Gauge dependence ambiguity and chemical potential in thermal 

$U(1)$ theory

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Abstract

In this letter we explore the dependence on the gauge fixing condition of several quantities in the $U(1)$ Higgs model at finite temperature and chemical potential. We compute the effective potential at the one loop level, using a gauge fixing condition that depends on $\mu$ and which allows to decouple the contributions of the different fields in the model. In this way we get the mass spectrum and the characterization of the phase transition, pointing out in each case how these quantities depend on the gauge fixing parameter $\xi$. When $\mu$ vanishes, we agree with previous results if $\xi = 0$. The gauge dependence problem is also analyzed from the perspective of the Nielsen identities.
I. INTRODUCTION

In spite of being a very simple case, the $U(1)$ Higgs model acquires a quite complicated structure when we go to the scenario of finite temperature and chemical potential $\mu$. For example, since the paper by Bernard [1], it is well known that at finite temperature the ghost fields cannot be factorized in a trivial way from the functional integral in the $U(1)$ case, as it does occur at zero temperature.

The gauge fixing condition at finite temperature is a long standing problem in field theory. Dolan and Jackiw [9] explored the gauge dependence of several quantities, in particular the critical temperature from the effective potential. Keeping in mind the validity of the one loop calculation up to order $1/\beta^2$, it is possible to restore the gauge invariance by expanding the effective potential up to this order.

The relationship between the gauge fixing condition and the temperature for the phase transition in the electroweak theory has been explored in [10]. The analysis of the phase transition at finite $T$, by the Landau method has been discussed in [11]. In particular, in the last article, the authors found a critical temperature that does not depend on the gauge parameter. However, this is due to the fact that a gauge condition has been chosen, which couples the Higgs with the gauge field. As we will see in this article, if we decouple the Higgs from the gauge field, which is another possibility for fixing the gauge condition, the critical temperature still depends on the gauge parameter $\xi$.

In the literature, people usually chooses the $\xi = 1$ Feynman gauge, or $\xi = 0$ Landau gauge, claiming convenience reasons. When discussing the effective potential in the frame of the Weinberg-Salam model, the unitary gauge is excluded because it leads to an erroneous result for the critical temperature and the pressure of the system [2]. It seems that as soon as temperature is turned on, the gauge invariance of the theory becomes a subtle subject. In [3], the authors emphasize that only periodic gauge functions ($\Lambda_p$) are allowed, i.e. those that satisfy $\Lambda_p(x^\mu + i\beta u^\mu) = \Lambda_p(x^\mu)$. In fact, the gauge fixing procedure and the gauge invariance of theories at finite temperature and density has not yet been fully clarified. In particular, if we expand the effective potential up to the order $1/\beta^2$ following [9], the gauge dependence is not removed when finite chemical potential is present.

In this letter, we will consider the $U(1)$ Higgs model in the presence of finite temperature and chemical potential. First, we calculate the effective potential at the one loop level, using
a gauge fixing condition that depends explicitly on the chemical potential, and which was used previously in a different context \cite{7,8}. In this way, we get non trivial dispersion relations for the different particles of our model. The role of the ghost fields is analyzed in detail for different cases. We compare our results, when $\mu = 0$, with previous calculations \cite{2}, been in agreement with the results by Kapusta when $\xi \to 0$. We calculate the critical temperature $T_c$ associated to the restoration of the $U(1)$ symmetry, emphasizing its gauge dependence. The relationship between the chemical potential and the gauge invariance problem is also analyzed.

II. THE $U(1)$ HIGGS MODEL

Our model is described by the following Lagrangian

$$\mathcal{L} = (D_\mu \phi)^* D^\mu \phi - m^2 (\phi^* \phi) - \lambda (\phi^* \phi)^2 - \frac{1}{4} F^{\mu \nu} F_{\mu \nu},$$  \hspace{1cm} (1)

where

$$D_\rho \phi = (\partial_\rho + ieA_\rho) \phi,$$  \hspace{1cm} (2)

$$F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$  \hspace{1cm} (3)

The set $\{\phi, \phi^*\}$, represents a pair of complex scalar fields. We will start in the broken phase ($m^2 < 0$), where we have a real non-vanishing vacuum expectation value $\nu$ for the $\phi$ field. The local gauge transformations $U$ that leave our Lagrangian invariant are the following

$$(A^\mu(x))^U = A^\mu(x) + \partial^\mu_2 \Lambda(x),$$  \hspace{1cm} (4)

$$\phi^U = \phi^U - \nu,$$

$$= \phi(x') - ie\Lambda(x')\phi(x') - \nu.$$

The partition function is given by

$$Z = N(\beta) \int_{\text{Per.}} D\phi D\phi^* \prod_\rho D A_\rho \exp \int_0^\beta d\tau \int d^3 x \mathcal{L}_{\text{eff}},$$  \hspace{1cm} (5)
where $Per$ in the functional integral indicates that we have to integrate over periodic euclidean fields configuration in the interval $(0, \beta)$.

The Lagrangian $\mathcal{L}_{\text{eff}}$ in the previous equation, incorporates the chemical potential through the recipe of considering $\mu$ as the zero component of a constant external gauge field $[4],$

\[
\mathcal{L}_{\text{eff}} = -(\partial_\rho - ieA_\rho + i\mu\delta_{\rho 0})\phi^*(\partial_\rho + ieA_\rho - i\mu\delta_{\rho 0})\phi
- m^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4}F_{\mu\nu}F_{\mu\nu}.
\] (6)

### III. ONE LOOP EFFECTIVE POTENTIAL

In order to calculate the effective potential, it is more convenient to express the complex $\phi$ field as a linear combination of real fields $\phi_1$ and $\phi_2$, such that $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$. $\phi_1$ acquires the non vanishing expectation value $\nu$.

We will expand the action up to second order in powers of the fields, using a Lagrangian $\mathcal{L}_J$ that incorporates external sources, $J_1$, $J_2$ and $J_\mu$,

\[
\mathcal{L}_J = \mathcal{L}_{\text{eff}} + \frac{J_1}{\beta\Omega}\phi_1 + \frac{J_2}{\beta\Omega}\phi_2 + \frac{J_\rho}{\beta\Omega}A_\rho.
\] (7)

In this way we get

\[
S = \int_0^{\beta} d\tau \int d^3x \mathcal{L}_T = S^{(0)} + S^{(1)} + S^{(2)} + ..., \tag{8}
\]

where

\[
S^{(0)} = \left[ \frac{\mu^2}{2}\nu^2 - \frac{m^2}{2}\nu^2 - \frac{\lambda}{4}\nu^4 \right] \beta\Omega + J_1\nu, \tag{9}
\]

and

\[
S^{(1)} = \int_0^{\beta} d\tau \int d^3x \{-i\mu\nu\delta_{\rho 0}\partial_\rho \phi_2 + \mu^2\nu\phi_1 - eA_\rho\delta_{\rho 0}i\mu\nu^2 - m^2\nu\phi_1 - \lambda\nu^3\phi_1
+ \frac{J_1}{\beta\Omega}\phi_1 + \frac{J_2}{\beta\Omega}\phi_2 + \frac{J_\rho}{\beta\Omega}A_\rho \}.
\] (10)

By shifting $\phi_1 \to \phi_1 + \nu$, we can suppress terms proportional to $\phi_1$ in $S^{(1)}$, since the quantum fields must have a zero vacuum expectation value. Choosing
\[
\frac{J_{\phi_1}}{\beta \Omega} = \nu(m^2 - \mu^2 + \lambda \nu^2), \quad (11)
\]

and

\[
\frac{J_{A_0}}{\beta \Omega} = e i \mu \nu^2, \quad (12)
\]

we can eliminate the terms proportional to \(\phi_1\) and \(A^0\) in \(S^{(1)}\).

The \(S^{(2)}\) can be written as a quadratic form

\[
S^{(2)} = -\frac{1}{2} \int_0^\beta d\tau \int d^3x (\phi_1, \phi_2, A_\rho) M \begin{pmatrix} \phi_1 \\ \phi_2 \\ A_\alpha \end{pmatrix}, \quad (13)
\]

where \(M\) is a matrix given by

\[
M = \begin{pmatrix} -\partial_\rho \partial_\rho - \mu^2 + m^2 + 3 \lambda \nu^2 & 2 i \mu \partial_\rho \delta_{\rho \alpha} & 2 i \mu \partial_\rho \nu \\ -2 i \mu \partial_\rho \delta_{\rho \alpha} & -\partial_\rho \partial_\rho - \mu^2 + m^2 + \lambda \nu^2 & e \nu \partial_\rho \\ 2 i \mu \delta_{\alpha \alpha} \nu & -e \nu \partial_\alpha & e^2 \delta_{\rho \alpha} \nu^2 + B_{\rho \alpha} \end{pmatrix}. \quad (14)
\]

The operator \(B_{\rho \alpha}\) is defined as

\[
B_{\rho \alpha} = -\partial_\lambda \partial_\lambda \delta_{\rho \alpha} + \partial_\rho \partial_\alpha. \quad (15)
\]

In the previous matrix we have undesirable mixing terms which couples \(\phi\) with \(A\). These terms can be eliminated through an appropriate gauge fixing condition given by

\[
F = (\partial_\rho + i2\mu \delta_{\rho \alpha}) A_\rho - i e \xi \nu (\phi_1 + i \phi_2). \quad (16)
\]

This gauge fixing condition was introduced previously \[\text{[7]}\] to calculate the contribution of each field to the the effective potential of the Weinberg-Salam model, and as a background field that maintains explicitly the gauge invariance \[\text{[8]}\].

From this condition, the Faddeev-Popov Lagrangian, at finite \(\mu\) can be read as

\[
\mathcal{L}_{FP} = \partial_\rho \bar{\eta}(\partial^\rho + 2 i \mu \delta_{\rho \alpha}) \eta - \xi \bar{\eta} e^2 \nu^2 \eta \quad (17)
\]

Finally, our total lagrangian is written as a sum of the different contributions \(\mathcal{L}_T = \mathcal{L}_J + \mathcal{L}_{GF} + \mathcal{L}_{FP}\). The ghost Lagrangian plays a important role when counting the degrees
of freedom of the model, but since they do not couple to the physical fields, their role will be discussed in the next section. The action term $S^{(2)}$ is defined by a new matrix $\tilde{M}$,

$$
\tilde{M} = \begin{pmatrix}
-\partial_\rho \partial_\rho - \mu^2 + m_1^2 & 2i\mu \partial_\rho \delta_{\rho 0} & 0 \\
-2i\mu \partial_\rho \delta_{\rho 0} & -\partial_\rho \partial_\rho - \mu^2 + m_2^2 & 0 \\
0 & 0 & e^2 \delta_{\rho a} \nu^2 + \tilde{B}_{\rho a}
\end{pmatrix},
$$

(18)

where we have introduced effective gauge dependent masses $m_1$, $m_2$ as

$$
m_1^2 = m^2 + 3\lambda \nu^2 + e^2 \nu^2 \xi, \hspace{1cm} (19)
$$

$$
m_2^2 = m^2 + \lambda \nu^2 + e^2 \nu^2 \xi. \hspace{1cm} (20)
$$

Note that both $m_1$ and $m_2$ depend on the gauge parameter $\xi$ this means, for an arbitrary value of $\xi$ there are no Goldstone boson in this model. We will come back to this point later.

The operator $\tilde{B}_{\rho a}$ is the extension of (15) including gauge dependent terms

$$
\tilde{B}_{\rho a} = -\partial_\rho \partial_\rho \delta_{\rho a} + \partial_\rho \partial_\rho (1 - \xi^{-1}) + 4\mu^2 \delta_{\rho 0} \delta_{\alpha 0} \xi^{-1}. \hspace{1cm} (21)
$$

Finally, the effective potential is given by

$$
\mathcal{V}_{\text{eff}} = -\frac{1}{\beta \Omega} (\ln Z) = -\frac{1}{\beta \Omega} (S^{(0)} + \ln Z_1), \hspace{1cm} (22)
$$

with

$$
Z_1 \equiv \int_{\text{Per.}} D\phi_1 D\phi_2 DA e^{S^{(2)}}, \hspace{1cm} (23)
$$

where $\Omega$ is the spatial volume.

In the momentum space, $\ln Z_1$ can be written as a sum over the Matsubara frequencies $\omega_n = 2\pi n / \beta$

$$
\ln Z_1 = -\frac{\Omega}{T} \sum_n \int \frac{d^3k}{(2\pi)^3} \ln \left\{ \det_{\rho a} ([k^2 + e^2 \nu^2] \delta_{\rho a} - k_\rho k_\alpha (1 - \xi^{-1}) - 4\mu^2 \delta_{\rho 0} \delta_{\alpha 0} \xi^{-1}) \right\} \\
-\frac{\Omega}{T} \sum_n \int \frac{d^3k}{(2\pi)^3} \ln \left\{ k^4 + k^2 (m_1^2 + m_2^2 - 2\mu^2) + 4\mu^2 \omega_n^2 + (m_1^2 - \mu^2) (m_2^2 - \mu^2) \right\},
$$

(24)
with $k^2 = \omega_n^2 + \mathbf{k}^2$ and where $\det_{\rho\alpha}$ refers to the determinant in the Lorentz indices. The polynomials in the equation above, since they are quadratic in $\omega_n^2$, can be factorized such that

$$
\ln Z_1 = -\frac{\Omega}{2} \sum_n \int \frac{d^3k}{(2\pi)^3} \ln\{(\omega_n^2 + x_1^2)(\omega_n^2 + x_2^2)\}
- \frac{\Omega}{2} \sum_n \int \frac{d^3k}{(2\pi)^3} \ln\{(\omega_n^2 + y_1^2)(\omega_n^2 + y_2^2)\} + \ln\{(\omega_n^2 + y_3^2)(\omega_n^2 + y_4^2)\},
$$

(25)

Using the identity

$$
\sum_n \ln(\omega_n^2 + X^2) = \beta X + 2 \ln(1 - e^{-\beta X}),
$$

(26)

we can decompose the effective potential as a sum of terms associated to the different fields in the model. In other words we have diagonalized the effective potential. Then

$$
V_{\text{eff}} = V_\phi + V_A,
$$

(27)

where $V_{\phi,A}$ is given by

$$
V = -\int \frac{d^3k}{(2\pi)^3} \frac{X(k)}{2} + \frac{\ln(1 - e^{-\beta X(k)})}{\beta},
$$

(28)

which is valid for each component of the fields.

In this way we can identify $x_{1,2}$ and $y_i$ ($i = 1..4$) with the energy spectra of these pseudo-particles

$$
x_1^2 = m_1^2 + \mathbf{k}^2 - \lambda \nu^2 + \mu^2 + \sqrt{4\mu^2(m_1^2 + \mathbf{k}^2 - \lambda \nu^2) + \lambda^2 \nu^4},
$$

$$
x_2^2 = m_2^2 + \mathbf{k}^2 + \lambda \nu^2 + \mu^2 - \sqrt{4\mu^2(m_2^2 + \mathbf{k}^2 + \lambda \nu^2) + \lambda^2 \nu^4}.
$$

(29)

Notice that if $\nu = 0$ and $\mu \neq 0$, (29) simplifies to the well known expression

$$
x_{1,2} = E_\pm^\phi = \sqrt{\mathbf{k}^2 + m^2} \pm \mu.
$$

(30)

Here, $m_1 = m_2 = m$ and there is no dependence on $\xi$. If $\mu = 0$ and $\nu \neq 0$, $x_1$ and $x_2$ depend on $\xi$.

For the gauge fields we find two non-anomalous massive “photonic” states which obeys the following dispersion relation
\[ y_1 = y_2 = E_\gamma = \sqrt{k^2 + e^2 \nu^2}. \] (31)

Notice that \( x_1 \) and \( x_2 \) describe the possible energy values for \( \phi_1 \) and \( \phi_2 \).

The pathological energies associated to the other two “photonic” degrees of freedom are given, in terms of \( \varepsilon = 1 - \xi \), by

\[
y_3^2 = k^2 + \frac{e^2 \nu^2}{2} (1 + \xi) + 2\mu^2 + \frac{1}{2} \sqrt{e^4 \nu^4 \varepsilon^2 + 8\mu^2 e^2 \nu^2 \varepsilon - 16\mu^2 (k^2 \xi^{-1} \varepsilon - \mu^2)}, \\
y_4^2 = k^2 + \frac{e^2 \nu^2}{2} (1 + \xi) + 2\mu^2 - \frac{1}{2} \sqrt{e^4 \nu^4 \varepsilon^2 + 8\mu^2 e^2 \nu^2 \varepsilon - 16\mu^2 (k^2 \xi^{-1} \varepsilon - \mu^2)}. \] (32)

We call these energies pathologicals, since they do depend explicitly on the gauge parameter \( \xi \).

Repeating the same previous analysis, if \( \nu = 0 \) and \( \mu \neq 0 \), with \( \xi = 1 \), we recover a photon with energy \( y_4 = |k| \).

On the other side, in the broken phase when \( \mu = 0 \), we have \( y_3 = \sqrt{k^2 + e^2 \nu^2} \) and \( y_4 = \sqrt{k^2 + e^2 \nu^2 \xi} \), i.e. we have three massive degrees of freedom \((y_1, y_2 \text{ and } y_3)\) for the photon and a gauge dependent contribution \((y_4)\). Now taking \( \mu \neq 0 \) and \( \xi = 1 \), we recover the three degrees of freedom for the massive photon: \( y_1, y_2 \text{ and } y_4 \).

As we will see in the next section, the Faddeev-Popov ghost fields will compensate these anomalous dispersion relations. In particular for \( \xi = 1 \), the situation is completely clear. This probably is one of the reasons why people prefer the Feynman Gauge \( \xi = 1 \).

**IV. GHOSTS AND EFFECTIVE POTENTIAL**

In order to get a proper counting of the degrees of freedom, we have to introduce the Faddeev-Popov ghost fields. This point is completely different with respect to what occurs in the U(1) theory at zero temperature, where the ghost can be factorized and do not contribute to the effective potential.

The functional integration over the ghost fields, represented by the complex Grassmann variables \((\eta, \bar{\eta})\), leaves

\[
\int D\eta D\bar{\eta} \exp \left\{ \int_0^\beta d^3x \bar{\eta} \left( -\frac{\delta F}{\delta A} \right) \eta \right\} = \Omega \sum \ln(\omega_n^2 + k^2 + e^2 \nu^2 \xi + 2i \mu \omega_n). \] (33)
If we write the argument of the logarithm in the polar representation

\[ \rho e^{i\varphi} = \omega_n^2 + k^2 + e^2\nu^2\xi + 2i\mu \omega_n, \] (34)

with

\[ \rho = \sqrt{(\omega_n^2 + k^2 + e^2\nu^2)^2 + 4\mu^2\omega_n^2}; \] (35)
\[ \varphi = \arctan \left( \frac{4\omega_n\mu}{\omega^2 + k^2 + e^2\nu^2} \right), \] (36)

we see that the sum over \( \varphi \) vanishes since \( \varphi(\omega_n) = -\varphi(-\omega_n) \). Factorizing \( \rho^2 = (\omega_n^2 + z_1^2)(\omega_n^2 + z_2^2) \), we find

\[ z_{1,2} = E_{\pm} = \sqrt{k^2 + e^2\nu^2\xi + \mu^2 \pm \mu}. \] (37)

If \( \mu = 0 \), it is interesting to notice that one of the ghost fields cancels the contribution from the nonphysical photon \( y_3 \).

Using (26), we can decompose the effective potential such that \( V_{\text{eff}} = V_{\text{tree}} + V_\phi + V_A + V_\eta \), where the different contributions are

- **Tree level contribution**

\[ V_{\text{tree}} = \left[ \frac{m^2}{2}\nu^2 - \frac{\mu^2}{2}\nu^2 + \frac{\lambda}{4}\nu^4 \right]; \] (38)

- **\( \phi \) contribution**

\[ V_\phi = \int \frac{d^3k}{(2\pi)^3} \left( \frac{x_1(k) + x_2(k)}{2} \right) + \ln(1 - e^{-\beta x_1(k)})(1 - e^{-\beta x_2(k)}) \frac{\ln(1 - e^{-\beta x_1(k)})(1 - e^{-\beta x_2(k)})}{\beta}; \] (39)

- **Massive photons contribution**

\[ V_A = \int \frac{d^3k}{(2\pi)^3} \left( \frac{y_1(k) + y_2(k)}{2} \right) + \ln(1 - e^{-\beta y_1(k)})(1 - e^{-\beta y_2(k)}) \frac{\ln(1 - e^{-\beta y_1(k)})(1 - e^{-\beta y_2(k)})}{\beta} + \sqrt{k^2 + e^2\nu^2} + \frac{2}{\beta} \ln(1 - e^{-\beta \sqrt{k^2 + e^2\nu^2}}); \] (40)

- **Ghost contribution**

\[ V_\eta = -\int \frac{d^3k}{(2\pi)^3} \left( \frac{z_1(k) + z_2(k)}{2} \right) + \ln(1 - e^{-\beta z_1(k)})(1 - e^{-\beta z_2(k)}) \frac{\ln(1 - e^{-\beta z_1(k)})(1 - e^{-\beta z_2(k)})}{\beta}. \] (41)
V. HIGH TEMPERATURE EXPANSION

Our previous results for the effective potential are interesting since we decouple the contributions for the different fields. However, the integrals cannot be calculated analytically, and therefore it is appealing to carry on a high temperature expansion of the effective potential, which will allow us to compare our expressions with well known results from the literature.

For the one loop effective potential in a high temperature expansion we find

\[
V_{\text{eff}} = \left[ \frac{m^2}{2} \nu^2 - \frac{\mu^2}{2} \nu^2 + \frac{\lambda}{4} \nu^4 \right] - \frac{2\pi^2 T^4}{45} + \frac{T^2}{12} \left\{ m^2 + 2\lambda \nu^2 + \frac{(3 + \xi) e^2 \nu^2}{2} + \xi^{-1} \mu^2 \right\}. \quad (42)
\]

The vacuum is defined by

\[
\lambda \nu(T)^2_{\text{min}} = \begin{cases} 
(\mu^2 + |m|^2) \left[ 1 - \frac{T^2}{T_c^2} \right], & \text{para } T \leq T_c; \\
0, & \text{para } T > T_c.
\end{cases}
\]

\[
T_c^2 = \frac{12(\mu^2 + |m|^2)}{4\lambda + 3e^2 + \xi e^2}, \quad (44)
\]

where \(T_c\) is the critical temperature where the symmetry is restored.

The critical reader at this moment could be surprised that the critical temperature, which is in principle an observable quantity, depends on the gauge parameter. Notice, however, that for finite chemical potential it is well known that the number of Goldstone bosons is lesser than the usual prediction \([12]\). This is in agreement with our results. In our case, both higgs fields acquire a mass. On the other hand, if \(\mu\) vanishes, we know that we must recover one Goldstone boson. This is not possible, unless that \(\xi\) also vanishes. In this case the \(R_\xi\) gauge becomes the Lorentz gauge. Our results is then in agreement with \([2]\) and \([10]\), when \(\mu = 0\), in the gauge defined as \(\xi = 0\) (Landau gauge). So this gauge seems to be the more appropriate to extend the calculations to the scenario with finite mu.

For finite \(\mu\), it seems that the one loop calculation of the effective potential is not enough for having a gauge independent result for the critical temperature.

In general, from previous work \([13]\) we know that the determination of critical values of parameters associated to phase transitions requires to go beyond the one loop approximation through an appropriate re-summation.
As it is well known, the occurrence of the phase transition can be inferred from the behavior of the isothermals of the effective potential. The pressure is defined as \( P = T \frac{\partial}{\partial V} \ln Z \), which implies \( P = -V_{\text{eff}} \), since the effective potential corresponds to the thermodynamical potential of the system.

We found for the pressure.

\[
P_\lt = \frac{2\pi^2 T^4}{45} + \frac{|m|^2 - \mu^2 \xi^{-1}}{12} T^2 + \frac{(|m|^2 + \mu^2)^2}{4\lambda} \left( 1 - \frac{T^2}{T_c^2} \right)^2,
\]
\[
P_\gt = \frac{2\pi^2 T^4}{45} + \frac{|m|^2 - \mu^2 \xi^{-1}}{12} T^2,
\]

where \( P_\lt \) (\( P_\gt \)) corresponds to the broken (symmetric) phase. Here we have the same difficulty, a gauge dependence, we found previously for the critical temperature.

The pressure and the entropy are continuous at \( T_c \). However the specific heat has a discontinuity which depends on \( \xi \) only through \( T_c \). This means that this is a second order phase transition. In fact, the behavior of the specific heat confirms this picture.

The gauge dependence problem can also be analyzed from the perspective of the Nielsen Identities. This method [14], related to the BRST symmetry transformations, is a procedure that allows to search for possible gauge dependence of physical quantities which are related to an explicitly gauge dependent effective action. If these identities are satisfied, we may have confidence that the results for the physical magnitudes will be gauge independent. The Nielsen identities are still valid for finite temperature [15]. An interesting discussion of the Nielsen identities for the generalized \( R_\xi \) gauge in the abelian Higgs model can be found in [16]. We will follow the strategy of this article to explore the validity of the Nielsen identities for our case.

The Nielsen identities arise from the following identity

\[
\xi \frac{\partial \Gamma}{\partial \xi} = \int d^d x \int d^d y \frac{\delta \Gamma}{\delta \phi_i(y)} \left\langle \Delta_i \eta(y) \bar{\eta}(x) \left[ \frac{F}{2} - \xi \frac{\partial F}{\partial \xi} \right] \right\rangle_{\Gamma},
\]

where we use the notation
\[ \langle O \rangle = e^\Gamma \int D\phi_i D\eta D\bar{\eta} O \exp \left( -S_F + \int d^d x \frac{\delta \Gamma}{\delta \phi_i} (\phi_i - \varphi_i) \right), \quad (48) \]

and where \( \eta, \bar{\eta} \) are the ghost fields, \( F \) is the gauge fixing condition, which in our case is given by eq. (16), \( \Gamma \) is the effective action, \( \xi \) is the gauge parameter and \( \Delta_i \) is the BRST transformation of the scalar fields \( \phi_1 \) and \( \phi_2 \) which is given by \( \delta \phi_i = \Delta_i \eta \) with \( \Delta_i = (-\phi_2, \phi_1) \).

In the finite temperature scenario, we have to keep in mind that the integrals in the previous equation, when going to the momentum space must be handled (d=4) as

\[ \int d^4 k \to \sum_n \int \frac{d^3 k}{(2\pi)^3}, \quad (49) \]

where we sum over Matsubara frequencies \( \omega_n = 2\pi n/\beta \).

Assuming for the classical components of the fields \( \phi_1, \phi_2, \phi_1 = \nu \) and \( \phi_2 = 0 \), the Nielsen identities can be expressed as

\[ \xi \frac{\partial V}{\partial \xi} = C \frac{\partial V}{\partial \nu}, \quad (50) \]

where \( C \) is given by

\[ C = \frac{1}{2} \int d^d x \int d^d y \langle \phi_2(y) \eta(y) \bar{\eta}(x) \left( (\partial_\rho + i2\mu \delta_\rho_0) A_\rho + i\epsilon \xi \nu (\phi_1 + i\phi_2) \right) \rangle, \quad (51) \]

and must be evaluated in a perturbative expansion, see [16].

We explored the validity of the Nielsen identities for the following cases

- \( \mu = 0, \nu = 0 \): It is very easy to see that the equation (50) is valid for any value of \( \xi \), in this case, confirming [15] the validity of the Nielsen identities for finite temperature.

- \( \mu = 0, \nu \neq 0 \) In this case the Nielsen identities are valid only if we take \( \xi = 0 \) and if we evaluate the derivatives in \( \nu = \nu_{\text{min}} \), i.e. in the classical value. Note that \( \xi = 0 \) is demanded by the Goldstone theorem, which implies \( m_2 = 0 \).

- \( \mu \neq 0, \nu = 0 \) and \( \mu \neq 0, \nu \neq 0 \). In these cases the Nielsen identities are not valid for any possible value of \( \xi \). In fact, the only non vanishing contribution to the left hand side of the Nielsen identities, according to eq. (50) is given by the “photonic” dispersion relations, eq. (32), for the \( \mu = 0, \nu \neq 0 \) case. When \( \mu \neq 0, \nu \neq 0 \), all degrees of freedom contribute to the derivatives that appear on the Nielsen identities.
However, in both cases the $\frac{\partial \mu}{\partial \xi}$ diverge when $\xi$ goes to zero. This means that we are not allowed to take the $\xi = 0$ gauge, where the Goldstone theorem is realized, in agreement with the result by [12].

To summarize we can see that the $\mu$ dependent gauge fixing condition we used here, and which was valuable for diagonalizing the effective potential in the Weinberg-Salam model for $\xi = 1$, [7], turn out to be unnatural when looking for gauge invariant predictions for physical quantities in the abelian Higgs model.

The gauge dependence problem is related to two different aspects:

- The necessity of having gauge transformation which are periodic in the temporal direction, with period $\beta$ and
- the occurrence of finite $\mu$.

As a conclusion, we would like to remark that the problem of the gauge fixing at finite temperature and/or density has not yet been solved. In the U(1) Higgs model we have explored in detail how this gauge dependence propagates through several physical relevant quantities. Nevertheless, when $\mu$ vanishes, we agree with previous results in the literature.

It can be shown that $\mu$ can be incorporated as a boundary condition for the Green function of fields confined to a finite spatial region, as it occurs in chiral bag models [5, 6]. Eventually, the construction of the theory should start by taking the theory in a bounded space region, with adequate $\mu$-dependent boundary condition. When taking then the limit where we go into the whole space, probably a remanent of the non-trivial boundary conditions will remain as a non-trivial dependence on the gauge fixing condition. This point will be explored in a future work.

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