Quantum Gravity as an Information Network  
Self-Organization of a 4D Universe

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I propose a quantum gravity model in which the fundamental degrees of freedom are pure information bits for both discrete space-time points and links connecting them. The Hamiltonian is a very simple network model consisting of a ferromagnetic Ising model for space-time vertices and an antiferromagnetic Ising model for the links. As a result of the frustration arising between these two terms, the ground state self-organizes as a new type of low-clustering, lattice-like graph with finite Hausdorff dimension. The model has three quantum phases: a mean field phase in which the spectral and Hausdorff dimensions coincide and are larger than 4. A fluctuations-dominated phase in which the Hausdorff dimension can only be 4 and the spectral dimension is lower than the Hausdorff dimension and a disordered phase in which there is no space-time interpretation. The large-scale dimension 4 of the universe is related to the upper critical dimension 4 of the Ising model. An ultraviolet fixed point at the lower critical dimension of the Ising model is conjectured to imply the absence of space-time at very small scales. At finite temperatures the universe emerges without big bang and without singularities from a ferromagnetic phase transition in which space-time itself forms out of a hot soup of information bits. When the temperature is lowered the universe unfolds and expands by lowering its connectivity, a mechanism I have called topological expansion. Topological expansion is associated with one emerging dimension describing the unfolding process. Quantum fluctuations about this semiclassical universes are elementary black holes and wormholes. The model admits, however, also macroscopic black hole configurations corresponding to graphs containing holes with no space time inside and around which there are Schwarzschild-like horizons with a lower spectral dimension and an entropy proportional to their “area”.

I. INTRODUCTION

Quantum gravity is probably the most important open problem in fundamental physics. Quantizing geometry fluctuations around classical solutions of the Einstein action leads to a perturbatively non-renormalizable quantum field theory. Typically, a breakdown of renormalizability signals the presence of new fundamental degrees of freedom at high energy. In the case of gravity, however, it looks like one has to add an infinity of new degrees of freedom in order to fix the problem. The mainstream approach to do so has its origins in string theory [1], which evolved over the last two decades into M-theory [2], a theory that shares many aspects with eleven-dimensional supergravity [3] and is considered the best candidate framework for a fundamental theory incorporating all known interactions of nature. The major problem of M-theory, however is that it is not even properly defined at the non-perturbative level. In string theory, mathematical consistency requires that the new degrees of freedom needed to get rid of the non-renormalizability problem necessarily bring with them many (as yet) unobserved companions, as well as new symmetries and dimensions. An entirely new approach seems to be needed to guide the search for the correct fundamental definition of M-theory.

Given the not (yet) obvious success of the string theory approach, it is natural to ponder if the problem of non-renormalizability might not have its origin in the expansion around a classical background. This is the idea behind background-independent approaches to quantum gravity, whose goal is to derive viable geometries from first principles. A first such approach is loop quantum gravity [4], in which the Einstein-Cartan action is formulated in terms of connections and vielbeins and quantised canonically in a fashion similar to gauge theories, taking as fundamental objects closed holonomies (loops). Despite its many successes and appealing structure at the Planck length, however, loop quantum gravity has had difficulty in crossing the bridge to the geometries expected in the semi-classical limit.

Not so for the Wilsonian approach based on causal dynamical triangulations (CDT) [5]. This approach is the gravity equivalent of lattice gauge theories; it is formulated by discretizing space-time in terms of (generalized) triangles, formulating the gravitational action via a modified Regge calculus [6] and summing over all possible triangulations to obtain the quantum theory. The idea is to look for non-Gaussian ultraviolet fixed points that would define quantum gravity in a non-perturbative sense. There is indeed compelling numerical evidence for the emergence of a de Sitter universe with finite Hausdorff dimension and scale-dependent spectral dimension [7]. The physics of these solutions is suggestive of a relation with a recent proposal [8] of a quantum gravity model in which time and space are fundamentally different at high energies, with the Lorentz symmetry emerg-
The idea that in a proper theory of quantum gravity space-time should emerge spontaneously is not new. Models in which space is dynamically generated in quantum models with a well-defined notion of time were already discussed in the 80s [10]. Ideas similar to those introduced in this paper have been previously considered in the so-called quantum graphity model [11]. The Hamiltonian of this model, however, is substantially different (and more complex) from the one proposed here and the derivation of the self-organization of a 4D universe-like graph from information combinatorics, is entirely missing. This crucially requires to be radical in treating the fundamental degrees of freedom: there are not only \(N(N-1)/2\) spin 1/2 degrees of freedom corresponding to edges, but the \(N\) vertices are spin 1/2 degrees of freedom themselves, one of the states denoting presence of space-time and the other absence of space-time (or anti-space-time to be brief).

The model is surprisingly predictive already at the level of topology. First of all it is the first model, to my knowledge, that can answer the question of “why the universe has dimension 4”. The answer is that we are in a fluctuations-dominated phase of a model with three possible phases. In this phase, 4 is the only possible Hausdorff dimension that a universe can have and its spectral dimension is always smaller than its Hausdorff dimension, exactly as predicted also in the CDT approach [3]. As I will show, the special value 4 for the large-scale dimension of the universe is related to the fact that 4 is the upper critical dimension of the Ising model [12]. The small-scale physics is associated, instead, with an ultraviolet fixed point at the lower critical dimension of the Ising model and describes information bits with no geometric space-time interpretation.

In addition, solving the model at finite temperatures predicts a topological cosmic expansion due to a decrease in the graph connectivity, i.e. the universe actually “unfolds” in this phase. There is no big bang singularity, the origin of the universe is a ferromagnetic phase transition in which the space-time information bits align and connect, i.e. space-time itself self-organizes. Above the critical temperature there is only a hot information soup with maximal entropy and no possible interpretation as space-time. During topological expansion the average distance on the graph scales exponentially with the temperature. A small drop in temperature causes a completely connected universe graph to unfold to a regular lattice of dimension 4. It is tempting to propose this topological expansion of the universe as an alternative to cosmic inflation, although a full investigation of this is beyond the scope of the present paper.

Finally, the model implies the emergence of a “special dimension” during topological expansion. The realization of a decreasing average connectivity in regular graphs is a break-up of links between vertices, corresponding to a spontaneous breaking of the link permutation symmetry. The consequence is that topological expansion turns the original undirected graph into a partially directed graph with one emerging dimension describing the “direction of unfolding”.

To close this section I would like to point out that the model proposed here can be viewed as an extension and generalization to dynamical topologies of the random lattice Ising model, introduced and solved exactly in [14] in two dimensions. It is well established that the critical behaviour of this model is completely different than its fixed lattice counterpart due to the emergence of a new
quantum degree of freedom corresponding to metric fluctuations in the local limit and that the model is equivalent to two-dimensional gravity coupled with Majorana fermions. The further modifications to these relations introduced by letting also the topology fluctuate will be the subject of a forthcoming publication and might shed a new light on fundamental aspects of M-theory.

This paper is organized as follows. In the next section I define the model, whose macroscopic properties are then derived in section 3. Section 4 is devoted to a discussion of quantum fluctuations whereas section 5 discusses the finite-temperature behaviour and topological expansion. In section 6 I describe inhomogeneous configurations corresponding to topological black holes. Finally, I draw my conclusions in the last section.

II. THE MODEL

Consider \( N \) spin \( 1/2 \) information bits \( s_i = \pm 1 \), for \( i = 1 \ldots N \) and \( N(N-1)/2 \) spin \( 1/2 \) information bits \( w_{ij} = 0, 1 \), for \( i, j = 1 \ldots N \). A value of \( s_i = +1 \) denotes the existence of space time, while \( s_i = -1 \) indicates the absence of space time (or the presence of anti-space-time). A value \( w_{ij} = 1 \) denotes a connection between bits \( s_i \) and \( s_j \), a value \( w_{ij} = 0 \) indicates that the two spins \( s_i \) and \( s_j \) are not connected.

All these information bits form a network with energy

\[
H = \frac{J_w}{2} \sum_{i \neq j} \sum_{k \neq j} w_{ik} w_{kj} - \frac{J_s}{2} \sum_{i \neq j} s_i w_{ij} s_j ,
\]

where the couplings are symmetric, \( w_{ij} = w_{ji} \) and vanish on the diagonal, \( w_{ii} = 0 \) and where \( J_w \) and \( J_s \) are two coupling constants with dimension energy (I use units in which \( \hbar = 1 \) and \( c = 1 \)). The coupling \( J_s \) determines the scale of space-time quantisation. The meaning of the remaining dimensionless coupling \( J_w/J_s \) will become clear below. For simplicity of presentation, from now on I shall measure all energies in units of \( J_s \) by setting everywhere \( J_s = 1 \): I will relabel the unique remaining dimensionless coupling of the model simply by \( J \),

\[
H = \frac{J}{2} \sum_{i \neq j} \sum_{k \neq j} w_{ik} w_{kj} - \frac{1}{2} \sum_{i \neq j} s_i w_{ij} s_j ,
\]

The second term in this energy function is the simplest network model. If the couplings \( w_{ij} \) would run over nearest neighbours of a lattice, it would be nothing else than the standard ferromagnetic Ising model. With no underlying lattice structure, as in the present case, and with the \( w_{ij} \) as independent random variables drawn from a Gaussian distribution centred at zero (so that there are both ferromagnetic and antiferromagnetic interactions), the model would describe the Sherrington-Kirkpatrick spin glass \([13]\). If the \( w_{ij} \) would be fixed variables encoding binary patterns, the model would describe Hopfield’s associative memory \([16]\). Finally, if the couplings \( w_{ij} \) would be uniformly drawn random adjacency matrices of degree 4, the model would be Kazakov’s random lattice Ising model in two dimensions \([14]\). The first term in the energy functional, on the other side, is simply an antiferromagnetic Ising model for the link spins. Note that this is an exact statement: when the vertex spins become aligned, the link energy functional involves only link nearest neighbours that share one common vertex.

The novelty with respect to the standard Ising model, spin glasses, associative memories or the random lattice Ising model is that, in the present case, the links are determined dynamically by a minimum energy principle, given itself by an antiferromagnetic Ising model. In particular, the extension with respect to Kazakov’s random lattice Ising model \([14]\) is that not only the lattice geometry is allowed to fluctuate but its topology too: the dimension of the ground state is dynamically determined.

The difference between the vertex ferromagnetic coupling and the link antiferromagnetic one creates ”link frustration” in the model. Indeed the vertex ferromagnetic coupling favours the creation of many links (positive values of \( w_{ij} \)) in a state with the majority of vertex spins aligned, corresponding to an incipient space-time: space-time points have a tendency to link together. However, due to the antiferromagnetic link coupling, creating many links costs energy and is energetically disfavoured. It is particularly nearest neighbour links, sharing a common vertex, that are disfavoured. As I now show, the compromise is to create links between space-time points but to avoid triangles involving nearest neighbours links. The number of triangles (or more precisely the clustering coefficient) is a topological measure of the Ricci curvature of a graph \([17]\). The result is thus a Ricci flat, lattice-like graph with power-law extension, exactly what one would expect for a discretized universe. The dimension of this universe, encoded in the power-law for the average distance is determined by the unique dimensionless coupling \( J \) of the model.

To show this, let me consider a \( k \)-regular graph, i.e. a configuration with \( s_i = +1 \), \( \forall i \) and such that each vertex has exactly \( k \) incident edges (degree \( k \) in graph parlance \([3]\)). When \( k \) is even, such a graph resembles locally a \( k/2 \)-dimensional lattice. I will now show that such a graph is a local minimum of the energy \([2]\).

To this end, suppose we are given such a graph and let me first change one single vertex spin from \( s_i = +1 \) to \( s_i = -1 \). The corresponding energy change \( \Delta E_i = k > 0 \) is positive. Changing vertex spins, thus costs energy. Let us now try to eliminate an existing connection, i.e. changing \( w_{ij} = 1 \) to \( w_{ij} = 0 \). In this case there are contributions from both terms in the energy function. Before the elimination, the existing connection contributed \( E_{ij} = J(k - 1) - (1/2) \). After the elimination, of course the contribution of this connection vanishes. The energy change due to the elimination of a connection \( w_{ij} \) is thus \( \Delta_{\text{elim}} E_{ij} = (1/2) - J(k - 1) \). Let me now consider adding a previously non-existent connection \( w_{ij} \). In this case it is the energy contribution before the addition that...
vanishes, while, after the addition I have an additional energy \( E_{ij} = JK - (1/2) \). The energy change for adding a connection is thus \( \Delta_{\text{add}} E_{ij} = JK - (1/2) \). Requiring that both eliminating and adding a connection costs energy we obtain the stability condition

\[
\frac{1}{2J} < k < 1 + \frac{1}{2J} .
\]

To proceed, let me compute the total energy of a \( k \)-regular graph on \( N \) vertices. This is easily obtained as

\[
E_{N,k} = N \left( \frac{J}{2} k(k-1) - \frac{k}{2} \right) .
\]

This expression is minimized when the vertex degree takes the value \( k = 1/2 + (1/2J) \) and this value of \( k \) satisfies the stability condition (3). Defining

\[
J = \frac{1}{4d-1} ,
\]

I have obtained the result that for any choice of integer \( d \) a 2\( d \)-regular graph is a local minimum of the energy and that this minimum is the one of lowest energy among all possible regular graphs. As anticipated, the links arrange themselves to form a locally square lattice with 2\( d \) edges at each vertex; the number of triangles is minimized. Thus, \( d \) plays the role of the "bare" dimension of the universe. I call this dimension "bare" since, as I will show below, the physical Hausdorff dimension of the universe can differ from \( d \).

Of course, I have not shown that 2\( d \)-regular graphs are the true global minima of the energy functional (2). There could be other, non-regular graphs, with even lower energies. That this is not so can be made plausible with the numerical methods introduced in the next section.

### III. NETWORK DYNAMICS AND THE PROPERTIES OF EMERGENT UNIVERSES

The foremost task of a background-independent quantum gravity model is to generate a macroscopic, semi-classical universe with properties that can be reconciled with what we know of our own universe, dressed by quantum fluctuations. In the present case, the full quantum model involves qubits rather than bits and the Hamiltonian should be formulated in terms of operators involving the Pauli matrices. If all interactions involve only the third Pauli matrix \( \sigma^3 \), however, one has to all effects a classical model on binary variables since all terms in the Hamiltonian commute. The network (2) has to be considered in this light and the ground state graphs are the emerging semi-classical universes. In this section I shall describe their macroscopic properties, quantum fluctuations will be addressed in the next section.

To study the properties of the ground states of (2) I will reverse a technique used to analyze neural networks such as the Hopfield model [16]. There, a network dynamics is posited and an energy functional is looked for that acts as a Lyapunov function, i.e. such that the energy minimum determines the fixed points of the network dynamics. Here I will do the contrary, namely I will look for a network dynamics whose fixed points correspond to minima of the energy (2).

To this end let me start form a random initial configuration and sequentially update the vertex and link spins according to the rule

\[
s_i(t+1) = \text{sign}_+ (h_i(t)) ,
\]

\[
h_i = \sum_{j \neq i} w_{ij}s_j ,
\]

\[
w_{ij}(t+1) = \Theta_\pm (h_{ij})
\]

\[
h_{ij} = \frac{1}{2}s_is_j - \sum_{j \neq i} w_{kj}(t) - \frac{J}{2} \sum_{j \neq i} w_{ik}(t) ,
\]

where \( \Theta \) denotes the Heaviside function and \( h_{ii} = 0 \). The subscripts "+" and "\( \pm \)" on the sign and \( \Theta \) functions indicate what is the rule to follow in the (rare) cases in which the argument is zero. The subscript "+" indicates that, in this case, the + sign has to be chosen. This favours positive vertex spin values: I have made this choice since I am interested to study the properties of large space-times with complete ferromagnetic order. The subscript "\( \pm \)" on the Heaviside function, instead indicates that a random choice will be made when the argument vanishes. This choice helps avoiding being stuck in sub-optimal minima during the evolution.

I will now show that the energy function (2) cannot increase along a sequential evolution (6) and (7): \( E(t+1) \leq E(t) \). To this end let me begin by considering the update of vertex spin \( s_i \). The corresponding contribution \( E_i \) to the energy changes according to

\[
E_i(t+1) = -s_i(t+1) \sum_{j \neq i} w_{ij}(t)s_j(t)
\]

\[
= -\text{sign}_+ (h_i(t)) h_i(t) = -|h_i(t)|
\]

\[
\leq -s_i(t)h_i(t) = E_i(t) .
\]

With the exact same procedure for the update of a link spin \( h_{ij} \) we obtain

\[
E_{ij}(t+1) = -w_{ij}(t+1)h_{ij}(t) = -\Theta_\pm (h_{ij}(t)) h_{ij}(t) .
\]

Now we have two possibilities

\[
\bullet \ h_{ij}(t) < 0 \rightarrow E_{ij}(t+1) = 0 \leq -w_{ij}(t)h_{ij}(t) = E_{ij}(t)
\]

\[
\bullet \ h_{ij}(t) > 0 \rightarrow E_{ij}(t+1) = -h_{ij}(t) = -|h_{ij}(t)| \leq -w_{ij}(t)h_{ij}(t) = E_{ij}(t)
\]

and in both cases we obtain \( E_{ij}(t+1) \leq E_{ij}(t) \), which proves the claim that the energy cannot increase during sequential evolution (note that, for both the sign and the Heaviside functions, the exact procedure on how to treat
the undefined cases in which the argument vanishes has no effect on this result). As a consequence, every minimum of the energy (2) is a fixed point of the sequential network evolution (6) and (7).

Every such fixed point defines an (undirected) graph. Several classes of graphs have been studied [8] extensively. The simplest such one is the class of regular graphs, for which every node has a fixed degree $k$, i.e. a fixed number $k$ of neighbours. Regular graphs are typically highly clustered, with the notable exception of $d$-dimensional hypercubic lattices, which have vanishing clustering coefficient since they do not contain any triangles, and in which the average distance scales as $N^{1/d}$.

This is the exact opposite of classical random graphs [18] in which connections between nodes are chosen randomly with a given fixed probability. Random graphs have small clustering $c_{\text{rg}} = \langle k \rangle / N$ where $\langle k \rangle$ is the degree expectation value and small average distance that scales logarithmically with $N$ (the so-called small world effect). In classical random graphs the degrees follow a Poisson distribution. In order to obtain graphs that better model real-world networks a class of generalized random graphs was introduced [19], in which connections are chosen randomly subject to the constraint that the degree distribution must follow a predetermined law. In generalized random graphs the clustering depends, of course, from the chosen input degree distribution. Actually, only the first two moments of this distribution matter,

$$c_{\text{grg}} = \frac{1}{N} \frac{\langle (k^2) \rangle - \langle k \rangle^2}{\langle k \rangle^3}.$$  \hspace{1cm} (10)

This reduces to $c_{\text{rg}}$ for a Poisson distribution, as expected. Generalized random graphs are also small worlds, the average distance remains a logarithmic function of $N$. These two classes describe static graphs, with a fixed number $N$ of nodes. The number of nodes grows, instead in the class of evolving graphs [20]. Many evolution rules have been considered [9], essentially all aimed at generating scale-free graphs with power-law degree distributions of various forms.

The graphs that minimize the energy (2) constitute yet another, novel category for which

- The number of nodes is not growing but fixed.
- There is no random assignment of connections but, rather, these are dynamically determined. The dynamics, however involves both connections and nodes.
- The dynamics is not an ad-hoc rule of how a graph is grown but is governed by an energy minimization principle.
- Contrary to generalized random graphs and typical evolving graphs, the degree distribution is not an input but an output.

I shall call such graphs dynamical graphs and, in the following, I shall describe their main properties.

In order to investigate these dynamical graphs, I have repeatedly started with random values of all the spins and let them evolve according to the dynamics (6) and (7) until a fixed point is reached. The properties of interest can then be extracted from the adjacency matrix of this graph. Finally, an average over the results form different runs is taken. Of course, care has to be taken to avoid ending up in fixed points corresponding to metastable, local minima of the energy function. Fortunately, these typically involve configurations for which not all space-time information bits take the same value 1/2 and can thus be recognized easily. Actually, it could also be possible that a fixed point cannot be reached since the ground state is degenerate and so the network dynamics would wander among the different ground states with the same energy. This would be no problem however, since anyone of these ground states would correspond to a viable universe.

As expected, I have observed that, for $d$ integer, the minimum energy configuration always corresponds to a $2d$-regular graph with exactly $2d$ edges for every vertex. By the degree sum formula $2e = \sum_{i \geq 3} i v_i$, with $e$ the number of edges and $v_i$ the number of vertices of degree $i$, we can derive that these graphs have exactly $d$N edges. The connectivity details of the ground state graph, like clustering coefficient, number of 4-cycles and others can indeed vary from run to run but they do correspond to finite size effects: the bulk properties convergence to unique values for large values of $N$.

The three most important quantities that I have measured are the clustering coefficient, the spectral dimension and the Hausdorff dimension of the emergent dynamical graphs. For simplicity (and due to limitations in computing power) I have restricted these measurements to the most interesting case of integer $d$, when the graphs are regular. As already anticipated, the clustering coefficient $c$ [2] is a measure of the Ricci curvature of the graph [17]. The spectral dimension $d_s$ measures the connectivity of the graph [21], the dimension that a particle moving on the graph would feel. It is defined via the scaling of the return probability $p_r(t)$ to the initial point after $t$ steps of a random walk on the graph [22].

$$p_r(t) \sim t^{-d_s/2}.$$  \hspace{1cm} (11)

For infinite graphs this scaling relation is valid in the limit $t \to \infty$. For finite $k$-regular graphs, the return time to the initial point is $t = 1/N$ [22] and thus $p_r(t) \to 1$ in the limit $t \to \infty$. The correct scaling is typically found in the intermediate region $1 \ll t \leq O(N^{4/k})$ where finite size effects are suppressed [22]. Finally, the intrinsic Hausdorff dimension $d_H$ measures how the graph volume scales with the average distance $< D >$ among points along the graph [4].

$$< D > \sim N^{1/d_H}.$$  \hspace{1cm} (12)

Let me stress that this is an intrinsic property of the graph that does not need an embedding in an extrinsic
Euclidean space to be defined and measured. It is the dimension that would be measured with clocks and rods by "inhabitants" of the graph.

For large values of $N$ the convergence to the ground states slows down considerably and so does the computing time needed to evaluate the quantities of interest. This difficulty is compounded by the increasing probability of ending up in local minima of the energy function. For simplicity and to adapt to the available computational resources, I have restricted the numerical computations to the case $d$ integer, for which the ground state graphs can be most easily recognized and to vertex numbers of up to a maximum of $N = 250$ ($N(N - 1)/2 = 31125$ possible links). As we will now see, this are anyway the most interesting cases.

The first important observation is that dynamical graphs are disordered for $J \geq 1/3$ or equivalently $d \leq 1$. In this phase of the model the space-time bits never align and the ground state is disordered with no space-time interpretation possible. The point $d = 1 (J = 1/3)$ corresponds thus to a first quantum phase transition, a quantitative change in behaviour of the ground state at zero temperature as a function of the model coupling constant $J$.

Dynamical graphs, emerging as ground states of (2), are low-clustering graphs. Their clustering coefficients scale as $1/N$, as is the case for random graphs and generalized random graphs. Any additional power posited in the $N$-dependence resulted in statistically non-significant parameters of the non-linear regression. In the following table, the clustering coefficients $c_d$ of dynamical graphs and their regression standard deviations are reported together with the clustering coefficients of generalized random graphs with the same, uniform degree distribution, obtained from [10].

\[
\begin{array}{cccccc}
\text{d} & 2 & 3 & 4 & 5 \\
\text{c}_d & 2.4 & 4.52 & 6.27 & 8.39 \\
\sigma_c & 0.15 & 0.09 & 0.22 & 0.17 \\
\text{c}_{arg} & 2.25 & 4.17 & 6.125 & 8.1 \\
\end{array}
\]

The two values are very close, although the values for dynamical graphs are slightly higher. This may be the result of the limited number of vertices in the numerical simulations, a limitation due to the available computational power. In any case it is evident that clustering is a finite size effect in dynamical graphs, which reduce to Ricci flat graphs [17] in the large $N$ limit.

As expected, the spectral dimension $d_s$ coincides with the bare dimension $d$, as is evident from the results in the following table.

\[
\begin{array}{cccc}
\text{d} & 2 & 3 & 4 \\
\text{d}_s & 2.005 & 2.99 & 4.04 \\
\sigma_s & 0.005 & 0.02 & 0.063 \\
\end{array}
\]

I cannot report the spectral dimension for $d = 5$ since the values of $N$ required for such a measurement were not accessible with the available computational resources.

Finally, the measurements of the Hausdorff dimensions are reported in the next table.

\[
\begin{array}{cccccc}
\text{d} & 2 & 3 & 4 & 5 \\
\text{d}_H & 4.28 & 4.32 & 4.26 & 5.22 \\
\sigma_H & 0.09 & 0.12 & 0.18 & 0.09 \\
\end{array}
\]

The results are again slightly higher than expected but there is a clear numerical evidence of a quantitative change of behaviour at $d = 4$ (equivalently $J = ~1/5$), corresponding to a second quantum phase transition. The indication is that, for $d \geq 4$, Hausdorff, spectral and bare dimension coincide, whereas for the whole range $1 < d < 4$ the spectral dimension is lower than the Hausdorff dimension, which is fixed at $d_H = 4$, the required value to describe our universe.

There is actually a plausible explanation the physical origin of the special role of the dimension $4$. The model [2] can be considered as a standard nearest-neighbour Ising model with dynamical topology and dimensionality $d$. As is well known [12], the Ising model has a lower critical dimension $d_{ic} = 1$ below which order is impossible due to strong fluctuations. As already mentioned this is fully confirmed in the present model, which has a disordered phase for $d \leq 1$. What is even more interesting, however, is that, as most statistical mechanics systems, the Ising model has also an upper critical dimensions $d_{uc}$ beyond which fluctuations can be neglected altogether and mean field theory becomes exact [12]. For the Ising model this upper critical dimension turns out to be $d_{uc} = 4$.

It is often stated that hyperscaling fails above the upper critical dimension. This is because, in the standard approach to finite size scaling [24], one considers that the correlation length is bounded by the system size $L$. It has been shown, however, that hyperscaling can be fully restored if one considers scaling in terms of Binder’s thermodynamic length $\xi_b$ [25],

\[
\xi_b \propto L^{d/d_{uc}} , \quad d > d_{uc} , \\
\xi_b \propto L , \quad d \leq d_{uc} .
\]

Recent results [26] have shown that this is because, in finite size samples, the correlation length is not commensurate with the physical length above the upper critical dimension. The thermodynamic length is the relevant quantity that permits a unified treatment of finite size scaling in all dimensions. It coincides with the physical length below the critical dimension but differs above it. One can now define also two volumes, $V = L^d$ and the effective volume $V_{eff} = \xi_d^{d_{uc}}$. These, instead, coincide above the upper critical dimension but differ below it. An alternative, but equivalent way to represent the incommensurability found in [24] when the number of spins, rather than the geometric size, is fixed, is to consider the effective volume as the volume actually dynamically occupied by the system in any number of dimensions. In terms of this effective volume, the physical length scales...
where I have already explicitly used $d_{ac} = 4$ for the Ising model. This is exactly what I have found for the Hausdorff dimension of model (2). In other words, the quantum phase transition of (2) at $d = 4$ (or equivalently at $J = 1/15$) is due to the change in critical behaviour of the Ising model at its upper critical dimension. For $d > 4$ the model is thus in a mean field phase with $d = d_s = d_H$, for $d < 4$, instead fluctuations become important and the model is in a fluctuations dominated phase in which $d_H = 4$ and $d_s < d_H$. The dimension 4 of the universe would thus be simply related to the fact that 4 is the upper critical dimension of the Ising model. Note that a spectral dimension lower than the Hausdorff dimension is also exactly what is predicted by the CDT approach [7]. On the graph sizes accessible to numerical simulations, I have not been able to observe a scale-dependence of the spectral dimension, as predicted by the CDT approach [7]. It seems, however natural to conjecture that the quantum phase transition at $d = 4$ corresponds to an infrared-stable critical point, especially in view of the fact that the Gaussian model is an infrared fixed point of the Ising model at $d = 4$. In this case the model (2) would predict the large-scale dimension 4 of the universe and realize at the same time the asymptotic safety scenario of quantum gravity [5]. The other quantum phase transition at $d = 1$, in fact, would necessarily imply the existence of an ultraviolet-stable fixed point either at $d = 1$ or in the intermediate region $1 < d < 4$. In both cases the spectral dimension would be scale-dependent. The latter case would correspond to a universe of lower dimensionality at short distances. The former, even more interesting case, would imply that there is no space-time at all at short distances, only fluctuating information bits, something like the “quarks of quantum gravity”. The fact that $T = 0$ is an ultraviolet fixed point of the one-dimensional Ising model [28] supports this picture.

To conclude this section I would like to stress an important point. Even if the Hamiltonian (2) is not different at the classical and quantum levels, as pointed out above, the ground state as a function of the coupling constant displays quantum behaviour, embodied in the two quantum phase transitions. This is one of the two possible realizations of quantum phase transitions: a Hamiltonian $H = H_0 + J H_1$ on a finite lattice can display non-analiticity in its ground state if $H_0$ and $H_1$ commute (as in the present model) leading to level-crossings at particular values of $J$ while the wave functions remain the same [23]. Essentially, this means that the dynamical lattice topology leads to the emergence of quantum degrees of freedom. This is not a new phenomenon: it is known that lattice fluctuations in the random lattice Ising model also lead to the emergence of a new quantum degree of freedom [14].
assuming the binary values,
\[
\text{Prob}(s_i(t+1) = +1) = f(h_i(t)),
\]
\[
\text{Prob}(w_{ij}(t+1) = +1) = f(h_{ij}(t)),
\]
where
\[
f(h) = \frac{1}{1 + e^{-\beta h}},
\]
is the Fermi function at temperature \( T = 1/\beta \). I will now show that the equilibrium reached by this stochastic update rule is the Boltzmann distribution corresponding to the energy function \( \mathcal{E} \). Therefore, assuming an equilibrium configuration, the two above rules will provide coupled mean field equations for mean space-time "magnetization" \( < s > \) and the average degree \( < k > \) of the graph vertices. This is a measure of the temperature dependence of the universe bare dimension.

Let us denote by \( F_s(s_i \rightarrow -s_i) \) the probability of a vertex spin flip and by \( F_w(w_{ij} \rightarrow 1 - w_{ij}) \) the probability of a link spin flip. As a consequence of (16) these are given by
\[
F_s(s_i \rightarrow -s_i) = \frac{e^{-\beta h_is_i}}{2 \cosh (\beta h_is_i)},
\]
\[
F_w(w_{ij} \rightarrow 1 - w_{ij}) = \frac{e^{-\beta h_is_{ij}}}{2 \cosh (\beta h_is_{ij})},
\]
where we have introduced \( s_{ij} = 2(w_{ij} - 1/2) \). Let us denote by \( D(...s_i...w_{ij}...) \) the probability distribution for activity patterns of all vertex and link spins. Due to the vanishing of the diagonal terms, \( w_{ii} = 0 \), and symmetry \( w_{ij} = w_{ji} \) of the link variables we will consider \( D \) as function of \( w_{ij} \) for \( i < j \) only. In equilibrium (denoted by "e"), the distribution \( D_e \) must satisfy the conditions
\[
F_s(s_i \rightarrow -s_i) D_e(...s_i...w_{ij}...)
\]
\[
= F_s(-s_i \rightarrow s_i) D_e(...s_i...w_{ij}...),
\]
\[
F_w(w_{ij} \rightarrow 0 \rightarrow w_{ij} = 1) D_e(...s_i...w_{ij}...),
\]
\[
= F_w(w_{ij} = 1 \rightarrow w_{ij} = 0) D_e(...s_i...w_{ij} = 1)...(19)
\]
Using (18) one obtains readily
\[
\frac{D_e(...s_i...w_{ij}...)}{D_e(...-s_i...w_{ij}...)} = e^{2\beta h_is_i},
\]
\[
\frac{D_e(...s_i...w_{ij} = 1...)}{D_e(...-s_i...w_{ij} = 0...)} = e^{2\beta h_{ij}}.
\]
Finally, one easily verifies that these conditions are met by the equilibrium probability distribution
\[
D_e(...s_i...w_{ij}...) = \frac{1}{Z} e^{-\beta H(...s_i...w_{ij}...)}.
\]
where \( H \) is the energy function \( (2) \) and \( Z \) the corresponding partition function. The factor of 2 in the exponent of (20) arises since all fundamental variables, both vertex spins \( s_i \) and link spins \( w_{ij} \) appear twice in the energy function, the latter due to the symmetry \( w_{ij} = w_{ji} \). The equilibrium distribution is thus the Boltzmann distribution corresponding to the energy function \( \mathcal{E} \).

Having established that (16) are the probabilities in the equilibrium Boltzmann distribution at inverse temperature \( \beta \) we can use them to compute the expectation values of both vertex and link spins in thermal equilibrium. Let us begin by the vertex spins. In this case, the thermal expectation value is
\[
<s_i> = 1/2 f(h_i) - 1/2 f(-h_i) = \tanh (\beta \sum_{j \neq i} w_{ij} s_j).
\]
In the mean field approximation, which becomes exact in large number of dimensions, we can bring thermal averages \( \langle ... \rangle \) inside non-linear functions. With the usual homogeneity Ansatz \( \langle s_i \rangle = \langle s \rangle \), independent of \( i \), we obtain
\[
\langle s \rangle = \tanh (\beta \langle k \rangle \langle s \rangle),
\]
where we have introduced the mean degree \( \langle k_i \rangle \) at vertex \( i \),
\[
\langle k_i \rangle = \sum_{j \neq i} \langle w_{ij} \rangle,
\]
and assumed that it is independent of \( i \), \( \langle k_i \rangle = \langle k \rangle, \forall i \).

One proceeds similarly for link spins. In the mean field approximation, the thermal average is given by
\[
\langle w_{ij} \rangle = f(h_{ij}) = f\left(1/2 \langle s \rangle^2 - J \langle k \rangle + J \langle w_{ij} \rangle\right).
\]
In this case, however, I am not interested in homogeneous solutions in the link variables, but, rather, only the average degree \( \langle w_{ij} \rangle \) is expected to be independent of the vertex label. I will therefore sum the left hand side of (25) over all \( j \neq i \) to obtain
\[
\langle k \rangle = \sum_{j \neq i} f\left(1/2 \langle s \rangle^2 - J \langle k \rangle + J \langle w_{ij} \rangle\right).
\]
At high temperatures one can expect the thermal average of the link spins to approach its limiting value \( 1/2 \) from below. I will thus approximate inside the Fermi function \( \langle w_{ij} \rangle \approx 1/2 - \epsilon_{ij} \), \( \epsilon_{ij} > 0 \) and neglect the deviations \( \epsilon_{ij} \), keeping however in mind that they are positive. This gives
\[
\langle k \rangle = f\left(-\frac{1}{2} \langle s \rangle^2 - J \left(\langle k \rangle - \frac{1}{2}\right)\right),
\]
where the superscript "n" means that 0 must be approached from below in the Fermi function. Equations (23) and (27) constitute the coupled mean field equations for "space-time magnetization" and mean degree, respectively. They describe topological fluctuations of space-time and its connectivity at inverse temperature \( \beta \).
and provide a correct picture of the thermodynamics of these emerging universes as long as \( \langle k \rangle \gg 1 \).

To solve these equations I will introduce the new variable \( x = (1/2)\langle s \rangle^2 - J (\langle k \rangle - (1/2)) \) and the new linear function

\[
g(x) = \frac{1}{N-1} \left( \frac{1}{2J} \langle s \rangle^2 + \frac{1}{2} \right) - \frac{1}{J(N-1)} x .
\]

and rewrite the mean field equations as

\[
\langle s \rangle = \tanh (\beta \langle k \rangle \langle s \rangle) ,
\]

\[
g(x) = f^-(x) .
\]

When \( \beta \langle k \rangle \leq 1 \), eq. (29) has the unique solution \( \langle s \rangle = 0 \). This is the case in the high temperature regime \( \beta \rightarrow 0 \) (for finite \( \langle k \rangle \)). In this limit, the Fermi function in (30) reduces to its limiting value 1/2, giving the solution \( g(x) = 1/2 \) or \( \langle k \rangle = \langle k \rangle_{\text{max}} = (N - 1)/2 \), which is indeed finite. This is the information phase of the model, a hot soup of information bits with no space-time interpretation. When \( \beta \langle k \rangle > 1 \), instead, two solutions of (29) exist, the same solution \( \langle s \rangle = 0 \) as before and a solution with finite \( \langle s \rangle > 0 \). Only the latter, however, is stable, as is well known from the mean field theory of the Ising model. In the low temperature regime \( \beta \rightarrow \infty \) (with \( \langle k \rangle \langle s \rangle > 0 \)) the stable solution becomes \( \langle s \rangle = 1 \). In this limit, the Fermi function approaches a Heaviside function at \( x = 0 \), always crossing the linear function \( g(x) \) when \( x \rightarrow 0^- \), which amounts to \( \langle k \rangle \rightarrow \langle k \rangle_{\text{min}} = 2d \). This is the topology phase of the model, in which the ground state of the model is a low-clustering, lattice-like universe with effective Hausdorff dimension \( d_{\text{eff}} = \langle k \rangle/2 \geq 4 \). Between the two phases there is a ferromagnetic phase transition in which space-time and the universe emerge spontaneously. The exact position of the phase transition can be determined by solving numerically the system of equations (29) and (30). For example, for \( d = 4 \) it is located at \( T_{cT} = \langle k \rangle_{cT} = 0.468N \).

In the topology phase of the model the average connectivity \( \langle k \rangle \) of the universe increases with decreasing temperature. This corresponds to a topological expansion in which the universe actually “unfolds” rather than expands. There is no big bang, space-time emerges collectively in a phase transition as a very tight ball of “hyperconnected” points with scaling factor \( D_{\text{min}} \propto N^{2/(k_c)r} \). Because of the very high connectivity at the phase transition, \( \langle k \rangle_{cT} = O(N) \) this is a finite quantity in the limit \( N \rightarrow \infty \). Note also that there is no singularity at the phase transition, the universe emerges with an average distance of \( O(1) \).

For very high and very low temperatures it is possible to derive how the average distance in the universe scales with the temperature. For very high temperatures, \( \beta \rightarrow 0 \) one can expand the exponential in the Fermi function, for very low temperatures \( \beta \rightarrow \infty \), instead, one can replace the Fermi function (30) with \( \exp(x) \) (bear in mind that \( x \) is a negative quantity). This gives

\[
\langle D \rangle \propto D_{\text{min}} N^{2/k_c} , \quad T \rightarrow \infty ,
\]

\[
\langle D \rangle \propto D_{\text{max}} N^{-aT} , \quad T \rightarrow 0 ,
\]

with \( a = (J + 1)^2/(4J^2) \log(2J(N - 1)/J + 1) \) and \( D_{\text{max}} \propto N^{1/d} \) for \( d > 4 \) and \( D_{\text{max}} \propto N^{1/4} \) for \( d \leq 4 \). The important point is that, during topological expansion, the average distances in the universe scale exponentially with the temperature (or its inverse). A tiny decrease in temperature causes a large increase in the “topological scale factor” of the universe. This starts out as an almost completely connected graph and expands exponentially (with temperature) to a lattice of dimension 4 (if \( d \leq 4 \)). It is tempting to suggest topological expansion as an alternative mechanism to inflation to obtain a large-scale homogenous universe, a mechanism that would not require reheating since the temperature would drop only logarithmically with the universe scale. A proper investigation of this, however, is beyond the scope of the present paper.

Topological expansion has another interesting consequence, as I now show. Contrary to “geometric” expansion in standard cosmology, it is not the length scale of the universe that increases but, rather, it is the topological neighbourhood relations between space-time points that change (there is no length scale at this point). During topological expansion the average degree

\[
\langle k \rangle = \frac{1}{N} \sum_i \sum_{j \neq i} \langle w_{ij} \rangle ,
\]

decreases with decreasing temperature, as derived above. There are two ways for this to happen. Either the thermal averages \( \langle w_{ij} \rangle \) decrease homogeneously or, instead, the decrease is inhomogeneous. Suppose one starts at a given temperature from a \( (d + 1) \)-dimensional graph, by which I mean a configuration in which for every vertex \( i \), \( 2(d + 1) \) links have values \( \langle w_{ij} \rangle \) close to 1 and all the rest of the links have expectation values close to 0 so that, in sum, \( \langle k_i \rangle = 2(d + 1) \). If the temperature is decreased to the value corresponding to a \( d \)-dimensional graph, all \( \langle w_{ij} \rangle \) could decrease by the same amount so that in the end \( \langle k_i \rangle = 2d \) or, otherwise, at the other end of the spectrum, two links could ”break up” so that in the end one has only \( 2d \) links. There is no way to tell from eqs. (29) to (30) that govern only the degree average over the graph. The homogeneous decrease, however is clearly incompatible with the inhomogenous ground state I have previously derived at \( T = 0 \). The only picture compatible with this \( T = 0 \) ground state and with a number \( 2d \) of links per vertex for all integer values of \( \langle k \rangle = 2d \) is a break-up of two links when the temperature is decreased between values of \( \langle k \rangle \) that differ by an even integer. This constitutes a spontaneous breaking of the local labelling permutation symmetry of links at each vertex.

The consequence of this permutation symmetry breaking is that topological expansion implies one local marked
dimension at each graph vertex. For every space-time vertex it is possible to mark which other two vertices were connected to it at higher connectivity but are no more. For each of these there will be a new shortest path connecting it to the space-time vertex under consideration in the universe with diminished connectivity. The first edges of these two new paths starting from the vertex under consideration can be marked as the “new version” of the old, now absent dimension. Finally, one can assign an orientation to this marked dimension (one of two possible choices). Topological expansion turns thus the original undirected graph into a partially directed graph $\mathbb{G}$. The directed dimension that arises from spontaneous link permutation symmetry breaking describes the unfolding of the universe graph. Topological expansion is associated with the spontaneous emergence of a preferred dimension on the universe graph.

Let me make a visual analogy for the simplest case of unfolding. To this end I will consider, for a moment, the graph as embedded in $(d+1)$-dimensional Euclidean space. To obtain the simplest configuration I will arrange the links that are breaking up in one particular dimension of this Euclidean space, with the rest spanning the orthogonal $d$ dimensions. One can think of this arrangement as akin to the bellows of an accordion, the $d$ fixed dimensions spanning the folds of the bellows. Lowering the temperature and breaking the links would be like stretching the bellows until they become “flat”. In this process, a marked point on the bellows folds experiences a flow in the direction of the stretch. This is the preferred dimension associated with this simplest unfolding. Of course, in general, the unfolding process can much more complex, like stretching out a crumpled sheet of newspaper. The general problem of characterising the (unfolding) paths on partially directed graphs has been addressed in $^{12,27}$.

VI. TOPOLOGICAL BLACK HOLES

Until now I have concentrated on emerging universes with homogeneous space-time “magnetization” and vertex degrees, both at $T = 0$ and at finite temperature in the mean field limit $(k) \gg 1$. In this section, instead, I will explore possibly inhomogeneous configurations that can emerge at generic, lower values of $(k)$.

To do so I will consider a Peierls droplet of $n \ll N$ wrong space-time spins $s_i = -1$ in an otherwise homogeneous universe of $N$ vertices. In the usual Ising model, the vertex connectivity is fixed and, when it is high enough, these droplets disappear below a finite critical temperature since the entropy of the droplet becomes lower than its internal energy, so that the free energy is dominated by the latter $^{29}$. This is given by the boundary contribution $E = E_h + \delta$ where $E_h$ is the energy of the homogeneous universe and $\delta$ is the number of links between spins of opposite signs. The internal energy is thus minimized by turning all wrong spins so that $\delta \to 0$. This is the standard Peierls argument to explain why all the spins align when lowering the temperature and it is exactly what happens in the homogenous mean field solution discussed in the previous section.

In the present model, instead, the links are themselves dynamical variables and there is another way to lower the internal energy of the droplet. Indeed, turning one single spin does not lower the internal energy but it actually increases it by $\Delta E = \langle k \rangle - 2$, where $(k)$ is the mean connectivity at the considered temperature, since for one “corrected” link there are now, on average $(k) - 1$ new wrong ones. For high enough temperatures, $(k) > 2$ and the energy cost is a positive quantity. Of course, if the links are given external parameters, the only way to lower the energy is a simultaneous turnaround of all the wrong spins. Here, however the system could simply severe the links between spins of opposite signs, thereby lowering the internal energy by $\Delta E = -\delta/2(1 + J((k) - 1))$. The resulting configuration is a universe represented by a graph containing a hole within which there is simply no space-time. This hole is represented by a disconnected subgraph for which all vertex spins have value $s_i = -1$. The original universe contains a Schwarzschild-like boundary on which the mean spectral dimension is lower by one than in the rest of the universe: this boundary is constituted by all $s_i = +1$ vertices for which a link to a $s_i = -1$ vertex inside the hole has been severed. I will call such a configuration a topological black hole. It is characterized by the absence of space-time inside the hole, by a lowered spectral dimension on its boundary and, at least for integer $d$, by an entropy proportional to the “area”, i.e. the number of vertices on the boundary. For $d = 2$ this is the original Peierls argument $^{12}$. The Peierls argument, however, can be generalized to any integer $d \geq 2$ $^{29}$, the entropy always scales as the number of vertices on the higher-dimensional contour around the hole, i.e. its “area”.

The question arises if a topological black hole is, on average, a stable configuration for temperatures down to $T = 0$. The only way to detect the presence of the topological black hole from within the main universe is through the lower spectral dimension on its boundary. And indeed, the lowest-energy instabilities come from vertices on the boundary forming links between themselves. Forming such a link would lower the system energy by $1/2$ via the ferromagnetic vertex term in the energy, on the other side, it would cost energy due to the antiferromagnetic link term. In total, one such link would cost

$$\Delta E = J((k) - 1) - \frac{1}{2}.$$  \hspace{1cm} (34)

Note that forming links to other vertices not on the boundary costs more energy since they possess already more links.

Topological black holes are thus, on average, stable configurations against radius connections if $^{44}$ is a positive quantity. Using $fJ = 1/(4d - 1)$ this stability con-
dition can formulated as
\[
\langle k \rangle > 2d + \frac{1}{2} = k(T = 0) + \frac{1}{2}.
\] (35)

This shows that topological black holes are typically stable down to very low temperatures. This, together with the presence of a Schwarzschild-type radius with entropy proportional to its area and lower spectral dimension gives topological black holes some of the essential traits of physical black holes. Note that such an identification would automatically answer the question of “what is inside a black hole”: there simply would be no inside from the space-time point of view.

\section{VII. CONCLUSIONS AND FUTURE DIRECTIONS}

The frustration in a combined ferromagnetic Ising model for vertices and antiferromagnetic model for links leads to the self-organization of these information bits into a ground state configuration representing a new type of graph with the correct topology of a universe. A whole quantum phase of the model predicts universes of Hausdorff dimension 4 with a lower spectral dimension, the dimension 4 being related to the upper critical dimension of the Ising model. Fluctuating large-scale 4D geometries can emerge at an infrared fixed point at this Ising upper critical dimension, the small-scale physics is associated with an ultraviolet fixed point at the lower critical dimension of the Ising model and describes fluctuating information bits with no space-time interpretation. At high temperatures such a universe emerges from a hot soup of information bits by an ordering phase transition with no big bang and no singularity and unfolds when the temperature is decreased by diminishing its topological connectivity. This topological expansion is accompanied by one special dimension along which the unfolding takes place. In this paper I have given numerical evidence for this picture, accompanied by plausibility arguments based on recent results on the Ising model. Of course further evidence is needed too fully confirm this picture of quantum gravity. Also a natural next step consists, of course, in studying the emergence of large-scale geometry and the Einstein equations governing it.

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