A new approach in index theory

M. Berkani 1

Received: 5 February 2020 / Accepted: 15 June 2020 / Published online: 27 October 2020
© Springer-Verlag GmbH Germany, part of Springer Nature 2020

Abstract
In this paper, we define an analytical index for a continuous family of Fredholm operators parameterized by a topological space $X$ into a Hilbert space $H$, as a sequence of integers, extending naturally the usual definition of the index and we prove the homotopy invariance of the index. We give also an extension of the Weyl theorem for normal continuous families and we prove that if $H$ is separable, then the space of B-Fredholm operators on $H$ is path connected.

Keywords B-Fredholm · Connected components · Fredholm · Homotopy · Index

Mathematics Subject Classification 47A53 · 58B05

1 Introduction

Let $L(H)$ be the Banach algebra of all bounded linear operators defined from an infinite dimensional separable Hilbert space $H$ to $H$, $K(H)$ the closed ideal of compact operators on $H$ and $L(H)/K(H)$ the Calkin Algebra. We write $N(T)$ and $R(T)$ for the nullspace and the range of an operator $T \in L(H)$. An operator $T \in L(H)$ is called [5, Definition 1.1] a Fredholm operator if both the nullity of $T$, $n(T) = \dim N(T)$ and the defect of $T$, $d(T) = \text{codim } R(T)$, are finite. The index $ind(T)$ of a Fredholm operator $T$ is defined by $ind(T) = n(T) - d(T)$. It is well known that if $T$ is a Fredholm operator, then $R(T)$ is closed.

Definition 1.1 [6] Let $T \in L(H)$ and let

$$\Delta(T) = \{n \in \mathbb{N} : \forall m \in \mathbb{N}, \ m \geq n \Rightarrow R(T^n) \cap N(T) \subseteq R(T^m) \cap N(T)\}.$$ 

Then the degree of stable iteration of $T$ is defined as $\text{dis}(T) = \inf \Delta(T)$ (with $\text{dis}(T) = \infty$ if $\Delta(T) = \emptyset$).

Define an equivalence relation $\mathcal{R}$ on the set $f \text{dim}(H) \times f \text{cod}(H)$, where $f \text{dim}(H)$ is the set of finite dimensional vector subspaces of $H$ and $f \text{cod}(H)$ is the set of finite codimension...
vector subspaces of $H$ by:

$$\text{(E}_1, F_1) \in \mathcal{R}(E'_1, F'_1) \iff \dim E_1 - \text{codim } F_1 = \dim E'_1 - \text{codim } F'_1,$$

Since $H$ is an infinite dimensional vector space, then the map:

$$\Psi : [\text{fdim}(H) \times \text{fcd}(H)] / \mathcal{R} \to \mathbb{Z},$$

defined by $\Psi((E_1, F_1)) = \dim E_1 - \text{codim } F_1$, where $(E_1, F_1)$ is the equivalence class of the couple $(E_1, F_1)$, is a bijection. Moreover $\Psi$ generates a commutative group structure on the set $[\text{fdim}(H) \times \text{fcd}(H)] / \mathcal{R}$ and $\psi$ is then a group isomorphism, where $\dim$ (resp. codim) is set for the dimension (resp. codimension) of a vector space.

Reformulating [3, Proposition 2.1], we obtain.

**Proposition 1.2** Let $H$ be a Hilbert space and let $T \in L(H)$. If there exists an integer $n$ such that $(N(T) \cap R(T^n), R(T) + N(T^n))$ is an element of $\text{fdim}(H) \times \text{fcd}(H)$, then $d = \text{dis}(T)$ is finite and for all $m \geq d$, $(N(T) \cap R(T^m), R(T) + N(T^m))$ is an element of $\text{fdim}(H) \times \text{fcd}(H)$ and

$$(N(T) \cap R(T^m), R(T) + N(T^m)) \in (N(T) \cap R(T^d), R(T) + N(T^d))$$

**Definition 1.3** [3, Definition 2.2] Let $H$ be a Hilbert space and let $T \in L(H)$. Then $T$ is called a B-Fredholm operator if there exists an integer $n$ such that $(N(T) \cap R(T^n), R(T) + N(T^n))$ is an element of $\text{fdim}(H) \times \text{fcd}(H)$. In this case the index $\text{ind}(T)$ is defined by:

$$\text{ind}(T) = \dim(N(T) \cap R(T^n)) - \text{codim}(R(T) + N(T^n))$$

From Proposition 1.2, the definition of the index of a B-Fredholm operator is independent of the choice of the integer $n$. Moreover, it extends the usual definition of the index of Fredholm operators, which are obtained if $n = 0$.

Note also that if $n$ is an integer such that $(N(T) \cap R(T^n), R(T) + N(T^n))$ is an element of $\text{fdim}(H) \times \text{fcd}(H)$, then from [4, Theorem 3.1], $R(T^n)$ is closed and the operator $T_n : R(T^n) \to R(T^n)$ defined by $T_n(x) = T(x)$ is a Fredholm operator whose index is equal to the index of $T$.

In the papers [1,2], the analytical index of a single or a family of elliptic operators is expressed in terms of K-theory, and involves in its construction vector bundles.

The first motivation of this work, is to give an alternative way to build an analytical index (called here simply index), for a continuous family of Fredholm operators parameterized by a topological space, in the form of a sequence of integers, as a natural extension of the usual index of a single Fredholm operator, which is an integer, avoiding the use of vector bundles.

The second motivation is the extension of Weyl’s theorem [7] to continuous families of normal operators.

Moreover, we will prove in the fourth section, that if $H$ is a separable Hilbert space, then the space $BFred(H)$ is path connected.

## 2 Index of continuous families of Fredholm operators

Consider now a family of Fredholm operators parametrized by a topological space $X$, that is a continuous map $T : X \to Fred(H)$, where $Fred(H)$ is the set of Fredholm operators, endowed with the norm topology of $L(H)$. We denote by $T_x$ the image $T(x)$ of an element $x \in X$. 
Define an equivalence relation $\sim$ on the space $\mathbb{X}$ by setting that $x \sim y$, if and only if $x$ and $y$ belongs to the same connected component of $\mathbb{X}$. Let $\mathbb{X}/\sim$ be the quotient space associated to this equivalence relation and let $C(\mathbb{X}, Fred(H))$ be the space of continuous maps from the topological space $\mathbb{X}$ into the topological space $Fred(H)$ and let the map:

$$q : C(\mathbb{X}, Fred(H)) \rightarrow \{(f \dim(H) \times f\text{cod}(H)) / R\}^{\mathbb{X}/\sim},$$

defined by $q(T) = (\overline{N(T_x)}, \overline{R(T_x)})_{\mathbb{X}/\sim}$ for all $T \in C(\mathbb{X}, Fred(H))$.

Define also the map

$$\Psi_{\mathbb{X}} : \{(f \dim(H) \times f\text{cod}(H)) / R\}^{\mathbb{X}/\sim} \rightarrow \mathbb{Z}^{n_\mathbb{X}}$$

by setting $\Psi_{\mathbb{X}}((y_\mathbb{X})_{\mathbb{X}/\sim}) = (\Psi(y_\mathbb{X}))_{\mathbb{X}/\sim}$ for all $(y_\mathbb{X})_{\mathbb{X}/\sim} \in \{(f \dim(H) \times f\text{cod}(H)) / R\}^{\mathbb{X}/\sim}$.

Here $n_\mathbb{X}$ stands for the cardinal of the connected components of the topological space $\mathbb{X}$, assuming that the space $\mathbb{X}$ has at most a countable connected components.

**Definition 2.1** The analytical index (or simply the index) of a continuous family of Fredholm operators $T : \mathbb{X} \rightarrow Fred(H)$, parameterized by a topological space $\mathbb{X}$ is defined by $ind(T) = \Psi_{\mathbb{X}}(q(T))$.

Explicitly, we have: $ind(T) = \Psi_{\mathbb{X}}(\overline{N(T_x)}, \overline{R(T_x)})_{\mathbb{X}/\sim} = (ind(T_x))_{\mathbb{X}/\sim}$.

Thus the index of a family of B-Fredholm operators $T$ is a sequence of integers in $\mathbb{Z}^{n_\mathbb{X}}$, which may be a finite sequence or infinite sequence, depending on the cardinal of the connected components of $\mathbb{X}$.

**Theorem 2.2** The index of a continuous family of Fredholm operators $T$ parameterized by a topological space $\mathbb{X}$ is well defined as an element of $\mathbb{Z}^{n_\mathbb{X}}$. In particular if $\mathbb{X}$ is reduced to a single element, then the index of $T$ is equal to the usual index of the Fredholm operator $T$.

**Proof** From the usual properties of the index [5, Theorem 3.11], we know that two Fredholm operators located in the same connected component of the set of Fredholm operators have the same index. Moreover, as $T$ is continuous, the image of a connected component of the topological space $\mathbb{X}$, is included in a connected component of the set of Fredholm operators. This shows that the index of a family of Fredholm operators is well defined, and it is clear that if $\mathbb{X}$ is reduced to a single element, the index of $T$ defined here is equal to the usual index of the single Fredholm operator $T$.

**Example 2.3** Let $\mathbb{X} = [-1, 1]$, $H = \mathbb{C}$ and $T : \mathbb{X} \rightarrow Fred(\mathbb{C})$ such that $T_x(z) = xz$ for all $z \in \mathbb{C}$ and $x \in \mathbb{X}$. Then from Definition 2.1, the index of the family $T$ is simply $ind(T) = 0$. But since the dimension of the kernel $N(T_x)$ presents a discontinuity in 0, to build an index using K-theory needs more tools.

**Definition 2.4** A continuous family $\mathcal{K}$ from $\mathbb{X}$ to $L(H)$ is said to be compact if $\mathcal{K}_x$ is compact for all $x \in \mathbb{X}$.

**Proposition 2.5**

i) Let $T \in C(\mathbb{X}, Fred(H))$ and let $\mathcal{K}$ be a continuous compact family from $\mathbb{X}$ to $L(H)$. Then $T + \mathcal{K}$ is a Fredholm family and $ind(T + \mathcal{K}) = ind(T)$.

ii) Let $S, T \in C(\mathbb{X}, Fred(H))$ be two Fredholm families, then the family $ST$ defined by $(ST)_x = S_xT_x$ is a Fredholm family and $ind(ST) = ind(S) + ind(T)$.

**Proof** This is clear from the usual properties of Fredholm operators.

**Theorem 2.6** Assume that $\mathbb{X}$ is a compact topological space. Then the set $KC(\mathbb{X}, L(H))$ of continuous compact families from $\mathbb{X}$ to $L(H)$ is a closed ideal in the Banach algebra $C(\mathbb{X}, L(H))$.
Proof Recall that \( \mathcal{C}(\mathbb{X}, L(H)) \) is a unital algebra with the usual properties of addition, scalar multiplication and multiplication defined by:

\[
(\lambda S + T)_x = \lambda S_x + T_x, \quad (ST)_x = S_x T_x, \quad \forall x \in \mathbb{X}, \lambda \in \mathbb{C}.
\]

The unit element of \( \mathcal{C}(\mathbb{X}, L(H)) \) is the constant function \( I \) defined by \( I_x = I_H \) the identity of \( H \), for all \( x \in \mathbb{X} \). Moreover as \( \mathbb{X} \) is compact, then if we set \( \| T \| = \sup_{x \in \mathbb{X}} \| T_x \| \), \( \forall T \in \mathcal{C}(\mathbb{X}, L(H)) \), then \( \mathcal{C}(\mathbb{X}, L(H)) \) equipped with this norm is a Banach algebra. Similarly \( \mathcal{C}(\mathbb{X}, L(H)/K(H)) \) equipped with the norm \( \| T \| = \sup_{x \in \mathbb{X}} \| PT_x \| \) is a unital Banach algebra, where \( P : L(H) \to L(H)/K(H) \) is the usual projection from \( L(H) \) onto the Calkin algebra \( L(H)/K(H) \).

It is clear that \( \mathcal{KC}(\mathbb{X}, L(H)) \) is an ideal of \( \mathcal{C}(\mathbb{X}, L(H)) \). Assume now that \( (\mathcal{T}_h)_n \) is a sequence in \( \mathcal{KC}(\mathbb{X}, L(H)) \) converging in \( \mathcal{C}(\mathbb{X}, L(H)) \) to \( T \). Then \((\mathcal{T}_h)_n\) converges to \( \mathcal{T}_x \), as each \( (\mathcal{T}_h)_n \) is compact, then \( T \in \mathcal{KC}(\mathbb{X}, L(H)) \). \( \square \)

Remark 2.7 In the same way as in the case of the Calkin algebra, Theorem 2.6 generates a new Banach algebra which is \( \mathcal{C}(\mathbb{X}, L(H))/\mathcal{KC}(\mathbb{X}, L(H)) \). Moreover, there is a natural injection \( \mathcal{II} : \mathcal{C}(\mathbb{X}, L(H))/\mathcal{KC}(\mathbb{X}, L(H)) \to \mathcal{C}(\mathbb{X}, L(H))/K(H) \) defined by \( \mathcal{II}(\mathcal{T}) = PT \), where \( \mathcal{T} \) is the equivalence class of the element \( T \) of \( \mathcal{C}(\mathbb{X}, L(H)) \) in \( \mathcal{C}(\mathbb{X}, L(H))/\mathcal{KC}(\mathbb{X}, L(H)) \) and \( P : L(H) \to L(H)/K(H) \) is the natural projection.

Open question: Given an element \( S \in \mathcal{C}(\mathbb{X}, L(H)/K(H)) \), does there exist a continuous family \( T \in \mathcal{C}(\mathbb{X}, L(H)) \) such that \( \mathcal{II}(T) = S \)?

Theorem 2.8 Assume that \( \mathbb{X} \) is a compact topological space and let \( T \in \mathcal{C}(\mathbb{X}, L(H)) \). Then \( T \) is a Fredholm family if and only if \( PT \) is invertible in the Banach algebra \( \mathcal{C}(\mathbb{X}, L(H)/K(H)) \).

Proof Assume that \( T \) is a Fredholm family, then for all \( x \in \mathbb{X} \), \( T_x \) is a Fredholm operator. Thus \( PT_x \) is invertible in \( L(H)/K(H) \). Let \( (PT_x)^{-1} \) be its inverse, then the family \( (PT)^{-1} \) defined by \( (PT)^{-1}(x) = (PT_x)^{-1} \) is a continuous family, because the inversion is a continuous map in the Banach algebra \( L(H)/K(H) \), and \( (PT)^{-1} \) is the inverse of \( PT \) in the Banach algebra \( \mathcal{C}(\mathbb{X}, L(H)/K(H)) \).

Conversely if \( PT \) is invertible in the Banach algebra \( \mathcal{C}(\mathbb{X}, L(H)/K(H)) \), then there exists \( S \in \mathcal{C}(\mathbb{X}, L(H)/K(H)) \) such that \( (PT)S = S(PT) = I \), where \( I \) is defined by \( I_x = I_H \), for all \( x \in \mathbb{X} \), \( I_H \) being the identity of \( H \). Thus \( (PT_x)S_x = S_x(PT_x) = PI_H \). Thus \( PT_x \) is invertible in the Calkin algebra \( L(H)/K(H) \), \( T_x \) is a Fredholm operator and \( T \in \mathcal{C}(\mathbb{X}, Fred(H)) \). \( \square \)

Definition 2.9 Let \( S, T \) be \( \in \mathcal{C}(\mathbb{X}, Fred(H)) \). We will say that \( S \) and \( T \) are Fredholm homotopic, if there exists a map \( \Phi : [0, 1] \times \mathbb{X} \to L(H) \) such that for all \( (t, x) \in [0, 1] \times \mathbb{X} \), \( (t, x) = S_x, \Phi(1, x) = T_x \) and \( \Phi(t, x) \) is a Fredholm operator.

Theorem 2.10 Let \( S, T \) be two Fredholm homotopic elements of \( \mathcal{C}(\mathbb{X}, Fred(H)) \). Then \( ind(T) = ind(S) \).

Proof Since \( S \) and \( T \) are Fredholm homotopic, there exists a continuous map \( h : \mathbb{X} \times [0, 1] \to Fred(H) \) such that such that \( h(x, 0) = S(x) \) and \( h(x, 1) = T(x) \) for all \( x \in \mathbb{X} \). For a fixed \( x \in \mathbb{X} \), the map \( h_x : [0, 1] \to Fred(H) \), defined by \( h_x(t) = h(x, t) \) is a continuous path in \( Fred(H) \) linking \( S \) to \( T \). Thus \( ind(S_x) = ind(T_x) \). So \( q(S) = q(T) \) and then \( ind(S) = ind(T) \). \( \square \)

Theorem 2.11 Let \( \mathbb{X} \) be a compact topological space. Then the index is a continuous locally constant function from \( \mathcal{C}(\mathbb{X}, Fred(H)) \) into the group \( \mathbb{Z}^{\mathbb{X}} \).
2. A disjoint union of compact convex sets in a topological vector space, satisfies the $H$-condition.

\[ \text{Definition 2.13} \]
This is the reason why we introduce the following class of topological spaces.

\[ \text{Proof} \]
Let $\mathbb{X} \in C(\mathbb{X}, Fred(H))$, then $\forall x \in \mathbb{X}, \exists \epsilon_x > 0$, such that $B(T_x, \epsilon_x) \subset Fred(H)$, because $Fred(H)$ is open in $L(H)$. Then the index is constant on $B(T_x, \epsilon_x)$, because $B(T_x, \epsilon_x)$ is connected. We have $\mathbb{X} \subset \bigcup_{x \in \mathbb{X}} T^{-1}(B(T_x, \frac{\epsilon_x}{2}))$. Since $\mathbb{X}$ is compact, there exists $x_1, \ldots, x_n$ in $\mathbb{X}$ such that $\mathbb{X} \subset \bigcup_{i=1}^n T^{-1}(B(T_{x_i}, \frac{\epsilon_{x_i}}{2}))$. Let $\epsilon = \min\{\frac{\epsilon_{x_i}}{2} | 1 \leq i \leq n\}$ the minimum of the $\frac{\epsilon_{x_i}}{2}, 1 \leq i \leq n$, and let $S \in C(\mathbb{X}, Fred(H))$ such that $\|T - S\| < \frac{\epsilon}{2}$. If $x \in \mathbb{X}$, then $\|T_x - S_x\| < \frac{\epsilon}{2}$ and there exists $i, 1 \leq i \leq n$, such that $x \in T^{-1}(B(T_{x_i}, \frac{\epsilon_{x_i}}{2}))$. Then $\|S_x - T_{x_i}\| \leq \|S_x - T_x\| + \|T_x - T_{x_i}\| < \epsilon/2 + \epsilon_{x_i}/2 \leq \epsilon_{x_i}$ and $S_x$ is a Fredholm operator. Therefore $S \in C(\mathbb{X}, Fred(H))$ and $C(\mathbb{X}, Fred(H))$ is open in $C(\mathbb{X}, L(H)).$

Alternatively, we can see that $C(\mathbb{X}, Fred(H)) = \Pi^{-1}(\{C(\mathbb{X}, L(H)/K(H))\}^{inv})$, where $\{C(\mathbb{X}, L(H)/K(H))\}^{inv}$ is the open group of invertible elements of the unital Banach algebra $C(\mathbb{X}, L(H)/K(H))$ and $\Pi : C(\mathbb{X}, L(H)) \rightarrow C(\mathbb{X}, L(H)/K(H))$ is the map defined by $\Pi(T) = PT$, for all $T \in C(\mathbb{X}, L(H)/K(H)).$

If $T$ is a Fredholm operator of index 0, then $T$ is Fredholm connected to the identity $I_H$ of $H$. However, it follows from [5, Remark 3.32], that there exists continuous Fredholm families with vanishing index, which are not Fredholm homotopic to the identity $I$ of $C(\mathbb{X}, L(H))$. This is the reason why we introduce the following class of topological spaces.

**Definition 2.13** Let $\mathbb{X}$ be a compact topological space. We will say that $\mathbb{X}$ satisfies the $\mathcal{H}$-condition, if every continuous Fredholm family of index 0 in $\mathbb{Z}^{\mathbb{X}}$, is Fredholm homotopic to the identity $I$ of $C(\mathbb{X}, L(H))$.

**Examples 2.14**
1. Let $\mathbb{X}$ be a compact contractible space. Then there exists $x_0 \in \mathbb{X}$, a continuous map $\Phi : [0, 1] \times \mathbb{X} \rightarrow \mathbb{X}$, such that $\Phi(x, 0) = x$, $\Phi(x, 1) = x_0$, for all $x \in \mathbb{X}$. If $T$ in $C(\mathbb{X}, L(H))$ is a continuous Fredholm family of index 0, then $T_{x_0}$ is a Fredholm operator of index 0. So $T_{x_0}$ is Fredholm connected to the identity $I_H$ of $H$. Hence there exist a map $\varphi : [0, 1] \rightarrow Fred(H)$, such that $\varphi(0) = T_{x_0}$, and $\varphi(1) = I_H$.

Define the map $\Psi : X \times [0, 1] \rightarrow Fred(H)$, by setting $\Psi(x, t) = \varphi(t)$, for all $(x, t) \in \mathbb{X} \times [0, 1]$. Since $\varphi$ is continuous, $\Psi$ is continuous.

Now let $\Gamma : X \times [0, 1] \rightarrow Fred(H)$, defined by setting $\Gamma(x, t) = T(\Phi(x, 2t))$, for all $(x, t) \in \mathbb{X} \times [0, 1]$, and $\Gamma(x, t) = \Psi(x, 2t - 1)$, for all $(x, t) \in \mathbb{X} \times [\frac{1}{2}, 1]$. Then $\Gamma$ is a continuous map from $X \times [0, 1] \rightarrow Fred(H)$, satisfying $\Gamma(x, 0) = T_x$ and $\Gamma(x, 1) = I_H$, for all $x \in \mathbb{X}$.

Hence $T$ is Fredholm homotopic to the identity $I$ and $\mathbb{X}$ satisfies the $\mathcal{H}$-condition.

2. A disjoint union of compact convex sets in a topological vector space, satisfies the $\mathcal{H}$-condition, as each convex set is contractible.

\[ \text{ Springer} \]
The $H$-condition is a necessary condition in order to have that the index is a bijective monoid isomorphism map from the homotopy equivalence classes $[X, Fred(H)]$ of continuous families of Fredholm operators into the group $\mathbb{Z}^{n_c}$, similarly to the Atiyah–Jänich theorem [5, Theorem 3.40]. In the case of a locally connected compact topological space, we show that the $H$-condition is also sufficient.

**Theorem 2.15** Let $X$ be a locally connected compact topological space. Then the index is a bijective monoid isomorphism map from the homotopy equivalence classes $[X, Fred(H)]$ of continuous families of Fredholm operators into the group $\mathbb{Z}^{n_c}$ if and only if $X$ satisfies the $H$-condition.

**Proof** Observe first that the index is a monoid homomorphism, since $\text{ind}(I) = 0$, and $\text{ind}(ST) = \text{ind}(S) + \text{ind}(T)$, for all $S, T$ in $C(X, Fred(H))$.

Assume now that the index is a bijective monoid isomorphism map from the homotopy equivalence classes $[X, Fred(H)]$ of continuous families of Fredholm operators into the group $\mathbb{Z}^{n_c}$. If $T$ is a continuous Fredholm family of index 0, then $T$ is Fredholm homotopic to the identity $I$, because the identity $I$ is of index 0.

Conversely assume that $X$ satisfies the $H$-condition and let $T \in C(X, Fred(H))$ such that $\text{ind}(T) = 0$. Then the Fredholm family $T$ is Fredholm homotopic to the identity $I$. Thus the index is injective.

As $X$ is a locally connected compact topological space, its connected components are clopen subsets of $X$ and their cardinal $n_c$ is finite. Let $C_1, \ldots, C_{n_c}$ be those connected components. To prove that the index is surjective, let $u = (n_1, n_2, \ldots, n_{n_c}) \in \mathbb{Z}^{n_c}$. On each connected component of $X$, take a Fredholm operator $T_i$ such that $\text{ind}(T_i) = n_i$, for $1 \leq i \leq n_c$ and define the map $T : X \to \mathbb{Z}^{n_c}$ by $T(x) = T_i$, if $x \in C_i$. Then it is clear that $T$ is continuous and that $\text{ind}(T) = (n_1, n_2, \ldots, n_{n_c})$. Thus the index is surjective. \[\square\]

We can conclude from Theorem 2.15, that in the case of a locally connected, compact topological space satisfying the $H$-condition, the index defined here, is up to an isomorphism the same as the index bundle defined in [1,2].

### 3 An application to Weyl theorem

In this section, by an application to Weyl’s theorem, we show that the index of continuous Fredholm families defined in Sect. 1 is a natural generalization of the usual index of a single Fredholm operator.

So we will extend Weyl theorem to continuous normal families of bounded linear operators. Recall that the classical Weyl’s theorem [7] asserts that if $T$ is a normal operator acting on a Hilbert space $H$, then the Weyl spectrum $\sigma_W(T)$ is exactly the set of all points in $\sigma(T)$ except the isolated eigenvalues of finite multiplicity, that is $$\sigma_W(T) = \sigma(T) \setminus E_0(T),$$

where $E_0(T)$ is the set of isolated eigenvalues of $T$ of finite multiplicity and $\sigma_W(T)$ is the Weyl spectrum of $T$, that is $\sigma_W(T) = \{ \lambda \in \mathbb{C} \mid T - \lambda I \text{ is not a Fredholm operator of index 0} \}$.

**Definition 3.1** Let $T \in C(X, L(H))$. Then $T$ is called a Weyl family if it is a Fredholm family of index 0 in $\mathbb{Z}^{n_c}$. 

\[\square\ Springer\]
The Weyl spectrum $\sigma_W(T)$ of $T$ is defined by $\sigma_W(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not a Weyl family}\}$.

**Definition 3.2** Let $T \in L(H)$.

1. The ascent $a(T)$ of $T$ is defined by $a(T) = \inf\{ n \in \mathbb{N} \mid N(T^n) = N(T^{n+1}) \}$, and the descent $\delta(T)$ of $T$, is defined by $\delta(T) = \inf\{ n \in \mathbb{N} \mid R(T^n) = R(T^{n+1}) \}$, with $\inf \emptyset = \infty$.
2. A complex number $\lambda$ is a pole of the resolvent of $T$ of finite rank if $0 < n(T - \lambda I) < \infty$ and $\max(\lambda - \lambda I, \delta(T - \lambda I)) < \infty$. Moreover, if this is true, then $a(T - \lambda I) = \delta(T - \lambda I)$.

**Definition 3.3** Let $T \in C(\mathbb{X}, L(H))$, then:

1. $E_0(T)$ is defined by $E_0(T) = \{ \lambda \in \sigma(T) \mid \exists A \subset \mathbb{X}, A \neq \emptyset : \lambda \in \bigcap_{x \in A} E_0(T_x), \lambda \notin \bigcup_{x \in \mathbb{X} \setminus A} \sigma(T_x) \}$, where $E_0(T_x)$ is the set of isolated eigenvalues of finite multiplicity of $T_x$ in $\sigma(T_x)$.
2. $\Pi_0(T)$ is defined by $\Pi_0(T) = \{ \lambda \in \sigma(T) \mid \exists A \subset \mathbb{X}, A \neq \emptyset : \lambda \in \bigcap_{x \in A} \Pi_0(T_x), \lambda \notin \bigcup_{x \in \mathbb{X} \setminus A} \sigma(T_x) \}$, where $\Pi_0(T_x)$ is the set of poles of $T_x$ of finite rank.

**Definition 3.4** Let $T \in C(\mathbb{X}, L(H))$. We will say that:

1. $T$ satisfies Browder’s theorem if $\sigma_W(T) = \sigma(T) \setminus E_0(T)$.
2. $T$ satisfies Weyl’s theorem if $\sigma_W(T) = \sigma(T) \setminus E_0(T)$.

**Theorem 3.5** Let $T \in C(\mathbb{X}, L(H))$. Then the following holds:

1. If $T_x$ satisfies Weyl’s theorem for all $x \in \mathbb{X}$, then $T$ satisfies Weyl’s theorem.
2. If $T_x$ satisfies Browder’s theorem for all $x \in \mathbb{X}$, then $T$ satisfies Browder’s theorem.

**Proof** 1. First, let us show that $\sigma_W(T) = \bigcup_{x \in \mathbb{X}} \sigma_W(T_x)$.

   If $\lambda \notin \sigma_W(T)$, then $\forall x \in \mathbb{X}, \text{ind}(T_x - \lambda I) = 0$. Thus $\lambda \notin \bigcup_{x \in \mathbb{X}} \sigma_W(T_x)$.

   Conversely if $\lambda \notin \bigcup_{x \in \mathbb{X}} \sigma_W(T_x)$, then $\forall x \in \mathbb{X}, \text{ind}(T_x - \lambda I) = 0$. So on each connected component $C$ of $\mathbb{X}$, we have $\text{ind}(T_x - \lambda I) = 0$, $\forall x \in C$. Using the definition of the index of a Fredholm family, we obtain that $\text{ind}(T_x - \lambda I) = 0$ in $\mathbb{Z}^n$.

   Moreover, as the inversion is a continuous map on the group of invertible elements of $L(H)$, then $\sigma(T) = \bigcup_{x \in \mathbb{X}} \sigma(T_x)$.

   Assume now that $\lambda \in \sigma(T)$ and $\lambda \notin \sigma_W(T)$. Then $\forall x \in \mathbb{X}, \lambda \notin \sigma_W(T_x)$.

   If $\lambda \notin E_0(T_x)$, as $T_x$ satisfies Weyl’s theorem, then $\lambda \notin \sigma(T_x)$. As $\lambda \in \sigma(T)$, then there exists $x_0 \in \mathbb{X}$ such that $\lambda \in \sigma(T_{x_0})$. So $\lambda \in E_0(T_{x_0})$. Let $A = \{ x \in \mathbb{X} \mid \lambda \in E_0(T_x) \}$, then $A \neq \emptyset$. Moreover $\lambda \in \bigcap_{x \in A} E_0(T_x)$ and $\lambda \notin \bigcup_{x \in \mathbb{X} \setminus A} \sigma(T_x)$. So $\lambda \in E_0(T)$ and $\sigma(T) \subset \sigma_W(T) \bigcup E_0(T)$.

   As we have always $\sigma_W(T) \bigcup E_0(T) \subset \sigma(T)$, then

   $$\sigma(T) = \sigma_W(T) \bigcup E_0(T).$$

   Suppose now that $\lambda \in \sigma_W(T) \bigcap E_0(T)$. Then $A = \{ x \in \mathbb{X} \mid \lambda \in E_0(T_x) \} \neq \emptyset$ and $\forall x \in A, \lambda \notin \sigma_W(T_x)$, because $T_x$ satisfies Weyl’s theorem. If $x \notin \mathbb{X} \setminus A$, then $\lambda \notin \sigma(T_x)$ and then $\lambda \notin \sigma_W(T_x)$. So $\lambda \notin \bigcup_{x \in \mathbb{X}} \sigma_W(T_x) = \sigma_W(T)$. But this is a contradiction because $\lambda \in \sigma_W(T)$. Therefore $\sigma_W(T) \bigcap E_0(T) = \emptyset$. Finally we have $\sigma_W(T) = \sigma(T) \setminus E_0(T)$ and so $T$ satisfies Weyl’s theorem.

2. We use the same proof as in the first part, just by replacing $E_0(T)$ by $\Pi_0(T)$.

**Definition 3.6** Let $T \in C(\mathbb{X}, L(H))$. Then $T$ is called a normal family if $\forall x \in \mathbb{X}, T_x$ is a normal operator.
We give now an extension of Weyl’s theorem to the case of normal families.

**Theorem 3.7** Let $T \in \mathcal{C}(X, L(H))$ be a normal family. Then $T$ satisfies Weyl’s theorem, that’s $\sigma_W(T) = \sigma(T) \setminus E_0(T)$.

**Proof** Since $T$ is a normal family, then $\forall x \in X$, $T_x$ is a normal operator. From [7], we know that a normal operator satisfies Weyl’s theorem. The theorem is then a consequence of Theorem 3.5.

\[ \square \]

## 4 The space of B-Fredholm operators

In this section, we prove that the space of B-Fredholm operators $\text{BFred}(H)$ on a separable Hilbert space $H$ is path connected, and the index function is not continuous on $\text{BFred}(H)$.

**Definition 4.1** Let $S$, $T$ be in $\text{BFred}(H)$. We will say that $S$ and $T$ are B-Fredholm homotopic, if there exists a continuous map $\Phi : [0, 1] \to \text{BFred}(H)$ such $\Phi(0) = S$ and $\Phi(1) = T$.

**Proposition 4.2** Let $S, T$ be two B-Fredholm operators acting on a Hilbert space having equal index. Then $S$ and $T$ are B-Fredholm path connected.

**Proof** Let $S, T \in \text{BFred}(H)$ such that $\text{ind}(S) = \text{ind}(T) = n$. Then From [4, Remark A] there exists $\epsilon > 0$ such if $\lambda \in \mathbb{C}$, $0 < |\lambda| < \epsilon$, both of $S - \lambda I$ and $T - \lambda I$ are Fredholm operator and $\text{ind}(S - \lambda I) = \text{ind}(T - \lambda I) = n$.

Let $\lambda$ such that $0 < |\lambda| < \epsilon$. Since $S - \lambda I$ and $T - \lambda I$ are Fredholm operators having the same index, then they are Fredholm path-connected.

Consider now the map $\phi : [0, 1] \to L(H)$, defined by $\phi(t) = S - t\lambda I$. Then $\phi$ is continuous, $\phi(0) = S$, $\phi(1) = S - \lambda I$ and for all $t \in [0, 1]$, $\phi(t)$ is a B-Fredholm operator. Thus $S$ is B-Fredholm path-connected to $S - \lambda I$. Similarly $T$ is B-Fredholm path connected to $T - \lambda I$. By the transitivity of the path-connectedness, $S$ and $T$ are B-Fredholm path-connected.

We prove now the main result of this section.

**Theorem 4.3** Let $H$ be a separable Hilbert space. Then the topological space $\text{BFred}(H)$ is path connected.

**Proof** Without loss of generality, we may assume that $H = l^2(\mathbb{C})$. Let $S, T$ be the operators defined on $H$ by:

- $S(x_1, x_2, \ldots, x_n, \ldots) = (x_1, 0, 0, 0, \ldots, 0, \ldots)$, $\forall x = (x_i)_i \in l^2(\mathbb{C})$,
- $T(x_1, x_2, \ldots, x_n, \ldots) = (x_1, x_3, x_4, x_5, x_6, \ldots)$, $\forall x = (x_i)_i \in l^2(\mathbb{C})$.

Then $S$ is a B-Fredholm operator of index 0 and $T$ is a Fredholm operator of index 1. Let $\Phi : [0, 1] \to L(H)$ be the map defined by:

- $\Phi(t)(x_1, x_2, \ldots, x_n, \ldots) = (x_1, tx_3, tx_4, tx_5, \ldots)$, for all $x = (x_i)_i \in l^2(\mathbb{C})$.

Then $\Phi$ is continuous and if $0 < t \leq 1$, $\Phi(t)$ is a Fredholm operator of index 1. So $\Phi$ is a continuous path of B-Fredholm operators such that $\Phi(0) = S$ and $\Phi(1) = T$. Thus $S$ and $T$ are B-Fredholm path connected.

As $S$ is of index 0, then from Proposition 4.2, $S$ is B-Fredholm path connected to the identity, because they have the same index. Thus $T$ is B-Fredholm path connected to the identity operator $I$. Consequently, $\forall n \in \mathbb{N}$, $T^n$ is path connected to the identity operator.
\( I = I^n \). We observe that \( \text{ind}(T^n) = n \text{ind}(T) = n \). Using the adjoint, we deduce that \( T^{*n} \) is B-Fredholm path connected to the identity operator \( I \), with \( \text{ind}(T^*) = -1 \) and \( \text{ind}(T^{*n}) = -n \).

Let \( U, V \) be two B-Fredholm operators of indexes \( m \geq 0 \) and \( p \leq 0 \) respectively. Again from Proposition 4.2, \( U \) is B-Fredholm path connected to \( T^m \) and \( V \) is B-Fredholm path connected to \( (T^*)^{-p} \). Hence \( U \) and \( V \) are B-Fredholm path connected to \( I \), and so they are B-Fredholm path connected. From here, we conclude that any two B-Fredholm operators are B-Fredholm path connected and the space \( BFred(H) \) is path connected. But the index function is highly discontinuous on \( BFred(H) \).

As a consequence of Theorem 4.3, we cannot define an analytical index for continuous B-Fredholm families as done for continuous Fredholm families.

Acknowledgements The author would like to thank the referee for his comments, which led to an important improvement of Theorem 2.15.

References

1. Atiyah, M.F., Singer, I.M.: The index of elliptic operators on compact manifolds. Bull. Am. Math. Soc. 69(3), 422–433 (1963)
2. Atiyah, M.F., Singer, I.M.: The index of elliptic operators: IV. Ann. Math. Sec. Ser. 93(1), 119–138 (1971)
3. Berkani, M.: On a class of quasi-Fredholm operators. Integr. Equ. Oper. Theory 34, 244–249 (1999)
4. Berkani, M.: Index of B-Fredholm operators and generalization of a Weyl Theorem. Proc. Am. Math. Soc. 130(6), 1717–1723 (2002)
5. Bleecker, D.D.: Booß–Bavnbek, Index Theory with Applications to Mathematics and Physics. International Press of Boston, Boston (2013)
6. Labrousse, J.P.: Les opérateurs quasi-Fredholm: une généralisation des opérateurs semi-Fredholm. Rend. Circ. Mat. Palermo 2(29), 161–258 (1980)
7. Weyl, H.: Über beschränkte quadratische Formen, deren Differenz vollstetig ist. Rend. Circ. Mat. Palermo 27, 373–392 (1909)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.