Local Hölder continuity for fractional nonlocal equations with general growth

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Abstract
We study generalized fractional $p$-Laplacian equations to prove local boundedness and Hölder continuity of weak solutions to such nonlocal problems by finding a suitable fractional Sobolev-Poincaré inequality.

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1 Introduction

In this paper we study the fractional nonlocal equation

$$\mathcal{L} u = 0 \quad \text{in} \quad \Omega$$

defined on a bounded domain $\Omega$ in $\mathbb{R}^n$ by

$$\mathcal{L} v(x) := \text{p.v.} \int_{\mathbb{R}^n} g \left( \frac{|v(x) - v(y)|}{|x - y|^s} \right) \frac{v(x) - v(y)}{|v(x) - v(y)|} K(x, y) \frac{dy}{|x - y|^s},$$

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where \(0 < s < 1, g: [0, \infty) \to [0, \infty)\) is a strictly increasing, continuous function that satisfies \(g(0) = 0, \lim_{t \to \infty} g(t) = \infty\) and

\[
1 < p \leq \frac{t g(t)}{G(t)} \leq q < \infty \quad \text{for some} \quad 1 < p \leq q, \quad \text{where} \quad G(t) := \int_0^t g(s) \, ds.
\]

(1.3)

Note that the inequality means that the growth of the function \(G\) is in between \(p\) and \(q\), which is naturally obtained from the \(p\)-th power function case. This condition covers not only the power case \(g(t) = t^{p-1},\) but also \(g(t) = t^{p-1} \log (e + t),\) \(g(t) = t^{p-1} + t^{q-1},\) and so on. For more examples of \(g\) and applications of problems with general growth, we refer to \([3, 27, 46]\).

\(K: \mathbb{R}^n \times \mathbb{R}^n \to (0, \infty)\) is a symmetric, i.e., \(K(x, y) = K(y, x)\), and measurable kernel that satisfies

\[
\frac{\lambda}{|x - y|^n} \leq K(x, y) \leq \frac{\Lambda}{|x - y|^n}, \quad x, y \in \mathbb{R}^n,
\]

for some \(0 < \lambda \leq \Lambda.\) Notice that the symmetricity condition on \(K\) is not necessary. However, by considering the kernel \(\tilde{K}(x, y) = \frac{K(x, y) + K(y, x)}{2},\) we shall always assume the symmetricity, see \([35, \text{Section 1.5}]\) for more details. A main point is that the function \(G\) is an \(N\)-function satisfying the \(\Delta_2\) and \(\nabla_2\) conditions (see the next section) and that a simple example of the kernel \(K(x, y)\) is when \(\lambda \leq a \leq \Lambda.\) Note in particular case when \(K(x, y) = |x - y|^{-n}, \) \(\mathcal{L}\) is the so-called \(s\)-fractional \(G\)-Laplace operator and we denote it by \(\mathcal{L} = (-\Delta)^s_G.\)

The goal of this paper is to establish local Hölder regularity for the nonlocal problem (1.1) without a priori boundedness assumption of a weak solution. In addition we will discuss the existence and uniqueness of a weak solution to (1.1) with Dirichlet boundary conditions.

An obvious example is \(g(t) = t\) and \(K(x, y) = |x - y|^{-n},\) in which case it reduces to the \(s\)-fraction Laplace operator \((-\Delta)^s\) and Caffarelli, Chan and Vasseur [5] proved Hölder regularity result for the corresponding parabolic fractional equation by the approach of De Giorgi modified to the nonlocal setting. We refer to \([5, 6, 30, 31, 36, 40, 47]\) for various regularity results, including the Harnack inequality, self-improving property and \(L^p\)-regularity, for weak solutions to the fractional nonlocal linear equations. On the other hand for the fractional \(p\)-Laplacian type equations, i.e., \(g(t) = t^{p-1}\) with \(1 < p < \infty,\) Di Castro, Kuusi and Palatucci [14] proved local Hölder regularity by employing the so-called tail (see the next section). We further refer to \([9, 10, 13, 19, 24, 25, 32–35, 38, 42, 43]\) for studies on the nonlocal nonlinear equations of the fractional \(p\)-Laplacian type.

A general non-autonomous fractional nonlocal operator can be written as

\[
\mathcal{L} v(x) := \text{p.v.} \int_{\mathbb{R}^n} h \left( x, y, \frac{|v(x) - v(y)|}{|x - y|^s} \right) v(x) - v(y) \frac{K(x, y)}{|x - y|^s} dy.
\]
If \( h(x, y, t) = t^{p-1} \), then we say that the operator or equation satisfies the \( p \)-growth condition. On the other hand, if \( h(x, y, t) \) has a more general structure, then we say that the operator or equation satisfies a non-standard growth condition. Typical examples of non-standard growth conditions include the variable growth condition: 
\[ h(x, y, t) = t^{p(x,y)-1}, \]
the double phase condition: 
\[ h(x, y, t) = t^{p-1} + a(x, y)t^{q-1}, \]
and the general growth condition: 
\[ h(x, y, t) = g(t). \]
Recently there has been a great deal of studies concerning fractional nonlocal equations with nonstandard growth conditions, in particular for Hölder regularity in \([7, 45]\) with the variable growth condition and in \([4, 12, 20]\) with the double phase condition, respectively.

We are mainly focusing on the general growth condition. The local one corresponding to the nonlocal equation (1.1) is the so called \( G \)-Laplace equation:
\[
\text{div} \left( g(|Du|) \frac{Du}{|Du|} \right) = 0 \quad \text{in } \Omega, \quad \text{where } g(t) = g'(t),
\]
for which Lieberman [37] proved the \( C^{0,\alpha} \) continuity of weak solutions under the condition (1.3). We also refer to \([11, 18, 28, 29, 39]\) and references therein for the regularity results for equation of the \( G \)-Laplacian type. In particular, the papers \([11, 39]\) deal with problems modeled by the \( G \)-Laplace equation with \( G \) violating the \( \Delta_2 \)-or \( \nabla_2 \)-condition. According to the local regularity results for the \( G \)-Laplace equation, \( C^{0,\alpha} \)-regularity has been a natural, interesting issue for the corresponding nonlocal equation (1.1), and has been studied in the recent papers \([8, 22, 23]\). The results in the above papers however are established with a strong Dirichlet boundary condition or the boundedness of a weak solution or a restrictive condition on \( g \) such that \( q < p^* \) in (1.3). In this paper, on the other hand, we prove the local Hölder regularity for any weak solution to (1.1) with the assumption (1.3) only, and so without boundary data and any bounded a priori assumption. Therefore, we first obtain the local boundedness of a weak solution with a suitable estimate (1.5). To this end, we focused on finding inequalities and embeddings on fractional Orlicz-Sobolev spaces \( W^{s,G} \). Especially, we proved an integral version of a fractional Sobolev-Poincaré inequality in \( W^{s,G} \) which plays a major role in the proof of the main result (see Lemma 4.1).

With the definition of a weak solution, the related function spaces and the tail to be introduced in details in the next section, we now state our main result.

**Theorem 1.1** Let \( 0 < s < 1 \). Suppose that \( u \in \mathbb{V}^{s,G}(\Omega) \cap L^s_x(\mathbb{R}^n) \) is a weak solution to (1.1) with (1.2), (1.3) and (1.4). Then \( u \in C^{0,\alpha}_{\text{loc}}(\Omega) \) for some \( \alpha \equiv \alpha(n, s, p, q, \lambda, \Lambda) \in (0, 1) \). Moreover, there exist positive constants \( c_b \) and \( c_h \) depending on \( n, s, p, q, \lambda, \Lambda \) such that for any \( B_r(x_0) \subseteq \Omega \),
\[
\|u\|_{L^\infty(B_{r/2}(x_0))} \leq c_b r^s G^{-1} \left( \int_{B_r(x_0)} G \left( \frac{|u|}{r^s} \right) dx \right) + r^s g^{-1}(r^s \text{Tail}(u; x_0, r/2))
\]
(1.5)
and
\[
[u]_{C^{0,\alpha}(B_{r/2}(x_0))} \leq \frac{c_h}{r^\alpha} \left[ r^s G^{-1} \left( \int_{B_r(x_0)} G \left( \frac{|u|}{r^s} \right) \, dx \right) + r^s g^{-1} \left( r^s \text{Tail}(u; x_0, r/2) \right) \right].
\]

(1.6)

**Remark 1.2** We can get the same results in Theorem 1.1 under a slightly weaker condition on \( g \) that \( g(0) = 0, \lim_{t \to \infty} g(t) = \infty \), and
\[
\frac{g(s)}{s^{p-1}} \leq L \frac{g(t)}{t^{p-1}} \quad \text{and} \quad \frac{g(t)}{t^{q-1}} \leq L \frac{g(s)}{s^{q-1}} \quad \text{for all} \quad 0 < s \leq t,
\]
for some \( 1 < p \leq q \) and \( L \geq 1 \). Note that the above inequality implies the \( \Delta_2 \)- and the \( \nabla_2 \)-conditions of the function \( t \mapsto t g(t) \). Under this condition, there exists an increasing continuous function \( \tilde{g} \) with \( \tilde{g}(0) = 0 \) and \( \lim_{t \to \infty} \tilde{g}(t) = \infty \) such that \( \tilde{g} \) satisfies (1.3) and \( \tilde{g} \approx g \) (i.e., there exists a positive constant \( c \geq 1 \) such that \( c^{-1} g \leq \tilde{g} \leq c g \)). See [27, Chapter 2] for details. Then we get the result in Theorem 1.1 with respect to \( \tilde{g} \). Therefore, by the equivalence, we can obtain the same estimates for \( g \). Nevertheless, we adapt the condition (1.3) instead of the above one, as the proof of the equivalence is rather technical and the condition (1.3) is simpler, practical and used widely.

Our proof of the theorem is based on the De Giorgi approach established in [14], in particular, for the fractional \( p \)-Laplacian type equations in the setting of fractional Sobolev space \( W^{s,p} \). On the other hand, to the fractional \( G \)-Laplacian type equations, this approach can not be directly applied, as \( G(st) \not\approx G(s)G(t) \) (this equivalence is true when \( G(t) = t^p \)). Indeed, we are forced to face a more complicated and delicate situation under which we need to make a very careful systematic analysis to overcome the complexity and difficulty coming from such a \( G \)-Laplacian type nonlocal problem. Moreover, an integral version of Sobolev-Poincaré type inequality plays an essential role in the process of De Giorgi iteration, which is not known in the fractional Orlicz-Sobolev space as of today, as far as we are concerned. Therefore in this paper we obtain this estimate in Lemma 4.1.

The paper is organized as follows. In the next section we introduce notation, functions spaces, weak solutions and fundamental inequalities that will be used throughout this paper. Section 3 is devoted to deriving two essential estimates for weak solutions to (1.1). One is a Caccioppoli type inequality and the other is a logarithmic estimate. In Sect. 4 we prove Theorem 1.1 by proving the local boundedness and then Hölder continuity of a weak solution. The final section includes the existence and uniqueness of weak solutions to (1.1) with the Dirichlet boundary condition.

## 2 Preliminaries

In this paper \( B_r(x_0) \) denotes the ball in \( \mathbb{R}^n \) with radius \( r > 0 \) centered at \( x_0 \in \mathbb{R}^n \). When the center is clear in the context, we write it by \( B_r \) for the sake of simplicity.
The average of an integrable function \( f \) on \( B_r \) is defined as

\[
(f)_{B_r} = \frac{1}{|B_r|} \int_{B_r} f \, dx.
\]

We denote by \( c \) to mean a universal constant that can be computed by given quantities such as \( n, s, p, q, \lambda, \Lambda \). This generic constant can vary from line to line.

### 2.1 Properties for function \( G \)

Throughout this paper we always assume that \( G \in C^1([0, \infty)) \) satisfies (1.3). Then \( G : [0, \infty) \to [0, \infty) \) is an N-function (nice Young function), i.e., it is increasing and convex, and satisfies

\[
\lim_{t \to 0^+} \frac{G(t)}{t} = 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{G(t)}{t} = \infty.
\]

We always assume \( G(1) = 1 \).

The conjugate function \( G^* : [0, \infty) \to [0, \infty) \) is defined by

\[
G^*(t) := \sup_{s \geq 0} (st - G(s)), \quad t \geq 0.
\]

Then we have from (1.3) that for every \( t \in [0, \infty) \),

\[
a^q G(t) \leq G(at) \leq a^p G(t) \text{ if } 0 < a < 1 \quad \text{and} \quad a^p G(t) \leq G(at) \leq a^q G(t) \text{ if } a > 1,
\]

and that

\[
a^p' G^*(t) \leq G^*(at) \leq a^q' G^*(t) \text{ if } 0 < a < 1
\]

\[
\text{and} \quad a^q' G^*(t) \leq G^*(at) \leq a^p' G^*(t) \text{ if } a > 1.
\]

(2.1)

where \( p' \) and \( q' \) are the Hölder conjugates of \( p \) and \( q \), respectively. Also we see that \( G \) satisfies the following \( \Delta_2 \)- and \( \nabla_2 \)-conditions (see [41, Proposition 2.3]):

(\( \Delta_2 \)) there exists a constant \( \kappa > 1 \) such that

\[
G(2t) \leq \kappa G(t) \quad \text{for all} \quad t \geq 0;
\]

(2.3)

(\( \nabla_2 \)) there exists a constant \( l > 1 \) such that

\[
G(t) \leq \frac{1}{2l} G(lt) \quad \text{for all} \quad t \geq 0,
\]

(2.4)

where the constants \( \kappa \) and \( l \) are to be determined by \( q \) and \( p \). Note that \( G \) satisfies the \( \nabla_2 \)-condition if and only if \( G^* \) does the \( \Delta_2 \)-condition. In addition, from the definition of the conjugate function, we have

\[
ts s \leq G(t) + G^*(s), \quad t, s \geq 0.
\]

(2.5)
From (2.1), we deduce that for every $\epsilon \in (0, 1)$

$$ts \leq \epsilon^{1-q}G(t) + \epsilon G^*(s), \quad t, s \geq 0,$$

which is Young’s inequality with $\epsilon$. We further have from (1.3) that

$$G^*(g(t)) = tg(t) - G(t) \leq (q - 1)G(t), \quad t \geq 0. \quad (2.7)$$

Also the convexity and (2.1) imply

$$2^{-1}(G(t) + G(s)) \leq G(t + s) \leq 2^{q-1}(G(t) + G(s)),$$

which will be used often later in this paper.

### 2.2 Fractional Orlicz–Sobolev spaces

For an open subset $U$ in $\mathbb{R}^n$, we denote by $\mathcal{M}(U)$ to mean the class of all real-valued measurable functions on $U$. For an N-function $G$ satisfying the $\Delta_2$ and $\nabla_2$ conditions, we define the Orlicz space $L^G(U)$ as

$$L^G(U) := \left\{ v \in \mathcal{M}(U) \mid \int_U G(|v(x)|) \, dx < \infty \right\},$$

which is a Banach space with the Luxemburg norm defined as

$$\|v\|_{L^G(U)} := \inf \left\{ \lambda > 0 \mid \int_U G\left(\frac{|v(x)|}{\lambda}\right) \, dx \leq 1 \right\}.$$

Then note that

$$\|v\|_{L^G(U)} \leq \int_U G(|v|) \, dx + 1 \quad (2.8)$$

We next let $0 < s < 1$ and define the fractional Orlicz-Sobolev space $W^{s,G}(U)$ as

$$W^{s,G}(U) := \left\{ v \in L^G(U) \mid \int_U \int_U G\left(\frac{|v(x) - v(y)|}{|x-y|^s}\right) \frac{dxdy}{|x-y|^n} < \infty \right\},$$

which is also a Banach space with the norm

$$\|v\|_{W^{s,G}(U)} := \|v\|_{L^G(U)} + [v]_{s,G,U},$$

where $[v]_{s,G,U}$ is the Gagliardo semi-norm defined by

$$[v]_{s,G,U} := \inf \left\{ \lambda > 0 \mid \int_U \int_U G\left(\frac{|v(x) - v(y)|}{\lambda|x-y|^s}\right) \frac{dxdy}{|x-y|^n} \leq 1 \right\}.$$
Thus we have
\[ [v]_{s,G,U} \leq \int_U \int_U G\left(\frac{|v(x) - v(y)|}{|x - y|^s}\right) \frac{dxdy}{|x - y|^n} + 1. \tag{2.9} \]

We introduce the function space to which weak solutions of (1.2) belong, see the next subsection for the concept of a weak solution. We write
\[ C_\Omega := (\Omega \times \mathbb{R}^n) \cup (\mathbb{R}^n \times \Omega). \tag{2.10} \]

Then the space \( W_{s,G}(\Omega) \) consists of all functions \( v \in \mathcal{M}(\mathbb{R}^n) \) with \( v|_\Omega \in L^G(\Omega) \) and
\[ \iint_{C_\Omega} G\left(\frac{|v(x) - v(y)|}{|x - y|^s}\right) \frac{dxdy}{|x - y|^n} < \infty. \]

Note that if \( v \in W_{s,G}(\Omega) \), then \( v|_\Omega \in W_{s,G}(\Omega) \).

### 2.3 Weak solution and tail

We first recall \( g \) with (1.3) and \( K \) with (1.4) to define a weak solution to (1.1).

**Definition 2.1** \( u \in W_{s,G}(\Omega) \) is a weak solution (resp. subsolution or supersolution) to (1.1) if
\[ \iint_{C_\Omega} g\left(\frac{|u(x) - u(y)|}{|x - y|^s}\right) \frac{u(x) - u(y)}{|u(x) - u(y)|} (\eta(x) - \eta(y)) K(x, y) \, dxdy = 0 \ (resp. \leq 0 \ or \ \geq 0) \]
for any \( \eta \in W_{s,G}(\Omega) \) (resp. nonnegative \( \eta \in W_{s,G}(\Omega) \)) such that \( \eta = 0 \) in \( \mathbb{R}^n \setminus \Omega \).

We next write
\[ L_s^g(\mathbb{R}^n) := \left\{ u \in \mathcal{M}(\mathbb{R}^n) : \int_{\mathbb{R}^n} g\left(\frac{|u(x)|}{(1 + |x|)^s}\right) \frac{dx}{(1 + |x|)^{n+s}} < \infty \right\}, \]
and the tail of \( u \in L_s^g(\mathbb{R}^n) \) for the ball \( B_R(x_0) \) is denoted by
\[ \text{Tail}(u; x_0, R) := \int_{\mathbb{R}^n \setminus B_R(x_0)} g\left(\frac{|u(x)|}{|x - x_0|^s}\right) \frac{dx}{|x - x_0|^{n+s}}. \tag{2.11} \]

We notice that \( u \in L_s^g(\mathbb{R}^n) \) if and only if \( \text{Tail}(u; x_0, R) < \infty \) for all \( x_0 \in \mathbb{R}^n \) and \( R > 0 \). Indeed, for \( x \in \mathbb{R}^n \setminus B_R(x_0) \), a direct computation leads to
\[ \frac{1 + |x|}{|x - x_0|} \leq 1 + \frac{1 + |x_0|}{R}. \]
Then it follows from (2.1) that

\[
\text{Tail}(u; x_0, R) = \int_{\mathbb{R}^n \setminus B_R(x_0)} g \left( \frac{|u(x)|}{(1 + |x|)^r} \right) \left( \frac{1 + |x|}{|x - x_0|} \right)^{n+s} \frac{dx}{(1 + |x|)^{n+s}} \leq \left( 1 + \frac{1 + |x_0|}{R} \right)^{n+q} \int_{\mathbb{R}^n \setminus B_R(x_0)} g \left( \frac{|u(x)|}{(1 + |x|)^r} \right) \frac{dx}{(1 + |x|)^{n+s}} < \infty.
\]

To show the converse relation, choose two different points \(x_1, x_2\) with \(|x_1| > 1, |x_2| > 1\), and let \(0 < R \leq \frac{|x_1 - x_2|}{4}\). Then we find that for \(x \in \mathbb{R}^n\)

\[
\frac{|x - x_i|}{1 + |x|} \leq 1 + \frac{|x_i| - 1}{1 + |x|} \leq |x_i|, \quad i = 1, 2.
\]

Therefore we can estimate as above that

\[
\int_{\mathbb{R}^n} g \left( \frac{|u(x)|}{(1 + |x|)^r} \right) \frac{dx}{(1 + |x|)^{n+s}} \leq \int_{\mathbb{R}^n \setminus B_R(x_1)} g \left( \frac{|u(x)|}{(1 + |x|)^r} \right) \frac{dx}{(1 + |x|)^{n+s}} + \int_{\mathbb{R}^n \setminus B_R(x_2)} g \left( \frac{|u(x)|}{(1 + |x|)^r} \right) \frac{dx}{(1 + |x|)^{n+s}} \leq |x_1|^{n+q} \int_{\mathbb{R}^n \setminus B_R(x_1)} g \left( \frac{|u(x)|}{|x - x_1|^r} \right) \frac{dx}{|x - x_1|^{n+s}} + |x_2|^{n+q} \int_{\mathbb{R}^n \setminus B_R(x_2)} g \left( \frac{|u(x)|}{|x - x_2|^r} \right) \frac{dx}{|x - x_2|^{n+s}} < \infty.
\]

\[\text{Remark 2.2} \quad \text{Observe that}
\]

\[
R^s g^{-1} \left( R^s \text{Tail}(u; x_0, R) \right) = R^s g^{-1} \left( R^s \int_{\mathbb{R}^n \setminus B_R(x_0)} g \left( \frac{|u(x)|}{|x - x_0|^r} \right) \frac{dx}{|x - x_0|^{n+s}} \right).
\]

In particular, if \(g(t) = t^{p-1}\), (2.12) is reduced to

\[
\left[ R^s \int_{\mathbb{R}^n \setminus B_R(x_0)} \frac{|u(x)|^{p-1}}{|x - x_0|^{n+sp}} \, dx \right]^{\frac{1}{p-1}},
\]

which is the tail used in [14]. In this paper we use (2.11) instead of (2.12) for simplicity.

\[\text{Remark 2.3} \quad 1. \text{ Note that } W^{s,G}(\mathbb{R}^n) \subset W^{s,G}(\Omega) \cap L^s(\mathbb{R}^n).
\]

2. Let \(\psi\) be an N-function satisfying \(g(t) \leq c\psi(t)\) for \(t \geq t_0\), where \(c\) and \(t_0\) are some positive constants. If \(u \in L^\psi(\mathbb{R}^n)\) or \(u \in L^\psi(B_R(0)) \cap L^\infty(\mathbb{R}^n \setminus B_R(0))\), then \(u \in L^s(\mathbb{R}^n)\).
3 Auxiliary estimates

In this section we derive two estimates for the weak solutions to (1.1) that play essential roles in the proof of the main theorem. The first one is a Caccioppoli type estimate. A similar Caccioppoli type estimate in the Orlicz setting can be also found in [8].

Proposition 3.1 (Caccioppoli type estimate) Let \( u \in W^{s, G}(\Omega) \cap L^q(\mathbb{R}^n) \) be a weak solution to (1.1). Then for any \( k \geq 0 \), \( B_r \equiv B_r(x_0) \subseteq \Omega \) and \( \phi \in C^\infty_0(B_r) \) with \( 0 \leq \phi \leq 1 \), we have

\[
\int_{B_r} \int_{B_r} G \left( \frac{|w_\pm(x) - w_\pm(y)|}{|x - y|^s} \right) \min \{ \phi^q(x), \phi^q(y) \} \frac{dx dy}{|x - y|^n} \\
\leq c \int_{B_r} \int_{B_r} G \left( \frac{|\phi(x) - \phi(y)|}{|x - y|^s} \right) \max \{ w_\pm(x), w_\pm(y) \} \frac{dx dy}{|x - y|^n} \\
+ c \int_{B_r} w_\pm(x) \phi^q(x) \frac{dx}{\sup_{y \in \text{supp } \phi} \int_{\mathbb{R}^n \setminus B_r} g \left( \frac{w_\pm(x)}{|x - y|^s} \right) dx} \\
\geq 0 \quad \text{for } x \in \supp \phi \setminus \Omega_1 \\
\equiv I + II.
\]

where \( w_\pm := (u - k)_\pm = \max \{ \pm(u - k), 0 \} \) and \( c > 0 \) depends on \( n, s, p, q, \lambda \) and \( \Lambda \).

Proof We only consider \( w_+ \), as the same argument can apply to \( w_- \). Take \( \eta := w_+ \phi^q \in W^{s, p}(\Omega) \) as a test function to find

\[
0 = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{u(x) - u(y)}{|u(x) - u(y)|} (\eta(x) - \eta(y)) K(x, y) \frac{dx dy}{|x - y|^s} \\
= \int_{B_r} \int_{B_r} g \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{u(x) - u(y)}{|u(x) - u(y)|} (\eta(x) - \eta(y)) K(x, y) \frac{dx dy}{|x - y|^s} \\
+ 2 \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} g \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{u(x) - u(y)}{|u(x) - u(y)|} \eta(x) K(x, y) \frac{dx dy}{|x - y|^s} \\
=: I + II.
\]

Note that \( \eta(x) = 0 \) for \( x \in B_r \cap \{ u(x) < k \} \). We divide the latter part of the proof into two steps.

Step 1. In this step we derive an estimate in terms of \( w_+ \) from (3.2). We first consider the integrand of \( I \) with respect to the measure, \( K(x, y) \frac{dx dy}{|x - y|^s} \). In the case when \( u(x) \geq u(y) \) for \( x, y \in B_r \), we have

\[Springer\]
\[
g \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{u(x) - u(y)}{|u(x) - u(y)|} (\eta(x) - \eta(y))
\]
\[
= g \left( \frac{u(x) - u(y)}{|x - y|^s} \right) (\eta(x) - \eta(y))
\]
\[
= \begin{cases} 
  g \left( \frac{w_+(x) - w_+(y)}{|x - y|^s} \right) (\eta(x) - \eta(y)) & \text{if } u(x) \geq u(y) \geq k, \\
  g \left( \frac{u(x) - u(y)}{|x - y|^s} \right) \eta(x) & \text{if } u(x) \geq k > u(y), \\
  0 & \text{if } k > u(x) \geq u(y)
\end{cases}
\]
\[
\geq g \left( \frac{w_+(x) - w_+(y)}{|x - y|^s} \right) (\eta(x) - \eta(y))
\]
\[
= g \left( \frac{w_+(x) - w_+(y)}{|x - y|^s} \right) \frac{w_+(x) - w_+(y)}{|w_+(x) - w_+(y)|} (\eta(x) - \eta(y)).
\]

(3.3)

On the other hand, in the case when \( u(x) < u(y) \) for \( x, y \in B_r \), we exchange the roles of \( x \) and \( y \) in (3.3) to obtain the same result. Then we recall the assumption (1.4) to get

\[
I \geq \lambda \int_{B_r} \int_{B_r} g \left( \frac{|w_+(x) - w_+(y)|}{|x - y|^s} \right) \frac{w_+(x) - w_+(y)}{|w_+(x) - w_+(y)|} (\eta(x) - \eta(y)) \frac{dxdy}{|x - y|^{n+s}}.
\]

(3.4)

Next, let us consider \( II \). Note that

\[
g \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{u(x) - u(y)}{|u(x) - u(y)|} \eta(x) \geq \begin{cases} 
  -g \left( \frac{w_+(y)}{|x - y|^s} \right) \eta(x) & \text{if } u(y) > u(x) \geq k, \\
  0 & \text{otherwise}.
\end{cases}
\]

Inserting this inequality into \( II \), we deduce

\[
II \geq -2\Lambda \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} g \left( \frac{w_+(y)}{|x - y|^s} \right) \eta(x) \frac{dxdy}{|x - y|^{n+s}}.
\]

(3.5)

We then combine (3.2), (3.4), and (3.5) to discover

\[
\int_{B_r} \int_{B_r} g \left( \frac{|w_+(x) - w_+(y)|}{|x - y|^s} \right) \frac{w_+(x) - w_+(y)}{|w_+(x) - w_+(y)|} (\eta(x) - \eta(y)) \frac{dxdy}{|x - y|^{n+s}} \leq \frac{2\Lambda}{\lambda} \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} g \left( \frac{w_+(y)}{|x - y|^s} \right) \eta(x) \frac{dxdy}{|x - y|^{n+s}}.
\]

(3.6)
Step 2. Set

\[ III := \int_{B_r} \int_{B_r} g \left( \frac{|w_+(x) - w_+(y)|}{|x - y|^s} \right) \frac{w_+(x) - w_+(y)}{|w_+(x) - w_+(y)|} \frac{w_+(x)\phi^q(x) - w_+(y)\phi^q(y)}{|x - y|^s} \]

and

\[ IV := \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} g \left( \frac{w_+(y)}{|x - y|^s} \right) \eta(x) \frac{dx dy}{|x - y|^{n+s}}. \]

Then we see from (3.6) that \( III \leq \frac{2\Lambda}{\lambda} IV \).

To estimate III, we first look at the integrand of III with respect to the measure \( \frac{dx dy}{|x - y|^s} \). Consider the following three cases:

1. \( w_+(x) > w_+(y) \) and \( \phi(x) \leq \phi(y) \),
2. \( w_+(x) > w_+(y) \) and \( \phi(x) > \phi(y) \),
3. \( w_+(x) \leq w_+(y) \).

In the case (1), we have

\[
\begin{align*}
&g \left( \frac{|w_+(x) - w_+(y)|}{|x - y|^s} \right) \frac{w_+(x) - w_+(y)}{|w_+(x) - w_+(y)|} \frac{w_+(x)\phi^q(x) - w_+(y)\phi^q(y)}{|x - y|^s} \\
&= g \left( \frac{w_+(x) - w_+(y)}{|x - y|^s} \right) \frac{w_+(x) - w_+(y)}{|x - y|^s} \phi^q(y) \\
&\quad - g \left( \frac{w_+(x) - w_+(y)}{|x - y|^s} \right) \phi^q(y) - \phi^q(x) w_+(x) \\
&\geq pG \left( \frac{w_+(x) - w_+(y)}{|x - y|^s} \right) \phi^q(y) \\
&\quad - qg \left( \frac{w_+(x) - w_+(y)}{|x - y|^s} \right) \phi^{q-1}(y) \phi(y) - \phi(x) w_+(x),
\end{align*}
\]

where we have used (1.3) and the following elementary inequality

\[ \phi^q(y) - \phi^q(x) \leq q \phi^{q-1}(y) (\phi(y) - \phi(x)). \]

We further estimate the second term on the right-hand side of (3.7). By using (2.6) and (2.7), we have that for \( \epsilon \in (0, 1) \),

\[
\begin{align*}
&g \left( \frac{w_+(x) - w_+(y)}{|x - y|^s} \right) \phi^{q-1}(y) \phi(y) - \phi(x) w_+(x) \\
&\leq \epsilon G^* \left( g \left( \frac{w_+(x) - w_+(y)}{|x - y|^s} \right) \phi^{q-1}(y) \right) + c(\epsilon) G \left( \frac{\phi(y) - \phi(x)}{|x - y|^s} - w_+(x) \right) \\
&\leq \epsilon (q - 1) G \left( \frac{w_+(x) - w_+(y)}{|x - y|^s} \right) \phi^q(y) + c(\epsilon) G \left( \frac{\phi(y) - \phi(x)}{|x - y|^s} w_+(x) \right). \quad (3.8)
\]

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For the last inequality, we have used (2.2) with \( a = \phi^{q-1}(y) \leq 1 \). Choosing \( \epsilon = \min \left\{ \frac{p}{2(q-1)}, \frac{1}{2} \right\} \) and plugging (3.8) into (3.7), we discover

\[
g \left( \frac{|w_+(x) - w_+(y)|}{|x - y|^s} \right) \frac{w_+(x) - w_+(y)}{|w_+(x) - w_+(y)|} \frac{w_+(x)\phi^q(x) - w_+(y)\phi^q(y)}{|x - y|^s} \geq \frac{p}{2} G \left( \frac{|w_+(x) - w_+(y)|}{|x - y|^s} \right) \min\{\phi^q(x), \phi^q(y)\} - cG \left( \frac{\phi(x) - \phi(y)}{|x - y|^s} \max\{w_+(x), w_+(y)\} \right).
\]

(3.9)

In the case (2), we use (1.3) to have

\[
g \left( \frac{|w_+(x) - w_+(y)|}{|x - y|^s} \right) \frac{w_+(x) - w_+(y)}{|w_+(x) - w_+(y)|} \frac{w_+(x)\phi^q(x) - w_+(y)\phi^q(y)}{|x - y|^s} \geq g \left( \frac{w_+(x) - w_+(y)}{|x - y|^s} \right) \frac{w_+(x) - w_+(y)}{|x - y|^s} \phi^q(x)
\]

\[
\geq pG \left( \frac{|w_+(x) - w_+(y)|}{|x - y|^s} \right) \min\{\phi^q(x), \phi^q(y)\}.
\]

Therefore, we also obtain the estimate (3.9) in this case. Moreover, since the integrand is invariant with the exchanging of \( x \) and \( y \), we again have the estimate (3.9) in the case (3). Consequently, we obtain

\[
III \geq c \int_{B_r} \int_{B_r} G \left( \frac{|w_+(x) - w_+(y)|}{|x - y|^s} \right) \min\{\phi^q(x), \phi^q(x)\} \frac{dxdy}{|x - y|^{n+s}} - c \int_{B_r} \int_{B_r} G \left( \frac{\phi(x) - \phi(y)}{|x - y|^s} \max\{w_+(x), w_+(y)\} \right) \frac{dxdy}{|x - y|^{n+s}}.
\]

To estimate IV, we first use Fubini’s theorem to find

\[
IV = \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} g \left( \frac{w_+(y)}{|x - y|^s} \right) \eta(x) \frac{dxdy}{|x - y|^{n+s}} \leq \int_{B_r} w_+(x)\phi^q(x) dx \left( \sup_{x \in \supp \phi} \int_{\mathbb{R}^n \setminus B_r} g \left( \frac{w_+(y)}{|x - y|^s} \right) \frac{dy}{|x - y|^{n+s}} \right).
\]

Hence we obtain (3.1), as \( III \leq cIV \). □

**Remark 3.2** In Proposition 3.1, the estimate (3.1) for \( w_+ \) (resp. \( w_- \)) still holds true when \( u \) is a weak subsolution (resp. supersolution) to (1.1).

The second one to be derived is a logarithmic estimate. This will be used in the proof of the decay estimate for the oscillation of weak solutions, Lemma 4.5. We need the following elementary inequality.

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Lemma 3.3 [14, Lemma 3.1] Let $q \geq 1$ and $\epsilon \in (0, 1]$. Then

$$|a|^q \leq (1 + c_q \epsilon)|b|^q + (1 + c_q \epsilon)^{1-q}|a-b|^q$$

for every $a, b \in \mathbb{R}^n$. Here $c_q > 0$ depends on $n$ and $q$.

Proposition 3.4 (Logarithmic estimate) Let $u \in \mathcal{W}^{s,G}(\Omega) \cap L^s_s(\mathbb{R}^n)$ be a weak supersolution to (1.1) with $u \geq 0$ in $B_R \equiv B_R(x_0) \subset \Omega$. Then for any $d > 0$ and $0 < r < \frac{R}{2}$, we have

$$\int_{B_r} \int_{B_r} |\log (u(x) + d) - \log (u(y) + d)| \frac{dx dy}{|x-y|^n} \leq cr^n + c r^{n+s} \frac{d}{g(d/r^s)} \text{Tail}(u_{-}; x_0, R)$$

for some $c = c(n, s, p, q, \lambda, \Lambda) > 0$. In addition, we have the estimate

$$\int_{B_r} |h - (h)_{B_r}| dx \leq cr^n \left[ 1 + \frac{r^s}{g(d/r^s)} \text{Tail}(u_{-}; x_0, R) \right],$$

where

$$h := \min \{(\log(a + d) - \log(u + d))_+, \log b\}, \quad a > 0 \text{ and } b > 1.$$ 

Proof Write $v(x) := u(x) + d$ and fix a cut-off function $\phi \in C^\infty_0(B_{3r}/2)$ such that $0 \leq \phi \leq 1$, $|D\phi| \leq 4/r$ and $\phi \equiv 1$ in $B_r$. Since $\frac{v}{G(v/r^s)}$ is nonnegative in $B_R$ and belongs to $\mathcal{W}^{s,G}(\Omega)$, we can take $\eta = \frac{v\phi^q}{G(v/r^s)}$ as a test function to find

$$0 \leq \int_{B_{2r}} \int_{B_{2r}} g \left( \frac{|v(x) - v(y)|}{|x-y|^s} \right) \frac{v(x) - v(y)}{|v(x) - v(y)|} (\eta(x) - \eta(y)) K(x, y) \frac{dx dy}{|x-y|^s}$$

$$+ 2 \int_{\mathbb{R}^n \setminus B_{2r}} \int_{B_{2r}} g \left( \frac{|u(x) - u(y)|}{|x-y|^s} \right) \frac{u(x) - u(y)}{|u(x) - u(y)|} \eta(x) K(x, y) \frac{dx dy}{|x-y|^s}$$

$$=: I + II.$$

We define

$$F = F(x, y) := g \left( \frac{|v(x) - v(y)|}{|x-y|^s} \right) \frac{v(x) - v(y)}{|v(x) - v(y)|} \eta(x) - \eta(y), \quad x, y \in B_{2r}.$$ 

Note that $F(x, y) = F(y, x)$. We divide the remaining proof into five steps.

Step I. We first assume that $v(y) \leq v(x) \leq 2v(y)$ for $x, y \in B_{2r}$ to assert that

$$F(x, y) \leq -\tilde{c} (\log v(x) - \log v(y)) \phi(x)^q + c \left( \frac{|x-y|}{r} \right)^s + c \left( \frac{|x-y|}{r} \right)^{(1-s)p}$$

(13.13)
for some small constant \( \tilde{c} > 0 \) and large constant \( c > 0 \) depending on \( n, p \) and \( q \). To prove this, let us suppose \( \phi(x) \geq \phi(y) \). By the definition of \( \eta \), we get

\[
F(x, y) = g \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \left( \frac{v(y)}{G(v(y)/r^s)} - \frac{v(y) \phi^q(x)}{|x - y|^s} \right) \phi^q(x) \\
+ g \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \frac{v(y)}{G(v(y)/r^s)} \phi^q(x) - \phi^q(y) \tag{3.14}
\]

\[
=: F_1(x, y) + F_2(x, y).
\]

Before estimating \( F_1 \) and \( F_2 \), we apply Mean Value Theorem to the mapping \( t \mapsto \frac{t}{G(t/r^s)} \) for \( v(y) \leq t \leq v(x) \) and use the inequality

\[
\left( \frac{t}{G(t/r^s)} \right)' = \frac{G(t/r^s) - (t/r^s)g(t/r^s)}{G^2(t/r^s)} \leq - \frac{p - 1}{G(t/r^s)} \tag{by (1.3)}.
\]

to find

\[
\frac{v(x)}{G(v(x)/r^s)} - \frac{v(y)}{G(v(y)/r^s)} \leq -(p - 1) \frac{v(x) - v(y)}{G(v(x)/r^s)}. \tag{3.15}
\]

We again apply Mean value theorem to the mapping \( t \mapsto t^q \) for \( \phi(y) \leq t \leq \phi(x) \) to have

\[
\phi^q(x) - \phi^q(y) \leq q \phi^{q-1}(x) (\phi(x) - \phi(y)). \tag{3.16}
\]

Putting (3.15) into \( F_1 \) and using (1.3) and the fact that \( v(x) \leq 2v(y) \), we have

\[
F_1 \leq - (p - 1) g \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \frac{v(x) - v(y)}{|x - y|^s} \phi^q(x) \\
\leq - c_1 G \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \frac{\phi^q(x)}{G(v(y)/r^s)} \tag{3.17}
\]

for some small constant \( c_1 = c_1(p, q) > 0 \).

We use (3.16) and recall (2.6) with \( \epsilon = \min \left\{ \frac{c_1}{2q(q-1)}, \frac{1}{2} \right\} \) and (2.7), to discover

\[
F_2 \leq q g \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \phi^{q-1}(x) \frac{\phi(x) - \phi(y)}{|x - y|^s} \frac{v(y)}{G(v(y)/r^s)} \\
\leq q \left[ \epsilon(q - 1) G \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \phi^q(x) + \epsilon^{1-q} G \left( \frac{\phi(x) - \phi(y)}{|x - y|^s} - v(y) \right) \right] \frac{1}{G(v(y)/r^s)} \\
\leq \left[ \frac{c_1}{2} G \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \phi^q(x) + c G \left( \frac{\phi(x) - \phi(y)}{|x - y|^s} v(y) \right) \right] \frac{1}{G(v(y)/r^s)} \tag{3.18}
\]

We then combine (3.14), (3.17), and (3.18) and use the fact that \( |D\phi| \leq 4/r \) and \( |x - y| \leq 4r \) for \( x, y \in B_{2r} \), to obtain (3.13).
We next suppose $\phi(x) < \phi(y)$. Using (3.15) and (1.3), we have

$$F(x, y) = g \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \left( \frac{\phi^q(x)v(x)}{G(v(x)/r^s)} - \frac{\phi^q(y)v(y)}{G(v(y)/r^s)} \right) \frac{1}{|x - y|^s}$$

$$\leq g \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \left( \frac{v(x)}{G(v(x)/r^s)} - \frac{v(y)}{G(v(y)/r^s)} \right) \frac{\phi^q(y)}{|x - y|^s}$$

$$\leq -cg \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \frac{\phi^q(x)}{G(v(y)/r^s)}.$$

Therefore we also have

$$F(x, y) \leq -cg \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \frac{\phi^q(x)}{G(v(y)/r^s)} + c \left( \frac{|x - y|}{r} \right)^{1-s} p. \quad (3.19)$$

In addition, by Mean Value Theorem,

$$\log v(x) - \log v(y) \leq \frac{v(x) - v(y)}{v(y)} = \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \frac{1}{G(v(y)/r^s)}$$

$$\leq \left\{ G \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \frac{1}{G(v(y)/r^s)} + 1 \right\} \frac{|x - y|^s}{r^s}$$

$$\leq cG \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \frac{1}{G(v(y)/r^s)} + \frac{|x - y|^s}{r^s}.$$

For the second inequality we have used the fact that $\frac{G(t)}{t}$ is increasing for $t$. This estimate and (3.19) imply finally (3.13).

**Step 2.** We now assume that $v(x) > 2v(y)$ for $x, y \in B_{2r}$ to claim that

$$F(x, y) \leq -\tilde{c}(\log v(x) - \log v(y))\phi^q(y)$$

$$+ c \left( \frac{|x - y|}{r} \right)^{s(p-1)} + c \left( \frac{|x - y|}{r} \right)^{(1-s)q} \quad (3.20)$$

for some small constant $\tilde{c} > 0$ and large constant $c > 0$ depending on $n$, $p$ and $q$. To this end, we recall the definition of $\eta$ to see that

$$F(x, y) = g \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \left( \frac{v(x)}{G(v(x)/r^s)} - \frac{v(y)}{G(v(y)/r^s)} \right) \frac{\phi^q(y)}{|x - y|^s}$$

$$+ g \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \frac{v(x)}{G(v(x)/r^s)} \frac{\phi^q(x) - \phi^q(y)}{|x - y|^s}$$

$$=: F_3(x, y) + F_4(x, y).$$
Since $\frac{I}{G(t)}$ is decreasing for $t$, we have

$$F_3 \leq g \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \left( \frac{2v(y)}{G(2v(y)/r^s)} - \frac{v(y)}{G(v(y)/r^s)} \right) \frac{\phi^q(y)}{|x - y|^s}$$

$$= g \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \frac{v(y)}{G(v(y)/r^s)} \left( 2 \frac{G(v(y)/r^s)}{G(2v(y)/r^s)} - 1 \right) \frac{\phi^q(y)}{|x - y|^s}$$

$$\leq - \left( 1 - \frac{1}{2^{p-1}} \right) g \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \frac{v(y)}{G(v(y)/r^s)} \frac{\phi^q(y)}{|x - y|^s}.$$ 

On the other hand, in light of Lemma 3.3, we find that for $\epsilon \in (0, 1)$,

$$F_4 \leq c_q \epsilon g \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \frac{v(x)}{G(v(x)/r^s)} \frac{\phi^q(y)}{|x - y|^s}$$

$$+ c\epsilon^{1-q} g \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \frac{v(x)}{G(v(x)/r^s)} \frac{|\phi(x) - \phi(y)|^q}{|x - y|^s}.$$

In addition, using the fact that $\frac{I}{G(t)}$ is decreasing for $t$, $2v(y) < v(x)$, $|D\phi| \leq c/r$, $|x - y| \leq 4r$ for $x, y \in B_{2r}$ and $tg(t) \leq qG(t)$, we discover

$$F_4 \leq c_q \epsilon g \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \frac{v(y)}{G(v(y)/r^s)} \frac{\phi^q(y)}{|x - y|^s}$$

$$+ c\epsilon^{1-q} r^s \left( \frac{v(x) - v(y)}{v(x)} \right)^{p-1} \left( \frac{r}{|x - y|} \right)^{x(q-1)} \frac{|\phi(x) - \phi(y)|^q}{|x - y|^s}$$

$$\leq c_q \epsilon g \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \frac{v(y)}{G(v(y)/r^s)} \frac{\phi^q(y)}{|x - y|^s} + c\epsilon^{1-q} \left( \frac{|x - y|}{r} \right)^{(1-s)q}.$$ 

We then choose $\epsilon = \min \left\{ \frac{1}{2c_q} \left( 1 - \frac{1}{2^{p-1}} \right), \frac{1}{2} \right\}$, and combine the above estimates to discover

$$F \leq -cg \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \frac{v(y)}{G(v(y)/r^s)} \frac{\phi^q(y)}{|x - y|^s} + c \left( \frac{|x - y|}{r} \right)^{(1-s)q}.$$ 

Note that

$$\frac{v(y)}{G(v(y)/r^s)} \frac{1}{|x - y|^s} \geq \frac{1}{4^s} \frac{v(y)/r^s}{G(v(y)/r^s)} \geq \frac{p}{4^s} \frac{1}{g(v(y)/r^s)}$$

to have

$$F(x, y) \leq -cg \left( \frac{v(x) - v(y)}{|x - y|^s} \right) \frac{\phi^q(y)}{g(v(y)/r^s)} + c \left( \frac{|x - y|}{r} \right)^{(1-s)q} \quad (3.21).$$
Moreover, since \( v(x) > 2v(y) \),

\[
\log v(x) - \log v(y) \leq \log 2(v(x) - v(y)) - \log v(y) \leq c \left( \frac{2(v(x) - v(y))}{v(y)} \right)^{p-1},
\]

where we have used the fact that \( \log t < \frac{t^{p-1}}{p-1} \). Note that

\[
\frac{g(s)}{s^{p-1}} \leq q \frac{G(s)}{s^p} \leq q \frac{G(t)}{t^p} \leq q \frac{g(t)}{p t^{p-1}} \quad \text{for any } t \geq s > 0,
\]

to discover

\[
\log v(x) - \log v(y) \leq c \left( \frac{(v(x) - v(y))|x - y|^s}{\log 2|v(y)/r^s|} \right)^{p-1} \leq cg \left( \frac{|x - y|}{g(v(y)/r^s)} \right)^{1} + c \left( \frac{|x - y|}{r} \right)^{s(p-1)}.
\]

This and (3.21) imply the estimate (3.20).

**Step 3.** We next estimate \( I \) in (3.12). We recall (3.11) when \( v(y) \leq v(x) \leq 2v(y) \), and (3.20) when \( v(x) > 2v(y) \), and use the fact \( F(x, y) = F(y, x) \), to discover that for every \( x, y \in B_{2r}, \)

\[
F(x, y) \leq -\tilde{c} |\log v(x) - \log v(y)| \min \{\phi(x), \phi(y)\}^q + c \left( \frac{|x - y|}{r} \right)^s + c \left( \frac{|x - y|}{r} \right)^{(1-s)p} + c \left( \frac{|x - y|}{r} \right)^{s(p-1)}.
\]

Then since \( \phi \equiv 1 \) in \( B_r \) and \( K(x, y) \) satisfies (1.4), we have

\[
I \leq -\tilde{c} \int_{B_r} \int_{B_r} |\log v(x) - \log v(y)| \frac{dxdy}{|x - y|^n}
+ c \int_{B_{2r}} \int_{B_{2r}} \left[ \left( \frac{|x - y|}{r} \right)^s + \left( \frac{|x - y|}{r} \right)^{(1-s)p} + \left( \frac{|x - y|}{r} \right)^{s(p-1)} \right] \frac{dxdy}{|x - y|^n}
\leq -\tilde{c} \int_{B_r} \int_{B_r} |\log v(x) - \log v(y)| \frac{dxdy}{|x - y|^n} + c r^n.
\]

(3.22)

**Step 4.** We next estimate \( II \). Observe that for \( x \in B_R \) and \( y \in \mathbb{R}^n \),

\[
g \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{u(x) - u(y)}{|u(x) - u(y)|} \leq g \left( \frac{u(x) - u(y)}{|x - y|^s} \right) \leq c \left[ g \left( \frac{u(x)}{|x - y|^s} \right) + g \left( \frac{u(y)}{|x - y|^s} \right) \right].
\]
Recalling supp $\phi \subset B_{3r/2}$, we have

$$II \leq c \int_{\mathbb{R}^n \setminus B_2} \int_{B_{3r/2}} g \left( \frac{(u(x) - u(y))^+}{|x - y|^s} \right) \frac{r^s}{g(v(x)/r^s)} \frac{dxdy}{|x - y|^{n+s}}$$

$$\leq c \int_{B_2 \setminus B_{3r/2}} \int_{B_{3r/2}} g \left( \frac{(u(x) - u(y))^+}{|x - y|^s} \right) \frac{r^s}{g(v(x)/r^s)} \frac{dxdy}{|x - y|^{n+s}}$$

$$+ c \int_{\mathbb{R}^n \setminus B_2} \int_{B_{3r/2}} g \left( \frac{u(x)}{|x - y|^s} \right) \frac{r^s}{g(v(x)/r^s)} \frac{dxdy}{|x - y|^{n+s}}$$

$$+ c \int_{\mathbb{R}^n \setminus B_2} \int_{B_{3r/2}} g \left( \frac{u(y_0)}{|x - y|^s} \right) \frac{r^s}{g(v(x)/r^s)} \frac{dxdy}{|x - y|^{n+s}}$$

$$=: II_1 + II_2 + II_3.$$ 

Since $u \geq 0$ in $B_R$ and $v = u + d$, we see that

$$(u(x) - u(y))^+ \leq v(x) \text{ and } u(x) \leq v(x), \quad x, y \in B_R.$$ 

Thus

$$II_1 \leq c \int_{B_2 \setminus B_{3r/2}} \int_{B_{3r/2}} g \left( \frac{(u(x) - u(y))^+}{r^s} \right) \frac{dxdy}{|x - y|^{n+s}} \leq cr^s \int_{B_{3r/2}} \int_{\mathbb{R}^n \setminus B_{2r}} \frac{dxdy}{|x - y|^{n+s}} \leq cr^n$$

and

$$II_2 \leq c \int_{\mathbb{R}^n \setminus B_2} \int_{B_{3r/2}} g \left( \frac{u(x)}{r^s} \right) \frac{dxdy}{|x - y|^{n+s}} \leq cr^s \int_{B_{3r/2}} \int_{\mathbb{R}^n \setminus B_{2r}} \frac{dxdy}{|x - y|^{n+s}} \leq cr^n.$$ 

Observing that for any $x \in B_{3r/2}$ and $y \in \mathbb{R}^n \setminus B_2$

$$\frac{|y - x_0|}{|x - y|} \leq 1 + \frac{|x - x_0|}{|x - y|} \leq 1 + \frac{3r/2}{2r - (3r/2)} = 4,$$

we find

$$II_3 \leq c \int_{\mathbb{R}^n \setminus B_2} \int_{B_{3r/2}} g \left( \frac{u(y_0)}{|y - x_0|^s} \right) \frac{r^s}{g(d/r^s)} \frac{dxdy}{|y - x_0|^{n+s}} \leq c \frac{r^{n+s}}{g(d/r^s)} \text{Tail}(u_-, x_0, R).$$

Consequently, we have

$$II \leq cr^n + c \frac{r^{n+s}}{g(d/r^s)} \text{Tail}(u_-, x_0, R).$$

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Inserting this estimate and (3.22) into (3.12), we get (3.10).

Step 5. Now we are ready to prove the estimate (3.11). Observe that

$$\int_{B_r} |h - (h)_{B_r}| dx \leq c \int_{B_r} \int_{B_r} |h(x) - h(y)| \frac{dxdy}{|x - y|^n}.$$  

Since $h(x)$ is a truncation of $\log v(x)$,

$$\int_{B_r} \int_{B_r} |h(x) - h(y)| \frac{dxdy}{|x - y|^n} \leq c \int_{B_r} \int_{B_r} |\log v(x) - \log v(y)| \frac{dxdy}{|x - y|^n}.$$  

Combining (3.10) and the above inequalities, we finally obtain (3.11). ⊓⊔

4 Proof of Theorem 1.1

4.1 Local Boundedness

This subsection is devoted to the proof of the local boundedness of weak solutions to (1.1) with the estimate (1.5) in Theorem 1.1. Key ingredients of the proof are the Caccioppoli type estimate, Proposition 3.1, and the Sobolev-Poincaré type inequality below. We notice that the Sobolev inequality and the Sobolev-Poincaré inequality for the fractional Orlicz-Sobolev space $W^{s,G}(B_r)$ are well known in terms of the Luxemburg norms. However, it does not directly imply a certain integral version of the Sobolev-Poincaré inequality. For the sake of completeness, we need to prove the following Sobolev-Poincaré inequality for functions in $W^{s,G}(B_r)$.

Lemma 4.1 (Sobolev-Poincaré inequality) Let $s \in (0, 1)$. Then there exists $\theta = \theta(n, s) > 1$ such that if $G$ is an N-function satisfying the $\Delta_2$ condition (2.3) and the $\nabla_2$ condition (2.4) with constants $\kappa$ and $l$, and $f \in W^{s,G}(B_r)$, then

$$\left( \int_{B_r} G \left( \frac{|f - (f)_{B_r}|}{r^s} \right)^\theta \right)^{\frac{1}{\theta}} \leq c \int_{B_r} \int_{B_r} G \left( \frac{|f(x) - f(y)|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^n},$$  

(4.1)

where $c = c(n, s, \kappa, l) > 0$.

Proof We first show that

$$|f(x) - (f)_{B_r}| \leq c \int_{B_r} \left( \int_{B_r} \frac{|f(y) - f(z)|}{|y - z|^{n+s}} dz \right) \frac{dy}{|x - y|^{n-s}}, \quad \text{a.e. } x \in B_r,$$

(4.2)

by using a standard chain argument (see for instance [17] and references therein). Fix any Lebesgue’s point $x \in B_r$ for $f$. For each $i \in \mathbb{N}_0$, set $r_i = 2^{-i} r$. Then there exists a sequence $\{B_i\}_{i=0}^\infty$ of balls in $B_r$ such that $x \in B_i$, $B_i \subset B_{2r_i}(x) \cap B_r$, $B^{i+1} \subset B^i$, $\ldots$
$r_i \leq \text{(the radius of $B^i$)} \leq 2r_i$. In particular, we can choose $B^0 = B_r$ and $B^i = B_{r_i}(x)$ for large $i$ with $r_i \leq \text{dist}(x, \partial B_r)$. Then,

$$\left| f(x) - (f)_{B_r} \right| \leq \sum_{i=0}^{\infty} \left| (f)_{B^{i+1}} - (f)_{B^i} \right| \leq \sum_{i=0}^{\infty} \int_{B^{i+1}} \left| f(y) - (f)_{B^i} \right| dy \leq c \sum_{i=0}^{\infty} r_i^{-n} \int_{B^i} \left| f(y) - (f)_{B^i} \right| dy \leq c \sum_{i=0}^{\infty} r_i^{-n+s} \int_{B^i} \int_{B^i} \frac{|f(y) - f(z)|}{|y - z|^{n+s}} dz dy.$$  

Set

$$h(y) := \int_{B_r} \left| f(y) - f(z) \right| \frac{1}{|y - z|^{n+s}} dz, \quad y \in B_r.$$  

Then

$$\left| f(x) - (f)_{B_r} \right| \leq c \sum_{i=0}^{\infty} r_i^{-n+s} \int_{B_2r_i(x) \cap B_r} h(y) dy \leq c \sum_{i=0}^{\infty} \left( \sum_{j=0}^{\infty} 2^{(n-s)i} r_i^{-n+s} \right) \int_{(B_{2r_j}(x) \setminus B_{2r_{j+1}}(x)) \cap B_r} h(y) dy \leq c \sum_{j=0}^{\infty} \int_{(B_{2r_j}(x) \setminus B_{2r_{j+1}}(x)) \cap B_r} h(y) \frac{1}{|x - y|^{n-s}} dy \leq c \int_{B_r} \frac{h(y)}{|x - y|^{n-s}} dy,$$  

and this is (4.2).

We next prove the desired estimate following the argument in [16, Theorem 7]. To do this, note that for $s > 0$ there exists $c(n, s) \geq 1$ such that

$$\frac{1}{c(n, s)} \leq r^{-s} \int_{B_r} \frac{1}{|x - y|^{n-s}} dy \leq c(n, s), \quad \text{for every } x \in B_r. \quad (4.3)$$  

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Using (4.2) with \( s \) replaced by \( \frac{s}{2} \) and Jensen’s inequality, we have

\[
\int_{B_r} G \left( \frac{|f(x) - (f)_{B_r}|}{r^s} \right)^\theta \, dx \\
\leq c \int_{B_r} G \left( r^{-s} \int_{B_r} \int_{B_r} \frac{|f(y) - f(z)|}{|y - z|^{n+s/2}} \, dz \, dy \right)^\theta \, dx \\
= c \int_{B_r} G \left( r^{-s} \int_{B_r} \int_{B_r} \frac{|f(y) - f(z)|}{|y - z|^{s}} \, dz \, dy \right)^\theta \, dx \\
\leq c \int_{B_r} \left[ r^{-s} \int_{B_r} \int_{B_r} G \left( \frac{|f(y) - f(z)|}{|y - z|^s} \right) \, dz \, dy \right]^\theta \, dx.
\]

Denote

\[
L := \int_{B_r} \int_{B_r} G \left( \frac{|f(y) - f(z)|}{|y - z|^s} \right) \, dz \, dy.
\]

Then recall the fact that \(|y - z| \leq 2r\) and use Jensen’s inequality and Fubini’s theorem, to discover

\[
\int_{B_r} G \left( \frac{|f(x) - (f)_{B_r}|}{r^s} \right)^\theta \, dx \\
\leq cL^\theta \int_{B_r} \left[ L^{-1} \int_{B_r} \frac{r^{-s/2}}{|x - y|^{n-s/2}} \left( \int_{B_r} G \left( \frac{|f(y) - f(z)|}{|y - z|^s} \right) \, dz \right)^\theta \, dy \right] \, dx \\
\leq cL^{\theta - 1} \int_{B_r} \left[ \frac{r^{-s/2}}{|x - y|^{n-s/2}} \left( \int_{B_r} G \left( \frac{|f(y) - f(z)|}{|y - z|^s} \right) \, dz \right)^\theta \, dy \right] \, dx \\
= cL^{\theta - 1} \int_{B_r} \left[ \frac{r^{-s/2}}{|x - y|^{n-s/2}} \right] \, dx \left( \int_{B_r} G \left( \frac{|f(y) - f(z)|}{|y - z|^s} \right) \, dz \, dy \right)^\theta.
\]

We now choose \( \theta = \theta(n, s) \) such that

\[
1 < \theta < \frac{n}{n - s/2}.
\]

From this choice and (4.3) with \( s \) replaced by \( s\theta/2 - n(\theta - 1) \), we discover

\[
\frac{1}{c} \leq r^{n(\theta - 1) - s\theta/2} \int_{B_r} \frac{1}{|x - y|^{(n-s/2)\theta}} \, dx \leq c \quad \text{for every} \ x \in B_r.
\]
Consequently,
\[
\int_{B_r} G\left( \frac{|f(x) - (f)_{B_r}|}{r^s} \right)^\theta \, dx \\
\leq c([B_r]^{-1}L)^{\theta-1} \int_{B_r} \int_{B_r} G\left( \frac{|f(y) - f(z)|}{|y-z|^s} \right) \, dz \, dy \\
\leq c \left( \int_{B_r} \int_{B_r} G\left( \frac{|f(y) - f(z)|}{|y-z|^s} \right) \, dz \, dy \right)^\theta.
\]

This finishes the proof. \(\Box\)

**Remark 4.2** In Lemma 4.1, we selected \(\theta > 1\) such that \(\theta \in (1, \frac{n}{n-s/2})\). This selection is not optimal and it is possible to consider a larger value \(\theta\). However the condition \(\theta > 1\) is enough in the proof of Theorem 4.4 below.

The following lemma will be used in the De Giorgi iteration.

**Lemma 4.3** [26, Lemma 7.1] Let \(\beta > 0\) and \(A_i\) be a sequence of real positive numbers such that

\[A_{i+1} \leq CB^i A_i^{1+\beta}\]

with \(C > 0\) and \(B > 1\). If \(A_0 \leq C^{-\frac{1}{\beta}} B^{-\frac{1}{\beta^2}}\), then we have

\[A_i \leq B^{-\frac{i}{\beta}} A_0\] hence, in particular, \(\lim_{i \to \infty} A_i = 0.\)

Now, we are ready to prove the local boundedness of weak solutions to (1.1).

**Theorem 4.4** Let \(u \in W^{s, G}(\Omega) \cap L^\infty_\text{loc}(\mathbb{R}^n)\) be a weak subsolution to (1.1) and \(B_r \Subset \Omega\). Then we have

\[
\sup_{B_{r/2}} u_+ \leq c_b r^s G^{-1} \left( \int_{B_r} G\left( \frac{u_+}{r^s} \right) \, dx \right) + r^s g^{-1}(r^s \text{Tail}(u_+; x_0, r/2)),
\]

(4.4)

where \(c_b = c_b(n, s, p, q, \lambda, \Lambda) > 0\). Moreover, if \(u\) is a weak solution to (1.1), then \(u \in L^\infty_\text{loc}(\Omega)\) and we have the estimate (1.5).

**Proof** Suppose that \(u\) is a weak subsolution. Fix \(B_r = B_r(x_0) \Subset \Omega\). For any \(j \in \mathbb{N}_0\), write

\[
r_j = (1 + 2^{-j})^\frac{r}{2}, \quad \tilde{r}_j = \frac{r_j + r_{j+1}}{2}, \quad B_j = B_{r_j}(x_0), \quad \tilde{B}_j = B_{\tilde{r}_j}(x_0),
\]

\[
k_j = (1 - 2^{-j})k, \quad \tilde{k}_j = \frac{k_j + k_{j+1}}{2}, \quad w_j = (u - k_j)_+ \text{ and } \tilde{w}_j = (u - \tilde{k}_j)_+.
\]
Note from the above setting that

\[ B_{j+1} \subset \tilde{B}_j \subset B_j, \quad k_j \leq \tilde{k}_j \leq k_{j+1} \quad \text{and} \quad w_{j+1} \leq \tilde{w}_j \leq w_j. \quad (4.5) \]

We take any cut-off functions \( \phi_j \in C_0^\infty(\tilde{B}_j) \) such that \( 0 \leq \phi_j \leq 1, \phi_j \equiv 1 \) in \( B_{j+1} \) and \( |D\phi_j| \leq 2^{j+4}/r \). Putting \( \phi_j \) into the Caccioppoli inequality (3.1) with \( w_+ = \tilde{w}_j \) (see Remark 3.2) and dividing the inequality by \( |B_{j+1}| \), we get

\[
\int_{B_{j+1}} \int_{B_{j+1}} G \left( \frac{|\tilde{w}_j(x) - \tilde{w}_j(y)|}{|x-y|^s} \right) \frac{dxdy}{|x-y|^n} \leq c \int_{B_j} \int_{B_j} G \left( \frac{|\phi_j(x) - \phi_j(y)|}{|x-y|^s} \right) \frac{dxdy}{|x-y|^n} \max \{ \tilde{w}_j(x), \tilde{w}_j(y) \} \frac{dxdy}{|x-y|^n} \\
+ c \int_{B_j} \tilde{w}_j(x) \phi_j^q(x) dx \sup_{y \in \text{supp} \phi_j} \int_{\mathbb{R}^n \setminus B_j} g \left( \frac{\tilde{w}_j(x)}{|x-y|^s} \right) \frac{dx}{|x-y|^{n+s}} \\
=: I + II. \quad (4.6)
\]

We first look at the first term \( I \) in the right-hand side of the above inequality. Since \( |\phi_j(x) - \phi_j(y)| \leq \|D\phi_j\|_{L^\infty} |x-y| \leq c 2^j |x-y|/r \), we find

\[
I \leq c \int_{B_j} \int_{B_j} G \left( 2^j r^{-1} |x-y|^{1-s} \max \{ \tilde{w}_j(x), \tilde{w}_j(y) \} \right) \frac{dxdy}{|x-y|^n} \\
\leq c 2^q 2^j \int_{B_j} \int_{B_j} G \left( \frac{\max \{ \tilde{w}_j(x), \tilde{w}_j(y) \}}{r^s} \right) \left( \frac{|x-y|}{r} \right)^{(1-s)p} \frac{dxdy}{|x-y|^n} \\
\leq c 2^q 2^j r^{-(1-s)p} \int_{B_j} G \left( \frac{\tilde{w}_j(x)}{r^s} \right) \left( \frac{dy}{|x-y|^{n-(1-s)p}} \right) dx \\
\leq c 2^q 2^j \int_{B_j} G \left( \frac{w_j(x)}{r^s} \right) dx. \quad (4.7)
\]

To estimate \( II \), we write

\[
II_1 = \int_{B_j} \tilde{w}_j(x) \phi_j^q(x) dx \quad \text{and} \quad II_2 = \sup_{y \in \text{supp} \phi_j} \int_{\mathbb{R}^n \setminus B_j} g \left( \frac{\tilde{w}_j(x)}{|x-y|^s} \right) \frac{dx}{|x-y|^{n+s}}.
\]

Since \( g \) is increasing and \( w_j \geq \tilde{k}_j - k_j \) in \( \{ u_j \geq \tilde{k}_j \} \), we have

\[
G \left( \frac{w_j}{r^s} \right) \geq \frac{1}{q} \frac{w_j}{r^s} g \left( \frac{w_j}{r^s} \right) \geq \frac{1}{q} \frac{\tilde{w}_j}{r^s} g \left( \frac{\tilde{k}_j - k_j}{r^s} \right) \geq c 2^{-(q-1)j} \frac{\tilde{w}_j}{r^s} g \left( \frac{k}{r^s} \right).
\]
Thus

$$II_1 \leq c \frac{2^{(q-1)j}}{r^s} \int_{B_j} G \left( \frac{w_j}{r^s} \right) dx.$$  \hspace{1cm} (4.8)

In order to estimate $II_2$, we notice that for $x \in \mathbb{R}^n \setminus B_j$ and $y \in \tilde{B}_j$,

$$\frac{|x - x_0|}{|x - y|} \leq \frac{|x - y| + |y - x_0|}{|x - y|} \leq 1 + \frac{\tilde{r}_j}{r_j - \tilde{r}_j} \leq 2^{j+4}.$$  

This and (4.5) imply

$$II_2 \leq \sup_{y \in B_j} \int_{\mathbb{R}^n \setminus Br/2} g \left( \frac{u_0}{|x - y|^s} \right) dx |x - y|^{n+s}$$

$$\leq c \frac{2^{(n+sq)j}}{r^s} \int_{\mathbb{R}^n \setminus Br/2} \left( \frac{u_+}{|x - x_0|^s} \right) dx |x - x_0|^{n+s}$$

$$= c \frac{2^{(n+sq)j}}{r^s} \text{Tail}(u_+; x_0, r/2).$$  \hspace{1cm} (4.9)

In light of (4.8) and (4.9), we deduce

$$II \leq c \frac{2^{(n+sq+q)j}}{r^s} \left( \int_{B_j} G \left( \frac{w_j}{r^s} \right) dx \right) \text{Tail}(u_+; x_0, r/2).$$  \hspace{1cm} (4.10)

Combining (4.6), (4.7), and (4.10), and applying the Sobolev–Poincaré inequality (4.1) to the left-hand side of (4.6), we have

$$\left( \int_{B_{j+1}} G^\theta \left( \frac{|	ilde{w}_j - (\tilde{w})_{B_{j+1}}|}{r^s_{j+1}} \right) dx \right)^{\frac{1}{\theta}}$$

$$\leq c \frac{2^{(n+sq+q)j}}{r^s} \left[ \int_{B_j} G \left( \frac{w_j}{r^s} \right) dx + \frac{r^s}{g(k/r^s)} \left( \int_{B_j} G \left( \frac{w_j}{r^s} \right) dx \right) \text{Tail}(u_+; x_0, r/2) \right]$$  \hspace{1cm} (4.11)
for some \( \theta = \theta(n, s) > 1 \). On the other hand, recalling the definition of \( r_{j+1} \) and using Jensen’s inequality and (4.5), we discover

\[
\left( \int_{B_{j+1}} G^\theta \left( \frac{\tilde{w}_j}{r^s} \right) \, dx \right)^{\frac{1}{\theta}} \leq c \left( \int_{B_{j+1}} G^\theta \left( \frac{\tilde{w}_j - (\tilde{w}_j)_{B_{j+1}}}{r^s} \right) \, dx \right)^{\frac{1}{\theta}} + c G \left( \frac{(\tilde{w}_j)_{B_{j+1}}}{r^s} \right)
\]

\[
\leq c \left( \int_{B_{j+1}} G^\theta \left( \frac{\tilde{w}_j - (\tilde{w}_j)_{B_{j+1}}}{r^s_{j+1}} \right) \, dx \right)^{\frac{1}{\theta}} + c \int_{B_j} G \left( \frac{w_j}{r^s} \right) \, dx.
\]

(4.12)

Let us estimate the left-hand side of (4.12). Notice that the relations in (4.5) yield

\[
G^\theta \left( \frac{\tilde{w}_j}{r^s} \right) \geq G^{\theta-1} \left( \frac{\tilde{w}_j}{r^s} \right) G \left( \frac{w_{j+1}}{r^s} \right) \geq G^{\theta-1} \left( \frac{k_{j+1} - \tilde{k}_j}{r^s} \right) G \left( \frac{w_{j+1}}{r^s} \right).
\]

Therefore it follows that

\[
G^{\frac{\theta-1}{\theta}} \left( \frac{k}{r^s} \right) \left( \int_{B_{j+1}} G \left( \frac{w_{j+1}}{r^s} \right) \, dx \right)^{\frac{1}{\theta}} \leq c 2^{qj} G^{\frac{\theta-1}{\theta}} \left( \frac{k_{j+1} - \tilde{k}_j}{r^s} \right) \left( \int_{B_{j+1}} G \left( \frac{w_{j+1}}{r^s} \right) \, dx \right)^{\frac{1}{\theta}} \leq c 2^{qj} \left( \int_{B_{j+1}} G^\theta \left( \frac{\tilde{w}_j}{r^s} \right) \, dx \right)^{\frac{1}{\theta}}.
\]

(4.13)

Taking into account (4.11), (4.12) and (4.13), we deduce that

\[
\int_{B_{j+1}} G \left( \frac{w_{j+1}}{r^s} \right) \, dx \leq c 2^{(n+sq+2q)j} \left[ \int_{B_j} G \left( \frac{w_j}{r^s} \right) \, dx + \frac{r^s}{g(k/r^s)} \left( \int_{B_j} G \left( \frac{w_j}{r^s} \right) \, dx \right) \right] \text{Tail}(u_+; x_0, r/2).
\]

(4.14)

Denote

\[
a_j := \frac{1}{G(k/r^s)} \int_{B_j} G \left( \frac{w_j}{r^s} \right) \, dx.
\]
Then (4.14) is identical to
\[
a_{j+1} \leq c_2 2^{(n+sq+2q)\theta j} \left[ 1 + \frac{r^s}{g(k/r^s)} \text{Tail}(u_+; x_0, r/2) \right]^\theta a_j^\theta
\]
for some \( c_2 > 0 \) depending on \( n, s, p, q, \lambda \) and \( \Lambda \). At this stage, choose
\[
k = r^s G^{-1} \left( c_3 \int_{B_r} G \left( \frac{|u_+|}{r^s} \right) dx \right) + r^s g^{-1} (r^s \text{Tail}(u_+; x_0, r/2)),
\]
where \( c_3 = (c_2 2^\theta)^{\frac{1}{\theta-1}} 2^{\frac{(n+sq+2q)\theta}{(\theta-1)^2}} \). Then we see that
\[
a_{j+1} \leq (c_2 2^\theta)^{2(n+sq+2q)\theta j} a_j^\theta \quad \text{and} \quad a_0 \leq c_3^{-1} = (c_2 2^\theta)^{-\frac{1}{\theta-1}} 2^{\frac{(n+sq+2q)\theta}{(\theta-1)^2}}.
\]
Set \( c_b = \max \left\{ c_3^{1/p}, c_3^{1/q} \right\} \). Since Lemma 4.3 implies \( a_j \to 0 \) as \( j \to \infty \), we discover
\[
\sup_{B_{r/2}} u_+ \leq k \leq c_b r^s G^{-1} \left( \int_{B_r} G \left( \frac{|u_+|}{r^s} \right) dx \right) + r^s g^{-1} (r^s \text{Tail}(u_+; x_0, r/2)),
\]
and this is (4.4).

If \( u \) is a weak solution, then \( -u \) is a weak subsolution. Then we have the estimate (4.4) with \( u_+ \) replaced by \((-u)_+ = u_-\). This completes the proof. \( \square \)

4.2 Hölder continuity

We complete the proof of Theorem 1.1 by obtaining (1.6). Let \( u \in \mathcal{W}^{s, G}(\Omega) \cap L^\infty(\mathbb{R}^n) \) be a weak solution to (1.1). Let \( B_r \equiv B_r(x_0) \subseteq \Omega \). For \( \alpha \in (0, 1) \), \( \sigma \in (0, 1) \) and \( i \in \mathbb{N}_0 \), we write
\[
r_i := \sigma^i \frac{r}{2} \quad \text{and} \quad B_i = B_{r_i}(x_0)
\]
and define
\[
v_i := \left( \frac{r_i}{r_0} \right)^\alpha v_0 = \sigma^{\alpha i} v_0 \quad \text{(4.16)}
\]
with
\[
v_0 := 2 \left( c_b r^s G^{-1} \left( \int_{B_r} G \left( \frac{|u|}{r^s} \right) dx \right) + r^s g^{-1} (r^s \text{Tail}(u; x_0, r/2)) \right), \quad \text{(4.17)}
\]
where \( c_b \) is as in (1.5).

For the proof of (1.6), it is enough to show the following oscillation decay estimate.
Lemma 4.5 Under the above setting, there exist small $\alpha, \sigma \in (0, 1)$ depending on $n, s, p, q, \lambda$ and $\Lambda$ such that for every $i \in \mathbb{N}_0$,

$$\text{osc } u := \sup_{B_i} u - \inf_{B_i} u \leq v_i. \quad (4.18)$$

Proof First of all, we assume that

$$\alpha \leq \frac{sp}{2(p-1)} \quad \text{and} \quad \sigma < \frac{1}{4}. \quad (4.19)$$

We prove this lemma by induction. Obviously, (4.18) holds true for $i = 0$ from (1.5) and the definition of $v_0$. Suppose that for some $j \geq 0$,

$$\text{osc } u \leq v_i \quad \text{for all } i \in \{0, 1, 2, \ldots, j\}, \quad (4.20)$$

and then we will prove (4.18) for $i = j + 1$. We define $u_j$ by

$$u_j := \begin{cases} u - \inf_{B_j} u, & \text{if } |2B_{j+1} \cap \{u \geq \inf_{B_j} u + v_j/2\}| \geq \frac{1}{2}|2B_{j+1}|, \\ v_j - (u - \inf_{B_j} u), & \text{if } |2B_{j+1} \cap \{u \leq \inf_{B_j} u + v_j/2\}| \geq \frac{1}{2}|2B_{j+1}|, \end{cases}$$

where $2B_{j+1} := B_{2r_{j+1}}(x_0)$. Then $u_j \geq 0$ in $B_j$ and

$$\frac{|2B_{j+1} \cap \{u_j \geq v_j/2\}|}{|2B_{j+1}|} \geq \frac{1}{2}. \quad (4.21)$$

We divide the remaining part of the proof into three steps.

Step 1. We first estimate $\text{Tail}(u_j; x_0, r_j)$. Define $T_1$ and $T_2$ as follows:

$$\text{Tail}(u_j; x_0, r_j) = \sum_{i=1}^{j} \int_{B_{i-1} \setminus B_i} g \left( \frac{|u_j(x)|}{|x-x_0|^s} \right) \frac{dx}{|x-x_0|^{n+s}}$$

$$+ \int_{\mathbb{R}^n \setminus B_0} g \left( \frac{|u_j(x)|}{|x-x_0|^s} \right) \frac{dx}{|x-x_0|^{n+s}}$$

$$=: T_1 + T_2. \quad (4.22)$$

Before estimating $T_1$ and $T_2$, observe that the definition of $u_j$ and the induction hypothesis (4.20) imply

$$\sup_{B_i} |u_j| \leq 2v_i \quad \text{for all } i \leq j. \quad (4.23)$$

Moreover, the local boundedness of $u$ implies

$$|u_j| \leq |u| + v_j + \sup_{B_j} |u| \leq |u| + 2v_0. \quad (4.24)$$
We now estimate $T_1$. Recall (4.23) to find

$$T_1 \leq \sum_{i=1}^{j} \int_{B_{i-1} \setminus B_i} g \left( \frac{\sup_{B_{i-1}} |u_j|}{|x-x_0|^s} \right) \frac{dx}{|x-x_0|^{n+s}}$$

$$\leq c \sum_{i=1}^{j} \int_{B_{i-1} \setminus B_i} g \left( \frac{v_{i-1}}{r_i} \right) \left( \frac{r_i}{|x-x_0|^s} \right)^{p-1} \frac{dx}{|x-x_0|^{n+s}}$$

$$= c \sum_{i=1}^{j} r_i^{s(p-1)} g \left( \frac{v_{i-1}}{r_i^s} \right) \int_{B_{i-1} \setminus B_i} \frac{dx}{|x-x_0|^{n+s}} \leq c \sum_{i=1}^{j} \frac{r_i}{r_i^s} g \left( \frac{v_{i-1}}{r_i^s} \right).$$

(4.25)

In order to estimate $T_2$, we write $\tilde{g}(t) := G(t)/t$. Note that (1.3) implies $p\tilde{g}(t) \leq g(t) \leq q\tilde{g}(t)$ and

$$(p-1) \frac{G(t)}{t^2} \leq \tilde{g}'(t) = \frac{tg(t) - G(t)}{t^2} \leq (q-1) \frac{G(t)}{t^2}. \quad (4.26)$$

Now set $h(t) := \tilde{g}(t^{1/(q-1)})$. Using (4.26), we get

$$0 \leq \frac{p-1}{q-1} \frac{G \left( t \frac{1}{q-1} \right)}{t \frac{1}{q-1} + 1} \leq h'(t) = \frac{1}{q-1} \tilde{g}' \left( t \frac{1}{q-1} \right) t \frac{1}{q-1} - 1 \leq \frac{G \left( t \frac{1}{q-1} \right)}{t \frac{1}{q-1} + 1} = \frac{h(t)}{t}$$

and so

$$\left( \frac{h(t)}{t} \right)' = \frac{th'(t) - h(t)}{t^2} \leq 0.$$

Therefore $h(t)$ is non-decreasing and $h(t)/t$ is non-increasing. We then set $\psi$ be the concave envelope of $h$ to conclude that $\frac{\psi}{2} \leq h \leq \psi$, see [44, Lemma 2.2] for details. Additionally, considering (4.24) and the inequality $p\tilde{g}(t) \leq g(t) \leq q\tilde{g}(t)$, we find

$$T_2 \leq c \int_{\mathbb{R}^n \setminus B_0} \tilde{g} \left( \frac{v_0}{|x-x_0|^s} \right) \frac{dx}{|x-x_0|^{n+s}} + c \int_{\mathbb{R}^n \setminus B_0} g \left( \frac{|u(x)|}{|x-x_0|^s} \right) \frac{dx}{|x-x_0|^{n+s}}$$

$$= c \int_{\mathbb{R}^n \setminus B_0} h \left( \left( \frac{v_0}{|x-x_0|^s} \right)^{q-1} \right) \frac{dx}{|x-x_0|^{n+s}} + c \text{Tail}(u; x_0, r_0)$$

$$\leq c \int_{\mathbb{R}^n \setminus B_0} \psi \left( \left( \frac{v_0}{|x-x_0|^s} \right)^{q-1} \right) \frac{dx}{|x-x_0|^{n+s}} + c \text{Tail}(u; x_0, r_0).$$
Now we use Jensen’s inequality with respect to the measure $\frac{dx}{|x-x_0|^{n+s}}$. Then

$$T_2 \leq \frac{c}{r_0^s} \psi \left( r_0^s \int_{\mathbb{R}^n \setminus B_0 \left( x_0 \right)} \left( \frac{v_0}{|x-x_0|^s} \right)^{q-1} \frac{dx}{|x-x_0|^{n+s}} \right) + c \text{Tail}(u; x_0, r_0)$$

$$\leq \frac{c}{r_0^s} \left( \left( \frac{v_0}{r_0^s} \right)^{q-1} \right) + c \text{Tail}(u; x_0, r_0)$$

$$\leq \frac{c}{r_0^s} \tilde{g} \left( \frac{v_0}{r_0^s} \right) + c \text{Tail}(u; x_0, r_0) \leq \frac{c}{r_0^s} \left( \frac{v_0}{r_0^s} \right) + c \text{Tail}(u; x_0, r_0).$$

We recall (4.17) to discover

$$T_2 \leq \frac{c}{r_0^s} \tilde{g} \left( \frac{v_0}{r_0^s} \right) \leq \frac{c}{r_1^s} \tilde{g} \left( \frac{v_0}{r_1^s} \right). \quad (4.27)$$

We combine (4.22), (4.25), and (4.27), and recall (4.15) and (4.16) to have

$$\text{Tail}(u; x_0, r_j) \leq \sum_{i=1}^{j} \frac{c}{r_i^s} \tilde{g} \left( \frac{v_{j-1}}{r_i^s} \right) = \sum_{i=1}^{j} \frac{c}{r_i^s} \tilde{g} \left( \frac{v_i}{r_j^s} \sigma^{(s-\alpha)(j-i+1)} \right)$$

$$\leq \frac{c}{r_j^{s+1}} \tilde{g} \left( \frac{v_j}{r_j^s} \sigma^{(sp-\alpha(p-1))(j-i+1)} \right)$$

$$\leq \frac{c}{r_j^{s+1}} \tilde{g} \left( \frac{v_j}{r_j^s} \sigma^{sp-\alpha(p-1)} \right) \cdot \sigma^{-sp-\alpha(p-1)}$$

$$\leq \frac{c}{r_j^{s+1}} \tilde{g} \left( \frac{v_j}{r_j^s} \sigma^{sp-\alpha(p-1)} \right),$$

by taking $\sigma > 0$ sufficiently small so that

$$\sigma^{sp-\alpha(p-1)} \leq \sigma^{\frac{sp}{2}} \leq \frac{1}{2}. \quad (4.29)$$

**Step 2.** In this step, we look at

$$\frac{|2B_{j+1} \cap \{ u_j \leq 2\epsilon v_j \}|}{|2B_{j+1}|}, \quad \text{where } \epsilon := \sigma^{\frac{sp-\alpha(p-1)}{q-1}} \leq \sigma^{\frac{sp}{2(q-1)}} < 1. \quad (4.30)$$

For $k > 0$ to be determined later, we write

$$v := \min \left\{ \left[ \log \left( \frac{v_j/2 + \epsilon v_j}{u_j + \epsilon v_j} \right) \right]_+, k \right\}.$$
Applying Proposition 3.4 with \( u = u_j, r = 2r_{j+1}, R = r_j, a = v_j/2, b = \exp(k) \) and \( d = \epsilon v_j \) and using (4.28), we find

\[
\int_{2B_{j+1}} |v - (v)_{2B_{j+1}}| \, dx \leq c \left[ 1 + \frac{g(v_j/r_{j+1}^s)}{g(\epsilon v_j/r_{j+1}^s)} \sigma^{sp-a(p-1)} \right] \leq c(1 + \epsilon^{1-q} \sigma^{sp-a(p-1)}) \leq c. \tag{4.31}
\]

On the other hand, using the fact \( \{ v = 0 \} = \{ u_j \geq v_j/2 \} \) and (4.21), we see

\[
k = \frac{1}{|2B_{j+1} \cap \{ u_j \geq v_j/2 \}|} \int_{2B_{j+1} \cap \{ v = 0 \}} k \, dx \leq \frac{2}{|2B_{j+1}|} \int_{2B_{j+1} \cap \{ v = k \}} (k - v) \, dx = 2[k - (v)_{2B_{j+1}}].
\]

Integrating the above inequality over \( 2B_{j+1} \cap \{ v = k \} \) and using (4.31), we get

\[
\frac{|2B_{j+1} \cap \{ v = k \}|}{|2B_{j+1}|} k \leq \frac{2}{|2B_{j+1}|} \int_{2B_{j+1} \cap \{ v = k \}} (k - v)_{2B_{j+1}} \, dx \leq \frac{2}{|2B_{j+1}|} \int_{2B_{j+1}} |v - (v)_{2B_{j+1}}| \, dx \leq c.
\]

Here we assume \( \sigma > 0 \) is sufficiently small so that

\[
\sqrt{\epsilon} = \sigma \frac{sp-a(p-1)}{2(q-1)} \leq \sigma \frac{sp}{4(q-1)} \leq \frac{1}{6}, \tag{4.32}
\]

and take

\[
k = \log \left( \frac{v_j/2 + \epsilon v_j}{3\epsilon v_j} \right) \geq \log \left( \frac{1}{6\epsilon} \right) \geq \frac{1}{2} \log \left( \frac{1}{\epsilon} \right),
\]

from which, together with (4.30), we discover

\[
\frac{|2B_{j+1} \cap \{ u_j \leq 2\epsilon v_j \}|}{|2B_{j+1}|} \leq \frac{c}{k} \leq \frac{c_4}{\log (1/\sigma)}
\]

for some \( c_4 > 0 \) depending on \( n, p, q, \lambda \) and \( \Lambda \).

**Step 3.** Finally, we prove (4.18) for \( i = j + 1 \). For any \( m \in \mathbb{N}_0 \), we write

\[
\rho_m = (1 + 2^{-m})r_{j+1}, \quad \tilde{\rho}_m = \frac{\rho_m + \rho_{m+1}}{2}, \quad B^m = B_{\rho_m}, \quad \tilde{B}^m = B_{\tilde{\rho}_m}, \quad k_m = (1 + 2^{-m})\epsilon v_j \quad \text{and} \quad w_m = (k_m - u_j)_+ = (u_j - k_m)_-.
\]
Note that \( r_{j+1} < \rho_m \leq 2r_{j+1} \) and \( \epsilon v_j < k_m \leq 2\epsilon v_j \). Take cut-off functions \( \phi_m \in C^\infty_0(\bar{B}^m) \) such that \( 0 \leq \phi_m \leq 1, \phi_m \equiv 1 \) in \( B^{m+1} \) and \( |D\phi_m| \leq 2^{m+4}/r_{j+1} \). Applying the Caccioppoli inequality (3.1) to \( w_- = w_m, \phi = \phi_m \) and \( B_r = B^m \), we have

\[
\int_{B^{m+1}} \int_{B^{m+1}} G\left( \frac{|w_m(x) - w_m(y)|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^n} \\
\leq c \int_{B^{m+1}} \int_{B^{m+1}} G\left( \frac{|\phi_m(x) - \phi_m(y)|}{|x - y|^s} \right) \max\{w_m(x), w_m(y)\} \frac{dxdy}{|x - y|^n} \\
+ c \int_{B^{m+1}} w_m(x)\phi_m^q(x)dx \left( \sup_{y \in B^m} \int_{\mathbb{R}^n \setminus B^m} g\left( \frac{w_m(x)}{|x - y|^s} \right) \frac{dx}{|x - y|^{n+s}} \right).
\]

(4.33)

As in the proof of the local boundedness, we use the Sobolev-Poincaré inequality (4.1), Jensen’s inequality and (4.33), to find

\[
I := \left( \int_{B^{m+1}} G^\theta \left( \frac{w_m}{\rho_{m+1}^s} \right) dx \right)^{\frac{1}{\theta}} \\
\leq c \left( \int_{B^{m+1}} G^\theta \left( \frac{w_m - (w_m)_{B^{m+1}}}{\rho_{m+1}^s} \right) dx \right)^{\frac{1}{\theta}} + c \left( \int_{B^{m+1}} G^\theta \left( \frac{(w_m)_{B^{m+1}}}{\rho_{m+1}^s} \right) dx \right)^{\frac{1}{\theta}} \\
\leq c \int_{B^{m+1}} \int_{B^{m+1}} G\left( \frac{|w_m(x) - w_m(y)|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^n} + cG\left( \frac{(w_m)_{B^{m+1}}}{\rho_{m+1}^s} \right) \\
\leq c \int_{B^{m}} \int_{B^{m}} G\left( \frac{|\phi_m(x) - \phi_m(y)|}{|x - y|^s} \max\{w_m(x), w_m(y)\} \right) \frac{dxdy}{|x - y|^n} \\
+ c \int_{B^{m}} w_m(x)\phi_m^q(x)dx \left( \sup_{y \in B^m} \int_{\mathbb{R}^n \setminus B^m} g\left( \frac{w_m(x)}{|x - y|^s} \right) \frac{dx}{|x - y|^{n+s}} \right) \\
+ c \int_{B^{m+1}} G\left( \frac{w_m}{\rho_{m+1}^s} \right) dx =: II + III + IV.
\]

(4.34)

We write

\[
A_m := \frac{|B^m \cap \{u_j \leq k_m\}|}{|B^m|}.
\]

From the definition of \( u_j, k_m \) and \( A_m \), we estimate \( I \) as follows:

\[
I \geq \frac{1}{|B^{m+1}|^{\frac{1}{\theta}} \left( \int_{B^{m+1} \cap \{u_j \leq k_m+1\}} G^\theta \left( \frac{k_m - k_m+1}{\rho_{m+1}^s} \right) dx \right)^{\frac{1}{\theta}}} = A^\frac{1}{\theta}_{m+1} G\left( \frac{k_m - k_m+1}{\rho_{m+1}^s} \right)
\]

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Moreover, since $|\phi_m(x) - \phi_m(y)| \leq c 2^{m} \frac{|x-y|}{r_{j+1}}$, we find

\[
II \leq c 2^{qm} \left| B^m \right| \int_{B^m} \int_{B^m} G \left( \frac{\max\{w_m(x), w_m(y)\}}{r_{j+1}} \right) \frac{|x-y|}{r_{j+1}}^{(1-s)p} \frac{dx dy}{|x-y|^n} \\
\leq c 2^{qm} \left| B^m \right| r_{j+1}^{-(1-s)p} \int_{B^m \cap \{u_j \leq k_m\}} \int_{B^m} G \left( \frac{k_m}{r_{j+1}^{1+s}} \right) \frac{dydx}{|x-y|^{n-(1-s)p}} \\
\leq c 2^{qm} \left| B^m \right| G \left( \frac{k_m}{r_{j+1}^{1+s}} \right) |B^m \cap \{u_j \leq k_m\}| \leq c 2^{qm} G \left( \frac{\epsilon v_j}{r_{j+1}^{1+s}} \right) A_m.
\]

(4.36)

As for III, set

\begin{align*}
III_1 &= \int_{B^m} w_m(x) \phi_0(x) \, dx \quad \text{and} \quad III_2 = \sup_{y \in B^m} \int_{\mathbb{R}^n \setminus B^m} g \left( \frac{w_m(x)}{|x-y|^s} \right) \frac{dx}{|x-y|^{n-s}}.
\end{align*}

Then we have

\[
III_1 \leq |B^m|^{-1} \int_{B^m \cap \{u_j \leq k_m\}} k_m dx \leq c \epsilon v_j A_m.
\]

Using the fact that $\frac{|x-x_0|}{|x-y|} \leq 1 + \frac{|y-x_0|}{|x-y|} \leq 1 + \frac{\tilde{\rho}_m}{\rho_m - \tilde{\rho}_m} \leq c 2^m$ for $x \in \mathbb{R}^n \setminus B^m$ and $y \in \tilde{B}^m$, we have

\[
III_2 \leq c \int_{\mathbb{R}^n \setminus B_{j+1}} g \left( \frac{w_m(x)}{|x-x_0|^s} \right) 2^{sm} |x-x_0|^{2(n+s)m} \frac{dx}{|x-x_0|^{n+s}} \leq c 2^{(n+sm)|\text{Tail}(w_m; x_0, r_{j+1})|}.
\]

Moreover, since $u_j \geq 0$ in $B_j$, we have $w_m \leq k_m \leq 2 \epsilon v_j$ in $B_j$ and $w_m \leq k_m + |u_j| \leq 2 \epsilon v_j + |u_j|$ in $\mathbb{R}^n \setminus B_j$. Then from (4.28), we see

\[
\text{Tail}(w_m; x_0, r_{j+1}) \leq c \int_{B_j \setminus B_{j+1}} g \left( \frac{\epsilon v_j}{|x-x_0|^s} \right) \frac{dx}{|x-x_0|^{n+s}} + c \text{Tail}(u_j; x_0, r_j) \\
\leq c \int_{\mathbb{R}^n \setminus B_{j+1}} g \left( \frac{\epsilon v_j}{r_{j+1}^s} \right) \frac{r_{j+1}^{(p-1)s}}{|x-x_0|^{n+s}} \frac{dx}{|x-x_0|^{n+s}} + c \int_{r_{j+1}^s} g \left( \frac{\epsilon v_j}{r_{j+1}^s} \right) \sigma^{p-\alpha(p-1)} \\
\leq c \frac{r_{j+1}^s}{r_{j+1}} g \left( \frac{\epsilon v_j}{r_{j+1}^s} \right).
\]
Therefore we obtain

\[ III \leq c 2^{(n + s q)m} \frac{\epsilon v_j}{r^{s+1}_j} g \left( \frac{\epsilon v_j}{r^{s+1}_j} \right) A_m \leq c 2^{(n + s q)m} G \left( \frac{\epsilon v_j}{r^{s+1}_j} \right) A_m. \]  

(4.37)

We recall the notation for IV to find

\[ IV \leq c G \left( \frac{\epsilon v_j}{r^{s+1}_j} \right) A_m. \]  

(4.38)

We finally combine (4.34), (4.35), (4.36), (4.37), and (4.38), to discover

\[ A_{m+1} \leq c 2^{(n + s q + 2q)\theta m} A_m^{1 + \beta}, \quad \text{where} \quad \beta = \theta - 1. \]

Recall that

\[ A_0 = \frac{|2B_{j+1} \cap \{ u_j \leq 2\epsilon v_j \}|}{|2B_{j+1}|} \leq \frac{c_4}{\log(1/\sigma)} \]

and choose \( \sigma > 0 \) sufficiently small such that

\[ \frac{c_4}{\log(1/\sigma)} \leq c^{-1/\beta} \frac{2^{-[n + s q + 2q] \theta / \beta^2}}{\sigma}. \]  

(4.39)

Here, we notice that the constant \( \sigma \) is determined from (4.19), (4.29), and (4.39), hence depends only on \( n, s, p, q, \lambda \) and \( \Lambda \). Then we apply Lemma 4.3 to see that \( \lim_{m \to \infty} A_m = 0 \), which implies

\[ u_j > \epsilon v_j \quad \text{in} \quad B_{j+1}. \]  

(4.40)

If \( u_j = u - \inf_{B_j} u \), then (4.40) implies \( \inf_{B_{j+1}} u \geq \epsilon v_j + \inf_{B_j} u \) and therefore

\[ \text{osc} u \leq \sup_{B_{j+1}} u - \inf_{B_j} u \leq \sup_{B_{j+1}} u - (\epsilon v_j + \inf_{B_j} u) = \text{osc} u - \epsilon v_j \leq (1 - \epsilon) v_j. \]

On the other hand, if \( u_j = v_j - (u - \inf_{B_j} u) \), then we have \( \sup_{B_{j+1}} u \leq (1 - \epsilon) v_j + \inf_{B_j} u \) from (4.40). Thus

\[ \text{osc} u \leq \sup_{B_{j+1}} u - \inf_{B_j} u \leq (1 - \epsilon) v_j. \]

Considering both cases, we obtain

\[ \text{osc} u \leq (1 - \epsilon) v_j = \frac{1 - \sigma^{sp-q(p-1)}}{\sigma^\alpha} v_{j+1} \leq \frac{1 - \sigma^{sp-q-1}}{\sigma^\alpha} v_{j+1} \leq v_{j+1}, \]
by taking $\alpha = \alpha(n, s, p, q, \lambda, \Lambda) > 0$ sufficiently small so that

$$\sigma^\alpha \geq 1 - \sigma^{\frac{sp}{q-1}}.$$ 

This completes the proof. \hfill \Box

### 5 Existence and uniqueness of a weak solution

In this section we prove the existence and uniqueness of a weak solution to (1.1) with Dirichlet boundary conditions. The proof is based on a direct method in the calculus of variations. We refer the reader to [2, 13, 21, 26] for the details. Before introducing the compact embedding in $W^{s,G}(\Omega)$, we need the following definition.

**Definition 5.1** Let $A$ and $B$ be two Young functions. We say $B$ grows essentially more slowly near infinity than $A$ if

$$\lim_{t \to \infty} \frac{B(\lambda t)}{A(t)} = 0 \quad (5.1)$$

for every $\lambda > 0$.

Note that the condition (5.1) is equivalent to

$$\lim_{t \to \infty} \frac{A^{-1}(t)}{B^{-1}(t)} = 0.$$

Let $s \in (0, 1)$ and let $A$ be a Young function such that

$$\int_0^1 \left( \frac{t}{A(t)} \right)^{\frac{s}{n-s}} dt < \infty \quad \text{and} \quad \int_1^\infty \left( \frac{t}{A(t)} \right)^{\frac{s}{n-s}} dt = \infty. \quad (5.2)$$

Then $A^s_{\frac{1}{s}}$ is given by

$$A^s_{\frac{1}{s}}(t) := A(H^{-1}(t)) \quad \text{for} \ t \geq 0, \quad (5.3)$$

where the function $H : [0, \infty) \to [0, \infty)$ obeys

$$H(t) := \left( \int_0^t \left( \frac{\tau}{A(\tau)} \right)^{\frac{s}{n-s}} d\tau \right)^{\frac{n-s}{n}} \quad \text{for} \ t \geq 0. \quad (5.4)$$

The following lemma is related to compact embedding of $W^{s,G}$.

**Lemma 5.2** [1, Theorem 3.5] Let $s \in (0, 1)$ and let $A$ be a Young function fulfilling (5.2). Let $A^s_{\frac{1}{s}}$ be the Young function defined as in (5.3). Assume that $B$ is a Young function. Then the following properties are equivalent.

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(1) B grows essentially more slowly near infinity than $A^{\frac{n}{s}}$.

(2) The embedding

$$W^{s,A}(U) \to L^B(U)$$

is compact for every bounded domain $U \subset \mathbb{R}^n$ with Lipschitz boundary.

From the above lemma, we see the following compact embedding result.

**Lemma 5.3** Suppose an N-function $G$ satisfies (1.3) and $0 < s < 1$. The embedding

$$W^{s,G}(U) \to L^G(U)$$

is compact for any bounded domain $U \subset \mathbb{R}^n$ with Lipschitz boundary.

**Proof** We first consider the case $0 < s < \frac{n}{q}$. Note that the condition (1.3) implies

$$G(t) \geq t^q \quad \text{for } 0 < t < 1 \quad \text{and} \quad G(t) \leq t^q \quad \text{for } t \geq 1.$$

Then recalling the assumption $\frac{(q-1)s}{n-s} < 1$, we get

$$\int_0^1 \left( \frac{t}{G(t)} \right)^{\frac{s}{n-s}} dt \leq c \int_0^1 t^{-\frac{(q-1)s}{n-s}} dt < \infty \quad (5.5)$$

and

$$\int_1^\infty \left( \frac{t}{G(t)} \right)^{\frac{s}{n-s}} dt \geq c \int_1^\infty t^{-\frac{(q-1)s}{n-s}} dt = \infty.$$

Thus $G$ satisfies (5.2) with $A = G$.

Next, we will check that G grows essentially more slowly near infinity than $G^{\frac{n}{s}}$. For $t > G(1) = 1$, we recall the definition of $G^{\frac{n}{s}}$ to see

$$\frac{G^{\frac{n}{s}}(t)}{G^{-1}(t)} = \frac{H(G^{-1}(t))}{G^{-1}(t)} = \frac{1}{G^{-1}(t)} \left( \int_0^1 \left( \frac{\tau}{G(\tau)} \right)^{\frac{s}{n-s}} d\tau \right)^{\frac{n-s}{n}}$$

$$+ \frac{1}{G^{-1}(t)} \left( \int_1^G \left( \frac{\tau}{G(\tau)} \right)^{\frac{s}{n-s}} d\tau \right)^{\frac{n-s}{n}}, \quad (5.6)$$
where $H$ is as in (5.4) with $A = G$. Let us consider the second term in the right-hand side. Since $\tau \leq G^{-1}(\tau)$ is non-increasing,

$$
\frac{1}{G^{-1}(t)} \left( \int_1^{G^{-1}(t)} \left( \frac{\tau}{G(\tau)} \right)^{\frac{s}{n-s}} \, d\tau \right)^{\frac{n-s}{n}} \leq \frac{1}{G^{-1}(t)} \left( \int_1^{G^{-1}(t)} \left( \frac{1}{G(1)} \right)^{\frac{s}{n}} \, (G^{-1}(t) - 1)^{\frac{n-s}{n}} \right) \leq \frac{(G^{-1}(t))^{\frac{n-s}{n}}}{G^{-1}(t)} = (G^{-1}(t))^{-\frac{s}{n}}.
$$

(5.7)

Gathering together (5.5), (5.6) and (5.7) gives

$$
\frac{G^{-1}_n(t)}{G^{-1}(t)} \leq \frac{c}{G^{-1}(t)} + (G^{-1}(t))^{-\frac{s}{n}},
$$

hence $\lim_{t \to \infty} \frac{G^{-1}_n(t)}{G^{-1}(t)} = 0$. Therefore, Lemma 5.2 directly implies the compact embedding from $W^{s,G}(U)$ to $L^G(U)$ when $s < \frac{n}{q}$.

On the other hand, for the case $s \geq \frac{n}{q}$, a simple modification to [15, Proposition 2.1] shows that the embedding $W^{s,G}(U) \to W^{	ilde{s},G}(U)$ is continuous, i.e., $\|u\|_{W^{s,G}(U)} \leq c\|u\|_{W^{s,G}(U)}$, for every $\tilde{s} \in (0, s)$. Now take any number $\tilde{s} \in (0, \frac{n}{q})$. Since $W^{s,G}(U) \subset W^{	ilde{s},G}(U)$ and $W^{	ilde{s},G}(U) \to L^G(U)$ is compact, the embedding $W^{s,G}(U) \to L^G(U)$ is also compact when $s \geq \frac{n}{q}$. \hfill \qedsymbol

We next recall the following Poincaré type inequality.

**Lemma 5.4** [21, Corollary 6.2] Let $U \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Suppose $G$ is an N-function satisfying (1.3). Then there exists a constant $c > 0$ depending on $n$, $s$, $p$, $q$ and $U$ such that

$$
\int_U G(|u|) \, dx \leq c \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \, dxdy,
$$

for every $s \in (0, 1)$ and $u \in W^{s,G}(U)$.

To prove the existence and uniqueness of weak solutions to (1.1), we consider the following energy functional

$$
I[v] := \int_{C_{\Omega}} G \left( \frac{|v(x) - v(y)|}{|x - y|^s} \right) K(x, y) \, dxdy,
$$

(5.8)

where $C_{\Omega}$ is from (2.10) and

$$
v \in \mathcal{A}_f(\Omega) := \{ v \in \mathbb{W}^{s,G}(\Omega) : v = f \text{ in } \mathbb{R}^n \setminus \Omega \}.
$$

We say $u \in \mathcal{A}_f(\Omega)$ is a minimizer of $I$ over $\mathcal{A}_f(\Omega)$ if $I[u] \leq I[v]$ for all $v \in \mathcal{A}_f(\Omega)$.
Theorem 5.5 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, the operator $\mathcal{L}$ and an N-function $G$ be given as in Sect. 1, and $f \in \mathbb{W}^{s,G}(\Omega)$. Then there exists a unique minimizer $u$ of $\mathcal{I}$ over $\mathcal{A}_f(\Omega)$. Moreover, a function $u \in \mathcal{A}_f(\Omega)$ is the minimizer of $\mathcal{I}$ over $\mathcal{A}_f(\Omega)$ if and only if it is the weak solution to

$$
\begin{cases}
\mathcal{L} u = 0 \quad \text{in } \Omega, \\
u = f \quad \text{in } \mathbb{R}^n \setminus \Omega.
\end{cases}
$$

(5.9)

Proof Step 1. We first prove the existence and uniqueness of minimizer of $\mathcal{I}$. Since $f \in \mathcal{A}_f(\Omega)$, $\mathcal{A}_f(\Omega)$ is nonempty. We now choose a minimizing sequence $\{u_m\}_{m \geq 1}$ in $\mathcal{A}_f(\Omega)$ so that $\mathcal{I}[u_m]$ is non-increasing and $\lim_{m \to \infty} \mathcal{I}[u_m] = \inf_{w \in \mathcal{A}_f(\Omega)} \mathcal{I}[w]$. Set $v_m := u_m - f$. Then $\{v_m\}_{m \geq 1} \subset \mathbb{W}^{s,G}(\Omega)$ and $v_m = 0$ in $\mathbb{R}^n \setminus \Omega$. Choose a ball $B \equiv B_R(0)$ such that $B \supset \Omega$. In order to use the compactness argument, we need to show that $\|v_m\|_{\mathbb{W}^{s,G}(B)}$ is bounded for $m$. Using (2.8), (2.9) and Lemma 5.4, and the fact that $v_m = 0$ in $\mathbb{R}^n \setminus \Omega$, we find

$$
\|v_m\|_{\mathbb{W}^{s,G}(B)} \leq \int_B G(|v_m|) \, dx + \int_B \int_B G\left(\frac{|v_m(x) - v_m(y)|}{|x - y|^s}\right) \frac{dxdy}{|x - y|^n} + 2 \\
\leq c \left[ \int_B \int_{C_\Omega} G\left(\frac{|v_m(x) - v_m(y)|}{|x - y|^s}\right) K(x, y) \, dxdy + 2 \right] \\
\leq c(\mathcal{I}[u_m] + \mathcal{I}[f] + 2) \leq c.
$$

Since $\mathcal{I}[u_m]$ is bounded, so is $\|v_m\|_{\mathbb{W}^{s,G}(B)}$. By the compactness result in Lemma 5.3, there exist a subsequence $\{v_{m_j}\}_{j \geq 1}$ and $v \in \mathbb{W}^{s,G}(B)$ such that

$$
\begin{align*}
v_{m_j} &\rightharpoonup v \quad \text{weakly in } \mathbb{W}^{s,G}(B), \\
v_{m_j} &\to v \quad \text{in } L^G(B), \quad \text{as } j \to \infty, \\
v_{m_j} &\to v \quad \text{a.e. in } B,
\end{align*}
$$

Now extend $v$ by zero outside $B$ and set $u = v + f$. Then we see that $u \in \mathcal{A}_f(\Omega)$ and $\lim_{j \to \infty} u_{m_j} = u$ a.e. in $\mathbb{R}^n$. Therefore $v \in \mathbb{W}^{s,G}(\Omega)$ such that $v = 0$ in $\mathbb{R}^n \setminus \Omega$ and so $v + f \in \mathcal{A}_f(\Omega)$. Then Fatou’s lemma implies

$$
\mathcal{I}[u] \leq \liminf_{j \to \infty} \mathcal{I}[u_{m_j}] = \inf_{w \in \mathcal{A}_f} \mathcal{I}[w].
$$

The uniqueness directly follows from the convexity of $G$. Indeed, to prove this, we first suppose that $u, v \in \mathcal{A}_f(\Omega)$ are two different minimizers of $\mathcal{I}$. Then $\mathcal{I}[u] = \mathcal{I}[v]$. Since $G$ is strictly convex, we have

$$
\mathcal{I}[u] \leq \frac{\mathcal{I}[u + v]}{2} < \frac{\mathcal{I}[u] + \mathcal{I}[v]}{2} = \mathcal{I}[u],
$$

which is a contradiction.
Step 2. We next show the equivalence between the minimizer of (5.8) and a weak solution to (5.9). Suppose \( u \) is the minimizer of (5.8). Then for any \( \eta \in W^{s,G}(\Omega) \) with \( \eta = 0 \) in \( \mathbb{R}^n \setminus \Omega \), \( I[u + \tau \eta] \) has a critical point at \( \tau = 0 \). Thus

\[
0 = \frac{d}{d\tau} I[u + \tau \eta] \bigg|_{\tau=0} = \int_{C_\Omega} \int_{C_\Omega} \frac{d}{d\tau} G \left( \frac{|u(x) - u(y) + \tau(\eta(x) - \eta(y))|}{|x - y|^s} \right) \bigg|_{\tau=0} dxdy
\]

\[
= \int_{C_\Omega} \int_{C_\Omega} g \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{u(x) - u(y)}{|u(x) - u(y)|} (\eta(x) - \eta(y)) K(x,y) dxdy
\]

Therefore \( u \) is a weak solution to (5.9).

On the other hand, suppose \( u \) is a weak solution to (5.9). Then for any \( v \in A_f(\Omega) \), we see that \( u - v \in W^{s,G}(\Omega) \) and that \( u - v = 0 \) in \( \mathbb{R}^n \setminus \Omega \). We then test \( \eta := u - v \) in the weak formulation of (5.9) to discover

\[
0 = \int_{C_\Omega} \int_{C_\Omega} g \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{u(x) - u(y)}{|u(x) - u(y)|} (\eta(x) - \eta(y)) K(x,y) dxdy
\]

\[
= \int_{C_\Omega} \int_{C_\Omega} g \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{u(x) - u(y)}{|u(x) - u(y)|} \eta(x) K(x,y) dxdy
\]

\[
- \int_{C_\Omega} \int_{C_\Omega} g \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{u(x) - u(y)}{|u(x) - u(y)|} \eta(y) K(x,y) dxdy
\]

\[
(5.10)
\]

Let us look at the integrand of the second term with respect to the measure \( K(x,y) dxdy \) on the right-hand side. From (2.5) and (2.7), we see

\[
\begin{align*}
&g \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{u(x) - u(y)}{|u(x) - u(y)|} \frac{v(x) - v(y)}{|v(x) - v(y)|} \leq g \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{|v(x) - v(y)|}{|x - y|^s} \\
&\leq G^s \left( g \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \right) + G \left( \frac{|v(x) - v(y)|}{|x - y|^s} \right) \\
&= \frac{|u(x) - u(y)|}{|x - y|^s} g \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) - G \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) + G \left( \frac{|v(x) - v(y)|}{|x - y|^s} \right) \\
&(5.11)
\end{align*}
\]

We combine (5.11) and (5.10) to conclude that \( I[v] \geq I[u] \). Therefore \( u \) is the minimizer of \( I \).

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Data availability Data sharing is not applicable to this article as no data sets were generated or analysed.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.
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