A note on causality in the bulk and stability on the boundary

Jan Troost *
Center for Theoretical Physics
MIT 77 Mass Ave
Cambridge, MA 02139 USA

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Abstract

By carefully analyzing the radial part of the wave-equation for a scalar field in $AdS_{d+1}$, we show that for a particular range of boundary conditions on the scalar field, the radial spectrum contains a bound state. Using the AdS/CFT correspondence, we interpret this peculiar phenomenon as being dual to an unstable double trace deformation of the boundary conformal field theory. We thus show how the bulk theory holographically detects whether a boundary perturbation is stable.

1 Introduction

In the study of the AdS/CFT correspondence [1][2][3], we have learned to translate holographically many characteristics of the $AdS_{d+1}$ bulk gravitational theory into features of the boundary conformal field theory, and vice versa. It is our intention in this paper to add an entry to the holographic dictionary. It was shown previously that a perturbation of the boundary conformal field theory with a double trace operator (see e.g. [4]), can be mimicked in the bulk $AdS_{d+1}$ theory by imposing particular radial boundary conditions on the dual scalar field [5][6] (see also [7][8][9][10][11]). Depending on the sign of the coefficient of the perturbation, the boundary theory will be stable or unstable after perturbation [5]. We ask how the bulk theory detects an unstable boundary deformation. Technically, the answer to this innocuous question lies hidden in a careful determination of the spectrum of the radial part of the wave equation in $AdS_{d+1}$, to which we turn.

2 Set-up

We write the $AdS_{d+1}$ metric in Poincare coordinates $(u, x^\mu)$ as:

$$ds^2 = \frac{L^2}{u^2}(du^2 + (dx^\mu)^2)$$

\*troost@mit.edu; Preprint MIT-CTP-3407
where \( x^\mu \) parametrizes a \( d \)-dimensional Minkowski space (with mostly plus signature) and the \( u \)-coordinate runs over the interval \([0, \infty[. \) The scale \( L \) sets the radius of the \( AdS_{d+1} \) space (and the negative cosmological constant). In the following we set \( L = 1 \). The \( u \)-coordinate is intuitively thought of as the inverse of a radial coordinate \((u = 1/r)\) with the boundary of the \( AdS \) space at \( u = 0 \). The wave-equation for a massive minimally coupled scalar field \( \Phi \) is:

\[
(\Box - m^2)\Phi(u, x^\mu) = 0
\]

(2)

where \( \Box \) denotes the wave-operator in \( AdS_{d+1} \). We will analyze the solutions to the wave-equation by using the method of separation of variables. We expand the solutions in multiples of Minkowski plane waves \( \phi(x^\mu) = e^{ik_\mu x^\mu} \) as \( \Phi = e^{ik_\mu x^\mu} u^{d-1} f(u) \). The Minkowski and the radial part of the wave-function then satisfy the equations:

\[
(\Box_M - \lambda)\phi(x^\mu) = 0
\]

\[
-f''(u) + \frac{\nu^2 - 1/4}{u^2} f(u) = \lambda f(u),
\]

(3)

where \( \Box_M \) denotes the wave-operator in Minkowski space, and \( \lambda = -k_\mu k^\mu \) can be thought off as a Minkowski mass squared. The parameter \( \nu \) is defined by the formula \( \nu = \sqrt{m^2 + d^2/4} \). The radial part of the wave-equation corresponds to an interesting, non-trivial Sturm-Liouville problem which we need to analyze in detail.

### 3 Sturm-Liouville

The key to the bulk interpretation of an unstable double trace boundary deformation will be the spectrum of the radial Sturm-Liouville problem. In this section, we will determine the spectrum rigorously.

The potential for the radial problem is \( q(u) = \frac{\nu^2 - 1/4}{u^2} \). It is well-known that at the end of the half-line (at \( u = 0 \)), we need to specify boundary conditions for the Sturm-Liouville problem when \( 0 \leq \nu < 1 \), because then the potential is not steep enough to provide for a single dynamical solution to the (one-dimensional) scattering problem. For these values of the mass squared \([12][13]\), we need to specify how a wave reflects from the end point of the half-line (i.e. at radial infinity in \( AdS_{d+1} \)) to determine its full evolution. (I.e. we need to choose a self-adjoint extension of the radial differential operator \([14]\). See also \([15][16]\).) We restrict to the interesting mass squared range \(-d^2/4 \leq m^2 < 1 - d^2/4 \) where \( 0 \leq \nu < 1 \). Note that we take the scalar field to satisfy the Breitenlohner-Freedman lower bound \([12][13]\).

For the analysis of the Sturm-Liouville problem we follow the rigorous textbook treatment in \([17]\). We refer to \([17]\) for a pedagogical explanation of the nomenclature and techniques involved in the following subsections. The two ends of the half-line are singular points for the Sturm-Liouville problem. To treat them rigorously, we cut the half-line into an interval \([0, \epsilon[ \) and another half-line \([\epsilon, \infty[ \) (see e.g. \([17]\) section 2.18). Physically, we introduce an infrared bulk cut-off \( \epsilon \) and study the behavior at radial infinity (small \( u \)) separately from the behavior of the wave-function in the far interior (large \( u \)). At the end, we will combine the solutions to the separate Sturm-Liouville problems. The following subsections contain the necessary technical manipulations to obtain the spectrum, which is summarized at the end of subsection 3.3.
3.1 Near the boundary

On the interval $]0, \epsilon[,$ there are two solutions to the radial equation which are both quadratically integrable. We are in the limit circle case for the singularity at $0$ (see e.g. [17] p.23-25). Given two normalizable solutions $\phi$ and $\theta,$ we then have to specify a boundary condition to pick a unique solution $\psi_1 = \theta + m_1(\lambda)\phi$ that satisfies these boundary conditions. Thus, the function $m_1(\lambda),$ which encodes the spectrum, depends on the boundary conditions. Some technical details follow. We define $\phi$ and $\theta$ to be the solutions to the differential equation satisfying the boundary conditions:

$$\begin{align*}
\phi(\epsilon, \lambda) &= 0 \quad \phi'(\epsilon, \lambda) = -1 \\
\theta(\epsilon, \lambda) &= 1 \quad \theta'(\epsilon, \lambda) = 0.
\end{align*}$$ (4)

They are given by (with $\lambda = s^2$):

$$\begin{align*}
\phi(u, \lambda) &= \frac{\pi}{2}u^{1/2}\epsilon^{1/2}(J_\nu(us)Y_\nu(\epsilon s) - Y_\nu(us)J_\nu(\epsilon s)) \\
\theta(u, \lambda) &= \frac{\pi}{2}u^{1/2}\epsilon^{1/2}s(J_\nu(us)Y'_\nu(\epsilon s) - Y_\nu(us)J'_\nu(\epsilon s)) + \frac{\phi(u, \lambda)}{2\epsilon}
\end{align*}$$ (5)

as is easily verified using the fact that $J_\nu(x)Y'_\nu(x) - J'_\nu(x)Y_\nu(x) = 2/(\pi x).$ We suppose in the following that $\nu \neq 1/2$ and $0 < \nu < 1.$ (The case $\nu = 0$ deserves a separate treatment. See e.g. [5][17].) We consider a solution with boundary condition $\cot \delta = -\frac{\theta'(\epsilon)}{\theta(\epsilon)}$ and define the limit circle as the limit of the circles (circumscribed by varying $\cot \delta$) as $\epsilon \to 0$:

$$l(\epsilon) = -\frac{\theta(\epsilon, \lambda)\cot \delta + \theta'(\epsilon, \lambda)}{\phi(\epsilon, \lambda)\cot \delta + \phi'(\epsilon, \lambda)}. $$ (6)

When we take the limit $\epsilon \to 0,$ with $\lambda = s^2$ fixed, we find it useful to define the quantity $c$ by the formula:

$$\frac{\epsilon^{1/2-\nu}\cot \delta + (1/2 - \nu)\epsilon^{-1/2-\nu}}{2^{-\nu}\Gamma(1-\nu)} \equiv c\frac{\epsilon^{1/2+\nu}\cot \delta + (1/2 + \nu)\epsilon^{-1/2+\nu}}{2^\nu\Gamma(1+\nu)}. $$ (7)

Using the fact that $\cot \delta = O(\frac{1}{\epsilon})$ in this limit, we find that (for the specified range of $\nu$) $l(\lambda)$ asymptotes to $m_1(\lambda)$:

$$m_1(\lambda) = -s\frac{cs^{-\nu}J'_\nu(\epsilon s) - s^{\nu}J'_{-\nu}(\epsilon s)}{cs^{-\nu}J_\nu(\epsilon s) - s^{\nu}J_{-\nu}(\epsilon s)} - \frac{1}{2\epsilon}. $$ (8)

Note that $c$ parametrizes the boundary condition at radial infinity, via $\cot \delta$ and that $m_1(\lambda)$ encodes the chosen normalizable solution as well as the spectrum of the Sturm-Liouville operator with the given boundary conditions (see e.g. [17] section 3.9).

3.2 The interior

We now turn to the analysis of the Sturm-Liouville problem on the half-line $]\epsilon, \infty[.$ When we regularize the Sturm-Liouville problem by adding a small positive imaginary part to $\lambda,$ the one normalizable solution is proportional to the first Hankel function $H^{(1)}_\nu(us).$ (Physically, the regularisation procedure...
picks out a particular contour prescription for the propagator \([16]\). That leads to the expression for the function \(m_2(\lambda)\) (as can easily be checked using the fact that the normalizable solution is \(\psi_2 = \theta + m_2 \phi\), and formulas \((5)\):

\[
m_2(\lambda) = -s \frac{H_{\nu}^{(1)'}(\epsilon s)}{H_{\nu}^{(1)}(\epsilon s)} - \frac{1}{2\epsilon} \tag{9}
\]

### 3.3 The full problem

Next we combine our results to treat the full Sturm-Liouville problem on the half-infinite interval with the two singular ends. We take \(\epsilon\) as the basic point. We have found the wave functions

\[
\psi_1 = \frac{u^{1/2} c J_\nu(\epsilon s) - s^{2\nu} J_\nu(\epsilon s)}{c^{1/2} c J_\nu(\epsilon s) - s^{2\nu} J_{-\nu}(\epsilon s)}
\]

\[
\psi_2 = \frac{u^{1/2} H^{(1)}(\epsilon s)}{c^{1/2} H^{(1)}(\epsilon s)}.
\]

To obtain the spectrum\(^1\), we need to study the function \(m_1(\lambda) - m_2(\lambda)\). For \(\lambda > 0\) (\(s\) real and positive) we obtain:

\[
-\text{Im} \frac{1}{m_1(\lambda) - m_2(\lambda)} = \frac{\pi \epsilon (c J_\nu(\epsilon s) - s^{2\nu} J_{-\nu}(\epsilon s))^2}{2 \left( c^2 - 2cs^{2\nu} \cos \nu \pi + s^{4\nu} \right)} \tag{11}
\]

such that we find a continuous spectrum (since the above expression varies continuously with \(\lambda\) and does not contain poles) for \(\lambda > 0\). For \(\lambda < 0\) (\(\lambda = s^2\), \(s = it\) and \(t\) real and positive) we find that:

\[
-\frac{1}{m_1(\lambda) - m_2(\lambda)} = \frac{\epsilon K_\nu(\epsilon t)(c I_\nu(\epsilon t) - t^{2\nu} I_{-\nu}(\epsilon t))}{c - t^{2\nu}} \tag{12}
\]

is real. Now, if \(c < 0\), the last expression is continuous with no poles and gives no contribution to the spectrum. When \(c > 0\), we find a pole, which indicates another (discrete) eigenvalue in the spectrum. The new eigenvalue \(\lambda = -c^{1/\nu}\) is central to our paper.

As a check on our formalism, note that the Green functions \(G(x, y; \lambda)\) can be written down as a combination of \(m_{1,2}\) and \(\psi_{1,2}\):

\[
G(x, y; \lambda) = \frac{\psi_2(x, \lambda) \psi_1(y, \lambda)}{m_2(\lambda) - m_1(\lambda)} \quad \text{for} \quad (y \leq x)
\]

\[
= \frac{\pi i}{2} \frac{1}{c - s^{2\nu} e^{-i\nu \pi}} x^{1/2} H^{(1)}_{\nu}(x s) y^{1/2} (c J_\nu(y s) - s^{2\nu} J_{-\nu}(y s)). \tag{13}
\]

It is understood in the previous expression that the role of \(\psi_1\) and \(\psi_2\) is reversed for \(y > x\). We note that the Green function agrees with the radial part of the Green function computed in \([18]\) (equation \((32)\)), after using the formulas \(\pi i H^{(1)}_{\nu}(iz) = 2e^{-i\nu \pi/2} K_\nu(z)\) and \(I_\nu(z) = e^{-i\nu \pi/2} J_\nu(iz)\) which connect different forms of the (modified) Bessel functions.

However, to obtain the Green function for the full bulk problem in \(AdS_{d+1}\), it is crucial to know the precise spectrum of eigenvalues \(\lambda\) over which we have to integrate the product of the Minkowski and radial Green functions. That is

\(^1\)For the general theory see \([17]\) p. 52-53 and section 3.9.
why we embarked on the long derivation of the spectrum above. We have found a continuous spectrum, with an extra discrete eigenvalue when \( c > 0 \), i.e. an extra discrete eigenvalue appears for a particular range of boundary conditions. The bound state is directly associated to the appearance of a pole in the Green function (13).

4 Holographic translation

We demonstrate now that the peculiar feature of the bulk theory uncovered above, provides the answer to the question we posed in the introduction. Recall that in the mass range \(-d^2/4 < m^2 < 1 - d^2/4\), we can approximate the scalar field near the boundary by:

\[
\Phi(u) \approx \alpha u^{\frac{d}{2}-\nu} + \beta u^{\frac{d}{2}+\nu}.
\]

We show how the values for \( c \) translate into properties of the coefficients \( \alpha \) and \( \beta \). We define the following quantity which encodes the boundary condition on the scalar field, but does not scale with \( \epsilon \):

\[
A = \epsilon \cot \delta.
\]

It satisfies the relation:

\[
c = \epsilon^{-2\nu} 2^{2\nu} \frac{\Gamma(1+\nu) 1/2 - \nu + A}{\Gamma(1-\nu) 1/2 + \nu + A}.
\]

Since we have that \( \epsilon > 0 \) and \( 0 < \nu < 1 \), it is easy to check that \( c \) is positive when \( A > \nu - 1/2 \) and when \( A < -\nu - 1/2 \), and that \( c \) is negative for \( -\nu - 1/2 < A < \nu - 1/2 \).

Recall that \( A = -\epsilon f'(\epsilon) / f(\epsilon) \) (and \( f(u) = \alpha u^{1/2-\nu} + \beta u^{1/2+\nu} \)), such that we find that for the boundary conditions consistent with the full \( AdS_{d+1} \) isometry group (i.e. boundary conformal symmetry), i.e. \( \beta = 0 \) or \( \alpha = 0 \), we obtain the limiting values \( A = \nu - 1/2 \) or \( A = -\nu - 1/2 \) respectively. When \( \alpha \neq 0 \), we can show that \( A > \nu - 1/2 \) for small \( \epsilon \) and \( \beta / \alpha > 0 \) while \( A < \nu - 1/2 \) for \( \beta / \alpha < 0 \).

In general we find that there is no bound state for \( \beta / \alpha > 0 \) and \( c < 0 \), while there is a bound state for \( c > 0 \) or in other words \( \beta / \alpha < 0 \). (Note that for \( \alpha \neq 0 \) we find that \( c = -2^{2\nu} \frac{\Gamma(1+\nu) \beta}{\Gamma(1-\nu) / \alpha} \) for small \( \epsilon \).)

We note in passing that is not difficult to formulate an action principle (with a boundary term) that gives rise to the boundary conditions we analyzed (see e.g. [11]).

Interpretation

In [5] it was argued that the boundary condition \( \alpha = g \beta \) for the quantization of a scalar field in \( AdS \) is dual to adding a double trace deformation for the operator coupling to the field \( \Phi \) in the boundary conformal field theory. The sign of \( g \) determines whether the boundary perturbation is stable or not. We have now seen that the sign appearing in the boundary condition also determines whether the bulk wave equation allows for a bound state with negative energy in the corresponding radial Schrödinger problem. The negative energy bound state corresponds to a wave in the Minkowski slices of the Poincare patch with a disturbing negative (Minkowski) mass squared \( \lambda \) (see (3)). This is the bulk holographic signal for the instability of the boundary theory.
5 Conclusions

We added an entry to the $AdS/CFT$ dictionary. An instability in the boundary conformal field theory, caused by perturbing it with a double trace deformation with the “wrong” sign (see e.g. appendix of [19]), is detected in the bulk theory by the appearance of a solution to the wave equation with tachyonic behavior in a Minkowski slice. Note that the tachyonic behavior appeared due to the boundary conditions solely and that the Breitenlohner-Freidman is satisfied by the mass squared of the field. This is a perhaps surprising feature of the $AdS_{d+1}$ scalar wave equation (and thus of the quantization of scalar fields in $AdS_{d+1}$.) The careful analysis of the Sturm-Liouville differential equation for the radial part of the scalar field was instrumental in laying bare this quaint feature of the $AdS/CFT$ correspondence.

The quantization of a scalar field in $AdS_{d+1}$ with non-conformal, stable boundary conditions found applications in the analysis of the flow of a bulk analogue of the central charge in the boundary conformal field theory [18] and the Legendre transform between conformal fixed points [20]. (See also [21].) We have shown that the stable quantization used in that analysis seems to be the only intuitively acceptable one from a bulk perspective (although more general boundary conditions exist).

A bulk theory is specified not only by the background geometry, but also by the boundary conditions on the propagating fields. Beyond analyzing which bulk geometries are allowed (see e.g.[22]), one has to specify which boundary conditions are acceptable. One can read our analysis as specifying the set of allowed boundary conditions in the simple case of pure $AdS_{d+1}$.

It is illuminating to see how the instability of the boundary theory is translated in acausal properties of the bulk theory. In general, it would be interesting to understand still better the role of causality in the $AdS/CFT$ dictionary, which is intimately related to getting a grip on truely Lorentzian aspects of the $AdS/CFT$ correspondence (see e.g.[23] and references thereto). The basic mechanism that we identified is an element in that broader study.

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