Research Article

Extending an Almost Complete Pair of Matrices to a Complete Triple

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Motivated by the concept of complete pairs, which was introduced by Krein and Langer, we present the concepts of an almost complete pair of matrices and a complete triple of matrices. It is proved that an almost complete pair of matrices can be extended to a complete triple. An application of the result to differential equations is also given.

1. Introduction

Let

\[ L(\lambda) = I\lambda^l + A_{l-1}\lambda^{l-1} + \cdots + A_0 \tag{1} \]

be a monic matrix polynomial, where \( A_i \in M_n(\mathbb{C}) \) for \( i = 0, 1, \ldots, l-1 \). If there exist a matrix \( S \) and a polynomial \( L_1(\lambda) \) of degree \( l-1 \) such that \( L(\lambda) = L_1(\lambda)(I\lambda - S) \), then \( I\lambda - S \) is said to be a monic right divisor of \( L(\lambda) \). As is well known, \( I\lambda - S \) is a monic right divisor of \( L(\lambda) \) if and only if \( S \) is a root of \( L(\lambda) \); that is,

\[ S^l + A_{l-1}S^{l-1} + \cdots + A_0 = 0. \tag{2} \]

Let \( C \) be the companion matrix of \( L(\lambda) \); that is,

\[ C = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -A_0 & -A_1 & -A_2 & \cdots & -A_{l-1} \end{pmatrix}. \tag{3} \]

Then we have \( \det(I\lambda - C) = \det(L(\lambda)) \). If \( S_1, S_2, \ldots, S_l \in M_n(\mathbb{C}) \) are the different \( l \) roots of \( L(\lambda) \), it is clear that the Vandermonde matrix

\[ W(S_1, S_2, \ldots, S_l) = \begin{pmatrix} I & I & I & \cdots & I \\ S_1 & S_2 & S_3 & \cdots & S_l \\ S_1^2 & S_2^2 & S_3^2 & \cdots & S_l^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_1^{l-1} & S_2^{l-1} & S_3^{l-1} & \cdots & S_l^{l-1} \end{pmatrix} \tag{4} \]

plays a role in the investigation of the spectral analysis of \( L(\lambda) \) [1].

**Definition 1.** We say that the \( l \)-tuple \( (S_1, S_2, \ldots, S_l) \) of complex square matrices of order \( n \) is complete if the Vandermonde matrix \( W(S_1, S_2, \ldots, S_l) \) is invertible. When \( l = 2 \) and \( l = 3 \), we call a complete \( l \)-tuple of matrices a complete pair and a complete triple, respectively.

Suppose that \( (S_1, S_2) \) is complete. Take \( P = S_2 - S_1 \). Then \( P \) is invertible. If we take

\[ A_1 = -PS_2P^{-1} - S_1, \quad A_2 = PS_2P^{-1}S_1, \tag{5} \]

then \( I\lambda - S_1, I\lambda - S_2 \), form a complete pair of right divisors of \( L(\lambda) \) (cf. Lemma 2.4 in [2]). The concept of complete pair was
introduced in [3, 4]. In [5], Lancaster gave some important applications of complete pairs to solutions of differential equations. Motivated by the results in [5], we are interested in the following question.

**Question 1.** Suppose that \( (S_1, S_2) \) is not a complete pair. Is it possible for us to find \( S_1 \) and a cubic matrix polynomial \( L(\lambda) \) such that \( (S_1, S_2, S_3) \) is a complete triple and

\[
\lambda I - S_1, \quad \lambda I - S_2, \quad \lambda I - S_3
\]

are right divisors of \( L(\lambda) \)?

If \( (S_1, S_2, S_3) \) is a complete triple, then

\[
\begin{pmatrix}
1 & 1 \\
S_1 & S_2 \\
S_1^2 & S_2^2
\end{pmatrix}
\]

is of full column rank. Note that if \( S_1, S_2 \in M_n(C) \) and the rank of \( S_1 - S_2 \) is less than \( n - 1 \), we cannot always find \( S_1 \) such that \( (S_1, S_2, S_3) \) is a complete triple. For example, let \( S_1, S_2 \in M_n(C) \) be defined as follows:

\[
S_1 = \begin{pmatrix} A_1 & \alpha \\ \beta & A_2 \end{pmatrix}
\]

where \( A_1, A_2 \) are square matrices of orders \( n - 2 \) and 2, respectively, and \( S_2 - S_1 \) is in the form

\[
\begin{pmatrix}
I_{n-2} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Then

\[
S_2^2 - S_1^2 = \begin{pmatrix} 2A_1 + I_{n-2} & \alpha \\ \beta & 0 \end{pmatrix}
\]

If \( \text{rank}(\alpha) = 1 \), then the last two columns of \( (S_2^2 - S_1^2) \) are linearly dependent. For any \( S_3 \in M_n(C) \),

\[
\begin{pmatrix}
1 & 1 & 1 \\
S_1 & S_2 & S_3 \\
S_1^2 & S_2^2 & S_3^2
\end{pmatrix}
\]

cannot be invertible. If \( \text{rank}(\alpha) = 2 \), then

\[
\begin{pmatrix}
1 & 1 \\
S_1 & S_2 \\
S_1^2 & S_2^2
\end{pmatrix}
\]

is of full column rank. From the condition that

\[
\begin{pmatrix}
1 & 1 \\
S_1 & S_2 \\
S_1^2 & S_2^2
\end{pmatrix}
\]

is of full column rank, we cannot conclude that \( \text{rank}(S_2 - S_1) \geq n - 1 \).

After the above discussion, we give the following definition.

**Definition 2.** For two matrices \( S_1, S_2 \in M_n(C) \), one says they form an almost complete pair if the matrix

\[
\begin{pmatrix}
1 & 1 & 1 \\
S_1 & S_2 & S_3 \\
S_1^2 & S_2^2 & S_3^2
\end{pmatrix}
\]

is of full column rank and the rank of \( S_1 - S_2 \) is \( n - 1 \). For simplification, one says \( (S_1, S_2) \) is an almost complete pair of matrices of order \( n \).

The following is the main result of this paper.

**Theorem 3.** Suppose that \( S_1, S_2 \in M_n(C) \), \( n \geq 2 \), and \( (S_1, S_2) \) is an almost complete pair. Then one can find a matrix \( S_3 \in M_n(C) \) such that \( (S_1, S_2, S_3) \) is a complete triple; that is,

\[
\begin{pmatrix}
1 & 1 & 1 \\
S_1 & S_2 & S_3 \\
S_1^2 & S_2^2 & S_3^2
\end{pmatrix}
\]

is invertible.

We prove this theorem in Section 2. In Section 3, we partially answer the question mentioned before and give an application of the main result to differential equations.

### 2. Proof of the Main Theorem

In this section, we prove Theorem 3. Note that there is an invertible matrix \( P \) such that \( P^{-1}(S_2 - S_1)P \) is the Jordan normal form of \( S_2 - S_1 \). Let \( Q = P \oplus P \oplus P \). Then the condition that

\[
X = \begin{pmatrix}
1 & 1 & 1 \\
S_1 & S_2 & S_3 \\
S_1^2 & S_2^2 & S_3^2
\end{pmatrix}
\]

is invertible is equivalent to that

\[
Q^{-1}XQ
\]

is invertible. So from the beginning, we can assume that \( S_2 - S_1 \) is in Jordan normal form.

First we prove a lemma.

**Lemma 4.** Assume \( S_1, S_2 \in M_n(C) \) and the pair \( (S_1, S_2) \) is an almost complete pair with \( S_2 - S_1 = I_{n-1} \oplus 0 \). Write

\[
S_1 = \begin{pmatrix} A & \alpha \\ \beta & s \end{pmatrix}
\]

where \( A \in M_{n-1}(C) \). Then \( \alpha \) is not a zero vector.
Proof. By some computations, we have
\[
S_2^2 - S_1^2 = \begin{pmatrix} 2A + I_{n-1} & \alpha \\ \beta & 0 \end{pmatrix}.
\] (19)

The last column of
\[
\begin{pmatrix} S_2 - S_1 \\ S_2^2 - S_1^2 \end{pmatrix}
\] (20)
is \((0, 0, \ldots, 0, \alpha, 0)^T\), which is a \(2n\)-dimensional column vector. By the definition of almost complete pair, \(\alpha\) is not a zero vector.

Now we prove the main theorem.

Proof. From now on, we write \(S_1 = \begin{pmatrix} a_1 & \cdots & a_n \\ \vdots & \ddots & \vdots \\ a_{n-1} & \cdots & a_n \end{pmatrix}\). We denote by \(P(i, j) \in M_{n-1}(\mathbb{R})\) the matrix which is obtained by exchanging the columns of \(i\) and \(j\) of \((n-1) \times (n-1)\) identity matrix.

Without loss of generality, we can assume \(S_2 - S_1\) is in the Jordan normal form.

Case 1. Firstly, we consider the case that
\[
S_2 - S_1 = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix}.
\] (21)

If \(a_{1n} \neq 0\), then we take
\[
S_3 = S_1 + \begin{pmatrix} 0 & 0 \\ tI_{n-1} & 0 \end{pmatrix}.
\] (22)

By some computation, the last columns of \(S_2^2 - S_1^2\) and \(S_2 - S_1\) are \((a_1, a_{2n}, \ldots, a_{n-1}, 0, 0)^T\) and \((0, ta_{1n}, ta_{2n}, \ldots, ta_{n-1}, 0)^T\), respectively, and the \(i\)th \((1 \leq i \leq n-2)\) column of \(S_3^2 - S_1^2\) is
\[
\begin{pmatrix} f_{i1}(t) \\ f_{i2}(t) \\ \vdots \\ f_{i,i+1}(t) \\ t^2 + f_{i,i+2}(t), f_{i,i+3}(t), \ldots, f_{in}(t) \end{pmatrix},
\] (23)
and the \((n - 1)\)th column of \(S_2^2 - S_1^2\) is
\[
\begin{pmatrix} f_{n-1,1}(t) \\ f_{n-2,1}(t) \\ \vdots \\ f_{n-1,n-1}(t) \\ f_{n-1,n}(t) \end{pmatrix},
\]
where
\[
f_{i1} = ta_{i,i+1}
\]
and
\[
f_{ij} = ta_{j,i+1} + a_{i,j-1} (1 \leq i \leq n -1, 2 \leq j \leq n).
\]
It is clear that
\[
\det \begin{pmatrix} S_2 - S_1 & S_3 - S_1 \\ S_2^2 - S_1^2 & S_3^2 - S_1^2 \end{pmatrix}
\] (24)
is a polynomial in \(t\). If \(a_{1n} \neq 0\), then the degree of this polynomial is \(2n - 2\) and the leading term of this polynomial is \(a_{1n}^2 t^{2n-2}\). If \(|t|\) is sufficiently large, we can get that
\[
X = \begin{pmatrix} I & I \\ S_1 & S_2 \\ S_2^2 & S_3 \\ S_3^2 & S_2 & S_3 \end{pmatrix}
\] (25)
is invertible. If \(a_{1n} = 0\), by Lemma 4, then there must be some \(a_{kn} \neq 0\), where \(2 \leq i \leq n - 1\). So
\[
\begin{pmatrix} P(1, i) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ \beta & 0 \end{pmatrix} \begin{pmatrix} P(1, i) & 0 \\ 0 & 1 \end{pmatrix}
\] (27)
where the first entry of the vector \(P(1, i)\alpha\) is \(a_{kn}\). Take \(S_3\) such that
\[
(P(1, i) \oplus 1) (S_3 - S_1) (P(1, i) \oplus 1) = \begin{pmatrix} 0 & 0 \\ tI_{n-1} & 0 \end{pmatrix}.
\] (28)

Then
\[
X = \begin{pmatrix} I & I \\ S_1 & S_2 & S_3 \\ S_2^2 & S_2 & S_3 \\ S_3^2 & S_2 & S_3 \end{pmatrix}
\] (29)
is invertible.

Case 2. Now we consider the case that \(S_2 - S_1\) has the Jordan normal form
\[
J(\lambda_1) \oplus J(\lambda_2) \oplus \cdots \oplus J(\lambda_k) \oplus 0,
\] (30)
where \(\lambda_i \neq 0\), \(1 \leq i \leq k\). If \(a_{1n} \neq 0\), then we take
\[
S_3 = S_1 + \begin{pmatrix} 0 & 0 \\ tI_{n-1} & 0 \end{pmatrix}.
\] (31)
If \(a_{1n} = 0\) and \(a_{kn} \neq 0\), \(1 < i < n\), then we take \(S_3\) such that
\[
(P(1, i) \oplus 1) (S_3 - S_1) (P(1, i) \oplus 1) = \begin{pmatrix} 0 & 0 \\ tI_{n-1} & 0 \end{pmatrix}.
\] (32)
In any case when \(|t|\) is sufficiently large, we can get that
\[
X = \begin{pmatrix} I & I \\ S_1 & S_2 & S_3 \\ S_2^2 & S_2 & S_3 \\ S_3^2 & S_2 & S_3 \end{pmatrix}
\] (33)
is invertible.

Case 3. Now we consider the case that \(0\) is not a single root of characteristic polynomial of \(S_2 - S_1\); that is, \(S_2 - S_1\) has the Jordan normal form:
\[
J(\lambda_1) \oplus J(\lambda_2) \oplus \cdots \oplus J(\lambda_s) \oplus J(0).
\] (34)
If \(n = 2\), then
\[
S_2 - S_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\] (35)
By Lemma 4, $a_{12} \neq 0$. Take
\[ S_3 = S_1 + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{36} \]
Then
\[ \det \begin{pmatrix} I & I & I \\ S_1 & S_2 & S_3 \\ S_1^2 & S_2^2 & S_3^2 \end{pmatrix} \] \[ = \det \begin{pmatrix} S_2 - S_1 & S_3 - S_1 \\ S_2^2 - S_1^2 & S_3^2 - S_1^2 \end{pmatrix} = -t^2 a_{12}^2 \neq 0. \tag{37} \]
We can conclude that
\[ X = \begin{pmatrix} I & I & I \\ S_1 & S_2 & S_3 \\ S_1^2 & S_2^2 & S_3^2 \end{pmatrix} \tag{38} \]
is invertible.
Firstly, we prove the case that the normal form itself is
\[ \begin{pmatrix} 0 & 0 \\ tI_{n-1} & 0 \end{pmatrix}. \tag{39} \]
If $a_{1n} \neq 0$, then we take
\[ S_3 = S_1 + \begin{pmatrix} t & 0 & 0 & 0 & \cdots & 0 & t \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ t & t & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & t & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & t & 0 \end{pmatrix}. \tag{40} \]
So
\[ (S_3 - S_1)^2 = \begin{pmatrix} t^2 & 0 & 0 & 0 & \cdots & 0 & t^2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ t^2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & t^2 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & t^2 & 0 \end{pmatrix}. \tag{41} \]
By some computation,
\[ \det X = \det \begin{pmatrix} I & I & I \\ S_1 & S_2 & S_3 \\ S_1^2 & S_2^2 & S_3^2 \end{pmatrix} \tag{42} \]
is a polynomial in $t$, where the leading term of it is $a_{1n} t^{2n-1}$. So if $a_{1n} \neq 0$, then $X$ is invertible when $|t|$ is sufficiently large. If $a_{1n} = 0$, then there must be an $i \in \{2, 3, \ldots, n-1\}$ such that $a_{in} \neq 0$. If $i \in \{2, 3, \ldots, n-2\}$, take $S_3$ such that
\[ \begin{pmatrix} P(1, i-1) \oplus 1 \end{pmatrix} (S_3 - S_1) \begin{pmatrix} P(1, i-1) \oplus 1 \end{pmatrix} \]
\[ = \begin{pmatrix} t & 0 & 0 & 0 & \cdots & 0 & t \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ t & t & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & t & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & t & 0 \end{pmatrix}. \tag{43} \]
When $|t|$ is sufficiently large, we have $X$ being invertible. If $i = n - 1$, that is to say, the last column of $S_3^2 - S_1^2$ is $(0, 0, \ldots, 0, a_{n-1} n)^T$, then we take
\[ S_3 = S_1 + \begin{pmatrix} t & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & t & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & t & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & t & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{44} \]
We have
\[ (S_3 - S_1)^2 = \begin{pmatrix} t^2 & 0 & 0 & 0 & \cdots & 2t & 0 \\ 0 & t^2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & t^2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & t^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & t^2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{45} \]
where the first entry of the last column of $S_3^2 - S_1^2$ is $a_{n-1} n$. So the leading term of the polynomial
\[ \det X = \det \begin{pmatrix} I & I & I \\ S_1 & S_2 & S_3 \\ S_1^2 & S_2^2 & S_3^2 \end{pmatrix} \tag{46} \]
is $a_{n-1}^2 t^{2n-3}$. Thus
\[ X = \begin{pmatrix} I & I & I \\ S_1 & S_2 & S_3 \\ S_1^2 & S_2^2 & S_3^2 \end{pmatrix} \tag{47} \]
is invertible when $|t|$ is sufficiently large.
Now we prove the general case that $S_2 - S_1$ has the Jordan normal form:
\[ J(\lambda_1) \oplus J(\lambda_2) \oplus \cdots \oplus J(\lambda_s) \oplus J(0), \tag{48} \]
where $\lambda_i \neq 0$, $1 \leq i \leq s$. If there is some $a_{ij} \neq 0$, $m + 1 \leq i \leq n$, we assume that $J(0)$ is a matrix of order $n - m$. For any square matrix $B$ of order $n - m$, we can construct a square matrix $A$ of order $n - m$ such that
\[ J(0) \oplus J(0) B^2 - J^2(0) (A + B)^2 - A^2 \]
is invertible. Take
\[ S_3 = S_1 + (tI_{n-m} \oplus A). \tag{50} \]
If $a_{in} = 0$, for all $m + 1 \leq i \leq n$, there must be some $j$ such that $1 \leq j \leq m$, $a_{jm} \neq 0$. Let $Y = XP$, where $P$ is the matrix which is obtained by exchanging the columns $m + 1$ and $n$ of $n \times n$ identity matrix. Then $\det X = - \det Y$ and
\[ Y = J(\lambda_1) \oplus J(\lambda_2) \oplus \cdots \oplus J(\lambda_s) \oplus 0 \oplus C, \tag{51} \]
where the square matrix $C$ of order $n - m + 1$ has the form
\[
\begin{pmatrix}
0 & 1 \\
I_{n-m} & 0
\end{pmatrix}.
\] (52)

Let $T$ be a matrix such that $T^{-1}CT$ has the Jordan normal form $J$. Then we can consider the matrix $Z = (I \otimes T^{-1})Y (I \otimes T)$. Note that $Z$ has the form
\[
Z = J(\lambda_1) \oplus J(\lambda_2) \oplus \cdots \oplus J(\lambda_i) \oplus 0 \oplus J.
\] (53)

Denote $S = J(\lambda_1) \oplus J(\lambda_2) \oplus \cdots \oplus J(\lambda_i) \oplus 0$ For given monic matrix polynomials was important (cf. [6–13]). The construction of a common multiple of them was pre-

3. Applications

Theorem 3 in Section 1, we can find $S_3 \in M_n(C)$ such that $(S_1, S_2, S_3)$ is a complete triple; that is,
\[
\begin{pmatrix}
I & I & I \\
S_1 & S_2 & S_3 \\
S_1^2 & S_2^2 & S_3^2
\end{pmatrix}
\] (55)
is invertible. The proof is complete.

The general solution of
\[
L(\frac{d}{dt})x(t) = \frac{d^2 x(t)}{dt^2} + \sum_{j=0}^{l-1} A_j \frac{d^{j+1} x(t)}{dt^{j+1}} = f(t).
\] (60)

Lemma 7 (Theorem 2.9 in [2]). The general solution of (60) is given by the formula
\[
x(t) = Xe^{rT}c + X \int_a^t e^{(t-s)T}Yf(s)ds, \quad t \in [a, b],
\] (61)

where $(X, Y, T)$ is a standard triple of $L(\lambda)$ and $c \in C^{nl}$ is arbitrary. In particular, the general solution of the homogeneous equation
\[
L(\frac{d}{dt})x(t) = 0
\] (62)
is given by the formula
\[
x(t) = Xe^{rT}c, \quad c \in C^{nl}.
\] (63)

Before we give the main result of this section, we introduce some notations. Suppose
\[
J_1 = X_1^{-1}S_1X_1, \quad J_2 = X_2^{-1}S_2X_2, \quad J_3 = X_3^{-1}S_3X_3
\] (64)
are the Jordan forms of $S_1, S_2, S_3$, respectively. Write
\[
X = (X_1 \ X_2 \ X_3), \quad J = J_1 \oplus J_2 \oplus J_3.
\] (65)

Theorem 8. Let $(S_1, S_2, S_3)$ be a complete triple of $L(\lambda) = 0$, where $L(\lambda) = \lambda^3 + A_2 \lambda^2 + A_1 \lambda + A_0$. Then every solution of
\[
L(\frac{d}{dt})u = f
\] (66)
is given by

\[ u(t) = e^{S_1 t} c_1 + e^{S_2 t} c_2 + e^{S_3 t} c_3 + X \int_a^t e^{(t-s)J} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} f(s) ds, \quad t \in [a, b], \]  

(67)

for some \( c_1, c_2, c_3 \in \mathbb{C}^n \), where

\[ \begin{pmatrix} Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} S_2 - S_1 & S_3 - S_1 \\ S_2^2 - S_1^2 & S_3^2 - S_1^2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ X_1^{-1} \end{pmatrix}, \]

\[ Y_1 = -X_1^{-1} X_2 Y_2 - X_1^{-1} X_3 Y_3. \]

In particular, every solution of

\[ L \left( \frac{d}{dt} \right) u = 0 \]  

(69)

has the form

\[ u(t) = e^{S_1 t} c_1 + e^{S_2 t} c_2 + e^{S_3 t} c_3, \]

(70)

where \( c_1, c_2, c_3 \in \mathbb{C}^n \).

**Proof.** Using Lemma 7, we take

\[ Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = X^{-1} \begin{pmatrix} I & I & I \\ S_1 & S_2 & S_3 \\ S_1^2 & S_2^2 & S_3^2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \]  

(71)

Then \((X, J, Y)\) is a standard triple; it is easy to get the conclusion.

**Remark 9.** The above theorem and its proof are motivated by those in [5] (cf. also Sections 2.4 and 2.5 in [2]).

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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