THE NON-PURE VERSION OF THE SIMPLEX AND THE BOUNDARY OF THE SIMPLEX

NICOLÁS A. CAPITELLI

Abstract. We introduce the non-pure versions of simplicial balls and spheres with minimum number of vertices. These are a special type of non-homogeneous balls and spheres (NH-balls and NH-spheres) satisfying a minimality condition on the number of maximal simplices. The main result is that minimal NH-balls and NH-spheres are precisely the simplicial complexes whose iterated Alexander duals converge respectively to a simplex or the boundary of a simplex.

1. Introduction

A simplicial complex $K$ of dimension $d$ is vertex-minimal if it is a simplex or it has $d+2$ vertices. It is not hard to see that a vertex-minimal homogeneous (or pure) complex of dimension $d$ is either an elementary starring $(\tau,a)\Delta^d$ of a $d$-simplex or the boundary $\partial\Delta^{d+1}$ of a $(d+1)$-simplex. On the other hand, a general non-pure complex with minimum number of vertices has no precise characterization. However, since vertex-minimal pure complexes are either balls or spheres, it is natural to ask whether there is a non-pure analogue to these polyhedra within the theory of non-homogeneous balls and spheres. NH-balls and NH-spheres are the non-necessarily pure versions of combinatorial balls and spheres. They are part of a general theory of non-homogeneous manifolds (NH-manifolds) recently introduced by G. Minian and the author [4]. The study of NH-manifolds was in part motivated by Björner and Wachs’s notion of non-pure shellability [2] and by their relationship with factorizations of Pachner moves between (classical) manifolds. NH-balls and NH-spheres share many of the basic properties of combinatorial balls and spheres and they play an equivalent role to these in the generalized non-pure versions of classical manifold theorems. In a recent work [5], the results of Dong and Santos-Sturmfels on the homotopy type of the Alexander dual of simplicial balls and spheres were generalized to the non-homogeneous setting: the Alexander dual of an NH-ball is a contractible space and the Alexander dual of an NH-sphere is homotopy equivalent to a sphere (see [6, 8]). It was also shown in [5] that non-homogeneous balls and spheres are the Alexander double duals of classical balls and spheres. This result establishes a natural connection between the pure and non-pure theories.

The purpose of this article is to introduce minimal NH-balls and NH-spheres, which are respectively the non-pure versions of vertex-minimal balls and spheres. Note that $\partial\Delta^{d+1}$ is not only the $d$-sphere with minimum number of vertices but also the one with minimum number of maximal simplices. For non-pure spheres, this last property is strictly stronger than vertex-minimality and it is convenient to define minimal NH-spheres as the ones with minimum number of maximal simplices. With this definition, minimal NH-spheres with the homotopy type of a $k$-sphere are precisely the non-pure spheres whose nerve is $\partial\Delta^{k+1}$, a property that also characterizes the boundary of simplices. On the other hand, an NH-ball $B$ is minimal if it is part of a decomposition of a minimal NH-sphere.

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i.e. if there exists a combinatorial ball $L$ with $B \cap L = \partial L$ such that $B + L$ is a minimal $NH$-sphere. This definition is consistent with the notion of vertex-minimal simplicial ball (see Lemma 1.1 below).

Surprisingly, minimal $NH$-balls and $NH$-spheres can be characterized independently of their definition by a property involving Alexander duals. Denote by $K^*$ the Alexander dual of a complex $K$ relative to the vertices of $K$. Put inductively $K^{*(0)} = K$ and $K^{*(m)} = (K^{*(m-1)})^*$. Thus, in each step $K^{*(i)}$ is computed relative to its own vertices, i.e. as a subcomplex of the sphere of minimum dimension containing it. We call $\{K^{*(m)}\}_{m \in \mathbb{N}_0}$ the sequence of iterated Alexander duals of $K$. The main result of the article is the following

**Theorem 1.1.**

(i) There is an $m \in \mathbb{N}_0$ such that $K^{*(m)} = \partial \Delta^d$ if and only if $K$ is a minimal $NH$-sphere.

(ii) There is an $m \in \mathbb{N}_0$ such that $K^{*(m)} = \Delta^d$ if and only if $K$ is a minimal $NH$-ball.

Note that $K^* = \Delta^d$ if and only if $K$ is a vertex-minimal $d$-ball which is not a simplex, so (ii) describes precisely all complexes converging to vertex-minimal balls. Theorem 1.1 characterizes the classes of $\Delta^d$ and $\partial \Delta^d$ in the equivalence relation generated by $K \sim K^*$.

2. Preliminaries

2.1. **Notations and definitions.** All simplicial complexes that we deal with are assumed to be finite. Given a set of vertices $V$, $|V|$ will denote its cardinality and $\Delta(V)$ the simplex spanned by its vertices. $\Delta^d = \Delta(\{0, \ldots, d\})$ will denote a generic $d$-simplex and $\partial \Delta^d$ its boundary. The set of vertices of a complex $K$ will be denoted $V_K$ and we set $\Delta_K := \Delta(V_K)$. A simplex is maximal or principal in a complex $K$ if it is not a proper face of any other simplex of $K$. We denote by $m(K)$ the number of principal simplices in $K$. A ridge is a maximal proper face of a principal simplex. A complex is pure or homogeneous if all its maximal simplices have the same dimension.

$\sigma \ast \tau$ will denote the join of the simplices $\sigma$ and $\tau$ (with $V_\sigma \cap V_\tau = \emptyset$) and $K \ast L$ the join of the complexes $K$ and $L$ (where $V_K \cap V_L = \emptyset$). By convention, if $\emptyset$ is the empty simplex and $\{\emptyset\}$ the complex containing only the empty simplex then $K \ast \{\emptyset\} = K$ and $K \ast \emptyset = \emptyset$. Note that $\partial \Delta^0 = \{\emptyset\}$. For $\sigma \in K$, $\text{lk}(\sigma, K) = \{\tau \in K : \tau \cap \sigma = \emptyset, \tau \ast \sigma \in K\}$ denotes its link and $\text{st}(\sigma, K) = \sigma \ast \text{lk}(\sigma, K)$ its star. The union of two complexes $K, L$ will be denoted by $K + L$. A subcomplex $L \subset K$ is said to be top generated if every principal simplex of $L$ is also principal in $K$.

$K \setminus L$ will mean that $K$ (simplicially) collapses to $L$. A complex is collapsible if it has a subdivision which collapses to a single vertex. The simplicial nerve $N(K)$ of $K$ is the complex whose vertices are the principal simplices of $K$ and whose simplices are the finite subsets of principal simplices of $K$ with non-empty intersection.

Two complexes are PL-isomorphic if they have a common subdivision. A combinatorial $d$-ball is a complex PL-isomorphic to $\Delta^d$. A combinatorial $d$-sphere is a complex PL-isomorphic to $\partial \Delta^{d+1}$. By convention, $\partial \Delta^0 = \{\emptyset\}$ is a sphere of dimension $-1$. A combinatorial $d$-manifold is a complex $M$ such that $\text{lk}(v, M)$ is a combinatorial $(d-1)$-ball or $(d-1)$-sphere for every $v \in V_M$. A $(d-1)$-simplex in a combinatorial $d$-manifold $M$ is a face of at most two $d$-simplices of $M$ and the boundary $\partial M$ of the complex is generated by the $(d-1)$-simplices which are face of exactly one $d$-simplex. Combinatorial $d$-balls and $d$-spheres are combinatorial $d$-manifolds. The boundary of a combinatorial $d$-ball is a combinatorial $(d-1)$-sphere.
2.2. Non-homogeneous balls and spheres. In order to make the presentation self-contained, we recall first the definition and some basic properties of non-homogeneous balls and spheres. For a comprehensive exposition of the subject, the reader is referred to [4] (see also [5, §2.3] for a brief summary).

NH-balls and NH-spheres are special types of NH-manifolds, which are the non-necessarily pure versions of combinatorial manifolds. NH-manifolds have a local structure consisting of regularly-assembled pieces of Euclidean spaces of different dimensions. In Figure 1 we show some examples of NH-manifolds and their underlying spaces. NH-manifolds, NH-balls and NH-spheres are defined as follows.

**Definition.** An NH-manifold (resp. NH-ball, NH-sphere) of dimension 0 is a manifold (resp. ball, sphere) of dimension 0. An NH-sphere of dimension \(-1\) is, by convention, the complex \(\{\emptyset\}\). For \(d \geq 1\), we define by induction

- An NH-manifold of dimension \(d\) is a complex \(M\) of dimension \(d\) such that \(\text{lk}(v, M)\) is an NH-ball of dimension \(0 \leq k \leq d - 1\) or an NH-sphere of dimension \(-1\) \(\leq k \leq d - 1\) for all \(v \in V_M\).
- An NH-ball of dimension \(d\) is a collapsible NH-manifold of dimension \(d\).
- An NH-sphere of dimension \(d\) and homotopy dimension \(k\) is an NH-manifold \(S\) of dimension \(d\) such that there exist a top generated NH-ball \(B\) of dimension \(d\) and a top generated combinatorial \(k\)-ball \(L\) such that \(B + L = S\) and \(B \cap L = \partial L\). We say that \(S = B + L\) is a decomposition of \(S\) and write \(\text{dim}_h(S)\) for the homotopy dimension of \(S\).

**Figure 1.** Examples of NH-manifolds. (a), (d) and (e) are NH-spheres of dimension 1, 3 and 2 and homotopy dimension 0, 2 and 1 respectively. (b) is an NH-ball of dimension 2 and (c), (f) are NH-balls of dimension 3. (g) is an NH-manifold which is neither an NH-ball nor an NH-sphere. The sequence (a)-(d) evidences how NH-manifolds are inductively defined.

The definitions of NH-ball and NH-sphere are motivated by the classical theorems of Whitehead and Newman (see e.g. [7] Corollaries 3.28 and 3.13). Just like for classical combinatorial manifolds, it can be seen that the class of NH-manifolds (resp. NH-balls, NH-spheres) is closed under subdivision and that the link of every simplex in an NH-manifold is an NH-ball or an NH-sphere. Also, the homogeneous NH-manifolds (resp. NH-balls, NH-spheres) are precisely the combinatorial manifolds (resp. balls, spheres). Globally, a connected NH-manifold \(M\) is (non-pure) strongly connected: given two principal simplices \(\sigma, \tau \in M\) there is a sequence of maximal simplices \(\sigma = \eta_1, \ldots, \eta_t = \tau\) such that \(\eta_i \cap \eta_{i+1}\) is a ridge of \(\eta_i\) or \(\eta_{i+1}\) for every \(1 \leq i \leq t - 1\) (see [4] Lemma 3.15)). In particular, NH-balls and NH-spheres of homotopy dimension greater that 0 are strongly connected.
Unlike for classical spheres, non-pure \(NH\)-spheres do have boundary simplices; that is, simplices whose links are \(NH\)-balls. However, for any decomposition \(S = B + L\) of an \(NH\)-sphere and any \(\sigma \in L\), \(lk(\sigma, S)\) is an \(NH\)-sphere with decomposition \(lk(\sigma, B) + lk(\sigma, L)\) (see \[4, Lemma 4.8\]). In particular, if \(\sigma \in B \cap L\) then \(lk(\sigma, B)\) is an \(NH\)-ball.

2.3. **The Alexander dual.** For a finite simplicial complex \(K\) and a ground set of vertices \(V \supseteq V_K\), the Alexander dual of \(K\) (relative to \(V\)) is the complex

\[
K^* \subset V = \{ \sigma \in \Delta(V) | \Delta(V - \sigma) \notin K \}.
\]

The main importance of \(K^*\) lies in the combinatorial formulation of Alexander duality: \(H_i(K^*) \cong H^{n-i-3}(K)\). Here \(n = |V|\) and the homology and cohomology groups are reduced (see e.g. \[1\]). In what follows, we shall write \(K^* := K^*\K\) and \(K^\tau := K^*\\tau\) if \(\tau = \Delta(V - V_K)\). With this convention, \(K^\tau = K^*\) if \(\tau = \emptyset\). Note that \((\Delta^d)^* = \emptyset\) and \((\partial\Delta^{d+1})^* = \{\emptyset\}\).

The relationship between Alexander duals relative to different ground sets of vertices is given by the following formula (see \[3, Lemma 3.1\]):

\[
K^\tau = \partial \tau \ast \Delta_K + \tau \ast K^*.
\]

Here \(K^*\) is viewed as a subcomplex of \(\Delta_K\). It is easy to see from the definition that \((K^*)^\tau|_{\Delta(V_K - V_{K^*})} = K\) and that \((K^\tau)^* = K\) if \(K \neq \Delta^d\) (see \[3, Lemma 3.1\]). The following result characterizes the Alexander dual of vertex-minimal complexes.

**Lemma 2.1** (\[3, Lemma 4.1\]). If \(K = \Delta^d + u \ast lk(u, K)\) with \(u \notin \Delta^d\), then \(K^* = lk(u, K)^\tau\) where \(\tau = \Delta(V_K - V_{\ast lk(u, K)})\).

It can be shown that \(K^\tau\) is an \(NH\)-ball (resp. \(NH\)-sphere) if and only if \(K^*\) is an \(NH\)-ball (resp. \(NH\)-sphere). This actually follows from the next result involving a slightly more general form of formula \(\ast\), which we include here for future reference.

**Lemma 2.2** (\[3, Lemma 3.5\]). If \(V_K \subset V\) and \(\eta \neq \emptyset\), then \(L := \partial \eta \ast \Delta(V) + \eta \ast K\) is an \(NH\)-ball (resp. \(NH\)-sphere) if and only if \(K\) is an \(NH\)-ball (resp. \(NH\)-sphere).

3. **Minimal \(NH\)-spheres**

In this section we introduce the non-pure version of \(\partial\Delta^d\) and prove part \((i)\) of Theorem \[1,1\]. Recall that \(m(K)\) denotes the number of maximal simplices of \(K\). We shall see that for a non-homogeneous sphere \(S\), requesting minimality of \(m(S)\) is strictly stronger than requesting that of \(V_S\). This is the reason why vertex-minimal \(NH\)-spheres are not necessarily minimal in our sense.

To introduce minimal \(NH\)-spheres we note first that any complex \(K\) with the homotopy type of a \(k\)-sphere has at least \(k + 2\) principal simplices. This follows from the fact that the simplicial nerve \(\mathcal{N}(K)\) is homotopy equivalent to \(K\).

**Definition.** An \(NH\)-sphere \(S\) is said to be minimal if \(m(S) = \dim_h(S) + 2\).

Note that, equivalently, an \(NH\)-sphere \(S\) of homotopy dimension \(k\) is minimal if and only if \(\mathcal{N}(S) = \partial\Delta^{k+1}\).

**Remark 3.1.** Suppose \(S = B + L\) is a decomposition of a minimal \(NH\)-sphere of homotopy dimension \(k\) and let \(v \in V_L\). Then \(lk(v, S)\) is an \(NH\)-sphere of homotopy dimension \(\dim_h(lk(v, S)) = k - 1\) and \(lk(v, S) = lk(v, B) + lk(v, L)\) is a valid decomposition (see §2.2). In particular, \(m(lk(v, S)) \geq k + 1\). Also, \(m(lk(v, S)) < k + 3\) since \(m(S) < k + 3\) and \(m(lk(v, S)) \neq k + 2\) since otherwise \(S\) is a cone. Therefore, \(m(lk(v, S)) = k + 1 = \dim_h(lk(v, S)) + 2\), which shows that \(lk(v, S)\) is also a minimal \(NH\)-sphere.
We next prove that minimal $NH$-spheres are vertex-minimal.

**Proposition 3.2.** If $S$ is a $d$-dimensional minimal $NH$-sphere then $|V_S| = d + 2$.

**Proof.** Let $S = B + L$ be decomposition of $S$ and set $k = \dim_h(S)$. We shall prove that $|V_S| \leq d + 2$ by induction on $k$. The case $k = 0$ is straightforward, so assume $k \geq 1$. Let $\eta \in B$ be a principal simplex of minimal dimension and let $\Omega$ denote the intersection of all principal simplices of $S$ different from $\eta$. Note that $\Omega \neq \emptyset$ since $\mathcal{N}(S) = \partial \Delta^{k+1}$ and let $u \in \Omega$ be a vertex. Since $\eta \notin L$ then $\Omega \subseteq L$ and $u \in L$. By Remark 3.1 $lk(u, S)$ is a minimal $NH$-sphere of dimension $d - 2 = d - 1$ and homotopy dimension $k - 1$. By inductive hypothesis, $|V_{lk(u, S)}| \leq d' + 2 \leq d + 1$. Hence, $st(u, S)$ is a top generated subcomplex of $S$ with $k + 1$ principal simplices and at most $d + 2$ vertices. By construction, $S = st(u, S) + \eta$. We claim that $V_\eta \subseteq V_{st(u, S)}$. Since $B = st(u, B) + \eta$, by strong connectivity there is a ridge $\sigma \in B$ in $st(u, B) \cap \eta$ (see §2.2). By the minimality of $\eta$ we must have $\eta = w \ast \sigma$ for some vertex $w$. Now, $\sigma \in st(u, B) \cap \eta \subset st(u, S) \cap \eta$; but $st(v, S) \cap \eta \neq \sigma$ since, otherwise, $S = st(u, S) + \eta \cap \eta st(u, S) \cup u$, contradicting the fact that $S$ has the homotopy type of a sphere. We conclude that $w \in st(u, S)$ since every face of $\eta$ different from $\sigma$ contains $w$. Thus, $|V_S| = |V_{st(u, S)}| + |V_\eta| = |V_{st(u, S)}| \leq d + 2$. □

This last proposition shows that, in the non-pure setting, requesting the minimality of $m(S)$ is strictly more restrictive than requesting that of $|V_S|$. For example, a vertex-minimal $NH$-sphere can be constructed from any $NH$-sphere and a vertex $u \notin S$ by the formula $S := \Delta_S + u \ast S$. It is easy to see that if $S$ is not minimal, neither is $S$.

**Remark 3.3.** By Proposition 3.2 a $d$-dimensional minimal $NH$-sphere $S$ may be written $S = \Delta^d + u \ast \text{lk}(u, S)$ for some $u \notin \Delta^d$. Note that for any decomposition $S = B + L$, the vertex $u$ must lie in $L$ (since this last complex is top generated). In particular, $\text{lk}(u, S)$ is a minimal $NH$-sphere by Remark 3.1.

As we mentioned above, the Alexander duals play a key role in characterizing minimal $NH$-spheres. We now turn to prove Theorem 1.1(i). We derive first the following corollary of Proposition 3.2.

**Corollary 3.4.** If $S$ is a minimal $NH$-sphere then $|V_{S^*}| < |V_S|$ and $\dim(S^*) < \dim(S)$.

**Proof.** $V_{S^*} \subset V_S$ follows from Proposition 3.2 since if $S = \Delta^d + u \ast \text{lk}(u, S)$ then $u \notin S^*$. In particular, this implies that $\dim(S^*) \neq \dim(S)$ since $S^*$ is not a simplex by Alexander duality. □

**Theorem 3.5.** Let $K$ be a finite simplicial complex and let $\tau$ be a simplex (possibly empty) disjoint from $K$. Then, $K$ is a minimal $NH$-sphere if and only if $K^\tau$ is a minimal $NH$-sphere. That is, the class of minimal $NH$-spheres is closed under taking Alexander dual.

**Proof.** Assume first that $K$ is a minimal $NH$-sphere and set $d = \dim(K)$. We proceed by induction on $d$. By Proposition 3.2 we can write $K = \Delta^d + u \ast \text{lk}(u, K)$ for $u \notin \Delta^d$. If $\tau = \emptyset$ then, by Lemma 2.1, $K^\tau = \text{lk}(u, K)^\rho$ for $\rho = \Delta(V_K - V_{st(u, K)})$. By Remark 3.3 $\text{lk}(u, K)$ is a minimal $NH$-sphere. Therefore, $K^\tau = \text{lk}(u, K)^\rho$ is a minimal $NH$-sphere by inductive hypothesis. If $\tau \neq \emptyset$, $K^\tau = \text{d} \ast \Delta_K + \tau \ast K^\tau$ by formula (2). In particular, $K^\tau$ is an $NH$-sphere by Lemma 2.2 and the case $\tau = \emptyset$. Now, by Alexander duality,

$$\dim_h(K^\tau) = |V_K \cup V_\tau| - \dim_h(K) - 3 = |V_K| + |V_\tau| - \dim_h(K) - 3 = \dim_h(K^\tau) + |V_\tau|.$$

On the other hand,

$$m(K^\tau) = m(\text{d} \ast \Delta_K + \tau \ast K^\tau) = m(\text{d}) + \dim_h(K^\tau) + 2,$$

where the last equality follows from the case $\tau = \emptyset$. This shows that $S^\tau$ is minimal.
Assume now that $K^\tau$ is a minimal $NH$-sphere. If $\tau \not= \emptyset$ then $K = (K^\tau)^*$ and if $\tau = \emptyset$ then $K = (K^\tau)^{\Delta (V_K - V_{K^*})}$ (see §2.3). In any case, the result follows immediately from the previous implication.

**Proof of Theorem 1.1** (i). Suppose first that $K$ is a minimal $NH$-sphere. By Theorem 3.5, every non-empty complex in the sequence $\{K^{(m)}\}^{m \in N_0}$ is a minimal $NH$-sphere. By Corollary 3.4, $|V_{K^{(m+1)}}| < |V_{K^{(m)}}|$ for all $m$ such that $K^{(m)} \not= \emptyset$. Therefore, $K^{(m_0)} = \emptyset$ for some $m_0 < |V_K|$ and hence $K^{(m_0-1)} = \partial \Delta^d$ for some $d \geq 1$.

Assume now that $K^{(m)} = \partial \Delta^d$ for some $m \in N_0$ and $d \geq 1$. We proceed by induction on $m$. The case $m = 0$ corresponds to the trivial case $K = \partial \Delta^d$. For $m \geq 1$, the result follows immediately from Theorem 3.5 and the inductive hypothesis.

4. **Minimal $NH$-balls**

We now develop the notion of minimal $NH$-ball. The definition in this case is a little less straightforward that in the case of spheres because there is no piecewise-linear-equivalence argument in the construction of non-pure balls. To motivate the definition of minimal $NH$-ball, recall that for a non-empty simplex $\tau \in K$ and a vertex $a \not\in K$, the elementary starring $(\tau, a)$ of $K$ is the operation which transforms $K$ in $(\tau, a)K$ by removing $\tau \ast lk(\tau, K) = st(\tau, K)$ and replacing it with $a \ast \partial \tau \ast lk(\tau, K)$. Note that when $\dim(\tau) = 0$ then $(\tau, a)K$ is isomorphic to $K$.

**Lemma 4.1.** Let $B$ be a combinatorial $d$-ball. The following statements are equivalent.

1. $|V_B| \leq d + 2$ (i.e. $B$ is vertex-minimal).
2. $B$ is an elementary starring of $\Delta^d$.
3. There is a combinatorial $d$-ball $L$ such that $B + L = \partial \Delta^{d+1}$.

**Proof.** We first prove that (1) implies (2) by induction on $d$. Since $\Delta^d$ is trivially a starring of any of its vertices, we may assume $|V_B| = d + 2$ and write $B = \Delta^d + u \ast lk(u, B)$ for $u \not\in \Delta^d$. Since $lk(u, B)$ is necessarily a vertex-minimal $(d - 1)$-combinatorial ball then $lk(u, B) = (\tau, a)\Delta^{d-1}$ by inductive hypothesis. It follows from an easy computation that $B$ is isomorphic to $(u \ast \tau, a)\Delta^d$.

We next prove that (2) implies (3). We have

$$B = (\tau, a)\Delta^d = a \ast \partial \tau \ast lk(\tau, \Delta^d) = a \ast \partial \tau \ast \Delta^{d \dim(\tau) - 1} = \partial \tau \ast \Delta^{d \dim(\tau)}.$$ 

Letting $L := \tau \ast \partial \Delta^{d \dim(\tau)}$ we get the statement of (3).

The other implication is trivial.

**Definition.** An $NH$-ball $B$ is said to be **minimal** if there exists a minimal $NH$-sphere $S$ that admits a decomposition $S = B + L$.

Note that if $B$ is a minimal $NH$-ball and $S = B + L$ is a decomposition of a minimal $NH$-sphere then, by Remark 3.1, $lk(v, B)$ is a minimal $NH$-ball for every $v \in B \cap L$ (see §2.2). Note also that the intersection of all the principal simplices of $B$ is non-empty since $\mathcal{N}(B) \subseteq \mathcal{N}(S) = \partial \Delta^{k+1}$. Therefore, $\mathcal{N}(B)$ is a simplex. The converse, however, is easily seen to be false.

The proof of Theorem 1.1 (ii) will follow the same lines as its version for $NH$-spheres.

**Proposition 4.2.** If $B$ is a $d$-dimensional minimal $NH$-ball then $|V_B| \leq d + 2$.

**Proof.** This follows immediately from Proposition 3.2 since $\dim(B) = \dim(S)$ for any decomposition $S = B + L$ of an $NH$-sphere.

**Corollary 4.3.** If $B$ is a minimal $NH$-ball then $|V_B| < |V_B|$ and $\dim(B^*) < \dim(B)$.
Proof. We may assume \( B \neq \Delta^d \). \( V_{B^*} \subseteq V_B \) by the same reasoning made in the proof of Corollary 3.4. Also, if \( \dim(B) = \dim(B^*) \) then \( B^* = \Delta^d \). By formula (39), \( B = (B^*)^0 = \partial \rho * \Delta^d \) where \( \rho = \Delta(V_B - V_{B^*}) \), which is a contradiction since \( |V_B| = d + 2 \). \( \square \)

**Remark 4.4.** The same construction that we made for minimal \( NH \)-spheres shows that vertex-minimal \( NH \)-balls need not be minimal. Also, similarly to the case of non-pure spheres, if \( B = \Delta^d + u * \text{lk}(u, B) \) is a minimal \( NH \)-ball which is not a simplex then for any decomposition \( S = B + L \) of a minimal \( NH \)-sphere we have \( u \in L \). In particular, since \( \text{lk}(u, S) = \text{lk}(u, B) + \text{lk}(u, L) \) is a valid decomposition of a minimal \( NH \)-sphere, then \( \text{lk}(u, B) \) is a minimal \( NH \)-ball (see Remark 3.3).

**Theorem 4.5.** Let \( K \) be a finite simplicial complex and let \( \tau \) be a simplex (possibly empty) disjoint from \( K \). Then, \( K \) is a minimal \( NH \)-ball if and only if \( K^\tau \) is a minimal \( NH \)-ball. That is, the class of minimal \( NH \)-balls is closed under taking Alexander dual.

**Proof.** Assume first that \( K \) is a minimal \( NH \)-ball and proceed by induction on \( d = \dim(K) \). The case \( \tau = \emptyset \) follows the same reasoning as the proof of Theorem 3.3 using the previous remarks. Suppose then \( \tau \neq \emptyset \). Since by the previous case \( K^\tau \) is a minimal \( NH \)-ball, there exists a decomposition \( \tilde{S} = K^\tau + \tilde{L} \) of a minimal \( NH \)-sphere. By Propositions 3.2 and 4.2 either \( K^\tau \) is a simplex (and \( V_{\tilde{S}} - V_{K^\tau} = \{ w \} \) is a single vertex) or \( V_{\tilde{S}} = V_{K^\tau} \subseteq V_K \). Let \( S := K^\tau + \tau * \tilde{L} \), where we identify the vertex \( w \) with any vertex in \( V_K - V_{K^\tau} \) if \( K^\tau \) is a simplex. We claim that \( S = K^\tau + \tau * \tilde{L} \) is a valid decomposition of a minimal \( NH \)-sphere. On one hand, formula (40) and Lemma 2.2 imply that \( K^\tau \) is an \( NH \)-ball and that

\[
S = \partial \tau * \Delta_K + \tau * K^\tau + \tau * \tilde{L} = \partial \tau * \Delta_K + \tau * \tilde{S}
\]

is an \( NH \)-sphere. Also,

\[
K^\tau \cap (\tau * \tilde{L}) = (\partial \tau * \Delta_K + \tau * K^\tau) \cap (\tau * \tilde{L})
\]

\[
= \partial \tau * \tilde{L} + \tau * (K^\tau \cap \tilde{L})
\]

\[
= \partial \tau * \tilde{L} + \tau * \partial \tilde{L}
\]

\[
= \partial(\tau * \tilde{L}).
\]

This shows that \( S = K^\tau + \tau * \tilde{L} \) is valid decomposition of an \( NH \)-sphere. On the other hand,

\[
m(S) = m(\partial \tau) + m(\tilde{S}) = \dim(\tau) + 1 + \dim(\tilde{L}) + 2 = \dim_k(S) + 2,
\]

which proves that \( S \) is minimal. This settles the implication.

The other implication is analogous to the corresponding part of the proof of Theorem 3.5. \( \square \)

**Proof of Theorem 1.1 (ii).** It follows the same reasoning as the proof of Theorem 1.1 (i) (replacing \( \emptyset \) with \( \emptyset \)). \( \square \)

If \( K^* = \Delta^d \) then, letting \( \tau = \Delta(V_K - V_{\Delta^d}) \neq \emptyset \), we have \( K = (K^*)^\tau = \partial \tau * \Delta^d = (\tau, v)\Delta^{d+\dim(\tau)} \). This shows that Theorem 1.1 (ii) characterizes all complexes which converge to vertex-minimal balls.

5. Further properties of minimal \( NH \)-balls and \( NH \)-spheres

In this final section we briefly discuss some characteristic properties of minimal \( NH \)-balls and \( NH \)-spheres.

**Proposition 5.1.** In a minimal \( NH \)-ball or \( NH \)-sphere, the link of every simplex is a minimal \( NH \)-ball or \( NH \)-sphere.
Therefore, using the inductive hypothesis, let
\[ f(S) \] be a well defined application on \( d \). We may assume \( K \neq \Delta^d \). Since for a non-trivial decomposition \( \sigma = w \ast \eta \) we have \( \text{lk}(\sigma, S) = \text{lk}(w, \text{lk}(q, S)) \), by an inductive argument it suffices to prove the case \( \sigma = v \in V_K \). We proceed by induction on \( d \). We may assume \( d \geq 1 \). Write \( K = \Delta^d + u \ast \text{lk}(u, K) \) where, as shown before, \( \text{lk}(u, K) \) is either a minimal \( NH \)-ball or a minimal \( NH \)-sphere. Note that this in particular settles the case \( v = u \). Suppose then \( v \neq u \). If \( v \notin \text{lk}(u, K) \) then \( \text{lk}(v, K) = \Delta^{d-1} \). Otherwise, \( \text{lk}(v, K) = \Delta^{d-1} + u \ast \text{lk}(v, \text{lk}(u, K)) \). By inductive hypothesis, \( \text{lk}(v, \text{lk}(u, K)) \) is a minimal \( NH \)-ball or \( NH \)-sphere. By Lemma 2.1
\[ \text{lk}(v, K)^* = \text{lk}(v, \text{lk}(u, K))^\rho, \]
and the result follows from Theorems 3.5 and 4.5. \( \square \)

For any vertex \( v \in K \), the deletion \( K - v = \{ \sigma \in K \mid v \notin \sigma \} \) is again a minimal \( NH \)-ball or \( NH \)-sphere. This follows from Proposition 2.3 Theorems 3.5 and 4.5 and the fact that \( \text{lk}(v, K^*) = (K - v)^* \) for any \( v \in V_K \) (see [3, Lemma 4.2 (1)]). Also, Remark 4.4 implies that minimal \( NH \)-balls are (non-pure) vertex-decomposable as defined by Björner and Wachs (see [5, §11]).

Finally, we make use of Theorems 3.5 and 4.5 to compute the number of minimal \( NH \)-spheres and \( NH \)-balls in each dimension.

**Proposition 5.2.** Let \( 0 \leq k \leq d \).

1. There are exactly \( \binom{d}{k} \) minimal \( NH \)-spheres of dimension \( d \) and homotopy dimension \( k \). In particular, there are exactly \( 2^d \) minimal \( NH \)-spheres of dimension \( d \).

2. There are exactly \( 2^d \) minimal \( NH \)-balls of dimension \( d \).

**Proof.** We first prove (1). An \( NH \)-sphere with \( d = k \) is homogeneous by [3, Proposition 2.4], in which case the result is obvious. Assume then \( 0 \leq k \leq d - 1 \) and proceed by induction on \( d \). Let \( S_{d,k} \) denote the set of minimal \( NH \)-spheres of dimension \( d \) and homotopy dimension \( k \). If \( S \in S_{d,k} \) it follows from Theorem 3.5 Corollary 3.4 and Alexander duality that \( S^* \) is a minimal \( NH \)-sphere with \( \dim(S^*) < d \) and \( \dim_h(S^*) = d - k - 1 \). Therefore, there is a well defined application
\[ S_{d,k} \xrightarrow{f} \bigcup_{i=d-k-1}^{d-1} S_{i,d-k-1} \]
sending \( S \) to \( S^* \). We claim that \( f \) is a bijection. To prove injectivity, suppose \( S_1, S_2 \in S_{d,k} \) are such that \( S_1^* = S_2^* \). Let \( \rho_i = \Delta(V_{S_i} - V_{S_i^*}) \) \( (i = 1, 2) \). Since \( |V_{S_1}| = d + 2 = |V_{S_2}| \) then \( \dim(\rho_1) = \dim(\rho_2) \) and, hence, \( S_1 = (S_1^*)^{\rho_1} = (S_2^*)^{\rho_2} = S_2 \). To prove surjectivity, let \( \tilde{S} \in S_{j,d-k-1} \) with \( d - k - 1 \leq j \leq d - 1 \). Taking \( \tau = \Delta^{d-j-1} \) we have \( S^* \in S_{d,k} \) and \( f(S^*) = \tilde{S} \) (see §2.3). Finally, using the inductive hypothesis,
\[ |S_{d,k}| = \sum_{i=d-k-1}^{d-1} |S_{i,d-k-1}| = \sum_{i=d-k-1}^{d-1} \binom{i}{d-k-1} = \binom{d}{k}. \]

For (2), let \( B_d \) denote the set of minimal \( NH \)-balls of dimension \( d \) and proceed again by induction on \( d \). The very same reasoning as above gives a well defined bijection
\[ B_d - \{ \Delta^d \} \xrightarrow{f} \bigcup_{i=0}^{d-1} B_i. \]
Therefore, using the inductive hypothesis,
\[ |B_d - \{ \Delta^d \}| = \sum_{i=0}^{d-1} |B_i| = \sum_{i=0}^{d-1} 2^i = 2^d - 1. \] \( \square \)
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