Negative phase velocity of electromagnetic waves and the cosmological constant

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Abstract

Examining the propagation of electromagnetic plane waves with the wavevector directed in opposition to the time–averaged Poynting vector in cosmological spacetime with piecewise uniform metric, we show that such negative–phase–velocity (NPV) propagation is possible in certain de Sitter spacetimes but not in anti–de Sitter spacetimes. This difference suggests the possibility of an optical/electromagnetic experiment to discern the cosmological constant of a four–dimensional universe stemming from a five–dimensional brane universe.

Keywords: General theory of relativity, Negative phase velocity, Poynting vector, cosmological constant

1 Introduction

Literature on the phenomenon of negative refraction by certain linear, homogeneous, isotropic, dielectric–magnetic materials began to record an impressive rate of growth from mid–2000, after the announcement of an experimental result that could not be explained in any other way [1]. The underlying reason for negative refraction is that the time–averaged Poynting vector and the wavevector of a plane wave are oppositely aligned in those materials, for which reason we call them negative–phase–velocity (NPV) materials [2]. The promise of industrial exploitation of negative refraction has midwifed the extension of theoretical studies to bianisotropic materials [2, 3, 4]. The hope is that nanotechnological routes will allow negative refraction in the optical regime, which would be a truly remarkable accomplishment.

The classical electromagnetic vacuum cannot support NPV plane waves, as it appears the same to all observers moving at constant relative velocities [5]. However, gravitational
fields due to nearby massive objects would certainly distort electromagnetic propagation, which points towards the possibility of gravitationally assisted NPV propagation in vacuum. Investigation of a special case showed that this possibility cannot be discounted in spacetime manifolds of limited extent wherein the metric can be assumed to be approximately uniform [6].

Our objective in this communication is to examine the facilitation of NPV propagation of electromagnetic waves by the cosmological constant. Applying the formalism presented elsewhere [6, 7], we consider both de Sitter and anti–de Sitter spacetimes [8, 9].

2 Cosmological spacetime

In a static (i.e., time-independent) cosmological spacetime, the matrix representation of the metric $g_{\alpha\beta}$ is expressed in Cartesian coordinates as§ [8]

$$
[g_{\alpha\beta}] = \begin{pmatrix}
m & 0 & 0 & 0 \\
0 & -\left(1 + \frac{\Lambda y^2}{3c^2m}\right) & -\frac{\Lambda xy}{3c^2m} & -\frac{\Lambda xz}{3c^2m} \\
0 & -\frac{\Lambda xy}{3c^2m} & -\left(1 + \frac{\Lambda y^2}{3c^2m}\right) & -\frac{\Lambda yz}{3c^2m} \\
0 & -\frac{\Lambda xz}{3c^2m} & -\frac{\Lambda yz}{3c^2m} & -\left(1 + \frac{\Lambda z^2}{3c^2m}\right)
\end{pmatrix}
$$

(1)

where

$$m = 1 - \frac{\Lambda (x^2 + y^2 + z^2)}{3c^2},
$$

(2)

and $c$ is the speed of light in vacuum in the absence of a gravitational field. If the cosmological constant $\Lambda$ is positive (negative), the spacetime is called de Sitter (anti-de Sitter) spacetime [9]. The apparent singularity at $m = 0$ arises because of an infelicitous choice of coordinate system. The metric can actually be extended to a geodesically complete space of constant curvature in other coordinate systems [10].

The electromagnetic response of vacuum in curved spacetime may be described by the constitutive relations of an equivalent, instantaneously responding, medium as per [6, 11, 12, 13]

$$
\begin{align*}
D &= \varepsilon_0 \gamma \cdot E \\
B &= \mu_0 \gamma \cdot H
\end{align*}
$$

(3)

wherein SI units are implemented. Here, $\varepsilon_0 = 8.854 \times 10^{-12}$ F m$^{-1}$, $\mu_0 = 4\pi \times 10^{-12}$ H m$^{-1}$, and $\gamma$ is the $3\times3$ dyadic equivalent of the metric $[\gamma_{\alpha\beta}]$ with components

$$
\gamma_{\alpha\beta} = -\frac{g^{\alpha\beta}}{g_{00}}.
$$

(4)

§Roman indexes take the values 1, 2 and 3; while Greek indexes take the values 0, 1, 2, and 3.
The constitutive relations (3) provide a global description of cosmological spacetime. Let us partition the global spacetime into neighbourhoods \([7, 14]\). We focus our attention upon a neighbourhood \(\mathcal{R}\) of the arbitrary location \((\tilde{x}, \tilde{y}, \tilde{z})\) wherein the nonuniform metric \(\gamma_{ab}\) may be approximated by the uniform metric \(\tilde{\gamma}_{ab}\). Thus, we have the uniform 3×3 dyadic representation

\[
\tilde{\gamma} \equiv [\tilde{\gamma}_{ab}] = \frac{1}{\tilde{m}} \begin{pmatrix}
1 - \frac{\Lambda \tilde{x}^2}{3c^2} & -\frac{\Lambda \tilde{x} \tilde{y}}{3c^2} & -\frac{\Lambda \tilde{x} \tilde{z}}{3c^2} \\
-\frac{\Lambda \tilde{x} \tilde{y}}{3c^2} & 1 - \frac{\Lambda \tilde{y}^2}{3c^2} & -\frac{\Lambda \tilde{y} \tilde{z}}{3c^2} \\
-\frac{\Lambda \tilde{x} \tilde{z}}{3c^2} & -\frac{\Lambda \tilde{y} \tilde{z}}{3c^2} & 1 - \frac{\Lambda \tilde{z}^2}{3c^2}
\end{pmatrix}
\]  

(5)
in \(\mathcal{R}\), where the constant

\[
\tilde{m} = 1 - \frac{\Lambda (\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2)}{3c^2}.
\]  

(6)

3 Plane waves in \(\mathcal{R}\)

Planewave solutions

\[
\begin{align*}
E &= \text{Re} \{E_o \exp [i (k \cdot \mathbf{r} - \omega t)]\}, \\
H &= \text{Re} \{H_o \exp [i (k \cdot \mathbf{r} - \omega t)]\},
\end{align*}
\]

(7)

(8)

are sought to the source–free Maxwell curl postulates

\[
\nabla \times E + \frac{\partial}{\partial t} B = 0,
\]

(9)

\[
\nabla \times H - \frac{\partial}{\partial t} D = 0
\]

(10)
in \(\mathcal{R}\). Here, \(k\) is the wavevector, \(\mathbf{r}\) is the position vector within the neighbourhood containing \((\tilde{x}, \tilde{y}, \tilde{z})\), \(\omega\) is the angular frequency, and \(t\) denotes the time; whereas \(i = \sqrt{-1}\), and \(E_o\) as well as \(H_o\) are complex–valued amplitudes. By combining (7)–(10) we find, after some manipulation, that

\[
W \cdot E_o = 0,
\]

(11)

where

\[
W = \left(k_o^2 \det \left[\tilde{\gamma}\right] - k \cdot \tilde{\gamma} \cdot k\right) \mathbf{I} + k k \cdot \tilde{\gamma},
\]

(12)

and \(k_o = \omega \sqrt{\varepsilon_o \mu_o}\). The corresponding dispersion relation \(\det [W] = 0\) is expressible as

\[
k_o^2 \det \left[\tilde{\gamma}\right] \left(k_o^2 \det \left[\tilde{\gamma}\right] - k \cdot \tilde{\gamma} \cdot k\right)^2 = 0.
\]

(13)

Thus, the wavevectors satisfy the condition

\[
k \cdot \tilde{\gamma} \cdot k = k_o^2 \det \left[\tilde{\gamma}\right],
\]

(14)
as long as \( \tilde{\gamma} \) is nonsingular.

Now we turn to the eigensolutions of (11). By virtue of (14), we have
\[
\mathbf{k} \cdot \tilde{\gamma} \cdot \mathbf{E}_0 = 0;
\]
(15)
hence, \( \mathbf{E}_0 \) is orthogonal to \( \mathbf{k} \cdot \tilde{\gamma} \). Without any loss of generality, let us choose the wavevector \( \mathbf{k} \) to lie along the \( z \) axis, i.e.,
\[
\mathbf{k} = k \hat{u}_z;
\]
(16)
with the unit vector \( \hat{u}_z \) lying along the Cartesian \( z \) axis. Thereby,
\[
\mathbf{k} \cdot \tilde{\gamma} = k(\tilde{\gamma}_1 \hat{u}_x + \tilde{\gamma}_2 \hat{u}_y + \tilde{\gamma}_3 \hat{u}_z),
\]
(17)
with
\[
\tilde{\gamma}_1 = -\frac{\Lambda \tilde{z} \tilde{z}}{3c^2\tilde{m}}, \quad \tilde{\gamma}_2 = -\frac{\Lambda \tilde{y} \tilde{z}}{3c^2\tilde{m}}, \quad \tilde{\gamma}_3 = \frac{\tilde{m}_z}{\tilde{m}}, \quad \tilde{m}_z = 1 - \frac{\Lambda \tilde{z}^2}{3c^2},
\]
(18)
and \( \hat{u}_x \) and \( \hat{u}_y \) being unit vectors lying along the Cartesian \( x \) and \( y \) axes, respectively.

Two linearly independent eigenvectors satisfying (15) are provided as
\[
e_1 = \tilde{\gamma}_2 \hat{u}_x - \tilde{\gamma}_1 \hat{u}_y,
\]
(19)
\[
e_2 = \tilde{\gamma}_1 \tilde{\gamma}_3 \hat{u}_x + \tilde{\gamma}_2 \tilde{\gamma}_3 \hat{u}_y - (\tilde{\gamma}_1 + \tilde{\gamma}_2) \hat{u}_z;
\]
(20)

hence, the general solution is given by
\[
\mathbf{E}_0 = C_1e_1 + C_2e_2,
\]
(21)
wherein \( C_1 \) and \( C_2 \) are arbitrary complex–valued constants. The corresponding expression for \( \mathbf{H}_0 \) follows from the Maxwell postulates as
\[
\mathbf{H}_0 = \frac{k}{\omega \mu_0} \left( C_1 \tilde{m} \mathbf{e}_2 - C_2 \tilde{m}_z \mathbf{e}_1 \right),
\]
(22)

The wavenumbers emerge straightforwardly from the dispersion equation (14). For \( \mathbf{k} \) aligned with the \( z \) axis we obtain the \( k \)–quadratic expression
\[
k^2 \tilde{m}_z \tilde{m}_z - \frac{k_0^2}{\tilde{m}_z} = 0,
\]
(23)
which yields the wavenumbers
\[
k = \pm k_0 \left( \tilde{m} \tilde{m}_z \right)^{-1/2}.
\]
(24)
The requirement that \( k \in \mathbb{R} \) imposes the condition
\[
\tilde{m} \tilde{m}_z > 0;
\]
(25)
in other words, both \( \tilde{m} \) and \( \tilde{m}_z \) must be of the same signs for real–valued \( k \).
4 NPV Condition

In order to establish the feasibility of NPV propagation, we turn to the time-averaged Poynting vector given by

\[ \mathbf{P} = \frac{1}{2} \text{Re} \left\{ \mathbf{E}_o \times \mathbf{H}_o^* \right\}. \]  
(26)

The general solution (21) and (22) delivers

\[ \mathbf{P} = \frac{k}{2 \omega \mu_0} \left( |C_1|^2 \hat{m} + |C_2|^2 \hat{m}_z \right) \mathbf{e}_1 \times \mathbf{e}_2. \]  
(27)

From (19) and (20) we have

\[ \mathbf{e}_1 \times \mathbf{e}_2 = (\gamma_1^2 + \gamma_2^2) \left( \gamma_1 \hat{x} + \gamma_2 \hat{y} + \gamma_3 \hat{z} \right); \]  
(28)

thus

\[ k \cdot (\mathbf{e}_1 \times \mathbf{e}_2) = \frac{\Lambda \hat{z}}{3c^2 \hat{m}} \left( \frac{\Lambda \hat{z}}{3c^2 \hat{m}} \right)^2 (\hat{x}^2 + \hat{y}^2) \]  
(29)

and

\[ k \cdot \mathbf{P} = \frac{1}{2 \omega \mu_0} \left( \frac{k \Lambda \hat{z}}{3c^2 \hat{m}} \right)^2 (\hat{x}^2 + \hat{y}^2) \left( |C_1|^2 + |C_2|^2 \frac{\hat{m}_z}{\hat{m}} \right) \hat{m}_z. \]  
(30)

The definition of NPV is that \( k \cdot \mathbf{P} < 0 \) [5, 6]. By exploiting (25), we see that \( k \cdot \mathbf{P} < 0 \) follows from \( \hat{m}_z < 0 \); i.e., NPV propagation arises provided that

\[ \Lambda > \frac{3c^2}{\hat{z}^2}. \]  
(31)

According to the NPV condition (31), the anti-de Sitter spacetime (i.e., \( \Lambda < 0 \)) does not support NPV propagation, unlike the de Sitter spacetime (i.e., \( \Lambda > 0 \)) which supports NPV propagation as long as the cosmological constant is sufficiently large. Equivalently, NPV propagation is feasible in de Sitter spacetime at sufficiently remote locations. We observe that de Sitter NPV propagation is supported at locations which lie outside the event horizon specified by \( x^2 + y^2 + z^2 = 3c^2/\Lambda \).

5 Applicability of NPV condition

Let us comment now on the applicability of the NPV condition (31). Suppose \( \delta \) is some representative linear dimension of the neighbourhood \( \mathcal{R} \). The uniform approximation, as implemented for our planewave analysis, rests upon the assumptions that \( \delta \) is (i) small compared with the curvature of the global spacetime; and (ii) large compared with electromagnetic wavelength as given by \( 2\pi/k \). To be specific, consider the Ricci scalar \( R \). For de Sitter spacetime we have [15]

\[ R = \frac{4\Lambda}{c^2}. \]  
(32)
As $R$ provides a measure of the inverse radius of spacetime curvature squared, the neighbour- 
hood $\mathcal{R}$ is such that

$$\frac{2\pi}{|k|} \ll \delta \ll \frac{c}{2}\sqrt{\rho|\Lambda|},$$

(33)

where $\rho$ is a proportionality constant. Thus, we see that the partition of global spacetime 
requires

$$|\Lambda| \ll \frac{c^2|k|^2\rho}{16\pi^2}.$$  

(34)

Since the linear dimensions $\delta$ of the neighbourhood $\mathcal{R}$ are chosen independently of the $\tilde{z}$ 
coordinate specifying the location of $\mathcal{R}$, there is no incompatibility between the conditions 
imposed on $\Lambda$ by the NPV inequality (31) and the uniform approximation inequalities (33).

After solving for the plane-wave propagation modes in each neighbourhood, the neighbour-
hood solutions may be stitched together to provide the global solution. The piecewise 
uniform approximation technique adopted here — which is described in detail elsewhere [14] — is commonly employed in solving differential equations with nonhomogeneous coefficients [16].

6 Concluding Remarks

We have theoretically examined the propagation of electromagnetic plane waves with the 
time–averaged Poynting vector directed in opposition to the wavevector in cosmological 
spacetime with piecewise uniform metric. Our derivations lead us to conclude that NPV 
propagation is possible for $\Lambda$ sufficiently large and positive (de Sitter spacetime); but it 
is impossible for negative $\Lambda$, i.e., when the spacetime is of the anti–de Sitter type. This 
difference between the two types of spacetimes should be added to the catalogue of known 
differences for scalar waves [17].

Recent string–inspired theories with large extra dimensions, known as brane–world mod-
els [18, 19, 20], suggest that gravitational interactions between particles on the brane in 
uncompactified bulk five-dimensional space can exhibit the correct four–dimensional brane 
behaviour, through finely tuning a relation between the bulk cosmological constant and 
the brane tension. In the original Randall–Sundrum model [19], the cosmological constant 
and the brane tension exactly cancel each other. More recent models, which admit the 
possibility of the cosmological constant and the brane tension not being in complete can-
cellation, yield a net cosmological constant on the four–dimensional brane that could be 
either positive or negative [21]. As a consequence of our results, one could envision an op-
tical/electromagnetic experiment, based upon the negative refraction associated with NPV 
propagation [3, 22], which may reveal whether or not a four–dimensional universe — deriv-
ing from a five–dimensional brane universe — has a sufficiently large positive cosmological 
constant.

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