The Fekete-Szegö Problem for Subclasses of Analytic Functions Associated With Touchard Polynomials

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Abstract. In this paper, we solve the Fekete-Szegö problem for a new subclasses of analytic functions defined by the integral operator $I(n,m)(f)$.

1. Introduction
Let $A$ be the family of all analytic functions $f$ defined on $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and $A_0$ be the family of functions $f \in A$ normalized by the conditions $f(0) = 0, f'(0) = 1$. Such functions $f \in A_0$ have the Taylor series expansion given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in U). \quad (1)$$

Also, let $A$ be the family of functions $f \in A_0$ which are univalent and $\phi(z)$ be an analytic function with positive real part on $A$ with $\phi(0) = 1, \phi'(0) > 0$ which maps the unit disk $U$ onto a region starlike with respect to 1 which is symmetric with respect to the real axis.

In 1994, Ma and Minda [1] introduced and studied the following classes:

Let $S^*(\phi)$ be the class of functions in $f \in A$ for which

$$\frac{zf'(z)}{f(z)} < \phi(z) \quad (z \in U),$$

and $C(\phi)$ be the class of functions in $f \in A$ for which

$$1 + \frac{zf''(z)}{f'(z)} < \phi(z) \quad (z \in U),$$

where $\prec$ denotes the subordination between analytic functions. They have obtained the Fekete-Szegö inequality for the functions in the class $C(\phi)$. Since $f \in C(\phi)$ if and only if $zf'(z) \in S^*(\phi)$, we get the Fekete-Szegö inequality for functions in the class $S^*(\phi)$. For a brief history of the Fekete-Szegö problem for class of starlike, convex, and close-to convex functions, see the paper by Srivastava et al. [2].

Recently, the author [3] introduce a series with Touchard polynomials coefficients after the second force as following:
\[ F_n(z, m) = z + \sum_{k=2}^{\infty} \frac{m^{k-1}(k-1)^n}{(k-1)!} e^{-m z^k}, \quad (m > 0, n \geq 0). \] (2)

It can be easily by ratio test showed that the above series is convergent and the radius of convergence is infinity.

The Hadamard product (or convolution) of \( f \) given by (1) and \( g = z + \sum_{k=2}^{\infty} b_k z^k \), given by

\[ f(z) \ast g(z) = (f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \]

Now, for \( m > 0 \) and \( n \geq 0 \), we introduce the operator \( I(n, m) : \mathcal{A} \rightarrow \mathcal{A} \) as following:

\[ I f(z) = I(n, m) f(z) = z + \sum_{k=2}^{\infty} \frac{m^{k-1}(k-1)^n}{(k-1)!} e^{-m z^k}, \quad (m > 0, n \geq 0). \]

**Definition 1.** Let \( \phi(z) \) be a univalent starlike function with respect to 1 which maps the unit disc \( U \) onto a region in the right half plane which is symmetric with respect to the real axis, \( \phi(0) = 1 \) and \( \phi'(0) > 0 \). For \( m > 0, n \geq 0 \) the class \( \mathcal{M}_n^m(\phi) \) consists of all functions \( f(z) \in \mathcal{A} \) satisfying the following subordination:

\[ \frac{z(I f(z))'}{I f(z)} \prec \phi(z), \quad (z \in U). \] (3)

A classical problem in geometric function theory of complex analysis, which was settled by Fekete and Szegö [4], is to find for each \( \nu \in [0, 1] \) the maximum value of the coefficient functional \( \Phi_\nu(f) \) given by

\[ \Phi_\nu(f) = |c_2 - \nu c_1^2| \]

over the class \( \mathcal{S} \) of univalent functions \( f \) in the open unit disk \( U \) (see, for details,[5], [6],[7]).

To prove our main result, we need the following:

**Lemma 1.** [1] If \( p_1(z) = 1 + c_1 z + c_2 z^2 + \ldots \) is an analytic function with positive real part in \( U \), then

\[ |c_2 - \nu c_1^2| \leq \begin{cases} 
-4\nu + 2 & \text{if } \nu \leq 0, \\
2 & \text{if } 0 \leq \nu \leq 1, \\
4\nu - 2 & \text{if } \nu \geq 1.
\end{cases} \]

When \( \nu < 0 \) or \( \nu > 1 \), the equality holds if and only if

\[ p_1(z) = \frac{(1+z)/(1-z)}{(1+z)/(1-z)} \]

or one of its rotations. If \( 0 < \nu < 1 \), then equality holds if and only if

\[ p_1(z) = \frac{(1+z^2)/(1-z^2)}{(1+z^2)/(1-z^2)} \]

or one of its rotations. If \( \nu = 0 \), the equality holds if and only if

\[ p_1(z) = \left( \frac{1}{2} + \frac{1}{2} \gamma \right) \frac{1+z}{1-z} + \left( \frac{1}{2} \gamma - \frac{1}{2} \right) \frac{1-z}{1+z} \quad (0 \leq \gamma \leq 1), \]
or one of its rotations. If \( \nu = 1 \), the equality holds if and only if
\[
\frac{1}{p_1(z)} = \left( \frac{1}{2} + \frac{1}{2} \gamma \right) \frac{1 + z}{1 - z} + \left( \frac{1}{2} + \frac{1}{2} \gamma \right) \frac{1 - z}{1 + z} \quad (0 \leq \gamma \leq 1).
\]

Also the above upper bound is sharp, it can be improved as follows when \( 0 < \nu < 1 \):
\[
|c_2 - \nu c_1^2| + \nu |c_1|^2 \leq 2 \quad (0 < \nu \leq 1/2)
\]
and
\[
|c_2 - \nu c_1^2| + (1 - \nu) |c_1|^2 \leq 2 \quad (1/2 < \nu \leq 1).
\]

2. Main results

We first consider the functional \( |a_3 - \mu a_2^2| \) for \( \mu \in \mathbb{C} \).

**Theorem 1.** Let \( \phi(z) = 1 + B_1 z + B_2 z^2 + ... \) where \( \phi(z) \in \mathcal{A} \) and \( \phi(0) > 0 \). If \( f(z) \) given by (1) belongs to the class \( M_n^m(\phi) \) and if \( \mu \in \mathbb{C} \), then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{B_3}{2^{m+1}e^{-m}} - \frac{B_1}{2^{m}e^{-m}} + \frac{B_1^2}{2^{2m}e^{-m}} & \text{if } \mu \leq \sigma_1; \\
\frac{B_3}{2^{m+1}e^{-m}} + \frac{B_1}{2^{m}e^{-m}} - \frac{B_1^2}{2^{2m}e^{-m}} & \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\
\frac{B_3}{2^{m+1}e^{-m}} + \frac{B_1}{2^{m}e^{-m}} - \frac{B_1^2}{2^{2m}e^{-m}} & \text{if } \mu \geq \sigma_2.
\end{cases}
\]

where
\[
\sigma_1 := \frac{m^2 e^{-2m} \{ (B_2 - B_1) + B_1^2 \}}{2^m m^2 e^{-m} B_1^2},
\]
\[
\sigma_2 := \frac{m^2 e^{-2m} \{ (B_2 + B_1) + B_1^2 \}}{2^m m^2 e^{-m} B_1^2}.
\]

The result is sharp.

**Proof.** If \( f(z) \in M_n^m(\phi) \), then there exists a Schwarz function \( w(z) \) which is analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) in \( U \) and such that
\[
\frac{z(I f(z))^\prime}{I f(z)} = \phi(w(z)).
\]

Define the function \( p_1(z) \) by
\[
p_1(z) = \frac{1 + w(z)}{1 - w(z)} = \phi(w(z)) = 1 + c_1 z + c_2 z^2 + ....
\]

Since \( w(z) \) is a Schwarz function, we see that \( \Re p_1(z) > 0 \) and \( p_1(0) = 1 \). Define the function \( p(z) \) by:
\[
p(z) = \frac{z(I f(z))^\prime}{I f(z)} = 1 + b_1 z + b_2 z^2 + ....
\]

In view of the equations (5), (6) and (7), we have
\[ p(z) = \phi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right) = \phi \left( \frac{c_1z + c_2z^2 + \ldots}{2 + c_1z + c_2z^2 + \ldots} \right) \]

\[ 1 + b_1z + b_2z^2 + \ldots = \phi \left( \frac{1}{2}c_1z + \frac{1}{2}(c_2 - \frac{1}{2}c_1^2)z^2 + \ldots \right) = 1 + B_1 \frac{1}{2}c_1z + B_1 \frac{1}{2}(c_2 - \frac{1}{2}c_1^2)z^2 + \ldots + B_2 \frac{1}{4}c_1^2z^2 + \ldots \]

Thus

\[ b_1 = \frac{1}{2}B_1c_1 \text{ and } b_2 = \frac{1}{2}B_1(c_2 - \frac{1}{2}c_1^2) + \frac{1}{4}B_2c_1^2. \]

Therefore we have

\[ a_3 - \mu a_2^2 = \frac{B_1}{2^{n+1}m^2e^{-m}} \left\{ c_2 - c_1^2 \left[ \frac{1}{2} \left( 1 - \frac{B_2}{B_1} \right) + \frac{2^n m^2 e^{-m} \mu - m^2 e^{-2m}}{m^2 e^{-2m} B_1} \right] \right\} \]

\[ = \frac{B_1}{2^{n+1}m^2e^{-m}} [c_2 - \nu c_1^2] \]

where

\[ \nu = \frac{1}{2} \left( 1 - \frac{B_2}{B_1} + \frac{2^n m^2 e^{-m} \mu - m^2 e^{-2m}}{m^2 e^{-2m} B_1} \right). \]

If \( \mu \leq \sigma_1 \), then by applying Lemma 1, we get

\[ |a_3 - \mu a_2^2| \leq \frac{B_2}{2^n m^2 e^{-m}} - \frac{\mu B_1^2}{m^2 e^{-2m}} + \frac{B_1^2}{2^n m^2 e^{-m}}, \]

which is the first part of assertion (4).

Similarly, if \( \mu \geq \sigma_2 \), we get

\[ |a_3 - \mu a_2^2| \leq -\frac{B_2}{2^n m^2 e^{-m}} + \frac{\mu B_1^2}{m^2 e^{-2m}} - \frac{B_1^2}{2^n m^2 e^{-m}}, \]

If \( \mu = \sigma_1 \), then equality holds if and only if

\[ p_1(z) = \left( \frac{1 + \gamma}{2} \right) \frac{1 + z}{1 - z} + \left( \frac{1 - \gamma}{2} \right) \frac{1 - z}{1 + z} \quad (0 \leq \gamma \leq 1; z \in U) \]

or one of its rotations.

Also, if \( \mu = \sigma_2 \), then

\[ \frac{1}{2} \left( 1 - \frac{B_2}{B_1} + \frac{2^n m^2 e^{-m} \mu - m^2 e^{-2m}}{m^2 e^{-2m} B_1} \right) = 0. \]

Therefore,

\[ \frac{1}{p_1(z)} = \left( \frac{1 + \gamma}{2} \right) \frac{1 + z}{1 - z} + \left( \frac{1 - \gamma}{2} \right) \frac{1 - z}{1 + z} \quad (0 < \gamma < 1; z \in U) \]
Finally, we see that

\[ |a_3 - \mu a_2^2| = \frac{B_1}{22n^2m^2e^{-m}} \left| c_2 - c_1^2 \left[ \frac{1}{2} \left( 1 - \frac{B_2}{B_1} \right) + \frac{2n^2m^2e^{-m}\mu - m^2e^{-2m}}{m^2e^{-2m}B_1} \right] \right| \]

and

\[ \max \left| \frac{1}{2} \left( 1 - \frac{B_2}{B_1} + \frac{2n^2m^2e^{-m}\mu - m^2e^{-2m}}{m^2e^{-2m}B_1} \right) \right| \quad (\sigma_1 \leq \mu \leq \sigma_2). \]

Therefore using Lemma 1, we get

\[ |a_3 - \mu a_2^2| = \frac{B_1 |c_1|}{22n^2m^2e^{-m}} \leq \frac{B_1}{2n^2m^2e^{-m}}, \quad (\sigma_1 \leq \mu \leq \sigma_2). \]

If \( \sigma_1 < \mu < \sigma_2 \), then we have

\[ p_1(z) = \frac{1 + \lambda z^2}{1 - \lambda z^2}, \quad (0 \leq \lambda < 1). \]

Our result now follows by an application of Lemma 1. To show that these bounds are sharp, we define the functions \( K^\phi_\delta (\delta = 2, 3, ...) \) by

\[ \frac{z(IK^\phi_\delta(z))'}{IK^\phi_\delta(z)} = \phi(\delta^{-1}), \quad K^\phi_\delta(0) = 0 = (K^\phi_\delta(0))' - 1 \]

and the function \( F_\gamma \) and \( G_\gamma \) \( (0 \leq \gamma \leq 1) \) by

\[ \frac{z(IF_\gamma(z))'}{IF_\gamma(z)} = \phi \left( \frac{z(z + \gamma)}{1 + \gamma z} \right), \quad F_\gamma(0) = 0 = (F_\gamma(0))' - 1 \]

and

\[ \frac{z(IG_\gamma(z))'}{IG_\gamma(z)} = \phi \left( -\frac{z(z + \gamma)}{1 + \gamma z} \right), \quad G_\gamma(0) = 0 = (G_\gamma(0))' - 1 \]

Clearly the functions \( K^\phi_\delta, F_\gamma, G_\gamma \in I(\phi) \). Also we write \( K^\phi := K^\phi_2 \). If \( \mu < \sigma_1 \) or \( \mu > \sigma_2 \), then the equality holds if and only if \( f \) is \( K^\phi \) or one of its rotations. When \( \sigma_1 < \mu < \sigma_2 \), the equality holds if and only if \( f \) is \( K^\phi \) or one of its rotations. If \( \mu = \sigma_1 \) then the equality holds if and only if \( f \) is \( F_\gamma \) or one of its rotations. If \( \mu = \sigma_2 \) then the equality holds if and only if \( f \) is \( G_\gamma \) or one of its rotations. \( \square \)

**Remark 1.** If \( \sigma_1 \leq \mu \leq \sigma_2 \), then in view of Lemma 1, Theorem 1 can be improved. Let \( \sigma_3 \) be given by

\[ \sigma_3 := \frac{m^2e^{-2m}(B_1^2 + B_2^2)}{2n^2m^2e^{-m}B_1^2} \]

If \( \sigma_1 \leq \mu \leq \sigma_3 \), then

\[ |a_3 - \mu a_2^2| + \frac{m^2e^{-2m}}{2n^2m^2e^{-m}B_1^2} \left[ B_1 - B_2 + \frac{2n^2m^2e^{-m}\mu - m^2e^{-2m}}{m^2e^{-2m}B_1} \right] |a_2|^2 \leq \frac{B_1}{2n^2m^2e^{-m}}. \]

If \( \sigma_3 \leq \mu \leq \sigma_2 \), then

\[ |a_3 - \mu a_2^2| + \frac{m^2e^{-2m}}{2n^2m^2e^{-m}B_1^2} \left[ B_1 + B_2 - \frac{2n^2m^2e^{-m}\mu - m^2e^{-2m}}{m^2e^{-2m}B_1} \right] |a_2|^2 \leq \frac{B_1}{2n^2m^2e^{-m}}. \]
Proof. For the values of \( \sigma_1 \leq \mu \leq \sigma_3 \), we have

\[
|a_3 - \mu a_2^2| + (\mu - \sigma_1)|a_2|^2 = \frac{B_1}{2n+1} \left( c_2 - \nu c_1^2 \right) + (\mu - \sigma_1) \frac{B_1^2}{4m^2 e^{-2m}} |c_1|^2
\]

\[
= \frac{B_1}{2n+1} \left( c_2 - \nu c_1^2 \right) + \left( \mu - \frac{m^2 e^{-2m} (B_2 - B_1 + B_1^2)}{2n^2 e^{-m} B_1^2} \right) \frac{B_1^2}{4m^2 e^{-2m}} |c_1|^2
\]

\[
= \frac{B_1}{(n+2)(n+2)(1+2\lambda)} \left\{ \frac{1}{2} \left[ |c_2 - \nu c_1^2| + \nu |c_1|^2 \right] \right\}
\]

Similarly, for the values of \( \sigma_3 \leq \mu \leq \sigma_2 \), we write

\[
|a_3 - \mu a_2^2| + (\sigma_2 - \mu)|a_2|^2 = \frac{B_1}{2n+1} \left( c_2 - \nu c_1^2 \right) + (\sigma_2 - \mu) \frac{B_1^2}{4m^2 e^{-2m}} |c_1|^2
\]

\[
= \frac{B_1}{2n+1} \left( c_2 - \nu c_1^2 \right) + \left( \mu - \frac{m^2 e^{-2m} (B_2 + B_1 + B_1^2)}{2n^2 e^{-m} B_1^2} \right) - \mu \right) \frac{B_1^2}{4m^2 e^{-2m}} |c_1|^2
\]

\[
= \frac{B_1}{(n+2)(n+2)(1+2\lambda)} \left\{ \frac{1}{2} \left[ |c_2 - \nu c_1^2| + (1 - \nu |c_1|^2) \right] \right\}
\]

\[
\leq \frac{B_1}{(n+2)(n+2)(1+2\lambda)}
\]

Thus, the proof of Remark 1 is evidently completed. \( \square \)

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