Hall current and thermophoresis effects on magnetohydrodynamic mixed convective heat and mass transfer thin film flow

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Abstract
The combined influences of Hall current and thermophoresis on a magnetohydrodynamic mixed convective heat and mass transfer flow of a thin film second-grade fluid with viscous dissipation and thermal radiation past a stretching sheet are analyzed. An external strong and uniform magnetic field is applied transversely to the stretching sheet. The surface of the stretching sheet is taken to move with a linear velocity and subject to the constant reference temperature and concentration. Entropy generation is introduced to investigate the irreversibility associated with flow, heat and mass transfer. The basic governing equations for the velocities, temperature and concentration of the fluid motion have been modeled by employing appropriate similarity transformations which result in high nonlinear coupled differential equations with physical conditions. The solution of the transformed systems of equations has been achieved by employing Homotopy Analysis Method (HAM) which lead to detailed expressions for the velocities, temperature and concentration components. The obtained results are compared with the published results in the tables 3–5 demonstrating an excellent agreement and correlation. Graphs are discussed to elucidate the effects of various parameters.

1. Introduction

The use of magnetic field of high intensity to an ionic liquid having less density, the conduction normal to the magnetic field is converted to curling of atomic particles and ions related to magnetic lines of force before occurring the clashing and a current induced perpendicular to both the electric and magnetic fields, is known as Hall effect. This effect is considered with heat or mass transfer analysis under the situation where the effect of the electromagnetic force is strong. Hall current is most prominent on the absolute value and orientation of the current density and thereby on the magnetic force term. Under the effects of Hall currents the convective flow problem with magnetic field is significant in view of engineering uses in electric transformers, transmission lines, refrigeration coils, power generators, MHD accelerators, nanotechnological processing, nuclear energy systems exploiting fluid metals, blood flow control and heating elements. In case of magnetic field of high strength and less density of the gas, the investigation of magnetohydrodynamic flows with Hall current have the best utilizations in the study of Hall accelerators and flight magnetohydrodynamic. Peristaltic flows have vast applications under the effects of applied magnetic field in the magnetohydrodynamic feature of blood, process of dialysis, oxygenation and hypothermia. Exploration of non-Newtonian fluid flows has been the focus of many scientists due to its vast applications in industries and engineering. Important applications are exist in food engineering, petroleum production, power engineering, in polymer solutions and in melt in the plastic processing industries. Researchers have keen interest in investigating non-Newtonian fluids. Meraj et al.[1] considered the two dimensional laminar flow of Jeffery fluid with variable property in a Darcy–Fröhlicher
porous space on a vertical sheet using Cattaneo–Christov heat flux theory. They used HAM to solve the nonlinear coupled system of equations and showed the convergence of the homotopy equations graphically and numerically as well as proved that flow and heat are higher in Fourier’s law of heat conduction cases in comparison with the Cattaneo–Christov heat flux model. Khan [2] investigated the second grade nanofluid flow containing gyrotactic microorganisms using semi analytical technique. He modeled and solved the complicated governing equations in which the investigations show that velocity reduces with the enhancement of bioconvection Rayleigh number, temperature decreases with increasing thermal slip parameter, nanoparticle concentration increases with Lewis number and microorganisms profile develops with the second grade nanofluid parameter. Waqas et al [3] analyzed the mixed convective flow of an incompressible Carreau fluid past a permeable moving surface. They used HAM to solve the nonlinear coupled system of equations and showed that the flow is high for larger ratio of rate constants while enhancing the values of suction parameter reduce velocity and temperature. Moreover, they presented a benchmark to authenticate the solutions. Khan et al [4] analyzed the comparison of two non-Newtonian thin film flows in the presence of actively controlled nanofluid model boundary conditions. Their research manifests that both the nanofluids have different responses for the effects of Schmidt number on the microorganism density function. Shehzad et al [5] presented a paper in which the three dimensional flow of a Maxwell fluid with variable property towards a moving space is discussed using Cattaneo–Christov model of heat and mass transportation theory in the presence of first order chemical reaction. Their numerical results show the convergent values of the developed solution and concentration as well as temperature are reduced in case of Cattaneo–Christov theory of heat and mass diffusion. Khan et al [6] elaborated to show the effects of of graphene nanoparticles on two dimensional magnetohydrodynamic unsteady flow and heat transfer in a thin film Eyring Powell nanofluid past a stretching sheet using velocity slip condition in which the contents of graphene nanoparticles increase simultaneously the thermal conductivity and stability when incorporated into the dispersion of water based liquid network. Zuhra et al [7] discussed the OHAM solution of a second grade nanofluid flow containing nanoparticles and gyrotactic microorganisms in two parallel plates. Their investigations express that on increasing the unsteadiness parameter $\beta$, the velocity enhances as the plates move apart which increases the space inside the channel. Palwasha et al [8] presented a research paper regarding the flow through porous medium of two non-Newtonian thin film nanofluids in the presence of nanoparticles, gyrotactic microorganisms under the effect of magnetic field and gravity. They showed that the system under study is strongly affected by the driving forces. Zuhra et al [9] discussed the MHD flow of a second grade nanofluid thin film flow containing nanoparticles and gyrotactic microorganisms along a convectively heated vertical solid surface. Their results show that the gyrotactic microorganisms increases by increasing the Peclet number. Palwasha et al [10] analyzed the study of two dimensional boundary layer thin film fluid flow with temperature dependent viscosity and thermal conductivity. Using HAM they showed that how three dimensional study is achieved from two dimensional study when there exist intensified magnetic field. Zuhra et al [11] considered one of the important nanoparticles namely graphene in the unsteady flow and heat transfer in thin liquid film non-Newtonian (Casson and Williamson) nanofluids dispersion over a stretching sheet considering the transverse magnetic field and non-uniform heat source/sink. Their study shows that in case of Casson nanofluid graphene-fluid dispersion, the liquid molecules near to nanoparticles surface construct layered structure and often act as thick liquid film. Sithole et al [12] addressed the entropy generation in MHD flow of a second grade nanofluid past a convectively heated stretching sheet considering nonlinear thermal radiation and viscous dissipation. In that study the temperature increases with increasing thermophoresis parameter and the entropy is increased by increasing the magnetic field parameter and Reynolds number. Rashidi et al [13] used optimal homotopy analysis method to study the convective flow of a third grade non-Newtonian fluid due to a linear stretching sheet implementing magnetic field effect and entropy generation. They discussed the effects of Reynolds number, dimensional temperature difference, Brinkman number, Hartmann number and other physical parameters on entropy generation.

Hall current effect become significant in strong magnetic field and can considerably affect the current density in hydromagnetic heat transfer. In several industrial and geophysical flows viscous dissipation effects also generate due to internal friction in viscous fluids which can affect temperature distributions. In the flow of liquid of electrically conduction nature including Hall currents, different complex phenomena like Alfvén waves in plasma flows, ionslip effects, joule heating etc (see Cramar and Pai [14]) are created by magnetic field. Krishnamurthy et al [15] analyzed the steady boundary layer flow and heat transfer of a nanofluid with fluid-particle suspension past an exponentially stretching surface in the presence of transverse magnetic field and viscous dissipation by using similarity transformations to reduce the governing equations in partial forms Abo-Eldahab and Elbarbary [16] reported the magnetohydrodynamic free convection fluid flow problem past a semi-infinite vertical plate under the Hall effect and mass transfer. Khan et al [17] explored the Brownian motion and thermophoresis effects on MHD mixed convective thin film second-grade nanofluid flow with Hall effect and heat transfer past a stretching sheet. In another study, Abo-Eldahab and Aziz [18] investigated the flow of a
micropolar fluid with Hall current, mixed convection and ohmic heating effects from a rotating cone with power-law fluid at a stretching surface. Gireesha et al. [19] explained the thermal radiation with Hall influences on fluid flow over a non-isothermal stretching surface immersed in porous non-uniform heat generation/absorption and liquid-particle dispersion. El Koumy et al. [20] analyzed the Hall current effect in peristaltic flow of a particle-fluid suspension. Hassain [21] reported the influences of Hall current on the transient natural convection MHD boundary layer flow with suction at the wall employing the Keller-Box numerical method. Raju and Rao [22] carried out the study of conducting and non-conducting walls for ionized hydromagnetic liquid in a parallel plate channel with Hall current effects and transfer of heat showing that the temperature distribution is not dependent on partial pressure of electron gas for the case of non-conducting walls. Sawaya et al. [23] evaluated experimentally the Hall parameter for electrolytic solutions in a closed loop thermo symphonic magnetohydrodynamic flow system in which one dimensional theoretical model with the measured open circuit voltage was used to analyze Hall parameter. The problem of Hall and ion-slip influences on rotating natural convection MHD flow is discussed by Bhargava and Takhar [24].

Radiative heat transfer confronts in power plant and industries. A sound knowledge is required in this regard. In power plant the thermal radiation operates at high temperature. Similarly the importance of thermal radiation in industries cannot be ignored. Heat transfer in the boundary layer flow past a stretching sheet under the effect of thermal radiation has a vast industrial utilizations in combustion, furnace design, nuclear reactor safety, fluidized bed heat exchangers, solar ponds, solar collectors, turbid water bodies and photochemical reactors. To this end, Umeshiaiah et al. [25] discussed the two dimensional boundary layer flow and nonlinear radiative heat transfer of an incompressible non-Newtonian Carreau fluid in the presence of non-uniform heat source/sink and viscous dissipation with convective boundary condition. They obtained interesting results by using Runge-Kutta-Fehlberg fourth-fifth order method with shooting method to solve the transformed equations. Kumar et al. [26] analyzed the effects of nonlinear thermal radiation and viscous dissipation on Casson nanoliquid past a melting moving surface by obtaining numerical solutions for the reduced ordinary differential equations employing Runge-Kutta-Fehlberg fourth-fifth order method. Ramesh et al. [27] investigated the theoretical study of steady stagnation point flow with heat transfer of nanofluid towards a stretching surface taking into account the effects of variable thickness, magnetic field and radiation by assuming the sheet is non-flat. Krishnamurthy et al. [28] carried out the analysis of convective heat transfer of steady two dimensional slip flow of nanofluid immersed in a porous medium past a stretching sheet in the presence of nonlinear thermal radiation in its own plane by observing that the temperature gradient at the surface increases for higher nanoparticles volume fraction, radiation parameter and Biot number. They discussed the skin friction coefficient and Nusselt number for three types of nanoparticles, aluminum oxide, copper oxide and titanium oxide.

An activity in which small particles are push forwarded from a high temperature place to a low temperature place is called as thermophoresis. Very small particles having the size of micron are accumulated on cold places due to thermophoresis. Khan et al. [29] investigated the effects of thermophoresis and thermal radiation with heat and mass transfer in a magnetohydrodynamic thin film second grade fluid of variable properties past a stretching sheet. Narayana and Sibanda [30] obtained similarity solution for the laminar flow of a thin film nanofluid past an unsteady stretching sheet using effective medium theory (EMT) based model for the thermal conductivity of the nanoliquid. Qasim et al. [31] analyzed the flow of thin film nanofluid past an unsteady stretching sheet with heat and mass transfer using Buongiorno’s model incorporating the effects of Brownian motion and thermophoresis. Applying analytical technique, Khan et al. [32] obtained interesting results for the thin film flow of a second-grade fluid through a porous medium past a stretching sheet with heat transfer. In another study, Khan et al. [33] analyzed the problem of magnetohydrodynamic thin film CuO-water sprayed on a stretching cylinder with heat transfer employing similarity solution.

Different types of studies can be seen in the literature about Hall current, thermophoresis, thermal radiation, viscous dissipation over stretching surfaces. There is no study to discuss the entropy generation, Hall current, mixed convection and thermophoresis effects on the MHD heat and mass transfer with thermal radiation properties of a thin film second grade fluid past a stretching sheet. Therefore the recent study is the first attempt to analyze the Hall current and thermophoresis impacts on the thin film movement of a non-Newtonian second grade liquid with entropy generation, heat and mass transfer, viscous dissipation and thermal radiation past a stretching sheet. Employing appropriate transformations the basic governing equations of the velocities, temperature and concentration are deduced in dimensionless form. The transformed equations have been solved using a powerful analytic tool HAM (Homotopy Analysis Method) [34–36]. Various graphs have been drawn to show the effects of all the pertinents parameters on velocities, temperature and concentration fields as well as entropy generation.
2. Methods

2.1. Basic equations

A motion of a magnetohydrodynamic, time independent, laminar, incompressible and an electrically conducting thin film second grade fluid past a stretching sheet in two dimensions is considered. The thermal radiation is taking place in the flow and the influences of thermophoresis are taken into account to observe the variation of mass deposition over stretching surface. Choosing x-axis along the surface and y-axis normal to it. The leading edge of the stretching sheet is considered as coincident with z-axis as depicted in figure 1. An external and uniform magnetic field of high intensity is imposed normally to the sheet along y-axis. Hall current affects the liquid motion of electrically conducting nature in the existence of magnetic field with high intensity, in general. The occurrence of Hall current produces a force in z-direction which generates a cross movement in that direction and so the motion is made in three-dimensions. To solve the problem, it is suppose that there is no change for velocities, temperature and concentration fields in z-direction because the motion quantities are independent of z variable. Mathematically generalized Ohm’s law involving Hall current can be written in the form as in [19].

\[
\frac{\nabla \times \mathbf{v}}{\nabla \cdot \mathbf{v}} + \frac{\omega_e \tau_e}{B_0} \left( \mathbf{j} \times \mathbf{B} \right) = \sigma \left( \mathbf{E} + \mathbf{V} \times \mathbf{B} \right),
\]

where \( \mathbf{V} = (u, v, w) \) is the velocity vector in which \( u, v \) and \( w \) are \( x, y \) and \( z \) components of the velocity vector \( \mathbf{V} \) respectively, \( \mathbf{j} = (j_x, j_y, j_z) \) is the current density vector having components \( j_x, j_y, j_z \) along \( x, y \) and \( z \) axes respectively, \( \mathbf{B} = (0, B_0, 0) \) is the magnetic induction vector, \( \mathbf{E} \) is the intensity vector of the electric field, \( \omega_e, \sigma, \tau_e \) and \( e \) are electron frequency, electrical conductivity, electron collision time and charge of electron respectively.

Since there exists no applied or polarization voltage on the flow field, so the electric field vector \( \mathbf{E} = 0 \). In the light of above used assumptions the generalized Ohm’s law for a weakly ionized gases result in \( j_y = 0 \) everywhere in the fluid so comparing the \( x, z \) components in equation (1) and simplifying for the current density having components \( j_x \) and \( j_z \) as

\[
j_x = \frac{\sigma B_0}{1 + m^2} (mu - w),
\]

\[
j_z = \frac{\sigma B_0}{1 + m^2} (mw + u),
\]

where \( m = \omega_e \tau_e \) is Hall parameter. Under these conditions the governing equations of continuity, momentum, energy and concentration are as in [17]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\]
\[
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\mu \partial^2 u}{\rho \partial y^2} + \frac{\alpha_1}{\rho} \left[ u \frac{\partial^3 u}{\partial x \partial y^2} + v \frac{\partial^3 u}{\partial y^3} + \frac{\partial u \partial^2 u}{\partial y \partial x^2} - \frac{\partial u \partial^2 u}{\partial x \partial y^2} + \frac{\partial w \partial^2 w}{\partial x \partial y^2} + \frac{\partial w \partial^2 w}{\partial x \partial y^2} \right]
\]
\[+ g_r \beta_T (T - T_\infty) + g_r \beta_C (C - C_\infty) - \frac{\sigma B^2 (u + mw)}{\rho (1 + m^2)}, \tag{5}\]
\[
\frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} = \frac{\mu \partial^2 w}{\rho \partial y^2} + \frac{\alpha_1}{\rho} \left[ u \frac{\partial^3 w}{\partial x \partial y^2} + v \frac{\partial^3 w}{\partial y^3} + \frac{\partial u \partial^2 u}{\partial y \partial x^2} + \frac{\partial w \partial^2 w}{\partial y \partial x^2} + \frac{\partial u \partial^2 w}{\partial x \partial y^2} + \frac{\partial w \partial^2 w}{\partial x \partial y^2} \right]
\]
\[+ \frac{\sigma B^2 (u^2 + w^2)}{C_i} + \frac{D_m K_T \partial^2 C}{C_i \partial y^2}, \tag{6}\]
\[
\frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = \frac{D_m \partial^2 C}{\partial y^2} + \frac{D_m K_T \partial^2 T}{T_m \partial y^2} - \frac{\partial \left( V_f C \right)}{\partial y}, \tag{7}\]
\[
\frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{K \partial^2 T}{C_p \partial y^2} - \frac{\partial q_r}{\partial x} \partial y + \mu \left[ \frac{\partial u}{\partial y} + \frac{\partial w}{\partial y} \right]^2
\]
\[+ \alpha_1 \left[ u \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + v \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + v \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right]
\]
\[+ \frac{\sigma B^2 (u^2 + w^2)}{C_i} + \frac{D_m K_T \partial^2 C}{C_i \partial y^2}, \tag{8}\]

where \(\alpha_1(>0)\) is the material parameter, \(\mu\) is the dynamic viscosity, \(\beta_T\) is the coefficient of thermal expansion, \(\rho\) is the density, \(\beta_C\) is the coefficient of volumetric expansion, \(g_r\) is the acceleration due to gravity, \(T\) is the fluid temperature, \(C_p\) is the specific heat at constant pressure, \(C_i\) is the concentration, \(T_\infty\) is the ambient fluid temperature, \(C_\infty\) is the ambient fluid concentration, \(q_r\) is the radiative heat flux, \(K\) is the thermal conductivity, \(K_T\) is the thermal diffusion ratio, \(D_m\) is the molecular diffusivity, \(C_i\) is the absorption susceptibility, \(T_m\) is the mean fluid temperature, \(V_f = \frac{k_v \partial T}{T_\infty \partial y}\) is the thermophoretic velocity in which \(k_v\) is the thermophoretic coefficient, \(\nu = \frac{\mu}{\rho}\) is the kinematic viscosity of the second grade fluid and \(T_\infty\) is some reference temperature [29].

Considering the stretching velocity along positive \(x\)-axis as
\[
U_w = bx,
\]
where \(b > 0\) is the stretching rate and the subscript \(w\) denotes properties at the wall. The temperature
\[
T_i(x) = T_0 - T_\infty \left[ \frac{bx^2}{2v} \right]
\]
is the surface temperature changing with the distance \(x\) from the slit in which \(T_0\) is the temperature at the slit and \(T_\infty\) is the reference temperature such that \(0 \leq T_\infty \leq T_0\).

Similarly the concentration
\[
C_i(x) = C_0 - C_\infty \left[ \frac{bx^2}{2v} \right]
\]
is the surface concentration varying with the distance \(x\) from the slit in which \(C_0\) is the concentration at the slit and \(C_\infty\) is the reference concentration such that \(0 \leq C_\infty \leq C_0\) [17, 29]. Initially the slit is fixed with the origin. An external force is applied to the slit to make it stretch at the rate \(b\) in the positive \(x\)-direction with velocity \(U_w\).

The boundary conditions for the thin film second grade fluid are
\[
\begin{align*}
    u &= U_w, & v &= 0, & w &= 0, & T &= T_0, & C &= C_0 & \text{at} & y = 0, \\
    \frac{\partial u}{\partial y} &= \frac{\partial w}{\partial y} = \frac{\partial T}{\partial y} = 0, & v &= \frac{d\delta}{dk}, & T &= T_\infty, & C &= C_\infty & \text{at} & y = \delta,
\end{align*}
\]

where \(\delta\) is the thickness of the liquid film. By Rosseland approximation [29] the radiative heat flux is defined as
\[
q_r = -\frac{4\sigma_1 \partial T^4}{3k_2 \partial y}, \tag{11}\]

where \(k_2\) and \(\sigma_1\) are the mean absorption coefficient and Stefan-Boltzmann constant respectively. By taking assumption that the differences in temperature of the motion are exist in a manner that \(T^4\) can be written as a linear combination of the temperature, using the Taylor’s theorem about \(T_\infty\) and ignoring higher order terms, yield
\[
T^4 = 4T_\infty^3 T - 3T^4, \tag{12}\]

generating the quantity
\[
\frac{\partial q_r}{\partial y} = -\frac{16T_\infty^3 \sigma_1 \partial^2 T}{3k_2 \partial y^2} \tag{13}\]
Introducing the transformations for nondimensional variables \( f, g, \theta, \phi \) and similarity variable \( \zeta \) as

\[
u = bx f'(\zeta), \quad v = -(bv)^{1/2} f(\zeta), \quad w = b x g(\zeta), \quad \zeta = \left[\frac{b}{v}\right]^{1/2} y, \quad (14)
\]

\[
T(x) = T_0 - T_{ref} \left[ \frac{bx^2}{2v} \right] \theta(\zeta), \quad C(x) = C_0 - C_{ref} \left[ \frac{bx^2}{2v} \right] \phi(\zeta), \quad \beta = \left[ \frac{b}{v} \right]^{1/2}, \quad (15)
\]

where \( \beta \) is the non-dimensional parameter for the thickness of the fluid film. Through equation (14), mass conservation equation (4) is automatically satisfied. With the help of equations (14) and (15) the equations (5)–(10) yield the following six equations (16)–(21)

\[
f'' + ff'\, f'' - f' + f' + f' = (2f'f'' - f'' + f' + f' + f' + f') + Gr\theta + Gm\phi - \frac{M(f' + mg)}{1 + m^2} = 0, \quad (16)
\]

\[
g'' + fg' + f'g + \gamma_1 \left[ (f'g' - f'g'g' - f') + \frac{M(f' + mg)}{1 + m^2} \right] = 0, \quad (17)
\]

\[
(1 + Nr)\theta'' + Pr \left[ 2f'\theta - f'\theta \right] + Br(4f'' + g'^2) + (f'' + g'^2) + \gamma_1 \left[ (f'g' - f'g'g' - f') + \frac{M(f' + mg)}{1 + m^2} \right] = PrD\phi'' = 0, \quad (18)
\]

\[
\phi'' + Sc \left[ f - \tau^2 \phi \right] + Sc \left[ S \tau - \tau^2 \phi \right] = 0, \quad (19)
\]

\[
f = g = 0, \quad f'(0) = \frac{\partial f}{\partial \zeta} = 0, \quad \text{at} \quad \zeta = 0, \quad (20)
\]

\[
f = g = 0, \quad f'(0) = \phi' = 0, \quad \text{at} \quad \zeta = \beta, \quad (21)
\]

where prime (') represents the derivative with respect to \( \zeta \). For \( \gamma_1 = M = 0 \), the hydromagnetic non-Newtonian effect is vanished.

The physical quantities of practical and engineering primary interest are the skin friction coefficients \( C_{f_x} \) (rate of shear stress in \( x \)-direction), \( C_{g_z} \) (rate of shear stress) in \( z \)-direction and the Nusselt number \(-\theta'(0)\), rate of heat transfer). The skin friction coefficients \( C_{f_x} \) and \( C_{g_z} \) are defined as

\[
C_{f_x} = \frac{\tau_{w_x}}{\rho U_w^2}, \quad C_{g_z} = \frac{\tau_{w_z}}{\rho U_w^2}. \quad (22)
\]

The equation defining the wall shear stress \( \tau_{w_x} \) in \( x \)-direction is

\[
\tau_{w_x} = \left[ \mu \frac{\partial u}{\partial y} + \rho \alpha \left( 2\frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \right] \bigg|_{y=0} \quad (23)
\]

and for \( \tau_{w_z} \) the equation in \( z \)-direction is

\[
\tau_{w_z} = \left[ \mu \frac{\partial u}{\partial y} + \rho \alpha \left( \frac{\partial u}{\partial x} \frac{\partial w}{\partial y} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial y} \right) \right] \bigg|_{y=0}. \quad (24)
\]

Using equations (23)–(24) the local dimensionless skin friction coefficients \( C_{f_x} \) in \( x \)-direction and \( C_{g_z} \) in \( z \)-direction are given by

\[
C_{f_x} = (Re_x)^{-1/2} \left[ 1 + 3\gamma_1 \right] f''(0), \quad (25)
\]

\[
C_{g_z} = (Re_x)^{-1/2} \left[ 1 + 3\gamma_1 \right] g'(0), \quad (26)
\]

where \( Re_x = \frac{U_xa}{\nu} \) is the local Reynolds number based on the stretching velocity.

The equation defining the Nusselt number \( Nu_{\infty} \) in terms of the free convection and thermal radiation is given by

\[
Nu_{\infty} = \left[ \frac{\lambda_{\infty}}{K(T_{\infty} - T_0)} \right] \bigg|_{y=0}, \quad (27)
\]

where \( q_w \) is the local wall heat flux. Due to the influence of free convection and thermal radiation on the rate of heat transfer, \( q_w \) is given by

\[
q_w = -K \left[ \frac{\partial T}{\partial y} \right]_{y=0} - \frac{4\sigma}{3K} \left[ \frac{\partial T^4}{\partial y} \right]_{y=0} \quad (28)
\]

So the rate of heat transfer in dimensionless form reduces to

\[
Nu_{\infty} = -\left( Re_x \right)^{1/2} (1 + Nr)\theta'(0). \quad (29)
\]
Similarly, the equation defining the Sherwood number $Sh_x$ is given by

$$Sh_x = \left[ \frac{xq_m}{D} \right]_{y=0} = \left[ \frac{xD\partial C}{\partial y} \right]_{y=0} = -(Re_x)^{\frac{1}{2}} \phi'(0),$$

(30)

where $q_m$ is the mass flux at the stretching surface. So the rate of mass transfer in dimensionless form reduces to

$$Sh_x = -(Re_x)^{\frac{1}{2}} \phi'(0).$$

(31)

### 3. Entropy generation

The present study is related to engineering and industrial processes so it is better to discuss the rate of entropy generation. The volumetric entropy generation of the fluid can be written as

$$E_{gen}^m = K \left[ \frac{\partial T}{\partial y} \right]^2 + \frac{16T_\infty^2 \sigma_1}{k_2} \left( \frac{\partial T}{\partial y} \right)^2 + \frac{\rho}{T_\infty} \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 + \left( \frac{\sigma B_0^2}{T_\infty} (u^2 + w^2) + \frac{RD}{C_\infty} \frac{\partial C}{\partial y} \right)^2$$

$$+ \frac{RD}{T_\infty} \left[ \frac{\partial T}{\partial x} \frac{\partial C}{\partial y} + \frac{\partial T}{\partial y} \frac{\partial C}{\partial x} \right],$$

(32)

where $R$ denotes the ideal gas constant and $D$ represents the diffusivity. In equation (32), the first term represents the irreversibility due to heat transfer with thermal radiation, the second term is entropy generation due to viscous dissipation, the third term is the local entropy generation due to the effect of magnetic field, the fourth and fifth terms are the irreversibilities due to diffusion effect. The characteristic entropy generation rate is

$$E_0^m = K(T_0 - T_\infty)^2 L^2 T_\infty,$$

(33)

where $L$ manifests the characteristic length. The Entropy generation $E_G$ is the ratio between the entropy generation rate $E_{gen}^m$ and the characteristic entropy generation rate $E_0^m$ i.e.

$$E_G = \frac{E_{gen}^m}{E_0^m}$$

(34)

Applying equations (14)–(15) the non-dimensional form of equation (34) for Entropy generation number $N_G$ is

$$N_G = Re(1 + Nr)(\theta')^2 + \frac{ReBr}{\theta_w} \left[ (f')^2 + (g')^2 \right] + \frac{ReBrM}{\theta_w} \left[ (f')^2 + (g')^2 \right] + Re\gamma_2 \left( \frac{\phi_w}{\theta_w} \right) \phi'^2$$

$$+ Re\gamma_2 \left( \frac{\phi_w}{\theta_w} \right) \theta' \phi',$$

(35)

where $Re = \frac{h^2}{\nu}$ is the Reynolds number, $\gamma_2 = \frac{RD_{C_\infty}}{K}$ is the diffusive constant parameter, $\theta_w = \frac{\Delta T}{T_\infty}$ and $\phi_w = \frac{\Delta C}{C_\infty}$ are respectively the temperature and concentration differences.

### 4. Problem solution through homotopy analysis method

Using the initial approximate values and auxiliary linear operators in the following form

$$f_0(\zeta) = \zeta, \quad g_0(\zeta) = 1, \quad \theta_0(\zeta) = 1, \quad \phi_0(\zeta) = 1,$$

(36)

$$L_f = f^n, \quad L_g = g^n, \quad L_\theta = \theta^n, \quad L_\phi = \phi^n$$

(37)

with the properties

$$L_f [C_1 + C_2\zeta + C_3\zeta^2] = 0, \quad L_g [C_4 + C_5\zeta] = 0, \quad L_\theta [C_6 + C_7\zeta] = 0, \quad L_\phi [C_8 + C_9\zeta] = 0,$$

(38)

where $C_i (i = 1–9)$ are the arbitrary constants.
4.1. Zeroth-order deformation problems

Introducing the nonlinear operator \( \mathbb{N} \) as

\[
\mathbb{N}_f[f(\zeta, p), g(\zeta, p), \theta(\zeta, p), \phi(\zeta, p)] = \frac{\partial^3 f(\zeta, p)}{\partial \zeta^3} + f(\zeta, p) \frac{\partial^2 f(\zeta, p)}{\partial \zeta^2} - \left( \frac{\partial f(\zeta, p)}{\partial \zeta} \right)^2
\]

\[
+ \gamma_1 \left[ 2 \frac{\partial f(\zeta, p)}{\partial \zeta} \frac{\partial^2 f(\zeta, p)}{\partial \zeta^2} \right] - f(\zeta, p) \frac{\partial^4 f(\zeta, p)}{\partial \zeta^4} + \left( \frac{\partial g(\zeta, p)}{\partial \zeta} \right)^2 + g(\zeta, p) \frac{\partial^2 g(\zeta, p)}{\partial \zeta^2}
\]

\[
+ Gr\theta(\zeta, p) + Gm\phi(\zeta, p) - \frac{M}{1 + m^2} \left[ \frac{\partial f(\zeta, p)}{\partial \zeta} + mg(\zeta, p) \right].
\]

(39)

\[
\mathbb{N}_g[f(\zeta, p), g(\zeta, p)] = \frac{\partial^2 g(\zeta, p)}{\partial \zeta^2} + f(\zeta, p) \frac{\partial g(\zeta, p)}{\partial \zeta} - \left( \frac{\partial g(\zeta, p)}{\partial \zeta} \right)^2
\]

\[
+ \gamma_1 \left[ 2 \frac{\partial g(\zeta, p)}{\partial \zeta} \frac{\partial^2 g(\zeta, p)}{\partial \zeta^2} \right] - f(\zeta, p) \frac{\partial^4 g(\zeta, p)}{\partial \zeta^4} + \left( \frac{\partial g(\zeta, p)}{\partial \zeta} \right)^2 + g(\zeta, p) \frac{\partial^2 g(\zeta, p)}{\partial \zeta^2}
\]

\[
+ M \left[ \frac{\partial f(\zeta, p)}{\partial \zeta} + mg(\zeta, p) \right].
\]

(40)

\[
\mathbb{N}_\theta[f(\zeta, p), \theta(\zeta, p)] = (1 + N_r) \frac{\partial \theta(\zeta, p)}{\partial \zeta} - p_r \left[ f(\zeta, p) \frac{\partial \theta(\zeta, p)}{\partial \zeta} \right]
\]

\[
+ Br \left[ \frac{\partial^2 f(\zeta, p)}{\partial \zeta^2} \left( \frac{\partial \theta(\zeta, p)}{\partial \zeta} \right)^2 + \left( \frac{\partial g(\zeta, p)}{\partial \zeta} \right)^2 \right]
\]

\[
+ \gamma_1 Br \left[ 2 \frac{\partial \theta(\zeta, p)}{\partial \zeta} \frac{\partial^2 \theta(\zeta, p)}{\partial \zeta^2} \right] - f(\zeta, p) \frac{\partial^4 \theta(\zeta, p)}{\partial \zeta^4} + \left( \frac{\partial g(\zeta, p)}{\partial \zeta} \right)^2 + g(\zeta, p) \frac{\partial^2 g(\zeta, p)}{\partial \zeta^2}
\]

\[
+ p_{Br} \frac{\partial^2 \phi(\zeta, p)}{\partial \zeta^2}.
\]

(41)

\[
\mathbb{N}_\phi[f(\zeta, p), \theta(\zeta, p), \phi(\zeta, p)] = \frac{\partial^2 \phi(\zeta, p)}{\partial \zeta^2} + Sc \left[ f(\zeta, p) - \frac{\partial \theta(\zeta, p)}{\partial \zeta} \right] \frac{\partial \phi(\zeta, p)}{\partial \zeta}
\]

\[
+ Sc \left[ Sr - \tau \phi(\zeta, p) \right] \frac{\partial^2 \phi(\zeta, p)}{\partial \zeta^2} - 2Sc \frac{\partial f(\zeta, p)}{\partial \zeta} \phi(\zeta, p)
\]

(42)

where \( p \) is an embedding parameter such that \( p \in [0, 1] \).

The equations of zeroth-order deformation are constructed as

\[
(1 - p)L_f[f(\zeta, p) - f_0(\zeta)] = phL[f(\zeta, p), g(\zeta, p), \theta(\zeta, p), \phi(\zeta, p)],
\]

(43)

\[
(1 - p)L_g[g(\zeta, p) - g_0(\zeta)] = phL[f(\zeta, p), g(\zeta, p)],
\]

(44)

\[
(1 - p)L_\theta[\theta(\zeta, p) - \theta_0(\zeta)] = phL[f(\zeta, p), \theta(\zeta, p)],
\]

(45)

\[
(1 - p)L_\phi[\phi(\zeta, p) - \phi_0(\zeta)] = phL[f(\zeta, p), \theta(\zeta, p), \phi(\zeta, p)],
\]

(46)

where \( h \) denotes the auxiliary non-zero parameter. Equation (43) has the boundary conditions

\[
f(0, p) = 0, \quad f'(0, p) = 1, \quad f''(\beta, p) = 0,
\]

(47)

Equation (44) has the boundary conditions

\[
g(0, p) = 0, \quad g'(\beta, p) = 0.
\]

(48)

Equation (45) has the boundary conditions

\[
\theta(0, p) = 1, \quad \theta'(\beta, p) = 0.
\]

(49)

Similarly equation (46) has the boundary conditions

\[
\phi(0, p) = 1, \quad \phi'(\beta, p) = 0.
\]

(50)

Considering the results for \( p = 0 \) and \( p = 1 \), one obtains the following

\[
p = 0 \Rightarrow f(\zeta, 0) = f_0(\zeta) \quad \text{and} \quad p = 1 \Rightarrow f(\zeta, 1) = f(\zeta),
\]

(51)

\[
p = 0 \Rightarrow g(\zeta, 0) = g_0(\zeta) \quad \text{and} \quad p = 1 \Rightarrow g(\zeta, 1) = g(\zeta),
\]

(52)

\[
p = 0 \Rightarrow \theta(\zeta, 0) = \theta_0(\zeta) \quad \text{and} \quad p = 1 \Rightarrow \theta(\zeta, 1) = \theta(\zeta),
\]

(53)

Similarly

\[
p = 0 \Rightarrow \phi(\zeta, 0) = \phi_0(\zeta) \quad \text{and} \quad p = 1 \Rightarrow \phi(\zeta, 1) = \phi(\zeta).
\]

(54)
By taking $p$ assumes the values from 0 to 1, $g(\zeta, p)$ becomes $g_0(\zeta)$ to $g(\zeta)$ when $p$ has the values from 0 to 1. Similarly, $\theta(\zeta, p)$ becomes $\theta_0(\zeta)$ to $\theta(\zeta)$ when $p$ assumes the values from 0 to 1. Exactly in the same manner for $p = 0, \phi(\zeta, 0) = \phi_0(\zeta)$ and for $p = 1, \phi(\zeta, 1) = \phi(\zeta)$. Using Taylor series expansion and equations (43)–(46) one obtains

\[
f(\zeta, p) = f_0(\zeta) + \sum_{m=1}^{\infty} f_m(\zeta) p^m, \quad f_m(\zeta) = \frac{\partial^m f(\zeta, p)}{\partial p^m}
\]

\[
g(\zeta, p) = g_0(\zeta) + \sum_{m=1}^{\infty} g_m(\zeta) p^m, \quad g_m(\zeta) = \frac{\partial^m g(\zeta, p)}{\partial p^m}
\]

\[
\theta(\zeta, p) = \theta_0(\zeta) + \sum_{m=1}^{\infty} \theta_m(\zeta) p^m, \quad \theta_m(\zeta) = \frac{\partial^m \theta(\zeta, p)}{\partial p^m}
\]

\[
\phi(\zeta, p) = \phi_0(\zeta) + \sum_{m=1}^{\infty} \phi_m(\zeta) p^m, \quad \phi_m(\zeta) = \frac{\partial^m \phi(\zeta, p)}{\partial p^m}
\]

The convergence of the series (55)–(58) is strongly dependent on $h$. Suppose $h$ is taken in such a manner that the series (55)–(58) converge at $p = 1$, then equations (55)–(58) result in

\[
f(\zeta) = f_0(\zeta) + \sum_{m=1}^{\infty} f_m(\zeta),
\]

\[
g(\zeta) = g_0(\zeta) + \sum_{m=1}^{\infty} g_m(\zeta),
\]

\[
\theta(\zeta) = \theta_0(\zeta) + \sum_{m=1}^{\infty} \theta_m(\zeta),
\]

\[
\phi(\zeta) = \phi_0(\zeta) + \sum_{m=1}^{\infty} \phi_m(\zeta).
\]

### 4.2. M-th-order deformation problems

By taking $m$ times derivative with respect to $p$ of equations (43), (47) then dividing by $m$ and substituting $p = 0$, develop the following simplifications

\[
L_f \left[ f_m(\zeta) - \chi_m f_m(\zeta) \right] = h \mathfrak{R}^f_m(\zeta),
\]

\[
f_m(0) = f_m(0) = 0,
\]

\[
\mathfrak{R}^f_m(\zeta) = f_m'' - \sum_{k=0}^{m-1} \frac{2f_m'}{k!} f_m' - \sum_{k=0}^{m-1} \left[ f_m' - f_m'' k \right] + \gamma_1 \sum_{k=0}^{m-1} \left[ f_m' - f_m'' k \right]
\]

\[
+ \mbox{Gr} h_m + \mbox{Gm} f_m - \frac{M}{1 + m} \left[ f_m' - mg_m \right]
\]

By taking $m$ times derivative with respect to $p$ of equations (44), (48) then dividing by $m$ and substituting $p = 0$, develop the following simplifications

\[
L_g \left[ g_m(\zeta) - \chi_m g_m(\zeta) \right] = h \mathfrak{R}^g_m(\zeta),
\]

\[
g_m(0) = g_m(0) = 0,
\]

\[
\mathfrak{R}^g_m(\zeta) = g_m'' - \sum_{k=0}^{m-1} \frac{2g_m'}{k!} g_m' - \sum_{k=0}^{m-1} \left[ g_m' - g_m'' k \right] + \gamma_1 \sum_{k=0}^{m-1} \left[ g_m' - g_m'' k \right]
\]

\[
+ \frac{M}{1 + m} \left[ m f_m' - g_m \right]
\]

By taking $m$ times derivative with respect to $p$ of equations (45), (49) then dividing by $m$ and substituting $p = 0$, develop the following simplifications

\[
L_\theta \left[ \theta_m(\zeta) - \chi_m \theta_m(\zeta) \right] = h \mathfrak{R}^\theta_m(\zeta),
\]

\[
\theta_m(0) = \theta_m(0) = 0,
\]
If \( f_i \) and \( g_m \) are the particular solutions, then the general solutions of equations (46), (50) then dividing by \( m \) and substituting \( p = 0 \), develop the following simplifications

\[
\mathcal{R}_m^0(\zeta) = (1 + Nr) \theta''_{m-1} - Pr \sum_{k=0}^{n-1} \left[ 2f'_{m-1-k} \theta_k - f_{m-1-k} \theta_k' \right] + \frac{M}{\gamma_1} \left[ \sum_{k=0}^{m-1} \left( f'_{m-1-k} f_k' + g_{m-1-k} g_k' \right) + \sum_{k=0}^{n-1} \left[ f''_{m-1-k} f_k'' + g''_{m-1-k} g_k'' \right] \right] + \frac{\gamma_1 M}{\gamma_2} \left[ \sum_{k=0}^{m-1} \left( f'_{m-1-k} f_k' - f_{k-1} f_k' \right) + \sum_{k=0}^{n-1} \left( g'_{m-1-k} g_k' \right) \right] \]

\[
\mathcal{R}_m^\alpha(\zeta) = \phi''_{m-1} + Sc \sum_{k=0}^{m-1} \left[ f_{m-1-k} \phi_k' \right] - Sc \sum_{k=0}^{n-1} \left[ \theta'_{m-1-k} \phi_k' \right] + Sc \sum_{k=0}^{n-1} \left[ \phi'_{m-1-k} \theta''_k \right]
\]

Similarly by taking \( m \) times derivative with respect to \( p \) of equations (46), (50) then dividing by \( m \) and substituting \( p = 0 \), develop the following simplifications

\[
L_p \left[ \phi_m(\zeta) - \lambda \phi_{m-1}(\zeta) \right] = h \mathcal{R}_m^\alpha(\zeta),
\]

\[
\phi_m(0) = \phi'_m(0) = 0,
\]

\[
\mathcal{R}_m^\alpha(\zeta) = \phi''_{m-1} + Sc \sum_{k=0}^{m-1} \left[ f_{m-1-k} \phi_k' \right] - Sc \sum_{k=0}^{n-1} \left[ \theta'_{m-1-k} \phi_k' \right] + Sc \sum_{k=0}^{n-1} \left[ \phi'_{m-1-k} \theta''_k \right],
\]

\[
\lambda_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1. \end{cases}
\]

If \( f_m^x(\zeta), g_m^x(\zeta), \theta_m^x(\zeta) \) and \( \phi_m^x(\zeta) \) are the particular solutions, then the general solutions of equations (63), (66), (69) and (72) are

\[
f_m(\zeta) = f_m^x(\zeta) + C_1 + C_2 \zeta + C_3 \zeta^2,
\]

\[
g_m(\zeta) = g_m^x(\zeta) + C_4 + C_5 \zeta,
\]

\[
\theta_m(\zeta) = \theta_m^x(\zeta) + C_6 + C_7 \zeta,
\]

\[
\phi_m(\zeta) = \phi_m^x(\zeta) + C_8 + C_9 \zeta.
\]

5. Results and discussion

The transformed equations (16)–(21) have been solved through the help of symbolic computation software MATHEMATICA employing HAM code. Various parameters are obtained during the nondimensionality of the basic governing equations. The effects of embedded parameters on the velocities \( f'(\zeta), g'(\zeta) \), temperature \( \theta(\zeta) \) and concentration \( \phi(\zeta) \) fields have been plotted in figures 6–26, 27–36 and 37–48 respectively. The graphs of skin friction coefficients \( f'(0), g'(0) \), rate of heat and mass transfer \( \theta'(0), \phi'(0) \) are sketched in figures 49–58. Similarly the entropy generation graphs are sketched in figures 59–64. The physical model of the problem is demonstrated in figure 1. Liao [34–36] introduced \( h \) curves for the convergence of the series solution to get the accurate results of the problems. Therefore, the admissible \( h \)-curves for \( f'(\zeta), g'(\zeta), \theta(\zeta) \) and \( \phi(\zeta) \) are drawn in the ranges of \(-0.25 \leq h \leq 0.0, -0.3 \leq h \leq -0.05, -0.25 \leq h \leq -0.05 \) and \(-0.27 \leq h \leq -0.1 \) in figures 2–5 respectively.

5.1. Velocity profile

Magnetic field has a dominant effect in the present study. For the values of \( M \) from 0.30 to 3.30, the magnetic field is intensified, whose effect is seen in figure 6 diminishing the axial velocity \( f'(\zeta) \). The axial flow is strongly decelerated with an increasing positive values of \( M \). It is observed that as the values of \( M \) increase, the axial velocity \( f'(\zeta) \) is decreasing, consequently a negative body force is produced due to the overall strength of applied magnetic field. Transverse velocity \( g(\zeta) \) is because of applied strong magnetic field, so when there is no magnetic filed \( (M = 0) \), then there is no transverse velocity and with the increase of magnetic field, a cross flow is automatically generated transversely due to Hall effect. One of the core subject of this study is Hall effect whose parameter is \( m \) which shows its contribution in figure 7. The Hall parameter \( m \) accelerates the axial velocity \( f'(\zeta) \). Since Hall parameter \( m \) is due to the strong magnetic field which is applied in perpendicular to the \( y \)-axis hence the Hall force is parallel to \( x \)-axis, consequently, flow increases in the direction of Hall force. Further, \( f'(\zeta) \) and \( g(\zeta) \) profiles tend to their classical hydrodynamic values when the Hall parameter approaches infinity because the magnetic force terms assume zero value for very high values of Hall parameter \( m \). The mixed
Figure 2. $h$ curve of $f'(\zeta)$.

Figure 3. $h$ curve of $g'(\zeta)$.

Figure 4. $h$ curve of $h'(\zeta)$.
convection is important part of the present analysis whose parameters $Gr$ (Grashof number) and $Gm$ (solutal Grashof number) show the effects of gravitational force in figures 8 and 9 respectively. Since the flow is along $x$-axis, so both the parameters amplify the velocity. Figure 10, shows the effect of second-grade fluid parameter $\gamma_1$ on the non-dimensional axial velocity profile $f' (\zeta)$. Due to the rising values of second-grade fluid parameter $\gamma_1$, the rheological characteristics of the second-grade fluid enhances which generates resistance to the flow. It is observed that axial and transverse velocity profiles are higher for ordinary viscous fluid. The non-dimensional velocity $f'(\zeta)$ for larger quantities of Prandtl number $Pr$ is illustrated in figure 11. It is shown that the large values of $Pr$ reduces the momentum boundary layer size resulting in decreasing the axial velocity $f'(\zeta)$.

Fluid flows become fast due to gravity in the presence of particles deposition. In such cases thickness of the fluid favors the velocity. Figure 13 shows the effect of thickness parameter $\beta$. The flow accelerates with the greater magnitude of thickness parameter $\beta$ because the mass of the fluid increase as a result momentum boundary layer thickness increases which causes acceleration in the flow. Flow speed is directly related to the
Figure 7. Velocity $f'(\zeta)$ graph for $h = -0.10, \gamma_1 = 1.00, Gr = 0.90, Gm = 0.80, M = 0.30, \beta = 0.10, Br = 0.50, Pr = 1.00, Nr = 0.10, Sr = 0.60, Sc = 0.50, \tau = 0.50$ and various values of $m$.

Figure 8. Velocity $f'(\zeta)$ graph for $h = -0.10, m = 0.50, \gamma_1 = 1.00, Gm = 0.80, M = 0.30, \beta = 0.40, Br = 0.50, Pr = 1.00, Nr = 0.10, Sr = 0.60, Sc = 0.50, \tau = 0.50$ and various values of $Gr$.

Figure 9. Velocity $f'(\zeta)$ graph for $h = -0.10, m = 0.50, Gr = 0.90, \beta = 0.40, M = 0.30, \gamma_1 = 1.00, Br = 0.50, Pr = 1.00, Nr = 0.10, Sr = 0.60, Sc = 0.50, \tau = 0.50$ and various values of $Gm$. 
Figure 10. Velocity $f'\zeta$ graph for $h = -0.10, m = 0.50, Gr = 0.90, Gm = 0.80, M = 0.30, \beta = 0.10, Br = 0.50, Pr = 1.00, Nr = 0.10, Sr = 0.60, Sc = 0.50, \tau = 0.50$ and various values of $\gamma_1$.

Figure 11. Velocity $f'\zeta$ graph for $h = 1.00, m = 0.50, Gr = 0.90, Gm = 0.80, M = 0.30, \gamma_1 = 1.00, Br = 0.50, \beta = 0.40, Nr = 0.10, Sr = 0.60, Sc = 0.50, \tau = 0.50$ and various values of $Pr$.

Figure 12. Velocity $f'\zeta$ graph for $h = -0.10, m = 0.50, Gr = 1.90, Gm = 1.80, M = 0.30, \gamma_1 = 1.00, Br = 0.50, Pr = 1.00, \beta = 0.10, Sr = 0.60, Sc = 0.50, \tau = 0.50$ and various values of $Nr$. 
mass of the fluid when it is under the gravitational force. Brinkman number $Br$ effect is shown in figure 14. This parameter boost the axial velocity $f'_z(\zeta)$ for the rising values of Brinkman number $Br$. The effect of Schmidt number $Sc$ is shown in figure 15. Schmidt number $Sc$ is the ratio of kinematic viscosity to molecular diffusivity so due to this characteristic axial velocity $f'_z(\zeta)$ decreases. Figure 16 demonstrates that axial velocity $f'_z(\zeta)$ increases by large quantities of the Soret number. The reason is that an increment in $Sr$ increases the difference of temperatures between the surface and the ambient fluid as a result reduction in the viscosity occurs and therefore axial velocity $f'_z(\zeta)$ accelerates. Figure 17 depicts that axial velocity $f'_z(\zeta)$ decreases by positive quantities of thermophoretic parameter $\tau$. It is observed that larger quantities of $\tau$ bring the reduction in the quantity of the mass transfer rate so $f'_z(\zeta)$ decreases automatically.

The transverse velocity $g(\zeta)$ is discussed from figure 18. Strong applied magnetic field generates the transverse velocity $g(\zeta)$. Figure 18 displays that when magnetic field parameter $M$ enhances, the transverse velocity $g(\zeta)$ also enhances. The Hall effect induces a cross flow $g(\zeta)$ in the direction of $z$-axis. Figure 19 depicts the important feature of this motion due to which the flow is three dimensional. This figure shows that how the Hall force is involved in the overall motion. When the Hall parameter $m$ increases the transverse velocity $g(\zeta)$ decreases as shown in figure 19. Moreover, it is observed that the transverse velocity $g(\zeta)$ vanishes when there is no Hall effect i.e. $g(\zeta) = 0$ when $m = 0$. It is due to the fact that motion in $z$-direction is because of Hall force. So Hall force is a cause of three dimensional flow and the flow becomes two dimensional when Hall force is absent. In the present study the mixed convection has significant contribution. It is obvious from figures 20 and 21 that

Figure 13. Velocity $f'_z(\zeta)$ graph for $h = -0.10, m = 0.50, Gr = 0.90, Gm = 0.80, M = 0.30, \gamma_1 = 1.00, Br = 0.50, Pr = 1.00, Nr = 0.10, Sr = 0.60, Sc = 0.50, \tau = 0.50$ and various values of $\beta$.

Figure 14. Velocity $f'_z(\zeta)$ graph for $h = -0.10, m = 0.50, Gr = 0.90, Gm = 0.80, M = 0.30, \gamma_1 = 1.00, \beta = 0.40, Pr = 1.00, Nr = 0.10, Sr = 0.60, Sc = 0.50, \tau = 0.50$ and various values of $Br$. 

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Figure 15. Velocity $f'(\zeta)$ graph for $h = -0.10, m = 0.50, Gr = 1.90, Gm = 1.80, M = 0.30, \gamma_1 = 1.00, Br = 0.50, Pr = 1.00, Nr = 0.10, Sr = 0.60, \beta = 0.10, \tau = 0.50$ and various values of Sc.

Figure 16. Velocity $f'(\zeta)$ graph for $h = -0.10, m = 0.50, Gr = 1.90, Gm = 1.80, M = 0.30, \gamma_1 = 1.00, Br = 0.50, Pr = 1.00, Nr = 0.10, \beta = 0.10, Sc = 0.50, \tau = 0.50$ and various values of Sr.

Figure 17. Velocity $f'(\zeta)$ graph for $h = -0.10, m = 0.50, Gr = 1.90, Gm = 1.80, M = 0.30, \gamma_1 = 1.00, Br = 0.50, Pr = 1.00, Nr = 0.10, Sr = 0.60, Sc = 0.50, \beta = 0.10$ and various values of $\tau$. 
Figure 18. Velocity $g(\zeta)$ graph for $h = -0.10, m = 0.50, Gr = 0.90, Gm = 0.80, \beta = 0.10, \gamma_1 = 1.00, Br = 0.50, Pr = 1.00, Nr = 0.10, Sr = 0.60, Sc = 0.50, \tau = 0.50$ and various values of $M$.

Figure 19. Velocity $g(\zeta)$ graph for $h = -0.10, \beta = 0.70, Gr = 0.90, Gm = 0.80, M = 3.30, \gamma_1 = 1.00, Br = 0.50, Pr = 1.00, Nr = 0.10, Sr = 0.60, Sc = 0.50, \tau = 0.50$ and various values of $m$.

Figure 20. Velocity $g(\zeta)$ graph for $h = -0.10, m = 0.50, \beta = 0.40, Gm = 0.80, M = 0.30, \gamma_1 = 1.00, Br = 0.50, Pr = 1.00, Nr = 0.10, Sr = 0.60, Sc = 0.50, \tau = 0.50$ and various values of $Gr$. 
by increasing quantities of the convection parameters $Gr$, $Gm$, the buoyancy forces do not allow the flow. The leading role of convection is played by the parameters $Gr$, $Gm$ having aptitude to depreciate the lateral velocity $g(\zeta)$, therefore it causes to shorten the boundary layer size associated to the transverse velocity $g(\zeta)$. Figure 22 displays the influence of second-grade fluid parameter $\gamma_1$ showing that the transverse velocity $g(\zeta)$ decreases for the dissimilar values of $\gamma_1$. The reason is that non-Newtonian effect overcomes the flow. Viscoelastic forces are strong enough to resist the motion. Figure 23 presents the influence of Prandtl number $Pr$. At constant pressure Prandtl number defines the ratio of the product of fluid viscosity and specific heat to the thermal conductivity using some reference point. At constant pressure the viscous forces are too much powerful to hold the fluid particles (atoms/molecules) therefore with increasing values of Prandtl number $Pr$ the transverse velocity $g(\zeta)$ depreciates. Figure 24 presents the role of film thickness parameter $\beta$. Since the flow is due to gravity therefore the transverse velocity $g(\zeta)$ becomes faster and faster with the greater quantities of film size parameter $\beta$. Figure 25 is prepared for the Brinkman number $Br$. It shows that with large values of Brinkman number $Br$ the transverse velocity $g(\zeta)$ increases. In fact, Brinkman number is the the ratio of the product of square of the initial velocity and dynamic viscosity to the product of the thermal conductivity and difference of the ambient fluid temperature and initial temperature so when once motion starts with initial velocity then the increasing values of Brinkman number favor the overall motion. Figure 26 reveals that the lateral velocity $g(\zeta)$ enhances for the larger values of the Schmidt number $Sc$. The increase in $Sc$ means lower molecular diffusivity, which causes to make shorten the concentration boundary layer and in turn increase the lateral velocity $g(\zeta)$.
Figure 23. Velocity $g(\zeta)$ graph for $h = -0.10, m = 0.50, Gr = 1.90, Gm = 1.80, M = 0.30, \gamma_1 = 1.00, Br = 0.50, \beta = 1.00, Nr = 0.10, Sr = 0.60, Sc = 0.50, \tau = 3.30$ and various values of $Pr$.

Figure 24. Velocity $g(\zeta)$ graph for $h = -0.10, m = 0.50, Gr = 0.90, Gm = 0.80, M = 0.30, \gamma_1 = 1.00, Br = 0.50, Pr = 1.00, Nr = 0.10, Sr = 0.60, Sc = 0.50, \tau = 0.30$ and various values of $\beta$.

Figure 25. Velocity $g(\zeta)$ graph for $h = -0.10, m = 0.50, Gr = 1.90, Gm = 1.80, M = 0.30, \gamma_1 = 4.00, \beta = 1.00, Pr = 1.00, Nr = 0.10, Sr = 0.60, Sc = 0.50, \tau = 3.30$ and various values of $Br$. 
5.2. Temperature profile

Temperature has the characteristic to diffuse into mass. As soon as it is exhaust it transferred abruptly from high energy level to low energy level. Figure 27 projects the influence of magnetic field parameter $M$ on the temperature profile $\theta(\zeta)$. The curves magnifies that for the values of $M$ from 0.30 to 3.30 the temperature $\theta(\zeta)$ amplifies. As it is accepted that the magnetic field has a heating effect so with greater values of $M$ the temperature $\theta(\zeta)$ remarkably enhances. Figure 28 shows the contribution of Hall parameter $m$. It is evident that $\theta(\zeta)$ amplifies for the big quantities of $m$. The effects of convection parameters $Gr$ and $Gm$ are shown in figures 29 and 30 respectively. Both of these parameters namely thermal Grashof number $Gr$ and solutal Grashof number $Gm$ have the same effect on the temperature $\theta(\zeta)$. For the positive values of $Gr$ and $Gm$ the temperature $\theta(\zeta)$ decreases in figures 29 and 30 respectively. The effect of second grade fluid parameter $\gamma_1$ is demonstrated in figure 31. The outcomes show that with the increase of non-Newtonian second grade fluid parameter $\gamma_1$, the nondimensional temperature $\theta(\zeta)$ decreases and therefore a reduction in the thermal boundary layer size exists. It is due to the reason that the viscosity of non-Newtonian fluid absorbs too much heat. Prandtl number $Pr$ shows its effect in figure 32. From the graph it is clear that temperature $\theta(\zeta)$ reduces with the rising values of Prandtl number $Pr$. Prandtl number fluids have relatively low thermal conductivity, which depreciates the conduction and thermal boundary layer size consequently, temperature of the fluid trim downs. Figure 33, projects that $\theta(\zeta)$ increases for the larger quantities of thermal radiation parameter $Nr$ and so gives rise in the size of the thermal boundary layer. This is due to the fact that, as radiation parameter $Nr$ increases, the mean Roseland absorption co-efficient $k_2$
Figure 28. Temperature $\theta(\zeta)$ graph for $h = -0.10, \beta = 0.50, Gr = 0.90, Gm = 0.80, M = 3.30, \gamma_1 = 1.00, Br = 0.50, Pr = 1.00, Nr = 0.10, Sr = 0.60, Sc = 0.50, \tau = 3.50$ and various values of $m$.

Figure 29. Temperature $\theta(\zeta)$ graph for $h = -0.10, m = 0.50, \beta = 0.40, Gm = 0.80, M = 0.30, \gamma_1 = 1.00, Br = 0.50, Pr = 1.00, Nr = 0.10, Sr = 0.60, Sc = 0.50, \tau = 0.50$ and various values of $Gr$.

Figure 30. Temperature $\theta(\zeta)$ graph for $h = -0.10, m = 0.50, \beta = 0.40, Gr = 0.90, M = 0.30, \gamma_1 = 1.00, Br = 0.50, Pr = 1.00, Nr = 0.10, Sr = 0.60, Sc = 0.50, \tau = 0.50$ and various values of $Gm$. 
Figure 31. Temperature $\theta(\zeta)$ graph for $h = -0.10$, $m = 0.50$, $Gr = 0.90$, $Gm = 0.80$, $M = 0.30$, $\beta = 0.10$, $Br = 0.50$, $Pr = 1.00$, $Nr = 0.10$, $Sr = 0.60$, $Sc = 0.50$, $\tau = 0.50$ and various values of $\gamma_1$.

Figure 32. Temperature $\theta(\zeta)$ graph for $h = -0.10$, $m = 0.50$, $\beta = 0.40$, $Gm = 0.80$, $Gr = 0.90$, $M = 0.30$, $\gamma_1 = 1.00$, $Br = 0.50$, $Pr = 1.00$, $Nr = 0.10$, $Sr = 0.60$, $Sc = 0.50$, $\tau = 0.50$ and various values of $Pr$.

Figure 33. Temperature $\theta(\zeta)$ graph for $h = -0.10$, $m = 0.50$, $Gr = 1.90$, $Gm = 1.80$, $M = 0.30$, $\gamma_1 = 1.00$, $Br = 0.50$, $Pr = 1.00$, $\beta = 0.10$, $Sr = 0.60$, $Sc = 0.50$, $\tau = 0.50$ and various values of $Nr$. 
decreases, this causes an increase in temperature profile. Generally, radiation occurs at high temperature. The effect of thin film parameter $\beta$ is shown in figure 34 which depicts that temperature distribution $\theta(\zeta)$ diminishes when $\beta$ assumes the values from 0.10 to 0.40. With the expansion of film size the temperature becomes low. The effect of Brinkman number $Br$ is shown in figure 35 demonstrating that temperature $\theta(\zeta)$ increases by the ways of heating due to friction. Figure 36 is plotted for the Schmidt number $Sc$. The plot demonstrates that the dimensionless temperature $\theta(\zeta)$ goes to low with larger quantities of $Sc$ resulting in the thinning of the thermal boundary layer. Figure 37 shows that temperature has decreasing behavior for the greater values of Dofour number since it has an inverse relation with temperature difference.

5.3. Concentration profile

The concentration is related to the types of fluids in special i.e. whether it is Newtonian fluid or non-Newtonian fluid. The magnetic field parameter $M$ shows its effect in figure 38. The fluid flow is regulated by magnetic field which shows that the concentration field $\phi(\zeta)$ increases with rising values of magnetic field parameter $M$. The Lorentz force due to strong magnetic field comes into play which controls the fluid flow. The behavior of Hall parameter $m$ in figure 39 is illustrated against $\phi(\zeta)$. Hall effect do not favor the concentration profile i.e. the concentration depreciates with increasing values of Hall parameter $m$. Hall force is attached to generate the cross flow in $z$-direction which support concentration slightly but not completely. From figures 40 and 41, prediction is realized that for the big quantities of convection parameters $Gr, Gm$ the concentration field $\phi(\zeta)$ decreases.

Figure 34. Temperature $\theta(\zeta)$ graph for $h = -0.10, m = 0.50, Gr = 0.90, Gm = 0.80, M = 0.30, \gamma_1 = 1.00, Br = 0.50, Pr = 1.00, Nr = 0.10, Sr = 0.60, Sc = 0.50, \tau = 0.50$ and various values of $\beta$.

Figure 35. Temperature $\theta(\zeta)$ graph for $h = -0.10, m = 0.50, \beta = 0.40, Gm = 0.80, Gr = 0.90, M = 0.30, \gamma_1 = 1.00, Br = 0.50, Pr = 1.00, Nr = 0.10, Sr = 0.60, Sc = 0.50, \tau = 0.50$ and various values of $Br$. 

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Figure 36. Temperature $\theta(\zeta)$ graph for $h = -0.10, m = 0.50, Gr = 1.90, Gm = 1.80, M = 0.30, \gamma_1 = 1.00, Br = 0.50, Pr = 1.00, Nr = 0.10, Sr = 0.60, \beta = 0.10, \tau = 0.50$ and various values of $Sc$.

Figure 37. Temperature $\theta(\zeta)$ graph for $h = -0.10, m = 0.50, Gr = 1.90, Gm = 1.80, M = 0.30, \gamma_1 = 1.00, Br = 0.50, Pr = 1.00, Nr = 0.10, Sr = 0.60, \beta = 0.10, \tau = 0.50, Sc = 0.50$ and various values of $Du$.

Figure 38. Concentration $\phi(\zeta)$ graph for $h = -0.10, m = 0.50, Gr = 1.90, Gm = 1.80, \beta = 0.40, \gamma_1 = 4.00, Br = 0.50, Pr = 1.00, Nr = 0.10, Sr = 0.60, Sc = 0.50, \tau = 3.50$ and various values of $M$. 
Figure 39. Concentration $\phi(\zeta)$ graph for $h = -0.10$, $m = 0.50$, $Gr = 0.90$, $Gm = 0.80$, $\gamma_1 = 1.00$, $Br = 0.50$, $Pr = 1.00$, $\beta = 0.10$, $Sr = 0.60$, $Sc = 0.50$, $\tau = 3.50$ and various values of $m$.

Figure 40. Concentration $\phi(\zeta)$ graph for $h = -0.10$, $m = 0.50$, $\beta = 0.10$, $Gm = 0.80$, $M = 3.30$, $\gamma_1 = 1.00$, $Br = 0.50$, $Pr = 1.00$, $Nr = 0.10$, $Sr = 0.60$, $Sc = 0.50$, $\tau = 3.50$ and various values of $Gr$.

Figure 41. Concentration $\phi(\zeta)$ graph for $h = -0.10$, $m = 0.50$, $Gr = 0.90$, $\beta = 0.10$, $M = 3.30$, $\gamma_1 = 1.00$, $Br = 0.50$, $Pr = 1.00$, $Nr = 0.10$, $Sr = 0.60$, $Sc = 0.50$, $\tau = 3.50$ and various values of $Gm$. 
Figure 42 projects that the non-dimensional concentration $\phi(\zeta)$ increases with larger values of second grade fluid parameter $\gamma_1$. Usually all the non-Newtonian liquids are high viscous and saturated compared to the Newtonian liquids. With increasing values of $\gamma_1$ the non-Newtonian effect increases as a result the concentration profile $\phi(\zeta)$ increases. In case of figure 43 concentration profile shows an increasing behavior against the Prandtl number $Pr$ owing to the thickening of the concentration boundary layer. Prandtl number is the ratio of the momentum diffusivity to thermal diffusivity. In fact Prandtl number empowers the viscous forces as a result viscosity of the fluid automatically increases which in turn enhances the concentration. Figure 44 expresses that by increasing the thermal radiation parameter $Nr$, the concentration profile $\phi(\zeta)$ increases. Thermal radiation has a direct relation with heat, so when radiation increases concentration is also increased. Generally, thermal radiation occurs at high temperature so concentration increases with increasing temperature. Figure 45 shows the influence of thin film parameter $\beta$. Increasing the values of $\beta$ the amount of solutes and other species in the solvent increases which automatically boost the concentration profile $\phi(\zeta)$. Figure 46 is plotted for the activity of thermophoretic parameter $\tau$. It is clear that the non-dimensional concentration profile $\phi(\zeta)$ becomes weak as the thermophoretic parameter $\tau$ varies from 0.50 to 2.00. Thermophoretic parameter relates to the temperature therefore when $\tau$ increases then the size of the concentration boundary layer decreases. Contribution of the Soret number $Sr$ is depicted in figure 47, showing that $\phi(\zeta)$ increases when the parameter $Sr$ increases while figure 48 shows that by varying Schmidt number $Sc$, the non-dimensional concentration profile $\phi(\zeta)$ decreases. The Schmidt number $Sc$ is an important parameter in mass transfer process as it characterizes the ratio of the
Figure 44. Concentration $\phi(\zeta)$ graph for $h = -0.10, m = 0.50, Gr = 1.90, Gm = 1.80, M = 0.30, \gamma_1 = 1.00, Br = 0.50, Pr = 1.00, Nr = 0.10, Sr = 0.60, Sc = 0.50, \beta = 0.10$ and various values of $\tau$.

Figure 45. Concentration $\phi(\zeta)$ graph for $h = -0.10, m = 0.50, Gr = 0.90, Gm = 0.80, M = 1.30, \gamma_1 = 1.00, Br = 0.50, Pr = 1.00, Nr = 0.10, \beta = 0.40, Sc = 0.50, \tau = 0.50$ and various values of $Sr$.

Figure 46. Concentration $\phi(\zeta)$ graph for $h = -0.10, m = 0.50, Gr = 0.90, Gm = 0.80, M = 1.30, \gamma_1 = 1.00, Br = 0.50, Pr = 1.00, Nr = 0.10, Sr = 0.60, \beta = 0.40, \tau = 0.50$ and various values of $Sc$. 

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thickness of the viscous and concentration boundary layers. The increase in Schmidt number $Sc$ significantly depreciates the concentration boundary layer. This decrease in solute concentration results in the solutal buoyancy effect, causing a small induced flow along the sheet surface. In figure 49 the influence of Brinkman number has been shown. It reveals that the concentration field is increasing with larger values of $Br$. The reason of this activity is that heat is added to the fluid which speeds up the chemical reaction.

5.4. Engineering and industrial quantities with tables showing comparison with the published experimental work

Graphs of key importance and dominant parameters are sketched to show the effects on skin friction coefficients, rate of heat and mass transfer. Figure 50 views the decreasing behavior of $f''(0)$ against the thin film parameter $\beta$ as the second grade fluid parameter $\gamma_1$ increases from 1.00 to 2.50. The reason is that in the presence of increasing layer, the motion faces less resistance due to gravity. Hall effect parameter $m$ shows its appearance in figure 51. The skin friction coefficient $f''(0)$ tends to minimum for the greater values of $m$ due to intensified magnetic field. In transverse direction the skin friction coefficient $g'(0)$ is acting. Figure 52 witnesses that $g'(0)$ decreases with the non-Newtonian characteristic of the fluid. The skin friction coefficient $g'(0)$ is due to Hall current effect. The influence of hall parameter $m$ is shown in figure 53 which increases the skin friction coefficient $g'(0)$. The present study is mainly concern with the strong applied magnetic field whose parameter $M$ favors the rate of heat transfer $\theta'(0)$ in figure 54. The effect of thermal radiation parameter $Nr$ in figure 55 demonstrates that rate of heat transfer $\theta'(0)$ is an increasing function of $Nr$ in the presence of...
Figure 49. Concentration $\phi(\zeta)$ graph for $h = -0.10$, $m = 0.50$, $Gr = 1.90$, $Gm = 0.80$, $M = 0.30$, $\gamma_1 = 1.00$, $Br = 0.50$, $Pr = 1.00$, $\beta = 0.40$, $Sr = 0.60$, $Sc = 0.50$, $\tau = 3.50$ and various values of $Nr$.

Figure 50. Skin friction coefficient $f''(0)$ graph for $h = -1.10$, $m = 0.50$, $Gr = 0.90$, $Gm = 0.80$, $M = 0.30$, $Nr = 0.10$, $Br = 0.50$, $Pr = 1.00$, $Sr = 0.60$, $Sc = 0.50$, $\tau = 0.50$ and various values of $\gamma_1$.

Figure 51. Skin friction coefficient $f''(0)$ graph for $h = -1.10$, $\gamma_1 = 1.00$, $Gr = 0.90$, $Gm = 0.80$, $M = 0.30$, $Nr = 0.10$, $Br = 0.50$, $Pr = 1.00$, $Sr = 0.60$, $Sc = 0.50$, $\tau = 0.50$ and various values of $m$. 
Figure 52. Skin friction coefficient $g'(0)$ graph for $h = -1.10$, $m = 0.50$, $Gr = 0.90$, $Gm = 0.80$, $M = 0.30$, $Nr = 0.10$, $Br = 0.50$, $Pr = 1.00$, $Sr = 0.60$, $Sc = 0.50$, $\tau = 0.50$ and various values of $\gamma_1$.

Figure 53. Skin friction coefficient $g'(0)$ graph for $h = -1.10$, $m = 0.50$, $Gr = 0.90$, $Gm = 0.80$, $M = 0.30$, $Nr = 0.10$, $Br = 0.50$, $Pr = 1.00$, $Sr = 0.60$, $Sc = 0.50$, $\tau = 0.50$ and various values of $m$.

Figure 54. Rate of heat transfer $\theta'(0)$ graph for $h = -1.10$, $m = 0.50$, $Gr = 0.90$, $Gm = 0.80$, $\gamma_1 = 1.00$, $Nr = 0.10$, $Br = 0.50$, $Pr = 1.00$, $Sr = 0.60$, $Sc = 0.50$, $\tau = 0.50$ and various values of $M$. 

thin film progress. Figure 56 provides that rate of heat transfer $\theta'(0)$ reduces with high values of Prandtl number $Pr$. The rate of mass transfer $\phi'(0)$ becomes fast with greater values of Soret number $Sr$ which is shown in figure 57. The effect of Schmidt number $Sc$ can be seen in figure 58. It shows that rate of mass transfer $\phi'(0)$ decreases with increasing $Sc$. Figure 59 reveals that thermophoretic parameter $\tau$ has decreasing effect on rate of mass transfer $\phi'(0)$.

Solution of the problems needs authentication. Due to this reason the obtained results are compared with the published work. To establish a correlation with the thin film literature some related studies are given from the published literature (see table 1). Table 1 shows the different published studies taken into account different models including the present study model. Table 2 contains the different non-dimensional parameters existing in equations (16)–(19). Skin friction coefficient, Nusselt and Sherwood numbers are presented through tables 3 and 4 and 5 which provide the verification for the achieved results. Table 3 presents the numerical values of film thickness $\beta$ and skin friction coefficient $f''(0)$ for the different published data and the present work. The values of $\beta$ and $f''(0)$ in table 3 have the same behaviors as the published results. Table 4 depicts the same tendency of the numerical values of the wall temperature $-\theta(0)$ and Nusselt number (wall temperature gradient) $-\theta'(0)$ for the different published literature and the present investigation in terms of increasing values of Prandtl number $Pr$. Similarly table 5 show the surface concentration and rate of mass transfer in terms of different values of magnetic field parameter $M$, Brinkman number $Br$ and thermophoresis parameter $\tau$. All the three tables

**Figure 55.** Rate of heat transfer $\theta'(0)$ graph for $h = -1.10$, $m = 0.50$, $Gr = 0.90$, $Gm = 0.80$, $\gamma_1 = 1.00$, $Br = 0.50$, $Pr = 1.00$, $Sr = 0.60$, $Sc = 0.50$, $\tau = 0.50$ and various values of $Nr$.

**Figure 56.** Rate of heat transfer $\theta'(0)$ graph for $h = -1.10$, $m = 0.50$, $Gr = 0.90$, $Gm = 0.80$, $M = 0.30$, $\gamma_1 = 1.00$, $Br = 0.50$, $Pr = 0.10$, $Sr = 0.60$, $Sc = 0.50$, $\tau = 0.50$ and various values of $Pr$.
Figure 57. Rate of mass transfer $\phi'(0)$ graph for $h = -1.10, m = 0.50, Gr = 0.90, Gm = 0.80, M = 0.30, \gamma_1 = 1.00, Br = 0.50, Nr = 0.10, Pr = 1.00, Sc = 0.30, \tau = 0.50$ and various values of Sr.

Figure 58. Rate of mass transfer $\phi'(0)$ graph for $h = -1.10, m = 0.50, Gr = 0.90, Gm = 0.80, M = 0.30, \gamma_1 = 1.00, Br = 0.50, Nr = 0.10, Pr = 1.00, Sr = 0.60, \tau = 0.50$ and various values of Sc.

Figure 59. Rate of mass transfer $\phi'(0)$ graph for $h = -1.10, m = 0.50, Gr = 0.90, Gm = 0.80, M = 0.30, \gamma_1 = 1.00, Br = 0.50, Nr = 0.10, Pr = 1.00, Sr = 0.60, Sc = 0.50$ and various values of $\tau$. 
Table 1. Literature related to the present study.

| Authors                  | Models                        | Film thickness | Viscous dissipation |
|--------------------------|-------------------------------|----------------|--------------------|
| Narayana and Sibanda [30] | Homogeneous                  | Considered     | Not Considered     |
| Qasim [31] *et al*       | Buongiorno’s                  | Considered     | Considered         |
| Khan [8] *et al*         | Non-Newtonian (Secondgrade)   | Considered     | Considered         |
| Present Study            | Non-Newtonian (Secondgrade)   | Considered     | Considered         |

Table 3. Various parameters in equations (16)–(21).

| Parameter names          | Symbols/Notations | Defined values |
|--------------------------|-------------------|----------------|
| Dimensionless second-grade fluid parameter | $\gamma_1$ | $\eta_0 \beta_1^2 \rho c T_0$ |
| Thermal Grashof number   | $Gr$              | $g k^2 \beta_1 (T_w - T_0)$ |
| Solutal Grashof number   | $Gm$              | $g k^2 \beta_1 (C_w - C_0)$ |
| Magnetic field parameter | $M$               | $\eta_0 \beta_1^2 \rho c T_0$ |
| Thermal radiation parameter | $Nr$           | $3 k \rho c T_0$ |
| Schmidt number           | $Sc$              | $\nu \rho c T_0$ |
| Soret number             | $Sr$              | $k \delta \beta_1 (T_w - T_0)$ |
| Thermophoretic parameter | $\tau$           | $T_0 (T_w - T_0)$ |
| Prandtl number           | $Pr$              | $\eta_0 \beta_1^2 \rho c T_0$ |
| Brinkman number          | $Br$              | $k (T_w - T_0)$ |
| Dofour number            | $Du$              | $k (T_w - T_0)$ |

Table 3. Comparison of values of film thickness $\beta$ and skin friction coefficient $f^\prime (0)$. Note: Narayana and Sibanda [30] have used the similarity transformations in his paper by evaluating experimentally $\beta^\prime (0)$. Qasim *et al* [31] has used the same similarity transformations by employing the Buongiorno’s model. Khan *et al* [8] has used the same similarity transformations and explored the non-Newtonian second grade model. The present study has non-Newtonian second grade model.

| Authors                  | Film thickness | Viscous dissipation |
|--------------------------|----------------|--------------------|
| Narayana, Sibanda [30]   | Considered     | Not Considered     |
| Qasim [31] *et al*       | Considered     | Considered         |
| Khan [8] *et al*         | Considered     | Considered         |
| Present results          | Considered     | Considered         |

numerically depict the published experimental results and the present study results which show an excellent agreement and correlation.

6. Entropy generation analysis

Entropy is also the important part of this study. Figure 60 shows diverse form of entropy generation. It is attributed to the high activity of Reynolds number number $Re$. It is depicted that entropy generation rate $N_{re}(\zeta)$ is high for the choice of the various values of $Re$ while the values of $Br$ make enhancement in entropy generation rate $N_{re}(\zeta)$ which has the same increasing behavior in figure 61. Entropy is controlled by the different values of Brinkman number $Br$ in figure 62. The various values of $Br$ show that the entropy is high. The differences in the
values of magnetic field parameter $M$ in figure 62 presents an interesting behavior in depreciating the entropy generation rate $N_G(\zeta)$ to show the changes. In figure 63 the non-dimensional parameter $\gamma_2$ can assume a sequence of high values (range from 0.50 to 1.20) with high effects observations. $\gamma_2$ is related to the diffusivity. It shows that entropy generation rate $N_G(\zeta)$ increases with increasing $\gamma_2$ where at the same time the temperature difference parameter $\theta_w$ reveals a unique activity, including the depreciation of entropy generation rate $N_G(\zeta)$ in

### Table 4. Comparison of values of surface temperature and wall temperature gradient with increasing values of Prandtl number. Note: Narayana and Sibanda [30] have used the similarity transformations in his paper and worked experimentally by evaluating $-\theta(0)$ and $-\beta\phi'(0)$. Qasim et al [31] has used the same similarity transformations by employing the Buongiorno’s model. Khan et al [8] has used the same similarity transformations and explored the non-Newtonian second grade model. The present study has non-Newtonian second grade model.

| Parameter | Narayana [30] | Qasim et al [31] | Khan et al [8] | Present Results |
|-----------|---------------|-----------------|----------------|-----------------|
| $Pr$      | $\theta(\beta)$ | $-\beta\phi'(0)$ | $\theta(\beta)$ | $-\beta\phi'(0)$ | $\theta(\beta)$ | $-\beta\phi'(0)$ |
| 0.01      | 0.9604803     | 0.0994744       | 0.9604803       | 0.0994744       | 0.996805        | 0.0992258         | 0.00898253       |
| 0.1       | 0.9692532     | 0.6616210       | 0.9692532       | 0.6616210       | 0.969343        | 0.437373          | 0.9258           | 0.151507         |
| 1.0       | 0.0978841     | 3.5959916       | 0.0978841       | 3.5959911       | 0.7607          | 5.15088           | 0.495089         | 1.20959         |
| 2.0       | 0.0248625     | 5.2591876       | 0.0248625       | 5.2591868       | 0.682131        | 3.7502            | 0.269180         | 1.97185        |
| 3.0       | 0.0083111     | 6.5337030       | 0.0083111       | 6.5337020       | 0.616268        | 2.7688            | 0.153675        | 2.53633        |
| 5.0       | 0.0013612     | 8.5584774       | 0.0013612       | 8.5584920       | 0.523808        | 1.93815           | 0.542203         | 3.38690         |
| 7.0       | 0.0002935     | 10.2067133      | 0.0002935       | 10.2067312      | 0.453264        | 1.17529           | 0.0202849        | 4.05250         |
| 10.0      | 0.0000390     | 12.2865044      | 0.0000390       | 12.2865270      | 0.370247        | 2.33462           | 0.00527341       | 4.87762         |

### Table 5. Comparison of values of surface concentration and Sherwood number with increasing values of $M$, $Br$ and $\tau$.

| Parameters | Qasim et al [31] | Present results |
|------------|-----------------|-----------------|
| $M$        | $Br$            | $\tau$          | $\phi(\beta)$ | $-\phi'(0)$    | $\phi(\beta)$ | $-\phi'(0)$ |
| 0.00       | 0.00            | 0.10            | 0.00120        | 4.69946        | 0.012312       | 0.046936    |
| 0.00       | 0.00            | 0.30            | 0.00055        | 5.63125        | 0.0012342      | 0.0643057   |
| 0.00       | 0.00            | 0.50            | 0.00042        | 5.73992        | 0.0012732      | 0.0833171   |
| 0.00       | 0.50            | 0.10            | 0.00309        | 4.96867        | 0.0213345      | 0.0490258   |
| 0.00       | 0.50            | 0.30            | 0.00117        | 6.85898        | 0.0432314      | 0.064105    |
| 0.00       | 0.50            | 0.50            | 0.00079        | 5.75820        | 0.0212543      | 0.0831835   |
| 0.50       | 0.00            | 0.10            | 0.00453        | 4.65665        | 0.124412       | 0.0469865   |
| 0.50       | 0.00            | 0.30            | 0.00208        | 7.59404        | 0.00275412     | 0.0758686   |
| 0.50       | 0.00            | 0.50            | 0.00160        | 5.70473        | 0.00324        | 0.0843827   |

Figure 60. Entropy generation rate $N_G(\zeta)$ graph for $h = 0.50$, $Br = 0.40$, $M = 0.50$, $\gamma_2 = 0.50$, $\theta_w = 0.50$, $\phi_w = 1.00$ and various values of $Re$. 
Figure 61. Entropy generation rate $N_{G}(\zeta)$ graph for $h = 0.50, Re = 0.50, M = 0.50, \gamma_2 = 0.50, \theta_w = 0.50, \phi_w = 1.00$ and various values of $Br$.

Figure 62. Entropy generation rate $N_{G}(\zeta)$ graph for $h = 0.50, Re = 0.50, Br = 0.40, \gamma_2 = 0.50, \theta_w = 0.50, \phi_w = 1.00$ and various values of $M$.

Figure 63. Entropy generation rate $N_{G}(\zeta)$ graph for $h = 0.50, Re = 0.50, Br = 0.40, M = 0.50, \theta_w = 0.50, \phi_w = 1.00$ and various values of $\gamma_2$. 
7. Conclusions

The study concerns with the analysis of Hall current and thermophoresis effects on two dimensional flow of a magnetohydrodynamic mixed convective heat and mass transfer in a thin film non-Newtonian second grade fluid past a stretching sheet with viscous dissipation, thermal radiation and entropy generation. The transformed equations have been solved via analytical technique Homotopy Analysis Method. The solution has been displayed through graphs and the influences of all the indispensable parameters on velocities as well as temperature and concentration fields have been discussed. Entropy generation rate is also discussed through graphs in terms of different parameters. Authentication of the study has been provided by the comparison values of skin friction coefficient, surface, rate of heat and mass transfer.
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Conflict of Interests

The authors declare that they have no competing interests.

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