Utility maximization under endogenous pricing

Thai Nguyen¹ and Mitja Stadje²

¹Université Laval, École d’Actuariat,
²University of Ulm, Faculty of Mathematics and Economics, Institute of Insurance Science and Institute of Financial Mathematics

Abstract: We study the expected utility maximization problem of a large investor who is allowed to make transactions on tradable assets in an incomplete financial market with endogenous permanent market impacts. The asset prices are assumed to follow a nonlinear price curve quoted in the market as the utility indifference curve of a representative liquidity supplier. We show that optimality can be fully characterized via a system of coupled forward-backward stochastic differential equations (FBSDEs) which corresponds to a non-linear backward stochastic partial differential equation (BSPDE). We show existence of solutions to the optimal investment problem and the FBSDEs in the case where the driver function of the representative market maker grows at least quadratically or the utility function of the large investor falls faster than quadratically or is exponential. Furthermore, we derive smoothness results for the existence of solutions of BSPDEs. Examples are provided when the market is complete or the utility function is exponential.

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1. Introduction

This paper studies a stochastic optimal control problem with feedback effect using coupled forward backward stochastic differential equations (FBSDEs) and backward stochastic partial differential equations (BSPDEs). Our findings demonstrate that FBSDEs and BSPDEs can serve to define optimal wealth and value functions in the context of investment with endogenous pricing and permanent price impacts. Furthermore, we establish the existence of optimal solutions for various utility functions, accompanied by solutions for associated FBSDEs. Additionally, we reveal that the value function exhibits smoothness when the market is complete or when the utility function is exponential.

To comprehend our market impact model, consider a substantial trader or financial investor dealing in risky assets like financial derivatives within the financial market. Traditional financial mathematics assumes traders are price takers, meaning asset prices and their stochastic processes are predefined and unaffected by trader activities. However, it’s widely acknowledged that significant buying or selling impacts prices by altering supply and asset volatility, as discussed in [9, 59, 26]. In this paper, we consider a model with (permanent) market impact and analyze a utility maximization problem for a large investor who is trading risky assets in a financial market where the trading influences the future prices, and the price curves are non-linear in volume. Thus, our model naturally encompasses phenomena such as nonlinearity in liquidation and market contractions resulting from illiquidity. Despite its significant nonlinearity, our model unexpectedly facilitates a comprehensive mathematical analysis of the optimal portfolio choice problem and enables the characterization of the optimal strategy, suitable for numerical computations. As far as we are aware, this paper

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marks a pioneering endeavor to address the utility maximization problem in the presence of fully non-linear endogenous market impact using robust expectations.

Our contribution can be summarized as follows: a) Building on [59, 26, 3, 4] we introduce a new type of intrinsically non-linear optimal control problem with a solid foundation in endogenous pricing. b) We show that the problem is well defined in a continuous-time framework which arises naturally as the limit of discrete-time trading with feedback effects. c) In general incomplete markets with endowments of the investor and the market maker, we characterize optimal solutions in terms of coupled FBSDEs and of BSPDEs, thereby highlighting the interplay between these approaches. Furthermore, the FBSDEs dependence on the utility function of the investor can be completely captured through the level of prudence the utility function induces. d) We give new existence results for the arising FBSDEs and BSPDEs. e) We furnish examples in scenarios where the market is complete, and/or the utility function adopts an exponential form. Notably, under the circumstances of linear pricing without market impact, our problem simplifies to the classical utility maximization problem.

The classical Merton utility maximization problem [46] with and without transaction costs has been extensively studied and we recommend [47] for an overview. In the absence of transaction costs and feedback effects, the classical solution methods are based on convex duality which first appeared in [8] and was extended for example in [7, 51, 33, 34, 14, 37] and (with endowment) in [49]. Despite their broad applicability, dual methods come with a drawback: they are not well-suited for numerical approximations. A somewhat newer approach in particular for exponential, power, or logarithmic utility functions is based on BSDEs with and without convex duality methods; see [53, 28, 29, 58, 43, 40] among others. BSDEs can, for instance through Monte Carlo simulation, be approximated efficiently. They offer advantages such as independence from convex hedging constraints and unlike PDEs a broader applicability beyond Markovian frameworks. In [27, 55, 54], the optimal strategy for an unhedgeable terminal condition and general utility functions can be described through a solution to a fully coupled forward backward system. By using dynamic programming, the authors in [44, 43, 45] show that the optimal strategy can be represented in terms of the value function related to the problem and its derivatives. Furthermore, under appropriate assumptions, the value function is characterized as the solution to a BSPDE.

For utility maximization problems where the trading itself influences the prices most works focus on temporary or transient price impact of exogenous-type; see [5] and the references therein. We note that our paper focuses on investigating the permanent price impact, rather than temporary market impact, and assumes that trades are fragmented into smaller portions, allowing their effects to dissipate gradually over time. This concept is discussed comprehensively in [23]. In this paper, we consider an endogenously based liquidity model where the permanent price impact is due to an inventory (supply-side) change of the security. It is shown in [22] that the profit and loss (P&L) process of any trading strategy can then be expressed as a non-linear stochastic integral or a g-expectation and the pricing and hedging can be done by solving a semi-linear PDE in the special case of a Markovian setting. [22] also provides a completeness condition under which any derivative can be perfectly replicated by a dynamic trading strategy. Contrary to exogenously based-liquidity models, an endogenously liquidity-based model may give better economic understanding on the liquidity risk. Non-utility based additive permanent market impact models are also studied in many other works, see for instance [10] and the references therein. Feedback effects with BSDEs are studied in [15]. In [20], a Nash equilibrium for a market impact game is characterized in terms of a fully coupled system of FBSDEs.

The closest price impact model to ours is the one introduced by Bank and Kramkov [3, 4] who
allow the Market Makers to have utility functions of von Neumann-Morgenstern type (see also the earlier works [59, 26]). They show the existence of the representative liquidity supplier and construct a nonlinear stochastic integral to describe the Large Trader’s P&L. Our utility function stems from time-consistent convex risk measures represented as a $g$-expectation, and in terms of decision theory, it represents an ambiguity-averse preference with the exponential utility being the only intersection to von Neumann-Morgenstern utility functions. The focus of this paper is on optimal portfolio choice which is not addressed in [3, 4, 22]. The special case of an exponential utility function for the investor and for the representative liquidity supplier is analyzed in [1] without considering coupled FBSDEs or BSPDEs.

We remark that contrary to the works on utility maximization above, the terminal wealth in our case lies in a space of feasible terminal conditions of $g$-expectations and depends on the strategy in a complex and non-convex way. Due to the non-linearity in the target function, the control variable and the feedback effects, the mathematical proofs are delicate. We first show that the wealth process corresponds to a non-linear Kunita integral where every admissible trading strategy induces a process, say $Z$, which is based on a parametric family of $Z^y$ processes from the solution of a parametric BSDE. We prove that trading can be extended to continuous time by deriving from appropriate assumptions the existence of a version of $y \mapsto Z^y$ which can be shown to be continuous in the trading strategy and satisfies suitable growth conditions. Through calculus of variations, we derive a coupled FBSDE for the optimal wealth process which is necessary and sufficient for a trading strategy to be optimal. Existence of a solution under conditions covering most known examples is shown by proving weak compactness of a space of suitable $Z$’s or wealth processes. To the best of our knowledge such a $Z$-based optimal control approach has not been used yet to show existence of a solution of a coupled FBSDE.

While for decoupled quadratic FBSDEs there is a rich theory available, for coupled FBSDEs the literature is much more sparse. For non-Markovian systems, existence for solutions of coupled FBSDEs for sufficiently small time horizons $T$ have been obtained by [16] using a contraction method. Well-posedness of the system has been investigated by [42] using the so-called decoupling field method introduced in [19]. [41] study the well-posedness for multi-dimensional and coupled systems of FBSDEs with a generator that can be separated into a quadratic and a subquadratic part. Using Malliavin calculus arguments [39] obtains local and global existence and uniqueness results for multidimensional coupled FBSDEs for monotone generators with arbitrary growth in the control variable. [31] obtains some recent results for Markovian FBSDEs with quadratic growth. Existence results for multi-dimensional coupled FBSDEs with diagonally quadratic generators can be found in [12]. [25] study equilibria of asset prices with quadratic transaction costs, and solve an arising coupled FBSDE using stochastic Riccati equations. In contrast to these works our forward process boasts a more versatile volatility term, denoted as $H$ in Equation (3.5), which depending on the utility function can fully depend on the variables $X, Y, M$, and $\omega$. Moreover, for the value function, we derive a dynamic programming principle and prove that under smoothness conditions the value function satisfies a certain BSPDE which can be connected to the previously derived FBSDE. The smoothness of the value function is then shown using duality methods. All results except in Section 7.2, are shown for an incomplete financial market with the randomness being generated by a $d$-dimensional Brownian motion.

The remainder of the paper is organized as follows. Section 2 outlines a model with permanent market impact, depicting hedging through $g$-expectations. We specify key assumptions validating our results and our expected utility maximization problem under permanent market impacts. Solutions are characterized by FBSDEs in Section 3. FBSDE existence results are derived in Section
The connection to BSPDEs is studied in Section 4. The results on regularity necessary for the existence of a solution of the BSPDEs together with examples in complete markets are presented in Section 5. The main proofs and extra technical results are reported in the appendix. Section 6 concludes.

2. The model setting

In a limit order book of specific asset, the roles of liquidity suppliers and liquidity demanders are not symmetric. Every liquidity supplier submits a price quote for a specific volume and trades with the other liquidity suppliers until an equilibrium is achieved. The remaining limit order form a price curve which is a nonlinear function in volume. Taking a Bertrand-type competition among liquidity suppliers into account, it would then be reasonable to begin with modelling the price curve as the utility indifference curve of a representative liquidity supplier. While Bank and Kramkov [3, 4] used Neumann-Morgenstern utility functions for the representative agent we will as in [22] use time-consistent convex risk measures instead. The exponential utility function assumed by Anthropelos et al. [1] is in the intersection of these two frameworks. Modulo a compactness assumption using a time-consistent convex risk measure is equivalent to using a $g$-expectation, providing a powerful stochastic calculus tool. A further advantage of our approach from an economic point of view is that ambiguity aversion is taken into account, see the discussion below. In the present paper, we therefore simply assume that there is a representative liquidity supplier, called the Market Maker, who quotes a price for each volume based on the utility indifference principle and her utility is a $g$-expectation with a cash-invariance property. The existence of the representative agent under such utility functions follows from Horst et al. [28]. While the cash invariance axiom might not be realistic for an individual investor, market makers are often financial institutions for which a cash invariance axiom seems more reasonable. If the driver of the $g$-expectation is a linear function, then the price curve becomes linear in volume and we recover the standard framework of financial engineering. For the individual investor we assume a strictly concave Neumann-Morgenstern utility function $U : \mathbb{R} \to \mathbb{R}$ satisfying standard conditions.

In Subsection 2.1 we first present our market setting. In Subsection 2.2 we describe the utility function of a market maker (which is given by a $g$-expectation) under some assumptions on the function $g$ and the traded securities. In Subsection 2.3 we give the pricing rule of a market maker which arises as utility indifference price. We further show that our assumptions lead to a workable structure of a space of $Z$’s which arise from the optimal control variables. Specifically, this space can be expressed as a non-empty (random) Cartesian product of intervals. Furthermore, the P&L process of a trader can be described as a linear stochastic integral and a non-linear Lebesgue integral in $Z$. In Subsection 2.4 we define the expected utility maximization problem of the Large Trader.

2.1. The basic setting

We assume zero risk-free rates meaning in particular that cash is risk free and does not have any bid-ask spread. Let $T > 0$ be the end of an accounting period. Each agent evaluates her utility based on her wealth at $T$. Consider $n$ securities $S = (S^1, \ldots, S^n)\top$ whose values at time $T$ are exogenously determined, where ‘$\top$’ denotes transpose. We denote the value by $S$ and regard it as an $\mathcal{F}_T$-measurable random variable defined on a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in [0, T]})$ with $(\mathcal{F}_t)$ being the completion of the filtration generated by a $d$-dimensional Brownian motion $W = (W^1, \ldots, W^d)\top$ with $n \leq d$ satisfying the usual conditions. The security $S^i$ can for instance be a zero-coupon bond, an asset backed security, or a derivative with an underlying which is not
traded. The price of this security at \( T \) is trivially \( S \), but the price at \( t < T \) should be \( \mathcal{F}_t \)-measurable and will be endogenously determined by a utility-based mechanism. There are two agents in our model: A Large Trader and a Market Maker. The Market Maker quotes a price for each volume of the security. She can be risk-averse and so her quotes can be nonlinear in volume and depend on her inventory of this security. The Large Trader refers to the quotes and makes a decision. She cannot avoid affecting the quotes by her trading due to the inventory consideration of the Market Maker, and seeks an optimal strategy under this endogenous market impact.

2.2. The evaluation method of the market maker and examples

As the pricing rule of the Market Maker, our model adopts the utility indifference principle, using an evaluation \( \Pi \) given by the solution of a \( g \)-expectation defined as follows. Let \( g(t, \omega, z) : [0, T] \times \Omega \times \mathbb{R}^d \to \mathbb{R} \) be a \( \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \) measurable function, where \( \mathcal{P} \) is the progressively measurable \( \sigma \) field, such that \( z \mapsto g(t, \omega, z) \) is a convex function with \( g(t, \omega, 0) = 0 \) for each \((t, \omega) \in [0, T] \times \Omega\). As usual the \( \omega \) will typically be suppressed. For a stopping time \( \tau \) we denote by \( \mathcal{D}_\tau \) a linear space of \( \mathcal{F}_\tau \)-measurable random variables to be specified below. We denote by \( L^0(\mathcal{F}_\tau) \) the space of \( \mathcal{F}_\tau \)-measurable random variables, by \( L^2(\mathcal{F}_\tau) = L^2(\Omega, \mathcal{F}_\tau, \mathcal{P}) \) the space of \( \mathcal{F}_\tau \)-measurable random variables which are square integrable with respect to \( \mathcal{P} \), by \( L^2(d\mathcal{P} \times dt) \) the space of progressively measurable processes which are square integrable with respect to \( d\mathcal{P} \times dt \) and by \( L^\infty(\mathcal{F}_\tau) \) the space of \( \mathcal{F}_\tau \)-measurable random variables which are bounded. Products of vectors will be understood as vector products. Equalities, inequalities and inclusions are understood a.s. unless stated otherwise. Throughout the rest of this paper, we always assume either one of the following conditions:

(Hg) \( g(t, z) = \alpha |z|^2 + l(t, z) \) where \( \alpha > 0 \) is convex in \( z \) and \( l \) is Lipschitz in \( z \) (also uniformly in \( t \) and \( \omega \)) and continuously differentiable in \( z \) with \( l(t, 0) = 0 \). Furthermore,

\[ \mathcal{D}_\tau = \{ H \in L^0(\mathcal{F}_\tau) \mid \mathbb{E}[\exp(|a|H)] < \infty \text{ for all } a > 0 \} \text{.} \]

(HL) \( g(t, z) \) is convex, Lipschitz in \( z \) (also uniformly in \( t \) and \( \omega \)) with \( g(t, 0) = 0 \) and \( \mathcal{D}_\tau = L^2(\Omega, \mathcal{F}_\tau, \mathcal{P}) \).

For a convex function \( g \), the subgradient of \( g \) at \( z \in \mathbb{R}^{1 \times d} \) is defined as

\[ \nabla g(t, z) := \{ y \in \mathbb{R}^d \mid g(t, \tilde{z}) - g(t, z) \geq (\tilde{z} - z)y \text{ for all } \tilde{z} \in \mathbb{R}^{1 \times d} \} \text{.} \]

If \( g \) is differentiable in \( z \), then the subgradient has only one element, the derivative of \( g \), henceforth, denoted by \( \nabla g \). It follows directly from the definition of a subgradient that a convex function \( g \) is minimized in a \( z \) if and only if \( 0 \in \nabla g(t, z) \). For a concave function \( f \) we define \( \nabla f := -\nabla(-f) \).

For a set \( M \subset \mathbb{R}^d \) we set \( a + bM := \{ a + bm \mid m \in M \} \) for \( a, b \in \mathbb{R} \). Next, we will define the utility dynamic evaluation method \( \Pi_t \) of the market maker. Let \( g \) be given as above.

It is well-known, see e.g. [36, 11] that under (HL) or (Hg) for each \( H \in \mathcal{D}_T \), there exist unique progressively measurable square integrable processes \( (\Pi(H), Z(H)) = (\Pi(H), Z^1(H), \ldots, Z^d(H)) \) such that for all \( t \in [0, T] \), it holds that

\[ H = \Pi_t(H) + \int_t^T g(s, Z_s(H)) ds - \int_t^T Z_s(H) dW_s, \tag{2.1} \]

where \( \int_t^T Z_s(H) dW_s = \sum_{j=1}^d \int_t^T \dot{Z}^j_s(H) dW_s^j \). Equation (2.1) in differential form can equivalently be written as

\[ d\Pi_s(H) = g(s, Z_s(H)) ds - Z_s(H) dW_s, \quad \Pi_T(H) = H. \tag{2.2} \]
(2.1) or (2.2) is also called a backward stochastic differential equation (BSDE) with driver function $g$ and terminal condition $H$. Sometimes $\Pi(H)$ is also called a $g$-expectation of $H$. Under our assumptions it is shown for instance in [32] that $\Pi$ is concave, cash invariant (meaning that $\Pi_t(H + m) = \Pi_t(H) + m$ for $m \in D_t$) and monotone (meaning that $\Pi_t(H_1) \leq \Pi_t(H_2)$ if $H_1 \leq H_2$). Furthermore, $\Pi$ satisfies the tower property (time-consistency).

Example 1 ($g$-expectation). 1. The simplest example is given by the case where $g$ is linear in $z$, i.e., $g(s, z) = -\eta_k z$ for a bounded progressively measurable process $\eta$. By the Girsanov Theorem,

$$\Pi_t(H) = E^Q[H|\mathcal{F}_t] \quad (2.3)$$

with $Q$ having the Radon-Nikodym derivative with stochastic logarithm $\eta$, i.e.,

$$\frac{dQ}{dP} := \exp\left\{\int_0^T -\eta_s dW_s - \frac{1}{2} \int_0^T |\eta_s|^2 ds\right\}. \quad (2.4)$$

In this case we set $D_t = L^2(\Omega, \mathcal{F}_t, P)$.

2. Another example of $g$-expectation is the following exponential utility-based equivalent:

$$\Pi_t(H) = -\frac{1}{\gamma} \log E^Q[\exp(-\gamma H)|\mathcal{F}_t],$$

with $D_t = \{H \in L^0(\mathcal{F}_t)|E[\exp(a|H|)] < \infty \text{ for all } a > 0\}$, where $\gamma > 0$ is a parameter of risk-aversion and $Q$ is defined through (2.4). In this case $\Pi$ is a $g$-expectation with driver function $g(t, z) = \frac{1}{2}\gamma|z|^2 - \eta_t z$, see Barrieu and El Karoui [6].

3. We remark that there is a close connection between $g$-expectations and convex risk measures, which (modulo a change of sign) are cash-invariant, monotone and convex, see e.g., Barrieu and El Karoui [6]. A $g$-expectation is a convex risk measure and satisfies time consistency if and only if $-g$ is convex and $g(t, 0) = 0$, see Jiang [32]. In particular, $-\Pi(-\cdot)$ is a convex risk measure. It is worth noting that under additional compactness or domination assumptions, also every time-consistent convex risk measure corresponds to a $g$-expectation. For these and other related results, see Coquet et al. [13]

2.3. The pricing rule and the trading with permanent market impact

Suppose that the Market Maker has initially an endowment whose cash-flow at time $T$ is represented by a bounded random variable $H_M$. Additionally, for each quantity of the risky assets $S = (S^1, \cdots, S^n)$ with $n \geq 1$ the Market Maker quotes a price. Below, we assume that $S$ is either square integrable if (HL) holds or bounded if alternatively (Hg) holds. If the Market Maker at time $t \in [0, T]$ is holding $z^i$ units of the security $S^i$ in question besides $H_M$, then her utility is measured as $\Pi_t(H_M + \sum_{i=1}^n z^i S^i) = \Pi_t(H_M + zS)$. According to the utility indifference principle, the Market Maker quotes a selling price for $y = (y^1, \ldots, y^n)$ units of the security by

$$P_t(z, y) := \text{essinf } \{p \in D_t : \Pi_t(H_M + zS - yS + p) \geq \Pi_t(H_M + zS)\}$$

$$= \Pi_t(H_M + zS) - \Pi_t(H_M + (z - y)S). \quad (2.5)$$

For the above equality we have used the cash invariance property. Note that in the risk-neutral case (2.3) in the previous example, we have $P_t(z, y) = \sum_{i=1}^n y^i E^Q[S^i|\mathcal{F}_t] = y E^Q[S|\mathcal{F}_t]$, where we used vector notation.

Let $\Theta_0$ be the set of simple $n$-dimensional left-continuous processes $\theta$ with $\theta_0 = 0$. The Large Trader is allowed to take any element $\theta \in \Theta_0$ as her trading strategy. The price for the $y$ units
of the security at time $t$ is $P_t(-\theta_t, y)$. This is because the Market Maker holds $-\theta_t$ units of the security due to the preceding trades with the Large Trader. Then the profit and loss (P&L) at time $T$ associated with $\theta \in \Theta_0$ (i.e., corresponding to the self-financing strategy $\theta$) of the Large Trader is given by

$$\mathcal{I}(\theta) := \theta_T S - \sum_{0 \leq t < T} P_t(-\theta_t, \Delta \theta_t). \quad (2.6)$$

Due to Proposition 2.4 below, $\mathcal{I}(\theta)$ has the form of a nonlinear stochastic integral studied in Kunita [38]. Note that in the risk-neutral case (2.3), $\mathcal{I}(\theta) = \theta_T S_T - \sum_{0 \leq t < T} \Delta \theta_t S_t = \int_0^T \theta_t dS_t$ by integration-by-parts, where $S_t = \mathbb{E}^Q[S_t | \mathcal{F}_t]$. For any $y \in \mathbb{R}^{1 \times n}$, using the BSDE representation (2.1), we set $\Pi(y, \omega, t, \theta) = H(M - yS)$. Note that for each fixed $y$, $\mathcal{Z}^y$ is only defined $d\mathbb{P} \times dt$ a.s. To extend trading to general predicting trading strategies we need more structure for $\mathcal{Z}^y$ and will from now on always assume that the following assumption holds:

**Assumption (H1) (Additive separability):** Define the diffusion process $\mathcal{R} = (\mathcal{R}^1, \ldots, \mathcal{R}^n)$ to be the solution to the following SDE: $\mathcal{R}^{i, t, r, \theta} = r^i$, $0 \leq u \leq t$, and

$$d\mathcal{R}^{i, t, r, \theta} = \mu^i(u, \mathcal{R}^{i, t, r, \theta}) du + \sigma^i u dW^i_u, \quad t \leq u \leq T, \quad i = 1, \ldots, n,$$

(2.7)

where $\mu^i : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable with respect to its second argument with bounded derivative, and $\sigma^i u$ is a deterministic bounded process taking values in $\mathbb{R}^{1 \times d}$. $\mathcal{R}^i$ could for instance reflect the returns or dividends of $n$ independent risky assets. We then assume that $S$ and $H_M$ are of the form

$$S = \left(s^1(\mathcal{R}^1_T), \ldots, s^n(\mathcal{R}^n_T)\right) \quad \text{and} \quad H_M = \sum_{i=1}^n h^{M,i}(\mathcal{R}^i_T) + H^{\perp}_M,$$

with $\mathcal{R}_0 = (r_0^1, \ldots, r_0^n)$, where $H^{\perp}_M = h^{\perp}(W^{n+1}, \ldots, W^d)$ is assumed to be measurable with respect to the filtration generated by $(W^{n+1}, \ldots, W^n)$ and $s^i, h^{M,i} : \mathbb{R} \to \mathbb{R}$ are functions in $C^1$ with derivatives growing at most polynomially and first derivatives bounded for $i = 1, \ldots, n$. Furthermore suppose that, $g$ does not depend on $\omega$, is three times differentiable with all derivatives continuous on $[0, T] \times \mathbb{R}^d$, and is of the form

$$g(t, z) = \sum_{i=1}^d g^i(t, z^i), \quad (2.8)$$

for functions $g^i : [0, T] \times \mathbb{R} \to \mathbb{R}$ with $g^i(t, 0) = 0$ satisfying (HL) or (Hg). The separation condition (2.8) is rather simple and can be checked immediately and is satisfied in many examples in the BSDE literature, see for instance [21]. It postulates that the risk due to each noise generator $W^i$ is penalized separately.

**Remark 2.1. In the case of a one dimensional risky asset, it is possible to, instead of (H1), assume that $g$ is positively homogenous, which is actually equivalent modulo a compactness assumption to $\Pi$ being a coherent risk measure, see for instance [13]. Since then the mathematical analysis is rather different complicating the exposition, we will not consider this case in this paper.**

**Proposition 2.2. There exists a $\mathbb{P} \otimes \mathbb{B}(\mathbb{R}^n)$- measurable, mapping $Z : \Omega \times [0, T] \times \mathbb{R}^n \to \mathbb{R}^d$ such that $Z^y(\omega) = Z(\omega, t, y) d\mathbb{P} \times dt$ almost surely for each fixed $y \in \mathbb{R}^n$. Furthermore, $Z$ can (and will in the sequel) be chosen such that the mapping $y \mapsto Z(\omega, t, y)$ is continuous $d\mathbb{P} \times dt$ a.s., and $

\|\sup_t |Z(t, \omega, y)|\|_\infty \leq K(1 + |y|)$ for some $K > 0.$
The next proposition shows that the image space of $Z^y$ can be expressed as a non-empty Cartesian product of random intervals. Denote the $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable image space of $Z$ as

$$\text{Im}(Z) = \{Z(t,\omega,y)\mid t \in [0,T], \omega \in \Omega, y \in \mathbb{R}^n\}.$$  

Proposition 2.3. $\text{Im}(Z)$ is a multi-dimensional non-empty (random) interval, $I = (I_t(\omega))_{t,\omega}$, of the form $I_t(\omega) := I_t^1(\omega) \times \ldots \times I_t^n(\omega) := [a^1_t(\omega), b^1_t(\omega)] \times \ldots \times [a^n_t(\omega), b^n_t(\omega)] \times \{Z_t^{n+1}(H_M(\omega))\} \times \ldots \times \{Z_t^{n}(H_M(\omega))\} \; d\mathcal{P} \times dt$ a.s. where $a^i \leq b^i$ are progressively measurable processes possibly taking the values $\pm \infty$, and all intervals may also be open or half-open. Therefore,

$$\text{Im}(Z) = I = (I_t(\omega))_{t,\omega}, \quad \text{and we write } \text{Im}(Z(t,\omega,\cdot)) = I_t(\omega).$$

Moreover, $Z(H_M)$ is bounded.

The next proposition shows how to extend trading from simple strategies to continuous time.

Proposition 2.4. Let $\mathcal{I}(\theta^m)$ be the Market Maker’s profit-loss defined in (2.6) for simple trading strategies $\theta^m \in \Theta_0$. If $\theta \in L^2(\mathcal{P} \times dt)$ and $\theta^m \to \theta$ in $L^2(\mathcal{P} \times dt)$, then we have

$$\lim_{m \to \infty} \mathcal{I}(\theta^m) = \mathcal{I}(\theta) := H_M - \Pi_0(H_M) - \int_0^T g(t, Z_t^\theta) dt + \int_0^T Z_t^\theta dW_t,$$

where $Z_t^\theta(\omega) := Z(\omega, t, \theta_t(\omega))$ and the limit holds in $L^2$ under (HL) and $L^1$ under (Hg).

Remark 2.5. The profit and loss until time $t$ measured in terms of its market value is given by

$$\mathcal{I}_t(\theta) = \Pi_t(H_M) - \Pi_0(H_M) - \int_0^t g(s, Z_s^\theta) ds + \int_0^t Z_s^\theta dW_s.$$

Remark 2.6. $(\Pi_t(H_M), Z_t(H_M))$ solves the BSDE

$$H_M = \Pi_t(H_M) + \int_t^T g(s, Z_s(H_M)) ds - \int_t^T Z_s(H_M) dW_s.$$

By definition $Z_0^1 = Z_1(H_M)$ and in particular $\mathcal{I}_t(0) = 0$. Finally we remark that if $H_M = 0$ (i.e. if the Market Maker does not have an endowment), we clearly have $(\Pi_t(H_M), Z_t(H_M)) = (0,0)$ since $g(t,0) = 0$.

2.4. Expected utility maximization

Motivated by Proposition 2.4 we define the set of admissible strategies as

$$\Theta := \{\theta : \Omega \times [0,T] \to \mathbb{R}^n \text{ progressively measurable with } \mathbb{E}\left[\int_0^T |Z_t^\theta|^2 dt\right] < \infty\}.$$  

We furthermore see that $Z \in L^2(\mathcal{P} \times dt)$ is admissible if there exists $\theta \in \Theta$ such that $Z = Z^\theta$. As can be seen from Proposition 2.4 and the definition of $\Theta$, $Z(t,\omega,y)$ plays a crucial role for our analysis.

Recall that $Z_t^\theta(\omega) = Z(\omega, t, \theta_t(\omega))$. By Proposition 2.4 and Remark 2.5 the portfolio value at time $t$ is given by $X_t^\theta = x_0 + \mathcal{I}_t(\theta)$ with

$$\mathcal{I}_t(\theta) := - \int_0^t g(s, Z_s^\theta) ds + \int_0^t Z_s^\theta dW_s + \Pi_t(H_M) - \Pi_0(H_M), \quad (2.9)$$
which can be considered as the gain/loss in the time interval $[0, t]$. Note that by definition of $\mathcal{I}_t(\theta)$ we have that $-(X_t - \Pi_t(H_M) + \Pi_0(H_M), Z^{\theta}_t)$ is a solution of a BSDE with driver $g$ and terminal condition $-(X_0^\theta - H_M + \Pi_0(H_M))$ (since it satisfies (2.2)). In particular, setting $t = 0$ we have, $x_0 = X_0^\theta = -\Pi_0(-X_0^\theta - H_M + \Pi_0(H_M))$. Furthermore, in case that $H_M = 0$ we have that $\Pi_t(H_M) = \Pi_0(H_M) = 0$ and therefore $x_0 = -\Pi_0(-X_0^\theta)$.

Below, we study the following utility maximization problem for the Large Trader

$$
\sup_{\theta \in \Theta} E[U(X_T^\theta + H_L)],
$$

(2.10)

where $\theta_t$ is the number of risky assets held at time $t$, and $X_0^\theta + H_L = x_0 + H_L$ is the initial endowment with $H_L \in L^\infty(\mathcal{F}_T)$. In (2.10) we use the convention that $E[U(X_T^\theta)] = -\infty$ if for the negative part we have $E[U^-(X_T^\theta)] = E[U(X_T^\theta)1_{U(X_T^\theta) < 0}] = -\infty$. We assume that the utility function $U$ is a strictly increasing, strictly concave and three times differentiable function. Note that problem (2.10) can be restated as $\sup_{\theta \in \Theta} E[U(x_0 + \mathcal{I}_T(\theta) + H_L)]$. The rest of the paper is dedicated to analyze problem (2.10).

3. An FBSDE approach

In this section we show first that an optimal solution of the utility maximization problem (2.10) necessarily involves a coupled FBSDE. Conversely, when suitable integrability conditions are met, the solution to a coupled FBSDE corresponds to an optimal strategy. Subsequently, we give existence results of the associated FBSDEs and discuss numerical approximations.

Recall that, $g(t, z) = \sum_{i=1}^d g(t, z^i)$. Below, we define the extension of the function $g^i$ beyond the image space of the $i$-th component of $Z$ as

$$
g^i(t, z^i) = \begin{cases}
g^i(t, z^i) & \text{if } z^i \in \overline{cl(\mathbb{I}_i)}, \\
\infty & \text{else},
\end{cases}
$$

(3.1)

$i = 1, \ldots, d$, where we denote $\overline{cl(\mathbb{I}_i)}$ as the pointwise closure of the set $\mathbb{I}_i(\omega)$ for each $(t, \omega)$ (see Proposition 2.3). Note that $\overline{g^i}$ and $g^i$ coincide on the set of attainable $z^i$, and for $i = 1, \ldots, n$ we have $\overline{cl(\mathbb{I}_i)} = [a^i_1, b^i_1]$, while for $i = n+1, \ldots, d$ we have $cl(\mathbb{I}_i) = [Z^i_t(H_M), Z^i_t(H_M)]$. Define $\overline{g} = \sum_{i=1}^n \overline{g^i} + \sum_{i=n+1}^d g^i$. The reason for this extension is that we can transform the problem (2.10) of maximizing over all strategies, into the problem of maximizing over all admissible $Z^\theta$. However, if the set $\mathbb{I}$ is not closed, the maximum of the latter problem may be attained in an admissible $Z$. In this case, the portfolio optimal problem (2.10) does not have a solution, see Theorem 3.3 and Remark 3.4 below. Hence, the boundary points of $\mathbb{I}$ play an important role in the analysis.

3.1. Necessary and sufficient conditions

The following theorem gives necessary optimality conditions.

**Theorem 3.1.** Suppose that $\theta^*$ is an optimal strategy of Problem (2.10), $E[U(X_T^{\theta^*} + H_L)] < \infty$ and $E[U^+(X_T^{\theta^*} + H_L)]^{1+\epsilon} < \infty$ with $\epsilon > 0$. Then there exists a continuous adapted process $\zeta$ with $\mathcal{I}_T = H_L$ such that $U'(X^{\theta^*} + \zeta)$ is a martingale process, and setting $M^\zeta_t := d\langle \zeta, W \rangle_t / dt$ for $i = 1, \ldots, n$, we have

$$
0 \in U''(X_t^{\theta^*} + \zeta_t)\left(Z_t^{\theta^*}, -Z_t^i(H_M) + M^\zeta_t\right) - U'(X_t^{\theta^*} + \zeta_t)\nabla g^i(t, Z_t^{\theta^*}) dt + dP \times dt \text{ a.s.}
$$

(3.2)
For a triple of adapted processes \((X, \zeta, M)\) (which solves the FBSDE system (3.5) below), we define \(\mathcal{H}(t, X_t, \zeta_t, M_t) := (\mathcal{H}^1(t, X_t, \zeta_t, M_t^1), \ldots, \mathcal{H}^n(t, X_t, \zeta_t, M_t^n), Z_t^{n+1}(H_M), \ldots, Z_t^n(H_M))\), with
\[
\mathcal{H}^i(t, X_t, \zeta_t, M_t^i) := a_t^i \vee \tilde{U}_t^{i-1}(H_t^i, \zeta_t^i, M_t^i)(0) \land b_t^i \quad \text{for } i = 1, \ldots, n, \tag{3.3}
\]
and \(\tilde{U}_t^{i}(H_t^i) := -U'(X + \zeta)g_z^i(t, H_t^i) + U''(X + \zeta)(H_t^i - Z_t^n(H_M) + M_t^i)\), see Proposition 2.3 for the definition of \(a_t^i\) and \(b_t^i\). Note that \(\mathcal{H}\) only takes values in \(cl(\mathbb{I}) = cl(Im(Z))\). Below we show that the optimal strategy can be characterized by a solution of a fully-coupled forward-backward system.

**Theorem 3.2.** Under the assumptions of Theorem 3.1, the optimal strategy for Problem (2.10) is characterized by
\[
\mathcal{Z}_t^{\theta, i} = \mathcal{H}^i(t, X_t, \zeta_t, M_t^i) \quad i = 1, \ldots, n, \tag{3.4}
\]
where the function \(\mathcal{H}^i\) is given by (3.3) and \((X, \zeta, M)\) is a triple of adapted processes which solves the following FBSDE
\[
\begin{align*}
X_t &= x_0 - \int_0^t g(s, \mathcal{H}(s, X_s, \zeta_s, M_s))ds + \int_0^t \mathcal{H}(s, X_s, \zeta_s, M_s)dW_s + \Pi_t(H_M) - \Pi_0(H_M), \\
\zeta_t &= H_L - \int_t^T M_s dW_s + \int_t^T g(s, Z_s(H_M)) - g(s, \mathcal{H}(s, X_s, \zeta_s, M_s))ds \\
&\quad + \int_t^T \frac{1}{2} \frac{U(3)}{U''} (X_s + \zeta_s) \mathcal{H}(s, X_s, \zeta_s, M_s) - Z_s(H_M) + M_s^2 ds
\end{align*}
\]  
\(-\frac{U(3)}{U''}\) is also called prudence in the decision theoretic literature. Note that the utility function of the large investor influences the FBSDE and the optimal solution only through the levels of the underlying prudence it induces.

**Example 2.** The functional \(\mathcal{H}\) defined above can be computed explicitly for special cases. For example:

- If \(g(t, z) = c_t z + d_t\) for some deterministic functions \(c, d\) and \(U\) is a CARA function, meaning that \(U(x) = -e^{-x\gamma}\), with a risk aversion coefficient \(\gamma \in (0, \infty)\), then we have \(\mathcal{H}^i(t, X, \zeta, M^i) = a_t^i \vee (-c_t^i/\gamma - M_t^i) \land b_t^i\), for \(i = 1, \ldots, n\), which is independent of \((X, \zeta)\).
- Similarly, if \(g(t, z) = c_t |z|^2/2\) and \(U\) is a CARA utility function we get \(\mathcal{H}^i(t, X, \zeta, M^i) = a_t^i \vee (-c^2_t/(\gamma + c_t^i)) \land b_t^i\), for \(i = 1, \ldots, n\), which is also independent of \((X, \zeta)\).

Let us study the other direction of Theorem 3.2. Below we show that an optimal strategy can be obtained from the solution of an FBSDE system. To begin, let
\[
\psi_1(x) := \frac{U'}{U''}(x) \leq 0 \quad \text{and} \quad \psi_2(x) := \frac{U(3)}{U''}(x).
\]

\(-1/\psi_1\) and \(-\psi_2\) are also called the risk aversion and the prudence of the investor, respectively.

**Theorem 3.3.** Suppose that \(\psi_1\) is bounded, and let \((X, \zeta, M)\) be a triple of adapted processes which solves the FBSDE (3.5), and satisfies
\[
\mathbf{E}[U'(X_T + H_L)^2] < \infty, \quad \mathbf{E}[|U(X_T + H_L)|] < \infty, \quad \mathbf{E}[\int_0^T |M_t|^2 dt] < \infty.
\]
Furthermore, assume that either (a) \(\sup_t U'(X_t + \zeta_t)\) is integrable or (b) \(\psi_2\) is bounded and in case that \(d > n\) additionally \(\psi_1\) is bounded away from zero. Then, the solution of the problem
\[
\mathbf{E}[U(X_T^2 + H_L)] \tag{3.6}
\]
\(Z\) takes values in \(cl(\mathbb{I})\), \(Z \in \mathcal{L}(dP \times ds)\).
is given by
\[
(Z^*_t)_t = ((H^1_t(t, \zeta, M^1_t), \ldots, H^n_t(t, \zeta, M^n_t), Z^{1+1}_t(H_M), \ldots, Z^d_t(H_M)))_t \in \text{cl}(\mathbb{I}).
\]
Furthermore, the optimal portfolio choice Problem (2.10) has a solution \( \theta^* \) if and only if there exists a version of \( Z^* \) such that \( Z^* \in \mathbb{I} \). In this case, \( \theta^*_t \) is given by (3.7) below.

**Remark 3.4.** Without the “cl”, Problem (3.6) corresponds to the optimal portfolio problem (2.10).

Now under the conditions of Theorem 3.3 above we always have therefore \( Z^* \in \text{cl}(\mathbb{I}) = \text{cl}(\text{Im}(Z)) \). On the other hand, \( Z^* \) might be in the boundary of \( \text{cl}(\mathbb{I}) \), but on a \( d\mathbb{P} \times dt \) non-zero set not in \( \mathbb{I} = \text{Im}(Z) \) itself. In this case an admissible optimal strategy \( \theta^* \) does not exist.

Using a measurable selection theorem, for \( i = 1, \ldots, n \), there exists a \( \mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \)-measurable function \( \tilde{Z}^i: \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \) such that \( Z^i(\omega, t, \tilde{Z}^i(\omega, t, z)) = z^i \) for \( z^i \in \mathbb{I}^i \). By the definition of \( H^i \) we have \( H^i(t, X^i_{\theta^*}, \zeta_t, M^i_t) \) \( \in \text{cl}(\mathbb{I}^i) = [a^i_1, b^i_2] \). If \( H^i(t, X^i_{\theta^*}, \zeta_t, M^i_t) \) actually belongs to \( \mathbb{I}^i \) for \( i = 1, \ldots, n \), the optimal strategy \( \theta^* \) is therefore given by
\[
\theta^*_t = \tilde{Z}^i(t, H^i(t, X^i_{\theta^*}, \zeta_t, M^i_t)), \quad i = 1, \ldots, n.
\]

The next proposition gives cases where it is optimal to invest everything in the riskless asset.

**Proposition 3.5.** Suppose that \( H_L = 0 \) and \( 0 \in \nabla g^i(t, Z^*_t(H_M)) \) for \( i = 1, \ldots, n \). Then the optimal terminal wealth is given by \( X^*_T = x_0 \). This means that it is optimal for the Large Trader to invest nothing, i.e., \( \theta^* = 0 \). In addition, the triple \( (X^*_t = x_0, \zeta^*_t = 0, M^*_t = 0) \) is a solution of the FBSDE system (3.5).

If one wants to compute the optimal solution numerically in the general case, a possible way suggested by our results is to: a) in a first step solve the coupled FBSDEs (3.5), to obtain the optimal \( Z^* \) given by equation (3.4). Then step b): once the optimal \( Z^* \) is found, invert the parameterized family of decoupled FBSDEs or semi-linear PDEs in order to find the corresponding \( \theta^* \) such that \( Z^* = Z^{\theta^*} \). For both steps there are by now numerical algorithms available, see [17], [60], [30], and [24], or the earlier work [19]. Note that step b) can actually be implemented by solving the decoupled FBSDEs for fixed \( \theta \), for values of \( \theta \) on a grid. Two further remarks are in order:

i) As shown in Section 5, in the case of a complete financial market step a) can be avoided. A complete financial market means that for any suitable integrable \( \mathcal{F}_T \)-measurable random variable \( H \) there exists \( (a, \theta) \in \mathbb{R} \times \Theta \) such that \( H = a + I_T(\theta) \), with \( I_T(\theta) \) corresponding to the P&L of the strategy \( \theta \). Sufficient conditions when such a completeness condition holds can be found in [22].

ii) The optimal \( Z^* \) in our setting becomes explicit (see Proposition 5.2 below) if the Market Maker uses an exponential utility function. It is worthwhile noting that in a setting without price impact the optimal strategy also typically becomes explicit only when the utility function is power.

### 3.2. Existence results & uniqueness for coupled FBSDEs

In this section we show that the FBSDE (3.5) admits a solution under appropriate assumptions on \( g \) and the utility function \( U \).

**Theorem 3.6.** Suppose that one of the following conditions holds:

(i) There exists \( K, \bar{c} > 0 \) such that \( |U'(x)|^{1+\bar{c}} \leq K(1 + |x| + |U(x)|) \) for all \( x \leq \text{const} \), and \( g \) grows at least quadratically, meaning that there exists \( K_1, K_2 > 0 \) such that
\[
g(t, z) \geq -K_1 + K_2|z|^2.
\]
\[(\text{i})\] There exists \(K, K_1, K_2 > 0\) such that \(|U'(x)|^{1+\varepsilon} \leq K(1+|x|^2+|U(x)|)\) and \(U(x) \leq K_1-K_2|x|^2\) both for all \(x \leq \text{const}\). Furthermore, there exists an \(w_0 \in \mathbb{R}\) such that for all \(x \leq w_0\), \(U\) is strictly increasing, and \(U(x)\) is constant for all \(x \geq w_0\).

\[(\text{iii})\] \(U\) is exponential, i.e., \(U(x) = a - be^{-x/\gamma}\) for \(a \in \mathbb{R}\) and \(b, \gamma > 0\).

Then there exists a solution to the FBSDE \((3.5)\).

**Remark 3.7.** It is shown in the appendix that actually under one of the conditions \((\text{i})-(\text{ii})\) the maximization problem \((3.6)\) above has a solution.

**Remark 3.8.** Although throughout the paper we require \(U\) besides being concave to be strictly increasing and three times continuously differentiable on \(\mathbb{R}\), for Theorem 3.6(ii) we actually only need these conditions on \((-\infty, w_0]\) as we will see in the proof that the optimal solution will only take values in this interval. The classical examples for a utility satisfying \((\text{ii})\) are quadratic utility functions.

### 4. Connection with BSPDEs

In this section we characterise the value function of our expected utility maximization problem \((2.10)\) by a BSPDE which results from a direct application of the Itô-Ventzel formula for regular families of semimartingales. Setting \(H_M = 0\) we first introduce

\[
\mathcal{I}_{s,t}(\theta) := -\int_s^t g(u, Z_u^\theta)du + \int_s^t Z_u^\theta dW_u, \quad 0 \leq s \leq t \leq T,
\]

which represents the total gain/loss of the strategy \(\theta\) in \([s,t]\. For any \(t \in [0,T]\) and \(x \in \mathbb{R}\) we define

\[
V(t,x) := \text{ess sup}_{\theta \in \Theta, \theta_s, \theta_t \in [t,T]} E \left[ U(x + \mathcal{I}_{t,T}(\theta) + H_L) \big| \mathcal{F}_t \right].
\]  

(\text{CV}) For any \(t \in [0,T]\) and \(x \in \mathbb{R}\), the supremum in \((4.1)\) is attained, i.e., there exists an admissible strategy \(\theta^*(x), s \in [t,T]\) such that \(V(t,x) = E \left[ U(x + \mathcal{I}_{t,T}(\theta^*(x)) + H_L) \big| \mathcal{F}_t \right]\).

Sufficient conditions for \((\text{CV})\) to hold are given in Theorem 3.6 (guaranteeing existence of an optimal \(Z^*\) solving Problem \((3.6)\)) and in Theorem 3.3.

**Proposition 4.1.** The value function \(V(t,x)\) is strictly concave with respect to \(x\) meaning that for all \(\lambda \in (0,1), V(t, \lambda x_1 + (1-\lambda)x_2) > \lambda V(t, x_1) + (1+\lambda)V(t, x_2)\) a.s. if \(x_1 \neq x_2\).

The following proposition shows a dynamic programming principle.

**Proposition 4.2.** Let \(\theta\) be admissible and \(s \in [0, T]\), then the process \(\{V(t,x + \mathcal{I}_{s,t}(\theta))\}, t \geq s\) is a supermartingale for all \(x \in \mathbb{R}\). Furthermore,

\[
V(s,x) = \text{ess sup}_{\theta \in \Theta, \theta_s, \theta_t \in [s,T]} E \left[ V(t,x + \mathcal{I}_{s,t}(\theta)) \big| \mathcal{F}_s \right]
\]

and a strategy \(\theta^*\) is optimal if and only if \(V(t,x + \mathcal{I}_{s,t}(\theta^*))\) is a martingale process for every \(s\).

We recall that for any \(x \in \mathbb{R}\), the process \(V(t,x), t \in [0,T]\) is a supermartingale admitting an RCLL modification (see e.g. Theorem 9 in [52]). Its Galtchouk-Kunita-Watanabe (GKW) decomposition is given by \(V(t,x) = V(0,x) - A(t,x) + \int_0^t \alpha(s,x) dW_s\), where \(A(t,x)\) is an increasing process, and \(\alpha\) is a progressively measurable and square integrable process. As in [43, 45], we define a regular family of semimartingales as follows:
Definition 4.3 (regular family of semimartingales). The process \( V(t, x) : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R} \) is a regular family of semimartingales if

(a) \( V(t, x) \) is twice continuously differentiable with respect to \( x \) for any \( t \in [0, T] \).
(b) For any \( x \in \mathbb{R} \), \( V(t, x), t \in [0, T] \) is a special semimartingale with progressively measurable finite variation part \( A(t, x) \) which admits the representation \( A(t, x) = \int_0^t b(s, x)ds \), where \( b(s, x) \) is progressively measurable, i.e.,

\[
V(t, x) = V(0, x) - \int_0^t b(s, x)ds + \int_0^t \alpha(s, x)dW_s.
\] (4.3)

(c) For any \( x \in \mathbb{R} \), the derivative process \( V_x(t, x) \) is a special semimartingale with decomposition

\[
V_x(t, x) = V_x(0, x) - \int_0^t b_x(s, x)ds + \int_0^t \alpha_x(s, x)dW_s,
\]

where \( \alpha_x \) and \( b_x \) denote the derivative of \( \alpha \) and \( b \) with respect to \( x \) respectively.

We in addition assume that the following condition holds for the coefficients of the regular family of semimartingales \( V(t, x), t \in [0, T] \):

\textbf{(CR)} The functions \( b(t, x), \alpha(t, x) \) and \( \alpha_x(t, x) \) in Definition 4.3 are continuous with respect to \( x \) and satisfy, for any constant \( c > 0 \),

\[
\mathbb{E} \left[ \int_0^T \max_{|x| \leq c} (|b(t, x)|, |\alpha(t, x)|^2, |\alpha_x(t, x)|^2)ds \right] < \infty.
\]

Condition (CR) is standard in the analysis of stochastic flows and typically satisfied in examples, see for instance Section 5. Below, \( \mathcal{V}^{1,2} \) denotes the class of all regular families of semimartingales \( V \) defined by Definition 4.3 whose coefficients \( b \) and \( \alpha \) satisfy Condition (CR). Recall also that a process \( V \) belongs to the class \( D \) if the family of processes \( V_t1_{\tau \leq T} \) for all stopping times \( \tau \) is uniformly integrable. The next lemma prepares the ground for showing that under appropriate smoothness conditions, the value function solves a BSPDE.

Lemma 4.4. There exists a unique progressively measurable process denoted by \( v(t, x) \) such that the supremum of

\[
\mathcal{L}^V(t, x) := \text{ess sup}_{Z \in \mathcal{C}_{\mathcal{L}}(M)} \left( -g(t, Z)V_x(t, x) + \frac{1}{2} |Z|^2V_{xx}(t, x) + Z\alpha_x(t, x) \right),
\] (4.4)

is attained at \( Z^* = v(t, x) \). In particular, \( v(t, x) \) satisfies the first order condition \( 0 \in \mathcal{U}^V(t, v(t, x), x) \), where \( \mathcal{U}^V(t, z, x) := -\nabla g^T(t, z)V_x(t, x) + zV_{xx}(t, x) + \alpha_x(t, x) \).

We can now present the two main theorems of this section.

Theorem 4.5. Suppose that there exists a modification of the value function, also denoted by \( V \), which is in \( \mathcal{V}^{1,2} \). Then \( V \) is a solution of the BSPDE

\[
V(t, x) = V(0, x) + \int_0^t \alpha(s, x)dW_s - \int_0^t \mathcal{L}^V(s, x)ds,
\] (4.5)

where \( V(T, x) = U(x + H_L) \) and the operator \( \mathcal{L} \) is defined by (4.4). Moreover, a strategy \( \theta^* \in \Theta \) with \( V(t, X_t^{\theta^*}) \) belonging to class \( D \) is optimal if and only if \( Z_t^{\theta^*} = v(t, X_t^{\theta^*}) \), i.e.

\[
0 \in \mathcal{U}^V(t, v(t, X_t^{\theta^*}), X_t^{\theta^*}) := -\nabla g^T(t, v(t, X_t^{\theta^*}))V_x(t, X_t^{\theta^*}) + v(t, X_t^{\theta^*})V_{xx}(t, X_t^{\theta^*}) + \alpha_x(t, X_t^{\theta^*}).
\] (4.6)

The optimal wealth process \( X_t^{\theta^*} \) is then characterized by the forward SDE

\[
X_t^{\theta^*} = x_0 - \int_0^t g(s, Z_s^{\theta^*})ds + \int_0^t Z_s^{\theta^*}dW_s.
\] (4.7)
We have seen that the value function of an optimal strategy can be characterized by a BSPDE (4.5)-(4.7). Differentiating this BSPDE (assuming all derivatives below exist) we obtain

\[
\begin{aligned}
V_x(t, x) &= V_x(0, x) + \int_0^t \alpha_x(s, x) dW_s - \int_0^t \mathcal{L}^V_x(s, x) ds, \\
X^\theta_t &= x_0 - \int_0^t g(s, Z^\theta_s) ds + \int_0^t Z^\theta_s dW_s.
\end{aligned}
\]

(4.8)

The following theorem gives the connection between the FBSDEs (3.5) and the BSPDE (4.5).

**Theorem 4.6.** Assume all the conditions of Theorem 4.5, that \((V_x(t, x), \alpha_x(t, x), \mathcal{L}^V_x(t, x), X^\theta_t)\) is a solution of the BSPDE (4.8) and that \(V_x(t, x)\) is a regular family of semimartingales. Let \(v(t, X^\theta_t)\) be the unique adapted maximizer process in (4.4) and taking only values in the interior of \(\bar{U}_t\). Then the triple \((X^\theta_t, \zeta_t, M_t)\) defined by

\[
\zeta_t = I(V_x(t, X^\theta_t)) - X^\theta_t, \\
M_t = \frac{v(t, X^\theta_t)V_{xx}(t, X^\theta_t) + \alpha_x(t, X^\theta_t)}{U''(X^\theta_t + \zeta_t)} - v(t, X^\theta_t),
\]

is a solution of the FBSDE (3.5).

**5. Regularity of the value function**

In this section we provide some results on the regularity (see Definition 4.3) of the dynamic value function assuming that \(H_L = 0\). These properties together with the results from the last section imply that the value function satisfies a corresponding BSPDE. We first consider the case of CARA utility functions.

**Proposition 5.1.** Assume \(U(x) = -\frac{1}{\gamma}e^{-\gamma x}\). Then, the value function in (4.1) satisfies conditions (a)-(c) in Definition 4.3 and can be written as \(V(t, x) = U(x)V_t\), where \(V_t\) satisfies the BSDE \(V_t = V_0 + \int_0^t \alpha_s dW_s - \int_0^t \mathcal{L}^V_s ds\), with terminal condition \(V_T = 1\), where the operator \(\mathcal{L}^V\) is defined by \(\mathcal{L}^V := \inf_{Z \in \Omega_t} \left(\gamma g(t, Z)V_t + \frac{1}{2} \gamma^2 |Z|^2 V_t - \gamma Z \alpha_t \right)\). Moreover, the value function satisfies the BSPDE (4.5) with \(\alpha(t, x) = U(x)\alpha\) and \(\mathcal{L}^V\) defined above.

Next let us consider the case of a quadratic \(g\) and \(H_M = H_L = 0\). We show that the optimal investment strategy and its FBSDE characterization can be explicitly determined in settings where the market is complete, i.e., for any \(X \in D_T\) there exists a strategy \(\theta \in \Theta\) such that \(X = -\Pi_0(-X) + \mathcal{I}_T(\theta)\). See [22] for sufficient conditions for market completeness. Furthermore, in this case \(V\) can be shown to be a regular family of semimartingales. To this end, we assume that the utility \(U\) in addition satisfies the Inada condition \(\lim_{x \to -\infty} U'(x) = \infty\) and \(\lim_{x \to +\infty} U'(x) = 0\).

We assume that the Market Maker evaluates the market risks in terms of an exponential utility certainty equivalent principle under an equivalent measure \(\mathcal{Q} \sim \mathcal{P}\). More precisely, for any \(X \in D_T\), we assume that the evaluation II of the market maker is given by \(\Pi_t(X) = -\frac{1}{\gamma} \log \left[ E^\mathcal{Q} [e^{-\gamma X} | \mathcal{F}_t] \right] \), where the constant \(\gamma > 0\) is the risk aversion and

\[
\frac{d\mathcal{Q}}{d\mathcal{P}} = \exp \left\{ -\frac{1}{2} \int_0^T |\eta_t|^2 dt - \int_0^T \eta_t dW_t \right\} := \xi_t,
\]

where \(\eta\) is a deterministic and bounded process. By Girsanov’s Theorem we have that \(W^\mathcal{Q}_t = W_t + \int_0^t \eta_t ds\) is a standard Brownian motion under \(\mathcal{Q}\). Using Itô’s lemma (see e.g. [22]) we observe that for any \(X \in D_T\), \(X = \Pi_t(X) + \int_t^T g(s, Z_s(X)) ds - \int_t^T Z_s(X) dW_s\), where \(g(t, z) = \frac{1}{2} \gamma |z|^2 - \eta_t z\),
see also Example 1.2. Next, we show that the expected utility maximization (2.10) in this special case can be solved by a martingale approach. By our assumptions, any terminal value $X_T \in \mathcal{D}_T$ can be hedged perfectly with an initial endowment $x_0 = -\Pi_0(-X_T)$ which for our $g$ by Example 1.2 in Section 3 is equivalent to $e^{\gamma x_0} = \mathbb{E}^{g}[e^{\gamma X_T}] = \mathbb{E}[e^{\gamma X_T} \xi_T]$.

Therefore, the problem of utility maximization (2.10) is equivalent to the following static optimization problem

$$\max_X \mathbb{E}[U(X)], \quad \text{s.t.} \quad \mathbb{E}[e^{\gamma X_T} \xi_T] \leq e^{\gamma x_0}.$$ 

Using that by the concavity of $U$, $\lim_{x \to +\infty} U'(x)e^{-\gamma x} = 0$ and $\lim_{x \to -\infty} U'(x)e^{-\gamma x} = +\infty$, we can conclude that $U'(x)e^{-\gamma x} \gamma^{-1}$ is a decreasing function on $\mathbb{R} \to \mathbb{R}_+$ whose inverse we denote by $f$. The next proposition gives an explicit solution of the optimal portfolio in case of a complete market. The proof of the first part can be done using Lagrangian techniques while the second part can be seen using Itô calculus.

**Proposition 5.2.** Assume that for any $\lambda > 0$, $\mathbb{E}[\gamma f(\lambda \xi_T)] < \infty$. The optimal terminal wealth of Problem (2.10) is then given by $X_T^\gamma := f(\lambda \xi_T)$, where $\lambda$ is determined such that the budget constraint $\mathbb{E}[e^{\gamma X_T} \xi_T] = e^{\gamma x_0}$ is met. The optimal strategy can be characterized as the strategy $\theta^\gamma$ that perfectly replicates the terminal optimal wealth $X_T^\gamma$, i.e., $X_t^\gamma = x_0 - \frac{\gamma}{2} \int_0^t |Z_t^\gamma|^2 \, ds + \int_0^t Z_t^\gamma \, dW_t$, $t \in [0, T], \quad \text{where}$

$$Z_t^\gamma := \frac{1}{\gamma} \left( \beta_t^\gamma + \eta_t \right) \quad (5.1)$$

with $\beta_t^\gamma$ being the progressively measurable process resulting from the martingale representation

$$R_t^\gamma := \mathbb{E}[U'(X_T^\gamma) | \mathcal{F}_t] = U'(X_T^\gamma) - \int_t^T \beta_s^\gamma \, dW_s. \quad (5.2)$$

Furthermore, define $\zeta_t^\gamma := I(R_t^\gamma) - \frac{\lambda}{\gamma} \log \left( \frac{R_t^\gamma}{\gamma \xi_t^\gamma} \right)$, and $M_t^\gamma := \frac{\beta_t^\gamma}{U'(X_t^\gamma + \zeta_t^\gamma)} - \frac{1}{\gamma} \left( \frac{\beta_t^\gamma}{R_t^\gamma} + \eta_t \right)$. Then the triple $(X_t^\gamma, \zeta_t^\gamma, M_t^\gamma)$ solves the FBSDE (3.5). Finally, if additionally $U(x) = -e^{-\gamma A x}$, for some constant $\gamma_A > 0$ then $Z_t^\gamma = \frac{\eta_t}{\gamma + \gamma_A}$.

For convenience, we have included the proof in the online version of the paper. In the sequel, let $\tilde{U}(y)$ be the “exponential”-conjugate of $U$, which is defined by

$$\tilde{U}(y) := \sup_x (U(x) - ye^{\gamma x}), \quad y > 0. \quad (5.3)$$

By the definition of $f$, we have $U'(f(y))e^{-\gamma f(y)} \gamma^{-1} = y$, for $y > 0$. It can be verified that $f'(y) < 0$ and that $\tilde{U}(y) = U(f(y)) - ye^{\gamma f(y)}$ is a continuously differentiable convex function and we have the following conjugate relation $U(x) = \inf_{y>0}(\tilde{U}(y) + ye^{\gamma x})$, and $\tilde{U}'(y) = -f'(y)$. Let

$$V(x) := \sup_{x \in L^2(d\mathbb{P} \times ds)} \mathbb{E} \left[ U \left( x + \int_0^T \frac{1}{2} |Z_t|^2 - \eta_t Z_t \right) dt - \int_0^T Z_t \, dW_t \right]$$

and the dual value function is given $\tilde{v}(y) := \mathbb{E}[\tilde{U}(y \xi_T)]$ for $y > 0$. Let $R_1(x) := -\frac{U'(x)}{U^\gamma(x)}$ and $R_2(x) := \frac{U''(x)}{U^\gamma(x)}$. The following result gives the optimal wealth explicitly through a duality approach with the linear terms used for instance in [37, 56] being replaced with an exponential function.

**Proposition 5.3.** Assume that for any $\lambda > 0$, $\mathbb{E}[\gamma f(\lambda \xi_T)] < \infty$. The following statements hold:
(i) The value functions $V(x)$ and $\widetilde{V}(y)$ are exponential-conjugate, i.e.

$$\widetilde{V}(y) = \sup_x (V(x) - ye^{\gamma x}), \quad V(x) = \inf_{y>0} (\widetilde{V}(y) + ye^{\gamma x}).$$

They are continuously differentiable concave (resp. convex) function defined and finite valued on $\mathbb{R}$ (resp. $\mathbb{R}_+$) and satisfy

$$\lim_{x \to -\infty} V'(x) = 0, \quad \lim_{x \to \infty} V'(x) = \infty, \quad \lim_{y \to 0} \widetilde{V}'(y) = -\infty, \quad \lim_{y \to \infty} \widetilde{V}'(y) = 0.$$

(ii) An optimal investment strategy $\theta^*$ exists, is unique and the following duality holds

$$\frac{1}{\gamma} U'(X_T^{\theta^*}(x))e^{-\gamma X_T^{\theta^*}(x)} = y \xi_T, \quad \text{or equivalently,} \quad e^{\gamma X_T^{\theta^*}(x)} = -\widetilde{U}'(y \xi_T),$$

where $X_T^{\theta^*}(x) = x - \int_0^T \langle \frac{1}{2} \beta Z_t^{\theta^*}, Z_t^{\theta^*} \rangle dt + \int_0^T Z_t^{\theta^*} dW_t$ and $y = \frac{1}{\gamma} V'(x)e^{-\gamma x}$.

(iii) For any $x \in \mathbb{R}$ and $y = \frac{1}{\gamma} V'(x)e^{-\gamma x}$, the process $e^{\gamma X_T^{\theta^*}(x)} = \mathbb{E}_Q[-\widetilde{U}(y \xi_T)|\mathcal{F}_t] = \mathbb{E}_Q[e^{\gamma X_T^{\theta^*}(x)}|\mathcal{F}_t]$ is a $\mathbb{Q}$-martingale.

We now provide sufficient conditions on the utility function $U$ which gives the smoothness needed for the existence of a solution of the BSPDE (4.5).

**Proposition 5.4.** Assume that the risk aversion $R_1(x)$ and the prudence $R_2(x)$ are bounded and bounded away from zero, and $\mathbb{E}[U(f(\lambda \xi_T))] < \infty$, and $\mathbb{E}[e^{\gamma f(\lambda \xi_T)} \xi_T] < \infty$, for any $\lambda > 0$. Assume further that $|F(x, y)| := |U'(f(\lambda(x) \exp(-\eta y - \frac{\gamma x^2}{2})))|$, $|\frac{\partial F}{\partial y}|$, $|\frac{\partial F}{\partial x^2 y^2}|$ and $|\frac{\partial F}{\partial x^2 y^2}|$ are bounded by $K(x) \exp(a |y|^2)$ with $0 < a < \frac{1}{\gamma^2}$ and $K$ a continuous function. Then there exists a modification of the value function in (4.1) satisfying conditions (a)-(c) in Definition 4.3.

6. Conclusion

We considered a continuous-time setting with permanent endogenous price impact induced by a change in the inventory of the market maker. We showed that trading in such a setting corresponds to non-linear stochastic integrals and arises naturally as the limit of discrete time trading. In general incomplete markets with endowments of the investor and the market maker, we then characterized optimal solutions in terms of coupled FBSDEs and BSPDEs, thereby highlighting the interplay between these approaches. Finally, we gave new existence results for the arising FBSDEs and BSPDEs and furnished examples in scenarios where the market is complete, and/or the utility function adopts an exponential form. Future works in this direction are to consider different probabilistic settings or other decision theoretic preferences for the large investor or the market maker.

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Appendix A: BMO’s

Consider the set of progressively measurable processes \( \phi \) satisfying \( \mathbb{E}[\int_0^T |\phi_t|^2 dt] < \infty \). The natural extension of the space of bounded processes is the space of BMO processes defined by

\[
BMO(\mathbb{P}) = \left\{ \phi \in \mathbb{F} \mid \exists C \text{ s.t. } \forall t, \mathbb{E} \left[ \int_t^T |\phi_s|^2 ds \bigg| \mathcal{F}_t \right] \leq C \right\}.
\]

The BMO norm of a process \( \phi \in BMO(\mathbb{P}) \) is defined as the smallest constant \( C \) in the above definition. The stochastic integral process \( \int_0^T \phi_s dW_s \) is called a \( BMO(\mathbb{P}) \) martingale if \( \phi \in BMO(\mathbb{P}) \). We remark that for \( \phi \in BMO(\mathbb{P}) \), the corresponding Doléans-Dade exponential is a Radon-Nikodym derivative giving rise to a measure, say \( \mathbb{Q} \), for which the Girsanov transformation is well-defined and for which the \( BMO \) norms are equivalent to the ones under \( \mathbb{P} \); see [6, 35].
Appendix B: Proofs of Section 2

Proof of Proposition 2.2 and Proposition 2.3. Let us first show Proposition 2.2. We will see in the proof that Proposition 2.3 will then follow immediately. Recall that $Z^{t,r}(H_M - yS)$ is defined through the BSDE

$$
\Pi^{t,r}_{\tilde{s}} \left( H_M - \sum_{i=1}^{n} y^i S^i \right) = \sum_{i=1}^{n} \left( h^{M,i}(R^{t,r}_{\tilde{s}}) - y^i s^i(R^{t,r}_{\tilde{s}}) \right) + h^{M,\bot}(W^{n+1}, \ldots, W^{d})
$$

$$
- \sum_{i=1}^{d} \int_{\tilde{s}}^{T} g^{i}(u, Z^{t,r}_{u}(H_M - \sum_{j=1}^{n} y^j S^j)) \text{d}u
$$

$$
+ \sum_{i=1}^{d} \int_{\tilde{s}}^{T} Z^{i,t,r}_{u}(H_M - \sum_{j=1}^{n} y^j S^j) \text{d}W^i_u, \quad i = 1, \ldots, n, \quad \tilde{s} \in [t, T],
$$

(B.1)

with $H^{i}_{M} = h^{M,\bot}(W^{n+1}, \ldots, W^{d})$. By definition $Z^{t,r}_{u} = Z^{0,r}_{u}(H_M - \sum_{j=1}^{n} y^j S^j)$. Note that by (H1), $g$ admits an additive decomposition over the components of $z$, and $R^i$ only depends on $W^i$ for $i = 1, \ldots, n$. Hence, we can solve the BSDE for each terminal condition $h^{M,i}(R^{t,r}_{\tilde{s}}) - y^i s^i(R^{t,r}_{\tilde{s}})$ for $i = 1, \ldots, n$, and $H^{M,\bot}(W^{n+1}, \ldots, W^{d})$ by itself regardless of the other components, and then add the different solutions up together. In other words, the solution of (B.1) has the additive composition

$$
\Pi^{t,r} \left( H_M - \sum_{i=1}^{n} y^i S^i \right) = \sum_{i=1}^{n} \Pi^{t,r,i} \left( h^{M,i}(R^{t,r}_{\tilde{s}}) - y^i s^i(R^{t,r}_{\tilde{s}}) \right) + \Pi^{t,r} \left( h^{M,\bot}(W^{d+1}, \ldots, W^{d}) \right)
$$

with

$$
Z^{i,t,r} \left( H_M - \sum_{i=1}^{n} y^i S^i \right) = \left( Z^{1,1,t,r}_{\tilde{s}}(h^{M,1}(R^{1,1,t,r}_{\tilde{s}}) - y^1 s^1(R^{1,1,t,r}_{\tilde{s}})) \right), \ldots, Z^{n,1,t,r}_{\tilde{s}}(h^{M,n}(R^{n,1,t,r}_{\tilde{s}}) - y^n s^n(R^{n,1,t,r}_{\tilde{s}}))
$$

$$
Z^{n+1,1,t,r} \left( H^{n}_{M} \right), \ldots, Z^{n,t,r} \left( H^{n}_{M} \right).
$$

(B.2)

Formally, this may be seen as the right-hand side above satisfies the BSDE (B.1) and each component in the sum depends on separate Brownian motions. Now from the definition of $R$ it follows by taking derivatives with respect to $r$ in (2.7) that $\nabla_{r} R^{i,t,r,i}$ and $(\nabla_{r} R^{i,t,r,i})^{-1}$ are bounded (by $e^{\|\mu_{i}\|_{\infty,T}}$) and bounded away from zero (by $e^{-\|\mu_{i}\|_{\infty,T}}$). Next, we will see in the sequel that for fixed $y^i$, $Z^{i,t,r,i}$ is actually uniformly bounded by $C_i := \|h^{M,i} - y^i s^i\|_{\infty} \|\nabla_{r} R^{i,t,r,i}\|_{\infty} \|\nabla_{r} R^{i,t,r,i})^{-1}\|_{\infty} \|\sigma^i\|_{\infty}$ for $i = 1, \ldots, n$. Define a new driver function with the first three derivatives bounded and coinciding with $g^i$ on $[0, T] \times [-C^i, C^i]$ for $i = 1, \ldots, n$. To simplify notation denote this new function again by $g^i$. We will show that the $Z$-part of this (new) BSDE is bounded by $C_i$, so that the solution for this (new) BSDE is also a solution for the (old) BSDE with the original driver, see for instance the proof of Theorem 4.1 in [18] for this approach.

It is well known that $Z^{i,t,r,i}$ can be characterized by taking formally the derivative with respect to $r^i$ in (B.1) (with the modified drivers $g^i$). Specifically, for $i = 1, \ldots, n$ denote by $(P^{n,t,r,i}, g^i, V^{n,t,r,i})$ the solution to the BSDE

$$
F^{i,t,r,i,y^i}_{\tilde{s}} = \left( h^{M,i}(R^{t,r}_{\tilde{s}}) - y^i s^i(R^{t,r}_{\tilde{s}}) \right) \nabla_{r} R^{i,t,r,i}
$$

$$
- \int_{\tilde{s}}^{T} g^{i}(u, F^{i,t,r,i,y^i}_{u}(\nabla_{r} R^{i,t,r,i})^{-1}\sigma^i)V^{i,t,r,i,y^i}_{u} \text{d}u + \int_{\tilde{s}}^{T} V^{i,t,r,i,y^i}_{u} \text{d}W^i_u.
$$

(B.3)
where $\int_0^T V_i^u \, dW_u$ is defined by $\sum_{1 \leq j \leq d} \int_0^T V_i^{i,j} \, dW_j$, with $V_i^{i,j}$ denoting the $j$-th entry of the $1 \times d$ dimensional process $V_i$ for $i = 1, \ldots, n$. Note that the index $y^i$ in $F$ and $V$ simply refers to the dependence of these process in $y^i$ through the terminal condition. We remark further that $(F_i^{t,r,y^i}, V_i^{t,r,y^i})$ is again the solution of a BSDE with driver function satisfying the quadratic growth conditions (H2) and (H3) in [36]. Hence, by Theorem 2.6 in [36] a unique solution to (B.3) exists which is bounded by $|h_r^{M,i} - y^i|^1 \|\nabla_r R_i^{t,r} \|_{\infty}$, and therefore also satisfies a uniform growth condition in $y$. Furthermore, Theorem 2.8 in [36] guarantees that if $y_m \to y^i$ as $m$ tends to infinity, then $F_i^{t,r,y_m}$ converges a.s. uniformly in $t$ to $F_i^{t,r,y^i}$. Thus, except for a fixed $\mathbb{P}$-zero set, say $\mathcal{N}$, we can for any $y^i \in \mathbb{Q}$ define $F_i^{t,r,y^i}$, continuously in $t$. Then we extend this parametrized process to non-rational $y^i$ by setting $F_i^{t,r,y^i}(\omega) = \lim_{m \to \infty} F_i^{t,r,y_m}(\omega)$, where the limit can be taken uniformly in time, and $(y_m)_{m \in \mathbb{N}}$ is a sequence of rational numbers converging to $y^i$ for each $\omega \in \Omega \setminus \mathcal{N}$. Note that $F_i^{t,r,y^i}(\omega)$ is well defined up to the $\mathbb{P}$-zero set $\mathcal{N}$ which is independent of $y$ and $t$. Setting on $\mathcal{N}$, $F_i^{t,r,y^i} = 0$, the mapping $y \mapsto F_i^{t,r,y^i}(\omega)$ is continuous for every $t$ and $\omega$. Now by Lemma 2.5 and Corollary 2.11 in [50] we have

$$Z_i^{t,r}(h_i^{M,i}(R_i^{t,r,i}) - y^i)^i(R_i^{t,r,i}) = -F_i^{t,r,y^i}(\nabla_r R_i^{t,r,i})^{-1} \sigma_i^i \quad \text{(B.4)}$$

which therefore is bounded by $C_i$ and hence a solution to the original BSDE. Setting then $\omega$-wise

$$Z_i(t, y^i) := -F_i^{t,0,0}(\nabla_r R_i^{t,0,0})^{-1} \sigma_i^i = Z_i^{t,0,0}(h_i^{M,i}(R_i^{t,0,0}) - y^i)^i(R_i^{t,0,0}) = Z_i^{t,0,0}(H_i - \sum_{i=1}^n y_i S_i)$$

for $i = 1, \ldots, n$ (see (B.2) for the last equation), and $Z_i(t, y^i) := Z_i(H_M) = Z_i^{t,0,0}(H_M^1) = Z_i^{t,0,0}(H_M - \sum_{i=1}^n y_i S_i)$ for $i = n+1, \ldots, d$ finishes the first part of the proof of Proposition 2.2. The last two equalities in both equations hold $L^2(d\mathbb{P} \times dt)$ a.s. It also follows directly from the remarks before about $F$, that $Z_i(t, y^i)$ is uniformly bounded by $C_i$ and that a linear growth condition of $Z_i$ in $y^i$ holds. Finally, for the proof of Proposition 2.3, note that for each $i$ the image of the mapping $y^i \to Z_i(t, y^i)$ is an interval (since the image of a continuous function mapping to $\mathbb{R}$ is an interval), which we can thus denote as $\mathbb{I}$ for $i = 1, \ldots, n$. Clearly also for $i = n+1, \ldots, d$, the image space of $Z_i(t, \cdot)$ must be the point $Z_i^{t,0,0}(H_M) = Z_i^{t,0,0}(H_M) = Z_i^{t,0,0}(H_M)$ (as $H_M$ is independent of $y$). That $Z(H_M)$ is bounded follows with an analogous argument as above with $C_i := ||h_i^{M,i}||_{\infty} \|\nabla_r R_i^{t,r,i} \|_{\infty} \|\nabla_r R_i^{t,r,i} \|_{\infty}^{-1} \|\sigma_i^i \|_{\infty}$, completing the proof of Proposition 2.3. □

Proof of Proposition 2.4. For the sake of exposition let us recall Lemma 1 in [22] showing the integral representation above for a fixed admissible strategy $\theta^m \in \Theta_0$. Denote the discontinuity points of $\theta^m \in \Theta_0$ by $0 \leq \tau_1 < \tau_2 < \cdots$. Let $l$ be the number of the discontinuity points, $\tau_0 = 0$ and $\tau_k = T$ for $k \geq l + 1$. By definition, using that $\Pi_T(H_M - \theta_T^m S) = H_M - \theta_T^m S$ and $\theta_0^m = 0$

$$\mathcal{I}(\theta^m) = \theta_T^m S - \sum_{0 \leq l < T} \left( \Pi_{l}(H_M - \theta_l^m S) - \Pi_{l}(H_M - \theta_l^m S) \right)$$

$$= \theta_T^m S - \sum_{j=1}^{l} \left( \Pi_{\tau_j}(H_M - \theta_{\tau_j}^m S) - \Pi_{\tau_j}(H_M - \theta_{\tau_j}^m S) \right)$$

$$= H_M - \Pi_0(H_M) - \sum_{j=0}^{l} \left( \Pi_{\tau_j+1}(H_M - \theta_{\tau_j+1}^m S) - \Pi_{\tau_j}(H_M - \theta_{\tau_j}^m S) \right).$$
Again, by definition, $\Pi_{t_{j+1}}(H_M - yS) - \Pi_{t_j}(H_M - yS) = \int_{t_j}^{t_{j+1}} g(s, Z^\theta_s)ds - \int_{t_j}^{t_{j+1}} Z^\theta_s dW_s$. Since $\theta^n$ is a simple left-continuous process, $\theta^n_{t_{j+1}}$ is $\mathcal{F}_t$ measurable and so, we can substitute $y = \theta^n_{t_{j+1}}$ to obtain

$$J(\theta^n) = H_M - \Pi_0(H_M) - \sum_{j=0}^{t} \left( \int_{t_j}^{t_{j+1}} g(s, Z^{\theta^n}_s)ds - \int_{t_j}^{t_{j+1}} Z^{\theta^n}_s dW_s \right)$$

$$= H_M - \Pi_0(H_M) - \int_0^T g(t, Z^{\theta^n}_t)dt + \int_0^T Z^{\theta^n}_t dW_t. \quad (B.5)$$

Now, by the continuity of $Z$ in $y$ shown in Proposition 2.2 we have that $Z^{\theta^n} \rightarrow Z^\theta$ in measure with respect to $dP \times dt$. As by Proposition 2.2 and the uniform integrability of $|\theta^n|^2, |Z^{\theta^n}|^2$ is uniformly integrable, the convergence holds actually in $L^2(dP \times dt)$ if $g$ is Lipschitz, and in $L^1(dP \times dt)$ if $g$ grows at most quadratically. Passing to the limit in (B.5) yields the proposition. \hfill \Box

Appendix C: Proofs of Section 3

Proof of Theorem 3.1. Clearly,

$$E[U(X_T^x + H_L)] = \sup_{\theta \in \Theta} E[U(X_T^\theta + H_L)] = \sup_{\mathcal{E} \in \mathcal{I}} E[U(X_T^{\mathcal{E}} + H_L)] = E[U(X_T^{Z^*} + H_L)], \quad (C.1)$$

where with a slight abuse of notation we write

$$X_T^{Z} := x_0 - \sum_{j=1}^{d} \int_0^T g^j(s, Z^{\mathcal{E}}_s)ds + \sum_{j=1}^{d} \int_0^T Z^j_s dW^j_s + H_M - \Pi_0(H_M).$$

Denote $Z^{*} := Z^{\theta^*} \in \mathcal{I} = I_1 \times \ldots \times I_n \times \{Z^{H_{H^M}} \times \ldots \times \{Z^{d}(H_M)\} \}$ (see Proposition 2.3) attaining the supremum in (C.1). Define $R_t := E[U'(X_T^{Z^*} + H_L)|\mathcal{F}_t]$ and $\zeta_t := I(R_t) - X_t^{Z^*}$ with $I = (U')^{-1}$. Then $\zeta$ is progressively measurable and $R$ is a martingale. By the (local) martingale representation theorem there exists a locally square integrable progressively measurable process $\beta$ taking values in $\mathbb{R}^{1 \times d}$ such that

$$R_t = U'(X_T^{Z^*} + H_L) - \int_t^T \beta_s dW_s. \quad (C.2)$$

Hence, $dR_t = \beta_t dW_t$ with terminal condition $R_T = U'(X_T^{Z^*} + H_L)$. By Doob’s Maximal inequality $\sup_s |R_s|$ is in $L^{1+\epsilon}$, and by the Burkholder-Davis Gundy (BDG) inequality $E[(\int_0^T |\beta_s|^2 ds)^{1+\epsilon/2}] < \infty$.

Note that $I(R_t) = X_T^{Z^*} + \zeta_t$ and $\zeta_T = H_L$. Applying Itô’s formula to $I(R_t)$ we have the following backward representation

$$X_t^{Z^*} + \zeta_t = X_T^{Z^*} + H_L - \int_t^T \beta_s dW_s - \frac{1}{2} \int_t^T \frac{U''}{U'^3} (X_t^{Z^*} + \zeta_s) d\langle R, R \rangle_s$$

$$= X_T^{Z^*} + H_L - \int_t^T \beta_s dW_s - \frac{1}{2} \int_t^T \frac{U''}{U'^3} (X_t^{Z^*} + \zeta_s) |\beta_s|^2 ds,$$

where $U''$ and $U^{(3)}$ are the second and the third order derivatives of $U$, respectively. Hence, $\zeta_t$ is a solution of the following BSDE

$$\zeta_t = H_L - \int_t^T \left( \frac{\beta_s}{U''(X_t^{Z^*} + \zeta_s)} - (Z_s^{Z^*} - Z_s(H^M)) \right) dW_s$$

$$+ \frac{1}{2} \int_t^T \left( |\beta_s|^2 \frac{U^{(3)}}{U'^3} (X_t^{Z^*} + \zeta_s) - 2g(s, Z_s^\theta) + 2g(s, Z_s(H^M)) \right) ds.$$


By construction the marginal utility process \( U'(X_t^Z + \zeta_t) = R_t \) is a martingale and by the definition of \( M_t^z \) above

\[
\beta_i^t = U''(X_t^Z + \zeta_t)(Z_t^{*,i} + M_t^z - Z_t^i(H_M)), \quad \text{for } i = 1, \ldots, d.
\] (C.3)

Plugging the last equation into the dynamics of \( \zeta \) we get

\[
\zeta_t = H_L - \int_t^T M_s^z dW_s + \frac{1}{2} \int_t^T \left( |Z_s^* - Z_s(H_M) + M_s^z |^2 \frac{U'(3)}{U'(1)}(X_s^Z + \zeta_s) \right.

\left. - 2(g(t, Z_t^*) - g(t, Z_t(H_M))) \right) ds.
\] (C.4)

Now by a measurable selection theorem (see e.g. Aumann [2]), we can choose a bounded \( n \)-dimensional process \( \delta = (\delta^1, \ldots, \delta^n) \) such that

(a) On \( Z_t^{*,i} \in (a_i^t, b_i^t) = \text{int}(\mathbb{I}) \), \( 0 < \delta_t^i \) and \( Z_t^{*,i} + \varepsilon \delta_t^i \notin \mathbb{I} \) for \( i = 1, \ldots, n \) for all \( 0 < \varepsilon < 1 \).

(b) On \( Z_t^{*,i} = a_i^t \) and \( a_i^t \neq b_i^t \), \( 0 < \delta_t^i \) and \( Z_t^{*,i} + \varepsilon \delta_t^i \notin \mathbb{I} \) for \( i = 1, \ldots, n \) for all \( 0 < \varepsilon < 1 \).

(c) On \( Z_t^{*,i} = b_i^t \) and \( a_i^t \neq b_i^t \), \( \delta_t^i < 0 \) and \( Z_t^{*,i} + \varepsilon \delta_t^i \notin \mathbb{I} \) for \( i = 1, \ldots, n \) for all \( 0 < \varepsilon < 1 \).

To see this, note that for \( i = 1, \ldots, n \) and fixed \((t, \omega)\) such that \( Z_t^{*,i}(\omega) \) is in the interior of the corresponding interval it is clearly possible to find a suitable \( \delta_t(\omega) \) satisfying (a). On the other hand, in case (b) or (c), \( Z_t^{*,i}(\omega) \) is equal to the boundary of the corresponding interval. Since the interval is non-degenerate (i.e., \( a_i^t(\omega) < b_i^t(\omega) \)) we can choose \( \delta_t(\omega) = \pm (b_i^t(\omega) - a_i^t(\omega)) \) depending if \( Z_t^{*,i}(\omega) \) is equal to the lower or the upper bound of the interval. If \( a_i^t(\omega) = b_i^t(\omega) \) we choose \( \delta_t(\omega) = 0 \).

To ensure that \( g^i_t(t, Z_t^{*,i}) \delta^i_t \) is bounded (which will be needed below), let us define the even smaller perturbation \( \delta^i_t := I_{Z_t^{*,i} \neq 0} \min(\delta^i_t, |Z_t^{*,i}|, \varepsilon^{-1}/\varepsilon) + I_{Z_t^{*,i} = 0} \delta^i_t \), for \( i = 1, \ldots, n \). Now suppose that the \( n \)-dimensional perturbation \( h_t = (h_1^t, \ldots, h_n^t) \) only takes values (\( \omega \)-wise) in \( \{0, \pm \delta_t^i e_i \} \) for a fixed \( i \) with \( i = 1, \ldots, n \) on the set where \( Z_t^{*,i} \in (a_i^t, b_i^t) = \text{int}(\mathbb{I}) \), while on the set, where for the same fixed \( i, Z_t^{*,i} \in \{a_i^t, b_i^t\} \), \( h_t \) takes values (\( \omega \)-wise) in \( \{0, \delta_t^i e_i \} \). Define then

\[
X^z_T := x_0 - \sum_{j=1}^n \int_0^T g^j(s, Z_{s}^{*,j} + \varepsilon h_{s}^j)ds + \sum_{j=1}^n \int_0^T (Z_{s}^{*,j} + \varepsilon h_{s}^j)dW_s^j

- \sum_{j=n+1}^d \int_0^T g^j(s, Z_{s}^{*,j})ds + \sum_{j=n+1}^d \int_0^T Z_{s}^{*,j}dW_s^j + H_M - \Pi_0(H_M).
\] (C.5)

As \( Z^z \) by construction takes only values in \( \mathbb{I} \), \( Z^z \) is admissible. Set \( \phi(\varepsilon) := U(X^z_T + H_L) \). Since the wealth process is a.s. concave in \( Z \), and \( U \) is increasing and concave, \( \phi(\varepsilon) \) is concave. Moreover,

\[
\phi'(\varepsilon) = U'(X^z_T + H_L) \left( -\sum_{j=1}^n \int_0^T g^j(t, Z_{s}^{*,j} + \varepsilon h_{s}^j)h_{s}^j dt + \sum_{j=1}^n \int_0^T h_{s}^j dW_s^j \right). \]

This implies that the function \( \Phi(\varepsilon) \) defined by

\[
\Phi(\varepsilon) := \frac{\phi(\varepsilon) - \phi(0)}{\varepsilon} = \frac{U(X^z_T + H_L) - U(X^z_T + H_L)}{\varepsilon},
\]

is decreasing with respect to \( \varepsilon \in (0, 1) \) and \( \lim_{\varepsilon \downarrow 0} \Phi(\varepsilon) = U'(X^z_T + H_L)\chi_T^{\varepsilon}(h) \) a.s., where \( \chi_T^{\varepsilon}(h) := -\sum_{j=1}^n \int_0^T g^j(t, Z_{s}^{*,j} + \varepsilon h_{s}^j)h_{s}^j dt + \sum_{j=1}^n \int_0^T h_{s}^j dW_s^j \). Since \( h_t \) takes values only in \( \{0, \pm \delta_t^i e_i \} \) for a fixed \( i \), we have by (Hg) or (HL) that \( g^i_t(t, Z_t^{*,i})h_t^i \) is uniformly bounded. Thus, by the BDG inequality for semimartingales all moments of \( X^z_T(h) \) exist. Since by assumption \( \mathbb{E}[U'(X^z_T + H_L)]^{1+\varepsilon} < \infty \), we obtain \( \mathbb{E}[\Phi(\varepsilon)] < \infty \). Note that \( \mathbb{E}[\Phi(\varepsilon)] \) is well-defined as
\[ E[\Phi(\varepsilon)^+] \leq E[(U'(X_T^{\varepsilon^*} + H_T)\tilde{\gamma}^{\theta^*}_T(h))^+] < \infty \text{ for all } \varepsilon > 0. \] Hence, by the monotone convergence theorem we conclude that

\[ 0 \geq \lim_{\varepsilon \downarrow 0} E[\Phi(\varepsilon)] = \lim_{\varepsilon \downarrow 0} E\left[ \frac{U(X_T^{\varepsilon^*} + H_L) - U(X_T^{\varepsilon^*} + H_L)}{\varepsilon} \right] = E[U'(X_T^{\varepsilon^*} + H_L)\tilde{\gamma}^{\theta^*}_T(h)]. \quad (C.6) \]

Next, for \( i = 1, \ldots, n \) define an increasing sequence of stopping times \( \tau_k := \inf\{\int_0^t |\beta_i^j\chi_t^j| dt \geq k\} \wedge T \) with \( P[\tau_k = T] \rightarrow 1 \) as \( k \rightarrow \infty \). Applying Itô’s lemma to the product \( U'(X_T^{\varepsilon^*} + \zeta_t)\chi_t^j(h) = R_t\tilde{\chi}_t^j(h) \) we have

\[
U'(X_{\tau_k}^{\varepsilon^*} + \zeta_{\tau_k})\chi_{\tau_k}^j(h) = \sum_{j=1}^n \left\{ \int_0^{\tau_k} h_i^j \left( \beta_i^j - U'(X_t^{\varepsilon^*} + \zeta_t)g_t^i(t, Z_t^{\varepsilon^*}) \right) dt + \int_0^{\tau_k} \left( \beta_i^j\chi_t^j(h) + U'(X_t^{\varepsilon^*} + \zeta_t)h_t^j \right) dW_t^j \right\} \\
= \int_0^{\tau_k} h_t^j \left( \beta_i^j - U'(X_t^{\varepsilon^*} + \zeta_t)g_t^i(t, Z_t^{\varepsilon^*}) \right) dt + \int_0^{\tau_k} \left( \beta_i^j\chi_t^j(h) + U'(X_t^{\varepsilon^*} + \zeta_t)h_t^j \right) dW_t^j, \quad (C.7)
\]

where we used that \( h^j = 0 \) for \( j \neq i \) in the last equation. Taking expectations and the limit on both sides and noting that \( \zeta_T = H_L \) leads to

\[
\lim_{k \rightarrow \infty} E\left[ \int_0^{\tau_k} h_t^j \left( \beta_i^j - U'(X_t^{\varepsilon^*} + \zeta_t)g_t^i(t, Z_t^{\varepsilon^*}) \right) dt \right] \\
= \lim_{k \rightarrow \infty} E\left[ U'(X_{\tau_k}^{\varepsilon^*} + \zeta_{\tau_k})\chi_{\tau_k}^j(h) \right] = E\left[ U'(X_T^{\varepsilon^*} + H_L)\tilde{\gamma}^{\theta^*}_T(h) \right] \leq 0, \quad (C.8)
\]

by \((C.6)\), for any bounded progressively measurable \( h \) taking values \( (\omega\text{-wise}) \) in \( \{0, \pm \delta_i^j(\omega)e_i\} \) for \( i = 1, \ldots, n \). In the last equality we have used uniform integrability, since \( \sup_\tau |R_t| \in L^{1+\varepsilon} \) and according to Condition 6 above and by the BDG inequality for semimartingales (see for instance Theorem 2, Chapter V in \([52]\)) all moments of \( \sup_\tau |\chi_t(h)| \) exist. Defining \( C_i := \{(t, \omega)|Z_t^{\varepsilon^*}(\omega) \in (a_i^j(\omega), b_i^j(\omega))\} \) and replacing \( h \) by \( h1_{C_i} \) and \( -h1_{C_i} \), we get a reverse inequality in \((C.8)\) and can conclude that

\[
0 = \lim_{k \rightarrow \infty} E\left[ \int_0^{\tau_k} I_{C_i} h_t^j \left( \beta_i^j - U'(X_t^{\varepsilon^*} + \zeta_t)g_t^i(t, Z_t^{\varepsilon^*}) \right) dt \right]. \quad (C.9)
\]

Now let \( A_t^i := (\beta_i^j - U'(X_t^{\varepsilon^*} + \zeta_t)g_t^i(t, Z_t^{\varepsilon^*})) \) and choose \( h_t^j = \delta_t^i 1_{A_t^i > 0} \). Note that \( A_t^i h_t^j 1_{[0, \tau_k]} \) is increasing in \( k \) so that by the monotone convergence theorem, the limit in \((C.9)\) can be taken inside. Hence, from \((C.9)\) we get \( A_t^i \leq 0, \, dP \times dt \) almost everywhere. Similarly choosing \( h_t^i = \delta_t^i 1_{A_t^i < 0} \) we get the reverse inequality and can conclude that for \( i = 1, \ldots, n \) on \( C_i \)

\[
\beta_i^j - U'(X_t^{\varepsilon^*} + \zeta_t)g_t^i(t, Z_t^{\varepsilon^*}) = 0, \quad dP \times dt \text{ a.s.} \quad (C.10)
\]

Finally, let us consider the case where \( Z^* \) attains the boundary of the intervals \( (a_t^i, b_t^i) \). On the set \( B_i = \{(t, \omega)|Z_t^{\varepsilon^*}(\omega) = b_t^i, \, a_t^i(\omega) < b_t^i(\omega)\} \) we have as in \((C.8)\)

\[
\lim_{k \rightarrow \infty} E\left[ \int_0^{\tau_k} 1_{B_i} \left( h_t^i \beta_t^i - U'(X_t^{\varepsilon^*} + \zeta_t)h_t^i g_t^i(t, Z_t^{\varepsilon^*}) \right) dt \right] \leq 0. \quad (C.11)
\]
Note that (C.11) holds for any bounded progressively measurable (perturbation) \( h^i \) taking values (\( \omega \)-wise) in \( \{0, \delta^i_1\} \). Now let \( A_t := \beta_i^t - U'(X_t^{Z^*} + \zeta_i)g^i(z, Z_t^{z,i}) \) and replace \( h^i \) in (C.11) by \( \delta^i_1 A_t < 0 \) (which by (c) above is non-positive on \( B_i \)). From (C.11) we get then by contradiction

\[
A_t = \beta_i^t - U'(X_t^{Z^*} + \zeta_i)g^i(t, Z_t^{z,i}) \geq 0, \quad dP \times dt \quad \text{a.s.} \tag{C.12}
\]
on the set \( B_i = \{(t, \omega)|Z_t^{z,i}(\omega) = \beta_i^t, a^i_t(\omega) < \beta^i_0(\omega)\} \). Similarly, on the set \( \{(t, \omega)|Z_t^{z,i}(\omega) = a^i_0, a^i_t(\omega) \leq \beta^i_0(\omega)\} \), we get

\[
A_t = \beta_i^t - U'(X_t^{Z^*} + \zeta_i)g^i(t, Z_t^{z,i}) \leq 0, \quad dP \times dt \quad \text{a.s.} \tag{C.13}
\]

In particular, the concave function, say \( f \), given by \( z^i \mapsto \beta_i z^i - U'(X_t^{Z^*} + \zeta_i)g^i(t, z^i) \) defined on the set \( I_i^0 \) attains its maximum in \( Z_{1}^{z^i} \in I_i^0 \). To see this we remark that if \( a^i_0 = \beta^i_0 \) then \( I_i^0 \) is a singleton so that clearly the only feasible value, \( Z_{1}^{z^i} \), must also be the maximum. The other cases follow from (C.10)-(C.12)-(C.13) by noting for instance that for \( a^i_0 = Z_{1}^{z^i} \) the one sided derivative of \( f \) is negative, while being positive on the set where \( \beta_0 = Z_{1}^{z^i} \) (corresponding to the right boundary point). Since by the definition of a subgradient, the concave function, \( f \), attains its maximum in \( z^* \) if and only if \( 0 \in \nabla f(z^*) \), we can conclude that indeed \( 0 \in \beta_i^t - U'(X_t^{Z^*} + \zeta_i)\nabla g^i(t, Z_t^{z,i}), \quad dP \times dt \quad \text{a.s.} \)

Using (C.2) leads to the desired conclusion. \( \square \)

The following proposition will be needed for the proof of Theorem 3.2 below.

**Proposition C.1.** \( \mathcal{H}^i(t, X, \zeta, M_i) : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \mapsto cl(I^i) \), defined in (3.3) above for \( i = 1, \ldots, n \) uniquely solve the equations

\[
0 \in -U'(X + \zeta)\nabla \tilde{g}^i(t, \mathcal{H}^i(t, X, \zeta, M^i)) + U''(X + \zeta)(\mathcal{H}^i(t, X, \zeta, M^i) - Z_{1}^{i}((H^M_i) + M^i)) \tag{C.14}
\]

Equality (C.14) holds for each \( (\omega, t, X, \zeta, M_1, \ldots, M_n) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \). Furthermore, if an optimal solution \( \theta^* \) exists and the assumptions of Theorem 3.1 hold, we have

\[
Z_t^{\theta^*} = \mathcal{H}(t, X_t^{\theta^*}, \zeta, M_t) := (\mathcal{H}^1(t, X_t^{\theta^*}, \zeta, M^1), \ldots, \mathcal{H}^n(t, X_t^{\theta^*}, \zeta, M^n), Z_t^{n+1}(H^M), \ldots, Z_t^d(H^M)), \tag{C.15}
\]

with \( M = (M^1, \ldots, M^n, M^{n+1}, \ldots, M^d) \).

**Proof.** Since \( g_i(t, \cdot) \) is decreasing and \( U''(X + \zeta) < 0 \), the continuous function

\[
\bar{U}^i_{X, \zeta, M^i}(\mathcal{H}^i) := -U'(X + \zeta)g^i_2(t, \mathcal{H}^i) + U''(X + \zeta)(\mathcal{H}^i - Z_{1}^{i}((H^M_i) + M^i)
\]
is bijective so that \( \mathcal{H}^i(t, X, \zeta, M^i) = a^i \vee (\bar{U}^i_{X, \zeta, M^i}(0) \wedge b^i) \in cl(I^i) \), is well defined for \( i = 1, \ldots, n \) and is the unique solution of (C.14), see also (3.1). Finally, by Theorem 3.1 and the bijectivity of \( \bar{U} \), if \( \theta^* \) is the optimal solution we actually must have \( Z_t^{\theta^*, i} = \mathcal{H}^i(t, X_t, \zeta, M_t) \), for \( i = 1, \ldots, n \). Recalling that by Proposition 2.3 we have that \( Z_t^{\theta^*, i} = Z_{1}^{i}(H^M) \) for \( i = n+1, \ldots, d \), we can conclude that (C.15) holds. \( \square \)

**Proof of Theorem 3.2.** \( (\zeta, M) \) is a solution of the BSDE (C.4) with driver

\[
M \to |Z_s^{\theta^*} - Z_s(H^M)| + M^2 \frac{U^{(3)}}{U''}(X_s^{\theta^*} + \zeta_s) - 2(g(s, Z_s^{\theta^*}) - g(s, Z_s(H^M))).
\]

Furthermore, by Theorem 3.1, \( Z^{\theta^*} \) satisfies (3.2). Using Proposition C.1 we obtain then immediately (C.15). The proof is completed by plugging this identity back into the BSDE and into the dynamics of \( X_t^{\theta^*} \). \( \square \)

Next let us show a technical lemma for proof of Theorem 3.3.
Lemma C.2. Suppose that $\psi := U'/U''$ is bounded. Then the function $\mathcal{H}(t, X, \zeta, M)$ satisfying (C.14) grows at most linearly in $M$, i.e., there exists $C > 0$ such that $|\mathcal{H}(t, X, \zeta, M)| \leq C(1 + |M|)$.

Proof. It is sufficient to check the first $n$ components of $\mathcal{H}(t, X, \zeta, M)$ because the last $n - d$ components of $\mathcal{H}(t, X, \zeta, M)$ are equal to $(Z^{n+1}(H_M), \ldots, Z^d(H_M))$, which by Proposition 2.3 are bounded. Denote therefore $\hat{\mathcal{H}} := (\mathcal{H}^1(t, X, \zeta, M^1), \ldots, \mathcal{H}^n(t, X, \zeta, M^n))$, $\hat{Z}(H_M) := (Z^1(H_M), \ldots, Z^n(H_M))$ and $M := (M^1, \ldots, M^n)$. We get from (C.14) that $\mathcal{H} = -M + \hat{Z}(H_M) + \psi_1 Y$ with $Y = (Y^1, \ldots, Y^n)$ and $Y^j \in \nabla \hat{g}^j(t, \vec{\zeta})$ for $i = 1, \ldots, n$. Using that $\psi_1 \leq 0$ and that by convexity $(\nabla \hat{g}^j(t, z) - \nabla \hat{g}^j(t, 0))z^j \geq 0$ (elementwise), multiplying both sides by $\hat{\mathcal{H}}$ yields for any $\bar{\gamma}_t = (\bar{\gamma}_t^1, \ldots, \bar{\gamma}_t^n)$ with $\bar{\gamma}_t^j \in \nabla \hat{g}^j(t, 0)$ being uniformly bounded by assumption (Hg) or (HL) that

$$|\mathcal{H}|^2 = -\mathcal{H} (\hat{M} - \hat{Z}(H_M))^\top + \psi_1 \hat{\mathcal{H}} Y$$

$$\leq \frac{|\hat{\mathcal{H}}|^2}{2a^2} + \frac{a^2|M| + \hat{Z}(H_M)|^2}{2} + \psi_1 \hat{\mathcal{H}} \bar{\gamma} \leq \frac{|\hat{\mathcal{H}}|^2}{2a^2} + a^2(|\bar{\gamma}| + |\bar{\gamma}|^2) \bar{\gamma}^2 \left( \frac{a^2}{2} + \frac{|\hat{\mathcal{H}}|^2}{2a^2} \right),$$

with $a > 0$. Since again by Proposition 2.3 $\hat{Z}(H_M)$ is bounded, we can choose a fixed and sufficiently large constant $C$ and the lemma follows.

Proof of Theorem 3.3. Denote $W = (\hat{W}, \bar{W})^\top$ with $\hat{W} = (W^1, \ldots, W^n)^\top$ and $\bar{W} = (W^{n+1}, \ldots, W^d)^\top$. Recall that $\bar{Z}_t := (\mathcal{H}^1(t, X, \zeta, M_t^1), \ldots, \mathcal{H}^n(t, X, \zeta, M_t^n), Z^{n+1}_t(H_M), \ldots, Z^d_t(H_M))$ and denote $\bar{M} := (M^{n+1}, \ldots, M^d)$. Since by Lemma C.2, $\mathcal{H}$ grows at most linearly in $M$, $E\int_0^T |\mathcal{H}(t, X_t, \zeta_t, M_t)|^2 dt < \infty$, and therefore $\bar{Z}^*$ taking values in $\mathbb{C}(\bar{I})$ is square-integrable. For an $n$-dimensional progressively measurable process $Y = (Y^1, \ldots, Y^n)$ with $Y_t^i \in \nabla \hat{g}^i(t, \bar{H}^i(t, X_t, \zeta_t, M_t^i))$ for $i = 1, \ldots, n$ (with $\bar{g}^i$ defined in (3.1)) we have

$$dU'(X_t + \zeta_t) = U''(X_t + \zeta_t)(Z^* + M_t - Z_t(H_M))dW_t$$

$$=: U'(X_t + \zeta_t)\left( Y_t d\bar{W}_t + \frac{U''(X_t + \zeta_t)}{U'(X_t + \zeta_t)} M_t d\bar{W}_t \right), \quad (C.16)$$

where we used (3.5) in the first, and Proposition C.1 in the second equation. Hence, $U'(X_t + \zeta_t)$ is always a local martingale. In case of assumption (a) in the theorem, $U'(X_t + \zeta_t)$ by assumption is actually a positive martingale. That the same holds in case of (b) is seen as follows: By Proposition 2.3, $Z(H_M)$ is bounded. Thus, given $X$, the couple $(\zeta, M)$ may be viewed as a solution of a quadratic FBSDE, and by Proposition 2.1 in [36], $\zeta$ is bounded. Hence, by Proposition 3.13 in Barrieu and El Karoui [6], $M$ is a BMO$(\mathcal{P})$ (see the Appendix A for a definition of BMOs). Lemma C.2 entails then that $Z^*$ is BMO$(\mathcal{P})$, and therefore $Y$ is BMO$(\mathcal{P})$. Since by assumption in case (b), either the second term in (C.16) is zero or if $d > n$, $U''/U' = 1/\psi_1$ is bounded, it follows by Kazamaki’s criterion that $U'(X_t + \zeta_t)$ is a positive martingale.

Therefore, by (C.16), $dU'(X_t + \zeta_t) = U'(X_t + \zeta_t)G_t dW_t$ for a d-dimensional process $G = (Y, \ldots, Y^n)$ and $U'(X_t + \zeta_t)$ is a positive martingale. This entails that $U'(X_t + \zeta_t)/E[U''(X_t + \zeta_t)] = \mathcal{E}\left( \int_0^T G_s dW_s \right)$, with $\mathcal{E}$ corresponding to the Doléans-Dade exponential. Furthermore, by Girsanov’s theorem $dW_t^{Q_0} = dW_t - G_t dt$ is a standard Brownian motion under the measure $Q_0$ defined by $dQ_0 := \frac{U''(X_t + \zeta_t)}{E[U''(X_t + \zeta_t)]}$. Now for the first $n$ components of the Brownian motion $W^{Q_0}$ we get

$$dW_t^{Q_0,i} = dW_t^i - G_t^i dt = dW_t^i - Y_t^i dt \quad \text{for } i = 1, \ldots, n.$$
Note that $g^i$ and $\tilde{g}^i$ coincide on the domain of $g^i$ for $i = 1, \ldots, d$. We observe that

$$
\tilde{X}_T - X_T = \int_0^T (\tilde{Z}_t - \mathcal{H}(t, X_t, \zeta_t, M_t))dW_t - \int_0^T (\tilde{g}(t, \tilde{Z}_t) - \tilde{g}(t, \mathcal{H}(t, X_t, \zeta_t, M_t)))dt
$$

$$
= \sum_{i=1}^n \int_0^T (\tilde{Z}_t^i - \mathcal{H}(t, X_t, \zeta_t, M_t^i))dW_t^{i, Q_0}
- \sum_{i=1}^n \int_0^T \left( \tilde{g}^i(t, \tilde{Z}_t^i) - \tilde{g}(t, \mathcal{H}(t, X_t, \zeta_t, M_t^i)) - (\tilde{Z}_t^i - \mathcal{H}(t, X_t, \zeta_t, M_t^i))Y_t^i \right)dt.
$$

By the definition of $\mathcal{H}$ the components $n+1, \ldots, d$ were canceled. Since $Y^i \in \nabla \tilde{g}^i(t, \mathcal{H}(t, X_t, \zeta_t, M_t))$ the last integral is always non-negative due to the convexity of $\tilde{g}^i$. Therefore,

$$
\mathbb{E}[U(\tilde{X}_T + H_L) - U(X_T + H_L)] \leq \mathbb{E}[U'(X_T + H_L)(\tilde{X}_T - X_T)]
= \mathbb{E}[U'(X_T + H_L)]\mathbb{E}^{Q_0}[\tilde{X}_T - X_T]
\leq \mathbb{E}[U'(X_T + H_L)]\sum_{i=1}^n \mathbb{E}^{Q_0}\left[\int_0^T (\tilde{Z}_t^i - \mathcal{H}(t, X_t, \zeta_t, M_t^i))dW_t^{i, Q_0}\right].
$$

By the BDG inequality and the Cauchy-Schwarz inequality (using that $U'(X_T + H_L)$ is square-integrable) the last expectation is zero, showing that $X$ is indeed optimal. \(\square\)

**Proof of Proposition 3.5.** It is straightforward to see that the triple $(X_t^* = x_0, \zeta_t^* = 0, M_t^* = 0)$ solves the FBSDE system (3.5), where $\mathcal{H} \equiv Z(H_M) = Z^0$ and satisfies (C.14). Clearly, the resulting constant process $U'(x_0)$ is a bounded strictly positive martingale and $Z^0$ is square-integrable. Consequently, Theorem 3.3 applies without needing the assumption that $\psi_1$ is bounded. \(\square\)

**Proof of Theorem 3.6.** To show (i) and (ii), consider the problem

$$
\sup_{Z \text{ takes values in } cl(I), Z \in L^2(dP \times dt)} \mathbb{E} \left[ U(X_T^Z + H_L) \right].
$$

(C.18)

For the rest of the proof we will call the $Z$’s over which the supremum in (C.18) is taken admissible. Looking at the proof of Theorem 3.2, to show Theorem 3.6 it is actually sufficient to prove that the supremum in (C.18) is attained for some $Z^*$ and that $\mathbb{E}[|U(X_T^{Z^*} + H_L)|] < \infty$ and $\mathbb{E}[|U'(X_T^{Z^*} + H_L)|^{1+\ell}] < \infty$. To prove that this holds in case of (i), we first show that the space over which the supremum in (C.18) is taken is weakly compact, and, in addition to being concave, the function to be optimized is upper semi-continuous.

**Proof of (i).**

First of all, note that $\mathbb{E}[X_T^Z]$ is finite for any admissible $Z$ by (Hg) or (HL). In particular $\mathbb{E}[U(X_T^Z + H_L)] < \infty$. Furthermore, since the mapping $Z \mapsto X_T^Z$ is concave in $Z$ and by Proposition 2.3, $\bar{I}$ is concave as well, we have

$$
U\left(X_T^{\lambda Z_1 + (1-\lambda)Z_2} + H_L\right) \geq U\left(\lambda X_T^{Z_1} + H_L + (1-\lambda)X_T^{Z_2} + H_L\right) \geq \lambda U(X_T^{Z_1} + H_L) + (1-\lambda)U(X_T^{Z_2} + H_L).
$$

It follows that also the mapping $Z \mapsto \mathbb{E}[U(X_T^Z + H_L)]$ from $L^2(dP \times ds)$ to $\mathbb{R} \cup \{-\infty\}$ is concave. Now, to show that a concave functional attains its maximum, it is sufficient to show that we may restrict the maximization problem to a weakly compact set, and that the functional is (strongly) upper semi-continuous.
Weak Compactness: Note that we may restrict ourselves to \( \mathcal{Z} \) such that 
\[
E\left[U(x_0 + H_L)\right] \leq E\left[U(X_T^F + H_L)\right] \leq a + \hat{K} E\left[X_T^F + H_L\right],
\]
for a certain \( a \) and \( \hat{K} > 0 \), where the second inequality holds as \( U \) grows at most linearly by concavity. Hence, we may restrict the optimization problem to wealth processes satisfying
\[
E[X_T^F + H_L] \geq \frac{E[U(x_0 + H_L)]}{\hat{K}} - \frac{a}{\hat{K}}. \tag{C.19}
\]

For a suitable \( \hat{K} > 0 \), the corresponding \( \mathcal{Z} \in L^2(d\mathbf{P} \times ds) \) of such a wealth process satisfies
\[
E\left[\int_0^T |Z_s|^2 ds\right] \leq \hat{K} \left(1 + E\left[\int_0^T g(s, Z_s) ds\right]\right)
\]
\[
= \hat{K} \left(1 + x_0 + E[H_M] - \Pi_0(H_M) + E\left[-X_T^F + \int_0^T g(s, Z_s) ds - \int_0^T Z_s dW_s - H_M + \Pi_0(H_M)\right]\right)
\]
\[
\leq \hat{K} \left(1 + x_0 + E[H_M] - \Pi_0(H_M) + \frac{a}{\hat{K}} + E[H_L] - \frac{E[U(x_0 + H_L)]}{\hat{K}}\right).
\]
The first inequality holds since \( g \) is assumed to grow at least quadratically. The last inequality holds by (C.19). Hence, there exists \( K' > 0 \) with \( E\left[\int_0^T |Z_s|^2 ds\right] \leq K' \) and we can set \( E\left[U(X_T^F + H_L)\right] = -\infty \) for any other \( \mathcal{Z} \) without affecting the maximization problem.

Upper semi-continuity: If \( \mathcal{Z}^m \to \mathcal{Z} \) in \( L^2(d\mathbf{P} \times ds) \) as \( m \) tends to infinity, then there exists a subsequence converging a.s., which guarantees that \( \mathcal{Z} \) takes values in \( cl(\mathcal{Z}) \) (pointwise closure). By (Hg) or (HL), \( |g(s, Z^m_s)| \leq K(1 + |Z^m_s|^2) \), which is uniformly integrable in \( L^1(d\mathbf{P} \times ds) \). Therefore,
\[
-K \leq \int_0^T g(s, Z^m_s) ds \xrightarrow{L^1} \int_0^T g(s, Z_s) ds \quad \text{and} \quad \int_0^T Z^m_s dW_s \xrightarrow{L^2} \int_0^T Z_s dW_s, \quad \text{as } m \to \infty.
\]

Thus, \( X_T^{Z^m} \to X_T^F \) in \( L^1 \) as \( m \to \infty \). Hence, denoting by \((\cdot)^+\) the positive part of a number (and the negative part similarly) we have \( (U(X_T^F + H_L))^{+} \leq (a + K(X_T^{Z^m} + H_L))^{+} \) is uniformly integrable. Denoting by \( m_j \) a subsequence which attains the limsup and by \( m_{j\ln} \) a subsequence of \( m_j \) along which \( X_T^{Z^{m_{j\ln}}} \) converges to \( X_T \) a.s., we obtain by Fatou’s generalized Lemma
\[
\limsup_m E[U(X_T^{Z^m} + H_L)] = \limsup_{m_{j\ln}} E[U(X_T^{Z^{m_{j\ln}}} + H_L)] \leq E[U(X_T^F + H_L)].
\]

This shows upper semi-continuity.

Upper semi-continuity together with the weak compactness of the optimization set guarantees for a concave functional the existence of an optimal \( Z^* \). Next note that by (Hg) or (HL), \( X_T^{Z^*} \in L^1 \) which entails that for the positive utility part we have \( E[U^{+}(X_T^{Z^*} + H_L)] < \infty \). Since \( \theta^* = 0 \) is an admissible strategy with corresponding wealth process \( x_0 \), we obtain \(-\infty < U(x_0) \leq E[U(X_T^{Z^*} + H_L)]\) so that we can not have that \( E[U^{+}(X_T^{Z^*} + H_L)] = -\infty \). Hence, overall \( E[U(X_T^{Z^*} + H_L)] < \infty \).

From this and from the fact that the first moment of the optimal wealth process exists, we obtain
from our growth conditions on $U'$ that also $\mathbb{E}[|U'(X^*_T + H_L)|^{1+\delta}] < \infty$. By Theorem 3.2 and the remark in the beginning of the proof, (i) follows.

**Proof of (ii).** Since by assumption $U(x) = U(w_0)$ for all $x \geq w_0$ we may assume that for the optimal $Z^*$ we have that $X^*_T + H_L \leq w_0$. This can be seen as follows: Recall that by Remark 2.6, $\mathcal{Z}^0 = \mathcal{Z}(H_M)$ corresponds to a $\mathcal{Z}$ induced by investing nothing, which by (2.9) entails that the corresponding wealth process $X$ is constant. Denote $\tau = \inf \{ t > 0 | X_t + H_L \geq w_0 \}$ and set

$$Z^*_t = Z^*_t1_{t \leq \tau} + Z_t(H_M)1_{t > \tau}.$$  \hfill (C.20)

Then $Z^*$ is admissible (i.e., square integrable and taking values in $c(l)$) and $X^*_T + H_L = X^*_{T\wedge \tau} + H_L \leq w_0$. Furthermore, clearly

$$\mathbb{E}[U(X^*_T + H_L)] = \mathbb{E}[U(X^*_T + H_L)1_{T < \tau}] + \mathbb{E}[U(w_0)1_{T \geq \tau}] \geq \mathbb{E}[U(X^*_T + H_L)1_{T < \tau}] + \mathbb{E}[U(X^*_T + H_L)1_{T \geq \tau}] = \mathbb{E}[U(X^*_T + H_L)],$$

so that $Z^*$ is optimal as well. Therefore, it is indeed sufficient to consider $\mathcal{Z}$ with $X^*_T + H_L \leq w_0$.

Next, assume first that (HL) holds and therefore $g$ is Lipschitz-continuous. Let $X^m_T$ be a sequence with $X^m_T \leq w_0 - H_L$ whose expected utility converges as an increasing sequence to the supremum in (C.18) as $m$ tends to infinity, i.e., $\mathbb{E}[U(X^m_T + H_L)] \leq \mathbb{E}[U(X^*_T + H_L)] \leq \ldots$ and

$$-\infty < v^* := \lim_m \mathbb{E}[U(X^m_T + H_L)] = \sup_{\mathcal{Z}}\mathbb{E}[U(X^\mathcal{Z}_T + H_L)].$$  \hfill (C.21)

Assume by contradiction that $||X^m_T||_2 \to \infty$ as $m \to \infty$. As $X^m_T$ is bounded from above, we have $||(X^m_T) - ||_2 \to \infty$ and consequently also $||(X^m_T + H_L) - ||_2 \to \infty$ as $m \to \infty$, since $H_L$ is bounded. Therefore, as $U^-$ decreases faster than quadratically,

$$\mathbb{E}[U(X^m_T + H_L)] \leq \mathbb{E}[-(U(X^m_T + H_L))] + U(w_0) \leq K'||(X^m_T + H_L)^-||_2^2 + K' + U(w_0) \to -\infty,$$

for some $K' > 0$, which is a contradiction to (C.21). The sequence $X^m_T$ must therefore be bounded in $L^2$ entailing that there exists a subsequence (for the sake of simplicity again denoted by $m$) which converges weakly to an $X^* \in L^2$.

**Claim A:** $X^*$ is admissible (i.e., $X^* = X^*_T$ for an admissible $Z^*$) and attains the supremum in (C.18).

Let us show that Claim A holds. By Mazur’s lemma there exists a function $N : \mathbb{N} \to \mathbb{N}$ and a sequence of non-negative numbers $\{\lambda(m)_k : k = m, \ldots, N(m)\}$ such that $\sum_{k=m}^{N(m)} \lambda(m)_k = 1$ and the sequence defined by the convex combination $\tilde{X}_t^m := \sum_{k=m}^{N(m)} \lambda(m)_k X^m_t$ has the property that $\tilde{X}_t^m$ converges in $L^2$ to $X^*$ as $m$ tends to infinity. By switching to a subsequence if necessary we may assume that additionally convergence holds a.s. Finally, as $X^m_T \leq w_0 - H_L$ we have that

$$\tilde{X}_t^m \leq w_0 - H_L, \quad \text{and therefore} \quad X^* \leq w_0 - H_L,$$  \hfill (C.22)

as well. Denote $\tilde{Z}^m := \sum_{k=m}^{N(m)} \lambda(m)_k Z^k$ which is in $L^2(d\mathbf{P} \times dt)$ since the $Z^k$ themselves are by construction admissible and hence in $L^2(d\mathbf{P} \times dt)$. Furthermore, by Proposition 2.3, $c(l)$ clearly is convex. Therefore, $\tilde{Z}^m$ takes values in $c(l)$ as a convex combination of elements of $c(l)$. Hence, under our assumptions $\tilde{Z}^m$ is admissible.

Next, note that by concavity of $-g$ we have that
\[ X_T^{\tilde{Z}_m} \geq \sum_{k=m}^{N(m)} \lambda(m)_k X_T^{\hat{Z}_k} = \hat{X}_T^m. \]  

Thus,
\[
\lim_{m} \inf \mathbb{E}[U(X_T^{\tilde{Z}_m} + H_L)] \geq \lim_{m} \inf \mathbb{E}[U(X_T^{\hat{X}_m} + H_L)] \geq \lim_{m} \inf \mathbb{E}[U(X_T^{Z_m} + H_L)] = v^*,
\]

where the last inequality follows as \( U \) is concave, \( \hat{X}_m \) is a convex combination of \( X_T^{\tilde{Z}_m}, X_T^{\tilde{Z}_{m+1}}, \ldots \), and by construction \( \mathbb{E}[U(X_T^{Z_m^*} + H_L)] \) increases in \( m \). Hence, since \( \tilde{Z}_m \) are admissible by the definition of \( v^* \) in (C.21), all equalities in (C.24) must be equalities. In particular, \( X_T^{\tilde{Z}_m^*} \) itself is an admissible maximizing sequence and the same holds true for the sequence which “stops” at \( w_0 - H_L, X_T^{\tilde{Z}_m^*,\tau_m^*} \) with \( \tau_m^* = \inf\{t > 0 | X_T^{\tilde{Z}_m^*} + H_L \geq w_0\} \) and \( \tilde{Z}_m^*,\tau_m^* = \tilde{Z}_m^* 1_{t \leq \tau_m^*} + Z_t(H_M) 1_{t > \tau_m^*} \) as defined in (C.20).

Furthermore, by (C.24) and Fatou’s lemma we have that
\[
v^* = \lim_{m} \inf \mathbb{E}[U(\hat{X}_T^m + H_L)] \leq \mathbb{E}[U(X^* + H_L)],
\]

showing that \( X^* \) is superoptimal. Now to show Claim A, we will first show the following claim B:

**Claim B:** \( X_T^{\tilde{Z}_m^*,\tau_m^*} \) converges to \( X^* \) in \( L^2 \).

To see this, the first key observation is that the admissible wealth process \( X_T^{\tilde{Z}_m^*,\tau_m^*} = X_T^{\tilde{Z}_m^*} 1_{\tau_m^* > T} + (w_0 - H_L) 1_{\tau_m^* \leq T} \) is either equal to \( X_T^{\tilde{Z}_m^*} \) (which, by (C.23), is greater or equal than \( \hat{X}_T^m \)) or is equal to \( w_0 - H_L \) (which by (C.22) is greater or equal than \( \hat{X}_T^m \) as well). Thus,
\[
\hat{X}_m^T \leq X_T^{\tilde{Z}_m^*,\tau_m^*} \leq w_0 - H_L.
\]

As \( \hat{X}_m^T \) converges to \( X^* \) in \( L^2 \) and \( H_L \) is uniformly bounded, \( |X_T^{\tilde{Z}_m^*,\tau_m^*}|^2 \) is uniformly integrable. Next, let us show that \( X_T^{\tilde{Z}_m^*,\tau_m^*} \) converges in probability to \( X^* \) as \( m \) tends to infinity which would prove Claim B. To this end, recall that \( \hat{X}_m^T \) converges to \( X^* \) a.s. and that therefore by (C.26)
\[
w_0 - H_L \geq \lim_{m} \inf X_T^{\tilde{Z}_m^*,\tau_m^*} \geq \lim_{m} \hat{X}_m^T = X^*.
\]

Clearly, on the set \( A := \{X^* = w_0 - H_L\} \) we must have that all inequalities in (C.27) are equalities. Since an analogous argument holds when taking the lim sup instead of the lim inf in (C.26), we have that
\[
\lim_{m} X_T^{\tilde{Z}_m^*,\tau_m^*} = X^*, \quad \text{on } A.
\]

Last, let us remark that also on \( A^c = \{X^* < w_0 - H_L\} \), (the complement of \( A \)) we actually must have that \( X_T^{\tilde{Z}_m^*,\tau_m^*} \) converges to \( X^* \) in probability. As shown in Lemma C.3 below, the main idea is that from (C.27) we already know that the lim inf of \( X_T^{\tilde{Z}_m^*,\tau_m^*} \) dominates \( X^* \) a.s. Observe that this domination can actually not be strict on \( A^c \), as this would be a contradiction that by (C.25) \( X^* \) is superoptimal. Now, the fact that \( X_T^{\tilde{Z}_m^*,\tau_m^*} \) converges in probability to \( X^* \) together with uniform integrability of \( |X_T^{\tilde{Z}_m^*,\tau_m^*}|^2 \) proves Claim B.

Next, let us complete proving Claim A. By Claim B, we have that the admissible terminal wealth \( X_T^{\tilde{Z}_m^*,\tau_m^*} \) converges to \( X^* \) in \( L^2 \). From well known results on the continuity of BSDEs with Lipschitz-continuous drivers in the terminal condition, it follows then that the tuple
\[
(\Pi_t(-X_T^{\tilde{Z}_m^*,\tau_m^*} + H_M - \Pi_0(H_M)), \tilde{Z}_t^{m,\tau_m^*})
\]
\[
= (-X_T^{\tilde{Z}_m^*,\tau_m^*} + H_M - \Pi_0(H_M) - \int_t^T g(t, \tilde{Z}_t^{m,\tau_m^*}) dt + \int_t^T \tilde{Z}_t^{m,\tau_m^*} dW_t, \tilde{Z}_t^{m,\tau_m^*})
\]
converges to

\[(\Pi_t(-X^* + H_M - \Pi_0(H_M)), \hat{Z}_t^*) = (-X^* + H_M - \Pi_0(H_M) - \int_t^T g(t, \hat{Z}_t^*)dt + \int_t^T \hat{Z}_t^*dW_t, \hat{Z}_t^*)\]

in \(\mathcal{S}^2 \times L^2(d\mathbb{P} \times ds)\) with \(\hat{Z}^*\) being the unique square-integrable process from (2.1) with terminal condition \(-X^* + H_M - \Pi_0(H_M)\). Note that \(X^* = X_T^{\hat{Z}^*}\) (plug in \(t = 0\) above and compare the first component of the tuple with (2.9)). By switching to a subsequence if necessary we may assume that \(\hat{Z}_i^{m,n} \) converge to \(\hat{Z}^* d\mathbb{P} \times ds\) a.s. so that \(\hat{Z}^*\) takes values in \(\text{cl}(I)\). In particular, \(X^*\) is an admissible portfolio of problem (C.18), and by (C.25) attains the optimum, showing (ii) the existence of an optimal \(Z^*\) in case the driver is Lipschitz.

Finally, if the driver is not Lipschitz, \((Hg)\) holds, and therefore the driver grows at least quadratically. In particular, (i) already shown above applies showing the existence of an optimal \(Z^*\). For both cases \((HL)\) and \((Hg)\) it can be checked as in (i) that the integrability conditions of \(U(X_T^{\hat{Z}^*} + H_L)\) and \(U'(X_T^{\hat{Z}^*} + H_L)\) are satisfied so that by Theorem 3.2, (ii) follows.

**Proof of (iii).** So assume that \(U\) is a CARA utility function, i.e. \(U(x) = a - be^{-x/\gamma}\) for \(a \in \mathbb{R}\) and \(b, \gamma > 0\). Then, from (C.14) we deduce that \(\mathcal{H}(t, X, \zeta, M) = \mathcal{H}(t, M)\) does not depend on \((X, \zeta)\). The FBSDE in (3.5) decouples and becomes then

\[\zeta_t = H_L - \int_t^T M_s dW_s + \int_t^T f(s, M_s)ds, \quad \zeta_T = H_L, \quad (C.29)\]

where, \(f(t, M) = \frac{\gamma}{T} \sum_{i=1}^n |\mathcal{H}_i(t, M^i) - Z_i^i(H_M)| + M^i|2 + \sum_{i=n+1}^d |M^i|^2 + g(t, \mathcal{H}(t, M))\). From Proposition 2.3, we have that \(Z_i^i(H_M)\) is bounded for \(i = 1, \ldots, d\). Furthermore, by Lemma C.2, \(\mathcal{H}\) grows at most linearly in \(M\), implying that \(f\) grows at most quadratically. The existence of a solution to the FBSDE follows then from Theorem 2.3 in [36].

**Lemma C.3.** In the proof of Theorem 3.6 (ii), \(X_T^{\hat{Z}^*,r^m}\) converges to \(X^*\) on \(A^c\) in probability

**Proof.** Let us show that on \(A^c = \{X^* + H_L < w_0\}\), \(X_T^{\hat{Z}^*,r^m}\) converges to \(X^*\) in probability. First of all note that if we could show that for any arbitrary \(\delta > 0\),

\[
\lim_m \mathbb{P}[X_T^{\hat{Z}^*,r^m} > X^* + \delta, X^* + H_L < w_0] = 0, \quad (C.30)
\]

then

\[
\lim_m \mathbb{P}[|X_T^{\hat{Z}^*,r^m} - X^*| > \delta, X^* + H_L < w_0] \leq \limsup_m \mathbb{P}[X_T^{\hat{Z}^*,r^m} > X^* + \delta, X^* + H_L < w_0] \\
+ \limsup_m \mathbb{P}[X_T^{\hat{Z}^*,r^m} < X^* - \delta, X^* + H_L < w_0] = 0,
\]

where the second term converges to zero by (C.27). So all what is left is to show (C.30). Assume by contradiction that (C.30) does not hold and we have that \(\mathbb{P}[X_T^{\hat{Z}^*,r^m} > X^* + \delta, X^* + H_L < w_0] > 2\varepsilon\), for a subsequence again indexed by \(m\) and an \(\varepsilon > 0\). Then there exists \(N_0 \in \mathbb{N}\) such that for all \(m > N_0\)

\[
\mathbb{P}[C_m] := \mathbb{P}\left[X_T^{\hat{Z}^*,r^m} > X^* + \delta, X^* + H_L < w_0 - \frac{1}{N_0}\right] > \varepsilon.
\]
Let $C_m^c$ denote the complement of $C_m$. Note that $U(x + y) - U(x)$ is decreasing in $x$ and increasing in $y$. Since $U(x) = const$ for $x + y \geq \omega_0$ this entails that on $X^* + H_L < w_0 - \frac{1}{N_0}$ we have $U(X^* + H_L + \delta) - U(X^* + H_L) \geq U(w_0) - U(w_0 - \min(\delta, \frac{1}{N_0}))$. Therefore,

$$\lim_{m} \mathbb{E}[U(X_T^{\hat{m},r_m} + H_L)] = \lim_{m} \left\{ \mathbb{E}[U(X_T^{\hat{m},r_m} + H_L)1_{C_m}] + \mathbb{E}[U(X_T^{\hat{m},r_m} + H_L)1_{C_m^c}] \right\}$$

$$\geq \lim_{m} \sup \left\{ \mathbb{E}[U(X^* + \delta + H_L)1_{C_m}] + \mathbb{E}[U(X_T^{\hat{m},r_m} + H_L)1_{C_m^c}] \right\}$$

$$\geq \lim_{m} \sup \left\{ \mathbb{E}[U(X^* + H_L)1_{C_m}] + \mathbb{E}[U(X_T^{\hat{m},r_m} + H_L)1_{C_m^c}] \right\} + \mathbb{E}[U(X_T^{\hat{m},r_m} + H_L)1_{C_m^c}]$$

$$> \lim_{m} \sup \left\{ \mathbb{E}[U(X^* + H_L)1_{C_m}] + \mathbb{E}[U(X_T^{\hat{m},r_m} + H_L)1_{C_m^c}] \right\}$$

$$\geq \lim_{m} \inf \left\{ \mathbb{E}[U(X^* + H_L)1_{C_m}] + \mathbb{E}[U(X^* + H_L)1_{C_m^c}] \right\} = \mathbb{E}[U(X^* + H_L)] \geq v^*,$$

which is a contradiction to the definition of $v^*$ in (C.21), where we used the definition of $C_m$ in the first, (C.27) in the fourth and (C.25) in the last inequality.

\[ \square \]

Appendix D: Proofs of Section 4

**Proof of Proposition 4.1.** Note that, $\mathbb{E}[U(x - \int_{t}^{T} g(s, Z_s)ds + \int_{t}^{T} Z_s dW_s + H_L)]$ is jointly concave in $(x, Z)$. By Proposition 2.3, $\mathbb{I}$, the set where $Z$ takes values in, is a convex set. Thus, $V$ is concave as maximum of a jointly concave function in one of its arguments over a convex set. To show the strict concavity let $\theta^1, \theta^2$ be the corresponding optimal strategy starting with initial wealth $x_1, x_2$ respectively, i.e.,

$$V(t, x_1) = \mathbb{E}[U(x_1 + \mathcal{I}_{t,T}(\theta^1) + H_L)|\mathcal{F}_t], \quad \text{and} \quad V(t, x_2) = \mathbb{E}[U(x_2 + \mathcal{I}_{t,T}(\theta^2) + H_L)|\mathcal{F}_t].$$

Let us assume that $\lambda V(t, x_1) + (1 - \lambda)V(t, x_2) = V(t, \lambda x_1 + (1 - \lambda)x_2)$ for some $\lambda \in [0, 1]$ on a non-zero $\mathcal{F}_t$-measurable set, say $A$. Due to the strict concavity of $U$ we deduce that $x_1 + \mathcal{I}_{t,T}(\theta^1) = x_2 + \mathcal{I}_{t,T}(\theta^2)$, on $A$ which leads to

$$x_2 - x_1 = -\int_{t}^{T}(g(s, Z_s^{\theta^1}) - g(s, Z_s^{\theta^2}))ds + \int_{t}^{T}(Z_s^{\theta^1} - Z_s^{\theta^2})dW_s = \int_{t}^{T}(Z_s^{\theta^1} - Z_s^{\theta^2})dW_s^G, \quad \text{(D.1)}$$

on $A$, where $W_t^G := W_t - \sum_{j=1}^{d} \int_{t}^{T} G^j(s, Z_s^{\theta^1}, Z_s^{\theta^2})ds$ and

$$G^j(s, z, \tilde{z}) := g(s, z^1, \ldots, z^{j}, \tilde{z}^{j+1}, \ldots, \tilde{z}^{d}) - g(s, z^1, \ldots, z^{j-1}, \tilde{z}^{j}, \ldots, \tilde{z}^{d}),$$

with $\frac{0}{0} := 0$. In case of Condition (Hl), $G$ is bounded. In case of Condition (Hg) we observe that $|G(s, Z_s^{\theta^1}, Z_s^{\theta^2})| \leq K(1 + |Z_s^{\theta^1}| + |Z_s^{\theta^2}|)$. Now it follows directly from (D.1) that $(Z_s^{\theta^1})_{t\leq s\leq T}$
and \((Z^\theta_s)_{t \leq s \leq T}\) are \(\text{BMO}(\mathbb{P}|A)\)'s (see Appendix A). Thus, we conclude by our growth conditions that \(G(s, Z^\theta_s, Z^\theta_{s+}) \in \text{BMO}(\mathbb{P}|A)\) as well. Hence, by Kazamaki's Theorem (see e.g. [6] [Th. 3.24, Chapter 3]), \(W^G\) is a Brownian motion under \(\mathbb{P}^G\) conditionally on \(A\) defined by \(d\mathbb{P}^G/d\mathbb{P}|A = E_T(-\int_t^T G(s, Z^\theta_s, Z^\theta_{s+})dW_s)/\mathbb{P}(A)\). Moreover, since \(\int_t^T (Z^\theta_s - Z^\theta_{s+})dW_s\) is a \(\text{BMO}(\mathbb{P}|A)\) martingale it follows from general properties of BMOs, see [6], that \(\int_t^T (Z^\theta_s - Z^\theta_{s+})dW_s^G\) is a \(\text{BMO}(\mathbb{P}^G|A)\) martingale. Taking expectations conditional with respect to \(\mathbb{P}^G|[A]\) on both sides in (D.1) we obtain that \(x_2 = x_1\) and the proof is complete. 

Proof of Proposition 4.2. Let \(\theta^0 \in \Theta\). Denote by \(\Theta(\theta^0, t, T)\) the set of all admissible strategies being equal to \(\theta^0\) until time \(t\), i.e., \(\theta \in \Theta(\theta^0, t, T)\) if \(\theta \in \Theta\) and \(\theta_s 1_{0 \leq s \leq t} = \theta^0_s 1_{0 \leq s \leq t}\). Let us show that the family

\[
\{ \Upsilon^\theta_t := \mathbb{E}[U(x + \mathcal{I}_{t,T}(\theta) + H_L)|\mathcal{F}_t], \theta \in \hat{\Theta}(\theta^0, t, T) \}
\]

admits the lattice property. Indeed, for \(\theta^1, \theta^2 \in \hat{\Theta}(\theta^0, t, T)\) we define

\[
\theta_s := \theta^0_s 1_{0 \leq s < t} + \left( \theta^1_s 1_{\Upsilon^\theta_t \geq \Upsilon^\theta_{t+}} + \theta^2_s 1_{\Upsilon^\theta_t < \Upsilon^\theta_{t+}} \right) 1_{T \geq s \geq t}.
\]

Note that for any \(y_1, y_2 \in \mathbb{R}^n\) and \(A \in \mathcal{F}_t\), we have \(Z_t(-y_1 S1_A - y_2 S1_{A^c}) = Z_t(-y_1 S)1_A + Z_t(-y_2 S)1_{A^c}\). It is then clear that

\[
\mathcal{Z}^\theta_{t+} = \mathcal{Z}^\theta_{t} + \left( \mathcal{Z}^\theta_{t+} 1_{\Upsilon^\theta_t \geq \Upsilon^\theta_{t+}} + \mathcal{Z}^\theta_{t+} 1_{\Upsilon^\theta_t < \Upsilon^\theta_{t+}} \right) 1_{T \geq s \geq t},
\]

and \(\theta\) is admissible. Since \(1_{\Upsilon^\theta_t \geq \Upsilon^\theta_{t+}}\) is \(\mathcal{F}_t\)-measurable we deduce that \(\Upsilon^\theta_t = 1_{\Upsilon^\theta_t \geq \Upsilon^\theta_{t+}} \Upsilon^\theta_t + 1_{\Upsilon^\theta_t < \Upsilon^\theta_{t+}} \Upsilon^\theta_t = \max(\Upsilon^\theta_t, \Upsilon^\theta_{t+})\). Noting that the lattice property allows to interchange essential supremum and conditional expectations, it follows that \(V(t, x + \mathcal{I}_{s,t}(\theta))\) is a supermartingale and (4.2) holds. Finally, the equivalence property can be seen as follows: if \(\theta^*\) is optimal then

\[
V(0, x) = \sup_{\theta \in \Theta} \mathbb{E}[U(x + \mathcal{I}_{0,T}(\theta) + H_L)] = \mathbb{E}[U(x + \mathcal{I}_{0,T}(\theta^*) + H_L)] = \mathbb{E}[V(T, x + \mathcal{I}_{0,T}(\theta^*))].
\]

Hence, the supermartingale process \(V(t, x + \mathcal{I}_{s,t}(\theta^*))\) is a martingale. Assume now that \(V(t, x + \mathcal{I}_{s,t}(\theta^*))\) is a martingale. We have then by the supermartingale property

\[
\sup_{\theta \in \Theta} \mathbb{E}[U(x + \mathcal{I}_{0,T}(\theta) + H_L)] \leq V(0, x) = \mathbb{E}[V(T, x + \mathcal{I}_{0,T}(\theta^*))] = \mathbb{E}[U(x + \mathcal{I}_{0,T}(\theta^*) + H_L)],
\]

which shows that \(\theta^*\) is optimal.

Proof of Lemma 4.4. The lemma follows as the right-hand side of (4.4) is a strictly concave, coercive function in \(Z\).

Proof of Theorem 4.5. Using Itô-Ventzel’s formula we can represent

\[
V(t, x + \mathcal{I}_{s,t}(\theta)) = V(s, x) + \int_s^t \left( G(u, x + \mathcal{I}_{s,u}(\theta), Z^\theta_u) - b(u, x + \mathcal{I}_{s,u}(\theta)) \right) du
\]

\[
+ \int_s^t \left( V_x(u, x + \mathcal{I}_{s,u}(\theta)) Z^\theta_u + \alpha(u, x + \mathcal{I}_{s,u}(\theta)) \right) dW_u,
\]

where \(G(u, p, z) := -g(u, z)V_x(u, p) + \frac{1}{2}|z|^2 V_{xx}(u, p) + \alpha(u, p)z\).
We recall from Proposition 4.2 that for any $x$, the value process $V(t, x + \mathcal{I}_{s,t}(\theta))$ is a supermartingale. Hence, the finite variation part in (D.2) is decreasing. Therefore, for any admissible strategy $\theta$ and $\epsilon > 0$
\[ \int_s^{s+\epsilon} b(u, x + \mathcal{I}_{s,u}(\theta))du \geq \int_s^{s+\epsilon} G(u, x + \mathcal{I}_{s,u}(\theta), Z^\theta_u)du. \]
Dividing both sides by $\epsilon$ and letting $\epsilon \to 0$ we obtain $b(s, x) \geq G(s, x, Z^\theta_s)$ d$\mathbb{P} \times dt$ a.s. Noting that $V_{xx} < 0$ and $V_x > 0$ a.e. we get from Lemma 4.4 for all $x$
\[ b(t, x) \geq \text{ess sup}_{Z \in \mathcal{C}(\mathbb{F}_t)} G(t, x, Z) = G(t, x, v(t, x)) = \mathcal{L}V(t, x), \] (D.3)
where $v(t, x)$ and $\mathcal{L}V(t, x)$ are defined by (4.4). Now assume that $\theta^*$ is an optimal strategy, i.e., $V(t, x + \mathcal{I}_{s,t}(\theta^*))$ is a martingale. Let $X^\theta_{s,t}(x) = x + \mathcal{I}_{s,t}(\theta^*)$ and note that $X^\theta_{0,t}(x) = x + \mathcal{I}_{0,t}(\theta^*) = X^\theta_t$. Using the Itô-Ventzel formula we get for any $s \in [0, t]$, $G(t, X^\theta_{s,t}(x), Z^\theta_t) - b(t, X^\theta_{s,t}(x)) = 0$, d$\mathbb{P} \times dt$ a.s. It follows from (D.3) then that $Z^\theta_t$ must be the maximizer of $G(t, X^\theta_{s,t}(x), Z)$ and therefore $Z^\theta_t = v(t, X^\theta_{s,t}(x))$. Hence, for any $s \in [0, t]$ we have $0 \in U^V(t, Z^\theta_t, X^\theta_{s,t}(x))$. Consequently, taking $s = 0$ and $s = t$ we obtain (4.6) and $b(t, x) = \mathcal{L}V(t, x)$, which leads to the BSPDE (4.5).

On the other hand, if $\bar{\theta}$ is a strategy satisfying (4.6)-(4.7), one can verify using the preceding steps that the value process $V(t, x + \mathcal{I}_{s,t}(\bar{\theta}))$ is a local martingale. Since it by assumption belongs to the class $D$, $V(t, x + \mathcal{I}_{s,t}(\bar{\theta}))$ is a martingale and $\bar{\theta}$ is optimal by Proposition 4.2.

\textbf{Proof of Theorem 4.6.} First, by the Itô-Ventzel formula, it can be seen directly that

\[ V_x(t, X^\theta_t) = V_x(0, x) + \int_0^t \left( Z^\theta_s V_{xx}(s, X^\theta_s) + \alpha_x(s, X^\theta_s) \right) dW_s \]
\[ + \int_0^t (- \mathcal{L}V_x(s, X^\theta_s) + \frac{1}{2} |Z^\theta_s|^2 V_{xx}(s, X^\theta_s) - g(s, Z^\theta_s)V_{xx}(s, X^\theta_s) + Z^\theta_s \alpha_{xx}(s, X^\theta_s))ds. \]

From Lemma D.1 below we observe that the finite variation term is zero, which implies that $V_x(t, X^\theta_t)$ is a local martingale whose decomposition is given by

\[ V_x(t, X^\theta_t) = V_x(0, x) + \int_0^t \left( Z^\theta_s V_{xx}(s, X^\theta_s) + \alpha_x(s, X^\theta_s) \right) dW_s. \]

Let $\zeta_t = I(V_x(t, X^\theta_t)) - X^\theta_t$. Clearly, $V_x(t, X^\theta_t) = U'(X^\theta_t + \zeta_t)$ and therefore

\[ I'(V_x(t, X^\theta_t)) = \frac{1}{U''(X^\theta_t + \zeta_t)}, \quad I''(V_x(t, X^\theta_t)) = -\frac{U^{(3)}(X^\theta_t + \zeta_t)}{(U'')^2(X^\theta_t + \zeta_t)}. \]

Note that $\zeta_T = I(U'(X^\theta_T + H_L)) - X^\theta_T = H_L$. By Itô’s formula we get

\[ d\zeta_t = \frac{Z^\theta_t V_{xx}(t, X^\theta_t) + \alpha_x(t, X^\theta_t)}{U''(X^\theta_t + \zeta_t)}dW_t - Z^\theta_t dW_t \]
\[ + \frac{1}{2} I''(V_x(t, X^\theta_t)) |Z^\theta_t V_{xx}(t, X^\theta_t) + \alpha_x(t, X^\theta_t)|^2 dt + g(t, Z^\theta_t)dt. \] (D.4)

Set

\[ M_t = \frac{v(t, X^\theta_t)V_{xx}(t, X^\theta_t) + \alpha_x(t, X^\theta_t)}{U''(X^\theta_t + \zeta_t)} - v(t, X^\theta_t). \] (D.5)
By (4.6) we have \( v(t, X_t^{\theta^*}) V_{xx}(t, X_t^{\theta^*}) + \alpha_x(t, X_t^{\theta^*}) \in \nabla g^I(t, v(t, X_t^{\theta^*})) V_x(t, X_t^{\theta^*}) \). Hence, it follows from (D.5) that \( 0 \in -\nabla g^I(t, v(t, X_t^{\theta^*})) U'(X_t^{\theta^*} + \zeta_t) + U''(X_t^{\theta^*} + \zeta_t)(M_t + v(t, X_t^{\theta^*})) \), and we have \( H(t, X_t^{\theta^*}, \zeta_t, M_t) = Z_t^{\theta^*} = v(t, X_t^{\theta^*}) \), where the last equation holds by Theorem 4.5. It is then straightforward to see using (D.4) that the triple \((X_t^{\theta^*}, \zeta_t, M_t)\) satisfies the FBSDE (3.5) with \( H_M = 0 \). \( \square \)

Lemma D.1. If \( v \) only takes values in the interior of \( \mathbb{I} \) we have

\[
L^V_x(t, x) = -g(t, v(t, x)) V_{xx}(t, x) + \frac{1}{2} v^2(t, x) V_{xxx}(t, x) + v(t, x) \alpha_{xx}(t, x).
\]

Proof. By Lemma 4.4 and since \( g \) is differentiable, \( v \) is characterized by

\[
-g_x(t, v(t, x)) V_x(t, x) + v(t, x) V_{xx}(t, x) + \alpha_x(t, x) = \mathcal{U}^V(t, v(t, x), x) = 0. \tag{D.6}
\]

Note that \( g_x \) is increasing, \( V_x > 0 \), and \( V_{xx} < 0 \). As \( v \) by assumption only takes values in the interior of \( \mathbb{I} \) all functions involved are continuously differentiable and the implicit function theorem gives that \( v \) is continuously differentiable in \( x \). Hence, taking the derivative of \( L^V \) in (4.4) w.r.t. \( x \) we obtain using (D.6)

\[
L^V_x(t, x) = v_x(t, x)(-g_x(t, v(t, x)) V_x(t, x) + v(t, x) V_{xx}(t, x) + \alpha_x(t, x))
\]

\[
- g(t, v(t, x)) V_x(t, x) + \frac{1}{2} v_x^2(t, x) V_{xx}(t, x) + v(t, x) \alpha_{xx}(t, x)
\]

\[
= -g(t, v(t, x)) V_{xx}(t, x) + \frac{1}{2} v_x^2(t, x) V_{xxx}(t, x) + v(t, x) \alpha_{xx}(t, x).
\]

Appendix E: Proofs of Section 5

Proof of Proposition 5.1. By (4.1) and (CV), the value function can be written as \( V(t, x) = U(x)V_t \)

where

\[
V_t : = \text{essinf}_{\theta \in \Theta, b, \alpha \in [t, T]} \mathbb{E}[e^{-\gamma T_t, x \theta}(\mathcal{F}_t)] = \mathbb{E}[e^{-\gamma T_t, x \theta} | \mathcal{F}_t] = e^{\gamma T_0, x \theta} \mathbb{E}[e^{-\gamma T_0, x \theta} | \mathcal{F}_t].
\]

By the martingale representation theorem and since \( \mathcal{I}_{0,t}(\theta^*) \) is an Itô-process, \( V_t \) is an Itô-process as well, and therefore, \( V(t, x) \) is a special semimartingale. Hence, for \( V(t, x) = U(x)V_t \), conditions (a)-(c) in Definition 4.3 are fulfilled. The definition of \( V_t \) and the separation structure allow us to rewrite the BSPDE (4.3) as \( V(t, x) = U(x)V_t \) with \( \alpha(t, x) = U(x) \alpha_t, b(t, x) = U(x)b_t \), where \( V \) satisfies a BSDE of the form \( V_t = V_0 - \int_0^t b_s ds + \int_0^t \alpha_s dW_s \) and \( V_T = 1 \). Repeating the steps in the proof of Theorem 4.5 (where, due to the separation structure, we can simply apply Itô’s Lemma rather than the Itô-Ventzel formula) we obtain the finite variation term \( L^V_t \) of the proposition. \( \square \)

Proof of Proposition 5.2. Observe first that the function \( \vartheta : (0, \infty) \rightarrow (0, \infty), \lambda \mapsto \vartheta(\lambda) = \mathbb{E}[e^{\gamma f(\lambda \xi_T)} \xi_T] \) by assumption is well-defined. Since \( f \) is continuous, surjective and decreasing we observe that \( \vartheta \) is continuous and surjective as well by the monotone convergence theorem. Hence, for any \( x_0 \in \mathbb{R} \), there exists \( \lambda > 0 \) such that \( \mathbb{E}[e^{\gamma X_t^* \xi_T}] = e^{\gamma x_0} \). Noting from the definition of \( f \) that

\[
U'(X_t^*) = U'(f(\lambda \xi_T)) = \gamma \lambda \xi_T e^{\gamma f(\lambda \xi_T)}, \tag{E.1}
\]

we derive

\[
X_t^* = \frac{1}{\gamma} \log \mathbb{E}^Q[e^{\gamma f(\lambda \xi_T)} | \mathcal{F}_t] = \frac{1}{\gamma} \log \mathbb{E}[^{-1} \xi_T e^{\gamma f(\lambda \xi_T)} | \mathcal{F}_t] = \frac{1}{\gamma} \log \mathbb{E}[[^\lambda \xi_T] U'(X_t^*) | \mathcal{F}_t],
\]
which implies that \( X_t^* = \frac{1}{\gamma} \log(R_t^*/\lambda \gamma \xi_t) \). Recall that \( d\xi_t = -\eta_t \xi_t dW_t \). By Itô’s lemma we get
\[
dX_t^* = \frac{1}{2\gamma} \left( |\eta_t|^2 - \frac{\beta_t^*}{|R_t^*|^2} \right) dt + \frac{1}{\gamma} \left( \frac{\beta_t^*}{R_t^*} + \eta_t \right) dW_t.
\]
Identifying the last SDE with the wealth dynamics we obtain (5.1). It remains to verify that \((X_t^*, \zeta_t^*, M_t^*)\) solves the FBSDE system (3.5). To this end, we consider
\[
\zeta_t^* := I(R_t^*) - \frac{1}{\gamma} \log(R_t^*/\lambda \gamma \xi_t),
\]
where \(R_t^*\) is defined by (5.2). First, it follows from (E.1) that
\[
\zeta_T^* = I(R_T^*) - \frac{1}{\gamma} \log(R_T^*/\gamma \lambda \xi_T) = I(U'(X_T^*)) - \frac{1}{\gamma} \log(U'(X_T^*)) = X_T^* - \frac{1}{\gamma} \log(e^{\gamma X_T^*}) = 0.
\]
Set
\[
\mathcal{H}(t, X_t^*, \zeta_t^*) := \frac{1}{\gamma} \left( \frac{\beta_t^*}{R_t^*} + \eta_t \right).
\]
It follows that
\[
X_t^* = x_0 - \int_0^t g(s, \mathcal{H}(s, X_s^*, \zeta_s^*)) ds + \int_0^t \mathcal{H}(s, X_s^*, \zeta_s^*) dW_s.
\]
Furthermore, applying Itô’s lemma for (E.2) we obtain that
\[
\zeta_t^* = -\int_t^T M_s^* dW_s + \frac{1}{2} \int_t^T \left( \frac{\beta_s^*}{U''(X_s^* + \zeta_s^*)} \mathcal{H}(s, X_s^*, \zeta_s^*) + M_s^* \right)^2 ds - \int_t^T g(s, \mathcal{H}(s, X_s^*, \zeta_s^*)) ds,
\]
where \(M_t^*\) is defined by
\[
M_t^* := \frac{\beta_t^*}{U''(X_t^* + \zeta_t^*)} - \mathcal{H}(t, X_t^*, \zeta_t^*).
\]
From (E.4) and (E.5) we conclude that the triple \((X_t^*, \zeta_t^*, M_t^*)\) solves the FBSDE system (3.5). To finish the proof we recall that \(R_t^* = U'(X_t^* + \zeta_t^*)\) and \(g_z(t, z) = \gamma z - \eta_t\). It then follows from (E.6) and (E.3) that
\[
-U'(X_t^* + \zeta_t^*) g_z(t, \mathcal{H}(t, X_t^*, \zeta_t^*)) + U''(X_t^* + \zeta_t^*) (\mathcal{H}(t, X_t^*, \zeta_t^*) + M_t^* ) = 0,
\]
which means that the optimality condition (C.14) is fulfilled. To finish, assuming now \(U(x) = -e^{-\gamma A x} \) we easily compute the inverse marginal utility as \(f(x) = -\frac{1}{\gamma + \gamma A} \log(\frac{\gamma x}{\gamma A})\). Therefore, the optimal terminal wealth is given by \(X_T^* = \frac{-1}{\gamma + \gamma A} \log(\frac{2 \lambda T}{\gamma A})\), where the multiplier \(\lambda\) is defined by
\[
x_0 = \frac{1}{\gamma} \log \left( E^Q [e^{\gamma X_T^*}] \right)
= \frac{1}{\gamma} \log \left( E^Q \left[ \frac{\lambda}{\gamma A} \log \left( \frac{\gamma \xi_T^*}{\gamma A} \right) \right] \right)
= \frac{-1}{\gamma + \gamma A} \log \left( \frac{\gamma \lambda}{\gamma A} \right) + \frac{-\gamma A}{2(\gamma + \gamma A)^2} \int_0^T |\eta_s|^2 ds.
\]
In the same way we obtain

\[ X_t^* = \frac{1}{\gamma} \log \left( E^Q [e^{\gamma X_T^*} | F_t] \right) = \frac{-1}{\gamma + \gamma A} \log \left( \frac{\gamma A}{\gamma A} \right) + \frac{-1}{\gamma + \gamma A} \left( -\int_t^T \eta_s dW_s^Q + \frac{1}{2} \int_t^T |\eta_s|^2 ds \right) + \frac{-\gamma A}{2(\gamma + \gamma A)^2} \int_t^T |\eta_s|^2 ds. \]  

(E.8)

Plugging (E.7) into the last equation we get

\[ X_t^* = x_0 - \frac{\gamma}{2} \int_0^t |\eta_s|^2 ds + \int_0^t \eta_s dW_s^Q, \]  

(E.9)

which implies \( Z_t^* = \frac{\eta_t}{\gamma + \gamma A} \). Now, consider the martingale process \( R_t^* := E[U'(X_T^*) | F_t] \). A direct calculation using (E.8) shows that \( R_t^* = U'(X_T^*) + \int_t^T R_s^* \gamma A \eta_s dW_s \), which means that \( \beta_s^* = -\frac{\gamma A}{\gamma + \gamma A} \eta_s \), implying \( R_t^* = -\frac{1}{\gamma + \gamma A} \int_t^T |\eta_s|^2 ds \), which means that \( M_t^* = 0 \). Finally, it is straightforward to check that the optimality condition (C.14) is fulfilled.

Lemma E.1. For \( x = f(y) \), which is the inverse of the decreasing function \( U'(x) e^{-\gamma x} \gamma^{-1} \) on \( \mathbb{R} \rightarrow \mathbb{R}_+ \), we have

(i) \( y U''(y) = \frac{\gamma}{R_1(x)+\gamma} e^{\gamma y} \),

(ii) \( y^2 U'''(y) = -\gamma^2 \left( \frac{\gamma}{R_1(x)+\gamma} \right)^2 e^{\gamma y} + \gamma \frac{R_1(x)R_3(x)+2\gamma R_1(x)-\gamma^2}{(R_1(x)+\gamma)^2} e^{\gamma y} \),

(iii) if, in addition, the relative risk aversion \( R_1(x) \) and \( R_2(x) \) are bounded and bounded away from zero and \( E[U(f(\lambda x)) < \infty \), and \( E[\gamma f(\lambda x)] < \infty \), for any \( \lambda > 0 \), then \( \tilde{V}(y) \) is well-defined and three times differentiable for \( y > 0 \).

Proof of Lemma E.1. It can be derived from (5.3) that \( \tilde{U}'(y) = -e^{\gamma y} \), \( \tilde{U}''(y) = -\gamma e^{\gamma y} x' \) and \( \tilde{U}'''(y) = -\gamma^2 e^{\gamma y} x'^2 - \gamma e^{\gamma y} x'' \) (suppressing the dependence of the optimal \( x \) on \( y \)). Using the relation \( \gamma y = U'(x) e^{-\gamma x} \) we obtain \( x' = \frac{\gamma y U'(x)}{U''(x)} \) and \( x'' = -\gamma^2 \frac{U'''(x) - 2U''(x) + \gamma U'(x)}{(U'(x))} e^{\gamma y} \), which implies the first two conclusions. For (iii), it suffices to observe that for \( y > 0 \), \( \tilde{V}(y) = E[\tilde{U}(y \xi_T)] \) and \( \tilde{V}'(y) = E[\xi_T \tilde{U}'(y \xi_T)] = E[-\xi_T \gamma e^{\gamma f(\lambda \xi_T)}] < \infty \) are well-defined by assumption. Moreover, using (i) and (ii), we can show that

\[ \tilde{V}''(y) = E[\xi_T^2 \tilde{U}''(y \xi_T)] = \frac{1}{y} E[\xi_T (y \xi_T) \tilde{U}''(y \xi_T)] = \frac{1}{y} E[\xi_T \frac{\gamma}{R_1(f(y \xi_T))} e^{\gamma f(y \xi_T)}] < \infty \]

and, similarly,

\[ \tilde{V}'''(y) = E[\xi_T^3 \tilde{U}'''(y \xi_T)] = \frac{1}{y^2} E[\xi_T (y \xi_T)^2 \tilde{U}'''(y \xi_T)] = \frac{1}{y^2} E[\left( \frac{\gamma}{R_1(x)} \right)^2 \xi_T e^{\gamma f(y \xi_T)}] - \gamma E\left[ \frac{-R_1(f(y \xi_T))R_2(f(y \xi_T)) + 2\gamma R_1(f(y \xi_T)) - \gamma^2}{\gamma^2(R_1(f(y \xi_T)) + \gamma)^2} \xi_T e^{\gamma f(y \xi_T)} \right] \]

Note that derivatives and expectations can be interchanged since by assumption, the relative risk aversion \( R_1(x) \) and \( R_2(x) \) are bounded and bounded away from zero, \( E[\gamma f(\lambda \xi_T)] < \infty \) for any \( \lambda > 0 \), and \( f \) is monotone.
Lemma E.2. Under the assumptions of Proposition 5.4 the inverse of the stochastic flow \( x \rightarrow X_t(x) \) exists and is three times continuously differentiable for any \( t \in [0,T] \). Furthermore, all coefficients of \( X_t(x) \) are differentiable and locally Lipschitz. Finally, denoting the stochastic integrand of \( X_t(x) \) by \( Z^*(t,x) \), \( Z^\ast \) and \( \frac{\partial Z^\ast}{\partial x} \) are locally Lipschitz.

**Proof of Lemma E.2.** By Proposition 5.2 we have that \( X_T(x) = f(\lambda(x)\xi_T) \) with \( \lambda(x) \) being uniquely determined by the budget constraint

\[
-\frac{1}{\gamma} \log \left( \mathbb{E}^Q[e^{-\gamma f(\lambda(x)\xi_T)}] \right) = x = 0, \tag{E.10}
\]

and \( X_t(x) = -\frac{1}{\gamma} \log \left( \mathbb{E}^Q[e^{-\gamma f(\lambda(x)\xi_T)}|\mathcal{F}_t] \right) \). Clearly, the derivative w.r.t. \( x \) of the left hand-side in (E.10) is not zero for any \( x \). Hence, the inverse function theorem entails that \( \lambda(x) \) is three times continuously differentiable (as \( f \) is). Furthermore, it follows directly by (E.10) that \( \lambda \) is strictly monotone and converges to \( \pm \infty \) as \( x \) converges to \( \pm \infty \). Note that by the definition of \( 0 \leq \xi_T \) and the fact that \( f \) is decreasing we have that \( \mathbb{E}[e^{\gamma f(\lambda(x)\xi_T})) < \infty \) for \( p \geq 1 \) and for every \( \lambda > 0 \). Hence, \( X_t(x) \) is invertible and three times continuously differentiable as well, which entails that also its inverse is three times continuously differentiable. Differentiability and local Lipschitz continuity of the coefficients of \( X_t(x) \) together with \( Z^* \) and \( \frac{\partial Z^*}{\partial x} \) being locally Lipschitz may be seen via expressing \( X_t(x) \) by the Feynman-Kac theorem, as a function of \( W_t \) and \( x \) which can be written as a differentiable integral over a Gaussian kernel, with derivatives differentiable in \( x \). \( \square \)

**Lemma E.3.** Under the assumptions of Proposition 5.4, the inverse stochastic flow \( \psi_t(x) := X_t^{-1}(x) \) exists and is an Itô’s process whose dynamics is given by

\[
dv_t(x) = a(t, \psi_t(x))dt + b(t, \psi_t(x))dW^Q_t, \tag{E.11}
\]

where \( b(t, \psi_t) = -Z^*(t, \psi_t)/X_t(t, \psi_t) \) and

\[
a(t, \psi_t(x)) = -\frac{1}{2} \frac{X_{xx}(t, \psi_t(x))(Z^*(t, \psi_t(x))^2}{X^3_x(t, \psi_t(x))} + \frac{\gamma Z^*(t, \psi_t(x))}{2 X_x(t, \psi_t(x))} + \frac{Z^*(t, \psi_t(x))Z^\ast_x(t, \psi_t(x))}{X^2_x(t, \psi_t(x))}.
\]

**Proof.** The proof is similar to that in [45] and [48]. By Lemma E.2 the coefficients of \( a(t, y) \) and \( b(t, y) \) defined above are locally Lipschitz continuous in \( y \). Therefore, by a truncating argument in [38], the SDEs (E.11) has a unique maximal solution up to an explosion time \( \tau_x \leq T \). Using Itô-Ventzel’s formula, it is straightforward to see that \( dX_t(\psi_t(x)) = 0 \) hence \( X_t(\psi_t(x)) = x \) a.s. for \( t \in [0, \tau_x) \). Since \( \psi_{\tau_x}(x) = \infty \) if \( \tau_x < T \) and \( X_t(\infty) = \infty \). On the other hand, by continuity \( X_t(\psi_t(x)) = x \) if \( t = \tau_x < T \). Consequently, we must have \( \tau_x = T \) a.s., meaning that \( \psi_t(x) = X_t^{-1}(x) \) for \( t \in [0,T] \). \( \square \)

**Proof of Proposition 5.3.** The proof is similar to that of Theorem 2.0 in [37] and is omitted. \( \square \)

Following the ideas of Lemma 4 in [57], the conditional version of Proposition 5.3 can be derived. In particular, one can show that

\[
-\frac{1}{\gamma} V'(t, X_t^{\theta^*}(x))e^{-\gamma X_t^{\theta^*}(x)} = y_\xi_t, \tag{E.12}
\]

where \( y = -\frac{1}{\gamma} V'(x)e^{-\gamma x} \). Set \( X_t(x) := X_t^{\theta^*}(x) = x + T_{0,t}^\theta \). Partially generalizing [45], we show below that for the BSPDE above regularity can be obtained. The optimal wealth is a strictly increasing continuous function of \( x \) \( \mathbb{P} \)-a.s. Hence, by Lemma E.2 an adapted inverse of \( X_t(x) \) exists.
Proof of Proposition 5.4. Define the martingale random fields \( M(t, x) := E[U(X_T(x))|\mathcal{F}_t] \), and \( \tilde{M}(t, x) := E[U'(X_T(x))|\mathcal{F}_t] \). Since \( X_T(x) \) is locally Lipschitz \( M \) and \( \tilde{M} \) can by Kolmogorov’s criterion be chosen up to a zero set which is independent of \( x \). By optimality using Theorem 4.5, \( V(t, X_t(x)) \) is a martingale and
\[
V(t, X_t(x)) = E[V(T, X_T(x))|\mathcal{F}_t] = E[U(X_T(x))|\mathcal{F}_t] = M(t, x).
\]
Moreover, it follows from (E.12) that \( \frac{1}{2}V'(t, X_t(x))e^{-\gamma X_t(x)} = y\xi_t \), which by Lemma E.2 and Proposition 5.3 implies that
\[
M'(t, x) = V'(t, X_t(x))X'_t(x) = \gamma y\xi_t e^{\gamma X_t(x)}X'_t(x) = V'(x)e^{-\gamma x}\xi_t e^{\gamma X_t(x)}X'_t(x).
\]
By (ii) and (iii) of Proposition 5.3 we obtain
\[
\tilde{M}(t, x) = E[U'(X_T(x))|\mathcal{F}_t] = E[\gamma y\xi_t e^{\gamma X_T(x)}|\mathcal{F}_t] = \gamma y\xi_t e^{\gamma X_t(x)} = V'(t, X_t(x))
\]
by using the identity (E.12). It follows that \( M'(t, x) = \tilde{M}(t, x)X'_t(x) \). From Lemma E.1, \( \tilde{V} \) is three-times differentiable, which implies that its conjugate (see Proposition 5.3) is also \( V(x) \) is three-times differentiable. Therefore, we can conclude that the martingale random fields \( M(t, x) \) and \( \tilde{M}(t, x) \) are two times differentiable. Let \( M'(t, x) = V'(x) + \int_0^t h(s, x)dW_s \) be the GKW decomposition of the martingale random field \( M'(t, x) \). Since \( V(t, x) = M(t, X_t^{-1}(x)) \) we obtain by Itô-Ventzel’s formula that
\[
V(t, x) = V(0, x) + \int_0^t M(ds, \psi_s) + \int_0^t M'(s, \psi_s)d\psi_s + \frac{1}{2} \int_0^t M''(s, \psi_s)d\langle \psi \rangle_s
\]
where \( \psi_t := X_t^{-1}(x) \) and \( \langle \cdot, \cdot \rangle \) denotes the covariation. From the assumption that \( zf'(z) \) is uniformly bounded, Lemma E.3 implied that the diffusion term \( \mathcal{Z}(t, x) \) of \( X_t(x) \) is \( \mathcal{F}_t \)-measurable and bounded and by [38] (chapter 3), the inverse stochastic flow \( \psi_t(x) := X_t^{-1}(x) \) exists and is an Itô process \( d\psi_t(x) = a(t, \psi_t(x))dt + k(t, \psi_t(x))dW_t \), where \( k(t, \psi_t) = -\mathcal{Z}(t, \psi_t(x))/X_t(x, \psi_t(x)) \) and
\[
a(t, \psi_t(x)) = -\frac{1}{2}X_{xx}(t, \psi_t(x)) + \frac{\mathcal{Z}(t, \psi_t(x))}{2X_t(x, \psi_t(x))} + \frac{\mathcal{Z}(t, \psi_t(x))\mathcal{Z}_x(t, \psi_t(x))}{X^2_t(x, \psi_t(x))}.
\]
Now, using Itô-Ventzel’s formula, we may see directly that the integrand of the finite variation term (in the definition of regular semimartingales) of \( V(t, x) \) is given by
\[
M'(t, \psi_t(x))a(t, \psi_t(x)) + \frac{1}{2}M''(t, \psi_t(x))k^2(t, \psi_t(x)) + h(t, \psi_t(x))k(t, \psi_t(x)),
\]
and hence (b) is confirmed.

Similarly, since \( V'(t, x) = \tilde{M}(t, X_t^{-1}(x)) \) and \( \tilde{M} \) is twice differentiable we can use Itô-Ventzel’s formula to obtain that \( V'(t, x) \) is a special semimartingale with progressively measurable finite variation and \( V'(t, x) \) admits the representation
\[
V'(t, x) = V'(0, x) + \int_0^t \mathcal{B}(s, x)ds + \int_0^t \mathcal{K}(s, x)dW_s \quad \text{(E.13)}
\]
for integrable processes \( \mathcal{B}, \mathcal{K} \) being continuous in \( x \). Integrating both sides of (E.13) with respect to the space argument on \([0, x]\) and applying the stochastic Fubini’s theorem we can conclude that (c) holds. Hence, conditions (a)-(c) are fulfilled and \( V(t, x) \) is a regular family of semimartingales. □