Gen-Oja: A Two-time-scale approach for Streaming CCA

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Abstract

In this paper, we study the problems of principal Generalized Eigenvector computation and Canonical Correlation Analysis in the stochastic setting. We propose a simple and efficient algorithm, Gen-Oja, for these problems. We prove the global convergence of our algorithm, borrowing ideas from the theory of fast-mixing Markov chains and two-time-scale stochastic approximation, showing that it achieves the optimal rate of convergence. In the process, we develop tools for understanding stochastic processes with Markovian noise which might be of independent interest.

1 Introduction

Canonical Correlation Analysis (CCA) and the Generalized Eigenvalue Problem are two fundamental problems in machine learning and statistics, widely used for feature extraction in applications including regression [19], clustering [9] and classification [20].

Originally introduced by Hotelling in [17], CCA is a statistical tool for the analysis of multi-view data that can be viewed as a “correlation-aware” version of Principal Component Analysis (PCA). Given two multidimensional random variables, the objective in CCA is to find a pair of linear transformations that maximize the correlation between the transformed variables.

Given access to samples \( \{ (x_i, y_i) \}_{i=1}^n \) of zero mean random variables \( X, Y \in \mathbb{R}^d \) with an unknown joint distribution \( P_{XY} \), CCA can be used to discover features expressing similarity or dissimilarity between \( X \) and \( Y \). Formally, CCA aims to find a pair of linear transformations that maximize the correlation between the transformed variables.

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In the context of covariance matrices, the objective of the generalized eigenvalue problem is to obtain the direction \( u \) or \( v \in \mathbb{R}^d \) maximizing discrepancy between \( X \) and \( Y \) and can be formulated as,

\[
\begin{aligned}
\max u^\top \mathbb{E}[YY^\top]u & \quad \text{s.t.} \quad v^\top \mathbb{E}[XX^\top]v = 1 \quad \text{and} \quad u^\top \mathbb{E}[YY^\top]u = 1. \\
\end{aligned}
\] (1)

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\[
\begin{aligned}
\arg \max_{v \neq 0} & \quad v^\top \mathbb{E}[XX^\top]v \\
& \quad \text{and} \quad \arg \max_{u \neq 0} & \quad u^\top \mathbb{E}[YY^\top]u.
\end{aligned}
\] (2)

More generally, given symmetric matrices \( A, B \), with \( B \) positive definite, the objective of the principal generalized eigenvector problem is to obtain a unit norm vector \( w \) such that \( Aw = \lambda Bw \) for \( \lambda \) maximal.

*Equal contribution.
CCA and the generalized eigenvalue problem are intimately related. In fact, the CCA problem can be cast as a special case of the generalized eigenvalue problem by solving for \( u \) and \( v \) in the following objective:

\[
\begin{pmatrix}
0 & \mathbb{E}[XY^\top] \\
\mathbb{E}[YX^\top] & 0
\end{pmatrix} \begin{pmatrix}
u \\
u
\end{pmatrix} = \lambda \begin{pmatrix}
\mathbb{E}[XX^\top] & 0 \\
0 & \mathbb{E}[YY^\top]
\end{pmatrix} \begin{pmatrix}
u \\
u
\end{pmatrix}.
\] (3)

The optimization problems underlying both CCA and the generalized eigenvector problem are non-convex in general. While they admit closed-form solutions, even in the offline setting a direct computation requires \( \mathcal{O}(d^3) \) flops which is infeasible for large-scale datasets. Recently, there has been work on solving these problems by leveraging fast linear system solvers [15, 2] while requiring complete knowledge of the matrices \( A \) and \( B \).

In the stochastic setting, the difficulty increases because the objective is to maximize a ratio of expectations, in contrast to the standard setting of stochastic optimization [27], where the objective is the maximization of an expectation. There has been recent interest in understanding and developing efficient algorithms with provable convergence guarantees for such non-convex problems. [18] and [28] recently analyzed the convergence rate of Oja’s algorithm [26], one of the most commonly used algorithm for streaming PCA.

In contrast, for the stochastic generalized eigenvalue problem and CCA problem, the focus has been to translate algorithms from the offline setting to the online one. For example, [12] proposes a streaming algorithm for the stochastic CCA problem which utilizes a streaming SVRG method to solve an online least-squares problem. Despite being streaming in nature, this algorithm requires a non-trivial initialization and, in contrast to the spirit of streaming algorithms, updates its eigenvector estimate only after every few samples. This raises the following challenging question:

**Is it possible to obtain an efficient and provably convergent counterpart to Oja’s Algorithm for computing the principal generalized eigenvector in the stochastic setting?**

In this paper, we propose a simple, globally convergent, two-line algorithm, Gen-Oja, for the stochastic principal generalized eigenvector problem and, as a consequence, we obtain a natural extension of Oja’s algorithm for the streaming CCA problem. Gen-Oja is an iterative algorithm which works by updating two coupled sequences at every time step. In contrast with existing methods [18], at each time step the algorithm can be seen as performing a step of Oja’s method, with a noise term which is neither zero mean nor conditionally independent, but instead is Markovian in nature. The analysis of the algorithm borrows tools from the theory of fast mixing of Markov chains [11] as well as two-time-scale stochastic approximation [6, 7, 8] to obtain an optimal (up to dimension dependence) fast convergence rate of \( \tilde{O}(1/n) \). Our main contribution can summarized in the following informal theorem (made formal in Section 5).

**Main Result (informal).** With probability greater than \( 4/5 \), one can obtain an \( \epsilon \)-accurate estimate of the generalized eigenvector in the stochastic setting using \( \tilde{O}(1/\epsilon) \) unbiased independent samples of the matrices. The multiplicative pre-factors depend polynomially on the inverse eigengap and the dimension of the problem.

**Notation:** We denote by \( \lambda_i(M) \) and \( \sigma_i(M) \) the \( i \)th largest eigenvalue and singular value of a square matrix \( M \). For any positive semi-definite matrix \( N \), we denote inner product in the \( N \)-norm by \( \langle \cdot, \cdot \rangle_N \) and the corresponding norm by \( \| \cdot \|_N \). We let \( \kappa_N = \frac{\lambda_{\max}(N)}{\lambda_{\min}(N)} \) denote the condition number of \( N \). We denote the eigenvalues of the matrix \( B^{-1}A \) by \( \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_d \) with \( (u_i)_{i=1}^d \) and \( (\bar{u}_i)_{i=1}^d \) denoting the corresponding right and left eigenvectors of \( B^{-1}A \) whose existence is guaranteed by Lemma 24 in Appendix G.3. We use \( \Delta \) to denote the eigengap \( \lambda_1 - \lambda_2 \).
2 Problem Statement

In this section, we focus on the problem of estimating principal generalized eigenvectors in a stochastic setting. The generalized eigenvector, \( v_i \), corresponding to a system of matrices \((A, B)\), where \( A \in \mathbb{R}^{d \times d} \) is a symmetric matrix and \( B \in \mathbb{R}^{d \times d} \) is a symmetric positive definite matrix, satisfies

\[
Av_i = \lambda_i Bv_i.
\]  

The principal generalized eigenvector \( v_1 \) corresponds to the vector with the largest value\(^1\) of \( \lambda_i \), or, equivalently, \( v_1 \) is the principal eigenvector of the non-symmetric matrix \( B^{-1}A \). The vector \( v_1 \) also corresponds to the maximizer of the generalized Rayleigh quotient given by

\[
v_1 = \arg \max_{v \in \mathbb{R}^d} \frac{v^\top Av}{v^\top Bv}.
\]  

In the stochastic setting, we only have access to a sequence of matrices \( A_1, \ldots, A_n \in \mathbb{R}^{d \times d} \) and \( B_1, \ldots, B_n \in \mathbb{R}^{d \times d} \) assumed to be drawn i.i.d. from an unknown underlying distribution, such that \( \mathbb{E}[A_i] = A \) and \( \mathbb{E}[B_i] = B \) and the objective is to estimate \( v_1 \) given access to \( O(d) \) memory.

In order to quantify the error between a vector and its estimate, we define the following generalization of the sine with respect to the \( B \)-norm as,

\[
\sin^2_B(v, w) = 1 - \left( \frac{v^\top Bw}{\|v\|_B \|w\|_B} \right)^2.
\]  

3 Related Work

**PCA.** There is a vast literature dedicated to the development of computationally efficient algorithms for the PCA problem in the offline setting (see \([24, 14]\) and references therein). In the stochastic setting, sharp convergence results were obtained recently by \([18, 28]\) for the principal eigenvector computation problem using Oja’s algorithm and later extended to the streaming k-PCA setting by \([1]\). They are able to obtain a \( O(1/n) \) convergence rate when the eigengap of the matrix is positive and a \( O(1/\sqrt{n}) \) rate is attained in the gap free setting.

**Offline CCA and generalized eigenvector.** Computationally efficient optimization algorithms with finite convergence guarantees for CCA and the generalized eigenvector problem based on Empirical Risk Minimization (ERM) on a fixed dataset have recently been proposed in \([15, 32, 2]\). These approaches work by reducing the CCA and generalized eigenvector problem to that of solving a PCA problem on a modified matrix \( M \) (e.g., for CCA, \( M = B^{-\frac{1}{2}} A B^{-\frac{1}{2}} \)). This reformulation is then solved by using an approximate version of the Power Method that relies on a linear system solver to obtain the approximate power method step. \([15, 2]\) propose an algorithm for the generalized eigenvector computation problem and instantiate their results for the CCA problem. \([21, 22, 32]\) focus on the CCA problem by optimizing a different objective:

\[
\min \frac{1}{2} \hat{\mathbb{E}}[|\phi^\top x_i - \psi^\top y_i|^2 + \lambda_x \|\phi\|_2^2 + \lambda_y \|\psi\|_2^2 \quad \text{s.t.} \quad \|\phi\|_\hat{\mathbb{E}[xx^\top]} = \|\psi\|_\hat{\mathbb{E}[yy^\top]} = 1,
\]

where \( \hat{\mathbb{E}} \) denotes the empirical expectation. The proposed methods utilize the knowledge of complete data in order to solve the ERM problem, and hence is unclear how to extend them to the stochastic setting.

\(^1\)Note that we consider here the largest signed value of \( \lambda_i \)
Algorithm 1: Gen-Oja for Streaming $Av = \lambda Bv$

Input: Time steps $T$, step size $\alpha_t$ (Least Squares), $\beta_t$ (Oja)

Initialize: $(w_0, v_0) \leftarrow$ sample uniformly from the unit sphere in $\mathbb{R}^d$, $\bar{v}_0 = v_0$

for $t = 1, \ldots, T$ do
- Draw sample $(A_t, B_t)$
- $w_t \leftarrow w_{t-1} - \alpha_t (B_t w_{t-1} - A_t v_{t-1})$
- $v'_t \leftarrow v_{t-1} + \beta_t w_t$
- $v_t \leftarrow \frac{v'_t}{\|v'_t\|_2}$

Output: Estimate of Principal Generalized Eigenvector: $v_T$

Stochastic CCA and generalized eigenvector. There has been a dearth of work for solving these problems in the stochastic setting owing to the difficulties mentioned in Section 1. Recently, [12] extend the algorithm of [32] from the offline to the streaming setting by utilizing a streaming version of the SVRG algorithm for the least squares system solver. Their algorithm, based on the shift and invert method, suffers from two drawbacks: a) contrary to the spirit of streaming algorithms, this method does not update its estimate at each iteration – it requires to use logarithmic samples for solving an online least squares problem, and, b) their algorithm critically relies on obtaining an estimate of $\lambda_1$ to a small accuracy for which it requires to burn a few samples in the process. In comparison, Gen-Oja takes a single stochastic gradient step for the inner least squares problem and updates its estimate of the eigenvector after each sample. Perhaps the closest to our approach is [4], who propose an online method by solving a convex relaxation of the CCA objective with an inexact stochastic mirror descent algorithm. Unfortunately, the computational complexity of their method is $O(d^2)$ which renders it infeasible for large-scale problems.

4 Gen-Oja

In this section, we describe our proposed approach for the stochastic generalized eigenvector problem (see Section 2). Our algorithm Gen-Oja, described in Algorithm 1, is a natural extension of the popular Oja’s algorithm used for solving the streaming PCA problem. The algorithm proceeds by iteratively updating two coupled sequences $(w_t, v_t)$ at the same time: $w_t$ is updated using one step of stochastic gradient descent with constant step-size to minimize $w^T B w - 2 w^T A v_t$ and $v_t$ is updated using a step of Oja’s algorithm. Gen-Oja has its roots in the theory of two-time-scale stochastic approximation, by viewing the sequence $w_t$ as a fast mixing Markov chain and $v_t$ as a slowly evolving one. In the sequel, we describe the evolution of the Markov chains $(w_t)_{t\geq0}, (v_t)_{t\geq0}$, in the process outlining the intuition underlying Gen-Oja and understanding the key challenges which arise in the convergence analysis.

Oja’s algorithm. Gen-Oja is closely related to the Oja’s algorithm [26] for the streaming PCA problem. Consider a special case of the problem, when each $B_t = I$. In the offline setting, this reduces the generalized eigenvector problem to that of computing the principal eigenvector of $A$. With the setting of step-size $\alpha_t = 1$, Gen-Oja recovers the Oja’s algorithm given by

$$v_t = \frac{v_{t-1} + \beta_t A_t v_{t-1}}{\|v_{t-1} + \beta_t A_t v_{t-1}\|}$$

This algorithm is exactly a projected stochastic gradient ascent on the Rayleigh quotient $v^T A v$ (with a step size $\beta_t$). Alternatively, it can be interpreted as a randomized power method on the matrix $(I + \beta_t A)[16]$.

Two-time-scale approximation. The theory of two-time-scale approximation forms the underlying basis for Gen-Oja. It considers coupled iterative systems where one component changes much
faster than the other [7, 8]. More precisely, its objective is to understand classical systems of the type:
\[
\begin{align*}
    x_t &= x_{t-1} + \alpha_t \left[ h(x_{t-1}, y_{t-1}) + \xi_t^1 \right] \quad (7) \\
    y_t &= y_{t-1} + \beta_t \left[ g(x_{t-1}, y_{t-1}) + \xi_t^2 \right], \quad (8)
\end{align*}
\]

where \(g\) and \(h\) are the update functions and \((\xi_t^1, \xi_t^2)\) correspond to the noise vectors at step \(t\) and typically assumed to be martingale difference sequences.

In the above model, whenever the two step sizes \(\alpha_t\) and \(\beta_t\) satisfy \(\beta_t/\alpha_t \to 0\), the sequence \(y_t\) moves on a slower timescale than \(x_t\). For any fixed value of \(y\) the dynamical system given by \(x_t\),
\[
x_t = x_{t-1} + \alpha_t [h(x_{t-1}, y) + \xi_t^1],
\]
converges to a solution \(x^*(y)\). In the coupled system, since the state variables \(x_t\) move at a much faster time scale, they can be seen as being close to \(x^*(y_t)\), and thus, we can alternatively consider:
\[
y_t = y_{t-1} + \beta_t \left[ g(x_t(y_{t-1}), y_{t-1}) + \xi_t^2 \right]. \quad (10)
\]

If the process given by \(y_t\) above were to converge to \(y^*\), under certain conditions, we can argue that the coupled process \((x_t, y_t)\) converges to \((x^*(y^*), y^*)\). Intuitively, because \(x_t\) and \(y_t\) are evolving at different time-scales, \(x_t\) views the process \(y_t\) as quasi-constant while \(y_t\) views \(x_t\) as a process rapidly converging to \(x^*(y_t)\).

Gen-Oja can be seen as a particular instance of the coupled iterative system given by Equations (7) and (8) where the sequence \(v_t\) evolves with a step-size \(\beta_t \approx 1/\alpha_t\), much slower than the sequence \(w_t\), which has a step-size of \(\alpha_t \approx 1/\log(t)\). Proceeding as above, the sequence \(v_t\) views \(w_t\) as having converged to \(B^{-1}Av_t + \xi_t\), where \(\xi_t\) is a noise term, and the update step for \(v_t\) in Gen-Oja can be viewed as a step of Oja’s algorithm, albeit with Markovian noise.

While previous works on the stochastic CCA problem required to use logarithmic independent samples to solve the inner least-squares problem in order to perform an approximate power method (or Oja) step, the theory of two-time-scale stochastic approximation suggests that it is possible to obtain a similar effect by evolving the sequences \(w_t\) and \(v_t\) at two different time scales.

**Understanding the Markov Process \(\{w_t\}\).** In order to understand the process described by the sequence \(w_t\), we consider the homogeneous Markov chain \((w_t^v)\) defined by
\[
w_t^v = w_{t-1}^v - \alpha (B_tw_{t-1}^v - Av_t), \quad (11)
\]
for a constant vector \(v\) and we denote its \(t\)-step kernel by \(\pi_t^v\) [23]. This Markov process is an iterative linear model and has been extensively studied by [29, 10, 5]. It is known that for any step-size \(\alpha \leq 2/R^2\), the Markov chain \((w_t^v)_{t \geq 0}\) admits a unique stationary distribution, denoted by \(\nu_v\). In addition,
\[
W_2^2(\pi_t^v(w_0, \cdot), \nu_v) \leq (1 - 2\mu(1 - \alpha R_B/2))^t \int_{\mathbb{R}^d} ||w_0 - w||_2^2 d\nu_v(w), \quad (12)
\]
where \(W_2^2(\lambda, \nu)\) denotes the Wasserstein distance of order 2 between probability measures \(\lambda\) and \(\nu\) (see, e.g., [31] for more properties of \(W_2\)). Equation (12) implies that the iterative linear process described by (11) mixes exponentially fast to the stationary distribution. This forms a crucial ingredient in our convergence analysis where we use the fast mixing to obtain a bound on the expected norm of the Markovian noise (see Lemma 1).

Moreover, one can compute the mean \(\bar{w}\) of the process \(w_t\) under the stationary distribution by taking expectation under \(\nu_v\) on both sides in equation (11). Doing so, we obtain, \(\bar{w} = B^{-1}Av\). Thus, in our setting, since the \(v_t\) process evolves slowly, we can expect that \(w_t \approx B^{-1}Av_t\), allowing Gen-Oja to mimic Oja’s algorithm.
5 Main Theorem

In this section, we present our main convergence guarantee for Gen-Oja when applied to the streaming generalized eigenvector problem. We begin by listing the key assumptions required by our analysis:

(A1) The matrices \((A_i)_{i \geq 0}\) satisfy \(\mathbb{E}[A_i] = A\) for a symmetric matrix \(A \in \mathbb{R}^{d \times d}\).

(A2) The matrices \((B_i)_{i \geq 0}\) are such that each \(B_i \succ 0\) is symmetric and satisfies \(\mathbb{E}[B_i] = B\) for a symmetric matrix \(B \in \mathbb{R}^{d \times d}\) with \(B \succeq \mu I\) for \(\mu > 0\).

(A3) There exists \(R \geq 0\) such that \(\max\{\|A_i\|, \|B_i\|\} \leq R\) almost surely.

Under the assumptions stated above, we obtain the following convergence theorem for Gen-Oja with respect to the \(\sin^2\beta\) distance, as described in Section 2.

Theorem 1 (Main Result). Fix any \(\delta > 0\) and \(\epsilon_1 > 0\). Suppose that the step sizes are set to

\[
\alpha_t = \frac{c}{\log(d^2 \beta + \epsilon_1)} \quad \text{and} \quad \beta_t = \frac{20\gamma^2 \lambda_1^2}{\Delta_\lambda^2 d^2 \log(1 + \frac{\delta}{100})},
\]

where \(\gamma = \max\{1, \frac{1}{\sqrt{\Delta_\lambda}}\}\). Suppose that the number of samples \(n\) satisfy

\[
\frac{d^2 \beta + n}{\log\min(1, 2\gamma \lambda_1 / \Delta_\lambda)} \geq \frac{cd}{\delta_1 \min(1, \lambda_1)} (d^2 \beta + n) \exp\left(\frac{c \lambda_1^2}{d^2}\right)
\]

Then, the output \(v_n\) of Algorithm 1 satisfies,

\[
\sin^2(\beta, u_1, v_n) \leq (2 + \epsilon_1)cd \frac{\|u_1\|^2}{d^2 \|u_1\|^2} \left(\frac{c \lambda_1^2}{d^2 \beta + n} + \frac{c d^2 \beta + \log^3(d^2 \beta)}{\Delta_\lambda^2 (d^2 \beta + n + 1)}\right) \exp\left(\frac{c \lambda_1^2}{d^2}\right),
\]

with probability at least \(1 - \delta\) with \(c\) depending polynomially on parameters of the problem \(\lambda_1, \kappa_B, R, \mu\). The parameter \(\delta_1 = 1 - 2\delta\).

The above result shows that with probability at least \(1 - \delta\), Gen-Oja converges in the \(B\)-norm to the right eigenvector, \(u_1\), corresponding to the maximum eigenvalue of the matrix \(B^{-1} A\). Further, Gen-Oja exhibits an \(\tilde{O}(1/n)\) rate of convergence, which is known to be optimal for stochastic approximation algorithms even with convex objectives [25].

Comparison with Streaming PCA. In the setting where \(B = I\), and \(A \succeq 0\) is a covariance matrix, the principal generalized eigenvector problem reduces to performing PCA on the \(A\). When compared with the results obtained for streaming PCA by [18], our corresponding results differ by a factor of dimension \(d\) and problem dependent parameters \(\lambda_1, \Delta_\lambda\). We believe that such a dependence is not inherent to Gen-Oja but a consequence of our analysis. We leave this task of obtaining a dimension free bound for Gen-Oja as future work.

Gap-independent step size: While the step size for the sequence \(v_n\) in Gen-Oja depends on eigen-gap, which is a priori unknown, one can leverage recent results as in [30] to get around this issue by using a streaming average step size.

6 Proof Sketch

In this section, we detail out the two key ideas underlying the analysis of Gen-Oja to obtain the convergence rate mentioned in Theorem 1: a) controlling the non i.i.d. Markovian noise term which is introduced because of the coupled Markov chains in Gen-Oja and b) proving that a noisy power method with such Markovian noise converges to the correct solution.
Controlling Markovian perturbations. In order to better understand the sequence \( v_t \), we rewrite the update as,

\[
v_t' = v_{t-1} + \beta_t w_t = v_{t-1} + \beta_t (B^{-1} Av_{t-1} + \xi_t),
\]

where \( \xi_t = w_t - B^{-1} Av_{t-1} \) is the prediction error which is a Markovian noise. Note that the noise term is neither mean zero nor a martingale difference sequence. Instead, the noise term \( \xi_t \) is dependent on all previous iterates, which makes the analysis of the process more involved. This framework with Markovian noise has been extensively studied by [6, 3].

From the update in Equation (13), we observe that Gen-Oja is performing an Oja update but with a controlled Markovian noise. However, we would like to highlight that classical techniques in the study of stochastic approximation with Markovian noise (as the Poisson Equation [6, 23]) were not enough to provide adequate control on the noise to show convergence.

In order to overcome this difficulty, we leverage the fast mixing of the chain \( w_t \) for understanding the Markovian noise. While it holds that \( \mathbb{E}[\|\xi_t\|_2] = \mathcal{O}(1) \) (see Appendix C), a key part of our analysis is the following lemma, the proof of which can be found in Appendix B.

**Lemma 1.** For any choice of \( k > 4\frac{\lambda_1(B)}{\mu s} \log(\frac{1}{\beta_{t+k}}) \), and assuming that \( \|w_s\| \leq W_s \) for \( t \leq s \leq t + k \) we have that

\[
\|\mathbb{E} [\xi_{t+k}|\mathcal{F}_t]\|_2 = \mathcal{O}(\beta_t k^2 c_s W_{t+k})
\]

Lemma 1 uses the fast mixing of \( w_t \) to show that \( \|\mathbb{E}[\xi_t]|\mathcal{F}_{t-r}\|_2 = \tilde{\mathcal{O}}(\beta_t) \) where \( r = \mathcal{O}(\log t) \), i.e., the magnitude of the expected noise is small conditioned on \( \log(t) \) steps in the past.

**Analysis of Oja’s algorithm.** The usual proofs of convergence for stochastic approximation define a Lyapunov function and show that it decreases sufficiently at each iteration. Oftentimes control on the per step rate of decrease can then be translated into a global convergence result. Unfortunately in the context of PCA, due to the non-convexity of the Raleigh quotient, the quality of the estimate \( v_t \) cannot be related to the previous \( v_{t-1} \). Indeed \( v_t \) may become orthogonal to the leading eigenvector. Instead [18] circumvent this issue by leveraging the randomness of the initialization and adopt an operator view of the problem. We take inspiration from this approach in our analysis of Gen-Oja.

Let \( G_t = w_t v_{t-1}^\top \) and \( H_t = \prod_{i=1}^{t} (I + \beta_i G_i) \), Gen-Oja’s update can be equivalently written as

\[
v_t = \frac{H_t v_0}{\|H_t v_0\|_2},
\]

pushing, for the analysis only, the normalization step at the end. This point of view enables us to analyze the improvement of \( H_t \) over \( H_{t-1} \) since allows one to interpret Oja’s update as one step of power method on \( H_t \) starting on a random vector \( v_0 \). We present here an easy adaptation of [18, Lemma 3.1] that takes into account the special geometry of the generalized eigenvector problem and the asymmetry of \( B^{-1} A \). The proof can be found in Appendix A.

**Lemma 2.** Let \( H \in \mathbb{R}^{d \times d}, (u_i)_{i=1}^d \) and \((\tilde{u}_i)_{i=1}^d \) be the corresponding right and left eigenvectors of \( B^{-1} A \) and \( w \in \mathbb{R}^d \) chosen uniformly on the sphere, then with probability \( 1-\delta \) (over the randomness in the initial iterate)

\[
\sin_B^2(u_i, H w) \leq \frac{C\log(1/\delta)}{\delta} \frac{\text{Tr}(HH^\top \sum_{j \neq i} \tilde{u}_j \tilde{u}_j^\top)}{\tilde{u}_i^\top HH^\top \tilde{u}_i},
\]

for some universal constant \( C > 0 \).

This lemma has the virtue of highly simplifying the challenging proof of convergence of Oja’s algorithm. Indeed we only have to prove that \( H_t \) will be close to \( \prod_{i=1}^{t} (I + \beta_i B^{-1} A) \) for \( t \) large enough which can be interpreted as an analogue of the law of large numbers for the multiplication of matrices. This will ensure that \( \text{Tr}(H_t H_t^\top \sum_{j \neq i} \tilde{u}_j \tilde{u}_j^\top) \) is relatively small compared to \( \tilde{u}_i^\top H_t H_t^\top \tilde{u}_i \) and be enough with Lemma 2 to prove Theorem 1. The proof follows the line of [18] with two additional tedious difficulties: the Markovian noise is neither unbiased nor independent of the previous iterates, and the matrix \( B^{-1} A \) is no longer symmetric, which is precisely why we consider the left eigenvector \( \tilde{u}_i \) in the right-hand side of Eq. (14). We highlight two key steps:
• First we show that $\mathbb{E} \Tr (H_1 H_1^\top \sum_{j \neq i} \tilde{u}_j \tilde{u}_j^\top )$ grows as $O(\exp(2\lambda_2 \sum_{i=1}^t \beta_i))$, which implies by Markov’s inequality the same bound on $\Tr (H_1 H_1^\top \sum_{j \neq i} \tilde{u}_j \tilde{u}_j^\top )$ with constant probability. See Lemmas 16 for more details.

• Second we show that $\text{Var} \tilde{u}_i^\top H_i H_i^\top \tilde{u}_i$ grows as $O(\exp(4\lambda_1 \sum_{i=1}^t \beta_i))$ and $\mathbb{E} \tilde{u}_i^\top H H^\top \tilde{u}_i$ grows as $O(\exp(2\lambda_1 \sum_{i=1}^t \beta_i))$ which implies by Chebyshev’s inequality the same bound for $\tilde{u}_i^\top H H^\top \tilde{u}_i$ with constant probability. See Lemmas 17 and 19 for more details.

7 Application to Canonical Correlation Analysis

Consider two random vectors $X \in \mathbb{R}^d$ and $Y \in \mathbb{R}^d$ with joint distribution $P_{XY}$. The objective of canonical correlation analysis in the population setting is to find the canonical correlation vectors $\phi, \psi \in \mathbb{R}^{d,d}$ which maximize the correlation

$$\max_{\phi, \psi} \frac{\mathbb{E}[(\phi^\top X) (\psi^\top Y)]}{\sqrt{\mathbb{E}[(\phi^\top X)^2] \mathbb{E}[(\psi^\top Y)^2]}}.$$

This problem is equivalent to maximizing $\phi^\top \mathbb{E}[XY^\top] \psi$ under the constraint $\mathbb{E}[(\phi^\top X)^2] = \mathbb{E}[(\psi^\top Y)^2] = 1$ and admits a closed form solution: if we define $T = \mathbb{E}[XX^\top]^{-1/2} \mathbb{E}[XY^\top] \mathbb{E}[YY^\top]^{-1/2}$, then the solution is $(\phi_*, \psi_*) = (\mathbb{E}[XX^\top]^{-1/2} \alpha_1 \mathbb{E}[YY^\top]^{-1/2} \beta_1)$ where $\alpha_1, \beta_1$ are the left and right principal singular vectors of $T$. By the KKT conditions, there exist $\nu_1, \nu_2 \in \mathbb{R}$ such that this solution satisfies the stationarity equation

$$\mathbb{E}[XY^\top] \psi = \nu_1 \mathbb{E}[XX^\top] \phi \quad \text{and} \quad \mathbb{E}[YY^\top] \phi = \nu_2 \mathbb{E}[YY^\top] \psi.$$

Using the constraint conditions we conclude that $\nu_1 = \nu_2$. This condition can be written (for $\lambda = \nu_1$) in the matrix form of Eq. (3). As a consequence, finding the largest generalized eigenvector for the matrices $(A, B)$ will recover the canonical correlation vector $(\phi, \psi)$. Solving the associated generalized streaming eigenvector problem, we obtain the following result for estimating the canonical correlation vector whose proof easily follows from Theorem 1 (setting $\gamma = 6$).

**Theorem 2.** Assume that $\max\{\|X\|, \|Y\|\} \leq R$ a.s., $\min\{\lambda_{\min}(\mathbb{E}[XX^\top]), \lambda_{\min}(\mathbb{E}[YY^\top])\} = \mu > 0$ and $\sigma_1(T) - \sigma_2(T) = \Delta > 0$. Fix any $\delta > 0$, let $\epsilon_1 \geq 0$, and suppose the step sizes are set to $\alpha_t = \frac{1}{2R^2 \log(d^2 + 1)}$ and $\beta_t = \frac{\Delta}{d \mu} \beta_{t-1}$ and

$$\beta = \max \left( \frac{200 (1 + \frac{B}{2R^2})}{\Delta^2 d^2 \log \left( \frac{1 + \delta/1000}{1 + \epsilon_1} \right)}, \frac{1}{\delta} \min(1, \lambda_1) \right) \frac{1}{\lambda_{\min}(XX^\top, YY^\top, XX^\top YY^\top)} \frac{1}{\log(1 + \frac{R}{\mu}) + \frac{\mu^2}{\mu^2}} \frac{\log^3(d^2 + 1)}{\lambda^2}\left(\frac{c\lambda^2}{d^2}\right)$$

Suppose that the number of samples $n$ satisfy

$$\frac{d^2 \beta + n}{\log \frac{\min(1, \lambda_1)}{n} \left( \frac{d^2 \beta + n}{\Delta^2} \right)} \geq \left( \frac{c d}{\delta^3 \min(1, \lambda_1)} \right) \log^3(d^2 + 1) \exp \left( \frac{c\lambda^2}{d^2} \right)$$

Then the output $(\phi_t, \psi_t)$ of Algorithm 1 applied to $(A, B)$ defined above satisfies,

$$\sin^2 \left( \frac{(\phi_*, \psi_*)}{\lambda} \right) \leq \frac{(2 + \epsilon_1) c d^2 \log \left( \frac{1}{\delta^3} \right)}{\lambda^2 \|u_1\|^2} \frac{\log^3(d^2 + 1)}{\Delta^2 (d^2 \beta + n + 1)}$$

with probability at least $1 - \delta$ with $c$ depending on parameters of the problem and independent of $d$ and $\Delta$ where $\delta_1 = \frac{\epsilon_1}{2(2 + \epsilon_1)}$.

We can make the following observations:

• The convergence guarantee are comparable with the sample complexity obtained by the ERM ($t = \mathcal{O}(d/\epsilon \Delta^2)$ for sub-Gaussian variables and $t = \mathcal{O}(1/(\epsilon \Delta^2 \mu^2))$ for bounded variables) [12] and matches the lower bound $t = \mathcal{O}(d/(\epsilon \Delta^2))$ known for sparse CCA [13].

• The sample complexity in [12] is better in term of the dependence on $d$. They obtain the same rates as the ERM. The comparison with [4] is meaningless since they are in the gap free setting and their computational complexity is $\mathcal{O}(d^2)$. 


8 Simulations

Here we illustrate the practical utility of Gen-Oja on a synthetic, streaming generalized eigenvector problem. We take $d = 20$ and $T = 10^6$. The streams $(A_t, B_t) \in (\mathbb{R}^{d \times d})^2$ are normally-distributed with covariance matrix $A$ and $B$ with random eigenvectors and eigenvalues decaying as $1/i$, for $i = 1, \ldots, d$. Here $R^2$ denotes the radius of the streams with $R^2 = \max\{\text{Tr} A, \text{Tr} B\}$. All results are averaged over ten repetitions.

**Comparison with two-steps methods.** In the left plot of Figure 1 we compare the behavior of Gen-Oja to different two-steps algorithms. Since the method by [4] is of complexity $O(d^2)$, we compare Gen-Oja to a method which alternates between one step of Oja’s algorithm and $\tau$ steps of averaged stochastic gradient descent with constant step size $1/(2R^2)$. Gen-Oja is converging at rate $O(1/t)$ whereas the other methods are very slow. For $\tau = 10$, the solution of the inner loop is too inaccurate and the steps of Oja are inefficient. For $\tau = 10000$, the output of the sgd steps is very accurate but there are too few Oja iterations to make any progress. $\tau = 1000$ seems an optimal parameter choice but this method is slower than Gen-Oja by an order of magnitude.

**Robustness to incorrect step-size $\alpha$.** In the middle plot of Figure 1 we compare the behavior of Gen-Oja for step size $\alpha \in \{\alpha, \alpha/8, \alpha/16\}$ where $\alpha_* = 1/R^2$. We observe that Gen-Oja converges at a rate $O(1/t)$ independently of the choice of $\alpha$.

**Robustness to incorrect step-size $\beta_t$.** In the right plot of Figure 1 we compare the behavior of Gen-Oja for step size $\beta_t \in \{\beta_*/t, \beta_*/16t, \beta_*/\sqrt{t}, \beta_*/16\sqrt{t}\}$ where $\beta_*$ corresponds to the minimal error after one pass over the data. We observe that Gen-Oja is not robust to the choice of the constant for step size $\beta_t \propto 1/t$. If the constant is too small, the rate of convergence is arbitrary slow. We observe that considering the streaming average of [30] on Gen-Oja with a step size $\beta_t \propto 1/\sqrt{t}$ enables to recover the fast $O(1/t)$ convergence while being robust to constant misspecification.

9 Conclusion

We have proposed and analyzed a simple online algorithm to solve the streaming generalized eigenvector problem and applied it to CCA. This algorithm, inspired by two-time-scale stochastic approximation achieves a fast $O(1/t)$ convergence. Considering recovering the $k$-principal generalized eigenvector (for $k > 1$) and obtaining a slow convergence rate $O(1/\sqrt{t})$ in the gap free setting are promising future directions. Finally, it would be worth considering removing the dimension dependence in our convergence guarantee.
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A Proof of Lemma 2

We prove here the Lemma 2 which is an easy adaptation of [18, Lemma 3.1]. We first recall it.

Lemma 3. Let $H \in \mathbb{R}^{d \times d}$, $(u_i)_{i=1}^d$ and $(\tilde{u}_i)_{i=1}^d$ the corresponding right and left eigenvectors of $B^{-1}A$ and $w \in \mathbb{R}^d$ chosen uniformly on the sphere, then with probability $1 - \delta$ (over the randomness in the initial iterate)

$$\sin_B^2(u_i, H w) \leq \frac{C \log(1/\delta)}{\delta} \frac{\text{Tr}(H H^T \sum_{j \neq i} \tilde{u}_j \tilde{u}_j^T)}{\tilde{u}_i^T H H^T \tilde{u}_i},$$

for some universal constant $C > 0$.

Proof. We follow the proof of [18]. Given a $B$-normalized right eigenvector $u_i$ of $B^{-1}A$ and $w = \frac{g}{\|g\|_2}$ for $g \sim \mathcal{N}(0, I)$, we consider:

$$\sin_B^2(u_i, H w) = 1 - \frac{(u_i^T B H w)^2}{w^T H^T B H w} = \frac{g^T H^T B^{1/2} \left[ I - B^{1/2} \sum_{i=1}^d (u_i u_i^T)^{1/2} \right] B^{1/2} H g}{g^T H^T B H g}.$$

Moreover following Lemma 24 and denoting by $\hat{u}_i$ the corresponding orthonormal family of eigenvectors of the symmetric matrix $B^{-1/2} A B^{-1/2}$, we have that $u_i = B^{1/2} \hat{u}_i$. This yields:

$$\left[ I - B^{1/2} u_i u_i^T B^{1/2} \right] = \left[ I - \hat{u}_i \hat{u}_i^T \right] = \sum_{j \neq i} \hat{u}_j \hat{u}_j^T$$

Using now that the left eigenvectors of $B^{-1}A$ are given by $\tilde{u}_i = B u_i$, we get

$$\sin_B^2(u_i, H w) = \frac{g^T H^T B^{1/2} \sum_{j \neq i} \tilde{u}_j \tilde{u}_j^T B^{1/2} H g}{g^T H^T B H g} = \frac{g^T H^T \sum_{j \neq i} \tilde{u}_j \tilde{u}_j^T H g}{g^T H^T B H g}.$$

We may bound the denominator by

$$g^T H^T B H g \geq g^T H^T B^{1/2} \sum_{i=1}^d \hat{u}_i \hat{u}_i^T B^{1/2} H g = g^T H^T \sum_{i=1}^d \hat{u}_i \hat{u}_i^T H g = (\tilde{u}_i^T H g)^2 \geq \frac{\delta}{C_1} \tilde{u}_i^T H H^T \tilde{u}_i,$$

where the last inequality follows as $\tilde{u}_i^T H g$ is a Gaussian random vector with variance $\|H^T \tilde{u}_i\|_2^2$.

We can also bound the numerator as

$$g^T H^T \sum_{j \neq i} \tilde{u}_j \tilde{u}_j^T H g \leq C_2 \log(1/\delta) \text{Tr}[H^T \sum_{j \neq i} \tilde{u}_j \tilde{u}_j^T H],$$

since $w^T H^T \sum_{j \neq i} \tilde{u}_j \tilde{u}_j^T H w$ is a $\chi^2$ random variable with $\text{Tr}[H^T \sum_{j \neq i} \tilde{u}_j \tilde{u}_j^T H]$ degrees of freedom. Therefore it exists a universal constant $C > 0$ such that

$$\sin_B^2(u_i, H w) \leq \frac{C \log(1/\delta)}{\delta} \frac{\text{Tr}[H^T \sum_{j \neq i} \tilde{u}_j \tilde{u}_j^T H]}{\tilde{u}_i^T H H^T \tilde{u}_i},$$

with probability $1 - \delta$. \hfill \qed
B Deviation bounds for fast-mixing Markov Chain

In this section, we prove an upper bound on \( \|E[\epsilon_t | \mathcal{F}_t] \|_2 \), where \( \epsilon_t = (w_t - B^{-1} Aw_{t-1})v_{t-1}^\top \) and \( \mathcal{F}_t = \sigma(w_0, \cdots, w_t) \) denotes the \( \sigma \)-algebra generated by \( w_0, \cdots, w_t \). For the purpose of this section, we denote the pointwise upperbound on \( \|w_t\|_2 \) by \( W_t \). To begin with, we consider bounding the error term considering a fixed step-size \( \alpha_t = \alpha \) in order to keep the analysis cleaner. In Lemma 6, we bound the deviation of chains with step-size \( \alpha_t = O(c/ \log(d^2 \beta + 1)) \) and fixed step size over a short horizon of length \( O(\log^2(1/\beta)) \).

In order to prove the requisite bound, consider the following Markov chain given by,

\[
\theta_{k+1} = \theta_k - \eta (f'(\theta_k) + \epsilon_{k+1}),
\]

where \( f : \mathbb{R}^d \to \mathbb{R} \) is some strongly convex function. We make use of the following proposition highlighting the fast-mixing property of constant step-size stochastic gradient descent from [11].

**Proposition 1.** For any step size \( \alpha \in (0, 2/L_0) \), the markov chain given by \((\theta_k)_{k \geq 0}\) defined by recursion (15), admits a unique stationary distribution \( \pi \in \mathcal{P}(\mathbb{R}^d) \). In addition, for all \( \theta \in \mathbb{R}^d, k \in \mathbb{N} \), we have,

\[
W_2^2(R^k(\theta, \cdot), \pi) \leq (1 - 2\mu_\theta \eta (1 - \eta L_0/2))^k \int_{\mathbb{R}^d} \| \theta - \theta' \|^2 d\pi(\theta'),
\]

where \( L_0 \) and \( \mu_\theta \) are the smoothness and the strong convexity parameters of \( f \) respectively.

Now, consider the Markov chain given by

\[
w^{k+1}_t = w^k_t - \alpha (B_k w^k_t - A_k v_t),
\]

where \( E[B_k] = B, E[A_k] = A, w^0_t = w_t \) where \( w_t \) is as given by Algorithm 1. Equation (17) represents the update step for the \( k^{th} \) step of a Markov chain starting at \( w_k \) and performing stochastic gradient updates on \( f_t(w) = 1/2w^\top B w - w^\top A v_t \).

For this function \( f_t \), the smoothness constant \( L = \lambda_B \). Further, proposition 1 guarantees the existence of a unique stationary distribution \( \pi \) and we have that under the stationary distribution,

\[
E_\pi[w^k_t] = B^{-1} A v_t.
\]

**Lemma 4.** For the Markov chain given by (17) with any step size \( \alpha \in (0, 2/\lambda_B) \), for any \( k > \frac{\log(\frac{\lambda_B}{\lambda_2})}{\mu_\alpha(1- \frac{\alpha \lambda_B}{2})} \), we have

\[
\|E[w^k_t - B^{-1} A v_t]|_{\mathcal{F}_t}\|_2^2 \leq \epsilon
\]

**Proof.** We know from (18), \( B^{-1} A v_t = E_\pi[w^k_t] \). Now, we consider the term \( \| E[w^k_t - B^{-1} A v_t]|_{\mathcal{F}_t}\|_2^2 \),

\[
\|E[w^k_t - B^{-1} A v_t]|_{\mathcal{F}_t}\|_2^2 = \|E[w^k_t] - E_\pi[w]|_{\mathcal{F}_t}\|_2^2
\]

\[
\leq \|E(R^k(w_t, \cdot), \pi)[w^k_t - w]|_{\mathcal{F}_t}\|_2^2
\]

\[
\leq W_2^2(R^k(\cdot, \cdot), \pi)
\]

\[
\leq (1 - 2\mu_\alpha(1 - \alpha \lambda_B/2))^k \lambda_2^2,
\]

where \( R^k(w_t, \cdot) \) denotes the \( k \)-step transition kernel of the Markov chain beginning from \( w_t \), \( \Gamma(R^k(w_t, \cdot), \pi) \) denotes any coupling of the distributions \( R^k(w_t, \cdot) \) and \( \pi \) and \( E_\Gamma(\cdot, \cdot) \) denotes the expectation under the joint distribution, conditioned on \( \mathcal{F}_t \). Now, \( \zeta_1 \) follows from Jenson’s inequality, \( \zeta_2 \) follows by setting \( \Gamma(R^k(w_t, \cdot), \pi) \) to the coupling attaining the infimum in the wasserstein bound and \( \zeta_3 \) follows by using proposition (1). The lemma now follows by setting \( k > \frac{\log(\frac{\lambda_B}{\lambda_2})}{\mu_\alpha(1- \frac{\alpha \lambda_B}{2})} \). [see, e.g., 31, for more properties of \( W_2 \)]
Deviation bound for \( \|v_t - v_{t+k}\|_2 \): We now bound the deviation of \( v_{t+k} \) from \( v_t \) if we execute \( k \) steps of the algorithm starting from \( v_t \),

\[
\|v_t - v_{t+k}\|_2 \leq \sum_{i=0}^{k-1} \|v_{t+i} - v_{t+i+1}\|_2. \tag{19}
\]

Now, for a single step of the algorithm, using the contractivity of the projection

\[
\|v_t - v_{t+1}\|_2 \leq \|v_t - \frac{v'_{t+1}}{\|v'_{t+1}\|}\|_2 \leq \|v_t - v_{t+1}'\|_2 \leq W_{t+1} \beta_{t+1}.
\]

Using the above bound in (19), we obtain,

\[
\|v_t - v_{t+k}\|_2 \leq W_{t+k} \sum_{i=0}^{k-1} \beta_{t+i+1} \leq W_{t+k} k \beta_t, \tag{20}
\]

by using the fact that \( \beta_t \) is a decreasing sequence.

Deviation bound for Coupled Chains: Consider the sequence \( (w_{t+i})_{i=0}^k \) as generated by Algorithm 1, assuming a constant step-size \( \alpha \), and the sequence \( (w_t^i)_{i=1}^k \) generated by the recurrence (17) in the case when both have the same randomness with respect to the sampling of the matrices \( A_{t+i}, B_{t+i} \). We now obtain a bound on \( \|\mathbb{E}[w_t^k - w_{t+k}]|\mathcal{F}_t\|_2 \).

\[
\|\mathbb{E}[w_t^k - w_{t+k}]|\mathcal{F}_t\|_2 = \| \mathbb{E} \left[ (I - \alpha B)^k (w_{t+k-1} - w_t) - \alpha A (w_t - w_{t+k-1}) \right] |\mathcal{F}_t \|_2
\]

\[
\leq \alpha \mathbb{E} \left[ \sum_{i=0}^{k-1} (I - \alpha B)^i A (w_t - w_{t+k-1-i}) |\mathcal{F}_t \right]
\]

\[
\leq \alpha \lambda A W_{t+k} k \sum_{i=0}^{k-1} (1 - \alpha \mu)^i \beta_{t+k-1-i}
\]

\[
\leq \frac{\lambda A W_{t+k} k \beta_t}{\mu}, \tag{21}
\]

where we expand the terms using the recursion and bound the geometric series by using that \( \alpha \mu \leq 1 \).

Lemma 5. For any choice of \( k > \frac{\log(\frac{1}{\beta_t})}{2\mu \alpha (1 - \alpha \mu)} \), we have that

\[
\|\mathbb{E}[\epsilon_{t+k}|\mathcal{F}_t]\|_2 \leq \left( \frac{\lambda A W_{t+k} k}{\mu} + \lambda_1 (1 + 2 W_{t+k} k) + W_{t+k}^2 \right) \beta_t = O(W_{t+k}^2 \beta_t)
\]

Proof. Consider the term \( \|\mathbb{E}[\epsilon_{t+k}|\mathcal{F}_t]\|_2 \),

\[
\|\mathbb{E}[\epsilon_{t+k}|\mathcal{F}_t]\|_2 = \|\mathbb{E}[(w_{t+k} - B^{-1} A v_{t+k-1}) v_{t+k-1}^T |\mathcal{F}_t]\|_2
\]

\[
\leq \|\mathbb{E}[(w_{t+k} - B^{-1} A v_{t+k-1}) v_{t+k-1}^T |\mathcal{F}_t]\|_2 + \|\mathbb{E}[(w_{t+k} - B^{-1} A v_{t+k-1})(v_{t+k-1} - v_t)^T |\mathcal{F}_t]\|_2.
\]

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We first analyze term (I) in the expansion above.

\[
\|E[(w_{t+k} - B^{-1} A v_{t+k-1}) v_t^T | F_t] \|_2 = \|E[(w_{t+k} - w_k^t) + (w_k^t - B^{-1} A v_t) + (B^{-1} A v_t - B^{-1} A v_{t+k-1})) v_t^T | F_t] \|_2 \\
\leq \|E[(w_{t+k} - w_k^t) | F_t] v_t^T \|_2 + \|E[(w_k^t - B^{-1} A v_t) | F_t] v_t^T \|_2 \\
\leq \|E[(w_{t+k} - w_k^t) | F_t] v_t^T \|_2 + \|E[(w_k^t - B^{-1} A v_t) | F_t] v_t^T \|_2 \\
+ |E[(B^{-1} A v_t - B^{-1} A v_{t+k-1}) | F_t] v_t^T \|_2 \\
\leq \lambda_A W_{t+k} k \beta_t + \lambda_1 \beta_t + \lambda_1 W_{t+k} k \beta_t \\
= (\frac{\lambda_A W_{t+k} k}{\mu} + \lambda_1 (1 + W_{t+k} k)) \beta_t, \quad (22)
\]

where \(\zeta_i\) follows from using lemma 4 with \(k > \frac{\log(\frac{1}{\beta_t})}{2\mu \alpha (1 - \frac{\lambda_1}{\lambda_A})}\), bound in (20) and bound in (21).

We now look at term (II) in the expansion.

\[
\|E[(w_{t+k} - B^{-1} A v_{t+k-1})(v_{t+k-1} - v_t) | F_t] \|_2 \leq (W_{t+k} + \lambda_1) \|v_{t+k-1} - v_t\| \\
\leq W_{t+k} (W_{t+k} + \lambda_1) k \beta_t. \quad (23)
\]

Combining the bounds in (22) and (23), we get the desired result.

The bound we proved above hold for any fixed fixed step-size \(\alpha\). However, in order to obtain the sharpest convergence result for our algorithm, we would require the step size \(\alpha_t = \frac{\log(\frac{1}{\beta_t})}{(2\mu)\beta_t + \lambda A}\) for some constant \(\beta\). We provide the following lemma which accomodates for this change.

In order to get a bound on the noise term with a logarithmically decaying step size, in addition to the previous analysis, we consider processes \((\tilde{w}_{t+i})_{i=1}^k\) and \((\tilde{v}_{t+i})_{i=1}^k\) which evolve with the same random matrices \(A_{t+i}\) and \(B_{t+i}\), but with a step size of \(\alpha_{t+i} = \alpha_t = \frac{\log(\beta_t)}{\lambda A}\).

**Pointwise bound on \(\|\tilde{w}_{t+k}\|\):** We can obtain a pointwise bound on \(\|\tilde{w}_{t+k}\|\) using the simple recursive evaluation:

\[
\|\tilde{w}_{t+k}\| \leq \|I - \alpha_t B_{t+k}\| \|\tilde{w}_{t+k-1}\| + \alpha_t \lambda A \\
\leq W_t + k \alpha_t \lambda A, \quad (24)
\]

where the final inequality follows from recursing on \(\|\tilde{w}_{t+k-1}\|\) and using the assumption that \(B_t \geq 0\).

**Deviation bound for \(\|v_{t+k} - \tilde{v}_{t+k}\|\):** We can obtain a bound on this quantity as follows:

\[
\|v_{t+k} - \tilde{v}_{t+k}\| \leq \|v_{t+k} - v'_{t+k}\| + \|v'_{t+k} - \tilde{v}_{t+k}\| + \|\tilde{v}_{t+k} - \tilde{v}'_{t+k}\| \\
\leq 2\beta_{t+k} W_{t+k} + 2\beta_{t+k} \|\tilde{w}_{t+k}\| + \|\tilde{v}_{t+k} - \tilde{w}_{t+k}\| + \|v_{t+k-1} - \tilde{v}_{t+k-1}\| \\
\leq 2 \left( \sum_{i=1}^{k} \beta_{t+i} (W_{t+i} + \|\tilde{w}_{t+i}\|) \right) + \sum_{i=1}^{k} \beta_{t+i} \|w_{t+i} - \tilde{w}_{t+i}\| \\
\leq 3\beta k (2W_{t+k} + k \alpha_t \lambda A), \quad (25)
\]

where the final bound is obtained using \(\|w_{t+k}\| \leq W_{t+k}\) and \(\|\tilde{w}_{t+k}\| \leq W_{t+k} + k \alpha_t \lambda A\) from Equation (24).
Lemma 6. For any choice of \( k > \frac{\log(\frac{1}{\alpha})}{2\mu \alpha} \) and \( \alpha \in (0, 2/\lambda_B) \) of the form \( \alpha = \frac{c}{d^2 \beta + t} \), we have that

\[
\|E[\epsilon_{t+k}|\mathcal{F}_t]\|_2 \leq \left( \frac{\alpha_B W_{t+k}^0}{\mu} + \lambda_1(1 + 2W_{t+k}^0) + W_{t+k}^2 \right) \beta_t \\
+ \frac{\lambda_B W_{t+k}^0 k\alpha \beta}{c \mu \gamma} + \frac{\lambda_A k \alpha \beta}{c \mu} + \frac{3\lambda_B \beta k(2W_{t+k}^0 + k\alpha \lambda_A)}{\mu}
+ (2W_{t+k}^0 + k\alpha \lambda_A)W_{t+k}k\beta_t.
\]

In other words, we get that \( \|E[\epsilon_{t+k}|\mathcal{F}_t]\|_2 = O(\beta k^2 \alpha_t W_{t+k}). \)

Proof. In continuation from Lemma 5, we consider bounding the deviation of the process \( \tilde{w}_{t+k} \) from the process \( w_{t+k} \). The extra components in the error term \( \epsilon_t \) remain the same and we ignore them for clarity of this lemma.

\[
\|E[(w_{t+k} - \tilde{w}_{t+k})v_{t+k-1}^T | \mathcal{F}_t]\|_2 \leq \|E[(w_{t+k} - \tilde{w}_{t+k})v_{t+k-1}^T | \mathcal{F}_t]\|_2 + \|E[(w_{t+k} - \tilde{w}_{t+k})(v_{t+k-1} - v_t)^T | \mathcal{F}_t]\|_2 \quad (26)
\]

We proceed by first analyzing term (I) in Equation (26).

\[
\|E[(w_{t+k} - \tilde{w}_{t+k})v_{t+k}^T | \mathcal{F}_t]\|_2 = \|E[E[(\alpha_t - \alpha_{t+1})(\alpha_t - \alpha_{t+1})Bw_{t+i-1} | \mathcal{F}_t]v_{t}^T]\|_2 + \|E[E[\alpha_t (I - \alpha_t B)A(v_{t+i-1} - \hat{v}_{t+i-1})]v_{t}^T]\|_2 + \|E[\sum_{i=1}^{k} (\alpha_t - \alpha_{t+1})(I - \alpha_t B)A(v_{t+i-1} - \hat{v}_{t+i-1})]v_{t}^T]\|_2
\]

\[
\leq \frac{\alpha_B W_{t+k}^0 + \lambda_A k \alpha_t}{\alpha_t \mu} + \frac{\lambda_A k \alpha_t}{\alpha_t \mu} + \frac{3\lambda_B \beta k(2W_{t+k}^0 + k\alpha \lambda_A)}{\mu}
\leq \frac{\lambda_B W_{t+k}^0 k\alpha t}{c \mu (d \beta + t)} + \frac{\lambda_A k \alpha_t}{c \mu (d \beta + t)} + \frac{3\lambda_B \beta k(2W_{t+k}^0 + k\alpha \lambda_A)}{\mu}
\]

where the second last inequality follows using Jensen’s inequality along with a triangle inequality and using the fact that \( B \succeq \mu I \) and the last equality follows from using the form of \( \beta_t = \frac{b}{d^2 \beta + t} \) for some constant \( b \).

We now consider term (II) in Equation (26).

\[
\|E[(w_{t+k} - \tilde{w}_{t+k})(v_{t+k-1} - v_t)^T | \mathcal{F}_t]\|_2 \leq (2W_{t+k}^0 + k\alpha \lambda_A)W_{t+k}k\beta_t, \quad (28)
\]

by using Jensen’s inequality along with bound (20). Combining (27) and (28) with (26), and using Lemma 5, we obtain the desired result. \( \square \)
Note that in order to prove the final convergence for Algorithm 1, we use the form of the step sizes $\alpha_t$ and $\beta_t$ as mentioned in this section.

In the following sections we denote by $\nu_t = \frac{1}{2\mu\alpha_t (1 - \alpha B R^2)} \log^2 \left( \frac{1}{\mu \nu_t} \right)$ and $A_t$ to be such that:

$$A_t \nu_t \beta_t \geq \|E [c_{t+k}] F_t \|$$

(29)

When $\alpha_t = \frac{1}{\log(d^2(1 + 2 + R^2_t))}$, $\nu_t$ will be $O(\log^2 (1/\beta_t))$ and when $\alpha_t$ is constant, $\nu_t$ will be $O(\log^2 (1/\beta_t))$.

C Controlling Markov Chain $w_t$

For the purpose of this section, we stick with bounds $R_A, R_B$ the maximum of which equals $R$ in the main paper. In this section we provide a bound on the norm of the markov chain $w_t$. We start by showing some lemmas that bring out the behavior of $w_t$.

We first expand $w_{t+1} = (I - \alpha B_{t+1})w_t + \alpha A_{t+1}v_t$ and use the Minkowski inequality on $L_2$-norm (denoted by $\|\|_{L_2}$) to obtain:

$$\|w_{t+1}\|_{L_2} \leq \|(I - \alpha B_t)w_t\|_{L_2} + \|\alpha A_{t+1}v_t\|_{L_2}$$

We directly have that $\|\alpha A_{t+1}v_t\|_{L_2} \leq \alpha R^2_A$ almost surely and we can directly compute for $\alpha < 1/R^2_B$:

$$\|(I - \alpha B_{t+1})w_t\|_{L_2}^2 = E[w_t^T (I - \alpha B_{t+1})^2 w_t] = E[w_t^T (I - 2\alpha B_{t+1} + \alpha^2 B_{t+1}^2) w_t]$$

$$\leq E[w_t^T (I - \alpha B_{t+1}) w_t] \leq E[w_t^T (I - \alpha \text{E}[B_{t+1}|F_{\cup}]) w_t] \leq (1 - \alpha \mu) E[\|w_t\|_2^2]$$

where (1) follows as $B_{t+1} \leq R^2_B I$. We obtain expanding the recursion (and using $\sqrt{1 - x} \leq 1 - x/2$ for $x \geq 0$):

$$\|w_t\|_{L_2} \leq (1 - \alpha \mu/2)^t \|w_0\|_{L_2} + \alpha R^2_A \sum_{i=0}^{t-1} (1 - \alpha \mu/2)^i$$

We conclude

$$\|w_t\|_{L_2} \leq (1 - \alpha \mu/2)^t \|w_0\|_{L_2} + \alpha R^2_A \frac{\mu}{\mu}.$$

We consider now $p \geq 3$. We expand again $w_{t+1} = (I - \alpha B_{t+1})w_t + \alpha A_{t+1}v_t$ and use now the Minkowski inequality on $L_p$-norm on $(\mathbb{R}^d, \|\|_2)$ (denoted by $\|\|_{L_p}$ and defined by $\|x\|_{L_p} = (\text{E}[\|x\|_2^p])^{1/p}$) to obtain:

$$\|w_{t+1}\|_{L_p} \leq \|(I - \alpha B_t) w_t\|_{L_p} + \|\alpha A_{t+1}v_t\|_{L_p}$$
We then compute for $\alpha < 1/R_B^2$
\[ ||(I - \alpha B_{t+1})w_t||_{L_p}^p = \mathbb{E}[(w_t^T (I - \alpha B_{t+1})^2 w_t)^{p/2}] = \mathbb{E}[(w_t^T (I - 2\alpha B_{t+1} + \alpha^2 B_{t+1}^2) w_t)^{p/2}] \leq \mathbb{E}[(w_t^T (I - \alpha B_{t+1}) w_t)^{p/2}] \leq \mathbb{E}[||w_t||_2^p \left(1 - \alpha \frac{w_t^T B_{t+1} w_t}{||w_t||_2^2}\right)^{p/2}] \leq (1) \mathbb{E}[||w_t||_2^p \left(1 - \alpha \frac{w_t^T B_{t+1} w_t}{2||w_t||_2^2} + \frac{\alpha^2 (p-2)}{8} \left(\frac{w_t^T B_{t+1} w_t}{||w_t||_2^2}\right)^2\right)] \leq (2) \mathbb{E}[||w_t||_2^p \left(1 - \alpha \frac{w_t^T B_{t+1} w_t}{2||w_t||_2^2} + \frac{\alpha^2 R_B^2 (p-2)}{8} \left(\frac{w_t^T B_{t+1} w_t}{||w_t||_2^2}\right)^2\right)] \leq (3) \mathbb{E}[||w_t||_2^p \left(1 - \alpha \frac{w_t^T B_{t+1} w_t}{2||w_t||_2^2} + \frac{\alpha^2 R_B^2 (p-2)}{4} \mu\right)]
\]
where (1) follows as $(1-x)^p \leq (1-px+p(p-1)/2x^2)$ for $x \in [0, 1]$, (2) follows as $w_t^T B_{t+1} w_t \leq R_B^2 ||w_t||_2^2$ and (3) follows as $\mathbb{E}[B_{t+1} F_t] = B \succcurlyeq \mu I$. Then using $(1-x)^{1/p} \leq 1 - x/p$ for $x \geq 0$ yields
\[ \| (I - \alpha B_{t+1}) w_t \|_{L_p} \leq \| w_t \|_{L_p} \left(1 - \frac{\alpha}{2} (1 - \alpha R_B^2 \frac{p-2}{4} \mu)\right). \]
Moreover
\[ \| \alpha A_{t+1} w_t \|_{L_p} \leq \alpha R_A^2 \quad \text{a.s.} \]
And therefore
\[ \| w_{t+1} \|_{L_p} \leq \| w_t \|_{L_p} \left(1 - \frac{\alpha}{2} (1 - \alpha R_B^2 \frac{p-2}{4} \mu)\right) + \alpha R_A^2. \] (30)
Let us denote by $\delta = \frac{\alpha}{2} (1 - \alpha R_B^2 \frac{p-2}{4} \mu)$, then we directly obtain expanding the recursion:
\[ \| w_t \|_{L_p} \leq (1 - \delta)^t \| w_0 \|_{L_p} + \alpha R_A^2 \sum_{i=0}^{t-1} (1 - \delta)^i. \]
We conclude for $\alpha \leq \frac{2}{R_B^2 (p-2)}$
\[ \| w_t \|_{L_p} \leq (1 - \mu \alpha / 4)^t \| w_0 \|_{L_p} + \frac{4 R_A^2}{\mu}. \]
This concludes the proof.

As a corollary, we conclude that:

**Corollary 1.** If $p \geq 3$, $w_0$ is sampled from the unit sphere, and $\alpha$ satisfies $\alpha \leq \min\left(\frac{2}{R_B^2 (p-2)}, \frac{4}{\mu}\right)$ then:
\[ \mathbb{E} [||w_t||_2^p] \leq \left(1 + \frac{4 R_A^2}{\mu}\right)^p. \] (31)

We can leverage corollary 1 to obtain the following control on the norms of $w_t$. As a warm up first we show that polynomial control on the norms of $w$ is possible.
Lemma 8. Let $\eta > 0$ and $b > 0$. If:

$$p = \frac{1 + a}{b}, \quad c \geq \left(1 + \frac{4R^2}{\eta^{1/p}}\right) \left(\frac{1}{\sum_{j=1}^{\infty} \frac{1}{j^{1+a}}}\right)^{1/p}$$

(32)

Then whenever $\alpha \leq \min\left(\frac{2}{R_B^2(p-2)}\right)$, we have that with probability $1 - \eta$, $\|w_t\| \leq ct^b$ for all $t \leq n$.

Proof. By Corollary 1 and Markov’s inequality:

$$\Pr\left(\|w_t\|^p \geq c^p t^{bp}\right) \leq \frac{\mathbb{E}[\|w_t\|^p]}{c^{bp}} \leq \left(1 + \frac{4R^2/\mu}{c}\right) \frac{1}{t^{bp}} \leq \eta \left(\frac{1}{\sum_{j=1}^{\infty} \frac{1}{j^{1+a}}}\right) \frac{1}{t^{bp}}$$

The first inequality follows by Markov, the second by Corollary 1 and the third by the definition of $c$ and $p$. Applying the union bound to all $w_t$ from $t = 1$ to $\infty$ yields the desired result. \qed

The lemma above implies that for any probability level $\eta$, whenever the step size $\alpha_t$ is a small enough constant, independent of time $t$, by picking $\alpha$ small enough, we can show pointwise control on the norms of $\|w_t\|$ with constant probability so that at time $t$, $\|w_t\| \leq ct^b$.

Notice that for a fixed $a$, $\sum_{j=1}^{\infty} \frac{1}{j^{1+a}}$ converges, and that in case $a \geq 1$, $\sum_{j=1}^{\infty} \frac{1}{j^{1+a}} < 10$ (an absolute constant).

We now proceed to show that in fact for any $\delta > 0$, there is a constant $C(\delta, \mu, R_B, R_A, \log(d))$ such that with probability $1 - \delta$, $w_t \in B(\delta, \mu, R_B, R_A, \log(d))$ for all $t$ whenever the step size is $\alpha_t = \frac{c}{\log(d^2 \beta + t)}$ with $\beta \geq 0$.

We start with the following observation:

Lemma 9. Let $t_0 \in \mathbb{N}$ and $t_1 = 2t_0$. Assume $\|w_{t_0}\| \leq B$. Then for all $t_0 + k \in [t_0 + \frac{8 \log(B) \log(d^2 \beta + t_0)}{\mu c}, \ldots, t_1]$, the following holds:

$$\mathbb{E}[\|w_{t_0+k}\|^{c_1 \log(t_1)}] \leq (1 + \frac{8R^2_A}{\mu})^{c_1 \log(t_1)}$$

Where $\alpha_{t_0+k} = \frac{c}{\log(d^2 \beta + t_0+k)}$, $t_0 \geq 2$. And $c, c_1$ are positive constants such that $c \leq \frac{1}{R_B^2+1}$.

Proof. Mimicking the proof of Lemma 7, the same result of said Lemma holds up to Equation 30 even if the step size $\alpha_{t_0+m} = \frac{c}{\log(d^2 \beta + t_0+m)}$, therefore for any $m$:

$$\|w_{t_0+m+1}\|_{L_p} \leq \|w_{t_0+m}\|_{L_p} \left(1 - \frac{\alpha_{t_0+m}}{2} \left(1 - \frac{\alpha_{t_0+m} R_B^2 p - 2}{4}\right) \mu\right) + \alpha_{t_0+k} R_A^2$$

Let $\delta_{t_0+m} = \frac{\alpha_{t_0+m}}{2} \left(1 - \frac{\alpha_{t_0+m} R_B^2 p - 2}{4}\right) \mu$, we obtain the recursion:

$$\|w_{t_0+m+1}\|_{L_p} \leq \|w_{t_0+m}\|_{L_p} \left(1 - \delta_{t_0+m}\right) + \alpha_{t_0+m} R_A^2$$

Which for any $k$ can be expanded to:

$$\|w_{t_0+k}\|_{L_p} \leq \prod_{m=0}^{k-1} \left(1 - \delta_{t_0+m}\right) \|w_{t_0}\|_{L_p} + R_A^2 \sum_{m'=0}^{k-1} \alpha_{t_0+m'} \prod_{j=m'+1}^{k-1} \left(1 - \delta_{t_0+j}\right)$$

We now show that we can substitute all instances of $\delta_{t_0+k}$ in the upper bound with a fixed quantity, which will allow us to bound the whole expression afterwards.

Notice that $\alpha_{t_0+k}$ is decreasing and that $\delta_{t_0+k} \geq \frac{2R^2}{\mu} \left(1 - 2\alpha_t R_B^2 \frac{p - 2}{4}\right)$, the later follows because by assumption $\alpha_{t_0+k} = \frac{c}{\log(d^2 \beta + t_0+k)} \leq \frac{2}{\log(d^2 \beta + t_1)} = 2\alpha_t$ (recall that $t_1 = 2t_0$, implying this is true as long as $t_0 \geq 2$) and therefore $\alpha_t \leq \alpha_{t_0+k} \leq 2\alpha_t$.
Define $\delta_{t_1} := \frac{\alpha_1}{2} \left(1 - 2\alpha_1, R_B^2 \frac{p-2}{4}\right) \mu$. As a consequence:

$$\|w_{t_0+k}\|_{L^p} \leq \prod_{i=0}^{k-1} (1 - \delta_{t_1}') \|w_{t_0}\|_{L^p} + 2R_A^{k+1} \sum_{m'=0}^{k-1} (1 - \delta_{t_1})^{m'}$$

$$\leq \prod_{i=0}^{k-1} (1 - \delta_{t_1}') \|w_{t_0}\|_{L^p} + 2R_A^{k+1} \frac{1}{\delta_{t_1}}$$

$$= (1 - \delta_{t_1}')^k \|w_{t_0}\|_{L^p} + 2R_A^{k+1} \frac{1}{\delta_{t_1}}$$

If $\alpha_{t_1} < \frac{1}{R_B(p-2)}$, then $\delta_{t_1}' > \frac{\alpha_1}{4} \mu$. Then:

$$\|w_{t_0+k}\|_{L^p} \leq (1 - \mu_{\alpha_{t_1}/4})^k \|w_{t_0}\|_{L^p} + 8 \frac{R_A^2}{\mu}$$

And therefore:

$$\mathbb{E} [\|w_{t_0+k}\|^p] \leq \left((1 - \mu_{\alpha_{t_1}/4})^k \|w_{t_0}\|_{L^p} + 8 \frac{R_A^2}{\mu}\right)^p$$

Notice that $(1 - \mu_{\alpha_{t_1}/4})^k \leq e^{-\frac{\mu_{\alpha_{t_1}}}{2} k}$ and therefore $(1 - \mu_{\alpha_{t_1}/4})^k \|w_{t_0}\|_{L^p} \leq 1$ whenever $-\mu_{\alpha_{t_1}} k/4 + \log(B) \leq 0$. Since $2 \log(d^2 \beta + t_0) \geq \log(d^2 \beta + t_1)$ (because $t_0 \geq 2$), the relationship $(1 - \mu_{\alpha_{t_1}/4})^k \|w_{t_0}\|_{L^p} \leq 1$ holds (at least) whenever $k \geq \frac{\log(B) \log(d^2 \beta + t_0)}{\mu c}$.

Recall that $p = c_1 \log(t_1)$. Since the above conditions require $\alpha_{t_1} < \frac{1}{R_B(p-2)}$ to hold, it is enough to ensure that:

$$\alpha_{t_1} = \frac{c}{\log(d^2 \beta + t_1)} \leq \frac{1}{R_B^2 \mu} = \frac{R_A^2}{R_B^2 c_1 \log(t_1)} \leq \frac{1}{R_B^2 (p-2)} = \frac{1}{R_B \log(t_1) - 2}$$

It is enough to take $c \leq \frac{1}{R_B c_1}$ to satisfy the bound. Putting all these relationships together:

$$\mathbb{E}[\|w_{t_0+k}\|^p] \leq \left(1 + 8 \frac{R_A^2}{\mu}\right)^p$$

For $p = c_1 \log(t_1)$ and for all $k$ such that $k \in \left[\frac{\log(B) \log(d^2 \beta + t_0)}{\mu c}, \ldots, t_0\right]$. \[\square\]

As a consequence of Lemma 9, we have the following corollary:

**Corollary 2.** Let $t_0 \in \mathbb{N}$ and $t_1 = 2t_0$. Assume $\|w_{t_0}\| \leq B$. Then for all $t_0 + k \in \left[t_0 + \frac{\log(B) \log(d^2 \beta + t_0)}{\mu c}, \ldots, t_1\right]$, the following holds:

$$\mathbb{E} \left[\|w_{t_0+k}\|^{c_1 \log(t_0+k)}\right] \leq \left(1 + 8 \frac{R_A^2}{\mu}\right)^{c_1 \log(t_0+k)}$$

Where $\alpha_{t_0+k} = \frac{c}{\log(d^2 \beta + t_0+k)} \mu$, $t_0 \geq 2$. And $c, c_1$ are positive constants such that $c \leq \frac{1}{R_B c_1}$.

The proof of this result follows the exact same template as the proof of Lemma 9, the only difference is the substitution of $p$ with the desired $c_1 \log(t_0 + k)$ wherever necessary.

Now we proceed to show that having control up to the $c_1 \log(t)$ moments for $\|w_t\|$ implies boundedness of $w_t$ with high probability:

**Lemma 10.** Assume $\mathbb{E} \left[\|w_t\|^{c_1 \log(t)}\right] \leq (1 + 8 \frac{R_A^2}{\mu})^{c_1 \log(t)}$, and $\delta > 0$, then for $B \geq 2 \left(1 + 8 \frac{R_A^2}{\mu}\right)^{\frac{1}{\delta}}$, we have:

$$\Pr (\|w_t\| \geq B) \leq \frac{1}{t^{c_1 \log(t)}}$$

Where log is base 2.
Proof. The proof follows from a simple application of Markov’s inequality:
\[
\Pr (\|w_t\| \geq B) \leq \Pr \left( \|w_t\| \geq B \delta c, \log(t) \right) \leq \frac{1}{t^c} \delta c, \log(t)
\]
This concludes the proof. □

We now show that if there is \( t_0 \) for which \( \|w_t\| \leq B \), for some large enough constant \( B \), then by leveraging Lemmas 9 and 10 then we can say that with any constant probability a large chunk of the \( w_t \) are bounded provided \( \alpha \) is time dependent \( \alpha_t \) with \( \alpha_t = \frac{c}{\log(d^2 + t)} \) for some constant \( c \).

**Lemma 11.** Let \( \delta > 0 \), define \( \eta := \frac{\sum_{j=1}^{\infty} \frac{1}{\delta^j}}{\delta} \), and let the step size \( \alpha_t = \frac{c}{\log(d^2 + t)} \) with \( c > 0 \) satisfying \( c \leq \frac{1}{B R_B^3} \). Assume there exists \( t_0 \geq 2 \) such that \( \|w_{t_0}\| \leq B \) with \( B \geq 2 \left( 1 + \frac{8 R_B^2}{\mu} \right) \eta \). Define \( t_1 = 2 t_0 \) and \( t_{i+1} = 2 t_i \) for all \( i \geq 1 \). With probability \( 1 - \delta \) it holds that for all \( t \geq t_0 \) such that \( t \in [t_i + \frac{2 \log(B) \log(d^2 + t_i)}{\mu} 2 R_B^2, \ldots, t_{i+1}] \) it follows that:
\[
\|w_t\| \leq B
\]

**Proof.** The proof is a simple application of Lemmas 9 and 10. Indeed, by Lemma 9 and the assumptions on \( w_{t_0} \) and the step size, conditioning on the event that \( w_{t_0} \leq B \), the \( 2 \log(t_1) \) moments (and in fact the \( 2 \log(t) \) moments as well) of \( \|w_t\| \) for \( t \in [t_0 + \frac{2 \log(B) \log(d^2 + t_0)}{\mu} 2 R_B^2, \ldots, t_1] \) are bounded by \( (1 + \frac{8 R_B^2}{\mu})^{2 \log(t_1)} \) (respectively \( (1 + \frac{8 R_B^2}{\mu})^{2 \log(t)} \) for the \( 2 \log(t) \) moments). This in turn implies by Lemma 10, that conditional on \( \|w_{t_0}\| \leq B \), for any \( t \in [t_0 + \frac{2 \log(B) \log(d^2 + t_0)}{\mu} 2 R_B^2, \ldots, t_1] \) the probability that \( \|w_t\| \) is larger than \( B \) is upper bounded by \( \frac{1}{t^c \eta^2} \leq \frac{1}{t^c \sum_{j=1}^{\infty} \delta^j} \) (this inequality follows because \( \eta \geq 1 \) and \( 2 \log(t) \geq 1 \) as well). Consequently, the probability that any \( \|w_t\| > B \) for \( t \in [t_0 + \frac{2 \log(B) \log(d^2 + t_0)}{\mu} 2 R_B^2, \ldots, t_1] \) can be bounded by the union bound as:
\[
\sum_{j=1}^{\infty} \frac{\delta}{\sum_{j=1}^{\infty} \frac{1}{\delta^j}} \sum_{i=0}^{\infty} \frac{1}{t^c} \leq \frac{\delta}{\sum_{j=1}^{\infty} \frac{1}{\delta^j}} \sum_{i=0}^{\infty} \frac{1}{t^c} \leq \frac{\delta}{\sum_{j=1}^{\infty} \frac{1}{\delta^j}} \sum_{i=0}^{\infty} \frac{1}{t^c}
\]

Conditioning on \( \|w_{t_1}\| \leq B \) and repeating the argument, for all \( i \), we obtain that the probability that there is any \( t \) such that \( \|w_t\| > B \) and \( t \in [t_i + \frac{2 \log(B) \log(d^2 + t_i)}{\mu} 2 R_B^2, \ldots, t_{i+1}] \) is at most:
\[
\sum_{j=1}^{\infty} \frac{\delta}{\sum_{j=1}^{\infty} \frac{1}{\delta^j}} \sum_{i=0}^{\infty} \frac{1}{t^c} \leq \frac{\delta}{\sum_{j=1}^{\infty} \frac{1}{\delta^j}} \sum_{i=0}^{\infty} \frac{1}{t^c}
\]

This concludes the proof. □

Now we show that in fact, for any \( \delta \in (0, 1) \), then, with probability \( 1 - \delta \), for all \( t \), all \( w_t \) are bounded (by a quantity that depends inversely on \( \delta \)). More formally:

**Lemma 12.** Define \( R_A \) and \( R_B \) such that \( R_A = R_B \geq \frac{1}{2} \). Let
\[
B = \max \left( 1 + \frac{1}{R_B}, (1 + \frac{8 R_A^2}{\mu}) \frac{\sum_{j=1}^{\infty} \frac{1}{\delta^j}}{\delta}, 2, (5 + 72 \frac{\log(1 + d^2 \beta)}{\mu^2} R_B^3)^2 \right),
\]
If \( \alpha_t = \frac{c}{\log(d^2 + t)} \) with \( c = \frac{1}{2 R_B^3} \) and \( \|w_0\| = 1 \), then with probability \( 1 - \delta \) for all \( t \):
\[
\|w_t\| \leq B + \frac{2 \log(B) R_B}{\mu} := C(\delta, \mu, R_B, R_A, \log(d))
\]

22
Proof. Let \( t_0 = \max \left( \frac{4}{3} * \frac{4 \log(1+d^2 \beta) \log(B) R_B^2}{\mu}, 2 \right) \). Define \( t_1 = 2t_0 \) and in general for all \( i \geq 1, t_i = 2t_{i-1} \).

- We start by showing that \( t_0 \geq 4 \frac{\log(B) \log(d^2 + t_0) R_B^2}{\mu} \), which will allow us to show that the interval \([t_0 + 4 \frac{\log(B) \log(d^2 + t_0) R_B^2}{\mu}, \cdots, t_1]\) is nonempty.

First notice that for all \( t \geq 1 \), (and in particular for all \( t \geq 2 \)), we have that:

\[
\frac{t}{\log_2(t)} \geq \frac{3}{4} t^{\frac{1}{2}}
\]

Therefore:

\[
\frac{t_0}{\log(t_0)} \geq \frac{3}{4} t_0^{1/2} \geq \max \left( \frac{4 \log(1+d^2 \beta) \log(B) R_B^2}{\mu}, 1 \right) \geq 4 \frac{\log(1+d^2 \beta) \log(B) R_B^2}{\mu}
\]

And therefore, since \( \log(t_0) \log(1+d^2 \beta) \geq \log(d^2 + t_0) \):

\[
t_0 \geq 4 \frac{\log(t_0) \log(1+d^2 \beta) \log(B) R_B^2}{\mu} \geq 4 \frac{\log(d^2 + t_0) \log(B) R_B^2}{\mu}
\]

Which implies the desired inequality.

- Now we see that \( \|w_t\| \leq B \) for all \( t \leq t_0 \).

We use a very rough bound on \( w_t \). Recall that \( w_t = (I - \alpha_{t-1}B_t)w_{t-1} + \alpha_{t-1}A_t v_t \). The following sequence of inequalities holds:

\[
\|w_t\| \leq \|I - \alpha_{t-1}B_t\| \|w_{t-1}\| + \frac{1}{2R_B^2} \|A_t\|
\]

\[
\leq \|w_{t-1}\| + \frac{1}{2R_B^2}
\]

This holds as long as \( \|I - \alpha_{t-1}B_t\| \leq 1 \), which is true since by assumption \( B_t \succeq 0 \) for all \( t \) and therefore \( \|\alpha_{t-1}B_t\| \leq \frac{1}{2R_B} \). The last inequality follows because \( R_B \geq \frac{1}{2} \).

Consequently, \( \|w_t\| \leq 1 + \frac{t_0}{2R_B} \) for \( t \leq t_0 \). We want to ensure \( B \geq 1 + \frac{t_0}{2R_B} \). Notice that:

\[
1 + \frac{t_0}{2R_B} = 1 + \max \left( \frac{4 \frac{\log(1+d^2 \beta) \log(B) R_B^2}{\mu}}{2R_B^2}, 1 \right)
\]

If \( t_0 = 1 \), this provides the condition \( B \geq 1 + \frac{1}{R_B} \). When the max defining \( t_0 \) is achieved at \( \left( \frac{4 \frac{\log(1+d^2 \beta) \log(B) R_B^2}{\mu}}{2R_B^2} \right)^2 \), we obtain the condition:

\[
1 + \left( \frac{4^4}{2 \cdot 3^2} \frac{\log^2(1+d^2 \beta) R_B^2}{\mu^2} \right) \leq B
\]

(33)

Since we already have \( B \geq 2 \), it follows that \( \log(B) \geq 1 \). And therefore, Equation 33 is satisfied as long as:

\[
\log^2(B) \left( 1 + \left( \frac{4^4}{2 \cdot 3^2} \frac{\log^2(1+d^2 \beta) R_B^2}{\mu^2} \right) \right) \leq B
\]

Notice that for all \( x \geq 1 \):

\[
\frac{x}{\log^2(x)} \geq \frac{1}{5} x^{1/2}
\]

Therefore, picking \( B \geq (5 + 72 \cdot \frac{\log^2(1+d^2 \beta) R_B^2}{\mu^2})^2 \geq (5 + 5 \cdot \frac{4^4}{2 \cdot 3^2} \cdot \frac{\log^2(1+d^2 \beta) R_B^2}{\mu^2})^2 \) guarantees that Equation 33 is satisfied, (since \( B \) is also greater than \( 1 \)).
• We can therefore invoke Lemma 11 to the sequence \( \{t_i\} \) and conclude that with probability 1 − δ for all \( t \) such that \( t \in [t_i + \frac{4 \log(B) \log(d^2 \beta + t_i)}{\mu}, \ldots, t_{i+1}] \) for some \( i \), we have \( \|w_t\| \leq B \) simultaneously for all such \( t \). This uses the fact that \( B \geq \left( 1 + \frac{8 R_B^2}{\mu} \right) \sum_{j=1}^\infty \frac{1}{\sigma_j^2} \).

• The final step is to show a bound on \( w_t \) for the remaining blocks.

For the remaining blocks notice that if \( \|w_{t_i}\| \leq B \), then by a crude bound since \( c_t = \frac{e}{\log(d^2 \beta + t_i)} \), with \( e = \frac{1}{2R_B^2} \), at each step starting from \( t_i \), \( w_t \) grows by at most an additive \( \frac{1}{2R_B \log(d^2 \beta + t_i)} \) factor:

\[
\|w_t\| \leq \|I - c_{t-1}B_t\|\|w_{t-1}\| + \frac{1}{2R_B^2 \log(d^2 \beta + t_i)} \|A_t\|
\leq \|w_{t-1}\| + \frac{1}{2R_B \log(d^2 \beta + t_i)}
\]

For all \( t \in [t_i + 1, \ldots, t_{i+1} + \frac{2 \log(B) \log(d^2 \beta + t_i)}{\mu} \|B\|] \).

Since \( \|w_{t_i}\| \leq B \), we have that \( \|w_t\| \leq B + \frac{2 \log(B) \|B\|}{\mu} \) for all \( t \in [t_i + 1, \ldots, t_{i+1} + \frac{2 \log(B) \log(d^2 \beta + t_i)}{\mu} \|B\|] \).

As desired.

\[ \square \]

**Notation for following sections:** Throughout the following sections we use the following notation:

We use the assumption that \( \|\mathbb{E}[\epsilon_t|\mathcal{F}_{t-r_t}]\| \leq A_t r_t \beta_t \) as proved in Section B where \( r_t \) is the mixing time window at time \( t \).

Also, as proved in Section C, we have that \( \|w_{t}\| \leq W_t \) and consequently:

\[
\|\epsilon_t\| \leq \|w_t - B^{-1}Av_{t-1}\| \leq W_t + \|B^{-1}A\| := B_{\epsilon_t}
\]

Additionally we also have that:

\[
\|G_t\| \leq \lambda_1 + B_{\epsilon_t} := G_t
\]

Notice that \( B_{\epsilon_t} \) and \( G_t \) are of the same order.

## D Analysis burn in times

In order to provide a convergence analysis for Algorithm 1, we use Lemma 3 and bound each of the terms appearing in it. To obtain those bounds, we use a mixing time argument that allows us to bound the expected error accumulated by terms of the form \( \beta_t \) \( \epsilon_t H_{t-1} H_{t-1}^\top + H_{t-1} H_{t-1}^\top \epsilon_t \).

To control terms of this kind we deal with the set \( \{t\} \) such that \( t \geq r_t \) and the set of \( \{t\} \) such that \( t < r_t \) differently. Let \( t_0 = \max t \) such that \( t < r_t \). This value \( t_0 \) is finite because \( r_t \) grows polylogarithmically.

Recall that \( r_t = O(\log^3(\frac{1}{\beta_t})) \) where \( \beta_t = \frac{b}{d^2 \beta + 1} \). We define \( r_t := \log^3(\frac{1}{\beta_t}) C_r \). Where \( C_r \) is a constant capturing all the missing dependencies between \( r_t \) and \( A, B \). Let’s start with an auxiliary lemma:

**Lemma 13.** Let \( c > 0 \) be some constant. If \( x \geq 6lc \) then, \( x^{\frac{3}{4}} \geq \log(cx) \).

**Proof.** Observe that \( x^{\frac{3}{4}} \geq \log(cx) \) iff \( \exp(x^{\frac{3}{4}}) \geq cx \). Let’s write the left hand side using its taylor series:

\[
\exp(x^{\frac{3}{4}}) = \sum_{i=0}^{\infty} \frac{x^{\frac{3}{4}}}{i!}
\]

Notice that \( \sum_{i=0}^{\infty} \frac{x^{\frac{3}{4}}}{i!} \geq \frac{x^2}{2!} \), which in turn implies that if \( \frac{x^2}{6!} \geq cx \) and therefore \( x \geq 6!c \), then \( \exp(x^{\frac{3}{4}}) \geq cx \), as desired. \( \square \)
We provide an upper bound for $t_0$:

\textbf{Lemma 14.} The breakpoint $t_0$ satisfies:

$$ t_0 = \max(B_1(b, C_r), C_r (\log(d^2 \beta) + \log(b) - 1)^3) $$

Where $B_1(b, C_r) := 1440 \frac{c^2}{b}$ is a constant dependent only on $b$ and $C_r$.

\textbf{Proof.} We would like to show $t_0$ satisfies the property that for all $t \geq t_0$, it follows that $t \geq C_r \log^3(\frac{1}{3t})$. This is true iff $t^\frac{1}{3} - C_r^\frac{1}{3} \log(\frac{1}{3t}) \geq 0$. The following sequence of equalities holds:

$$ t^\frac{1}{3} - C_r^\frac{1}{3} \log(\frac{1}{3t}) = t^\frac{1}{3} - C_r^\frac{1}{3} \log(d^2 \beta + t) + \log(b)C_r^\frac{1}{3} $$

$$ = t^\frac{1}{3} - C_r^\frac{1}{3} \log(\frac{d^2 \beta + t}{t}) - C_r^\frac{1}{3} \log(t) + \log(b)C_r^\frac{1}{3} $$

$$ = t^\frac{1}{3} - C_r^\frac{1}{3} \log(\frac{d^2 \beta}{t}) + 1 - C_r^\frac{1}{3} \log(t) + \log(b)C_r^\frac{1}{3} $$

We now massage this expression by considering two cases and making use of the following inequality: For $\log(1 + x) \leq \log(x) + 1$ if $x \geq 1$

**Case 1** : $t \geq d^2 \beta$

This implies that $\log(\frac{d^2 \beta}{t} + 1) \leq \log(1 + 1) = 1$. The following inequalities hold:

$$ t^\frac{1}{3} - C_r^\frac{1}{3} \log(\frac{d^2 \beta}{t} + 1) - C_r^\frac{1}{3} \log(t) + \log(b)C_r^\frac{1}{3} \geq t^\frac{1}{3} - C_r^\frac{1}{3} - C_r^\frac{1}{3} \log(t) + \log(b)C_r^\frac{1}{3} $$

$$ = t^\frac{1}{3} - C_r^\frac{1}{3} \left(1 - \log(b) + \log(t)\right) $$

$$ = t^\frac{1}{3} - C_r^\frac{1}{3} \left(\log\left(\frac{2}{b}\right) + \log(t)\right) $$

$$ = t^\frac{1}{3} - C_r^\frac{1}{3} \left(\log\left(\frac{2b}{t}\right)\right) $$

Let $t = C_r h$. Substituting into the previous equation, we would like to find a condition for $h$ such that $t^\frac{1}{3} - C_r^\frac{1}{3} \left(\log\left(\frac{2b}{t}\right)\right) = C_r^\frac{1}{3} \left(h^\frac{1}{3} - \log\left(\frac{2C_r b}{h}\right)\right) \geq 0$. This follows as long as $h \geq \frac{6! 2C_r b}{b} = 1440 \frac{c^2}{b}$, by Lemma 13. Let $B_1(b, C_r) = 1440 \frac{c^2}{b}$.

We conclude that as long as we have $t \geq B_1(b, C_r)$ for some constant $B_1(b, C_r)$ depending on $\gamma$ and $C_r$, we can guarantee that $t^\frac{1}{3} - C_r^\frac{1}{3} \left(\log\left(\frac{2b}{t}\right)\right) \geq 0$.

**Case 2** : $t < d^2 \beta$

This implies that $\log(\frac{d^2 \beta}{t} + 1) \leq \log(\frac{d^2 \beta}{t}) + 1$. The following inequalities hold:

$$ t^\frac{1}{3} - C_r^\frac{1}{3} \log(\frac{d^2 \beta}{t} + 1) - C_r^\frac{1}{3} \log(t) + \log(b)C_r^\frac{1}{3} \geq t^\frac{1}{3} - C_r^\frac{1}{3} \log\left(\frac{d^2 \beta}{t}\right) - C_r^\frac{1}{3} \log(t) + \log(b)C_r^\frac{1}{3} $$

$$ = t^\frac{1}{3} - C_r^\frac{1}{3} \log\left(\frac{d^2 \beta}{t}\right) - C_r^\frac{1}{3} \log(t) + \log(b)C_r^\frac{1}{3} $$

And therefore the last expression is greater than zero if $t \geq C_r \log(d^2 \beta) + \log(b) - 1)^3 \geq 0$. As a consequence we get that as long as $t \geq t_0 = \max(B_1(b, C_r), C_r (\log(d^2 \beta) + \log(b) - 1)^3)$ we have that $t \geq C_r \log\left(\frac{1}{3t}\right)$ as desired. \qed

Throughout the next sections, we use $t_0$ to denote this breakpoint.
E  Analysis for Gen-Oja

In this section, we provide bounds on expectations of various terms appearing in Lemma 3 which are required to obtain a convergence bound for Gen-Oja.

E.1 Upper Bound on Operator Norm of \( \mathbb{E} \left[ H_t H_t^T \right] \)

We start by showing an upper bound for \( \| \mathbb{E} \left[ H_t H_t^T \right] \| \).

Lemma 15. For all \( t \geq 0 \):

\[
\| \mathbb{E} \left[ H_t H_t^T \right] \| \leq \exp \left( 2 \sum_{i=1}^{t} \beta_i \lambda_1 + \beta_i^2 \text{d}r_i \right) \left( A_i + B_{t, i}^2 G_i + B_{t, i}^2 \right) C^{(1)} + \sum_{j=1}^{f_0} \beta_j 2dB_{e_j}
\]

Where \( C^{(1)} \) is a constant. Assuming that for all \( t \geq 0 \), \( \beta t G_t < \frac{1}{t} \).

Proof. We start by substituting the identity: \( H_t = (I + \beta t G_t)H_{t-1} = (I + \beta t B^{-1} A + \beta t \epsilon_t)H_{t-1} \)
into the expectation:

\[
\mathbb{E} \left[ H_t H_t^T \right] = \mathbb{E} \left[ (I + \beta t B^{-1} A + \beta t \epsilon_t)H_{t-1} H_{t-1}^T (I + \beta t B^{-1} A + \beta t \epsilon_t)^T \right] \\
= (I + \beta t B^{-1} A) \mathbb{E} \left[ H_{t-1} H_{t-1}^T \right] (I + \beta t B^{-1} A)^T + \beta t \mathbb{E} \left[ \epsilon_t H_{t-1} H_{t-1}^T + H_{t-1} H_{t-1}^T \epsilon_t \right] + \beta t^2 \mathbb{E} \left[ \epsilon_t H_{t-1} H_{t-1}^T + H_{t-1} H_{t-1}^T \epsilon_t \right]
\]

If we assume to have a series of upper bounds \( \theta_1 \leq \cdots \leq \theta_{t-1} \) such that:

\[
\mathbb{E} \left[ B_u B_u^T \right] \leq \theta_u I 
\]

The following inequality holds:

\[
(I + \beta t B^{-1} A) \mathbb{E} \left[ H_{t-1} H_{t-1}^T \right] (I + \beta t B^{-1} A)^T \leq \theta_{t-1} (I + \beta t B^{-1} A)(I + \beta t B^{-1} A)^T
\]

Furthermore, we show how that \( (I + \beta t B^{-1} A)(I + \beta t B^{-1} A)^T \leq (1 + \beta t \lambda_1)^2 I \):

Indeed, let \( v \) be an eigenvector of \( B^{-\frac{1}{2}} AB^{-\frac{1}{2}} \) with eigenvalue \( \lambda \) and denote \( \tilde{v} = B^{\frac{1}{2}} v \). We show that \( \tilde{v} \) is an eigenvector of \( (I + \beta t B^{-1} A)(I + \beta t B^{-1} A)^T \) with eigenvalue \( (1 + \beta t \lambda)^2 \):

\[
\tilde{v}^T (I + \beta t B^{-1} A)(I + \beta t B^{-1} A)^T \tilde{v} = v^T (B^{\frac{1}{2}} + \beta t B^{-\frac{1}{2}} A)(B^{\frac{1}{2}} + \beta t B^{-\frac{1}{2}} A)^T v \\
= v^T (B^{\frac{1}{2}} + \beta t B^{-\frac{1}{2}} A)B^{-\frac{1}{2}} B^{\frac{1}{2}} B^{-\frac{1}{2}} B^{\frac{1}{2}} (B^{\frac{1}{2}} + \beta t B^{-\frac{1}{2}} A)^T v \\
= v^T (I + \beta t B^{-\frac{1}{2}} AB^{-\frac{1}{2}})B(I + \beta t B^{-\frac{1}{2}} AB^{-\frac{1}{2}})^T v \\
= (1 + \beta t \lambda)^2 v^T B v \\
= (1 + \beta t \lambda)^2 \tilde{v}^T \tilde{v}
\]

As a consequence, we conclude the set of eigenvalues of \( (I + \beta t B^{-1} A)(I + \beta t B^{-1} A)^T \) equals \( \{(1 + \beta t \lambda)^2\}_{i=1}^{d} \), since the set of eigenvalues of \( B^{-\frac{1}{2}} AB^{-\frac{1}{2}} \) equals \( \{\lambda_i\}_{i=1}^{d} \), the set of eigenvalues of \( B^{-1} A \). Therefore we conclude that

\[
(I + \beta t B^{-1} A)(I + \beta t B^{-1} A)^T \leq (1 + \beta t \lambda_1)^2 I
\]
We proceed to bound the remaining terms.

\[
\mathbb{E} \left[ \epsilon_t H_{t-1} H_{t-1}^\top \epsilon_t^\top \right] \leq \mathbb{E} \left[ \| \epsilon_t \| \| H_{t-1} H_{t-1}^\top \| \| \epsilon_t^\top \| \right]
\]

\[
\leq B_t^2 \mathbb{E} \left[ \| H_{t-1} H_{t-1}^\top \| \right]
\]

\[
\leq B_t^2 \mathbb{E} \left[ \text{Tr}(H_{t-1} H_{t-1}^\top) \right]
\]

\[
\leq d B_t^2 \| \mathbb{E} \left[ H_{t-1} H_{t-1}^\top \right] \|
\]

\[
\leq d B_t^2 \theta_{t-1}
\]

(38)

The first step is a consequence of Cauchy Schwartz, the second step because of the uniform boundedness of \( \epsilon_t \) and the last step is true because \( H_{t-1} H_{t-1}^\top \) is a positive semidefinite matrix.

**Terms with a single \( \epsilon_t \):** Let \( H_{t-1} = \prod_{j=t-r_t+1}^{t-1} (I + \beta_j G_j) H_{t-r_t} \).

Define \( H_{t-1}^{t-r_t+1} := \prod_{j=t-r_t+1}^{t-1} (I + \beta_j G_j) \) and \( L_{t-1}^{t-r_t+1} := H_{t-1}^{t-r_t+1} - I \).

In order to control this term we start by bounding \( \| L_{t-1}^{t-r_t+1} \| \). For this we use a crude bound.

\[
L_{t-1}^{t-r_t-1} = \sum_{k=1}^{r_t} \left( \sum_{i_1 \cdots i_k \in \{t-r_t-1, \ldots, t-1\}} \left[ \prod_{j=1}^{k} \beta_{i_j} G_{i_j} \right] \right)
\]

(39)

For any \( k \in \{1, \ldots, r_t\} \):

\[
\left\| \sum_{i_1 \cdots i_k \in \{t-r_t-1, \ldots, t-1\}} \left[ \prod_{j=1}^{k} \beta_{i_j} G_{i_j} \right] \right\| \leq \sum_{i_1 \cdots i_k \in \{t-r_t-1, \ldots, t-1\}} \left[ \prod_{j=1}^{k} \| \beta_{i_j} G_{i_j} \| \right]
\]

\[
\leq \sum_{i_1 \cdots i_k \in \{t-r_t-1, \ldots, t-1\}} \left[ r_t G_{i} \beta_{t-r_t} \right]^k
\]

The first follows from the triangle inequality, the second because of the uniform boundedness assumptions at the beginning of the section and the third because \( \left( \frac{r_t}{k} \right) \leq r_t^k \).

For all \( t \geq 0 \), since the step size condition holds:

\[
[r_t G_{t} \beta_{t-r_t}]^k \leq [2 r_t G_{t} \beta_{t}]^k \leq 2 r_t G_{t} \beta_{t} \leq \frac{1}{2}
\]

Putting these rough bounds together we conclude that:

\[
\| L_{t-1}^{t-r_t-1} \| \leq \sum_{k=1}^{r_t} [2 r_t G_{t} \beta_{t}]^k = \left[ 2 r_t G_{t} \beta_{t} \right] \frac{1 - [2 r_t G_{t} \beta_{t}]^k}{1 - [2 r_t G_{t} \beta_{t}]} \leq 2 \left[ 2 r_t G_{t} \beta_{t} \right] = 4 r_t G_{t} \beta_{t},
\]

(40)

where we have used that \( 1/(1-x) \leq 2x \) for \( x \in [0, 1/2] \). We can write \( H_t = (I+L_{t-1}^{t-r_t+1}) H_{t-r_t} = H_{t-r_t} + L_{t-1}^{t-r_t+1} H_{t-r_t} \). Substituting this equation into \( \mathbb{E} \left[ \epsilon_t H_{t-1} H_{t-1}^\top + H_{t-1} H_{t-1}^\top \epsilon_t \right] \) gives:

\[
\mathbb{E} \left[ \epsilon_t H_{t-1} H_{t-1}^\top + H_{t-1} H_{t-1}^\top \epsilon_t \right] = \mathbb{E} \left[ \epsilon_t (H_{t-r_t} + L_{t-1}^{t-r_t+1} H_{t-r_t}) (H_{t-r_t} + L_{t-1}^{t-r_t+1} H_{t-r_t})^\top \right]
\]

\[
+ \mathbb{E} \left[ (H_{t-r_t} + L_{t-1}^{t-r_t+1} H_{t-r_t}) (H_{t-r_t} + L_{t-1}^{t-r_t+1} H_{t-r_t})^\top \epsilon_t \right]
\]

\[
= \mathbb{E} \left[ \epsilon_t H_{t-r_t} H_{t-r_t}^\top + \mathbb{E} \left[ \epsilon_t H_{t-r_t} H_{t-r_t}^\top (L_{t-1}^{t-r_t+1})^\top \right] \right]
\]

\[
+ \mathbb{E} \left[ \epsilon_t L_{t-1}^{t-r_t+1} H_{t-r_t} H_{t-r_t}^\top \right] + \mathbb{E} \left[ \epsilon_t L_{t-1}^{t-r_t+1} H_{t-r_t} H_{t-r_t} (L_{t-1}^{t-r_t+1})^\top \right]
\]

\[
+ \mathbb{E} \left[ H_{t-r_t} H_{t-r_t}^\top \epsilon_t \right] + \mathbb{E} \left[ H_{t-r_t} H_{t-r_t}^\top (L_{t-1}^{t-r_t+1})^\top \epsilon_t \right]
\]

\[
+ \mathbb{E} \left[ L_{t-1}^{t-r_t+1} H_{t-r_t} H_{t-r_t}^\top \right] + \mathbb{E} \left[ L_{t-1}^{t-r_t+1} H_{t-r_t} H_{t-r_t} (L_{t-1}^{t-r_t+1})^\top \epsilon_t \right]
\]
We focus first on bounding the terms of this expansion containing $L_{t-1}^{t-r_t+1}$. We analyze the term

$$
\mathbb{E} \left[ \varepsilon_t L_{t-1}^{t-r_t+1} H_{t-r_t} H_{t-r_t}^\top \right].
$$

$$
\| \mathbb{E} \left[ \varepsilon_t L_{t-1}^{t-r_t+1} H_{t-r_t} H_{t-r_t}^\top \right] \| \leq \mathbb{E} \left[ \| \varepsilon_t \| \| L_{t-1}^{t-r_t+1} \| \| H_{t-r_t} H_{t-r_t}^\top \| \right]
$$

\[
\leq B_{\varepsilon t} 4 r_t G_t \beta_t \mathbb{E} \left[ \| H_{t-r_t} H_{t-r_t}^\top \| \right]
\]

\[
\leq B_{\varepsilon t} 4 r_t G_t \beta_t \mathbb{E} [\text{Tr}(H_{t-r_t} H_{t-r_t}^\top)]
\]

\[
\leq B_{\varepsilon t} 4 r_t G_t \beta_t d ||E[H_{t-r_t} H_{t-r_t}]||
\]

All other terms containing $L_{t-1}^{t-r_t+1}$ can be bounded in the same way. Combining these terms, we obtain the following bound for the sum of all these terms:

$$
\mathbb{E} \left[ \varepsilon_t H_{t-1} H_{t-1}^\top + H_{t-1} H_{t-1}^\top \varepsilon_t^\top \right] \leq B_{\varepsilon t} 4 G_t d r_t \beta_t (4 + 8 G_t r_t \beta_t)
$$

$$
= 16 B_{\varepsilon t} G_t d r_t \beta_t + 32 d B_{\varepsilon t} G_t^2 r_t \beta_t^2
\]

$$
\leq 8 d B_{\varepsilon t} G_t \beta_t + 16 B_{\varepsilon t} G_t d \beta_t r_t
\]

$$
= 8 d B_{\varepsilon t} G_t \beta_t (2 r_t + 1)
$$

The last inequality holds because of the step size condition. It remains to bound the terms $E \left[ \varepsilon_t H_{t-r_t} H_{t-r_t}^\top \right]$ and $E \left[ H_{t-r_t} H_{t-r_t}^\top \varepsilon_t^\top \right].$

By assumption, we know $\|\mathbb{E} [\varepsilon_t | \mathcal{F}_{t-r_t}]\| \leq A_t \beta_t r_t$ and therefore:

$$
\mathbb{E} \left[ \varepsilon_t H_{t-r_t} H_{t-r_t}^\top \right] \leq A_t r_t \beta_t \mathbb{E} \left[ \| H_{t-r_t} H_{t-r_t}^\top \| \right]
$$

\[
\leq A_t r_t \beta_t \mathbb{E} [\text{Tr}(H_{t-r_t} H_{t-r_t}^\top)]
\]

\[
\leq d \cdot A_t r_t \beta_t \mathbb{E} \left[ \| H_{t-r_t} H_{t-r_t}^\top \| \right]
\]

\[
\leq d \cdot A_t r_t \beta_t \theta_{t-r_t}
\]

\[
\leq d \cdot A_t r_t \beta_t \theta_{t-1}
\]

Combining the last bounds we get that whenever $t > t_0$:

$$
\mathbb{E} \left[ \varepsilon_t H_{t-1} H_{t-1}^\top + H_{t-1} H_{t-1}^\top \varepsilon_t^\top \right] \leq (8 d B_{\varepsilon t} G_t \beta_t (2 r_t + 1) + d A_t r_t \beta_t) \theta_{t-1}
$$

(41)

Also, whenever $t \leq t_0$, we have that:

$$
\mathbb{E} \left[ \varepsilon_t H_{t-1} H_{t-1}^\top + H_{t-1} H_{t-1}^\top \varepsilon_t^\top \right] \leq 2 d B_{\varepsilon t} \theta_{t-1}
$$

(42)

Combining the bound of equation 41 with equations 36, 38, and 42 yields for $t > t_0$

$$
\| \mathbb{E} \left[ H_t H_t^\top \right] \| \leq \theta_{t-1} \| (I + \beta_t B^{-1} A)(I + \beta_t B^{-1} A)^\top \| + \theta_{t-1} \beta_t^2 (8 d B_{\varepsilon t} G_t (2 r_t + 1) + d A_t r_t)
$$

$$
+ \theta_{t-1} \beta_t^2 d B_{\varepsilon t}^2
\]

\[
\leq \theta_{t-1} \left( 1 + 2 \beta_t \lambda_1 + \beta_t^2 \| \Lambda_1 + d B_{\varepsilon t}^2 \| + \beta_t^2 (8 d B_{\varepsilon t} G_t (2 r_t + 1) + d A_t r_t) \right)
\]

where $\Lambda_1 = \lambda_1^2$. This gives us a recursion of the form:

$$
\theta_t = \theta_{t-1} \left( 1 + 2 \beta_t \lambda_1 + \beta_t^2 d r_t (A_t + B_{\varepsilon t} G_t) \mathcal{C}^{(1)} \right)
$$

(43)
where $C^{(1)}$ is the smallest constant depending on $\Lambda_1$ such that:

$$dr_t(A_t + B_{c_t}G_t + B_{c_t}^2)C^{(1)} \geq \Lambda_1 + dB_{c_t}^2 + 8dB_{c_t}G_t(2r_t + 1) + dA_tr_t$$ (44)

Similarly, whenever $t \leq t_0$, we have that

$$\|E[H_tH_t^\top]\| \leq \theta_{t-1}\|(I + \beta_tB^{-1}A)(I + \beta_tB^{-1}A)^\top\| + \theta_{t-1}\beta_t * 2dB_{c_t}$$

$$+ \theta_{t-1}\beta_t^2dB_{c_t}^2$$

$$\leq \theta_{t-1}\left(1 + 2\beta_t\lambda_1 + \beta_t^2(A_1 + dB_{c_t}^2) + \beta_t * 2dB_{c_t}\right)$$

$$\leq \theta_{t-1}\left(1 + 2\beta_t\lambda_1 + \beta_t^2dr_t(A_t + B_{c_t}G_t + B_{c_t}^2)c^{(1)} + \beta_t * 2dB_{c_t}\right)$$

Using the inequality $(1 + x) \leq \exp(x)$ for $x \geq 0$, and noting that $\theta_0 = 1$ we obtain the desired result:

$$\theta_t \leq \exp\left(\sum_{i=1}^{t} 2\beta_i\lambda_1 + \beta_i^2dr_i(A_i + B_{c_i}G_i + B_{c_i}^2)c^{(1)} + \sum_{j=1}^{t_0} \beta_j 2dB_{c_j}\right)$$

\[\square\]

### E.2 Orthogonal Subspace: Upper Bound on Expectation of $\text{Tr}(V_\perp^\top H_tH_t^\top V_\perp)$

In this section, we provide a bound on $E[\text{Tr}(V_\perp^\top H_tH_t^\top V_\perp)]$.

**Lemma 16.** For all $t > 0$ and $\beta_t$ is such that $\beta_tG_tr_t < \frac{1}{4}$ (which can be obtained by appropriately controlling the constant $\beta$ in the step size),

$$E\left[\text{Tr}(V_\perp^\top H_tH_t^\top V_\perp)\right]$$

$$\leq \exp\left(\sum_{j=1}^{t} 2\beta_j\lambda_2 + \beta_j^2\lambda_2^2\right)\left(|\text{Tr}(V_\perp V_\perp^\top)| + d|V_\perp V_\perp^\top|_2\sum_{i=1}^{t} \left(r_i\beta_i^2S_iC^{(2)} + 1(i \leq t_0)\beta_t*2dB_{c_i}\right)\cdot\right.$$  

$$\exp\left(\sum_{j=1}^{i} 2\beta_j(\lambda_1 - \lambda_2) + \beta_j^2(S_jdr_jc^{(1)} - \lambda_2^2) + \sum_{j=1}^{\min(i,t_0)} \beta_j 2*dB_{c_j}\right)\right)$$

where the $V_\perp$ matrix contains in its columns $\tilde{u}_2,\ldots,\tilde{u}_d$, where each $\tilde{u}_1 = Bu_i$ is the unnormalized left eigenvector of the matrix $B^{-1}A$ and $S_i = (A_i + B_{c_i}G_i + B_{c_i}^2)$ for all $i$.  

**Proof.** Let $\gamma_t = E[\text{Tr}(V_\perp^\top H_tH_t^\top V_\perp)]$. By definition:

$$\gamma_t = \text{Tr}(E\left[H_tH_t^\top\right]V_\perp V_\perp^\top)$$

$$= \text{Tr}(E\left[H_{t-1}H_{t-1}^\top\right](I + \beta_tB^{-1}A)^\top V_\perp V_\perp^\top(I + \beta_tB^{-1}A))$$

$$+ \text{Tr}(\beta_tE\left[\varepsilon_tH_{t-1}H_{t-1}^\top + H_{t-1}H_{t-1}^\top\varepsilon_t^\top\right] V_\perp V_\perp^\top + \beta_t^2E\left[\varepsilon_tH_{t-1}H_{t-1}^\top\varepsilon_t^\top\right] V_\perp V_\perp^\top)$$

We focus on term $\blacklozenge$:

$$(I + \beta_tB^{-1}A)^\top V_\perp V_\perp^\top(I + \beta_tB^{-1}A) = V_\perp V_\perp^\top + \beta_t(B^{-1}A)^\top V_\perp V_\perp^\top + \beta_tV_\perp V_\perp^\top(B^{-1}A)$$

$$+ \beta_t^2(B^{-1}A)^\top V_\perp V_\perp^\top B^{-1}A$$

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Analysis of $\mathbf{\Phi}_1$: We begin by noting that the columns of $V_{\perp}$ contain the vectors $\bar{u}_i$ which are the unnormalized left eigenvectors of $B^{-1}A$ and therefore,

$$V_{\perp}^T (B^{-1}A) = V_{\perp}^T \Lambda,$$

where $\Lambda$ is a diagonal matrix with $\Lambda_{i,i} = \lambda_{i+1} \forall i = 2 \ldots d$. Noting that $V_{\perp} V_{\perp}^T \Lambda \preceq \lambda_2 V_{\perp} V_{\perp}^T$, we obtain,

$$\mathbf{\Phi}_1 \preceq V_{\perp} V_{\perp}^T (1 + 2 \beta_1 \lambda_2). \quad (45)$$

Following a similar argument, we obtain that,

$$(B^{-1}A)^T V_{\perp} V_{\perp}^T B^{-1}A \preceq \lambda_2^2 V_{\perp} V_{\perp}^T. \quad (46)$$

Combining Eqs (45) and (46), we obtain,

$$\mathbf{\Phi} \leq \text{Tr} \left( \mathbb{E}[H_{t-1}^T H_{t-1}^T] V_{\perp} V_{\perp}^T (1 + 2 \beta_t \lambda_2 + \beta_t^2 \lambda_2^2) \right)$$

The terms corresponding to $\mathbf{\Box}$ can also be bounded by bonding the operator norms of its two constituent expectations. In the same way as in Lemma 15, let $H_{t-1} = (I + G_{t-1}^T) H_{t-1}$. Note that $V_{\perp} V_{\perp}^T \preceq \|V_{\perp} V_{\perp}^T\|_2 I$ and we bound the normalized term $\mathbf{\Box}/\|V_{\perp} V_{\perp}^T\|_2$.

$$\text{Tr} \left( \mathbb{E} \left[ \epsilon_t H_{t-1}^T H_{t-1}^T + H_{t-1}^T H_{t-1}^T \epsilon_t \right] V_{\perp} V_{\perp}^T \right) \leq \text{Tr} \left( \mathbb{E} \left[ \epsilon_t H_{t-1}^T H_{t-1}^T + H_{t-1}^T H_{t-1}^T \epsilon_t \right] \right)$$

$$= \text{Tr} \left( \mathbb{E} \left[ H_{t-r_t}^T H_{t-r_t}^T (\epsilon_t + \epsilon_t^T) \right] \right) + \text{Tr} \left( \mathbb{E} \left[ H_{t-r_t}^T H_{t-r_t}^T \left( (L_{t-1}^{t-r_t+1})^T \epsilon_t + \epsilon_t^T L_{t-1}^{t-r_t+1} \right) \right] \right)$$

$$+ \text{Tr} \left( \mathbb{E} \left[ H_{t-r_t}^T H_{t-r_t}^T \left( (L_{t-1}^{t-r_t+1})^T \epsilon_t L_{t-1}^{t-r_t+1} + (L_{t-1}^{t-r_t+1})^T \epsilon_t L_{t-1}^{t-r_t+1} \right) \right] \right)$$

Recall that $\|L_{t-1}^{t-r_t+1}\| \leq 4 r_t G_t \beta_t$. As a consequence:

$$(L_{t-1}^{t-r_t+1})^T \epsilon_t \leq 2 B \epsilon_t + 4 r_t G_t \beta_t I = 8 B \epsilon_t r_t G_t \beta_t I$$

$$(L_{t-1}^{t-r_t+1})^T \epsilon_t \leq 2 B \epsilon_t + 4 r_t G_t \beta_t I = 8 B \epsilon_t r_t G_t \beta_t I$$

$$(L_{t-1}^{t-r_t+1})^T \epsilon_t L_{t-1}^{t-r_t+1} + (L_{t-1}^{t-r_t+1})^T \epsilon_t L_{t-1}^{t-r_t+1} \leq 2(4 r_t G_t \beta_t)^2 B \epsilon_t I \leq 8 B \epsilon_t r_t G_t \beta_t I$$

The second inequality in the last line follows from the step size condition. Therefore:

$$\Gamma_2 + \Gamma_3 + \Gamma_4 \leq 32 B \epsilon_t r_t G_t \beta_t \text{Tr} \left( \mathbb{E} \left[ H_{t-r_t}^T H_{t-r_t}^T \right] \right)$$

$$\leq 32 B \epsilon_t r_t G_t \beta_t \text{d} \| \mathbb{E} \left[ H_{t-r_t}^T H_{t-r_t}^T \right] \|$$

$$\leq 32 B \epsilon_t r_t G_t \beta_t \text{d} \beta_t \epsilon_t$$

$$\leq 32 B \epsilon_t r_t G_t \beta_t \text{d} \beta_t \epsilon_t$$

We proceed to bound $\Gamma_1$. We know that $||\mathbb{E} \left[ \epsilon_t F_{t-r_t} \right] || = A \beta_t r_t$ and therefore $\mathbb{E} \left[ \epsilon_t + \epsilon_t^T F_{t-r_t} \right] \leq 2 A \beta_t r_t I$

$$\Gamma_1 = \text{Tr} \left( \mathbb{E} \left[ H_{t-r_t}^T H_{t-r_t}^T (\epsilon_t + \epsilon_t^T) \right] \right) = \mathbb{E} \left[ \text{Tr} \left( H_{t-r_t}^T H_{t-r_t}^T \mathbb{E} \left[ \epsilon_t + \epsilon_t^T F_{t-r_t} \right] \right) \right]$$

$$\leq 2 A \beta_t r_t \mathbb{E} \left[ H_{t-r_t}^T H_{t-r_t}^T \right]$$
And consequently:

\[
\begin{align*}
\text{where the first inequality follows because} & \\
\text{On the other hand, for} & \\
\text{bound obtained in the previous lemma for} & \\
\text{with} & \\
\text{and the second term in} & \\
\text{for all} & \\
\text{where the first inequality follows because} & \\
\text{and the second term in} & \\
\text{Let} C^{(2)} \text{ be a constant such that} & \\
\text{The last inequalities follow from the same argument as in equation 41, where} \theta_{t-1} \text{ is the upper bound obtained in the previous lemma for } \|E[H_{t-1}H_{t-1}^\top]\|. & \\
\text{As a consequence, whenever } t \geq t_0 \text{ the first term in } \overline{\square}/\|V_\perp V_\perp^\top\|_2 \text{ can be bounded by:} & \\
\text{For the case when } t < t_0: & \\
\text{And the second term in } \overline{\square}/\|V_\perp V_\perp^\top\|_2 \text{ can be bounded for all } t: & \\
\text{Let } C^{(2)} \text{ be a constant such that } dr_t(A_t + B_{et}G_t + B_{et}^2)C^{(2)} \geq 32dB_{et}r_tG_t + 2dB_{et}^2r_t + dB_{et}^2. & \\
\text{The last inequalities follow from the same argument as in equation 38. We conclude that whenever } t > t_0: & \\
\text{Combining } \bigtriangledown \text{ with } \square, \text{ whenever } t > t_0: & \\
\text{On the other hand, for } t \leq t_0: & \\
\text{And consequently:} & \\
\gamma_t = \bigtriangledown + \square \leq \gamma_{t-1}(1 + 2\beta_t\lambda_2 + \beta_t\lambda_2^2) + dr_t\beta_t^2(A_t + B_{et}G_t + B_{et}^2)C^{(2)}(2)\theta_{t-1}\|V_\perp V_\perp^\top\|_2 & \\
\gamma_t = \bigtriangledown + \square \leq \gamma_{t-1}(1 + 2\beta_t\lambda_2 + \beta_t\lambda_2^2) + (dr_t\beta_t^2(A_t + B_{et}G_t + B_{et}^2)C^{(2)} + \beta_t2dB_{et})\theta_{t-1}\|V_\perp V_\perp^\top\|_2 & \\
\end{align*}
\]
Using the bound for \( \theta_{t-1} \) in Lemma 15 and the inequality \( 1 + x \leq e^x \):

\[
\gamma_t \leq \exp(2\beta_t \lambda_2 + \beta^2_t \lambda_2^2) \gamma_{t-1} + \\
\|V_\perp V_\perp^T\|_2 \left( d\epsilon_t^2 (A_t + B_{\epsilon_t} G_t + B_{\epsilon_t}^2) C^{(2)} + 1(t \leq t_0) \beta_t * 2dB_{\epsilon_t} \right) \\
\exp \left( \sum_{j=1}^{\min(t,t_0)} \beta_j 2 * dB_{\epsilon_j} \right),
\]

After doing recursion we obtain the upper bound,

\[
\gamma_t \leq \sum_{i=1}^{t} \left[ \|V_\perp V_\perp^T\|_2 \left( d\epsilon_t^2 (A_t + B_{\epsilon_t} G_t + B_{\epsilon_t}^2) C^{(2)} + 1(i \leq t_0) \beta_i * 2dB_{\epsilon_i} \right) \exp \left( \sum_{j=i+1}^{t} \beta_j 2 * dB_{\epsilon_j} \right) \right]
\]

Where \( \gamma_0 = \text{Tr}(V_\perp V_\perp^T) \). Let \( S_i = (A_i + B_{\epsilon_i} G_i + B_{\epsilon_i}^2) \)

\[
\gamma_t \leq \exp \left( \sum_{j=1}^{t} \beta_j 2 * dB_{\epsilon_j} \right) \left( \text{Tr}(V_\perp V_\perp^T) + d\|V_\perp V_\perp^T\|_2 \sum_{i=1}^{t} \left( r_i \beta_i^2 S_i C^{(2)} + 1(i \leq t_0) \beta_i * 2dB_{\epsilon_i} \right) \right)
\]

\[
\exp \left( \sum_{j=1}^{t} \beta_j (\lambda_1 - \lambda_2) + \beta^2_j (S_j \delta_j C^{(1)} - \lambda_2^2) + \sum_{j=1}^{\min(t,t_0)} \beta_j 2 * dB_{\epsilon_j} \right)
\]

\[\square\]

### E.3 Lower Bound on Expectation of \( \tilde{u}_1^T H_t H_t^T \tilde{u}_1 \)

**Lemma 17.** For all \( t \geq 0 \) and \( \beta_t \geq 0 \) we have,

\[
\mathbb{E}[\tilde{u}_1 H_t H_t^T \tilde{u}_1] \geq \|\tilde{u}_1\|_2^2 \exp \left( \sum_{i=1}^{t} \beta_i \lambda_1 - \epsilon_t \beta_i^2 \lambda_2^2 \right) - d\|\tilde{u}_1\|_2^2 \sum_{i=1}^{t} \left( \beta_i^2 r_i (A_i + B_{\epsilon_i} G_i + B_{\epsilon_i}^2) C^{(2)} \right)
\]

\[+ \beta_i 1(t \leq t_0)(B_{\epsilon_i}) \exp \left( \sum_{j=1}^{\min(t-1,t_0)} \beta_j \lambda_1 + \beta_j^2 d\epsilon_j (A_j + B_{\epsilon_j} G_j + B_{\epsilon_j}^2) C^{(1)} + \sum_{j=1}^{\min(t-1,t_0)} \beta_j 2 * dB_{\epsilon_j} \right),\]

where \( \tilde{u}_1 \) is the unnormalized left eigenvector corresponding to the maximum eigenvalue \( \lambda_1 \) of \( (B^{-1}A)^T \).

**Proof.** Let \( \gamma_t \triangleq \mathbb{E}[v^T H_t H_t^T v] \) where \( v = \tilde{u}_1 / \|\tilde{u}_1\|_2 \) be the normalized left eigenvector and \( \Sigma = B^{-1}A \). Since \( H_t = (I + \beta_t G_t) \), we can obtain a bound on \( \gamma_t \) as,

\[
\gamma_t = \mathbb{E}[v^T (I + \beta_t G_t) H_{t-1} H_{t-1}^T (I + \beta_t G_t) v] \\
= \mathbb{E}[v^T (I + \beta_t \Sigma) H_{t-1} H_{t-1}^T (I + \beta_t \Sigma) v] + \beta_t \mathbb{E}[v^T (\epsilon_t H_{t-1} H_{t-1}^T + H_{t-1} H_{t-1}^T \epsilon_t^T) v] \\
+ \beta_t^2 \mathbb{E}[v^T (\epsilon_t + \Sigma) H_{t-1} H_{t-1}^T (\epsilon_t + \Sigma)^T v] - \beta_t^2 \mathbb{E}[v^T \Sigma H_{t-1} H_{t-1}^T \Sigma^T v] \\
\geq \mathbb{E}[v^T H_{t-1} H_{t-1}^T v] + \beta_t \mathbb{E}[v^T \Sigma H_{t-1} H_{t-1}^T v] + \beta_t \mathbb{E}[v^T H_{t-1} H_{t-1}^T \Sigma^T v]
\]

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where $\zeta_1$ follows since $(\epsilon_t + \Sigma)H_{t-1}^T \epsilon_t$ is a positive semi-definite matrix and $\zeta_2$ follows since $v$ is the top left eigenvector of $\Sigma$. Now, in order to bound term (I), we note that

$$\mathbb{E}[v^T(\epsilon_t H_{t-1}^T + H_{t-1}^T \epsilon_t^T)v] \geq -\|\mathbb{E}[\epsilon_t H_{t-1}^T + H_{t-1}^T \epsilon_t^T]\|_2.$$  

Using the bound obtained in (41), we get that for $t > t_0$,

$$\mathbb{E}[v^T(\epsilon_t H_{t-1}^T + H_{t-1}^T \epsilon_t^T)v] \geq -\beta_t (8dB_{\epsilon_t} G_t (2r_t + 1) + dA_t r_t) \theta_{t-1},$$

and for $t \leq t_0$, we have from equation (42)

$$\mathbb{E}[v^T(\epsilon_t H_{t-1}^T + H_{t-1}^T \epsilon_t^T)v] \geq -2dB_{\epsilon_t} \theta_{t-1},$$

Where $\theta_{t-1}$ is defined as in 15. We next use the bound from lemma 15 to lower bound $-\theta_{t-1}$,

$$\mathbb{E}[v^T(\epsilon_t H_{t-1}^T + H_{t-1}^T \epsilon_t^T)v] \geq -\beta_t (8dB_{\epsilon_t} G_t (2r_t + 1) + dA_t r_t + I(t \leq t_0)(2dB_{\epsilon_t})).$$

Recall that in Lemma 16 we defined $C^{(2)}$ as a constant such that: $dr_t(A_t + B_{\epsilon_t} G_t + B_{\epsilon_t}^2)C^{(2)} \geq 32dB_{\epsilon_t} r_t G_t + 2dA_t r_t + dB_{\epsilon_t}^2$, therefore:

$$\mathbb{E}[v^T(\epsilon_t H_{t-1}^T + H_{t-1}^T \epsilon_t^T)v] \geq -\left(\beta_t dr_t(A_t + B_{\epsilon_t} G_t + B_{\epsilon_t}^2)C^{(2)} + I(t \leq t_0)(2dB_{\epsilon_t})\right).$$

Substituting the above in equation (48), we obtain the following recursion,

$$\gamma_t \geq (1 + 2\lambda_1 \beta_t) \gamma_{t-1} - \left(\beta_t dr_t(A_t + B_{\epsilon_t} G_t + B_{\epsilon_t}^2)C^{(2)} + I(t \leq t_0)(2dB_{\epsilon_t})\right).$$

Using the inequality $1 + x \geq \exp(x - x^2)$ for all $x \geq 0$, along with $\gamma_0 = 1$, we obtain,

$$\gamma_t \geq \exp\left(\sum_{i=1}^{t} 2\beta_i \lambda_1 - 4\beta_i^2 \lambda_1^2\right) - d \sum_{i=1}^{t} \left(\left(\beta_i^2 r_t(A_t + B_{\epsilon_t} G_t + B_{\epsilon_t}^2)C^{(2)} + \beta_i I(t \leq t_0)(B_{\epsilon_t})\right)\right).$$

which concludes the proof of the lemma.
E.4 Upper Bound on Variance of $\tilde{u}_1^T H_t H_t^T \tilde{u}_1$

In this section, we provide an upper bound on $\mathbb{E} \left[ (v^T H_t H_t^T v)^2 \right]$ which will be later used in order to lower bound the requisite term using the Chebyshev Inequality. We first prove an upper bound on $\mathbb{E} \left[ \text{Tr}(H_t H_t^T H_t H_t^T) \right]$ and use this in the next lemma to obtain the requisite bounds.

Lemma 18. For all $t \geq 0$:

$$\mathbb{E} \left[ \text{Tr}(H_t H_t^T H_t H_t^T) \right] \leq d \exp \left( \sum_{i=1}^{t} 4\lambda_i \beta_i + dr_i (A_i + B_i^2 + B_t \epsilon_i G_t) C(3) \beta_i^2 + \frac{\min(t, t_0)}{10} \beta_i \frac{(101/100)^3 B_t \epsilon_i}{} \right)$$

As long as $\beta_t$ satisfies that for all $t$, $\|I + \beta_t B_t^{-1} A\| \leq \frac{101}{100}$, $\beta_t r_t G_t < \frac{1}{4}$, and $\beta_t r_t B_t < \frac{1}{4}$.

Proof. We start by substituting the identity: $H_t = (I + \beta_t G_t) H_{t-1} = (I + \beta_t B_t^{-1} A + \beta_t \epsilon_t) H_{t-1}$.

Substituting this decomposition into the trace we want to bound we obtain:

$$\mathbb{E} \left[ \text{Tr}(H_t H_t^T H_t H_t^T) \right] = \text{Tr} \mathbb{E} \left[ (I + \beta_t G_t) H_{t-1} H_{t-1}^T (I + \beta_t G_t) H_{t-1} H_{t-1}^T (I + \beta_t G_t)\right]$$

Expanding $\bullet$ yields:

$$\bullet = (I + \beta_t B_t^{-1} A)^T (I + \beta_t B_t^{-1} A) (I + \beta_t B_t^{-1} A)^T (I + \beta_t B_t^{-1} A)$$

$$+ \beta_t (I + \beta_t B_t^{-1} A)^T (I + \beta_t B_t^{-1} A) (I + \beta_t B_t^{-1} A)^T (I + \beta_t B_t^{-1} A)$$

$$+ \beta_t (I + \beta_t B_t^{-1} A)^T (I + \beta_t B_t^{-1} A) \epsilon_t (I + \beta_t B_t^{-1} A)^T (I + \beta_t B_t^{-1} A)$$

$$+ \epsilon_t (I + \beta_t B_t^{-1} A)^T (I + \beta_t B_t^{-1} A) (I + \beta_t B_t^{-1} A)^T (I + \beta_t B_t^{-1} A)$$

$$\leq \frac{\gamma_1}{2} \left( I + \beta_t B_t^{-1} A \right)^2 + \frac{\gamma_2}{3} \left( I + \beta_t B_t^{-1} A \right)^3 + \frac{\gamma_3}{4} \left( I + \beta_t B_t^{-1} A \right)^4$$

where $\gamma_1$ follows from triangle, and $\gamma_2, \gamma_3$ from the step size condition. Recall that:

$$\text{Tr} \mathbb{E} \left[ H_{t-1} H_{t-1}^T H_{t-1} H_{t-1}^T \right] = \text{Tr} \mathbb{E} \left[ H_{t-1} H_{t-1}^T H_{t-1} H_{t-1}^T (\bullet + (\bullet_1^{(1)} + \bullet_2^{(2)}) + \bullet_3) \right]$$
Since, as shown in equation Equation 37 we have that \((I + \beta_t B^{-1}A)(I + \beta_t B^{-1}A)^T \preceq (1 + \beta_t \lambda_1)^2 I\). then, \(\mathbf{\phi}_1 \preceq (1 + \beta_t \lambda_1)^4 I\) (this is because \((I + \beta_t B^{-1}A)(I + \beta_t B^{-1}A)^T\) and \((I + \beta_t B^{-1}A)^T (I + \beta_t B^{-1}A)\) have the same eigenvalues. And therefore \(\mathbf{\phi}_1 \preceq (1 + \beta_t \lambda_1)^4 I \preceq (1 + 4\beta_t \lambda_1 + 11\beta_t^2 \max(\lambda_1^1, 1)) I\) and \(\mathbf{\phi}_3 \preceq 8\beta_t^2 B_{\epsilon_t} I\), thus implying:

\[
\text{Tr} \mathbb{E} \left[ H_{t-1} H_{t-1}^T H_{t-1} H_{t-1}^T (\mathbf{\phi}_1 + \mathbf{\phi}_3) \right] \leq (1 + 4\beta_t \lambda_1 + 11\beta_t^2 \max(\lambda_1^1, 1) + 8\beta_t^2 B_{\epsilon_t}^2) \text{Tr} \mathbb{E} \left[ H_{t-1} H_{t-1}^T H_{t-1} H_{t-1}^T \right]
\]

It only remains to bound the term \(\text{Tr} \mathbb{E} \left[ H_{t-1} H_{t-1}^T H_{t-1} H_{t-1}^T (\mathbf{\phi}_2^{(1)} + \mathbf{\phi}_2^{(2)}) \right]\). Notice that \(\mathbf{\phi}_2^{(1)} + \mathbf{\phi}_2^{(2)}\) is a symmetric matrix. Therefore, whenever \(t \leq t_0\):

\[
\|\mathbf{\phi}_2^{(1)} + \mathbf{\phi}_2^{(2)}\| \leq 2\beta_t B_{\epsilon_t} \|I + \beta_t B^{-1}A\|^3 \leq 2\beta_t \left(\frac{101}{100}\right)^3 B_{\epsilon_t}
\]

And also whenever \(t \leq t_0\):

\[
\text{Tr} \mathbb{E} \left[ H_{t-1} H_{t-1}^T H_{t-1} H_{t-1}^T (\mathbf{\phi}_2^{(1)} + \mathbf{\phi}_2^{(2)}) \right] \leq 2\beta_t B_{\epsilon_t} \|I + \beta_t B^{-1}A\|^3 \\
\leq 2(1.01)^3 \beta_t \text{Tr} \mathbb{E} \left[ H_{t-1} H_{t-1}^T H_{t-1} H_{t-1}^T \right]
\]

We will use similar arguments to what we used in previous sections to bound these types of terms for the case when \(t > t_0\):

Let \(H_{t-1} = (I + L_{t-1}^{t-r_t+1})H_{t-r_t}\) as in Lemma 15, therefore:

\[
\text{Tr} \mathbb{E} \left[ H_{t-1} H_{t-1}^T H_{t-1} H_{t-1}^T (\mathbf{\phi}_2^{(1)} + \mathbf{\phi}_2^{(2)}) \right] = \text{Tr} \mathbb{E} \left[ (I + L_{t-1}^{t-r_t+1}) H_{t-r_t} H_{t-r_t}^T (I + L_{t-1}^{t-r_t+1})^T \right] \\
(I + L_{t-1}^{t-r_t+1}) H_{t-r_t} H_{t-r_t}^T (I + L_{t-1}^{t-r_t+1})^T (\mathbf{\phi}_2^{(1)} + \mathbf{\phi}_2^{(2)})
\]

We can now expand the right hand side of the last equation into different types of terms. We start by bounding the term that does not contain any \(L_{t-1}^{t-r_t+1}\) nor \((L_{t-1}^{t-r_t+1})^T\). It is easy to see that \(\|\mathbb{E} \left[ \mathbf{\phi}_2^{(1)} + \mathbf{\phi}_2^{(2)} | F_{t-r_t} \right]\| \leq \beta_t^2 A_t \epsilon_r * \left(\frac{101}{100}\right)^3 * 4\). This follows because \(\|\mathbb{E} [\epsilon_r | F_{t-r_t}]\| \leq \mathcal{A}_t \beta_t \epsilon_r\), and an operator bound on each of the remaining 3 terms in each of the four factors by \(\|I + \beta_t B^{-1}A\| \leq \frac{101}{100}\). With these observations and using the fact that \(\mathbf{\phi}_2^{(1)} + \mathbf{\phi}_2^{(2)}\) is a symmetric matrix, we can bound the following term:

\[
\mathbb{E} \left[ \text{Tr} \left( H_{t-r_t} H_{t-r_t}^T H_{t-r_t} H_{t-r_t}^T (\mathbf{\phi}_2^{(1)} + \mathbf{\phi}_2^{(2)}) \right) | F_{t-r_t} \right] \\
= \text{Tr} \left( H_{t-r_t} H_{t-r_t}^T H_{t-r_t} H_{t-r_t}^T \mathbb{E} \left[ \mathbf{\phi}_2^{(1)} + \mathbf{\phi}_2^{(2)} | F_{t-r_t} \right] \right) \\
\leq \beta_t^2 A_t \epsilon_r * \left(\frac{101}{100}\right)^3 * 4 \text{Tr}(H_{t-r_t} H_{t-r_t}^T H_{t-r_t} H_{t-r_t}^T) \\
\leq \beta_t^2 A_t \epsilon_r * 5 \text{Tr}(H_{t-r_t} H_{t-r_t}^T H_{t-r_t} H_{t-r_t}^T)
\]

For the terms of \(\Gamma\) containing \(L_{t-1}^{t-r_t+1}\) components we use a simple bound. Notice that \(\|\mathbf{\phi}_2^{(1)} + \mathbf{\phi}_2^{(2)}\| \leq \beta_t \left(\frac{101}{100}\right)^3 * 4 * B_{\epsilon_t}\). And recall just as in Equation 40, \(\|L_{t-1}^{t-r_t+1}\| \leq \epsilon_r \mathcal{G}_t \beta_t\) and therefore \(\|(L_{t-1}^{t-r_t+1})^T L_{t-1}^{t-r_t+1}\| \leq 16\epsilon_r^2 \mathcal{G}_t^2 \beta_t^2\). We look at the term containing four copies of \(L_{t-1}^{t-r_t+1}\) terms:

Let \(O_1 = \text{Tr} \left( L_{t-1}^{t-r_t+1} H_{t-r_t} H_{t-r_t}^T (L_{t-1}^{t-r_t+1})^T L_{t-1}^{t-r_t+1} H_{t-r_t} H_{t-r_t}^T (L_{t-1}^{t-r_t+1})^T (\mathbf{\phi}_2^{(1)} + \mathbf{\phi}_2^{(1)}) \right)\).

\[
O_1 \leq \beta_t \left(\frac{101}{100}\right)^3 * 4 * B_{\epsilon_t} \text{Tr} \left( L_{t-1}^{t-r_t+1} H_{t-r_t} H_{t-r_t}^T (L_{t-1}^{t-r_t+1})^T L_{t-1}^{t-r_t+1} H_{t-r_t} H_{t-r_t}^T (L_{t-1}^{t-r_t+1})^T (\mathbf{\phi}_2^{(1)} + \mathbf{\phi}_2^{(1)}) \right) \\
= \beta_t \left(\frac{101}{100}\right)^3 * 4 * B_{\epsilon_t} \text{Tr} \left( (H_{t-r_t} H_{t-r_t}^T (L_{t-1}^{t-r_t+1})^T L_{t-1}^{t-r_t+1} H_{t-r_t} H_{t-r_t}^T (L_{t-1}^{t-r_t+1})^T L_{t-1}^{t-r_t+1}) \right)
\]

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where the last inequality follows from the step size conditions. We now look at the following term in $\Gamma$ that has three $L_i^{-r_i+1}$ terms:

$$O_2 = \text{Tr}(H_{t-r_i}^T H_{t-r_i}^T (L_i^{-r_i+1})^T L_i^{-r_i+1} H_{t-r_i} H_{t-r_i}^T (L_i^{-r_i+1})^T (\phi_2^{(1)} + \phi_2^{(2)})) + \text{Tr}(L_i^{-r_i+1} H_{t-r_i} H_{t-r_i}^T (L_i^{-r_i+1})^T L_i^{-r_i+1} H_{t-r_i} H_{t-r_i}^T (\phi_2^{(1)} + \phi_2^{(2)}))$$

Since $\|L_i^{-r_i+1})^T (\phi_2^{(1)} + \phi_2^{(2)}) + (\phi_2^{(1)} + \phi_2^{(2)}) L_i^{-r_i+1}\| \leq \beta_2^2 (\frac{101}{100})^3 B_{c_t} r_t G_t$. Using a similar series of inequalities as in the case above we obtain a bound:

$$O_0 = \text{Tr}(H_{t-r_i}^T H_{t-r_i}^T (L_i^{-r_i+1})^T L_i^{-r_i+1} H_{t-r_i} H_{t-r_i}^T (L_i^{-r_i+1})^T (\phi_2^{(1)} + \phi_2^{(2)})) + \text{Tr}(L_i^{-r_i+1} H_{t-r_i} H_{t-r_i}^T (L_i^{-r_i+1})^T L_i^{-r_i+1} H_{t-r_i} H_{t-r_i}^T (\phi_2^{(1)} + \phi_2^{(2)}))$$

The last inequality $\gamma_i$ follows from the step size conditions.

$$O_2 = \text{Tr}(L_i^{-r_i+1} H_{t-r_i} H_{t-r_i}^T (L_i^{-r_i+1})^T L_i^{-r_i+1} H_{t-r_i} H_{t-r_i}^T (L_i^{-r_i+1})^T (\phi_2^{(1)} + \phi_2^{(2)})) + \text{Tr}(L_i^{-r_i+1} H_{t-r_i} H_{t-r_i}^T (L_i^{-r_i+1})^T L_i^{-r_i+1} H_{t-r_i} H_{t-r_i}^T (\phi_2^{(1)} + \phi_2^{(2)}))$$

Since $\|L_i^{-r_i+1})^T (\phi_2^{(1)} + \phi_2^{(2)}) + (\phi_2^{(1)} + \phi_2^{(2)}) L_i^{-r_i+1}\| \leq \beta_2^2 (\frac{101}{100})^3 * 4 * B_{c_t} r_t^2 G_t^2$:

$$O_2 = \text{Tr}(L_i^{-r_i+1} H_{t-r_i} H_{t-r_i}^T (L_i^{-r_i+1})^T L_i^{-r_i+1} H_{t-r_i} H_{t-r_i}^T (L_i^{-r_i+1})^T (\phi_2^{(1)} + \phi_2^{(2)})) + \text{Tr}(L_i^{-r_i+1} H_{t-r_i} H_{t-r_i}^T (L_i^{-r_i+1})^T L_i^{-r_i+1} H_{t-r_i} H_{t-r_i}^T (\phi_2^{(1)} + \phi_2^{(2)}))$$

The last inequality $\gamma_i$ follows from the step size conditions.
Then, we have that,

\[ \text{The last inequality follows from the step size conditions. This finalizes the analysis for the components in } \Gamma \text{ having three } L_{t-1}^{t-r_t+1} \text{ terms.} \]

We now look at a generic term in \( \Gamma \) with two \( L_{t-1}^{t-r_t+1} \) terms. Their sum equals:

\[
\text{We look at the components of } \Gamma \text{ having exactly two } L_{t-1}^{t-r_t+1} \text{ terms: Let }
\]

\[ O_3 = \text{Tr}(H_{t-r_t} H_{t-r_t}^T (L_{t-1}^{t-r_t+1})^T L_{t-1}^{t-r_t+1} H_{t-r_t} H_{t-r_t}^T (\|^{(1)} + \|^{(2)})). \]

Then, we have that,

\[ O_3 \leq d \| L_{t-1}^{t-r_t+1} \|^2 \| H_{t-r_t} H_{t-r_t}^T \|^2 (\|^{(1)} + \|^{(2)})\]

\[ \leq d \beta_t^2 * 4 * r_t^2 * G_t^3 * (101) \beta_t^3 * 4 * B_{\epsilon t} \| (H_{t-r_t} H_{t-r_t}^T) \|^2 \]

\[ \leq d \beta_t^2 17G_t B_{\epsilon t} \text{Tr}(H_{t-r_t} H_{t-r_t}^T H_{t-r_t} H_{t-r_t}^T) \]

The last inequality follows from a the step size conditions plus the fact that trace is larger than operator norm for a PSD matrix.

We now look at a generic term in \( \Gamma \) with one \( L_{t-1}^{t-r_t+1} \) term: Let

\[ O_4 = \text{Tr}(H_{t-r_t} H_{t-r_t}^T (L_{t-1}^{t-r_t+1})^T H_{t-r_t} H_{t-r_t}^T (\|^{(1)} + \|^{(2)})). \]

Then, we have that,

\[ O_4 \leq d \| L_{t-1}^{t-r_t+1} \|^2 \| H_{t-r_t} H_{t-r_t}^T \|^2 (\|^{(1)} + \|^{(2)})\]

\[ \leq 4r_t G_t \beta_t (101)^3 * 4B_{\epsilon t} \| (H_{t-r_t} H_{t-r_t}^T) \|^2 \]

\[ \leq 17d \beta_t^2 r_t G_t B_{\epsilon t} \text{Tr}(H_{t-r_t} H_{t-r_t}^T H_{t-r_t} H_{t-r_t}^T) \]

Since there is a single term of type \( O_1 \), four of type \( O_2 \), six of type \( O_3 \) and four of type \( O_4 \), we obtain the bound whenever \( t > t_0 \):

\[ \text{Tr} \left[ H \right] \cdot E \left[ \left( H_{t-1} H_{t-1}^T H_{t-1} H_{t-1}^T \right) \right] \leq \beta_t^2 (5r_t A_t + 55B_{\epsilon t} G_t r_t + 23dG_t B_{\epsilon t} r_t). \]

\[ E \left[ \text{Tr}(H_{t-r_t} H_{t-r_t}^T H_{t-r_t} H_{t-r_t}^T) \right] \]

Therefore we obtain the following recursion:

\[ \text{Tr}(E \left[ H H^T \right]) \leq \text{Tr} E \left[ H_{t-1} H_{t-1}^T H_{t-1} H_{t-1}^T \right] \]

\[ \leq (1 + 4\beta_t \lambda_1 + 11\beta_t^2 \max(\lambda_t^2, 1) + 8\beta_t^2 B_{\epsilon t}^2) \text{Tr} E \left[ H_{t-1} H_{t-1}^T H_{t-1} H_{t-1}^T \right] + \]

\[ \beta_t^2 (5r_t A_t + 55B_{\epsilon t} G_t r_t + 23dG_t B_{\epsilon t} r_t + 1(t \leq t_0)2\beta_t (101)^3 B_{\epsilon t}). \]
\[ \mathbb{E} \left[ \text{Tr}(H_{t-r_t} H_{t-r_t}^T H_{t-r_t} H_{t-r_t}^T) \right] \]

Let \( C^{(3)} \) be a constant such that:
\[ dr_t(A_t + B_{\epsilon_t}^2 + B_{\epsilon_t} G_t) C^{(3)} \geq (5 r_t A_t + 55 B_{\epsilon_t} G_t r_t + 23 d G_t B_{\epsilon_t} r_t) + 11 \max(\lambda_1^2, 1) + 8 B_{\epsilon_t}^2 \]

Let \( \{\eta_i\} \) be a sequence of increasing upper bounds for \( \mathbb{E} \left[ \text{Tr}(H_{t} H_{t}^T H_{t} H_{t}^T) \right] \). In other words,
\[ \mathbb{E} \left[ \text{Tr}(H_{t} H_{t}^T H_{t} H_{t}^T) \right] \leq \eta_i \forall i \]

And \( \eta_0 \leq \eta_1 \leq \eta_2 \leq \cdots \), where \( \eta_0 = d \). Let \( C^{(3)} = E_\epsilon + D^{(3)}_\epsilon + 11 \max(\lambda_1^2, 1) \). We can obtain a recursion of the form:
\[ \eta_t \leq (1 + 4 \beta_t \lambda_1 + \beta_t^2 dr_t(A_t + B_{\epsilon_t}^2 + B_{\epsilon_t} G_t) C^{(3)} + 1(t \leq t_0)2 \beta_t (\frac{101}{100})^3 B_{\epsilon_t}) \eta_{t-1} \]

We conclude by applying the inequality \( 1 + x \leq \exp(x) \) for \( x > 0 \) and the initial condition \( \eta_0 = d \):
\[ \eta_t \leq d \exp(\sum_{i=1}^{t} 4 \lambda_1 \beta_i + \beta_i^2 dr(A_t + B_{\epsilon_t}^2 + B_{\epsilon_t} G_t) C^{(3)} \beta_i^2 + \sum_{j=1}^{\min(t,t_0)} \beta_j 2 * (\frac{101}{100})^3 B_{\epsilon_j}) \]

\[ \square \]

Lemma 19. For \( t > 0 \), we have that
\[ \mathbb{E} \left[ (\tilde{u}_t^T H_{t} H_{t}^T \tilde{u}_t)^2 \right] \]
\[ \leq \|\tilde{u}_1\|^4 \exp(\sum_{i=1}^{t} 4 \lambda_1 \beta_i + 11 \lambda_1^2 \beta_i^2) + \|\tilde{u}_1\|^4 \sum_{i=1}^{t} \left( \beta_i^2 dr(A_t + B_{\epsilon_t}^2 + B_{\epsilon_t} G_t) U_2 + 1(i \leq t_0) \beta_4 B_{\epsilon_t} \right) \exp \left( \sum_{j=1}^{\min(t,t_0)} 4 \lambda_1 \beta_j + dr(A_j + B_{\epsilon_j}^2 + G_j B_{\epsilon_j}) C^{(3)} \beta_j^2 + \sum_{j=1}^{\min(t,t_0)} \beta_j 2 * (\frac{101}{100})^3 B_{\epsilon_j} \right) \]

where \( \tilde{u}_1 \) is the unnormalized left eigenvector corresponding to the maximum eigenvalue \( \lambda_1 \) of \( B^{-1} A \). As long as \( \beta_t \) follows that \( \|I + \beta_t B^{-1} A\| \leq \frac{101}{100}, \beta_t B_{\epsilon_t} < 1 \)

Proof. As in the previous lemma, we let \( v = \tilde{u}_1/\|\tilde{u}_1\| \) denote the normalized left principal eigenvector. Let \( H_t = (I + \beta_t G_t) H_{t-1} = (I + \beta_t B^{-1} A + \beta_t \epsilon_t) H_{t-1} \). The desired expectation can be written as:
\[ \mathbb{E} \left[ (v^T H_{t} H_{t}^T v)^2 \right] = \mathbb{E} \left[ v^T (I + \beta_t G_t) H_{t-1} H_{t-1}^T (I + \beta_t G_t) v v^T (I + \beta_t G_t) H_{t-1} H_{t-1}^T (I + \beta_t G_t) v \right] \]
\[ = \mathbb{E} \left[ v^T (I + \beta_t B^{-1} A) H_{t-1} H_{t-1}^T (I + \beta_t B^{-1} A) v v^T (I + \beta_t B^{-1} A) H_{t-1} H_{t-1}^T (I + \beta_t B^{-1} A) v \right] \]
\[ + \mathbb{E} [\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4] \]

where \( \Gamma_i \) is the collection of terms in the expansion of \( \mathbb{E} \left[ (v^T H_{t} H_{t}^T v)^2 \right] \) that have exactly \( i \) terms of the form \( \epsilon_t \).

Since \( v \) is a left eigenvector of \( B^{-1} A \), the term \( \Gamma_0 \) can be written as follows:
\[ \mathbb{E} [\Gamma_0] = (1 + \beta_t \lambda_1)^4 \mathbb{E} \left[ v^T H_{t-1} H_{t-1}^T v v^T H_{t-1} H_{t-1}^T v \right] \]
\[
|v^\top \beta_t \epsilon_t H_{t-1} H_{t-1}^\top \beta_t \epsilon_t vv^\top (I + \beta_t B^{-1} A) H_{t-1} H_{t-1}^\top (I + \beta_t B^{-1} A)^\top v| \leq \beta_t^2 \|H_{t-1} H_{t-1}^\top\|^2 B_{\epsilon_t}^2 (\frac{101}{100})^2 \\
\leq 2 B_{\epsilon_t}^2 \beta_t^2 \text{Tr}(H_{t-1} H_{t-1}^\top H_{t-1} H_{t-1}^\top)
\]

By a similar argument, and using the step size conditions \(\beta_t B_{\epsilon_t} < 1\), we can bound each of the terms in \(\Gamma_2, \Gamma_3\) and \(\Gamma_4\) and obtain (using the fact that \(\beta_t < 1\)):

\[
\Gamma_2 + \Gamma_3 + \Gamma_4 \leq \beta_t^2 B_{\epsilon_t} \text{Tr}(H_{t-1} H_{t-1}^\top H_{t-1} H_{t-1}^\top) \tag{50}
\]

For some universal constant \(U_t\) depending on \(\frac{101}{100}\) and the number of component terms in \(\Gamma_2, \Gamma_3\), and \(\Gamma_4\). Therefore,

\[
\mathbb{E}[\Gamma_2 + \Gamma_3 + \Gamma_4] \leq \beta_t^2 B_{\epsilon_t}^2 U_t \mathbb{E}[\text{Tr}(H_{t-1} H_{t-1}^\top H_{t-1} H_{t-1}^\top)] \\
\leq \beta_t^2 B_{\epsilon_t}^2 U_t \text{exp}\left(\sum_{i=1}^{t-1} 4 \lambda_1 \beta_i + dr_i (A_i + B_{\epsilon_i} + B_{\epsilon_i} G_i) C(3) \beta_i^2 \right) + \sum_{j=1}^{\min(t, t_0)} 2 \beta_j \left(\frac{101}{100}\right)^2 B_{\epsilon_j}
\]

**Bounding expectation of \(\Gamma_1\):** We start by bounding the expectation of \(\Gamma_1\) whenever \(t \leq t_0\). Let’s look at a generic term from \(\Gamma_1\):

\[
Z := v^\top (I + \beta_t B^{-1} A) H_{t-1} H_{t-1}^\top \beta_t \epsilon_t vv^\top (I + \beta_t B^{-1} A) H_{t-1} H_{t-1}^\top (I + \beta_t B^{-1} A)^\top v
\]

We bound this term naively:

\[
\|Z\| \leq \beta_t \|I + \beta_t B^{-1} A\|^3 \|H_{t-1} H_{t-1}^\top\|^2 B_{\epsilon_t} \\
\leq \beta_t \left(\frac{101}{100}\right)^3 \text{Tr}(H_{t-1} H_{t-1}^\top H_{t-1} H_{t-1}^\top) B_{\epsilon_t}
\]

There are exactly 4 terms of type \(Z\). Now we proceed to bound the expectation of \(\Gamma_1\) whenever \(t > t_0\): Let’s look at a generic term from \(\Gamma_1\):

\[
v^\top (I + \beta_t B^{-1} A) H_{t-1} H_{t-1}^\top \beta_t \epsilon_t vv^\top (I + \beta_t B^{-1} A) H_{t-1} H_{t-1}^\top (I + \beta_t B^{-1} A)^\top v \tag{51}
\]

In the same way as in previous lemmas, in order to obtain a bound for this term, we write \(H_{t-1} = (I + L_{t-1}^{t-r_t+1}) H_{t-r_t}\) and substitute this equality in Equation 51. Recall that \(\|L_{t-1}^{t-r_t+1}\| \leq 4 r_t G_i \beta_t\).

In this expansion, we bound all terms that have at least one \(L_{t-1}^{t-r_t+1}\) using a simple bound. Let’s look at a generic such term and bound it:

\[
\Lambda := |v^\top (I + \beta_t B^{-1} A) L_{t-1}^{t-r_t+1} H_{t-r_t} H_{t-r_t}^\top \beta_t \epsilon_t vv^\top (I + \beta_t B^{-1} A) H_{t-r_t} H_{t-r_t}^\top (I + \beta_t B^{-1} A)^\top v|
\]

\[
\Lambda \leq 4 r_t G_i \beta_t^2 \left(\frac{101}{100}\right)^3 \|H_{t-r_t} H_{t-r_t}^\top\|^2 B_{\epsilon_t} \\
\leq 2 r_t G_i \beta_t^2 B_{\epsilon_t} \left(\frac{101}{100}\right)^3 \text{Tr}(H_{t-r_t} H_{t-r_t}^\top H_{t-r_t} H_{t-r_t}^\top)
\]

\[
39
\]
And therefore:

\[
\mathbb{E}[\diamondsuit] \leq 2r_t \mathbb{E}[\beta_t B_t] \left( \frac{101}{100} \right)^3 \exp\left( \sum_{i=1}^{t-r_t} 4\lambda_1 \beta_i + dr_t(A_i + B^2_{e_t} + B_{e_t} G_i)C^{(3)} \beta_t^2 \right) \\
\leq 2r_t \mathbb{E}[\beta_t B_t] \left( \frac{101}{100} \right)^3 \exp\left( \sum_{i=1}^{t-1} 4\lambda_1 \beta_i + dr_t(A_i + B^2_{e_t} + B_{e_t} G_i)C^{(3)} \beta_t^2 \right)
\]

Using the step size condition, \( \beta_t r_t \leq \frac{1}{4} \), all of the remaining terms with at least one \( L_{t-1}^{-r_t+1} \) can be upper bounded by an expression of order \( O(\beta_t^2 r_t \mathbb{E}[B_t \text{Tr}(H_{t-1} H_{t-1}^T H_{t-r_t}^T H_{t-r_t}^T)]) \). This procedure will handle the terms in \( \Gamma_1 \) that after the substitution \( H_{t-1} = (I + L_{t-1}^{-r_t+1}) H_{t-r_t} \) have at least one \( L_{t-1}^{-r_t+1} \).

The only terms remaining to bound are those coming from \( \Gamma_1 \), such that after substituting \( H_{t-1} = (I + L_{t-1}^{-r_t+1}) H_{t-r_t} \) do not involve any \( L_{t-1}^{-r_t+1} \). Let’s look at a generic such term and bound its expectation:

\[
\diamondsuit := \mathbb{E}\left[ v^T (I + \beta_t B^{-1} A) H_{t-r_t} H_{t-r_t}^T \beta_{cts} v \right] = \mathbb{E}\left[ v^T (I + \beta_t B^{-1} A) H_{t-r_t} H_{t-r_t}^T \beta_{cts} v \right]
\]

Recall that \( \|\mathbb{E}[\epsilon_t | \mathcal{F}_{t-r_t}]\| \leq \mathcal{A}_t \beta_t r_t \). We bound \( \diamondsuit \) by first bounding the norm of the conditional expectation of \( \diamondsuit_1 \):

\[
\|\mathbb{E}[\diamondsuit_1 | \mathcal{F}_{t-r_t}]\| \leq \beta_t^2 O(r_t) \left( \frac{101}{100} \right)^3 \left\| H_{t-r_t} H_{t-r_t}^T \right\|^2 \\
\leq \beta_t^2 A_t r_t \left( \frac{101}{100} \right)^3 \text{Tr}(H_{t-r_t} H_{t-r_t}^T H_{t-r_t} H_{t-r_t}^T)
\]

And therefore:

\[
\diamondsuit = \mathbb{E}[\diamondsuit_1] \leq \mathbb{E}[\|\mathbb{E}[\diamondsuit_1 | \mathcal{F}_{t-r_t}]\|] \\
\leq \beta_t^2 A_t r_t \left( \frac{101}{100} \right)^3 \exp\left( \sum_{i=1}^{t-r_t} 4\lambda_1 \beta_i + dr_t(A_i + B^2_{e_t} + B_{e_t} G_i)C^{(3)} \beta_t^2 + \sum_{j=1}^{\min(t-r_t,t_0)} \beta_j 2 * \left( \frac{101}{100} \right)^3 B_{e_j} \right) \\
\leq \beta_t^2 A_t r_t \left( \frac{101}{100} \right)^3 \exp\left( \sum_{i=1}^{t-1} 4\lambda_1 \beta_i + dr_t(A_i + B^2_{e_t} + B_{e_t} G_i)C^{(3)} \beta_t^2 + \sum_{j=1}^{\min(t,t_0)} \beta_j 2 * \left( \frac{101}{100} \right)^3 B_{e_j} \right)
\]

The last inequality follows from the results of 18. Combining all these bounds yields for all \( t \) we have:

\[
\mathbb{E}[\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4] \leq \left( \beta_t^2 \exp((t - t_0) \beta_t A_t + B^2_{e_t} + B_{e_t} G_i)) \mathcal{U}_2 + 1(t \leq t_0) \beta_t 4 * B_{e_t} \right) \left( \frac{101}{100} \right)^3 \\
\exp\left( \sum_{i=1}^{t-1} 4\lambda_1 \beta_i + dr_t(A_i + B^2_{e_t} + B_{e_t} G_i)C^{(3)} \beta_t^2 + \sum_{j=1}^{\min(t,t_0)} \beta_j 2 * \left( \frac{101}{100} \right)^3 B_{e_j} \right)
\]

where \( \mathcal{U}_2 \) is an absolute constant depending on \( \frac{101}{100} \), and the number of terms in \( \Gamma_1, \Gamma_2, \ldots, \Gamma_4 \).

Combining all these terms we get a recursion of the form:

\[
\mathbb{E}\left[ (v^T H_t H_t^T v)^2 \right] \leq \exp(4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2) \mathbb{E}\left[ (v^T H_{t-1} H_{t-1}^T v)^2 \right] + \left( \beta_t^2 \exp((t - t_0) \beta_t A_t + B^2_{e_t} + B_{e_t} G_i)) \mathcal{U}_2 + 1(t \leq t_0) \beta_t 4 * B_{e_t} \right) \left( \frac{101}{100} \right)^3 \\
\exp\left( \sum_{i=1}^{t-1} 4\lambda_1 \beta_i + dr_t(A_i + B^2_{e_t} + B_{e_t} G_i)C^{(3)} \beta_t^2 \right)
\]

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We reproduce the bounds that we will be requiring in this section from the previous ones. We begin where we have merged previous explicit constants into $c$ which will be useful in our analysis. We first prove the following upper bound:

As desired.

Note that as mentioned before in Section 5.4, we have, restating the bound from Lemma 5.3, we have,

After applying recursion on this equation we obtain:

$$
\mathbb{E} \left[ (v^\top H_t H_t^\top v)^2 \right] \leq \exp\left( \sum_{i=1}^{t} 4\lambda_1 \beta_i + 11\lambda_1^2 \beta_i^2 \right) + \sum_{i=1}^{t} \left( \beta_i^2 dr_i(A_i + B_{e_i}^2 + B_{e_i} G_i)U_2 + 1(i \leq t_0) \beta_i 4B_{e_i} \left( \frac{101}{100} \right)^3 \right) \exp\left( \sum_{j=1}^{i} 4\lambda_1 \beta_j + dr_j(A_j + B_{e_j}^2 + G_j B_{e_j})O(3) \beta_j^2 + \sum_{j=1}^{\min(t,t_0)} \beta_j 2\left( \frac{101}{100} \right)^3 B_{e_j} \right)
$$

As desired.

\[\square\]

### F Convergence Analysis and Main Result

We reproduce the bounds that we will be requiring in this section from the previous ones. We begin by reproducing the lower bound of Lemma 5.3.

$$
\mathbb{E} \left[ \frac{\lVert \hat{u}_n H_n H_n^\top \hat{u}_n \rVert_2^2}{\lVert \hat{u}_n \rVert_2^4} \right] \geq \exp\left( \sum_{i=1}^{n} 2\beta_i \lambda_2 - 4\beta_i^2 \lambda_2^2 \right) - d \sum_{t=1}^{n} c_1 \left( \beta_i^2 r_t + \beta_i \mathbb{1}(t \leq t_0) \right) \exp\left( \sum_{i=1}^{t} 2\beta_i \lambda_2 + c_2 \beta_i^2 dr_i + c_3 \sum_{i=1}^{t_0} \beta_i d \right) \right). \tag{53}
$$

where we have merged previous explicit constants into $c_1, c_2$ and $c_3$, which throughout the course of this section might assume different values. Restating the bound from Lemma 5.4, we have,

$$
\mathbb{E} \left[ \frac{(\hat{u}_n^\top H_n H_n^\top \hat{u}_n)^2}{\lVert \hat{u}_n \rVert_2^4} \right] \leq \exp\left( \sum_{i=1}^{n} 4\lambda_1 \beta_i + 11\lambda_1^2 \beta_i^2 \right) + c_1 \sum_{t=1}^{n} \left( (d\beta_i^2 r_t + 1)(t \leq t_0) \beta_i \right) \exp\left( \sum_{i=1}^{t} 4\lambda_1 \beta_i + c_2 dr_i \beta_i^2 + c_3 \sum_{i=1}^{t_0} \beta_i \right). \tag{54}
$$

Note that as mentioned before in Section B, the term $r_t = O(\log^3(\beta_i^{-1}))$ and $t_0 = O(\log^3(d^2 \beta))$. In the following, we substitute the step size $\beta_t = \frac{b}{d^2 \beta + t}$, where $b, \beta$ are constants, implying that $r_t = O(\log^3(d^2 \beta + t))$.

**Bounds on partial sums of series:** We begin by obtaining bounds on partial sums of some series which will be useful in our analysis. We first prove the following upper bound:

$$
\sum_{i=1}^{t} 4\beta_i \lambda_1 = 4b \lambda_1 \sum_{i=1}^{t} \frac{1}{d^2 \beta + i} = 4b \lambda_1 \sum_{i=d^2 \beta + 1}^{t} \frac{1}{i} \leq 4b \lambda_1 \log \left( \frac{d^2 \beta + t}{d^2 \beta} \right). \tag{55}
$$
We next have the following lower bound:

\[
\sum_{i=1}^{t} 4\beta_i \lambda_1 = 4b \lambda_1 \sum_{i=1}^{t} \frac{1}{d^2 \beta + i} = 4b \lambda_1 \sum_{i=d^2 \beta + 1}^{d^2 \beta + t} \frac{1}{i} \geq 4b \lambda_1 \log \left( \frac{d^2 \beta + t + 1}{d^2 \beta + 1} \right). \tag{56}
\]

We can obtain the following bound on the squared terms:

\[
c \sum_{i=1}^{t} \beta_i^2 \log^3 (d^2 \beta + i) = c \sum_{i=1}^{t} \frac{\log^3 (d^2 \beta + i)}{(d^2 \beta + i)^2} = c \sum_{i=d^2 \beta + 1}^{d^2 \beta + t} \frac{\log^3 (i)}{i^2} \leq c \int_{d^2 \beta}^{\infty} \frac{\log^3 (x)}{x^2} dx \leq c \frac{\log^3 (d \beta)}{d^2 \beta},
\]

where \(c\) is a constant which changes with inequality. Next, we proceed by bounding the excess terms in the exponent corresponding to the summation over the \(t_0\) terms.

\[
c \sum_{i=1}^{t_0} \beta_i \leq cb \log \left( \frac{d^2 \beta + t_0}{d^2 \beta} \right) \leq ct_0 \leq \frac{c \log^3 (d \beta)}{d^2 \beta} \leq c d^2 \beta,
\tag{57}
\]

where the last inequality follows since \(\frac{\log^3 (x)}{x^2} \leq 2\).

**Bounds on** \(\mathbb{E}[v^T H_n H_n^T v]\) and \(\mathbb{E}[(v^T H_n H_n^T v)^2]\): We first proceed by providing upper bounds on Term (I) in (53) and Term (II) in (54).

\[
d \sum_{t=1}^{n} c_1 \left( (d^2 \beta_i r_t + \beta_i \mathbb{I}(t \leq t_0)) \exp \left( \sum_{i=1}^{t} 2 \beta_i \lambda_1 + c_2 \beta_i^2 dr_i + c_3 \sum_{i=1}^{t_0} \beta_i d_1 \right) \right) \leq cd \sum_{t=1}^{n} \left( (d^2 \beta_i r_t + \beta_i \mathbb{I}(t \leq t_0)) \left( \frac{d^2 \beta + t}{d^2 \beta} \right)^{2b \lambda_1} \right).
\]

Similarly term (II) by:

\[
c_1 \sum_{t=1}^{n} \left( (d^2 \beta_i r_t + \mathbb{I}(t \leq t_0)) \exp \left( \sum_{i=1}^{t} 4 \lambda_1 \beta_i + c_2 d r_i \beta_i^2 + c_3 \sum_{i=1}^{t_0} \beta_i \right) \right) \leq c \sum_{t=1}^{n} (d^2 \beta_i r_t + \beta_i \mathbb{I}(t \leq t_0)) \left( \frac{d^2 \beta + t}{d^2 \beta} \right)^{4b \lambda_1}.
\]

**Lemma 20.** For any \(\delta_1 \in (0, 1)\) and \(n\) satisfying:

\[
\frac{d^2 \beta + n}{\log_{\text{max}(1,2b\lambda_1)} (d^2 \beta + n)} \geq \max \left( \frac{\exp \left( \frac{c \lambda^2}{d^2} \right)}{\delta_1} \right)^{1/2b \lambda_1} (d^2 \beta + 1),
\]

\[
\frac{cd^2 \beta + 1}{\delta_1} \exp \left( \frac{c \lambda^2}{d^2} \right) \left( \left( 1 + \frac{1}{d^2 \beta} \right)^{2b \lambda_1} + d^2 \beta \right), \frac{c_2 \beta^2}{\delta_1} \exp \left( \frac{c \lambda^2}{d^2} \right)
\]

we have that

\[
\mathbb{E}[\tilde{u}_1^T H_n H_n^T \tilde{u}_1] \geq (1 - \delta_1) \exp \left( \sum_{t=1}^{n} 2 \beta_t \lambda_1 - 4b \beta_t^2 \lambda_1^2 \right),
\]

where \(c\) depends polynomially on \(b, \beta, \lambda_1\).
Proof. We consider the term $\mathbb{E}[\tilde{u}_1^\top H_n H_n^\top \tilde{u}_1]$ from Equation (53),

$$\frac{\mathbb{E}[\tilde{u}_1^\top H_n H_n^\top \tilde{u}_1]}{\|\tilde{u}_1\|_2^2} \geq \exp\left(\sum_{t=1}^n 2\beta_t \lambda_1 - 4\beta_t^2 \lambda_1^2\right) - cd \sum_{t=1}^n (\beta_t^2 r_t + \beta_t \mathbb{I}(t \leq t_0)) \left(\frac{d^2 \beta + t}{d^2 \beta}\right)^{2b\lambda_1}$$

$$+ \delta_1 \exp \left(\sum_{t=1}^n 2\beta_t \lambda_1 - 4\beta_t^2 \lambda_1^2\right)$$

$$\geq (1 - \delta_1) \exp \left(\sum_{t=1}^n 2\beta_t \lambda_1 - 4\beta_t^2 \lambda_1^2\right) - \frac{c}{d^{4b\lambda_1-1}} \sum_{t=1}^n (\beta_t^2 r_t + \beta_t \mathbb{I}(t \leq t_0)) (d^2 \beta + t)^{2b\lambda_1}$$

$$+ \delta_1 \exp \left(- \frac{c \lambda_1^2}{d^2}\right) \left(\frac{d^2 \beta + n + 1}{d^2 \beta + 1}\right)^{2b\lambda_1}$$

$$\geq (1 - \delta_1) \exp \left(\sum_{t=1}^n 2\beta_t \lambda_1 - 4\beta_t^2 \lambda_1^2\right) - \frac{c}{d^{4b\lambda_1-1}} \sum_{t=1}^n (\beta_t^2 r_t + \beta_t \mathbb{I}(t \leq t_0)) (d^2 \beta + t)^{2b\lambda_1}$$

$$+ \delta_1 \exp \left(- \frac{c \lambda_1^2}{d^2}\right) \left(\frac{d^2 \beta + n + 1}{d^2 \beta + 1}\right)^{2b\lambda_1} - \frac{c}{d^{4b\lambda_1-1}} \sum_{t=1}^{t_0} (d^2 \beta + t)^{2b\lambda_1-1}$$

$$\geq (1 - \delta_1) \exp \left(\sum_{t=1}^n 2\beta_t \lambda_1 - 4\beta_t^2 \lambda_1^2\right) - \frac{c}{d^{4b\lambda_1-1}} \sum_{t=1}^n (\beta_t^2 r_t + \beta_t \mathbb{I}(t \leq t_0)) (d^2 \beta + t)^{2b\lambda_1}$$

$$+ \delta_1 \exp \left(- \frac{c \lambda_1^2}{d^2}\right) \left(\frac{d^2 \beta + n + 1}{d^2 \beta + 1}\right)^{2b\lambda_1} - \frac{c}{2b\lambda_1 d^{4b\lambda_1-1}} (d^2 \beta + \log^3(d\beta))^{2b\lambda_1}$$

$$\geq (1 - \delta_1) \exp \left(\sum_{t=1}^n 2\beta_t \lambda_1 - 4\beta_t^2 \lambda_1^2\right) - \frac{c}{d^{4b\lambda_1-1}} \sum_{t=1}^n (\beta_t^2 r_t + \beta_t \mathbb{I}(t \leq t_0)) (d^2 \beta + t)^{2b\lambda_1}$$

$$+ \delta_1 \exp \left(- \frac{c \lambda_1^2}{d^2}\right) \left(\frac{d^2 \beta + n + 1}{d^2 \beta + 1}\right)^{2b\lambda_1} - \frac{c}{d^{4b\lambda_1-1} d^{4b\lambda_1-1}} (d^2 \beta + t)^{2b\lambda_1-1}$$

$$\geq (1 - \delta_1) \exp \left(\sum_{t=1}^n 2\beta_t \lambda_1 - 4\beta_t^2 \lambda_1^2\right) - \frac{c}{d^{4b\lambda_1-1}} \sum_{t=1}^n (\beta_t^2 r_t + \beta_t \mathbb{I}(t \leq t_0)) (d^2 \beta + t)^{2b\lambda_1-2}$$

$$+ \delta_1 \exp \left(- \frac{c \lambda_1^2}{d^2}\right) \left(\frac{d^2 \beta + n + 1}{d^2 \beta + 1}\right)^{2b\lambda_1} - \frac{c}{d^{4b\lambda_1-1}} \sum_{t=1}^n (d^2 \beta + t)^{2b\lambda_1-2}$$

$$\geq (1 - \delta_1) \exp \left(\sum_{t=1}^n 2\beta_t \lambda_1 - 4\beta_t^2 \lambda_1^2\right) - \frac{c}{d^{4b\lambda_1-1}} \sum_{t=1}^n (d^2 \beta + t)^{2b\lambda_1-2}$$

$$+ \delta_1 \exp \left(- \frac{c \lambda_1^2}{d^2}\right) \left(\frac{d^2 \beta + n + 1}{d^2 \beta + 1}\right)^{2b\lambda_1} - \frac{c}{d^{4b\lambda_1-1}} \sum_{t=1}^n (d^2 \beta + t)^{2b\lambda_1-2}$$

$$+ \delta_1 \exp \left(- \frac{c \lambda_1^2}{d^2}\right) \left(\frac{d^2 \beta + n + 1}{d^2 \beta + 1}\right)^{2b\lambda_1} - \frac{c}{d^{4b\lambda_1-1}} \sum_{t=1}^n (d^2 \beta + t)^{2b\lambda_1-2}$$

where $\zeta_1$ from using $\sum_{i=1}^n \gamma_i \leq n^{\gamma+1}/\gamma + 1$ for $\gamma > -1$ and $\zeta_2$ follows from the fact that $\log^3(x) \leq cx$. We now consider the following three cases:

**Case 1:** $2b\lambda_1 < 1$

In this case we can lower bound the term $\mathbb{E}[\tilde{u}_1^\top H_n H_n^\top \tilde{u}_1]/\|\tilde{u}_1\|_2^2$ as,

$$\frac{\mathbb{E}[\tilde{u}_1^\top H_n H_n^\top \tilde{u}_1]}{\|\tilde{u}_1\|_2^2} \geq (1 - \delta_1) \exp \left(\sum_{t=1}^n 2\beta_t \lambda_1 - 4\beta_t^2 \lambda_1^2\right) - \frac{c}{d^{4b\lambda_1-1}} \sum_{t=1}^n (d^2 \beta + t)^{2b\lambda_1}$$

$$+ \delta_1 \exp \left(- \frac{c \lambda_1^2}{d^2}\right) \left(\frac{d^2 \beta + n + 1}{d^2 \beta + 1}\right)^{2b\lambda_1} - \frac{c}{d^{4b\lambda_1-1}} \sum_{t=1}^n (d^2 \beta + t)^{2b\lambda_1-2}$$

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where $\zeta_1$ follows by using that

$$\frac{d^2 \beta + n}{\log^{3/2b\lambda_1}(d^2 \beta + n)} \geq \left( \frac{d^1}{\delta_1} \right)^{1/2b\lambda_1} (d^2 \beta + 1).$$

**Case 2:** $2b\lambda_1 > 1$

In this case, we can lower bound the term $\frac{E[\hat{u}_1^TH_n H_n^T \hat{u}_1]}{\|\hat{u}_1\|_2^2}$ as,

$$\begin{align*}
\frac{E[\hat{u}_1^TH_n H_n^T \hat{u}_1]}{\|\hat{u}_1\|_2^2} &\geq (1 - \delta_1) \exp \left( \sum_{i=1}^n 2\beta_i \lambda_1 - 4\beta_i^2 \lambda_1^2 \right) - \frac{c \log^3(d^2 \beta + n)(d^2 \beta + n)^{2b\lambda_1-1}}{d^{4b\lambda_1-1}} \frac{1}{2b\lambda_1 - 1} \\
&\quad + \delta_1 \exp \left( - \frac{c' \lambda_1^2}{d^2} \right) \left( \frac{d^2 \beta + n + 1}{d^2 \beta + 1} \right)^{2b\lambda_1} - c\beta^{2b\lambda_1} d \\
&\geq \left( \frac{d^2 \beta + n}{d^2 \beta + 1} \right)^{2b\lambda_1} \left( \delta_1 \exp \left( - \frac{c' \lambda_1^2}{d^2} \right) - c\beta^{2b\lambda_1} d \left( \frac{d^2 \beta + 1}{d^2 \beta + n} \right) \right) - cd\beta^{2b\lambda_1} \left( 1 + \frac{1}{d^2 \beta} \right)^{2b\lambda_1} \frac{\log^3(d^2 \beta + n)}{d^2 \beta + n} + (1 - \delta_1) \exp \left( \sum_{i=1}^n 2\beta_i \lambda_1 - 4\beta_i^2 \lambda_1^2 \right)
\end{align*}$$

where $\zeta_1$ follows by using that

$$\frac{d^2 \beta + n}{\log^{3/2b\lambda_1}(d^2 \beta + n)} \geq \left( \frac{d^1}{\delta_1} \right)^{1/2b\lambda_1} (d^2 \beta + 1).$$

**Case 3:** $2b\lambda_1 = 1$

In this case, we can lower bound the term $\frac{E[\hat{u}_1^TH_n H_n^T \hat{u}_1]}{\|\hat{u}_1\|_2^2}$ as,

$$\begin{align*}
\frac{E[\hat{u}_1^TH_n H_n^T \hat{u}_1]}{\|\hat{u}_1\|_2^2} &\geq (1 - \delta_1) \exp \left( \sum_{i=1}^n 2\beta_i \lambda_1 - 4\beta_i^2 \lambda_1^2 \right) - \frac{c \log^4(d^2 \beta + n)}{d} \\
&\quad + \delta_1 \exp \left( - \frac{c' \lambda_1^2}{d^2} \right) \left( \frac{d^2 \beta + n + 1}{d^2 \beta + 1} \right) - c\beta d \\
&\geq (1 - \delta_1) \exp \left( \sum_{i=1}^n 2\beta_i \lambda_1 - 4\beta_i^2 \lambda_1^2 \right)
\end{align*}$$
where \( \zeta_1 \) follows from using
\[
\frac{d^2 \beta + n}{\log^4(d^2 \beta + n)} \geq \frac{c \beta^2 d^3 \exp \left( \frac{c \lambda_1^2}{d^2 \beta} \right)}{\delta_1}.
\]

\[\square\]

**Lemma 21.** For any \( \delta_2 \in (0, 1) \) and \( n \) satisfying.
\[
\frac{d^2 \beta + n}{\log^4(1/4b\lambda_1)}(d^2 \beta + n) \geq \max \left( \frac{c(d^2 \beta + 1)}{(\delta_2 \log^3(d\beta))^{4b\lambda_1}}, \frac{c^{4b\lambda_1}}{\delta_2}(d^2 \beta + n) \right),
\]
we have that,
\[
\mathbb{E} \left[ \frac{(\tilde{u}_1^T H_n H_n^T \tilde{u}_1)^2}{\|\tilde{u}_1\|_2^2} \right] \leq (1 + \delta_2) \exp \left( \sum_{t=1}^n 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right),
\]
where \( c \) depends polynomially on \( b, \beta, \lambda_1, \Delta_\lambda \).

**Proof.** We consider the term \( \mathbb{E} \left[ (\tilde{u}_1^T H_n H_n^T \tilde{u}_1)^2 \right] \) from Equation (54),
\[
\mathbb{E} \left[ (\tilde{u}_1^T H_n H_n^T \tilde{u}_1)^2 \right] \leq \exp \left( \sum_{t=1}^n 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right) + c \sum_{t=1}^n (d\beta_t^2 r_t + \beta_t \mathbb{I}(t \leq t_0)) \left( \frac{d^2 \beta + t}{d^2 \beta} \right)^{4b\lambda_1}
\]
\[
= c \sum_{t=1}^n (d\beta_t^2 r_t + \beta_t \mathbb{I}(t \leq t_0)) \left( \frac{d^2 \beta + t}{d^2 \beta} \right)^{4b\lambda_1} - \delta_2 \exp \left( \sum_{t=1}^n 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right)
+ (1 + \delta_2) \exp \left( \sum_{t=1}^n 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right)
\]
\[
= c \sum_{t=1}^n (d\beta_t^2 r_t) \left( \frac{d^2 \beta + t}{d^2 \beta} \right)^{4b\lambda_1} - \delta_2 \exp \left( \sum_{t=1}^n 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right)
+ cb \sum_{t=1}^{t_0} \frac{1}{d^2 \beta + t} \left( \frac{d^2 \beta + t}{d^2 \beta} \right)^{4b\lambda_1} + (1 + \delta_2) \exp \left( \sum_{t=1}^n 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right)
\]
\[
= c \sum_{t=1}^n (d\beta_t^2 r_t) \left( \frac{d^2 \beta + t}{d^2 \beta} \right)^{4b\lambda_1} - \delta_2 \exp \left( \sum_{t=1}^n 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right)
+ \frac{cb}{(d^2 \beta)^{4b\lambda_1}} \sum_{t=1}^{t_0} (d^2 \beta + t)^{4b\lambda_1-1} + (1 + \delta_2) \exp \left( \sum_{t=1}^n 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right)
\]
\[
\leq c \sum_{t=1}^n (d\beta_t^2 r_t) \left( \frac{d^2 \beta + t}{d^2 \beta} \right)^{4b\lambda_1} - \delta_2 \exp \left( \sum_{t=1}^n 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right) + \frac{c^{4b\lambda_1}}{\lambda_1}
+ (1 + \delta_2) \exp \left( \sum_{t=1}^n 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right)
\]
\[
\leq \frac{cd b^2}{(d^2 \beta)^{4b\lambda_1}} \sum_{t=1}^n \log^3(d^2 \beta + t)(d^2 \beta + t)^{4b\lambda_1-2} - \delta_2 \exp \left( \sum_{t=1}^n 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right)
+ \frac{c^{4b\lambda_1}}{\lambda_1} + (1 + \delta_2) \exp \left( \sum_{t=1}^n 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right)
\]
\[
\begin{align*}
&\leq \frac{cd_b \log^3(d^2 \beta + n)}{(d^2 \beta)^{4b\lambda_1}} \sum_{t=1}^{n} (d^2 \beta + t)^{4b\lambda_1 - 2} - \delta_2 \exp \left( \sum_{t=1}^{n} 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right) + \frac{c_{4b\lambda_1}}{\lambda_1} \\
&+ (1 + \delta_2) \exp \left( \sum_{t=1}^{n} 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right) \\
&\leq \frac{cd_b \log^3(d^2 \beta + n)}{(d^2 \beta)^{4b\lambda_1}} \sum_{t=1}^{n} (d^2 \beta + t)^{4b\lambda_1 - 2} - \delta_2 \left( \frac{d^2 \beta + n + 1}{d^2 \beta + 1} \right)^{4b\lambda_1} + \frac{c_{4b\lambda_1}}{\lambda_1} \\
&+ (1 + \delta_2) \exp \left( \sum_{t=1}^{n} 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right),
\end{align*}
\]

where \(\zeta_1\) follows by using the fact that \(\sum_{i=1}^{n} i^\gamma \leq n^{\gamma+1}/\gamma + 1\) for \(\gamma > -1\). We consider now the following three cases as before:

**Case 1: \(4b\lambda_1 < 1\)**

In this case, we can upper bound the term \(\mathbb{E}[\|\mathbf{u}_1^\top \mathbf{H}_n \mathbf{H}_n^\top \mathbf{u}_1\|_2^2]\) as,

\[
\mathbb{E}[\|\mathbf{u}_1^\top \mathbf{H}_n \mathbf{H}_n^\top \mathbf{u}_1\|_2^2] \leq \frac{cb \log^3(d^2 \beta + n)}{d^2 \beta} - \delta_2 \left( \frac{d^2 \beta + n + 1}{d^2 \beta + 1} \right)^{4b\lambda_1} + \frac{c_{4b\lambda_1}}{\lambda_1} \\
+ (1 + \delta_2) \exp \left( \sum_{t=1}^{n} 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right) \\
\leq (1 + \delta_2) \exp \left( \sum_{t=1}^{n} 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right),
\]

where \(\zeta_1\) follows from using that

\[
\frac{d^2 \beta + n}{\log^{4b\lambda_1} (d^2 \beta + n)} \geq \frac{c(d^2 \beta + 1)}{(\delta_2 \log^3(d^2 \beta))^{4b\lambda_1}}.
\]

**Case 2: \(4b\lambda_1 > 1\)**

In this case, we can upper bound the term \(\mathbb{E}[\|\mathbf{u}_1^\top \mathbf{H}_n \mathbf{H}_n^\top \mathbf{u}_1\|_2^2]\) as,

\[
\mathbb{E}[\|\mathbf{u}_1^\top \mathbf{H}_n \mathbf{H}_n^\top \mathbf{u}_1\|_2^2] \leq \frac{cb \log^3(d^2 \beta + n)}{(d^2 \beta)^{4b\lambda_1}} \sum_{t=1}^{n} (d^2 \beta + t)^{4b\lambda_1 - 2} - \delta_2 \left( \frac{d^2 \beta + n + 1}{d^2 \beta + 1} \right)^{4b\lambda_1} + \frac{c_{4b\lambda_1}}{\lambda_1} \\
+ (1 + \delta_2) \exp \left( \sum_{t=1}^{n} 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right) \\
\leq \frac{cb \log^3(d^2 \beta + n)}{(d^2 \beta)^{4b\lambda_1}} \sum_{t=1}^{n} (d^2 \beta + t)^{4b\lambda_1 - 1} - \delta_2 \left( \frac{d^2 \beta + n + 1}{d^2 \beta + 1} \right)^{4b\lambda_1} + \frac{c_{4b\lambda_1}}{\lambda_1} \\
+ (1 + \delta_2) \exp \left( \sum_{t=1}^{n} 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right) \\
\leq (1 + \delta_2) \exp \left( \sum_{t=1}^{n} 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right),
\]

where \(\zeta_2\) follows by using that

\[
\frac{d^2 \beta + n}{\log^4(d^2 \beta + n)} \geq \frac{c_{4b\lambda_1}}{\delta_2}(d^2 \beta + 1).
\]
Case 3: $4b\lambda_1 = 1$

In this case, we can upper bound the term $\frac{\mathbb{E}[\langle \tilde{u}_1^\top H_n H_n^\top \tilde{u}_1 \rangle]^2}{\|\tilde{u}_1\|_2^4}$ as,

$$
\mathbb{E}[\langle \tilde{u}_1^\top H_n H_n^\top \tilde{u}_1 \rangle]^2 \leq \frac{cd^2 \log^3(d^2 \beta + n)}{(d^2 \beta)} \sum_{t=1}^{n} (d^2 \beta + t)^{-1} - \delta_2 \left( \frac{d^2 \beta + n + 1}{d^2 \beta + 1} \right) + \frac{c}{\lambda_1}
$$

$$
+ (1 + \delta_2) \exp \left( \sum_{t=1}^{n} 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right)
$$

$$
\leq \frac{cd^2 \log^3(d^2 \beta + n)}{(d^2 \beta)} \log \left( \frac{d^2 \beta + n + 1}{d^2 \beta + 1} \right) + \frac{c}{\lambda_1}
$$

$$
+ (1 + \delta_2) \exp \left( \sum_{t=1}^{n} 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right)
$$

$$
\leq (1 + \delta_2) \exp \left( \sum_{t=1}^{n} 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right),
$$

where $\zeta_1$ holds due to

$$
\frac{d^2 \beta + n}{\log^4(d^2 \beta + n)} \geq \frac{cd}{\delta_2}.
$$

\[\square\]

F.1 Convergence Theorem

We begin by restating the bound obtained on $\mathbb{E} \left[ \text{Tr}(V_\perp^\top H_n H_n^\top V_\perp) \right]$ in Lemma 16.

$$
\mathbb{E} \left[ \text{Tr}(V_\perp^\top H_n H_n^\top V_\perp) \right] \leq \exp \left( \sum_{t=1}^{n} 2\beta_t \lambda_2 + \beta_t^2 \lambda_2^2 \right)
$$

$$
\left( \text{Tr}(V_\perp V_\perp^\top) + cd\|V_\perp V_\perp^\top\|_2 \sum_{t=1}^{n} (r_t \beta_t^2 + 1)(t \leq t_0) \beta_t d \exp \left( 2 \sum_{i=1}^{t} \beta_i (\lambda_1 - \lambda_2) + cd \beta_t^2 r_t + c \sum_{i=1}^{t} \beta_i d \right) \right)
$$

$$
\left. \zeta_1 \leq \exp \left( \sum_{t=1}^{n} 2\beta_t \lambda_2 + \beta_t^2 \lambda_2^2 \right) \left( \text{Tr}(V_\perp V_\perp^\top) \right) \right) \quad (58)
$$

$$
+ cd\|V_\perp V_\perp^\top\|_2 \sum_{t=1}^{n} (r_t \beta_t^2 + 1)(t \leq t_0) \beta_t d \exp \left( 2 \sum_{i=1}^{t} \beta_i (\lambda_1 - \lambda_2) + cd \beta_t^2 r_t \right), \quad (59)
$$

where $\zeta_1$ follows from using Equation (57).

Theorem 3 (Convergence Theorem). Let $\delta > 0$ and the step sizes $\beta_t = \frac{b}{d^2 \beta_0 + t}$. The output $v_n$ of Algorithm 1 for $n$ satisfying the assumption in Lemma 20 and 21 is an $\epsilon$-approximation to $u_1$ with probability at least $1 - \delta$ where,

$$
\sin^2 \left( \frac{u_1^\top v_n}{\epsilon} \right) \leq \frac{d\|V_\perp V_\perp^\top\|_2}{Q} \exp \left( 5\lambda_2^2 \sum_{t=1}^{n} \beta_t^2 \right) \left( \exp \left( -2\Delta_\lambda \sum_{t=1}^{n} \beta_t \right) \right)
$$

$$
+ c \sum_{t=1}^{n} (r_t \beta_t^2 + 1)(t \leq t_0) \beta_t d \exp \left( -2\Delta_\lambda \sum_{i=t+1}^{n} \beta_i \right),
$$

where $\Delta_\lambda = \lambda_1 - \lambda_2$ and

$$
Q = \frac{2\beta^2 \|\tilde{u}_1\|_2^2}{(2 + \epsilon_1)c \log(1/\delta)} \left( 1 - \frac{1}{\sqrt{\delta}} \right) \left( (1 + \epsilon_1) \exp \left( 19 \sum_{i=1}^{n} \beta_i^2 \lambda_i^2 \right) - 1 \right),
$$

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The constant $c$ occurring in the equations, as before depends polynomially on problem dependent parameters $b, \lambda_1, \Delta_\lambda$ and the parameters $\frac{\delta_1}{2} = \frac{\epsilon_1}{2t+1}$.

**Proof.** First, using the Chebychev’s inequality, we have:

$$P \left[ \left\| \tilde{u}_1^T H_n H_n^T \tilde{u}_1 - E \left[ \tilde{u}_1^T H_n H_n^T \tilde{u}_1 \right] \right\| \geq 1 \right] \leq \frac{1}{\delta} \sqrt{\text{Var}[\tilde{u}_1^T H_n H_n^T \tilde{u}_1]} \leq \delta.$$  

With probability greater than $1 - \delta$, we have,

$$\tilde{u}_1^T H_n H_n^T \tilde{u}_1 \geq E \left[ \tilde{u}_1^T H_n H_n^T \tilde{u}_1 \right] - \frac{1}{\sqrt{\delta}} \sqrt{\text{Var}[\tilde{u}_1^T H_n H_n^T \tilde{u}_1]} = E \left[ \tilde{u}_1^T H_n H_n^T \tilde{u}_1 \right] \left( 1 - \frac{1}{\sqrt{\delta}} \sqrt{\frac{E[(\tilde{u}_1^T H_n H_n^T \tilde{u}_1)^2]}{E[\tilde{u}_1^T H_n H_n^T \tilde{u}_1]^2}} - 1 \right)$$

(60)

Now, using Lemma 21, we have that,

$$\frac{E[(\tilde{u}_1^T H_n H_n^T \tilde{u}_1)^2]}{\|\tilde{u}_1\|^2} \leq (1 + \delta_2) \exp \left( \sum_{t=1}^{n} 4\lambda_1 \beta_t + 11\lambda_1^2 \beta_t^2 \right)$$

(61)

and using Lemma 20, we have,

$$\frac{E[\tilde{u}_1^T H_n H_n^T \tilde{u}_1]}{\|\tilde{u}_1\|^2} \geq (1 - \delta_1) \exp \left( \sum_{t=1}^{n} 2\beta_t \lambda_1 - 4\beta_t^2 \lambda_1^2 \right),$$

squaring the above, we obtain,

$$\frac{E[\tilde{u}_1^T H_n H_n^T \tilde{u}_1]^2}{\|\tilde{u}_1\|^4} \geq (1 - \delta_1') \exp \left( \sum_{t=1}^{n} 4\beta_t \lambda_1 - 8\beta_t^2 \lambda_1^2 \right),$$

(62)

where $\delta_1' = 2\delta_1$. Setting $\delta_1' = \frac{\epsilon_1}{2t+1}$ and substituting bounds (61) and (62) in (60), we obtain,

$$\tilde{u}_1^T H_n H_n^T \tilde{u}_1 \geq \frac{2\|\tilde{u}_1\|^2}{2 + \epsilon_1} \exp \left( \sum_{t=1}^{n} 2\beta_t \lambda_1 - 4\beta_t^2 \lambda_1^2 \right) \left( 1 - \frac{1}{\sqrt{\delta}} \sqrt{\frac{E[(\tilde{u}_1^T H_n H_n^T \tilde{u}_1)^2]}{E[\tilde{u}_1^T H_n H_n^T \tilde{u}_1]^2}} - 1 \right).$$

Further, using the Equation (58) along with Markov’s inequality, we have with probability atleast $1 - \delta$

$$\text{Tr}(V_{\perp}^T H_n H_n^T V_{\perp}) \leq \frac{1}{\delta} \exp \left( \sum_{t=1}^{n} 2\beta_t \lambda_2 + \beta_t^2 \lambda_2^2 \right) \left( \text{Tr}(V_{\perp}^T V_{\perp}^T) \right.  

\left. + cd\|V_{\perp}^T V_{\perp}^T\|_2 \sum_{t=1}^{n} \left( r_t \beta_t^2 \right) \text{I}(t \leq t_0) \beta_t d \right) \exp \left( 2 \sum_{t=1}^{n} \beta_t (\lambda_1 - \lambda_2) + cd \beta_t^2 r_t \right).$$

Combining the above with Lemma 2, we have that the output $v_n$ of Algorithm 1 is an $\epsilon$-approximation to $u_1$ with probability atleast $1 - \delta$.

$$\epsilon \leq \frac{c \log(1/\delta) (2 + \epsilon_1) \exp \left( \sum_{t=1}^{n} -2\beta_t \lambda_1 + 4\beta_t^2 \lambda_1^2 \right) \text{Tr}(V_{\perp}^T H_n H_n^T V_{\perp})}{2\delta\|\tilde{u}_1\|^2} \right) \left( 1 - \frac{1}{\sqrt{\delta}} \sqrt{1 - \frac{1}{\epsilon_1} \exp \left( 19 \sum_{t=1}^{n} \beta_t^2 \lambda_1^2 \right) - 1} \right)$$

$$\leq \frac{d\|V_{\perp} V_{\perp}^T\|_2}{Q} \exp \left( 5\lambda_1^2 \sum_{t=1}^{n} \beta_t^2 \right) \left( \exp \left( -2\Delta_\lambda \sum_{t=1}^{n} \beta_t \right) \right)$$
and satisfies,
\[ + c \sum_{t=1}^{n} (r_t \beta_t^2 + \|t \leq t_0\| \beta_t d) \exp \left( -2\Delta \sum_{i=t+1}^{n} \beta_i \right), \]

where \( \Delta = \lambda_1 - \lambda_2 \) and
\[ Q = \frac{2\delta^2 \|\tilde{u}_1\|_2^2}{(2 + \epsilon_1)c \log(1/\delta)} \left( 1 - \frac{1}{\sqrt{\delta}} \left( (1 + \epsilon_1) \exp \left( 19 \sum_{t=1}^{n} \beta_t^2 \lambda_t^2 \right) - 1 \right) \right). \]

\[ \Box \]

F.2 Main Result

In this section, we state our main theorem and instantiate the parameters of our algorithm.

**Theorem 4 (Main Result).** Fix any \( \delta > 0 \) and \( \epsilon_1 > 0 \). Suppose that the step sizes are set to \( \alpha_t = \frac{\delta}{\log(d^2 \beta + t)} \) and \( \beta_t = \frac{\gamma}{\Delta \lambda (d^2 \beta + t)} \), for \( \gamma > 1/2 \) and
\[ \beta = \max \left( \frac{20\gamma^2 \lambda_2}{\Delta \lambda d^2 \log \left( \frac{1 + \delta/100}{1 + \epsilon_1} \right)}, \frac{200 \left( \frac{R}{\mu} + \frac{R^3}{\mu^2} + \frac{R^4}{\mu^2} \right) \log(1 + \frac{R^2}{\mu} + \frac{R^4}{\mu^2})}{\delta \Delta \lambda} \right). \]

Suppose that the number of samples \( n \) satisfy the assumptions of Lemma 20 and 21. Then, the output \( v_n \) of Algorithm 1 satisfies,
\[ \sin_B^2(u_1, v_n) \leq \left( 2 + \epsilon_1 \right) cd \|\tilde{u}_1\|_2^2 \log \left( \frac{1}{\delta} \right) \left( \frac{d^2 \beta + 1}{d^2 \beta + n + 1} \right)^{2\gamma} \left( \frac{d^2 \beta + \log^3(d^2 \beta + n)}{d^2 \beta + n + 1} \right)^{2\gamma}, \]

with probability at least \( 1 - \delta \) with \( c \) depending polynomially on parameters of the problem \( \lambda_1, \kappa_B, R, \mu \).

The parameters \( \delta_1, \delta_2 \) are set as \( \delta_1 = \frac{\epsilon_1}{2(2\gamma + 1)} \) and \( \delta_2 = \frac{\epsilon_1}{2 + \epsilon_1} \).

**Proof.** With the step size \( \beta_t = \frac{b}{d^2 \beta + t} \), we set the parameter \( b = \frac{\gamma}{\lambda_1 - \lambda_2} \) and thus get \( \beta_t = \frac{\gamma}{\Delta \lambda (d^2 \beta + t)} \). Now, we have that
\[ \sum_{t=1}^{n} \beta_t^2 \leq \frac{\gamma^2}{\Delta \lambda d^2 \beta} \]
and using the assumption that \( \frac{\gamma^2 \lambda_2}{\Delta \lambda d^2 \beta} \leq \frac{1}{19} \log \left( \frac{1 + \delta/100}{1 + \epsilon_1} \right) \), we obtain,
\[ \sqrt{\left( (1 + \epsilon_1) \exp \left( 19 \sum_{t=1}^{n} \beta_t^2 \lambda_t^2 \right) - 1 \right)} \geq \frac{9}{10} \implies Q \geq \frac{c d^2 \|\tilde{u}_1\|_2^2}{(2 + \epsilon_1) \log(1/\delta)}. \] (63)

Using previous bounds on sums of partial harmonic sums, we have that,
\[ \sum_{t=1}^{n} \beta_t \geq \frac{\gamma}{\Delta \lambda} \log \left( \frac{d^2 \beta + n + 1}{d^2 \beta + 1} \right) \quad \text{and} \quad \sum_{i=t+1}^{n} \beta_i \geq \frac{\gamma}{\Delta \lambda} \log \left( \frac{d^2 \beta + n + 1}{d^2 \beta + t + 1} \right). \]

Using these bounds, we obtain,
\[ \exp \left( -2\Delta \sum_{t=1}^{n} \beta_t \right) \leq \left( \frac{d^2 \beta + 1}{d^2 \beta + n + 1} \right)^{2\gamma}. \] (64)
In order to bound the remaining terms from Theorem 3, we note that,

$$
c_n \sum_{i=1}^n (r_i \beta_i^2 + \|t \leq t_0\| \beta_i d) \exp \left( -2\Delta \sum_{i=t+1}^n \beta_i \right) \leq c_n \sum_{i=1}^n \frac{r_i \gamma^2}{(\Delta \gamma)(d^2 \beta + t)} \left( \frac{d^2 \beta + t + 1}{2d^2 \beta + n + 1} \right)^{2\gamma} + cd \sum_{i=1}^t \gamma \frac{d^2 \beta + t + 1}{2d^2 \beta + n + 1} \frac{2\gamma}{\Delta \gamma},
$$

where the last bounds holds for any $\gamma > 1/2$. Substituting bounds (63),(64) and (65) in the result of Theorem 3, we obtain that the output $v_n$ of Algorithm 1 satisfies,

$$
\sin^2_{B}(u_1, v_n) \leq \frac{(2 + \epsilon_1)cd}{d^2 \|u_1\|_2^2} \left( \sum_{t=1}^d \bar{u}_t \bar{u}_t^\top \log \left( \frac{1}{\bar{u}_t^\top \bar{u}_t} \right) \frac{d^2 \beta + t + 1}{d^2 \beta + n + 1} \right) \frac{2\gamma}{\Delta \gamma} \frac{d^2 \beta + n}{d^2 \beta + n + 1} + \frac{cd}{\Delta \gamma} \left( \frac{d^2 \beta + n}{d^2 \beta + n + 1} \right)^{2\gamma}.
$$

G Auxiliary Properties

G.1 Useful Trace Inequalities

In this section we enumerate some useful inequalities.

**Lemma 22.**

1. $\langle A, B \rangle \leq \langle A, C \rangle$ for PSD matrices $A, B, C$ with $B \preceq C$.

2. $\text{Tr}(A^\top B) \leq \frac{1}{2} \text{Tr}(A^\top A + B^\top B)$ for all matrices $A, B \in \mathbb{R}^{m \times n}$.

As a consequence:

**Corollary 3.** $\langle A, B \rangle \leq \langle A, C \rangle$ for a PSD matrix $A$ and $B \preceq C$, with $B$ and $C$ symmetric.

**Proof.** If $B$ is PSD, the result follows immediately from the previous lemma. Otherwise let $\lambda_{\min}$ be the smallest eigenvalue of $B$. Let $B' = B + |\lambda_{\min}|I$ and $C' = C + |\lambda_{\min}|I$. The matrices $B'$ and $C'$ are PSD and satisfy $B' \preceq C'$. The result follows by applying the lemma above and rearranging the terms. 

G.2 Useful spectral norm Inequalities

In this section we enumerate some useful inequalities.

**Lemma 23.** If $0 \preceq B \preceq C$ and symmetric then $0 \preceq ABA^\top \preceq ACA^\top$.

As a consequence:

**Corollary 4.** If $0 \preceq B \preceq C$ and symmetric then $\|ABA^\top\| \leq \|ACA^\top\|$.

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G.3 Properties concerning Eigenvectors of $B^{-1}A$

In this subsection, we highlight some important properties concerning the left and right eigenvectors of the matrix under consideration $B^{-1}A$.

As before, we let $\tilde{u}_1, \ldots, \tilde{u}_d$ denote the left eigenvectors and $u_1, \ldots, u_d$ denote the right eigenvectors of $B^{-1}A$.

**Lemma 24.** The right eigenvectors of the matrix $B^{-1}A$ satisfy the following:

$$u_i^\top Bu_j = 0 \quad \text{if } i \neq j.$$  

**Proof.** Consider the symmetric matrix $C = B^{-1/2}AB^{-1/2}$. Let $u_i^C, \ldots, u_d^C$ be the eigenvectors of $C$. Notice that if $u_i^C$ is an eigenvector of $C$ with eigenvalue $\lambda_i$, then

$$B^{-1/2}(B^{-1/2}AB^{-1/2})u_i^C = \lambda_i B^{-1/2}u_i^C,$$

implying that $B^{-1/2}u_i^C$ is a right eigenvector of $B^{-1}A$, $u_i$. Therefore the eigenvector of $C$ are related to the right eigenvectors of $B^{-1}A$ as $B^{1/2}u_i = u_i^C$. Further, since the matrix $C$ is symmetric, its eigenvectors can be taken to form an orthogonal basis, and hence,

$$(u_i^C)^\top u_j^C = u_i^\top Bu_j = 0 \quad \text{if } i \neq j.$$  

$\square$

**Lemma 25.** Let $u_1$ denote the top right eigenvector of $B^{-1}A$. Then,

$$\tilde{u}_i^\top u_1 = 0 \quad \text{for all } i \geq 2,$$

where $\tilde{u}_i$ represent the left eigenvectors of the matrix $B^{-1}A$.

**Proof.** We begin by noting that the left and right eigenvectors of the matrix $B^{-1}A$ are related as $\tilde{u}_i = Bu_i$, which follows from,

$$(B^{-1}A)B^{-1}\tilde{u}_i = B^{-1}(AB^{-1})\tilde{u}_i = \lambda_i B^{-1}\tilde{u}_i$$

As a consequence $B^{-1}\tilde{u}_i$ is a right eigenvector of $B^{-1}A$ and the lemma now follows from using Lemma 24.  

$\square$

As a consequence of Lemma 25, we have the following corollary relating the orthogonal subspace of $u_1$ to the left eigenvectors $\tilde{u}_2, \ldots, \tilde{u}_d$.

**Corollary 5.** If $\lambda_1$ has multiplicity 1, the space orthogonal to $u_1$ is spanned by the vectors $\{\tilde{u}_2, \ldots, \tilde{u}_d\}$.  

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