Polarized image of synchrotron radiations of hotspots in Schwarzschild-Melvin black hole spacetime

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Abstract

We revisit the innermost stable circular orbits (ISCOs) of charged particles and study the polarized images of synchrotron radiations emitted from such orbiting hotspots on the equatorial plane in Schwarzschild-Melvin black hole spacetime. We obtain a constraint on the magnetic field to retain ISCOs for charged particles. In particular, we identify a critical value for the strength of the magnetic field $B_c$ that the circular orbits can have positive and negative angular momentums above $B_c$ while only one branch survive below $B_c$. Furthermore, we investigate and discuss the primary and secondary images of circularly orbiting charged hotspots carrying the information of polarization directions observed by distant observers.
1 Introduction

The Event Horizon Telescope (EHT) Collaboration have recently released a polarized image of photon rings around the supermassive black hole at the centre of M87 galaxy \[1, 2\], which reveals that a significant magnetic field exists in the vicinity of M87*. Surrounding charged particles would be affected by the magnetic field and strong gravitational field simultaneously so that lots of interesting phenomena emerge in the environment containing a black hole. Moreover, the reason why EHT is able to observe polarized photon ring structures is that electrons accelerated by a magnetic field emit synchrotron radiations at high velocities and synchrotron radiations exhibit excellent polarization properties. The converse also applies: from the polarization information encoded in the polarized image of photon rings we can infer the magnetic field configuration for a given spacetime.

In order to understand the observations theoretically, simulations of polarized emission has played an essential role in the study of astrophysics \[3–13\]. But beyond that, some analytical models are also presented to gain an insight into the polarized black hole image. Along this line, an analytical treatment of high-spin Kerr black hole spacetime was proposed in \[14\]. In \[15\], EHT Collaboration developed a model of an equatorial magnetized fluid orbiting a Schwarzschild black to understand polarimetric images of black holes. Soon, this model has been generalized to a Kerr black hole spacetime and other black hole geometries arise from modified theory of gravity in \[16–18\]. Also, In virtue of such a model, one can investigate the polarized image for arbitrary emission radius, magnetic field geometry, equatorial fluid velocity, and observer inclination to match the astroobservation. However, such models are not particularly satisfactory for physicists, since some parameters in such models are empirical rather than determined by the basic physical equations. Recently, in \[19\], authors built a toy model to study the polarized images of synchrotron radiations of charged hotspots (or thin free electron gas, equivalently) in Schwarzschild black hole spacetime. This model is obviously very simplified by ignoring the thermal motions and interactions of fluids, and thus, the calculated results are not comparable with real observational data. Nevertheless, each detail of the model is based on fundamental physical laws and no parameters are manually placed. As a result, this model can be seen as a first step and must be an important element of the final complete theory. In addition, this model offers an efficient way to decouple and characterize the effects of the magnetic and spacetime on black hole images over a complex case of multiple parameter coupling.

More precisely, in \[19\] they focused on the Wald magnetic fields, that is, the Schwarzschild black hole is immersed in vertical magnetic fields, which satisfy the Maxwell equation and the
solution is firstly found by Wald in [20]. Moreover, the Wald magnetic field are assumed to have no back reaction to the spacetime geometry. On the other hand, if a vertical magnetic field becomes strong and the back reaction cannot be ignored, such a black hole is called the Schwarzschild-Melvin black hole (SMBH) which is an exact solution of Einstein-Maxwell theory found by Ernst [21]. The SMBH spacetime is not asymptotically flat due to the existence of magnetic field at infinity and the SMBH can be regarded as a Schwarzschild black hole embedded in a Melvin universe [22], so that, this type of magnetic fields is also dubbed Melvin magnetic fields. The SMBH has been studied in various aspects, such as the motion of charged particles, neutral particles and photons on the equatorial plane [23–29], shadows and lensing of SMBH [30, 31]. In this work, we would like to investigate the influence of the Melvin magnetic fields on the polarized image of synchrotron radiations emitted from an equatorial charged hotspots employing the methods in [19]. We assume the charged particles to move in a circular orbit outside the innermost stable circular orbit (ISCO) on the equatorial plane. Thus, our research starts with the ISCO in the SMBH spacetime and we find some interesting properties of ISCOs that were not found in the literature passingly. Nevertheless, our main attention is focused on the shape of the image and the polarization directions of the synchrotron radiations of the orbiting source.

The remaining part of this paper is organized as follows. In Sec. 2 we introduce the SMBH spacetime and revisit the ISCO of charged particles. In Sec. 3 we set up the methods for calculating the polarized image of synchrotron radiations emitted from charged particles of circular motion observed by distant observers in SMBH spacetime. In Sec. 4 we show the details of results. In Sec. 5 we summarize and conclude this work.

2 SMBH spacetime and ISCO of charged particles

In this section, we would like to give a necessary review of the SMBH spacetime and revisit the ISCO of charged particles in this background. In the quasi-isotropic coordinate \( \{ t, r, \theta, \phi \} \), the metric of the SMBH spacetime reads [21]

\[
 ds^2 = -\Lambda^2(r, \theta) f(r) dt^2 + \frac{\Lambda^2(r, \theta)}{f(r)} dr^2 + r^2 \Lambda^2(r, \theta) d\theta^2 + \frac{r^2 \sin^2 \theta}{\Lambda^2(r, \theta)} d\phi^2 ,
\]

where

\[
 \Lambda(r, \theta) = 1 + \frac{B^2 r^2 \sin^2 \theta}{4} , \quad f(r) = 1 - \frac{2M}{r} ,
\]

\( M \) is the mass of the BH and \( B \) is the strength of the magnetic field around the BH. The corresponding electromagnetic potential takes in this form

\[
 A_a = \frac{Br^2 \sin^2 \theta}{2\Lambda} (d\phi)_a .
\]
It’s worth noting that the magnetic field around the SMBH is very strong so that it has a non-trivial back reaction to the background spacetime. Moreover, one can notice that the SMBH is not asymptotically flat due to the strong magnetic field. When $B = 0$, the SMBH spacetime reduce to the usual Schwarzschild spacetime and it returns to the so-called Melvin universe with $M = 0$. As long as $M$ is not equal to zero, Inevitably one always finds the coordinate $r = 0$ to be the singularity of the spacetime and $r = 2M$ is a coordinate singularity which corresponds to an apparent horizon and it is believed that there is a BH inside the apparent horizon. In the following, we would like to focus on the spacetime containing a SMBH that is, $M \neq 0$. For the sake of convenience, we are going to set $M = 1$ without loss of generality.

![Figure 1: A diagram of the orbiting charged particles around the SMBH black hole on the equatorial plane, which includes borrowings in [19]. In our convention, we set $B > 0$ and $e < 0$.](image)

Next, Let us consider the ISCO of charged particles moving in the SMBH spacetime. We assume the 4–velocity of a charged particle is denoted by $Z^a$, which is in general given by

$$Z^a = i \left( \frac{\partial}{\partial t} \right)^a + \dot{r} \left( \frac{\partial}{\partial r} \right)^a + \dot{\theta} \left( \frac{\partial}{\partial \theta} \right)^a + \dot{\phi} \left( \frac{\partial}{\partial \phi} \right)^a,$$

where overdots mean derivative with respect to the proper time $\tau$. Noting that $\left( \frac{\partial}{\partial t} \right)^a$ and $\left( \frac{\partial}{\partial \phi} \right)^a$ are Killing vectors of the SMBH spacetime, we have two corresponding conserved quantities,

$$E = -Z_t = \Lambda^2 f \dot{t}, \quad L = Z_\phi + eA_\phi = \frac{r^2 \sin^2 \theta}{\Lambda^2} \dot{\theta} + eB \frac{r^2 \sin^2 \theta}{2 \Lambda},$$

which are the energy per unit mass and the angular momentum per unit mass of the charged particles, respectively. And, $e$ is the charge per unit mass of the particle. We want to stress that
since charged particles would be acted upon by Lorentz force in the magnetic field, the term $eAφ$ appeared in Eq. (2.5) cannot be omitted, while this term needs to be dropped for neutral particles and photons\(^1\). On the other hand, due to the $Z_2$ symmetry of the spacetime, one is allowed to choose $B > 0$ to study the effects of magnetic fields on charged particles. Moreover, considering the invariance of Eqs. (2.1) and (2.3) under the transformation $B → -B$, $φ → -φ$, in terms of studying the orbits of charged particles in a certain magnetic field, there is no essential difference between $e > 0$ and $e < 0$. Note that we are going to investigate the polarized image of synchrotron radiations of electrons, we might as well set $e < 0$ in the following. A diagram is given in Fig. 1 to make the explanation to be clearer. In particular, we want to stress that for $L > 0$ we have $dφ/dt < 0$ from Eq. (2.5), while for $L < 0$ the sign of $dφ/dt$ is not uncertain, since $dφ/dt$ has the same sign with $-\left(\frac{L - \frac{eBr^2\sin^2θ}{2Λ}}{2Λ}\right)$ with $\frac{eBr^2\sin^2θ}{2Λ}$ being negative in our convention. Thus, we would like to use the terminology of $L > 0$ and $L < 0$ orbits instead of retrograde and prograde orbits in this work.

In addition, in this work, we would like to focus our attentions on the equatorial orbits, thus, we have $θ = \frac{π}{2}$. Then, from the normalization condition of the 4-velocity, that is, $Z^aZ_a = -1$, we find

$$\Lambda^4\dot{r}^2 = E^2 - V(r), \quad (2.6)$$

where

$$V(r) = \Lambda^4 f \left[ \frac{1}{r^2} \left( L - \frac{eBr^2\sin^2θ}{2Λ} \right)^2 + \frac{1}{Λ^2} \right], \quad (2.7)$$

is the so-called effective potential of the charged particles in the radial motion. Since $\dot{r}^2 ≥ 0$ is always true, we always have $E^2 - V(r) ≥ 0$ and the condition for equality gives the turning points in the radial motion. For circular orbits with fixed orbit radii, we have $r = \text{const}$ and $\dot{r} = 0$, which correspond to $E^2 - V(r) = 0$ and $\partial_r V(r) = 0$. Note that not all the circular orbits are stable, in order to find a stable circular orbit, the condition $\partial_{\dot{r}}^2 V(r) ≥ 0$ has to be imposed. Combining the equations $E^2 - V(r) = 0$, $\partial_r V = 0$ and $\partial_{\dot{r}}^2 V = 0$, one can obtain the radius of the ISCO. In order to know more details of the ISCO, we would like to study the behavior of the effective potential carefully.

\(^1\)In classical physics, the trajectories of photons are unaffected by magnetic fields, however, if the effect of quantum electrodynamics is included, the dispersion relationship of photons changes and photons no longer travel along geodesics, see examples in [32, 33].
Figure 2: Constraints on $e$ and $B$ from the existence of physical solutions to the equation $\partial_r V = 0$.

Considering that $V(r) > 0$ is always true, so that there must be solutions to the equation $V(r) - E^2 = 0$. Then, we turn to the equation $\partial_r V = 0$. The explicit expression of $\partial_r V$ can be found in this form

$$\frac{\partial_r V}{4\Lambda} = XL^2(4 + B^2 r^2)^2 - 4YeBLr^2 + 4Zr^2$$

(2.8)

where

$$X = 3B^2 r^3 - 5B^2 r^2 - 4r + 12,$$

$$Y = B^4 r^4 (3r - 5) + 4B^2 r^2 (3r - 4) + 16,$$

$$Z = B^4 e^2 r^4 (3r - 5) + 4B^2 r^2 [2r - 3 + (r - 1)e^2] + 16.$$  

(2.9)

Figure 3: The variation of $L$ with respect to $B$ from 0 to 0.34 with a fixed $e = -2$ for ISCOs.
To make sure the existence of circular orbits, we should ensure the equation $\partial_r V = 0$ has solutions outside the horizon $r_h = 2$. Note that the right side of the Eq. (2.8) is a quadratic polynomial for $l$, thus, we have the discriminant of the quadratic equation

$$
\Delta = 16Y^2e^2B^2r^4 - 16XZr^2(4 + B^2r^2)^2 \geq 0,
$$

holds for $r > 2$, which gives us a constraint on $e$ and $B$. This inequality is too complicated to solve analytically, so that, we numerically obtain the result shown as the red line in Fig. 2. Under and including the red line, we have $\Delta \geq 0$ and for the region above the red line we have $\Delta < 0$. In addition, we note that $Y \geq 0$ and $Z \geq 0$ always hold when $r > 2$, as a result, when $L > 0$ we can see that the latter two terms of the right side of the Eq. (2.8) are positive. Therefore, if the $\partial_r V = 0$ is satisfied, we conclude the first term of the right side of the Eq. (2.8) must be negative which implies $X < 0$ for $r > 2$. From this condition, we find $B < B_c = \frac{2}{5}\sqrt{\frac{11}{115}}(169 - 38\sqrt{19}) \simeq 0.189$ which is shown as the blue line in Fig. 2. Very interestingly, in [30], the authors found that $B_c$ is also a critical value of the magnetic field strength and no light rings exists outside the apparent horizon for $B > B_c$ while both the stable and unstable light rings exist for $B < B_c$. As mentioned above, we can come to the conclusion that when the values of $(e, B)$ are in the region between the red and blue line, we only have $L < 0$ circular orbits. And in the region between $B = B_c$ and $B = 0$, both $L > 0$ and $L < 0$ circular orbits exist. We give an example shown in Fig. 3 for $e = -2$.

Figure 4: The radius of the ISCO with respect to $B$ for $e = -2$. $B$ goes from 0 to 0.34.

On the other hand, we can rewrite $\partial_r V$ in this form

$$
\frac{32r^4\partial_r V}{\Lambda} = 3PB^4r^7 - 5PB^4r^6 + 4(8Q + 2B^2L^2 + 3P)B^2r^5 - 4(12Q + 3B^2L^2 + 4P)B^2r^4 \\
+ 16B^2L^2r^3 + 16(4Q + P)r^2 - 64L^2r + 192L^2,
$$

(2.11)

Roughly speaking, Light rings are circular photon orbits in a spacetime. We suggest readers to see [34–38] if interested in more properties of light rings in stationary spacetimes.
where

\[ P = (-2e + BL)^2, \quad Q = 1 - e^2. \tag{2.12} \]

From \( V = E^2 \), we can obtain

\[ \Lambda \left( \frac{L^2}{\Lambda^2 \lambda^2 f} - \frac{L^2}{r^2} + \frac{B^2 L^2}{4 \Lambda} \right) = \frac{P}{4} + \frac{Q}{\Lambda}, \tag{2.13} \]

where we have introduced a new parameter \( \lambda^2 = \frac{L^2}{E^2} \), thus, we can see that when \( Q = 0 \), each term in the right hand of the Eq. (2.11) contains a factor \( L^2 \) which can be dropped to solve \( \partial_r V = 0 \). Therefore, we find that apart from the parameter \( B \), \( r \) is only related to \( \lambda^2 \), so that, it is interesting to find that in this case \( Q = 0 \), that is \( e = -1 \) in our convention, no matter what values of \( L \) and \( E \) are, \( |L/E| \) is the same for a circular orbit. On the contrary, when \( e \neq -1 \), the value \( L/E \) is different for a circular orbit.

Moreover, we observe that the right hand of the Eq. (2.11) is \( 192 L^2 > 0 \) when \( r = 0 \), and it approached \( \pm \infty \) when \( r \to \pm \infty \), which means \( \partial_r V = 0 \) at least has a minus root for \( r \). Considering the constraint shown in Fig. (2), we have \( \partial_r V = 0 \) at least has a root \( r > 2 \) when \( e \) and \( B \) are satisfied to the corresponding restrain. As a result, we know that under this constrain, we must have \( \partial^2_r V > 0 \) for \( r > 2 \). Furthermore, considering the continuity of function \( \partial_r V(r) \), we are always able to find the ISCO under the condition. As a consequence, on the basis of those facts mentioned above we can reach the following conclusion, the constraints shown in Fig. (2) are necessary and sufficient conditions for the existence of the ISCO. An example of the radius of ISCO with respect to \( B \) is given in Fig. 4 with \( e = -2 \).

3 Synchrotron radiations of charged particles in SMBH spacetime

In this section, we move to investigate the synchrotron radiations of charged particles orbiting the SMBH on the equatorial plane. The synchrotron radiations of charged particles have good polarization properties, and the polarization direction can reflect the feature of the magnetic fields surrounded the black hole. In order to find the polarized information of synchrotron radiations on the screen of observers, a possible way to know the polarizations at the position of the source. In this work, we would like to focus on the study of polarization directions employing the method proposed in [19], where they found the polarization vector \( f^\mu \) of synchrotron radiations from charged particles in a curved spacetime reads

\[ f^\mu = N^{-1} \left[ g_{\mu\alpha} (k_\beta D_\tau Z^\beta Z^\alpha) \right], \tag{3.1} \]
where $N$ is the normalized factor which is not important to the direction, $k_{\alpha}$ is the 4-momentum of the radiations, $Z^\alpha$ is the 4-velocity of the charged particles, $D_\tau$ is defined as the derivative operator along the vector $Z^\alpha$, that is, $D_\tau = Z^\alpha \nabla_\alpha$, $\tau$ is the proper time and $\bar{g}_{\mu\alpha}$ is the so-called bi-vector of geodesic parallel displacement which can be recognized as the metric $g_{\mu\alpha}$ here in practice.

### 3.1 Null geodesics in SMBH spacetime

In sec. 2 we have obtained the 4-velocity of charged particles in stable circular orbits on the equatorial plane and the 4-acceleration $D_\tau Z^\alpha$ can be determined straightforwardly. The 4-momentum $k^\alpha$ can be obtained considering null geodesics originating from the source at $(t_s, r_s, \theta_s = \pi/2, \phi_s)$ and reaching the observer at $(t_o, r_o, \theta_o, \phi_o)$. Note that the equation of motion along null geodesics cannot be separated in SMBH spacetime, one has to employ the numerical ray-tracing method to find the trajectory connecting the source and the observer and determine $k^\alpha$ at the source.

![null geodesics in SMBH spacetime](image)

Figure 5: The main content of the picture is taken from [39], which shows the projection of the photon’s momentum $\vec{K}$ in the observer’s frame. $(\hat{x}, \hat{y})$ are the Cartesian coordinates glued to the screen of the observer.

Similar with the timelike particles, the components of 4-momentum $k^\mu$ in the quasi-isotropic coordinate takes

$$k^\mu = (t', r', \theta', \phi'),$$

where $t'$ denotes derivative with respect to the proper time $s$. In SMBH spacetime, in order to do the numerical geodesic evolution more conveniently, we obtain the null geodesic equations in the Hamiltonian canonical formulism

$$k'_\mu = -\frac{\partial H}{\partial x^\mu}, \quad x^\mu = \frac{\partial H}{\partial k'_\mu}$$

(3.3)
where $H$ and $x^\mu$ are the Hamiltonian and positions of the photons, respectively. In particular, considering the Killing vectors $\partial_t$ and $\partial_\phi$ in SMBH spacetime, we are allowed to define the constants along the null geodesics

$$\omega = -k_t = \Lambda^2 f t', \quad l = k_\phi = \frac{r^2 \sin^2 \theta}{\Lambda^2} \phi'$$

which are the energy and angular momentum, respectively.

Since optical paths are reversible, we can solve the Eq. (3.3) using backward ray-tracing method, that is, we can take $(x^\mu, -k^\mu)$ observed at the observer as the initial values of the Eq. (3.3). In order to achieve this aim, at first we build a local rest frame in the neighborhood of the observer, of which the tetrad reads

$$\hat{e}_t(t) = \Lambda^{-1} f^{-1/2} \partial_t, \quad \hat{e}_r(r) = \Lambda^{-1} f^{1/2} \partial_r, \quad \hat{e}_\theta(\theta) = \Lambda^{-1} r^{-2} \sin^{-1} \theta \partial_\theta, \quad \hat{e}_\phi(\phi) = \Lambda^{-1} r^{-2} \sin^{-1} \theta \partial_\phi$$

where $\hat{e}_t(t)$ is timelike and the others are spacelike. The 4-momentum $k^\mu$ can be measured by the observer and we find

$$k(t) = -k_t \hat{e}_t(t), \quad k(i) = k_\mu \hat{e}_\mu(i), \quad i = r, \theta, \phi.$$  

In addition, we introduce the 3-momentum vector $\vec{K}$ of $k^\mu$, seen as in Fig. 5. Since $k^\mu$ is a null vector, we have $|\vec{K}| = k(t)$. Then on the screen of the observer, we can define the apparent position of the image of the photons in the Cartesian coordinates $(\alpha, \beta)$ as

$$\alpha = -r_o \tan \Phi, \quad \beta = r_o \sin \Theta.$$  

where, the angles $\Theta$ and $\Phi$ are defined as

$$\sin \Theta = \frac{k^\theta}{|\vec{K}|}, \quad \tan \Phi = \frac{k_{(\phi)}}{k_{(r)}}.$$  

Therefore, plunging the Eq. (3.8) into the Eq. (3.7), we can obtain the final expressions of $\alpha$ and $\beta$ in this form

$$\alpha = -r_o \frac{k_{(\phi)}}{k_{(r)}}, \quad \beta = r_o \frac{k_{(\theta)}}{k(t)}.$$  

Then, inversely combining with the Eq. (3.6) we can find the components of the 4-momentum $k^\mu$,

$$k^t = \Lambda^{-2} f^{-1} \omega,$$

$$k^r = \frac{\sqrt{r_o^2 - \beta^2}}{\sqrt{r_o^2 + \alpha^2}} \Lambda^{-2} \omega,$$

$$k^\theta = \beta r_o^{-2} \Lambda^{-2} f^{-1/2} \omega,$$

$$k^\phi = -\alpha \frac{\sqrt{r_o^2 - \beta^2}}{\sqrt{r_o^2 + \alpha^2}} r_o^{-2} \sin^{-1} \theta_o f^{-1/2} \omega,$$

(3.10)
in the coordinates \((t,r,\theta,\phi)\). Now, we have obtained all the initial values \((x^\mu, -k_\mu)\) at the observer. Then we are able to solve the null geodesics using the Hamiltonian canonical formulism numerically and find the momentum vector \(k^\mu\) at the source \((t_s, r_s, \theta_s, \phi_s)\). It needs to be emphasized that the light rays may cross the equatorial plane many times before reaching the source. We introduce \(m\) to denote number of the times and we can see that the value of \(m\) will give the \(m+1\)th image on the observer’s screen. In the present work, we would like to show the primary and secondary images of the source.

3.2 Polarization propagation and Penrose-Walker constant in SMBH spacetime

In this subsection, we move to calculate the polarization propagation along null geodesics in SMBH spacetime. Considering the SMBH spacetime is of type D in the Petrov classification like the Schwarzschild spacetime, there exists the Penrose Walker constant \([40]\) which takes

\[
\kappa = [ (k \cdot l)(f \cdot n) - (k \cdot n)(f \cdot l) - (k \cdot m)(f \cdot \bar{m}) + (k \cdot \bar{m})(f \cdot m) ] \Psi_2^{-\frac{1}{2}} \\
= 2(\Lambda^2 k^t f^r - \Lambda^2 k^r f^t + ir^2 \sin \theta k^\theta f^\phi - ir^2 \sin \theta f^\theta k^\phi) \Psi_2^{-\frac{1}{2}} \\
= K_1 + iK_2
\]  

(3.11)

where \(\{l^\mu, n^\mu, m^\mu, \bar{m}^\mu\}\) is a special complex null tetrad, in the second “\(=\)”, they are chosen as

\[
l^\mu = \frac{1}{\sqrt{2}} [ e^\mu_{(t)} - e^\mu_{(\phi)} ] , \quad n^\mu = \frac{1}{\sqrt{2}} [ e^\mu_{(t)} + e^\mu_{(\phi)} ] , \\
m^\mu = \frac{1}{\sqrt{2}} [ e^\mu_{(r)} - ie^\mu_{(\theta)} ] , \quad \bar{m}^\mu = \frac{1}{\sqrt{2}} [ e^\mu_{(r)} + ie^\mu_{(\theta)} ]
\]  

(3.12)

and \(\Psi_2\) is the non-zero Weyl scalar

\[
\Psi_2 = \frac{R_{abcd} m^a n^b \bar{m}^c \bar{m}^d}{4(8 - B^2 r^2 + B^2 r^2 \cos 2\theta) \left[ 3B^2(r - 1)^2 \cos 2\theta + B^2 r^3 + (8 + 3B^2 r^2) \right]} r^3(4 + B^2 r^2 \sin^2 \theta)^{\frac{3}{2}}
\]  

(3.13)

In addition, in the third “\(=\)” we have introduced two parameters

\[
K_1 = 2\Lambda^2 (k^t f^r - k^r f^t) \Psi_2^{-1/3} , \quad K_2 = 2r^2 \sin \theta (k^\theta f^\phi - f^\theta k^\phi) \Psi_2^{-1/3}
\]  

(3.14)

which are the real and imaginary parts of \(\kappa\), respectively. Then in the Cartesian coordinates \((\alpha, \beta)\), we have the polarization vector

\[
f_\alpha \equiv f^{(\phi)} = \frac{r_\alpha \sin \theta_\alpha f^\phi}{\Lambda} = \frac{\delta}{\alpha^2 \gamma^2 + \beta^2} \left( \gamma^2 \alpha K_1 + \beta K_2 \right), \\
f_\beta \equiv -f^{(\theta)} = -r_\alpha \Lambda f^\theta = \frac{\delta}{\alpha^2 \gamma^2 + \beta^2} \left( \gamma \beta K_1 - \gamma \alpha K_2 \right)
\]  

(3.15)
where
\[ \delta = \frac{r \Lambda f^{1/2} \Psi^{1/3}}{2\omega}, \quad \gamma = \frac{\sqrt{r_0^2 - \beta^2}}{\sqrt{r_0^2 + \alpha^2}}. \] (3.16)

For a large \( r_\alpha \), we approximately have \( \gamma = 1 \) which will significantly simplify the expressions of \( f_\alpha \) and \( f_\beta \).

4 Results

In this section, we are ready to find out the results of the polarized images of the charged source orbiting the SMBH on the equatorial plane. In our work, on one hand, we would like to consider \( B \) is small so that the radius of the ISCO in SMBH spacetime is close to the radius of the ISCO in Schwarzschild black hole spacetime. On the other hand, as found in [30], the shadow and image become weird when \( B > B_c \), thus, we take \( B = 0.02 \) in the following.

Figure 6: Primary polarized images of charged particles orbiting around the SMBH on the equatorial plane observed at the inclination angle \( \theta_o = 0^\circ \) and \( 17^\circ \). We set \( e = -2 \) and the radii of ISCO are 5.42 for \( L < 0 \) and 5.47 for \( L > 0 \), which are shown as the innermost circles in the plot, respectively. The outer circle next to the innermost one is \( r_{ISCO} + 1 \) and so on. Five orbits are given for each case in the plot.
Figure 7: Primary polarized images of charged particles orbiting around the SMBH on the equatorial plane observed at the inclination angle $\theta_o = 30^\circ$ and $60^\circ$. The other parameters are the same with those in Fig. 6.

In Fig. 6 and Fig. 7, we give the primary polarized images of orbiting charged particles with the charge being $e = -2$ on the equatorial plane in the SMBH spacetime observed by observers at the inclination angle $\theta_o = 0^\circ$, $17^\circ$, $30^\circ$, and $60^\circ$. The radius of ISCO is $5.42$ for charged particles with a negative $L$ while for a positive $L$, the ISCO is located at $r_{\text{ISCO}} = 5.47$ which is a little bigger than that for a negative $L$. From these polarized images, we can draw some interesting conclusions. At first, we can see that the image of charged particle becomes oblate when the inclination angle $\theta_o$ increases. The reason is that the SMBH spacetime is non-rotating and the existence of the magnetic field makes the image more flat. Similar observations of black hole shadows can be also found in [32], where the non-rotating black holes immersed in Wald magnetic fields are considered. Secondly, we find that the polarization directions are different for $L < 0$ and $L > 0$ orbits at the same inclination angle. If we look at the polarization direction from the bottom up, the polarization are pointing in the opposite directions, that is, one is pointing to the left and the other to the right. This finding can help us to distinguish the sigh of the angular momentum of equatorial orbits of charged particles through their polarization directions of synchrotron radiations. Moreover, one can see that the rotations of polarization directions become larger as the increasing of the inclination angle $\theta_o$. In addition, our results reveal small differences of the polarization directions between the inner orbit and the outer orbit at the same Cartesian coordinates $(\alpha, \beta)$. 
Figure 8: Secondary polarized images of charged particles orbiting around the SMBH on the equatorial plane observed at the inclination angle $\theta_0 = 0^\circ$, $17^\circ$ and $30^\circ$. The other parameters are the same with those in Fig. 8.

Figure 9: Secondary polarized images of charged particles orbiting around the SMBH on the equatorial plane observed at the inclination angle $\theta_0 = 60^\circ$. The parameters of charge and ISCO are the same with those in Fig. 6 except that we only show two orbits for each case in the plot.

In Fig. 8 and Fig. 9, we give the secondary polarized images of orbiting charged particles with the charge being $e = -2$ on the equatorial plane in the SMBH spacetime observed by
observers at the inclination angle $\theta_o = 0^\circ, 17^\circ, 30^\circ$ and $60^\circ$. The difference of the radii of two adjacent orbits is 1 and we can see that the images of two adjacent orbits are very close on the $\alpha - \beta$ plane, some parts even overlap when the inclination angle $\theta_o = 60^\circ$, thus, we would like to show only two orbits for each plot so that it would help people to better tell apart the images and corresponding polarization directions. From these plots, we can find the secondary images and corresponding polarization directions of charged particles behave similarly with the primary images. The qualitative conclusions of primary images can be applied to the secondary images. Besides, there exists a significant distinction is that the difference of polarization directions between two adjacent orbits in secondary images are larger than those in the primary images with the same radii of orbits and Cartesian coordinates $(\alpha, \beta)$.

Figure 10: Primary and secondary polarized images of charged particles orbiting around the SMBH on the equatorial plane observed at the inclination angle $\theta_o = 17^\circ$. The charge of the particle is $e = -1$ and the radius of the ISCO is 5.66. We show five orbits for primary images and two orbits for secondary images. The difference of radii of two adjacent orbits is 1.
Recall that the circular orbits have a same $|L/E|$ when the strength of the magnetic field is given and the charge of the particle takes $e = -1$. We would like to see the polarized images of orbiting charged particles. In Fig. [10] we show the primary and secondary polarized images of charged particles in SMBH spacetime at $\theta_o = 17^\circ$. The radius of the ISCO in SMBH spacetime is 5.66. From the Fig. [10] we can see that there is no special features for $e = -1$ even though the value of $|L/E|$ is the same for $L > 0$ and $L < 0$ orbits.

5 Summary and Discussion

In this paper, we have revisited the ISCO of charged particles and studied the polarized images of synchrotron radiations of charged hotspots orbiting on the equatorial plane in SMBH spacetime based on the synchrotron radiation model proposed in [19]. For the part of ISCOs of charged particles, we found the constrain of the strength of the magnetic field $B$ that an ISCO exists. Interestingly, we identified a critical $B_c$ that only $L < 0$ orbits survive under $B_c$ and there are both $L > 0$ abd $L < 0$ orbits over $B_c$. In particular, our critical $B_c$ is exactly the critical value appeared in [30], where they found there are no light ring when $B > B_c$ and light rings exist when $B < B_c$.

For the part of synchrotron radiations of charged hotspots, based on the formula in Eq. 3.1 given in [19], we obtained the polarized images of equatorial orbiting charged particles employing the numerical backward ray-tracing method and Penrose-Walker constant in SMBH spacetime. Some general characteristics of the shape and the polarizations of images were summarized in the last section. It’s worth noting that our results cannot be directly comparable with the results in [15], where they considered the polarized images of fluids on the equatorial plane and the magnetic field configurations can be arbitrarily given. In particular, their fluids can be assumed to include charged particles in a complex state of motion and the underlying physical mechanism is ignored. However, although our model is simple, all the physical mechanisms are clear and can play a theoretical base of understanding synchrotron radiations near black holes.

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