Development of Partition of Unity Finite Element Method for the Modelling of Acoustic Waves Propagating through Porous Domain

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Abstract. The numerical simulation of propagating sound fields in two dimensions using Partition of Unity Finite Element Method (PUFEM) technique is implemented in this work. PUFEM is one of the most practical methods developed to alleviate high computational cost of conventional Finite Element Method when we deal with the medium and high frequency problem. The main idea of PUFEM is to expand the finite element approximation space by incorporating a group of wave functions which are solutions of the governing equation into the target problem. With the intention of further accelerating the computational process, an advanced 2D Exact Integration Technique (EIT) for accelerating the computation process of highly oscillating integrals is proposed and the related PUFEM triangular element is created. The performance and practical interest of this new PUFEM element combined with EIT are evaluated for the wave propagation problem where a porous absorber is included.

1. Introduction

Due to the excessive demands of computational cost and the dispersion error problems, the conventional Finite element Method is not sufficient for solving the medium and high frequency acoustic problems. The primary reason lies in the fact that at least 7 to 10 nodal points are required to fully describe the single wavelength in the context of traditional FEM. To tackle this limitation, some new deterministic prediction methods are proposed based on the theory of wave based method where the dynamic field variable is expanded with a set of wave equations, these wave equations are exactly the solutions to the governing partial differential equation. These new deterministic numerical methods include the Wave-Based Methods[1], the Variational Theory of Complex Rays[2], the Ultra-Weak formulation[3], the Discontinuous Galerkin Method[4] as well as the Partition of Unity Finite Element Method (PUFEM)[5]. In comparison with the conventional FEM, these methods allow us to greatly benefit from the computational efficiency and reduce the required degrees of freedom to achieve the accuracy level of simulation results. Among these advanced wave based methods, PUFEM enjoys the advantage of being easily adapted to the common FEM mesh, and the applications of this method to the various wave problems can be found in the literature[6,7,8].

Although PUFEM contributes to a remarkable reduction of computational cost and time for obtaining the element matrices, it is still difficult to compute the integration of highly oscillatory wave functions because a large number of integration points will be involved. To alleviate this limitation, several efficient integration techniques were proposed and the efficiency of PUFEM has been largely improved. A semi-analytical method for computing the integration of the oscillatory functions through
the use of quadrilateral finite elements was proposed by Sugimoto[9]. In another approach, Kacimi[10] used an explicit closed-form solution to evaluate two-dimensional oscillatory integrals. However, to the best of author’s knowledge, no similar work has been investigated for the numerical simulation of acoustic wave transmitting through porous domains in the framework of both PUFEM and two-dimensional exact integration technique.

In this paper, a 2D Exact Integration Technique (EIT) for accelerating the computing process of high frequency wave transmission phenomenon is introduced and explained with all necessary details. With this semi-analytical integration technique, it becomes more efficient and convenient to evaluate the system coefficient matrices of medium and high frequency wave problems. In addition, a numerical model involving acoustic transmission problem of practical interest is established for the validation purpose. Finally, we also investigate the efficiency performance of this numerical model using newly-developed PUFEM triangular element.

2. PUFEM for 2D Acoustic Waves

The sound pressure \( p \) in the governing equation of the wave problems refers to the classical Helmholtz equation[5],

\[
\Delta p + \kappa^2 p = 0
\]  

in which \( \omega \) is the harmonic angular frequency, \( \kappa = \omega / c \) denotes the wavenumber. By applying the weighted residual scheme on the above governing equation and proceeding to the integration by parts, we then arrive at the weak form of the Helmholtz equation as [8],

\[
\int_{\Omega} (\nabla p \cdot \nabla (\delta p) - \kappa^2 p \delta p) d\Omega - \int_{\Gamma} \frac{\partial p}{\partial n} \delta p d\Gamma = 0
\]  

where \( \Omega \) is a closed two-dimensional domain with boundary \( \Gamma \) in figure 1. PUFEM provides a basic numerical structure in which a part of the analytical solutions of PDE can be incorporated into the shape function. For the sake of convenience, it is common to employ plane waves as the expansion basis for sound pressure field, therefore, the wave expansion process can be written as,

\[
p(x) = \sum_{j=1}^{3} N_j \sum_{q=1}^{Q_j} A_{jq} \exp(i\kappa d_{jq} \cdot x)
\]  

where \( p \) is the sound pressure field, it is also the function of \( x \), \( Q_j \) represents the number of employed plane wave directions attached to each node of PUFEM element in the wave expansion process, \( d_{jq} \) denotes a group of directional vectors of the adopted plane wave basis on \( j^{th} \) node, \( N_j \) are the shape functions. Figure 2 depicts the two-dimensional plane wave expansion process of PUFEM where a set of plane wave basis consists of 4 plane waves are used in the nodal enrichment process of PUFEM element, namely, \( Q_j = 4 \). Besides, \( A_{jq} \) is the unknowns for the PUFEM discretization process, which represents the amplitude of each plane wave on the \( j^{th} \) node of PUFEM triangular element.

![Figure 1. Different types of boundary condition of interior sound problem.](image-url)
Figure 2. Triangle PUFEM element enriched by 4 plane waves on each node.

By substituting the weak form of governing equation (2) with the plane wave basis in equation (3), we can derive the PUFEM system coefficient matrices as,

\[
A_{pum} = -\kappa^2 (1 + \mathbf{d}' \cdot \mathbf{d}^*) \int_{\Omega_e} N_i N_j \phi d\Omega + i\kappa \mathbf{d}' \cdot \nabla N_j \int_{\Omega_e} N_i \phi d\Omega
\]

we can observe that the matrix coefficients consist of 4 integrals over an element \(\Omega_e\), where we introduce \(\phi = \exp(i\kappa(\mathbf{d}' + \mathbf{d}^*) \cdot \mathbf{x})\). Here, \(\mathbf{d}'\) and \(\mathbf{d}^*\) denote, respectively, the directional vectors of the trial function and test function. Then, due to the fact that the shape functions are linear over the triangle element, the integration domain can be easily reduced to closed-form expressions by applying the Green theorem, thus only the information of node coordinates will become necessary for the final evaluation of the system matrices. To be more precise, the use of Green theorem shift the integration domain of equation (1) from the original surface area \(\Omega_e\) to its linear boundary \(\Gamma = \partial \Omega_e\), and only the first three terms need to be taken from the infinite series for reach the sufficient accuracy, which yields,

\[
\int_{\Omega_e} F \phi d = -\int_{\Gamma_e} \left( \epsilon F \mathbf{d} + \epsilon^2 \nabla F + \epsilon^3 \Delta F \mathbf{d} \right) \cdot \mathbf{n} \phi d\Gamma
\]

where \(\epsilon = i/\kappa\) and \(F\) represents the polynomial functions of the which the maximum order is two, this means that this function can be a constant, a linear or a quadratic function. Through the use of above exact integration scheme, we finally arrive at the analytical formulas for the evaluation of system matrices of PUFEM and the high-order numerical quadrature method can be avoided. The efficiency of the PUFEM triangular element combined with the exact integration technique will be shown in the next section.

3. Numerical Example

Firstly, the main focus of this section is on the validation of the PUFEM element through a simplified numerical application taken from the reference [7]. Next, we aim at comparing the computational efficiency (CPU time) of the new developed PUFEM triangular element to the classical Gaussian quadrature method. Finally, we investigate the numerical performances of PUFEM element with EIT through an example of standing wave tube where porous material is involved. The program and corresponding code are developed in Matlab and all the numerical cases are implemented on the Pilcam win64 server with the core of Xeon 2.67GHz.

3.1. The Establishment of Numerical Model

In this section, the geometry of the this numerical example in shown figure 3, it is constituted of a common standing wave tube of which the total length \(L\) is equal to 0.15 m, the whole length is divided into three sections, the middle part consists of porous absorber and is depicted in gray whereas the rest
merely contain the air. In order to simulate a wave transmission process in the standing wave tube, an incident wave boundary condition expressed by the formula \( \frac{\partial p_x}{\partial n_x} = 1 \) is applied on the left side of the built model. To take into account the sound absorbing effect in the established model, the second region started from the position \( x_1 = L/3 \) is filled with porous material and denoted as porous domain in gray as shown in figure 3. After propagating through the porous region, the sound wave enter into the air region again and this region is supported and fixed by a rigid wall on the end. Besides, the mesh partitioning of the model is also presented in figure 3 where we can observe that each region of this tube is partitioned into two triangular elements. Note that \( h_{\text{max}} \) is an important factor for determining the required number of plane wave functions used in the wave field approximation process, and it is normally equal to the length of longest edge of generated PUFEM mesh. For this case, \( h_{\text{max}} \approx \sqrt{0.05^2 + 0.03^2} \approx 0.06 \text{m}. \)

![Figure 3. Generated PUFEM mesh for the current problem](image)

### 3.2. Performance of the Method

With the aim of simulating medium and high frequency wave transmission problem, the non-dimensional quantity \( \kappa \lambda h_{\text{max}} \) has to be set as a relative large number, here we set \( \kappa \lambda h_{\text{max}} = 50 \) in which \( \kappa \lambda \) indicates the wavenumber in the air domain. Through the calculation, we can roughly estimate that the corresponding frequency \( f \approx 45,000 \text{ Hz} \), and around eight wavelengths are contained in each PUFEM element in the current mesh grid. Figure 4 and 5 present the pressure field and the computed sound pressure level (in dB) in the test tube with respect to its horizontal axis. The simulation results show that the porous absorber in the middle region of the tube greatly reduce the sound pressure of incident wave to a very low level. In this case, the wavenumber in the porous region \( \kappa_p = \kappa \lambda (1 + 0.1i) \).

![Figure 4. Analytical solution (top) and PUFEM solution (bottom) of sound pressure field in the standing wave tube, for \( \kappa \lambda h_{\text{max}} = 50 \) and \( f = 45,000 \text{ Hz} \).](image)
The second observation from figure 4 and figure 5 is that the computational results using PUFEM technique are very close to the analytical solution. The top concern of this numerical test is to compare the numerical benefit (CPU time) of EIT with the counterpart of standard quadrature method, and at least 10 Gauss points are adopted for each wavelength in the current numerical quadrature. In order to independently analyze low frequency problems and high frequency problems, we separate the frequency range into two categories, low frequency range is from 900 Hz to 4,000 Hz and high frequency range is from 6,000 Hz to 45,000 Hz. For the sake of fairness, two integration methods are compared for solving the PUFEM system coefficient matrices of same numerical cases.

![Figure 5](image1.png)

**Figure 5.** SPL in (dB) in the middle section of standing wave tube, for $\kappa h_{\text{max}} = 50$, corresponding to $f \approx 45,000$ Hz, the PUFEM solution is shown by red cross symbol and the analytical solution is shown by blue curve.

It is shown that, in the low frequency range, the efficiency of exact integration technique is lower than Gauss-Legendre quadrature method, this is because that the number of Gauss points required for the integrals to reach a certain level of accuracy for low frequency problem $(f < 4000)$ Hz is small. However, exact integration scheme has the property of frequency independent, the elapsed CPU time, therefore, will not be largely affected by the increase of frequency. More precisely, the elapsed CPU time of EIT will be only influenced by the plane wave functions in the adopted plane wave basis. In contrast, the consuming time for the quadrature method increases severely with the growing number of gauss points for high frequency wave problem due to the “thumb rule” of 10 nodes per wavelength.

![Figure 6](image2.png)

**Figure 6.** Comparison between two integration schemes in terms of computational time: EIT (circle) and quadrature method (triangular).
4. Conclusion
In this paper, the PUFEM triangular element combining with exact integration technique is carried out and applied to solve a high frequency wave transmission problem where absorbing material is taken into account. The concrete process of implementing Exact Integration Technique is introduced which allows us to evaluate the system coefficient matrices in an efficient manner, and to remarkably reduce the computational cost while maintaining high accuracy level of the simulation results. The simulation results based on PUFEM technique have been verified and validated by comparing with the analytical solutions of the simple tube case. Moreover, the numerical performance of the Exact Integration Technique is also investigated by comparing to the classical Gauss-Legendre quadrature method, it turns out that the developed two-dimensional EIT is more efficient than the traditional quadrature method in terms of computational cost when medium and high frequency wave problems are involved.

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