LOCALIZING ALGEBRAS AND INVARIANT SUBSPACES

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Abstract. It is shown that the algebra \( L^\infty(\mu) \) of all bounded measurable functions with respect to a finite measure \( \mu \) is localizing on the Hilbert space \( L^2(\mu) \) if and only if the measure \( \mu \) has an atom. Next, it is shown that the algebra \( H^\infty(D) \) of all bounded analytic multipliers on the unit disc fails to be localizing, both on the Bergman space \( A^2(D) \) and on the Hardy space \( H^2(D) \). Then, several conditions are provided for the algebra generated by a diagonal operator on a Hilbert space to be localizing. Finally, a theorem is provided about the existence of hyperinvariant subspaces for operators with a localizing subspace of extended eigenoperators. This theorem extends and unifies some previously known results of Scott Brown and Kim, Moore and Pearcy, and Lomonosov, Radjavi and Troitsky.

1. Introduction

Let \( B(E) \) denote the algebra of all bounded linear operators defined on a Banach space \( E \). A subspace \( X \subseteq B(E) \) is said to be localizing provided that there is a closed ball \( B \subseteq E \) such that \( 0 \notin B \) and such that for every sequence \( (x_n) \) in \( B \) there is a subsequence \( (x_{n_j}) \) and a sequence \( (x_j) \) in \( X \) such that \( \|x_j\| \leq 1 \) and such that the sequence \( (x_j x_{n_j}) \) converges in norm to some nonzero vector. This notion was introduced by Lomonosov, Radjavi, and Troitsky [11] as a side condition to build invariant subspaces for bounded linear operators on Banach spaces.

Recall that the commutant of an operator \( T \in B(E) \) is the subalgebra \( \{T\}' \subseteq B(E) \) of all operators that commute with \( T \). A subspace \( F \subseteq E \) is said to be invariant under an operator \( T \in B(E) \) provided that \( TF \subseteq F \). A subspace \( F \subseteq E \) is said to be invariant under a subalgebra \( R \subseteq B(E) \) if \( F \) is invariant under every \( R \in R \). A subspace \( F \subseteq E \) is said to be hyperinvariant under an operator \( T \in B(E) \) if \( F \) is invariant under the commutant \( \{T\}' \). A subalgebra \( R \subseteq B(E) \) is said to be transitive if the only closed subspaces invariant under \( R \) are the trivial ones, namely, \( F = \{0\} \) and \( F = E \). It is easy to see that this is equivalent to saying that for every \( x \in E \setminus \{0\} \), the subspace \( \{Rx : R \in R\} \) is dense in \( E \).

We shall denote by \( \text{ball}(X) \) the unit ball of a subspace \( X \subseteq B(E) \). Also, we shall denote by \( \sigma \) the weak operator topology on \( B(E) \). Recall that for a convex subset of \( B(E) \), the closure in the weak operator topology agrees with the closure in the strong operator topology.

Lomonosov, Radjavi, and Troitsky [11] proved among other results the following

Theorem 1.1. If \( T \) is a nonzero quasinilpotent operator on a Banach space and its commutant \( \{T\}' \) is a localizing algebra, then \( T \) has a nontrivial hyperinvariant subspace.

They also made the observation that any operator algebra containing a nonzero compact operator is a localizing algebra. This observation can be refined as follows.

Proposition 1.2. Let \( X \) be a subspace of \( B(E) \) such that the closure of its unit ball in the weak operator topology contains a nonzero compact operator. Then \( X \) is localizing.

Proof. Let \( T \in \text{ball}(X) \) be a nonzero compact operator. We may assume without loss of generality that \( \|T\| = 1 \). Let \( x_0 \in E \) be a vector with the property that \( \|x_0\| = 1 \) and \( \|Tx_0\| \geq 3/4 \). Consider the closed
ball $B = \{x \in E : \|x-x_0\| \leq 1/4\}$. It is clear that $0 \notin B$. Now, let $(x_n)$ be any sequence in $B$. Since $T$ is compact, there is a subsequence $(x_{n_j})$ such that $(Tx_{n_j})$ converges in norm to $y \in E$, say. Since the closure in the weak operator topology of a convex set agrees with the closure in the strong operator topology, for every $j \geq 1$, there is an operator $X_j \in \text{ball}(X)$ such that $\|Tx_{n_j} - X_jx_{n_j}\| \leq 1/j$. It follows that $\|y - X_jx_{n_j}\| \to 0$ as $j \to \infty$. Finally, we show that $y \neq 0$. Notice that $\|Tx_n - Tx_0\| \leq \|x_n - x_0\| \leq 1/4$, and since $\|Tx_0\| \geq 3/4$, we conclude that $\|Tx_n\| \geq 1/2$ for all $n \geq 1$, so that $\|y\| \geq 1/2$, as we wanted. □

The first author [8] provided an example of a weakly closed, localizing algebra of bounded operators on the Banach space $C[0,1]$ that does not contain any nonzero compact operators. As a matter of fact, the example is the algebra of all multiplication operators by continuous functions on the unit interval, and as it turns out, this is the uniformly closed unital algebra generated by the position operator. It is natural to ask if such a construction can be carried out on a Hilbert space. This question can be formulated more precisely as follows.

**Problem 1.3.** Is there a Hilbert space $H$ and a localizing algebra $R \subseteq \mathcal{B}(H)$ such that $\overline{\text{ball}(R)}$ does not contain any nonzero compact operators?

The first part of this paper (sections 2, 3, and 4) initiates the investigation of some properties of localizing algebras bearing this question in mind, although we have not been able to solve it.

The notions of extended eigenvalue and eigenoperator became popular back in the 1970s when searching for invariant subspaces of operators on Banach spaces. A complex scalar $\lambda \in \mathbb{C}$ is said to be an extended eigenvalue for an operator $T \in \mathcal{B}(E)$ if there exists a nonzero operator $X \in \mathcal{B}(E)$ such that $TX = \lambda XT$. Such an operator $X$ is called an extended eigenoperator for $T$ associated with $\lambda$. The following extension of Lomonosov's invariant subspace theorem [10] was obtained by Scott Brown [4], and independently by Kim, Moore and Pearcy [7].

**Theorem 1.4.** If a nonscalar operator $T \in \mathcal{B}(E)$ has a compact eigenoperator then $T$ has a nontrivial hyperinvariant subspace.

The special case that $T$ commutes with a nonzero compact operator is the original result of Lomonosov. Recently, the concepts of extended eigenvalue and eigenoperator have received a considerable amount of attention, both in the context of invariant subspaces [9] and in the study of extended eigenvalues and extended eigenoperators for some special classes of operators [1] [2] [3] [6] [12] [14]. The second part of this paper (section 5) provides a result that extends and unifies Theorem 1.1 and Theorem 1.4. Our result can be stated as follows.

**Theorem 1.5.** Let $T \in \mathcal{B}(E)$ be a nonzero operator, let $\lambda \in \mathbb{C}$ be an extended eigenvalue of $T$ such that the subspace $X$ of all associated extended eigenoperators is localizing and suppose that either

\begin{enumerate}
  \item $|\lambda| < 1$,
  \item $|\lambda| > 1$, or
  \item $|\lambda| = 1$ and $T$ is quasinilpotent.
\end{enumerate}

Then $T$ has a nontrivial hyperinvariant subspace.

Since the commutant $\{T^*\}'$ is the family of all extended eigenoperators associated with the extended eigenvalue $\lambda = 1$, it follows that Theorem 1.5 is an extension of Theorem 1.1 to the case of extended eigenvalues with $|\lambda| = 1$. On the other hand, it follows from Proposition 1.2 that Theorem 1.5 is an extension of Theorem 1.4 at least for extended eigenvalues with $|\lambda| \neq 1$.

The paper is organized as follows. In section 2, it is shown that if the algebra $L^\infty(\mu)$ of all bounded measurable functions with respect to a finite measure $\mu$ is localizing on the Hilbert space $L^2(\mu)$ then the measure $\mu$ must have an atom. In section 3, it is shown that the algebra $H^\infty(\mathbb{D})$ of all bounded analytic multipliers on the unit disc fails to be localizing, both on the Bergman space $A^2(\mathbb{D})$ and on the Hardy space $H^2(\mathbb{D})$. In section 4, some conditions are given for the algebra generated by a diagonal operator
on a Hilbert space to be localizing. In section 3, a proof of Theorem 1.5 is provided, and an example is given to illustrate that Theorem 1.5 is more general than Theorem 1.4.

2. Abelian selfadjoint localizing algebras

The question arises of whether the closure in the weak operator topology of the unit ball of a localizing algebra of operators on a Hilbert space must contain a nonzero compact operator. First, we consider the case of an abelian selfadjoint algebra. Once again, recall that the closure in the weak operator topology of a convex set agrees with the closure in the strong operator topology. Kaplansky’s density theorem [5] is the assertion that if $\mathcal{R}$ is a selfadjoint algebra of operators on a Hilbert space then the strong closure of the unit ball of $\mathcal{R}$ is the unit ball of the strong closure of $\mathcal{R}$. See the book of Takesaki [15, Theorem 4.8] for another reference on Kaplansky’s density theorem. This result is the key to the following

**Proposition 2.1.** If $\mathcal{R}$ is a selfadjoint algebra of operators on a Hilbert space $H$ such that its closure in the weak operator topology is a localizing algebra, then $\mathcal{R}$ itself is a localizing algebra.

**Proof.** Suppose that the closure in the weak operator topology $\mathcal{R}'$ is a localizing algebra and let $B$ be a closed ball as in the definition. Take a sequence of vectors $(x_n)$ in $B$, extract a subsequence $(x_{n_j})$, and find a sequence of operators $(T_{j_k})$ in $\overline{\text{ball}(\mathcal{R}''')}$ and a nonzero vector $y \in H$ such that $\|y - T_{j_k}x_{n_j}\| \to 0$ as $j \to \infty$. It follows from Kaplansky’s density theorem that for every $j \geq 1$ there is an operator $R_j \in \text{ball}(\mathcal{R})$ such that $\|T_{j_k}x_{n_j} - R_jx_{n_j}\| \leq 1/j$. Thus, $\|y - R_jx_{n_j}\| \to 0$ as $j \to \infty$, as we wanted. □

Let $(\Omega, \Sigma, \mu)$ be a finite measure space. We identify every function $\varphi$ in $L^\infty(\mu)$ with the multiplication operator $M_\varphi$ defined on $L^2(\mu)$ by the expression $(M_\varphi f)(\omega) = \varphi(\omega)f(\omega)$, and we regard $L^\infty(\mu)$ as a subalgebra of $\mathcal{B}(L^2(\mu))$ under this identification.

**Theorem 2.2.** If the measure $\mu$ contains no atoms then the algebra $L^\infty(\mu)$ is not localizing on the Hilbert space $L^2(\mu)$.

**Proof.** Let $f_0 \in L^2(\mu)$ and consider a closed ball $B = \{ f \in L^2(\mu) : \|f - f_0\|_2 \leq \varepsilon \}$. We must show that the algebra $L^\infty(\mu)$ and the ball $B$ do not satisfy the condition in the definition of a localizing algebra. First of all, there is a $\delta > 0$ such that, for each measurable subset $A \subseteq \Omega$, the condition $\mu(A) < \delta$ implies $\|f_0 \cdot \chi_A\|_2 < \varepsilon$, or equivalently, $\|f_0 \cdot \chi_{A^c} - f_0\|_2 < \varepsilon$. Since $\mu$ has no atoms, we may construct a sequence $(A_n)$ of independent, measurable subsets of $\Omega$ such that $\mu(A_n) = \delta/2$ for each $n \geq 1$. Then, we define a sequence of functions $(f_n)$ inside the ball $B$ by the expression $f_n = f_0 \cdot \chi_{A_n}$. Suppose that there is a subsequence $(f_{n_j})$, a function $f \in L^2(\mu)$, and a sequence of functions $(\varphi_j)$ in $L^\infty(\mu)$ such that $\|\varphi_j\|_\infty \leq 1$ and $\|f - \varphi_jf_{n_j}\|_2 \to 0$ as $j \to \infty$. Thus, it suffices to show that $f = 0$. Now, extract a further subsequence $(\varphi_{j_k}f_{n_{j_k}})$ that converges almost everywhere to $f$. Next, apply the Borel-Cantelli Lemma to the sequence $(A_{n_{j_k}})$, and conclude that there is a measurable subset $Z \subseteq \Omega$ such that $\mu(Z) = 0$ and such that the set $\{ k \geq 1 : \omega \in A_{n_{j_k}} \}$ is infinite for every $\omega \in Z^c$. It follows that $f$ vanishes almost everywhere, as we wanted. □

Since any maximal abelian, selfadjoint algebra can be represented as a function algebra $L^\infty(\mu)$, as a consequence of Theorem 2.2 we get the following

**Corollary 2.3.** Let $H$ be an infinite dimensional separable Hilbert space and let $\mathcal{R}$ be a maximal abelian selfadjoint subalgebra of $\mathcal{B}(H)$. The following conditions are equivalent:

1. $\mathcal{R}$ is a localizing algebra,
2. $\mathcal{R}$ contains a rank one operator,
3. $\mathcal{R}$ contains a nonzero finite rank operator,
4. $\mathcal{R}$ contains a nonzero compact operator.
See the book of Radjavi and Rosenthal [13] Corollary 7.14 for a reference on the representation of maximal abelian selfadjoint algebras. We finish this section with an example of a probability measure \( \mu \) and a subalgebra \( \mathcal{R} \subseteq L^\infty(\mu) \) that fails to be localizing although its closure \( \overline{\mathcal{R}}^w \) in the weak operator topology is a localizing algebra. This example goes to show that the assumption that the algebra \( \mathcal{R} \) is selfadjoint cannot be dropped from the hypotheses of Proposition 2.1.

**Example 2.4.** Let \( \{z_k\} \) be a sequence complex scalars in the open unit disc \( \mathbb{D} \) such that \( \{z_k : k \geq 1\} \) \( \subset \partial \mathbb{D} \) and \( \{z_k : k \geq 1\} \cap [0, 1/2] = \emptyset \). Then, let \( \lambda \) denote the Lebesgue measure on the real line and consider the probability measure

\[
\mu = \lambda_{[0,1/2]} + \sum_{k=1}^{\infty} 2^{-k-1} \delta_{z_k}.
\]

Finally, consider the subalgebra \( \mathcal{R} \subseteq L^\infty(\mu) \) defined as \( \mathcal{R} = \{p(M_x) : p \text{ is a polynomial}\} \). We start with a result that is an immediate consequence of the maximum modulus principle.

**Lemma 2.5.** \( \|p\|_{L^\infty(\mu)} = \sup\{|p(z)| : z \in \partial \mathbb{D}\} \) for every polynomial \( p \).

The next result is a condition for an operator \( T \in \mathcal{B}(L^2(\mu)) \) to belong to the weak closure of \( \overline{\mathcal{B}(R)} \).

**Lemma 2.6.** \( T \in \overline{\mathcal{B}(R)}^w \) if and only if there is some \( \varphi \in \mathcal{B}(H^\infty(\mathbb{D})) \) such that \( T = M_\varphi \).

**Proof.** Notice that \( L^\infty(\mu) \) is closed in the weak operator topology and that the weak operator topology restricted to \( L^\infty(\mu) \) agrees with weak-* topology \( \sigma(L^\infty, L^1) \). Thus, given an operator \( T \in \overline{\mathcal{B}(R)}^w \), there is a function \( \psi \in L^\infty(\mu) \) such that \( T = M_\psi \), and there exists a sequence of polynomials \( \{p_n\} \) such that \( \|p_n\|_{L^\infty} \leq 1 \) and \( p_n \to \psi \) in the weak-* topology. Then, it follows from Montel’s theorem that there is a subsequence \( \{p_{n_j}\} \) and there is a function \( \varphi \in \mathcal{B}(H^\infty(\mathbb{D})) \) such that \( p_{n_j} \to \varphi \) uniformly on compact subsets of \( \mathbb{D} \). Hence, \( p_{n_j} \to \varphi \) almost everywhere, and it follows from the bounded convergence theorem that \( p_{n_j} \to \varphi \) in the weak-* topology. Therefore, \( \varphi = \psi \) and \( T = M_\varphi \), as we wanted. Conversely, suppose that there is some \( \varphi \in \mathcal{B}(H^\infty(\mathbb{D})) \) such that \( T = M_\varphi \), and let \( \varphi_r(z) = \varphi(rz) \) for \( 0 < r < 1 \), so that \( \varphi_r(z) \to \varphi(z) \) as \( r \to 1^- \) for all \( z \in \mathbb{D} \), and it follows from the bounded convergence theorem that \( \varphi_r \to \varphi \) in the weak-* topology. Since \( \varphi_r \in \mathcal{B}(A(\mathbb{D})) \), we have \( M_{\varphi_r} \in \overline{\mathcal{B}(R)} \), so that \( M_\varphi \in \overline{\mathcal{B}(R)}^w \). □

**Theorem 2.7.** If the sequence \( \{z_k\} \) satisfies the Blaschke condition

\[
\sum_{k=1}^{\infty} (1 - |z_k|) < \infty
\]

then the algebra \( \overline{\mathcal{R}}^w \) contains a rank one operator and therefore it is localizing.

**Proof.** Start with a \( B \in \mathcal{B}(H^\infty(\mathbb{D})) \) such that \( B(z_k) = 0 \) for all \( k \geq 1 \) and \( B(x) \neq 0 \) for all \( x \in [0, 1/2] \). Notice that the family \( \{p \cdot B : p \text{ is a polynomial}\} \) is dense in \( C[0,1/2] \) since the polynomials are dense and the multiplication operator \( M_B \) is invertible on \( C[0,1/2] \). Now, every function \( \varphi \in C[0,1/2] \) can be extended to a function \( \tilde{\varphi} \in L^\infty(\mu) \) given by the expression

\[
\tilde{\varphi}(z) = \begin{cases} 
\varphi(z), & \text{if } z \in [0,1/2], \\
0, & \text{if } z \in \{z_k : k \geq 1\}.
\end{cases}
\]

We claim that \( M_{\tilde{\varphi}} \in \overline{\mathcal{R}}^w \) for every \( \varphi \in C[0,1/2] \). Indeed, let \( \{p_n\} \) be a sequence of polynomials such that \( p_n \cdot B \to \varphi \) uniformly on \( [0,1/2] \). Since \( p_n \cdot B \) vanishes identically on \( \{z_k : k \geq 1\} \), it follows that \( p_n \cdot B \to \tilde{\varphi} \) on \( L^\infty(\mu) \), and since \( p_n \cdot B \in H^\infty(\mathbb{D}) \), we obtain \( M_{p_n \cdot B} \in \overline{\mathcal{R}}^w \), and we conclude that \( M_{\tilde{\varphi}} \in \overline{\mathcal{R}}^w \). Next, take a \( B_1 \in H^\infty(\mathbb{D}) \) such that \( B_1(z_1) = 1 \) and \( B_1(z_k) = 0 \) for all \( k \geq 2 \). Then, consider the function defined as \( \varphi = B_1 - B_1 \cdot \chi_{[0,1/2]} \). Since \( B_1 \in \overline{\mathcal{R}}^w \) and since \( B_1 \cdot \chi_{[0,1/2]} = B_1|_{[0,1/2]} \in \overline{\mathcal{R}}^w \), we obtain \( M_\varphi \in \overline{\mathcal{R}}^w \). Notice that \( M_\varphi \) is a rank one operator, since \( \varphi(z) = 1 \) for \( z = z_1 \) and \( \varphi(z) = 0 \) for \( z \neq z_1 \). □
We have shown so far that the algebra $\mathcal{R}'$ is localizing because it contains a rank one operator. This is all we need for our construction, although something stronger can be said, namely, that $\mathcal{R}' = L^\infty(\mu)$. Since the measure $\mu$ contains many atoms, there are many rank one operators in $\mathcal{R}'$.

**Lemma 2.8.** If the sequence $(z_k)$ satisfies the Blaschke condition

$$\sum_{k=1}^{\infty} (1 - |z_k|) < \infty$$

then the algebra $\mathcal{R}$ is dense in $L^\infty(\mu)$ with respect to the weak operator topology.

**Proof.** We claim that for every sequence of scalars $\alpha = (\alpha_k)$ with $\alpha_k = 0$ for all $k > N$ there is a $\varphi \in \mathcal{R}'$ such that $\varphi(x) = 0$ for all $x \in [0,1/2]$ and such that $\varphi(z_k) = \alpha_k$ for all $k \geq 1$. Indeed, take a sequence $(B_k)$ in $H^\infty(D)$ such that $B_k(z_j) = 1$ if $j = k$ and $B_k(z_j) = 0$ if $j \neq k$. Then, the required conditions are fulfilled by the function

$$\varphi(z) = \sum_{k=1}^{N} \alpha_k B_k(z).$$

Next, we claim that for every $\psi \in C[0,1/2]$ there is a $\Phi \in \mathcal{R}'$ such that $\Phi(x) = \psi(x)$ for all $x \in [0,1/2]$ and $\Phi(z_k) = \alpha_k$ for all $k \geq 1$. Indeed, let $\varphi$ be a function as above and notice that the function $\Phi = \varphi + \psi$ does the job. Finally, we show that $\mathcal{R}$ is dense in $L^\infty(\mu)$ with respect to the weak operator topology. Take a function $\varphi \in L^\infty(\mu)$. There is a sequence of functions $(\psi_n)$ in $C[0,1/2]$ such that $\psi_n \to \varphi|_{[0,1/2]}$ in the weak-* topology. Then, consider for every $n \geq 1$ the sequence of scalars $\alpha^n = (\alpha_k^n)_{k \geq 1}$ defined by

$$\alpha^n_k = \begin{cases} \varphi(z_k), & \text{if } 1 \leq k \leq n, \\ 0, & \text{if } k > n. \end{cases}$$

Let $\Phi_n \in \mathcal{R}'$ be the function associated with $\psi_n$ and $\alpha^n$, that is, $\Phi_n = \varphi + \tilde{\psi}_n$, and notice that $\Phi_n \to \varphi$ in the weak-* topology, as we wanted. □

**Theorem 2.9.** The algebra $\mathcal{R}$ is not localizing on the Hilbert space $L^2(\mu)$.

**Proof.** Let $f_0 \in L^2(\mu)$ and consider a closed ball $B = \{ f \in L^2(\mu) : \|f - f_0\|_2 \leq \varepsilon \}$. We proceed by contradiction. Suppose that the algebra $\mathcal{R}$ and the ball $B$ satisfy the condition in the definition of a localizing algebra.

Claim: $f_0 = 0$ almost everywhere on $[0,1/2]$. Indeed, there is a $\delta > 0$ such that, for every Borel subset $A \subseteq [0,1/2]$, the condition $\mu(A) < \delta$ implies $\|f_0 \cdot \chi_A\|_2 < \varepsilon$. Then, we may construct a sequence $(A_n)$ of independent Borel subsets of $[0,1/2]$ such that $\mu(A_n) = \delta/2$ for each $n \geq 1$. Next, we define a sequence of functions $(f_n)$ inside the ball $B$ by the expression $f_n = f_0 - f_0 \cdot \chi_{A_n}$. Now, there exists a subsequence $(f_{n_j})$, a sequence $(\varphi_j)$ in ball($\mathcal{R}$) and a nonzero $g \in L^2(\mu)$ such that $\varphi_j f_{n_j} \to g$ in $L^2(\mu)$. Then, apply Montel’s theorem to obtain another subsequence $(\varphi_{j_k})$ and a function $\varphi \in H^\infty(D)$ such that $\varphi_{j_k} \to \varphi$ uniformly on compact subsets of $\mathbb{D}$. Since $\varphi_{j_k} f_{n_{j_k}} \to g$ almost everywhere, we may extract a further subsequence $(\varphi_{j_{k_l}})$ such that $\varphi_{j_{k_l}} f_{n_{j_{k_l}}} \to g$ almost everywhere. Next, apply the Borel-Cantelli Lemma to the sequence $(A_{n_{j_{k_l}}})$, and conclude that there is a measurable subset $Z_1 \subseteq [0,1/2]$ such that $\mu(Z_1) = 0$ and such that the set $\{ l \geq 1 : x \in A_{n_{j_{k_l}}} \}$ is infinite for every $x \in Z_1$. It follows that $g(x) = 0$ almost everywhere on $[0,1/2]$. Then, apply once again the Borel-Cantelli Lemma to the sequence $(A_{n_{j_{k_l}}})$, and conclude that there is a measurable subset $Z_2 \subseteq [0,1/2]$ such that $\mu(Z_2) = 0$ and such that the set $\{ l \geq 1 : x \in A_{n_{j_{k_l}}^{\circ}} \}$ is infinite for every $x \in Z_2$. It follows that $f_0(x) \varphi(x) = g(x) = 0$ almost everywhere on $[0,1/2]$. Since $\varphi$ is an analytic function, there are two possibilities: either $\varphi \equiv 0$ or $f_0 \equiv 0$ almost everywhere on $[0,1/2]$. Now we prove that the first possibility leads to a contradiction. We have $|f_{n_{j_k}}| \leq |f_0|$, so that
\[ |\varphi_{jk} f_{n_{jk}}| \leq |\varphi_{jk}| |f_0| \to 0, \] and it follows from the bounded convergence theorem that \( \varphi_{jk} f_{n_{jk}} \to 0 \) in \( L^2(\mu) \). Hence, \( g = 0 \), and we arrived at a contradiction. The proof of our claim is now complete.

Now consider the sequence \( \{f_n\} \) in \( B \) defined as \( f_n = f_0 + \chi_{A_n} \). Since \( R \) is localizing, there exists a subsequence \( \{f_{n_j}\} \), a sequence \( \{\varphi_j\} \) in \( \text{ball}(R) \), and a nonzero \( g \in L^2(\mu) \) such that \( \varphi_j f_{n_j} \to g \) in \( L^2(\mu) \). Then, apply Montel’s theorem to obtain another subsequence \( \{\varphi_{jk}\} \) and a function \( \varphi \in H^\infty(\mathbb{D}) \) such that \( \varphi_{jk} \to \varphi \) uniformly on compact subsets of \( \mathbb{D} \). Next, extract a further subsequence \( \{\varphi_{jk}\} \) such that \( \varphi_{jk} f_{n_{jk}} \to g \) almost everywhere on \( \mathbb{D} \). Then, apply the Borel-Cantelli Lemma to the sequence \( \{A_{n_{jk}}\} \), and conclude that there is a measurable subset \( Z_1 \subseteq [0, 1/2] \) such that \( \mu(Z_1) = 0 \) and such that the set \( \{l \geq 1 : x \in A_{n_{jk}}\} \) is infinite for every \( x \in Z_1 \). Hence, \( \varphi_{jk} f_{n_{jk}} \to 0 \) almost everywhere on \([0, 1/2] \). Then, apply once again the Borel-Cantelli lemma to the sequence \( \{A_{n_{jk}}\} \), and conclude that there is a measurable subset \( Z_2 \subseteq [0, 1/2] \) such that \( \mu(Z_2) = 0 \) and such that the set \( \{l \geq 1 : x \in A_{n_{jk}}\} \) is infinite for every \( x \in Z_2 \). Hence, \( \varphi_{jk} f_{n_{jk}} \to \varphi(x) \) almost everywhere on \([0, 1/2] \). Therefore, we get \( \varphi(x) = 0 \) almost everywhere on \([0, 1/2] \), and since \( \varphi \) is analytic, we conclude that \( \varphi(z) = 0 \) for all \( z \in \mathbb{D} \). Finally, \( |\varphi_{jk} f_{n_{jk}}| \leq (1 + |f_0|)|\varphi_{n_{jk}}| \to 0 \), so that it follows from the bounded convergence theorem that \( g = 0 \), and a contradiction has arrived. \[ \square \]

3. Bounded analytic multipliers on the Bergman space and the Hardy space

We now consider algebras of multiplication operators on Hilbert spaces of analytic functions. We identify every bounded analytic function \( \varphi \in H^\infty(\mathbb{D}) \) with the multiplication operator \( M_{\varphi} \) defined either on the Bergman space \( A^2(\mathbb{D}) \) or on the Hardy space \( H^2(\mathbb{D}) \) by the expression \( (M_{\varphi}f)(z) = \varphi(z)f(z) \). Then, \( H^\infty(\mathbb{D}) \) becomes a subalgebra of both \( B(A^2(\mathbb{D})) \) and \( B(H^2(\mathbb{D})) \) under this identification.

**Theorem 3.1.** The algebra \( H^\infty(\mathbb{D}) \) of all bounded analytic functions on the unit disc is not localizing on the Bergman space \( A^2(\mathbb{D}) \).

**Proof.** Let \( f_0 \in A^2(\mathbb{D}) \) and consider a closed ball \( B = \{f \in A^2(\mathbb{D}) : \|f - f_0\|_2 \leq \epsilon\} \). We must show that the algebra \( H^\infty(\mathbb{D}) \) and the ball \( B \) do not satisfy the condition in the definition of a localizing algebra. First of all, consider the orthonormal basis \( \{e_n\} \) of \( A^2(\mathbb{D}) \) formed by the monomials \( e_n(z) = (n+1)^{1/2}z^n \). Next, consider the sequence \( \{f_n\} \) in \( B \) defined by the expression \( f_n = f_0 + \epsilon e_n \). Now, suppose that there is a subsequence \( \{f_{n_j}\} \), a function \( g_0 \in A^2(\mathbb{D}) \), and a sequence of functions \( \{\varphi_j\} \) in \( H^\infty(\mathbb{D}) \) such that \( \|\varphi_j\|_\infty \leq 1 \) and \( \|\varphi_j f_{n_j} - g_0\|_2 \to 0 \) as \( j \to \infty \). Thus, it suffices to show that \( g_0 = 0 \). Then, apply Montel’s theorem to extract a further subsequence \( \{\varphi_{jk}\} \) that converges uniformly on compact sets to some \( \varphi \in H^\infty(\mathbb{D}) \). The bounded convergence theorem gives \( \|\varphi_{jk} f_0 - \varphi f_0\|_2 \to 0 \) as \( k \to \infty \). Hence, \( \varphi g_0 = 0 \). Notice that \( \|\varphi_j(z)e_{n_j}(z)\| \leq (n_j+1)^{1/2}|z|^{n_j} \to 0 \) as \( j \to \infty \) for each \( z \in \mathbb{D} \), and this gives \( g_0 = \varphi f_0 \) and \( \varphi_j e_{n_j} \to 0 \). Since our aim is to prove that \( g_0 = 0 \), it is enough to show that \( \varphi = 0 \). Fix an integer \( m \geq 0 \) and use Cauchy’s integral formula for the derivatives to get, for each \( 1/2 \leq r < 1 \) and each \( j \geq 1 \),

\[
|\varphi_j^{(m)}(0)| \leq \frac{m!}{2\pi r^m} \int_0^{2\pi} |\varphi_j(re^{i\theta})| \, d\theta \leq \frac{m!2^m}{(2\pi)^{1/2}} \left( \int_0^{2\pi} |\varphi_j(re^{i\theta})|^2 \, d\theta \right)^{1/2}.
\]

Since \( |e_{n_j}(re^{i\theta})|^2 = (n_j+1)r^{2n_j} \), squaring both sides in the above inequality gives

\[
|\varphi_j^{(m)}(0)|^2 \cdot \frac{2(n_j+1)r^{2n_j+1}}{(m!)^2 4^m} \leq \frac{1}{\pi} \int_0^{2\pi} |\varphi_j(re^{i\theta})e_{n_j}(re^{i\theta})|^2 r \, d\theta.
\]

Now, integrating this inequality over the interval \( 1/2 \leq r < 1 \) leads to

\[
|\varphi_j^{(m)}(0)|^2 \cdot \frac{2(n_j+1)}{(m!)^2 4^m} \int_{1/2}^{1} r^{2n_j+1} \, dr \leq \frac{1}{\pi} \int_{1/2}^{1} \int_0^{2\pi} |\varphi_j(re^{i\theta})e_{n_j}(re^{i\theta})|^2 r \, d\theta \, dr.
\]
and from here we obtain the estimate

\[ |\varphi^{(m)}(0)|^2 \cdot \frac{1 - 1/4^n + 1}{(m!)^24m} \leq \frac{1}{\pi} \int_0^{2\pi} |\varphi_j(\text{e}^{i\theta})| e_{n_j}(\text{e}^{i\theta})|^2 \, d\theta \, dr = \|\varphi_j e_{n_j}\|^2. \]

Finally, passing a subsequence \((\varphi_{j_k})\) and taking limits as \(k \to \infty\) yields

\[ \lim_{k \to \infty} |\varphi^{(m)}(0)| \cdot \frac{(1 - 1/4^n + 1)^{1/2}}{(m!)^{2m}} \leq \lim_{k \to \infty} \|\varphi_{j_k} e_{n_{j_k}}\|^2 = 0, \]

so that \(\varphi^{(m)}(0) = 0\) for each \(m \geq 0\), that is, \(\varphi = 0\), as we wanted.

**Theorem 3.2.** The algebra \(H^\infty(D)\) of all bounded analytic functions on the unit disc is not localizing on the Hardy space \(H^2(D)\).

Before we proceed with the proof of Theorem 3.2, we state and prove several lemmas. We shall denote by \(\mu\) the normalized Haar measure on the torus \(T = \{z \in \mathbb{C} : |z| = 1\}\). Also, we shall denote by \((e_n)\) the orthonormal basis in \(L^2(T)\) of the functions defined by the expression \(e_n(z) = z^n\) for every \(n \in \mathbb{Z}\). Finally, for every measurable set \(B \subseteq T\), we shall consider the preimages \(e_n^{-1}(B) = \{z \in T : z^n \in B\}\).

**Lemma 3.3.** If \(A, B \subseteq T\) is any pair of measurable sets then we have

\[ \lim_{n \to \infty} \mu(A \cap e_n^{-1}(B)) = \mu(A)\mu(B). \]

**Proof.** First of all, it is plain that \(\chi_{e_n^{-1}(B)}(z) = \chi_B(z^n)\) for every \(z \in T\), and from this fact it follows that the Fourier coefficients for the characteristic function of the preimage \(e_n^{-1}(B)\) are given by the expression

\[ \hat{\chi}_{e_n^{-1}(B)}(m) = \begin{cases} \hat{\chi}_B(m/n), & \text{if } m \text{ is a multiple of } n, \\ 0, & \text{otherwise.} \end{cases} \]

Next, use Parseval’s identity to obtain

\[ \mu(A \cap e_n^{-1}(B)) = \int_T \chi_A(z) \cdot \chi_{e_n^{-1}(B)}(z) \, d\mu(z) \]

\[ = \sum_{m \in \mathbb{Z}} \hat{\chi}_A(m) \cdot \hat{\chi}_{e_n^{-1}(B)}(m) = \sum_{k \in \mathbb{Z}} \hat{\chi}_A(nk) \cdot \hat{\chi}_B(k) \]

\[ = \hat{\chi}_A(0) \cdot \hat{\chi}_B(0) + \sum_{k \in \mathbb{Z} \setminus \{0\}} \hat{\chi}_A(nk) \cdot \hat{\chi}_B(k) \]

\[ = \mu(A)\mu(B) + \sum_{k \in \mathbb{Z} \setminus \{0\}} \hat{\chi}_A(nk) \cdot \hat{\chi}_B(k). \]

Finally, use the Cauchy-Schwartz inequality to conclude that

\[ |\mu(A \cap e_n^{-1}(B)) - \mu(A)\mu(B)| \leq \|\chi_B\|_2 \left( \sum_{k \in \mathbb{Z} \setminus \{0\}} |\hat{\chi}_A(nk)|^2 \right)^{1/2} \]

\[ \leq \mu(B)^{1/2} \left( \sum_{|k| \geq |n|} |\hat{\chi}_A(k)|^2 \right)^{1/2}, \]

and notice that the last expression approaches zero as \(n \to \infty\), as we wanted.

The following result has the same flavour as the Borel-Cantelli Lemma, which cannot be applied here because the measurable sets under consideration are not necessarily independent.
Lemma 3.4. Let \( A \subseteq \mathbb{T} \) be a measurable set with \( \mu(A) > 0 \), let \( (n_j) \) be an increasing sequence of positive integers, and let \( A_j = \{ z \in \mathbb{T} : z^{n_j} \in A \} \). Then we have

\[
\mu \left( \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} A_j \right) = 1.
\]

Proof. Taking complements, the above statement is equivalent to saying that

\[
\mu \left( \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} (\mathbb{T} \setminus A_j) \right) = 0.
\]

Hence, it suffices to show for every \( k \geq 1 \) that

\[
\mu \left( \bigcap_{j=k}^{\infty} (\mathbb{T} \setminus A_j) \right) = 0.
\]

Fix \( k_0 \geq 1 \), and for each \( k \geq k_0 \), consider the quantity

\[
\alpha_k = \mu \left( \bigcap_{j=k}^{k_0} (\mathbb{T} \setminus A_j) \right).
\]

Then, \((\alpha_k)\) is a decreasing sequence of nonnegative numbers. Set \( \alpha = \lim_{k \to \infty} \alpha_k \). We must show that \( \alpha = 0 \).

Fix \( k \geq k_0 \) and notice that, for each \( l \geq k \),

\[
\bigcap_{j=k}^{l} (\mathbb{T} \setminus A_j) \subseteq \left( \bigcap_{j=k_0}^{k} (\mathbb{T} \setminus A_j) \right) \cap (\mathbb{T} \setminus A_l). \]

We have \( \mathbb{T} \setminus A_l = e^{-1/n_l}(\mathbb{T} \setminus A) \), so that Lemma 3.3 can be applied to obtain

\[
\alpha = \lim_{l \to \infty} \mu \left( \bigcap_{j=k_0}^{l} (\mathbb{T} \setminus A_j) \right) \leq \lim_{l \to \infty} \mu \left[ \left( \bigcap_{j=k_0}^{k} (\mathbb{T} \setminus A_j) \right) \cap (\mathbb{T} \setminus A_l) \right]
\]

\[
= \mu \left( \bigcap_{j=k_0}^{k} (\mathbb{T} \setminus A_j) \right) \cdot \mu(\mathbb{T} \setminus A) = \alpha_k \cdot \mu(\mathbb{T} \setminus A).
\]

Finally, taking limits as \( k \to \infty \) leads to the inequality \( \alpha \leq \alpha \cdot \mu(\mathbb{T} \setminus A) \), and since \( \mu(\mathbb{T} \setminus A) < 1 \), we conclude that \( \alpha = 0 \), as we wanted. \( \square \)

Lemma 3.5. If \( 0 < \delta < 1 \) then there is an open set \( U \supseteq \overline{\mathbb{D}} \) and there is a holomorphic function \( h \in H(U) \) such that

1. \( h(1) = 0 \),
2. \( |h(z)| \leq 1 \) for each \( z \in \mathbb{D} \),
3. \( \mu(\{ z \in \mathbb{T} : |h(z) - 1| > \delta \}) < \delta \).

Proof. Let \( 0 < r < 1 \) to be chosen later on, and consider the function \( h \) defined by the expression

\[
h(z) = \frac{r(1-z)}{1-rz}.
\]

It is plain that \( h \) is holomorphic on \( \mathbb{C} \setminus \{1/r\} \supseteq \mathbb{D} \) and that \( h(1) = 0 \). Since \( h \) is a Moebius transformation, it is easy to check that \( h(\overline{\mathbb{D}}) \) is a disc of radius \( r/(1+r) \) centered at \( r/(1+r) \). It follows that \( |h(z)| \leq 1 \) for each \( z \in \mathbb{D} \). Now, consider the arc \( A = \{ 1 \} < \pi/2 \} \) and notice that \( \mu(A) = \delta/2 \). Thus, it
suffices to show that \{ z \in \mathbb{T} : |h(z) - 1| > \delta \} \subseteq A for a suitable choice of \( r \). Consider the compact set \( K = \{ rz : r \in [0, 1], z \in \mathbb{T}\setminus A \} \). Since \( 1 \notin K \), there is an \( \eta > 0 \) such that \( |1 - rz| \geq \eta \) for every \( r \in [0, 1] \) and for every \( z \in \mathbb{T}\setminus A \). Thus, for each \( z \in \mathbb{T}\setminus A \) we have
\[
|h(z) - 1| = \frac{1 - r}{|1 - rz|} \leq \frac{1 - r}{\eta} < \delta,
\]
as long as \( r \) is chosen to be close enough to 1.

Proof of Theorem \([3,5]\) Let \( f_0 \in H^2(\mathbb{D}) \) and consider the closed ball \( B = \{ f \in H^2(\mathbb{D}) : ||f - f_0|| \leq \varepsilon \} \). We must show that the algebra \( H^\infty(\mathbb{D}) \) and the ball \( B \) do not satisfy the conditions in the definition. Let \( \delta > 0 \) to be chosen later on, let \( h \) be a holomorphic function as in Lemma \([3,5]\) and define a sequence of functions \( (f_n) \) in \( H^2(\mathbb{D}) \) by the expression
\[
f_n(z) = f_0(z)h(z^n), \quad z \in \mathbb{D}, \quad n \geq 1.
\]
We claim that \( f_n \in B \) for each \( n \geq 1 \), provided that \( \delta > 0 \) is suitably chosen. Indeed, consider the measurable sets
\[
A = \{ z \in \mathbb{T} : |h(z) - 1| > \delta \} \quad \text{and} \quad A_n = \{ z \in \mathbb{T} : |h(z^n) - 1| > \delta \},
\]
and notice that \( A_n = c_n^{-1}(A) \), so that \( \mu(A_n) = \mu(A) < \delta \). Then we get
\[
||f_n - f_0||_2^2 = \int_{\mathbb{T}} |h(z^n) - 1|^2 \cdot |f_0(z)|^2 \, d\mu(z)
\leq \int_{\mathbb{T}\setminus A_n} \delta^2 |f_0(z)|^2 \, d\mu(z) + \int_{A_n} 4|f_0(z)|^2 \, d\mu(z)
\leq \delta^2 ||f_0||_2^2 + \int_{A_n} 4|f_0(z)|^2 \, d\mu(z).
\]
Now, choose \( \delta > 0 \) such that \( \delta^2 ||f_0||_2^2 < \varepsilon^2/2 \), and with the property that, for each measurable set \( B \subseteq \mathbb{T} \), the condition \( \mu(B) < \delta \) implies that
\[
\int_B 4|f_0(z)|^2 \, d\mu(z) < \varepsilon^2/2.
\]
Hence, \( ||f_n - f_0||_2 < \varepsilon^2 \), and the proof of our claim is over. Next, suppose that there is a subsequence \( (f_{n_j}) \), a sequence \( (\varphi_j) \) in \( H^\infty(\mathbb{D}) \), and a function \( g \in H^2(\mathbb{D}) \) such that \( ||g - \varphi_j f_{n_j}||_2 \to 0 \) as \( j \to \infty \). Then, it suffices to show that \( g(z) = 0 \) for almost every \( z \in \mathbb{T} \). We may assume, extracting a subsequence if necessary, that \( \varphi_j(z) f_{n_j}(z) \to g(z) \) as \( j \to \infty \) for almost every \( z \in \mathbb{T} \). Thus, there is a measurable set \( N_0 \subseteq \mathbb{T} \) such that \( \mu(N_0) = 0 \) and such that, for every \( z \in \mathbb{T}\setminus N_0 \), we have
\[
|g(z)| = \liminf_{j \to \infty} |\varphi_j(z)| \cdot |f_{n_j}(z)| 
\leq \liminf_{j \to \infty} |f_{n_j}(z)| 
= |f_0(z)| \cdot \liminf_{j \to \infty} |h(z^{n_j})|.
\]
Since \( h \) is continuous at \( z = 1 \), for every integer \( m \geq 1 \) there is an open set \( G_m \subseteq \mathbb{T} \) such that \( 1 \in G_m \) and \( |h(z)| < 1/m \) for each \( z \in G_m \). Now, apply Lemma \([3,4]\) to get a measurable set \( N_m \subseteq \mathbb{T} \) with \( \mu(N_m) = 0 \) and such that \( z^{n_j} \in G_m \) infinitely often for each \( z \in \mathbb{T}\setminus N_m \). Therefore, \( \liminf |h(z^{n_j})| \leq 1/m \) as \( j \to \infty \) for each \( z \in \mathbb{T}\setminus N_m \). Finally, consider the countable union of measurable sets
\[
N = \bigcup_{m=0}^{\infty} N_m,
\]
and notice that \(\mu(N) = 0\). If \(z \in \mathbb{T} \setminus N\) then \(|g(z)| \leq |f_0(z)|/m\) for every integer \(m \geq 1\). We conclude from this inequality that \(g(z) = 0\) for every \(z \in \mathbb{T} \setminus N\), as we wanted. \(\square\)

4. Algebras generated by diagonal operators

Now we turn our attention to the algebra generated by a single normal operator \(T\). The spectral theorem ensures that there is measure \(\mu\) of compact support on the Borel subsets of the complex plane such that \(T\) is unitarily equivalent to a multiplication by a bounded measurable function on \(L^2(\mu)\). Then the algebra generated by \(T\) may be regarded as a subalgebra of \(L^\infty(\mu)\), and in view of Theorem 2.2, if such an algebra is localizing then the measure \(\mu\) must have an atom. Now we focus on the extreme case that \(\mu\) is a purely atomic measure, so that \(T\) is a diagonal operator.

Let \((x_j)\) be a sequence of complex numbers in the closed unit disc. Consider the diagonal operator \(T = \text{diag}(x_j)\), that is, \(Te_j = x_je_j\), where \((e_j)\) is an orthonormal basis of an infinite dimensional, separable complex Hilbert space \(H\) and \(j\) runs through the non negative integers. Suppose that \(x_j \neq x_k\) whenever \(j \neq k\). Then, let \(R = \{p(T) : p\text{ is a polynomial}\}\) denote the unital algebra generated by \(T\). In this section, some conditions are given for the algebra \(R\) to be localizing.

**Proposition 4.1.** If \(|x_{j_0}| = 1\) for some \(j_0 \geq 0\) then \(\text{ball}(R) = \mathbb{D}\) contains a rank one operator.

**Proof.** Consider the sequence of polynomials \((p_n)\) defined by the expression
\[
p_n(z) := \left(\frac{x_{j_0}z + 1}{2}\right)^n.
\]
Then \(\|p_n\|_\infty \leq 1\), so that \(p_n(T) \in \text{ball}(R)\). Moreover, \((p_n)\) converges pointwise to the function \(f\) defined by \(f(z) = 0\) if \(z \neq x_{j_0}\) and \(f(x_{j_0}) = 1\). Therefore, the sequence of operators \((p_n(T))\) converges in the weak operator topology to the rank one operator \(e_{j_0} \otimes e_{j_0}\). \(\square\)

Recall that the spectrum of \(T\) is the compact set
\[
\sigma(T) = \{x_j : j \geq 0\}.
\]

**Proposition 4.2.** If \(\sigma(T)\) has empty interior and \(\mathbb{C} \setminus \sigma(T)\) is connected, then \(\text{ball}(R) = \mathbb{D}\) is the set of all diagonal operators of the form \(\text{diag}(\lambda_j)\), for some \((\lambda_j) \in \ell_\infty\) with \(||(\lambda_j)||_\infty \leq 1\). In particular, \(\text{ball}(R)\) contains a nonzero compact operator.

**Proof.** We prove the non trivial inclusion. Let \((\lambda_j) \in \ell_\infty\) with \(||(\lambda_j)||_\infty \leq 1\), and for every \(n \geq 1\), choose a continuous function \(f_n : \sigma(T) \to \mathbb{C}\) with \(f_n(z_j) = \lambda_j\) whenever \(1 \leq j \leq n\) and \(|f(z)| \leq 1\) for all \(z \in \sigma(T)\). It follows from Mergelyan’s theorem that for every \(n \geq 1\) there is a polynomial \(p_n(z)\) such that \(|p_n(z)| \leq 1\) for each \(z \in \sigma(T)\) and such that \(|p_n(z_j) - \lambda_j| < 1/n\) whenever \(1 \leq j \leq n\). Finally, the sequence of diagonal operators \((p_n(T))\) lies inside \(\text{ball}(R)\) and it converges to the diagonal operator \(\text{diag}(\lambda_j)\) in the strong operator topology. \(\square\)

The rest of this section deals with diagonal operators \(T\) with the property that \(\sigma(T) \supseteq \partial\mathbb{D}\). We make this assumption because it allows us to control the norm of an operator in the algebra generated by \(T\). Indeed, if \(p\) is a polynomial then it follows from the maximum modulus principle that
\[
\|p(T)\| = \sup\{|p(z)| : z \in \sigma(T)\} = \sup\{|p(z)| : z \in \mathbb{D}\} = \|p\|_\infty.
\]
Notice that Proposition 4.1 allows us to discard the case \(|x_{j_0}| = 1\) for some \(j_0 \geq 0\), so that from now on we shall assume \(|x_j| < 1\) for all \(j \geq 0\).

**Proposition 4.3.** The closure of the unit ball of \(R\) in the weak operator topology is the set of all diagonal operators of the form \(\text{diag}(f(z_j))\), where \(f \in H^\infty(\mathbb{D})\) and \(\|f\|_\infty \leq 1\).
Proof. First, let \( R \in \text{ball}(\mathcal{R})^\sigma \). Since \( H \) is separable, the weak operator topology is metrizable on bounded subsets of \( \mathcal{B}(H) \), and therefore, there exists a sequence of polynomials \((p_n)\) such that \( \|p_n(T)\| \leq 1 \) and \( p_n(T) \to R \) in the weak operator topology. Now, \( p_n(T) \) is a diagonal operator with diagonal sequence \( \langle p_n(z_j) \rangle \) so that \( \|p_n\| = \|p_n(T)\| \leq 1 \). Then, it follows from Montel’s theorem that there is a subsequence \((p_{n_k})\) that converges uniformly on compact subsets of \( \mathbb{D} \) to some function \( f \in H^\infty(\mathbb{D}) \) with \( \|f\|_\infty \leq 1 \). Therefore,

\[
\langle Re_j, e_l \rangle = \lim_{k \to \infty} \langle p_{n_k}(T)e_j, e_l \rangle = \lim_{k \to \infty} \langle p_{n_k}(z_j)e_j, e_l \rangle = \langle f(z_j)e_j, e_l \rangle.
\]

Thus, \( R = \text{diag}(f(z_j)) \), as we wanted. Next, let \( f \in H^\infty(\mathbb{D}) \) with \( \|f\|_\infty \leq 1 \), and let \( R = \text{diag}(f(z_j)) \). Then, there is a sequence of polynomials \((p_n)\) such that \( \|p_n\| \leq 1 \) and \( p_n \to f \) uniformly on compact subsets of \( \mathbb{D} \). We can take for instance the sequence of polynomials \( p_n = F_n \ast f \), where \((F_n)\) is the sequence of the Fejér kernels, that is,

\[
p_n(z) = \sum_{j=0}^n \left(1 - \frac{j}{n+1} \right) \hat{f}(j)z^j.
\]

Thus, \( \|p_n(T)\| \leq 1 \) and for every \( j, k \geq 0 \) we have

\[
\lim_{n \to \infty} \langle p_n(T)e_j, e_k \rangle = \lim_{n \to \infty} \langle p_n(z_j)e_j, e_k \rangle = \langle f(z_j)e_j, e_k \rangle = \langle Re_j, e_k \rangle.
\]

This shows that \( p_n(T) \to R \) in the weak operator topology, so that \( R \in \text{ball}(\mathcal{R})^\sigma \), as we wanted. \( \square \)

**Corollary 4.4.** The following conditions are equivalent:

1. \( \text{ball}(\mathcal{R})^\sigma \) contains a non zero compact operator,
2. there exists \( f \in H^\infty(\mathbb{D}) \) such that \( \|f\|_\infty \leq 1 \), with \( f(z_j) \neq 0 \) for some \( j \geq 0 \) and \( \lim_{j \to \infty} f(z_j) = 0 \).

Consider the set \( \sigma(T)' \) of all cluster points of the spectrum of \( T \). The meaning of the following result is that when the part of \( \sigma(T)' \) in the open unit disc is large enough, the algebra \( \mathcal{R} \) fails to be localization.

**Proposition 4.5.** Suppose that there is a sequence \((w_p)\) of distinct points in \( \sigma(T)' \cap \mathbb{D} \) such that

\[
\sum_{p=1}^\infty (1 - |w_p|) = \infty.
\]

Then \( \mathcal{R} \) fails to be a localization algebra.

**Proof.** We proceed by contradiction. Suppose \( \mathcal{R} \) is a localization algebra and let \( B = \{ x \in H : \|x - x_0\| \leq \varepsilon \} \) be a ball as in the definition. We may assume without loss of generality that \( x_0 \in H \) has finite support, say \( \text{supp}(x_0) \subseteq [0, M] \). Then, for every \( p \geq 1 \) there is a subsequence \( (z_{j_p,q}) \) such that \( \lim_{q \to \infty} z_{j_p,q} = w_p \) for all \( p \geq 1 \). Moreover, the indices \( j_{p,q} \) can be chosen in such a way that \( j_{p,q} > M \) for all \( p, q \geq 1 \) and \( j_{p,q} \neq j_{s,t} \) if \( (p, q) \neq (s, t) \). Now, let \((\alpha_p)\) be a sequence of positive real numbers such that \( \sum_{p=1}^\infty \alpha_p^2 < \varepsilon^2 \), and consider the sequence of vectors \((x_q)\) in the ball \( B \) defined by \( x_q := x_0 + y_q \), where

\[
y_q := \sum_{p=1}^\infty \alpha_p e_{j_{p,q}}.
\]

Notice that \( x_0 \perp y_q \), because \( \text{supp}(x_0) \subseteq [0, M] \) and \( j_{p,q} > M \). Since \( \mathcal{R} \) is a localization algebra, there is a sequence of polynomials \((f_k)\) such that \( \|f_k(T)\| \leq 1 \), and there is a subsequence \((x_{q_k})\) such that \( (f_k(T)x_{q_k}) \) converges in norm to some vector \( y \neq 0 \). Since \( \|f_k\|_\infty \leq 1 \), using Montel’s theorem we may assume by extracting a subsequence if necessary that \( (f_k) \) converges uniformly on compact sets to some function \( f \in H^\infty(\mathbb{D}) \). Consider the diagonal operator \( f(T) := \text{diag}(f(z_j)) \). Since the vector \( x_0 \) has finite support, the sequence \( (f_k(T)x_0) \) converges in norm to \( f(T)x_0 \). Thus, the sequence \( (f_k(T)y_{q_k}) \) converges in norm.
Notice that \( y_q \to 0 \) weakly. Hence, \( f_k(T)y_{q_k} \to 0 \) weakly, and we may conclude that \( \|f_k(T)y_{q_k}\| \to 0 \). Therefore, we have \( y = f(T)x_0 \). Finally, it follows from the bounded convergence theorem that
\[
\sum_{p=1}^{\infty} \alpha_p^2 |f(w_p)|^2 = \lim_{k \to \infty} \sum_{p=1}^{\infty} \alpha_p^2 |f_k(z_{j_p,q_k})|^2 = \lim_{k \to \infty} \|f_k(T)y_{q_k}\|^2 = 0,
\]
and from this identity we get \( f(w_p) = 0 \) for all \( p \geq 1 \). Finally, the condition \( \sum_{p=1}^{\infty} (1 - |w_p|) = \infty \) implies \( f(z) = 0 \) for all \( z \in \mathbb{D} \). Hence, \( y = 0 \), and the contradiction has arrived.

We finish this section with a statement of Problem 1.3 for the special case of the algebra \( \mathcal{R} \) generated by a single diagonal operator.

**Problem 4.6.** Let \( \mathcal{R} \) be the algebra generated by a single diagonal operator on an infinite dimensional, separable complex Hilbert space. Suppose that \( \mathcal{R} \) is localizing. Does \( \text{ball}(\mathcal{R})^\sigma \) contain a rank one operator, or at least, a nonzero compact operator?

## 5. Extended eigenvalues and invariant subspaces

The first author [8] obtained a simple proof of Theorem 1.1 that is reminiscent of Hilden’s proof of a special case of the Lomonosov original result [10] and that can be adapted to prove Theorem 1.5.

**Proof of Theorem 1.5.** We proceed by contradiction. Assume the commutant \( \{T\}' \) is a transitive algebra. Since \( \ker T \) is invariant under \( \{T\}' \) and since \( T \neq 0 \), we must have \( \ker T = \{0\} \), so that \( T \) is injective. Then, let \( B \subseteq E \) be a closed ball that makes a localizing subspace out of \( \mathcal{X} \). We claim that there is some constant \( c > 0 \) such that for every \( x \in B \) there is an \( x \in \mathcal{X} \) such that \( \|X\| \leq c \) and \( TXx \in B \). Otherwise, for every \( n \in \mathbb{N} \) there is an \( x_n \in B \) such that the condition \( X \in \mathcal{X} \) and \( TXx_n \in B \) implies \( \|X\| > n \). Since \( \mathcal{X} \) is localizing, there is a sequence \( (x_{n_j}) \) and there is a sequence \( (X_{j}) \) in \( \mathcal{X} \) with \( \|X_{j}\| \leq 1 \), and such that \( (X_{j}x_{n_j}) \) converges in norm to some nonzero vector \( x \in E \). Therefore, \( (X_{j}x_{n_j}) \) converges in norm to \( Tx \). Since \( T \) is injective, we have \( Tx \neq 0 \). Since \( \{T\}' \) is transitive, there is an \( R \in \{T\}' \) such that \( RTx \in \text{int} B \). Hence, there is some \( j_0 \geq 1 \) such that \( RTX_{j}x_{n_j} \in B \) for all \( j \geq j_0 \). Since \( RT = TR \), we have \( TRX_{j}x_{n_j} \in B \) for all \( j \geq j_0 \). Since \( RX_{j} \in \mathcal{X} \), the choice of the sequence \( (x_{n_j}) \) implies \( \|RX_{j}\| > n_j \) for all \( j \geq j_0 \). Finally, this leads to a contradiction, because \( \|RX_{j}\| \leq \|R\| \) for all \( j \geq 1 \). This completes the proof of our claim.

Start with a vector \( x_0 \in B \) and choose an operator \( X_1 \in \mathcal{X} \) such that \( \|X_1\| \leq c \) and \( TX_1x_0 \in B \). Now choose another operator \( X_2 \in \mathcal{X} \) such that \( \|X_2\| \leq c \) and \( TX_2TX_1x_0 \in B \). Continue this ping pong game to obtain a sequence of vectors \( x_n \in B \) and a sequence of operators \( (X_n) \) in \( \mathcal{X} \) such that \( \|X_n\| \leq c \) and such that
\[
x_n = TX_n \cdots TX_1x_0 = \lambda^{n(n+1)/2} X_n \cdots X_1 T^n x_0.
\]
Then, let \( d = \min\{\|x\| : x \in B\} \). It is plain that \( d > 0 \) because \( 0 \notin B \). Assume \( |\lambda| \leq 1 \). We get
\[
d \leq \|x_n\| \leq c^n |\lambda|^{n(n+1)/2} \cdot \|T^n\| \cdot \|x_0\|,
\]
and this gives information on the spectral radius of \( T \), namely,
\[
r(T) = \lim_{n \to \infty} \|T^n\|^{1/n} \geq \lim_{n \to \infty} \frac{1}{c|\lambda|^{(n+1)/2}}.
\]
If \( |\lambda| < 1 \) then we get \( r(T) = \infty \), and if \( |\lambda| = 1 \) and \( T \) is quasinilpotent then we get \( r(T) \geq 1/c \). In both cases we obtain a contradiction. Finally, assume \( |\lambda| > 1 \). Notice that
\[
x_n = TX_n \cdots TX_1x_0 = \lambda^{-n(n-1)/2} T^n X_n \cdots X_1 x_0.
\]
From this identity we get
\[
d \leq \|x_n\| \leq c^n |\lambda|^{-n(n-1)/2} \cdot \|T^n\| \cdot \|x_0\|,
\]
and once again this gives information on the spectral radius of $T$, namely,

$$r(T) = \lim_{n \to \infty} \|T^n\|^{1/n} \geq \lim_{n \to \infty} \frac{|\lambda|^{(n-1)/2}}{c} = \infty.$$ 

A contradiction has arrived. \hfill \Box

The following is an example of an operator $T$ on a Banach space $E$ such that the family of all extended eigenoperators associated with some extended eigenvalue of $T$ is a localizing subspace of $B(E)$ although it does not contain any nonzero compact operators.

**Example 5.1.** Let $E = C[0, 1]$ be the Banach space of continuous functions on the unit interval endowed with the supremum norm. Then, consider the position operator $M \in B(E)$ defined as $(Mf)(t) = tf(t)$. We shall show that the set of extended eigenvalues of the position operator $M$ associated with some extended eigenvalue of $T$ is an extended eigenoperator of $T$.

**Lemma 5.2.** Let $\lambda \in \mathbb{C}$ be an extended eigenvalue of $M$ and let $X \in B(E)$ be an extended eigenoperator associated with $\lambda$. Then $\lambda \neq 0$ and there is a nonzero function $\varphi \in C[0, 1]$ so that for every polynomial $p$,

$$(Xp)(t) = \varphi(t)p(t/\lambda).$$

**Proof.** We have $M \varphi = \lambda X \varphi$. Notice that $\lambda \neq 0$ because otherwise $M \varphi = 0$, and since $M$ is injective, it follows that $X = 0$. Then we have $XM^n = \lambda^{-n}M^n X$, and setting $\varphi = X1$ we get $Xt^n = \varphi(t)(t/\lambda)^n$. Hence, the desired identity follows by linearity. Notice that the function $\varphi$ does not vanish identically, for otherwise $Xp = 0$ for every polynomial $p$, and it follows from the Weierstrass theorem that $X = 0$. \hfill \Box

**Lemma 5.3.** If $\lambda \in \mathbb{C}$ is an extended eigenvalue of $M$ then $\lambda \in (0, \infty)$. 

**Proof.** We know that $\lambda \neq 0$ and there is a nonzero function $\varphi \in C[0, 1]$ such that for every polynomial $p$, $(Xp)(t) = \varphi(t)p(t/\lambda)$. Let $t_0 \in [0, 1]$ such that $\varphi(t_0) \neq 0$. Since $\varphi$ is continuous, we may assume without loss of generality that $t_0 \neq 0$. We proceed by contradiction. If $\lambda \notin [0, \infty)$ then $t_0/\lambda \notin [0, 1]$ and it follows from Runge’s theorem that for every $n \geq 1$ there is a polynomial $p_n$ such that $|p_n(t)| \leq 1$ for every $t \in [0, 1]$ and such that $|p_n(t_0/\lambda)| \geq n$. Hence, $\|p_n\|_\infty \leq 1$ but $\|Xp_n\|_\infty \geq |\varphi(t_0)|n$, and this is a contradiction, because $X$ is a bounded operator. \hfill \Box

**Lemma 5.4.** If $\lambda \in (0, \infty)$ then $\lambda$ is an extended eigenvalue of $M$, and moreover, if $\varphi \in C[0, 1]$ is some nonzero function such that $\varphi(t) = 0$ for all $t \in [0, 1] \cap (\lambda, \infty)$, then the operator $X$ defined by

$$(Xf)(t) = \begin{cases} 
\varphi(t)f(t/\lambda), & \text{if } t \in [0, 1] \cap [0, \lambda], \\
0, & \text{if } t \in [0, 1] \cap (\lambda, \infty),
\end{cases} \quad (*)$$

is an extended eigenoperator of $M$ associated with $\lambda$. Conversely, if $X$ is an extended eigenoperator of $M$ associated with $\lambda$ then there is some nonzero function $\varphi \in C[0, 1]$ such that $\varphi(t) = 0$ for every $t \in [0, 1] \cap (\lambda, \infty)$ and such that $X$ is given by the above expression.

**Proof.** Let us suppose that an operator $X$ is given by the expression $(*)$. Notice that $X$ is well defined since $t/\lambda \in [0, 1]$ for all $t \in [0, 1] \cap [0, \lambda]$, and $Xf$ is continuous since $\varphi(\lambda) = 0$ in the case $\lambda \in (0, 1)$. Also, it is clear that $X$ is linear and bounded, with $\|X\| \leq \|\varphi\|_\infty$. Then, for every $f \in C[0, 1]$, we have

$$(M_X f)(t) = t \varphi(t)f(t/\lambda) = \lambda \varphi(t)(t/\lambda)f(t/\lambda) = \lambda(XMf)(t)$$

if $t \in [0, 1] \cap [0, \lambda]$ and $(M_X f)(t) = 0 = (XMf)(t)$ if $t \in [0, 1] \cap (\lambda, \infty)$, so that $\lambda$ is an extended eigenvalue of $M_X$ and $X$ is an extended eigenoperator associated with $\lambda$. Conversely, if $X$ is an extended eigenoperator of $M$ associated with $\lambda$ then it follows from Lemma 5.2 that there is some nonzero function $\varphi \in C[0, 1]$ such that $(Xp)(t) = \varphi(t)p(t/\lambda)$ for every polynomial $p$. We need to show that, when $\lambda \in (0, 1)$, we have
Theorem 5.5. The family $X$ of all extended eigenoperators of $M_\lambda$ associated with an extended eigenvalue $\lambda \in (0, \infty)$ is a localizing subspace of $B(C[0,1])$ and it does not contain any nonzero compact operators.

Proof. First, we show that $X$ is localizing. Consider the closed ball $B = \{ f \in C[0,1]: \|f - 1\|_\infty \leq 1/2 \}$. Take a sequence $(f_n)$ in $B$. Notice that $1/2 \leq |f_n(t)| \leq 3/2$ for every $t \in [0,1]$. Suppose that $\lambda \in [1,\infty)$ and let $(\varphi_n)$ be the sequence of functions defined by the expression

$$\varphi_n(t) = \frac{1}{2f_n(t/\lambda)}.$$ 

Then $\varphi_n \in C[0,1]$ and $\|\varphi_n\|_\infty \leq 1$. Consider the sequence $(X_n)$ in $X$ defined by $(X_n f)(t) = \varphi_n(t) f(t/\lambda)$. Then $\|X_n\| \leq 1$ and $(X_n f_n)(t) = 1/2$ for all $t \in [0,1]$. Now, suppose that $\lambda \in (0,1)$ and let $(\varphi_n)$ denote the sequence of functions defined by the expression

$$\varphi_n(t) = \begin{cases} \lambda - t/2f_n(t/\lambda), & \text{if } 0 \leq t < \lambda, \\ 0, & \text{if } \lambda \leq t \leq 1. \end{cases}$$

Then $\varphi_n \in C[0,1]$ and $\|\varphi_n\|_\infty \leq 1$. Consider the sequence $(X_n)$ in $X$ defined by the expression

$$(X_n f)(t) = \begin{cases} \varphi_n(t) f(t/\lambda), & \text{if } 0 \leq t \leq \lambda, \\ 0, & \text{if } \lambda < t \leq 1, \end{cases}$$

so that $\|X_n\| \leq 1$ and $(X_n f_n)(t) = \max\{0, (\lambda - t)/2\}$ for all $t \in [0,1]$ and all $n \geq 1$. In both cases we conclude that the family $X$ is a localizing subspace of $B(C[0,1])$.

Next, we show that $X$ does not contain any nonzero compact operators. Take an operator $X \in X \setminus \{0\}$ and let $\varphi \in C[0,1]$ be a nonzero function such that $X$ is given by the expression $(*)$. Since $\varphi$ is continuous and it does not vanish identically, and since $\varphi(t) = 0$ for all $t > \lambda$, there is some $\delta > 0$ and there is an open interval $I \subseteq [0,1]$ with $\lambda I \subseteq [0,1]$ and such that $|\varphi(t)| \geq \delta$ for all $t \in I$. Consider the infinite dimensional, closed subspaces

$$E = \{ f \in C[0,1]: f(t) = 0 \text{ for all } t \in [0,1] \setminus I \},$$

$$F = \{ f \in C[0,1]: f(t) = 0 \text{ for all } t \in [0,1] \setminus \lambda I \}.$$ 

Notice that $X E \subseteq F$. We claim that the restriction $X|_E: E \to F$ is onto, so that $X$ cannot be compact. Indeed, let $g \in F$ and consider the function defined by

$$f(t) = \begin{cases} g(\lambda t)/|\varphi(\lambda t)|, & \text{if } t \in I, \\ 0, & \text{if } t \in [0,1] \setminus I. \end{cases}$$

It is easy to see that $f \in E$ and $g = X f$, as we wanted. \qed

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References

[1] A. Biswas and S. Petrovic, On extended eigenvalues of operators, Integral Equations Operator Theory 57 (2007), 593–598. MR 2234256, Zbl 1119.47019.

[2] A. Biswas, A. Lambert and S. Petrovic, Extended eigenvalues and the Volterra operator, Glasgow Math. J. 44 (2002), 521–534. MR 1956558, Zbl 1037.47013.

[3] P. S. Bourdon and J. H. Shapiro, Intertwining relations and extended eigenvalues for analytic Toeplitz operators, Illinois Journal of Mathematics 52 (2008), 1007–1030. MR 2546021, Zbl 1174.47003.
[4] S. Brown, Connections between an operator and a compact operator that yield hyperinvariant subspaces, J. Operator Theory 1 (1979), 1–21. MR 526293, Zbl 0451.47002.

[5] I. Kaplansky, A theorem on rings of operators, Pacific J. Math. 1 (1951), 227–232. MR 0050181, Zbl 0043.11502.

[6] M. T. Karaev, On extended eigenvalues and extended eigenvectors of some operator classes, Proc. Amer. Math. Soc. 143 (2006), 2383–2392. MR 2213712, Zbl 1165.47302.

[7] H. W. Kim, R. Moore and C. M. Pearcy, A variation of Lomonosov’s theorem, J. Operator Theory 2 (1979), 131–140. MR 553868, Zbl 0433.47010.

[8] M. Lacruz, A note on transitive localizing algebras, Proc. Amer. Math. Soc. 137 (2009), 3421–3423. MR 2515411, Zbl 1177.47011.

[9] A. Lambert, Hyperinvariant subspaces and extended eigenvalues, New York J. Math. 10 (2004), 83–88. MR 0420305, Zbl 1090.47003.

[10] V. I. Lomonosov, Invariant subspaces for operators commuting with compact operators, Funkcional. Anal. i Prilozen. 7 (1973), 55–56 (Russian). English translation: Funct. Anal. Appl. 7 (1973), 213–214. MR 0420305, Zbl 0293.47003.

[11] V.I. Lomonosov, H. Radjavi and V.G. Troitsky, Sesquitransitive and localizing operator algebras, Integral Equations Operator Theory 60 (2008), 405–418. MR 2392834, Zbl 1143.47006.

[12] S. Petrovic, On the extended eigenvalues of some Volterra operators, Integral Equations Operator Theory 57 (2007), 593–598. MR 2313287, Zbl 1126.47017.

[13] H. Radjavi and P. Rosenthal, Invariant Subspaces, Springer-Verlag, New York, 1973. MR0367682, Zbl 0269.47003.

[14] S. Shkarin, Compact operators without extended eigenvalues, J. Math. Anal. Appl. 332 (2007), 445–462. MR 2319675, Zbl 1121.47012.

[15] M. Takesaki, Theory of Operator Algebras I, Springer-Verlag, New York, 1979. MR 0548728, Zbl 0436.46043.

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