FUNDAMENTAL GROUP AND PLURIDIFFERENTIALS ON COMPACT KÄHLER MANIFOLDS

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Abstract. A compact Kähler manifold $X$ is shown to be simply-connected if its ‘symmetric cotangent algebra’ is trivial. Conjecturally, such a manifold should even be rationally connected. The relative version is also shown: a proper surjective connected holomorphic map $f : X \to S$ between connected manifolds induces an isomorphism of fundamental groups if its smooth fibres are as above, and if $X$ is Kähler.

1. Introduction

We shall show:

Theorem 1.1. Let $X$ be a connected compact Kähler manifold. Suppose that for all $p \geq 1$ and $k \geq 1$ there is no non-zero global section of the sheaf $S^k\Omega^p_X$. Then $X$ is simply connected.

This theorem refines a former result of [5] with the very same statement, but with $\otimes^k\Omega^p_X$ in place of $S^k\Omega^p_X$. The proof of 1.1 is obtained by refining the proof of [5], which rests on $L^2$-methods à la Poincaré-Atiyah-Gromov.

The 'uniruledness conjecture' below implies easily (see §3) that $X$ should, in fact, be rationally connected, hence simply-connected, by [3]. Theorem 1.1 above permits to bypass this conjecture, as far as the fundamental group is concerned. It is usually quite easy to verify the vanishings of all $S^k\Omega^p_X$, while constructing sufficiently many rational curves requires the characteristic $p > 0$ methods introduced by S. Mori, no characteristic zero proof being presently known.

The weaker assumption that $H^0(X, S^k\Omega^1_X) = \{0\}$ for every $k \geq 1$ implies (see [2]) that all linear representations of the fundamental group $\pi_1(X) \to GL_n(K)$, $K$ a field, have finite image. This raises the question of whether the condition $H^0(X, S^k\Omega^1_X) = \{0\}$ for every $k \geq 1$ might imply that $\pi_1(X)$ is finite, instead of trivial. Enriques

\footnote{By a theorem of Kodaira, any $X$ as above is actually projective.}
surfaces (examples of general type also exist) indeed show that simple-
connectedness may then fail.

In contrast to the condition $H^0(X, S^k \Omega^p_X) = \{0\}$ for every $k \geq 1$
and $p \geq 1$, the condition $H^0(X, S^k \Omega^1_X) = \{0\}$ for every $k \geq 1$
does not seem however to have an even conjectural geometric interpreta-
tion in the frame of bimeromorphic classification of compact Kähler mani-
folds.

The theorem above has a relative version, shown in section §4 below:

\begin{corollary}
Let $f : X \to S$ be a proper holomorphic map with con-
ected fibres between connected complex manifolds. Assume that $X$
admits a Kähler metric, and that $f_*(S^k(\Omega^p_X/S)) = 0$ for every $k \geq 1$
and $p \geq 1$. Then $f_* : \pi_1(X) \to \pi_1(S)$ is an isomorphism of groups.
\end{corollary}

Note that the conclusion of corollary may fail for a projective
morphism $f : X \to S$ with smooth fibres simply-connected, because of
the possible presence of multiple fibres. Consider indeed an Enriques
surface $Y$ and its $K3$ universal cover $Y' \to Y = Y'/\mathbb{Z}_2$. Let $C \to \mathbb{P}^1 = C/\mathbb{Z}_2$
be the 2-sheeted cover defined by a hyperelliptic curve $C$. Now let
$X \to S := \mathbb{P}^1$ be deduced from the first projection $X' := C \times Y' \to C$
by taking the equivariant quotient by the involution $u \times v$ acting freely
on $X'$, $u$ and $v$ being the involutions on $Y'$ and $C$ respectively deduced
from the $\mathbb{Z}_2$ covers above. Here $S = \mathbb{P}^1$ is simply connected although
$\pi_1(X)$ is a $\mathbb{Z}_2$ extension of $\pi_1(C)$ and the smooth fibres of $f$ are simply-
connected.

\section{Proof of theorem}

As in \cite{5}, the proof goes in two steps: show first that $\pi_1(X)$ is finite
(this is the main step, established below), and then show, using Serre's
covering trick, that $\pi_1(X)$ is in fact trivial.

We start by establishing this second step. Let $\pi : X' \to X$ be a
finite Galois étale cover of $X$ of group $G$ and degree $d$. The Euler
characteristic of the structural sheaf of $X$
\begin{equation}
\chi(X, \mathcal{O}_X) := \sum_{i=0}^{\dim X} (-1)^i \cdot h^i(X, \mathcal{O}_X)
\end{equation}

is equal to 1, since by Serre’s duality $h^i(X, \mathcal{O}_X) = h^0(X, \Omega^i_X)$, and
the latter is zero for $i \neq 0$ by hypothesis.

\footnote{Hopf surfaces $X$ have $H^0(X, S^k \Omega^p_X) = \{0\}, \forall k > 1, p > 1$, showing that the Kähler assumption cannot be removed in \cite{1} since $\pi_1(X) \cong \mathbb{Z}$.}

\footnote{These hypothesis should imply that $f$ is projective, locally above $S$.}
Now, if $\omega \in H^0(X', \Omega^i_{X'})$, the product of the $g^*\omega$ for $g \in G$ defines an element of $H^0(X', S^d\Omega^i_{X'})$ invariant by the action of $G$. We obtain in this way a global section of $S^d\Omega^i_X$, which is non zero if $\omega$ is non zero. Thus it follows from the hypothesis that we must also have $\chi(X', \mathcal{O}_{X'}) = 1$.

From the multiplicativity of the Euler characteristic (see lemma 2.1 below), we get:

$$1 = \chi(X', \mathcal{O}_{X'}) = d \cdot \chi(X, \mathcal{O}_X),$$

and $d$ is then necessarily equal to 1.

**Lemma 2.1.** Let $X' \to X$ be a finite étale covering of degree $d$ of compact complex analytic spaces. Then

$$\chi(X', \mathcal{O}_{X'}) = d \cdot \chi(X, \mathcal{O}_X).$$

**Proof.** When $X$ is projective, an elementary proof due to Kleiman is given in [12], exemple 1.1.30. In general, it is an easy consequence of the theorem of Riemann-Roch-Hirzebruch, which is proved in [14] for compact complex analytic spaces.

To complete the proof of theorem 1.1, we need to show that the fundamental group of $X$ is finite. Equivalently, we have to show the

**Theorem 2.2.** Let $X$ be a connected compact Kähler manifold with infinite fundamental group. Then there exists $p \geq 1$ and $k \geq 1$ such that $H^0(X, S^k\Omega^p_X) \neq \{0\}$.

**Proof.** Let $p : \tilde{X} \to X$ be the universal cover of $X$. The fundamental group $\Gamma := \pi_1(X)$ acts on $\tilde{X}$. The choice of a Kähler metric on $X$ induces a complete Kähler metric on $\tilde{X}$. Denote by $\mathcal{H}^k_{(2)}(\tilde{X})$ the Hilbert space of $L^2$-harmonic complex-valued forms of degree $k$ on $\tilde{X}$. Recall that a $p$-form $\alpha$ is called harmonic if $\Delta \alpha = 0$, where $\Delta := d \circ d^* + d^* \circ d$ and $d^* := - * \circ d \circ $. Moreover, a $L^2$ $p$-form $\alpha$ is harmonic if and only if $d\alpha = 0$ and $d^*\alpha = 0$ (the metric being complete), if and only if $\bar{\partial}\alpha = 0$ and $\bar{\partial}^*\alpha = 0$ (the metric being complete and Kähler), see [9].

The decomposition in types gives rise to a orthogonal sum

$$\mathcal{H}^k_{(2)}(\tilde{X}) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{(2)}(\tilde{X}).$$

The space $\mathcal{H}^{p,q}_{(2)}(\tilde{X})$ consists of the $L^2$-holomorphic $p$-forms on $\tilde{X}$.

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4We shall only need the case when $X$ is a divisor with normal crossings in a complex Kähler manifold in the proof of corollary 1.
The Hilbert spaces $\mathcal{H}^{p,q}_{(2)}(\tilde{X})$ might be infinite dimensional. Nevertheless, using the isometric action of $\Gamma$ on them, one can associate to them a non-negative real number $\dim_{\Gamma}(\mathcal{H}^{p,q}_{(2)}(\tilde{X}))$ (cf. [1]). This number is zero if and only if $\mathcal{H}^{p,q}_{(2)}(\tilde{X}) = \{0\}$.

By Atiyah’s $L^2$-index theorem (cf. [1, 9]), we know that

$$\chi(\mathcal{X}, \mathcal{O}_\mathcal{X}) = \chi_{(2)}(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}) := \sum_{q=0}^{\dim \mathcal{X}} (-1)^q \cdot \dim_{\Gamma}(\mathcal{H}^{0,q}_{(2)}(\tilde{X}))$$

Observe that there are no non-zero $L^2$-holomorphic functions on $\tilde{\mathcal{X}}$.

Indeed, the metric being complete, any harmonic function is closed, hence locally constant. By hypothesis $\tilde{\mathcal{X}}$ is non-compact, and any constant $L^2$ function has to be zero.

Let us distinguish two cases. Suppose first that $\chi(\mathcal{X}, \mathcal{O}_\mathcal{X}) = 0$. Since $\dim \mathcal{H}^0(\mathcal{X}, \mathcal{O}_\mathcal{X}) = 1$, Hodge symmetry shows that $\mathcal{H}^0(\mathcal{X}, \mathcal{O}_\mathcal{X}) \neq \{0\}$, for some (odd) $p \geq 1$, and the theorem is proved in this case. If, now, $\chi(\mathcal{X}, \mathcal{O}_\mathcal{X}) \neq 0$, it follows from the discussion above that there exists $p \geq 1$ such that $\mathcal{H}^{0,p}_{(2)}(\tilde{X}) \neq \{0\}$. By conjugation $\mathcal{H}^{p,0}_{(2)}(\tilde{X}) \neq \{0\}$, hence we get a non-zero $L^2$-holomorphic $p$-form for some $p \geq 1$.

The rest of the proof consists, following [9], in constructing from this $L^2$ section a non-zero $\Gamma$-invariant section of some $S^k\mathcal{O}_\tilde{\mathcal{X}}^p$. This can be done using a construction which goes back to Poincaré, that we now describe in a general setting.

. Let $\mathcal{M}$ be a complex manifold and $E$ be a holomorphic vector bundle on $\mathcal{M}$. Let $\Gamma$ be a countable discrete group acting on $\mathcal{M}$ and suppose that the action of $\Gamma$ lifts to an action on $E$. Let $h_E$ be a $\Gamma$-invariant continuous hermitian metric on $E$. Let $\Phi : \mathbb{P}(E) \to \mathcal{M}$ denote the projective bundle of hyperplanes in $E$ and $\mathcal{O}_E(1) \to \mathbb{P}(E)$ be the tautological line bundle endowed with the induced hermitian metric $h_L$. By functoriality the group $\Gamma$ acts on $\mathbb{P}(E)$ and $\mathcal{O}_E(1)$, and all the maps considered above are $\Gamma$-equivariant. As $\Phi_*(\mathcal{O}_E(k)) = S^kE$ for all $k \geq 1$ (where $\mathcal{O}_E(k)$ denotes the line bundle $\mathcal{O}_E(1)^{\otimes k}$), there is a $\Gamma$-equivariant identification between the space of holomorphic sections $H^0(\mathbb{P}(E), \mathcal{O}_E(k)) = H^0(M, S^kE)$ under which $L^q$ holomorphic sections are identified for all $q \geq 1$.

To any $L^1$ holomorphic section $s$ of $E$ we can associate a $\Gamma$-invariant section of $S^kE$ for all $k \geq 1$ (the so-called Poincaré series) as follows:

$$P_k(s)(x) := \sum_{\gamma \in \Gamma} \gamma^* s^k(\gamma \cdot x)$$
As $s$ is $L^1$, this series converges absolutely to a $\Gamma$-invariant holomorphic section of $S^k E$.

Moreover, if $s$ is not the zero section, then $P_k(s)$ is non-zero for infinitely many $k \geq 1$. Indeed, the preceding construction shows that we need only to consider the case where $E$ is a line bundle. The assertion is then a consequence of the following lemma.

**Lemma 2.3.** (See Lemma 3.2.A from [9]) Let $\{a_i\}$ be an $l^1$-sequence of complex numbers, not all zero. Then there are infinitely many $k \geq 1$ such that $\sum_i a_i^k \neq 0$.

Now recall that in the case where $\chi(X, \mathcal{O}_X) \neq 0$, we showed the existence of a non-zero $L^2$ section of $\Omega^{p}_{X}$ for some $p > 0$. If we see $s$ as a section of the tautological line bundle $\mathcal{O}_{\Omega^{p}_{X}}(1)$ on the projectified bundle of $\Omega^{p}_{X}$, then $s^\otimes k$ is a non-zero $L^1$ section of $\mathcal{O}_{\Omega^{p}_{X}}(1)$ for any $k \geq 2$. Applying the averaging construction just described to $s^\otimes 2$, we get a non-zero $\Gamma$-invariant section of some $\mathcal{O}_{\Omega^{2p}_{X}}(2k)$, giving a non-zero section of $S^{2k}\Omega^{p}_{X}$, as claimed. This concludes the proof. □

**Remark.** For any compact connected Kähler manifold $X$ with infinite fundamental group, let $P(X)$ (resp. $P_2(X)$) be the set of integers $p$ such that $H^0(X, S^k\Omega^{p}_{X}) \neq \{0\}$ for some $k > 0$ (resp. such that $H^0_2(X', S^k\Omega^{p}_{X'}) \neq \{0\}$ for some $k > 0$ and some infinite connected étale cover $X'$ of $X$). The arguments above show that $P_2(X) \subset P(X)$. Complex tori show that this inclusion can be strict.

### 3. A criterion for rational connectedness.

Recall the following consequence of the ‘Abundance Conjecture’

**Conjecture.** (‘uniruledness’ conjecture) Let $X$ be a connected compact Kähler manifold. Then $X$ is uniruled (i.e. covered by rational curves) if and only if $H^0(X, K_X^\otimes k) = \{0\}$ for all $k > 0$.

Consider also the following conjecture:

**Conjecture.** Let $X$ be a connected compact Kähler manifold. Then $X$ is rationally connected (i.e. any two generic points are joined by some rational curve) if and only if $H^0(X, S^k\Omega^{p}_{X}) = 0$, for every $k > 0$ and $p > 0$.

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5 We thank C. Mourougane for observing that in our first version, our construction appeared to give a section of $S^k(S^2(\Omega^{p}_{X}))$, instead of $S^{2k}(\Omega^{p}_{X})$.

6 A weaker form, usually attributed to D. Mumford, claims the same conclusion assuming that $H^0(X, (\Omega^{1}_{X})^\otimes k) = \{0\}$ for all $k > 0$. 
In [6] a weaker form of Conjecture 3 is established: X is rationally connected if \( H^0(X, S^k\Omega^p_X \otimes A) = 0 \), for every \( k > k(A) \), every \( p > 0 \), and some ample line bundle \( A \) on \( X \).

For both conjectures, the “only if” part is easy. The second conjecture implies theorem 1.1 above, since rationally connected manifolds are simply connected [3].

Let us show that the first conjecture implies the second. First, a Kähler manifold \( X \) as in the second conjecture has \( h^2,0(X) = 0 \), so it is projective algebraic by Kodaira’s projectivity criterion. Now consider the so-called ‘rational quotient’ \( r_X : X \rightarrow R \) (constructed in [4] and in [11], where it is called the ‘MRC’-fibration), which has rationally connected fibres and non-uniruled base \( R \) (by [7]). Assuming that \( r := \dim(R) > 0 \), we get a contradiction, since by the first conjecture there exists a non-zero \( s \in H^0(R, K_R^\otimes k) \), for some \( k > 0 \), which lifts to \( X \) as a non-zero section of \( H^0(X, S^k\Omega^p_X) \). Thus \( r = 0 \) and \( X \) is rationally connected.

Remark. For any compact connected Kähler manifold, let \( r^-(X) := \max\{p \geq 0 | \exists k > 0, H^0(X, S^k\Omega^p_X) \neq \{0\} \} \). Let \( r(X) := \dim(R), R \) as above. The preceding arguments show that \( r(X) \geq r^-(X) \), and the uniruledness conjecture is equivalent to the equality: \( r(X) = r^-(X) \).

4. Proof of corollary 1

The corollary is an easy consequence of the theorem and the following, the proof and statement of which are inspired by [10], theorem 5.2:

**Theorem 4.1.** Let \( f : X \rightarrow S \) be a proper holomorphic map with connected fibres between connected complex manifolds. Assume that \( X \) admits a Kähler metric and that there exists a smooth fibre \( X_s \) of \( f \) which is simply-connected and satisfies \( H^p(X_s, O_{X_s}) = 0 \) for all \( p > 0 \). Then \( f_* : \pi_1(X) \rightarrow \pi_1(S) \) is an isomorphism of groups.

**Proof.** First observe that all the smooth fibres \( X_s \) of \( f \) are simply-connected and satisfy \( H^p(X_s, O_{X_s}) = 0 \) for all \( p > 0 \). Indeed, the restriction of \( f \) to its smooth locus \( S^o \subset S \) is topologically a locally trivial fiber bundle by Ehresmann’s lemma, and the dimension of \( H^p(X_s, O_{X_s}) \) is locally constant for \( s \in S^o \), as follows from the theory of variations of Hodge structures.

Let us first consider the following special case: \( X \) is a connected complex Kähler manifold, \( f : X \rightarrow \Delta \) is a proper holomorphic map with connected fibres, smooth outside \( 0 \in \Delta \). Recall that in this situation \( X_0 \) is a retract of \( X \). We have to show that the fundamental group of
X (which is isomorphic to $\pi_1(X_0)$) is trivial. By blowing-up $X$, one can ensure that $X_0$ has only simple normal crossings (i.e. the irreducible components of the corresponding reduced divisor are smooth and meet transversally); this does not change the fundamental group of $X$. By ([10], lemma 5.2.2) the fundamental group of $X$ is finite cyclic, say of order $d$. Let $\pi : \tilde{X} \to X$ be a universal cover of $X$ and $g : \tilde{X} \to \Delta$ be the Stein factorization of $f \circ \pi$ so that:

$$
\begin{array}{c}
\tilde{X} \\
\downarrow \pi \\
X \\
\downarrow \downarrow \downarrow \\
\Delta \\
\end{array}
$$

The fibre $\tilde{X}_t$ of $g$ at any $t \neq 0$ is isomorphic to $X_{td}$, hence $H^p(\tilde{X}_t, \mathcal{O}_{\tilde{X}_t}) = H^p(X_{td}, \mathcal{O}_{X_{td}}) = 0$ for $t \neq 0$ and $p > 0$, and the sheaves $R^pg_*\mathcal{O}_X$ are generically zero for all $p > 0$. Being torsion-free (see [16], theorem 2.11), they are in fact zero on $\Delta$. Using Leray's spectral sequence, this implies that $H^p(\tilde{X}, \mathcal{O}_{\tilde{X}}) = H^p(\Delta, g_*\mathcal{O}_{\tilde{X}}) = 0$ for $p > 0$. Applying the lemma [4.2] below, it follows that $H^p(\tilde{X}_{0,\text{red}}, \mathcal{O}_{\tilde{X}_{0,\text{red}}}) = 0$ for all $p > 0$, hence $\chi(\tilde{X}_{0,\text{red}}, \mathcal{O}_{\tilde{X}_{0,\text{red}}}) = 1$. By multiplicativity of the holomorphic Euler characteristic in finite étale cover (see lemma 2.1), $d = 1$ and $X$ is simply-connected.

**Lemma 4.2.** (Steenbrink, see [17] lemma 2.14 and [10] lemma 5.2.3) Let $X$ be a complex Kähler manifold and let $D \subset X$ be a reduced divisor such that $D$ as a complex space is proper and has normal crossing only. Assume moreover that $D$ is topologically a retract of $X$. Then the restriction maps $H^p(X, \mathcal{O}_X) \to H^p(D, \mathcal{O}_D)$ are surjective for all $p \geq 0$.

**Proof.** Fix a $p \geq 0$. Since $D$ is topologically a retract of $X$, the map $H^p(X, \mathbb{C}) \to H^p(D, \mathbb{C})$ is an isomorphism. On the other hand, as $D$ is a union of compact Kähler manifolds crossing transversally, $H^p(D, \mathbb{C})$ admits a canonical mixed Hodge structure (see [8] section 4) whose Hodge filtration $H^p(D, \mathbb{C}) = F^0H^p(D, \mathbb{C}) \supseteq F^1H^p(D, \mathbb{C}) \supseteq \cdots$ satisfies $Gr^p F H^p(D, \mathbb{C}) \cong H^p(D, \mathcal{O}_D)$, see [17] section (1.5). It follows that

\[\text{in this reference the morphism is supposed projective but the same proof works for a proper morphism assuming that the total space admits a Kähler metric. See also [15], corollary 11.18.}\]
the map $H^p(D, \mathcal{C}) \to H^p(D, \mathcal{O}_D)$ is surjective. The following commutative diagram

$$
\begin{array}{ccc}
H^p(X, \mathcal{C}) & \longrightarrow & H^p(X, \mathcal{O}_X) \\
\downarrow & & \downarrow \\
H^p(D, \mathcal{C}) & \longrightarrow & H^p(D, \mathcal{O}_D)
\end{array}
$$

shows that $H^p(X, \mathcal{O}_X) \to H^p(D, \mathcal{O}_D)$ is surjective. □

We now reduce the general case to this special case. First, because of the following diagram, theorem 4.1 for $f$ follows from the corresponding statement for the restriction of $f$ to an open $U := S - T$, if the codimension in $S$ of $T$, Zariski closed in $S$, is at least 2:

$$
\begin{array}{ccc}
\pi_1(f^{-1}(U)) & \longrightarrow & \pi_1(U) \\
\downarrow & & \downarrow \\
\pi_1(X) & \longrightarrow & \pi_1(S)
\end{array}
$$

On the other hand, any $s \in S$ admits a contractible neighborhood $U$ in $S$ such that $f^{-1}(U)$ is homeomorphic to $U \times f^{-1}(s)$ (see for example [13]). From this, one easily sees that the theorem 4.1 for $f : X \to S$ follows if all fibres $X_s$ are simply-connected, at least for $s$ outside a codimension $\geq 2$ closed subvariety by the preceding observation.

Let $D \subset S$ be the proper closed subset of points $s$ for which $X_s$ is not smooth. By removing a codimension $\geq 2$ subvariety of $S$, one can assume that $D$ is a smooth divisor in $S$. Now, an easy application of Sard’s lemma shows that for $s \in D$ outside a proper subvariety $Z \subset D$, there exists a small disk $\Delta_s$ crossing $D$ transversally at $s$ such that $f^{-1}(\Delta_s)$ is smooth. For any $s \in D - Z$, the restriction of $f$ to $\Delta_s$ satisfies the assumptions of the special case of theorem 4.1 that we showed above, hence $\pi_1(X_s) = \pi_1(f^{-1}(\Delta_s)) = \{1\}$. □

Let us now explain how the theorems 1.1 and 4.1 imply the corollary. First observe that for fixed $k > 0$ and $p > 0$, the dimension of $H^0(X_s, (S^k \Omega^p_X)_{|X_s})$ is constant on a non empty Zariski open subset of $S$, and this dimension has to be zero by the flat base change theorem. It follows that $H^0(X_s, S^k \Omega^p_{X_s}) = 0$ for all $k > 0$ and $p > 0$ for a general smooth fibre $X_s$ of $f$. By theorem 1.1 this implies that a general smooth fibre of $f$ is simply connected; hence every smooth fibre is simply connected. The same argument shows that, in particular, for all $p > 0$, $h^0(X_s, \mathcal{O}_{X_s}) = 0$, for $s \in S$ generic, and so: $h^p(X_s, \mathcal{O}_{X_s}) = 0$. □
by Hodge symmetry. We can thus apply theorem 4.1 to conclude the proof of corollary 1.

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