Dependence of acoustic surface gravity on geometric configuration of matter for axisymmetric background flow in Schwarzschild metric

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In black hole evaporation process, the mass of the hole anti-correlates with the Hawking temperature enabling us to infer that the smaller mass holes will have higher surface gravity. For analogue Hawking effects, however, the acoustic surface gravity is determined by the local value of the dynamical velocity of the stationary background fluid flow and the speed of propagation of the characteristic perturbation embedded in the background fluid, as well as their space derivatives evaluated along the direction normal to the acoustic horizon, respectively. The mass of the analogue system - whether classical or quantum - does not directly contribute to extremise the value of the associated acoustic surface gravity. For general relativistic axisymmetric background fluid flow in the Schwarzschild metric, we show that the initial boundary conditions describing such axisymmetrically accreting matter flow influence the maximization scheme of the acoustic surface gravity as well as the corresponding characteristic temperature. Aforementioned background flow onto astrophysical black hole can assume three distinct geometric configurations. Identical set of initial boundary conditions can lead to entirely different phase space behaviour of the stationary flow solutions, as well as the salient features of the associated relativistic acoustic geometry. It is thus important to investigate how the acoustic surface gravity for the aforementioned classical analogue system gets influenced by the geometric configuration of the stationary axisymmetric matter flow described by various astrophysically relevant thermodynamic equations of state. Our work is useful to study the effect of gravity on the non-conventional classical features in Hawking like effect in a dispersive medium as is expected to be observed in the limit of a strong dispersion relation.

I. INTRODUCTION

Black hole analogues are fluid dynamical analogue of the black hole space time as perceived in the general theory of relativity \cite{1-8}, and such analogue systems can be constructed for a small linear perturbation propagating through a dissipationless, irrotational, barotropic transonic fluid. Contemporary research in the field of analogue gravity phenomena has gain widespread currency since it opens up the possibility of understanding the salient features of the horizon effects directly through the experimentally realizable physical configuration within the laboratory set up.

Conventional works in this direction, however, concentrate on systems not directly subjected to the gravitational force; for such systems, gravity like effects are manifested as emergent phenomena. In such cases, only the Hawking like effects can be studied and no direct connection can be made to such effects with the general relativistic Hawking effects since such non gravitating systems do not include any source of strong gravity capable of producing the Hawking radiation.

Motivated by the intention of studying whether (and how) the gravity like effects emerges from a physical system which itself is under the influence of a strong gravitational field, a series of recent work made successful attempt to explain how the acoustic geometry may be constructed for spherically and axially symmetric hydrodynamic flow onto astrophysical black holes \cite{9-16}. Such accreting black holes represent systems which simultaneously contain gravitational as well as acoustic horizons and are shown to be natural examples of large scale classical analogue systems found in the university. Such systems also allow one to study the influence of the original (background) black hole space time metric on the embedded (perturbative) acoustic metric.

Following Unruh’s pioneering work \cite{1}, the surface gravity $\kappa$ as well as the analogue temperature $T_{AH}$ can be found to be proportional to the speed of propagation of the acoustic perturbation $c_s$ and the space gradient $\partial/\partial \eta$ (taken along the normal to the acoustic horizon) of the bulk flow velocity $u_\perp$ measured along the direction normal to the acoustic horizon

$$[T_{AH}, \kappa] \propto \left( \frac{1}{c_s} \frac{\partial u_\perp}{\partial \eta} \right)_{r_h}, \quad \text{(1)}$$

where the subscript $r_h$ indicates that the quantity has been evaluated on the acoustic horizon $r_h$. The acoustic horizon is the surface defined, for the stationary flow configuration, by the equation

$$u_\perp^2 - c_s^2 = 0, \quad \text{(2)}$$
\( c_s \) was assumed to be position independent in eq. (1). For position dependent sound speed, the surface gravity as well as the analogue temperature can be obtained as [3]

\[
[T_{AH}, \kappa] \propto \left[ c_s \frac{\partial}{\partial \eta} (c_s - u_\perp) \right]_{r_h}.
\]

(3)

Expression for the surface gravity as well as the analogue temperature as defined by eq. (1) and eq. (3) corresponds to the flat background flow geometry of Minkowskian type and its relativistic generalization may be obtained as [4, 11, 12]

\[
\kappa = \left[ \sqrt{\chi^\mu \chi_\mu} \frac{\partial}{\partial \eta} (u_\perp - c_s) \right]_{r_h},
\]

(4)

where \( \chi^\mu \) is the Killing field which is null on the corresponding acoustic horizon. In subsequent sections, we will show that the explicit expression for the norm of \( \chi^\mu \) can be evaluated in terms of of the values of the corresponding metric elements evaluated on the acoustic horizon and on the value of the specific parameters governing the dynamical state of matter (specific angular momentum parameterizing the degree of rotation of an axisymmetric flow, for example).

For Newtonian as well as for the general relativistic fluid, one thus needs to calculate the location of the acoustic horizon \( r_h \), as well as to evaluate the expression for the normal bulk flow velocity \( u_\perp \) and the speed of the propagation of the acoustic perturbation \( c_s \) along with their space gradients normal to the acoustic horizon to compute the value of the surface gravity as well as the corresponding analogue temperature. For axially symmetric non-self gravitating hydrodynamic low angular momentum inviscid accretion onto a non rotating black hole, \( r_h \) and \([u, c_s, du/dr, dc_s/dr]_{r_h}\) can be evaluated using the initial boundary conditions defined by the triad \([E, \lambda, \gamma] \) for the polytropic accretion and the diad \([T, \lambda]\) for the isothermal accretion, where \( E, \lambda, \gamma \) and \( T \) are the specific total conserved energy, specific conserved angular momentum, the adiabatic index \( (\gamma = c_p/c_v \) where \( c_p \) and \( c_v \) are the specific heats at constant pressure and volume, respectively), and the bulk ion temperature of the accreting matter, respectively. For generalized pseudo-Schwarzschild potentials, such calculations has recently been presented [17] for matter flow in three different geometric configuration, viz., disc accretion with constant flow thickness (hereafter constant height flow), quasi-spherical accretion in conical configuration (hereafter conical flow), and for axisymmetric flow maintained in the hydrostatic equilibrium along the vertical direction (hereafter vertical equilibrium flow).

In our present work, we intend to demonstrate how the estimation of the surface gravity gets influenced by the geometric configuration of matter for general relativistic axisymmetric hydrodynamic flow onto a Schwarzschild black hole. Next section describes the main motivations behind our work.

**II. ON FLOW GEOMETRY DEPENDENCE OF THE ACOUSTIC SURFACE GRAVITY**

Corresponding to the acoustic surface gravity, the analogue temperature can be estimated as

\[
T_{AH} = \frac{\kappa}{2\pi}.
\]

(5)

Since the value of the acoustic surface gravity depends on the location of the acoustic horizon \( r_h \) and on the value of \([u, c_s, du/dr, dc_s/dr] \) evaluated at \( r_h \), the analogue temperature \( T_{AH} \) can be functionally expressed as

\[
T_{AH} = T_{AH} \left[ r_h, (c_s, du/dr, dc_s/dr)_{r_h} \right].
\]

(6)

As mentioned earlier, in subsequent sections we will demonstrate that for axisymmetric general relativistic accretion in the Schwarzschild metric, the value of \( r_h \) and that of \([u, c_s, du/dr, dc_s/dr] \) can be expressed in terms of \([E, \lambda, \gamma] \) and \([T, \lambda]\) for the adiabatic and for the isothermal fluid, respectively. Hence

\[
T_{AH}^{\text{adi}} = T_{AH}^{\text{adi}} (E, \lambda, \gamma), \quad \text{and} \quad T_{AH}^{\text{iso}} = T_{AH}^{\text{iso}} (T, \lambda)
\]

(7)

where \( T_{AH}^{\text{adi}} \) and \( T_{AH}^{\text{iso}} \) stands for the analogue temperature for the adiabatic and for the isothermal accretion, respectively.

The expression for the general relativistic Hawking temperature \( T_H \) can be obtained as [18]

\[
T_H = \frac{\hbar c^3}{8\pi GM_{BH}k_B} \approx \frac{1.227 \times 10^{23} \text{kg}}{M_{BH}} \approx 6.17 \times 10^{-8} \frac{M_\odot}{M_{BH}} \text{\& K}
\]

(8)
where $M_\odot$ and $M_{BH}$ stands for the Solar mass and the black hole mass, respectively, other symbols for the fundamental constants carrying their usual meaning.

Eq. (8) provides an obvious anti-correlation between the black hole mass and the Hawking temperature in the sense that one requires a black hole of reasonably small, e.g., a primordial black hole of cosmological origin, to maximize the observable Hawking effect. The extremization of the observable Hawking effect can thus be parameterized by the mass of the black hole, $M_{BH}$ only.

One can not have such a straightforward (anti) correlation available for the analogue temperature with the mass parameter of the system to obviate that the acoustic black hole of microscopic dimension will indeed produce a larger analogue temperature. Extremization of $T_{AH}$ (and hence that of $\kappa$) nonlinearly depends on $[E_c, \lambda, \gamma]$ and on $[T, \lambda]$ for the adiabatic and the isothermal flow, respectively. One thus needs to explore the three dimensional parameter space spanned by $[E, \lambda, \gamma]$ and the two dimensional parameter space spanned by $[T, \lambda]$, for the adiabatic as well as for the isothermal accretion, respectively, to apprehend what initial set of boundary conditions prefers the extremization of $T_{AH}$ and hence might help to obtain the observable signature of the analogue radiation. One also needs to understand which flow configuration out of the three, constant height flow, conical flow and flow in vertical equilibrium respectively, discussed in subsequent sections, favours the production of reasonably large analogue temperature.

Hawking like effects in a dispersive medium can manifest non-conventional classical features observable within the laboratory set up [19–24]. Origin of such non-trivial features may be attributed to the modified dispersion relation close to the acoustic horizon. In the limit of the strong dispersion relation where the background flow has non-constant velocity gradient, the measure of the analogue temperature, unlike the original Hawking one, depends on the frequency of the propagating perturbation embedded in the background flow. The deviation of the Hawking like effects in a dispersive medium from the original Hawking effect is thus quite sensitive to the spatial velocity gradient corresponding to the stationary solutions of the background fluid flow. Whereas the Hawking radiation caused by a linear velocity profile is insensitive to the dispersion relation, the dispersion relation near the acoustic horizon strongly influences the Hawking spectra for the non linear velocity profile of the background fluid, and the phase integral method to determine the associated analogue temperature ceases to be valid if the velocity gradient tends to diverge [23, 24]. As we will show in the subsequent sections, the expression for the acoustic surface gravity is an analytical function of the spatial gradient of the steady state bulk velocity of the background fluid. Such velocity gradient influences the universality (as well as the departure from it) of the Hawking like radiation, and various other properties of the anomalous scattering of the acoustic mode due to the modified dispersion relation at the acoustic horizon.

Acoustic surface gravity estimated in our present work can further be used to study the modified dispersion relation as well as the non universal features of the classical Hawking like process for curved background flow subjected to the gravitational force. In addition to the spatial velocity gradient, the expression for the surface gravity for the accreting black hole system is a function of the non zero space gradient of the speed of propagation of the acoustic perturbation as well. Hence for the axisymmetric accretion, the position dependent sound speed will contribute to the modified dispersion relation.

Hereafter, the main task boils down to the identification of the location of the acoustic horizon and to the evaluation of $[\kappa, u, c_s, \frac{dc_s}{du}, \frac{du}{dr}]$ as defined on that horizon, in terms of the initial boundary conditions governing the flow. This will enable us to provide a ‘calibration space’ spanned by various astrophysically relevant parameters governing the flow, for which the extremization of $\kappa$ as well as of $T_{AH}$ can be performed for axisymmetric accretion of a particular geometric configuration and described by a specific equation of state. We then perform similar operation for accretion with various other flow geometries and described by the different equation of states. This will provide a comprehensive idea about the influence of initial boundary conditions governing the flow, and, primarily as we are interested in the present work, on the nature of the geometric configuration of the black hole accretion flow on the extremization process of the analogue surface gravity and thus on the associated Hawking like temperature. Also the detail nature of the deviation of the associated Hawking like effects from the universal behaviour due to the modified dispersion relation in background flow under strong gravity environment should depend on the geometric configuration of the flow.

In next section, we will describe the general space time structure on which the flow will be studied.

### III. CONFIGURATION OF THE BACKGROUND FLUID FLOW

In our present work, we consider low angular momentum non self-gravitating axisymmetric inviscid hydrodynamic accretion onto a Schwarzschild black hole. At the outset it is important to understand that the low angular momentum inviscid flow is not a theoretical abstraction. For astrophysical systems, such sub-Keplerian weakly rotating flows are exhibited in various physical situations, such as detached binary systems fed by accretion from OB stellar winds.
is considered in a Boyer-Lindquist [38] line element normalized for velocity perpendicular to the set of timelike hypersurfaces the form (corresponding to the accreting black hole) as well as the acoustic (corresponding to the transonic fluid flow).

\[ v^\mu = \frac{c_s^2}{\rho} \nabla_\mu \rho (g^{\mu\nu} + v^\mu v^\nu) = 0, \quad \text{(9)} \]

\( v^\mu \) being the velocity vector field defined on the manifold constructed by the family of streamlines. The normalization condition for such velocity field yields \( v^\mu v_\mu = -1 \), and \( c_s \) is the speed of propagation of the acoustic perturbation embedded inside the bulk flow. \( \rho \) is the local rest mass energy density. The local timelike Killing fields \( \xi^\mu \equiv (\partial/\partial t)^\mu \) and \( \phi^\mu \equiv (\partial/\partial \phi)^\mu \) are the generators of the stationarity (constant specific flow energy is the outcome) and axial symmetry, respectively. Hereafter, any relevant distance will be scaled in units of \( GM_{BH}/c^2 \) and any velocity will be scaled by the velocity of light in vacuum, \( c \), respectively, where \( M_{BH} \) represents the mass of the black hole.

In general, the acoustic ergosphere and the acoustic event horizon do not co-incide. However, for a radial flow onto a sink placed at the origin of a stationary axisymmetric geometry they do (see, e.g., [4, 11] for detail discussion), since only the radial component of the flow velocity \( u = u(r) \) remains non zero everywhere. In this work we consider accretion flow with radial advective velocity \( u \) confined on the equatorial plane. The flow will be assumed to have finite radial spatial velocity \( u \), the advective flow velocity as designated in usual astrophysics literature [36, 37], defined on the equatorial plane of the axisymmetric matter configuration. We focus on stationary solutions of the fluid dynamic equations (to determine the stationary background geometry) and hence consider only the spatial part of such advective velocity. Considering \( v \) to be the magnitude of the three velocity, \( u \) is the component of three velocity perpendicular to the set of timelike hypersurfaces \( \Sigma_c \) defined by \( v^2 = \text{constant} \).

For a transonic flow as perceived within the aforementioned configuration, the collection of the sonic points (where the radial Mach number, the ratio of the advective velocity and the speed of propagation of the acoustic perturbation in the radial direction, becomes unity) at a specified radial distance forms the acoustic horizon, the generators of which are the phonon trajectories. An axially symmetric transonic black hole accretion can thus be considered as a natural example of the classical analogue gravity model which contains two different horizons, the gravitational (corresponding to the accreting black hole) as well as the acoustic (corresponding to the transonic fluid flow).

To describe the flow structure in further detail, we first consider the energy momentum tensor of an ideal fluid of the form

\[ T^{\mu\nu} = (e + p)v^\mu v^\nu + pg^{\mu\nu}, \quad \text{(10)} \]

is considered in a Boyer-Lindquist [38] line element normalized for \( G = c = M_{BH} = 1 \) and \( \theta = \pi/2 \) as defined below [39]

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\frac{r^2}{\Delta} dt^2 + \frac{A}{r^2} \left( d\phi - \omega dt \right)^2 + \frac{r^2}{\Delta} dr^2 + dz^2, \quad \text{(11)} \]

where

\[ \Delta = r^2 - 2a^2, A = r^4 + r^2 a^2 + 2r^2 a^2, \omega = 2ar/A, \quad \text{(12)} \]

\( a \) being the Kerr parameter related to the black holes spin angular momentum. The required metric elements are:

\[ g_{rr} = \frac{r^2}{\Delta}, \quad g_{tt} = \left( \frac{A\omega^2}{r^2} - \frac{r^2\Delta}{A} \right), \quad g_{\phi\phi} = \frac{A}{r^2}, \quad g_{t\phi} = g_{\phi t} = \frac{-A\omega}{r^2}. \quad \text{(13)} \]

The specific angular momentum \( \lambda \) (angular momentum per unit mass) and the angular velocity \( \Omega \) can thus be expressed as

\[ \lambda = \frac{v_\phi}{v_t}, \quad \Omega = \frac{v_\phi}{v_t} = -\frac{g_{t\phi}}{g_{\phi\phi} + \lambda g_{t\phi}}. \quad \text{(14)} \]
We also define

\[ B = g_{\phi\phi} + 2\lambda g_{t\phi} + \lambda^2 g_{tt}, \]  

which will be used in the subsequent sections to calculate the value of the acoustic surface gravity.

For flow onto a Schwarzschild black hole, one can obtain the respective metric elements (and hence, the expression for \( \lambda \) and \( B \) thereof) by substituting \( a = 0 \) in eq. (12 - 15). We construct a Killing vector \( \chi^\mu = \xi^\mu + \Omega^{\phi} \phi^\mu \) where the Killing vectors \( \xi^\mu \) and \( \phi^\mu \) are the two generators of the temporal and axial isometry groups, respectively. Once \( \Omega \) is computed at the acoustic horizon \( r_h \), \( \chi^\mu \) becomes null on the transonic surface. The norm of the Killing vector \( \chi^\mu \) may be computed as

\[ \sqrt{|\chi^\mu \chi_\mu|} = \sqrt{g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi}} = \frac{\sqrt{\Delta B}}{g_{\phi\phi} + \lambda g_{t\phi}}. \]  

Hence the explicit form of the acoustic surface gravity for relativistic flow onto a Schwarzschild black hole would look like

\[ \kappa = \left[ \frac{\sqrt{r^2 - 2r}}{r^2 (1 - c_s^2)} \sqrt{g_{\phi\phi} + \lambda^2 g_{tt}} \right] \left( \frac{du}{dr} - \frac{dc_s}{dr} \right) \bigg|_{r_h}. \]  

Knowledge of \( u, c_s, du/dr \) and \( dc_s/dr \) as evaluated at the sonic point is thus sufficient to calculate \( \kappa \) for a fixed set of values of \([E, \lambda, \gamma]\) for flow described by a particular barotropic equation of state and having a specific geometric configuration. In the next section, we study the transonic flow model for constant height accretion, conical flow and flow in vertical equilibrium for polytropic as well as for isothermal accretion. We thus calculate the acoustic surface gravity for six different flow models – three geometric configurations for two different equations of state, to be more specific, and will compare the values of \( \kappa \) for three different geometric matter configuration in each category, i.e, for polytropic as well as for the isothermal accretion.

In the present work, the equation of state of the form

\[ p = K\rho^\gamma \]  

is considered to describe the polytropic accretion, where the polytropic index \( \gamma \) (which is equal to the ratio of the specific heats at constant pressure and at constant volume, \( C_p \) and \( C_v \), respectively) of the accreting material is assumed to be constant throughout the flow in the steady state. A more realistic flow model, however, perhaps requires the implementation of a non constant polytropic index having a functional dependence on the radial distance of the form \( \gamma \equiv \gamma(r) \) [40–44]. We, nevertheless, have performed our calculations for a reasonably wide spectrum of \( \gamma \) and thus believe that all astrophysically relevant polytropic indices are covered in our analysis. The proportionality constant \( K \) in eq. (18) is a measure of the specific entropy of the accreting fluid provided no additional entropy generation takes place. The specific enthalpy \( h \) is formulated as

\[ h = \frac{p + \epsilon}{\rho}, \]  

where the energy density \( \epsilon \) includes the rest mass density and internal energy and is defined as

\[ \epsilon = \rho + \frac{p}{\gamma - 1}. \]  

The adiabatic sound speed \( c_s \) is defined as

\[ c_s^2 = \left( \frac{\partial p}{\partial \epsilon} \right)_{\text{constant enthalpy}} \]  

At constant entropy, the enthalpy can be expressed as

\[ h = \frac{\partial \epsilon}{\partial \rho}, \]  

and hence

\[ h = \frac{\gamma - 1}{\gamma - (1 + c_s^2)} \]
We also have studied the isothermal accretion flow described by the equation of state
\[ p = \rho c_s^2 = \frac{R}{\mu} \rho T = \frac{\rho \kappa_B T}{\mu m_H} \] (24)
\( R, K, \gamma, \kappa_B, T, \mu \) and \( m_H \) are the universal gas constant, the entropy per particle, the Boltzmann constant, the isothermal flow temperature, the reduced mass and the mass of the Hydrogen atom, respectively. \( c_s \) is the above equation represents the isothermal sound speed which is position independent. For isothermal accretion, the space gradient of the bulk advective velocity only, and not of the speed of propagation of the acoustic perturbation, contributes to the estimation of the acoustic surface gravity.

IV. THE FIRST INTEGRALS OF MOTION

For polytropic flow, vanishing of the four divergence of the energy momentum tensor provides the general relativistic version of the Euler equation
\[ T^\mu_{;\nu} = 0. \] (25)
whereas the corresponding continuity equation is obtained from
\[ (\rho v^\mu) = 0. \] (26)

A. Integral solution of the linear momentum conservation equation

Contracting eq. (25) with \( \phi^\mu \) one obtains (since \( \phi^\nu p^\nu = 0, \phi^\mu = \delta^\mu_\phi, \) and \( g_\mu\lambda; \nu = 0 \))
\[ \left[ \phi^\mu h v^\nu \right]_{;\nu} = 0. \] (27)
Since \( \phi^\mu h v^\mu = h v_\phi, h v_\phi, \) the angular momentum per baryon for the axisymmetry flow is conserved. Contraction of eq. (25) with \( \xi^\mu \) provides
\[ \xi^\mu \left[ \xi^\mu T^\nu_{;\nu} = 0 \right] \] (28)
from where the quantity \( h v_t \) comes out as one of the first integrals of motion of the system. \( h v_t \) is actually the relativistic version of the Bernoulli’s constant [45] and can be identified with the total specific energy of the general relativistic ideal fluid \( E \) (see, e.g., [46] and references therein) scaled in units of the rest mass energy.

The angular velocity \( \Omega \), as defined in terms of the specific angular momentum
\[ \lambda = -\frac{v^\phi}{v_t} \] (29)
is expressed as
\[ \Omega = \frac{v^\phi}{v_t} = -\frac{g_{\phi\phi} + g_{t\phi} \lambda}{g_{\phi\phi} + g_{t\phi} \lambda} \] (30)
From the normalization condition \( v^\mu v_\mu = -1 \) one obtains
\[ v_t = \sqrt{\frac{g_{t\phi}^2 - g_{tt} g_{\phi\phi}}{(1 - \lambda \Omega)(1 - u^2)(g_{\phi\phi} + \lambda g_{t\phi})}} \] (31)
It is to be noted that the specific energy \( E \) remains a first integral of motion for polytropic accretion for all possible geometric configuration of infalling matter, and the corresponding expression for such conserved energy remains of the following form
\[ E = \frac{\gamma - 1}{\gamma - \left(1 + c_s^2\right)^{\gamma/2}} \sqrt{\frac{g_{t\phi}^2 - g_{tt} g_{\phi\phi}}{(1 - \lambda \Omega)(1 - u^2)(g_{\phi\phi} + \lambda g_{t\phi})}} \] (32)
where the exact form for $E$ will depend on the space time structure (and not on the matter geometry since the accretion is assumed to non self gravitating) appearing in the expression for $E$ through the metric elements. As will be shown in the subsequent paragraphs, for isothermal flow, the total specific flow energy does not remain constant for obvious reason, rather the first integral of motion obtained by integrating the relativistic Euler equation has a different algebraic form which can not be identified with the total energy of the system. In general, the time independent part of the Euler equation being a first order homogeneous differential equation, it’s integral solution will provide a constant of motion (first integral of motion) for whatever equation of state is used to describe the accreting matter. Such first integral of motion, however, can not formally be identified with the total energy of the background fluid flow for any equation of state other than the polytropic one.

For isothermal flow, the system has to dissipate energy to keep the temperature constant. The isotropic pressure is proportional to the energy density through

$$p = c_s^2 \rho$$  \hspace{1cm} (33)

From the time part of eq. (25), one obtains

$$\frac{dv_t}{v_t} = -\frac{dp}{p + \epsilon}$$  \hspace{1cm} (34)

Using the definition of enthalpy, the above equation may be re-written as

$$\frac{dv_t}{v_t} = -\frac{1}{h} \frac{dp}{d\rho} \frac{d\rho}{\rho}$$  \hspace{1cm} (35)

Since the isothermal sound speed can be defined as (see, e.g., [47] and references therein)

$$c_s = \sqrt{\frac{1}{h} \frac{dp}{d\rho}}$$  \hspace{1cm} (36)

we obtain

$$\ln v_t = -c_s^2 \ln \rho + A, \text{ where } A \text{ is a constant}$$  \hspace{1cm} (37)

Which further implies that

$$v_t \rho^{c_s^2} = C_{iso}$$  \hspace{1cm} (38)

Hence $C_{iso}$ is the first integral of motion for the isothermal flow, which is not to be confused with the total conserved specific energy $E$.

**B. Integral solution of the mass conservation equation**

For $g \equiv \det(g_{\mu\nu})$, the mass conservation equation (26) implies

$$\frac{1}{\sqrt{-g}} \left( \sqrt{-g} \rho v^\mu \right)_\mu = 0,$$ \hspace{1cm} (39)

which further leads to

$$\left[ \left( \sqrt{-g} \rho v^\mu \right)_{\mu} \right] d^4x = 0 \hspace{1cm} (40)$$

$s\sqrt{-g}d^4x$ being the co-variant volume element. We assume that there is no convection current along any non equatorial direction, and hence no non-zero terms involving $v^\theta$ (for spherical polar co-ordinate) or $v^z$ (for flow studied within the framework of cylindrical co-ordinate) should become significant. This assumption leads to the condition

$$\partial_r (\sqrt{-g}\rho v^r) dr d\theta d\phi = 0,$$ \hspace{1cm} (41)

for the stationary background flow studied using the spherical polar co-ordinate $(r, \theta, \phi)$ and

$$\partial_r (\sqrt{-g}\rho v^r) dr dz d\phi = 0,$$ \hspace{1cm} (42)
for such flow studied using the cylindrical co-ordinate \((r, \phi, z)\).

We integrate eq. (41) for \(\phi = 0 \to 2\pi\) and \(\theta = -H_0 \to H_0; \pm H_0\) being the value of the polar co-ordinates above and below the equatorial plane, respectively, for a local flow half thickness \(H\) to obtain the conserved mass accretion rate \(M\) in the equatorial plane. The integral solution of the mass conservation equation, the mass accretion rate \(M\) is thus the second first integral of motion for our stationary background fluid configuration. For conical wedge shaped flow studied in the spherical polar co-ordinate, \(2H/r\) remains constant. Flow with such geometric configuration was first studied by [48] and followed by [49] for pseudo-Schwarzschild flow geometry under the influence of the Paczyński & Wiita [50] pseudo-Schwarzschild Newtonian like black hole potential. The relativistic version for such flow has further been studied by [47, 51–59]. In a similar fashion, eq. (42) can be integrated for \(z = -H_e \to H_e\) (where \(H_e\) is the local half thickness of the flow) symmetrically over and below the equatorial plane for axisymmetric accretion studied using the cylindrical polar co-ordinate to obtain the corresponding mass accretion rate on the equatorial plane. On contrary to the other first integral of motion (the integral solution of the Euler equation), the expression for the mass accretion rate does not depend on the equation of state, but is different for different geometric configuration of the matter distribution. The general expression for the mass accretion rate can be provided as

\[
M = \rho v' A(r)
\]

\(A(r)\) being the two dimensional surface area having surface topology \(\mathbb{R}^1 \times \mathbb{R}^1\) or \(S^1 \times S^1\) through which the inward mass flux is estimated in the steady state. For \(S^1 \times S^1\) (and for not so large value of \(\theta\)), \(A(r) = 4\pi H_0 r^2\), and for \(\mathbb{R}^1 \times \mathbb{R}^1\) (axisymmetric accretion studied using the cylindrical co-ordinate), \(A(r) = 4\pi H_0 r\).

In standard literature of accretion astrophysics, the local flow thickness for an inviscid axisymmetric flow described by a polytropic as well as an isothermal equation of state, can have three different geometric configuration as has already been mentioned in previous sections, see, e.g., [15] and references therein for further detail. A constant flow thickness (flow resembles a right circular disc with a symmetry about the Z axis, and in absence of any convection current at any non equatorial direction, any plane orthogonal to the symmetry axis is isomorphic with the equatorial plane as long as the formulation of the steady state solutions are concerned) is considered for simplest possible flow configuration where the disc height \(H\) is not a function of the radial distance [11]. In its next variant, the axisymmetric accretion can have a conical wedge shaped structure [47–49, 51–53, 56–59] where \(H\) is directly proportional to the radial distance as \(H = A_h r\), and hence the geometric constant \(A_h\) is determined from the measure of the solid angle subtended by the flow. For the hydrostatic equilibrium in the vertical direction, (see, e.g., [35, 36, 46] and references therein for further detail about such geometric configuration) expression for the local flow thickness can have a rather complex dependence on the local radial distance and on the local speed of propagation of the acoustic perturbation embedded inside the accretion flow. In such configurations, the background flow is assumed to have a radius dependence local flow thickness with its central plane coinciding with the equatorial plane of the black hole. Gravity is balanced by the component of the gas pressure along the vertical direction. The thermodynamic flow variables are averaged over the half thickness of the disc \(H\). For general relativistic flow in a Kerr metric (whose Schwarzschild equivalent can be found out by substituting the Kerr parameter \(a = 0\) in the respective expression), first ever work was by [39] where the expression for such disc height was formulated. Results of [39] was later modified by [60]. Several other works (see, e.g., [61, 62] and references therein) provided various other forms of the relativistic disc height for flow in vertical equilibrium. We modify the disc height (for the polytropic as well as the isothermal equation of state, can have three different flow geometries and will study the transonic flow structure to find out the exact location of the acoustic horizon \(r_0\), the value of the advective and the acoustic velocity and their space gradients evaluate don the acoustic horizon in terms of the three parameter set \([E_i, \lambda, \gamma]\) describing the initial boundary condition for the stationary background configuration. We first describe the aforementioned formalism in detail for the axisymmetric flow with constant height, and provide the expression for the mass and the entropy accretion rate, the location of \(r_0\) and \([u, c_s, du/dr, dc_s/dr]_r\), which will be required to estimate the value of the acoustic surface gravity \(\kappa\) in terms of \([E_i, \lambda, \gamma]\). We then provide such expressions for other two flow models.
V. STATIONARY TRANSONIC SOLUTIONS FOR AXISYMMETRIC BACKGROUND FLOW

A. polytropic accretion

The explicit expression for the conserved energy takes the following form

\[ E = \frac{\gamma - 1}{(\gamma - (1 + c_s^2))} \left( \frac{1 - \frac{u^2}{r^2}}{(1 - \frac{u^2}{r^2})(1 - u^2)} \right) \]  

(44)

1. Flow with constant thickness

As stated in the paragraphs preceding eq. (43), we integrate the continuity equation to obtain the conserved mass accretion rate to be

\[ \dot{M} = 2\pi \rho \frac{u}{\sqrt{1 - u^2}} r H \]  

(45)

\( H \) being the constant disc height.

Equations (44 – 45) can not directly be solved simultaneously since it contains three unknown variables \( u, c_s, \) and \( \rho \), all of which are functions of the radial distance \( r \). Any one of the triad \([u, c_s, \rho]\) has to be eliminated in terms of the other two. We are, however, interested to study the radial Mach number profile to identify the location of the acoustic horizon (the radial distance at which \( M \) becomes unity), and hence the study of the radial variation of \( u \) and \( c_s \) are of prime interest in this case. We would thus like to express \( \rho \) in terms of \( c_s \) and other related constant quantities.

To accomplish the aforementioned task, we make a transformation \( \dot{\Xi} = \dot{M} \frac{K^{\gamma - 1}_s}{\gamma - 1} \). Employing the definition of the sound speed \( c_s^2 = \left( \frac{\gamma}{\gamma - 1} \right)_s \text{Constant Entropy} \) as well as the equation of state used to describe the flow, the expression for \( \dot{\Xi} \) can further be elaborated as

\[ \dot{\Xi} = 2\pi \frac{u}{\sqrt{1 - u^2}} r c_s^{\gamma - 1} \left( \frac{\gamma - 1}{\gamma - (1 + c_s^2)} \right) \frac{1}{\gamma - 1} H \]  

(46)

The entropy per particle \( \sigma \) is related to \( K \) and \( \gamma \) as [63]

\[ \sigma = \frac{1}{\gamma - 1} \log K + \frac{\gamma}{\gamma - 1} + \text{constant} \]

where the constant depends on the chemical composition of the accreting material. The above equation implies that \( K \) is a measure of the specific entropy of the accreting matter. We thus interpret \( \dot{\Xi} \) as the measure of the total inward entropy flux associated with the accreting material and label \( \dot{\Xi} \) to be the stationary entropy accretion rate.

The concept of the entropy accretion rate was first introduced in [48, 49] to obtain the stationary transonic solutions of the low angular momentum non relativistic axisymmetric accretion under the influence of the Paczyński and Wiita [50] pseudo-Schwarzschild potential onto a non rotating black hole.

The conservation equations for \( E, \dot{M} \) and \( \dot{\Xi} \) may simultaneously be solved to obtain the complete accretion profile on the radial Mach number vs radial distance phase space, see, e.g., [12, 46] for the depiction of several such phase portraits.

The relationship between the space gradient of the acoustic velocity and that of the advective velocity can now be established by differentiating eq. (46)

\[ \frac{dc_s}{dr} = -\frac{\gamma - 1}{2} \left( \frac{1}{c_s} + \frac{\mu}{1 - u^2} \right) \frac{du}{dr} + \left( \frac{1}{r} + \frac{1}{r^2(1 - \frac{r}{r^2})} \right) \]  

(47)

Differentiation of eq. (44) with respect to the radial distance \( r \) provides another relation between \( dc_s/dr \) and \( du/dr \). We substitute \( dc_s/dr \) as obtained from eq. (47) into that relation and finally obtain the expression for the space gradient
of the advective velocity as

\[ \frac{du}{dr} = \frac{c_s^2 \left( \frac{1}{r} + \frac{1}{r^0(1-\frac{2}{r})} \right) - f_2(r, \lambda)}{(1 - c_s^2) \frac{u}{1-u} - \frac{c_s^2}{u}} = \frac{N_1}{D_1} \]  

(48)

Where

\[ f_2(r, \lambda) = -\frac{\lambda^2}{r^3} \left\{ \frac{1}{r} - \frac{3}{r^2} \right\} + \frac{1}{r^2(1 - \frac{2}{r})} \]  

(49a)

We define another quantity \( f_1(r, \lambda) \) which shall be used later,

\[ f_1(r, \lambda) = \frac{3}{r} + \frac{\lambda^2}{r^3} \left\{ \frac{1}{r} - \frac{3}{r^2} \right\} \]  

(49b)

Eq. (47–48) can now be identified with a set of non-linear first order differential equations representing autonomous dynamical systems [64], and their integral solutions provide phase trajectories on the radial Mach number \( M \) vs the radial distance \( r \) plane. The ‘regular’ critical point conditions for these integral solutions are obtained by simultaneously making the numerator and the denominator of eq. (48) vanish. The aforementioned critical point conditions may thus be expressed as

\[ [u = c_s]_{r_c}, \quad [c_s]_{r_c} = \sqrt{\frac{f_2(r_c, \lambda)}{\frac{1}{r_c^0(1 - \frac{2}{r_c})}} + \frac{1}{r_c^0}} \]  

(50)

Since in this work we deal with a transonic fluid in real space for which the flow is continuous all throughout, we consider only the ‘regular’ or ‘smooth’ critical point, for which \( u, c_s \) as well as their space derivatives remain regular and does not diverge. Such a critical point may be of saddle type allowing a transonic solution to pass through it, or may be of centre type through which no physical transonic solution can be constructed. Other categories of critical point include a ‘singular’ one for which \( u, c_s \) are continuous but their derivatives diverge. All such classifications has been discussed in detail and the criteria for a critical point to qualify as a ‘regular’ one which is associated with a physical acoustic horizon has been found out in [11].

Equation (50) provides the critical point condition but not the location of the critical point(s). It is necessary to solve eq. (44) under the critical point condition for a set of initial boundary conditions as defined by \([E, \lambda, \gamma, \cdot] \). The value of \( c_s \) and \( u \), as obtained from eq. (50), may be substituted at eq. (44) to obtain the following 11th degree algebraic polynomial for \( r = r_c, r_c \) being the location of the critical point

\[ a_0 + a_1 r_c + a_2 r_c^2 + a_3 r_c^3 + a_4 r_c^4 + a_5 r_c^5 + a_6 r_c^6 + a_7 r_c^7 + a_8 r_c^8 + a_9 r_c^9 + a_{10} r_c^{10} + a_{11} r_c^{11} = 0 \]  

(51)
where, the coefficients \(a_i\) are given by

\[
\begin{align*}
    a_0 &= -4E^2(5 - 3\gamma)^2\lambda^6 \\
    a_1 &= 8E^2\left(40 - 49\gamma + 15\gamma^2\right)\lambda^6 \\
    a_2 &= -\lambda^4\left(108 + 401E^2\lambda^2 + \gamma^2\left(108 + 157E^2\lambda^2\right) - 2\gamma\left(108 + 251E^2\lambda^2\right)\right) \\
    a_3 &= \lambda^4\left(324(-1 + \gamma)^2 + E^2\left(-240 + 247\lambda^2 - 4\gamma\left(-81 + 79\lambda^2\right) + \gamma^2\left(-108 + 101\lambda^2\right)\right)\right) \\
    a_4 &= \lambda^4\left(-387(-1 + \gamma)^2 + E^2\left(564 - 753\lambda^2 + \gamma^2\left(270 - 32\lambda^2\right) + \gamma\left(-786 + 98\lambda^2\right)\right)\right) \\
    a_5 &= \lambda^4(-108 + (230 - 502E^2)\lambda^2 + 9E^2\lambda^4 + 2\gamma^2(-54 + (115 - 126E^2)\lambda^2 + 2E^2\lambda^4) \\
    a_6 &= \lambda^2(-2(-1 + \gamma)^2(-135 + 34\lambda^2)^2 + E^2\left(-84 + 200\lambda^2 + \gamma\left(180 - 290\lambda^2\right) + \gamma^2\left(-81 + 104\lambda^2\right)\right) \\
    a_7 &= \lambda^2(4(-1 + \gamma)^2(-63 + 2\lambda^2) + E^2\left(153 - 16\lambda^2 - 30(-6 + \lambda^2) + 4\gamma\left(-87 + 11\lambda^2\right)\right) \\
    a_8 &= -27 - 4\left(-26 + 31E^2\right)\lambda^2 + \gamma^2\left(-27 + (104 - 96E^2)\lambda^2\right) + \gamma\left(54 + 16(-13 + 14E^2)\lambda^2\right) \\
    a_9 &= 2\left((-1 + \gamma)^2(-27 + 8\lambda^2) + E^2\left(-4 + 14\lambda^2 + \gamma\left(12 - 24\lambda^2 + \gamma^2\left(-9 + 10\lambda^2\right)\right)\right) \\
    a_{10} &= 4(-1 + \gamma)(9 - 9\gamma + E^2(-4 + 6\gamma)) \\
    a_{11} &= -8\left(-1 + E^2\right)(-1 + \gamma)^2
\end{align*}
\]

(52)

A particular set of values of \([E, \lambda, \gamma, a]\) will then provide the numerical solution for the algebraic expression to obtain the exact value of \(r_c\). Astrophysically relevant domain for such initial boundary conditions are [46] defined by

\[
[1 \leq E \leq 2, 0 < \lambda \leq 2, 4/3 \leq \gamma \leq 5/3]
\]

(53)

Once the value of \(r_c\) is computed for an astrophysically relevant set of \([E, \lambda, \gamma, a]\), the nature of the critical point(s) can also be studied to confirm whether it is a saddle type or a centre type critical point [64]. For accretion with constant height, the critical point condition reveals that the advective velocity and the sound velocity are same at the critical point. Hence the critical point \(r_c\) and the acoustic horizon \(r_h\) coincides for this flow geometry.

For an astrophysically relevant set of \([E, \lambda, \gamma]\) the critical point(s) of the phase trajectory can be identified, and a linearization study in the neighbourhood of these critical points(s) may be performed [64] to develop a classification scheme to identify the nature of the critical point(s). Since viscous transport of angular momentum has not been taken into account in the present work, such critical points are either of saddle type through which a stationary transonic flow solution can be constructed, or of an unphysical centre type which does not allow any transonic solution on phase portrait to pass through it. A complete understanding of the background stationary transonic flow topologies on the phase portrait will require a numerical integration of the non analytically solvable non linear couple differential equations describing the space gradient of the advective velocity as well as the speed of propagation of the acoustic perturbation embedded within the stationary axisymmetric background spacetime.

For a particular set of \([E, \lambda, \gamma]\), solution of eq. (51) provides either no real positive root lying out side the gravitational black hole horizon implying that no acoustic horizon forms outside the black hole event horizon (non availability of the transonic solution) for that value of \([E, \lambda, \gamma]\) or provides one, two or three (at most) real positive roots lying outside the black hole event horizon. Typically, if the number of root is one, the critical point is of saddle type and a mono-transonic flow profile is obtained with a single acoustic horizon for obvious reason. Solution containing two critical saddle points implies the presence of a homoclinic orbit[85] on the phase plot and hence such solutions are excluded.

Stationary configuration with three critical points requires a somewhat detail understanding. Although a full description is available in [46], it is perhaps be not unjustified to provide a brief account over here for the sake of completeness. One out of the aforementioned three critical points is of centre type which is circumscribed by two saddle type critical points. With reference to the gravitational horizon, one of these saddle points forms sufficiently close to it, even closer than the innermost circular stable orbit (ISCO) in general, and is termed as the inner type critical. The other saddle type point, termed as the outer saddle type point, is usually formed at a fairly large distance away from the gravitational horizon. The inner critical point thus forms in a region of substantially strong gravitational field whereas the outer type critical point, in many cases, is formed in a region of asymptotically flat spacetime since, depending on the choice of \([E, \lambda, \gamma]\), such a critical point can be located at a distance (from the gravitational horizon) \(10^6GM_{BH}/c^2\) or even more. The centre type critical point, termed as the middle critical point because of the fact that
forms usually at a length scale ranging from 10 to 10^{-4} in units of GM_{BH}/c^2, depending on the initial boundary conditions as determined by \([\text{call}\ E, \lambda, \gamma] \). For certain values of the initial boundary condition \([E, \lambda, \gamma] \in \{E, \lambda, \gamma\}\) thus provides the multi-critical behaviour of stationary transonic solution. The parameter space spanned by \([E, \lambda, \gamma] \) can further be classified into two different subspaces for which the representative phase portraits are topologically different. Such subspaces are characterized by the relative values of the stationary entropy accretion rate \(\dot{E}\) evaluated at the inner and the outer critical points, respectively. For \(\dot{E}_{\text{inner}} > \dot{E}_{\text{outer}}\), transonic accretion can have only one saddle type (inner) critical point and the homoclinic orbit forms through the outer saddle type critical point. Hence only \([E, \lambda, \gamma]_{\text{inner}} > \dot{E}_{\text{outer}}\) provides the multi-critical accretion configuration for which two saddle type and one center type (delimited between two such saddle type points) are available. As already been mentioned, a physically acceptable transonic solution for inviscid accretion cannot be constructed through a centre type critical point. A multi-critical flow with three critical points is thus a theoretical abstraction whereas a bi-transonic accretion is a practically realizable configuration where the stationary transonic accretion solution passes through one inner and one outer saddle type sonic points. For flow geometries providing the isomorphism between the critical and the sonic points, such sonic points define the acoustic horizons. For flow configuration which does not allow such isomorphism, a sonic point can be identified on the integral spaces spanned by \(\dot{E}\). The critical acoustic velocity gradient \(\frac{\partial \dot{E}}{\partial r}|_{\text{crit}}\) can be obtained from eq. (47) and by evaluating other quantities in eq. (47) at \(r_c\). Both \(\frac{dc_s}{dr}|_{r_c}\) and \(\frac{dc_s}{dr}|_{r_c}\) can be reduced to an algebraic
expression in \( r \), with real coefficients that are complicated functions of \([E, \lambda, \gamma]\). Once \( r_s \) is known for a set of values of \([E, \lambda, \gamma]\), the critical slope, i.e., the space gradient for \( u \) at \( r_s \) for the advective velocity can be computed as a pure number, which may either be a real (for stationary transonic accretion solution to exist) or an imaginary (no transonic solution can be found) number.

To obtain the Mach number vs radial distance phase plot for the stationary transonic accretion flow, one needs to simultaneously integrate the set of coupled differential equations (47 – 48) for a specific set of initial boundary conditions determined by \([E, \lambda, \gamma]\). The initial value of the space gradient of the advective velocity, i.e., the critical velocity gradient evaluated at the critical point and provided in eq. (54) and the critical space gradient of the sound speed can be numerically iterated using the fourth order Runge - Kutta method [76] to obtain the integral solutions both for the mono-transonic as well as for the multi-transonic flow. Details of such numerical integration scheme, along with the representative phase plots are available in \([12, 16, 46]\). It is, however, to be noted that for flow geometries where the saddle type critical points and the sonic points are isomorphic (a centre type critical point does not allow any transonic solution to pass through hence it does not have any corresponding sonic point), numerical construction of the integral stationary solution is not required to calculate the corresponding acoustic surface gravity. Since the critical surface and the acoustic horizon is identical, value of \([E, \lambda, \gamma]\) for the respective flow configuration. In such case, for axisymmetric flow with constant height for example, the surface gravity can be computed as

\[
\kappa_{\text{Adiabatic}} = \left| \frac{r - 2}{r^2 (1 - c_s^2)} \sqrt{r^2 - \lambda^2 (1 - \frac{2}{r}) \left[ \eta_1 \frac{du}{dr} + \sigma_1 \right]} \right|_{r_s}
\]

where

\[
\eta_1 = 1 + \frac{\gamma - (1 + V^2)}{2 (1 - V^2)}, \quad \sigma_1 = \left( \frac{\gamma - 1}{2} \right) \left( \frac{1}{\eta} + \frac{1}{\theta - (1 + \eta)} \right)
\]

For certain flow geometries, however, the critical points and the sonic points may not be isomorphic. As a result, the acoustic horizon does not form on the critical surface for such flow configuration and hence \([u, c_s, du/dr, dc_s/dr]_{r_s}\) can not be used to evaluate the corresponding \( \kappa \). As will be demonstrated in the subsequent sections, axisymmetric flow in hydrostatic equilibrium in the vertical direction (for both polytropic as well as for isothermal accretion) is an example of the aforementioned situation. The location of the sonic point, as will be manifested from section VA 3 and VB 3, will always be located at a radial distance \( r_{\text{sonic}} < r_{\text{critical}} \) (hereafter we will designate a sonic point as \( r_s \) instead of \( r_{\text{sonic}} \)). Such \( r_s \) is to be found out by integrating the expression of \((du/dr)\) and \((dc_s/dr)\) and by locating the radial co ordinate on the equatorial plane for which the Mach number becomes exactly unity. Since the condition \( u^2 - c_s^2 = 0 \) is satisfied at \( r_s \) and not at \( r_s \), \([u, c_s, du/dr, dc_s/dr]_{r_s}\) is to be used to calculate the corresponding value of the acoustic surface gravity instead of \([u, c_s, du/dr, dc_s/dr]_{r_{\text{sonic}}}\) for those particular flow geometries.

It is relevant to note that the absolute value of the (constant) disc thickness \( H \) does not enter anywhere in the expression of the acoustic surface gravity (and hence, into the calculation of the Hawking like temperature). Similar result will be obtained in the subsequent sections where it will be demonstrated that the geometrical factor (the solid angle) representing the angular opening of the conical flow does not show up in the expression for \( \kappa \) or \( T_{\text{AH}} \) as well. This implies that it is only the geometrical configuration of the matter (non self gravitating) and not the absolute measure of the flow thickness (for constant height flow) or the ratio of the local height to the local radial distance (for conical flow) which influences the computation of the acoustic surface gravity and related analogue temperature. It is, however, observed in section VA 3 and VB 3 that this may not be the case where the radius flow thickness itself is found to be a function of the speed of propagation of acoustic perturbation.

2. Quasi-spherical flow in conical equilibrium

The expressions for the mass and the entropy accretion rate are

\[
\dot{M} = \Lambda \rho \frac{u \sqrt{1 - \frac{2}{r^2}}}{\sqrt{1 - u^2}}
\]

and

\[
\dot{S} = \Lambda \frac{u \sqrt{1 - \frac{2}{r^2}}}{\sqrt{1 - u^2}} c_s^2 \left( \frac{\gamma - 1}{\gamma - (1 + c_s^2)} \right) \frac{1}{\gamma - 1}
\]
respectively, where, $\Lambda$ is the geometric factor determining the exact shape of the flow, over which integration of the continuity equation is performed. The space gradient of the sound velocity and the advective velocity comes out to be

$$\frac{dc_s}{dr} = -\frac{\gamma - 1}{2} \left( \frac{u}{r - u} \right) \frac{du}{dr} + \left( \frac{\beta + \frac{1}{r(1-\gamma)}}{c_s^2 - 1} \right)$$

and

$$\frac{du}{dr} = -\frac{c_s^2 \left( \frac{\beta}{r(1-\gamma)} + f_s(r, \lambda) \right)}{1 - u^2 (c_s^2 - 1) + \frac{c_s^2}{u}}$$

respectively. The critical point conditions are as follows

$$[u = c_s]_c = \frac{f_s(r_c, \lambda)}{\sqrt{\frac{\beta}{r_c(1-\gamma)}}}$$

As is obvious, the critical points and the sonic points are isomorphic for this flow geometry as well.

The critical point $r_c$ is found out by solving the following 11th degree polynomial numerically

$$a_0 + a_1 r_c + a_2 r_c^2 + a_3 r_c^3 + a_4 r_c^4 + a_5 r_c^5 + a_6 r_c^6 + a_7 r_c^7 + a_8 r_c^8 + a_9 r_c^9 + a_{10} r_c^{10} + a_{11} r_c^{11}$$

where the coefficients $a_i$ are found to be

$$a_0 = -4\mathcal{E}^2(5 - 3\gamma)^2 \lambda^6$$
$$a_1 = 8\mathcal{E}^2 \left( 40 - 49\gamma + 15\gamma^2 \right) \lambda^6$$
$$a_2 = -\lambda^4 \left( 108 + 401\mathcal{E}^2 \lambda^2 + \gamma^2 \left( 108 + 157\mathcal{E}^2 \lambda^2 \right) - 2\gamma \left( 108 + 251\mathcal{E}^2 \lambda^2 \right) \right)$$
$$a_3 = \lambda^4 \left( 324(-1 + \gamma)^2 + \mathcal{E}^2 \left( -240 + 247\lambda^2 - 4\gamma \left( -81 + 79\lambda^2 \right) + \gamma^2 \left( -108 + 101\lambda^2 \right) \right) \right)$$
$$a_4 = \lambda^4 \left( -387(1 + \gamma)^2 + \mathcal{E}^2 \left( 564 - 75\lambda^2 + \gamma^2 \left( 270 - 32\lambda^2 \right) + \gamma \left( -786 + 98\lambda^2 \right) \right) \right)$$
$$a_5 = \lambda^2(-108 + \left( 230 - 502\mathcal{E}^2 \right) \lambda^2 + 9\mathcal{E}^2 \lambda^4 + 2\gamma^2 \left( -54 + \left( 115 - 126\mathcal{E}^2 \right) \lambda^2 + 2\mathcal{E}^2 \lambda^4 \right)$$
$$a_6 = \lambda^2 \left( -2(-1 + \gamma)^2 \left( -135 + 34\lambda^2 \right) + \mathcal{E}^2 \left( -84 + 200\lambda^2 + \gamma \left( 180 - 290\lambda^2 \right) + \gamma^2 \left( -81 + 104\lambda^2 \right) \right) \right)$$
$$a_7 = \lambda^2 \left( 4(-1 + \gamma)^2 \left( -63 + 2\lambda^2 \right) + \mathcal{E}^2 \left( \gamma^2 \left( 153 - 16\lambda^2 \right) - 30 \left( -6 + \lambda^2 \right) + 4\gamma \left( -87 + 11\lambda^2 \right) \right) \right)$$
$$a_8 = -27 - 4 \left( -26 + 31\mathcal{E}^2 \right) \lambda^2 + \gamma^2 \left( -27 + \left( 104 - 96\mathcal{E}^2 \right) \lambda^2 \right) + \gamma \left( 54 + 16 \left( -13 + 14\mathcal{E}^2 \right) \lambda^2 \right)$$
$$a_9 = 2 \left( -1 + \gamma \right) \left( -27 + 8\lambda^2 \right) + \mathcal{E}^2 \left( -4 + 14\lambda^2 + \gamma \left( 12 - 24\lambda^2 \right) + \gamma^2 \left( -9 + 10\lambda^2 \right) \right)$$
$$a_{10} = 4(-1 + \gamma) \left( 9 - 9\gamma + \mathcal{E}^2(-4 + 6\gamma) \right)$$
$$a_{11} = -8 \left( -1 + \mathcal{E}^2 \right)(-1 + \gamma)^2$$

The corresponding space gradients of $u$ evaluated on the acoustic horizon, are calculated as

$$\left( \frac{du}{dr} \right)_c = \left[ \frac{\alpha_2 + \sqrt{\alpha_2^2 + \Gamma_2 \beta_2}}{\Gamma_2} \right] \bigg|_{r_c}$$

where

$$\alpha_2 = -\frac{c(-3 + 2\gamma) \left( 1 + \gamma \right) \left( 1 - \gamma \right)}{(-1 + \gamma) (-2 + r) r}, \quad \beta_2 = \frac{\beta''}{(-2 + r)^2 r^2 \left( r^3 - (-2 + r) \lambda^2 \right)^2}, \quad \Gamma_2 = \frac{\gamma - 3u_c^2 + 1}{(1 - u_c^2)^2}$$

(66)
respectively. The critical point condition is found out to be

\[ u = \sqrt{\frac{1}{1 + (\gamma + 1)(\frac{1}{c_s^2})}} \]

\[ = \sqrt{\frac{f_2(r_c, \lambda)}{f_1(r_c, \lambda) + f_2(r_c, \lambda)}} \]  \tag{74}

It is obvious from eq. (74) that \( u = c_s \) at \( r_c \), and hence the Mach number at the critical point is

\[ M_c = \sqrt{\frac{2}{\gamma + 1}} \frac{f_1(r_c, \lambda)}{f_1(r_c, \lambda) + f_2(r_c, \lambda)} \]  \tag{75}
and not equal to unity in general (since $\gamma < 1$ for the polytropic accretion), and always remains less than one. For an astrophysically relevant set of initial boundary condition, the separation between the critical and the sonic points (the radial distance at which Mach number becomes unity) can be as high as several hundred gravitational radii. Hence, for polytropic flow in vertical equilibrium, the sonic points are isomorphic to the critical points in general, neither numerically, nor topologically, and we categorically distinguish the critical surfaces (with radius $r_c$) with the sonic surface (with radius $r_h$), i.e., with the acoustic horizons. In the existing literature on non general relativistic transonic disc accretion, the Mach number at the critical point turns out to be a function of $\gamma$ only, and can be expressed as [15]

$$
M_c = \sqrt{\frac{2}{\gamma + 1}} \tag{76}
$$

which, however, does not depend on the location of the critical point and depends only on the value of the adiabatic index chosen to describe the flow. However, the quantity $M_c$ as expressed through eq. (75) turns out to be a function of $r_c$, and hence, generally, it takes different values for different $r_c$ for transonic accretion. The Mach numbers at the critical (saddle type) points are thus functional, and not functions, of $[E, \lambda, \gamma]$ The difference between the radii of the critical point and the sonic point is found to be a complicated functional of $[E, \lambda, \gamma]$, the form of which can not be expressed analytically, rather can easily be evaluated using numerical integrations of the flow equation through the procedure we describe in subsequent paragraphs. It is worth emphasizing that the distinction between critical and sonic points is a direct manifestation of the non-trivial functional dependence of the disc thickness (for flow in hydrostatic equilibrium in the vertical direction) on the fluid velocity, the sound speed and the radial distance, i.e., on the disc geometry as well as the equation of state in general.

The critical point $r_c$ can be found by numerically solving the following $8^{th}$ degree polynomial

$$
a_0 + a_1 r_c + a_2 r_c^2 + a_3 r_c^3 + a_4 r_c^4 + a_5 r_c^5 + a_6 r_c^6 + a_7 r_c^7 + a_8 r_c^8 = 0 \tag{77}
$$

where the coefficients $a_i$ are calculated as

$$
a_0 = 64(-2 + \gamma)^2E^2\lambda^4$$
$$a_1 = -32(18 - 19\gamma + 5\gamma^2)E^2\lambda^4$$
$$a_2 = 4\lambda^2((121 - 134\gamma + 37\gamma^2)E^2\lambda^2 + 60(-1 + \gamma)^2)$$
$$a_3 = 4\lambda^2((-45 + 52\gamma - 15\gamma^2)E^2\lambda^2 + 4(-34 + 22E^2 + \gamma(68 - 37E^2) + \gamma^2(-34 + 13E^2)))$$
$$a_4 = \lambda^2((5 - 3\gamma)(E^2\lambda^2 - 4(-115 + 147E^3 + \gamma(230 - 244E^2) + \gamma^2(-115 + 89E^2)))$$
$$a_5 = -2(-60(-1 + \gamma)^2 + (86 - 163E^2 + \gamma^2(86 - 99E^2) + 2\gamma(-86 + 133E^2))\lambda^2 - 60(-1 + \gamma)^2)$$
$$a_6 = (-12(-1 + \gamma)(17 - 11E^2 + \gamma(-17 + 13E^2))\lambda^2 + (-384 + 121E^2 + \gamma(768 - 286E^2) + \gamma^2(-384 + 169E^2))$$
$$a_7 = -12(-1 + \gamma)(17 - 11E^2 + \gamma(-17 + 13E^2))$$
$$a_8 = 36(-1 + \gamma)^2(-1 + E^2)$$

The critical advective gradient can be calculated as

$$
\left(\frac{d\mu}{d\lambda}\right)_c = \left[ \frac{-\delta_3}{\Gamma_3} + \sqrt{\frac{\delta_3^2 + \Gamma_3\delta_3}{\Gamma_3}} \right]_{r_c} \tag{78}
$$
where the -ve sign is to be taken for accretion. Various quantities in eq. (78) can be defined as

\[ \Gamma_3 = \frac{4y - (3y - 1)u_c^2}{(y + 1)(1 - u_c^2)^2} \]  

\[ \delta_3 = \frac{u_c \left( 2(y - 1) - (3y - 1)u_c^2 \right)}{(y + 1)(1 - u_c^2)^2} \]

\[ \beta_3 = -\frac{f_1(r_c, \lambda)u_c^2 \left[ 2(y - 1) - (3y - 1)u_c^2 \right]}{(y + 1)(1 - u_c^2)^2} + \frac{2u_c^2}{1 - u_c^2} \left[ \frac{df_1(r_c, \lambda)}{dr} \right]_c - \frac{4}{\gamma + 1} \left[ \frac{df_2(r_c, \lambda)}{dr} \right]_c \]

The overall scheme to calculate \( \kappa \) for such flow configuration would be the following: For a certain value of initial boundary condition as determined by \([E, \lambda, \gamma]\), the critical point(s) \( r_c \) is determined by numerically solving eq. (77), and the corresponding \([u, c_s, du/dr]_c\) are then calculated from eq. (74) and eq. (78), respectively. \([dc_s/dr]_c\) is then obtained by substituting the value of \([du/dr]_c\) in eq. (72) and by evaluating the other quantities in eq. (72) on the critical point \( r_c \). \([u, c_s, du/dr, dc_s/dr]_c\) are then used as initial value to numerically integrate eq. (72 – 73) simultaneously up to the radial distance \( r_s \) where \( u = c_s \) condition holds. The value of \([du/dr]_c \) and \([dc_s/dr]_c\) are then computed at that point and the quad \([u, c_s, du/dr, dc_s/dr]_c\) is then used to evaluate the value of the corresponding acoustic surface gravity as

\[ k_{\text{Vertical Equilibrium}}^{\text{Adiabatic}} = \left| \frac{r - 2}{r^2 (1 - c_s^2)} \sqrt{r^2 - \lambda^2} \left( 1 - \frac{2}{r} \right) \left[ \eta_3 \frac{du}{dr} + \sigma_3 \right] \right|_r \]  

where

\[ \eta_3 = 1 + \frac{c_s}{H} \left( \frac{\gamma - (1 + c_s^2)}{\gamma + 1} \right), \quad \sigma_3 = \frac{c_s f_1 (\gamma - (1 + c_s^2))}{\gamma + 1} \]

B. Isothermal accretion

As already been mentioned in the preceding paragraphs, the integral solution of the time independent general relativistic Euler equation for background isothermal flow provides a first integral of motion independent of the geometric configuration of the background non self gravitating flow. The explicit form of such conserved quantity is

\[ \xi = \frac{r^2 (r - 2)}{(r^2 - (r - 2)\lambda^2(1 - u^2))^{2/3}} \]

For polytropic accretion, we demonstrated the necessity of the introduction of the stationary accretion rate since the two primary first integrals of motion \([E, M]\) could not be solved simultaneously – owing to the fact that that such system of equations could not be presented in closed form. The primary reason behind the aforementioned issues has been the position dependence of the speed of propagation of the acoustic perturbation. We thus had to replace \( \rho \) in terms of the stationary polytropic sound speed \( c_s \). For isothermal flow, however, the sound speed remains a position independent constant following the Clayperon-Mendeleev equation [65, 66]

\[ c_s = \sqrt{\frac{k_B}{\mu m_H} T} \]

\( k_B \) being the Boltzmann’s constant, \( m_H \approx m_p \) is the mass of hydrogen atom, and \( \mu \) is the mean molecular weight. Since for isothermal flow the bulk flow temperature \( T \) remains constant, one can directly differentiate the steady mass accretion rate to express \( dp/dr \) in terms of \( du/dr \). Expression for \( dp/dr \) can then be substituted by differentiating the other first integral of motion as defined in eq. (82) to obtain the space gradient of the advective velocity.

1. Flow with constant height \( H \)

Mass accretion rate is found as

\[ \dot{M} = 2\pi H \frac{u \sqrt{1 - u^2}}{\sqrt{1 - u^2} H} \]
where $H$ is the constant disc height. Using eq. (82) and eq. (84), the space gradient of the advective velocity is thus found as

$$
\frac{du}{dr} = \frac{(2r^3 - 2(r - 2)\lambda^2 + (1 - r)(2r^3 + 4\lambda^2 - 2r\lambda^2)c_s^2)u(u^2 - 1)}{(2 - r)(-2r^3 - 4\lambda^2 + 2r\lambda^2)(u^2 - c_s^2)}
$$

The critical point conditions come out to be

$$
u^2 = c_s^2 = \frac{-r^3 + (r - 2)\lambda^2}{r^3 - r^4 + (r - 2)(r - 1)\lambda^2}
$$

The critical and the sonic points are thus seen to be isomorphic. Since $c_s \propto T^{\frac{1}{2}}$, a two parameter input $[T, \lambda]$ can be used to analytically solve the following fourth degree polynomial

$$
2c_s^2r^4 - 2(1 + c_s^2)r^3 - 2\lambda^2(c_s^2 - 1)r^2 - 2\lambda^2(4 - 3c_s^2)r - 4\lambda^2(c_s^2 - 2)r = 0
$$

(87)

to find out the location of the critical point(s) $r_c$ (and hence, the location of the acoustic horizon(s) $r_h$). The critical velocity gradient can be found as

$$
\left(\frac{du}{dr}\right)_{r_c} = \left[ \frac{\alpha_1^{\text{iso}} + \sqrt{\alpha_1^{\text{iso}2} + 4\beta_1^{\text{iso}}\Gamma_1^{\text{iso}}} }{2\Gamma_1^{\text{iso}}} \right]_{r_c}
$$

where the $-$ve sign has to be considered for the accretion solution. In eq. (88),

$$
\alpha_1^{\text{iso}} = \left[ -3(c_s^2 - 1) \right] \left[ (1 + c_s^2(r_c - 1)) r^3 - (2 + c_s^2(r - 1) - r)(r - 2)\lambda^2 \right]_{r_c}
$$

$$
\beta_1^{\text{iso}} = \left[ c_s(1 - c_s^2) \right] \left[ (-3 + c_s^2(4r - 3)) r^2 + (c_s^2(3 - 2r) + 2(r - 2)\lambda^2) \right]_{r_c}
$$

$$
\Gamma_1^{\text{iso}} = \left[ 2r(r - 2)(r^3 - (r - 2)\lambda^2) c_s \right]_{r_c}
$$

The corresponding acoustic surface gravity can thus be computed as

$$
K_{\text{isothermal}}^{\text{Constant Height}} = \frac{r - 2}{r^2(1 - c_s^2)} \sqrt{r^2 - \lambda^2} \left( 1 - \frac{2}{r} \right) \left( \frac{du}{dr} \right)_{r_c}
$$

(90)

Note that once the initial boundary conditions are specified through $[T, \lambda]$, the analogue surface gravity and the corresponding Hawking like temperature can be estimated completely analytically without using any numerical technique since the the location of the acoustic surface horizon can be obtained by solving a 4th degree polynomial.

2. Conical Wedge Shaped Flow

Mass accretion rate is found as

$$
\dot{M} = \Lambda \rho \sqrt{1 - \frac{2}{r}} \frac{\sqrt{1 - \frac{2}{r}} u}{\sqrt{1 - u^2}}
$$

(91)

\[where \ \Lambda \ \text{is the geometric constant (solid angle) describing the flow profile. Using eq. (82) and eq. (91), the space gradient of the advective velocity is thus found as}

$$
\frac{du}{dr} = \frac{\left[ 2r^3 - 2(r - 2)\lambda^2 + (3 - 2r)(2r^3 + 4\lambda^2 - 2r\lambda^2)c_s^2 \right]u(u^2 - 1)}{(2 - r)(-2r^3 - 4\lambda^2 + 2r\lambda^2)(u^2 - c_s^2)}
$$

(92)

\[The critical point conditions comes out to be]

$$
u^2 = c_s^2 = \frac{-r^3 + (r - 2)\lambda^2}{3r^3 - 2r^4 + 6\lambda^2 - 7r\lambda^2 + 2r^2\lambda^2}
$$

(93)
The radial Mach number becomes unity. Solve the following fourth degree polynomial

\[ 4c_s^2 r^4 - 2(3c_s^2 + 1) r^3 - 2\lambda^2 (2c_s^2 - 1) r^2 + 2\lambda^2 \left(7c_s^2 - 4\right)r - 4\lambda^2 \left(3c_s^2 - 2\right) \bigg|_{r_c} = 0 \]  

(94)

to find out the location of the critical point(s) \( r_c \) (and hence, the location of the acoustic horizon(s) \( r_h \)). The critical velocity gradient can thus be found as

\[ \left(\frac{du}{dr}\right)_c = \left[ \frac{a_2^{iso} + \sqrt{a_2^{iso} + 4b_2^{iso}\Gamma_1^{iso}}}{2\Gamma_1^{iso}} \right]_{r_c} \]  

(95)

where the \(-ve\) sign has to be considered for the accretion solution. In eq. (95),

\[ a_2^{iso} = \left[\left(3c_s^2 - 1\right)\left(-1 + c_s^2 \left(2r_c - 3\right)\right) r^3 - \left(2 + c_s^2 \left(2r - 3\right) - r\right) \left(r - 2\right) \lambda^2\right]_{r_c} \]  

(96)

\[ b_2^{iso} = \left[c_s \left(1 - c_s^2\right) \left(-3 + c_s^2 \left(8r - 9\right)\right) r^2 + \left(c_s^2 \left(7 - 4r\right) + 2 \left(r - 2\right) \lambda^2\right)\right]_{r_c} \]

The acoustic surface gravity can thus be expressed as

\[ \kappa_{\text{Conical Flow}}^{\text{Isothermal}} = \left[\frac{r - 2}{r^2 (1 - c_s^2)} \sqrt{r^2 - \lambda^2 \left(1 - \frac{2}{r}\right)} \left(\frac{du}{dr}\right)_{r_c} \right] \]  

(97)

Here also \( \kappa \) has been calculated completely analytically.

### 3. Flow in the hydrostatic equilibrium in the vertical direction

As described in section VA.3, we compute the flow thickness following the work of [62] for this category of the geometric configuration of the flow of non self gravitating axisymmetric accretion. For isothermal flow, however, the sound speed is constant and hence the disc height can be computed as

\[ H(r)^{iso} = \frac{r c_s \sqrt{2(r^3 - (r - 2)\lambda^2)(1 - u^2)}}{\lambda \sqrt{r - 2}} \]  

(98)

to obtain the mass accretion rate as

\[ \dot{M} = 4\pi \rho \frac{r^2 u c_s}{\lambda} \sqrt{2(r^3 - (r - 2)\lambda^2)} \]  

(99)

The space gradient of the advective velocity can thus be found out

\[ \frac{du}{dr} = \frac{\left[r^3 - (r - 2)^2 \lambda^2 + (2 - r)(4r^3 + 5\lambda^2 - 3r\lambda^2)c_s^2\right] u(u^2 - 1)}{\frac{4}{r}(r - 2) (-2r^3 - 4\lambda^2 + 2r^2 \lambda^2) \left[c_s^2 - (1 + c_s^2) u^2\right]} \]  

(100)

leading to the following form of the critical point condition

\[ u^2 = \frac{c_s^2}{1 + c_s^2} = \frac{-r^3 + (r - 2)^2 \lambda^2}{8r^3 - 4r^4 + 10\lambda^2 - 11r\lambda^2 + 3r^2 \lambda^2} \bigg|_{r_c} \]  

(101)

As discussed in the section VA.3, the sonic point and the critical points are not isomorphic for flow in vertical equilibrium for general relativistic isothermal flow in the Schwarzschild metric. For isothermal axisymmetric flow in vertical equilibrium under the influence of the Newtonian or the pseudo-Schwarzschild black hole potential, however, the sonic surface coincides with the critical surface in general [15]. For relativistic flow, we thus need to integrate eq. (100) along the flow line to obtain the location of the acoustic horizon \( r_h \) (as the radial distance where the radial Mach number becomes unity).
Note, however, that there is a fundamental difference in between the non isomorphism of the critical points and the sonic points for the polytropic and for the isothermal flow in vertical equilibrium. From eq. (101), one obtains

\[
M_c = \left[ \frac{1}{1 + \frac{c_s^2}{3}} \right]_{r_c}
\]

where \(M_c\) stands for the Mach number evaluated at the critical point. Since the temperature of any isothermal flow remains invariant, \(\frac{1}{1 + \frac{c_s^2}{3}}\) is a constant at all spatial point of the flow. The deviation of \(M_c\) from unity (\(M_c = 1\) would imply the isomorphism between the critical and the sonic points) becomes fixed once the initial boundary conditions determined by \([T, \lambda]\) are specified. Such deviation increases for hotter flow since \(c_s \propto \sqrt{T}\) and tends to obliterate as the flow temperature approaches absolute zero. One can define an effective sound speed \(c_s^e = \frac{c_s}{\sqrt{1 + \frac{c_s^2}{3}}}\) to obtain the transonic flow structure for which a critical point becomes identical with a sonic point. An effective acoustic geometry is thus to be constructed where the constant speed of propagation of the acoustic perturbation would be \(\frac{c_s}{\sqrt{1 + \frac{c_s^2}{3}}}\) instead of \(c_s\). For polytropic flow in vertical equilibrium, however, eq. (75) implies that such simplified formulation of the effective acoustic geometry is not amenable since Mach number evaluated at the critical point itself is a nonlinear function of the critical point, and hence for the same initial boundary condition defined by \([\tilde{E}, \lambda, \gamma]\), the amount of deviation of \(M_c\) from unity will be different for different critical points for a multi-critical accretion. As a result, \((r_c - r_s)\) will be different for the inner and the outer saddle type critical points. On the Mach number radial distance phase portrait, the inner and the outer critical points are not ‘co linear’ in the sense that the line joining them is not parallel to the X axis since \(M_{c,\text{inner}} \neq M_{c,\text{outer}}\) for obvious reason. Same is true for the non collinearity of their corresponding sonic points as well. For isothermal accretion in vertical equilibrium, however, \(r_s^{\text{inner}}\) and \(r_s^{\text{outer}}\) are co-linear, as well as their corresponding sonic points \([r_s^{\text{inner}}, r_s^{\text{outer}}]\).

The value of the critical point(s) can be found by analytically solving the following 4th degree polynomial

\[
\left[4r_c^2c_s^2 - r^3(1 + 8c_s^2) - r^2\lambda^2(-1 + 3c_s^2) - r\lambda^2(2 - 2c_s^2) - 2\lambda^2 (2 - 2c_s^2)\right]_{r_c} = 0
\]

using the two parameter set \([T, \lambda]\) as has been done for other geometric configurations. The critical point deviation of the advective velocity can further be calculated as

\[
\left(\frac{du}{dr}\right)_c = \left[\frac{\alpha_3^{\text{iso}}}{2\Gamma_3^{\text{iso}}} + \frac{\sqrt{\beta_3^{\text{iso}} + 4\Gamma_3^{\text{iso}}}}{2\Gamma_3^{\text{iso}}} \right]_{r_c}
\]

where the –ve sign has to be considered for the accretion solution. In eq. (104),

\[
\alpha_3^{\text{iso}} = \left[-\left((-1 + 4c_s^2(r - 2))r^3 - 2(1 + c_s^2)(3r - 5) - r\right)\lambda^2\left(\frac{3c_s^2}{1 + c_s^2} - 1\right)\right]_{r_c}
\]

\[
\beta_3^{\text{iso}} = \left[\left((-3 + 8c_s^2(2r - 3))r^2 + (c_s^2(2 - 2r) + 2(2 - 2r)^2)\lambda^2\right)\left(\frac{c_s}{1 + c_s^2}\right)\right]_{r_c}
\]

\[
\Gamma_3^{\text{iso}} = \left[2r(r - 3)(r^3 - r^2\lambda^2)c_s\sqrt{1 + c_s^2}\right]_{r_c}
\]

For a fixed value of \([T, \lambda]\) as the accretion parameters, we find \(\left[u_1\right]_{r_c}\) as defined by eq. (101) and \(\left(du/dr\right)_{r_c}\) as defined by eq. (104), we integrate eq. (100) starting from the critical point (as provided by the analytical solution of eq. (103)) up to the radial distance where the radial Mach number becomes unity. Once that sonic point \(c_s\) is identified (as the acoustic horizon), \(u\) and \(du/dr\) are evaluated at that point and \(\left[u, du/dr\right]_{r_c}\) is then used to calculate the corresponding acoustic surface gravity as

\[
k_{\text{Vertical Equilibrium}} = \left[r - 2 \sqrt{r^2 - \lambda^2(\frac{2}{r} - 1)}\left(\frac{du}{dr}\right)_{r_c}\right]
\]

\[
k_{\text{Isothermal}} = \left[\frac{r - 2 \sqrt{r^2 - \lambda^2(\frac{2}{r} - 1)}}{r^2(1 - c_s^2)}\left(\frac{du}{dr}\right)_{r_c}\right]
\]

VI. DEPENDENCE OF ACOUSTIC SURFACE GRAVITY ON FLOW GEOMETRY – POLYTROPIC ACCRETION

In this section we will manifest how the geometric configuration of the non self-gravitating axially symmetric stationary background flow in a Schwarzschild metric influences the computation of the acoustic surface gravity for
polytropic accretion onto astrophysical black holes. We first construct the parameter space determined by the initial boundary conditions to show the parameter dependence of the multi-critical flow behaviour, and then will describe how one can pick up certain regions of \([E, \lambda, \gamma]\) space for which mono-transonic accretion can be available for all three different geometric configurations.

### A. The parameter space classification

As manifested through eq. (51,64,77), the critical point(s) are completely determined once \([E, \lambda, \gamma]\) is specified. A three dimensional parameter space spanned by \([E, \lambda, \gamma]\) and bounded by \(1 < E < 2, 0 < \lambda \leq 2, 4/3 \leq \gamma \leq 5/3\) can thus be explored to understand the dependence of the multi-critical behaviour on initial boundary conditions.

For the sake of convenience, a two dimensional projection of such a three dimensional parameter space will be analyzed. \(^3\)C\(_2\) allowed combinations of such projection defining parameters are available. In this work, we prefer to project the \([E, \lambda, \gamma]\) space on a \([E, \lambda]\) plane by keeping the value of the adiabatic constant to a fixed value \(\gamma = 4/3\). Although such \([E, \lambda]\) projections can be studied for any other values lying in the range \(4/3 \leq \gamma \leq 5/3\) as well.

In figure 1, we study the \(E - \lambda\) plane for three different flow geometries. Variation of \(E - \lambda\) branches for flow in hydrostatic equilibrium along the vertical direction, conical flow and flow with constant thickness are represented by solid red lines, dashed green lines, and dotted blue lines, respectively. Hereafter, we will follow the aforementioned colour scheme to show results corresponding to the flow geometries discussed above. For flow with constant height, \(A_1A_2A_3A_4\) represents the region of \([E, \lambda]\) for which eq. (31) provides three real positive roots lying outside the gravitational horizon. For region \(A_1A_2A_3\), one finds \(\Xi_{inner} > \Xi_{outer}\) and accretion is multi-critical. A subspace of \(A_1A_2A_3\) allows shock formation and such subspace provides true multi-transonic accretion where the stationary transonic solution passing through the outer sonic point joins with the stationary transonic solution through the inner sonic point through a discontinuous energy preserving shock of Rankine-Hugoniot type. Such shocked multi-transonic solution contains two smooth transonic (from sub to super) transitions at two regular sonic points (of saddle type) and a discontinuous transition (from super to sub) at the shock location. Such configuration possesses multiple acoustic horizons, two regular black hole type at two sonic points and one ‘irregular’ white hole type at the shock location. The acoustic gravity corresponding to the white hole, however, can not be computed since the advective velocity and sound speed becomes non-differentiable at such discontinuities. The analogue Hawking temperature thus becomes formally infinite at the shock location, which is in accordance with the finding reported in [23] where the formalism for obtaining the Hawking like temperature fails to provide acceptable results for very large value of the space gradient of the background flow velocity normal to the acoustic horizon (advective velocity for our case).

The aforementioned discontinuity would be smeared out provided the viscosity and other dissipative effects would be included in the background fluid system (so that the middle center type critical point would become a critical point of spiral type) and one would obtain a finite but large value of the acoustic surface gravity evaluated at the white hole horizon which can be considered as an astrophysical manifestation of the general result reported in [77].

On the other hand, the region \(A_1A_2A_4\) represents the subset of \([E, \lambda, \gamma]\)\(_{mc}\) (where ‘mc’ stands for ‘multi critical’) for which \(\Xi_{inner} < \Xi_{outer}\) and hence incoming flow can have only one critical point of saddle type and the background flow possesses one acoustic horizon at the inner saddle type sonic point. The boundary \(A_1A_3\) between these two regions represents the value of \([E, \lambda, \gamma]\) for which multi-critical accretion is characterized by \(\Xi_{inner} = \Xi_{outer}\) and hence the transonic solutions passing through the inner and the outer sonic points are completely degenerate, leading to the formation of a heteroclinic orbit\(^{[86]}\) on the phase portrait. Such flow configuration can not be used to study the analogue properties as we believe since it does not have uniqueness in forming the acoustic horizons. Such flow pattern is subjected to instability and turbulence as well\([46]\).

Similar analysis can be carried out for other two flow geometries as well, \(B_1B_2B_3B_4\) and \(C_1C_2C_3C_4\) represents \([E, \lambda, \gamma]\)\(_{mc}\) for the quasi-spherical conical flow and for flow in vertical equilibrium, respectively. for \(B_1B_2B_3B_4\), \(B_1B_2B_3\) represents the multi-critical flow with \(\Xi_{inner} > \Xi_{outer}\) and \(B_1B_2B_4\) represents such region for \(\Xi_{inner} < \Xi_{outer}\), \(B_1B_3\) being the interface between them representing the values of \([E, \lambda, \gamma]\) for which only the hetero-clinic connections are obtained. Similar classifications can also be made for the region \(C_1C_2C_3C_4\) as well.

We would like to pick up a range of \([E, \lambda, \gamma]\) for which all three flow configurations will provide mono-transonic accretion. Moreover, we are interested mainly in the stationary mono-transonic solutions passing though the inner sonic point since the acoustic surface gravity evaluated at the inner acoustic horizon is of the order of magnitude (upto about \(10^5\) or even higher) higher than the acoustic surface gravity evaluated at the outer acoustic horizon. In addition, the location of the outer acoustic horizon, and hence the value of the acoustic surface gravity evaluated on it, are not much sensitive to the variation of \([E, \lambda, \gamma]\) compared to its counterparts corresponding to the inner acoustic horizon.

Out of the three parameters \(E, \lambda\) and \(\gamma\), we choose one parameter, say \(\lambda\), to vary by keeping the values of \([E, \gamma]\) fixed, for three different flow configurations for the mono-transonic flow through the inner acoustic horizon to obtain...
the $[\kappa - \lambda]$ variation. Three different $[\kappa - \lambda]$ variations for three different flow geometries will then be compared to examined the influence of the accreting matter configuration on the computation of the acoustic surface gravity. We then perform the same operation for other two parameters $E$ and $\lambda$, to obtain the $[\kappa - E]$ and $[\kappa - \gamma]$ variation for three different geometries by keeping $[\lambda, \gamma]$ and $[E, \lambda]$ invariant, respectively. Finally, we will do similar exercise for isothermal accretion of three different flow geometries to obtain and to compare the $[\kappa - \lambda]$ and $[\kappa - T]$ variations, respectively.

![Figure 1](image_url)

**FIG. 1:** $E - \lambda$ plane for three different flow geometries for adiabatic accretion for fixed value of $\gamma = 4/3$. Variation of $E - \lambda$ branches for flow in hydrostatic equilibrium along the vertical direction, conical flow and flow with constant thickness are represented by solid red lines, dashed green lines, and dotted blue lines, respectively. See section VIA for further detail about the parameter space classification.

### B. Variation of $\kappa$ with $[E, \lambda, \gamma]$

In figure 2., for a fixed set of $[E = 1.2, \gamma = 4/3][87]$, we plot the location of the inner type acoustic horizon (the inner sonic point $r_s$) as a function of the specific angular momentum $\lambda$ of the flow for mono-transonic stationary accretion solution for three different geometric configurations of matter as considered in the present work. It is observed that the location of the acoustic horizon anti-correlates with the specific angular momentum of the background axisymmetric flow. This is somewhat obvious because for greater amount of rotational energy content of the flow, accretion starts with smaller advective velocity and has to approach very close to the black hole event horizon to acquire the dynamical velocity sufficiently large to overcome the acoustic velocity so that the smooth transition from the subsonic to the supersonic state takes place.

For a specified initial boundary condition (as determined by $[E, \lambda, \gamma]$) describing the flow, one observes

$$r_s^{\text{vertical}} < r_s^{\text{conical}} < r_s^{\text{constant height}}$$

(107)

This indicates that for same set of $[E, \lambda, \gamma]$, the acoustic horizon for accretion in hydrostatic equilibrium along the vertical direction forms at the closest proximity of the black hole event horizon and hence the relativistic acoustic geometry at the neighbourhood of such acoustic horizons are subjected to extremely strong gravity space time. One thus intuitively concludes that among all three glow configurations considered in this work, the Hawking like effects will be more pronounced for axisymmetric background flow in hydrostatic equilibrium along the vertical direction. This intuitive conclusion is further supported by results represented in figure 3 where we have studied the variation
of the acoustic surface gravity \( \kappa \) as a function of the flow angular momentum \( \lambda \) for same set of initial boundary conditions as well as for the span of \( \lambda \) for which figure 2 has been obtained. For identical initial boundary conditions as determined by \([\mathcal{E}, \lambda, \gamma]\), one obtains

\[
\kappa_{\text{vertical}} > \kappa_{\text{conical}} > \kappa_{\text{constant height}}
\] (108)

For a fixed value of \([\lambda = 2.89, \gamma = 4/3]\), the variation of the location of the inner acoustic horizons (the inner sonic points \( r_s \)) as a function of the specific flow energy \( \mathcal{E} \) is plotted in figure 4. \( r_s \) anti-co-relates with \( \mathcal{E} \) for obvious reasons. Since at effective infinity (at a very large distance away from the accretor) the total specific energy is essentially determined by the thermal energy of the flow, a large value of \( \mathcal{E} \) (‘hot’ accretion) corresponds to a high value of the sound speed \( c_s \), to begin with and hence the subsonic to the supersonic transition takes place quite close to the black hole where the bulk flow velocity (the advective velocity \( u \)) becomes large enough to overcome the sound speed. Once again, flow in vertical equilibrium produces the acoustic horizons located at a relatively stronger gravity region, and hence the analogue effect should be more pronounced for such geometric configuration of the flow. Results presented in figure 5, where the acoustic surface gravity \( \kappa \) has been plotted as a function of the specific energy of the flow for same set of \([\lambda, \gamma]\) used to draw figure 4, asserts such conclusion. For a fixed value of \([\mathcal{E} = 1.12, \lambda = 3.3]\), in figure 6 we find that the location of the acoustic horizon anti-correlates with \( \gamma \) and hence the acoustic surface gravity \( \kappa \) co-rrelates with \( \gamma \) as expected (as shown in figure 7). Similar result can be obtained for any set of \([\mathcal{E}, \lambda]\) for which mono transonic stationary accretion passing through the inner type sonic point can be obtained for all three flow configurations considered in this work. Here too the acoustic surface gravity for accretion in hydrostatic equilibrium along the vertical direction is maximum (compared to the conical flow and flow with constant thickness) for the same set of initial boundary conditions. We thus obtaine

\[
\begin{align*}
\kappa_{\text{vertical}} &> \kappa_{\text{conical}} > \kappa_{\text{constant height}} \\
r_s^{\text{vertical}} &< r_s^{\text{conical}} < r_s^{\text{constant height}}
\end{align*}
\] (109)

here as well.

FIG. 2: For \([\mathcal{E} = 1.12, \gamma = 4/3]\), variation of the location of the acoustic horizons as a function of the flow angular momentum \( \lambda \) for stationary mono-transonic adiabatic accretion passing through the inner sonic point \( r_s \) for three different geometric configuration of the flow considered in this work. \( r_s - \lambda \) curves for flow in hydrostatic equilibrium along the vertical direction, conical flow and flow with constant thickness are represented by solid red lines, dashed green lines, and dotted blue lines, respectively.
FIG. 3: For same initial boundary condition used to obtain figure 2, variation of acoustic surface gravity $\kappa$ with the flow angular momentum $\lambda$ for stationary mono-transonic adiabatic accretion passing through the inner sonic point $r_s$ for three different geometric configuration of the flow considered in this work. $\kappa - \lambda$ curves for flow in hydrostatic equilibrium along the vertical direction, conical flow and flow with constant thickness are represented by solid red lines, dashed green lines, and dotted blue lines, respectively.

FIG. 4: For $[\lambda = 2.89, \gamma = 4/3]$, variation of the location of the acoustic horizons as a function of the specific energy of the flow $E$ for stationary mono-transonic adiabatic accretion passing through the inner sonic point $r_s$ for three different geometric configuration of the flow considered in this work. $r_s - \lambda$ curves for flow in hydrostatic equilibrium along the vertical direction, conical flow and flow with constant thickness are represented by solid red lines, dashed green lines, and dotted blue lines, respectively.
FIG. 5: For same initial boundary conditioned used to obtain figure 4, variation of acoustic surface gravity $\kappa$ with the specific energy of the flow $\mathcal{E}$ for stationary mono-transonic adiabatic accretion passing through the inner sonic point $r_s$ for three different geometric configuration of the flow considered in this work. $\kappa - \mathcal{E}$ curves for flow in hydrostatic equilibrium along the vertical direction, conical flow and flow with constant thickness are represented by solid red lines, dashed green lines, and dotted blue lines, respectively.

FIG. 6: For $[\mathcal{E} = 1.12, \lambda = 3.3]$, variation of the location of the acoustic horizons as a function of the adiabatic index $\gamma$ for stationary mono-transonic adiabatic accretion passing through the inner sonic point $r_s$ for three different geometric configuration of the flow considered in this work. $r_s - \gamma$ curves for flow in hydrostatic equilibrium along the vertical direction, conical flow and flow with constant thickness are represented by solid red lines, dashed green lines, and dotted blue lines, respectively.
FIG. 7: For same initial boundary conditioned used to obtain figure 6, variation of acoustic surface gravity $\kappa$ with the adiabatic index $\gamma$ has been plotted for stationary mono-transonic adiabatic accretion passing through the inner sonic point $r_s$ for three different geometric configuration of the flow considered in this work. $\kappa - \gamma$ curves for flow in hydrostatic equilibrium along the vertical direction, conical flow and flow with constant thickness are represented by solid red lines, dashed green lines, and dotted blue lines, respectively.

VII. DEPENDENCE OF ACOUSTIC SURFACE GRAVITY ON FLOW GEOMETRY – ISOTHERMAL ACCRETION

This section will illustrate how the acoustic surface gravity for isothermal accretion depends on the constant flow temperature (ion temperature for a single component flow) as well as on flow angular momentum for three different matter geometries considered in this work. The $[T, \lambda]$ parameter space will first be constructed to manifest the multi-critical flow behaviour. Certain subset of the entire $[T, \lambda]$ will then be chosen for which isothermal accretion in all three matter geometries will have mono transonic solutions constructed through the inner sonic point.

A. The parameter space classification

Unlike the polytropic accretion, parameter space corresponding to the isothermal one is essentially two dimensional, which consists of subspaces for which the mono and the multi-critical solutions may be obtained. Figure 8 depicts the parameter space division labeled following the scheme introduced in section VIA. For constant height flow, conical flow and vertical equilibrium flow, $A_1'A_2'A_3'A_4'$, $B_1'B_2'B_3'B_4'$, and $C_1'C_2'C_3'C_4'$, represents the $[T, \lambda]$ regions for which eq. (87), eq. (94) and eq. (103) will provide three real physical roots located outside the gravitational horizon, respectively. It is to be noted that since eq. (87), eq. (94) and eq. (103) are all $4^{th}$ degree polynomials, figure 8 can be obtained completely analytically by analytically solving the representative equations using the Ferrari’s method [83].

Similar to the polytropic accretion, the wedge shaped region $[T, \lambda]_{mc}$, where ‘mc’ stands for ‘multi critical’, has two subsections divided by a distinct boundary. However, unlike the entropy accretion rate $\dot{S}$ for the polytropic flow, the first integral of motion $\xi$ as defined in eq. (82) determines the characteristic features of various subspaces of $[T, \lambda]_{mc}$. For constant height flow, $A_1'A_2'A_3'A_4'$ region is subdivided into $A_1'A_2'A_3'$ for which $\xi_{in} > \xi_{out}$ and $A_1'A_3'A_4'$ for which $\xi_{in} < \xi_{out}$ with the boundary line $A_1'A_3'$ on which $\xi_{in} = \xi_{out}$. $[T, \lambda]_{A_1'A_3'A_4'}$ provides the multi critical integral solutions for which the homoclinic orbit is constructed through the inner critical point whereas $[T, \lambda]_{A_1'A_2'A_4'}$ produces the multi critical solution for which accretion is mono transonic and the corresponding homoclinic orbit is constructed through the outer critical point. $[T, \lambda]_{A_1'A_3'}$ provides the heteroclinic orbits for which one obtains degenerate accretion solutions.
For a certain subset of \( [T, \lambda]_{A'1A'2A'3} \in [T, \lambda]_{mc} \), temperature preserving shock may form to provide true multi transonicity. Such stationary solutions contain two acoustic black hole horizons at the inner and the outer sonic points and an acoustic white hole solution at the shock location. We, however, will not perform the shock finding analysis in the present work.

In a similar spirit, subdivisions in multi critical parameter spaces for the conical and the vertical equilibrium can also be obtained.

![Graph of T - \( \lambda \) plane for three different flow geometries for isothermal accretion. Variation of \( T - \lambda \) branches for flow in hydrostatic equilibrium along the vertical direction, conical flow and flow with constant thickness are represented by solid red lines, dashed green lines, and dotted blue lines, respectively. See section VII A for further detail about the parameter space classification.](image)

**B. Variation of \( \kappa \) with \([\lambda, T]\)**

In figure 9, we plot the variation of the location of the acoustic horizon (the inner sonic point \( r_s \) with the constant specific angular momentum of the flow \( \lambda \). For obvious reasons (as described in section VI B), \( r_s \) anti-correlates with \( \lambda \). For accretion flow in hydrostatic equilibrium along the vertical direction, entire range of sonic points produced (for the domain of \( \lambda \) considered in this work) lie in the very close proximity of the black hole event horizon. This indicates that for the same set of initial boundary conditions describing the flow, the Hawking like effects will be maximally pronounced for such flow model. Such conclusion is further reinforced from results presented in figure 10 where we have plotted the acoustic surface gravity \( \kappa \) as a function of the flow angular momentum \( \lambda \) to obtain

\[
\kappa_{\text{vertical}} > \kappa_{\text{conical}} > \kappa_{\text{constant height}}
\]  

(110)

Figure 11 represents the variation of the location of the acoustic horizon with constant flow temperature \( T \) expressed in units of \( 10^{10} \) degree Kelvin and denoted by \( T_{10} \). Since the position independent sound speed \( c_s \propto T^{\frac{3}{2}} \), the sonic point \( r_s \) anti-correlates with the flow temperature \( T \). In figure 12 we plot the variation of the acoustic surface gravity \( \kappa \) as a function of \( T \) in units of \( T_{10} \). Hotter flow produces higher Hawking like temperature since \( T_{AH\text{ocK}} \).
FIG. 9: For $T_{10} = 0.04$, variation of the location of the acoustic horizons as a function of the flow angular momentum $\lambda$ for stationary mono-transonic isothermal accretion passing through the inner sonic point $r_s$ for three different geometric configuration of the flow considered in this work. $r_s - \lambda$ curves for flow in hydrostatic equilibrium along the vertical direction, conical flow and flow with constant thickness are represented by solid red lines, dashed green lines, and dotted blue lines, respectively.

FIG. 10: For same initial boundary conditioned used to obtain figure 9, variation of acoustic surface gravity $\kappa$ with the flow angular momentum $\lambda$ for stationary mono-transonic isothermal accretion passing through the inner sonic point $r_s$ for three different geometric configuration of the flow considered in this work. $\kappa - \lambda$ curves for flow in hydrostatic equilibrium along the vertical direction, conical flow and flow with constant thickness are represented by solid red lines, dashed green lines, and dotted blue lines, respectively.
FIG. 11: For $\lambda = 3.3$, variation of the location of the acoustic horizon with the constant flow temperature $T$ (in units of $10^{10}$ degree Kelvin and denoted as $T_{10}$) for stationary mono-transonic isothermal accretion passing through the inner sonic point $r_s$ for three different geometric configuration of the flow considered in this work. $r_s - T_{10}$ curves for flow in hydrostatic equilibrium along the vertical direction, conical flow and flow with constant thickness are represented by solid red lines, dashed green lines, and dotted blue lines, respectively.

FIG. 12: For same initial boundary conditioned used to obtain figure 11, variation of acoustic surface gravity $\kappa$ with the constant flow temperature $T$ (in units of $10^{10}$ degree Kelvin and denoted as $T_{10}$) for stationary mono-transonic isothermal accretion passing through the inner sonic point $r_s$ for three different geometric configuration of the flow considered in this work. $\kappa - T_{10}$ curves for flow in hydrostatic equilibrium along the vertical direction, conical flow and flow with constant thickness are represented by solid red lines, dashed green lines, and dotted blue lines, respectively.
VIII. CONCLUDING REMARKS

For analogue effects, the mass of the system itself can not directly be anti-correlated to the acoustic surface gravity (as well as with the associated Hawking like temperature) since $\kappa$ is obtained as a complicated non linear functional [88] of the initial boundary conditions describing the flow profile. Since same set of initial boundary conditions may provide significantly different phase portrait for integral stationary solutions for different geometric configurations of the background matter flow, it is necessary to study the influence of the matter geometry on the determination of the corresponding relativistic acoustic geometry. In this work, we accomplish such task by studying the dependence of the value of $\kappa$ on the geometric configurations of the background matter flow as well as on various astrophysically relevant initial boundary conditions governing such flow described by different thermodynamic equations of state. In this way we intended to provide a reference space spanned by fundamental accretion parameters to apprehend under which astrophysically relevant scenario the analogue Hawking temperature can assume its maximum value.

We found that the Hawking like effects become more pronounced in the relatively stronger gravity region. Irrespective of the equation of state as well as the initial boundary conditions, the acoustic surface gravity for stationary mono-transonic solutions assumes its maximum value when the acoustic horizons are formed at very close proximity of the black hole event horizon. Among all three geometric configurations of the background axisymmetric flow considered in this work, flow in hydrostatic equilibrium along the vertical direction produces the acoustic horizons of smallest radius and hence the corresponding surface gravity and the Hawking like temperature becomes maximum for such flow configuration. This is true for both the adiabatic as well as the isothermal accretion. It has also been observed that hotter flow (adiabatic flow parameterized by large value of $E$ or isothermal flow parameterized by high temperature) produces the larger value of the acoustic surface gravity since for such flow the inner type sonic points are formed very close to the black hole event horizon. Similar effects are observed for flow with large large values of $\lambda$ and $\gamma$. One thus concludes that relatively faster rotating hotter flows are responsible to maximize the analogue effects for axisymmetric background flow in Schwarzschild metric.

As already clarified in section II, non universal features of Hawking like effects in a dispersive media depends on the value of the space gradient of the background flow velocity as well as the speed of propagation of perturbation for fluid flow with position dependent sound speed, the aforementioned calibration space will also be useful to point out the relevance of certain astrophysical configuration to simulate the set up where such deviation can be a maximum. This will certainly be useful to study the effect of gravity on the non-conventional classical features in Hawking like effect as is expected to be observed in the limit of a strong dispersion relation - no such work has been reported in the literature yet.

This work does not report any analysis of the shock formation phenomena in background fluid flow. Our preliminary calculations with a shocked multi-transonic accretion in various flow profile considered here (not reported in the paper), however, asserts that $\kappa_{\text{in}}$ and $\kappa_{\text{mout}}$ evaluated at the inner and the outer sonic point of the stationary multi-transonic shocked flow respectively, does have the same qualitative feature compared to acoustic surface gravity evaluated for mono-transonic solutions passing through the inner and the outer type sonic points, respectively, and the value of $\kappa_{\text{in}}$ is always approximately $10^5$ times (or more) higher than the corresponding value of $\kappa_{\text{mout}}$ for all three different geometric configurations of the matter flow described by both the thermodynamic equations of state, the adiabatic as well as the isothermal, respectively. This indicates that irrespective of the flow topology (i.e., whether mono or multi transonic), the measure of the acoustic surface gravity is essentially dominated by the flow properties close to the gravitational horizon.

For adiabatic as well as for the isothermal flows in hydrostatic equilibrium in the vertical direction, the critical point and the sonic points are found to be non overlapping, and an integral solution of the flow equations are needed to obtain the location of the acoustic surface gravity. Note that whereas for the adiabatic flow it is true for pseudo-Schwarzschild accretion under the influence of the modified potentials, isothermal accretion within such modified Newtonian framework does not discriminate between a critical and a sonic point [17]. It is, however, difficult to conclude anything about the universality of such phenomena since the corresponding expression for the flow thickness has been derived using a set of idealized assumptions. A more realistic flow thickness may be derived by by employing the non-LTE radiative transfer [78, 79] or by taking recourse to the Grad-Shafranov equations for the MHD flow [80–82].

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[1] Unruh, W. G. 1981, Phys. Rev. Lett. 46, 1351
[2] Unruh, W. G. 1995, Phys. Rev. D. 51, 2827
[3] Visser, M. 1995, Class. Quant. Grav. 15, 1767
[4] Bilić, N. 1999, Class. Quant. Grav. 16, 3953
[5] Novello, Visser & Volovik (ed.) 2002, Artificial Black Holes. World Scientific, Singapore.
[6] Cardoso, V., 2005, Acoustic black holes, physics/0503042
[7] Barceló, C., Liberati, S., and Visser, M., 2005, ‘Analogue Gravity’, Living Reviews in Relativity, Vol. 8, no. 12, webservice http://relativity.livingreviews.org/Articles/lrr-2005-12/, also at gr-qc/0505065
[8] Unruh, W., & Schützhold, R. (ed), 2007, Quantum Analogues: From Phase Transitions to Black Holes and Cosmology, Springer, Lecture Note in Physics, Vol. 718
[9] Das, T. K. 2004, Class. Quant. Grav. 21, 5253
[10] Dasgupta, S., Bilić, N., and Das, T. K., 2005,
[11] Abraham, H., Bilić, N., Das, T. K., Bilic, N., and Dasgupta, S. 2005, Classical and Quantum Gravity, 23, 1767
[12] Das, T. K., Bilic, N., & Dasgupta, S. 2007, JCAP, 6
[13] Mach, P., & Malec, E., 2008, Physical Review D, vol. 78, Issue 12, id. 124016 9
[14] Mach, P., 2009, Reports on Mathematical Physics, vol. 64, issue1-2, pp. 257-269
[15] Nag, S., Acharya, S., Ray, A. K., Das, T. K., 2012, New Astronomy, Volume 17, Issue 3, p. 289-295
[16] Pu, H-Y., Maity, I., Das, T. K., & Chang, H. K., 2012, Class. Quant. Grav. 29, 245020
[17] Bilić, N., Choudhary, A., Das, T. K., & Nag, S., 2012, arXiv:1205.5506 [gr-qc]
[18] Helfer A D, 2003, Rept. Prog.Phys., 66, 943
[19] Rousseaux, G., Mathis, C., MaÅ issa, P., Philbin, T. G., & Leonhardt, U., 2008, New Journal of Physics, Volume 10, Issue 5, pp. 053015
[20] Rousseaux, G., MaÅ issa, P., Mathis, C., Coullet, P., Philbin, T. G., & Leonhardt, U., 2010, New Journal of Physics, Volume 12, Issue 9, pp. 095018
[21] Jannes, G., Piquet, R., Maissa, P., Mathis, C., & Rousseaux, G., 2011, Physical Review E, 83, 056312
[22] Weinfurther, S., Tedford, E. W., Pennice, M. C. J., Unruh, W. G., & Lawrence, G. A., 2011, Physical Review Letters, vol. 106, Issue 2, id. 021302
[23] Leonhardt, U., & Robertson, S., 2012, New Journal of Physics, 14, 053003
[24] Robertson, S. J., 2012, Journal of Physics B, Volume 45, Issue 16, pp. 163001
[25] Illarionov, A.F., & Sunyaev, R. A. 1975a, A & A, 39, 205
[26] Liang, E. P. T., Thomson, K. A., 1980, ApJ, 240, 271
[27] Bisikalo, A. A., Boyarchuk, V. M., Chechetkin, V. M., Kuznetsov, O. A., & Molteni, D. 1998, MNRAS, 300, 39
[28] Illarionov, A. F. 1988, Soviet Astron., 31, 618
[29] Ho, L. C. 1989, in Observational Evidence For Black Holes in the Universe, ed. S. K. Chakrabarti (Dordrecht: Kluwer), 153
[30] Igumenshchev, I. V. & Abramowicz, M. A. 1999, MNRAS, 303, 309
[31] Beloborodov, A. M., & Illarionov, A. F. 1991, MNRAS, 323, 167
[32] Igumenshchev, I. V., & Beloborodov, A. M., 1997, MNRAS, 284, 767
[33] Proga, D., & Begelman, M. C., 2003, ApJ, 582, 69
[34] Janiuk, A., Szajder, M., Moscibrodzka, M., & Proga, D., 2009, ApJ, 705, 1503
[35] Matsumoto, R., Kato, S., Fukue, J., Okazaki, A. T., 1984, PASJ, 36, 71
[36] Frank, J., King, A., Raine, D., 2002, Accretion Power in Astrophysics, Cambridge University Press, Cambridge
[37] Kato, S., Fukue, J., & Mineshige, S., 1998, Black Hole Accretion Disc, Kyoto University Press.
[38] Boyer, R. H., & Lindquist, R. W. 1967, J. Math. Phys. 8, 265
[39] Novikov, I., & Thorne, K. S. 1973, in Black Holes,eds. c. De Witt and B. De Witt(Gordon and Breach, New York).
[40] Meliani, Z., Sauty, C., Tsinganos, K., & Vlahakis, N., 2004, A & A, Volume 425, pp. 773
[41] Ryu, D. & Chattopadhyay, I., 2006, Astrophys. J. Suppl. Series, 166(1), 410
[42] Migone, A. & McKinney, J.C., 2007, MNRAS, Volume 378, pp. 1118
[43] Mondal, S. & Basu, P., 2011, Class. Quant. Grav., 28, 235004
[44] Mukhopadhyay, B. & Dutta, P., 2012, New Astronomy, 17, 51
[45] Anderson, M. 1989, MNRAS, 239, 19
[46] Das, T. K., & Czerny, B., 2012b, New Astronomy, 17, 254
[47] Yuan, F., Dong, S., & Lu, J-F., 1996, Ap&SS 246, 197
[48] Abramowicz, M. A. & Zurek, W.H., 1981, ApJ, 246, 314
[49] Blaes, O., 1987, MNRAS, 227, 975
[50] Paczyński, B., Wiita P. J., 1980, A&A, 88, 23
[51] Lu, J. F. 1985, A & A, 148, 176
[52] Lu, J. F. 1986, Gen. Rel. Grav. 18, 45L
[53] Lu, J. F., Yu, K. N., & Young, E. C. M., 1995, A & A, 304, 662
[54] Gammie, C. F., & Popham, R., 1998, ApJ, 498, 313
[55] Popham, R., & Gammie, C. F., 1998, ApJ, 504, 419
[56] Lu, J. F., Yu, K. N., Yuan, F., Young, E. C. M., 1997, A & A, 321, 665
[57] Lu, J. F., Yu, K. N., Yuan, F., Young, E. C. M., 1997, Astrophysical Letters and Communications, 35, 389
[58] Lu, J. F., & Yuan, F., 1998, MNRAS, 295, 96
[59] Lu, J. F., & Gu, W. M., 2004, Chin. Phys. Lett., 21, 2551
[60] Riffert, H., & Herold, H., 1995, ApJ, 450, 508
[61] Abramowicz, M. A., Lanza, A., & Percival, M. J. 1997, ApJ, 479, 179
[62] Landau, L. D., & Lifshitz, E. M., 1987, Fluid Mechanics, Butterworth-Heinemann, Oxford
[63] Salas, M. D., The curious events leading to the theory of shock waves, Invited lecture given at the 17th Shock Interaction Symposium, Rome, Italy, 4 - 8 September, 2006, 2006, Shock Waves, 16, 477
[64] Yang, R. X., Kafatos, M., 1995, A&A, 295, 238
[65] Press, W. H., Teukolsky, S. A., Vetterling, W. T., & Flannery, B. P., 2007, Numerical Recipes: The Art of Scientific Computing (3rd ed.), New York: Cambridge University Press.
[66] Beskin, V. S., Tchekhovskoy, A., 2005, A & A, 433, 619
[67] Beskin, V. S., & Tchekhovskoy, A., 2009, MHD Flows in Compact Astrophysical Objects: Accretion, Winds and Jets. Springer.
[68] Beskin, V. S., 1997, Phys. Usp. 40, 659
[69] Beskin, V. S., & Tchekhovskoy, A., 2005, A & A, 433, 619
[70] Beskin, V. S., Tchekhovskoy, A., 2009, MHD Flows in Compact Astrophysical Objects: Accretion, Winds and Jets. Springer.
[71] Kurosh, A. G., 1972, Higher Algebra, Mir Publication, Moscow
[72] Details about such geometric configuration can further be found in [15].
[73] A homoclinic orbit or a homoclinic connection is a bi-asymptotic trajectory converging to a saddle like orbit at time goes to positive or negative infinity. For our stationary systems, a homoclinic orbit on a phase portrait is realized as an integral solution that re-connects a saddle type critical point to itself and embarrasses the corresponding centre type critical point.
[74] Heteroclinic orbits are the trajectories defined on a phase portrait which connects two different saddle type critical points. Integral solution configuration on phase portrait characterized by heteroclinic orbits are topologically unstable [67–69].
[75] Such a high value of $E$ (‘hot’ accretion) has been considered to ensure that mono-transonic stationary solution passing through the inner sonic point is obtained for all three different geometrical configuration of the axisymmetric matter considered in this work.
[76] $κ$ is actually functions of $[u, c_s, du/dr, dc_s/dr]$ for adiabatic flow and $[u, c_s, du/dr]$ of isothermal flow. $[u, c_s, du/dr, dc_s/dr]_{adu}$ and $[u, c_s, du/dr]_{iso}$ are non linear functions of $[E, λ, γ]_{adu}$ and $[T, λ]_{iso}$ respectively, and hence $κ_{adu}$ and $κ_{iso}$ are functionals of $[E, λ, γ]_{adu}$ and $[T, λ]_{iso}$, respectively.