Automorphisms of Classical Geometries
in the Sense of Klein

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Abstract

In this note, we compute the group of automorphisms of Projective, Affine
and Euclidean Geometries in the sense of Klein.

As an application, we give a simple construction of the outer automorphism
of $S_6$.

1 Introduction

Let $P_n$ be the set of 1-dimensional subspaces of a $n + 1$-dimensional vector space $E$
over a (commutative) field $k$. This standard definition does not capture the "structure"
of the projective space although it does point out its automorphisms: they are
projectivizations of semilinear automorphisms of $E$, also known as Staudt projectivities.

A different approach (v. gr. [1]), defines the projective space as a lattice (the
lattice of all linear subvarieties) satisfying certain axioms. Then, the so named
Fundamental Theorem of Projective Geometry ([1], Thm 2.26) states that, when $n > 1$,
collineations of $P_n$ (i.e., bijections preserving alignment, which are the automorphisms
of this lattice structure) are precisely Staudt projectivities.

In this note we are concerned with geometries in the sense of Klein: a geometry
is a pair $(X, G)$ where $X$ is a set and $G$ is a subgroup of the group Biy$(X)$ of all
bijections of $X$. In Klein’s view, Projective Geometry is the pair $(P_n, PGL_n)$, where
$PGL_n$ is the group of projectivizations of $k$-linear automorphisms of the vector space
$E$ (see Example 2.2). The main result of this note is a computation, analogous to the
aforementioned theorem for collineations, but in the realm of Klein geometries:

**Theorem 3.4** The group of automorphisms of the Projective Geometry $(P_n, PGL_n)$ is
the group of Staudt projectivities, for any $n \geq 1$.

Let us remark this statement covers the case $n = 1$ of the projective line, which
usually requires a separated treatment ([1], [5]).

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Also, we compute in Theorems 3.10 and 3.14 the group of automorphisms for other classical geometries – namely Affine and Euclidean Geometries.

Finally, as an application, we use Theorem 3.4 to give a description of the outer automorphism of $S_6$, which is simple in comparison with other constructions existing in the literature (see [2, 3] or [4]).

2 Preliminaries

Let us firstly introduce the main definitions and examples that will be used later on:

**Definition 2.1.** A *geometry* in the sense of Klein is a pair $(X, G)$ where $X$ is a set and $G$ is a subgroup of the group $\text{Biy}(X)$ of all bijections $X \to X$.

The *concepts* of a geometry are those notions invariant with respect to the action of the structural group $G$.

**Example 2.2.** The examples that will appear in this note are:

- The *Projective Geometry* $(\mathbb{P}_n, \text{PGL}_n)$, where $\mathbb{P}_n$ is the set of 1-dimensional vector subspaces of a $k$-vector space $E$ of dimension $n + 1$ and $\text{PGL}_n$ is the group of projectivizations of $k$-linear automorphisms of $E$ with the obvious action on $\mathbb{P}_n$.

- The *Affine Geometry* $(A_n, \text{Aff}_n)$, where $A_n$ is the complement of an hyperplane $H$ in $\mathbb{P}_n$, called *hyperplane at infinity*, and $\text{Aff}_n$ is the subgroup of $\text{PGL}_n$ consisting of all projectivities $\varphi: \mathbb{P}_n \to \mathbb{P}_n$ preserving the hyperplane at infinity (i.e., such that $\varphi(H) = H$), with the action by restriction on $A_n$.

- The *Euclidean Geometry* $(\mathbb{A}_n(\mathbb{R}), \text{Mot}_n)$, where $\mathbb{A}_n(\mathbb{R})$ denotes the affine space over the real numbers, endowed with a positive definite, non-singular metric $g$ on its vector space of directions, and $\text{Mot}_n$ stands for the group of *motions*; i.e., the group of those affinities $\varphi: \mathbb{A}_n(\mathbb{R}) \to \mathbb{A}_n(\mathbb{R})$ whose tangent linear map $\tilde{\varphi}$ satisfy $\tilde{\varphi}^* g = g$.

**Definition 2.3.** An *isomorphism* between two geometries $(X, G)$ and $(X', G')$ is a bijection $f: X \to X'$ such that the map

$$\phi: \text{Biy}(X) \to \text{Biy}(X'), \quad \phi(g) := \varphi \circ g \circ \varphi^{-1}$$

preserves the structural groups, $\phi(G) = G'$; that is to say, such that $\phi G \phi^{-1} = G'$. An *automorphism* of a geometry $(X, G)$ is an isomorphism of geometries $f: (X, G) \to (X, G)$. With the composition of maps, the automorphisms of a geometry $(X, G)$ are a group.

In other words, the group of automorphisms of a geometry $(X, G)$ is the normalizer of $G$ inside the group $\text{Biy}(X)$ of all bijections of $X$. 

2
3 Automorphisms of classical geometries

3.1 Projective Geometry

Let \( \mathbb{P}_n \) be the set of 1-dimensional subspaces of a \((n+1)\)-dimensional \(k\)-vector space \(E\) and let:

\[
\pi: E - \{0\} \rightarrow \mathbb{P}_n , \quad e \mapsto \langle e \rangle,
\]

be the projectivization map. Recall that linear subvarieties of \( \mathbb{P}_n \) (lines, planes,...) are defined as the projectivization of linear subspaces of \(E\).

**Definition 3.1.** Three points \( p_1, p_2, p_3 \in \mathbb{P}_n \) are collinear if there exists a line \( L \) passing through them.

A bijection \( \varphi: \mathbb{P}_n \rightarrow \mathbb{P}_n \) is a collineation if it transforms lines into lines; that is, it maps collinear points into collinear points.

**Definition 3.2.** A semilinear automorphism \( f: E \rightarrow E \) is a bijection such that there exists an automorphism of fields \( h: k \rightarrow k \) satisfying

\[
f(\lambda e + \mu v) = h(\lambda)f(e) + h(\mu)f(v) , \quad \forall e, v \in E, \quad \lambda, \mu \in k .
\]

A bijection \( \Phi: \mathbb{P}_n \rightarrow \mathbb{P}_n \) is a Staudt projectivity if it is the projectivization of a semilinear automorphism.

Observe that elements in \( \text{PGL}_n \) are precisely the Staudt projectivities with associated automorphism of fields \( h = \text{Id} \).

For what follows, it will be useful to characterize collineations in terms of projectivities; to this end, let us consider the following sets:

\[
P_{p_1, p_2}(p_3) := \{ p \in \mathbb{P}_n - \{p_3\} : \exists \varphi \in \text{PGL}_n , \varphi(p_1) = p_1, \varphi(p_2) = p_2, \varphi(p_3) = p \} .
\]

**Lemma 3.3.** Three different points \( p_1, p_2, p_3 \in \mathbb{P}_n \) are collinear if and only if

\[
P_{p_1, p_2}(p_3) = P_{p_1, p_3}(p_2) = P_{p_2, p_3}(p_1) .
\]

**Proof:** If \( p_1, p_2, p_3 \) are collinear, then \( P_{p_1, p_2}(p_3) \) equals the complement of \( p_1, p_2, p_3 \) in the line passing through them.

Conversely, if \( p_1, p_2, p_3 \) are not collinear, the set \( P_{p_1, p_2}(p_3) \) is the complement of the line \( L \) joining \( p_1 \) and \( p_2 \) (apart from \( p_3 \)). Then, any point \( q \in L, q \neq p_1, p_2 \) satisfies that \( q \notin P_{p_1, p_2}(p_3) \) and \( q \in P_{p_1, p_3}(p_2) \cap P_{p_2, p_3}(p_1) \).

**Theorem 3.4.** The group of automorphisms of the Projective Geometry \((\mathbb{P}_n, \text{PGL}_n)\) is the group of Staudt projectivities, for any \( n \geq 1 \).

That is to say, the group of Staudt projectivities is the normalizer of the group \( \text{PGL}_n \) in the group of all bijections of \( \mathbb{P}_n \).

**Proof:** Let \( \Phi: \mathbb{P}_n \rightarrow \mathbb{P}_n \) be a Staudt projectivity, with associated automorphism of fields \( h: k \rightarrow k \).

If \( \varphi \in \text{PGL}_n \) then \( \Phi \varphi \Phi^{-1} \) is a Staudt projectivity, with associated automorphism \( h \circ \text{Id} \circ h^{-1} = \text{Id} \), so that \( \Phi \text{PGL}_n \Phi^{-1} \subseteq \text{PGL}_n \). Since \( \Phi^{-1} \) is also a Staudt projectivity,
it also holds $\Phi^{-1}\text{PGL}_n \Phi \subseteq \text{PGL}_n$ and the reverse inclusion follows. We conclude that $\Phi$ belongs to the normalizer of $\text{PGL}_n$ in the group $\text{Biy}(\mathbb{P}_n)$.

Now, let us prove that any automorphism $\varphi$ of $(\mathbb{P}_n, \text{PGL}_n)$ is a Staudt projectivity.

$n \geq 2$. As we already mentioned, in this case Staudt projectivities are precisely collineations ([I], Thm 2.26), and hence it is enough to check that $\varphi$ is a collineation. Since $\varphi$ preserves elements in $\text{PGL}_n$ by hypothesis, it easily follows that

$$\varphi(P_{p_1, p_2}(p_3)) = P_{\varphi(P_{p_1}), \varphi(P_{p_2})}(\varphi(p_3)),$$

so Lemma 3.3 allows to deduce that $\varphi(p_1), \varphi(p_2), \varphi(p_3)$ are collinear whenever so are $p_1, p_2, p_3$.

$n = 1$. Let us fix a projective reference $(p_0, p_\infty, p_1)$ in $\mathbb{P}_1$, and write $p_0' := \varphi(p_0)$, $p_1' := \varphi(p_1)$, $p_\infty' := \varphi(p_\infty)$.

In the affine line $\mathbb{A}_1 = \mathbb{P}_1 - \{p_\infty\}$, consider the origin $p_0$ and the unit point $p_1$, thus inducing a bijection $\mathbb{A}_1 \simeq k$. Let $p_\lambda$ denote the point corresponding to $\lambda \in k$ (and analogously in the affine line $\mathbb{A}_1' = \mathbb{P}_1 - \{p_\infty'\}$).

The composition $k \simeq \mathbb{A}_1 \xrightarrow{\varphi} \mathbb{A}_1' \simeq k$ defines a bijection $h: k \to k$ that is an automorphism of the field $k$ (Lemma 3.3) and such that $\varphi(p_\lambda) = p_{h(\lambda)}'$.

Therefore, if $(e_0, e_1)$ is a basis of $E$ normalized to the reference $(p_0, p_\infty, p_1)$ and $(e_0', e_1')$ is a basis normalized to the reference $(p_0', p_\infty', p_1')$, then $\varphi$ coincides with the projectivization of the semilinear map $f(\lambda e_0 + \mu e_1) = h(\lambda) e_0' + h(\mu) e_1'$.

□

**Lemma 3.5.** The map $h: k \to k$ defined in the proof above is an automorphism of fields.

**Proof:** By definition, $h(0) = 0$ and $h(1) = 1$, so we only have to show that $h$ is compatible with additions and products.

Let $\tau_\mu$ and $\sigma_\mu$ denote the translation by $\mu \in k$ in $\mathbb{A}_1$ and the homothety with center at $p_0$ and ratio $\mu$, respectively (and analogously, $\tau_\mu'$ and $\sigma_\mu'$ in $\mathbb{A}_1'$).

Let us first check the equality $\varphi \tau_\mu \varphi^{-1} = \tau_\mu'$: since $p_\infty$ is the only point that is fixed by $\tau_\mu$, its image $p_\infty'$ is the unique fixed point of $\varphi \tau_\mu \varphi^{-1}$; hence this composition is a translation in $\mathbb{A}_1'$ (it is an homography by hypothesis), and it suffices to see that it transforms $p_0'$ into $p_{h(\mu)}'$:

$$\varphi \tau_\mu \varphi^{-1}(p_0') = \varphi \tau_\mu(p_0) = \varphi(p_\mu) = p_{h(\mu)}'.$$

Using this equality, $\varphi \tau_\mu = \tau_\mu'$, and applying it to $p_\lambda$:

$$\varphi \tau_\mu(p_{\lambda}) = \varphi(p_{\lambda + \mu}) = p_{h(\lambda + \mu)}'$$

$$\tau_\mu' \varphi(p_{\lambda}) = \tau_\mu'(p_{h(\lambda)}) = p_{h(\lambda) + h(\mu)}'$$

it follows that $h(\lambda + \mu) = h(\lambda) + h(\mu)$.

In a similar way, $\varphi \sigma_\mu \varphi^{-1}$ can be proved to be the homothety $\sigma_{h(\mu)}'$: since $p_0$ and $p_\infty$ are fixed points of $\sigma_\mu$, both $p_0'$ and $p_\infty'$ are fixed points of $\varphi \sigma_\mu \varphi^{-1}$, and hence this composition is a homothety in $\mathbb{A}_1'$ with center at $p_0'$ that satisfies:
\[ \varphi \sigma \nu \varphi^{-1}(p_1') = \varphi \sigma \nu (p_1) = \varphi (p_\nu) = p'_{h(\mu)} . \]

We conclude \( h(\lambda \mu) = h(\lambda) h(\mu) \) by using the equality \( \varphi \sigma \mu = \sigma'_{h(\mu)} \nu \):

\[ \varphi \sigma \mu (p_\lambda) = \varphi (p_{\lambda \mu}) = p'_{h(\lambda \mu)} \]

\[ \sigma'_{h(\mu)} \nu (p_\lambda) = \sigma'_{h(\mu)} (p'_{h(\lambda)}) = p'_{h(\lambda) h(\mu)} . \]

\[ \square \]

### The outer automorphism of \( S_6 \)

As an application of Theorem 3.4, let us construct an automorphism of \( S_6 \) which is not inner; i.e., which does not coincide with conjugation by an element of \( S_6 \).

Let \( k \) be the field with 5 elements. The projective line over \( k \) has 6 elements, so that there are \( 6 \cdot 5 \cdot 4 \) homographies and \( \text{PGL}_1 \) is a subgroup of \( S_6 \) of index 6. Since the identity is the unique automorphism of the field \( k = \mathbb{Z}/5\mathbb{Z} \), Theorem 3.4 states that \( \text{PGL}_1 \) coincides with its normalizer in \( S_6 \); hence it has 6 conjugated subgroups \( H_1 = \text{PGL}_1, H_2, \ldots, H_6 \).

Any permutation \( \tau \in S_6 \) of the projective line defines, by conjugation, a permutation \( F(\tau) \) of this set \( \{H_1, \ldots, H_6\} \). Thus, we obtain an automorphism

\[ S_6 = \text{Bi}y(P_1) \xrightarrow{F} \text{Per}m(\{H_1, \ldots, H_6\}) = S_6 \]

such that \( F(\text{PGL}_1) \) is contained in the stabilizer of \( H_1 \); i.e., in the subgroup of all permutations fixing the element \( H_1 \) (in fact they coincide, since both subgroups have index 6).

As the image of a stabilizer under an inner automorphism is the stabilizer of another point, we conclude that \( F \) cannot be inner, since no point of the projective line is fixed by the group of all homographies \( \text{PGL}_1 \).

### 3.2 Affine Geometry

Let \( \mathbb{A}_n \) be the complementary of an hyperplane \( H \) on \( \mathbb{P}_n \), and let \( \text{Aff}_n \) be the group of affinites; i.e., the group of projectivities \( \varphi : \mathbb{P}_n \to \mathbb{P}_n \) such that \( \varphi (H) = H \).

Linear subvarieties of \( \mathbb{A}_n \) (affine lines, affine planes,...) are defined as the restriction of affine subvarieties on \( \mathbb{P}_n \).

**Lemma 3.6.** Let \( \varphi : \mathbb{A}_n \to \mathbb{A}_n \) be a collineation of an affine space of dimension \( n > 1 \) over a field \( k \neq \mathbb{F}_2 \).

If \( \Pi \subset \mathbb{A}_n \) is an affine plane, then \( \varphi(\Pi) \) is contained in some affine plane.

*Proof:* Let \( L_1, L_2 \) be two lines in \( \Pi \) intersecting at a point \( z \). Their images \( \varphi (L_1) \) and \( \varphi (L_2) \) are lines with one point in common, so both lie in some affine plane \( \Pi' \).

For any point \( p \in \Pi - (L_1 \cup L_2) \), let \( L_p \subset \Pi \) be any line passing through \( p \), not parallel neither to \( L_1 \) nor to \( L_2 \), and such that \( z \notin L_p \). As \( L_p \) intersects both \( L_1 \) and \( L_2 \), it follows that \( \varphi (p) \in \varphi (L_p) \subset \Pi' \).

\[ \square \]

If the base field has 2 elements, then there not exist 3 different affine collinear points; in that case all bijections of \( \mathbb{A}_n \) are in fact collineations.
Definition 3.7. A Staudt projectivity \( \Phi: \mathbb{P}_n \rightarrow \mathbb{P}_n \) is called a Staudt affinity if it preserves the hyperplane at infinity; i.e., if \( \Phi(H) = H \).

Lemma 3.8. Let \( \varphi: \mathbb{A}_n \rightarrow \mathbb{A}_n \) be a collineation of an affine space of dimension \( n \geq 1 \) over a field \( k \neq \mathbb{F}_2 \).

There exists a unique Staudt affinity \( \Phi: \mathbb{P}_n \rightarrow \mathbb{P}_n \) such that \( \varphi = \Phi|_{\mathbb{A}_n} \).

Proof: We first define \( \Phi \) on the points at infinity: if \( p \in H \), let \( \Phi(p) \) be the point at infinity of \( \varphi(L_p) \), where \( L_p \) is any line passing through \( p \). If \( L'_p \) is another line parallel to \( L_p \), Lemma 3.6 shows that \( \varphi(L_p) \) and \( \varphi(L'_p) \) are contained in some affine plane; since they do not intersect, they are parallel and both have the same point at infinity \( \Phi(p) \).

Then, it is enough to check that \( \Phi \) is a collineation on the hyperplane at infinity (for collineations correspond with Staudt projectivities, [1] Thm 2.26): if three points \( p_1, p_2, p_3 \) of the infinity are collinear, they lie in the direction of some affine plane \( \Pi \). By Lemma 3.8, \( \varphi(\Pi) \) is contained in some affine plane \( \Pi' \), so that \( \Phi(p_1), \Phi(p_2), \Phi(p_3) \in H \) are in the direction of \( \Pi' \) and are thus collinear.

Let us now characterize collinear points in terms of affinities. Consider:

\[ A_{p_1, p_2}(p_3) := \{ p \in \mathbb{A}_n | \{ p_1 \} \cap \{ p \} \neq \emptyset \} \]

Since any affinity fixing \( p_1 \) and \( p_2 \) also fixes the point at infinity of the line passing through them, it has to be the identity on such line. Hence,

Lemma 3.9. Three different points \( p_1, p_2, p_3 \in \mathbb{A}_n \) are collinear if and only if

\[ A_{p_1, p_2}(p_3) = A_{p_1, p_3}(p_2) = A_{p_2, p_3}(p_1) = \emptyset \]

Theorem 3.10. The group of automorphisms of the Affine Geometry \((\mathbb{A}_n, \text{Aff}_n)\) over a field \( k \neq \mathbb{F}_2 \) is the group of Staudt affinities, for any \( n \geq 1 \).

Proof: On the one hand, it is trivial to check that Staudt affinities are indeed automorphisms, arguing as in the proof of Theorem 3.4.

On the other hand, let us prove that any automorphism of \((\mathbb{A}_n, \text{Aff}_n)\) is indeed a Staudt affinity:

\( n \geq 2 \). If \( \varphi \in \text{Bi}(\mathbb{A}_n) \) is in the normalizer of \( \text{Aff}_n \), it preserves affinities and so \( \varphi(A_{p_1, p_2}(p_3)) = A_{\varphi(p_1), \varphi(p_2)}(\varphi(p_3)). \) By Lemma 3.9 it follows that \( \varphi \) is a collineation of \( \mathbb{A}_n \), and hence it coincides with the restriction to \( \mathbb{A}_n \) of a unique Staudt affinity (Lemma 3.8).

\( n = 1 \). Let \( \varphi \in \text{Bi}(\mathbb{A}_1) \) be in the normalizer of \( \text{Aff}_1 \), and fix a projective reference \((p_0, p_\infty, p_1)\) such that \( \mathbb{A}_1 = \mathbb{P}_1 - \{p_\infty\} \). Via the induced bijection \( \mathbb{A}_1 \cong k \), let \( p_\lambda \) denote the point corresponding to \( \lambda \in k \). Consider \( p_0' = \varphi(p_0) \) and \( p_1' = \varphi(p_1) \) as another origin and unit point, and let \( p_\lambda' \) denote the point with coordinate \( \lambda \in k \) via the corresponding bijection \( \mathbb{A}_1 \cong k \).

The composition \( k \cong \mathbb{A}_1 \overset{\varphi}{\rightarrow} \mathbb{A}_1 \cong k \) defines a bijection \( h: k \rightarrow k \) such that

\[ \varphi(p_\lambda) = p_{h(\lambda)}' \].
A similar proof to that of Lemma 3.5 shows that $h$ is an automorphism of the field $k$.

Then, it is easy to check that the bijection $\Phi: \mathbb{P}_1 \rightarrow \mathbb{P}_1$ defined as $\Phi|_{\mathbb{A}_1} := \varphi$, $\Phi(p_\infty) := p_\infty$, is a Staudt affinity (with associated automorphism $h$).

\[\square\]

### 3.3 Euclidean Geometry

Let $\mathbb{A}_n(\mathbb{R})$ be the affine space over the real numbers, endowed with a positive definite, non-singular metric $g$ on its vector space of directions. Let $\text{Mot}_n$ denote the group of motions; i.e., those affinities $\varphi: \mathbb{A}_n(\mathbb{R}) \rightarrow \mathbb{A}_n(\mathbb{R})$ whose tangent linear map $\vec{\varphi}$ preserves the metric $g$.

**Lemma 3.11.** If an affinity $\varphi: \mathbb{A}_n(\mathbb{R}) \rightarrow \mathbb{A}_n(\mathbb{R})$ preserves motions (i.e., $\varphi\text{Mot}_n\varphi^{-1} = \text{Mot}_n$), then it maps perpendicular lines into perpendicular lines.

**Proof:** Assume there exists perpendicular lines $L_1, L_2$ such that $\varphi(L_1)$ and $\varphi(L_2)$ are not perpendicular.

Let $H_1$ be the hyperplane perpendicular to $L_2$ passing through $L_1$. The symmetry $\sigma$ with respect to $H_1$ is a motion such that $\varphi\sigma\varphi^{-1}$ is not a motion, for this composition (which is not the identity), fixes the hyperplane $\varphi(H_1)$ and preserves the oblique line $\varphi(L_2)$.

\[\square\]

**Definition 3.12.** An affinity $\varphi: \mathbb{A}_n(\mathbb{R}) \rightarrow \mathbb{A}_n(\mathbb{R})$ is a similarity if there exists $\lambda \in \mathbb{R}$ such that $\vec{\varphi}^* g = \lambda^2 g$.

Observe that motions are a particular case of similarities, for which $\lambda = 1$.

The Fundamental Theorem of Euclidean Geometry ([1]) characterizes similarities as those affinities $\varphi: \mathbb{A}_n(\mathbb{R}) \rightarrow \mathbb{A}_n(\mathbb{R})$ that map perpendicular lines into perpendicular lines.

Analogously to what is made in previous sections, collinear points may be characterized in terms of motions, considering the sets:

$$
\mathcal{M}_{p_1, p_2}(p_3) := \{ p \in \mathbb{A}_n - \{p_3\} : \exists \varphi \in \text{Mot}_n \ , \ \varphi(p_1) = p_1, \ \varphi(p_2) = p_2, \ \varphi(p_3) = p \}.
$$

**Lemma 3.13.** Three different points $p_1, p_2, p_3 \in \mathbb{A}_n(\mathbb{R})$ are collinear if and only if

$$
\mathcal{M}_{p_1, p_2}(p_3) = \mathcal{M}_{p_1, p_3}(p_2) = \mathcal{M}_{p_2, p_3}(p_1) = \emptyset.
$$

**Theorem 3.14.** The group of automorphisms of the Euclidean Geometry $(\mathbb{A}_n(\mathbb{R}), \text{Mot}_n)$ is the group of similarities, for any $n > 1$.

**Proof:** If $\varphi: \mathbb{A}_n(\mathbb{R}) \rightarrow \mathbb{A}_n(\mathbb{R})$ is a similarity, it is routine to check that, for any motion $\tau \in \text{Mot}_n$, the composition $\varphi\tau\varphi^{-1}$ is also a motion.

On the other hand, if $\varphi \in \text{Bi}(\mathbb{A}_n(\mathbb{R}))$ is a bijection that preserves motions, then $\varphi(\mathcal{M}_{p_1, p_2}(p_3)) = \mathcal{M}_{\varphi(p_1), \varphi(p_2)}(\varphi(p_3))$ and, using Lemma 3.13, it follows that $\varphi$ is a collineation of $\mathbb{A}_n(\mathbb{R})$.
As the identity is the unique automorphism of the field $\mathbb{R}$, Lemma 3.8 then assures that $\varphi$ is an affinity.

Finally, as $\varphi$ is an affinity that preserves motions, it also preserves perpendicular lines (Lemma 3.11) and we can assure that $\varphi$ is indeed a similarity.

$\square$

The above theorem is false when $n = 1$: a counterexample is any bijection $\varphi: \mathbb{R} \to \mathbb{R}$ that respects the addition, v. gr., any $\mathbb{Q}$-linear automorphism $\varphi$ of $\mathbb{R}$: if $\phi \in \text{Mot}_1$, then $\phi(x) = \pm x + b$ and it follows that $(\varphi^{-1}\phi\varphi)(x) = \varphi^{-1}(\pm\varphi(x) + b) = \pm x + \varphi^{-1}(b)$, so that $\varphi$ belongs to the normalizer of $\text{Mot}_1$.

However, it is not difficult to show that any continuous automorphism $\varphi$ of the euclidean line $(A_1(\mathbb{R}), \text{Mot}_1)$ is a similarity: $\varphi(x) = ax + b$; that is, the group of similarities is the normalizer of $\text{Mot}_1$ in the group of all homeomorphisms of $A_1(\mathbb{R})$.

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**References**

[1] Artin, E. Geometric Algebra, Wiley Classics Library Edition, New York (1988).

[2] Howard, B., Millson, J., Snowden, A., Vakil, L. A description of the outer automorphism of $S_6$ and the invariants of six points in projective space, J. Combin. Theory Ser. A 115, (2008) 1296-1303

[3] Janusz, G., Rotman, J. Outer automorphism of $S_6$, Amer. Math. Monthly, 89, 6, (1982) 407-410

[4] Lam, T. Y., Leep, D. Combinatorial structure of the automorphism group of $S_6$, Exposition. Math., 11 (1993) 289–308

[5] Requejo, B. and Sancho, J.B. A Characterization of the Projective Line, Proc. of the AMS, Vol.133, 10, (2005) 3097–3101