A semi-discrete approximation for first-order stationary mean field games

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Abstract

We provide an approximation scheme for first-order stationary mean field games with a separable Hamiltonian. First, we discretize Hamilton-Jacobi equations by discretizing in time, and then prove the existence of minimizing holonomic measures for mean field games. At last, we obtain two sequences of solutions \( \{u_i\} \) of discrete Hamilton-Jacobi equations and minimizing holonomic measures \( \{m_i\} \) for mean field games and show that \( (u_i, m_i) \) converges to a solution of the stationary mean field games.

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Mean field games \cite{15,16,17,18} consists of studying the global behavior of systems composed of infinitely many agents which interact in a symmetric manner. A first-order mean field games model is a coupled system of partial differential equations, one Hamilton-Jacobi equation and one continuity equation. Here, we focus on the first-order ergodic (or stationary) mean field games system

\begin{equation}
H(x, Du) = F(x, m) + c(m) \quad \text{in} \quad \mathbb{T}^d, \tag{1.1a}
\end{equation}

\begin{equation}
\text{div} \left( m \frac{\partial H}{\partial p}(x, Du) \right) = 0 \quad \text{in} \quad \mathbb{T}^d, \tag{1.1b}
\end{equation}

\begin{equation}
\int_{\mathbb{T}^d} m \, dx = 1. \tag{1.1c}
\end{equation}

This system arises in the study of the long-time behavior problem of first-order mean field games with finite horizon \cite{10}. In this work, we aim to study a semi-discrete in time approximation of the first-order ergodic mean field games system \eqref{sys}. We are concerned with the convergence of the discrete scheme. In a forthcoming paper we will deal with a fully discrete approximation problem for \eqref{sys}, where space discretization will be added. See \cite{9}, \cite{11} for semi-discrete and fully discrete approximation schemes for first-order evolutionary mean field games with finite horizon, respectively. We refer the readers to \cite{1,2,3,4,5,6,8} and the references therein for numerical methods and convergence results of different discrete schemes for second-order mean field games.

### 1.1 Assumptions and main results

Let $H(x, p) : \mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a $C^2$ Hamiltonian satisfying Tonelli conditions, where $\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$ denotes the standard flat torus. The associated Lagrangian is defined by

$$L(x, v) := \sup_{v \in \mathbb{R}^d} \left( \langle p, v \rangle - H(x, p) \right), \quad (x, v) \in \mathbb{T}^d \times \mathbb{R}^d.$$ 

Then $L$ satisfies:

**(L1) Strict convexity:** for each $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$, $\frac{\partial^2 L}{\partial v^2}(x, v)$ is positive definite;

**(L2) Superlinearity:** for each $K > 0$, there is $C(K) \in \mathbb{R}$ such that

$$L(x, v) \geq K|v| + C(K), \quad \forall (x, v) \in \mathbb{T}^d \times \mathbb{R}^d.$$ 

Let $\mathcal{P}(\mathbb{T}^d)$ denote the set of probability measures on $\mathbb{T}^d$. Let the coupling term $F : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \rightarrow \mathbb{R}$ be a function, satisfying the following assumptions:

**(F1) for each $m \in \mathcal{P}(\mathbb{T}^d)$, the function $x \mapsto F(x, m)$ is of class $C^2$, and there is a constant $F_\infty > 0$ such that

$$\|F(\cdot, m)\|_\infty, \|D_x F(\cdot, m)\|_\infty \leq F_\infty, \quad \forall m \in \mathcal{P}(\mathbb{T}^d),$$
where \( \| \cdot \|_\infty \) denotes the supremum norm;

(F2) \( F(\cdot, \cdot) \) and \( D_x F(\cdot, \cdot) \) are continuous on \( \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \);

(F3) there is a constant \( \text{Lip}(F) > 0 \) such that
\[
|F(x, m_1) - F(x, m_2)| \leq \text{Lip}(F)d_1(m_1, m_2), \quad \forall x \in \mathbb{T}^d, \forall m_1, m_2 \in \mathcal{P}(\mathbb{T}^d),
\]
where the distance \( d_1 \) is the Kantorovich-Rubinstein distance.

**Example 1.** Let \( F(x, m) = f(x)g(m) \), where \( f : \mathbb{T}^d \to \mathbb{R} \) is of class \( C^2 \), and \( g : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R} \) is Lipschitz. Then
\[
\|F(\cdot, m)\|_\infty = \|f(\cdot)g(m)\|_\infty, \quad \|D_x F(\cdot, m)\|_\infty = \|Df(\cdot)g(m)\|_\infty \leq F_\infty, \quad \forall m \in \mathcal{P}(\mathbb{T}^d),
\]
for some \( F_\infty > 0 \); \( F(\cdot, \cdot) = f(\cdot)g(\cdot) \) and \( D_x F(\cdot, \cdot) = Df(\cdot)g(\cdot) \) are continuous on \( \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \); for each \( x \in \mathbb{T}^d \), each \( m_1, m_2 \in \mathcal{P}(\mathbb{T}^d) \),
\[
|F(x, m_1) - F(x, m_2)| = |f(x)(g(m_1) - g(m_2))| \leq \|f\|_\infty \text{Lip}(g)d_1(m_1, m_2).
\]

**Definition 1.** A solution of the mean field games system (1.1) is a couple \((u, m) \in C(\mathbb{T}^d) \times \mathcal{P}(\mathbb{T}^d)\) such that (1.1a) is satisfied in viscosity sense and (1.1b) is satisfied in distributions sense.

**Remark 1.** Let us recall the definition of viscosity solutions of (1.1a) and the one of solutions of (1.1b) in distributions sense here.

A function \( u : \mathbb{T}^d \to \mathbb{R} \) is called a viscosity subsolution of equation (1.1a), if for every \( C^1 \) function \( \varphi : \mathbb{T}^d \to \mathbb{R} \) and every point \( x_0 \in \mathbb{T}^d \) such that \( u - \varphi \) has a local maximum at \( x_0 \), we have
\[
H(x_0, D\varphi(x_0)) \leq F(x_0, m) + c(m);
\]

A function \( u : \mathbb{T}^d \to \mathbb{R} \) is called a viscosity supersolution of equation (1.1a), if for every \( C^1 \) function \( \psi : \mathbb{T}^d \to \mathbb{R} \) and every point \( y_0 \in \mathbb{T}^d \) such that \( u - \psi \) has a local minimum at \( y_0 \), we have
\[
H(y_0, D\psi(y_0)) \geq F(y_0, m) + c(m);
\]

A function \( u : \mathbb{T}^d \to \mathbb{R} \) is called a viscosity solution of equation (1.1a) if it is both a viscosity subsolution and a viscosity supersolution.

We say that a measure \( m \in \mathcal{P}(\mathbb{T}^d) \) satisfies (1.1b) in the sense of distributions, if
\[
\int_{\mathbb{T}^d} \left< Df(x), \frac{\partial H}{\partial p}(x, Du(x)) \right> dm(x) = 0, \quad \forall f \in C^\infty(\mathbb{T}^d).
\]

For each \( m \in \mathcal{P}(\mathbb{T}^d) \), \( H_m(x, p) := H(x, p) - F(x, m) \) is a Tonelli Hamiltonian defined on \( \mathbb{T}^d \times \mathbb{R}^d \). Denote by \( L_m \) the associated Lagrangian, i.e.,
\[
L_m(x, v) = L(x, v) + F(x, m), \quad (x, v) \in \mathbb{T}^d \times \mathbb{R}^d.
\]
Denote by \( \Phi_t^{L_m} \) and \( \Phi_t^{H_m} \) the Euler-Lagrange flow of \( L_m \) and the Hamiltonian flow of \( H_m \), respectively.
Remark 2. Assume (L1), (L2) and (F1).

(i) We used $c(m)$ in (1.1) to denote the Mañé critical value [20] of $H_m$. It is well known that for any given $m \in \mathcal{P}(\mathbb{T}^d)$, $c(m)$ is the unique real number $k$ such that equation $H_m(x, Du(x)) = k$ has viscosity solutions.

(ii) Let us recall the notion of Mather measures for Tonelli Lagrangians introduced by Mather in [21]. A measure $\mu \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ is called a Mather measure for $L_m$, if it satisfies

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} L_m(x, v) d\mu = -c(m),$$

where the minimum is taken over the set of all Borel probability measures on $\mathbb{T}^d \times \mathbb{R}^d$ invariant under the Euler-Lagrange flow $\Phi^t_{L_m}$. Let $\mathcal{P}^\ell(\mathbb{T}^d \times \mathbb{R}^d)$ be the set of probability measures on the Borel $\sigma$-algebra of $\mathbb{T}^d \times \mathbb{R}^d$ such that $\int_{\mathbb{T}^d \times \mathbb{R}^d} |v| d\mu < +\infty$. Define the set of closed measures on $\mathbb{T}^d \times \mathbb{R}^d$ as

$$\mathcal{K}(\mathbb{T}^d \times \mathbb{R}^d) := \left\{ \mu \in \mathcal{P}^\ell(\mathbb{T}^d \times \mathbb{R}^d) : \int_{\mathbb{T}^d \times \mathbb{R}^d} v D\varphi(x) d\mu = 0, \forall \varphi \in C^1(\mathbb{T}^d) \right\}.$$ 

A closed measure $\mu$ satisfying $\int_{\mathbb{T}^d \times \mathbb{R}^d} L_m(x, v) d\mu = -c(m)$ is a Mather measure.

(iii) Let $m_0 \in \mathcal{P}(\mathbb{T}^d)$ be such that there is a Mather measure $\mu_0$ for $L_{m_0}$ with $m_0 = \pi^*\mu_0$, where $\pi : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d$ denotes the canonical projection, and $\pi^*\mu_0$ denotes the push-forward of $\mu_0$ through $\pi$. Let $u_0$ be any viscosity solution of $H(x, Du) = F(x, m_0) + c(m_0)$. Then $m_0$ satisfies $\text{div} (m_0 \frac{\partial H}{\partial p}(x, Du_0)) = 0$ in distributions sense. See [14] for details.

Remark 3. A $\mathbb{Z}^d$-periodic function $u \in C(\mathbb{R}^d)$ is a viscosity solution of (1.1a) if and only if

$$u(y) - c(m)t = \inf_{x \in \mathbb{R}^d} \left( u(x) + h^m_t(x, y) \right), \quad \forall y \in \mathbb{R}^d, \forall t > 0, \quad (1.2)$$

where

$$h^m_t(x, y) := \inf_{\gamma} \int_0^t L_m(\gamma, \dot{\gamma}) ds, \quad (1.3)$$

where the infimum is taken among the continuous and piecewise $C^1$ paths $\gamma : [0, t] \to \mathbb{R}^d$ with $\gamma(0) = x$, $\gamma(t) = y$. See, for instance, [12] for a proof. We call $h^m_t(x, y)$ the minimal action function and the curves achieving the infimum in (1.3) minimizing curves of $L_m$ with the action $h^m_t(x, y)$.

For each $\tau > 0$ and each $m \in \mathcal{P}(\mathbb{T}^d)$, define the discrete action function by

$$\mathcal{L}_{\tau, m}(x, y) := \tau \left( L(x, \frac{y - x}{\tau}) + F(x, m) \right), \quad \forall x, y \in \mathbb{R}^d.$$
According to \cite[Theorem 4.3]{13} and \cite[Theorem 9]{22}, under assumptions (L1), (L2) and (F1) one can deduce that for each $\tau > 0$, each $m \in \mathcal{P}(\mathbb{T}^d)$, there is a unique constant $\tilde{L}(\tau, m) \in \mathbb{R}$, such that the discrete Lax-Oleinik equation
\begin{equation}
 u_{\tau,m}(y) + \tau \tilde{L}(\tau, m) = \inf_{x \in \mathbb{R}^d} \left( u_{\tau,m}(x) + L_{\tau,m}(x,y) \right), \quad \forall y \in \mathbb{R}^d,
\end{equation}
has continuous $\mathbb{Z}^d$-periodic solutions $u_{\tau,m}$, and $\tilde{L}(\tau, m) \to -c(m)$ as $\tau \to 0$. The authors of \cite{22} showed the convergence of a subsequence of solutions of discrete Lax-Oleinik equations.

Mañé \cite{19} introduced the notion of holonomic measures in his study of Mather theory. It has great advantage of dealing with different Lagrangians at the same time. Here we will use a discrete version of the notion of holonomic measures.

**Definition 2.** We say that a probability measure $\mu \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ is $\tau$-holonomic, provided
\[ \int_{\mathbb{T}^d \times \mathbb{R}^d} \varphi(x + \tau v) d\mu(x, v) = \int_{\mathbb{T}^d \times \mathbb{R}^d} \varphi(x) d\mu(x, v) \]
for any $\varphi \in C(\mathbb{T}^d)$. The set of $\tau$-holonomic measures is denoted by $\mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d)$.

In view of \cite[Definition 3.5 and Theorem 4.3]{13}, we know that for each $\tau > 0$, each $m \in \mathcal{P}(\mathbb{T}^d)$,
\begin{equation}
 \tilde{L}(\tau, m) = \min_{\mu \in \mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d)} \int_{\mathbb{T}^d \times \mathbb{R}^d} L_m(x, v) d\mu,
\end{equation}
where the minimum is taken over $\mathcal{P}_\tau(\mathbb{T}^d \times \mathbb{R}^d)$. A measure $\mu$ attaining the minimum is called a minimizing $\tau$-holonomic measure for $L_m$.

The main result of the present paper is stated as follows.

**Theorem 1.** Assume (L1), (L2) and (F1)-(F3). Then

(i) For each $\tau > 0$, there is $m \in \mathcal{P}(\mathbb{T}^d)$ such that there exists a minimizing $\tau$-holonomic measure $\eta_{\tau,m}$ for the Lagrangian $L_m$ with
\[ m = \pi^\# \eta_{\tau,m}. \]
Such a measure $m$ is denoted by $m_\tau$ (maybe not unique).

(ii) There is a subsequence $\tau_i \to 0$, a subsequence $m_{\tau_i} \overset{w^*}{\to} m_0$, and a subsequence $u_{\tau_i, m_{\tau_i}}$ solutions of (1.4) such that $u_{\tau_i, m_{\tau_i}}$ converges to $u_0$ uniformly on $\mathbb{T}^d$ and $(u_0, m_0)$ is a solution of (1.1).

**Remark 4.** Outline of the proof of Theorem 1:

(i) First, we discretize the continuous Lax-Oleinik equation (1.2) by discretizing in time. Analyzing the properties of solutions to the discrete Lax-Oleinik equation (1.4) is our starting point. Our discrete scheme is the mean field games analogue of the approximation scheme for Hamilton-Jacobi equations $H(x, Du) = c(H)$ considered in \cite{13, 22}, where $c(H)$ is the Mañé critical value of $H$. 
(ii) Next, we study the tightness of minimizing $\tau$-holonomic measures for $L_m$ and introduce the notion of minimizing $\tau$-holonomic measures for mean field games. Based on the tightness result we get the existence of minimizing $\tau$-holonomic measures for mean field games by using Kakutani fixed point theorem.

(iii) At last, we get a convergent subsequence $(u_{\tau_i,m_{\tau_i}},m_{\tau_i})$ whose limit is a solution of (1.1). Theorem 7 can be regarded as a selection type result for (1.1).

1.2 Notations and definitions

Now we introduce the symbols used in this paper. Denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{R}^d$ the $d$-dimensional real Euclidean space, by $\langle p, v \rangle$ or $pv$ the Euclidean scalar product of $p$ and $v$, by $|\cdot|$ the usual norm in $\mathbb{R}^d$, and by $B_R$ the open ball with center 0 and radius $R$. Let $\Omega \subset \mathbb{T}^d \times \mathbb{R}^d$. cl($\Omega$) stands for its closure. We identify the tangent bundle $T\mathbb{T}^d$ and the cotangent bundle $T^*\mathbb{T}^d$ with $\mathbb{T}^d \times \mathbb{R}^d$, $C^k (\mathbb{T}^d)$ ($k \in \mathbb{N}$) stands for the function space of $k$-times continuously differentiable functions on $\mathbb{T}^d$, and $C^\infty (\mathbb{T}^d) := \bigcap_{k=0}^\infty C^k (\mathbb{T}^d)$. The spatial gradient of $F$ is denoted by $D_x F = \frac{\partial F}{\partial x} = (D_{x_1} F, \ldots, D_{x_d} F)$, where $D_{x_i} F = \frac{\partial F}{\partial x_i}, i = 1, 2, \ldots, d$. Given a metric space $(X, d)$ we denote by $\mathcal{B}(X)$ the Borel $\sigma$-algebra on $X$ and by $\mathcal{P}(X)$ the set of Borel probability measures on $(X, \mathcal{B}(X))$. The support of a measure $\mu \in \mathcal{P}(X)$, denoted by $\text{supp}(\mu)$, is the closed set defined by

$$\text{supp}(\mu) := \{ x \in X : \mu(V_x) > 0 \text{ for each open neighborhood } V_x \text{ of } x \}.$$  

Let $X$ be a Polish space (complete, separable metric spaces, equipped with their Borel $\sigma$-algebra) endowed with a distance $d$. As mentioned above, we denote by $\mathcal{P}(X)$ the space of Borel probability measures, $\mu_k \in \mathcal{P}(X)$ converges weakly to $\mu$ if for all $\varphi \in C_b(X)$ (i.e., $\varphi$ is bounded and continuous), $\int_X \varphi d\mu_k$ converges to $\int_X \varphi d\mu$ as $k \to +\infty$. This defines a separable, Hausdorff topology on $\mathcal{P}(X)$, called the weak topology.

Prokhorov theorem (see, for instance, [23]) ensures that a subset $S$ of $\mathcal{P}(X)$ is relatively weakly compact if and only if it is tight, i.e. for all $\varepsilon > 0$ there is a compact subset $K_\varepsilon$ of $X$ such that for all $\mu \in S, \mu(X \setminus K_\varepsilon) \leq \varepsilon$.

If $X$ is locally compact, then Riesz theorem identifies the space $M(X)$ of measures, normed by total variation, with the dual of the space $C_0(X)$ of continuous functions going to 0 at infinity. Then one can introduce the “weak-* topology” on $\mathcal{P}(X)$. At the level of probability measures, weak and weak-* convergences are equivalent.

Let $p \geq 0$ be a nonnegative real number. Denote by $\mathcal{P}_p(X)$ the set of probability measures with finite moments of order $p$, i.e. those measures $\mu$ such that for some (and thus any) $x_0 \in X$,

$$\int d(x_0, x)^p \, d\mu(x) < +\infty.$$  

If $d$ is bounded, then $\mathcal{P}_p(X)$ coincides with $\mathcal{P}(X)$. Given $\mu, \nu \in \mathcal{P}_p(X)$, those probability measures $\pi \in \mathcal{P}(X \times X)$ that satisfy

$$\pi(A \times X) = \mu(A), \quad \pi(X \times A) = \nu(A)$$  

(1.6)
for all measurable subsets $A$ of $X$, are said to have marginals $\mu$ and $\nu$. Let $\Pi(\mu, \nu) := \{\pi \in P(X \times X) : (1.6) \text{ holds for all measurable } A\}$. Define the Monge-Kantorovich distance of order $p$ between $\mu$ and $\nu$ by

$$d_p(\mu, \nu) = \left( \inf_{\pi \in \Pi(\mu, \nu)} \int_{X \times X} d(x, y)^p d\pi(x, y) \right)^{1/p}.$$ 

The Monge-Kantorovich distance of order 1 will be also called the Kantorovich-Rubinstein distance.

Let us recall a very useful fact (see, for example, [23]): let $p \in (0, +\infty)$, let $(\mu_k)_{k \in \mathbb{N}}$ be a sequence of probability measures in $P_p(X)$, and let $\mu \in P(X)$. Then, the following two statements are equivalent: (i) $d_p(\mu_k, \mu) \to 0$, as $k \to +\infty$ (ii) $\mu_k$ converges weakly to $\mu$ as $k \to +\infty$, and $\mu_k$ satisfies the tightness condition: for some (and thus any) $x_0 \in X$,

$$\lim_{R \to +\infty} \limsup_{k \to +\infty} \int_{d(x_0, x) \geq R} d(x_0, x)^p d\mu_k(x) = 0.$$ 

The rest of the paper is organized as follows. We provide some preliminary results in Section 2. Section 3 is devoted to the existence of minimizing holonomic measures for mean field games. We show the convergence of the approximation scheme and that the limit functions are solutions of (1.1) in Sections 4 and 5.

## 2 A priori estimates

In this part we provide some preliminary results. These results can be regarded as mean field games analogues of a priori estimates for Hamilton-Jacobi equations without the coupling term considered in [13, 22]. For completeness sake, we prove our versions here. The key point is that the estimates are uniform on $m \in \mathcal{P}(\mathbb{T}^d)$.

**Lemma 1.** For each $D > 0$, there is $C(D) > 0$ such that for each $0 < \tau < 1$, each $m \in \mathcal{P}(\mathbb{T}^d)$, each $x, y \in \mathbb{R}^d$ with $|x - y| \leq \tau D$, and each minimizing curve $\gamma_{x,y}^m : [0, \tau] \to \mathbb{R}^d$ of $L_m$ with the action $h^m_{\tau}(x, y)$, there hold

$$|\dot{\gamma}_{x,y}^m(s)|, |\ddot{\gamma}_{x,y}^m(s)| \leq C(D), \quad \forall s \in [0, \tau].$$

**Proof.** Fix $D > 0$. For each $0 < \tau < 1$, each $x, y \in \mathbb{R}^d$ with $|x - y| \leq \tau D$, let $\ell_{x,y}$ be a segment connecting $x$ and $y$

$$\ell_{x,y} : [0, \tau] \to \mathbb{R}^d, \quad \ell_{x,y}(s) := x + s \frac{y - x}{\tau}.$$ 

Then for each $m \in \mathcal{P}(\mathbb{T}^d)$,

$$\int_0^\tau \left( L(\ell_{x,y}(s), \dot{\ell}_{x,y}(s)) + F(\ell_{x,y}(s), m) \right) ds = \int_0^\tau \left( L(\ell_{x,y}(s), \frac{y - x}{\tau}) + F(\ell_{x,y}(s), m) \right) ds$$

$$\leq \tau \left( \max_{\ell \in \mathbb{T}^d} L(x, v) + F_\infty \right)$$

$$=: \tau C_1(D).$$
Since $L$ is superlinear in $v$, then there is $R > 0$ such that for any $v \in \mathbb{R}^d$ with $|v| > R$,

$$L(x, v) + F(x, m) > C_1(D), \quad \forall x \in \mathbb{T}^d, \forall m \in \mathcal{P}(\mathbb{T}^d).$$

Let

$$\Sigma_R := \{(x, v) \in \mathbb{T}^d \times \mathbb{R}^d : |v| \leq R\}.$$

Obviously, $\Sigma_R$ is a compact subset of $\mathbb{T}^d \times \mathbb{R}^d$. By the compactness of $\Sigma_R$ and $\mathcal{P}(\mathbb{T}^d)$, the continuous dependence of the solutions on the initial condition and a parameter and (F1), one can deduce that there is $R_1 > 0$ independent of $\tau$ and $m$ such that

$$\Phi_s^{L_m}(\Sigma_R) \subset \Sigma_{R_1} := \{(x, v) \in \mathbb{T}^d \times \mathbb{R}^d : |v| \leq R_1\}$$

for all $s \in [-1, 1]$ and all $m \in \mathcal{P}(\mathbb{T}^d)$.

For any minimizing curve $\gamma_{x,y}^m : [0, \tau] \to \mathbb{R}^d$ of $L_m$ with the action $h_m^\tau(x, y)$, we assert that $|\dot{\gamma}_{x,y}^m(s)| \leq R_1$ for all $s \in [0, \tau]$. Otherwise, there would be $s_0 \in [0, \tau]$ such that $|\dot{\gamma}_{x,y}^m(s_0)| > R_1$. We define a curve $\tilde{\gamma}$ in $\mathbb{T}^d$ by $\tilde{\gamma} := \pi\gamma_{x,y}^m$. Since $\gamma_{x,y}^m$ is a minimizing curve, then we know that $(\tilde{\gamma}(s), \dot{\tilde{\gamma}}(s)) \subset \mathbb{T}^d \times \mathbb{R}^d$ is a solution of the Lagrangian system generated by $L_m$. In view of $|\dot{\tilde{\gamma}}(s_0)| = |\dot{\gamma}_{x,y}^m(s_0)| > R_1$, one can deduce that

$$(\tilde{\gamma}(s), \dot{\tilde{\gamma}}(s)) \notin \Sigma_R \quad \forall s \in [0, \tau].$$

So,

$$|\dot{\tilde{\gamma}}(s)| = |\dot{\gamma}_{x,y}^m(s)| > R$$

for all $s \in [0, \tau]$. Thus, we have that

$$L(\gamma_{x,y}^m(s), \dot{\gamma}_{x,y}^m(s)) + F(\gamma_{x,y}^m(s), m) > C_1(D), \quad \forall s \in [0, \tau]$$

implying that

$$\int_0^\tau L(\gamma_{x,y}^m(s), \dot{\gamma}_{x,y}^m(s)) + F(\gamma_{x,y}^m(s), m) ds > C_1(D)\tau \geq \int_0^\tau \left(L(\ell_{x,y}(s), \dot{\ell}_{x,y}(s)) + F(\ell_{x,y}(s), m)\right) ds,$$

a contradiction.

At last, note that

$$\ddot{\gamma}_{x,y}^m = \frac{\partial^2 L}{\partial v^2}(\gamma_{x,y}^m, \dot{\gamma}_{x,y}^m)^{-1}\left(\frac{\partial L}{\partial x}(\gamma_{x,y}^m, \dot{\gamma}_{x,y}^m) + \frac{\partial F}{\partial x}(\gamma_{x,y}^m, m) - \frac{\partial^2 L}{\partial x \partial v}(\gamma_{x,y}^m, \dot{\gamma}_{x,y}^m)\right),$$

which finishes the proof.

\[\square\]

**Proposition 1.** For each $D > 0$, there is $\tilde{C}(D) > 0$ such that if $\tau \in (0, 1]$, $x, y \in \mathbb{R}^d$ with $|x - y| \leq \tau D$, then

$$|h_{\tau}^m(x, y) - L_{\tau,m}(x, y)| \leq \tau^2 \tilde{C}(D), \quad \forall m \in \mathcal{P}(\mathbb{T}^d).$$
Proof. Fix $D > 0$. Let $C(D)$ be the constant given by Lemma 1. Let $\tau \in (0, 1]$ and $x, y \in \mathbb{R}^d$ with $|x - y| \leq \tau D$. Let $\gamma_{x,y}^m$ be a minimizing curve $\gamma_{x,y}^m : [0, \tau] \to \mathbb{R}^d$ of $L_m$ with the action $h^m(x, y)$. Then by Lemma 1 we get that $|\dot{\gamma}_{x,y}^m(s)|, |\ddot{\gamma}_{x,y}^m(s)| \leq C(D)$ for all $s \in [0, \tau]$. For any $s \in [0, \tau]$, we have that

$$|\gamma_{x,y}^m(s) - x| = |\gamma_{x,y}^m(s) - \gamma_{x,y}^m(0)| \leq \tau C(D), \quad |\dot{\gamma}_{x,y}^m(s) - \dot{\gamma}_{x,y}^m(0)| \leq \tau C(D),$$

$$|\frac{y - x}{\tau} - \frac{\dot{\gamma}_{x,y}^m(0)}{\tau}| = \left| \frac{\gamma_{x,y}^m(\tau) - \gamma_{x,y}^m(0) - \dot{\gamma}_{x,y}^m(0)\tau}{\tau} \right| \leq \tau C(D),$$

$$|\ddot{\gamma}_{x,y}^m(s) - \frac{y - x}{\tau}| \leq |\ddot{\gamma}_{x,y}^m(s) - \ddot{\gamma}_{x,y}^m(0)| + |\ddot{\gamma}_{x,y}^m(0) - \frac{y - x}{\tau}| \leq 2\tau C(D).$$

So, we get that

$$|h^m_{\tau}(x, y) - L_{\tau,m}(x, y)| \leq \int_0^\tau \left| L(\gamma_{x,y}^m(s), \dot{\gamma}_{x,y}^m(s)) + F(\gamma_{x,y}^m(s), m) - L(x, \frac{y - x}{\tau}) - F(x, m) \right| ds \leq C_2(D)\tau^2 + F_\infty C(D)\tau^2 =: \tilde{C}(D)\tau^2.$$

We use the symbol $A^m_{\tau}(x, y)$ to denote $h^m_{\tau}(x, y)$ or $L_{\tau,m}(x, y)$ in the following four propositions, which means these results hold for both $h^m_{\tau}(x, y)$ and $L_{\tau,m}(x, y)$. The first one is a direct consequence of assumptions (L1), (L2), (F1) and Lemma 6. We omit the proof here.

**Proposition 2.** $A^m_{\tau}(x, y)$ satisfies the following properties:

(i) for each $D > 0$,

$$\inf_{m \in \mathcal{P}(\mathbb{T}^d)} \inf_{\tau \in [0, 1]} \inf_{x,y \in \mathbb{R}^d} \frac{1}{\tau} A^m_{\tau}(x, y) > -\infty, \quad \sup_{m \in \mathcal{P}(\mathbb{T}^d)} \sup_{\tau \in (0, 1]} \sup_{|y - x| \leq \tau D} \frac{1}{\tau} A^m_{\tau}(x, y) < +\infty;$$

(ii)

$$\lim_{D \to +\infty} \inf_{\tau \in (0, 1]} \inf_{|y - x| \geq \tau D} \frac{A^m_{\tau}(x, y)}{|x - y|} = +\infty$$

uniformly on $m \in \mathcal{P}(\mathbb{T}^d)$;

(iii) for each $D > 0$, there exists a constant $C(D) > 0$ such that for each $\tau \in (0, 1]$, for each $x, y, z \in \mathbb{R}^d$, and each $m \in \mathcal{P}(\mathbb{T}^d)$,

(iii') if $|y - x| \leq \tau D$ and $|z - x| \leq \tau D$, then $|A^m_{\tau}(x, z) - A^m_{\tau}(x, y)| \leq C(D)|z - y|,$

(iii’’) if $|z - x| \leq \tau D$ and $|z - y| \leq \tau D$, then $|A^m_{\tau}(x, z) - A^m_{\tau}(y, z)| \leq C(D)|y - x|.$

The following result comes from [13], where the authors dealt with Hamilton-Jacobi equations without coupling term $F(x, m).$
Proposition 3. (i) For each $\tau > 0$ and each $m \in \mathcal{P}(\mathbb{T}^d)$, there exists a unique constant $\bar{A}_\tau^m$ such that equation

$$u_{\tau,m}(y) + \bar{A}_\tau^m = \inf_{x \in \mathbb{R}^d} \left( u_{\tau,m}(x) + A_\tau^m(x,y) \right), \quad \forall y \in \mathbb{R}^d, \quad (2.1)$$

admits a continuous $\mathbb{Z}^d$-periodic solution $u_{\tau,m}$.

(ii) $\bar{A}_\tau^m$ can be represented by

$$\bar{A}_\tau^m = \lim_{k \to +\infty} \inf_{\tau,m \in \mathbb{R}^d} \frac{1}{k} \sum_{i=0}^{k-1} A_\tau^m(z_i, z_{i+1}). \quad (2.2)$$

Remark 5. Let $\tilde{L}_{\tau,m} := \bar{A}_\tau^m$ when $A_\tau^m(x,y) = L_{\tau,m}(x,y)$. In view of (1.4) and (2.1), one can deduce that $\frac{\tilde{L}_{\tau,m}}{\tau} = \bar{L}(\tau,m)$.

Proposition 4. There exist constants $C, D > 0$ such that if $\tau \in (0,1]$ and $m \in \mathcal{P}(\mathbb{T}^d)$ and $u_{\tau,m}$ is a solution of (2.1), then

(i) $u_{\tau,m}$ is Lipschitz and $\text{Lip}(u_{\tau,m}) \leq C$,

(ii) $\forall y \in \mathbb{R}^d, x \in \arg\min_{x \in \mathbb{R}^d} \{ u_{\tau}(x) + A_\tau^m(x,y) \} \Rightarrow |y - x| \leq \tau D$.

Proof. Let

$$C_1 := \sup_{\tau \in (0,1], |y-x| \leq \tau, m \in \mathcal{P}(\mathbb{T}^d)} \frac{A_\tau^m(x,y) - \bar{A}_\tau^m}{\tau},$$

$$D := \inf \left\{ D' > 1 : \inf_{\tau \in (0,1], |y-x| > \tau D', m \in \mathcal{P}(\mathbb{T}^d)} \frac{A_\tau^m(x,y) - \bar{A}_\tau^m}{|y-x|} > C_1 \right\},$$

$$C := \max \left\{ C_1, \sup_{|y-x|, |z-x| \leq \tau(D+1), \tau \in (0,1], m \in \mathcal{P}(\mathbb{T}^d)} \frac{A_\tau^m(x,y) - A_\tau^m(x,z)}{|z-y|} \right\}.$$

Notice that the above three constants $C_1, D$ and $C$ are well defined since $A_\tau^m(x,y)$ satisfies (i), (ii), (iii) in Proposition [2] and $\bar{A}_\tau^m$ has the representation formula (2.2).

First, we show if $|x - y| > \tau$, then $u_{\tau,m}(y) - u_{\tau,m}(x) \leq C_1 |y - x|$. In fact, by choosing $n \geq 2$ such that $(n-1)\tau < |y-x| \leq n\tau$ and by choosing $x_i = x + \frac{k}{n}(y-x)$, we obtain $n\tau \leq 2|y-x|$,

$$u_{\tau,m}(x_{i+1}) - u_{\tau,m}(x_i) \leq A_\tau^m(x_i, x_{i+1}) - \bar{A}_\tau^m,$$

$$u_{\tau,m}(y) - u_{\tau,m}(x) \leq n\tau \sup_{|z-z'| \leq \tau} \frac{A_\tau^m(z, z') - \bar{A}_\tau^m}{\tau} \leq C_1 |y - x|.$$

Second, we prove (ii). Let $y \in \mathbb{R}^d$ and take $x$ satisfying

$$u_{\tau,m}(y) - u_{\tau,m}(x) = A_\tau^m(x,y) - \bar{A}_\tau^m.$$
Assume by contradiction that $|y - x| > \tau D$. Then the first step of the proof may be used and we obtain that

$$C_1|y - x| \geq u_{\tau, m}(y) - u_{\tau, m}(x) = A^m_\tau(x, y) - A^m_\tau > C_1|y - x|,$$

a contradiction.

Third, we end the proof of (i). Let $y, z \in \mathbb{R}^d$ with $|z - y| \leq \tau$. Let $x$ be a point satisfying $u_{\tau, m}(y) - u_{\tau, m}(x) = A^m_\tau(x, y) - A^m_\tau$. Then $|y - x| \leq \tau D$, $|z - x| \leq \tau (D + 1)$,

$$u_{\tau, m}(z) - u_{\tau, m}(x) \leq A^m_\tau(x, z) - A^m_\tau,$$

$$u_{\tau, m}(z) - u_{\tau, m}(y) \leq A^m_\tau(x, z) - A^m_\tau(x, y) \leq C|z - y|.$$

By changing the roles of $z$ and $y$, we just have proved that $\text{Lip}(u_{\tau, m}) \leq C$.

\[\square\]

**Proposition 5.** For each constant $\kappa > 0$, there exist constants $D_\kappa, C_\kappa > 0$, such that if $\varphi$ is any $\mathbb{Z}^d$-periodic Lipschitz function satisfying $\text{Lip}(\varphi) \leq \kappa$, $\tau \in (0, 1]$, and $m \in \mathcal{P}(\mathbb{T}^d)$, then

(i) $\forall y \in \mathbb{R}^d, x \in \arg\min_{x \in \mathbb{R}^d} \{\varphi(x) + A^m_\tau(x, y)\} \Rightarrow |y - x| \leq \tau D_\kappa$

(ii) $\|\inf_{x \in \mathbb{R}^d} (\varphi(x) + A^m_\tau(x, \cdot)) - u(\cdot)\|_{\infty} \leq \tau C_\kappa$.

**Proof.** (i) Let $\kappa > 0$. Define

$$D_\kappa := \inf \left\{ D' > 1 : \inf_{\tau \in (0, 1], |y - x| > \tau D', m \in \mathcal{P}(\mathbb{T}^d)} \frac{A^m_\tau(x, y) - A^m_\tau(y, y)}{|y - x|} > \kappa \right\}.$$

Let $\varphi$ be a periodic function satisfying $\text{Lip}(\varphi) \leq \kappa$ and $y$ be any point in $\mathbb{R}^d$. Let $x$ be a point realizing the minimum of $\min_x \{\varphi(x) + A^m_\tau(x, y)\}$. Assume by contradiction that $|y - x| > \tau D_\kappa$, then

$$A^m_\tau(x, y) - A^m_\tau(y, y) > \kappa|y - x|.$$

On the other hand, we have $\varphi(x) + A^m_\tau(x, y) \leq \varphi(y) + A^m_\tau(y, y)$ and

$$\kappa|y - x| \geq \varphi(y) - \varphi(x) \geq A^m_\tau(x, y) - A^m_\tau(y, y),$$

a contradiction.

(ii) Consider the case $A^m_\tau(x, y) = \mathcal{L}_{\tau, m}(x, y)$ first. For any given $y \in \mathbb{R}^d$, let $x_0 \in \mathbb{R}^d$ be a point satisfying $x_0 \in \arg\min_{x \in \mathbb{R}^d} \{\varphi(x) + \mathcal{L}_{\tau, m}(x, y)\}$. Then by (i) we get that $|y - x_0| \leq \tau D_\kappa$. Hence,

$$\left| \min_{x \in \mathbb{R}^d} \{\varphi(x) + \mathcal{L}_{\tau, m}(x, y)\} - \varphi(y) \right| = \left| \varphi(x_0) - \varphi(y) + \tau L(x_0, y - x_0) + \tau F(x_0, m) \right|$$

$$\leq \kappa|x_0 - y| + \tau \max_{|v| \leq D_\kappa} |L(x, v)| + \tau F_\infty$$

$$\leq \tau(\kappa D_\kappa + \max_{|v| \leq D_\kappa} |L(x, v)| + F_\infty)$$

$$= : \tau C_\kappa.$$
Next, consider the case $A^m_T(x, y) = h^m_T(x, y)$. For any given $y \in \mathbb{R}^d$, let $x_0 \in \mathbb{R}^d$ be a point satisfying $x_0 \in \arg \min_{x \in \mathbb{R}^d} \{ \varphi(x) + h^m_T(x, y) \}$. Then by (i) we get that $|y - x_0| \leq \tau D_\kappa$. Let $\gamma^m_{x_0, y}$ be a minimizing curve of $L_m$ with the action $h^m_T(x_0, y)$. Then,

$$
\left| \min_{x \in \mathbb{R}^d} \{ \varphi(x) + h^m_T(x_0, y) \} - \varphi(y) \right| = \left| \varphi(x_0) - \varphi(y) + \int_0^T L(\gamma^m_{x_0, y}(s), \dot{\gamma}^m_{x_0, y}(s)) + F(\gamma^m_{x_0, y}(s), m) \, ds \right|
\leq \kappa |x_0 - y| + \tau \max_{|v| \leq C(D_\kappa)} |L(x, v)| + \tau F_\infty
\leq \tau (\kappa D_\kappa + \max_{|v| \leq C(D_\kappa)} |L(x, v)| + F_\infty)
=: \tau C_\kappa.
$$

\[\Box\]

3 Minimizing holonomic measures for mean field games

**Definition 3.** For each $\tau > 0$, each $m \in \mathcal{P}(\mathbb{T}^d)$, the set

$$
\mathcal{M}_\tau(L_m) = \text{cl} \left( \bigcup \{ \text{supp}(\mu) : \mu \text{ is a minimizing } \tau\text{-holonomic measure for } L_m \} \right),
$$

is called $\tau$-Mather set for $L_m$.

A function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is called a $\tau$-sub-action with respect to $L_m$ if $\varphi(x)$ is $\mathbb{Z}_d$-periodic, continuous and satisfies

$$
\tau \tilde{L}(\tau, m) \leq \tau L_m(x, v) + \varphi(x) - \varphi(x + \tau v), \quad \forall (x, v) \in \mathbb{T}^d \times \mathbb{R}^d.
$$

It is straightforward to check that any solution $u_{\tau, m}$ of (1.4) is a $\tau$-sub-action with respect to $L_m$.

Define the sets

$$
\mathcal{N}_\tau(L_m, u_{\tau, m}) := \{(x, v) \in \mathbb{T}^d \times \mathbb{R}^d : \tau L_m(x, v) = u_{\tau, m}(x + \tau v) - u_{\tau, m}(x) + \tau \tilde{L}(\tau, m) \}.
$$

By [13, Proposition 6.3] we have $\mathcal{M}_\tau(L_m) \subset \mathcal{N}_\tau(L_m, u_{\tau, m})$.

**Proposition 6.** There are a compact subset $\mathcal{K} \subset \mathbb{T}^d \times \mathbb{R}^d$ and a constant $\tau_0 > 0$, such that $\mathcal{M}_\tau(L_m) \subset \mathcal{K}$ for all $0 < \tau < \tau_0$ and all $m \in \mathcal{P}(\mathbb{T}^d)$.

**Proof.** We show that $\mathcal{N}_\tau(L_m, u_{\tau, m})$ is a bounded subset of $\mathbb{T}^d \times \mathbb{R}^d$. Note that for any $(x, v) \in \mathcal{N}_\tau(L_m, u_{\tau, m})$, we have

$$
\tau L_m(x, v) = u_{\tau, m}(x + \tau v) - u_{\tau, m}(x) + \tau \tilde{L}(\tau, m) \leq C |v| + \tau \tilde{L}(\tau, m),
$$

where $C$ is a constant depending on $\kappa$, $\tau$ and $M$.
which implies that

\[ L(x, v) + F(x, m) \leq C|v| + \bar{L}(\tau, m) \leq C|v| + \min_{\mu} \int_{T^d \times \mathbb{R}^d} L(x, v) d\mu + F_\infty \]

\[ \leq C|v| + \bar{L}(\tau) + F_\infty, \]

where \( C \) independent of \( \tau \) and \( m \) is the common Lipschitz constant of \( u_{\tau,m} \), and the minimum is taken over \( \mathcal{P}_\tau(T^d \times \mathbb{R}^d) \). Since \( \bar{L}(\tau) \to -c(H) \) as \( \tau \to 0 \), then there is a constant \( R_1 > 0 \) and \( \tau_0 > 0 \), such that \( |\bar{L}(\tau)| \leq R_1 \) for all \( \tau \in (0, \tau_0) \). Recall that \( L \) is superlinear in \( v \). Then by (3.1), there is a constant \( R_2 > 0 \) such that

\[ |v| \leq R_2. \]

Hence, we have proved that \( \mathcal{N}_\tau(L_m, u_{\tau,m}) \subset T^d \times B_{R_2} \) for all \( \tau < \tau_0 \) and all \( m \in \mathcal{P}(T^d) \).

**Proposition 7.** For each \( 0 < \tau < \tau_0 \), there is \( m \in \mathcal{P}(T^d) \) such that there exists a minimizing \( \tau \)-holonomic measure \( \mu_{\tau,m} \) for the Lagrangian \( L_m \) with

\[ m = \pi^* \mu_{\tau,m}. \]

We call such a measure \( m \) minimizing \( \tau \)-holonomic measure for mean field games (1.1) and denote it by \( m_\tau \) (maybe not unique).

**Proof.** For each \( \tau > 0 \), define a set-valued map as follows:

\[ \Psi : \mathcal{P}(T^d) \to \mathcal{P}(T^d) \]

\[ m \mapsto \Psi(m) := \{ \pi^* \mu : \mu \text{ is a minimizing } \tau \text{-holonomic measure for } L_m \}. \]

We will use Kakutani fixed point theorem to get a fixed point of the map \( \Psi \). So, we only need to check: (i) \( \mathcal{P}(T^d) \) is convex and compact; (ii) \( \Psi \) is upper semicontinuous with nonempty closed convex values.

It is clear that the metric space \( (\mathcal{P}(T^d), d_1) \) is convex and compact due to Prokhorov theorem. By \cite{13}, Proposition 3.7], for each \( 0 < \tau < \tau_0 \) and each \( m \in \mathcal{P}(T^d) \), there exists a minimizing \( \tau \)-holonomic measure for \( L_m \) and thus \( \Psi(m) \) is nonempty. In view of Proposition 6 it is direct to check that \( \Psi(m) \) is closed. The convexity of \( \Psi(m) \) follows from the definition of \( \Psi \).

Next, we show: if \( m_i \xrightarrow{w^*} m_0 \), \( \eta_i \in \Psi(m_i) \) and \( \eta_i \xrightarrow{w^*} \eta_0 \), then \( \eta_0 \in \Psi(m_0) \). By definition, there is a sequence of minimizing \( \tau \)-holonomic measures \( \{ \mu_{m_i} \} \) for \( L_{m_i} \) such that

\[ \eta_i = \pi^* \mu_{m_i}. \]

From Proposition 6 if necessary passing to a subsequence, we have

\[ \mu_{m_i} \xrightarrow{w^*} \mu_0. \]
In view of (3.2), (3.3) and \( \eta_i \overset{w^*}{\longrightarrow} \eta_0 \), one can get that

\[ \eta_0 = \pi \mu_0. \]

So, we only need to show that \( \mu_0 \) is a minimizing \( \tau \)-holonomic measure for \( L_{m_0} \). By (1.5),

\[ \bar{L}(\tau, m_i) = \min_{\mu} \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, v) + F(x, m_i) d\mu, \]

and

\[ \bar{L}(\tau, m_0) = \min_{\mu} \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, v) + F(x, m_0) d\mu, \]

where the minimum is taken over \( \mathcal{P}_\tau (\mathbb{T}^d \times \mathbb{R}^d) \). Thus, we get that

\[ |\bar{L}(\tau, m_i) - \bar{L}(\tau, m_0)| \leq \text{Lip}(F) d_1(m_i, m_0) \rightarrow 0, \quad i \rightarrow +\infty. \]

Since \( \mu_{m_i} \) are minimizing \( \tau \)-holonomic measures, by (3.3) and the compactness of the supports of the measures we deduce that \( \mu_0 \) is also \( \tau \)-holonomic, and the proof is complete.

\[ \square \]

## 4 Convergence to Hamilton-Jacobi equations

For each \( \tau > 0 \), consider solutions of the discrete Lax-Oleinik equation

\[ u_{\tau, m_\tau}(y) + \tau \bar{L}(\tau, m_\tau) = \inf_{x \in \mathbb{R}^d} \left( u_{\tau, m_\tau}(x) + \mathcal{L}_{\tau, m_\tau}(x, y) \right), \quad \forall y \in \mathbb{R}^d, \quad (4.1) \]
Proposition 8. There is a subsequence $\tau_i \to 0$, a subsequence $m_{\tau_i} \xrightarrow{w^*} m_0$, and a subsequence $u_{\tau_i,m_{\tau_i}}$ solutions of (4.1) such that $u_{\tau_i,m_{\tau_i}}$ converges to $u_0$ uniformly on $\mathbb{T}^d$. Moreover, $u_0$ is a viscosity solution of

$$H(x, Du) = F(x, m_0) + c(m_0).$$

Proof. For each $m \in \mathcal{P}(\mathbb{T}^d)$, define two kinds of one-parameter operators $\xi^m_\tau$ and $T^m_\tau$ as follows:

$$\xi^m_\tau \varphi(y) = \inf_{x \in \mathbb{R}^d} \left( \varphi(x) + \mathcal{L}_{\tau,m}(x, y) \right),$$

and

$$T^m_\tau \varphi(y) = \inf_{x \in \mathbb{R}^d} \left( \varphi(x) + h^m_\tau(x, y) \right).$$

We claim there exists a constant $C > 0$ such that for each small $\tau > 0$, each $m \in \mathcal{P}(\mathbb{T}^d)$, and each solution $u$ of equation (1.4) with $u(0) = 0$,

$$\|T^m_\tau u - \xi^m_\tau u\|_\infty \leq \tau^2 C.$$ 

In fact, the above estimate is a consequence of Propositions 1, 4 and 5. More precisely, from Propositions 1, 4 and 5 there exist positive constants $D$ and $C$ such that for each $\tau \in (0, 1)$, each $m \in \mathcal{P}(\mathbb{T}^d)$ and each solution $u$ of equation (1.4), we have that

- Lip$(u) \leq C$, $\|u\|_\infty \leq C$;
- $\forall y \in \mathbb{R}^d, \ x \in \text{arg min}_{x \in \mathbb{R}^d} \{u(x) + \mathcal{L}_{\tau,m}(x, y)\} \Rightarrow |y - x| \leq \tau D$;
- $\forall y \in \mathbb{R}^d, \ x \in \text{arg min}_{x \in \mathbb{R}^d} \{u(x) + h^m_\tau(x, y)\} \Rightarrow |y - x| \leq \tau D$;
- $\|T^m_\tau u - u\|_\infty \leq \tau C$;
- for each $x, y \in \mathbb{R}^d, |y - x| \leq \tau D \Rightarrow |h^m_\tau(x, y) - \mathcal{L}_{\tau,m}(x, y)| \leq \tau^2 C$.

For each $y$ and $x \in \text{arg min}_{x \in \mathbb{R}^d} \{u(x) + \mathcal{L}_{\tau,m}(x, y)\}$, we have

$$T^m_\tau u(y) \leq u(x) + h^m_\tau(x, y) \leq u(x) + \mathcal{L}_{\tau,m}(x, y) + \tau^2 C \leq \xi^m_\tau u(y) + \tau^2 C.$$

On the other hand, if $x \in \text{arg min}_{x \in \mathbb{R}^d} \{u(x) + h^m_\tau(x, y)\}$,

$$T^m_\tau u(y) = u(x) + h^m_\tau(x, y) \geq u(x) + \mathcal{L}_{\tau,m}(x, y) - \tau^2 C \geq \xi^m_\tau u(y) - \tau^2 C.$$

Therefore, the above claim is true.

By the Lipschitz estimate, for each $\tau \in (0, 1]$ and each $m \in \mathcal{P}(\mathbb{T}^d)$, one can choose solutions $u_{\tau,m}$ of (1.4) such that $u_{\tau,m}(0) = 0$ and thus $u_{\tau,m}$ is uniformly bounded in $\tau \in (0, 1]$ and $m \in \mathcal{P}(\mathbb{T}^d)$. Thus, by Ascoli-Arzela theorem and Prokhorov theorem, we can choose a subsequence $m_{\tau_i} \xrightarrow{w^*} m_0 \in \mathcal{P}(\mathbb{T}^d)$ and a subsequence $u_{\tau_i,m_{\tau_i}} \to u_0$ uniformly on $\mathbb{T}^d$. For brevity, we use $m_i, u_i$ to denote $m_{\tau_i}$ and $u_{\tau_i,m_{\tau_i}}$ respectively in the following.
Let $t > 0$ be fixed, and $N_i$ be integers such that $N_i \tau_i \leq t < (N_i + 1) \tau_i$. The non-expansiveness property of the Lax-Oleinik operator $T_t^{m_0}$ implies

$$\|T_t^{m_0} u - T_{N_i \tau_i}^{m_i} u_i\|_\infty \leq \|T_t^{m_0} u - T_{N_i \tau_i}^{m_0} u_i\|_\infty + \|T_{N_i \tau_i}^{m_0} u_i - T_{N_i \tau_i}^{m_i} u_i\|_\infty.$$ 

Note that by Proposition 5, we get

$$\|T_t^{m_0} u - T_{N_i \tau_i}^{m_0} u_i\|_\infty \leq \|T_t^{m_0} u - u\|_\infty + \|u - u_i\|_\infty \to 0, \ i \to +\infty.$$

Notice that

$$\|T_{N_i \tau_i}^{m_0} u_i - T_{N_i \tau_i}^{m_0} u_i\|_\infty = \inf_{x \in \mathbb{R}^d} \left( u_i(x) + h_{N_i \tau_i}^{m_0}(x, \cdot) \right) - \inf_{x \in \mathbb{R}^d} \left( u_i(x) + h_{N_i \tau_i}^{m_0}(x, \cdot) \right)$$

$$\leq N_i \tau_i \text{Lip}(F)d_1(m_i, m_0) \to 0, \ i \to +\infty.$$

So, we have proved that

$$\|T_t^{m_0} u - T_{N_i \tau_i}^{m_0} u_i\|_\infty \to 0, \ i \to +\infty.$$

Consider

$$\tilde{L}_{\tau_i, m_i} + c(m_0) = \frac{\tilde{L}_{\tau_i, m_i}}{\tau_i} + \frac{\tilde{L}_{\tau_i, m_0}}{\tau_i} + \frac{\tilde{L}_{\tau_i, m_0}}{\tau_i} + c(m_0).$$

Note that

$$\left| \frac{\tilde{L}_{\tau_i, m_i}}{\tau_i} - \frac{\tilde{L}_{\tau_i, m_0}}{\tau_i} \right| = \left| \tilde{L}(\tau_i, m_i) - \tilde{L}(\tau_i, m_0) \right|$$

$$= \left| \min_{\mu} \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, v) + F(x, m_i)d\mu - \min_{\mu} \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, v) + F(x, m_0)d\mu \right|$$

$$\leq \text{Lip}(F)d_1(m_i, m_0) \to 0, \ i \to +\infty,$$

and by [22] Theorem 9,

$$\left| \frac{\tilde{L}_{\tau_i, m_0}}{\tau_i} + c(m_0) \right| \leq \tau_i C.$$

Thus, we obtain that

$$\left| \frac{\tilde{L}_{\tau_i, m_i}}{\tau_i} + c(m_0) \right| \leq \text{Lip}(F)d_1(m_i, m_0) + \tau_i C. \quad (4.2)$$

The previous claim $\|T_{\tau_i}^{m_i} u_i - T_{\tau_i}^{m_i} u_i\|_\infty \leq \tau_i^2 C$ and (4.2) imply that

$$\|T_{\tau_i}^{m_i} u_i - u_i + \tau_i c(m_0)\|_\infty \leq \|T_{\tau_i}^{m_i} u_i - u_i - \tilde{L}_{\tau_i, m_i}\|_\infty + \|\tilde{L}_{\tau_i, m_i} + \tau_i c(m_0)\| \leq \tau_i \text{Lip}(F)d_1(m_i, m_0) + 2\tau_i^2 C.$$ 

By iterating this inequality, we have

$$\|T_{N_i \tau_i}^{m_i} u_i - u_i + N_i \tau_i c(m_0)\|_\infty \leq N_i (\tau_i \text{Lip}(F)d_1(m_i, m_0) + 2\tau_i^2 C) \leq t(\text{Lip}(F)d_1(m_i, m_0) + 2\tau_i C).$$

Since $u_i + N_i \tau_i c(m_0) \to u_0 + tc(m_0)$, we get

$$T_t^{m_0} u_0 = u_0 - tc(m_0), \ \forall t > 0.$$
5 Convergence to continuity equations

**Proposition 9.** There is a measure $\mu_0 \in \mathcal{P}(T^d \times \mathbb{R}^d)$ such that $\mu_{\tau_i, m_{\tau_i}} \overset{w^*}{\to} \mu_0$ with $m_0 = \pi^*_d \mu_0$, where $\tau_i$, $m_{\tau_i}$ and $m_0$ are as in Proposition 8.

**Proof.** By Proposition 6, the sequence of $\mu_{\tau_i, m_{\tau_i}}$ is tight. In view of Prokhorov theorem, there is $\mu_0 \in \mathcal{P}(T^d \times \mathbb{R}^d)$ such that $\mu_{\tau_i, m_{\tau_i}} \overset{w^*}{\to} \mu_0$. Note that

$$m_{\tau_i} = \pi^*_d \mu_{\tau_i, m_{\tau_i}}, \quad m_{\tau_i} \overset{w^*}{\to} m_0, \quad i \to +\infty.$$  

One can deduce that $m_0 = \pi^*_d \mu_0$. \hfill \Box

**Proposition 10.** $\mu_0$ is a Mather measure for $L_{m_0}$.

**Proof.** First we prove that

$$\int_{T^d \times \mathbb{R}^d} L_{m_0}(x, v) d\mu_0 = -c(m_0). \quad (5.1)$$

Recall that

$$\bar{L}(\tau_i, m_{\tau_i}) = \int_{T^d \times \mathbb{R}^d} L(x, v) + F(x, m_{\tau_i}) d\mu_{\tau_i, m_{\tau_i}}.$$  

Note that

$$\left| \int_{T^d \times \mathbb{R}^d} L(x, v) + F(x, m_{\tau_i}) d\mu_{\tau_i, m_{\tau_i}} - \int_{T^d \times \mathbb{R}^d} L(x, v) + F(x, m_0) d\mu_0 \right|$$

$$\leq \left| \int_{T^d \times \mathbb{R}^d} L(x, v) d\mu_{\tau_i, m_{\tau_i}} - \int_{T^d \times \mathbb{R}^d} L(x, v) d\mu_0 \right| + \left| \int_{T^d \times \mathbb{R}^d} F(x, m_{\tau_i}) d\mu_{\tau_i, m_{\tau_i}} - \int_{T^d \times \mathbb{R}^d} F(x, m_0) d\mu_{\tau_i, m_{\tau_i}} \right|$$

$$+ \left| \int_{T^d \times \mathbb{R}^d} F(x, m_0) d\mu_{\tau_i, m_{\tau_i}} - \int_{T^d \times \mathbb{R}^d} F(x, m_0) d\mu_0 \right|.$$  

Since $\mu_{\tau_i, m_{\tau_i}} \overset{w^*}{\to} \mu_0$ and $m_{\tau_i} \overset{w^*}{\to} m_0$, then the first and the third terms in the right hand side of the above inequality go to 0 as $i \to +\infty$. We take care of the second term as follows:

$$\left| \int_{T^d \times \mathbb{R}^d} F(x, m_{\tau_i}) d\mu_{\tau_i, m_{\tau_i}} - \int_{T^d \times \mathbb{R}^d} F(x, m_0) d\mu_{\tau_i, m_{\tau_i}} \right| \leq \text{Lip}(F) \|m_{\tau_i} - m_0\|_{L^1} \to 0,$$

as $i \to +\infty$. So, we get that

$$\bar{L}(\tau_i, m_{\tau_i}) \to \int_{T^d \times \mathbb{R}^d} L(x, v) + F(x, m_0) d\mu_0.$$  

To finish the proof of (5.1), it suffices to show that

$$\bar{L}(\tau_i, m_{\tau_i}) \to -c(m_0). \quad (5.2)$$
Note that
\[
|\bar{L}(\tau_i, m_{\tau_i}) + c(m_0)| \\
\leq |\bar{L}(\tau_i, m_{\tau_i}) - \bar{L}(\tau_i, m_0)| + |\bar{L}(\tau_i, m_0) - c(m_0)| \\
\leq \sup_{\mu} \left| \int_{T^d \times \mathbb{R}^d} L(x, v) + F(x, m_{\tau_i})d\mu - \int_{T^d \times \mathbb{R}^d} L(x, v) + F(x, m_0)d\mu \right| + |\bar{L}(\tau_i, m_0) - c(m_0)| \\
\leq \sup_{\mu} \int_{T^d \times \mathbb{R}^d} |F(x, m_{\tau_i}) - F(x, m_0)|d\mu + |\bar{L}(\tau_i, m_0) - c(m_0)| \\
\leq \text{Lip}(F) d_1(m_{\tau_i}, m_0) + |\bar{L}(\tau_i, m_0) - c(m_0)|,
\]
where the supremum is taken over $\mathcal{P}_{\tau_i}(T^d \times \mathbb{R}^d)$. Letting $i \rightarrow +\infty$, we get (5.2).

Next, we only need to show that $\mu_0$ is a closed measure, i.e.,
\[
\int_{T^d \times \mathbb{R}^d} vD\varphi(x)d\mu_0 = 0, \quad \forall \varphi \in C^1(T^d).
\]  
(5.3)

Since $C^2(T^d)$ is a dense subset of $C^1(T^d)$, it suffices to show (5.3) holds for each $\varphi \in C^2(T^d)$.

For each $\tau$ and each $\varphi \in C^2(T^d)$, define
\[
\Delta \varphi_{\tau}(x, v) := \frac{\varphi(x + \tau v) - \varphi(x)}{\tau}, \quad \forall (x, v) \in T^d \times \mathbb{R}^d.
\]

It is clear that
\[
\lim_{\tau \rightarrow 0} \Delta \varphi_{\tau}(x, v) = vD\varphi(x), \quad \forall (x, v) \in T^d \times \mathbb{R}^d.
\]

In fact, for any compact subset $\mathcal{K}'$ of $T^d \times \mathbb{R}^d$, one can deduce that
\[
\lim_{\tau \rightarrow 0} \Delta \varphi_{\tau}(x, v) = vD\varphi(x)
\]  
(5.4)
uniformly on $\mathcal{K}'$.

Note that
\[
\left| \int_{T^d \times \mathbb{R}^d} \Delta \varphi_{\tau_i}(x, v)d\mu_{\tau_i, m_{\tau_i}} - \int_{T^d \times \mathbb{R}^d} vD\varphi(x)d\mu_0 \right| \\
\leq \left| \int_{T^d \times \mathbb{R}^d} \Delta \varphi_{\tau_i}(x, v)d\mu_{\tau_i, m_{\tau_i}} - \int_{T^d \times \mathbb{R}^d} vD\varphi(x)d\mu_{\tau_i, m_{\tau_i}} \right| + \left| \int_{T^d \times \mathbb{R}^d} vD\varphi(x)d\mu_{\tau_i, m_{\tau_i}} - \int_{T^d \times \mathbb{R}^d} vD\varphi(x)d\mu_0 \right|.
\]

Recall that $\mu_{\tau_i, m_{\tau_i}} \xrightarrow{w^*} \mu_0$ as $i \rightarrow +\infty$ and Corollary 6 and (5.4). We get that
\[
\lim_{i \rightarrow +\infty} \int_{T^d \times \mathbb{R}^d} \Delta \varphi_{\tau_i}(x, v)d\mu_{\tau_i, m_{\tau_i}} = \int_{T^d \times \mathbb{R}^d} vD\varphi(x)d\mu_0.
\]

Since $\mu_{\tau_i, m_{\tau_i}}$ is a minimizing $\tau_i$-holonomic measure, then
\[
\frac{1}{\tau_i} \int_{T^d \times \mathbb{R}^d} \varphi(x + \tau_i v) - \varphi(x)d\mu_{\tau_i, m_{\tau_i}} = 0.
\]
So, we have that
\[ 0 = \lim_{i \to +\infty} \int_{\mathbb{T}^d \times \mathbb{R}^d} \Delta \varphi_{\tau_i}(x,v) d\mu_{\tau_i,m_{\tau_i}} = \int_{\mathbb{T}^d \times \mathbb{R}^d} vD\varphi(x) d\mu. \]
The proof is complete.

The last result of this paper is well known, see for example [14].

**Proposition 11.** \( m_0 \) is a solution of \( \text{div}\left(m \frac{\partial H}{\partial p}(x, Du_0)\right) = 0 \) in the sense of distributions.

**Proof.** Let \( \Phi_t^{H_{m_0}} \) denote the Hamiltonian flow of \( H_{m_0} \). For any \( x \in \text{supp}(m_0) \), let \( \gamma_t(x) = \pi \circ \Phi_t^{H_{m_0}}(x, Du_0(x)) \). Then, we have that
\[ \frac{d}{dt} \gamma_t(x) = \frac{\partial H_{m_0}}{\partial p}(\gamma_t(x), Du_0(\gamma_t(x))). \]

Since the map \( \pi : \text{supp}(\mu_0) \to \text{supp}(m_0) \) is one-to-one and its inverse is given by \( x \mapsto (x, Du_0(x)) \) on \( \text{supp}(m_0) \), then \( \gamma_t : \text{supp}(m_0) \to \text{supp}(m_0) \) is a bijection for each \( t \in \mathbb{R} \). Note that, for each \( t \in \mathbb{R} \) and any function \( f \in C^1(\mathbb{T}^d) \), we get that
\[
\int_{\text{supp}(m_0)} f(\gamma_t(x)) dm_0 = \int_{\text{supp}(m_0)} f \circ \gamma_t(x) d\pi^\sharp \mu_0 \\
= \int_{\text{supp}(\mu_0)} f \circ \gamma_t(\pi(x,p)) d\mu_0 \\
= \int_{\text{supp}(\mu_0)} f(\pi \circ \Phi_t^{H_{m_0}}(x,p)) d\mu_0 \\
= \int_{\text{supp}(\mu_0)} f(\pi(x,p)) d\mu_0 \\
= \int_{\text{supp}(\mu_0)} f(x) dm_0.
\]

Here, the first equality holds since \( m_0 = \pi^\sharp \mu_0 \), the second one holds by the property of the push-forward, the third holds since \( \gamma_t \) is a bijection, the fourth one comes from the \( \Phi_t^{H_{m_0}} \)-invariance property of \( \mu_0 \), and the last one is again due to the property of the push-forward. So, for any function \( f \in C^1(\mathbb{T}^d) \) and any \( t \in \mathbb{R} \), one can deduce that
\[
0 = \frac{d}{dt} \int_{\mathbb{T}^d} f(\gamma_t(x)) dm_0(x) = \int_{\mathbb{T}^d} \langle Df(\gamma_t(x)), \frac{\partial H_{m_0}}{\partial p}(\gamma_t(x), Du_0(\gamma_t(x))) \rangle dm_0(x) \\
= \int_{\mathbb{T}^d} \langle Df(x), \frac{\partial H_{m_0}}{\partial p}(x, Du_0(x)) \rangle dm_0(x).
\]
Hence, \( m_0 \) satisfies the continuity equation which completes the proof.

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