An integral Relationship for a Fractional One-phase Stefan Problem

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Abstract:
A one-dimensional fractional one-phase Stefan problem with a temperature boundary condition at the fixed face is considered. An integral relationship between the temperature and the free boundary is obtained which is equivalent to the fractional Stefan condition. Moreover, an exact solution of similarity type expressed in terms of Wright functions is given.

Keywords: Stefan Problem, Fractional diffusion equation, Riemann–Liouville derivative, equivalent integral relationship.

AMS: Primary: 35R35, 26A33, 35C05. Secondary: 33E20, 80A22.

1 Introduction

The free boundary problems for the one-dimensional diffusion equation are problems linked to the processes of melting and freezing. In these problems the diffusion, considered as a heat flow, is expressed in terms of instantaneous local flow of temperature, and a latent heat-type condition at the interface connecting the velocity of the free boundary and the heat flux of the temperatures in both phases. This kind of problems have been widely studied in the last 50 years (see [1, 3, 5, 8, 9, 13, 20, 28, 30, 31]).

In this paper a fractional Stefan problem is considered. That is, a problem governed by a fractional diffusion equation (FDE) with two unknown functions: a two variables function \( u = u(x,t) \) and a free boundary \( s = s(t) \).

When the order of the FDE is less than 1, the process is called anomalous diffusion and it has been studied by numerous authors [11, 14, 16, 18, 19, 22, 23, 38]. For example it has been observed that the behaviour of proteins is subdiffusive due to molecular crowding [3]. Also, Mainardi studied in [21] the application to the theory of linear viscoelasticity.

Fractional free boundary problems where studied in [2, 4, 15, 17, 27, 34] and it has been observed that growth of frost on a cooled plate can be superdiffusive [29]. Voller et al. [35] state that if we consider an ideal non local flow and we replace it in a heat balance equation, we derive in sub-diffusion, modeled by a fractional diffusion equation involving the fractional Caputo derivative in time of order \( \alpha \in (0,1) \). Following this idea, we consider the non local flow given by

\[
q = -k \frac{RL D_t^{1-\alpha}}{x} \frac{\partial}{\partial x} u(x,t),
\]

in which the fractional derivative is the Riemann–Liouville derivative respect on time of order \( \alpha \in (0,1) \) defined by

\[
RL D_t^{1-\alpha} u(x,t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t \left( \frac{d}{(t-\tau)^{1-\alpha}} \right) u(x,\tau) d\tau, \quad \alpha \in (0,1)
\]

where \( \Gamma \) is the Gamma function defined by \( \Gamma(x) = \int_0^\infty w^{x-1} e^{-w} dw \). Note that the Fourier law

\[
q'(x,t) = -k \frac{\partial}{\partial x} u(x,t)
\]
that says that the heat flux is proportional to the temperature gradient is recovered recovered when the value $\alpha = 1$ is considered in (1).

Now, if we replace equation (1) in the heat balance equation, the following fractional diffusion equation is obtained:

$$\frac{\partial}{\partial t} u(x, t) = \lambda^2 \frac{\partial}{\partial x} \left( \int_0^1 D_t^{1-\alpha} \frac{\partial}{\partial x} u(x, t) \right).$$  \hspace{1cm} (3)

The classical Stefan condition derived from the Rankine-Hugoniot conditions in a one-phase Stefan problem is given by

$$\frac{d}{dt} s(t) = q^r(x, t) \bigg|_{(s(t))}, \quad 0 < t \leq T.$$  \hspace{1cm} (4)

So, replacing the non local flux (1) in (4) we obtain the following “fractional Stefan condition”:

$$\frac{d}{dt} s(t) = -k \int_0^1 D_t^{1-\alpha} \frac{\partial}{\partial x} u(x, t) \bigg|_{(s(t))}, \quad 0 < t \leq T.$$  \hspace{1cm} (5)

Therefore, our problem is given by:

(i) $\frac{\partial}{\partial x} u(x, t) = \lambda^2 \frac{\partial}{\partial x} \left( \int_0^1 D_t^{1-\alpha} \frac{\partial}{\partial x} u(x, t) \right), \quad 0 < x < s(t), 0 < t < T, 0 < \alpha < 1,

(ii) $u(x, 0) = f(x), \quad 0 \leq x \leq b = s(0),

(iii) $u(0, t) = g(t), \quad 0 < t \leq T,$

(iv) $u(s(t), t) = 0, \quad 0 < t \leq T, C \geq 0,$

(v) $\frac{d}{dt} s(t) = -k \int_0^1 D_t^{1-\alpha} \frac{\partial}{\partial x} u(x, t) \bigg|_{(s(t))}, \quad 0 < t \leq T,$

where $f$ and $g$ are non-negative continuous functions defined in $(0, T]$.

In the study of classical Stefan problems, the validity of a so called integral Stefan relationship (which is equivalent to the Stefan condition (4)), was demonstrated for temperature, heat flow, and convective condition (a Robin type condition) at the fixed face $x = 0$ in [6], [7] and [32], respectively. This integral relationship is fundamental to obtain results of existence and uniqueness, continuous and monotonous dependence on data and asymptotic behavior of the solution.

The purpose of this work is to obtain this integral condition for the fractional Stefan problem (3), predicting future results of uniqueness or asymptotic behavior of the free boundary. This result states that if the pair $\{u, s\}$ is a solution of problem (3), then (under certain conditions) the following integral condition for the free boundary $s(t)$ and the function $u(x, t)$ is verified:

$$s^2(t) = \bigg( 0^2 + 2 \int_0^1 D_t^{1-\alpha} g(\tau) d\tau + 2 \int_0^b z f(z) dz - 2 \int_0^{s(t)} z u(z, t) dz - 2 \int_0^1 D_t^{1-\alpha} u(x, t) \bigg|_{(s(\tau), \tau)} d\tau \bigg).$$  \hspace{1cm} (7)

The paper is present as follows. In Section 2 some basic concepts of fractional calculus that will be used later are given. In Section 3 the integral relationship (3) is proved by using the Green’s Theorem and it is also proved that, under certain hypothesis, this integral relationship is equivalent to the fractional Stefan condition (3). Finally we present an exact solution to a particular case of problem (5) in terms of the Wright functions and the relationship already obtained is checked.

2 Preliminaries

2.1 Basics of Fractional Calculus

Definition 1. Let $[a, b] \subset \mathbb{R}$ and $\alpha \in \mathbb{R}^+$ be such that $n-1 < \alpha \leq n$.

1. If $f \in L^1[a, b]$ we define the fractional Riemann–Liouville integral of order $\alpha$ to

$$aI^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} f(\tau) d\tau.$$
It is known that if 

Remark 1.

Then if we consider a function 

defined by

The following properties involving the fractional integral \( s \) and derivatives hold:

The Mainardi function \( \psi \) will be presented in the next section.

In this subsection we present some special functions that will be part of the explicit solution

2.2 The special functions involved

In particular, if \( 0 < \alpha < 1 \), then \( aI_{a}^{\alpha}(RLD^{\alpha}f)(t) = f(t) - \frac{a}{\Gamma(\alpha)(t-a)^{1-\alpha}} \). 

3. If \( f \in AC^{n}[a,b] \), then \( RL\ a D_{a}^{\alpha}f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(1+k-\alpha)}(t-a)^{k-\alpha} + C_{a}D_{a}^{\alpha}f(t) \).

Remark 1. It is known that if \( f \) and \( g \) are functions supported in \([0,\infty)\), then the convolution of \( f \) and \( g \) is defined by

\[
(f * g)(t) = \int_{0}^{t} f(t-\tau)g(\tau)d\tau.
\]

Then if we consider a function \( f \) supported in \([0,\infty)\) and \( \chi_{\alpha} \) is the locally integrable function defined by

then we have the following properties:

\[
aI_{a}^{\alpha}f(t) = (\chi_{\alpha} * f)(t) \tag{9}
\]

\[
RLD_{a}^{\alpha}f(t) = \frac{d^{n}}{dt^{n}}(\chi_{n-\alpha} * f)(t) \tag{10}
\]

\[
C_{a}D_{a}^{\alpha}f(t) = (\chi_{n-\alpha} * \frac{d^{n}}{dt^{n}}f(t)) \tag{11}
\]

2.2 The special functions involved

In this subsection we present some special functions that will be part of the explicit solution that will be presented in the next section.

Definition 2. For every \( z \in \mathbb{C} \), \( \rho > -1 \) and \( \beta \in \mathbb{R} \) the Wright function is defined by

\[
W(z;\rho,\beta) = \sum_{k=0}^{\infty} \frac{z^{k}}{k!\Gamma(k\rho + \beta)}.
\]

The Mainardi function \( M_{\rho} \) is a special case of the Wright function defined by

\[
M_{\rho}(z) = W(-z,-\rho,1-\rho) = \sum_{n=0}^{\infty} \frac{(-z)^{n}}{n!\Gamma(-\rho n + 1 - \rho)}, \quad z \in \mathbb{C}, \ \rho < 1.
\]
Remark 2. Function \((1.2)\) was presented by E. M. Wright at the beginning of the XX Century, and he studied its asymptotic behaviour in \([30]\) and \([37]\). It is known that:

1. The Wright function is an entire function if \(\rho > -1\).
2. The derivative of the Wright function can be computed as
\[
\frac{\partial}{\partial z} W(z, \rho, \beta) = W(z, \rho, \rho + \beta).
\]
3. Some particular cases are:
   - A Gaussian function: \(\frac{1}{\sqrt{\pi}} e^{-x^2} = W (-2x, -\frac{1}{2}, \frac{1}{2}) = M_{1/2}(2x)\).
   - The error function: \(\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi = 1 - W (-2x, -\frac{1}{2}, 1)\).
   - And the complementary \(\text{erfc}\) function:
\[
\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\xi^2} d\xi = W (-2x, -\frac{1}{2}, 1).
\]
4. \[
\lim_{x\to\infty} W (-x, -\frac{\alpha}{2}, \beta) = 0 \quad \text{for all } \alpha \in (0, 1), \quad \beta > 0.
\]

The next two propositions were proved in \([25]\).

Proposition 2. If \(0 < \alpha < 1\), then:
1. \(M_{\alpha/2}(x)\) is a positive and strictly decreasing positive function in \(\mathbb{R}^+\) such that \(M_{\alpha/2}(x) < \frac{1}{\Gamma(1-\frac{\alpha}{2})}\) for all \(x > 0\).
2. \(W (-x, -\frac{\alpha}{2}, 1)\) is a positive and strictly decreasing function in \(\mathbb{R}^+\) such that \(0 < W (-x, -\frac{\alpha}{2}, 1) < 1\), for all \(x > 0\).
3. \(1 - W (-x, -\frac{\alpha}{2}, 1)\) is a positive and strictly increasing function in \(\mathbb{R}^+\) such that \(0 < 1 - W (-x, -\frac{\alpha}{2}, 1) < 1\), for all \(x > 0\).

Proposition 3. If \(x \in \mathbb{R}_0^+\) and \(\alpha \in (0, 1)\) then:
1. \[
\lim_{\alpha \to 1} M_{\alpha/2}(2x) = M_{1/2}(2x) = \frac{e^{-x^2}}{\sqrt{\pi}};
\]
2. \[
\lim_{\alpha \to 1} [1 - W (-2x, -\frac{\alpha}{2}, 1)] = \text{erf}(x).
\]

3 The fractional Stefan Problem

Let us now study problem \([10]\), which is the main goal of this paper. The following two regions will be considered: \(\Omega_T = \{(x,t)/0 < x < s(t), 0 < t \leq T\}\) and \(\partial_p \Omega_T = \{(0,t), 0 < t \leq T\} \cup \{(s(t),t), 0 < t \leq T\} \cup \{(x,0), 0 \leq x \leq b\}\), where the latter is called parabolic boundary.

Definition 3. A pair \(\{u, s\}\) is a solution of problem \([10]\) if

1. \(u\) is defined in \([0, b] \times [0, T]\) where \(b_0 := \max\{s(t), 0 \leq t \leq T\}\).
2. \(u \in C(D_T) \cap C^2_s(D_T)\), such that \(u \in AC^1((0,T))\) where \(AC^1((0,T)) := \{f(x,\cdot): f \in AC^1(0,T)\text{ for every fixed } x \in [0,b_0]\}\).
3. \(u\) is continuous in \(D_T \cup \partial_p D_T\) except perhaps at \((0,0)\) and \((b,0)\) where
\[
0 \leq \liminf_{(x,t) \to (b,0)} u(x,t) \leq \limsup_{(x,t) \to (b,0)} u(x,t) < +\infty
\]
and
\[
0 \leq \liminf_{(x,t) \to (0,0)} u(x,t) \leq \limsup_{(x,t) \to (0,0)} u(x,t) < +\infty.
\]
4. \( s \in C^1(0,T) \).

5. There exists \( 0 \leq t < T \) such that if \( R^l_0 \) is a solution of the fractional diffusion equation for the Caputo derivative \( \alpha \), then it verifies the FDE in the region \( \Omega_T \).

Remark 3. We request \( u \) to be defined in \([0,b_0] \times [0,T]\) since the fractional derivative \( R^l_0 \) involves values \( \frac{\partial}{\partial x} u(x,\tau) \) for all \( \tau \) in \([0,t]\), but we ask \( u \) to verify the FDE in the region \( \Omega_T \).

Remark 4. Similar problems for the fractional derivative in the Caputo sense were studied in [2, 22, 26, 27, 33, 34]. For example, the formulation given in [25] is:

\[
\begin{align*}
(\text{i}) & \quad C^l_D x u(x,t) = \lambda^2 \frac{\partial}{\partial x} u(x,t), \quad 0 < x < s(t), 0 < t < T, 0 < \alpha < 1, \lambda > 0 \\
(\text{ii}) & \quad u(x,0) = f(x), \quad 0 \leq x \leq b = s(0), \\
(\text{iii}) & \quad u(0,t) = g(t), \quad 0 < t \leq T, \\
(\text{iv}) & \quad u(s(t),t) = 0, \quad 0 < t \leq T, \\
(\text{v}) & \quad C^l_D x u(x,t) = -k u_x(s(t),t), \quad 0 < t \leq T.
\end{align*}
\]

(16)

We assert that problem (16) is similar to problem (6) because from Proposition 1 it results that if \( u \) is a solution of the fractional diffusion equation for the Caputo derivative (16 – i), then \( u \) verifies the FDE (6 – i).

However, in general, the converse of the previous statement is not true because the fractional integral of Riemann-Liouville is not the inverse operator of the Riemann-Liouville derivative of equal order.

Also is worth noting that if we apply the integral operator \( R^l_0 \) to both members of equation (16 – v), we get equation

\[
\frac{d}{dt} u(t) = -k R^l_0 [ \frac{\partial}{\partial x} u(s(t),t) ] , \quad 0 < t \leq T,
\]

which is different to equation (5) unless \( \alpha = 1 \).

3.1 An exact solution

Without loss of generality, hereinafter the values \( \lambda = 1 \) and \( k = 1 \) are taken. Consider \( b = 0 \) and a constant boundary condition in (6 – iii). Namely, let the following fractional Stefan problem be:

\[
\begin{align*}
(\text{i}) & \quad \frac{\partial}{\partial x} u(x,t) = \frac{\partial}{\partial x} [ R^l_0 (\alpha) \frac{\partial}{\partial x} u(x,t) ] , \quad 0 < x < s(t), 0 < t < T, 0 < \alpha < 1, \\
(\text{ii}) & \quad u(0,t) = 1, \quad 0 < t \leq T, \\
(\text{iii}) & \quad u(s(t),t) = 0, \quad 0 < t \leq T, s(0) = 0, \\
(\text{iv}) & \quad \frac{\partial}{\partial s} u(s(t),t) = - R^l_0 (\alpha) \frac{\partial}{\partial x} u(x,t) |_{(s(t),t)}, \quad 0 < t \leq T.
\end{align*}
\]

(17)

In order to find an exact solution to problem (17), the next lemma proved in [10] will be used:

Lemma 1. Let \( c(x,t) \) be a solution of the time–fractional diffusion equation for the Caputo derivative (10 – i) such that:

For every \( (x,t) \), the function \( F(x,t) = \int_{-\infty}^{\infty} c(x,t) d\xi \) is well defined, (18)

\[
\lim_{x \to \infty} \frac{\partial}{\partial x} [ x , t ) = 0,
\]

\[
\left| \frac{\partial}{\partial \tau} c(x,\tau) \right| \leq g(\xi) \in L^1 (x,\infty),
\]

\[
\frac{\partial}{\partial \tau} c(x,\tau) (t-\tau)^{\alpha} \in L^1((x,\infty) \times (0,t)).
\]
Then \( \int_{\xi}^{\infty} c(\xi, t) d\xi \) is a solution to the time fractional diffusion equation for the Caputo derivative (16) – (i).

**Remark 5.** The factor 2 appearing in the next functions (24) and (26) was considered with the aim to recover the Gaussian and erf functions when we make \( \alpha \rightarrow 1 \) (according to Proposition 3).

**Theorem 1.** The pair given by

\[
\begin{align*}
u(x, t) = 1 - \frac{1}{1 - W(-2\xi, -\frac{\alpha}{2}, 1)} \left[ 1 - W \left( -\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1 \right) \right], \\
s(t) = 2\xi t^{\alpha/2},
\end{align*}
\]

where \( \xi \) is the unique solution to the equation

\[
2x \left[ 1 - W \left( -2x, -\frac{\alpha}{2}, 1 \right) \right] = 2x W \left( -2x, -\frac{\alpha}{2}, 1 \right) + W \left( -2x, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2} \right), \quad x > 0
\]

is a solution to problem (17).

**Proof.** We know that

\[
u(x, t) = a + b \left[ 1 - W \left( -\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1 \right) \right]
\]

is a solution of fractional diffusion equation for the Caputo derivative (16) – (i) for all \( a \in \mathbb{R}, \ b \in \mathbb{R} \) (see [19] or [27]). Then, applying Remark 4, it results that \( u \) is a solution to equation (17) – (i). From (17) – (ii) we obtain

\[
1 = u(0, t) = a + b \left[ 1 - W \left( 0, -\frac{\alpha}{2}, 1 \right) \right] = a,
\]

and from (17) – (iii) we get

\[
u(s(t), t) = 1 + b \left[ 1 - W \left( -\frac{s(t)}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1 \right) \right] = 0.
\]

Note that (27) must be verified for all \( t > 0 \), so we will ask for \( s(t) \) to be proportional to \( t^{\alpha/2} \), that is to say

\[
s(t) = 2\xi t^{\alpha/2} \quad \text{for some} \quad \xi > 0.
\]

Replacing (28) in (27) and taking into account Proposition 2 it follows that \( b = -k \frac{1}{1 - W(-2\xi, -\frac{\alpha}{2}, 1)} \) and then (22) holds.

With the aim of use the fractional Stefan condition (17) – (iv), the fractional derivative \( \frac{\partial}{\partial t} \mathcal{D}_{t}^{\alpha-\alpha} \left( \frac{\partial}{\partial x} u(x, t) \right) \) must be computed.

From (15) and using estimates made in [10], it yields that, for every \( x > 0 \), \( w_{1}(x, t) = W \left( -\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1 \right) \) is under the assumptions of Lemma 1. Clearly, \( w_{2}(x, t) = x \) is a solution to the FDE (17) – (i).

Then, using the linearity of the Caputo derivative [16] and the principle of superposition we can state that the function defined by

\[
v(x, t) = - \left[ 1 - \frac{1}{1 - W(-2\xi, -\frac{\alpha}{2}, 1)} \right] x + \frac{t^{\alpha/2}}{1 - W(-2\xi, -\frac{\alpha}{2}, 1)} W \left( -\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2} \right)
\]

is a solution of the FDE such that \( \frac{\partial}{\partial x} v(x, t) = -u(x, t) \) for all \( x > 0, t > 0 \). Hence

\[
\frac{\partial}{\partial t} v(x, t) = \frac{\partial}{\partial x} \left( \mathcal{D}_{t}^{1-\alpha} \frac{\partial}{\partial x} v(x, t) \right) = \frac{\partial}{\partial x} \left( \mathcal{D}_{t}^{1-\alpha} (-u(x, t)) \right) = -\mathcal{D}_{t}^{1-\alpha} \frac{\partial}{\partial x} u(x, t), \quad x > 0, \ t > 0.
\]
Derivating $v$ with respect to the $t$ variable and using (30) it yields
\[ 0 \mathcal{D}_t^{1-\alpha} \partial_x u(x, t) = \frac{\alpha/2}{1 - W\left(-2\xi, -\frac{\alpha}{2}, 1\right)} \left[ x W\left(-\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right) + t^{\alpha/2-1} W\left(-\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2}\right) \right]. \]

Replacing (28) and (31) into the fractional Stefan condition (17) and (iv) it results that $\xi$ must verify the following equation:
\[ 2x \left[ 1 - W\left(-2x, -\frac{\alpha}{2}, 1\right) \right] = 2x W\left(-2x, -\frac{\alpha}{2}, 1\right) + W\left(-2x, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2}\right). \] (32)

Define functions $H(x) = x \left[ 1 - W\left(-x, -\frac{\alpha}{2}, 1\right) \right]$ and $G(x) = xW\left(-x, -\frac{\alpha}{2}, 1\right) + W\left(-x, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2}\right)$ in $\mathbb{R}$. From Proposition 2, $H$ is an increasing function such that $H(0) = 0$. On the other hand, $G$ is a decreasing function in $\mathbb{R}^+$ such that $G(0) = \frac{1}{1/(1+\frac{\alpha}{2})} > 0$ due to $G'(x) = -x M_{\alpha/2}(x) < 0$ for all $x > 0$ and $\alpha \in (0, 1)$. Then, we can assert that there exists a unique positive solution $\xi$ such that $H(2\xi) = G(2\xi)$.

3.2 An integral relationship between $u$ and $s$

The next Theorem provides an integral relationship between the free boundary $s$ and function $u$, obtained from the fractional Stefan condition (9).

**Theorem 2.** Let $\{u, s\}$ be a solution of problem (9) such that $\frac{\partial^2}{\partial x \partial t} u(x, t) \in C^1(\Omega_T)$, $g \in AC^1(0, T)$ and $\mathcal{D}_t^{1-\alpha} g$ and $\mathcal{D}_t^{1-\alpha} u(x, t)|_{s(t), \tau} \in L^1(a, b)$. Then the following integral relationship for the free boundary $s(t)$ and the function $u(x, t)$
\[ s^2(t) = b^2 + 2 \int_0^t \mathcal{D}_t^{1-\alpha} g(\tau) d\tau + 2 \int_0^b z f(z) dz - 2 \int_0^t z u(z, t) dz - 2 \int_0^t \mathcal{D}_t^{1-\alpha} u(x, t)|_{s(\tau), \tau} d\tau \]

is verified.

**Proof.** Recall the Green identity:
\[ \int_{\partial \Omega} P dt + Q dx = \int_{\Omega} \left( Q_t - p_x \right) dA, \]
where $\Omega$ is an open simply connected region, $\partial \Omega$ is a positively oriented, piecewise smooth, simple closed curve, and the field $F = (P, Q)$ is $C^1$ in an open set containing $\Omega$.

Consider the functions $P$ and $Q$ defined by
\[ P(x, t) = -x \mathcal{D}_t^{1-\alpha} u_{x}(x, t) + \mathcal{D}_t^{1-\alpha} u(x, t) \] (34)
\[ Q(x, t) = -x u(x, t). \] (35)
and the region
\[ \Omega_\epsilon = \{(x, \tau) \in \mathbb{R}^2 / \epsilon < \tau < t, 0 < x < s(\tau)\}, \epsilon > 0. \]

Note that
\[ \mathcal{D}_t^{1-\alpha} u_x(x, t) = \frac{\partial}{\partial t} \mathcal{D}_t^{1-\alpha} (u_x(x, t)) = \frac{\partial}{\partial t} (\chi_{1-\alpha}(t) * u_x(x, t)), \]
where $\chi_\alpha$ (defined in eq. (33) ) is an $L^1_{loc}$ function. Then the convolution inherits all the properties of $u_x$ and, taking into account that $u$ is under the assumptions of Definition 3 and that $\frac{\partial^2}{\partial x \partial t} u(x, t) \in C^1(\Omega_T)$, the field $F$ have all the regularity required in $\Omega_\epsilon$.

Also, due to the regularity of the field $F$, the derivatives $\partial/\partial x$ and $\mathcal{D}_t^{1-\alpha}$ commute. Then, applying Green’s Theorem and taking into account that $u$ verifies (30)-(i), we get
\[
\int_{\partial \Omega} P \, dt + Q \, dx = \int_{\partial \Omega} \left[ -x RL D_1^{1-\alpha} u(x, \tau) + RL D_1^{1-\alpha} u(x, \tau) \right] \, d\tau - x u(x, t) \, dx \\
= \int_{\partial \Omega} \left[ -x u(x, \tau) + RL D_1^{1-\alpha} u(x, \tau) + \frac{\partial}{\partial x} \left( RL D_1^{1-\alpha} u(x, \tau) \right) \right] \, d\tau \, dx \\
- \frac{\partial}{\partial x} \left( RL D_1^{1-\alpha} u(x, \tau) \right) \right] \, d\tau \, dx \\
= \int_{\partial \Omega} x \left[ RL D_1^{1-\alpha} u(x, \tau) - u_i(x, \tau) \right] \, d\tau \, dx = 0.
\] (36)

Consider \( \partial \Omega' = \partial \Omega_1 \cup \partial \Omega_2 \cup \partial \Omega_3 \cup \partial \Omega_4 \) where \( \partial \Omega_1 = \{(0, \tau), \epsilon \leq \tau \leq t\} \), \( \partial \Omega_2 = \{(z, \epsilon), 0 \leq z \leq s(\epsilon)\} \), \( \partial \Omega_3 = \{(s(\tau), \tau), \epsilon \leq \tau \leq t\} \) and \( \partial \Omega_4 = \{(z, 0), 0 \leq z \leq s(t)\} \).

Integrating over the contour \( \partial \Omega' \) (positively oriented) we get:

\[
\int_{\partial \Omega_1} P \, dt + Q \, dx = \int_{t}^{t} RL D_1^{1-\alpha} u(0, \tau) \, d\tau = -\int_{t}^{t} RL D_1^{1-\alpha} g(\tau) \, d\tau,
\] (37)

\[
\int_{\partial \Omega_2} P \, dt + Q \, dx = \int_{0}^{s(\epsilon)} -zu(z, \epsilon) \, dz,
\] (38)

\[
\int_{\partial \Omega_3} P \, dt + Q \, dx = \int_{t}^{t} \left[ -s(\tau) RL D_1^{1-\alpha} u(x, \tau) \right]_{(s(\tau), \tau)} - s(\tau) u(s(\tau), \tau) s'(\tau) \, d\tau \\
+ \int_{t}^{t} RL D_1^{1-\alpha} u(x, \tau)_{(s(\tau), \tau)} \, d\tau \\
= \int_{t}^{t} s(\tau) s'(\tau) \, d\tau + \int_{t}^{t} RL D_1^{1-\alpha} u(x, \tau)_{(s(\tau), \tau)} \, d\tau \\
= \frac{s^2(t)}{2} - \frac{s^2(\epsilon)}{2} + \int_{t}^{t} RL D_1^{1-\alpha} u(x, \tau)_{(s(\tau), \tau)} \, d\tau,
\] (39)

\[
\int_{\partial \Omega_4} P \, dt + Q \, dx = \int_{0}^{s(t)} zu(z, t) \, dz.
\] (40)

Join (36), (37), (38), (39) and (40), it results that

\[
-\int_{t}^{t} RL D_1^{1-\alpha} g(\tau) \, d\tau - \int_{0}^{s(\epsilon)} zu(z, \epsilon) \, dz + \frac{s^2(t)}{2} - \frac{s^2(\epsilon)}{2} + \int_{t}^{t} RL D_1^{1-\alpha} u(x, \tau)_{(s(\tau), \tau)} \, d\tau \\
+ \int_{0}^{s(t)} zu(z, t) \, dz = 0.
\] (41)

Taking the limit when \( \epsilon \searrow 0 \) we get the integral relationship (33), i.e. the thesis holds.

\[\square\]

**Remark 6.** If we take \( \alpha = 1 \) in the integral relationship (33) we get

\[
s^2(t) = b^2 + 2 \int_{0}^{t} g(\tau) \, d\tau - 2 \int_{0}^{b} z f(z) \, dz - 2 \int_{0}^{s(t)} zu(z, t) \, dz - 2 \int_{0}^{t} u(s(\tau), \tau) \, d\tau,
\]

and using condition \( \text{iii} \), it results that

\[
s^2(t) = b^2 + 2 \int_{0}^{t} g(\tau) \, d\tau + 2 \int_{0}^{b} z f(z) \, dz - 2 \int_{0}^{s(t)} zu(z, t) \, dz,
\] (42)
where (42) is the classical integral relationship for the free boundary when a classical Stefan problem is considered (see [47], Lemma 17.1.1).

It was also proven in [91] that condition (42) is equivalent to the Stefan condition

$$\frac{d}{dt}s(t) = -\frac{\partial}{\partial x}u(s(t), t), \quad \forall t > 0. \quad (43)$$

Hence, it is natural to wonder if the “fractional Stefan condition” (43) and the “fractional integral relationship” (42) are equivalent too.

**Theorem 3.** Let \(\{u, s\}\) be a solution of problem \{(i), (ii), (iii), (iv), (33)\} such that \(\frac{\partial}{\partial t}u(x, t) \in C^1(\Omega_T), g \in AC^1(0, T), RL D_t^{1-\alpha}g, RL D_t^{1-\alpha}u(x, t)\}_{s(t), t} \in L^1(0, b)\) and \(s(t) > 0\) for all \(t > 0\). Then functions \(s = s(t)\) and \(u = u(x, t)\) verify the fractional Stefan condition (43).

**Proof.** Reasoning as in Theorem 2 we can state that the equalities (36), (37), (38) and (39) hold. Then, taking the limit when \(\epsilon \to 0\) it results that

$$-\int_0^t RL D_t^{1-\alpha}g(\tau)d\tau - \int_0^b zf(z)d\tau - \int_0^t s(\tau)RL D_t^{1-\alpha}\frac{\partial}{\partial x}u(x, t)\bigg|_{(s(\tau), \tau)}d\tau +$$

$$+ \int_0^t RL D_t^{1-\alpha}u(x, t)\bigg|_{(s(\tau), \tau)}d\tau + \int_0^t zu(z, t)dz = 0. \quad (44)$$

Multiplying (44) by 2 and using hypothesis (33) it yields that

$$-2 \int_0^t s(\tau)RL D_t^{1-\alpha}\frac{\partial}{\partial x}u(x, t)\bigg|_{(s(\tau), \tau)}d\tau = s^2(t) - b^2. \quad (45)$$

Differentiating both sides of equation (45) with respect to the \(t\)-variable and being \(s(t) > 0\) for all \(t > 0\), the thesis holds.

**Remark 7.** The hypothesis \(s(t) > 0\) for all \(t > 0\) in the previous Theorem is not necessary in the classical Stefan problem. In fact, the Stefan condition (29) join with the maximum principle imply that \(u\) is a decreasing function of the \(x\)-variable for every \(t > 0\), leading function \(s\) to be a non-decreasing function of \(t\).

However this simple tool can not be considered in this case, because decreasing functions may have a positive Riemann-Liouville derivative. For example, let \(\alpha \in (0, 1)\) be and consider \(\gamma \in (0, 1)\) such that \(0 < \alpha - \gamma\). Function \(f(t) = t^{-\gamma}\), \(t > 0\) is a decreasing function in \(\mathbb{R}^+\) and

$$RL D_t^{1-\alpha}(t^{-\gamma}) = \frac{\Gamma(-\gamma + 1)}{\Gamma(-\gamma - (1 - \alpha) + 1)}t^{-\gamma + \alpha - 1} = \frac{\Gamma(-\gamma + 1)}{\Gamma(\alpha - \gamma)}t^{-\gamma + \alpha - 1} > 0$$

for all \(t > 0\).

### 3.3 Example

The solution (22), (23) to problem (17) given in Theorem 1 verify the integral relationship (33). In this case, \(g(t) = 1\) for all \(t > 0\), \(b = 0\), \(s(t) = 2\xi^{\alpha/2}\) and \(u(x, t) = 1 - W\left(-\frac{1}{\Gamma(\alpha)}, -\frac{\xi t^{\alpha/2}}{\Gamma(\alpha)}\right)\) where \(\xi\) is the unique solution to equation (24).

The fractional derivative of Riemann-Liouville of a constant is easy to compute (see [23]) and it is given by

$$RL D_t^{1-\alpha}1 = \frac{\Gamma(\alpha - (1 - \alpha) + 1)}{\Gamma(\alpha)}t^{\alpha - 1} = \frac{\Gamma(1 - \alpha) + 1}{\Gamma(\alpha)}t^{\alpha - 1}. \quad (46)$$

Integrating (10) from 0 to \(t\), and using the Gamma function property \(z\Gamma(z) = \Gamma(z + 1)\), we have

$$\int_0^t RL D_t^{1-\alpha}1d\tau = \frac{\Gamma(1 - \alpha) + 1}{\Gamma(\alpha + 1)}. \quad (47)$$
Applying two times Lemma 1 to function $W\left(-\frac{x}{\sqrt[\alpha]{t}}, -\frac{\alpha}{2}, 1\right)$, we get that function $w(x, t) = W\left(-\frac{x}{\sqrt[\alpha]{t}}, -\frac{\alpha}{2}, 1 + \alpha\right) t^\alpha$ is a solution to the fractional diffusion equation for the Caputo derivative (16) and therefore is a solution to (16) and (17) such that $\frac{\partial}{\partial t} w(x, t) = W\left(-\frac{x}{\sqrt[\alpha]{t}}, -\frac{\alpha}{2}, 1\right)$. Then

$$\frac{\partial}{\partial t} w(x, t) = RL D_t^{1-\alpha} \frac{\partial^2}{\partial x^2} w(x, t) = RL D_t^{1-\alpha} W\left(-\frac{x}{\sqrt[\alpha]{t}}, -\frac{\alpha}{2}, 1\right).$$

(48)

Using (48) and the linearity of the Riemann-Liouville derivative, it results that

$$RL D_t^{1-\alpha} u(x, t) = RL D_t^{1-\alpha} \left(1 - \frac{1}{1 - W\left(-2\xi, -\frac{\alpha}{2}, 1\right)}\right) + \frac{1}{1 - W\left(-2\xi, -\frac{\alpha}{2}, 1\right)} \frac{\partial}{\partial t} w(x, t).$$

Hence

$$RL D_t^{1-\alpha} u(x, t) \bigg|_{(2\xi^{\alpha/2}, t)} = \left(1 - \frac{1}{1 - W\left(-2\xi, -\frac{\alpha}{2}, 1\right)}\right) t^{\alpha-1} \frac{\Gamma(\alpha+1)}{\xi^{\alpha} - \frac{\alpha}{2}, 1} \left[W\left(-2\xi, -2\xi + 1 + \frac{\alpha}{2}\right) + W\left(-2\xi, -\frac{\alpha}{2}, 1 + \alpha\right)\right].$$

(49)

Integrating (49) from 0 to t,

$$\int_0^t RL D_t^{1-\alpha} u(x, t) \bigg|_{(2\xi^{\alpha/2}, t)} d\tau = \left(1 - \frac{1}{1 - W\left(-2\xi, -\frac{\alpha}{2}, 1\right)}\right) \frac{t^{\alpha}}{\xi^{\alpha} - \frac{\alpha}{2}, 1} \left[W\left(-2\xi, -2\xi + 1 + \frac{\alpha}{2}\right) + W\left(-2\xi, -\frac{\alpha}{2}, 1 + \alpha\right)\right].$$

(50)

Integrating by parts, the next computation follows:

$$\int_0^{2\xi^{\alpha/2}} w(z, t) dz = 2\xi^{2\alpha} \left(1 - \frac{1}{1 - W\left(-2\xi, -\frac{\alpha}{2}, 1\right)}\right) \frac{W\left(-2\xi, -\frac{\alpha}{2}, 1 + \alpha\right) t^{\alpha}}{1 - W\left(-2\xi, -\frac{\alpha}{2}, 1\right)} +

\left[-\frac{2\xi t^{\alpha}}{1 - W\left(-2\xi, -\frac{\alpha}{2}, 1\right)} \frac{W\left(-2\xi, -\frac{\alpha}{2}, 1 + \alpha\right)}{\Gamma(1 + \alpha)}\right].$$

(51)

Taking account (47), (50) and (51), it follows that

$$2 \int_0^t RL D_t^{1-\alpha} 1 d\tau - 2 \int_0^t RL D_t^{1-\alpha} u(x, t) \bigg|_{(2\xi^{\alpha/2}, t)} d\tau = 2 \int_0^{2\xi^{\alpha/2}} w(z, t) dz =

-4\xi^2 t^{\alpha} + \frac{4\xi^{2\alpha}}{1 - W\left(-2\xi, -\frac{\alpha}{2}, 1\right)} + \frac{2\xi t^{\alpha}}{1 - W\left(-2\xi, -\frac{\alpha}{2}, 1\right)} W\left(-2\xi, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2}\right)

= 2\xi^{\alpha} \left[-2\xi + \frac{2\xi}{1 - W\left(-2\xi, -\frac{\alpha}{2}, 1\right)} W\left(-2\xi, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2}\right)ight].$$

(52)

Finally, taking into account that $\xi$ verify equation (24), it results that

$$2 \int_0^t RL D_t^{1-\alpha} 1 d\tau - 2 \int_0^t RL D_t^{1-\alpha} u(x, t) \bigg|_{(2\xi^{\alpha/2}, t)} d\tau - 2 \int_0^{2\xi^{\alpha/2}} w(z, t) dz = 4\xi^2 t^{\alpha} = s^2(t),$$

and the fractional integral relationship (33) is satisfied.
4 Conclusions

We have considered a fractional Stefan problem by using the fractional Riemann-Liouville derivative. We have obtained an integral relationship between the fractional temperature and the fractional free boundary, which is equivalent to the fractional Stefan condition. We have also shown an exact solution to the fractional Stefan problem, which verifies the integral relationship.

Acknowledgments

The present work has been sponsored by the Projects PIP N° 0275 from CONICET–Univ. Austral, and by AFOSR–SOARD Grant FA9550 – 14 – 1 – 0122.

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