NEW PERSPECTIVES IN COMPLEX GENERAL RELATIVITY

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Abstract. In complex general relativity, Lorentzian space-time is replaced by a four-complex-dimensional complex-Riemannian manifold, with holomorphic connection and holomorphic curvature tensor. A multisymplectic analysis shows that the Hamiltonian constraint is replaced by a geometric structure linear in the holomorphic multimomenta, providing some boundary conditions are imposed on two-complex-dimensional surfaces. On studying such boundary conditions, a link with the Penrose twistor programme is found. Moreover, in the case of real Riemannian four-manifolds, the local theory of primary and secondary potentials for gravitino fields, recently proposed by Penrose, has been applied to Ricci-flat backgrounds with boundary. The geometric interpretation of the differential equations obeyed by such secondary potentials is related to the analysis of integrability conditions in the theory of massless fields, and might lead to a better understanding of twistor geometry. Thus, new tools are available in complex general relativity and in classical field theory in real Riemannian backgrounds.

1. Introduction

Within the framework of relativistic theories of gravitation, the main aim of the Penrose twistor programme is to provide a purely geometric tool for solving the Einstein equations within a holomorphic, conformally invariant formalism.\(^1\)\(^2\) For this purpose, the Lorentzian space-time of Einstein’s general relativity is replaced by a four-complex-dimensional complex-Riemannian manifold with holomorphic metric, holomorphic connection and holomorphic curvature tensor. Such a programme succeeds in studying anti-self-dual space-times with or without a cosmological constant, as well as in the analysis of the massless free-field equations of classical field theory. However, the main problem is that, unless half of the conformal curvature vanishes (i.e. the self-dual Weyl spinor \(\tilde{\psi}_{ABCD}^\prime\)), no twistor-space description of general relativity can be given. More recently, in the work appearing in Refs. 3–4, Lorentzian and complex general relativity have been studied within a multisymplectic framework. The multisymplectic analysis is very helpful to study the gravitational field, viewed as a constrained system, in a manifestly covariant way. This means that no restrictive assumption on the space-time topology is made, and the invariance group remains the
full diffeomorphism group of four-dimensional space-time. Note that the analysis in Refs. 3–4 differs substantially from the recent approaches to canonical gravity, where one takes instead complex tetrads on a four-real-dimensional Lorentzian manifold.

The resulting description of complex general relativity is outlined in the following section, following Ref. 4, while new perspectives for twistor theory in real Riemannian four-manifolds with boundary are described in Secs. 3 and 4. These sections deal with a recent analysis of Rarita-Schwinger potentials in curved backgrounds, and are relevant both for twistor theory and for the understanding of consistent supergravity theories.

2. Multimomenta for Complex General Relativity

In our geometric framework one starts from a one-jet bundle $J^1$ which, in local coordinates, is described by a holomorphic coordinate system, with holomorphic tetrad $e^a\hat{a}$, holomorphic connection one-form $\omega^a\hat{b}\hat{c}$, multivelocities corresponding to $e^a\hat{a}$ and multivelocities corresponding to $\omega^a\hat{b}\hat{c}$, both of holomorphic nature. The intrinsic form of the field equations, which is a generalization of a mathematical structure already existing in classical mechanics, leads to the complex vacuum Einstein equations $R_{ab} = 0$, and to a condition on the covariant divergence of the multimomenta. Moreover, the multimomentum map, when evaluated on a section of $J^1$ and integrated on an arbitrary three-complex-dimensional surface $\Sigma_c$, leads to the holomorphic equations corresponding to the constraint equations of the Lorentzian theory, and reflects the invariance of complex general relativity under all holomorphic coordinate transformations. The Hamiltonian constraint is then replaced by a geometric structure linear in the holomorphic multimomenta providing two boundary terms in these holomorphic equations can be set to zero. For this purpose, one of the following three conditions should hold:

(i) $\Sigma_c$ has no boundary;

(ii) the holomorphic multimomenta $\tilde{p}^{ab}_{\hat{c}\hat{d}} \equiv (\text{det } e) \left( e^a\hat{c}^b e^b_{\hat{d}} - e^a_{\hat{d}} e^b\hat{c} \right)$, defined as the derivatives of the Lagrangian with respect to the multivelocities corresponding to the connection one-form, vanish at $\partial \Sigma_c$ (and hence everywhere on $\Sigma_c$, by virtue of a well-known theorem in complex analysis);

(iii) denoting by $\lambda^{ab}$ the elements of the algebra $o(4, C)$ corresponding to the gauge group, their spinorial version $\lambda^{AA'BB'}$, and the spinor form of the connection one-form, vanish at $\partial \Sigma_c$. On using two-component spinor notation, these boundary conditions take the form

$$\Lambda^{(CD)} = 0 \quad \tilde{\Lambda}^{(C'D')} = 0 \quad \text{at } \partial \Sigma_c, \quad (2.1)$$

$$\Omega_f^{(CD)} = 0 \quad \tilde{\Omega}_f^{(C'D')} = 0 \quad \text{at } \partial \Sigma_c. \quad (2.2)$$
where \( \lambda^{C'C'D'D'} = \Lambda^{(CD)} \epsilon^{C'D'} + \tilde{\Lambda}^{(C'D')} \epsilon^{CD} \), and \( \omega_f^{C'C'D'D'} = \Omega_f^{(CD)} \epsilon^{C'D'} + \tilde{\Omega}_f^{(C'D')} \epsilon^{CD} \).

The boundary conditions (2.2) may be replaced by the condition \( u^{A'A'} = 0 \) at \( \partial \Sigma_c \), where \( u \) is a vector field describing holomorphic coordinate transformations on complex space-time. In other words the work in Ref. 4 shows that, if \( \Sigma_c \) has a boundary (which cannot be ruled out), the holomorphic multimomenta should vanish on the whole of \( \Sigma_c \), to avoid having restrictions at \( \Sigma_c \) on the spinor fields expressing the invariance of the theory under all holomorphic coordinate transformations.

Interestingly, to ensure that the holomorphic multimomenta vanish at \( \partial \Sigma_c \), one obtains conditions which admit, as a subset, the totally null two-complex-dimensional surfaces known as \( \alpha \)-surfaces and \( \beta \)-surfaces. The integrability condition for \( \alpha \)-surfaces is the vanishing of the self-dual Weyl spinor, and hence the multisymplectic formalism enables one to recover the anti-self-dual space-time relevant for twistor theory. However, if \( \partial \Sigma_c \) is not totally null, the resulting theory does not correspond to twistor theory, and one has to study the topology and the geometry of the space of two-complex-dimensional surfaces \( \partial \Sigma_c \) in the generic case. Moreover, one has to solve a set of equations which are now linear in the holomorphic multimomenta, both in classical and in quantum gravity (as we said before, they correspond to the constraint equations of the Lorentzian theory). In the classical holomorphic theory, such equations take the form\(^4\)

\[
\int_{\Sigma_c} \lambda^{\hat{c}\hat{d}} \left( D_a \tilde{p}^{ab} \right)_{\hat{c}\hat{d}} \, d^3x_b = 0 \quad ,
\]

\[
\int_{\Sigma_c} \text{Tr} \left[ \tilde{p}^{af} \Omega_{ad} - \frac{1}{2} \tilde{p}^{ab} \Omega_{ab} \delta^f_d \right] u^d \, d^3x_f = 0 \quad .
\]

With our notation, \( \Omega_{ab} \) is the holomorphic curvature of the holomorphic connection one-form \( \omega_a^{\hat{c}\hat{d}} \). Moreover, \( D \) is a connection which annihilates the internal-space metric \( \eta_{ab} \).\(^4\) It should be emphasized that these equations, resulting from the holomorphic version of the multimomentum map, cannot be related to a Cauchy problem as in the Lorentzian theory. Hence their interpretation, as well as the proposal to set to zero the integral of the holomorphic multimomentum map on an arbitrary three-complex-dimensional surface,\(^4\) deserve further thinking.

Interestingly, the analysis in Ref. 4 shows that a deep link exists between complex space-times which are not anti-self-dual and two-complex-dimensional surfaces which are not totally null. In other words, on going beyond twistor theory, the analysis of two-complex-dimensional surfaces still plays a key role. However, we do not yet know how to write and solve the operator version of the Eqs. (2.3)–(2.4), and the question is, of course, crucial for the whole multisymplectic programme.
3. Local Theory of Spin-$\frac{3}{2}$ Potentials

Penrose has recently proposed a new definition of twistors as charges for massless spin-$\frac{3}{2}$ fields in Ricci-flat Riemannian manifolds.\cite{Penrose1986, Penrose1987} We now show that the Penrose formalism can be applied to Ricci-flat backgrounds with boundary, which are relevant for one-loop quantum cosmology and for the analysis of consistent supergravity theories.

The basic ideas of the local theory of Rarita-Schwinger potentials are as follows.\cite{Penrose1986} The independent spinor-valued one-forms $\left(\psi_a^A, \tilde{\psi}_a^{A'}\right)$ occurring in the action functional of supergravity are obtained from the tetrad and from some spinor fields as

\[ \psi_a^A = \Gamma^{C'B}_B \, e^B_C \sigma_a , \quad (3.1) \]
\[ \tilde{\psi}_a^{A'} = \gamma^{C'A'} B', e^B_C \sigma_a . \quad (3.2) \]

By virtue of the spinor Ricci identities and of the local equations

\[ \Gamma^A_B B' = \nabla^B_B' \alpha^A , \quad (3.3) \]
\[ \gamma^{A'B'}_B = \nabla^B_B' \tilde{\alpha}^{A'} , \quad (3.4) \]

the primary potentials $\gamma$ and $\Gamma$ obey the differential equations

\[ \epsilon^{B'C'} \nabla_A (A' \gamma^A_B)_C = -3\Lambda \tilde{\alpha}^{A'} , \quad (3.5) \]
\[ \nabla^{B'(B} \gamma^{A)_C} = \Phi^{A'B'}_{C'} \tilde{\alpha}_L , \quad (3.6) \]
\[ \epsilon^{BC} \nabla_A (A' \Gamma^{A')}_B)_C = -3\Lambda \alpha^A , \quad (3.7) \]
\[ \nabla^{B'(B} \Gamma^{A')}_B = \tilde{\Phi}^{A'B'L} C \alpha_L , \quad (3.8) \]

where the spinor fields $\Phi$ and $\tilde{\Phi}$ describe the trace-free Ricci spinor. Note that $\left(\alpha_A, \tilde{\alpha}_{A'}\right)$ are a pair of independent and anticommuting spinor fields. The primary potentials are subject to the gauge transformations

\[ \tilde{\gamma}^{A}_B C' \equiv \gamma^{A}_B C' + \nabla^{A}_B \lambda_{C'} , \quad (3.9) \]
\[ \tilde{\Gamma}^{A'}_B C \equiv \Gamma^{A'}_B C + \nabla^{A'}_B \nu_C , \quad (3.10) \]

where the spinor fields $\nu_B$ and $\lambda_{B'}$ are freely specifiable. Thus, the gauge transformations (3.9)–(3.10) are compatible with the field equations if and only if the scalar curvature and the trace-free part of the Ricci tensor vanish, which implies that the background geometry has to be Ricci-flat. A set of secondary potentials is now introduced locally by requiring that (cf. Eqs. (3.3)–(3.4))

\[ \gamma^{A'B'}_B \equiv \nabla_{BB'} \rho_{A'B'B} , \quad (3.11) \]
\[ \Gamma_{AB}^{C'} \equiv \nabla_{BB'} \theta_A^{C'B'} \quad . \tag{3.12} \]

If one now inserts Eqs. (3.11)–(3.12) into Eqs. (3.5)–(3.8) one finds that, providing the following conditions hold (see appendix):

\[ \nabla^{B'(F} \rho_{B')L} = 0 \quad , \tag{3.13} \]
\[ \nabla^{B(F'} \theta_B^{A)L'} = 0 \quad , \tag{3.14} \]

the secondary potentials obey the equations\(^2\)

\[ \psi^{ABLM} \rho_{(LM)C'} = 0 \quad , \tag{3.15} \]
\[ \psi^{A'B'L'M'} \theta_{(L'M')C} = 0 \quad . \tag{3.16} \]

Interestingly, no further restriction on the curvature of the background is obtained providing the symmetric parts of the secondary potentials vanish, i.e. \( \rho_{(AB)}^{C'} = 0 \), \( \theta_{(A'B')}^{C} = 0 \), and providing \( \alpha_A, \tilde{\alpha}_{A'} \) obey the Weyl equations

\[ \nabla^{AA'} \alpha = 0 \quad , \quad \nabla^{AA'} \tilde{\alpha}_{A'} = 0 \quad . \tag{3.17} \]

This implies that the secondary potentials take the form

\[ \rho_A^{CB} = \epsilon^{CB} \tilde{\alpha}_{A'} \quad , \tag{3.18} \]
\[ \theta_A^{C'B'} = \epsilon^{C'B'} \alpha_A \quad . \tag{3.19} \]

However, if one wants to consider secondary potentials in their complete form, Eqs. (3.15) and (3.16) imply that only flat Euclidean four-space can be studied, since otherwise one would obtain secondary potentials which depend explicitly on the curvature of the background, and this is inconsistent.

Moreover, the gauge freedom is restricted by the presence of boundaries, since the boundary conditions should be preserved by the gauge transformations (3.9)–(3.10). On imposing the boundary conditions motivated by local supersymmetry\(^2\)

\[ \sqrt{2} \epsilon n_A^{A'} \psi^A_i = \pm \tilde{\psi}^{A'}_i \quad \text{at} \quad \partial M \quad , \tag{3.20} \]

one thus finds the following restrictions on the spinor fields \( \nu_B \) and \( \lambda_{B'} \):

\[ \sqrt{2} \epsilon n_A^{A'} (\nabla^{AC'} \nu^B) e_{BC'i} = \pm (\nabla^{CA'} \lambda^{B'}) e_{CB'i} \quad \text{at} \quad \partial M \quad . \tag{3.21} \]

4. Open Problems

Although we have presented only a very brief outline of the local theory of spin-\( \frac{3}{2} \) potentials, many interesting questions arise already at this stage:
(i) Can one find a complete two-spinor description of massive spin-$\frac{3}{2}$ potentials in Einstein backgrounds with non-vanishing cosmological constant? For this purpose, one has to introduce a new covariant derivative, which differs from the original one by a term proportional to the curved-space $\gamma$-matrices. The two-component spinor formulation of the resulting set of equations for spin-$\frac{3}{2}$ potentials is highly non-trivial, and is being investigated by myself and G. Pollifrone.

(ii) Can one relate Eqs. (3.13)–(3.14) to the theory of integrability conditions relevant for massless fields in curved backgrounds (see appendix)? What happens when such equations do not hold?

(iii) Is there an underlying global theory? In the affirmative case, what are the key features of the global theory?

(iv) Can one define twistors as charges$^2$ for spin $\frac{3}{2}$ in Ricci-flat backgrounds with boundary?

(v) Can one reconstruct the Riemannian four-geometry from the twistor space, or from whatever is going to replace twistor space?

The solution of these problems might improve our understanding of the geometric properties relevant for classical and quantum gravity. When combined with the ideas described in the first part of this paper, these investigations seem to suggest that a new synthesis is in sight in relativistic theories of gravitation.

Appendix

The local theory of Rarita-Schwinger potentials leads naturally to the consideration of Eq. (3.13) (and similarly for Eq. (3.14)), since the insertion of Eq. (3.11) into Eq. (3.5) yields in the Ricci-flat case

$$
\varepsilon_{FL} \nabla_{AA'} \nabla^{B(F} \rho_{B')A} + \frac{1}{2} \nabla^{A'} \nabla^{B'M} \rho_{B'(AM)} + \nabla_{AM} \rho_{A'}^{(AM)} + \frac{3}{8} \nabla \rho_{A'} = 0 \quad (A.1)
$$

Thus, if Eq. (3.13) holds, Eq. (A.1) reduces to an identity by virtue of Ricci-flatness.

In the original approach by Penrose,$^5$ one describes Rarita-Schwinger potentials in flat space-time in terms of a rank-3 vector bundle with local coordinates $\left(\eta_A, \zeta\right)$, and an operator $\Omega_{AA'}$ whose action is defined by

$$
\Omega_{AA'}(\eta_B, \zeta) \equiv \left(\mathcal{D}_{AA'} \eta_B, \mathcal{D}_{AA'} \zeta - \eta^C \rho_{A'AC}\right) \quad (A.2)
$$

where $\mathcal{D}$ is the flat Levi-Civita connection of Minkowski space-time. The gauge transformations are then

$$
(\tilde{\eta}_B, \tilde{\zeta}) \equiv (\eta_B, \zeta + \eta_A \xi^A) \quad (A.3)
$$
\[ \tilde{\rho}_{A'AB} \equiv \rho_{A'AB} + \mathcal{D}_{AA'} \xi_B \quad . \]  
(A.4)

For the operator \( \Omega_{AA'} \) defined in Eq. (A.2), the integrability condition on \( \beta \)-planes turns out to be

\[ \mathcal{D}^{A'(A} \rho_{A'B)C} = 0 \quad . \]  
(A.5)

It now remains to be seen whether, in a curved background, an operator can be defined (cf. Eq. (A.2)) whose integrability condition on \( \beta \)-surfaces is indeed given by Eq. (3.13) (cf. Eq. (A.5)).

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