A Topological Spin Chern Pump

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(Dated: May 7, 2014)

We propose a one-dimensional electron model with parameters modulated adiabatically in closed cycles, which can continuously pump spin to leads. By defining the spin-polarized Wannier functions, we show that the spin pump is protected by the spin Chern numbers, so that it is stable to perturbations violating the time-reversal symmetry and spin conservation. Our work demonstrates the possibility and principle to realize topological spin pumps independent of any symmetries, and also suggests a possible way to experimentally observe the bulk topological invariants.

PACS numbers: 72.25.-b, 73.43.-f, 73.23.-b, 75.76.+j

The quantum Hall (QH) effect discovered in 1980 [1] is the first example of topological state in the field of condensed matter physics. Since then, there has been continuously strong interest in topological phenomena of condensed matter systems. Laughlin [2] interpreted the integer QH effect as a quantum charge pump. Increasing the magnetic flux by a single flux quantum that threads a looped QH ribbon constitutes a cycle of the pump due to gauge invariance, transferring an integer-quantized amount of charge from one edge of the ribbon to the other. Thouless, Kohmoto, Nightingale, and Nijs [3] showed that the QH state can be classified by a topological invariant, the Chern number. Thouless and Niu [4, 5] also established a general relation between the Chern number and the charge pumped during a period of slow variation of potential in the Schrödinger equation.

Recently, an important discovery was the topological insulator, a new quantum state of matter existing in nature. Different from the QH systems, the topological insulators preserve the time-reversal (TR) symmetry. Two-dimensional topological insulators, also called the quantum spin Hall (QSH) systems, have a bulk band gap and a pair of gapless helical edge states traversing the bulk gap. When electron spin is conserved, the topological properties of the QSH systems can be easily understood, as a QSH system can be viewed as two independent QH systems without Landau levels. [10] When the spin conservation is destroyed, unconventional topological invariants are needed to classify the QSH systems. The $Z_2$ index [11] and the spin Chern numbers [12, 13] have been proposed to describe the QSH systems. While the two different invariants are found to be equivalent to each other for TR-invariant systems, [13, 14] they lead to controversial predictions when the TR symmetry is broken. The definition of the $Z_2$ index explicitly relies on the presence of TR symmetry, suggesting that the QSH state turns into a trivial insulator once the TR symmetry is broken. However, calculations [13] based upon the spin Chern numbers showed that the nontrivial topological properties of the QSH systems remain intact when the TR symmetry is broken, as long as the band gap and spin spectrum gap stay open. The nonzero spin Chern numbers guarantee that the edge states must appear on the sample boundary, [16] which could be either gaped or gapless, depending on symmetries or spatial distributions of the edge states. [17] This prediction was supported by the recent experimental observation of the QSH effect in InAs/GaSb bilayers under broken TR symmetry. [18]

Spin pumps promise broad applications in spintronics, e.g., the resulting spin battery is the spintronic analog of the charge battery in conventional electronics. Topological spin pumps [19-21] are expected to have an advantage over other approaches, [22-27] being insusceptible to environmental perturbations. When spin $s_z$ is conserved, the idea of the Thouless charge pump was extended to construct quantized adiabatic spin pumps. [19, 20] However, Fu and Kane argued [21] that, unlike the charge, the spin does not obey a fundamental conservation law, and they introduced a more general concept of the $Z_2$ pump. In the $Z_2$ pump, while the amount of spin pumped per cycle is not integer-quantized in the absence of spin conservation, the pumping process is protected by a $Z_2$ topological invariant, provided that the TR symmetry is present. So far, in existing proposals, either spin conservation or TR symmetry is necessary for constructing topological spin pumps, which greatly restricts their practical applications. Robust spin pumps protected by topology alone, independent of any symmetries, are still awaited.

In this Letter, we predict another intriguing effect resulting from the spin Chern numbers, namely, topological spin pumping. A one-dimensional electron model with parameters modulated adiabatically in closed cycles is proposed, which can continuously pump spin into leads. By defining the spin-polarized Wannier functions (SPWFs), we reveal that the spin pumping effect is a direct manifestation of the nontrivial topological properties of the electron wavefunctions, characterized by nonzero spin Chern numbers. In contrast to the $Z_2$ pump, this spin Chern pump remains to be robust in the presence of magnetic impurities, which destroy both the TR symmetry and spin conservation. Our work demonstrates the possibility and principle to implement robust topologi-
high spin pumps independent of any symmetries, and also suggests a possible way to observe the bulk topological invariants experimentally.

We consider a one-dimensional electron model with Hamiltonian \[ H_P = \sum_{\langle i,j \rangle} t_{i,j} c_i^\dagger c_j + g(t) \sum_i (-1)^i c_i^\dagger s_z c_i, \] (1)

where \( c_i^\dagger = (c_{i+1}^\dagger, c_i^\dagger) \) are the creation operators in the spinor representation for electrons with up and down spins on site \( i \), \( t_{i,j} \) is the periodically varying hopping integral between the nearest neighboring sites, given by \( t_{i,i+1} = t_{i+1,i} = t_0 + (-) t_1 \cos \omega_0 t \) for \( i \) on the \( A \) (\( B \)) sublattices, \( g(t) = g_0 \sin \omega_0 t \) is the Zeeman splitting energy, and \( s_z \) is the Pauli matrix acting on the electron spin. A possible experimental realization of this model is illustrated in Fig. 1. It is easy to see that Eq. (1) preserves the TR symmetry, i.e., \( H_P(-t) = \Theta H_P(t) \Theta \) with \( \Theta \) as the ordinary TR operator. For an infinitely long chain of atoms, the eigenenergies of Eq. (1) can be obtained by the Fourier transform, yielding \( E(k_x) = \pm \sqrt{g^2(t) + 4t_0^2 \cos^2 \frac{k_x a_0}{2} + \alpha^2(t) \sin^2 \frac{k_x a_0}{2}}, \) (2)

with \( \alpha(t) = 2 t_1 \cos \omega_0 t \) and \( a_0 \) the lattice constant. Given \( t_0 \gg t_1, g_0 > 0 \), the system has a middle band gap between \( \pm \sqrt{\alpha^2(t) + g^2(t)} \), which is finite at any time. In the adiabatic limit, on the torus of \( k_x \) and \( t \), one can define the spin Chern numbers \( C_\pm \) in a standard way. \[1^2, 13\] and obtain \( C_\pm = \pm 1 \).

In what follows we want to set up a relation of the nontrivial topological properties of the system to the spectral flow of the centers of mass of the SPWFs. In order to show the robustness of nontrivial topological properties, we introduce magnetic impurities with randomly oriented classical spins into the system. The Hamiltonian is given by \( H_I = V_0 \sum_{\alpha} c_i^\dagger \mathbf{s} \cdot \mathbf{m}_\alpha c_i \), where \( \alpha \) runs over all the impurity sites, and \( \mathbf{m}_\alpha \) is a unit vector in the direction of the \( \alpha \)-th impurity spin. Apparently, the presence of the magnetic disorder destroys both the spin conservation and TR symmetry of the system.

We diagonalize the total Hamiltonian \( H_P + H_I \) numerically, and the eigenenergies and eigenstates are denoted as \( E_n \) and \( | \varphi_n \rangle \). By using the same procedure as calculating the spin Chern numbers, the occupied valence bands can be partitioned into two spin sectors by diagonalizing the projected spin operator \( P_{\pm} \) with \( P = \sum E_n < E_P | \varphi_n \rangle \langle \varphi_n | \) as the projection operator to the occupied space. If the spin is conserved, the eigenvalues of \( P_{\pm} \) have only two values: 1 or -1. When the spin conservation is broken weakly, there is still a finite gap in the eigen-spectrum of \( P_{\pm} \), which naturally divides the spectrum into two sectors: spin-up and spin-down sectors. The eigenstates of \( P_{\pm} \) for the two spin sectors are denoted by \( | \psi_{m,\pm} \rangle \). By definition, \( | \psi_{m,\pm} \rangle \) are essentially the maximally spin-polarized states. Then we can construct the Wannier functions \[29, 30\] \( | \phi_{m,\pm} \rangle \) for the spin-up and spin-down sectors, respectively, which are called the SPWFs.

The evolution of the centers of mass \( \langle z \rangle \) of the SPWFs with time is plotted in Fig. 2 for two different disorder strengths. It is found that for relatively weak magnetic disorder, all the centers of \( | \chi_{m,\pm} \rangle \) move rightwards, each center on average shifting a lattice constant per cycle, and those of \( | \chi_{m,-} \rangle \) move in the opposite direction, as shown in Fig. 2(a). According to the general theory, \[30\] the total displacement of the centers of \( | \chi_{m,+} \rangle \) \( (\langle \chi_{m,-} \rangle) \)}
per cycle divided by the length of the system is equal to the
spin Chern number \( C_+ = 1 \) (\( C_- = -1 \)) for the spin-
up (spin-down) sector. For strong magnetic disorder, the
ordered movement of the Wannier centers is interrupted,
though rearrangement of some centers still happens lo-

cally, as shown in Fig. 2(b), indicating that the system
becomes topologically trivial (\( C_- = 0 \)).

It is worth pointing out that the space spanned by the
SPWFs \( |x_{m,\pm}\rangle \) is identical to that spanned by \( |\varphi_n\rangle \)
for \( E_n < E_F \), namely, \( \sum_{m} |x_{m,\pm}\rangle \langle x_{m,\pm}| = P \). The SPWFs are just another equivalent representa-
tion of the occupied space. Therefore, the \( N \) electrons occu-
pying the energy eigenstates \( |\varphi_n\rangle \) for \( E_n < E_F \) may also
be equivalently considered as two groups: \( N/2 \) electrons occu-
pying \( |x_{m,\pm}\rangle \) and \( N/2 \) electrons occupying \( |x_{m,-}\rangle \).
The counter flows of the centers of the SPWFs observed
in Fig. 2(a), as a consequence of the nonzero spin Chern
numbers, represent the true movements of the electrons
in the spin-up and spin-down sectors with time. Without
nontrivial spectral flows become visible only if the occupied space
is properly partitioned, as has been done above. It is
expected that if leads are strongly connected to the two
ends of the atomic chain, the opposite movements of the
electrons in the spin-up and spin-down sectors will extend
into the leads, transferring spin to the leads continuously.
The system becomes a topological spin pump.

We now consider a sufficiently long pump for \( x < 0 \)
and a lead for \( x > 0 \), which are in good contact with each other. In order to obtain a transparent analytical
expression for the pumped spin, we expand the Hamilto-
nian Eq. (1) for the pump around \( k = \pi h/a \), where the
band gap is minimal, yielding

\[
H_P = \alpha(t)\sigma_x + v_F p_x \sigma_y + g(t) s_z \sigma_z ,
\]

where \( v_F = t_0 a / \hbar \), \( p_x = k_x - \pi h / a_0 \), \( \sigma_{x(y,z)} \) are the
Pauli matrices associated with the \( AB \) sublattices. The Hamiltonian for the lead is taken to be

\[
H_L = v_F E_x \sigma_y .
\]

The Fermi level is set to be \( E_F = 0 \), inside the bulk band
gap of the pump. It is assumed that within the decay
length of electron wavefunctions into the pump, there
exists only one magnetic impurity at \( x = 0 \) with the
potential taken as \( H_I = V(x)s_z \). Here, \( V(x) \) is modeled
as a square potential centered at \( x = 0 \) with height \( V_0 \n
and width \( d \). If \( d \) is much smaller than the decay length
of wavefunctions, by taking the \( d \rightarrow 0 \) limit and keeping
\( U_0 = V_0 d \) finite, it can be shown that the scattering effect
of the impurity potential is equivalent to imposing a uni-
tary boundary condition for the electron wavefunctions

\[
\Psi(x = 0^+) = S \Psi(x = 0^-) ,
\]

where \( S = e^{-i\phi \sigma_y} \) with \( \phi = U_0 / \hbar v_F \). Without the
impurity (\( \phi = 0 \)), Eq. (5) will reduce to the ordinary
continuity condition \( \Psi(x = 0^+) = \Psi(x = 0^-) \).

Calculation of the spin pumped into the lead per cycle
amounts to solving the scattering problem of an electron
incident at the Fermi level from the lead. We first
consider the case, where the spin of the incident electron
is parallel to the \( z \) axis. On the bases \( |\uparrow, 1\rangle , |\uparrow, -1\rangle,
| \downarrow, 1\rangle , | \downarrow, -1\rangle \) with the kets as the eigenstates of \( s_z \) and
\( \sigma_z \), the wavefunction in the lead is given by

\[
\Psi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix} ,
\]

for \( x > 0 \), and that in the pump is given by

\[
\Psi(x) = C_1 \begin{pmatrix} \sin \frac{x}{2} \\ 0 \\ \cos \frac{x}{2} \end{pmatrix} e^{\gamma x} + C_2 \begin{pmatrix} 0 \\ 0 \\ -\sin \frac{x}{2} \end{pmatrix} e^{\gamma x} ,
\]

for \( x < 0 \). Here, \( \varphi = \text{Arg}\{a(t) + ig(t)\} \), and \( \gamma = \sqrt{\alpha^2(t) + g^2(t) / 2 \hbar v_F} \). By substituting Eqs. (6) and (7)
into Eq. (5), it is straightforward to derive for the reflection
amplitudes \( r_{\uparrow\uparrow}^* = -\cos(2\phi) \cos(\varphi) + i \sin(2\phi) \) and
\( r_{\downarrow\uparrow}^* = i \sin(2\phi) \cos(\varphi) \). Similarly, by considering the case,
where the spin of the incident electron is antiparallel to the
\( z \) axis, one can obtain \( r_{\uparrow\downarrow}^* = r_{\downarrow\uparrow}^* \) and \( r_{\downarrow\downarrow}^* = r_{\uparrow\uparrow}^* \).

The \( s_z \)-component of the pumped spin per cycle in unit
of \( \hbar / 2 \) is given by

\[
\Delta s_z = -\frac{1}{2\pi i} \oint_T dt \left[ r_{\uparrow\uparrow}^* \frac{dr_{\uparrow\uparrow}}{dt} - r_{\downarrow\downarrow}^* \frac{dr_{\downarrow\downarrow}}{dt} \right] + \frac{1}{2} \oint_T r_{\uparrow\uparrow}^* \frac{dr_{\downarrow\downarrow}}{dt} ,
\]

with \( T = 2\pi / \omega_0 \) as a period of the pump. Here, the third
and fourth terms in the integrand have no contribution,
and \( r_{\uparrow\downarrow}^* \) is always imaginary. Due to \( r_{\uparrow\downarrow} = r_{\uparrow\uparrow}^* \), the first and second terms make an equal contribution.

Therefore, \( \Delta s_z = \frac{1}{2} \oint_T r_{\uparrow\uparrow}^* \frac{dr_{\uparrow\uparrow}}{dt} \), which can be further evaluated to be

\[
\Delta s_z = 2 - 4\phi^2 + \mathcal{O}(\phi^4) ,
\]

for 0 \ll 1. Similarly, one can find \( \Delta s_x = \Delta s_y = 0 \). In
Eq. (9), \( \Delta s_z \) is quantized to be 2 at \( \phi = 0 \), and there is
a small deviation from the quantized value for small \( \phi \),
being consistent with the analysis of the SPWFs. The
small deviation arises from the destruction of the spin
conservation by the magnetic impurity, rather than the
breaking of the TR symmetry. Such a deviation occurs
as well in the absence of the magnetic impurity, if the
Rashba spin-orbit coupling is included, which destroys
the spin conservation but preserves the TR sym-
metry. Physically, it is because the electron wavefunction
\( \Psi(x) \), given by Eqs. (6) and (7), is not an eigenstate of \( s_z \)
when \( \phi \neq 0 \). Moreover, the direction of the spin polariza-
tion of the wavefunction varies with time (\( \phi \) is a function
of \( t \), and so the quantized value cannot be recovered by properly choosing the spin quantization axis.

To get some more insight into the spin pump, we plot the trajectories of \( r_\uparrow \uparrow \) in a cycle on the complex plane in Fig. 3, for three different \( \phi \). Each trajectory is a closed orbit simply because the Hamiltonian is periodic in time. At \( \phi = 0 \), the orbit of \( r_\uparrow \uparrow \) is a unit circle, and increasing \( \phi \) deforms the orbit. In general, one can find that \( \Delta s_z \) equals to the area enclosed by the trajectory of \( r_\uparrow \uparrow \) divided by \( \pi/2 \). Therefore, any small perturbation may cause a small deformation of the trajectory from the unit circle, but can neither stop the spin pumping nor change the sign of \( \Delta s_z \). This reflects the topological stability of the spin pump from another aspect.

Finally, we wish to make a comment on the \( Z_2 \) spin pump proposed by Fu and Kane. These authors studied the same model as Eq. (1), and showed that for a system with closed ends, there are a pair of bound states localized near each end with energy levels crossing the bulk energy gap with time going on. The end states exhibit level crossing at the TR invariant point \( t = T/2 \), forming a Kramers doublet. For the system “weakly” coupled to leads, there occurs a resonance in reflection amplitudes, when the Kramers degenerate end states appear. Such a resonance structure allows spin to be pumped into the leads. However, if the TR symmetry is broken, an energy gap will open at the level-crossing point, and the spin pumping will be stopped. Therefore, they concluded that the TR symmetry plays a crucial role in the \( Z_2 \) spin pump. Apparently, the \( Z_2 \) pump essentially reveals the properties of the end states. An important difference of the present spin Chern pump from the \( Z_2 \) pump is the “strong” connection between the pump and leads, which allows the electrons to move freely between the pump and leads, without appearance and participation of the end states. The spin pumping process in the spin Chern pump is guaranteed by the spin Chern numbers alone, and hence robust against symmetry-breaking perturbations, as has been shown above. It reveals the bulk topological invariants directly.

This work was supported by the State Key Program for Basic Researches of China under grants numbers 2014CB921103 (LS), 2011CB922103 and 2010CB923400 (DYX), the National Natural Science Foundation of China under grant numbers 11225420 (LS), 11174125, 91021003 (DYX) and a project funded by the PAPD of Jiangsu Higher Education Institutions. We also thank the US NSF grants numbers DMR-0906816 and DMR-1205734 (DNS).

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**FIG. 3**: Trajectories of \( r_\uparrow \uparrow \) on the complex plane for three different strengths of impurity scattering potential.

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