THE FOUR-FERMI MODEL IN THREE DIMENSIONS
AT NON-ZERO DENSITY AND TEMPERATURE

Simon HANDS

Department of Physics and Astronomy, University of Glasgow, Glasgow G12 8QQ, U.K.
and
Theory Division, CERN, CH-1211 Geneva 23, Switzerland

Aleksandar KOCIĆ

Theory Division, CERN, CH-1211 Geneva 23, Switzerland

John B. KOGUT

Department of Physics, University of Illinois at Urbana-Champaign
1110 West Green Street, Urbana, IL 61801, U.S.A.

Abstract

The Four Fermi model with discrete chiral symmetry is studied in three dimensions at non-zero chemical potential and temperature using the Hybrid Monte Carlo algorithm. The number of fermion flavors is chosen large ($N_f = 12$) to compare with analytic results. A first order chiral symmetry restoring transition is found at zero temperature with a critical chemical potential $\mu_c$ in good agreement with the large $N_f$ calculations. The critical index $\nu$ of the correlation length is measured in good agreement with analytic calculations. The two dimensional phase diagram (chemical potential vs. temperature) is mapped out quantitatively. Finite size effects on relatively small lattices and non-zero fermion mass effects are seen to smooth out the chiral transition dramatically.
1. Introduction

The behavior of symmetries at finite temperatures and densities is one of the most outstanding and relevant problems in many current areas of particle physics; e.g. cosmology, relativistic heavy-ion collisions, and the quark-gluon plasma [1]. In recent years we have witnessed revived interest in the chiral symmetry restoration transition in QCD. The problem of symmetry breaking and its restoration is intrinsically non-perturbative. Therefore, the number of available techniques is limited and most of our knowledge about the phenomenon comes from lattice simulations. Due to the enormous complexity of QCD, studies have so far been done on lattices of modest size and have been unable to yield quantitative claims as far as the order of the transition is concerned. This is unfortunate since several studies suggest that the high temperature phase of QCD has a number of interesting features [2]. In addition, only very slow progress has been made in lattice simulations at finite chemical potential, which is the regime of direct relevance to real physics [3].

In this paper we will approach the general problem of chiral symmetry restoration at finite temperature and density in a three-dimensional toy model in order to understand what ingredients might play a decisive role in more complex systems like gauge theories. We have simplified our model as much as possible in order to produce the highest quality data and learn what range of parameters we need for studies of more realistic cases. We have thus chosen to study the Gross-Neveu model [4], in which the chiral symmetry is discrete. The Lagrangian is

\[ \mathcal{L} = \sum_{j=1}^{N_f} \left[ \bar{\psi}^{(j)} \gamma \psi^{(j)} - \frac{g^2}{2N_f} (\bar{\psi}^{(j)} \psi^{(j)})^2 \right]. \]  

(1.1)

Here \( \psi^{(j)} \) is a four component spinor, \( N_f \) is the number of flavors of elementary fermion, and the discrete \( Z_2 \) chiral symmetry is \( \psi \rightarrow \gamma_5 \psi, \bar{\psi} \rightarrow -\bar{\psi} \gamma_5 \). Our choice of model also reflects our interest in its behavior at zero temperature and density [5,6]: in less than four dimensions the model has a non-trivial renormalization group fixed point, also characterized by the spontaneous breaking of the chiral symmetry at a critical coupling \( g_c^2 \). It is thus a toy model for the study of non-trivial strongly coupled theories. Although the theory is non-renormalizable in a standard perturbation expansion in the coupling \( g^2 \), its \( 1/N_f \) expansion about the fixed point \( g_c^2 \) is renormalizable. We have argued elsewhere [6] that this is closely related to the fact that the theory’s critical indices satisfy hyperscaling. Physically this means that the theory has a divergent correlation length at the fixed point and this length sets the scale for the theory’s low-energy phenomena. To \( O(1/N_f) \) the theory’s critical indices are [6]

\[ \nu = 1 + \frac{8}{3N_f \pi^2}; \quad \delta = 2 + \frac{8}{N_f \pi^2}; \quad \beta_m = 1; \quad \gamma = 1 + \frac{8}{N_f \pi^2}; \quad \eta = 1 - \frac{16}{3N_f \pi^2}, \]  

(1.2)

in the standard notation of classical statistical mechanics [7].
There are several motivations for studying such a simple model. In four dimensions the four-fermi model is believed to be an effective theory of quarks and gluons at intermediate energies. The degrees of freedom are light mesons and quarks. As far as finite temperature and density is concerned, the low temperature regime will be dominated by the lightest particles and, if the restoration temperature is of the order of 100MeV, then the contribution of the heavier particles like $\rho$ mesons will be exponentially suppressed. In that sense, the universal properties of chiral symmetry restoration in QCD could be well described by an effective theory like the Nambu – Jona-Lasinio model. In three dimensions, however, the four-fermi model is actually renormalizable, and precise analytic predictions are available from an expansion in $1/N_f$. Although we consider the simplest such model in this paper, extensions to more realistic models where the chiral symmetry is continuous are straightforward [4]. One might think of a yet more drastic simplification, and consider the model in two dimensions. However, in this case there are conceptual problems; eg. in the $Z_2$ case the symmetry restoration is now dominated by the materialization of kink – anti-kink states, which are composite states of the fundamental fermion fields, which are not probed in the $1/N_f$ expansion. In two dimensions the extension to continuous symmetries is also plagued by the non-existence of massless Goldstone bosons [8].

The major technical barrier to progress in simulating non-vanishing baryon number densities in QCD is the absence of a probabilistic interpretation of the path integral measure due to the action becoming complex once the chemical potential $\mu \neq 0$. In the model considered here, the action remains real even after the introduction of $\mu$, which means we can study the physics of the high-density regime using standard Monte Carlo techniques. In addition, it has fewer degrees of freedom than QCD, or even the Nambu – Jona-Lasinio model, and hence can be studied with greater precision on much bigger lattices than are presently used for QCD thermodynamics. In general, simulations focus on the evaluation of the order parameter $<\bar{\psi}\psi>$ which requires the inversion of the Dirac operator. Most numerical algorithms which simulate the effect of virtual quark – anti-quark pairs in the vacuum also require the inversion of this operator; this is the most computer-intensive step in such simulations. In gauge theories, this operator is singular in the chiral limit (ie. the matrix to be inverted becomes ill-conditioned), and the simulations have to be done using a finite bare mass. Information about the chiral limit is then obtained by extrapolating the finite mass data to $m = 0$. This last step is severely constrained by the lattice volume; $m$ can not be taken arbitrarily small, since otherwise the Compton wavelength of the Goldstone pion associated with the symmetry breaking would exceed the lattice size producing severe finite volume effects [9]. Consequently, lattice QCD data, always taken at finite mass, show only a crossover rather than a real phase transition. In our model, we are dealing with a Yukawa-like coupling (see below), so that the fermion matrix to be inverted has non-zero diagonal elements even when
the bare mass is set to zero. The order parameter (the inverse Dirac operator) is thus not singular and the simulations can be done in the chiral limit directly. In this way, we can in principle explore the systematics of finite mass effects, finite volume effects, and the validity of various extrapolation procedures. This might teach us how to use the QCD data better.

The mean field theory description of the transition, to be presented in Sec. 3, predicts a first order transition for \( T = 0 \) and a continuous transition for \( T > 0 \). Both analytic and numerical work in this model is aided by the introduction of an auxiliary scalar field \( \sigma \), so eqn. (1.1) becomes

\[
\mathcal{L} = \sum_{j=1}^{N_f} \left[ \bar{\psi}^{(j)} \partial \psi^{(j)} + \sigma \bar{\psi}^{(j)} \psi^{(j)} \right] + \frac{N_f}{2g^2} \sigma^2,
\]

and the Lagrangian becomes quadratic in the fermion field. The ground state expectation value of \( \sigma \) serves as a convenient order parameter for the theory’s critical point. Using the standard lattice regularization scheme to be discussed in Sec.2, the bulk critical coupling is found to be \( 1/g_0^2 = 0.975 - 1.000 \) for \( N_f = 12 \) when the chemical potential is set to zero as in eqn. (1.3) \[5\]. Simulation of \( 20^3 \) lattices at various couplings close to the critical point (within its scaling region as determined in ref. \[5\]) showed that \( \Sigma = \langle \sigma \rangle \) jumps \textit{discontinuously} to zero as \( \mu \) is increased and that the induced fermion density also jumps \textit{discontinuously} through the transition. In addition, the dependence of \( \mu_c \) on the coupling allows us to determine the correlation length exponent \( \nu \) for \( N_f = 12 \):

\[
\nu = 1.05(10),
\]

in good agreement with the large \( N_f \) prediction (1.2). Furthermore the magnitude of \( \mu_c \) itself is found to be in good agreement with the mean field result \( \mu_c = \Sigma_0 \), where \( \Sigma_0 \) is the value of the vacuum expectation of the scalar field at zero temperature and chemical potential. This result indicates that materialization of the fermion itself drives the symmetry restoration transition: this is not the case in the two-dimensional Gross-Neveu model, where kink – anti-kink states materialize at the transition \[10,11\]. The fact that \( \mu_c \) agrees with the mean field result is indirect evidence that such exotic fermion states in which the energy per constituent is smaller than the energy of a single fermion state do not occur in the three dimensional model \[12\].

We also simulated the model at both non-zero \( \mu \) and non-zero temperature \( T \). In ref. \[5\] we confirmed that the \( \mu = 0, T \neq 0 \) symmetry restoring transition occurs at \( T_c/\Sigma_0 \simeq 0.72 \) in good agreement with mean field predictions, and that this transition is second order. Here we map out the phase diagram in the \((\mu, T)\) plane.

The simplicity of this model and the efficiency of the simulation algorithm allowed us to address two technical issues of interest to lattice gauge theorists studying QCD in extreme environments. First, four-
dimensional QCD simulations are presently restricted to relatively small lattices. We show here that if the lattice is taken relatively small (12$^3$ as opposed to 20$^3$) the non-zero $\mu$ transition is smeared out and all evidence for a discontinuous transition is lost. Furthermore, as discussed above, QCD cannot be simulated directly in the chiral limit. We show here that, even on relatively large lattices and at couplings chosen in the scaling region, even very small bare fermion masses $m$ smooth the transition dramatically. We learn that it would be very difficult indeed to find evidence for a discontinuous transition by approaching the chiral limit $m = 0$ of the theory via $m \neq 0$ simulations.

This paper is organized into several sections. In Sec. 2 we briefly discuss the lattice formulation of the model and its simulation algorithm. More detail on these issues has already been provided in ref [5]. In Sec. 3 we present the mean field analysis of the theory – this overlaps to some extent with the results of ref. [12]. In Sec. 4 the simulation study of the zero-temperature model at non-zero chemical potential is presented. In Sec. 5 the model is studied at both $\mu \neq 0$ and $T \neq 0$. Finally, in Sec. 6 we consider finite volume effects by simulating on a 12$^3$ lattice, and finite $m$ effects on relatively large lattices. In both cases the discontinuous nature of the transition driven by non-zero chemical potential is lost. Sec. 7 summarizes our work.

2. Lattice Formulation of the Gross-Neveu Model

The Gross-Neveu model in its bosonised form (1.3) may be formulated on a space-time lattice using the following action:

$$S = \sum_{i=1}^{N_f/2} \left( \sum_{x,y} \bar{\chi}_i(x)M_{x,y}\chi_i(y) + \frac{1}{8} \sum_x \bar{\chi}_i(x)\chi_i(x) \sum_{\tilde{x},x} \sigma(\tilde{x}) \right) + \frac{N_f}{4g^2} \sum_{\tilde{x}} \sigma^2(\tilde{x}), \quad (2.1)$$

where $\chi_i, \bar{\chi}_i$ are Grassmann-valued staggered fermion fields defined on the lattice sites, the auxiliary scalar field $\sigma$ is defined on the dual lattice sites, and the symbol $<\tilde{x},x>$ denotes the set of 8 dual lattice sites $\tilde{x}$ surrounding the direct lattice site $x$. The lattice spacing $a$ has been set to one for convenience. The fermion kinetic operator $\mathcal{M}$ is given by:

$$\mathcal{M}_{x,y} = \frac{1}{2} \left[ \delta_{y,x+\hat{0}e^\mu} - \delta_{y,x-\hat{0}e^{-\mu}} \right] + \frac{1}{2} \sum_{\nu=1,2} \eta_\nu(x) \left[ \delta_{y,x+\hat{\nu}} - \delta_{y,x-\hat{\nu}} \right], \quad (2.2)$$

where $\eta_\nu(x)$ are the Kawamoto-Smit phases $(-1)^{x_0 + \cdots + x_{\nu-1}}$. The influence of the chemical potential $\mu$ is manifested through the timelike links, following [13]: only fermion loops which wrap around the timelike direction are affected by its inclusion.

The Gross-Neveu model in two dimensions was first formulated using auxiliary fields on the dual sites in reference [14]. We can motivate this particular scheme by considering a unitary transformation to fields
$u$ and $d$ [15]:

$$u_i^a(Y) = \frac{1}{4\sqrt{2}} \sum_A \Gamma_A^a \chi_i(A; Y),$$

$$d_i^a(Y) = \frac{1}{4\sqrt{2}} \sum_A B_A^a \chi_i(A; Y).$$

(2.3)

Here $Y$ denotes a site on a lattice of twice the spacing of the original, and $A$ is a lattice vector with entries either 0 or 1, which ranges over the corners of the elementary cube associated with $Y$, so that each site on the original lattice corresponds to a unique choice of $A$ and $Y$. The $2 \times 2$ matrices $\Gamma_A$ and $B_A$ are defined by

$$\Gamma_A = \tau_1^{A_1} \tau_2^{A_2} \tau_3^{A_3},$$

$$B_A = (-\tau_1)^{A_1} (-\tau_2)^{A_2} (-\tau_3)^{A_3},$$

(2.4)

where the $\tau_\nu$ are the Pauli matrices. Now, if we write

$$q_i^a(Y) = \left(\frac{u_i^a(Y)}{d_i^a(Y)}\right)^a,$$

(2.5)

and interpret $q$ as a 4-spinor with two flavors counted by the latin index $a$, then the fermion kinetic term of the action (2.1) may be recast in Fourier space as follows:

$$S_{\text{kin}} = \int \frac{d^3k}{(2\pi)^3} \sum_i \sum_{\nu=1,2} \frac{i}{2} \left\{ \bar{q}_i(k)(\gamma_\nu \otimes 1_2)q_i(k) \sin 2k_\nu + \bar{q}_i(k)(\gamma_5 \otimes \tau_\nu)q_i(k)(1 - \cos 2k_\nu) \right\}$$

$$+ \frac{1}{2} \left\{ \bar{q}_i(k)(\gamma_0 \otimes 1_2)q_i(k) \left[ i \sin 2k_0 \cosh \mu + (1 + \cos 2k_0) \sinh \mu \right] \right\}$$

$$+ \bar{q}_i(k)(\gamma_5 \otimes \tau_3^*)q_i(k) \left[ i(1 - \cos 2k_0) \cosh \mu + \sin 2k_0 \sinh \mu \right],$$

(2.6)

where

$$(\gamma_\nu)_{\alpha\beta} = \begin{pmatrix} \tau_\nu & -\tau_\nu \\ -\tau_\nu & \tau_\nu \end{pmatrix}_{\alpha\beta} : \ (\gamma_0)_{\alpha\beta} = \begin{pmatrix} \tau_3 & -\tau_3 \\ -\tau_3 & \tau_3 \end{pmatrix}_{\alpha\beta} : \ (\gamma_5)_{\alpha\beta} = \begin{pmatrix} i1_2 & -i1_2 \\ i1_2 & -i1_2 \end{pmatrix}_{\alpha\beta},$$

(2.7)

the second set of $(2 \times 2)$ matrices in the direct product act on the flavor indices, and the momentum integral extends over the range $k_\nu \in (-\pi/2, \pi/2)$. At non-zero temperature the lattice has finite extent in the temporal direction, and $\int dk_0$ is replaced by a sum over $N_T/2$ modes, where $N_T$ is the number of lattice spacings in the time direction, and antiperiodic boundary conditions are imposed on the fermion fields. In the classical continuum limit lattice spacing $a \to 0$, the flavor non-diagonal terms vanish as $O(a)$, and we recover the standard Euclidian form $\bar{q}_j(\theta + \mu \gamma_0)q_j$, where the flavor index $j$ now runs from 1 to $N_f$.

Similarly, it is straightforward to show that the interaction terms can be rewritten (with obvious notation):

$$S_{\text{int}} = \sum_Y \left( \sum_A \sigma(A; \bar{Y}) \right) \bar{q}_i(Y)(1_4 \otimes 1_2)q_i(Y) + O(a),$$

(2.8)

where the $O(a)$ terms contain non-covariant and flavor non-singlet terms. If we used a formulation in which the $\sigma$ fields lived on the direct lattice sites, then such non-covariant terms would contribute at $O(a^0)$ [14].
Thus we see that the lattice action (2.1) reproduces the bosonised Gross-Neveu model at non-zero density, at least in the classical continuum limit. Most importantly, (2.1) has a discrete global invariance under
\[ \chi_i(x) \mapsto (-1)^{x_0+x_1+x_2}\chi_i(x); \quad \bar{\chi}_i(x) \mapsto -(-1)^{x_0+x_1+x_2}\bar{\chi}_i(x); \quad \sigma(\tilde{x}) \mapsto -\sigma(\tilde{x}), \]
(2.9a)

ie.
\[ q_i(Y) \mapsto (\gamma_5 \otimes 1)q_i(Y); \quad \bar{q}_i(Y) \mapsto -\bar{q}_i(Y)(\gamma_5 \otimes 1); \quad \sigma(\tilde{x}) \mapsto -\sigma(\tilde{x}). \]
(2.9b)

It is this symmetry, corresponding to the continuum form \( \psi \mapsto \gamma_5\psi, \bar{\psi} \mapsto -\bar{\psi}\gamma_5 \), which is spontaneously broken at strong coupling, signalled by the appearance of a non-vanishing condensate \( \langle \bar{\chi}\chi \rangle \) or equivalently \( \langle \bar{q}(1_4 \otimes 1_2)q \rangle \). We shall see in the next section that to leading order in \( 1/N_f \) the lattice formulation (2.1) gives predictions in agreement with those of the continuum.

The action (2.1) was numerically simulated using the hybrid Monte Carlo algorithm [16], in which the Grassmann fields are replaced by real bosonic pseudofermion fields \( \phi(x) \) governed by the action
\[
S = \sum_{x,y} \frac{N_f}{2} \sum_{i,j=1}^2 \phi_i(x)(M_i^2 M)^{-1}_{xyij}\phi_j(y) + \frac{N_f}{4g^2} \sum_{\tilde{x}} \sigma^2(\tilde{x}),
\]
(2.10)

where
\[
M_{xyij} = M_{xy} \delta_{ij} + \delta_{xy} \delta_{ij} \frac{1}{8} \sum_{\tilde{x},x} \sigma(\tilde{x}).
\]
(2.11)

Note \( M \) is strictly real. Integration over \( \phi \) yields the functional measure \( \sqrt{\det(M_i^2 M)} = \det M \) if the determinant of \( M \) is positive semi-definite. This condition is fulfilled if \( N_f/2 \) is an even number, even for \( \mu \neq 0 \). The problem of complex determinants associated with simulating gauge theories at finite density do not appear. Further details of our simulation procedure are given in [5].

As well as measuring the expectation value of the scalar field \( \langle \sigma \rangle \) in the simulation, which for our purposes is the order parameter of the transition, we also monitored the chiral condensate \( \langle \bar{\psi}\psi \rangle \), the energy density \( \langle \epsilon \rangle \), and the fermion number density \( \langle n \rangle \), which are defined by
\[
-\langle \bar{\psi}\psi \rangle = \frac{1}{V} \text{tr} S_F = \frac{1}{V} \langle \text{tr} M^{-1} \rangle,
\]
\[
\langle \epsilon \rangle = \frac{1}{V_s} \frac{1}{\beta} \frac{\partial \ln Z}{\partial \beta} = \frac{1}{V} \text{tr} \partial_\beta \gamma_0 S_F = \frac{1}{2V} \langle \sum_x e^\mu M^{-1}_{x,x+0} - e^{-\mu} M^{-1}_{x,x-0} \rangle,
\]
\[
\langle n \rangle = -\frac{1}{V_s \beta} \frac{1}{\mu} \frac{\partial \ln Z}{\partial \mu} = \frac{1}{V} \text{tr} \gamma_0 S_F = \frac{1}{2V} \langle \sum_x e^\mu M^{-1}_{x,x+0} + e^{-\mu} M^{-1}_{x,x-0} \rangle.
\]
(2.12)

Here \( V_s \) is the spatial volume, \( \beta \) the inverse temperature, and \( V = V_s \beta \) the overall volume of spacetime. The final expression in each case is the quantity measured in the simulation, using a noisy estimator to calculate the matrix inverses.
3. Mean Field Analysis

In this section we calculate the order parameter $\Sigma = \langle \sigma \rangle$ as a function of coupling $g$, chemical potential $\mu$, and temperature $T$ ($\equiv 1/\beta$), to leading order in $1/N_f$. Since the limit $N_f \to \infty$ suppresses fluctuations around the saddle point solution, this is equivalent to a mean field treatment. We will work both in the continuum and using the specific lattice regularisation (2.1). If diagrams of $O(1/N_f)$ and beyond are ignored, the only contribution to $\Sigma$ comes from a simple fermion loop tadpole, and we determine $\Sigma$ self-consistently using the gap equation:

$$\Sigma = -g^2 \langle \bar{\psi} \psi \rangle = g^2 \frac{\text{tr} S_F(\mu, T, \Sigma)}{V},$$

(3.1)

where in the Euclidean formulation the fermi propagator $S_F$ is given by

$$S_F^{-1}(k; \mu, T, \Sigma) = i\gamma_0 (k_0 - i\mu) + \sum_{\nu=1,2} i k_\nu \gamma_\nu + \Sigma.$$ 

(3.2)

For non-zero temperatures the allowed values of $k_0$ are quantised as

$$k_0 = (2n - 1)\pi T, \quad n \in \mathbb{Z},$$

(3.3)

ie. with antiperiodic boundary conditions in the finite temporal direction. Collecting together equations (3.1-3) we arrive at

$$\frac{1}{g^2} = 4T \sum_{n=-\infty}^{\infty} \int \frac{d^2 p}{(2\pi)^2} \frac{1}{((2n-1)\pi T - i\mu)^2 + p^2 + \Sigma^2}.$$ 

(3.4)

The manipulations from here are standard [10,12]. First one resums over $n$ using the Poisson formula:

$$\frac{1}{g^2} = 4T \sum_{m=-\infty}^{\infty} \int \frac{d^2 p}{(2\pi)^2} \int_0^{\infty} \frac{d\phi}{2\pi} \frac{e^{2\pi im\phi}}{((2\phi - 1)\pi T - i\mu + iE)((2\phi - 1)\pi T - i\mu - iE)}.$$ 

(3.5)

Here $E = \sqrt{p^2 + \Sigma^2}$. If $\mu < \Sigma$, then the poles in the integrand lie on opposite sides of the integration contour for all values of $p$, and the integral over $\phi$ is easily performed to yield

$$\frac{1}{g^2} = \int_{\Sigma}^{\infty} \frac{dE}{\pi} \left[ 1 - \frac{1}{e^{\beta(E - \mu)} + 1} - \frac{1}{e^{\beta(E + \mu)} + 1} \right].$$

(3.6)

If, however, $\mu > \Sigma$ we must take care, since for certain values of $p$, both poles in the integrand of (3.5) will lie to the same side of the contour. We find in this case

$$\frac{1}{g^2} = \int_\mu^{\infty} \frac{dE}{\pi} \left[ 1 - \frac{1}{e^{\beta(E - \mu)} + 1} \right] + \int_{\Sigma}^{\mu} \frac{dE}{\pi} \frac{1}{e^{\beta(E - \mu)} + 1} - \int_{\Sigma}^{\infty} \frac{dE}{\pi} \frac{1}{e^{\beta(E + \mu)} + 1}. $$

(3.7)

We now eliminate $g$ in favour of $\Sigma_0$, the order parameter at zero temperature and chemical potential, using equation (3.6) at $T = \mu = 0$. This gives an implicit equation for $\Sigma$ in terms of a physical scale $\Sigma_0$, with no reference to any UV cutoff. For $\mu < \Sigma$ we obtain

$$\Sigma_0 - \Sigma = T \left( \ln(1 + e^{-\beta(\Sigma - \mu)}) + \ln(1 + e^{-\beta(\Sigma + \mu)}) \right).$$

(3.8a)
while for $\mu > \Sigma$:

$$\Sigma_0 - \mu = T \left( \ln(1 + e^{-\beta(\mu - \Sigma)}) + \ln(1 + e^{-\beta(\mu + \Sigma)}) \right). \quad (3.8b)$$

In fact, these two equations are identical solutions for $\Sigma(\mu, T)$, and also demonstrate that curves of $\Sigma(\mu)$ at constant $T$ are symmetric under reflection in the line $\Sigma = \mu$. This result is peculiar to three spacetime dimensions. Equation (3.8a) was first derived in [12].

Equation (3.8) gives a complete solution for $\Sigma(\mu, T)$ in terms of $\Sigma_0$. Since $\Sigma \to 0_+$ smoothly, the symmetry-restoring transition is second order throughout the $(\mu, T)$ plane, except for one isolated point, as we shall see. To obtain the equation for the critical line in this plane we set $\Sigma = 0$ in (3.8) to get the curve:

$$1 - \frac{\mu}{\Sigma_0} = 2 \frac{T}{\Sigma_0} \ln(1 + e^{-\beta \mu}). \quad (3.9)$$

At zero chemical potential, therefore, we predict a chiral symmetry-restoring transition at a critical temperature

$$T_c = \frac{\Sigma_0}{2 \ln 2} \simeq 0.72 \Sigma_0. \quad (3.10)$$

The gap equation in the broken phase in this limit is the $\mu = 0$ limit of (3.8):

$$\Sigma_0 - \Sigma = 2T \ln(1 + e^{-\beta \Sigma}). \quad (3.11)$$

At zero temperature we find $\Sigma = \Sigma_0$ independent of $\mu$ up to a critical value

$$\mu_c = \Sigma_0, \quad (3.12)$$

at which point there is a discontinuous drop to zero, i.e. at this isolated point the transition is first order.

For small excursions into the $(\mu, T)$ plane we find from (3.8)

$$\frac{\partial \Sigma}{\partial \mu} \bigg|_{T\to 0} = \lim_{T\to 0} \frac{\sinh \beta \mu}{\sinh \beta \Sigma} \simeq -e^{\beta(\mu - \Sigma)}, \quad (3.13)$$

ie. the slope of the surface $\Sigma(\mu, T)$ diverges in an essentially singular way as $\mu \to \mu_c$, $T \to 0$. So, the mean field analysis predicts a first order transition for $T = 0$, which becomes second order as soon as $T > 0$.

Using a similar route we can also calculate the fermion number density $< n >$ in the broken phase starting from (2.12). For $\mu < \Sigma$ we find

$$< n > = \frac{\Sigma T}{\pi} \ln \left( \frac{1 + e^{-\beta(\Sigma - \mu)}}{1 + e^{-\beta(\Sigma + \mu)}} \right) - \frac{2T^2}{\pi} \sum_{k=1}^{\infty} (-1)^k \frac{e^{-\beta k \Sigma}}{k^2} \sinh \beta k \mu, \quad (3.14)$$

whereas for $\mu > \Sigma$:

$$< n > = \frac{\mu^2 - \Sigma^2}{2\pi} + \frac{\Sigma T}{\pi} \ln \left( \frac{1 + e^{-\beta(\mu - \Sigma)}}{1 + e^{-\beta(\mu + \Sigma)}} \right) - \frac{2T^2}{\pi} \sum_{k=1}^{\infty} (-1)^k \frac{(1 - e^{-\beta k \mu}) \cosh \beta k \Sigma}{k^2}. \quad (3.15)$$
In the symmetric phase we recover the usual expression for a two-dimensional relativistic free fermi gas:

\[ <n> = \frac{\mu^2}{2\pi} - \frac{2T^2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k (1 - e^{-\beta k\mu})}{k^2}. \]  

(3.16)

In the limit \( T \to 0 \) we see that fermion density is strongly suppressed for \( \mu < \Sigma \), then jumps discontinuously and continues to rise quadratically with \( \mu \) as soon as \( \mu \) exceeds \( \Sigma \). As required, \( <n> \) vanishes for all \( T \) when \( \mu = 0 \).

We have also studied the gap equation using an explicit UV regularisation defined by the lattice action (2.1). This is useful for comparison with the numerical results, particularly so that we can determine whether any differences with the continuum predictions arise from genuine \( 1/N_f \) corrections (ie. departure from mean field behaviour), or simply from the fact that on a finite lattice it is impossible to attain the thermodynamic limit. This latter point arises because lattice simulations at non-zero temperature are generally accomplished using a system with \( N_\tau \) lattice points in the temporal direction, with \( N_\tau \ll N \), the spatial dimension. In the work presented here \( N_\tau \) ranges from 6 to 12, for \( N = 36 \). Clearly the main effect is that the sum over Matsubara frequencies in the lattice gap equation is truncated at a rather small value of \( n \).

Using the free fermion action in the form (2.6) we evaluate equation (3.1) to yield the lattice gap equation on a system of infinite spatial extent:

\[ \frac{1}{g^2} = \frac{2}{N_\tau} \int_{\pi/2}^{\pi/2} d^2k \sum_{n} \frac{1}{2} \left\{ 1 - \cos \left( \frac{2\pi(2n-1)}{N_\tau} \right) \cosh 2\mu - i \sin \left( \frac{2\pi(2n-1)}{N_\tau} \right) \sinh 2\mu \right\} + \sum_{\nu=1}^{2} \sin^2 k\nu + \Sigma^2, \]

(3.17)

where \( \sum_n \) defines a sum running from \( -\frac{N_\tau}{4} + \frac{1}{2} \) to \( \frac{N_\tau}{4} - \frac{1}{2} \) if \( N_\tau/2 \) is odd or \( -\frac{N_\tau}{4} + 1 \) to \( \frac{N_\tau}{4} \) if \( N_\tau/2 \) is even (note \( N_\tau \) must be even in order for staggered fermions to be defined). The integral over \( k \) can be done via Schwinger parameterisation to yield

\[ \frac{1}{g^2} = \frac{2}{N_\tau} \int_0^{\infty} d\alpha \alpha e^{-\alpha(\frac{\pi}{2} + \Sigma^2)} I_0^2(\alpha) \sum_n \exp \left[ \frac{\alpha}{2} \cos \left( \frac{2\pi(2n-1)}{N_\tau} \right) \cosh 2\mu \right] \cos \left[ \frac{\alpha}{2} \sin \left( \frac{2\pi(2n-1)}{N_\tau} \right) \sinh 2\mu \right], \]

(3.18)

which is now in a form suitable for numerical quadrature. \( I_0 \) is the modified Bessel function. Unfortunately the integral over \( \alpha \) in (3.18) only converges for \( 2\mu \leq \cosh^{-1}(\sec 2\pi/N_\tau) \): for values of \( \mu \) greater than this, although (3.17) is convergent, it is no longer possible to cast it into convenient form.

Figure 3.1 shows a comparison of \( \Sigma(\mu) \) calculated using the continuum solution (3.8) and the lattice solution (3.18) evaluated at inverse coupling \( 1/g^2 = 0.75 \) on a lattice with \( N_\tau = 6, 8, 10 \) and 12. In practice we solve (3.18) for \( 1/g^2 \) as a function of \( \Sigma \) and invert using interpolation. The continuum solution sets \( \beta = N_\tau \) and uses a value for \( \Sigma_0 \) given by the lattice gap equation at zero temperature and chemical potential:

\[ \frac{1}{g^2} = \int_0^{\infty} d\alpha e^{-\alpha(\frac{\pi}{2} + \Sigma_0^2)} I_0^2(\alpha). \]

(3.19)
We see that the agreement between the two is fair, with the lattice results always lying below the continuum ones. As discussed above, we ascribe this difference to the finite number of thermal modes available on the lattice.

4. Non-zero Chemical Potential Simulation Results

We first studied the theory at non-zero chemical potential on a symmetric lattice. In ref. [5] we obtained accurate results at vanishing chemical potential for lattices ranging in size from $8^3$ through $20^3$. The vacuum expectation value of $\sigma$, and its susceptibility were measured over a range of coupling extending from $1/g^2 \sim 0.5$ to 1.2. Good agreement with large $N_f$ scaling laws were found for $1/g^2$ ranging from 0.70 to 1.1 and the critical point in the infinite volume limit was estimated to be $1/g_2^* = 0.975 - 1.00$. So, the scaling window of the lattice formulation lying in the chirally asymmetric phase was seen to be $0.70 \leq 1/g^2 \leq 1.00$. These results led us to simulate the model with $\mu \neq 0$ on a $20^3$ lattice at couplings $1/g^2 = 0.70, 0.75$ and 0.80. Short exploratory runs indicated that larger lattices would be needed to push $(1/g^2)_{\text{critical}}$ closer to the critical point. In Table 1 we show the simulation results for various $\mu$ at $1/g^2 = 0.70$, measurements of $\Sigma = \langle \sigma \rangle$, the energy density $\langle \epsilon \rangle$, and the induced ground state fermion number density $\langle n \rangle$ are recorded. We also show the number of trajectories of the hybrid Monte Carlo algorithm at each point. The huge number of trajectories relative to state-of-the-art lattice QCD simulations with dynamical fermions was possible because of the three dimensional character of the model and its relatively simple form, a random scalar field coupled to a fermionic scalar density. This last feature led to a conjugate gradient routine which converged with an order of magnitude fewer sweeps than typically needed in lattice QCD simulations. The results recorded in Table 1 are plotted in Figure 2 where we see a jump discontinuity in $\Sigma$ vs. $\mu$ of 0.275 as $\mu$ varies from 0.39375 to 0.3941. Note from the table that we were particularly careful to accumulate good statistics near the transition. In Figure 3 we show the induced fermion number $\langle n \rangle$ plotted against $\mu$ which shows a clear jump over the same coupling range. We believe that the nonvanishing values of $\langle n \rangle$ recorded for $\mu < 0.3941$ are finite size (temperature) effects. Finite size effects will be discussed further in Section 6 below. The error bars in the table and plotted in the figures come from standard binning procedures. It was possible to "confirm" these error estimates in many cases by running the algorithm for an extra several thousand trajectories and reproducing average values and variances.

As shown in Table 2 the simulation was repeated at $1/g^2 = 0.75$, slightly closer to the bulk critical point. The critical chemical potential shifted to $\mu_c = 0.32 - 0.3225$ and discontinuities were again observed in $\Sigma$ vs. $\mu$ (Figure 4) and $\langle n \rangle$ vs. $\mu$ (Figure 5). The discontinuities were slightly more difficult to measure because of the proximity to the critical point which reduces the size of physical observables when measured
in lattice units.

Runs were also completed at $1/g^2 = 0.80$ and a value of $\mu_c = 0.250(5)$ was measured. Less extensive data was taken here as reflected in the larger error bar. We identified $\mu_c$ by plotting time histories of $\Sigma$ ($\Sigma$ vs. computer time) and noting that for $\mu < \mu_c$ maintained a non-zero value while for $\mu > \mu_c$ evolved to zero and fluctuated around it. These results were less quantitative than the $1/g^2 = 0.70$ and 0.75 simulations because critical slowing down was affecting the efficiency of the algorithm. In fact, we abandoned an attempt to study $1/g^2 = 0.85$ because of severe critical slowing down and the possibility of large finite size effects invalidating an estimate of $\mu_c$ on a $20^3$ lattice.

It is interesting to analyze the measurements of $\mu_c$ at $1/g^2 = 0.70, 0.75$ and 0.80 for two additional purposes. The first is to estimate $\mu_c$ in physical rather than lattice units. The natural way to do this is to record the ratio of $\mu_c$ to $\Sigma_0$ measured at the same coupling as the finite-$m$ transition. At large $N_f$, $\Sigma_0$ is essentially the dynamical fermion mass and Mean Field theory predicts that $\mu_c/\Sigma_0 = 1.0$. From ref.[5] we have the values of $\Sigma_0$ on a $20^3$ lattice ($\mu = 0$) at $1/g^2 = 0.70(\Sigma_0 = 0.432), 1/g^2 = 0.75(\Sigma_0 = 0.346)$ and $1/g^2 = 0.80(\Sigma_0 = 0.262)$. The ratios $\mu_c/\Sigma_0$ at each coupling are plotted in Figure 6. The error bars come almost exclusively from the $\mu_c$ measurements – the $20^3 \mu = 0$ measurements of $\Sigma_0$ recorded in ref.[5] were very accurate indeed. The $1/g^2 = 0.80$ result for $\mu_c$ is particularly uncertain. Nonetheless, as the critical point is approached the curve strongly suggests that $\mu_c/\Sigma_0$ approaches unity in accord with Mean Field theory, although a departure downwards from this value as a result of $1/N_f$ corrections cannot be ruled out.

Our last use for this data is for a calculation of the critical index $\nu$, the critical index of the correlation length. Since the critical chemical potential $\mu_c$ is a dimensionful parameter coupled to a conserved current in the Lagrangian, it undergoes no renormalization due to $1/N_f$ corrections, and should scale as a physical mass as the critical point is approached, i.e. vanishing in lattice units with the exponent $\nu$

$$\mu_c = C(1/g^2_*-1/g^2)^\nu \quad (4.1)$$

From ref.[5] $1/g^2_* = 0.975 - 1.00$. In Figure 7 we show a plot of ln $\mu_c$ vs. ln $\left(1/g^2 - 1/g^2_*\right)$ with choice $1/g^2_* = 0.975$. A linear fit is good and it gives $\nu = 1.00(5)$ to be compared with the first two terms of the large-$N_f$ expansion ($\nu = 1 + 8/3N_f\pi^2 = 1.0225.\ldots$ for $N_f = 12$). A similar plot for $1/g^2_* = 1.00$ gives $\nu = 1.10(5)$. So, in summary we have

$$\nu = 1.05(10) \quad (4.2)$$

in excellent agreement with the large-$N_f$ analysis. This analysis and Eq. (4.2) could certainly be pursued more systematically with greater control, but consistency between the analytic and numerical approaches to
this problem is our only goal here.

A systematic search for $1/N_f$ effects in our simulation results, or in more accurate simulations on larger lattices that might be done in the future, would require more calculations than done here. For example, if we take our results for $\mu_c/\Sigma_0$ at face value, i.e. an increasing function of $1/g^2$ as $1/g^2 \rightarrow 1/g_\ast^2$, then we would predict the exponent $\nu$ to be smaller than the exponent $\beta_m$ governing the scaling of $\Sigma_0$, independently of any estimate of the exact value of $1/g_\ast^2$. This is at variance with the $O(1/N_f)$ corrections of eq.(1.2) which predict

$$
\frac{\mu_c(g)}{\Sigma_0(g)} \sim \frac{(1/g_\ast^2 - 1/g^2)^\nu}{(1/g_\ast^2 - 1/g^2)^{\beta_m}} \sim (1/g_\ast^2 - 1/g^2)^{\frac{8}{3N_f}\pi^2}
$$

(4.3)

Numerically $8/3N_f\pi^2 = 0.0225$ for $N_f = 12$, and since $1/g_\ast^2 - 1/g^2$ varies from 0.3 to 0.2 over the region of couplings explored here, the right hand side of eq.(4.3) varies by less than a percent. Clearly much greater precision is needed before we can interpret the trend of Figure 7 to $1/N_f$ effects, which would require large corrections at $O(1/N_f^2)$ and beyond.

5. Non-Zero Chemical Potential and Temperature

In ref.[5] we studied the four Fermi model at non-zero temperature by simulating it on asymmetric lattices $N_T \times N^2$ with $N \gg N_T$ and $N_T$ ranging from 4 to 12. A second order chiral transition was discovered with a critical temperature measured in physical units of $T_c/\Sigma_0 \approx 0.70(5)$. The result was in good agreement with Mean Field theory which predicts $T_c/\Sigma_0 = (2 \ln 2)^{-1} = 0.721$. This result and the zero temperature, non-zero chemical potential result of Section 4, were then extended to map out the $\mu - T$ phase diagram. We simulated $6 \times 36^2, 8 \times 36^2, 10 \times 36^2$ and $12 \times 36^2$ lattices and obtained $\mu_c$ on each of these lattices by measuring $\Sigma$ vs. $\mu$. Each lattice was simulated at a coupling $1/g^2 = 0.75$ because this coupling lay in the scaling window of the lattice Lagrangian and because of the success of Section 4. The curves of $\Sigma$ vs. $\mu$ are shown in Figure 8. The statistics at each point are comparable with Table 1 and the error bars on each point in the figure are smaller than the symbols themselves. The precise values of $\mu_c$ for each $N_T$ are recorded in Table 3 with error bars. As usual, vacuum tunnelling (flipflops between $\pm \Sigma$ values) limited our precision on $\mu_c$ measurements. It is interesting to compare Fig. 8 with its Mean Field counterpart in Fig. 1. The qualitative features of both plots are identical, but the scale of $\Sigma$ and $\mu$ values coming from the computer experiment are consistently below the Mean Field predictions.

The results in Table 3 can be converted to physical points on a $\mu_c/\Sigma_0$ vs. $T_c/\Sigma_0$ phase diagram by recalling the value of $\Sigma_0$ at $1/g^2 = 0.75$ at zero temperature, $\Sigma_0 = 0.346$. A bit of arithmetic and the fact that the temperature is related to the temporal extent of the lattice $N_T$, $T = N_T^{-1}$, produces the phase diagram
of Figure 9. The solid line is the Mean Field phase boundary between the low-$T$, low-$\mu$ symmetry broken phase and the chiral symmetry restored phase. Note that the point at $T_c/\Sigma_0 = 0.482$ and $\mu_c/\Sigma_0 = 0.564$ comes from the smallest lattice $N_\tau = 6$ and is subject to the largest finite size correction. The data point at $m = 0$ comes from ref.[5].

It would be interesting to determine the order of the phase transition in Figure 9 away from the $\mu = 0$ and $T = 0$ boundaries to test the Mean Field prediction of a line of second order transitions inside the plane. Our attempts to do this used time histories of $\Sigma$ to search for two-state signals either visually or through histogramming. The results were not conclusive, however. In light of finite size effects to be discussed below, we believe that larger $N_\tau$ values would actually be needed to accomplish this goal.

6. Finite Size and Finite Mass Effects

Finally, we did two additional simulations motivated as much by lattice QCD simulations as our interest in three dimensional four Fermi models. It is clear from the lattice Mean Field discussion in Section 3 above that even in the large $N_f$ limit where fluctuations are suppressed relatively large lattices are needed to simulate the theory’s actual critical behavior. We show this effect in Figure 10 which shows the induced fermion number plotted against $\mu$ for a $12^3$ lattice at $1/g^2 = 0.70$. The discontinuous transition seen on a $20^3$ lattice in Figure 3 is replaced by a relatively smooth curve. A careful search for metastability in the region $0.35 < \mu < 0.40$ revealed none.

It is also of interest to add a small explicit chiral symmetry breaking fermion bare mass to the Lagrangian and record its tendency to smooth out the transition. In fact, even a very small bare fermion mass obscures the first order transition of Figure 3 entirely. This effect is shown in Figure 11 where the theory is again simulated on a $20^3$ lattice at $1/g^2 = 0.70$ and the induced fermion charge is measured with the bare mass of the dynamical fermion chosen to be either $m = 0.01$ or 0.005 in lattice units. A comparison with Figure 3 shows that $<n(\mu, m)>$ decreases as $m$ is increased from zero; in either phase this reflects the fact that more energy is always needed to excite fermion states from the vacuum once a bare mass is introduced. The average value of the sigma field is shown in Figure 12 for the same conditions. In neither case did we find any evidence for metastability, although, as shown in the figures, a very fine grid of $\mu$ values were simulated. One can also use the $m = 0.01$ and 0.005 results in Figure 12 to linearly extrapolate the average values of $\Sigma$ to zero fermion mass at each value of $\mu$. One can read off the figure that this procedure predicts a critical point at $\mu_c = 0.42 - 0.43$, considerably higher than the actual value of 0.3933(8) shown in Figure 1. A more sophisticated extrapolation method appears required to achieve quantitative results. QCD enthusiasts may find similar problems in their more complicated systems.
7. Conclusions

In this study we have presented results of numerical simulations of an interacting relativistic field theory near its continuum limit at non-vanishing baryon-number density on far larger systems than have been possible hitherto [3]. The main result of our work is that the data we have is in excellent qualitative agreement with analytic predictions based on the $1/N_f$ expansion. We observe that in the regime $1/g^2 < 1/g_*^2$ where chiral symmetry is spontaneously broken at zero temperature and density, for sufficiently large chemical potential $\mu$ there is a phase transition which restores the symmetry. For $T = 0$ the transition is first order: for $T > 0$ the transition appears continuous. Both results are consistent with the leading order $1/N_f$ prediction that the transition at $(\mu/\Sigma_0 = 1, T = 0)$ is an isolated first order point; however, the possibility of first order behavior persisting for values of $T > 0$ cannot be excluded. In all cases our measured values of $\Sigma(\mu, T)$ fall slightly below the predictions of Sec. 2 – it will require further systematic study to establish whether this is due to finite volume effects or to genuine $1/N_f$ corrections.

It is interesting to contrast this situation with that of the two-dimensional Gross-Neveu model, where the leading order $1/N_f$ expansion predicts a very rich phase diagram with a tricritical point in the $(\mu, T)$ plane separating first and second order symmetry restoring transitions [17]. Once quantum fluctuations are switched on, however, (ie. $N_f < \infty$), the symmetry restoration is dominated by condensation of kink – anti-kink states, which are not present in the $1/N_f$ expansion, and the transition becomes first order everywhere apart from an isolated second-order point at $\mu = 0$ [11].

We can speculate on whether the first order nature of the $T = 0$ transition of the three-dimensional model remains stable as $T$ is increased from zero, ie. whether there is in fact a tricritical point somewhere in the $(\mu, T)$ plane in this case, around which $\Sigma$ exhibits power-law scaling rather than the essential singularity predicted in eqn. (3.13). Studies on much larger systems, enabling us to probe sufficiently low temperatures free of finite volume effects, would be required to locate such a point unambiguously. However, we note supporting evidence for the existence of such a point from a comparison on Figs. 3 and 10, showing plots of $<n>$ vs. $\mu$ on $20^3$ and $12^4$ systems respectively. The major difference in the plots occurs in the broken phase, where $<n>$ is considerably suppressed on the larger system, supporting the assertion made in Sec. 4 that the signal in this phase is a finite volume effect (mean field theory predicts $<n> = 0$ for $T = 0$ in the broken phase, eqn. (3.14)). Even on the $20^3$ lattice the presence of a signal shows that the system is experiencing a small but non-zero effective temperature: however the symmetry-restoring transition on this lattice is clearly first order, suggesting that the discontinuous nature of the transition is stable some way into the $(\mu, T)$ plane. Studies of the $O(1/N_f)$ corrections would be of value here: a first order transition would manifest itself in an extra non-trivial solution of the gap equation for $\mu, T > 0$, implying an extra extremum
in the effective potential for \( \Sigma [17] \).

By studying the scaling of the critical chemical potential \( \mu_c \) as a function of coupling, we have been able to extract the correlation length critical exponent \( \nu \). This is unusual, because we have only measured bulk properties of the system, and not any two-point correlations. Of course, our analysis implicitly relies on the theory’s renormalizability [5,6]. Our result is in good agreement with the leading order \( 1/N_f \) prediction \( \nu = 1 \), and our precision is slightly better than that obtained through finite-size scaling studies [5]. As always in these measurements, the major uncertainty is the location of the bulk critical point \( 1/g_2^2 \). Clearly from our data much greater precision will be required to probe \( O(1/N_f) \) corrections.

Finally, our studies on small volumes and with non-zero bare masses highlight the difficulties that must be faced when trying to understand the critical behavior of the chiral/thermodynamic limit by extrapolation from systems away from these limits. In this case the first order nature of the zero temperature transition is obscured – this should be a warning for the lattice QCD community (if one is still needed!) that a quantitative understanding of the behavior of QCD at non-vanishing density lies along a difficult road.

Acknowledgments

The calculations reported here used the resources of the National Energy Research Supercomputer Center, the Pittsburgh Supercomputer Center and the Materials Research Laboratory at the University of Illinois at Urbana-Champaign under grant NSF DMR89-20538. JBK is supported in part by NSF-PHY87-01775. SJH is supported in part by an SERC Advanced Fellowship, and we would like to thank Bill Wyld for helpful discussions in the early stages of this work.
References

[1] D.A. Kirzhnits and A.D. Linde, Ann. Phys. 101 (1976) 195.
[2] C. Bernard, T.A. DeGrand, C. DeTar, S. Gottlieb, A. Krasnitz, M.C. Ogilvie, R.L. Sugar and D. Toussaint, Phys. Rev. Lett. 68 (1992) 2125;
T.H. Hansson and I. Zahed, Nucl. Phys. B374 (1992) 277;
G.E. Brown, A.D. Jackson, H.A. Bethe and P.M. Pizzochero, SUNY preprint NTG-92-06 (1992).
[3] I.M. Barbour, talk presented at Lat '91, Tsukuba, Japan (to appear in Nucl. Phys. B(Proc. Suppl.)).
[4] D.J. Gross and A. Neveu, Phys. Rev. D20 (1974) 3235.
[5] S.J. Hands, A. Kocić and J.B. Kogut, CERN preprint.
[6] S.J. Hands, A. Kocić and J.B. Kogut, Phys. Lett. B273 (1991) 111.
[7] S.-K. Ma, Modern Theory of Critical Phenomena (Benjamin, New York, 1976).
[8] N.D. Mermin and H. Wagner, Phys. Rev. Lett. 17 (1966) 1133;
S. Coleman, Commun. Math. Phys. 31 (1973) 259.
[9] T. Jolicour and A. Morel, Nucl. Phys. B262 (1985) 627;
J. Gasser and H. Leutwyler, Phys. Lett. 188B (1987) 477.
[10] R.F. Dashen, S.-K. Ma and R. Rajamaran, Phys. Rev. D11 (1975) 1499.
[11] F. Karsch, J.B. Kogut and H.W. Wyld, Nucl. Phys. B280 [FS18] (1987) 289.
[12] B. Rosenstein, B.J. Warr and S.H. Park, Phys. Rev. D39 (1989) 3088.
[13] J.B. Kogut, H. Matsuoka, S.H. Shenker, J. Shigemitsu, D.K. Sinclair, M. Stone and H.W. Wyld, Nucl. Phys. B225 [FS9] (1983) 93;
P. Hasenfratz and F. Karsch, Phys. Lett. B125 (1983) 308.
[14] Y. Cohen, S. Elitzur and E. Rabinovici, Nucl. Phys. B220 (1983) 102.
[15] C.J. Burden and A.N. Burkitt, Europhys. Lett. 3 (1987) 545.
[16] S. Duane, A.D. Kennedy, B.J. Pendleton and D. Roweth, Phys. Lett. B195 (1987) 216.
[17] U. Wolff, Phys. Lett. 157B (1985) 303.
Table Captions

1. Chemical Potential $\mu$, ground state expectation value of the sigma field $\Sigma$, energy density $\epsilon$ and ground state expectation value of the induced fermion number $<n>$ on a $20^3$ lattice at coupling $1/g^2 = 0.70$. The number of trajectories for the hybrid Monte Carlo algorithm is recorded in the last column.

2. Same as Table 1 except $1/g^2 = 0.75$.

3. Measurement of $\Sigma$ vs. $\mu$ on $N_t \times N^2$ ($N = 36$) lattices. The coupling $1/g^2 = 0.75$. 
| µ   | Σ      | ε      | < n > | Trajectories |
|-----|--------|--------|-------|--------------|
| .10 | .430(1)| .287(1)| .00013| 2500         |
| .20 | .429(1)| .288(1)| .00015| 2500         |
| .30 | .423(1)| .290(1)| .0013 | 2500         |
| .35 | .401(1)| .295(1)| .0036 | 1500         |
| .36 | .396(2)| .297(1)| .005(1)| 1500        |
| .37 | .373(3)| .299(2)| .006(1)| 1500        |
| .38 | .351(4)| .304(3)| .008(1)| 1500        |
| .39 | .318(6)| .313(3)| .013(1)| 3000        |
| .3925| .307(8)| .316(3)| .015(2)| 3000        |
| .39375| .275(8)| .321(3)| .017(2)| 6000        |
| .3941 | .00 | .334(3)| .026(2)| 6000        |
| .395 | .00 | .335(2)| .026(2)| 3000        |
| .40  | .00 | .342(2)| .029(2)| 1500        |
Table 2

| $\mu$ | $\Sigma$ | $\varepsilon$ | $< n >$ | Trajectories |
|-------|----------|--------------|----------|--------------|
| .10   | .346(1)  | .300(1)      | .00035(70) | 2500         |
| .20   | .340(2)  | .302(1)      | .00032(65) | 1500         |
| .30   | .282(3)  | .312(2)      | .006(2)   | 1500         |
| .31   | .261(7)  | .316(2)      | .009(1)   | 1500         |
| .32   | .221(9)  | .322(2)      | .012(1)   | 3000         |
| .3225 | .00      | .331(3)      | .016(1)   | 3000         |
| .325  | .00      | .330(3)      | .015(1)   | 3000         |
| .33   | .00      | .333(3)      | .017(1)   | 1500         |
| .335  | .00      | .336(3)      | .020(1)   | 3000         |
| .34   | .00      | .337(1)      | .021(1)   | 3000         |
| \( \mu \) | \( \Sigma \) | \( \mu \) | \( \Sigma \) | \( \mu \) | \( \Sigma \) | \( \mu \) | \( \Sigma \) |
|------|------|------|------|------|------|------|------|
| 0    | .259(2) | 0    | .318(3) | 0    | .335(2) | 0    | .342(2) |
| .10  | .236(3) | .10  | .310(3) | .20  | .307(2) | .10  | .338(2) |
| .15  | .181(3) | .20  | .260(4) | .25  | .258(3) | .20  | .322(2) |
| .16  | .167(3) | .25  | .175(4) | .26  | .243(3) | .25  | .290(3) |
| .17  | .140(3) | .26  | .126(4) | .27  | .202(3) | .26  | .281(3) |
| .18  | .090(8) | .27  | .063(20) | .28  | .192(3) | .28  | .241(3) |
| .19  | .050(15) | .28  | .00  | .285  | .122(4) | .30  | .138(7) |
| .20  | .00  | .29  | .100(10) | .31  | .030(15) | .30  | .00  | .32  | .00  |

Table 3

\( N_{\tau} = 6 \) \hspace{1cm} \( N_{\tau} = 8 \) \hspace{1cm} \( N_{\tau} = 10 \) \hspace{1cm} \( N_{\tau} = 12 \)
Figure Captions

1. Mean Field predictions for Σ vs. μ curves for 1/g^2 = 0.75. Solid lines are solutions for the continuum Eq.(3.8) for \( N_\tau = 6, 8, 10 \) and 12 reading from the bottom left to the top right. The dotted lines are solutions of the discrete Eq.(3.18) (where convergent) for \( N_\tau = 6, 8, 10 \) and 12. The difference between the two sets of curves gives a measure of discretization effects.

2. Chiral order parameters Σ vs. chemical potential μ on a 20^3 lattice at coupling 1/g^2 = 0.70.

3. Induced ground state fermion number < n > vs. μ on a 20^3 lattice at 1/g^2 = 0.70. The dashed line is the mean field prediction.

4. Chiral order parameters Σ vs. chemical potential μ on a 20^3 lattice at coupling 1/g^2 = 0.75.

5. Induced ground state fermion number < n > vs. μ on a 20^3 lattice at 1/g^2 = 0.75.

6. \( \mu_c/\Sigma_0 \) vs. 1/g^2 for a 20^3 lattice.

7. ln(μ_c) vs. ln(1/g^2 - 1/g^2) for 1/g^2 = 0.70, 0.75 and 0.80 on a 20^3 lattice with the bulk transition 1/g^2 = 0.975.

8. Σ vs. μ for asymmetric lattices \( N_\tau \times N^2 \) with \( N_\tau = 6, 8, 10 \) and 12 and \( N = 36 \) at coupling 1/g^2 = 0.75.

9. The phase diagram, \( \mu_c/\Sigma_0 \) vs. \( T_c/\Sigma_0 \), for the three dimensional Four-Fermi model. The solid Mean Field line separates the chirally broken phase at low μ and T from the symmetric phase at large μ and T.

10. The induced fermion number < n > vs. μ at coupling 1/g^2 = 0.70 on a "small" 12^3 lattice.

11. < n > vs. μ at 1/g^2 = 0.70 on a 20^3 lattice for the theory with a bare fermion mass term, \( m = 0.01 \) and 0.005 in lattice units.

12. Σ vs. μ for the same parameters as in Fig. 11.