]

\[ \times t^{j_2-k_2} \frac{(1/t; q)_{j_2-k_2}(u_2q^{2k_2}; q)_{j_2-k_2}}{(q; q)_{j_2-k_2}(tu_2q^{2k_2+1}; q)_{j_2-k_2}} \frac{1-u_2q^{2j_2}}{1-u_2q^{2k_2}}, \]

\[ g_{kl} = (t; q)_{k_1-l_1} \frac{(u_1q^{k_1+l_1+1}; q)_{k_1-l_1}}{(q; q)_{k_1-l_1}(tu_1q^{k_1+l_1}; q)_{k_1-l_1}} t^{k_2-l_2} \frac{(u_2q^{k_2+l_2+1}; q)_{k_2-l_2}}{(q; q)_{k_2-l_2}(tu_2q^{k_2+l_2}; q)_{k_2-l_2}}. \]
Proof of Theorem 5. For \( j = (j_1, j_2) \) and \( k = (k_1, k_2) \), we define
\[
\begin{align*}
a_j &= Q_{(\lambda_1 + j_1 - j_2)\varepsilon_1} \, Q_{(\lambda_2 - j_1 - 2)\varepsilon_1}, \\
b_k &= Q_{(\lambda_1 + k_1 - 2)\varepsilon_1 + (\lambda_2 - k_1 - 2)\varepsilon_2}.
\end{align*}
\]
If in Theorem 3 we perform the substitutions \( \lambda_1 \mapsto \lambda_1 + l_1 - l_2 \), \( \lambda_2 \mapsto \lambda_2 - l_1 - l_2 \), we easily obtain
\[
a_l = \sum_{k \geq l} (t/q)^{k_2 - l_2} g_{kl} \, b_k,
\]
with \( u_1 = q^{\lambda_1 - \lambda_2} \) and \( u_2 = q^{-\lambda_1 - \lambda_2 t^{-2n}} \). This yields
\[
b_k = \sum_{j \geq k} (t/q)^{j_2 - k_2} f_{jk} \, a_j.
\]
We conclude by the substitution \( k_1 = k_2 = 0 \).

Our expansion of \( Q_{\lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2} \) is very similar to the one obtained by Jing and Józefiak [6] for the two-row Macdonald polynomials associated with \( A_n \). The latter is also a consequence of Bressoud’s matrix inverse.

For \( t = q \) we recover the following classical result due to Hermann Weyl [25].

**Corollary.** The irreducible characters \( \chi_{\lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2} \) of the symplectic group \( \text{Sp}(2n, \mathbb{C}) \) are given by
\[
\chi_{\lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2} = h_{\lambda_1} h_{\lambda_2} + h_{\lambda_1} h_{\lambda_2 - 2} - h_{\lambda_1 + 1} h_{\lambda_2 - 1} - h_{\lambda_1 - 1} h_{\lambda_2 - 1},
\]
with \( h_k \) the complete functions defined by
\[
\sum_{k \geq 0} u^k h_k = \prod_{i=1}^{n} \frac{1}{(1 - u x_i)(1 - u/x_i)}.
\]

**Proof.** If \( t = q \) we have \( Q_{r \varepsilon_1} = h_r \) and \( P_{\lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2} = Q_{\lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2} \). In the summation of Theorem 5, the only non-zero contributions correspond to \( i, j = 0, 1 \).

\[\square\]

### 8 A rational identity

It remains to prove Theorem 4, which will be done in Section 10. For that purpose, two ingredients will be needed. Firstly the language of \( \lambda \)-rings, a powerful way to handle series, which we briefly recall in Section 9. Secondly a remarkable rational identity, which we present in this section.

This multivariate identity depends on two sets of indeterminates \( x = (x_1, \ldots, x_n) \) and \( u = (u_1, \ldots, u_r) \). In this paper we shall only use it for \( r = 2 \). However in view of its own interest, we prove it in full generality.

By specialization of the indeterminates \( x \) or \( u \), we may obtain basic \( q \)-hypergeometric identities. We present some examples of the latter at the end of this paper, in Section 12.
We start from the following identity, which was proved independently in [20, Lemma 2] and [11, Theorem 5]. Let \( x = (x_1, \ldots, x_n) \) and \( u \) be \( n+1 \) indeterminates. We have

\[
\sum_{\sigma \in (-1,+1)^n} \prod_{i=1}^n \frac{1 - tx_i^{2\sigma_i}}{1 - x_i^{2\sigma_i}} \prod_{1 \leq i < j \leq n} \frac{1 - tx_i^{\sigma_i} x_j^{\sigma_j}}{1 - x_i^{\sigma_i} x_j^{\sigma_j}} = \prod_{i=1}^{n-1} (t^i + 1) \left( t^n + \prod_{i=1}^n \frac{1 - tx_i}{1 - ux_i} \frac{1 - tu/x_i}{1 - u/x_i} \right).
\]

Recalling the definition (4.1) of \( \Phi_{\pi} \), writing \( T_i \) for the operator \( x_i \rightarrow 1/x_i \) and \( T = (1 + T_1) \cdots (1 + T_n) \), this identity can be written more compactly

\[
T \left( \Phi_{\pi} \prod_{i=1}^n \frac{1 - tu/x_i}{1 - u/x_i} \right) = \prod_{i=1}^{n-1} (t^i + 1) (t^n + R(u)), \tag{8.1}
\]

with

\[
R(u) = \prod_{i=1}^n \frac{1 - tx_i}{1 - ux_i} \frac{1 - tu/x_i}{1 - u/x_i}.
\]

In particular for \( u = 0 \) we have

\[
T (\Phi_{\pi}) = \prod_{i=1}^n (t^i + 1). \tag{8.2}
\]

Both properties correspond to the cases \( r = 0, 1 \) of the following result.

**Theorem 6.** Let \( x = (x_1, \ldots, x_n) \) and \( u = (u_1, \ldots, u_r) \). We have

\[
T \left( \Phi_{\pi} \prod_{k=1}^r \prod_{i=1}^n \frac{1 - tu_k/x_i}{1 - u_k/x_i} \right) = (-t; t)_{n-r} \sum_{I \subseteq \{1,\ldots, r\}} c_I \prod_{i \in I} R(u_i), \tag{8.3}
\]

with

\[
c_I = i^{n(|I|-|I|)} \prod_{1 \leq i < j \leq r} \frac{1 - v_i v_j}{1 - tv_i v_j},
\]

defining \( v_i = u_i \) if \( i \in I \) and \( v_i = 1/tu_i \) if \( i \notin I \).

**Remark:** The value of the factor \((-t; t)_{n-r}\) is \( \prod_{k=1}^{n-r} (t^k + 1) \) or \( \prod_{k=0}^{r-n-1} (t^{-k} + 1)^{-1} \) according to the sign of \( n - r \).

**Proof.** (i) **First case:** \( r \leq n \). The proof is done by induction on \( r \), starting from the case \( r = 1 \) given by (7.1). Both sides of the identity are rational functions of \( u_r \) having poles firstly at \( u_r = x_i \) and \( u_r = 1/x_i \) for \( i = 1, \ldots, n \), secondly at \( u_r = u_i \) and \( u_r = 1/tu_i \) for \( i = 1, \ldots, r - 1 \).
Their constant terms are equal as a consequence of the inductive hypothesis. Actually, specifying the dependence on \( r \), for any set \( I \subset \{1, \ldots, r-1\} \) we have

\[
(c_I^{(r)} + c_{I^c r}^{(r)})|_{u_r=0} = (t^{n-r+1} + 1)c_I^{(r-1)}.
\]

Thus at \( u_r = 0 \) we obtain the identity written for \( r - 1 \). Therefore it is sufficient to prove that both sides of the identity have the same residue at each of their poles.

In a first step, we consider the poles \( u_r = x_i \) and \( u_r = 1/x_i \). By symmetry, the equality of residues has only to be checked for some \( x_i \), say at \( u_r = x_n \) or \( u_r = 1/x_n \). We shall only do it at \( u = x_n \), the proof at \( u = 1/x_n \) being similar.

For any indeterminates \((a_1, \ldots, a_m)\) we have

\[
\prod_{i=1}^{m} \frac{tu - a_i}{u - a_i} = t^m + (t - 1) \sum_{i=1}^{m} \frac{a_i}{u - a_i} \prod_{j=1, j \neq i}^{m} \frac{t a_i - a_j}{a_i - a_j}.
\]

This decomposition as a sum of partial fractions is actually a Lagrange interpolation [9, Section 7.8]. We first apply it to \( R(u) \). Its residue at \( u = x_n \) is given by

\[
x_n(t-1) \prod_{j=1}^{n-1} \frac{tx_n - x_j}{x_n - x_j} \prod_{j=1}^{n} \frac{1 - tx_n x_j}{1 - x_n x_j}.
\]

The residue of the right-hand side at \( u_r = x_n \) is therefore

\[
x_n(t-1) \prod_{k=1}^{n-r} (t^k + 1) \left( \sum_{I \subset \{1, \ldots, r-1\}} c_{I^c r} |_{u_r=x_n} \prod_{i \in I} R(u_i) \right) \prod_{j=1}^{n-1} \frac{tx_n - x_j}{x_n - x_j} \prod_{j=1}^{n} \frac{1 - tx_n x_j}{1 - x_n x_j}.
\]

Then we apply the Lagrange interpolation to

\[
\prod_{i=1}^{n} \frac{tu_r - x_i^{\sigma_i}}{u_r - x_i^{\sigma_i}},
\]

on the left-hand side of the identity. Only fractions with \( \sigma_n = 1 \) contribute to the residue at \( u_r = x_n \). Thus it can be written as

\[
x_n(t-1) \frac{1 - tx_n^2}{1 - x_n^2} \prod_{k=1}^{n-1} \frac{1 - tu_k/x_n}{1 - u_k/x_n} \sum_{\sigma \in (-1,+1)^{n-1}} \left( \prod_{k=1}^{n-1} \prod_{l=1}^{n-1} \frac{1 - tu_k x_l^{-\sigma_l}}{1 - u_k x_l^{-\sigma_l}} \right) \times \prod_{i=1}^{n-1} \frac{1 - tx_i^{2\sigma_i}}{1 - x_i^{2\sigma_i}} \frac{1 - tx_n x_i^{-\sigma_i}}{1 - x_n x_i^{-\sigma_i}} \frac{1 - tx_i^{\sigma_i} x_n}{1 - x_i^{\sigma_i} x_n} \prod_{1 \leq i < j \leq n-1} \frac{1 - x_i^{\sigma_i} x_j^{\sigma_j}}{1 - x_i^{\sigma_i} x_j^{\sigma_j}}.
\]

Now the product

\[
\prod_{i=1}^{n-1} \frac{1 - tx_n x_i}{1 - x_n x_i} \frac{1 - tx_n/x_i}{1 - x_n/x_i},
\]

18
is obviously invariant under any $T_i$, and can be cancelled on both sides. Therefore, writing $x_n = z$, we are led to prove the following equality, for $r - 1$ indeterminates ($u_1, \ldots, u_{r-1}$) and $n - 1$ variables ($x_1, \ldots, x_{n-1}$),

$$
\prod_{k=1}^{n-r} (t^k + 1) \left( \sum_{I \subset \{1, \ldots, r-1\}} c_I \left|_{u_r} = z \right. \prod_{i \in I} \left( \frac{1 - tu_i z}{1 - u_i z} \frac{1 - tu_i/z}{1 - u_i/z} R(u_i) \right) \right)
= \prod_{k=1}^{r-1} \frac{1 - tu_k/z}{1 - u_k/z} T \left( \Phi \prod_{k=1}^{r-1} \prod_{i=1}^{n-1} \frac{1 - tu_k/x_i}{1 - u_k/x_i} \right).
$$

Specifying the dependence on $n, r$ it is equivalent to check that for any $I \subset \{1, \ldots, r-1\}$ we have

$$
c_I^{(n,r)} = c_I^{(n-1,r-1)} \prod_{i \in I} \frac{1 - u_i u_r}{1 - tu_i u_r} \prod_{j \notin I} \frac{1 - tu_j/u_r}{1 - u_j/u_r},
$$

which is obvious.

In a second step, we consider the values $u_r = u_i$ and $u_r = 1/tu_i$ for $i = 1, \ldots, r - 1$, which are poles for the right-hand side of the identity, but not for its left-hand side. Therefore we have to prove that their residue is zero. The proofs being similar, we shall only do it for $u_r = 1/tu_i$.

The only $c_I$ having a pole at $u_r = 1/tu_i$ correspond to sets $I$ such that $r, i \in I$ or $r, i \notin I$. The sum of both contributions brings a factor

$$
t^{n(r-|I|)-1} \frac{1 - t^2 u_i u_r}{1 - tu_i u_r} + t^{n(r-|I|-2)} \frac{1 - u_i u_r}{1 - tu_i u_r} R(u_i) R(u_r),
$$

for any set $I \subset \{1, \ldots, r-2\}$. Since $R(u_i) R(1/tu_i) = t^{2n}$, its residue is

$$
t - 1 \frac{1}{tu_i} \left( t^{n(r-|I|-1)} - t^{n(r-|I|-2)} t^{2n}/t \right) = 0.
$$

Summing all contributions, the residue of the right-hand side at $u_r = 1/tu_i$ is zero. This achieves the proof for $r \leq n$.

(ii) Second case: $n < r$. The proof is strictly identical to the previous one except that, once (7.4) obtained, it is done by induction on $n$ rather than on $r$. Therefore it remains to prove the identity for $n = 0$, which writes as

$$
\sum_{I \subset \{1, \ldots, r\}} c_I = \prod_{k=0}^{r-1} (t^{-k} + 1).
$$

The left-hand side is a rational function of $u_r$ having poles at $u_r = u_i$ and $u_r = 1/tu_i$ for $i = 1, \ldots, r - 1$. Exactly as above, its residues at these poles are shown to be zero. Thus it only remains to show (7.5) at $u_r = 0$. 

19
This is done by induction on \( r \) starting from the obvious case \( r = 1 \). Specifying the dependence on \( n \), for any \( I \subset \{1, \ldots, r - 1\} \) we have

\[
(c^{(r)}_I + c^{(r)}_{I \cup \{r\}})|_{u_r = 0} = (t^{1-r} + 1)c^{(r-1)}_I.
\]

Therefore at \( u_r = 0 \) we obtain (7.5) written for \( r - 1 \). \( \square \)

In this paper we shall only use Theorem 6 for \( r = 2 \).

**Corollary.** For any two indeterminates \( u, v \), we have

\[
T \left( \Phi \prod_{i=1}^n \frac{1-tu/x_i}{1-u/x_i} \frac{1-tv/x_i}{1-v/x_i} \right) = \prod_{i=1}^{n-2} (t^i + 1) \times \left( t^{2n-1} \frac{1-t^2uv}{1-tuv} + t^n \frac{u-tv}{u-v} R(u) + t^{n-1} \frac{v-tu}{v-u} R(v) + \frac{1-uv}{1-tuv} R(u) R(v) \right).
\]

(8.6)

### 9 \( \lambda \)-rings

In the next section, Theorem 4 will be proved by using the powerful language of \( \lambda \)-rings. Here we only give a short survey of this theory. More details and other applications may be found, for instance, in [9, 10] and in some examples of [15] (see pp. 25, 43, 65 and 79).

Let \( A = \{a_1, a_2, a_3, \ldots\} \) be a (finite or infinite) set of independent indeterminates, called an alphabet. We introduce the generating functions

\[
E_u(A) = \prod_{a \in A} (1 + ua), \quad H_u(A) = \prod_{a \in A} \frac{1}{1-ua}, \quad P_u(A) = \sum_{a \in A} \frac{a}{1-ua},
\]

whose development defines symmetric functions known as elementary functions \( e_k(A) \), complete functions \( h_k(A) \), and power sums \( p_k(A) \), respectively. Each of these three sets generate algebraically the symmetric algebra \( S(A) \).

We define an action \( f \rightarrow f[\cdot] \) of \( S(A) \) on the ring \( R[A] \) of polynomials in \( A \) with real coefficients. Since the power sums \( p_k \) generate \( S(A) \), it is enough to define the action of \( p_k \) on \( R[A] \). Writing any polynomial as \( \sum c_P cP \), with \( c \) a real constant and \( P \) a monomial in \( (a_1, a_2, a_3, \ldots) \), we define

\[
p_k \left[ \sum c_P cP \right] = \sum cP^k.
\]

This action extends to \( S[A] \). For instance we obtain

\[
E_u \left[ \sum c_P cP \right] = \prod_{c_P} (1 + uP)^c, \quad H_u \left[ \sum c_P cP \right] = \prod_{c_P} (1 - uP)^{-c}.
\]

More generally, we can define an action of \( S(A) \) on the ring of rational functions, and even on the ring of formal series, by writing

\[
p_k \left( \frac{\sum cP}{\sum dQ} \right) = \frac{\sum cP^k}{\sum dQ^k}.
\]
with $c, d$ real constants and $P, Q$ monomials in $(a_1, a_2, a_3, \ldots)$. This action still extends to $S(A)$.

If we write $A^t = \sum_i a_i$, we have $p_k[A^t] = \sum_i a_i^k$ by definition. Thus $p_k[A^t] = p_k(A)$, which yields that for any symmetric function $f$, we have $f[A^t] = f(A)$. In particular

$$f(1, q, q^2, \ldots, q^{m-1}) = f\left[\frac{1 - q^m}{1 - q}\right], \quad f(1, q, q^2, q^3, \ldots) = f\left[\frac{1}{1 - q}\right].$$

Moreover for any formal series $P, Q$ we have

$$h_r[P + Q] = \sum_{k=0}^r h_{r-k}[P] h_k[Q], \quad e_r[P + Q] = \sum_{k=0}^r e_{r-k}[P] e_k[Q].$$

Or equivalently

$$H_u[P + Q] = H_u[P] H_u[Q], \quad E_u[P + Q] = E_u[P] E_u[Q]$$

$$H_u[P - Q] = H_u[P] H_u[Q]^{-1}, \quad E_u[P - Q] = E_u[P] E_u[Q]^{-1}.$$

As an application, for a finite alphabet $X = \{x_1, x_2, \ldots, x_m\}$ and two indeterminates $a, q$ we may write

$$H_u\left[\frac{aX^t}{1 - q}\right] = \prod_{i \geq 0} H_u[\frac{aq^i X^t}{1 - q}] = \prod_{k=1}^m \prod_{i \geq 0} H_u[\frac{aq^i x_k}{1 - auq^i x_k}] = \prod_{k=1}^m \frac{1}{(ax_k; q)_\infty}.$$

Since

$$H_u\left[\frac{1 - t}{1 - q} X^t\right] = H_u\left[\frac{X^t}{1 - q}\right] \left(H_u\left[\frac{tX^t}{1 - q}\right]\right)^{-1},$$

we obtain

$$H_u\left[\frac{1 - t}{1 - q} X^t\right] = \sum_{r \geq 0} u^r h_r\left[\frac{1 - t}{1 - q} X^t\right] = \prod_{i=1}^m \frac{(tx_i; q)_\infty}{(ux_i; q)_\infty}.$$

## 10 Action of the Macdonald operator

We are now in a position to prove Theorem 4. Recall that in [20] and [11, Theorem 4] it was shown independently that the generating function of the polynomials $Q_{(r)}$ is given by

$$\sum_{r \in \mathbb{N}} u^r Q_{(r)} = \prod_{i=1}^n \frac{(tu/x_i; q)_\infty}{(ux_i; q)_\infty} \frac{(tu/x_i; q)_\infty}{(u/x_i; q)_\infty} = H_u\left[\frac{1 - t}{1 - q} X^t\right],$$

with $X = \{x_1, \ldots, x_n\} \cup \{1/x_1, \ldots, 1/x_n\}$, so that $X^t = \sum_{i=1}^n (x_i + 1/x_i)$. 

21
Proof of Theorem 4. We have to compute the action of the Macdonald operator $E_\pi$ on products $Q_{\lambda_1}Q_{\lambda_2}$, hence the generating function

$$\sum_{\lambda_1, \lambda_2 \in \mathbb{N}} u^{\lambda_1} v^{\lambda_2} E_\pi(Q_{\lambda_1}Q_{\lambda_2}) = E_\pi \left( H_u \left[ \frac{1-t}{1-q} \right] H_v \left[ \frac{1-t}{1-q} \right] \right).$$

By the definition of $E_\pi$ and $X^\dagger$, this is

$$\sum_{\sigma \in \{-1,+1\}^n} \prod_{i=1}^n \frac{1-tx_i^{2\sigma_i}}{1-x_i^{2\sigma_i}} \prod_{1 \leq i < j \leq n} \frac{1-tx_i^{\sigma_i}x_j^{\sigma_j}}{1-x_i^{\sigma_i}x_j^{\sigma_j}} \times H_u \left[ \frac{1-t}{1-q} \sum_{i=1}^n \left( x_i q^{\sigma_i/2} + \frac{1}{x_i q^{\sigma_i/2}} \right) \right] H_v \left[ \frac{1-t}{1-q} \sum_{i=1}^n \left( x_i q^{\sigma_i/2} + \frac{1}{x_i q^{\sigma_i/2}} \right) \right].$$

Now we have the relations

$$xq^{\sigma/2} + \frac{1}{xq^{\sigma/2}} = q^{\frac{\sigma}{2}} \left( x + \frac{1}{x} \right) + q^{-\frac{\sigma}{2}} (1-q)x^{-\sigma},$$

which are checked separately for $\sigma = \pm 1$. They imply

$$H_u \left[ \frac{1-t}{1-q} \sum_{i=1}^n \left( x_i q^{\sigma_i/2} + \frac{1}{x_i q^{\sigma_i/2}} \right) \right] = H_u \left[ \frac{1-t}{1-q} q^{\frac{\sigma}{2}} X^\dagger + (1-t)q^{-\frac{\sigma}{2}} \sum_{i=1}^n x_i^{-\sigma_i} \right]$$

$$= H_u \left[ (1-t)q^{-\frac{\sigma}{2}} \sum_{i=1}^n x_i^{-\sigma_i} \right] H_u \left[ \frac{1-t}{1-q} q^{\frac{\sigma}{2}} X^\dagger \right]$$

$$= \prod_{i=1}^n \frac{1-q^{-\frac{1}{2}} tx_i^{-\sigma_i}}{1-q^{-\frac{1}{2}} ux_i^{-\sigma_i}} \times H_u \left[ \frac{1-t}{1-q} q^{\frac{1}{2}} X^\dagger \right].$$

Finally we have

$$\sum_{\lambda_1, \lambda_2 \in \mathbb{N}} u^{\lambda_1} v^{\lambda_2} E_\pi(Q_{\lambda_1}Q_{\lambda_2}) = \sum_{\sigma \in \{-1,+1\}^n} \prod_{i=1}^n \frac{1-tx_i^{2\sigma_i}}{1-x_i^{2\sigma_i}} \prod_{1 \leq i < j \leq n} \frac{1-tx_i^{\sigma_i}x_j^{\sigma_j}}{1-x_i^{\sigma_i}x_j^{\sigma_j}} \times \prod_{i=1}^n \frac{1-q^{-\frac{1}{2}} tx_i^{-\sigma_i}}{1-q^{-\frac{1}{2}} ux_i^{-\sigma_i}} \times H_u \left[ \frac{1-t}{1-q} q^{\frac{1}{2}} X^\dagger \right] H_v \left[ \frac{1-t}{1-q} q^{\frac{1}{2}} X^\dagger \right].$$

The right hand-side is

$$T \left( \Phi_\pi \prod_{i=1}^n \frac{1-q^{-\frac{1}{2}} tu/x_i}{1-q^{-\frac{1}{2}} tu/x_i} \times \frac{1-q^{-\frac{1}{2}} tv/x_i}{1-q^{-\frac{1}{2}} tv/x_i} \right) H_u \left[ \frac{1-t}{1-q} q^{\frac{1}{2}} X^\dagger \right] H_v \left[ \frac{1-t}{1-q} q^{\frac{1}{2}} X^\dagger \right].$$
Therefore we may apply the $r = 2$ case (8.6) of Theorem 6, with $q^{-\frac{1}{z}} u$ and $q^{-\frac{1}{z}} v$ instead of $u$ and $v$. We obtain
\[
\sum_{\lambda_1, \lambda_2 \in \mathbb{N}} u^{\lambda_1} v^{\lambda_2} E_n(Q(\lambda_1)Q(\lambda_2)) = \prod_{i=1}^{n-2} (t^i + 1) H_u \left[ \frac{1-t}{1-q} q^{\frac{1}{2}} X^i \right] H_v \left[ \frac{1-t}{1-q} q^{\frac{1}{2}} X^i \right] \\
- \left( t^{2n-1} \frac{1-t^2 uv/q}{1-tuv/q} + t^{n-1} \frac{u-tv}{u-v} R(q^{-\frac{1}{2}} u) \right. \\
\left. + t^{n-1} \frac{v-tu}{v-u} R(q^{-\frac{1}{2}} v) + \frac{1-uv/q}{1-tuv/q} R(q^{-\frac{1}{2}} u) R(q^{-\frac{1}{2}} v) \right). 
\]

But we have
\[
R(q^{-\frac{1}{2}} u) H_u \left[ \frac{1-t}{1-q} q^{\frac{1}{2}} X^i \right] = \prod_{i=1}^{n} \left( 1-q^{-\frac{1}{2}} tu/x_i \right) H_u \left[ \frac{1-t}{1-q} q^{\frac{1}{2}} X^i \right] \\
= H_u \left[ (1-t) q^{-\frac{1}{2}} X^i \right] H_u \left[ \frac{1-t}{1-q} q^{\frac{1}{2}} X^i \right] \\
= H_u \left[ \frac{1-t}{1-q} q^{\frac{1}{2}} X^i + (1-t) q^{-\frac{1}{2}} X^i \right] \\
= H_u \left[ \frac{1-t}{1-q} q^{\frac{1}{2}} X^i \right] .
\]

Finally writing $A = (1-t)/(1-q) X^i$ for a better display, we have
\[
\sum_{\lambda_1, \lambda_2 \in \mathbb{N}} u^{\lambda_1} v^{\lambda_2} E_n(Q(\lambda_1)Q(\lambda_2)) = \prod_{i=1}^{n-2} (t^i + 1) \\
\times \left( t^{2n-1} \frac{1-t^2 uv/q}{1-tuv/q} H_u[q^{\frac{1}{2}} A] H_v[q^{\frac{1}{2}} A] + t^{n-1} \frac{u-tv}{u-v} H_u[q^{-\frac{1}{2}} A] H_v[q^{\frac{1}{2}} A] \right. \\
\left. + t^{n-1} \frac{v-tu}{v-u} H_u[q^{\frac{1}{2}} A] H_v[q^{-\frac{1}{2}} A] + \frac{1-uv/q}{1-tuv/q} H_u[q^{-\frac{1}{2}} A] H_v[q^{-\frac{1}{2}} A] \right). 
\]

We may compute the series expansion of the right-hand side by using
\[
H_u[q^{\frac{1}{2}} A] = \sum_{r \geq 0} u^r q^{\frac{r}{2}} Q_r. 
\]

Then if we write (4.2) as
\[
e_{(\lambda_1, \lambda_2)} = (t^n q^{\lambda_1/2} + q^{-\lambda_1/2})(t^{n-1} q^{\lambda_2/2} + q^{-\lambda_2/2}) \prod_{i=1}^{n-2} (t^i + 1), 
\]
we have
\[
\sum_{\lambda_1, \lambda_2 \in \mathbb{N}} u^{\lambda_1} v^{\lambda_2} e_{(\lambda_1, \lambda_2)} Q(\lambda_1)Q(\lambda_2) = \prod_{i=1}^{n-2} (t^i + 1) \left( t^{2n-1} H_u[q^{\frac{1}{2}} A] H_v[q^{\frac{1}{2}} A] \right. \\
\left. + t^{n-1} H_u[q^{-\frac{1}{2}} A] H_v[q^{\frac{1}{2}} A] + t^n H_u[q^{\frac{1}{2}} A] H_v[q^{-\frac{1}{2}} A] + H_u[q^{-\frac{1}{2}} A] H_v[q^{-\frac{1}{2}} A] \right). 
\]
Thus we obtain

\[
\sum_{\lambda_1, \lambda_2 \in \mathbb{N}} u^{\lambda_1} v^{\lambda_2} (E_\pi - e_{(\lambda_1, \lambda_2)}) Q_{(\lambda_1)} Q_{(\lambda_2)} = (1 - t) \prod_{i=1}^{n-2} (t^i + 1) \sum_{r,s \in \mathbb{N}} u^r v^s Q_r Q_s \\
\times \left( \frac{uv/q}{1 - tuv/q} (q^{1/2(r+s)} t^{2n} - q^{-1/2(r+s)}) + t^{n-1} \frac{v}{u - v} (q^{1/2(-r+s)} - q^{-1/2(r-s)}) \right).
\]

Identification of coefficients achieves the proof. \(\square\)

**Remark:** Starting from Theorem 6, the same method allows to obtain the polynomials \(E_\pi(Q_{(\lambda_1)} \ldots Q_{(\lambda_n)})\) for \(r \geq 2\). However their generating function involves \(2^n\) series and becomes quickly intricate.

### 11 The root system \(B_2\)

It is still unclear whether a similar method might be used for the root system \(B_n\). However this result is easy to get for \(n = 2\), since Macdonald polynomials of type \(B_2\) and \(C_2\) are in bijective correspondence.

The set of positive roots of the root system \(B_2\) is the union of \(R_1 = \{\varepsilon_1 + \varepsilon_2, \varepsilon_1 - \varepsilon_2\}\) and \(R_2 = \{\varepsilon_1, \varepsilon_2\}\). The fundamental weights of \(B_2\) are \(\varpi_1 = \varepsilon_1, \varpi_2 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2)\). The dominant weights \(\lambda \in P^+\) are vectors \(\lambda = \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2\) where \(\lambda_1, \lambda_2\) are all in \(\mathbb{N}\) or in \(\mathbb{N}/2\) and \(\lambda_1 \geq \lambda_2\). The weight \(\varpi_2\) is minuscule and \(\varpi_1\) is quasi-minuscule.

The dual root system \(C_2\) has one minuscule weight \(\tilde{\pi} = \varepsilon_1\), with Weyl group orbit \(\{\pm \varepsilon_1, \pm \varepsilon_2\}\). We have

\[
\Phi_{\tilde{\pi}} = \frac{1 - tx_1 x_2}{1 - x_1 x_2} \frac{1 - tx_1/x_2}{1 - x_1/x_2} \frac{1 - tx_1}{1 - x_1}.
\]

The translation operator \(T_{\tilde{\pi}}\) acts on \(A\) by \(T_{\tilde{\pi}} f(x_1, x_2) = f(q x_1, x_2)\). The Macdonald operator \(E_{\tilde{\pi}}\) writes as

\[
E_{\tilde{\pi}} f = \sum_{\sigma = \pm 1} \frac{1 - tx_1 x_2}{1 - x_1 x_2} \frac{1 - tx_1/x_2}{1 - x_1/x_2} \frac{1 - tx_1}{1 - x_1} f(q^\sigma x_1, x_2) \\
+ \sum_{\sigma = \pm 1} \frac{1 - tx_1 x_2}{1 - x_1 x_2} \frac{1 - tx_1/x_2}{1 - x_1/x_2} \frac{1 - tx_2}{1 - x_2} f(x_1, q^\sigma x_2).
\]

The Macdonald polynomial \(P_{\lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2}\) is defined, up to a constant, by

\[
E_{\tilde{\pi}} P_{\lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2} = (t^3 q^\lambda_1 + q^{-\lambda_1} + t^3 q^\lambda_2 + t q^{-\lambda_2}) P_{\lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2}.
\]

By comparison with the Macdonald operator of \(C_2\), we easily obtain the following bijective correspondence between Macdonald polynomials of type \(B_2\) and \(C_2\). Defining \(y_1 = x_1 x_2\) and \(y_2 = x_1/x_2\), we have

\[
P_{\lambda_1 \varepsilon_1 \lambda_2 \varepsilon_2}^{(C)}(x; q, t, T) = P_{\lambda_2 \varpi_1 + (\lambda_1 - \lambda_2) \varpi_2}^{(B)}(y; q, T, t) \\
= P_{(\lambda_1 + \lambda_2) \varepsilon_1/2 + (\lambda_1 - \lambda_2) \varepsilon_2/2}^{(B)}(y; q, T, t).
\]

24
The following results are obvious consequences. We normalize Macdonald polynomials attached to $B_2$ by

$$Q_{\lambda_1 \omega_1 + \lambda_2 \omega_2} = \frac{(t; q)_{\lambda_1}}{(q; q)_{\lambda_1}} \frac{(t; q)_{\lambda_2}}{(q; q)_{\lambda_2}} \frac{(q^{\lambda_1 \omega_2}; q)_{\lambda_1}}{(q^{\lambda_2 \omega_1}; q)_{\lambda_1}} P_{\lambda_1 \omega_1 + \lambda_2 \omega_2}.$$  

The generating function of the polynomials $Q_{\omega_2}(y_1, y_2)$ is given by

$$\sum_{r \in \mathbb{N}} u^r Q_{\omega_2}(x_1 x_2; x_1/x_2) = \frac{(tux_1; q)_\infty}{(ux_1; q)_\infty} \frac{(tu/x_1; q)_\infty}{(u/x_1; q)_\infty} \frac{(tux_2; q)_\infty}{(ux_2; q)_\infty} \frac{(tu/x_2; q)_\infty}{(u/x_2; q)_\infty}.$$  

**Theorem 7.** For any partition $\lambda = (\lambda_1, \lambda_2)$ we have

$$Q_{\lambda_1 \omega_2} Q_{\lambda_2 \omega_2} = \sum_{(i,j) \in \mathbb{N}^2 \atop 0 \leq i+j \leq \lambda_2} c_{ij}(\lambda_1, \lambda_2)|_{n=2} Q_{(\lambda_1+\lambda_2-2j)\varepsilon_1/2+(\lambda_1-\lambda_2+2i)\varepsilon_2/2}.$$  

Conversely

$$Q_{(\lambda_1+\lambda_2)\varepsilon_1/2+(\lambda_1-\lambda_2)\varepsilon_2/2} = \sum_{(i,j) \in \mathbb{N}^2 \atop 0 \leq i+j \leq \lambda_2} C_{ij}(\lambda_1, \lambda_2)|_{n=2} Q_{(\lambda_1+i-j)\omega_2} Q_{(\lambda_2-i-j)\omega_2}.$$  

The extension to any root system of rank 2 is an interesting problem. Of course the only remaining case is $R = G_2$, which might be investigated by using the Pieri formulas of van Diejen and Ito [4].

## 12 Basic hypergeometric identities

A referee has observed that the multivariate identity of Theorem 6 leads to basic $q$-hypergeometric identities by specialization of the indeterminates $x$ and $u$. In this section we present some of these corollaries, and explicit their connection with known results.

We adopt the notation of [5] and write

$$r+1 \phi_r \left[ \frac{a_1, a_2, \ldots, a_{r+1}}{b_1, b_2, \ldots, b_r}; q, z \right] = \sum_{i \geq 0} \frac{(a_1; q)_i \cdots (a_{r+1}; q)_i}{(b_1; q)_i \cdots (b_r; q)_i} \frac{z^i}{(q; q)_i}.$$  

A first identity is obtained by the “principal specialization” of the $u$ indeterminates, i.e. $u_i = zt^{i-1}$ ($1 \leq i \leq r$).

**Theorem 8.** Let $x = (x_1, \ldots, x_n)$. We have the very-well-poised $q$-hypergeometric sum

$$q^{nr-n} \left(\frac{q^2; q^2}{q_2^2; q_2^2}\right)_r \prod_{\sigma \in (-1, +1)^n} \prod_{i=1}^n \frac{1 - q x_{2i}^{-\sigma_i}}{1 - x_{2i}^{-\sigma_i}} \prod_{1 \leq i < j \leq n} \frac{1 - q x_i^{\sigma_i} x_j^{\sigma_j}}{1 - x_i^{\sigma_i} x_j^{\sigma_j}}.$$  

$$= \frac{z^2, -qz, -q^{-1}z, q^{-r}, \{q z x_i, q z / x_i\}, q, -q^{-r-n}}{z, -z, q^{r+1} z^2, \{z x_i, z / x_i\}, q, -q^{-r-n}}.$$  

$$\left[ z^2, q z, -z, -q^{-r}, \{q z x_i, q z / x_i\}, q, -q^{-r-n} \right].$$
Proof. With \( u_i = zt^{i-1} \) (\( 1 \leq i \leq r \)), on the right-hand side of (8.3) only the subsets \( I \) of the form \( I = \emptyset \) and \( I = \{1, 2, \ldots, m\} \) with \( 1 \leq m \leq r \) have nonvanishing contributions. Actually if there exist \( j = i - 1 \) with \( i \in I \) and \( j \notin I \), we have \( v_i = zt^{i-1} \) and \( v_j = t^{-i}/z \) so that \( v_iv_j = 1 \).

Thus the right-hand side of (8.3) writes as

\[
(-t;t)_{n-r} \sum_{m=0}^{r} t^{n(r-m)} c_m \prod_{i=1}^{n} \frac{(tzx_i; t)_m}{(zx_i; t)_m},
\]

with

\[
c_m = t^{-(r-m)} \prod_{1 \leq i < j \leq m} \frac{1 - t^{i+j-2}z^2}{1 - t^{i+j-1}z^2} \prod_{m+1 \leq i < j \leq r} \frac{1 - t^{i+j-2}z^2}{1 - t^{i+j-1}z^2} \prod_{m+1 \leq j < r} \frac{1 - t^{i-j-1}}{1 - t^{i-j}}
\]

\[
= t^{-(r-m)} \prod_{i=1}^{m} \frac{1 - t^{2i-1}z^2}{1 - t^{2m+1}z^2} \prod_{i=m+1}^{r} \frac{1 - t^{-i}z^2}{1 - t^{-2i}z^2} \prod_{i=1}^{r} \frac{1 - t^{-i-r-1}}{1 - t^{-i-r}}
\]

\[
= (-1)^m t^{(m+1)-(r-m)} \prod_{i=1}^{r} \frac{1 - t^{-i}z^2}{1 - t^{-2i}z^2} \prod_{i=1}^{r} \frac{1 - t^{-i}z^2}{1 - t^{-2i}z^2} \prod_{i=1}^{r} \frac{1 - t^{-i-r}z^2}{1 - t^{-2i}z^2} \prod_{i=1}^{r} \frac{1 - t^{-i-r}z^2}{1 - t^{-2i}z^2}.
\]

Hence the statement. \( \square \)

In Theorem 8 we may perform a principal specialization of the \( x \) indeterminates, i.e. \( x_i = wz_i^{-1} \) (\( 1 \leq i \leq n \)). This leads to the following transformation between a \( \psi_3 \) and a very-well-poised \( \psi_5 \).

**Theorem 9.** We have

\[
\frac{(qw^2, q^2)_n}{(w^2, q)_n} \frac{(q^{-r}w/z; q)_n}{(w/z; q)_n} \psi_3 \left[ \frac{w/z, q^{-r}wz, w^2/q, q^{-n}}{wz, q^{-r}w/z, q^n w^2} ; q, -q^{-n+r+1} \right]
\]

\[
= q^{-\binom{r}{2}} (-q; q)_{n-r} \frac{(qz^2; q^2)_r}{(q z^2; q)_r} \psi_5 \left[ \frac{z^2, qz, -qz, q^n wz, qz/w, q^{-r}}{z, -z, w, q^{-1-n}z/w, q^{r+1}z^2} ; q, -q^{-n} \right].
\]

Proof. Due to

\[
\prod_{i=1}^{n} \frac{(q^i w z; q)_m}{(q^{i-1} w z; q)_m} \frac{(q^{i-1} w z; q)_m}{(q^{i-1} w z; q)_m} = \frac{(q^n wz; q)_m}{(wz; q)_m} \frac{(q^r w; q)_m}{(z; q)_m},
\]

in Theorem 8 the right-hand side writes as

\[
(-q; q)_{n-r} q^{-\binom{r}{2}} \frac{(qz^2; q^2)_r}{(qz^2; q)_r} \psi_5 \left[ \frac{z^2, qz, -qz, q^n wz, qz/w, q^{-r}}{z, -z, w, q^{-1-n}z/w, q^{r+1}z^2} ; q, -q^{-n} \right].
\]

In the sum on the left-hand side, when \( \sigma_i = +1 \) for some \( i \) only terms with \( \sigma_{i+1} = +1 \) have a non-zero contribution. Actually if \( \sigma_{i+1} = -1 \) the last factor \( 1 - q x_i^{\sigma_i} x_{i+1}^{\sigma_{i+1}} \) vanishes. Therefore the non-zero terms are obtained for \( \sigma_i = -1 \) (\( 1 \leq i \leq m \)) and \( \sigma_i = +1 \) (\( m+1 \leq i \leq n \)), with \( 0 \leq m \leq n \).
Therefore the left-hand side writes as
\[
\sum_{m=0}^{n} \prod_{i=1}^{m} \frac{1 - q^{3-2i}/w^2}{1 - q^{2-2i}/w^2} \prod_{i=m+1}^{n} \frac{1 - q^{2i-1}w^2}{1 - q^{1-i}w^2} \times \prod_{i=1}^{m} \frac{1 - q^{2-2i}/w^2}{1 - q^{2-2i-m}/w^2} \prod_{i=1}^{m} \frac{1 - q^{n-i+1}}{1 - q^{m-i+1}} \prod_{i=m+1}^{n} \frac{1 - q^{i+n-1}w^2}{1 - q^{2i-1}w^2}.
\]

This is easily transformed to
\[
\sum_{m=0}^{n} (-1)^m q^{(m+1)+r(n-m)+(n+1)-(m+1)} \times \frac{(w^2/q; q^2)_m}{(q^{m-1}w^2; q)_m} \frac{(q^r w z; q)_m}{(w^2; q^2)_n} \frac{(q^s w^2; q)_n}{(w^2; q^2)_m} \frac{(q^{-r}/w; q)_m}{(w^2; q^2)_n} \frac{(q^{-s}/w^2; q)_n}{(w^2; q^2)_m} \quad 4^{\sigma_3} \left[ w/z, q^r w z, w^2/q, q^{-n} \right] \left[ w^2, q^{-r}/w z, q^s w^2 \right] ; q^{-q^{r-1}}.
\]

Schlosser has given another proof by combining two identities of [3], firstly Exercise 2.13 (ii) p.60, with \((a, b, c, d) \equiv (w^2/q, w/z, q^r w z, q^{-n})\), then Equ. (2.11.1) p.53, with \((a, b, c, d, e, f) \equiv (w^2 q^{n-r-1}, w q^{-1/2}, w q^{n-r}/z, -w q^{-1/2}, w z q^n, q^{-r})\).

An extension of Theorem 8 may be obtained by “multiple principal specialization”, defined as follows. Given \(s\) positive integers \(k = (k_1, \ldots, k_s)\) summing to \(|k| = r\), we consider the \(s\) intervals \([k_{i-1} + 1, k_i]\) where \(k_i\) denotes the partial sum \(\sum_{j=1}^{i} k_j\). Given \(r\) indeterminates \(u = (u_1, \ldots, u_r)\), their multiple principal specialization is defined by
\[
u_i = z_i t^{i-1-k_{i-1}}; \quad i \in [k_{i-1} + 1, k_i] \quad (1 \leq l \leq s).
\]

References may be found in [3] Section 4.1.

**Theorem 10.** Let \(x = (x_1, \ldots, x_n)\), \(s\) positive integers \(k = (k_1, \ldots, k_s)\) and \(z = (z_1, \ldots, z_s)\).

We have the basic hypergeometric sum
\[
\sum_{\sigma \in \{1, +\}^n} \prod_{i=1}^{n} \frac{1 - q x_i^{\sigma_i}}{1 - x_i^{2 \sigma_i}} \prod_{i=1}^{s} \frac{1 - q^{k_i} z_i x_i^{-\sigma_i}}{1 - z_i x_i^{-\sigma_i}} \prod_{1 \leq i < j \leq n} \frac{1 - q x_i^{\sigma_i} x_j^{\sigma_j}}{1 - x_i^{\sigma_i} x_j^{\sigma_j}}
\]
\[
= q^{n|k|-\binom{|k|}{2}} (-q; q)_{n-|k|} \prod_{1 \leq a < b \leq s} \frac{(q^{k_a+1} z_a z_b; q)_{k_a}}{(q z_a z_b; q)_{k_a}} \prod_{a=1}^{s} \frac{(q^{a^2}; q^2)_{k_a}}{(q z_a; q)_{k_a}}
\]
\[
\times \left( \sum_{m \leq k} \prod_{i=1}^{n} \frac{(q^m z_i; q^m)_{m_i}}{(x_i z_i; q)_{m_i}} \prod_{i=1}^{s} \frac{(q z_i; q)_{m_i}}{(z_i; q)_{m_i}} \right),
\]

with
\[
c_m = \prod_{1 \leq a < b \leq s} \frac{q^{m_a} z_a - q^{m_b} z_b}{z_a - z_b} \prod_{1 \leq a < b \leq s} \frac{1 - q^{m_a+m_b} z_a z_b}{1 - z_a z_b} \prod_{a, b=1}^{s} \frac{(q^{-k_b} z_a / z_b; q)_{m_a}}{(q^{-k_b} z_a; q)_{m_a}} \frac{(z_a z_b; q)_{m_a}}{(q^{1+k_b} z_a z_b; q)_{m_a}}.
\]

27
Proof. We write (8.3) with \( u_i \) defined as above. By the same argument as in the proof of Theorem 8, on the right-hand side only the subsets \( I \) of the form \( I = \bigcup_{l=1}^s I_l \), with \( I_l = \{k_{l-1} + i, 1 \leq i \leq m_l \} \) and \( 0 \leq m_l \leq k_l \), have nonvanishing contributions. We denote \( J_l = \{k_{l-1} + i, m_l + 1 \leq i \leq k_l \} \) the complement of \( I_l \) in \([k_{l-1} + 1, k_l]\).

Then the right-hand side of (8.3) writes as

\[
(-t; t)^{n-|k|} \prod_{m \leq k} c_m \prod_{i=1}^n \frac{(tx_i z_i; t)_{m_i}}{(x_i z_i; t)_{m_i}} \frac{(tz_i/x_i; t)_{m_i}}{(z_i/x_i; t)_{m_i}}.
\]

The coefficient \( c_m \) is the product of the following contributions. If \( i \in I_a \) then either \( j \in I_b \) for \( b \geq a \), or \( j \in J_b \) for \( b > a \). Such cases contribute to

\[
\prod_{i=1}^{m_a} \prod_{1 \leq a < b \leq s} \frac{1 - t^{-i}z_a z_b}{1 - t^{-i-k_{a-1}} z_a/z_b} \prod_{i=1}^{s} \frac{1 - t^{2i-1} z_a^2}{1 - t^{-i-k_{a-1}} z_a^2} \prod_{i=1}^{s} \frac{1 - t^{i-k_{a-1}}}{1 - t^{-i-m_{a-1}} z_a}. 
\]

If \( i \in J_a \) then either \( j \in I_b \) for \( b > a \), or \( j \in J_b \) for \( b > a \). Such cases contribute to

\[
\prod_{i=m_a+1}^{k_a} \prod_{1 \leq a < b \leq s} \frac{1 - t^{-i}z_a z_b}{1 - t^{-i-m_{b-1}} z_a/z_b} \prod_{i=1}^{s} \frac{1 - t^{k_{b+1}+i} z_a z_b}{1 - t^{k_{b+1}+i} z_a z_b} \prod_{i=1}^{s} \frac{1 - t^{k_{b+1}+i} z_a^2}{1 - t^{2i} z_a^2}.
\]

The statement follows by easy, but tedious, transformations.

The multidimensional series at the right-hand side involves the factor

\[
\prod_{1 \leq a < b \leq s} \frac{q^{m_a} z_a - q^{m_b} z_b}{z_a - z_b} \prod_{1 \leq a < b \leq s} \frac{1 - q^{m_a+m_b} z_a z_b}{1 - z_a z_b}.
\]

Such series are called \( C_n \) basic hypergeometric series. There is a rich litterature in this domain. Many references may be found in the bibliography of [22]-[24].

A (messy) multiple principal specialization of the \( x \) indeterminates might be performed on Theorem 10. Here we shall only consider a simple principal specialization \( x_i = wq^{i-1} \) \((1 \leq i \leq n)\). A proof strictly parallel to Theorem 9 leads to the following basic hypergeometric identity.

**Theorem 11.** We have

\[
\frac{(qw^2; q^2)_n}{(w; q)_n} \prod_{l=1}^{s} \frac{(g^{k_l} w/z_l; q)_{2s+2}}{(w/z_l; q)_{2s+2}} \prod_{1 \leq a < b \leq s} \frac{q^{g^{k_{a+1}} z_a z_b; q)}{q^{g_{a+1} z_b; q)_{ka}} \prod_{a=1}^{s} \frac{q^{g^2 z_a^2; q^2)_{ka}}{q^{g^2 z_b^2; q^2)_{ka}} \times (\sum_{m \leq k} (-1)^m c_m q^{(|k|-n)m} \prod_{l=1}^{s} \frac{(q^n w z_l; q)_{m_i}}{(w z_l; q)_{m_i}} \frac{(q z_l w; q)_{m_l}}{(q^{1-n} z_l w; q)_{m_l}}},
\]

with \( c_m \) as in Theorem 10.
Schlosser has pointed out that Theorem 10 is already known, and corresponds to a particular case of an identity due to Rosengren [21] Corollary 4.4, p.342.

Indeed in this result, if we substitute \( l = (l_1, \ldots, l_n) \rightarrow k = (k_1, \ldots, k_s), \ y = (y_1, \ldots, y_n) \rightarrow m = (m_1, \ldots, m_s), \ c = (c_1, \ldots, c_p) \rightarrow x = (x_1, \ldots, x_n), \) and specialize \( b = -d = q^2 \) and \((m_1, \ldots, m_p) = (1, \ldots, 1),\) we recover Theorem 10.

Incidentally the identity of [21] has a symmetrical form since it writes the left-hand side of Theorem 10 as the following \( C_n \) basic hypergeometric series

\[
\prod_{i=1}^{n} \frac{1 - qx_i^2}{1 - x_i^2} \prod_{l=1}^{s} \frac{1 - q^{-k_l}x_l/z_l}{1 - x_l/z_l} \prod_{1 \leq i < j \leq n} \frac{1 - qx_i x_j}{1 - x_i x_j} \times \sum_{\sigma \in (0,1)^n} (-1)^{|\sigma|} d_\sigma \frac{q^{(n-|k|+1)|\sigma|}}{1 - q^{k_1}} \prod_{i=1}^{n} \prod_{l=1}^{s} \frac{(q^{k_l} x_l z_l; q)_{\sigma_l}}{(x_l z_l; q)_{\sigma_l}} (\frac{x_l / z_l}{q})_{\sigma_l},
\]

with

\[
d_\sigma = \prod_{1 \leq a < b \leq n} \frac{x_a q^{a_b} - x_b q^{a_b}}{x_a - x_b} \prod_{1 \leq a \leq b \leq n} \frac{1 - x_a x_b q^{a_b - 1}}{1 - x_a x_b / q} \times \prod_{a=1}^{n} \frac{(x_a^2 / q; q^2)_{\sigma_a}}{(q x_a^2 / q^2)_{\sigma_a}} \prod_{a,b=1}^{n} \frac{(x_a x_b / q; q)_{\sigma_a}}{(q x_a / x_b; q)_{\sigma_a} (q x_b / x_a; q)_{\sigma_a}}.
\]

The connection of our multivariate identity with [21] and \( C_n \) basic hypergeometric series opens several interesting questions.

Firstly, since [21] has an \( A_n \) counterpart [23], is it also the case for Theorem 6? More generally does such an identity exist for any root system? Secondly, since the result of [21] depends on indeterminates \((m_1, \ldots, m_p)\) and has a symmetrical form, is such a generalization also possible for Theorem 6? Finally would it be even possible to generalize Theorem 6 in the framework of elliptic functions [8, 22]?

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