Corrected mathematical model of transverse oscillations of ropes and shafts of mine hoists

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Abstract: The oscillating processes of the rope and the shaft in small movements provoke fretting wear and corrosion at the attachment point. On the basis of the Fourier method, an improved model of oscillating processes has been developed that significantly extends the functionality of the variable separation method. The model adequacy is confirmed.

1. Introduction
Oscillation and wave processes in loaded long ropes and elastic shafts lead to a sharp increase in dynamic loads and intensive wear and tear of ropes, mechanical connections, shafts, bearings, power electromechanical equipment, emergency failure of hoisting vessels, discharge guides, electric motors of mine hoist. The probability of accidents with forced shutdown of the production process increases. Negative processes are intensified by the imposition of elastic transverse oscillations. Increased dynamic loads through the drum affect the elastic shaft. The shaft vibrates in the transverse direction from the radial impact of the rope on the drum.

The oscillating processes are accompanied by fretting wear [1, 2, 3, 4, 5] and the corrosion it provokes in the places of fixed joints of the elastic shaft with the bearings and the drum. Destructive effect of fretting has on bearings, gears, connecting clutches, drums, and fretting is accompanied by corrosion and the appearance of cracks. Negative effects of the fretting are exacerbated by impacts in the gaps in a number of moving joints of gearboxes, couplings. The research of fretting takes place both abroad [2, 3], and in Russia [1, 4, 5], but this problem has been practically not touched upon in relation to the slurry control, except for the work [5], although the fretting wear test is provided for by a special standard [4].

The oscillating processes are well studied for free-finished beams [6, 7], but the rope and shaft have significant differences. The ends of the shaft and rope are fixed in contrast to the beams, and their breakage at the outlet of the fastening is physically impossible. Mathematically, this results in not only a bending function but also a derivative of the bending function being equal to zero at the shaft ends. Bending moments lead to uneven loading of the bearings along their longitudinal axis, pressure on the end parts increases, which contributes to the phenomenon of fretting.

Note the paradoxical situation in the bending theory of beams. Static bending and oscillating processes are described by different equations, although bending is in essence an established state of oscillating processes after their attenuation, the processes of oscillation and static bending should logically be described by a single equation. It should be noted that the study of oscillatory processes, as a rule, does not take into account the damping of oscillations, obviously this was the justification for the application of separate equations.
Thus, the problem of mathematical description of transverse oscillations pinched at the ends of the rope and the shaft remains unsolved. Thus, in the sources [6, 7] is not provided equality to zero derivatives of bends at the ends of the shaft, and the theory of bending is constructed using moments bending the beam for its ends and it is not specified how the bending moments are created from the impact of concentrated force causing transverse vibrational processes of the rope and the shaft. Nor is the Fourier method able to account for the braking effect of external processes on the damping of vibrations.

2. Results and discussion

In work [8] the author, on the basis of the formulated idea of mirror symmetry and the introduction of the intermediate second coordinate, created a new more universal mathematical model of the shaft and rope (1), which are affected perpendicular to the longitudinal axis concentrated load. The model describes forced and free oscillations, and the account of internal viscous friction provides the account of their attenuation.

\[
\frac{E}{J} \frac{\partial^2 h(x,t)}{\partial x^2} = m L_b \frac{\partial^2 h(x,t)}{\partial t^2} + B \frac{\partial h(x,t)}{\partial t} + \begin{cases} 
P \left( \frac{L_b-a}{L_b} x + C_1 \right), & 0 \leq x \leq a, \\
- \frac{a}{L_b} x + C_2, & a \leq x \leq L_b,
\end{cases}
\]

(1)

where \(E\) - Young's modulus, \(J\) - moment of inertia of rope and shaft, \(m\) - mass, \(h\) - is the current bending value at \(x\), \(L_b\) - length of rope and shaft, \(B = C_t L_b \eta D\), \(C_t L_b = C_t\) - coefficients of frontal resistance and internal viscous friction, \(D\) - rope or shaft diameter, \(\eta\) - dynamic air viscosity [6, 7, 8].

The research will start with free (natural) rope and shaft oscillations. There is no force \(P\) in the initial state, the rope and the shaft are at rest without bending and without oscillations. To cause the shaft (rope) to bend, apply force \(P\). The oscillating process described in the model (1) develops. After the transient process has subsided, the shaft (rope) is in an established state, which we call static bending. If we remove force \(P\), i.e. take \(P=0\), the function (1) will take the form of

\[
\frac{E}{P} h(x) = \begin{cases} 
\frac{L_b-a}{6L_b} x^3 - \frac{L_b^2-a^2}{12L_b} x^2, & 0 \leq x \leq a, \\
- \frac{a}{6L_b} x^3 + \frac{a(4L_b+a)}{12L_b} x^2 - \frac{a(L_b+a)}{6} x + \frac{L_b a^2}{12}, & a \leq x \leq L_b,
\end{cases}
\]

(3)

where \(x = a\) is where \(P\) force is applied to a rope or shaft.

The model is illustrated in figure 1, which shows all the parameters used.
The figure shows: \( P \) - concentrated force, \( P_1 \) and \( P_2 \) - fastener reaction, \( L_b \) - rope and shaft length, \( a \) - distance from the beginning of the coordinates to the point of force application \( P \), \( h \) - bending at different values of the abscissa \( x \) coordinate.

A universal and rather simple method for solving differential equations with partial derivatives (2) is the method of separation of variables (Fourier method). But according to the sources [12, 13], the classical method uses the sinus division of functions of initial conditions in a Fourier series, which is applicable only in the case of initial conditions described by odd functions. However, the initial conditions (3) include, in addition to the odd ones, even functions, which leads to a general decomposition in a series

\[
EJh(x,0) = \frac{f_{01}}{2} + \sum_{k=1}^{\infty} \left[ f_{1k}\cos(kx) + f_{2k}\sin(kx) \right], \quad 0 \leq x \leq L_b, k = 1, 2, 3, ..., \infty, \tag{4}
\]

which does not contain the summand \( \frac{f_{01}}{2} \) when decomposed by sinus [12]. When decomposing into a Fourier row of initial conditions (3), as will be shown below, the row has form (4).

Solutions to equation (2) will be sought as a product of two functions, but unlike [12, 13] with the addition of the constant component \( A = \text{constant} \)

\[
h(x, t) = A + X(x)T(t) \tag{5}
\]

Since equation (2) includes only derivatives from the function \( h(x,t) \), and derivatives from a constant number are equal to zero, the summation \( A \) does not affect the fairness of the solution, at the same time, the developed approach significantly extends the range of application of the Fourier method in relation to functions, the expansion of which contains the term \( \frac{f_{01}}{2} \).

By differentiating twice (5) and substituting the obtained derivatives in (2) we obtain

\[
EJ\ddot{X}(x)T(t) = mL_b\ddot{T}(t)X(x) + B\dot{T}(t)X(x),
\]

Divide the left and right parts of the equations by \( EJT(t)X(x) \), according to [12].

\[
\frac{\ddot{X}(x)}{\dot{X}(x)} = \frac{mL_b\ddot{T}(t) + B\dot{T}(t)}{EJT(t)} = -\lambda = \text{const}, \tag{6}
\]

From equality (6) get:

\[
\ddot{X}(x) + \lambda X(x) = 0, \tag{7,a}
\]

\[
\dot{T}(t) + \frac{B}{mL_b}\ddot{T}(t) + \frac{\lambda EJ}{mL_b}T(t) = 0, \tag{7,b}
\]
The differential equation (7,a) corresponds to the characteristic equation 
\[ p^2 + \lambda = 0 \]
with two imaginary roots 
\[ p = \pm \sqrt{-\lambda} \], 
which the solution corresponds
\[ X(x) = fF_1 \sin(\sqrt{\lambda} x) + F_2 \cos(\sqrt{\lambda} x) \]  
(8,a)

Differential equation (7,b) corresponds to a characteristic polynomial of
\[ p^2 + \frac{B}{mL_b} p + \frac{EJ}{mL_b} = 0 \],  
whence
\[ p = -\frac{B}{2mL_b} \pm \sqrt{\frac{B^2}{4(mL_b)^2} - \frac{\lambda EJ}{mL_b}} = \alpha \pm i\beta \],  
(8,b)

where \( \alpha = -\frac{B}{2mL_b}, \beta = \pm \sqrt{\alpha^2 - \frac{\lambda EJ}{mL_b}} \).

All constants included in the expression (8,b) are positive, so the first summand \( \alpha \) is always negative, and since the Young's modulus for steel \( E = 210 \cdot 10^9 \) is a very large number, there is \( \alpha^2 < \frac{\lambda EJ}{mL_b} \), the roots are complex and the solution of the equation (7,b) is oscillatory
\[ T(t) = e^{\alpha t} (F_3 \cos \beta t + F_4 \sin \beta t) \]

By substituting in (5) the function \( T(t) \) and \( X(x) \) from (8,b), we get the function
\[ h(x, t) = A + \left[ F_1 \cos \left( \frac{\sqrt{\lambda} x}{L_b} \right) + F_2 \sin \left( \frac{\sqrt{\lambda} x}{L_b} \right) \right] e^{\alpha t} [F_3 \cos (\beta t) + F_4 \sin (\beta t)]. \]  
(9)

In the developed theory, the coefficients of a series are determined by comparing a series (9) with a series obtained by decomposition into a static bending function (3). For the row coefficients to coincide, they must be decomposed at the same interval [12, 14]. For these reasons, the value of the parameter \( \lambda \) and coefficients \( F_{1k}, F_{2k}, F_{3k} \) and \( F_{4k} \), in contrast to the classical Fourier method, let us define not by boundary conditions with the beginning of coordinates in the midpoint of the period, but by the natural coordinate system with the beginning at \( x = 0 \) in a segment equal to the length of the shaft (rope). In this case, according to [12, 14] we have \( 2l = L_b \), from where \( l = L_b/2 \) and from the period of repetition of \( L_b \) in trigonometric functions (9) follows \( \sqrt{\lambda} = \frac{\pi k}{L_b} \), \( k = 1, 2, 3, ..., \infty \), whose substitution in (9) and (7,b) gives
\[ EJh(x, t) = A + \left[ F_{1k} \cos \left( \frac{2\pi k}{L_b} x \right) + F_{2k} \sin \left( \frac{2\pi k}{L_b} x \right) \right] e^{\alpha t} [F_{3k} \cos (\beta t) + F_{4k} \sin (\beta t)], \]
from where it follows that the wave equation (2) has an infinite number of solutions.

The sum of the first square brackets is multiplied by the sum of the second ones.
\[ EJh(x, t) = \frac{f_{01}}{2} + e^{\alpha t} \left[ f_{1k} \cos (\beta t) + f_{3k} \sin (\beta t) \right] \cos \left( \frac{2\pi k}{L_b} x \right) + \]
\[ + \left[ f_{2k} \cos (\beta t) + f_{4k} \sin (\beta t) \right] \sin \left( \frac{2\pi k}{L_b} x \right), \]
where \( f_{01} = A, f_{1k} = F_{1k} F_{3k}, f_{2k} = F_{2k} F_{3k}, f_{3k} = F_{1k} F_{4k}, f_{4k} = F_{2k} F_{4k}, \alpha = -\frac{B}{2mL_b}, \beta = \pm \sqrt{\alpha^2 - \frac{(2\pi k)^2 EJ}{mL_b^2}}. \)

Summarizing the solutions for the totality \( k = 1, 2, 3, ..., \infty \), according to [12] we get the general solution of the wave function (2) in the form of a series of two variables \( x \) and \( t \)
\[ EJh(x, t) = \frac{f_{01}}{2} + e^{\alpha t} \sum_{k=1}^{\infty} \left[ f_{1k} \cos (\beta t) + f_{3k} \sin (\beta t) \right] \cos \left( \frac{2\pi k}{L_b} x \right) + \]
\[ + \left[ f_{2k} \cos(\beta t) + f_{4k} \sin(\beta t) \right] \sin\left(\frac{2\pi k}{L_b} x\right) \] \quad k=1, 2, 3, \ldots, \infty. \quad (10)

At \( t = 0 \), function (10) describes the initial state and takes the form of

\[ EJh(x, 0) = \frac{f_{01}}{2} + \sum_{k=1}^{\infty} \left[ f_{1k} \cos\left(\frac{2\pi k}{L_b} x\right) + f_{2k} \sin\left(\frac{2\pi k}{L_b} x\right) \right], \quad 0 \leq x \leq L_b, \quad (11) \]

The series (10) and static bending function (3) at \( 0 \leq x \leq L_b \) describe the same process. If the static bending function (3) is expanded into a Fourier row on the interval \( 0 \leq x \leq L_b \), we obtain a row coinciding with the row (11), which makes it possible to determine the coefficients \( f_{01}, f_{1k} \) and \( f_{2k} \).

Compose the initial expressions of the set of functions and parameters to calculate the coefficients of the Fourier series (11) in relation to the static bending function (3)

\[
\begin{align*}
 f_{01} &= \frac{2}{L_b} \left[ \int_0^a \left( \frac{b-a}{6L_b} x^3 - \frac{b-a}{12L_b} x^2 \right) dx + \int_a^b \left( -\frac{a}{6L_b} x^3 + \frac{4a}{12L_b} x^2 \right) dx \right] = \\
 &= \frac{2}{L_b} \left[ \frac{b}{24L_b} x^4 - \frac{b}{6L_b} x^3 + \frac{a^2}{36L_b} x^2 - \frac{L_b a^2}{12} x + \frac{L_b a^2}{12} \right] = \\
 &= \frac{2}{L_b} \left[ \frac{b}{24L_b} - \frac{a}{9} \right] = \frac{b}{36L_b} - \frac{a}{9} + \frac{L_b a^2}{12} = \frac{2}{L_b} \left( \frac{L_b a^2}{9} + \frac{L_b a^3}{36} + \frac{5a^3}{72} \right), \\
 f_{01} &= \frac{-3}{36L_b} \left( L_b^2 b + 2L_b a^2 - a^3 \right). \quad (13)
\end{align*}
\]

To calculate the coefficients \( f_{1k} \) and \( f_{2k} \) from (11) let us calculate the undefined integrals checking their correctness by differentiation

\[
\begin{align*}
\int x^3 \cos\left(\frac{2\pi k}{L_b} x\right) dx &= \left( \frac{L_b^2 x^2}{2k^2 \pi^2} - \frac{3L_b^3 x}{8k^4 \pi^4} \right) \cos\left(\frac{2\pi k}{L_b} x\right) + \left( \frac{L_b x^3}{2k \pi} - \frac{3L_b^3 x}{8k^3 \pi^3} \right) \sin\left(\frac{2\pi k}{L_b} x\right), \\
\int x^2 \cos\left(\frac{2\pi k}{L_b} x\right) dx &= \left( \frac{L_b^2 x^3}{2k^2 \pi^2} - \frac{L_b x^2}{2k \pi} \right) \cos\left(\frac{2\pi k}{L_b} x\right) + \left( \frac{L_b^3 x}{4k \pi^3} \right) \sin\left(\frac{2\pi k}{L_b} x\right), \\
\int x \cos\left(\frac{2\pi k}{L_b} x\right) dx &= \left( \frac{L_b^2 x^2}{2k^2 \pi^2} - \frac{L_b x}{2k \pi} \right) \cos\left(\frac{2\pi k}{L_b} x\right) + \left( \frac{L_b^3 x}{4k \pi^3} \right) \sin\left(\frac{2\pi k}{L_b} x\right), \\
\int x^3 \sin\left(\frac{2\pi k}{L_b} x\right) dx &= \left( \frac{L_b^2 x}{2k^2 \pi^2} - \frac{L_b x^2}{2k \pi} \right) \sin\left(\frac{2\pi k}{L_b} x\right) + \left( \frac{L_b^3 x^3}{4k^3 \pi^3} \right) \cos\left(\frac{2\pi k}{L_b} x\right), \\
\int x^2 \sin\left(\frac{2\pi k}{L_b} x\right) dx &= \left( \frac{L_b^2 x^2}{2k^2 \pi^2} - \frac{L_b x^3}{2k \pi} \right) \sin\left(\frac{2\pi k}{L_b} x\right) - \left( \frac{L_b^3 x^3}{4k^3 \pi^3} \right) \cos\left(\frac{2\pi k}{L_b} x\right), \\
\int x \sin\left(\frac{2\pi k}{L_b} x\right) dx &= \frac{L_b^2}{2k^2 \pi^2} \sin\left(\frac{2\pi k}{L_b} x\right) - \left( \frac{L_b x^2}{2k \pi} \right) \cos\left(\frac{2\pi k}{L_b} x\right).
\end{align*}
\]
Let us move on to certain integrals and put them in (12, b).

\[
f_{1k} = \frac{2}{L_b} \left( \frac{L_b - a}{16L_b^2} \left[ \left( \frac{3L_b^2 x^2}{8k^2 \pi^2} - \frac{3L_b^4}{8k^4 \pi^4} \right) \cos \left( \frac{2k\pi x}{L_b} \right) + \left( \frac{L_b x^3}{2k\pi} - \frac{3L_b^3 x}{4k^3 \pi^3} \right) \sin \left( \frac{2k\pi x}{L_b} \right) \right]_0^a + \frac{l_b^2}{2k^2 \pi^2} \cos \left( \frac{2k\pi x}{L_b} \right) - \left( \frac{L_b}{2k\pi} x^2 - \frac{l_b^3}{2k^3 \pi^3} \right) \sin \left( \frac{2k\pi x}{L_b} \right) \right)_0^a + \frac{a}{6L_b} \left[ \left( \frac{3L_b^2 x^2}{8k^2 \pi^2} - \frac{3L_b^4}{8k^4 \pi^4} \right) \cos \left( \frac{2k\pi x}{L_b} \right) - \left( \frac{L_b x^3}{2k\pi} - \frac{3L_b^3 x}{4k^3 \pi^3} \right) \sin \left( \frac{2k\pi x}{L_b} \right) \right]_a^L_b + 4L_b a + a^2 \frac{L_b^2}{2k^2 \pi^2} \cos \left( \frac{2k\pi x}{L_b} \right) + \left( \frac{L_b x^3}{2k\pi} - \frac{3L_b^3 x}{4k^3 \pi^3} \right) \sin \left( \frac{2k\pi x}{L_b} \right) \right]_a^L_b + \frac{l_b a + a^2}{6} \left[ - \frac{l_b^2}{2k^2 \pi^2} \cos \left( \frac{2k\pi x}{L_b} \right) + \frac{L_b a}{2k\pi} \sin \left( \frac{2k\pi x}{L_b} \right) \right]_a^L_b \right) \right]

Reveal the limits.

\[
f_{1k} = \frac{2}{L_b} \left( \frac{L_b - a}{16L_b^2} \left[ \left( \frac{3L_b^2 x^2}{8k^2 \pi^2} - \frac{3L_b^4}{8k^4 \pi^4} \right) \cos \left( \frac{2k\pi x}{L_b} \right) + \left( \frac{L_b x^3}{2k\pi} - \frac{3L_b^3 x}{4k^3 \pi^3} \right) \sin \left( \frac{2k\pi x}{L_b} \right) \right]_0^a + \frac{l_b^2}{2k^2 \pi^2} \cos \left( \frac{2k\pi x}{L_b} \right) - \left( \frac{L_b}{2k\pi} x^2 - \frac{l_b^3}{2k^3 \pi^3} \right) \sin \left( \frac{2k\pi x}{L_b} \right) \right)_0^a + \frac{a}{6L_b} \left[ \left( \frac{3L_b^2 x^2}{8k^2 \pi^2} - \frac{3L_b^4}{8k^4 \pi^4} \right) \cos \left( \frac{2k\pi x}{L_b} \right) - \left( \frac{L_b x^3}{2k\pi} - \frac{3L_b^3 x}{4k^3 \pi^3} \right) \sin \left( \frac{2k\pi x}{L_b} \right) \right]_a^L_b + 4L_b a + a^2 \frac{L_b^2}{2k^2 \pi^2} \cos \left( \frac{2k\pi x}{L_b} \right) + \left( \frac{L_b x^3}{2k\pi} - \frac{3L_b^3 x}{4k^3 \pi^3} \right) \sin \left( \frac{2k\pi x}{L_b} \right) \right]_a^L_b + \frac{l_b a + a^2}{6} \left[ - \frac{l_b^2}{2k^2 \pi^2} \cos \left( \frac{2k\pi x}{L_b} \right) + \frac{L_b a}{2k\pi} \sin \left( \frac{2k\pi x}{L_b} \right) \right]_a^L_b \right) \right]

Let us open the brackets and group the summed up on trigonometric functions:

\[
f_{1k} = \frac{2}{L_b} \left( \frac{L_b - a}{16L_b^2} \left[ \left( \frac{3L_b^2 x^2}{8k^2 \pi^2} - \frac{3L_b^4}{8k^4 \pi^4} \right) \cos \left( \frac{2k\pi x}{L_b} \right) + \left( \frac{L_b x^3}{2k\pi} - \frac{3L_b^3 x}{4k^3 \pi^3} \right) \sin \left( \frac{2k\pi x}{L_b} \right) \right]_0^a + \frac{l_b^2}{2k^2 \pi^2} \cos \left( \frac{2k\pi x}{L_b} \right) - \left( \frac{L_b}{2k\pi} x^2 - \frac{l_b^3}{2k^3 \pi^3} \right) \sin \left( \frac{2k\pi x}{L_b} \right) \right)_0^a + \frac{a}{6L_b} \left[ \left( \frac{3L_b^2 x^2}{8k^2 \pi^2} - \frac{3L_b^4}{8k^4 \pi^4} \right) \cos \left( \frac{2k\pi x}{L_b} \right) - \left( \frac{L_b x^3}{2k\pi} - \frac{3L_b^3 x}{4k^3 \pi^3} \right) \sin \left( \frac{2k\pi x}{L_b} \right) \right]_a^L_b + 4L_b a + a^2 \frac{L_b^2}{2k^2 \pi^2} \cos \left( \frac{2k\pi x}{L_b} \right) + \left( \frac{L_b x^3}{2k\pi} - \frac{3L_b^3 x}{4k^3 \pi^3} \right) \sin \left( \frac{2k\pi x}{L_b} \right) \right]_a^L_b + \frac{l_b a + a^2}{6} \left[ - \frac{l_b^2}{2k^2 \pi^2} \cos \left( \frac{2k\pi x}{L_b} \right) + \frac{L_b a}{2k\pi} \sin \left( \frac{2k\pi x}{L_b} \right) \right]_a^L_b \right) \right]

Let us do the multiplications and bring similar members and group the members.

\[
f_{1k} = \frac{L_b}{4k^3 \pi^3} \left[ \frac{l_b}{2k\pi} \left( 1 - \cos \left( \frac{2k\pi x}{L_b} \right) \right) + \frac{l_b - 2a}{6} \sin \left( \frac{2k\pi x}{L_b} \right) \right] + \frac{l_b a + a^2}{6} \frac{L_b^2}{2k^2 \pi^2} \sin \left( \frac{2k\pi x}{L_b} \right) \right]_a^L_b \right) \right]

Let us proceed to the determination of \( f_{2k} \) using the ratio (12, c). In the ratio (12, c) let us substitute integrals from (14) by transforming undefined integrals into certain

\[
f_{2k} = \frac{2}{L_b} \left( \frac{L_b - a}{6L_b^2} \left[ \left( \frac{3L_b^2 x^2}{4k^2 \pi^2} - \frac{3L_b^4}{8k^4 \pi^4} \right) \sin \left( \frac{2k\pi x}{L_b} \right) - \left( \frac{L_b x^3}{2k\pi} - \frac{3L_b^3 x}{4k^3 \pi^3} \right) \cos \left( \frac{2k\pi x}{L_b} \right) \right]_0^a + \frac{l_b^2}{2k^2 \pi^2} \cos \left( \frac{2k\pi x}{L_b} \right) - \left( \frac{L_b}{2k\pi} x^2 - \frac{l_b^3}{2k^3 \pi^3} \right) \sin \left( \frac{2k\pi x}{L_b} \right) \right)_0^a + \frac{a}{6L_b} \left[ \left( \frac{3L_b^2 x^2}{4k^2 \pi^2} - \frac{3L_b^4}{8k^4 \pi^4} \right) \sin \left( \frac{2k\pi x}{L_b} \right) - \left( \frac{L_b x^3}{2k\pi} - \frac{3L_b^3 x}{4k^3 \pi^3} \right) \cos \left( \frac{2k\pi x}{L_b} \right) \right]_a^L_b + 4L_b a + a^2 \frac{L_b^2}{2k^2 \pi^2} \cos \left( \frac{2k\pi x}{L_b} \right) + \left( \frac{L_b x^3}{2k\pi} - \frac{3L_b^3 x}{4k^3 \pi^3} \right) \sin \left( \frac{2k\pi x}{L_b} \right) \right]_a^L_b + \frac{l_b a + a^2}{6} \left[ - \frac{l_b^2}{2k^2 \pi^2} \cos \left( \frac{2k\pi x}{L_b} \right) + \frac{L_b a}{2k\pi} \sin \left( \frac{2k\pi x}{L_b} \right) \right]_a^L_b \right) \right]

\[
(14)
\]
After performing conversions similar to those performed at the output of the $f_1k$ function, the result will be

$$f_{2k} = \frac{1}{4k^2\pi^2} \left[ L_b^2(1 - \cos \frac{2\pi k}{L_b} a) \right] + \frac{L_b a}{6\pi k} \left[ \frac{L_b^2(1 - 2L_b a^2 - 2a^2)}{2k^2\pi^2} + a^2 \right] \sin \left( \frac{2\pi k}{L_b} a \right).$$  (15)

It remains to determine the coefficients $f_{3k}$ and $f_{4k}$ of the series (10). For this purpose, let us use the fact that the time derivative from the initial conditions (3) at $t = 0$ is equal to zero, therefore, it should be equal to zero and the derivative from the series (10). Thus, we have

$$E \frac{dh(x,0)}{dx} = e^{\alpha t} \sum_{k=1}^{\infty} \left[ \alpha f_{1k} \cos \left( \frac{2\pi k}{L_b} x \right) + \alpha f_{2k} \sin \left( \frac{2\pi k}{L_b} x \right) \right] \cos(\beta t) +$$

$$+ \left[ \alpha f_{3k} \cos \left( \frac{2\pi k}{L_b} x \right) + \alpha f_{4k} \sin \left( \frac{2\pi k}{L_b} x \right) \right] \sin(\beta t) -$$

$$- \beta f_{1k} \cos \left( \frac{2\pi k}{L_b} x \right) + \beta f_{2k} \sin \left( \frac{2\pi k}{L_b} x \right) \sin(\beta t) +$$

$$+ \beta f_{3k} \cos \left( \frac{2\pi k}{L_b} x \right) + \beta f_{4k} \sin \left( \frac{2\pi k}{L_b} x \right) \cos(\beta t), \quad k = 1, 2, 3, \ldots, \infty.$$  (16)

By substituting $t = 0$ we get

$$E \frac{dh(x,0)}{dx} = \sum_{k=1}^{\infty} \left[ \alpha f_{1k} \cos \left( \frac{2\pi k}{L_b} x \right) + \alpha f_{2k} \sin \left( \frac{2\pi k}{L_b} x \right) + \beta f_{3k} \cos \left( \frac{2\pi k}{L_b} x \right) + \beta f_{4k} \sin \left( \frac{2\pi k}{L_b} x \right) \right].$$

Let us group the components by trigonometric functions.

$$E \frac{dh(x,0)}{dx} = \sum_{k=1}^{\infty} \left[ \alpha f_{1k} \cos \left( \frac{2\pi k}{L_b} x \right) + \alpha f_{2k} \sin \left( \frac{2\pi k}{L_b} x \right) + \beta f_{3k} \cos \left( \frac{2\pi k}{L_b} x \right) \right] \cos(\beta t) +$$

$$+ \left[ \alpha f_{3k} \cos \left( \frac{2\pi k}{L_b} x \right) + \alpha f_{4k} \sin \left( \frac{2\pi k}{L_b} x \right) \right] \sin(\beta t) -$$

$$- \beta f_{1k} \cos \left( \frac{2\pi k}{L_b} x \right) + \beta f_{2k} \sin \left( \frac{2\pi k}{L_b} x \right) \sin(\beta t) +$$

$$+ \beta f_{3k} \cos \left( \frac{2\pi k}{L_b} x \right) + \beta f_{4k} \sin \left( \frac{2\pi k}{L_b} x \right) \cos(\beta t), \quad k = 1, 2, 3, \ldots, \infty.$$  (17)
3. Conclusion

To demonstrate the model adequacy the author has developed a program model in C++Builder environment. Figure 2,a shows one curve that displays the static bending and the same curve simultaneously displays the initial state of the oscillating process for parameter \( a = Lb/2 \). The curves are exactly the same. Figures 2,b and 2,c show the intermediate positions recorded during the oscillating process for the case when the concentrated force is applied at point \( a = Lb/2 \).

Figure 3,a shows two curves that are fully coincident with each other. One of the coincident curves shows a static bend, and the other one shows the initial state of the oscillating process for the parameter \( a = Lb/4 \). The full coincidence of the curves convincingly confirms the initial state (17) representing the oscillatory process and represented by the series (17) at \( t = 0 \) fully coincide with the curve of the function of the initial conditions (3). Figure 3,b shows the intermediate positions recorded during the oscillating process for \( a = Lb/4 \).

![Figure 2,a](image1.png)

Figure 2,a. Matching static bending and oscillation graphs at \( a=Lb/2 \) and \( t=0 \).

![Figure 2,b](image2.png)

Figure 2,b. Oscillation intermediate position for \( a=Lb/2 \)

![Figure 2,c](image3.png)

Figure 2,c. Second intermediate position with oscillations for \( a=Lb/2 \)

![Figure 3,a](image4.png)

Figure 3,a. Matching charts of static bending and oscillation at \( a=Lb/4 \) and \( t=0 \)

![Figure 3,b](image5.png)

Figure 3,b. Intermediate oscillation position for \( a=Lb/4 \)

![Figure 4,a](image6.png)

Figure 4,a. Matching static bending and oscillation graphs at \( a=3Lb/4 \) and \( t=0 \).
Figure 4.a like in the previous figures shows also one curve that represents the static bending and simultaneously the initial state of the oscillating process for parameter $a=3L_b/4$. The curves are completely identical. Figure 4.b shows the intermediate positions recorded during the oscillating process for $a=3L_b/4$.

Figure 4.b. Oscillation intermediate position for $a=3L_b/4$

Thus, the extended universal mathematical model providing research of both dynamic and established processes of a wider class of initial conditions and with a new implementation approach is offered. The model is based on the Fourier method with expansion of initial conditions and on even and arbitrary functions. The method provides application of rows with constant component with coefficient $f_{0i}$. The mathematical reasoning has been reworked in detail, and the auxiliary program developed in C++ allows to demonstrate effectively the possibilities of the proposed extended model.

4. References

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