On the Superstring Realization of the
Yang Monopole

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Abstract

Based on the result of string/string duality, we construct the six dimensional Yang monopole in terms of Type IIA wrapped D-branes. In particular, we show that all the information of such a magnetic solution can be encoded in the K3 surface compactification in the presence of D2 and D4-branes wrapping its non trivial cycles. We give a geometrical and physical interpretations for the \{+1, −1\} Yang monopole charges. Lifting to eleven dimensions, we relate this Type IIA configuration with the heterotic M-theory one, given in hep-th/0607193. The nature of the black Yang monopole is also discussed.

Keywords: Yang Monopole, Superstring theory, Dualities, M-theory, K3 surface and Black Hole.

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1 Introduction

The Yang monopole \cite{1} was constructed as a generalization of the Dirac monopole \cite{2}. The latter measures the flux of the magnetic two-form field $F$, charged under the $U(1)$ gauge group, integrated over a two-sphere that encloses the origin. The flux is quantized and the monopole can have an arbitrary integer magnetic charge. The charge is given by integrating the first Chern class of the gauge bundle $F/2\pi$, over the two-sphere $S^2$. This integer corresponds to the different ways the $U(1)$ connection can wrap the circle that constitutes the equator of the two-sphere. The origin is singular and near it an ultraviolet divergence occurs for the energy of the field configuration.

The Yang monopole is characterized by the flux of the four-form field $Tr F \wedge F$, charged under the $SU(2)$ gauge group, across the four dimensional sphere that covers the origin in a $5 + 1$ dimensional space-time. It corresponds to the conformal mapping onto $S^4$ of the BPST \cite{3} Euclidean instanton solution. Again, the origin is singular but now the energy of this solitonic configuration is well behaved in the UV regime, although IR divergences linearly appear. The total energy inside a four sphere is proportional to its radius. Also the flux is quantized but now, the magnetic charge of the Yang monopole can take only two values \{+1, −1\} \cite{1}. This charge, which may correspond to the self-dual and anti-self-dual BPST instanton configurations respectively, is given by the integral over $S^4$ of the second chern class $Tr(F \wedge F)/8\pi^2$.

The Yang monopole can be easily generalized to higher even dimensional space-time in the following way \cite{4, 5}. In $2n + 2$ dimensions we take the trace of the $2n$-form $F^n$, charged under the $SO(2n)$ gauge group. Then we integrate the flux of this form on the $2n$-sphere surrounding a singular point where the generalized monopole is located. Explicit solutions can be systematically obtained \cite{6}.

Recently, it has been suggested the possibility of considering the Yang Monopole as placed at the end of a string with an energy given by the product of the tension and the length of the string \cite{7}. This fact has opened a new line for studying the Yang monopole in string theory. In this regard, a heterotic M-theory realization has been proposed in \cite{8}. Recall that M-theory is an eleven dimensional theory containing M2-branes and its magnetic dual M5-branes. It also contains M9-branes supporting, each one, an $E_8$ gauge bundle of the heterotic superstring in ten dimensions \cite{9}. In this way, $E_8 \times E_8$ heterotic superstring can be interpreted in M-theory as a M2-brane suspended between two M9-branes \cite{10, 11}. It has been shown \cite{8} that M5-brane may have boundaries on M9-branes, where the boundary is a 4-brane with an infinite tension so its centre of mass

\footnote{Although for not use in our discussion, KK monopoles are involved, which can be identified after circle compactification with D6-branes in type IIA superstring theory.}
is not free to move. This boundary has been identified with the Yang monopole, which is a singular configuration of the $SU(2)$ Yang Mills gauge theory in six dimensions. In the heterotic M-theory picture, there are two Yang monopoles. They correspond to the ends of the oriented M5-brane which stretches between two M9-branes. Each monopole (each end) is charged under an $SU(2)$ subgroup of $E_8$ with the topological charges \{+1, −1\} respectively. On the other hand, a matrix model of the Yang monopole has been given in [12].

According to brane physics, $SU(2)$ Yang Mills in six dimensions can be embedded into Type II superstring theory in, at least, two different realizations which are connected by the T-duality transformations [13, 14, 15, 16]. For instance, $SU(2)$ gauge symmetry in Type IIB superstring can be identified with the gauge theory living in the word-volume of two coincide D5-branes moving on flat spaces. Alternatively, $SU(2)$ gauge group can be obtained from Type IIA superstring compactified on a singular K3 surface in the presence of D2-branes wrapped around the collapsing 2-cycles. This way of constructing a gauge theory from the singular limit of a Calabi-Yau compactification is known as the geometric engineering method [17]. At this level, one might naturally ask the following questions. Is there any realization of the Yang monopole in terms of Type II D-branes? and, in this case, what will be the relevant geometry which can explain its properties?

In this paper, we address these questions using the duality between Type IIA superstring compactified on the K3 surface and heterotic superstring on $T^4$. In particular, we will give a Type IIA geometric realization of the Yang monopole in six dimensions. First, we get the $SU(2)$ gauge group as an enhanced gauge symmetry corresponding to singular limit of the K3 surface, where the singularity arises from shrinking 2-cycles inside K3. Then we will show that the Yang monopole can come up as D-branes wrapping the K3 non-trivial cycles, in such a way that the above properties are encoded in the K3 surface features. We also give a K3 dual interpretation for the result reported in [8].

## 2 Type IIA superstring construction of the Yang monopole

Our starting setup is Type IIA superstring on a K3 surface. This compactification gives a six dimensions model and reduces the number of supercharges from thirty two to sixteen. The total moduli space of this theory is

$$\mathcal{M} = \mathcal{M}^{\text{stringy}} \times \mathbb{R}^+ = \frac{SO(20,4)}{SO(20) \times SO(4)} \times \mathbb{R}^+, \quad (2.1)$$

where $\mathbb{R}^+$ describes the choice of the superstring coupling which is a real parameter (the dilaton). $\mathcal{M}^{\text{stringy}}$ parameterizes the geometric moduli space of the $K3$ surface and the
values of the NS-NS $B$ fields on $K3$. Note that the R-R fields do not enter in this moduli space because the Betti numbers $b_1$ and $b_3$ of $K3$ are zero. The same physical moduli space appears naturally when heterotic superstring is compactified on $T^4$. The origin of the first factor in (2.1) comes from the values of the metric, $B$ field and non-abelian gauge vector over $T^4$ and the second factor is the dilaton field. More details on the six dimensional duality can be found in [18].

Based on this duality and the result of [8], we show that the Yang monopole in six dimensions can be embedded in Type IIA superstring compactified on a $K3$ surface in the presence of wrapped D-branes. For this purpose, let us take a local description of the $K3$ surface where the manifold develops a $su(2)$ singularity (known as $A_1$ singularity). This singularity corresponds to a vanishing two-sphere. Near such a singular point, the $K3$ surface can be identified with the asymptotically locally Euclidean (ALE) space which is algebraically given by

$$f(x, y, z) = xy - z^2 = 0,$$

where $(x, y, z)$ are complex variables. This geometry is singular at $x = y = z = 0$ since it is the only solution of $f = df = 0$. It has a nice physical representation as the target space of the two-dimensional $N = 2$ linear sigma model with only one $U(1)$ gauge symmetry and three chiral fields $\phi_i$, ($i = 1, 2, 3$) with vector charge $\vec{q} = (1, -2, 1)$. This vector satisfies the local Calabi-Yau condition

$$\sum_i q_i = 1 - 2 + 1 = 0.$$  

(2.3)

The coordinates of the ALE space in (2.2) can then be expressed in terms of the following $U(1)$ gauge invariants

$$x = \phi_1^2 \phi_2, \quad y = \phi_3^2 \phi_2, \quad z = \phi_1 \phi_2 \phi_3.$$  

(2.4)

Equation (2.2) is related to the D-term in the bosonic potential $V(\phi_1, \phi_2, \phi_3)$ in supersymmetric theories with four supercharges:

$$V(\phi_1, \phi_2, \phi_3) = (|\phi_1|^2 - 2|\phi_2|^2 + |\phi_3|^2 - R)^2,$$

(2.5)

where $R$ is the $U(1)$ Fayet-Iliopoulos (FI) parameter. The presence of this FI term resolves the singularity of the potential $V(\phi_1, \phi_2, \phi_3)$. Geometrically, this corresponds to replacing the singular point $x = y = z = 0$ by the 2-sphere defined by

$$S^2 : \quad |\phi_1|^2 + |\phi_3|^2 = R.$$  

(2.6)

For the ALE geometry $A_n$ the gauge group is $U(1)^n$. In this way, one sees that the $U(1)$ Cartan subgroup of the $SU(2)$ symmetry of the singularity of $K3$ carries the gauge symmetry of the $N = 2$ supersymmetric linear sigma model.
which is the only non-trivial 2-cycle on which we can wrap D2-branes. Note that $\phi_2$ defines the non compact direction of the $A_1$ ALE space. The theory is ten dimensional Type IIA superstring on a K3 surface with an infinite volume. To get the SU(2) gauge symmetry only the local piece containing the $S^2$ is needed. Now, the system consists of Type IIA D2-branes wrapping around $S^2$. This gives a pair of massive vectors $W^\pm$, one for each of the two possible ways of the wrapping. The masses of these particles are proportional to the volume of the 2-sphere. They are charged under the $U(1)$ gauge field obtained by decomposing the type IIA three form in terms of the harmonic form on the 2-sphere and the one form gauge field in the K3 transverse six dimensional space-time. In the limit where the 2-sphere shrinks, the $W^\pm$ particles become massless and, together with the one form gauge field, generate the $SU(2)$ adjoint representation. A geometric realization of $SU(2)$ gauge symmetry in six dimensions is then obtained \[17\]. This will be identified with the gauge symmetry of our Yang monopole.

We have obtained the electrically charged sector, associated to D2-branes wrapping 2-cycles in a K3 surface. Lifting consistently to 11 dimensions, the M2-brane is encountered. It is responsible of the electric particles in the compactification of M-theory. The dual M5-brane, when reduced to ten dimensions, gives rise to the D4-brane which is the magnetically charged object in Type IIA superstring. The D4-brane is in essence, the responsible for the magnetically charged sector in six dimensions as shown below. Based on the result of M-theory/superstring dualities in seven and six dimensions, the magnetic Yang monopole can be identified with D4-branes, totally wrapped on the K3 surface. As consequence, they generate the magnetic objects in the six dimensional space-time. This is expected from the fact that the D4-brane is the only magnetic object in Type IIA superstring theory which can be obtained from the M5-brane and gives a zero dimensional particle after wrapping the K3 surface. In this way, we conjecture that all Yang monopole properties should be derived from the K3 surface data. Since the gauge group origin is linked to the singular limit of the geometry, we expect that the magnetic properties can also be encoded in the K3 surface.

We will show that the charges $\{+1, -1\}$ can have different compatible K3 surface interpretations. First, the different ways in which D4-branes are wrapped on K3 surfaces are classified by the fourth homotopy group of K3. As seen before, in order to construct the $SU(2)$ gauge group, it is necessary to work with a local K3 with a singularity $A_1$. The deformed geometry is given by the product of the complex C plane and a two sphere $S^2$. Since $\Pi_q(X \times Y) = \Pi_q(X) \times \Pi_q(Y)$, we have the following remarkable relation

$$\pi_4(A_1) \sim \pi_4(S^2) = Z_2. \quad (2.7)$$

\[4\] The scenario of type IIA superstring theory in six dimensions is lifted to seven dimensions. The D2-branes are replaced by M2-branes, and we obtain an $N = 2$ gauge theory in seven dimensions.
The two charges of the Yang monopole are related to the two ways the geometry allows a D4-brane to wrap on it.

Another possible interpretation takes into account the fact that a locally K3 surface (2.2) has a $\mathbb{Z}_2$ symmetry acting as follows

$$x \rightarrow wx, \quad y \rightarrow \bar{w}y, \quad z \rightarrow wz,$$

(2.8)

where $w^2 = 1$. Wrapping D4-branes over such a geometry gives two configurations. Each one corresponds to an equivalent class of the $\mathbb{Z}_2$ group, which can be identified with the center of the $SU(2)$ gauge symmetry in six dimensions. The wrapped D4-brane over the K3 surface is sensitive to this $\mathbb{Z}_2$ symmetry and, thus, it carries $\{+1, -1\}$ charges. A general value of the charge, which corresponds to a non spherical solution, can be obtained by wrapping an arbitrary number of D4-branes on K3. These solutions are the multistanton configuration on the $S^4$.

It is known that the energy of the Yang monopole diverges linearly in space-time. This fact is not manifest in our geometric construction. We believe that this property can be understood in the effective field theory description.

3 Relation with the heterotic M-theory Yang monopole configuration

As explained in the introduction, the authors of [8] have suggested a Yang monopole representation with two $SU(2)$ gauge factors obtained by breaking the $E_8 \times E_8$ heterotic gauge symmetry in ten dimensions. This breaking, $E_8 \rightarrow E_7 \times SU(2)$, could be related with the fact that the extremes of the M5-branes are located on the M9-branes and they are just the core of the Yang monopole. On the Type IIA side however there is only one $SU(2)$ factor which comes from a D2-brane wrapped around the collapsing $S^2$ inside the K3 surface. Lifting to M-theory the nature of this difference is appreciated. Reduction from 11 to 6 dimensions with sixteen supercharges can be performed in two dual ways depending on the action of the $\mathbb{Z}_2$ symmetry on the five dimensional internal space $S^1 \times T^4$. In the heterotic realization of M-theory, the $\mathbb{Z}_2$ symmetry acts on the $S^1$ factor giving rise the segment between the two M9-branes, while in the type IIA, M-theory, the symmetry acts on the $T^4$ factor producing the $K3$ geometry. However, since these two M-theory compactifications are dual in six dimensions, the two above string Yang monopole realizations should be connected. In what follows, we will argue on how they could be related.

The K3 surface has two possible constructions as the target space of a sigma model. They depend on the R-symmetry of the supercharges. Previously, we have mainly con-
cerned with $N = 2$ sigma model, where the R-symmetry is supported by a $U(1)$ group, and the $K3$ target space gets manifested as a Khaler manifold. Now, let us use the other realization of $K3$ where the manifold is hyperkhaler and the corresponding sigma model involves eight supercharges and has a $SU(2)$ R-symmetry. Then, the $\{+1, -1\}$ charges of the Yang monopole can be explained by physical arguments when the K3 surface is constructed in terms of $N = 4$ sigma model. This is related to the heterotic M-theory configuration where the Yang monopole has two copies in the boundaries of the M5-brane suspended between two M9-branes. These copies, with charges +1 and −1, might be understood as two hypermultiplets appearing in the hyperkhaler quotient construction of the $A_1$ local manifestation of $K3$.

The six dimensional $SU(2)$ Yang Mills theory can also be obtained from a K3 surface that is realized in terms of $N = 4$ supersymmetric sigma model. This sigma model has only one $U(1)$ gauge group, two hypermultiplets with charges $(q_1, q_2)$, and one isotriplet FI coupling $\vec{\xi} = (\xi_1, \xi_2, \xi_3)$ \cite{20, 21}. The sigma model gauge symmetry is related to the Cartan subgroup of the six dimensional gauge group. In this construction, the K3 surface is expressed by the vanishing condition of the following D-terms

$$
\sum_{i=1}^{2} q_i (\phi_i^\alpha \bar{\phi}_i^\beta + \phi_i^\beta \bar{\phi}_i^\alpha) - \vec{\xi} \sigma^\alpha = 0.
$$

(3.1)

The double index $(i, \alpha)$ of the scalars refers to the component field doublets $(\alpha)$ of the two hypermultiplets $(i)$, and $\sigma$ are the traceless $2 \times 2$ Pauli matrices. The condition under which the gauge theory flows in the infrared to 2d $N = 4$ superconformal field theory, which is also the condition to have a hyperkhaler Calabi-Yau background, is

$$
q_1 + q_2 = 0.
$$

(3.2)

This equation has different solutions that can be seen as redefinitions of the coupling constant $\vec{\xi}$. Due to its conformal invariance, the theory does not get affected by redefinitions of $\vec{\xi}$, so the charge can be fixed to −1 and +1.

Let us discuss the construction of $K3$ in this case. The starting point consists of two hypermultiplets with four scalars each. They can be expressed as $\mathbb{R}^4 \times \mathbb{R}^4$. The gauge invariance of each hypermultiplet (with +1 and −1 charge) under the $U(1)$ symmetry, together with the invariance under the $SU(2)$ R-symmetry that rotates the supercharges, enables us to express the $K3$ locally as the following homogenous space

$$
\frac{\mathbb{R}^4 \times \mathbb{R}^4}{U(1) \times SU(2)}.
$$

(3.3)

There is a $Z_2$ symmetry that interchanges the two hypermultiplets (the two $\mathbb{R}^4$ factors). We interpret the two $\mathbb{R}^4$ factors with their corresponding charges as the two
Yang monopole copies which are the boundaries of the M5-branes on the two M9-branes. At this stage, we can ask the following question, which is the role of the $SU(2)$ $R$-symmetry in this construction? The answer of this question lies on the association of the $R$-symmetry with the instantonic nature of the M5-brane in the context of the heterotic M-theory picture.

4 Discussion

In this paper, a Type IIA geometric realization of the Yang monopole in six dimensions is given. For this purpose, it has been used the result of the duality between Type IIA superstring compactified on the K3 surface and heterotic superstring on $T^4$. The $SU(2)$ gauge symmetry of the Yang monopole has been considered as the enhanced gauge symmetry corresponding to shrinking 2-cycles inside the K3 surface. We have shown that the Yang monopole comes up by wrapping D-branes on the K3 non-trivial cycles. In this way, the properties of the Yang monopole have been encoded in the K3 surface features. Then we have analyzed how this realization can be related to the result of [8] in which a heterotic M-theory picture is involved.

The Yang monopole interpretation given in this paper opens some interesting questions in the connection with the type IIA superstring black hole physics in six dimensions. In this case, the black hole, the black string and the black membrane appear in the same footing. This is a consequence of the fact that the physical dimensions of two dual configurations are related by $p + q = D - 4 = 2$, where $p$ and $q$ are the spatial dimensions of the branes and $D = 6$ is the dimension of the space-time where they live. It should be interesting to connect our D-brane Yang monopole construction with the six dimensional black solution of Einstein equations. This will be reported elsewhere.

When the gravitational interaction is taken into account, the Yang monopole curves the space-time which develops event horizon. The geometry is similar to the one of the Reiner-Nostrom black hole, and it has been also obtained in the presence of a cosmological constant[6]. In [22], this geometry has been analyzed for the Nariai type solutions when the cosmological and black hole horizons coalesce. Extremal geometries can be candidates for BPS type solutions where the attractor mechanism could apply. We hope the results of this paper can be used to address this point.

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