Finding Exact Values For Infinite Sums

COSTAS EFTHIMIOU
Department of Physics
Tel Aviv University
Tel Aviv, 69978 Israel

Introduction In the 1995 February issue of Math Horizons I. Fisher posed the following problem [1]:

From the well-known results,

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{and} \quad \sum_{n=1}^{+\infty} \frac{1}{n^2 + n} = 1 ,
\]

it follows that

\[
1 < \sum_{n=1}^{\infty} \frac{1}{n^2 + n/2} < \frac{\pi^2}{6} .
\]

Find the exact value of the convergent sum.

This paper offers a solution method that allows one to find exact values for a large class of convergent series of rational terms.

In the next section, we first illustrate the method for a special case. We then describe the general result pointing out further generalizations of the method and we finally end with a brief discussion.

A Special Case Consider the series

\[
S(a, b) = \sum_{n=1}^{\infty} \frac{1}{(n + a)(n + b)} ,
\]

where \( a \neq b \) and neither \( a \) nor \( b \) is a negative integer. Sums of this form arise often in problems dealing with Quantum Field Theory (p. 89ff., Ref. [2]).

Decomposing each term of (1) in partial fractions gives

\[
S(a, b) = \frac{1}{a - b} \sum_{n=1}^{\infty} \left( \frac{1}{n + b} - \frac{1}{n + a} \right) .
\]

Now we use the identity

\[
\frac{1}{A} = \int_{0}^{\infty} e^{-Ax} dx , \quad A > 0 .
\]

Therefore for \( a, b > -1 \) we have

\[
S(a, b) = \frac{1}{a - b} \lim_{N \to +\infty} \sum_{n=1}^{N} \int_{0}^{\infty} e^{-nx} \left( e^{-bx} - e^{-ax} \right) dx
\]

\[
= \lim_{N \to +\infty} \int_{0}^{\infty} \frac{e^{-bx} - e^{-ax}}{a - b} \frac{e^{-x} (1 - e^{-Nx})}{1 - e^{-x}} dx
\]

\[
= \frac{1}{a - b} \int_{0}^{\infty} \frac{e^{-bx} - e^{-ax}}{a - b} \frac{e^{-x}}{1 - e^{-x}} dx .
\]
In deriving the last result, we made use of the monotone convergence theorem (p. 318, Ref. [3]). In particular, the integrand in the second line of equation (4) consists of a non-decreasing sequence of non-negative functions and therefore we can swap the indicated operations of taking the limit and performing the integration.

Making the change of variable \( t = e^{-x} \) in the integral of equation (4) gives a more symmetric result:

\[
S(a, b) = \frac{1}{a-b} \int_0^1 \frac{t^b - t^a}{1-t} dt .
\]  \hspace{1cm} (5)

Some comments are in order here:

- Although the integral

\[
\int_0^1 \frac{t^b}{1-t} dt
\]

diverges, the integral of equation (5) converges for \( a \neq b \) and \( a, b > -1 \).

- The problem of Fisher that appeared in *Math Horizons* corresponds to \( a = 1/2, b = 0 \). In this case, we have

\[
\sum_{n=1}^{\infty} \frac{1}{n(n+1/2)} = 2 \int_0^1 \frac{dt}{1+\sqrt{t}} .
\]

After the change of variables \( t = u^2 \), it is easy to calculate the integral:

\[
\sum_{n=1}^{\infty} \frac{1}{n(n+1/2)} = 4 \int_0^1 \left( 1 - \frac{1}{1+u} \right) du = 4(1 - \ln 2) \simeq 1.227 .
\]

- When the two numbers \( a \) and \( b \) differ by an integer \( k \), i.e. \( a = b + k \), then the sum (1) “telescopes” and it can be easily calculated from (2):

\[
S(a, a-k) = \frac{1}{k} \sum_{j=1}^{k} \frac{1}{j+a} .
\]

This can be used as a consistency check of formula (5). Indeed

\[
S(a, a - k) = \frac{1}{k} \int_0^1 \frac{t^k - t^a}{t-1} dt = \frac{1}{k} \int_0^1 \sum_{i=0}^{k-1} t^{i+a} dt = \frac{1}{k} \sum_{i=0}^{k-1} \frac{t^{i+a+1}}{i+a+1} \biggr|_0^1
\]

\[
= \frac{1}{k} \sum_{j=1}^{k} \frac{1}{j+a} ,
\]

in agreement with the last result.

Now we note that one can express the result (5) in another equivalent form — namely, using the well known representation (see p. 258, Ref. [4]) of the digamma function \( \psi(z) \):

\[
\psi(z) \equiv \frac{d}{dz} \ln \Gamma(z) = -\gamma - \int_0^1 \frac{t^{z-1} - 1}{1-t} dt , \hspace{1cm} (6)
\]
where \( \gamma \) is the Euler-Mascheroni constant and \( \Gamma(z) \) is the gamma function. This is motivated by the fact that part of the integrand in the r.h.s. of equation (6) is similar to the integrand in (5). In fact, we have

\[
S(a, b) = \frac{\psi(b + 1) - \psi(a + 1)}{b - a}.
\]  

(7)

There are many useful identities involving the digamma function (p. 258, Ref. [4]). For example

\[
\psi(1 + z) = \psi(z) + \frac{1}{z}.
\]  

(8)

Moreover, the exact value of \( \psi(z) \) is known for several values of \( z \):

\[
\psi(1) = -\gamma, \quad \psi(1/2) = -\gamma - 2 \ln 2.
\]  

(9)

Equations (8) and (9) can be used to evaluate \( S(a, b) \) exactly for many values of \( a \) and \( b \). For example, if \( a = 1/2, b = 0 \) we find

\[
S(0, 1/2) = 2 \left[ \psi(3/2) - \psi(1) \right] = 2 \left[ \psi(1/2) + 2 - \psi(1) \right] = 2 \left[ -\gamma - 2 \ln 2 + 2 + \gamma \right] = 4 (1 - \ln 2),
\]

in agreement with our previous result.

When \( a = b \) in (1), the sum can still be calculated. We consider two approaches. The first approach is to repeat the calculations presented above but observing that the basic equation (3) has now to be modified in the form

\[
\frac{1}{A^2} = \int_0^\infty x e^{-Ax} \, dx, \quad A > 0.
\]

Following the same reasoning, we find

\[
\sum_{n=1}^{\infty} \frac{1}{(n+a)^2} = -\int_0^1 \frac{t^a \ln t}{1-t} \, dt.
\]  

(10)

Alternatively, we can obtain the same result by taking the limit \( b \to a \) in (5):

\[
\sum_{n=1}^{\infty} \frac{1}{(n+a)^2} = \lim_{b \to a} \int_0^1 \frac{t^b - t^a}{b-a} \frac{-1}{1-t} \, dt.
\]  

(11)

Without loss of generality, we can assume that \(-1 < a < b\). We notice that for \( 0 \leq t \leq 1/2 \)

\[
\left| \frac{t^b - t^a}{1-t} \right| \leq 2t^b,
\]

while for \( 1/2 < t \leq 1 \) the integrand of (5) is bounded; let \( M \) be its supremum in this subdomain. The function

\[
g(t) = \begin{cases} \frac{2t^b}{b-a}, & \text{if } 0 \leq t \leq 1/2, \\ M, & \text{if } 1/2 < t \leq 1. \end{cases}
\]

3
is integrable in \([0, 1]\) and therefore the dominated convergence theorem (p. 167, 321, Ref. [3]) can be used to interchange the operations of the integral and the limit in (11):

\[
\sum_{n=1}^{\infty} \frac{1}{(n+a)^2} = \int_0^1 \lim_{b \to a} \frac{t^b - t^a}{b - a} \frac{-1}{1-t} \, dt = -\int_0^1 \frac{t^a \ln t}{1-t} \, dt.
\]

Also, from equation (7) we find

\[
\sum_{n=1}^{\infty} \frac{1}{(n+a)^2} = \frac{d\psi(z)}{dz} \bigg|_{z=a+1}.
\]

The functions

\[
\psi^{(n)}(z) = \frac{d^{n+1}}{dz^{n+1}} \ln \Gamma(z) = -\int_0^1 dt \frac{t^{z-1}(\ln t)^n}{1-t}, \quad n = 1, 2, \ldots (12)
\]

are known as polygamma functions (p. 260, Ref. [4]). Several identities for the polygamma functions are known (p. 258ff., Ref. [4]). For example

\[
\psi^{(n)}(1) = (-1)^{n+1} n! \zeta(n+1),
\]

\[
\psi^{(n)}(1/2) = (-1)^{n+1} n! \left(2^{n+1} - 1\right) \zeta(n+1),
\]

\[
\psi^{(n)}(z+1) = \psi^{(n)}(z) + \frac{(-1)^n n!}{z^{n+1}},
\]

where \(\zeta(z)\) is the zeta function:

\[
\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad \text{Re} z > 1.
\]

As an application of (11), we obtain the well known result

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = -\int_0^1 \frac{\ln t}{1-t} \, dt = \frac{\pi^2}{6} \simeq 1.645.
\]

Formula (11) also implies the less well known result

\[
\sum_{n=1}^{\infty} \frac{1}{(n+1/2)^2} = -\int_0^1 \frac{\sqrt{t} \ln t}{1-t} \, dt = 3 \zeta(2) - 4 = \frac{\pi^2}{2} - 4 \approx 0.935.
\]

**The General Case** After our preceding discussion, we can now establish a more general result. Let

\[
S = \sum_{n=1}^{\infty} \frac{Q_{N-2}(n)}{P_N(n)},
\]

where \(Q_{N-2}(n), P_N(n)\) are two polynomials in \(n\) of degree \(N-2\) and \(N\) respectively. We shall assume that \(P_N(n)\) is expressible in the form

\[
P_N(n) = (n+a_1)^{m_1}(n+a_2)^{m_2} \ldots (n+a_k)^{m_k},
\]
with all \( a_i, \ i = 1, 2, \ldots, k \) distinct real numbers none of which is a negative integer. This ensures the convergence of \( S \). Then for any polynomial

\[
Q_{N-2}(n) = c_{N-2} n^{N-2} + c_{N-3} n^{N-3} + \ldots + c_0,
\]

the sum \( S \) is written in terms of partial fractions:

\[
S(a_1, \ldots a_k; c_0, \ldots, c_{N-2}) = \sum_{n=1}^{\infty} \sum_{i=1}^{k} \sum_{j=1}^{m_i} \frac{A_{ij}}{(n + a_i)^j},
\]

where the constants \( A_{ij} \) are uniquely determined by the partial fraction decomposition of each summand. In particular, notice that since there is no term of degree \( N - 1 \) in \( Q_{N-2}(n) \),

\[
\sum_{i=1}^{k} A_{i1} = 0. \quad (13)
\]

Using the identity

\[
\frac{1}{A^L} = \frac{1}{(L-1)!} \int_{0}^{\infty} x^{L-1} e^{-Ax} \, dx, \quad (14)
\]

we write the series in an integral form valid only if \( a_i > -1, \ \forall i \):

\[
S(a_1, \ldots a_k; c_0, \ldots, c_{N-2}) = \sum_{n=1}^{\infty} \sum_{i=1}^{k} \sum_{j=1}^{m_i} \frac{A_{ij}}{(j-1)!} \int_{0}^{\infty} x^{j-1} e^{-(n+a_i)x} \, dx. \quad (15)
\]

Working in a similar fashion as in the derivation of equation (14), we find

\[
S(a_1, \ldots a_k; c_0, \ldots, c_{N-2}) = \sum_{i=1}^{k} \sum_{j=2}^{m_i} \frac{A_{ij}}{(j-1)!} \int_{0}^{\infty} x^{j-1} e^{-(a_i+1)x} \, dx
\]

\[
+ \sum_{i=1}^{k} A_{i1} \int_{0}^{\infty} \frac{e^{-(a_i+1)x} - 1}{1 - e^{-x}} \, dx,
\]

where we have taken extra care for the \( j = 1 \) term (by using the condition (13)) in order to guarantee the convergence of the corresponding integral. This is our result in an integral form. We can also express it in terms of the polygamma functions (12):

\[
S(a_1, \ldots a_k; c_0, \ldots, c_{N-2}) = \sum_{i=1}^{k} \sum_{j=1}^{m_i} \frac{(-1)^j}{(j-1)!} A_{ij} \psi^{(j-1)}(a_i + 1), \quad (16)
\]

where we have defined \( \psi^{(0)}(z) \equiv \psi(z) \).

As a straightforward application of our method, let us consider the following examples:

**Example 1:**

\[
S(a, -a) = \sum_{n=1}^{+\infty} \frac{1}{n^2 - a^2},
\]
where $a$ is not a positive integer and satisfies the inequality $a > -1$. Using the formula (16) we find
\[ S(a, -a) = \frac{\psi(a + 1) - \psi(-a + 1)}{2a}, \quad a \neq 0. \]
This result can be further simplified if we make use of the functional relation
\[ \psi(z + 1) = \psi(z) + \pi \cot(\pi z), \]
in conjunction with (8). Then
\[ S(a, -a) = \frac{1}{2a} \left[ 1 - \pi \cot(\pi a) \right]. \]

Example 2:
\[ S(a, \ldots, a) = \sum_{n=1}^{+\infty} \frac{1}{(n + a)^N} = \frac{(-1)^N}{(N-1)!} \psi^{(N-1)}(a + 1), \]
where $N \geq 2$ and $a > -1$.

Example 3:
\[ S(0, 0, 3/2) = \sum_{n=1}^{+\infty} \frac{1}{n^2(n + 1/2)}. \]

Using formula (16), we find
\[ S(0, 0, 3/2) = 4 \psi(1) + 2 \psi^{(1)}(1) - 4 \psi(3/2) = \frac{\pi^2}{3} - 8 (1 - \ln 2) = 0.835. \]

Example 4:
\[ S(1, 1, 1/2) = \sum_{n=1}^{+\infty} \frac{1}{(n + 1)^2(n + 1/2)}. \]

Using the formula (16) we find
\[ S(1, 1, 1/2) = \left[ 4 \psi(2) - 2 \psi^{(1)}(2) - 4 \psi(3/2) \right] = 2 (4 \ln 2 - 1) - \frac{\pi^2}{3} = 0.255. \]

Finally, the reader is invited to write down the values for other infinite sums of the form presented above.

**Discussion** Before we finish, we would like to point out that the identities (3) and (14) we used in our derivations express the quantities $1/A$ and $1/A^L$ as the Laplace transforms of $1$ and $x^{L-1}$ respectively. In general, if $f(s)$ is the Laplace transform of $g(x)$,
\[ f(s) = \int_{0}^{\infty} e^{-sx} g(x) \, dx, \]
then the sum
\[ S_I = \sum_{n \in I} f(n), \]
where $I \subset \mathbb{Z}$, can be written in the form

$$S_I = \int_{0}^{\infty} g(x) \left( \sum_{n \in I} e^{-nx} \right) \, dx,$$

assuming that the operations of summation and integration are interchangable. Assuming moreover that the sum inside the parenthesis can be performed explicitly, we have thus obtained an integral representation of $S_I$.

The Laplace transform has been proved a very valuable tool in the solution of differential equations. Unfortunately, in the summation of series, the Laplace transform does not enjoy the same popularity. In this paper, we have tried to present some of the limitless possibilities that the method offers. We propose our reader to solve the following problem:

**Problem**

(i) Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n + a} = \int_{0}^{1} \frac{t^a}{1 + t} \, dt.$$

(ii) Using the previous result, derive the well known result

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2 = 0.693,$$

and the less known result

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n + 1/2} = 2 - \frac{\pi}{2} = 0.429.$$

We hope that this will motivate him/her to explore more aspects of the method presented in this paper and establish many additional results.

**Acknowledgments** The author would like to thank the referees of the paper for their valuable comments and help during the revision of the initial version of the paper. Also, he thanks the Cornell High Energy Group where the preliminary version of this paper was written.

**REFERENCES**

[1] I. Fischer, Problem 23 in Problem Section, *Math Horizons*, February 1995.
[2] P. Ramond, *Field Theory: A Modern Primer*, Addison-Wesley, New York, USA, 1994.
[3] W. Rudin, *Principles of Mathematical Analysis*, 3rd edition, McGraw-Hill, Inc., New York, 1976.
[4] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover Publications, New York, USA, 1972.