REGULARITY CRITERIA FOR WEAK SOLUTIONS OF THE NAVIER-STOKES SYSTEM IN GENERAL UNBOUNDED DOMAINS

REINHARD FARWIG AND PAUL FELIX RIECHWALD

Fachbereich Mathematik
Technische Universität Darmstadt
64289 Darmstadt, Germany

Dedicated to our colleagues Paolo Secchi and Alberto Valli on the occasion of their 60th birthdays

ABSTRACT. We consider weak solutions of the instationary Navier-Stokes system in general unbounded smooth domains \( \Omega \subset \mathbb{R}^3 \) and discuss several criteria to prove that the weak solution is locally or globally in time a strong solution in the sense of Serrin. Since the usual Stokes operator cannot be defined on all types of unbounded domains we have to replace the space \( L^q(\Omega) \), \( q > 2 \), by \( \tilde{L}^q(\Omega) = L^q(\Omega) \cap L^2(\Omega) \) and Serrin’s class \( L^r(0, T; L^q(\Omega)) \) by \( L^r(0, T; \tilde{L}^q(\Omega)) \) where \( 2 < r < \infty \), \( 3 < q < \infty \) and \( \frac{2}{r} + \frac{3}{q} = 1 \).

1. Introduction. We consider the instationary Navier-Stokes system

\[
\begin{align*}
\frac{du}{dt} - \Delta u + \text{div}(u \otimes u) + \nabla p &= f \quad \text{in} \quad (0, T) \times \Omega, \\
\text{div } u &= 0 \quad \text{in} \quad (0, T) \times \Omega, \\
u &= 0 \quad \text{on} \quad (0, T) \times \partial \Omega, \\
u(0) &= u_0 \quad \text{at} \quad t = 0,
\end{align*}
\]

in a general unbounded domain \( \Omega \subset \mathbb{R}^3 \) with uniform \( C^2 \)-boundary and a finite time interval \( (0, T) \). Here \( u = (u_1, u_2, u_3) \) denotes the unknown velocity field, \( p \) an associated pressure, \( f \) a given external force of the form \( f = f_1 + \text{div } f_2 \), and \( u_0 \) the initial value of \( u \) at time \( t = 0 \). For simplicity, the viscosity is set to \( \nu = 1 \). A precise definition of domains with uniform \( C^2 \)-boundary can be found in Definition 2.1 below.

A problem in this setting is the unboundedness of the underlying domain \( \Omega \). Due to counter-examples by M.E. Bogovskij and V.N. Maslennikova [2, 3] the Helmholtz decomposition of vector fields in \( L^q(\Omega) \), \( 1 < q < \infty \), on an unbounded smooth domain may fail unless \( q = 2 \). By analogy, a bounded Helmholtz projection \( P_q \) with the properties required to define the Stokes operator \( A_q = -P_q \Delta \) when \( q \neq 2 \) may not exist. Therefore, in [5, 7, 8, 9, 10] H. Kozono, H. Sohr and the first author of this article introduced the spaces

\[
\tilde{L}^q(\Omega) := \begin{cases} 
L^q(\Omega) + L^2(\Omega), & \text{if } 1 \leq q < 2, \\
L^q(\Omega) \cap L^2(\Omega), & \text{if } 2 \leq q \leq \infty.
\end{cases}
\]

2010 Mathematics Subject Classification. Primary: 35B65, 35Q30; Secondary: 76D05.

Key words and phrases. Navier-Stokes system, weak solutions, general unbounded domains, regularity criteria, uniqueness.
The corresponding norm is defined as $\|u\|_{L_q} = \max\{\|u\|_q, \|u\|_2\}$ when $q \geq 2$, and as $\inf\{\|u_1\|_q + \|u_2\|_2 : u = u_1 + u_2, u_1 \in L^q(\Omega), u_2 \in L^2(\Omega)\}$ when $1 \leq q < 2$.

For bounded domains we have that $\tilde{L}^q(\Omega) = L^q(\Omega)$ with equivalent norms. We note that functions in $\tilde{L}^q(\Omega)$ locally behave like $L^q$-functions, but globally exploit $L^2$-properties. By analogy, function spaces like $\tilde{L}_2^q(\Omega)$ of solenoidal vector fields and $\tilde{W}^{k,q}(\Omega)$ of weakly differentiable functions will be defined.

As shown in [7] a Helmholtz projection $\tilde{P}_q : \tilde{L}^q(\Omega)^n \to \tilde{L}_q^q(\Omega)$ is well defined, allowing to define a closed Stokes operator $\tilde{A}_q = -\tilde{P}_q \Delta$ with domain $\tilde{D}_q := \tilde{W}^{2,q}(\Omega) \cap \tilde{W}_0^{1,q}(\Omega) \cap \tilde{L}_q^q(\Omega)$ dense in $\tilde{L}_q^q(\Omega)$. The operator $\tilde{A}_q$ has similar properties as the usual Stokes operator $A_{q,t}$, which generates an analytic semigroup $e^{-t\tilde{A}_q}$, $t \geq 0$, enjoys the property of bounded imaginary powers and maximal regularity; for details and further properties of these function spaces and operators we refer to [5, 7, 8, 9, 10] and [14, 15] as well as to Sect. 2.

Our first result shows that a weak solution in the sense of Leray and Hopf must coincide with a very weak solution; for the definition and existence of very weak solutions see Definition 2.2 and Theorem 2.3 in Sect. 2 below. The negative Sobolev space $\mathcal{T}^{-1, r, q}(T, \Omega)$ will also be explained in Sect. 2, cf. (18). This result will be important for showing regularity of weak solutions.

**Theorem 1.1.** Let $0 < T < \infty$, let $\Omega \subset \mathbb{R}^3$ be a uniform $C^2$-domain, and

$$2 < r < \infty, \quad 3 < q < \infty, \quad \frac{2}{r} + \frac{3}{q} = 1.$$ 

Given data $u_0 \in L_2^q(\Omega), f_1 \in L^1(0,T;L_2^2(\Omega)), f_2 \in L^2(0,T;L_2^2(\Omega))$ let $\tilde{u}$ be a weak solution to the Navier-Stokes equations in the sense of Leray and Hopf, i.e.,

$$\tilde{u} \in L^\infty(0,T;L_2^2(\Omega)) \cap L^2(0,T;W_{0,loc}^{1,2}(\Omega))$$

solves (1) in the sense of distributions, and let $\tilde{u}$ satisfy the energy inequality.

Furthermore, assume that the data functional $\mathcal{F}$ defined by

$$\langle \mathcal{F}, \phi \rangle = (u_0, \phi(0))_\Omega + (f_1, \phi(T) - \phi(0))_\Omega - (f_2, \nabla \phi)_{T,\Omega}$$

lies in $\mathcal{T}^{-1, r, q}(T, \Omega)$ and that there exists a very weak solution $u \in L^r(0,T;\tilde{L}_q^q(\Omega))$ of the Navier-Stokes equations to this data sharing the additional important property $u \in L^q(0,T;\tilde{L}_2^2(\Omega))$.

Then $u = \tilde{u}$ almost everywhere on $(0,T)$.

A typical theorem on local regularity at a point $t \in (0,T)$ in the sense that there exists $\delta > 0$ such that $u \in L^r(t-\delta, t+\delta;\tilde{L}^q(\Omega))$ with exponents $r, q$ satisfying (Serrin’s condition) $\frac{2}{r} + \frac{3}{q} = 1$ reads as follows.

**Theorem 1.2.** Let $\Omega \subset \mathbb{R}^3$ be a uniform $C^2$-domain, $0 < T < \infty$, and

$$\frac{16}{5} \leq r \leq 16, \quad \frac{24}{7} \leq q \leq 8, \quad \frac{2}{r} + \frac{3}{q} = 1, \quad \frac{1}{\gamma_1} = \frac{1}{q} - \frac{2}{3}, \quad \frac{1}{\gamma_2} = \frac{1}{q} - \frac{1}{3}. \quad (3)$$

Assume data $u_0 \in L_2^q(\Omega)$ and

$$f_1 \in L^{2/7}(0,T;L_2^2(\Omega)) \cap L_{loc}^{\gamma_1}(0,T;\tilde{L}_q^{\gamma_1}(\Omega)), \quad (4)$$

$$f_2 \in L^{2/3}(0,T;L_2^2(\Omega)) \cap L_{loc}^{\gamma_2}(0,T;\tilde{L}_q^{\gamma_2}(\Omega)) \quad (5)$$

and that $u$ is a weak solution in the sense of Leray and Hopf satisfying the strong energy inequality

$$\frac{1}{2}\|u(t)\|_{L_q^2(\Omega)}^2 + \int_s^t \|\nabla u\|_{L_q^2(\Omega)}^2 \, d\tau \leq \frac{1}{2}\|u(s)\|_{L_q^2(\Omega)}^2 + \int_s^t ((f_1, u) - (f_2, \nabla u)) \, d\tau$$
for a.a. $s \in (0, T)$ (including $s = 0$) and all $s \leq t < T$. Moreover, choose exponents $1 \leq r_0 \leq r$ and $3 < q_0 \leq q$.

Then there exists a constant $\eta = \eta(\Omega, q, r, q_0, r_0, T) > 0$ with the following property: If for a point $t \in (0, T)$

$$\liminf_{\delta \to 0^+} \frac{1}{\delta^\alpha} \int_{t-\delta}^t \|u(\tau)\|_{L^q(\Omega)}^q d\tau \leq \eta,$$

where $\alpha = \frac{r_0}{2}(\frac{2}{r_0} + \frac{3}{q_0} - 1)$, then $t$ is a regular point.

In the case where $q_0 = q$ this condition reads

$$\liminf_{\delta \to 0^+} \frac{1}{\delta^{1-r_0/r}} \int_{t-\delta}^t \|u(\tau)\|_{L^q(\Omega)}^q d\tau \leq \eta$$

and is even a necessary condition.

In particular, the left-side Serrin-condition $u \in L^r(t-\delta, t; \tilde{L}^q(\Omega))$ for some $\delta > 0$ is sufficient for $t$ to be a regular point.

Actually, the constant $\eta$ depending on the domain $\Omega$ only depends on the so-called type of the domain $(\text{type}(\Omega))$; this notion will be explained in Sect. 2.

A consequence of Theorem 1.2 is a criterion based on the kinetic energy $E_{\text{kin}}(t) = \frac{1}{2}\|u(t)\|_{L^2}^2$.

**Theorem 1.3.** Let $\Omega \subseteq \mathbb{R}^3$ be a uniform $C^2$-domain, $0 < T < \infty$, $u_0 \in L^2_+(\Omega)$, and let $u$ be a weak solution satisfying the strong energy inequality (for simplicity we assume $f_1 = 0, f_2 = 0$).

There is a constant $\eta = \eta(\text{type}(\Omega), T)$ with the following property: If for a point $t \in (0, T)$ and some $\mu > 0$ the left-sided $\frac{1}{2}$-Hölder continuity

$$\sup_{0 < \delta \leq \mu} \frac{|E_{\text{kin}}(t) - E_{\text{kin}}(t - \delta)|}{\delta^{1/2}} \leq \eta$$

(7)

holds, then $t$ is a regular point of $u$ in the sense $u \in L^4(t - \delta, t + \delta; \tilde{L}^6(\Omega))$.

In particular, for $\alpha \in \left(\frac{1}{2}, 1\right]$ the left-sided $\alpha$-Hölder condition

$$\sup_{0 < \delta \leq \mu} \frac{|E_{\text{kin}}(t) - E_{\text{kin}}(t - \delta)|}{\delta^\alpha} < \infty$$

implies the left-sided $\frac{1}{2}$-Hölder continuity (7) with smallnes $\eta$.

For further results on uniqueness and local or global regularity of weak solutions we refer to Sect. 3.

2. Preliminaries.

**Definition 2.1.** A domain $\Omega \subset \mathbb{R}^n$ is called uniform $C^2$-domain if there are constants $\alpha, \beta, K > 0$ such that for all $x_0 \in \partial\Omega$ there exist - after an orthogonal and an affine coordinate transform - a function $h$ on the closed ball $B_r(0) \subseteq \mathbb{R}^{n-1}$ of class $C^2$ and a neighborhood $U_{\alpha, \beta, h}(x_0)$ of $x_0$ with the following properties: $\|h\|_{C^2} \leq K$ and $h(0) = 0, h'(0) = 0$; moreover,

$U_{\alpha, \beta, h}(x_0) := \{(y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |y'| < \alpha, |h(y') - y_n| < \beta\}$,

$U_{\alpha, \beta, h}^-(x_0) := \{(y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |y'| < \alpha, h(y') - \beta < y_n < h(y')\}$

$= \Omega \cap U_{\alpha, \beta, h}(x_0)$,

$\partial \Omega \cap U_{\alpha, \beta, h}(x_0) = \{(y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : h(y') = y_n\}$. 


for all \( f \) generating an analytic semigroup \( \{e^{\lambda t} \mid \lambda \in \mathbb{C} \} \) does not depend only on \( \alpha, \beta \) and \( K \), but in no other way on \( \Omega \).

For spaces of Sobolev-type we proceed analogously to the definition in (2): For \( k \in \mathbb{N} \) and \( 1 \leq q \leq \infty \) we let

\[
W^{k,q}(\Omega) := \begin{cases} W^{k,2}(\Omega) + W^{k,q}(\Omega), & 1 \leq q < 2, \\ W^{k,2}(\Omega) \cap W^{k,q}(\Omega), & 2 \leq q \leq \infty. \end{cases}
\] (8)

Similarly, we define the spaces \( \tilde{W}^{1,q}(\Omega) \), \( 1 < q < 2 \) and \( 2 \leq q < \infty \), based on the classical Sobolev spaces \( W^{1,q}_0(\Omega) \) and \( W^{1,2}_0(\Omega) \).

The \( \tilde{L}^q \)-spaces satisfy \( (\tilde{L}^q(\Omega))^* = \tilde{L}^q(\Omega) \), and we have the Sobolev embeddings (see [15]): Let \( m \in \mathbb{N} \), \( 1 \leq q < \infty \) and \( \Omega \subseteq \mathbb{R}^n \) be a uniform \( C^2 \)-domain. Then

\[
\tilde{W}^{m,q}(\Omega) \hookrightarrow \tilde{L}^r(\Omega)
\]

if either \( q \leq r \leq \infty \) and \( mq > n \), or \( q \leq r < \infty \) and \( mq = n \), or \( q \leq r \leq \frac{mq}{n-mq} \) and \( mq < n \).

Concerning the Helmholtz projection on \( \tilde{L}^q(\Omega) \) for a domain \( \Omega \subseteq \mathbb{R}^n \) of uniform type \( C^2 \) (here the uniform type \( C^1 \) suffices, cf. [7]), we first define

\[
\tilde{L}^q(\Omega) := \begin{cases} L^q(\Omega) + L^q_\sigma(\Omega), & 1 < q < 2, \\ L^q(\Omega) \cap L^q_\sigma(\Omega), & 2 \leq q < \infty, \end{cases}
\] (9)
equipped with the norm of \( \tilde{L}^q(\Omega) \), and gradient spaces by

\[
\tilde{G}_q(\Omega) := \begin{cases} G_q(\Omega) + G_2(\Omega), & 1 < q < 2, \\ G_q(\Omega) \cap G_2(\Omega), & 2 \leq q < \infty, \end{cases}
\] (10)

which are based on the gradient spaces \( G_r(\Omega) = \{ \nabla p \in L^r(\Omega) : p \in L^r_{\text{loc}}(\Omega) \} \). The norm in \( \tilde{G}_q(\Omega) \) is denoted by \( \| \cdot \|_{\tilde{G}_q(\Omega)} := \| \cdot \|_{\tilde{L}^q(\Omega)} \).

The space \( \tilde{L}^q(\Omega) \) admits the direct algebraic and topological decomposition

\[
\tilde{L}^q(\Omega) = \tilde{L}^q_\sigma(\Omega) \oplus \tilde{G}_q(\Omega)
\]
yielding a projection \( \tilde{P}_q \) from \( \tilde{L}^q(\Omega) \) onto \( \tilde{L}^q_\sigma(\Omega) \) with operator norm bounded by a constant \( c = c(q, \text{type}(\Omega)) \), see [7]. We have the duality relations \( (\tilde{P}_q)^* = \tilde{P}_{q'} \) and \( (\tilde{L}^q_\sigma(\Omega))^* = \tilde{L}^{q'}_\sigma(\Omega) \). Using the Helmholtz projection \( \tilde{P}_q \) we can define the Stokes operator \( \tilde{A}_q \), \( 1 < q < \infty \), for a uniform \( C^2 \)-domain \( \Omega \subseteq \mathbb{R}^n \) with domain

\[
\mathcal{D}(\tilde{A}_q) := \begin{cases} \mathcal{D}_q + \mathcal{D}_2, & 1 < q < 2, \\ \mathcal{D}_q \cap \mathcal{D}_2, & 2 \leq q < \infty, \end{cases}
\] (11)

where \( \mathcal{D}_q := L^q_\sigma(\Omega) \cap W^{1,q}_0(\Omega) \cap W^{2,q}(\Omega) \). Then \( \tilde{A}_q := -\tilde{P}_q \Delta : \mathcal{D}(\tilde{A}_q) \subseteq \tilde{L}^q_\sigma(\Omega) \rightarrow \tilde{L}^q_\sigma(\Omega) \) is a densely defined closed operator in \( \tilde{L}^q_\sigma(\Omega) \) satisfying \( (\tilde{A}_q)^* = \tilde{A}_{q'} \) and generating an analytic semigroup \( e^{-t\tilde{A}_q}, t \geq 0 \), with bound

\[
\|e^{-t\tilde{A}_q}f\|_{\tilde{L}^q(\Omega)} \leq Ce^{\delta t}\|f\|_{\tilde{L}^q(\Omega)}
\]
for all \( f \in \tilde{L}^q_\sigma(\Omega) \) and \( t \geq 0 \) with a constant \( C = C(q, \delta, \text{type}(\Omega)) \), \( \delta > 0 \), see [10]. It is unknown whether the usual resolvent estimate for the infinitesimal generator \( \tilde{A}_q \) of the semigroup \( e^{-t\tilde{A}_q} \) holds uniformly in the resolvent parameter \( \lambda \) in a sector of \( \mathbb{C} \) as \(|\lambda| \to 0\). Therefore, the semigroup may increase exponentially fast. Note that
from time to time we will omit the symbols $\Omega$ and $T$ for domain and length of the time interval, respectively, when this data is known from the context.

For an external force $f \in L^r(0, T; \dot{L}_q^\theta(\Omega))$ and an initial value $u_0 \in D(\dot{A}_q)$ (for simplicity) consider the abstract Cauchy problem

$$u_t + \dot{A}_q u = f, \quad u(0) = u_0.$$  

It is known that there exists a unique solution $u \in L^r(0, T; D(\dot{A}_q)) \cap W^{1, r}(0, T; \dot{L}_q^\theta(\Omega))$ which can be represented by the variation of constants formula

$$u(t) = e^{-t\dot{A}_q} u_0 + \int_0^t e^{-(t-\tau)\dot{A}_q} f(\tau) \, d\tau \quad \text{for } 0 \leq t \leq T. \quad (12)$$

Moreover, it satisfies the maximal regularity estimate

$$\|u\|_{L^r(0, T; D(\dot{A}_q))} + \|u_t\|_{L^r(0, T; \dot{L}_q^\theta)} \leq C(\|u_0\|_{D(\dot{A}_q)} + \|f\|_{L^r(0, T; \dot{L}_q^\theta)}) \quad \text{with a constant } C = C(q, r, T, \text{type}(\Omega)) > 0, \text{ cf. } (9, \text{Theorem 1.4}).$$

A further crucial property of the Stokes operator is the fact that $1 + \dot{A}_q$ admits bounded imaginary powers, see [14]. Hence complex interpolation methods can be used to describe domains of fractional powers $(1 + \dot{A}_q)^\alpha$ for $0 \leq \alpha \leq 1$ let the domain of $(1 + \dot{A}_q)^\alpha$ be denoted by

$$\dot{D}_q^\alpha = D((1 + \dot{A}_q)^\alpha), \quad (13)$$

equipped with the norm $\|(1 + \dot{A}_q)^\alpha\|_{\dot{L}_q}$. If $-1 \leq \alpha < 0$ define $\dot{D}_q^\alpha$ as the completion of $\dot{L}_q^\theta(\Omega)$ in the norm $\|(1 + \dot{A}_q)^\alpha\|_{\dot{L}_q}$. These spaces are reflexive and satisfy the duality relation $(\dot{D}_q^\alpha)^* = \dot{D}_q^{-\alpha}$. As special cases we get that

$$\dot{D}_q^1 = D(\dot{A}_q), \quad \dot{D}_q^{1/2} = \dot{W}_0^{1, q}(\Omega) \cap \dot{L}_q^\theta(\Omega)$$

(with norm $\|(1 + \dot{A}_q)^{1/2}\|_{\dot{L}_q}$ equivalent to $\|\cdot\|_{\dot{W}^{1, q}_q}$). Moreover, for $-1 \leq \alpha \leq \beta \leq 1$, we obtain that $[\dot{D}_q^\alpha, \dot{D}_q^\beta]_\theta = \dot{D}_q^\gamma$ where $(1 - \theta)\alpha + \theta\beta = \gamma, \theta \in (0, 1)$. This result implies the following embedding and decay estimate ([15, Proposition 3, Theorem 1]): Let $1 < q \leq r < \infty$, and $\alpha := \frac{\alpha}{2(\frac{1}{q} - \frac{1}{r})}$. Then

$$\|u\|_{\dot{L}_r^\gamma(\Omega)} \leq C\|(1 + \dot{A}_q)^\alpha u\|_{\dot{L}_q^\theta(\Omega)}, \quad \alpha \leq 1, \quad (14)$$

$$\|e^{-t\dot{A}_q} f\|_{\dot{L}^\gamma(\Omega)} \leq C e^{\delta t}(1 + t)^{-\alpha} \|f\|_{\dot{L}_q^\theta(\Omega)}, \quad (15)$$

for every $u \in \dot{D}_q^\alpha$ and $f \in \dot{L}_q^\theta(\Omega)$, respectively, and for any $t > 0$ and $\delta > 0$; here $C = C(\text{type}(\Omega), r, q, \delta) > 0$.

For a discussion of spaces of initial values in Proposition 2 below it is reasonable to consider also Lorentz spaces over $\dot{L}^q(\Omega)$ and their solenoidal subspaces. First we define for $1 < q < \infty$, $1 \leq \rho \leq \infty$ the Lorentz spaces

$$\dot{L}_q^{\theta, \rho}(\Omega) := \begin{cases} \dot{L}_q^{\theta, \rho}(\Omega) + L^2(\Omega), & 1 < q < 2, \\ \dot{L}_q^{\theta, \rho}(\Omega) \cap L^2(\Omega), & 2 < q < \infty, \end{cases}$$

letting the case $q = 2$ undefined; here $L_q^{\theta, \rho}(\Omega)$ denotes a usual Lorentz space, cf. [17, Ch. 1.18.6]. Next we define for $1 < q < \infty$, $q \neq 2$, and $1 \leq \rho < \infty$ the spaces

$$\dot{L}_q^{\theta, \rho}(\Omega) := C_{0, \alpha}^{\infty}(\Omega) \|\cdot\|_{\dot{L}^{\theta, \rho}}.$$
Then, by [15, Corollary 1], $1 < q, r, s < \infty$ with $r \neq q, s \neq 2$, satisfying $\frac{1}{2} = \frac{1-\theta}{q} + \frac{\theta}{r}$ with some $0 < \theta < 1$, and for $1 \leq \rho < \infty$ we get that

\[ (\tilde{L}_s^q(\Omega), \tilde{L}_s^r(\Omega))_{\eta, \rho} = \tilde{L}_s^{s\rho}(\Omega). \]  

Finally, we provide some details on the theory of very weak solutions in the context of general unbounded smooth domains using the spaces $\tilde{L}^q(\Omega)$. For more details we refer to [11]; the case of exterior domains has been discussed in [6]. The abstract external force field $F$ in Definition 2.2 below combines the initial value $u_0$ and an external force $f = f_1 + \text{div} f_2$ as follows:

\[ \langle F, \phi \rangle = (u_0, \phi(0)) + (f_1, \phi)_{T, \Omega} - (f_2, \nabla \phi)_{T, \Omega}. \]  

In (17) the expression $(\cdot, \cdot)_{T, \Omega}$ denotes the usual duality product for functions on $\Omega \times (0, T)$ whereas $(\cdot, \cdot)$ denotes the corresponding duality product on $\Omega$. The test function $\phi$ is taken from the space

\[ \mathcal{T}^{1, r', q'}(\Omega) := \{ \phi \in L^{r'}(0, T; \tilde{D}^{1, q'}_w) \cap W^{1, r'}(0, T; \tilde{L}^{q'}(\Omega)) : \phi(T) = 0 \} \]  

equipped with the norm

\[ \|\phi\|_{\mathcal{T}^{1, r', q'}} := \|\phi\|_{L^{r'}(0, T; \tilde{L}^1)} + \|\phi\|_{L^{r'}(0, T; \tilde{D}^{1, q'})}. \]

The dual space to $\mathcal{T}^{1, r', q'}(\Omega)$ is denoted by $\mathcal{T}^{-1, r, q}(\Omega)$.

**Definition 2.2.** Let $\Omega \subseteq \mathbb{R}^3$ be a uniform $C^2$-domain, $0 < T < \infty$ and $2 < r < \infty$, $3 < q < \infty$ and $2/r + 3/q = 1$. For an external force $F \in \mathcal{T}^{-1, r, q}(\Omega)$ we call $u \in L^r(0, T; \tilde{L}^q(\Omega))$ a very weak solution to the Navier-Stokes system with data $F$ if the conditions

\[ - (u, \phi_t)_{T, \Omega} - (u, \Delta \phi)_{T, \Omega} - (u \otimes u, \nabla \phi)_{T, \Omega} = \langle F, \phi \rangle, \]  

\[ (u, \nabla \psi)_{T, \Omega} = 0 \]  

hold for all test functions $\phi \in \mathcal{T}^{1, r', q'}(\Omega)$ and $\nabla \psi \in L^{r'}(0, T; \tilde{L}^{q'}(\Omega))$.

**Remark 1.** (i) Concerning very weak solutions to the nonstationary Stokes system the nonlinear term $(u \otimes u, \nabla \phi)_{T, \Omega} \in (19)$ is omitted. In that case $1 < q, r < \infty$ can be chosen arbitrarily, ignoring the Serrin condition $2/r + 3/q = 1$.

(ii) For more concrete conditions on the data functional $F$, including the higher dimensional case $n \geq 3$ and the case of nonhomogeneous boundary data as well as nonsolenoidal velocity fields, we refer to Propositions 2.4, 3.4, 4.5, and Corollary 4.6 in [11].

**Theorem 2.3 (Existence of Very Weak Solutions).** (Cf. [11, Theorem 1.3]) Let $\Omega \subseteq \mathbb{R}^3$ be a uniform $C^2$-domain and let $0 < T < \infty$. Assume that $F \in \mathcal{T}^{-1, r, q}(\Omega)$, where $2 < r < \infty$, $3 < q < \infty$, and $\frac{2}{r} + \frac{3}{q} = 1$.

(i) There exists an $\eta = \eta(\text{type}(\Omega), q, T) > 0$ with the following property: if

\[ \|F\|_{\mathcal{T}^{-1, r, q}(\Omega)} \leq \eta, \]  

then there exists a unique very weak solution $u \in L^r(0, T; \tilde{L}^q(\Omega))$ to the Navier-Stokes system with data $F$ in the sense of Definition 2.2. The a priori estimate

\[ \|u\|_{L^r(0, T; \tilde{L}^q(\Omega))} \leq C\|F\|_{\mathcal{T}^{-1, r, q}(\Omega)} \]  

holds with a constant $C = C(\text{type}(\Omega), q, T)$.

(ii) There exists a $T' \in (0, T)$ such that there is a unique very weak solution $u \in L^r(0, T'; \tilde{L}^q(\Omega))$ to the Navier-Stokes system with data $F|_{[0, T']} \in \mathcal{T}^{-1, r, q}(T', \Omega)$. 

In the case of more regular data \( u_0, f_1, f_2 \) for \( F \) as in (17) the very weak solution \( u \) has the integral representation (variation of constants formula)
\[
  u(t) = (u_1(t) + u_2(t)) + u_3(t)
\]
\[
  = e^{-t \tilde{A}_4} u_0 + \int_0^t e^{-(t-\tau) \tilde{A}_4} \tilde{P} f_1(\tau) \, d\tau
\]
\[
  - \int_0^t \tilde{A}_4^{1/2} e^{-(t-\tau) \tilde{A}_4} (\tilde{A}_4^{-1/2} \tilde{P} \div (-f_2 + u \otimes u)) \, d\tau
\]
for \( 0 \leq t \leq T \). The term \( \tilde{A}_4^{-1/2} \tilde{P} \div F \) in (22) is defined in the sense of distributions (with solenoidal vector fields as test functions)
\[
  (\tilde{A}_4^{-1/2} \tilde{P} \div F, \tilde{A}_4^{1/2} \varphi)_\Omega = -(F, \nabla \varphi)_\Omega, \quad \varphi \in C_0^\infty(\Omega).
\]

For the application to questions of regularity we need that the very weak solution in Theorem 2.3 is contained in \( L^4(0, T; L^4(\Omega)) \) as well. The following Proposition describes conditions on the data under which this property will hold.

**Proposition 1.** Let \( \Omega \subset \mathbb{R}^3 \) be a uniform \( C^2 \)-domain, \( 0 < T < \infty \), and let the exponents \( r, q, \gamma_1, \gamma_2 \) satisfy (3). Assume data \( u_0 \in L^2_\sigma(\Omega) \) such that
\[
e^{-\tau \tilde{A}_4} u_0 \in L^4(0, T; \bar{L}^4(\Omega)) \cap L^r(0, T; \bar{L}^q(\Omega)),
\]
and either \( f_1 \in L^{4/3}(0, T; L^{2}(\Omega)) \) or \( f_1 \in L^{r}(0, T; \bar{L}^{\gamma_1}(\Omega)) \) together with the condition
\[
f_1 \in L^4(0, T; \bar{L}^{12/11}(\Omega)) \cap L^{8/7}(0, T; L^2(\Omega)),
\]
and either \( f_2 \in L^4(0, T; L^2(\Omega)) \) (and \( q \leq 6 \)) or \( f_2 \in L^r(0, T; \bar{L}^{\gamma_2}(\Omega)) \) together with the condition
\[
f_2 \in L^4(0, T; \bar{L}^{12/7}(\Omega)) \cap L^{8/3}(0, T; L^2(\Omega)).
\]

Then there exists a constant \( \eta = \eta(\text{type}(\Omega), q, T) > 0 \) with the following property: if
\[
  \int_0^T \| e^{-\tau \tilde{A}_4} u_0 \|_{L^q(\Omega)}^r \, d\tau \leq \eta
\]
and
\[
  \| f_1 \|_{L^r(0, T; \bar{L}^{\gamma_1}(\Omega))} \leq \eta \quad \text{or} \quad \| f_1 \|_{L^{4/3}(0, T; L^{2}(\Omega))} \leq \eta,
\]
and
\[
  \| f_2 \|_{L^r(0, T; \bar{L}^{\gamma_2}(\Omega))} \leq \eta \quad \text{or} \quad \| f_2 \|_{L^4(0, T; L^{2}(\Omega))} \leq \eta \quad \text{if} \quad q \leq 6,
\]
then there is a very weak solution \( u \in L^r(0, T; \bar{L}^q(\Omega)) \) to (19) with data \( F \) as in (17) additionally satisfying \( u \in L^4(0, T; L^4(\Omega)) \).

**Proof.** The result is a special case of [11, Corollary 4.6]. The main idea of the proof is to show that not only \( F \in \mathcal{T}^{-1, r, q}(T, \Omega) \), but also \( F \in \mathcal{T}^{-1, 4, 4}(T, \Omega) \), and to apply [11, Theorem 4.4]. \( \square \)

**Remark 2.** We note that the \( L^4(\bar{L}^4) \)-condition in (24) is satisfied when \( u_0 \in \bar{L}^\alpha(\Omega) \) where \( \frac{2q}{r} < \gamma \leq 4 \); it suffices to apply (15).

The next proposition describes assumptions on \( u_0 \) to guarantee the condition \( F \in \mathcal{T}^{-1, r, q}(T, \Omega) \) for Serrin exponents \( r \) and \( q \); the results are part of [11, Proposition 2.4].
Let $\Omega \subseteq \mathbb{R}^3$ be a uniform $C^2$-domain, $0 < T < \infty$, and let Serrin exponents $2 < r < \infty$, $3 < q < \infty$, $\frac{2}{3} + \frac{3}{q} = 1$ be given. Then the following conditions on $u_0$ are sufficient for $F$ defined by $(F, \phi) = (u_0, \phi(0))$ to be contained in the data space $T^{-1, r, q}(T, \Omega)$.

The “optimal” condition in terms of real interpolation theory is

$$u_0 \in \left( \tilde{D}^{-1}_q, \tilde{L}^q_3(\Omega) \right)_{1/r', r'}$$

i.e. $u_0 \in \tilde{D}^{-1}_q$ and $\int_0^T \|e^{-t\hat{A}_v}u_0\|_{L^q}^r \, dt < \infty$. In particular, the conditions $u_0 \in \tilde{L}^q_3(\Omega)$ and $\int_0^T \|e^{-t\hat{A}_v}u_0\|_{L^q}^r \, dt < \infty$ where $1 < \rho < \infty$ imply that $u_0 \in \left( \tilde{D}^{-1}_q, \tilde{L}^q_3(\Omega) \right)_{1/r', r'}$.

Moreover, $u_0 \in \tilde{L}^3_3(\Omega)$ and, if even $r \geq 3$, $u_0 \in \tilde{L}^3_3(\Omega)$ are sufficient conditions. Finally, the $L^2$-conditions $u_0 \in L^2_2(\Omega)$ together with

$$\int_0^T \left\| (1 + \tilde{A}_2) \tilde{A}_2 e^{-t\tilde{A}_2}u_0 \right\|_{L^2}^r \, dt < \infty$$

are sufficient. This inegrability condition is satisfied when $u_0 \in \tilde{D}^{1/4}_2$.

3. Proofs and further results.

3.1. Identification of weak and very weak solutions. It is well-known that Serrin’s condition $u \in \tilde{L}^r_2(0, T; L^q(\Omega))$, $2 < r < \infty$, $3 < q < \infty$, $\frac{2}{r} + \frac{3}{q} = 1$, is a sufficient condition for a weak solution $u$ in the sense of Leray and Hopf to be unique and regular, see e.g. [16, Ch. V, Theorems 1.5.1, 1.8.1, 1.8.2]. Therefore, the following definition is useful:

Definition 3.1. Let $u$ be a weak solution to the Navier-Stokes system on $\Omega \times (0, T)$ in the sense of Leray and Hopf. Then a point $t \in (0, T)$ is called regular point of $u$ if there exist $\delta > 0$ (with $\delta \leq \min(t, T - t)$) and exponents $2 < r < \infty$ and $3 < q < \infty$ with $\frac{2}{r} + \frac{3}{q} = 1$ such that $u \in L^r(t - \delta, t + \delta; L^q(\Omega))$.

Proof of Theorem 1.1. We will show that the very weak solution $u$ is also a weak solution in the sense of Leray and Hopf. Using Serrin’s uniqueness criterion we conclude that $u$ and $\tilde{u}$ coincide almost everywhere.

First we show that $u$ admits the integral representation (variation of constants formula) $u = (u_1 + u_2) + u_3$, see (22). Let $t_0 \in (0, T)$ be a Lebesgue point of the Bochner functions $u, u_1, u_2$ and $u_3$, and let $\psi \in C^\infty_{0, \varepsilon}(\Omega)$ be arbitrary. Choose $\varepsilon > 0$ small enough and define

$$v_\varepsilon(t) := \frac{1}{2\varepsilon} \chi_\varepsilon(t) \psi$$

where $\chi_\varepsilon$ denotes the characteristic function of $[t_0 - \varepsilon, t_0 + \varepsilon] \subseteq (0, T)$. Moreover, we consider the strong solution $\phi_\varepsilon \in T^{1, r', q'}(T, \Omega)$ of the backward Stokes problem $-(\phi_\varepsilon)_t + \tilde{A}_v \phi_\varepsilon = v_\varepsilon$ with initial value $\phi_\varepsilon(0) = 0$. Recall that $\phi_\varepsilon$ has the representation, cf. (12),

$$\phi_\varepsilon(t) = \int_0^{T-t} e^{-(T-t-\tau)} \tilde{A}_v v_\varepsilon(T - \tau) \, d\tau, \quad t \in [0, T].$$

We then find that

$$u(t_0, \psi) = \frac{1}{2\varepsilon} \lim_{\varepsilon \to 0} \int_{t_0 - \varepsilon}^{t_0 + \varepsilon} (u(t), \psi) \, dt = \lim_{\varepsilon \to 0} (u, v_\varepsilon)_{T, \Omega},$$

$$u_i(t_0, \psi) = \lim_{\varepsilon \to 0} (u_i, v_\varepsilon)_{T, \Omega}, \quad i = 1, 2, 3.$$
For $u_2$ we see that
\[
(u_2, v_\varepsilon)_{T, \Omega} = \int_0^T (u_2(T - \tau), v_\varepsilon(T - \tau)) \, d\tau \\
= \int_0^T \left( \int_0^{T-\tau} e^{-(T-\tau-t)\tilde{A}_2} \tilde{P}_2 f_1(t) \, dt \, , \, v_\varepsilon(T - \tau) \right) \, d\tau \\
= \int_0^T \int_0^{T-\tau} (f_1(t), e^{-(T-\tau-t)\tilde{A}_2} v_\varepsilon(T - \tau)) \, dt \, d\tau.
\]
Changing order of integration we get that
\[
(u_2, v_\varepsilon)_{T, \Omega} = \int_0^T \left( f_1(t) \, , \, \int_0^{T-t} e^{-(T-t-\tau)\tilde{A}_2} \tilde{\varepsilon} \varepsilon v_\varepsilon(T - \tau) \, d\tau \right) \, dt \\
= \int_0^T (f_1(t), \tilde{\phi}_\varepsilon(t)) \, dt \\
= (f_1, \tilde{\phi}_\varepsilon)_{T, \Omega}.
\]
By analogy, we prove that $u_3$ satisfies $(u_1, v_\varepsilon)_{T, \Omega} = (u_0, \phi_\varepsilon(0))$. Finally, for $u_3$ we find, using the abbreviation $F := -f_2 + u \otimes u$, that
\[
(u_3, v_\varepsilon)_{T, \Omega} = \int_0^T (u_3(T - \tau), v_\varepsilon(T - \tau)) \, d\tau \\
= -\int_0^T \left( \tilde{A}_{1/2}^{1/2} \int_0^{T-\tau} e^{-(T-\tau-t)\tilde{A}_2} (\tilde{\varepsilon} \varepsilon \varepsilon + \tilde{\varepsilon} \varepsilon \varepsilon + \tilde{\varepsilon} \varepsilon \varepsilon) \, dt \, , \, v_\varepsilon(T - \tau) \right) \, d\tau.
\]
Since $v_\varepsilon(T - \tau) \in C^\infty_{0, \sigma}(\Omega)$ and the semigroup $e^{\tau \tilde{A}_2}$ commutes with fractional powers of $\tilde{A}_2$, we can continue by
\[
(u_3, v_\varepsilon)_{T, \Omega} = -\int_0^T \int_0^{T-\tau} ((\tilde{A}^{-1/2} \tilde{P} \varepsilon \varepsilon \varepsilon + \tilde{\varepsilon} \varepsilon \varepsilon \varepsilon + \tilde{\varepsilon} \varepsilon \varepsilon) \, dt \, , \, \tilde{A}^{1/2} e^{-(T-\tau-t)\tilde{A}_2} v_\varepsilon(T - \tau) \right) \, d\tau \\
= \int_0^T \int_0^{T-\tau} (F(t), \tilde{A}^{1/2} e^{-(T-\tau-t)\tilde{A}_2} v_\varepsilon(T - \tau) \, dt \, d\tau
\]
Moreover, changing the order of integration, we get that
\[
(u_3, v_\varepsilon)_{T, \Omega} = \int_0^T \int_0^{T-t} ((F(t), \tilde{A}^{1/2} e^{-(T-\tau-t)\tilde{A}_2} v_\varepsilon(T - \tau)) \, d\tau \, dt \\
= \int_0^T \left( F(t) \, , \, \tilde{A}^{1/2} e^{-(T-\tau-t)\tilde{A}_2} v_\varepsilon(T - \tau) \right) \, dt \\
= \int_0^T (F(t), \tilde{\phi}_\varepsilon(t)) \, dt \\
= (f_1, \tilde{\phi}_\varepsilon)_{T, \Omega}.
\]
Altogether, using the definition of the very weak solution $u$ and the test function $\phi_\varepsilon$, cf. (17), (19), we find that
\[
(u_1 + u_2 + u_3, v_\varepsilon)_{T, \Omega} = (u_0, \phi_\varepsilon(0)) + (f_1, \phi_\varepsilon)_{T, \Omega} - (f_2, \tilde{\nabla} \phi_\varepsilon)_{T, \Omega} + (u \otimes u, \tilde{\nabla} \phi_\varepsilon)_{T, \Omega} \\
= -(u, (\phi_\varepsilon))_{T, \Omega} + (u, \tilde{\phi}_q \phi_\varepsilon)_{T, \Omega} \\
= (u, v_\varepsilon)_{T, \Omega}.
\]
Now we let $\varepsilon \to 0$ to see that $(u(t_0), \psi) = (u_1(t_0) + u_2(t_0) + u_3(t_0), \psi)$ for all $\psi \in C_{0,\sigma}^\infty(\Omega)$. This proves the integral formula

$$u(t) = u_1(t) + u_2(t) + u_3(t)$$

for almost all $t \in (0, T)$.

Now we argue by [16, Ch. V, Theorems 2.4.1, 2.3.1] that $u \in L_{\text{loc}}^1(0, T; W_{0,\sigma}^{1,2}(\Omega))$ and that it is even a weak solution to the Navier-Stokes equations in the sense of Leray and Hopf with data $u_0, f_1, f_2$. Here we also used that $(u \otimes u, \nabla \phi) = (u \cdot \nabla u, \Phi)$ and the crucial assumption that $u \in L^4(0, T; L^4(\Omega))$.

To finish the proof we remark that $u$ as very weak solution is contained in Serrin’s uniqueness class $L^\ast(0, T; L^3)$. Since the weak solution $\tilde{u}$ satisfies the energy inequality, due to [16, Ch. V, Theorem 1.5.1], $u$ coincides with $\tilde{u}$. \hfill $\square$

This theorem can be used to prove the short-time existence and uniqueness of regular (strong) solutions.

**Corollary 1.** Let $\Omega \subseteq \mathbb{R}^3$ be a uniform $C^2$-domain and $0 < T < \infty$. Assume that $u_0 \in L^2(\Omega)$ or $u_0 \in D^{1/4}_2$, $f_1 \in L^{1/3}(0, T; L^2(\Omega))$, and $f_2 \in L^4(0, T; L^2(\Omega))$.

Then there exists $0 < T' \leq T$ such that there exists a unique weak solution to the Navier-Stokes system with data $u_0, f_1, f_2$ in the sense of Leray-Hopf on the interval $[0, T')$, which satisfies the energy inequality. This solution satisfies also $u \in L^r(0, T'; L^q(\Omega))$ for all exponents $4 \leq r \leq 16$, $\frac{2}{r} \leq q \leq 6$, $\frac{3}{2} + \frac{3}{q} = 1$. Thus all $t \in (0, T')$ are regular points.

**Proof.** Since $f_1 \in L^1(0, T; L^2(\Omega))$, $f_2 \in L^2(0, T; L^2(\Omega))$ we conclude from [16, Ch. V, Theorems 2.4.1, 2.3.1] that there exist weak solutions satisfying the energy inequality. Moreover, by Proposition 1 we get the existence of a very weak solution $u \in L^r(0, T'; L^q(\Omega))$ on some time interval $[0, T')$, which has the additional property $u \in L^3(0, T; L^4(\Omega))$. Now Theorem 1.1 implies that $u$ coincides on $[0, T')$ with each given weak solution.

It is clear that $u \in L^r(0, T'; L^q(\Omega))$ for all exponents $(r, q)$ as in the formulation of the corollary. \hfill $\square$

3.2. **Local and global regularity criteria.** First we present an abstract result which will be the key to all regularity criteria in the sequel.

**Lemma 3.2.** Let $\Omega \subseteq \mathbb{R}^3$ be a uniform $C^2$-domain, $0 < T < \infty$ and let data $u_0 \in L^2(\Omega)$, $f_1 \in L^{8/7}(0, T; L^2(\Omega))$ and $f_2 \in L^{13/3}(0, T; L^2(\Omega))$ be given. Let $u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W_{0,\sigma}^{1,2}(\Omega))$ be a weak solution to the Navier-Stokes equations with the given data, satisfying the strong energy inequality.

Fix some exponents $r, q, \gamma_1, \gamma_2$ satisfying (3). Then there is a constant $\eta = \eta(\text{type}(\Omega), T, q) > 0$ with the following property: If $0 < t_0 < t_1 < T$ are given such that $u(t_0) \in L^4(\Omega)$ and

$$\int_{t_0}^{t_1-t_0} \|e^{-\tau A_2} u(t_0)\|_{L^r(\Omega)}^r \, d\tau \leq \eta,$$

$$\|f_1\|_{L^r(t_0, t_1; L^{\gamma_1}(\Omega))} \leq \eta, \quad \|f_2\|_{L^r(t_0, t_1; L^{\gamma_2}(\Omega))} \leq \eta,$$

then $u \in L^r(t_0, t_1; L^q(\Omega))$. Hence each $t \in (t_0, t_1)$ is a regular point.

**Remark 3.** Note that the existence of a weak solution satisfying the strong energy inequality as in Lemma 3.2 has not yet been proven for the general data. Only the case $f_2 = 0$ and $f_1 \in L^{5/4}(0, T; L^2(\Omega))$ is treated, cf. [5, Theorem 2.7].
Proof of Lemma 3.2. Define the data functional \( F \in T^{-1,r,q}(t_1 - t_0, \Omega) \) by
\[
(F, \phi) = (u(t_0), \phi(0)) + \int_0^{t_1 - t_0} \left( (f_1(t_0 + \tau), \phi(\tau)) - (f_2(t_0 + \tau), \nabla \phi(\tau)) \right) d\tau
\]
for every \( \phi \in T^{1,r',q'}(t_1 - t_0, \Omega) \). By the assumptions on \( f_1 \) and \( f_2 \) and also by the assumption that \( u(t_0) \in \tilde{L}^4(\Omega) \). We can use Proposition 1 to find a very weak solution \( v \in L^r(0, t_1 - t_0; \tilde{L}^q(\Omega)) \) with the additional property \( v \in L^4(0, t_1 - t_0; \tilde{L}^4(\Omega)) \).

Now we define \( \tilde{u}(\tau) := u(\tau + t_0) \). It is readily seen that \( \tilde{u} \) is a weak solution on \( [0, t_1 - t_0] \) in the sense of Leray and Hopf with data \( u(t_0) \), \( f_1(\cdot + t_0) \), \( f_2(\cdot + t_0) \) satisfying the (usual) energy inequality since \( u(t_0) \in \tilde{L}^4(\Omega) \). Now Theorem 1.1 implies that \( \tilde{u} = v \in L^r(0, t_1 - t_0; \tilde{L}^q(\Omega)) \). This proves that \( u \in L^r(t_0, t_1; \tilde{L}^q(\Omega)) \) finishing the proof. \( \Box \)

Another lemma on local uniqueness of weak solutions will be convenient:

Lemma 3.3. Let \( \Omega \subseteq \mathbb{R}^3 \) be a uniform \( C^2 \)-domain, \( 0 < T < \infty \). Let \( u \) and \( v \) be weak solutions in the sense of Leray and Hopf to the Navier-Stokes equations with initial data \( u_0 \in L^2(\Omega) \) and \( v_0 \in L^2(\Omega) \), respectively, and vanishing external forces \( f_1 = 0, f_2 = 0 \). Assume that \( u \) and \( v \) are weakly continuous with values in \( L^2(\Omega) \), and choose Serrin exponents \( \frac{16}{5} \leq r \leq 16, \frac{24}{7} \leq q \leq 8, \frac{2}{r} + \frac{3}{q} = 1 \).

If at some point \( t_0 \in [0, T) \) the following conditions hold:

\begin{itemize}
  \item \( u(t_0) = v(t_0) \) and \( u(t_0) \in \tilde{L}^2(\Omega) \) or \( u(t_0) \in \tilde{L}^{3,r}(\Omega) \) or \( u(t_0) \in \tilde{D}^{1/4}_2 \).
  \item For every \( t \geq t_0 \) it holds that
    \[
    \frac{1}{2} \| u(t) \|_2^2 + \int_{t_0}^{t} \| \nabla u \|_2^2 d\tau \leq \frac{1}{2} \| u(t_0) \|_2^2;
    \]
    \[
    \frac{1}{2} \| v(t) \|_2^2 + \int_{t_0}^{t} \| \nabla v \|_2^2 d\tau \leq \frac{1}{2} \| v(t_0) \|_2^2.
    \]
\end{itemize}

Then there exists \( \delta > 0 \) such that \( u = v \in L^r(t_0, t_0 + \delta; \tilde{L}^q(\Omega)) \).

Proof. First of all note that the functional \( F \) defined by \( \phi \mapsto (u(t_0), \phi(0)) \) is contained in \( T^{-1,r,q}(T, \Omega) \cap T^{-1,4,4}(T, \Omega) \) by Proposition 2 and Remark 2. It follows by Theorem 2.3 (ii) that there exists a very weak solution \( w \in L^r(0, \delta; \tilde{L}^q(\Omega)) \) with some \( \delta > 0 \) to the initial datum \( u(t_0) \). Since \( F \in T^{-1,4,4}(T, \Omega) \) we may conclude from Proposition 1 that even \( w \in L^4(0, \delta; \tilde{L}^4(\Omega)) \). Note that the functions \( \tilde{u}(t) := u(t + t_0) \) and \( \tilde{v}(t) := v(t + t_0) \) are both weak solutions in the sense of Leray and Hopf on \( [0, \delta] \) with initial value \( u(t_0) = v(t_0) \). Moreover, by assumption, they both satisfy the (usual) energy inequality. It follows by Theorem 1.1, that \( \tilde{u} = w = \tilde{v} \) on \( [0, \delta] \). Consequently, \( u = v \in L^r(t_0, t_0 + \delta; \tilde{L}^q(\Omega)) \), finishing the proof. \( \Box \)

As an application we get a result which can be considered as extension of Serrin's uniqueness theorem. It generalizes the uniqueness result for weak solutions in the class \( L^\infty(0, T; \tilde{L}^3(\Omega)) \) of [12, 13] from bounded domains to general unbounded domains.

Theorem 3.4. Let \( \Omega \subseteq \mathbb{R}^3 \) be a uniform \( C^2 \)-domain, \( 0 < T < \infty \), and \( u_0 \in L^2(\Omega) \). For simplicity assume that \( f_1 = 0 \) and \( f_2 = 0 \). Choose Serrin exponents \( \frac{16}{5} \leq r \leq 16, \frac{24}{7} \leq q \leq 8, \frac{2}{r} + \frac{3}{q} = 1 \).
Let \( u \) and \( v \) both be weak solutions to the Navier-Stokes equations in the sense of Leray and Hopf with initial value \( u_0 \). Assume that \( u \) satisfies the strong energy inequality and that \( v \in L^\infty_{\text{loc}}([0, T); L^3(\Omega)) \).

Then \( u = v \) a.e. in \([0, T)\) and for every \( t \in [0, T) \) there is a \( \delta(t) > 0 \) such that \( u = v \in L^r(t, t + \delta(t); \tilde{L}^q_2(\Omega)) \).

**Proof.** First note that \( v(t) \) is even contained in the solenoidal space \( \tilde{L}^3_2(\Omega) \) for a.a. \( t \). Since \( v \in L^\infty_{\text{loc}}([0, T); L^3(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega)) \), we see from Hölder’s inequality that \( v \in L^4_{\text{loc}}([0, T); L^4(\Omega)) \). By [16, Theorem V.1.4.1] we also know that \( v \) satisfies the energy equality

\[
\frac{1}{2} \|v(t)\|^2 + \int_s^t \|\nabla v(\tau)\|^2 d\tau = \frac{1}{2} \|v(s)\|^2
\]

for all \( s \leq t \in [0, T) \), and \( v \colon [0, T) \to L^2_\sigma(\Omega) \) is strongly continuous after a possible redefinition on a null set. Then standard density and reflexivity arguments imply that \( v \colon [0, T) \to L^2_\sigma(\Omega) \) is weakly continuous.

In particular, \( u_0 = v(0) \in L^2_\sigma(\Omega) \). By Lemma 3.3, we find that \( u = v \) on a small interval \([0, \delta)\), \( \delta > 0 \), and that \( u = v \in L^r(0, \delta; \tilde{L}^q_2(\Omega)) \). Now we let

\[
T_* := \sup\{0 \leq T' \leq T : \, u = v \text{ in } [0, T')\}.
\]

By what we just noted, \( T_* \geq \delta > 0 \). So let us assume that \( 0 < T_* < T \).

By the weak continuity of both \( u \) and \( v \) with values in \( L^2_\sigma(\Omega) \), it follows that \( u(T_* = v(T_*)) \) and by weak continuity of \( v \) with values in \( L^2_\sigma(\Omega) \) that even \( u(T_* = v(T_*)) \in \tilde{L}^2_\sigma(\Omega) \).

Since \( u \) satisfies the strong energy inequality we find \( (t_j)_j \subset (0, T_* \) with \( t_j \nearrow T_* \) such that

\[
\frac{1}{2} \|u(t_j)\|^2 + \int_{t_j}^{t} \|\nabla u(\tau)\|^2 d\tau \leq \frac{1}{2} \|u(t_j)\|^2
\]

for all \( t_j \geq T_* \). For the term \( \frac{1}{2} \|u(t_j)\|^2 \) we get, using the energy equality for \( v \), that

\[
\frac{1}{2} \|u(t_j)\|^2 = \frac{1}{2} \|v(t_j)\|^2 = \frac{1}{2} \|v(T_*)\|^2 + \int_{t_j}^{T_*} \|\nabla v(\tau)\|^2 d\tau
\]

\[
\leq \frac{1}{2} \|u(T_*)\|^2 + \int_{t_j}^{T_*} \|\nabla v(\tau)\|^2 d\tau.
\]

Hence, taking the limit \( j \to \infty \), we find the energy inequality

\[
\frac{1}{2} \|u(t)\|^2 + \int_{T_*}^{t} \|\nabla u(\tau)\|^2 d\tau \leq \frac{1}{2} \|u(T_*)\|^2
\]

for all \( t > T_* \).

Now we again use Lemma 3.3 to find some \( \delta_1 > 0 \) such that \( u = v \in L^r(T_*, T_* + \delta_1; \tilde{L}^q_2(\Omega)) \) contradicting the definition of \( T_* \). Hence \( T_* = T \).

Finally, since \( v \) satisfies the energy equality and \( v(t) \in \tilde{L}^3_2(\Omega) \) for all \( t \), Lemma 3.3 proves that \( u = v \in L^r(t, t + \delta(t); \tilde{L}^q_2(\Omega)) \) for all \( t \in [0, T) \) and some \( \delta(t) > 0 \).

A slight modification of the above proof shows the following result:

**Corollary 2.** Under the basic assumptions of Theorem 3.4 let \( u \) and \( v \) be weak solutions to the Navier-Stokes equations with initial data \( u_0 \in L^2_\sigma(\Omega) \). Assume that \( u \) satisfies the strong energy inequality and that \( v(t) \in \tilde{L}^3_2(\Omega) \) or \( v(t) \in \tilde{D}^{1/4}_2 \) or
Lemma 3.2 and choose $\delta > 0$ small enough to fulfill the last two conditions in Lemma 3.2, i.e.,

$$\|f_1\|_{L^r(t-\delta_0,t+\delta_0;L^{\gamma_1})} \leq \eta', \quad \|f_2\|_{L^r(t-\delta_0,t+\delta_0;L^{\gamma_2})} \leq \eta'.$$

Next, for every $\delta > 0$ we find $t_0 \in (t-\delta,t)$ such that $u$ satisfies the energy inequality starting at $t_0$, $u(t_0) \in L^q(\Omega)$ and

$$(t + \delta - t_0)^{1-\alpha} \|u(t_0)\|_{L^q} \leq \frac{1}{\delta} \int_{t-\delta}^{t} (t + \delta - \tau)^{1-\alpha} \|u(\tau)\|_{L^q} d\tau.$$

Moreover, let $t_1 = t + \delta$. Now we use the $L^r-L^q$-estimate (15) to find that

$$\int_{t}^{t_1} \|e^{-rA_t} u(t_0)\|_{L^q} d\tau \leq C \int_{t}^{t_1} \left(\tau - \frac{3(1/q_0-1/q)}{2}\right) \|u(t_0)\|_{L^q} d\tau = C(t_1 - t_0)^{r(1-\alpha)/r_0} \|u(t_0)\|_{L^q},$$

with $\alpha = \frac{2}{3} \left(\frac{2}{q_0} + \frac{3}{q_0} - 1\right)$. Then we can continue by

$$\int_{t}^{t_1} \|e^{-rA_t} u(t_0)\|_{L^q} d\tau \leq C(t_1 - t_0)^{1-\alpha} \|u(t_0)\|_{L^q}^{r/r_0},$$

since $t + \delta - \tau \leq 2\delta$. By assumption (6) we find $0 < \delta \leq \delta_0$ such that the right-hand term in the last inequality above is smaller than $\eta'$, if only the new constant $\eta > 0$ is chosen small enough (set $\eta := \eta^{r_0/r/C}$). This proves that for this choice of $t_0$ and $t_1$ the first condition in Lemma 3.2 is satisfied. Since $\delta \leq \delta_0$ also the second and third condition is fulfilled. Thus Lemma 3.2 proves regularity of the point $t$.

Next we prove that the condition (6) is necessary in case $q_0 = q$. Let $t$ be a regular point of $u$. Then, for any $1 \leq r_0 \leq r$ we get by Hölder’s inequality

$$\frac{1}{\delta^{1-r_0/r}} \int_{t-\delta}^{t} \|u(\tau)\|_{L^q}^{r_0} d\tau \leq \left( \int_{t-\delta}^{t} \|u(\tau)\|_{L^q}^{r} d\tau \right)^{r_0/r} \to 0$$

for $\delta \to 0+$. This proves that the condition (6) is necessary in case $q_0 = q$.

It is only left to show that $u \in L^r(t - \varepsilon', t; L^{\gamma_1})$ is sufficient for $t$ to be regular. This is easily seen by choosing $q_0 = q$ and $r_0 = r$. Now $\alpha = 0$, and (6) reads

$$\liminf_{\delta \to 0+} \int_{t-\delta}^{t} \|u(\tau)\|_{L^q}^r d\tau \leq \eta,$$

which is obviously satisfied by the assumption in view of Lebesgue’s theorem on dominated convergence. This finishes the proof of Theorem 1.2. \qed
The next global regularity result is a consequence of Theorem 1.2.

**Corollary 3.** Let $\Omega \subseteq \mathbb{R}^3$ be a uniform $C^2$-domain, $0 < T < \infty$, let the exponents $r, q, \gamma_1, \gamma_2$ satisfy (3) and the data $f_1, f_2$ satisfy (4) as in Theorem 1.2. Assume that $u$ is a weak solution in the sense of Leray and Hopf with initial value $u_0 \in \dot{L}^q(\Omega)$, $3 < q_1 \leq q$, satisfying the strong energy inequality.

Assume two more exponents $1 \leq r_0 \leq r$ and $3 < q_0 \leq q$ are given. Then there exists a constant $\eta = \eta(\text{type}(\Omega), q, q_0, r, r_0, T)$ such that the conditions

$$
\|f_1\|_{L^r(0,T;\dot{L}^{\gamma_1}(\Omega))} + \|f_2\|_{L^r(0,T;\dot{L}^{\gamma_2}(\Omega))} \leq \eta
$$

and

$$
\|u\|_{L^q(0,T;\dot{L}^{\gamma_1}(\Omega))} \leq \eta\|u_0\|_{\dot{L}^{\gamma_1}(\Omega)}^\alpha, \quad \alpha := \frac{q_1}{q_1 - \frac{3}{r_0} - \frac{3}{q_0}},
$$

imply that $u \in L^r(0, T; \dot{L}^q(\Omega)).$

**Proof.** First of all note that the choice of data and Proposition 1 imply the existence of a very weak solution $\hat{u}$ at least on some possibly small finite interval $(0, \delta_1)$ with the additional property $\hat{u} \in L^4(0, \delta_1; L^2(\Omega)).$

Let us investigate the dependence of $\delta_1$ on the norm $\|u_0\|_{\dot{L}^{\gamma_1}}$. By the $L^r$-$\dot{L}^q$-estimate (15) we see that

$$
\int_0^{\delta_1} \|e^{-\hat{A}_q}u_0\|_{L^q}^r d\tau \leq C \int_0^{\delta_1} \tau^{-3r(1/q_1 - 1/q)/2} d\tau \|u_0\|_{\dot{L}^{\gamma_1}}^r = C_0 \|u_0\|_{\dot{L}^{\gamma_1}}^r \delta_1^{r(1-3/q_1)/2}.
$$

Let $\eta' > 0$ be a constant implying the existence of a very weak solution $\hat{u}$ on $(0, \delta_1)$ under the conditions

$$
\|f_1\|_{L^r(0,\delta_1;\dot{L}^{\gamma_1})} + \|f_2\|_{L^r(0,\delta_1;\dot{L}^{\gamma_2})} \leq \eta',
$$

$$
\int_0^{\delta_1} \|e^{-\hat{A}_q}u_0\|_{L^q}^r d\tau \leq \eta',
$$

cf. Theorem 2.3. Assume already that the constant $\eta$, which is to be chosen in this proof, is smaller than $\eta'$. Then, in order to fulfill the last inequality, we may choose

$$
\delta_1 := C_1 \|u_0\|_{\dot{L}^{\gamma_1}}^{2q_1/(3-q_1)},
$$

where $C_1 = (\eta'/C_0)^{2q_1/(r(3-q_1))}$ is a constant which depends only on type$(\Omega)$, $r$, $q$, $q_1$. The very weak solution $\hat{u}$ on $(0, \delta_1)$ must coincide almost everywhere with $u$ by Theorem 1.1. Hence we proved so far that $u \in L^r((0, \delta_1); \dot{L}^q(\Omega))$.

Now let $t \in [\delta_1, T)$. In this case choose $t_1 = t - \delta_1/2$. As in the previous proof we see that there exists a $t_0 \in (t_1 - \delta_1/2, t)$ such that $u(t_0) \in \dot{L}^{\gamma_1}(\Omega), u$ satisfies the energy inequality starting at $t_0$ and such that

$$
(t_1 - t_0)^{(1-3/q_0)/2} \|u(t_0)\|_{\dot{L}^{\gamma_1}} \leq \frac{2}{\delta_1} \int_{t_1}^{t_0} (t_1 - \tau)^{(1-3/q_0)/2} \|u(\tau)\|_{\dot{L}^{\gamma_1}} \|u(\tau)\|_{\dot{L}^{\gamma_1}} d\tau.
$$

Hence we proceed to get that

$$
\int_0^{t_1 - t_0} \|e^{-\hat{A}_q}u(t_0)\|_{L^q}^r d\tau \leq C \int_0^{t_1 - t_0} \tau^{-3r(1/q_0 - 1/q)/2} \|u(t_0)\|_{\dot{L}^{\gamma_1}}^r d\tau \leq C \left((t_1 - t_0)^{(1-3/q_0)/2} \|u(t_0)\|_{\dot{L}^{\gamma_1}}\right)^{r/q_0},
$$

where $C = C_1 \|u(t_0)\|_{\dot{L}^{\gamma_1}}^{2q_1/(3-q_1)}$. The proof is complete.
Here the first term can be estimated using the energy equality by
\[ \eta > 0 \] smallness in (6) will here be denoted by
\[ \eta > 0 \] in Theorem 1.2 we choose
\[ \eta > 0 \] the theorem and
\[ \eta > 0 \] is a weak solution satisfying the strong energy inequality.
From Theorem 1.2 we can also derive regularity conditions
that satisfies the energy estimate with starting point
\[ t \rightarrow 0^+ \]  is a regular point of
\[ t \rightarrow 0^+ \]  with a fixed constant
\[ C_2 = C_2(\text{type}(\Omega), q, r, q_0, q_1, r_0, T) \] Hence, if the constant
\[ \eta > 0 \] is chosen small enough we can apply Lemma 3.2 to find that
\[ t \rightarrow 0^+ \]  is a regular point. Now a compactness argument finishes the proof.

3.3. Energy criteria. From Theorem 1.2 we can also derive regularity conditions
in terms of \( \|u\|_2 \) or \( \|\nabla u\|_2 \), i.e., in terms describing physical energies. For simplicity
we assume \( f_1 = 0 \) and \( f_2 = 0 \).

**Theorem 3.5.** Let \( \Omega \subseteq \mathbb{R}^3 \) be a uniform \( C^2 \)-domain, \( 0 < T < \infty \). Assume
that \( u \) with initial data \( u_0 \in L^2_\gamma(\Omega) \) is a weak solution satisfying the strong energy
inequality.

There is a constant \( \eta = \eta(\text{type}(\Omega), T) \) such that the following holds: If for a
point \( t \in (0, T) \) it holds that
\[
\liminf_{\delta \to 0^+} \frac{1}{\delta^{3/2}} \int_{t-\delta}^t \|\nabla u(\tau)\|_{L^2(\Omega)}^2 d\tau \leq \eta,
\]
then \( t \) is a regular point of \( u \) in the sense that \( u \in L^4(t-\varepsilon, t+\varepsilon; \tilde{L}^6(\Omega)) \).

**Proof.** In Theorem 1.2 we choose \( q = q_0 = 6, r = 4, r_0 = 2 \). The constant for smallness in (6) will here be denoted by \( \eta' \) so that we have to show that
\[
\liminf_{\delta \to 0^+} \frac{1}{\delta^{3/2}} \int_{t-\delta}^t \|u(\tau)\|_{L^4}^4 d\tau \leq \eta'.
\]
We estimate by Sobolev’s embedding theorem
\[
\|u(\tau)\|_{L^6} \leq \|u(\tau)\|_{L^2} + C\|\nabla u(\tau)\|_{L^2}
\]
and find
\[
\frac{1}{\delta^{3/2}} \int_{t-\delta}^t \|u(\tau)\|_{L^4}^2 d\tau \leq \frac{1}{\delta^{3/2}} \int_{t-\delta}^t \|u(\tau)\|_{L^2}^2 d\tau + \frac{C}{\delta^{3/2}} \int_{t-\delta}^t \|\nabla u(\tau)\|_{L^2}^2 d\tau.
\]
Here the first term can be estimated using the energy equality by \( \delta^{1/2}\|u_0\|_{L^2}^2 \) which
tends to zero for \( \delta \to 0 \). The second term is smaller than \( \eta'/2 \) for small \( \delta > 0 \) if
only the new constant \( \eta > 0 \) is chosen small enough. So Theorem 1.2 implies the result.

**Proof of Theorem 1.3.** To use Theorem 3.5 let \( \eta' > 0 \) be the constant from that
theorem and \( \eta := \eta' \). Choose a sequence \( \delta_k \searrow 0 \) as \( k \to \infty \) with the property that
\( u \) satisfies the energy estimate with starting point \( t - \delta_k \) for all \( k \in \mathbb{N} \). For this
sequence we can estimate
\[
\frac{1}{\delta_k^{3/2}} \int_{t-\delta_k}^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \leq \frac{1}{\delta_k^{3/2}} \left( \frac{1}{2} \|u(t-\delta_k)\|_{L^2}^2 - \frac{1}{2} \|u(t)\|_{L^2}^2 \right).
\]
As $k \to \infty$ the right-hand side is bounded by $\eta = \eta'$ so that the preceding theorem shows that $t$ is a regular point. This finishes the proof.

REFERENCES

[1] H. Amann, Linear and Quasilinear Parabolic Problems. Vol. I: Abstract Linear Theory, Monographs in Mathematics, 89, Birkhäuser, Basel-Boston-Berlin, 1995.
[2] M. E. Bogovskij and V. N. Maslennikova, Elliptic boundary value problems in unbounded domains with noncompact and nonsmooth boundaries, Sem. Mat. Fis. Milano, 56 (1986), 125–138.
[3] M. E. Bogovskij, Decomposition of $L_p(\Omega; \mathbb{R}^n)$ into the direct sum of subspaces of solenoidal and potential vector fields, Math. Dokl., 33 (1986), 161–165.
[4] R. Farwig, G. P. Galdi and H. Sohr, A new class of weak solutions of the Navier-Stokes equations with nonhomogeneous data, J. Math. Fluid Mech., 8 (2006), 423–444.
[5] R. Farwig, H. Kozono and H. Sohr, An $L^q$-approach to Stokes and Navier-Stokes equations in general domains, Acta Math., 195 (2005), 21–53.
[6] R. Farwig, H. Kozono and H. Sohr, Very weak solutions of the Navier-Stokes equations in exterior domains with nonhomogeneous data, J. Math. Soc. Japan, 59 (2007), 127–150.
[7] R. Farwig, H. Kozono and H. Sohr, On the Helmholtz decomposition in general unbounded domains, Arch. Math., 88 (2007), 239–248.
[8] R. Farwig, H. Kozono and H. Sohr, The Stokes resolvent problem in general unbounded domains, in Kyoto Conference on the Navier-Stokes Equations and their Applications, RIMS Kökyüroku Bessatsu, Res. Inst. Math. Sci., B1, Kyoto, 2007, 79–91.
[9] R. Farwig, H. Kozono and H. Sohr, Maximal regularity of the Stokes operator in general unbounded domains, in Functional Analysis and Evolution Equations. The Günter Lumer Volume (eds. H. Amann, W. Arendt, M. Hieber, F. Neubrander, S. Nicaise and J. von Below), Birkhäuser Verlag, Basel, 2008, 257–272.
[10] R. Farwig, H. Kozono and H. Sohr, On the Stokes operator in general unbounded domains, Hokkaido Math. J., 38 (2009), 111–136.
[11] R. Farwig and P. F. Riechwald, Very weak solutions to the Navier-Stokes system in general unbounded domains, J. Evol. Equ., 15 (2015), 253–279.
[12] R. Farwig, H. Sohr and W. Varnhorn, Extensions of Serrin’s uniqueness and regularity conditions for the Navier-Stokes equations, J. Math. Fluid Mech., 14 (2012), 529–540.
[13] H. Kozono and H. Sohr, Remark on uniqueness of weak solutions to the Navier-Stokes equations, Analysis, 16 (1996), 255–271.
[14] P. C. Kunstmann, $H^\infty$-calculus for the Stokes operator on unbounded domains, Arch. Math., 91 (2008), 178–186.
[15] P. F. Riechwald, Interpolation of sum and intersection spaces of $L^q$-type and applications to the Stokes problem in general unbounded domains, Ann. Univ. Ferrara Sez. VII Sci. Mat., 58 (2012), 167–181.
[16] H. Sohr, The Navier-Stokes Equations. An Elementary Functional Analytic Approach, Birkhäuser Verlag, Basel, 2001.
[17] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, North-Holland Publ., Amsterdam, 1978.