Total systemic risk statistics

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Abstract Systemic risk is a critical factor not only for financial markets but also for risk management. In this paper, we consider a special class of risk statistics, named total systemic risk statistics. Our result provides a new approach for dealing with systemic risk. By further developing the properties related to total systemic risk statistics, we are able to derive dual representation for such risk.

Keywords risk statistics · systemic risk

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1 Introduction

Research on systemic risk is a popular topic in both quantitative and theoretical research, and systemic risk models have attracted considerable attention. The quantitative calculation of risk involves two problems: choosing an appropriate systemic risk model and allocating systemic risk to individual institutions. This has led to further research on systemic risk.

In a seminal paper, Artzner et al. [4,5] first introduced the class of coherent risk measures. However, traditional risk measures may fail to describe the characteristics of systemic risk. This concept has promoted the study of systemic risk measures. Systemic risk measures were axiomatically introduced by Chen et al. [8]. Other studies of systemic risk measures include those of Acharya et al. [11], Armenti et al. [2], Biagini et al. [3], Brunnermeier and Cheridito [7], Feinstein et al. [9], Gauthier et al. [10], Tarashev et al. [13], and the references therein.

From the statistical point of view, the behaviour of a random variable can be characterized by its observations, the samples of the random variable. [11] and [12] first introduced the class of natural risk statistics, the corresponding representation results are also derived. An alternative proof of the representation result of the natural risk statistics was also derived by [9]. Later, [13] obtained representation results for convex risk statistics, and the corresponding results for quasiconvex risk statistics were obtained by [14]. However, all of these risk statistics are designed to quantify risk of simple financial position (i.e. a random variable) by its samples. A natural question is determining how to quantify systemic risk by its samples.

The main focus of this paper is a new class of risk statistics for portfolios, named total systemic risk statistics. In this context, we measure the systemic risk into two steps. Our results illustrate that each total systemic risk statistic can be decomposed into a clustering function and a simple-systemic risk statistic, which provides a new approach for addressing systemic risk. By further developing the axioms related to
total systemic risk statistics, we are able to derive their dual representations.

The remainder of this paper is organized as follows. In Sect. 2, we briefly introduce some preliminaries. In Sect. 3, we develop the definitions related to systemic risk statistics. Sect. 4 discusses a new measurement of total systemic risk statistics. Finally, in Sect. 5 we consider the dual representations of total systemic risk statistics.

2 Preliminaries

Let \( \mathbb{R}^d \) be the d-dimensional Euclidean space, \( d \geq 1 \). For any \( x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{R}^d \), \( x \leq y \) means \( x_i \leq y_i \) for any \( 1 \leq i \leq d \). For any positive integer \( k \), the element \( X \) in product Euclidean space \( \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d} \) is denote by \( X := (X_1^1, X_2^1, X_2^2, \ldots, X_d^1, \ldots, X_d^d) \). For any \( X, Y \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d} \) and \( 1 \leq i \leq d \), \( X \preceq Y \) means \( \sum_{j=1}^{k_i} X_j^1 \leq \sum_{j=1}^{k_i} Y_j^1 \). From now on, the addition and multiplication are all defined pointwise, \( (X, Y) = \sum_{i=1}^{d} (\sum_{j=1}^{k_i} X_j^1 \sum_{j=1}^{k_i} Y_j^1) \). For any \( X \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d} \), \( X_{[k]} := (0, \ldots, 0, X_1^1, \ldots, X_d^1, 0, \ldots, 0) \in \mathbb{R}^{k_1} \times \ldots \times \mathbb{R}^{k_i-1} \times \mathbb{R}^{k_i} \times \mathbb{R}^{k_{i+1}} \times \ldots \times \mathbb{R}^{k_d} \).

3 The definition of systemic risk statistics

During the most financial markets, systemic risk is defined as involving the risk of break down among institutions and other market participants in a chain-like fashion that has the potential to affect the entire financial system negatively. More concretely, the risk of ‘domino effect’ certainly seems central to the concept of systemic risk, as does the risk of some triggering event that causes the first domino to fall. In this section, we state the definitions related to systemic risk statistics.

Definition 31 A simple-systemic risk statistic is a function \( \varrho : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\} \) that satisfies the following properties,

A1 Monotonicity: for any \( x, y \in \mathbb{R}^d \), \( x \geq y \) implies \( \varrho(x) \geq \varrho(y) \);

A2 Convexity: for any \( x, y \in \mathbb{R}^d \) and \( \lambda \in [0, 1] \), \( \varrho(\lambda x + (1 - \lambda)y) \leq \lambda \varrho(x) + (1 - \lambda)\varrho(y) \).

Remark 31 The properties A1 – A2 are very well known and have been studied in detail in the study of risk statistics (see for instance).

Definition 32 A clustering function is a function \( \phi : \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d} \rightarrow \mathbb{R}^d \) that satisfies the following properties,

B1 Monotonicity: for any \( X, Y \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d} \), \( X \succeq Y \) implies \( \phi(X) \succeq \phi(Y) \);

B2 Convexity: for any \( X, Y \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d} \) and \( \lambda \in [0, 1] \), \( \phi(\lambda X + (1 - \lambda)Y) \leq \lambda \phi(X) + (1 - \lambda)\phi(Y) \);

B3 Correlation: for any \( X \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d} \), there exists a simple-systemic risk statistic \( \varrho \) such that \( (\varrho \circ \phi)(X) \), \( (\varrho \circ \phi)(X_{[k]}) \), \( (\varrho \circ \phi)(X_{[k]}) \) = \( \phi(X) \).

Definition 33 A total systemic risk statistic is a function \( \rho : \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d} \rightarrow \mathbb{R} \) that satisfies the following properties,

C1 Monotonicity: for any \( X, Y \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d} \), \( X \succeq Y \) implies \( \rho(X) \geq \rho(Y) \);

C2 Convexity: for any \( X, Y \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d} \) and \( \lambda \in [0, 1] \), \( \rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y) \);

C3 Statistical convexity: for any \( X, Y, Z \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d} \), \( \lambda \in [0, 1] \) and \( 1 \leq i \leq d \), if \( \rho(Z_{[k]}) = \lambda \rho(X_{[k]}) + (1 - \lambda)\rho(Y_{[k]}) \), then \( \rho(Z) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y) \).

Remark 32 The properties C1 and C2 can be interpreted in the same way as in the definition of simple-systemic risk statistics.

We will see in the following section that each total systemic risk statistic can be decomposed into a simple-systemic risk statistic \( \varrho \) and a clustering function \( \phi \). In other words, the following section will show that the measurement of total systemic risk statistics can be simplified into two steps.
4 How to measure the systemic risk

Theorem 41 A function \( \rho : \mathbb{R}^k \times \mathbb{R}^k \times \ldots \times \mathbb{R}^k \rightarrow \mathbb{R} \) is a total systemic risk statistic if and only if there exists a clustering function \( \phi : \mathbb{R}^k \times \mathbb{R}^k \times \ldots \times \mathbb{R}^k \rightarrow \mathbb{R}^d \) and a simple-systemic risk statistic \( \rho : \mathbb{R}^d \rightarrow \mathbb{R} \) such that \( \rho \) is the composition of \( \rho \) and \( \phi \), i.e.,

\[
\rho(X) = (\rho \circ \phi)(X) \quad \text{for all } X \in \mathbb{R}^k \times \mathbb{R}^k \times \ldots \times \mathbb{R}^k.
\]

Proof. We first show the ‘only if’ part. Suppose \( \rho \) is a systemic risk statistic and define a function \( \phi \) by

\[
\phi(X) := (\rho(X_{[k_1]}), \rho(X_{[k_2]}), \ldots, \rho(X_{[k_d]}))
\]

for any \( X \in \mathbb{R}^k \times \mathbb{R}^k \times \ldots \times \mathbb{R}^k \). Since \( \rho \) satisfies the convexity \( \textbf{C2} \), it follows

\[
\phi(\lambda X + (1 - \lambda)Y) = (\rho(\lambda X_{[k_1]} + (1 - \lambda)Y_{[k_1]}), \ldots, \rho(\lambda X_{[k_d]} + (1 - \lambda)Y_{[k_d]}))
\]

\[
\leq \lambda (\rho(X_{[k_1]}), \ldots, \rho(X_{[k_d]})) + (1 - \lambda)(\rho(Y_{[k_1]}), \ldots, \rho(Y_{[k_d]}))
\]

\[
= \lambda \phi(X) + (1 - \lambda)\phi(Y)
\]

for any \( X,Y \in \mathbb{R}^k \times \mathbb{R}^k \times \ldots \times \mathbb{R}^k \) and \( \lambda \in [0,1] \). Thus, \( \phi \) satisfies the convexity \( \textbf{C2} \). Similarly, the monotonicity \( \textbf{B1} \) of \( \phi \) can also be implied by the monotonicity \( \textbf{C1} \) of \( \rho \). Next, we consider a function \( \rho : \phi(\mathbb{R}^k \times \mathbb{R}^k \times \ldots \times \mathbb{R}^k) \rightarrow \mathbb{R} \), which is defined by

\[
\rho(x) := \rho(X) \quad \text{where } X \in \mathbb{R}^k \times \mathbb{R}^k \times \ldots \times \mathbb{R}^k \text{ with } \phi(X) = x.
\]

Thus, we immediately know that \( \phi \) satisfies the correlation \( \textbf{B3} \), which means \( \phi \) defined above is a clustering function. Next, we want to show that the \( \rho \) defined above is a simple-systemic risk statistic. Suppose \( x,y \in \phi(\mathbb{R}^k \times \mathbb{R}^k \times \ldots \times \mathbb{R}^k) \) with \( x \geq y \), there exists \( X,Y \in \mathbb{R}^k \times \mathbb{R}^k \times \ldots \times \mathbb{R}^k \) such that \( \phi(X) = x, \phi(Y) = y \). Then, we have \( \phi(X) \geq \phi(Y) \), which means \( X \geq Y \) by the monotonicity of \( \phi \). Thus, it follows again from the property \( \textbf{C1} \) of \( \rho \) that

\[
\rho(x) = \rho(X) \geq \rho(Y) = \rho(y)
\]

which implies \( \rho \) satisfies the monotonicity \( \textbf{A1} \). Let \( x,y \in \phi(\mathbb{R}^k \times \mathbb{R}^k \times \ldots \times \mathbb{R}^k) \) with \( \phi(X) = x, \phi(Y) = y \) for any \( X,Y \in \mathbb{R}^k \times \mathbb{R}^k \times \ldots \times \mathbb{R}^k \), which implies \( \phi(x) = \rho(X) \) and \( \phi(y) = \rho(Y) \). We also consider \( z := \lambda x + (1 - \lambda)y \) for any \( \lambda \in [0,1] \). Thus, from the definition of \( \rho \), there exists a \( Z \in \mathbb{R}^k \times \mathbb{R}^k \times \ldots \times \mathbb{R}^k \) such that

\[
\rho(\lambda x + (1 - \lambda)y) = \rho(Z)
\]

with \( \rho(Z) = \lambda x + (1 - \lambda)y \). Hence, from the statistic convexity \( \textbf{C3} \) of \( \rho \), we know that

\[
\rho(\lambda x + (1 - \lambda)y) = \rho(Z)
\]

\[
\leq \lambda \rho(X) + (1 - \lambda)\rho(Y)
\]

\[
= \lambda \phi(x) + (1 - \lambda)\phi(y),
\]

which implies the convexity \( \textbf{A2} \) of \( \rho \). Thus, \( \rho \) is a simple-systemic risk statistic and from \( \textbf{4.12} \) and \( \textbf{4.13} \), we have \( \rho = \rho \circ \phi \). Next, we will show the ‘if’ part. Suppose \( \phi \) is a clustering function and \( \rho \) is a simple-systemic risk statistic. Furthermore, define \( \rho = \rho \circ \phi \). Since \( \rho \) and \( \phi \) are monotone and convex, it is not hard to check that \( \rho \) satisfies monotonicity \( \textbf{C1} \) and convexity \( \textbf{C2} \). Now, suppose \( X,Y,Z \in \mathbb{R}^k \times \mathbb{R}^k \times \ldots \times \mathbb{R}^k \) which satisfies

\[
\rho(Z_{[k_i]}) = \lambda \rho(X_{[k_i]}) + (1 - \lambda)\rho(Y_{[k_i]})
\]

for any \( \lambda \in [0,1] \). Then, the property \( \textbf{B3} \) of \( \phi \) implies

\[
\phi(Z) = \lambda \phi(X) + (1 - \lambda)\phi(Y).
\]

Thus, we have

\[
\rho(Z) = \rho(\lambda \phi(X) + (1 - \lambda)\phi(Y))
\]

\[
\leq \lambda \rho(X) + (1 - \lambda)\rho(Y),
\]

which yields \( \rho \) satisfies the property \( \textbf{C3} \). Thus, the \( \rho \) defined above is a systemic risk measure. \( \square \)
Remark 41  Theorem not only provide a decomposition result for systemic risk measure, but also propose a idea to deal with the systemic risk on a market with uncertainty and volatility. More concretely, we first use the convex certain function $\phi$ to convert the uncertainty of systemic risk into certainty, then we quantify the simplified risk by the simple-systemic risk measure. This means that a regulator who deal with the measurement of this systemic risk can construct a reasonable systemic risk measure by choosing an appropriate certain function and an appropriate simple-systemic risk measure. The certain function should reflect his preferences towards the uncertainty and volatility of the financial markets.

In the following section, we derive the dual representation of the total systemic risk statistics with the help of the acceptance sets of $\phi$ and $\varrho$.

5 Dual representation

Before we study the dual representation of the total systemic risk statistics on $\mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d}$, the acceptance sets should be defined. Since each total systemic risk statistic $\rho$ can be decomposed into a clustering function $\phi$ and a simple-systemic risk statistic $\varrho$, we only need to define the acceptance sets of $\phi$ and $\varrho$, i.e.

$$A_\phi := \{(c, x) \in \mathbb{R} \times \mathbb{R}^d : \varrho(x) \leq c\}$$

and

$$A_\varrho := \{(y, X) \in \mathbb{R}^d \times (\mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d}) : \phi(X) \leq y\}.$$  \hspace{1cm} (5.1)

We will see later on that these acceptance sets can be used to provide the total systemic risk statistics on $\mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d}$. The following properties are needed in the subsequent study.

Definition 51  Let $M$ and $N$ be two ordered linear spaces. A set $A \subset M \times N$ satisfies f-monotonicity if $(m, n) \in A$, $q \in N$ and $n \geq q$ imply $(m, q) \in A$. A set $A \subset M \times N$ satisfies b-monotonicity if $(m, n) \in A$, $p \in M$ and $p \geq m$ imply $(p, n) \in A$.

Proposition 51  Suppose $\rho = \varrho \circ \phi$ is a total systemic risk statistic with a clustering function $\phi : \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d} \to \mathbb{R}^d$ and a simple-systemic risk statistic $\varrho : \mathbb{R}^d \to \mathbb{R}$. The corresponding acceptance sets $A_\phi$ and $A_\varrho$ are defined by (5.1) and (5.2). Then, $A_\phi$ and $A_\varrho$ are convex sets and they satisfy the f-monotonicity and b-monotonicity.

Proof.  It is not hard to check the properties by the definition of $\phi$ and $\varrho$. \hfill \square

The next proposition provides the primal representation of total systemic risk statistics on $\mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d}$ at the point of acceptance sets.

Proposition 52  Suppose $\rho = \varrho \circ \phi$ is a total systemic risk statistic with a clustering function $\phi : \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d} \to \mathbb{R}^d$ and a simple-systemic risk statistic $\varrho : \mathbb{R}^d \to \mathbb{R}$. The corresponding acceptance sets $A_\phi$ and $A_\varrho$ are defined by (5.1) and (5.2). Then, for any $X \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d}$,

$$\rho(X) = \inf \{c \in \mathbb{R} : (c, x) \in A_\phi, (x, X) \in A_\varrho\}$$ \hspace{1cm} (5.3)

where we set $\inf \emptyset = +\infty$.

Proof.  Since $\rho = \varrho \circ \phi$, we have

$$\rho(X) = \inf \{c \in \mathbb{R} : (\varrho \circ \phi)(X) \leq c\}.$$ \hspace{1cm} (5.4)

By the definition of $A_\varrho$, we know that

$$\rho(x) = \inf \{c \in \mathbb{R} : (c, x) \in A_\varrho\}$$ \hspace{1cm} (5.5)

for any $x \in \mathbb{R}^d$. Then, from (5.4) and (5.5),

$$\rho(X) = \inf \{c \in \mathbb{R} : (c, \phi(X)) \in A_\varrho\}.$$
It is not hard to check that
\[ \{ c \in \mathbb{R} : (c, \phi(X)) \in \mathcal{A}_\phi \} = \{ c \in \mathbb{R} : (c, x) \in \mathcal{A}_\phi, (x, X) \in \mathcal{A}_\phi \}. \]

Thus,
\[ \rho(X) = \inf \{ c \in \mathbb{R} : (c, x) \in \mathcal{A}_\phi, (x, X) \in \mathcal{A}_\phi \}. \]

\[ \square \]

Now, with the help of Proposition 52, we will introduce the main result of this section: the dual representation of the total systemic risk statistics on \( \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d} \).

**Theorem 51** Suppose \( \rho = \varrho \circ \phi \) is a total systemic risk statistic characterized by a continue clustering function \( \phi \) and a continue simple-systemic risk statistic \( \varrho \). Then, for any \( X \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d} \), \( \rho(X) \) is of the following form
\[ \rho(X) = \sup_{(\tilde{\gamma}, \tilde{X}) \in \mathcal{P}} \{ (\tilde{\gamma}, X) - \alpha(\tilde{\gamma}, \tilde{X}) \} \]
(5.6)
where \( \alpha : \mathbb{R}^d \times (\mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d}) \to \mathbb{R} \) is defined by

\[ \alpha(\tilde{\gamma}, \tilde{X}) := \sup_{(c, x) \in \mathcal{A}_\phi} \{ -c - (\tilde{\gamma}, (y - x)) + (\tilde{\gamma}, Y) \} \]

and
\[ \mathcal{P} := \{ (\tilde{\gamma}, \tilde{X}) \in \mathbb{R}^d \times (\mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d}), \alpha(\tilde{\gamma}, \tilde{X}) < \infty \}. \]

**Proof.** By Proposition 52 we have
\[ \rho(X) = \inf \{ c \in \mathbb{R} : (c, x) \in \mathcal{A}_\phi, (x, X) \in \mathcal{A}_\phi \} \]
for any \( X \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d} \). Furthermore, we can rewritten it by
\[ \rho(X) = \inf_{(c, x) \in \mathcal{P}} \{ c + I_{\mathcal{A}_\phi}(c, x) + I_{\mathcal{A}_\phi}(x, X) \} \]
(5.7)
where the indicator function of a set \( A \in \mathcal{X} \times \mathcal{Y} \) is defined by
\[ I_{\mathcal{A}_\phi}(a, b) := \begin{cases} 0, & (a, b) \in \mathcal{X} \times \mathcal{Y} \\ \infty, & \text{otherwise}. \end{cases} \]

From Proposition 51 we know that \( \mathcal{A}_\phi \) and \( \mathcal{A}_\phi \) are convex sets. Thus,
\[ I'_{\mathcal{A}_\phi}(\tilde{c}, \tilde{x}) = \sup_{(\tilde{\gamma}, \tilde{X}) \in \mathcal{A}_\phi} \{ \tilde{c}(\tilde{\gamma}, \tilde{x}) \}, \quad \tilde{c} \in \mathbb{R}, \tilde{x} \in \mathbb{R}^d \]
and
\[ I'_{\mathcal{A}_\phi}(\tilde{\gamma}, \tilde{X}) = \sup_{(\tilde{\gamma}, \tilde{X}) \in \mathcal{A}_\phi} \{ \tilde{\gamma} + \langle \tilde{\gamma}, \tilde{X} \rangle \}, \quad \tilde{\gamma} \in \mathbb{R}^d, \tilde{X} \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d}. \]

On the other hand, since \( \varrho \) is continue, it follows that \( \mathcal{A}_\phi \) is closed. Thus, by the duality theorem for conjugate functions, we have
\[ I_{\mathcal{A}_\phi}(c, x) = I'_{\mathcal{A}_\phi}(c, x) \]
\[ = \sup_{(\tilde{\gamma}, \tilde{X}) \in \mathcal{A}_\phi} \{ \tilde{\gamma} + \langle \tilde{\gamma}, \tilde{x} \rangle - I'_{\mathcal{A}_\phi}(\tilde{\gamma}, \tilde{x}) \} \]
\[ = \sup_{(\tilde{\gamma}, \tilde{x}) \in \mathcal{P}} \{ \tilde{c}(\tilde{\gamma}, \tilde{x}) - \sup_{(\tilde{\gamma}, \tilde{x}) \in \mathcal{A}_\phi} \{ \tilde{c}(\tilde{\gamma}, \tilde{x}) \} \}. \]

Similarly, we have
\[ I_{\mathcal{A}_\phi}(x, X) = I'_{\mathcal{A}_\phi}(x, X) \]
\[ = \sup_{(\tilde{\gamma}, \tilde{X}) \in \mathcal{P}} \{ \langle \tilde{\gamma}, \tilde{X} \rangle - I'_{\mathcal{A}_\phi}(\tilde{\gamma}, \tilde{X}) \} \]
\[ = \sup_{(\tilde{\gamma}, \tilde{X}) \in \mathcal{P}} \{ \langle \tilde{\gamma}, \tilde{X} \rangle - \sup_{(\tilde{\gamma}, \tilde{X}) \in \mathcal{P}} \{ \langle \tilde{\gamma}, \tilde{X} \rangle \} \}. \]
Thus, we know that
\[
\rho(X) = \inf_{(c,x) \in \mathbb{R} \times \mathbb{R}^d} \left\{ c + I_{A^c}(c,x) + I_{A^c}(x,X) \right\}
\]
\[
= \inf_{(c,x) \in \mathbb{R} \times \mathbb{R}^d} \sup_{(\tilde{c},\tilde{x}) \in \mathbb{R} \times \mathbb{R}^d} \left\{ c(1 + \tilde{c}) + \langle \tilde{x} + \tilde{y}, x \rangle + \langle \tilde{X}, X \rangle - I'_{A^c}(\tilde{c}, \tilde{x}) - I'_{A^c}(\tilde{y}, \tilde{X}) \right\}.
\]

By the Theorem 7 of Rockafellar (1974), because of the continuity of \( \rho \) and the continuity of \( \phi \), we can interchange the supremum and the infimum above, i.e.
\[
\rho(X) = \sup_{(\tilde{c},\tilde{x}) \in \mathbb{R} \times \mathbb{R}^d} \left\{ c(1 + \tilde{c}) + \langle \tilde{x} + \tilde{y}, x \rangle + \langle \tilde{X}, X \rangle - I'_{A^c}(\tilde{c}, \tilde{x}) - I'_{A^c}(\tilde{y}, \tilde{X}) \right\}
\]
\[
= \sup_{(\tilde{y},\tilde{X}) \in \mathbb{R}^d \times (\mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d})} \left\{ \langle \tilde{X}, X \rangle - \sup_{(y,Y) \in A_\alpha} \left\{ - c - \langle \tilde{y}, (y-x) \rangle + \langle \tilde{X}, Y \rangle \right\} \right\}.
\]

With \( \alpha(\tilde{y}, \tilde{X}) \) defined by
\[
\alpha(\tilde{y}, \tilde{X}) := \sup_{(y,Y) \in A_\alpha} \left\{ - c - \langle \tilde{y}, (y-x) \rangle + \langle \tilde{X}, Y \rangle \right\}
\]
and
\[
\mathcal{P} := \{(\tilde{y}, \tilde{X}) \in \mathbb{R}^d \times (\mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \ldots \times \mathbb{R}^{k_d}), \alpha(\tilde{y}, \tilde{X}) < \infty \},
\]
it immediately follows that
\[
\rho(X) = \sup_{(\tilde{y},\tilde{X}) \in \mathcal{P}} \left\{ \langle \tilde{X}, X \rangle - \alpha(\tilde{y}, \tilde{X}) \right\}.
\]

\[\square\]

**Remark 51** Note that, the proof of Theorem 51 above utilized the primal representation of total systemic risk statistics in Proposition 52, which means that the acceptance sets \( A_\alpha \) and \( A_{\phi} \) played a vital role. Thus, the dual representation of total systemic risk statistics \( \rho \) still dependent on the clustering function \( \phi \) and the simple-systemic risk statistic \( \phi \).

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