Entropy zero area preserving diffeomorphisms of $S^2$

John Franks,* Michael Handel†

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Abstract

In this paper we formulate and prove a structure theorem for area preserving diffeomorphisms of $S^2$ with zero entropy. As an application we relate the existence of faithful actions of a finite index subgroup of the mapping class group of a closed surface $\Sigma_g$ on $S^2$ by area preserving diffeomorphisms to the existence of finite index subgroups of bounded mapping class groups $\text{MCG}(S, \partial S)$ with non-trivial first cohomology.

1 Introduction and Statement of Results

For any smooth area form $\mu$ on $S^2$, let $\text{Diff}_\mu(S^2)$ be the group of $C^\infty$ diffeomorphisms of $S^2$ that preserve $\mu$. In particular they preserve the orientation of $S^2$. Elements of $\text{Diff}_\mu(S^2)$ are said to preserve area. Surface diffeomorphisms with positive entropy have been studied from both the hyperbolic dynamical systems point of view and the Nielsen-Thurston point of view. In this paper we formulate and prove a structure theorem for $F \in \text{Diff}_\mu(S^2)$ with zero entropy. The assumptions that $S = S^2$ and that $F$ preserves area are made to simplify the problem.

Suppose that $F \in \text{Diff}_\mu(S^2)$ has zero topological entropy and that $\text{Fix}(F)$ is the set of fixed points for $F$. To enhance the topology of the ambient surface, we consider $\mathcal{M} = S^2 \setminus \text{Fix}(F)$ and $f = F|_{\mathcal{M}}: \mathcal{M} \to \mathcal{M}$.

Every $x \in \mathcal{M}$ has a neighborhood $B$ that is a free disk, meaning that $B$ is an open disk and that $f(B) \cap B = \emptyset$. A very weak notion of recurrence for a point $x \in \mathcal{M}$ is to require that there be $n \neq 0$ and a free disk $B$ that contains both $x$ and $f^n(x)$. Note that this is equivalent to saying there is a free disk $B$ which intersects $\text{orb}(x)$, the orbit of $x$, in at least two points. We will call such points free disk recurrent and denote the set of these points by $\mathcal{W}_0$. Clearly, if either $\alpha(F, x)$ or $\omega(F, x)$ contains a point which is not in $\text{Fix}(F)$ then $x \in \mathcal{W}_0$. In particular the set $\mathcal{W}_0$ contains the full measure subset of $\mathcal{M}$ consisting of birecurrent points. The set $\mathcal{W}_0$ is open and dense.

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in \( \mathcal{M} \). However, its components may not be equal to the interior of their closure, and this is a property which is technically useful. Consequently we define the larger set \( \mathcal{W} \) of \emph{weakly free disk recurrent} points as follows.

**Definition 1.1.** A point \( x \in \mathcal{M} \) is \emph{free disk recurrent} for \( f \) provided there is an \( f \)-free disk \( B \) in \( \mathcal{M} \), such that \( \text{orb}(x) \), the orbit of \( x \) intersects \( B \) in at least two points. The set of free disk recurrent points in \( \mathcal{M} \) is denoted \( \mathcal{W}_0 \). If \( \mathcal{W}_0 \) is a component of \( \mathcal{W} \) and \( x \in \mathcal{M} \) is in \( \text{int} ( \text{cl}_\mathcal{M}(\mathcal{W}_0)) \), then we say that \( x \) is \emph{weakly free disk recurrent}. The set of weakly free disk recurrent points in \( \mathcal{M} \) is denoted \( \mathcal{W} \).

There are several immediate consequences of the definition which we record for later use.

**Remark 1.2.** The set \( \mathcal{W} \subset \mathcal{M} \) is open, dense, and contains all birecurrent points for \( f \). The complement of \( \mathcal{W} \) is a closed set with measure zero. If any point of \( \omega(x, F) \) or \( \alpha(x, F) \) is not in \( \text{Fix}(F) \) then a free disk containing that point will contain infinitely many points of \( \text{orb}(x) \) so \( x \) is free disk recurrent.

There are certain elements of \( \text{Diff}_\mu(S^2) \) which are trivial from the point of view of their periodic points. These include \( F \in \text{Diff}_\mu(S^2) \) of finite order and \( F \) for which \( \text{Per}(F) \) contains only two points. It is known that an area preserving \( F \) must have at least two fixed points (see [24]). In the case that \( \text{Per}(F) \) contains exactly two points those points must be fixed since \( F \) must have a fixed point and the complement of those points may be compactified to form a closed annulus and \( f \) extends to this closed annulus with every point of that annulus having the same irrational rotation number. This is an interesting topic to investigate but is not addressed in this article.

The main building block in our structure theorem is a partition of \( \mathcal{W} \) into countably many disjoint \( f \)-invariant annuli.

**Theorem 1.3.** Suppose \( F \in \text{Diff}_\mu(S^2) \) has entropy zero. Let \( f = F|_\mathcal{M} \) where \( \mathcal{M} = S^2 \setminus \text{Fix}(F) \). Then there is a countable collection \( \mathcal{A} \) of pairwise disjoint open \( f \)-invariant annuli such that

1. The union \( \bigcup_{U \in \mathcal{A}} U \) is the the set \( \mathcal{W} \) of weakly free disk recurrent points for \( f \).

2. If \( U \in \mathcal{A} \) and \( z \) is in the frontier of \( U \) in \( S^2 \), there are components \( F_+(z) \) and \( F_-(z) \) of \( \text{Fix}(F) \) so that \( \omega(F, z) \subset F_+(z) \) and \( \alpha(F, z) \subset F_-(z) \).

3. For each \( U \in \mathcal{A} \) and each component \( C_\mathcal{M} \) of the frontier of \( U \) in \( \mathcal{M} \), \( F_+(z) \) and \( F_-(z) \) are independent of the choice of \( z \in C_\mathcal{M} \).

**Remark 1.4.** If \( h : S^2 \to S^2 \) commutes with \( F \) then it permutes the open annuli in the family \( \mathcal{A} \). This is because each \( U \in \mathcal{A} \) is a component of \( \mathcal{W} \) and \( h \) is a conjugacy from \( F \) to itself and hence preserves \( \mathcal{W} \).
To see how the elements of $\mathcal{A}$ arise, consider the special case that $F$ is the time one map of an area preserving flow $\phi_t$. Given $x \in \mathcal{M}$, choose a free disk neighborhood $B$ of $x$ which is also a flow box for $\phi_t$. It is an easy consequence of the Poincare-Bendixson theorem that if the flow line for $\phi_t$ that contains $x$ returns to $B$ it closes up into a simple closed curve $\rho_x$. In particular, in this case the subsets $\mathcal{W}_0$ and $\mathcal{W}$ are equal and coincide with the union of the periodic orbits of the flow which lie in $\mathcal{M}$. Denote the isotopy class of $\rho_x$ in $\mathcal{M}$ by $[\rho_x]$. It is clear that $\rho_x$ depends only on the orbit of $x$ and not $x$ itself and that if $z \in B$ is sufficiently close to $x$ then $\rho_x$ and $\rho_z$ cobound an annulus in $M$; in particular $[\rho_x] = [\rho_z]$. In this case $U = \{y \in \mathcal{W} : [\rho_y] = [\rho_x]\}$ is the element of $\mathcal{A}$ that contains $x$.

For a second special case suppose that $f$ is isotopic to the identity. Given $x \in \mathcal{W}_0$, choose $B$ and $n$ as in the definition of weakly free disk recurrent. If $f_t : \mathcal{M} \rightarrow \mathcal{M}$ is an isotopy between $f_0 = \text{identity}$ and $f_1 = f$ then the path $\mu_x \subset \mathcal{M}$ defined by $\mu_x(t) = f_t(x)$ connects $x$ to $f(x)$. The path $\mu_x \cdot \mu_{f(x)} \cdot \ldots \cdot \mu_{f^{n-1}(x)}$ can be closed by adding a path in $B$ connecting $f^n(x)$ to $x$. Up to homotopy in $\mathcal{M}$, this closed path is a multiple of some non-repeating closed path $\rho_x$. Using the hypothesis that $F$ has entropy zero, one can show (see [10]) that the homotopy class of $\rho_x$ is represented by a simple closed curve, (also written $\rho_x$) that is independent of $B, n$ and the choice of isotopy $f_t$. It is easy to see that if $z \in B$ is sufficiently close to $x$ then $[\rho_x] = [\rho_z]$. As in the previous case, $U = \{y \in \mathcal{W} : [\rho_y] = [\rho_x]\}$ is the element of $\mathcal{A}$ that contains $x$.

In the general case, we make use of the fact (Theorem 1.2, Lemma 6.3 and Remark 6.4, all in [10]) that $f$ is isotopic to a composition of Dehn twists along a finite set of simple closed curves $\mathcal{R}$. Cutting along the elements of $\mathcal{R}$ produces a decomposition of $\mathcal{M}$ into subsurfaces $\mathcal{M}_i$ such that $f|_{\mathcal{M}_i} : \mathcal{M}_i \rightarrow \mathcal{M}$ is isotopic to the inclusion $\mathcal{M}_i \hookrightarrow \mathcal{M}$. The main technical work in this proof is showing that each $\mathcal{M}_i$ is realized, in a suitable sense, by an $f$-invariant subsurface; see section 9. One then defines $\mathcal{A}$ in a fashion similar to the second special case.

Theorem 1.3 can be applied to $F^q$ for each $q \geq 2$. This gives a countable collection $\mathcal{A}(q)$ of pairwise disjoint open $F^q$-invariant annuli that (see Proposition 12.2) refines $\mathcal{A}$ in the sense that each $V_j \in \mathcal{A}(q)$ is contained in some $U_i \in \mathcal{A}$. This renormalization process can be iterated with $\mathcal{A}(q)$ playing the role of $\mathcal{A}$ and so on. The $V_j$'s may be essential or inessential in $U_i$. In the limit, the former lead to twist-map-like behavior and the latter to solenoid-like behavior when they are nested infinitely often. It is important to note that replacing $F$ with $F^q$ changes the set of fixed points and hence changes $\mathcal{M}$ and changes the free disk recurrent points of $\mathcal{M}$.

We are interested in partitioning $cl(U)$ into sets analogous to the periodic orbits in the case of the time one map of a flow. In particular we would like the rotation number to be constant on these sets. The two components of the frontier of $U$ can be somewhat problematic since such a component could be a single point or could be a complicated fractal. To deal with this issue we introduce a natural compactification $U_c$ of $U$ and a natural extension $f_c$ of $f$ (see Definition 2.7 in the next section). The compactification of an end described there is either the prime end compactification or
the compactification obtained by blowing up a fixed point, whichever is appropriate. The set of two rotation numbers of \( f_c \) on the two circles added in the compactification will be denoted by \( \rho(\partial U_c) \).

We are now prepared to state the second of our main results. It describes the finer structure of the dynamics of \( f \) on one of the annuli in \( \mathcal{A} \). The proof is based on renormalization and the details are in Section 12.

\[ \textbf{Theorem 1.5.} \text{ Suppose } F \in \text{Diff}_\mu(S^2) \text{ has entropy zero. Let } f = F|_\mathcal{M} \text{ where } \mathcal{M} = S^2 \setminus \text{Fix}(F) \text{ and let } \mathcal{A} \text{ be as in Theorem 1.3. Then for each } U \in \mathcal{A} \text{ there is a partition of } cl(U) \subset S^2 \text{ into a family } \mathcal{C} \text{ of closed } F\text{-invariant sets with the following properties:} \]

- Each \( C \in \mathcal{C} \) is compact and connected.
- There are two elements of \( \mathcal{C} \) (which may coincide), called ends and denoted \( C_0 \) and \( C_1 \), each of which contains a component of the frontier of \( U \). Every element of \( \mathcal{C} \) which is not an end is a subset of \( U \) and is called interior. Each interior \( C \) is essential in \( U \), i.e. its complement (in \( U \)) has two components and it separates \( C_0 \) and \( C_1 \).
- The rotation number \( \rho_f(x) \in \mathbb{R}/\mathbb{Z} \) is well defined and constant on any interior \( C \). In fact, each \( C \) is a connected component of a level set of \( \rho_f(x) \). Moreover, \( \rho_f(x) \) is is continuous on \( U \) and has a unique continuous extension to \( cl(U) \). The value of this extension on \( C_i \), \( i = 0, 1 \) is the element of \( \rho(\partial U) \) corresponding to the component of the frontier of \( U \) contained in \( C_i \).

We emphasize the fact that the \( C \) containing a point \( x \) may be very different for \( F^q \) than for \( F \), and, in fact, will be undefined for \( F^q \) if \( x \) is a periodic point whose period divides \( q \).

\[ \text{Remark 1.6.} \text{ There are some degenerate cases where both Theorem 1.3 and Theorem 1.5 remain true, but are not particularly interesting. For example, if } F \text{ is the identity then } \mathcal{M} \text{ and } \mathcal{A} \text{ are empty and the results are vacuously true. Slightly more interesting is the case when } \text{Fix}(F) \text{ contains only two points and all points of the annulus } U = \mathcal{M} = S^2 \setminus \text{Fix}(F) \text{ have the same rotation number, (e.g. if } F \text{ is an isometry of some smooth metric). Then } U = \mathcal{M} \text{ is an } f\text{-invariant annulus. In this case } \mathcal{A} \text{ contains the single annulus } U = \mathcal{M} \text{ and (as we show in Proposition 2.10)} \text{ every point of } U \text{ is free disk recurrent. Also in this case Theorem 1.5 remains true since the two “ends” } C_0 \text{ and } C_1 \text{ coincide, there are no interior } C\text{'s, and } C_0 = C_1 = S^2. \]

The sets in \( \mathcal{C} \) are the generalizations of the closed orbits foliating \( U \) in the special case that \( F \) is the time one map of a flow. Of course in the general case \( C \in \mathcal{C} \) can be considerably more complicated. The main example constructed in 13 shows \( C \) can be a pseudo-circle. It is also possible for \( \mathcal{C} \) to have interior.

A heuristic picture of one possibility in the case that \( \rho_f|_C \) is rational is an essential “necklace” in \( U \) consisting of a periodic orbit of saddle periodic points each joined to
the next by a stable manifold (which is the unstable manifold of the next) and by an unstable manifold (which is the stable manifold of the next). This pair, stable and unstable, bound a “bead”, an open disk. The diffeomorphism \( f \) permutes the beads and has a periodic orbit with one point in each bead. The set \( C \) containing any \( x \) in one of the beads will be the entire necklace. For such a \( C \) there is an \( n \) such that \( f^n \) will fix each bead and each saddle point joining them.

We now turn to applications. Recall that a group \( G \) is a \textit{indicable} if there exists a non-trivial homomorphism \( G \to \mathbb{Z} \). For finitely generated groups this is equivalent to \( H^1(G, \mathbb{Z}) \neq 0 \) and equivalent to the abelianization of \( G \) being infinite. If a finite index subgroup of \( G \) is indicable then we say that \( G \) is \textit{virtually indicable}.

Denote the centralizer of \( f \in \text{Diff}_\mu(S^2) \) by \( Z(f) \). As an application of Theorem 1.3 and Theorem 1.5 we prove

\textbf{Theorem 1.7.} If \( f \in \text{Diff}_\mu(S^2) \) has infinite order then each finitely generated infinite subgroup \( H \) of \( Z(f) \) is virtually indicable.

One might expect that Theorem 1.7 is proved by first proving the existence of a finite index subgroup \( H_0 \) of \( H \) with global fixed points and then applying the Thurston stability theorem (see also Theorem 3.4 of [9]) to produce a non-trivial homomorphism from \( H_0 \) to \( \mathbb{Z} \). This is easy to do (see Proposition 13.1) in the case that \( f \) has positive entropy but fails when \( f \) has zero entropy. Indeed, there are examples (see 13.2) for which no finite index subgroup of \( Z(f) \) has a global fixed point. We prove Theorem 1.7 by analyzing the possible ways in which the existence of global fixed points can fail and by showing that each allows one to define a non-trivial homomorphism to \( \mathbb{Z} \).

As an application of Theorem 1.7 we have the following result about mapping class groups.

\textbf{Corollary 1.8.} If \( \Sigma_g \) is the closed orientable surface of genus \( g \geq 2 \) then at least one of the following holds.

1. No finite index subgroup of \( \text{MCG}(\Sigma_g) \) acts faithfully on \( S^2 \) by area preserving diffeomorphisms.

2. For all \( 1 \leq k \leq g - 1 \), there is an indicable finite index subgroup \( \Gamma \) of the bounded mapping class group \( \text{MCG}(S_k, \partial S_k) \) where \( S_k \) is the surface with genus \( k \) and connected non-empty boundary.

Corollary 1.8 relates to the following well known questions about mapping class groups.

\textbf{Question 1.9.} Does \( \text{MCG}(\Sigma_g) \), or any of its finite index subgroups, act faithfully on a closed surface \( S \) by diffeomorphisms? by area preserving diffeomorphisms?

\textbf{Question 1.10 (Ivanov).} Does every finite index subgroup \( \Gamma \) of \( \text{MCG}(\Sigma_g) \) satisfy \( H^1(\Gamma, \mathbb{R}) = 0 \)?
Question 1.9 is motivated in part by the sections problem (see Problem 6.5 and Question 6.7) of Farb’s survey/problem list [3] on the mapping class group: which subgroups of \( \text{MCG}(\Sigma_g) \) lift to \( \text{Diff}(\Sigma_g) \)? It is also motivated by the analogy between mapping class groups and higher rank lattices and the fact (\[23\],\[10\], \[11\]) that every action of a non-uniform irreducible higher rank lattice on \( \Sigma_g \) by area preserving diffeomorphisms factors through a finite group; see Question 12.4 of Fisher’s survey article [4] on the Zimmer program.

Question 1.10 is Problem 2.11 of [18]; see also [16] and [19]. Corollary 1.8-(1) is a negative answer to the area preserving, \( S = S^2 \) case of Question 1.9. The answer to Question 1.10 is no for genus 2 (see [21]) but is unknown for for genus at least three. Presumably a positive answer to Question 1.10 would imply that Corollary 1.8-(2) does not hold and so imply that Corollary 1.8-(1) does hold.

2 Area preserving annulus maps.

We will make use of a number of results on area preserving homeomorphisms and diffeomorphisms of the annulus which we cite here.

If \( A = S^1 \times [0, 1] \) is the annulus its universal covering space is \( \tilde{A} = \mathbb{R} \times [0, 1] \). We will denote by \( p_1 \) the projection, \( p_1 : \mathbb{R} \times [0, 1] \to \mathbb{R} \), of \( A \) onto its first factor.

**Definition 2.1.** If \( f : A \to A \) is an orientation preserving homeomorphism isotopic to the identity and \( \tilde{f} \) is a lift to \( \tilde{A} \) then the translation number \( \tau_f(\tilde{x}) \) of \( \tilde{x} \in \tilde{A} \) is defined (if it exists) to be

\[
\lim_{n \to \infty} \frac{p_1(\tilde{f}^n(\tilde{x})) - p_1(\tilde{x})}{n}.
\]

The rotation number \( \rho_f(x) \) of \( x \in A \) is defined to be the projection of \( \tau_f(\tilde{x}) \) in \( T^1 = \mathbb{R}/\mathbb{Z} \) for any lift \( \tilde{f} \) of \( f \) and any lift \( \tilde{x} \) of \( x \).

There are several results about these numbers which we will need.

**Theorem 2.2.** Suppose \( f : A \to A \) is an area preserving homeomorphism of the closed annulus which is isotopic to the identity and let \( \tilde{f} : \tilde{A} \to \tilde{A} \) be a lift to its universal covering space. Then \( \tau_f(\tilde{x}) \) exists for almost all \( \tilde{x} \in \tilde{A} \). Moreover if there exist points \( \tilde{x}, \tilde{y} \in \tilde{A} \) with \( \tau_f(\tilde{x}) = s \) and \( \tau_f(\tilde{y}) = t \) then for any rational in lowest terms \( p/q \in [s, t] \) there is a point \( \tilde{z} \in \tilde{A} \) such that \( \tau_f(\tilde{z}) = p/q \) and \( \tilde{z} \) is the lift of a periodic point for \( f \) with period \( q \).

**Proof.** The fact that \( \tau_f(\tilde{x}) \) exists for almost all \( \tilde{x} \) is a standard consequence of the Birkhoff ergodic theorem applied to the function \( \phi(x) = p_1(\tilde{f}(\tilde{x})) - p_1(\tilde{x}) \). The remainder of the result is Corollary 2.4 of [6].
Definition 2.3. If \( f : A \to A \) is an area preserving homeomorphism isotopic to the identity and \( \tilde{f} \) is a lift to \( \tilde{A} \), then the translation interval \( \tau(\tilde{f}) \) of \( \tilde{f} \) is the smallest closed interval that contains \( \{\tau_f(\tilde{x}) : \tilde{x} \in \tilde{A}\} \). The rotation interval \( \rho(f) \) of \( f \) is defined to be the projection of \( \tau(f) \) in \( \mathbb{T}^1 = \mathbb{R}/\mathbb{Z} \) for any lift \( \tilde{f} \) of \( f \).

The terminology of the previous definition is appropriate in light of the following theorem.

Theorem 2.4 ([14]). Suppose \( f : A \to A \) is an area preserving homeomorphism of the closed annulus which is isotopic to the identity and let \( \tilde{f} : \tilde{A} \to \tilde{A} \) be a lift to its universal covering space. Then the set \( I \) of \( r \in \mathbb{R} \) such that there is \( \tilde{x} \in \tilde{A} \) with \( \tau_f(\tilde{x}) = r \) is closed and hence \( I \) is the closed interval \( \tau(\tilde{f}) \).

The following result can be found as Theorem (3.3) of [5].

Proposition 2.5 ([5]). Suppose \( f : A \to A \) is an area preserving homeomorphism of the closed annulus which is isotopic to the identity and let \( \tilde{f} : \tilde{A} \to \tilde{A} \) be a lift to its universal covering space and \( \tau(\tilde{f}) = [r, s] \) its translation interval. Then for all \( \tilde{x} \in \tilde{A} \)

\[
r \leq \liminf_{n \to \infty} \frac{p_1(\tilde{f}^n(\tilde{x})) - p_1(\tilde{x})}{n} \leq \limsup_{n \to \infty} \frac{p_1(\tilde{f}^n(\tilde{x})) - p_1(\tilde{x})}{n} \leq s.
\]

Proposition 2.6. Suppose \( f : A \to A \) is an area preserving homeomorphism of the closed annulus which is isotopic to the identity. If there is a subset \( \tilde{Y} \subset \tilde{A} \) with Lebesgue measure \( \mu(Y) > 0 \) and such that \( \rho_f(x) = 0 \) for almost all \( x \in Y \) then \( f \) has a fixed point in the interior of \( A \).

Proof. Since \( \mu(Y) > 0 \) there is a small open disk \( D \) whose closure is in the interior of \( A \) with \( \mu(Y \cap D) > 0 \). If \( f \) has no fixed point in \( D \) then by making \( D \) smaller we may assume it is a free disk. We let \( X = Y \cap D \). Let \( r : X \to X \) be the first return map so \( r(x) = f^n(x) \) where \( n \) is the smallest positive integer such that \( f^n(x) \in X \). The function \( r \) is well defined for almost all \( x \in X \), so deleting a set of measure 0 from \( X \) we may assume it defined for all \( x \in X \).

Let \( \tilde{D} \) be a lift of \( D \). If \( \tilde{X} \) is the set of lifts to \( \tilde{D} \) of points in \( X \) then there is a positive measure subset \( \tilde{X}_0 \subset \tilde{X} \) and a lift \( \tilde{f} \) of \( f \) such that \( \tau(\tilde{x}, \tilde{f}) = 0 \) for all \( \tilde{x} \in \tilde{X}_0 \), where \( \tilde{X}_0 \) is the lift of \( X_0 \) to \( \tilde{D} \).

Suppose the first return time for \( x \) is \( n \), so \( r(x) = f^n(x) \). Then \( \tilde{f}^n(\tilde{x}) \in T^k(\tilde{D}) \) for a unique integer \( k \). We define \( h(x, \tilde{f}) \), the homological displacement of \( x \), to be \( k \). It depends on \( \tilde{f} \) but not on the choice of lift \( \tilde{D} \) of \( D \).

It suffices to prove that \( h(x, \tilde{f}) = 0 \) for some \( x \in X_0 \) because then \( \tilde{x} \) is contained in a periodic disk chain (see Proposition (1.3) of [5]) and \( \tilde{f} \) has a fixed point. We note that if there are \( x, y \in X_0 \) such that \( h(x, \tilde{f}) > 0 \) and \( h(y, \tilde{f}) < 0 \) then \( \tilde{f} \) has a fixed point. This is a consequence of Theorem (2.1) of [5] since there are both positive and negative recurring disk chains for \( f \). Hence we may assume \( h(x, \tilde{f}) \) has a constant sign.
A result of [8] shows that if
\[ B = \bigcup_{n \in \mathbb{Z}} f^n(X_0) \]
then
\[ \int_{X_0} h(x, \tilde{f}) \, d\mu = \int_B \tau(x, \tilde{f}) \, d\mu. \]
Since \( \tau(x, \tilde{f}) = 0 \) for all \( x \in X_0 \) we conclude that \( \int_{X_0} h(x, \tilde{f}) \, d\mu = 0 \). Since \( h \) has constant sign it follows that \( h(x, \tilde{f}) = 0 \) for almost all \( x \in X_0 \).

Suppose \( U \subset S^2 \) is an open \( f \)-invariant annulus. We would like to compactify \( U \) to a closed annulus for which \( f \) has a natural extension. The annulus \( U \) has two ends which we compactify separately in a way depending on the nature of the end. We say that an end of \( U \) is singular if the complementary component of \( U \) that it determines is a single, necessarily fixed, point \( x \in S^2 \). In this case we compactify that end by blowing up \( x \) to obtain a circle on which \( f \) acts by the projectivization of \( Df_x \). If the end is not singular we will take the prime end compactification (see Mather [20] for properties). In either case we obtain a closed annulus \( U_c \) whose interior is naturally identified with \( U \) in such a way that \( f|_U \) extends to a homeomorphism \( f_c : U_c \to U_c \).

**Notation 2.7.** We will call \( U_c \) the annular compactification of \( U \) and \( f_c : U_c \to U_c \) the annular compactification of \( f|_U \). If there is no ambiguity about the choice of \( f \) we will denote \( \rho(f_c) \) by \( \rho(U) \) and the two rotation numbers of the restriction of \( f_c \) to its boundary circles by \( \rho(\partial U_c) \).

**Lemma 2.8.** Let \( f \) be an area and orientation preserving diffeomorphism of either \( S^2 \) or the closed disk \( D^2 \). Suppose \( U \) is an open \( f \)-invariant annulus and \( f_c : U_c \to U_c \) is the extension of \( f \) to its annular compactification. If there is a point \( x \in U_c \) with \( \rho(f_c)(x) = 0 \) then there is a fixed point \( \bar{x} \) of \( f \) in \( \text{cl}(U) \). If \( x \) lies in the component \( X \) of \( \partial U_c \) corresponding to a component \( \bar{X} \) of the frontier of \( U \) then \( \bar{x} \in \bar{X} \).

**Proof.** The function \( f_c \) has a fixed point by Theorem (2.2). If it is in \( \text{int}(U_c) = U \) we are done. So we may assume it is in a boundary component \( X \) of \( U_c \). If \( X \) corresponds to a singular end of \( U \) then the fixed point corresponding to that end is \( \bar{X} \subset \text{cl}(U) \). Otherwise \( X \) is the prime end compactification of an end of \( U \) and each prime end \( x \in X \) is defined by a sequence of “cross-cuts” \( \{\gamma_n\} \) where each \( \gamma_n \) is a Jordan arc whose interior is in \( U \) and whose endpoints are in the frontier of \( U \). They satisfy

\begin{enumerate}
  \item The limit \( \lim_{n \to \infty} \text{diam}(\gamma_n) = 0 \).
  \item Each \( \gamma_n \) has two complementary components in \( U \), one of which is an annulus and the other of which is an open disk which we will denote \( D_n \).
\end{enumerate}
(3) The disk $D_{n+1}$ is a subset of $D_n$ and $\bigcap_n D_n = \emptyset$.

Two such sequences of cross-cuts $\{\gamma_n\}$ and $\{\gamma'_m\}$ determine the same prime end if for each $n$ there is an $m$ with $D_{m}^\prime \subset D_n$ and for each $m$ an $n$ with $D_n \subset D'_m$.

Let $\{\gamma_n\}$ determine a prime end in $X$ which is fixed by $f_c$. Then from the fact that $f$ preserves area it follows that $f(\gamma_n) \cap \gamma_n \neq \emptyset$. For $n \geq 1$ choose $x_n \in \text{int}(\gamma_n)$. From the properties above it follows that any point in the limit set of the sequence $\{x_n\}$ is a fixed point of $f$. It is clearly in $X$. \hfill \Box

**Corollary 2.9.** Let $\mathcal{G}$ be a group of area and orientation preserving diffeomorphisms of $S^2$ or the closed disk $D^2$. Suppose $U$ is an open $\mathcal{G}$-invariant annulus and $\mathcal{G}_c$ is the group of homeomorphisms of $g_c : U_c \to U_c$, for $g \in \mathcal{G}$. If there is a point $x \in \text{Fix}(\mathcal{G}_c)$ then $\text{cl}(U)$ contains a point $\bar{x}$ of $\text{Fix}(\mathcal{G})$. If $x$ lies in the component $X$ of $\partial U_c$ corresponding to a component $\bar{X}$ of the frontier of $U$ then $\bar{x} \in \bar{X}$.

**Proof.** If $x$ is a point of $\text{Fix}(\mathcal{G}_c)$ and $x \in \text{int}(U_c) = U$ we are done. So we may assume it is in a boundary component $X$ of $U_c$. If $X$ corresponds to a singular end of $U$ then the point corresponding to that end is in $\text{cl}(U) \cap \text{Fix}(\mathcal{G})$. Otherwise $X$ is the prime end compactification of an end of $U$. Let $\{\gamma_n\}$ be a sequence of cross-cuts that determine a prime end in $X$ which is in $\text{Fix}(\mathcal{G}_c)$. Then, as in the previous lemma, the fact that each $g \in \mathcal{G}$ preserves area implies that $g(\gamma_n) \cap \gamma_n \neq \emptyset$. For $n \geq 1$ choose $x_n \in \text{int}(\gamma_n)$. From the properties above it follows that any point in the limit set of the sequence $\{x_n\}$ is in $\text{Fix}(g)$. Since this is independent of the choice of $g \in \mathcal{G}$ it follows that any point in the limit set is in $\text{Fix}(\mathcal{G})$. \hfill \Box

**Proposition 2.10.** Suppose $f : A \to A$ is an area preserving homeomorphism of the closed annulus which is isotopic to the identity and suppose every point of $A$ has the same rotation number. Let $U = \text{int}(A)$. Then either $f$ has a fixed point in $U$ or every point of $U$ is free disk recurrent for $f|_U$.

**Proof.** If the rotation number of all points of $A$ is $0$, then Proposition 2.6 implies that $f$ has a fixed point in $U$. Hence we may assume the common rotation number of the points of $A$ is non-zero and consequently $\text{Fix}(f) = \emptyset$. Suppose $x \in U$ and $z \in \omega(x) \subset A$. If $z \in U$, then any free disk containing $z$ intersects $\text{orb}(x)$ in infinitely many points. If $z \in \partial A$ let $V$ be a free half disk neighborhood of $z$ in $A$ and let $V_0 = V \cap U$. Then $\text{orb}(x)$ intersects $V_0$ infinitely often. \hfill \Box

### 3 Planar topology

In this section we record and prove two useful elementary results.

Recall that by the Riemann mapping theorem, every open, unbounded, connected, simply connected subset of $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^2$. A closed set $X \subset \mathbb{R}^2$ is said to separate two subsets $A$ and $B$ of $\mathbb{R}^2$ provided $A$ and $B$ are contained in different components of $\mathbb{R}^2 \setminus X$.
Lemma 3.1. If $A$ and $B$ are disjoint closed connected subsets of $\mathbb{R}^2$ then they are separated by a simple closed curve or a properly embedded line.

Proof. Choose a smooth function $\phi : \mathbb{R}^2 \to [0, 1]$ such that $\phi(A) = 0$ and $\phi(B) = 1$ and a regular value $c \in (0, 1)$. Then $\phi^{-1}(c)$ is a countable union of properly embedded lines and simple closed curves. Each component of $\phi^{-1}(c)$ has a collar neighborhood which is disjoint from the other components.

Let $U$ denote the component of the complement of $\phi^{-1}(c)$ which contains $B$ and let $X$ denote the frontier of $U$. Then $X$ also separates $A$ and $B$ and $X$ consists of a countable subcollection of the components of $\phi^{-1}(c)$, each of which is also a component of $X$. The set $U$ is the component of $\mathbb{R}^2 \setminus X$ which contains $B$. Each component $L$ of $X$ separates $\mathbb{R}^2$ into two open sets, one of which contains $B$ and $X \setminus L$ and the other of which is disjoint from $X$ and $B$.

Let $V$ be the component of $\mathbb{R}^2 \setminus X$ containing $A$. Consider a curve $\gamma$ running from a point of $A$ to a point of $B$ and let $L_0$ be the first component of $X$ which $\gamma$ intersects. The component $L_0$ is independent of the choice of $\gamma$, since $L_0$ separates $A$ from all other components of $X$. It follows that $A$ and $B$ are in different components of the complement of $L_0$ since otherwise they could be joined by a $\gamma$ which does not intersect $L_0$.

Lemma 3.2. If $U \subset \mathbb{R}^2$ is open and connected then each component $Z$ of the complement of $U$ has connected frontier and connected complement.

Proof. The complement of $Z$ is the union of $U$ with some of its complementary components and is therefore connected. If the frontier $W$ of $Z$ is not connected then by Lemma 3.1 there is a separation of $W$ by a set $Y \subset \mathbb{R}^2$ that is either a simple closed curve or a properly embedded line. Note that both components of $\mathbb{R}^2 \setminus Y$ must intersect both the interior of $Z$ and $\mathbb{R}^2 \setminus Z$. Since $Y$ is disjoint from the frontier $W$ of $Z$, it is contained in either the interior of $Z$ or in $\mathbb{R}^2 \setminus Z$. In the former case $Y$ separates $Z$ and in the latter case $Y$ separates $\mathbb{R}^2 \setminus Z$. This contradicts the fact that $Z$ and $\mathbb{R}^2 \setminus Z$ are connected and so proves that the frontier of $Z$ is connected.

4 Hyperbolic Structures

In this section we establish notation and recall standard results about hyperbolic structures on surfaces.

Suppose that $M$ is a connected open subset of $S^2$ that is not homeomorphic to either the open disk or open annulus. A closed curve $\alpha$ in $M$ is essential if it is not freely homotopic to a point and is peripheral if it is freely homotopic into arbitrarily small neighborhoods of an end of $M$. If $M$ has infinitely many ends then it can be written as an increasing union of finitely punctured compact connected subsurfaces $M_i$ whose boundary components determine essential non-peripheral homotopy classes in $M$. We may assume that boundary curves in $M_{i+1}$ are not parallel to boundary.
curves in \( M_i \). It is straightforward (see [2]) to put compatible hyperbolic structures on the \( M_i \)'s whose union defines a complete hyperbolic structure on \( M \). Of course \( M \) also has a complete hyperbolic structure when it only has finitely many ends. All hyperbolic structures in this paper are assumed to be complete.

We use the Poincaré disk model for the hyperbolic plane \( H \). In this model, \( H \) is identified with the interior of the unit disk and geodesics are segments of Euclidean circles and straight lines that meet the boundary in right angles. A choice of hyperbolic structure on \( M \) provides an identification of the universal cover \( \tilde{M} \) of \( M \) with \( H \). Under this identification covering translations become isometries of \( H \) and geodesics in \( M \) lift to geodesics in \( H \). The compactification of the interior of the unit disk by the unit circle induces a compactification of \( H \) by the ‘circle at infinity’ \( S_\infty \). Geodesics in \( H \) have unique endpoints on \( S_\infty \). Conversely, any pair of distinct points on \( S_\infty \) are the endpoints of a unique geodesic.

Each covering translation \( T : H \to H \) extends to a homeomorphism (also called) \( T : H \cup S_\infty \to H \cup S_\infty \). The fixed point set of a non-trivial \( T \) is either one or two points in \( S_\infty \). We denote these point(s) by \( T^+ \) and \( T^- \), allowing the possibility that \( T^+ = T^- \). If \( T^+ = T^- \), then \( T \) is said to be \textit{parabolic}; a root-free parabolic covering translation with fixed point \( P \) is sometimes written \( T_P \). If \( T^+ \) and \( T^- \) are distinct, then \( T \) is said to be \textit{hyperbolic} and we may assume that \( T^+ \) is a sink and \( T^- \) is a source; the unoriented geodesic connecting \( T^- \) and \( T^+ \) is called the \textit{axis} of \( T \). A root-free covering translation with axis \( \tilde{\gamma} \) is sometimes denoted \( T_{\tilde{\gamma}} \).

Each essential non-peripheral \( \alpha \subset M \) is homotopic to a unique closed geodesic \( \gamma \). For each lift \( \tilde{\alpha} \subset H \), the homotopy between \( \alpha \) and \( \gamma \) lifts to a bounded homotopy between \( \tilde{\alpha} \) and a lift \( \tilde{\gamma} \) of \( \gamma \) which is the axis of a hyperbolic covering translation \( \tilde{T} \). The ends of both lines \( \tilde{\alpha} \) and \( \tilde{\gamma} \) converge to \( T^- \) and \( T^+ \).

Similarly, each essential peripheral \( \alpha \) is homotopic to a (non-unique) horocycle \( \gamma \). For each lift \( \tilde{\alpha} \subset H \), the homotopy between \( \alpha \) and \( \gamma \) lifts to a bounded homotopy between \( \tilde{\alpha} \) and a lift \( \tilde{\gamma} \) of \( \gamma \). Both ends of both lines \( \tilde{\alpha} \) and \( \tilde{\gamma} \) converge to a single point \( T^{\pm} \) that is the unique fixed point of a parabolic covering translation \( T \).

Suppose now that \( f : M \to M \) is a homeomorphism. Identify \( H \) with \( \tilde{M} \) and write \( \tilde{f} : H \to H \) for lifts of \( f : M \to M \) to the universal cover. If \( f : M \to M \) and \( g : M \to M \) are homotopic and the homotopy between \( f \) and \( g \) lifts to a homotopy between \( \tilde{f} : H \to H \) and \( \tilde{g} : H \to H \) then we say that \( \tilde{f} \) and \( \tilde{g} \) are equivariantly homotopic. A proof of the following fundamental result of Nielsen theory appears in Proposition 3.1 of [13].

**Proposition 4.1.** Every lift \( \tilde{f} : H \to H \) extends uniquely to a homeomorphism (also called) \( \tilde{f} : H \cup S_\infty \to H \cup S_\infty \). If \( \tilde{f} \) and \( \tilde{g} \) are equivariantly homotopic lifts of \( f : M \to M \) and \( g : M \to M \) then \( \tilde{f}|_{S_\infty} = \tilde{g}|_{S_\infty} \).

For any extended lift \( \tilde{f} : H \cup S_\infty \to H \cup S_\infty \) there is an associated \textit{action} \( \tilde{f}_\# \) on geodesics [horocycles] in \( H \) defined by sending the geodesic with endpoints \( P \) and \( Q \) to the geodesic [horocycle] with endpoints \( \tilde{f}(P) \) and \( \tilde{f}(Q) \). The action \( \tilde{f}_\# \) projects to
an action $f_\#$ on geodesics [horocycles] in $M$. Proposition 4.1 implies that $f_\#$ depends only on the isotopy class of $f$.

The following results are well known and follow easily from the definitions.

**Lemma 4.2.** Suppose for $i = 1, 2$, that $\alpha_i$ is an essential closed curve homotopic to the geodesic [horocycle] $\gamma_i$. Then $f(\alpha_1)$ is freely homotopic to $\alpha_2$ if and only if $f_\#(\gamma_1) = \gamma_2$.

**Lemma 4.3.** For any extended lift $\tilde{f} : H \cup S_\infty \to H \cup S_\infty$ and extended covering translation $T : H \cup S_\infty \to H \cup S_\infty$, the following are equivalent:

1. $\tilde{f}$ commutes with $T$.
2. $\tilde{f}$ fixes $T^+$ or $T^-$.
3. $\tilde{f}$ fixes $T^+$ and $T^-$.

## 5 An intermediate proposition

Recall that $F \in \text{Diff}_\mu(S^2)$ has entropy zero. By Theorem 1.2, Lemma 6.3 and Remark 6.4 of [10], $F$ is isotopic rel $\text{Fix}(F)$ to a composition $F_1$ of non-trivial Dehn twists along the elements of a finite (possibly empty) set $R_F$ of disjoint simple closed curves. By the main theorem of [1], each component $M$ of $M = S^2 \setminus \text{Fix}(F)$ is $F$-invariant. Let $f = F|_M : M \to M$ and let $R$ be the subset of $R_F \cap M$ consisting of elements that are non-peripheral in $M$. The restriction to $M$ of the isotopy from $F$ to $F_1$ can be modified to an isotopy from $f$ to a a composition of non-trivial Dehn twists along the elements of $R$. The fact that $F$ preserves $\mu$ implies that $M$ has $f$-recurrent points. The Brouwer plane translation theorem and the fact that $\text{Fix}(f) = 0$ imply that $M$ is not an open disk.

The proof of the first item of Theorem 1.3 requires renormalization and so is completed after the proofs of the second and third items. To clarify the logic we introduce Proposition 5.1 which contains the second and third items of Theorem 1.3 plus two additional properties of the annuli.

**Proposition 5.1.** Suppose that $M$ is a component of $M = S^2 \setminus \text{Fix}(F)$ and that $f = F|_M : M \to M$. Then there is a countable collection $\mathcal{A}$ of pairwise disjoint open $f$-invariant annuli such that

1. For each compact set $X \subset M$ there is a constant $K_X$ such that any $f$-orbit that is not contained in some $U \in \mathcal{A}$ intersects $X$ in at most $K_X$ points. In particular each $x \in \mathcal{B}(f)$ is contained in some $U \in \mathcal{A}$.

2. For each $U \in \mathcal{A}$ and $z$ in the frontier of $U$ in $S^2$, there are components $F_+(z)$ and $F_-(z)$ of $\text{Fix}(F)$ so that $\omega(F, z) \subset F_+(z)$ and $\alpha(F, z) \subset F_-(z)$. 

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(3) For each $U \in \mathcal{A}$ and each component $C_M$ of the frontier of $U$ in $M$, $F_+(z)$ and $F_-(z)$ are independent of the choice of $z \in C_M$.

(4) If $U \in \mathcal{A}$, and $f_c : U_c \to U_c$ is the extension to the annular compactification of $U$, then each component of $\partial U_c$ corresponding to a non-singular end of $U$ contains a fixed point of $f_c$.

In the special case that $M$ is an annulus, let $\mathcal{A}$ be the single annulus $M$. Items (1) - (3) are obvious and item (4) follows from Lemma 5.1 of [10]. The constructions and analysis needed for the case that $M$ is not an annulus are carried out in sections 6 through 11. The final formal proof of Proposition 5.1 occurs at the end of section 11.

6 The endpoint maps $\tilde{\alpha}$ and $\tilde{\omega}$ and annular covers

In this section we begin the proof of Proposition 5.1 in the case that $M$ has at least three ends.

Equip $M$ with a complete hyperbolic structure as described in section 4. The full pre-image in $H$ of the reducing set $\mathcal{R}$ is denoted $\tilde{\mathcal{R}}$. The closure of a component of $H \setminus \tilde{\mathcal{R}}$ is called a domain. If $\mathcal{R} = \emptyset$ then $H$ is the unique domain but otherwise there are infinitely many domains. The subgroup of covering translations that preserves a domain $\tilde{C}$ is denoted $\text{Stab}(\tilde{C})$. For each domain $\tilde{C}$, there is an associated lift $\tilde{f}_C : H \to H$ that commutes with each $T \in \text{Stab}(\tilde{C})$. This lift is also characterized by the property that $\text{Fix}(f_{|\infty})$ is equal to the intersection of the closure of $\tilde{C}$ with $S_\infty$; thus $\text{Fix}(f_{|\infty})$ is a Cantor set if $\mathcal{R} \neq \emptyset$ and all of $S_\infty$ if $\mathcal{R} = \emptyset$.

Lemma 6.1. For each lift $\tilde{f}$ of $f$ and each $\tilde{x} \in H$, $\alpha(\tilde{f}, \tilde{x})$ and $\omega(\tilde{f}, \tilde{x})$ are single points in $S_\infty$.

Proof. The Brouwer translation theorem implies that $\omega(\tilde{f}, \tilde{x}) \subset S_\infty$. We assume that $\omega(\tilde{f}, \tilde{x})$ is not a single point and argue to a contradiction. It must be the case that $\omega(\tilde{f}, \tilde{x}) \subset S_\infty \cap \text{Fix}(\tilde{f})$. If not, a non-fixed point $z \in \omega(\tilde{f}, \tilde{x})$ would have a free neighborhood whose intersection with $H$ would be a free disk visited by the orbit of $\tilde{x}$ more than once (indeed infinitely often). According to Proposition (1.3) of [5] this implies $\tilde{f}$ has a fixed point in $H$ – a contradiction. Since $\omega(\tilde{f}, \tilde{x})$ consists of fixed points it is straightforward to see that it is also connected.

If $\text{Fix}(\tilde{f})$ does not contain an interval we are done. Otherwise, every covering translation with one endpoint in this interval commutes with $\tilde{f}$ and so preserves $\text{Fix}(\tilde{f})$. It follows that $\text{Fix}(\tilde{f}) = S_\infty$. A proof of the lemma in this special case is given in Proposition 9.1 of [10].

In addition to lifts of $f$ to the universal cover $H$ we will also use lifts of $f$ to infinite cyclic covers.
Definitions 6.2. Suppose that $\sigma$ is a closed geodesic that is either equal to an element of $\mathcal{R}$ or disjoint from every element of $\mathcal{R}$. For each lift $\tilde{\sigma}$, let $T_{\tilde{\sigma}}$ be a root free covering translation with axis $\tilde{\sigma}$. Choose a domain $\tilde{C}$ that contains $\tilde{\sigma}$. (If $\sigma \in \mathcal{R}$ then there are two choices but otherwise there is just one.) Since $\tilde{f}_{\tilde{C}}$ fixes the ends of $\tilde{\sigma}$, it commutes with $T_{\tilde{\sigma}}$ by Lemma (4.3). The annular cover $A_{\sigma}$ is the closed annulus that is the quotient space of $(H \cup S_{\infty}) \setminus T_{\tilde{\sigma}}^{\pm}$ by the action of $T_{\tilde{\sigma}}$ and $f_{\sigma} : A_{\sigma} \to A_{\sigma}$ is the homeomorphism induced by $\tilde{f}_{\tilde{C}}$. If $\alpha(\tilde{f}_{\tilde{C}}, \tilde{x})$ is not an endpoint of $\tilde{\sigma}$ then $\alpha(f_{\sigma}, \hat{x})$ is a single point in $\partial A_{\sigma}$ and similarly for $\omega(f_{\sigma}, \hat{x})$.

Similarly, if $\sigma$ is a lift of a horocycle $\sigma$ then both ends of $\sigma$ converge to a point $P \in S_{\infty}$ and there is a root free covering translation $T_{P}$ that preserves $\sigma$. Let $\tilde{C}$ be the unique domain that contains $\tilde{\sigma}$. In this case, the annular cover $A_{\sigma}$ is the half-open annulus that is the quotient space of $(H \cup S_{\infty}) \setminus P$ by the action of $T_{P}$ and the boundary is a single circle denoted $\partial A_{\sigma}$. As in the previous case, $\tilde{f}_{\tilde{C}}$ induces a homeomorphism $f_{\sigma} : A_{\sigma} \to A_{\sigma}$. The end of $A_{\sigma}$ corresponding to $P$ projects homeomorphically to the end of $M$ circumscribed by $\sigma$. We can compactify this end exactly as in Definition [2.7] to form a closed annulus $A^\omega_{\sigma}$. There is an extension of $f_{\sigma}$ (also called $f_{\sigma}$) to a homeomorphism of $A^\omega_{\sigma}$.

As the notation suggests, $f_{\sigma}$ is independent of the choice of $\tilde{C}$ and, up to conjugacy, the choice of lift $\tilde{\sigma}$. The former follows from the fact that if $\tilde{C}_1$ and $\tilde{C}_2$ contain $\tilde{\sigma}$ then $\tilde{f}_{\tilde{C}_1}$ and $\tilde{f}_{\tilde{C}_2}$ differ by an iterate of $T_{\tilde{\sigma}}$ and the latter from the fact that if $\tilde{\sigma}$ is replaced with $S(\tilde{\sigma})$ for some covering translation $S$ then $\tilde{C}$ is replaced by $S(\tilde{C})$ and $T_{\tilde{\sigma}}$ is replaced by $S T_{\tilde{\sigma}} S^{-1}$.

Lemma 6.3. Suppose that $\sigma$ is a horocycle or a closed geodesic that is either equal to an element of $\mathcal{R}$ or disjoint from every element of $\mathcal{R}$.

(1) For each closed geodesic $\sigma$, $\text{Fix}(f_{\sigma}|_{\partial A_{\sigma}})$ intersects both components of $\partial A_{\sigma}$. If $\sigma \in \mathcal{R}$ then $f_{\sigma}$ is isotopic rel $\text{Fix}(f_{\sigma}|_{\partial A_{\sigma}})$ to a Dehn twist of the same index that $f$ twists around $\sigma$. If $\sigma \not\in \mathcal{R}$ then $f_{\sigma}$ is isotopic rel $\text{Fix}(f_{\sigma}|_{\partial A_{\sigma}})$ to the identity.

(2) For each horocycle $\sigma$, $\text{Fix}(f_{\sigma}|_{\partial A_{\sigma}}) \neq \emptyset$.

Proof. Suppose at first that $\sigma$ is a lift of the closed geodesic $\sigma$.

If $\sigma \not\in \mathcal{R}$ then the closure of the domain $\tilde{C}$ that contains $\tilde{\sigma}$ intersects both components of $S_{\infty} \setminus \tilde{\sigma}^{\pm}$. The points in this intersection are fixed by $\tilde{f}_{\tilde{C}}$ and project to fixed points for $f_{\sigma}$ in both components of $\partial A_{\sigma}$, all of which are in the same Nielsen class for $f_{\sigma}$. It follows that $f_{\sigma}$ is isotopic rel $\text{Fix}(f_{\sigma}|_{\partial A_{\sigma}})$ to the identity.

If $\sigma \in \mathcal{R}$ and $\tilde{C}_1$ and $\tilde{C}_2$ are the domains that contain $\tilde{\sigma}$ then points in the intersection of the closure of $\tilde{C}_1$ with $S_{\infty}$ are fixed by $\tilde{f}_{\tilde{C}_1}$ and project to fixed points for $f_{\sigma}$ in one component of $\partial A_{\sigma}$ and points in the intersection of the closure of $\tilde{C}_2$ with $S_{\infty}$ are fixed by $\tilde{f}_{\tilde{C}_2}$ and project to fixed points for $f_{\sigma}$ in the other component of $\partial A_{\sigma}$. If $f$ twists with degree $k$ around $\sigma$ then $\tilde{f}_{\tilde{C}_1}$ and $\tilde{f}_{\tilde{C}_2}$ differ by $T_{\tilde{\sigma}}^{k}$ so $f_{\sigma}$ is isotopic rel $\text{Fix}(f_{\sigma}|_{\partial A_{\sigma}})$ to a Dehn twist of index $k$. This completes the proof of (1).

The proof for (2) is similar. \hfill \qed
7 Reducing Arcs in Annular Covers

In this section we recall, adapt and improve definitions and results from section 10 of 10, where the assumption is that \( F \) is periodic point free and isotopic rel \( \text{Fix}(F) \) to the identity as opposed to our current assumption that \( F \) has zero entropy and is isotopic rel \( \text{Fix}(F) \) to a composition of Dehn twists on the elements of \( \mathcal{R} \).

**Notation 7.1.** We assume throughout this section that \( f_\sigma : A_\sigma \to A_\sigma \) is as in Definition 7.2 and that \( \hat{x}_1, \ldots, \hat{x}_r \) are points in the interior of \( A_\sigma \) such that the \( \alpha(f_\sigma, \hat{x}_i) \)'s are distinct points in \( \partial A_\sigma \) and the \( \omega(f_\sigma, \hat{x}_i) \)'s are distinct points in \( \partial A_\sigma \). Let \( N = \text{int}(A_\sigma) \), let \( \hat{X} \subset N \) be the union of the \( f_\sigma \)-orbits of the \( \hat{x}_i \)'s and let \( N_{\hat{X}} = N \setminus \hat{X} \) equipped with a complete hyperbolic structure.

A properly embedded line \( \ell \subset N_{\hat{X}} \) is essential if it is not properly isotopic into arbitrarily small neighborhoods of some end of \( N_{\hat{X}} \). Each essential \( \ell \) is properly isotopic to a unique geodesic. The action of \( f_\sigma \) on isotopy classes of properly embedded lines in \( N_{\hat{X}} \) is captured by the map \( f_\sigma# \) on geodesics defined in section 4.

If an embedded path \( \beta \subset N \) has endpoints in \( \hat{X} \) but is otherwise disjoint from \( \hat{X} \) then the interior of \( \beta \) determines a properly embedded line \( \ell \subset N_{\hat{X}} \). Proper isotopy of \( \ell \) in \( N_{\hat{X}} \) corresponds to isotopy rel \( \hat{X} \) of \( \beta \) in \( N \). If \( \ell \) is essential [resp. a geodesic in \( N_{\hat{X}} \)] then we say that \( \beta \) is essential [resp. a geodesic rel \( \hat{X} \) in \( N \)]. There is an induced map \( f_\sigma# \) on geodesics rel \( \hat{X} \) in \( N \) such that \( f_\sigma#(\beta) \) is the unique geodesic path in the isotopy class rel \( \hat{X} \) of \( f_\sigma(\beta) \).

**Definition 7.2.** An arc \( \beta' \subset N \) connecting \( \hat{x} \in \hat{X} \) to \( f_\sigma(\hat{x}) \) is called a translation arc for \( x \) if \( f_\sigma(\beta') \cap \beta' = f_\sigma(\hat{x}) \). The geodesic rel \( \hat{X} \beta \subset N \) determined by \( \beta' \) is called a translation arc geodesic for \( x \) relative to \( \hat{X} \).

**Lemma 7.3.** For each \( 1 \leq i \leq r \) there are translation arc geodesics \( \hat{\beta}_i^+ \) and \( \hat{\beta}_i^- \) for some points in the orbit of \( \hat{x}_i \) such that

1. \( \hat{\beta}_i^+ = \bigcup_{j=0}^{\infty} f_\sigma#(\hat{\beta}_i^+) \) is an embedded ray that converges to \( \omega(f_\sigma, \hat{x}_i) \).
2. \( \hat{\beta}_i^- = \bigcup_{j=0}^{\infty} f_\sigma#(\hat{\beta}_i^-) \) is an embedded ray that converges to \( \alpha(f_\sigma, \hat{x}_i) \).
3. The \( \hat{\beta}_i^\pm \)'s are all disjoint.

**Proof.** Items (1) and (2) are proved explicitly and item (3) is unstated but proved implicitly in Lemma 10.6 of 10 under the hypotheses that \( \text{Per}(f_\sigma) = \emptyset \) and \( \mathcal{R} = \emptyset \). The former is only used to conclude that \( \text{Fix}(f_\sigma) = \emptyset \) and the latter is only used to guarantee that \( f_{\sigma^*} \) commutes with \( T_{\hat{x}} \). Since these hold in our context, the proof given in 10 applies in this context as well.

**Remark 7.4.** Assume the notation of Lemma 7.3. For each \( i \) the map \( f_\sigma# \) defined on geodesics rel \( \hat{X} \) induces a self-map of \( B_i^\pm \) that is conjugate to a standard translation of \( [0, \infty) \) into itself. The analogous statement also holds for \( B_i^- \) with respect to \( f_\sigma^{-1}# \).
In the next definition we recall the main items associated to the $\hat{\beta}_i^\pm$'s. Further details can be found in section 10 of [10].

**Definition 7.5.** Assume the notation of Lemma 7.3. Let $V_i^\pm$ be the neighborhood of $B_i^\pm$ that has connected geodesic boundary rel $X$, intersects $X$ exactly in $B_i^\pm \cap X$ and deformation retracts to $B_i^\pm$. The nested intersection $V_i^+ \supset f_{\sigma\#}(V_i^+) \supset f_{\sigma\#}^2(V_i^+) \supset \ldots$ is empty and similarly for $V_i^- \supset f_{\sigma\#}^{-1}(V_i^-) \supset f_{\sigma\#}^{-2}(V_i^-) \supset \ldots$ We say that $V_i^\pm$ is the translation neighborhood determined by $\hat{\beta}_i^\pm$. Item (3) in Definition 7.3 implies that the $V_i^\pm$'s are disjoint.

The subsurface $W = N \setminus (X \cup (\bigcup_{i=1}^n V_i^\pm))$ is finitely punctured. We write $\partial W = \partial_+ W \cup \partial_- W$ where $\partial_+ W = \bigcup_{i=1}^n \partial V_i^+$. Then $f_{\sigma\#}(\partial_+ W) \cap W = \emptyset$ and $\partial_- W \cap f_{\sigma\#}(W) = \emptyset$. We say that $W$ is the Brouwer subsurface determined by the $\hat{\beta}_i^\pm$'s.

Let $RH(W, \partial_+ W)$ be the set of non-trivial relative homotopy classes $[\tau]$ determined by embedded arcs $(\tau, \partial \tau) \subset (W, \partial_+ W)$. Denote $\tau$ with its orientation reversed by $-\tau$ and $[-\tau]$ by $-\tau$. Let $\mathcal{T}_i \subset RH(W, \partial_+ W)$ consist of one representative $(\{\tau\} \text{ or } -\{\tau\})$ of each unoriented homotopy class that is represented by a component of $f_{\sigma\#}(\hat{\beta}_i^-) \cap W$ for some $n > 0$. Since the $f_{\sigma\#}(\hat{\beta}_i^-)$'s are disjoint and simple and since $W$ is finitely punctured, $\mathcal{T}_i$ is finite. We say that $\mathcal{T}_i$ is the fitted family determined by the $\hat{\beta}_i^\pm$'s.

If $f_{\sigma\#}(\hat{\beta}_i^-) \subset V_i^+$ for some $n$ then $B_i = \cup_{j=-\infty}^\infty f_{\sigma\#}(\hat{\beta}_i^-)$ is a properly embedded $f_{\sigma\#}$-invariant line in $N$ whose endpoints converge to $\alpha(f_\sigma, \hat{x}_i)$ and $\omega(f_\sigma, \hat{x}_i)$ by Lemma 7.3 and the assumption that $V_i^+$ is a translation neighborhood.

To analyze the limit set of $\hat{\beta}_i$ when no such $n$ exists, we make use of a map that assigns to each finite set $\mathcal{T} \subset RH(W, \partial_+ W)$ another finite set $f_{\sigma\#}(\mathcal{T}) \cap W$. We abuse notation slightly and write $\mathcal{T} = \{[\tau_i]\}$ where each $[\tau_i] \in RH(W, \partial_+ W)$; we do not assume that the $[\tau_i]$'s are distinct.

Choose a homeomorphism $h : N \to N$ that is isotopic to $f_{\sigma}$ rel $\hat{X}$ such that $h(L) = f_{\sigma\#}(L)$ for each component $L$ of $\partial W$. For any arc $\tau \subset W$ with endpoints on $\partial_+ W$, $h(\tau)$ is an arc in $h(W) = f_{\sigma\#}(W)$ with endpoints on $f_{\sigma\#}(\partial_+ W)$; in particular, $h(\tau) \cap \partial_- W = \emptyset$ and $\partial h(\tau) \cap W = \emptyset$. Let $f_{\sigma\#}(\tau) \subset f_{\sigma\#}(W)$ be the geodesic arc that is isotopic rel endpoints to $h(\tau)$. The components $\tau_1, \ldots, \tau_r$ of $f_{\sigma\#}(\tau) \cap W$ are arcs in $W$ with endpoints in $\partial_+ W$. Define $f_{\sigma\#}([\tau]) \cap W = \{[\tau_1], \ldots, [\tau_r]\}$. It is shown in [15] (see pages 249 - 250) that $f_{\sigma\#}([\tau]) \cap W$ is well defined.

More generally if $\mathcal{T}$ is a finite collection of elements of $RH(W, \partial_+ W)$ then we define $f_{\sigma\#}(\mathcal{T}) \cap W = \cup_{\tau \in \mathcal{T}} f_{\sigma\#}([\tau]) \cap W$. Note that $f_{\sigma\#}(\cdot) \cap W$ can be iterated. Recursively define $(f_{\sigma\#})^n([\tau]) \cap W = (f_{\sigma\#})^{n-1}(f_{\sigma\#}([\tau]) \cap W) \cap W$.

The next lemma states that one gets the same answer by either iterating the intersection operator or by first iterating $f_\sigma$ and then applying the intersection operator once.

**Lemma 7.6.** For all $\tau \in RH(W, \partial_+ W)$, $(f_{\sigma\#})^n([\tau]) \cap W = (f_\sigma^n)([\tau]) \cap W$.

**Proof.** This is Lemma 5.4 of [15]. \qed

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We next record some properties of the fitted family $\mathcal{T}_i$.

**Corollary 7.7.** The fitted family $\mathcal{T}_i = \{[\tau_1], \ldots, [\tau_m]\}$ satisfies

(a) $\tau_j$ and $\tau_k$ are disjoint and $[\tau_j] \neq \pm [\tau_k]$ for $j \neq k$

(b) each element of $f_{\sigma \#}(\mathcal{T}_i) \cap W$ is $\pm [\tau_j]$ for some $1 \leq j \leq m$.

**Proof.** The first item is by construction and the second follows from Lemma 7.6.

We used the assumption that $F : S^2 \to S^2$ has zero entropy to conclude that $F$ and hence $f : M \to M$ is isotopic to a composition of Dehn twists along disjoint simple closed curves. Lemma 7.8 below (c.f. Theorem 5.5(b) of [15]) is the only other place in which the entropy zero hypothesis is applied.

We say that $[\tau] \in RH(W, \partial_+ W)$ disappears under iteration if $(f_{\sigma \#})^n([\tau]) \cap W = \emptyset$ for some $n > 0$ and that $\mathcal{T}_i$ disappears under iteration if each element in $\mathcal{T}_i$ does.

**Lemma 7.8.** Assume that $\hat{\beta}_i^\pm$ are translation arc geodesics as in Lemma 7.3 that $W$ is the associated Brouwer subsurface and that $\mathcal{T}_i$ is the $i^{th}$ associated fitted family. Then for each $n \geq 1$ and $[\tau] \in RH(W, \partial_+ W)$, $(f_{\sigma \#})^n([\tau]) \cap W$ contains at most one element that equals either $[\tau]$ or $-[\tau]$.

**Proof.** We assume to the contrary that there exist $[\tau]$ and $n \geq 1$ such $(f_{\sigma \#})^n([\tau]) \cap W$ contains at least two elements that equal either $[\tau]$ or $-[\tau]$ and argue to a contradiction. The obvious induction argument on $k$ implies that $(f_{\sigma \#})^k([\tau]) \cap W$ contains at least $2^k$ elements that equal $\pm [\tau]$.

Let $L_1$ and $L_2$ be the components of $\partial_+ W$ that contain the endpoints of $\tau$.

As a first case suppose that $L_1 = L_2$ and that $\tau$ and the interval in $L_1$ connecting the endpoints of $\tau$ bound a disk $D$ in $N$. Choose an element $\hat{x} \in \hat{X} \cap D$ and a compact essential subannulus $A \subset A_{\hat{x}}$ that separates $\hat{x}$ from $L_1$. There are at least $2^k$ subpaths $\mu_j$ of $(f_{\sigma}^k)(\tau)$ that cross $A$. Each $\mu_j$ projects to a path in $M$ whose distance in the metric inherited from $S^2$ is bounded below by a constant that does not depend on $j$. Letting $\tau_M \subset M$ be the projected image of $\tau \subset N$ it follows that the length of $f_{\sigma}^k(\tau_M)$ in the metric on $M$ inherited from $S^2$ grows exponentially in $k$. This contradicts part (2) of Theorem 2 of [22] and the fact that $F$ has entropy zero.

In the case that the ends of $L_1$ and $L_2$ converge to distinct components of $\partial A_{\sigma}$, the same argument works with respect to a compact essential subannulus $A \subset A_{\sigma}$ that separates $L_1$ and $L_2$.

The third case is that $L_1 = L_2$ and that $\tau$ and the interval in $L_1$ connecting the endpoints of $\tau$ define a simple closed curve that is essential in $A_{\sigma}$. Choose a compact essential annulus $A \subset A_{\sigma}$ that is disjoint from $L_1 \cup \tau$. Choose half-disks whose frontier consist of intervals $I_1$ and $I_2$ in the component of $\partial A_{\sigma}$ that contains the endpoints of $L_1$ and half-circles $\tilde{\rho}_1$ and $\tilde{\rho}_2$ that project to the same simple closed curve $\rho \subset M$. We may assume that the half-disks are disjoint from each other and from $L_i$. Choose
thickened arcs $J_1$ and $J_2$ connecting $\tilde{\rho}_1$ and $\tilde{\rho}_2$ to the far end of $A$. (In other words $J_1$ and $J_2$ overlap with $A$ in rectangles that cross $A$.)

There are at least $2^k$ subpaths $\mu_j$ of $(f_\sigma^{kn})(\tau)$ such that each $\mu_j$ either crosses $J_1$, crosses $J_2$, crosses $A$ or is an arc with one endpoint on $\tilde{\rho}_1$ and the other on $\tilde{\rho}_2$. We claim that $\mu_j$ projects to a path in $M$ whose distance in the metric inherited from $S^2$ is bounded below by some constant that is independent of $j$. If $\mu_j$ crosses $J_1$, $J_2$ or $A$ then this follows from the compactness of these sets. In the remaining case $\mu_j$ projects to an arc in $M$ with endpoints on $\rho$ that is not homotopic rel endpoints into $\rho$ and again the claim is clear. The proof of the lemma in this case concludes as in the previous cases.

The fourth and final case is that $L_1 \neq L_2$ have endpoints in the same component of $\partial A_\sigma$. The obvious modification of the argument from the third case applies here. □

The next lemma is based on Theorem 5.5(c) of [15]. See also Lemma 10.8 of [10].

Suppose that $W$ is a Brouwer subsurface, that $\tau \in RH(W, \partial_+ W)$ and that both endpoints of $\tau$ are contained in a single component $L$ of $\partial_+ W$. If the simple closed curve that is the union of $\tau$ with an interval in $L$ does not bound a disk in $N$ then we say that $\tau$ is essential.

**Lemma 7.9.** Assume that $\tilde{\beta}_i^\pm$ are translation arc geodesics as in Lemma 7.3 that $W$ is the associated Brouwer subsurface and that $T$ is an associated fitted family that does not disappear under iteration. Then there exists $[\tau] \in T$ such that $f_\sigma^\#([\tau]) \cap W = \{[\tau], \pm[\sigma_1], \ldots, \pm[\sigma_m]\}$ where each $[\sigma_j] \in T$ disappears under iteration. If $\tau$ has both endpoints on the same component $L$ of $\partial_+ W$ then $[\tau]$ is essential.

**Proof.** Let $\Gamma$ be the directed graph with one vertex for each element of $T$ and one oriented edge from the vertex corresponding to $\tau_i \in T$ to the vertex corresponding to $\tau_j \in T$ for each occurrence of $\pm[\tau_j]$ in $f_\sigma^\#([\tau_i]) \cap W$. Lemma 7.3 implies that each vertex of $\Gamma$ is contained in at most one non-repeating oriented closed path in $\Gamma$. There is a partial order on the vertices of $\Gamma$ defined by $v_1 > v_2$ if there is an oriented path in $\Gamma$ from $v_1$ to $v_2$ but no oriented path from $v_2$ to $v_1$. Among all vertices that are contained in an oriented closed path, choose one, $v$, that is smallest with respect to the partial order. Let $[\tau]$ correspond to $v$ and let $n$ be the length of the unique oriented non-repeating closed path $\rho$ through $v$. Then $(f_\sigma^n)^\#([\tau]) \cap W = \{\epsilon[\tau], \pm[\sigma_1], \ldots, \pm[\sigma_m]\}$ for some $[\sigma_j] \neq [\tau]$ in $T$ and $\epsilon = \pm 1$. We claim that each $[\sigma_j]$ disappears under iteration. If not then the vertex of $\Gamma$ corresponding to $[\sigma_j]$ is on the oriented closed path containing $\tau$ so there would be distinct paths in $\Gamma$ from $\tau$ to $\sigma_j$, each of which could be completed to a closed path from $\tau$ to itself.

If $n = 1$, let $\tau' = \tau$; otherwise let $v'$ be the vertex following $v$ in $\rho$ and let $\tau'$ be the corresponding element of $T$. Thus $\tau'$ is the unique element of $f_\sigma^\#([\tau]) \cap W$ that does not disappear under iteration.

For the remainder of the proof we make use of the end (pages 253 - 254) of the proof of Theorem 5.5 of [15]. The case that $\tau$ has endpoints on distinct components
of $\partial_+ W$ is treated in the last two paragraphs of that proof. (The labels $\tau^k$ in the diagram on the bottom of page 253 are incorrect; they should be $\tau^*_i$.) The argument given there applies without change in our present context so we may assume that $\tau$ has both endpoints on the same component, say $L$, of $\partial_+ W$.

The endpoints of $f_{\sigma\#}(\tau)$ are contained in the component of $W$ bounded by $L$. If $f_{\sigma\#}(\tau)$ intersects any other component of the complement of $W$ then at least two elements of $f_{\sigma\#}(\tau) \cap W$ would be represented by paths with endpoints on distinct components of $\partial_+ W$. Since no such paths disappear under iteration, this can not happen and we conclude that each element of $f_{\sigma\#}(\tau) \cap W$, and in particular $\tau'$, has both endpoints on $L$.

Both ends of $L$ converge to the same end of $N$. The argument given in the first and second paragraphs on page 253 of [15] (which is a proof by contradiction) carries over without change to this context and proves that $\tau$ is essential. By symmetry, $\tau'$ is also essential.

If $\tau \neq \tau'$ then either $\tau$ is contained in the annulus bounded by $\tau'$ and an interval in $L$ or $\tau'$ is contained in the annulus bounded by $\tau$ and in interval in $L$. There is a punctured rectangle $D$ bounded by $\tau, \tau'$ and intervals in $L$. There are only finitely many punctures contained in disks bounded by an element of $\mathcal{T}$ and an arc in $L$. Thus, for all sufficiently large $k$, $f_{\sigma}^{kn}(D)$ does not contain any punctures in $W$. It follows that $(f_{\sigma}^{kn})_{\#}([\tau]) \cap W = (f_{\sigma}^{kn})_{\#}([\tau']) \cap W$. But $\tau$ [resp. $\tau'$] is the unique element of $(f_{\sigma}^{kn})_{\#}([\tau]) \cap W$ [resp. $(f_{\sigma}^{kn})_{\#}([\tau']) \cap W$] that does not disappear under iteration. This contradicts the assumption that $\tau \neq \tau'$. We conclude that $\tau = \tau'$ and hence that $n = 1$. Since $f_{\sigma}$ is orientation preserving and $f_{\sigma\#}(L)$ is parallel to $L$, it follows that $\epsilon = 1$.

**Definitions 7.10.** Suppose that $W$ is a Brouwer subsurface, that $\mathcal{T}$ is a fitted family and that $\tau \in \mathcal{T}$. If $L_1$ and $L_2$ are the components of $\partial_+ W$ that contain the initial and terminal endpoints $x_1$ and $x_2$ of $\tau$ respectively, and if some component of the complement of $L_1 \cup L_2 \cup \tau$ is contractible in $N_X$ then we say that $[\tau]$ is peripheral. Equivalently, $\tau$ is peripheral if there are rays $R_1 \subset L_1$ and $R_2 \subset L_2$ whose initial points are $x_1$ and $x_2$ such that the line $R_1^{-1} \tau R_2$ can be properly isotoped rel $X$ into any neighborhood of some end of $N$.

Our next result is based on Lemma 6.4 of [15].

**Lemma 7.11.** Suppose that $\hat{\beta}_i^{\pm}$ are translation arc geodesics as in Lemma 7.3, that $W$ is the associated Brouwer subsurface, that $\mathcal{T}$ is an associated fitted family and that $\tau \in \mathcal{T}$ is non-peripheral and satisfies $f_{\sigma\#}(\tau) \cap W = \{[\tau], [\sigma_1], \ldots, [\sigma_m]\}$ where each $[\sigma_j]$ disappears under iteration. Let $L_1$ and $L_2$ be the (possibly equal) components of $\partial_+ W$ that contain the initial and terminal endpoints $v_1$ and $v_2$ of $\tau$. Then there are rays $R_1 \subset L_1$ and $R_2 \subset L_2$ such that $R_1^{-1} \tau R_2$ is isotopic to an $f_{\sigma\#}$-invariant geodesic line $\mu$.

**Proof.** To simplify the notation somewhat let $h = f_{\sigma}$.
Choose a lift $\tilde{\tau}$ of $\tau$ in the compactified universal cover $H \cup S_\infty$ of $N_X$ and for $k = 1, 2$, let $\tilde{L}_k$ be the lift of $L_k$ that contains the endpoint $\tilde{v}_k$ of $\tilde{\tau}$ that projects to $v_k$. Since $\tau$ is not peripheral, $\tilde{L}_1$ and $\tilde{L}_2$ do not have a common endpoint. The endpoints of $\tilde{L}_1$ and the endpoints of $\tilde{L}_2$ bound disjoint intervals $J_1$ and $J_2$ in $S_\infty$.

Since $[\tau] \in h_\#([\tau]) \cap W$, there is a lift $\tilde{h} : H \cup S_\infty \to H \cup S_\infty$ such that $\tilde{L}_1$ separates $\tilde{h}_\#(L_1)$ from $\tilde{L}_2$ and $\tilde{L}_2$ separates $\tilde{h}_\#(L_2)$ from $\tilde{L}_1$. The sequences $\{h_\#^n(L_k) : n = 0, 1, \ldots\}$ have no accumulation points in $N_X$ so the sequences $\{\tilde{h}_\#^n(\tilde{L}_k) : n = 0, 1, \ldots\}$ converge to distinct single points $P_k \in S_\infty$. The geodesic $\tilde{\rho}$ connecting $P_1$ to $P_2$ in $H$ projects to an $h_\#$-invariant geodesic $\rho \subset N_X$.

For all $k > 1$, $h_k(\tau) \cap W$ is the union of $\tau$ with $\bigcup_{j=1}^{n_1} \bigcup_{i=0}^{k-1} h_i(\sigma_j) \cap W$. Since each $\sigma_j$ disappears under iteration, there exists $k_0$ such that $h_k(\tau) \cap W = h_k(\tau) \cap W$ for all $k \geq k_0$. In particular, $\rho \cap W$ is a finite union of disjoint arcs.

Given $n > 0$, we could replace the translations arc geodesics $\tilde{h}_\#^n$ by their images under $h_\#$. The resulting Brouwer subsurface $W_n$ satisfies $\partial_- W_n = \partial_- W$ and $\partial_+ W_n = h_\#(\partial_+ W)$. The resulting fitted family $T_n$ is the $h_\#$-image of $T$. Repeating the above construction of $\tilde{\rho}$ using $\tilde{h}_\#(\tau)$, $\tilde{h}_\#(\tilde{L}_1)$ and $\tilde{h}_\#(\tilde{L}_2)$, we still get $\tilde{\rho}$. Thus $\rho \cap W_n$ is a finite union of disjoint arcs for all $n \geq 0$. This proves that $\rho$ is a properly embedded line.

It remains to show that $P_i$ is an endpoint of $J_i$. This is equivalent to one of the endpoints of $\tilde{L}_i$ being a fixed point of $\tilde{h}$ and so also equivalent to $\tilde{h}_\#(\tilde{L}_1)$ sharing an endpoint with $\tilde{L}_1$. Let $V_1^+$ be the component of the complement of $W$ bounded by $L_1$. The closure $S$ of $V_1^+ \setminus h_\#(V_1^+)$ is a once-punctured strip. Let $\mu$ be the subpath of $\tilde{h}_\#(\tau)$ connecting $\tilde{L}_1$ to $\tilde{h}_\#(\tilde{L}_1)$. The projected image $\mu \subset h_\#(W)$ is disjoint from $h_\#(V_1^+)$ and connects $L_1$ to $h_\#(L_1)$. If $\mu$ is entirely contained in $S$ then one of the complementary components of $\mu$ in $S$ is unpunctured. There are rays $R' \subset L_1$ and $R'' \subset h_\#(L_1)$ so that $R'^{-1}\mu R''$ is peripheral. Lifting this to a line that contains $\tilde{\tau}$ shows that $\tilde{L}_1$ and $\tilde{h}_\#(\tilde{L}_1)$ have a common endpoint and so completes the proof in this case.

To complete the proof we assume that $\mu$ is not entirely contained in $S$ argue to a contradiction. There is a component $\mu_0$ of $\mu \cap S$ that has both endpoints in $L_1$. Let $\tilde{S}_0$ be the subpath of $\tilde{\mu}$ that projects to $\mu_0$, let $\tilde{S}_0$ be the component of the full pre-image of $S$ that contains $\tilde{\mu}_0$ and let $\tilde{L}_1'$ and $\tilde{L}_1''$ be the components of the frontier of $\tilde{S}_0$ that contain the endpoints of $\tilde{\mu}_0$. As the notation indicates, both $\tilde{L}_1'$ and $\tilde{L}_1''$ are lifts of $L_1$. By construction, there is a subpath $\tilde{\rho}_0$ of $\tilde{\rho}$ that connects $\tilde{L}_1'$ and $\tilde{L}_1''$. The projection of $\tilde{\rho}_0$ is a subpath $\rho_0$ of $\rho$ that is contained in $S$ and has both endpoints on $L_1$.

By the same logic there is a subpath $\rho_n$ of $\rho$ that is contained in the closure $S_n$ of $h_\#^n(V_1^+) \setminus h_\#^{n+1}(V_1^+)$ with both endpoints on $h_\#^n(L_1)$. Since $\rho$ is an embedded line and each $S_n$ is a once punctured strip no component of $\rho \cap S_n$ has both endpoints in $h_\#^{n+1}(L_1)$. But this means that once $\rho$ crosses from $S_{n+1}$ to $S_n$ it cannot re-enter $S_{n+1}$ without first entering $W$. Since this happens only finitely many times, we have a contradiction to the existence of $\rho_n$ for all $n$. 

\[\square\]
**Definition 7.12.** An embedded arc $\rho \subset A_\sigma \setminus \hat{X}$ with endpoints in $\text{Fix}(f_\sigma) \cap \partial A_\sigma$ is a **reducing arc** for $f_\sigma$ rel $\hat{X}$ if it is $f_\sigma$-invariant up to isotopy rel $\hat{X}$ and rel its endpoints and is non-peripheral in the sense that it is not homotopic rel endpoints and rel $\hat{X}$ into $\partial A_\sigma$.

The following lemma is similar to Proposition 10.10 of [10]. The conclusions of the lemma are more detailed than those of that proposition and apply to $f_\sigma$ and not just some iterate of $f_\sigma$.

**Lemma 7.13.** Let $\alpha = \bigcup_{i=1}^r \alpha(f_\sigma, \hat{x}_i)$ and $\omega = \bigcup_{i=1}^r \omega(f_\sigma, \hat{x}_i)$. Assume that for each component $\partial_i A$ of $\partial A$, $\alpha_i = \alpha \cap \partial_i A$ and $\omega_i = \omega \cap \partial_i A$ have the same cardinality $c_i$ and that if $c_i > 1$ then the elements of $\alpha_i$ and $\omega_i$ alternate around $\partial_i A$. Then

1. There is a reducing arc $\rho$ for $f_\sigma$ with respect to $\hat{X}$.

2. If $r = 1$ or if $\sigma \in \mathcal{R}$ then

3. For each $1 \leq i \leq r$, $\alpha(f_\sigma, \hat{x}_i)$ and $\omega(f_\sigma, \hat{x}_i)$ belong to the same component of $\partial A_\sigma$.

4. For each $1 \leq i \leq r$, there is a reducing arc $\rho_i$ whose endpoints are $\alpha(f_\sigma, \hat{x}_i)$ and $\omega(f_\sigma, \hat{x}_i)$.

5. For each $1 \leq i \leq r$ there is a translation arc geodesic $\hat{\beta}_i$ for $\hat{x}_i$ such that $\hat{\beta}_i = \bigcup_{j=-\infty}^{\infty} h^j_\#(\hat{\beta}_i)$ is a properly embedded line whose ends converge to $\alpha(f_\sigma, \hat{x}_i)$ and to $\omega(f_\sigma, \hat{x}_i)$.

6. The $\hat{\beta}_i$'s are disjoint.

7. If $r = 1$

8. There is a unique translation arc geodesic for $\hat{x}_1$.

**Remark 7.14.** Item (6) holds when $\sigma \in \mathcal{R}$. We limit our proof to the $r = 1$ case because this is all we need.

**Proof.** Choose translation arc geodesics $\hat{\beta}_i^\pm$ as in Lemma 7.3. Let $V_i^\pm$, $W$ and $T_i$ be the associated translation neighborhoods, Brouwer subsurface and fitted families. We may assume that both ends of $\partial V_i^-$ converge to $\alpha(f_\sigma, \hat{x}_i)$ and that both ends of $\partial V_i^+$ converge to $\omega(f_\sigma, \hat{x}_i)$. After performing an isotopy rel $X \cup \partial A_\sigma$ we may assume that $f_\sigma(L_i) = f_\sigma_\#(L_i)$ for each $L_i = \partial V_i^\pm$. In particular, the ends of each $L_i$ are asymptotic to the ends of $f_\sigma(L_i)$.

If $T_i$ disappears under iteration then $\hat{\beta}_i = \bigcup_{n=-\infty}^{\infty} f_\sigma_\#^n(\hat{\beta}_i^\pm)$ is a properly embedded $f_\sigma_\#$-invariant line whose ends converge to $\alpha(f_\sigma, \hat{x}_i)$ and $\omega(f_\sigma, \hat{x}_i)$. A reducing arc $\rho \subset W$ with endpoints $\alpha(f_\sigma, \hat{x}_i)$ and $\omega(f_\sigma, \hat{x}_i)$ is obtained by pushing $\hat{\beta}_i$ off of itself into a non-contractible complementary component of $N \setminus \hat{X}$. 

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Suppose next that \( \mathcal{T}_i \) does not disappear under iteration. Let \( [\tau] \in \mathcal{T}_i \) satisfy the conclusions of Lemma 7.9. Let \( L_p \) and \( L_q \) be the components of \( \partial_1 W \) containing the initial endpoint \( v_p \) and terminal endpoint \( v_q \) of \( \tau \) respectively. Our hypotheses on \( \alpha_1 \) and \( \omega_1 \) imply that each component of the complement of \( L_p \cup L_q \cup \tau \) either intersects \( \hat{X} \) or contains an essential subannulus of \( A_\sigma \). This proves that that \( \tau \) is non-peripheral.

Lemma 7.11 implies that there are rays \( R_p \subset L_p \) and \( R_q \subset L_q \) such that \( R_p^{-1} \tau R_q \) is properly isotopic rel \( \hat{X} \) to a geodesic line \( \mu \) that is \( f_{\sigma} \)-invariant. To see that \( \mu \) is the interior of a reducing arc \( \rho \) with endpoints \( \alpha(f_{\sigma}, \hat{x}_p) \) and \( \omega(f_{\sigma}, \hat{x}_q) \) we need only check that the isotopy rel \( \hat{X} \) between \( \mu \) and \( f_{\sigma}(\mu) \) can be performed relative to \( \hat{X} \cup \partial A_\sigma \).

This follows from the fact that the ends of \( \mu \) are asymptotic to the ends of \( f_{\sigma}(\mu) \). We have now completed the proof of (1).

If \( r = 1 \) then (2) follows from our assumption on \( c_1 \). If \( \sigma \in \mathcal{R} \) then Lemma 6.3 implies that no reducing arc can connect the two components of \( \partial A_\sigma \) and hence that (2) is satisfied. This completes the proof of (2).

For the remainder of the proof we assume that either \( r = 1 \) or \( \sigma \in \mathcal{R} \). We prove that each \( \mathcal{T}_i \) disappears under iteration by assuming that some \( \mathcal{T}_i \) does not disappear under iteration and arguing to a contradiction. By (2) we may assume that the endpoints of \( L_p \) and \( L_q \) belong to belong to the same component \( \partial_1 A_\sigma \) of \( \partial A_\sigma \). If \( L_p \neq L_q \) then \( c_l \geq 2 \) and \( \alpha_l \) and \( \omega_l \) alternate around \( \partial_1 A_\sigma \). But then \( \rho \) must separate \( \alpha(f_{\sigma}, \hat{x}_i) \) from \( \omega(f_{\sigma}, \hat{x}_i) \) for some \( i \) which contradicts the fact that \( \rho \) is a reducing arc.

In the remaining case, \( p = q \). Let \( Y \) and \( Z \) be the components of the complement of \( \rho \) with \( Y \) containing the open end of \( A_\sigma \) if \( \sigma \) is a horocycle and containing the component of \( \partial A_\sigma \) that does not contain the endpoints of \( L_p \) otherwise. Since \( \rho \) is non-peripheral, \( Z \) must contain at least one orbit in \( \hat{X} \). From \( \rho \cap \hat{B}^*_1 = \emptyset \) it follows that \( \rho \cap \tau = \emptyset \). Combined with the fact that \( \tau \) is essential (see Definition 7.10) and the fact that \( \rho \) is disjoint from \( L_p \), it follows that \( V_p \) is contained in \( Y \) and so contains at least one orbit in \( \hat{X} \); in particular, \( r > 1 \). This completes the proof that \( \mathcal{T}_i \) disappears under iteration in the \( r = 1 \) case so we may assume by induction that if one works relative to \( \hat{X} \cap Y \) or relative to \( \hat{X} \cap Z \) then \( \mathcal{T}_i \) disappears under iteration. In other words, if \( \hat{x}_i \in Y \) (the argument for \( \hat{x}_i \in Z \) is symmetric) then for all sufficiently large \( n, f_{\sigma, n}(\hat{\beta}_i^-) \) is isotopic rel \( \hat{X} \cap Y \) to an arc \( \gamma_{i,n} \subset V_i \subset Y \). Since \( \rho \) is a reducing arc, \( f_{\sigma, n}(\hat{\beta}_i^-) \subset Y \). It is a standard fact that the isotopy rel \( \hat{X} \cap Y \) of \( f_{\sigma, n}(\hat{\beta}_i^-) \) to \( \gamma_{i,n} \) can be taken with support in \( Y \). It follows that this isotopy is rel \( \hat{X} \) which implies that \( f_{\sigma, n}(\hat{\beta}_i^-) \subset V_i \) in contradiction to the assumption that \( \mathcal{T}_j \) does not disappear under iteration.

We have now proved that each \( \mathcal{T}_i \) disappears under iteration. Items (3) - (5) are immediate.

To verify (6), denote the translation arc geodesics that make up \( \hat{B}_1 \) by \( \hat{\beta}_{1,m} \) where \( f_{\sigma, n}(\hat{\beta}_{1,m}) = \hat{\beta}_{1,m+1} \) and where \( \hat{\beta}_{1,0} \) is a translation arc geodesic for \( \hat{x}_1 \). We assume that there is a translation arc geodesic \( \hat{\tau} \neq \hat{\beta}_{1,0} \) for \( \hat{x}_1 \) and argue to a contradiction. Let \( \hat{\mu} \) be the maximum initial segment of \( \hat{\tau} \) whose interior is disjoint from \( \hat{B}_1 \) and let
\[\dot{y}\] be the terminal endpoint of \(\dot{\mu}\). Let \(\dot{\nu}\) be the maximum initial segment of \(\dot{f}_{\sigma, \hat{x}}(\hat{\tau})\) whose interior is disjoint from \(\hat{B}_1\) and let \(\hat{z}\) be the terminal endpoint of \(\dot{\nu}\). If \(\dot{y} \in \hat{X}\) then \(\hat{z} = \hat{f}(\dot{y})\); otherwise \(\dot{y}\) is in the interior of some \(\hat{\beta}_{1,m}\) and \(\dot{z}\) is in the interior of \(\hat{\beta}_{1,m+1}\).

If \(\dot{y} \notin \hat{\beta}_{1,-1} \cup \hat{\beta}_{1,0}\) then the endpoints of \(\dot{\mu}\) and \(\dot{\nu}\) are linked in \(\hat{B}_1\) in contradiction to the fact that the interiors of \(\dot{\mu}\) and \(\dot{\nu}\) are disjoint and lie on the same side of \(\hat{B}_1\). We may therefore assume that \(\dot{y} \in \hat{\beta}_{1,-1} \cup \hat{\beta}_{1,0}\). In this case, the endpoints of \(\dot{\mu}\) and \(\dot{\nu}\) bound intervals \(I_\mu\) and \(I_\nu\) in \(\hat{B}_1\) that meet in at most one point. It follows that either the simple closed curve \(\dot{\mu} \cup I_\mu\) or the simple closed curve \(\dot{\nu} \cup I_\nu\) is inessential in \(A\) and so bounds a disk that is disjoint from \(\hat{X}\) in contradiction to the fact that these simple closed curves are composed of two geodesic segments. This completes the proof that \(\hat{\beta}_{1,0}\) is unique and hence the proof of the lemma.

We conclude this section with three corollaries of Lemma 7.13.

**Corollary 7.15.** If \(\sigma \in \mathcal{R}\) then there do not exist \(\hat{x}_1, \hat{x}_2 \in A_\sigma\) such that \(\alpha(f_{\sigma}, \hat{x}_1)\) and \(\omega(f_{\sigma}, \hat{x}_2)\) are contained in the same component of \(\partial A_\sigma\) and \(\alpha(f_{\sigma}, \hat{x}_1)\) and \(\omega(f_{\sigma}, \hat{x}_2)\) are contained in the other component of \(\partial A_\sigma\).

**Proof.** This follows from item (2) of Lemma 7.13 with \(r = 2\) and \(c_0 = c_1 = 1\).

**Corollary 7.16.** If \(\alpha(\hat{f}_C, \hat{x}) \neq \omega(\hat{f}_C, \hat{x})\), then the geodesic \(\hat{\gamma}\) with initial endpoint \(\alpha(\hat{f}_C, \hat{x})\) and terminal endpoint \(\omega(\hat{f}_C, \hat{x})\) projects to a simple geodesic in \(C\).

**Proof.** Let \(\hat{f} = \hat{f}_C\). If the corollary fails then there exists a covering translation \(T\) such that \(T(\hat{\gamma}) \cap \hat{\gamma} \neq \emptyset\). Assuming without loss that \(\hat{\gamma} \subset \text{int}(\hat{C})\) we have \(T(\text{int}(\hat{C})) \cap \text{int}(\hat{C}) \neq \emptyset\) which implies that \(T \in \text{Stab}(\hat{C})\) commutes with \(\hat{f}\). If \(T\) is hyperbolic, let \(\hat{\sigma}\) be its axis; otherwise let \(\hat{\sigma}\) be a horocycle that is preserved by \(T\). Define \(f_{\sigma}: A_\sigma \to A_\sigma\) as in Definition 6.2. Since the endpoints \(\{\alpha(\hat{f}, \hat{x}), \omega(\hat{f}, \hat{x})\}\) of \(\hat{\gamma}\) link the endpoints of \(T(\hat{\gamma})\), it follows that \(\alpha(\hat{f}, \hat{x})\) and \(\omega(\hat{f}, \hat{x})\) are contained in the same component of \(S_\infty \setminus \{T^\pm\}\). Thus \(\alpha(f_{\sigma}, \hat{x})\) and \(\omega(f_{\sigma}, \hat{x})\) are contained in the same component, say \(\partial_1 A_\sigma\), of \(\partial A_\sigma\). By Lemma 7.13 applied with \(r = 1\) and \(\hat{x}_1 = \hat{x}\), there is a line \(\hat{B}_1 \subset \text{int}(A_\sigma)\) that contains the \(f_{\sigma}\)-orbit \(\hat{X}\) of \(\hat{x}\), that is \(f_{\sigma}\)-invariant up to isotopy rel \(\hat{X}\) and whose ends converge to \(\alpha(f_{\sigma}, \hat{x})\) and \(\omega(f_{\sigma}, \hat{x})\). The lift \(\hat{B}_1 \subset H\) of \(\hat{B}_1\) that contains \(\hat{x}\) has endpoint set \(\{\alpha(\hat{f}, \hat{x}), \omega(\hat{f}, \hat{x})\}\). The endpoints set \(T\{\alpha(\hat{f}, \hat{x}), \omega(\hat{f}, \hat{x})\}\) of \(T(\hat{B}_1)\) links \(\{\alpha(\hat{f}, \hat{x}), \omega(\hat{f}, \hat{x})\}\) since these are also the endpoint sets of \(\hat{\gamma}\) and \(T(\hat{\gamma})\). But then \(\hat{B}_1\) and \(T(\hat{B}_1)\) intersect in contradiction to the fact that \(\hat{B}_1\) is an embedded line.

The next corollary states that if there is twisting across an annular cover then orbits that start and end on one boundary component can not get to close to the other.

**Corollary 7.17.** Suppose that \(h: A \to A\) is either
(1) \( f_\sigma : A_\sigma \to A_\sigma \) for some \( \sigma \in \mathcal{R} \).

or

(2) \( f_\sigma : A'_\sigma \to A'_\sigma \) for some horocycle \( \sigma \) corresponding to an isolated end of \( M \).

Let \( \partial_0 A \) and \( \partial_1 A \) be the components of \( \partial A \). In case (2) assume that \( \partial_0 A \) is the unique component of \( \partial A_\sigma \) and that if \( \text{Fix}(f_\sigma|_{\partial_1 A}) \neq \emptyset \) then \( f_\sigma \) is not isotopic to the identity rel \( \text{Fix}(f_\sigma|_{\partial A}) \). Then there is a neighborhood of \( \partial_1 A \) that is disjoint from the \( h \) orbit of any \( \hat{x} \) for which both \( \alpha(h, \hat{x}) \) and \( \omega(h, \hat{x}) \) are contained in \( \partial_0 A \).

Proof. If the corollary fails then there exist \( \hat{x}_t \to \hat{P} \in \partial_1 A \) with \( \alpha(h, \hat{x}_t), \omega(h, \hat{x}_t) \in \partial_0 A \). After replacing \( h \) by some \( h^n \) we may assume that the rotation number of \( h|_{\partial_1 A} \) is less than \( \frac{1}{2} \). In particular there is an interval \( J_3 \) in \( \partial_1 A \) that contains \( \hat{P}, h(\hat{P}), h^2(\hat{P}) \) and \( h^3(\hat{P}) \) in that order. Additionally we may assume (Corollary [13]) that if \( \eta \) is a path connecting a fixed point in \( \partial_0 A \) to \( \hat{P} \) then \( h(\eta) \) is not homotopic rel endpoints to the path obtained by concatenating \( \eta \) with the subpath \( J_1 \subset J_3 \) connecting \( \hat{P} \) to \( h(\hat{P}) \). Let \( J_2 \subset J_3 \) be the subpath connecting \( \hat{P} \) to \( h^2(\hat{P}) \).

Choose contractible neighborhoods \( \hat{U}_i \) of \( J_i \) in \( A \) such that \( \hat{U}_1 \subset \hat{U}_2 \subset \hat{U}_3 \) and such that \( h(\hat{U}_i) \subset \hat{U}_{i+1} \). Choose lifts \( P \in \hat{U}_1 \subset \hat{U}_2 \subset \hat{U}_3 \) in \( H \cup S_\infty \) and let \( \hat{h} \) be the lift of \( h \) such that \( \hat{h}(P) \in \hat{U}_2 \). After passing to a subsequence, \( \hat{x}_t \to \hat{P} \) lifts to a sequence \( \tilde{x}_t \to P \) such that \( \tilde{x}_t, \hat{h}(\tilde{x}_t) \in \hat{U}_1 \) for all \( t \). Recall that a translation arc for \( \tilde{x} \) is a path from \( \tilde{x}_t \) to \( \hat{h}(\tilde{x}_t) \) that intersects its \( \hat{h} \)-image only in \( \hat{h}(\tilde{x}_t) \). By Lemma 4.1 of [15], there is a translation arc \( \hat{\delta}_t \subset \hat{U}_2 \) for \( \tilde{x}_t \). Let \( \delta_t \subset U_2 \) be the projected image of \( \hat{\delta}_t \). Since \( \hat{h}(\delta_t) \subset \hat{U}_3 \), \( h(\delta_t) \subset \delta_t \) and \( \delta_t \) is the projected image of \( \hat{h}(\delta_t) \). Thus \( \delta_t \subset \hat{U}_2 \) is a translation arc for \( \tilde{x}_t \). We now fix such a \( \tilde{x}_t \) and drop the \( t \) subscript.

Assume the notation of Lemma [7.13] applied with \( r = 1 \) and \( \tilde{x}_1 = \tilde{x} \). The homotopy streamline \( \hat{B}_1 \) can be thought of as an arc \( \hat{\mu} \) with initial endpoint \( \alpha(\hat{h}, \tilde{x}) \) and terminal endpoint \( \omega(\hat{h}, \tilde{x}) \). Let \( \hat{\mu}_0 \) be the initial subpath of \( \hat{\mu} \) that ends with \( \tilde{x} \) and let \( \hat{\nu} \subset \hat{U}_1 \) be a path connecting \( \tilde{x} \) to \( \hat{P} \). The path \( \hat{\eta} = \hat{\mu}_0 \hat{\nu} \) connects \( \alpha(\hat{h}, \tilde{x}) \in \partial_0 A \) to \( \hat{P} \in \partial_1 A \). By the uniqueness part of Lemma [7.13] (6), \( \hat{\delta} \) is isotopic rel \( X \) to the subpath of \( \hat{\mu} \) connecting \( \hat{x} \) to \( h(\hat{x}) \). It follows that the path \( \hat{\eta}^{-1} h(\hat{\eta}) \) connecting \( P \) to \( h(P) \) is homotopic rel endpoints to \( \hat{\nu}^{-1} \hat{h}(\hat{\nu}) \subset \hat{U}_2 \). Hence \( h(\hat{\eta}) \) is homotopic rel endpoints to \( \eta J_1 \). This contradiction completes the proof.

\[ \square \]

8 \( \omega \)-lifts

We assume throughout this section that \( \mathcal{R} \neq \emptyset \).

The direction (left or right) of twisting on \( \sigma \in \mathcal{R} \) and the choice of a domain \( \hat{C} \) containing a lift \( \hat{\sigma} \) induce an orientation on \( \sigma \). We sometimes write \( \hat{\sigma}_C \) for \( \hat{\sigma} \) equipped with this orientation. We say that two geodesics in \( H \) are anti-parallel if their initial endpoints are separated in \( S_\infty \) by their terminal endpoints.
Lemma 8.1. The orientations on $\hat{\sigma}$ induced from the two domains that contain it are opposite. If $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are components of $\partial \hat{C}$ that project to the same element of $R$ then $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are anti-parallel with the orientations induced from $\hat{C}$.

Proof. The first statement follows from the definitions. The second also uses the assumption that $M$ has genus zero.

Lemma 8.2. Suppose that $\hat{C}_1$ and $\hat{C}_2$ are domains with intersection $\hat{\sigma} \subset \hat{R}$ and that $\hat{f}_i = \hat{f}_{\hat{C}_i}$. If $\hat{\omega}(\hat{f}_1, \tilde{x}) \neq \hat{\sigma}^+_{\hat{C}_1}$ then $\hat{\omega}(\hat{f}_2, \tilde{x}) = \hat{\sigma}^+_{\hat{C}_2} = \hat{\sigma}^-_{\hat{C}_1}$. Symmetrically, if $\hat{\alpha}(\hat{f}_1, \tilde{x}) \neq \hat{\sigma}^-_{\hat{C}_1}$ then $\hat{\alpha}(\hat{f}_2, \tilde{x}) = \hat{\sigma}^-_{\hat{C}_2} = \hat{\sigma}^+_{\hat{C}_1}$.

Proof. Let $T_{\hat{\sigma}}$ be the covering translation with axis $\hat{\sigma}$ and orientation induced by $\hat{C}_2$. Then $\hat{f}_2^n = T_{\hat{\sigma}}^dn\hat{f}_1^n$ where $d > 0$ is the degree of Dehn twisting about $R$. By hypothesis and by Lemma 8.1 $\hat{\omega}(\hat{f}_1, \tilde{x}) \neq \hat{\sigma}^+_{\hat{C}_1} = \hat{\sigma}^-_{\hat{C}_2} = T_{\hat{\sigma}}^n$. Since $\hat{f}_1^n(\tilde{x})$ converges to $\hat{\omega}(\hat{f}_1, \tilde{x})$ it follows that $T_{\hat{\sigma}}^dn\hat{f}_1^n(\tilde{x}) \to T_{\hat{\sigma}}^\pm$. This proves that $\hat{\omega}(\hat{f}_2, \tilde{x}) = \hat{\sigma}^\pm_{\hat{C}_2}$.

Lemma 8.3. There is a constant $D_1 > 0$ so that if $\text{dist}(\tilde{x}, \hat{C}) > D_1$ and $\hat{\sigma}$ is the component of $\partial \hat{C}$ closest to $\tilde{x}$, then at least one of $\hat{\alpha}(\hat{f}_{\hat{C}}, \tilde{x})$ and $\hat{\omega}(\hat{f}_{\hat{C}}, \tilde{x})$ is an endpoint of $\hat{\sigma}$.

Proof. For any $\tilde{x}$ both $\alpha(\hat{f}_{\hat{C}}, \tilde{x})$ and $\hat{\omega}(\hat{f}_{\hat{C}}, \tilde{x})$ lie in $\text{Fix}(f_{\hat{C}})$, i.e. in points of $S_\infty$ which correspond to ends of $\hat{C}$. Two of the points of $\text{Fix}(f_{\hat{C}})$ are ends of $\hat{\sigma}$ and all the others lie on one side of $\hat{\sigma}$. Consider the annular cover $A_{\hat{\sigma}}$ and the induced map $f_{\sigma}: A_{\sigma} \to A_{\sigma}$. If the lemma is false then for any $D_1 > 0$ we can find $\tilde{x}$ such that

- $\hat{\sigma}$ is the component of $\partial \hat{C}$ closest to $\tilde{x}$
- Neither $\alpha(f_{\sigma}, \tilde{x})$ nor $\hat{\omega}(f_{\sigma}, \tilde{x})$ is an endpoint of $\hat{\sigma}$.
- $\text{dist}(\tilde{x}, \hat{C}) > D_1$.

From the second item we conclude that both $\alpha(f_{\sigma}, \tilde{x})$ and $\hat{\omega}(f_{\sigma}, \tilde{x})$ lie in $\partial A_{\sigma}$, and, in fact, in the same component of $\partial A_{\sigma}$ since their lifts lie on the same side of $\hat{\sigma}$. From the first and third item we conclude that the $\tilde{x}$’s lie arbitrarily close to the other component of $\partial A_{\sigma}$ in contradiction to Corollary 7.17 applied to $f_{\sigma}$.

Lemma 8.4. Suppose that $\hat{C}_1$ and $\hat{C}_2$ are domains with intersection $\hat{\sigma} \subset \hat{R}$ and that $\hat{f}_i = \hat{f}_{\hat{C}_i}$. Let $\partial_1 A_{\sigma}$ be the boundary component that contains points that lift into the closure of $\hat{C}_i$. If neither $\alpha(\hat{f}_1, \tilde{x})$ nor $\hat{\omega}(\hat{f}_2, \tilde{x})$ is an endpoint of $\hat{\sigma}$ then $\alpha(f_{\sigma}, \tilde{x}) \in \partial_1 A_{\sigma}$ and $\hat{\omega}(f_{\sigma}, \tilde{x}) \in \partial_2 A_{\sigma}$.

Proof. This is an immediate consequence of the definitions.

Corollary 8.5. For all $\tilde{x} \in H$ exactly one of the following hold.
(1) There is a unique domain \( \tilde{C} \) such that \( \omega(f_{\tilde{C}}, \tilde{x}) \) is not an endpoint of a component of \( \partial \tilde{C} \).

(2) There is a unique component \( \tilde{\sigma} \) of \( \tilde{\mathcal{R}} \) such that both \( \omega(f_{\tilde{C}_1}, \tilde{x}) \) and \( \omega(f_{\tilde{C}_2}, \tilde{x}) \) are the two endpoints of \( \tilde{\sigma} \) where \( \tilde{C}_1 \) and \( \tilde{C}_2 \) are the two domains that contain \( \tilde{\sigma} \) in their boundaries.

**Proof.** Lemma \[8.2\] and the obvious induction argument imply that the two items are mutually exclusive and also imply the uniqueness parts of the items. It therefore suffices to find \( \tilde{C} \) satisfying (1) or \( \tilde{\sigma} \) satisfying (2).

Choose a domain \( \tilde{C}'_1 \). If \( \omega(f_{\tilde{C}'_1}, \tilde{x}) \) is not an endpoint of a component of \( \partial \tilde{C}'_1 \) we are done. Otherwise, let \( \tilde{C}'_2 \) be the domain whose intersection with \( \tilde{C}'_1 \) is the component \( \tilde{\sigma}_1 \) of \( \tilde{\mathcal{R}} \) that contains \( \omega(f_{\tilde{C}'_1}, \tilde{x}) \). If \( \tilde{\omega}(\tilde{C}'_2, \tilde{x}) \) is either not the endpoint of a component of \( \partial \tilde{C}'_2 \) or is an endpoint of \( \tilde{\sigma}_1 \) we are done. Otherwise, let \( \tilde{C}'_3 \) be the domain whose intersection with \( \tilde{C}'_2 \) is the component \( \tilde{\sigma}_2 \) of \( \tilde{\mathcal{R}} \) that contains \( \omega(f_{\tilde{C}'_2}, \tilde{x}) \). Iterating this procedure we either reach the desired conclusion or produce domains \( \tilde{C}'_k \) such that \( \omega(f_{\tilde{C}'_k}, \tilde{x}) \) is an endpoint of \( \tilde{\sigma}_k = \tilde{C}'_k \cap \tilde{C}'_{k+1} \). By Lemma \[8.3\], \( \alpha(f_{\tilde{C}'_1}, \tilde{x}) \) is an endpoint of \( \tilde{\sigma}_{k-1} \) for all sufficiently large \( k \). Let \( f_k : A_{\sigma_k} \to A_{\sigma_k} \) be the annulus map induced from \( \tilde{\sigma}_k \) and let \( \partial_- A_{\sigma_k} \) and \( \partial_+ A_{\sigma_k} \) be the components of \( \partial A_{\sigma_k} \) that contain points that lift into the closure of \( \tilde{C}'_k \) and \( \tilde{C}'_{k+1} \) respectively. Lemma \[8.4\] implies that \( \alpha(f_k, \tilde{x}) \in \partial_- A_{\sigma_k} \) and \( \omega(f_k, \tilde{x}) \in \partial_+ A_{\sigma_k} \).

Since \( \mathcal{R} \) consists of a finite set of simple closed geodesics there is an \( l \) which is the first integer greater than \( k \) such that \( \sigma_l = \sigma_k \). Since \( M \) has genus zero, the arc in \( A_{\sigma_k} \) connecting \( \alpha(f_k, \tilde{x}) \) to \( \omega(f_k, \tilde{x}) \) and the arc connecting \( \alpha(f_l, \tilde{x}) \) to \( \omega(f_l, \tilde{x}) \) cross \( \sigma_k \) in opposite directions in contradiction to Lemma \[7.15\] The process therefore terminates after finitely many steps. \( \square \)

**Definition 8.6.** If Corollary \[8.5\] (1) is satisfied then we say that \( \tilde{C} \) is the \( \omega \)-domain for \( \tilde{x} \) and \( f_{\tilde{C}} \) is the \( \omega \)-lift for \( \tilde{x} \). Otherwise, Corollary \[8.5\] (2) is satisfied and we say that \( \tilde{C}_1 \) and \( \tilde{C}_2 \) are the \( \omega \)-domains for \( \tilde{x} \) and \( f_{\tilde{C}_1} \) and \( f_{\tilde{C}_2} \) are the \( \omega \)-lifts for \( \tilde{x} \).

**Corollary 8.7.** Let \( D_1 \) be the constant of Lemma \[8.3\]

(1) If \( \tilde{C} \) is the unique \( \omega \)-domain for \( \tilde{x} \) then \( \hat{f}^n_{\tilde{C}}(\tilde{x}) \in N_{D_1}(\tilde{C}) \) for all sufficiently large \( n \).

(2) If \( \tilde{C}_1 \) and \( \tilde{C}_2 \) are \( \omega \)-domains for \( \tilde{x} \) with intersection \( \tilde{\sigma} \in \tilde{\mathcal{R}} \) then \( \hat{f}^n_{\tilde{C}_i}(\tilde{x}) \in N_{D_1}(\tilde{C}_1 \cup \tilde{C}_2) \) for \( i = 1, 2 \) and all sufficiently large \( n \).

**Proof.** If \( \tilde{C} \) is the unique \( \omega \)-domain for \( \tilde{x} \) and (1) fails then there exist arbitrarily large \( n \) such that \( \hat{f}^n_{\tilde{C}}(\tilde{x}) \notin N_{D_1}(\tilde{C}) \). Since \( \hat{f}^n_{\tilde{C}}(\tilde{x}) \to \omega(f_{\tilde{C}}, \tilde{x}) \) we may assume that \( \alpha(f_{\tilde{C}}, \tilde{x}) \) is not an endpoint of the component of \( \partial \tilde{C} \) that is closest to \( \hat{f}^n_{\tilde{C}}(\tilde{x}) \). This contradicts Lemma \[8.3\] and so completes the proof of (1).
Assume the notation of (2) and that (2) fails for \( i = 1 \); the \( i = 2 \) case is symmetric. There exist arbitrarily large \( n \) such that \( \hat{f}^{n}_{\hat{C}_{1}}(\hat{x}) \notin N_{D_{1}}(\hat{C}_{1} \cup \hat{C}_{2}) \). Lemma \( \mathbf{5.3} \) implies that \( \tilde{\sigma} \) is the component of \( \partial \hat{C}_{1} \) that is closest to \( \hat{f}^{n}_{\hat{C}_{1}}(\hat{x}) \) for all sufficiently large \( n \). Since \( \hat{f}_{\hat{C}_{1}} \) and \( \hat{f}_{\hat{C}_{2}} \) differ by an iterate of \( T_{\hat{\sigma}} \) and since \( T_{\hat{\sigma}} \) preserves both \( \hat{C}_{1} \) and \( \hat{C}_{2} \), \( \hat{f}^{n}_{\hat{C}_{2}}(\hat{x}) \notin N_{D_{1}}(\hat{C}_{1} \cup \hat{C}_{2}) \) and \( \tilde{\sigma} \) is not the component of \( \partial \hat{C}_{2} \) that is closest to \( \hat{f}^{n}_{\hat{C}_{2}}(\hat{x}) \). This contradicts Lemma \( \mathbf{5.3} \) and so completes the proof of (2).

We record the following observation for easy reference.

**Lemma 8.8.** If \( \hat{f}^{k}_{\hat{C}}(\hat{x}) \in N_{D}(\hat{C}) \) for some \( D > 0 \) and all sufficiently large \( k \) then \( \hat{C} \) is an \( \omega \)-domain for \( \hat{x} \).

**Proof.** It suffices to show that if \( \omega(\hat{f}_{\hat{C}}, \hat{x}) \) is an endpoint of \( \tilde{\sigma} \in \hat{\mathcal{R}} \) and \( \hat{C}' \) is the other domain whose frontier contains \( \hat{\sigma} \) then \( \omega(\hat{f}_{\hat{C}'}, \hat{x}) \) is an endpoint of \( \hat{\sigma} \). The covering translation \( T_{\hat{\sigma}} \) preserves \( N_{D}(\hat{C}) \). Since \( \hat{f}^{k}_{\hat{C}'} \) and \( \hat{f}^{k}_{\hat{C}} \) differ by an iterate of \( T_{\hat{\sigma}} \), it follows that \( \hat{f}^{k}_{\hat{C}'}(\hat{x}) \in N_{D}(\hat{C}) \) for all sufficiently large \( k \) and hence that \( \omega(\hat{f}_{\hat{C}'}, \hat{x}) \) is an endpoint of \( \hat{\sigma} \).

\[ \square \]

9 Domain Covers

We assume throughout this section that \( \mathcal{R} \neq \emptyset \).

Let \( \hat{C} \) be a domain and let \( C \) be its image in \( S \). Recall that \( \text{Stab}(\hat{C}) \) is the subgroup of covering translations that preserve \( \hat{C} \) and that elements of \( \text{Stab}(\hat{C}) \) commute with \( \hat{f}_{\hat{C}} \). We can not restrict \( f \) to \( C \) because \( C \) is not \( f \)-invariant and we can not replace \( f \) by an isotopic map that preserves \( C \) because we might lose the entropy zero property. Instead we lift to the \( \pi_{1}(C) \) cover \( \hat{C} \) of \( S \). More precisely we make the following definitions.

**Definitions 9.1.** Define \( \hat{C}_{\text{core}} \subset \hat{C} \) to be the quotient spaces of \( \hat{C} \subset H \) by the action of \( \text{Stab}(\hat{C}) \) and \( \hat{f}_{\hat{C}} : \hat{C} \to \hat{C} \) to be the homeomorphism induced by \( \hat{f}_{\hat{C}} \). Up to conjugacy, \( \hat{f}_{\hat{C}} : \hat{C} \to \hat{C} \) is independent of the choice of lift \( \hat{C} \) of \( C \).

**Standing Notation 9.2.** Our convention will be that if \( \hat{x} \in \hat{C} \) then its image in \( M \) is \( x \) and its image in \( \hat{C} \) is \( \hat{x} \).

Note that \( \hat{C}_{\text{core}} \) is homeomorphic to \( C \) and (topologically) \( \hat{C} \) is obtained from \( \hat{C}_{\text{core}} \) by adding collar neighborhoods to each component of \( \partial \hat{C}_{\text{core}} \). Note also that \( \hat{f}_{\hat{C}} \) is isotopic to the identity.

If \( \hat{C} \) is both an \( \alpha \)-domain and an \( \omega \)-domain for \( \hat{x} \) then we say that \( \hat{C} \) is a *home domain* for \( \hat{x} \). Denote the set of birecurrent points for \( f \) by \( \mathcal{B}(f) \) and the full pre-image of \( \mathcal{B}(f) \) by \( \hat{\mathcal{B}}(f) \). The following proposition, whose proof is delayed until the end of the section, is the main result of this section.
Proposition 9.3. If $\tilde{C}$ is an $ω$-domain for $\tilde{x} ∈ B(\tilde{f})$ then $\tilde{x} ∈ B(\tilde{f}_C)$ and $\tilde{C}$ is a home domain for $\tilde{x}$. Moreover if $\tilde{ω}(\tilde{f}_C, \tilde{x})$ is an endpoint of $\tilde{σ} ∈ \tilde{R}$ then $\tilde{α}(\tilde{f}_C, \tilde{x})$ is also an endpoint of $\tilde{σ}$.

The following definition is key to the proof of Proposition 9.3

Definition 9.4. A covering translation $T : H → H$ is a near-cycle of period $m$ for $\tilde{x} ∈ H$ with respect to $\tilde{f}_C$ if there is a free disk $U$ for $f$ and a lift $\tilde{U}$ that contains $\tilde{x}$ such that $\tilde{f}^m_C(\tilde{x}) ∈ T(\tilde{U})$. If $m$ is irrelevant then we simply say that $T$ is a near-cycle for $\tilde{x} ∈ H$ with respect to $\tilde{f}_C$.

Remark 9.5. It is an immediate consequence of the definitions that if $T : H → H$ is a near-cycle of period $m$ with respect to $\tilde{f}_C$ for $\tilde{x}$ then it is also a near-cycle of period $m$ with respect to $\tilde{f}_C$ for all points in a neighborhood of $\tilde{x}$. Moreover, it is clear that by shrinking the free disk $U$ slightly to $U_0$, we may assume that $cl(U_0)$ is contained in a free disk and we still have $\tilde{f}^m_C(\tilde{x}) ∈ T(\tilde{U}_0)$.

Remark 9.6. A point $\tilde{x} ∈ H$ has at least one near cycle if and only if its image $x ∈ M$ is free disk recurrent.

The following lemma is essentially the same as Lemma 10.5 of [10]. We reprove it here because our assumptions have changed.

Lemma 9.7. If $T ∈ Stab(\tilde{C})$ is a near-cycle for $\tilde{x} ∈ H$ with respect to $\tilde{f}_C$ then $\tilde{α}(\tilde{f}_C, \tilde{x})$ and $\tilde{ω}(\tilde{f}_C, \tilde{x})$ can not both lie in the same component of $S_∞ \setminus \{T^+, T^-\}$.

Proof. If $T$ is parabolic let $\tilde{σ}$ be a horocycle preserved by $T$; otherwise let $\tilde{σ}$ be the axis of $T$. From $T ∈ Stab(\tilde{C})$ it follows that $\tilde{σ}$ is either an element of $\tilde{R}$ or disjoint from $\tilde{R}$. Let $f_\sigma : A_\sigma → A_\sigma$ be as in Definition 6.2. Assume the notation of Lemma 7.13 applied with $r = 1$ and $\tilde{x}_1$ the image of $\tilde{x}$ in $A_\sigma$. The lifts $\tilde{B}_1$ and $\tilde{B}_1'$ of $\tilde{B}_1$ that contain $\tilde{x}$ and $T(\tilde{x})$ respectively are disjoint and $\tilde{f}$-invariant up to isotopy rel the orbits of $\tilde{x}$ and $T(\tilde{x})$. Lemma 8.7-(2) of [10] implies that $\tilde{B}_1$ and $\tilde{B}_1'$ have parallel orientations. The lemma now follows from the fact that the endpoints of $\tilde{B}_1$ are $\tilde{α}(\tilde{f}_C, \tilde{x})$ and $\tilde{ω}(\tilde{f}_C, \tilde{x})$ and the endpoints of $\tilde{B}_1'$ are $T\tilde{α}(\tilde{f}_C, \tilde{x})$ and $T\tilde{ω}(\tilde{f}_C, \tilde{x})$.

Lemma 9.8. Suppose $\tilde{C}$ is an $ω$-domain for $\tilde{x}$, that $ω(\tilde{f}_C, \tilde{x})$ is an endpoint of $\tilde{σ} ∈ \tilde{R}$ and that $\tilde{x}$ is $\tilde{f}_C$-recurrent. Then every near cycle $T ∈ Stab(\tilde{C})$ for a point in the $\tilde{f}_C$-orbit of $\tilde{x}$ is hyperbolic with axis $\tilde{σ}$.

Proof. To simplify notation we write $\tilde{f} = \tilde{f}_C$. There is no loss in assuming that $T$ is a near cycle for $\tilde{x}$. Let $U$ be the free disk with respect to which $T$ is defined, let $\tilde{U}$ be the lift of $U$ containing $\tilde{x}$ and let $n$ satisfy $\tilde{f}^n(\tilde{x}) ∈ T(\tilde{U})$. There is a neighborhood $x ∈ V ⊂ U$ such that $\tilde{f}^n(V) ⊂ U$. Let $\tilde{V}$ be the lift of $V$ contained in $\tilde{U}$.

If $\tilde{α}(\tilde{f}, \tilde{x})$ is an endpoint of $\tilde{σ}$ then Lemma 9.7 and the fact that $\tilde{σ} ⊂ \partial \tilde{C}$ complete the proof. Suppose then that $\tilde{α}(\tilde{f}, \tilde{x})$ is not an endpoint of $\tilde{σ}$ and in particular, $\tilde{α}(\tilde{f}, \tilde{x}) \neq \tilde{ω}(\tilde{f}, \tilde{x})$.
Lemma 9.9. Suppose that \( T \) is parabolic then each \( T_i \) is finite. We may therefore replace the former with the latter in the hypotheses of this lemma. In this case the \( T_i \)'s are iterates of a single parabolic covering translation and there is a neighborhood of \( \omega(f, \tilde{x}) \) that is moved off of itself by each \( T_i \). This contradicts \( \lim_{n \to \infty} f^n(\tilde{x}) = \omega(f, \tilde{x}) \). We conclude that \( T \) and \( T_i \) are hyperbolic. Let \( A_T \) be the axis of \( T \) and \( A_i = S_i(A_T) \) the axis of \( T_i \).

We claim that \( A_i \neq \tilde{\sigma} \). This is obvious if \( A_T \) is not an element of \( \tilde{\mathcal{R}} \) so we assume that \( A_T \) is an element of \( \tilde{\mathcal{R}} \) and that \( \tilde{\sigma} = A_i = S_i(A_T) \) for some \( S_i \in \text{Stab}(\tilde{C}) \) and argue to a contradiction. Keeping in mind that \( \tilde{\sigma} \) and \( A_T \) are distinct components of the frontier of \( \tilde{C} \), Lemma 9.7 implies that \( \alpha(f, \tilde{x}) \) is an endpoint of \( A_T \). The axis of \( S_i \) is contained in \( \tilde{C} \) and is not \( \tilde{\sigma} \) or \( A_T \). It follows that the axis of \( S_i \) is disjoint from \( A_T \) and \( \tilde{\sigma} \) and has no endpoints in common with either. Since \( \tilde{\sigma} = S_i(A_T) \), the axis of \( S_i \) does not separate \( A_T \) from \( \tilde{\sigma} \). This contradicts Lemma 9.7 applied to the near cycle \( S_i \) and so completes the proof that \( A_i \neq \tilde{\sigma} \).

After passing to a subsequence we may assume that either the \( A_i \)'s are all the same or all different. In the former case, \( T_i \) is independent of \( i \) and there is a neighborhood of \( \omega(f, \tilde{x}) \) that is moved off of itself by each \( T_i \). As above this contradicts the fact that \( T_i(f^{n_i}(\tilde{x})) \to \omega(f, \tilde{x}) \). We may therefore assume that the \( A_i \)'s are distinct lifts of a closed curve in \( \tilde{M} \) and hence, after passing to a subsequence, converge to some point \( Q \in S_\infty \). If \( Q \neq \omega(f, \tilde{x}) \) then there is a neighborhood of \( \omega(f, \tilde{x}) \) that is moved off of itself by each \( T_i \) and we have a contradiction. Thus \( Q = \omega(f, \tilde{x}) \).

For sufficiently large \( i \) the endpoints of \( A_i \) are contained in a neighborhood of \( \omega(f, \tilde{x}) \) that does not contain \( \alpha(f, \tilde{x}) \) and does not contain the other endpoint of \( \tilde{\sigma} \). Since \( A_i \) is disjoint from \( \tilde{\sigma} \), it does not separate \( \alpha(f, \tilde{x}) \) from \( \omega(f, \tilde{x}) \). This contradicts Lemma 9.7 since neither \( \alpha(f, \tilde{x}) \) nor \( \omega(f, \tilde{x}) \) is an endpoint of \( A_i \). \( \square \)

**Lemma 9.9.** Suppose that \( U \) is a free disk, that \( x \in U \) is recurrent [birecurrent] with respect to \( f \) and that the set of lifts of \( U \) to \( H \) that intersect \( \{ f^k_C(\tilde{x}) : k \geq 0 \} \) is finite up to the action of \( \text{Stab}(\tilde{C}) \). Then \( \tilde{x} \in \tilde{C} \) is recurrent [birecurrent] with respect to \( \tilde{f} : \tilde{C} \to \tilde{C} \).

**Proof.** The set of lifts of \( U \) to \( H \) that intersect \( \{ f^k_C(\tilde{x}) : k \geq 0 \} \) is finite up to the action of \( \text{Stab}(\tilde{C}) \) if and only if the set of lifts of \( U \) to \( \tilde{C} \) that intersect \( \{ \tilde{f}^k_C(\tilde{x}) : k \geq 0 \} \) is finite. We may therefore replace the former with the latter in the hypotheses of this lemma.

Suppose that \( x \) is recurrent. We must prove that \( \tilde{x} \) is recurrent and that if \( x \) is recurrent with respect to \( f^{-1} \) then \( \tilde{x} \) is recurrent with respect to \( \tilde{f}^{-1} \).

Let \( U_1, \ldots, U_m \) be the lifts of \( U \) to \( \tilde{C} \) that intersect \( \{ \tilde{f}^k_C(\tilde{x}) : k \geq 0 \} \) and let \( \tilde{x}_j \in U_j \) be the corresponding lifts of \( x \). We may assume that \( \tilde{x}_1 = \tilde{x} \). Choose a sequence \( n_i \to \infty \) such that \( f^{n_i}(x) \to x \) and such that each \( f^{n_i}(x) \in U \). After passing to a subsequence we may assume that \( f^{n_i}(\tilde{x}_1) \in U_s \) where \( s \) is independent of \( i \). Then
\( \bar{f}_C^n(x) \rightarrow \bar{x} \) and we are done if \( s = 1 \). Otherwise we may assume that \( s = 2 \). Since \( \bar{x}_2 \) is in the \( \omega \)-limit set of \( \bar{x}_1 \), each point in \( \{ \bar{f}_C^k(\bar{x}_2) : k \geq 0 \} \) that projects to \( U \) is contained in some \( \bar{U}_j \). We may therefore apply the previous argument with \( \bar{x}_2 \) in place of \( \bar{x}_1 \). After passing to a further subsequence we may assume that \( \bar{f}_C^n(\bar{x}_2) \rightarrow \bar{x}_t \) where \( t \neq 2 \) because \( \bar{f}_C^n(\bar{x}_1) \) is the unique point in \( \bar{U}_2 \) that projects to \( f^n(x) \). If \( t = 1 \) then \( \bar{x}_1 \) is in the \( \omega \)-limit set of \( \bar{x}_1 \) and we are done. Otherwise we may assume \( t = 3 \). After iterating this argument at most \( m \) times, we have shown that \( \bar{x} \) is recurrent.

From the recurrence of \( \bar{x} \), it follows that a lift of \( U \) to \( \bar{C} \) intersects \( \{ \bar{f}_C^k(\bar{x}) : k \geq 0 \} \) if and only if it intersects \( \{ \bar{f}_C^k(\bar{x}) : k \geq 0 \} \). In particular, the set of lifts of \( U \) to \( \bar{C} \) that intersect \( \{ \bar{f}_C^{-k}(\bar{x}) : k \geq 0 \} \) is finite. If \( x \) is recurrent with respect to \( f \) then by the above argument \( \bar{x} \in \bar{C} \) is recurrent with respect to \( \bar{f}^{-1} : \bar{C} \rightarrow \bar{C} \) as desired.

**Remark 9.10.** If \( \bar{U} \) is a lift of a disk \( U \) and \( T_1, T_2 \) are covering translations then \( T_1(\bar{U}) \) and \( T_2(\bar{U}) \) are in the same \( \text{Stab}(\bar{C}) \)-orbit if and only if \( T_2T_1^{-1} \in \text{Stab}(\bar{C}) \). Thus a collection of lifts \( \{ T_m(\bar{U}) \} \) of \( U \) is finite up to the action of \( \text{Stab}(\bar{C}) \) is finite if and only if the \( T_m \)'s determine only finitely many right cosets of \( \text{Stab}(\bar{C}) \).

**Proof of Proposition 9.3** Let \( U \) be a free disk that contains \( x \) and let \( \bar{U} \) be the lift that contains \( \bar{x} \).

As a first case suppose that \( \bar{\omega}(\bar{f}_C, \bar{x}) \) is not an endpoint of an element of \( \bar{R} \). Lemma \[8.7\](1) implies that for some \( D \) and all \( k \geq 0 \), \( \bar{f}_D^k(\bar{x}) \in N_D(\bar{C}) \) or equivalently, \( \bar{f}_C^k(\bar{x}) \in N_D(\bar{C}_{\text{core}}) \). Since \( N_D(\bar{C}_{\text{core}}) \) is compact and by Remark \[9.5\] we may assume the closure of \( U \) is contained in a disk in \( M \), it follows that \( \{ \bar{f}_C^k(\bar{x}) : k \geq 0 \} \) intersects only finitely many lifts of \( U \). Equivalently, \( \{ \bar{f}_C^k(\bar{x}) : k \geq 0 \} \) intersects only finitely many lifts of \( U \) to \( H \) up to the action of \( \text{Stab}(\bar{C}) \). Lemma \[9.9\] implies that \( \bar{x} \) is birecurrent under \( \bar{f}_C \). It follows that \( \bar{f}_C^n(\bar{x}) \in N_D(\bar{C}_{\text{core}}) \) for all \( k \) and hence that \( \bar{f}_C^n(\bar{x}) \in N_D(\bar{C}) \) for all \( k \). Lemma \[8.8\] applied to \( \bar{f}^{-1} \) implies that \( \bar{C} \) is a home domain.

We assume now that \( \omega(\bar{f}, \bar{x}) \) is an endpoint of \( \bar{\sigma} \in \bar{R} \) and that \( \bar{C}_1 \) and \( \bar{C}_2 \) are the two domains that contain \( \bar{\sigma} \) in their frontier. We will treat \( \bar{C}_1 \) and \( \bar{C}_2 \) symmetrically and prove that the proposition holds for \( \bar{C} = \bar{C}_1 \) and \( \bar{C} = \bar{C}_2 \). Denote \( \bar{f}_{\bar{C}_1} \) by \( \bar{f}_1 \) and \( \bar{f}_{\bar{C}_2} \) by \( \bar{f}_2 \). When near cycles are defined with respect to \( \bar{f}_1 \) we refer to them as \( \bar{f}_1 \)-near cycles. Let \( S \) be a root-free covering translation with axis \( \bar{\sigma} \). Lemma \[8.7\](2) implies that \( \bar{f}_1^k(\bar{x}), \bar{f}_2^k(\bar{x}) \in N_D(\bar{C}_1 \cup \bar{C}_2) \) for some \( D \) and all \( k \geq 0 \). We may assume without loss that \( U \subset N_D(\bar{C}_1) \cap N_D(\bar{C}_2) \).

After interchanging \( \bar{C}_1 \) with \( \bar{C}_2 \) if necessary, we may assume by Lemma \[8.2\] that \( \alpha(\bar{f}_1, \bar{x}) \) is an endpoint of \( \bar{\sigma} \). Lemma \[9.7\] implies that every \( \bar{f}_1 \)-near cycle \( T \in \text{Stab}(\bar{C}_1) \) for a point in the \( \bar{f}_1 \)-orbit of \( \bar{x} \) is an iterate of \( S \). We will apply this as follows. If \( T_1 \) and \( T_2 \) are \( \bar{f}_1 \)-near cycles for \( \bar{x} \) and if \( T_1T_2^{-1} \) (which is a near cycle for a point in the \( \bar{f}_1 \)-orbit of \( \bar{x} \)) is an element of \( \text{Stab}(\bar{C}_1) \) then \( T_1T_2^{-1} \) is an iterate of \( S \). In particular, if \( T_1 \) and \( T_2 \) determine the same right coset of \( \text{Stab}(\bar{C}_1) \) then they also determine the same right coset of \( \text{Stab}(\bar{C}_2) \),
Let $\mathcal{U}_1$ be the set of lifts of $U$ that intersect $N_D(\tilde{C}_i)$ and contain $\tilde{f}_2^k(\tilde{x})$ for some $k \geq 0$. To prove that $\tilde{x}$ is $\tilde{f}_2$-birecurrent it suffices by Lemma 9.9 to prove that $\mathcal{U}_1 \cup \mathcal{U}_2$ is finite up to the action of $\text{Stab}(\tilde{C}_2)$. As above, the compactness of $N_D(\tilde{C}_{\text{core}})$ implies that $\mathcal{U}_1$ is finite up to the action of $\text{Stab}(\tilde{C}_i)$.

Each element of $\mathcal{U}_i$ has the form $T(U)$ for some covering translation $T$; let $T_i$ be the set of all such $T$. Each $T_m \in T_i$ is an $\tilde{f}_2$-near cycle for $\tilde{x}$. Since $\tilde{f}_2$ and $\tilde{f}_1$ differ by an iterate of $S$, there exists $j_m$ such that $S^{j_m}T_m$ is an $\tilde{f}_1$-near cycle for $\tilde{x}$. Remark 9.10 implies that $\{S^{j_m}T_m\}$ determines only finitely many right cosets of $\text{Stab}(\tilde{C}_1)$ which, as observed above, implies that $\{S^{j_m}T_m\}$ determines only finitely many right cosets of $\text{Stab}(\tilde{C}_2)$. Since $\{S^{j_m}T_m\}$ and $\{T_m\}$ determine the same right cosets of $\text{Stab}(\tilde{C}_2)$, we have shown that $\{T_m\}$ determines only finitely many right cosets of $\text{Stab}(\tilde{C}_2)$. A second application of Remark 9.10 completes the proof that $\tilde{x}$ is $\tilde{f}_2$-birecurrent.

Since $x$ is recurrent with respect to $f^{-1}$ there are near cycles $T'_j$ with respect to $\tilde{f}_2^{-1}$ for $\tilde{x}$ such that $T'_j(\tilde{x}) \to \alpha(\tilde{f}_2, \tilde{x})$. From the fact that $T'_j$ is a near cycle with respect to $\tilde{f}_2$ for a point in the orbit of $\tilde{x}$ and hence is an iterate of $S$ we conclude that $\alpha(\tilde{f}_2, \tilde{x})$ is also an endpoint of $\tilde{\sigma}$. This completes the proof for $\tilde{C}_2$.

Now that we have established that $\alpha(\tilde{f}_2, \tilde{x})$ is an endpoint of $\tilde{\sigma}$, this same argument can be applied to $\tilde{C}_1$.

We conclude this section by strengthening Corollary 8.7.

**Corollary 9.11.** Suppose that $x \in \mathcal{B}(f)$ and that $D_1$ is the constant of Lemma 8.3.

1. If $\tilde{C}$ is the unique home domain for $\tilde{x}$ then $\tilde{f}_C^n(\tilde{x}) \in N_{D_1}(\tilde{C})$ for all $n$.

2. If $\tilde{C}_1$ and $\tilde{C}_2$ are home domains for $\tilde{x}$ with intersection $\tilde{\sigma} \in \tilde{\mathcal{R}}$ then $\tilde{f}_C^n(\tilde{x}) \in N_{D_1}(\tilde{C}_1 \cup \tilde{C}_2)$ for all $n$.

**Proof.** Suppose that $\tilde{C}$ is the unique home domain for $\tilde{x}$ and that $\tilde{f}_C^n(\tilde{x}) \notin N_{D_1}(\tilde{C})$. Choose $\epsilon$ greater than the distance from $\tilde{f}_C^n(\tilde{x})$ to $N_{D_1}(\tilde{C})$. Proposition 9.3 implies that $\tilde{x} \in \mathcal{B}(\tilde{f}_C)$ and hence that there exist arbitrarily large $k$ and $S_k \in \text{Stab}(\tilde{C})$ such that the distance from $\tilde{f}_C^n(\tilde{x})$ to $S_k\tilde{f}_C^n(\tilde{x})$ is less than $\epsilon$. Since $S_k$ preserves distance to $\tilde{C}$, $\tilde{f}_C^n(\tilde{x}) \notin N_{D_1}(\tilde{C})$. This contradicts Corollary 8.7 and so completes the proof of (1).

Assuming the notation of (2), suppose that $\tilde{f}_C^n(\tilde{x}) \notin N_{D_1}(\tilde{C}_1 \cup \tilde{C}_2)$. There is no loss in assuming that $\tilde{f}_C^n(\tilde{x})$ is closer to $\tilde{C}_1$ than $\tilde{C}_2$. If $S_k \in \text{Stab}(\tilde{C}_1)$ then $S_k\tilde{f}_C^n(\tilde{x})$ is closer to $\tilde{C}_1$ than $\tilde{C}_2$ and has distance greater than $D_1$ from $\tilde{C}_1$. The argument given for (1) therefore applies in this context as well.

**10 Two compactifications**

We now return to the general case, allowing the possibility that $\mathcal{R} = \emptyset$. 

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If $\mathcal{R} = \emptyset$ then $H$ is the only domain and $\tilde{f}_H$ commutes with all covering translations and fixes every point in $S_\infty$. For consistency of notation we define $H$ to be the home domain for each $\tilde{x}$, define $M$ to be $M$ and define $\tilde{f} : \tilde{C} \to \tilde{C}$ to be $f : M \to M$.

Suppose that $\tilde{C}$ is the home domain for $\tilde{x}$. The diffeomorphism $\tilde{f} : \tilde{C} \to \tilde{C}$ is isotopic to the identity and so satisfies one of the two hypotheses needed to apply results from sections 10 and 11 of [10]. The other hypothesis, that $f$ have no periodic points, is used only to prove Lemma 10.8 of [10] (which is then applied in other proofs). We proved in Lemma 7.9 that $\tilde{f} : \tilde{C} \to \tilde{C}$ satisfies the conclusions of Lemma 10.8 of [10] and hence that the results of [10] apply to $\tilde{f} : \tilde{C} \to \tilde{C}$.

The one subtlety in applying these results is that when $\mathcal{R} \neq \emptyset$, two different compactifications of the universal cover of $\tilde{C}$ are being used.

In the **extrinsic compactification**, the universal cover of $\tilde{C}$ is metrically identified with the universal cover $\tilde{M}$ of $M$, which is metrically identified with $H$ and is compactified by $S_\infty$. The covering translations of the universal cover of $\tilde{C}$ are identified with the subgroup $\text{Stab}(\tilde{C})$ of covering translations of the universal cover of $M$; the closure in $S_\infty$ of the fixed points of the elements of $\text{Stab}(\tilde{C})$ is a Cantor set $K$ whose convex hull projects to $C_{\text{core}} \subset \tilde{C}$.

In the **intrinsic compactification**, $\tilde{C}$ is viewed without regard to $M$ and is equipped with a hyperbolic structure in which the ends corresponding to the components of $\partial C$ are cusps. The universal cover of $\tilde{C}$ is then metrically identified with $H$ and compactified with $S_\infty$. In this case, the set of fixed points of covering translations is dense in $S_\infty$. Topologically the intrinsic compactification of the universal cover is obtained from the extrinsic compactification by collapsing the closure of each component of $S_\infty \setminus K$ to a point.

We have defined $\tilde{C}$ using the extrinsic metric so that geodesics in $\tilde{C}_{\text{core}}$ correspond exactly to geodesics in $C \subset M$. If one considers $\tilde{f} : \tilde{C} \to \tilde{C}$ as a homeomorphism of a punctured surface without reference to $M$ and applies results from [10] then the intrinsic metric is used. To help separate the two, write $g : N \to N$ for $\tilde{f} : \tilde{C} \to \tilde{C}$ when $\tilde{C}$ has the intrinsic metric. Since $g$ is isotopic to the identity there is a preferred lift $\tilde{g} : \tilde{N} \to \tilde{N}$ to the universal cover that commutes with all covering translations. The ‘identity map’ $p : \tilde{M} \to \tilde{N}$ conjugates $\tilde{f}_{\tilde{C}} : \tilde{M} \to \tilde{M}$ to $\tilde{g} : \tilde{N} \to \tilde{N}$. The homeomorphism $p$, which is not an isometry, extends over the compactifying circles but not by a homeomorphism; it collapses the closure of each component of $S_\infty \setminus K$ to a point. In particular, $p|_K$ identifies a pair of points if and only if they bound a component of $\partial \tilde{C}$. The relevance to us is that if $\omega(\tilde{f}_{\tilde{C}}, \tilde{x})$ and $\alpha(\tilde{f}_{\tilde{C}}, \tilde{x})$ bound a component of $\partial \tilde{C}$ then $\omega(\tilde{g}, p(\tilde{x})) = \alpha(\tilde{g}, p(\tilde{x}))$.

We now turn to the applications.

**Definition 10.1.** Assume that $\tilde{C}$ is a home domain for a lift $\tilde{x}$ of $x \in M$. A point $P \in S_\infty$ **projects to an isolated puncture** if some (and hence every) ray in $\tilde{M}$ that converges to $P$ projects to a ray in $M$ that converges to $c$. We say that $x$ **rotates about an isolated puncture** $c$ if for some (and hence all) lifts $\tilde{x} \in H$ there exists $P \in S_\infty$ that projects to $c$ and a parabolic covering translation $T$ that fixes $P$ such
that every near cycle \( S \in \text{Stab}(\bar{C}) \) for every \( \tilde{f}_C^{\pm}(\tilde{x}) \) is a positive iterate of \( T \).

**Lemma 10.2.** Suppose that \( x \in \mathcal{B}(f) \), that \( \bar{C} \) is a home domain for a lift \( \tilde{x} \) and that 
\[
\alpha(\tilde{f}_C, \tilde{x}) = \omega(\tilde{f}_C, \tilde{x}) = P.
\]
Then \( P \) projects to an isolated puncture \( c \) and \( x \) rotates about \( c \).

**Proof.** If \( \mathcal{R} = \emptyset \) then this is Lemma 11.2 of [10].

Suppose then that \( \mathcal{R} \neq \emptyset \) and assume notation as above. Since 
\[
\alpha(\tilde{f}_C, \tilde{x}) = \omega(\tilde{f}_C, \tilde{x}) = P,
\]
it follows that \( \alpha(\tilde{g}, p(\tilde{x})) = \omega(\tilde{g}, p(\tilde{x})) = p(P) \). By Lemma 11.2 of [10], \( p(P) \) projects to an isolated puncture \( c' \) in \( \bar{C} \) and there is a parabolic covering translation \( T' \) that fixes \( p(P) \) such every near cycle for every point in the orbit of \( p(\tilde{x}) \) is a positive iterate of \( T' \). If \( c' \) does not correspond to an end of \( \bar{C} \) determined by a component of \( \partial \bar{C} \) then we are done.

To complete the proof we assume that \( c' \) corresponds to an end of \( \bar{C} \) determined by a component of \( \partial \bar{C} \) and argue to a contradiction. In this case the parabolic \( T' : \hat{N} \to \hat{N} \) corresponds to a hyperbolic \( T \in \text{Stab}(\bar{C}) \) and every near cycle \( S \in \text{Stab}(\bar{C}) \) for every point in the \( \tilde{f}_C \)-orbit of \( \tilde{x} \) is a positive iterate of \( T \). Since (Proposition 9.3) \( \tilde{x} \in \mathcal{B}(\tilde{f}) \), there are near cycles \( S^+_i \in \text{Stab}(\bar{C}) \) such that \( S^+_i(\tilde{x}) \to \omega(\tilde{f}_C, \tilde{x}) = P \) and near cycles \( S^-_i \) such that \( S^-_i(\tilde{x}) \to \alpha(\tilde{f}_C, \tilde{x}) = P \). Since \( S^+_i \) is a positive iterate of \( T \) and \( S^-_i \) is a negative iterate of \( T \), this is impossible. \( \square \)

**Definition 10.3.** If \( \bar{C} \) is a home domain for \( \tilde{x} \in \hat{M} \) and \( \alpha(\tilde{f}_C, \tilde{x}) \neq \omega(\tilde{f}_C, \tilde{x}) \) let 
\( \tilde{\gamma}(\tilde{x}, \bar{C}) \) be the oriented geodesic with endpoints \( \alpha(\tilde{f}_C, \tilde{x}) \) and \( \omega(\tilde{f}_C, \tilde{x}) \); we say that \( \tilde{x} \) tracks \( \tilde{\gamma}(\tilde{x}, \bar{C}) \) under iteration by \( \tilde{f}_C \). Let \( \gamma(x) \subset M \) be the unoriented geodesic that is the projected image of \( \tilde{\gamma}(\tilde{x}) \); we say that \( x \) tracks \( \gamma(x) \). Note that \( \gamma(x) \) is independent of the choice of lift \( \tilde{x} \) and the choice of home lift for \( \tilde{x} \); the latter would not be true if we imposed an orientation on \( \gamma(x) \).

**Lemma 10.4.** If \( x \in \mathcal{B}(f) \) tracks \( \gamma(x) \) then \( \gamma(x) \) is a simple closed curve. If in addition \( y \in \mathcal{B}(f) \) tracks \( \gamma(y) \) then \( \gamma(x) \) and \( \gamma(y) \) are either disjoint or equal.

**Proof.** If \( \mathcal{R} = \emptyset \) then this follows from Lemmas 10.2 and 11.6 of [10] and the fact that \( M \) has genus zero.

Suppose then that \( \mathcal{R} \neq \emptyset \). Choose a lift \( \tilde{x} \) and home domain \( \bar{C} \) for \( \tilde{x} \). We may assume without loss that \( \bar{C} \) is also a home domain for a lift \( \tilde{y} \) of \( y \). By Lemma 9.3 \( \tilde{x}, \tilde{y} \in \mathcal{B}(\tilde{f}_C) \).

Assume notation as in the beginning of this section. If \( \alpha(\tilde{g}, p(\tilde{x})) = \omega(\tilde{g}, p(\tilde{x})) \) then \( \alpha(\tilde{f}_C, \tilde{x}) \) and \( \omega(\tilde{f}_C, \tilde{x}) \) bound a component of \( \partial \bar{C} \). In this case \( \gamma(x) \in \mathcal{R} \) and the lemma is obvious. We may therefore assume that \( \alpha(\tilde{g}, p(\tilde{x})) \neq \omega(\tilde{g}, p(\tilde{x})) \) and similarly for \( \tilde{y} \). As in the \( \mathcal{R} = \emptyset \) case, the lemma follows from Lemmas 10.2 and 11.6 of [10] and the fact that \( M \) has genus zero. \( \square \)

**Corollary 10.5.** Suppose that \( x \in \mathcal{B}(f) \), that \( \bar{C} \) is a home domain for a lift \( \tilde{x} \) and that \( \tilde{x} \) tracks \( \tilde{\gamma} \). Then every \( \tilde{f}_C \)-near cycle \( S \in \text{Stab}(\bar{C}) \) for a point in the orbit of \( \tilde{x} \) is an iterate of \( T_{\tilde{\gamma}} \).
Proof. We may assume without loss that $S$ is a near cycle for $\tilde{x}$. There exist $m \neq 0$ and a lift $\tilde{U}$ of a free disk $U \subset M$ such that $\tilde{x} \in \tilde{U}$ and $f^m(\tilde{x}) \in S(\tilde{U})$. Let $\tilde{y} = S(\tilde{x})$. Since $S \in \text{Stab}(\tilde{C})$, $S$ commutes with $f_{\tilde{C}}$. Thus $S(\tilde{\gamma}) \subset \tilde{C}$ is the oriented geodesic connecting $\alpha(f_{\tilde{C}}, \tilde{y})$ to $\omega(f_{\tilde{C}}, \tilde{y})$. Lemma 10.3 implies that $\tilde{\gamma}$ and $S(\tilde{\gamma})$ are disjoint or equal (up to perhaps a change of orientation). In the latter case we are done so we assume the former and argue to a contradiction. Since $\tilde{y}$ and $f^m_{\tilde{C}}(\tilde{x})$ are contained in the free disk $\tilde{U}$, Lemma 8.7(2) of [10] implies that the $\tilde{\gamma}$ and $S(\tilde{\gamma})$ are parallel. Since $M$ has genus zero there is an anti-parallel translate $S'(\tilde{\gamma})$ that separates $\tilde{\gamma}$ and $S(\tilde{\gamma})$. We have $S' \in \text{Stab}(\tilde{C})$ because $S'(\tilde{\gamma}) \subset \tilde{C}$. Thus $S'(\tilde{\gamma})$ is the oriented geodesic connecting $\alpha(f_{\tilde{C}}, \tilde{z})$ to $\omega(f_{\tilde{C}}, \tilde{z})$ for $\tilde{z} = S'(\tilde{x})$. This contradicts Lemmas 8.9 and 8.10 of [10] and so completes the proof. \hfill \Box

11 The Set of Annuli $A$

Definitions 11.1. Let $\Gamma$ be the set of simple closed curves that are tracked by at least one element of $B(f)$. For each lift $\tilde{\gamma}$ of $\gamma \in \Gamma$, choose a domain $\tilde{C}$ that contains $\tilde{\gamma}$ and let $\tilde{U}(\tilde{\gamma})$ be the set of points in $H$ which have a neighborhood $\tilde{V}$ such that every point in $\tilde{V} \cap \tilde{B}(f)$ tracks $\tilde{\gamma}$. We say that $\tilde{C}$ is a home domain for $\tilde{U}(\tilde{\gamma})$, that $\tilde{\gamma}$ is the defining parameter of $\tilde{U}(\tilde{\gamma})$ and that $T_{\tilde{\gamma}}$ is the covering translation associated to $\tilde{U}(\tilde{\gamma})$.

For each $\gamma \in \Gamma$ define $U(\gamma)$ to be the projected image of $\tilde{U}(\tilde{\gamma})$ for any lift $\tilde{\gamma}$. We say that $C$ is a home domain for $U(\gamma)$ and that $\gamma$ is the defining parameter of $U(\gamma)$.

Remark 11.2. As the notation suggests, $\tilde{U}(\tilde{\gamma})$ depends only on $\tilde{\gamma}$ and not on the choice of $\tilde{C}$. Indeed, if $\tilde{C}$ is not unique then $\tilde{\gamma} \in \tilde{R}$ and every element of $\tilde{V} \cap \tilde{B}(f)$ has exactly two home domains $C$ and $C'$ (where $C'$ is the other domain that contains $\tilde{\gamma}$) and both $\{\alpha(f_{\tilde{C}}, \tilde{z}), \omega(f_{\tilde{C}}, \tilde{z})\}$ and $\{\alpha(f_{\tilde{C'}}, \tilde{z}), \omega(f_{\tilde{C'}}, \tilde{z})\}$ are contained in $\{\tilde{\gamma}^\pm\}$. $U(\gamma)$ is well defined because $\tilde{U}(S(\tilde{\gamma})) = S\tilde{U}(\tilde{\gamma})$ for any covering translation $S$.

Definitions 11.3. Let $\mathcal{C}$ be the set of punctures in $M$ for which there is at least one element of $B(f)$ that rotates about $c$. For each $P \in S_\infty$ that projects to $c \in \mathcal{C}$, let $\tilde{C}$ be the unique domain whose closure contains $P$ and let $\tilde{U}(P)$ be the set of points in $H$ for which there is a neighborhood $\tilde{V}$ such that every point in $\tilde{V} \cap \tilde{B}(f)$ rotates about $P$. We say that $\tilde{C}$ is the home domain for $\tilde{U}(P)$, that $P$ is the defining parameter of $\tilde{U} = \tilde{U}(P)$ and that $T_p$ is the covering translation associated to $\tilde{U}(P)$.

For each $c \in \mathcal{C}$ define $U(c)$ to be the projected image of $\tilde{U}(P)$ for any puncture $P$ that projects to $c$. We say that $C$ is the home domain for $U(c)$ and that $c$ is the defining parameter of $U(\gamma)$. As in the previous remark, $U(c)$ is well defined.

Let $\mathcal{A}$ be the set of all $\tilde{U}(\tilde{\gamma})$’s and $\tilde{U}(P)$’s and let

$$\tilde{U} = \bigcup_{\tilde{\gamma}} \tilde{U}(\tilde{\gamma}) \cup \bigcup_P \tilde{U}(P).$$
Let \( \mathcal{A} \) be the set of all \( U(\gamma) \)'s and \( U(c) \)'s and let \( \mathcal{U} \) be the projection of \( \tilde{U} \) into \( M \).

**Lemma 11.4.**

1. Each \( \tilde{U} \in \tilde{\mathcal{A}} \) is open and invariant by both \( T \) and \( \tilde{f}_C \) where \( \tilde{C} \) is a home domain for \( \tilde{U} \) and \( T \) is the covering translation associated to \( \tilde{U} \).
2. If \( \tilde{U}, \tilde{U}' \in \tilde{\mathcal{A}} \) have different defining parameters then \( \tilde{U} \cap \tilde{U}' = \emptyset \).
3. If \( \tilde{U} \in \tilde{\mathcal{A}} \) and \( S \) is a covering translation then \( S(\tilde{U}) \cap \tilde{U} \neq \emptyset \) if and only if \( S \) is an iterate of the covering translation associated to \( \tilde{U} \).
4. Each \( U \in \mathcal{A} \) is open and \( f \)-invariant; if \( U_1 \) and \( U_2 \) have different defining parameters then \( U_1 \cap U_2 = \emptyset \).

**Proof.** (1) and (2) are immediate from the definitions. (3) follows from (2) and the fact that \( S \) maps the defining parameter for \( \tilde{U} \) to the defining parameter for \( S(\tilde{U}) \). (4) follows from (1) - (3).

**Remark 11.5.** If \( h : M \to M \) commutes with \( f \) then \( h \) permutes the elements of \( \mathcal{A} \). We observed in Remark 1.4 that this is an easy consequence of part (1) of Theorem 1.3. However, we wish to use this fact before proving part (1) of that theorem. Therefore we observe that it also follows from what we have shown about the elements of \( \mathcal{A} \) and their defining parameters. To see this, suppose that \( \tilde{C} \) is a home domain for \( \tilde{x} \in H \) and that \( \tilde{x} \) tracks \( \gamma \) under iteration by \( \tilde{f}_C \). Choose a lift \( \tilde{h} : H \to H \) of \( h \). Then \( \tilde{h}\tilde{f}_C\tilde{h}^{-1} = \tilde{f}_{\tilde{C}'} \) for some domain \( \tilde{C}' \) that is a home domain for \( \tilde{h}(\tilde{x}) \) and \( \tilde{h}(\tilde{x}) \) tracks \( \tilde{h}(\gamma) \) under iteration by \( \tilde{f}_{\tilde{C}'} \). This proves that \( h(U(\gamma)) = U(h(\gamma)) \). A similar argument applies to the \( U(P)'s \).

To further analyze the elements of \( \mathcal{A} \), we need the following observations.

**Lemma 11.6.**

1. If \( \tilde{C} \) is not a home domain for \( \tilde{y} \in \tilde{B}(f) \) then \( \alpha(\tilde{f}_C, \tilde{y}) \) and \( \omega(\tilde{f}_C, \tilde{y}) \) are both endpoints of the component of \( \partial \tilde{C} \) that is closest to the home domain for \( \tilde{y} \).
2. If \( \tilde{y} \in \tilde{B}(f) \), \( \tilde{C} \) is any domain and either \( \alpha(\tilde{f}_C, \tilde{y}) \) or \( \omega(\tilde{f}_C, \tilde{y}) \) is an endpoint of a frontier component \( \tilde{\sigma} \) of \( \tilde{C} \) then both \( \alpha(\tilde{f}_C, \tilde{y}) \) and \( \omega(\tilde{f}_C, \tilde{y}) \) are endpoints of \( \tilde{\sigma} \).

**Proof.** Item (1) follows from the existence of a home domain for \( \tilde{y} \), Lemma 8.2 and the obvious induction argument on the number of domains that separate \( \tilde{C} \) from a home domain for \( \tilde{y} \). Item (2) follows from (1) if \( \tilde{C} \) is not a home domain and from Lemma 10.2 and Lemma 9.3 otherwise.

We next show that \( \mathcal{B}(f) \subset \mathcal{U} \).

**Lemma 11.7.** If either \( \alpha(f, y) \neq \emptyset \) or \( \omega(f, y) \neq \emptyset \) then \( y \) is contained in an element \( U \) of \( \mathcal{A} \). In particular, each \( y \in \mathcal{B}(f) \) is contained in some \( U \in \mathcal{A} \).
Proof. The two cases are symmetric so we may assume that \( \omega(f, y) \neq \emptyset \). Choose \( z \in \omega(f, y) \) and a free disk neighborhood \( V \) of \( z \). After replacing \( y \) by some \( f^k(y) \), we may assume that \( y \in V \). Since \( z \in \omega(f, y) \) there exist \( m_i \to \infty \) such that \( f^{m_i}(y) \to z \) and such that each \( f^{m_i}(y) \in V \). Choose lifts \( \tilde{y}, \tilde{z} \in \tilde{V} \).

By Corollary \( 9.11 \) the distance between a point in \( \tilde{B}(\tilde{f}) \) and a home domain for that point is uniformly bounded. It follows that there are only finitely many home domains for elements \( \tilde{x}_l \in \tilde{B}(f) \cap \tilde{V} \) and so we may choose a sequence \( \tilde{x}_l \to \tilde{y} \) all of which have the same home domain(s) \( \tilde{C} \) and \( \tilde{C}' \), where we allow the possibility that \( \tilde{C} = \tilde{C}' \). By Corollary \( 9.11 \) the distance between \( \tilde{f}^{m_i}_C(\tilde{x}_l) \) and \( \tilde{C} \cup \tilde{C}' \) is uniformly bounded. It follows that the distance between \( \tilde{f}^{m_i}_C(\tilde{y}) \) and \( \tilde{C} \cup \tilde{C}' \) is uniformly bounded. After passing to a subsequence of the \( m_i \)'s and interchanging \( \tilde{C} \) and \( \tilde{C}' \) if necessary, we may assume that the distance between \( \tilde{f}^{m_i}_C(\tilde{y}) \) and \( \tilde{C} \) is uniformly bounded.

Let \( S_i \) be the covering translation such that \( \tilde{f}^{m_i}_C(\tilde{y}) \in S_i(\tilde{V}) \) and note that the distance between \( S_i(\tilde{z}) \) and \( \tilde{C} \) is uniformly bounded. Up to the action of \( \text{Stab}(\tilde{C}) \), the number of translates of \( \tilde{z} \) that have uniformly bounded distance from \( \tilde{C} \) is finite. We may therefore choose \( k > j \) such that \( S = S_k S_j^{-1} \in \text{Stab}(\tilde{C}) \). Let \( \tilde{W} = S_j(\tilde{V}) \) and let \( \tilde{W}' \subset \tilde{W} \) be a neighborhood of \( \tilde{f}^{m_j}_C(\tilde{y}) \) such that \( \tilde{f}^{m_k-m_j}(\tilde{W}') \subset S(\tilde{W}) \). Then \( S \) is a \( \tilde{f}_C \)-near cycle for every point in \( \tilde{W}' \) and in particular for \( \tilde{f}^{m_i}_C(\tilde{x}_l) \) for all sufficiently large \( l \). Choose such an \( \tilde{f}^{m_i}_C(\tilde{x}_l) \) and denote it simply by \( \tilde{x} \).

To prove that \( \tilde{f}^{m_j}_C(\tilde{y}) \), and hence \( \tilde{y} \), is contained in an element of \( \tilde{U} \) with home domain \( \tilde{C} \) it suffices to show that if \( \tilde{w} \in \tilde{B}(\tilde{f}) \cap \tilde{W}' \) then \( \tilde{C} \) is a home domain for \( \tilde{w} \) and \( \{ \alpha(\tilde{f}_C, \tilde{x}), \omega(\tilde{f}_C, \tilde{x}) \} = \{ \alpha(\tilde{f}_C, \tilde{w}), \omega(\tilde{f}_C, \tilde{w}) \} \).

We proceed with a case analysis. As a first case suppose that \( \tilde{x} \) tracks a geodesic \( \tilde{\gamma}(\tilde{x}) \). Corollary \( 10.5 \) implies that \( S \) is an iterate of \( T_{\tilde{\gamma}(\tilde{x})} \). As a first subcase suppose that \( \tilde{C} \) is a home domain for \( \tilde{w} \). Since \( T_{\tilde{\gamma}(\tilde{x})} \) is not parabolic, Lemma \( 10.2 \) implies that \( \tilde{w} \) tracks some geodesic \( \tilde{\gamma}(\tilde{w}) \). Corollary \( 10.5 \) implies that \( \tilde{\gamma}(\tilde{w}) = \tilde{\gamma}(\tilde{x}) \) as desired.

The remaining subcase is that \( \tilde{C} \) is not a home domain for \( \tilde{w} \). Lemma \( 11.6 \) implies that \( \{ \alpha(\tilde{f}_C, \tilde{w}), \omega(\tilde{f}_C, \tilde{w}) \} \) is contained in the set of endpoints for some \( \tilde{\sigma} \) in the frontier of \( \tilde{C} \). Lemma \( 9.7 \) then implies that \( \tilde{\sigma} = \tilde{\gamma}(\tilde{x}) \). Let \( \tilde{C}' \) be the other domain that contains \( \tilde{\gamma}(\tilde{x}) \). Since some iterate of \( T_{\tilde{\gamma}(\tilde{x})} \) is a near cycle for \( \tilde{w} \) with respect to \( \tilde{f}_C \), the same is true with respect to \( \tilde{f}_{\tilde{C}'} \). Lemma \( 9.7 \) implies that \( \{ \alpha(\tilde{f}_{\tilde{C}'}, \tilde{w}), \omega(\tilde{f}_{\tilde{C}'}, \tilde{w}) \} \cap \{ \tilde{\gamma}(\tilde{x}) \} \neq \emptyset \) and Lemma \( 11.6 \) implies that both \( \alpha(\tilde{f}_{\tilde{C}'}, \tilde{w}) \) and \( \omega(\tilde{f}_{\tilde{C}'}, \tilde{w}) \) are endpoints of \( \tilde{\gamma}(\tilde{x}) \). This contradicts the assumption that \( \tilde{C} \) is not home domain for \( \tilde{w} \) and so proves that the second subcase never occurs.

By Lemma \( 10.2 \) the only remaining case is that \( \alpha(\tilde{f}_C, \tilde{x}) = \omega(\tilde{f}_C, \tilde{x}) = P \) and that \( S \) is an iterate of \( T_P \). Lemma \( 9.7 \) implies that \( P \in \{ \alpha(\tilde{f}_{\tilde{C}'}, \tilde{w}), \omega(\tilde{f}_{\tilde{C}'}, \tilde{w}) \} \), Lemma \( 11.6 \) implies that \( \tilde{C} \) is a home domain for \( \tilde{w} \) and Lemma \( 10.4 \) implies that \( \alpha(\tilde{f}_{\tilde{C}'}, \tilde{w}) = \omega(\tilde{f}_{\tilde{C}'}, \tilde{w}) = P \).

Lemma 11.8. Let \( Y = M \setminus \mathcal{U} \) and let \( \tilde{Y} \subset H \) be the full pre-image of \( Y \).
(1) For each $\tilde{y} \in \tilde{Y}$ there is a domain $\tilde{C}$ that is the unique $\alpha$-domain, unique $\omega$-domain and unique home domain for $\tilde{y}$. Moreover, $\tilde{C}$ is a home domain for all points in a neighborhood of $\tilde{y}$.

(2) If $\tilde{C}$ is the home domain for $\tilde{y} \in \tilde{Y}$ then $\tilde{y}$ has no $\tilde{f}_C$-near cycles in $\text{Stab}(\tilde{C})$.

(3) For any compact subset $X \subset M$ there is a constant $K_X$ such that for each $y \in Y$, $f(i)(y) \in X$ for at most $K_X$ values of $i$.

(4) There exists $\epsilon > 0$ so that if $\tilde{y}_1, \tilde{y}_2 \in \tilde{Y}$ and $\text{dist}(\tilde{y}_1, \tilde{y}_2) < \epsilon$ then $\tilde{y}_1$ and $\tilde{y}_2$ have the same home domain.

Proof. Suppose at first that $\mathcal{R} = \emptyset$. Items (1) and (4) are obvious. Every neighborhood of $\tilde{y}$ contains points in $\tilde{B}(f)$ that are contained in distinct elements of $\mathcal{U}$. Lemma 10.2 and Lemma 10.5 imply that such points have no common near cycles. Item (2) therefore follows from Remark 9.5. Item (3) follows from item (2) and the fact that every compact set has a finite cover by free disks.

We now assume that $\mathcal{R} \neq \emptyset$. Write $M$ as an increasing sequence of compact connected subsurfaces $M_1 \subset M_2 \subset \ldots$ such that each complementary component of each $M_i$ is unbounded and such that

$$N_{D_1}\mathcal{R} \subset M_i \subset M_i \cup f(M_i) \subset \text{int} M_{i+1}$$

for each $i$.

Lemma 11.7 implies that $f^n(y)$ converges to some puncture $c$ of $M$. After replacing $y$ by some point in its forward orbit, we may assume that $f^n(y) \in H \setminus M_2$ for all $i \geq 0$. Given a lift $\tilde{y}$, let $\tilde{C}$ be the domain that contains $\tilde{y}$ and let $\tilde{W}_1$ be the component of $H \setminus \tilde{M}_1$ that contains $\tilde{y}$. We claim that $\tilde{f}_C(\tilde{y}) \in \tilde{W}_1$. Choose a ray $\tilde{\rho} \subset H \setminus \tilde{M}_2$ that connects $\tilde{y}$ to some $Q \in S_\infty$ and note that $Q$ is in the closure of $\tilde{C}$ and hence fixed by $\tilde{f}_C$ because $\tilde{\rho}$ does not cross any element of $\tilde{\mathcal{R}}$. Then $\tilde{f}_C(\tilde{\rho}) \subset H \setminus \tilde{M}_1$ connects $\tilde{f}_C(\tilde{y})$ to $Q$. The claim now follows from the fact that $\tilde{W}_1$ is the unique component of $H \setminus \tilde{M}_1$ whose closure contains $Q$.

Since $f^n(y) \in H \setminus M_2$ for all $i \geq 0$ the previous argument can be iterated to show that $\tilde{f}_C^n(\tilde{y}) \in \tilde{W}_1$ for all $i \geq 0$. There exists $n_2$ so that $f^n(y) \in M \setminus M_3$ for all $n \geq n_2$. Let $\tilde{W}_2$ be the component of $H \setminus \tilde{M}_2$ that contains $\tilde{f}_C^{n_2}(\tilde{y})$. By the same argument, $\tilde{f}_C^n(\tilde{y}) \in \tilde{W}_2$ for all $n \geq n_2$. Continuing in this manner, we can choose a decreasing sequence of components $\tilde{W}_i$ of $H \setminus \tilde{M}_i$ such that for all $k$, $\tilde{f}_C^n(\tilde{y}) \in \tilde{W}_k$ for all sufficiently large $n$. It follows that $\tilde{f}_C^n(\tilde{y}) \to P$ where $P$ necessarily projects to some end $c$. This proves that $\tilde{f}_C$ is the unique $\omega$-lift for $\tilde{y}$. Corollary 9.11 implies that $\tilde{C}$ is a home domain for every point in $\tilde{B}(f) \cap \tilde{W}_1$.

By the symmetric argument applied to $f^{-1}$, there is a unique domain $\tilde{C}^*$ that is an $\alpha$-domain for $\tilde{y}$; moreover there is a neighborhood of $\tilde{y}$ such that $\tilde{C}^*$ is a home domain for every birecurrent point in this neighborhood. To complete the proof of
(1) it suffices to prove that $\tilde{C} = \tilde{C}^*$. If $\tilde{C} \neq \tilde{C}^*$, then both $\tilde{C}$ and $\tilde{C}^*$ are home domains for every birecurrent point in in a neighborhood of $\tilde{y}$. But then $y \in U(\sigma)$ where $\sigma = \tilde{C} \cap \tilde{C}^*$ contradicting the assumption that $y$ is not contained in any $U \in U$.

Every neighborhood of $\tilde{y}$ contains points in $\tilde{B}(f)$ that are contained in distinct elements of $U$. Lemma 10.2 and Lemma 10.3 imply that such points have no common $\tilde{f}_C$-near cycle in $\text{Stab}(\tilde{C})$. Item (2) now follows from Remark 9.2.

Any compact $X \subset M$ has a cover by finitely many, say $K$, free disks. Since $N_{D_1}(\tilde{C}_{\text{core}})$ is a compact subset of $\tilde{C}$, there is a constant $L$ so that for each of these $K$ free disks $B$, there are at most $L \text{Stab}(\tilde{C})$-orbits of lifts of $B$ to $H$ that intersect $N_{D_1}(\tilde{C})$. From (1) and Lemma 8.3 it follows that $\tilde{f}_C^i(\tilde{y}) \in N_{D_1}(\tilde{C})$ for all $i$. Item (2) therefore implies that there are at most $K_X = KL$ values of $i$ such that $\tilde{f}^i(y) \in X$. This proves (3).

Applying (3) to $X = M_2$ we have that for each $y \in Y$ there exists $0 \leq i \leq K_{M_2}$ such $\tilde{f}^i(y_1) \in M \setminus M_2$. Choose $\epsilon > 0$ so that

- $z \notin M_2 \implies N_\epsilon(z) \cap M_1 = \emptyset$.
- $z \in M_2, 1 \leq i \leq K_{M_2}$ and $\tilde{f}^i(z) \notin M_2 \implies \tilde{f}^i(N_\epsilon(z)) \cap M_1 = \emptyset$.

Suppose now that $\tilde{C}'$ is the home domain for $\tilde{y}_1$ and that $\text{dist}(\tilde{y}_1, \tilde{y}_2) < \epsilon$. There exists $0 \leq i \leq K_{M_2}$ such that $\tilde{f}_C^i(\tilde{y}_1)$ and $\tilde{f}_C^i(\tilde{y}_2)$ belong to the same component of $H \setminus M_1$. The domain that contains $\tilde{f}_C^i(\tilde{y}_1)$ and $\tilde{f}_C^i(\tilde{y}_2)$ is the home domain for these points so it is $\tilde{C}'$. This implies that $\tilde{C}'$ is also the home domain for $\tilde{y}_2$. \hfill $\square$

**Corollary 11.9.** Suppose that $\tilde{V}$ is a component of $\tilde{U} \in \tilde{A}$ and that the union $\tilde{V}'$ of $\tilde{V}$ with all of its bounded complementary components has finite area. Then each point in the frontier $\text{fr}(\tilde{V'})$ of $\tilde{V}$ has the same home domain.

**Proof.** Choose $\epsilon > 0$ as in Corollary 11.8(4). Since $\tilde{V}'$ is simply connected it is the union of an increasing sequence of compact disks $\{B_i, i = 1 \ldots \infty\}$. Since $\tilde{V}'$ has finite area we may assume that each $\partial B_i \subset N_\epsilon(\text{fr}(\tilde{V}'))$. For any pair of points $\tilde{y}_1, \tilde{y}_2 \in \text{fr}(\tilde{V}')$ there exists $k$ such that $\partial B_k$ intersects both $N_\epsilon(\tilde{y}_1)$ and $N_\epsilon(\tilde{y}_2)$. It follows that $N_\epsilon(\text{fr}(\tilde{V}'))$ is connected. Lemma 11.8(4) implies that each point in $\text{fr}(\tilde{V}')$ has the same unique home domain $\tilde{C}$ and hence, by Lemma 8.3 that $\text{fr}(\tilde{f}_C^i(\tilde{V}')) = \tilde{f}_C^i(\text{fr}(\tilde{V}')) \subset N_{D_1}(\tilde{C})$ for all $i$. It follows that $\tilde{f}_C^i(\tilde{V}') \subset N_{D_1}(\tilde{C})$ for all $i$ which implies by Lemma 8.3 that $\tilde{C}$ is a home domain for every point in $\tilde{V}'$. \hfill $\square$

Our next arguments make use of a projection to $\mathbb{R}$ defined with respect to the defining parameter of $\tilde{U} \in \tilde{U}$. If $\tilde{U} = \tilde{U}(\tilde{\gamma})$ choose a parameterization of the annulus $A_\gamma$ (see Definition 6.2) as $S^1 \times [0, 1]$. Lift this to a parameterization of $H \setminus \tilde{\gamma}^\pm$ as $\mathbb{R} \times [0, 1]$ and let $\pi : H \setminus \tilde{\gamma}^\pm \rightarrow \mathbb{R}$ be projection onto the $\mathbb{R}$ coordinate. (Alternatively, one can define this directly as orthogonal projection onto $\tilde{\gamma}$ parameterized as $\mathbb{R}$.) If $\tilde{U} = \tilde{U}(P)$ where $P$ projects to an isolated end $M$ with horocycle $\tau$ define $\pi : H \setminus P \rightarrow$
as above using the compactified annular cover \( A_p^c = A_c^c \). In both case we say that
the \( \pi \) is the projection associated to \( \tilde{U} \).

**Remark 11.10.** If \( \tilde{\sigma} \in \tilde{\mathcal{R}} \) then by Lemma 11.8(1), the home domain for a point in the frontier of \( \tilde{U}(\tilde{\sigma}) \) is one of the two home domains of \( \tilde{U}(\tilde{\sigma}) \).

**Corollary 11.11.** Suppose that \( T \) is the covering translation associated to \( \tilde{U} \in \tilde{A} \), that \( \pi \) is projection associated to \( \tilde{U} \) and that \( \tilde{C} \) is a home domain for a component \( \tilde{V} \) of \( \tilde{U} \). Given \( p, q > 0 \) define \( \tilde{g} = T^{-p} \tilde{f}_C^q \). Then there exists \( r > 0 \) so that \( \pi(\tilde{g}^r(\tilde{y})) < \pi(\tilde{y}) - 1 \) for all \( \tilde{y} \) in the frontier of \( \tilde{V} \) whose home domain is \( \tilde{C} \).

**Proof.** After replacing \( f \) with \( T^q \) we may assume that \( q = 1 \). Increasing \( p \) makes the desired inequality easier to satisfy so we may assume that \( p = 1 \) and \( \tilde{g} = T^{-1} \tilde{f}_C \). The goal is to prove that \( \pi(\tilde{f}_C^r(\tilde{y})) < \pi(\tilde{y}) + 1 \)
for all sufficiently large \( r \). There is a constant \( B \) so that \( \pi(\tilde{f}(\tilde{y})) < \pi(\tilde{y}) + B \) for all \( \tilde{y} \).

Choose compact subsurfaces \( M_1 \subset M_2 \subset M \) such that

1. Each component \( X \) of \( M \setminus M_2 \) and each component \( Z \) of \( M \setminus M_1 \) is unbounded.

2. For each component \( X \) of \( M \setminus M_2 \) there is a component \( Z \) of \( M \setminus M_1 \) such that \( X \cup f(X) \subset Z \).

We first consider the case that \( \tilde{U} = \tilde{U}(\tilde{\gamma}) \). We may assume that

3. \( M \setminus M_1 \) does not contain a simple closed curve that is isotopic to \( \gamma \)

and hence that for any component \( \tilde{Z} \) of \( H \setminus \tilde{M}_1 \), the diameter of \( \pi(\tilde{Z}) \) is at most the length \( L \) of a fundamental domain of \( \gamma \).

Suppose that \( f^j(\tilde{y}) \notin M_2 \) for \( 0 \leq j \leq J \). Let \( X_0 \) be the component of \( M \setminus M_2 \) that contains \( y \) and let \( Z_0 \) be the component of \( M \setminus M_1 \) satisfying \( X_0 \cup f(X_0) \subset Z_0 \). Lift these to \( \tilde{y} \in \tilde{X}_0 \subset \tilde{Z}_0 \). Then \( \tilde{f}_C(\tilde{y}) \in \tilde{Z}_0 \). There is a component \( X_1 \subset Z_0 \) of \( M \setminus M_2 \) that contains \( f \). Lift this to \( \tilde{f}_C(\tilde{y}) \in \tilde{X}_1 \subset \tilde{Z}_0 \). Repeating this argument shows that \( \tilde{f}_C^j(\tilde{y}) \in \tilde{Z}_0 \) for \( 0 \leq j \leq J \) and hence that \( \pi(\tilde{f}_C^j(\tilde{y})) < \pi(\tilde{y}) + L \).

Since \( y \) is in the frontier of \( U \) it is not in any element of \( A \). Hence by Lemma 11.8(3) there is a constant \( K \) such that there are at most \( K \) values of \( j \) with \( f^j(y) \in M_2 \). The inequality (11.1) therefore holds for \( r > KA + (K + 1)L + 1 \).

The second and last case is that \( \tilde{U} = \tilde{U}(P) \). One component \( Z_1 \) of \( M \setminus M_1 \) is a neighborhood of the isolated end of \( M \) that lifts to \( P \). Let \( A \) [resp. \( A^c \)] be the [resp. compactified] annular cover corresponding to \( P \) and let \( \partial A^c \) be the component of \( \partial A^c \) that is not part of \( A \). As a first subcase suppose that either \( \text{Fix}(\tilde{f}|_{\partial A^c}) = \emptyset \) or \( \tilde{f} \) is isotopic to a non-trivial Dehn twist relative to fixed points in both components of \( \partial A^c \). By Corollary 7.17 we may replace (3) by

(3') \( \tilde{Z}_1 \cap \text{fr}(\tilde{V}) = \emptyset \).
Since the other components of $M \setminus M_1$ do not contain simple closed curves that are isotopic to horocycles corresponding to $P$, the proof given in the first case carries over to this subcase.

It remains to consider the subcase that $\hat{f}$ is isotopic to the identity relative to fixed points in both components of $\partial A^c$. In this subcase we replace (3) by

(3') if $J \leq 10$ and $\tilde{f}^j(\tilde{y}) \in \tilde{Z}_1$ for all $0 \leq j \leq J$ then $\pi(\tilde{f}^J(\tilde{y})) < \pi(\tilde{y}) + 1$

and choose $r > KA + (K + 1)L + 1 + r/10$.

\[ \square \]

**Lemma 11.12.** Suppose that $U \in A$.

(1) $U$ is an open annulus.

(2) If $U = U(\gamma)$ then each essential simple closed curve in $U$ is isotopic to $\gamma$. If $U = U(P)$ then each essential simple closed curve in $U$ is isotopic to a horocycle surrounding the isolated end of $M$ corresponding to $P$.

(3) Assume that the ends of $U$ are non-singular. If $U = U(\gamma)$ then both components of $\partial U^c$ of $U$ have fixed points. If $U = U(P)$ then the frontier of $U$ in $M$ is connected and there is a fixed point in the corresponding component of $\partial U^c$.

**Proof.** Choose $\tilde{U} \in \tilde{A}$ projecting to $U$ and let $T$ be the covering translation associated to $\tilde{U}$. By Lemma 11.4-(3), (1) and (2) follow from the following three properties that we prove below.

(a) each component $\tilde{V}$ of $\tilde{U}$ is simply connected.

(b) each component $\tilde{V}$ of $\tilde{U}$ is $T$-invariant.

(c) $\tilde{U}$ is connected.

We first prove that each component $\tilde{V}$ of $\tilde{U}$ is unbounded by assuming that $\tilde{V}$ is bounded and arguing to a contradiction. Let $\tilde{f} = \tilde{f}_C$ where (Corollary 11.9) $\tilde{C}$ is a home domain for each point in the frontier of $\tilde{V}$. Since $\tilde{f}$ preserves area there exists $q \geq 0$ and a covering translation $S$ so that $\tilde{f}^q(\tilde{V}) \cap S(\tilde{V}) \neq \emptyset$. Lemma 11.4-(3) implies that $S = T^p$ for some $p \in \mathbb{Z}$. After replacing $T$ with $T^{-1}$ if necessary we may assume that $p > 0$. From the fact that $\tilde{f}^q(\tilde{V})$ and $S(\tilde{V})$ are both components of $\tilde{U}$, it follows that $\tilde{f}^q(\tilde{V}) = S(\tilde{V}) = T^p(\tilde{V})$. Thus $\tilde{V}$ is $\tilde{g}$-invariant where $\tilde{g} = T^{-p} \tilde{f}^q$. If $p = 0$ then $\tilde{f}$ has bounded orbits (since we are assuming $\tilde{V}$ is bounded) and hence fixed points by the Brouwer plane translation theorem. Since $\tilde{f}$ is fixed point free, $p \neq 0$. This contradicts Corollary 11.11 and so completes the proof that each component $\tilde{V}$ of $\tilde{U}$ is unbounded.

If (a) fails then some component of the complement of $\tilde{V}$ is bounded and intersects some $\tilde{U}' \neq \tilde{U}$. This contradicts the above fact that no component of $\tilde{U}'$ is bounded, and hence proves (a).
We next assume that (b) fails and argue to a contradiction. A subsurface that contains a curve homotopic to an iterate of $\gamma$ contains a curve isotopic to $\gamma$. Thus $T^p(V) \neq V$ for all $p \neq 0$. Lemma \[11.4\](3) implies that $V$ is moved off itself by every covering translation. In particular, $V$ has finite area because the covering projection into $M$ is injective on $V$. Define $\tilde{f} = \tilde{f}_C$ where $\tilde{C}$ is a home domain for each point in the frontier of $V$. As in the previous argument, there exists $p$ and $q \geq 0$ so that $\tilde{f}^q(V) = T^p(V)$. If $p = 0$, then $V$ has recurrent points, and hence fixed points for $\tilde{f}$, which is impossible. Thus $p \neq 0$ and we assume without loss that $p > 0$. Also by hypothesis, $q \neq 0$.

Let $\pi$ be the projection associated to $\tilde{U}$ and let $\tilde{g} = T^{-p} \tilde{f}^q$. Then $\tilde{g}(V) = \tilde{V}$ and by Corollary \[11.1\] there is an $r > 0$ such that $\pi(\tilde{g}^r(\tilde{y})) < \pi(\tilde{y}) - 1$ for every $\tilde{y}$ in $\partial \tilde{V}$. Consequently, by continuity, there is $\delta > 0$ such that every $\tilde{x} \in \tilde{V}$ which is within $\delta$ of $\partial \tilde{V}$ satisfies $\pi(\tilde{g}^r(\tilde{x})) < \pi(\tilde{x}) - 1$. Since $\tilde{V}$ has finite area there exists $N > 0$ such that $\tilde{V}_N = \{\tilde{x} \in \tilde{V} \mid \pi(\tilde{x}) < -N\}$ contains no ball of diameter $\delta$, and hence every point of $\tilde{V}_N$ must be within $\delta$ of $\partial \tilde{V}$. We conclude the $\tilde{g}^r(\tilde{V}_N)$ is a subset of $\tilde{V}_{N-1}$. Note that $\tilde{V}_N$ is non-empty since $\tilde{V}$ is $\tilde{g}$-invariant and $\lim_{n \to \infty} \pi(\tilde{g}^n(\tilde{y})) = -\infty$ for all $\tilde{y} \in \partial \tilde{V}$. Since $\tilde{V}_N$ is open and $\tilde{V}_{N-1}$ is a proper subset the area of $\tilde{V}_{N-1}$ is strictly smaller than that of $\tilde{V}_N$ contradicting the fact that $\tilde{g}$ is area preserving. This completes the proof of (b).

We have now proved that each component of $U$ contains a simple closed curve that is essential in $M$ and that all such simple closed curves in $U$ are in the same isotopy class. Moreover if $U \neq U'$ then $U$ and $U'$ do not contain isotopic simple closed curves. If (c) fails then there is an unpunctured annulus $A$ whose boundary curves are in $U$ and whose interior contains a component $V'$ of some $U' \neq U \in \mathcal{U}$. But every simple closed curve in $V'$ that is essential in $M$ is parallel to the boundary components of $A$. This contradiction implies that $U$ and hence $\tilde{U}$ is connected. This completes the proof of (1) and (2). This same argument proves that if $U$ corresponds to an isolated end of $M$ then $U$ contains a neighborhood of that isolated end and so has connected complement and connected frontier.

We now consider (3). Choose a component $Y$ of the frontier of $U$ and let $\partial_0 U_c$ be the corresponding component of $\partial U_c$. We will assume that $\rho(\partial_0 U_c) > 0$ and argue to a contradiction. A symmetric argument implies that $\rho(\partial_0 U_c)$ is not negative and so is 0, which is equivalent to $\partial_0 U_c$ having a fixed point.

Choose an accessible point $z \in Y$ and a degree one closed path $\mu$ with embedded interior in $U$ and with both endpoints at $z$. Let $\tilde{\mu}_0$ be a lift of the interior of $\mu$ to $\tilde{U}$. Since $\mu$ has degree one, the ends of $\tilde{\mu}_0$ converge to lifts $\tilde{z}$ and $T(\tilde{z})$ of $z$ in $\tilde{Y}$. Denote the bounded area component of $\tilde{U} \setminus \tilde{\mu}_0$ by $D_0$. For each $k$, let $\tilde{\mu}_k = T^k(\tilde{\mu}_0)$ and $D_k = T^k(D_0)$.

Choose $0 < p/q < \rho(\partial_0 U)$ and let $\tilde{g} = T^{-p} \tilde{f}^q$. If $\nu$ is a cross cut in $D_k$ then the $\pi$-image of the endpoints of $\tilde{g}^l(\nu)$ decrease linearly in $j$ by Corollary \[11.11\]. On the other hand, for arbitrarilily large $j$ one can choose $\nu$ so that $\tilde{g}^l(\nu)$ is a cross cut in $D_l$ where $l$ increases linearly in $j$. It follows that $\pi(D_k)$ is not bounded below.
Choose $N$ so that $\pi(\tilde{\mu}_0) > -N$. For all $n > N$ and $k > 0$ consider all cross cuts $\tau_{k,n} \subset D_k$ such that $\pi(\tau_{k,n}) = -n$. (In other words, $\tau_{k,n}$ is a component of $\pi^{-1}(n) \cap D_k$.) Let $E(\tau_{k,n})$ be the complementary component of $\tau_{k,n}$ that is contained in $D_k$ and let $d_{k,n}$ be the maximum area of all such $E(\tau_{k,n})$. Then

$$d_{k,n} = d_{k+1,n-1} < d_{k+1,n}$$

The equality follows from the fact that $\tau_{k+1,n-1} = T(\tau_{k,n}) \subset D_{k+1}$ is a cross cut with $\pi(\tau_{k+1,n-1}) = -n + 1$. The inequality follows from the fact that each $E(\tau_{k,n})$ is contained in some $E(\tau_{k,n-1})$.

Fix $k$ and choose $\tau_{k,n}$ so that $d_{k,n} = E(\tau_{k,n})$. Since $D_k$ has finite area, we have $\lim_{n \to \infty} d_{k,n} = 0$. Arguing as in the proof of (2), there exists $r > 0$ and $N' > N$ such that $\pi(\tilde{g}(\tau_{k,n})) < -n - 1$ for all $n > N'$. Our choice of $N$ guarantees that $\tilde{g}(\tau_{k,n}) \cap \mu_l = \emptyset$ for $l \geq k$. Since $p/q < \rho(\partial_0 U)$ it follows that $\tilde{g}(E(\tau_{k,n}))$ is contained in $D_l$ for some $l \geq k$ and hence that $\tilde{g}(E(\tau_{k,n}))$ is contained in some $E(\tau_{l,n+1})$. This contradicts the fact that $d_{l,n+1} < d_{k,n}$ for all $l \geq k$.

We are now able to prove Proposition 5.1

**Proposition 5.1** Suppose that $M$ is a component of $M = S^2 \setminus \text{Fix}(F)$ and that $f = F|_M : M \to M$. Then there is a countable collection $\mathcal{A}$ of pairwise disjoint open $f$-invariant annuli such that

1. For each compact set $X \subset M$ there is a constant $K_X$ such that any $f$-orbit that is not contained in some $U \in \mathcal{A}$ intersects $X$ in at most $K_X$ points. In particular each $x \in \mathcal{B}(f)$ is contained in some $U \in \mathcal{A}$.

2. For each $U \in \mathcal{A}$ and $z$ in the frontier of $U$ in $S^2$, there are components $F_+(z)$ and $F_-(z)$ of $\text{Fix}(F)$ so that $\omega(F,z) \subset F_+(z)$ and $\alpha(F,z) \subset F_-(z)$.

3. For each $U \in \mathcal{A}$ and each component $C_M$ of the frontier of $U$ in $M$, $F_+(z)$ and $F_-(z)$ are independent of the choice of $z \in C_M$.

4. If $U \in \mathcal{A}$, and $f_c : U_c \to U_c$ is the extension to the annular compactification of $U$, then each component of $\partial U_c$ corresponding to a non-singular end of $U$ contains a fixed point of $f_c$.

**Proof of Proposition 5.1** The case in which $M$ has less than three ends is proved in section 5 so we may assume that $M$ has at least three ends.

The index set $\mathcal{A}$ is defined in Definition 11.3. Lemma 11.12-(1) states that each $U \in \mathcal{A}$ is an open annulus. Lemma 11.8-(3) implies (1) which implies (2). Item (4) follows from Lemma 11.12-(3).

To prove (3), suppose that $\tilde{Z}$ is a component of the frontier of $\tilde{U}$ in $H$. If three oriented simple closed curves in $S^2$ bound disjoint disks then at least one pair of them have anti-parallel orientations. Part (2) of Lemma (8.7) of [10] therefore implies that
if a free disk in $H$ intersects distinct $\tilde{U}_1, \tilde{U}_2$ and $\tilde{U}_3$ then one of these annuli separates the other two. It follows that each $\tilde{z} \in \tilde{Z}$ has a neighborhood that intersects exactly one component $V_z$ of the complement of the closure of $\tilde{U}$. The open set $V_z = V_{\tilde{z}}$ depends only on $\tilde{z}$ and not on $\tilde{z}$. In particular, $\tilde{Z}$ is contained in the frontier of $V_{\tilde{Z}}$.

Let $W$ be the component of the complement of $\tilde{U}$ that contains $V_{\tilde{Z}}$ and so contains $\tilde{Z}$. Lemma (3.2) implies that the frontier of $W$ is connected and hence is contained in a component of the frontier of $\tilde{U}$. Thus $\tilde{Z}$ is the frontier of $W$. Lemma (3.2) implies that the complement of $\tilde{Z}$ in $H$ has exactly two components. It follows that the intersection $B$ of $S_\infty$ with the closure of $\tilde{Z}$ cannot have more than two components: if it did there would be two components of $S_\infty \setminus B$ with neighborhoods contained in the same component of $H \setminus \tilde{Z}$ and so there would be a line in $H \setminus \tilde{Z}$ that separates $\tilde{Z}$ which is impossible. The components of $B$ must be points because translates of $\tilde{U} = \tilde{U}(\tilde{z})$ by covering translations will have ends of their parametrizing geodesics which are dense in $S_\infty$.

By Lemma (11.9) and Lemma (11.8(1)) there exists $\tilde{C}$ that is a home domain for each point in a neighborhood of $\tilde{Z}$. Since $W$ contains elements of $A$ that intersect this neighborhood, $W$ is $\tilde{f}_C$-invariant. It follows that $\tilde{Z}$ is $\tilde{f}_C$-invariant.

If $B$ is a single point $P$ then $P$ is also the intersection of $S_\infty$ with the closure of one of the complementary components of $\tilde{Z}$. The only possibility is that this complementary component is $\tilde{U}(P)$. But this contradicts (2) applied to the projection $Z$ in $M$ of $\tilde{Z}$ since $Z$ is disjoint from $U(c)$ which is a neighborhood of the end corresponding to $P$. We conclude that the limit set of $\tilde{Z}$ in $S_\infty$ is a pair of points, say $a$ and $b$.

Choose a small disk $\tilde{D}$ that projects to a free disk $D \subset M$ and contains an arc $\beta_0$ that has endpoints in distinct components of $H \setminus \tilde{Z}$. Extend $\beta_0$ to a properly embedded line $\beta \subset H$ that intersects $\tilde{Z}$ only in $(\beta_0 \cap \tilde{Z})$ and separates $a$ from $b$. Let $\tilde{Z}_a$ be the set of points in $\tilde{Z}$ that are in the same component of the complement of $\beta$ as $a$; define $\tilde{Z}_b$ similarly. Perform an isotopy rel $\tilde{Z} \cup S_\infty$ from $\tilde{f}_C$ to a homeomorphism $g$ such that $g(\beta) \cap \beta = \emptyset$. After interchanging $a$ and $b$ we may assume that $g(\beta)$ separates $b$ from $\beta$. Since $g|_{\tilde{Z}} = \tilde{f}_C|_{\tilde{Z}}$ it follows that $\tilde{f}_C(\tilde{Z}_b) \subset Z_b$ which implies that $b = \omega(\tilde{z}, \tilde{f}_C)$ for each $\tilde{z} \in Z_b$. Varying the location of $\tilde{D}$ we conclude that $b = \omega(\tilde{z}, \tilde{f}_C)$ for all $\tilde{z} \in \tilde{Z}$. Applying the same argument to $\tilde{f}_C^{-1}$ we see that $a = \alpha(\tilde{z}, \tilde{f}_C)$ for all $\tilde{z} \in \tilde{Z}$ proving (3).

12 Renormalization.

In this section we study the finer structure of $f|_U$, the restriction of $f$ to one of the annuli $U \in \mathcal{A}$.

For each $q \geq 1$ let $\mathcal{M}_q = S^2 \setminus \text{Fix}(F^q) \subset S^2 \setminus \text{Fix}(F) = \mathcal{M}$. Recall that by the main theorem of [1], each component $M$ of $\mathcal{M}$ is $F$-invariant and similarly each component $M_q$ of $\mathcal{M}_q$ is $F^q$-invariant. Let $\mathcal{A}(q)$ be the family of open $F^q$-invariant
annuli obtained by applying Proposition 12.1 to the restriction of $F^q$ to a component $M_q$ of $\mathcal{M}_q$ that is contained in the component $M$ of $\mathcal{M}$.

Lemma 12.1. (1) If $V \in A(q)$ is essential in $M$ then $V$ is $f$-invariant.

(2) If $V$ is any $f$-invariant open annulus in $M$ then $V$ is essential in $M$.

Proof. Suppose $V \in A(q)$ is essential in $M$. Since $f$ commutes with $f^q$, $f(V)$ is an element of $A(q)$ (see Remark (11.5)) and hence $f(V)$ is either equal to or disjoint from $V$. Since $V$ is essential in $M$, $f(V)$ is essential in $M$. If $f(V)$ is disjoint from $V$ then since $f(V)$ is essential in $M$, $f$ maps one component of the complement of $V$ to a proper subset of itself contradicting the fact that $f$ preserves area. This proves (1).

If an $f$-invariant open annulus $V \subset M$ is inessential then one of its complementary components is contained in $M$. The union of $V$ and this component is an $f$-invariant open disk which, by the Brouwer plane translation theorem, must contain a point of $\text{Fix}(f) \cap M$. This contradiction implies that $V$ is essential in $M$ and so verifies (2).

The following proposition shows that elements of the family $A(q)$ refine the elements of $A$.

Proposition 12.2. Each $V \in A(q)$ is a subset of some $U \in A$. Moreover, each $f$-invariant open annulus in $M$ is contained in some $U \in A$.

Proof. Assume at first that $V$ is an essential annulus in $M$. If $V \in A(q)$, then $V$ is $f$-invariant by Lemma (12.1) Let $\alpha$ be a simple closed loop in $V$ representing a generator of its fundamental group and let $\gamma$ be the simple closed geodesic or a horocycle in $M$ that is freely homotopic to $\alpha$. Since $V$ is $f$-invariant, $\gamma$ is isotopic to $f(\gamma)$ and in particular does not cross any reducing curves.

Choose a lift $\tilde{\gamma} \subset H$ of $\gamma$ and let $T = T_{\tilde{\gamma}}$ be a primitive covering translation that preserves $\tilde{\gamma}$. If $\gamma$ is not itself a reducing curve then $\tilde{\gamma}$ lies in a unique domain $\tilde{C}$. The lift $\tilde{f}_1 = \tilde{f}_{\tilde{C}}$ of $f$ commutes with $T$ and fixes the endpoints $\tilde{\gamma}^\pm$. If $\gamma$ is in $R$ then $\tilde{\gamma}$ is the common frontier of two domains $\tilde{C}_1$ and $\tilde{C}_2$. Let $\tilde{f}_j$, $j = 1, 2$ be the lift which fixes the ends of $\tilde{C}_j$. In this case too $\tilde{f}_j$ commutes with $T$ and fixes $\tilde{\gamma}^\pm$.

The homotopy from $\gamma$ to $\alpha$ lifts to a homotopy between $\tilde{\gamma}$ and a lift $\tilde{\alpha}$ of $\alpha$. Since the lifted homotopy moves points a uniformly bounded distance and commutes with $T$, the ends of $\tilde{\alpha}$ converge to $\tilde{\gamma}^\pm$ and $\tilde{\alpha}$ is $T$-invariant. The components of the full pre-image of $V$ are copies of the universal cover of $V$; we refer to each component as a lift of $V$. The lift $\tilde{V}$ of $V$ that contains $\tilde{\alpha}$ is $T$-invariant and is the only lift of $V$ whose closure contains $\tilde{\gamma}^\pm$. Since $\tilde{f}_j$ fixes $\tilde{\gamma}^\pm$ it follows that $\tilde{V}$ is $\tilde{f}_j$-invariant.

Suppose $x \in V$ is birecurrent. Let $\tilde{x}$ be a lift of $x$ which lies in $\tilde{V}$. Choose a small free disk $D \subset V$ with diameter $\epsilon$ containing $x$. Suppose that $f^n(x) \in D$. Then if $\tilde{D}$ is the lift of $D$ containing $\tilde{x}$, we have $\tilde{f}^{kn}(\tilde{x}) \in T^k(\tilde{D})$ for some $k \in \mathbb{Z}$.
Since $x$ is recurrent there is a sequence $\{n_i\}$ tending to infinity such that $f_j^{n_i}(x) \in D$ and hence a sequence $\{k_i\}$ such that $\tilde{f}_j^{n_i}(\tilde{x}) \in T^{k_i}(\tilde{D})$. It follows that $\tilde{f}_j^{n_i}(\tilde{x})$ has distance at most $\epsilon$ from $T^{k_i}(\tilde{x})$. We conclude that

$$\lim_{i \to \infty} \tilde{f}_j^{n_i}(\tilde{x}) = \lim_{i \to \infty} T^{k_i}(\tilde{x}) = \tilde{\gamma}^+ \text{ or } \tilde{\gamma}^-.$$ 

This shows that $\omega(\tilde{f}_j, \tilde{x}) = \tilde{\gamma}^+$ or $\tilde{\gamma}^-$. A similar argument shows $\alpha(\tilde{f}_j, \tilde{x}) = \tilde{\gamma}^+$ or $\tilde{\gamma}^-$. Hence $\tilde{x} \in U(\tilde{\gamma})$. Since $x \in V$ was an arbitrary recurrent point we conclude that $V \subset U$ where $U \in \mathcal{A}$ is the projection of $\tilde{U}(\tilde{\gamma})$ in $M$. This completes the proof when $V$ is essential. Note that by Lemma (12.1) any $f$-invariant $V$ in $M$ must be essential so we have proved the “moreover” part of the result.

We are left with the case that $V \in \mathcal{A}(q)$ is inessential in $M$. An essential closed curve in $V$ bounds a closed disk in $M$ and we let $W$ be the union of $V$ and this disk. Since $W$ is open and invariant under $f^q$ there is a periodic point $p \in W \cap \text{Fix}(f^q)$ by the Brouwer plane translation theorem. Choose a lift $\tilde{p}$ of $p$ and let $\tilde{C}$ be a home domain for $\tilde{p}$, let $\tilde{W}$ be the lift of $W$ that contains $\tilde{p}$ and let $\tilde{U}$ be the element of $\tilde{\mathcal{A}}$ that contains $\tilde{p}$. It suffices to prove that $\tilde{W} \subset U$ and for this it suffices to show that if $\tilde{w} \in \tilde{W}$ is a lift of $w \in W \cap \mathcal{B}(f)$ then $\omega(\tilde{f}_{\tilde{C}}, \tilde{w}) = \omega(\tilde{f}_{\tilde{C}}, \tilde{p})$.

Choose a path $\rho \subset \tilde{U}$ connecting $w$ to $p$. If $K$ is a number greater than the length of $\rho$ then there is a sequence $n_i \to \infty$ such that $f^{n_i}(w)$ can be joined to $p$ by a path in $W$ of length at most $K$. Thus

$$\omega(\tilde{f}_{\tilde{C}}, \tilde{w}) = \lim \tilde{f}^{n_i}_{\tilde{C}}(\tilde{w}) = \lim \tilde{f}^{n_i}_{\tilde{C}}(\tilde{p}) = \omega(\tilde{f}_{\tilde{C}}, \tilde{p})$$

as desired. \qed

**Remark 12.3.** $V \in \mathcal{A}(q)$ is essential in $\mathcal{M}$ if and only if it is essential in the unique $U \in \mathcal{A}$ containing it. In this case we will simply say that $V$ is essential.

Recall (see Definition (2.7)) that for any open $f$-invariant annulus $V \subset M$ there is a natural annular compactification of $V$ denoted $V_c$ and an extension of $f$ to the closed annulus $f_c: V_c \to V_c$. To simplify notation we will denote the rotation interval $\rho(f_c)$ by $\rho(V)$ when there is no ambiguity about the choice of diffeomorphism $f$ but various annuli $V$ are under consideration.

The next lemmas provide information about the translation and rotation intervals of the extension $f_c: U_c \to U_c$ of $f$ to the annular compactification of $U$.

**Lemma 12.4.** Suppose that $q$ is prime and that $x \in U$ is not contained in any $V \in \mathcal{A}(q)$.

1. The rotation number, $\rho_f(x)$, with respect to $f_c$ is well defined.

2. If $\omega(f_c, x)$ contains a point of $U$ then $\rho_f(x) = p/q$ for some $0 < p < q$.

3. If $\omega(f_c, x)$ is contained in a component of $\partial U_c$ corresponding to a non-singular end of $U$ then $\rho_f(x) = 0$.  

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Lemma 12.5. Suppose that $U \in \mathcal{A}$ and that $X$ is a component of $\partial U_c$ corresponding to a non-singular end. Then the translation number $\tau(f_c|_X)$ of any lift of $f_c$ restricted to the universal covering space $\tilde{X}$ is an integer $p$. Moreover $\tau(f_c)$ is a non-trivial interval containing $p$ as an endpoint and having length at most 1.

Proof. No integer can be in the interior of $\tau(f_c)$. To see this we suppose to the contrary that an integer (which without loss we assume is 0) is in the interior of $\tau(f_c)$ and show this leads to a contradiction. In this case by Theorem 2.7 there would be periodic

(4) $\omega(f_c, x)$ is contained in a component $B$ of $\partial U_c$ corresponding to a singular end of $U$ then $\rho(f_c) = \rho(B)$.

Proof. Each point in $U \setminus \text{Fix}(f^q)$ has a compact neighborhood in $U \setminus \text{Fix}(f^q)$. Proposition 5.1-(1) therefore implies that $\omega(f^q, x) \subset \text{Fix}(f^q)$ and hence that $\omega(f^q_c, x) \subset \text{Fix}(f^q_c) \cup \partial U_c$. Since each component of $\text{Fix}(f^q_c) \cup \partial U_c$ is $f^q_c$-invariant, $\omega(f^q_c, x)$ is contained in a component $K$ of $\text{Fix}(f^q_c) \cup \partial U_c$.

The rotation number $\rho(f_c)$ is well defined and constant on each component of $\partial U_c$. It is also well defined and locally constant on $\text{Fix}(f^q_c)$. Since both sets are closed, $\rho(f_c)$ is locally constant on their union and hence constant on $K$, say $\rho(f_c, K) = \rho_K$. It follows that $\rho(f^q_c, K) = q\rho_K$.

In fact, more is true. There is a lift $\tilde{f}_c : \tilde{U}_c \to \tilde{U}_c$ of $f$ such that $\tau_{\tilde{f}_c}(\tilde{y}) = \rho_K$ for each $\tilde{y}$ that projects into $K$. Let $p_1 : \tilde{U}_c \to \mathbb{R}$ be the projection used to define $\tau_{\tilde{f}_c}$. Then for any $k \in \mathbb{Z}$

$$|p_1\tilde{f}^\alpha(q_{\tilde{y}}) - p_1\tilde{y} - kq\rho_K| < 1$$

for any $\tilde{y}$ that projects into $K$. For any fixed $k$, this inequality holds for any point $\tilde{z}$ that projects into a neighborhood, say $W_k$, of $K$. Suppose that $z$ and the forward $f^q$-orbit of $z$ is contained in $W_k$. Then by applying the above inequality with $\tilde{y}$ equal, in order, to $\tilde{z}, \tilde{f}_c^\alpha(\tilde{z}), \tilde{f}_c^{2\alpha}(\tilde{z}), \ldots, \tilde{f}_c^{(q-1)\alpha}(\tilde{z})$ we see that

$$|p_1\tilde{f}^\alpha(j_{\tilde{z}}) - p_1\tilde{z} - qjk\rho_K| < j$$

for all $j$. This proves that

$$q\rho_K - 1/k \leq \lim \inf \frac{p_1(\tilde{f}^{nq}(\tilde{z})) - p_1(\tilde{z})}{n} \leq \lim \sup \frac{p_1(\tilde{f}^{nq}(\tilde{z})) - p_1(\tilde{z})}{n} \leq q\rho_K + 1/k$$

for all $z$ with $\omega(f^q_c, z) \subset W_k$. Since $\omega(f^q_c, x) \subset W_k$ for all $k$ it follows that $\rho(f_c, x) = q\rho_K$ and hence that $\rho(f_c) = \rho_K$.

If $K$ contains a point $y$ in the interior of $U_c$ then $y \in \text{Fix}(f^q_c)$ and $\rho(f_c, x) = p/q$ for some $0 \leq p < q$. If $p = 0$ then there is a lift $\tilde{f} : \tilde{U} \to \tilde{U}$ and a lift $\tilde{y} \in \text{Fix}(f^q)$ in contradiction to the Brouwer translation theorem applied to $\tilde{f}$ and the fact that $\text{Fix}(\tilde{f}) = \emptyset$. Thus $0 < p < q$.

If $K$ is a component of $\partial U_c$ corresponding to a non-singular end then $\rho_K = 0$ by Proposition 5.1-(4).
points in $U$ with both positive and negative rotation numbers. Theorem (2.1) of [5] then implies that $f$ has a fixed point in the open annulus $U$, which is a contradiction.

By part (4) of Proposition 5.1, $f_c$ has a fixed point in $X$. It follows that the translation number of the lift $\tilde{f}_c|\tilde{X} : \tilde{X} \to \tilde{X}$ is an integer, say $p$. Hence $p \in \tau(\tilde{f}_c)$. There is a point in the interior of $U$ with a well defined non-integer translation number. This is because almost all points of $U$ have a well defined translation number by Theorem (2.2) and if these were all equal to $p$ then Proposition (2.6) would imply $U$ contains a fixed point – a contradiction. Since $p \in \tau(\tilde{f}_c)$ and no integer can be in its interior, it follows that $\tau(\tilde{f}_c)$ is non-trivial, $p$ is one endpoint and it must be contained in either $[p, p + 1]$ or $[p - 1, p]$.

**Corollary 12.6.** Suppose $q > 1$ and $V \in \mathcal{A}(q)$ is a subset of $U \in \mathcal{A}$ and $U$ has a non-singular end. If $V$ is $f$-invariant and $f_c : V_c \to V_c$ is the extension to its annular compactification then for any lift $\tilde{f}_c$ to the universal covering there is $p \in \mathbb{Z}$ such that $\tau(\tilde{f}_c)$ is a non-trivial subinterval of $[p/q, (p + 1)/q]$ and contains at least one of its endpoints.

**Proof.** The fact that $U$ has a non-singular end implies that $V$ does also. The result now follows from Lemma (12.5) since

$$\tau(\tilde{f}_c) = \frac{\tau(\tilde{f}_c^q)}{q}$$

and

$$r = \liminf_{n \to \infty} \frac{p_1(\tilde{f}_c^n(\tilde{x})) - p_1(\tilde{x})}{n}$$

and

$$s = \limsup_{n \to \infty} \frac{p_1(\tilde{f}_c^n(\tilde{x})) - p_1(\tilde{x})}{n}$$

Lemma 12.5 and Proposition 2.5 imply that $s - r \leq 1/q$ for all $q$ and hence that $r = s$. This proves that $x$ has a well defined backward rotation number. Similar arguments shows that $x$ has a well defined forward rotation number and a well defined rotation number.

**Lemma 12.8.** If $V \in \mathcal{A}(q)$ is a subset of $U \in \mathcal{A}$, then

(1) If $V$ is essential in $U$, then every $x \in V$ has the same rotation number in $V_c$ as in $U_c$. 47
(2) The annulus $V$ is inessential in $U$ if and only if all recurrent points in $V$ have the same rotation number for $f_c : U_c \to U_c$. In this case there is $p \in \mathbb{Z}$ with $0 < p < q$, such that the common rotation number is $p/q$.

**Remark 12.9.** In fact if $V \in \mathcal{A}(q)$ is inessential in $U$ then all points of $V$ have the same rotation number, not just the recurrent points. This will follow when we show as part of Proposition [12.18] that $\rho_f$ is continuous on $U$, since the recurrent points are dense in $V$.

**Proof.** If $V$ is essential in $U$ then it is $f$-invariant by Lemma [12.1]. The inclusion of $V$ in $U$ induces an isomorphism on the fundamental group and any point $x \in V$ has the same rotation number in $V$ and $U$. Since $\rho(V_c)$ is non-trivial by Lemma [12.5], points of $V$ have a non-trivial interval of rotation numbers in $U$.

If $V$ is inessential in $U$ there is a component $X$ of its complement in $S^2$ contained in $U$. The set $W = V \cap X$ is an open disk invariant under $f^q$. Since $f$ is area preserving and $f^q(W) = W$, by the Brouwer plane translation theorem there is a point $x_0 \in W \cap \text{Fix}(f^q)$. The rotation number of $x_0$ is $p/q$ for some $0 < p < q$ by Lemma [12.4](1). If $x \in V$ is recurrent there is a constant $K_x$ and a sequence $\{n_i = k_i q\}$ such that the distance in $U_c$ from $f^{n_i}(x)$ to $f^{n_i}(x_0)$ is less than $K_x$. Hence the distance in the covering space $\tilde{U}_c$ from $\tilde{f}^{n_i}(\tilde{x})$ to $\tilde{f}^{n_i}(\tilde{x}_0)$ is less than $K_x$, where $\tilde{f}, \tilde{x},$ and $\tilde{x}_0$ are lifts of $f, x,$ and $x_0$ respectively. It follows that $x$ and $x_0$ have the same rotation number in $U$, namely $p/q$.

**Corollary 12.10.** If $x \in U$ is not asymptotic to an end of $U$ then, with at most two exceptions, for each prime $q$ there is $V \in \mathcal{A}(q)$ which is essential in $U$ and contains $x$.

**Proof.** If $x \in V_1 \in \mathcal{A}(q_1)$ and $x \in V_2 \in \mathcal{A}(q_2)$ for distinct primes $q_1$ and $q_2$ then $V_1 \cap V_2$ is an open set and so contains a recurrent point $x'$. Since the rotation number of $x'$ with respect to $f_c : U_c \to U_c$ cannot have both the form $p/q$ and $p'/q'$, Lemma [12.8]-(2) implies that either $V$ or $V'$ is essential in $U$. We conclude that there is at most one $q$ for which $x$ is contained in an inessential element of $\mathcal{A}(q)$. The corollary then follows from Lemma [12.1] which implies that there is at most one $q$ for which $x$ is not contained in any $V \in \mathcal{A}(q)$.

**Lemma 12.11.** Suppose $U \in \mathcal{A}$. If $V \in \mathcal{A}(q)$ is essential in $U$ and $\rho(V)$ is disjoint from $\rho(\partial U_c)$, then $\text{cl}(V) \subset U$. Moreover, if $x \in U$ and $\rho_f(x) \notin \rho(\partial U_c)$, then for every sufficiently large prime $q$ there exists an essential $V \in \mathcal{A}(q)$ such that $x \in V$, $\text{cl}(V) \subset U$.

**Proof.** Let $X$ be one component of the complement of $V$ in $U$ and define $W = U \setminus X$. Then $W$ is an open subannulus of $U$. One of the components of its frontier, which we denote $Y$, coincides with a component of the frontier of $U$ and the other is the common frontier of $V$ and $X$. The rotation interval $\rho(W)$ contains $\rho(V)$ and the number $r_0 \in \rho(\partial U_c)$ corresponding to the end of $U$ determined by $Y$. The number
Lemma 12.13. Suppose that $r_0 \notin \rho(V)$ and we assume without loss that it is less than $v = \min \rho(V)$. Choose a rational number (in lowest terms) $r_1 = p_1/q_1$ in $(r_0, v)$. By Theorem 2.2 there is a periodic point $x \in W$ with rotation number $r_1$ and period $q_1$. If $q_2$ is a sufficiently large prime, then $1/q_2 < v - r_1$ and, by Corollary 12.10, the point $x$ lies in some $V_0 \in \mathcal{A}(q_2)$ which is essential in $U$. Corollary 12.6 implies that $\rho(V_0)$ and $\rho(V)$ are disjoint and hence that $V_0 \cap V = \emptyset$.

Every point of $V$ is separated by $V_0$ from the component $Y$ of the frontier of $U$. Hence $\overline{cl}(V)$ does not intersect $Y$. The same argument shows $\overline{cl}(V)$ does not intersect the other component of the frontier of $U$. It follows that $\overline{cl}(V) \subset U$.

To prove the moreover part of this lemma suppose $x \in U$ and $\rho_f(x) \notin \rho(\partial U_c)$. Then by Corollary 12.10 for every sufficiently large prime $q$, there is an essential $V \in \mathcal{A}(q)$ that contains $x$. Also we may assume $1/q$ is less than the distance from $\rho_f(x)$ to $\rho(\partial U_c)$. Since $\rho(V)$ has length at most $1/q$ and contains $\rho_f(x)$, it is disjoint from $\rho(\partial U_c)$ and so $\overline{cl}(V) \subset U$ by the first part of this lemma. 

We will write $|\rho(V)|$ for the length of the interval $\rho(V)$.

**Lemma 12.12.** Suppose that $Y$ is a component of the frontier of $U$ in $S^2$ and that $\{V_i\}$ is an infinite sequence of distinct essential elements of $\mathcal{A}(q)$ such that $V_{i+1}$ separates $V_i$ from $Y$. Then

$$
\lim_{i \to \infty} |\rho(V_i)| = 0.
$$

**Proof.** Let $W_i$ be the open annulus that is the union of $V_1, V_i$ and a closed annulus bounded by an essential curve in $V_1$ and and essential curve in $V_i$. The complementary components of $W_i$ in $U$ [resp. $S^2$] are the component of $U \setminus V_i$ [ resp. $S^2 \setminus V_i$] that contains $Y$ and the component of $U \setminus V_1$ [ resp. $S^2 \setminus V_1$] that contains the other component of the frontier of $U$. Let $W \subset U$ be the union $\cup_i W_i$. Since the nested intersection of compact connected sets is compact and connected, $W$ is open and has two complementary components in $S^2$ so must be an open annulus.

The boundary component $B$ of $\partial W$, corresponding to the end of $W$ that is disjoint from $V_1$ has arbitrarily small $f_c$-invariant neighborhoods. It follows that if $x_i \in V_i$ is periodic, then the rotation number of $x_i$ with respect to $f$ converges to the rotation number $a$ of the restriction of $f_c$ to $B$. Theorem 2.2 therefore implies that the interval $\rho(V_i)$ converges to the point $a$ and so has length tending to zero. 

**Lemma 12.13.** Suppose $\rho(U)$ is non-trivial and $Y$ is a component of the frontier of $U$. Let $a$ be the rotation number of the component of $\partial U_c$ corresponding to $Y$.

1. If $q$ is any sufficiently large prime, then $Y$ is a frontier component of a (necessarily unique) essential $V_0(q) \in \mathcal{A}(q)$ with $V_0(q) \subset U$.

2. $a \in \rho(V_0(q))$.

3. If $a \neq p/q$ for some $0 < p < q$ then $\overline{cl}_U(V_0(q')) \subset V_0(q)$ for all sufficiently large $q'$.
Proof. The first step in the proof of (1) is to prove that for all sufficiently large \( q \) there exists a (necessarily unique) essential \( V_0(q) \in \mathcal{A}(q) \) that is not separated from \( Y \) by any other essential element in \( \mathcal{A}(q) \).

By the moreover part of Lemma [12.11] we may assume that there exists an essential \( V_1 \in \mathcal{A}(q) \) whose closure is contained in \( U \). Let \( \rho(\partial U_c) = \{ a, b \} \). We may assume that \( q \) is so large that neither \( a \) nor \( b \) has the form \( p/q \) with \( 0 < p < q \). Choose \( \delta < |a - p/q|, |b - p/q| \) for all \( 0 < p < q \).

The proof is by contradiction: assuming that no such \( V_0(q) \) exists we will inductively define an infinite sequence \( \{ V_i \} \) of distinct essential elements of \( \mathcal{A}(q) \) such that \( V_{i+1} \) separates \( V_i \) from \( Y \) and such that \( |\rho(V_i)| > \delta/2 \) in contradiction to Lemma [12.12]. It suffices to assume that \( V_1, \ldots, V_{i-1} \) have been defined for \( i \geq 2 \), and define \( V_i \). By assumption, there exists \( V_i^* \in \mathcal{A}(q) \) that separates \( V_{i-1} \) from \( Y \).

Since \( V_i^* \) is contained between two open essential annuli in \( U \), each component of its frontier is contained in the interior of \( U \). Lemma [12.4] implies that the rotation number of the restriction of \( f_c \) to a component of \( \partial V_i^* \) has the form \( p/q \) with \( 0 < p < q \).

Choose an essential closed curve \( \alpha \) in \( V_i^* \) and let \( W_i \) be the union of \( V_i^* \) with the component of \( U \setminus \alpha \) that does not contain \( V_{i-1} \). Then \( W_i \) is an open annulus whose frontier components are \( Y \) and a component of the frontier of \( V_i^* \). Theorem [2.2] implies that \( W_i \) contains a periodic point \( z \) whose rotation number has distance less than \( \delta/2 \) from \( a \) and so is not of the form \( p/q \). In particular, \( z \in \mathcal{M}_q \) and so is contained in some \( V_i \in \mathcal{A}(q) \) by Lemma [12.4]. Proposition [12.8] (2) implies that \( V_i \) is essential and hence separates \( Y \) from \( V_{i-1} \). The rotation number of the restriction of \( f_c \) to a component of \( \partial V_i^{*e} \) has the form \( p/q \) with \( 0 < p < q \) for the same reason that components of \( \partial V_i^{*e} \) satisfy this property. Since \( z \in V_i \), it follows that \( |\rho(V_i)| > \delta/2 \). This completes the induction step and hence the proof that there is a unique essential \( V_0(q) \in \mathcal{A}(q) \) that is not separated from \( Y \) by any essential element of \( \mathcal{A}(q) \).

For the remainder of the proof we view \( V_0(q) \) as an essential open sub-annulus of \( U_c \). Let \( \partial_0 U_c \) be the component of \( \partial U_c \) corresponding to \( Y \).

Since there exists \( V_1 \in \mathcal{A}(q) \) whose closure is contained in \( U \), the component \( B(q) \) of \( \text{fr}(V_0(q)) \) which is separated from \( \partial_0 U_c \) by \( V_0(q) \) is contained in \( U \). Lemma [12.4] implies that if \( q \neq q' \) then \( B(q) \cap B(q') = \emptyset \). Let \( W(q) \) be the open sub-annulus of \( U_c \) bounded by \( \partial_0 U_c \) and \( B(q) \) and note that either \( \text{cl}_{U}(W(q)) \subset W(q') \) or \( \text{cl}_{U}(W(q')) \subset W(q) \).

Theorem [2.2] implies that \( W(q) \) contains a periodic point \( w \) whose rotation number is arbitrarily close to \( a \) and in particular is not of the form \( p/q \). Lemma [12.4] and Proposition [12.8] (2) imply that \( w \) is contained in some element of \( \mathcal{A}(q) \) that is essential and hence must be \( V_0(q) \). Theorem [2.4] implies that \( a \in \rho(V_0(q)) \) which verifies (2). The same argument shows that \( V_0(q') \) contains a point in \( W(q) \) and hence that \( B(q') \) is not contained in the component of \( U_c \setminus V_0(q') \) that contains \( \partial_0 U_c \). Assuming without loss that \( q' > 2/\delta \), \( \rho(V_0(q')) \) does not contain any element of the form \( p/q \) with \( 0 < p < q \). It follows that \( B(q) \) is disjoint from \( V_0(q') \) and so is contained in the
component of $U_c \setminus V_0(q')$ that is separated from $\partial_0 U_c$ by $B(q')$. In other words, $c_l U \cap W(q') \subset W(q)$. Item (3) will follow once we prove that $W(q) = V_0(q)$ (which is equivalent to showing that $\partial_0 U_c$ is the boundary component $B'(q)$ of $V_0(q)$ which is not $B(q)$).

We claim that if $P$ is an open set in $W(q) \setminus V_0(q)$ then $P \cap W(q') = \emptyset$. If this fails then the open set $P \cap W(q')$ would contain a birecurrent point $x$, which by Lemma 11.7 is contained in some, necessarily inessential, $V \in A(q)$. Lemma 12.4 and Proposition 12.8 therefore imply that $\rho(x)$ has the form $p/q$ with $0 < p < q$. It follows that $x$ is not contained in $V_0(q')$ but is contained in an essential element of $A(q')$. This contradicts the assumption that $x \in W(q')$ and so verifies the claim.

One immediate consequence of the claim is that $\partial_0 U_c \subset B'(q)$. Another is that each point in $B'(q)$ has a neighborhood that does not intersect any element of $A(q)$ other than $V_0(q)$. It follows that $\partial_0 U_c \subset B'(q) \subset \partial_0 U_c \cup \text{Fix}(f^q)$. Since (c.f. the proof of Lemma 12.4) rotation number is constant on each component of $\partial_0 U_c \cup \text{Fix}(f^q) = \partial_0 U_c \cup \text{Fix}(f^q)$, it must be zero on $B'(q)$. Since $\text{Fix}(f) = \emptyset$ this implies that $B'(q) \cap \text{Fix}(f^q) = \emptyset$ and hence that $B'(q) = \partial_0 U_c$.

\[ \square \]

**Remark 12.14.** In the following definition we assume that $\rho(U)$ is non-trivial. If $\rho(U)$ is trivial then both ends of $U$ are singular by Lemma 12.9 so $\text{Fix}(F)$ is a pair of points and $U$ is the unique element of $A$.

**Definitions 12.15.** Suppose that $\rho(U)$ is non-trivial, that $Y_0$ and $Y_1$ are the frontier components of $U \in A$ and that $a_i$ is the rotation number of the component of $\partial U_c$ determined by $Y_i$. Choose $Q = Q(U)$ so that:

- If $a_i = p/q$ for some $0 < p < q$ then $q < Q$.

- For every prime $q > Q$ there is an essential element $V \in A(q)$ such that $cl(V) \subset U$. (See Lemma 12.11.)

- For every prime $q > Q$ there are distinct elements $V_0(q), V_1(q) \in A(q)$ (as in Lemma 12.13) that are contained in $U$ such that $\text{fr}(V_i(q)) = Y_i \cup B_i(q)$ for $i = 0, 1$ where $B_0(q) \cap B_1(q) = \emptyset$.

Define

$$\hat{Y}_i = \bigcap_{q > Q} \text{cl}(V_i(q)).$$

and $\hat{U} = \text{cl}(U) \setminus (\hat{Y}_0 \cup \hat{Y}_1) \subset U$.

**Lemma 12.16.** Assume notation as above.

1. $\hat{Y}_i$ is well-defined, i.e., independent of the choice of $Q$.

2. $\hat{U}$ and $\hat{U} \cup (\hat{Y}_i \cap U)$ are essential open $f$-invariant annuli.
(3) If $a_i = 0$ then $\hat{Y}_i \cap U$ has measure 0.

(4) If $a_i \neq 0$ then $\hat{Y}_i \cap U \subset \mathcal{W}_0$.

(5) $\rho(y) = a_i$ for each $y \in \hat{Y}_i \cap U$.

**Remark 12.17.** For any $x \in \hat{U}$ the $\omega$-limit set $\omega(F, x)$ is contained in $U$ because the orbit of $x$ is separated from the frontier of $U$ by $V_0(q) \cup V_1(q)$ for some $q$ Lemma 12.4(1) therefore implies that every point in $\hat{U}$ has non-zero rotation number. If $a_0 = a_1 = 0$ then item (5) of Lemma 12.16 implies that $\hat{U}$ is the subset of $U$ with non-zero rotation number.

**Proof.** By part (3) of Lemma 12.13 there is a sequence of primes $q_j \to \infty$ such that

$$\hat{Y}_i = \bigcap_{q_j} \text{cl}(V_i(q_j))$$

and

$$V_i(q_{j+1}) \cup B_i(q_{j+1}) \subset V_i(q_j)$$

for all $j$, where $B_i(q)$ denotes the component of the frontier of $V_i(q)$ which lies in $U$. This proves that $\hat{Y}_i$ does not depend on the choice of $Q$ in its definition and so is well-defined.

Let $Z_i(q)$ be the component of $S^2 \setminus B_i(q)$ that contains $Y_i$. Then $\text{cl}(Z_i(q_{j+1})) = Z_i(q_{j+1}) \cup B_i(q_{j+1}) \subset Z_i(q_j)$ for the sequence of primes $\{q_j\}$ chosen above. Define

$$\hat{Z}_i = \bigcap_{q_j} \text{cl}(Z_i(q_j)) = \bigcap_{q \geq Q} \text{cl}(Z_i(q))$$

Then $\hat{Z}_0$ and $\hat{Z}_1$ are disjoint, compact, connected sets and $\hat{U} = \text{cl}(U \setminus (\hat{Y}_0 \cup \hat{Y}_1)) = S^2 \setminus (\hat{Z}_0 \cup \hat{Z}_1)$. Thus $\hat{U}$ is an open subsurface of $S^2$ with two ends and hence an annulus. The set $\hat{U}$ separates $\hat{Z}_0$ and $\hat{Z}_1$ each of which contains a point of $\text{Fix}(F)$. Hence $\hat{U}$ is essential in $\mathcal{M}$ and therefore in $U$. It is $f$-invariant by part (1) of Lemma 12.1. Having verified that $\hat{U}$ is an essential open $f$-invariant annulus, the same is true for $\bar{U} \cup (\hat{Y}_i \cap U)$, which is the union of $\hat{U}$ with the essential sub-annulus of $U$ bounded by $Y_i$ and an essential simple closed curve in $\hat{U}$. This completes the proof of (2).

Item (5) follows from the fact that $a_i \in \rho(V_0(q))$ and the fact that $|\rho(V_0(q))| \leq 1/q$. Item (3) then follows from Proposition 2.6 and that fact that $\text{Fix}(f) = \emptyset$.

For (4) suppose that $a_i \neq 0$ and that $x \in \hat{Y}_i$. If the $\omega$-limit set of $x$ contains a point in $\hat{U}$ then $x \in \mathcal{W}_0$ by Remark 12.2. Otherwise there is a non-fixed point $z$ in $\omega(x, f_c)$. If $D_0$ is a disk neighborhood of $z$ then $D_0 \cap U$ is a free disk that the orbit of $x$ intersects more than once and again $x \in \mathcal{W}_0$. \[ \square \]

We are now prepared to complete the proof of Theorem 1.3. For the definition of free disk recurrent and weakly free disk recurrent see Definition 1.1.
Theorem 1.3. Suppose $F \in \text{Diff}_*(S^2)$ has entropy zero. Let $f = F|_\mathcal{M}$ where $\mathcal{M} = S^2 \setminus \text{Fix}(F)$. Then there is a countable collection $\mathcal{A}$ of pairwise disjoint open $f$-invariant annuli such that

1. $\mathcal{U} = \bigcup_{U \in \mathcal{A}} U$ is the the set $\mathcal{W}$ of weakly free disk recurrent points for $f$.
2. If $U \in \mathcal{A}$ and $z$ is in the frontier of $U$ in $S^2$, there are components $F_+(z)$ and $F_-(z)$ of $\text{Fix}(F)$ so that $\omega(F,z) \subset F_+(z)$ and $\alpha(F,z) \subset F_-(z)$.
3. For each $U \in \mathcal{A}$ and each component $C_M$ of the frontier of $U$ in $\mathcal{M}$, $F_+(z)$ and $F_-(z)$ are independent of the choice of $z \in C_M$.

Proof. By items (2) and (3) of Proposition 5.1 it suffices to prove (1).

Given $x \notin \mathcal{U}$, choose a lift $\tilde{x} \in H$ and let $\tilde{C}$ be the home domain for $\tilde{x}$. If there is a free disk $D$ and $n > 0$ such that $x, f^n(x) \in D$, then there is a lift $\tilde{D}$ of $D$ that contains $\tilde{x}$ and a covering translation $T$ such that $f^n\tilde{C}(x) \in T(D)$. Thus $T$ is an $f_\tilde{C}$-near cycle for $\tilde{x}$. Since $\tilde{x}$ and $f^n\tilde{C}(\tilde{x})$ are both contained in $\tilde{U}$, $T$ preserves $U$ and so preserves $\tilde{C}$ in contradiction to Lemma 11.3(2). We conclude that there is no such $n$ and $D$ which proves that $x \notin \mathcal{W}_0$. Thus $\mathcal{W}_0 \subset \mathcal{U}$ and the interior of the closure in $\mathcal{M}$ of any component of $\mathcal{W}_0$ is contained in some $U \in \mathcal{A}$ by Lemma 11.4(5). This proves that $\mathcal{W} \subset \mathcal{U}$.

To prove the converse note that the $\omega$-limit set of any point in $\tilde{U}$ lies in $U$ and hence contains points that are not fixed by $f$. Remark 1.2 therefore implies that $\tilde{U} \subset \mathcal{W}_0$. If both $a_0$ and $a_1$ are non-zero then $U = \tilde{U} \cup (U \cap (\tilde{Y}_0 \cup \tilde{Y}_1)) \subset \mathcal{W}_0 \subset \mathcal{W}$ by item (4) of Lemma 12.16. If both $a_0$ and $a_1$ are zero then $\tilde{U}$ is dense in $U$ by item (3) of Lemma 12.16. Thus $U \subset \text{cl}(\tilde{U}) \subset \mathcal{W}$ since $\tilde{U}$ is a connected (item (2) of Lemma 12.16) subset of $\mathcal{W}_0$. For the remaining case we may assume that $a_0 = 0$ and $a_1 \neq 0$. Then $U \subset \text{cl}(\tilde{U} \cup (\tilde{Y}_1 \cap U)) \subset \mathcal{W}$ because $\tilde{U} \cup (\tilde{Y}_1 \cap U)$ is a connected subset of $\mathcal{W}_0$.

Proposition 12.18. Each $x \in \tilde{U}$ is contained in an $f$-invariant set $C(x) \subset U$ with the following properties:

1. Each $C(x)$ is compact, connected and essential in $U$, i.e. its complement in $U$ has two components and it separates the ends of $U$.
2. If $x, y \in \tilde{U}$ then $C(x)$ and $C(y)$ are equal or disjoint.
3. The rotation number $\rho$ is constant on $C(x)$. In fact, each $C(x)$ is a connected component of a level set of the function $\rho$.
4. $\rho$ is continuous on $U$ and extends continuously to $\text{cl}(U)$ assigning (in the notation of Definition 12.15) $a_i$ to $\tilde{Y}_i$.
Proof. The definition of \( C(x) \) is very similar to the definition of \( \hat{Y}_i \) in Definition \((12.15)\). We specify \( Q = Q(x) \) by a series of largeness conditions. By Lemma \((12.13)\) and the assumption that \( x \in \hat{U} \), we may assume that \( x \notin V_0(q) \cup V_1(q) \) for \( q \geq Q \). In particular, \( \omega(x) \subset U \). We may also assume that \( \rho(x) \neq p/q \) for \( q \geq Q \) and \( 0 < p < q \). Lemma \((12.4)\) therefore implies that \( x \) is contained in some \( V(q,x) \in A(q) \) which is essential by for Lemma \((12.8)\). Since \( x \notin V_0(q) \cup V_1(q) \), we have \( \text{cl}(V(q,x)) \subset U \).

Define \[
C(x) = \bigcap_{q>Q} \text{cl}(V(q,x)).
\]
Given \( q \) let \( \delta \) be the minimum value of \( |p/q - \rho(x)| \) for \( 0 < p < q \). If \( q' > 1/\delta \) then \( \rho(V(q',x)) \) does not contain \( p/q \) for \( 0 < p < q \) and so does not contain any points in the frontier of \( V(x,q) \). It follows that \( \text{cl}(V(q',x)) \subset V(q,x) \). We may therefore choose a sequence of primes \( q_j \to \infty \) such that
\[
C(x) = \bigcap_{q_j} \text{cl}(V(q_j,x))
\]
and
\[
\text{cl}(V(q_{j+1},x)) \subset V(q_j,x)
\]
for all \( j \). This proves that \( C(x) \) does not depend on the choice of \( Q \) in its definition and so is well-defined.

Items (1) and (2) are obvious from the construction. The first part of (3) follows from the fact that \( |\rho(V(q,x))| \leq 1/q \).

If \( y \in V(q,x) \) then \( V(q,x) \) is a neighborhood of \( y \) on which \( \rho(x) \) is in \( \rho(V_k) \). This proves \( \rho \) is continuous at \( y \). Hence \( \rho \) is continuous on \( \hat{U} \).

Assume notation as in Definition \((12.15)\). The sets \( V_i(q) \) form a neighborhood basis of \( \hat{Y}_i \) and \( \hat{Y}_i \) contains \( Y_i \). Since for \( x \in V_i(q) \), both \( a_i \) and \( \rho_j(x) \) are in \( \rho(V_i(q)) \) and \( |\rho(V_i(q))| \leq 1/q \), it follows that the extension of \( \rho \) to \( \hat{Y}_i \) assigning the value \( a_i \) to every point of \( Y_i \) is also continuous. This completes the proof of (4).

Finally observe that if \( y \notin C(x) \) then there exists \( q > Q \) such that \( y \notin \text{cl}(V(q,x)) \). Then the frontier of \( V(q,x) \) separates \( C(x) \) and \( y \) and this frontier consists of points whose rotation number is not equal to \( \rho(x) \) so \( y \) is not in the same connected component as \( x \) of the level set of \( \rho \). It follows that \( C(x) \) is a connected component of this level set, proving part (3).

If \( x \) is in a component of the frontier of \( \hat{U} \) it is natural to define \( C(x) \) to be the set \( \hat{Y}_i \) (as in Definition \((12.15)\)) which contains it. This set \( C(x) \) is compact and connected. As noted in the previous theorem the function \( \rho_f \) extends continuously, but this \( C(x) \) may not necessarily separate the sphere \( S^2 \). In fact it may be a single point.

We have completed most of the work needed to prove the following.
Theorem 1.5. Suppose $F \in \text{Diff}_\mu(S^2)$ has entropy zero. Let $f = F|_\mathcal{M}$ where $\mathcal{M} = S^2 \setminus \text{Fix}(F)$ and let $\mathcal{A}$ be as in Theorem 1.4. Then for each $U \in \mathcal{A}$ there is a partition of $\text{cl}(U) \subset S^2$ into a family $\mathcal{C}$ of closed $F$-invariant sets with the following properties:

- Each $C \in \mathcal{C}$ is compact and connected.
- There are two elements of $\mathcal{C}$ (which may coincide), called ends and denoted $C_0$ and $C_1$, each of which contains a component of the frontier of $U$. Every element of $\mathcal{C}$ which is not an end is a subset of $U$ and is called interior. Each interior $C$ is essential in $U$, i.e. its complement (in $U$) has two components and it separates $C_0$ and $C_1$.
- The rotation number $\rho_f(x) \in \mathbb{R}/\mathbb{Z}$ is well defined and constant on any interior $C$. In fact, each $C$ is a connected component of a level set of $\rho_f(x)$. Moreover, $\rho_f(x)$ is continuous on $U$ and has a unique continuous extension to $\text{cl}(U)$. The value of this extension on $C_i$, $i = 0, 1$ is the element of $\rho(\partial U_c)$ corresponding to the component of the frontier of $U$ contained in $C_i$.

Proof. We first define the two end elements of $\mathcal{C}$ by letting $C_0 = \hat{Y}_0$ and $C_1 = \hat{Y}_1$. We define the collection $\mathcal{C}$ to consist of the these two sets together with the closed sets $C(x)$ for $x \in \hat{U}$. Proposition 12.16 and Proposition 12.18 (2) imply this is a partition of $U$. Proposition 12.18 (1) says that each $C$ is compact and connected and that each interior $C$ is essential and separates $U$. Parts (3) and (4) of Proposition 12.18 complete the proof.

13 The proof of Theorem 1.7.

Recall that a group $G$ is called indicable if there is a non-trivial homomorphism $\phi : G \to \mathbb{Z}$. We say $G$ is virtually indicable if it has a finite index subgroup which is indicable.

Proposition 13.1. Suppose that $S$ is a surface and $F : S \to S$ is $C^{1+\epsilon}$ and has positive topological entropy. Then every finitely generated infinite subgroup $H$ of the centralizer $Z(F)$ of $F$ is virtually indicable and has a finite index subgroup that has a global fixed point.

Proof. A result of Katok [17] asserts that $F^q$ has a hyperbolic fixed point $p$ for some $q \geq 1$. The orbit of $p$ under $H$ consists of hyperbolic fixed points of $F^q$ at which the derivative of $DF^q$ has the same eigenvalues as $DF^q_p$. If the $H$ orbit of $p$ were infinite, continuity of the derivative would imply that at any limit point of this orbit $DF^q$ would have the same eigenvalues and in particular would be hyperbolic. But this is impossible since hyperbolic fixed points are isolated. We conclude the orbit of $p$ under $H$ is finite and hence that the subgroup $H_0$ of $H$ that fixes $p$ has finite index.
After passing to a further finite index subgroup we may assume that $Dh_p$ has positive eigenvalues and the same eigenspaces as $DF_p$ for each $h \in H_0$. For each eigenspace the function which assigns to $h$ the log of the eigenvalue of $Dh_p$ on that eigenspace is a homomorphism from $H_0$ to $\mathbb{R}$. If this is non-trivial we are done. Otherwise both eigenvalues are 1 for each $Dh_p$. Hence in the appropriate basis

$$Dh_p = \begin{pmatrix} 1 & r_h \\ 0 & 1 \end{pmatrix}$$

for some $r_h \in \mathbb{R}$. The function $h \mapsto r_h$ defines a homomorphism from $H_0$ to $\mathbb{R}$, so we are done unless $r_h = 0$ for all $h \in H_0$. But in this latter case $Dh_p = I$ for all $h \in H_0$ so we may apply the Thurston stability theorem ([25]; see also Theorem 3.4 of [9]) to conclude there is a non-trivial homomorphism from $H_0$ to $\mathbb{R}$. 

In the case of an entropy zero diffeomorphism $F$ the analogous result does not hold, even when we restrict to the group of area preserving diffeomorphisms.

**Examples 13.2.** Let $S = S^2$ be the unit sphere in $\mathbb{R}^3$. Let $F : S \to S$ be a diffeomorphism whose restriction to each of the level sets $z = c$ is a rotation of that circle and with the property that $F = id$ for all points $(x, y, z)$ with $|z| \geq 3/4$. We assume that $F$ is not the identity on the equator $z = 0$. Let $g : S \to S$ be rotation about the $z$-axis by an angle which is an irrational multiple of $\pi$. Let $h : S \to S$ be a diffeomorphism supported in the interior of the disks $|z| > 3/4$ with the property that $h$ preserves area and the $h$-orbits of $(0, 0, 1)$ and $(0, 0, -1)$ are infinite. Let $G$ be the group of all rotations about the $z$-axis through angles which are rational multiples of $\pi$.

1. The group $H$ generated by $g$ and $h$ lies in the centralizer $Z(F)$ of $F$ but has no finite index subgroup with a global fixed point.

2. The group $G$ is a subgroup of $Z(g)$. Every element of $G$ has finite order so there are no non-trivial homomorphisms from any subgroup of $G$ to $\mathbb{R}$ and hence $G$ is not virtually indicable.

The first example above shows that we cannot generalize Proposition (13.1) to the centralizer of a diffeomorphism $F$ with zero entropy, even in the group of area preserving diffeomorphisms. The second example shows the necessity of the hypothesis of finitely generated in the following.

**Theorem 1.7.** If $F \in \text{Diff}_\mu(S^2)$ has infinite order then each finitely generated infinite subgroup $H$ of $Z(F)$ is virtually indicable.

**Proof.** The case that $F$ has positive entropy is covered by Proposition (13.1) so we need only consider the case when $F$ has entropy zero. We assume that every finite index subgroup of $H$ admits only the trivial homomorphism to $\mathbb{R}$ and show this leads to a contradiction.

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Recall that $\mathcal{A}$ denotes the family of $F$-invariant annuli guaranteed by Theorem (1.3). There is a special case when every $U \in \mathcal{A}$ has trivial rotation interval $\rho(U)$. We postpone this until after the more general case.

Let $U \in \mathcal{A}$ be an element of $\mathcal{A}$ with the rotation interval $\rho(U)$ non-trivial. By Remark (1.4) each $h \in H$ permutes the open annuli in the family $\mathcal{A}$. In particular, for any $h \in H$ the open annuli $U$ and $h(U)$ must be disjoint or equal.

Since elements of $H$ preserve area the $H$ orbit of the open set $U$ must be finite. We let $H'$ be the finite index subgroup of $H$ which leaves $U$ invariant. Let $x \in U$ be a point with $\rho_F(x) = \lambda$ which is irrational. Such a point exists by Theorem (2.3) since the rotation interval of $F_c : U_c \to U_c$ is non-trivial. The set $C(x)$ was defined in Proposition (12.18) to be the intersection of a family $\{V_n\}$ of essential $F$-invariant annuli containing $x$ with the property that the rotation interval (under $F|_U$) is contained in an interval $I_n$ with $\bigcap I_n = \{\lambda\}$. The annulus $V_n$ is an element of $\mathcal{A}(q)$ for some $q$, and, as above, $h(V_n)$ is either equal to $V_n$ or disjoint from it. However, $h$ cannot map an essential open subannulus of $U$ to a disjoint subannulus since that would require that it map an open subset of $U$ onto a proper subset contradicting the property that $h$ preserves area.

Choose one component, $V_n^+$, of the complement of $C(x)$ in $V_n$ in such a way that $V_{n+1}^+ \subset V_n^+$, i.e., always choose the component on the same side of $C(x)$. Let $\bar{A}_n$ denote $V_n^+$ with its ends compactified by the prime end compactification. Let $\partial^+ A$ denote the circle of prime ends added to the end corresponding to $C(x)$. The natural identification of these circles for different $n$ is reflected in the notation which is independent of $n$. Let $A_n = V_nA^+ \cup \partial^+ A$, i.e. $V_n^+$ with only one end (the one corresponding to $C(x)$) compactified. Then $A_{n+1} \subset A_n$ and

$$\bigcap_{n>0} A_n = \partial^+ A.$$ 

Let $\bar{f} : \bar{A}_n \to \bar{A}_n$ and $\bar{h} : \bar{A}_n \to \bar{A}_n$ denote the natural extensions of $F$ and $h \in H'$ to $\bar{A}_n$.

The rotation number $\rho(\bar{f}|_{\partial^+ A})$ of the restriction of $\bar{f}$ to $\partial^+ A$ must be $\lambda$. This is because if it were not and $p/q$ is between $\rho(\bar{f}|_{\partial^+ A})$ and $\lambda$ then by Theorem (2.2) applied to $\bar{A}_n$ there would be periodic points in the interior of $\bar{A}_n \subset \bar{A}_n$ with rotation number $p/q$ for all $n$, a contradiction.

For each $n$ there is a homomorphism $\phi_n : H' \to S^1 = \mathbb{R}/\mathbb{Z}$ given by $h \mapsto \rho_n(h|_{A_n})$ where $\rho_n(h|_{A_n})$ denotes the mean rotation number of $h$ on the annulus $A_n$ (see (1.1) of [7]). Let $H''$ denote the subgroup of $H'$ which is the kernel of the canonical homomorphism from $H'$ to its abelianization. Then $\rho_n(h|_{A_n}) = 0$ for all $h \in H''$. Also the abelianization of $H'$ must be finite since this is one of the equivalent conditions for $H'$ not to be indicable. Therefore $H''$ has finite index in $H'$ (and hence in $H$).

Since $\rho_n(\bar{h}|_{A_n}) = 0$ for each $n$ and each $\bar{h} \in H''$ we conclude from Theorem (2.1) of [7] that it has a fixed point $y_n$ in $\bar{A}_n$ for all $n$. Let $B$ be the closed disk which is the union of $\partial^+ A$ and the component of the complement of $C(x)$ in $S^2$ which contains $V_n^+$. 57
Lemma (2.8) applied to the open annulus $V_n^+$ as a subset of $B$ implies that $cl_B(V_n^+)$ contains a fixed point $x_n$ of $\tilde{h}$.

Taking the limit of a subsequence we note that each $\tilde{h} \in H''$ has a fixed point in $\partial^+ A$. But the rotation number of $\tilde{f}$ on $\partial^+ A$ is irrational so $\tilde{f}$ has a unique invariant minimal set which is the omega limit set $\omega(x, \tilde{f})$ for each $x \in \partial^+ A$. Since $\tilde{f}$ preserves $Fix(\tilde{h})$ we conclude this minimal set is in $Fix(\tilde{h})$ for every $\tilde{h} \in H''$.

We have found a prime end (in fact infinitely many) in $\partial^+ A$ which is fixed for each $\tilde{h} \in H''$. It follows from Corollary (2.9) that there is a point of $Fix(H'')$ in $cl(V_n^+)$ for each $n$. Taking the limit of a subsequence again we find a point of $Fix(H'')$ which lies in $\bigcap_n cl(V_n^+) = C(x)$.

Choosing an infinite collection $\{\lambda_i\}$ of distinct irrationals in the rotation interval of $F|U$ and repeating the construction we obtain an infinite collection of global fixed points for $H''$ with distinct rotation numbers for $F|U$. They must possess a limit point in $Fix(H'')$.

Proposition (3.1) of [12] asserts that if there is an accumulation point of $Fix(H'')$ then there is a homomorphism from $H''$ to $\mathbb{R}$. So $H''$ is indicable.

We are left with addressing the special case that every $U \in \mathcal{A}$ has trivial $\rho(U)$. Note that when $\rho(U)$ is trivial, by Proposition (2.6) it cannot consist of the single element $0 \in \mathbb{R}/\mathbb{Z}$. Note also that if every $\rho(U)$ is trivial then $Fix(F)$ has precisely two components and $\mathcal{A}$ contains only one element. This is because there is at least one $U \in \mathcal{A}$ and its complement has two components, and if $Fix(F)$ has three components, two of them must be in one component of the complement of $U$ which implies the corresponding end of $U$ is non-singular. Non-singular ends always imply $\rho(U)$ is non-trivial by Lemma (12.5).

Moreover, the two components of $Fix(F)$ must be single points since for an end with more than one fixed point in its frontier, the compactification of that end will be the prime end compactification and $F_c : U_c \to U_c$ will be the identity on that boundary component. This would imply that $\rho(U) = \{0\}$, a contradiction to the remarks above. It follows that there is only one $U \in \mathcal{A}$ and it is the complement of the two fixed points of $F$.

As a consequence of Theorem (2.2) and Proposition (2.6) we conclude that almost all points of $S^2$ have the same rotation number (with respect to the two fixed points). First assume that number is rational, say $p/q$ in lowest terms. In that case $g = F^q$ necessarily has more than 2 components to its fixed point set (by Theorem (2.2) again). We consider $H \subset Z(g)$ and the argument from the first part of this theorem can be applied.

Finally suppose that $\rho(U)$ is a single irrational $\lambda$. Passing to a subgroup of index 2 if necessary we may assume that $H$ fixes the two fixed points of $F$. Blowing up the two points we obtain a homeomorphism (also called $F$) on an annulus. The restriction to the boundary component corresponding to the fixed point $x$ is conjugate to the projectivization of $DF_x$. It must have rotation number $\lambda$ since otherwise there would be an additional periodic point by Theorem (2.2).
This map on the boundary circle is the projectivization of an element of $SL(2, \mathbb{R})$, i.e. a fractional linear transformation. Since its rotation number is irrational, in appropriately chosen coordinates it is an irrational rotation. It follows that the restrictions of blow-ups of elements of $H$ to this circle are rotations, since the centralizer of an irrational rotation consists of rotations. Therefore this group of restrictions, which is finitely generated because it is the image under a homomorphism of a finitely generated group, is abelian. Since since it admits no non-trivial homomorphisms to $\mathbb{R}$ it must be finite. We conclude there is a finite index subgroup $H'$ of $H$ whose restrictions to the boundary circle are all the identity. It follows that for any $h \in H'$ we have $Dh_x = I$ (since the projectivization of $Dh_x$ is the identity and $det(Dh_x) = 1$).

We can then apply the Thurston stability theorem [25] to arrive at a contradiction to the assumption that there are no non-trivial homomorphisms from $H'$ to $\mathbb{R}$. □

We now provide the proof of Corollary (1.8).

Corollary 1.8. If $\Sigma_g$ is the closed orientable surface of genus $g \geq 2$ then at least one of the following holds.

(1) No finite index subgroup of $\operatorname{MCG}(\Sigma_g)$ acts faithfully on $S^2$ by area preserving diffeomorphisms.

(2) For all $1 \leq k \leq g - 1$, there is an indicable finite index subgroup $\Gamma$ of the bounded mapping class group $\operatorname{MCG}(S_k, \partial S_k)$ where $S_k$ is the surface with genus $k$ and connected non-empty boundary.

Proof. We assume that (1) fails, i.e., that there is a finite index subgroup $G$ of $\operatorname{MCG}(\Sigma_g)$ which acts faithfully on $S^2$ by area preserving diffeomorphisms, and show that this implies (2).

Suppose $1 \leq k \leq g - 1$ and $S$ is the compact surface with genus $k$ and a connected non-empty boundary, $\partial S$. We assume $S$ is embedded in $\Sigma_g$ with $\partial S$ a separating closed curve and let $S'$ be the closure of complement of $S$, a surface with genus $g - k$ and boundary $\partial S$. There is a natural embedding of $\operatorname{MCG}(S, \partial S)$ into $\operatorname{MCG}(\Sigma_g)$ obtained by extending a representative of an element of $\operatorname{MCG}(S, \partial S)$ to all of $\Sigma_g$ by letting it be the identity on the complement of $S$. Similarly there is a natural embedding of $\operatorname{MCG}(S', \partial S')$ into $\operatorname{MCG}(\Sigma_g)$. If $\Gamma_0$ and $\Gamma'_0$ are the images of these two embeddings it is clear that every element of $\Gamma_0$ commutes with every element of $\Gamma'_0$ since they have representatives in $\operatorname{Diff}(\Sigma_g)$ which commute.

We let $\Gamma_1 = \Gamma_0 \cap G$ and $\Gamma'_1 = \Gamma'_0 \cap G$. Since $\Gamma'_1$ has finite index in $\operatorname{MCG}(S', \partial S)$ it contains an element $\gamma$ of infinite order. Suppose $\phi : G \to \operatorname{Diff}_\mu(S^2)$ is the injective homomorphism defining the action of $G$. Let $F = \phi(\gamma)$. Then $\phi(\Gamma'_1)$ is in the centralizer $Z(F)$. According to Theorem (1.7) $\Gamma_1$ is virtually indicable. Therefore there is an indicable $\Gamma$ of finite index in $\Gamma_1$. □
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