Roots and (re)sources of value (in)definiteness versus contextuality.  
A contribution to the Pitowsky Volume in memory of Itamar Pitowsky (1950–2010)

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In Itamar Pitowsky’s reading of the Gleason and the Kochen-Specker theorems, in particular, his Logical Indeterminacy Principle, the emphasis is on the value indefiniteness of observables which are not within the preparation context. This is in stark contrast to the prevalent term contextuality used by many researchers in informal, heuristic yet omni-realistic and potentially misleading ways. This paper discusses both concepts and argues in favor of value indefiniteness in all but a single quantum context, defined by a maximal observable, or in terms of its spectral decomposition as the associated orthonormal basis.

Keywords: Value indefiniteness, Pitowsky’s Logical Indeterminacy Principle, Quantum mechanics, Gleason theorem, Kochen-Specker theorem, Born rule

I. INTRODUCTION

An upfront caveat seems in order: The following is a rather subjective narrative of my reading of Itamar Pitowsky’s thoughts about classical value indeterminacy on quantum logical structures of observables, amalgamated with my current thinking on related issues. I have never discussed these matters with Itamar Pitowsky explicitly; therefore the term “my reading” should be taken rather literally; namely as taken from his publications. In what follows classical value indefiniteness on collections of (intertwined) quantum observables will be considered a consequence, or even a synonym, of what he called indeterminacy. Whether or not this identification is justified is certainly negotiable; but in what follows this is taken for granted.

The term value indefiniteness has been stimulated by recursion theory [1–3], and in particular by partial functions [4] – indeed the notion of partiality has not diffused into physical theory formation, and might even appear alien to the very notion of functional value assignments – and yet it appears to be necessary [5–7] if one insists (somewhat superficially) on classical interpretations of quantized systems.

Value indefiniteness/indeterminacy will be contrasted with some related interpretations and approaches, in particular, with contextuality. Indeed, I believe that contextuality was rather foreign to Itamar Pitowsky’s thinking: the term “contextuality” appears marginally – as in “a different context” – in his book Quantum Probability - Quantum Logic [8], nowhere in his reviews on Boole-Bell type inequalities [8–10], and mostly with reference to contextual quantum probabilities in his late writings [11]. The emphasis on value indefiniteness/indeterminacy was, I believe, independently shared by Asher Peres as well as Ernst Specker.

I met Itamar Pitowsky [12] personally rather late; after he gave a lecture entitled “All Bell Inequalities” in Vienna [13] on September 6th, 2000. Subsequent discussions resulted in a joint paper [14] (stimulating further research [15, 16]). It presents an application of his correlation polytope method [8–10, 17, 18] to more general configurations than had been studied before. Thereby semi-automated symbolic as well as numeric computations have been used. But the violations of what Boole called [19, p. 229] “conditions of possible experience,” obtained through solving the hull problem of classical correlation polytopes was just one route to quantum indeterminacy pursued by Itamar Pitowsky. One could identify at least two more routes he contributed to: One approach compares differences of classical with quantum predictions through conditions and constraints imposed by certain intertwined configurations of observables which I like to call quantum clouds [20]. And another approach pushes these predictions to the limit of logical inconsistency; such that any attempt of a classical description fails relative to the assumptions.

II. STOCHASTIC VALUE INDEFINITENESS/INDETERMINACY BY BOOLE-BELL TYPE CONDITIONS OF POSSIBLE EXPERIENCE

Since I have reviewed this subject exhaustively [21, Sect. 12.9] (see also Ref. [22]) I will just sketch it to obtain a taste for its relevance for quantum indeterminacy. As is often the case in mathematical physics the method seems to have been envisioned independently a couple of times. Froissart [23, 24] might have been the first explicitly proposing it as a method to generalized Bell-type inequalities. I suggested its usefulness for non-Boolean cases [25] with “enough” two-valued states; preferable sufficiently many to allow a proper distinction/separation of all observables. Consideration of the pentagon/pentagram logic – that is, five cyclically intertwined contexts/blocks/Boolean subalgebras/cliques/orthonormal bases rendered new predictions which could be used to differentiate classical from quantized systems [26–30].

The basic idea to obtain all classical predictions – including classical probabilities, expectations as well as consistency constraints thereof – associated with (mostly complementary; that is, non-simultaneously measurable) collections of observables is quite straightforward: Figure out all “extreme” cases or states which would be classically allowed. Then construct
all classically conceivable situations by forming suitable combinations of the former.

Formally this amounts to performing the following steps [8–10, 17, 18]:

- Contemplate about some concrete structure of observables and their interconnections in intertwining observables – the quantum cloud.
- Find all two-valued states of that quantum cloud. (In the case of “contextual inequalities” [31] include all variations of true/1 and false/0, irrespective of exclusivity; thereby often violating the Kolmogorovian axioms of probability theory even within a single context.)
- Depending on one’s preferences, form all (joint) probabilities and expectations.
- For each of these two-valued states, evaluate the joint probabilities and expectations as products of the single particle probabilities and expectations they are formed of (this reflects statistical independence of the constituent observables).
- For each of the two-valued states, form a tuple containing these relevant (joint) probabilities and expectations.
- Interpret this tuple as a vector.
- Consider the set of all such vectors – there are as many as there are two-valued states, and their dimension depends on the number of (joint) probabilities and expectations considered – and interpret them as vertices forming a convex polytope.
- The convex combination of all conceivable two-valued states yields the surface of this polytope; such that every point inside its convex hull corresponds to a classical probability distribution.
- Determine the conditions of possible experience by solving the hull problem – that is, by computing the hyperplanes which determine the inside–versus–outside criteria for that polytope. These then can serve as necessary criteria for all classical probabilities and expectations considered.

The systematic application of this method yields necessary criteria for classical probabilities and expectations which are violated by the quantum probabilities and expectations. A caveat: these criteria involve multiple summands which are not all simultaneously measurable. Therefore, different terms, when evaluated experimentally, correspond to different, complementary measurement configurations. They are obtained at different times, and on different particles.

Empirical findings are too numerous to even attempt a just appreciation of all the efforts that went into testing classicality. There is overwhelming evidence that the quantum predictions are correct; and that they violate Boole’s conditions of possible classical experience [32] relative to the assumptions (basically non-contextual realism and locality).

So, if Boole’s conditions of possible experience are violated, then they can no longer be considered appropriate for any reasonable ontology forcing “reality” upon them. This includes the realistic [33] existence of hypothetical counterfactual observables: “unperformed experiments seem to have no consistent outcomes” [34]. The inconsistency of counterfactuals (in Specker’s scholastic terminology infuturabilities [35, 36]) provides a connection to value indefiniteness/indeterminacy – at least, and let me again repeat earlier provisos, relative to the assumptions. More of this, piled higher and deeper, has been supplied by Itamar Pitowsky, as will be discussed later.

III. INTERLUDE: QUANTUM PROBABILITIES FROM PYTHAGOREAN “VIEWS ON VECTORS”

Quantum probabilities are vector based. At the same time it is classical if it needs to be classical: namely among mutually commuting observables; in particular, corresponding to projection operators which are either orthogonal (exclusive) or identical (inclusive). It is “contextual” (I assume he had succumbed to the prevalent nomenclature at the time) according to Itamar Pitowsky’s late late writings [11] if it needs not be classical: namely among non-commuting observables.

Thereby, classical probability theory is maintained for simultaneously co-measurable (that is, non-complementary) observables. This essentially amounts to the validity of the Kolmogorov axioms of probability theory of such observables within a given context/block/Boolean subalgebra/clique/orthonormal basis, whereby the probability of an event associated with an observable

- is a non-negative real number between 0 and 1;
- is 1 for an event associated with an observable occurring with certainty (in particular, by considering any observable or its complement); as well as
- additivity of probabilities for events associated with mutually exclusive observables.

Sufficiency is assured by an elementary geometric argument [37] which is based upon the Pythagorean theorem; and which can be used to explicitly construct vector-based probabilities satisfying the aforementioned Kolmogorov axioms within contexts: Suppose a pure state of a quantized system is formalized by the unit state vector |ψ⟩. Consider some orthonormal basis $\mathcal{B} = \{|e_1\rangle, \ldots, |e_n\rangle\}$ of $\mathcal{V}$. Then the square $P'_\psi(e_i) = |\langle \psi | e_i \rangle|^2$ of the length/norm $\sqrt{\langle \psi | e_i \rangle \langle e_i | \psi \rangle}$ of the orthogonal projection $(\langle e_i | \psi \rangle | e_i \rangle)$ of that unit vector |ψ⟩ along the basis element |e_i⟩ can be interpreted as the probability of the event associated with the 0 – 1-observable (proposition) associated with the basis vector |e_i⟩ (or rather the orthogonal projector $E_i = |e_i\rangle \langle e_i|$) associated with the dyadic product of the basis vector |e_i⟩; given a quantized physical system which has been prepared to be in a pure state |ψ⟩. Evidently, $1 \leq P'_\psi(e_i) \leq 1$, and $\sum_{i=1}^n P'_\psi(e_i) = 1$. In that Pythagorean way, every context, formalized by an orthonormal basis $\mathcal{B}$, “grants a (probabilistic) view” on the pure state |ψ⟩.
It can be expected that these Pythagorean-style probabilities are different from classical probabilities almost everywhere – that is, for almost all relative measurement positions. Indeed, for instance, whereas classical two-partite correlations are linear in the relative measurement angles, their respective quantum correlations follow trigonometric functions – in particular, the cosine for “singlets” [38]. These differences, or rather the vector-based Pythagorean-style quantum probabilities, are the “root cause” for violations of Boole’s aforementioned conditions of possible experience in quantum setups.

Because of the convex combinations from which they are derived, all of these conditions of possible experience contain only linear constraints [19, 39–59]. And because linear combinations of linear operators remain linear, one can identify the terms occurring in conditions of possible experience with linear self-adjoint operators, whose sum yields a self-adjoint operator, which stands for the “quantum version” of the respective conditions of possible experience. This operator has a spectral decomposition whose min-max eigenvalues correspond to the quantum bounds [60, 61], which thereby generalize the Tsirelson bound [62]. In that way, every condition of possible experience which is violated by the quantum probabilities provides a direct criterion for non-classicality.

**IV. CLASSICAL VALUE INDETERMINENESS/INDETERMINACY BY DIRECT OBSERVATION**

In addition to the “fragmented, explosion view” criteria allowing “nonlocality” via Einstein separability [63] among its parts, classical predictions from quantum clouds – essentially intertwined (therefore the Hilbert space dimensionality has to be greater than two) arrangements of contexts – can be used as a criterion for quantum “supremacy”. Thereby it is sufficient to observe of a single outcome of a quantized system which directly contradicts the classical predictions.

One example of such a configuration of quantum observables forcing a “one-zero rule” [64] because of a true-implies-false set of two-valued classical states (TIFS) [65] is the “Specker bug” logic [66, Fig. 1, p. 182] called “cat’s cradle” [11, 67] by Itamar Pitowsky (see also Refs. [68, Fig. B.1, p. 64], [69, p. 588-589], [70, Sects. IV, Fig. 2] and [71, p. 39, Fig. 2.4.6] for early discussions), as depicted in Fig. 1.

For such configurations, it is often convenient to represent both its labels as well as the classical probability distributions in terms of a partition logic [72] of the set of two-valued states – in this case, there are 14 such classical states. Every maximal observable is characterized by a context. The atoms of this context are labeled according to the indices of the two-valued measure with the value 1 on this atom. The axioms of probability theory require that, for each two-valued state, and within each context, there is exactly one such atom. As a result, as long as the set of two-valued states is separating [73, Theorem 0], one obtains a set of partitions of the set of two-valued states; each partition corresponding to a context.

Classically, if one prepares the system to be in the state \(\{1, 2, 3\}\) – standing for either one of the classical two-valued states 1, 2 or 3 or their convex combinations – then there is no chance that the “remote” target state \(\{7, 10, 13\}\) can be observed. A direct observation of quantum “supremacy” is then possibly by some faithful orthogonal representation (FOR) [74–77] of this graph. In the particular Specker but/cats cradle configuration, an elementary geometric argument [78, 79] forces the relative angle between the quantum states \(\{|1, 2, 3\}\) and \(\{|7, 10, 13\}\) in three dimensions to be not smaller than \(\arctan(2/\sqrt{3})\), so that the quantum prediction of the occurrence of the event associated with state \(\{|1, 2, 3\}\), if the system was prepared in state \(\{|1, 2, 3\}\) is that the probability can be at most \(\langle\{|1, 2, 3\}\{|7, 10, 13\}\rangle^2 = \cos^2 \left(\arctan(2/\sqrt{3})\right) = \frac{1}{9}\). That is, on the average, if the system was prepared in state \(\{|1, 2, 3\}\) at most one of 9 outcomes indicates that the system has the property associated with the observable \(\{|7, 10, 13\}\\rangle\langle\{|7, 10, 13\}\}\). The occurrence of a single such event indicates quantum “supremacy” over the classical prediction of non-occurrence.

This limitation is only true for the particular quantum cloud involved. Similar arguments with different quantum clouds resulting in TIFS can be extended to arbitrary small relative angles between preparation and measurement states, so that the relative quantum “supremacy” can be made arbitrarily high [7]. Classical value indefiniteness/indeterminacy comes naturally: because (relative to the assumptions) the existence of such classical definite values would enforce non-occurrence of outcomes which are nevertheless observed in quantized systems.

Very similar arguments against classical value definiteness can be inferred from quantum clouds with true-implies-true sets of two-valued states (TITS) [30, 65, 68-70, 80-90]. There the quantum “supremacy” is in the non-occurrence of outcomes which classical predictions mandate to occur.
V. CLASSICAL VALUE INDEFINITENESS/INDETERMINACY PILED HIGHER AND DEEPER: THE LOGICAL INDETERMINACY PRINCIPLE

For the next and final stage of classical value indefiniteness/indefiniteness on quantum clouds (relative to the assumptions) one can combine two logics with simultaneous classical TIFS and TITS properties at the same terminals. That is, suppose one is preparing the same state and measuring the same observable, while contemplating the simultaneous counterfactual existence of two different quantum clouds of intertwined contexts interconnecting those state and measured observable – albeit those quantum clouds induce contradicting classical predictions. In such a setup the only consistent choice (relative to the assumptions) is to abandon classical value definiteness/determinacy. Because the assumption of classical value definiteness/determinacy for any such logic, therefore, yields a complete contradiction, thereby eliminating prospects for hidden variable models [5, 7, 20] satisfying the assumptions.

Indeed, suppose that a quantized system is prepared in some pure quantum state. Then Itamar Pitowsky’s [91, 92] indeterminacy principle states that – relative to the assumptions, in particular, global classical value definiteness for all observables involved – any other distinct (non-collinear) observable which is not orthogonal can neither occur nor not occur. This can be seen as an extension of both Gleason’s theorem [37, 93] as well as the Kochen–Specker theorem [73] implying and utilizing the non-existence of any two-valued global truth assignments on even finite quantum clouds.

For the sake of a concrete example consider the two TIFS and TITS clouds – that is, logics with 35 intertwined binary observables (propositions) in 24 contexts – depicted in Fig. 2 [94]. They represent quantum clouds with the same terminal points \( \{1\} \equiv \{1'\} \) and \( \{2,3,4,5,6,7\} \equiv \{1',3',4',5'\} \), forcing the latter ones (that is, \( \{2,3,4,5,6,7\} \) and \( \{1',3',4',5'\} \)) to be false/0 and true/1, respectively, if the former ones (that is, \( \{1\} \equiv \{1'\} \)) are true/1.

Formally, the only two-valued states on the logics depicted in Figs. 2(a) and 2(b) which allow \( v(\{1\}) = v(\{1'\}) = 1 \) requires that \( v(\{2,3,4,5,6,7\}) = 0 \) but \( v(\{1',3',4',5'\}) = 1 - v(\{2,3,4,5,6,7\}) \), respectively. However, both these logics have a faithful orthogonal representation [7, Table. 1, p. 102201-7] in terms of vectors which coincide in \( \{|\{1\}\} = |\{1'\}| \), as well as in \( \{|2,3,4,5,6,7\} = |\{1',3',4',5'\}| \), and even in all of the other adjacent observables.

The combined logic, which features 37 binary observables (propositions) in 26 contexts has no more a classical interpretation in terms of a partition logic, as the 8 two-valued states enumerated in Table I cannot mutually separate [73, Theorem 0] the observables 2, 13, 15, 16, 17, 25, 27 and 36, respectively.

It might be amusing to keep in mind that, because of non-separability [73, Theorem 0] of some of the binary observables (propositions), there does not exist a proper partition logic. However, there exist generalized urn [95, 96] and finite automata [97–99] model realisations thereof: just consider urns “loaded” with balls which have no colored symbols on them; or no such balls at all, for the binary observables (propositions) 2, 13, 15, 16, 17, 25, 27 and 36. In such cases it is no more possible to empirically reconstruct the binary logic; yet if an underlying logic is assumed then – at least as long as there still are truth assignments/two-valued states on the logic – ”reduced” probability distributions can be defined, urns can be loaded, and automata prepared, which conform to the classical predictions from a convex combination of these truth assignments/two-valued states – thereby giving rise to

![FIG. 2. (a) TIFS cloud, and (b) TITS cloud with only a single overlaid classical value assignment if the system is prepared in state \( |\{1\}| \) [94]. (c) The combined cloud from (a) and (b) has no value assignment allowing 36 \( = \{\} \) to be true/1; but still allows 8 classical value assignments enumerated by Table I, with overlaid partial coverage common to all of them. A faithful orthogonal realization is enumerated in Ref. [7, Table. 1, p. 102201-7].](image-url)
TABLE I. Enumeration of the 8 two-valued states on 37 binary observables (propositions) of the combined quantum clouds/logics depicted in Figs. 2(a) and 2(b). Row vector indicate the state values on the observables, column vectors the values on all states per the respective observable.

VI. THE “MESSAGE” OF QUANTUM (IN)DETERMINACY

At the peril of becoming, as expressed by Clauser [32], “evangelical,” let me “sort things out” from my own very subjective and private perspective. (Readers adverse to “evangelical, ” let me “sort things out” from my own very subjective, global) truth assignments [91, 92] as well as for local admissibility rules allowing partial (as opposed to total, global) truth assignments [5, 7], such arguments can be extended to cover all terminal states which are neither collinear nor orthogonal. One could point out that, insofar as a fixed state has to be prepared the resulting value indefiniteness/indeterminacy is state dependent. One may indeed hold that the strongest indication for quantum value indefiniteness/indeterminacy is the total absence/non-existence of two-valued states, as exposed in the Kochen-Specker theorem [73]. But this is rather a question of nominalistic taste, as both cases have no direct empirical testability; and as has already been pointed out by Clifton in a private conversation in 1995: “how can you measure a contradiction?”

“reduced” conditions of experience via hull computations.

For global/local truth assignments [91, 92] as well as for local admissibility rules allowing partial (as opposed to total, global) truth assignments [5, 7], such arguments can be extended to cover all terminal states which are neither collinear nor orthogonal. One could point out that, insofar as a fixed state has to be prepared the resulting value indefiniteness/indeterminacy is state dependent. One may indeed hold that the strongest indication for quantum value indefiniteness/indeterminacy is the total absence/non-existence of two-valued states, as exposed in the Kochen-Specker theorem [73]. But this is rather a question of nominalistic taste, as both cases have no direct empirical testability; and as has already been pointed out by Clifton in a private conversation in 1995: “how can you measure a contradiction?”

Thus, in maintaining rationality one needs to grant oneself – or rather one is forced to accept – the abandonment of at least some or all assumptions made. Some options are exotic; for instance, Itamar Pitowsky’s suggestions to apply paradoxical set decompositions to probability measures [17, 104], or to allow only unconnected (non-intertwined) contexts whose observables are dense [105–107]. Some are quite straightforward, and we shall concentrate our further attention to these latter ones [64].

A. Simultaneous definiteness of counterfactual, complementary observables, and abandonment of context independence

Suppose one insists on the simultaneous definite omnexistence of mutually complementary, and therefore necessarily counterfactual, observables. One straightforward way to cope with the aforementioned findings is the abandonment of context-independence of intertwining observables.

There is no indication in the quantum formalism which would support such an assumption, as the respective projection operators do not in any way depend on the contexts involved. However, one may hold that the outcomes are context dependent as functions of the initial state and the context measured [108–110]; and that they actually “are real” and not just “idealistically occur in our imagination;” that is, being “mental through-and-through” [111]. Early conceptualizations of context-dependence aka contextuality can be found in Bohr’s remark (in his typical Nostradamus-like style) [112] on “the impossibility of any sharp separation between the behavior of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear.” Bell, referring to Bohr, suggested [113], Sec. 5) that “the result of an observation may reasonably depend not only on the state of the system (including hidden variables) but also on the complete disposition of the apparatus.”

However, the common, prevalent, use of the term “contextuality” is not an explicit context-dependent form, as suggested by the realist Bell in his earlier quote, but rather a situation where the classical predictions of quantum clouds are vi-
olated. More concretely, if experiments on quantized systems violate certain Boole-Bell type classical bounds or direct classical predictions, the narratives claim to have thereby “proven contextuality” (e.g., see Refs. [29, 31, 114–117] and Ref. [90] for a “direct proof of quantum contextuality”). What if we take Bell’s proposal of a context dependence of valuations – and consequently, “classical” contextual probability theory – seriously? One of the consequences would be the introduction of an uncountable multiplicity of counterfactual observables. An example to illustrate this multiplicity – comparable to de Witt’s view of Everett’s relative state interpretation [118] – is the uncountable set of orthonormal bases of $\mathbb{R}^3$ which are all interconnected at the same single intertwining element. A continuous angular parameter characterizes the angles between the other elements of the bases, located in the plane orthogonal to that common intertwining element. Contextuality suggests that the value assignment of an observable (proposition) corresponding to this common intertwining element needs to be both true/1 and false/0, depending on the context involved, or whenever some quantum cloud (collection of intertwining observables) demands this through consistency requirements.

Indeed, the introduction of multiple quantum clouds would force any context dependence to also implicitly depend on this general perspective – that is, on the respective quantum cloud and its faithful orthogonal realization, which in turn determines the quantum probabilities via the Born-Gleason rule: Because there exist various different quantum clouds as “pathways interconnecting” two observables, context dependence needs to vary according to any concrete connection between the prepared and the measured state.

A single context participates in an arbitrary, potentially infinite, multiplicity of quantum clouds. This requires this one context to “behave very differently” when it comes to contextual value assignments. Alas, as quantum clouds are hypothetical constructions of our mind and therefore “mental through-and-through” [111], so appears context dependence: as an idealistic concept, devoid of any empirical evidence, created to rescue the desideratum of omni-realistic existence.

Pointedly stated, contextual value assignments appear both utterly ad hoc and arbitrary – like a deus ex machina “saving” the desideratum of a classical omni-value definite reality, whereby it must obey quantum probability theory without grounding it (indeed, in the absence of any additional criterion or principle there is no reason to assume that the likelihood of true/1 and false/0 is other than 50:50); as well as highly discontinuous. In this latter, discontinuity respect, context dependence is similar to the earlier mentioned breakup of the intertwine observables by reducing quantum observables to disconnected contexts [105–107].

B. Abandonment of omni-value definiteness of observables in all but one context

Nietzsche once speculated [119, 120] that what he has called “slave morality” originated from superficially pretending that – in what later Blair (aka Orwell) called [121] “doublespeak” – weakness means strength. In a rather similar sense the lack of comprehension – Planck’s “sign-post” – and even the resulting inconsistencies tended to become reinterpreted as an asset: nowadays consequences of the vector-based quantum probability law are marketed as “quantum supremacy” – a “quantum magic” or “hocus-pocus” [122] of sorts.

Indeed, future centuries may look back at our period, and may even call it a second “renaissance” period of scholasticism [35]. In years from now historians of science will be amused about our ongoing queer efforts, the calamities and “magic” experienced through our painful incapacity to recognize the obvious – that is, the non-existence and therefore value indefiniteness/indeterminacy of certain counterfactual observables – namely exactly those mentioned in Itamar Pitowsky’s indeterminacy principle.

This principle has a positive interpretation in terms of a value definiteness of a single context. In terms of Hilbert space quantum mechanics, this amounts to the claim that the only value definite entity can be a single orthonormal basis/maximal operator. All other “observables” grant an, albeit necessarily stochastic, value indefinite/indeterministic, view on this basis/maximal operator.

Where might this type of stochasticism arise from? It could well be that it is introduced by interactions with the environment; and through the many uncontrollable and, for all practical purposes [123], huge number of degrees of freedom in unknown states.

The finiteness of physical resources needs not prevent the specification of a particular vector or context. Because any other context needs to be operationalized within the physically feasible means available to the respective experiment: it is the measurable coordinate differences which count; not the absolute locatedness relative to a hypothetical, idealistic absolute frame of reference which cannot be accessed operationally.

Finally, as the type of context envisioned to be value definite can be expressed in terms of vector spaces equipped with a scalar product – in particular, by identifying a context with an orthonormal bases or (the spectral decomposition of) a maximal observable – one may ask how one could imagine the origin of such entities? Abstractly vectors and vector spaces could originate from a great variety of very different forms; such as from systems of solutions of ordinary linear differential equations. Any investigation into the origins of the quantum mechanical Hilbert space formalism itself might, if this turns out to be a progressive research program [124], eventually yield to a theory indicating operational physical capacities beyond quantum mechanics.

VII. BIOGRAPHICAL NOTES ON ITAMAR PITOWSKY

I am certainly not in the position to present a view of Itamar Pitowsky’s thinking. Therefore I shall make a few rather anecdotal observations. First of all, he seemed to me as one of the most original physicists I have ever met – but that might be a triviality, given his opus. One thing I realized was that he exhibited a – sometimes maybe even unconscious, but
sometimes very outspoken – regret that he was working in a philosopher’s department. I believe he considered himself rather a mathematician or theoretical physicist. To this I responded that being in a philosophy department might be rather fortunate because there one could “go wild” in every direction; allowing much greater freedom than in other academic realms. But, of course, this had no effect on his uneasiness.

He was astonished that I spent a not so little money (means relative to my investment capacities) in an Israeli internet startup company which later flopped, depriving me of all but a fraction of what I had invested. He told me that, at least at that point, many startups in Israel had been put up intentionally only to attract money from people like me; only to collapse later.

A late project of his concerned quantum bounds in general; maybe in a similar – graph theoretical and at the time undirected to quantum – way as Grötschel, Lovász and Schrijver’s theta body [125, 126]. The idea was not just deriving absolute [62] or parameterized, continuous [60, 61] bounds for existing classical conditions of possible experience obtained by hull computations of polytopes; but rather genuine quantum bounds on, say, Einstein-Podolsky-Rosen type setups.

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