ASYMPTOTIC STABILITY OF SHOCK PROFILES AND RAREFACTION WAVES UNDER PERIODIC PERTURBATIONS FOR 1-D CONVEX SCALAR VISCOUS CONSERVATION LAWS

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Abstract. We study the asymptotic stability of shock profiles and rarefaction waves under periodic perturbations for 1-D convex scalar viscous conservation laws. For the shock profile, the solution converges to the background shock profile with a constant shift in $L^p$ norm as $t \to +\infty$, with an exponential decay rate. And the shift is essentially due to the viscosity and the periodic perturbation, which may not be zero in general. However, the shift will disappear as viscosity vanishes. The idea is to construct an approximate solution which tends to two periodic solutions as $x \to \pm \infty$ respectively, and then use the anti-derivative variable method and the maximum principle. Moreover, for the rarefaction wave, the solution converges to the background rarefaction wave in $L^p$ norm as $t \to +\infty$.

1. Introduction

The 1-D convex scalar viscous conservation law reads:

$$
\partial_t u + \partial_x f(u) = \partial_x^2 u, \quad \forall x \in \mathbb{R}, \quad t > 0,
$$

here the flux $f(u)$ is smooth and strictly convex. A shock profile $\phi(x-st)$ is a classical traveling wave solution to the equation (1.1), satisfying the ODE:

$$
\begin{align*}
\phi'' &= f'(\phi)\phi' - s\phi', \\
\lim_{x \to -\infty} \phi(x) &= \overline{u}_l, \quad \lim_{x \to +\infty} \phi(x) = \overline{u}_r,
\end{align*}
$$

where $\overline{u}_l > \overline{u}_r$ are two given constants and $s$ is the shock speed defined by the Rankine-Hugoniot condition:

$$
s = \frac{f(\overline{u}_l) - f(\overline{u}_r)}{\overline{u}_l - \overline{u}_r}.
$$

The existence of the shock profile $\phi$ can follow from the center-manifold theorem in Kopell-Howard [14]. And it was also shown that $\phi$ tends to the end states $\overline{u}_l$ and $\overline{u}_r$ exponentially fast as $x \to -\infty$ and $+\infty$ respectively. A rarefaction wave $u^R(x,t)$ centered at the origin is

$$
u^R(x,t) = \begin{cases} 
\overline{u}_l, & \text{if } \frac{x}{t} < f'(\overline{u}_l); \\
(f')^{-1}(\frac{x}{t}), & \text{if } f'(\overline{u}_l) \leq \frac{x}{t} \leq f'(\overline{u}_r); \\
\overline{u}_r, & \text{if } \frac{x}{t} > f'(\overline{u}_r),
\end{cases}
$$

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where \( \overline{u}_I < \overline{u}_r \) are two given constants. Recall that \( u^R \) is an entropy weak solution to the inviscid conservation law \( \partial_t u + \partial_x f(u) = 0 \).

In this paper we are concerned with the asymptotic stability of shock profiles and rarefaction waves under arbitrary periodic perturbation \( w_0(x) \in L^\infty(\mathbb{R}) \), which makes the initial data \( \phi(x) + w_0(x) \) or \( u^R(x,0) + w_0(x) \) keep oscillating as \( |x| \to \infty \). In order to assume without loss of generality in the case of shock profile that \( w_0(x) \) has zero average:

\[
(1.4) \quad \frac{1}{p} \int_0^p w_0(x)dx = 0,
\]

we will consider the following more general initial data.

- For the shock profile \( \phi \), we consider the initial data

\[
(1.5) \quad u(x,0) = u_0(x) = \phi(x) + w_0(x) + v_0(x), \quad \forall x \in \mathbb{R},
\]

where \( w_0 \in L^\infty \) is periodic, \( v_0(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) decays exponentially as \( |x| \to \infty \), say, there exist constants \( A_0, \beta_0 > 0 \), such that

\[
(1.6) \quad |v_0(x)| \leq A_0 e^{-\beta_0 |x|}, \quad \text{a.e. } x \in \mathbb{R}.
\]

Then one can assume without loss of generality in (1.5) that the perturbation \( w_0 \) satisfies (1.4) and \( v_0 \) has zero mass:

\[
(1.7) \quad \int_{-\infty}^{+\infty} v_0(x)dx = 0.
\]

In fact, for \( \overline{w} = \frac{1}{p} \int_0^p w_0(x)dx \) and \( \delta = \int_{-\infty}^{+\infty} v_0(x)dx \), one can let \( \tilde{\phi}(x) \) be the shock profile solving (1.1) connecting the states \( \overline{u}_I + \overline{w} \) as \( x \to -\infty \) and \( \overline{u}_r + \overline{w} \) as \( x \to +\infty \). Then one has that \( u_0 = \tilde{\phi} + (w_0 - \overline{w}) + v_0 + (\phi + \overline{w} - \tilde{\phi}) \), where \( \phi + \overline{w} - \tilde{\phi} \) decays exponentially as \( |x| \to \infty \). Then one can choose a shift \( x_0 \) to make the integral

\[
\int_{-\infty}^{+\infty} \left( \phi(x) + \overline{w} - \tilde{\phi}(x + x_0) \right)dx + \int_{-\infty}^{+\infty} v_0(x)dx = 0.
\]

Thus, it holds that \( u_0(x) = \tilde{\phi}(x + x_0) + \tilde{w}_0(x) + \tilde{v}_0(x) \), where \( \tilde{w}_0(x) = w_0(x) - \overline{w} \) satisfies (1.4), and \( \tilde{v}_0(x) = \phi(x) + \overline{w} - \tilde{\phi}(x + x_0) + v_0(x) \) decays exponentially as \( |x| \to \infty \) and satisfies (1.7).

- For the rarefaction wave \( u^R \), we also consider the kind of initial data

\[
(1.8) \quad u(x,0) = u_0(x) = u^R(x,0) + w_0(x) + v_0(x), \quad \forall x \in \mathbb{R}.
\]

Similarly, by adding a constant on \( \overline{u}_I \) and \( \overline{u}_r \), one can assume without loss of generality that \( w_0 \) satisfies (1.4). And there is no need here to assume that \( v_0 \) satisfies (1.7), since rarefaction waves with different centers are asymptotic equivalent in \( L^\infty(dx) \).

The asymptotic stability of the shock profiles and rarefaction waves in viscous conservation laws have been widely studied so far. Far-reaching results have been obtained not only for the 1-D scalar case, see Hopf [10], Il’in-Oleinik [12], Nishihara [22], Harabetian [7], Freistühler-Serre [2] and Howard [11] for details, but also for the Multi-D scalar case, such as Goodman [5], Xin [26], Hoff-Zumbrun [8, 9] and Kenig-Merle [13]. Besides, there are also many results for the 1-D system case, such as Matsumura-Nishihara [21], Goodman [4], Liu [18, 19], Xin [25], Liu-Xin [20] and Szepessy-Xin [24].
However, in the literatures above, the perturbations are all in $L^1$, and Il’in-Oleńik [12] even constructed a counter example to show that the initial perturbation around the shock profile being not summable may result in instability. The perturbations which keep oscillating as $|x| \to \infty$ have never been studied before. For periodic perturbations, the authors in [27] used the generalized characteristic method to achieve the asymptotic stability of shock waves and rarefaction waves in the inviscid case. And in this paper, we can also prove the corresponding asymptotic stability for the viscous case.

For the shock profile under periodic perturbations, the solution is proved to converge to the background shock profile with a constant shift $\phi(x-st+X_\infty)$ as $t \to +\infty$, where $X_\infty$ is the limit of $X(t) - st$ (see the definition in (2.14)). Although $X(t)$ depends on the background shock profile $\phi$, its limit $X_\infty$ only depends on $\varpi_t, \varpi_r, f, w_0(x)$, and may not be zero in general (see Theorem 2.6). And if we consider the vanishing viscosity limit $\nu \to 0+$ in the equation $\partial_t u + \partial_x f(u) = \nu \partial_x^2 u$, the corresponding constant shift $X_\nu^\infty$ converges to zero, which is compatible with the result in [27], namely, for the inviscid conservation laws, the shock wave under a periodic perturbation satisfying (1.4) converges to itself with no shift as $t \to +\infty$. For the rarefaction wave under a periodic perturbation, it is proved that the solution tends to the background one in $L^\infty$ norm as $t \to +\infty$ in both inviscid and viscous cases.

In this paper, we use the method of maximum principle and construct auxiliary functions, which follows the idea of [12]. The main difficulty is that the solution subtracted by the shock profile is not integrable anymore, leading to the non-existence of its anti-derivative variable, which plays a crucial role in the study of the long time behavior of the shock profile under $L^1$ perturbations. In fact, if one considers the equation of $u - \phi$, there is a zero-order term with its coefficient $f''(\phi)\phi' < 0$, which breaks the maximum principle and voids the energy method.

The novelty of our proof is that we find an approximate solution $\psi_{X(t)}(x,t)$, where $X(t)$ is a shift function (see (2.14)), such that the difference $u - \psi_X$ is integrable and has zero mass at any $t \geq 0$. Therefore, it is possible to study the equation of the anti-derivative of the difference $u - \psi_X$. Although $\psi_X$ is not a solution to (1.1), its source term (see (2.10)) decays exponentially both in space and in time, which makes it possible to construct an auxiliary function and use the maximum principle to achieve our main result.

2. MAIN RESULTS

It is well-known from [23, 16] that the equation (1.1) generates a semi-group \{ $S_t : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$; $t \geq 0$ \} to give that, for any initial data $u_0 \in L^\infty(\mathbb{R})$, the function $u(x,t) := S_t u_0$ is the unique bounded solution to (1.1), which is smooth for $t > 0$, and satisfies the initial condition in the weak sense: for any continuous functions $\varphi(x,t)$ compactly supported in $\mathbb{R} \times [0, +\infty)$, there holds

\[(2.1) \quad \int_{\mathbb{R}} [\varphi(x,t) u(x,t) - \varphi(x,0) u_0(x)] \, dx \to 0, \quad \text{as } t \to 0.\]

Here the semi-group $S_t$ satisfies the following classical Co-properties:

- (Comparison) If $u_0, v_0 \in L^\infty(\mathbb{R})$ and $u_0 \leq v_0$ almost everywhere, then $S_t u_0 \leq S_t v_0$ for any $x \in \mathbb{R}$, $t > 0$.
- (Contraction) If $u_0, v_0 \in L^\infty(\mathbb{R})$ and $u_0 - v_0 \in L^1(\mathbb{R})$, then $S_t u_0 - S_t v_0 \in L^1(\mathbb{R})$ and $\|S_t u_0 - S_t v_0\|_{L^1(\mathbb{R})}$ is non-increasing with respect to $t$. 

Here the constants which is finite and may not be zero in general. More precisely,
\begin{equation}
(2.2) \quad \partial_x u(x,t) \leq \frac{E}{t}, \quad \forall x \in \mathbb{R}, \ t > 0.
\end{equation}

Moreover, there exists a constant $E > 0$ depending on $f, \|u_0\|_{L^\infty}$, such that
\begin{equation}
(2.3) \quad u_l := S_t(\overline{u}_l + w_0), \quad u_r := S_t(\overline{u}_r + w_0),
\end{equation}

then by the comparison property, it holds that if $\overline{u}_l > \overline{u}_r$,
\begin{equation}
(2.4) \quad u_l(x,t) \geq u_r(x,t), \quad \forall x \in \mathbb{R}, \ t > 0.
\end{equation}

Now we give the main theorems of this paper as follows:

**Theorem 2.1.** When $\overline{u}_l > \overline{u}_r$, for the initial data (1.5), if the periodic perturbation $w_0(x) \in L^\infty(\mathbb{R})$ with period $p > 0$ satisfies (1.4) and the perturbation $v_0(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfies (1.6) and (1.7), then there exist constants $C > 0, \mu > 0$ such that the unique bounded solution $u$ to (1.1), (1.5) satisfies
\begin{equation}
(2.5) \quad \sup_{x \in \mathbb{R}} \|u(x,t) - \phi(x - st - X_\infty)\| \leq Ce^{-\mu t}, \quad \forall t \geq 0,
\end{equation}

with the constant shift $X_\infty$ defined by
\begin{equation}
(2.6) \quad X_\infty := \frac{1}{p(\overline{u}_l - \overline{u}_r)} \left\{ \int_0^{+\infty} \int_0^p [f(u_l) - f(\overline{u}_l)] \, dx dt - \int_0^{+\infty} \int_0^p [f(u_r) - f(\overline{u}_r)] \, dx dt \right\},
\end{equation}

which is finite and may not be zero in general. More precisely,

1. for the viscous Burgers’ equation, i.e. $f(u) = u^2/2$, it holds that $X_\infty = 0$;
2. for any periodic perturbation $w_0$ satisfying (1.4), if
\begin{equation}
0 < \|w_0\|_{L^\infty(\mathbb{R})} < (\overline{u}_l - \overline{u}_r)/2,
\end{equation}

then there exists a smooth and strictly convex flux $f$, such that $X_\infty \neq 0$.

Here the constants $C$ and $\mu$ depend on $p, f, \overline{u}_l, \overline{u}_r, A_0, \beta_0, \|w_0\|_{L^\infty}$.

**Remark 2.2.** By (2.6), the constant shift $X_\infty$ only depends on $f, \overline{u}_l, \overline{u}_r, w_0$. Meanwhile, when considering the vanishing viscosity limit $\nu \to 0+$ in the equation $\partial_t u^\nu + \partial_x f(u^\nu) = \nu \partial_x^2 u^\nu$, although Theorem 2.1 shows that the corresponding constant shift $X_\infty^\nu$ may not be 0 in general, it can be proved that $X_\infty^\nu$ converges to 0 as $\nu \to 0+$, which is compatible with the result obtained in [27]. See the detailed explanations in Section 7.

**Theorem 2.3.** When $\overline{u}_l < \overline{u}_r$, for the initial data (1.8), if the periodic perturbation $w_0(x) \in L^\infty(\mathbb{R})$ with period $p > 0$ satisfies (1.4), and the perturbation $v_0(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfies (1.6), then the unique bounded solution $u$ to (1.1), (1.8) satisfies
\begin{equation}
\sup_{x \in \mathbb{R}} |u(x,t) - u^R(x,t)| \to 0 \quad \text{as} \ t \to \infty.
\end{equation}
Before giving the proof, we firstly introduce some notations.

**Notations:** For the shock profile φ, we define the function:

\[(2.7)\]
\[g(x) := \frac{\phi(x) - \overline{u}_r}{\overline{u}_l - \overline{u}_r}.\]

Then it is easy to check that \( g \in C^\infty(\mathbb{R}) \) is decreasing with \( 0 < g < 1 \), and \( g(x) \to 1 \) as \( x \to -\infty \), \( g(x) \to 0 \) as \( x \to +\infty \).

For any \( C^1 \) curve \( \xi(t) : [0, +\infty) \to \mathbb{R} \), denote the shifted function:

\[g_\xi(x) := g(x - \xi(t)),\]

with the derivatives:

\[g_\xi^{(k)}(x) := g^{(k)}(x - \xi(t)), \quad k \geq 1.\]

Motivated by the formula of the shifted viscous shock profile:

\[\phi_\xi(x) := \phi(x - \xi(t)) = \overline{u}_l \, g_\xi(x) + \overline{u}_r (1 - g_\xi(x)),\]

in this paper we construct the approximate solution as

\[(2.8)\]
\[\psi_\xi(x, t) := u_l(x, t) \, g_\xi(x) + u_r(x, t) (1 - g_\xi(x)).\]

Note that for \( \psi_\xi \), the shift function \( \xi(t) \) only appears in \( g \). Then \( \psi_\xi \) satisfies the equation:

\[(2.9)\]
\[\partial_t \psi_\xi + \partial_x f(\psi_\xi) - \partial_x q^2 \psi_\xi = h_\xi,\]

where the source term \( h_\xi \) can be calculated directly by the equations of \( u_l \) and \( u_r \), which is given by

\[(2.10)\]
\[h_\xi = \partial_x [f(\psi_\xi) - f(u_l) \, g_\xi + f(\psi_\xi) - f(u_r) \, (1 - g_\xi) - 2 \partial_x (u_l - u_r) g_\xi' - (u_l - u_r) (g_\xi' \, \xi' + g_\xi'')].\]

By easy calculations on (2.10), one can have that

\[(2.11)\]
\[h_\xi = \partial_x \left[ \left( f(\psi_\xi) - f(u_l) \right) \, g_\xi + \left( f(\psi_\xi) - f(u_r) \right) (1 - g_\xi) + 2 \partial_x (u_l - u_r) g_\xi' - (u_l - u_r) \left( f'(\psi_\xi) \, g_\xi' - g_\xi' \, \xi' - g_\xi'' \right) \right].\]

And

\[h_\xi = \left( f'(\psi_\xi) - f'(u_l) \right) \partial_x u_l \, g_\xi + \left( f'(\psi_\xi) - f'(u_r) \right) \partial_x u_r (1 - g_\xi)
- 2 \partial_x (u_l - u_r) g_\xi' + (u_l - u_r) \left( f'(\psi_\xi) \, g_\xi' - g_\xi' \, \xi' - g_\xi'' \right),\]

then by \( g_\xi'' = f'(\phi_\xi) g_\xi' - s g_\xi' \) (see (3.1)), it holds that

\[(2.12)\]
\[h_\xi = \left( f'(\psi_\xi) - f'(u_l) \right) \partial_x u_l \, g_\xi + \left( f'(\psi_\xi) - f'(u_r) \right) \partial_x u_r (1 - g_\xi)
- 2 \partial_x (u_l - u_r) g_\xi' + (u_l - u_r) \left( f'(\psi_\xi) \, g_\xi' - f'(\psi_\xi) \right) \partial_x u_r (1 - g_\xi)
- 2 \partial_x (u_l - u_r) g_\xi' + (u_l - u_r) \left( f'(\psi_\xi) \, g_\xi' - f'(\psi_\xi) \right) g_\xi'.\]

The formulas (2.11) and (2.12) will be used later.

The terms appearing in the square brackets of (2.11) vanish as \(|x| \to \infty\). Then regarding the equation

\[(2.13)\]
\[\partial_t (u - \psi_X) + \partial_x \left( f(u) - f(\psi_X) \right) = \partial_x^2 (u - \psi_X) - h_X,\]
we can formally choose a curve $X(t)$ to make $\int_{\mathbb{R}} h_X(x,t) dx = 0$. By integrating (2.11) in $x$, the curve $X(t)$ is determined by the ODE:

$$
\begin{aligned}
X'(t) &= \int_{-\infty}^{+\infty} \left[ (u_l - u_r) g'_X + (f(u_l) - f(u_r)) g'_X \right] dx, \quad t > 0, \\
X(0) &= 0,
\end{aligned}
$$

where the initial data $X(0) = 0$ is chosen to make $\int_{\mathbb{R}} (u - \psi_X(0))(x,0) dx = 0$. Then by Cauchy-Lipschitz theorem, (2.14) has a unique $C^1$ solution $X(t) : [0, +\infty) \to \mathbb{R}$, see Lemma 4.5 for details. Then formally, by integrating (2.13) with respect to $x$, one can have that the perturbation $u - \psi_X$ always has zero mass at any time $t \geq 0$, which can make its anti-derivative variable decays exponentially as $|x| \to +\infty$. Besides, the choice of $X(t)$ can make the source term $h_X(x,t)$ and its anti-derivative variable $\int_{-\infty}^{x} h_X(y,t) dy$ decay exponentially fast both in space and in time (see Proposition 4.6 for details), which plays an important role in our proof of Theorem 2.1.

Under the notations above, there hold the following results.

**Proposition 2.4.** Under the assumptions of Theorem 2.1, there exist constants $C > 0$ and $\alpha > 0$ such that the unique $C^1$ solution $X(t)$ to (2.14) satisfies

$$
|X(t) - st - X_{\infty}| \leq Ce^{-\alpha t}, \quad \forall \ t \geq 0,
$$

where $X_{\infty}$ is defined in (2.6). Moreover, if $f(u) = u^2/2$, then at each time $t_k = kp/(\overline{u}_l - \overline{u}_r)$, $k = 0, 1, 2, \cdots$, it holds that

$$
X(t_k) = st_k.
$$

Here $\alpha$ only depends on $p$, and $C$ depends on $p, \overline{u}_l, \overline{u}_r, \|w_0\|_{L^\infty}$.

**Remark 2.5.** Although (2.14) implies that $X(t)$ depends on the viscous shock $\phi$ (since for any number $\delta \in \mathbb{R}$, $\phi(x + \delta)$ is a viscous shock connecting $\overline{u}_l$ as $x \to -\infty$ and $\overline{u}_r$ as $x \to +\infty$, and thus the curve $X(t)$ may depend on $\delta$), the limit of $X(t) - st$, which is $X_{\infty}$ given in (2.6), does not depend on $\phi$, and it only depends on $p, \overline{u}_l, \overline{u}_r, f$ and the periodic perturbation $w_0$.

**Theorem 2.6.** Under the assumptions of Theorem 2.1, there exist constants $C > 0$ and $\mu > 0$ such that the unique bounded solution $u$ of (1.1), (1.5) satisfies

$$
\sup_{x \in \mathbb{R}} |u(x,t) - \psi_X(t)(x,t)| \leq Ce^{-\mu t}, \quad \forall \ t \geq 0.
$$

Moreover, if $f(u) = u^2/2$ and the $L^1$ perturbation $v_0(x) \equiv 0$ in (1.5), then at each time $t_k = kp/(\overline{u}_l - \overline{u}_r)$, $k = 0, 1, 2, \cdots$, it holds that

$$
u(x, t_k) = \psi_{st_k}(x, t_k), \quad \forall x \in \mathbb{R}.
$$

Here the constants $C$ and $\mu$ depend on $p, \overline{u}_l, \overline{u}_r, A_0, \beta_0, \|w_0\|_{L^\infty}$.

**Remark 2.7.** By (2.16) and (2.18), one has that for the Burgers’ equation, if $v_0(x) \equiv 0$, then $u(x, t_k) = \psi_X(u_k)(x, t_k)$, $\forall k \geq 0$. So the result (2.17) is compatible with (2.18), and it implies that the approximate solution $\psi_X$ is a good choice to approach the real solution.

This paper proceeds as follows: In Section 3, we state some properties on $g(x)$ and the stability of periodic solutions with exponential decay rates in time. In Section 4, Theorem 2.6 is proved by 4 steps, and we place each step in one subsection. Moreover,
the existence and uniqueness of \( X(t) \) can be found in Lemma 4.5. In Section 5 we prove Proposition 2.4. In Section 6, Theorem 2.1 can be verified from Proposition 2.4, Theorem 2.6, and a strictly convex flux \( f \) is constructed to make \( X_\infty \neq 0 \). In Section 7, we give the vanishing viscosity limit for the shift function, which is related to the results obtained in [27]. At last, the proof of Theorem 2.3 is presented in Section 8.

3. Preliminaries

In this section, we give the preliminaries on the decay rates of \( g, g' \) as \( |x| \to \infty \), and the exponential decay rates of periodic solutions to the viscous conservation laws as \( t \to \infty \).

By integrating the equation (1.2), the shock profile \( \phi \) satisfies that
\[
\phi' = f(\phi) - f(\overline u_t) - s(\phi - \overline u_t),
\]
which implies that
\[
\phi' = (\phi - \overline u_t) \left( \frac{f(\phi) - f(\overline u_t)}{\phi - \overline u_t} - s \right) = (\phi - \overline u_r) \left( \frac{f(\phi) - f(\overline u_r)}{\phi - \overline u_r} - s \right).
\]
thus one has that the function \( g(x) \) defined in (2.7) satisfies the equation:
\[
g' = \frac{1}{\overline u_t - \overline u_r} \left[ f((\overline u_t - \overline u_r)g + \overline u_r) - f(\overline u_r) \right] - sg
\]
\[
= \frac{1}{\overline u_t - \overline u_r} \left[ f(\overline u_t g + \overline u_r (1 - g)) - f(\overline u_t g - f(\overline u_r) (1 - g)) \right].
\]

Proposition 3.1. The function \( g \) satisfies
(i) There exist positive constants \( \beta_1, \beta_2 \) depending on \( f, \overline u_t, \overline u_r \) such that
\[
\beta_1 \leq \frac{-g'(x)}{(\overline u_t - \overline u_r)g(x)(1 - g(x))} \leq \beta_2, \quad \forall x \in \mathbb{R}.
\]
(ii) With the inequality (3.2), there exist a constant \( C > 0 \) depending on \( f, \overline u_t, \overline u_r \) such that
\[
\frac{1}{C} e^{-\beta_2 x} \leq g(x) \leq C e^{-\beta_1 x}, \quad \forall x > 0,
\]
\[
\frac{1}{C} e^{\beta_2 x} \leq 1 - g(x) \leq C e^{\beta_1 x}, \quad \forall x < 0,
\]
\[
\frac{1}{C} e^{\beta_2 |x|} \leq -g'(x) \leq C e^{-\beta_1 |x|}, \quad \forall x \in \mathbb{R}.
\]

The proof can be found in [6]. And we give a simplified proof for the scalar viscous conservation law, which is placed in the Appendix A.

Proposition 3.2. For any periodic initial data \( u_0(x) \in L^\infty(\mathbb{R}) \) with its average \( \overline u = \frac{1}{p} \int_0^p u_0(x) \, dx \), there exists a constant \( \alpha > 0 \), such that for any integers \( k, l \geq 0 \), the periodic solution \( u(x,t) \) to (1.1) satisfies that
\[
\| \partial_x^k \partial_t^l (u - \overline u) \|_{L^\infty(\mathbb{R})} \leq C e^{-\alpha t}, \quad \forall t \geq 1,
\]
here \( \alpha \) depends only on \( p \), and \( C \) depends on \( k, l, p, f, \| u_0 - \overline u \|_{L^\infty} \).

The proof of Proposition 3.2 can be obtained by standard energy estimates and the Poincaré inequality on \([0, p]\). We place it in the Appendix B. By Proposition 3.2, one has the following corollary
Corollary 3.3. The periodic solutions \( u_l \) and \( u_r \) defined in (2.3) satisfy that for any integers \( k, l \geq 0 \),
\[
\|e^{t\xi}(u_l - \bar{u}_l)\|_{L^\infty(\mathbb{R})}, \|e^{t\xi}(u_r - \bar{u}_r)\|_{L^\infty(\mathbb{R})} \leq Ce^{-\alpha t}, \quad \forall t \geq 1,
\]
here the constant \( C > 0 \) depends on \( k, l, p, f, \bar{u}_l, \bar{u}_r, \|w_0\|_{L^\infty} \). Moreover, by (3.5), there exists a large time \( T_0 \geq 1 \), such that
\[
\frac{1}{2}(\bar{u}_l - \bar{u}_r) \leq u_l(x, t) - u_r(x, t) \leq \frac{3}{2}(\bar{u}_l - \bar{u}_r), \quad \forall t \geq T_0, \ x \in \mathbb{R},
\]
here the constant \( T_0 \) depends on \( p, f, \bar{u}_l, \bar{u}_r, \|w_0\|_{L^\infty} \).

In the following part of this paper, for convenience, one can choose a small constant \( 0 < \beta \leq \min\{\beta_0, \beta_1\} \), such that
\[
|v_0(x)| \leq A_0e^{-\beta|x|}, \quad a.e. \ x \in \mathbb{R},
\]
and without special description, we will use \( C \) as the generic positive constant only depending on \( p, f, \bar{u}_l, \bar{u}_r, A_0, \beta_0 \) and \( \|w_0\|_{L^\infty(\mathbb{R})} \).

4. Proof of Theorem 2.6

The proof of Theorem 2.6 consists of 4 steps. Firstly, at any \( t > 0 \), the behaviors of the solution \( u(x, t) \) as \( |x| \to \infty \) is studied (Corollary 4.2). We further extend the estimates in Corollary 4.2 to time-independent ones (Proposition 4.4). In the third step, motivated by [12], the equation of the anti-derivative variable of \( u - \psi_x \) is studied and the comparison principle is applied to prove (2.17). Finally, we will prove (2.18) by using the Hopf formula introduced in [10].

4.1. Behaviors of \( u(x, t) \) as \( |x| \to \infty \).

Denote \( K^t(x) := \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}} \) to be the heat kernel.

Lemma 4.1. If \( u_0, \tilde{u}_0 \in L^\infty(\mathbb{R}) \) and there exist constants \( A > 0, \ \delta \in \mathbb{R} \), such that
\[
|u_0(x) - \tilde{u}_0(x)| \leq Ae^\delta x, \quad a.e. \ x \in \mathbb{R},
\]
then there exists a constant \( t_0 > 0 \) such that
\[
|S_t u_0 - S_t \tilde{u}_0| \leq 2\lceil \frac{\delta}{2} \rceil Ae^\delta x, \quad \forall x \in \mathbb{R}, \ t > 0,
\]
here \( t_0 \) depends on \( |\delta|, f, \|u_0, \tilde{u}_0\|_{L^\infty} \), and \( \lceil \cdot \rceil \) represents the ceiling function (mapping \( x \) to the smallest integer greater than or equal to \( x \)).

Proof. As quoted by [2], the solution \( S_t u_0 \) can be obtained by constructing the following approximating sequence
\[
\begin{align*}
u^{(1)} &= K^t * u_0, \\
u^{(n+1)} &= K^t * u_0 - \int_0^t \partial_x K^r(\cdot) * f(u^{(n)}(\cdot, t - \tau)) d\tau, \quad n = 1, 2, 3, \ldots
\end{align*}
\]
where “*” represents the convolution operation with respect to the space variable.

Suppose that \( \{\tilde{u}^{(n)}\}_{n=1}^\infty \) is the approximating sequence induced by \( \tilde{u}_0 \), which is constructed in the same way with \( u^{(n)} \). Therefore, one has that
\[
|u^{(1)} - \tilde{u}^{(1)}| = |K^t * (u_0 - \tilde{u}_0)| \leq \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} Ae^{\delta(x-y)}dy
\]
\[
\begin{align*}
|u^{(2)}(t) - \tilde{u}^{(2)}(t)| & \leq |K(t) * (u_0 - \tilde{u}_0)| + \left| \int_0^t \partial_x K(\cdot) \cdot \left[ f\left(u^{(1)}(\cdot, t - \tau)\right) - f\left(\tilde{u}^{(1)}(\cdot, t - \tau)\right) \right] d\tau \right| \\
& \leq A e^{\delta x + \delta t} + \frac{1}{\sqrt{2\pi t}} \int_0^t \frac{|y|}{2\pi} e^{-\frac{x^2}{4\tau}} C_0 A e^{\delta (x - y) + \delta (t - \tau)} dy d\tau \\
& \leq A e^{\delta x + \delta t} + C_0 A e^{\delta x + \delta t} \int_0^t \frac{1}{\sqrt{2\pi \tau}} \frac{|y|}{2\pi} e^{-\frac{(x + 2\tau)^2}{4\tau}} dy d\tau \\
& \leq A e^{\delta x + \delta t} + C_0 A e^{\delta x + \delta t} \int_0^t \left( \sqrt{\frac{2}{\pi \tau}} + |\delta| \right) e^{-\frac{x^2}{\pi \tau}} dy d\tau \\
& = A e^{\delta x + \delta t} + C_0 A e^{\delta x + \delta t} \int_0^t \left( \sqrt{\frac{2}{\pi \tau}} + |\delta| \right) d\tau \\
& \leq A e^{\delta x + \delta t} \left(1 + \frac{2\sqrt{2}}{\sqrt{\pi}} C_0 \sqrt{t} + C_0 |\delta| t\right),
\end{align*}
\]

here \(C_0 := \max\{|f'(u)| : |u| \leq \|u_0, \tilde{u}_0\|_{L^\infty(\mathbb{R})}\}.\) By induction, one has that for \(t > 0\) small,

\[
|u^{(n)}(t) - \tilde{u}^{(n)}(t)| \leq A e^{\delta x + \delta t} \left[ 1 + \left( \frac{2\sqrt{2}}{\sqrt{\pi}} C_0 \sqrt{t} + C_0 |\delta| t \right) + \cdots + \left( \frac{2\sqrt{2}}{\sqrt{\pi}} C_0 \sqrt{t} + C_0 |\delta| t \right)^{n-1} \right] \\
\leq A e^{\delta x + \delta t} \frac{1}{1 - \left( \frac{2\sqrt{2}}{\sqrt{\pi}} C_0 \sqrt{t} + C_0 |\delta| t \right)}.
\]

Therefore, letting \(n \to +\infty\), there exist a small enough \(t_0 = t_0(|\delta|, C_0)\) with

\[
\frac{e^{\delta x t_0}}{1 - \left( \frac{2\sqrt{2}}{\sqrt{\pi}} C_0 \sqrt{t_0} + C_0 |\delta| t_0 \right)} < 2,
\]

such that \(|S_t u_0 - S_t \tilde{u}_0| \leq 2 A e^{\delta x}\) for any \(x \in \mathbb{R}, t \in (0, t_0)\). At time \(t = kt_0, k = 1, 2, 3, \ldots\), one can take \(S_{kt_0} u_0, S_{kt_0} \tilde{u}_0\) instead of \(u_0, \tilde{u}_0\) as the initial data and then repeat the same estimates as above in the interval \([kt_0, (k + 1)t_0]\). It concludes that for any \(x \in \mathbb{R}, t > 0, |S_t u_0 - S_t \tilde{u}_0| \leq 2 \left( \frac{1}{t_0} \right) A e^{\delta x}.
\]

**Corollary 4.2.** There exists a function \(A(t) > 0\) which is bounded in any compact subset of \([0, +\infty),\) such that

\[
|u(x, t) - u_l(x, t)| \leq A(t) e^{\beta x}, \quad |u(x, t) - u_r(x, t)| \leq A(t) e^{-\beta x}, \quad \forall x \in \mathbb{R}, \ t > 0,
\]

\[
|u(x, t) - \psi_\xi(x, t)| \leq A(t) e^{\beta \xi} e^{-\beta |x - \xi|}, \quad \forall \xi, x \in \mathbb{R}, \ t > 0,
\]

where \(A(t)\) depends on \(p, f, \bar{u}_l, \bar{u}_r, A_0, \beta_0, \|w_0\|_{L^\infty}\).

**Proof.** By substituting \(u_0 = \phi + w_0 + v_0, \tilde{u}_0 = \bar{u}_l + w_0\) with \(\delta = \beta\) or \(\delta = -\beta\) in Lemma 4.1, one has that

\[
|u(x, t) - u_l(x, t)| \leq A(t) e^{\beta x} \quad \text{and} \quad |u(x, t) - u_r(x, t)| \leq A(t) e^{-\beta x}.
\]

Then by the definition of \(\psi_\xi\) in (2.8), one has

\[
u(x, t) - \psi_\xi(x, t) = (u - u_l)(x, t) (g(x - \xi) + (u - u_r)(x, t) (1 - g(x - \xi)) \]
then by Proposition 3.1 and that \( u_t \) and \( u_r \) are bounded, it holds that
\[
|u(x, t) - \psi_q(x, t)| \leq A(t) \left\{ e^{\beta t}e^{-\beta|\xi|} + e^{-\beta t}e^{-\beta|x-\xi|} \right\} \leq A(t)e^{\beta|\xi|}e^{-\beta|x-\xi|}.
\]

\[
\square
\]

4.2. Time-independent estimates.

By Corollary 4.2, for any \( x \in \mathbb{R}, t \geq 0 \), one can define the functions:
\[
U_t(x, t) := \int_{-\infty}^{x} (u - u_t)(y, t)dy,
\]
(4.1)
\[
U_r(x, t) := \int_{x}^{\infty} (u - u_r)(y, t)dy,
\]
(4.2)
and for any \( C^\infty \) curve \( \xi(t) : [0, +\infty) \to \mathbb{R} \), one can define:
\[
\check{U}_\xi(x, t) := \int_{-\infty}^{x} (u - \psi_\xi)(y, t)dy,
\]
(4.3)
\[
\check{U}_\xi(x, t) := \int_{x}^{+\infty} (u - \psi_\xi)(y, t)dy.
\]
(4.4)

Lemma 4.3. The functions \( U_t, U_r, \check{U}_\xi \) and \( \check{U}_\xi \) defined above satisfy the following equations:
\[
\partial_t U_t - \partial_x^2 U_t = f(u_t) - f(u), \quad \forall x \in \mathbb{R}, \; t > 1,
\]
(4.5)
\[
\partial_t U_r - \partial_x^2 U_r = f(u) - f(u_r), \quad \forall x \in \mathbb{R}, \; t > 1,
\]
(4.6)
\[
\partial_t \check{U}_\xi - \partial_x^2 \check{U}_\xi = f(\psi_\xi) - f(u) - \int_{-\infty}^{x} h_\xi(y, t)dy, \quad \forall x \in \mathbb{R}, \; t > 1,
\]
(4.7)
\[
\partial_t \check{U}_\xi - \partial_x^2 \check{U}_\xi = f(u) - f(\psi_\xi) - \int_{x}^{+\infty} h_\xi(y, t)dy, \quad \forall x \in \mathbb{R}, \; t > 1,
\]
(4.8)
where the derivatives appearing in these equations are all continuous in \( \mathbb{R} \times [1, +\infty) \).

Proof. Here we only prove (4.5) and (4.7), since the proof of the other two is similar.

(1). To prove (4.5), for any \( T > 1 \), one considers the following linear heat equation:
\[
\begin{cases}
\partial_t V - \partial_x^2 V = f(u_t) - f(u), & \forall x \in \mathbb{R}, \; 1 < t \leq T, \\
V(x, 1) = U_t(x, 1).
\end{cases}
\]
(4.9)

By Corollary 4.2, it holds that
\[
|f(u_t) - f(u)| \leq C(T)e^{\beta x}, \quad \forall x \in \mathbb{R}, \; 1 < t \leq T,
\]
(4.10)
\[
|U_t(x, 1)| \leq Ce^{\beta x}, \quad \forall x \in \mathbb{R},
\]
Note that
\[
e^{\beta|x|} \leq C(T)e^{\frac{1}{T}|x|^2}, \quad \forall x \in \mathbb{R},
\]
and the initial data \( U_t(x, 1) \) and the source term \( f(u_t) - f(u) \) are smooth in \( \mathbb{R} \), and in \( \mathbb{R} \times [1, T] \) respectively. Therefore, by the standard parabolic theory (see [3, Chapter 1, Theorem 12]), the function
\[
V(x, t) = \int_{0}^{t-1} K^\tau(\cdot) * (f(u_t) - f(u))(\cdot, t - \tau)d\tau + K^{t-1}(\cdot) * U_t(\cdot, 1)
\]
(4.11)
solves (4.9) and all the derivatives of \( V \) appearing in the equation exist and are all continuous in \( \mathbb{R} \times [1, T] \). By (4.10) and (4.11), it is easy to verify that \( V(x, t) \) vanishes as \( x \to -\infty \). And by (4.11) and the equations of \( u \) and \( u_i \), it holds that

\[
\partial_x V(x, t) = \int_0^{t-1} \partial_x K^\tau(\cdot) \ast (f(u_i) - f(u))(\cdot, t - \tau) d\tau + K^{t-1}(\cdot) \ast (u - u_i)(\cdot, 1)
\]

\[
= (u - u_i)(x, t), \quad \forall x \in \mathbb{R}, \ 1 \leq t \leq T,
\]

which implies that \( V(x, t) = U_i(x, t), \forall x \in \mathbb{R}, \ 1 \leq t \leq T \). And since \( T > 1 \) is arbitrary chosen, (4.5) holds.

(2). To prove (4.7), by (2.11), since \( g' \) and \( g'' \) are integrable, it holds that

\[
\int_{-\infty}^{x} h_{\xi} \ dy = (f(\psi_{\xi}) - f(u_i))g_{\xi} + (f(\psi_{\xi}) - f(u_r))(1 - g_{\xi}) - 2(u_i - u_r)g'_{\xi}
\]

\[
+ \int_{-\infty}^{x} (f(u_i) - f(u_r))g'_{\xi} \ dy - \int_{-\infty}^{x} (u_i - u_r)g'_{\xi} \ dy + \int_{-\infty}^{x} (u_i - u_r)g''_{\xi} \ dy.
\]

Then by Proposition 3.2, \( \int_{-\infty}^{x} h_{\xi} \ dy \) is smooth. And for any \( T > 1 \), by Proposition 3.1 and Corollary 4.2, and note that \( \psi_{\xi} - u_i = -(u_i - u_r)(1 - g_{\xi}) \), it holds that

\[
\left| \int_{-\infty}^{x} h_{\xi} \ dy \right| \leq C(T)\left[ 1 - g_{\xi}(x) + |g'_{\xi}(x)| + \int_{-\infty}^{x} |g'_{\xi}| dy + \int_{-\infty}^{x} |g''_{\xi}| dy \right].
\]

By \( g' < 0 \), one has \( \int_{-\infty}^{x} |g'_{\xi}| dy = 1 - g_{\xi} \). And since there exists \( x_0 \in \mathbb{R} \), such that \( g'' < 0 \) for \( x < x_0 \), thus \( \int_{-\infty}^{x} |g''_{\xi}| dy = -g''_{\xi} \) if \( x < x_0 \). Hence, by Proposition 3.1, it holds that

\[
|\bar{U}_{\xi(1)}(x, 1)| \leq Ce^{\beta x}, \quad \forall x \in \mathbb{R}.
\]

And by Corollary 4.2, one can easily verify that

\[
|\bar{U}_{\xi(1)}(x, 1)| \leq Ce^{\beta x}, \quad \forall x \in \mathbb{R}.
\]

Then consider the problem:

\[
\begin{align*}
\bar{\partial}_t \bar{V} - \bar{\partial}_x^2 \bar{V} &= \bar{H} := f(\psi_{\xi}) - f(u) - \int_{-\infty}^{x} h_{\xi}(y, t) dy, \quad \forall x \in \mathbb{R}, \ 1 < t \leq T, \\
\bar{V}(x, 1) &= \bar{U}_{\xi(1)}(x, 1).
\end{align*}
\]

By Corollary 4.2 and (4.12), \( \bar{H} \) satisfies that

\[
|\bar{H}(x, t)| \leq C(T)e^{\beta x}, \quad \forall x \in \mathbb{R}, \ 1 < t \leq T.
\]

Thus, since the initial data \( \bar{U}_{\xi(1)}(x, 1) \) is smooth in \( \mathbb{R} \), satisfying (4.13), and the source term \( \bar{H} \) is smooth in \( \mathbb{R} \times (1, T] \), satisfying (4.15), then similar to the proof in (1), one can verify that the function

\[
\bar{V}(x, t) = \int_0^{t-1} K^\tau(\cdot) \ast \bar{H}(\cdot, t - \tau) d\tau + K^{t-1}(\cdot) \ast \bar{U}_{\xi(1)}(\cdot, 1)
\]

solves (4.14) and all the derivatives of \( \bar{V} \) appearing in the equation exist and are all continuous in \( \mathbb{R} \times [1, T] \). By (4.15) and (4.16), it is easy to verify that \( \bar{V}(x, t) \) vanishes as \( x \to -\infty \). And by (4.16) and the equation of \( \psi_{\xi} \), one has that

\[
\partial_t (u - \psi_{\xi}) - \partial_x^2 (u - \psi_{\xi}) = \partial_x [f(\psi_{\xi}) - f(u) - \int_{-\infty}^{x} h_{\xi}(y, t) dy] = \partial_x \bar{H}.
\]
Hence,
\[
\partial_x \tilde{V}(x, t) = \int_0^{t-1} \partial_x K^\tau(\cdot) \ast \tilde{H}(\cdot, t - \tau) d\tau + K^{t-1}(\cdot) \ast (u - \psi_\xi)(\cdot, 1)
= (u - \psi_\xi)(x, t), \quad \forall x \in \mathbb{R}, \quad 1 < t \leq T,
\]
which implies that \( \tilde{V} = \tilde{U}_\xi. \) \( \square \)

For \( \varepsilon_0 > 0 \) small enough, one can have that
\[
(4.17) \quad f'(\overline{u}_l - 2\varepsilon_0) - s > 0 \quad \text{and} \quad f'(\overline{u}_r + 2\varepsilon_0) - s < 0,
\]
then for this \( \varepsilon_0, \) by Proposition 3.2, there exists \( T_1 > T_0 \) large enough, such that
\[
(4.18) \quad u_l(x, t) > \overline{u}_l - \varepsilon_0 \quad \text{and} \quad u_r(x, t) < \overline{u}_r + \varepsilon_0, \quad \forall x \in \mathbb{R}, \quad t > T_1,
\]
here \( T_1 \) depends on \( p, f, \overline{u}_l, \overline{u}_r, \|w_0\|_{L^\infty} \).

And for the later use, we denote
\[
(4.19) \quad a(v, w) := \int_0^1 f'(w + \rho(v - w)) \ d\rho,
(4.20) \quad b(v, w) := \int_0^1 f''(w + \rho(v - w)) \ d\rho.
\]

Proposition 4.4. There exists \( T_2 > T_1, \) such that for any \( \varepsilon > 0, \) there exists \( N_\varepsilon > 0 \) such that
\[
(4.21) \quad |u(x, t) - u_l(x, t)| \leq \varepsilon, \quad \forall t > T_2, \quad x < st - N_\varepsilon,
(4.22) \quad |u(x, t) - u_r(x, t)| \leq \varepsilon, \quad \forall t > T_2, \quad x > st + N_\varepsilon,
\]
here \( T_2 \) and \( N_\varepsilon \) depend on \( p, f, \overline{u}_l, \overline{u}_r, A_0, \beta_0, \|w_0\|_{L^\infty}. \)

Proof. For convenience, we only give the proof of (4.21), since the proof of (4.22) is similar. And the proof will be stated in four steps.

**Step 1.** In the first step, we will prove that there exist \( T_2 > T_1 \) and \( N_\varepsilon > 0, \) such that
\[
(4.23) \quad U_l(x, t) < \varepsilon, \quad \forall t > T_2, \quad x < st - N_\varepsilon,
\]
where \( U_l \) is defined in (4.1).

By (4.17), one can denote the constant:
\[
(4.24) \quad \tilde{\beta} := f'(\overline{u}_l - 2\varepsilon_0) - s > 0.
\]
For a constant \( M > 1 \) which will be determined later, define the function:
\[
(4.25) \quad \tilde{u}(x, t) := Me^{\beta(x-st)} + u_l(x, t), \quad \forall x \in \mathbb{R}, \quad t > 0.
\]
Then by the equation of \( u_l, \) one has
\[
(4.26) \quad \partial_t \tilde{u} - \partial_x^2 \tilde{u} + \partial_x f(\tilde{u}) = \tilde{h} := -\tilde{\beta} s M e^{\beta(x-st)} - \tilde{\beta}^2 M e^{\beta(x-st)} + \partial_x [f(\tilde{u}) - f(u_l)].
\]
Since \( f \) is strictly convex, then by (4.18), one can obtain
\[
f'(\tilde{u}) \geq f'(u_l) \geq f'(\overline{u}_l - \varepsilon_0), \quad \forall x \in \mathbb{R}, \quad t \geq T_1.
\]
So for the given constants \( M, \ t \geq T_1, \)
(1). if \( x \) satisfies \( Me^{\beta(x-st)} \gg 1 \), it holds that
\[
\partial_x [f(\tilde{u}) - f(u_t)] = f'(\tilde{u}) \tilde{M} e^{\beta(x-st)} + [f'(\tilde{u}) - f'(u_t)] \partial_x u_t \\
\geq [f'(\tilde{u}) - f'(u_t)] (\tilde{\beta} - |\partial_x u_t|) + f'(\bar{u}_t - \varepsilon_0) \tilde{M} e^{\beta(x-st)}.
\]
then by (4.26) and (3.5), one has
\[
\tilde{h} \geq [f'(\tilde{u}) - f'(u_t)](\tilde{\beta} - |\partial_x u_t|) + \tilde{\beta} Me^{\beta(x-st)}[f'(\bar{u}_t - \varepsilon_0) - (s + \tilde{\beta})] \\
\geq [f'(\tilde{u}) - f'(u_t)](\tilde{\beta} - Ce^{-\alpha t}) + \tilde{\beta} Me^{\beta(x-st)}[f'(\bar{u}_t - \varepsilon_0) - f'(\bar{u}_t - 2\varepsilon_0)] \\
\geq [f'(\tilde{u}) - f'(u_t)](\tilde{\beta} - Ce^{-\alpha t}).
\]
(2). if \( x \) satisfies \( Me^{\beta(x-st)} < 1 \), one has \( 0 < \tilde{u} - u_t \leq 1 \), then it holds that
\[
\partial_x [f(\tilde{u}) - f(u_t)] = f'(\tilde{u}) \tilde{M} e^{\beta(x-st)} + [f'(\tilde{u}) - f'(u_t)] \partial_x u_t \\
= f'(\tilde{u}) \tilde{M} e^{\beta(x-st)} + f''(\cdot)(\tilde{u} - u_t) \partial_x u_t \\
\geq f'(\bar{u}_t - \varepsilon_0) \tilde{M} e^{\beta(x-st)} - CM e^{\beta(x-st)}|\partial_x u_t|.
\]
then one has
\[
\tilde{h} \geq \tilde{\beta} Me^{\beta(x-st)}[f'(\bar{u}_t - \varepsilon_0) - (s + \tilde{\beta})] - CM e^{\beta(x-st)}|\partial_x u_t| \\
\geq Me^{\beta(x-st)} \{ \tilde{\beta} [f'(\bar{u}_t - \varepsilon_0) - (s + \tilde{\beta})] - Ce^{-\alpha t} \} \\
\geq Me^{\beta(x-st)} \{ \tilde{\beta} [f'(\bar{u}_t - \varepsilon_0) - f'(\bar{u}_t - 2\varepsilon_0)] - Ce^{-\alpha t} \} \\
\geq Me^{\beta(x-st)} \{ \tilde{\beta} \varepsilon_0 - Ce^{-\alpha t} \},
\]
where \( c_0 := \min f'' > 0 \).

As a result of (1) and (2), there exists a constant \( T_2 > T_1 \), depending on \( p, f, \bar{u}_t, \bar{u}_r, \|w_0\|_{L^\infty} \), which is independent of \( M \), such that
\[
\tilde{h} = \partial_t \tilde{u} - \partial_x^2 \tilde{u} + \partial_x f(\tilde{u}) > 0, \quad \forall \ x \in \mathbb{R}, \ t \geq T_2.
\]
Denote \( \tilde{U}(x, t) := \int_{-\infty}^x (u - \tilde{u})(y, t) dy \), then similar to the proof of Lemma 4.3, one can prove that
\[
\partial_t \tilde{U} - \partial_x^2 \tilde{U} + [f(u) - f(\tilde{u})] = -\int_{-\infty}^x \tilde{h}(y, t) dy, \quad \forall \ x \in \mathbb{R}, \ t > T_2,
\]
which implies that
\[
\partial_t \tilde{U} - \partial_x^2 \tilde{U} + a(u, \tilde{u}) \partial_x \tilde{U} < 0, \quad \forall \ x \in \mathbb{R}, \ t > T_2,
\]
here \( a(u, v) \) is defined by (4.19).

By Corollary 4.2, there exists \( N > 0 \), which depends on \( \varepsilon, T_2, p, f, \bar{u}_t, \bar{u}_r, A_0, \beta_0, \|w_0\|_{L^\infty} \), such that
\[
|U_t(x, T_2)| = \left| \int_{-\infty}^x (u - u_t)(y, T_2) dy \right| \leq \varepsilon, \quad \forall \ x \leq -N, \tag{4.28}
\]
and by Corollary 4.2 again, one can denote the constant \( B \) as
\[
B := \sup_{x \geq N} \left| \int_{-N}^x (u - u_r)(y, T_2) dy \right| < +\infty. \tag{4.29}
\]
Thus, for the given constants $\tilde{\beta}, T_2, N$ and $B$, one can choose $M > 0$ large enough, such that
\[
\int_{-\infty}^{-N} (u_t - \tilde{u})(y, T_2)dy = -\frac{M}{\beta} e^{-\tilde{\beta}(N + st_2)} < -B.
\]
Then we claim that
\[
\textbf{Claim 1:} \quad \tilde{U}(x, T_2) = \int_{-\infty}^{x} (u - \tilde{u})(y, T_2)dy \leq \varepsilon, \quad \forall x \in \mathbb{R}.
\]
In fact, if $x \leq -N$, then by (4.28),
\[
\int_{-\infty}^{x} (u - \tilde{u})(y, T_2)dy \leq \int_{-\infty}^{x} (u - u_t)(y, T_2)dy \leq \varepsilon.
\]
And if $x > -N$, then by (4.28) and (4.29),
\[
\int_{-\infty}^{x} (u - \tilde{u})(y, T_2)dy = \int_{-\infty}^{-N} (u - u_t) + \int_{-N}^{x} (u - \tilde{u}) + \int_{-N}^{x} (u_r - \tilde{u}) \leq \varepsilon - B + B + 0 = \varepsilon,
\]
which completes the proof of Claim 1.

Combing (4.27) and Claim 1, and using the maximum principle ([12, Lemma 1]), one can obtain that
\[
\tilde{U}(x, t) \leq \varepsilon, \quad \forall x \in \mathbb{R}, \ t \geq T_2,
\]
which implies that
\[
U_t(x, t) = \int_{-\infty}^{x} (u_t - u)(y, t)dy = \tilde{U}(x, t) + \frac{M}{\beta} e^{\tilde{\beta}(x - st)} \leq 2\varepsilon,
\]
if $t \geq T_2$ and $x - st < -N\varepsilon$ with $N\varepsilon > 0$ large enough.

**Step 2.** In this step, it is aimed to construct a $C^\infty$ curve
\[
\xi(t) : [T_2, +\infty) \to \mathbb{R},
\]
such that the approximate solution $\psi_\xi$ defined by (2.8) satisfy that
\[
h_\xi = \partial_t \psi_\xi + \partial_x f(\psi_\xi) - \partial_x^2 \psi_\xi < 0, \quad \forall x \in \mathbb{R}, \ t > T_2.
\]

For two given constants $M > 0$ and $d > 0$, where $M$ will be determined in this step, and $d$ will be determined in the next step, define a $C^\infty$ curves $\xi(t) : [T_2, +\infty) \to \mathbb{R}$, which solves the following ODE:
\[
\left\{ \begin{array}{l}
\xi'(t) = s - Me^{-at}, \quad \forall t > T_2, \\
\xi(T_2) = -d.
\end{array} \right.
\]
Then by (4.31), there exists a constant $D > d$, which depends on $\alpha, T_2, d, M$, such that
\[
\xi(t) \in [st - D, st + D], \quad \forall t \geq T_2.
\]
Then we calculate the source term $h_\xi$. By (2.12) and (3.6), it holds that for $t \geq T_2$,
\[
h_\xi = (u_t - u_r) \left( \xi'(t) - s + f'(\phi_\xi) - f'(\psi_\xi) + 2\frac{\partial_x u_t - \partial_x u_r}{u_t - u_r} \right) |g'| \\
- (u_t - u_r) \left( b(\psi_\xi, u_l) \partial_x u_t - b(\psi_\xi, u_r) \partial_x u_r \right) g_\xi (1 - g_\xi),
\]
where $b$ is defined in (4.20). Then by Corollary 3.3 and (3.2), one has that for $t \geq T_2$,

\[
\begin{aligned}
    h_\xi &\leq (u_l - u_r) \left( \xi'(t) - s + f'(\phi_\xi) - f'(\psi_\xi) + 2 \frac{\partial_x u_l - \partial_x u_r}{u_l - u_r} \right) |g_\xi^t| \\
    &\quad + C(u_l - u_r) \left( |\partial_x u_l| + |\partial_x u_r| \right) |g_\xi^t| \\
    &\leq (u_l - u_r) |g_\xi^t| \left[ -M e^{-\alpha t} + C|\phi_\xi - \psi_\xi| + C(|\partial_x u_l| + |\partial_x u_r|) \right],
\end{aligned}
\]

where $C > 0$ depends on $f, \bar{u}_l, \bar{u}_r, p, \|w_0\|_{L^\infty(\mathbb{R})}$.

Note that

\[
|\phi_\xi - \psi_\xi| \leq |\bar{u}_l - u_l| g_\xi + |\bar{u}_r - u_r|(1 - g_\xi) \leq C e^{-\alpha t},
\]

then if $M > 0$ is large enough,

\[
h_\xi \leq (u_l - u_r) |g_\xi^t| (-M + C)e^{-\alpha t} < 0, \quad \forall x \in \mathbb{R}, \; t > T_2.
\]

Therefore, by choosing $M$ large enough, which depends on $f, \bar{u}_l, \bar{u}_r, p, \|w_0\|_{L^\infty(\mathbb{R})}$, (4.30) is fulfilled with the $\xi$ constructed in (4.31).

**Step 3.** In this step, we will prove that there exists $N_\varepsilon > 0$, such that

\[
U_l(x, t) > -\varepsilon, \quad \forall t > T_2, \; x < st - N_\varepsilon.
\]

For the constants $N$ and $B$ defined in (4.28) and (4.29), one can choose the constant $d = \xi(T_2) > 0$ (which is in (4.31)) large enough, such that

\[
\begin{aligned}
    &\int_{-\infty}^{-N} (u_l - \psi_{-d})(x, T_2)dx + \int_{-N}^{+\infty} (u_r - \psi_{-d})(x, T_2)dx \\
    &= \int_{-\infty}^{-N} (u_l - u_r)(x, T_2)(1 - g(x + d))dx - \int_{-N}^{+\infty} (u_l - u_r)(x, T_2) g(x + d)dx \\
    &> B.
\end{aligned}
\]

Then we claim that

**Claim 2:** There holds that

\[
\bar{U}_{\xi(T_2)}(x, T_2) = \int_{-\infty}^{x} (u - \psi_{-d})(y, T_2)dy \geq -\varepsilon, \quad \forall x \in \mathbb{R}.
\]

Here $\bar{U}_\xi$ is defined in (4.3).

In fact, if $x \leq -N$, then by (4.28), it holds that

\[
\int_{-\infty}^{x} (u - \psi_{-d})(y, T_2)dy \geq \int_{-\infty}^{x} (u - u_l)(y, T_2)dy \geq -\varepsilon.
\]

And if $x > -N$, then by (4.28), (4.29) and (4.34), it holds that

\[
\begin{aligned}
    \int_{-\infty}^{x} (u - \psi_{-d}) &= \int_{-\infty}^{-N} (u - u_l) + \int_{-N}^{x} (u - \psi_{-d}) + \int_{-N}^{x} (u - u_r) + \int_{-N}^{x} (u_r - \psi_{-d}) \\
    &\geq -\varepsilon + \int_{-\infty}^{-N} (u_l - \psi_{-d}) - B + \int_{-N}^{+\infty} (u_r - \psi_{-d}) \\
    &\geq -\varepsilon,
\end{aligned}
\]

which completes the proof of Claim 2.
By Lemma 4.3, one has that
\[ \partial_t \bar{U}_\xi - \partial_x^2 \bar{U}_\xi = f(\psi_\xi) - f(u) - \int_{-\infty}^{x} h_\xi(y, t)dy, \quad \forall x \in \mathbb{R}, \ t \geq T_2, \]
then by (4.30), it holds that
\[ (4.36) \quad \partial_t \bar{U}_\xi - \partial_x^2 \bar{U}_\xi + a(u, \psi_\xi)\partial_x \bar{U}_\xi > 0, \quad \forall x \in \mathbb{R}, \ t \geq T_2, \]
here \( a \) is defined by (4.19). Then by (4.35), (4.36) and using the maximum principle, one can obtain
\[ \bar{U}_\xi(x, t) \geq -\epsilon, \quad \forall x \in \mathbb{R}, \ t \geq T_2. \]
Then by (3.6) and (4.32), it holds that
\[ (4.37) \quad \left| \int_{x_1}^{x_2} (u - u_l)(y, t)dy \right| \leq 2\epsilon. \]
And by (2.2) and (3.5), there exists a positive number \( M_1 > 0 \), depending on \( p, f, \bar{u}_l, \bar{u}_r, A_0, \beta_0, \| w_0 \|_{L^p} \), such that
\[ (4.38) \quad \partial_x (u - u_l) \leq M_1, \quad \forall \ t > T_2, \ x \in \mathbb{R}. \]
Then (4.21) holds if the following Claim is true.

**Claim 3:** For any \( t > T_2, \ x < st - N_\epsilon - \frac{3\sqrt{\epsilon}}{\sqrt{M_1}} \), it holds that
\[ (4.39) \quad |u(x, t) - u_l(x, t)| \leq 3\sqrt{M_1}\epsilon. \]
In fact, if there exist \( t_0 > T_2 \) and \( x_0 < st_0 - N_\epsilon - \frac{3\sqrt{\epsilon}}{\sqrt{M_1}} \), such that
\[ u(x_0, t_0) - u_l(x_0, t_0) < -3\sqrt{M_1}\epsilon. \]
Then for any \( x \in (x_0, x_0 + \frac{3\sqrt{\epsilon}}{\sqrt{M_1}}) \), it holds that
\[ (u - u_l)(x, t_0) - (u - u_l)(x_0, t_0) = \partial_x (u - u_l)(\cdot, t_0)(x - x_0). \]
Then by (4.38), one has
\[ (u - u_l)(x, t_0) < -3\sqrt{M_1}\epsilon + M_1(x - x_0). \]
The problem \( \text{Lemma 4.5.} \)

If \( \frac{\partial B}{\partial x} \) is Lipschitz with respect to \( \frac{\partial B}{\partial x} \), which is smooth in \( \frac{\partial B}{\partial x} \) constructed below, then by considering the interval \( (x_0 - \frac{3\sqrt{\varepsilon}}{M_1}, x_0) \) instead. So Claim 3 is true.

4.3. The anti-derivative of \( u - \psi_X \).

In this part, we follow the idea of Il’in and Oleı̈nik [12] to prove (2.17). The idea is to consider the equation of the anti-derivative of \( u - \psi_X \), to construct an auxiliary function (\( \Theta(x) \) constructed below) and to use the maximal principle.

Firstly, we prove the existence and uniqueness of \( X(t) \) solving the ODE (2.14).

**Lemma 4.5.** The problem (2.14) has a unique \( C^1 \) solution \( X(t) : [0, +\infty) \rightarrow \mathbb{R} \), which is smooth in \( \{ t \geq 1 \} \), and satisfies

\[
|X'(t) - s| \leq Ce^{-\alpha t}, \quad \forall \ t \geq 0,
\]

here \( C > 0 \) depends on \( p, f, \bar{u}_t, \bar{m}_r, \| u_0 \|_{L^p} \).

**Proof.** Denote \( F(\xi, t) \) to be the right hand side of equation (2.14) with \( X \) replaced by \( \xi \). By Proposition 3.1, it holds that

\[
\int_{-\infty}^{+\infty} (u_l - u_r)(x, t)g'_{\psi_X}(x) \, dx \leq -C \int_{-\infty}^{+\infty} (u_l - u_r)(x, t)e^{-\beta_2|x-X|} \, dx
\]

\[
= -C \int_{-\infty}^{+\infty} (u_l - u_r)(x, t) \sum_{m \in \mathbb{Z}} e^{-\beta_2|x+mp-X|} \, dx
\]

\[
\leq -C \int_{0}^{p} (u_l - u_r)(x, t)e^{-\beta_2p} \, dx
\]

\[
= -C(\bar{u}_t - \bar{m}_r)e^{-\beta_2p} < 0.
\]

So the denominator of \( F(\xi, t) \) has a negative upper bound uniformly for all \( \xi \in \mathbb{R}, t \geq 0 \). Since for any \( k \geq 1 \), \( g^{(k)} \) is integrable and \( u_l, u_r \) are bounded, then the numerator of \( F \) is also bounded, and it’s similar to obtain the uniform bound of \( \partial_\xi F \). Thus \( F \) is Lipschitz with respect to \( \xi \), so the Cauchy-Lipschitz theorem verifies that the solution \( X(t) \in C^1 \) exists and is unique. And by (3.5), one can obtain the smoothness of \( F(\xi, t) \) on \( \{ t \geq 1 \} \). Thus the curve \( X(t) \) is also smooth on \( \{ t \geq 1 \} \).

By (3.3) and (3.5), there hold that

\[
\int_{-\infty}^{+\infty} (u_l - u_r)g''_{\psi_X}(x) \, dx = \int_{-\infty}^{+\infty} (u_l - u_r + \bar{u}_t) \, dx = O(e^{-\alpha t}),
\]

\[
\int_{-\infty}^{+\infty} (f(u_l) - f(u_r))g'_{\psi_X}(x) \, dx = -f(\bar{u}_t) + f(\bar{m}_r)
\]

\[
+ \int_{-\infty}^{+\infty} \left( f(u_l) - f(\bar{u}_t) - f(u_r) + f(\bar{m}_r) \right) g'_{\psi_X}(x) \, dx
\]

\[
= -f(\bar{u}_t) + f(\bar{m}_r) + O(e^{-\alpha t}),
\]
and similarly,
\[
\int_{-\infty}^{+\infty} (u_l - u_r) g_{X(t)}' dx = -\overline{u}_l + \overline{u}_r + O(e^{-\alpha t}).
\]
Therefore, (4.40) holds.

Since \( g' \) and \( g(1 - g) \) are integrable, one can define the anti-derivative variable of \(-h_X\) in (2.10):

\[
(4.41) \quad H(x, t) := - \int_{-\infty}^{x} h_X(y, t) \, dy, \quad \forall t \geq 0, \ x \in \mathbb{R}..
\]

Then by (2.11), one has that for any \( x \in \mathbb{R}, \ t > 0, \)
\[
H(x, t) = - (f(\psi_X) - f(u_l)) g_X - (f(\psi_X) - f(u_r)) (1 - g_X)
\]
\[
+ 2(u_l - u_r) g'_X - \int_{-\infty}^{x} (f(u_l) - f(u_r))(y, t) \, g'_X(y) dy
\]
\[
+ X'(t) \int_{-\infty}^{x} (u_l - u_r)(y, t) \, g'_X(y) dy - \int_{-\infty}^{x} (u_l - u_r)(y, t) \, g''_X(y) dy.
\]

And by Corollary 4.2, one can define the anti-derivative variable of \( u - \psi_X \):
\[
(4.43) \quad U(x, t) := \tilde{U}_X(x, t) = \int_{-\infty}^{x} (u - \psi_X)(y, t) dy, \quad \forall t \geq 0, \ x \in \mathbb{R}.
\]

Proposition 4.6. The functions \( H(x, t) \) and \( a(u, \psi_X) \) are smooth in \( \mathbb{R} \times [1, +\infty) \), and \( U(x, t) \) solves the equation
\[
(4.44) \quad \partial_t U - \partial_x^2 U + a(u, \psi_X) \partial_x U = H, \quad \forall x \in \mathbb{R}, \ t \geq 1,
\]
where \( a(u, v) \) is defined by (4.19), and the derivatives of \( U \) appearing in (4.44) are all continuous in \( \mathbb{R} \times [1, +\infty) \). Moreover, \( H(x, t) \) satisfies
\[
(4.45) \quad |H(x, t)| \leq C_1 e^{-\alpha t} e^{-\beta|x-X(t)|}, \quad \forall x \in \mathbb{R}, \ t \geq 0,
\]
and \( U(x, t) \) satisfies
\[
(4.46) \quad |U(x, t)| \leq A_1(t) e^{-\beta|x-X(t)|}, \quad \forall x \in \mathbb{R}, \ t \geq 0,
\]
here \( C_1 > 0 \) is a constant, and \( A_1(t) > 0 \) is bounded in any compact subset of \([0, +\infty)\), both of which depend on \( p, f, \overline{u}_l, \overline{u}_r, A_0, \beta_0, \|w_0\|_{L^2} \).

Proof. The smoothness of \( H(x, t) \) and \( a(u, \psi_X) \) can be derived easily from (3.5) and the smoothness of \( u, u_l, u_r \) and \( \psi_X \). And by Lemma 4.3, \( U = \tilde{U}_X \) solves (4.44) and all the derivatives of \( U \) appearing in (4.44) are all continuous in \( \mathbb{R} \times [1, +\infty) \).

Then we will prove (4.45). Note that there exists \( x_0 \in \mathbb{R} \), such that if \( x < x_0, \) \( g''(x) < 0 \), and if \( x > x_0, \) \( g''(x) > 0 \). Then for \( x < X(t) + x_0 \), by (4.42) and Proposition 3.1, one has
\[
H(x, t) = - (f(\psi_X) - f(u_l)) - \left( f(\overline{u}_l) - f(\overline{u}_r) + O(e^{-\alpha t}) \right) (1 - g_X) +
\]
\[
+ 2(\overline{u}_l - \overline{u}_r + O(e^{-\alpha t})) g'_X - \int_{-\infty}^{x} \left( f(\overline{u}_l) - f(\overline{u}_r) + O(e^{-\alpha t}) \right) g'_X(y) dy
\]
\[
+ (s + O(e^{-\alpha t})) \int_{-\infty}^{x} (\overline{u}_l - \overline{u}_r + O(e^{-\alpha t})) g''_X(y) dy
\]
\[ -\int_{-\infty}^{x} (\overline{u}_t - \overline{u}_r + O(e^{-\alpha t})) \, g''_X(y) dy. \]

Since \( x < X(t) + x_0 \), \( \int_{-\infty}^{x} |g''_X| = -\int_{-\infty}^{x} g''_X = -g''_X \), and \( 1 - g_X(x) + |g'_X(x)| = O(e^{\beta(x-X(t))}) \). Then

\[
H(x, t) = - (f(\psi_X) - f(u_t)) - \left( f(\overline{u}_t) - f(\overline{u}_r) \right) (1 - g_X) + 2(\overline{u}_t - \overline{u}_r) \, g'_X \\
+ \left( f(\overline{u}_t) - f(\overline{u}_r) \right) (1 - g_X) - s(\overline{u}_t - \overline{u}_r)(1 - g_X) \\
= - (f(\psi_X) - f(u_t)) + (f(\phi_X) - f(\overline{u}_l)) \\
= a(\psi_X, u_t)(\psi_X - u_t) + a(\phi_X, \overline{u}_l)(\phi_X - \overline{u}_l) \\
= (\overline{u}_t - \overline{u}_r)(1 - g_X) \left( a(\psi_X, u_t) - a(\phi_X, \overline{u}_l) \right) + O(e^{-\alpha t}e^{\beta(x-X(t))}),
\]

here \( a \) is defined in (4.19), and it is easy to verify that

\[ a(\psi_X, \overline{u}_l) - a(\phi_X, u_t) = O(e^{-\alpha t}), \]

which implies that if \( x < X(t) + x_0 \), then \( H(x, t) = O(e^{-\alpha t}e^{\beta(x-X(t))}) \). And for \( x > X(t) + x_0 \), \( \int_{-\infty}^{x} |g''_X| = \int_{x}^{\infty} g''_X = -g''_X \), and \( g_X(x) + |g'_X(x)| = O(e^{-\beta(x-X(t))}) \). And by the ODE (2.14) satisfied by \( X(t) \), it is easy to verify that for any \( t > 0, x \in \mathbb{R}, \)

\[
H(x, t) = - (f(\psi_X) - f(u_t)) \, g_X - (f(\psi_X) - f(u_r)) (1 - g_X) \\
+ 2(u_t - u_r) \, g'_X + \int_{x}^{\infty} (f(u_t) - f(u_r))(y, t) \, g'_X(y) dy \\
- X'(t) \int_{x}^{\infty} (u_t - u_r)(y, t) \, g'_X(y) dy + \int_{x}^{\infty} (u_t - u_r)(y, t) \, g''_X(y) dy.
\]

Then by similar arguments as above, one can prove that if \( x < X(t) + x_0 \), \( H(x, t) = O(e^{-\alpha t}e^{-\beta(x-X(t))}) \). Hence, one can get (4.45).

To prove (4.46), we firstly claim that

**Claim:** \( \int_{-\infty}^{+\infty} [u(x, t) - \psi_X(x, t)] dx = 0, \quad \forall t \geq 0. \)

Indeed, for any \( N > 0 \), one can choose a cut-off function \( \varphi_N(x) \in C_0^\infty(\mathbb{R}) \) satisfying \( \varphi_N(x) = 1, \) if \( |x| < N, \) and \( \varphi_N(x) = 0, \) if \( |x| > N + 1. \) Then by multiplying \( \varphi_N(x) \) on each side of (2.13) and doing the integration by parts, one can have that for any \( t > 0, \)

\[
\int_{-\infty}^{+\infty} (u - \psi_X)(x, t) \varphi_N(x) dx = \int_{-\infty}^{+\infty} (u - \psi_X)(x, 0) \varphi_N(x) dx + \int_{0}^{t} \int_{-\infty}^{+\infty} (u - \psi_X) \varphi''_N dx d\tau \\
+ \int_{0}^{t} \int_{-\infty}^{+\infty} [f(u) - f(\psi_X)] \varphi'_N dx d\tau - \int_{0}^{t} \int_{-\infty}^{+\infty} h_X \varphi_N dx d\tau \\
= \int_{-\infty}^{+\infty} \varphi_N(x) dx + \int_{0}^{t} \int_{-\infty}^{+\infty} (u - \psi_X) \varphi''_N dx d\tau \\
+ \int_{0}^{t} \int_{-\infty}^{+\infty} [f(u) - f(\psi_X)] \varphi'_N dx d\tau + \int_{0}^{t} \int_{-\infty}^{+\infty} \partial_x H \varphi_N dx d\tau.
\]
(4.48) \[ \int_{-\infty}^{+\infty} v_0(x) \varphi_N(x) dx + \int_0^{+\infty} (u - \psi_X) \varphi_N'' dx \]
\[ + \int_0^{+\infty} \int_{-\infty}^{+\infty} [f(u) - f(\psi_X)] \varphi_N' dx \, d\tau - \int_0^{+\infty} H \varphi_N' dx \, d\tau. \]

Thus, by applying Corollary 4.2, (3.7) and (4.45) in (4.48), one can let \( N \to +\infty \) and use the dominated convergence theorem to get that
\[ \int_{-\infty}^{+\infty} (u - \psi_X)(x, t) dx = \int_{-\infty}^{+\infty} v_0(x) dx = 0, \quad \forall t > 0, \]
which proves the claim.

Hence, for any \( x \in \mathbb{R}, t > 0, \)
\[ (4.49) \quad U(x, t) = \int_{-\infty}^{x} (u - \psi_X)(y, t) dy = - \int_{x}^{+\infty} (u - \psi_X)(y, t) dy. \]

By Lemma 4.5, one has \( |X(t)| \leq C + |s|t, \) then by Corollary 4.2, one has that
\[ |u(x, t) - \psi_X(x, t)| \leq A(t)e^{\beta |X(t)|}e^{-\beta x - X(t)} \leq A_1(t)e^{-\beta |x - X(t)|}, \]
so combined with (4.49), one can obtain (4.46).

Denote the positive constant \( \varepsilon_1 := \min\{\frac{f'(\nu_l) - s}{2}, \frac{f'(\nu_r) - s}{2}\} > 0. \)

**Lemma 4.7.** There exist positive constants \( N_0 \) and \( T_3 > T_2, \) such that
\[ (4.50) \quad a(u, \psi_X)(x, t) - X'(t) > \varepsilon_1 > 0, \quad \forall x - X(t) < -N_0, \ t > T_3, \]
\[ (4.51) \quad a(u, \psi_X)(x, t) - X'(t) < -\varepsilon_1 < 0, \quad \forall x - X(t) > N_0, \ t > T_3, \]
where \( N_0, T_3 \) depend on \( p, f, \nu_l, \nu_r, A_0, \beta_0, \|w_0\|_{L^\infty}. \)

**Proof.** Here we only give the proof of (4.50), since (4.51) is similar to prove.
By (4.19), \( a(u, \psi_X) = \int_0^t \int_{-\infty}^{+\infty} f'(\psi_X + \rho(u - \psi_X)) \, d\rho, \) and then one has that
\[ \left| \psi_X + \rho(u - \psi_X) - \nu_l \right| \]
\[ = \left| u_l g_X + u_r (1 - g_X) + \rho(u - u_l) + \rho(u_l - u_r)(1 - g_X) - \nu_l \right| \]
\[ = \left| (u_l - \nu_l) - (1 - \rho)(u_l - u_r)(1 - g_X) + \rho(u - u_l) \right| \]
\[ \leq |u_l - \nu_l| + \frac{3}{2} |\nu_l - \nu_r| (1 - g_X) + |u_l - u_r|. \]

By Lemma 4.5, one has \( |X(t) - st| \leq C, \) and thus, combining Proposition 3.1, Proposition 3.2 with Proposition 4.4, for any \( \varepsilon > 0, \) there exist \( T_\varepsilon > T_2 \) and \( N_\varepsilon > 0 \) such that
\[ |\psi_X + \rho(u - \psi_X) - \nu_l| < \varepsilon, \quad \forall t > T_\varepsilon, \ x - X(t) < -N_\varepsilon. \]
Therefore, since \( f \) is smooth, it holds that
\[ \left| f'(\psi_X + \rho(u - \psi_X)) - f'(\nu_l) \right| \leq C \varepsilon, \quad \forall t > T_\varepsilon, \ x - X(t) < -N_\varepsilon. \]
Then it follows from Lemma 4.5 that
\[ a(u, \psi_X) - X' > f'(\nu_l) - C \varepsilon - s + [s - X'(t)] \]
\[ \geq f'(\nu_l) - C \varepsilon - s - Ce^{-at} \]
\[ \geq \frac{f'(\nu_l) - s}{2}, \quad \forall t > T_3, \ x - X(t) < -N_0, \]
if \( \varepsilon \) is small enough, and \( N_0 > N_\varepsilon, \ T_3 > T_\varepsilon \) are large enough.

Denote the linear operator \( L \) as
\[
L := \partial_t - \partial_x^2 + a(u, \psi_X)\partial_x.
\]

Therefore, by Proposition 4.6, it holds that \( LU = H \) on \( \{ t \geq 1 \} \). Given the constant \( N_0 \) given in Lemma 4.7, we can define a convex \( C^2 \) function \( \theta \) on \( \mathbb{R} \) and the auxiliary function \( \Theta \) as \([12]\):
\[
(4.53) \quad \begin{aligned}
\theta(x) := \begin{cases}
\cosh(\gamma x), & |x| \leq N_0, \\
\theta \in C^2, \ 0 \leq \theta'' \leq \gamma^2 \cosh(\gamma x), & N_0 < |x| \leq N_0 + 1, \\
\text{linear function}, & |x| > N_0 + 1,
\end{cases}
\end{aligned}
\]
\[
\Theta(x) := e^{-\delta \theta(x)},
\]
where \( \gamma, \delta > 0 \) are two constants to be determined.

**Lemma 4.8.** There exist positive constants \( \gamma, \delta \) and \( \mu \), such that the auxiliary function \( \Theta \) defined above satisfies that
\[
L(\Theta(x - X(t))) \geq 2\mu \ \Theta(x - X(t)), \quad \forall x \in \mathbb{R}, \ t > T_3.
\]

Here \( \gamma, \delta \mu \) depend on \( p, f, \bar{\tau}, \bar{\pi}, A_0, \beta_0, \| w_0 \|_{L^\infty} \).

**Proof.** In the following we denote \( \zeta := x - X(t) \) for simplicity.

By (4.52), it holds that
\[
L(\Theta(\zeta)) = e^{-\delta \theta(\zeta)} \left[ \delta X'(t)\theta'(\zeta) - \delta^2 (\theta'(\zeta))^2 + \delta \theta''(\zeta) - \delta a(u, \psi_X)\theta'(\zeta) \right]
\]
\[
= \delta \Theta(\zeta) \left[ \theta''(\zeta) - (a(u, \psi_X) - X'(t))\theta'(\zeta) - \delta (\theta'(\zeta))^2 \right]
\]
\begin{enumerate}
\item If \( |\zeta| < N_0 \), then \( \theta(\zeta) = \cosh(\gamma \zeta) \). Therefore,
\[
\theta''(\zeta) - (a(u, \psi_X) - X'(t))\theta'(\zeta) - \delta (\theta'(\zeta))^2
\]
\[
= \gamma^2 \cosh(\gamma \zeta) - (a(u, \psi_X) - X'(t))\gamma \sinh(\gamma \zeta) - \delta \gamma^2 \left( \sinh(\gamma \zeta) \right)^2
\]
\[
= \gamma \cosh(\gamma \zeta) \left[ \gamma \left( 1 - \sinh(\gamma \zeta) \tanh(\gamma \zeta) \right) - (a(u, \psi_X) - X'(t)) \tanh(\gamma \zeta) \right].
\]
\end{enumerate}

Since \( a(u, \psi_X) - X' \) is bounded, one can choose \( \gamma > 0 \) large enough firstly, such that
\[
|a(u, \psi_X) - X'| < \frac{\gamma}{4}, \quad \forall x \in \mathbb{R}, \ t > 0,
\]

here \( \gamma \) depends on \( p, f, \bar{\tau}, \bar{\pi}, A_0, \beta_0, \| w_0 \|_{L^\infty} \). For \( |\zeta| < N_0, |\sinh(\gamma \zeta)| < e^{\gamma N_0} \), then one can choose \( \delta = \delta(\gamma, N_0) > 0 \) small enough, such that
\[
1 - \delta e^{\gamma N_0} \geq \frac{1}{2}.
\]

Hence, since \( |\tanh(\gamma \zeta)| \leq 1 \) and \( \cosh(\gamma \zeta) \geq 1 \), one has that for \( |\zeta| < N_0, \ t > T_3 \),
\[
L(\Theta(\zeta)) \geq \delta \Theta(\zeta) \cdot \gamma \cosh(\gamma \zeta) \cdot \frac{\gamma}{4} \geq \frac{\delta \gamma^2}{4} \Theta(\zeta).
\]
(2) If $\zeta > N_0$, then $0 < k_1 \leq \theta'(\zeta) \leq k_2$, where $k_1 = \gamma \sinh(\gamma N_0)$ and $k_2 = \gamma \sinh(\gamma (N_0 + 1))$, since $0 \leq \theta''(\zeta) \leq \gamma^2 \cosh(\gamma \zeta)$ for $\zeta > N_0$. It follows from the fact $\theta'' \geq 0$, and Lemma 4.7 that
\[
\theta''(\zeta) - \left(a(u, \psi_X) - X'(t)\right)\theta'(\zeta) - \delta(\theta'(\zeta))^2 \geq - \left(a(u, \psi_X) - X'(t)\right)\theta'(\zeta) - \delta(\theta'(\zeta))^2 \\
\geq \theta'(\zeta)\left(\varepsilon_1 - \delta(\theta'(\zeta))\right)
\]
One can choose $\delta = \delta(\gamma, N_0, \varepsilon_1) > 0$ small enough such that for $\zeta > N_0$,
\[
\delta \theta'(\zeta) \leq \delta k_2 \leq \frac{\varepsilon_1}{2},
\]
then one has
\[
L(\Theta(\zeta)) \geq \delta \Theta(\zeta) \cdot k_1 \cdot \frac{\varepsilon_1}{2} \geq \frac{\delta \varepsilon_1 k_1}{2} \Theta(\zeta).
\]
(3) For the case $\zeta < -N_0$, one can also obtain (4.54) by the similar proof in (2).

Combining (1), (2) and (3), the proof of this lemma is completed by choosing $\gamma$ sufficiently large, $\delta$ sufficiently small and $\mu = \min\{\frac{\delta \varepsilon_1^2}{8}, \frac{\delta \varepsilon_1 k_1}{4}\}$.

**Proof of (2.17).** Set
\[
Z(x, t) := M_2 e^{-\mu t} \Theta(x - X(t)) \pm U(x, t),
\]
where $\mu$ is the constant in Lemma 4.8, which can be actually chosen small enough, so that $0 < \mu \leq \min\{1, \alpha\}$. And $M_2 > 0$ is a constant to be determined. Then by (4.45) and Lemma 4.8, it holds that
\[
LZ = M_2 e^{-\mu t}(-\mu)\Theta(x - X(t)) + M_2 e^{-\mu t} L\Theta(x - X(t)) \pm H \\
\geq M_2 e^{-\mu t}(-\mu + 2\mu)\Theta(x - X(t)) \pm H \\
\geq \mu M_2 e^{-\mu t} e^{-\delta \theta(x - X(t))} - C_0 e^{-\mu t} e^{-\beta |x - X(t)|} \\
\geq e^{-\mu t}\left(\mu M_2 e^{-\delta \theta(x - X(t))} - C_0 e^{-\beta |x - X(t)|}\right).
\]
By (4.46), one has
\[
Z(x, T_3) \geq M_2 e^{-\mu T_3} e^{-\delta \theta(x - X(T_3))} - C(T_3) e^{-\beta |x - X(T_3)|}.
\]
(1) If $|x - X(t)| \leq N_0 + 1$, then
\[
\mu M_2 e^{-\delta \theta(x - X(t))} - C_0 e^{-\beta |x - X(t)|} \geq \mu M_2 e^{-\delta \theta(N_0 + 1)} - C_0, \\
M_2 e^{-\mu T_3} e^{-\delta \theta(x - X(T_3))} - C(T_3) e^{-\beta |x - X(T_3)|} \geq M_2 e^{-\mu T_3} e^{-\delta \theta(N_0 + 1)} - C(T_3)
\]
Therefore, by letting $M_2 > \max\{\frac{C_0}{\mu} e^{\delta \theta(N_0 + 1)}, C(T_3) e^{\mu T_3} e^{\delta \theta(N_0 + 1)}\}$, it holds that $LZ > 0$ and $Z(x, T_3) > 0$.
(2) If $|x - X(t)| > N_0 + 1$, then $\theta$ is linear and
\[
|\theta'(x - X(t))| < k_2 = \gamma \sinh(\gamma (N_0 + 1)).
\]
Therefore, by $\theta(x) \leq \cosh(x)$ for $|x| > N_0 + 1$, it holds that
\[
\theta(x - X(t)) < k_2 |x - X(t)| + \cosh(\gamma (N_0 + 1)).
\]
Then one has that
\[\mu M_2 e^{-\theta(x-X(t))} - C_0 e^{-\beta|x-x(t)|}\]
\[\geq \mu M_2 e^{-\delta \cosh(\gamma (N_0 + 1))} e^{-\delta k_2|x-x(t)|} - C_0 e^{-\beta|x-x(t)|},\]
\[M_2 e^{-\mu T_3 e^{-\theta(x-X(T_3))}} - C(T_3) e^{-\beta|x-x(T_3)|}\]
\[\geq M_2 e^{-\mu T_3 e^{-\delta \cosh(\gamma (N_0 + 1))} e^{-\delta k_2|x-x(T_3)|}} - C(T_3) e^{-\beta|x-x(T_3)|}\]

Then by choosing \(\delta\) small enough with \(\delta k_2 \leq \beta\), and \(M_2\) large enough with
\[M_2 > \max\left\{ \frac{C_0}{\mu} e^{\delta \cosh(\gamma (N_0 + 1))}, C(T_3) e^{\mu T_3 e^{\delta \cosh(\gamma (N_0 + 1))}} \right\},\]
one can obtain that \(LZ > 0\) and \(Z(x, T_3) > 0\).

By combining (1) and (2), if \(\delta\) is small and \(M_2\) is large, then \(LZ > 0\) and \(Z(x, T_3) > 0\) for any \(x \in \mathbb{R}, t \geq T_3\).

Therefore, the maximum principle implies that \(Z(x, t) > 0\) for any \(x \in \mathbb{R}, t \geq T_3\), that is,
\[|U(x, t)| = M_2 e^{-\mu t} \Theta(x - X(t)) < M_2 e^{-\mu t} \quad \forall x \in \mathbb{R}, t \geq T_3.\]

Hence, by the definition (4.43) of \(U\), one has that for any \(x_1 < x_2, t \geq T_3\),
\[\int_{x_1}^{x_2} \left( u(y, t) - \psi_X(y, t) \right) dy = |U(x_2, t) - U(x_1, t)| \leq 2M_2 e^{-\mu t}.\]

By (2.2), (3.3) and (3.5), there exists a constant \(M_3 > 0\), such that for any \(x \in \mathbb{R}, t \geq T_3\),
\[\partial_x(u - \psi_X) = \partial_x(u - u_i) g_X + \partial_x(u - u_r)(1 - g_X) - (u_l - u_r) g_X' \leq M_3,\]
here \(M_3\) depends on \(p, f, \bar{u}_i, \bar{u}_r, A_0, \beta_0, \|w_0\|_{L^\infty}\).

With (4.55) and (4.56), we claim that
\[\text{Claim: } |u(x, t) - \psi_X(x, t)| < 3\sqrt{M_2 M_3} e^{-\frac{\alpha t}{2}}, \quad \forall x \in \mathbb{R}, t \geq T_3.\]

Indeed, if there exists \((x_0, t_0)\) with \(x_0 \in \mathbb{R}, t_0 \geq T_3\) such that
\[u(x_0, t_0) - \psi_X(x_0, t_0) < -3\sqrt{M_2 M_3} e^{-\frac{\alpha t_0}{2}}.\]

Then for any \(x \in (x_0, x_1)\), where \(x_1 := x_0 + 3\sqrt{M_2 M_3} e^{-\frac{\alpha t_0}{2}}\), by (4.56), one can obtain that
\[|u(x, t_0) - \psi_X(x, t_0)| < |u(x_0, t_0) - \psi_X(x_0, t_0)| \leq M_3(x - x_0),\]
Then
\[\int_{x_0}^{x_1} [u(x, t_0) - \psi_X(x, t_0)] dx\]
\[\leq \int_{x_0}^{x_1} \left( u(x, t_0) - \psi_X(x_0, t_0) \right)(x_1 - x_0) + M_3 \frac{(x_1 - x_0)^2}{2}\]
\[\leq -3\sqrt{M_2 M_3} e^{-\frac{\alpha t_0}{2}} \cdot 3\sqrt{\frac{M_2}{M_3}} e^{-\frac{\alpha t_0}{2} + M_3 \frac{9M_2}{2} e^{-\mu t_0}}\]
\[= -\frac{9}{2} M_2 e^{-\mu t_0} < -2M_2 e^{-\mu t_0},\]
which contradicts with (4.55). If we assume \(u(x, t_0) - \psi_X(x, t_0) > 3\sqrt{M_2 M_3} e^{-\frac{\alpha t_0}{2}}\) for some point \((x_0, t_0)\), the contradiction arguments are similar by considering the interval \((x_0 - 3\sqrt{M_2 M_3} e^{-\frac{\alpha t_0}{2}}, x_0)\) instead.
Therefore, the claim above is proved, and by combining the fact that \( u \) and \( \psi_x \) are both bounded, one can finish the proof of (2.17). \( \square \)

4.4. **Proof of** (2.18).

To finish the proof of Theorem 2.6, it remains to prove (2.18). When \( f(u) = u^2/2 \), (1.1) is the Burgers’ equation. In [10], Hopf introduced the well-known Hopf transformation to compute the explicit formula of the solution to (1.1) with the initial data \( u_0 \), which is given by:

\[
(4.57) \quad u(x, t) = \frac{\int_{-\infty}^{+\infty} \frac{x-y}{t} \exp\left\{ -\frac{(x-y)^2}{4t} - \frac{1}{2} \int_0^y u_0(\eta) d\eta \right\} d\eta}{\int_{-\infty}^{+\infty} \exp\left\{ -\frac{(x-y)^2}{4t} - \frac{1}{2} \int_0^y u_0(\eta) d\eta \right\} d\eta}, \quad \forall x \in \mathbb{R}, \ t > 0.
\]

Since \( u_0 \) is bounded, then by integration by part on the numerator of (4.57), it holds that

\[
(4.58) \quad u(x, t) = \frac{\int_{-\infty}^{+\infty} u_0(y) \exp\left\{ -\frac{(x-y)^2}{4t} - \frac{1}{2} \int_0^y u_0(\eta) d\eta \right\} d\eta}{\int_{-\infty}^{+\infty} \exp\left\{ -\frac{(x-y)^2}{4t} - \frac{1}{2} \int_0^y u_0(\eta) d\eta \right\} d\eta}, \quad \forall x \in \mathbb{R}, \ t > 0.
\]

The viscous shock \( \phi \) with the end states \( \overline{u}_l, \overline{u}_r \) of the Burgers’ equation has the explicit formula

\[
\phi(x) = \frac{\overline{u}_l + \overline{u}_r}{2} - \frac{\overline{u}_l - \overline{u}_r}{2} \tanh\left( \frac{\overline{u}_l - \overline{u}_r}{4} x \right).
\]

Denote

\[
\lambda := \frac{\overline{u}_l - \overline{u}_r}{4},
\]

thus the corresponding \( g \) defined in (2.7) is given by

\[
(4.59) \quad g(x) = \frac{1 - \tanh(\lambda x)}{2} = \frac{e^{-\lambda x}}{e^{\lambda x} + e^{-\lambda x}},
\]

where

\[
(4.60) \quad \phi(x) = \overline{u}_l g(x) + \overline{u}_r (1 - g(x)).
\]

And one also has

\[
(4.61) \quad \int_0^x g(y) \, dy = \frac{1}{2} \int_0^x [1 - \tanh(\lambda y)] \, dy = \frac{1}{2\lambda} \log \frac{e^{\lambda x} + e^{-\lambda x}}{2} + \frac{1}{2\lambda} \log 2
\]

\[
= \frac{1}{2\lambda} \log(1 - g(x)) + \frac{1}{2\lambda} \log 2,
\]

and similarly,

\[
(4.62) \quad \int_0^x [1 - g(y)] \, dy = -\frac{1}{2\lambda} \log g(x) - \frac{1}{2\lambda} \log 2.
\]

If the initial data \( u_0(x) = \phi(x) + w_0(x) \) with \( \int_0^\infty w_0(x) \, dx = 0 \), by taking (4.60) into (4.58), it holds that

\[
(4.63) \quad u(x, t) = \frac{P_l(x, t) + P_r(x, t)}{Q_l(x, t) + Q_r(x, t)},
\]

where the two terms in the numerator are

\[
P_l(x, t) = 2 \int_{-\infty}^{+\infty} \left[ \overline{u}_l + w_0(y) \right] g(y) \exp\left\{ -\frac{(x-y)^2}{4t} - \frac{1}{2} \int_0^y u_0(\eta) d\eta \right\} \, dy,
\]

\[
P_r(x, t) = 2 \int_{-\infty}^{+\infty} \left[ \overline{u}_r + w_0(y) \right] [1 - g(y)] \exp\left\{ -\frac{(x-y)^2}{4t} - \frac{1}{2} \int_0^y u_0(\eta) d\eta \right\} \, dy,
\]

\[
P_r(x, t) = 2 \int_{-\infty}^{+\infty} \left[ \overline{u}_r + w_0(y) \right] [1 - g(y)] \exp\left\{ -\frac{(x-y)^2}{4t} - \frac{1}{2} \int_0^y u_0(\eta) d\eta \right\} \, dy,
\]
and the two terms in the denominator are

\[
Q_l(x, t) := 2 \int_{-\infty}^{+\infty} g(y) \exp \left\{ -\frac{(x - y)^2}{4t} - \frac{1}{2} \int_{0}^{y} u_0(\eta) \, d\eta \right\} \, dy,
\]

\[
Q_r(x, t) := 2 \int_{-\infty}^{+\infty} [1 - g(y)] \exp \left\{ -\frac{(x - y)^2}{4t} - \frac{1}{2} \int_{0}^{y} u_0(\eta) \, d\eta \right\} \, dy.
\]

By (4.62), \( \lambda = \frac{\overline{r} - \overline{p}}{4} \), and \( u_0 = \overline{u}_l g + \overline{u}_r (1 - g) + w_0 = \overline{u}_l + w_0 - (\overline{u}_l - \overline{u}_r) (1 - g) \), one has that

\[
g(y) \exp \left\{ -\frac{1}{2} \int_{0}^{y} u_0(\eta) \, d\eta \right\} = g(y) \exp \left\{ 2\lambda \int_{0}^{y} [1 - g(\eta)] \, d\eta \right\} \exp \left\{ -\frac{1}{2} \int_{0}^{y} [\overline{u}_l + w_0(\eta)] \, d\eta \right\} = \frac{1}{2} \exp \left\{ -\frac{1}{2} \int_{0}^{y} [\overline{u}_l + w_0(\eta)] \, d\eta \right\}.
\]

And by (4.61), it’s similar to prove that

\[
[1 - g(y)] \exp \left\{ -\frac{1}{2} \int_{0}^{y} u_0(\eta) \, d\eta \right\} = \frac{1}{2} \exp \left\{ -\frac{1}{2} \int_{0}^{y} [\overline{u}_r + w_0(\eta)] \, d\eta \right\}.
\]

Hence, one has that

\[
P_l(x, t) = \int_{-\infty}^{+\infty} [\overline{u}_l + w_0(y)] \exp \left\{ -\frac{(x - y)^2}{4t} - \frac{1}{2} \int_{0}^{y} [\overline{u}_l + w_0(\eta)] \, d\eta \right\} \, dy,
\]

\[
P_r(x, t) = \int_{-\infty}^{+\infty} [\overline{u}_r + w_0(y)] \exp \left\{ -\frac{(x - y)^2}{4t} - \frac{1}{2} \int_{0}^{y} [\overline{u}_r + w_0(\eta)] \, d\eta \right\} \, dy,
\]

\[
Q_l(x, t) = \int_{-\infty}^{+\infty} \exp \left\{ -\frac{(x - y)^2}{4t} - \frac{1}{2} \int_{0}^{y} [\overline{u}_l + w_0(\eta)] \, d\eta \right\} \, dy,
\]

\[
Q_r(x, t) = \int_{-\infty}^{+\infty} \exp \left\{ -\frac{(x - y)^2}{4t} - \frac{1}{2} \int_{0}^{y} [\overline{u}_r + w_0(\eta)] \, d\eta \right\} \, dy.
\]

Moreover, by using the Hopf formula (4.58) on \( u_l(x, t) \) and \( u_r(x, t) \), one can have that

\[
(4.64) \quad u_l(x, t) = \frac{P_l(x, t)}{Q_l(x, t)}, \quad u_r(x, t) = \frac{P_r(x, t)}{Q_r(x, t)}.
\]

And if \( t = t_k = \frac{kp}{\overline{u}_l - \overline{u}_r} = \frac{kp}{4\lambda} \), then it holds that

\[
Q_r(x, t_k) = e^{2\lambda x + 4\lambda^2 t_k} \int_{-\infty}^{+\infty} \exp \left\{ -\frac{(y - x)^2}{4t_k} - \frac{1}{2} \int_{0}^{y} [\overline{u}_l + w_0(\eta)] \, d\eta \right\} \, dy
\]

\[
= e^{2\lambda x + 4\lambda^2 t_k} \int_{-\infty}^{+\infty} \exp \left\{ -\frac{(y - x - 4\lambda t_k)^2}{4t_k} - \frac{1}{2} \int_{0}^{y} [\overline{u}_l + w_0(\eta)] \, d\eta \right\} \, dy
\]

\[
= e^{2\lambda x + 4\lambda^2 t_k} \int_{-\infty}^{+\infty} \exp \left\{ -\frac{(y - x)^2}{4t_k} - \frac{1}{2} \int_{0}^{y+4\lambda t_k} [\overline{u}_l + w_0(\eta)] \, d\eta \right\} \, dy
\]

\[
\times \exp \left\{ -\frac{1}{2} \int_{y}^{y+4\lambda t_k} w_0(\eta) \, d\eta \right\} \, dy.
\]
Since the average \( \int_0^x w_0(x) dx = 0 \) and \( 4\lambda t_k = kp \), thus it holds that
\[
Q_r(x, t_k) = e^{2\lambda(x-st_k)}Q_l(x, t_k), \quad \forall x \in \mathbb{R}.
\]
And it’s similar to prove that
\[
P_r(x, t_k) = e^{2\lambda(x-st_k)}P_l(x, t_k) - 4\lambda e^{2\lambda(x-st_k)}Q_l(x, t_k), \quad \forall x \in \mathbb{R}.
\]
Hence, by (4.63), one has that
\[
u(x, t_k) = \frac{P_l(x, t_k) + e^{2\lambda(x-st_k)}P_l(x, t_k) - 4\lambda e^{2\lambda(x-st_k)}Q_l(x, t_k)}{Q_l(x, t_k) + e^{2\lambda(x-st_k)}Q_l(x, t_k)}
\]
(4.65)
\[
= \frac{P_l(x, t_k)}{Q_l(x, t_k)} - 4\lambda \left( 1 - g(x-st_k) \right),
\]
where \( g \) is defined in (4.59). Meanwhile, by (4.64), there holds that
\[
u_l(x, t_k) = \frac{P_l(x, t_k)}{Q_l(x, t_k)}, \quad \nu_r(x, t_k) = \frac{P_l(x, t_k) - 4\lambda Q_l(x, t_k)}{Q_l(x, t_k)} = \frac{P_l(x, t_k)}{Q_l(x, t_k)} - 4\lambda,
\]
then combined with (4.65), one can prove (2.18).

5. Proof of Proposition 2.4

By Lemma 4.5, it remains to prove (2.15) and (2.16) to finish the proof of Proposition 2.4.

5.1. Proof of (2.15).

For \( y \in [0, p] \), \( N \in \mathbb{N}^* \), define the domain
\[
\Omega_y^N := \{ (x, \tau) : X(t) - Np + y \leq x \leq X(t) + Np + y, \; 0 \leq \tau \leq t \},
\]
and denote \( \Gamma^N \) to be its boundary. We fix \( y \) and \( t \) firstly, and denote the four parts of \( \Gamma^N \) to be \( \Gamma_0^N, \Gamma_r^N, \Gamma_l^N, \Gamma_i^N \), respectively. See Figure 1.
By integration by parts,  

\[ \int_{\Omega_N^y} (\text{equation of } u_l) g_X + (\text{equation of } u_r)(1 - g_X) \, dx \, d\tau \Rightarrow 
\]

\[ -X'(t) (u_l - u_r) g'_X + \left\{ \left[ f(u_l) - \partial_x u_l \right] - \left[ f(u_r) - \partial_x u_r \right] \right\} g'_X \, dx \, d\tau 
\]

(5.1)

\[ = A_N^y(y, t) - A_N^y(0, y) - B_N^y(y, t), \]

where

\[ A_N^y(y, t) := \int_{X(t) - Np + y}^{X(t) + Np + y} \left[ u_l(x, t) g_X(x) + u_r(x, t)(1 - g_X(x)) \right] \, dx, \]

\[ A_N^y(0, y) := \int_{-Np + y}^{-Np + y} \left[ u_l(x, 0) g(x) + u_r(x, 0)(1 - g(x)) \right] \, dx, \]

\[ B_N^y(y, t) := \int_{0}^{t} \left\{ \left[ f(u_l) - \partial_x u_l \right] g_X + \left[ f(u_r) - \partial_x u_r \right] (1 - g_X) \right\} (X(\tau) - Np + y, \tau) \, d\tau 
\]

\[ - \int_{0}^{t} X'(\tau) \left\{ u_l g_X + u_r \left[ 1 - g_X \right] \right\} (X(\tau) - Np + y, \tau) \, d\tau, \]

\[ B_N^y(0, y) := \int_{0}^{t} \left\{ \left[ f(u_l) - \partial_x u_l \right] g_X + \left[ f(u_r) - \partial_x u_r \right] (1 - g_X) \right\} (X(\tau) + Np + y, \tau) \, d\tau 
\]

\[ - \int_{0}^{t} X'(\tau) \left\{ u_l g_X + u_r \left[ 1 - g_X \right] \right\} (X(\tau) + Np + y, \tau) \, d\tau. \]

The equation (2.14) of \( X'(t) \) implies that the left hand side of (5.1) converges to 0 as \( N \to \infty \). Then it remains to evaluate the right hand side as \( N \to \infty \).

(i) The integrals on \( \Gamma_0^N, \Gamma_t^N \).

Denote \( w_l(x, t) := u_l(x, t) - \bar{u}_l \), and \( w_r(x, t) := u_r(x, t) - \bar{u}_r \), then since \( w_0 \) is of average zero on each period, one has that

\[ A_N^y(y, t) - A_N^y(0, y) \]

\[ = \int_{X(t) - Np + y}^{X(t) + Np + y} \left[ u_l(x, t) g_X(x) + u_r(x, t)(1 - g_X(x)) \right] \, dx 
\]

\[ - \int_{-Np + y}^{-Np + y} \left[ (\bar{u}_l + w_0(x)) g(x) + (\bar{u}_r + w_0(x))(1 - g(x)) \right] \, dx 
\]

\[ = \int_{-Np + y}^{-Np + y} \left[ w_l(x + X, t) g(x) + \bar{u}_l g(x) + w_r(x + X, t)(1 - g(x)) + \bar{u}_r(1 - g(x)) \right] \, dx 
\]

\[ - \int_{-Np + y}^{-Np + y} \left[ \bar{u}_l g(x) + \bar{u}_r(1 - g(x)) \right] \, dx 
\]

\[ = \int_{-Np + y}^{-Np + y} \left[ w_l(x + X, t) g(x) + w_r(x + X, t)(1 - g(x)) \right] \, dx =: J_N^y(y, t). \]

Since both of \( w_l \) and \( w_r \) have zero average, it holds that

\[ J_N^y = \int_{-Np + y}^{y} (w_r - w_l)(x + X, t) \, dx + \int_{y}^{Np + y} (w_l - w_r)(x + X, t) \, dx, \]

then by (3.3) and (3.5), one has that for \( t > 0, \ y \in [0, p] \),

\[ J(y, t) = \lim_{N \to \infty} J_N^y(y, t) \]
Thus one has that
\[ (5.2) \quad \int_{-\infty}^{y} (w_r - w_l)(x + X, t) \left[ 1 - g(x) \right] \, dx + \int_{y}^{\infty} (w_l - w_r)(x + X, t) \, g(x) \, dx = O(e^{-\alpha t}), \]
where the bound in \( O(e^{-\alpha t}) \) depends on \( f, \overline{u}_l, \overline{u}_r, p, \|w_0\|_{L^\infty}. \)

**ii) The integrals on \( \Gamma_N^l, \Gamma_N^p.**

Since \( u_l \) and \( u_r \) are periodic, it holds that
\[
B_N^l = \int_0^t \left\{ [f(u_l) - \partial_x u_l](X(\tau) + y, \tau) \, g(-Np + y) + [f(u_r) - \partial_x u_r](X(\tau) + y, \tau) \, [1 - g(-Np + y)] \right\} \, d\tau
- \int_0^t X'(\tau) \left\{ u_l(X(\tau) + y, \tau)g(-Np + y) + u_r(X(\tau) + y, \tau)[1 - g(-Np + y)] \right\} \, d\tau.
\]
Then by (3.3), as \( N \to \infty, \)
\[
B_N^l(y, t) \to \int_0^t \left\{ [f(u_l) - \partial_x u_l](X(\tau) + y, \tau) \, d\tau \right\} X'(\tau) u_l(X(\tau) + y, \tau) \, d\tau,
\]
thus one has that
\[
\lim_{N \to \infty} \int_0^y B_N^l(y, t) \, dy = \int_0^t \int_0^p f(u_l)(X(\tau) + y, \tau) \, d\tau \right\} X'(\tau) u_l(X(\tau) + y, \tau) \, d\tau,
\]
\[
= \int_0^t \int_0^p f(u_l(y, \tau)) \, dyd\tau - p\overline{u}_l X(t).
\]
And it’s similar to prove that
\[
\lim_{N \to \infty} \int_0^y B_N^p(y, t) \, dy = \int_0^t \int_0^p f(u_r)(y, \tau) \, dyd\tau - p\overline{u}_r X(t).
\]

With the calculations in (i) and (ii), one can integrate the equation (5.1) with respect to \( y \) over \( [0, p] \), and then let \( N \to \infty \), which yields that
\[
\int_0^p J(y, t) \, dy + p(\overline{u}_l - \overline{u}_r)X(t) - \int_0^t \int_0^p \left[ f(u_l) - f(u_r) \right] \, dyd\tau = 0, \quad t \geq 0.
\]
And note that
\[
\int_0^t \int_0^p \left[ f(u_l) - f(u_r) \right] \, dyd\tau = \int_0^t \int_0^p \left[ f(u_l) - f(\overline{u}_l) \right] \, dyd\tau
- \int_0^t \int_0^p \left[ f(u_r) - f(\overline{u}_r) \right] \, dyd\tau + p \left[ f(\overline{u}_l) - f(\overline{u}_r) \right] t,
\]
thus, it holds that
\[
X(t) - st = \frac{1}{p(\overline{u}_l - \overline{u}_r)} \left\{ \int_0^t \int_0^p \left[ f(u_l) - f(\overline{u}_l) \right] \, dyd\tau
- \int_0^t \int_0^p \left[ f(u_r) - f(\overline{u}_r) \right] \, dyd\tau - \int_0^p J(y, t) \, dy \right\},
\]
(5.3)
thus, by (3.5) and (5.2), one can obtain that \( X(t) - st = O(e^{-at}) \), which proves (2.15).

5.2. \( X(t) \) for Burgers’ equation.

For the Burgers’ equation (1.1) with \( f(u) = u^2/2 \), one can use the Galilean transform to verify that

\[
(5.4) \quad w_l(x, t) = w_r(x - (\bar{u}_l - \bar{u}_r)t, t), \quad \forall x \in \mathbb{R}, t \geq 0,
\]

where \( w_l = u_l - \bar{u}_l, w_r = u_r - \bar{u}_r \). Therefore, for any \( t > 0 \), it holds that

\[
\int_0^t \int_0^1 \left\{ [f(u_l) - f(\bar{u}_l)] - [f(u_r) - f(\bar{u}_r)] \right\} \, dx \, d\tau
= \int_0^t \int_0^1 \left\{ \left[ 2\bar{u}_l + w_l(x, \tau) \right] w_l(x, \tau) - \left[ 2\bar{u}_r + w_r(x, \tau) \right] w_r(x, \tau) \right\} \, dx \, d\tau
\]

\[
= \int_0^t \int_0^1 \frac{1}{2} w_l^2(x, \tau) \, dx \, d\tau
= \int_0^t \int_0^1 \frac{1}{2} w_r^2(x, \tau) \, dx \, d\tau
\]

\[
= \int_0^t \int_{-(\bar{u}_l - \bar{u}_r)t}^{-(\bar{u}_r - \bar{u}_r)t} \frac{1}{2} w_r^2(x, \tau) \, dx \, d\tau
= \int_0^t \int (\bar{u}_l - \bar{u}_r)^2 \, dx \, d\tau = 0,
\]

here the second equality holds since the averages of \( w_l \) and \( w_r \) are zero.

Then if \( (\bar{u}_l - \bar{u}_r)t_k = kp \) for any \( k \geq 0 \), by (5.4), one has \( w_l(x, t_k) \equiv w_r(x, t_k) \). So the term \( J \) in (5.2) satisfies that

\[
(5.6) \quad J(y, t_k) \equiv 0.
\]

Then by applying (5.5) and (5.6) in (5.3), one has \( X(t_k) = st_k \). The proof of Proposition 2.4 is finished.

6. PROOF OF THEOREM 2.1

Proof. Since there holds that

\[
|\phi(x - X(t)) - \psi(x - X(t))| \leq |u_l(x, t) - \bar{u}_l| + |u_r(x, t) - \bar{u}_r|,
\]

thus by Corollary 3.3 and Theorem 2.6, one can obtain that

\[
|u(x, t) - \phi(x - X)| \leq |u(x, t) - \psi(x - X)| + |\phi(x - X) - \psi(x - X)|
\leq Ce^{-\mu t}, \quad \forall x \in \mathbb{R}, t \geq 0.
\]

Then by (2.15), one can easily prove (2.5). Hence, it remains to prove the properties (1) and (2) of \( X_x \) stated in Theorem 2.1.

1. If \( f(u) = u^2/2 \), then by applying (5.5) into (5.3), and by (5.2), one can obtain that \( X_x = 0 \) easily.

2. Given any periodic perturbation \( w_0 \) with zero average and \( 0 < \| w_0 \|_{L^\infty(\mathbb{R})} < (\bar{u}_l - \bar{u}_r)/2 \), it holds that \( \bar{u}_l + \| w_0 \|_{L^\infty(\mathbb{R})} < \bar{u}_l - \| w_0 \|_{L^\infty(\mathbb{R})} \), then one can construct a smooth and strictly convex function \( f \) such that \( f(u) = \frac{1}{2n} u^2 \) when \( u \leq \bar{u}_l + \| w_0 \|_{L^\infty(\mathbb{R})} \) and \( f(u) = \frac{1}{2} u^2 \) when \( u \geq \bar{u}_l - \| w_0 \|_{L^\infty(\mathbb{R})} \), where \( n \) is a positive number to be determined later, see Figure 2.
Since for any \( x \in \mathbb{R}, t > 0 \), it holds that
\[
 u_l(x,t) \geq \inf_x u_l(x,0) \geq \overline{u}_l - \|w_0\|_{L^\infty(\mathbb{R})}
\]
\[
 u_r(x,t) \leq \sup_x u_r(x,0) \leq \overline{u}_r + \|w_0\|_{L^\infty(\mathbb{R})}.
\]

Then by the fact that \( w_l = u_l - \overline{u}_l, w_r = u_r - \overline{u}_r \) have zero average, one has
\[
 \int_0^\infty \int_0^p [f(u_l) - f(\overline{u}_l)] \, dx \, d\tau = \frac{1}{2} \int_0^\infty \int_0^p (u_l^2 - \overline{u}_l^2) \, dx \, d\tau
\]
\[
 = \frac{1}{2} \int_0^\infty \int_0^p (\overline{u}_l + w_l)^2 - \overline{u}_l^2 \, dx \, d\tau
\]
\[
 = \frac{1}{2} \int_0^\infty \int_0^p w_l^2 \, dx \, d\tau,
\]
and similarly,
\[
 \int_0^\infty \int_0^p [f(u_r) - f(\overline{u}_r)] \, dx \, d\tau = \frac{1}{2n} \int_0^\infty \int_0^p (u_r^2 - \overline{u}_r^2) \, dx \, d\tau = \frac{1}{2n} \int_0^\infty \int_0^p w_r^2 \, dx \, d\tau.
\]

Since \( w_0 \) is not zero function, the solution \( u_l(x,t) \) with the initial data \( \overline{u}_l + w_0(x) \) cannot be a constant in \( \mathbb{R} \times [0, +\infty) \), which means the integral of (6.1) is positive. And more importantly, this integral is independent of \( n \), since no matter what \( n \) is, the range of \( u_l(x,t) \) is always in the interval where \( f(u) = u^2/2 \), which means that \( u_l(x,t) \) is actually the solution to the viscous Burgers’ equation.

On the other side, for the solution \( u_r(x,t) \), by (B.2), it holds that
\[
 \int_0^{+\infty} \int_0^p w_r^2 \, dx \, d\tau \leq C,
\]
here \( C \) is independent of \( f \), which only depends on \( p, \|w_0\|_{L^\infty} \). Hence, by (6.2), it holds that
\[
 \int_0^\infty \int_0^p [f(u_r) - f(\overline{u}_r)] \, dx \, d\tau \leq \frac{C}{2n}.
\]

By (6.1) and (6.3), one can choose \( n \) sufficiently large, such that
\[
 \int_0^\infty \int_0^p [f(u_l) - f(\overline{u}_l)] \, dx \, d\tau > \int_0^\infty \int_0^p [f(u_r) - f(\overline{u}_r)] \, dx \, d\tau,
\]
which implies that \( X_\infty \neq 0 \).
The proof of Theorem 2.1 is finished.

7. Vanishing viscosity limit for the shift function

If we add the viscosity coefficient $0 < \nu \leq 1$ in the equation (1.1):

\begin{equation}
\partial_t u^\nu + \partial_x f(u^\nu) = \nu \partial_x^2 u^\nu, \quad \forall x \in \mathbb{R}, \ t > 0,
\end{equation}

then it is well known that for any initial data $u_0 \in L^\infty$, the viscous solution $u^\nu$ converges almost everywhere to the inviscid entropy solution $u^0 \in Lip([0, +\infty); L^1_{loc}(\mathbb{R}))$, which solves (7.1) with $\nu = 0$:

\begin{equation}
\partial_t u^0 + \partial_x f(u^0) = 0, \quad a.e. x \in \mathbb{R}, \ t > 0.
\end{equation}

Then for any $0 \leq \nu \leq 1$, one can denote $u^\nu_l$ and $u^\nu_r$ to be the periodic solutions to (7.1) with the corresponding periodic initial data:

\[ u^\nu_l(x, 0) = \overline{u}_l + w_0(x), \quad u^\nu_r(x, 0) = \overline{u}_r + w_0(x). \]

**Claim:** There exists a constant $C > 0$ independent of $\nu$, which only depends on $f, \overline{u}_l, \overline{u}_r, \|w_0\|_{L^\infty}$, such that

\begin{equation}
\sup_x |u^\nu_l - \overline{u}_l|, \ \sup_x |u^\nu_r - \overline{u}_r| \leq \frac{C}{1 + t}, \quad \forall \ t > 0, \ 0 \leq \nu \leq 1.
\end{equation}

In fact, (7.3) can be verified from the entropy condition proved by Oleinik [23], which gives that there exists a constant $E > 0$ independent of $\nu$, such that for $0 < \nu \leq 1$, the solution $u^\nu$ of (7.1) with $L^\infty$ initial data satisfies

\[ \partial_x u^\nu(x, t) \leq \frac{E}{t}, \quad \forall t > 0, \ x \in \mathbb{R}. \]

Then, since $\int_0^p \partial_x u^\nu_l(x, t) dx = 0$, then by (2.2), it holds that for any $t > 0$,

\[ \int_{x \in (0, p), \partial_x u^\nu_l < 0} |\partial_x u^\nu_l(x, t)| dx = \int_{x \in (0, p), \partial_x u^\nu_l > 0} \partial_x u^\nu_l(x, t) dx \leq \frac{pE}{t}, \]

which implies that for $0 < \nu \leq 1$,

\[ \sup_x |u^\nu_l - \overline{u}_l| \leq \int_0^p |\partial_x u^\nu_l(x, t)| dx \leq \frac{2pE}{t}. \]

For $u^\nu_r$, the proof is similar. And since $u^\nu_l, u^\nu_r$ converge to $u^0_l, u^0_r$ almost everywhere respectively, (7.3) is proved.

For $\nu > 0$, the viscous shock profile for (7.1) is $\phi^\nu(x - st) = \phi((x - st)/\nu)$, and the shifted function $g^\nu_\xi$ defined in (2.7) becomes

\begin{equation}
\phi^\nu(x) = \phi(x - \xi) := \frac{\phi^\nu(x - \xi - \overline{u}_r)}{\overline{u}_l - \overline{u}_r} = \phi \left( \frac{x - \xi}{\nu} \right),
\end{equation}

and the approximate solution defined in (2.8) is

\[ \psi^\nu_\xi := u^\nu_l g^\nu_\xi + u^\nu_r (1 - g^\nu_\xi), \]

with the corresponding source term:

\[ h^\nu_\xi := \partial_t \psi^\nu_\xi + \partial_x f(\psi^\nu_\xi) - \nu \partial_x^2 \psi^\nu_\xi .\]
satisfying
\[
\begin{align*}
    h_r^\xi = \partial_x \left( f(u_r^\nu) - f(u_l^\nu) g \left( \frac{x - \xi}{\nu} \right) - f(u_l^\nu) (1 - g \left( \frac{x - \xi}{\nu} \right)) - 2 \frac{\nu}{\nu} g' \left( \frac{x - \xi}{\nu} \right) \right) \\
    + \left( f(u_r^\nu) - f(u_l^\nu) \right) \frac{\nu}{\nu} g' \left( \frac{x - \xi}{\nu} \right) - (u_r^\nu - u_l^\nu) \frac{\nu}{\nu} g' \left( \frac{x - \xi}{\nu} \right) \xi' + (u_r^\nu - u_l^\nu) \frac{\nu}{\nu} g'' \left( \frac{x - \xi}{\nu} \right).
\end{align*}
\]

Thus, by integrating the equality above in \(x\), one can choose the curve \(X^\nu\) to make \(\int_R h_r^\nu dx = 0\), which is determined by

\[
\begin{align*}
    (X^\nu)' &= F^\nu(X^\nu, t), \\
    X^\nu(0) &= 0,
\end{align*}
\]

where
\[
F^\nu(\xi, t) := \frac{\int_{-\infty}^{\xi} [(u_r^\nu - u_l^\nu) \frac{\nu}{\nu} g' \left( \frac{x - \xi}{\nu} \right) + (f(u_r^\nu) - f(u_l^\nu)) \frac{\nu}{\nu} g' \left( \frac{x - \xi}{\nu} \right)] dx}{\int_{-\infty}^{\xi} (u_r^\nu - u_l^\nu) \frac{\nu}{\nu} g' \left( \frac{x - \xi}{\nu} \right) dx}.
\]

Similar to the proof in Section 5.1, one can calculate that

\[
\begin{align*}
    X^\nu(t) - st &= \frac{1}{p(\overline{u}_l - \overline{u}_r)} \left\{ \int_0^t \int_0^p \left[ f(u_r^\nu) - f(\overline{u}_l) \right] dx d\tau \right. \\
    &\quad - \int_0^t \int_0^p \left[ f(u_l^\nu) - f(\overline{u}_r) \right] dx d\tau \left. \right\} - \frac{1}{p(\overline{u}_l - \overline{u}_r)} \int_0^p J^\nu(y, t) dy,
\end{align*}
\]

where

\[
J^\nu(y, t) = \int_{-\infty}^y (u_r^\nu - u_l^\nu)(x + X^\nu(t), t) \left[ 1 - g \left( \frac{x}{\nu} \right) \right] dx
\]

\[
+ \int_{y}^{\infty} (u_r^\nu - u_l^\nu)(x + X^\nu(t), t) g \left( \frac{x}{\nu} \right) dx.
\]

with \(u_r^\nu := u_r^\nu - \overline{u}_l \) and \(u_l^\nu := u_l^\nu - \overline{u}_r\).

For the viscous case \(\nu > 0\), Theorem 2.1 shows that as \(t \to +\infty\), the viscous shock profile under a periodic perturbation is itself with a constant shift \(X^\nu_\infty\), which is the limit of \(X^\nu(t) - st\) as \(t \to +\infty\). However, as the viscosity vanishes, we will show that \(X^\nu_\infty\) converges to zero, which is compatible with the result obtained in [27], namely, in the inviscid case \(\nu = 0\), for any strictly convex flux \(f\), if the initial periodic perturbation is bounded and has zero average, then the solution tends to the background shock wave with no shift. The vanishing viscosity limit about \(X^\nu\) is given in the following theorem:

**Theorem 7.1.** For any smooth and strictly convex flux \(f\), if the periodic perturbation \(w_0 \in L^\infty\) satisfies (1.4), the corresponding shift curve \(X^\nu(t)\) defined in (7.5) satisfies that

\[
X^\nu(t) = st + o(1) + O \left( \frac{1}{1 + t} \right), \quad \forall t \geq 0,
\]

where \(o(1)\) tends to zero as \(\nu \to 0^+\), with the bound in \(o(1)\) independent of \(t\), and the bound in \(O \left( \frac{1}{1 + t} \right)\) is independent of \(\nu\). Moreover, if additionally \(w_0\) is assumed to have bounded total variation on each period, then the term \(o(1) = O(\nu^{1/5})\), where the bound in \(O(\nu^{1/5})\) is still independent of \(t\).
Proof. Regarding (7.6), we firstly estimate the term \( J^\nu \) defined in (7.7). By (7.3), for any \( y \in (0, p) \) and \( t \geq 0 \), it holds that

\[
|J^\nu(y, t)| \leq \int_{-\infty}^{0} |w^\nu(x) - w^\nu_0|(x + X^\nu, t)[1 - g(x)] \, dx + \int_{y}^{0} |w^\nu(x) - w^\nu_0|(x + X^\nu, t)[1 - g(x)] \, dx + \int_{y}^{+\infty} |w^\nu_0 - w^\nu|g(x) \, dx
\]

which implies that

\[
\int_{0}^{p} J^\nu(y, t) \, dy = O\left(\frac{1}{1+t}\right), \quad \forall t \geq 0,
\]

where the bound in \( O\left(\frac{1}{1+t}\right) \) is independent of \( \nu \).

Secondly, using Taylor’s expansion, there holds that

\[
\int_{0}^{p} [f(u^\nu_\nu) - f(\overline{u}_i)] \, dx = \int_{0}^{p} \left(f'(\overline{u}_i)(u^\nu_\nu - \overline{u}_i) + \frac{1}{2} f''(\cdot)(u^\nu_\nu - \overline{u}_i)^2\right) \, dx
\]

then by (7.3) and the strict convexity of \( f \), one can obtain

\[
0 < \int_{0}^{p} [f(u^\nu_\nu) - f(\overline{u}_i)] \, dx \leq \frac{C}{(1+t)^2}.
\]

And it’s similar to prove

\[
0 < \int_{0}^{p} [f(u^\nu_\nu) - f(\overline{u}_\nu)] \, dx \leq \frac{C}{(1+t)^2}.
\]

Thus, by applying (7.8) – (7.10) into (7.6), one has that

\[
X^\nu(t) - st = X^\nu_{\infty} + O\left(\frac{1}{1+t}\right), \quad \forall t \geq 0,
\]

where

\[
X^\nu_{\infty} := \frac{1}{p(\overline{u}_i - \overline{u}_\nu)} \left\{ \int_{0}^{+\infty} \int_{0}^{p} [f(u^\nu_\nu) - f(\overline{u}_i)] \, dx \, d\tau - \int_{0}^{+\infty} \int_{0}^{p} [f(u^\nu_\nu) - f(\overline{u}_\nu)] \, dx \, d\tau \right\}
\]

with the bound in \( O\left(\frac{1}{1+t}\right) \) independent of \( \nu \). And by (7.9) and (7.10), one can use the dominated convergence theorem to obtain that, as \( \nu \to 0+ \),

\[
X^\nu_{\infty} \longrightarrow X^0_{\infty} := \frac{1}{p(\overline{u}_i - \overline{u}_\nu)} \left\{ \int_{0}^{+\infty} \int_{0}^{p} [f(u^0_\nu) - f(\overline{u}_i)] \, dx \, dt - \int_{0}^{+\infty} \int_{0}^{p} [f(u^0_\nu) - f(\overline{u}_\nu)] \, dx \, dt \right\},
\]

then \( X^\nu_{\infty} = X^0_{\infty} + o(1) \). If the initial data \( w_0 \) has bounded total variation on each period: \( TV_{[0, p]} w_0 < +\infty \), then it can be derived from Kruzhkov’s theory (see [17],...
that, the viscous solutions $u_i^\nu$, $i = l, r$, tends to the inviscid entropy solutions $u_i^0$ in $L^1$ norm with the following rate:

$$\int_0^p |u_i^\nu(x, t) - u_i^0(x, t)| dx \leq C(t/\nu)^{1/2} TV_{[0, p]}w_0, \quad \forall t > 0, \ 0 \leq \nu \leq 1,$$

where $C > 0$ is independent of $\nu$ and $t$. And since for $i = l, r$, and given any $T_\nu > 0$, one has that

$$\int_0^{+\infty} \int_0^p [f(u_i^\nu) - f(\overline{u}_i)] dx dt - \int_0^{+\infty} \int_0^p [f(u_i^0) - f(\overline{u}_i)] dx dt = \int_0^{T_\nu} \int_0^p [f(u_i^\nu) - f(u_i^0)] dx dt + \int_0^{T_\nu} \int_0^p [f(u_i^\nu) - f(\overline{u}_i)] dx dt - \int_0^{+\infty} \int_0^p [f(u_i^0) - f(\overline{u}_i)] dx dt,$$

then by combining (7.9), (7.10) and (7.14), one can obtain that

$$\left| \int_0^{+\infty} \int_0^p [f(u_i^\nu) - f(\overline{u}_i)] dx dt - \int_0^{+\infty} \int_0^p [f(u_i^0) - f(\overline{u}_i)] dx dt \right| \leq C \left\{ \nu^{1/2} \int_0^{T_\nu} t^{1/2} dt + \int_0^{+\infty} \frac{1}{(1 + t)^2} dt \right\} \leq C \left( \nu^{1/2} T_\nu^{3/2} + T_\nu^{-1} \right),$$

thus by choosing $T_\nu = \nu^{-1/5}$, one can obtain

$$\left| \int_0^{+\infty} \int_0^p [f(u_i^\nu) - f(\overline{u}_i)] dx dt - \int_0^{+\infty} \int_0^p [f(u_i^0) - f(\overline{u}_i)] dx dt \right| \leq C \nu^{1/5}.$$ 

So by (7.12) and (7.13), one can have that if $TV_{[0, p]}w_0 < +\infty$, then

$$X_\nu^\nu = X_\nu^0 + O(\nu^{1/5}),$$

where the bound depends on $f, \overline{u}, \overline{\nu}$, and $TV_{[0, p]}w_0$.

Then it only remains to show $X_\nu^0 = 0$ to finish the proof of Theorem 7.1.

For any constant $\overline{u}$, let $u(x, t)$ denote the periodic entropy solution to the inviscid conservation law (7.2) with $\nu = 0$, satisfying the initial data $u(x, 0) = \overline{u} + w_0(x)$.

For the periodic perturbation $w_0$, since it satisfies (1.4), the anti-derivative variable $\int_0^x w_0(y) dy$ is continuous and periodic with the period $p$. Then one can choose a constant $x_0 \in [0, p)$ such that

$$\int_0^{x_0} w_0(y) dy = \min_{x \in [0, p]} \int_0^x w_0(y) dy,$$

which is equivalent to

$$\int_{x_0}^x w_0(y) dy \geq 0, \quad \forall x \in \mathbb{R}.$$ 

By the formula of $X_\nu^0$ in (7.13), if one can show that

$$\int_0^{+\infty} \int_0^p [f(u) - f(\overline{u})] dx dt = \int_0^{x_0} \int_0^x w_0(y) dy dx,$$

then $X_\nu^0 = 0$ holds. To prove (7.15), by Theorem 14.1.1 in [1], the periodic solution $u$ takes the constant value $\overline{u}$ along the straight line $x = x_0 + f'(\overline{u})t$. Then given any
y ∈ (x₀, x₀ + p) and t > 0, denote the domain:
\[ \Omega_{(y,t)} := \{(x, \tau) : x₀ + f'(\overline{u}) \tau < x < y + f'(\overline{u}) \tau, \ 0 < \tau < t\}. \]
Then by integrating the equation \( \partial_t u + \partial_x f(u) = 0 \) in \( \Omega_{(y,t)} \), one can obtain by the divergence theorem that
\[
0 = -\int_{x₀}^{y} (\overline{u} + w₀(x)) \ dx + \int_{0}^{t} \left[ f(u) - f'(\overline{u})u \right] (y + f'(\overline{u}) \tau, \tau) \ d\tau \\
- \int_{0}^{t} \left[ f(\overline{u}) - f'(\overline{u})\overline{u} \right] \ d\tau + \int_{x₀ + f'(\overline{u})t}^{y + f'(\overline{u})t} (u(x, t) \ dx)
\]
(7.17) \[
= -\int_{y}^{x₀ + p} w₀(x) \ dx + \int_{x₀}^{t} \left[ f(u) - f'(\overline{u})u \right] (y + f'(\overline{u}) \tau, \tau) \ d\tau \\
- f'(\overline{u}) \int_{0}^{t} (u - \overline{u})(y + f'(\overline{u}) \tau) \ d\tau + \int_{x₀ + f'(\overline{u})t}^{y + f'(\overline{u})t} (u(x, t) - \overline{u}) \ dx.
\]
Since for any \( y \in \mathbb{R}, t \geq 0, \int_{y}^{y+p}(u - \overline{u})(x, t) \ dx = 0 \), then by integrating (7.17) with respect to \( y \) in the interval \((x₀, x₀ + p)\), one can have that
\[
0 = -\int_{x₀}^{x₀+p} \int_{y}^{x₀ + p} w₀(x) \ dxdy + \int_{x₀}^{t} \int_{y}^{x₀ + p} \left[ f(u) - f'(\overline{u})u \right] (y + f'(\overline{u}) \tau, \tau) \ d\tau d\tau \\
+ \int_{x₀}^{x₀+p} \int_{x₀ + f'(\overline{u})t}^{y + f'(\overline{u})t} (u(x, t) - \overline{u}) \ dxdy
\]
Since \( \int_{x₀}^{y} w₀(x)dx \) is periodic with respect to \( y \), and
\[
\int_{x₀}^{x₀+p} \int_{x₀ + f'(\overline{u})t}^{y + f'(\overline{u})t} (u(x, t) - \overline{u}) \ dxdy = O\left(\frac{1}{1+t}\right),
\]
then it holds that
(7.18) \[
\int_{0}^{t} \int_{0}^{p} \left[ f(u(x, \tau)) - f(\overline{u}) \right] \ dxd\tau = \int_{0}^{p} \int_{y}^{x₀} w₀(x) \ dxdy + O\left(\frac{1}{1+t}\right).
\]
Similar to the proof of (7.9), one has that
\[
\int_{0}^{p} \left[ f(u(x, \tau)) - f(\overline{u}) \right] \ dx = O\left(\frac{1}{(1+\tau)^2}\right).
\]
Then by letting \( t \rightarrow +\infty \) in (7.18) and using the dominated convergence theorem, one can finish the proof. \( \square \)

8. Proof of Theorem 2.3

The proof of Theorem 2.3 follows directly from the one in [12]. To make this paper complete, we still place it here. The proof consists of two steps. The first step is to prove a time-independent estimate of the solution \( u \), when converging to the two periodic solutions \( u_\ell \) or \( u_r \) as \( x \rightarrow \pm \infty \), just like Proposition 4.4. Step 2 is to construct an auxiliary function and use the maximal principle to finish the proof.

Proposition 8.1. For any \( \varepsilon > 0 \), there exist \( N_\varepsilon > 0 \) and \( T_\varepsilon > 0 \) such that
(8.1) \[
|u(x, t) - \overline{u}_\ell| \leq \varepsilon, \quad \forall t > T_\varepsilon, \ x - f'(\overline{u}_\ell) t < -N_\varepsilon,
\]
(8.2) \[
|u(x, t) - \overline{u}_r| \leq \varepsilon, \quad \forall t > T_\varepsilon, \ x - f'(\overline{u}_r) t > N_\varepsilon.
\]
Proof. It suffices to prove (8.1), since the proof of (8.2) is similar. For any \( \varepsilon > 0 \), there exists \( T > 0 \) such that \( |u - \bar{u}| < \varepsilon / 2 \), for any \( x \in \mathbb{R}, t \geq T \). And since \( f \) is strictly convex, there exists \( B > 0 \) such that, for any \( 0 < \sigma < 2 \| u_0 \|_{L^\infty} \),
\[
f'(\bar{u} + \sigma) - f'(\bar{u}_l) > B\sigma.
\]
For the constant \( B > 0 \) and \( \beta > 0 \) in Corollary 4.2, one can choose \( 0 < \delta < \varepsilon \) small enough such that \( B\delta / 2 < \beta \).

(1) By Corollary 4.2 and that \( \| u \|_{L^\infty(dx, dt)} \leq \| u_0 \|_{L^\infty} \), one can choose \( M > 0 \) such that
\[
M e^{\frac{B\delta}{2}|x-f'(\bar{u})|} + \frac{\varepsilon}{2} + \bar{u}_l > u(x, T), \quad x \in \mathbb{R}.
\]
Define
\[
v(x, t) := M e^{\frac{B\delta}{2}|x-f'(\bar{u})|} + \frac{\varepsilon}{2} + \bar{u}_l - u(x, t),
\]
then one has that \( v(x, T) > 0 \), and
\[
(8.3) \quad \partial_x^2 v - \partial_t v - f'(u)\partial_x v = M e^{\frac{B\delta}{2}|x-f'(\bar{u})|} \left[ \frac{B\delta}{2} + f'(\bar{u}_l) - f'(u) \right].
\]
Note that if \( v < 0 \), then by the definition of \( v \), one has \( \bar{u}_l - u < -\frac{\varepsilon}{2} \), which implies that
\[
f'(\bar{u}_l) - f'(u) < A(\bar{u}_l - u) \leq -\frac{B\varepsilon}{2},
\]
then by (8.3) and \( 0 < \delta < \varepsilon \),
\[
\partial_x^2 v - \partial_t v - f'(u)\partial_x v < 0.
\]
Therefore, it follows from the maximum principle ([12, Lemma 1]) that \( v(x, t) \geq 0 \), for any \( x \in \mathbb{R}, t \geq T \). Then by choosing \( N := \frac{2}{B\delta} \ln \frac{\varepsilon}{2M} \), one has that for any \( t > T \) and \( x - f'(\bar{u}_l) t < -N \),
\[
u(x, t) \leq \bar{u}_l + \frac{\varepsilon}{2} + M e^{\frac{B\delta}{2}|x-f'(\bar{u})|} < \bar{u}_l + \varepsilon.
\]

(2) On the other hand, for the initial data (1.8) and any \( \varepsilon > 0 \), one can assume \( u^\varepsilon(x, t) \) to be the solution to (1.1) satisfying the initial data,
\[
(8.4) \quad u^\varepsilon(x, 0) = \phi^\varepsilon(x) + w_0(x) + v_0(x),
\]
where \( \phi^\varepsilon \) is the shock profile connecting \( \bar{u}_l \) as \( x \to -\infty \) and \( \bar{u}_l - \varepsilon / 2 \) as \( x \to +\infty \). Then by \( u^\varepsilon(x, 0) \leq u_0(x) \) and the comparison principle, one has
\[
(8.5) \quad u^\varepsilon(x, t) \leq u(x, t), \quad \forall x \in \mathbb{R}, t \geq 0.
\]
And by Theorem 2.1, there exists \( T > 0 \), such that for any \( x \in \mathbb{R}, t > T \),
\[
(8.6) \quad u^\varepsilon(x, t) \geq -\varepsilon / 2 + (\bar{u}_l - \varepsilon / 2) \geq \bar{u}_l - \varepsilon.
\]
Then by (8.5) and (8.6), the proof of (8.1) is finished. \( \square \)

Proof of Theorem 2.3. It is equivalent to prove that for any \( \varepsilon > 0 \), there exists \( T > 0 \), such that
\[
(8.7) \quad \sup_{x \in \mathbb{R}} |u(x, t) - u^R(x, t)| < \varepsilon, \quad \forall t > T.
\]
For the constants \( N_\varepsilon \) and \( T_\varepsilon \) in Proposition 8.1, one can define two constants
\[
x_0 := N_\varepsilon \frac{f'(\bar{u}_l) + f'(\bar{u}_r)}{f'(\bar{u}_l) - f'(\bar{u}_r)}; \quad t_0 := \frac{-2N_\varepsilon}{f'(\bar{u}_l) - f'(\bar{u}_r)} > 0,
\]
and the region
\( \Omega_\epsilon := \{(x, t) : f'(\overline{u}_t)t - N_\epsilon < x < f'(\overline{u}_r)t + N_\epsilon, \ t > T_\epsilon \} \).

Then the shifted rarefaction wave \( \tilde{u}^R(x, t) := u^R(x - x_0, t + t_0) \) satisfies that
\[
\tilde{u}^R(x, t) = \begin{cases} 
\overline{u}_t, & \text{for } x - f'(\overline{u}_t)t \leq -N_\epsilon, \\
(f')^{-1}(\frac{x - x_0}{t + t_0}), & \text{in } \Omega_\epsilon, \\
\overline{u}_r, & \text{for } x - f'(\overline{u}_r)t \geq N_\epsilon.
\end{cases}
\]  
(8.8)

Therefore, Proposition 8.1 implies that for any \( x < f'(\overline{u}_t)t - N_\epsilon \) or \( x > f'(\overline{u}_r)t + N_\epsilon, \ t \geq T_\epsilon \), it holds that
\[
|u(x, t) - \tilde{u}^R(x, t)| < \epsilon.
\]  
(8.9)

Define
\[
Z(x, t) := (t + t_0)^{\kappa} \left[ u(x, t) - \tilde{u}^R(x, t) \right],
\]
where \( 0 < \kappa < 1 \) is a constant to be determined. Then one has
\[
\partial_x^2 Z - f'(u)\partial_x Z - \partial_t Z = \left[ f''(v)\partial_x \tilde{u}^R - \frac{\kappa}{t + t_0} \right] Z - (t + t_0)^{\kappa} \partial_x^2 \tilde{u}^R, \quad \text{in } \Omega_\epsilon,
\]
where \( v \) is the function satisfying
\[
f'(u) - f'(\tilde{u}^R) = f''(v)(u - \tilde{u}^R).
\]
For \( (x, t) \in \Omega_\epsilon \), it holds that \( f''(v)\partial_x \tilde{u}^R = f''(v)\frac{1}{\partial_x \left[ \frac{1}{f'(u)} \right]} > \frac{\omega}{t + t_0} \) for some \( \omega > 0 \).

Then \( 0 < \kappa < 1 \) can be chosen small enough such that
\[
f''(v)\partial_x \tilde{u}^R - \frac{\kappa}{t + t_0} > \frac{\omega}{t + t_0} > 0.
\]  
(8.12)

Note that for any \( x \in \mathbb{R} \),
\[
|Z(x, T_\epsilon)| \leq M_4(T_\epsilon + t_0)^{\kappa},
\]
where \( M_4 = \|u_0\|_{L^\infty} + \|u^R\|_{L^\infty} \). And by (8.9), one has that for any \( t \geq T_\epsilon \),
\[
|Z(x, t)| \leq \epsilon(t + t_0)^{\kappa},
\]
if \( x = f'(\overline{u}_t)t - N_\epsilon \) or \( f'(\overline{u}_r)t + N_\epsilon \). Now if for some point \( (x, t) \in \Omega_\epsilon \),
\[
Z(x, t) = \sup_{\Omega_\epsilon} Z,
\]
and it satisfies that
\[
Z(x, t) \geq \max \left\{ M_4(T_\epsilon + t_0)^{\kappa}, \epsilon(t + t_0)^{\kappa}, \frac{M_4}{\omega} (T_\epsilon + t_0)^{\kappa - 1} \right\} > 0,
\]  
(8.13)

then by (8.11)–(8.13), one has that at \( (x, t) \),
\[
\partial_x^2 Z - f'(u)\partial_x Z - \partial_t Z \geq \frac{\omega}{t + t_0} M_4 (T_\epsilon + t_0)^{\kappa - 1} - (t + t_0)^{\kappa} \frac{M_5}{(t + t_0)^2} > 0,
\]
where \( M_5 = \max_{\xi \in [\overline{u}_t, \overline{u}_r]} \|(f')^{-1}(\xi)\|^\prime \). Therefore, the maximal principle implies that for any \( (x, t) \in \Omega_\epsilon \),
\[
Z(x, t) \leq \max \left\{ M_4(T_\epsilon + t_0)^{\kappa}, \epsilon(t + t_0)^{\kappa}, \frac{M_4}{\omega} (T_\epsilon + t_0)^{\kappa - 1} \right\}.
\]
And it is similar to prove that for any \( (x, t) \in \Omega_\epsilon \),
\[
Z(x, t) \geq \min \left\{ -M_4(T_\epsilon + t_0)^{\kappa}, -\epsilon(t + t_0)^{\kappa}, -\frac{M_4}{\omega} (T_\epsilon + t_0)^{\kappa - 1} \right\}.
\]
As a result, by choosing a large $T > T_0$, one has that

\begin{equation}
|u(x, t) - u^R(x - x_0, t + t_0)| \leq \max \left\{ \frac{M_4(T_0 + t_0)^\kappa}{(t + t_0)^\kappa}, \frac{M_4}{(T_0 + t_0)^{1-\kappa}(t + t_0)} \right\} \leq \varepsilon,
\end{equation}

for any $f'(\bar{u})t - N_\varepsilon < x < f'(\bar{u})t + N_\varepsilon$ and $t > T$. Moreover, since $u^R$ is Lipschitz continuous and $\frac{x - x_0}{t + t_0} \to 0$ as $t \to \infty$, then combining Proposition 8.1 and (8.14), (8.7) can be proved easily.

\appendix
\section*{Appendix A. Proof of Proposition 3.1}

\begin{proof}(i). Since $f$ is smooth, for any $x > y, 0 \leq \rho \leq 1, z = \rho x + (1 - \rho)y$, one has

\begin{align*}
f(z) - \left[ \rho f(x) + (1 - \rho)f(y) \right]
&= \rho(z - x) \int_0^1 f'(\tau z + (1 - \tau)x) \, d\tau \\
&\quad + (1 - \rho)(z - y) \int_0^1 f'(\tau z + (1 - \tau)y) \, d\tau \\
&= \rho(1 - \rho)(x - y) \int_0^1 \int_0^1 (y - x)(1 - \tau)f''(\tilde{\tau})[\tau z + (1 - \tau)y] \\
&\quad + (1 - \tilde{\tau})[\tau z + (1 - \tau)x] \, d\tilde{\tau} d\tau.
\end{align*}

Therefore,
\begin{align*}
\frac{1}{2} \min_{u \in [y, x]} f''(u) &\leq -\frac{f(z) - \left[ \rho f(x) + (1 - \rho)f(y) \right]}{\rho(1 - \rho)(x - y)^2} \\
&\leq \frac{1}{2} \max_{u \in [y, x]} f''(u).
\end{align*}

Then (3.2) follows by substituting $x = \bar{u}_l, y = \bar{u}_r, \rho = g$ and $z = \bar{u}_l g + \bar{u}_r (1 - g)$, and applying the definition (3.1).

(ii). Integrating the equation (3.2) yields

\begin{align*}
\beta_1 x &\leq \ln \frac{1 - g(x)}{g(x)} - \beta_3 \leq \beta_2 x, \quad x > 0,
\\
\beta_2 x &\leq \ln \frac{1 - g(x)}{g(x)} - \beta_3 \leq \beta_1 x, \quad x < 0,
\end{align*}

where $\beta_3 = \ln \frac{1 - g(0)}{g(0)}$. And then
\begin{align*}
\frac{1}{1 + e^{\beta_2 x + \beta_3}} &\leq g(x) \leq \frac{1}{1 + e^{\beta_1 x + \beta_3}}, \quad x > 0,
\\
\frac{e^{\beta_2 x + \beta_3}}{1 + e^{\beta_2 x + \beta_3}} &\leq 1 - g(x) \leq \frac{e^{\beta_1 x + \beta_3}}{1 + e^{\beta_1 x + \beta_3}}, \quad x < 0.
\end{align*}

Therefore, (3.3) follows, and $C$ depends on $\beta_1, \beta_2, \beta_3$. \qed
Appendix B. Proof of Proposition 3.2

Proof. By multiplying \( u - \overline{u} \) on each side of (1.1) and integrating on \([0, p]\), it holds that

\[
\frac{d}{dt} \int_0^p (u - \overline{u})^2(x, t) \, dx + 2 \int_0^p (\partial_x u)^2(x, t) \, dx = 0, \, \forall t > 0. \tag{B.1}
\]

By Poincaré inequality on \([0, p]\), there exists a constant \( \alpha > 0 \), which depends only on \( p \), such that

\[
\int_0^p (\partial_x u)^2(x, t) \, dx \geq \frac{\alpha}{2} \int_0^p (u - \overline{u})^2(x, t) \, dx,
\]

then by (B.1), one has

\[
\int_0^p (u - \overline{u})^2(x, t) \, dx \leq C_0 e^{-\alpha t}, \, \forall \, t \geq 0. \tag{B.2}
\]

Here \( C_0 = \int_0^p (u_0 - \overline{u})^2 \, dx \) depends on \( p, \|u_0\|_{L^\infty} \).

Claim 1: For any integer \( k \geq 1 \),

\[
\int_0^p (\partial_x^k u)^2(x, t) \, dx \leq C, \, \forall \, t \geq 1,
\]

where \( C \) depends on \( k, p, f, \|u_0\|_{L^\infty} \).

To prove Claim 1, for each \( k \geq 0 \), let \( t_k := \frac{1}{2} - \frac{1}{k+3} \), then we define the smooth functions \( \zeta_k(t) : [0, +\infty) \to [0, 1] \), are increasing and satisfy

\[
\zeta_k(t) = \begin{cases} 0, & t \in [0, t_k] \\ 1, & t \in [1, +\infty) \end{cases}, \quad \text{and} \quad \zeta_k'(t) + \zeta_k(t) \leq B_k \zeta_{k-1}(t), \, \forall t \geq 0,
\]

here \( B_k > 0 \) is a constant depending only on \( k \). See Figure 3.

![Figure 3](image_url)

Then we prove the Claim 1 by the induction method. We will prove that for each \( k \geq 1 \), there exists a constant \( C > 0 \) depending on \( k, p, f, \|u_0\|_{L^\infty} \), such that

\[
\int_0^p (\partial_x^{k-1}(u - \overline{u}))^2 \, dx + \int_0^t \zeta_{k-1}(\tau) \int_0^p (\partial_x^k u)^2 \, dx \, d\tau \leq C, \, \forall t > t_k. \tag{B.3}
\]

In fact, when \( k = 1 \), (B.3) holds by (B.1). And we assume that (B.3) holds for \( k = 1, 2, \cdots, m \) with \( m \geq 1 \), then we will prove that (B.3) also holds for \( k = m + 1 \).

By taking the derivative \( \partial_x^m \) in (1.1) and multiplying \( \zeta_m \partial_x^m u \) on each side, one can obtain

\[
\partial_t (\zeta_m (\partial_x^m u)^2) - \zeta_m (\partial_x^m u)^2 + \partial_x \left( \zeta_m \partial_x^m u \partial_x^m f(u) \right) - \zeta_m \partial_x^{m+1} u \partial_x^m f(u)
\]
and then by integrating in $x$, and by Cauchy-Schwartz inequality, it holds that

\[
\frac{d}{dt} \int_0^p \zeta_m(t)(\partial_x^mu_0)^2 \, dx + \zeta_m(t) \int_0^p (\partial_x^{m+1}u_0)^2 \, dx \\
\leq C(\zeta_m + \zeta_{m-1}) \sum_{k=1}^m \int_0^p (\partial_x^k u_0)^2 \, dx \\
\leq CB_m \zeta_{m-1}(t) \int_0^p (\partial_x^k u_0)^2 \, dx, \quad \forall t > 0,
\]

thus one can have that

\[
\int_0^p \zeta_m(t)(\partial_x^mu_0)^2 \, dx + \int_0^t \zeta_m(\tau) \int_0^\tau (\partial_x^{m+1}u_0)^2 \, dx d\tau \\
\leq C_m \sum_{k=1}^m \int_0^t \zeta_{k-1}(\tau) \int_0^\tau (\partial_x^k u_0)^2 \, dx d\tau, \quad \forall t > 0,
\]

here $C_m > 0$ depends on $f, p, \|u_0\|_{L^p}$. Then for any $t > t_m$, by (B.3) for $k = 1, 2, \ldots, m$, the right hand side of (B.5) is bounded by a constant if $t > t_m$, so one can prove (B.3) holds for $k = m + 1$. Thus, by the induction method, (B.3) holds for any $k \geq 1$ and any $t \geq 1$, which completes the proof of Claim 1.

Then by Sobolev inequality and Claim 1, and combined with the equation (1.1), one can have that for any integers $k, l \geq 0$,

\[
\|\partial_t^l \partial_x^k (u - \overline{u})\|_{L^p(\mathbb{R})} \leq C_{k,l}, \quad \forall t \geq 1.
\]

Since for each $k \geq 0$, $\zeta_k(t) = 1$ and $\zeta_k'(t) = 0$ for all $t \geq 1$, then by (B.3) and (B.4), it holds that

\[
\frac{d}{dt} \int_0^p \left(\partial_x^k(u - \overline{u})\right)^2 \, dx + \int_0^p (\partial_x^{k+1}u_0)^2 \, dx \leq C \sum_{l=1}^k \int_0^p (\partial_x^l u_0)^2 \, dx, \quad \forall t \geq 1.
\]

here $C > 0$ depends on $k, p, f, \|u_0\|_{L^p}$.

**Claim 2:** For each $k \geq 0$, there holds that

\[
\int_0^p \left(\partial_x^k(u - \overline{u})\right)^2 (x, t) \, dx \leq Ce^{-\alpha t}, \quad \forall t \geq 1,
\]

where $C > 0$ depends on $k, p, f, \|u_0\|_{L^p}$.

To prove Claim 2, we also use the induction method. For $k = 0$, (B.7) can proved by (B.2). Thus, one can assume that for $k = 0, 1, \ldots, m - 1$ with $m \geq 1$, Claim 2 is true. Then for $k = m$, by (B.6) with $k = m$, one has that

\[
\frac{d}{dt} \int_0^p (\partial_x^m u_0)^2 \, dx \leq C_m \sum_{k=1}^m \int_0^p (\partial_x^k u_0)^2 \, dx \\
\leq Ce^{-\alpha t} + C_m \int_0^p (\partial_x^m u)^2 \, dx, \quad \forall t \geq 1,
\]
here $C, C_m > 0$ depends on $m, p, f, \|u_0\|_{L^x}$, and one can choose $C_m$ to satisfy $C_m > \alpha$. And by \eqref{B.6} with $k = m - 1$, one can have that
\begin{equation}
\frac{d}{dt} \int_0^p \left( \partial_x^{m-1}(u - \overline{u}) \right)^2 dx + \int_0^p (\partial_x^m u)^2 dx \leq Ce^{-\alpha t}, \quad \forall t \geq 1.
\end{equation}
Then by multiplying $2C_m$ on \eqref{B.9} and then adding it to \eqref{B.8}, one can obtain
\begin{equation}
\frac{d}{dt} \left[ 2C_m \int_0^p \left( \partial_x^{m-1}(u - \overline{u}) \right)^2 dx + \int_0^p (\partial_x^m u)^2 dx \right] + C_m \int_0^p (\partial_x^m u)^2 dx \leq Ce^{-\alpha t}, \quad \forall t \geq 1.
\end{equation}
Thus, by denoting
$$E_m(t) := 2C_m \int_0^p \left( \partial_x^{m-1}(u - \overline{u}) \right)^2 dx + \int_0^p (\partial_x^m u)^2 dx,$$
and by multiplying $2C_m^2$ on \eqref{B.7} with $k = m - 1$, it holds that
\begin{equation}
E_m'(t) + C_mE_m(t) \leq C_m' e^{-\alpha t}, \quad \forall t \geq 1.
\end{equation}
Since $C_m > \alpha$, one can easily obtain that $E_m(t) \leq Ce^{-\alpha t}$, where $C > 0$ depends on $m, p, f, \|u_0\|_{L^x}$. The proof of Claim 2 is finished.

Then by Sobolev inequality and Claim 2, and combined with the equation \eqref{1.1}, one can have that for any integers $k, l \geq 0$,
$$\|\partial_t^k \partial_x^l (u - \overline{u})\|_{L^x(\mathbb{R})} \leq Ce^{-\alpha t}, \quad \forall t \geq 1,$$
which finishes the proof of Proposition 3.2. \qed

\section*{References}
\begin{enumerate}
\item Constantine M. Dafermos, \textit{Hyperbolic conservation laws in continuum physics}, fourth ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 325, Springer-Verlag, Berlin, 2016. MR 3468916
\item Freistühler, Heinrich and Serre, Denis, \textit{$l^1$ stability of shock waves in scalar viscous conservation laws}, Communications on Pure and Applied Mathematics \textbf{51} (1998), no. 3, 291–301.
\item Avner Friedman, \textit{Partial differential equations of parabolic type}, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.
\item Jonathan Goodman, \textit{Nonlinear asymptotic stability of viscous shock profiles for conservation laws}, Archive for Rational Mechanics and Analysis \textbf{95} (1986), no. 4, 325–344.
\item \textit{Stability of viscous scalar shock fronts in several dimensions}, Trans. Amer. Math. Soc. \textbf{311} (1989), no. 2, 683–695.
\item Jonathan Goodman and Zhouping Xin, \textit{Viscous limits for piecewise smooth solutions to systems of conservation laws}, Archive for Rational Mechanics and Analysis \textbf{121} (1992), no. 3, 235–265.
\item Eduard Harabetian, \textit{Rarefactions and large time behavior for parabolic equations and monotone schemes}, Communications in Mathematical Physics \textbf{114} (1988), no. 4, 527–536.
\item David Hoff and Kevin Zumbrun, \textit{Asymptotic behavior of multidimensional scalar viscous shock fronts}, Indiana University Mathematics Journal \textbf{49} (2000), no. 2, 427–474.
\item \textit{Pointwise green’s function bounds for multidimensional scalar viscous shock fronts}, J. Differential Equations \textbf{183} (2002), no. 2, 368–408.
\item Eberhard Hopf, \textit{The partial differential equation $u_t + uu_x = \mu u_{xx}$}, Communications on Pure and Applied Mathematics \textbf{3} (1950), no. 3, 201–230.
\item Peter Howard, \textit{Pointwise green’s function approach to stability for scalar conservation laws}, Communications on Pure and Applied Mathematics \textbf{52} (1999), no. 10, 1295–1313.
\end{enumerate}
12. A. M. Il’in and O. A. Oleinik, *Asymptotic behavior of solutions of the cauchy problem for some quasilinear equations for large values of time*, Matematicheskii Sbornik 51(93) (1960), no. 2, 191–216.

13. C. E. Kenig and F. Merle, *Asymptotic stability and biouville theorem for scalar viscous conservation laws in cylinders*, Communications on Pure and Applied Mathematics 59 (2006), no. 6, 769–796.

14. N. Kopell and L. N. Howard, *Bifurcations and trajectories joining critical points*, Advances in Mathematics 18 (1975), no. 3, 306–358.

15. Heinz-Otto Kreiss, *Fourier expansions of the solutions of the Navier-Stokes equations and their exponential decay rate*, Analyse mathématique et applications, Gauthier-Villars, Montrouge, 1988, pp. 245–262. MR 956963

16. S N Kruzkov, *First order quasilinear equations with several independent variables*, Mat. Sb. (N.S.) 81 (123) (1970), 228–255.

17. N. N. Kuznetsov, *Accuracy of some approximate methods for computing the weak solutions of a first-order quasi-linear equation*, USSR Computational Mathematics and Mathematical Physics 16 (1976), no. 6, 105–119.

18. Tai-Ping Liu, *Nonlinear stability of shock waves for viscous conservation laws*, Bullentin (New Series) of the American Mathematical Society 12 (1985), no. 2, 233–236.

19. , *Pointwise convergence to shock waves for viscous conservation laws*, Communications on Pure and Applied Mathematics 50 (1997), no. 11, 1113–1182.

20. Tai-Ping Liu and Zhouping Xin, *Nonlinear Stability of Rarefaction Waves for Compressible Navier Stokes Equations*, Communications in Mathematical Physics 118 (1988), 451–465.

21. Akitaka Matsumura and Kenji Nishihara, *On the stability of travelling wave solutions of a one-dimensional model system for compressible viscous gas*, Japan J. Appl. Math. 2 (1985), no. 1, 17–25.

22. Kenji Nishihara, *A note on the stability of travelling wave solutions of burgers’ equation*, Japan Journal of Applied Mathematics 2 (1985), no. 1, 27–35.

23. O. A. Oleinik, *Discontinuous solutions of non-linear differential equations*, Uspehi Mat. Nauk (N.S.) 12 (1957), no. 3(75), 3–73.

24. Anders Szepessy and Zhouping Xin, *Nonlinear stability of viscous shock waves*, Archive for Rational Mechanics and Analysis 122 (1993), no. 1, 53–103.

25. Zhouping Xin, *Asymptotic stability of rarefaction waves for 2×2 viscous hyperbolic conservation laws*, Journal of Differential Equations 73 (1988), no. 1, 45–77.

26. , *Asymptotic stability of planar rarefaction waves for viscous conservation laws in several dimensions*, Trans. Amer. Math. Soc. 319 (1990), no. 2, 805–820.

27. Zhouping Xin, Qian Yuan, and Yuan Yuan, *Asymptotic stability of shock waves and rarefaction waves under periodic perturbations for 1-D convex scalar conservation laws*, arXiv:1809.09308 (2018).

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