ABSTRACT. We describe a general procedure for constructing new Sasaki metrics of constant scalar curvature from old ones. Explicitly, we begin with a regular Sasaki metric of constant scalar curvature on a $2n + 1$-dimensional compact manifold $M$ and construct a sequence, depending on four integer parameters, of rays of constant scalar curvature (CSC) Sasaki metrics on a compact Sasaki manifold $M_{l_1,l_2,w}$ of dimension $2n + 3$. We also give examples which show that the CSC rays are often not unique on a fixed strictly pseudoconvex CR manifold or a fixed contact manifold.

1. Introduction

The purpose of this paper is to present a general geometric construction that combines the Sasaki join construction of [BG00, BGO07] with the Hamiltonian 2-form formalism of [ACG06, ACGTF04, ACGTF08] to construct many new Sasaki metrics of constant scalar curvature. This method has already been used by the authors in special cases [BTF13a, BTF13c, BTF14a, BTF13b]. The method is the following: consider a regular Sasaki manifold $M$ with its ‘Boothby-Wang circle bundle’ $S^1 \to M \to N$ over the Kähler manifold $N$. For each pair of relatively prime positive integers $(l_1, l_2)$ we form the Sasaki join $M_{l_1,l_2,w}$ of $M$ with a weighted 3-sphere $S^3_w$ (cf. [BG08], Example 7.1.12), where the components of the weight vector $w = (w_1, w_2)$ are relatively prime integers satisfying $w_1 \geq w_2$. The latter has a 2-dimensional Sasaki cone $t^+_w$ we call the $w$-Sasaki cone. Now we can deform within $t^+_w$ to obtain other Sasakian structures. The quasi-regular ones will fiber over a ruled orbifold $(S_n, \Delta)$ with the following structure. $S_n$ is a $\mathbb{CP}^1$-bundle over $N$ with an orbifold structure on its fibers giving rise to a branch divisor $\Delta$. The orbifold $(S_n, \Delta)$ is a projectivization...
\( P(1 \oplus L_n) \) where \( L_n \) is certain line bundle over \( N \), and it admits a Hamiltonian 2-form. The explicit nature of this formalism allows us to obtain extremal (or constant scalar curvature) Kähler orbifold metrics on \((S_n, \Delta)\). Then by a well known procedure we obtain extremal (constant scalar curvature) Sasaki metrics on the join \( M_{l_1,l_2,w} \). This approach was initiated in [BTF14a] for the case \( l_2 = 1 \) (the case of arbitrary \( l_2 \) is being worked out in [Ch14]), and continued in [BTF13b] for Sasaki-Einstein metrics.

Our main theorem is:

**Theorem 1.1.** Let \( M_{l_1,l_2,w} = M \ast_{l_1,l_2} S^3_w \) be the \( S^3_w \)-join with a regular Sasaki manifold \( M \) which is an \( S^1 \)-bundle over a compact Kähler manifold \( N \) with constant scalar curvature. Then for each vector \( w = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \) with relatively prime components satisfying \( w_1 > w_2 \) there exists a Reeb vector field \( \xi_v \) in the 2-dimensional \( w \)-Sasaki cone on \( M_{l_1,l_2,w} \) such that the corresponding ray of Sasakian structures \( S_a = (a^{-1} \xi_v, a\eta_v, \Phi, g_a) \) has constant scalar curvature.

Suppose in addition that the scalar curvature of \( N \) satisfies \( s_N \geq 0 \), then we obtain more information about extremal Sasaki metrics. In fact, we have

**Theorem 1.2.** Suppose that in addition to the hypothesis of Theorem 1.1 the scalar curvature of \( N \) satisfies \( s_N \geq 0 \), then the \( w \)-Sasaki cone is exhausted by extremal Sasaki metrics. In particular, if the Kähler structure on \( N \) admits no Hamiltonian vector fields, then the entire Sasaki cone \( \kappa \) of the join \( M_{l_1,l_2,w} = M \ast_{l_1,l_2} S^3_w \) can be represented by extremal Sasaki metrics.

A particular example of interest when the hypothesis of the last sentence of Theorem 1.2 is satisfied is when \( N \) is an algebraic K3 surface. In this case there are many choices of complex structures and many choices of line bundles. But in all cases \( M = 21 \#(S^2 \times S^3) \). It is interesting to contemplate the possible diffeomorphism types of the 7-manifolds \( 21 \#(S^2 \times S^3) \ast_{l_1,l_2} S^3_w \) in this case.

We also give examples where there are more than one CSC ray in the \( w \)-Sasaki cone. Indeed, generally we have

**Theorem 1.3.** Suppose that in addition to the hypothesis of Theorem 1.1 the scalar curvature of \( N \) satisfies \( s_N > 0 \). Then for sufficiently large \( l_2 \) there are at least three CSC rays in the \( w \)-Sasaki cone of the join \( M_{l_1,l_2,w} \).

In particular, Theorem 5.6 below gives a countable infinity of inequivalent contact structures on the two \( S^5 \)-bundles over \( S^2 \) such that there
are at least three CSC rays of Sasaki metrics. However, the bouquet phenomenon which is related to distinct underlying CR structures and appears for $S^3$-bundles over Riemann surfaces \cite{Boy11, Boy13, BTF13a, BTF14a} seems not to occur in these more general cases. This is related to the topological rigidity of the Boothby-Wang base space as discussed briefly in Section 2.2 below. The non-uniqueness described in Theorem 1.3 occurs on a fixed strictly pseudoconvex CR manifold and a fixed contact manifold. The former also illustrates non-uniqueness on the sub-Riemannian level.

It should be mentioned that generally the CSC rays are most often irregular, that is the closure of a generic Reeb orbit is a torus of dimension greater than one. It has been shown recently that irregular Sasakian structures have irreducible transverse holonomy \cite{HS12}, and that the corresponding Kähler cone is K-semistable \cite{CS12}. Also the non-uniqueness phenomenon of CSC Sasakian structures was first shown to occur for the case of $S^3$-bundles over $S^2$ by a different method in \cite{Leg11}. Theorem 1.3 shows that this is fairly common.

2. Ruled Manifolds

In this section we consider ruled manifolds of the following form. Let $(N, \omega_N)$ be a compact Kähler manifold with primitive integer Kähler class $[\omega_N]$, that is, a Hodge manifold. Consider a rank two complex vector bundle of the form $E = \mathbb{L} \oplus L$ where $L$ is a complex line bundle on $N$ and $\mathbb{L}$ denotes the trivial bundle. By a ruled manifold we shall mean the projectivization $S = \mathbb{P}(\mathbb{L} \oplus L)$. We can view $S$ as a compactification of the complex line bundle $L$ on $N$ by adding the ‘section at infinity’. For $x \in N$ we let $(c, z)$ denote a point of the fiber $E_x = \mathbb{L} \oplus L_x$. There is a natural action of $\mathbb{C}^*$ (hence, $S^1$) on $E$ given by $(c, z) \mapsto (c, \lambda z)$ with $\lambda \in \mathbb{C}^*$. The action $z \mapsto \lambda z$ is a complex irreducible representation of $\mathbb{C}^*$ determined by the line bundle $L$. Such representations (characters) are labeled by the integers $\mathbb{Z}$. Thus, we write $L = L_n$ for $n \in \mathbb{Z}$ and refer to $n$ as the ‘degree’ of $L$.

2.1. A Construction of Ruled Manifolds. We now give a construction of such manifolds. Let $S^1 \to M \to N$ be the circle bundle over $N$ determined by the class $[\omega_N] \in H^2(N, \mathbb{Z})$. We denote the $S^1$-action by $(x, u) \mapsto (x, e^{i\theta} u)$. Now represent $S^3 \subset \mathbb{C}^2$ as $|z_1|^2 + |z_2|^2 = 1$ and consider an $S^1$-action on $M \times S^3$ given by $(x, u; z_1, z_2) \mapsto (x, e^{i\theta} u; e^{i\mu \theta} z_1, e^{i\mu \theta} z_2)$. There is also the standard $S^1$-action on $S^3$ given by $(z_1, z_2) \mapsto (e^{i\chi} z_1, e^{i\chi} z_2)$ giving a $T^2$-action on $M \times S^3$ defined by

\begin{equation}
(x, u; z_1, z_2) \mapsto (x, e^{i\theta} u; e^{i\chi} z_1, e^{i(\chi + n\theta)} z_2).
\end{equation}
Lemma 2.1. The quotient by the $T^2$-action of Equation (1) is the projectivization $S_n = \mathbb{P}(\mathbb{L} \oplus L_n)$.

Proof. First we see from (1) that the action is free, so there is a natural bundle projection $(M \times S^3)/T^2 \rightarrow N$ defined by $\pi(x, [u; z_1, z_2]) = x$ where the bracket denotes the $T^2$ equivalence class. The fiber is $\pi^{-1}(x) = [u; z_1, z_2]$ which since $u$ parameterizes a circle is identified with $S^3/S^1 = \mathbb{C}P^1$. This bundle is trivial if and only if $n = 0$ and $n$ labels the irreducible representation of $S^1$ on the line bundle $L_n$. □

We can take the line bundle $L_1$ to be any primitive line bundle in Pic$(N)$. In particular, we are interested in the taking $L_1$ to be the line bundle associated to the primitive cohomology class $[\omega_N] \in H^2(N, \mathbb{Z})$. Then we have

Lemma 2.2. The following relation holds: $c_1(L_n) = n[\omega_N]$.

Proof. Equation (1) implies that the $S^1$-action on the line bundle $L_n$ is given by $z \mapsto e^{in\theta}z$. But we know that the definition of $M$ that it is the unit sphere in the line bundle over $N$ corresponding to $n = 1$, and this corresponds to the class $[\omega_N]$, that is $c_1(L_1) = [\omega_N]$. Thus, $c_1(L_n) = n[\omega_N]$. □

2.2. Ruled Manifolds with known Diffeomorphism Type. There are several cases when the diffeomorphism type of the ruled manifold can be ascertained. First we have the case when $N = \Sigma_g$ a Riemann surface of genus $g$. It is well known [MS98] that in this case there are precisely two diffeomorphism types. They are distinguished by their second Stiefel-Whitney class. This gives rise to inequivalent Kähler structures belonging to the same underlying symplectic structure (up to symplectomorphism). It also gives rise to non-conjugate maximal tori in the symplectomorphism group, a fact that was exploited in [Boy11, Boy13, BP14, BTF13a, BTF14a].

On the other hand it appears that this phenomenon changes in higher dimension. It is still known to occur as witnessed by the polygon spaces of [HT03] and described in Example 8.5 of [Boy13]. However, it has been shown recently [CMS10, CPS12] that for $N = \mathbb{C}P^p$ with $p > 1$ the two ruled manifolds $S_n$ and $S_{n'}$ are diffeomorphic if and only if $|n'| = |n|$. Indeed, the diffeomorphism type is determined completely by its cohomology ring which takes the form

\[ H^*(S_n, \mathbb{Z}) = \mathbb{Z}[x_1, x_2]/(x_1^{p+1}, (x_2(nx_1 + x_2)) \]

where $x_1, x_2$ have degree 2. So the Hirzebruch-type phenomenon involving inequivalent complex structures on the same manifold does not generalize when $p > 1$. 
2.3. The Admissible Construction. We will now assume that \( n \) from Section 2.1 is non-zero and \((N, \omega_N)\) defines a Kähler structure with CSC Kähler metric \( g_N \). Then \((\omega_N, g_N) := (2n\pi \omega_N, 2n\pi g_N)\) satisfies that \((g_N, \omega_N)\) or \((-g_N, -\omega_N)\) is a Kähler structure (depending on the sign of \( n \)). In either case, we let \((\pm g_N, \pm \omega_N)\) refer to the Kähler structure. We denote the real dimension of \( N \) by \( 2d_N \) and write the scalar curvature of \( \pm g_N \) as \( \pm 2d_N s_N \). [So, if e.g. \(-g_N\) is a Kähler structure with positive scalar curvature, \( s_N \) would be negative.]

Now Lemma 2.2 implies that \( c_1(L_n) = [\omega_N / 2\pi] \). Then, following [ACGTF08], the total space of the projectivization \( S_n = \mathbb{P}(\mathbb{I} \oplus L_n) \) is called admissible.

On these manifolds, a particular type of Kähler metric on \( S_n \), also called admissible, can now be constructed [ACGTF08]. We shall describe this construction in Section 3 where we will use it to prove Theorems 1.1 and 1.2 of the Introduction. An admissible Kähler manifold is a special case of a Kähler manifold admitting a so-called Hamiltonian 2-form [ACG06]. More specifically, the admissible metrics as described in section 4 admit a Hamiltonian 2-form of order one.

Remark 2.3. In the special case where \((N, \omega_N)\) is Kähler-Einstein with Kähler metric \( g_N \) and Ricci form \( \rho_N = 2\pi I_N \omega_N \), where \( I_N \) denotes the Fano index, there is a simple relationship between the value of \( s_N \) and the value of \( n \). Since the (scale invariant) Ricci form is given by \( \rho_N = s_N \omega_N \), it is easy to see that \( s_N = I_N / n \). For the general CSC case this will be more complicated and will need to be handled case by case. We do know that if we write \( s_N = p_n / n \), then \( p_n \leq d_N + 1 \) (see Remark 1 in [ACGTF08]).

3. The \( S^3_w \)-Join Construction

The join construction was first introduced in [BG00] for Sasakian-Einstein manifolds, and later generalized to any quasi-regular Sasakian manifolds in [BG00b] (see also Section 7.6.2 of [BG08]). However, as pointed out in [BTF13a] it is actually a construction involving the orbifold Boothby-Wang construction [BW58, BG00], and so applies to quasi-regular strict contact structures. Although it is quite natural to do so, we do not need to fix the transverse (almost) complex structure. Moreover, in [BTF14a] it was shown that in the special case of \( S^3 \)-bundles over Riemann surfaces a twisted transverse complex structure on a regular Sasakian manifold can be realized by a product transverse complex structure on a certain quasi-regular Sasakian structure in the same Sasaki cone.
We consider a generalization of the join construction used in previous work [BTF13a, BTF14a, BTF13b]. We refer to [BGO07, BG08] for a thorough discussion of the join construction. Here we let $M$ be a regular Sasakian manifold with constant scalar curvature, and consider the join $M_{l_1, l_2, w} = M \star_{l_1, l_2} S^3_w$ with the weighted 3-sphere $S^3_w$ (a sphere with a weighted circle action) where both $w = (w_1, w_2)$ and $l = (l_1, l_2)$ are pairs of relatively prime positive integers. We can assume that the weights $(w_1, w_2)$ are ordered, namely they satisfy $w_1 \geq w_2$. Furthermore, $M_{l_1, l_2, w}$ is a smooth manifold if and only if \( \gcd(l_2, l_1 w_1 w_2) = 1 \) which is equivalent to \( \gcd(l_2, w_i) = 1 \) for $i = 1, 2$. Henceforth, we shall assume these conditions.

The join is constructed from the following commutative diagram

\[
\begin{array}{ccc}
M \times S^3_w & \xrightarrow{\pi_L} & \pi_2 \downarrow M_{l_1, l_2, w} \\
\downarrow \pi_2 & & \pi_1 \\
N \times \mathbb{C}P^1[w] & \xleftarrow{\pi_1} & \\
\end{array}
\]

where the $\pi$s are the obvious projections. Here $M$ has a regular contact form $\eta_1$ with Reeb vector field $\xi_1$, and $S^3_w$ has the weighted contact form $\eta_w$ with Reeb vector field $\xi_w = w_1 H_1 + w_2 H_2$ where $H_i$ is the infinitesimal generators of the $S^1$ action on $S^3 = \{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1 \}$

given by sending $z_i$ to $e^{i\theta} z_i$. The circle projection $\pi_L$ is generated by the vector field

\[
L_{l_1, l_2, w} = \frac{1}{2l_1} \xi_1 - \frac{1}{2l_2} \xi_w.
\]

Moreover, the 1-form $\eta_{l_1, l_2, w} = l_1 \eta_1 + l_2 \eta_w$ on $M \times S^3$ passes to the quotient $M_{l_1, l_2, w}$ and gives it a contact structure. The Reeb vector field of $\eta_{l_1, l_2, w}$ is the vector field

\[
\xi_{l_1, l_2, w} = \frac{1}{2l_1} \xi_1 + \frac{1}{2l_2} \xi_w.
\]

The base orbifold $N \times \mathbb{C}P^1[w]$ has a natural Kähler structure, namely the product structure, and this induces a Sasakian structure $S_{l_1, l_2, w} = (\xi_{l_1, l_2, w}, \eta_{l_1, l_2, w}, \Phi, g)$ on $M_{l_1, l_2, w}$. The transverse complex structure $J = \Phi|_{M_{l_1, l_2, w}}$ is the lift of the product complex structure on $N \times \mathbb{C}P^1[w]$.

It follows from Proposition 7.6.7 of [BG08] that the join $M_{l_1, l_2, w}$ can also be realized as a fiber bundle over $N$ associated to the principal $S^1$-bundle $M \to N$ with fiber the lens space $L(l_2; l_1 w_1, l_1 w_2)$. It is easy
to see that the join of extremal (CSC) Sasaki metrics gives an extremal (CSC) Sasaki metric induced by the product extremal (CSC) Kähler metrics. Thus, since weighted projective spaces have extremal orbifold metrics, we can take the Sasakian structure $S_{l_1, l_2, w}$ to be extremal. However, most of the CSC Sasaki metrics of interest in this work are not induced by the product of CSC Kähler metrics.

3.1. The First Chern Class. Let us compute the first Chern class of our induced contact structure $\mathcal{D}_{l_1, l_2, w}$ on $M \ast_{l_1, l_2} S^3_w$. The orbifold first Chern class of the base is

$$c_1^{orb}(N \times \mathbb{C}P^1[w]) = c_1(N) + \frac{w}{w_1 w_2} PD(D)$$

as an element of $H^2(N \times \mathbb{C}P^1[w], Q) \approx H^2(N, Q) \oplus H^2(\mathbb{C}P^1[w], Q)$ where $D$ a divisor given by $z_1 = 0$ or $z_2 = 0$. The Kähler form on $N \times \mathbb{C}P^1[w]$ is $\omega_{l_1, l_2} = l_1 \omega_N + l_2 \omega_w$ where $\omega_w$ is the standard Kähler form on $\mathbb{C}P^1[w]$ such that $p^* [\omega_w]$ is a positive generator in $H^2(\mathbb{C}P^1[w], \mathbb{Z})$ where $p : \mathbb{B}CP^1[w] \rightarrow \mathbb{C}P^1[w]$ is the natural orbifold classifying projection, that is, $\mathbb{B}CP^1[w]$ is the classifying space associated to any groupoid representing the orbifold $\mathbb{C}P^1[w]$ [HaeS]. Pulling $\omega_{l_1, l_2}$ back to the join $M_{l_1, l_2, w} = M \ast_{l_1, l_2} S^3_w$ we have $\pi^* \omega_{l_1, l_2} = d\eta_{l_1, l_2, w}$ implying that $l_1 \pi^* [\omega_N] + l_2 \pi^* [\omega_w] = 0$ in $H^2(M_{l_1, l_2, w}, \mathbb{Z})$. So taking $\pi^* [\omega_N] = l_2 \gamma$ and $\pi^* [\omega_w] = -l_1 \gamma$ for some generator $\gamma \in H^2(M_{l_1, l_2, w}, \mathbb{Z})$ we have

$$c_1(\mathcal{D}_{l_1, l_2, w}) = \pi^* c_1(N) - l_1 |w| \gamma.$$ 

Taking the mod 2 reduction gives the second Stiefel-Whitney class of $M_{l_1, l_2, w}$, viz.

$$w_2(M_{l_1, l_2, w}) = \pi^* w_2(N) - \rho(l_1 |w| \gamma)$$

where $\rho$ is the reduction mod 2 map. This implies

**Corollary 3.1.** If $l_1$ is even or if $w_i$ are both odd for $i = 1, 2$, then $M_{l_1, l_2, w}$ is a spin manifold if and only if $N$ is a spin manifold. On the other hand if both $l_1$ and $|w|$ are odd, then $M_{l_1, l_2, w}$ is a spin manifold if and only if $N$ is not a spin manifold.

Equation (7) reduces further in the special case that $[\omega_N]$ is monotone. Actually we are interested in a generalization. We say that $[\omega_N]$ is quasi-monotone if $c_1(N) = J_N[\omega_N]$ for some integer $J_N$. Here $J_N$ is the *Fano index* when $J_N$ is positive (the monotone case) and the *canonical index* when it is negative. We also allow the case $J_N = 0$. So when $[\omega_N]$ is quasi-monotone we have

$$c_1(\mathcal{D}_{l_1, l_2, w}) = (l_2 J_N - l_1 |w|) \gamma.$$
From the homotopy exact sequence of the fibration $S^1 \to M \to N$ we see that if $M$ is 2-connected then $N$ is quasi-monotone.

**Example 3.2.** Let $M = S^{2p+1}$ so we have the Hopf fibration with $N = \mathbb{CP}^p$ which is monotone with $\mathcal{J}_N = p + 1$. In this case Equation (9) becomes

$$c_1(\mathcal{D}_{l_1, l_2, w}) = (l_2(p + 1) - l_1|w|)\gamma. \quad (10)$$

The cohomology ring for this case was computed in [BTF13b], viz.

$$H^*(M_{l_1, l_2, w}, \mathbb{Z}) \approx \mathbb{Z}[x, y]/(w_1 w_2 l_1 x^2, x^{p+1}, x^2 y, y^2)$$

where $x, y$ are classes of degree 2 and $2p+1$, respectively. The topology of this case is studied further in [BTF14b], and more specific information about the Sasakian geometry is treated in Section 5.1 below.

### 3.2. The Sasaki Cone.

Since for any Sasakian structure $\mathcal{S}$ the Reeb vector field lies in the center of the Lie algebra $\mathfrak{aut}(\mathcal{S})$ of the Sasaki automorphism group $\mathfrak{Aut}(\mathcal{S})$, it follows from the join construction that the Lie algebra $\mathfrak{aut}(\mathcal{S}_{l_1, l_2, w})$ of the automorphism group of the join satisfies $\mathfrak{aut}(\mathcal{S}_{l_1, l_2, w}) = \mathfrak{aut}(\mathcal{S}_1) \oplus \mathfrak{aut}(\mathcal{S}_w)$ where $\mathcal{S}_1$ is the Sasakian structure on $M$, and $\mathcal{S}_w$ is the Sasakian structure on $S^3_w$. Now the unreduced Sasaki cone [BGS08] $\mathcal{T}^+$ of $\mathcal{S} = (\xi, \eta, \Phi, g)$ is by definition the positive cone in the Lie algebra $\mathfrak{t}$ of a maximal torus in $\mathfrak{Aut}(\mathcal{S})$, i.e.

$$\mathfrak{t}^+ = \{ X \in \mathfrak{t} \mid \eta(X) > 0 \}. \quad (11)$$

Thus, the Sasaki cone $\mathfrak{t}^+_{l_1, l_2, w}$ of the join $M_{l_1, l_2, w}$ satisfies

$$\mathfrak{t}^+_{l_1, l_2, w} = \{ X \in \mathfrak{t}_{l_1, l_2, w} \mid \eta_{l_1, l_2, w}(X) > 0 \} = \mathfrak{t}^+_1 + \mathfrak{t}^+_w. \quad (12)$$

If the Lie algebra of a maximal torus of the automorphism group of $\mathcal{S}_1$ has dimension $k$, then $\dim \mathfrak{t}^+_1 = k + 2$, since the $\mathfrak{t}_w$ has dimension 2. However, in this paper we are mainly concerned with the 2-dimensional subcone $\mathfrak{t}_w^+$, which we call the $w$-Sasaki cone, of the full Sasaki cone $\mathfrak{t}^+_{l_1, l_2, w}$. We are also interested in the full reduced Sasaki cone $\mathcal{K}$ which is $\mathfrak{t}^+/\mathcal{W}$ where $\mathcal{W}$ is the Weyl group of the Sasaki automorphism group $\mathfrak{Aut}(\mathcal{S})$. One can think of $\mathcal{K}$ as the moduli space of Sasakian structures with a fixed underlying CR structure $(\mathcal{D}, \mathcal{J})$.

### 3.3. The Tori Actions.

Consider the action of the 3-dimensional torus $T^3$ on the product $M \times S^3_w$ defined by

$$(x, u; z_1, z_2) \mapsto (x, e^{il_2 \theta} u; e^{i(\phi_1 - l_1 w_1 \theta)} z_1, e^{i(\phi_2 - l_1 w_2 \theta)} z_2). \quad (13)$$

The Lie algebra $\mathfrak{t}_3$ of $T^3$ is generated by the vector fields $L_{l_1, l_2, w}, H_1, H_2$. 

Following the analysis of Section 3.3 of [BTF13b] we see that our join $M_{l_1,l_2,w}$ is the fiber bundle

$$M_{l_1,l_2,w} = M \times S^1 \times L(l_2; l_1 w_1, l_1 w_2)$$

over the Kähler manifold $N$ associated to the principal $S^1$-bundle $M \rightarrow N$ with fiber the lens space $L(l_2; l_1 w_1, l_1 w_2)$ over the Kähler manifold $N$. The $S^1$ action on the lens space is accomplished in two stages. First, represent $L(l_2; l_1 w_1, l_1 w_2)$ as a $\mathbb{Z}_{l_2}$ quotient of $S^3_w$, then the residual $S^1/\mathbb{Z}_{l_2} \approx S^1$ action is

$$\begin{align*}
(x, u; z_1, z_2) & \mapsto (x, e^{i\theta} u; [e^{-i\frac{l_1 w_1}{l_2} \theta} z_1, e^{-i\frac{l_1 w_2}{l_2} \theta} z_2]).
\end{align*}$$

The brackets in Equation (14) denote the equivalence class defined by $(z'_1, z'_2) \sim (z_1, z_2)$ if $(z'_1, z'_2) = (\lambda^{l_1 w_1} z_1, \lambda^{l_1 w_2} z_2)$ for $\lambda^{l_2} = 1$.

Next consider the $T^2$ action of $S^1_\phi \times (S^1/\mathbb{Z}_{l_2})$ on $M \times L(l_2; l_1 w_1, l_1 w_2)$ given by

$$\begin{align*}
(x, u; z_1, z_2) & \mapsto (x, e^{i\theta} u; [e^{i(u_1 \phi - \frac{l_1 w_1}{l_2} \theta)} z_1, e^{i(u_2 \phi - \frac{l_1 w_2}{l_2} \theta)} z_2]),
\end{align*}$$

This gives rise to the commutative diagram

$$\begin{align*}
\begin{array}{ccc}
M \times L(l_2; l_1 w_1, l_1 w_2) & \xrightarrow{\pi_L} & M_{l_1,l_2,w} \\
\downarrow \pi_B & & \downarrow \pi_v \\
B_{l_1,l_2,v,w} & \xleftarrow{\pi_v} & M_{l_1,l_2,w}
\end{array}
\end{align*}$$

where $B_{l_1,l_2,v,w}$ is a bundle over $N$ with fiber a weighted projective space, and $\pi_B$ denotes the quotient projection by $T^2$. The Lie algebra of this $T^2$ is generated by

$$L_w = \frac{1}{2l_1} \xi_M - \sum_{j=0}^{q} \frac{1}{2l_2} w_j H_j, \quad \xi_v = \sum_j v_j H_j,$$

where $\xi_M$ denotes the Reeb vector field of the regular Sasakian structure on $M$. Note that $\xi_v$ is a Reeb vector field in the $w$-Sasaki cone of $M_{l_1,l_2,w}$.

Let us analyze the behavior of the $T^2$ action given by Equation (15). We shall see that it it not generally effective. First we notice that the $S^1_\phi$ action is free since it is free on the first factor. Next we look for fixed points under a subgroup of the circle $S^1_\phi$. Thus, we impose

$$\begin{align*}
(e^{i u_1 \phi} z_1, e^{i u_2 \phi} z_2) & = (e^{-2\pi \frac{l_1 w_1}{l_2} ri} z_1, e^{-2\pi \frac{l_1 w_2}{l_2} ri} z_2)
\end{align*}$$
for some $r = 0, \ldots, l_2 - 1$. If $z_1 z_2 \neq 0$ we must have

$$v_1 \phi = 2\pi (-\frac{l_1 w_1 r}{l_2} + k_1), \quad v_2 \phi = 2\pi (-\frac{l_1 w_2 r}{l_2} + k_2)$$

for some integers $k_1, k_2$ which in turn implies

$$l_1 r (w_2 v_1 - w_1 v_2) = l_2 (k_2 v_1 - k_1 v_2).$$

This gives

$$r = \frac{l_2 k_2 v_1 - k_1 v_2}{l_1 w_2 v_1 - w_1 v_2}$$

which must be a nonnegative integer less than $l_2$. We can also solve Equations (18) for $\phi$ by eliminating $\frac{lr}{l_2}$ giving

$$\phi = 2\pi \frac{k_1 w_2 - k_2 w_1}{w_2 v_1 - w_1 v_2}.$$  

Next we write (19) as

$$r = \left(\frac{l_2}{\gcd(|w_2 v_1 - w_1 v_2|, l_2)}\right) \left(\frac{k_2 v_1 - k_1 v_2}{l_1 \gcd(|w_2 v_1 - w_1 v_2|, l_2)}\right).$$

Since $v_1$ and $v_2$ are relatively prime, we can choose $k_1$ and $k_2$ so that the term in the last parentheses is 1. This determines $r$ as

$$r = \frac{l_2}{\gcd(|w_2 v_1 - w_1 v_2|, l_2)}$$

Now suppose that $z_2 = 0$. Then generally we have $e^{iv_1 \phi} = e^{-2\pi \frac{l_1 w_1 r}{l_2} i}$ for some $r = 0, \ldots, l_2 - 1$ or equivalently $r = 1, \ldots, l_2$. This gives

$$\phi = 2\pi (-\frac{l_1 w_1 r}{v_1 l_2} + \frac{k}{v_1}).$$

A similar computation at $z_1 = 0$ gives

$$\phi = 2\pi (-\frac{l_1 w_2 r'}{v_2 l_2} + \frac{k'}{v_2}).$$

We are interested in when regularity can occur. For this we need the minimal angle at the two endpoints to be equal. This gives

$$-\frac{l_1 w_2 r'}{v_2 l_2} + \frac{k'}{v_2} = -\frac{l_1 w_1 r}{v_1 l_2} + \frac{k}{v_1}$$

for some choice of integers $k, k'$ and nonnegative integers $r, r' < l_2$. This gives

$$\frac{-l_1 w_2 r' + k' l_2}{v_2} = -\frac{l_1 w_1 r + k l_2}{v_1}.$$
3.4. Periods of Reeb Orbits. We assume that \( w \neq (1, 1) \). We want to determine the periods of the orbits of the flow of the Reeb vector field defined by the weight vector \( v = (v_1, v_2) \). In particular, we want to know when there is a regular Reeb vector field in the \( w \)-Sasaki cone.

Let us now generally determine the minimal angle, hence the generic period of the Reeb orbits, on the dense open subset \( Z \) defined by \( z_1 z_2 \neq 0 \). For convenience we set \( s = \gcd(|w_2 v_1 - w_1 v_2|, l_2) \) in which case (22) becomes \( r = l_2/s \).

**Lemma 3.3.** The minimal angle on \( Z \) is \( \frac{2\pi}{s} \). Thus, \( S^1_\phi/Z_s \) acts freely on the dense open subset \( Z \).

**Proof.** We choose \( k_1, k_2 \) in Equation (21) so that the last parentheses equals 1. This gives

\[
\frac{l_1 w_2 v_1 - w_1 v_2}{s} = k_2 v_1 - k_1 v_2.
\]

Rearranging this becomes

\[
(sk_2 - l_1 w_2)v_1 = (sk_1 - l_1 w_1)v_2.
\]

Since \( v_1 \) and \( v_2 \) are relatively prime this equation implies \( sk_i = l_1 w_i + mv_i \) for \( i = 1, 2 \) and some integer \( m \). Putting this into Equation (20) gives \( \phi = \frac{2\pi m}{s} \), so the minimal angle is \( \frac{2\pi}{s} \).

We now investigate the endpoints defined by \( z_2 = 0 \) and \( z_1 = 0 \).

**Proposition 3.4.** The following hold:

1. The period on \( Z \), namely \( \frac{2\pi}{s} \), is an integral multiple of the periods at the endpoints. Hence, \( S^1_\phi/Z_s \) acts effectively on \( M_{l_1, l_2, w} \).
2. The period at the endpoint \( z_j = 0 \) is \( \frac{2\pi}{v_i l_2} \) where \( i \equiv j + 1 \mod 2 \). So the endpoints have equal periods if and only if \( v = (1, 1) \).
3. The \( w \)-Sasaki cone contains a regular Reeb vector field if and only if \( l_2 \) divides \( w_1 - w_2 \), and in this case it is given by \( v = (1, 1) \).

**Proof.** A Reeb vector field will be regular if and only if the period of its orbit is the same at all points. We know that it is \( \frac{2\pi}{s} \) on \( Z \). We need to determine the minimal angle at the endpoints. From Equation (23) the angle at \( z_2 = 0 \) is

\[
\phi = 2\pi \left( -\frac{l_1 w_1 r + kl_2}{v_1 l_2} \right).
\]

Now \( \gcd(l_2, l_1 w_1) = 1 \), so we can choose \( k \) and \( r \) such that numerator of the term in the large parentheses is 1. This gives period \( \frac{2\pi}{v_1 l_2} \). Similarly, at \( z_1 = 0 \) we have the period \( \frac{2\pi}{v_2 l_2} \). So the period is the same at the
endpoints if and only if \( v_1 = v_2 \) which is equivalent to \( \mathbf{v} = (1, 1) \) since \( v_1 \) and \( v_2 \) are relatively prime which proves (2).

Moreover, the period is the same at all points if and only if
\[
(26) \quad \mathbf{v} = (1, 1), \quad l_2 = s = \gcd(|w_2v_1 - w_1v_2|, l_2).
\]
But the last equation holds if and only if \( l_2 \) divides \( w_1 - w_2 \) proving (3).

(1) follows from the fact that for each \( i = 1, 2 \), \( v_il_2 \) is an integral multiple of \( \gcd(|w_2v_1 - w_1v_2|, l_2) = s \). \( \square \)

In contrast to the 2-dimensional Sasaki cones in \([BTF14a]\), not every \( w \)-Sasaki cone has a regular Reeb vector field. Nevertheless, it does have a special Reeb vector field, namely that given by \( \mathbf{v} = (1, 1) \). For this there can be, as usual, two branch divisors, but they have the same ramification index, namely \( m = l_2/s \). We refer to this Reeb field as almost regular. Clearly, there is precisely one almost regular Reeb vector field in each \( w \)-Sasaki cone of \( M_{l_1,l_2,w} \).

**Example 3.5.** Regular Reeb vector fields. As stated in (c) of Proposition 3.3, when \( l_2 \) divides \( w_1 - w_2 \) we always have a regular Reeb vector field in the \( w \)-Sasaki cone by taking \( \mathbf{v} = (1, 1) \). (This was the case in \([BTF14a]\) where \( l_2 = 1 \).) We obtain \( M_{l_1,l_2,w} \) as a principle \( S^1 \) bundle over the smooth quotient \( B_{l_1,l_2-1,w} = S_n = \mathbb{P}(\mathbb{L}^L_m) \) with \( n = l_1w_1 - w_2 \).

3.5. \( B_{l_1,l_2,v,w} \) as a Log Pair. We follow the analysis in Section 3 of \([BTF14a]\). We have the action of the 2-torus \( S^2_0/\mathbb{Z}_s \times (S^1_0/\mathbb{Z}_t) \) on \( M \times L(l_2; l_1w_1, l_1w_2) \) given by Equation (15), and denoted by \( \mathcal{A}_{v,l,w} \), whose quotient space is \( B_{l_1,l_2,v,w} \). It follows from Equation (15) that \( B_{l_1,l_2,v,w} \) is a bundle over \( N \) with fiber a weighted projective space of complex dimension one. By (1) of Proposition 3.3 the generic period is an integral multiple, say \( m_i \), of the period at the divisor \( D_i \). Thus, for \( i = 1, 2 \) we have
\[
(27) \quad m_i = v_il_2/s = v_il.
\]
Note that from its definition \( m = \frac{L}{s} \), so \( m_i \) is indeed a positive integer.

It is the ramification index of the branch divisor \( D_i \). We think of \( D_1 \) as the zero section and \( D_2 \) as the infinity section of the bundle \( B_{l_1,l_2,v,w} \). Thus, \( B_{l_1,l_2,v,w} \) is a fiber bundle over \( N \) with fiber \( \mathbb{CP}^1[v_1,v_2]/\mathbb{Z}_m \approx \mathbb{CP}^1 \). The isomorphism is simply \( [z_1, z_2] \mapsto [z_1^{m_2}, z_2^{m_1}] \) where the brackets denote the obvious equivalence classes on \( \mathbb{CP}^1[v_1,v_2]/\mathbb{Z}_m \). The complex structure of \( B_{l_1,l_2,v,w} \) is the projection of the transverse complex structure on \( M_{l_1,l_2,w} \) which in turn is the lift of the product complex
structure on $N \times \mathbb{CP}^1[w]$. However, $B_{l_1,l_2,v,w}$ is not generally a product as a complex orbifold, nor even topologically.

Now we can follow the analysis leading to Lemma 3.14 of \cite{BTF14a}. So we define the map

$$h_v : M \times L(l_2; l_1 w_1, l_1 w_2) \longrightarrow M \times L(l_2; l_1 w_1 v_2, l_1 w_2 v_1)$$

by

$$h_v(x, u; [z_1, z_2]) = (x, u; [z_1^{m_2}, z_2^{m_1}]).$$

It is a $mv_1v_2$-fold covering map. Similar to \cite{BTF14a} we get a commutative diagram:

$$
\begin{array}{c c c}
M \times L(l_2; l_1 w_1, l_1 w_2) & \overset{A_{v,l,w}(\lambda, \tau)}{\longrightarrow} & M \times L(l_2; l_1 w_1, l_1 w_2) \\
\downarrow h_v & & \downarrow h_v \\
M \times L(l_2; l_1 w'_1, l_1 w'_2) & \overset{A_{(1,1),l,w'}(\lambda, \tau^{m_1v_1v_2})}{\longrightarrow} & M \times L(l_2; l_1 w'_1, l_1 w'_2),
\end{array}
$$

where $w' = (v_2 w_1, v_1 w_2)$ and $\tau = e^{i\theta}, \lambda = e^{i\phi}$. So $B_{l_1,l_2,v,w}$ is the log pair $(B_{l_1,l_2,1,w'}, \Delta)$ with branch divisor

$$\Delta = (1 - \frac{1}{m_1})D_1 + (1 - \frac{1}{m_2})D_2,$$

where $B_{l_1,l_2,1,w'}$ is a $\mathbb{CP}^1$-bundle over $N$. Now $B_{l_1,l_2,v,w}$ is the quotient $(M \times L(l_2; l_1 w_1, l_1 w_2))/A_{v,l,w}(\lambda, \tau)$, and $B_{l_1,l_2,1,w}$ is the quotient $(M \times L(l_2; l_1 w'_1, l_1 w'_2))/A_{(1,1),l,w'}(\lambda, \tau^{v_1v_2})$. So $h_v$ induces a map $h_v : B_{l_1,l_2,v,w} \longrightarrow B_{l_1,l_2,1,w'}$ defined by

$$h_v([x, u; [z_1, z_2]]) = [x, u; [z_1^{m_2}, z_2^{m_1}]],$$

where the outer brackets denote the equivalence class with respect to the corresponding $T^2$ action. We have

**Lemma 3.6.** The map $h_v : B_{l_1,l_2,v,w} \longrightarrow B_{l_1,l_2,1,w'}$ defined by Equation (31) is a biholomorphism.

**Proof.** The map is ostensibly holomorphic. Now $h_v$ is the identity map on $M$ and a $v_1v_2$-fold covering map on the corresponding lens spaces. From the commutative diagram (29) the induced map $h_v$ is fiber preserving and is a bijection on the fibers with holomorphic inverse. \qed

Lemma 3.6 allows us to consider the orbifold $B_{l_1,l_2,v,w}$ as the log pair $(B_{l_1,l_2,1,w'}, \Delta)$ where $\Delta$ is given by Equation (31). Notice, as mentioned above, when $v = (1,1)$ we have an almost regular Reeb vector field. Here the orbifold structure can be non-trivial, namely, $B_{l_1,l_2,(1,1),w} =$
\[(B_{l_1,l_2,w}, \Delta)\] where \(m_1 = m_2 = m = \frac{l_2}{s}\) and the branch divisor is given by
\[
\Delta = (1 - \frac{1}{m})(D_1 + D_2).
\]

The \(T^2\) action \(\mathcal{A}_{(1,1),l,w} : M \times L(l_2; l_1 w_1', l_1 w_2') \longrightarrow M \times L(l_2; l_1 w_1', l_1 w_2')\) is given by
\[
(x, u; z_1, z_2) \mapsto (x, e^{i\theta} u; [e^{i\theta\frac{1}{l_2}} z_1, e^{i\theta\frac{1}{l_2}} z_2]).
\]

Defining \(\chi = \phi - \frac{l_1 w_1'}{l_2}\) gives
\[
(x, u; z_1, z_2) \mapsto (x, e^{i\theta} u; [e^{i\chi z_1}, e^{i\chi z_2}]).
\]

The analysis above shows that this action is generally not free, but has branch divisors at the zero \((z_2 = 0)\) and infinity \((z_1 = 0)\) sections with ramification indices both equal to \(m\).

Equation (33) tells us that the \(T^2\)-quotient space \(B_{l_1,l_2,1,w}\) is the projectivization of the holomorphic rank two vector bundle \(E = \mathbb{I} \oplus L_n\) over \(N\) where \(\mathbb{I}\) denotes the trivial line bundle and \(L_n\) is a line bundle of ‘degree’ \(n = \frac{1}{s}(w_1 v_2 - w_2 v_1)\) with \(s = \gcd(|w_1 v_2 - w_2 v_1|, l_2)\). So \(S_n = \mathbb{P}(\mathbb{I} \oplus L_n)\) is a smooth projective algebraic variety. Next we identify \(N\) with the zero section \(D_1\) of \(L_n\), and note that \(c_1(L_n)\) is just the restriction of the Poincaré dual of \(D_1\) to \(D_1\), i.e. \(PD(D_1)|_{D_1} = c_1(L_n)\).

Summarizing we have

**Theorem 3.7.** Let \(M_{l_1,l_2,w} = M_{\ast l_1,l_2,S^3_w}\) be the join as described in the beginning of the section with the induced contact structure \(D_{l_1,l_2,w}\). Let \(v = (v_1, v_2)\) be a weight vector with relatively prime integer components and let \(\xi_v\) be the corresponding Reeb vector field in the Sasaki cone \(\mathbb{I}_w^+\). Then the quotient of \(M_{l_1,l_2,w}\) by the flow of the Reeb vector field \(\xi_v\) is a projective algebraic orbifold written as a the log pair \((S_n, \Delta)\) where \(S_n\) is the total space of the projective bundle \(\mathbb{P}(\mathbb{I} \oplus L_n)\) over the Kähler manifold \(N\) with \(n = l_1 \left(\frac{w_1 v_2 - w_2 v_1}{s}\right)\), \(\Delta\) the branch divisor
\[
\Delta = (1 - \frac{1}{m_1})D_1 + (1 - \frac{1}{m_2})D_2,
\]

with ramification indices \(m_1 = v_1 \frac{1}{s} = v_1 m\) and divisors \(D_1\) and \(D_2\) given by the zero section \(\mathbb{I} \oplus 0\) and infinity section \(0 \oplus L_n\), respectively. The fiber of the orbifold \((S_n, \Delta)\) is the orbifold \(\mathbb{CP}[v_1, v_2]/\mathbb{Z}_m\).

Next we focus on the projective bundle \(S_n = \mathbb{P}(\mathbb{I} \oplus L_n)\). From Equation (33) we see that \(S_n\) is a fiber bundle over \(N\) with fiber \(\mathbb{CP}^1\) associated to the principle \(S^1\)-bundle \(M \longrightarrow N\). We want to determine the Kähler class \([\omega_B]\) of the orbifold \(B_{l_1,l_2,v,w} = (S_n, \Delta)\) induced by the
projection \( M_{l_1,l_2,w} \to B_{l_1,l_2,v,w} \). First consider the following commutative diagram:

\[
\begin{array}{ccc}
M \times L(l_2;l_1w_1,l_1w_2) & \to & M_{l_1,l_2,w} \\
\downarrow \pi_L & & \downarrow \pi_{l_1,l_2,w} \\
N \times \mathbb{CP}^1[w] & \to & (S_n, \Delta) \\
\downarrow p_w & & \downarrow p_v \\
N & & N
\end{array}
\]

where \( p_w, p_v \) are the obvious projections. Second, note that we have the following lemma

**Lemma 3.8.** [BTF13b] For the log pair \((S_n, \Delta)\) with

\[
\Delta = (1 - 1/m_1)D_1 + (1 - 1/m_2)D_2
\]

the orbifold Chern class equals

\[
c_{orb}^1(S_n, \Delta) = p_v^* c_1(N) + \frac{1}{m_1}PD(D_1) + \frac{1}{m_2}PD(D_2).
\]

By (7)

\[
\pi_v^* c_{orb}^1(S_n, \Delta) = c_1(D_{l_1,l_2,w}) = (p_w \circ \pi_w)^* c_1(N) - l_1|w|\gamma.
\]

So from Lemma 3.8 we have

\[
(p_v \circ \pi_v)^* c_1(N) + \frac{1}{m_1} \pi_v^* PD(D_1) + \frac{1}{m_2} \pi_v^* PD(D_2) = (p_w \circ \pi_w)^* c_1(N) - l_1|w|\gamma.
\]

We also know that (see e.g. Section 1.3 in [ACGT08])

\[
PD(D_1) - PD(D_2) = np_v^*[\omega_N]
\]

and so

\[
\pi_v^* PD(D_1) - \pi_v^* PD(D_2) = n(p_v \circ \pi_v)^*[\omega_N].
\]

From the commutative diagram (35) we see that

\[
(p_v \circ \pi_v)^*[\omega_N] = (p_w \circ \pi_w)^*[\omega_N] = l_2\gamma.
\]

and

\[
(p_v \circ \pi_v)^* c_1(N) = (p_w \circ \pi_w)^* c_1(N),
\]

so we get the system

\[
\frac{1}{m_1} \pi_v^* PD(D_1) + \frac{1}{m_2} \pi_v^* PD(D_2) = -l_1|w|\gamma.
\]

\[
\pi_v^* PD(D_1) - \pi_v^* PD(D_2) = l_2\gamma
\]
which implies that $\pi^*_v PD(D_1) = \frac{m_1 l_1 |w|}{m_2 - l_1 |w|} = -m_1 l_1 w_2 \gamma$ and $\pi^*_v PD(D_2) = -m_2 l_1 w_1 \gamma$.

We are now ready to prove the following lemma

**Lemma 3.9.** The induced Kähler class on $B_{l_1,l_2,v,w} = (S_n, \Delta)$ takes the form

$$k_1 p^*_v [\omega_B] + k_2 PD(D_1)$$

for some positive integers $k_1, k_2$.

**Proof.** From the commutative diagram (35) we see that on degree 2 homology $\ker \pi^*_B = (\pi_v \circ \pi_L)^*$ has dimension 2. We claim that $p^*_v [\omega_B]$ and $PD(D_1)$ span $\ker \pi^*_B$. To see this we note that from the definition of the join, that $p^*_v [\omega_B]$ is in $\ker \pi^*_B$. Moreover,

$$(p_v \circ \pi_v \circ \pi_L)^* : H^2(N, \mathbb{R}) \to H^2(M \times L(l_2; l_1 w_1, l_1 w_2), \mathbb{R})$$

has a one dimensional kernel. So it must be spanned by $[\omega_B]$. Since $p^*_v [\omega_B]$ is in $\ker \pi^*_B$ and $(p_v \circ \pi_v)^*[\omega_B] = l_2 \gamma$, we must have that $\pi^*_L \gamma = 0$. It follows that $PD(D_1)$ is also in the kernel of $\pi^*_B$ and since it is clearly independent of $p^*_v [\omega_B]$ we conclude that $p^*_v [\omega_B]$ and $PD(D_1)$ span $\ker \pi^*_B$.

The induced Kähler class on $B_{l_1,l_2,v,w} = (S_n, \Delta)$ is clearly in $\ker \pi^*_B$ and so the lemma follows. $\square$

In view of Lemma 3.9 we write the induced Kähler class $[\omega_B]$ on $(S_n, \Delta)$ as

$$[\omega_B] = k_1 p^*_v [\omega_B] + k_2 PD(D_1) \quad (36)$$

**Lemma 3.10.** The following hold:

1. $k_2 = l_2$
2. $k_1 = m_1 l_1 w_2$

**Proof.** Since we know that $p^*_v [\omega_B]$ is a trivial class in $M_{l_1,l_2,w}$ and $(p_v \circ \pi_v)^*[\omega_B] = l_2 \gamma$ while $p^*_v PD(D_1) = -m_1 l_1 w_2 \gamma$, we see immediately that $k_1 l_2 - k_2 m_1 l_1 w_2 = 0$ and since $\gcd(k_1, k_2) = m = l_2/s$, we conclude that $k_2 = l_2$ while $k_1 = m_1 l_1 w_2$. $\square$

In the almost regular case this process can be inverted. Given positive integers $n, m, k_1, k_2$ with $m = \gcd(k_1, k_2)$ we can determine the relatively prime positive integers $w_1, w_2$ from the equation

$$\frac{w_2}{w_1} = \frac{k_1}{nk_2 + k_1}$$
and the relatively prime positive integers \( l_1, l_2 \) from
\[
\frac{l_1}{l_2} = \frac{n}{m(w_1 - w_2)}.
\]
This gives an analog of diagram (32) of [BTF14a] together with its interpretation depicted in the diagram
\[
\xymatrix{ M_{l_1,l_2,w} \ar@{<->}[rr] \ar[ld] \ar[rd] & & N \times \mathbb{CP}(w) \ar[ld] \ar[rd] \ar@{<->}[rr] & & (S_n, \Delta). }
\]
Thus, we can view \( M_{l_1,l_2,w} \) in two ways. First, the southwest arrow describes an \( S^1 \) orbibundle over the Kähler orbifold \( N \times \mathbb{CP}^1[w] \) with its product structure. Second the southeast arrow describes the Kähler structure of a \( \mathbb{CP}^1 \)-bundle over \( N \) with twisted complex structure and a mild orbifold structure on the fibers given as a quotient by an almost regular Reeb vector field. Note that in (32) of [BTF14a] the southeast arrow is the quotient by a regular Reeb vector field.

4. Admissible CSC constructions

We now pick up the thread from Section 2.3 and describe the construction (see also [ACGTF08]) of admissible Kähler metrics on \( S_n \) (in fact, more generally on log pairs \( (S_n, \Delta) \)). Consider the circle action on \( S_n \) induced by the natural circle action on \( L_n \). It extends to a holomorphic \( \mathbb{C}^* \) action. The open and dense set \( S_{n0} \subset S_n \) of stable points with respect to the latter action has the structure of a principal circle bundle over the stable quotient. The hermitian norm on the fibers induces via a Legendre transform a function \( z : S_{n0} \to (-1, 1) \) whose extension to \( S_n \) consists of the critical manifolds \( z^{-1}(-1) = P(0 \oplus L_n) \) and \( z^{-1}(1) = P(1 \oplus 0) \). Letting \( \theta \) be a connection one form for the Hermitian metric on \( S_{n0} \), with curvature \( d\theta = \omega_{N_n} \), an admissible Kähler metric and form are given up to scale by the respective formulas
\[
g = \frac{1 + r \Theta}{\theta} g_{N_n} + \frac{d\Theta^2}{\Theta(\Theta)} + \Theta(\Theta)\theta^2, \quad \omega = \frac{1 + r \Theta}{\theta} \omega_{N_n} + d\Theta \wedge \theta,
\]
valid on \( S_{n0} \). Here \( \Theta \) is a smooth function with domain containing \( (-1, 1) \) and \( r \), is a real number of the same sign as \( g_{N_n} \) and satisfying \( 0 < |r| < 1 \). The complex structure yielding this Kähler structure is given by the pullback of the base complex structure along with the requirement \( Jd\Theta = \theta \). The function \( \Theta \) is hamiltonian with \( K = J \text{grad} \Theta \) a Killing vector field. In fact, \( \Theta \) is the moment map on \( S_n \) for the circle action, decomposing \( S_n \) into the free orbits \( S_{n0} = \Theta^{-1}((-1, 1)) \) and
the special orbits $D_1 = z^{-1}(1)$ and $D_2 = z^{-1}(-1)$. Finally, $\theta$ satisfies $\theta(K) = 1$.

**Remark 4.1.** Note that

$$\phi := -\frac{(1 + r_J)}{r^2} \omega_{N_n} + \omega d_J \wedge \theta$$

is a Hamiltonian 2-form of order one.

We can now interpret $g$ as a metric on the log pair $(S_n, \Delta)$ with

$$\Delta = (1 - 1/m_1)D_1 + (1 - 1/m_2)D_2$$

if $\Theta$ satisfies the positivity and boundary conditions

$$\Theta(\bar{z}) > 0, \quad -1 < \bar{z} < 1,$$

$$\Theta(\pm 1) = 0,$$

$$\Theta'(-1) = 2/m_2, \quad \Theta'(1) = -2/m_1.$$  \hspace{1cm} (39)

**Remark 4.2.** This construction is based on the symplectic viewpoint where different choices of $\Theta$ yields different complex structures all compatible with the same fixed symplectic form $\omega$. However, for each $\Theta$ there is an $S^1$-equivariant diffeomorphism pulling back $J$ to the original fixed complex structure on $S_n$ in such a way that the Kähler form of the new Kähler metric is in the same cohomology class as $\omega$ [ACGTF08]. Therefore, with all else fixed, we may view the set of the functions $\Theta$ satisfying (39) as parametrizing a family of Kähler metrics within the same Kähler class of $(S_n, \Delta)$.

The Kähler class $\Omega_r = [\omega]$ of an admissible metric is also called *admissible* and is uniquely determined by the parameter $r$, once the data associated with $S_n$ (i.e. $d_{N_n}, s_{N_n}, g_{N_n}$ etc.) is fixed. In fact, up to scale

$$\Omega_r = [\omega_{N_n}]/r + 2\pi PD[D_1 + D_2],$$  \hspace{1cm} (40)

where $PD$ denotes the Poincaré dual. The number $r$, together with the data associated with $S_n$ will be called *admissible data*.

Define a function $F(\bar{z})$ by the formula $\Theta(\bar{z}) = F(\bar{z})/p(\bar{z})$, where $p(\bar{z}) = (1 + r_J)^{d_N}$. Since $p(\bar{z})$ is positive for $-1 \leq \bar{z} \leq 1$, conditions (39)
are equivalent to the following conditions on $F(\zeta)$:

\[
F(\zeta) > 0, \quad -1 < \zeta < 1,
\]

(41)

\[
F(\pm 1) = 0,
\]

\[
F'(\pm 1) = 2 \frac{p(\pm 1)}{m_2}, \quad F'(1) = -2 \frac{p(1)}{m_1}.
\]

4.1. The CSC condition. From [ACG06] we have that the scalar curvature of an admissible metric given by (38) equals

\[
\text{Scal} = \frac{2d_Ns_{N_n}r}{1 + \zeta} - \frac{F''(\zeta)}{p(\zeta)}.
\]

Thus the CSC condition is equivalent the ODE

\[
F''(\zeta) = (2d_Ns_{N_n}r - k(1 + r_3)) (1 + r_3)^{d_N-1},
\]

where $k$ is a constant (equal to $\text{Scal}$ when (43) is solved). Notice that if (43) has a solution such that the boundary conditions from (41) holds, then it will also follow that $F(\zeta) > 0$ for $-1 < \zeta < 1$ and thus all of (41) is satisfied. To see this, merely observe that since $|r| < 1$ for $0 < -1 < \zeta < 1$, then $\text{Scal}$ can change sign at most once over the interval $-1 < \zeta < 1$. Together with this fact, the endpoint conditions rule out any possibility of $F(\zeta)$ being zero for any $-1 < \zeta < 1$.

Integrating and using the conditions of $F'(\pm 1)$ in (41), we immediately get that

\[
F'(\zeta) = \left(2s_{N_n} - k \frac{1}{r(d_N + 1)}(1 + r\zeta)\right) (1 + r\zeta)^{d_N} + c,
\]

where

\[
c = \frac{2(1 - r^2)^{d_N} (m_2(1 - r) + m_1(1 + r) - 2m_1m_2s_{N_n})}{m_1m_2 ((1 + r)^{d_N+1} - (1 - r)^{d_N+1})}
\]

and

\[
k = \frac{2(d_N + 1)r (m_2(1 + r)^{d_N} + m_1s_{N_n}) - m_1(1 - r)^{d_N}(-1 + m_2s_{N_n})}{m_1m_2 ((1 + r)^{d_N+1} - (1 - r)^{d_N+1})}
\]

Now we have a solution to (43), namely

\[
F(\zeta) = \int_{-1}^{\zeta} \left(\left(2s_{N_n} - k \frac{1}{r(d_N + 1)}(1 + rt)\right) (1 + rt)^{d_N} + c\right) d\zeta,
\]
satisfying (41) iff
\[
\int_{-1}^{1} \left( (2s_{Nn} - k \frac{1}{r(d_N + 1)}(1 + r \zeta) \right) (1 + r \zeta)^{d_N} + c \right) \, d \zeta = 0.
\]

Now we integrate (46) to arrive at the equation

\[
2s_{Nn} \left( (1 + r)^{d_N+1} - (1 - r)^{d_N+1} \right) \frac{k \left( (1 + r)^{d_N+2} - (1 - r)^{d_N+2} \right)}{r(d_N + 1)} d \zeta + 2c = 0.
\]

Thus the existence of an admissible CSC Kähler metric on the log pair \((S_n, \Delta)\) correspond to solving all three equations (44), (45), and (47).

4.2. Extremal Kähler metrics. If we generalize equation (43) to the condition that \(Scal\) from (42) is a linear function of \(\zeta\), then we obtain the equation

\[
F''(\zeta) = \left( 2d_N s_{Nn} r + (\alpha \zeta + \beta)(1 + r \zeta) \right),
\]

where \(\alpha\) and \(\beta\) are constants. It is well known that this corresponds to extremal Kähler metrics (see e.g. [ACGT08]). Moreover, similarly to the smooth case, one easily sees (by integrating and solving for \(A\) and \(B\)) that (48) has a unique solution \(F(\zeta)\) satisfying the endpoint conditions of (41). Finally, if the Kähler form \(\omega_N\) on \(N\) is assumed to have positive scalar curvature, this polynomial \(F(\zeta)\) also satisfies the positivity condition of (41) by the standard root-counting argument introduced by Hwang [Hwa94] and Guan [Gua95]. For completeness we give the root-counting argument for this special case: Assume for contradiction that the positivity condition of (41) fails. Then, due to the endpoint conditions on \(F(\zeta)\), the function \(F(\zeta)\) has at least two relative maxima and at least one relative minimum inside the interval \((-1, 1)\). Thus, in the interval \((-1, 1)\), the concavity of \(F(\zeta)\) changes at least twice, i.e. \(F''(\zeta)\) is zero at least twice. Since \((1 + r \zeta)^{d_N-1} > 0\) for \(-1 < \zeta < 1\), we see that this implies that the second order polynomial \(P(\zeta) = (2d_N s_{Nn} r + (\alpha \zeta + \beta)(1 + r \zeta))\) has two roots inside \((-1, 1)\) and further the concavity changes exactly twice. Thus \(F(\zeta)\) has two relative maxima at \(\zeta = a_1\) and \(\zeta = a_3\) and one relative minimum at \(\zeta = a_2\) such that \(-1 < a_1 < a_2 < a_3 < 1\) and the roots \(r_1, r_2\) of \(P(\zeta)\) are such that \(a_1 < r_1 < a_2 < r_2 < a_3\). Moreover, \(P(a_1) < 0\) and \(P(a_3) < 0\). Now we observe that \(P(-1/r) = 2d_N s_{Nn} r\) and thus if \(s_{Nn} r \geq 0\), we see that \(P(\zeta)\) must have one more root in either \([-1/r, a_1]\) (if \(r > 0\)) or \((a_3, -1/r]\) (if \(r < 0\)). Obviously \(P(\zeta)\) cannot have three roots and so we have a contradiction. We conclude that the positivity condition of (41) must be satisfied.
This yields the following proposition which also proves Theorem 1.2 as we shall see below.

**Proposition 4.3.** Assume that the scalar curvature \( s_N \) of \((N, \omega_N)\) is non-negative. For any log pair \((S_n, \Delta)\), any admissible Kähler class on \(S_n\) contains an admissible extremal metric which is smooth in the orbifold sense on \((S_n, \Delta)\).

5. CSC and Extremal Rays

In order to finish the proofs of Theorem 1.1 and Theorem 1.2, we now connect Section 4 with Section 3. Assume \(M_{l_1, l_2, w} = M \star_{l_1, l_2} S^3_w\) is the join as described in the beginning of the Section 3 with the induced contact structure \(D_{l_1, l_2, w}\). Let \(v = (v_1, v_2)\) be a weight vector with relatively prime integer components and let \(\xi_v\) be the corresponding Reeb vector field in the \(w\)-Sasaki cone \(t^+_w\). Let the log pair \((S_n, \Delta)\) be the quotient of \(M_{l_1, l_2, w}\) by the flow of the Reeb vector field \(\xi_v\). Using Theorem 3.7 we have

\[
\Omega_r = 4\pi \left( \frac{n(1-r)}{2r} [\omega_N] + PD(D_1) \right)
\]

and so \([\omega_B]\) given by Lemma 3.10 is indeed admissible, where \(r\) is such that

\[
\frac{n(1-r)}{2r} = k_1/k_2 = m_1 l_1 w_2 / l_2
\]

which is equivalent to

\[
r = \frac{w_1 v_2 - w_2 v_1}{w_1 v_2 + w_2 v_1}
\]

Notice that together with Proposition 4.3 this proves that when the scalar curvature \(s_N\) of \((N, \omega_N)\) is non-negative, then each ray in the \(w\)-Sasaki cone can be represented by an extremal Sasaki metric (see the discussion below about irregular rays). Consequently this proves Theorem 1.2.

Assuming \(w_1 > w_2\), the existence of a quasi-regular CSC ray in the \(w\)-Sasaki cone \(t^+_w\) corresponds to showing that for a choice of \((v_1, v_2) \neq (w_1, w_2)\) and the values of \(m_i, n, r\) given as above, the equation system (44), (45), and (47) is solved. Recall that what we mean by this is that the Kähler class of the corresponding quotient metric, \([\omega_B]\), contains an admissible CSC Kähler metric. Thus the Sasaki structure
given by \((v_1, v_2)\) is CSC up to contact isotopy. [See the discussion above and below Lemma 8.1 in [BTF13a].]

For practical reasons, we are setting \(s_N = A/n = \frac{A}{l_1(w_1v_2 - w_2v_1)}\), where, by Remark 2.3, \(A \leq d_N + 1\) (If \(\omega_N\) is Kähler-Einstein, \(A\) is just \(J_N\) as introduced in Remark 2.3). Notice that then the value \(s\) cancels from the equation system (44), (45), and (47). In fact we have (assuming \((v_1, v_2) \neq (w_1, w_2)\)), that the system is equivalent to the equation \(f(b) = 0\) for \(b > 0\), where \(b = \frac{w_2}{v_1} \neq \frac{w_1}{v_1}\) and

\[
(49) \quad f(b) = w_1^{2(d_N+1)}b^{2d_N+3}(Al_2 + l_1(d_N + 1)w_2 - b(d_N + 1)l_1w_1)
- w_1^{d_N+2}w_2^{d_N}b^{d_N+3}((d_N + 1)(A(d_N + 1)l_2 - l_1((d_N + 1)w_1 + (d_N + 2)w_2)))
+ w_1^{d_N+1}w_2^{d_N+1}b^{d_N+2}((2Ad_N(d_N + 2)l_2 - (3 + 5d_N + 2d_N^2)l_1(w_1 + w_2)))
- w_1^{d_N}w_2^{d_N+2}b^{d_N+1}((d_N + 1)(A(d_N + 1)l_2 - l_1((d_N + 2)w_1 + (d_N + 1)w_2)))
+ w_2^{2(d_N+1)}(b(Al_2 + l_1(d_N + 1)w_1) - (d_N + 1)l_1w_2).
\]

Since \(f(b)\) is a polynomial which is formally defined for any real value of \(b\) and

\[
f\left(\frac{w_2}{w_1}\right) = f'(\frac{w_2}{w_1}) = f''\left(\frac{w_2}{w_1}\right) = 0
\]

while

\[
f''\left(\frac{w_2}{w_1}\right) = 3(d_N + 1)(d_N + 2)l_1w_1^{d_N}w_2^{d_N}(w_1 - w_2) > 0
\]

and

\[
\lim_{b \to +\infty} f(b) = -\infty,
\]

we see that there is at least one solution \(b \in \left(\frac{w_2}{w_1}, +\infty\right)\) to \(f(b) = 0\).

Unless, this solution is a rational number, it does not correspond to a quasi-regular CSC ray. However following Remark 5.4 in [BTF14a], the denseness of the rational rays in \(t_2^2\) implies that an irrational solution \(b > 0\) to \(f(b) = 0\), corresponds to an irregular CSC ray. Thus the proof of Theorem 1.1 is complete. \(\square\)

**Remark 5.1.** In the special case when \(\omega_N\) is Kähler-Einstein, \(J_N > 0\) and \(l_1 = \frac{J_N}{\gcd(w_1, w_2, J_N)}\) and \(l_2 = \frac{w_1 + w_2}{\gcd(w_1 + w_2, J_N)}\) we saw in [BTF13b] that we have the Sasaki-Einstein ray discovered by the physicists [GMSW04]. While the majority of these are irregular we do know some of them to be quasi-regular (see e.g. Example 5.5 in [BTF13b]). As the examples
in the next subsection will illustrate we can also produce many non-Einstein quasi-regular CSC rays in all dimensions.

5.1. A Special Case: \( N = \mathbb{CP}^p \). Recall Example 3.2 in which case \( A = \mathbb{F}_N = p + 1 \). Now if we let \((l_1, l_2)\) be any relatively prime pair of positive integers except \((\frac{p+1}{\gcd(|w|, p+1)}, \frac{|w|}{\gcd(|w|, p+1)})\) we know that the CSC ray from the proof of Theorem 1.1 is not Sasaki-Einstein (see Lemma 3.1 in \([BTF13b]\)), and by Equation (10) \( c_1(\mathcal{D}_{l_1,l_2,w}) \neq 0 \). Again, for the majority of choices of \((w_1, w_2)\), the CSC ray discovered will be irregular. However we can produce quite a lot of quasi-regular CSC rays as the example below shows.

This case is studied in much more depth in \([BTF14b]\). In particular, it is shown there that if \( p, l_1 \) and \( w \) are fixed, there is only a finite number of diffeomorphism types among the manifolds \( M_{l_1,l_2,w} \). So for each \( p > 1, l_1, w_1, w_2 \) with \( w_1 > w_2 \), there exists a smooth 2\( p+3 \)-dimensional manifold \( M_{l_1,w} \) which admits a countably infinite number of contact structures of Sasaki type each with a compatible Sasaki metric of constant scalar curvature.

Example 5.2. For example with \( A = p + 1, l_1 = \frac{p}{\gcd(|w|, p+1)} \), and \( l_2 = \frac{|w|}{\gcd(|w|, p+1)} \) we set \( t = \frac{w_2}{w_1} \) and \( b = kt \) in \( f(b) \) above. Then the equation \( f(b) = 0 \) is equivalent to

\[
t = \frac{p - (p+1)k + kp + k^{p+1}}{k(1 - (p+1)k + pk^{p+1})}.
\]

It is a straightforward calculus exercise to check that for \( k > 1 \) we get a solution \( 0 < t < 1 \) as predicted in the previous section (so \( b = kt > t = \frac{w_2}{w_1} \)) and \( w_1 > w_2 \). In particular, if we pick a rational \( k > 1 \) we get a rational \( t \). This value of \( t \) will determine \((w_1, w_2)\) and then with \((v_1, v_2)\) such that \( v_2/v_1 = b = kt \) we have our CSC quasi-regular Sasaki metric.

Example 5.3. Let us assume that \( p = 2 \) (hence \( A = 3 \)), \( l_1 = 1 \) and \( w = (3, 2) \). So to have smooth 7-manifolds \( M^7_{1,l_2,(3,2)} \) we must have \( \gcd(l_2, 6) = 1 \). Then \( f(b) = 3(2 - 3b)g(b) \), where

\[
g(b) = 81b^5 - 27(l_2 - 4)b^4 - 54(l_2 - 2)b^3 + 36(l_2 - 1)b^2 + 8(l_2 - 6)b - 16.
\]

Now \( g(2/3) = -32 < 0 \) and \( \lim_{b \to +\infty} g(b) = +\infty \), justifying the solution \( g(b) = 0 \) in the interval \((2/3, +\infty)\) as already established. Notice however that \( g(0) = -16 < 0 \) and \( g(1/3) = (13l_2 - 115)/3 \). So for any \( l_2 \geq 9 \) with \( \gcd(l_2, 6) = 1 \), we have two additional solutions to \( g(b) = 0 \) in the interval \((0, 2/3)\). Furthermore, one can check that the other two
solutions are negative, so there are exactly 3 rays of CSC Sasaki metrics in the \( w \)-cone. It can also be checked that for \( l_2 = 1, 5, 7 \), there is only one solution to \( g(b) = 0 \). So sometimes, but not always, there will be more than one CSC ray in the \( w \)-Sasaki cone. We thus have

**Proposition 5.4.** For each \( l_2 \geq 9 \) relatively prime to 6 there are exactly three distinct constant scalar curvature rays in the \( w \)-Sasaki cone of the toric contact 7-manifold \( (M^7_{1,l_2,(3,2)}, D_{1,l_2,(3,2)}) \).

It also follows from our results in [BTF14b] that infinitely many of the manifolds \( M^7_{1,l_2,(3,2)} \) are diffeomorphic. Thus, there exists an infinite subsequence \( s_j \subset \{ l_2 \} \) of the integers \( l_2 \geq 9 \) giving distinct contact structures \( D_{s_j} \) of Sasaki type occurring on the same 7-manifold all containing three rays of CSC Sasaki metrics in their \( w \)-Sasaki cone.

**Example 5.5.** Wang-Ziller manifolds. In the calculus analysis we have done on \( f(b) \) so far, we have assumed that \( w_1 > w_2 \). For arguments sake let us assume that \( w_1 = w_2 = 1 \) in which case our manifolds \( M^7_{l_1,l_2,(1,1)} = M^7_{l_2,l_1} \), a Wang-Ziller manifold [WZ90]. If we assume that \( N = \mathbb{C}P^2 \) and pick \( l_1 = 1 \), we know from Proposition 2.3 of Wang and Ziller that \( M^7_{l_2,1} \) is \( S^2 \times S^5 \) when \( l_2 \) is even and the non-trivial \( S^5 \)-bundle over \( S^2 \), which we denote by \( S^2 \times S^5 \), when \( l_2 \) is odd. So there are exactly these two diffeomorphism types. Moreover, we know that we have at least one CSC ray, namely the regular ray in the \( S^1 \)-bundle over the product \( N \times \mathbb{C}P^1 \). This case corresponds to \( b = 1 \), although \( f(b) \) has no geometric meaning for \( b = 1 \). However, we also get that

\[
g(b) = (1 + (3 - l_2)b - 4b^2l_2 + 6b^2 + (3 - l_2)b^3 + b^4).
\]

Now we observe that \( g(0) = 1 > 0 \), \( g(1) = 2(7 - 3l_2) \), and \( \lim_{b \to +\infty} g(b) = +\infty. \) So for \( l_2 \geq 3 \) \( f(b) \) has at least 2 roots not equal to 1 ; one in the interval \((0,1)\) and one in the interval \((1, +\infty)\). Thus we have at least three CSC rays in this case as well.

Now the Wang-Ziller manifolds are toric, in fact, they are homogeneous, and in our case \( M^7_{l_2,1} \) have a four-dimensional Sasaki cone, and when \( l_2 \geq 3 \) the \( w \)-Sasaki cone (i.e. the 2-dimensional Sasaki cone associated with \( S^3 \)) has three CSC rays, one regular and the other two irregular or quasi-regular. Notice also in our case the first Chern class of the contact bundle is \( c_1(D_{1,l_2}) = (3l_2 - 2)\gamma. \) Summarizing we have

**Theorem 5.6.** The 7-manifolds \( S^2 \times S^5 \) and \( S^2 \times_{\gamma} S^5 \) admit countably infinite inequivalent toric contact structures \( D_{1,l_2} \) of Reeb type with \( l_2 \) even for the former and \( l_2 \) odd for the latter. Furthermore, when \( l_2 \geq 3 \)
these contact structures admit three distinct rays of Sasaki metrics with constant scalar curvature.

As $l_2$ varies the contact structures are clearly inequivalent as contact structures, not just as toric contact structures.

Remark 5.7. Theorem 5.6 generalizes to arbitrary $p$ as we shall see in [BTF14b].

5.2. Multiple CSC Rays. The multiple CSC rays in Proposition 5.4 and Theorem 5.6 illustrate a somewhat common phenomenon that was first illustrated in the case of quadrilateral toric structures ($S^2$-bundles over $S^2$) by Legendre [Leg11]. Consider $f(b)$ in (49). As already stated, any positive solution $b \neq \frac{w_2}{w_1}$ to the equation $f(b) = 0$ corresponds to a CSC ray in the $w$-Sasaki cone. So far we know that, assuming $w_1 > w_2$, there is at least one solution in the interval $(\frac{w_2}{w_1}, +\infty)$. Since

$$f\left(\frac{w_2}{w_1}\right) = f'\left(\frac{w_2}{w_1}\right) = f''\left(\frac{w_2}{w_1}\right) = 0$$

while

$$f'''\left(\frac{w_2}{w_1}\right) = 3(d_N + 1)(d_N + 2)l_1w_1^{d_N}w_2^{d_N}(w_1 - w_2) > 0$$

we know that for $b < \frac{w_2}{w_1}$ sufficiently close to $\frac{w_2}{w_1}$, we have $f(b) < 0$. Further it is easy to see that $f(0) < 0$. Now we notice that $f\left(\frac{w_2}{2w_1}\right)$ is a linear function of $l_2$ with slope equal to

$$\frac{Aw_1^{2d_N+3}}{2^{2d+3}w_1} \left[ 1 + 2d_N \left(2^{d_N+2} - (d_N^2 + 2d_N + 5)\right) \right].$$

When $A > 0$, which is equivalent to the scalar curvature $s_N$ of $(N, \omega_N)$ being positive, then this slope is positive and thus for sufficiently large value of $l_2$ we have that $f\left(\frac{w_2}{2w_1}\right) > 0$. In that case we have at least two more roots; one in the interval $(0, \frac{w_2}{w_1})$ and one in the interval $(\frac{w_2}{2w_1}, \frac{w_2}{w_1})$. As Example 5.5 illustrates, even if $w_1 = w_2 = 1$, we can have several CSC rays in the $w$-Sasaki cone. This proves Theorem 1.3.

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