Kostka functions associated to complex reflection groups

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Abstract. Kostka functions $K_{\lambda,\mu}^\pm(t)$ associated to complex reflection groups are a generalization of Kostka polynomials, which are indexed by a pair $\lambda, \mu$ of $r$-partitions of $n$ (and by the sign $+$, $-$). It is expected that there exists a close relationship between those Kostka functions and the intersection cohomology associated to the enhanced variety $X$ of level $r$. In this paper, we study combinatorial properties of $K_{\lambda,\mu}^\pm(t)$ based on the geometry of $X$. In particular, we show that in the case where $\mu = (\cdots,-,\cdots,\mu(r))$ (and for arbitrary $\lambda$), $K_{\lambda,\mu}^-(t)$ has a Lascoux-Schützenberger type combinatorial description.

Introduction

In 1981, Lusztig gave a geometric interpretation of Kostka polynomials in the following sense; let $V$ be an $n$-dimensional vector space over an algebraically closed field, and put $G = GL(V)$. Let $P_n$ be the set of partitions of $n$. Let $O_\lambda$ be the unipotent class in $G$ labelled by $\lambda \in P_n$, and $K = IC(O_\lambda, \bar{Q}_l)$ the intersection cohomology associated to the closure $O_\lambda$ of $O_\lambda$. Let $K_{\lambda,\mu}(t)$ be the Kostka polynomial indexed by $\lambda, \mu \in P_n$, and $\tilde{K}_{\lambda,\mu}(t) = t^{n(\mu)} K_{\lambda,\mu}(t^{-1})$ the modified Kostka polynomial (see 1.1 for the definition $n(\mu)$). Lusztig proved that

$$\tilde{K}_{\lambda,\mu}(t) = t^{n(\lambda)} \sum_{i \geq 0} \dim(\mathcal{H}_x^{2i}K) t^i$$

for $x \in O_\mu \subset O_\lambda$, where $\mathcal{H}_x^{2i}K$ is the stalk at $x$ of the $2i$-th cohomology sheaf $\mathcal{H}^{2i}K$ of $K$. (0.1) implies that $K_{\lambda,\mu}(t) \in \mathbb{Z}_{\geq 0}[t]$.

Let $P_{n,r}$ be the set of $r$-tuple of partitions $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ such that $\sum_{i=1}^r |\lambda^{(i)}| = n$ (we write $|\lambda^{(i)}| = m$ if $\lambda^{(i)} \in P_m$). In [S1], [S2], Kostka functions $K_{\lambda,\mu}^\pm(t)$ associated to complex reflections groups (depending on the signs $+$, $-$) are introduced, which are apriori rational functions in $t$ indexed by $\lambda, \mu \in P_{n,r}$. In the case where $r = 2$ (in this case $K_{\lambda,\mu}^-(t) = K_{\lambda,\mu}^+(t)$), it is proved in [S2] that $K_{\lambda,\mu}^\pm(t) \in \mathbb{Z}[t]$. In this case, Achar-Henderson [AH] proved that those (generalized) Kostka polynomials have a geometric interpretation in the following sense; under the previous notation, consider the variety $X = G \times V$ on which $G$ acts naturally. Put $X_{uni} = G_{uni} \times V$, where $G_{uni}$ is the set of unipotent elements in $G$. $X_{uni}$ is a $G$-stable subset of $X$, and is isomorphic to the enhanced nilpotent cone introduced by [AH]. It is known by [AH], [T] that $X_{uni}$ has finitely many $G$-orbits, which are naturally parametrized by $P_{n,2}$. They proved in [AH] that the modified Kostka polynomial $\tilde{K}_{\lambda,\mu}^\pm(t)$ ($\lambda, \mu \in P_{n,2}$), defined in a similar way as in the original case,
can be written as in (0.1) in terms of the intersection cohomology associated to the closure $\overline{O}_\lambda$ of the $G$-orbit $O_\lambda \subset X_{\text{uni}}$.

In the case where $r = 2$, the interaction of geometric properties and combinatorial properties of Kostka polynomials was studied in [LS]. In particular, it was proved that in the special case where $\mu = (-, \mu^{(2)})$ (and for arbitrary $\lambda \in P_{n,2}$), $K_{\lambda,\mu}(t)$ has a combinatorial description analogous to Lascoux-Schützenberger theorem for the original Kostka polynomials ([M, III, (6.5)]).

We now consider the variety $X = G \times V^{r-1}$ for an integer $r \geq 1$, on which $G$ acts diagonally, and let $X_{\text{uni}} = G_{\text{uni}} \times V^{r-1}$ be the $G$-stable subset of $X$. The variety $X$ is called the enhanced variety of level $r$. In [S4], the relationship between Kostka functions $K_{\lambda,\mu}(t)$ indexed by $\lambda, \mu \in P_{n,r}$ and the geometry of $X_{\text{uni}}$ was studied. In contrast to the case where $r = 1, 2$, $X_{\text{uni}}$ has infinitely many $G$-orbits if $r \geq 3$. A partition $X_{\text{uni}} = \bigsqcup_{\lambda \in P_{n,r}} X_\lambda$ into $G$-stable pieces $X_\lambda$ was constructed in [S3], and some formulas expressing the Kostka functions in terms of the intersection cohomology associated to the closure of $X_\lambda$ were obtained in [S4], though it is a partial generalization of the result of Achar-Henderson for the case $r = 2$.

In this paper, we prove a formula (Theorem 2.6) which is a generalization of the formula in [AH, Theorem 4.5] (and also in [FGT (11)]) to arbitrary $r$. Combined this formula with the results in [S4], we extend some results in [LS] to arbitrary $r$. In particular, we show in the special case where $\mu = (-, \ldots, -, \mu^{(r)}) \in P_{n,r}$ (and for arbitrary $\lambda \in P_{n,r}$) that $K_{-\lambda,\mu}(t)$ has a Lasacoux-Schützenberger type combinatorial description.

1. Review on Kostka functions

1.1. First we recall basic properties of Hall-Littlewood functions and Kostka polynomials in the original setting, following [M]. Let $\Lambda = \Lambda(y) = \bigoplus_{n \geq 0} \Lambda^n$ be the ring of symmetric functions over $\mathbb{Z}$ with respect to the variables $y = (y_1, y_2, \ldots)$, where $\Lambda^n$ denotes the free $\mathbb{Z}$-module of symmetric functions of degree $n$. We put $\Lambda^Q = \mathbb{Q} \otimes_\mathbb{Z} \Lambda$, $\Lambda^Q_n = \mathbb{Q} \otimes_\mathbb{Z} \Lambda^n$. Let $s_\lambda$ be the Schur function associated to $\lambda \in P_n$. Then $\{s_\lambda \mid \lambda \in P_n\}$ gives a $\mathbb{Z}$-basis of $\Lambda^n$. Let $p_\lambda \in \Lambda^n$ be the power sum symmetric function associated to $\lambda \in P_n$,

$$p_\lambda = \prod_{i=1}^k p_{\lambda_i},$$

where $p_m$ denotes the $m$-th power sum symmetric function for each integer $m > 0$. Then $\{p_\lambda \mid \lambda \in P_n\}$ gives a $\mathbb{Q}$-basis of $\Lambda^Q_n$. For $\lambda = (1^{n_1}, 2^{n_2}, \ldots) \in P_n$, define an integer $z_\lambda$ by

$$z_\lambda = \prod_{i \geq 1} i^{m_i} m_i !.$$  

(1.1.1)

Following [M, I], we introduce a scalar product on $\Lambda^Q$ by $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda$. It is known that $\{s_\lambda\}$ form an orthonormal basis of $\Lambda$. 


Let $P_\lambda(y;t)$ be the Hall-Littlewood function associated to a partition $\lambda$. Then 
$$\{P_\lambda \mid \lambda \in \mathcal{P}_n\}$$
gives a $\mathbb{Z}[t]$-basis of $\Lambda^n[t] = \mathbb{Z}[t] \otimes_{\mathbb{Z}} \Lambda^n$, where $t$ is an indeterminate. 
Kostka polynomials $K_{\lambda,\mu}(t) \in \mathbb{Z}[t]$ ($\lambda, \mu \in \mathcal{P}_n$) are defined by the formula 
\begin{equation}
(1.1.2)
  s_\lambda(y) = \sum_{\mu \in \mathcal{P}_n} K_{\lambda,\mu}(t) P_\mu(y;t).
\end{equation}

Recall the dominance order $\lambda \geq \mu$ in $\mathcal{P}_n$, which is defined by the condition 
$\sum_{j=1}^i \lambda_j \geq \sum_{j=1}^i \mu_j$ for each $i \geq 1$. For each partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, we define an 
integer $n(\lambda)$ by $n(\lambda) = \sum_{i=1}^k (i-1) \lambda_i$. It is known that $K_{\lambda,\mu}(t) = 0$ unless $\lambda \geq \mu$, 
and that $K_{\lambda,\mu}(t)$ is a monic of degree $n(\mu) - n(\lambda)$ if $\lambda \geq \mu$ ([M, III, (6.5)]). Put 
$\widetilde{K}_{\lambda,\mu}(t) = t^{n(\mu)} K_{\lambda,\mu}(t^{-1})$. Then $\widetilde{K}_{\lambda,\mu}(t) \in \mathbb{Z}[t]$, which we call the modified Kostka 
polynomial.

For $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{P}_n$ with $\lambda_k > 0$, we define $z_\lambda(t) \in \mathbb{Q}(t)$ by 
\begin{equation}
(1.1.3)
  z_\lambda(t) = z_\lambda \prod_{i \geq 1} (1 - t^{\lambda_i})^{-1},
\end{equation}
where $z_\lambda$ is as in (1.1.1). Following [M, III], we introduce a scalar product on 
$\Lambda_\mathbb{Q}(t) = \mathbb{Q}(t) \otimes_{\mathbb{Z}} \Lambda$ by $\langle P_\lambda, p_\mu \rangle = z_\lambda(t) \delta_{\lambda,\mu}$. Then $P_\lambda(y;t)$ form an orthogonal basis 
of $\Lambda[t] = \mathbb{Z}[t] \otimes_{\mathbb{Z}} \Lambda$. In fact, they are characterized by the following two properties 
([M, III, (2.6) and (4.9)]); 
\begin{equation}
(1.1.4)
  P_\lambda(y;t) = s_\lambda(y) + \sum_{\mu < \lambda} w_{\lambda\mu}(t) s_\mu(y)
\end{equation}
with $w_{\lambda\mu}(t) \in \mathbb{Z}[t]$, and 
\begin{equation}
(1.1.5)
  \langle P_\lambda, P_\mu \rangle = 0 \text{ unless } \lambda = \mu.
\end{equation}

1.2. We fix a positive integer $r$. Let $\Xi = \Xi(x) \simeq \Lambda(x^{(1)}) \otimes \cdots \otimes \Lambda(x^{(r)})$ be the ring of symmetric functions over $\mathbb{Z}$ with respect to variables $x = (x^{(1)}, \ldots, x^{(r)})$, 
where $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, \ldots)$. We denote it as $\Xi = \bigoplus_{n \geq 0} \Xi^n$, similarly to the case 
of $\Lambda$. Let $\mathcal{P}_{n,r}$ be as in Introduction. For $\lambda \in \mathcal{P}_{n,r}$, we define a Schur function 
$s_\lambda(x) \in \Xi^n$ by 
\begin{equation}
(1.2.1)
  s_\lambda(x) = s_{\lambda^{(1)}}(x^{(1)}) \cdots s_{\lambda^{(r)}}(x^{(r)}).
\end{equation}
Then $\{s_\lambda \mid \lambda \in \mathcal{P}_{n,r}\}$ gives a $\mathbb{Z}$-basis of $\Xi^n$. Let $\zeta$ be a primitive $r$-th root of unity 
in $\mathbb{C}$. For an integer $m \geq 1$ and $k$ such that $1 \leq k \leq r$, put 
$$p_{m}^{(k)}(x) = \sum_{j=1}^{r} \zeta^{(k-1)(j-1)} p_{m}(x^{(j)}),$$
where \( p_m(x^{(j)}) \) denotes the \( m \)-th power sum symmetric function with respect to the variables \( x^{(j)} \). For \( \lambda \in \mathcal{P}_{n,r} \), we define \( p_\lambda(x) \in \Xi^n_C = \Xi^n \otimes Z \ C \) by

\[
(1.2.2) \quad p_\lambda(x) = \prod_{k=1}^r \prod_{j=1}^{m_k} p_{\lambda_{j}}^{(k)}(x),
\]

where \( \lambda^{(k)} = (\lambda_1^{(k)}, \ldots, \lambda_m^{(k)}) \) with \( \lambda_m^{(k)} > 0 \). Then \( \{p_\lambda \mid \lambda \in \mathcal{P}_{n,r}\} \) gives a \( C \)-basis of \( \Xi^n_C \). For a partition \( \lambda^{(k)} \) as above, we define a function \( z_{\lambda^{(k)}}(t) \in C(t) \) by

\[
z_{\lambda^{(k)}}(t) = \prod_{j=1}^{m_k} (1 - \xi^{k-1} t^{\lambda_{j}})^{-1}.
\]

For \( \lambda \in \mathcal{P}_{n,r} \), we define an integer \( z_{\lambda} \) by \( z_{\lambda} = \prod_{k=1}^r r^{m_k} z_{\lambda^{(k)}} \), where \( z_{\lambda^{(k)}} \) is as in (1.1.1). We now define a function \( z_\lambda(t) \in C(t) \) by

\[
(1.2.3) \quad z_\lambda(t) = z_{\lambda} \prod_{k=1}^r z_{\lambda^{(k)}}(t).
\]

Let \( \Xi[t] = Z[t] \otimes Z \Xi \) be the free \( Z[t] \)-module, and \( \Xi_C(t) = C(t) \otimes Z \Xi \) be the \( C(t) \)-space. Then \( \{p_\lambda(x) \mid \lambda \in \mathcal{P}_{n,r}\} \) gives a basis of \( \Xi^n_C(t) \). We define a sesquilinear form on \( \Xi^n_C(t) \) by

\[
(1.2.4) \quad \langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} z_\lambda(t).
\]

We express an \( r \)-partition \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \) as \( \lambda^{(k)} = (\lambda_1^{(k)}, \ldots, \lambda_m^{(k)}) \) with a common \( m \), by allowing zero on parts \( \lambda_j^{(i)} \), and define a composition \( c(\lambda) \) of \( n \) by

\[
c(\lambda) = (\lambda_1^{(1)}, \ldots, \lambda_1^{(r)}, \lambda_2^{(1)}, \ldots, \lambda_2^{(r)}, \ldots, \lambda_m^{(1)}, \ldots, \lambda_m^{(r)}).
\]

We define a partial order \( \lambda \geq \mu \) on \( \mathcal{P}_{n,r} \) by the condition \( c(\lambda) \geq c(\mu) \), where \( \geq \) is the dominance order on the set of compositions of \( n \) defined in a similar way as in the case of partitions. We fix a total order \( \lambda > \mu \) on \( \mathcal{P}_{n,r} \) compatible with the partial order \( \lambda > \mu \).

The following result was proved in Theorem 4.4 and Proposition 4.8 in [S1], combined with [S2, §3].

**Proposition 1.3.** For each \( \lambda \in \mathcal{P}_{n,r} \), there exist unique functions \( P^\pm_\lambda(x;t) \in \Xi^n_Q(t) \) (depending on the signs \( +, - \)) satisfying the following properties.

(i) \( P^\pm_\lambda(x;t) \) can be written as

\[
P^\pm_\lambda(x;t) = s_\lambda(x) + \sum_{\mu < \lambda} u^\pm_{\lambda,\mu}(t)s_\mu(x)
\]

with \( u^\pm_{\lambda,\mu}(t) \in Q(t) \).
(ii) \( \langle P_\lambda^-, P_\mu^+ \rangle = 0 \) unless \( \lambda = \mu \).

1.4. \( P_\lambda^\pm(x; t) \) are called Hall-Littlewood functions associated to \( \lambda \in \mathcal{P}_{n,r} \). By Proposition 1.3, for \( \varepsilon \in \{+, -\} \), \( \{ P_\lambda^\varepsilon \mid \lambda \in \mathcal{P}_{n,r} \} \) gives a \( \mathbb{Q}(t) \)-basis for \( \Xi_{\mathbb{Q}}(t) \). For \( \lambda, \mu \in \mathcal{P}_{n,r} \), we define functions \( K_{\lambda, \mu}^\pm(t) \in \mathbb{Q}(t) \) by

\[
(1.4.1) \quad s_\lambda(x) = \sum_{\mu \in \mathcal{P}_{n,r}} K_{\lambda, \mu}^\pm(t) P_\mu^\pm(x; t).
\]

\( K_{\lambda, \mu}^\pm(t) \) are called Kostka functions associated to complex reflection groups since they are closely related to the complex reflection group \( S_n \times (\mathbb{Z}/r\mathbb{Z})^n \) (see [S1, Theorem 5.4]). For each \( \lambda \in \mathcal{P}_{n,r} \), by putting \( n(\lambda) = n(\lambda^{(1)}) + \cdots + n(\lambda^{(r)}) \), we define an \( a \)-function \( a(\lambda) \) on \( \mathcal{P}_{n,r} \) by

\[
(1.4.2) \quad a(\lambda) = r \cdot n(\lambda) + |\lambda^{(2)}| + 2|\lambda^{(3)}| + \cdots + (r - 1)|\lambda^{(r)}|.
\]

We define modified Kostka functions \( \tilde{K}_{\lambda, \mu}^\pm(t) \) by

\[
(1.4.3) \quad \tilde{K}_{\lambda, \mu}^\pm(t) = t^{a(\mu)} K_{\lambda, \mu}^\pm(t^{-1}).
\]

Remark 1.5. In the case where \( r = 1 \), \( P_\lambda^\pm(x; t) \) coincides with the original Hall-Littlewood function given in 1.1. In the case where \( r = 2 \), it is proved by [S2, Prop. 3.3] that \( P_{\lambda^-}^- = P_{\lambda^+}^+ \), hence \( K_{\lambda, \mu}^- = K_{\lambda, \mu}^+ \in \mathbb{Z}[t] \). Moreover it is shown that \( K_{\lambda, \mu}^\pm(t) \in \mathbb{Z}[t] \), which is a monic of degree \( a(\mu) - a(\lambda) \). Thus \( \tilde{K}_{\lambda, \mu}^\pm(t) \in \mathbb{Z}[t] \). As mentioned in Introduction \( \tilde{K}_{\lambda, \mu}^\pm(t) \) has a geometric interpretation, which imples that \( K_{\lambda, \mu}^\pm(t) \), and so \( P_{\lambda}^\pm(x; t) \) are independent of the choice of the total order \( \prec \) on \( \mathcal{P}_{n,r} \). In the case where \( r \geq 3 \), it is not known whether Hall-Littlewood functions do not depend on the choice of the total order \( \prec \), whether \( K_{\lambda, \mu}^\pm(t) \) are polynomials in \( t \).

2. Enhanced variety of level \( r \)

2.1. Let \( V \) be an \( n \)-dimensional vector space over an algebraic closure \( k \) of a finite field \( F_q \), and \( G = GL(V) \simeq GL_n \). Let \( B = TU \) be a Borel subgroup of \( G \), \( T \) a maximal torus and \( U \) the unipotent radical of \( B \). Let \( W = N_G(T)/T \) be the Weyl group of \( G \), which is isomorphic to the symmetric group \( S_n \). By fixing an integer \( r \geq 1 \), put \( \mathcal{B} = G \times V^{r-1} \) and \( \mathcal{B}_{\text{uni}} = G_{\text{uni}} \times V^{r-1} \), where \( G_{\text{uni}} \) is the set of unipotent elements in \( G \). The variety \( \mathcal{B} \) is called the enhanced variety of level \( r \). We consider the diagonal action of \( G \) on \( \mathcal{B} \). Put \( \mathcal{B}_{n,r} = \{ \mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}_{\geq 0}^r \mid \sum m_i = n \} \). For each \( \mathbf{m} \in \mathcal{B}_{n,r} \), we define integers \( p_i = p_i(\mathbf{m}) \) by \( p_i = m_1 + \cdots + m_i \) for \( i = 1, \ldots, r \). Let \( (M_i)_{1 \leq i \leq n} \) be the total flag in \( V \) whose stabilizer in \( G \) coincides with \( B \). We define varieties
\[ \tilde{X}_m = \{(x, v, gB) \in G \times V^{r-1} \times G/B \mid g^{-1}xg \in B, g^{-1}v \in \prod_{i=1}^{r-1} M_{p_i} \}, \]

\[ \mathcal{X}_m = \bigcup_{g \in G} g(B \times \prod_{i=1}^{r-1} M_{p_i}), \]

and the map \( \pi_m : \tilde{X}_m \to \mathcal{X}_m \) by \((x, v, gB) \mapsto (x, v)\). We also define the varieties

\[ \tilde{X}_{m, \text{uni}} = \{(x, v, gB) \in G_{\text{uni}} \times V^{r-1} \times G/B \mid g^{-1}xg \in U, g^{-1}v \in \prod_{i=1}^{r-1} M_{p_i} \}, \]

\[ \mathcal{X}_m = \bigcup_{g \in G} g(U \times \prod_{i=1}^{r-1} M_{p_i}), \]

and the map \( \pi_{m, 1} : \tilde{X}_{m, \text{uni}} \to \mathcal{X}_{m, \text{uni}} \), similarly. Note that in the case where \( m = (n, 0, \ldots, 0) \), \( \tilde{X}_m \) (resp. \( \tilde{X}_{m, \text{uni}} \)) coincides with \( \tilde{X} \) (resp. \( \tilde{X}_{\text{uni}} \)). In that case, we denote \( \tilde{X}_m, \pi_m, \) etc. by \( \tilde{X}, \pi, \) etc. by omitting the symbol \( m \). (Note: here we follow the notation in [S4], but, in part, it differs from [S3]. In [S3], our \( \pi_m, \pi_{m, 1} \) are denoted by \( \pi^{(m)}, \pi_1^{(m)} \) for the consistency with the exotic case).

2.2. In [S3, 5.3], a partition of \( \mathcal{X}_{\text{uni}} \) into pieces \( X_\lambda \) is defined

\[ \mathcal{X}_{\text{uni}} = \bigsqcup_{\lambda \in \mathcal{P}_{n, r}} X_\lambda, \]

where \( X_\lambda \) is a locally closed, smooth irreducible, \( G \)-stable subvariety of \( \mathcal{X}_{\text{uni}} \). If \( r = 1 \) or \( 2 \), \( X_\lambda \) is a single \( G \)-orbit. However, if \( r \geq 3 \), \( X_\lambda \) is in general a union of infinitely many \( G \)-orbits.

For \( m \in \mathcal{P}_{n, r} \), let \( W_m = S_{m_1} \times \cdots \times S_{m_r} \) be the Young subgroup of \( W = S_n \). For \( m \in \mathcal{P}_{n, r} \), we denote by \( \mathcal{P}(m) \) the set of \( \lambda \in \mathcal{P}_{n, r} \) such that \(|\lambda^{(i)}| = m_i \). The (isomorphism classes of) irreducible representations (over \( \bar{\mathbb{Q}}_l \)) of \( W_m \) are parametrized by \( \mathcal{P}(m) \). We denote by \( V_\lambda \) an irreducible representation of \( W_m \) corresponding to \( \lambda \), namely \( V_\lambda = V_{\lambda^{(1)}} \otimes \cdots \otimes V_{\lambda^{(r)}} \), where \( V_\mu \) denotes the irreducible representation of \( S_n \) corresponding to the partition \( \mu \) of \( n \). (Here we use the parametrization such that \( V_{(n)} \) is the trivial representation of \( S_n \)). The following results were proved in [S3].

**Theorem 2.3** ([S3, Thm. 4.5]). Put \( d_m = \dim \mathcal{X}_m \). Then \((\pi_m)_* \mathcal{Q}_l[d_m] \) is a semisimple perverse sheaf equipped with the action of \( W_m \), and is decomposed as

\[ (\pi_m)_* \mathcal{Q}_l[d_m] \cong \bigoplus_{\lambda \in \mathcal{P}(m)} V_\lambda \otimes \text{IC}(\mathcal{X}_m, \mathcal{L}_\lambda)[d_m], \]

where \( \mathcal{L}_\lambda \) is a simple local system on a certain open dense subvariety of \( \mathcal{X}_m \).
Theorem 2.4 ([S3, Thm. 8.13, Thm. 7.12]). Put \( d'_m = \dim \mathcal{X}_{m, \text{uni}} \).

(i) \( (\pi_{m,1})_* \mathcal{Q}_t[d'_m] \) is a semisimple perverse sheaf equipped with the action of \( W_m \), and is decomposed as

\[
(\pi_{m,1})_* \mathcal{Q}_t[d'_m] \cong \bigoplus_{\lambda \in \mathcal{P}(m)} V_{\lambda} \otimes \text{IC}(\overline{\mathcal{X}}_{\lambda}, \mathcal{Q}_t)[\dim X_{\lambda}].
\]

(ii) We have \( \text{IC}(\mathcal{X}_{m, \text{uni}}, \mathcal{L})|_{\mathcal{X}_{m, \text{uni}}} \cong \text{IC}(\overline{\mathcal{X}}_{\lambda}, \mathcal{Q}_t)[\dim X_{\lambda} - d'_m] \).

2.5. For a partition \( \lambda \), we denote by \( \lambda^t \) the dual partition of \( \lambda \). For \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \in \mathcal{P}(m) \), we define \( \lambda^t \in \mathcal{P}(m) \) by \( \lambda^t = ((\lambda^{(1)})^t, \ldots, (\lambda^{(r)})^t) \). Assume that \( \lambda \in \mathcal{P}(m) \). We write \( (\lambda^{(i)})^t \) as \( (\mu_1^{(i)} \leq \mu_2^{(i)} \leq \cdots \leq \mu_{\ell_i}^{(i)}) \), in the increasing order, where \( \ell_i = \lambda^{(i)} \). For each \( 1 \leq i \leq r, 1 \leq j < \ell_i \), we define an integer \( n(i, j) \) by

\[
n(i, j) = (|\lambda^{(1)}| + \cdots + |\lambda^{(i-1)}|) + \mu_1^{(i)} + \cdots + \mu_j^{(i)}.
\]

Let \( Q = Q_\lambda \) be the stabilizer of the partial flag \( (M_{n(i, j)}) \) in \( G \), and \( U_Q \) the unipotent radical of \( Q \). In particular, \( Q \) stabilizes the subspaces \( M_{n(i, j)} \). Let us define a variety \( \tilde{\mathcal{X}}_{\lambda} \) by

\[
\tilde{\mathcal{X}}_{\lambda} = \{(x, v, gQ) \in G_{\text{uni}} \times V^{r-1} \times G/Q \mid g^{-1}xg \in U_Q, g^{-1}v \in \prod_{i=1}^{r-1} M_{n(i, j)} \}.
\]

We define a map \( \pi_{\lambda} : \tilde{\mathcal{X}}_{\lambda} \to \mathcal{X}_{\text{uni}} \) by \( (x, v, gQ) \mapsto (x, v) \). Then \( \pi_{\lambda} \) is a proper map. Since \( \tilde{\mathcal{X}}_{\lambda} \cong G \times^Q (U_Q \times \prod_i M_{n(i, j)}) \), \( \tilde{\mathcal{X}}_{\lambda} \) is smooth and irreducible. It is known by [S3, Lemma 5.6] that \( \dim \tilde{\mathcal{X}}_{\lambda} = \dim X_{\lambda} \) and that \( \text{Im} \pi_{\lambda} \) coincides with \( \overline{\mathcal{X}}_{\lambda} \), the closure of \( X_{\lambda} \) in \( \mathcal{X}_{\text{uni}} \).

For \( \lambda, \mu \in \mathcal{P}_n \), let \( K_{\lambda, \mu} = K_{\lambda, \mu}(1) \) be the Kostka number. We have \( K_{\lambda, \mu} = 0 \) unless \( \lambda \geq \mu \). For \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \), \( \mu = (\mu^{(1)}, \ldots, \mu^{(r)}) \in \mathcal{P}(m) \), we define an integer \( K_{\lambda, \mu} \) by

\[
K_{\lambda, \mu} = K_{\lambda^{(1)}, \mu^{(1)}} K_{\lambda^{(2)}, \mu^{(2)}} \cdots K_{\lambda^{(r)}, \mu^{(r)}}.
\]

We define a partial order \( \lambda \geq \mu \) in \( \mathcal{P}_{n,r} \) by the condition \( \lambda^{(i)} \geq \mu^{(i)} \) for \( i = 1, \ldots, r \). Hence \( \lambda \geq \mu \) implies that \( \lambda, \mu \in \mathcal{P}(m) \) for a common \( m \). We have \( K_{\lambda, \mu} = 0 \) unless \( \lambda \geq \mu \). Note that \( \lambda \geq \mu \) implies that \( \mu^t \geq \lambda^t \). We show the following theorem. In the case where \( r = 2 \), this result was proved by [AH, Thm. 4.5].

Theorem 2.6. Assume that \( \lambda \in \mathcal{P}_{n,r} \). Then \( (\pi_{\lambda})_* \mathcal{Q}_t[\dim X_{\lambda}] \) is a semisimple perverse sheaf on \( \overline{\mathcal{X}}_{\lambda} \), and is decomposed as

\[
(\pi_{\lambda})_* \mathcal{Q}_t[\dim X_{\lambda}] \cong \bigoplus_{\mu \preceq \lambda} K_{\mu^t, \lambda^t} \mathcal{Q}_t \otimes \text{IC}(\overline{\mathcal{X}}_{\mu^t}, \mathcal{Q}_t)[\dim X_{\mu^t}].
\]
2.7. The rest of this section is devoted to the proof of Theorem 2.6. First we consider the case where \( r = 1 \). Actually, the result in this case is contained in [AH]. Their proof (for \( r = 2 \)) depends on the result of Spaltenstein [Sp] concerning the “Springer fibre” \((\pi_\lambda)^{-1}(z)\) for \( z \in \mathfrak{X}_\lambda \) in the case \( r = 1 \). In the following, we give an alternate proof independent of [Sp] for the later use. Let \( Q \) be a parabolic subgroup of \( G \) containing \( B, M \) the Levi subgroup of \( Q \) containing \( T \) and \( U_Q \) the unipotent radical of \( Q \). (In this stage, this \( Q \) is independent of \( Q \) in 2.5.) Let \( W_Q \) be the Weyl subgroup of \( W \) corresponding to \( Q \). Let \( G_{\text{reg}} \) be the set of regular semisimple elements in \( G \), and put \( T_{\text{reg}} = G_{\text{reg}} \cap T \). Consider the map \( \psi : \tilde{G}_{\text{reg}} \to G_{\text{reg}}, \) where

\[
\tilde{G}_{\text{reg}} = \{(x, gT) \in G_{\text{reg}} \times G/T \mid g^{-1}xg \in T_{\text{reg}}\}
\]

and \( \psi : (x, gT) \mapsto x \). Then \( \psi \) is a finite Galois covering with group \( W \). We also consider a variety

\[
\tilde{G}_{\text{reg}}^M = \{(x, gM) \in G_{\text{reg}} \times G/M \mid g^{-1}xg \in M_{\text{reg}}\},
\]

where \( M_{\text{reg}} = G_{\text{reg}} \cap M \). The map \( \psi \) is decomposed as

\[
\psi : \tilde{G}_{\text{reg}} \xrightarrow{\psi'} \tilde{G}_{\text{reg}}^M \xrightarrow{\psi''} G_{\text{reg}},
\]

where \( \psi' : (x, gT) \mapsto (x, gM) \), \( \psi'' : (x, gM) \mapsto x \). Here \( \psi' \) is a finite Galois covering with group \( W_Q \). Now \( \psi_* \tilde{Q}_l \) is a semisimple local system on \( G_{\text{reg}} \) such that \( \text{End}(\psi_* \tilde{Q}_l) \simeq \tilde{Q}_l[W] \), and is decomposed as

\[
(2.7.1) \quad \psi_* \tilde{Q}_l \simeq \bigoplus_{\rho \in W^\wedge} \rho \otimes L_{\rho},
\]

where \( L_{\rho} = \text{Hom}_W(\rho, \psi_* \tilde{Q}_l) \) is a simple local system on \( G_{\text{reg}} \). We also have

\[
(2.7.2) \quad \psi'_* Q_l \simeq \bigoplus_{\rho' \in W_Q^\wedge} \rho' \otimes L_{\rho'},
\]

where \( L_{\rho'} \) is a simple local system on \( \tilde{G}_{\text{reg}}^M \). Hence

\[
(2.7.3) \quad \psi_* \tilde{Q}_l \simeq \psi''_* \psi'_* Q_l \simeq \bigoplus_{\rho' \in W_Q^\wedge} \rho' \otimes \psi''_* L_{\rho'}.
\]

(2.7.3) gives a decomposition of \( \psi_* \tilde{Q}_l \) with respect to the action of \( W_Q \). Comparing (2.7.1) and (2.7.3), we have
(2.7.4) 
\[ \psi_* \mathcal{L}'' \simeq \bigoplus_{\rho \in \mathcal{W}^\times} \mathcal{Q}_t^{(\rho, \rho')} \otimes \mathcal{L}_{\rho'} \]

where \((\rho : \rho')\) is the multiplicity of \(\rho'\) in the restricted \(W_Q\)-module \(\rho\).

We consider the map \(\pi : \tilde{G} \to G\), where

\[ \tilde{G} = \{(x, gB) \in G \times G/B \mid g^{-1}xg \in B\} \simeq G \times^B B, \]
and \(\pi : (x, gB) \mapsto x\). We also consider

\[ \tilde{G}^Q = \{(x, gQ) \in G \times G/Q \mid g^{-1}xg \in Q\} \simeq G \times^Q Q. \]

The map \(\pi\) is decomposed as

\[ \pi : \tilde{G} \xrightarrow{\pi'} \tilde{G}^Q \xrightarrow{\pi''} G, \]

where \(\pi' : (x, gB) \mapsto (x, gQ)\), \(\pi'' : (x, gQ) \mapsto x\). It is well-known ([L1]) that

(2.7.5) 
\[ \pi_* \mathcal{Q}_t \simeq \bigoplus_{\rho \in \mathcal{W}^\times} \rho \otimes \text{IC}(G, \mathcal{L}_\rho). \]

Let \(B_M = B \cap M\) be the Borel subgroup of \(M\) containing \(T\). We consider the following commutative diagram

(2.7.6) 
\[
\begin{array}{ccc}
G \times^B B & \xleftarrow{\tilde{p}} & G \times (Q \times^B B) & \xrightarrow{\tilde{q}} & M \times^{B_M} B_M \\
\pi' \downarrow & & \downarrow r & & \downarrow \pi^M \\
G \times^Q Q & \xleftarrow{p} & G \times Q & \xrightarrow{q} & M,
\end{array}
\]

where under the identification \(G \times^B B \simeq G \times^Q (Q \times^B B)\), the maps \(p, \tilde{p}\) are defined by the quotient by \(Q\). The map \(q\) is a projection to the \(M\)-factor of \(Q\), and \(\tilde{q}\) is the map induced from the projection \(Q \times B \to M \times B_M\). \(\pi^M\) is defined similarly to \(\pi\) replacing \(G\) by \(M\). The map \(r\) is defined by \((g, h \star x) \mapsto (g, hxh^{-1})\). (We use the notation \(h \star x \in Q \times^B B\) to denote the \(B\)-orbit in \(Q \times B\) containing \((h, x)\).) Here all the squares are cartesian squares. Moreover,

(a) \(p\) is a principal \(Q\)-bundle.

(b) \(q\) is a locally trivial fibration with fibre isomorphic to \(G \times U_Q\).

Thus as in [S4, (1.5.2)], for any \(M\)-equivariant simple perverse sheaf \(A_1\) on \(M\), there exists a unique (up to isomorphism) simple perverse sheaf \(A_2\) on \(\tilde{G}^Q\) such that \(p^*A_2[a] \simeq q^*A_1[b]\), where \(a = \dim Q\) and \(b = \dim G + \dim U_Q\).
By using the cartesian squares in (2.7.6), and by (2.7.2), we see that 
\[ \pi^\prime_\ast \bar{Q} \simeq \text{IC}(\tilde{G}^Q, \psi^\prime_\ast \bar{Q}) \], and \( \pi^\prime_\ast \bar{Q} \) is decomposed as

\[ (2.7.7) \pi^\prime_\ast \bar{Q} \simeq \bigoplus_{\rho' \in W^\wedge} \bar{Q}^{(\rho'; \rho)} \otimes \text{IC}(G, L') \].

By comparing (2.7.4) and (2.7.7), we have

\[ (2.7.8) \pi^\prime\prime_\ast \text{IC}(\tilde{G}^Q, L') \simeq \bigoplus_{\rho' \in W^\wedge} \bar{Q}_l^{(\rho'; \rho)} \otimes \text{IC}(\tilde{G}, \bar{Q}_l \otimes \text{IC}(G, L')) \].

Note that if \( \rho = V_\lambda \) for \( \lambda \in \mathcal{P}_n \), we have

\[ (2.7.9) \text{IC}(G, L_\rho) \mid_{G\text{uni}} \simeq \text{IC}(O_\lambda^{\rho}, \bar{Q}_l \mid_{\dim O_\lambda^{\rho} - 2\nu_G}) \]

by [BM], where \( \nu_G = \dim U \). Hence by restricting on \( G\text{uni} \), we have

\[ (2.7.10) \pi^\prime\prime_\ast \text{IC}(\tilde{G}^Q, L') \mid_{G\text{uni}} \simeq \bigoplus_{\lambda \in \mathcal{P}_n} \bar{Q}_l^{(V_\lambda : \rho')} \otimes \text{IC}(O_\lambda^{\rho}, \bar{Q}_l \mid_{\dim O_\lambda^{\rho}}) \].

## 2.8.

Now assume that \( W_Q \simeq S_\mu \) for a partition \( \mu \), where we put \( S_\mu = S_{\mu_1} \times \cdots \times S_{\mu_k} \) if \( \mu = (\mu_1, \ldots, \mu_k) \in \mathcal{P}_n \). Take \( \rho' = \varepsilon \) the sign representation of \( W_Q \).

We have

\[ (2.8.1) (V_\lambda : \varepsilon) = (V_{\lambda'} : 1_W) = K_{\lambda', \mu}, \]

where \( 1_W \) is the trivial representation of \( W_Q \).

The restriction of the diagram (2.7.6) to the “unipotent parts” makes sense, and we have the commutative diagram

\[ \begin{array}{ccc} G \times B U & \xleftarrow{p_1} & G \times Q (Q \times B U) \xrightarrow{q_1} M \times B_M U_M \\ \downarrow & & \downarrow \\ G \times Q \text{uni} & \xrightarrow{p_1} & G \times Q \text{uni} \xrightarrow{q_1} M\text{uni}, \end{array} \]

where \( U_M \) is the unipotent radical of \( B_M \), and \( Q\text{uni}, M\text{uni} \) are the set of unipotent elements in \( Q, M \), respectively. \( p_1, q_1 \) have similar properties as (a), (b) in 2.7. We consider \( \text{IC}(M, L_\varepsilon^M) \) on \( M \), where \( L_\varepsilon^M \) is the simple local system on \( M_{\text{reg}} \) corresponding to \( \varepsilon \in W^\wedge \). Then by (2.7.6), we see that

\[ p^\ast \text{IC}(\tilde{G}^Q, L_\varepsilon^M) \simeq q^\ast \text{IC}(M, L_\varepsilon^M). \]

By applying (2.7.9) to \( M \), \( \text{IC}(M, L_\varepsilon^M) \mid_{M\text{uni}} \simeq \text{IC}(\overline{G}, \bar{Q}_l \mid_{\dim \overline{G} - 2\nu_M}) \), where \( \overline{G} \) is the orbit in \( M\text{uni} \) corresponding to \( \varepsilon \) under the Springer correspondence, and \( \nu_M \)
is defined similarly to $\nu_G$. It is known that $\mathcal{O}_{\nu_G}$ is the orbit $\{e\} \subset M_{\text{uni}}$, where $e$ is the identity element in $M$. Hence $\text{IC}(M, \mathcal{L}_e^M)|_{M_{\text{uni}}}$ coincides with $\mathcal{Q}_l[-2\nu_M]$ supported on $\{e\}$. It follows, by (2.8.2)

(2.8.3) The restriction of $\text{IC}(\tilde{G}_Q, \mathcal{L}_e^Q)$ on $G \times Q_{\text{uni}}$ coincides with $i_* \mathcal{Q}_l[-2\nu_M]$, where $i : G \times Q \to G \times Q_{\text{uni}}$ is the closed embedding.

We define a map $\pi_Q : G \times Q U_Q \to G_{\text{uni}}$ by $g \ast x \mapsto gxg^{-1}$. Put $\tilde{G}_1^Q = G \times Q U_Q$.

**Proposition 2.9.** Under the notation as above,

(i) $\pi'_* \text{IC}(\tilde{G}_Q, \mathcal{L}_e^Q)[2\nu_G]|_{G_{\text{uni}}} \simeq (\pi_Q)_* \mathcal{Q}_l[\dim \tilde{G}_1^Q]$.

(ii) We have

$$(\pi_Q)_* \mathcal{Q}_l[\dim \tilde{G}_1^Q] \simeq \bigoplus_{\mu \in \mathcal{P}_n \atop \mu \leq \lambda} \mathcal{Q}_l^{K_{\lambda, \mu}} \otimes \text{IC}(\mathcal{O}_{\lambda}, \mathcal{Q}_l)[\dim \mathcal{O}_{\lambda}].$$

**Proof.** Note that $2\nu_G - 2\nu_M = 2 \dim U_Q = \dim \tilde{G}_1^Q$. Thus by (2.8.3),

(2.9.1) $\text{IC}(\tilde{G}_Q, \mathcal{L}_e^Q)[2\nu_G]|_{G \times Q_{\text{uni}}} \simeq i_* \mathcal{Q}_l[\dim \tilde{G}_1^Q]$.

By applying the base change theorem to the cartesian square

$$
\begin{array}{ccc}
G \times Q_{\text{uni}} & \longrightarrow & G \times Q \\
\pi'_1 & & \pi'' \\
\downarrow & & \downarrow \\
G_{\text{uni}} & \longrightarrow & G,
\end{array}
$$

we obtain (i) from (2.9.1) since $\pi_Q = \pi'_1 \circ i$. Then (ii) follows from (i) by using (2.7.10) and (2.8.1). \hfill \Box

**2.10.** Returning to the setting in 2.5, we consider the case where $r$ is arbitrary. We fix $m \in \mathcal{O}_{n,r}$, and let $P = P_m$ be the parabolic subgroup of $G$ containing $B$ which is the stabilizer of the partial flag $(M_{p_i})_{1 \leq i \leq r}$. Let $L$ be the Levi subgroup of $P$ containing $T$, and $B_L = B \cap L$ the Borel subgroup of $L$ containing $T$. Let $U_L$ be the unipotent radical of $B_L$. Put $\overline{M}_{p_i} = M_{p_i}/M_{p_{i-1}}$ for each $i$, under the convention $M_{p_0} = 0$. Then $L$ acts naturally on $\overline{M}_{p_i}$, and by applying the definition of $\pi_{m,1} : \mathcal{F}_{m,\text{uni}} \to \mathcal{F}_{m,\text{uni}}$ to $L$, we can define

$$
\mathcal{F}_{m,\text{uni}}^L \simeq L \times B_L (U_L \times \prod_{i=1}^{r-1} \overline{M}_{p_i}),
$$

$$
\mathcal{F}_{m,\text{uni}}^L \simeq \bigcup_{g \in L} g(U_L \times \prod_{i=1}^{r-1} \overline{M}_{p_i}) = L_{\text{uni}} \times \prod_{i=1}^{r-1} \overline{M}_{p_i},
$$

and the map $\pi_{m,1}^L : \mathcal{F}_{m,\text{uni}}^L \to \mathcal{F}_{m,\text{uni}}^L$ similarly.
Let $Q = Q_\lambda$ be as in 2.5 for $\lambda \in \mathcal{P}(m)$. Thus we have $B \subset Q \subset P$, and $Q_L = Q \cap L$ is a parabolic subgroup of $L$ containing $B_L$. We consider the following commutative diagram

\[
\begin{array}{c}
\tilde{X}_{\text{uni}}^{L,Q} \quad \tilde{X}_{\text{uni}}^P \quad \tilde{X}_{\text{uni}}^{L,Q} \quad \tilde{X}_{\text{uni}}^P \quad \tilde{X}_{\text{uni}}^{P,Q} \quad \tilde{X}_{\text{uni}}^L \quad \tilde{X}_{\text{uni}}^{P,Q} \quad \tilde{X}_{\text{uni}}^L \\
\downarrow \alpha'_i \quad \downarrow r'_i \quad \downarrow \beta'_i \\
\tilde{X}_{\text{uni}}^{P,Q} \quad \tilde{X}_{\text{uni}}^P \quad \tilde{X}_{\text{uni}}^{L,Q} \quad \tilde{X}_{\text{uni}}^P \quad \tilde{X}_{\text{uni}}^{P,Q} \quad \tilde{X}_{\text{uni}}^L \quad \tilde{X}_{\text{uni}}^{P,Q} \quad \tilde{X}_{\text{uni}}^L \\
\downarrow \alpha''_i \quad \downarrow r''_i \quad \downarrow \beta''_i \\
\tilde{X}_{\text{uni}}^P \quad \tilde{X}_{\text{uni}}^P \quad \tilde{X}_{\text{uni}}^P \quad \tilde{X}_{\text{uni}}^P \\
\downarrow \pi''_i \\
\tilde{X}_{\text{uni}},
\end{array}
\]

(2.10.1)

where, by putting $P_{\text{uni}} = L_{\text{uni}} U_P$ (the set of unipotent elements in $P$),

\[
\begin{align*}
\tilde{X}_{\text{uni}}^P &= \bigcup_{g \in P} g(U \times \prod_i M_{p_i}) = P_{\text{uni}} \times \prod_i M_{p_i}, \\
\tilde{X}_{\text{uni}}^{P,Q} &= G \times \tilde{X}_{\text{uni}}^P = G \times (P_{\text{uni}} \times \prod_i M_{p_i}), \\
\tilde{X}_{\text{uni}}^{P,Q} &= P \times B (U \times \prod_i M_{p_i}), \\
\tilde{X}_{\text{uni}}^{Q} &= G \times Q (Q_{\text{uni}} \times \prod_i M_{p_i}), \\
\tilde{X}_{\text{uni}}^{P,Q} &= P \times Q (Q_{\text{uni}} \times \prod_i M_{p_i}).
\end{align*}
\]

$\tilde{X}_{\text{uni}}^{L,Q_L}$ is a similar variety as $\tilde{X}_{\text{uni}}^{P,Q}$ defined with respect to $(L, Q_L)$, namely,

\[
\tilde{X}_{\text{uni}}^{L,Q_L} = L \times Q_L ((Q_L)_{\text{uni}} \times \prod_i M_{p_i}).
\]

The maps are defined as follows; under the identification $\tilde{X}_{\text{uni}} \simeq G \times B (U \times \prod_i M_{p_i}),$ $\alpha'_i, \alpha''_i$ are the natural maps induced from the inclusions $G \times (U \times \prod_i M_{p_i}) \rightarrow G \times (Q_{\text{uni}} \times \prod_i M_{p_i}) \rightarrow G \times (P_{\text{uni}} \times \prod_i M_{p_i}).$ $\pi''_i : g \times (x, v) \mapsto (gxg^{-1}, gv).$ $q_1$ is defined by $(g, x, v) \mapsto (\bar{x}, \bar{v})$, where $x \mapsto \bar{x}, v \mapsto \bar{v}$ are natural maps $P \rightarrow L, \prod_i M_{p_i} \rightarrow \prod_i M_{p_i}$. $\tilde{q}_1$ is the composite of the projection $G \times \tilde{X}_{\text{uni}}^P \rightarrow \tilde{X}_{\text{uni}}^{P,Q}$ and the map $\tilde{X}_{\text{uni}} \rightarrow \tilde{X}_{\text{uni}}^{L,Q_L}$ induced from the projection $P \times (U \times \prod_i M_{p_i}) \rightarrow L \times (U_L \times \prod_i M_{p_i}).$ $\tilde{q}_1$ is defined similarly by using the map $\tilde{X}_{\text{uni}}^{P,Q} \rightarrow \tilde{X}_{\text{uni}}^{L,Q_L}$ induced from the projection $P \times (Q_{\text{uni}} \times \prod_i M_{p_i}) \rightarrow L \times ((Q_L)_{\text{uni}} \times \prod_i M_{p_i}).$ $p_1$ is the quotient by $P.$ $\tilde{p}_1$ and $\tilde{p}_1$ are also quotient by $P$ under the identifications $\tilde{X}_{\text{uni}} \simeq G \times \tilde{X}_{\text{uni}}^{P,Q}, \tilde{X}_{\text{uni}} \simeq G \times \tilde{X}_{\text{uni}}^{P,Q},$ $\tilde{p}'_1$ is defined similarly to $\alpha'_1$ and
\( \beta''_1 \) is defined similarly to \( \pi''_1 \). \( r'_1 \) is the natural map induced from the injection \( P \times (U \times \prod M_{p_i}) \to P \times (Q_{\text{uni}} \times \prod M_{p_i}) \), and \( r''_1 \) is the natural map induced from the map \( P \times Q (Q_{\text{uni}} \times \prod M_{p_i}) \to P_{\text{uni}} \times \prod M_{p_i}, g \ast (x, v) \mapsto (gxg^{-1}, gv) \).

Put \( \pi'_1 = \alpha'_1 \circ \beta'_1 : \mathcal{F}_{\text{uni}} \to \mathcal{F}_{\text{uni}}^P \). We have \( \beta''_1 \circ \beta'_1 = \pi''_1 \), and the diagram (2.10.1) is the refinement of the diagram (6.3.2) in [S4] (see also the diagram (1.5.1) in [S4]). In particular, the map \( p_1 \) is a principal \( P \)-bundle, and the diagram \( q_1 \) is a locally trivial fibration with fibre isomorphic to \( \prod M_{p_i} \). Moreover, all the squares appearing in (2.10.1) are cartesian squares. Hence the diagram (2.10.1) satisfies similar properties as in the diagram (2.8.2).

Note that \( L \simeq G_1 \times \cdots \times G_r \), with \( G_i = GL(M_{p_i}) \). Then \( Q_L \) can be written as \( Q_L \simeq Q_1 \times \cdots \times Q_r \), where \( Q_i \) is a parabolic subgroup of \( G_i \). We have

\[
\mathcal{F}_{\text{uni}} = \prod_{i=1}^r (\tilde{G}_i)_{\text{uni}} \times V,
\]
\[
\mathcal{F}_{\text{uni}}^L = \prod_{i=1}^r (\tilde{G}_i)^Q_1 \times V,
\]
\[
\mathcal{F}_{\text{uni}}^L = \prod_{i=1}^r (G_i)_{\text{uni}} \times V,
\]

where \((\tilde{G}_i)_{\text{uni}}, (\tilde{G}_i^Q_1)_{\text{uni}}, \) etc. denote the unipotent parts of \( \tilde{G}_i, \tilde{G}_i^Q_1, \) etc. as in (2.8.2). The maps \( \beta'_1, \beta''_1 \) are induced from the maps \((\tilde{G}_i)_{\text{uni}} \to (\tilde{G}_i^Q_1)_{\text{uni}}, (\tilde{G}_i^Q_1)_{\text{uni}} \to (G_i)_{\text{uni}}, \) and those maps coincide with the maps \( \beta'_1, \beta''_1 \) in 2.7 defined with respect to \( G_i \). Note that \( W_{Q_i} \simeq S(\lambda^{(i)}) \) for each \( i \) by the construction of \( Q = Q_\lambda \) in 2.5. Put

\[
\mathcal{F}_1^Q = G \times Q (U_Q \times \prod M_{p_i}),
\]
\[
\mathcal{F}_1^L = L \times Q (U_{Q_i} \times \prod M_{p_i}),
\]

and let \( i_Q : \mathcal{F}_1^Q \hookrightarrow \mathcal{F}_{\text{uni}}^Q, i_{Q_L} : \mathcal{F}_1^L \hookrightarrow \mathcal{F}_{\text{uni}}^L \) be the closed embeddings. Let \( \pi_{Q_L} : \mathcal{F}_1^L \to \mathcal{F}_{\text{uni}}^L \) be the restriction of \( \beta''_1 \). Let \( \mathcal{O}_L^{(i)} = \mathcal{O}_{(i)^{Q_1}}^{(i)} \times \cdots \times \mathcal{O}_{(i)^{Q_1}}^{(i)} \) be the \( L \)-orbit in \( \mathcal{F}_{\text{uni}}^L \), where \( \mathcal{O}_{(i)^{Q_1}}^{(i)} \) is the \( G_i \)-orbit in \((G_i)_{\text{uni}} \times M_{p_i}\) of type \((\mu^{(i)}), \emptyset)\). Note that if we denote by \( \mathcal{O}_{(i)}^{(i)} \) the \( G_i \)-orbit in \((G_i)_{\text{uni}}\) of type \( \mu^{(i)} \), we have \( \text{IC}(\mathcal{O}_{(i), Q_i}) \simeq \text{IC}(\mathcal{O}_{(i)^{Q_1}, Q_i}) \otimes Q_i \) (the latter term \( Q_i \) denotes the constatn sheaf on \( M_{p_i} \)). Hence the decomposition of \( \pi_{Q_L}^{(i)^{Q_1}} \) into simple components is described by considering the factors \( \text{IC}(\mathcal{O}_{(i)^{Q_1}, Q_i}) \). In particular, by Proposition 2.9, we have

\[
(\pi_{Q_L}^{(i)^{Q_1}})_* Q_i [\dim \mathcal{F}_1^{L, Q_L}] \simeq \bigoplus_{\mu \geq \lambda} \mathcal{Q}_t^{\mu^t, \lambda^t} \otimes \text{IC}(\mathcal{O}_{(i)^{Q_1}, Q_i}) [\dim \mathcal{O}_{(i)^{Q_1}}^{(i)}].
\]

By using the diagram (2.10.1), we see that

\[
\mathcal{g}_1(i_{Q_L})_* Q_i [\dim \mathcal{F}_1^{L, Q_L}] \simeq \mathcal{p}_1(i_Q)_* Q_i [\dim \nabla^L_\lambda].
\]
It follows, again by using the diagram (2.10.1), we have

\[(\alpha''_1)_*(i_Q)_*\bar{Q}_l[\dim \tilde{X}_\lambda] \simeq \bigoplus_{\mu \subseteq \lambda} \bar{Q}_l^{K_{\mu^{', \lambda'}}} \otimes B_\mu,\]

where $B_\mu$ is the simple perverse sheaf on $\tilde{\mathcal{X}}_m^{P, \text{uni}}$ characterized by the property that

\[p_1^*B_\mu[a'] \simeq q_1^*\text{IC}(\bar{\mathcal{O}}_\mu, \bar{Q}_l)[b' + \dim \mathcal{O}_L^L]\]

with $a' = \dim P$, $b' = \dim G + \dim U_P + \dim \prod_{i=1}^{r-2} M_{p_i}$.

On the other hand, by Proposition 1.6 in [S4], we have

\[(\pi''_1)_*A_\mu \simeq \text{IC}(\mathcal{X}_m, \bar{Q}_l)[d_m],\]

where $\pi'' : \tilde{\mathcal{X}}_m^P = G \times P (P \times \prod M_{p_i}) \to \mathcal{X}_m$ is an analogous map to $\pi''_1$, and $A_\mu$ is a simple perverse sheaf on $\tilde{\mathcal{X}}_m^P$ such that the restriction of $A_\mu$ on $\tilde{\mathcal{X}}_m^{P, \text{uni}}$ coincides with $B_\mu$, up to shift. Thus by Theorem 2.4 (ii), we have

\[(\pi''_1)_*B_\mu \simeq \text{IC}(\mathcal{X}_m, \bar{Q}_l)[\dim \mathcal{X}_\mu].\]

Since $\pi_\lambda = \pi''_1 \circ \alpha''_1 \circ i_Q$, by applying $(\pi''_1)_*$ on both sides of (2.10.3), we obtain the formula (2.6.1). This completes the proof of Theorem 2.6.

### 3. \(G^F\)-invariant functions on the enhanced variety and Kostka functions

#### 3.1.

We now assume that $G$ and $V$ are defined over $\mathbb{F}_q$, and let $F : G \to G$, $F : V \to V$ be the corresponding Frobenius maps. Assume that $B$ and $T$ are $F$-stable. Then $X_\lambda$ and $\tilde{X}_\lambda$ have natural $\mathbb{F}_q$-structures, and the map $\pi_\lambda : \tilde{X}_\lambda \to \mathcal{X}_\lambda$ is $F$-equivariant. Thus one can define a canonical isomorphism $\varphi : F^*K_\lambda \simeq K_\lambda$ for $K_\lambda = (\pi_\lambda)_*\bar{Q}_l$. By using the decomposition in Theorem 2.6, $\varphi$ can be written as

$\varphi = \sum_{\mu} \sigma_\mu \otimes \varphi_\mu$, where $\sigma_\mu$ is the identity map on $\bar{Q}_l^{K_{\mu^{', \lambda'}}}$ and $\varphi_\mu : F^*L_\mu \simeq L_\mu$ is the isomorphism induced from $\varphi$ for $L_\mu = \text{IC}(\mathcal{X}_\mu, \bar{Q}_l)$. (Note that $\dim X_\lambda - \dim X_\mu$ is even if $\mu \subseteq \lambda$ by [S4, Prop. 4.3], so the degree shift is negligible). We also consider the natural isomorphism $\phi_{\mu :} : F^*L_\mu \simeq L_\mu$ induced from the $\mathbb{F}_q$-structure of $X_\mu$. By using a similar argument as in [S4, (6.1.1)], we see that

\[\varphi_\mu = q^{d_\mu} \phi_{\mu},\]

where $d_\mu = n(\mu)$. We consider the characteristic function $\chi_{L_\mu}$ of $L_\mu$ with respect to $\phi_{\mu}$, which is a $G^F$-invariant function on $\mathcal{X}_\mu^F$. 
3.2. Take $\mu, \nu \in \mathcal{P}_{n,r}$, and assume that $\nu \in \mathcal{P}(m)$. For each $z = (x, v) \in X_\mu$ with $v = (v_1, \ldots, v_{r-1})$, we define a variety $\mathcal{G}_{\nu, z}$ by

$$
\mathcal{G}_{\nu, z} = \{(W_{pi}) : x\text{-stable flag } | \ v_i \in W_{pi} \ (1 \leq i \leq r-1), \ x|_{W_{pi}/W_{pi-1}}: \nu^{(i)} \ (1 \leq i \leq r)\}.
$$

If $z \in X^F_\mu$, the variety $\mathcal{G}_{\nu, z}$ is defined over $\mathbb{F}_q$. Put $g_{\nu, z}(q) = |\mathcal{G}_{\nu, z}|$. Let $\tilde{K}_{\lambda, \mu}(t)$ be the modified Kostka polynomial indexed by partitions $\lambda, \mu$. The following result is a generalization of Proposition 5.8 in [AH].

**Proposition 3.3.** Assume that $\lambda, \mu \in \mathcal{P}_{n,r}$. For each $z \in X^F_\mu$, we have

$$
|\pi^{-1}(z)^F| = \sum_{\nu \in \mathcal{P}_{n,r}} |\mathcal{G}_{\nu, z}| \prod_i |\pi^{-1}(x_i)^F|,
$$

where $\pi_{\lambda^{(i)}} : \tilde{\phi}_{\lambda^{(i)}} \rightarrow \tilde{\phi}_{\lambda^{(i)}}$ is a similar map as $\pi_{\lambda}$ applied to the case $r = 1$, by replacing $G$ by $G_i = GL(M_{pi})$, and $x_i = x|_{M_{pi}}$ has Jordan type $\nu^{(i)}$. It is known by [L1] that $q^{n(\xi^{(i)})} \chi_{\xi^{(i)}}(x_i) = \tilde{K}_{\xi^{(i)}, \nu^{(i)}}(q)$ for a partition $\xi^{(i)}$ of $m_i$. It follows, by applying (3.3.1) to the case where $r = 1$, and by the Grothendieck’s fixed point formula, we have

$$
|\pi^{-1}(x_i)^F| = \sum_{\xi^{(i)} \leq \lambda^{(i)}} K_{\xi^{(i)}, \lambda^{(i)}} \tilde{K}_{\xi^{(i)}, \nu^{(i)}}(q).
$$

Then (3.3.2) implies that

$$
\chi_{\lambda^{(i)}} = |\pi^{-1}(z)^F| = \sum_{\nu \in \mathcal{P}_{n,r}} g_{\nu, z}(q) \sum_{\xi \leq \lambda} K_{\xi^{(i)}, \lambda^{(i)}} \tilde{K}_{\xi^{(i)}, \nu^{(i)}}(q) \cdots \tilde{K}_{\nu^{(r)}, \nu^{(r)}}(q).
$$

**Proof.** Let $\chi_{\lambda^{(i)}, \varphi}$ be the characteristic function of $K_{\lambda}$ with respect to $\varphi$. By Theorem 2.6 together with (3.1.1), we have

$$
\chi_{\lambda^{(i)}, \varphi} = \sum_{\xi \leq \lambda} K_{\xi^{(i)}, \lambda^{(i)}} q^{n(\xi)} \chi_{\xi^{(i)}}.
$$

On the other hand, by the Grothendieck’s fixed point formula, we have $\chi_{\lambda^{(i)}, \varphi}(z) = |\pi^{-1}(z)^F|$ for $z \in X^F_\lambda$. Then if $z = (x, v) \in X^F_\mu$,
Remark 3.4. In general, $X_\mu$ consists of infinitely many $G$-orbits. Hence the value $g_{\nu,z}(q)$ may depend on the choice of $z \in X^F_\mu$. However, if $X_\mu$ is a single $G$-orbit, then $X^F_\mu$ is also a single $G^F$-orbit, and $g_{\nu,z}(q)$ is constant for $z \in X^F_\mu$, in which case, we denote $g_{\nu,z}(q)$ by $g^\mu_{\nu}(q)$. In what follows, we show in some special cases that there exists a polynomial $g^\mu_{\nu}(t) \in \mathbb{Z}[t]$ such that $g^\mu_{\nu}(q)$ coincides with the value at $t = q$ of $g^\mu_{\nu}(t)$.

3.5. We consider the special case where $\mu \in \mathcal{P}(m')$ is such that $m'_i = 0$ for $i = 1, \ldots, r - 2$. In this case, $X_\mu$ consists of a single $G$-orbit. In particular, for $\lambda \in \mathcal{P}_{n,r}$, dim $\mathcal{H}^z_2 IC(\overline{X}_\lambda, Q_t)$ does not depend on the choice of $z \in X_\mu$. We define a polynomial $IC^{-}_{\lambda,\mu}(t) \in \mathbb{Z}[t]$ by

$$IC^{-}_{\lambda,\mu}(t) = \sum_{i \geq 0} \dim \mathcal{H}^z_2 IC(\overline{X}_\lambda, Q_t)t^i.$$  

The following result was proved in [S4].

Proposition 3.6 ([S4, Prop. 6.8]). Let $\lambda, \mu \in \mathcal{P}_{n,r}$, and assume that $\mu$ is as in 3.5.

(i) Assume that $z \in X^F_\mu$. Then $H^1 IC(\overline{X}_\lambda, Q_t) = 0$ if $i$ is odd, and the eigenvalues of $\phi_0$ on $H^2 IC(\overline{X}_\lambda, Q_t)$ are $q^i$. In particular, $\chi_{\lambda}(z) = IC^{-}_{\lambda,\mu}(q)$.

(ii) $K^{-}_{\lambda,\mu}(t) = t^{\alpha(\lambda)} IC^{-}_{\lambda,\mu}(t^r)$.

As a corollary, we have the following result, which is a generalization of [AH, Prop. 5.8] (see also [LS, Prop. 3.2]).

Corollary 3.7. Assume that $\mu$ is as in 3.5.

(i) There exists a polynomial $g^\mu_{\nu}(t) \in \mathbb{Z}[t]$ such that $g^\mu_{\nu}(q)$ coincides with the value at $t = q$ of $g^\mu_{\nu}(t)$.

(ii) We have

$$K^{-}_{\lambda,\mu}(t) = t^{\alpha(\lambda)} \sum_{\nu \leq \lambda} g^\mu_{\nu}(t^r) \widetilde{K}_{\lambda(1),\nu(1)}(t^r) \cdots \widetilde{K}_{\lambda(r),\nu(r)}(t^r).$$

Proof. By Proposition 3.6 (i) and Proposition 3.3, we have

$$IC^{-}_{\lambda,\mu}(q) = q^{-\alpha(\lambda)} \sum_{\nu \leq \lambda} g^\mu_{\nu}(q) \widetilde{K}_{\lambda(1),\nu(1)}(q) \cdots \widetilde{K}_{\lambda(r),\nu(r)}(q).$$

By fixing $\mu$, we consider two sets of functions $\{IC^{-}_{\lambda,\mu}(q) \mid \lambda \in \mathcal{P}_{n,r}\}$ and $\{g^\mu_{\nu}(q) \mid \nu \in \mathcal{P}_{n,r}\}$. If we notice that $\widetilde{K}_{\lambda(1),\nu(1)}(q) \cdots \widetilde{K}_{\lambda(r),\nu(r)}(q) = q^{\alpha(\lambda)}$ for $\nu = \lambda$, (3.7.2) shows that the transition matrix between those two sets is unitriangular. Hence $g^\mu_{\nu}(q)$ is determined from $IC^{-}_{\lambda,\mu}(q)$, and a similar formula makes sense if we replace $q$ by $t$. This implies (i). (ii) now follows from (3.7.2) by replacing $q$ by $t$. \qed
3.8. In what follows, we assume that \( \mu \) is of the form \( \mu = (-, \ldots, -, \xi) \) with \( \xi \in \mathcal{P}_n \). In this case, \( g^\xi_\mu(t) \) coincides with the polynomial \( G^\xi_{\mu(1), \ldots, \mu(r)}(t) \) obtained from \( G^\xi_{\mu(1), \ldots, \mu(r)}(\delta) \) discussed in [M, II, 2]. On the other hand, we define a polynomial \( f^\xi_{\mu(1), \ldots, \mu(r)}(t) \) by

\[
P_{\mu(1)}(y; t) \cdots P_{\mu(r)}(y; t) = \sum_{\xi \in \mathcal{P}_n} f^\xi_{\mu(1), \ldots, \mu(r)}(t) P_\xi(y; t).
\]

(3.8.1)

In the case where \( r = 2 \), \( g^\xi_{\mu(1), \mu(2)}(t) \) coincides with the Hall polynomial, and a simple formula relating it with \( f^\xi_{\mu(1), \mu(2)}(t) \) is known ([M, III (3.6)]). In the general case, we also have a formula

\[
g^\xi_{\mu(1), \ldots, \mu(r)}(t) = t^{n(\xi) - n(\nu)} f^\xi_{\mu(1), \ldots, \mu(r)}(t^{-1}).
\]

(3.8.2)

The proof is easily reduced to [M, III (3.6)].

For partitions \( \lambda, \nu^{(1)}, \ldots, \nu^{(r)} \), we define an integer \( c^\lambda_{\nu^{(1)}, \ldots, \nu^{(r)}} \) by

\[
\sum_{\lambda} c^\lambda_{\mu(1), \ldots, \mu(r)} s_\lambda.
\]

In the case where \( r = 2 \), \( c^\lambda_{\mu(1), \mu(2)} \) coincides with the Littlewood-Richardson coefficient.

For \( \lambda \in \mathcal{P}_{n,r} \), put

\[
b(\lambda) = a(\lambda) - r \cdot n(\lambda) = |\lambda^{(2)}| + 2|\lambda^{(3)}| + \cdots + (r - 1)|\lambda^{(r)}|.
\]

(3.8.3)

The following lemma is a generalization of [LS, Lemma 3.4].

**Lemma 3.9.** Let \( \lambda, \mu \in \mathcal{P}_{n,r} \), and assume that \( \mu = (-, \ldots, -, \xi) \). Then we have

\[
K^-_{\lambda, \mu}(t) = t^{b(\mu) - b(\lambda)} \sum_{\nu \subset \lambda} f^\xi_{\mu(1), \ldots, \mu(r)}(t) K_{\lambda(1), \mu(1)}(t^{r}) \cdots K_{\lambda(r), \mu(r)}(t^{r}),
\]

(3.9.1)

\[
K^-_{\lambda, \mu}(t) = t^{b(\mu) - b(\lambda)} \sum_{\eta \in \mathcal{P}_n} c^\eta_{\lambda(1), \ldots, \lambda(r)} K_{\eta, \xi}(t)
\]

(3.9.2)

**Proof.** The formula (3.7.1) can be rewritten as

\[
K^-_{\lambda, \mu}(t) = t^{a(\mu) - a(\lambda) + rn(\lambda)} \sum_{\nu \subset \lambda} t^{-rn(\nu)} g^\xi_{\nu(1), \ldots, \nu(r)}(t^{-r}) K_{\lambda(1), \mu(1)}(t^{r}) \cdots K_{\lambda(r), \mu(r)}(t^{r}).
\]

(3.9.3)

Substituting (3.8.2) into (3.9.3), we obtain (3.9.1). Next we show (3.9.2). One can write as
\[ s_{\lambda(t)}(y) = \sum_{\mu(t)} K_{\lambda(t),\mu(t)}(t)P_{\mu(t)}(y; t). \]

Hence

(3.9.4)
\[ s_{\lambda(1)}(y) \cdots s_{\lambda(r)}(y) = \sum_{\nu \in \mathcal{P}_n,r} K_{\lambda(1),\nu(1)}(t) \cdots K_{\lambda(r),\nu(r)}(t)P_{\nu(1)}(y; t) \cdots P_{\nu(r)}(y; t) \]
\[ = \sum_{\nu \in \mathcal{P}_n,r} \sum_{\xi \in \mathcal{P}_n} f_{\xi(1),\ldots,\nu(r)}(t)K_{\lambda(1),\nu(1)}(t) \cdots K_{\lambda(r),\nu(r)}(t)P_{\xi}(y; t). \]

On the other hand,

(3.9.5)
\[ s_{\lambda(1)}(y) \cdots s_{\lambda(r)}(y) = \sum_{\eta \in \mathcal{P}_n} c_{\lambda(1),\ldots,\lambda(r)}^{(1)} s_{\eta}(y) \]
\[ = \sum_{\eta \in \mathcal{P}_n} c_{\lambda(1),\ldots,\lambda(r)}^{(1)} \sum_{\xi \in \mathcal{P}_n} K_{\eta,\xi}(t)P_{\xi}(y; t). \]

By comparing (3.9.4) and (3.9.5), we have an equality for each \( \xi \in \mathcal{P}_n; \)
\[ \sum_{\eta \in \mathcal{P}_n} c_{\lambda(1),\ldots,\lambda(r)}^{(1)} K_{\eta,\xi}(t) = \sum_{\nu \in \mathcal{P}_n,r} f_{\nu(1),\ldots,\nu(r)}(t)K_{\lambda(1),\nu(1)}(t) \cdots K_{\lambda(r),\nu(r)}(t). \]

Combining this with (3.9.1), we obtain (3.9.2). The lemma is proved. \( \square \)

**3.10.** Let \( \eta' = \lambda' - \theta' \), \( \eta'' = \lambda'' - \theta'' \) be skew diagrams, where \( \theta', \theta'' \subseteq \lambda', \lambda'' \subseteq \lambda'' \) are partitions. We define a new skew diagram \( \eta' * \eta'' = \lambda - \theta \) as follows; write the partitions \( \lambda', \lambda'' \) as \( \lambda' = (\lambda'_1, \ldots, \lambda'_{k'}) \), \( \lambda'' = (\lambda''_1, \ldots, \lambda''_{k''}) \) with \( \lambda'_{k'} > 0 \), \( \lambda''_{k''} > 0 \). Put \( a = \lambda''_{k''} \). We define a partition \( \lambda = (\lambda_1, \ldots, \lambda_{k' + k''}) \) by
\[ \lambda_i = \begin{cases} 
\lambda'_i + a & \text{for } 1 \leq i \leq k', \\
\lambda''_{i-k'} & \text{for } k' + 1 \leq i \leq k' + k''.
\end{cases} \]

Write partitions \( \theta', \theta'' \) as \( \theta' = (\theta'_1, \ldots, \theta'_{k'}) \), \( \theta'' = (\theta''_1, \ldots, \theta''_{k''}) \) with \( \theta'_{k'} \geq 0 \), \( \theta''_{k''} \geq 0 \). We define a partition \( \theta = (\theta_1, \ldots, \theta_{k' + k''}) \), in a similar way as above, by
\[ \theta_i = \begin{cases} 
\theta'_i + a & \text{for } 1 \leq i \leq k', \\
\theta''_{i-k'} & \text{for } k' + 1 \leq i \leq k' + k''.
\end{cases} \]

We have \( \theta \subseteq \lambda \), and the skew diagram \( \eta' * \eta'' = \lambda - \theta \) can be defined.

For \( \lambda, \mu \in \mathcal{P}_n \), let \( \text{SST} \) be the set of semistandard tableaux of shape \( \lambda \) and weight \( \mu \). Let \( \lambda \in \mathcal{P}_{n,r} \). An \( r \)-tuple \( T = (T^{(1)}, \ldots, T^{(r)}) \) is called a semistandard tableau of shape \( \lambda \) if \( T^{(i)} \) is a semistandard tableau of shape \( \lambda^{(i)} \) with respect to
the letters \( \{1, \ldots, n\} \). We denote by \( \text{SST}(\lambda) \) the set of semistandard tableaux of shape \( \lambda \). For \( \lambda \in \mathcal{P}_{n,r} \), let \( \tilde{\lambda} \) be the skew diagram \( \lambda^{(1)} \ast \lambda^{(2)} \ast \cdots \ast \lambda^{(r)} \). Then \( T \in \text{SST}(\lambda) \) is regarded as a usual semistandard tableau \( \tilde{T} \) associated to the skew diagram \( \tilde{\lambda} \). Assume \( \pi \in \mathcal{P}_n \). We say that \( T \in \text{SST}(\lambda) \) has weight \( \pi \) if the corresponding tableau \( \tilde{T} \) has shape \( \tilde{\lambda} \) and weight \( \pi \). We denote by \( \text{SST}(\lambda, \pi) \) the set of semistandard tableaux of shape \( \lambda \) and weight \( \pi \).

3.11. In [M, I, (9.4)], a bijective map \( \Theta \)

\[
\Theta : \text{SST}(\tilde{\lambda}, \pi) \cong \coprod_{\nu \in \mathcal{P}_n} (\text{SST}^0(\tilde{\lambda}, \nu) \times \text{SST}(\nu, \pi))
\]

was constructed, where \( \text{SST}^0(\tilde{\lambda}, \nu) \) is the set of tableau \( T \) such that the associated word \( w(T) \) is a lattice permutation (see [M, I, 9] for the definition). Under the identification \( \text{SST}(\tilde{\lambda}, \pi) \cong \text{SST}(\lambda, \pi) \), the subset \( \text{SST}^0(\tilde{\lambda}, \nu) \) of \( \text{SST}(\lambda, \nu) \) is also defined. Then we can regard \( \Theta \) as a bijection with respect to the set \( \text{SST}(\lambda, \pi) \) (and \( \text{SST}^0(\lambda, \nu) \)).

In the case where \( r = 2 \), it is shown in [LS, Cor. 3.9] that \( |\text{SST}^0(\lambda, \nu)| \) coincides with the Littlewood-Richardson coefficient \( c^{(1)}_{\lambda, \pi} \). A similar argument can be applied also to the general case, and we have

**Corollary 3.12.** Assume that \( \lambda \in \mathcal{P}_{n,r}, \nu \in \mathcal{P}_n \). Then we have

\[
|\text{SST}^0(\lambda, \nu)| = c^{(1)}_{\lambda, \pi}.
\]

3.13. For a semistandard tableau \( S \), the charge \( c(S) \) is defined as in [M, III, 6]. It is known that Lascoux-Schützenberger Theorem ([M, III, (6.5)]) gives a combinatorial description of Koskta polynomials \( K_{\lambda, \mu}(t) \) in terms of semistandard tableaux,

\[
K_{\lambda, \mu}(t) = \sum_{S \in \text{SST}(\lambda, \mu)} t^{c(S)}.
\]

In the case where \( r = 2 \), a similar formula was proved for \( K_{\lambda, \mu}(t) \) in [LS, Thm. 3.12], in the special case where \( \mu = (-, \mu'') \). Here we consider \( K_{\lambda, \mu}(t) \) for general \( r \). Assume that \( \lambda \in \mathcal{P}_{n,r} \) and \( \xi \in \mathcal{P}_n \). For \( T \in \text{SST}(\lambda, \xi) \), we write \( \Theta(T) = (D, S) \) with \( S \in \text{SST}(\nu, \xi) \) for some \( \nu \). We define a charge \( c(T) \) of \( T \) by \( c(T) = c(S) \). We have the following theorem. Note that the proof is quite similar to [LS].

**Theorem 3.14.** Let \( \lambda, \mu \in \mathcal{P}_{n,r} \), and assume that \( \mu = (-, \ldots, -, \xi) \). Then

\[
K_{\lambda, \mu}^-(t) = t^{b(\mu) - b(\lambda)} \sum_{T \in \text{SST}(\lambda, \xi)} t^{c(T)}.
\]

**Proof.** We define a map \( \Psi : \text{SST}(\lambda, \xi) \to \coprod_{\nu \in \mathcal{P}_n} \text{SST}(\nu, \xi) \) by \( T \mapsto S \), where \( \Theta(T) = (D, S) \). Then by Corollary 3.12, for each \( S \in \text{SST}(\nu, \xi) \), the set \( \Psi^{-1}(S) \)
has the cardinality $c_{\lambda(1),\ldots,\lambda(r)}^\xi$, and by definition, any $T \in \Psi^{-1}(S)$ has the charge $c(T) = c(S)$. Hence
\[
\sum_{T \in \text{SST}(\lambda, \xi)} t^{c(T)} = \sum_{\nu \in \mathcal{P}_n} \sum_{S \in \text{SST}(\nu, \xi)} c_{\lambda(1),\ldots,\lambda(r)}^\nu t^{c(S)} = \sum_{\nu \in \mathcal{P}_n} c_{\lambda(1),\ldots,\lambda(r)}^\nu K_{\nu, \xi}(t).
\]
The last equality follows from (3.13.1). The theorem now follows from (3.9.2). □

**Corollary 3.15.** Under the assumption of Theorem 3.14, we have
\[
K_{\lambda, \mu}^-(1) = |\text{SST}(\lambda, \xi)|.
\]

3.16. In the rest of this section, we shall give an alternate description of the polynomial $g_{\nu}(t)$ in the case where $\mu = (-,\ldots,-,\xi)$. For $\nu \in \mathcal{P}_{n,r}$, put $R_{\nu}(x; t) = P_{\nu(1)}(x^{(1)}; t^r) \cdots P_{\nu(r)}(x^{(r)}; t^r)$. Then $\{R_{\nu} \mid \nu \in \mathcal{P}_{n,r}\}$ gives a basis of $\Xi^n[t]$. We define functions $h_{\nu}(t) \in \mathbb{Q}(t)$ by the condition that
\[
(3.16.1) \quad R_{\nu}(x; t) = \sum_{\mu \in \mathcal{P}_{n,r}} h_{\nu}(t) P_{\mu}^-(x; t).
\]

The following formula is a generalization of Proposition 4.2 in [LS].

**Proposition 3.17.** Assume that $\mu = (-,\ldots,-,\xi)$. Then
\[
h_{\nu}(t) = t^{a(\mu)-a(\nu)} g_{\nu}(t^r).
\]

**Proof.** The proof is quite similar to that of [LS, Prop. 4.2]. For $\lambda \in \mathcal{P}_{n,r}$, we have
\[
s_{\lambda}(x) = s_{\lambda(1)}(x^{(1)}) \cdots s_{\lambda(r)}(x^{(r)})
\]
\[
= \prod_{i=1}^r \sum_{\nu(1)} K_{\lambda(1),\nu(1)}(t^r) P_{\nu(1)}(x^{(i)}; t^r)
\]
\[
= \sum_{\nu} \prod_{i=1}^r K_{\lambda(1),\nu(1)}(t^r) K_{\lambda(r),\nu(r)}(t^r) \sum_{\mu \in \mathcal{P}_{n,r}} h_{\nu}(t) P_{\mu}^-(x; t)
\]
\[
= \sum_{\mu \in \mathcal{P}_{n,r}} \left( \sum_{\nu} \prod_{i=1}^r K_{\lambda(1),\nu(1)}(t^r) K_{\lambda(r),\nu(r)}(t^r) h_{\nu}(t) \right) P_{\mu}^-(x; t).
\]

Since $s_{\lambda}(x) = \sum_{\mu \in \mathcal{P}_{n,r}} K_{\lambda, \mu}^-(t) P_{\mu}^-(x; t)$, by comparing the coefficients of $P_{\mu}^-(x; t)$, we have
Now assume that $\mu = (-,\ldots,-,\xi)$. If we notice that $K_{\lambda^{(i)},\mu^{(i)}}(t^r) \neq 0$ only when $|\lambda^{(i)}| = |\mu^{(i)}|$, (3.9.3) implies that

$$K_{\lambda,\mu}^{-}(t) = \sum_{\nu \in \mathcal{P}_{n,r}} h_{\mu}^{\nu}(t)K_{\lambda^{(1)},\mu^{(1)}}(t^r) \cdots K_{\lambda^{(r)},\mu^{(r)}}(t^r).$$

Since $(K_{\lambda^{(1)},\mu^{(1)}}(t^r) \cdots K_{\lambda^{(r)},\mu^{(r)}}(t^r))_{\lambda,\mu \in \mathcal{P}_{n,r}}$ is a unitriangular matrix, the proposition follows by comparing (3.17.1) and (3.17.2). □

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