THE DIRAC OPERATOR UNDER COLLAPSE TO A SMOOTH LIMIT SPACE

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ABSTRACT. Let \((M_i, g_i)_{i \in \mathbb{N}}\) be a sequence of spin manifolds with uniform bounded curvature and diameter that converges to a lower dimensional Riemannian manifold \((B, h)\) in the Gromov-Hausdorff topology. Lott showed that the spectrum converges to the spectrum of a certain first order elliptic differential operator \(D\) on \(B\).

In this article we give an explicit description of \(D\). We conclude that \(D\) is self-adjoint and characterize the special case where \(D\) is the Dirac operator on \(B\).

1. Introduction

Let \(\mathcal{M}(n, d)\) be the space of closed \(n\)-dimensional Riemannian manifolds \((M, g)\) with \(|\sec| \leq 1\) and \(\text{diam} \leq d\). Gromov proved that any sequence in \(\mathcal{M}(n, d)\) contains a subsequence that converges with respect to the Gromov-Hausdorff distance to a compact metric space, \([\text{Gro}81]\). There it can happen that the dimension of the limit space is strictly less than \(n\). This phenomenon is called collapsing. One of the first nontrivial examples of collapse with bounded curvature and diameter was pointed out by Marcel Berger in 1962. He considered the Hopf fibration \(S^1 \to S^3 \to S^2\). Scaling the fibers by an \(\varepsilon > 0\) one obtains a collapsing sequence with bounded sectional curvature that converges to a round two-sphere of radius \(\frac{1}{2}\) as \(\varepsilon \to 0\).

The structure of collapse in \(\mathcal{M}(n, d)\) was studied by Cheeger, Fukaya and Gromov, \([\text{CG}86, \text{CG}90, \text{Fuk}87, \text{Fuk}88, \text{Fuk}89, \text{CFG}92]\). Roughly speaking the authors showed that a sequence \((M_i, g_i)_{i \in \mathbb{N}}\) that collapses to a space \(B\) can be approximated by a sequence of singular fibrations \(f_i : (M_i, \tilde{g}_i) \to B\) such that the fibers are infranilmanifolds and \(f_i\) are almost everywhere Riemannian submersions.

An interesting question is now, how do the spectra of geometric operators behave under collapse? One would like to know if the spectrum converges and how the limit spectrum is related to the spectrum of the corresponding geometric operator on the limit space.

For the Laplacian on functions, Fukaya showed that if a sequence \((M_i, g_i)_{i \in \mathbb{N}}\) in \(\mathcal{M}(n, d)\) converges with respect to the measured Gromov-Hausdorff topology to a compact metric space \(B\), then the spectrum of the Laplacian on \((M_i, g_i)_{i \in \mathbb{N}}\) converges to the spectrum of the Laplacian on \(B\) with respect to the limit measure, \([\text{Fuk}87a]\). In general
this limit measure is different to the usual volume measure and the spectrum of the associated Laplacian differs from the spectrum of the Laplacian with respect to the standard measure, see Example 5.3.

In [Lot02b, Lot02c], Lott generalized this behavior to the Laplacian on \( p \)-forms. Combining the results of [Lot02b, Lot02c] with Bochner-type formulas for Dirac-type operators, Lott proved similar results for the spectrum of Dirac-type operators on G-Clifford bundles, where \( G \in \{ \text{SO}(n), \text{Spin}(n) \} \). The results of [Lot02a] can be briefly summarized as follows. Let \((M_i, g_i)_{i \in \mathbb{N}}\) be a sequence of manifolds with a \( G \)-structure in \( \mathcal{M}(n, d) \) that converges to a lower dimensional space \( B \).

Then the spectrum of the Dirac operator restricted to the subspace of spinors that are “invariant” along the fibers of the fibrations \( M_i \to B \) converges to the spectrum of an elliptic first order differential operator \( \mathcal{D}^B = \sqrt{\Delta} + V \) acting on a Spin-Clifford bundle over \( B \). Here \( \Delta \) is the Laplacian with respect to the limit measure and \( V \) is a symmetric potential. The remaining part of the spectrum goes to \( \pm \infty \) in the limit \( i \to \infty \).

The aim of this paper is to characterize the limit operator \( \mathcal{D}^B \) in more detail. We restrict our attention to collapsing sequences of spin manifolds with smooth limit space. This paper is a continuation of [Roo17] where we discussed the special case of collapsing sequences of spin manifolds losing one dimension in the limit. In this special case any collapsing sequence can be approximated by a sequence of \( S^1 \)-bundles with a local isometric \( S^1 \)-action. In the general case we have to deal with fiber bundles whose fibers are infranilmanifolds. Let \((M_i, g_i)_{i \in \mathbb{N}}\) be a collapsing sequence of spin manifolds with smooth limit space \( B \). In this situation we show that a part of the Dirac spectrum converges to the spectrum of an elliptic first order self-adjoint differential operator

\[
\mathcal{D} = \bar{D}^T + \frac{1}{2} \gamma(\hat{Z}) + \frac{1}{2} \gamma(\hat{A})
\]

acting on a twisted spinor bundle over \( B \), where \( \gamma \) denotes Clifford multiplication. In the above equation \( \bar{D}^T \) is a twisted Dirac operator that depends on the holonomy of the vertical distribution of the fiber bundles \( M_i \to B \), \( \hat{Z} \) depends on the intrinsic curvature of the fibers and the \( \hat{A} \)-term depends on the integrability of the horizontal distributions.

These three conditions are independent of each other as we show in Section 3 via various examples. We want to emphasize that we do not need to consider the measured Gromov-Hausdorff topology. In particular, everything is defined with respect to the standard volume measure on the limit space \( B \).

The paper is structured as follows. We start with an introduction of infranilmanifolds. In Section 3 we explain how the results of [CFG92] imply that any collapsing sequence in \( \mathcal{M}(n, d) \) can be approximated by a sequence of fiber bundles whose fibers are infranilmanifolds \( Z \) and the structure group lies in group of the affine diffeomorphisms, \( \text{Aff}(Z) \).
In that section we also study the different operators that will be used to define the limit-operator $D$ and show that they can be bounded appropriately. In Section 4 we first show that the spin structure on the total space of a fibration $M \to B$ does not imply that $B$ has a spin structure or is even orientable. Nevertheless we show how the space of “invariant” spinors on $M$ can be realized as spinors of a twisted spinor bundle over $B$. There we also explain in detail how spinor bundles on Riemannian submersions interact with each other. In Section 5 we combine all the results of the previous sections to prove the main result about the characterization of the convergent part of the Dirac spectra. As a conclusion we characterize the special case, where the convergent part of the Dirac spectra converges to the spectrum of the Dirac operator on the limit space. In Example 5.3 we show that these results are special for the spin case and can be not extended to the Dirac operator acting on differential forms. In the appendix we shortly discuss the continuity of the Dirac spectra under a $C^1$-change of the metric following, [Now13].

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2. Infranilmanifolds

In this section we recall the basic properties and definitions of infranilmanifolds. For a thorough introduction to infranilmanifolds we refer to [Dek17], see also [CFG92 Section 3] and [Lot02c Section 3].

Let $N$ be a simply-connected nilpotent Lie group. There is a canonical flat connection $\nabla^{\text{aff}}$ defined by the requirement that all left-invariant vector fields are parallel. Let $\text{Aff}(N)$ denote the subgroup of $\text{Diff}(N)$ that preserves $\nabla^{\text{aff}}$. It follows that $\text{Aff}(N)$ is isomorphic to the semi-product $N_L \rtimes \text{Aut}(N)$, where $N_L$ denotes the left-action of $N$ on itself and $\text{Aut}(N)$ is the automorphism group of $N$.

By results of Gromov [Gro78] and Ruh [Ruh82] any compact infranilmanifold $Z$ is a quotient $\Gamma \backslash N$ of a simply-connected nilpotent Lie group $N$ by a discrete subgroup $\Gamma$ of $\text{Aff}(N)$. Furthermore $\hat{\Gamma} := \Gamma \cap N_L$ is a subgroup of finite index by the generalized first Bieberbach Theorem (see for instance [Dek17 Theorem 3.4]). There are the following two
short exact sequences

\[ 1 \longrightarrow NL \longrightarrow \text{Aff}(N) \xrightarrow{p} \text{Aut}(N) \longrightarrow 1 , \]

\[ 1 \longrightarrow \hat{\Gamma} \longrightarrow \Gamma \longrightarrow p(\Gamma) \longrightarrow 1 . \]

It follows that \( Z \) is finitely covered by the nilmanifold \( \hat{Z} := \hat{\Gamma} \backslash N \). The deck transformation group is given by \( F := p(\Gamma) \). Since \( \Gamma \) is a subgroup of \( \text{Aff}(N) \) it follows that the flat connection \( \nabla^{\text{aff}} \) on \( N \) descends to a well-defined flat connection on \( \hat{Z} \) and \( Z \). Let \( n \) be the Lie algebra of \( N \). The space of affine parallel vector fields on \( N \) and \( \hat{Z} \) is isomorphic to \( n \). For the infranilmanifold \( Z \) the space of affine parallel vector fields is isomorphic to the subspace of \( n \) consisting of elements that are invariant under the induced action of \( F \) on \( n \). We denote this space by \( n^F \). Obviously \( n^F \) is finite dimensional.

Let \( g \) be a left-invariant metric on \( N \). For a local orthonormal frame \((e_1, \ldots, e_k)\) of \((N, g)\) the structural constants of the Lie algebra \( n \) of \( N \) are given by

\[
[e_a, e_b] = \sum_{c=1}^{k} \tau_{ab}^c e_c.
\]

The Christoffel symbols of \( \nabla^{\text{aff}} \) vanish since \( g \) is parallel with respect to \( \nabla^{\text{aff}} \). To calculate the Christoffel symbols of the Levi-Civita connection we use the Koszul formula,

\[
\Gamma_{ab}^c = \frac{1}{2} \left( \tau_{ab}^c - \tau_{ac}^b - \tau_{bc}^a \right).
\]

It follows that \( \nabla^{\text{aff}} \) is identical to the Levi-Civita connection if and only if \( N \) is abelian, i.e. \( N \) is isometric to the additive group \((\mathbb{R}^k, +)\) with the euclidean metric. Therefore we fix the following terminology for tensor fields on a Riemannian infranilmanifold \((Z, g)\): A tensor field \( X \) on \( Z \) is affine parallel if it is parallel with respect to \( \nabla^{\text{aff}} \), i.e. \( X \) lifts to a left-invariant tensor field on \( N \). We call a tensor field \( X \) parallel if it is parallel with respect to the Levi-Civita connection.

In the remainder of this article we are interested in Riemannian infranilmanifolds \((Z, g)\) with an affine parallel metric \( g \). Let \( g_N \) be the lift of \( g \) to \( N \). Since \( g \) is affine parallel the lift \( g_N \) is a left-invariant metric on \( N \). Hence, the oriented orthonormal frame bundle is trivial. Thus, there is a trivial spin structure given by

\[
N \times \text{Spin}(k) \rightarrow N \times \text{SO}(k).
\]

As the metric \( g \) on \( Z \) is affine parallel it follows that \( \Gamma \) is a discrete group of isometries of \((N, g_N)\). Therefore, the oriented orthonormal
bundle of $Z$ is isomorphic to
\[ P_{SO}Z \cong \Gamma \backslash (N \times SO(k)). \]

At this point we want to remark that there are examples of infranilmanifolds that are not spin, e.g. the Kleinian Bottle. An infranilmanifold $Z$ is spin if and only if $F \subset SO(k)$ and if there exists a lift
\[
\begin{array}{ccc}
\Gamma & \overset{\tilde{\rho}}{\longrightarrow} & F \\
& & \downarrow \\
& Spin(k) & \hookrightarrow SO(k).
\end{array}
\]

The different spin structures of $Z$ correspond to different lifts of the map $\Gamma \rightarrow SO(k)$ to $\Gamma \rightarrow Spin(k)$. These are labeled by
\[ \text{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z}) \cong H^1(\Gamma, \mathbb{Z}/2\mathbb{Z}) \cong H^1(Z, \mathbb{Z}/2\mathbb{Z}). \]

The corresponding $Spin(k)$-principal bundle is given by
\[ P_{Spin}(Z) \cong \Gamma \backslash (N \times Spin(k)). \]

Let $\theta_k : Spin(k) \rightarrow \text{Aut}(\Sigma_k)$ be the canonical spinor representation, where $\Sigma_k$ is a complex vector space of with $\dim_{\mathbb{C}}(\Sigma_k) = 2^\lfloor \frac{k}{2} \rfloor$. The spinor bundle of $Z$ is defined as
\[ \Sigma Z = P_{Spin}Z \times_{\theta_k} \Sigma_k = \Gamma \backslash (N \times \Sigma_k). \]

Recall the affine connection $\nabla^{aff}$ on $Z$ that is induced by the canonical flat connection $\nabla^{aff}$ on $N$ for which all left-invariant vector fields are parallel. Since the metric $g$ on $Z$ to be affine parallel $\nabla^{aff}$ induces a connection on $P_{SO}Z$ and on $P_{Spin}Z$. For brevity, we continue to write $\nabla^{aff}$ for these induced connections. In this article we are mainly interested in the space of affine parallel spinors on an infranilmanifold $(Z, g)$ with a fixed spin structure and affine parallel metric $g$. First we observe that the space of affine parallel spinors is isomorphic to
\[ \Sigma^F_k = \{ \nu \in \Sigma_k : (\theta_k \circ \tilde{\rho}(\gamma))(\nu) = \nu, \ \forall \gamma \in \Gamma \}. \]

Since $\bar{\Gamma} \subset N_{\bar{F}}$ it follows that $\rho(\bar{\Gamma})$ takes values in $\{ \pm 1 \}$. Here $\pm 1$ are the two pre-images of the identity $\text{Id} \in SO(k)$ under the double covering $Spin(k) \rightarrow SO(k)$. We conclude that $\Sigma^F_\Gamma = \{ 0 \}$ if there exists a $\gamma \in \Gamma$ such that $\tilde{\rho}(\gamma) = -1$. If $\tilde{\rho}(\gamma) = 1$ then $\Sigma^F_k = \Sigma^F_k$, where the latter is the space of all elements in $\Sigma_k$ that are fixed by the action of the finite group $F$. Since $\Sigma^F_k \subset \Sigma_k$, the space of affine parallel spinors on $Z$ is finite dimensional.
3. Geometry of affine fiber bundles

Cheeger, Fukaya, and Gromov studied the structure of collapsing sequences with bounded sectional curvature in great detail and full generality in [CFG92]. There, they showed that any collapsing sequence in $\mathcal{M}(n, d)$ with smooth limit space can be approximated by a sequence of Riemannian affine fiber bundles.

**Definition 3.1.** A fiber bundle $f : (M, g) \to (B, h)$ is called a Riemannian affine fiber bundle if

- $f$ is a Riemannian submersion,
- for each $p$ the fiber $Z_p := f^{-1}(p)$ is an infranilmanifold with an induced affine parallel metric $\hat{g}_p$,
- the structure group lies in $\text{Aff}(Z)$.

Let $(M_i, g_i)_{i \in \mathbb{N}}$ be a sequence in $\mathcal{M}(n, d)$ converging to a Riemannian manifold $(B, h)$ in the Gromov-Hausdorff topology. By [Fuk88, Theorem 10.1] the metric $h$ on $B$ is $C^{1, \alpha}$. Furthermore, for $i$ sufficiently large, there is a fibration $f_i : (M_i, g_i) \to (B, h)$ such that the fiber is an infranilmanifold $Z_i$ and such that $f_i$ is a $\tau(d_{GH}(M_i, B))$-almost Riemannian submersion, i.e.

$$e^{-\tau(d_{GH}(M_i, B))} \leq \frac{\|df_i(X)\|}{\|X\|} \leq e^{\tau(d_{GH}(M_i, B))}$$

for any vector field $X$ orthogonal to the fibers. Here $\tau(\cdot)$ is a smooth function satisfying $\lim_{\epsilon \to 0}(\tau(\epsilon)) = 0$.

As mentioned in the beginning of this section, the results of [CFG92] show that we can approximate the sequence $f_i : (M_i, g_i) \to (B, h)$ by a collapsing sequence of Riemannian affine fiber bundles $f_i : (M_i, \hat{g}_i) \to (B, h_i)$. To explain the existence of the metrics $\hat{g}_i$ in more detail, we fix a fiber bundle $f_i : (M_i, g_i) \to (B, h)$ for some sufficiently large $i$. For simplicity we drop the index $i$ in the following.

As done in [CFG92], we first apply the smoothing result by Abresch (see for instance [CFG92, Theorem 1.12]): For any positive $\delta$ there is a smooth Riemannian metric $\bar{g}$ such that

- $\|g - \bar{g}\|_{C^1} < \delta$,
- $\|\nabla^i R\|_{C^0} < A_i(n, \delta)$ for some positive constants $A_i(n, \delta)$.

By [Ron96, Proposition 2.5] there is a positive constant $C(n)$ such that

$$|\sec(g) - \sec(\hat{g})| \leq C(n)\delta$$

for any sufficiently small positive $\delta$. Now we can apply [CFG92, Proposition 3.6 and 4.9] to the fibration $f_i : (M_i, \hat{g}_i) \to (B, h_i)$. We obtain another metric $\tilde{g}_i$ with $\|\tilde{g} - \hat{g}\|_{C^1} \leq C(n)d_{GH}(M, B)$ for a positive constant $C(n)$ such that $f : (M, \tilde{g}) \to (B, h)$ is a Riemannian affine fiber bundle for an induced metric $\hat{h}$.

Letting $\delta$ go to 0 as $i$ goes to infinity, we summarize this discussion in the following theorem.
Theorem 3.2. Let \((M_i, g_i)_{i \in \mathbb{N}}\) be a collapsing sequence in \(\mathcal{M}(n,d)\). Suppose that this sequence converges to a Riemannian manifold \((B, h)\) with respect to the Gromov-Hausdorff topology. Then, for any \(i\) sufficiently large, there is a metric \(\tilde{g}_i\) on \(M_i\) and a metric \(\tilde{h}_i\) on \(B\) such that

\[
\lim_{i \to \infty} \|g_i - \tilde{g}_i\|_{C^1} = 0 = \lim_{i \to \infty} \|h - \tilde{h}_i\|_{C^1}
\]

and \(f_i : (M_i, \tilde{g}_i) \to (B, \tilde{h}_i)\) is a Riemannian affine fiber bundle.

Thus, we prove the relevant bounds for Riemannian affine fiber bundles \(f : (M, g) \to (B, h)\) with infranil fiber \(Z\). From now on, we set \(\dim(B) = n\) and \(\dim(Z) = k\) for positive integers \(n, k\). Since \(f\) is a Riemannian submersion \(TM = H \oplus V\), where \(H\) is the horizontal distribution isomorphic to \(f^*TB\) and \(V = \ker(df)\) is the vertical distribution. The relations between the curvatures of \((M, g), (B, h)\) and the fibers \((Z, \tilde{g})\) are given by O’Neill’s formulas, see for instance [Bes08, Theorem 9.28]. These formulas involve the two tensors \(T\) and \(A\) defined via

\[
T(X, Y) := \left(\nabla_X Y^V\right)^H + \left(\nabla_X Y^H\right)^V,
\]

\[
A(X, Y) := \left(\nabla_X H Y^V\right)^H + \left(\nabla_X H Y^H\right)^V,
\]

for all vector fields \(X, Y \in \Gamma(TM)\). Here \(X^V, X^H\) denote the vertical, resp. horizontal part. Roughly speaking, the \(T\)-tensor is related to the second fundamental form of the fibers and the \(A\)-tensor vanishes if and only if the horizontal distribution \(H\) is integrable. In the remainder of this section many calculations are carried out in a local orthonormal frame.

Definition 3.3. Let \(f : (M, g) \to (B, h)\) be a Riemannian affine fiber bundle. For all \(x \in M\) a local orthonormal frame \((\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_k)\) is a split orthonormal frame if \((\tilde{\xi}_1, \ldots, \tilde{\xi}_n)\) is the horizontal lift of a local orthonormal frame \((\xi_1, \ldots, \xi_n)\) around \(p = f(x) \in B\) and \((\zeta_1, \ldots, \zeta_k)\) are locally defined affine parallel vector fields tangent to the fibers.

Here and subsequently we label the vertical components \(a, b, c, \ldots\), and the horizontal components \(\alpha, \beta, \gamma, \ldots\). The Christoffel symbols with respect to a split orthonormal frame \((\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_k)\) can be calculated with the Koszul formula:

\[
\Gamma^c_{ab} = \Gamma^c_{ba},
\]

\[
\Gamma^b_{aa} = -\Gamma^b_{ba} = g(T(\zeta_a, \zeta_b), \xi_a),
\]

\[
\Gamma^b_{aa} = g([\xi_a, \xi_a], \zeta_b) + g(T(\zeta_a, \zeta_a), \zeta_b),
\]

\[
\Gamma^a_{a\beta} = -\Gamma^\beta_{aa} = g(A(\xi_a, \xi_\beta), \zeta_a),
\]

\[
\Gamma^\gamma_{a\beta} = \tilde{\Gamma}_{a\beta}.
\]
Here $\hat{\Gamma}_{ab}^c$ are the Christoffel symbols of the fiber $(Z, \hat{g})$, see (1), and $\hat{\Gamma}_{\alpha\beta}^\gamma$ are the Christoffel symbols of $(B, h)$ with respect to $(\hat{\xi}_1, \ldots, \hat{\xi}_n)$. For later use we need to consider the following two operators characterized by their action on vector fields $X, Y$.

\[
\nabla^Z_X Y := (\nabla_X V Y^V)^V,
\]

\[
\nabla^V_X Y := (\nabla_X H Y^V)^V.
\]

We observe that for each $p \in B$, $\nabla^Z$ restricted to a fiber $Z_p$ is the Levi-Civita connection with respect to the induced metric $\hat{g}_p$ on $Z_p$. Since $\hat{g}_p$ is by assumption affine parallel, it follows that $\nabla^Z$ preserves the space of affine parallel vector fields. Let $Z := \nabla^Z - \nabla^{\text{aff}}$. This is a one-form with values in $\text{End}(TZ)$, where we view $TZ$ as a vector bundle over $M$. Further $Z = 0$ if and only if the induced metric $\hat{g}_p$ is flat for all $p \in B$.

Next, we view $\nabla^V$ as a connection of the vertical distribution $\mathcal{V}$. It is immediate that $\nabla^V$ also preserves the space of affine parallel vector fields.

We recall from Section 2 that the space of affine parallel vector fields on an infranilmanifold $Z$ is isomorphic to $n^F$. In particular, it is finite dimensional. Thus, there is a vector bundle $P$ over $B$ such that, for any $p \in B$, the fiber $P_p$ is given by all affine parallel vector fields of the infranilmanifold $Z_p$. By the discussion above we conclude that $Z$ descends to a well-defined operator on $P$ and $\nabla^V$ induces a connection of $P$. In addition, there is an $A \in \Omega^2(B, P)$ characterized by

\[
A(X, Y) = A(\hat{X}, \hat{Y}),
\]

for any vector fields $X, Y$. Here $\hat{X}$ denotes the horizontal lift. It will be shown that exactly these three operators, $\nabla^V$, $Z$, and $A$ contribute additionally to the limit operator. To do so, we need to extract subsequences such that these three operators converge on $B$ in the $C^0$-topology. Our strategy is to prove uniform $C^1(B)$-bounds. Then it follows from the compactness of the embedding $C^{0, \alpha} \hookrightarrow C^1$, for $\alpha \in [0, 1)$ that we can extract convergent subsequences. Let $f : M \to B$ be a Riemannian affine fiber bundle. The $C^1(B)$-bounds on $\nabla^V$, $Z$, and $A$ will depend on the following three bounds

\[
\|A\|_{\infty} \leq C_A,
\]

\[
\|T\|_{\infty} \leq C_T,
\]

\[
\|R^M\|_{\infty} \leq C_R.
\]

We show in the following lemma that such constants exist uniformly for any sequence $(M_i, g_i)_{i \in \mathbb{N}}$ in $\mathcal{M}(n+k, d)$ converging to an $n$-dimensional Riemannian manifold $(B, h)$. It follows from [Ron96, Proposition 2.5] that the sectional curvature of $(M_i, g_i)$ only changes slightly if we pass to invariant metrics in the sense of Theorem 3.2. Hence, there is no
loss of generality in assuming that the metrics \( g_i \) are invariant from the beginning.

**Lemma 3.4.** Let \( (M_i, g_i)_{i \in \mathbb{N}} \) be a sequence in \( \mathcal{M}(n + k, d) \) converging to an \( n \)-dimensional Riemannian manifold \( (B, h) \). Suppose further that the metrics \( g_i \) are invariant for all \( i \). Then there are positive constants \( C_A(n, k, B, h), C_T(n, k, B, h) \) such that

\[
\|A\|_\infty \leq C_A, \quad \|T\|_\infty \leq C_T,
\]

for all \( i \in \mathbb{N} \).

**Proof.** By [Fuk88, Theorem 10.1] and the assumption that \( g_i \) is an invariant metric, it follows that for any \( i \in \mathbb{N} \) there is a map \( f_i : (M_i, g_i) \to (B, h) \) defining a Riemannian affine fiber bundle. Applying [Fuk88, Lemma 7.8] to the sequence \( (B, h_i) \) it follows that there is a positive constant \( \Lambda \) such that for all \( i \in \mathbb{N} \),

\[
|\text{sec}^{h_i}| \leq \Lambda.
\]

Hence, it follows with O’Neill’s formula, see for instance [Bes08, (9.29c)], that

\[
\|A_i\|_\infty \leq \frac{n(n-1)}{6}(|\text{sec}^{g_i}| + |\text{sec}^{h_i}|) \leq \frac{n(n-1)}{6}(1 + \Lambda) =: C_A.
\]

The uniform bound on the \( T \)-tensor follows directly from [CFG92, Theorem 2.6]. \( \square \)

In order to obtain \( C^1(B) \) bounds for \( \nabla^V, Z, \) and \( A \), we fix a Riemannian affine fiber bundle \( f : (M, g) \to (B, h) \). First we deal with \( \nabla^V \). Let \( (e_1, \ldots, e_k) \) be a locally affine parallel frame for the vertical distribution \( \mathcal{V} \) such that \([X, e_a] = 0\) for any horizontal vector field \( X \). We write \( \langle \cdot, \cdot \rangle \) for a locally defined metric on \( \mathcal{V} \) characterized by \( \langle e_a, e_b \rangle = \delta_{ab} \). There is a unique positive definite symmetric operator \( W \) satisfying

\[
(3) \quad g(U, V) = \langle W(U), W(V) \rangle,
\]

for all vertical vector fields \( U, V \). Recall that the induced metric on the fiber is affine parallel. Hence, \( W \) is affine parallel as well. Let \( (\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_k) \) be a split orthonormal frame where \( \zeta_a := W^{-1}(e_a) \).

A short computation shows that

\[
g(T(\zeta_a, \zeta_a), \zeta_b) = \frac{1}{2}(W^{-1}(\zeta_a(W) + \zeta_a(W)W^{-1})e_a, e_b).
\]
It follows that
\[
\Gamma^b_{\alpha a} = g([\xi_\alpha, \zeta_a], \zeta_b) + g(T(\zeta_\alpha, \xi_\alpha), \zeta_b) \\
= g(\xi_\alpha(W^{-1})e_a, \zeta_b) + \frac{1}{2}((W^{-1}\xi_\alpha(W) + \xi_\alpha(W)W^{-1})e_a, e_b) \\
= \frac{1}{2}((W^{-1}\xi_\alpha(W) - \xi_\alpha(W)W^{-1})e_a, e_b) =: (\mathcal{W}_{\xi_\alpha}e_a, e_b).
\]

By abuse of notation, we use the same letter \(W\) for the connection one-form of \(\nabla^V\). As discussed above \(\nabla^V\) induces a well-defined connection on the vector bundle \(P \to B\). To show that for a collapsing sequence \((M_i, g_i)_{i \in \mathbb{N}}\) in \(\mathcal{M}(n, d)\) as in Lemma 3.4 there is a subsequence such that \((\nabla^V_{M_i})_{i \in \mathbb{N}}\) converge in \(C^0\), it suffices to derive a uniform \(C^1\)-bound on the connection forms \(\mathcal{W}_i\).

**Lemma 3.5.** Let \(f : M \to B\) be a Riemannian affine fiber bundle such that
\[
\|T\|_{\infty} \leq C_T, \\
\|A\|_{\infty} \leq C_A, \\
\|R^M\|_{\infty} \leq C_R,
\]
then
\[
\|\mathcal{W}_{\xi_\alpha}\|_{\infty} \leq 2C_T, \\
\|\xi_\beta(\mathcal{W}_{\xi_\alpha})\| \leq C(C_T, C_A, C_R),
\]
where \((\xi_1, \ldots, \xi_n)\) is a horizontal orthonormal frame.

**Proof.** Let \((\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_k)\) be a split orthormal frame such that \(\zeta_a = W^{-1}e_a\), as above. In particular, \(W\) can be viewed as a field of symmetric positive definite matrices.

For the first inequality, we calculate
\[
\|T\|^2 = \sum_{a=1}^{n} \sum_{a,b=1}^{k} g(T(\zeta_\alpha, \xi_\alpha), \zeta_b)^2 \\
= \frac{1}{4} \sum_{a=1}^{n} \sum_{a,b=1}^{k} ((\xi_\alpha(W)W^{-1} + W^{-1}\xi_\alpha(W))e_a, e_b)^2 \\
= \frac{1}{4} \sum_{a=1}^{n} \|\xi_\alpha(W)W^{-1} + W^{-1}\xi_\alpha(W)\|^2 \\
= \frac{1}{2} \sum_{a=1}^{n} \text{tr} (W^{-1}\xi_\alpha(W))^2 + \text{tr} (W^{-2}\xi_\alpha(W)^2).
\]
As \(W^{-1}\) and \(\xi_\alpha(W)\) are symmetric, it follows that
\[
\text{tr} (W^{-2}\xi_\alpha(W)^2) = \|W^{-1}\xi_\alpha(W)\|^2 \geq 0.
\]
Since $W^{-1}$ is also symmetric and positive definite it has a unique symmetric positive-definite square root $C$, i.e. $C^2 = W^{-1}$. Replacing $W^{-1}$ by $C^2$ leads to

$$\text{tr} \left( (W^{-1}\xi_\alpha(W))^2 \right) = \text{tr}(C^2\xi_\alpha(W)C^2\xi_\alpha(W)) = \|C\xi_\alpha(W)C\|^2 \geq 0.$$  

Thus,

$$\frac{1}{2} \sum_{\alpha=1}^n \|W^{-1}\xi_\alpha(W)\|^2 \leq \|T\|^2 \leq C_T. \quad (4)$$

It follows immediately that

$$\|W_{\xi_\alpha}\|_{\infty} = \frac{1}{2}\|W^{-1}\xi_\alpha(W) - \xi_\alpha(W)W^{-1}\| \leq 2C_T.$$  

For the second inequality we fix a point $p \in B$. Suppose that $(\xi_1, \ldots, \xi_n)$ is the horizontal lift of an orthonormal frame parallel in $p$. All following calculations are done with respect to $p$. We compute

$$\xi_\beta(W_{\xi_\alpha}) = \frac{1}{2} \left( W^{-1}\xi_\beta\xi_\alpha(W) - \xi_\beta\xi_\alpha(W)W^{-1} \right.$$

$$\left. + \xi_\alpha(W)W^{-1}\xi_\beta(W)W^{-1} - W^{-1}\xi_\beta(W)^{-1}\xi_\alpha(W) \right).$$

By the inequality (4), it remains to bound the second derivatives. A straight forward calculation shows that

$$g((\nabla_{\xi_\alpha}T)(\xi_\alpha, \xi_\beta), \xi_\alpha) = \frac{1}{2} \left( (\xi_\beta\xi_\alpha(W)W^{-1} + W^{-1}\xi_\beta\xi_\alpha(W) \right.$$

$$\left. + \xi_\beta(W)W^{-1}\xi_\alpha(W)W^{-1} - W^{-1}\xi_\beta(W^{-1}\xi_\alpha(W) \right.$$  

$$\left. - \xi_\beta(W)W^{-1}\xi_\alpha(W) \right.$$  

$$\left. + W^{-1}\xi_\alpha(W)\xi_\beta(W)W^{-1} \right) \xi_\alpha, \xi_\beta) \quad (5)$$

for all $1 \leq a, b \leq k$. Since $\xi_\beta\xi_\alpha(W) = \xi_\alpha\xi_\beta(W)$, it follows from (4) that

$$\chi_{\alpha,\beta,i,j} := g((\nabla_{\xi_\alpha}T)(\xi_\alpha, \xi_\beta), \xi_\alpha) - g((\nabla_{\xi_\beta}T)(\xi_\alpha, \xi_\beta), \xi_\alpha)$$

is bounded by

$$\|\chi_{\alpha,\beta,i,j}\| \leq 8C_T^2.$$  

Next, we apply one of O'Neill’s formula (see for instance [Bes08, 9.32]) to derive that

$$\chi_{\alpha,\beta,i,j} = g((\nabla_{\xi_\beta}A)(\xi_\alpha, \xi_\alpha), \xi_j) + g((\nabla_{\xi_\beta}A)(\xi_\beta, \xi_\alpha), \xi_i).$$
Inserting this equality in \[\text{Bes08}, \text{9.28d}\] we obtain that
\[
g(R^M(\zeta_j, \zeta_\beta, \zeta_\alpha)) = g((\nabla_{\zeta_j}A)(\zeta_\beta, \zeta_\alpha), \zeta_j) - g((\nabla_{\zeta_j}A)(\zeta_\beta, \zeta_\alpha), \zeta_j) + g(A(\zeta_\beta, \zeta_j), A(\zeta_\alpha, \zeta_j)) + g(T(\zeta_j, \zeta_\beta), T(\zeta_\alpha, \zeta_j)) + g(T(\zeta_j, \zeta_\beta), T(\zeta_\alpha, \zeta_j)).
\]

Next, we consider \[\text{Bes08}, \text{9.28c}\],
\[
g(R(\zeta_j, \zeta_\beta)\zeta_\alpha, \zeta_i) = g((\nabla_{\zeta_j}T)(\zeta_i, \zeta_\beta), \zeta_\alpha)) - g(T(\zeta_j, \zeta_\beta), T(\zeta_i, \zeta_\alpha)) + g((\nabla_{\zeta_j}A)(\zeta_\beta, \zeta_\alpha), \zeta_j) + g(A(\zeta_\beta, \zeta_j), A(\zeta_\alpha, \zeta_i)).
\]

It follows that
\[
|g((\nabla_{\zeta_j}T)(\zeta_i, \zeta_\beta), \zeta_\alpha)| \leq C_R + C_T^2 + 2C_A^2 + C_1 =: C_2.
\]

Using this bound and the inequality \([4]\), we conclude from \([5]\) that
\[
\|W^{-1}\xi_\beta \xi_\alpha(W) + \xi_\beta \xi_\alpha(W)W^{-1}\| \leq C_2 + 4C_T^2 := C_3.
\]

Using the same strategy as in the proof of the inequality \([4]\), it follows that
\[
\|W^{-1}\xi_\beta \xi_\alpha(W)\| \leq C_3.
\]

Collecting everything so far, the claim follows from
\[
\|\xi_\beta(W \xi_\alpha)\| \leq C_3 + 2C_T^2.
\]

The condition for $\nabla^V$ to be gauge equivalent to the trivial connection is that the holonomy $\text{Hol}(\nabla^V)$ is trivial, see for instance \[\text{Bau14}, \text{Section 4.3}\]. In the following examples we show that $\nabla^V$ can be the trivial connection although the $T$-tensor is nontrivial.

**Example 3.6.** Let $M = B \times \mathbb{T}^k$ be the trivial $\mathbb{T}^k$-bundle over a Riemannian manifold $(B, h)$. In this situation the vertical distribution $\mathcal{V}$ is the trivial vector bundle $B \times \mathbb{R}^k$. For any $i \in \mathbb{N}$ we endow $M$ with the Riemannian product metric
\[
g_i := h \oplus \frac{1}{i^2} u^2 \hat{g},
\]
where $u : B \to \mathbb{R}$ is a fixed smooth function and $\hat{g}$ is the standard flat metric on $\mathbb{T}^k$. Then $(M, g_i)_{i \in \mathbb{N}}$ is a collapsing sequence with bounded sectional curvature and diameter. Consider the Riemannian submersions $f_i : (M, g_i) \to (B, h)$. The fibers are embedded flat tori and the
horizontal distribution is integrable for all \(i \in \mathbb{N}\). The \(T\)-tensor is, for all \(i \in \mathbb{N}\), given by

\[
T_i(U, V) = \frac{\text{grad}(u)}{u} g_i(U, V)
\]

for any two vertical vectors \(U, V\). We claim that the induced connection \(\nabla^V\) is trivial with respect to an isometric trivialization. To see this claim, we adapt the notation of Lemma 3.5. Let \(e^i_1, \ldots, e^i_k\) be an orthonormal frame for \(\hat{g}_i\). Any such choice induces a global vertical frame on \((M, g_i)\). For \(W_i\) defined as in (3) we obtain

\[
W_i = \frac{u}{i} \text{Id}.
\]

Hence,

\[
(W_i)_X = \frac{1}{2}(W_i^{-1}X(W_i) - X(W_i)W_i^{-1}) = \frac{1}{2}\left(\frac{i}{u} X(u) - \frac{X(u)}{u} i\right) \text{Id} = 0.
\]

Therefore, \(W_i = 0\) for all \(i \in \mathbb{N}\), although the \(T\)-tensor is non-trivial. In particular, \(\nabla^V\) is the trivial connection on the trivial vector bundle \(\mathcal{V} = B \times \mathbb{R}^k\).

Next, we state an example of a collapsing sequence such that the corresponding connections \(\nabla^V_i\) do not converge to a connection that is gauge equivalent to the trivial connection.

**Example 3.7.** Consider the two-dimensional torus \(\mathbb{T}^2\) and choose an element of \(\text{Aut}(\mathbb{T}^2) \cong \text{GL}(2, \mathbb{Z})\), e.g.

\[
H := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.
\]

Let \(C := [0, 1] \times \mathbb{T}^2\) be the cylinder over \(\mathbb{T}^2\) and set

\[
M := C/\sim,
\]

where we identify \((0, x)\) with \((1, Hx)\) for all \(x \in \mathbb{T}^2\). This defines a non-trivial \(\mathbb{T}^2\)-bundle \(f : M \to S^1\). There is a Riemannian metric \(g = h \oplus \hat{g}\) on \(M\) such that \(h\) is the standard metric on \(S^1\) and \((\hat{g}_\phi)_{\phi \in S^1}\) is a family of flat metrics on \(\mathbb{T}^2\). Then \((M, g_i)_{i \in \mathbb{N}}\) with \(g_i := h \oplus \frac{1}{i} \hat{g}\) defines a collapsing sequence with bounded sectional curvature and diameter such that the induced connection \(\nabla^V_i\) is never trivial.

Next we consider \(Z = \nabla^Z - \nabla^\text{aff} \in \Omega^1(\text{End}(TZ))\). Recall that \(\nabla^Z\) restricted to a fiber \(Z_p\) is the Levi-Civita connection of \((Z_p, \hat{g}_p)\), where \(\hat{g}_p\) is the induced affine parallel metric. We also recall that \(Z = 0\) if and only if the induced metric \(\hat{g}_p\) is flat for all \(p \in B\). The following
Example shows that for Riemannian affine fiber bundles with non-flat fibers the one-form $Z$ is nontrivial.

**Example 3.8.** Let $M = \Gamma \setminus N$ be a nilmanifold, where $N$ is the 3-dimensional Heisenberg group

$$N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

and

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}.$$

The Lie algebra $\mathfrak{n}$ of $N$ is given by

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

We fix the basis

$$X := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and let $X^*, Y^*, Z^*$ be the dual basis. For any $i \in \mathbb{N}$ we consider the affine parallel metric

$$g_i := \frac{1}{i^2} X^* \cdot X^* + \frac{1}{i^2} Y^* \cdot Y^* + \frac{1}{i^4} Z^* \cdot Z^*.$$

It is not hard to check that $(Z, g_i)_{i \in \mathbb{N}}$ defines a collapsing sequence with bounded curvature that converges to a point as $i$ goes to infinity. For each $i \in \mathbb{N}$ we consider the orthonormal frame $(e_1, e_2, e_3)$ defined by

$$e_1 = iX, \quad e_2 = iY, \quad e_3 = i^2 Z.$$

The Koszul formula shows that the Christoffel symbol $\Gamma^3_{12}(i) = \frac{1}{2}$ for all $i \in \mathbb{N}$. Therefore, the Levi-Civita connection $\nabla^{Z_i}$ does not converge to the affine connection $\nabla^{aff}$ as $i \to \infty$, i.e. $Z_i$ does not vanish in the limit. That $Z_i \neq 0$ for all $i \in \mathbb{N}$ does also follow from the fact that

$$\left( (\Gamma \setminus N, g_1) \rightarrow (\Gamma \setminus N, g_i) \right),$$

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & ix & i^2 z \\ 0 & 1 & iy \\ 0 & 0 & 1 \end{pmatrix}$$

is an $i^4$-fold isometric covering.
Next we want to identify $Z = \nabla^Z - \nabla^\text{aff}$ acting on affine parallel vector fields with an operator over $B$. Since $M \to B$ is a Riemannian affine fiber bundle, the induced metric $g_p$ on the fiber $Z_p$ is affine parallel. Therefore the space of affine parallel vector fields is invariant under the action of $Z \in \Omega^1(\text{End}(TZ))$, where we view $TZ$ as a vector bundle over $M$. Hence, there is an induced operator $Z$ on the vector bundle $P \to B$. In addition, we want to remark that the actions of $\nabla^Z$ and $Z$ coincide on the space of affine parallel vector fields.

**Lemma 3.9.** Let $f : M \to B$ be a Riemannian affine fiber bundle such that

$$\|T\|_\infty \leq C_T,$$
$$\|A\|_\infty \leq C_A,$$
$$\|R^M\|_\infty \leq C_R,$$

then

$$\|Z\| \leq C(k, C_T, C_R),$$
$$\|X(Z)\| \leq C(k, C_t, C_A, C_R)\|X\|,$$

for all horizontal vector fields $X$.

**Proof.** Let $(\zeta_1, \ldots, \zeta_k)$ be as in the split orthonormal frame, see Definition 3.3. Since $(\zeta_1, \ldots, \zeta_k)$ is affine parallel, their structural coefficients $\tau_{ab}^c$ are defined via

$$[\zeta_a, \zeta_b] = \sum_{c=1}^k \tau_{ab}^c \zeta_c.$$

These are the structural constants of the Lie algebra $\mathfrak{n}$ of the nilpotent Lie group $N$ that covers $Z$.

We recall from [1] that

$$\Gamma_{ab}^c = \tilde{\Gamma}_{ab}^c = \frac{1}{2} (\tau_{ab}^b - \tau_{ac}^b - \tau_{bc}^a).$$

By [Lot02c, Lemma 1],

$$\sum_{a,b,c=1}^k (\tau_{ab}^c)^2 = -4 \text{scal}(Z).$$

Thus,

$$\|Z\|^2 = \sum_{a,b,c=1}^k (\Gamma_{ab}^c)^2$$

$$= \frac{3}{4} \sum_{a,b,c=1}^k (\tau_{ab}^c)^2 + 2 \sum_{a,b,c=1}^k (\tau_{ac}^b \tau_{bc}^a - \tau_{ab}^c \tau_{ac}^b - \tau_{ab}^c \tau_{bc}^a)$$

$$= -3 \text{scal}(Z),$$
because \( \sum_{i=a,b,c=1}^{k} \tau_{ab}^i \tau_{ac}^i = 0 \) as \( n \) is nilpotent. The first inequality follows from O’Neill’s formula,

\[
\left| \text{scal}(Z) \right| \leq \sum_{a,b=1} \left| \text{sec}^Z(\zeta_a, \zeta_b) \right|
\]

\[
= \sum_{a,b=1} \left| \text{sec}^M(\zeta_a, \zeta_b) - |T(\zeta_a, \zeta_b)|^2 + g(T(\zeta_a, \zeta_a), T(\zeta_b, \zeta_b)) \right|
\]

\[
\leq k^2 (C_R + 2C_T^2).
\]

The second inequality is also proven in local coordinates. We have that

\[
|X(Z)|^2 = \sum_{a,b,c=1}^k (X(\Gamma_{ab}^c))^2
\]

for any horizontal vector field \( X \). We calculate

\[
|X(\Gamma_{ab}^c)| = |g(\nabla_X \nabla_{\zeta_a} \zeta_b, \nabla_X \zeta_c) - g(\nabla_{\zeta_a} \zeta_b, \nabla_X \zeta_c)|
\]

\[
= |g(R^M(X, \zeta_a) \zeta_b + \nabla_{[X, \zeta_a]} \zeta_b, \zeta_c) - g(\nabla_X \zeta_b, \nabla_{\zeta_a} \zeta_c) - g(\nabla_{\zeta_a} \zeta_b, \nabla_X \zeta_c)|
\]

By the assumptions and Lemma 3.5, we conclude

\[
|g(\nabla_{[X, \zeta_a]} \zeta_b, \zeta_c)| \leq \|[X, \zeta_a] ||Z||
\]

\[
\leq |\nabla^X_{\zeta_a} T(\zeta_a, X)||Z||
\]

\[
\leq 3C_T||Z|| ||X||,
\]

and

\[
|g(\nabla_X \zeta_b, \nabla_{\zeta_a} \zeta_c)| \leq |g(\nabla^Y_X \zeta_b, \nabla^Z_{\zeta_a} \zeta_c)| + |g(A(X, \zeta_a), T(\zeta_a, \zeta_c))|
\]

\[
\leq (2C_T ||Z|| + C_A C_T) ||X||.
\]

Hence,

\[
|X(\Gamma_{ab}^c)| \leq \left( C_R + 7C_T ||\omega|| + 2C_A C_T \right) ||X||.
\]

Finally we consider \( A \in \Omega^2(B, P) \). This two-form on a Riemannian affine fiber bundle \( f : M \to B \) is characterized by the property \( f^* A = A |_{\mathcal{H} \times H} \). In the following example we see that this tensor can be nonzero while \( Z = 0 \) and \( \nabla^V \) is trivial.

**Example 3.10.** Let \( f : (M, g) \to (B, h) \) be an \( S^1 \)-principal bundle such that \( f \) is a Riemannian submersion with totally geodesic fibers of length \( 2\pi \). Suppose further that the curvature form \( A \) of the \( S^1 \)-principal bundle is nontrivial. Note that for any \( i \) the cyclic subgroup \( \mathbb{Z}_i < S^1 \) acts on \( M \) by isometries. The sequence \( (M/\mathbb{Z}_i, g_i)_{i \in \mathbb{N}} \) converges with bounded sectional curvature and diameter to \( (B, h) \). Here \( g_i \) is the induced quotient metric. In this sequence the fibers are embedded
flat manifolds and the holonomy of the vertical bundle is trivial for all $i$. However, the $A$-tensor of $(M/\mathbb{Z}_i; g_i) \to (B, h)$ is given by

$$A_i(X, Y) = -\frac{1}{2} A(X, Y)V,$$

where $X, Y$ are horizontal vector fields and $V$ is a vertical vector field of unit length. In particular, $A_i = A$ for all $i \in \mathbb{N}$.

A weaker version of the following lemma is proven in [Roo17], where the behavior of Dirac eigenvalues on collapsing sequences, whose limit space is of one dimension less, are studied. The generalization of the $C^1$-bound on $A$ to general collapsing sequences with smooth limit space is proved similar to [Roo17, Lemma 2.7]. For the sake of completeness we give a detailed proof.

**Lemma 3.11.** Let $f : M \to B$ be a Riemannian affine fiber bundle. Then there exists an $A \in \Omega^2(B, P)$ such that $A|_{\mathcal{H} \times \mathcal{H}} = f^* A$. If in addition,

$$\|A\|_\infty \leq C_A,$$

$$\|T\|_\infty \leq C_T,$$

$$\|R^M\|_\infty \leq C_R,$$

then

$$\|A\|_{C^1(B)} \leq C(k, n, C_A, C_T, C_R).$$

**Proof.** As the locally defined left translations by $N$ on $Z$ are isometries $A|_{\mathcal{H} \times \mathcal{H}}$ is affine parallel. Hence, there is an $A \in \Omega^2(B, P)$ such that $A|_{\mathcal{H} \times \mathcal{H}} = f^* A$.

It remains to bound $A$ in $C^1(B)$. We use a split orthonormal frame, see Definition 3.3 and write

$$(f^* A)(X, Y) = A|_{\mathcal{H} \times \mathcal{H}}(X, Y) = \sum_{a=1}^{k} g(A(X, Y), \zeta_a)\zeta_a$$

$$=: \sum_{a=1}^{k} A^a(X, Y)\zeta_a =: \sum_{a=1}^{k} (f^* A^a)(X, Y)\zeta_a.$$ 

Then,

$$\|A\|_{C^1(B)} \leq \sum_{a=1}^{k} (\|A^a\|_\infty + \|\nabla (A^a)\|_\infty)$$

$$\leq k C_A + \sum_{a=1}^{k} \|\nabla (A^a)\|_\infty.$$ 

We calculate the second term pointwise. Let $p \in B$ be arbitrary and $x \in f^{-1}(p)$. Suppose further that $(\xi_1, \ldots, \xi_k)$ is the horizontal lift of a
local orthonormal frame \((\hat{\xi}_1, \ldots, \hat{\xi}_n)\) that is parallel in \(p \in B\). Using (2), we calculate

\[
|\nabla(A^a)|^2 = \frac{1}{2} \sum_{\alpha,\beta,\gamma=1}^n |(\nabla_{\hat{\xi}_\alpha} A^a)(\hat{\xi}_\beta, \hat{\xi}_\gamma)|^2
\]

\[
= \frac{1}{2} \sum_{\alpha,\beta,\gamma=1}^n \left| g\left(\nabla_{\hat{\xi}_\alpha} A(\xi_\beta, \xi_\gamma), \xi_\alpha\right) \right|^2
\]

\[
= \frac{1}{2} \sum_{\alpha,\beta,\gamma=1}^n \left| g\left(R^M(\xi_\beta, \xi_\gamma)\xi_\alpha, \xi_\alpha\right) - g\left(A(\xi_\beta, \xi_\gamma), T(\xi_\alpha, \xi_\gamma)\right) + g\left(A(\xi_\gamma, \xi_\alpha), T(\xi_\alpha, \xi_\beta)\right) + g\left(A(\xi_\alpha, \xi_\beta), T(\xi_\alpha, \xi_\gamma)\right) \right|^2
\]

\[
\leq \frac{1}{2} \sum_{\alpha,\beta,\gamma=1}^n (C_R + 3C_AC_T)^2.
\]

\[\square\]

4. Spin structures on affine fiber bundles

In this section we study Riemannian affine fiber bundles \(f : (M, g) \to (B, h)\) where \((M, g)\) is a spin manifold with a fixed spin structure. First we discuss whether the spin structure on \(M\) induces a structure on \(B\) or on the fibers. The main observation of the first subsection is that there is an induced spin structure on each fiber \(Z_p, p \in B\). Moreover we cannot determine whether there is an induced structure on \(B\).

In the second subsection we derive formulas for the connection and the Dirac operator with respect to a split orthonormal frame, see Definition 3.3. As the affine connection \(\nabla^{aff}\) lifts to the spinor bundle \(\Sigma M\) the notion of affine parallel spinors is well-defined. The Dirac operator leaves the space of affine parallel spinors invariant. The main result of the second subsection is that the space of \(L^2\)-affine parallel spinors is isometric to a twisted Clifford bundle over the base manifold \(B\).

4.1. Induced structures. Let \(f : (M, g) \to (B, h)\) be a Riemannian affine fiber bundle with infranil fiber \(Z\). In the remainder of this section we assume that \(M\) is a spin manifold with a fixed spin structure. Then there is an induced spin structure on any fiber \(Z_p, p \in B\). This follows from the fact that any fiber \(Z_p\) is an embedded oriented submanifold with trivial normal bundle. Furthermore, each path in \(B\) connecting two points \(p, q \in B\) induces an isomorphism between the induced spin structure on \(Z_p\) and the induced spin structure on \(Z_q\).

However, there is in general no induced spin structure on \(B\), as can be seen in Example 4.6. There are even examples of Riemannian affine fiber bundles \(M \to B\) where \(M\) is spin and \(B\) is nonorientable.
Example 4.1. Let $M := U(1) \times \mathbb{Z}_2 S^2$, where $\mathbb{Z}_2$ acts on $U(1)$ via complex conjugation and on $S^2$ via the antipodal map. Then $M$ is spin and $f : M \to \mathbb{R}P^2$ is a non-trivial $S^1$-bundle.

Therefore, we also have to deal with pin$^\pm$ structures. Loosely speaking, pin$^\pm$ structures are a generalization of spin structures to a non-orientable setting. In the following, we briefly sketch the basic definitions and properties of pin$^\pm$-structures. For further details, we refer to [KT90] and [Gil89, Appendix A].

The double cover $\text{Spin}(n) \to \text{SO}(n)$ can be extended to a double cover of $\text{O}(n)$ in two inequivalent ways, called $\rho^+: \text{Pin}^+(n) \to \text{O}(n)$ and $\rho^- : \text{Pin}^-(n) \to \text{O}(n)$. As topological spaces $\text{Pin}^+(n)$ and $\text{Pin}^-(n)$ are both isomorphic to $\text{Spin}(n) \sqcup \text{Spin}(n)$ but the group structure of $\text{Pin}^+(n)$ and $\text{Pin}^-(n)$ is different. To see this, we consider the subgroup $\{\text{Id}, r\} \subset \text{O}(n)$, where $r$ is a reflection along an hyperplane. Then

$$\rho^+ \{\text{Id}, r\} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z};$$
$$\rho^- \{\text{Id}, r\} \cong \mathbb{Z}/4\mathbb{Z}.$$

The notion of pin$^\pm$ structures is an extension of the definition of spin structures to the double covers $\text{Pin}^\pm(n) \to \text{O}(n)$.

Definition 4.2. A pin$^\pm$ structure on an $n$-dimensional Riemannian manifold $(M, g)$ is a Pin$^\pm$-principal bundle $P_{\text{Pin}^\pm}B$ that is a double cover of the orthonormal frame bundle $P_{\text{O}}B$, compatible with the double cover $\text{Pin}^\pm(n) \to \text{O}(n)$.

Example 4.3.

The real projective space $\mathbb{R}P^n$ is

$$\begin{cases} 
\text{pin}^+, & \text{if } n = 4k, \\
\text{pin}^-, & \text{if } n = 4k+2, \\
\text{spin}, & \text{if } n = 4k+3.
\end{cases}$$

Similar to spin structures, the existence of a pin$^\pm$ structure is a topological property characterized by the vanishing of specific Stiefel-Whitney classes. The proof of the following theorem can be found in [KT90, Lemma 1.3].

Theorem 4.4. A manifold $M$ admits a pin$^+$ structure if the second Stiefel-Whitney class $w_2(M)$ vanishes and a pin$^-$ structure if the Stiefel-Whitney classes satisfy the equation $w_2(M) + w_1(M)^2 = 0$. The topological obstruction for a spin structure is the vanishing of $w_2(M)$ and $w_1(M)$.

It is an immediate consequence that an orientable manifold is spin if and only if it has a pin$^\pm$ structure. Equivalently we see that a manifold is spin if and only if it is pin$^+$ and pin$^-$. 
The $i$-th Stiefel-Whitney class $w_i$ of a vector bundle $E \oplus F$ satisfies
\[ w_i(E \oplus F) = \sum_{k=1}^{i} w_i(E) \cup w_{i-k}(F), \]
where $\cup$ is the cup product. Together with the above characterization of the various structures, we conclude the following lemma, see [Gil89, Lemma A.1.5].

**Lemma 4.5.** Let
\[ 0 \to V_1 \to V_2 \to V_3 \to 0 \]
be a short exact sequence of real vector bundles over a manifold $M$. For any permutation $\{i, j, k\}$ of $\{1, 2, 3\}$, we have
1. if $V_i$ and $V_j$ are spin, there is an induced spin structure on $V_k$,
2. If $V_i$ is spin and $V_j$ is pin$^\pm$, there is an induced pin$^{\mp}$ structure on $V_k$,
3. if $V_i$ is pin$^\pm$ and $V_j$ is pin$^\mp$ and $V_k$ is orientable, then there is an induced spin structure on $V_k$.

Let $f: (M, g) \to (B, h)$ be a Riemannian affine fiber bundle and assume that $M$ is spin. We have the following short exact sequence
\[ 0 \to f^*TB \to TM \to V \to 0. \]
Here $V = \ker(df)$ is the vertical distribution. Applying Lemma 4.5 we conclude that $f^*TB$ is spin if and only if $V$ is spin and that $f^*TB$ is pin$^\pm$ if and only if $V$ is pin$^{\mp}$. But a spin or pin$^\pm$ structure on $f^*TB$ does not induce a corresponding structure on $B$ itself as can be seen in the following example.

**Example 4.6.** Consider $S^5 \to \mathbb{C}P^2$. Then $f^*w_2(\mathbb{C}P^2) \in H^2(S^5, \mathbb{Z}_2)$. But $H^2(S^5, \mathbb{Z}_2)$ is trivial. Hence $f^*w_2(\mathbb{C}P^2) = 0$ although $\mathbb{C}P^2$ is not spin.

Nevertheless, if $B$ is spin or pin$^\pm$ then the structure can be pulled back to $f^*TB$. Therefore, if $B$ is pin$^\pm$ and $V$ is pin$^{\mp}$ then there is an induced spin structure on $M$ by Lemma 4.5.

4.2. **Spin structures with affine parallel spinors.** Let $f: (M, g) \to (B, h)$ be a fixed Riemannian affine fiber bundle. We set $n := \dim(B)$ and $k := \dim(Z)$. Suppose that $(M, g)$ has a fixed spin structure. In what follows we want to derive explicit formulas for the connection and the Dirac operator on the spinor bundle $\Sigma M$ with respect to a split orthonormal frame $(\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_k)$, see Definition 3.3. This has the advantage that we can explicitly see how the horizontal and vertical parts are involved. In order to do so, we first discuss spin bundles on Riemannian submersions in detail.
Recall the canonical spin representation $\rho_n : \text{Spin}(n) \to \text{GL}(\Sigma_n)$, where $\Sigma_n$ is a vector space of complex dimension $2^{[\frac{n}{2}]}$. Hence,

$$\dim_{\mathbb{C}}(\Sigma_{n+k}) = \begin{cases} 2 \dim_{\mathbb{C}}(\Sigma_n) \dim_{\mathbb{C}}(\Sigma_k), & \text{if } n \text{ and } k \text{ are odd}, \\ \dim_{\mathbb{C}}(\Sigma_n) \dim_{\mathbb{C}}(\Sigma_k), & \text{else}. \end{cases}$$

If $n$ or $k$ is even, there is a vector space isomorphism

$$\Sigma_{n+k} \cong \Sigma_n \otimes \Sigma_k; \quad (6)$$

Here $\Sigma_n \otimes \Sigma_k$ is to be understood as the tensor product of two complex vector spaces. We discuss the behavior of the Clifford multiplication under this isomorphism later in this section.

Counting dimensions, it follows that such an isomorphism cannot exist if $n$ and $k$ are both odd. In that case, we proceed as follows:

Here and subsequently, $\gamma : \mathbb{R}^{n+k} \to \text{GL}(\Sigma_{n+k})$ denotes the Clifford multiplication. Let $(e_1, \ldots, e_{n+k})$ be the standard basis of $\mathbb{R}^{n+k}$. We define

$$\omega^C_n := i^{\left[\frac{n+1}{2}\right]} \gamma(e_1) \cdots \gamma(e_n).$$

It follows easily that $(\omega^C_n)^2 = \text{Id}$. Therefore, the action of $\omega^C_n$ decomposes $\Sigma_{n+k}$ into the two eigenspaces $\Sigma_{n+k}^+$ and $\Sigma_{n+k}^-$ with respect to the eigenvalues $\pm 1$. The operator

$$\omega^C_k := i^{\left[\frac{k+1}{2}\right]} \gamma(e_{n+1}) \cdots \gamma(e_{n+k})$$

anticommutes with $\omega^C_n$. Hence, the action of $\omega^C_k$ defines an involution

$$\omega^C_k : \Sigma_{n+k}^\pm \to \Sigma_{n+k}^{\mp}.$$

As an immediate consequence, we conclude that

$$\dim_{\mathbb{C}}(\Sigma_{n+k}^\pm) = \frac{1}{2} \dim_{\mathbb{C}}(\Sigma_{n+k}) = \dim_{\mathbb{C}}(\Sigma_n) \dim_{\mathbb{C}}(\Sigma_k).$$

Therefore, we obtain that

$$\Sigma_{n+k} = \Sigma_{n+k}^+ \oplus \Sigma_{n+k}^-; \quad (7)$$

if $n$ and $k$ are odd.

Next we want to determine how Clifford multiplication with vectors in $\mathbb{R}^{n+k}$ are “separated” into Clifford multiplications on $\Sigma_n$ and $\Sigma_k$. For all natural numbers $p$ and $q$ the Clifford algebra $\text{Cl}(p+q)$ is canonical isomorphic to the graded tensor product $\text{Cl}(p) \hat{\otimes} \text{Cl}(q)$, endowed with the multiplication

$$(a \hat{\otimes} \varphi) \cdot (b \hat{\otimes} \psi) = (-1)^{\deg(\varphi) \deg(b)} (a \cdot b) \hat{\otimes} (\varphi \cdot \psi).$$

It is a general fact that the complex vector space $\Sigma_p$ is a $\mathbb{Z}/2\mathbb{Z}$-graded $\text{Cl}(p)$ space if and only if $p$ is even.
If $n$ or $k$ is even, the multiplication of the graded tensor product $\mathcal{Cl}(n) \otimes \mathcal{Cl}(k)$ carries over to $\Sigma^+_n \otimes \Sigma^+_k$. In the remaining case, $n$ and $k$ odd, we recall that $\Sigma_{n+k} = \Sigma^+_n \oplus \Sigma^-_{n+k}$ together with the involution $\omega^C_n : \Sigma^+_n \rightarrow \Sigma^-_{n+k}$. Since $\omega^C_n$ anticommutes with $\omega^C_k$ it follows that Clifford multiplication with vectors $v \in \text{Span}\{e_1, \ldots, e_n\}$ acts as $\gamma(v)$ on $\Sigma^+_n$ and as $-\gamma(v)$ on $\Sigma^-_{n+k}$. On the other hand, Clifford multiplication with any vector in $\text{Span}\{e_{n+1}, \ldots, e_{n+k}\}$ interchanges the eigenspaces $\Sigma^+_{n+k}$ and commutes with $\omega^C_k$. Using the isomorphisms (6) and (7) combined with the above discussion, we obtain the following identifications for Clifford multiplication with vectors $(x, v) \in \mathbb{R}^n \times \mathbb{R}^k$:

\[
\gamma((x, v)) (\psi \otimes \nu) \approx \begin{cases} 
(\gamma(x) \psi) \otimes \nu + \bar{\psi} \otimes (\gamma(v) \nu), & \text{if } n \text{ is even,} \\
(\gamma(x) \psi) \otimes \bar{\nu} + \psi \otimes (\gamma(v) \nu), & \text{if } k \text{ is even,} \\
(\gamma(x) \psi^+ \otimes -\gamma(x) \psi^-) \otimes \nu + (\psi^- \otimes \psi^+) \otimes (\gamma(v) \nu), & \text{if } n \text{ and } k \text{ are odd.}
\end{cases}
\]

Here $\bar{\psi}$ means the complex conjugation with respect to the $\mathbb{Z}_2$ grading on $\Sigma_p$, whenever $p$ is even. In the case, where $n$ and $k$ are even, both possibilities are isomorphic to each other.

We return to the case of a Riemannian affine fiber bundle $f : M \rightarrow B$ where $M$ is a spin manifold with a fixed spin structure. Applying the above discussion pointwise we conclude that

\[
\Sigma M \cong \begin{cases} 
f^*(\Sigma B) \otimes \Sigma \mathcal{V}, & \text{if } n \text{ or } k \text{ is even,} \\
(f^*(\Sigma^+ B) \oplus f^*(\Sigma^- B)) \otimes \Sigma \mathcal{V}, & \text{if } n \text{ and } k \text{ are odd.}
\end{cases}
\]

As discussed in the previous section, in general the base manifold $B$ and the vertical distribution $\mathcal{V}$ are not spin. The spinor bundles $\Sigma B$ and $\Sigma \mathcal{V}$ are only defined locally. Nevertheless their tensor product is globally well-defined. The rules for Clifford multiplication (8) carries over and distinguishes between the action of horizontal and vertical vector fields. For simplicity we write

\[
\Sigma \mathcal{B} := \begin{cases} 
\Sigma B, & \text{if } n \text{ or } k \text{ is even,} \\
\Sigma^+ B \oplus \Sigma^- B, & \text{if } n \text{ and } k \text{ are odd.}
\end{cases}
\]

Next we calculate the spinorial connection on $\Sigma M$ with respect to a split orthonormal frame $(\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_k)$. The spinorial connection $\nabla^M$ on $M$ is given by

\[
\nabla^M_X \Phi = X(\Phi) + \frac{1}{4} \sum_{i,j=1}^{n+k} g(\nabla_X e_i, e_j) \gamma(e_i) \gamma(e_j) \Phi,
\]

for any vector field $X$ and spinor $\Phi$, where $(e_1, \ldots, e_n)$ is a local orthonormal frame. By (9), any spinor $\Phi$ on $M$ can be locally written
as a linear combination $\Phi = \sum_i f^i \phi_i \otimes \nu_i$. Straightforward calculations lead to the following formulas for the spinorial connection and the Dirac operator.

**Lemma 4.7.** Let $f : (M, g) \to (B, h)$ be a Riemannian affine fiber bundle. Suppose that $M$ is a spin manifold with a fixed spin structure. With respect to a split orthonormal frame $(\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_k)$, any spinor $\Phi = f^* \phi \otimes \nu$ satisfies the following identities.

$$\nabla^M_{\xi_\alpha} \Phi = (f^* \nabla^B_{\xi_\alpha} \varphi) \otimes \nu + f^* \varphi \otimes \nabla^V_{\xi_\alpha} \nu + \frac{1}{2} \sum_{\beta=1}^n \gamma(\xi_\beta) \gamma(A(\xi_\alpha, \xi_\beta)) \Phi$$

$$\nabla^M_{\zeta_a} \Phi = f^* \varphi \otimes \nabla^Z_{\zeta_a} \nu + \frac{1}{2} \sum_{b=1}^k \gamma(\zeta_b) \gamma(T(\zeta_a, \zeta_b)) \Phi + \frac{1}{4} \sum_{\alpha=1}^n \gamma(\xi_\alpha) \gamma(A(\xi_\alpha, \zeta_a)) \Phi$$

$$D^M \Phi = \sum_{\alpha=1}^n \gamma(\xi_\alpha) \nabla^T_{\xi_\alpha} \Phi + \sum_{a=1}^k \gamma(\zeta_a) \nabla^Z_{\zeta_a} \Phi - \frac{1}{2} \sum_{a=1}^k \gamma(T(\zeta_a, \zeta_a)) \Phi$$

$$+ \frac{1}{2} \sum_{\alpha, \beta=1}^n \gamma(A(\xi_\alpha, \xi_\beta)) \gamma(\xi_\alpha) \gamma(\xi_\beta) \Phi$$

$$=: D^T \Phi + D^Z \Phi - \frac{1}{2} \sum_{a=1}^k \gamma(T(\zeta_a, \zeta_a)) \Phi + \frac{1}{2} \gamma(A) \Phi.$$

Here $\nabla^B$, $\nabla^V$ and $\nabla^Z$ are the induced connections by the respective connections on $TM$, defined in Section 3, and $D^T, D^Z$ the associated Dirac operators.

From the discussion in Section 2, it follows that the affine connection $\nabla^\text{aff}$ lifts to $\Sigma M$. The space of affine parallel spinors is defined as

$$S^\text{aff} := \{ \Phi \in L^2(\Sigma M) : \nabla^\text{aff} \Phi = 0 \}.$$
Lemma 4.8. Let $f : M \to B$ be a Riemannian affine fiber bundle such that $M$ is spin. Then there is an isometry

$$Q : L^2(\Sigma B \otimes \mathcal{P}) \to S^{\text{aff}}.$$ 

In addition, taking a split orthonormal frame around $x$,

$$\nabla^M_{\xi_\alpha} Q(\Phi) = Q(\nabla^T \Phi) + \frac{1}{2} \sum_{\beta=1}^n \gamma(\xi_\beta) \gamma(A(\xi_\alpha, \xi_\beta))Q(\Phi) - \frac{\xi_\alpha(\text{vol}(Z_{f(x)}))}{2 \text{vol}(Z_{f(x)})}Q(\tilde{\Phi}).$$

Proof. First we recall that $\Sigma M \cong f^*(\Sigma B) \otimes \Sigma V$, where $\Sigma B$ and $\Sigma V$ are in general only defined locally. Let $f^{-1}(U) \cong U \times Z$ be a trivializing neighborhood on which $\Sigma B$ and $\Sigma V$ exist. Then any spinor $\Phi$ restricted to $f^{-1}(U)$ can be written as $\Phi = \sum_j f^* \varphi_j \otimes \nu_j$. By linearity it suffices to consider elementary tensors $f^* \varphi \otimes \nu$. Observe that

$$\nabla^{\text{aff}}(f^* \varphi \otimes \nu) = f^* \varphi \otimes \nabla^{\text{aff}} \nu.$$ 

Thus, a spinor is affine parallel if and only if $\nu$, restricted to any fiber, is an affine parallel spinor of $Z_f(x)$. Hence, we obtain on any trivializing neighborhood, an isomorphism

$$Q^{-1} : S^{\text{aff}} \to L^2(\Sigma B \otimes \mathcal{P})$$

$$(f^* \varphi \otimes \nu)_x \mapsto \sqrt{\text{vol}(Z_{f(x)})} \varphi_{f(x)} \otimes \nu_{f(x)},$$

where $\nu_{f(x)}$ denotes the restriction of $\nu$ to the fiber $Z_{f(x)}$. Due to the factor $\sqrt{\text{vol}(Z_{f(x)})}$ it is evident that $Q^{-1}$ defines an isometry. By the construction of the vector bundle $\mathcal{P}$ it follows that $Q^{-1}$ extends to a well-defined global isometrie. Taking its inverse gives the desired map.

It remains to show the identity for the spinorial connection. By the discussion in Section 3 it follows that $S^{\text{aff}}$ is invariant under the action of $\nabla^T_X$, for any horizontal vector field $X$. Using Lemma 4.7 the explicit formula follows from a straightforward calculation.

Corollary 4.9. Let $f : M \to B$ be a Riemannian affine fiber bundle such that $S^{\text{aff}}$ is not empty. With respect to a split orthonormal frame $(\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_k)$, any spinor $\Phi \in S^{\text{aff}}$ satisfies

$$D^M \Phi = Q \circ \tilde{D}^T \circ Q^{-1} \Phi + \frac{1}{2} \sum_{a,b,c=1}^k \gamma(\zeta_a) \gamma(\zeta_b) \gamma(\zeta_c) \Gamma_{ab}^c \Phi + \frac{1}{2} \gamma(A) \Phi$$

$$=: Q \circ \tilde{D}^T \circ Q^{-1} \Phi + \frac{1}{2} \gamma(Z) \Phi + \frac{1}{2} \gamma(A) \Phi.$$ 

Here $\tilde{D}^T$ is the Dirac operator on $\Sigma B \otimes \mathcal{P}$ associated to the connection $\nabla^T$.

Proof. Let $\Phi$ be an affine parallel spinor. It is easily seen that $S^{\text{aff}}$ is invariant under the action of the Dirac operator $D^M$. With respect to
a split orthonormal frame \((\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_k)\), see Definition \([X3]\) we obtain
\[
D^M \Phi = \sum_{\alpha=1}^n \gamma(\xi_\alpha) \nabla_{\xi_\alpha} \Phi + \sum_{i=1}^k \gamma(\zeta_i) \nabla_{\zeta_i} \Phi
\]
\[
= \sum_{\alpha=1}^n \gamma(\xi_\alpha) \nabla_{\xi_\alpha} \left( Q \circ Q^{-1}(\Phi) \right) + \frac{1}{4} \sum_{a=1}^k \gamma(\zeta_a) \left( \sum_{b,c=1}^k \Gamma_{ab}^c \gamma(\xi_b) \gamma(\zeta_c) \Phi \right)
\]
\[
- \frac{1}{2} \sum_{a=1}^k \gamma(T(\xi_a, \zeta_a)) \Phi + \frac{1}{4} \sum_{a=1}^k \sum_{\alpha=1}^n \gamma(\zeta_a) \gamma(\xi_\alpha) \gamma(A(\xi_\alpha, \zeta_a)) \Phi
\]
\[
= Q \circ \tilde{D}^T \circ Q^{-1} \Phi - \frac{1}{2} \gamma \left( \frac{\text{grad}(\text{vol}(Z_f(x)))}{\text{vol}(Z_f(x))} \right) \Phi
\]
\[
+ \frac{1}{2} \sum_{a,b,c=1}^k \gamma(\zeta_a) \gamma(\zeta_b) \gamma(\zeta_c) \Gamma_{ab}^c \Phi + \frac{1}{2} \gamma(A) \Phi - \frac{1}{2} \sum_{a=1}^k \gamma(T(\xi_a, \zeta_a)) \Phi
\]
\[
= Q \circ \tilde{D}^T \circ Q^{-1} \Phi + \frac{1}{2} \gamma(Z) \Phi + \frac{1}{2} \gamma(A) \Phi.
\]

Here we used Lemma \([4,7]\) Lemma \([4,8]\) and the formulas for the Christoffel symbols, see \([2]\). The last line follows from the fact (see for instance \([GLP99, \text{Lemma } 1.17.2]\)) that
\[
\sum_{i=1}^k T(\zeta_i, \zeta_i) = - \text{grad}(\text{ln}(\text{vol}(Z_p))).
\]

\[\square\]

5. THE DIRAC OPERATOR ON COLLAPSING AFFINE FIBER BUNDLES

Let \((M_i, g_i)_{i \in \mathbb{N}}\) be a collapsing sequence of Riemannian spin manifolds in \(\mathcal{M}(n + k, d)\) converging to a Riemannian manifold \((B, h)\). Let \(S^\text{aff}_i \subset L^2(\Sigma M_i)\) be the subspace of affine parallel spinors. For each \(i \in \mathbb{N}\) we have the splitting
\[
L^2(\Sigma M_i) = S^\text{aff}_i \oplus (S^\text{aff}_i)^\perp,
\]
which is invariant under the action of the Dirac operator \(D^{M_i}\). Combining the Schrödinger Lichnerowicz formula
\[
D^2 = \nabla^* \nabla + \text{scal}
\]
with the results of \([Lot02a, Lot02b]\), Lott showed that any eigenvalue \(\lambda_i\) of \(D^{M_i}_{|S^\text{aff}_i}^\perp\) goes to \(\pm \infty\) as \(i\) tends to \(\infty\), \([Lot02a, \text{Theorem } 2]\). Furthermore, the spectrum of \(D^{M_i}_{|S^\text{aff}_i}^\perp\) converges to an elliptic first order differential operator \(\mathcal{D} = \sqrt{\Delta + V}\) on \(\Sigma B \otimes \mathcal{P}\). Here \(\Delta\) denotes the Laplacian of \(L^2(\Sigma B \otimes \mathcal{P}, \chi \text{dvol}^{B})\), where \(\chi \text{dvol}\) is the weak-* limit of
the push-forwards of the normalized Riemannian measures on $M_i$ and $V$ is a symmetric potential that arises as a weak-$\ast$ limit of $(\text{scal}_{M_i})_{i \in \mathbb{N}}$.

In the next theorem, we give an explicit description of the limit operator $\mathcal{D}$ without using the Schrödinger-Lichnerowicz formula. Furthermore, we show that $\mathcal{D}$ is an elliptic first order self-adjoint operator with respect to the standard measure. In the following we use the notation introduced in Section 3 and Section 4.2.

**Theorem 5.1.** Let $(M_i, g_i)_{i \in \mathbb{N}}$ be a sequence of spin manifolds in $\mathcal{M}(n+k,d)$ that converges to a smooth $n$-dimensional Riemannian manifold $(B,h)$. Suppose further that for almost all $i \in \mathbb{N}$ the space $S_{\text{aff}}(i)$ of affine parallel spinors on $M_i$ is nontrivial. Then there is a subsequence $(M_i, g_i)_{i \in \mathbb{N}}$ such that the spectrum of $D_{|S_{\text{aff}}(i)}$ converges uniformly with respect to the arsinh-topology to the spectrum of the elliptic self-adjoint first order differential operator

$$\mathcal{D} : L^2(\Sigma B \otimes \mathcal{P}) \to L^2(\Sigma B \otimes \mathcal{P})$$

$$\Phi \mapsto \mathcal{D}^T \Phi + \frac{1}{2} \gamma(\mathcal{Z}_{\infty}) \Phi + \frac{1}{2} \gamma(\mathcal{A}_{\infty}) \Phi,$$

where

$$\Sigma B := \begin{cases} 
\Sigma B, & \text{if } n \text{ or } k \text{ is even,} \\
\Sigma^+ B \oplus \Sigma^- B, & \text{if } n \text{ and } k \text{ are odd.}
\end{cases}$$

Further,

1. $\mathcal{P}$ represents the affine parallel spinors of the fibers $Z_i$,
2. $\mathcal{D}^T$ is the Dirac operator on $\Sigma B \otimes \mathcal{P}$ with respect to the connection $\nabla^T = \nabla^B \otimes \nabla^V$, where $\nabla^B$ is the spinorial connection and $\nabla^V$ is the limit of $\nabla^V_i$,
3. $\mathcal{Z}_{\infty}$ is the limit of $Z_i = \nabla_{Z_i} - \nabla_{\text{aff}}$,
4. $\mathcal{A}_{\infty}$ is the limit of $(A_i)_{i \in \mathbb{N}}$.

In particular, $\mathcal{D}$ is self-adjoint with respect to the standard measure.

**Proof.** First we apply Theorem 3.2 to obtain a sequence $f_i : (M_i, g_i) \to (B, h_i)$ of Riemannian affine fiber bundles. Since $\lim_{i \to \infty} \|g_i - \tilde{g}_i\|_{C^1} = 0$, it follows from Theorem A.4 that the spectra of the Dirac operator $\mathcal{D}_i$ on $(M_i, g_i)$ and $\tilde{\mathcal{D}}_i$ on $(M_i, \tilde{g}_i)$ are close in the arsinh-topology, i.e.

$$d_a(\sigma(D_i), \sigma(\tilde{D}_i)) \leq C\|g_i - \tilde{g}_i\|_{C^1}. \quad (10)$$

Let $S_{\text{aff}}(i)$ be the space of affine parallel spinors in $L^2(\Sigma M_i) (L^2(\Sigma M_i))$. It was shown in [Lot02a, Theorem 2 and 3] that $\sigma(D_i|_{S_{\text{aff}}(i)})$ has a well-defined limit as $i$ goes to infinity. We conclude from (10) that the spectra $\sigma(D_i|_{S_{\text{aff}}(i)})$ and $\sigma(\tilde{D}_i|_{S_{\text{aff}}(i)})$ have the same limit. Thus, it remains to prove the statement for the sequence $f_i : (M_i, g_i) \to (B, \tilde{h}_i)$ of Riemannian affine fiber bundles. In this setting we can apply Corollary
4.9 to obtain

$$\tilde{D}_i = Q \circ \tilde{D}_i^T \circ Q^{-1} + \frac{1}{2} \gamma(\tilde{Z}_i) + \frac{1}{2} \gamma(\tilde{A}_i).$$

From the discussion in Section 3 it follows that $\gamma(\tilde{Z}_i)$ and $\gamma(\tilde{A}_i)$ act diagonally with respect to the splitting $L^2(\Sigma M_i) = \tilde{S}_i^{aff} \oplus (\tilde{S}_i^{aff})^\perp$. Thus, there are well-defined operators $\tilde{Z}_i$ and $\tilde{A}_i$ such that

$$\tilde{D}_i|_{\tilde{S}_i^{aff}} = Q \circ \left( \tilde{D}_i^T + \frac{1}{2} \gamma(\tilde{Z}_i) + \frac{1}{2} \gamma(\tilde{A}_i) \right) \circ Q^{-1}.$$

Here $\tilde{D}_i^T$ is the spinorial connection of $\Sigma_i B$, the spinor bundle of $(B, \tilde{h}_i)$ and $\tilde{D}_i^T$ is the induced connection of $\tilde{V}_i$. Since $Q : L^2(\Sigma_i B \otimes \mathcal{P}_i) \to \tilde{S}_i^{aff}$ is an isometry, it follows that $\tilde{D}_i|_{\tilde{S}_i^{aff}}$ is isospectral to an operator acting on a dense subspace of $L^2(\Sigma_i B \otimes \mathcal{P}_i)$.

Now, we are going to extract a subsequence such that the spectrum $\sigma(\tilde{D}_i^T + \frac{1}{2} \gamma(\tilde{Z}_i) + \frac{1}{2} \gamma(\tilde{A}_i))$ converges as claimed in the theorem. By abuse of notation, we will use the same index $i$ for any subsequence we extract. The following identifications are based on the constructions in [Lot02a, Section 4].

Let $\tilde{P}_i$ be the Spin$(n + k)$-principal bundle of $(M_i, \tilde{g}_i)$. There exists an isomorphism

$$\left( L^2(\tilde{P}_i) \otimes \Sigma_{n+k} \right)^{Spin(n+k)} \cong L^2(\Sigma M_i),$$

where the right-hand side denotes the Spin$(n + k)$-invariant subspace of $L^2(\tilde{P}_i) \otimes \Sigma_{n+k}$. Applying the $G$-equivariant version of Gromov’s compactness theorem (see for instance [Fuk88, Lemma 1.13]) we obtain a subsequence such that the sequence $(\tilde{P}_i)_{i \in \mathbb{N}}$ converges to a Riemannian manifold $(B', \tilde{h}')$ in the Spin$(n + k)$-equivariant Gromov-Hausdorff distance. In particular, $(B', \tilde{h}')/Spin(n + k)$ is isometric to $(B, h)$. Furthermore, there is a sequence $(h'_i)_{i \in \mathbb{N}}$ of Riemannian metrics on $B'$ such that $\tilde{P}_i \to (B', h'_i)$ is a Spin$(n + k)$-equivariant Riemannian affine fiber bundle and $(B', h'_i)/Spin(n + k)$ is isometric to $(B, \tilde{h}_i)$, i.e.

$$\tilde{P}_i \quad \longrightarrow \quad (B', h'_i) \quad \longrightarrow \quad (M_i, \tilde{g}_i) \quad \longrightarrow \quad (B, \tilde{h}_i).$$

Next we define $L^2(B', h'_i) \otimes \Sigma_{n+k}$ to be the subspace of $L^2(\tilde{P}_i) \otimes \Sigma_{n+k}$ consisting of functions that are constant along the fibers of $\tilde{P}_i \to$


(B', h'_i). It follows immediately that

\[ S_{i}^{\text{aff}} \cong L^2(\Sigma_i \mathcal{B} \otimes \mathcal{P}_i) \cong (L^2(\mathcal{B}', h'_i) \otimes \Sigma_{n+k})^{\text{Spin}(n+k)}. \]

Since \( \Sigma_{n+k} \) is a fixed complex vector space, the spaces \( L^2(\Sigma_i \mathcal{B} \otimes \mathcal{P}_i) \) are pairwise isometric to each other. In particular, there is a Clifford bundle \( \Sigma \mathcal{B} \otimes \mathcal{P} \) such that \( \Sigma \mathcal{B} \) is the locally defined spinor bundle of \( (\mathcal{B}, h) \), and isometries

\[ \Theta_i : L^2(\Sigma \mathcal{B} \otimes \mathcal{P}) \to L^2(\Sigma_i \mathcal{B} \otimes \mathcal{P}_i). \]

It follows from Theorem A.1 that the operators

\[ D_i := \Theta_i^{-1} \circ \left( D_i^T + \frac{1}{2} \gamma(\tilde{Z}_i) + \frac{1}{2} \gamma(\tilde{A}_i) \right) \circ \Theta_i \]

are closed and densely defined on \( H^1(\Sigma \mathcal{B} \otimes \mathcal{P}) \). We continue to write \( D_i^T, \gamma(\tilde{Z}_i) \) and \( \gamma(\tilde{A}_i) \) for the corresponding operators acting on the sections of \( \Sigma \mathcal{B} \otimes \mathcal{P} \).

Combining Lemma 3.4 with Lemma 3.5, Lemma 3.9 and Lemma 3.11, it follows that the connection one-forms of \( \nabla_i^T \) and the zeroth order operators \( \gamma(\tilde{Z}_i) \) and \( \gamma(\tilde{A}_i) \) are uniformly bounded in \( C^1(\mathcal{B}) \). Since \( C^{0,\alpha} \hookrightarrow C^1 \) is a compact embedding for \( \alpha \in [0,1) \), there is a further subsequence such that \( (D_i)_{i \in \mathbb{N}} \) defines a convergent family of operators satisfying the assumptions of Theorem A.4. In particular, it follows that \( \sigma(D_i) \) converges to \( \sigma(D) \) uniformly in the arsinh-topology. □

From the above theorem, we can now discuss under which assumptions the limit operator \( D \) corresponds to the Dirac operator on \( \Sigma \mathcal{B} \).

**Corollary 5.2.** Let \((M_i, g_i)_{i \in \mathbb{N}}\) be a sequence of spin manifolds in \( \mathcal{M}(n + k, d) \) that converges to a smooth \( n \)-dimensional Riemannian manifold \( \mathcal{B} \) such that

\[
\limsup_{i \to \infty} \| \text{Hol}(V_i) - \text{Id} \|_\infty = 0,
\]

\[
\limsup_{i \to \infty} \left( \sup_{p \in \mathcal{B}} \| \text{scal}(Z_p^i) \|_\infty \right) = 0,
\]

\[
\limsup_{i \to \infty} \| A_i \|_\infty = 0,
\]

then there is a subsequence such that there is an induced spin structure on \( \mathcal{B} \) and the spectrum of the Dirac operator \( D_{i \in \mathcal{S}^{\text{aff}}} \) converges to the spectrum of \( D^\mathcal{B} \) if \( n \) or \( k \) is even, and to the spectrum of \( D^\mathcal{B} \oplus -D^\mathcal{B} \) if \( n \) and \( k \) are odd. Each eigenvalue is counted \( \text{rank}(\mathcal{P}) \)-times.

**Proof.** From the above theorem we conclude that \( D \) equals \( D^\mathcal{B} \), resp. \( D^\mathcal{B} \oplus -D^\mathcal{B} \) if

1. \( V \) is trivial and \( \nabla^V_\infty \) is the trivial connection,
2. \( Z_\infty = 0 \),
3. \( A_\infty = 0 \).
Regarding the first point, it is known that this condition is equivalent to the statement that the holonomy is trivial. In this case it follows that $P$ is a trivial vector bundle. Thus, there is an induced spin structure on $B$. Therefore, we do not have to assume that $V_i$ is spin for almost all $i$.

Since $\|Z_i\| \leq 3\|\text{scal}Z_i\|$, see Lemma \ref{lem:3.9}, the second condition implies that the limit $Z_\infty$ is identical zero.

Finally, it is immediate that $A_\infty = 0$ is equivalent to the third stated condition $\limsup_i \|A_i\| = 0$ since $\|A_i\| = \|A_i\|$.

For the last statement, let $l := \text{rank}(P)$ and $\varphi$ be an eigenspinor of $D^B$, resp. $D^B \oplus -D^B$, and $(\rho_1, \ldots, \rho_l)$ be a global frame of $P$. Then $\varphi \otimes \rho_1, \ldots, \varphi \otimes \rho_l$ are eigenspinors for $D$ with the same eigenvalue. Therefore, each eigenvalue is counted $l$-times. Furthermore, any spinor in $\Sigma B \otimes P$ can be written as

$$\Phi = \sum_{i=1}^l \varphi_i \otimes \rho_i.$$

In conclusion, there are three geometric obstructions for a convergence to the Dirac operator on the base space. As discussed in the examples of Section \ref{sec:3} these geometric obstructions are all independent from each other.

At this point we want to remark that the results of \cite{Lot02a} also include the Dirac operator on differential forms, i.e. the operator $D = d + d^* : \Omega^*(M) \to \Omega^*(M)$.

In contrast to this our refined knowledge can not be extended to the Dirac operator on differential forms. This can be seen in the following example, which was also discussed in \cite[Example 6.3]{Roo17}.

**Example 5.3.** Consider the torus $T^2 = \{(e^{is}, e^{it}) : s, t \in \mathbb{R}\}$ with the Riemannian metric

$$g_\varepsilon := ds^2 + \varepsilon^2 c(s)^2 dt^2,$$

for some fixed positive function $c : S^1 \to \mathbb{R}_+$. Then $\lim_{\varepsilon \to 0}(T^2, g_\varepsilon) = (S^1, ds)$. Note that the integrability tensor $A_\varepsilon$ is $0$ for all $\varepsilon$ but the $T$-tensor is characterized by $\frac{c'(s)}{c(s)}$.

We endow $(T^2, g_\varepsilon)$ with the spin structure induced by the pullback of a chosen spin structure on $S^1$. By Theorem \ref{thm:5.1} the spectrum of the Dirac operator $D_\varepsilon$ on $(T^2, g_\varepsilon)$ restricted to the $S^1$-invariant spinors converges to the spectrum of the Dirac operator $D^{S^1}$ on $S^1$.

Next we take a look at the Dirac operator on forms. In that case the space of affine parallel forms is given by

$$S^{\text{aff}} := \{f \in C^\infty(T^2) : \frac{\partial}{\partial t} f = 0\} \cup \{\alpha ds \in \Omega^1(T^2) : \frac{\partial}{\partial t} \alpha = 0\}.$$
The Dirac operator $D_\varepsilon = d + d^*$ on forms acts on $(f + \alpha ds) \in \mathcal{S}^{aff}$ as follows:

$$D_\varepsilon(f(s) + \alpha(s)ds) = \frac{\partial}{\partial s}f(s)ds - c(s)^{-1}\frac{\partial}{\partial s}(c(s)\alpha(s)).$$

We observe that $(D_\varepsilon)|_{\mathcal{S}^{aff}}$ is independent of $\varepsilon$. In particular, as $\varepsilon$ goes to zero, $(D_\varepsilon)|_{\mathcal{S}^{aff}}$ converges to a first order differential operator $D_0$ on $\Omega^*(S^1)$. The eigenvalue problem for $D_0$ reduces to the following eigenvalue problem:

$$\lambda^2 f(s) = -c(s)^{-1}\frac{\partial}{\partial s}\left(c(s)\frac{\partial}{\partial s}f\right). \quad (11)$$

For a generic choice of $c(s)$ we obtain a different spectrum than for the classical Dirac operator $D^{S^1}$ on $S^1$. For example, if $c(s) = e^{\cos(s)}$ then one can calculate numerically, adapting the program given at the first pages of [Str], that the first eigenvalues are approximately given by

$$\lambda_0 = 0, \lambda_1 = 0, 918, \lambda_2 = 1, 293.$$ 

In particular, $\lambda_1, \lambda_2 \neq 1$.

We can do a similar calculation for the Laplacian acting on functions and obtain that

$$\Delta_\varepsilon f(s) = -c(s)^{-1}\frac{\partial}{\partial s}\left(c(s)\frac{\partial}{\partial s}f\right).$$

Since this equation is independent of $\varepsilon$ it converges to a second order differential operator on $S^1$. Furthermore, we obtain the same eigenvalue problem as in (11). In particular, the spectrum of $\Delta_0$ differs from the spectrum of $\Delta^{S^1} = -\frac{\partial^2}{\partial s^2}$.

**APPENDIX A. CONTINUITY OF DIRAC SPECTRA**

In Section 3 we observed that any collapsing sequence $(M_i, g_i)_{i \in \mathbb{N}}$ in $\mathcal{M}(n + k, d)$ with smooth limit space $(B, h)$ can be approximated by a sequence of Riemannian affine fiber bundles $f_i : (M_i, g_i) \to (B, h_i)$ such that

$$\lim_{i \to \infty} \|\tilde{g}_i - g_i\|_{C^1} = \lim_{i \to \infty} \|h_i - h\|_{C^1} = 0.$$ 

Since we are interested in Dirac eigenvalues, we have to verify that the Dirac spectrum is continuous with respect to a $C^1$-change of metric. Although this behavior is well-known we want to discuss this topic in more detail following [Now13]. Furthermore we formulate an explicit consequence of [Now13, Theorem 4.10] that is crucial for the proof of our main theorem. Let $M$ be an $n$-dimensional spin manifold with a fixed topological spin structure. A topological spin structure is a double covering $P_{\text{GL}^+}(M) \to P_{\text{GL}^+}(M)$ of the $\text{GL}^+_n$-principal bundle $P_{\text{GL}^+}(M)$ consisting of oriented frames that is compatible with the
group double covering $\tilde{GL}_+(n) \to GL_+(n)$. Let $\mathcal{R}(M)$ denote the space of all Riemannian metrics on $M$ endowed with the $C^1$-topology. For any $g \in \mathcal{R}(M)$, the topological spin structure restricts to a spin structure of $(M, g)$. Let $\Sigma_g M$ be the spinor bundle of $(M, g)$. The relation between the spinor bundles of different metrics was studied by Bourguignon and Gauduchon [BG92]. We summarize one of their results in the next theorem.

**Theorem A.1.** Let $g \in \mathcal{R}(M)$ be a fixed metric. For every $h \in \mathcal{R}(M)$, there exists an isometry of Hilbert spaces

$$\Theta_h : L^2(\Sigma_g M) \to L^2(\Sigma_h M),$$

such that the operator

$$\Theta_h^{-1} \circ D^h \circ \Theta_h : L^2(\Sigma_g M) \to L^2(\Sigma_g M)$$

is closed, densely defined on $H^1(\Sigma_g M)$, and isospectral to $D^h$. Furthermore, the map

$$\mathcal{R}(M) \to B(H^1(\Sigma_g M), L^2(\Sigma_g M))$$

$$h \mapsto \Theta_h^{-1} \circ D^h \circ \Theta_h$$

is continuous. Here, $B(., .)$ denotes the space of bounded linear operators endowed with the operator norm.

Following an idea of Lott, [Lot02a], Nowaczyk studied the continuity of the eigenvalues with respect to the arsinh-topology.

**Definition A.2.** On $\mathbb{R}^\mathbb{Z}$, let the metric $d_a$ be defined by

$$d_a(u, v) := \sup_{j \in \mathbb{Z}} | \text{arsinh}(u(j)) - \text{arsinh}(v(j)) |$$

for all $u, v \in \mathbb{R}^\mathbb{Z}$. The topology induced by $d_a$ is called the arsinh-topology.

In this setting, the continuity result for Dirac eigenvalues of [Now13] reads as

**Theorem A.3** ([Now13, Main Theorem 2]). Let $M$ be a spin manifold with a fixed topological spin structure. There exists a family of functions $(\lambda_j \in C^0(\mathcal{R}(M), \mathbb{R}))_{j \in \mathbb{Z}}$ such that the sequence $(\lambda_j(g))_{j \in \mathbb{Z}}$ are the eigenvalues of $D^g$. In addition, the sequence $\text{arsinh}(\lambda_j)$ of $d_a$ is equicontinuous and non-decreasing.

In fact, the above theorem is a restriction of a far more general continuity statement proven in [Now13, Section 4].

**Theorem A.4.** Let $H$ be an Hilbert space, The spectrum of a family of unbounded self-adjoint operators $T : E \to C(H)$ is continuous in the arsinh-topology if

1. there exists a dense subspace $Z \subset H$, such that $\text{dom} T_e = Z$ for all $e \in E$. 

there exists a norm $|.|$ on $Z$ such that $T_e : Z \to H$ is bounded and the graph norm of $T_e$ is equivalent to $|.|$.
(3) $E$ is a topological space,
(4) the map $E \to B(Z,H), e \mapsto T_e$ is continuous.

By Theorem A.1 all the conditions of the above theorem hold for Dirac operators.

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