POINTWISE WEYL LAWS FOR SCHRÖDINGER OPERATORS WITH SINGULAR POTENTIALS

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Abstract. We consider the Schrödinger operators $H_V = -\Delta_g + V$ with singular potentials $V$ on general $n$-dimensional Riemannian manifolds and study the eigenvalues and eigenfunctions under this perturbation. These singular potentials appear naturally in physics, most notably the Coulomb potential $|x|^{-1}$. Sogge and the first author [14] proved the sharp Weyl laws for these $H_V$ with potentials in the Kato class, which is the minimal assumption to ensure that $H_V$ is essentially self-adjoint and bounded from below and the eigenfunctions of $H_V$ are bounded. Later, Frank-Sabin [9] studied the problem on the pointwise Weyl laws for these $H_V$ in three dimensions by extending the method of Avakumović [2], while it is unknown how to reconstruct this argument in other dimensions. In this paper, we completely solve this problem in any dimensions by using a different argument. First, we establish the pointwise Weyl law for potentials in the Kato class on any $n$-dimensional manifolds. This extends the 3-dimensional results of Frank-Sabin [9] by a different method. Second, we prove that the pointwise Weyl law with the standard sharp error term $O(\lambda^{n-1})$ holds for potentials in $L^n(M)$. This extends the classical results for smooth potentials by Avakumović [2], Levitan [13] and Hörmander [12] to critically singular potentials. In three dimensions, this $L^3$ condition also naturally appears in Boccato-Brennecke-Cenatiempo-Schlein [6] on the ground state energy of the Hamilton operator in the Gross-Pitaevskii regime. These two results are sharp, and our proof exploits Li-Yau’s heat kernel bounds and Blair-Sire-Sogge’s eigenfunction estimates.

The purpose of this paper is to study the pointwise Weyl Law for the Schrödinger operators $H_V = -\Delta_g + V$ on compact $n$-dimensional ($n \geq 2$) Riemannian manifolds $(M,g)$ without boundary. We shall assume throughout that the potentials $V$ are real-valued. Moreover, we shall assume that $V \in \mathcal{K}(M)$, which is the Kato class. Recall that $\mathcal{K}(M)$ is all $V$ satisfying

$$\lim_{\delta \to 0} \sup_{x \in M} \int_{d_g(x,y) < \delta} |V(y)| W_n(d_g(x,y)) dy = 0,$$

where

$$W_n(r) = \begin{cases} r^{2-n}, & n \geq 3 \\ \log(2 + r^{-1}), & n = 2 \end{cases}$$

and $d_g, dy$ denote geodesic distance, the volume element on $(M,g)$. For later use, note that $L^p(M) \subset \mathcal{K}(M) \subset L^1(M)$ for all $p > \frac{n}{2}$. The Kato class $\mathcal{K}(M)$ and $L^{n/2}(M)$ share the same critical scaling behavior, while neither one is contained in the other one for $n \geq 3$. For instance, singularities of the type $|x|^{-\alpha}$ for $\alpha < 2$ are allowed for both classes. These singular potentials appear naturally in physics, most notably the Coulomb
potential $|x|^{-1}$ in three dimensions. See e.g. Simon [23] for a detailed introduction to the Schrödinger operators with potentials in the Kato class and their physical motivations.

As was shown in [5] (see also [23]) the assumption that $V$ is in the Kato class is needed to ensure that the Schrödinger operator $H_V$ is essentially self-adjoint and bounded from below, and the eigenfunctions of $H_V$ are bounded, which is an obvious requirement for a pointwise Weyl law to hold. Although the Schrödinger operators with potentials in $L^{n/2}(M)$ are also self-adjoint and bounded from below for $n \geq 3$, the eigenfunctions for these potentials need not be bounded. Moreover, for potentials in the Kato class, the associated eigenfunctions are continuous by the heat kernel estimates of Li-Yau [20] and Sturm [29]. Since $M$ is compact, the spectrum of $H_V$ is discrete. Assuming, as we may, that $H_V$ is a positive operator, we shall write the spectrum of $\sqrt{H_V}$ as

$$\{\tau_k\}_{k=1}^\infty,$$

where the eigenvalues, $\tau_1 \leq \tau_2 \leq \cdots$, are arranged in increasing order and we account for multiplicity. For each $\tau_k$ there is an eigenfunction $e_{\tau_k} \in \text{Dom}(H_V)$ (the domain of $H_V$) so that

$$H_V e_{\tau_k} = \tau_k^2 e_{\tau_k}.$$ 

We shall always assume that the eigenfunctions are $L^2$-normalized, i.e.,

$$\int_M |e_{\tau_k}(x)|^2 dx = 1.$$

After possibly adding a constant to $V$ we may, and shall, assume throughout that $H_V$ is bounded below by one, i.e.,

$$\|f\|_2^2 \leq \langle H_V f, f \rangle, \quad f \in \text{Dom}(H_V).$$

Also, to be consistent, we shall let

$$H^0 = -\Delta_g$$

be the unperturbed operator. The corresponding eigenvalues and associated $L^2$-normalized eigenfunctions are denoted by $\{\lambda_j\}_{j=1}^\infty$ and $\{e^0_j\}_{j=1}^\infty$, respectively so that

$$H^0 e^0_j = \lambda_j^2 e^0_j, \quad \text{and} \quad \int_M |e^0_j(x)|^2 dx = 1.$$

Both $\{e_{\tau_k}\}_{k=1}^\infty$ and $\{e^0_j\}_{j=1}^\infty$ are orthonormal bases for $L^2(M)$. Recall (see e.g. [27]) that if $N^0(\lambda)$ denotes the Weyl counting function for $H^0$ then one has the sharp Weyl law

$$N^0(\lambda) := \# \{j : \lambda_j \leq \lambda\} = (2\pi)^{-n} \omega_n \text{Vol}_g(M) \lambda^n + O(\lambda^{n-1}),$$

where $\omega_n = \pi^{\frac{n}{2}} / \Gamma(\frac{n}{2} + 1)$ denotes the volume of the unit ball in $\mathbb{R}^n$ and $\text{Vol}_g(M)$ denotes the Riemannian volume of $M$. This result is due to Avakumović [2] and Levitan [18], and it was generalized to general self-adjoint elliptic pseudo-differential operators by Hörmander [12]. The sharpness of (0.6) means that it cannot be improved for the standard sphere. The original Weyl law was proved by Weyl [30] for a compact domain in $\mathbb{R}^n$ over a hundred years ago. See Arendt, Nittka, Peter and Steiner [1] for historical background on this famous problem and its solution by Weyl.
Recall that
\begin{equation}
N^0(\lambda) := \# \{ j : \lambda_j \leq \lambda \} = \int_M \sum_{\lambda_j \leq \lambda} |e_j^0(x)|^2 \, dx.
\end{equation}

The Weyl law (0.6) can be obtained from the following sharp pointwise Weyl law
\begin{equation}
\sum_{\lambda_j \leq \lambda} |e_j^0(x)|^2 = (2\pi)^{-n} \omega_n \lambda^n + O(\lambda^{n-1}).
\end{equation}

It is due to Avakumović [2], following earlier partial results of Levitan [18], [19]. The error term $O(\lambda^{n-1})$ is also sharp on the standard sphere. Proofs are presented in several texts, including Hörmander [11] and Sogge [26], [27]. The pointwise Weyl law for a compact domain in $\mathbb{R}^n$ is due to Carleman [7]. Similar results for compact manifolds with boundary are due to Seeley [21], [22].

Recently, Huang-Sogge [14] proved that if $V \in \mathcal{K}(M)$, then the sharp Weyl law of the same form still holds for the Schrödinger operators $H_V$, i.e.
\begin{equation}
N_V(\lambda) := \# \{ k : \tau_k \leq \lambda \} = (2\pi)^{-n} \omega_n \text{Vol}_g(M) \lambda^n + O(\lambda^{n-1}).
\end{equation}

For $H_V$ with smooth potentials, the pointwise Weyl law of the form (0.8) follows from Hörmander [12]. So it is natural to study the pointwise Weyl law for $H_V$ with singular potentials.

Let $P^0 = \sqrt{H^0}$ and $P_V = \sqrt{H_V}$. We denote the indicator function of the interval $[-\lambda, \lambda]$ by $1_\lambda(\tau)$, and write
\begin{align*}
1_\lambda(P^0)(x, x) &= \sum_{\lambda_j \leq \lambda} |e_j^0(x)|^2, \\
1_\lambda(P_V)(x, x) &= \sum_{\tau_k \leq \lambda} |e_{\tau_k}(x)|^2.
\end{align*}

When $n = 3$, Frank-Sabin [9] proved that if $V \in \mathcal{K}(M)$, then as $\lambda \to \infty$ and uniformly in $x \in M$
\begin{equation}
1_\lambda(P_V)(x, x) = (2\pi)^{-3} \omega_3 \lambda^3 + o(\lambda^3).
\end{equation}

They pointed out that the error term can not be replaced by $O(\lambda^{3-\delta})$, for any $\delta > 0$. Moreover, [9] proved that if $V$ satisfies a stronger condition, the sharp pointwise law may hold. Indeed, if $V$ satisfies for some $\varepsilon' > 0$
\begin{equation}
\sup_{x \in M} \int_{d_s(y, x) < \varepsilon'} \frac{|V(y)|}{d_g(y, x)^2} \, dy < \infty,
\end{equation}
then uniformly in $x \in M$
\begin{equation}
1_\lambda(P_V)(x, x) = (2\pi)^{-3} \omega_3 \lambda^3 + O(\lambda^2).
\end{equation}

Note that the condition (0.12) is satisfied by $V \in L^q(M)$, for any $q > 3$. For comparison, for any $q < 3$, they showed that the sharp pointwise Weyl law (0.13) fails to hold for some $V \in L^q$. So $q = 3$ is the threshold for the validity of the sharp pointwise Weyl law on the $L^q$ scale. The proof of [9] extends the method of Avakumović [2], which relies on Tauberian theorems and parametrix estimates. To our knowledge, it is unknown how to
reconstruct this argument in other dimensions, see [9] Remark 4.5. So it is an interesting open problem to determine the pointwise Weyl law for the Schrödinger operators with critically singular potentials on general \( n \)-dimensional manifolds.

In this paper, we completely solve this open problem in any dimensions. Our proof extends the wave equation method in [14], [27] to get around the difficulties in [9].

**Theorem 1.** Let \( n \geq 2 \) and \( V \in K(M) \). Then for any fixed \( \varepsilon > 0 \) there exists a \( \Lambda(\varepsilon, V) < \infty \) such that for \( \lambda > \Lambda(\varepsilon, V) \), we have

\[
\sup_{x \in M} |I_\lambda(P_V)(x, x) - (2\pi)^{-n}\omega_n\lambda^n| \leq C_V \varepsilon \lambda^n.
\]

Here \( C_V > 0 \) is a constant independent of \( \lambda \) and \( \varepsilon \).

So for potentials in the Kato class, as \( \lambda \to \infty \) and uniformly in \( x \in M \),

\[
(0.14) \sum_{\tau_k \leq \lambda} |\tau_k(x)|^2 = (2\pi)^{-n}\omega_n\lambda^n + o(\lambda^n).
\]

The 3-dimensional case is due to Frank-Sabin [9], while other dimensions of Theorem 1 are new.

**Theorem 2.** Let \( n \geq 2 \) and \( V \in L^n(M) \). Then for \( \lambda \geq 1 \)

\[
\sup_{x \in M} |I_\lambda(P_V)(x, x) - (2\pi)^{-n}\omega_n\lambda^n| \leq C_V \lambda^{n-1}.
\]

Here \( C_V > 0 \) is a constant independent of \( \lambda \).

In other words, for potentials in \( L^n(M) \), uniformly in \( x \in M \),

\[
(0.15) \sum_{\tau_k \leq \lambda} |\tau_k(x)|^2 = (2\pi)^{-n}\omega_n\lambda^n + O(\lambda^{n-1}).
\]

Theorem 2 is new in all dimensions. In three dimensions, either (0.12) or \( L^3(M) \) can ensure the error term is \( O(\lambda^{n-1}) \), while neither of them can imply the other one. Moreover, it is worth mentioning that this \( L^3 \) condition also naturally appears in the study of the ground state energy of the Hamilton operator in the Gross-Pitaevskii regime, see Boccato-Brennecke-Cenatiempo-Schlein [6].

These two theorems are sharp, by the explicit examples studied in the recent work [17] of the authors. Specifically, the sharpness of Theorem 1 means the error term \( o(\lambda^n) \) cannot be replaced by \( O(\lambda^{n-\delta}) \) for any \( \delta > 0 \). The sharpness of Theorem 2 means that the condition \( L^n(M) \) cannot be replaced by \( L^p(M) \) for any \( p < n \). In [17], we consider the singular potentials

\[
V(x) = \rho(d_g(x, x_0))d_g(x, x_0)^{-2+\eta}, \quad 0 < \eta < 1,
\]

where \( x_0 \in M \) is fixed, \( d_g \) is the Riemannian distance function on \((M, g)\), and \( \rho \) is a smooth cutoff function nonvanishing at zero. This \( V \) is clearly in \( K(M) \), and it belongs to \( L^q(M) \), for all \( q < \frac{2n}{n-2} \). The pointwise Weyl law for \( H_V \) with these \( V \) is expected to have a sharp error term \( \approx \lambda^{n-\eta} \). In [17], we proved this sharp bound on the flat torus \( M = T^n \) for any dimensions \( n \). See also Frank-Sabin [9] for another different proof for the sharpness in three dimensions. Recall that \( L^p(M) \subset K(M) \subset L^1(M) \) for all \( p > \frac{n}{n} \), and that the Kato class can ensure the boundedness of eigenfunctions, while \( L^{\frac{n}{n}} \) cannot (see
From the discussion above, we can see that Kato class is exactly the border for the existence of the pointwise Weyl law, and \( p = n \) is the threshold for the validity of the sharp pointwise Weyl law (with the error term \( O(\lambda^{n-1}) \)) on the \( L^p \) scale. If \( V \in L^p(M) \) \( (\frac{n}{2} < p < n) \), one can easily modify the argument in the proof of Theorem 2 to obtain the sharp error term \( O(\lambda^{n-2+\frac{2}{p}}) \).

The main strategy of the proof is using Fourier analysis and the wave equation techniques to estimate the difference between the classical kernel \( \mathbf{1}_\lambda(P^0)(x,x) \) and the Schrödinger kernel \( \mathbf{1}_\lambda(P_V)(x,x) \). We reduce it to estimating the difference between their smooth approximations, by the Fourier inversion formula, Duhamel’s principle and Sogge’s \( L^p \)-spectral projection bounds. To deal with the difference of two kernels, the main difficulty is to handle the “mixed terms” with two kinds of frequencies from \( P^0 \) and \( P_V \). To get around this, we must design new efficient frequency decompositions, and estimate the terms carefully by Li-Yau’s heat kernel bounds and the theory of pseudo-differential operators.

It is likely that the sharp pointwise Weyl laws for \( H_V \) can be improved under some global geometric conditions on the manifolds (see e.g. [8], [3], [10] for \( V = 0 \)). Moreover, it is interesting to investigate the Weyl laws for \( H_V \) on compact manifolds with boundary (see Seeley [21], [22] for \( V = 0 \)). We are working in progress on these problems. See also [13], [4], [15], [16] for recent related works.

The paper is organized as follow. In Section 1, we prove Theorem 1 by assuming Lemma 3 and Lemma 4. In Section 2, we prove Lemma 3 and Lemma 4 follows by repeating the same argument. In Section 3, we prove Theorem 2 by assuming Lemma 9. In Section 4, we prove Lemma 9. In the Appendix, we prove two lemmas used in the proof of Lemma 9. Throughout the paper, \( A \gtrless B \) (or \( A \lesssim B \)) means \( A \leq CB \) (or \( A \geq CB \)) for some implicit constant \( C > 0 \) that may change from line to line. \( A \approx B \) means \( A \lesssim B \) and \( A \gtrsim B \). All implicit constants \( C \) are independent of the parameters \( \lambda, \varepsilon, \lambda_j, \tau_k \).

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1. Proof of Theorem 1

Let \( 0 < \varepsilon < 1 \) and \( \lambda \geq \varepsilon^{-1} \). Fix an even real-valued function \( \rho \in C^\infty(\mathbb{R}) \) satisfying

\[
\rho(t) = 1 \text{ on } [-\delta_0/2, \delta_0/2] \text{ and } \text{supp } \rho \subset (-\delta_0, \delta_0),
\]

where we assume that \( \delta_0 < \text{Inj} M \) (the injectivity radius of \( M \)). For \( \tau > 0 \), let

\[
h(\tau) = \frac{1}{\pi} \int \rho(\varepsilon\lambda t) \frac{\sin \lambda t}{t} \cos t\tau dt.
\]

Then for \( \tau > 0 \)

\[
|h(\tau) - \mathbf{1}_\lambda(\tau)| \lesssim (1 + (\varepsilon\lambda)^{-1}|\tau - \lambda|)^{-N}, \forall N,
\]

and

\[
|\partial_x^j h(\tau)| \lesssim (\varepsilon\lambda)^{-j}(1 + (\varepsilon\lambda)^{-1}|\tau - \lambda|)^{-N}, \forall N, j = 1, 2, \ldots.
\]
Let us fix a non-negative function \( \chi \in \mathcal{S}(\mathbb{R}) \) satisfying:

\[
\chi(\tau) \geq 1, \quad |\tau| \leq 1, \quad \text{and} \quad \hat{\chi}(t) = 0, \quad |t| \geq \frac{1}{2}.
\]

Let

\[
(1.3) \quad \tilde{\chi}_\lambda(\tau) = \frac{\varepsilon \lambda}{\pi} \int \hat{\chi}(\varepsilon \lambda t) e^{it\lambda} \cos \tau t dt = \chi((\varepsilon \lambda)^{-1}(\lambda - \tau)) + \chi((\varepsilon \lambda)^{-1}(\lambda + \tau)).
\]

We have for \( \tau > 0 \)

\[
(1.4) \quad |\partial^j \tilde{\chi}_\lambda(\tau)| \lesssim (\varepsilon \lambda)^{-j}(1 + (\varepsilon \lambda)^{-1}|\tau - \lambda|)^{-N}, \forall N, \quad j = 0, 1, 2, \ldots.
\]

The key lemmas for Theorem 1 are the following.

**Lemma 3.** There exists a \( \Lambda(\varepsilon, V) < \infty \) such that for any \( \lambda > \Lambda(\varepsilon, V) \), we have

\[
\sup_{x \in \mathcal{M}} |(h(P_V) - h(P^0))(x, x)| \lesssim \varepsilon \lambda^n.
\]

**Lemma 4.** There exists a \( \Lambda(\varepsilon, V) < \infty \) such that for any \( \lambda > \Lambda(\varepsilon, V) \), we have

\[
\sup_{x \in \mathcal{M}} |(\tilde{\chi}_\lambda(P_V) - \tilde{\chi}_\lambda(P^0))(x, x)| \lesssim \varepsilon \lambda^n.
\]

We postpone the proof of these lemmas. The following two lemmas will be used several times in the proof.

**Lemma 5** (Spectral projection bounds, [25]). For \( \lambda \geq 1 \), we have

\[
\|1_{[\lambda, \lambda + 1)}(P^0)\|_{L^2 \to L^p} \lesssim \lambda^{\sigma(n, p)}, \quad 2 \leq p \leq \infty,
\]

where \( \sigma(n, p) = \max\{\frac{n-1}{2}(\frac{1}{2} - \frac{1}{p}), \frac{n-1}{2} - \frac{n}{p}\} \).

These \( L^p \)-spectral projections bounds can be viewed as the generalized Tomas-Stein restriction estimates on manifolds. They were first obtained by Sogge [25], and recently extended to the Schrödinger operators with critically singular potentials by Blair-Sire-Sogge [5]. These bounds are sharp on any closed manifolds. See [26, Chapter 5].

**Lemma 6** (Heat kernel bounds, [20], [29]). If \( V \in \mathcal{K}(\mathcal{M}) \), then for \( 0 < t \leq 1 \), there is a uniform constant \( c = c_{M,V} > 0 \) so that

\[
e^{-tH_V}(x, y) \lesssim \begin{cases} 
1^{-n/2} e^{-cd_g(x,y)/t^2}, & \text{if } d_g(x,y) \leq \text{Inj}(\mathcal{M})/2 \\
1, & \text{otherwise}.
\end{cases}
\]

Here \( \text{Inj}(\mathcal{M}) \) is the injectivity radius of \( \mathcal{M} \).

The heat kernels bounds were first obtained by Li-Yau [20] for smooth potentials, and extended to the Kato class by Sturm [29]. Note that

\[
\sum_{\tau \leq \lambda} |e_{\tau_k}(x)|^2 \lesssim \sum_{\tau \leq \lambda} e^{-\lambda^{-2} \tau^2} |e_{\tau_k}(x)|^2 = e^{-\lambda^{-2} H_V(x, x)},
\]

so we have the following eigenfunction bounds.

**Corollary 7** (Rough eigenfunction bounds). If \( V \in \mathcal{K}(\mathcal{M}) \), then for \( \lambda \geq 1 \)

\[
\sup_{x \in \mathcal{M}} \sum_{\tau \leq \lambda} |e_{\tau_k}(x)|^2 \leq C_V \lambda^n.
\]
Using the classical pointwise Weyl Law for $P^0$ (see e.g. [27]), we have

$$h(P^0)(x, x) = \frac{\omega_n}{(2\pi)^n} \lambda^n + O(\varepsilon \lambda^n).$$

Then by Lemma 3, we get

$$h(P_V)(x, x) = \frac{\omega_n}{(2\pi)^n} \lambda^n + O(\varepsilon \lambda^n).$$

We claim that there exists a $\Lambda(\varepsilon, V) < \infty$ such that for any $\lambda > \Lambda(\varepsilon, V)$ we have

$$\chi((\varepsilon \lambda)^{-1}(\lambda - P_V))(x, x) = O(\varepsilon \lambda^n).$$

Indeed, it follows from

$$\chi(((\varepsilon \lambda)^{-1}(\lambda - P_V))(x, x) = O(\varepsilon \lambda^n).$$

By (1.3), it suffices to show

$$\tilde{\chi}_\lambda(P_V)(x, x) = O(\varepsilon \lambda^n),$$

as Corollary 7 implies

$$\chi(((\varepsilon \lambda)^{-1}(\lambda + P_V))(x, x) = O(\varepsilon N \lambda^n), \forall N.$$

By the classical $L^\infty$ spectral projection bounds (Lemma 5), we know $\tilde{\chi}_\lambda(P^0)(x, x) = O(\varepsilon \lambda^n)$, so we obtain (1.5) by Lemma 4. This proves the claim.

As a corollary of this claim, we have for $j \geq 0$ and $\lambda > \Lambda(\varepsilon, V)$,

$$\sum_{\tau_k \in [(1+\varepsilon)^{-j-1}\lambda, (1+\varepsilon)^{-j}\lambda]} |e_{\tau_k}(x)|^2 = O(\varepsilon((1+\varepsilon)^{-j}\lambda)^n),$$

and for $j \geq 0$ and $\lambda > \Lambda(\varepsilon, V)$

$$\sum_{\tau_k \in [(1+\varepsilon)^{j}\lambda, (1+\varepsilon)^{j+1}\lambda]} |e_{\tau_k}(x)|^2 = O(\varepsilon((1+\varepsilon)^{j}\lambda)^n).$$

Moreover, by Corollary 7

$$\sum_{\tau_k \in [1, \Lambda(\varepsilon, V)]} |e_{\tau_k}(x)|^2 \lesssim \Lambda(\varepsilon, V)^n.$$
Hence if \( \lambda > \Lambda_1(\varepsilon, V) \) with \( \varepsilon \Lambda_1^n \) > \( \Lambda^n \), then
\[
\left| \mathbf{I}_\lambda(P_V) - h(P_V) \right|(x, x) \\
\leq \sum_k \left| \mathbf{I}_\lambda(\tau_k) - h(\tau_k) \right| |e_{\tau_k}(x)|^2 \\
\lesssim \sum_k (1 + (\varepsilon \lambda)^{-1} |\tau_k - \lambda|)^{-N} |e_{\tau_k}(x)|^2 \\
\lesssim \sum_{\tau_k \in [L, \Lambda(\varepsilon, V)]} |e_{\tau_k}(x)|^2 \\
+ \sum_{(1+\varepsilon)^{-j-1} \lambda > \Lambda(\varepsilon, V)} \sum_{\tau_k \in [(1+\varepsilon)^{-j-1} \lambda, (1+\varepsilon)^{-j-1} \lambda]} (1 + (\varepsilon \lambda)^{-1} |(1 + \varepsilon)^{-j} \lambda - \lambda|)^{-N} |e_{\tau_k}(x)|^2 \\
+ \sum_{j=0}^{\infty} (1 + (\varepsilon \lambda)^{-1} |(1 + \varepsilon)^{-j} \lambda - \lambda|)^{-N} \sum_{\tau_k \in [(1+\varepsilon)^{-j-1} \lambda, (1+\varepsilon)^{-j} \lambda]} |e_{\tau_k}(x)|^2 \\
\lesssim \Lambda(\varepsilon, V)^n + \sum_{j=0}^{\infty} (1 + (\varepsilon \lambda)^{-1} |(1 + \varepsilon)^{-j} \lambda - \lambda|)^{-N} \varepsilon ((1 + \varepsilon)^j \lambda)^n \\
+ \sum_{j=0}^{\infty} (1 + (\varepsilon \lambda)^{-1} |(1 + \varepsilon)^{-j} \lambda - \lambda|)^{-N} \varepsilon ((1 + \varepsilon)^j \lambda)^n \\
\lesssim \varepsilon \lambda^n.
\]
So for \( \lambda > \Lambda_1(\varepsilon, V) \),
\[
\mathbf{I}_\lambda(P_V)(x, x) = \frac{\omega_n}{(2\pi)^n} \lambda^n + O(\varepsilon \lambda^n).
\]

To complete the proof, we only need to prove Lemma 3 and Lemma 4. For simplicity, we shall only give the proof of Lemma 3 here, since \( \tilde{\chi}_\lambda(\tau) \) satisfies (1.4) that is analogous to the estimate (1.7) of \( h(\tau) \), Lemma 4 follows by repeating the same argument.

2. Proof of Lemma 3

First, we follow the reduction argument in (14). Let
\[
\cos tP_0^0(x, y) = \sum_j \cos t\lambda_j e_j^0(x)e_j^0(y).
\]
It is the kernel of the solution operator for \( f \to (\cos tP_0^0) f = u^0(t, x) \), where \( u^0(t, x) \) solves the wave equation
\[
(\partial^2_t + H_0^0) u^0(x, t) = 0, (x, t) \in M \times \mathbb{R}, \ u^0|_{t=0} = f, \ \partial_t u^0|_{t=0} = 0.
\]

Similarly,
\[
(\cos(tP_V))(x, y) = \sum_k \cos t\tau_k e_{\tau_k}(x)e_{\tau_k}(y)
\]
is the kernel of \( f \to \cos(tP_V) f = u_V(x, t) \), where \( u_V \) solves the wave equation
\[
(\partial^2_t + H_V^0) u_V(x, t) = 0, (x, t) \in M \times \mathbb{R}, \ u_V|_{t=0} = f, \ \partial_t u_V|_{t=0} = 0.
\]
By Duhamel’s principle,
\[ \cos tP_V(x, y) - \cos tP^0(x, y) = -\sum_{\lambda_j, \tau_k} \int_M \int_0^t \frac{\sin(t-s)\lambda_j}{\lambda_j} \cos s\tau_k \left( e_{\lambda_j}^0(x)e_{\tau_k}^0(y)\right) V(z) dz ds \]
\[ = \sum_{\lambda_j, \tau_k} \int_M \frac{\cos t\lambda_j - \cos t\tau_k}{\lambda_j^2 - \tau_k^2} e_{\lambda_j}^0(x)e_{\tau_k}^0(y) e_{\tau_k}(z) V(z) dz. \]

Recall that
\[ h(\tau) = \frac{1}{\pi} \int \rho(\varepsilon\lambda t) \sin \lambda t \cos \tau dt. \]

We have
\[ h(P_V)(x, y) - h(P^0)(x, y) = \sum_{\lambda_j, \tau_k} \int_M \frac{h(\lambda_j) - h(\tau_k)}{\lambda_j^2 - \tau_k^2} e_{\lambda_j}^0(x)e_{\tau_k}^0(y) e_{\tau_k}(x) V(y) dy \]
\[ \leq \varepsilon \lambda^a. \]

Decompose the sum into two parts with respect to the frequencies \( \tau_k \):
\[ \sum_{\lambda_j, \tau_k} = \sum_{\tau_k \leq 10\lambda} \sum_{\lambda_j} + \sum_{\tau_k > 10\lambda} \sum_{\lambda_j} \]

We will use the following lemma several times.

**Lemma 8** (Kernel estimates of PDO). Let \( \mu \in \mathbb{R} \), and \( m \in C^\infty(\mathbb{R}) \) belong to the symbol class \( S^\mu \), that is, assume that
\[ \left| \left( \frac{d}{dt} \right)^\alpha m(t) \right| \leq C_\alpha (1 + |t|)^{\mu - \alpha} \ \forall \alpha. \]

Then \( m(P^0) \) is a pseudo-diﬀerential operator of order \( \mu \). Moreover, if \( R \geq 1 \), then the kernel of the operator \( m(P^0/R) \) satisﬁes for all \( N \in \mathbb{N} \)
\[ |m(P^0/R)(x, y)| \leq \begin{cases} CR^n (Rd_g(x, y))^{-n-\mu} (1 + Rd_g(x, y))^{-N}, & n + \mu > 0 \\ CR^n \log(2 + (Rd_g(x, y))^{-1}) (1 + Rd_g(x, y))^{-N}, & n + \mu = 0 \\ CR^n (1 + Rd_g(x, y))^{-N}, & n + \mu < 0. \end{cases} \]

See [26] Theorem 4.3.1] [28 Prop.1 on page 241] for the proof. In the lemma, we mean that the inequalities hold near the diagonal (so that \( d_g(x, y) \) is well-deﬁned) and that outside the neighborhood of the diagonal we have \( |m(P^0/R)(x, y)| \lesssim R^{-N} \) for all \( N \).

We ﬁrst deal with the low-frequency part \( \tau_k \leq 10\lambda \).

**1. Low-frequency** \( (\tau_k \leq 10\lambda) \).

Let
\[ M_0 : \lambda_j \mapsto \frac{h(\lambda_j) - h(\tau_k)}{\lambda_j^2 - \tau_k^2} \]
and \( P_\ell (\tau) := h(\tau_k) - \sum_{j=0}^{\ell} \frac{1}{j!} h^{(j)}(\tau)(\tau_k - \tau)^j \) for \( \ell = 0, 1, 2, \ldots \). Then by induction we have
\[
(2.3) \quad \frac{\partial^\ell}{\partial \tau^\ell} \left( \frac{h(\tau) - h(\tau_k)}{\tau - \tau_k} \right) = \frac{\ell!}{(\tau_k - \tau)^{\ell+1}} P_\ell(\tau), \quad \ell = 0, 1, 2, \ldots
\]
Since
\[
(2.4) \quad P_\ell(\tau) = \frac{1}{(\ell + 1)!} h^{(\ell+1)}(\tau + \theta(\tau_k - \tau))(\tau_k - \tau)^{\ell+1}
\]
for some \( 0 \leq \theta \leq 1 \), by (1.2) we have for |\( \tau - \tau_k \) | \( \leq \frac{1}{2} \tau \),
\[
(2.5) \quad |\partial^\ell \left( \frac{h(\tau) - h(\tau_k)}{\tau - \tau_k} \right) | \leq |h^{(\ell+1)}(\tau + \theta(\tau_k - \tau))| \lesssim \varepsilon^{-1-\ell}(1+\tau)^{-1-\ell}.
\]
So when |\( \lambda_j - \tau_k \) | \( \leq \frac{1}{2} \lambda_j \), \( \lambda_j + \theta(\tau_k - \lambda_j) \approx \lambda_j \), by (1.2), we have for \( \ell = 0, 1, 2, \ldots \),
\[
(2.6) \quad |\partial^\ell_{\lambda_j} M_0(\lambda_j) | \lesssim \begin{cases} \varepsilon^{-1-\ell}(1+\lambda_j)^{-2-\ell}, & \text{if } \lambda_j \geq 2\lambda \\ \varepsilon^{-1-\ell}(1+\lambda_j)^{-2-\ell}, & \text{if } \lambda_j \leq 2\lambda \end{cases}
\]
When |\( \lambda_j - \tau_k \) | \( > \frac{1}{2} \lambda_j \), it follows directly from (1.2) that
\[
(2.7) \quad |\partial^\ell_{\lambda_j} M_0(\lambda_j) | \lesssim \begin{cases} \varepsilon^{-1-\ell}(1+\lambda_j)^{-2-\ell}, & \text{if } \lambda_j > 2\tau_k \\ \varepsilon^{-1-\ell}(1+\tau_k)^{-2-\ell}, & \text{if } \lambda_j < \frac{1}{2} \tau_k \end{cases}
\]
The implicit constants in (2.8), (2.9), (2.10) are independent of \( \tau_k \). Now we have shown \( \tilde{M}_0 \in S^{-2} \), and satisfies
\[
(2.8) \quad |\partial^\ell_{\lambda_j} \tilde{M}_0(\lambda_j) | \lesssim \varepsilon^{-1-\ell}(1+\lambda_j)^{-2-\ell}, \quad \ell = 0, 1, 2, \ldots
\]
Then by Lemma 8, we have for some constant \( n_0 > 0 \) (only dependent on \( n \)),
\[
(2.9) \quad |\sum_{\lambda_j} \frac{h(\lambda_j) - h(\tau_k)}{\lambda_j^2 - \tau_k^2} e_j^0(x)e_j^0(y) | \lesssim \varepsilon^{-n_0} W_n(d_g(x,y)).
\]
Additionally, by (2.8) and (2.7), it is straightforward to check that for \( \tau_k \leq 10\lambda \), if we let \( \tilde{M}_0(\mu) = \tau_k^2 M_0(\tau_k \mu) \), then
\[
(2.10) \quad |\partial^\ell_{\lambda_j} \tilde{M}_0(\mu) | \lesssim \varepsilon^{-1-\ell}(1+\mu)^{-2-\ell}, \quad \ell = 0, 1, 2, \ldots.
\]
Thus, using (2.7), since \( M_0(\mu) = \tau_k^{-2} \tilde{M}_0(\mu/\tau_k) \), we also have
\[
(2.11) \quad |\sum_{\lambda_j} \frac{h(\lambda_j) - h(\tau_k)}{\lambda_j^2 - \tau_k^2} e_j^0(x)e_j^0(y) | \lesssim \varepsilon^{-n_0} \tau_k^{-2} W_n(\tau_k d_g(x,y))(1 + \tau_k d_g(x,y))^{-\sigma}, \quad \forall \sigma.
\]
We may write \( V = V_{\leq N} + V_{> N} \), where
\[
V_{\leq N}(x) = \begin{cases} V(x), & \text{if } |V(x)| \leq N, \\ 0, & \text{otherwise.} \end{cases}
\]
One the one hand, by the fact that \( V \in K(M) \subset L^1(M) \) we can choose \( N(\varepsilon, V) < \infty \) and \( \delta(\varepsilon, V) > 0 \) such that
\[
\int_{d_g(x,y) \leq \delta} |V_{> N}(y)| W_n(d_g(x,y))dy < \varepsilon^{n_0+1},
\]
In the following three subsections, we show that they are all 
Here \( \beta \) (2.16) the support of \( \beta \) then \( \tau \) (2.15).
\[
\int_{d_k(x,y) > \delta} |V_N(y)| W_n(d_k(x,y)) dy < \varepsilon^{n_0+1}.
\]
Recall that Corollary 4 gives \( \sum_{\tau_k \leq 10\lambda} |e_{\tau_k}(x)e_{\tau_k}(y)| \lesssim \lambda^n \). So by using (2.10) we get
\[
(2.12) \quad | \sum_{\tau_k \leq 10\lambda} \sum_{\lambda_j} \int_M \frac{h(\lambda_j) - h(\tau_k)}{\lambda_j^2 - \tau_k^2} e_j^0(x)e_j^0(y)e_{\tau_k}(x)e_{\tau_k}(y)V_N(y) dy | \lesssim \varepsilon \lambda^n.
\]
On the other hand, since Corollary 7 also gives \( \sum_{\tau_k \geq 2m} |e_{\tau_k}(x)e_{\tau_k}(y)| \lesssim 2^{mn} \), by using (2.11) we have
\[
(2.13) \quad | \sum_{\tau_k \leq 10\lambda} \sum_{\lambda_j} \int_M \frac{h(\lambda_j) - h(\tau_k)}{\lambda_j^2 - \tau_k^2} e_j^0(x)e_j^0(y)e_{\tau_k}(x)e_{\tau_k}(y)V_N(y) dy | \lesssim \sum_{m \in \mathbb{N}: 2^{m_0} \leq 10\lambda} N \varepsilon^{-n_0} 2^{mn} \cdot 2^{-2m} \lesssim N \varepsilon^{-n_0} \lambda^{-2} \log \lambda.
\]
Thus the proof of (2.11) is complete for the low-frequency part \( \tau_k \leq 10\lambda \) if we choose \( \Lambda(\varepsilon,V) \) such that \( N \varepsilon^{-n_0} \Lambda^{-2} \log \Lambda \leq \varepsilon \Lambda^n \).

Next, we only need to deal with high-frequency part \( \tau_k > 10\lambda \).

2. High-frequency \( (\tau_k > 10\lambda) \).

Choose smooth cut-off functions such that
\[
\beta(\lambda_j \lesssim \lambda) + \sum_{2^{m_0} \geq 5\lambda} \beta(\lambda_j \approx 2^m) = 1, \quad \lambda_j \geq 1,
\]
\[
\beta(\tau_k \lesssim 2^m) + \beta(\tau_k \gtrsim 2^m) = 1, \quad 2^m \geq 5\lambda.
\]
Here \( \beta(\lambda_j \lesssim \lambda) \) is supported on \( \{ \lambda_j < 5\lambda \} \), and \( \beta(\lambda_j \approx 2^m) \) is supported on \( \{ 2^{m_0-2} < \lambda_j < 2^m \} \), and \( \beta(\tau_k \gtrsim 2^m) \) is supported on \( \{ \tau_k > 2^m \} \). Thus, if \( 2^m \geq 5\lambda \) and \( \tau_k > 10\lambda \), then \( \tau_k > 2\lambda_j \) on the support of \( \beta(\lambda_j \approx 2^m) \beta(\tau_k \gtrsim 2^m) \). If \( \tau_k > 10\lambda \), then \( \tau_k > 2\lambda_j \) on the support of \( \beta(\lambda_j \lesssim \lambda) \).

For each fixed \( \tau_k > 10\lambda \), let
\[
M(\tau) = \frac{h(\tau) - h(\tau_k)}{\tau^2 - \tau_k^2},
\]
where \( h(\tau) \) is defined as in (1.1). We shall decompose the sum into three parts,
\[
(2.14) \quad \int_M \sum_{\lambda_j \geq 5\lambda} \sum_{\tau_k > 10\lambda} M(\lambda_j) \beta(\lambda_j \approx 2^m) \beta(\tau_k \lesssim 2^m) e_j^0(x)e_j^0(y)e_{\tau_k}(x)e_{\tau_k}(y)V(y) dy.
\]
\[
(2.15) \quad \int_M \sum_{\lambda_j \geq 5\lambda} \sum_{\tau_k > 10\lambda} M(\lambda_j) \beta(\lambda_j \approx 2^m) \beta(\tau_k \gtrsim 2^m) e_j^0(x)e_j^0(y)e_{\tau_k}(x)e_{\tau_k}(y)V(y) dy.
\]
\[
(2.16) \quad \int_M \sum_{\lambda_j} \sum_{\tau_k > 10\lambda} M(\lambda_j) \beta(\lambda_j \lesssim \lambda) e_j^0(x)e_j^0(y)e_{\tau_k}(x)e_{\tau_k}(y)V(y) dy.
\]
In the following three subsections, we show that they are all \( O(\varepsilon \lambda^n) \) for large \( \lambda \).
1. **Estimate of (2.14).** First, by (12) and mean value theorem, it is not hard to see that

$$M_1 : \lambda_j \mapsto \sum_{2^m \geq 5\lambda} \frac{h(\lambda_j) - h(\tau_k)}{\lambda^2_j - \tau_k^2} \beta(\lambda_j \approx 2^m) \beta(\tau_k \gtrsim 2^m)$$

is a symbol in $S^{-2}$, and satisfies

$$|\partial_{\lambda_j}^k M_1(\lambda_j)| \lesssim (1 + \lambda_j)^{-2-\ell} (1 + (\varepsilon \lambda)^{-1} \tau_k)^{-N}, \quad \forall \lambda_j, \ell = 0, 1, 2, \ldots,$$

where we used the fact that $|\tau_k - \lambda| \approx \tau_k$ if $\tau_k > 10\lambda$. Then by Lemma 5 the kernel satisfies

$$|M_1(P^0)(x, y)| \lesssim W_n(d_y(x, y))(1 + (\varepsilon \lambda)^{-1} \tau_k)^{-N}, \quad \forall \lambda$$

and Corollary 7 gives

$$\sum_{\tau_k > 10\lambda} (1 + (\varepsilon \lambda)^{-1} \tau_k)^{-N}|e_{\tau_k}(x)e_{\tau_k}(y)| \lesssim \sum_{\tau_k > 10\lambda} (1 + (\varepsilon \lambda)^{-1} \tau_k)^{-N} 2^{n\ell} \lesssim \varepsilon^N \lambda^n \leq \varepsilon \lambda^n.$$

Thus using $V \in \mathcal{K}(M)$ we get

$$\left|\int_M \sum_{2^m \geq 5\lambda} \sum_{\lambda_j} \sum_{\tau_k > 10\lambda} M_1(\lambda_j)e^0_j(x)e^0_j(y)e_{\tau_k}(x)e_{\tau_k}(y)V(y)dy\right| \lesssim \varepsilon \lambda^n.$$

2. **Estimate of (2.15).** Second, by our construction of the cut-off functions, we have $\tau_k > 2\lambda_j$ on the support of $\beta(\lambda_j \approx 2^m) \beta(\tau_k \gtrsim 2^m)$ if $2^m \geq 5\lambda$ and $\tau_k > 10\lambda$. Then

$$M_{21} : \lambda_j \mapsto \sum_{2^m \geq 5\lambda} \frac{h(\tau_k)}{\lambda^2_j - \tau_k^2} \beta(\lambda_j \approx 2^m) \beta(\tau_k \gtrsim 2^m)$$

is a symbol in $S^{-2}$ and satisfies

$$|\partial_{\lambda_j}^k M_{21}(\lambda_j)| \lesssim (1 + \lambda_j)^{-2-\ell} (1 + (\varepsilon \lambda)^{-1} \tau_k)^{-N}, \quad \forall \lambda_j, \ell = 0, 1, 2, \ldots.$$

Hence by Lemma 5 the kernel satisfies

$$|M_{21}(P^0)(x, y)| \lesssim W_n(d_y(x, y))(1 + (\varepsilon \lambda)^{-1} \tau_k)^{-N},$$

and again Corollary 7 gives

$$\sum_{\tau_k > 10\lambda} (1 + (\varepsilon \lambda)^{-1} \tau_k)^{-N}|e_{\tau_k}(x)e_{\tau_k}(y)| \lesssim \varepsilon^N \lambda^n \leq \varepsilon \lambda^n.$$

Thus using $V \in \mathcal{K}(M)$ we get

$$\left|\int_M \sum_{2^m \geq 5\lambda} \sum_{\lambda_j} \sum_{\tau_k > 10\lambda} M_{21}(\lambda_j)e^0_j(x)e^0_j(y)e_{\tau_k}(x)e_{\tau_k}(y)V(y)dy\right| \lesssim \varepsilon \lambda^n.$$

Moreover, if we write

$$\frac{-h(\lambda_j)}{\lambda^2_j - \tau_k^2} = \int_0^\infty h(\lambda_j)e^{t(\lambda^2_j - \tau_k^2)} dt = \int_0^{2^{-2m}} h(\lambda_j)e^{t(\lambda^2_j - \tau_k^2)} dt - \frac{h(\lambda_j)e^{-2^{2m}(\lambda^2_j - \tau_k^2)}}{\lambda^2_j - \tau_k^2},$$

then

$$M_{22} : \lambda_j \mapsto \frac{h(\lambda_j)e^{-2^{2m}(\lambda^2_j - \tau_k^2)}}{\lambda^2_j - \tau_k^2} \beta(\lambda_j \approx 2^m) \beta(\tau_k \gtrsim 2^m)$$

is a symbol in $S^{-2}$ and satisfies

$$|\partial_{\lambda_j}^k M_{22}(\lambda_j)| \lesssim (1 + \lambda_j)^{-2-\ell} (1 + (\varepsilon \lambda)^{-1}2^m)^{-N}, \quad \forall \lambda, \ell = 0, 1, 2, \ldots.$$
Hence by Lemma 7, the kernel satisfies
\[ |M_{22}(P^0)(x, y)| \lesssim W_n(d_g(x, y))(1 + (\varepsilon \lambda)^{-1}2^m)^{-N}. \]

By Corollary 7,
\[ \sum_{\tau_k \geq 2^m} e^{-2^{-m}t^2} |e_{\tau_k}(x)e_{\tau_k}(y)| \lesssim \sum_{\ell \in \mathbb{N}: 2^\ell \geq 2^m} e^{-2^{-m}2^\ell} 2^{n\ell} \lesssim 2^{nm}. \]

Thus using \( V \in \mathcal{K}(M) \) we get
\[
\int_M \sum_{2^m \geq 5\lambda} \sum_{\lambda_j \tau_k > 10\lambda} M_{22}(\lambda_j) e_j^0(x)e_j^0(y)e_{\tau_k}(x)e_{\tau_k}(y)V(y)\]
\[
\lesssim \sum_{2^m \geq 5\lambda} 2^{nm}(1 + (\varepsilon \lambda)^{-1}2^m)^{-N}
\lesssim \varepsilon^N \lambda^n \leq \varepsilon \lambda^n.
\]

Furthermore, by heat kernel bounds Lemma 9 and \( \lambda_j \approx 2^m \)

\[
(2.17) \quad |\int_0^{2^{-2m}} e^{\lambda_j^2} \sum_{\tau_k} e^{-t\tau_k^2} e_{\tau_k}(x)e_{\tau_k}(y)dt|
\]
\[
\lesssim \int_0^{2^{-2m}} |\sum_{\tau_k} e^{-t\tau_k^2} e_{\tau_k}(x)e_{\tau_k}(y)|dt
\]
\[
\lesssim \int_0^{2^{-2m}} t^{\frac{1}{2}} e^{-c d_g(x, y)^2/t} dt
\]
\[
\lesssim \begin{cases} 
\log(2 + (2^n d_g(x, y))^{-1}), & n = 2 \\
 d_g(x, y)^{2-n}, & n \geq 3 
\end{cases} \lesssim W_n(d_g(x, y)).
\]

Moreover, by Corollary 9
\[
\sum_{\lambda_j} |h(\lambda_j)\beta(\lambda_j \approx 2^m)||e_j^0(x)e_j^0(y)|| \lesssim 2^{nm}(1 + (\varepsilon \lambda)^{-1}2^m)^{-N}, \ \forall N.
\]

Thus using \( V \in \mathcal{K}(M) \) we get
\[
|\int_M \sum_{2^m \geq 5\lambda} \sum_{\lambda_j} \sum_{\tau_k} \int_0^{2^{-2m}} h(\lambda_j)e^{t(\lambda_j^2 - \tau_k^2)} dt \beta(\lambda_j \approx 2^m) e_j^0(x)e_j^0(y)e_{\tau_k}(x)e_{\tau_k}(y)V(y) dy|
\]
\[
\lesssim \sum_{2^m \geq 5\lambda} 2^{nm}(1 + (\varepsilon \lambda)^{-1}2^m)^{-N}
\lesssim \varepsilon^N \lambda^n \leq \varepsilon \lambda^n.
\]

Since \( 1 = \beta(\tau_k \lesssim 2^m) + \beta(\tau_k \gtrsim 2^m) \) when \( 2^m \geq 5\lambda \), it remains to estimate
\[
\int_M \sum_{2^m \geq 5\lambda} \sum_{\lambda_j} \sum_{\tau_k} \int_0^{2^{-2m}} M_{23}(\lambda_j) e_j^0(x)e_j^0(y)e_{\tau_k}(x)e_{\tau_k}(y)V(y) dy,
\]
where
\[ M_{23}(\lambda_j) = \int_{0}^{2^{-2m}} h(\lambda_j) e^{t(\lambda_j^2 - \tau_k^2)} \beta(\lambda_j \approx 2^m) \beta(\tau_k \lesssim 2^m) \, dt \]

Indeed, \( M_{23} \) is a symbol in \( S^{-2} \) which satisfies
\[ |\partial^\ell_j M_{23}(\lambda_j)| \lesssim (1 + \lambda_j)^{-2-\ell} (1 + (\varepsilon \lambda)^{-1}2^m)^{-N}, \quad \forall N, \, \ell = 0, 1, 2, \ldots \]

So by Lemma 8 the kernel satisfies
\[ |M_{23}(P^0)(x,y)| \lesssim W_n(d_y(x,y))(1 + (\varepsilon \lambda)^{-1}2^m)^{-N}. \]

By Corollary 7
\[ \sum_{\tau_k \leq 2^m} |e_{\tau_k}(x)e_{\tau_k}(y)| \lesssim 2^{mn}. \]

Thus
\[ |\int M_{23}(\lambda_j) e^0_j(x)e^0_j(y)e_{\tau_k}(x)e_{\tau_k}(y)V(y)dy| \lesssim \sum_{2^m \geq 5\lambda} 2^{mn}(1 + (\varepsilon \lambda)^{-1}2^m)^{-N} \lesssim \varepsilon^N \lambda^n \lesssim \varepsilon \lambda^n. \]

3. **Estimate of \( (2.16) \).** By our construction of the cut-off functions, we have \( \tau_k > 2\lambda_j \) on the support of \( \beta(\lambda_j \lesssim \lambda) \) if \( \tau_k > 10\lambda \), so

\[ M_{31} : \lambda_j \mapsto \frac{h(\tau_k)}{\lambda_j^2 - \tau_k^2} \beta(\lambda_j \lesssim \lambda) \]

is a symbol in \( S^{-2} \) and satisfies
\[ |\partial^\ell_j M_{31}(\lambda_j)| \lesssim (1 + \lambda_j)^{-2-\ell} (1 + (\varepsilon \lambda)^{-1}\tau_k)^{-N}, \quad \forall N, \, \ell = 0, 1, 2, \ldots \]

Then by Lemma 8 the kernel satisfies
\[ |M_{31}(P^0)(x,y)| \lesssim W_n(d_y(x,y))(1 + (\varepsilon \lambda)^{-1}\tau_k)^{-N}, \]

and again Corollary 7 implies
\[ \sum_{\tau_k > 10\lambda} (1 + (\varepsilon \lambda)^{-1}\tau_k)^{-N} |e_{\tau_k}(x)e_{\tau_k}(y)| \lesssim \varepsilon^N \lambda^n \lesssim \varepsilon \lambda^n. \]

Thus using \( V \in K(M) \) we get
\[ |\int M_{31}(\lambda_j)e^0_j(x)e^0_j(y)e_{\tau_k}(x)e_{\tau_k}(y)V(y)dy| \lesssim \varepsilon \lambda^n. \]
Moreover, we write
\[-\frac{h(\lambda_j)}{\lambda_j - \tau_k} 1(\tau_k > 10\lambda) \beta(\lambda_j < \lambda) = \int_0^\infty h(\lambda_j) e^{t(\lambda_j^2 - \tau_k^2)} dt 1(\tau_k > 10\lambda) \beta(\lambda_j < \lambda)\]
\[= \int_0^{\lambda^{-2}} h(\lambda_j) e^{t(\lambda_j^2 - \tau_k^2)} dt \beta(\lambda_j < \lambda)\]
\[-\int_0^{\lambda^{-2}} h(\lambda_j) e^{t(\lambda_j^2 - \tau_k^2)} dt 1(\tau_k < 10\lambda) \beta(\lambda_j < \lambda)\]
\[-\frac{h(\lambda_j) e^{\lambda^{-2}(\lambda_j^2 - \tau_k^2)}}{\lambda_j^2 - \tau_k^2} 1(\tau_k > 10\lambda) \beta(\lambda_j < \lambda)\]
\[:= M_{34}(\lambda_j) - M_{33}(\lambda_j) - M_{32}(\lambda_j).\]

In the following, we handle these three parts separately. First, let
\[\bar{M}_{32} : \mu \mapsto \frac{h(\mu \lambda) e^{-\mu^2}}{\mu^2 - (\tau_k/\lambda)^2} \beta(\mu < 1) 1(\tau_k > 10\lambda).\]

It is a symbol in $S^{-2}$ and satisfies
\[|\partial^\ell \bar{M}_{32}(\mu)| \lesssim \varepsilon^{-\ell} (1 + \mu)^{-2-\ell}, \quad \ell = 0, 1, 2, \ldots,\]

Here $\beta(\mu < 1)$ is supported on $\{\mu \leq \delta\}$. Thus by Lemma 8 the kernel satisfies for some constant $n_0 > 0$ (only dependent on $n$)
\[(2.18) \quad |M_{32}(P^0)(x, y)| = \lambda^{-2} |M_{32}(P^0/\lambda)(x, y)|\]
\[\lesssim \begin{cases} \varepsilon^{-n_0} \log(2 + (\lambda d_g(x, y)^{-1})(1 + \lambda d_g(x, y))^{-N}, & n = 2 \\ \varepsilon^{-n_0} d_g(x, y)^{2-n}(1 + \lambda d_g(x, y))^{-N}, & n \geq 3 \end{cases}\]
\[\lesssim \varepsilon^{-n_0} \tau_n(d_g(x, y))(1 + \lambda d_g(x, y))^{-N}, \quad \forall N.\]

By Corollary [7]
\[\sum_{\tau_k > 10\lambda} e^{-\lambda^{-2}\tau_k^2} |e_{\tau_k}(x) e_{\tau_k}(y)| \lesssim \sum_{\ell \in \mathbb{N}, 2^\ell > 10\lambda} e^{-\lambda^{-2}2^n\ell} 2^n \lesssim \lambda^n.\]

Fix $N = 1$ in (2.18). Thus, using (2.18) and $V \in \mathcal{K}(M)$ we may choose $\Lambda(\varepsilon, V) < \infty$ so that for $\lambda > \Lambda(\varepsilon, V)$
\[(2.19) \quad \int_M \lambda^{-2} |M_{32}(P^0/\lambda)(x, y)V(y)| dy\]
\[\lesssim \varepsilon^{-n_0} \int_{d_g(x, y) \leq \lambda^{-1/2}} \tau_n(d_g(x, y))|V(y)| dy + \varepsilon^{-n_0} \lambda^{-1/2} \int_M \tau_n(d_g(x, y))|V(y)| dy\]
\[\lesssim \varepsilon.\]

So we have
\[(2.20) \quad |\int_M \sum_{\lambda_j} \sum_{\tau_k > 10\lambda} M_{32}(\lambda_j) e_{\lambda_j}^0(x) e_{\lambda_j}^0(y) e_{\tau_k}(x) e_{\tau_k}(y) V(y) dy| \lesssim \varepsilon \lambda^n.\]
Second, if $\tau_k \leq 10\lambda$, then
\[ \tilde{M}_{33}: \mu \mapsto \int_0^1 h(\lambda \mu) e^{t(\mu^2 - (\tau_k / \lambda)^2)} \, dt \beta(\mu, \Lambda, \Lambda) \mathbf{1}(\tau_k \leq 10\lambda) \]
is a symbol in $S^{-2}$ and satisfies
\[ |\partial^\ell_x \tilde{M}_{33}(\mu)| \lesssim \varepsilon^{-\ell} (1 + \mu)^{-2-\ell}, \quad \ell = 0, 1, 2, \ldots. \]
So by Lemma 8, the kernel satisfies for some constant $n_0 > 0$ (only dependent on $n$)
\begin{equation}
|M_{33}(P^0)(x, y)| = \lambda^{-2} |\tilde{M}_{33}(P^0 / \lambda)(x, y)| \lesssim \begin{cases} 
\varepsilon^{-n_0} \log(2 + (\lambda d_g(x, y))^{-1}) (1 + \lambda d_g(x, y))^{-N}, & n = 2 \\
\varepsilon^{-n_0} d_g(x, y)^{2-n} (1 + \lambda d_g(x, y))^{-N}, & n \geq 3 
\end{cases}, \quad \forall N.
\end{equation}

By Corollary 7
\[ \sum_{\tau_k \leq 10\lambda} |e_{\tau_k}(x)e_{\tau_k}(y)| \lesssim \lambda^n. \]
Thus, as in (2.19), using (2.21) and $V \in \mathcal{K}(M)$ we may choose $\Lambda(\varepsilon, V) < \infty$ so that for $\lambda > \Lambda(\varepsilon, V)$
\begin{equation}
|\int_M \sum_{\lambda_j} \sum_{\tau_k \leq 10\lambda} M_{33}(\lambda_j) e^0_j(x)e^0_j(y)e_{\tau_k}(x)e_{\tau_k}(y)V(y)dy| \lesssim \varepsilon \lambda^n.
\end{equation}

Third, by heat kernel bounds Lemma 6 and $\lambda_j \lesssim \lambda$
\begin{equation}
|\int_0^{\lambda^{-2}} e^{t\lambda^2} \sum_{\tau_k} e^{-t\tau^2} e_{\tau_k}(x) e_{\tau_k}(y) dt| \lesssim \int_0^{\lambda^{-2}} |\sum_{\tau_k} e^{-t\tau^2} e_{\tau_k}(x) e_{\tau_k}(y)| dt \lesssim \int_0^{\lambda^{-2}} t^{-\frac{1}{2}} e^{-cd_g(x, y)^2 / t} dt \lesssim \begin{cases} 
\log(2 + (\lambda d_g(x, y))^{-1}) (1 + \lambda d_g(x, y))^{-N}, & n = 2 \\
\lambda d_g(x, y)^{2-n} (1 + \lambda d_g(x, y))^{-N}, & n \geq 3 
\end{cases}, \quad \forall N.
\end{equation}

By Corollary 7
\begin{equation}
\sum_{\lambda_j} |h(\lambda_j) \beta(\lambda_j \lesssim \lambda)| |e^0_j(x)e^0_j(y)| \lesssim \lambda^n.
\end{equation}
Again, as in (2.19), using (2.23) and $V \in \mathcal{K}(M)$ we may choose $\Lambda(\varepsilon, V) < \infty$ so that for $\lambda > \Lambda(\varepsilon, V)$
\begin{equation}
|\int_M \sum_{\lambda_j} \sum_{\tau_k} M_{34}(\lambda_j) e^0_j(x)e^0_j(y)e_{\tau_k}(x)e_{\tau_k}(y)V(y)dy| \lesssim \varepsilon \lambda^n.
\end{equation}

So the proof is complete.
3. Sharp pointwise Weyl Law

In this section, we prove Theorem 2: the sharp pointwise Weyl Law for $-\Delta_g + V$

$$\mathbf{1}_\lambda(P_V)(x,x) = \frac{\omega_n}{(2\pi)^n} \lambda^n + O(\lambda^{n-1}).$$

From now on, we fix $\epsilon = \lambda^{-1}$ in the definitions of $h(\tau)$ and $\tilde{\chi}_\lambda(\tau)$ in (1.1), (1.3). They satisfy the following rapid decay properties: for $\tau > 0$

$$|h(\tau) - \mathbf{1}_\lambda(\tau)| \lesssim (1 + |\tau - \lambda|)^{-N}, \forall N,$$

(3.1)

$$|\partial_\tau^j h(\tau)| \lesssim (1 + |\tau - \lambda|)^{-N}, \forall N, j = 1, 2, \ldots$$

(3.2)

$$|\partial_\tau^j \tilde{\chi}_\lambda(\tau)| \lesssim (1 + |\tau - \lambda|)^{-N}, \forall N, j = 0, 1, 2, \ldots$$

We shall need the following lemma whose proof we postpone to the next section.

**Lemma 9.** Let $n \geq 2$ and $V \in L^n(M)$. Then

$$\sup_{x \in M} |(h(P_V) - h(P^0))(x,x)| \leq C_V \lambda^{n-1}.$$  

**Lemma 10** (Spectral projection bounds for $H_V$, [5]). Let $n \geq 2$, if $V \in K(M) \cap L^{n/2}(M)$, then for $\lambda \geq 1$, we have

$$\sup_{x \in M} \mathbf{1}_{[\lambda,\lambda+1]}(P_V)(x,x) \lesssim \lambda^{n-1}.$$  

The condition $L^{n/2}$ in Lemma 10 can be dropped when $n = 2, 3$, see [5], [9].

Using the classical pointwise Weyl Law for $P^0$ (see e.g. [27])

$$h(P^0)(x,x) = \frac{\omega_n}{(2\pi)^n} \lambda^n + O(\lambda^{n-1}).$$

Then by Lemma 9 we get

$$h(P_V)(x,x) = \frac{\omega_n}{(2\pi)^n} \lambda^n + O(\lambda^{n-1}).$$

Since $V \in L^n(M) \subset K(M) \cap L^{n/2}(M)$, we have

$$\mathbf{1}_{[\lambda,\lambda+1]}(P_V)(x,x) = O(\lambda^{n-1}),$$

which follows from the spectral projection bounds for $H_V$ (Lemma 10).

Hence

$$\left| \mathbf{1}_\lambda(P_V) - h(P_V)(x,x) \right| \leq \sum_{k} |\mathbf{1}_\lambda(\tau_k) - h(\tau_k)||e_{\tau_k}(x)|^2 \lesssim \sum_{\mu=0}^{\infty} (1 + |\mu - \lambda|)^{-N} \sum_{\tau_k \in [\mu, \mu+1]} |e_{\tau_k}(x)|^2 \lesssim \sum_{\mu=0}^{\infty} (1 + |\mu - \lambda|)^{-N} \cdot (1 + \mu)^{n-1} \lesssim \lambda^{n-1}.$$
So
\[ \psi_P(x, x) = \frac{\alpha}{(2\pi)^n} \lambda^n + O(\lambda^{n-1}). \]
To complete the proof, we only need to prove Lemma [3].

4. Proof of Lemma [3]

By the same reduction argument using the Duhamel’s principle as in Section 2, it suffices to show
\[ |\sum_{\lambda_j} \sum_{\tau_k} \int_M h(\lambda_j) - h(\tau_k) \frac{\tau_k^2 - \tau_k^2}{\lambda_j^2 - \tau_k^2} e_j^0(x) e_j^0(y) e(\tau_k(x) e(\tau_k(y) V(y) dy) \lesssim \|V\|_{L^n(M)} \lambda^{n-1}. \]
Split the sum into two parts
\[ \sum_{\lambda_j} \sum_{\tau_k} = \sum_{\tau_k \leq 10\lambda} \sum_{\lambda_j} + \sum_{\tau_k > 10\lambda} \sum_{\lambda_j} \]
We first deal with the high-frequency part. Note that when \( \tau_k > 10\lambda \), we may simply fix \( \varepsilon = 10^{-1} \) and get the desired bound \( O(\lambda^{n-1}) \), except in \( 2.20, 2.22, 2.25 \). The reason is that the kernel estimates in these three terms have some negative powers of \( \varepsilon \), which are not good enough if \( \varepsilon = 10^{-1} \). To get around the difficulty, we need to use the condition \( V \in L^n(M) \).

First, we handle \( 2.25 \) by using Hölder inequality and the estimates \( 2.23 \) we get
\[ \int_M \sum_{\lambda_j} \sum_{\tau_k} \int_0^{10^{-2}} h(\lambda_j) e^{c(\lambda^2 - \tau^2)} dt \cdot \sum_{\tau_k} e^{-c\tau^2} e(\tau_k(x) e(\tau_k(y) V(y) dy) \lesssim \|V\|_{L^n(M)} \lambda^{n-1}, \]
where in the last inequality we used \( 2.23 \) and the fact that
\[ \|\lambda^{n-2} W_n(\lambda d_g(x, \cdot)) (1 + \lambda d_g(x, \cdot))^{-N} \|_{L^n(M)} \lesssim \lambda^{-1}, \text{ if } N > n. \]
To handle \( 2.20 \) and \( 2.22 \), we need to use the following lemma, whose proof can be found in the Appendix. Throughout this paper, we use the convention that \( L^\infty(M) \) means \( L^\infty(M) \) if \( n = 2 \).

Lemma 11. Let \( I \subset \mathbb{R}_+ \) and for eigenvalues \( \tau_k \in I \) assume that \( \delta_{\tau_k} \in [0, \delta] \). Then if \( m \in C^1(\mathbb{R}_+ \times M) \), and \( V \in L^n(M) \), we have
\[ \int_M \sum_{\tau_k \in I} m(\tau_k y) a_k V(y) e(\tau_k y) dy \lesssim \|V\|_{L^n(M)} \cdot \left( \|m(0, \cdot)\|_{L^\infty(M)} + \int_0^\delta \|\delta_j m(s, \cdot)\|_{L^\infty(M)} ds \right) \times \sum_{\tau_k \in I} |a_k|^2 \frac{1}{2}. \]
Second, decompose \((10\lambda, \infty) = \bigcup_{\ell \geq 0} I_{\ell}\), where

\[
I_{\ell} = (10 \cdot 2^{\ell} \lambda, 10 \cdot 2^{\ell+1} \lambda].
\]

Then by classical Sobolev estimates

\[
\left\| \sum_{\lambda_j \in I_{\ell}} \frac{h(\lambda_j) e^{\lambda^{-2} \lambda_j^2}}{\lambda_j^2 - (10 \cdot 2^{\ell} \lambda)^2} \beta(\lambda_j \lesssim \lambda) e \left( (0) \right) \right\|_{L^{\infty}(\mathcal{M})} \lesssim \lambda \left\| \sum_{\lambda_j \in I_{\ell}} \frac{h(\lambda_j) e^{\lambda^{-2} \lambda_j^2}}{\lambda_j^2} \beta(\lambda_j \lesssim \lambda) e \left( (0) \right) \right\|_{L^2(\mathcal{M})} \lesssim (2^{\ell} \lambda)^{-2} \lambda n/2 + 1,
\]

and similarly, for \(s \in I_{\ell}\)

\[
\left\| \frac{\partial}{\partial s} \sum_{\lambda_j \in I_{\ell}} \frac{h(\lambda_j) e^{\lambda^{-2} \lambda_j^2}}{\lambda_j^2 - s^2} \beta(\lambda_j \lesssim \lambda) e \left( (0) \right) \right\|_{L^{\infty}(\mathcal{M})} \lesssim (2^{\ell} \lambda)^{-3} \lambda n/2 + 1.
\]

By Lemma \([11]\) with \(\delta = 10 \cdot 2^{\ell} \lambda\), and Corollary \([7]\), we have

\[
\left| \int_{\mathcal{M}} \sum_{\lambda_j \in I_{\ell}} \sum_{\tau_k \in I_{\ell}} \frac{h(\lambda_j)}{\lambda_j^2 - \tau_k^2} e^{\lambda^{-2} (\lambda_j^2 - \tau_k^2)} \beta(\lambda_j \lesssim \lambda) e \left( (0) \right) e \left( (0) \right) e \left( (0) \right) e \left( (0) \right) V(y) dy \right| \lesssim \left\| V \right\|_{L^n(\mathcal{M})} \cdot (2^{\ell} \lambda)^{-2} \lambda n/2 \cdot \lambda \cdot \left( \sum_{\tau_k \in I_{\ell}} |e^{-\lambda^{-2} \tau_k^2} e \left( (0) \right)|^2 \right)^{1/2} \lesssim \left\| V \right\|_{L^n(\mathcal{M})} \cdot (2^{\ell} \lambda)^{-2} \lambda n/2 \cdot \lambda \cdot e^{-2^{\ell} \cdot (2^{\ell} \lambda)^{n/2}} \lesssim \left\| V \right\|_{L^n(\mathcal{M})} \lambda^{n-1} e^{-2^{\ell} \cdot 2^{(n/2-2)\ell}}.
\]

Summing over \(\ell\), we get

\[
\left| \int_{\mathcal{M}} \sum_{\lambda_j \in I_{\ell}} \sum_{\tau_k > 10 \lambda} \frac{h(\lambda_j)}{\lambda_j^2 - \tau_k^2} e^{\lambda^{-2} (\lambda_j^2 - \tau_k^2)} \beta(\lambda_j \lesssim \lambda) e \left( (0) \right) e \left( (0) \right) e \left( (0) \right) e \left( (0) \right) V(y) dy \right| \lesssim \left\| V \right\|_{L^n(\mathcal{M})} \lambda^{n-1}.
\]

Third, for \(\tau_k \leq 10 \lambda\),

\[
\left\| \sum_{\lambda_j \in I_{\ell}} \int_0^{\lambda^{-2}} h(\lambda_j) e^{\lambda^2 (\lambda_j^2 - \tau_k^2)} d\tau \beta(\lambda_j \lesssim \lambda) e \left( (0) \right) \right\|_{L^2(\mathcal{M})} \lesssim \lambda^{-2} \lambda n/2
\]

and for \(s \in [1, 10 \lambda]\)

\[
\left\| \frac{\partial}{\partial s} \sum_{\lambda_j \in I_{\ell}} \int_0^{\lambda^{-2}} h(\lambda_j) e^{\lambda^2 (\lambda_j^2 - s^2)} d\tau \beta(\lambda_j \lesssim \lambda) e \left( (0) \right) \right\|_{L^2(\mathcal{M})} \lesssim \lambda^{-3} \lambda n/2
\]
By Lemma 11 with $\delta = 10\lambda - 1$, and using Sobolev estimates and Corollary 7 as before, we have

$$
\left| \int_M \sum_{\lambda_j, \tau_k \in [1, 10\lambda]} \int_0^{h(\lambda_j)e^{(\lambda_j^2 - \tau_k^2)\lambda}} \beta(\lambda_j \lesssim \lambda)e_j^0(x)e_j^0(y)e_{\tau_k}(x)e_{\tau_k}(y)V(y)dy \right|
\lesssim \|V\|_{L^\infty(M)} \cdot \lambda^{-2}\lambda^{n/2} \cdot \lambda \cdot (\sum_{\tau_k \in [1, 10\lambda]} |e_{\tau_k}(x)|^2)^{1/2}
\lesssim \|V\|_{L^\infty(M)} \cdot \lambda^{-2}\lambda^{n/2} \cdot \lambda \cdot \lambda^{n/2}
\lesssim \|V\|_{L^\infty(M)} \lambda^{n-1}.
$$

So we finish dealing with (2.20) and (2.22). As a result, we obtain

$$
\left| \sum_{\lambda_j, \tau_k > 10\lambda} \int_M \frac{h(\lambda_j) - h(\tau_k)}{\lambda_j^2 - \tau_k^2} e_j^0(x)e_j^0(y)e_{\tau_k}(x)e_{\tau_k}(y)V(y)dy \right| \lesssim \|V\|_{L^\infty(M)} \lambda^{n-1}.
$$

Therefore, we only need to deal with $\tau_k \leq 10\lambda$. We want to show

$$
\left| \sum_{\lambda_j, \lambda_j / 2 \leq \tau_k \leq 10\lambda} \int_M \frac{h(\lambda_j) - h(\tau_k)}{\lambda_j^2 - \tau_k^2} e_j^0(x)e_j^0(y)e_{\tau_k}(x)e_{\tau_k}(y)V(y)dy \right| \lesssim \|V\|_{L^\infty(M)} \lambda^{n-1},
$$

as well as

$$
\left| \sum_{\lambda_j, \tau_k < \lambda / 2} \int_M \frac{h(\lambda_j) - h(\tau_k)}{\lambda_j^2 - \tau_k^2} e_j^0(x)e_j^0(y)e_{\tau_k}(x)e_{\tau_k}(y)V(y)dy \right| \lesssim \|V\|_{L^\infty(M)} \lambda^{n-1}.
$$

1. Low-frequency ($\tau_k < \lambda/2$). First, we prove the low-frequency estimate (4.9). We may choose smooth cut-off functions to decompose

$$
1 = \beta(\lambda_j \lesssim \lambda) + \beta(\lambda_j \approx \lambda) + \beta(\lambda_j \gtrsim \lambda)
$$

where $\beta(\lambda_j \lesssim \lambda)$ is supported on $\{\lambda_j < 3\lambda/4\}$, and $\beta(\lambda_j \gtrsim \lambda)$ is supported on $\{\lambda_j > 2\lambda\}$. By the rapid decay properties of $h(\tau)$ for $\tau > 0$

$$
|h(\tau) - \mathbf{1}(\tau)| \lesssim (1 + |\tau - \lambda|)^{-N}, \quad \forall N,
$$

$$
|h_\ell(\tau)| \lesssim (1 + |\tau - \lambda|)^{-N}, \quad \forall N, \quad \ell = 1, 2, ...
$$

we have for any $\sigma > 0$

$$
\left| \sum_{\lambda_j, \tau_k < \lambda / 2} \int_M \frac{h(\lambda_j) - h(\tau_k)}{\lambda_j^2 - \tau_k^2} \beta(\lambda_j \lesssim \lambda)e_j^0(x)e_j^0(y)e_{\tau_k}(x)e_{\tau_k}(y)V(y)dy \right| \lesssim \|V\|_{L^\infty(M)} \lambda^{-\sigma},
$$

$$
\left| \sum_{\lambda_j, \tau_k < \lambda / 2} \int_M \frac{1 - h(\tau_k)}{\lambda_j^2 - \tau_k^2} \beta(\lambda_j \approx \lambda)e_j^0(x)e_j^0(y)e_{\tau_k}(x)e_{\tau_k}(y)V(y)dy \right| \lesssim \|V\|_{L^\infty(M)} \lambda^{-\sigma},
$$
Then by Hölder inequality and Corollary 7

\[ \left| \sum_{\lambda_j} \sum_{\tau_k < \lambda/2} \int_M \frac{h(\lambda_j)}{\lambda_j^2 - \tau_k^2} \beta(\lambda_j \geq \lambda) e_j^0(x) e_j^0(y) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dy \right| \]

\[ \lesssim \| V \|_{L^n(M)} \lambda^{-\sigma}. \]

Here we use the mean-value theorem and the rough eigenfunction bounds (Corollary 7). So it remains to show

\[ (4.10) \quad \left| \sum_{\lambda_j} \sum_{\tau_k < \lambda/2} \int_M \frac{h(\lambda_j)}{\lambda_j^2 - \tau_k^2} \beta(\lambda_j \geq \lambda) e_j^0(x) e_j^0(y) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dy \right| \]

\[ \lesssim \| V \|_{L^n(M)} \lambda^{n-1}, \]

and

\[ (4.11) \quad \left| \sum_{\lambda_j} \sum_{\tau_k < \lambda/2} \int_M \frac{h(\lambda_j) - 1}{\lambda_j^2 - \tau_k^2} \beta(\lambda_j \approx \lambda) e_j^0(x) e_j^0(y) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dy \right| \]

\[ \lesssim \| V \|_{L^n(M)} \lambda^{n-1}. \]

To deal with (4.10), we note that if \( \tau_k \leq \lambda/2 \), then

\[ \tilde{M}_4 : \mu \mapsto \frac{1}{\mu^2 - (\tau_k/\lambda)^2} \beta(\mu \geq 1) \]

is a symbol in \( S^{-2} \) and satisfies

\[ |\partial^\ell \tilde{M}_4(\mu)| \lesssim (1 + \mu)^{-2-\ell}, \quad \ell = 0, 1, 2, \ldots. \]

Here \( \beta(\mu \geq 1) \) is supported on \( \{ \mu > 2 \} \). So we get the kernel estimate

\[ (4.12) \quad \left| \sum_{\lambda_j} \sum_{\tau_k < \lambda/2} \int_M \frac{h(\lambda_j)}{\lambda_j^2 - \tau_k^2} \beta(\lambda_j \geq \lambda) e_j^0(x) e_j^0(y) \right| \]

\[ = \lambda^{-2} \tilde{M}_4(P^0/\lambda)(x, y) \]

\[ \lesssim \begin{cases} 
\log(2 + (\lambda d_g(x, y))^{-1})(1 + \lambda d_g(x, y))^{-N}, & n = 2 \\
\langle d_g(x, y)^{2-n} (1 + \lambda d_g(x, y))^{-N}, & n \geq 3. 
\end{cases} \]

Then by Hölder inequality and Corollary 7

\[ \left| \sum_{\lambda_j} \sum_{\tau_k < \lambda/2} \int_M \frac{h(\lambda_j)}{\lambda_j^2 - \tau_k^2} \beta(\lambda_j \geq \lambda) e_j^0(x) e_j^0(y) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dy \right| \]

\[ \lesssim \lambda^n \cdot \| V \|_{L^n(M)} \cdot \| \lambda^{-2} \tilde{M}_4(P^0/\lambda)(x, \cdot) \|_{L^{\frac{n}{n-1}}} \]

\[ \lesssim \| V \|_{L^n(M)} \lambda^{n-1}. \]

Here in the last step we use (4.5) again. Moreover, to deal with (4.11), we need to use Lemma 11 with \( \delta = \lambda/2 - 1 \). Since by standard Sobolev estimates

\[ \| \sum_{\lambda_j} \frac{h(\lambda_j) - 1}{\lambda_j^2 - \tau_k^2} \beta(\lambda_j \approx \lambda) e_j^0(x) e_j^0(y) \|_{\frac{\sqrt{n}}{n-1}(M)} \lesssim \lambda^{-2} \lambda^{n/2+1}, \]

and for \( s \in [1, \lambda/2] \)

\[ \| \frac{\partial}{\partial s} \sum_{\lambda_j} \frac{h(\lambda_j) - 1}{\lambda_j^2 - s^2} \beta(\lambda_j \approx \lambda) e_j^0(x) e_j^0(y) \|_{\frac{\sqrt{n}}{n-1}(M)} \lesssim \lambda^{-3} \lambda^{n/2+1}, \]
by using Corollary \[\text{[4]}\] we get
\[
\left| \sum_{\lambda_j} \sum_{\tau_k < \lambda/2} \int_M \frac{h(\lambda_j) - h(\tau_k)}{\lambda_j - \tau_k} \beta(\lambda_j \approx \lambda) e_j^0(x) e_j^0(y) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dy \right|
\[
\lesssim \|V\|_{L^p(M)} \lambda^{-2} \lambda^{n/2} \cdot \lambda \cdot \left( \sum_{\tau_k \in [1, \lambda/2]} |e_{\tau_k}(x)|^2 \right)^{\frac{1}{2}}
\[
\lesssim \|V\|_{L^p(M)} \lambda^{-2} \lambda^{n/2} \cdot \lambda \cdot \lambda^{n/2}
\[
\lesssim \|V\|_{L^p(M)} \lambda^{n-1}.
\]
So we finish the proof of \[\text{[4.9]}\].

2. Middle-frequency \((\lambda/2 \leq \tau_k \leq 10\lambda)\). Next, we prove the middle-frequency estimate \[\text{[4.8]}\]. We choose smooth cut-off functions to decompose
\[
1 = \beta(|\lambda_j - \tau_k| \leq 1) + \sum_{\ell \in \mathbb{N}, 2^\ell \leq \frac{100}{\tau_k}} \beta(|\lambda_j - \tau_k| \approx 2^\ell) + \beta(|\lambda_j - \tau_k| \gtrsim \lambda).
\]
Here \(\beta(x \leq 1)\) is supported on \(\{x < \frac{1}{2}\}\), and \(\beta(x \approx 2^\ell)\) is supported on \(\{2^\ell - 2 < x < 2^\ell\}\).

Let
\[
K_{\tau_k}(x, y) = \sum_{\lambda_j} \frac{h(\lambda_j) - h(\tau_k)}{\lambda_j - \tau_k} e_j^0(x) e_j^0(y)
\]
\[
K_{\tau_k,0}(x, y) = \sum_{\lambda_j} \frac{h(\lambda_j) - h(\tau_k)}{\lambda_j - \tau_k} \beta(|\lambda_j - \tau_k| \leq 1) e_j^0(x) e_j^0(y)
\]
\[
K_{\tau_k,\ell}(x, y) = \sum_{\lambda_j} \frac{h(\lambda_j) - h(\tau_k)}{\lambda_j - \tau_k} \beta(|\lambda_j - \tau_k| \approx 2^\ell) e_j^0(x) e_j^0(y)
\]
\[
R_{\tau_k,\ell}(x, y) = \sum_{\lambda_j} \frac{1}{\lambda_j - \tau_k} \beta(|\lambda_j - \tau_k| \approx 2^\ell) e_j^0(x) e_j^0(y)
\]
\[
K_{\tau_k,\infty}(x, y) = \sum_{\lambda_j} \frac{h(\lambda_j) - h(\tau_k)}{\lambda_j - \tau_k} \beta(|\lambda_j - \tau_k| \gtrsim \lambda) e_j^0(x) e_j^0(y)
\]
\[
R_{\tau_k,\infty}(x, y) = \sum_{\lambda_j} \frac{1}{\lambda_j - \tau_k} \beta(|\lambda_j - \tau_k| \gtrsim \lambda) e_j^0(x) e_j^0(y)
\]
\[
K_{\tau_k,\ell}(x, y) = \sum_{\lambda_j} \frac{h(\lambda_j) - 1}{\lambda_j - \tau_k} \beta(|\lambda_j - \tau_k| \approx 2^\ell) e_j^0(x) e_j^0(y)
\]
\[
K_{\tau_k,\infty}(x, y) = \sum_{\lambda_j} \frac{h(\lambda_j) - 1}{\lambda_j - \tau_k} \beta(|\lambda_j - \tau_k| \gtrsim \lambda) e_j^0(x) e_j^0(y)
\]
When \(\tau_k \in (\lambda, 10\lambda]\), we decompose
\[
K_{\tau_k} = K_{\tau_k,0} + \sum_{2^\ell \leq \lambda/100} (K_{\tau_k,\ell} - R_{\tau_k,\ell} h(\tau_k)) + K_{\tau_k,\infty} - R_{\tau_k,\infty} h(\tau_k)
\]
and when \(\tau_k \in [\lambda/2, \lambda]\), we decompose
\[
K_{\tau_k} = K_{\tau_k,0} + \sum_{2^\ell \leq \lambda/100} (K_{\tau_k,\ell} - R_{\tau_k,\ell} (1 - h(\tau_k))) + K_{\tau_k,\infty} + R_{\tau_k,\infty} (1 - h(\tau_k))
\]
Moreover, we decompose
\[ 1 = \eta(\lambda_j \gtrsim \tau_k) + \eta(\lambda_j \lessapprox \tau_k) \]
where \( \eta(\lambda_j \gtrsim \tau_k) \) is supported on \( \{ \lambda_j \geq 2\tau_k \} \), and then write
\[ K_{\tau_k,\infty}^- = H_{\tau_k,\infty}^- + \tilde{K}_{\tau_k,\infty}^- \]
where
\[ H_{\tau_k,\infty}(x, y) = -\sum_{\lambda_j} \frac{\eta(\lambda_j \gtrsim \tau_k)}{\lambda_j^2 - \tau_k^2} \beta(|\lambda_j - \tau_k| \gtrsim \lambda) e_j^0(x) e_j^0(y) \]
\[ \tilde{K}_{\tau_k,\infty}(x, y) = \sum_{\lambda_j} \frac{h(\lambda_j)}{\lambda_j^2 - \tau_k^2} \beta(|\lambda_j - \tau_k| \gtrsim \lambda) e_j^0(x) e_j^0(y) \]
\[ -\sum_{\lambda_j} \frac{\eta(\lambda_j \lessapprox \tau_k)}{\lambda_j^2 - \tau_k^2} \beta(|\lambda_j - \tau_k| \lessapprox \lambda) e_j^0(x) e_j^0(y). \]

For \( \ell \in \mathbb{N} \) with \( 2^{\ell} \leq \lambda/100 \), let for \( j = 0, 1, 2, \ldots \)
\[ I_{-j}^-(\lambda - (j + 1)2^\ell, \lambda - j2^\ell) \quad \text{and} \quad I_{+j}^+(\lambda + j2^\ell, \lambda + (j + 1)2^\ell). \]

By the spectral projection estimates and the rapid decay property of \( h(\lambda_j) \), we have the following lemma.

**Lemma 12.** If \( \ell \in \mathbb{Z}_+, \ 2^{\ell} \leq \lambda/100, \) and \( j = 0, 1, 2, \ldots \), we have for each \( N \in \mathbb{N} \)
\[ \|K_{\tau,\ell}^+(\lambda, \cdot)\|_{L^2(M)} \|\frac{\partial}{\partial \tau} K_{\tau,\ell}^+(\lambda, \cdot)\|_{L^2(M)} \leq \lambda^{\frac{3}{2} - 1}(1 + j)^{-N}, \quad \tau \in I_{\ell,j}^- \cap [\lambda/2, 10\lambda]. \]

Also,
\[ \|K_{\tau,0}^+(\lambda, \cdot)\|_{L^2(M)} \|\frac{\partial}{\partial \tau} K_{\tau,0}^+(\lambda, \cdot)\|_{L^2(M)} \leq \lambda^{\frac{3}{2} - 1}(1 + j)^{-N}, \quad \tau \in I_{0,j}^+ \cap [\lambda/2, 10\lambda], \]

Moreover, we also have for \( 1 \leq 2^{\ell} \leq \lambda/100 \) and \( \tau \in [\lambda/2, 10\lambda] \)
\[ \|R_{\tau,\ell}(x, \cdot)\|_{L^2(M)} \|\frac{\partial}{\partial \tau} R_{\tau,\ell}(x, \cdot)\|_{L^2(M)} \leq \lambda^{\frac{3}{2} - 1}, \]
\[ |R_{\tau,\ell}(x, y)| \leq \begin{cases} \log(2 + (\lambda d_y(x, y))^{-1})(1 + \lambda d_y(x, y))^{-N}, & n = 2 \\ d_y(x, y)^{2-n}(1 + \lambda d_y(x, y))^{-N}, & n \geq 3. \end{cases} \]

Finally, we have for \( \tau \in (\lambda, 10\lambda] \)
\[ \|K_{\tau,\infty}^+(\lambda, \cdot)\|_{L^2(M)} \|\frac{\partial}{\partial \tau} K_{\tau,\infty}^+(\lambda, \cdot)\|_{L^2(M)} \leq \lambda^{\frac{3}{2} - 1}, \]
and for \( \tau \in [\lambda/2, \lambda] \)
\[ \|\tilde{K}_{\tau,\infty}^-(x, \cdot)\|_{L^2(M)} \|\frac{\partial}{\partial \tau} \tilde{K}_{\tau,\infty}^-(x, \cdot)\|_{L^2(M)} \leq \lambda^{\frac{3}{2} - 1}, \]
\[ |H_{\tau,\infty}(x, y)| \leq \begin{cases} \log(2 + (\lambda d_y(x, y))^{-1})(1 + \lambda d_y(x, y))^{-N}, & n = 2 \\ d_y(x, y)^{2-n}(1 + \lambda d_y(x, y))^{-N}, & n \geq 3. \end{cases} \]
We postpone the proof of Lemma 12 to the Appendix.

By Lemma 11 with \( \delta = 2 \ell \) along with the Lemma 12 and Lemma 10 we have

\[
(4.21) \quad \left| \sum_{\tau_k \in T_{\ell,j} \cap \{\lambda/2,10\lambda\}} \int K_{\pm}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)V(y) \, dy \right| \\
\leq \|V\|_{L^\infty(M)} \cdot \sup_x \left\{ \|K_{\lambda,32\ell}(x,\cdot)\|_{L^{2\infty}(M)} + \int_{T_{\ell,j}} \|\Delta_x K_{\pm}(x,\cdot)\|_{L^{2\infty}(M)} \, ds \right\} \\
\times \left( \sum_{\tau_k \in T_{\ell,j} \cap \{\lambda/2,10\lambda\}} |e_{\tau_k}(x)|^2 \right)^{1/2} \\
\lesssim \lambda^{1/2} - 1(1 + j)^{-N} \cdot \lambda^{\frac{3}{4} - 2}\ell^2 / 2 \cdot \|V\|_{L^\infty(M)} \\
\lesssim \lambda^{n-\frac{3}{2}} 2^{\ell/2}(1 + j)^{-N} \cdot \|V\|_{L^\infty(M)}.
\]

If we sum over \( j = 0,1,2,\ldots, \) we see that (4.21) yields

\[
(4.22) \quad \left| \sum_{\lambda < \tau_k \leq 10\lambda} \int K_{\tau_k,\ell}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)V(y) \, dy \right| \\
+ \left| \sum_{\lambda/2 \leq \tau_k \leq \lambda} \int K_{\tau_k,\ell}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)V(y) \, dy \right| \lesssim \|V\|_{L^\infty(M)} \lambda^{n-\frac{3}{2}} 2^{\ell/2}.
\]

If we take \( \delta = 1 \) in Lemma 11 this argument also gives

\[
(4.23) \quad \left| \sum_{\lambda < \tau_k \leq 10\lambda} \int K_{\tau_k,0}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)V(y) \, dy \right| \\
+ \left| \sum_{\lambda/2 \leq \tau_k \leq \lambda} \int K_{\tau_k,0}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)V(y) \, dy \right| \lesssim \|V\|_{L^\infty(M)} \lambda^{n-\frac{3}{2}}.
\]

If we take \( \delta = \lambda \) in Lemma 11 this argument also gives

\[
(4.24) \quad \left| \sum_{\lambda < \tau_k \leq 10\lambda} \int K_{\tau_k,\infty}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)V(y) \, dy \right| \lesssim \|V\|_{L^\infty(M)} \lambda^{n-1}.
\]

Similarly,

\[
(4.25) \quad \left| \sum_{\lambda/2 < \tau_k \leq \lambda} \int K_{\tau_k,\infty}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)V(y) \, dy \right| \lesssim \|V\|_{L^\infty(M)} \lambda^{n-1}.
\]

By (4.10), if we repeat the argument above, we have

\[
(4.26) \quad \left| \sum_{\lambda < \tau_k \leq 10\lambda} \int R_{\tau_k,\ell}(x,y)h(\tau_k)e_{\tau_k}(x)e_{\tau_k}(y)V(y) \, dy \right| \\
+ \left| \sum_{\lambda/2 \leq \tau_k \leq \lambda} \int R_{\tau_k,\ell}(x,y)(1 - h(\tau_k))e_{\tau_k}(x)e_{\tau_k}(y)V(y) \, dy \right| \lesssim \|V\|_{L^\infty(M)} \lambda^{n-\frac{3}{2}} 2^{\ell/2}.
\]
Moreover, by using (4.20), we have \( \|H_{_{\tau,\infty}}(x, \cdot)\|_{L^{\infty}R(M)} \lesssim \lambda^{-1} \) for \( \tau \in [\lambda/2, 10\lambda] \), and then
\[
\left| \sum_{\lambda/2 \leq \tau_k \leq \lambda} \int H_{_{\tau_k,\infty}}(x, y) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dy \right| \\
\lesssim \|V\|_{L^\infty(M)} \cdot \lambda^{-1} \sum_{\tau_k \in [\lambda/2, 10\lambda]} |e_{\tau_k}(x) e_{\tau_k}(\cdot)|_{\infty} \\
\lesssim \|V\|_{L^\infty(M)} \lambda^{n-1}.
\]
Similarly,
\[
(4.27) \quad \left| \sum_{\lambda < \tau_k \leq 10\lambda} \int R_{\tau_k,\infty}(x, y) h(\tau_k) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dy \right| \\
+ \left| \sum_{\lambda/2 \leq \tau_k \leq \lambda} \int R_{\tau_k,\infty}(x, y)(1 - h(\tau_k)) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dy \right| \leq \|V\|_{L^\infty(M)} \lambda^{n-1}.
\]
Hence, using the estimates above and summing over \( \ell \), we obtain (4.8).

5. Appendix: Proof of Lemmas

We give the proof of Lemma 12 and Lemma 11. They are essentially analogous to the lemmas in [14], but we prove them here for the sake of completeness.

Proof of Lemma 12. To prove Lemma 12, we shall need the fact that, by Lemma 5, for any fixed \( \ell \) with \( 1 \leq 2^\ell \leq \lambda/100 \), we have the following spectral projection estimates (Lemma 5)
\[
(5.1) \quad \|1_{\lambda, \lambda+2^\ell}(P^0)\|_{L^2 \rightarrow L^{\infty}} \lesssim 2^{\ell/2} \lambda^{1/2}.
\]

To prove the first inequality we note that if \( \tau \in I_{\ell,j}^{+} \cap [1, 10\lambda] \) then \( |\lambda_i - \tau| \leq 2^{\ell+1} \) if \( \beta(2^{-\ell}(\lambda_i - \tau)) \neq 0 \), and, in this case, we also have \( h(\lambda_i) - 1 = O((1 + j)^{-N}) \) if \( \tau \in I_{\ell,j}^{-} \) and \( h(\lambda_i) = O((1 + j)^{-N}) \) if \( \tau \in I_{\ell,j}^{+} \). Therefore, we have
\[
\|K_{\tau,\ell}^{\pm}(\cdot, y)\|_{L^{\infty}R(M)} \lesssim 2^{\ell/2} \lambda^{1/2} \|K_{\tau,\ell}^{\pm}(\cdot, y)\|_{L^2(M)} \\
\lesssim 2^{\ell/2} \lambda^{1/2} (1 + j)^{-N} 2^{-\ell} \lambda^{-1} \left( \sum_{\{i: |\lambda_i - \tau| \leq 2^{\ell+1}\}} |e_i^0(y)|^2 \right)^{1/2} \\
\lesssim (1 + j)^{-N} 2^{-\ell/2} \lambda^{-1/2} \sum_{\mu \in \mathbb{N}: |\mu - \tau| \leq 2^{\ell+1}} \mu^{n-1} \right)^{1/2} \\
\leq (1 + j)^{-N} \lambda^{n/2} \lambda^{-1}.
\]
In particular, if \( n = 2 \), the same argument implies that
\[
\|K_{\tau,\ell}^{\pm}(\cdot, y)\|_{L^\infty(M)} \leq (1 + j)^{-N},
\]
which proves the first part of (4.14). The other inequality in (4.14) follows from this argument since
\[
\frac{\partial}{\partial \tau} \beta(|\lambda_i - \tau| \approx 2^k) \approx 2(2^{-2^k \lambda^{-1}}),
\]
due to the fact that we are assuming that \(2^k \leq \lambda/100\).

This argument also gives us (4.15) if we use the fact that \(\tau \rightarrow (h(\tau) - h(\mu))/\tau^2 - \mu^2)\) is smooth and use the fact that
\[
\partial^k_x (\beta(|\lambda_i - \tau| \leq 1)(h(\lambda_i) - h(\tau))/\tau - \tau) = O((1 + j)^{-N}), \quad k = 0, 1, \tau \in I_{0,1},
\]
and we conclude that (4.17) is valid. □

To prove (4.18) we use the fact that for \(k = 0, 1\) we have for \(\tau \in (\lambda/2, 10\lambda]\)
\[
\left| \left( \frac{\partial}{\partial \tau} \right)^k (\beta(|\lambda_i - \tau| \approx \lambda)) \right| h(\lambda_i) \right| \lesssim \left\{ \begin{array}{ll}
\lambda^{-2-k} & \text{if } \lambda_i \leq \lambda \\
\lambda^{-2-k}(1 + |\lambda_i - \lambda|)^{-N} & \text{if } \lambda_i > \lambda.
\end{array} \right.
\]
Thus for \(k = 0, 1\), by (5.1)
\[
\| (\lambda \partial x)^k K_{\tau,\infty}(\cdot, y) \|_{L^\infty_x \Lambda_\tau(M)} \lesssim \lambda \cdot \left( \sum_{\lambda_i \leq 2\lambda} \lambda^{-4} |e_j^0(y)|^2 \right)^{\frac{1}{2}} + \sum_{j \in \mathbb{N}, 2^s > 2\lambda} 2^{s} \sum_{\lambda_i \leq 2^s} 2^{-4s}(1 + |\lambda_j - \lambda|)^{-N} |e_j^0(y)|^2 \right)^{\frac{1}{2}}
\lesssim \lambda^{1-\frac{\tau}{2}},
\]
as desired if \(N > 2n\). Similarly, \(\tilde{K}_{\tau,\infty}^\pm\) satisfies the bounds in (4.19).

Moreover, we can conclude from Lemma 8 that \(H_{\tau,\infty}^:\) satisfies the bounds in (4.20).
It just remains to prove the bounds in (4.10) for the \(R_{\tau,\infty}(x, y)\) and that in (4.14) for \(R_{\tau,\infty}(x, y)\). The former just follows from the proof of (4.14).

To prove the remaining inequality, (4.17), we note that
\[
R_{\tau,\infty}(x, y) = R_{\tau,\infty}^0(x, y) - H_{\tau,\infty}^-(x, y),
\]
if
\[
R_{\tau,\infty}^0(x, y) = \sum_i \eta(\lambda_i \lesssim \tau) \beta(|\lambda_i - \tau| \approx \lambda) \frac{e_i^0(x)e_i^0(y)}{\lambda_i^2 - \tau^2}.
\]
Since \(H_{\tau,\infty}^-\) satisfies (4.20), and Lemma 8 shows that
\[
|R_{\tau,\infty}^0(x, y)| \lesssim \tau^{-2}(1 + \tau d_g(x, y))^{-N}
\]
we conclude that (4.17) is valid. □

Proof of Lemma 7. We shall use the fact that
\[
m(\delta \tau; y) = m(0, y) + \int_0^\delta \mathbf{1}_{[0, \delta \tau]}(s) \frac{d}{ds} m(s, y) \, ds,
\]
where \(\mathbf{1}_{[0, \delta \tau]}(s)\) is the indicator function of the the interval \([0, \delta \tau] \subset [0, \delta]\). Therefore, by Hölder’s inequality and Minkowski’s inequality, the left side of (4.6) is dominated by
\[ \|V\|_{L^\infty(M)} \text{ times} \]
\[ \left( \int_M |m(0, y) \cdot \sum_{\tau_k \in I} a_k e_{\tau_k}(y)|^\frac{n-1}{n} \, dy \right)^{\frac{n}{n-1}} \]
\[ + \left( \int_M \left| \sum_{\tau_k \in I} \int_0^\delta \mathbf{1}_{[0, \delta_{\tau_k}]}(s) \frac{\partial}{\partial s} m(s, y) a_k e_{\tau_k}(y) \, ds \right|^\frac{n}{n-1} \, dy \right)^{\frac{n}{n-1}} \]
\[ \leq \|m(0, \cdot)\|_\frac{n}{n-2} \cdot \left\| \sum_{\tau_k \in I} a_k e_{\tau_k} \right\|_2 + \int_0^\delta \left( \left\| \frac{\partial}{\partial s} m(s, \cdot) \right\|_\frac{n}{n-2} \cdot \left\| \sum_{\tau_k \in I} \mathbf{1}_{[0, \delta_{\tau_k}]}(s) a_k e_{\tau_k} \right\|_2 \right) \, ds \]
\[ \leq \|m(0, \cdot)\|_\frac{n}{n-2} + \int_0^\delta \left( \left\| \frac{\partial}{\partial s} m(s, \cdot) \right\|_\frac{n}{n-2} \right) \times \left( \sum_{\tau_k \in I} |a_k|^2 \right)^{\frac{1}{2}} \]

as desired. \hfill \square

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