Dynamics of Space at High Energies

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Abstract. The dynamics of space endowed by a metric of the 3-dimensional sphere in the
framework of $f(R)$–gravity acting in $D = 4$ from the creation at high energies is studied.
Spaces of finite size are found as a result of exact solution of the classical equations of motion.
Generalization of the theory to other dimensions $D = 3$ and $D = 5$ are also considered.

1. Introduction
Historically, the first detailed Starobinsky model [1] of inflation is based on the theory of gravity
with added the quadratic term of scalar curvature. In this theory the Friedmann equation is
modified for large values of the Hubble parameter which leads to a cosmological solution with
a scale factor growing exponentially during a certain period of evolution. This model also has
a post-inflationary heating up mechanism. As a result the evolution of the Universe enters a
hot stage. The Starobinsky model is quite successful. However, it describes evolution starting
from a certain energy scale. In this paper it is proposed to study the dynamics of a three-
dimensional (and another dimensions) sphere at energy scales exceeding the inflationary using
$f(R)$–gravity. Naturally, this is not the only way to satisfy the observed inflation data. This can
be achieved, as works [2] and [3] demonstrate, using a completely different approach: to study
pure multidimensional gravity with higher derivatives.

2. Set up
Let us consider the theory described by the action and the corresponding equations of motion

$$S = \frac{1}{2} M_{Pl}^{D-2} \int d^Dx \sqrt{|g_D|} f(R) , \quad f'_R(R) R_{MN} - \frac{1}{2} f(R) g_{MN} + \left[ \nabla_M \nabla_N - g_{MN} \Box \right] f'_R(R) = 0 .$$

(1)

At a constant value of curvature $R(t) = \text{const} \equiv R_c$ from this we have

$$f'_R(R_c) R_{MN} - \frac{1}{2} f(R_c) g_{MN} = 0 \quad \Rightarrow \quad f'_R(R_c) R_c - 2 f(R_c) = 0 .$$

(2)

The our aim is to find the form of the $f(R)$–function which leads to asymptotics with large
and small curvature and, as a consequence, large and compactified spaces. If there are several
asymptotics, then we should find out the range of initial conditions leading to each of them.

Let the initial conditions for the system (1) on the function $\alpha(t)$ and on the derivative of the
function $R(t)$ be given by constant

$$\alpha(0) = \alpha_0 \sim \ln H_{\text{sub--Planck}}^{-1} , \quad \dot{\alpha}(0) = \alpha_1 \sim H_{\text{sub--Planck}}, \quad \dot{R}(0) = R_1$$

(3)
and the initial values for function $\alpha(t)$ are limited by $H_{\text{inf}} \sim 0.1$. Then with the choice of the metric of the three-dimensional sphere and the corresponding Ricci scalar

$$ds^2 = dt^2 - e^{2\alpha(t)}(dx^2 + \sin^2 x \, dy^2 + \sin^2 x \, \sin^2 y \, dz^2); \quad R(t) = 12\dot{\alpha}^2(t) + 6\dot{\alpha}(t) + 6e^{-2\alpha(t)}$$

(4)

the initial value of the curvature $R(0) = R_0$ will be determined by solution of the following equation for different $f(R)$–functions from [1].

3. Results

3.1. $R^2$–model

We will look at a solution with $f(R)$–function chosen as

$$f(R) = aR^2 + R + c \rightarrow, \quad R_c = -2c.$$  

(5)

For the function [5] we get one asymptotics, but this helps to understand the characteristic features of the solution to the system [1]. The initial value of the curvature $R_0$ will be determined by solutions of the following equation

$$R_0^2 - 12(\alpha_1^2 + e^{-2\alpha_0})R_0 - 12\alpha_1 R_1 - \frac{6}{a}(\alpha_1^2 + e^{-2\alpha_0}) - \frac{c}{a} = 0.$$  

(6)

Let’s check “Set up” on Starobinsky model. Then the parameters in [5] are

$$a = \frac{1}{6m^2}, \quad c = 0, \quad \text{with} \quad m \sim 1.5 \cdot 10^{-5} m_{Pl} \left(\frac{50}{N_e}\right), \quad N_e = 55 \div 60.$$  

(7)

The initial conditions on the functions $\alpha(t)$ and $R(t)$ be given as

$$\alpha_0 = -\ln H_{\text{inf}} = \ln 10^6 \sim 13.8, \quad \alpha_1 = H_{\text{inf}} \sim 10^{-6}, \quad R_1 = 0, \quad R_+(0) = 2.9 \cdot 10^{-11}. \quad (8)$$

As a result in the figure [1] we see that curvature oscillations lead to a slowdown in the growth of $\alpha(t)$ and thus not an infinitely large size of the sphere. All relevant quantities, such as: the area size $\alpha(t)$, the duration of the inflationary stage $t \sim 10^{10}$ (the beginning of the curvature $R(t)$ oscillations) the amplitude of this oscillations, the value of the Hubble parameter i.e. $\ddot{\alpha}(t)$ are in accordance with the predictions of inflation theory and experimental data. Let’s continue the construction of the numerical solution up to the current age of the Universe $t_{Univ} \sim 10^{61}$. Then we get

$$R(t_{Univ}) \sim 10^{-122} \quad \text{and} \quad \ddot{\alpha}(t_{Univ}) \sim 10^{-61}. \quad (9)$$

But the final area size is much closer to the visible size of the Universe:

$$l_{\text{Starobinsky}} \sim 10^{61} = \frac{\alpha_0(t_{Univ})}{10^{61}} \sim 10^5.$$  

(10)

3.2. $R^3 + R^2$–model

Now we solve a system with a choice of $f(R)$–function as

$$f(R) = \frac{A}{3}R^3 + aR^2 + R \rightarrow, \quad R_c = 0, \pm \frac{\sqrt{3}}{\sqrt{A}}.$$  

(11)

The initial value of the curvature $R_0$ will be determined by solutions of the following equation

$$2AR_0^3 - \left(18A(\alpha_1^2 + e^{-2\alpha_0}) - 3a\right)R_0^2 - 36a \left(\alpha_1^2 + e^{-2\alpha_0}\right)R_0 - 36\left(A R_0 + a\right)\alpha_1 R_1 - 18\left(\alpha_1^2 + e^{-2\alpha_0}\right) = 0.$$  

(12)
Let us set the function parameters and initial conditions as

\[ A = 10^{-4}, \quad a = 450, \quad \alpha_0 = 2.5, \quad \alpha_1 = 0.1, \quad R_0 = 0.2, \quad R_1 = 0. \]  

We see in the figure that in this case, like an illustrative example of Starobinsky $R^2$-model, the sphere slows down its growth rate with the beginning of curvature oscillations. A distinctive feature from the previous result is the initial conditions. It turns out that even such a very small area as $\alpha \sim 2.5$ can be grown to the size of the visible Universe only by the gravitational effect.

This is one of the options for solving a system with such a choice of the $f(R)$-function. The behavior is significantly influenced by the values of the coefficients in front of the powers of the Ricci scalar and especially by the initial conditions. In the future we will study in more detail the regions of parameters for which the dynamics are significantly different and that lead to each of the possible asymptotics.

### 3.3. Other dimensions

We repeat similar calculations for two-dimensional ($D = 3$) and four-dimensional ($D = 5$) spheres. The aim is to see how the solution will change depending on the dimension of the sphere with fixed coefficients for a three-dimensional sphere. As a result of the solution in
the figure 3 we obtain in the case $D = 3$ unstable configurations or more compact size in $D = 5$. But the solution is sensitive to changes in the initial conditions. It is possible to obtain a stable configuration for a two-dimensional sphere with a slight change in the initial conditions.

![Figure 3](image)

**Figure 3.** The result of solving a system (1) assuming (4) with (11) and (13) for $D = 3$, $R_0 = 0.04$ (a) and $D = 5$, $R_0 = 0.4$ (b).

4. Conclusion

It is shown that when considering $f(R)$–gravity starting from sub-Planck scales it seems possible to obtain three-dimensional space of large and "finite" size which can be identified with the visible part of the Universe. The calculation approach was tested on Starobinsky $R^2$–model. Also it was calculating for another dimensions in $R^3 + R^2$–model. In the future we will study more complex metrics in this way. The aim is to find conditions which lead to the large size of main space and small size of an extra dimensions. Interesting results on this topic have already been obtained in [4]. A large three-dimensional expanding sub-space was found within the framework of the multidimensional theory of gravity with higher derivatives, while the other remains constant.

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