Boundary Reflection Matrix
for $D_{4}^{(1)}$ Affine Toda Field Theory

J. D. Kim

Department of Mathematical Sciences,
University of Durham, Durham DH1 3LE, U.K.

H. S. Cho

Institute for Theoretical Physics,
20, Marshall Terrace, Durham DH1 2HX, U.K.

ABSTRACT

We present one loop boundary reflection matrix for $d_{4}^{(1)}$ Toda field theory defined on a half line with the Neumann boundary condition. This result demonstrates a nontrivial cancellation of non-meromorphic terms which are present when the model has a particle spectrum with more than one mass. Using this result, we determine uniquely the exact boundary reflection matrix which turns out to be ‘non-minimal’ if we assume the strong-weak coupling ‘duality’. 

*jideog.kim@durham.ac.uk
†On leave of absence from Korea Advanced Institute of Science and Technology
I. Introduction

The exact $S$-matrix for integrable quantum field theory defined on a full line has been conjectured using the symmetry principles such as Yang-Baxter equation, unitarity, crossing relation, real analyticity and bootstrap equation. This program entirely relies on the assumed quantum integrability of the model as well as the fundamental assumptions such as strong-weak coupling ‘duality’ and ‘minimality’.

In order to determine the exact $S$-matrix uniquely, Feynman’s perturbation theory has been used and shown to agree well with the conjectured ‘minimal’ $S$-matrices. In perturbation theory, $S$-matrix is extracted from the four-point correlation function with LSZ reduction formalism. Especially, the singularity structures were examined in terms of Landau singularity, of which odd order poles are interpreted as coming from the intermediate bound states.

About a decade ago, integrable quantum field theory defined on a half line ($-\infty < x \leq 0$) was studied using symmetry principles under the assumption that the integrability of the model remains intact. The boundary Yang-Baxter equation, unitarity relation for boundary reflection matrix $K^b(\theta)$ which is conceived to describe the scattering process off a wall was introduced. Recently, boundary crossing relation and boundary bootstrap equation was introduced. Subsequently, some exact boundary reflection matrices have been conjectured for affine Toda field theory (ATFT).

In order to determine the boundary reflection matrix uniquely, we have developed a method in the framework of the Lagrangian quantum field theory with a boundary. The idea is to extract the boundary reflection matrix directly from the two-point correlation function in the coordinate space.

Using this formalism, we determined the exact boundary reflection matrix for sinh-Gordon model ($a_1^{(1)}$ affine Toda theory) and Bullough-Dodd model ($a_2^{(2)}$ affine Toda theory) with the Neumann boundary condition modulo ‘a universal mysterious factor half’. If we assume the strong-weak coupling ‘duality’, these solutions are unique. Above two models have a particle spectrum with only one mass. On the other hand, when the theory has a particle spectrum with more than one mass, each one
loop contribution from different types of Feynman diagrams has non-meromorphic terms.

In this paper, we evaluate one loop boundary reflection matrix for $d_4^{(1)}$ affine Toda field theory and show a remarkable cancellation of non-meromorphic terms among themselves. This result also enables us to determine the exact boundary reflection matrix uniquely under the assumption of the strong-weak coupling ‘duality’. The boundary reflection matrix has singularities which can be accounted for by a new type of singularities of Feynman diagrams for a theory defined on a half line.

In section II, we review the formalism developed in ref.[18]. In section III, we present one loop result for $d_4^{(1)}$ affine Toda theory and determine the exact boundary reflection matrix. We also present the complete set of solutions of the boundary bootstrap equations. Finally, we make conclusions in section IV.

II. Boundary Reflection Matrix

The action for affine Toda field theory defined on a half line ($-\infty < x \leq 0$) is given by

$$S(\Phi) = \int_{-\infty}^{0} dx \int_{-\infty}^{\infty} dt \left( \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - \frac{m^2}{\beta^2} \sum_{i=0}^{r} n_i e^{\beta \alpha_i \cdot \Phi} \right),$$

(1)

where

$$\alpha_0 = - \sum_{i=1}^{r} n_i \alpha_i, \quad n_0 = 1.$$

The field $\phi^a$ ($a = 1, \cdots, r$) is $a$-th component of the scalar field $\Phi$, and $\alpha_i$ ($i = 1, \cdots, r$) are simple roots of a Lie algebra $g$ with rank $r$ normalized so that the universal function $B(\beta)$ through which the dimensionless coupling constant $\beta$ appears in the $S$-matrix takes the following form:

$$B(\beta) = \frac{1}{2\pi} \frac{\beta^2}{(1 + \beta^2/4\pi)}.$$

(2)

The $m$ sets the mass scale and the $n_i$s are the so-called Kac labels which are characteristic integers defined for each Lie algebra.

Here we consider the model with no boundary potential, which corresponds to the Neumann boundary condition: $\frac{\partial \phi^a}{\partial x} = 0$ at $x = 0$. This case is believed to be
quantum stable in the sense that the existence of a boundary does not change the structure of the spectrum.

In classical field theory, it is quite clear how we extract the boundary reflection matrix. It is the coefficient of reflection term in the two-point correlation function namely it is 1.

\[
G_N(t', x'; t, x) = G(t', x'; t, x) + G(t', x'; t, -x) = \int \frac{d^2p}{(2\pi)^2} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-iw(t'-t)}(e^{ik(x'-x)} + e^{ik(x'+x)}).
\]  

We may use the \( k \)-integrated version.

\[
G_N(t', x'; t, x) = \int \frac{dw}{2\pi} \frac{1}{2k} e^{-iw(t'-t)}(e^{ik|x'-x|} + e^{-ik(x'+x)}), \quad \tilde{k} = \sqrt{w^2 - m^2}.
\]  

We find that the unintegrated version is very useful to extract the asymptotic part of the two-point correlation function far away from the boundary.

In quantum field theory, it also seems quite natural to extend above idea in order to extract the quantum boundary reflection matrix directly from the quantum two-point correlation function. This idea has been pursued in ref. [18] to extract one loop boundary reflection matrix.

To compute two-point correlation functions at one loop order, we follow the idea of the conventional perturbation theory [19, 20, 21]. That is, we generate relevant Feynman diagrams and then evaluate each of them by using the zero-th order two-point function for each line occurring in the Feynman diagrams.

At one loop order, there are three types of Feynman diagram contributing to the two-point correlation function as depicted in Figure 1.
For a theory defined on a full line which has translational symmetry in space and time direction, Type I, II diagrams have logarithmic infinity independent of the external energy-momenta and are the only divergent diagrams in 1+1 dimensions. This infinity is usually absorbed into the infinite mass renormalization. Type III diagrams have finite corrections depending on the external energy-momenta and produces a double pole to the two-point correlation function.

The remedy for these double poles is to introduce a counter term to the original Lagrangian to cancel this term (or to renormalize the mass). In addition, to maintain the residue of the pole, we have to introduce wave function renormalization. Then the renormalized two-point correlation function remains the same as the tree level one with renormalized mass $m_o$, whose ratios are the same as the classical value. This mass renormalization procedure can be generalized to arbitrary order of loops.

Now let us consider each diagram for a theory defined on a half line. Type I diagram gives the following contribution:

$$
\int_{-\infty}^{0} dx_1 \int_{-\infty}^{\infty} dt_1 G_N(t, x; t_1, x_1) \ G_N(t', x'; t_1, x_1) \ G_N(t_1, x_1; t_1, x_1).
$$

From Type II diagram, we can read off the following expression:

$$
\int_{-\infty}^{0} dx_1 dx_2 \int_{-\infty}^{\infty} dt_1 dt_2 G_N(t, x; t_1, x_1) \ G_N(t', x'; t_1, x_1) \ G_N(t_1, x_1; t_2, x_2) \ G_N(t_2, x_2; t_2, x_2).
$$

Type III diagram gives the following contribution:

$$
\int_{-\infty}^{0} dx_1 dx_2 \int_{-\infty}^{\infty} dt_1 dt_2 G_N(t, x; t_1, x_1) \ G_N(t', x'; t_2, x_2) \ G_N(t_2, x_2; t_1, x_1).
$$
After the infinite as well as finite mass renormalization, the remaining terms coming from type I,II and III diagrams can be written as follows with different $I_i$ functions:

\[
\int \frac{dw \, dk \, dk'}{2\pi \, 2\pi \, 2\pi} e^{-iw(t' - t)} e^{i(kx + k'x')} \frac{i}{w^2 - k^2 - m_a^2 + i\varepsilon} \frac{i}{w^2 - k'^2 - m_a^2 + i\varepsilon} I_i(w, k, k').
\]

(8)

Contrary to the other terms which resemble those of a full line, this integral has two spatial momentum integration.

In the asymptotic region far away from the boundary, these terms can be evaluated up to exponentially damped term as $x, x'$ go to $-\infty$, yielding the following result for the elastic boundary reflection matrix $K_a(\theta)$ defined as the coefficient of the reflected term of the two-point correlation function.

\[
\int \frac{dw}{2\pi} e^{-iw(t' - t)} \frac{1}{2\bar{k}} (e^{i\bar{k}|x' - x|} + K_a(w) e^{i\bar{k}(x' + x)}) , \quad \bar{k} = \sqrt{w^2 - m_a^2}.
\]

(9)

$K_a(\theta)$ is obtained using $w = m_a \cosh \theta$.

Here we list each one loop contribution to $K_a(\theta)$ from the three types of diagram depicted in Figure 1

\[K_a^{(I)}(\theta) = \frac{1}{4m_a s h \theta} \left( \frac{1}{2\sqrt{m_a^2 s^2 \theta + m_b^2}} + \frac{1}{2m_b} \right) C_1 S_1,
\]

(10)

\[K_a^{(II)}(\theta) = \frac{1}{4m_a s h \theta} \left( \frac{-i}{(4m_a^2 s^2 \theta + m_b^2)2\sqrt{m_a^2 s^2 \theta + m_c^2}} + \frac{-i}{2m_b^2 m_c} \right) C_2 S_2,
\]

(11)

\[K_a^{(III)}(\theta) = \frac{1}{4m_a s h \theta} (4I_3(k_1 = 0, k_2 = \bar{k}) + 4I_3(k_1 = k, k_2 = 0)) C_3 S_3,
\]

(12)

where ‘a universal mysterious factor half’ is included. $C_i, S_i$ denote numerical coupling factors and symmetry factors, respectively. $I_3$ is defined by

\[I_3 \equiv \frac{1}{4} \left( \frac{i}{2\bar{w}_1(\bar{w}_1 - \bar{w}_1^+)(\bar{w}_1 - \bar{w}_1^-)} + \frac{i}{(\bar{w}_1^+ - \bar{w}_1)(\bar{w}_1^+ + \bar{w}_1)(\bar{w}_1^+ - \bar{w}_1^-)} \right),
\]

(13)

where

\[\bar{w}_1 = \sqrt{k_1^2 + m_b^2}, \quad \bar{w}_1^+ = w + \sqrt{k_2^2 + m_c^2}, \quad \bar{w}_1^- = w - \sqrt{k_2^2 + m_c^2}.
\]

(14)
It should be remarked that this term should be symmetrized with respect to \( m_b, m_c \)
with a half.

Let us remark a few interesting points. Firstly, above expressions have non-mmeromorphic terms when the theory has a mass spectrum with more than one mass. Secondly, they have singularities which are absent for the same Feynman diagrams from the theory on a full line. Later, we will see a nontrivial cancellation of non-mmeromorphic terms and the fact that the new type of singularities accounts for the singularities of the exact boundary reflection matrix.

### III. \( d_4^{(1)} \) affine Toda theory

We have to fix the normalization of roots so that the standard \( B(\beta) \) function takes the form given in Eq. (2).

We use the Lagrangian density given as follows.

\[
\mathcal{L} = \sum_{i=1}^{4} \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i - V(\phi), \tag{15}
\]

\[
V(\phi) = \frac{1}{2} m^2 (2\phi_1^2 + 6\phi_2^2 + 2\phi_3^2 + 2\phi_4^2)
+ \frac{1}{\sqrt{2}} m^2 \beta (-\phi_1^2 \phi_2 - \phi_3^2 \phi_2 - \phi_3^2 \phi_4 + \phi_2^3 - 2\phi_1 \phi_3 \phi_4)
+ \frac{1}{24} m^2 \beta^2 (\phi_1^4 + \phi_3^4 + \phi_4^4 + 9\phi_2^4 + 6\phi_1^2 \phi_2^2 + 6\phi_1^2 \phi_3^2 + 6\phi_1^2 \phi_4^2
+ 6\phi_2^2 \phi_3^2 + 6\phi_2^2 \phi_4^2 + 6\phi_3^2 \phi_4^2 + 24\phi_1 \phi_2 \phi_3 \phi_4) + O(\beta^3).
\]

The scattering matrix of this model is given by the following:\[3\].

\[
S_{11}(\theta) = S_{33} = S_{44} = \{1\}\{5\}, \quad S_{22} = \{1\}\{5\}\{3\}\{3\},
S_{12}(\theta) = S_{32} = S_{42} = \{2\}\{4\}, \quad S_{13} = S_{14} = S_{34} = \{3\},
\]

\[
\{x\} = \frac{(x - 1)(x + 1)}{(x - 1 + B)(x + 1 - B)}, \quad \{x\} = \frac{sh(\theta/2 + i\pi x/2h)}{sh(\theta/2 - i\pi x/2h)}.
\]

Here \( B \) is the same function defined in Eq. (3). For this model, \( h = 6 \) and from now on we set \( m = 1 \). Due to the triality symmetry among \( \phi_1, \phi_3 \) and \( \phi_4 \), we have only to consider one of the light particles and the heavy particle. We choose \( \phi_1 \) and \( \phi_2 \).
First, we consider the light particle. For type I diagram, there are four possible configurations three of which yield identical contribution. We follow the notation of Figure 1. For $b = \phi_1, \phi_3$ and $\phi_4$,

$$K_{1}^{(I-1)} = \frac{1}{4\sqrt{2}sh\theta} \left( \frac{1}{2\sqrt{2}ch\theta} + \frac{1}{2\sqrt{2}} \right) \times (-i\beta^2) \times \frac{1}{2} \times 3. \quad (17)$$

For $b = \phi_2$,

$$K_{1}^{(I-2)} = \frac{1}{4\sqrt{2}sh\theta} \left( \frac{1}{2\sqrt{2}sh^2\theta + 6} \right) \times (-i\beta^2) \times \frac{1}{2} \times 2. \quad (18)$$

For type II diagram, there are also four possible configurations three of which yield identical contribution. For $b = \phi_2, c = \phi_1, \phi_3$ and $\phi_4$,

$$K_{1}^{(II-1)} = \frac{1}{4\sqrt{2}sh\theta} \left( \frac{1}{2\sqrt{2}sh\theta + 6} \right) \times (-i\beta^2) \times \frac{1}{2} \times 2 \times 3. \quad (19)$$

For $b = c = \phi_2$,

$$K_{1}^{(II-2)} = \frac{1}{4\sqrt{2}sh\theta} \left( \frac{1}{2\sqrt{2}sh^2\theta + 6} \right) \times (-i\beta^2) \times \frac{1}{2} \times 6. \quad (20)$$

For type III diagram, there are two possible configurations. For $b = \phi_1, c = \phi_2$, when $k_1 = 0, k_2 = \bar{k}$,

$$\bar{w}_1 = \sqrt{2}, \quad \bar{w}_1^+ = \sqrt{2}ch\theta + \sqrt{2}sh^2\theta + 6, \quad \bar{w}_1^- = \sqrt{2}ch\theta - \sqrt{2}sh^2\theta + 6, \quad (21)$$

and when $k_1 = \bar{k}, k_2 = 0$,

$$\bar{w}_1 = \sqrt{2}ch\theta, \quad \bar{w}_1^+ = \sqrt{2}ch\theta + \sqrt{6}, \quad \bar{w}_1^- = \sqrt{2}ch\theta - \sqrt{6}. \quad (22)$$

For the symmetrized configuration of above, when $k_1 = 0, k_2 = \bar{k}$,

$$\bar{w}_1 = \sqrt{6}, \quad \bar{w}_1^+ = 2\sqrt{2}ch\theta, \quad \bar{w}_1^- = 0, \quad (23)$$

and when $k_1 = \bar{k}, k_2 = 0$,

$$\bar{w}_1 = \sqrt{2}sh^2\theta + 6, \quad \bar{w}_1^+ = \sqrt{2}ch\theta + \sqrt{2}, \quad \bar{w}_1^- = \sqrt{2}ch\theta - \sqrt{2}. \quad (24)$$
Inserting above data into Eq. (12), we obtain after some algebra,

\[
K^{(III - 1)}_1 = \frac{i\beta^2}{4\sqrt{2}sh\theta}\left(\frac{1}{4\sqrt{2}(2ch\theta + 1)} + \frac{1}{12\sqrt{2}ch\theta} - \frac{1}{12\sqrt{2}(2ch\theta + \sqrt{3})}\right) \\
- \frac{1}{4\sqrt{2}(2ch\theta - 1)} + \frac{1}{12\sqrt{2}(2ch\theta - \sqrt{3})} = \frac{1}{4\sqrt{2}ch\theta(4ch^2\theta - 3)} \\
+ \frac{4ch^2\theta + 2}{2\sqrt{2}sh^2\theta + 6(8ch^2\theta - 2)}. \tag{25}
\]

For \( b = \phi_3, \ c = \phi_4, \) when \( k_1 = 0, k_2 = \bar{k}, \)

\[
\bar{w}_1 = \sqrt{2}, \quad \bar{w}_1^+ = 2\sqrt{2}ch\theta, \quad \bar{w}_1^- = 0, \tag{26}
\]

and when \( k_1 = \bar{k}, k_2 = 0, \)

\[
\bar{w}_1 = \sqrt{2}ch\theta, \quad \bar{w}_1^+ = \sqrt{2}ch\theta + \sqrt{2}, \quad \bar{w}_1^- = \sqrt{2}ch\theta - \sqrt{2}. \tag{27}
\]

Inserting above data into Eq. (12), we obtain after some algebra,

\[
K^{(III - 2)}_1 = \frac{i\beta^2}{4\sqrt{2}sh\theta}\left(\frac{1}{(2ch\theta - 1)} - \frac{1}{ch\theta(4ch^2\theta - 1)} + \frac{1}{ch\theta} - \frac{1}{(2ch\theta + 1)}\right)\frac{1}{2\sqrt{2}}. \tag{28}
\]

Adding the above contributions as well as the tree result 1, we get

\[
K_1(\theta) = 1 + \frac{i\beta^2}{24}\left(\frac{sh\theta}{ch\theta} - \frac{sh\theta}{ch\theta - 1} + \frac{2sh\theta}{2ch\theta + \sqrt{3}} - \frac{2sh\theta}{2ch\theta - 1}\right) + O(\beta^4). \tag{29}
\]

The non-meromorphic terms exactly cancel among themselves.

Now, we consider the heavy particle. For type I diagram, there are four possible configurations three of which yield identical contribution. For \( b = \phi_1, \phi_3 \) and \( \phi_4, \)

\[
K^{(I - 1)}_2 = \frac{1}{4\sqrt{6}sh\theta}\left(\frac{1}{2\sqrt{6}sh^2\theta + 2} + \frac{1}{2\sqrt{2}}\right) \times \left(-\frac{i\beta^2}{4}\right) \times 2 \times 3. \tag{30}
\]

For \( b = \phi_2, \)

\[
K^{(I - 2)}_2 = \frac{1}{4\sqrt{6}sh\theta}\left(\frac{1}{2\sqrt{6}sh^2\theta + 2} + \frac{1}{2\sqrt{6}}\right) \times \left(-\frac{3i\beta^2}{8}\right) \times 12. \tag{31}
\]

For type II diagram, there are also four possible configurations three of which yield identical contribution. For \( b = \phi_2, c = \phi_1, \phi_3 \) and \( \phi_4, \)

\[
K^{(II - 1)}_2 = \frac{1}{4\sqrt{6}sh\theta}\left(\frac{1}{(24sh^2\theta + 6)2\sqrt{6}sh^2\theta + 2} + \frac{-i}{12\sqrt{2}}\right) \times \left(\frac{1}{2}\beta^2\right) \times 6 \times 3. \tag{32}
\]
For $b = c = \phi_2$, 
\[ K_2^{(III-2)} = \frac{1}{4\sqrt{6}sh\theta} \left( \frac{1}{24sh^2\theta + 6} - \frac{i}{2\sqrt{6}ch\theta} + \frac{-i}{12\sqrt{6}} \right) \times \left( \frac{-1}{2} \beta^2 \right) \times 18. \] (33)

For type III diagram, there are four possible configurations three of which yield identical contributions. For $b = c = \phi_1, \phi_3$ and $\phi_4$, when $k_1 = 0, k_2 = \bar{k}$, 
\[ \bar{w}_1 = \sqrt{2}, \quad \bar{w}_1^+ = \sqrt{6}ch\theta + \sqrt{6}sh^2\theta + 2, \quad \bar{w}_1^- = \sqrt{6}ch\theta - \sqrt{6}sh^2\theta + 2, \] (34)
and when $k_1 = \bar{k}, k_2 = 0$,
\[ \bar{w}_1 = \sqrt{6}sh^2\theta + 2, \quad \bar{w}_1^+ = \sqrt{6}ch\theta + \sqrt{2}, \quad \bar{w}_1^- = \sqrt{6}ch\theta - \sqrt{2}. \] (35)

Inserting above data into Eq. (12), we obtain after some algebra,
\[ K_2^{(III-1)} = \frac{3i\beta^2}{4\sqrt{6}sh\theta} \left( \frac{-1}{4\sqrt{6}(2ch\theta + \sqrt{3})} + \frac{1}{4\sqrt{6}(2ch\theta - \sqrt{3})} + \frac{2(ch^2\theta - 1)}{2\sqrt{6}sh^2\theta + 2(4ch^2\theta - 3)} \right). \] (36)

For $b = c = \phi_2$, when $k_1 = 0, k_2 = \bar{k}$,
\[ \bar{w}_1 = \sqrt{6}, \quad \bar{w}_1^+ = 2\sqrt{6}ch\theta, \quad \bar{w}_1^- = 0, \] (37)
and when $k_1 = \bar{k}, k_2 = 0$,
\[ \bar{w}_1 = \sqrt{6}ch\theta, \quad \bar{w}_1^+ = \sqrt{6}ch\theta + \sqrt{6}, \quad \bar{w}_1^- = \sqrt{6}ch\theta - \sqrt{6}. \] (38)

Inserting above data into Eq. (12), we obtain after some algebra,
\[ K_2^{(III-2)} = \frac{9i\beta^2}{4\sqrt{6}sh\theta} \left( \frac{1}{(2ch\theta - 1)} - \frac{1}{ch\theta(4ch^2\theta - 1)} + \frac{1}{ch\theta} - \frac{1}{(2ch\theta + 1)} \right) \frac{1}{12\sqrt{6}}. \] (39)

Adding the above contributions as well as the tree result 1, we get
\[ K_2(\theta) = 1 + \frac{i\beta^2}{24} \left( \frac{sh\theta}{ch\theta} - \frac{sh\theta}{ch\theta - 1} - \frac{4sh\theta}{2ch\theta - \sqrt{3}} + \frac{2sh\theta}{2ch\theta + \sqrt{3}} + \frac{2sh\theta}{2ch\theta + 1} \right) + O(\beta^4). \] (40)

Once again, the non-meromorphic terms exactly cancel among themselves.
This boundary reflection matrix up to one loop order satisfies the boundary bootstrap equations up to $\beta^2$ order.

\[ K_4(\theta) = K_1(\theta + i\pi 2/6)K_3(\theta - i\pi 2/6)S_{13}(2\theta), \] \hspace{1cm} (41)
\[ K_2(\theta) = K_2(\theta + i\pi 2/6)K_2(\theta - i\pi 2/6)S_{22}(2\theta), \]
\[ K_2(\theta) = K_1(\theta + i\pi/6)K_1(\theta - i\pi/6)S_{11}(2\theta), \]
\[ K_1(\theta) = K_3(\theta) = K_4(\theta). \]

If we consider all possible fusings as above, the boundary crossing unitarity relations are automatically satisfied.

The exact boundary reflection matrix is determined uniquely if we assume the strong-weak coupling ‘duality’.

\[ K_1(\theta) = [1/2][3/2][5/2]^{3/2}[7/2][9/2], \] \hspace{1cm} (42)
\[ K_2(\theta) = [1/2][3/2][5/2][7/2][9/2], \]

where
\[ [x] = \frac{(x - 1/2)(x + 1/2)}{(x - 1/2 + B/2)(x + 1/2 - B/2)}. \] \hspace{1cm} (43)

On the other hand, the most general solution can be written in the following form under the assumption of the strong-weak coupling ‘duality’.

\[ K_1(\theta) = [1/2]^{a_1}[3/2]^{b_1}[5/2]^{c_1}[7/2]^{d_1}[9/2]^{e_1}[11/2]^{f_1}, \] \hspace{1cm} (44)
\[ K_2(\theta) = [1/2]^{a_2}[3/2]^{b_2}[5/2]^{c_2}[7/2]^{d_2}[9/2]^{e_2}[11/2]^{f_2}. \]

Inserting the above into the boundary bootstrap equations, we can obtain linear algebraic relations among the exponents. Solving this system of equations yields the following result.

\[ a_1 = free, \quad b_1 = free, \quad c_1 = a_1 + b_1, \]
\[ d_1 = a_1 + b_1 - 1, \quad e_1 = b_1, \quad f_1 = a_1 - 1, \]
\[ a_2 = -a_1 + b_1 + 1, \quad b_2 = 2a_1 + b_1, \quad c_2 = a_1 + 2b_1, \]
\[ d_2 = a_1 + 2b_1 - 1, \quad e_2 = 2a_1 + b_1 - 2, \quad f_2 = -a_1 + b_1. \] \hspace{1cm} (45)
IV. Conclusions

In this paper, we computed the boundary reflection matrix for $d_4^{(1)}$ affine Toda field theory up to one loop order in order to demonstrate a remarkable cancellation of non-meromorphic terms which are always present for each diagram when the model has a particle spectrum with more than one mass.

Using this result, we also determined the exact boundary reflection matrix under the assumption of the strong-weak coupling ‘duality’, which turned out to be ‘non-minimal’. We also presented the complete set of solutions of the boundary bootstrap equations. Finally, we remark that Feynman diagrams which have no singularity for the theory on a full line produce a new type of singularities.

Acknowledgement

One(JDK) of the authors would like to thank Ed Corrigan and Ryu Sasaki. He is also grateful to Choonkyu Lee, Soonkwon Nam and Changrim Ahn for encouragement. This work was supported by Korea Science and Engineering Foundation and in part by the University of Durham.
References

[1] Al.B. Zamolodchikov and A.B. Zamolodchikov, Ann. Phys. 120 (1979) 253.

[2] A.E. Arinshtein, V.A. Fateev and A.B. Zamolodchikov, Phys. Lett. B 87 (1979) 389.

[3] H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, Nucl. Phys. B 338 (1990) 689.

[4] P. Christe and G. Mussardo, Int. J. Mod. Phys. A 5 (1990) 4581.

[5] G.W. Delius, M.T. Grisaru and D. Zanon, Nucl. Phys. B 382 (1992) 365.

[6] H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, Nucl. Phys. B 356 (1991) 469.

[7] H.W. Braden and R. Sasaki, Phys. Lett. B 255 (1991) 343; Nucl. Phys. B 379 (1992) 377.

[8] H.S. Cho, J.D. Kim and I.G. Koh, J. Math. Phys. 33 (1992) 2889.

[9] H.W. Braden, H.S. Cho, J.D. Kim, I.G. Koh and R. Sasaki, Prog. Theor. Phys. 88 (1992) 1205.

[10] R. Sasaki and F.P. Zen, Int. J. Mod. Phys. A 8 (1992) 115.

[11] R.J. Eden, P.V. Landshoff, D.I. Olive and J.C. Polkinghorne, The analytic S matrix, (Cambridge University Press 1966).

[12] I.V. Cherednik, Theor. Math. Phys. 61 (1984) 977.

[13] S. Ghoshal and A.B. Zamolodchikov, Int. J. Mod. Phys. A 9 (1994) 3841; Int. J. Mod. Phys. A 9 (1994) 4353.

[14] A. Fring and R. Köberle Nucl. Phys. B 421 (1994) 159; Nucl. Phys. B 419 (1994) 647.
[15] S. Ghoshal, Int. J. Mod. Phys. A 9 (1994) 4801, hep-th/9310188.

[16] R. Sasaki, YITP/U-93-33, hep-th/9311027 in Interface between Physics and Mathematics, eds. W. Nahm and J-M. Shen, World Scientific (1994) 201.

[17] E. Corrigan, P.E. Dorey, R.H. Rietdijk and R. Sasaki, Phys. Lett. B 333 (1994) 83.

[18] J.D. Kim, “Boundary Reflection Matrix in Perturbative Quantum Field Theory”, DTP/95-11, hep-th/9504018, to appear in Phys. Lett. B.

[19] K. Symanzik, Nucl. Phys. B 190 (1981) 1.

[20] H.W. Diehl and S. Dietrich, Z. Phys. B 50 (1983) 117.

[21] M. Benhamou and G. Mahoux, Nucl. Phys. B 305 (1988) 1.