A Regularization of Quantum Gravity

Wolfgang Beirl and Bernd A. Berg

(E-mails: wb1@gate.net, berg@hep.fsu.edu)

Department of Physics, The Florida State University, Tallahassee, FL 32306

(March 8, 2001)

We re-examine results of the Liouville theory and provide arguments that a negative bare cosmological constant is essential to define two-dimensional quantum gravity. From this we are naturally led to a regularization of quantum gravity within the Regge approach such that it is described by small fluctuations around equilateral triangles, whose average link length approaches zero in the continuum limit. We investigate a model based on this idea numerically and present evidence for the desired long-range correlations. Interestingly, the approach might generalize to higher dimensions. The picture of an inflated balloon, which is often used to demonstrate the properties of an expanding classical universe, seems to be valuable to understand quantum gravity as well.

PACS: 04.60-m.

I. INTRODUCTION

In recent years, substantial progress has been achieved in understanding two-dimensional quantum gravity based on the Liouville theory [1], see [2] for a review. Unfortunately it was not possible to extend these results to higher dimensions. In the light of these difficulties we re-examine the results of Polyakov, starting with the path integral of two-dimensional gravity for a fixed topology and with $D$ additional matter fields $X^i$:

$$ Z_D(\mu_0) = \int Dg \, DX^i \times \exp \left( -\mu_0 \int d^2x \sqrt{g} - \int d^2x \sqrt{g} g^{ab} \partial_a X^i \partial_b X^i \right). $$ (1)

Following the procedure of Polyakov, we ‘fix a gauge’ using coordinates $x$ so that $g_{ab} = e^{\phi} h_{ab}$, where $h$ is an external background metric. Using an appropriate regularization, this leads to the well-known path integral

$$ Z_D(\mu_s) = \int D\phi(x) \times \exp \left[ \frac{26 - D}{48\pi} \int d^2x \left( \frac{1}{2} (\partial \phi)^2 + \mu_s e^\phi + R \phi \right) \right] $$ (2)

for the Liouville field $\phi(x)$, with $R$ being the curvature of the metric $h$, which is chosen so that $R$ is a constant depending on the topology [3]. The conformal anomaly is present for $D \neq 26$ where a scalar field appears in the quantum theory although there are no physical degrees of freedom in the classical theory. (There are no field equations for gravity in two dimensions.) The parameter $\mu_s$ is not equal to the bare cosmological constant $\mu_0$ and the evaluation of the necessary counter-terms reveals [3]

$$ \mu_0 = \frac{D - 2}{8\pi \epsilon} + \mu_s \left( \frac{26 - D}{48\pi} \right) $$ (3)

where $\epsilon$ is the cut-off parameter of the regularization procedure. Usually the path integral (2) is the starting point for further examination. We emphasize only that for $R < 0$ the interaction term $R \phi + \mu_s e^\phi$ has a true minimum at $\phi_0 = \ln(-R/\mu_s)$ and using $\phi(x) = \phi_0 + \eta(x)$ one can approximate the vacuum functional by

$$ Z_D(\mu_s) = \int D\eta \times \exp \left[ \frac{26 - D}{48\pi} \int d^2x \left( \frac{1}{2} (\partial \eta)^2 + (c_0 \eta^2 + c_1 \eta^3 + \ldots) \right) \right] $$ (4)

with $c_0 = -R$ etc. [3].

We highlight the fact that for $D < 2$ relation (3) indicates that the bare cosmological constant $\mu_0$ is negative and infinite in the limit $\epsilon \to 0$. While the path integral (1) seems to be highly divergent for $\mu_0 \to -\infty$, the Polyakov procedure transforms it so that the scalar Liouville field results in two dimensions. In this paper we want to understand how this is possible, using the framework of the Regge calculus [3]. Subsequently, we touch briefly on the prospects of generalizing our regularization procedure to higher dimensions. For our investigations of quantum gravity in the Regge regularization we use a simple measure, thus defining a model. Although it is not clear whether this model can be fully equivalent to the Liouville theory in two dimensions, it shares some of its remarkable properties and exhibits a phase transition which appears to define a continuum theory. However, it might be that two dimensional gravity is a poor testing ground for the general case [4]. In higher dimensions the Einstein Hilbert action should dominate the physics of quantum gravity and since, without proper regularization, it is unbounded in the Euclidean sector, its behavior

*Following established notation [4], in the present context $D$ denotes the number of matter fields or dimension of the embedding space.
needs to be determined before one can discuss the subtleties of the correct measure.

In the next section we define the model and obtain analytical results including properties of its phase transition. Numerical simulations are subsequently performed in section II. They yield long-range correlations as needed for the continuum limit. Conclusions follow in section V.

II. THE MODEL

For $D = 0$ we can write the path integral (1) as

$$Z(\mu_0) = \int Dg \, e^{-\mu_0 A[g]}$$

(5)

where $A[g]$ is the total area of the geometry. The integral (5) is defined only after proper regularization as the following example illustrates: Consider a smooth geometry of finite total area. Then, assume that a "spike" of length $L$ and circumference $u$ grows out of this area. The spike can have arbitrarily large length $L$, but, if $u$ is sufficiently small, it would not be suppressed by the action of the path integral due to its infinitesimal area $\sim uL$. Indeed, the path integral would be dominated by ill-defined geometries, unless we introduce a regularization procedure, which provides for a cutoff to prohibit $L$ from becoming arbitrarily large. A proper regularization procedure introduces counter-terms, which prevent degenerate geometries and lead to a well-defined path integral (5), as we will discuss in the following for a model of pure gravity.

We consider a triangulation of fixed topology (e.g a 2-torus), with $N_2$ triangles and $N_1$ links. We assume that the triangulation is sufficiently regular and, following the Regge approach, consider the link lengths $x$ as the variables of this model. We use a cutoff $a$ on the link lengths and the simple local measure $\prod_l dx_l$ to obtain the path integral

$$Z(\mu_0, a) = \int_0^a \prod_l dx_l \, e^{-\mu_0 A[x]}$$

(6)

where the total area $A[x]$ of the geometry is the sum of all triangle areas,

$$A = \sum_l A_l,$$

and the integration range restricts the link lengths to $0 < x < a$. The above integral is obviously well-defined for all $\mu_0$ as long as $a$ is finite.

It is not clear whether the partition function (3) can be in the same universality class as (3), i.e. that different universality will be properly implemented by our simple measure in a continuum limit. It is known, however, that the Regge approach approximates Einstein gravity in the classical limit (3). We reach the continuum limit formally by increasing the number of links $N_1$ and decreasing their lengths, $a \to 0$.

To proceed further, we re-write the path integral (6) as

$$Z(\mu_0, a, n) = \int_0^{\infty} \prod_l dx_l \, \exp \left( -\mu_0 A[x] - \sum_l (x_l/a)^n \right)$$

(7)

i.e. replacing the cutoff with a counter-term, which is equivalent in the limit $n \to \infty$.

Let us qualitatively examine how the Regge lattice might behave for different values of $\mu_0$. A positive cosmological constant, $\mu_0 > 0$, tends to suppress large areas, while the counter-term restricts the link lengths. In the limit $\mu_0 \to +\infty$ the expectation value of the total area will tend to 0, but it is not clear whether the link lengths will approach 0 as well. It is possible that their expectation value remains finite and this would result in a 'crumpled' lattice with collapsed triangles. The numerical simulations presented in the following section suggest that this indeed happens and it is not clear if a reasonable continuum limit can be found for $\mu_0 > 0$. Previous numerical investigations (3) have been mostly confined to this range of a positive cosmological constant.

On the other hand, if $\mu_0 < 0$, the cosmological term tends to inflate the area of the lattice, while the counter-term limits the link lengths. Therefore, equilateral triangles are preferred and for $\mu_0 \to -\infty$ we have to expect a lattice which consists of (almost) equilateral triangles, with small fluctuations around the maximum link length $a$; other configurations cannot significantly contribute to the path integral. The situation is thus similar to an inflated balloon, where the pressure of the air is the analogue to the cosmological term, while the role of the rubber molecules is played by the Regge links. If we inflate the balloon enough, the rubber becomes locally flat and elastic. While it is immediately clear that the Regge lattice is locally flat for regular triangulations in two dimensions (3), if $\mu_0$ is sufficiently negative, we must examine in the following whether the lattice fluctuations are able to reproduce a scalar field as in the Liouville theory. To this purpose we consider first the path integral for $\mu_0 < 0$ and $n > 2$. The special case of $n = 2$ is then discussed separately.

A. $n > 2$

If $n$ is large but finite and $\mu_0$ is sufficiently negative, only lattice configurations with link lengths $x \sim a$ contribute significantly to the path integral. Configurations with $x >> a$ are suppressed by the term $-(x/a)^n$, while configurations with $x << a$ cannot contribute significantly due to the area term $-\mu_0 A > 0$. We thus replace $x_l$ with $a(1+\xi_l)$ where the variable $\xi_l$ determines the fluctuation of link $l$ around the length $a$. The path integral becomes

$$Z(\mu_0, a, n) = \int_0^{\infty} \prod_l a(1+\xi_l) \, \exp \left( -\mu_0 a^2 \sum_l (1+\xi_l)^n \right)$$
\[ Z(\mu_0, a, n) = a^{N_1} \int \prod_l d\xi_l \times \]

\[ \exp \left( -\mu_0 a^2 \sum_l A_l [1 + \xi_l] - \sum_l (1 + \xi_l)^n \right) \] (8)

and only configurations \( \xi \sim 0 \) contribute significantly. Notice that the cut-off \( a \) appears as a simple, unphysical multiplier of the bare cosmological constant, which allows us to remove it from the path integral by rescaling \( \mu_0 \).

The area of a single triangle with the links \( a, b, c \) is of the form

\[ A_l [1 + \xi_l] = A_0 + A_1 (\xi_a + \xi_b + \xi_c) - A_2 (\xi_a^2 + \xi_b^2 + \xi_c^2) \]

\[ - B_2 (\xi_a - \xi_b)^2 + (\xi_a - \xi_c)^2 + (\xi_b - \xi_c)^2 + O(\xi^3) \] (9)

with positive constants

\[ A_0 = \frac{\sqrt{3}}{4}, \quad A_1 = \frac{1}{2\sqrt{3}}, \quad A_2 = \frac{1}{12\sqrt{3}}, \quad \text{and} \quad B_2 = \frac{1}{6\sqrt{3}}. \] (10)

The cosmological constant term, i.e. the total area of the geometry, can therefore be expanded as

\[ \sum_l A_l [1 + \xi_l] = A_0 N_2 + 2 A_1 \sum_l \xi_l - 2 A_2 \sum_l \xi_l^2 \]

\[ - B_2 \sum_{[mn]} (\xi_m - \xi_n)^2 + O(\xi^3) \] (11)

where the sum over \([mn]\) indicates a sum over neighboring links. Correspondingly we expand the regulating link term as

\[ \sum_l (1 + \xi_l)^n = N_1 + n \sum_l \xi_l + \frac{n(n - 1)}{2} \sum_l \xi_l^2 + O(\xi^3) \] (12)

and insert both expansions in the path integral. We assume equilibrium of the link lengths around \( a \) and for consistency we have to set \( \mu_0 a^2 2 A_1 = -n \), so that the first order term in the action vanishes. This determines the bare cosmological constant as

\[ \mu_0 = -\frac{n}{2a^2 A_1} \] (13)

so that it is negative and tends to \(-\infty\) for \( a \to 0 \). (Notice the similarity with relation (3.).)

The path integral becomes

\[ Z(n) = \text{const} \int \prod_l d\xi_l \exp \left[ - \left( \frac{n(n - 1)}{2} + n \frac{A_2}{A_1} \right) \sum_l \xi_l^2 \right. \]

\[ \left. - n \frac{B_2}{A_1} \sum_{mn} (\xi_m - \xi_n)^2 + O(\xi^3) \right] \] (14)

where we now understand \( n \) as a coupling parameter which is not necessarily an integer. While the first term guarantees that \(|\xi| \ll 1\) for large enough \( n \), the second term can obviously be interpreted as the discretization of the kinetic term of a scalar field \( \xi \), similarly to the field \( \eta \) of the path integral (3).

To proceed further, we discretize the variables \( \xi_l \) using \( \epsilon \sigma_l \) instead, with \( \sigma_l \) being an integer \( \pm 1, \pm 3, \pm 5, \ldots \), so that the difference between two neighbouring values of \( \xi \) equals \( 2\epsilon \). The path integral is then replaced by a sum over different configurations \( \sigma_l \)

\[ Z(\beta, \epsilon) = \sum_{[\sigma]} \exp \left[ -\beta^2 \epsilon^{-2} \sum_l \sigma_l^2 \right. \]

\[ \left. - \frac{B_2}{A_1} \sum_{mn} (\sigma_m - \sigma_n)^2 + O(\epsilon^3) \right] \] (15)

where we use \( \beta = ne^2 \) and assume that \( n \) is large enough that the coupling of the first term in (14) is essentially \( n^2/2 \).

The limit \( \epsilon \to 0 \) leads back to the path integral (14) and, if we adjust \( n \) such that \( \beta \) remains finite, the \( \beta^2 \epsilon^{-2} \) term of (13) will suppress all higher values of \( \sigma_l \). It is then sufficient to perform a summation of all configurations \( \sigma_l = +1, -1 \) and the system becomes equivalent to an Ising model. Therefore, in this limit the model exhibits a second order phase transition at a certain critical coupling \( \beta_c \), which leads us to the equivalence with an interacting scalar field with S-matrix \(-1\) \( \Pi \). Although this field is in a different universality class than the Liouville theory, the associated long-range correlations demonstrate that our ‘quantum balloon’ is indeed elastic as we had hoped for. In previous investigations it has been proposed to approximate quantum gravity by Ising models and numerical simulations of such models have been performed \( \Pi \). Our calculation demonstrates that these Ising models follow naturally in a certain limit \( (\mu_0 \to -\infty, n \to \infty) \) of the Regge approach.

The occurrence of different universality classes is certainly related to the sensitive dependence of two dimensional quantum gravity on the definition of the measure. The above calculation is supposed to depend on the details of the limit \( (\mu_0 \to -\infty, n \to \infty) \) and, furthermore, we also do not know the universality classes for general \( n \). In principle, this could be determined by calculating the critical exponents numerically. So far, we did not investigate this issue in detail. It is expected to be of minor importance in higher dimensions due to the presence and dominance of the Einstein-Hilbert term there.

**B. \( n = 2 \)**

Having discussed the case \( n > 2 \) and a particular limit \( n \to \infty \), we examine the special case \( n = 2 \) in the following. The reason for our interest in this case is that
the critical point is found for a finite value of \( \mu_0 \), which allows for easier numerical investigations.

We use the dimensionless variables

\[
q_i = \left(\frac{x_i}{a}\right)^2
\]

with the measure \( \prod_i dq_i \) instead of \( \prod_i dx_i \) for reasons of analytical simplicity. Using the rescaled coupling parameter

\[
\mu = \mu_0 a^{-2}
\]

the path integral \( Z(\mu) \) becomes

\[
Z(\mu) = \int_0^\infty \prod_l dq_l \exp \left( -\mu A[q] - \sum_i q_i \right) .
\]

This path integral is well defined as long as \( \mu > \mu_c \) and the value of the critical coupling \( \mu_c \) is found from the configuration with all link lengths equal, \( q_i = q \). The action reduces to \( -\mu(\sqrt{3}/2)N_2 - N_1)q \) in this case and the exponent remains finite only if \( \mu > \mu_c = -(2/\sqrt{3})(N_1/N_2) \).

In two dimensions we have \( N_1 = 3N_2/2 \) and

\[
\mu_c = -\sqrt{3}
\]

is independent of the lattice size. Although the lattice is expected to fluctuate heavily at the critical point, we have nevertheless determined \( \mu_c \) by assuming that configurations around equilateral triangles dominate the path integral. Unfortunately, we cannot prove this assumption, but it is confirmed by our numerical simulations (see the next section) and it is self-consistent as we show in the following.

We substitute the \( N_1 \) variables \([q_1, ..., q_{N_1}]\) by the \( N_1 \) variables \([q, \xi_1,...,\xi_{N_1-1}]\) defined by

\[
q_1 = q(1 + \xi_1), \ q_2 = q(1 + \xi_2), \ldots
\]

and

\[
q_{N_1} = q \left( 1 - \sum_{i=1}^{N_1-1} \xi_i \right) ,
\]

so that \( \sum_i q_i = N_1 q \). The total area \( A[q] \) can then be written as

\[
A[q] = (\sqrt{3}/2) N_2 q \tau[\xi]
\]

where the function \( \tau \) depends on the variables \( \xi_i \) only and varies between 0 and 1. It equals 1 if all triangles are equilateral (maximum area) and it is 0 if all triangles collapse (minimum area). The path integral becomes

\[
Z(\mu) = \int_0^\infty dq q^{N_1-1} \times \prod_l d\xi_l \exp \left( -\frac{3}{2} \mu N_2 q \tau[\xi] - N_1 q \right)
\]

because the functional determinant is a constant, when it is evaluated under the assumption of small fluctuations around equilateral triangles, which allows us to ignore the triangle constraints. Integrating out \( q \), one arrives at

\[
Z(\mu) = \frac{\Gamma(N_1)}{N_1^{N_1}} \int \prod_l d\xi_l \frac{1}{(1 + \sqrt{3}/2 \mu N_2 \tau[\xi])^{N_1}} .
\]

We are interested to determine the expectation value of

\[
< \tau > \text{ for } \mu \to \mu_c = -\frac{2}{\sqrt{3}} \frac{N_1}{N_2}
\]

and obviously the behavior of the function \( \tau[\xi] \) near \( \tau = 1 \) is essential in this case. We use the fact that \( \tau \) has a maximum at \( \xi = 0 \) and expand it as

\[
\tau[\xi] = 1 - B N_1^{-1} \sum_i \xi_i^2 - \sum_{ij} B_{ij} \xi_i \xi_j + O(\xi^3)
\]

with a constant \( B > 0 \) and a constant matrix \( B_{ij} \). In a second step we introduce spherical coordinates \( \xi_i = r \sin \phi_1 \sin \phi_2 \ldots \) and thus arrive at

\[
Z(\mu) = \text{const} \int \prod d\phi dr r^{N_1-2} \times \int dr r^{N_1-2} \frac{J(\phi)}{(B + \sum_{ij} B_{ij} f_{ij}(\phi))^{N_1}}
\]

if we neglect terms of order \( O(\xi^3) \). The function \( J(\phi) \) denotes the Jacobian for the spherical coordinates and \( f_{ij} \) denotes a matrix of functions which depend on the angles \( \phi \) only such that \( \xi_i \xi_j = r^2 f_{ij}(\phi) \). At \( \mu = \mu_c \) the path integral becomes

\[
Z(\mu) = \text{const} \int \prod d\phi \frac{J(\phi)}{(B + \sum_{ij} B_{ij} f_{ij}(\phi))^{N_1}} \times \int dr r^{N_1-2} \frac{J(\phi)}{(B + \sum_{ij} B_{ij} f_{ij}(\phi))^{N_1}}
\]

which we can write as \( \text{const} \int dr r^{-1}(N_1+2) \) in order to calculate the expectation value \( < \tau > \). Obviously the path integral has a strong singularity at \( r = 0 \) and the expectation value of \( r \) is zero at \( \mu_c \). This, of course, implies that \( < \tau > = 1 \).

In order to determine \( < \tau > \) for \( \mu = \mu_c + \delta \), the first momentum \( \int dr f(r) r/ \int dr f(r) \) needs to be calculated for the function

\[
f(r) = r^{N_1-1}/(\delta + B'r^2)^{N_1},
\]

where \( B' \) equals \( B + \sum_{ij} B_{ij} f_{ij}[\phi] \). The function \( f \) has a maximum at \( r_{\text{max}} = \sqrt{(N_1-1)\delta/(N_1B' + B')} \) and falls off sharply away from it. We conclude that \( < \tau > \sim \text{const} \delta^2 \) where the constant depends on the integration of \( B' \) over the spherical angles \( \phi \). Therefore,
< \tau > \sim 1 - \text{const} \left( \mu - \mu_c \right)^{\frac{1}{2}} \tag{28}

for lattice configurations fluctuating around equilateral triangles, providing for the self-consistency of our initial assumption.

An eventual, non-trivial continuum limit \( \mu \to \mu_c \) is quite subtle. The numerical investigations of section \( \square \) suggest the following picture: At \( \mu_c \), the path integral is dominated by small fluctuations around equilateral triangles. The link lengths show correlations which for \( \mu \to \mu_c \) are consistent with a diverging correlation length (mass to zero). Considering the ratio \( r = \min(x_i)/\max(x_i) \), one finds that the expectation value \( < r > \) shows a strong finite size effect, in contrast to \( < \tau > \) which is essentially independent of \( N_1 \). The continuum limit is then found for \( (N_1 \to \infty, \mu \to \mu_c) \), so that \( < \tau > \to 1 \) and \( < r > \to 0 \), which indicates that the triangles are (almost) equilateral yet over large distances the geometry of a typical lattice is non-trivial.

C. Higher dimensions

Let us briefly comment on the situation in higher dimensions. We focus on the physically interesting case of four dimensions and, because of the importance of numerical calculations, on \( n = 2 \). We consider the path integral

\[
Z(\mu) = \int Dq \exp \left( -\mu V[q] - \sum \delta_i q_i^2 \right) \tag{29}
\]

with total 4-volume \( V[q] \) and a regulating counter-term. We use again a simple, local measure for simplicity and we ignore the Einstein-Regge action as well as other possible terms in the action. We introduce variables \( p \) and \( \xi \) so that \( pN_1 = \sum \delta_i q_i^2 \) and \( q_i = p^{\frac{1}{2}} (1 + \xi_i) \) and we use the function \( \nu[\xi] \) as we used \( \tau \) for two dimensions, so that \( V[q] = \sum \nu_i \xi_i = \nu[\xi] \sqrt{V_0 N_1} \) where the constant \( V_0 \) is chosen so that the maximum of \( \nu \) equals 1. The path integral becomes

\[
Z(\mu) = \int_0^\infty dp \left( \frac{(N_1-2)}{2} \right)^{\frac{1}{2}} \times \int \Pi \delta_i \exp \left( -\mu N_1 V_0 \nu[\xi] - N_1 p \right). \tag{30}
\]

Proceeding along the same lines as for the two-dimensional case, one concludes that \( < \nu > \to 1 \) as \( \mu \) approaches the critical value \( \mu_c \). At the critical point lattice configurations with equilateral 4-simplices contribute significantly only, indicating long-range correlations and a non-trivial phase transition as in two dimensions.

Unfortunately, a regular triangulation with equilateral simplices in four dimensions does not provide for a locally flat geometry. Also, the ratio \( N_4/N_1 \) is not constant and some properties of our regularization depend on the details of the triangulation. Clearly, a mechanism is needed which would favor locally flat geometries, independent of the triangulation and we hope that the Einstein-Hilbert action will achieve this. Indeed we can show that our regularization is sensitive to the Einstein-Hilbert action near the critical point, by adding

\[
S_{EH}(q) = \sum \delta_i \delta_i \tag{31}
\]

to the action in equation (29). Using the gravitational coupling \( \beta \) we write

\[
\beta S_{EH}(q) = -\beta p^{1/2} s(\xi) \tag{32}
\]

where \( s(\xi) \) is bounded because the deficit angles \( -n\pi < \delta_i < 2\pi \) are bounded \( (n \) depends on the details of the triangulation) for a given triangulation.

One can expand the action for small \( \beta \) as

\[
\exp(-\beta p^{1/2} s(\xi)) = 1 - \beta p^{1/2} s(\xi) + O(\beta^2)
\]

and evaluate the path integral equation (30) with this correction. Performing the integration of the variable \( p \) one finds that the integrand in the remaining path integral is modified by a factor

\[
[1 - \beta C(N_1) s(\xi)] / [1 - (\mu/\mu_c) \nu(\xi)]
\]

where \( C(N_1) \) is a constant, which depends slightly on \( N_1 \). Near the critical point we can set \( \xi = 0 \) to approximate \( \beta_c \) as the gravitational coupling which leads to

\[
[1 - \beta_c C(N_1) s(0)] / [1 - (\mu/\mu_c) \nu(0)] = 0.
\]

Of course, this does not imply that \( Z = 0 \) at \( \beta_c \), rather we have to conclude that our approximation for small \( \beta \) breaks down. It is now important to see that \( \beta_c \to 0 \) for \( \mu \to \mu_c \), which means that our model becomes very sensitive to the Einstein-Hilbert term near the critical point. Numerical simulations could clarify the relevance of this effect and in general the phase structure for the two coupling parameters \( \mu \) and \( \beta \). This is left as a project for future investigations.

III. NUMERICAL RESULTS

To support the assumptions as well as the results of our analytical calculations in the previous section, we perform numerical simulations of our two dimensional model. We use the path integral (3) for \( n = 2 \) on a 2-torus. Our consideration which leads to \( \mu_c = -\sqrt{\frac{3}{5}} \) of equation (13) is not affected by the change of the measure from the path integral (8) to (8). We have generated data on \( 8 \times 8, 16 \times 16 \) and \( 32 \times 32 \) lattices at \( \mu = 5.0, 2.0, 1.0, 0.0, -1.0, -1.5, -1.7 \) and \(-1.72 \). Each data point relies on at least 1M sweeps and measurements are taken every 10 sweeps to reduce auto-correlation of the data.
the accuracy of the plots data from 8 × 8, 16 × 16 and 32 × 32 lattices fall on top of one another.

In figure 1 the expectation value
\[ \langle \tau \rangle = 2\langle A/x^2 \rangle / (N_2 \sqrt{3}) \] (lower curve)
is shown together with
\[ 2\langle A \rangle / (N_2 \sqrt{3} \langle x_t \rangle^2) \] (upper curve)
for various values of the cosmological constant. There are almost no finite size effects for these quantities. Within the accuracy of the plot our data from 8 × 8, 16 × 16 and 32 × 32 lattices fall on top of one another. The curve for \( \langle \tau \rangle \) decreases for positive values and tends towards zero for increasing bare cosmological constant (\( \mu \rightarrow \infty \)), indicating a crumpled lattice consisting of collapsed triangles. The ratio is finite for negative values of the cosmological constant and approaches for \( \mu \rightarrow 0 \).

We compute the link-link correlation functions defined by
\[ C_l(t) = \frac{\langle (x_0 x_t) - \langle x_0 \rangle \langle x_t \rangle \rangle}{\langle (x_0 x_t) \rangle - \langle x_0 \rangle \langle x_t \rangle} \] (34)
where, using the regular triangulation [11], two face to face links pointing in the \( \hat{x} \) direction and separated by \( t \) lattice spacings in the \( \hat{t} \) direction are correlated. Note that \( t \) is not necessarily proportional to the geometrical distance between the links, especially for large \( t \), since we know that the geometry fluctuates according to our measurement of \( \langle \tau \rangle \).

Figure 2 compares our 32 × 32 and 64 × 64 results and shows indeed a strong correlation of the link lengths for

\[ r = \langle x_{l,\text{min}} / x_{l,\text{max}} \rangle \] for lattice sizes 8 × 8, 16 × 16 and 32 × 32.

We are now confident that the assumptions of the previous sections are essentially correct, leading to the picture of a lattice with highly correlated links near the critical point. In the following we check this conclusion, investigating the expected long-range correlations directly. We employ that we know the exact value [11] of \( \mu_c \) to control the approach to the critical point. To improve the numerical stability we add a \( n = 3 \) term \( f x_l^3 \) and investigate the partition function
\[ Z(\mu, f) = \int_0^\infty \prod_l dx_l \times \exp \left( -\mu \sum_l A_l[x] - \sum_l (x_l^2 + f x_l^3) \right). \] (33)
We take \( \mu = \mu_c \) and are interested to establish critical behavior for \( f \to 0 \). As long as \( f > 0 \), the integral (33) is well-defined for all \( \mu \). However, for \( f = 0 \) the lattice has a finite area for \( \mu > \mu_c \), only as discussed in subsection 11.

\[ \langle \tau \rangle = \langle x_{l,\text{min}} / x_{l,\text{max}} \rangle \] for our three lattice sizes. As we assumed, the expectation value of this ratio decreases with increasing lattice size and the results show that even a modest lattice size (32×32) is sufficient for substantial variations of the overall geometry near the critical point (\( \langle \tau \rangle < 0.1 \) for \( \langle \tau \rangle > 0.9 \)).
$f \to 0$, even for large $t$. However, irregular finite size effects are present as well (e.g. for $f = 10^{-6}$) and a full-scale FSS analysis, which is beyond the scope of this paper, seems necessary to clarify the critical behavior of our system.

For the $32 \times 32$ lattice we performed also calculation at higher values of $f$. The long distance correlations disappear then quickly, as we show in figure 4 (note the change in ordinate scale between figure 3 and 4).

In order to determine the mass of the ‘particle’ associated with the lattice fluctuations, we extracted mass values using so called cosh fits

$$C_{l}(t) = A \left( e^{-mt} + e^{-m(N-t)} \right)$$

(35)

from the correlation functions. Figure 3 depicts our results from the $32 \times 32$ and $64 \times 64$ lattices in the limit $f \to 0$, including also $f = 10^{-4}$ from our larger $f$ values on the $32 \times 32$ lattice. The mass values agree down to $f = 10^{-6}$; for $f = 10^{-7}$ we find then disagreement, which is expected when the small lattice can now longer accommodate the correlation length. Again, we emphasize that our data are currently not sufficient for a complete FSS analysis of the model.

After we have demonstrated numerical results in two dimensions, we are ready to have a peep at higher dimensions. Figure 4 shows very preliminary results of numerical calculations in four dimensions. We performed these calculations without gravitational coupling (i.e. $\beta = 0$) near $\mu = \mu_c$ and include a term $\epsilon \sum q_l^2$ in the partition function (29) for improving convergence.

The figure shows the results with error bars from two independent simulation runs on a $4 \times 4 \times 4 \times 10$ lattice, measuring the correlations for the long direction. Each simulation performed 10k sweeps through the lattice and sampled every 10th configuration. Depicted is the correlation function

$$C_{q}(t) = \frac{\left( \langle q_0 q_t \rangle - \langle q_0 \rangle \langle q_t \rangle \right)}{\left( \langle q_0 q_0 \rangle - \langle q_0 \rangle \langle q_0 \rangle \right)}$$

(36)

for the links forming the edges of the hypercubes in the lattice. The correlations are of course measured in the long direction. Although the figure suffers from large statistical errors, it exhibits the desired long-range correlations.

IV. CONCLUSIONS

We defined a class of models of fluctuating geometries, based on the Regge approach, to illustrate basic properties of the Polyakov path integral in two dimensions. Although we used a simple, local measure we find evidence for the existence of a non-trivial phase transition with long-range correlations and the emergence of a scalar field. We demonstrated the importance of a negative bare cosmological constant, which may open the door to investigations of related systems in higher dimensions. Our calculations indicate that one could expect a non-trivial phase transition with long-range correlations in four dimensions as well and numerical simulations with negative cosmological constant in higher dimensions are the obvious next step to investigate these models further.

ACKNOWLEDGMENTS

This work was in part supported by the US Department of Energy under contract DE-FG02-97ER40608.
The simulations were done on workstations of the FSU HEP group.

[1] A.M. Polyakov, Phys. Lett. B 103, 207 (1981).
[2] Brian Hatfield, *Quantum Field Theory of Point Particles and Strings*, Addison-Wesley, Reading, Massachusetts, 1998.
[3] Equation (22.124) of [2].
[4] E. Martinec, *An Introduction to 2d Gravity and Solvable String Models* [hep-th/9112019], in *Strong Theory and Quantum Gravity 91* (J. Harvey, R. Iengo, K.S. Narain, S. Randjbar-Daemi and H. Verlinde, editors), Proceedings of the 1991 Triest Spring School, World Scientific, p.1-19.
[5] T. Regge, Nuovo Cimento 9, 558 (1961).
[6] W. Beirl and B.A. Berg, Nucl. Phys. B 452, 415 (1995).
[7] B. Ambjorn, Comm. Math. Phys. 121, 351 (1989).
[8] R. Friedberg and T.D. Lee, Nucl. Phys. B 242, 145 (1984); G. Feinberg, R. Friedberg, T.D. Lee and H.C. Ren, Nucl. Phys. B 245, 343 (1984); H. Cheeger, W. Müller and R. Schrader, Commun. Math. Phys. 92, 405 (1984).
[9] M. Gross and H. Hamber, Nucl. Phys. B364, 703 (1991); C. Holm and W. Janke, Phys. Lett. B335, 143 (1994).
[10] Depending on the incidence matrix, one may or may not have non-zero deficit angles. It is a consequence of the lattice regularity combined with Euler’s theorem that the average curvature over a properly defined unit cell will always be zero. For the numerical investigations of this paper we use the standard incidence matrix. Starting from the square lattice the triangularization is obtained by supplemented each square with one diagonal in $\hat{x} + \hat{y}$ direction. There are then six links emerging at each site, such that for equilateral triangles all deficit angles are zero.
[11] M. Sato, T. Miwa, and Jimbo, Proc. Japan Acad. 53A, 6 (1979); B. Berg, M. Karowski and P. Weisz, Phys. Rev. D19 (1979) 2477.
[12] W. Beirl, P. Homolka, B. Krishnan, M. Markum and J. Riedler, Nucl. Phys. B (Proc. Suppl.) 42, 710 (1995); J. Riedler, W. Beirl, E. Bittner, A. Hauke, P. Homolka and H. Markum Class. Quant. Grav. 16, 1163 (1999).