Shear Viscosity of a Gluon Plasma in Perturbative QCD

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Abstract

We calculate the shear viscosity (η) to entropy density (s) ratio η/s of a gluon plasma in kinetic theory including the gg → gg and gg → ggg processes. Due to the suppressed contribution to η in the gg → gg forward scattering, it is known that the gluon bremsstrahlung gg ↔ ggg process also contributes at the same order (O(αs−2)) in perturbative QCD. Using the Gunion-Bertsch formula for the gg → ggg matrix element which is valid for the limit of soft bremsstrahlung, we find that the result is sensitive to whether the same limit is taken for the phase space. Using the exact phase space, the gg ↔ ggg contribution becomes more important to η than gg → gg for αs ≥ 2 × 10−3. Therefore, at αs = 0.1, η/s ≃ 1.0, between 2.7 obtained by Arnold, Moore and Yaffe (AMY) and 0.5 obtained by Xu and Greiner. If the soft bremsstrahlung limit is imposed on the phase space such that the recoil effect from the bremsstrahlung gluon is neglected, then the correction from the gg ↔ ggg process is about 10-30% of the total which is close to AMY’s prediction. This shows that the soft bremsstrahlung approximation is not as good as previously expected.
I. INTRODUCTION

One of the most surprising discoveries at the Relativistic Heavy Ion Collider (RHIC) is that the hot and dense matter (believed to be a quark gluon plasma (QGP), see [1, 2, 3, 4] for reviews) formed in collisions appears to be a near-perfect fluid [5, 6, 7, 8, 9, 10, 11, 12, 13]. The remnant of the non-central collisions shows collective motion (elliptic flow) with a shear viscosity ($\eta$) to entropy density ($s$) ratio $\eta/s = 0.1\pm0.1$(theory)$\pm0.08$(experiment) [14]. This $\eta/s$ ratio is close to a conjectured minimum bound $1/4\pi$ [15], which is motivated by uncertainty principle [16] and gauge/string duality [17, 18, 19, 20]. Since smaller $\eta/s$ implies stronger particle interactions, contrary to the conventional picture, the QGP produced at RHIC tends to be a strongly interacting fluid instead of a weakly interacting gas.

However, a recent perturbative QCD calculation of $\eta/s$ of a gluon plasma by Xu and Greiner (XG) [21] indicates that the gluon elastic scattering $gg \rightarrow gg$ does not give the dominant contribution. They found that $\eta/s$ for the gluon bremsstrahlung process $gg \leftrightarrow ggg$ is about 1/7 of that for $gg \rightarrow gg$, which means the contribution to the shear viscosity from $gg \leftrightarrow ggg$ is 7 times as important as that from $gg \rightarrow gg$. This would bring $\eta/s$ down to 1/4$\pi$ when strong coupling constant $\alpha_s \simeq 0.6$. This implies that the near-perfect QGP might still be described by perturbative QCD and that the conventional picture could still be valid. Their conclusion is quite different from an earlier study by Arnold, Moore and Yaffe (AMY) [22] (for a recent review, see, e.g., [23]). AMY found that $gg \leftrightarrow ggg$ only contributes at 10% level for the three flavor quark diffusion constant for $\alpha_s < 0.3$. For comparison, XG have $\eta/s \simeq 0.5$ at $\alpha_s = 0.1$, while AMY have $\eta/s \simeq 2.7$ (note that only $\eta$ was computed in [22], the free gluon $s$ is inserted by us for comparison).

Both approaches of XG and AMY are based on kinetic theory. However, the main points of differences are: 1) A parton cascade model [24] is used by XG to solve the Boltzmann equation. Since the bosonic nature of gluons is hard to implement in real time simulations in this model, gluons are treated as a Boltzmann gas (i.e. a classical gas). For AMY, the Boltzmann equation is solved in a variation method without taking the Boltzmann gas approximation. 2) The $Ng \leftrightarrow (N+1)g$ processes, $N = 2, 3, 4 \ldots$, are approximated by the effective $g \leftrightarrow gg$ splitting in AMY where the two gluons are nearly
collinear with a small splitting angle, while the \( gg \leftrightarrow ggg \) process is used in XG where
the bremsstrahlung gluon is soft but it can have a large splitting angle with its mother gluon. More specifically, in XG, the Gunion-Bertsch formula \([25]\) for the \( gg \rightarrow ggg \) matrix element squared in Eq. (12) is used. This formula is valid for the limit of soft but not necessarily collinear gluon bremsstrahlung. For the phase space, XG uses the exact phase space for the three gluon configurations (called “three-body-like” phase space in this paper). In principle, if the soft bremsstrahlung limit is a good approximation of the \( gg \rightarrow ggg \) process, one should be able to impose the same limit to the phase space as well and get approximately the same result. In this limit, the recoil effect from the bremsstrahlung gluon is neglected, and the phase space (for the two hard gluons) is called “two-body-like” here.

In this paper, we will perform a third independent calculation for comparison. We will use the same inputs on the Gunion-Bertsch formula for the gluon scattering amplitudes (modulo a factor 2 in Eq. (12)) with the soft gluon bremsstrahlung approximation as XG but we will solve the Boltzmann equation variationally as AMY without taking the Boltzmann gas approximation. We will also test the robustness of the soft gluon bremsstrahlung approximation by comparing the results with the two- and three-body-like phase space.

II. KINETIC THEORY

Using the Kubo formula, \( \eta \) can be calculated through the linearized response function of a thermal equilibrium state

\[
\eta = -\frac{1}{5} \int_{-\infty}^{0} dt' \int_{-\infty}^{t'} dt \int dx^3 \langle [T^{ij}(0), T^{ij}(x, t)] \rangle,
\]

where \( T^{ij} \) is the spatial part of the off-diagonal energy momentum tensor. In a leading order (LO) expansion of the coupling constant, there are an infinite number of diagrams \([26]\). However, it is proven that the summation of the LO diagrams in a weakly coupled \( \phi^4 \) theory \([26]\) or in hot QED \([27]\) is equivalent to solving the linearized Boltzmann equation with temperature-dependent particle masses and scattering amplitudes. The conclusion is expected to hold in weakly coupled systems and can as well be used to compute the LO transport coefficients in QCD-like theories \([22, 28]\), hadronic gases \([29, 30, 31, 32, 33, 34]\) and weakly coupled scalar field theories \([26, 35, 36]\).

The Boltzmann equation of a hot gluon plasma describes the evolution of the color
averaged gluon distribution function \( f = f(x, p, t) \equiv f_p(x) \) (a function of space, time and momentum) as \[37, 38, 39, 40, 41\]

\[
\frac{p^\mu}{E_p} \partial_\mu f_p = C[f],
\]

(2)

where \( E_p = p \) for massless gluons. The driving force for the evolution is the particle scattering in the microscopic theory described by the collision term \( C \) which is a functional of \( f \). It is known that, to compute \( \eta \) to LO in the coupling constant \( \alpha_s \), we need to include \( gg \rightarrow gg \) and \( gg \leftrightarrow ggg \) scattering in \( C \) \[40, 42\]. We will show this more explicitly later.

In thermal equilibrium, the gluon distribution \( f_p^0 \) is static, homogeneous and isotropic and hence \( \partial_\mu f_p^0 = 0 \) at every \( x \). This implies \( C[f^0] = 0 \) or detailed balance whose solution is just the Bose-Einstein distribution function \( f_p^0 = 1/(e^{E_p/T} - 1) \). When the system is not in thermal equilibrium, there will be momentum flow due to the breakdown of detailed balance. The momentum flow can be characterized by a velocity field \( V(x) \). The deviation from thermal equilibrium can be characterized by the inhomogeneity of \( V(x) \) or the derivative expansions of \( V(x) \). For simplicity, we work in the comoving frame of the fluid element at point \( x \) with \( V = 0 \) and to the order of first derivatives of \( V \). Thus the distribution function can be parametrized as

\[
f_p = f_p^0 [1 - \chi_p (1 + f_p^0)],
\]

(3)

where

\[
\chi_p = \left[ A(p) \nabla \cdot V + B(p) \hat{p}_i \hat{p}_j \nabla [i V_j] \right] / T,
\]

(4)

and where the symmetric traceless combinations \( \hat{p}_i \hat{p}_j = \hat{p}^i \hat{p}^j - \delta_{ij}/3 \) and \( \nabla [i V_j] = (\nabla_i V_j + \nabla_j V_i) / 2 - \nabla \cdot V \delta_{ij} / 3 \). Note that the time derivatives do not appear because they can be related to the spatial derivatives by virtue of the conservation of energy momentum tensor. Analogously the deviation of the energy momentum tensor away from its equilibrium value can be parametrized by the bulk (\( \zeta \)) and shear (\( \eta \)) viscosities

\[
\delta T_{ij} = \zeta \delta_{ij} \nabla \cdot V - 2\eta \nabla [i V_j].
\]

(5)

Using the definition in kinetic theory \( T_{\mu\nu} = N_g \int \frac{d^3p}{(2\pi)^3} \frac{p\mu p\nu}{E_p} f_p(x) \), one obtains

\[
\eta = \frac{N_g \beta}{15} \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{E_p} f_p^0 (1 + f_p^0) B(p),
\]

(6)

where \( N_g = 16 \) is the gluon polarization and color degeneracy.
Following the standard procedure (see e.g. [26]) and making use of the Boltzmann equation satisfied by $B(p)$, Eq. (3) can be recast into

$$
\eta = \frac{N_5^2 \beta}{80} \int \prod_{i=1}^{4} \frac{d^3 p_i}{(2\pi)^3 2E_i} |M_{12-34}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)(1 + f_1^0)(1 + f_2^0)f_3^0 f_4^0 \\
\times [B_{ij}(p_4) + B_{ij}(p_3) - B_{ij}(p_2) - B_{ij}(p_1)]^2 \\
+ \frac{N_5^2 \beta}{120} \int \prod_{i=1}^{5} \frac{d^3 p_i}{(2\pi)^3 2E_i} |M_{12-345}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4 - p_5)(1 + f_1^0)(1 + f_2^0)f_3^0 f_4^0 f_5^0 \\
\times [B_{ij}(p_5) + B_{ij}(p_4) + B_{ij}(p_3) - B_{ij}(p_2) - B_{ij}(p_1)]^2,
$$

where $B_{ij}(p) \equiv B(p)(\hat{p}_i \hat{p}_j - \frac{1}{3}\delta_{ij})$ and $M_{12-34}$ and $M_{12-345}$ are amplitudes for $gg \to gg$ and $gg \to ggg$ processes (or called 22 and 23 processes in this paper), respectively. One useful observation is that the right hand sides of Eqs. (6) and (7) correspond to integrations over both sides of the Boltzmann equation, or equivalently, a projection of the Boltzmann equation. It is certainly easier to solve the projected equation than the Boltzmann equation itself. However, there would be an infinite number of solutions satisfying the projected equation, even though the true solution is unique, corresponding to that which gives the largest $\eta$ (see e.g. [22]). This makes solving $\eta$ a variational problem.

To solve for $B(p)$, we assume it to be a smooth function which can be expanded in a set of orthogonal polynomials,

$$
B(z) = z^y \sum_{r=0}^{\infty} b_r B^{(r)}(z),
$$

where $z = \beta |p|$, $B^{(r)}(z)$ is a polynomial up to $z^r$ and the overall factor $z^y$ will be chosen by trial and error to get the fastest convergence [31]. The $B^{(r)}(z)$ polynomials can be constructed using the condition

$$
\int \frac{d^3 p}{(2\pi)^3} f_0^0(1 + f_0^0)|p| z^y B^{(r)}(z) B^{(s)}(z) = T^4 \delta_{rs}.
$$

One can solve the coefficients $b_r$ by equating Eqs. (5) and (7). Then, $\eta$ is just proportional to $b_0$ according to Eqs. (6) and (9). For practical reasons, one uses the approximation $B(z) = z^y \sum_{r=0}^{n-1} b_r B^{(r)}(z)$ where $n$ is a finite, positive integer. It can be proved that $\eta$ is an increasing function of $n$. Thus, one can systematically approach the true value of $\eta$. For $y = 1$, the series converges rapidly. From $n = 2$ to 3, $\eta$ only changes by $\sim 1\%$.

In vacuum, $|M_{12-34}|^2 = (12\pi \alpha_s)^2(3 - tu/s^2 - su/t^2 - st/u^2)/2$ (see e.g. [43]). In medium, $s = O(T^2)$. The most singular part of $|M_{12-34}|^2$ comes from the colinear region.
\[ t \approx 0 \text{ or } u \approx 0 \] which can be regularized by the Hard-Thermal-Loop (HTL) dressed propagators for gluons. However XG only used the Debye mass \( m_D = (4\pi\alpha_s)^{1/2} T \) as the regulator just as done in Ref. [44], so for the sake of comparison between AMY and XG, we also use \( m_D \) as the regulator for soft and collinear divergences in this paper. We will use the HTL gluon propagators in one of our future study. Thus, we consider the near collinear approximation

\[
|M_{12\rightarrow 34}|^2 \approx -\frac{(12\pi\alpha_s)^2}{2} \left( su/t^2 + st/u^2 \right) \bigg|_{t=0 \text{ or } u=0}, \tag{10}
\]

In the center-of-mass (CM) frame, we can use the crossed symmetry between the \( u \)- and \( t \)-channels and just use two times of the forward angle, \( t \)-channel contribution for the sum of the forward \( (t \)-channel) and backward \( (u \)-channel) angle contributions

\[
|M_{12\rightarrow 34}|^2 \approx -(12\pi\alpha_s)^2 s u/t^2 \bigg|_{t=0} \approx (12\pi\alpha_s)^2 \frac{s^2}{(q_T^2 + m_D^2)^2} \bigg|_{q^2=0}, \tag{11}
\]

where \( q_T \) is the transverse (with respect to \( p_1 \)) component of \( q = p_2 - p_4 \). Because small \( q_T \) could also come from large \( |q| \) through the \( u \)-channel, it is important to note that when using Eq. (11) to calculate the collisional integral, we only pick up the near forward scattering (around \( t = q^2 \approx 0 \)) to avoid double counting.

For the \( gg \rightarrow ggg \) process, we will take the approximation that the bremsstrahlung gluon is very soft (zero rapidity limit) and \( \sqrt{s} \) is much bigger than all transverse momenta. Then the exact result of Ref. [43] reduces to the Gunion-Bertsch formula [25],

\[
|M_{12\rightarrow 345}|^2 \approx -\frac{(12\pi\alpha_s)^2}{2} \sum_{\text{perm}(3,4,5)} \frac{s^2}{(q_T^2 + m_D^2)^2} \frac{48\pi\alpha_s q_T^2}{k_T^2\left[(q_T - k_T)^2 + m_D^2\right]} \bigg|_{q^2=0}, \tag{12}
\]

where we have inserted the regulator \( m_D^2 \) as in Ref. [44]. Here \( k_T \) is the transverse component of the bremsstrahlung gluon momentum \( (p_5) \) and \( q_T \) is still the transverse component of \( q = p_2 - p_4 \). The three final state gluons are identical particles. Thus, there are 3! permutations of \( (p_3, p_4, p_5) \), each gives the same contribution. As explained above in the 22 case, we need to be careful about using the \( q_T \) variable. Small \( q_T \) could mean either the forward \( (t \approx 0) \) or backward \( (u \approx 0) \) scattering. In the convention adopted for Eq. (12), one can only pick up the near forward scattering (around \( t = q^2 \approx 0 \)) but not the backward scattering otherwise double counting will happen. Our \( |M_{12\rightarrow 345}|^2 \) is derived from the exact result of Ref. [43], where Lorentz invariant Mandelstam variables are used so there is no this ambiguity, after taking the soft bremsstrahlung limit.
is also consistent with the Gunion-Bertsch formula as explicitly demonstrated in App. A. Effectively, the above treatment of collisional integrals leads to a factor 2 difference in the $gg \rightarrow ggg$ contribution to $\eta$ from that of XG (ours is one half of XG’s).

Naively the $gg \rightarrow gg$ collision rate is $\propto \int dq^2 T |M_{12 \rightarrow 34}|^2 = O(\alpha_s)$ and the $gg \rightarrow ggg$ rate is $\propto \int dq_T^2 dk_T^2 |M_{12 \rightarrow 345}|^2 = O(\alpha_s^2)$ (as will be discussed below, $k_T^2$ has an $O(\alpha_s)$ infrared (IR) cut-off). Thus, the $gg \rightarrow gg$ process seems more important than $gg \leftrightarrow ggg$. However, this is incorrect. In $gg \rightarrow gg$, the amplitude is the largest in the forward and backward scatterings. But there is no contribution to $\eta$ in these cases since there is no momentum redistribution. Mathematically, we have the additional suppression factor $\left[ B_{ij}(p_4) + B_{ij}(p_3) - B_{ij}(p_2) - B_{ij}(p_1) \right]^2 \simeq O(q_T^2)$ in Eq. (7), while no similar suppression in $gg \leftrightarrow ggg$. Thus, the $gg \rightarrow gg$ collision rate is proportional to $\int dq_T^2 |M_{12 \rightarrow 34}|^2 q_T^2 = O(\alpha_s^2 \log \alpha_s)$, which is of the same order as $O(\alpha_s^2)$ of $gg \leftrightarrow ggg$, up to a logarithm.

This power counting can be used to argue that other processes such as $ggg \rightarrow ggg$ and $gg \rightarrow gggg$ (called 33 and 24 processes) are higher order under the assumption that the most important contribution to $\eta$ comes from the configurations with at most two hard gluons in the initial or the final states. Under this momentum configuration, one observes in Eq. (12) that adding a soft gluon to the 22 process yields a factorizable form for the 23 matrix element squared. Schematically,

$$|M_{23}|^2 / |M_{22}|^2 \simeq O(\alpha_s p_T^{-2}), \quad (13)$$

where $p_T$ denotes the small momentum scale with $p_T \simeq O(q_T) \simeq O(k_T)$. Analogously, adding a soft gluon to the 23 process yields

$$|M_{33(24)}|^2 / |M_{23}|^2 \simeq O(\alpha_s p_T^{-2}). \quad (14)$$

Thus, the 33(24) collision rate is smaller than that of 23 by a factor of $\int dp_T^2 |M_{33(24)}|^2 / |M_{23}|^2 = O(\alpha_s \log \alpha_s)$. This argument can be generalized to other processes as well. Thus, 22 and 23 are the only processes in the LO under this assumption.

The phase space of the 3-gluon state plays an important role in the collisional integral in Eq. (7) for $gg \leftrightarrow ggg$, which is controlled by the delta-functions for energy-momentum conservation. Since we use the Gunion-Bertsch formula, Eq. (12), which is valid for soft gluon bremsstrahlung, it is consistent to apply the same condition for energy-momentum configuration of the 3-gluon state. This can be done by neglecting the recoil effect due to
the soft gluon bremsstrahlung, i.e. neglecting the momentum of the soft gluon inside the
delta-functions as is done in App. A. Therefore, the phase space for the two near collinear
gluons in 3-gluon state is 2-body-like. Additionally the exact phase space is 3-body-like if
the momentum of the soft gluon is kept and treated in equal footing as the other gluons
in the delta-functions. We will see that using the 3-body-like or 2-body-like phase space
makes a significant difference in the shear viscosity.

III. LEADING-LOG RESULT

In the leading-log (LL) approximation, one just needs to focus on the small $q_T$ contri-
bution from the $gg \rightarrow gg$ process. After performing the small $q_T$ expansion to Eq. (17),
we obtain ($g^2 = 4\pi \alpha_s$)

$$\eta_{LL} \approx 27.1 \frac{T^3}{g^4 \ln(1/g)};$$

which agrees with that of [28] very well. Using the entropy density for non-interacting
 gluons, $s = N_g \frac{2\pi^2}{45} T^3$, we obtain

$$\frac{\eta_{LL}}{s} \approx \frac{3.9}{g^4 \ln(1/g)}.$$  

This will be used to check our numerical result later. In contrast, we take the Boltzmann
gas approximation ($f_p^0 = e^{-E_p/T}$) used by XG, we get $\eta_{LL} \approx 44.7 T^3 g^{-4} \ln^{-1}(1/g)$ and
$s = N_g \frac{4\pi^2}{15} T^3$, which would yield $\eta_{LL}/s \approx 6.9 g^{-4} \ln^{-1}(1/g)$. Thus, the error from taking
the Boltzmann gas approximation for the LL result of $\eta/s$ is $\sim 80\%$, where $\sim 70\%$ comes
from $\eta$ and $\sim 10\%$ comes from $s$. This suggests that the quantum nature of gluons could
play an important role on transport coefficients, even though they might not be important
for thermodynamic quantities. In weak coupling regime, e.g. $\alpha_s = 10^{-3}$, the XG result in
[21] gives $\eta_{22}/s \approx 5.6 \times 10^3$ while the LL result gives $\eta_{22}/s \approx 2 \times 10^4$, which shows a factor
4 difference. But the difference from the LL result can be narrowed in Israel-Stewart
theory [46].

IV. TREATMENT OF $gg \leftrightarrow ggg$

As mentioned above, both $gg \rightarrow gg$ and $gg \leftrightarrow ggg$ are needed to compute $\eta$ to the
leading order ($O(\alpha_s^{-2})$). For the treatment of the 23 process, we consider three cases, (a)
with the 3-body-like phase space for three gluons and with the LPM effect as the cutoff for the soft gluon; (b) with the 3-body-like phase space and but with \( m_D \) as the regulator for the soft gluon; (c) with the 2-body-like phase space and with \( m_D \) as the regulator for the soft gluon.

In case (a), the scale of the \( k_T \) cut-off is set by the Landau-Pomeranchuk-Migdal (LPM) effect, as in Refs. [21] and [22]. Ref. [47] gives an intuitive explanation of the LPM effect: for the bremsstrahlung gluon with transverse momentum \(|k_T|\), the mother gluon has a transverse momentum uncertainty \(\sim |k_T|\) and a size uncertainty \(\sim 1/|k_T|\). It takes the bremsstrahlung gluon the formation time \(t \sim 1/|k_T|v_T \sim E_k/|k_T|^2\) to fly far enough from the mother gluon to be resolved as a radiation. But if the formation time is longer than the mean free path \(l_{mf p} \approx O(\alpha_s^{-1})\), then the radiation is incomplete and it would be resolved as \(gg \rightarrow gg\) instead of \(gg \rightarrow ggg\). Thus, the resolution scale is set by \(t \leq l_{mf p}\). This yields the condition \(|k_T|^2 \geq E_k/l_{mf p} \approx O(\alpha_s)\) which is confirmed through carefully derivations in Ref. [48].

Here the mean free path \(l_{mf p}\) is given by the collision rate \(R \simeq 1/l_{mf p}\) which sets the scale of the LPM effect is computed via the detailed balance rate. After integration, the Boltzmann equation of Eq.(2) can be written as

\[
\frac{dn}{dt} = n (R_{gain} - R_{loss}),
\]

where we have used \(n = \int \frac{d^3p}{(2\pi)^3} f_p\). Then the collision rate is the detailed balance rate in thermal equilibrium,

\[
R \equiv R_{gain} = R_{22} + R_{23} + R_{32},
\]

where

\[
R_{22} = \frac{N_g}{2n} \int \prod_{i=1}^{4} \frac{d^3p_i}{(2\pi)^3} 2E_i |M_{12;34}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \\
\times f_1^0 f_2^0 (1 + f_3^0) (1 + f_4^0),
\]

\[
R_{23} = \frac{N_g}{6n} \int \prod_{i=1}^{5} \frac{d^3p_i}{(2\pi)^3} 2E_i |M_{12;345}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4 - p_5) \\
\times f_1^0 f_2^0 f_3^0 (1 + f_4^0) (1 + f_5^0) (1 + f_5^0),
\]

\[
R_{32} = \frac{3}{2} R_{23}.
\]

Note that our definition of \(R\) is the same as that of XG. The phase space for three gluons is 3-body-like in \(R_{23}\). And, as mentioned above, only near forward scattering
FIG. 1: (color online) \( R_{22} \) and \( R_{23} \) of Eq. (19) shown as functions of \( \alpha_s \) for BE and Boltzmann gas.

(around \( t = q^2 \approx 0 \)) is included to avoid double counting (see App. A), which gives an additional factor 1/2 compared to XG [21]. We have computed \( R \) self-consistently since \( R_{23} \) also depends on \( R \). Our \( R_{22} \) and \( R_{23} \), together with the results for the Boltzmann gas approximation \( (f_p^0 = e^{-E_p/T}, 1 + f_p^0 \to 1) \), are shown in Fig. 1. Our \( R_{22} \), which uses Bose-Einstein (BE) statistics, is close to the Boltzmann gas result. Our \( R_{23} \), however, gets an enhancement for \( \alpha_s \lesssim 0.04 \) from the enhancement factor \( (1 + f_p^0) \) which is inversely proportional to the soft gluon’s bremsstrahlung energy. This enhancement in \( R \) makes the \( gg \leftrightarrow ggg \) contribution to \( \eta \) smaller in the BE case than the Boltzmann gas. The enhancement disappears at higher \( \alpha_s \) where \( R/T \), and hence the IR cut-off, becomes bigger.

In case (b) and (c), we introduce an IR cut-off \( m_D \) by replacing the \( 1/k_T^2 \) factor in Eq. (12) with \( 1/(k_T^2 + m_D^2) \). Thus, \( m_D \) not only screens the intermediate states but also the external states. This is motivated by demanding the optical theorem to be valid in the medium, even though it need not be the case when the system exchanges particles from a thermal bath. Thus, if the propagator in the loop is screened, then the bremsstrahlung gluon is also screened. This very naive treatment gives \( |k_T|^2 \gtrsim m_D^2 = O(\alpha_s) \), which is consistent with the first treatment in the \( \alpha_s \) counting.
V. NUMERICAL RESULTS AND DISCUSSIONS

We show in Fig. 2 the comparison between $\eta$ computed with $gg \to gg$ alone (denoted as $\eta_{22}$) and $\eta$ computed with $gg \to gg$ and $gg \leftrightarrow ggg$ (denoted as $\eta_{22+23}$). In computing the $23$ contribution in case (a) and (b) with the 3-body-like phase space for three gluons, we use different treatments of $k_T$ cut-offs: in case (a) we use $R = R_{22} + 2.5R_{23}$ as the cut-off, where $R_{23}$ is self-consistently determined (the blue dashed line in Fig. 1), while in case (b) we use $m_D$ as the regulator. For these two cases we find that adding $gg \leftrightarrow ggg$ reduces $\eta$ by $\sim 30\%$ at $\alpha_s = 10^{-3}$ where the contribution from $gg \leftrightarrow ggg$ is about $1/2$ of that from $gg \leftrightarrow gg$. The correction is the largest, $\sim 75\%$, at $\alpha_s = 0.1$. This means the $gg \leftrightarrow ggg$ contribution is about 3 times that of $gg \to gg$. The behavior shown here is different from that of XG which shows $\eta_{22+23}/\eta_{22} \sim 1/8 \sim 12.5\%$, meaning that the $gg \leftrightarrow ggg$ contribution is about 7 times as large as $gg \to gg$, for a wide range of $\alpha_s$ ($\alpha_s = 10^{-3} - 0.7$). The difference between our result and XG’s is largely due to the factor 2 difference in collisional integrals for the $gg \leftrightarrow ggg$ process and the BE statistics versus the Boltzmann gas approximation used. But we do see the dominance of $gg \leftrightarrow ggg$ over $gg \to gg$ when $\alpha_s \gtrsim 2 \times 10^{-3}$, as asserted by XG.

For case (c) with the 2-body-like phase space for three gluons the effect of the $23$ process is about 10-30\%, which is close to AMY’s result in the whole range of $\alpha_s$. Since our result changes dramatically after imposing the soft bremsstrahlung approximation, it means this approximation is not as good as previously expected. Thus, it is important to go beyond this approximation to obtain an accurate $\eta$.

In Fig. 3, $\eta/s$ as a function of $\alpha_s$ is shown for different cases: the LL result $\eta_{LL}/s$ of Eq. (16), $\eta_{22}/s$, and $\eta_{22+23}/s$ for two different $k_T$ cut-offs for the 3-body-like phase space and that for the 2-body-like phase space. When $\alpha_s \to 0$, all these curves should converge to the LL result. But at $\alpha_s = 10^{-3}$, we have $\ln(1/g) = 2.2$, which is not large enough to dominate the contribution. This is the reason for the deviations of the numerical results from the LL one in the current range of $\alpha_s$. However, the agreement between the $\eta_{LL}/s$ and $\eta_{22}/s$ is a good check to our numerical calculations which are carried out by the Monte Carlo method for multi-dimensional integrations. The power of $\alpha_s$ dependence of these curves are close to $(-2)$ as expected. At $\alpha_s = 0.1$, with both $k_T$ cut-offs for the 3-body-like phase space, the full result $\eta_{22+23}/s \simeq 1.0$ is between 2.7 of AMY and 0.5 of XG. At $\alpha_s = 0.3$ and 0.6, we have $\eta_{22+23}/s \simeq 0.22$ and 0.15, respectively, which are larger
FIG. 2: (color online) $\eta_{22+23}/\eta_{22}$ shown as a function of $\alpha_s$ for the 3-body-like and 2-body-like phase space (PS) of three gluons. There are two different treatments of the cut-off of the bremsstrahlung gluon momentum $k_T$ for the 3-body-like phase space.

than 0.13 and 0.076 obtained by XG. It is also interesting to note the good agreement using two different cut-offs for the bremsstrahlung gluon momentum. For the 2-body-like phase space the correction from the 23 process is small and $\eta_{22+23}/\eta_{22} \approx (70\% \sim 90\%)\eta_{22}/s$, which is close to AMY’s result.

In summary, we have calculated the shear viscosity over entropy density $\eta/s$ of a gluon plasma in kinetic theory. Due to the suppressed contribution to $\eta$ in the $gg \rightarrow gg$ forward scattering, the gluon bremsstrahlung $gg \leftrightarrow ggg$ process also contributes at the same order ($O(\alpha_s^{-2})$) in perturbative QCD. We find that the $gg \leftrightarrow ggg$ contribution becomes more important to $\eta$ than $gg \rightarrow gg$ for $\alpha_s \gtrsim 2 \times 10^{-3}$ for the 3-body-like phase space for the three-gluons state. At $\alpha_s = 0.1$, $\eta/s \simeq 1.0$ which is between 2.7 obtained by Arnold, Moore and Yaffe [22] and 0.5 obtained by Xu and Greiner [21]. Our $\eta/s$ is about 2 times as large as that of Xu and Greiner for $\alpha_s \gtrsim 0.1$, largely due to the factor 2 difference in collisional integrals for the $gg \leftrightarrow ggg$ process and the Bose-Einstein statistics versus the Boltzmann gas approximation used. We have observed that using $m_D$ as the regulator for transverse momentum of the soft bremsstrahlung gluon agrees well with that using the rate as the cut-off for the LPM effect in $\eta$ for the current range of $\alpha_s$. In dealing with the
FIG. 3: (color online) $\eta/s$ versus $\alpha_s$ for (a) the leading-log result in Eq. (15), (b) the result of the 22 process only, the full result with 22+23 processes for the 3-body-like phase space (PS) of three gluons where the $k_T$ cut-off is set by (c) $m_D$ or (d) the LPM effect, and (e) the full result with 22+23 processes for the 2-body-like phase space (PS).

23 process it is consistent to implement the soft gluon condition in the energy-momentum configuration of the three-gluons state that there is one soft gluon, which results in the 2-body-like phase space for the three-gluons state, since we use the Gunion-Bertsch formula for the 23 matrix element which is valid only for soft gluon bremsstrahlung. In this case we obtain results close to AMY’s. To test which is the correct description for the phase space of three gluons in the 23 process, or in other words, to test if the Gunion-Bertsch formula is still valid for general 3-body-like momentum configurations, a further and comprehensive study with the exact matrix element is needed.

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APPENDIX A: THE CROSS SECTION FOR 23 FROM GUNION-BERTSCH FORMULA

In the center-of-mass frame of 1 and 2, the cross section is written by,
\[
\sigma_{23} = \frac{1}{2s} \frac{1}{3!} \int d^3k_i \left| M_{12;345} \right|^2 (2\pi)^4 \delta^4(k_1 + k_2 - k_3 - k_4 - k_5)
\]
\[
= \frac{27}{\pi^2} \alpha_s^3 \int d^3k_3 \left( \frac{1}{q^2_T + m_D^2} \right) \delta(E_1 + E_2 - E_3 - E_4) \int d^2k_T dy \frac{q^2_T}{k^2_T[(q_T - k_T)^2 + m_D^2]}
\]
\[
= \frac{27}{\pi^2} \alpha_s^3 \int d^2q_T \left( \frac{1}{q^2_T + m_D^2} \right)^2 \int d^2k_T dy \frac{q^2_T}{k^2_T[(q_T - k_T)^2 + m_D^2]}.
\]  
\tag{A1}

Since we use the Gunion-Bertsch formula for soft gluon bremsstrahlung, we assume the 5th gluon is soft, so we made the approximation in the second equality of Eq. (A1),
\[
\delta^4(k_1 + k_2 - k_3 - k_4 - k_5) \approx \delta^4(k_1 + k_2 - k_3 - k_4),
\]
\tag{A2}
which means the phase space is dominated by the 22 process. We also used \( E_3 = E_4 = E_1 = E_2 = \sqrt{s}/2 \) and
\[
\int d^3k_3 \delta(E_1 + E_2 - E_3 - E_4) = \frac{1}{2} \int d^3q \delta(E_1 - E_3)
= \frac{1}{2} \int d^2q_T dq \delta(E_1 - \sqrt{(E_1 + q_z)^2 + q^2_T})
= \int d^2q_T \frac{E_1}{\sqrt{E^2_1 - q^2_T}} \approx \int d^2q_T
\]
\tag{A3}
where \( k_3 = k_1 + q \). Note that a factor of 2 is given from the two roots for \( q_z \) in the equation \( E_1 = \sqrt{(E_1 + q_z)^2 + q^2_T} \), i.e. \( q_z = -E_1 \pm \sqrt{E^2_1 - q^2_T} \) which correspond to forward and backward solution \( q_z = -\sqrt{s}, 0 \) or \( t = -s, 0 \) at \( q_T = 0 \). Eq. (A1) is 2 times as large as that derived in Ref. [44]. One has to choose the forward scattering and get the factor 1/2,
\[
\int_{\text{forward}} d^3k_3 \delta(E_1 + E_2 - E_3 - E_4) = \frac{1}{2} \int d^2q_T.
\]  
\tag{A4}
Then the differential cross section from Eq. (A1) becomes
\[
\frac{d\sigma_{23}}{d^2q_T d^2k_T dy} = \frac{27}{2\pi^2} \alpha_s^3 \frac{1}{(q^2_T + m_D^2)^2} \frac{q^2_T}{k^2_T[(q_T - k_T)^2 + m_D^2]},
\]  
\tag{A5}
which reproduces the result in [44].

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We calculate 27 points of \( \alpha_s \), \( \alpha_s = 0.01n/(4\pi), 0.1n/(4\pi), n/(4\pi) \) with \( n = 1, 2, ..., 9 \).

The number of event samples for each data point is \( 10^{10} \). The fluctuations for \( \eta_{22+23} \) with LPM cutoff for very small coupling constant, \( \alpha_s \lesssim 3 \times 10^{-3} \), are quite large. The \( \alpha_s \lesssim 3 \times 10^{-3} \) part of the red solid curves in Fig. 1 and Fig. 2 are fitted results and subject to change for larger numbers of event samples. The results outside this range converge very well for current event samples.

[50] In Fig. 22 of [23], Arnold has shown the LO \( \eta/s \) with estimated higher order error from two different calculations that should agree at LO but differ at higher order. We see that at \( \alpha_s = 0.125 \), the error is \( \sim 20\% \) (4\( \pi \eta/s = 27 \pm 6 \)). The higher order error is more significant at larger \( \alpha_s \). This error, although significant, is still consistent with an \( O(\alpha_s) \) error.