Extension maps

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Abstract

We define extension maps as maps that extend a system (through adding ancillary systems) without changing the state in the original system. We show, using extension maps, why a completely positive operation on an initially entangled system results in a non positive mapping of a subsystem. We also show that any trace preserving map, either positive or negative, can be decomposed in terms of an extension map and a completely positive map.

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1 Introduction

This paper ties in with previous papers on non completely positive maps [4] and the dynamical evolution of initially open entangled systems [5].

It is well known [1] that any completely positive linear map of a quantum system can be written as a unitary operation of an extended system where the original quantum state is coupled separably with an ancillary system (subscript \(e\)) in a known state:

\[
\Lambda \rho = \text{Tr}_e[\rho \otimes |e_0\rangle \langle e_0| u^\dagger] \quad (1)
\]

However it was shown in a recent paper [5] that dynamics of initially entangled systems need not be given by completely positive maps. Here we will outline the problem more concisely.

We wish to consider the linear transformation of a subspace of a system that undergoes a unitary evolution:

\[
\Lambda \rho = \text{Tr}_e[u E(\rho) u^\dagger] \quad (2)
\]

\(E\) is what we call an extension map. An extension map should satisfy:

\[
\text{Tr}_e[E(\rho)] = \rho \quad (3)
\]

This construction of using an extension map simplifies the problem - the map \(\Lambda\) is completely positive if and only if the extension map \(E\) is completely positive, since the unitary transformation \(u\) is by definition completely positive.

For example, to write a completely positive map as a larger unitary transformation [1], the extension map is chosen to be:

\[
E(\rho) = \rho \otimes |0_e\rangle \langle 0_e| \quad (4)
\]

For the problem outlined in [5], we wish to understand the mapping of a subsystem of a larger system, which is initially in a pure entangled state and undergoes unitary evolution. We find that the extension map has to be either non-linear (ie. like purification [3]), or if linear, can only be positive on certain density matrices. To demonstrate, consider any pure entangled state \(|\Phi\rangle\), the density matrix of a subsystem would be mixed. For example, in a \(2 \otimes 2\) system, we could have:

\[
E(\lambda_1 |\phi_1\rangle \langle \phi_1| + \lambda_2 |\phi_2\rangle \langle \phi_2|) = |\Phi\rangle \langle \Phi| \quad (5)
\]

If the extension map is linear, then we can expand:

\[
\lambda_1 E(|\phi_1\rangle \langle \phi_1|) + \lambda_2 E(|\phi_2\rangle \langle \phi_2|) = |\Phi\rangle \langle \Phi| \quad (6)
\]

The RHS is a pure state, which is convexly non-decomposable. Therefore \(E\) cannot be positive on all states, in this example it cannot be positive on \(|\phi_1\rangle\) and/or \(|\phi_2\rangle\).
This explains why a non-positive map results from what should be a completely positive operation (unitary transformation); it stems from the choice of a non-positive extension map.

2 Extension maps for non positive maps

An interesting question would be to ask: given a non positive linear map $\Lambda$, can a corresponding unitary evolution and extension map be found? In this section we construct a solution to this problem.

Let us consider only trace preserving maps:

$$Tr[\Lambda \rho] = \sum_{rs}^N \sum_{r'}^{N} \Lambda_{r',r,s} \rho_{rs} = Tr[\rho]$$  \hspace{1cm} (7)

$$\sum_{r'}^{N} \Lambda_{r',r,s} = \delta_{rs}$$ \hspace{1cm} (8)

We can write the map $\Lambda$ as the difference of two completely positive hermitian maps. Let us write the $\Lambda$ in its canonical decomposition:

$$\Lambda = \sum_{i} \lambda_i L_i \times L_i^\dagger$$ \hspace{1cm} (9)

We simply group the positive eigenvalues/matrices and the negative eigenvalues/matrices to define:

$$\Lambda^{(+)} = \sum_{i|\lambda_i>0} \lambda_i L_i \times L_i^\dagger$$ \hspace{1cm} (10)

$$\Lambda^{(-)} = \sum_{i|\lambda_i<0} |\lambda_i| L_i \times L_i^\dagger$$ \hspace{1cm} (11)

Note that by definition, both $\Lambda^{(+)}$ and $\Lambda^{(-)}$ are completely positive. And we have:

$$\Lambda = \Lambda^{(+)} - \Lambda^{(-)}$$ \hspace{1cm} (12)

Let us define matrices $J$ and $K$ as follows (this construction was first used in [4]):

$$J_{sr} = \sum_{r'}^{N} \Lambda^{(+)}_{r',r,s}$$ \hspace{1cm} (13)

$$K_{sr} = \sum_{r'}^{N} \Lambda^{(-)}_{r',r,s}$$ \hspace{1cm} (14)

First we note that the matrix $J$ (and similarly $K$) is hermitian:

$$J_{sr} = \sum_{r'}^{N} \sum_{i|\lambda_i>0} \lambda_i \{L_i\}_{r'} \{L_i\}^\dagger_{sr'} = J^*_{rs}$$ \hspace{1cm} (15)
Next we note that $J$ cannot be singular if the map $\Lambda$ is trace preserving, since the trace preserving condition is simply:

$$J - K = 1; J > 0, K \geq 0 \quad (16)$$

$J$ and $K$ are partially traced matrices of $\Lambda^+$ and $\Lambda^-$ which are completely positive, therefore $J$ and $J$ must be positive. Given $J = 1 + K$ and the matrices are all positive, it follows that $J$ cannot be singular.

However it is possible that $K$ is singular. Let us write down the canonical decomposition of $K$ (keeping in mind $K$ is hermitian):

$$K = \sum q k_q |\psi_q\rangle\langle\psi_q|; k_q > 0 \quad (17)$$

We can define a pseudo-inverse of matrix $K$ as:

$$K^{-1} = \sum q k_q^{-1} |\psi_q\rangle\langle\psi_q| \quad (18)$$

$$K^{-1} K = \sum q |\psi_q\rangle\langle\psi_q| \equiv \Psi \quad (19)$$

Let us also define an orthonormal set of eigenvectors $|\phi_q\rangle$ spanning the singular subspace of $K$:

$$K|\phi_q\rangle = 0 \quad (20)$$

We can show that $\Lambda^-$ must destroy all information in this subspace. Let us consider:

$$Tr[\Lambda^-(|\phi_q\rangle\langle\phi_q|)] = \sum_{rs} K_{sr}\{\phi_q\}_r^*\{\phi_q\}_s = 0 \quad (21)$$

Since $\Lambda^-$ is completely positive and hermitian, and $|\phi_q\rangle\langle\phi_q|$ is positive, if the trace of the result is zero then the result itself must be zero.

Next let us show:

$$\Lambda^-(|\phi_q\rangle\langle\psi_r|) = \Lambda^-(|\psi_r\rangle\langle\phi_q|) = 0 \quad (22)$$

Let us define the matrices:

$$A_{mn} = \Lambda^{(-)}_{m\phi_q,n\phi_q}$$
$$B_{mn} = \Lambda^{(-)}_{m\phi_q,n\psi_r}$$
$$D_{mn} = \Lambda^{(-)}_{m\psi_r,n\psi_r} \quad (23)$$

and the matrix:

$$Z = \begin{bmatrix} A & B \\ B^\dagger & D \end{bmatrix} \quad (24)$$
Z is a submatrix of $\Lambda^{(-)}$ so it is non-negative. We showed that $A = 0$ in equation 21, therefore it follows $B = B^\dagger = 0$ otherwise $Z$ would be negative.

This gives us a very useful result:

$$\Lambda^{(-)}(\rho) = \Lambda^{(-)}(\Psi\rho)$$

which we obtain by inserting the identity before $\rho$:

$$1 = \sum_u |\psi_u\rangle\langle\psi_u| + \sum_q |\phi_q\rangle\langle\phi_q|$$

Now we can move on to the main result – defining an extension map and unitary evolution for a non completely positive map $\Lambda$. Let us define the extension map:

$$E(\rho) = J\rho \otimes |e_+\rangle\langle e_+| - K\rho \otimes |e_-\rangle\langle e_-|$$

This satisfies the condition:

$$Tr_e[E(\rho)] = (J - K)\rho = \rho$$

We note that the dimension of the space needed for this extended state is $dim(J) + dim(K)$, and $dim(J) = N$ since we have noted that $J$ cannot be singular.

Then let us define a map $\Omega$:

$$\Omega(\rho \otimes |e_+\rangle\langle e_+|) = \Lambda^{(+)}(J^{-1}\rho) \otimes |e_+\rangle\langle e_+|$$

$$\Omega(\rho \otimes |e_-\rangle\langle e_-|) = \Lambda^{(-)}(K^{-1}\rho) \otimes |e_-\rangle\langle e_-|$$

$\Omega$ is completely positive since all its components, $\Lambda^{(+)}$, $\Lambda^{(-)}$, $J$ and $K$, are positive. It is also trace preserving, since for the $(+)$ component:

$$Tr[\Lambda^{(+)}(J^{-1}\rho)] = \Lambda^{(+)}_rJ^{-1}\rho J^{-1}_r = Tr[\rho]$$

For the $(-)$ component, singularities in $\Lambda^{(-)}$ poses a minor problem:

$$Tr[\Lambda^{(-)}(K^{-1}\rho)] = Tr[\Psi\rho]$$

However, we note that after applying the extension map the domain of states is $S\rho$, and $\Psi(S\rho) = S\rho$, so this component map is trace preserving on this domain.

Therefore the map $\Omega$ is completely positive and trace preserving. Putting $\Omega$ and $E$ together we have:

$$Tr_e[\Omega(E(\rho))] = \Lambda(\rho)$$

It is a known procedure to write the completely positive and trace preserving map $\Omega$ as a unitary transformation $u$ in an extended space. The dimension of this space is $dim(J) * l_{(+)} + dim(K) * l_{(-)}$, where $l_{(+)}$ and $l_{(-)}$ are the number of eigenmatrices of $\Lambda^{(+)}$ and $\Lambda^{(-)}$ respectively, and $l_{(+)} + l_{(-)} \leq N^2$. 

5
3 Conclusions

We showed that any trace preserving map, whether positive or negative, can be expanded in terms of an extension map and a completely positive map. We also described a procedure to make any map trace preserving.

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