ON TREE-PARTITION-WIDTH

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Abstract. A tree-partition of a graph $G$ is a proper partition of its vertex set into ‘bags’, such that identifying the vertices in each bag produces a forest. The tree-partition-width of $G$ is the minimum number of vertices in a bag in a tree-partition of $G$. An anonymous referee of the paper by Ding and Oporowski [J. Graph Theory, 1995] proved that every graph with tree-width $k \geq 3$ and maximum degree $\Delta \geq 1$ has tree-partition-width at most $24k\Delta$. We prove that this bound is within a constant factor of optimal. In particular, for all $k \geq 3$ and for all sufficiently large $\Delta$, we construct a graph with tree-width $k$, maximum degree $\Delta$, and tree-partition-width at least $(\frac{1}{8} - \epsilon)k\Delta$. Moreover, we slightly improve the upper bound to $5(\frac{k}{2} + 1)(\frac{7}{2}\Delta - 1)$ without the restriction that $k \geq 3$.

1. Introduction

A graph $H$ is a partition of a graph $G$ if:

- each vertex of $H$ is a set of vertices of $G$ (called a bag),
- every vertex of $G$ is in exactly one bag of $H$, and
- distinct bags $A$ and $B$ are adjacent in $H$ if and only if some edge of $G$ has one endpoint in $A$ and the other endpoint in $B$.

The width of a partition is the maximum number of vertices in a bag. Informally speaking, the graph $H$ is obtained from a proper partition of $V(G)$ by identifying the vertices in each part, deleting loops, and replacing parallel edges by a single edge.

If a forest $T$ is a partition of a graph $G$, then $T$ is a tree-partition of $G$. The tree-partition-width of $G$, denoted by $\text{tpw}(G)$, is the minimum width of a tree-partition of $G$. Tree-partitions were independently introduced by Seese [23] and Halin [19], and have since been widely investigated [6, 7, 12, 13, 17, 24]. Applications of tree-partitions include graph drawing [9, 14, 15, 23], graph colouring [2], partitioning graphs into subgraphs with only small components [1], monadic second-order logic [20], and network emulations [3, 4, 8, 18]. Planar-partitions and other more general structures have also recently been studied [11, 25].
What bounds can be proved on the tree-partition-width of a graph? Let $\text{tw}(G)$ denote the tree-width of a graph $G$. Seese \footnote{A graph is chordal if every induced cycle is a triangle. The tree-width of a graph $G$ can be defined to be the minimum integer $k$ such that $G$ is a subgraph of a chordal graph with no clique on $k + 2$ vertices. This parameter is particularly important in algorithmic and structural graph theory; see \cite{3, 21} for surveys.} \footnote{3} proved the lower bound,

$$2 \ \text{tpw}(G) \geq \text{tw}(G) + 1.$$ 

In general, tree-partition-width is not bounded from above by any function solely of tree-width. For example, wheel graphs have bounded tree-width and unbounded tree-partition-width \footnote{A graph is chordal if every induced cycle is a triangle. The tree-width of a graph $G$ can be defined to be the minimum integer $k$ such that $G$ is a subgraph of a chordal graph with no clique on $k + 2$ vertices. This parameter is particularly important in algorithmic and structural graph theory; see \cite{3, 21} for surveys.} \footnote{3}. However, tree-partition-width is bounded for graphs of bounded tree-width and bounded degree \footnote{A graph is chordal if every induced cycle is a triangle. The tree-width of a graph $G$ can be defined to be the minimum integer $k$ such that $G$ is a subgraph of a chordal graph with no clique on $k + 2$ vertices. This parameter is particularly important in algorithmic and structural graph theory; see \cite{3, 21} for surveys.} \footnote{3}. The best known upper bound is due to an anonymous referee of the paper by Ding and Oporowski \footnote{A graph is chordal if every induced cycle is a triangle. The tree-width of a graph $G$ can be defined to be the minimum integer $k$ such that $G$ is a subgraph of a chordal graph with no clique on $k + 2$ vertices. This parameter is particularly important in algorithmic and structural graph theory; see \cite{3, 21} for surveys.} \footnote{3}, who proved that

$$\text{tpw}(G) \leq 24 \text{tw}(G) \Delta(G)$$

whenever $\text{tw}(G) \geq 3$ and $\Delta(G) \geq 1$. Using a similar proof, we make the following improvement to this bound without the restriction that $\text{tw}(G) \geq 3$.

**Theorem 1.** Every graph $G$ with tree-width $\text{tw}(G) \geq 1$ and maximum degree $\Delta(G) \geq 1$ has tree-partition-width

$$\text{tpw}(G) < \frac{5}{2}(\text{tw}(G) + 1)\left(\frac{7}{2} \Delta(G) - 1\right).$$

Theorem 1 is proved in Section 2. Note that Theorem 1 can be improved in the case of chordal graphs. In particular, a simple extension of a result by Dujmović et al. \footnote{A graph is chordal if every induced cycle is a triangle. The tree-width of a graph $G$ can be defined to be the minimum integer $k$ such that $G$ is a subgraph of a chordal graph with no clique on $k + 2$ vertices. This parameter is particularly important in algorithmic and structural graph theory; see \cite{3, 21} for surveys.} \footnote{3} implies that

$$\text{tpw}(G) \leq \text{tw}(G)\left(\Delta(G) - 1\right)$$

for every chordal graph $G$ with $\Delta(G) \geq 2$; see \footnote{A graph is chordal if every induced cycle is a triangle. The tree-width of a graph $G$ can be defined to be the minimum integer $k$ such that $G$ is a subgraph of a chordal graph with no clique on $k + 2$ vertices. This parameter is particularly important in algorithmic and structural graph theory; see \cite{3, 21} for surveys.} \footnote{3} for a simple proof. Nevertheless, the following theorem proves that $O(\text{tw}(G)\Delta(G))$ is the best possible upper bound, even for chordal graphs.

**Theorem 2.** For every $\epsilon > 0$ and integer $k \geq 3$, for every sufficiently large integer $\Delta \geq \Delta(k, \epsilon)$, for infinitely many values of $N$, there is a chordal graph $G$ with $N$ vertices, tree-width $\text{tw}(G) \leq k$, maximum degree $\Delta(G) \leq \Delta$, and tree-partition-width

$$\text{tpw}(G) \geq \left(\frac{1}{8} - \epsilon\right)\text{tw}(G)\Delta(G).$$

Theorem 2 is proved in Section 3. Note that Theorem 2 is for $k \geq 3$. For $k = 1$, every tree is a tree-partition of itself with width 1. For $k = 2$, we prove that the upper bound $O(\Delta(G))$ is again best possible; see Section 4.

2. Upper Bound

In this section we prove Theorem 1. The proof relies on the following separator lemma by Robertson and Seymour \footnote{A graph is chordal if every induced cycle is a triangle. The tree-width of a graph $G$ can be defined to be the minimum integer $k$ such that $G$ is a subgraph of a chordal graph with no clique on $k + 2$ vertices. This parameter is particularly important in algorithmic and structural graph theory; see \cite{3, 21} for surveys.} \footnote{3}.

**Lemma 1 \footnote{A graph is chordal if every induced cycle is a triangle. The tree-width of a graph $G$ can be defined to be the minimum integer $k$ such that $G$ is a subgraph of a chordal graph with no clique on $k + 2$ vertices. This parameter is particularly important in algorithmic and structural graph theory; see \cite{3, 21} for surveys.}.** For every graph $G$ with tree-width at most $k$, for every set $S \subseteq V(G)$, there are edge-disjoint subgraphs $G_1$ and $G_2$ of $G$ such that $G_1 \cup G_2 = G$, $|V(G_1) \cap V(G_2)| \leq k + 1$, and $|S - V(G_i)| \leq \frac{2}{3}|S - (V(G_1) \cap V(G_2))|$ for each $i \in \{1, 2\}$.
Theorem 1 is a corollary of the following stronger result.

**Lemma 2.** Let $\alpha := 1 + 1/\sqrt{2}$ and $\gamma := 1 + \sqrt{2}$. Let $G$ be a graph with tree-width at most $k \geq 1$ and maximum degree at most $\Delta \geq 1$. Then $G$ has tree-partition-width

$$\text{tpw}(G) \leq \gamma(k+1)(3\gamma\Delta - 1).$$

Moreover, for each set $S \subseteq V(G)$ such that

$$(\gamma + 1)(k+1) \leq |S| \leq 3(\gamma + 1)(k+1)\Delta,$$

there is a tree-partition of $G$ with width at most

$$\gamma(k+1)(3\gamma\Delta - 1),$$

such that $S$ is contained in a single bag containing at most $\alpha|S| - \gamma(k+1)$ vertices.

**Proof.** We proceed by induction on $|V(G)|$.

**Case 1.** $|V(G)| < (\gamma + 1)(k+1)$: Then no set $S$ is specified, and the tree-partition in which all the vertices are in a single bag satisfies the lemma. Now assume that $|V(G)| \geq (\gamma + 1)(k+1)$, and without loss of generality, $S$ is specified.

**Case 2.** $|V(G) - S| < (\gamma + 1)(k+1)$: Then the tree-partition in which $S$ is one bag and $V(G) - S$ is another bag satisfies the lemma. Now assume that $|V(G) - S| \geq (\gamma + 1)(k+1)$.

**Case 3.** $|S| \leq 3(\gamma + 1)(k+1)$: Let $N$ be the set of vertices in $G$ that are adjacent to some vertex in $S$ but are not in $S$. Then $|N| \leq \Delta|S| \leq 3(\gamma + 1)(k+1)\Delta$. If $|N| < (\gamma + 1)(k+1)$ then add arbitrary vertices from $V(G) - (S \cup N)$ to $N$ until $|N| \geq (\gamma + 1)(k+1)$. This is possible since $|V(G) - S| \geq (\gamma + 1)(k+1)$.

By induction, there is a tree-partition of $G - S$ with width at most $\gamma(k+1)(3\gamma\Delta - 1)$, such that $N$ is contained in a single bag. Create a new bag only containing $S$. Since all the neighbours of $S$ are in a single bag, we obtain a tree-partition of $G$. (S corresponds to a leaf in the pattern.) Since $|S| \geq (\gamma + 1)(k+1)$, it follows that $|S| \leq \alpha|S| - \gamma(k+1)$ as desired. Now $|S| \leq 3(\gamma + 1)(k+1) < \gamma(k+1)(3\gamma\Delta - 1)$. Since the other bags do not change we have the desired tree-partition of $G$.

**Case 4.** $|S| \geq 3(\gamma + 1)(k+1)$: By Lemma 1 there are edge-disjoint subgraphs $G_1$ and $G_2$ of $G$ such that $G_1 \cup G_2 = G$, $|V(G_1) \cap V(G_2)| \leq k+1$, and $|S - V(G_i)| \leq \frac{2}{3}|S - (V(G_1) \cap V(G_2))|$ for each $i \in \{1,2\}$. Let $Y := V(G_1) \cap V(G_2)$. Let $a := |S \cap Y|$ and $b := |Y - S|$. Thus $a + b \leq k+1$. Let $p_i := |(S \cap V(G_i)) - Y|$. Then $p_1 \leq 2p_2$ and $p_2 \leq 2p_1$. Let $S_i := (S \cap V(G_i)) \cup Y$. Note that $|S_i| = p_i + a + b$.

Now $p_1 + p_2 + a = |S| \geq 3(\gamma + 1)(k+1)$. Thus $3p_i + a \geq 3(\gamma + 1)(k+1)$ and $3p_i + 3a + 3b \geq 3(\gamma + 1)(k+1)$. That is, $|S_i| \geq (\gamma + 1)(k+1)$ for each $i \in \{1,2\}$.

Now $p_1 + p_2 + a \leq 3(\gamma + 1)(k+1)\Delta$. Thus $\frac{3}{2}p_i + a \leq 3(\gamma + 1)(k+1)\Delta$ and $p_i \leq 2(\gamma + 1)(k+1)\Delta$. Thus $p_i + a + b \leq 2(\gamma + 1)(k+1)\Delta + (k+1)$. Hence $|S_i| = p_i + a + b < 3(\gamma + 1)(k+1)\Delta$.

Thus we can apply induction to the set $S_i$ in the graph $G_i$ for each $i \in \{1,2\}$. We obtain a tree-partition of $G_1$ with width at most $\gamma(k+1)(3\gamma\Delta - 1)$, such that $S_i$ is contained in a single bag $T_i$ containing at most $\alpha|S_i| - \gamma(k+1)$ vertices.

Construct a partition of $G$ by uniting $T_1$ and $T_2$. Each vertex of $G$ is in exactly one bag since $V(G_1) \cap V(G_2) = Y \subseteq S_i \subseteq T_i$. Since $G_1$ and $G_2$ are edge-disjoint, the pattern of
Figure 1. Illustration of Case 4.

this partition of $G$ is obtained by identifying one vertex of the pattern of the tree-partition of $G_1$ with one vertex of the pattern of the tree-partition of $G_2$. Since the patterns of the tree-partitions of $G_1$ and $G_2$ are forests, the pattern of the partition of $G$ is a forest, and we have a tree-partition of $G$.

Moreover, $S$ is contained in a single bag $T_1 \cup T_2$ and

$$|T_1 \cup T_2| = |T_1| + |T_2| - |Y|$$

$$\leq \alpha |S_1| - \gamma (k + 1) + \alpha |S_2| - \gamma (k + 1) - (a + b)$$

$$= \alpha (p_1 + a + b) - \gamma (k + 1) + \alpha (p_2 + a + b) - \gamma (k + 1) - (a + b)$$

$$= \alpha (p_1 + p_2 + a) - 2\gamma (k + 1) + (\alpha - 1)a + (2\alpha - 1)b$$

$$\leq \alpha |S| - 2\gamma (k + 1) + (2\alpha - 1)(a + b)$$

$$\leq \alpha |S| - 2\gamma (k + 1) + (2\alpha - 1)(k + 1)$$

$$= \alpha |S| - \gamma (k + 1).$$

Thus $|T_1 \cup T_2| \leq \alpha \cdot 3(\gamma + 1)(k + 1)\Delta - \gamma (k + 1) = \gamma (k + 1)(3\gamma \Delta - 1)$. Since the other bags do not change we have the desired tree-partition of $G$. □

3. General Lower Bound

The remainder of the paper studies lower bounds on the tree-partition-width. The graphs employed are chordal. We first show that tree-partitions of chordal graphs can be assumed to have certain useful properties.

**Lemma 3.** Every chordal graph $G$ has a tree-partition $T$ with width $\text{tpw}(G)$, such that for every independent set $S$ of simplicial\(^4\) vertices of $G$, and for every bag $B$ of $T$, either $B = \{v\}$ for some vertex $v \in S$, or the induced subgraph $G[B - S]$ is connected.

\(^4\)A vertex is simplicial if its neighbourhood is a clique.
Proof. Let $T_0$ be a tree-partition of a chordal graph $G$ with width $\text{tpw}(G)$. Let $T$ be the partition of $G$ obtained from $T_0$ by replacing each bag $B$ of $T_0$ by bags corresponding to the connected components of $G[B]$. Then $T$ has width at most $\text{tpw}(G)$.

To prove that $T$ is a forest, suppose on the contrary that $T$ contains an induced cycle $C$. Since each bag in $C$ induces a connected subgraph of $G$, $G$ contains an induced cycle $D$ with at least one vertex from each bag in $C$. Since $G$ is chordal, $D$ is a triangle. Thus $C$ is a triangle, implying that the vertices in $D$ were in distinct bags in $T_0$ (since the bags of $T$ that replaced each bag of $T_0$ form an independent set). Hence the bags of $T_0$ that contain $D$ induce a triangle in $T_0$, which is the desired contradiction since $T_0$ is a forest. Hence $T$ is a forest.

Let $S$ be an independent set of simplicial vertices of $G$. Consider a bag $B$ of $T$. By construction, $G[B]$ is connected. First suppose that $B \subseteq S$. Since $S$ is an independent set and $G[B]$ is connected, $B = \{v\}$ for some vertex $v \in S$.

Now assume that $B - S \neq \emptyset$. Suppose on the contrary that $G[B - S]$ is disconnected. Thus $B \cap S$ is a cut-set in $G[B]$. Let $v$ and $w$ be vertices in distinct components of $G[B - S]$ such that the distance between $v$ and $w$ in $G[B]$ is minimised. (This is well-defined since $G[B]$ is connected.) Since $S$ is an independent set, every shortest path between $v$ and $w$ in $G[B]$ has only two edges. That is, $v$ and $w$ have a common neighbour $x$ in $B \cap S$. Since $x$ is simplicial, $v$ and $w$ are adjacent. This contradiction proves that $G[B - S]$ is connected. $\square$

The next lemma is the key component of the proof of Theorem 2. For integers $a < b$, let $[a, b] := \{a, a + 1, \ldots, b\}$ and $[b] := [1, b]$.

**Lemma 4.** For all integers $k \geq 2$ and $\Delta \geq 3k + 1$, for infinitely many values of $N$ there is a chordal graph $G$ with $N$ vertices, tree-width $\text{tw}(G) = 2k - 1$, maximum degree $\Delta(G) \leq \Delta$, and tree-partition-width $\text{tpw}(G) > \frac{1}{2}k(\Delta - 3k)$.

**Proof.** Let $n$ be an integer with $n > \max\{\frac{1}{2}k(\Delta - 3k), 2\}$. Let $H$ be the graph with vertex set $\{(x, y) : x \in [n], y \in [k]\}$, where distinct vertices $(x_1, y_1)$ and $(x_2, y_2)$ are adjacent if and only if $|x_1 - x_2| \leq 1$. The set of vertices $\{(x, y) : y \in [k]\}$ is the $x$-column. The set of vertices $\{(x, y) : x \in [n]\}$ is the $y$-row. Observe that each column induces a $k$-vertex clique, and each row induces an $n$-vertex path.

Let $C$ be an induced cycle in $H$. If $(x, y)$ is a vertex in $C$ with $x$ minimum then the two neighbours of $(x, y)$ in $C$ are adjacent. Thus $C$ is a triangle. Hence $H$ is chordal. Observe that each pair of consecutive columns form a maximum clique of $2k$ vertices in $H$. Thus $H$ has tree-width $2k - 1$. Also note that $H$ has maximum degree $3k - 1$.

An edge of $H$ between vertices $(x, y)$ and $(x + 1, y)$ is horizontal. As illustrated in Figure 2 construct a graph $G$ from $H$ as follows. For each horizontal edge $vw$ of $H$, add $\left\lceil \frac{1}{2}(\Delta - 3k) \right\rceil$ new vertices, each adjacent to $v$ and $w$. Since $H$ is chordal and each new vertex is simplicial, $G$ is chordal. The addition of degree-2 vertices to $H$ does not increase the maximum clique size (since $k \geq 2$). Thus $G$ has clique number $2k$ and tree-width $2k - 1$. Since each vertex of $H$ is incident to at most two horizontal edges, $G$ has maximum degree $3k - 1 + 2\left\lceil \frac{1}{2}(\Delta - 3k) \right\rceil \leq \Delta$.

Observe that $V(G) - V(H)$ is an independent set of simplicial vertices in $G$. By Lemma 3 $G$ has a tree-partition $T$ with width $\text{tpw}(G)$, such that for every bag $B$ of $T$, either $B = \{v\}$
Figure 2. The graph $G$ with $k = 4$, $\Delta = 15$, and $n = 9$.

for some vertex $v$ of $G - H$, or the induced subgraph $H[B]$ is connected. Since $G$ is connected, $T$ is a (connected) tree. Let $U$ be the tree-partition of $H$ induced by $T$. That is, to obtain $U$ from $T$ delete the vertices of $G - H$ from each bag, and delete empty bags. Since $H$ is connected, $U$ is a (connected) tree. By Lemma 3, each bag of $U$ induces a connected subgraph of $H$.

Suppose that $U$ only has two bags $B$ and $C$. Then one of $B$ and $C$ contains at least $\frac{1}{2}nk$ vertices. Since $k \geq 2$, we have $\text{tpw}(G) \geq \frac{1}{2}nk > \frac{1}{4}k(\Delta - 3k)$, as desired. Now assume that $U$ has at least three bags.

Consider a bag $B$ of $U$. Let $\ell(B)$ be the minimum integer such that some vertex in $B$ is in the $\ell(B)$-column, and let $r(B)$ be the maximum integer such that some vertex in $B$ is in the $r(B)$-column. Since $H[B]$ is connected, there is a path in $B$ from the $\ell(B)$-column to the $r(B)$-column. By the definition of $H$, for each $x \in [\ell(B), r(B)]$, the $x$-column contains a vertex in $B$. Let $I(B)$ be the closed real interval from $\ell(B) - \frac{1}{2}$ to $r(B) + \frac{1}{2}$. Observe that two bags $B$ and $C$ of $U$ are adjacent if and only if $I(B) \cap I(C) \neq \emptyset$. Thus $\{I(B) : B \text{ is a bag of } U\}$ is an interval representation of the tree $U$. Every tree that is an interval graph is a caterpillar; see [16] for example. Thus $U$ is a caterpillar.

Let $\preceq$ be the relation on the set of non-leaf bags of $U$ defined by $A \preceq B$ if and only if $\ell(A) \leq \ell(B)$ and $r(A) \leq r(B)$. We claim that $\preceq$ is a total order. It is immediate that $\preceq$ is reflexive and transitive. To prove that $\preceq$ is antisymmetric, suppose on the contrary that $A \preceq B$ and $B \preceq A$ for distinct non-leaf bags $A$ and $B$. Thus $\ell(A) = \ell(B)$ and $r(A) = r(B)$.

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5A caterpillar is a tree such that deleting the leaves gives a path.
Since $U$ has at least three bags, there is a third bag $C$ that contains a vertex in the $(\ell(A) - 1)$-column or in the $(r(A) + 1)$-column. Thus $\{A, B, C\}$ induce a triangle in $U$, which is the desired contradiction. Hence $\leq$ is antisymmetric. To prove that $\leq$ is total, suppose on the contrary that $A \not\leq B$ and $B \not\leq A$ for distinct non-leaf bags $A$ and $B$. Now $A \not\leq B$ implies that $\ell(A) > \ell(B)$ or $r(A) > r(B)$. Without loss of generality, $\ell(A) > \ell(B)$. Thus $B \not\leq A$ implies that $r(B) > r(A)$. Hence the interval $[\ell(A), r(A)]$ is strictly within the interval $[\ell(B), r(B)]$ at both ends. For each $x \in [\ell(A), r(A)]$, every vertex in the $x$-column is in $A \cup B$, as otherwise $U$ would contain a triangle (since each column is a clique in $H$). Moreover, every vertex in the $(\ell(A) - 1)$-column or in the $(r(A) + 1)$-column is in $B$, as otherwise $U$ would contain a triangle (since the union of consecutive columns is a clique in $H$). Thus every neighbour of every vertex in $A$ is in $B$. That is, $A$ is a leaf in $U$. This contradiction proves that $\leq$ is a total order on the set of non-leaf bags of $U$.

Suppose that $U$ has a 4-vertex path $(A, B, C, D)$ as a subgraph.

Thus $B$ and $C$ are non-leaf bags. Without loss of generality, $B \prec C$. If every column contains vertices in both $B$ and $C$, then $B$ and $C$ and any other bag would induce a triangle in $U$ (since each column induces a clique in $H$). Thus some column contains a vertex in $B$ but no vertex in $C$, and some column contains a vertex in $C$ but no vertex in $B$. Let $p$ be the maximum integer such that some vertex in $B$ is in the $p$-column, but no vertex in $C$ is in the $p$-column. Let $q$ be the minimum integer such that some vertex in $C$ is in the $q$-column, but no vertex in $B$ is in the $q$-column. Now $p < q$ since $B \prec C$.

We claim that the $(p + 1)$-column contains a vertex in $C$. If not, then the $(p + 1)$-column contains no vertex in $B$ by the definition of $p$. Thus $r(B) = p$ since $H[B]$ is connected. Since $B$ is adjacent to $C$ in $U$, $\ell(C) \leq r(B) + 1 = p + 1$. In particular, the $(p + 1)$-column contains a vertex in $C$. Since $H[C]$ is connected, for $x \in [p + 1, q]$, each $x$-column contains a vertex in $C$. In fact, $\ell(C) = p + 1$ since the $p$-column contains no vertex in $C$. By symmetry, for $x \in [p, q - 1]$, each $x$-column contains a vertex in $B$, and $r(C) = q - 1$.

The union of the $p$-column and the $(p + 1)$-column only contains vertices in $B \cup C$, as otherwise $U$ would contain a triangle (since the union of two consecutive columns is a clique in $H$). By the definition of $p$, no vertex in the $p$-column is in $C$. Thus every vertex in the $p$-column is in $B$. By symmetry, every vertex in the $q$-column is in $C$. Now for each $y \in [k]$, the vertices $(p, y), (p + 1, y), \ldots, (q, y)$ are all in $B \cup C$, the first vertex $(p, y)$ is in $B$, and the last vertex $(q, y)$ is in $C$. Thus $(x, y) \in B$ and $(x + 1, y) \in C$ for some $x \in [p, q - 1]$. That is, in every row of $H$ there is a horizontal edge with one endpoint in $B$ and the other in $C$.

Thus there are at least $k$ horizontal edges with one endpoint in $B$ and the other in $C$ (now considered to be bags of $T$). For each such horizontal edge $vw$, each vertex of $G - H$ adjacent to $v$ and $w$ is in $B \cup C$, as otherwise $T$ would contain a triangle. There are $\left\lfloor \frac{1}{2}(\Delta - 3k) \right\rfloor$ such vertices of $G - H$ for each of the $k$ horizontal edges between $B$ and $C$. Thus $|B \cup C| \geq \frac{1}{2}k(\Delta - 3k)$. Thus one of $B$ and $C$ has at least $\frac{1}{2}k(\Delta - 3k)$ vertices. Hence $tpw(G) \geq \frac{1}{4}k(\Delta - 3k)$ as desired.

Now assume that $U$ has no 4-vertex path as a subgraph.

A tree is a star if and only if it has no 4-vertex path as a subgraph. Hence $U$ is a star. Let $R$ be the root bag of $U$. If $R$ contains a vertex in every column then $|R| \geq n$, implying
\[ \text{tpw}(G) \geq n \geq 1/4k(\Delta - 3k), \] as desired. Now assume that for some \( x \in [n] \), the \( x \)-column of \( H \) contains no vertex in \( R \). Let \( B \) be a bag containing some vertex in the \( x \)-column. The \( x \)-column induces a clique in \( H \), the only bag in \( U \) that is adjacent to \( B \) is \( R \), and \( R \) contains no vertex in the \( x \)-column. Thus every vertex in the \( x \)-column is in \( B \). Since \( R \) is the only bag in \( U \) adjacent to \( B \), there are at least \( k \) horizontal edges with one endpoint in \( B \) and the other endpoint in \( R \). As in the case when \( U \) contained a 4-vertex path, we conclude that \( \text{tpw}(G) \geq 1/4k(\Delta - 3k) \) as desired. \( \square \)

Proof of Theorem 2. Let \( \ell := \lceil k/2 \rceil \). Thus \( \ell \geq 2 \). By Lemma 4, for each integer \( \Delta \geq \Delta(k, \epsilon) := \max\{3\ell + 1, \frac{3\epsilon}{k}\} \), there are infinitely many values of \( N \) for which there is a chordal graph \( G \) with \( N \) vertices, tree-width \( \text{tw}(G) = 2\ell - 1 \leq k \), maximum degree \( \Delta(G) \leq \Delta \), and tree-partition-width \( \text{tpw}(G) > \frac{1}{4}\ell(\Delta - 3\ell) \), which is at least \((\frac{1}{8} - \epsilon)k\Delta \) since \( \Delta \geq \frac{3\epsilon}{k} \). \( \square \)

A domino tree decomposition is a tree decomposition in which each vertex appears in at most two bags. The domino tree-width of a graph \( G \), denoted by \( \text{dtw}(G) \), is the minimum width of a domino tree decomposition of \( G \). Domino tree-width behaves like tree-partition-width in the sense that \( \text{dtw}(G) \geq \text{tw}(G) \), and \( \text{dtw}(G) \) is bounded for graphs of bounded tree-width and bounded degree \( [7] \). The best upper bound is

\[ \text{dtw}(G) \leq (9\text{tw}(G) + 7) \Delta(G) (\Delta(G) + 1) - 1, \]

which is due to Bodlaender \([6]\), who also constructed a graph \( G \) with

\[ \text{dtw}(G) \geq \frac{1}{12} \text{tw}(G) \Delta(G) - 2. \]

Tree-partition-width and domino tree-width are related in that every graph \( G \) satisfies

\[ \text{dtw}(G) \geq \text{tpw}(G) - 1, \]

as observed by Bodlaender and Engelfriet \([7]\). Thus Theorem 2 provides examples of graphs \( G \) with

\[ \text{dtw}(G) \geq \left(\frac{1}{8} - \epsilon\right) \text{tw}(G) \Delta(G). \]

This represents a small constant-factor improvement over the above lower bound by Bodlaender \([6]\).

4. Lower Bound for Tree-width 2

We now prove a lower bound on the tree-partition-width of graphs with tree-width 2.

Theorem 3. For all odd \( \Delta \geq 11 \) there is a chordal graph \( G \) with tree-width 2, maximum degree \( \Delta \), and tree-partition-width \( \text{tpw}(G) \geq \frac{2}{3}(\Delta - 1) \).

Proof. As illustrated in Figure 3, let \( G \) be the graph with

\[ V(G) := \{r\} \cup \{v_i : i \in [\Delta]\} \cup \{w_{i,\ell} : i \in [\Delta - 1], \ell \in \left[\frac{1}{2} (\Delta - 3)\right]\} \]

and

\[ E(G) := \{rv_i : i \in [\Delta]\} \cup \{v_iv_{i+1} : i \in [\Delta - 1]\} \cup \{v_iw_{i,\ell}, v_{i+1}w_{i,\ell} : i \in [\Delta - 1], \ell \in \left[\frac{1}{2} (\Delta - 3)\right]\}. \]

\( \text{tpw}(G) \geq n \geq 1/4k(\Delta - 3k), \) as desired. Now assume that for some \( x \in [n] \), the \( x \)-column of \( H \) contains no vertex in \( R \). Let \( B \) be a bag containing some vertex in the \( x \)-column. The \( x \)-column induces a clique in \( H \), the only bag in \( U \) that is adjacent to \( B \) is \( R \), and \( R \) contains no vertex in the \( x \)-column. Thus every vertex in the \( x \)-column is in \( B \). Since \( R \) is the only bag in \( U \) adjacent to \( B \), there are at least \( k \) horizontal edges with one endpoint in \( B \) and the other endpoint in \( R \). As in the case when \( U \) contained a 4-vertex path, we conclude that \( \text{tpw}(G) \geq 1/4k(\Delta - 3k) \) as desired. \( \square \)

A domino tree decomposition is a tree decomposition in which each vertex appears in at most two bags. The domino tree-width of a graph \( G \), denoted by \( \text{dtw}(G) \), is the minimum width of a domino tree decomposition of \( G \). Domino tree-width behaves like tree-partition-width in the sense that \( \text{dtw}(G) \geq \text{tw}(G) \), and \( \text{dtw}(G) \) is bounded for graphs of bounded tree-width and bounded degree \([7]\). The best upper bound is

\[ \text{dtw}(G) \leq (9\text{tw}(G) + 7) \Delta(G) (\Delta(G) + 1) - 1, \]

which is due to Bodlaender \([6]\), who also constructed a graph \( G \) with

\[ \text{dtw}(G) \geq \frac{1}{12} \text{tw}(G) \Delta(G) - 2. \]

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Observe that $G$ has maximum degree $\Delta$. Clearly every induced cycle of $G$ is a triangle. Thus $G$ is chordal. Observe that $G$ has no 4-vertex clique. Thus $G$ has tree-width 2.

![Figure 3. Illustration for Theorem 3 with $\Delta = 13$.](image)

Let $T$ be the tree-partition of $G$ from Lemma 3. Then $T$ has width $\text{tpw}(G)$, and every bag induces a connected subgraph of $G$. Let $R$ be the bag containing $r$. Let $B_1, \ldots, B_d$ be the bags, not including $R$, that contain some vertex $v_i$. Thus $R$ is adjacent to each $B_j$ (since $r$ is adjacent to each $v_i$). Since $\{w_{i,\ell} : i \in [\Delta - 1], \ell \in [\frac{1}{2}(\Delta - 3)]\}$ is an independent set of simplicial vertices, by Lemma 3 for each $j \in [d]$, the vertices $\{v_1, v_2, \ldots, v_\Delta\} \cap B_j$ induce a (connected) subpath of $G$.

First suppose that $d = 0$. Then the $\Delta + 1$ vertices $\{r, v_1, \ldots, v_\Delta\}$ are contained in one bag $R$. Thus $\text{tpw}(G) \geq \Delta + 1 \geq \frac{2}{3}(\Delta - 1)$.

Now suppose that $d = 1$. Thus $\{r, v_1, \ldots, v_\Delta\} \subseteq R \cup B_1$. In addition, at least one edge $v_iv_{i+1}$ has one endpoint in $R$ and the other endpoint in $B_1$. Thus $w_{i,\ell} \in R \cup B_1$ for each $\ell \in [\frac{1}{2}(\Delta - 3)]$. Hence $1 + \Delta + \frac{1}{2}(\Delta - 3)$ vertices are contained in two bags. Thus one bag contains at least $\frac{1}{3}(3\Delta - 1)$ vertices, and $\text{tpw}(G) \geq \frac{1}{4}(3\Delta - 1) \geq \frac{2}{3}(\Delta - 1)$.

Finally suppose that $d \geq 2$. Since $\{v_1, v_2, \ldots, v_\Delta\} \cap B_j$ induce a subpath in each bag $B_j$, we can assume that $\{v_1, v_2, \ldots, v_\Delta\} \cap B_j = \{v_i : i \in [f(j), g(j)]\}$, where

$$1 \leq f(1) \leq g(1) < f(2) \leq g(2) < \cdots < f(d) \leq g(d) \leq \Delta.$$ 

Distinct $B_j$ bags are not adjacent (since $T$ is a tree). Thus $v_{f(j)-1} \in R$ for each $j \in [2, d]$. Similarly, $v_{g(j)+1} \in R$ for each $j \in [d - 1]$. Thus $w_{f(j)-1,\ell} \in R \cup B_j$ for each $j \in [2, d]$ and $\ell \in [\frac{1}{2}(\Delta - 3)]$. Similarly, $w_{g(j),\ell} \in R \cup B_j$ for each $j \in [d - 1]$ and $\ell \in [\frac{1}{2}(\Delta - 3)]$.

Hence the bags $R, B_1, \ldots, B_d$ contain at least

$$1 + \Delta + 2(d - 1) \cdot \frac{1}{2}(\Delta - 3)$$

vertices. Therefore one of these bags has at least

$$(1 + \Delta + (d - 1)(\Delta - 3))/(d + 1)$$

vertices, which is at least $\frac{2}{3}(\Delta - 1)$. Hence $\text{tpw}(G) \geq \frac{2}{3}(\Delta - 1)$. \qed

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Figure 4. Illustration for Theorem 3 with $\Delta = 19$ and $d = 4$.

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