Quantum Schrödinger bridges

Dedicated to Anders Lindquist on the occasion of his 60th birthday

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Abstract

Elaborating on M. Pavon, J. Math. Phys. 40 (1999), 5565-5577, we develop a simplified version of a variational principle within Nelson stochastic mechanics that produces the von Neumann wave packet reduction after a position measurement. This stochastic control problem parallels, with a different kinematics, the problem of the Schrödinger bridge. This gives a profound meaning to what was observed by Schrödinger in 1931 concerning Schrödinger bridges: “Merkwürdige Analogien zur Quantenmechanik, die mir sehr des Hindenkens wert erscheinen”.

1 Introduction: Schrödinger’s problem

In 1931/32 [1, 2], Schrödinger considered the following problem. A cloud of $N$ Brownian particles in $\mathbb{R}^n$ has been observed having at time $t_0$ an empirical distribution approximately equal to $\rho_0(x)dx$. At some later
time $t_1$, an empirical distribution approximately equal to $\rho_1(x)dx$ is observed. Suppose that $\rho_1(x)$ considerably differs from what it should be according to the law of large numbers ($N$ is large), namely

$$\int_{t_0}^{t_1} p(t_0, y, t_1, x) \rho_0(y) dy,$$

where

$$p(s, y, t, x) = [2\pi(t-s)]^{-\frac{1}{2}} \exp \left[ -\frac{|x-y|^2}{2(t-s)} \right], \quad s < t,$$

is the transition density of the Wiener process. It is apparent that the particles have been transported in an unlikely way. But of the many unlikely ways in which this could have happened, which one is the most likely?

In modern probabilistic language, this is a problem of large deviations of the empirical distribution [3]. By discretization and passage to the limit, Schrödinger computed the most likely intermediate empirical distribution as $N \to \infty$. It turned out that the optimal random evolution, the Schrödinger bridge from $\rho_0$ to $\rho_1$ over Brownian motion, had at each time a density $\rho(\cdot, t)$ that factored as $\rho(x, t) = \phi(x, t) \hat{\phi}(x, t)$, where $\phi$ and $\hat{\phi}$ are a $p$-harmonic and a $p$-coharmonic functions, respectively. That is

$$\phi(t, x) = \int p(t, x, t_1, y) \phi(t_1, y) dy,$$

$$\hat{\phi}(t, x) = \int p(t_0, y, t, x) \hat{\phi}(t_0, y) dy.$$

The existence and uniqueness of a pair $(\phi, \hat{\phi})$ satisfying [1]–[2] and the boundary conditions $\phi(x, t_0) \hat{\phi}(x, t_0) = \rho_0(x)$, $\phi(x, t_1) \hat{\phi}(x, t_1) = \rho_1(x)$ was guessed by Schrödinger on the basis of his intuition. He was later shown to be quite right in various degrees of generality by Fortet [4], Beurlin [5], Jamison [6], Föllmer [3]. Jamison showed, in particular, that the Schrödinger bridge is the unique Markov process \{x(t)\} in the class of reciprocal processes (one-dimensional Markov fields) introduced by Bernstein [7] having as two-sided transition density

$$q(s, x; t, y; u, z) = \frac{p(s, x; t, y)p(t, y; u, z)}{p(s, x; u, z)}, \quad s < t < u,$$

2
namely $q(s, x; t, y; u, z)dy$ is the probability of finding the process $x$ in the volume $dy$ at time $t$ given that $x(s) = x$ and $x(u) = z$. Schrödinger was struck by the following remarkable property of the solution: The Schrödinger bridge from $\rho_1$ to $\rho_0$ over Brownian motion is just the time reversal of the Schrödinger bridge from $\rho_0$ to $\rho_1$. In Schrödinger’s words: “Abnormal states have arisen with high probability by an exact time reversal of a proper diffusion process”. This led him to entitle [1]: “On the reversal of natural laws” A few years later, Kolmogorov wrote a paper on the subject with a very similar title [8]. Moreover, the fact that the Schrödinger bridge has density $\rho(x, t) = \phi(x, t) \hat{\phi}(x, t)$ resembles the fact that in quantum mechanics the density may be expressed as $\rho(x, t) = \psi(x, t) \bar{\psi}(x, t)$. This analogy has inspired various attempts to construct a stochastic reformulation of quantum mechanics [9]-[12] starting from [1, 2, 7]. In order to discuss a more general Schrödinger bridge problem, we recall in the next session some essential facts on the kinematics of finite-energy diffusions as presented in [13, 14, 15, 16].

2 Elements of Nelson-Föllmer kinematics of finite energy diffusions

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. A stochastic process $\{\xi(t); t_0 \leq t \leq t_1\}$ mapping $[t_0, t_1]$ into $L^2_n(\Omega, \mathcal{F}, P)$ is called a finite-energy diffusion with constant diffusion coefficient $I_n \sigma^2$ if the path $\xi(\omega)$ belongs a.s. to $C([t_0, t_1]; \mathbb{R}^n)$ (n-dimensional continuous functions) and

$$\xi(t) - \xi(s) = \int_s^t \beta(\tau)d\tau + \sigma w_+(s, t), \quad t_0 \leq s < t \leq t_1,$$

where the forward drift $\beta(t)$ is at each time $t$ a measurable function of the past $\{\xi(\tau); 0 \leq \tau \leq t\}$, and $w_+(\cdot, \cdot)$ is a standard, n-dimensional Wiener difference process with the property that $w_+(s, t)$ is independent of $\{\xi(\tau); 0 \leq \tau \leq s\}$. Moreover, $\beta$ must satisfy the finite-energy condition

$$E \left\{ \int_{t_0}^{t_1} \beta(\tau) \cdot \beta(\tau)d\tau \right\} < \infty.$$  

We recall the characterizing properties of the n-dimensional Wiener difference process $w_+(s, t)$, see [13] Chapter 11 and [15] Section 1. It is a process such that $w_+(t, s) = -w_+(s, t)$, $w_+(s, u) + w_+(u, t) = w_+(s, t)$,
and that \( w_+(s,t) \) is Gaussian distributed with mean zero and variance \( I_n|s-t| \). Moreover, (the components of) \( w_+(s,t) \) and \( w_+(u,v) \) are independent whenever \([s,t]\) and \([u,v]\) don’t overlap. Of course, \( w_+(t) := w_+(t_0,t) \) is a standard Wiener process such that \( w_+(s,t) = w_+(t) - w_+(s) \).

In [14], Föllmer has shown that a finite-energy diffusion also admits a reverse-time differential. Namely, there exists a measurable function \( \gamma(t) \) of the future \( \{\xi(\tau); t \leq \tau \leq t_1\} \) called backward drift, and another Wiener difference process \( w_- \) such that
\[
\xi(t) - \xi(s) = \int_s^t \gamma(\tau)d\tau + \sigma w_-(s,t), \quad t_0 \leq s < t \leq t_1.
\] (5)

Moreover, \( \gamma \) satisfies
\[
E\left\{\int_{t_0}^{t_1} \gamma(\tau) \cdot \gamma(\tau)d\tau\right\} < \infty,
\] (6)
and \( w_-(s,t) \) is independent of \( \{\xi(\tau); t \leq \tau \leq t_1\} \). Let us agree that \( dt \) always indicates a strictly positive variable. For any function \( f \) defined on \([t_0, t_1]\), let
\[
d_+ f(t) = f(t + dt) - f(t)
\]
be the forward increment at time \( t \), and
\[
d_- f(t) = f(t) - f(t - dt)
\]
be the backward increment at time \( t \). For a finite-energy diffusion, Föllmer has also shown in [14] that the forward and backward drifts may be obtained as Nelson’s conditional derivatives, namely
\[
\beta(t) = \lim_{dt \downarrow 0} E\left\{\frac{d_+ \xi(t)}{dt}|\xi(\tau), t_0 \leq \tau \leq t\right\},
\]
and
\[
\gamma(t) = \lim_{dt \downarrow 0} E\left\{\frac{d_- \xi(t)}{dt}|\xi(\tau), t \leq \tau \leq t_1\right\},
\]
the limits being taken in \( L^2_\mathcal{F}(\Omega, \mathcal{F}, P) \). It was finally shown in [14] that the one-time probability density \( \rho(\cdot,t) \) of \( \xi(t) \) (which exists for every \( t > t_0 \)) is absolutely continuous on \( \mathbb{R}^n \) and the following duality relation holds \( \forall t > 0 \)
\[
E\{\beta(t) - \gamma(t)|\xi(t)\} = \sigma^2 \nabla \log \rho(\xi(t),t), \quad \text{a.s.} \quad (7)
\]
Remark 1 It should be observed that in the study of reverse-time differentials of diffusion processes, initiated by Nelson in [17] and Nagasawa in [18], see [19, 20] and references therein, important results have obtained by A. Linquist and G. Picci in the Gaussian case [21, 22] without assumptions on the reverse-time differential. In particular, their results on Gauss-Markov processes have been crucial in order to develop a strong form of stochastic realization theory [21–24] together with a variety of applications [24–30].

Corresponding to (3) and (5) are two change of variables formulas. Let $f : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$ be twice continuously differentiable with respect to the spatial variable and once with respect to time. Then, if $\xi$ is a finite-energy diffusion satisfying (3) and (5), we have

$$f(\xi(t), t) - f(\xi(s), s) = \int_s^t \left( \frac{\partial}{\partial \tau} + \beta(\tau) \cdot \nabla + \frac{\sigma^2}{2} \Delta \right) f(\xi(\tau), \tau) d\tau$$

$$+ \int_s^t \sigma \nabla f(\xi(\tau), \tau) \cdot d_+ w_+(\tau), \quad (8)$$

$$f(\xi(t), t) - f(\xi(s), s) = \int_s^t \left( \frac{\partial}{\partial \tau} + \gamma(\tau) \cdot \nabla - \frac{\sigma^2}{2} \Delta \right) f(\xi(\tau), \tau) d\tau$$

$$+ \int_s^t \sigma \nabla f(\xi(\tau), \tau) \cdot d_- w_-(\tau). \quad (9)$$

The stochastic integrals appearing in (8) and (9) are a (forward) Itô integral and a backward Itô integral, respectively, see [15] for the details.

3 Schrödinger bridges

The solution to the Schrödinger problem can be obtained by solving a stochastic control problem. The Kullback-Leibler pseudo-distance between two probability densities $p(\cdot)$ and $q(\cdot)$ is defined by

$$H(p, q) := \int_{\mathbb{R}^n} \log \frac{p(x)}{q(x)} p(x) dx.$$

This concept can be considerably generalized. Let $\Omega := C([t_0, t_1], \mathbb{R}^n)$ denote the family of $n$-dimensional continuous functions, let $W_x$ denote Wiener measure on $\Omega$ starting at $x$, and let

$$W := \int W_x dx$$
be stationary Wiener measure. Let \( \mathbb{D} \) be the family of distributions on \( \Omega \) that are equivalent to \( W \). For \( Q, P \in \mathbb{D} \), we define the relative entropy \( H(Q, P) \) of \( Q \) with respect to \( P \) as

\[
H(Q, P) = E_Q[\log \frac{dQ}{dP}].
\]

It then follows from Girsanov’s theorem that \([16, 14, 3]\)

\[
H(Q, P) = H(q(t_0), p(t_0)) + E_Q \left[ \int_{t_0}^{t_1} \frac{1}{2} [\beta^Q(t) - \beta^P(t)] \cdot [\beta^Q(t) - \beta^P(t)] dt \right]
\]

\[
= H(q(t_1), p(t_1)) + E_Q \left[ \int_{t_0}^{t_1} \frac{1}{2} [\gamma^Q(t) - \gamma^P(t)] \cdot [\gamma^Q(t) - \gamma^P(t)] dt \right] .
\]

(10)

Here \( q(t_0) \) is the marginal density of \( Q \) at \( t_0 \), \( \beta^Q \) and \( \gamma^Q \) are the forward and the backward drifts of \( Q \), respectively. Now let \( \rho_0 \) and \( \rho_1 \) be two everywhere positive probability densities. Let \( \mathbb{D}(\rho_0, \rho_1) \) denote the set of distributions in \( \mathbb{D} \) having the prescribed marginal densities at \( t_0 \) and \( t_1 \).

Given \( P \in \mathbb{D} \), we consider the following problem:

\[
\text{Minimize } H(Q, P) \text{ over } \mathbb{D}(\rho_0, \rho_1).
\]

In view of (10), this is a stochastic control problem. It is connected through Sanov’s theorem \([3, 32]\) to a problem of large deviations of the empirical distribution, according to Schrödinger original motivation. Namely, if \( X^1, X^2, \ldots \) is an i.i.d. sequence of random elements on \( \Omega \) with distribution \( P \), then the sequence \( P^n \left[ \frac{1}{n} \sum_{i=1}^{n} \delta_{X^i} \right] \) satisfies a large deviation principle with rate function \( H(\cdot, P) \).

If there is at least one \( Q \) in \( \mathbb{D}(\rho_0, \rho_1) \) such that \( H(Q, P) < \infty \), it may be shown that there exists a unique minimizer \( Q^* \) in \( \mathbb{D}(\rho_0, \rho_1) \) called the Schrödinger bridge from \( \rho_0 \) to \( \rho_1 \) over \( P \). If (the coordinate process under) \( P \) is Markovian with forward drift field \( b^P_+ (x, t) \) and transition density \( p(\sigma, x, \tau, y) \), then \( Q^* \) is also Markovian with forward drift field

\[
b^Q_+ (x, t) = b^P_+ (x, t) + \nabla \log \phi(x, t),
\]

where the (everywhere positive) function \( \phi \) solves together with another function \( \hat{\phi} \) the system \([11]-[2]\) with boundary conditions

\[
\phi(x, t_0)\hat{\phi}(x, t_0) = \rho_0(x), \quad \phi(x, t_1)\hat{\phi}(x, t_1) = \rho_1(x).
\]
Moreover, $\rho(x, t) = \phi(x, t) \hat{\phi}(x, t), \forall t \in [t_0, t_1]$. This result has been suitably extended to the case where $P$ is non-Markovian in [31]. For a survey of the theory of Schrödinger bridges with an extended bibliography see [32].

Consider now the following simpler problem: We have a reference stochastic model $P \in \mathbb{D}$. We think of $P$ as modeling the macroscopic evolution of a thermodynamic system. Suppose we observe at time $t_1$ the (everywhere positive) density $\rho_1$ different from the marginal density of $P$. Thus we need to solve the following optimization problem

$$\text{Minimize } H(Q, P) \text{ over } Q \in \mathbb{D}(\rho_1).$$

where $\mathbb{D}(\rho_1)$ denotes the set of distributions in $\mathbb{D}$ having density $\rho_1$ at $t_1$. Let us assume that $H(\rho_1, p(t_1)) < \infty$. In view of (10), this stochastic control problem can be trivially solved. The unique solution is given by the distribution $Q^*$ having backward drift $\gamma^P(t)$ and marginal density $\rho_1$ at time $t_1$. Thus, the result of measurement at time $t_1$ leads to the replacement of the stochastic model $P$ with $Q^*$. Notice that the backward drift $\gamma^P(t)$ is perfectly preserved by this procedure. Symmetrically, if we were to change the initial distribution at time $t_0$, the procedure would preserve the forward drift $\beta^P(t)$.

4 Elements of Nelson’s stochastic mechanics

Nelson’s stochastic mechanics is a quantization procedure for classical dynamical systems based on diffusion processes. Following some early work by Feynman [33] and others, Nelson and Guerra elaborated a clean formulation starting from 1966 [31, 13, 35], showing that the Schrödinger equation could be derived from a continuity type equation plus a Newton type law, provided one accepted a certain definition for the stochastic acceleration. In analogy to classical mechanics, the Newton-Nelson law was later shown to follow from a Hamilton-like stochastic variational principle [36, 37]. Other versions of the variational principle have been proposed in [38, 39, 40, 41].

Consider the case of a nonrelativistic particle of mass $m$. Let $\{\psi(x, t); t_0 \leq$
\( t \leq t_1 \) be the solution of the Schrödinger equation

\[
\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m} \Delta \psi - \frac{i}{\hbar} V(x)\psi,
\]

such that

\[
||\nabla \psi||_2^2 \in L^1_{\text{loc}}[t_0, +\infty).
\]

This is Carlen’s finite action condition. Under these hypotheses, the Nelson measure \( P \in \mathcal{D} \) may be constructed on path space, \[12\], \[13\], \[39, Chapter IV\], and references therein. Namely, letting \( \Omega := C([t_0, t_1], \mathbb{R}^n) \) the \( n \)-dimensional continuous functions on \([t_0, t_1]\), under the probability measure \( P \), the canonical coordinate process \( x(t, \omega) = \omega(t) \) is an \( n \)-dimensional Markov diffusion process \( \{x(t); t_0 \leq t \leq t_1\} \), called Nelson’s process, having (forward) Ito differential

\[
dx(t) = \left[ \frac{\hbar}{m} \nabla \left( \Re \log \psi(x(t), t) + \Im \log \psi(x(t), t) \right) \right] dt + \sqrt{\frac{\hbar}{m}} dw(t), \tag{13}
\]

where \( w \) is a standard, \( n \)-dimensional Wiener process. Moreover, the probability density \( \rho(\cdot, t) \) of \( x(t) \) satisfies

\[
\rho(x, t) = |\psi(x, t)|^2, \quad \forall t \in [t_0, t_1]. \tag{14}
\]

Following Nelson \[13\], \[38\], for a finite-energy diffusion with stochastic differentials \[3\]-\[5\], we define the current and osmotic drifts, respectively:

\[
v(t) = \frac{\beta(t) + \gamma(t)}{2}, \quad u(t) = \frac{\beta(t) - \gamma(t)}{2}.
\]

Clearly \( v \) is similar to the classical velocity, whereas \( u \) is the velocity due to the “noise” which tends to zero when the diffusion coefficient \( \sigma^2 \) tends to zero. In order to obtain a unique time-reversal invariant differential \[40\], we take a complex linear combination of \( (3)-(5) \), obtaining

\[
x(t) - x(s) = \int_s^t \left[ \frac{1 - i}{2} \beta(\tau) + \frac{1 + i}{2} \gamma(\tau) \right] d\tau \\
+ \frac{\sigma}{2} \left[ (1 - i)(w_+(t) - w_+(s)) + (1 + i)(w_-(t) - w_-(s)) \right].
\]

Let us define the quantum drift

\[
v_q(t) := \frac{1 - i}{2} \beta(t) + \frac{1 + i}{2} \gamma(t) = v(t) - iu(t),
\]
and the quantum noise
\[ w_q(t) := \frac{1-i}{2} w_+(t) + \frac{1+i}{2} w_-(t). \]

Hence,
\[ x(t) - x(s) = \int_s^t v_q(\tau) d\tau + \sigma[w_q(t) - w_q(s)]. \tag{15} \]

This representation enjoys the time reversal invariance property. It has been crucial in order to develop a Lagrangian and a Hamiltonian dynamics formalism in the context of Nelson’s stochastic mechanics in \[40\, 44\, 45\]. Notice that replacing (3)-(5) with (15), we replace the pair of real drifts \((v, u)\) by the unique complex-valued drift \(v - iu\) that tends correctly to \(v\) when the diffusion coefficient tends to zero.

5 Quantum Schrödinger bridges

We now consider the same problem as at the end of Section 3. We have a reference stochastic model \(P \in \mathbb{D}\) given by the Nelson measure on path space that has been constructed through a variational principle \[37\, 38\, 40\]. This Nelson process \(x = \{x(t); t_0 \leq t \leq t_1\}\) has an associated solution \(\{\psi(x, t); t_0 \leq t \leq t_1\}\) of the Schrödinger equation in the sense that the quantum drift of \(x\) is \(v_q(t) = \hbar^{-1} m \nabla \log \psi(x(t), t)\) and the one-time density of \(x\) satisfies \(\rho(x, t) = |\psi(x, t)|^2\). Suppose a position measurement at time \(t_1\) yields the probability density \(\rho_1(x) \neq |\psi(x, t_1)|^2\). We need a suitable variational mechanism that, starting from \((P, \rho_1)\), produces the new stochastic model in \(\mathbb{D}(\rho_1)\). It is apparent that the variational problem of Section 3 is not suitable as it preserves the backward drift. Since in stochastic mechanics both differentials must be granted the same status, we need to change both drifts as little as possible given the new density \(\rho_1\) at time \(t_1\). Thus, we employ the differential (15), and consider the variational problem:

Extremize on \((\tilde{x}, \tilde{v}_q) \in (\mathbb{D}(\rho_1) \times \mathcal{V})\) the functional

\[ J(\tilde{x}, \tilde{v}_q) := E \left\{ \frac{1}{2} \log \frac{\tilde{\rho}_1(\tilde{x}(t_1))}{\rho(\tilde{x}(t_1), t_1)} + \int_{t_1}^{t_2} \frac{1}{2\hbar} (v_q(\tilde{x}(t), t) - \tilde{v}_q(t)) \cdot (v_q(\tilde{x}(t), t) - \tilde{v}_q(t)) dt \right\}. \]
subject to: $\tilde{x}$ has quantum drift (velocity) $\tilde{v}_q$.
Here $v_q(x, t) = \frac{\hbar}{im} \nabla \log \psi(x, t)$ is quantum drift of Nelson reference process, and $\mathbb{D}(\rho_1)$ is family of finite-energy, $\mathbb{R}^n$-valued diffusions on $[t_0, t_1]$ with diffusion coefficient $\frac{\hbar}{m}$, and having marginal $\rho_1$ at time $t_1$. Moreover, $\mathcal{V}$ denotes the family of finite-energy, $C^n$ - valued stochastic processes on $[t_0, t_1]$. Following the same variational analysis as in [15], we get a Hamilton-Jacobi-Bellman type equation

$$\frac{\partial \varphi}{\partial t} + v_q(x, t) \cdot \nabla \varphi(x, t) - \frac{i\hbar}{2m} \Delta \varphi(x, t) = \frac{i\hbar}{2m} \nabla \varphi(x, t) \cdot \nabla \varphi(x, t), \quad (16)$$

with terminal condition $\varphi(x, t_1) = \frac{1}{2} \log \frac{\rho_1(x)}{\rho(x, t_1)}$. Then $\tilde{x} \in \mathbb{D}(\rho_1)$ with quantum drift

$$v_q(\tilde{x}(t), t) + \frac{\hbar}{mi} \nabla \varphi(\tilde{x}(t), t)$$

solves the extremization problem. Write $\psi(x, t_1) = \rho(x, t_1)^{\frac{1}{2}} \exp[\frac{\rho_1(x)}{\rho(x, t_1)} S(x, t_1)]$, and let $\{\tilde{\psi}(x, t)\}$ be solution of Schrödinger equation [11] on $[t_0, t_1]$ with terminal condition

$$\tilde{\psi}(x, t_1) = \rho_1(x)^{\frac{1}{2}} \exp[\frac{\rho_1(x)}{\rho(x, t_1)} S(x, t_1)].$$

Next, notice that for $t \in [t_0, t_1]$

$$\left[ \frac{\partial}{\partial t} + v_q(x, t) \cdot \nabla - \frac{i\hbar}{2m} \Delta \right] \left( \frac{\tilde{\psi}}{\psi} \right) = 0, \quad \frac{\tilde{\psi}}{\psi}(x, t_1) = \left( \frac{\rho_1(x)}{\rho(x, t_1)} \right)^{\frac{1}{2}},$$

where $v_q(x, t) = \frac{\hbar}{im} \nabla \log \psi(x, t)$. It follows that $\varphi(x, t) := \log \frac{\tilde{\psi}}{\psi}(x, t)$ solves [16], and the corresponding quantum drift is

$$v_q(\tilde{x}(t), t) + \frac{\hbar}{mi} \nabla \varphi(\tilde{x}(t), t) = \frac{\hbar}{mi} \nabla \log \tilde{\psi}(\tilde{x}(t), t).$$

Thus, new process after measurement at time $t_1$ (quantum Schrödinger bridge) is just the Nelson process associated to another solution $\tilde{\psi}$ of the same Schrödinger equation. Invariance of phase at $t_1$ follows from the variational principle.
6 Collapse of the wavefunction

Consider the case where measurement at time $t_1$ only gives the information that $x$ lies in subset $D$ of configuration space $\mathbb{R}^n$ of the system. The density $\rho_1(x)$ just after measurement is

$$\rho_1(x) = \frac{\chi_D(x)\rho(x,t_1)}{\int_D \rho(x',t_1)dx'},$$

where $\rho(x,t_1)$ is density of Nelson reference process $x$ at time $t_1$. Let $A$ be subspace of $L^2(\mathbb{R}^n)$ of functions with support in $D$. Then $A^\perp$ is subspace of $L^2(\mathbb{R}^n)$ functions with support in $D^c$. Decompose $\psi(x,t_1)$ as

$$\psi(x,t_1) = \chi_D(x)\psi(x,t_1) + \chi_D^c(x)\psi(x,t_1) = \psi_1(x) + \psi_2(x),$$

with $\psi_1 \in A$ and $\psi_2 \in A^\perp$. The probability $p_1$ of finding particle in $D$ is

$$p_1 = \int_D |\psi(x,t_1)|^2dx = \int_{\mathbb{R}^n} |\psi_1(x)|^2dx.$$ 

If the result of the measurement at time $t_1$ is that the particle lies in $D$, the variational principle replaces $\{x(t)\}$ with $\{\tilde{x}(t)\}$ and, consequently, replaces $\psi(x,t_1) = \psi_1(x) + \psi_2(x)$ with $\tilde{\psi}(x,t_1)$ where

$$\tilde{\psi}(x,t_1) = \rho(x)^\frac{1}{2} \exp\left[\frac{i}{\hbar}S(x,t_1)\right] = \frac{\psi_1(x)}{||\psi_1||_2}.$$ 

Postulating the variational principle of the previous section (rather than the invariance of the phase at $t_1$), we have therefore obtained the so-called “collapse of the wavefunction”, see e.g. [46] and references therein. The collapse is instantaneous, precisely as in the orthodox theory. It occurs “when the result of the measurement enters the consciousness of the observer” [47]. We mention here that, outside of stochastic mechanics, there exist alternative stochastic descriptions of (non instantaneous) quantum state reduction such as those starting from a stochastic Schrödinger equation, see e.g. [48] and references therein.

7 Conclusion and outlook

We shall show elsewhere [49] that the variational principle of section 5 may be replaced by two stochastic differential games with real velocities.
with an appealing classical interpretation. We shall also show that, using Nelson’s observation in [38, 15] and this variational principle, it is possible to obtain a completely satisfactory classical probabilistic description of the two-slit experiment.

If the variational mechanism described here can be extended to the case where both the initial and final quantum states are varied, it would provide a general approach to the steering problem for quantum systems (extending [50]) that has important applications in quantum computation [51], control of molecular dynamics [52] and many other fields.

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