Analytical solutions for two inhomogeneous cosmological models with energy flow and dynamical curvature

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22.05.2018

Abstract

We present analytical solutions for two different relativistic generalizations of our recently introduced nonrelativistic cosmological model with dynamical curvature (NRCM). These relativistic models are characterized by two inequivalent extensions of the FLWR metric with a time-dependent curvature function and a geodesic fluid flow. Both contain two functions of time, a scale factor $a(t)$ and a curvature function $K(t)$. The inhomogeneous solutions of the corresponding Einstein equations will agree in leading order at small distances with the NRCM if $a(t)$ and $K(t)$ are each identical with those determined in the NCRM. Then the metric is completely fixed by three constants. The arising energy momentum tensor contains a nontrivial energy flow vector. Volume averaging leads to explicit expressions for the effective scale factor and Hubble function. The large scale (relativistic) corrections to the NCRM results are small for the particular case of scenario 2 presented in [1].

1 Introduction

There is no doubt that the present universe goes through a phase of (real or apparent) accelerated expansion (cp. [2] and the literature cited therein). Almost all observations are in good agreement with the standard cosmological model, however some observations are in disagreement with it (see [3]).

Two alternative strategies to the standard model are under discussion.

In the first category one introduces some kind of "new physics" by changing Einsteins equations (EEs) either by modifying the geometrical part of the EEs (called modified gravity), or by changing the matter part.

In the second category one considers accelerated expansion as an apparent effect due to averaging over inhomogeneities in the Universe (called backreaction, see [4] for a recent review).

For cosmological models based on averaging over inhomogeneities, one comes to the conclusion that the present day cosmic acceleration is due to a negative spatial curvature [5], [6]. A comparison of such backreaction effects with observations has been undertaken in [7]. Furthermore numerical solutions of Einsteins equations for a Silent Universe show "that the spatial curvature emerges due to nonlinear evolution of cosmic structures" [8].

Inhomogeneous cosmological models containing a time-dependent curvature function have been first introduced by Stephani [9] (see also [10] and the literature cited therein). Comparisons of such models with observational data have been performed quite recently for the case of spherical symmetry and a centrally placed observer [11], [12]. But in these papers one has used some ad hoc assumptions on the functional form for the remaining three free functions. Hence the predictive power of the used Stephani models [11], [12] is rather low.

Recently we have shown that a nonrelativistic cosmological model (NRCM) introduced in [13], reviewed in [14]) and derived as the nonrelativistic limit (approximation at sub-Hubble scales) of a general
relativistic model exhibits a dynamical curvature function with a negative value at the present cosmological epoch \([15],[16]\). In a very recent paper \([1]\) we have fixed the three constants (initial conditions) of the model by adjusting them in two different ways to a second order polynomial fit by Montenari and Rsnen \([17]\) to the observed expansion rate \(H(z)\). In the particular case of scenario 2 in \([1]\) we obtain for the curvature function the prediction \(k(z) \sim -1\) for \(z \lesssim 2\).

In the present paper we consider two different relativistic generalizations of the NRCM. These models are characterized by two inequivalent extensions of the FLWR metric with a time dependent curvature function and a geodesic fluid flow. Each of both models contain two functions of time, a scale factor \(a(t)\) and the curvature function \(K(t)\). The solutions of the corresponding Einstein equations turn out to be inhomogeneous. They will agree in leading order at small distances with the NRCM if \(a(t)\) and \(K(t)\) are each identical with those found in the NRCM. Therefore the metric as well as the energy momentum tensor become completely fixed. To compare the outcome of these models with observational results we consider a volume average of our inhomogeneous analytical solutions.

The paper is organized as follows: In section 2 we summarize the essentials of the NRCM. In section 3 we describe the two relativistic generalizations. Variant 1 consists of a shear-free fluid with isotropic pressure (subsection 3.1). A shearing model with anisotropic pressure (variant 2) will be discussed in subsection 3.2. Spatial averaging and the size of the large-scale (relativistic) corrections will be considered in section 4. We finish the paper with some concluding remarks (section 5).

## 2 Summary of the nonrelativistic cosmological model (NRCM)

In the following we summarize the essentials of our NRCM with dynamical curvature (for details we refer to \([1]\) and the literature cited therein).

Here and throughout the whole paper we will consider only spherical symmetric geometry. Then the fluid flow is irrotational.

We start with Einstein’s equations (EEs) for a self-gravitating geodesic fluid (velocity field \(u^\mu\); we use units \(c = 1 = 8\pi G\))

\[
G_{\mu\nu} = T_{\mu\nu} \tag{1}
\]

with an energy-momentum tensor (EMT) containing in the comoving frame only energy density and an energy flow vector \(q_\mu(u^\mu q_\mu = 0)\)

\[
T_{\mu\nu} = \rho u^\mu u^\nu + q_\mu u^\nu + q_\nu u^\mu \tag{2}
\]

In the nonrelativistic and shear-free limit \([15]\) (or at small distances \([16]\)) we obtain from the EEs, after having eliminated the energy flow vector, the following system of two coupled ordinary differential equations for the cosmological scale factor \(a(t)\) and the active gravitational mass density \(\rho(t)\)

\[
\dot{\rho} = -\frac{6\ddot{a}}{a} \tag{3}
\]

and

\[
\dot{\rho} + 3\frac{\dot{a}}{a} \rho + \frac{6K_1}{a^2} = 0 \tag{4}
\]

where the constant \(K_1\) measures the strength of the energy flow (see \([1]\)).

For the curvature function \(K(t) := \frac{\dot{a}^2}{a} R^* \) (\(R^*\) is the spatial curvature) we obtain from the Hamiltonian constraint (we define \(\dot{\rho} := a^2 \dot{\rho}/6\))

\[
K(t) = -\dot{a}^2 + \frac{2\dot{\rho}}{a} \tag{5}
\]

The dynamical system (3, 4) possesses two constants of motion \(Q_i, (i = 2, 3)\)

\[
Q_2 = K_1 \dot{a} - \frac{1}{2} \rho^2, \quad Q_3 = -\frac{\dot{\rho}^3}{6} - Q_2 \dot{\rho} + \frac{K_1^2}{a} \tag{6}
\]
On the solution space of (3, 4) the $Q_i$ take constant values $K_i$ which are determined by the initial values of $\rho$ and of the Hubble function $H := \dot{a}/a$. Introducing the redshift $z = a^{-1} - 1$ instead of the time $t$ as independent variable, we get finally from (6) analytic expressions for the Hubble function $H(z)$ and for the curvature function $K(z)$. In dimensionless units

\[ k_1 := \frac{K_1}{H_0^2}, \quad k_2 := \frac{K_2}{H_0^4}, \quad k_3 := \frac{K_3}{H_0^6} \]

\[ h(z) := \frac{H(z)}{H_0}, \quad k(z) := \frac{K(z)}{H_0^2} \]  

(7)

we get [1]

- A cubic equation for $h(z)$

\[ (k_1^2(1 + z) - k_3)^2 = \frac{2}{9} \left( \frac{k_1 h(z)}{1 + z} - k_2 \right) \left( \frac{k_1 h(z)}{1 + z} + 2k_2 \right)^2 \]  

(8)

- and $k(z)$ in terms of $h(z)$

\[ k(z) = - \left( \frac{h(z)}{1 + z} \right)^2 \pm 2^{3/2}(1 + z) \left( \frac{k_1 h(z)}{1 + z} - k_2 \right)^{\frac{1}{2}} \]  

(9)

with the + sign for $z > z_t$ and the – sign for $z < z_t$.

The transition redshift $z_t$

\[ 1 + z_t = \frac{k_3}{k_1^2} \]  

(10)

marks the changeover from the decelerating phase ($z > z_t$) to the accelerating phase ($z < z_t$) of the Universe [14].

3 Relativistic generalizations of the NRCM

Unfortunately we did not succeed to get an analytic solution for the full relativistic EES with the EMT (2) in spherical symmetric space-time. But in order to get in leading order at small distances the results described in section 2, it is not necessary to use the EMT (2). Instead we start with an extended EMT containing in addition nontrivial pressure terms

\[ T_{\mu\nu} = (\rho + p_t)u_\mu u_\nu + p_t g_{\mu\nu} + (p_r - p_t)s_\mu s_\nu + q_\mu u_\nu + q_\nu u_\mu \]  

(11)

where $p_r$ denotes the radial pressure, $p_t$ the transversal pressure and $s_\mu$ is an unit space-like vector with $u^\mu s_\mu = 0$. But then the pressure terms must behave for small distances such, that in leading order the NRCM results (see section 2) are valid. Of course such a requirement has no unique answer. To realize it we have to distinguish between two options:

- Variant 1 (V1): The fluid flow is shear-free.

  Our aim is to find a solution of the EEs with a time varying curvature function. But, as shown in [16], this requires for a shear-free and geodesic fluid flow necessarily nontrivial pressure terms in the EMT.

  In the following section 3.1 we consider only the case of isotropic pressure.

- Variant 2 (V2): The fluid flow has non-vanishing shear.

  This case will be considered in section 3.2.
3.1 Shear-free model with isotropic pressure (V1)

We consider only the case of isotropic pressure \( (p_r = p_t =: p) \). Then the EMT (11) may be written as

\[
T_{\mu\nu} = \rho u_\mu u_\nu + ph_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu
\]

(12)

where \( h_{\mu\nu} := u_\mu u_\nu + g_{\mu\nu} \) projects onto the space orthogonal to \( u^\mu \).

The geodesic and shear-free fluid motion allows the consideration of a co-moving coordinate system with \( u^\mu = \delta_0^\mu \) which may be written as (cp. [18])

\[
ds^2 = -dt^2 + B^2(t, r)(dr^2 + r^2 d\Omega^2)
\]

(13)

The condition of the isotropy of the pressure, inserted into the EEs (1), leads to a differential equation for \( F := 1/B \) with respect to \( x := r^2 \) [19], [20]

\[
\frac{d^2}{dx^2} F(t, x) = 0
\]

(14)

whose solution is conveniently be written as [20]

\[
B(t, r) = \frac{a(t)}{1 + \frac{r^2}{4} K(t)}
\]

(15)

We note that vanishing anisotropy of the pressure implies that the space-time is conformal flat (cp. [18]).

Insertion of (15) into the line element (13) leads to a metric which differs from the usual FLRW metric only by the time-dependence of \( K [20] \).

For the moment the functions \( a(t) \) and \( K(t) \) are not yet specified, but later we will see that they can be identified with the corresponding functions of the NRCM (cp. section 2).

Introducing the metric (13) with (15) together with the EMT (12) into the EEs (1) we obtain (cp. [20]) for the energy flow \( q \) \( (q_\mu = q s_\mu, s_\mu = (0, B)) \)

\[
q = -\frac{r \dot{K}}{a(1 + \frac{r^2}{4} K)}
\]

(16)

for the energy density \( \rho \)

\[
\rho = \frac{3K}{a^2} + \frac{1}{3} \Theta^2
\]

(17)

and for the pressure \( p \)

\[
p = -\frac{2}{3} \dot{\Theta} - \frac{1}{3} \Theta^2 - \frac{K}{a^2}
\]

(18)

with the volume expansion \( \Theta := \nabla_\mu u^\mu = 3 \frac{\dot{a}}{a} \) given by

\[
\Theta = 3 \left( \frac{\dot{a}}{a} - \frac{\frac{\dot{a}^2}{a^2} + \frac{\ddot{K}}{2}}{1 + \frac{r^2}{4} K} \right)
\]

(19)

where a dot represents differentiation with respect to time \( t \).

So, the spatial scalar curvature \( R^* \), obtained from the Hamiltonian constraint \( R^* = 2\rho - \frac{2}{3} \Theta^2 \), turns out to be only a function of time (cp. [21], [18])

\[
R^* = \frac{6K(t)}{a^2(t)}
\]

(20)

Let us now look at the behavior of the dynamical quantities at small distances \( r \):
For the energy density $\rho$ we obtain from (17) and (19)
\[ \rho \to 0^{+} = 3 \left( \frac{K}{a^2} + \left( \frac{\dot{a}}{a} \right)^2 \right) + 0(r^2). \] (21)
To bring (21) in leading order for small distances into agreement with the NRCM result (3) we have to require
\[ K = -(2\ddot{a} + \dot{a}^2). \] (22)
Next we have to consider the local energy conservation equation
\[ \dot{\rho} + \Theta(\rho + p) + \frac{1}{r^2 B^3} (r^2 q B^2)' = 0 \] (23)
where a prime represents differentiation with respect to $r$.

The last term in (23), the expansion $\Theta$ and the pressure $p$ respectively behave at small distances as
\[ \frac{1}{r^2 B^3} (r^2 B^2)' \to 0^{+} = -\frac{3K}{a^2} + 0(r^2), \quad \Theta \to 0^{+} = 3\frac{\dot{a}}{a} + 0(r^2) \] (24)
and
\[ p = -\left( \frac{2}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{K}{a^2} \right) + \frac{r^2}{2} \left( 3\frac{\dot{a}}{a} K + \dddot{a} \right) + 0(r^2). \] (25)

Therefore, to bring (23) at leading order at small distances in agreement with the NRCM result (4) we have to require according to (24), (25) and taking into account (22)
\[ \dddot{a} = -\frac{2K_1}{a^3}, \quad K_1 = \text{const.}. \] (26)

From the behavior of the pressure for small distances, eq. (25), we observe that the conditions (22) and (26) may be formulated exclusively in terms of $p(t, x)$ for $x \to 0$
\[ p(t, 0) = 0 \quad \text{and} \quad p'(t, 0) = 0 \] (27)
Combining (22) and (26) we obtain
\[ (\ddot{a} a^2)' = \frac{K_1}{a^2} \] (28)
which is identical with the differential equation for $a(t)$ obtained in the NRCM (eliminate $\rho$ in (4) by means of (3)).

The aforementioned results lead finally to the following conclusions:

• Variant 1 of the relativistic model agrees in leading order at small distances with the NRCM if and only if the pressure $p(t, x)$ satisfies the conditions (A)
\[ p(t, 0) = 0 \quad (A1) \quad \text{and} \quad p'(t, 0) = 0 \quad (A2) \]
which implies (28). So, due to the results presented in section 2 (see equations (8) and (9)) we obtain exact analytic expressions for $\dot{a}$ and $K$ as functions of the NRCM scale factor $a$. Then $a(t)$ may be obtained by quadrature (cp.[13], appendix A).

Conclusion:

With the known expressions for $a(t)$ and $K(t)$ our metric (13), (15) is completely fixed in terms of the three constants $K_i (i = 1, 2, 3)$. Therefore, by using (22) and (26) in (16) - (18) we obtain the following inhomogeneous solution of Einsteins equations
\[ q = \frac{4}{ar} f \] (29)
\[ \rho = 3 \left( \frac{K}{a^2} + \left( \frac{\dot{a}}{a} + f \right)^2 \right) \]  

(30)

and

\[ p = -5f^2 \]  

(31)

where we have defined

\[ f(t, r) := \frac{Kt}{a^3(t)(1 + \frac{r^2}{4}K(t))} \]  

(32)

In terms of \( f \) we get for the volume expansion

\[ \Theta = 3 \left( \frac{\dot{a}}{a} + f \right). \]  

(33)

These results exhibit the following interesting features:

- The large-scale (relativistic) corrections are determined by only one function \( f(t, r) \) which is proportional to the energy flow and vanishes at the coordinate origin.
- The pressure turns out to be negative.
- The spatial curvature shows no relativistic correction (see eq. (20)).

Readers who are more familiar with the system of evolution and constraint equations (cp. [18] for our case) instead of the EEs may easily check that the results (29) – (31) and (33) for \( q, \rho, p \) and \( \Theta \) together with (22) and (26) identically satisfy the conservation equations

\[ \dot{\rho} + \Theta(\rho + p) + \frac{q'}{B} + 2q \frac{(rB)'}{rB^2} = 0 \]  

(34)

and

\[ \dot{q} + \frac{q'}{B} + \frac{4}{3} \Theta q = 0 \]  

(35)

the Raychaudhuri-Ehlers equation

\[ \dot{\Theta} + \frac{1}{3} \Theta^2 + \frac{1}{2} (\rho + 3p) = 0 \]  

(36)

as well as the constraint equations

\[ \rho' = \Theta qB \]  

(37)

and

\[ \frac{2}{3} \Theta' = qB \]  

(38)

3.2 Shearing model with anisotropic pressure (V2)

We are not interested to study the most general shearing model with anisotropic pressure. Instead we will define our model by a metric which is as close as possible to the FLWR metric. This metric is given in co-moving coordinates with \( u^\mu = \delta^\mu_0 \) by

\[ ds^2 = -dt^2 + a^2(t) \left( \frac{dv^2}{1 - K(t)r^2} + r^2 d\Omega^2 \right). \]  

(39)

This is the usual FLRW metric except that the constant curvature is replaced by a function of time \( K(t) \). Insofar it is similar to the metric (13), (15) considered in section 3.1. Both are equivalent for \( K = \text{const.} \) as can be seen by the coordinate transformation

\[ r \rightarrow \frac{r}{1 + \frac{r^2}{4}K} \]  

(40)
which converts the metric (39) into the metric (13), (15). But for a time dependent curvature $K(t)$ both metrics are inequivalent [22]. The easiest way to see this is by considering the shear of the co-moving fluid, which for the metric (39) is different from zero and proportional to the time derivative of $K(t)$.

The metric (39) appeared already as a subcase of a more general metric in [22], it has later been considered as an effective metric for cosmological models with spatial averaging [23], [7].

Introducing the metric (39) and the EMT (11) into the EEs (1) we obtain (see [24])

$$\rho = 3 \left( \left( \frac{\dot{a}}{a} \right)^2 + \frac{K(t)}{a^2} \right) + \frac{\dot{a}}{a} \frac{\ddot{K}(t)r^2}{1 - K(t)r^2}$$

$$p_r = -2\frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 - \frac{K(t)}{a^2}$$

$$p_t = -2\frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 - \frac{K(t)}{a^2} - 3\frac{\dddot{K}(t)r^2}{2a(1 - K(t)r^2)}$$

$$- \frac{3}{4} \frac{\dddot{K}(t)r^2}{(1 - K(t)r^2)^2} - \frac{1}{2}\frac{\dddot{K}(t)r^2}{1 - K(t)r^2}$$

and

$$q = -\frac{1}{a} \frac{\dddot{K}(t)r}{(1 - K(t)r^2)^{1/2}}$$

Furthermore, volume expansion $\Theta$ and shear $\sigma$ defined by the decomposition of the covariant derivative of $u_\mu$ ($\sigma^2 := \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}$)

$$\nabla_\mu u_\nu = \frac{1}{3} \Theta h_{\mu\nu} + \sigma_{\mu\nu}$$

are given by

$$\Theta = 3\frac{\dot{a}}{a} + \frac{1}{2}\frac{\dddot{K}r^2}{1 - Kr^2}$$

and

$$\sigma = \frac{1}{2\sqrt{3}} \frac{\dddot{K}r^2}{1 - Kr^2}.$$

Then the spatial curvature $R^*$, obtained from the Hamiltonian constraint $R^* = 2\rho - \frac{2}{3}\Theta^2 + 2\sigma^2$, is again a function of time only and is given by the same expression (20) as for the model V1

$$R^* = \frac{6\dddot{K}(t)}{a^2(t)}.$$

Now, by looking for the small distance behavior we could duplicate in detail the considerations from section 3.1 applied to the present case. We will not do so, but instead we shorten the discussion and start with the

**Supposition:**

Variant 2 of the relativistic model agrees in leading order at small distances with the NRCM if the total pressure $p(t, x)$ ($p = 1/3(p_r + 2p_t)$) satisfies the conditions (A)

$$p(t, 0) = 0 \quad (A1) \quad \text{and} \quad p'(t, 0) = 0 \quad (A2)$$
Proof:

- (A1) leads to (cp. (22))

\[ K = -(2\dddot{a} + \dot{a}^2), \]  

and therefore the energy density \( \rho \) (41) approaches for \( r \to 0 \) the expression given by eq. (3).

- The local energy conservation equation, which reads in our case (cp. eq. (34) in [25]; \( B^2 := \frac{a^2}{1-Kr^2} \))

\[ \dot{\rho} + (\rho + p_r) \frac{\dot{B}}{B} + 2(\rho + p_t) \frac{\dddot{a}}{a} + \frac{q'}{B} + 2q \frac{1}{rB} = 0 \]  

approaches for \( r \to 0 \), according to (A2) together with (49) and (44), the NRCM form eq. (4).

Note that (A2) takes again the same explicit form as for the model V1 (cp. (26))

\[ \dddot{K} + 3\frac{\dot{a}}{a} \dot{K} = 0 \quad \text{with the solution} \quad \dot{K} = -\frac{2K_1}{a^3}, \ K_1 = \text{const.} \]  

so that (49) together with (51) imply again the NRCM condition (28). Hence we get the same analytic expressions for \( a(t) \) and \( K(t) \) as for V1 and obtain finally the following inhomogeneous solution of Einstein’s equations

\[ \rho = -6 \frac{\dddot{a}}{a} - 2 \frac{\dot{a}^2}{a} \]  

\[ p_r = 0 \]  

\[ p_t = -3g^2 \]  

and

\[ q = \frac{2K_1r}{a^4(1-Kr^2)^{3/2}} \]  

where we have defined

\[ g(t, r) := \frac{K_1r^2}{a^4(t)(1-K(t)r^2)}. \]  

In terms of \( g \) we obtain for the volume expansion

\[ \Theta = 3 \frac{\dddot{a}}{a} - g \]  

and for the shear

\[ \sigma = -\frac{g}{\sqrt{3}}. \]  

These results exhibit some interesting features which are similar to those found for V1 (cp. section 3.1)

- The large-scale (relativistic) corrections are determined by only one function \( g(t, r) \) which is proportional to the strength \( K_1 \) of the energy flow and vanishes in accordance with the NRCM at the coordinate origin.

- The radial pressure vanishes.

- The transversal pressure turns out to be negative.

- The spatial curvature shows no relativistic correction.
• The electric part $E_{\mu\nu}$ of the Weyl tensor, which may be written as

$$E_{\mu\nu} = E(s_\mu s_\nu - \frac{1}{3} h_{\mu\nu})$$

is different from zero.

From the shear evolution equation, which in our case reads (cp. eq. (44) in [26]; note the different normalization of $\sigma$)

$$\sqrt{3}\dot{\sigma} + \sigma^2 + \frac{2}{\sqrt{3}} \Theta \sigma = -(E + \frac{1}{2} p_t)$$

we obtain

$$E = -\frac{\dot{\sigma}}{\sigma} - \frac{3}{2} \sigma^2 .$$

For our solutions (52)–(55) we may check again all the evolution and constraint equations for the kinematical and matter variables following from the Einstein equations. Because of the large number of these equations for a shearing anisotropic fluid (cp. [27]) we have restricted these checks to the two conservation equations

• The local energy conservation equation (50),

• the momentum conservation equation, which reads in our case (cp. eq. (35) in [25])

$$\dot{q} + 2q \left( \frac{\dot{a}}{a} - g \right) - \frac{2p_t}{rB} = 0 .$$

Both are identically satisfied.

4 Spatial averaging and the size of the large-scale corrections

Because we have found for both of our models V1 and V2 inhomogeneous solutions of Einstein’s equations we must perform some spatial averaging before we are able to compare our findings with observational results. For reasons of simplicity we consider in the following exclusively model V2.

Spatial averaging of one scalar field $\psi$ over a sphere of radius $R$ is performed in the usual way (cp. [5])

$$< \psi(r,t) >_R := \frac{4\pi}{V_R} \int_0^R \psi(r,t) J(r,t) r^2 dr$$

where $J$ denotes the square root of the 3-metric determinant, given in our case by

$$J(r,t) = \frac{a^3(t)}{(1 - K(t) r^2)^{1/2}} .$$

Furthermore we define an effective volume scale factor $a_R$ by [5]

$$a_R(t) := \left( \frac{V_R(t)}{V_R(0)} \right)^{1/3}$$
4.1 Size of the relativistic corrections

We restrict our considerations to the case $K(t) < 0$, relevant for the present day Universe (see [3]). Then we can take the limit $R \to \infty$ in the averaging procedure (63). For that we need the asymptotic behavior of the following integrals

$$\int_0^R \frac{r^{2+n}}{(1 + |K| r^2)^{1+n}} \sim \frac{1}{2} \frac{R^2}{|K|^{1+n}}$$

Hence we obtain for the most important effective quantities

- **Scale factor**
  $$a_\infty(z) = \frac{1}{1 + z} \left( \frac{k(0)}{k(z)} \right)^{1/6}$$

- **Hubble function**
  $$H_\infty := \frac{1}{3} < \Theta >_\infty = H_0 \left( k(z) + \frac{1}{3} \frac{k_1}{k(z)} (1 + z)^3 \right) .$$

By using (66) we obtain the identity

$$H_\infty = \frac{\dot{a}_\infty}{a_\infty}$$

which would not hold for a finite averaging radius $R$.

To illustrate the size of the relativistic corrections, we consider the more interesting case of scenario 2 from [1] which shows an almost constant behavior of the curvature function $k(z) \sim -1$ for all $z \lesssim 2$. In this case the correction for the scale factor is negligible. We get for the highest considered $z$-value, $z = 2.33$

$$\left( \frac{k(0)}{k(2.33)} \right)^{1/6} = 1.0035$$

and for the corresponding effective Hubble function

$$H_\infty(2.33) = 213.47 - 1.67 = 211.8 .$$

The first number is the NRCM result and the second number gives the relativistic correction.

To be complete we note also the results for the averaged transversal pressure

$$< p_t >_\infty = -3H_0^2 \frac{k_1^2}{k(z)} (1 + z)^6$$

and the magnitude $E$ of the electric Weyl tensor

$$< E >_\infty = H_0^2 h(z) \frac{k_1}{k(z)} (1 + z)^3 + \frac{1}{2} < p_t >_\infty .$$

We recall that the curvature function $K(t)$ experiences no correction.

**We conclude:** The relativistic corrections for scenario 2 from [1] are small for the considered $z$-values $z \lesssim 2$. 

5 Concluding remarks

In this paper we have constructed two different relativistic generalizations of a nonrelativistic cosmological model with dynamical curvature (NRCM) whose solutions are fixed by three constants (initial conditions). These relativistic models rely on two inequivalent extensions of the FLRW metric containing besides a time dependent curvature function $K(t)$ a scale factor $a(t)$ as free functions. The corresponding Einstein equations are supposed to contain an energy momentum tensor with nontrivial pressure terms and energy flow. Then we have required that the now inhomogeneous solutions of the EEs agree in leading order at small distances with those of the NRCM. In technical terms this has been achieved by the demand of vanishing isotropic pressure and its first derivative at $x := r^2 = 0$. In conclusion $a(t)$ and $K(t)$ will agree with their counterparts in the NRCM. Hence they are fixed by the three constants appearing in the NRCM and, therefore, we have obtained exact analytic solutions of the EEs for each of the two relativistic models.

Our models are complementary to the Stephani models [9], [10]. The latter are characterized by a perfect fluid with an accelerating fluid flow whereas our models rely on an imperfect fluid (nontrivial energy flow) with a geodesic fluid flow.

It is interesting to use our new relativistic models in a cosmological context. Already the NRCM, used as a cosmological toy-model, has produced enjoying results for expansion rate and curvature function [1]. We have shown in section 4 of the present paper, that the large-scale (relativistic) corrections to these NRCM results are small for the most interesting case of scenario 2 from [1]. This should encourage us to proceed on this way.

Acknowledgements

I’m grateful to Thomas Buchert for discussions and valuable hints.

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