A Systematic Algorithm for Quantum Boolean Circuits Construction

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Abstract

To build a general-purpose quantum computer, it is crucial for the quantum devices to implement classical boolean logic. A straightforward realization of quantum boolean logic is to use auxiliary qubits as intermediate storage. This inefficient implementation causes a large number of auxiliary qubits to be used. In this paper, we have derived a systematic way of realizing any general $m$-to-$n$ bit combinational boolean logic using elementary quantum gates. Our approach transforms the $m$-to-$n$ bit classical mapping into a $t$-bit unitary quantum operation with minimum number of auxiliary qubits, then a variation of Toffoli gate is used as the basic building block to construct the unitary operation. Finally, each of these building blocks can be decomposed into one-bit rotation and two-bit control-U gates. The efficiency of the network is taken into consideration by formulating it as a constrained set partitioning problem.

1 Introduction

Since Feynman and Deutsch introduced the idea and theoretical model of quantum computer in the early 1980’s, a great deal of research effort has been focused on the topic of quantum information science. The discovery of Shor’s prime factorization and Grover’s fast database search algorithm have made quantum computing the most rapidly expanding research field recently. For a quantum algorithm to be useful, it is crucial that the algorithm should be able to be implemented using quantum gates. Not long after Deutsch proposed his theoretical model of quantum computer, he showed that a three-bit quantum gate is universal and capable of realizing any unitary operation.
A few years later, it was shown [3, 4] that two-bit gates are sufficient to implement any unitary operation. This makes experimental implementation of quantum circuits more practical.

Another approach that pushes the computing technology to its theoretical limits is called nanotechnology. Nanotechnology, combining physics and computer science, uses nanometer scale devices as the fundamental building block of electronic circuits. Just like a classical computer is built out of universal classical gates, a quantum computer can be built using nanoscale quantum gates. Various silicon-based nanoscale devices have been proposed as candidates for quantum computer [9, 10, 11, 12, 13, 14]. It is believed that scalable computation can be achieved using solid state quantum logic devices. However, to build a general-purpose computer, it is necessary for these nanoscale quantum devices to be able to have the ability to implement classical boolean logic.

A straightforward realization of quantum boolean logic is to use auxiliary qubits as intermediate storage. This inefficient implementation causes a large number of auxiliary qubits to be used. In this paper, we have derived a systematic way of realizing any general m-to-n bit combinational boolean logic using elementary quantum gates. Our approach transforms the m-to-n bit classical mapping into a t-bit unitary quantum operation with minimum number of auxiliary qubits, then a variation of Toffoli gate is used as the basic building block to construct the unitary operation. Finally, each of these building blocks can be decomposed into one-bit rotation and two-bit control-U gates. The efficiency of the network is taken into consideration by formulating it as a constrained set partitioning problem.

The rest of this paper is organized as follows. Section 2 describes the relation between permutation and our building block – $T(S, R, I)$ gate. The problem and algorithm are defined in section 3.1 and 3.2, the optimal solution is then derived in section 3.3 and 3.4. Finally, conclusions are given in section 4.

## 2 Gate Representation and Permutation

A Toffoli [15] gate consists of two control bits, $a$ and $b$, which do not change their values, and a target bit $c$ which changes its value only if $a = b = 1$. The gate can be written as:

$$\hat{a} = a$$
\[
\hat{b} = b \\
\hat{c} = (a \land b) \oplus c
\]  

where $\oplus$ denotes exclusive-or and $\land$ stands for logical AND. The three-bit Toffoli gate is a universal gate. A variation of the three-bit Toffoli gate is $n$-bit Toffoli gate, indicated by

\[
T(S, R, I) \quad S, R, I \in \{0, 1\}^n, \quad \Delta(I, \{0\}^n) = 1, \quad S \land R = R \land I = S \land I = 0
\]

where $\Delta(x, y)$ is the Hamming distance between $x$ and $y$, $\land$ stands for bit-wise logical AND operation. The function of a generalized Toffoli gate is similar to that of a three-bit Toffoli gate. All input bits are left unchanged while the target bit is inverted conditionally. In the notation shown above, $S$ and $R$ are indicators that, if expressed in binary digits, mark the position of control bits. The bits that are set in $S$ specify the control bits that have to be 1’s to activate the logic. Similarly, the bits that are set in $R$ specify the bits that have to be 0’s to activate the logic. $I$ simply represents the target bit to be inverted when the conditions of $S$ and $R$ are satisfied. Those bits that are not specified in either $S$, $R$, or $I$ are don’t care bits. Assuming $n$-bit input $X = x_{n-1}x_{n-2}\cdots x_1x_0$ and target bit $x_r$, the operation of a $n$-bit Toffoli gate, $T(S, R, I)$, can be written as:

\[
\hat{x}_i = x_i, \quad i = 0, 1, \ldots, r - 1, r + 1, \ldots, n - 1 \\
\hat{x}_r = (\land_{i=0}^{n-1} ((s_i \land x_i) \lor (r_i \land \bar{x}_i) \lor (\bar{s}_i \land \bar{r}_i))) \oplus x_r
\]

Using this notation, a three-bit Toffoli gate can be represented as $T_t(110, 000, 001)$, and a control-not gate is written as $T_{cn}(10, 00, 01)$.

Since the time evolution of any quantum transformation is a unitary and logically reversible process, thus any quantum boolean logic can be represented using permutation. A permutation is a one-to-one and onto mapping from a finite order set onto itself. A typical permutation $P$ is represented using the symbol

\[
P = \left( \begin{array}{cccccc}
a & b & c & d & e & f \\
d & e & c & a & f & b \\
\end{array} \right)
\]

This permutation changes $a \rightarrow d$, $d \rightarrow a$, $b \rightarrow e$, $e \rightarrow f$, and $f \rightarrow b$. The state $c$ stays unchanged. A permutation can also be expressed as disjoint cycles. A cycle includes its members in a list like

\[
C = (e_1, e_2, \ldots, e_{n-1}, e_n).
\]

The order of the elements describes the permutation. For example, in Eq.(5), the cycle takes $e_1 \rightarrow e_2$, $e_2 \rightarrow e_3$, $\ldots$, $e_{n-1} \rightarrow e_n$, and finally $e_n \rightarrow e_1$. The number of elements in a cycle
is called its length. A cycle with length 1 is called a trivial cycle, which does not change anything. A cycle of length 2 is called a transposition. Using this notation, the same permutation $P$ shown in Eq. (4) can be written as

$$P = (a, d)(c)(b, e, f) = (a, d)(b, e, f)$$  \hspace{1cm} (6)

Note that a trivial cycle is generally not shown in a permutation.

For each permutation $P$, there always exists a permutation $P^{-1}$ that puts the object back into their place. $P^{-1}$ can be derived simply by interchanging the two rows of $P$ or, if cycles are used, reversing the order of the components in each cycle. A permutation that does not change the order of the objects is called an identity, indicated by $E$. If two permutations, $P_1$ and $P_2$, are performed successively, we called this the product of $P_1$ and $P_2$. Following the convention, we write the first permutation on the right hand side as $P = P_2P_1$. Clearly, $EP = PE = P$ and $PP^{-1} = P^{-1}P = E$. Permutations do not commute, i.e. $P_1P_2 \neq P_2P_1$ for general $P_1$ and $P_2$.

A quantum boolean logic gate can then be expressed using the notation described above. For example, a control-not gate is indicated by $P_{cn} = (10, 11)$. Since it changes $10 \rightarrow 11$ and $11 \rightarrow 10$, leaving all other states unchanged. Similarly, a three-bit Toffoli gate is indicated by $P_t = (110, 111)$.

3 Quantum Boolean Logic Construction

3.1 Problem Description

The problem of transforming any $m$-to-$n$ bit combinational boolean logic into quantum operation can be formalized as follows:

**Problem** : Given a classical $m$-to-$n$ bit combinational boolean logic

$$C : A(\{0, 1\}^m) \rightarrow B(\{0, 1\}^n),$$  \hspace{1cm} (7)

and an integer $p \ (0 \leq p \leq m)$, construct a $t$-bit permutation

$$Q : \Psi(\{0, 1\}^t) \rightarrow \Psi(\{0, 1\}^t)$$  \hspace{1cm} (8)

such that for each classical mapping $\alpha = \alpha_0\alpha_1\cdots\alpha_{m-1} \in A \ (\alpha_i \in \{0, 1\})$ and $\beta = C(\alpha) = \beta_0\beta_1\cdots\beta_{n-1} \ (\beta_i \in \{0, 1\})$, there exist two states $\psi = \psi_0\psi_1\cdots\psi_{m-1}\cdots\psi_{t-1} \in \Psi$
and $\phi = \phi_0\phi_1 \cdots \phi_{t-1} \in \Psi$ satisfying:

1. $\psi_i = \alpha_i$, for $i = 0, 1, \ldots, m - 1$
2. $\psi_i = 0$, for $i = m, m + 1, \ldots, t - 1$
3. $Q(\psi) = \phi$
4. $\phi_i = \alpha_i$, for $i = 0, 1, \ldots, p - 1$
5. $\phi_i = \beta_{i-p}$, for $i = p, p + 1, \ldots, p + n - 1$

The construction process is described in the following sections.

### 3.2 Building the Quantum Transformation Table

For any classical combinational boolean logic, a *Classical Transformation Table* can be used to describe the behavior of the circuits. Taking an $m$-to-$n$ bit circuits as example, a classical transformation table consists of two parts, a $2^m$-by-$m$ $\alpha$ table for input, and a $2^m$-by-$n$ $\beta$ table for output. In the $\alpha$ table, there are $2^m$ rows, numbering from $\alpha[0][\ast]$ to $\alpha[2^m-1][\ast]$, and $m$ columns, numbering from $\alpha[\ast][0]$ to $\alpha[\ast][m-1]$. Similarly, there are $2^m$ rows in the $\beta$ table, numbering from $\beta[0][\ast]$ to $\beta[2^m-1][\ast]$, and $n$ columns, numbering from $\beta[\ast][0]$ to $\beta[\ast][n-1]$. Each row of the $\alpha$ table contains an $m$-bit input pattern, the same row of the $\beta$ table contains the corresponding $n$-bit output.

As in the classical case, a *Quantum Transformation Table* is used to describe a $t$-bit quantum combinational boolean logic. A quantum transformation table consists of two parts, a $2^t$-by-$t$ $\psi$ table for input, and a $\phi$ table of the same size for output. Both tables have $t$ bits in width, corresponding to $t$ input qubits, numbering from $\psi[\ast][0]$ to $\psi[\ast][t-1]$, and $t$ output qubits, numbering from $\phi[\ast][0]$ to $\phi[\ast][t-1]$. Similarly, both $\psi$ and $\phi$ are of length $2^t$, numbering from $\psi[0][\ast]$ to $\psi[2^t-1][\ast]$, and $\phi[0][\ast]$ to $\phi[2^t-1][\ast]$, corresponding to all combination of state patterns. Each row of the $\psi$ table contains a $t$-bit input pattern, the same row of the $\phi$ table contains the corresponding $t$-bit output. Because the quantum operation is a reversible unitary transformation, the $2^t$ rows in the $\phi$ table are simply a permutation of the input patterns.

The steps to build the quantum transformation table that based on the classical circuits

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5 We use the notation $A[i][\ast]$ to denote the $i$-th row, starting from column 0, all the way to the end. The notation $A[i][m : n]$, denotes the $i$-th row, from column $m$ to column $n$. Similar notations are used to denote column and block.
is shown below:

**Step I. Preserve the input qubits.**
We define the *preserved* bits to be the input bits that have to stay unchanged after the operation, while *volatile* bits are input bits that can be over-written by output bits. Preserved bits can be used as inputs for other circuits again. Without loss of generality, assume qubits 0 to \( p - 1 \) (\( 0 \leq p \leq m \)) are the bits to be preserved and qubits \( p \) to \( m - 1 \) are volatile bits. Note that \( p \) can be zero, in which case no input bit is preserved. Now prepare two empty tables, \( \psi \) and \( \phi \), which are both of size \( 2^m \)-by-\( m \). For each row \( i \) (\( 0 \leq i \leq 2^m - 1 \)), copy \( \alpha[i][0:m-1] \) to \( \psi[i][0:m-1] \). If \( p \neq 0 \), also copy the preserved bits from \( \alpha[i][0:p-1] \) to \( \phi[i][0:p-1] \), where \( 0 \leq i \leq 2^m - 1 \).

**Step II. Assign the output qubits.**
Since qubit 0 to \( p - 1 \) are used to preserve the input bits, assign qubit \( p \) to \( p + n - 1 \) to hold the output bits. Expand the width of the \( \phi \) table whenever needed. For each row \( i \) (\( 0 \leq i \leq 2^m - 1 \)), copy \( \beta[i][0:n-1] \) to \( \phi[i][p:p+n-1] \).

**Step III. Distinguish each output state.**
For a unitary quantum evolution, the quantum transformation table needs to be one-to-one and onto. For any two patterns \( x, y \in \{0,1\}^{p+n} \) in \( \phi \), if \( x \neq y \), then set \( d = 0 \), go to step IV. Otherwise, set

\[
d = \lceil \log_2 M \rceil \tag{9}
\]

where \( M \) is the maximum number of occurrences for a single pattern. Add extra \( d \) columns (numbering from \( \phi[*][p+n] \) to \( \phi[*][p+n+d-1] \)) to the \( \phi \) table. Expand the width of the \( \phi \) table whenever needed. For each row \( i \) that has a repeated pattern, assign a unique \( d \)-bit pattern to \( \phi[i][p+n:p+n+d-1] \), so that each row in the \( \phi \) table has a different bit pattern. Note that input bits are good candidates that can be used to distinguish the output patterns.

**Step IV. Add auxiliary qubits**
If \( m = p + n + d \), no auxiliary qubit is needed. The total number of qubits, \( t \), equals \( m \), go to Step V. Otherwise, if \( m < p + n + d \), set \( a = (p + n + d) - m \) and add \( a \) auxiliary qubits to the \( \psi \) table (numbering from \( \psi[*][m] \) to \( \psi[*][m+a-1] \)). Assign these qubits to be all 0’s. The total number of qubits, \( t \), equals \( p + n + d \).

**Step V. Expand the quantum transformation table**
If auxiliary qubits are used, expand both $\psi$ and $\phi$ tables to be $2^t$ rows in length. For the $\psi$ table, repeat the original block $2^a$ times and, for each block, fill in the auxiliary qubits with a unique $a$-bit pattern. For the $\phi$ table, leave the new entries blank.

Based on the constraints derived from the classical boolean circuit, the quantum transformation table is now partially constructed. The permutation can be completed simply by filling in the blanks and make it a one-to-one and onto mapping. However, to implement the quantum operation efficiently, the permutation should be carefully selected based on the elementary gate count. To do this, the gate count evaluation function is introduced in the next section.

### 3.3 Implementation and Gate Count Evaluation

The rules that are used to implement an arbitrary permutation is summarized as follows.

**Proposition I.** Given any two states $p$ and $q$ with $\Delta(p, q) = 1$, the transposition $U = (p, q)$ can be implemented using $T(S, R, I)$, where

$$S = p \land q, \quad R = \bar{p} \land \bar{q}, \quad I = p \oplus q.$$  \hspace{1cm} (10)

This proposition shows how a transposition of two adjacent states can be implemented using one $T(S, R, I)$ gate. Note that the $T(S, R, I)$ gate can be further decomposed into one-bit rotation and two-bit control-U gates [8].

With necessary modification, Proposition I. can be generalized to implement a transposition of two non-adjacent states as follows:

**Proposition II.** Given any two general states $p$ and $q$, with $\Delta(p, q) = d$, the transposition $U = (p, q)$ can be done using $2d - 1$ adjacent state transpositions.

The implementation of a transposition with distance $d$ can be done in the following way. Assume, in binary expression,

$$p = b_0b_1b_2 \cdots b_{t_1} \cdots b_{t_{d-1}} \cdots b_{t_d} \cdots b_{n-1}$$  \hspace{1cm} (11)

$$q = b_0b_1b_2 \cdots \bar{b}_{t_1} \cdots \bar{b}_{t_{d-1}} \cdots \bar{b}_{t_d} \cdots b_{n-1}$$  \hspace{1cm} (12)

where $b_t \in \{0, 1\}$. Then the transposition $U = (p, q)$ can be constructed as follows:
(i) Find a list of states, \( s_1, s_2, \ldots, s_{d-1} \), between \( p \) and \( q \), such that for \( 1 \leq i \leq d-2 \),

\[
\Delta(p, s_i) = \Delta(s_{i+1}, s_i) = \Delta(s_{d-1}, q) = 1 \tag{13}
\]

An example of the list is shown as follows:

\[
\begin{align*}
p &= b_0 b_1 b_2 \cdots b_t \cdots b_{t_{d-1}} \cdots b_t \cdots b_{n-1} \\
s_1 &= \bar{b}_0 b_1 b_2 \cdots \bar{b}_t \cdots b_{t_{d-1}} \cdots b_t \cdots b_{n-1} \\
s_2 &= \bar{b}_0 b_1 b_2 \cdots \bar{b}_t \cdots \bar{b}_{t_{d-1}} \cdots b_t \cdots b_{n-1} \\
& \vdots \\
s_{d-1} &= \bar{b}_0 b_1 b_2 \cdots \bar{b}_t \cdots \bar{b}_{t_{d-1}} \cdots \bar{b}_t \cdots b_{n-1} \\
q &= \bar{b}_0 b_1 b_2 \cdots \bar{b}_t \cdots \bar{b}_{t_{d-1}} \cdots \bar{b}_t \cdots b_{n-1} \\
\end{align*}
\tag{14}
\]

(ii) For the list \( p, s_1, s_2, \ldots, s_{d-1}, q \), perform the following adjacent state transpositions:

\[
(p, s_1)(s_1, s_2)(s_2, s_3) \cdots (s_{d-2}, s_{d-1})(s_{d-1}, q)(s_{d-2}, s_{d-1}) \cdots (s_2, s_3)(s_1, s_2)(p, s_1) \tag{15}
\]

All the transpositions shown above are performed on two adjacent states and hence can be implemented using \( T(S, R, I) \) gates as described in Proposition I.

Once the transposition of two arbitrary states can be performed. A general cycle of length \( n \) can be constructed. For a trivial cycle, no gates are needed. For a cycle of length 2, the implementation can be easily derived using Proposition I. and Proposition II. For a cycle of length \( n \) (\( n \geq 3 \)), the following rules are used:

**Proposition III.** Given a general cycle \( C = (p_0, p_1, p_2, \ldots, p_{n-1}) \), \( C \) can be constructed using \( n-1 \) transpositions:

\[
\begin{align*}
C &= (p_0, p_1, p_2, \ldots, p_{n-1}) \\
 &= (p_1, p_0)(p_2, p_1) \cdots (p_{n-2}, p_{n-3})(p_{n-1}, p_{n-2}) \\
\end{align*}
\tag{16}
\]

Each of these transpositions can be decomposed into \( T(S, R, I) \) gates using Proposition I. and Proposition II.

**Proposition IV.** A permutation consists of one or multiple disjoint cycles. Since disjoint cycles commute, so each cycle in the permutation can be implemented individually.
Given a general cycle \( C = (p_0, p_1, p_2, \ldots, p_{n-1}) \), the distances between any two states \( d_i = \Delta(p_i, p_{i+1}) \) for \( i = 0, 1, \ldots, n - 2 \), and \( d_{n-1} = \Delta(p_{n-1}, p_0) \). Assume \( d_m \geq d_i \) for every \( i \), the minimum number of \( T(S, R, I) \) gates for \( C \) can be achieved using the following transpositions:

\[
(p_{m+1}, p_{m+2})(p_{m+2}, p_{m+3}) \cdots (p_{n-2}, p_{n-1})(p_{n-1}, p_0)(p_0, p_1)(p_1, p_2) \cdots (p_{m-1}, p_m) \tag{17}
\]

and the total \( T(S, R, I) \) gate count is

\[
\Omega_C^T = \sum_{i=0}^{n-1} (2d_i - 1) - (2d_m - 1) \tag{18}
\]

Each of these \( T(S, R, I) \) gates can be further decomposed into one-bit rotation and two-bit control-U gates [8]. This results in a network with \( \Omega_C^E \) elementary quantum gates.

In general, the permutation is a product of disjoint cycles,

\[
P = C_0C_1C_2 \cdots C_{n-2}C_{n-1} \tag{19}
\]

The gate count for \( P \) is then

\[
\Omega_P^E = \sum_{i=0}^{n-1} \Omega_{C_i}^E \tag{20}
\]

To build an efficient circuit, the permutation table has to be constructed with minimum \( \Omega_P^E \). This problem is described in the next section.

### 3.4 Complete the Permutation with Minimum Gate Count

Define the digraph \( G = (V, E) \), where

\[
V = \{v_i \mid v_i = \psi[i][*], 0 \leq i \leq 2^t - 1\}
\]

\[
E = \{(v_s, v_d) \mid v_s = \psi[i][*], v_d = \phi[i][*], 0 \leq i \leq 2^t - 1\} \tag{21}
\]

The digraph has \( 2^t \) vertices, corresponding to each of the \( 2^t \) rows in the \( \psi \) table. An edge is defined from \( v_s \) to \( v_d \) if it is possible for \( Q \) to map \( v_s \) to \( v_d \). The \( \cong \) is used to denote, when only \( u \) \((u < t)\) bits are specified in \( \phi[i][*] \), all states that are compatible to the current entry. This results in \( 2^{t-u} \) edges to be generated for each of the possible \((v_s, v_d)\) pairs. Filling in the \( t-u \) blank bits in the \( \phi \) table selects one of the possible edges and delete others.

Using the digraph \( G \), the problem is equivalent to finding a set of disjoint cycles that cover all the vertices in \( V \) with minimum elementary gate count. This is formulated as
follows:

**Problem**: Given a digraph $G = (V, E)$ and the cost $\Omega^E_{C_i}$ associated with each cycle $C_i$, find a family of sets $S = \{S_i \mid S_i = \{v_i^0, v_i^1, \ldots, v_i^{n_i-1}\}, v_j^i \in V\}$ and corresponding cycles $C = \{C_i \mid C_i = (v_i^0, v_i^1, \ldots, v_i^{n_i-1}), v_j^i \in V\}$ with minimum

$$\sum_{C_i \in C} \Omega^E_{C_i}$$ (22)

subject to:

(1) $\bigcup_{S_i \in S} S_i = V$
(2) $\bigcap_{S_i \in S} S_i = \emptyset$

The problem is essentially a *constrained set partitioning problem*, with each partition being a cycle. There are many set partitioning problems that have been studied in graph theory [16] and operations research related works [17]. A simple but effective algorithm is described here to demonstrate how the elementary gate count is minimized.

**Step I. Enumerate all cycles.**
Given the graph $G(V, E)$ described in the quantum transformation table, list all cycles $C_i$ ($i = 0, 1, 2, \ldots$) in the graph. This can be done in the following way:

(i) Select a target edge $(v_\psi, v_\phi)$, list all cycles containing the edge. To find all cycles containing $(v_\psi, v_\phi)$, just list all paths from $v_\phi$ to $v_\psi$, then cycles can be found by concatenating any path from $v_\phi$ to $v_\psi$ with the edge $(v_\psi, v_\phi)$.

(ii) Delete the target edge in (i). If there is any edge left in $G$, go to (i), otherwise all cycles are found. For each cycle $C_i$, calculate the elementary gate count $\Omega^E_{C_i}$.

**Step II. Initialization.**
Let $X = \{x_i\}$ be a $n \times 1$ matrix with $x_i = 1$ if $C_i$ is included in the solution, and $x_i = -1$ if it has been excluded. Initially set $x_i = 0$ for each $i$. Also let $A = \{a_{ij}\}$ be an $m \times n$ matrix with $a_{ij} = 1$ if $v_i \in C_j$, and $a_{ij} = 0$ if $v_i \notin C_j$.

**Step III. Reduction (optional).**
The optional reduction process makes the optimization task easier. Although there are many effective rules, only three reductions are described here. Let $S$ be an $n \times 1$ matrix with all elements set to 1’s and $R = \{r_i\} = AS$ be the $m \times 1$ matrix that describes the coverage of the vertices.

(i) If $r_i = 0$ for any $i$, no solution exists.
(ii) For any \( i \), if \( r_i = 1 \) and \( a_{ij} = 1 \), then mark \( C_j \) as included.

(iii) If \( C_j \) denotes any cycle that has been included, then all \( C_k \) with \( C_k \cap C_j \neq \emptyset \) must be marked as excluded.

The reduction rules can be applied over again until no further reduction is possible.

**Step IV. Search the optimal solution.**

A depth-first search algorithm is used here to search the optimal solution.

(i) Set the initial elementary gate count to be \( M = \infty \).

(ii) If all vertices are covered, update \( M \) and record \( X \) in case \( \Omega_{C_j}^F < M \), return. Otherwise, for each \( C_j \) that has not been marked, update \( X \) to include \( C_j \), apply the reduction rules as described in Step III, then recursively call step (ii) with the parameter \( X \).

After these steps are done, the selected cycles are recorded and the optimal gate count is in \( M \).

### 4 Conclusions

We have derived a systematic way of realizing any general \( m \)-to-\( n \) bit combinational boolean logic using elementary quantum gates. Our approach transforms the \( m \)-to-\( n \) bit classical mapping into a \( t \)-bit unitary quantum operation with minimum number of auxiliary qubits. The efficiency of the network is taken into consideration by formulating it as a constrained set partitioning problem. This method can be used to transform classical combinational logic into its quantum version, which is crucial for a general-purpose quantum computer.

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