CALIBRATED REPRESENTATIONS OF TWO BOUNDARY TEMPERLEY-LIEB ALGEBRAS

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Abstract. The two boundary Temperley-Lieb algebra $TL_k$ arises in the transfer matrix formulation of lattice models in Statistical Mechanics, in particular in the introduction of integrable boundary terms to the six-vertex model. In this paper, we classify and study the calibrated representations—those for which all the Murphy elements (integrals) are simultaneously diagonalizable—which, in turn, corresponds to diagonalizing the transfer matrix in the associated model. Our approach is founded upon the realization of $TL_k$ as a quotient of the type $C_k$ affine Hecke algebra $H_k$. In previous work, we studied this Hecke algebra via its presentation by braid diagrams, tensor space operators, and related combinatorial constructions. That work is directly applied herein to give a combinatorial classification and construction of all irreducible calibrated $TL_k$-modules and explain how these modules also arise from a Schur-Weyl duality with the quantum group $U_q\mathfrak{gl}_2$.

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1. Introduction

The paper [DR] studied the calibrated representations of affine Hecke algebras of type C with unequal parameters and developed their combinatorics and their role in Schur-Weyl duality. This paper applies that information to the study of two boundary Temperley-Lieb algebras. The two boundary Temperley-Lieb algebras appear in statistical mechanics for analysis of spin chains with generalized boundary conditions [GP, GNPR]. The spectrum of the Hamiltonian for these spin chains with boundaries can be determined via the representation theory of the two boundary Temperley-Lieb algebras. In fact, the need to understand the representation theory of the two boundary Temperley-Lieb algebra better was a primary motivation for our preceding papers [Dau, DR] on two boundary Hecke algebras.
In the first part of this paper, Section 2, we review the definition and structure of the two boundary Hecke algebra $H_k$ (the affine Hecke algebra of type C with unequal parameters). Following this brief review we carefully analyze certain idempotents which, as we prove in Theorem 3.1, generate the ideal that one must quotient by to obtain the two boundary Temperley-Lieb algebra from the two boundary Hecke algebra. It is the expression of these idempotents in terms of the intertwiner presentation of $H_k$ (see Proposition 2.3) that will eventually provide understanding of the weights that can appear in two boundary Temperley-Lieb modules (the possible eigenvalues of the “Murphy elements” $W_i$—see equation (2.9) and §2.4).

In Section 3 we define the two boundary Temperley-Lieb algebra (or symplectic blob algebra) $TL_k$ following [GN, GMP07, GMP08, GMP12, Rec, KMP16, GMP17], and review the diagram algebra calculus for these algebras. Part of our contribution is to extend this calculus to make its connection to the diagrammatic calculus of the Hecke algebra $H_k$ via braids. In Theorem 3.2 we use these diagrammatics to give a proof of a result of [GN] that provides an expansion of a certain central element of $H_k$ inside $TL_k$. Using the Hecke algebra point of view, this result enables us to understand that the center of $TL_k$ is a polynomial ring in one variable $Z(TL_k) = C[Z]$, and that $TL_k$ is of finite rank over this center. In retrospect, the algebra $H_k$ has a similar structure and so perhaps this should not be surprising but, nonetheless, it is pleasant to see it come out in such a vivid and explicit form.

We have used a different normalization of the parameters of the two boundary Hecke and Temperley-Lieb algebra from those used in [GN, GMP12]. Our normalization will be helpful, for example, for future applications of these algebras to the theory of Macdonald polynomials and to the study of the exotic nilpotent cone. In both of these cases the affine Hecke algebra of type $C_n$ plays an important role: the Koornwinder polynomials are the Macdonald polynomials for type $(C'_n, C_n)$ [M03], and the K-theory of the Steinberg variety of the exotic nilpotent cone provides a geometric construction of the representations of the two boundary Hecke and Temperley-Lieb algebras at unequal parameters (see [Kat]).

The calibrated representations are the irreducible representations of the two boundary Hecke algebra for which a large family of commuting operators (integrals, or Murphy elements) have a simple (joint) spectrum. This property makes these representations particularly attractive, and the detailed combinatorics of these representations has been worked out in [DR]. In Section 4 we use the detailed analysis of the idempotents done in Section 2 to determine exactly which calibrated irreducible representations of the two boundary Hecke algebra are representations of the two boundary Temperley-Lieb algebra (Theorem 4.3). In consequence, we obtain a full classification of the calibrated irreducible representations of the two boundary Temperley-Lieb algebras.

As explained in [DR], there is a Schur-Weyl type duality between the two boundary Hecke algebra and the quantum group $U_q\mathfrak{gl}_n$. The classical Schur-Weyl duality between $U_q\mathfrak{gl}_n$ and the finite Hecke algebra of type A becomes a Schur-Weyl duality for the finite Temperley-Lieb algebra when $n = 2$. In Theorem 5.1 we show that at $n = 2$ the Schur-Weyl duality of [DR] gives a Schur-Weyl duality for the two boundary Temperley-Lieb algebra. This method (coming from R-matrices for the quantum group $U_q\mathfrak{gl}_2$) provides many many irreducible calibrated representations of the two boundary Temperley-Lieb algebra $TL_k$. Using our results from Section 4, we determine exactly which irreducible calibrated representations of $TL_k$ occur in the Schur-Weyl duality context.

The seeds of this work were sown in a conversation between Pavel Pyatov, Arun Ram and Vladimir Rittenberg at the Max Planck Institut in Bonn in 2006. Vladimir was the leader and provided the inspiration by introducing us to spin chains with boundaries. The seed has now grown from a concept into fully formed and fruitful mathematics. We thank all the institutions which have
supported our work on this paper, particularly the University of Melbourne, the Australian Research Council (grants DP1201001942 and DP130100674), the National Science Foundation (grant DMS-1162010), the Simons Foundation (grant #586728), ICERM (Institute for Computational and Experimental Research in Mathematics, 2013 semester on Automorphic Forms, Combinatorial Representation Theory and Multiple Dirichlet Series) and the Max Planck Institut in Bonn.

2. The two boundary Hecke algebra $H_k$

The two boundary Hecke algebra is often called the affine Hecke algebra of type $(C, C^\vee)$. In this section we review the definitions of $H^\text{ext}_k$ following our previous paper [DR]. In particular, we will need the basic diagrammatics and the “Bernstein” presentation with a Laurent polynomial ring $\mathbb{C}[W_1^\pm, \ldots, W_k^\pm]$ and intertwiners $\tau_1, \ldots, \tau_k$. After this review we define the idempotent elements $p^{(1)}(1\ 3), p^{(\emptyset, 1^2)}, p^{(2)}(1\ 2), p^{(\emptyset, 1^2)}, p^{(1^2, \emptyset)}$, which we will need to quotient by in order to obtain the two boundary Temperley-Lieb algebra. We derive expressions of these elements in terms of the different choices of generators: the braid generators $T_i$, the cap/cup generators $e_i$, and the intertwiner generators $\tau_i$ and $W_j$.

2.1. Graph notation for braid relations. For generators $g, h$, encode relations graphically by

\[
\begin{align*}
g & \quad h \quad \text{means } gh = hg, \\
g \quad h \quad \text{means } ghg = hgh, \quad & \quad (2.1) \\
g \quad h \quad \text{means } ghgh = hghg.
\end{align*}
\]

For example, the group of signed permutations,

\[
W_0 = \left\{ \text{ bijections } w: \{-k, \ldots, -1, 1, \ldots, k\} \to \{-k, \ldots, -1, 1, \ldots, k\} \right\}, \quad (2.2)
\]

has a presentation by generators $s_0, s_1, \ldots, s_{k-1}$, with relations

\[
\begin{align*}
s_0 & = s_1 s_2 s_3 \ldots s_{k-2} s_{k-1} \quad \text{ and } \quad s_i^2 = 1 \quad \text{for } i = 0, 1, 2, \ldots, k-1. \quad (2.3)
\end{align*}
\]

2.2. The two boundary braid group. The two boundary braid group is the group $B_k$ generated by $\bar{T}_0, \bar{T}_1, \ldots, \bar{T}_k$, with relations

\[
\begin{align*}
\bar{T}_0 & \quad \bar{T}_1 \quad \bar{T}_2 \quad \bar{T}_{k-2} \quad \bar{T}_{k-1} \quad \bar{T}_k \quad \bar{T}_0 \quad \text{.} \quad (2.4)
\end{align*}
\]

Pictorially, the generators of $B_k$ are identified with the braid diagrams

\[
\begin{align*}
\bar{T}_k &= \left[ \begin{array}{cccccc}
\quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\end{array} \right], \quad \bar{T}_0 = \left[ \begin{array}{ccccccc}
\quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\end{array} \right], \quad \text{and}
\end{align*}
\]

\[
\begin{align*}
\bar{T}_i &= \left[ \begin{array}{ccccccc}
\quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\end{array} \right] \quad \text{for } i = 1, \ldots, k-1, \quad (2.5)
\end{align*}
\]
and the multiplication of braid diagrams is given by placing one diagram on top of another (multiplying generators left-to-right corresponds to stacking diagrams top-to-bottom).

In some applications (notably to the Schur-Weyl duality of [DRI §5]), it is useful to move the rightmost pole to the left by conjugating by the diagram

\[
\sigma = \begin{array}{|c|c|c|c|c|c|}
\hline
& & & & & \\
\hline
& & & & & \\
\hline
\end{array}
\].

(2.6)

Define

\[
T_i = \sigma T_i \sigma^{-1} = \begin{array}{|c|c|c|c|c|}
\hline
& & & & \\
\hline
& & & & \\
\hline
\end{array}, \quad Y_1 = \sigma \bar{T}_0 \sigma^{-1} = \begin{array}{|c|c|c|c|c|}
\hline
& & & & \\
\hline
& & & & \\
\hline
\end{array},
\]

(2.7)

and

\[
X_1 = T_1^{-1} T_2^{-1} \cdots T_{k-1}^{-1} \sigma \bar{T}_k \sigma^{-1} T_{k-1} \cdots T_1 = \begin{array}{|c|c|c|c|c|c|}
\hline
& & & & & \\
\hline
& & & & & \\
\hline
& & & & & \\
\hline
& & & & & \\
\hline
\end{array}.
\]

(2.8)

Define

\[
Z_1 = X_1 Y_1 \quad \text{and} \quad Z_i = T_{i-1} T_{i-2} \cdots T_1 Y_1 T_1 \cdots T_{i-1} = \begin{array}{|c|c|c|c|c|c|}
\hline
& & & & & \\
\hline
& & & & & \\
\hline
& & & & & \\
\hline
& & & & & \\
\hline
\end{array},
\]

(2.9)

for \( i = 2, \ldots, k \). Let

\[
P = \begin{array}{|c|c|c|c|}
\hline
& & & \\
\hline
& & & \\
\hline
\end{array}.
\]

The extended affine braid group is the group \( \mathcal{B}_{k}^{\text{ext}} \) generated by \( \mathcal{B}_{k} \) and \( P \) with the additional relations

\[
PX_1 P^{-1} = Z_1^{-1} X_1 Z_1, \quad PY_1 P^{-1} = Z_1^{-1} Y_1 Z_1, \quad \text{and} \quad PT_i P^{-1} = T_i \quad \text{for} \quad i = 1, \ldots, k - 1.
\]

(2.10)

(2.11)

The element

\[
Z_0 = P Z_1 \cdots Z_k \quad \text{is central in} \quad \mathcal{B}_{k}^{\text{ext}}
\]

(2.12)

since the group \( \mathcal{B}_{k}^{\text{ext}} \) is a subgroup of the braid group on \( k + 2 \) strands, and \( Z_0 \) is the generator of the center of the braid group on \( k + 2 \) strands (see [GM Theorem 4.2]). So if \( \mathcal{D} = \{ Z_0^j \mid j \in \mathbb{Z} \} \) then \( \mathcal{B}_{k}^{\text{ext}} = \mathcal{D} \times \mathcal{B}_{k} \), with \( \mathcal{D} \cong \mathbb{Z} \).

2.3. The extended affine Hecke algebra \( H_{k}^{\text{ext}} \) of type \( C_k \). Fix \( a_1, a_2, b_1, b_2, t^{\frac{1}{2}} \in \mathbb{C}^\times \) and let

\[
t_{k}^{\frac{1}{2}} = a_1^{\frac{1}{2}} a_2^{-\frac{1}{2}}, \quad t_0^{\frac{1}{2}} = b_1^{\frac{1}{2}} b_2^{-\frac{1}{2}}.
\]

(2.13)

The extended two boundary Hecke algebra \( H_{k}^{\text{ext}} \) with parameters \( t_{k}, t_0^{\frac{1}{2}} \) and \( t_{k}^{\frac{1}{2}} \) is the quotient of \( \mathcal{B}_{k}^{\text{ext}} \) by the relations

\[
(X_1 - a_1)(X_1 - a_2) = 0, \quad (Y_1 - b_1)(Y_1 - b_2) = 0, \quad \text{and} \quad (T_i - t_{k}^{\frac{1}{2}})(T_i + t_{k}^{\frac{1}{2}}) = 0, \quad \text{(H)}
\]
for \(i = 1, \ldots, k - 1\). Let

\[
T_0 = b_1^{-\frac{1}{2}}(-b_2)^{-\frac{1}{2}}Y_1, \quad T_k = a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}T_{k-1} \cdots T_2T_1X_1T_1^{-1}T_2^{-1} \cdots T_{k-1}^{-1}.
\] (2.14)

Then

\[
\begin{array}{cccccc}
& T_0 & T_1 & T_2 & \cdots & T_{k-2} & T_{k-1} & T_k \\
\circ & \circ & \circ & \cdots & \circ & \circ & \circ \end{array}
\]

and

\[
(T_0 - \frac{1}{t_0})(T_0 + t_0^{-\frac{1}{2}}) = 0, \quad (T_i - \frac{1}{t_0})(T_i + t_i^{-\frac{1}{2}}) = 0, \quad (T_k - \frac{1}{t_k})(T_k + t_k^{-\frac{1}{2}}) = 0,
\] (2.15)

for \(i \in \{1, \ldots, k-1\}\).

Let \(a, a_0, a_k \in \mathbb{C}^\times\) and define

\[
a_0e_0 = T_0 - t_0^{-\frac{1}{2}}, \quad ae_i = T_i - t_i^{-\frac{1}{2}}, \quad a_ke_k = T_k - t_k^{-\frac{1}{2}},
\] (2.16)

for \(i \in \{1, \ldots, k-1\}\). The relations in (2.15) are equivalent to

\[
T_0e_0 = -t_0^{-\frac{1}{2}}e_0, \quad T_ie_i = -t_i^{-\frac{1}{2}}e_i, \quad T_ke_k = -t_k^{-\frac{1}{2}}e_k,
\] (2.17)

and to

\[
e_0^2 = -\frac{1}{a_0}(t_0^{-\frac{1}{2}} + t_0^{-\frac{1}{2}})e_0, \quad e_i^2 = -\frac{1}{a}(t_i^{-\frac{1}{2}} + t_i^{-\frac{1}{2}})e_i, \quad e_k^2 = -\frac{1}{a_k}(t_k^{-\frac{1}{2}} + t_k^{-\frac{1}{2}})e_k,
\] (2.18)

for \(i \in \{1, \ldots, k-1\}\).

Remark 2.1. For \(i \in \{1, \ldots, k-2\}\), using \(T_i = ae_i + t_i^{\frac{1}{2}}\) to expand \(T_iT_{i+1}T_i\) and \(T_{i+1}T_iT_{i+1}\) in terms of the \(e_i\) shows that in the presence of the relations (11),

\[
T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}
\]

is equivalent to

\[
a^3e_ie_{i+1}e_i - ae_i = a^3e_{i+1}e_i - ae_{i+1}.
\]

Similarly, \(T_0T_1T_0T_1\) is equivalent to

\[
a_0^2a^2e_0e_1e_0e_1 = a_0a(t_0^{-\frac{1}{2}}t_0^{\frac{1}{2}} + t_0^{\frac{1}{2}}t_0^{-\frac{1}{2}})e_0e_1 = a_0a^2e_1e_0e_1e_0 - a_0a(t_0^{-\frac{1}{2}}t_0^{\frac{1}{2}} + t_0^{\frac{1}{2}}t_0^{-\frac{1}{2}})e_1e_0.
\]

In the case that \(a_0a^2 = a_0a(t_0^{-\frac{1}{2}}t_0^{\frac{1}{2}} + t_0^{\frac{1}{2}}t_0^{-\frac{1}{2}})\) then

\[
T_0T_1T_0T_1 = T_1T_0T_1T_0
\]

is equivalent to

\[
e_0e_1e_0e_1 - e_0e_1 = e_1e_0e_1e_0 - e_1e_0.
\]

In the case that \(a^3 = a\) then

\[
T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}
\]

is equivalent to

\[
e_i e_{i+1} e_i - e_i = e_{i+1} e_i - e_{i+1}.
\]

This is the explanation for why the favorite choices of \(a, a_0\) and \(a_k\) satisfy

\[
a = \pm 1, \quad a_0a = t_0^{-\frac{1}{2}}t_0^{\frac{1}{2}} + t_0^{\frac{1}{2}}t_0^{-\frac{1}{2}} = [t_0t_0^{-1}] \quad \text{and} \quad a_ka = t_k^{-\frac{1}{2}}t_k^{\frac{1}{2}} + t_k^{\frac{1}{2}}t_k^{-\frac{1}{2}} = [t_kt_k^{-1}],
\]

where we use the notation

\[
[t^s] = [t_0^{s_0} t_1^{s_1} \cdots t_k^{s_k}] = \left\{ \frac{t_0^{s_0} t_1^{s_1} \cdots t_k^{s_k}}{t_0^{s_0} t_1^{s_1} \cdots t_k^{s_k}} \right\} = \frac{|s|}{|s|}.
\] (2.19)
2.4. The Bernstein presentation of $H_k^{\text{ext}}$. The Murphy elements for $H_k^{\text{ext}}$ are

$$W_j = T_1^{-1}T_2^{-1} \cdots T_{k-1}^{-1}T_kT_{k-1} \cdots T_1T_0$$

for $j \in \{2, \ldots, k\}$. Let

$$W_0 = PW_1 \cdots W_k.$$  

**Theorem 2.2.** (See, for example, [DR Theorem 2.2.]) Fix $t_0, t_k, t \in \mathbb{C}^\times$ and use notations for relations as defined in (2.1). The extended affine Hecke algebra $H_k^{\text{ext}}$ defined in (1) is presented by generators, $T_0, T_1, \ldots, T_{k-1}, W_0, W_1, \ldots, W_k$ and relations

$$W_0 \in Z(H_k^{\text{ext}}), \quad T_0 \quad T_1 \quad T_2 \quad T_{k-2} \quad T_{k-1}; \quad \text{(B1)}$$

$$W_iW_j = W_jW_i, \quad \text{for } i, j = 0, 1, \ldots, k; \quad \text{(B2)}$$

$$T_0W_j = W_jT_0, \quad \text{for } j \neq 1; \quad \text{(B3)}$$

$$T_iW_j = W_jT_i \quad \text{for } i = 1, \ldots, k-1 \quad \text{and } j = 1, \ldots, k \quad \text{with } j \neq i, i+1; \quad \text{(B4)}$$

$$(T_0 - t_0^2)(T_0 + t_0^2) = 0, \quad \text{and } \quad (T_i - t_i^2)(T_i + t_i^{-2}) = 0 \quad \text{for } i = 1, \ldots, k-1; \quad \text{(H)}$$

for $i = 1, \ldots, k-1$,

$$T_iW_i = W_{i+1}T_i + (t_i^{1/2} - t_i^{-1/2})W_iW_{i+1}^{-1}, \quad T_iW_{i+1} = W_iT_i + (t_i^{1/2} - t_i^{-1/2})W_iW_{i+1}^{-1}; \quad \text{(C1)}$$

and

$$T_0W_1 = W_1^{-1}T_0 + \left( (t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})W_1^{-1} \right) W_1 - W_1^{-1} \cdot \frac{1 - W_1^{-2}}{1 - W_i^{-2}}. \quad \text{(C2)}$$

The two boundary Hecke algebra $H_k$ with parameters $t_0^{1/2}, t_k^{1/2}$ and $t_0^{-1/2}$ is the subalgebra of $H_k^{\text{ext}}$ generated by $T_0, T_1, \ldots, T_k$. Then

$$H_k^{\text{ext}} = H_k \otimes \mathbb{C}[W_0^{\pm 1}] \quad \text{as algebras,} \quad \text{(2.20)}$$

and, as proved for example in [DR Theorem 2.3], the element

$$Z = W_1 + W_1^{-1} + W_2 + W_2^{-1} + \cdots + W_k + W_k^{-1} \quad \text{is central in } H_k^{\text{ext}}. \quad \text{(2.21)}$$

2.5. The elements $\tau_i$. Define

$$\tau_0 = T_0 - \frac{(t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})W_1^{-1}}{1 - W_i^{-2}}, \quad \text{and } \quad \tau_i = T_i - \frac{t_i^{1/2} - t_i^{-1/2}}{1 - W_iW_{i+1}^{-1}}, \quad \text{(2.22)}$$

for $i \in \{1, \ldots, k-1\}$. Evoking the notation of [DR §3], reviewed later in §4.1, let

$$f_{2\varepsilon_i} = (1 - W_i^{-1})(1 - W_i^{-2}) = 1 - W_i^{-2},$$

$$f_{\varepsilon_i - r_2} = (1 - t_0^{-1/2}t_k^{1/2}W_i^{-1}), \quad f_{\varepsilon_i - r_1} = (1 + t_0^{-1/2}t_k^{1/2}W_i^{-1}), \quad \text{(2.23)}$$

$$f_{-\varepsilon_i - r_2} = (1 - t_0^{1/2}t_k^{-1/2}W_i), \quad f_{-\varepsilon_i - r_1} = (1 + t_0^{1/2}t_k^{-1/2}W_i),$$

$$f_{\varepsilon_i - \varepsilon_j} = 1 - W_iW_j^{-1}, \quad f_{\varepsilon_i - \varepsilon_j + 1} = 1 - tW_iW_j^{-1},$$

for $i, j \in \{1, \ldots, k\}$. Then

$$a_0 e_0 = \tau_0 - t_0^{1/2}f_{\varepsilon_i - r_2}f_{\varepsilon_i - r_1}f_{2\varepsilon_i} \quad \text{and } \quad a_i e_i = \tau_i - t_i^{-1/2}f_{\varepsilon_i - \varepsilon_i + 1}f_{2\varepsilon_i}, \quad \text{(2.24)}$$
and, as proved in [DR, Proposition 2.4],
\[ \tau_0 \quad \tau_1 \quad \tau_2 \quad \tau_{k-2} \quad \tau_{k-1} \quad \text{and} \]
\[ \tau_0^2 = W_1^{-2}t_0^{-1}f_{\varepsilon_1-r_1}f_{\varepsilon_1-r_1}f_{\varepsilon_1-r_2}f_{\varepsilon_1-r_2}f_{\varepsilon_1-r_2}, \quad W_1\tau_0 = \tau_0W_1^{-1}, \quad W_r\tau_0 = \tau_0W_r, \]
\[ \tau_i^2 = t^{-1}f_{\varepsilon_i-\varepsilon_{i+1}+1}f_{\varepsilon_{i+1}-\varepsilon_i+1}, \quad W_i\tau_i = \tau_iW_{i+1}, \quad W_{i+1}\tau_i = \tau_iW_i, \quad W_j\tau_i = \tau_iW_j, \]
for \( r, j \in \{1, \ldots, k\} \) with \( r \neq 1 \) and \( j \neq i, i+1 \).

2.6. The elements \( p_i^{(13)} \), \( p_0^{(0,12)} \) and \( p_0^{(12,0)} \). Fix \( i \in \{1, \ldots, k-2\} \). Let
\[ HS_3 \] be the subalgebra of \( H_k^{\text{ext}} \) generated by \( T_i \) and \( T_{i+1} \), and let
\[ HB_2 \] be the subalgebra of \( H_k^{\text{ext}} \) generated by \( T_0 \) and \( T_1 \).

The idempotents \( p_i^{(13)} \) in \( HS_3 \) and the idempotents \( p_0^{(0,12)} \) and \( p_0^{(12,0)} \) in \( HB_2 \) are uniquely determined by the equations
\[ (p_i^{(13)})^2 = p_i^{(13)}, \quad (p_0^{(0,12)})^2 = p_0^{(0,12)}, \quad (p_0^{(12,0)})^2 = p_0^{(12,0)}, \quad (2.25) \]
and
\[ T_i p_i^{(13)} = -t^{-\frac{1}{2}}p_i^{(13)}, \quad T_i+1 p_i^{(13)} = -t^{-\frac{1}{2}}p_i^{(13)}, \quad (2.26) \]
\[ T_0 p_0^{(0,12)} = -t^{-\frac{1}{2}}p_0^{(0,12)}, \quad T_1 p_0^{(0,12)} = -t^{-\frac{1}{2}}p_0^{(0,12)}, \quad T_0 p_0^{(12,0)} = t^{-\frac{1}{2}}p_0^{(12,0)}, \quad T_1 p_0^{(12,0)} = -t^{-\frac{1}{2}}p_0^{(12,0)}. \]

The conditions in (2.26) are equivalent to
\[ a e_i p_i^{(13)} = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}})p_i^{(13)}, \quad a e_{i+1} p_i^{(13)} = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}})p_i^{(13)}, \quad (2.27) \]
\[ a_0 e_0 p_0^{(0,12)} = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}})p_0^{(0,12)}, \quad a e_0 p_0^{(0,12)} = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}})p_0^{(0,12)}, \quad a a_0 p_0^{(12,0)} = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}})p_0^{(12,0)}. \]

Proposition 2.3. Let \( p_i^{(13)} \), \( p_0^{(0,12)} \) and \( p_0^{(12,0)} \) be as defined in (2.25) and (2.26) and let
\[ N = t^{-\frac{1}{2}}(1 + t)(1 + t + t^2) \quad \text{and} \quad N_0 = N_0' = t_0^{-1}(1 + t_0)(1 + t + t_0t). \]

Then the expansions of these idempotents in terms of the three favored generating sets is given by
\[ N p_i^{(13)} = T_i T_{i+1} T_i - t^{-\frac{1}{2}} T_i T_{i+1} T_i - t^{-\frac{1}{2}} T_{i+1} T_i + t T_i + t T_{i+1} - t^{-\frac{1}{2}} \]
\[ = a^3 e_i e_{i+1} e_i - a e_i = a^3 e_{i+1} e_i e_{i+1} - a e_{i+1} \]
\[ = \tau_0 \tau_{i+1} \tau_i - t^{-\frac{1}{2}} \tau_{i+1} \tau_i \frac{f_{\varepsilon_{i+1}-\varepsilon_{i+1}+1}}{f_{\varepsilon_{i+1}-\varepsilon_{i+2}}} + t^{-\frac{1}{2}} \tau_{i+1} \tau_i \frac{f_{\varepsilon_{i+1}-\varepsilon_i+1}}{f_{\varepsilon_{i+1}-\varepsilon_i}} \]
\[ + t^{-1} \tau_i \frac{f_{\varepsilon_{i+1}-\varepsilon_{i+2}+1} f_{\varepsilon_{i+2}-\varepsilon_i+1}}{f_{\varepsilon_{i+1}-\varepsilon_{i+2}} f_{\varepsilon_{i+2}-\varepsilon_i+1}} + t^{-1} \tau_{i+1} \frac{f_{\varepsilon_{i+2}-\varepsilon_i+1} f_{\varepsilon_{i+1}-\varepsilon_i+1}}{f_{\varepsilon_{i+2}-\varepsilon_i} f_{\varepsilon_{i+1}-\varepsilon_i+1}} \]
\[ - t^{-\frac{3}{2}} \frac{f_{\varepsilon_{i+1}-\varepsilon_{i+2}+1} f_{\varepsilon_{i+2}-\varepsilon_i+1} f_{\varepsilon_{i+1}-\varepsilon_i+1}}{f_{\varepsilon_{i+1}-\varepsilon_{i+2}} f_{\varepsilon_{i+2}-\varepsilon_i} f_{\varepsilon_{i+1}-\varepsilon_i+1}}, \]
\[ N_{0p_0}^{(0,12)} = T_0T_1T_0T_1 - t_0^{-\frac{3}{4}}T_0T_1T_0 - t_0^{-\frac{1}{2}}T_0T_1T_0 + t_0^{-\frac{1}{4}}T_0T_1T_0 - t_0^{-\frac{1}{2}}T_0T_1T_0 - t_0^{\frac{1}{2}}T_0T_1T_0 - t_0^{\frac{3}{4}}T_0T_1T_0 + t_0^t + t_0^t + t_0^t \]

\[ = a_0^2a^2e_0e_1e_0e_1 - a_0a(t_0^{-\frac{1}{2}}t_0^{\frac{1}{2}} + t_0^{\frac{1}{2}}t_0^{-\frac{1}{2}})e_0e_0 = a_0^2a^2e_0e_1e_0e_1 - a_0a(t_0^{-\frac{1}{2}}t_0^{\frac{1}{2}} + t_0^{\frac{1}{2}}t_0^{-\frac{1}{2}})e_0e_0 \]

\[ = \tau_0\tau_1\tau_0\tau_1 - t_0^{\frac{1}{2}}\tau_1\tau_0\tau_1 f_{e_1-r_2}f_{e_1-r_1} - t_0^{-\frac{1}{4}}\tau_0\tau_1\tau_0 f_{e_2-e_1+1} \]

\[ + t_0^{\frac{1}{2}}t_0^{-\frac{1}{4}}\tau_0 T_{17} f_{e_2-e_1+1} f_{e_1-r_2}f_{e_1-r_1} - t_0^{\frac{1}{2}}t_0^{-\frac{1}{4}}\tau_1\tau_0 f_{e_2-e_1+1} f_{e_2-r_2}f_{e_2-r_1}f_{e_2-e_1+1} \\
- t_0^{\frac{1}{2}}t_0^{-\frac{1}{4}}\tau_1 f_{e_2-e_1+1} f_{e_2-r_2}f_{e_2-r_1}f_{e_2-e_1+1} + t_0^{\frac{1}{2}}t_0^{-\frac{1}{4}}\tau_1 T_{17} f_{e_2-e_1+1} f_{e_2-r_2}f_{e_2-r_1}f_{e_2-e_1+1} \]

\[ + t_0^{\frac{1}{2}}t_0^{-\frac{1}{4}}\tau_1 T_{17} f_{e_2-e_1+1} f_{e_2-r_2}f_{e_2-r_1}f_{e_2-e_1+1} \]

and

\[ N_{0p_0}^{(12,0)} = T_0T_1T_0T_1 + t_0^{\frac{1}{2}}T_0T_1T_0 - t_0^{-\frac{1}{4}}T_0T_1T_0 - t_0^{-\frac{1}{2}}T_0T_1T_0 - t_0^{-\frac{3}{4}}T_0T_1T_0 - t_0^{-\frac{1}{2}}T_0T_1T_0 + t_0^{\frac{1}{2}}T_0T_1T_0 + t_0^{\frac{3}{4}}T_0T_1T_0 + t_0^t \]

\[ = (a_0^2a^2e_0e_1e_0e_1 - a_0a(t_0^{-\frac{1}{2}}t_0^{\frac{1}{2}} + t_0^{\frac{1}{2}}t_0^{-\frac{1}{2}})e_0e_1) - (a_0^2a^2e_0e_1e_0e_1 - a(t_0^{-\frac{1}{2}}t_0^{\frac{1}{2}} + t_0^{\frac{1}{2}}t_0^{-\frac{1}{2}})e_0e_0) \]

\[ = \tau_0\tau_1\tau_0\tau_1 - t_0^{\frac{1}{2}}\tau_1\tau_0\tau_1 W_1^{\frac{1}{2}} f_{e_1-r_2}f_{e_1-r_1} - t_0^{-\frac{1}{4}}\tau_0\tau_1\tau_0 f_{e_2-e_1+1} \]

\[ + t_0^{\frac{1}{2}}t_0^{-\frac{1}{4}}\tau_0 T_{17} f_{e_2-e_1+1} f_{e_1-r_2}f_{e_1-r_1} - t_0^{\frac{1}{2}}t_0^{-\frac{1}{4}}\tau_1\tau_0 f_{e_2-e_1+1} f_{e_2-r_2}f_{e_2-r_1}f_{e_2-e_1+1} \]

\[ - t_0^{\frac{1}{2}}t_0^{-\frac{1}{4}}\tau_1 T_{17} W_1^{\frac{1}{2}} f_{e_2-e_1+1} f_{e_2-r_2}f_{e_2-r_1}f_{e_2-e_1+1} \]

\[ + t_0^{\frac{1}{2}}t_0^{-\frac{1}{4}}\tau_1 T_{17} f_{e_2-e_1+1} f_{e_2-r_2}f_{e_2-r_1}f_{e_2-e_1+1} \]

\[ + t_0^{\frac{1}{2}}t_0^{-\frac{1}{4}}\tau_1 T_{17} f_{e_2-e_1+1} f_{e_2-r_2}f_{e_2-r_1}f_{e_2-e_1+1} \]

**Proof.** The expressions in terms of \( T_i \) are proved by using the relations \( T_i^2 = (t_i^2 - t_i^{-2})T_i + 1 \) and \( T_0^2 = (t_0^2 - t_0^{-2})T_0 + 1 \) to show that the equations in (2.26) are satisfied. In view of the conditions (2.25), using the equations (2.26) to compute the product of the expansion in terms of the \( T_i \) with each element \( p_i^{(1)}, p_0^{(0,12)} \) and \( p_0^{(12,0)} \) respectively, determines the normalizing constants

\[ N = -t_i^{-\frac{1}{4}} - t_i^{-\frac{1}{2}} - t_i^{-\frac{3}{4}} - t_i^{-\frac{1}{2}} - t_i^\frac{1}{2} - t_i^2 = t_i^{-\frac{1}{4}}(1 + t)(1 + t + t^2), \] and

\[ N_0 = N'_0 = t_0^{-\frac{1}{2}}t^{-1} + t_0^{-1} + 1 + t_0 + t + t_0t = t_0^{-\frac{1}{2}}t^{-1}(1 + t_0)(1 + t)(1 + t_0). \]

Checking the conditions (2.27) verifies that the expressions in terms of the \( e_i \) for the elements \( Np_i^{(13)}, Np_0^{(0,12)}, \) and \( Np_0^{(12,0)} \) are correct. Similarly, using the expressions for \( aq_0e_0 \) and \( ae_i \) in terms of \( \tau_i \) given in (2.21) to check these same conditions verifies that the expressions for the elements \( Np_i^{(13)}, Np_0^{(0,12)} \) and \( Np_0^{(12,0)} \) in terms of the \( \tau_i \) are correct. \( \square \)
2.7. Setting up the relation \(a_k e_k e_{k-1} e_k e_{k-1} - a(t_k^{-\frac{1}{2}}t_k^{\frac{1}{2}} + t_k^{\frac{1}{2}}t_k^{-\frac{1}{2}})e_{k-1} = 0\). As in [DR] Remark 2.3, let \(w_A\) be the longest element of \(W A_k = \langle s_1, \ldots, s_{k-1}\rangle\). Let

\[
T_{0^\vee} = T_{w_A}^{-1} T_k T_{w_A} = a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}} = a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}} T_0^{-1},
\]

and note that \(T_{w_A}^{-1} T_{k-1} T_{w_A} = T_1\). Then

\[
(T_{0^\vee} - t_k^{\frac{1}{2}})(T_{0^\vee} + t_k^{\frac{1}{2}}) = 0 \quad \text{and} \quad T_{0^\vee} T_1 T_{0^\vee} T_1 = T_1 T_{0^\vee} T_1 T_{0^\vee}.
\]

Let \(HB_2^\vee\) be the subalgebra of \(H_k^{\text{ext}}\) generated by \(T_{0^\vee}\) and \(T_1\) and define idempotents \(p_{0^\vee}^{(0,1^2)}\) and \(p_{0^\vee}^{(1^2,0)}\) in \(HB_2^\vee\) by the equations

\[
(p_{0^\vee}^{(0,1^2)})^2 = p_{0^\vee}^{(0,1^2)}, \quad (p_{0^\vee}^{(1^2,0)})^2 = p_{0^\vee}^{(1^2,0)}; \quad (2.28)
\]

and

\[
T_{0^\vee} p_{0^\vee}^{(0,1^2)} = -t_k^{-\frac{1}{2}} p_{0^\vee}^{(0,1^2)}, \quad T_1 p_{0^\vee}^{(0,1^2)} = -t_k^{\frac{1}{2}} p_{0^\vee}^{(0,1^2)},
\]

\[
T_{0^\vee} p_{0^\vee}^{(1^2,0)} = t_k^{\frac{1}{2}} p_{0^\vee}^{(1^2,0)}, \quad \text{and} \quad T_1 p_{0^\vee}^{(1^2,0)} = -t_k^{\frac{1}{2}} p_{0^\vee}^{(1^2,0)}. \quad (2.29)
\]

Let \(a_k \in \mathbb{C}^\times\) and define

\[
a_k e_{0^\vee} = T_{0^\vee} - t_k^{\frac{1}{2}}, \quad \text{so that} \quad e_{0^\vee} = T_{w_A} e_k T^{-1}_{w_A} \quad \text{and} \quad e_1 = T_{w_A} e_{k-1} T^{-1}_{w_A}. \quad (2.30)
\]

The conditions in \((2.29)\) are equivalent to

\[
a_k e_{0^\vee} p_{0^\vee}^{(0,1^2)} = -t_k^{-\frac{1}{2}} t_k^{\frac{1}{2}} p_{0^\vee}^{(0,1^2)}, \quad a_k e_{0^\vee} p_{0^\vee}^{(1^2,0)} = 0, \quad \text{and} \quad ae_1 p_{0^\vee}^{(0,1^2)} = -(t_k^{\frac{1}{2}} + t_k^{-\frac{1}{2}}) p_{0^\vee}^{(0,1^2)}, \quad (2.31)
\]

Using \(a_k e_{0^\vee} = W_1 T_0^{-1} - t_k^{\frac{1}{2}} = W_1 (T_0 - (t_0^{\frac{1}{2}} + t_0^{-\frac{1}{2}})) - t_k^{\frac{1}{2}} = W_1 (\tau_0 + t_0^{\frac{1}{2}} - c_{\alpha_0} - (t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})) - t_k^{\frac{1}{2}}\), a short computation gives

\[
a_k e_{0^\vee} = \tau_0 W_1^{-1} - t_0^{\frac{1}{2}} W_1^{-1} f_{\varepsilon_1 - r_2} f_{-\varepsilon_1 - r_3} f_{2\varepsilon_1}. \]
And, with $N_k = t_k^{-1}t^{-1}(1 + t_k)(1 + t)(1 + t_k t)$, we have

$$N_k p_0^{2,8} = a_k^2\epsilon_0^\vee\epsilon_1\epsilon_0^\vee\epsilon_1 - a_k a(t_k^{-2} + t_k^{2}t^{-2})\epsilon_0^\vee\epsilon_1 = a_k^2\epsilon_0^\vee\epsilon_1\epsilon_0^\vee\epsilon_1 - a_k a(t_k^{-2} + t_k^{2}t^{-2})\epsilon_1\epsilon_0^\vee
$$

$$= \tau_0 \tau_1 \tau_1 (W_1 W_2)^{-1} - t_0^{-\frac{1}{2}} \tau_0 \tau_1 \tau_1 (W_1 W_2)^{-1} \frac{f_{\epsilon_1 - r_2 f_{\epsilon_1 - r_1}}}{f_{\epsilon_2 - \epsilon_1}} + t_0^{-\frac{1}{2}} \tau_0 \tau_1 \tau_0 (W_1 W_2)^{-1} \frac{f_{\epsilon_2 - \epsilon_1 + 1}}{f_{\epsilon_2 - \epsilon_1}}$$

$$- t_0^{-\frac{1}{2}} \tau_0 \tau_0 (W_1 W_2)^{-1} \frac{f_{\epsilon_2 - \epsilon_1 + 1}}{f_{\epsilon_2 - \epsilon_1}} + t_0^{-1} \tau_0 (W_1 W_2)^{-1} \frac{f_{\epsilon_2 - \epsilon_1 + 1}}{f_{\epsilon_2 - \epsilon_1}} + t_0^{-1} \tau_0 (W_1 W_2)^{-1} \frac{f_{\epsilon_2 - \epsilon_1 + 1}}{f_{\epsilon_2 - \epsilon_1}},$$

and

$$N_k p_0^{1,8} = (a_k^2\epsilon_0^\vee\epsilon_1\epsilon_0^\vee\epsilon_1 - a_k a(t_k^{-2} + t_k^{2}t^{-2})\epsilon_0^\vee\epsilon_1) - (a_k^2\epsilon_0^\vee\epsilon_1\epsilon_0^\vee\epsilon_1 - a(t_k^{-2} + t_k^{2}t^{-2})\epsilon_1\epsilon_0^\vee)
$$

$$= \tau_0 \tau_1 \tau_1 (W_1 W_2)^{-1} - t_0^{-\frac{1}{2}} \tau_0 \tau_1 \tau_1 (W_1 W_2)^{-1} \frac{f_{\epsilon_1 - r_2 f_{\epsilon_1 - r_1}}}{f_{\epsilon_2 - \epsilon_1}} + t_0^{-\frac{1}{2}} \tau_0 \tau_1 \tau_0 (W_1 W_2)^{-1} \frac{f_{\epsilon_2 - \epsilon_1 + 1}}{f_{\epsilon_2 - \epsilon_1}}$$

$$- t_0^{-\frac{1}{2}} \tau_0 \tau_0 (W_1 W_2)^{-1} \frac{f_{\epsilon_2 - \epsilon_1 + 1}}{f_{\epsilon_2 - \epsilon_1}} + t_0^{-1} \tau_0 (W_1 W_2)^{-1} \frac{f_{\epsilon_2 - \epsilon_1 + 1}}{f_{\epsilon_2 - \epsilon_1}} + t_0^{-1} \tau_0 (W_1 W_2)^{-1} \frac{f_{\epsilon_2 - \epsilon_1 + 1}}{f_{\epsilon_2 - \epsilon_1}},$$

in analogy with (and with the same proof as) Proposition 2.3.

3. The two boundary Temperley-Lieb algebra $TL_k$

In this section we define the two boundary Temperley-Lieb algebra $TL_k$ (also called the symplectic blob algebra, see [GMP07, GMP08, GMP12, Reh, KMP16, GMP17]) and review its diagrammatic calculus. We extend the diagrammatic calculus to make clear the relationship to the two boundary Hecke algebra and to set the stage for the proof of Theorem 3.2. Although Theorem 3.2 takes the form of a computation, it is a computation that has amazing consequences as it determines the relationship between the center of $H^\text{ext}_k$ and the center of $TL_k$. The center of $H^\text{ext}_k$ is a ring of symmetric functions (see [DR, Theorem 2.3]) and the center of $TL_k$ turns out to be a polynomial ring $\mathbb{C}[Z]$ in a single variable $Z$. We shall see that, in the same way that $H^\text{ext}_k$ is finite rank over its center, the algebra $TL_k$ is finite rank over $\mathbb{C}[Z]$. However, whereas the former has the easily
classified rank of \((2^k k!)^2\) over its center, the rank of \(\text{TL}_k\) is as yet unclassified combinatorially. For example, \(\dim(\text{TL}_k(b)) = 5, 19, 84, 335, \) and 1428, for \(k = 1, 2, 3, 4,\) and 5, respectively.

3.1. The extended two boundary Temperley-Lieb algebra \(\text{TL}_k^{\text{ext}}\). Let \(H_k^{\text{ext}}\) be the extended two boundary Hecke algebra as defined in (2.15). The extended two boundary Temperley-Lieb algebra \(\text{TL}_k^{\text{ext}}\) is the quotient of \(H_k^{\text{ext}}\) by the relations

\[
p_{0}^{(0,12)} = p_{0}^{(12,0)}, \quad p_{0}^{(0,12)} = p_{0}^{(12,0)} \quad \text{and} \quad p_{i}^{(13)} = 0 \quad \text{for} \quad i \in \{1, \ldots, k - 2\}.
\]

**Theorem 3.1.** The algebra \(\text{TL}_k^{\text{ext}}\) is the quotient of \(H_k^{\text{ext}}\) by the relations

\[
a_{k}a_{k}^{-2}e_{k-1}e_{k-1} - a(t_{k}^{-\frac{1}{2}}t_{k}^{\frac{1}{2}} + t_{k}^{\frac{1}{2}}t_{k}^{-\frac{1}{2}})e_{k-1} = 0, \quad a_{0}a_{2}^{2}e_{0}e_{0}e_{1} - a(t_{0}^{-\frac{1}{2}}t_{0}^{\frac{1}{2}} + t_{0}^{\frac{1}{2}}t_{0}^{-\frac{1}{2}})e_{1} = 0,
\]

and

\[
a_{i}^{3}e_{i+1}e_{i} - a_{i} = a_{i}^{3}e_{i+1}e_{i} - a_{i+1} = 0 \quad \text{for} \quad i \in \{1, \ldots, k - 2\}.
\]

**Proof.** Let \(F_{i} = a_{i}^{3}e_{i+1}e_{i} - a_{i} = a_{i}^{3}e_{i+1}e_{i} - a_{i+1} \) for \(i \in \{1, \ldots, k - 2\},\)

\[
F_{k} = a_{k}a_{k}^{-2}e_{k-1}e_{k-1} - a(t_{k}^{-\frac{1}{2}}t_{k}^{\frac{1}{2}} + t_{k}^{\frac{1}{2}}t_{k}^{-\frac{1}{2}})e_{k-1}, \quad \text{and} \quad F_{0} = a_{0}a_{2}^{2}e_{0}e_{0}e_{1} - a(t_{0}^{-\frac{1}{2}}t_{0}^{\frac{1}{2}} + t_{0}^{\frac{1}{2}}t_{0}^{-\frac{1}{2}})e_{1}.
\]

By Proposition (2.3),

\[
N_{0}p_{0}^{(12,0)} = e_{0}F_{0}, \quad N_{0}p_{0}^{(0,12)} = (e_{0} - 1)F_{0}, \quad F_{0} = N_{0}(p_{0}^{(12,0)} - p_{0}^{(0,12)}), \quad \text{and} \quad N_{0}p_{i}^{(13)} = F_{i};
\]

and, by (2.30),

\[
T_{w_{A}}F_{k}T_{w_{A}}^{-1} = N_{0}^{(0,12)}(0,12), \quad T_{w_{A}}^{-1}(0,12)T_{w_{A}} = e_{k}F_{k}, \quad \text{and} \quad T_{w_{A}}^{-1}(0,12)T_{w_{A}} = (e_{k} - 1)F_{k}.
\]

Thus, provided \(N, N_{0}\) and \(N_{k}\) are invertible, the ideal \(H_k^{\text{ext}}F_{k}H_k^{\text{ext}}\) is the same as the ideal generated by \(p_{0}^{(12,0)}\) \(p_{0}^{(0,12)}\); the ideal \(H_k^{\text{ext}}F_{0}H_k^{\text{ext}}\) is the same as the ideal generated by \(p_{0}^{(12,0)}\) \(p_{0}^{(0,12)}\); and \(H_k^{\text{ext}}p_i^{(13)}H_k^{\text{ext}} = H_k^{\text{ext}}F_iH_k^{\text{ext}}\).

3.2. The two boundary Temperley-Lieb algebra \(\text{TL}_k\). The two boundary Temperley-Lieb algebra \(\text{TL}_k\) is the subalgebra of \(\text{TL}_k^{\text{ext}}\) generated by \(a_{0}e_{0}, a_{1}e_{1}, \ldots, a_{k-1}e_{k-1}, a_{k}e_{k}\) as defined in (2.10). As in (2.12) and (2.20), where \(B_k^{\text{ext}} = B_k \times D\) and \(H_k^{\text{ext}} = H_k \otimes \mathbb{C}[W_{0}^{\pm 1}]\), the extended two boundary Temperley-Lieb algebra is

\[
\text{TL}_k^{\text{ext}} = \text{TL}_k \otimes \mathbb{C}[W_{0}^{\pm 1}], \quad \text{as algebras, where} \quad W_0 = PW_1 \cdots W_k.
\]

3.3. Diagrammatic calculus for \(\text{TL}_k\). Pictorially, identify

\[
T_k = \begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}, \quad T_0 = \begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}, \quad T_i = \begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}, \quad e_0 = \begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}, \quad e_k = \begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}, \quad \text{and} \quad a_{e_i} = \begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array},
\]

for \(i \in \{1, \ldots, k - 1\}\). Recall the notation

\[
[x] = x^\frac{1}{2} + x^{-\frac{1}{2}}
\]
from (2.19). With $i \in \{1, \ldots, k - 1\}$, the relations (2.16), (2.17) and (2.18) are

\[
\begin{align*}
T_0 &= a_0 e_0 + t_0^1, & T_i &= ae_i + t_i^2, & T_k &= a_k e_k + t_k^1, \\
\overline{a_0} &= a_0 \overline{t_0} + t_0^{1/2} & \overline{a_i} &= \overline{a} + t^{1/2} & \overline{a_k} &= a_k \overline{a} + t_k^{1/2}
\end{align*}
\]

\[
\begin{align*}
T_0 e_0 &= e_0 T_0 = -t_0^{-1/2} e_0, & T_i (ae_i) &= (ae_i) T_i = -t^{-1/2} (ae_i), & T_k e_k &= e_k T_k = -t_k^{-1/2} e_k,
\end{align*}
\]

\[
\begin{align*}
T_0^{-1} e_0 &= e_0 T_0^{-1} = -t_0^{1/2} e_0, & T_i^{-1} (ae_i) &= (ae_i) T_i^{-1} = -t^{1/2} (ae_i), & T_k^{-1} e_k &= e_k T_k^{-1} = -t_k^{1/2} e_k,
\end{align*}
\]

\[
\begin{align*}
e_0^2 &= -\frac{[t_0]}{a_0} e_0, & (ae_i)^2 &= -[t] (ae_i), & e_k^2 &= -\frac{[t_k]}{a_k} e_k,
\end{align*}
\]

In the quotient by $(ae_i)(ae_{i+1})(ae_i) = ae_i$, we have

\[
\begin{align*}
ac_i T_{i+1} T_i &= a T_{i+1} T_i e_{i+1} = t^{1/2} a^2 e_i e_{i+1}, & ac_i T_{i+1}^{-1} T_i &= a T_{i+1}^{-1} T_i^{-1} e_{i+1} = t^{-1/2} a^2 e_i e_{i+1}, \\
ae_{i+1} T_i T_{i+1} &= a T_i T_{i+1} e_i = t^{1/2} a^2 e_{i+1} e_i, & ae_{i+1} T_i^{-1} T_{i+1} &= a T_i^{-1} T_{i+1}^{-1} e_i = t^{-1/2} a^2 e_{i+1} e_i,
\end{align*}
\]

which are proved by using $T_i^{\pm 1} = ae_i + t^{\pm 1/2}$ to expand both sides in terms of $e_i$.

When $a_0 (ae_1) e_0 (ae_1) - \left[ t_0 t^{-1} \right] (ae_1) = 0$ and $a_k (ae_{k-1}) e_k (ae_{k-1}) - \left[ t_k t^{-1} \right] (ae_{k-1}) = 0$, then

\[
\begin{align*}
(\overline{a_1}) T_0 T_1 &= t^{1/2} (ae_1) T_0^{-1}, & T_1 T_0 (ae_1) &= t^{1/2} T_0^{-1} (ae_1), \\
(\overline{a_{k-1}}) T_k T_{k-1} &= t^{1/2} (ae_{k-1}) T_k^{-1}, & T_{k-1} T_k (ae_{k-1}) &= t^{1/2} T_k^{-1} (ae_{k-1}),
\end{align*}
\]

\[
(\overline{a_1}) T_1 T_0 T_1 T_2 = t^{3/2} (ae_2) T_1^{-1} T_0^{-1} T_1^{-1},
\]

\[
(\overline{a_1}) T_1 T_0 T_1 T_2 = t^{3/2} (ae_2) T_1^{-1} T_0^{-1} T_1^{-1},
\]
\[(ae_1)T_0(ae_1) = -t^{1/2}(t_0^{1/2} - t_0^{-1/2})(ae_1), \quad (ae_{k-1})T_k(ae_{k-1}) = -t^{1/2}(t_k^{1/2} - t_k^{-1/2})(ae_{k-1}), \quad (3.4)\]

and

\[e_0T_1^{-1}T_0^{-1}e_0 = -t^{-1/2}[t_0]e_0(ae_1)e_0 - t^{-1/2}t_0^{1/2}e_0^2.\]

### 3.4. TLk as a diagram algebra.

Using the pictorial notation, the algebra TLk has a basis (see [GMPT12, Theorem 3.4]) of non-crossing diagrams with k dots in the top row, k dots in the bottom row, edges connecting pairs of dots, an even number of left boundary to right boundary edges, and

\[(-1)^\#\{\text{left boundary edges}\} = 1 \quad \text{and} \quad (-1)^\#\{\text{right boundary edges}\} = 1.\]

For example,

\[d_1 = \quad \text{and} \quad d_2 = \]

are both basis elements of TLk. Multiplication of basis elements can be computed pictorially by vertical concatenation, with self-connected loops and strands with both ends on the left or on the right replaced by constant coefficients according to the following local rules:

\[\bigcirc = -\lfloor t \rceil, \quad \text{if even \# connections} \quad = \frac{[t_0t^{-1}]}{a_0}, \quad \text{if even \# connections} \quad = \frac{[t_kt^{-1}]}{a_k},\]

\[\bigcirc = -\lfloor t_0 \rceil \quad \text{if odd \# connections} \quad = \frac{[t_k]}{a_k}.\]

For example with \(d_1\) and \(d_2\) as above,

\[d_1d_2 = \quad = \quad = (-[t])(\frac{[t_k]}{a_k})(\frac{[t_kt^{-1}]}{a_k})\]

(where the dashed strand is removed with a coefficient of \(\frac{[t_0]}{a_k}\), and the thick strand is removed with a coefficient of \(\frac{[t_k]}{a_k}\)).
3.5. The through-strand filtration of $TL_k$. A through-strand is an edge that connects a top vertex to a bottom vertex. Define the ideals

$$TL_k^{(\leq j)} = \mathbb{C}\text{-span}\{\text{diagrams with } \leq j \text{ through-strands}\}.$$  

Then the algebra $TL_k$ is filtered by ideals as

$$TL_k = TL_k^{(\leq k)} \supseteq TL_k^{(\leq k-1)} \supseteq \cdots \supseteq TL_k^{(\leq 1)} \supseteq TL_k^{(\leq 0)} \supseteq 0.$$ (3.5)

If

$$TL_k^{(j)} = \frac{TL_k^{(\leq j)}}{TL_k^{(\leq j-1)}},$$

then $\dim(TL_k^{(j)}) < \infty$, for $j \geq 1$, and $\dim(TL_k^{(\leq 0)}) = \infty$, as there can be an arbitrarily large number of edges which connect the left and right sides in diagrams with no through strands:

![Diagram](image)

3.6. The elements $I_1$ and $I_2$. As in [GN, §3.2], define

$$I_1 = \begin{cases} (ae_1)(ae_3)\cdots(ae_{k-1}), & \text{if } k \text{ is even}, \\ (ae_1)(ae_3)\cdots(ae_{k-2})e_k, & \text{if } k \text{ is odd}, \end{cases} = \begin{cases} \ldots & \text{if } k \text{ is even}, \\ \ldots & \text{if } k \text{ is odd}, \end{cases}$$ (3.6)

and

$$I_2 = \begin{cases} e_0(ae_2)\cdots(ae_{k-2})e_k, & \text{if } k \text{ is even}, \\ e_0(ae_2)\cdots(ae_{k-1}), & \text{if } k \text{ is odd}. \end{cases} = \begin{cases} \ldots & \text{if } k \text{ is even}, \\ \ldots & \text{if } k \text{ is odd}. \end{cases}$$ (3.7)

Up to a constant multiple the elements $I_1$ and $I_2$ are idempotents and

$$I_1I_2I_1 = \begin{cases} \ldots & \text{if } k \text{ is even}, \\ \ldots & \text{if } k \text{ is odd}, \end{cases} \quad I_2I_1I_2 = \begin{cases} \ldots & \text{if } k \text{ is even}, \\ \ldots & \text{if } k \text{ is odd}. \end{cases}$$

Proposition [3.2] gives another striking formula for the elements $I_1I_2I_1$ and $I_2I_1I_2$.

3.7. The element $ZI_1$ in $TL_k$. Conceptually, the diagram

$$F = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

would be a central element of $H_k$

if it represented a true element of the algebra $H_k$. Though the diagram $F$ does not naturally represent an element of $H_k$, the diagrams

$$D^{\text{even}} = I_1(T^{-1}_0(ae_2)(ae_4)\cdots(ae_{k-2})T_k)I_1 = \begin{cases} \ldots, & \text{and} \end{cases}$$

$$D^{\text{odd}} = I_2(T^{-1}_1T_0^{-1}T_1^{-1}(ae_3)(ae_5)\cdots(ae_{k-2})T_k)I_2 = \begin{cases} \ldots, \end{cases}$$
do appear in the algebra $TL_k$ and play an important role in the proof of the following theorem. See also [GN Thm. 4.1], using Remark 3.4 below as a guide.

**Theorem 3.2.** Let $Z = W_1 + W_1^{-1} + \cdots + W_k + W_k^{-1}$ which, as noted in (2.21) is a central element of $H_k$. As elements of $TL_k$,

if $k$ is even, then $D^\text{even} = a_0 a_k I_2 I_2 I_1 + [t_0 t_k^{-1}] I_1$, and $Z I_1 = [k] D^\text{even}$, and

if $k$ is odd, then $D^\text{odd} = t^{-\frac{1}{2}} \left( \frac{[t_0]}{a_0} \right) (a_0 a_k I_2 I_2 - [t_0 t_k^{-1}] I_2)$ and $t^{-\frac{1}{2}} \left( -\frac{[t_0]}{a_0} \right) Z I_2 = [k] D^\text{odd}$.

**Proof.** Case: $k$ even. Let

$$L^\text{even} = I_1 \left( (ae_2)(ae_4) \cdots (ae_{k-2}) e_k \right) I_1 = \begin{array}{cccc} \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \end{array} = \left( \frac{[t_k t^{-1}]}{a_k} \right) I_1,$$

$$M^\text{even} = I_1 \left( e_0 (ae_2)(ae_4) \cdots (ae_{k-2}) \right) I_1 = \begin{array}{cccc} \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \end{array} = \left( \frac{[\overline{t_0} t^{-1}]}{a_0} \right) I_1,$$

$$P^\text{even} = I_1 \left( (ae_2)(ae_4) \cdots (ae_{k-2}) \right) I_1 = \begin{array}{cccc} \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \end{array} = -[t] I_1.$$

Using $T_0^{-1} = a_0 e_0 + t_0^{\frac{k}{2}}$ for the left pole and $T_k = a_k e_k + t_k^{\frac{k}{2}}$ for the right pole,

$$D^\text{even} = a_0 a_k I_2 I_2 I_1 + a_0 t_k^{\frac{k}{2}} M^\text{even} + a_k t_0^{-\frac{k}{2}} L^\text{even} + t_0^{-\frac{k}{2}} t_k^{\frac{k}{2}} P^\text{even}$$

$$= a_0 a_k I_2 I_2 I_1 + (t_k^{\frac{k}{2}} [t_0 t^{-1}] + t_0^{-\frac{k}{2}} [t_k t^{-1}] - t_0^{-\frac{k}{2}} t_k^{\frac{k}{2}} [t]) I_1$$

$$= a_0 a_k I_2 I_2 I_1 + [t_0 t_k^{-1}] I_1,$$

which completes the proof of the first statement.

Using $(ae_1) T_1^{-1} = (-t^{\frac{1}{2}})(ae_1)$ and $(ae_1) T_0 T_0 (ae_1) = t^{\frac{1}{2}} (ae_1) T_0^{-1} (ae_1)$ gives

$$R^\text{even} = I_1 \left( T_1^{-1}(ae_2)(ae_4) \cdots (ae_{k-2}) T_k T_0 \right) I_2 = \begin{array}{cccc} \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \end{array} = (-t^{\frac{1}{2}}) t^{\frac{k}{2}} D^\text{even},$$

and using $T_{k-1} (ae_{k-1}) = (-t^{\frac{k}{2}})(ae_{k-1})$ and $T_{k-1} T_k^{-1} (ae_{k-1}) = t^{\frac{1}{2}} T_k (ae_{k-1})$ gives

$$S^\text{even} = I_1 \left( T_0 (ae_2)(ae_4) \cdots (ae_{k-2}) T_{k-1}^{-1} T_{k-1} \right) I_1 = \begin{array}{cccc} \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \end{array} = (-t^{-\frac{1}{2}}) t^{\frac{k}{2}} D^\text{even}.$$
Working left to right removing loops,
\begin{align*}
I_1 W_{1+2i} I_1 &= (t^{\frac{1}{2}} t^{\frac{1}{2}} (-[t]) )^{i} (t^{-\frac{1}{2}} t^{\frac{1}{2}} (-[t]) )^{\frac{1}{2} - 1 - i} R_{\text{even}} = (-[t])^{\frac{1}{2} - 1} t^{\frac{1}{2} - i + \frac{1}{2}} (-t^\frac{1}{2}) D_{\text{even}}, \\
I_1 W_{1+2i}^{-1} I_1 &= (t^{-\frac{1}{2}} t^{-\frac{1}{2}} (-[t]) )^{i} (t^{-\frac{1}{2}} t^{\frac{1}{2}} (-[t]) )^{\frac{1}{2} - 1 - i} S = (-[t])^{\frac{1}{2} - 1} t^{-(i + \frac{1}{2})} (-t^{-\frac{1}{2}}) D_{\text{even}},
\end{align*}
for \( i \in \{0, \ldots, \frac{k}{2} - 1 \} \). Since \( I_1 W_{1+2i} I_1 \) and \( I_1 W_{1+2i}^{-1} I_1 \) only differ by two twists (similarly \( I_1 W_{1+2i} I_1 \) and \( I_1 W_{1+2i}^{-1} I_1 \) only differ by two twists) the relations \( T_{i \pm 1} (ae_i) = (ae_i) T_{i \pm 1} = (-t^{\pm \frac{1}{2}}) (ae_i) \) give
\begin{align*}
I_1 W_{2+2i} I_1 &= (-t^{\frac{1}{2}})((-t^{\frac{1}{2}}) t^{-1} I_1 W_{1+2i} I_1 = (-[t])^{\frac{1}{2} - 1} t^{i + \frac{1}{2}} (-t^{-\frac{1}{2}}) D_{\text{even}} \quad \text{and} \\
I_1 W_{2+2i}^{-1} I_1 &= (-t^{\frac{1}{2}})(-t^{\frac{1}{2}}) I_1 W_{1+2i}^{-1} I_1 = (-[t])^{\frac{1}{2} - 1} t^{-(i + \frac{1}{2})} (-t^{\frac{1}{2}}) D_{\text{even}}, \quad \text{for } i \in \{0, \ldots, \frac{k}{2} - 1 \}.
\end{align*}
Thus
\begin{align*}
(-[t])^{\frac{k}{2}} Z I_1 &= Z I_1^2 = I_1 Z I_1 = \sum_{i=0}^{\frac{k}{2} - 1} I_1(W_{1+2i} + W_{2+2i} + W_{1+2i}^{-1} + W_{2+2i}^{-1}) I_1 \\
&= -(-[t])^{\frac{k}{2} - 1} D_{\text{even}} \sum_{i=0}^{\frac{k}{2} - 1} \left( (t^{i + \frac{1}{2}} (t^{\frac{1}{2}} + t^{-\frac{1}{2}}) + t^{-(i + \frac{1}{2})} (t^{\frac{1}{2}} + t^{-\frac{1}{2}}) \right) \\
&= (-[t])^{\frac{k}{2}} D_{\text{even}} \left( \frac{t^{\frac{k}{2}} - t^{-\frac{k}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} \right) = (-[t])^{\frac{k}{2}} [k] D_{\text{even}}.
\end{align*}

Case: \( k \) odd. Let
\begin{align*}
L_{\text{odd}} &= I_2 ((ae_3)(ae_5) \cdots (ae_{k-2}) e_k) I_2 = \frac{\cdots \cdots \cdots}{\cdots \cdots \cdots} = (-[t_0]) \left( \frac{a_k t^{-1}}{a_0} \right) I_2, \\
M_{\text{odd}} &= I_2 ((ae_1)(ae_3) \cdots (ae_{k-2})) I_2 = \frac{\cdots \cdots \cdots}{\cdots \cdots \cdots} = (-[t_0]) \left( \frac{a_k}{a_0} \right) I_2, \quad \text{and} \\
P_{\text{odd}} &= I_2 ((ae_3)(ae_5) \cdots (ae_{k-2})) I_2 = \frac{\cdots \cdots \cdots}{\cdots \cdots \cdots} = (-[t_0]) \left( \frac{a_k}{a_0} \right) (-[t]) I_2.
\end{align*}
Using \( e_0 T_{1^{-1}} T_{0^{-1}} T_{1^{-1}} e_0 = -t^{-\frac{1}{2}} [t_0] e_0 (ae_1) e_0 - t^{-1} t_0^\frac{1}{2} e_0^2 \) and \( T_k = a_k e_k + t_k^\frac{1}{2} \) gives
\begin{align*}
D_{\text{odd}} &= -t^{-\frac{1}{2}} [t_0] a_k I_2 I_1 I_2 - t^{-1} t_0^\frac{1}{2} a_k L_{\text{odd}} - t^{-\frac{1}{2}} [t_0] t_k^\frac{1}{2} M_{\text{odd}} - t^{-1} t_0^\frac{1}{2} t_k^\frac{1}{2} P_{\text{odd}} \\
&= t^{-\frac{1}{2}} \left( \frac{[t_0]}{a_0} \right) \left( a_0 a_k I_2 I_1 I_2 + (-t^{-\frac{1}{2}} t_0^\frac{1}{2} [t_k t^{-1}] - t_k^\frac{1}{2} [t_0] + t^{-\frac{1}{2}} t_0^\frac{1}{2} [t]) I_2 \right) \\
&= t^{-\frac{1}{2}} \left( \frac{[t_0]}{a_0} \right) \left( a_0 a_k I_2 I_1 I_2 - [t_0 t_k^{-1}] I_2 \right),
\end{align*}
which completes the proof of the first statement.
Using \( (ae_2) T_{2^{-1}} = -t^{\frac{1}{2}} (ae_2) \) and \( T_2 T_1 T_0 (ae_2) = t^{\frac{3}{2}} T_{1^{-1}} T_{0^{-1}} T_{1^{-1}} (ae_2) \),
\begin{align*}
R_{\text{odd}} &= I_2 (T_{2^{-1}} (ae_3)(ae_5) \cdots (ae_{k-2}) T_k T_2 T_1 T_0) I_2 = \frac{\cdots \cdots \cdots}{\cdots \cdots \cdots} = (-t^{\frac{3}{2}}) t^{\frac{3}{2}} D_{\text{odd}}.
\end{align*}
Using $T_{k-1}(ae_{k-1}) = -t^{- \frac{1}{2}}(ae_{k-1})$ and $T_{k-1}^{-1}T_k^{-1}(ae_{k-1}) = t^{- \frac{1}{2}}T_k(ae_{k-1})$ gives

$$S^{\text{odd}} = I_2(T_1^{-1}T_0^{-1}(ae_3)(ae_5)\cdots(ae_{k-2})T_{k-1}^{-1}T_{k-1}^{-1}I_2 = \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right) = (-t^{- \frac{1}{2}})t^{- \frac{1}{2}}D^{\text{odd}}.$$

Pictorially,

$$I_2W_{1+2}I_2 = \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right), \quad I_2W_{2+2}I_2 = \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right),$$

$$I_2W_{1+2}I_2 = \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right), \quad \text{and} \quad I_2W_{2+2}I_2 = \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right).$$

Working left to right removing loops,

$$I_2W_{2+2}I_2 = (t^{\frac{1}{2}}t^{\frac{1}{2}}(-[t]))^i(t^{-\frac{1}{2}}t^{\frac{1}{2}}(-[t]))^{\frac{k-1}{2}+1-i}R^{\text{odd}} = (-[t])^{\frac{k-1}{2}+i}(-t)^iD^{\text{odd}},$$

$$I_2W_{3+2}I_2 = (t^{\frac{1}{2}}t^{\frac{1}{2}}(-[t]))^{i}t^{-\frac{1}{2}}t^{\frac{1}{2}}(-[t])^{\frac{k-1}{2}+1-i}S^{\text{odd}} = (-[t])^{\frac{k-1}{2}+i}(-t)^{-i-1}D^{\text{odd}},$$

for $i \in \{0, \ldots, \frac{k-1}{2} - 1\}$. Since $I_2W_{2+2}I_2$ and $I_2W_{3+2}I_2$ only differ by two twists (similarly $I_2W_{1+2}I_2$ and $I_2W_{2+2}I_2$ only differ by two twists) the relations $T_{i}^{ \pm 1}e_i = e_iT_{i}^{ \mp 1} = (t^{\mp \frac{1}{2}})e_i$ give

$$I_2W_{3+2}I_2 = (-t^{\frac{1}{2}})(-t^{\frac{1}{2}})I_2W_{2+2}I_2 = (-[t])^{\frac{k-3}{2}}t^{i+1}(-1)D^{\text{odd}},$$

$$I_2W_{3+2}I_2 = (-t^{\frac{1}{2}})(-t^{\frac{1}{2}})I_2W_{2+2}I_2 = (-[t])^{\frac{k-3}{2}}(-)t^{-i-1}(-1)D^{\text{odd}},$$

for $i \in \{0, \ldots, \frac{k-1}{2} - 1\}$.

Next,

$$I_2W_{1}I_2 = \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right) = (-t_0^{\frac{1}{2}})(-[t])^{\frac{k-1}{2}+1}I_2((ae_1)(ae_3)\cdots(ae_{k-2})T_k)I_2, \quad \text{and}$$

$$I_2W_{1}^{-1}I_2 = \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right) = (-t_0^{\frac{1}{2}})(-[t])^{\frac{k-1}{2}+1}I_2((ae_1)(ae_3)\cdots(ae_{k-2})T_k^{-1})I_2.$$

Using $-t_{0}^{-\frac{1}{2}}T_{k} - t_{0}^{-\frac{1}{2}}T_{k}^{-1} = -t_{0}^{-\frac{1}{2}}(ak_{e_{k}} + t_{k}^{\frac{1}{2}}) - t_{0}^{-\frac{1}{2}}(ak_{e_{k}} + t_{k}^{-\frac{1}{2}}) = -[t_0]ak_{e_k} - [t_0t_k^{-1}]$,.

$$I_2(W_{1} + W_{1}^{-1})I_2 = (-[t])^{\frac{k-1}{2}}(\frac{-[t_0]}{a_0}[a_0I_2I_2 - [t_0]t_k^{-1}]M^{\text{odd}})$$

$$= (-[t])^{\frac{k-1}{2}}(-[t_0]a_0I_2I_2 - [t_0]t_k^{-1})(\frac{-[t_0]}{a_0})I_2)$$

$$= (-[t])^{\frac{k-1}{2}}(\frac{-[t_0]}{a_0})(a_0a_0I_2I_2 - [t_0]t_k^{-1})I_2 = -(t+1)(-[t])^{\frac{k-1}{2}}D^{\text{odd}}.$$
Thus
\[
\left(-\frac{[t_0]}{a_0}\right)(- [t]) \frac{k+1}{2} Z I_2 = Z I_2^2 = I_2 Z I_2
\]
\[
= I_2(W_1 + W_1^{-1})I_2 + \sum_{i=0}^{k+1} I_2(W_{2+2i} + W_{3+2i} + W_{2+2i}^{-1} + W_{3+2i}^{-1})I_2
\]
\[
= -(t+1(-[t]) \frac{k+3}{2} t^{\frac{k+1}{2}} D^{\text{odd}} + (-[t]) \frac{k+3}{2} \sum_{i=0}^{k+1} (t^{i+1} - t^{-(i+1)})(-t-1) D^{\text{odd}})
\]
\[
= -(t) \frac{k+3}{2} (t+1) D^{\text{odd}} \left(1 + \sum_{i=0}^{k+1} (t^{i+1} - t^{-(i+1)})\right) = (-[t]) \frac{k+1}{2} t^{\frac{k+1}{2}} D^{\text{odd}} [k].
\]

Corollary 3.3. Let \( Z = W_1 + W_1^{-1} + \cdots + W_k + W_k^{-1} \) and let \( I_1 \) and \( I_2 \) be as defined in (3.6) and (3.7). If \( k \) is even, then
\[
a_0 a_k I_1 I_2 I_1 = \left(\frac{1}{k} Z - [t_0 t_k t^{-1}]\right) I_1 \quad \text{and} \quad a_0 a_k I_2 I_1 I_2 = \left(\frac{1}{k} Z - [t_0 t_k t^{-1}]\right) I_2.
\]
If \( k \) is odd, then
\[
a_0 a_k I_1 I_2 I_1 = \left(\frac{1}{k} Z + [t_0 t_k^{-1}]\right) I_1 \quad \text{and} \quad a_0 a_k I_2 I_1 I_2 = \left(\frac{1}{k} Z + [t_0 t_k^{-1}]\right) I_2.
\]

Proof. As observed in the proof of Theorem 3.2, the products \( I_1 Z I_1 \) and \( I_2 Z_2 \) reduce to computation of the diagram with a single string going around all the poles \( (D^{\text{even}} \ or \ D^{\text{odd}}) \). These diagrammatics

Then, computing \((I_1 I_2 I_1)^2\) in two different ways, we have
\[
I_1 I_2 I_1 I_2 I_1 = C I_1 , \quad I_1 I_2 I_1 = (C_1 Z + C_2) I_1 , \quad I_2 = D I_2 , \quad I_1 I_2 I_1 = (D_1 Z + D_2) I_2 .
\]
which indicates that \( C_1 Z + C_2 = D_1 Z + D_2 \).

Theorem 3.2 gives that, if \( k \) is even, then
\[
a_0 a_k I_1 I_2 I_1 = D^{\text{even}} - [t_0 t_k t^{-1}] I_1 = \frac{1}{k} Z I_1 - [t_0 t_k t^{-1}] I_1,
\]
and if \( k \) is odd, then
\[
a_0 a_k I_2 I_1 I_2 = t^{\frac{k}{2}} \left(\frac{a_0}{-t_0}\right) D^{\text{odd}} + [t_0 t_k^{-1}] I_2 = \frac{1}{k} Z I_2 + [t_0 t_k^{-1}] I_2.
\]

Remark 3.4. Comparison to de Gier-Nichols. Let us explain how to relate the constants in Corollary 3.3 and Proposition 4.4 to the values which appear in [GN]. Let
\[
\frac{1}{T_0} = -i q^{e_1}, \quad \frac{1}{T_1} = q^{-1}, \quad \frac{1}{T_k} = -i q^{e_2},
\]
\[
T_0 = -i g_0, \quad T_1 = -g_i, \quad T_k = -i g_k,
\]
\[
e_0 = e_0, \quad e_1 = e_1, \quad e_k = e_k.
\]
Then
\[(g_0 - q^{\omega_1})(g_0 - q^{-\omega_1}) = 0, \quad (g_i + q^{-1})(g_i - q) = 0, \quad (g_k - q^{\omega_1})(g_k - q^{-\omega_1}) = 0,\]
as in [GN, Definitions 2.4, 2.6, and 2.8], and
\[g_0 = q^{\omega_1} - (q^{1+\omega_1} - q^{-(1+\omega_1)})e_0, \quad g_i = e_i - q^{-1}, \quad g_k = q^{\omega_2} - (q^{1+\omega_2} - q^{-(1+\omega_2)})e_k,\]
as in [GN, (5)]. Following [GN, Definitions 2.8 and (9)],
\[\gamma_0^{(C)} = g_1^{-1} \cdots g_{k-1}^{-1} g_k g_{k-1} \cdots g_2 g_1 g_0 = (-1)^{k-1}(i)(-i)(-1)^{k-1}T_1^{-1} \cdots T_{k-1}^{-1} T_k T_{k-1} \cdots T_1 T_0 = -W_1,\]
\[\gamma_i^{(C)} = g_i \gamma_{i-1}^{(C)} g_i = (-1)^2 T_i(-W_i)T_i = -W_{i+1} \quad \text{for } i \in \{1, \ldots, k-1\}, \text{ and}\]
\[Z_k = \sum_{i=0}^{k-1} (J_i^{(C)} + (J_i^{(C)})^{-1}) = -(W_1 + W_1^{-1} + \cdots + W_k + W_k^{-1}) = -Z.\]

Use the notation \([x] = \frac{t^x - t^{-x}}{t^x - t^{-\frac{x}{2}}} = \frac{q^x - q^{-x}}{q^x - q^{-\frac{x}{2}}}\) and let \(a_0, a\) and \(a_k\) take the favorite values from Remark [2.1] so that
\[a = -1, \quad a_0 = -[t_0 t^{-1}], \quad \text{and } a_k = -[t_k t^{-1}], \quad \text{and set } \theta = c + \frac{k-1}{2} \quad \text{and } z = [t^\theta] [k],\]
as in Proposition [4.3] Following [GN] Theorem 4.1 and remembering that \(Z_k = -Z\), let
\[\Theta = \theta + \frac{1}{\log q} i \pi \quad \text{so that } -[k][t^\Theta] = -[k](t^\frac{\theta}{2} + t^{-\frac{\theta}{2}}) = [k](-q^{-\theta} - q^\theta) = [k](q^{-\theta + \frac{1}{\log q} i \pi}) = [k](q^{-\Theta} + q^\Theta) = [k] [2\Theta] [k].\]

Note that
\[a_0 a_k = [t_0 t^{-1}][t_k t^{-1}] = (t_0^\frac{\theta}{2} t^{-\frac{\theta}{2}} + t_0^{\frac{1}{2}} t^{-\frac{1}{2}})(t_k^\frac{\theta}{2} t^{-\frac{\theta}{2}} + t_k^{\frac{1}{2}} t^{-\frac{1}{2}})\]
\[= (i^{-1}q^{-\omega_1^{-1}} + iq^{\omega_1^{-1}})(i^{-1}q^{-\omega_2^{-1}} + iq^{\omega_2^{-1}}) = -[\omega_1 + 1][\omega_2 + 1](q - q^{-1})^2.\]

Then the constant \(b\) that appears in [GN] Definition 3.6 and Theorem 4.1 to make \(I_1 I_2 I_1 = b I_1\) and \(I_2 I_1 I_2 = b I_2\) as operators on a simple \(TL_k\)-module is computed from Corollary [3.3] as follows:
\[b = \frac{1}{a_0 a_k} \frac{[t^\Theta]}{[t_0 t^{-1}][t_k t^{-1}]} = \frac{[t^\Theta]}{[t_0 t^{-1}][t_k t^{-1}]} = \frac{[t^\Theta] - [t_0 t^{-1}]}{[t_0 t^{-1}][t_k t^{-1}]}\]
\[= \frac{(q^\Theta - q^{-\Theta}) - (-ig^{\omega_1})(-i q^{\omega_2})q + (i q^{-\omega_1})(i q^{-\omega_2})q^{-1}}{[\omega_1 + 1][\omega_2 + 1](q - q^{-1})^{-2}}\]
\[= \frac{q^\Theta + q^{-\Theta} - q^{\omega_1 + \omega_2 + 1} - q^{-\omega_1 + \omega_2 + 1}}{[\omega_1 + 1][\omega_2 + 1](q - q^{-1})^{-2}}\]
\[= \frac{(q^{\omega_1 + \omega_2 + 1 + \Theta})^\frac{1}{2} - (q^{\omega_1 + \omega_2 + 1 + \Theta})^{-\frac{1}{2}}}{[\omega_1 + 1][\omega_2 + 1](q - q^{-1})^{-2}}\]
\[= \frac{[(\omega_1 + \omega_2 + 1 + \Theta)][(\omega_1 + \omega_2 + 1 - \Theta)]}{[\omega_1 + 1][\omega_2 + 1]} \quad \text{when } k \text{ is even, and}\]
that the formulas for the elements of the two boundary Temperley-Lieb algebras $TL^\text{ext}$ are given by

$$W = \frac{[\theta]}{a_0 a_k} \frac{[t_0 t_k^{-1}]}{[t_0 t_k^{-1}][t_k^{-1}]} = \frac{[\theta]}{[t_0 t_k^{-1}][t_k^{-1}]} = \frac{[\theta]}{[t_0 t_k^{-1}][t_k^{-1}]}$$

for $\pi = (\pi_1, \ldots, \pi_{2k}) = (\pi_1, \ldots, \pi_{2k}) \in \mathbb{C}^k$ and define

$$Z(c) = \{\varepsilon_i \mid c_i = 0\} \cup \{\varepsilon_j - \varepsilon_i \mid 0 < i < j \text{ and } c_j - c_i = 0\},$$

$$P(c) = \{\varepsilon_i \mid c_i \in \{\pm r_1, \pm r_2\}\} \cup \{\varepsilon_j - \varepsilon_i \mid 0 < i < j \text{ and } c_j - c_i = \pm 1\} \cup \{\varepsilon_j + \varepsilon_i \mid 0 < i < j \text{ and } c_j + c_i = \pm 1\}.$$ (4.2)

where $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is an orthonormal basis for the weights corresponding to $\mathfrak{gl}_n$ (see [DR, §3]). A local region is a pair $(c, J)$ with $c \in \mathbb{C}^k$ and $J \subseteq P(c)$. The set of standard tableaux of shape $(c, J)$ is

$$\mathcal{F}(c, J) = \{w \in W_0 \mid R(w) \cap Z(c) = \emptyset, R(w) \cap P(c) = J\}$$ (4.4)

(see the following section for a visualization of this set as fillings of box arrangements). A skew local region is a local region $(c, J)$, $c = (c_1, \ldots, c_k)$, such that

if $w \in \mathcal{F}(c, J)$ then $w c = ((w c)_1, \ldots, (w c)_n)$ satisfies

$$(w c)_1 \neq 0, \quad (w c)_2 \neq 0, \quad (w c)_1 \neq -(w c)_2,$$

$$(w c)_i \neq (w c)_{i+1} \quad \text{for } i = 1, \ldots, k-1, \quad \text{and} \quad (w c)_i \neq (w c)_{i+2} \text{ for } i = 1, \ldots, k-2.$$ (4.5)

The following theorem constructs and classifies the calibrated irreducible representations of $H_k^\text{ext}$.
Theorem 4.1. [DR] Theorem 3.3] Assume \( t^\frac{1}{2}, t^\frac{1}{2}_0, \) and \( t^\frac{3}{2}_k \) are invertible, \( t^\frac{1}{2} \) is not a root of unity, and
\[
\begin{align*}
& t^\frac{1}{2}_0 t^\frac{1}{2}_k - t^\frac{1}{2}_0 t^\frac{3}{2}_k, -t^\frac{1}{2}_0 t^\frac{3}{2}_k \not\in \{ -1, -t^\pm\frac{1}{2}, -t^\pm\frac{1}{2}, t^\pm 1, -t^\pm 1 \} \quad \text{and} \quad t^\frac{1}{2}_0 t^\frac{3}{2}_k \neq (-t^\frac{1}{2}_0 t^\frac{3}{2}_k)^\pm 1.
\end{align*}
\]
Let \( r_1, r_2 \) be as in (4.1).
(a) Let \((c, J)\) be a skew local region and let \( z \in \mathbb{C}^\times \). Define
\[
H^{(z, c, J)}_k = \text{span}_\mathbb{C} \{ v_w \mid w \in \mathcal{F}(c, J) \},
\]
such that the symbols \( v_w \) are a labeled basis of the vector space \( H^{(z, c, J)}_k \). Let
\[
\gamma_i = -t^{c_i} \quad \text{for} \quad i = 1, 2, \ldots, k, \quad \text{and} \quad \gamma_0 = z^{\gamma_{w^{-1}(1)} \cdots \gamma_{w^{-1}(k)}}.
\]
Then the following formulas make \( H^{(z, c, J)}_k \) into an irreducible \( H^{\text{ext}}_k \)-module:
\[
\begin{align*}
PW_1 \cdots W_k v_w &= z v_w, \quad P v_w = \gamma_0 v_w, \quad W_i v_w = \gamma_{w^{-1}(j)} v_w, \\
T_i v_w &= [T_i]_{w w} v_w + \sqrt{-(T_i)_{w w} - t^\frac{1}{2}} (T_i)_{w w} + t^{-\frac{1}{2}} v_{s_i w}, \quad \text{for} \quad i = 1, \ldots, k - 1, \\
T_0 v_w &= [T_0]_{w w} v_w + \sqrt{-(T_0)_{w w} - t_0\frac{1}{2}} (T_0)_{w w} + t_0\frac{1}{2} v_{s_0 w},
\end{align*}
\]
where \( v_{s_i w} = 0 \) if \( s_i w \not\in \mathcal{F}(c, J) \), and
\[
[T_i]_{w w} = \frac{t^\frac{1}{2} - t^{-\frac{1}{2}}}{1 - \gamma_{w^{-1}(i)}\gamma_{w^{-1}(i+1)}} \quad \text{and} \quad [T_0]_{w w} = \frac{(t^\frac{1}{2}_0 - t^{-\frac{1}{2}}_0) + (t^\frac{1}{2}_0 - t^{-\frac{1}{2}}_0) \gamma_{w^{-1}(1)}}{1 - \gamma_{w^{-1}(1)}}.
\]
(b) The map
\[
\mathbb{C}^\times \times \{ \text{skew local regions } (c, J) \} \leftrightarrow \{ \text{irreducible calibrated } H^{\text{ext}}_k \text{-modules} \}
\]
\[
(z, c, J) \longmapsto H^{(z, c, J)}_k
\]
is a bijection.

4.2. Configurations of boxes. Let \((c, J)\) be a local region with \( c = (c_1, \ldots, c_k) \),
\[
c \in \mathbb{Z}^k \quad \text{or} \quad c \in (\mathbb{Z} + \frac{1}{2})^k, \quad \text{and} \quad 0 \leq c_1 \leq \cdots \leq c_k.
\]
Start with an infinite arrangement of NW to SE diagonals, numbered consecutively from \( \mathbb{Z} \) or \( \mathbb{Z} + \frac{1}{2} \), increasing southwest to northeast (see Example 4.2). The configuration \( \kappa \) of boxes corresponding to the local region \((c, J)\) has \( 2k \) boxes (labeled \( \text{box}_{-k}, \ldots, \text{box}_{-1}, \text{box}_{1}, \ldots, \text{box}_{k} \)) with the following conditions.

(\(\kappa_1\)) Location: box \( i \) is on diagonal \( c_i \), where \( c_{-i} = -c_i \) for \( i \in \{-k, \ldots, -1\} \).
(\(\kappa_2\)) Same diagonals: box \( i \) is NW of box \( j \) if \( i < j \) and box \( i \) and box \( j \) are on the same diagonal.
(\(\kappa_3\)) Adjacent diagonals:
If \( \varepsilon_j - \varepsilon_i \in J \), then box \( j \) is NW (strictly north and weakly west) of box \( i \):
\[
\begin{array}{c}
\text{box } i \\
\text{box } j
\end{array}
\]
If \( \varepsilon_j - \varepsilon_i \in P(c) - J \), then box \( j \) is SE (weakly south and strictly east) of box \( i \):
\[
\begin{array}{c}
\text{box } i \\
\text{box } j
\end{array}
\]
(\(\kappa_4\)) Markings: There is a marking on each of the diagonals \( r_1, -r_1, r_2 \) and \( -r_2 \).
If \( \varepsilon_i \in J \), box \( i \) is NW of the marking on diagonal \( c_i \):
\[
\begin{array}{c}
\text{box } i \\
\text{marking}
\end{array}
\]
If \( \varepsilon_i \in P(c) - J \), then box \( i \) is SE of the marking in diagonal \( c_i \):
\[
\begin{array}{c}
\text{marking} \\
\text{box } i
\end{array}
\]
A standard filling of the boxes of $\kappa$ is a bijective function $S: \kappa \to \{-k, \ldots, -1, 1, \ldots, k\}$ such that

1. Symmetry: $S(\text{box}_{-i}) = -S(\text{box}_i)$.
2. Same diagonals:
   - If $0 < i < j$ and box$_i$ and box$_j$ are on the same diagonal then $S(\text{box}_i) < S(\text{box}_j)$.
3. Adjacent diagonals:
   - If $0 < i < j$, box$_i$ and box$_j$ are on adjacent diagonals, and box$_j$ is NW of box$_i$, then $S(\text{box}_j) < S(\text{box}_i)$.
   - If $0 < i < j$, box$_i$ and box$_j$ are on adjacent diagonals, and box$_j$ is SE of box$_i$, then $S(\text{box}_j) > S(\text{box}_i)$.
4. Markings:
   - If box$_i$ is on a marked diagonal and is SE of the marking, then $S(\text{box}_i) > 0$.
   - If box$_i$ is on a marked diagonal and is NW of the marking, then $S(\text{box}_i) < 0$.

The identity filling of a configuration $\kappa$ is the filling $F$ of the boxes of $\kappa$ given by $F(\text{box}_i) = i$, for $i = -k, \ldots, -1, 1, \ldots, k$. The identity filling of $\kappa$ is usually not a standard filling of $\kappa$ (see Example 4.2).

**Example.** Let $k = 4$, $r_1 = 1$, and $r_2 = 3$. Consider $c = (2, 2, 3)$. Then

$$Z(c) = \{\varepsilon_1 - \varepsilon_2\} \quad \text{and} \quad P(c) = \{\varepsilon_3, \varepsilon_3 - \varepsilon_1, \varepsilon_3 - \varepsilon_2\}.$$ 

The box configurations corresponding to $J = \{\varepsilon_3 - \varepsilon_2\}$ and $J = \{\varepsilon_3, \varepsilon_3 - \varepsilon_1, \varepsilon_3 - \varepsilon_2\}$ (filled with their identity fillings) are

For both configurations, the identity filling is not a standard filling. Examples of standard fillings of the configuration corresponding to $J = \{\varepsilon_2 - \varepsilon_3\}$ include

$$1 2 \quad , \quad 3 2 \quad \text{and} \quad 2 1 3 \quad , \quad 3 2 1 \quad \text{but not} \quad 1 2 3.$$ 

**Proposition 4.2.** Let $\kappa$ be a configuration of boxes corresponding to a local region $(c, J)$ with $c \in \mathbb{Z}^k$ or $c \in (\mathbb{Z} + \frac{1}{2})^k$. For $w \in W_0$ let $S_w$ be the filling of the boxes of $\kappa$ given by

$$S_w(\text{box}_i) = w(i), \quad \text{for} \quad i = -k, \ldots, -1, 1, \ldots, k.$$ 

The map

$$\mathcal{F}(c, J) \quad w \quad \mapsto \quad \{\text{standard fillings} \ S \text{ of the boxes of } \kappa \}$$

is a bijection.
4.3. **Calibrated representations of** $\text{TL}^{\text{ext}}_k$. The following theorem determines which calibrated irreducible representations of $H_k^{\text{ext}}$ are $\text{TL}^{\text{ext}}_k$-modules. In Theorem 4.3 the answer is stated in terms of the configuration of boxes $\kappa$. By (κ1)–(κ4) of Section 4.2 the local region $(c, J)$ is determined by $\kappa$. See Theorem 4.1 for the explicit conversion from $\kappa$ to $(c, J)$ for the irreducible calibrated $\text{TL}_k$-modules.

**Theorem 4.3.** Assume that if $r_1, r_2 \in \mathbb{Z}$ or $r_1, r_2 \in \mathbb{Z} + \frac{1}{2}$, then $r_2 > r_1 + 1$. Let $\kappa$ be the configuration of boxes corresponding to a skew local region $(c, J)$ with $c \in \mathbb{Z}^k$ or $c \in (\mathbb{Z} + \frac{1}{2})^k$. The irreducible calibrated $H_k^{\text{ext}}$-module $H_k^{(z,c,J)}$ is a $\text{TL}^{\text{ext}}_k$-module if and only if $\kappa$ is a 180° rotationally symmetric skew shape with two rows of $k$ boxes each (with or without markings),

\[
\begin{array}{c}
\text{or} \\
\end{array}
\]

(4.12)

Proof. Let $P = \{P_0^{(0,1)}, P_0^{(2,2)}, P_0^{(3,3)}, P_0^{(1,2)}, P_1^{(1)}, P_2^{(2)}, \ldots, P_{k-2}^{(2)}\}$ so that $\text{TL}_k$ is the quotient of $H_k$ by the ideal generated by the set $P$. For $w \in \mathcal{F}(c,J)$ let $S_w$ be the standard tableau of shape $\kappa$ corresponding to $w$ as given in Proposition 4.2. For $j \in \{-k, -1, 1, \ldots, k\}$,

$(\omega c)_j$ is the diagonal number of $S_w(j)$,

Step 1: Rewriting of the conditions for $p_{\omega} w = 0$. By the construction of $H_k^{(z,c,J)}$ in Theorem 4.1 the module $H_k^{(z,c,J)}$ has basis $\{v_w \mid w \in \mathcal{F}(c,J)\}$ and, if $w \in \mathcal{F}(c,J)$ then

$$
\tau_i v_w = 0 \quad \text{if and only if} \quad (\omega c)_i = (\omega c)_i \pm 1,
$$

and

$$
v_{\omega} = 0 \quad \text{if and only if} \quad (\omega c)_i = r_2, \quad \text{and}
$$

$$
v_{\epsilon_i - \epsilon_j + 1} v_w = 0 \quad \text{if and only if} \quad (\omega c)_i = (\omega c)_j - 1.
$$

Let $i \in \{1, \ldots, k - 2\}$. Using the expansion of $p_i^{(1)}$ in terms of the $\tau_i$ from Proposition 4.2

$$
p_i^{(1)} v_w = \tau_i \tau_{i+1} \tau_{i+2} v_w - t^{-\frac{1}{2}} \tau_i \tau_{i+1} \frac{f_{\epsilon_i+1 - \epsilon_i} + 1}{f_{\epsilon_i+1 - \epsilon_i}} v_w - t^{-\frac{1}{2}} \tau_i \tau_{i+1} \frac{f_{\epsilon_i+1 - \epsilon_i} + 1}{f_{\epsilon_i+1 - \epsilon_i}} v_w
$$

we consider the condition $p_i^{(1)} v_w = 0$ term-by-term. First, $\tau_i \tau_{i+1} \tau_{i+2} v_w = 0$ exactly when $(\omega c)_i = (\omega c)_i \pm 1$ or $(s_i w c)_{i+2} = (s_i w c)_{i+2} \pm 1$ or $(s_{i+1} s_i w)_{i+1} = (s_{i+1} s_i w)_{i+1} = \pm 1$, i.e. when

$(\omega c)_i = (\omega c)_i \pm 1$ or $(\omega c)_{i+2} = (\omega c)_{i+2} \pm 1$ or $(\omega c)_{i+1} = (\omega c)_{i+1} \pm 1$.

Next, $-t^{-\frac{1}{2}} \frac{f_{\epsilon_i+1 - \epsilon_i} + 1}{f_{\epsilon_i+1 - \epsilon_i}} v_w = 0$ exactly when

$(\omega c)_{i+1} = (\omega c)_{i+1} + 1$ or $(\omega c)_{i+2} = (\omega c)_{i+2} - 1$ or $(\omega c)_{i+1} = (\omega c)_{i+1} + 1$.

Thus $-t^{-\frac{1}{2}} \frac{f_{\epsilon_i+1 - \epsilon_i} + 1}{f_{\epsilon_i+1 - \epsilon_i}} v_w = 0$ already implies $\tau_i \tau_{i+1} \tau_{i+2} v_w = 0$, and similarly for the other terms in the expansion of $p_i^{(1)} v_w = 0$. Thus $p_i^{(1)} v_w = 0$ if and only if

$(\omega c)_i = (\omega c)_{i+1} - 1$ or $(\omega c)_i = (\omega c)_{i+2} - 1$ or $(\omega c)_{i+1} = (\omega c)_{i+2} - 1$.  

(4.13)
Similarly, $p_0^{(0,1^2)}v_w = 0$ if and only if
\[(wc)_1 \in \{r_1, r_2\} \text{ or } (wc)_2 \in \{r_1, r_2\} \text{ or } (wc)_2 = (wc)_1 + 1 \text{ or } (wc)_2 = (wc)_{-1} + 1; \quad (4.14)\]
$p_0^{(1^2,0)}v_w = 0$ if and only if
\[(wc)_1 \in \{-r_1, -r_2\} \text{ or } (wc)_2 \in \{-r_1, -r_2\} \text{ or } (wc)_2 = (wc)_1 + 1 \text{ or } (wc)_2 = (wc)_{-1} + 1; \quad (4.15)\]
$p_0^{(0,1^2)}v_w = 0$ if and only if
\[(wc)_1 \in \{-r_1, -r_2\} \text{ or } (wc)_2 \in \{-r_1, -r_2\} \text{ or } (wc)_2 = (wc)_1 + 1 \text{ or } (wc)_2 = (wc)_{-1} + 1; \quad (4.16)\]
and $p_0^{(0^2,0)}v_w = 0$ if and only if
\[(wc)_1 \in \{r_1, -r_2\} \text{ or } (wc)_2 \in \{r_1, -r_2\} \text{ or } (wc)_2 = (wc)_1 + 1 \text{ or } (wc)_2 = (wc)_{-1} + 1. \quad (4.17)\]

Step 2: If $\kappa$ is as in (4.12) and $w \in F^{(c,J)}$ and $p \in P$ then $pv_w = 0$. Assume $\kappa$ has the form given in (4.12) and let $w \in F^{(c,J)}$. Since $\kappa$ has only two rows the positions of $(-2, -1, 1, 2)$ in $S_w$ take one of the following forms:

\[\begin{array}{c|c|c}
(wc)_{-1} & (wc)_1 & (wc)_{-1} \\
\hline
-2 & 1 & -1 \\
\hline
(wc)_1 & 1 & 2 \\
\hline
\end{array}\]

\[\begin{array}{c|c|c}
(wc)_1 < -\frac{1}{2}, & (wc)_1 > \frac{1}{2}, & (wc)_1 = -\frac{1}{2}, \\
\hline
-2 & -1 & 1 \\
\hline
1 & 2 & 0 \\
\hline
\end{array}\]

\[\begin{array}{c|c|c}
(wc)_1 \geq \frac{1}{2}, & (wc)_1 < \frac{1}{2}, & (wc)_1 = \frac{1}{2}. \\
\hline
-2 & -1 & 1 \\
\hline
1 & 2 & 0 \\
\hline
\end{array}\]

\[\begin{array}{c|c|c}
(wc)_i, & (wc)_{i+1} & (wc)_i, \quad \text{or } i \text{ and } i + 2 \text{ are in the same row.} \\
\hline
i & i+1 & i \quad \text{or } i \text{ and } i + 2 \text{ are in the same row.} \\
\hline
\end{array}\]

Thus, by (4.13), $p_1v_w = 0$. This completes the proof that if $\kappa$ is of the form (4.12) then $H_k^{(z,c,J)}$ is a $TL_k^{\text{ext}}$-module.

Step 3: If $\kappa$ is not as in (4.12) then there exists $w \in F^{(c,J)}$ and $p \in P$ such that $pv_w \neq 0$. Let $2k$ be the number of boxes in $\kappa$. The proof is by induction on $k$.

First, if $k = 2$, then the condition (4.13) does not apply. If $c = (r_1, r_2)$ then there are 8 possibilities for $wc$: $(r_1, r_2), (-r_1, r_2), (r_1, -r_2), (-r_1, -r_2), (r_1, r_2), (-r_2, r_1), (r_2, -r_1)$ and $(-r_2, -r_1)$. None of these satisfy all of the conditions (4.14)–(4.17). If $c = (c_1, c_1 + 1)$, then $s_1c = (c_1 + 1, c_1)$ does not satisfy (4.13) and $s_0s_1s_0s_1c = (-c, -c - 1)$ does not satisfy (4.17). Thus that only the shaded local regions in Figure 1 can have $pv_w = 0$ for all $p \in P$ and all $w \in F^{(c,J)}$. For all of these, $\kappa$ is as in (4.12).

Next, assume $k > 2$ and proceed inductively. If $H_k^{(z,c,J)}$ is a calibrated $TL_k^{\text{ext}}$-module then $\text{Res}_{TL_k^{\text{ext}}} (H_k^{(z,c,J)})$ is calibrated $TL_{k-1}^{\text{ext}}$-module. This means that if $S_w$ is a standard tableau of shape $\kappa$ and $S'_w$ is $S_w$ except with the boxes $S_w(k)$ and $S_w(-k)$ removed and $\kappa'$ is the shape of $S'_w$, then...
then \( \kappa' \) must be as in (4.12) and have only two rows. The box \( S_w(k) \) is in a SE corner of \( \kappa \) and the box \( S_w(-k) \) is in a NW corner of \( \kappa \).

Given that \( \kappa' \) has only two rows and \( \kappa \) is obtained from \( \kappa' \) by adding boxes that could contain \( k \) and \( -k \) in a standard tableau, the following are possibilities that we discard for \( \kappa \):

\[
\begin{array}{c}
-2 \\
k-1 \\
k-2 \\
k
\end{array}
\]

and

\[
\begin{array}{c}
-k \\
k-1 \\
k \\
k-2 \\
k-1 \\
k
\end{array}
\]

Namely, in each case there is a standard tableau that has \( k - 2 \), \( k - 1 \) and \( k \) in positions that do not satisfy the conditions in (4.13). Thus, in these cases, there exists an \( S_w \) of shape \( \kappa \) for which \( p_{k-2} v_w \neq 0 \). Further, in the case

\[
\begin{array}{c}
-k \\
\end{array}
\]

the shape \( \kappa \) does not satisfy the \((wc)_{k-2} \neq (wc)_k \) from (4.5) and the module \( H_k^{(z,c,J)} \) is not calibrated. In summary, unless \( \kappa \) is of the form given in (4.12)

\[
\begin{array}{c}
-k \\
k
\end{array}
\]

or

\[
\begin{array}{c}
-k \\
k
\end{array}
\]

then either \( H_k^{(z,c,J)} \) is not calibrated or there exists an \( S_w \) of shape \( \kappa \) for which \( p_{k-2} v_w \neq 0 \). \( \square \)
Figure 1. Calibrated representations of $H_2$ have regular central character. For each $(c, J)$ the corresponding configuration of boxes $\kappa$ is displayed in the local region of chambers corresponding to the elements of $\mathcal{F}(c, J)$; only the boxes on positive diagonals are shown, since they determine $\kappa$ when $c$ is regular. The local regions marked in blue are those that factor through the Temperley-Lieb quotient.
The following proposition determines the action of the central element \( Z \) on each of the irreducible calibrated \( TL_k^{\text{ext}} \)-modules.

**Proposition 4.4.** Let \( Z = W_1 + W_1^{-1} + \cdots + W_k + W_k^{-1} \) be the central element of \( TL_k^{\text{ext}} \) studied in Theorem 3.2. Assume that \( c = (c, c+1, \ldots, c+k-1) \) and \( H_k^{(z,c,J)} \) is an irreducible calibrated \( TL_k^{\text{ext}} \) as in Theorem 4.3. If \( v \in H_k^{(z,c,J)} \), then

\[
Z v = [t^\theta][k] v, \quad \text{where} \quad \theta = c + \frac{k-1}{2}, \quad [t^\theta] = t_\frac{k}{2} + t_\frac{1}{2} \quad \text{and} \quad [k] = \frac{t_\frac{k}{2} - t_\frac{1}{2}}{t_1^\frac{k}{2} - t_1^\frac{1}{2}}.
\]

**Proof.** Let \( v \in H_k^{(z,c,J)} \) be such that \( W_i v = q^{c+i-1} \) for \( i \in \{1, \ldots, k\} \). Then \( Z v_w = z v_w \) where

\[
z = t^{-(c+k-1)} + \cdots + t^{-(c+1)} + t^{-c} + t^{c+1} + \cdots + t^{c+k-1} = (t^{c+\frac{k-1}{2}} + t^{-(c+\frac{k-1}{2})})(t^{-\frac{k-1}{2}} + \cdots + t^{-\frac{1}{2}}) = (t_\frac{k}{2} + t_\frac{1}{2}) \frac{t_\frac{k}{2} - t_\frac{1}{2}}{t_1^\frac{k}{2} - t_1^\frac{1}{2}} = [t^\theta][k].
\]

Since \( Z \) is a central element of \( HL_k^{\text{ext}} \) and \( H_k^{(z,c,J)} \) is a simple \( H_k^{\text{ext}} \)-module, Schur’s lemma implies that if \( v \in H_k^{(z,c,J)} \) then \( Z v = z v \). \( \square \)

5. SCHUR-WEYL DUALITY BETWEEN \( TL_k^{\text{ext}} \) AND \( U_q gl_2 \)

In this section we show that the Schur-Weyl duality studied in [DR] provides calibrated irreducible representations of the two boundary Temperley-Lieb algebra. We classify these representations using the combinatorial classification of irreducible calibrated \( TL_k^{\text{ext}} \) modules obtained in Theorem 4.3. We follow the combinatorics of [Dau, §4] and [DR, §5]. Note that similar constructions hold for replacing \( gl_2 \) with \( sl_2 \)—see, for example, [Dau, §4]

The irreducible finite dimensional representations \( L(\lambda) \) of \( U_q gl_2 \) are indexed by \( \lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2 \) with \( \lambda_1 \geq \lambda_2 \). By the Clebsch-Gordan formula or the Littlewood-Richardson rule (see [Mac, (5.16)])

\[
L(a,0) \otimes L(b,0) = L(a+b,0) \oplus L(a+b+1,1) \oplus \cdots \oplus L(a+1,b-1) \oplus L(a,b),
\]

and

\[
L(\lambda_1, \lambda_2) \otimes L(1,0) = \begin{cases} L(\lambda_1+1, \lambda_2) \oplus L(\lambda_1, \lambda_2+1), & \text{if } \lambda_1 > \lambda_2, \\ L(\lambda_1+1, \lambda_2), & \text{if } \lambda_1 = \lambda_2. \end{cases}
\]

Let \( a, b \in \mathbb{Z}_{\geq 0} \) with \( a \geq b \) and fix the simple \( U_q gl_2 \)-modules

\[
M = L(a,0), \quad N = L(b,0) \quad V = L(1,0).
\]

We identify \( (\lambda_1, \lambda_2) \in \mathbb{Z}^2 \) with a left-justified arrangement of boxes with \( \lambda_i \) boxes in the \( i \)-th row. As in [DR] (5.28)] the shifted content of a box in row \( i \) and column \( j \) as

\[
\tilde{c}(\text{box}) = j - i - \frac{1}{2}(a + b - 2)
\]

i.e. the shifted content is its diagonal number, where the box in the upper left corner has shifted content \(-\frac{1}{2}(a + b - 2)\).

For \( j \in \mathbb{Z}_{\geq -1} \) let \( \mathcal{P}^{(j)} \) be an index set for the irreducible \( U_q gl_2 \)-modules that appear in \( M \otimes N \otimes V^{\otimes j} \). In particular,

\[
\mathcal{P}^{(-1)} = \{a,0\}, \quad \mathcal{P}^{(0)} = \{(a+b-j,j) \mid j = 0, 1, \ldots, b\} \quad \text{and} \quad \mathcal{P}^{(j)} = \{(a+b+j-\ell, \ell) \mid 0 \leq \ell \leq \frac{1}{2}(j+a+b)\}, \text{ for } j \geq 1.
\]

Following [DR, §5.4], the associated Bratteli diagram has
vertices on level $j$ labeled by the partitions in $P^{(j)}$,
an edge $(a,0) \rightarrow \mu$ for each $\mu \in P^{(0)}$, and
for each $j \geq 0$, $\mu \in P^{(j)}$ and $\lambda \in P^{(j+1)}$, there is
an edge $\mu \rightarrow \lambda$ if $\lambda$ is obtained from $\mu$ by adding a box.

The case when $a = 6$ and $b = 3$ is illustrated in Figure 2.
Assume $q \in \mathbb{C}^\times$ and $a > b + 2$ so that the generality condition $(a + 1) - (b + 1) \notin \{0, \pm 1, \pm 2\}$ of
[DR, Theorem 5.5] is satisfied. Define
\begin{equation}
 r_1 = \frac{1}{2}(a-b) \quad \text{and} \quad r_2 = \frac{1}{2}(a+b+2),
\end{equation}
and let $H_k^{\text{ext}}$ be the extended two boundary Hecke algebra with parameters $t_0^2$, $t_k^2$, and $t_k^{\frac{1}{2}}$ given by
\begin{equation}
 t_k^{\frac{1}{2}} = q, \quad t_0 = -t^{r_2-r_1} = -q^{(b+1)}, \quad \text{and} \quad t_k = -t^{r_2+r_1} = -q^{2(a+1)},
\end{equation}
so that $-t_k^2t_0^{-\frac{1}{2}} = -t^r$ and $t_k^2t_0^{\frac{1}{2}} = -t^{2r}$ as in [DR, (3.5), (5.21)]. By [DR, Theorem 5.4 and (5.21)] there are

commuting actions of $U_q\mathfrak{gl}_2$ and $H_k^{\text{ext}}$ on $M \otimes N \otimes V^{\otimes k}$, where the $H_k^{\text{ext}}$ action is given via R-matrices for the quantum group $U_q\mathfrak{gl}_2$.

**Theorem 5.1.** Let $a, b \in \mathbb{Z}_{\geq 0}$ with $a > b + 2$. Let $q \in \mathbb{C}^\times$ not a root of unity and let $H_k^{\text{ext}}$ be the two boundary Hecke algebra with parameters $t_0^2$, $t_k^2$, and $t_k^{\frac{1}{2}}$ as in (5.4). Let $U_q\mathfrak{gl}_2$ be the Drinfeld-Jimbo quantum group corresponding to $\mathfrak{gl}_2$ and let $M$, $N$ and $V$ be the simple $U_q\mathfrak{gl}_2$-modules given in (5.1). Then the $H_k^{\text{ext}}$ action factors through $TL_k^{\text{ext}}$ and, as $(U_q\mathfrak{gl}_2, TL_k^{\text{ext}})$-bimodules,
\begin{equation}
 M \otimes N \otimes V^{\otimes k} \cong \bigoplus_{\lambda \in P^{(k)}} L(\lambda) \otimes B_k^\lambda \quad \text{with} \quad B_k^{(a+b+k-\ell,\ell)} \cong H^{(z,c,J)},
\end{equation}
where $z = (-1)^k q^{(a+b-\ell)(a+b-\ell-1)+\ell(\ell-3)-a(a-1)-b(b-1)-k(a+b-2)}$ and $(c,J)$ is the local region corresponding to the configuration $\kappa$ of $2k$ boxes
\begin{equation}
 \ell + 1 - r_2 - k \quad \text{with} \quad \ell - r_2 \quad \text{and} \quad r_2 - 1 + k - \ell
\end{equation}
that has $k$ boxes in each row, the shifted content of the leftmost box in the first row is $r_2 - \ell$, the shifted content of the leftmost box in the second row is $\ell + 1 - r_2 - k$. Between the rows there are blue markers in diagonals with shifted content $\pm r_1$ and there are red markers in diagonals with shifted content $\pm r_2$, as pictured. Explicitly, $c = (c_1, c_2, \ldots, c_k)$ is the sequence of
absolute values of $c$, $c+1$, $\ldots$, $c+k-1$, where $c = \frac{1}{2}(a+b) - \ell + 1$,
arranged in increasing order; and $J$ is the union of
\begin{equation}
 J_1 = \begin{cases} 
 \emptyset, & \text{if } a \geq b \geq \ell, \\
 \{e_{\ell-b}\}, & \text{if } a \geq \ell > b, \\
 \{e_{a-b}\}, & \text{if } \ell > a > b,
\end{cases}
\end{equation}
and
\[ J_2 = \begin{cases} 
\emptyset, & \text{if } \frac{1}{2}(a + b + 2) > \ell, \\
\{\varepsilon_2 - \varepsilon_1, \varepsilon_4 - \varepsilon_3, \ldots, \varepsilon_{2\ell-a-b} - \varepsilon_{2\ell-a-b-1}\}, & \text{if } \ell \geq \frac{1}{2}(a + b + 2) \text{ and } a + b \text{ even,} \\
\{\varepsilon_3 - \varepsilon_2, \varepsilon_5 - \varepsilon_4, \ldots, \varepsilon_{2\ell-a-b} - \varepsilon_{2\ell-a-b-1}\}, & \text{if } \ell \geq \frac{1}{2}(a + b + 2) \text{ and } a + b \text{ odd.}
\end{cases} \]

Proof. Fix \( \lambda = (a + b + k - \ell, \ell) \in \mathcal{P}(k) \). The sum of the contents of the boxes in \( \lambda \) is
\[
\sum_{\text{box} \in \lambda} c(\text{box}) = (0 + 1 + \ldots + (a + b + k - \ell - 1)) + (-1 + 0 + \ldots + \ell - 2) = \frac{1}{2}(a + b + k - \ell - 1)(a + b + k - \ell) + \frac{1}{2}\ell(\ell - 3).
\]

By [DR] Theorem 5.5 and (5.35), \( B_k^{\lambda} \cong H_k^{(z,c,J)} \) where
\[
z = (-1)^k q^{2c_0}, \quad \text{where} \quad c_0 = -\frac{1}{2}(k(a + b - 2) + a(a - 1) + b(b - 1)) + \sum_{\text{box} \in \lambda} c(\text{box}),
\]
and \( c \) and \( J \) and the corresponding configuration \( \kappa \) of 2k boxes are determined as follows.

Place markers at the NW corner of the boxes at positions \((1, a + b + 1), (2, a + 1), (2, b + 1), \) and \((3, 1)\) so that these markers are in the diagonals with shifted contents \( \pm r_1 \) and \( \pm r_2 \).

\[
\lambda = (a + b + k - \ell, \ell) = \begin{array}{|c|c|c|}
\hline
 a & b & k - \ell \\
\hline
 b & \ell - b & \ell - r_2 \\
\hline
 \ell - r_2 - 1 + k - \ell & & \\
\hline
\end{array}
\]

Following [DR] (5.27), let
\[
S_{\text{max}}^{(0)} = \begin{cases} 
(a + b - \ell, \ell), & \text{if } a \geq b \geq \ell, \\
(a, b), & \text{if } a \geq \ell \geq b
\end{cases}
\]
(since \( a \geq b \) we are in the left case of [DR] (5.15)) with \( c = d = 1 \) so that \( \mu^c = \min(\ell, b) \) and \( S_{\text{max}}^{(0)} = \mu = (a + b - \mu^c, \mu^c) \):

\[
\begin{array}{c|c|c|}
\hline
 a + b - \ell & k & | \\
\hline
| & \text{if } a \geq b \geq \ell & | \\
\hline
| & \text{if } a \geq \ell \geq b & | \\
\hline
\end{array}
\]

By [DR] (5.35), the corresponding configuration of boxes is \( \kappa = \text{rot}(\lambda/S_{\text{max}}^{(0)}) \cup \lambda/S_{\text{max}}^{(0)} \), as pictured above in (5.5).

To determine \( (c, J) \), use the conditions \((\kappa_1)\)–\((\kappa_4)\) of Section 4.2 which specify the relation between \( \kappa \) and \( (c, J) \). First index the boxes of \( \kappa \) with \(-k, \ldots, -1, 1, \ldots, k\) by diagonals, left to right, and NW to SE along diagonals. The sequence \( c = (c_1, \ldots, c_k) \) with \( 0 \leq c_1 \leq c_2 \leq \cdots \leq c_k \) is the sequence of the absolute values of the shifted contents of boxes in the first row of \( \kappa \). Next, the set \( J \) is determined as follows.

1. By \((\kappa_4)\), the set \( J \) contains \( \varepsilon_i \) if \( i > 0 \) and box \( x_0 \) is NW of the marker in the diagonal with shifted content \( r_1 \) or \( r_2 \) in \( \kappa \). This occurs on diagonal \( r_1 \) whenever \( \ell > b \) (marked in blue),
\[
\varepsilon_{\ell-b} \in J \text{ if } a \geq \ell > b \quad \text{and} \quad \varepsilon_{a-b} \in J \text{ if } \ell > a \geq b;
\]
and \( J \) contains no roots of the form \( \varepsilon_j \) when \( a \geq b \geq \ell 
.

2. By \((\kappa 3)\), the set \( J \) contains \( \varepsilon_j - \varepsilon_i \) if \( j > i > 0 \) and box \( i \) and box \( j \) are in the same column of \( \kappa \) (so that box \( i \) and box \( j \) are in adjacent diagonals and box \( j \) is NW of box \( i \)). This occurs exactly when \( 0 \geq r_2 - \ell = \frac{1}{2}(a + b + 2) - \ell \). If \( \ell \geq \frac{1}{2}(a + b + 2) \) and \( a + b \) is even then the boxes indexed \( 1, 3, \ldots, 1 + 2(\ell - \frac{1}{2}(a + b + 2)) = 2\ell - (a + b + 1) \) are in the second row directly below boxes of index \( 2, 4, \ldots, 2\ell - a - b \). If \( \ell \geq \frac{1}{2}(a + b + 2) \) and \( a + b \) is odd then boxes \( 2, 4, \ldots, 2(\ell - \frac{1}{2}(a + b + 1)) \), directly below boxes of index \( 3, 5, \ldots, 2\ell - a - b \):

\[
\begin{array}{cccccccc}
-1 & -2 & -3 & 1 & 4 & \cdots & \cdots & k \\
0 & -1 & -2 & 1 & 3 & \cdots & \ast & 2\ell - a - b - 1 \\
\end{array}
\]

if \( a + b \) is even,

\[
\begin{array}{cccccccc}
-1 & -2 & -3 & 1 & 4 & \cdots & \cdots & k \\
0 & -1 & -2 & 1 & 3 & \cdots & \ast & 2\ell - a - b - 1 \\
\end{array}
\]

if \( a + b \) is odd.

So \( J \) contains

\[
\varepsilon_2 - \varepsilon_1, \varepsilon_4 - \varepsilon_3, \ldots, \varepsilon_{2\ell - a - b} - \varepsilon_{2\ell - a - b - 1} \quad \text{if} \quad \ell \geq \frac{1}{2}(a + b + 2) \quad \text{and} \quad a + b \quad \text{is even, or}
\]

\[
\varepsilon_3 - \varepsilon_2, \varepsilon_5 - \varepsilon_4, \ldots, \varepsilon_{2\ell - a - b} - \varepsilon_{2\ell - a - b - 1} \quad \text{if} \quad \ell \geq \frac{1}{2}(a + b + 2) \quad \text{and} \quad a + b \quad \text{is odd.}
\]

3. Also by \((\kappa 3)\), the set \( J \) contains \( \varepsilon_j + \varepsilon_i \) if \( j > i > 0 \), and box \( j \) is directly above box \( -i \), which does not occur.

In this way \( c \) and \( J \) are determined from \( \kappa \). Since all of these \( H_k^{(z,c,J)} \) satisfy the conditions of Theorem 4.3, it follows that the \( H_k^{\text{ext}} \)-action on \( M \otimes N \otimes V^{\otimes k} \) factors through \( TL_k^{\text{ext}} \).

\[ \square \]

**Remark 5.2.** The dimension of \( B_k^{(a + b + k - \ell, \ell)} \) is the number of paths in the Bratteli diagram from a shape on level 0 to the shape \( \lambda = (a + b + k - \ell, \ell) \) on level \( k \). Summing over the shapes on level 0 for which there is a path to \( \lambda \) gives

\[
\dim(B^{(a + b + k - \ell, \ell)}) = \sum_{c = \max(0, k - \ell)}^{\min(b, \ell)} f^{\lambda/(a + b - c, c)},
\]

where \( f^{\lambda/\mu} \) is the number of standard tableaux of skew shape \( \lambda/\mu \). If \( \ell \leq a + b - c \) then the second row of \( \lambda/(a + b - \ell, \ell) \) does not overlap the first row and thus

\[
f^{\lambda/(a + b - c, c)} = \binom{k}{\ell - c} \quad \text{if} \quad \ell \leq a + b - c.
\]

Since \( c \leq \min(b, \ell) \), the case \( \ell > a + b - c \) can occur only when \( \ell > a \geq b \), in which case

\[
(a + b + k - \ell, \ell)/(a + b - c, c) =
\]

\[
\begin{array}{cccccc}
\ast & a + b - c & k - \ell + c & \ell & (a + b - c) \\
\end{array}
\]
so that
\[ f(a+b+k-\ell)/(a+b-c,c) = \sum_{j=\ell-(a+b-c)}^{k+\ell-c} f(k-j,j) = \sum_{j=\ell-(a+b-c)}^{\min(k-(\ell-c),\ell-c)} \binom{k}{j} - \binom{k}{j-1} \]
\[ = \binom{k}{\ell-c} - \binom{k}{\ell-(a+b-c)-1}. \]

Namely, the first equality comes from the Pieri formula and the expansion of a skew Schur function by Littlewood-Richardson coefficients (see [Mac] (5.16)) for the Pieri formula and [Mac] (5.2) and (5.3) for Littlewood-Richardson coefficients) and the second equality comes from the number of standard tableaux of a two row shape as given, for example, in [GJK] Theorem 2.8.5 and Lemma 2.8.4.

The following examples reference the node label styles in Figure 2.

**Example.** Let \( a = 7 \) and \( b = 3 \). The markers are in the diagonals with shifted contents \( \pm r_1 \) and \( \pm r_2 \), where \( r_1 = 2 \) and \( r_2 = 6 \). An example where \( \ell > a \geq b \): Let \( \ell = 11 \) and \( k = 14 \), then
\[
\lambda = (13,11) = \begin{array}{cccccccccccc}
\bullet & | & \bullet & | & \bullet & | & \bullet & | & \bullet & | & \bullet & | & \bullet
\end{array}
\]
with \( S_{\text{max}}^{(0)} = (7,3) \).

The boxes of \( \lambda/S_{\text{max}}^{(0)} \) have

shifting contents:
\[
\begin{array}{cccccccc}
3 & 4 & 5 & 6 & 7 & 8
\end{array}
\]

Then \( c \) is the rearrangement of the absolute values of \( (-2,-1,0,1,2,3,4,5,6,7,8) \) into increasing order and \( J = \{\varepsilon_4, \varepsilon_2 - \varepsilon_1, \varepsilon_4 - \varepsilon_3, \varepsilon_6 - \varepsilon_5, \varepsilon_8 - \varepsilon_7, \varepsilon_{10} - \varepsilon_9, \varepsilon_{12} - \varepsilon_{11}\} \). The configuration of boxes \( \kappa \) corresponding to \( (c,J) \) has indexing of boxes
\[
\begin{array}{cccccccc}
-11 & -9 & -7 & -5 & -3 & -1 & 2 & 4 & 6 & 8 & 10 & 12 & 13 & 14
\end{array}
\]
and
\[
\begin{array}{cccccccc}
14 & 13 & 12 & 10 & 8 & 6 & 4 & 2 & 1 & 3 & 5 & 7 & 9 & 11
\end{array}
\]

**Example.** Let \( a = 6 \) and \( b = 3 \) to take advantage of the setting and notation of Figure 2. The markers are in the diagonals with shifted contents \( \pm r_1 \) and \( \pm r_2 \), where \( r_1 = \frac{3}{2} \) and \( r_2 = \frac{11}{2} \).

(1) An example where \( \ell > a \geq b \): Let \( \ell = 8 \) and \( k = 9 \), then
\[
\lambda = (10,8) = \begin{array}{cccccccc}
\bullet & | & \bullet & | & \bullet & | & \bullet & | & \bullet & | & \bullet & | & \bullet
\end{array}
\]
with \( S_{\text{max}}^{(0)} = (6,3) \).

The boxes of \( \lambda/S_{\text{max}}^{(0)} \) have

shifting contents:
\[
\begin{array}{cccc}
\begin{array}{cc}
3 & 4 \\
2 & 1
\end{array}
\end{array}
\begin{array}{cccc}
\begin{array}{cc}
7 & 5 \\
6 & 3
\end{array}
\end{array}
\]

Then \( c \) is the rearrangement of the absolute values of \( (-\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}) \) into increasing order and \( J = \{\varepsilon_3, \varepsilon_3 - \varepsilon_2, \varepsilon_5 - \varepsilon_4, \varepsilon_7 - \varepsilon_6\} \). The configuration of boxes \( \kappa \) corresponding to \( (c,J) \) has indexing of boxes
\[
\begin{array}{cccccccc}
-6 & -4 & -2 & 1 & 3 & 5 & 7 & 8 & 9
\end{array}
\]
\[
\begin{array}{cccccccc}
-9 & -8 & -7 & -5 & -3 & -1 & 2 & 4 & 6
\end{array}
\]
with \( P(c) = \{\varepsilon_3, \varepsilon_9, \varepsilon_7 + \varepsilon_1, \varepsilon_3 - \varepsilon_2, \varepsilon_3 - \varepsilon_1, \varepsilon_5 - \varepsilon_4, \varepsilon_5 - \varepsilon_3, \varepsilon_6 - \varepsilon_4, \varepsilon_6 - \varepsilon_3, \varepsilon_7 - \varepsilon_6, \varepsilon_7 - \varepsilon_5, \varepsilon_8 - \varepsilon_7, \varepsilon_9 - \varepsilon_8\} \).
(2) An example with $a \geq \ell > b$: Let $k = 3$ and $\ell = 5$, so that $a + b + k - \ell = 7$.

\[
\lambda = (7, 5) = \begin{array}{ccc}
\ast & \ast & \\
\ast & \ast & \\
\ast & \ast & \\
\end{array}
\quad \text{with } S_{\text{max}}^{(0)} = (6, 3).
\]

The boxes of $\lambda/S_{\text{max}}^{(0)}$ have

shifted contents: \[
\begin{array}{ccc}
\ast & \ast & \\
\ast & \ast & \\
\ast & \ast & \\
\end{array}
\]

Then $c$ is the rearrangement of the absolute values of $(\frac{1}{2}, \frac{3}{2}, \frac{5}{2})$ in increasing order and $J = \{\varepsilon_2\}$. The configuration of boxes $\kappa$ corresponding to $(c, J)$ is

\[
\begin{array}{ccc}
1 & 2 & 3 \\
-3 & -2 & -1 \\
\end{array}
\quad \text{with } P(c) = \{\varepsilon_2, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2\}.
\]

(3) An example with $a \geq b \geq \ell$: Let $k = 3$ and $\ell = 2$, so that $a + b + k - \ell = 10$. Then

\[
\lambda = (10, 2) = \begin{array}{ccc}
\ast & \ast & \\
\ast & \ast & \\
\ast & \ast & \\
\ast & \ast & \\
\end{array}
\quad \text{with } S_{\text{max}}^{(0)} = (7, 2).
\]

The boxes of $\lambda/S_{\text{max}}^{(0)}$ have

shifted contents: \[
\begin{array}{ccc}
\ast & \ast & \\
\ast & \ast & \\
\ast & \ast & \\
\ast & \ast & \\
\end{array}
\]

Then $c$ is the rearrangement of the absolute values of $(\frac{7}{2}, \frac{9}{2}, \frac{11}{2})$ in increasing order and $J = \emptyset$. The configuration of boxes $\kappa$ corresponding to $(c, J)$ is

\[
\begin{array}{ccc}
-3 & -2 & -1 \\
1 & 2 & 3
\end{array}
\quad \text{with } P(c) = \{\varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2\}.
\]
Figure 2. The Temperley-Lieb Bratteli diagram for $a = 6$ and $b = 3$, levels 0–9. Partitions $\lambda = (a + b + k - \ell, \ell)$ are labeled by $\ell$. The dimension formulas are consequences of Remark 5.2.

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