THE INTEGRAL CHOW RING OF THE STACK OF AT MOST 1-NODAL RATIONAL CURVES

DAN EDIDIN AND DAMIANO FULGHESU

Abstract. We give a presentation for the stack $\mathcal{M}_{0,1}^{\leq 1}$ of rational curves with at most 1 node as the quotient by $GL_3$ of an open set in a 6 dimensional irreducible representation. We then use equivariant intersection theory to calculate the integral Chow ring of this stack.

1. Introduction

The integral Chow ring of a smooth quotient stack was defined in the paper [Ed-Gr2]. Subsequently, Kresch [Kre] extended the definition to smooth stacks which admit stratifications by quotient stacks. In [Ed-Gr2] the integral Chow rings of the stacks of elliptic curves $\mathcal{M}_{1,1}$ and $\mathcal{M}_{1,1}$ were computed. In [Vis] Vistoli calculated the integral Chow ring of $\mathcal{M}_2$. All of these calculations use presentations for the stacks as quotient stacks.

In this paper we turn our attention to stacks of unpointed rational curves. Unlike stacks of curves of positive genus, they are not Deligne-Mumford, since $\mathbb{P}^1$ has an infinite automorphism group. In [Fulg] the second author introduced the stratification of the stack of rational curves $\mathcal{M}_0$ by the number of nodes. The stack $\mathcal{M}_0$ of smooth rational curves is the classifying stack $B PGL_2$ whose Chow ring was computed by Pandharipande [Pan]. The focus of this paper is $\mathcal{M}_{0,1}^{\leq 1}$, the stack of rational curves with at most one node.

Although it is not a Deligne-Mumford stack, $\mathcal{M}_{0,1}^{\leq 1}$ admits a relatively easy description as a quotient. On a rational curve $C$ with at most one node the dual of the dualizing sheaf embeds the curve as a plane conic. This observation allows us to prove that (Proposition 6) that $\mathcal{M}_{0,1}^{\leq 1}$ is a quotient by $GL_3$ of an open set in the representation $H^0(\omega_{\mathbb{P}^2}(-1))$. We can then calculate the Chow ring of the stack as the $GL_3$-equivariant Chow ring of this open set and obtain the following result.

\footnote{that is to say of arithmetic genus 0}
Let $\mathcal{C} \to \mathcal{M}_0^{\leq 1}$ be the universal curve and let $E = \pi_*\omega_{\mathcal{C}}$. The vector bundle $E$ is locally free of rank 3, so we denote the Chern classes of $E$ by $c_1, c_2, c_3$ respectively.

**Theorem 1.**

$$A^*\mathcal{M}_0^{\leq 1} = \mathbb{Z}[c_1, c_2, c_3]/(4c_3, 2c_1c_3, c_1^2c_3)$$

For any algebraic space the $\text{GL}_3$-equivariant Chow ring injects into the $\text{T}$-equivariant Chow ring, where $\text{T}$ is a maximal torus. Thus we may check relations in $\text{T}$-equivariant Chow ring. This allows us to use the localization theorem for torus actions and reduce the problem to one of polynomial interpolation. In the last section of the paper we use the techniques developed for the calculation of $A^*(\mathcal{M}_0^{\leq 1})$ to compute Chow rings of stacks of quadrics in $\mathbb{P}^{n-1}$ and classifying spaces for certain extensions of $\text{SO}(n)$ by a finite group.

**Remark 2.** If a rational curve $C$ has two or more nodes then $\omega_C^\vee$ is no longer ample and there is no obvious presentation for the stack $\mathcal{M}_0^{\leq n}$ of rational curves with at most $n$ nodes as a quotient stack. In fact there is some evidence that $\mathcal{M}_0^{\leq n}$ is not a quotient stack since the second author proved $\text{Fulg}$ that if $n \geq 2$ the projection from the universal curve $\mathcal{C}_0^{\leq n} \to \mathcal{M}_0^{\leq n}$ is not representable in the category of schemes. However, using the intersection theory developed by Kresch for stacks stratified by quotient stacks, the second author $\text{Fulg}$ was able to compute the rational Chow rings of the stacks $\mathcal{M}_{0}^{\leq 2}$ and $\mathcal{M}_{0}^{\leq 3}$, but it is unclear how to determine their integral Chow rings.

2. **Some facts about equivariant Chow groups**

Equivariant Chow groups were defined in the paper $\text{Ed-Gr2}$. We briefly recall some basic facts and notation that we will use in our computation.

We work over an arbitrary field $K$. Let $G$ be a linear algebraic group. For any algebraic space $X$ we denote the direct sum of the equivariant Chow groups by $A_G^*(X)$. If $X$ is smooth then there is a product structure on equivariant Chow groups and we denote the equivariant Chow ring by the notation $A_G^*(X)$. Following standard notation we denote the equivariant Chow ring of a point by $A_G^*$. Flat pullback via the morphism $X \to \text{Spec} K$ makes the equivariant Chow groups $A_G^*(X)$ into an $A_G^*$-module. When $X$ is smooth the equivariant Chow ring $A_G^*(X)$ becomes an $A_G^*$-algebra.

The relation between equivariant Chow rings and Chow rings of quotient stacks is given by the following result.
Proposition 3. [Ed-Gr2, Propositions 17, 19] If $F = [X/G]$ then the equivariant Chow ring $A^*_G(X)$ is independent of the presentation for $F$ and may be identified with the operational Chow ring of $F$.

If $V$ is a $G$-module, then $V$ defines a $G$-equivariant vector bundle over $\text{Spec } K$. Consequently a representation $V$ of rank $r$ has Chern classes $c_1, \ldots, c_r \in A^*_G$. If $X$ is a smooth algebraic space then we will view the Chern classes as elements of the equivariant Chow ring $A^*_G(X)$ via the pullback $A^*_G \to A^*_G(X)$.

Remark 4. For the presentation of the stack $\mathcal{M}^{\leq 1}_G$ we work over $\text{Spec } \mathbb{Z}$. For the calculation of Chow rings we work over an arbitrary field.

2.1. Equivariant Chow rings for $GL_n$ actions. Let $T = \mathbb{G}_m^n$ be a maximal torus. Because $GL_n$ is special the restriction homomorphism $A^*_{GL_n} \to A^*_T$ is injective and the image consists of the classes invariant under the action of the Weyl group $W(T, GL_n) = S_n$. If $E$ is the standard representation of $GL_n$ then the total character of the $T$-module $E$ decomposes into a sum of linearly independent characters $\lambda_1 + \lambda_2 + \ldots + \lambda_n$ and we get $A^*_T = \mathbb{Z}[t_1, \ldots, t_n]$ where $t_i = c_1(\lambda_i)$. The Weyl group $S_n$ acts on $A^*_T$ by permuting the $t_i$’s and as result $A^*_{GL_n} = \mathbb{Z}[c_1, \ldots, c_n]$ where $c_i = c_i(E)$ [Ed-Gr1].

More generally ([Ed-Gr2, Proposition 3.6], [Br, Theorem 6.7]) if $X$ is an algebraic space then the restriction map $A^*_{GL_n}(X) \to A^*_T(X)$ is an injective homomorphism of $A^*_{GL_n}$-modules. Likewise if $X$ is smooth, the restriction map $A^*_{GL_n}X \to A^*_TX$ is an injective homomorphism of $A^*_{GL_n}$-algebras. In both cases the images consist of the elements invariant under the natural action of the Weyl group.

As a result of this discussion, we may view $A^*_{GL_n}X$ as a subalgebra of $A^*_T(X)$. In particular we may check a formula in $A^*_{GL_n}X$ by restricting to $A^*_T X$.

If $V$ is a representation of rank $r$ of $GL_n$ then then the total character of the $T$-module $V$ decomposes as sum of characters $\mu_1 + \ldots + \mu_r$. Let $l_i = c_1(\mu_i)$. We refer to the classes $l_1, \ldots, l_r$ as the Chern roots of $V$ and view them as elements in $A^*_T X$. Any symmetric polynomial in the Chern roots is an element of $A^*_{GL_n}X$. The following easy lemma is proved for torus actions in [Ed-Gr2, Section 3.3] and follows in general from the projective bundle theorem [Ful, Example 8.3.4].

---

2This means that every $GL_n$-torsor is locally trivial in the Zariski topology.
Lemma 5. Let $V$ be an $r$-dimensional representation of $\text{GL}_n$ and let $\mathbb{P}(V)$ be the projective space of lines in $V$. Then
\[
A^*_\text{GL}_n(\mathbb{P}(V)) = A^*_\text{GL}_n[\xi]/(\xi^r + C_1\xi^{r-1} + \ldots C_r)
\]
and
\[
A^*_T(\mathbb{P}(V)) = A^*_T[\xi]/\left(\prod_{i=1}^r (\xi + l_i)\right)
\]
where $\xi = c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$ and $C_1, \ldots, C_r \in A^*_\text{GL}_n$ (resp. $l_1, \ldots, l_r$) are the equivariant Chern classes (resp. Chern roots) of the representation $V$.

3. A Presentation of $\mathcal{M}_0^{\leq 1}$ as a Quotient

Let $\mathcal{M}_0^{\leq 1}$ be the stack (defined over $\text{Spec } \mathbb{Z}$) of rational curves with at most one node.

Proposition 6. Let $X = \text{Sym}^2 E^* \setminus \Delta_2$ where $E$ is the defining representation of $\text{GL}_3$, $\Delta_2$ is the second degeneracy locus of $\text{Sym}^2 E^*$ (that is to say the locus of quadratic forms of rank $\leq 1$). Then the stack $\mathcal{M}_0^{\leq 1}$ is represented by the quotient stack
\[
[X/\text{GL}_3]
\]
where the action of $\text{GL}_3$ is given by
\[
(A \cdot Q)(x) = (\det A)Q(A^{-1}x).
\]

Remark 7. Proposition gives another proof that $\mathcal{M}_0^{\leq 1}$ is an algebraic stack.

Consider the natural action of $\text{GL}_3$ on $\mathbb{P}^2 = \text{Proj}(\text{Sym} E^*)$. If $L$ is a $\text{GL}_3$-equivariant line bundle on $\mathbb{P}^2$ then the space of global sections $H^0(\mathbb{P}^2, L)$ has a natural $\text{GL}_3$-module structure. If $k > 0$ then $H^0(\mathbb{P}^2, \mathcal{O}(k)) = \text{Sym}^k E^*$. The bundle $\omega_{\mathbb{P}^2}(-1)$ is non-equivariantly isomorphic to $\mathcal{O}_{\mathbb{P}^2}(2)$ but $H^0(\mathbb{P}^2, \omega_{\mathbb{P}^2}(-1)) = \text{Sym}^2 E^* \otimes \det E$ (cf. [Har, Exercise III.8.4]). To prove the proposition we will show that
\[
\mathcal{M}_0^{\leq 1} = \left[\left(H^0(\mathbb{P}^2, \omega_{\mathbb{P}^2}(-1))\right) \setminus \Delta_2\right] / \text{GL}_3.
\]

Before we prove Proposition we need an easy lemma.

Let $C \to T$ be a rational curve with at most one node and let $\omega_\pi$ be the dualizing sheaf on $C$.

Lemma 8. The $\mathcal{O}_T$-module $\pi_*\omega_\pi^\vee$ is locally free of rank 3 and its formation commutes with base change.
Proof of Lemma 8. Since formation of the dualizing sheaf commutes with base change and $\pi : C \to T$ is a curve it suffices, by the theorem of cohomology and base change, to prove that for every geometric point $t \to T$, $\dim H^1(C_t, \omega_{C/T}^\vee) = 0$ and $\dim H^0(C_t, \omega_{C/T}^\vee) = 3$. (Here $\pi_t : C_t \to t$ is the restriction of $\pi : C \to T$.)

From Serre duality $H^1(C_t, \omega_{C/T}^\vee) = H^0(C_t, \omega_{C/T}^{\otimes 2})^\vee$. Now, when the fiber is isomorphic to $\mathbb{P}^1$ we have $\omega_{C/T}^{\otimes 2} = \mathcal{O}(-4)$ and we do not have global sections different from 0. When $C_t$ is singular we have that the restriction of $\omega_{C/T}^{\otimes 2}$ to each of the two components is $(\mathcal{O}(-1))^{\otimes 2} = \mathcal{O}(-2)$ and consequently the restriction of sections to these components must be 0. Similarly we can verify that $h^0(C_t, \omega_{C/T}^\vee) = 3$. □

Proof of Proposition 6. Let $Y$ be the category fibered in groupoids whose objects are pairs $(C \xrightarrow{\pi} T, \varphi)$ where $\varphi$ is an isomorphism of $\mathcal{O}_T$ sheaves $\varphi : H^0(\mathbb{P}^2, \mathcal{O}(1)) \otimes \mathcal{O}_T \cong \pi_\ast \omega_{C/T}^\vee$ and the arrows being the obvious ones. Each object of $Y$ has only trivial automorphisms, so $Y$ is in fact a functor.

The action of $\text{GL}_3$ on $H^0(\mathbb{P}^2, \mathcal{O}(1))$ induces a natural left $\text{GL}_3$ action on $Y$. Specifically, if $A \in \text{GL}_3(T)$ then

$$A \cdot (C \xrightarrow{\pi} T, \varphi) = (C \xrightarrow{\pi} T, \varphi \circ A^{-1})$$

We now construct a $\text{GL}_3$-equivariant isomorphism from $Y \to X$ (or more precisely to the functor $\text{Hom}(\underline{\_}, X)$). Clearly $\mathcal{M}_0^{<1}$ is canonically isomorphic to the quotient $[Y/ \text{GL}_3]$ so our proposition will follow.

Given an object $(C \xrightarrow{\pi} T, \varphi)$ in $Y$ we have a morphism

$$C \xrightarrow{i} \mathbb{P}(\pi_\ast \omega_{C/T}^\vee).$$

Composing with the isomorphism $\varphi$ we get a map

$$C \xrightarrow{i} \mathbb{P}^2_T.$$ 

Next we observe that the map $i$ is an embedding. To see that it suffices to check at geometric points of $T$. When the fiber $C_t \xrightarrow{\pi} t$ (at a geometric point $t \to T$) is $\mathbb{P}^1$, we have

$$\pi_\ast \omega_{C/T}^\vee = H^0(\mathbb{P}^1, \mathcal{O}(2))$$

and the embedding is given by the complete linear system $|\mathcal{O}(2)|$.

When the fiber has 2 components then the dualizing sheaf has degree $-1$ on each component so $\omega_{C/T}^\vee$ is $\mathcal{O}(1)$ on each component. The space of global sections is the 3 dimensional subspace of sections of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ that agree on the singular point. In particular this means that the linear system $|\omega_{C/T}^\vee|$ maps each component of $C_t$ to a line in $\mathbb{P}^2$. 
Moreover the images of components cannot coincide as \(|\omega^\vee|\) separates the complement of the intersection of the two components.

Let \(I \subset \mathcal{O}_{\mathbb{P}^2_T}\) be the ideal sheaf of \(C\). The inclusion \(I \subset \mathcal{O}_{\mathbb{P}^2_T}\) gives a global section of \(I^\vee = \mathcal{O}_{\mathbb{P}^2_T}(C)\) over \(T\). By adjunction,
\[
\omega^\vee_T = \omega^\vee_{\mathbb{P}^2_T}(-C) \otimes \mathcal{O}_C.
\]

The isomorphism of global sections
\[
\pi_*(\omega^\vee_T) = H^0(\mathbb{P}^2, \mathcal{O}(1)) \otimes \mathcal{O}_T
\]
gives a \(\text{GL}_3\)-equivariant isomorphism
\[
(1) \quad p_*(\omega^\vee_{\mathbb{P}^2_T}(-C)) = H^0(\mathbb{P}^2, \mathcal{O}(1)) \otimes \mathcal{O}_T
\]
where \(p: \mathbb{P}^2_T \to T\) is the projection.

The line bundles \(\omega^\vee_{\mathbb{P}^2_T}(-C)\) and \(\mathcal{O}_{\mathbb{P}^2_T}(1)\) on \(\mathbb{P}^2_T\) both restrict to \(\mathcal{O}_{\mathbb{P}^2}(1)\) on the fibers of \(p\) so a priori they differ by the pullback of a line bundle on \(T\). The equivariant isomorphism \((1)\) of global sections then implies that they are in fact isomorphic (with their canonical linearizations) as \(G\)-line bundles on \(\mathbb{P}^2_T\). Twisting by \(\mathcal{O}_{\mathbb{P}^2_T}(C) \otimes \mathcal{O}_{\mathbb{P}^2_T}(-1)\) and taking global sections gives a \(\text{GL}_3\)-equivariant isomorphism of locally free \(\mathcal{O}_T\)-modules
\[
p_*(\omega^\vee_{\mathbb{P}^2_T}(-1)) \to p_*(\mathcal{O}_{\mathbb{P}^2_T}(C)).
\]
Thus the data \((C \to T, \varphi)\) determines a global section of \(\omega^\vee_{\mathbb{P}^2_T}(-1)\) over \(T\) whose restriction to each fiber is not in \(\Delta_2\). This gives our desired \(\text{GL}_3\)-equivariant map \(Y \to X\).

A \(\text{GL}_3\)-equivariant inverse morphism \(X \to Y\) is given as follows: As noted above, giving a \(T\)-valued point \(T \to X\) is the same as giving a global section, \(s\), of \(\omega^\vee_{\mathbb{P}^2_T}(-1)\) over \(T\) whose restriction to each fiber is not in \(\Delta_2\). Let \(C \subset \mathbb{P}^2_T\) be the subscheme defined by the cokernel of \(s^\vee\). On each fiber of \(\pi\) the subscheme defined by the cokernel of \(s^\vee\) is subscheme of \(\mathbb{P}^2\) cut out by the quadratic form which is the restriction of \(s\) to that fiber. Since we assume that the restriction of \(s\) is not in \(\Delta_2\) it follows that \(C \to \mathbb{P}^2_T\) is a family of at most 1-nodal rational curves of degree 2 such that \(\mathcal{O}_{\mathbb{P}^2_T}(C) = \omega^\vee_{\mathbb{P}^2_T}(-1)\). Applying adjunction again we see that \(\omega^\vee = \mathcal{O}_C(1)\). Taking global sections over \(T\) we see that
\[
\pi_*(\omega^\vee) = \pi_* \mathcal{O}_C(1) = H^0(\mathbb{P}^2, \mathcal{O}(1)) \otimes \mathcal{O}_T.
\]

4. Computation of \(A^*(\mathcal{M}_{0}^{\leq 1})\)

Since \(\mathcal{M}_{0}^{\leq 1} = [X/\text{GL}_3]\) where \(X = \text{Sym}^2 E/\Delta_2\) we may calculate the integral Chow ring of \(\mathcal{M}_{0}^{\leq 1}\) as \(A^*_{\text{GL}_3}X\).
Let $\mathbb{P}^5 = \mathbb{P}(\text{Sym}^2 E^*)$. As $X$ is homogeneous for the action of $\mathbb{G}_m$ with weight -1, we have that $Z := \mathbb{P}(X)$ is a well defined open subscheme of $\mathbb{P}^5 := \mathbb{P}(\text{Sym}^2 (E^*))$ and there is an induced action of $\text{GL}_3$ on $Z$. Since the determinant acts trivially on $\mathbb{P}^5$ the projection $p : X \to Z$ is $\text{GL}_3$-equivariant and makes $X$ into the total space of the principal $\mathbb{G}_m$-bundle on $Z$ associated to the line bundle

$$D \otimes \mathcal{O}(-1)$$

where $D$ is the determinant of the standard representation of $\text{GL}_3$ and $\mathcal{O}(-1)$ is the tautological bundle on $\mathbb{P}^5$.

**Lemma 9.** Let $H$ denote the first Chern class of $\mathcal{O}(1)$ on $\mathbb{P}^5$. Then the pull-back

$$A_{\text{GL}_3}^*(Z) \xrightarrow{p^*} A_{\text{GL}_3}^*(X)$$

is surjective, and its kernel is generated by $c_1 - H$.

**Proof.** If $p : X \to Z$ is a $G$-equivariant $\mathbb{G}_m$-bundle let $\mathcal{L}$ be the associated line bundle. Then $X$ is the complement of the 0-section in $\mathcal{L}$ so the basic exact sequence for equivariant Chow groups implies that

$$A_G^*(Z)/c_1(\mathcal{L}) \cong A_G^*(X).$$

In our case, the kernel of $p^*$ is generated by $c_1(D \otimes \mathcal{O}(-1)) = c_1 - H$. $\Box$

### 4.1. The equivariant Chow ring of $Z$

Lemma 9 reduces the calculation of $A_{\text{GL}_3}^*(X)$ to that of $A_{\text{GL}_3}^*(Z)$.

Consider the following embedding

$$i : E^* \to \text{Sym}^2 E^*$$

and its passage to the quotient

$$i : \mathbb{P}(E^*) \to \mathbb{P}^5$$

Clearly the map $i$ is $\text{GL}_3$ equivariant and its image is $\mathbb{P}(\Delta_2)$. Let $j : Z \to \mathbb{P}^5$ be the open inclusion. Then the basic exact sequence of $A^*\text{GL}_3$-modules

$$A_{\text{GL}_3}^*(\mathbb{P}(E^*)) \xrightarrow{i^*} A_{\text{GL}_3}^*(\mathbb{P}^5) \xrightarrow{j^*} A_{\text{GL}_3}^*Z \to 0$$

implies that $A_{\text{GL}_3}^*(Z) = A_{\text{GL}_3}^*(\mathbb{P}^5)/(\text{Im } i^*)$.

Let $l_1, l_2, l_3$ denote the Chern roots of $E^*$, so $l_i = -t_i$. Then the Chern roots of $\text{Sym}^2 E^*$ are $2l_1, 2l_2, 2l_3, l_1 + l_2, l_1 + l_3, l_2 + l_3$. By definition
of $c_1, c_2, c_3$ we have

$$
c_1 = -(l_1 + l_2 + l_3) \\
c_2 = l_1 l_2 + l_1 l_3 + l_2 l_3 \\
c_3 = -l_1 l_2 l_3
$$

Applying Lemma 5 we see that

$$A^*_{\text{GL}_3}(\mathbb{P}^5) = \mathbb{Z}[c_1, c_2, c_3, H]/P(H)$$

where $P(H)$ is the product of linear factors

$$(H + 2l_1)(H + 2l_2)(H + 2l_3)(H + l_1 + l_2)(H + l_1 + l_3)(t + l_2 + l_3)$$

which can rewritten as the product

$$(H^3 - 2c_1 H^2 + 4c_2 H - 8c_3)(H^3 - 2c_1 H^2 + (c_1^2 + c_2) H + c_3 - c_1 c_2).$$

Let $K$ be the first chern class of $O_{\mathbb{P}(E)}(1)$. Then as above we have

$$A^*_{\text{GL}_3}(\mathbb{P}(E^*)) = \mathbb{Z}[c_1, c_2, c_3, K]/(K^3 - c_1 K^2 + c_2 K - c_3).$$

In particular, $A^*_{\text{GL}_3}(\mathbb{P}(E^*))$ is generated by the classes $1, K, K^2$ as a module over $A^*_{\text{GL}_3}$.

4.2. Calculation via localization. To complete our calculation we must calculate the images of these classes. There are a number of ways to do this. The method we use is localization for the action of the maximal torus $T = G_3^m \subset \text{GL}_3$.

Because the restriction map $A^*_{\text{GL}_3}(\mathbb{P}^5) \to A^*_T(\mathbb{P}^5)$ is injective we can view $1, K, K^2$ as classes in $A^*_T(\mathbb{P}(E^*))$ and compute $i_* 1, i_* K, i_* K^2$ in $A^*_T(\mathbb{P}^5)$. Since $i_*$ is $\text{GL}_3$-equivariant these images will automatically lie in the submodule $A^*_{\text{GL}_3}(\mathbb{P}^5)$.

Now $A^*_T(\mathbb{P}^5)$ is a free $A^*_T$-module generated by the classes $1, H, \ldots H^5$. Thus any formula for $i_* K^n$ as a linear combination of the classes $H^i$ with coefficients in $A^*_T$ can be obtained after tensoring with $\mathbb{Q}$ and also localizing at the multiplicative set of homogeneous elements of positive degree in $A^*_T$. This allows us to use the explicit localization theorem [Ed-Gr3, Theorem 2].

Theorem 10 (Explicit localization). Let $X$ be a smooth variety equipped with an action of a torus $T$ and let $Q$ be $((A^*_T)^+)^{-1} A^*_T$. For every $\alpha$ in $A^*_T(X) \otimes Q$ we have

$$\alpha = \sum_F i_F^* \frac{i_F^* \alpha}{c^\text{top}(N_F X)}$$

where the sum is over the components $F$ of the locus of fixed points for the action of $T$ and $i_F$ denotes the inclusion $F \to X$. 
Let $\lambda_1, \lambda_2, \lambda_3$ be the basis for the characters of $T = G_m^3$ where $\lambda_i$ corresponds to projection to the $i$-th factor. Then then the total character of the $T$-module $E^*$ decomposes as the sum of characters
\[ \lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1}. \]

Hence the Chern roots $l_1, l_2, l_3$ of $E^*$ are $-c_1(\lambda_1), -c_1(\lambda_2), -c_2(\lambda_3)$, respectively. Likewise then the total character of the $T$-module $\text{Sym}^2 E^*$ decomposes as
\[ \lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} + \lambda_1^{-1}\lambda_2^{-1} + \lambda_1^{-1}\lambda_3^{-1} + \lambda_2^{-1}\lambda_3^{-1}. \]

Fix coordinates $[X_0: X_1: X_2]$ on $\mathbb{P}^2 := \mathbb{P}(E^*)$ so that $T$ acts by
\[ t \cdot [X_0: X_1: X_2] = [\lambda_1^{-1}(t)X_0: \lambda_2^{-1}(t)X_1: \lambda_3^{-1}(t)X_2] \]
and coordinates $[Z_0: Z_1: Z_2: Z_3: Z_4: Z_5]$ on $\mathbb{P}^5$ so that $T$ acts via the characters
\[ \lambda_1^{-2}, \lambda_2^{-2}, \lambda_3^{-2}, \lambda_1^{-1}\lambda_2^{-1}, \lambda_1^{-1}\lambda_3^{-1}, \lambda_2^{-1}\lambda_3^{-1} \]
on the respective coordinates.

With these actions $\mathbb{P}^2$ has three fixed points
\[ P_0 := [1, 0, 0] \]
\[ P_1 := [0, 1, 0] \]
\[ P_2 := [0, 0, 1] \]
and $\mathbb{P}^5$ has 6 fixed points $Q_0, \ldots, Q_5$ defined analogously. With our choices of coordinates, $i_*[P_j] = [Q_j]$ for $j = 0, 1, 2$.

Applying explicit localization to $\mathbb{P}^2$ we have
\[ 1 = \sum_{k=0}^2 i_{P_k}^* \frac{i_{P_k}^* 1}{c_{\text{top}}^T(T_{P_k}\mathbb{P}^2)} \]
\[ K = \sum_{k=0}^2 i_{P_k}^* \frac{i_{P_k}^* K}{c_{\text{top}}^T(T_{P_k}\mathbb{P}^2)} \]
\[ K^2 = \sum_{k=0}^2 i_{P_k}^* \frac{i_{P_k}^* K^2}{c_{\text{top}}^T(T_{P_k}\mathbb{P}^2)} \]

Let us compute the class $c_{\text{top}}^T(T_{P_0}\mathbb{P}^2)$. Local coordinates for $P_0$ are $x = X_1/X_0, y = X_2/X_0$, these coordinates are the same for the tangent space at $P_0$, so the action of $G_m^3$ on $T_{P_0}\mathbb{P}^2$ is
\[ G_m^3 \cdot \mathbb{C}^2 \rightarrow \mathbb{C}^2 \]
\[ t \cdot (x, y) \mapsto ((\lambda_1\lambda_2^{-1})(t)x, (\lambda_1\lambda_3^{-1})(t)y) \]
Consequently
\[ c_{\text{top}}^T(T_{P_0}\mathbb{P}^2) = (l_2 - l_1)(l_3 - l_1) \]
similarly we have
\[ c^T_{\text{top}}(T_{P_1}\mathbb{P}^2) = (l_1 - l_2)(l_3 - l_2) \]
\[ c^T_{\text{top}}(T_{P_2}\mathbb{P}^2) = (l_1 - l_3)(l_2 - l_3) \]

Now observe that a generator for \( i^*_{P_k} \mathcal{O}(1) \) is the dual form \( X_k \), so we have
\[ i^*_{P_0} K = -l_1 \]
\[ i^*_{P_1} K = -l_2 \]
\[ i^*_{P_2} K = -l_3 \]

Therefore we obtain (after taking the pushforward to \( \mathbb{P}^5 \))
\[ i_* 1 = \frac{[Q_0]}{(l_2 - l_1)(l_3 - l_1)} + \frac{[Q_1]}{(l_1 - l_2)(l_3 - l_2)} + \frac{[Q_2]}{(l_1 - l_3)(l_2 - l_3)} \]
\[ i_* K = \frac{-l_1 [Q_0]}{(l_2 - l_1)(l_3 - l_1)} + \frac{-l_2 [Q_1]}{(l_1 - l_2)(l_3 - l_2)} + \frac{-l_3 [Q_2]}{(l_1 - l_3)(l_2 - l_3)} \]
\[ i_* K^2 = \frac{l_1^2 [Q_0]}{(l_2 - l_1)(l_3 - l_1)} + \frac{l_2^2 [Q_1]}{(l_1 - l_2)(l_3 - l_2)} + \frac{l_3^2 [Q_2]}{(l_1 - l_3)(l_2 - l_3)} \]

The point \( Q_0 \) is the complete intersection of the hyperplanes cut out by the coordinate functions \( Z_1, Z_2, Z_3, Z_4, Z_5 \). The \( T \)-equivariant fundamental classes of these hyperplanes can be computed from the weights of the \( T \)-action. For example, since \( T \) acts via the character \( \lambda^2 \) on the coordinate \( Z_1 \) we see that \( [V(Z_1)] = (H + 2l_2) \) since \( l_2 = c_1(\lambda^2) \). Multiplying out we obtain
\[ [Q_0] = (H + 2l_2)(H + 2l_3)(H + l_1 + l_2)(H + l_1 + l_3)(H + l_2 + l_3). \]

Similar calculations show that
\[ [Q_1] = (H + 2l_1)(H + 2l_3)(H + l_1 + l_2)(H + l_1 + l_3)(H + l_2 + l_3) \]
\[ [Q_2] = (H + 2l_1)(H + 2l_2)(H + l_1 + l_2)(H + l_1 + l_3)(H + l_2 + l_3). \]

After straight-forward computations and substituting with Chern classes we obtain
\[ i_* 1 = 4(H^3 - 2c_1H^2 + (c_1^2 + c_2)H + (c_3 - c_1c_2)) \]
\[ i_* K = 2H(H^3 - 2c_1H^2 + (c_1^2 + c_2)H + (c_3 - c_1c_2)) \]
\[ i_* K^2 = H^2(H^3 - 2c_1H^2 + (c_1^2 + c_2)H + (c_3 - c_1c_2)) \]

Finally we substitute \( H \to c_1 \) and we see that the relations in \( A^*_{\text{GL}} X \) are generated by \( 4c_3, 2c_1c_3, c_1^2, c_3 \). Therefore,
\[ A^*_{\text{GL}}(\mathfrak{M}_0^{\leq 1}) = \mathbb{Z}[c_1, c_2, c_3]/(4c_3, 2c_1c_3, c_1^2c_3). \]
5. CHOW RINGS OF THE STACK OF REDUCED QUADRICS AND
FINITE EXTENSIONS OF $SO_n$

Let $E$ be the defining representation of $GL_n$ and $\Delta_j$ the degeneracy
locus in $\text{Sym}^2 E^*$ of matrices with rank at most $n - j$. Consider
the following action of $GL_n$ on $\text{Sym}^2 E^*$:

(2) $(A \cdot Q)(x) = (\det A)^k Q(A^{-1}x)$

with $k \in \mathbb{Z}$. Each subscheme $\Delta_j$ is invariant for this action and we set
$X_j := \text{Sym}^2 E^* \setminus \Delta_j$. Let $\mathcal{X}_{j,k}$ denote the quotient $[X_j/GL_n]$ where the
action of $GL_n$ is given by (2).

The localization method of Section 4 can be generalized to compute
$A^*(\mathcal{X}_{n-1,k})$. Note that the stack $\mathcal{X}_{n-1,0}$ is the stack of reduced quadrics
in $\mathbb{P}^{n-1}$.

On the other hand, if $j = 1$ then $\mathcal{X}_{1,k}$ is the quotient by $GL_n$ of the
open subset $X_1$ of nondegenerate quadratic forms in $\text{Sym}^2 E^*$. Since
$X_1$ is a homogeneous space for the action of $GL_n$ the quotient $\mathcal{X}_{1,k}$ is
the classifying stack $BO(n, k)$ where $O(n, k)$ is the stabilizer of any
nondegenerate quadratic form. In our case we may identify the stabi-
zer of a nondegenerate quadratic form with the closed subgroup $O(n, k) \subset GL_n$ defined by the condition $(\det A)^k I = A A^t$. The groups $O(n, k)$ are extensions of $SO(n, k)$ by the cyclic groups $\mu_{nk-2}$ and the
techniques of [Pan] can be used to calculate the Chow ring of their
classifying stacks.

5.1. Computation of $A^*(\mathcal{X}_{n-1,k})$. Let

$$e_{k,n} = c_{\text{top}} \left( (\det E)^{\otimes k} \otimes \wedge^2 E^* \right) \in A^*_{GL_n}$$

Proposition 11. $A^*(\mathcal{X}_{n-1,k}) = \mathbb{Z}[c_1, \ldots, c_n]/I$ where $I$ is the ideal
generated by the classes

$$\{2^{n-1-r}(kc_1)^r e_{k,n}\}_{r=0,\ldots,n-1}.$$

In particular the Chow ring of the stack of reduced quadrics is

$$A^*(\mathcal{X}_{n,0}) = \mathbb{Z}[c_1, \ldots, c_n]/ \left( 2^{n-1} c_{\text{top}}(\wedge^2 E^*) \right).$$

Remark 12. Observe if $k$ is even then $I$ is generated by the single
relation $2^{n-1} e_{k,n}$.

Proof. Let $X := X_{n-1}$. Since $X$ is an open set in a representation
of $GL_n$ we can express $A^*_{GL_n}(X)$ as a quotient of the polynomial ring
$\mathbb{Z}[c_1, \ldots, c_n]$. Let $N = \binom{n+1}{2} - 1$, $\mathbb{P}^N := \mathbb{P}(\text{Sym}^2 E^*)$ and $Z := \mathbb{P}(X)$. With the action given by (2) the projection $\pi : X \to Z$ is a $\mathbb{G}_m$
torsor corresponding to the line bundle $(\det E^{\otimes k}) \otimes \mathcal{O}_{\mathbb{P}^N}(-1)$. Thus the
pullback $A^*_{\text{GL}_n}(Z) \xrightarrow{\pi^*} A^*_{\text{GL}_n}(X)$ is surjective and its kernel is generated by $kc_1 - H$ where $H = c_1(\mathcal{O}_{\mathbb{P}^N}(1))$.

Moreover we have that

$$A^*_{\text{GL}_n}(Z) = A^*_{\text{GL}_n}(\mathbb{P}^N)/(\text{Im } i_*)$$

where $i$ is the second Veronese embedding

$$i : \mathbb{P}(E^*) \to \mathbb{P}^N$$

induced by

$$i : E^* \to \text{Sym}^2 E^*$$

$$\phi \mapsto \phi^2$$

If we denote by $l_1, \ldots, l_n$ the Chern roots of $E^*$, then the Chern roots of $\text{Sym}^2 E^*$ are

$$\{2l_i\}_{i=1,\ldots,n} \cup \{l_i + l_j\}_{1 \leq i < j \leq n}.$$ 

By definition of $c_1, \ldots, c_n$ we have

$$c_k = (-1)^k s_k$$

where $s_k$ is the $k$th symmetric polynomial in $l_1, \ldots, l_n$. Set

$$P(H) = \prod_{i=1}^{n} (H + 2l_i)$$

$$R(H) = \prod_{1 \leq i < j \leq n} (H + l_i + l_j)$$

Because $P(H)$ and $R(H)$ are both symmetric in the $l_i$ they can be expressed as polynomials with coefficients in $\mathbb{Z}[c_1, \ldots, c_n]$. In particular we have

$$P(H) = \sum_{i=0}^{n} (-2)^i c_i H^{n-i}.$$ 

Now applying Lemma 5 we see that

$$A^*_{\text{GL}_n}(\mathbb{P}^N) = \mathbb{Z}[c_1, \ldots, c_n, H]/P(H)R(H)$$

Likewise, if we let $K = c_1(\mathcal{O}_{\mathbb{P}(E^*)}(1))$, then $A^*_{\text{GL}_n}(\mathbb{P}(E^*))$ is generated by the classes $1, K, K^2, \ldots, K^{n-1}$.

Let $T$ be the maximal torus in $\text{GL}_n$ consisting of diagonal matrices. Applying the explicit localization formula as in Section 4 we can compute $i_* K^r \in A^*_{\text{GL}_n}(\mathbb{P}^N)$ and we obtain the following formula

$$i_* K^r = R(H) \sum_{j=1}^{n} \left( \prod_{k \neq j} \frac{H + 2l_k}{(l_k - l_j)} \right) \cdot (-l_j)^r.$$ 

(3)
Viewing the sum in right-hand side of (3) as a polynomial of degree $n$ in $H$ we can simplify by applying the Lagrange Interpolation Formula. In fact the sum is the unique polynomial of degree $n-1$ in $H$ which when evaluated at $H = -2l_m$ (for each of $m = 1, \ldots, n$) equals $(2)^{n-1}(-l_m)^r = (2)^{n-1-i}(-2l_m)^r$. Therefore,

$$i^*K^i = 2^{n-1-r}H^iR(H).$$

Moreover, the polynomial $P(H)R(H)$ is in the ideal generated by the $i^*K^r$'s. Consequently (after substituting $H \rightarrow kc_1$) we obtain that relations in $A^*_{\text{GL}_n}X$ are generated by

$$\{2^{n-1-r}(kc_1)^r R(kc_1)\}_{r=0,\ldots,n-1}.$$

Next, observe that

$$R(kc_1) = \prod_{1 \leq i < j \leq n} (kc_1 + l_i + l_j)$$

is the top Chern class of the $\text{GL}_n$-module $(\det E)^{\otimes k} \otimes \wedge^2 E^*$. In particular when $k = 0$ all relations vanish, except for the single relation $2^{n-1}c_{\text{top}}(\wedge^2 E^*)$.

5.2. **Computation of $A^*(X_{1,k})$.** Define polynomials $\alpha_i(H)$ in the ring $\mathbb{Z}[c_1, \ldots, c_n][H]$ by the formulas

$$\alpha_i(H) = \sum_{j=0}^{i-1} \binom{n-j}{i-j} (-1)^j c_j H^{i-j} - 2c_i$$

when $i$ is odd and

$$\alpha_i(H) = \sum_{j=0}^{i-1} \binom{n-j}{i-j} (-1)^j c_j H^{i-j}$$

when $i$ is even.

**Proposition 13.**

$$A^*(X_{1,k}) = A^*_{O(n,k)} = \mathbb{Z}[c_1, \ldots, c_n]/(\alpha_1(kc_1), \ldots, \alpha_n(kc_1)).$$

**Remark 14.** Observe that $\alpha_i(0) = 0$ when $i$ is even and $\alpha_i(0) = -2c_i$ when $i$ is odd, so that when $k = 0$ we recover Panharipande’s presentation of $A^*_{O(n)}$.

**Proof.** As above, let $N = \binom{n+1}{2} - 1$, $\mathbb{P}^N := \mathbb{P}((\text{Sym}^2 E^*))$ and $Z_1 := \mathbb{P}(X_1)$ With the action given by (2), the map $X_1 \rightarrow Z_1$ is the $\mathbb{G}_m$-torsor corresponding to the line bundle $(\det E^{\otimes k}) \otimes O_{\mathbb{P}^N}(-1)$. Hence,

$$A^*_{\text{GL}_n}(X_1) = A^*_{\text{GL}_n}(Z_1)/(H - kc_1).$$
We now calculate \( A^*_\text{GL}_n(Z_1) \) using Pandharipande’s technique. Since 
\( Z_1 = \mathbb{P}^N \setminus \mathbb{P}(\Delta_1) \), 
\( A^*_\text{GL}_n(Z_1) = A^*_\text{GL}_n(\mathbb{P}^N)/\text{Im } j_* \), 
where \( j: \mathbb{P}(\Delta_1) \to \mathbb{P}^N \) is the inclusion.

Applying [Pan] Lemma 2] with base \( M \) a Chow approximation to 
the classifying space \( B \text{GL}_n \) we see that
\[
A^*_\text{GL}_n(Z_1) = \mathbb{Z}[c_1, \ldots, c_n, H]/I
\]
where \( I \) is generated by \( \beta'_1, \ldots, \beta'_n \) defined as
\[
\frac{e^{\text{GL}_n}(E^* \otimes \mathcal{O}(1))}{e^{\text{GL}_n}(E)} = 1 + \beta'_1 + \cdots + \beta'_n + \ldots.
\]
where \( e^{\text{GL}_n} \) refers to the total equivariant Chern class. As above we 
can compute by restricting to torus action. Then
\[
P := c^T(E^* \otimes \mathcal{O}(1)) = \prod_{i=1}^n (1 + H + l_i)
\]
\[
= \sum_{j=0}^n (-1)^j c_j (1 + H)^{n-j}
\]
\[
R := c^T(E) = \prod_{i=1}^n (1 - l_i)
\]
\[
= 1 + c_1 + c_2 + \cdots + c_n.
\]
where by convention \( c_0 = 1 \).

We are looking for \( \beta'_i \) such that
\[
P = (1 + \beta'_1 + \cdots + \beta'_n + \ldots)R,
\]

Let \( P_i \) be the sum of the terms of \( P \) of degree \( i \). Arguing by induction 
we see that the ideal generated by \( \beta'_1, \ldots, \beta'_n \) is the same as that 
generated by
\[
\{ \alpha_i := P_i - c_i \}_{i=1,\ldots,n}
\]
More precisely for each \( i = 1, \ldots, n \), if \( i \) is odd we have
\[
\alpha_i = \sum_{j=0}^{i-1} \binom{n-j}{i-j} (-1)^j c_j H^{i-j} - 2c_i
\]
and if \( i = 2m \) is even we have
\[
\alpha_i = \sum_{j=0}^{i-1} \binom{n-j}{i-j} (-1)^j c_j H^{i-j}.
\]

Viewing the \( \alpha_i \)'s as polynomials in \( H \), and substituting \( H \to kc_1 \) we 
conclude \( A^*_\text{GL}_n(X_1) = \mathbb{Z}[c_1, \ldots, c_n]/(\alpha_1(kc_1), \ldots, \alpha_n(kc_1)) \) \( \square \)
Remark 15. There are values of $k$ and $n$ for which some of the generators $\alpha_1, \ldots, \alpha_n$ can be eliminated from the ideal of relations. For example if $n = 3$ we have $\alpha_2 = c_1\alpha_1$ for every $k$. On the other hand, when $n = 4$ and $k = 1$ we have

$$(\alpha_1, \ldots, \alpha_4) = (2c_1, c_1^2, 2c_3, c_1c_3)$$

and when $k = 3$

$$(\alpha_1, \ldots, \alpha_4) = (10c_1, 5c_1^2, c_1^3 + 6c_1c_2 - 2c_3, c_1^2c_2 - c_1c_3).$$

In both of these cases no generator may be eliminated.

References

[Br] M. Brion: Equivariant Chow groups for torus actions; Trans. Groups, 2 n. 2. 225-267 (1997).
[Ed-Gr1] D. Edidin, W. Graham: Characteristic classes in the Chow ring; J. Algebraic Geom. 6 n.3, 431-443 (1997).
[Ed-Gr2] D. Edidin, W. Graham: Equivariant intersection theory; Inv. Math. 131, 595-634 (1998).
[Ed-Gr3] D. Edidin, W. Graham: Localization in equivariant intersection theory and the Bott residue formula. Amer. J. Math. 120 no. 3, 619–636 (1998).
[Fulg] D. Fulghesu, PhD Thesis, Scuola Normale Superiore, Pisa 2005.
[Ful] W. Fulton: Intersection theory Springer-Verlag (Berlin), 1998.
[Har] R. Hartshorne: Algebraic Geometry Springer-Verlag New York, 1977.
[Kre] A. Kresch: Cycle groups for Artin Stacks Inv. Math. 138 495-536 (1999).
[Pan] R. Pandharipande: Equivariant Chow rings of $O(k), SO(2k + 1)$ and $SO(4)$; J. Reine Angew. Math. 496, 131-148 (1998).
[Vis] A. Vistoli: The Chow ring of $\mathcal{M}_2$ (Appendix to [Ed-Gr2]); Inv. Math. 131 635-644 (1998).