Time-dependent Spinor field in a static cylindrically symmetric space-time

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Within the scope of a static cylindrically symmetric space-time we study the behavior of a nonlinear spinor field that depends on time and radial coordinates. It is found that the presence of nontrivial non-diagonal components of the energy-momentum tensor (EMT) imposes some restrictions on both the metric functions and the spinor field. While for the time independent spinor field there occur three way restrictions resembling the Bianchi type-I Universe, in this case things become more complicated.

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I. INTRODUCTION

From the physical point of view the spherically symmetric solutions to the Einstein equations with nontrivial energy-momentum tensor present major interest, since this class of space-time is compatible with asymptotic flatness. Another class of asymptotically flat space-time is the axially symmetric ones. Due to mathematical difficulties related to axially symmetric space-time many authors consider cylindrically symmetric space-time as a preliminary step, though the later one is not asymptotically flat [1]. The cylindrical symmetric solutions localized in the vicinity of the axis of symmetry, such as vortices was studied in [2]. As it was shown in [3] even the simplest nonlinear equation of complex scalar field can provide with not only particle-like solutions, but also string-like solutions which was described by a cylindrical symmetric space-time. These solutions may describe realistic objects such as super-conducting threads - fluxons [4] or light beams [5] and serve as reasonable approximation to objects with a toroidal structure where large toroid radius is approximately replaced by a closed string segment [6].

Within the framework of a static cylindrically symmetric space-time Levi-Civita discovered a class of solutions of Einstein field equations in vacuum [7]. Further this was developed by many authors. An interacting system of scalar, electromagnetic and gravitational fields within the scope of cylindrical symmetric space-time was studied in [8, 9]. The corresponding equations were exactly solved and soliton-like solutions as a whole and droplet-like solutions in particular, were obtained. There has been increasing interest in the study of neutrino stars in astrophysics. In doing so static cylindrically symmetric solutions to the Einstein field equation with elastic matter was obtained in [10]. The possibility of forming anisotropic compact stars from cosmological constant as one of the competent candidates of dark energy with cylindrical symmetry was discussed in [11]. A comprehensive review of cylindrical symmetric systems in GR can be found in [12].
In most cases perfect fluid and scalar fields are considered as the the source of the corresponding gravitational fields. In recent years some alternatives to those are also being exploited. Since the spinor field is very sensitive to the gravitational one and can generate different kinds of fluids and dark energy, many authors consider this field in cosmological problems [13–15].

Lately spinor field is being used in various fields that includes gravity as well. The coupled Einstein-Dirac equations for a static, spherically symmetric system of two fermions in a singlet spinor state are derived and soliton-like solutions to these equations were found by Finster et al [16]. A comparative analysis of three different types of solitonic solutions of GR-matter systems, which can be interpreted as explicit realizations of Wheeler’s geon concept for matter fields of spin 0, 1/2 and 1, emphasizing the mathematical similarities and clarifying the physical differences, particularly between the bosonic and fermionic cases, is also presented. Beside these Einstein-Dirac system was studied in [18, 19].

In a recent paper [20] we have studied the behavior of a nonlinear spinor field within the scope of a static cylindrical-symmetric space-time. In that paper we have considered the spinor field that depends on time and radial coordinates within the framework of a static cylindrical-symmetric space-time.

In this report we will extend that study and consider the spinor field that depends on time and radial coordinates within a framework of a static cylindrical-symmetric space-time.

II. BASIC EQUATIONS AND THEIR SOLUTIONS

The spinor field Lagrangian we take in the form [13]
\[ L_{sp} = \frac{i}{2} \left[ \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi \right] - m_{sp} \bar{\psi} \psi - \lambda F(K), \] (2.1)

with the nonlinear term being some arbitrary function of the invariant \( K = \{ I, J, I+J, I-J \} \) i.e., \( F = F(K) \), where \( I = S^2 = (\bar{\psi} \psi)^2 \) and \( J = P^2 = (i \bar{\psi} \gamma^5 \psi)^2 \).

The spinor field equations corresponding to the Lagrangian (2.1) are
\[ i \gamma^\mu \nabla_\mu \psi - \left( m_{sp} + \mathcal{D} \right) \psi - i \mathcal{G} \gamma^5 \psi = 0, \] (2.2a)
\[ i \nabla_\mu \bar{\psi} \gamma^\mu + \left( m_{sp} + \mathcal{D} \right) \bar{\psi} + i \mathcal{G} \bar{\psi} \gamma^5 = 0, \] (2.2b)

where we denote \( \mathcal{D} = 2\lambda SF_K K_I \) and \( \mathcal{G} = 2\lambda PF_K K_J \), with \( F_K = dF/dK \), \( K_I = dK/dI \) and \( K_J = dK/dJ \). In view of (2.2) it can be shown that
\[ L_{sp} = \lambda \left( 2KF_K - F \right). \] (2.3)

The energy-momentum tensor (EMT) of the spinor field is obtained from
\[ T_{\mu}^\rho = \frac{i}{4} g^{\rho \nu} \left( \bar{\psi} \gamma_\mu \nabla_\nu \psi + \bar{\psi} \gamma_\nu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma_\nu \psi - \nabla_\nu \bar{\psi} \gamma_\mu \psi \right) - \delta_\mu^\rho L_{sp}. \] (2.4)

\[ \nabla_\nu \psi = \partial_\nu \psi - \Omega_\mu \psi, \quad \nabla_\nu \bar{\psi} = \partial_\nu \bar{\psi} + \Omega_\mu \bar{\psi}, \] (2.5)

\[ T_{\mu}^\rho \] can be written as
\[ T_{\mu}^\rho = \frac{i}{4} g^{\rho \nu} \left( \bar{\psi} \gamma_\mu \partial_\nu \psi + \bar{\psi} \gamma_\nu \partial_\mu \psi - \partial_\mu \bar{\psi} \gamma_\nu \psi - \partial_\nu \bar{\psi} \gamma_\mu \psi \right) - \frac{i}{4} g^{\rho \nu} \left( \gamma_\mu \Omega_\nu + \Omega_\nu \gamma_\mu + \gamma_\nu \Omega_\mu + \Omega_\mu \gamma_\nu \right) \psi - \delta_\mu^\rho L_{sp}. \] (2.6)
Here $\Omega_\nu$ is the spinor affine connection constructed from the metric function. Let us consider the static cylindrically symmetric space-time given by the metric element \[10\]

\[ ds^2 = e^{2\gamma} dt^2 - e^{2\alpha} du^2 - e^{2\beta} d\phi^2 - e^{2\mu} dz^2, \tag{2.7} \]

where the spacetime coordinates are $x^\mu = \{t, u, \phi, z\}$ and $\gamma, \alpha, \beta$ and $\mu$ are $C^2$ functions of $u$ only.

In analogy to standard spherically symmetric spacetime, it is customary to define the radial coordinate $r$ in such a way that co-efficient of $d\phi^2$ is equal to $r^2$, i.e. $e^{2\beta} = r^2$. This transformation is called tangential gauge \[11\].

The spinor affine connection $\Omega_\mu$ is defined as

\[ \Omega_\mu = \frac{1}{4} g_{\rho \sigma} \left( \partial_\mu e_{\tau}^{(b)} e_{\rho}^{(b)} - \Gamma_{\mu \tau}^{\rho} \right) \gamma^\rho \gamma^\tau, \tag{2.8} \]

where $e_{\tau}^{(b)}$ and $e_{\rho}^{(b)}$ are the tetrad vectors such that $e_{\tau}^{(b)} e_{\rho}^{(b)} = \delta_\tau^\rho$ and $e_{\rho}^{(a)} e_{\rho}^{(a)} = \delta_\beta^\alpha$. From (2.8) we find the spinor affine connection corresponding to the metric (2.7):

\[ \Omega_0 = -\frac{1}{2} e^{\gamma - \alpha} \gamma^0 \gamma^1, \quad \Omega_1 = 0, \quad \Omega_2 = \frac{1}{2} e^{\beta - \alpha} \beta^1 \gamma^2 \gamma^1, \quad \Omega_3 = \frac{1}{2} e^{\mu - \alpha} \mu^1 \gamma^3 \gamma^1, \tag{2.9} \]

where prime (') denotes differentiation with respect to $u$.

Let us now consider the case when the spinor field depends on $x^0 = t$ and $x^1 = u$ such that $\psi = e^{i\alpha t} \psi(u)$ and $\bar{\psi} = e^{i\alpha t} \bar{\psi}(u)$.

The spinor field equations (2.2) in this case read

\[ \psi' + \frac{1}{2} \tau' \psi - i \omega e^{\alpha - \gamma} \gamma^0 \gamma^1 \psi - i D_1 e^{\alpha} \gamma^1 \psi - \mathcal{G} e^{\alpha} \gamma^3 \gamma^1 \psi = 0, \tag{2.10a} \]

\[ \bar{\psi}' + \frac{1}{2} \bar{\tau}' \bar{\psi} - i \omega e^{-\alpha - \gamma} \gamma^0 \gamma^1 + i D_1 e^{-\alpha} \bar{\gamma}^3 \bar{\gamma}^1 + \mathcal{G} e^{-\alpha} \bar{\gamma}^5 \bar{\gamma}^1 + 0, \tag{2.10b} \]

where we denote $D_1 := (m_{\text{sp}} + D)$ and

\[ \tau = \gamma + \beta + \mu. \tag{2.11} \]

Taking into account that $S = \bar{\psi} \psi, P = i \bar{\psi} \gamma^5 \psi$ and denoting $\bar{\psi}^\mu = \bar{\psi}^\mu \psi, \bar{A}^\mu = i \bar{\psi} \gamma^5 \bar{\psi}^\mu \psi$ and $\bar{Q}^{\mu \nu} = \bar{\psi} \gamma^\mu \gamma^\nu \psi$ from the foregoing equations one finds the following relations

\[ \bar{\psi} \gamma^0 \psi' - \psi' \gamma^0 \psi = 2 \omega e^{\alpha - \gamma} \gamma^1 = 0, \tag{2.12a} \]
\[ \bar{\psi} \gamma^1 \psi' - \psi' \gamma^1 \psi = 2 \omega e^{-\alpha - \gamma} \gamma^0 + 2 e^{\alpha} D_1 S + 2 e^{\alpha} \mathcal{G} P = 0, \tag{2.12b} \]
\[ \psi \gamma^2 \psi' - \psi' \gamma^2 \psi = 0, \tag{2.12c} \]
\[ \psi \gamma^3 \psi' - \psi' \gamma^3 \psi = 0, \tag{2.12d} \]

and
The foregoing system gives the following first integrals

\begin{align}
S'_0 - 2t\omega e^{\alpha - \gamma} Q^{01}_0 - 2\mathcal{G} e^{\alpha} \bar{A}_0^1 &= 0, \\
P'_0 - 2t\omega e^{\alpha - \gamma} Q^{03}_0 - 2\mathcal{D} e^{\alpha} \bar{A}_0^1 &= 0, \\
\bar{v}_0^\mu - 2t\mathcal{D} e^{\alpha} \bar{Q}_0^{01} - 2t\mathcal{G} e^{\alpha} \bar{Q}_0^{03} &= 0, \\
\bar{v}_0^{1\prime} &= 0, \\
\bar{v}_0^{2\prime} + 2\omega e^{\alpha - \gamma} \bar{A}_0^3 + 2t\mathcal{D} e^{\alpha} \bar{Q}_0^{12} - 2t\mathcal{G} e^{\alpha} \bar{Q}_0^{03} &= 0, \\
\bar{v}_0^{3\prime} + 2\omega e^{\alpha - \gamma} \bar{A}_0^2 - 2t\mathcal{D} e^{\alpha} \bar{Q}_0^{03} + 2t\mathcal{G} e^{\alpha} \bar{Q}_0^{02} &= 0, \\
\bar{A}_0^0 &= 0, \\
\bar{A}_0^{1\prime} + 2\mathcal{D} e^{\alpha} P_0 + 2\mathcal{G} e^{\alpha} S_0 &= 0, \\
\bar{A}_0^{2\prime} - 2\omega e^{\alpha - \gamma} \bar{v}_0^1 &= 0, \\
\bar{A}_0^{3\prime} + 2\omega e^{\alpha - \gamma} \bar{v}_0^2 &= 0, \\
\bar{Q}_0^{01\prime} - 2t\omega e^{\alpha - \gamma} S_0 + 2t\mathcal{D} e^{\alpha} \bar{v}_0^0 &= 0, \\
\bar{Q}_0^{02\prime} - 2t\mathcal{G} e^{\alpha} \bar{v}_0^3 &= 0, \\
\bar{Q}_0^{03\prime} + 2t\mathcal{G} e^{\alpha} \bar{v}_0^2 &= 0, \\
\bar{Q}_0^{12\prime} - 2t\mathcal{D} e^{\alpha} \bar{v}_0^2 &= 0, \\
\bar{Q}_0^{13\prime} - 2t\mathcal{D} e^{\alpha} \bar{v}_0^3 &= 0, \\
\bar{Q}_0^{23\prime} - 2t\omega e^{\alpha - \gamma} P_0 + 2t\mathcal{G} e^{\alpha} \bar{v}_0^0 &= 0,
\end{align}

where we denote \( S_0 = Se^\tau, P_0 = Pe^\tau, \bar{v}_0^\mu = \bar{v}_0, \bar{A}_0^\mu = \bar{A}_0 e^\tau \) and \( \bar{Q}^{\mu\nu} = \bar{Q}^{\mu\nu} e^\tau \).

The foregoing system gives the following first integrals

\begin{align}
\bar{A}_0^0 &= C_1, \\
\bar{v}_0^1 &= C_2, \\
(S_0)^2 + (P_0)^2 + (\bar{A}_0^1)^2 - (\bar{v}_0^0)^2 - (\bar{Q}_0^{01})^2 - (\bar{Q}_0^{03})^2 &= C_3^2, \\
(\bar{v}_0^2)^2 + (\bar{v}_0^3)^2 + (\bar{A}_0^2)^2 + (\bar{A}_0^3)^2 + (\bar{Q}_0^{02})^2 + (\bar{Q}_0^{03})^2 + (\bar{Q}_0^{12})^2 - (\bar{Q}_0^{13})^2 &= C_4^2,
\end{align}

where \( C_i \) are the integration constants. From (2.14) it is clear that each term in these equalities, i.e. \( S_0, P_0, \bar{A}_0^\mu, \bar{v}_0^\mu, \bar{Q}_0^{\mu\nu} \) are constants.

Finally we have the following nontrivial components of EMT

\begin{align}
T_0^0 &= \omega e^{-\gamma} \bar{v}_0 + \lambda (F - 2KF), \\
T_1^1 &= m_{sp} s + \lambda F - \omega e^{-\gamma} \bar{v}_0, \\
T_2^2 &= \lambda (F - 2KF), \\
T_3^3 &= \lambda (F - 2KF), \\
T_1^0 &= -\omega e^{\alpha - 2\gamma} \bar{v}_1, \\
T_2^0 &= -\frac{\omega}{2} e^{\beta - 2\gamma} \bar{v}_2 - \frac{1}{4} (\gamma' - \beta') e^{\beta - \gamma - \alpha} \bar{A}_3^3, \\
T_3^0 &= -\frac{\omega}{2} e^{\mu - 2\gamma} \bar{v}_3 - \frac{1}{4} (\gamma' - \mu') e^{\mu - \gamma - \alpha} \bar{A}_2^2, \\
T_3^0 &= \frac{1}{4} (\beta' - \mu') e^{\mu - \beta - \alpha} \bar{A}_0^0.
\end{align}
From (2.16) one sees, that $T_0^0 \neq T_2^2$ and

$$T_0^0 + T_1^1 = m_{sp}S + 2\lambda (F - KF_k), \quad (2.16a)$$

$$T_0^0 - T_1^1 = 2\omega e^{-\tau}v^0 - m_{sp}S - 2\lambda KF_k. \quad (2.16b)$$

Before dealing with the Einstein system of equations let us remark that the Bianchi identity, i.e., $G_{\mu;\nu}^\nu = 0$ leads to $T_{\mu;\nu}^\nu = 0$. Indeed, on account of (2.16) we find

$$T_{\mu;\nu}^\nu = T_{\mu;\nu}^\nu + \Gamma_{\rho \nu}^\nu T_{\mu}^\rho - \Gamma_{\mu \nu}^\rho T_{\rho}^\nu$$

$$= m_{sp}e^{-\tau} \frac{d}{du} (Se^\tau) + \lambda F_k e^{-2\tau} \frac{d}{du} (Ke^{2\tau}) - \omega e^{-\gamma - \tau} \frac{d}{du} (v^0 e^\tau) = 0. \quad (2.17)$$

As it was mentioned earlier $Se^\tau = S_0$, $Ke^{2\tau} = K_0$ and $v^0 e^\tau = v_0^0$ are constants, hence the conservation law $T_{\mu;\nu}^\nu = 0$ is fulfilled identically.

Since, for the metric (2.7) the Einstein tensor has only diagonal components, so let us first consider the diagonal equations of Einstein system

$$e^{-2\alpha} [\gamma' \beta' + \beta' \mu' + \mu' \gamma'] = m_{sp}S_0 e^{-\tau} + \lambda F - \omega e^{-\gamma - \tau} v_0^0, \quad (2.18a)$$

$$e^{-2\alpha} [\gamma' + \mu'' + \gamma^2 + \mu^2 + \gamma' \beta - \alpha' \gamma - \alpha' \mu] = \lambda (F - 2KF_k), \quad (2.18b)$$

$$e^{-2\alpha} [\gamma' + \beta'' + \gamma^2 + \beta^2 + \gamma' \beta - \alpha' \gamma - \alpha' \beta] = \lambda (F - 2KF_k), \quad (2.18c)$$

$$e^{-2\alpha} [\beta'' + \mu'' + \beta^2 + \mu^2 + \beta' \mu - \alpha' \beta - \alpha' \mu] = \omega e^{-\gamma - \tau} v_0^0 + \lambda (F - 2KF_k). \quad (2.18d)$$

Subtraction of (2.18b) from (2.18a) and (2.18c) from (2.18d) yields, respectively

$$\beta'' - \gamma'' + \beta^2 - \gamma^2 + \mu' (\beta' - \gamma') - \alpha' (\beta' - \gamma') = \omega e^{2\alpha - \gamma - \tau} v_0^0, \quad (2.19a)$$

$$\mu'' - \gamma'' + \mu^2 - \gamma^2 + \beta' (\mu' - \gamma') - \alpha' (\mu' - \gamma') = \omega e^{2\alpha - \gamma - \tau} v_0^0. \quad (2.19b)$$

For the non-diagonal components of Einstein equations we have

$$0 = -\omega e^{\alpha - 2\gamma - \tau} v_0^1, \quad (2.20a)$$

$$0 = -\frac{\omega}{2} e^{\beta - 2\gamma - \tau} v_0^2 - \frac{1}{4} (\gamma' - \beta') e^{\beta - \gamma - \alpha - \tau} A_0^3, \quad (2.20b)$$

$$0 = -\frac{\omega}{2} e^{\mu - 2\gamma - \tau} v_0^2 - \frac{1}{4} (\gamma' - \mu') e^{\mu - \gamma - \alpha - \tau} A_0^2, \quad (2.20c)$$

$$0 = \frac{1}{4} (\beta' - \mu') e^{\mu - \beta - \alpha - \tau} A_0^2. \quad (2.20d)$$

From (2.20b) and (2.20c) we have

$$\beta' - \gamma' = 2\omega c_1 e^{\alpha - \gamma}, \quad (2.21)$$

$$\mu' - \gamma' = 2\omega c_2 e^{\alpha - \gamma}, \quad (2.22)$$

where we denote $c_1 = v_0^2 / A_0^3$ and $c_2 = v_0^3 / A_0^2$.

In view of (2.21) and (2.22) the eqns. (2.19a) and (2.19b) can be written as

$$\beta'' - \gamma'' - (\beta' - \gamma') \left( \alpha' - \tau' - \frac{v_0^0}{2c_1} e^{\alpha - \tau} \right) = 0, \quad (2.23a)$$

$$\mu'' - \gamma'' - (\mu' - \gamma') \left( \alpha' - \tau' - \frac{v_0^0}{2c_2} e^{\alpha - \tau} \right) = 0, \quad (2.23b)$$
with the solutions

\[ \beta' = \gamma' + N_1e^{\alpha - \tau - (\bar{v}_0/2c_1)} \int e^{(\alpha - \tau)} du, \]  
\[ \mu' = \gamma' + N_2e^{\alpha - \tau - (\bar{v}_0/2c_2)} \int e^{(\alpha - \tau)} du, \]

with \( N_1 \) and \( N_2 \) being the constants.

Then on account of (2.21), we find

\[ \gamma' = \frac{1}{3} \left[ \tau' - e^{\alpha - \tau} \left( N_1e^{-(\bar{v}_0/2c_1)} \int e^{(\alpha - \tau)} du + N_2e^{-(\bar{v}_0/2c_2)} \int e^{(\alpha - \tau)} du \right) \right]. \]  

(2.26)

Note that (2.20a) leads to \( \bar{v}_0^1 = 0 \) while (2.20d) allows two possibilities: (i) \( A_0^0 = 0 \) and/or (ii) \( \beta' - \mu' = 0 \). As one sees from (2.21) and (2.22), the second condition is valid only if \( c_1 = c_2 = N_1 = N_2 \) in the above equations.

Note that to determine the metric functions \( \gamma, \beta \) and \( \mu \) we have to derive the equation for \( \tau \). Summation of (2.18b)-(2.18d) and 3 times (2.18a) gives the equation for \( \tau \). Hence finally we obtain the system of equations for defining the metric functions that takes into account the restrictions those occur due to nontrivial non-diagonal components of the Einstein equations.

\[ \tau'' + \tau'^2 - \tau' \alpha' = \frac{k}{2}e^{2\alpha} \left[ 3m_{sp}S_0 e^{-\tau} + 3\lambda (2F - K\dot{F}) - 2\omega \bar{v}_0 e^{-\gamma - \tau} \right], \]  

(2.27a)

\[ \gamma' = \frac{1}{3} \left[ \tau' - e^{\alpha - \tau} \left( N_1e^{-\int\bar{v}_0 u e^{\alpha - \tau}} + N_2e^{-\int\bar{v}_0 u e^{(\alpha - \tau)}} \right) \right], \]  

(2.27b)

\[ \beta' = \frac{1}{3} \left[ \tau' + e^{\alpha - \tau} \left( 2N_1e^{-\int\bar{v}_0 u e^{\alpha - \tau}} - N_2e^{-\int\bar{v}_0 u e^{(\alpha - \tau)}} \right) \right], \]  

(2.27c)

\[ \mu' = \frac{1}{3} \left[ \tau' + e^{\alpha - \tau} \left( -N_1e^{-\int\bar{v}_0 u e^{\alpha - \tau}} + 2N_2e^{-\int\bar{v}_0 u e^{(\alpha - \tau)}} \right) \right]. \]  

(2.27d)

As one sees, here we don’t have any equation for defining \( \alpha \). So we need some additional conditions. We will solve the equations for determining the metric functions (2.21), (2.22), (2.26) and (2.27a) numerically. The nonlinear term \( F(K) \) we choose in the form \( F(K) = K^n \) that gives \( (2F - K\dot{F}) = (2 - n)K^{n-1} = (2 - n)K_0^n e^{-2n\tau} \).

Further we consider two cases.

Case 1. In what follows we consider the harmonic radial coordinate \( u \). In this case the following condition holds for the metric functions [21]

\[ \alpha = \gamma + \beta + \mu. \]  

(2.28)

Taking into account that in this case \( \tau = \alpha \) for a more general solution to the Einstein equations with massive and nonlinear spinor field as source we rewrite it in the form of Cauchy:

\[ \tau' = \eta, \]  

(2.29a)

\[ \eta' = \frac{k}{2}e^{2\tau} \left[ 3m_{sp}S_0 e^{-\tau} + 3\lambda (2 - n)K_0^n e^{-2n\tau} - 2\omega \bar{v}_0 e^{-\gamma - \tau} \right], \]  

(2.29b)

\[ \gamma' = \frac{1}{3} \left[ \eta - \left( N_1e^{-\int\bar{v}_0 u} + N_2e^{-\int\bar{v}_0 u} \right) \right], \]  

(2.29c)

\[ \beta' = \frac{1}{3} \left[ \eta + \left( 2N_1e^{-\int\bar{v}_0 u} - N_2e^{-\int\bar{v}_0 u} \right) \right], \]  

(2.29d)

\[ \mu' = \frac{1}{3} \left[ \eta + \left( -N_1e^{-\int\bar{v}_0 u} + 2N_2e^{-\int\bar{v}_0 u} \right) \right]. \]  

(2.29e)

Though the choice of harmonic radial coordinate given by (2.28) essentially simplifies the initial system (2.27), nevertheless it is hard to find exact solutions to the system (2.29). That is
why we solve the system (2.29) numerically. Right now our goal is to exhibit some qualitative solutions. Taking it into account we modify the system of equations as simple as possible. In doing so in (2.29) and in (2.30) that will follow later, we set \( S_0 = 1, K_0 = 1, \bar{v}_0 = 1 \). We also set \( c_1^0 = 1, c_2^0 = 2, N_1 = 2, N_2 = 3, m_{sp} = 1, \omega = 1 \) and \( \lambda = 1 \). The values the corresponding functions on the axis of symmetry, i.e. at \( u = 0 \) are set to be trivial: \( \tau_0 = 0, \eta_0 = 0, \gamma_0 = 0, \beta_0 = 0, \mu_0 = 0 \). We consider two different values of \( n \). In case of \( n = 5 \) the nonlinear term takes the form \(-e^{-10\tau}\), while for \( n = -2 \) the nonlinear term becomes \( 2e^{4\tau}\).

As one can see from (2.29c), (2.29d) and (2.29e) depending on the value of \( c_1 \) and \( c_2 \) they might vary relative to \( \tau \), whereas the behavior of \( \tau \) depends on not only the choice of spinor field nonlinearity, but also on the choice of coordinate condition. In Fig. 1 and Fig. 2 we illustrate the behavior of the metric functions for the spinor field nonlinearity being a power law of invariant \( K \).

While the Fig. 1 corresponds to the case with \( n = 5 \), Fig. 2 corresponds to the case with \( n = -2 \).

![FIG. 1. Evolution of the metric functions for n = 5](image1)

![FIG. 2. Evolution of the metric functions for n = -2](image2)

Case 2: As a second choice let us consider the quasiblogal coordinate \( \alpha = -\gamma \) [22]. In this case we have

\[
\tau' = \eta,
\]

\[
\eta' = -\frac{4}{3} \eta^2 + \frac{1}{3} \eta e^{-\gamma - \tau} \left( N_1 e^{-c_1^0 \int e^{-\gamma - \tau} du} + N_2 e^{-c_2^0 \int e^{-\gamma - \tau} du} \right) + \frac{K}{2} e^{-2\gamma} \left[ 3m_{sp} e^{-\tau} + 3\lambda (2 - n)K_0^\eta e^{-2n\tau} - 2\omega \bar{v}_0^e e^{-\gamma - \tau} \right],
\]

\[
\gamma' = \frac{1}{3} \left[ \eta - e^{-\gamma - \tau} \left( N_1 e^{-c_1^0 \int e^{-\gamma - \tau} du} + N_2 e^{-c_2^0 \int e^{-\gamma - \tau} du} \right) \right],
\]

\[
\beta' = \frac{1}{3} \left[ \eta + e^{-\gamma - \tau} \left( 2N_1 e^{-c_1^0 \int e^{-\gamma - \tau} du} - N_2 e^{-c_2^0 \int e^{-\gamma - \tau} du} \right) \right],
\]

\[
\mu' = \frac{1}{3} \left[ \eta + e^{-\gamma - \tau} \left( -N_1 e^{-c_1^0 \int e^{-\gamma - \tau} du} + 2N_2 e^{-c_2^0 \int e^{-\gamma - \tau} du} \right) \right]
\]

As in previous case the system (2.30) is also solved numerically. As was mentioned earlier, like the foregoing case we set \( S_0 = 1, K_0 = 1, \bar{v}_0 = 1 \). We also set \( c_1^0 = 1, c_2^0 = 2, N_1 = 2, N_2 = 3, m_{sp} = 1, \omega = 1 \) and \( \lambda = 1 \). Analagous to the previous case we consider that the values the
corresponding functions on the axis of symmetry, i.e. at $u = 0$ they are trivial: $\tau_0 = 0$, $\eta_0 = 0$, $\gamma_0 = 0$, $\beta_0 = 0$, $\mu_0 = 0$. We consider two different values of $n$. In case of $n = 5$ the nonlinear term takes the form $-e^{-10\tau}$ while for $n = -2$ the nonlinear term becomes $2e^{4\tau}$. In Fig. 3 we draw the metric functions those correspond to the nonlinear term being a power law of invariant $K$ with $n = 5$, while Fig. 4 corresponds to the case with $n = -2$.

![Graph 1](image1)

**FIG. 3.** Evolution of the metric functions for $n = 5$

![Graph 2](image2)

**FIG. 4.** Evolution of the metric functions for $n = -2$

### III. CONCLUSIONS

We studied a system of nonlinear spinor field minimally coupled to a static cylindrically symmetric space-time. We consider the spinor field that depends on time and radial coordinates. The time dependence is taken in such a way that the invariants of the spinor field remain time independent. The energy-momentum tensor (EMT) of the spinor field possesses nontrivial non-diagonal components. While in case of a static spinor field the presence of non-diagonal components of the EMT imposes three-way restrictions on the space-time geometry and/or on the components of the spinor field similar to the Bianchi type-I cosmological model [20], in case of a time dependent spinor field things become more complicated. Moreover, while in a static spherically symmetric space-time the presence of non-trivial non-diagonal components of EMT of the spinor field has no effect on the space-time geometry [23], in static cylindrically symmetric space-time it influences both the space-time geometry and the components of the spinor field. Unlike the static case in the present model we have $T_{0}^{0} \neq T_{2}^{2}$. It should be noted that the expressions $(T_{0}^{0} + T_{1}^{1})$ and $(T_{0}^{0} - T_{1}^{1})$ can be both positive and negative, depending on the type of nonlinearity.

**DAS:** No datasets were generated or analysed during the current study

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