PARACONTROLLED QUASI-GEOSTROPHIC EQUATION WITH SPACE-TIME WHITE NOISE

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Abstract. We study the stochastic dissipative quasi-geostrophic equation with space-time white noise on the two-dimensional torus. This equation is highly singular and basically ill-posed in its original form. The main objective of the present paper is to formulate and solve this equation locally in time in the framework of paracontrolled calculus when the differential order of the main term, the fractional Laplacian, is larger than \(7/4\). No renormalization has to be done for this model.

Keywords: Stochastic partial differential equation; Paracontrolled calculus; Quasi-geostrophic equation; white noise;
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1. Introduction

The dissipative 2D quasi-geostrophic equation (QGE for short) is a partial differential equation (PDE) which describes geophysical fluid dynamics on a two-dimensional space (see [19] for example). It has the fractional Laplacian as its main term. This type of QGE has been intensively studied (see [2, 3, 4, 5, 13, 14, 16] among many others) and has its stochastic counterpart (see [12, 20, 24] among others).

In this paper we study the following stochastic dissipative QGE with additive space-time white noise \(\xi\) on the two-dimensional torus \(T^2 = \mathbb{R}^2/\mathbb{Z}^2\) with a given initial condition:

\[
\partial_t u = \frac{(-\Delta)^{\theta/2} u + R^+ u \cdot \nabla u + \xi}{2}, \quad t > 0, \ x \in T^2.
\]

Here, (i) \(\nabla = (\partial_1, \partial_2)\) is the usual gradient on \(T^2\), (ii) \(R^\perp = (R_2, -R_1)\) with \(R_j\) being the \(j\)th Riesz transform on \(T^2\) (\(j = 1, 2\)), (iii) the dot stands for the standard inner product on \(\mathbb{R}^2\), (iv) \(\xi = \xi(t, x)\) is a (generalized) centered Gaussian random field associated with \(L^2(\mathbb{R} \times T^2)\) whose covariance is heuristically given by \(E[\xi(t, x)\xi(s, y)] = \delta(t - s)\delta(x - y)\), where \(\delta\) stands for Dirac’s delta function at 0.

Although stochastic QGE (1.1) may look natural and innocent at first glance, it does not even make sense in the classical sense since the regularity of \(\xi\) is quite bad. Let us take a quick look at this ill-definedness in the best case (\(\theta = 2\)). If the nonlinear term \(R^\perp u \cdot \nabla u\) were absent, the solution of (1.1) would be an Ornstein–Uhlenbeck process. It is a well-known Gaussian process and its space regularity at a fixed time is \(-\kappa\) for every small \(\kappa > 0\). Therefore, it is natural to guess that the regularity of \(u(t, \cdot)\) is \(-\kappa\) at best. Then, the regularities of \(R^\perp u(t, \cdot)\) and \(\nabla u(t, \cdot)\) are \(-\kappa\) and \(-1 - \kappa\) at best, respectively, and their (inner) product therefore cannot be defined.

Recently, three theories were invented (see [7, 8, 15]) and it became possible to study very singular stochastic PDEs of this kind. Now, studying singular stochastic PDEs is certainly one of the cutting edges in both probability theory and PDE theory. In the present paper we will use the so-called Gubinelli–Imkeller–Perkowski’s paracontrolled calculus [7] to solve (1.1) locally in time when \(7/4 < \theta \leq 2\).

Before stating our main result (Theorem 1.1 below), let us fix some notation. We denote by \(C^\alpha = B^\alpha_{\infty\infty}\) the Besov-Hölder space on \(T^2\) of regularity \(\alpha \in \mathbb{R}\). We set \(q := 2 - \frac{5}{2\theta}\) for \(11/6 < \theta \leq 2\).
and $q := 5 - \frac{8}{\theta} - \frac{2(\theta - 1)}{\theta}$ for $7/4 < \theta \leq 11/6$. The value of $q$ is a little bit messy and is not very important. However, one should note that $\frac{3}{2} \theta - 2 - \theta q + (\theta - 1)\kappa'$, the regularity of the initial condition, is smaller than that of the stationary Ornstein–Uhlenbeck process $X$ (i.e., the stationary solution of $\partial_t X = (-\Delta)^{\theta/2} X - X + \xi$). For example, if $\theta = 2$ then the Besov space in Theorem 1.1 is $C^{-\frac{1}{2}+\kappa'}$. Hence, the initial condition can be a distribution of regularity approximately $-1/2$. (The smaller $\theta$ is, the less regular everything becomes.)

Let $\chi: \mathbb{R}^2 \to \mathbb{R}$ be a smooth function with compact support such that $\chi(0) = 1$ and write $\chi^\varepsilon(x) = \chi(\varepsilon x)$ for $0 < \varepsilon < 1$. Mollify $\xi$ using $\chi^\varepsilon$ and denote it by $\xi^\varepsilon$. See (7.2) (and Remark 7.2) for the precise definition of $\xi^\varepsilon$. Then, $\xi^\varepsilon$ is a smooth noise and converges to $\xi$ as $\varepsilon \searrow 0$ in an appropriate sense. Since $\xi^\varepsilon$ is smooth, the following QGE driven by $\xi^\varepsilon$ is well-posed (up to the explosion time) in the classical mild sense:

\begin{equation}
\partial_t u^\varepsilon = -(-\Delta)^{\theta/2} u^\varepsilon + R^\perp u^\varepsilon \cdot \nabla u^\varepsilon + \xi^\varepsilon, \quad t > 0, x \in \mathbb{T}^2.
\end{equation}

Due to the structure of QGE, no renormalization is needed for (1.2). This phenomenon is rare among singular stochastic PDEs, but has already happened in [23] for the three-dimensional Navier-Stokes equation with space-time white noise and also in [6] for the stochastic Burgers equation.

Now we are in the position to state our main result. The proof of the following theorem is immediate from Propositions 5.2, 6.1 and Theorem 7.1:

**Theorem 1.1.** Let $7/4 < \theta \leq 2$ and then take sufficiently small $\kappa' > 0$ for this $\theta$. Assume that $u_0 \in C^{\frac{3\theta - 2 - \theta q + (\theta - 1)\kappa'}}$. Then, for every $0 < \varepsilon < 1$, there exist a random time $T^\varepsilon_\ast \in (0,1]$ and a unique process $u^\varepsilon$ defined up to time $T^\varepsilon_\ast$ such that

- $u^\varepsilon$ solves (1.2) on $[0, T^\varepsilon_\ast]$ with the initial condition $u_0$,
- $T^\varepsilon_\ast$ converges to some a.s. positive random time $T_\ast$ in probability,
- $u^\varepsilon$ converges to some $C^{\frac{3\theta - 2 - \theta q + (\theta - 1)\kappa'}}$-valued process $u$ defined on $[0, T_\ast)$ in the following sense:

$$
\lim_{\varepsilon \searrow 0} \sup_{0 \leq s \leq T^\varepsilon_\ast/2} \| u^\varepsilon_s - u_s \|_{C^{\frac{3\theta - 2 - \theta q + (\theta - 1)\kappa'}}} = 0
$$

in probability. Here, we understand $\sup_{0 \leq s \leq T^\varepsilon_\ast/2} \| u^\varepsilon_s - u_s \|_{C^{\frac{3\theta - 2 - \theta q + (\theta - 1)\kappa'}}} = \infty$ on the event $\{T^\varepsilon_\ast < T_\ast/2\}$. Furthermore, $u$ is independent of the choice of $\chi$.

**Remark 1.2.** The restriction $\theta > 7/4$ does not seem essential. A quick and heuristic computation shows that QGE with space-time white noise is subcritical in the sense of [8] if and only if $\theta > 4/3$. So, while the case $\theta \leq 4/3$ looks hopeless, there is a possibility that the case $4/3 < \theta \leq 7/4$ can be solved. However, as soon as $\theta$ gets smaller than $7/4$, computations get much more complicated. For example, more “symbols” are needed to define drivers. Moreover, a higher order version of paracontrolled calculus developed in [11] is probably needed, too. For these reasons we restrict ourselves to the case $7/4 < \theta \leq 2$, although the case of smaller $\theta$ still looks quite interesting.

As far as the authors know, the only paper which deals with a singular stochastic PDE with the fractional Laplacian $(-\Delta)^{\theta/2}$ is Gubinelli–Imkeller–Perkowski [7], in which paracontrolled calculus was invented. They studied a Burgers-like equation on $\mathbf{T}$ with $\theta > 5/3$. (It is almost certain that two other theories for singular stochastic PDE, Hairer’s regularity structure theory [8] and Kupiainen’s renormalization group approach [15], also work very well for singular stochastic PDEs with the fractional Laplacian. At the moment, however, there seems to be no literature which actually elaborates to solve QGE with white noise or other concrete examples of singular stochastic PDEs with fractional Laplacian with these theories.)
In this paper we slightly refine the argument in [7] in the following way: First, we will use Mourrat–Weber’s method [17] to formulate our paracontrolled QGE. In their work, a solution is defined to be a fixed point of a certain integration map in a Besov space-valued function space consisting of functions in time which may explode near $t = 0$. A merit of doing so is that we become able to treat a relatively bad initial condition. In turn, it allows us to use a stationary Ornstein–Uhlenbeck process, even though Ornstein–Uhlenbeck process in the original problem (in the mild form) starts at 0. Thanks to the stationarity, enhancing white noise become less cumbersome and can be done with a fractional generalization of Gubinelli–Perkowski’s enhancing method developed in [6].

The organization of this paper is as follows: Section 2 is a preliminary section. We recall basic results on Besov spaces and paradifferential calculus on the torus. The main aim is to study the heat semigroup associated to the fractional Laplacian and give a Schauder-type estimate for this semigroup. In Section 3 we formulate paracontrolled QGE for $\theta \in (7/4, 2]$, following the method in [17]. In the first half we define a driver of this equation. In the latter half we introduce a Besov space-valued function space in time in which solutions live. As usual for the mild formulation, a solution of our paracontrolled QGE is defined to be a fixed point of a suitable integration map. In Section 4 we estimate this integration map. This is the most important part in solving the equation. Using these estimates, we prove the local well-posedness of our paracontrolled QGE for $\theta \in (7/4, 2]$ in Section 5. In Section 6, we prove that for a nice driver a solution in the paracontrolled sense coincides with a solution in the classical mild sense. So far, everything is deterministic. Section 7 is the probabilistic part and devoted to enhancing the space-time white noise $\xi$ to a stationary random driver of paracontrolled QGE (Theorem 7.1). In Subsections 7.1 we generalize the enhancing method in [6] to the fractional case in a rather general setting. In Subsections 7.2–7.7 we then prove Theorem 7.1 with this method.

2. Preliminaries

In this section we introduce the Besov-Hölder space $C^\alpha = B^\alpha_{\infty, \infty}$, $\alpha \in \mathbb{R}$, and paradifferential calculus on the $d$-dimensional torus $\mathbb{T}^d$. Unlike the counterpart in most of the preceding works on paracontrolled calculus, we study the heat semigroup generated by the fractional Laplacian. In particular, a fractional version of the Schauder estimate is provided.

2.1. Besov spaces and paraproduct. We introduce Besov spaces over $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and recall their basic properties. Let $\mathcal{D} = \mathcal{D}(\mathbb{T}^d, \mathbb{R})$ be the space of all smooth $\mathbb{R}$-valued functions on $\mathbb{T}^d$. We denote by $\mathcal{D}'$ the dual of $\mathcal{D}$, that is, the space of Schwartz distributions in $\mathbb{T}^d$.

We set $e_k(x) = e^{2\pi i k \cdot x}$ for every $k \in \mathbb{Z}^d$ and $x \in \mathbb{T}^d$. The Fourier transform $\mathcal{F} f$ for $f \in \mathcal{D}$ is defined by $\mathcal{F} f(k) = \int_{\mathbb{T}^d} e_{-k}(x) f(x) dx$ and its inverse $\mathcal{F}^{-1} g$ for a rapidly decreasing sequences $\{g(k)\}_{k \in \mathbb{Z}^d}$ is defined by $\mathcal{F}^{-1} g = \sum_{k \in \mathbb{Z}^d} g(k) e_k$. For a rapidly decreasing smooth function $\phi : \mathbb{R}^d \to \mathbb{C}$, we set $\phi(D) f = \mathcal{F}^{-1} \phi \mathcal{F} f = \sum_{k \in \mathbb{Z}^d} \phi(k) \hat{f}(k) e_k$.

We denote by $\{\rho_m\}_{m = -1}^\infty$ a dyadic partition of unity, that is, it satisfies the following: (1) each $\rho_m : \mathbb{R}^d \to [0, 1]$ is radial and smooth, (2) $\text{supp}(\rho_{-1}) \subset B(0, 2^{1/2})$, $\text{supp}(\rho_0) \subset B(0, 2^{1/2}) \setminus B(0, 2^{3/2})$, (3) $\rho_m(\cdot) = \rho_0(2^{-m} \cdot)$ for $m \geq 0$, (4) $\sum_{m = -1}^\infty \rho_m(\cdot) = 1$. Here $B(0,r) = \{x \in \mathbb{R}^d ; |x| < r\}$. The Littlewood-Paley blocks (or the Littlewood-Paley operator) $\{\triangle_m\}_{m = -1}^\infty$ is defined by $\triangle_m = \rho_m(D)$. As usual we set $S_j f = \sum_{m \leq j - 1} \triangle_m f$. We note that

$$\sharp\{k \in \mathbb{Z}^d : \rho_j(k) > 0\} = O(2^m)$$

for each $j \in \mathbb{N}_0 = \{0, 1, \ldots\}$.
We are ready to define the Besov space $B^\alpha_{pq} = B^\alpha_{pq}(\mathbb{T}^d)$ for $\alpha \in \mathbb{R}$, $1 \leq p, q \leq \infty$. For $f \in \mathcal{D}'$, the $B^\alpha_{pq}$-norm $\|f\|_{B^\alpha_{pq}}$ is defined by

$$
\|f\|_{B^\alpha_{pq}} = \begin{cases} 
(\sum_{m=-1}^{\infty} (2^{m\alpha} \|\Delta_m f\|_{L^p(\mathbb{T}^d)})^q)^{\frac{1}{q}} & \text{(if } 1 \leq q < \infty), \\
\sup_{m \geq 1} 2^{m\alpha} \|\Delta_m f\|_{L^p(\mathbb{T}^d)} & \text{(if } q = \infty).
\end{cases}
$$

We set $B^\alpha_{pq} = \{f \in \mathcal{D}'; \|f\|_{B^\alpha_{pq}} < \infty\}$. See [21] for more about this function space. For simplicity, we will write $C^\alpha = C^\alpha(\mathbb{T}^d) = B^\alpha_{\infty\infty}(\mathbb{T}^d)$ and sometimes call it the Hölder-Zygmund class of order $\alpha$. We note that $\mathcal{D}' = \bigcup_{\alpha \in \mathbb{R}} C^\alpha$.

A couple of helpful remarks may be in order.

**Remark 2.1.**

1. Let $f \in \mathcal{D}'$. By using the Young inequality one can verify that the following norm is equivalent to the original one:

$$
\|f\|_{B^\alpha_{pq}^*} = \left(\sum_{m=-1}^{\infty} (2^{m\alpha} \|\Delta_m f\|_{L^p(\mathbb{T}^d)})^q\right)^{\frac{1}{q}}
$$

for $\alpha \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. See [21].

2. Our space $B^\alpha_{pq}$ may be different from the popular space $B^\alpha_{pq}$ which is defined to be the closure of $\mathcal{D}$ in $B^\alpha_{pq}$. However it causes no troubles, anyway. Some prefer to use the notation $b^\alpha_{pq}$ to denote the Besov space $B^\alpha_{pq}$.

In this paper we will mainly use $C^\alpha = B^\alpha_{\infty\infty}$. The following inequalities are frequently used results on these Banach spaces:

**Proposition 2.2.** We have the following:

1. Let $\alpha \leq \beta$, and let $f \in C^\beta$. Then $f \in C^\alpha$ and $\|f\|_{C^\alpha} \lesssim \|f\|_{C^\beta}$.

2. Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $0 \leq \nu \leq 1$. Then $\|f\|_{C^{(1-\nu)\alpha_1 + \nu\alpha_2}} \lesssim \|f\|_{C^\alpha_1}^\nu \|f\|_{C^\alpha_2}^{1-\nu}$ for all $f \in C^{\alpha_1} \cap C^{\alpha_2}$.

**Proposition 2.2** (1) and (2) are referred to as the embedding inequality and the interpolation inequality, respectively. We use the fact that $2^\alpha \lesssim 2^j\beta$ for $j \geq -1$ to prove the embedding inequality, while we use $2^{m\alpha} \|\Delta_m f\|_{L^\infty(\mathbb{T}^d)} \leq (2^{m\alpha}) \|\Delta_m f\|_{L^\infty(\mathbb{T}^d)}^{1-\nu} \cdot (2^{m\alpha_2}) \|\Delta_m f\|_{L^\infty(\mathbb{T}^d)}^\nu$ to prove the interpolation inequality.

We recall that the derivative stands for the smoothness order of functions.

**Lemma 2.3.** Let $u \in C^{\alpha+1}$ with $\alpha \in \mathbb{R}$. Then $\|\partial_j u\|_{C^\alpha} \lesssim \|u\|_{C^{\alpha+1}}$ for all $j = 1, 2, \ldots, d$.

**Proof.** The proof being simple, we recall it. We write the left-hand side out in full:

$$
\|\partial_j u\|_{C^\alpha} = \sup_{m \geq -1} 2^{m\alpha} \|\partial_j \Delta_m u\|_{L^\infty(\mathbb{T}^d)} = \sup_{m \geq -1} 2^{m\alpha+m} \|\partial_j (\Delta_{m-1} + \Delta_m + \Delta_{m+1}) \Delta_m u\|_{L^\infty(\mathbb{T}^d)}.
$$

We observe that $2^{-m} \partial_j (\Delta_{m-1} + \Delta_m + \Delta_{m+1}) = \tau(2^{-m} D)$ for some $\tau \in C^\alpha_{C^\infty}$. As a result, we obtain

$$
\|\partial_j u\|_{C^\alpha} \lesssim \sup_{m \geq -1} 2^{m\alpha+m} \|\Delta_m u\|_{L^\infty(\mathbb{T}^d)} = \|u\|_{C^{\alpha+1}},
$$

as required. \qed

**Proposition 2.4.** Let $\alpha > 0$. Then $C^\alpha \subset L^\infty$. 

Proof. Let $f \in C^\alpha$. Simply observe

$$f = \sum_{j=0}^{\infty} \triangle_j f$$

in $L^\infty$. □

Now we introduce the paraproduct and the resonant product. For every $f \in C^\alpha$, $g \in C^\beta$, we define the paraproduct by

$$f \otimes g = \sum_{m_1 \geq m_2 + 2} \triangle_{m_1} f \triangle_{m_2} g, \quad f \otimes g = \sum_{m_1 + 2 \leq m_2} \triangle_{m_1} f \triangle_{m_2} g;$$

and the resonant product by

$$f \otimes g = \sum_{|m_1 - m_2| \leq 1} \triangle_{m_1} f \triangle_{m_2} g.$$ 

Observe that $f g = f \otimes g + f \otimes g + f \otimes g$ at least formally. We have to establish that these definitions make sense. Here in Proposition 2.5 we collect the cases where these definitions are justified.

The following are basic properties of the paraproduct and the resonant.

**Proposition 2.5** (Paraproduct and resonant estimate). Let $\alpha, \beta \in \mathbb{R}$.

1. For all $f \in L^\infty(\mathbb{T}^d)$ and $g \in C^\beta$, $\|f \otimes g\|_{C^\beta} \lesssim \|f\|_{L^\infty(\mathbb{T}^d)} \|g\|_{C^\beta}$.
2. If $\alpha < 0$, then $\|f \otimes g\|_{C^{\alpha+\beta}} \lesssim \|f\|_{C^\alpha} \|g\|_{C^\beta}$ for $f \in C^\alpha$ and $g \in C^\beta$.
3. Assume $\alpha + \beta > 0$. Then $\|f \otimes g\|_{C^{\alpha+\beta}} \lesssim \|f\|_{C^\alpha} \|g\|_{C^\beta}$ for $f \in C^\alpha$ and $g \in C^\beta$.

**Proof.** The corresponding assertions to the ones over $\mathbb{R}^d$ are well known. Since $C^\alpha(\mathbb{T}^d) = C^\alpha(\mathbb{R}^d) \cap D'(\mathbb{T}^d)$, we can readily transplant these results over $\mathbb{R}^d$ into the ones over $\mathbb{T}^d$. □

A couple of remarks may be in order.

**Remark 2.6.** Let $\alpha, \beta \in \mathbb{R}$, and let $f \in C^\alpha$ and $g \in C^\beta$.

1. We need $\alpha + \beta > 0$ for Proposition 2.5(3). This assumption allows us not to take into account the “so called” moment condition as in [24].
2. Let $\varepsilon > 0$. In view of Proposition 2.2(1) and Proposition 2.5(2) the definition $f \otimes g$ makes sense as an element in $C^{\min(\alpha,-\varepsilon) + \beta}$. As a result, the element $f \otimes g$ and hence $f \otimes g$ make sense for any $f, g \in D'$.

We convert our observation to the form we use below.

**Corollary 2.7.** Let $\alpha, \beta \in \mathbb{R}$, and let $f \in C^\alpha$, $g \in C^\beta$ and $h \in L^\infty(\mathbb{T}^d)$.

1. $\|f \otimes h\|_{C^\alpha} \lesssim \|f\|_{C^\alpha} \|h\|_{L^\infty(\mathbb{T}^d)}$.
2. Assume $\beta > 0$. Then $\|f \otimes g\|_{C^\alpha} \lesssim \|f\|_{C^\alpha} \|g\|_{C^\beta}$.
3. Assume $\alpha + \beta > 0$ and $\alpha \beta \neq 0$. Then $\|fg\|_{C^{\min(\alpha,\beta)}} \lesssim \|f\|_{C^\alpha} \|g\|_{C^\beta}$.

**Proof.**

1. Simply swap the role of $f$ and $g$ in Proposition 2.5(1).
2. Combine $L^\infty \subset C^\beta$ and Proposition 2.5(1).
3. If $\min(\alpha, \beta) < 0$, then use Proposition 2.5(2) and (3) as well as Proposition 2.4 and Corollary 2.7(2). If $\min(\alpha, \beta) > 0$, then use Proposition 2.5(1) and (3) as well as Proposition 2.4 and Corollary 2.7(1). □

Now we record an identity on the partial derivative $\partial_j$ on the torus.
Lemma 2.8. Whenever \( f, g \in D' \) and \( l = 1, 2, \ldots, d \), \( \partial_l (f \otimes g) = (\partial_l f) \otimes g + f \otimes (\partial_l g) \).

Concerning the paraproduct and the resonant product, the following trilinear form is significant: Define the map \( C \) by

\[
C(f, g, h) = (f \otimes g) \circ h - f(g \circ h)
\]

for \( f, g, h \in D \). We prove the following estimate:

**Proposition 2.9** (Commutator estimates). Let \( \alpha, \beta, \gamma \in \mathbb{R} \) satisfy \( 0 < \alpha < 1 \), \( \beta + \gamma < 0 \) and \( \alpha + \beta + \gamma > 0 \). Then \( C \) uniquely extends to a continuous trilinear map from \( C^\alpha \times C^\beta \times C^\gamma \) to \( C^{\alpha + \beta + \gamma} \) and the map satisfies

\[
||C(f, g, h)||_{C^{\alpha + \beta + \gamma}} \lesssim ||f||_{C^\alpha}||g||_{C^\beta}||h||_{C^\gamma}
\]

for all \( f \in C^\alpha \), \( g \in C^\beta \) and \( h \in C^\gamma \).

**Proof.** We invoke [4] Lemma 2.4, where a counterpart to \( R^d \) is shown. Since \( B^s_{pq}(T^d) = D' \cap B^s_{pq}(R^d) \) with the identical norms. Thus the desired estimate follows from [4] Lemma 2.4. \( \square \)

Next we provide two lemmas on the Riesz transform \( R_l \) on the torus. Denote by \( R_l \) the \( l \)th Riesz transform for \( l = 1, 2, \ldots, d \), which is defined by

\[
R_l h = \sqrt{-1} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{k_l}{|k|} \hat{h}(k)e_k \quad (h \in D').
\]

**Lemma 2.10.** Let \( u \in C^\alpha \) with \( \alpha \in \mathbb{R} \). Then \( ||R_l u||_{C^\alpha} \lesssim ||u||_{C^\alpha} \) for all \( l = 1, 2, \ldots, d \).

**Proof.** Since \( 1 \in \ker(R_l) \), we may handle \( (1 - \rho_{-1}(D))u \) instead of \( u \) itself. The proof being simple once again, we recall it. We write the left-hand side out in full:

\[
||R_l(1 - \rho_{-1}(D))u||_{C^\alpha} = \sup_{m \geq 1} 2^{ma} ||R_l(1 - \rho_{-1}(D))\Delta_m u||_{L^\infty(T^d)}
\]

\[
= \sup_{m \geq 1} 2^{ma} ||R_l(1 - \rho_{-1}(D))|\Delta_{m-1} + \Delta_m + \Delta_{m+1}\rangle\Delta_m u||_{L^\infty(T^d)}
\]

\[
\leq \sup_{m \geq 1} 2^{ma} ||R_l(\Delta_{m-1} + \Delta_m + \Delta_{m+1})\Delta_m u||_{L^\infty(T^d)}.
\]

We observe that \( R_l(\Delta_{m-1} + \Delta_m + \Delta_{m+1}) = \eta_l(2^{-m}D) \) for some \( \eta_l \in C^\infty \) if \( m \gg 1 \). As a result, we obtain

\[
||R_l(1 - \rho_{-1}(D))u||_{C^\alpha} \lesssim \sup_{m \geq 1} 2^{ma} ||\Delta_m u||_{L^\infty(T^d)} = ||u||_{C^\alpha},
\]

as required. \( \square \)

If we consider the commutator, the postulate on \( \alpha \) is loosened. In fact, we needed \( \alpha < 0 \) in Proposition 2.5(2), while Lemma 2.11 requires \( \alpha < 1 \).

**Lemma 2.11.** Let \( \alpha < 1 \) and \( \beta \in \mathbb{R} \), and let \( f \in C^\alpha \) and \( g \in C^\beta \). Then \( R_l(f \otimes g) - f \otimes R_l g \in C^{\alpha + \beta} \), and it satisfies

\[
||R_l(f \otimes g) - f \otimes R_l g||_{C^{\alpha + \beta}} \lesssim ||f||_{C^\alpha}||g||_{C^\beta}.
\]

**Proof.** Let \( j \geq 5 \) and \( l = 1, 2, \ldots, n \). We disregard the lower frequency terms because we can incorporate them later easily. We observe that \( R_l \Delta_j \) has a kernel \( K_j = K_{j,l} \), that is

\[
\Delta_j (R_l(f \otimes g) - f \otimes R_l g) = \int_{T^d} K_j(x - y)(S_j f(y) - S_j f(x))\Delta_j g(y) dy
\]

and \( F K_j \in C^\infty \) satisfies the scaling relation: \( F K_j(2^{-j+5} \cdot) = F K_j \). By the mean value theorem, we have

\[
|S_j f(y) - S_j f(x)| \leq ||\nabla S_j f||_{L^\infty(T^d)}|x - y| = 2^{-j}||\nabla S_j f||_{L^\infty(T^d)} \cdot 2^j |x - y|.
\]
Since $\alpha < 1$, we have
\[
\|\nabla S_j f\|_{L^\infty(\mathbb{T}^d)} \leq \sum_{m=-1}^{j+2} \|\nabla S_j \Delta_m f\|_{L^\infty(\mathbb{T}^d)} \lesssim \sum_{m=-1}^{j+2} 2^m \|\Delta_m f\|_{L^\infty(\mathbb{T}^d)} \lesssim 2^{j(1-\alpha)} \|f\|_{C^\alpha}.
\]
Since
\[
\int_{\mathbb{R}^d} 2^j |x - y| \cdot |K_j(x - y)| \, dy = \int_{\mathbb{R}^d} 2^5 |y| \cdot |K_5(y)| \, dy < \infty \quad (x \in \mathbb{R}^d),
\]
we have
\[
\|\Delta_j (R_i(f \otimes g) - f \otimes R_i g)\|_{L^\infty(\mathbb{T}^d)} \lesssim 2^{-j(\alpha + \beta)} \|f\|_{C^\alpha} \|g\|_{C^\beta}.
\]

2.2. Fractional heat semigroup. In this subsection we study effects of the heat semigroup generated by the fractional Laplacian $(-\Delta)^{\theta/2}$ for $0 < \theta \leq 2$.

For later considerations, we need the following simple fact deduced from the Minkowski inequality:

**Proposition 2.12.** Let $1 \leq p \leq \infty$, and let $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{T}^d)$. Then
\[
\|f \ast g\|_{L^p(\mathbb{T}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^p(\mathbb{T}^d)}.
\]

Although the multiplier of $(-\Delta)^{\theta/2}$ has singularity at $\xi = 0$, as the following lemma shows, this singularity does not affect so much.

**Proposition 2.13.** The function
\[
x \in \mathbb{R}^d \mapsto K^\dagger(x) = \int_{\mathbb{R}^d} \exp(-|2\pi \xi|^\theta) \exp(2\pi \sqrt{-1} x \cdot \xi) \, d\xi \in C
\]
is an integrable function.

**Proof.** Let $u > 0$. We set
\[
K^\dagger_u(x) = \int_{\mathbb{R}^d} ((\rho_{-1}(u\xi) - \rho_{-1}(2u\xi)) \exp(-|2\pi \xi|^\theta) \exp(2\pi \sqrt{-1} x \cdot \xi) \, d\xi \quad (x \in \mathbb{R}^d).
\]
We decompose
\[
K^\dagger_u = \sum_{j=-\infty}^{\infty} K^\dagger_{2^{-j}}.
\]
We claim that we have an important bound for $K^\dagger_u$:
\[
|K^\dagger_u(x)| \lesssim u^{-d} \quad (x \in \mathbb{R}^d).
\]
In fact,
\[
|K^\dagger_u(x)| \leq \int_{\mathbb{R}^d} |\rho_{-1}(u\xi) - \rho_{-1}(2u\xi)| \, d\xi = O(u^{-d}).
\]
Let $u < 1$. Then we have
\[
|\partial^\alpha (\rho_{-1}(u\xi) - \rho_{-1}(2u\xi)) \exp(-|2\pi \xi|^\theta)| \lesssim u^L \chi_{x \sim u^{-1}}(\xi)
\]
for all $\alpha \in \mathbb{N}_0^d$ and $L \in \mathbb{N}$. As a consequence
\[
(|x|^{d-1} + |x|^{d+1}) |K^\dagger_u(x)| \lesssim u^{L-d}
\]
for all $u \in (0, 1]$ and $L \in \mathbb{N}$. Consequently,
\[
\|K^\dagger_u\|_{L^1(\mathbb{R}^d)} \lesssim u^{L-d}
\]
for any \( L \in \mathbb{N} \), where the implicit constant depends on \( L \). As a consequence,
\[
\left\| \sum_{j=0}^{\infty} K_{2^{-j}}^\dagger \right\|_{L^1(\mathbb{R}^d)} \lesssim \sum_{j=0}^{\infty} 2^{-j(L-d)} \sim 1.
\]

Let \( u > 1 \). Fix \( x \in \mathbb{R}^d \). We write
\[
L_u^\dagger(x) = \int_{\mathbb{R}^d} (\rho_{-1}(u\xi) - \rho_{-1}(2u\xi)) \exp(-|2\pi\xi|^\theta - 1) \exp(2\pi \sqrt{-1} x \cdot \xi) \, d\xi,
\]
\[
M_u^\dagger(x) = \int_{\mathbb{R}^d} (\rho_{-1}(u\xi) - \rho_{-1}(2u\xi)) \exp(2\pi \sqrt{-1} x \cdot \xi) \, d\xi.
\]
Then we decompose
\[
K_u^\dagger(x) = L_u^\dagger(x) + M_u^\dagger(x).
\]
We remark that (2.5)
\[
\left\| \sum_{j=-\infty}^{0} M_{2^{-j}}^\dagger \right\|_{L^1(\mathbb{R}^d)} \lesssim 1
\]
by the use of the Fourier transform, which maps \( S(\mathbb{R}^d) \) to itself continuously. Next, keeping in mind that \( \theta > 0 \) and that
\[
\exp(-|2\pi\xi|^\theta - 1) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} |2\pi\xi|^k \theta,
\]
we observe
\[
|\partial^\alpha [(\rho_{-1}(u\xi) - \rho_{-1}(2u\xi)) (\exp(-|2\pi\xi|^\theta - 1))]| \lesssim_\alpha \chi_{\xi \sim u^{-1}} (\xi) u^{-\theta+|\alpha|}.
\]
Thus,
\[
|x|^N |L_u^\dagger(x)| \lesssim u^{-\theta+N} \int_{\xi \sim u^{-1}} \, d\xi \sim u^{-d-\theta+N}
\]
for any integer \( N \). As a result,
\[
(2.7)
\]
\[
\left\| \sum_{j=-\infty}^{0} L_{2^{-j}}^\dagger(x) \right\| \lesssim 1.
\]
Meanwhile, (2.6) can be interpolated and we have
\[
|x|^{n+\frac{d}{2}} |L_u^\dagger(x)| \lesssim u^{-\frac{d}{2}}.
\]
Recall that \( \theta > 0 \). We consider the case where \( u = 2^{-j} \). If we add this estimate over \( j \), then we have
\[
|x|^{n+\frac{d}{2}} \left| \sum_{j=-\infty}^{0} L_{2^{-j}}^\dagger(x) \right| \lesssim 1.
\]
If we combine (2.7) with this estimate, then we obtain
\[
(1 + |x|^{n+\frac{d}{2}}) \left| \sum_{j=-\infty}^{0} L_{2^{-j}}^\dagger(x) \right| \lesssim 1,
\]
together with (2.6) proving the integrability of \( K^\dagger \).\( \square \)
We also need the following analogy to Proposition 2.12.

**Proposition 2.14.** Let \( \theta > 0 \). Then the function
\[
x \in \mathbb{R}^d \mapsto \int_{\mathbb{R}^d} |\xi|^{-\theta} (\exp(-|2\pi |\xi|^\theta) - 1) \exp(-2\pi \sqrt{-1}x \cdot \xi) \, d\xi \in \mathbb{C}
\]
is integrable.

**Proof.** The singularity at \( \xi = 0 \) can be dealt with similar to Proposition 2.13. Let
\[
G_\theta^*(\xi) = \sum_{j=0}^{\infty} \tau^*(2^{-j}\xi)|\xi|^{-\theta} (\exp(-|2\pi |\xi|^\theta) - 1), \quad (\xi \in \mathbb{R}^d),
\]
where \( \tau^* \) is supported near \( |\xi| = 1 \). We need to show that
\[
\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} G_\theta^*(\xi) \exp(-2\pi \sqrt{-1}x \cdot \xi) \, d\xi \right| \, dx < \infty.
\]
To this end, we fix \( \xi \in \mathbb{R}^d \). For \( \alpha \in \mathbb{N}_0^d \) we observe
\[
\partial^\alpha [\tau^*(2^{-j}\xi)|\xi|^{-\theta} (\exp(-|2\pi |\xi|^\theta) - 1)] = O(|\xi|^{-\theta-|\alpha|}),
\]
where the implicit constant in \( O \) depends on \( \alpha \in \mathbb{N}_0^d \). Thus,
\[
\| \partial^\alpha [\tau^*(2^{-j}\xi)|\xi|^{-\theta} (\exp(-|2\pi |\xi|^\theta) - 1)] \|_{L^1(\mathbb{R}^d)} = 2^j(2^{-\theta}-|\alpha|),
\]
which implies that
\[
|x|^L \left| \int_{\mathbb{R}^d} \tau^*(2^{-j}\xi)|\xi|^{-\theta} (\exp(-|2\pi |\xi|^\theta) - 1) \exp(-2\pi \sqrt{-1}x \cdot \xi) \, d\xi \right| \lesssim 2^{j(2^{-\theta}-L)}
\]
for any \( L \in \mathbb{N}_0 \). As a consequence, using this estimate for \( L = 0, d+1 \), we obtain
\[
\left| \int_{\mathbb{R}^d} \tau^*(2^{-j}\xi)|\xi|^{-\theta} (\exp(-|2\pi |\xi|^\theta) - 1) \exp(-2\pi \sqrt{-1}x \cdot \xi) \, d\xi \right| \lesssim \frac{2^j(2^{-\theta})}{1 + 2^j|x|^{d+1}}.
\]
If we add this estimate over \( j \in \mathbb{N}_0 \) and integrate this over \( \mathbb{R}^d \), then we conclude that the function is integrable. \( \square \)

**Lemma 2.15.** Let \( T > 0 \). Assume \( 1 \leq p, q \leq \infty \), \( s \in \mathbb{R} \), \( 0 < \theta \leq 2 \) and \( \delta \geq 0 \). Then for all \( f \in B^s_{pq} \) and \( 0 < t \leq T \), \( \| e^{-t(-\Delta)^{\theta/2}} f \|_{B^{s+\delta}_{pq}} \lesssim t^{-\delta} \| f \|_{B^{s}_{pq}} \).

**Proof.** We assume \( q < \infty \); otherwise we can readily modify the proof below. The case where \( \delta = 0 \) is clear in view of Proposition 2.13 since the function \( e^{-t(-\Delta)^{\theta/2}} f \) can be written as \( e^{-t(-\Delta)^{\theta/2}} f \) is integrable. Based on Remark 2.1, it suffices to show
\[
\| e^{-t(-\Delta)^{\theta/2}} f \|_{B^{s+\delta}_{pq}} \sim \left( \sum_{j=-1}^{\infty} (2^{j(s+\delta)} \| e^{-t(-\Delta)^{\theta/2}} \Delta_j^2 f \|_{L^p(\mathbb{T}^d)})^q \right)^{1/q} \lesssim t^{-\delta} \| f \|_{B^{s}_{pq}}.
\]
We notice
\[
e^{-t(-\Delta)^{\theta/2}} \Delta_j^2 f = c_d \mathcal{F}^{-1} \left( e^{-t(2\pi |\cdot|^\theta) \varphi(2^{-j} \cdot)} \right) * \Delta_j f
\]
and hence thanks to Proposition 2.12
\[
\| e^{-t(-\Delta)^{\theta/2}} \Delta_j^2 f \|_{L^p(\mathbb{T}^d)} \leq c_d \left\| \mathcal{F}^{-1} \left( e^{-t(2\pi |\cdot|^\theta) \varphi(2^{-j} \cdot)} \right) \right\|_{L^1(\mathbb{R}^d)} \| \Delta_j f \|_{L^p(\mathbb{T}^d)}.
\]
Here we denoted by $c_d$ some unimportant positive constants, whose precise value is irrelevant in the proof. Since

$$F^{-1}\left(\frac{e^{-t|2\pi|^\theta} \varphi(2^{-j} \cdot)}{t^{1/\theta}}\right)(x) = t^{-1/\theta}F^{-1}\left(\frac{e^{-t|2\pi|^\theta} \varphi(2^{-j} t^{-1/\theta} \cdot)}{t^{1/\theta}}\right)(x) \in \mathbb{R}^d,$$

we have

$$\left\| F^{-1}\left(\frac{e^{-t|2\pi|^\theta} \varphi(2^{-j} \cdot)}{t^{1/\theta}}\right) \right\|_{L^1(\mathbb{R}^d)} = \left\| F^{-1}\left(\frac{e^{-t|2\pi|^\theta} \varphi(2^{-j} t^{-1/\theta} \cdot)}{t^{1/\theta}}\right) \right\|_{L^1(\mathbb{R}^d)}.$$

Note that

$$\left\| F^{-1} f \right\|_{L^1(\mathbb{R}^d)} \leq \left\| \Delta^d f \right\|_{L^1(\mathbb{R}^d)} + C \left\| f \right\|_{L^1(\mathbb{R}^d)},$$

Thus, if $2^j t^{1/\theta} \geq 1$, then we obtain

$$\left\| F^{-1}\left(\frac{e^{-t|2\pi|^\theta} \varphi(2^{-j} \cdot)}{t^{1/\theta}}\right) \right\|_{L^1(\mathbb{R}^d)} \lesssim (2^j t^{1/\theta})^{-N}$$

for any $N \in \mathbb{N}$.

Likewise, if we start from

$$F^{-1}\left(\frac{e^{-t|2\pi|^\theta} \varphi(2^{-j} \cdot)}{t^{1/\theta}}\right)(x) = 2^j n F^{-1}\left(\frac{e^{-|2^{j+1} t^{1/\theta}|^\theta} \varphi}{t^{1/\theta}}\right)(2^j x),$$

then we obtain

$$\left\| F^{-1}\left(\frac{e^{-t|2\pi|^\theta} \varphi(2^{-j} \cdot)}{t^{1/\theta}}\right) \right\|_{L^1(\mathbb{R}^d)} \lesssim 1,$$

whenever $2^j t^{1/\theta} \leq 1$. Thus, since $\delta > 0$, it follows that

$$\left(\sum_{j=-\infty}^{\infty} 2^{j(\delta+\theta)} \|e^{-t(-\Delta)^{\theta/2}} \Delta^j f\|_{L^p(T^d)}\right)^{1/\theta} \lesssim \|f\|_{B^s_{p,q}} \left(\sum_{j=-\infty}^{\infty} 2^{j\delta} \min(1, (2^j t^{1/\theta})^{-N})q\right)^{1/\theta},$$

$$= t^{-\delta} \|f\|_{B^s_{p,q}} \left(\sum_{j=-\infty}^{\infty} 2^{j\delta} \min(1, (2^j t^{1/\theta})^{-N})q\right)^{1/\theta},$$

$$\leq t^{-\delta} \|f\|_{B^s_{p,q}} \left(\sum_{j=-\infty}^{\infty} 2^{j\delta} \min(1, (2^j t^{1/\theta})^{-N})q\right)^{1/\theta},$$

as was to be shown. \(\square\)

**Lemma 2.16.** Let $T > 0$. Assume $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, $0 < \theta \leq 2$ and $0 \leq \delta \leq 1$. Then for all $f \in B^s_{p,q}$ and $0 < t \leq T$, $\|e^{-t(-\Delta)^{\theta/2}} - 1\|_{B^{-s-\delta}_{p,q}} \lesssim t^\delta \|f\|_{B^s_{p,q}}$.

**Proof.** If $\delta = 0$, then simply use the fact that the integral kernel of $e^{-t(-\Delta)^{\theta/2}}$ is integrable; see Proposition 2.13. So, we need to consider the opposite endpoint case; suppose $\delta = 1$. The matter is reduced to investigating

$$(2.8) \left\| F^{-1}[(e^{-t|2\pi|^\theta} - 1) \varphi(2^{-j} \cdot)] \right\|_{L^1(\mathbb{R}^d)}$$
for $0 < t \leq T$ and $j \in \mathbb{N}$. If $2^j t^{1/\theta} \geq 1$, then we go through the same argument as above to have
\[
\left\| \mathcal{F}^{-1}[(e^{-t(2\pi|\cdot|^\theta) - 1}\varphi(2^{-j} \cdot)) - (e^{-t(2\pi|\cdot|^\theta) - 1}\varphi(2^{-j} \cdot))] \right\|_{L^1(\mathbb{R}^d)} = O((2^j t^{1/\theta})^{-N})
\]
for all $N \in \mathbb{N}$.

If $t^{1/\theta} 2^j \leq 1$, we need to handle (2.8) more carefully. As before, we use
\[
\left\| \mathcal{F}^{-1}[(e^{-t(2\pi|\cdot|^\theta) - 1}\varphi(2^{-j} \cdot)) - (e^{-t(2\pi|\cdot|^\theta) - 1}\varphi(2^{-j} t^{-1/\theta} \cdot))] \right\|_{L^1(\mathbb{R}^d)} = O((2^j t^{1/\theta})^{-N})
\]
Note that
\[
|\nabla^l [e^{-|2\pi\xi|^\theta} - 1]| = O(|\xi|^\theta - l) \quad (|\xi| \to 0)
\]
for all $l \in \mathbb{N} \cup \{0\}$. Thus,
\[
\| \nabla^l [e^{-|2\pi\xi|^\theta} - 1] \|_{L^1(B(8r) \setminus B(r))} = O(r^{\theta + n - l}).
\]

Hence
\[
|x|^{n+1} |\mathcal{F}^{-1}[(e^{-|2\pi\xi|^\theta} - 1)\varphi(2^{-j} t^{-1/\theta} \cdot)(x)]| \lesssim (2^j t^{1/\theta})^{\theta - 1}
\]
and
\[
|x|^n |\mathcal{F}^{-1}[(e^{-|2\pi\xi|^\theta} - 1)\varphi(2^{-j} t^{-1/\theta} \cdot)(x)]| \lesssim (2^j t^{1/\theta})^\theta.
\]
Interpolating between (2.9) and (2.10), we obtain
\[
|x|^{n+\theta/2} |\mathcal{F}^{-1}[(e^{-|2\pi\xi|^\theta} - 1)\varphi(2^{-j} t^{-1/\theta} \cdot)(x)]| \lesssim (2^j t^{1/\theta})^{\theta/2}.
\]

Meanwhile,
\[
|\mathcal{F}^{-1}[(e^{-|2\pi\xi|^\theta} - 1)\varphi(2^{-j} t^{-1/\theta} \cdot)(x)]| \lesssim \| (e^{-|2\pi\xi|^\theta} - 1)\varphi(2^{-j} t^{-1/\theta} \cdot) \|_{L^1(\mathbb{R}^d)} \lesssim (2^j t^{1/\theta})^{\theta + n}.
\]
Thus,
\[
|\mathcal{F}^{-1}[(e^{-|2\pi\xi|^\theta} - 1)\varphi(2^{-j} t^{-1/\theta} \cdot)(x)]| \lesssim \min\left((2^j t^{1/\theta})^{\theta + n}, (2^j t^{1/\theta})^{\theta/2}, |x|^{-n - \theta/2}\right).
\]
As a result,
\[
\left\| \mathcal{F}^{-1}[(e^{-t(2\pi|\cdot|^\theta) - 1}\varphi(2^{-j} \cdot)) - (e^{-t(2\pi|\cdot|^\theta) - 1}\varphi(2^{-j} t^{-1/\theta} \cdot))] \right\|_{L^1(\mathbb{R}^d)} \lesssim (2^j t^{1/\theta})^\theta.
\]

Thus, if we mimic the proof of Lemma 2.15 we obtain the desired result. \(\square\)

**Lemma 2.17.** Let $T > 0$. Assume $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, $0 \leq \theta \\leq 2$, $0 \leq \delta \leq 1$ and $\eta \in [0, \infty)$. Then $\|e^{-t_2(-\Delta)^{\theta/2}} f - e^{-t_1(-\Delta)^{\theta/2}} f\|_{B^s_{pq}} \leq |t_2 - t_1|^\delta t_1^{-\eta} \|f\|_{B^s_{pq} + \delta - \eta}$ for all $f \in B^s_{pq} + \delta - \eta$ and $0 \leq t_1 \leq t_2 \leq T$.

**Proof.** Combining Lemmas 2.15 and 2.16 we obtain
\[
\|e^{-t_2(-\Delta)^{\theta/2}} f - e^{-t_1(-\Delta)^{\theta/2}} f\|_{B^s_{pq}} = \|e^{-(t_2-t_1)(-\Delta)^{\theta/2}} - e^{-t_1(-\Delta)^{\theta/2}} f - e^{-t_1(-\Delta)^{\theta/2}} f\|_{B^s_{pq}} \lesssim |t_2 - t_1|^\delta \|e^{-t_1(-\Delta)^{\theta/2}} f\|_{B^s_{pq} + \delta} \lesssim |t_2 - t_1|^\delta t_1^{-\eta} \|f\|_{B^s_{pq} + \delta - \eta}.
\]

\(\square\)

In Lemmas 2.15 and 2.16 we stated our results in terms of the Besov space $B^s_{pq}$ with $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$. Here we record the results which we actually use. We present results on smoothing effects of semigroup $\{e^{-t(-\Delta)^{\theta/2}}\}_{t \geq 0}$ on Hölder-Besov spaces.

**Proposition 2.18 (Effects of heat semigroup).** Let $\alpha \in \mathbb{R}$ and $0 < \theta < 2$.

(1) Let $\delta \geq 0$. Then for all $f \in C^\alpha \|e^{-t(-\Delta)^{\theta/2}} f\|_{C_0 + \delta} \lesssim t^{-\delta} \|f\|_{C^\alpha}$ uniformly in $t > 0$.  

Let $\delta \in [0, 1]$. Then for all $f \in C^\alpha \| (e^{-t(-\Delta)^{\theta/2}} - 1) f \|_{C^\alpha} \lesssim t^\delta \| f \|_{C^\alpha}$ uniformly in $t > 0$.

Let $A$ be an operator acting on $\mathcal{D}'$ and let $f \in \mathcal{D}'$. Define the commutator $[A, f \otimes g]$ generated by paraproduct and $A$ by

$$[A, f \otimes g] = A(f \otimes g) - f \otimes Ag$$

for $g \in \mathcal{D}'$ whenever this is well defined.

**Proposition 2.19** (Commutators generated by paraproduct and heat semigroup). We let $\alpha < 1$, $\beta \in \mathbb{R}$, and let $0 < \theta \leq 2$ and $a < 1$. Then

$$|||e^{-t(-\Delta)^{\theta/2}} f \otimes g|||_{C^{\alpha+\beta-a\theta}} \lesssim t^a \| f \|_{C^\alpha} \| g \|_{C^\beta}$$

for $f \in C^\alpha$ and $g \in C^\beta$ uniformly over $t > 0$.

**Proof.** Let $j \geq 5$. We disregard the lower frequency terms because we can incorporate them later easily. Let

$$K_{j,t}^+(x) = \int_{\mathbb{R}^d} \exp(-t|2\pi \xi|^\theta)(\rho_{j-1}(\xi) + \rho_j(\xi) + \rho_{j+1}(\xi)) \exp(2\pi \sqrt{-1}t|x| \cdot \xi) \, d\xi$$

for $j \in \mathbb{N}$ and $x \in \mathbb{R}^d$. We observe that $e^{-t(-\Delta)^{\theta/2}} \Delta_j - e^{-t(-\Delta)^{\theta/2}} \Delta_{j+1}$ has a kernel $K_{j,t}$, that is, for some suitable constant $c_d$,

$$\sum_{k=j-1}^{j+1} \Delta_k(e^{-t(-\Delta)^{\theta/2}} f \otimes g) - f \otimes e^{-t(-\Delta)^{\theta/2}} g = c_d \int_{\mathbb{R}^d} K_{j,t}^+(x-y)(S_j f(y) - S_j f(x)) \Delta_j g(y) \, dy.$$  

Let $x, y \in \mathbb{R}^d$. By the mean value theorem, we have

$$|S_j f(y) - S_j f(x)| \leq \|\nabla S_j f\|_{L^\infty(\mathbb{T}^d)} |x-y| = 2^{-j} \|\nabla S_j f\|_{L^\infty(\mathbb{T}^d)} \cdot 2^j |x-y|.$$  

We notice

$$K_{j,t}^+(x) = \frac{1}{t^{d/\theta}} \int_{\mathbb{R}^d} \exp(-|2\pi \xi|^\theta)(\rho_{j+1}(t^{-1/\theta} \xi) + \rho_j(t^{-1/\theta} \xi) + \rho_{j-1}(t^{-1/\theta} \xi)) \exp(2\pi \sqrt{-1}t^{-1/\theta} x \cdot \xi) \, d\xi.$$  

We set

$$K_{j,t,-}^+(x) := K_{j,t}^+(x) - \frac{1}{t^{d/\theta}} \sum_{l=-1}^{1} \int_{\mathbb{R}^d} \rho_{j+1-l}(t^{-1/\theta} \xi) \exp(2\pi \sqrt{-1}t^{-1/\theta} x \cdot \xi) \, d\xi.$$  

If $2^{-j} t^{-1/\theta} \geq 1$, then for all $N = 0, 1, \ldots$

$$|K_{j,t,-}^+(x)| \lesssim_N |t^{-1/\theta} x|^{-N}(2^{-j} t^{-1/\theta})^{-d-\theta+N}$$

from (2.6). In particular, by considering the case of $N(N - d - 1) = 0$,

$$|K_{j,t,-}^+(x)| \lesssim t^{-d/\theta}(2^{-j} t^{-1/\theta})^{-d-\theta} \min(1, |t^{-1/\theta} x|^{-d-1}(2^{-j} t^{-1/\theta})^{d+1}).$$  

Consequently,

$$\|K_{j,t,-}^+\|_{L^1(\mathbb{R}^d)} \lesssim t^{-d/\theta} \int_{\mathbb{R}^d} (2^{-j} t^{-1/\theta})^{-d-\theta} \min(1, |t^{-1/\theta} x|^{-d-1}(2^{-j} t^{-1/\theta})^{d+1}) \, dx$$

$$= t^{-d/\theta} \int_{\mathbb{R}^d} (2^{-j} t^{-1/\theta})^{-d-\theta} \min(1, 2^j x|^{-d-1}) \, dx$$

$$\approx 2^j t/\theta \int_{\mathbb{R}^d} \min(1, |x|^{-d-1}) \, dx$$

$$\sim 2^j t/\theta.$$
Thus we have
\[
\| (\Delta_j^{-1} + \Delta_j + \Delta_{j+1}) (e^{-t(-\Delta)^{\theta/2}} (f \otimes g) - f \otimes e^{-t(-\Delta)^{\theta/2}} g) \|_{L^\infty(T^d)} \\
= \| (\Delta_j^{-1} + \Delta_j + \Delta_{j+1}) ((e^{-t(-\Delta)^{\theta/2}} - 1) (f \otimes g) - f \otimes (e^{-t(-\Delta)^{\theta/2}} - 1) g) \|_{L^\infty(T^d)} \\
\lesssim 2^{-j(\alpha+\beta+\theta)t} \| 2^j \alpha \nabla S_j f \|_{L^\infty(T^d)} \| g \|_{C^\beta} \\
\lesssim 2^{-j(\alpha+\beta+\theta)t} \| f \|_{C^\alpha} \| g \|_{C^\beta} \\
\leq 2^{-j(\alpha+\beta+\theta)t} \| f \|_{C^\alpha} \| g \|_{C^\beta}.
\]

Here for the last inequality, we used \(a < 1\). If \(2^{-j} t^{-1/\theta} \leq 1\), then the situation is simpler; we simply use
\[
|K_{j,t}^\dagger(x)| \leq 2^{jd} \exp(-2^j t)(1 + |2^j x|)^{-d-1} \quad (x \in \mathbb{R}^d).
\]

\[\square\]

Let \(0 < \theta \leq 2\). We set \(P_t^{\theta/2} = e^{-t(-\Delta)^{\theta/2} + 1} = e^{-t} e^{-t(-\Delta)^{\theta/2}}\). If \(f\) is given by the Fourier series
\[
f = \sum_{k \in \mathbb{Z}^d} a_k e_k,
\]
then
\[
P_t^{\theta/2} f = \sum_{k \in \mathbb{Z}^d} e^{-t(2\pi|k|)^{\theta/2} + 1} a_k e_k.
\]

**Remark 2.20.** It is easy to see that the results on the semigroup in this subsection (Lemmas 2.15, 2.16, 2.17 and Propositions 2.18, 2.19) still hold with possibly different positive constants even if \(e^{-t(-\Delta)^{\theta/2}}\) is replaced by \(P_t^{\theta/2}\).

### 2.3. Besov space-valued function spaces and the fractional version of the Schauder estimate

In this subsection we consider \(C^\alpha\)-valued functions in time and prove the fractional version of the Schauder estimate, which will play a key role in solving our QGE equation.

We introduce several function spaces as follows:

**Definition 2.21.** Let \(T > 0\), \(0 < \theta \leq 2\), \(\alpha \in \mathbb{R}\), \(\eta \geq 0\) and \(\delta \in (0,1]\).

- \(C((0,T], C^\alpha)\) is defined to the space of all continuous functions defined on \((0,T]\) which assume its value in \(C^\alpha\).
- \(C_T C^\alpha\) is the space of all continuous functions \(u \in C((0,T], C^\alpha)\) for which the norm
  \[
  \| u \|_{C_T C^\alpha} = \sup_{0 < t \leq T} \| u_t \|_{C^\alpha}
  \]
  is finite.
- \(C_T^\delta C^\alpha\) is the subspace of all \(\delta\)-Hölder continuous functions \(u \in C((0,T], C^\alpha)\) from \([0,T]\) to \(C^\alpha\) for which the seminorm
  \[
  \| u \|_{C_T^\delta C^\alpha} = \sup_{0 < s < t \leq T} \frac{\| u_t - u_s \|_{C^\alpha}}{|t - s|^\delta}
  \]
  is finite. Given \(u \in C_T^\delta C^\alpha\), define \(u_0 \in C^\alpha\) uniquely by continuity.
- \(E_T^\eta C^\alpha\) is defined uniquely by continuity.
"
• $E_T^{\eta,\delta}C^\alpha = \{ u \in C((0,T],C^\alpha); \|u\|_{E_T^{\eta,\delta}C^\alpha} < \infty \}$, where

$$\|u\|_{E_T^{\eta,\delta}C^\alpha} = \sup_{0<s<t<\infty} s^\eta \|u_t - u_s\|_{C^\alpha}.$$

• We define $L_T^{\alpha,\delta} = C_T C^\alpha \cap C_T^{\delta} C^{\alpha - \delta}$, where the norm of the left-hand side is given by the intersection norm.

• We define $L_T^{\eta,\alpha,\delta} = E_T^{\eta} C^\alpha \cap E_T^{\eta,\delta} C^{\alpha - \delta} \cap C_T C^{\alpha - \eta}$, where the norm of the left-hand side is given by the intersection norm.

Lemmas 2.15 and 2.17 correspond to $E_T^{\eta}C^\alpha$ and $E_T^{\eta,\delta}C^\alpha$, respectively.

**Remark 2.22.** We introduced the norms on the spaces $E_T^{\eta}C^\alpha$ and $E_T^{\eta,\delta}C^\alpha$ in order to control explosion at $t = 0$. The definitions of $L_T^{\alpha,\delta}$ and $L_T^{\eta,\alpha,\delta}$ look natural from the viewpoint of the time-space scaling of the fractional heat operator $\partial_t + (-\Delta)^{\theta/2}$. Note that $L_T^{\alpha,\delta}$ and $L_T^{\eta,\alpha,\delta}$ actually depend on $\theta$, while the other spaces in Definition 2.21 do not.

To understand the structure of the above function spaces, we prove the following embedding properties:

**Proposition 2.23.** Let $T > 0$. Let $\alpha \in \mathbb{R}$, $0 < \theta \leq 2$, $\eta > 0$ and $0 < \delta' \leq \delta < \infty$.

1. We have $L_T^{\alpha,\delta} = C_T C^\alpha \cap C_T^{\delta} C^{\alpha - \delta} \subset L_T^{\alpha,\delta'} = C_T C^\alpha \cap C_T^{\delta'} C^{\alpha - \delta'}$.

2. We have $L_T^{\eta,\alpha,\delta} = E_T^{\eta} C^\alpha \cap E_T^{\eta,\delta} C^{\alpha - \delta} \cap C_T C^{\alpha - \eta} \subset L_T^{\eta,\alpha,\delta'} = E_T^{\eta} C^\alpha \cap E_T^{\eta,\delta'} C^{\alpha - \delta'} \cap C_T C^{\alpha - \eta}$.

3. Let $\gamma \in [\alpha - \eta, \alpha]$. Then $v_t \in C^\gamma$ and $\|v_t\|_{C^\gamma} \lesssim t^{-\theta \delta' / \theta - \gamma} \|v\|_{L_T^{\eta,\alpha,\delta}}$ for every $v \in L_T^{\eta,\alpha,\delta}$ and every $0 < t < T$.

**Proof.** In principle, the proof hinges on the interpolation; see Proposition 2.22. The proof of each statement in Proposition 2.23 is made up to two steps: the first step is to deduce endpoint inequalities. We consider the case “$\delta' = 0$” and $\delta' = \delta$ in (1) and (2). In (3) we consider the cases $\gamma = \alpha - \delta$ and $\gamma = \alpha$. The second step is to interpolate between them.

1. It suffices to show $C_T C^\alpha \cap C_T^{\delta} C^{\alpha - \delta} \subset E_T^{\eta} C^{\alpha - \delta'}$. Set $\nu = \delta' / \delta \in (0,1)$, so that $\alpha - \delta' = (\alpha - \delta) \nu + \alpha(1 - \nu)$. Let $0 < s < t < T$. For every $v \in L_T^{\alpha,\delta}$, we have

$$\|v_t - v_s\|_{C^{\alpha - \delta}} \leq \|v_t - v_s\|_{C^{\alpha - \delta}}^{\nu} \|v_t - v_s\|_{C^\alpha}^{1 - \nu} \leq \{(t - s)^\delta \|v\|_{C_T^{\delta} C^{\alpha - \delta}}\}^{\nu} \|v\|_{C_T^{\alpha}}^{1 - \nu} \leq (t - s)^{\delta'} \|v\|_{L_T^{\eta,\alpha}}.$$

2. It suffices to show $E_T^{\eta} C^\alpha \cap E_T^{\eta,\delta} C^{\alpha - \delta} \subset E_T^{\eta,\delta'} C^{\alpha - \delta'}$. Fix $v \in L_T^{\eta,\alpha,\delta}$ and $0 < s < t < T$. We list two inequalities which we interpolate between.

$$\|v_t - v_s\|_{C^{\alpha - \delta}} \leq s^{-\eta} \|t - s\|^\delta \|v\|_{E_T^{\eta,\delta} C^{\alpha - \delta}};$$

$$\|v_t - v_s\|_{C^{\alpha - \delta}} \leq \|v_t\|_{C^\alpha} + \|v_s\|_{C^\alpha} \leq t^{-\eta} \|v\|_{E_T^{\eta} C^\alpha} + s^{-\eta} \|v\|_{E_T^{\eta} C^\alpha} \leq 2s^{-\eta} \|v\|_{E_T^{\eta} C^\alpha}.$$

Hence, for $0 < \nu = \delta' / \delta < 1$, we have

$$\|v_t - v_s\|_{C^{\alpha - \delta'}} \leq \|v_t - v_s\|_{C^{\alpha - \delta}}^{\nu} \|v_t - v_s\|_{C^\alpha}^{1 - \nu} \leq \{(s^{-\eta} t^{-\delta} \|v\|_{E_T^{\eta,\delta} C^{\alpha - \delta}}\}^{\nu} \{s^{-\eta} \|v\|_{E_T^{\eta} C^\alpha}\}^{1 - \nu} \leq s^{-\eta} t^{-\delta'} \|v\|_{E_T^{\eta,\alpha,\delta}},$$

as required.
(3) Let \( v \in \mathcal{L}^{\eta,\alpha,\delta}_T \). Since \( v \in \mathcal{E}^{\alpha-\delta}_T \), we have \( \|v_t\|_{C^{0,\alpha-\delta}} \leq t^{-\delta}\|v\|_{C^\alpha_T} \). Since \( v \in \mathcal{E}^{\eta}_T \), we have \( \|v_t\|_{C^\alpha} \leq t^{-\eta}\|v\|_{C^\alpha_T} \). Take \( \nu \in [0,1] \) such that \( \gamma = (\alpha - \theta \eta)(1 - \nu) + \alpha \nu \) and use Proposition 2.2 to obtain

\[
\|v_t\|_{C^\alpha} \leq \|v_t\|_{C^{0,\alpha-\delta}} \|v_t\|_{C^\alpha} \leq \|v\|_{C^\alpha_T}^{1-\nu} \{t^{-\eta}\|v\|_{C^\alpha_T}\}^{\nu} \leq t^{-\eta \nu}\|v\|_{\mathcal{L}^{\eta,\alpha,\delta}_T}.
\]

Noting that \( \gamma = \frac{\tau - (\alpha - \theta \eta)}{\theta} \), we obtain the desired result.

\[
\square
\]

In the course of the proof of (2) we proved:

**Corollary 2.24.** Let \( T > 0 \). Let \( \alpha \in \mathbb{R} \), \( 0 < \theta \leq 2 \), \( \eta > 0 \) and \( 0 < \delta' \leq \delta < \infty \). Then for every \( v \in \mathcal{L}^{\eta,\alpha,\delta}_T \) and \( 0 < s \leq t \leq T \), \( \|v_t - v_s\|_{C^{0,\delta'}} \leq s^{-\eta}(t - s)^{\delta'}\|v\|_{\mathcal{L}^{\eta,\alpha,\delta}_T} \).

Now we give a fractional version of the Schauder estimate. We define

\[
(2.12) \quad I[u]_t = \int_0^t P_t^{\frac{\theta}{2}} u_s \, ds
\]

whenever the right-hand side makes sense (dependency on \( \theta \) is suppressed). We denote by \( \mathcal{B}(x,y) \), \( x, y > 0 \) the beta function. We will repeatedly use the following fact:

**Proposition 2.25.** Let \( a_1, a_2 \in (0,1) \) and \( t > 0 \). Then

\[
\int_0^t s^{a_1}(t - s)^{-a_2} \, ds = \mathcal{B}(1 - a_1, 1 - a_2) t^{1-a_1-a_2} \asymp t^{1-a_1-a_2}.
\]

**Proposition 2.26 (Fractional Schauder estimates).** Let \( \alpha, \beta, \gamma, \delta \in \mathbb{R} \), \( \theta \in (0,2] \), \( \eta \in [0,1) \) and \( T \in (0,1] \).

1. Let \( f \in C^\alpha \). Then \( \|P_t^{\theta/2} f\|_{\mathcal{L}^{\beta-\gamma}_T} \lesssim \|f\|_{C^\alpha} \) for every \( \alpha < \beta \) and \( \delta \in [0,1] \).

2. Assume

\[
\alpha < \beta, \quad \alpha \leq \gamma < \alpha - \theta \eta + \theta, \quad \gamma \leq \beta < \alpha + \theta, \quad 0 < \delta \leq \frac{\beta - \alpha}{\theta}.
\]

Then \( \|I[u]\|_{\mathcal{L}^{\beta-\gamma}_T} \lesssim T^{\frac{\theta}{2}(\beta - \gamma)} \|u\|_{\mathcal{E}^{\beta}_T} \) for all \( u \in \mathcal{E}^{\beta}_T \).

**Proof.**

1. Recall that

\[
\mathcal{L}^{\beta-\gamma}_T = \mathcal{E}^{\beta}_T \cap C_T C^\alpha \cap \mathcal{E}^{\beta-\gamma}_T
\]

by definition. Since \( \beta = \alpha + \theta \cdot \frac{\beta - \alpha}{\theta} > \alpha \) and \( 0 < T \leq 1 \), we have

\[
(2.13) \quad \|P_t^{\theta/2} f\|_{\mathcal{L}^{\beta-\gamma}_T} \lesssim \sup_{0 < t < T} t^{\frac{\theta}{2}(\beta - \gamma)} \|P_t^{\theta/2} f\|_{C^\beta} \lesssim \|f\|_{C^\alpha}
\]

by Lemma 2.15. The estimate \( \|P_t^{\theta/2} f\|_{C_T C^\alpha} \lesssim \|f\|_{C^\alpha} \) is also a consequence of Lemma 2.15.

Finally, when \( T \geq t > s > 0 \), we deduce from Lemma 2.17 that

\[
\|e^{-t(-\Delta)^{\theta/2}} f \cdot e^{-s(-\Delta)^{\theta/2}} f\|_{C^{\beta-\delta s}} \lesssim (t-s)^{\delta s^{\frac{\beta - \alpha}{\theta}}} \|f\|_{C^\alpha}.
\]

From this, we can easily see that \( \|P_t^{\theta/2} f - P_s^{\theta/2} f\|_{C^{\beta-\delta s}} \) satisfies the same estimate with a different constant. Therefore, \( P_t^{\theta/2} f \in \mathcal{E}^{\beta-\gamma}_T \). Thus, we have shown (1).
(2) Recall that
\[ \mathcal{E}^{\frac{\alpha - \beta}{\eta}}_T \subset C^\beta \cap C_T^\gamma \cap \mathcal{E}^{\frac{\alpha - \beta}{\eta}}_T C^{3-\theta \delta} \]
by definition. Now we check that \( I[u] \in C_T^\gamma \). Let \( 0 \leq t \leq T \). By Lemma 2.15,
\[ \|I[u]_t\|_{C_T^\gamma} \leq \int_0^t \| e^{-(t-s)(-\Delta)^{\theta/2}} u_s \|_{C_T^\gamma} ds \lesssim \int_0^t (t-s)^{\frac{\alpha-\eta}{\beta}} \| u_s \|_{C_0} ds. \]
Using Proposition 2.25, \( 0 \leq t \leq T \) and \( \gamma \in [\alpha, \alpha - \theta \eta + \theta] \), we obtain
\[ \|I[u]_t\|_{C_T^\gamma} \lesssim \| u \|_{\mathcal{E}^\beta_T C^\alpha} \int_0^t (t-s)^{\frac{\alpha-\eta}{\beta}} s^{-\eta} ds \sim t^{\frac{\alpha-\eta}{\beta}} \| u \|_{\mathcal{E}^\beta_T C^\alpha} \leq T^{\frac{\alpha-\eta}{\beta}} \| u \|_{\mathcal{E}^\beta_T C^\alpha}. \]
In a similar way, using Lemma 2.15 and Proposition 2.25, as well as the assumptions \( 0 \leq t \leq T, \beta < \alpha + \theta \) and \( \eta < 1 \) we obtain
\[ t^{\frac{\alpha-\eta}{\beta}} \| I[u]_t \|_{C^\beta} \leq t^{\frac{\alpha-\eta}{\beta}} \int_0^t \| e^{-(t-s)(-\Delta)^{\theta/2}} u_s \|_{C^\beta} ds \]
\[ \leq t^{\frac{\alpha-\eta}{\beta}} \int_0^t (t-s)^{\frac{\alpha-\eta}{\beta}} \| u_s \|_{C_\beta} ds \]
\[ \lesssim t^{\frac{\alpha-\eta}{\beta}} \| u \|_{\mathcal{E}^\beta_T C^\alpha} \int_0^t (t-s)^{\frac{\alpha-\eta}{\beta}} s^{-\eta} ds \]
\[ \lesssim t^{\frac{\alpha-\eta+\theta-\eta}{\beta}} \| u \|_{\mathcal{E}^\beta_T C^\alpha} \leq T^{\frac{\alpha-\eta+\theta-\eta}{\beta}} \| u \|_{\mathcal{E}^\beta_T C^\alpha}. \]
Hence, \( \mathcal{E}^{\frac{\alpha - \beta}{\eta}}_T \subset C^\beta \)-norm of \( I[u] \) is dominated by a constant multiple of \( T^{\frac{\alpha-\eta+\theta-\eta}{\beta}} \| u \|_{\mathcal{E}^\beta_T C^\alpha} \).

Finally, we estimate \( \mathcal{E}^{\frac{\alpha - \beta}{\eta}}_T \subset C^\beta \)-norm of \( I[u]. \) Again by Lemma 2.15,
\[ \| I[u]_t - I[u]_s \|_{C^{3-\theta \delta}} = \left\| \int_s^t \left( P^{\theta/2}_{t-r} u_r \right) dr + \int_0^s \left( P^{\theta/2}_{t-s} - I \right) P^{\theta/2}_{s-r} u_r dr \right\|_{C^{3-\theta \delta}} \]
\[ \lesssim \| u \|_{\mathcal{E}^\beta_T C^\alpha} \int_s^t \left( t-r \right)^{\frac{\alpha-\eta}{\beta}} r^{-\eta} dr + \left\| \int_0^s \left( P^{\theta/2}_{t-s} - I \right) P^{\theta/2}_{s-r} u_r dr \right\|_{C^{3-\theta \delta}}. \]
We estimate the second term of the most right-hand side. By Proposition 2.18(2),
\[ \left\| \left( P^{\theta/2}_{t-s} - I \right) P^{\theta/2}_{s-r} u_r \right\|_{C^{3-\theta \delta}} \lesssim (t-s)^{\delta} \| P^{\theta/2}_{s-r} u_r \|_{C^\beta}. \]
By Lemma 2.15 and the definition of the norm \( \mathcal{E}^\beta_T C^\alpha \),
\[ \left\| \int_0^s \left( P^{\theta/2}_{t-s} - I \right) P^{\theta/2}_{s-r} u_r dr \right\|_{C^{3-\theta \delta}} \lesssim (t-s)^{\delta} \int_0^s \left( s-r \right)^{\frac{\alpha-\eta}{\beta}} \| u_r \|_{C_\beta} dr \]
\[ \leq (t-s)^{\delta} \int_0^s \left( s-r \right)^{\frac{\alpha-\eta}{\beta}} r^{-\eta} \| u \|_{\mathcal{E}^\beta_T C^\alpha} dr. \]
Recall that we are assuming \( 0 \leq \eta < 1 \) and \( \gamma < \alpha - \theta \eta + \theta \). Let \( 0 < s < t \leq T \). If we integrate this estimate against \( r \) and use Proposition 2.25 once again, we obtain
\[ (t-s)^{-\delta} \left\| \int_0^s \left( P^{\theta/2}_{t-s} - I \right) P^{\theta/2}_{s-r} u_r dr \right\|_{C^{3-\theta \delta}} \lesssim \int_0^s \left( s-r \right)^{\frac{\alpha-\eta}{\beta}} r^{-\eta} \| u \|_{\mathcal{E}^\beta_T C^\alpha} dr \]
\[ \sim s^{1-\eta-\frac{\alpha-\eta}{\beta}} \| u \|_{\mathcal{E}^\beta_T C^\alpha} \]
\[ \leq T^{\frac{\alpha+\eta-\theta-\eta}{\beta}} s^{-\frac{\alpha-\eta}{\beta}} \| u \|_{\mathcal{E}^\beta_T C^\alpha}. \]
So, we need to estimate the first term of the most right-hand side. We calculate

\[
(t - s)^{-\frac{\alpha}{\theta}} \theta^{\frac{\alpha}{\theta}} \left| I[u]_{t} - I[u]_{s} \right|_{L^{\infty}} \lesssim (t - s)^{-\frac{\alpha}{\theta}} \theta^{\frac{\alpha}{\theta}} \int_{s}^{t} (t - r)^{-\frac{\alpha}{\theta}} r^{-\eta + \frac{\alpha}{\theta}} dr \cdot \|u\|_{L^{\infty}_{t}C^{\theta}}.
\]

Since

\[
\frac{\beta - \theta \delta - \alpha}{\theta} < 1, \quad \eta < \frac{\alpha - \gamma}{\theta} + 1 < \frac{\beta - \gamma}{\theta} + 1,
\]

we are in the position of using Proposition 2.25 to have

\[
(t - s)^{-\frac{\alpha}{\theta}} \theta^{\frac{\alpha}{\theta}} \left| I[u]_{t} - I[u]_{s} \right|_{L^{\infty}} \lesssim (t - s)^{-\frac{\alpha}{\theta}} \theta^{\frac{\alpha}{\theta}} \int_{s}^{t} (t - r)^{-\frac{\alpha}{\theta}} r^{-\eta + \frac{\alpha}{\theta}} dr \cdot \|u\|_{L^{\infty}_{t}C^{\theta}}.
\]

we obtain

\[
(t - s)^{-\frac{\alpha}{\theta}} \theta^{\frac{\alpha}{\theta}} \left| I[u]_{t} - I[u]_{s} \right|_{L^{\infty}} \lesssim \left( t^{-\frac{\alpha}{\theta}} + \frac{\beta - \gamma}{\theta} \right) \|u\|_{L^{\infty}_{t}C^{\theta}}.
\]

Thus

\[
(t - s)^{-\frac{\alpha}{\theta}} \theta^{\frac{\alpha}{\theta}} \left| I[u]_{t} - I[u]_{s} \right|_{L^{\infty}} \lesssim \frac{\alpha - \theta \eta + \beta - \gamma}{\theta} \|u\|_{L^{\infty}_{t}C^{\theta}}.
\]

This completes the proof.

\[\square\]

3. Definition of Drivers and Solutions

From now on, we work on \( \mathbb{R}^{2} \). Let \( T > 0 \) be arbitrary and let \( 7/4 < \theta \leq 2 \). Once chosen, \( \theta \) will be fixed throughout. (Hence, the dependence on \( \theta \) will often be implicit.) For this \( \theta \), we take \( 0 < \kappa < \kappa' \) sufficiently small. More precisely, we basically assume

\[
0 < \kappa < \kappa' \ll 1, \quad \frac{1}{3} < \kappa \ll \frac{2}{3}.
\]

The exact values of \( \kappa \) and \( \kappa' \) are not important at all.

3.1. Definition of a Driver of Paraconstrained QGE. In this subsection we define a driver of the paraconstrained QGE.

Definition 3.1. An element

\[
\mathbf{X} = (X, V, Y, Z, W, \tilde{Z}, \tilde{W})
\]

of the product Banach space

\[
C_{T}C^{0,1-\kappa}_{T} \times (C_{T}C^{\frac{1}{2},2-\kappa}_{T})^{2} \times L^{2,3-\kappa}_{T} \times C_{T}C^{\frac{1}{2},5-\kappa}_{T} \times (C_{T}C^{\frac{3}{2},-4-\kappa}_{T})^{2} \times C_{T}C^{\frac{1}{2},5-\kappa}_{T} \times (C_{T}C^{0,4-\kappa}_{T})^{2}
\]

is called a driver of paraconstrained QGE.
is said to be a driver if the following relation holds:

\[ V_t = F^{\theta/2}_t V_0 + I[\nabla X]_t, \quad t \in [0, T]. \]

The set of all drivers is denoted by \( \mathcal{X}_T^\kappa \). The norm \( \|X\|_{\mathcal{X}_T^\kappa} \) is defined to be the sum of the norms of all components.

Assuming \( 7/4 < \theta \leq 2 \), we can suppose that \( \kappa, \kappa' > 0 \) are chosen so that \( 2\theta - 3 - \kappa', 2\theta - 3 - \kappa > 0 \). This means that the condition on \( Y \) makes sense.

As usual, we wrote \( \nabla = (\partial_1, \partial_2) \). Dependency on \( \theta \) is suppressed for notational simplicity. Note that the third component \( Y \) is assumed to have Hölder continuity in time. Let \( t \geq 0 \) and \( \kappa'' > 0 \). Since \( u \in C_T C^{\frac{1}{2}\theta - 1 - \kappa} \to I[\nabla u]_t \in C_T C^{\frac{1}{2}\theta - 2 - \kappa - \kappa''} \) and \( V_0 \in C_T C^{\frac{1}{2}\theta - 2 - \kappa} \subset C_T C^{\frac{1}{2}\theta - 2 - \kappa - \kappa''} \) are both continuous thanks to Proposition \( 2.23 \), we learn that \( \mathcal{X}_T^\kappa \) is a closed subset of the product Banach space above.

**Lemma 3.2.** Let \( X = (X, V, Y, Z, W, \hat{Z}, \hat{W}) \) be a driver. Then \( \|\nabla X_t\|_{C^{\frac{1}{2}\theta - 2 - \kappa}} \lesssim \|X_t\|_{C^{\frac{1}{2}\theta - 1 - \kappa}} \lesssim \|X\|_{\mathcal{X}_T^\kappa}. \)

**Proof.** Simply resort to Lemma \( 2.3 \).

Here is a simple remark on the \( Y \)-component of a driver.

**Lemma 3.3.** Let \( Y \in L_T^{2\theta - 3 - \kappa, \frac{2\theta - 3 - \kappa}{\theta}} = C_T C^{2\theta - 3 - \kappa} \cap C_T^{2\theta - 3 - \kappa} C^0 \) with \( 7/4 < \theta \leq 2 \) and \( 0 < \kappa < \kappa' < 1 \). Then \( \|Y_t - Y_s\|_{C^{\kappa' - \kappa}} \lesssim (t - s)^{2\theta - 3 - \kappa'} \|Y\|_{L^{2\theta - 3 - \kappa, \frac{2\theta - 3 - \kappa}{\theta}}}. \)

**Proof.** Since \( 0 < \kappa < \kappa' \), we see from Proposition \( 2.23(1) \) that \( L_T^{2\theta - 3 - \kappa, \frac{2\theta - 3 - \kappa}{\theta}} \subset L_T^{2\theta - 3 - \kappa, \frac{2\theta - 3 - \kappa'}{\theta}} \subset C_T^{2\theta - 3 - \kappa, \frac{2\theta - 3 - \kappa'}{\theta}} \). This inclusion implies

\[ \|Y_t - Y_s\|_{C^{\kappa' - \kappa}} \lesssim (t - s)^{2\theta - 3 - \kappa'} \|Y\|_{C_T^{2\theta - 3 - \kappa, \frac{2\theta - 3 - \kappa'}{\theta}}} \lesssim (t - s)^{2\theta - 3 - \kappa'} \|Y\|_{L^{2\theta - 3 - \kappa, \frac{2\theta - 3 - \kappa}{\theta}}}. \]

If the regularity of \( X \) is nice enough, we can enhance \( X \) to a driver \( \hat{X} \) in a very natural way. This driver is called the natural enhancement of \( X \).

**Example 3.4.** Let \( \alpha > 2 \) and \( \beta > 1 \). Let \( X \in C_T C^\alpha, Y_0 \in C^\beta \), and \( V_0 \in (C^\beta)^2 \) be given. Then, keeping in mind that

\[ X \in C_T C^\alpha \subset C_T \theta^{-2 - \kappa}, \]

we can define a driver \( \hat{X} = (X, V, Y, Z, W, \hat{Z}, \hat{W}) \) by setting

\[ V = F^{\theta/2}_t V_0 + I[\nabla X]_t \in C_T C^\beta \cap C_T C^\alpha \subset C_T \theta^{-1 - \kappa}, \]

\[ Y = F^{\theta/2}_t Y_0 + I[R^\perp X \cdot \nabla X] \subset F^{\theta/2}_t Y_0 + I[R_2 X \cdot \partial_1 X - R_1 X \cdot \partial_2 X] \in C_T C^{\min(\alpha - 1, \beta)}, \]

\[ Z = R^\perp Y \circ \nabla X = [R_2 Y \cdot \partial_1 X - R_1 Y \cdot \partial_2 X] \in C_T C^{2\alpha - 2} \subset C_T \theta^{-5 - \kappa}, \]

\[ W = R^\perp V \circ \nabla X = [R_2 V_1 \cdot \partial_1 X - R_1 V_1 \cdot \partial_2 X] \in C_T C^{2\alpha - 2} \subset C_T \theta^{-5 - \kappa}, \]

\[ \hat{Z} = \nabla Y \cdot R^\perp X = R_2 Y \cdot R_2 X - R_2 Y \cdot R_1 X \in C_T C^{2\alpha - 2} \subset C_T \theta^{-5 - \kappa}, \]

\[ \hat{W} = R^\perp X \circ \nabla V = [R_2 X \cdot \partial_1 V_1 - R_1 X \cdot \partial_2 V_1] \in C_T C^{2\alpha - 2} \subset C_T \theta^{-5 - \kappa}. \]
Here, we write $R^L = (R_2, -R_1)$ and $V = (V_1, V_2)$. Moreover, since
\[
\sup_{0 < t \leq T} \| R^L X_t \cdot \nabla X_t \|_{C^0} < \infty,
\]
we have
\[
\| I[R^L X \cdot \nabla X] - I[R^L X \cdot \nabla X] \|_{C^0} = O(|s - t|) \quad (0 < s \leq t \leq T).
\]
Thus,
\[
I[R^L X \cdot \nabla X] \in C^1_T C^0 \subset C^2_{T, \theta} \frac{2\theta-3-\kappa}{\theta}(\mathcal{F}).
\]
As a result, $Y \in \mathcal{L}^{2\theta-3-\kappa, \frac{2\theta-3-\kappa}{\theta}}_{T, \theta}$. So, it follows that $X = (X, V, Y, Z, \dot{Z}, \dot{W})$ is a driver.

The most important example of such $X$ we have in mind is the sample path of the approximation of the stationary OU-like process

\[
(3.2) \quad X_t^\varepsilon = \int_{-\infty}^{t} P_{t-s}^{\theta/2} \xi_s ds =: \mathcal{I}[\xi]^\varepsilon, \quad t \in \mathbb{R},
\]
where $\xi^\varepsilon$ stands for the smooth approximation of the space-time white noise $\xi$ on $\mathbb{R} \times \mathbb{T}^2$, respectively. Moreover, if we choose
\[
V_0 = \int_{-\infty}^{0} P_{0-s}^{\theta/2} \nabla X_s ds, \quad Y_0 = \int_{-\infty}^{0} P_{0-s}^{\theta/2} [R^L X_s \cdot \nabla X^\varepsilon] ds,
\]
Then, in this case we have $V^\varepsilon = \mathcal{I}[\nabla X^\varepsilon]$ and $Y^\varepsilon = \mathcal{I}[R^L X^\varepsilon \cdot \nabla X^\varepsilon]$. Using these, we define the other four symbols as follows:
\[
\begin{align*}
Z^\varepsilon &= R^L Y^\varepsilon \otimes \nabla X^\varepsilon = R^L \{ \mathcal{I}[R^L X^\varepsilon \cdot \nabla X^\varepsilon] \} \otimes \nabla X^\varepsilon, \\
W^\varepsilon &= R^L V^\varepsilon \otimes \nabla X^\varepsilon = R^L \{ \mathcal{I}[\nabla X^\varepsilon] \} \otimes \nabla X^\varepsilon, \\
\dot{Z}^\varepsilon &= \nabla V^\varepsilon \cdot R^L X^\varepsilon = \mathcal{I}[R^L X^\varepsilon \cdot \nabla X^\varepsilon] \cdot R^L X^\varepsilon, \\
\dot{W}^\varepsilon &= R^L X^\varepsilon \otimes \nabla V^\varepsilon = R^L X^\varepsilon \otimes \nabla \mathcal{I}[\nabla X^\varepsilon].
\end{align*}
\]
The random driver $X^\varepsilon = (X^\varepsilon, V^\varepsilon, Y^\varepsilon, Z^\varepsilon, \dot{Z}^\varepsilon, \dot{W}^\varepsilon)$ obtained in this way is called the stationary natural enhancement of the smooth approximation $X^\varepsilon$ given in (3.2).

Now, we summarize some data on the driver for the readers’ convenience.

| Symbol | Regularity | Space | Component of $X^\varepsilon$ | Order |
|--------|------------|-------|----------------------------|-------|
| $X$    | $\frac{3}{2}(\theta - 2)$ | $C_T C^{\frac{3}{2}(\theta - 2) - \kappa}_{\theta}$ | $X^\varepsilon$ | 1 |
| $V$    | $\frac{3}{2}\theta - 2$ | $(C_T C^{\frac{3}{2}(\theta - 2) - \kappa}_{\theta})^2$ | $V^\varepsilon := \mathcal{I}[\nabla X^\varepsilon]$ | 1 |
| $Y$    | $2\theta - 3$ | $C^{2\theta-3-\kappa, \frac{2\theta-3-\kappa}{\theta}}_{T, \theta}$ | $Y^\varepsilon := \mathcal{I}[R^L X^\varepsilon \cdot \nabla X^\varepsilon]$ | 2 |
| $Z$    | $\frac{3}{2}(\theta - 2)$ | $C_T C^{\frac{3}{2}(\theta - 2) - \kappa}_{\theta}$ | $R^L Y^\varepsilon \otimes \nabla X^\varepsilon$ | 3 |
| $W$    | $2(\theta - 2)$ | $(C_T C^{2(\theta - 2) - \kappa}_{\theta})^2$ | $R^L V^\varepsilon \otimes \nabla X^\varepsilon$ | 2 |
| $\dot{Z}$ | $\frac{3}{2}(\theta - 2)$ | $C_T C^{\frac{3}{2}(\theta - 2) - \kappa}_{\theta}$ | $\nabla Y^\varepsilon \cdot R^L X^\varepsilon$ | 3 |
| $\dot{W}$ | $2(\theta - 2)$ | $(C_T C^{2(\theta - 2) - \kappa}_{\theta})^2$ | $R^L X^\varepsilon \otimes \nabla V^\varepsilon$ | 2 |

In the above table, “Regularity $\alpha$” means that the space regularity of the symbol is $\alpha - \kappa$ for sufficiently small $\kappa > 0$. “Component of $X^\varepsilon$” indicates how the stationary natural enhancement of the smooth approximation of $X^\varepsilon$ is defined. “Order” shows the order of (inhomogeneous) Wiener chaos the corresponding component of $X^\varepsilon$ belongs to.
For instance, the “$Y$-component” of a driver has the space regularity $2\theta - 3 - \kappa$. Precisely, it belongs to the Banach space $\mathcal{L}_T^{2\theta - 3 - \kappa}$. The “$Y$-component” of $X^z$, the stationary natural enhancement of the smooth approximation of the stationary O-U-like process, is defined to be $\mathcal{I}[R^k X^z \cdot \nabla X^z]$. As a functional of white noise $\xi$, $\mathcal{I}[R^k X^z \cdot \nabla X^z]_{t,x}$ belongs to the second order (inhomogeneous) Wiener chaos for each $(t,x)$.

### 3.2. Banach space for solutions of paracontrolled QGE.

In this subsection we introduce the Banach space on which the fixed point problem for our paracontrolled QGE is formulated, so that solutions of QGE will belong to this Banach space.

Let $7/4 < \theta \leq 2$. Write

$$\rho := \frac{4\theta - 7}{10^{100\theta}}.$$  

**Definition 3.5.** We set

$$\mathcal{D}_{T}^{\kappa,\kappa'} := \mathcal{L}_T^{q - \kappa} \mathcal{C}_T^{2\theta - 2 - \kappa,1 - \frac{q'}{q}} \times \mathcal{L}_T^{q' - \kappa' + \kappa} \mathcal{C}_T^{\frac{5}{7} - \theta - \kappa,1 - \kappa'},$$

where

$$\langle q, q' \rangle := \begin{cases} (2 - \frac{5}{2\theta}, 1) & \frac{4}{7} < \theta \leq 2, \\ (5 - \frac{8}{\theta} - 2\rho, 7 - \frac{11}{\theta} - 3\rho) & \frac{7}{4} < \theta \leq \frac{11}{7}. \end{cases}$$

Its norm is denoted by $\|(v, w)\|_{\mathcal{D}_{T}^{\kappa,\kappa'}}$ for $(v, w) \in \mathcal{D}_{T}^{\kappa,\kappa'}$.

An element of this Banach space is usually denoted by $(v, w)$. The parameter $\rho$ is defined to be sufficiently small so as to justify computation below. Note that

$$0 < q < 1, \quad 0 < q' \leq 1.$$  

**Remark 3.6.** For the readers’ convenience, we compare the space regularities of the components $(X, V, Y, Z, W, \hat{Z}, \hat{W})$ of a driver and those of $v$ and $w$. If $3/2 < \theta < 2$, it holds that

$$\frac{5}{2} (\theta - 2) < 2(\theta - 2) < \frac{1}{2} (\theta - 2) < 0 < 2\theta - 3 < \frac{3}{2} \theta - 2 < \frac{7}{2} \theta - 5 < 3\theta - 4.$$  

By (3.6), one can easily see that, for every $\theta \in (7/4, 2]$,

$$\frac{3}{2} \theta - 2 - \theta q < \min \left\{ \frac{7}{2} \theta - 5 - \theta q', \frac{1}{2} (\theta - 2) \right\}, \quad q' \geq \frac{5}{\theta} - 3 + 2q.$$  

Here we give a couple of remarks on these function spaces. We now make a comment on $\mathcal{D}_{T}^{\kappa,\kappa'}$.

**Remark 3.7.** Let $7/4 < \theta \leq 2$. Recall that

$$\mathcal{L}_T^{q' - \kappa' + \kappa} \mathcal{C}_T^{\frac{5}{7} - \theta - \kappa'} \subset \mathcal{L}_T^{q' - \theta - \kappa'} \mathcal{C}_T^{\hat{T} - \theta' - \kappa'} \subset \mathcal{L}_T^{q' - \kappa' + \kappa} \mathcal{C}_T^{\hat{T} - \theta - \kappa'} \subset \mathcal{L}_T^{q' - \kappa' + \kappa} \mathcal{C}_T^{\hat{T} - \theta - \kappa'} \subset \mathcal{L}_T^{q' - \kappa' + \kappa} \mathcal{C}_T^{\hat{T} - \theta - \kappa'} \subset \mathcal{L}_T^{q' - \kappa' + \kappa} \mathcal{C}_T^{\hat{T} - \theta - \kappa'}.$$  

**Proposition 3.8.** Let $(u, v) \in \mathcal{D}_{T}^{\kappa,\kappa'}$ with $0 < \kappa < \kappa' \ll 1$, and let $q$ be as above.
(1) \[ ||R^1(v_t - v_s)||_{C^{(\theta-1),\kappa'}} + ||v_t - v_s||_{C^{(\theta-1),\kappa'}} \lesssim \|s^{-q+2\kappa'}(t-s)^{\frac{3}{2} - \frac{3}{3}}\|v||_{L_t^{q-\kappa',\frac{3}{2} \theta - 2 - \kappa', 1 - \frac{3}{3}}}. \] In particular, \[ ||R^1(v_t - v_s)||_{L^\infty(T^2)} + ||v_t - v_s||_{L^\infty(T^2)} \lesssim \|s^{-q+2\kappa'}(t-s)^{\frac{3}{2} - \frac{3}{3}}\|v||_{L_t^{q-\kappa',\frac{3}{2} \theta - 2 - \kappa', 1 - \frac{3}{3}}}. \]

(2) \[ ||R^1(w_t - w_s)||_{C^{\frac{7}{2} \theta - 6}, 0} + ||w_t - w_s||_{C^{\frac{7}{2} \theta - 6}} \lesssim \|s^{-q+2\kappa'-\kappa}(t-s)^{\frac{1}{2} - \kappa}\|v||_{L_t^{q-\kappa,\frac{7}{2} \theta - 5 - \kappa', 1 - \frac{1}{2}}} \] In particular, \[ ||R^1(w_t - w_s)||_{L^\infty(T^2)} + ||w_t - w_s||_{L^\infty(T^2)} \lesssim \|s^{-q+2\kappa'-\kappa}(t-s)^{\frac{1}{2} - \kappa}\|v||_{L_t^{q-\kappa,\frac{7}{2} \theta - 5 - \kappa', 1 - \frac{1}{2}}} \]

**Proof.** We concentrate on (2) since (2) is somewhat more delicate than (1). Since \(\frac{7}{2} \theta - 6 > \frac{1}{3}\), we have only to show the estimate on Hölder-Zygmund spaces. By Lemma 2.10 we have only to handle \(||w_t - w_s||_{C^{\frac{7}{2} \theta - 6}}\). It remains to use Corollary 2.24. \[ \square \]

**Proposition 3.9.** Under the above assumption on \(\kappa\) and \(\kappa'\), for every \(\gamma_1, \gamma_2\) satisfying \[ \frac{3}{2} \theta - 2 - \theta q + (\theta - 1)\kappa' \leq \gamma_1 \leq \frac{3}{2} \theta - 2 - \kappa', \quad \frac{7}{2} \theta - 5 - \theta q' - \theta \kappa \leq \gamma_2 \leq \frac{7}{2} \theta - 5 - \theta \kappa', \]

and for every \((v, w) \in D\kappa^{q, \kappa'}\), we have

(1) \[ ||v_t||_{C^{\gamma_1}} \lesssim t^{-\frac{1}{2}} \left( \gamma_1 - \left(\frac{3}{2} \theta - 2 - \theta q + (\theta - 1)\kappa'\right) \right) ||v||_{L_t^{q,\kappa',\frac{3}{2} \theta - 2 - \kappa', 1 - \frac{3}{3}}} \]

(2) \[ ||w_t||_{C^{\gamma_2}} \lesssim t^{-\frac{1}{3}} \left( \gamma_2 - \left(\frac{3}{2} \theta - 5 - \theta q' - \theta \kappa\right) \right) ||w||_{L_t^{q,\kappa',\frac{3}{2} \theta - 5 - \kappa', 1 - \frac{1}{2}}} \]

**Proof.** Use Proposition 2.23 (3). \[ \square \]

We can assume \(\kappa'\) is small enough to have

(3.7) \[ \frac{3}{2} \theta - 2 - \theta q + (\theta - 1)\kappa' < 0 < \frac{3}{2} \theta - 2 - \kappa', \quad \frac{7}{2} \theta - 5 - \theta q' - \theta \kappa < 0 < \frac{7}{2} \theta - 5 - \theta \kappa', \]

since \(0 < q < 1\).

**Corollary 3.10.** Assume (3.7).

(1) \[ ||v_t||_{C^{(\theta-1),\kappa'-\kappa}} \lesssim t^{\frac{1}{2}} \left( \frac{3}{2} \theta - 2 - \theta q + (\theta - 1)\kappa \right) ||v||_{L_t^{q,\kappa',\frac{3}{2} \theta - 2 - \kappa', 1 - \frac{3}{3}}} \]

\[ ||v_t||_{L^\infty(T^2)} + ||R^1 v_t||_{L^\infty(T^2)} \lesssim t^\frac{1}{2} \left( \frac{3}{2} \theta - 2 - \theta q + (\theta - 1)\kappa \right) ||v||_{L_t^{q,\kappa',\frac{3}{2} \theta - 2 - \kappa', 1 - \frac{3}{3}}} \]

(2) \[ ||w_t||_{C^{(\theta-1),\kappa'-\kappa}} \lesssim t^\frac{1}{3} \left( \frac{3}{2} \theta - 5 - \theta q' + (\theta - 1)\kappa + \kappa' \right) ||w||_{L_t^{q,\kappa'+\kappa,\frac{3}{2} \theta - 5 - \kappa', 1 - \frac{1}{2}}} \]

**Proof.** We concentrate on (1), the proof of (2) being similar. From Proposition 3.9 (1) with \(\gamma_1 = (\theta - 1)(\kappa' - \kappa) > 0\) we deduce the desired estimate. It remains to use Lemma 2.10 to have

\[ ||v_t||_{L^\infty(T^2)} + ||R^1 v_t||_{L^\infty(T^2)} \lesssim ||v_t||_{C^{(\theta-1),\kappa'-\kappa}} \]

\[ \square \]
Corollary 3.11. Let \((v, w) \in D_T^{\kappa, \kappa'}\). Assume (3.7). If
\[
\frac{7}{2} - 5 - \theta q' - \theta \kappa \leq \gamma \leq \frac{3}{2} - 2 - \kappa'
\]
Then we have \(\|v_t + w_t\|_{C^\gamma} + \|R^\perp(v_t + w_t)\|_{C^\gamma} \lesssim t^{\gamma/2}(\gamma - \frac{5}{2} + 2 + \theta q - (\theta - 1)\kappa')\|(v, w)\|_{D_T^{\kappa, \kappa'}}\). In particular, \(\|v_t + w_t\|_{C^{(\kappa' - \kappa)(\theta - 1)}} + \|R^\perp(v_t + w_t)\|_{C^{(\kappa' - \kappa)(\theta - 1)}} \lesssim t^{\frac{7}{2}\gamma/2 - 2 - \theta q - (\theta - 1)\kappa'}\|(v, w)\|_{D_T^{\kappa, \kappa'}}\). 

Proof. From (3.8), we can combine two estimates in Proposition 3.9. In particular, by letting \(\gamma = (\kappa' - \kappa)(\theta - 1)\) we obtain the second estimate. \(\Box\)

We transform Corollary 3.11 into the form in which we use.

Corollary 3.12. Let \((v, w) \in D_T^{\kappa, \kappa'}\). Let \(\Phi = R^\perp(Y + v + w)\). Then one has
\[
\|\Phi_t\|_{L^\infty(T^2)} \lesssim \|\Phi_t\|_{C^{2\kappa - 3 - \kappa}} \lesssim \|Y_t\|_{C^{2\kappa - 3 - \kappa}} + t^{\frac{3}{2} - \frac{2}{3} - \kappa'}\|v, w\|_{D_T^{\kappa, \kappa'}} \quad (t > 0).
\]

Lemma 3.13. Under assumptions on Lemmas 2.10 and 3.3 and Proposition 3.8 we have
\[
\|\Phi_t - \Phi_s\|_{L^\infty(T^2)} \lesssim (t - s)^{\frac{3}{2} - \frac{2}{3} - \kappa'}|Y|_{C^{2\kappa - 3 - \kappa}} + s^{-q + \kappa'}(t - s)^{\frac{3}{2} - \kappa'}\|v, w\|_{D_T^{\kappa, \kappa'}} \quad (t > s)\quad (3.9)
\]

Proof. By the definition of the function spaces and Lemmas 2.10 and 3.3 we have
\[
\|R^\perp(Y_t - Y_s)\|_{L^\infty(T^2)} \lesssim \|Y_t - Y_s\|_{C^{\kappa - \kappa}} \lesssim (t - s)^{\frac{3}{2} - \frac{2}{3} - \kappa'}|Y|_{C^{2\kappa - 3 - \kappa}}.
\]

Meanwhile by the definition of the function spaces, Lemma 2.10 and Proposition 3.8 we have
\[
\|R^\perp(v_t - v_s)\|_{L^\infty(T^2)} \lesssim \|v_t - v_s\|_{C^{(\theta - 1)\kappa}} \lesssim s^{-q + \kappa'}(t - s)^{\frac{3}{2} - \kappa'}|v, w|_{D_T^{\kappa, \kappa'}}
\]
\[
\|R^\perp(w_t - w_s)\|_{L^\infty(T^2)} \lesssim \|w_t - w_s\|_{C^{2\kappa - 3 - \kappa}} \lesssim s^{-q + \kappa' - \kappa}(t - s)^{\frac{3}{2} - \kappa'}|v, w|_{D_T^{\kappa, \kappa'}}
\]

Consequently, (3.9) follows. \(\Box\)

3.3. Integration maps for paracontrolled QGE. Let \(X = (X, V, Y, Z, W, \hat{Z}, \hat{W}) \in X_{T}^{\overline{\theta}}\) and \(v_0 \in C^\alpha\) for some \(\alpha \in \mathbb{R}\). For \((v, w) \in D_T^{\kappa, \kappa'}\) we set
\[
\Phi = R^\perp(Y + v + w)
\]
as before and
\[
\text{com}(v, w)_t = P_t^{\theta/2}v_0 + I[\Phi \otimes \nabla X]_t - \Phi_t \otimes V_t.
\]

Define a mapping \(F\) on \(D_T^{\kappa, \kappa'}\) by
\[
F(v, w) = \Phi \otimes \nabla X
\]
and a mapping \(G\) on \(D_T^{\kappa, \kappa'}\) by
\[
G(v, w) = \Phi \otimes \nabla X + Z + R^\perp w \otimes \nabla X + \Phi \cdot W + \{R^\perp(\Phi \otimes V) - \Phi \otimes R^\perp V\} \otimes \nabla X
\]
\[
+ C(\Phi, R^\perp V, \nabla X) + R^\perp \text{com}(v, w) \otimes \nabla X
\]
\[
+ \hat{Z} + R^\perp X \cdot \nabla w + R^\perp X \cdot \{\nabla \Phi \otimes V\} + R^\perp X(\otimes \otimes)\{\Phi \otimes \nabla V\} + \Phi \cdot \hat{W}
\]
\[
+ C(\Phi, \nabla V, R^\perp X) + R^\perp X \cdot \nabla \text{com}(v, w) + \Phi \cdot \nabla(Y + v + w) + (X + Y + v + w).
\]
Definition 3.15. The precise meanings of the simplified symbols used in the definitions of \( f, g, h \) are as follows: In this remark, \( \mathcal{M} \) is defined by

\[
\mathcal{M} : (v, w) \mapsto (\mathcal{M}^1(v, w), \mathcal{M}^2(v, w)) \quad (0 < t \leq T)
\]

for every initial value \((v_0, w_0)\). We will use the fractional Schauder estimate to show that this is a well-defined map from \( \mathcal{D}^{\kappa,\kappa'} \) to itself and has good property. Note the map \( \mathcal{M} \) depends on the driver \( X \in \mathcal{X}^\kappa \) and the initial value \((v_0, w_0)\). To make this dependency explicit, we will sometimes write \( \mathcal{M}_{X,(v_0,w_0)}(v, w) \) for \( \mathcal{M}(v, w) \).

We interpret QGE equation as a fixed point problem for \( \mathcal{M} : (v, w) \mapsto (\mathcal{M}^1(v, w), \mathcal{M}^2(v, w)) \) as follows:

**Definition 3.15.** For every \((v_0, w_0) \in \mathcal{C}^{2\theta - 2 - \eta q + (\theta - 1)\kappa'} \times \mathcal{C}^{2\theta - 5 - \eta q' - \theta \kappa} \) and \( X \in \mathcal{X}^\kappa \), we consider the system

\[
\begin{cases}
  v_t = \mathcal{M}^1(v, w)_t, \\
  w_t = \mathcal{M}^2(v, w)_t
\end{cases} \quad (0 < t \leq T).
\]

If there exist \( T_* > 0 \) and \((v, w) \in \mathcal{D}^{\kappa,\kappa'}\) satisfying (3.14), then \((v, w)\) is called a local solution to the paracontrolled QGE on \([0, T_*] \) with initial condition \((v_0, w_0)\).
About the regularity of \((v_0, w_0)\), one should note that due to (3.6) the regularity of \(v_0\) is worse than that of the first component \(X\) of a driver \(X\) (if \(0 < \kappa < \kappa'\) are chosen small enough).

### 4. Estimates of Integration Map \(\mathcal{M}\)

In this section we estimate the integration map \(\mathcal{M} = \mathcal{M}_{X, (v_0, w_0)} = (\mathcal{M}^1, \mathcal{M}^2)\) using the various inequalities we proved in the preceding sections.

**Proposition 4.1.** Let \(T \in (0, 1]\) and \(0 < \kappa < \kappa' \ll 1\) be as in (3.1). Then, for every \((v_0, w_0) \in C_{\frac{3}{2}\theta-2-\theta q+(\theta-1)\kappa'}^{\frac{3}{2}\theta-5-\theta q'-\theta \kappa} \times C_{\frac{3}{2}\theta-5-\theta q'-\theta \kappa}^{\frac{3}{2}\theta-2-\theta q+(\theta-1)\kappa'}\) and \(X \in X_T^\kappa\), the map \(\mathcal{M} = \mathcal{M}_{X, (v_0, w_0)} : D_T^{\kappa, \kappa'} \to D_T^{\kappa, \kappa'}\) is well defined. Moreover, there exist positive constants \(K_1, K_2\) and a such that the following estimate holds: For every \((v, w) \in D_T^{\kappa, \kappa'}\),

\[
\| \mathcal{M}(v, w) \|_{D_T^{\kappa, \kappa'}} \leq K_1 \left( \|v_0\|_{C_{\frac{3}{2}\theta-2-\theta q+(\theta-1)\kappa'}^{\frac{3}{2}\theta-5-\theta q'-\theta \kappa}} + \|w_0\|_{C_{\frac{3}{2}\theta-5-\theta q'-\theta \kappa}^{\frac{3}{2}\theta-2-\theta q+(\theta-1)\kappa'}} \right) + K_2 T^a \left(1 + \|v\|_{C_{\frac{3}{2}\theta-2-\theta q+(\theta-1)\kappa'}^{\frac{3}{2}\theta-5-\theta q'-\theta \kappa}} + \|(v, w)\|_{D_T^{\kappa, \kappa'}} + \|(v, w)\|_{D_T^{\kappa, \kappa'}} \right).
\]

Here, \(K_1\) and a depend only on \(\kappa, \kappa'\) and \(K_2\) depends only on \(\kappa, \kappa'\) and \(X\). More precisely, \(K_2\) is given by an at most third-order polynomial in \(\|X\|_{X_T^\kappa}\) for fixed \(\kappa, \kappa'\).

The aim of this section is to prove the above proposition. The proof is immediate from Propositions 4.4 and 4.7 below. In the following subsections (in particular, in Lemma 4.3 and Proposition 4.7), we implicitly assume that \((v_0, w_0) \in C_{\frac{3}{2}\theta-2-\theta q+(\theta-1)\kappa'}^{\frac{3}{2}\theta-5-\theta q'-\theta \kappa} \times C_{\frac{3}{2}\theta-5-\theta q'-\theta \kappa}^{\frac{3}{2}\theta-2-\theta q+(\theta-1)\kappa'}\), \(X \in X_T^\kappa\), \(0 < t \leq T \leq 1\) and \(0 < \kappa < \kappa' \ll 1\) are as in (3.1).

### 4.1. Estimates of \(\mathcal{M}^1\)

First, we estimate \(F\).

Let

\[
\eta := \frac{1}{\theta} \left(\frac{3}{2} \theta - 2 - \theta q + (\theta - 1)\kappa\right).
\]

**Lemma 4.2.** Assume \(2\theta - 3 - \kappa > (\theta - 1)(\kappa' - \kappa)\). Then we have

\[
\| \Phi_t \|_{L^\infty(T^2)} \lesssim \|X\|_{X_T^\kappa} + t^{-\eta} \|(v, w)\|_{D_T^{\kappa, \kappa'}}.
\]

**Proof.** Simply use Corollary 3.12 and \(\|Y_t\|_{C^{2\theta-3-\kappa}} \leq \|X\|_{X_T^\kappa}\).

**Lemma 4.3.** For any \((v, w) \in D_T^{\kappa, \kappa'}\) and \(0 < t \leq T\), we have \(F(v, w)_t \in C_{\frac{3}{2}\theta-2-\kappa}^{\frac{3}{2}\theta-5-\theta q'-\theta \kappa}\) and

\[
\| F(v, w)_t \|_{C_{\frac{3}{2}\theta-2-\kappa}^{\frac{3}{2}\theta-5-\theta q'-\theta \kappa}} \leq K (\|X\|_{X_T^\kappa} + t^{-\eta} \|(v, w)\|_{D_T^{\kappa, \kappa'}}) \|X\|_{X_T^\kappa},
\]

where \(K > 0\) is a constant depending only on \(\kappa\) and \(\kappa'\). In particular, \(F(v, w) \in \mathcal{E}_T^{0} C_{\frac{3}{2}\theta-2-\kappa}^{\frac{3}{2}\theta-2-\kappa}\) with the estimate

\[
\| F(v, w) \|_{C_{\frac{3}{2}\theta-2-\kappa}^{\frac{3}{2}\theta-2-\kappa}} \lesssim 1 + \|(v, w)\|_{D_T^{\kappa, \kappa'}}.
\]

**Proof.** By Proposition 2.5 (1), we obtain

\[
\| F(v, w)_t \|_{C_{\frac{3}{2}\theta-2-\kappa}^{\frac{3}{2}\theta-2-\kappa}} \lesssim \| \Phi_t \|_{L^\infty(T^2)} \| \nabla X_t \|_{C_{\frac{3}{2}\theta-2-\kappa}^{\frac{3}{2}\theta-2-\kappa}}.
\]

Thus it remains to combine Lemmas 3.2 and 4.2.

Next, we estimate \(\mathcal{M}^1(v, w)\) by using the fractional Schauder estimate.
Proposition 4.4. The map $\mathcal{M}^1 : \mathcal{D}^{\kappa,\kappa'}_T \to \mathcal{L}^{q-\kappa', \frac{2q}{\theta} - 2 - \kappa', 1 - \frac{\theta}{\theta'}}_T$ is well defined. Moreover, there exist positive constants $K_1, K_2$ such that the following estimate holds: For every $(v, w) \in \mathcal{D}^{\kappa,\kappa'}_T$,
\[
\| \mathcal{M}^1(v, w) \|_{\mathcal{L}^{q-\kappa', \frac{2q}{\theta} - 2 - \kappa', 1 - \frac{\theta}{\theta'}}_T} \leq K_1 \| v_0 \|_{C^{\frac{2q}{\theta} - 2 - \theta + (\theta - 1)\kappa} \mathcal{L}^{q-\kappa', \frac{2q}{\theta} - 2 - \kappa', 1 - \frac{\theta}{\theta'}}_T} + K_2 T^\theta \frac{1}{(\theta - 1)\kappa'} \left( 1 + \| (v, w) \|_{\mathcal{D}^{\kappa,\kappa'}_T} \right).
\]
Here, $K_1$ depends only on $\kappa, \kappa'$ and $K_2$ depends only on $\kappa, \kappa'$ and $X$. More precisely, $K_2$ is given by an at most second-order polynomial in $\| X \|_{\mathcal{X}_T^1}$ for fixed $\kappa, \kappa'$.

Proof. In Lemma 4.3, we showed $F(v, w) \in \mathcal{E}^\theta_\kappa C^{\frac{2q}{\theta} - 2 - \kappa}$, where $\eta$ is given by (4.1). We refine this estimate. We now invoke Proposition 2.26(2) with
\[
\alpha = \frac{1}{2} \theta - 2 - \kappa \leq \gamma = \frac{3}{2} \theta - 2 - \theta q + (\theta - 1)\kappa' < \beta = \frac{3}{2} \theta - 2 - \kappa', \quad \delta = 1 - \frac{\kappa'}{\theta}.
\]
Since $\alpha - \theta q + \theta - \gamma = \frac{3}{2} \theta - 2 - (\theta - 1)\kappa' - (2 - \theta)\kappa > 0$, we are in the position of using Proposition 2.26(2). Combining Proposition 2.26(2) with Lemma 4.3, we have
\[
\| I[F(v, w)] \|_{\mathcal{L}^{q-\kappa', \frac{2q}{\theta} - 2 - \kappa', 1 - \frac{\theta}{\theta'}}_T} \lesssim T^\theta \left\{ \frac{3}{2} \theta - 2 - (\theta - 1)\kappa' - (2 - \theta)\kappa \right\} \| F(v, w) \|_{\mathcal{L}^{q-\kappa', \frac{2q}{\theta} - 2 - \kappa}_T} \lesssim T^\theta \left\{ \frac{3}{2} \theta - 2 - (\theta - 1)\kappa' - (2 - \theta)\kappa \right\} \left( 1 + \| (v, w) \|_{\mathcal{D}^{\kappa,\kappa'}_T} \right).
\]

It remains to handle $F^{\theta/2} v_0$. We use Proposition 2.26(1) with
\[
\alpha = \frac{3}{2} \theta - 2 - \theta q + (\theta - 1)\kappa', \quad \beta = \frac{3}{2} \theta - 2 - \kappa', \quad \delta = 1 - \frac{\kappa'}{\theta}
\]
to obtain
\[
\| F^{\theta/2} v_0 \|_{\mathcal{L}^{q-\kappa', \frac{2q}{\theta} - 2 - \kappa', 1 - \frac{\theta}{\theta'}}_T} \lesssim \| v_0 \|_{C^{\frac{2q}{\theta} - 2 - \theta + (\theta - 1)\kappa'}},
\]
which completes the proof. \(\square\)

4.2. Estimates of $\mathcal{M}^2$. In this subsection we control $\mathcal{M}^2$. First, we estimate the comutator $\text{com}(v, w)$. From the assumption of the parameters we have
\[
(4.2) \quad -\frac{1}{2} \theta + 2 + 2\kappa' - \kappa > 0,
\]
\[
(4.3) \quad \max \left\{ -\frac{7}{2} q + \frac{6}{\theta}, -2 + q' + \frac{3}{\theta}, -2 + \frac{1}{\theta} + q, -2 + \frac{4}{\theta} + q, -2 + q - \frac{7}{\theta} \right\} < \frac{5}{\theta} - 3 + 2q,
\]
\[
(4.4) \quad \left( \frac{4}{\theta} - 1 + \frac{\kappa + \kappa'}{\theta} \right) - 2 - \frac{3}{\theta} - \frac{\kappa'}{\theta} < 1.
\]

Lemma 4.5. For any $(v, w) \in \mathcal{D}^{\kappa,\kappa'}_T$ and $0 < t \leq T$, we have
\[
\| \text{com}(v, w)_t \|_{\mathcal{L}^{q-\kappa', \frac{2q}{\theta} + 2 + \kappa'}_T} \leq K_1 t^{\frac{\theta}{3} - 3 + 2q - \frac{\theta - 1}{\theta}} \left( 1 + \| v_0 \|_{C^{\frac{2q}{\theta} - 2 - \theta + (\theta - 1)\kappa'}}, + \| (v, w) \|_{\mathcal{D}^{\kappa,\kappa'}_T} \right).
\]

Here, $K_1$ is a positive constants depending only on $\kappa, \kappa'$ and $\| X \|_{\mathcal{X}_T^1}$, which is given by an at most second-order polynomial in $\| X \|_{\mathcal{X}_T^1}$. \(\square\)
Proof. We write
\[ I = P_{t}^{\theta/2}v_0, \quad II = \Phi_t \otimes P_{t}^{\theta/2}V_0, \quad III = \int_{0}^{t}(\Phi_t - \Phi_s) \otimes P_{t-s}^{\theta/2}[\nabla X_s]ds, \quad IV = \int_{0}^{t}[P_{t-s}^{\theta/2}, \Phi_s] \otimes \nabla X_s ds, \]
so that
\[ (4.5) \quad \text{com}(v, w)_t = P_{t}^{\theta/2}v_0 + I[\Phi \otimes \nabla X]_t - \Phi_t \otimes V_t = I - II - III + IV. \]

We estimate I, . . ., IV.

Estimate of I. We have
\[ \|P_{t}^{\theta/2}v_0\|_{C^{-\frac{\theta}{2}+2+\kappa'}} \lesssim t^{-(\theta+\frac{2}{\theta}+q+\frac{\theta}{\theta-1}+\kappa')}\|v_0\|_{C^{\frac{\theta}{2}-\theta+q+(\theta-1)\kappa'}} \]
from Proposition 2.18(1). It remains to use (4.3) to control the negative power in the right-hand side above.

Estimate of II. We use Proposition 2.5(1) and Corollary 3.12 to have
\[
\|\Phi_t \otimes P_{t}^{\theta/2}V_0\|_{C^{-\frac{\theta}{2}+2+\kappa'}} \lesssim \|\Phi_t\|_{L^\infty}\|P_{t}^{\theta/2}V_0\|_{C^{-\frac{\theta}{2}+2+\kappa'}}
\lesssim \{\|Y_t\|_{C^{\theta-3-\kappa}} + t^{\frac{3}{2} - \frac{2}{\theta} - q + \frac{\theta - 1}{\theta-1}\kappa}(v, w)_{D_{\kappa'}^{\kappa'}}\} \|P_{t}^{\theta/2}V_0\|_{C^{-\frac{\theta}{2}+2+\kappa'}}.
\]

We use Proposition 2.18(1) to have
\[
\|\Phi_t \otimes P_{t}^{\theta/2}V_0\|_{C^{-\frac{\theta}{2}+2+\kappa'}} \lesssim \{\|Y_t\|_{C^{\theta-3-\kappa}} + t^{\frac{3}{2} - \frac{2}{\theta} - q + \frac{\theta - 1}{\theta-1}\kappa}(v, w)_{D_{\kappa'}^{\kappa'}}\} t^{2 - \frac{2}{\theta} - \kappa' + \kappa'}\|V_0\|_{C^{\frac{\theta}{2}-2-\kappa'}}.
\]

Since
\[
\frac{3}{2} - \frac{2}{\theta} - q < 0,
\]
we have
\[
\|\Phi_t \otimes P_{t}^{\theta/2}V_0\|_{C^{-\frac{\theta}{2}+2+\kappa'}} \lesssim \{\|X\|_{X_T} + t^{\frac{3}{2} - \frac{2}{\theta} - q + \frac{\theta - 1}{\theta-1}\kappa}(v, w)_{D_{\kappa'}^{\kappa'}}\} t^{2 - \frac{2}{\theta} - \kappa' + \kappa'}\|V_0\|_{C^{\frac{\theta}{2}-2-\kappa'}}
\leq \{\|X\|_{X_T} + ||(v, w)||_{D_{\kappa'}^{\kappa'}}\} t^{\frac{3}{2} - \frac{2}{\theta} - q + \frac{\theta - 1}{\theta-1}\kappa + 2 - \frac{2}{\theta} - \kappa' + \kappa'}\|V_0\|_{C^{\frac{\theta}{2}-2-\kappa'}}
= \{\|X\|_{X_T} + ||(v, w)||_{D_{\kappa'}^{\kappa'}}\} t^{\frac{3}{2} - \frac{2}{\theta} - q + \frac{\theta - 1}{\theta-1}\kappa - \kappa' + \kappa'}\|V_0\|_{C^{\frac{\theta}{2}-2-\kappa'}}.
\]

This term satisfies the desired estimate due to (4.3).

Estimate of III. From Lemma 2.3 and Proposition 2.18(1) we can also see that
\[ (4.6) \quad \|P_{t-s}^{\theta/2}[\nabla X_s]\|_{C^{-\frac{\theta}{2}+2+\kappa'}} \lesssim (t-s)^{-\left(\frac{3}{2} - \frac{2}{\theta} - \kappa'\right)}\|\nabla X_s\|_{C^\frac{\theta}{2}-2-\kappa'} \lesssim (t-s)^{-\left(\frac{3}{2} - \frac{2}{\theta} - \kappa'\right)}\|X\|_{X_T}.
\]

We use Proposition 2.5(1) to have
\[ \|III\|_{C^{-\frac{\theta}{2}+2+\kappa'}} \lesssim \int_{0}^{t}\|\Phi_t - \Phi_s\|_{L^\infty}\|P_{t-s}^{\theta/2}\nabla X_s\|_{C^{-\frac{\theta}{2}+2+\kappa'}} ds \]
keeping in mind that \(-\frac{1}{2}\theta + 2 + \kappa' > 0\).

If we combine (3.10) and (4.6), then we obtain
\[
\|III\|_{C^{-\frac{\theta}{2}+2+\kappa'}} \lesssim \|X\|_{X_T}\int_{0}^{t}\left\{\left(t-s)^{\frac{3}{2} - \frac{2}{\theta} - \kappa'}\|X\|_{X_T} + s^{-\kappa'}(t-s)^{\frac{3}{2} - \kappa'}\|(v, w)\|_{D_{\kappa'}^{\kappa'}} + s^{-\kappa'}(t-s)^{\frac{3}{2} - \kappa'}\|(v, w)\|_{D_{\kappa'}^{\kappa'}}\right\} ds
= \|X\|_{X_T}\int_{0}^{t}\left\{\left(t-s)^{\frac{3}{2} - \frac{2}{\theta} - \kappa' + \kappa'}\|X\|_{X_T} + s^{-\kappa'}(t-s)^{\frac{3}{2} - \kappa'}\|(v, w)\|_{D_{\kappa'}^{\kappa'}}\right\} ds
\]
Since we have (3.5), (4.4) and (4.7), we are in the position of using Proposition 2.25 to have
\[
\left\| \mathbf{X} \right\|_{C^{1,2-\kappa}} \lesssim \left\| \mathbf{X} \right\|^{(1-\theta)2q + \frac{\theta - 1}{\theta} \kappa'} + t^{2-\theta - \frac{\theta (1+\theta) \kappa}{\theta}} \left\| \mathbf{X} \right\| \left\| (v, w) \right\|_{D_{2,\kappa}}. 
\]
Thus, from (4.3) we conclude
\[
\left\| \mathbf{X} \right\|_{C^{1,2-\kappa}} \lesssim \left\| \mathbf{X} \right\|^{(1-\theta)2q + \frac{\theta - 1}{\theta} \kappa'} + t^{2-\theta - \frac{\theta (1+\theta) \kappa}{\theta}} \left\| \mathbf{X} \right\| \left\| (v, w) \right\|_{D_{2,\kappa}}. 
\]

Estimate of IV. We write
\[
\mathbf{I} \mathbf{V}_{Y} = \int_{0}^{t} \left[C_{\theta}^{\theta/2}, R \mathbf{Y}_{s} \right] \nabla \mathbf{X}_{s} ds, \quad \mathbf{I} \mathbf{V}_{v+w} = \int_{0}^{t} \left[C_{\theta}^{\theta/2}, R \mathbf{Y}_{s} \right] \nabla \mathbf{X}_{s} ds \]
We use Proposition 2.19 and \(-3 + \frac{7}{\theta} < 1\) to have
\[
\left\| \mathbf{I} \mathbf{V}_{Y} \right\|_{C^{1,2-\kappa}} \lesssim \int_{0}^{t} (t-s)^{3-\frac{7}{\theta} + \frac{\theta - 1}{\theta} \kappa'} \left\| R \mathbf{Y}_{s} \right\|_{C^{2,3-\kappa}} \left\| \nabla \mathbf{X}_{s} \right\|_{C^{2,2-\kappa}} ds \lesssim \mathbf{X}^{2}_{1}. 
\]
\[
\left\| \mathbf{I} \mathbf{V}_{v+w} \right\|_{C^{1,2-\kappa}} \lesssim \left\| \mathbf{X} \right\|_{C^{1,2-\kappa}} \left\| (v, w) \right\|_{D_{2,\kappa}} \int_{0}^{t} (t-s)^{3-\frac{7}{\theta} + \frac{\theta - 1}{\theta} \kappa'} ds \lesssim t^{2-\theta - \frac{\theta (1+\theta) \kappa}{\theta}} \left\| \mathbf{X} \right\|_{C^{1,2-\kappa}} \left\| (v, w) \right\|_{D_{2,\kappa}}. 
\]
Using (4.3) we obtain the desired estimate. \(\square\)

Now we estimate \(G\). Arithmetic shows
\[
\frac{1}{2} - 2 - \kappa < 0,
\]
\[
-\frac{5}{2} \theta + 5 + \theta \kappa' > 0 \quad \frac{5}{2} \theta - 5 - \kappa', \quad -\frac{5}{2} \theta + 5 + \theta \kappa' + \frac{5}{2} \theta - 5 - \kappa' = (\theta - 1) \kappa' > 0,
\]
\[
\frac{1}{2} + \frac{q - 1}{\theta} + \frac{2 - \theta}{\theta} \kappa' \leq \frac{5}{2} \theta - 3 + 2q - \frac{\theta - 1}{\theta} \kappa',
\]
\[
\frac{2}{\theta} - 1 + q + \kappa \leq \frac{5}{2} \theta - 3 + 2q - \frac{\theta - 1}{\theta} \kappa',
\]
\[
\frac{6}{\theta} - \frac{7}{2} + q + \frac{3 - \theta}{\theta} \kappa' \leq \frac{5}{2} \theta - 3 + 2q - \frac{\theta - 1}{\theta} \kappa',
\]
\[
-\frac{1}{\theta} \left\{ \frac{3}{2} \theta - 2 - \theta q + (\theta - 1) \kappa \right\} \leq \frac{5}{2} \theta - 3 + 2q - \frac{\theta - 1}{\theta} \kappa',
\]
\[
\frac{7}{2} \theta - 5 - \theta q - \theta \kappa \leq 0 \leq -2 \theta + 4 + 2 \kappa' \leq \frac{3}{2} \theta - 2 - \kappa'
\]
if $0 < \kappa < \kappa' \ll 1$. Since $4\theta > 7$ and $0 < \kappa < \kappa' \ll 1$, it holds

\begin{align}
-4 + \frac{7}{\theta} + q' + \frac{\kappa' + \theta\kappa}{\theta} < \frac{5}{\theta} - 3 + 2q - \frac{\theta - 1}{\theta} \kappa',
\end{align}

(4.14)

\begin{align}
(2\theta - 3 - \kappa) + (2\theta - 4 - \kappa) = 4\theta - 7 - 2\kappa > 0.
\end{align}

(4.15)

**Lemma 4.6.** For any $(v, w) \in \mathcal{D}_{T'}^{\kappa, \kappa'}$ and $0 < t \leq T$, we have

\[
\|G(v, w)\|_{L^1_T} \lesssim \kappa \left(1 + \|v_0\|_{C^T_{\theta - 2} + \|v\|_{\mathcal{D}_{T'}^{\kappa, \kappa'}} + \|w\|_{\mathcal{D}_{T'}^{\kappa, \kappa'}} + \|v\|_{\mathcal{D}_{T'}^{\kappa, \kappa'}} + \|w\|_{\mathcal{D}_{T'}^{\kappa, \kappa'}}\right).
\]

Here, $K$ is a positive constant depending only on $\kappa$, $\kappa'$, and $\|X\|_{X_T^1}$ and it is given by a third-order polynomial in $\|X\|_{X_T^1}$. In particular,

\[
\|G(v, w)\|_{\mathcal{D}_{T'}^{\kappa, \kappa'}} \leq K \left(1 + \|v_0\|_{C^T_{\theta - 2} + \|v\|_{\mathcal{D}_{T'}^{\kappa, \kappa'}} + \|w\|_{\mathcal{D}_{T'}^{\kappa, \kappa'}} + \|v\|_{\mathcal{D}_{T'}^{\kappa, \kappa'}} + \|w\|_{\mathcal{D}_{T'}^{\kappa, \kappa'}}\right).
\]

Noteworthy in Lemma 4.6 is the fact that we have

\[
\frac{5}{\theta} - 3 + 2q - \frac{\theta - 1}{\theta} \kappa' < 1.
\]

We set

\begin{align}
T_1 &= \Phi \cdot \nabla (Y + v + w), \quad T_2 = C(\Phi, R^1V, \nabla X) + C(\Phi, \nabla V, R^1X), \quad T_3 = \Phi \otimes \nabla X,
T_4 &= R^1w \otimes \nabla X, \quad T_5 = R^1X \cdot \nabla w, \quad T_6 = \{R^1(\Phi \otimes V) - \Phi \otimes R^1V\} \otimes \nabla X,
T_7 &= R^1X \cdot \{\nabla \Phi \otimes V\}, \quad T_8 = R^1X(\otimes + \otimes)\{\Phi \otimes \nabla V\} + \Phi \cdot W + \Phi \cdot W,
T_9 &= Z + \tilde{Z} + X + Y + v + w, \quad T_{10} = R^1 \text{com}(v, w) \otimes \nabla X + R^1X \cdot \nabla \text{com}(v, w),
\end{align}

so that

\[
G(v, w) = \sum_{j=1}^{10} T_j.
\]

In Step $j$ below, we estimate $T_j$.

**Proof.** [Step 1] First, we calculate $T_1$. Insert the definition of $\Phi$ to obtain 9 terms. Among the 9 terms, the most difficult one is $R^1v \cdot \nabla v$, which determines the indices of our solution spaces. We can use Corollary 2.7(3) in view of (4.13) to have

\[
\|R^1v_t \cdot \nabla v_t\|_{C^T_{\theta - 5} + \kappa'} \lesssim \|R^1v_t\|_{C^T_{\theta - 5} + \kappa'} \|\nabla v_t\|_{C^T_{\theta - 5} + \kappa'}.
\]

As before, from Lemmas 2.3 and 2.10,

\begin{align}
\|R^1v_t \cdot \nabla v_t\|_{C^T_{\theta - 5} + \kappa'} \lesssim \|v_t\|_{C^T_{\theta - 5} + \kappa'} \|\nabla v_t\|_{C^T_{\theta - 5} + \kappa'}.
\end{align}

(4.17)

From Proposition 3.9(1), we deduce

\[
\|v_t\|_{C^T_{\theta - 5} + \kappa'} \leq t^{-\frac{1}{\theta} \left(4\theta + 7\theta q + \kappa'\right)} \|v\|_{L^q_T \mathcal{D}_{T'}^{\theta - 2} + \kappa', 1}, \quad \|v_t\|_{C^T_{\theta - 4} + \kappa'} \leq t^{-\frac{1}{\theta} \left(\theta - 2\theta q - \kappa'\right)} \|v\|_{L^q_T \mathcal{D}_{T'}^{\theta - 2} + \kappa', 1}.\]

Inserting the estimate into (4.17), we have

\begin{align}
\|R^1v_t \cdot \nabla v_t\|_{C^T_{\theta - 5} + \kappa'} \lesssim t^{-\frac{1}{\theta} \left(3\theta - 2\theta q - \frac{\theta - 1}{\theta} \kappa'\right)} \|v\|^{2}_{L^q_T \mathcal{D}_{T'}^{\theta - 2} + \kappa', 1}.\]
\end{align}

(4.18)
Note that $\alpha$ commutator estimate.

From (4.15) we are in the position of using Corollary 2.7(3) to have

$$\text{L}^1_t \cdot \nabla \text{Y}_t \|_{\mathcal{C}^{2\theta - 5 - \kappa'}} \leq \| R^1_t \cdot \nabla \text{Y}_t \|_{\mathcal{C}^{2\theta - 4 - \kappa}}.$$ 

From (4.15) we are in the position of using Corollary 2.7(3) to have

$$\| R^1_t \cdot \nabla \text{Y}_t \|_{\mathcal{C}^{2\theta - 4 - \kappa}} \lesssim \| R^1_t \|_{\mathcal{C}^{2\theta - 4 - \kappa}} \| \nabla \text{Y}_t \|_{\mathcal{C}^{2\theta - 4 - \kappa}}.$$

Note that the condition $\theta > 7/4$ is used here. If we use Lemmas 2.3 and 2.10, then we obtain

$$\| R^1_t \cdot \nabla \text{Y}_t \|_{\mathcal{C}^{2\theta - 5 - \kappa'}} \lesssim \| \text{Y}_t \|_{\mathcal{C}^{2\theta - 3 - \kappa}} \leq \| \text{X}_t \|_{\mathcal{C}^{2\theta - 1 - \kappa'}}.$$

In a similar way, thanks to Proposition 3.9

$$\| R^1_t \cdot \nabla \text{Y}_t \|_{\mathcal{C}^{2\theta - 5 - \kappa'}} \lesssim \| R^1_t \|_{\mathcal{C}^{2\theta - 4 - \kappa'}} \| \nabla \text{Y}_t \|_{\mathcal{C}^{2\theta - 4 - \kappa}}$$

$$\lesssim \| v_t \|_{\mathcal{C}^{2\theta + 5 + \theta \kappa'}} \| \nabla \text{Y}_t \|_{\mathcal{C}^{2\theta - 4 - \kappa}}$$

$$\lesssim \| v_t \|_{\mathcal{C}^{2\theta + 5 + \theta \kappa'}} \| \text{Y}_t \|_{\mathcal{C}^{2\theta - 3 - \kappa}}$$

$$\lesssim \| v_t \|_{\mathcal{C}^{2\theta + 5 + \theta \kappa'}} \| \text{X}_t \|_{\mathcal{C}^{2\theta - 1 - \kappa'}}.$$

In view of (4.14), this estimate is better than (4.18). The term $R^2 \cdot \nabla \text{Y}_t$ admits the same estimate.

We also have

$$\| R^1_t \cdot \nabla v_t \|_{\mathcal{C}^{2\theta - 5 - \kappa'}} \lesssim \| R^1_t \|_{\mathcal{C}^{2\theta - 3 - \kappa}} \| \nabla v_t \|_{\mathcal{C}^{2\theta - 4 - \kappa}}$$

$$\lesssim \| \text{X}_t \|_{\mathcal{C}^{2\theta - 1 - \kappa'}} \| \nabla v_t \|_{\mathcal{C}^{2\theta - 4 - \kappa}}$$

$$\lesssim \| v_t \|_{\mathcal{C}^{2\theta + 5 + \theta \kappa'}} \| \text{X}_t \|_{\mathcal{C}^{2\theta - 1 - \kappa'}}.$$

From (4.10) and (4.11) we obtain the desired estimate for $R^1_t \cdot \nabla v_t$. The term $R^2 \cdot \nabla w_t$ can be done in the same way.

Combining these all, we have the desired estimate for $\Phi \cdot \nabla (Y + v + w)$.

$$\| (T_1)t \|_{\mathcal{C}^{2\theta - 5 - \kappa'}} \leq t^{-(\frac{3}{2} - 2\theta + \frac{4}{2} - \frac{3\theta - 2 - \kappa'}{4})} + \| (v, w) \|_{\mathcal{D}^{\alpha, \kappa'}}.$$

**[Step 2]** We estimate $C(\Phi, R^1 V, \nabla X)$ and $C(\Phi, \nabla V, R^1 X)$ with the help of Proposition 2.9, the commutator estimate.

To this end we set

$$\alpha = -2\theta + 4 + 2\kappa', \quad \beta = \frac{3}{2} - 2 - \kappa, \quad \gamma = \frac{1}{2} - 2 - \kappa.$$

Note that $\alpha \in (0, 1)$, that $\beta + \gamma < 0$ and that $\alpha + \beta + \gamma = 2\kappa' - 2\kappa > 0$. Thus, we deduce

$$\| C(\Phi t, R^1 V_t, \nabla X_t) \|_{\mathcal{C}^{2(\alpha - \kappa')}} \lesssim \| \Phi t \|_{\mathcal{C}^{2\theta + 4 + 2\kappa'}} \| R^1 V_t \|_{\mathcal{C}^{2\theta - 2 - \kappa}} \| \nabla X_t \|_{\mathcal{C}^{2\theta - 2 - \kappa}}.$$
from Proposition 2.9. Using Lemmas 2.3 and 2.10, we obtain
\[
\|C(\Phi_t, \mathcal{R}^1 V_t, \nabla X_t)\|_{C^2(\kappa' - \kappa)} \lesssim \|\Phi_t\|_{C^{2g+2+2\kappa'}} \|V_t\|_{C^2\theta-2-\kappa} \|X_t\|_{C^2\theta-1-\kappa}.
\]
Using Corollary 3.12, we estimate \(C(\Phi, \mathcal{R}^1 V, \nabla X)\) as follows:
\[
\|C(\Phi_t, \mathcal{R}^1 V_t, \nabla X_t)\|_{C^2(\kappa' - \kappa)} \lesssim \{\|X\|_{t_0} t^{-\left(\frac{2}{3} - \frac{3}{2} + q + \frac{3}{2} \theta \kappa'\right)}\} \|X\|^2_{t_0} \lesssim t^{-\left(\frac{2}{3} - \frac{3}{2} + q + \frac{3}{2} \theta \kappa'\right)} \{\|X\|_{t_0} + \|(v, w)\|_{D^\kappa_{\kappa'}}\} \|X\|^2_{t_0}.
\]
As a result, from (4.11) we obtain
\[
\|C(\Phi_t, \mathcal{R}^1 V_t, \nabla X_t)\|_{C^2(\kappa' - \kappa)} \lesssim \{\|X\|_{t_0} t^{-\left(\frac{2}{3} - \frac{3}{2} + q + \frac{3}{2} \theta \kappa'\right)}\} \|X\|^2_{t_0} \enspace \text{(4.20)}
\]
In a similar way, we set
\[
\alpha' = -2\theta + 4 + 2\kappa', \quad \beta' = \frac{3}{2} - 3 - \kappa, \quad \gamma' = \frac{1}{2} - 1 - \kappa.
\]
Note that \(\alpha' + \beta' + \gamma' = 2\kappa' - 2\kappa > 0\) and \(\beta' + \gamma' = 2\theta - 4 - \kappa'.\) Thus using Proposition 2.9 and Lemmas 2.3 and 2.10, we have
\[
\|C(\Phi_t, \nabla V_t, \mathcal{R}^1 X_t)\|_{C^2(\kappa' - \kappa)} \lesssim \{\|X\|_{t_0} t^{-\left(\frac{2}{3} - \frac{3}{2} + q + \frac{3}{2} \theta \kappa'\right)}\} \|X\|^2_{t_0} \enspace \text{(4.21)}
\]
From \(2(\kappa' - \kappa) > \frac{2}{3}(\theta - 2) - \kappa'\), (4.20) and (4.21) we deduce the desired estimate of \(C(\Phi, \mathcal{R}^1 V, \nabla X)\) and \(C(\Phi, \nabla V, \mathcal{R}^1 X)\):
\[
\|(T_2)_t\|_{\frac{2}{3}(\theta - 2) - \kappa - \kappa'} \lesssim t^{-\left(\frac{3}{2} - 3 + q + \frac{3}{2} \theta \kappa'\right)} \{\|X\|_{t_0} + \|(v, w)\|_{D^\kappa_{\kappa'}}\} \|X\|^2_{t_0}.
\]

**Step 3** Since we have (4.7), we can use Proposition 2.5 (2) to have
\[
\|\Phi_t \otimes \nabla X_t\|_{C^2\theta+5-\kappa'} \lesssim \|\Phi_t\|_{C^2\theta+3-\kappa'} \|
\]
By Lemma 2.3
\[
\|\Phi_t \otimes \nabla X_t\|_{C^2\theta+5-\kappa'} \lesssim \|\Phi_t\|_{C^2\theta+3-\kappa'} \|\Phi_t\|_{C^2\theta+1-\kappa'}
\]
From Lemma 4.2
\[
\|\Phi_t \otimes \nabla X_t\|_{C^2\theta+5-\kappa'} \lesssim \{\|X\|_{t_0} t^{-\left(\frac{1}{3} + q + \frac{3}{2} \theta \kappa'\right)}\} \|X\|^2_{t_0} \enspace \text{(4.9)}
\]
From (4.9), we conclude that the estimate of \(\|\Phi_t \otimes \nabla X_t\|_{C^2\theta+5-\kappa'} \) is valid:
\[
\|(T_3)_t\|_{C^2\theta+5-\kappa'} \lesssim \{\|X\|_{t_0} t^{-\left(\frac{3}{2} - 3 + q + \frac{3}{2} \theta \kappa'\right)}\} \|X\|^2_{t_0} \enspace \text{(4.10)}
\]

**Step 4** Since we have (4.11), we can use Proposition 2.5 (2) to have
\[
\|\Phi_t \otimes \nabla X_t\|_{C^2\theta+5-\kappa'} \lesssim \|\Phi_t\|_{C^2\theta+3-\kappa'} \|\Phi_t\|_{C^2\theta+1-\kappa'}
\]
From Lemma 4.2
\[
\|\Phi_t \otimes \nabla X_t\|_{C^2\theta+5-\kappa'} \lesssim \{\|X\|_{t_0} t^{-\left(\frac{1}{3} + q + \frac{3}{2} \theta \kappa'\right)}\} \|X\|^2_{t_0} \enspace \text{(4.9)}
\]
From (4.9), we conclude that the estimate of \(\|\Phi_t \otimes \nabla X_t\|_{C^2\theta+5-\kappa'} \) is valid:
\[
\|(T_3)_t\|_{C^2\theta+5-\kappa'} \lesssim \{\|X\|_{t_0} t^{-\left(\frac{3}{2} - 3 + q + \frac{3}{2} \theta \kappa'\right)}\} \|X\|^2_{t_0} \enspace \text{(4.10)}
\]
[Step 4] From Proposition 2.5(3), a basic property of the resonant, we see that
\[ ||R^+ w_t \otimes \nabla X_t||_{C^{\theta - \kappa}} \lesssim ||R^+ w_t||_{C^{-\frac{1}{2}\theta + 2+\kappa'}} ||\nabla X_t||_{C^{\frac{1}{2}\theta - 2-\kappa}} \lesssim t^{-\frac{1}{2}\theta + q + \frac{1}{8}\kappa'} ||(v, w)||_{D_T^{\alpha,\nu}} ||X||_{L_T^{q}}.\]

Thus, from (4.14),
\[ ||(T_4)_t||_{C^{\frac{1}{2}\theta - 5 - \kappa - \kappa}} \lesssim ||R^+ w_t \otimes \nabla X_t||_{C^{\theta - \kappa}} \lesssim t^{-\frac{1}{2}\kappa - 3 + 2q - \frac{5}{8}\kappa'} ||(v, w)||_{D_T^{\alpha,\nu}} ||X||_{L_T^{q}}.\]

[Step 5] We recall
\[ \eta = -\frac{1}{\theta} \left(\frac{3}{2} \theta - 2 - \theta q + (\theta - 1)\kappa\right) \leq \frac{6}{\theta} - \frac{7}{2} + q - \frac{3}{\theta} \kappa'.\]

See (4.11).

By Corollary 2.7(3), a basic property of the usual product, and Lemma 4.2,
\[ ||\Phi_t \cdot W_t||_{C^{2\theta - 4 - \kappa}} + ||\hat{\Phi}_t \cdot \hat{W}_t||_{C^{2\theta - 4 - \kappa}} \lesssim ||\Phi_t||_{C^{2\theta - 4 + 2\kappa'}} ||W_t||_{C^{2\theta - 4 + 2\kappa'}} \lesssim \{||X||_{L_T^{q}} + t^{-\frac{5}{2} + \theta q + \frac{3}{8} \kappa'} ||(v, w)||_{D_T^{\alpha,\nu}} \} ||X||_{L_T^{q}} \]
\[ \lesssim t^{-\frac{5}{2} + \theta q + \frac{3}{8} \kappa'} \{||X||_{L_T^{q}} + ||(v, w)||_{D_T^{\alpha,\nu}} \} ||X||_{L_T^{q}}.\]

Since \( \frac{1}{2} \theta - 1 - \kappa < 0 < 2\kappa' - 2\kappa \) and \( \frac{3}{2} \theta - 1 - \kappa = \frac{1}{2} \theta + 1 + \kappa' > 0 > \frac{1}{2} \theta - 1 - \kappa \), we are in the position of using Corollary 2.7(3) to have
\[ ||R^+ X_t \cdot \nabla w_t||_{C^{\frac{1}{2}\theta - 1 - \kappa}} \lesssim ||R^+ X_t \cdot \nabla w_t||_{C^{2\theta - 2 - \kappa}} \lesssim ||R^+ X_t||_{C^{-\frac{1}{2}\theta + 1 + \kappa'}} ||\nabla w_t||_{C^{-\frac{1}{2}\theta + 2 + \kappa'}}.\]

As before we use Lemmas 2.3 and 2.10 to obtain
\[ ||R^+ X_t \cdot \nabla w_t||_{C^{\frac{1}{2}\theta - 1 - \kappa}} \lesssim ||X||_{C^{\frac{1}{2}\theta - 1 - \kappa}} ||w||_{C^{-\frac{1}{2}\theta + 1 + \kappa'}}.\]

It follows from Proposition 5.2(2) that
\[ ||R^+ X_t \cdot \nabla w_t||_{C^{\frac{1}{2}\theta - 1 - \kappa}} \lesssim t^{-\frac{1}{2} + \frac{3}{2}\theta - 2 - \frac{1}{8}\kappa'} ||(v, w)||_{D_T^{\alpha,\nu}} ||X||_{L_T^{q}}.\]

Thus, from (4.14),
\[ ||R^+ X_t \cdot \nabla w_t||_{C^{\frac{1}{2}\theta - 5 - \kappa - \kappa}} \lesssim ||R^+ X_t \cdot \nabla w_t||_{C^{\frac{1}{2}\theta - 1 - \kappa}} \lesssim t^{-\frac{1}{2} + \frac{3}{2}\theta - 2q - \frac{1}{8}\kappa'} ||(v, w)||_{D_T^{\alpha,\nu}} ||X||_{L_T^{q}}.\]

In total, we have
\[ ||(T_5)_t||_{C^{\frac{1}{2}\theta - 5 - \kappa - \kappa}} \lesssim t^{-\frac{1}{2} + \frac{3}{2}\theta - 2q - \frac{1}{8}\kappa'} ||(v, w)||_{D_T^{\alpha,\nu}} ||X||_{L_T^{q}}.\]

[Step 6] Since (4.2) is satisfied, we are in the position of using Lemma 2.11. The result is:
\[ ||R^+ (\Phi_t \otimes V_t) - \Phi_t \otimes R^+ V_t||_{C^{-\frac{1}{2}\theta + 2 + 2\kappa' - \kappa}} \lesssim ||\Phi_t||_{C^{-\frac{1}{2}\theta + 2 + 2\kappa'} \otimes V_t||_{C^{\frac{1}{2}\theta - 2 - \kappa}}.\]

If we use Lemma 4.2, then we obtain
\[ ||R^+ (\Phi_t \otimes V_t) - \Phi_t \otimes R^+ V_t||_{C^{-\frac{1}{2}\theta + 2 + 2\kappa' - \kappa}} \lesssim \{||X||_{L_T^{q}} + t^{-\frac{3}{2} + \theta q + \frac{5}{8} \kappa'} ||(v, w)||_{D_T^{\alpha,\nu}} \} ||X||_{L_T^{q}}.\]

Since \( \kappa' > \kappa \), we deduce from Proposition 2.5(3) and Lemma 3.2 that
\[ ||(T_6)_t||_{C^{\frac{1}{2}(\kappa' - \kappa)}} \lesssim ||R^+ (\Phi_t \otimes V_t) - \Phi_t \otimes R^+ V_t||_{C^{-\frac{1}{2}\theta + 2 + 2\kappa' - \kappa}} \lesssim ||X||_{L_T^{q}} ||X||_{L_T^{q}}.\]
Since \( \kappa - \kappa' > \frac{5}{2}(\theta - 2) - \kappa - \kappa' \), we deduce the desired estimate.

**Step 7** By Corollary 2.7(3), a basic property of the usual product
\[
\| R^1 X_t \cdot (\nabla \Phi_t \otimes V_t) \|_{C^{\theta - 1 - \kappa}} \lesssim \| R^1 X_t \|_{C^{\theta - 1 - \kappa}} \| \nabla \Phi_t \otimes V_t \|_{C^{-\frac{1}{2} + \kappa' - \kappa}}.
\]
Since \( \kappa' > \kappa \), this implies
\[
\| R^1 X_t \cdot (\nabla \Phi_t \otimes V_t) \|_{C^{\theta - 1 - \kappa}} \lesssim \| R^1 X_t \|_{C^{\theta - 1 - \kappa}} \| \nabla \Phi_t \otimes V_t \|_{C^{\theta - 1 + \kappa' - \kappa}}.
\]
By Proposition 2.5(2) and Lemma 2.10 we have
\[
\| R^1 X_t \cdot (\nabla \Phi_t \otimes V_t) \|_{C^{\theta - 1 - \kappa}} \lesssim \| X_t \|_{C^{\theta - 1 - \kappa}} \| \nabla \Phi_t \otimes V_t \|_{C^{-2\theta + 3 + 2\kappa' - \kappa}} \lesssim \| X_t \|_{C^{\theta - 1 - \kappa}} \| \nabla \Phi_t \otimes V_t \|_{C^{\theta - 1 + \kappa' - \kappa}}^2.
\]
If we use Lemma 4.2, then we obtain
\[
\| R^1 X_t \cdot (\nabla \Phi_t \otimes V_t) \|_{C^{\theta - 1 - \kappa}} \lesssim \{ \| X \|_{X^1_t} + t^\kappa (\| v \|_{D^\kappa_t') \} \| X \|_{X^1_t}^2.
\]
As a result from (4.11)
\[
\| R^1 X_t \cdot (\nabla \Phi_t \otimes V_t) \|_{C^{\theta - 1 - \kappa}} \lesssim \{ \| X \|_{X^1_t} + t^\kappa (\| v \|_{D^\kappa_t') \} \| X \|_{X^1_t}^2.
\]
Since \( \frac{1}{2} - 1 - \kappa > \frac{1}{2}(\theta - 2) - \kappa - \kappa' \), we deduce the desired estimate:
\[
\| (T_t)_l \|_{C^{\theta - 1 - \kappa}} \lesssim t^\kappa (\| v \|_{D^\kappa_t') \} \| X \|_{X^1_t} + \| (v, w) \|_{D^\kappa_t') \} \| X \|_{X^1_t}^2.
\]

**Step 8** Note that
\[
\frac{1}{2} - 1 - \kappa < 0, \quad \frac{3}{2} - 3 - \kappa < 0.
\]
So we are in the position of using Proposition 2.5(2) to have
\[
\| R^1 X_t (\otimes + \otimes) \{ \Phi_t \otimes \nabla V_t \} \|_{C^{\theta - 4 - (\kappa + \kappa')}} \lesssim \| R^1 X_t \|_{C^{\theta - 1 - \kappa}} \| \Phi_t \otimes \nabla V_t \|_{C^{-3 - \kappa}}.
\]
Proposition 2.5(1) yields
\[
\| R^1 X_t (\otimes + \otimes) \{ \Phi_t \otimes \nabla V_t \} \|_{C^{\theta - 4 - (\kappa + \kappa')}} \lesssim \| R^1 X_t \|_{C^{\theta - 1 - \kappa}} \| \Phi_t \|_{L^\infty} \| \nabla V_t \|_{C^{\theta - 3 - \kappa}}.
\]
From Lemmas 2.3 and 2.10 we obtain
\[
\| R^1 X_t (\otimes + \otimes) \{ \Phi_t \otimes \nabla V_t \} \|_{C^{\theta - 4 - (\kappa + \kappa')}} \lesssim \| X_t \|_{C^{\theta - 1 - \kappa}} \| \Phi_t \|_{L^\infty} \| V_t \|_{C^{\theta - 3 - \kappa}}.
\]
Thus from Lemma 4.2
\[
\| R^1 X_t (\otimes + \otimes) \{ \Phi_t \otimes \nabla V_t \} \|_{C^{\theta - 4 - (\kappa + \kappa')}} \lesssim \{ \| X \|_{X^1_t} + t^\kappa (\| v \|_{D^\kappa_t') \} \| X \|_{X^1_t}^2.
\]
As a result from (4.12),
\[
\|R^\perp X_t (\Theta + \Theta) \{ \Phi_t \otimes \nabla V_t \} \|_{C^{2\theta-4-\kappa',\nu}} \lesssim \| \mathbf{X} \|_{X^\perp_t} + t^{-\left( \frac{3}{\theta} - 3 + 2q - \frac{2q-1}{\sigma} \right)} \| (v, w) \|_{D^{\kappa',\nu}_t} \| \mathbf{X} \|_{X^\perp_t}^2 \\
\lesssim t^{-\left( \frac{3}{\theta} - 3 + 2q - \frac{2q-1}{\sigma} \right)} \{ \| \mathbf{X} \|_{X^\perp_t}^2 + \| (v, w) \|_{D^{\kappa',\nu}_t} \} \| \mathbf{X} \|_{X^\perp_t}^2.
\]
Since \(2\theta - 4 - (\kappa + \kappa') > \frac{5}{2} (\theta - 2) - \kappa - \kappa'\), we deduce the desired estimate:
\[
\| (T_9)_t \|_{C^{2(\theta-2)-\kappa',\nu}} \lesssim t^{-\left( \frac{3}{\theta} - 3 + 2q - \frac{2q-1}{\sigma} \right)} \{ \| \mathbf{X} \|_{X^\perp_t}^2 + \| (v, w) \|_{D^{\kappa',\nu}_t} \} \| \mathbf{X} \|_{X^\perp_t}^2.
\]

**[Step 9]** From Corollary 3.11 with \(\gamma = 0\) we deduce
\[
\| (T_9)_t \|_{C^{2(\theta-2)-\kappa',\nu}} \lesssim \| \mathbf{X} \|_{X^\perp_t}^2 + \| u_t + w_t \|_{L^\infty} \lesssim \| \mathbf{X} \|_{X^\perp_t}^2 + t^{\frac{3}{2} (\frac{3}{\theta} - 2 - \theta q + (\theta - 1)\kappa)} \| (v, w) \|_{D^{\kappa',\nu}_t}.
\]
As a result from (4.12),
\[
\| (T_9)_t \|_{C^{2(\theta-2)-\kappa',\nu}} \lesssim \| (T_9)_t \|_{C^{2(\theta-2)-\kappa}} \lesssim \| \mathbf{X} \|_{X^\perp_t}^2 + t^{-\left( \frac{3}{\theta} - 3 + 2q - \frac{2q-1}{\sigma} \right)} \| (v, w) \|_{D^{\kappa',\nu}_t} \lesssim t^{-\left( \frac{3}{\theta} - 3 + 2q - \frac{2q-1}{\sigma} \right)} \{ \| \mathbf{X} \|_{X^\perp_t}^2 + \| (v, w) \|_{D^{\kappa',\nu}_t} \}.
\]

**[Step 10]** We can estimate the terms involving the commutator by using Lemma 4.5. Indeed,
\[
\| R^\perp \text{com}(v, w)_\epsilon \otimes \nabla X_t \|_{C^{\kappa',\nu}} \lesssim \| R^\perp \text{com}(v, w)_\epsilon \|_{C^{-\frac{3}{2}+\sigma+\kappa',\nu}} \| \nabla X_t \|_{C^{\frac{3}{2}-2-\kappa',\nu}} \\
\lesssim \| \text{com}(v, w)_\epsilon \|_{C^{-\frac{3}{2}+\sigma+\kappa',\nu}} \| \mathbf{X} \|_{X^\perp_t},
\]
\[
\| R^\perp X_t \cdot \nabla \text{com}(v, w)_\epsilon \|_{C^{\kappa',\nu}} \lesssim \| R^\perp X_t \|_{C^{\frac{3}{2}-1-\kappa',\nu}} \| \nabla \text{com}(v, w)_\epsilon \|_{C^{-\frac{3}{2}+\sigma+\nu',\nu}} \\
\lesssim \| \text{com}(v, w)_\epsilon \|_{C^{-\frac{3}{2}+\sigma+\nu',\nu}} \| \mathbf{X} \|_{X^\perp_t}.
\]
We can use Lemma 4.5 to handle \( \| \text{com}(v, w)_\epsilon \|_{C^{-\frac{3}{2}+\sigma+\kappa',\nu}} \). The result is
\[
\| R^\perp \text{com}(v, w)_\epsilon \otimes \nabla X_t \|_{C^{\kappa',\nu}} + \| R^\perp X_t \cdot \nabla \text{com}(v, w)_\epsilon \|_{C^{\kappa',\nu}} \lesssim \| K_t \|^{-\left( \frac{3}{\theta} - 3 + 2q - \frac{2q-1}{\sigma} \right)} (1 + \| v_0 \|_{C^{\frac{3}{2}-2q+\theta q + (\theta - 1)\kappa',\nu}} + \| (v, w) \|_{D^{\kappa',\nu}_t}).
\]
Since \( \min\{\kappa' - \kappa, \frac{3}{2}(\theta - 1) - \kappa\} > \frac{5}{2}(\theta - 2) - \kappa - \kappa' \), we deduce the desired estimate:
\[
\| (T_{10})_t \|_{C^{2(\theta-2)-\kappa',\nu}} \lesssim t^{-\left( \frac{3}{\theta} - 3 + 2q - \frac{2q-1}{\sigma} \right)} (1 + \| v_0 \|_{C^{\frac{3}{2}-2q+\theta q + (\theta - 1)\kappa',\nu}} + \| (v, w) \|_{D^{\kappa',\nu}_t}).
\]
Thus, we have estimated all the terms \( T_1, T_2, \ldots, T_{10} \) in the definition of \( G(v, w) \).

Now we are in the position to estimate \( M_2 \).

**Proposition 4.7.** The map \( \mathcal{M}_2 : D^{\kappa',\kappa'} \to \mathcal{D}^{4-\kappa'+\kappa, \frac{3}{2} \theta - 5 - \theta \kappa', 1-\kappa'} \) is well defined. Moreover, there exist positive constants \( K_1, K_2 \) and a such that the following estimate holds: For every \( (v, w) \in D^{\kappa',\kappa'} \)
\[
\| \mathcal{M}_2(v, w) \|_{\mathcal{D}^{4-\kappa'+\kappa, \frac{3}{2} \theta - 5 - \theta \kappa', 1-\kappa'}} \leq K_1(\| v_0 \|_{C^{\frac{3}{2}-2q+\theta q + (\theta - 1)\kappa',\nu}}) \\
+ K_2 T^a \left( 1 + \| v_0 \|_{C^{\frac{3}{2}-2q+\theta q + (\theta - 1)\kappa',\nu}}^3 + \| (v, w) \|_{D^{\kappa',\nu}_t}^3 + \| (v, w) \|_{D^{\kappa',\nu}_t} \right).
\]
Here, $K_1$ and the constant $a$ depend only on $\kappa, \kappa'$ and $K_2$ depends only on $\kappa, \kappa'$ and $X$. More precisely, $K_2$ is given by an at most third-order polynomial in $\|X\|_{\mathcal{X}^p}$ for fixed $\kappa, \kappa'$.

**Proof.** First, using Proposition 2.26(1) with Proof.

By Lemma 4.6, the estimate of \[\|w_0\|_{\mathcal{E}_T^{\theta-\delta/q-\theta \kappa'}} \leq \|w_0\|_{\mathcal{E}_T^{\theta-\delta/q-\theta \kappa}}.\]

Next, we use Proposition 2.26(2), the Schauder estimate with \[\alpha = \frac{5}{2}(\theta - 2) - (\kappa + \kappa'), \quad \beta = \frac{7}{2}\theta - 5 - \theta \kappa', \quad \gamma = \beta - (q' - \kappa' + \kappa), \quad \delta = 1 - \kappa'.\]

It is straightforward to check that $\alpha \leq \gamma < \alpha - \theta \eta + \theta - \gamma$, $\gamma \leq \beta < \alpha + \theta$ and $0 < \delta \leq (\beta - \alpha)/\theta$ due to the assumption $\kappa/\kappa' < 3/4$. In particular,

\[\alpha - \theta \eta + \theta - \gamma = \begin{cases} (\theta - 2)\kappa' + (\theta - 1)\kappa & \text{if } \frac{11}{7} < \theta \leq 2, \\ (\theta + (\theta - 2)\kappa' + (\theta - 1)\kappa & \text{if } \frac{7}{4} < \theta \leq \frac{11}{4} \end{cases}\]

and hence

\[\alpha - \theta \eta + \theta - \gamma > \frac{-\kappa' + 3\kappa}{4}.\]

The right-hand side is positive since $\kappa/\kappa' > 1/3$ is assumed.

Setting $a := \theta^{-1}(\alpha - \theta \eta + \theta - \gamma) > 0$, we see from Proposition 2.26(2), the fractional version of the Schauder estimate that

\[\|I(G(v, w))\|_{\mathcal{E}_T^{\theta-\delta/q-\theta \kappa'}} \leq T^a \|G(v, w)\|_{\mathcal{E}_T^{\frac{5}{2}\theta - 3 + 2q - \frac{11}{11} \theta \kappa' - (\theta - 2)\kappa'}}.\]

By Lemma 4.6 the estimate of $G(v, w)$ we prove the proposition. \qed

5. Local well-posedness of paracontrolled QGE

At the beginning of this section we prove the local Lipschitz continuity of the integration map $\mathcal{M}$.

Next, we prove the local well-posedness of our paracontrolled QGE, that is, the equation 3.14 admits a unique solution locally in time for every driver and initial condition and the solution depends continuously on these input data. (These parts are a kind of routine once the estimate of $\mathcal{M}$ as in Proposition 4.1 is established. So, some of our proofs may be sketchy in this section.)

In the following proposition the local Lipschitz continuity of the integration map is given. The positive constant $a = a(\kappa, \kappa') > 0$ the same as in Proposition 4.1.

**Proposition 5.1.** Let $T \in (0, 1]$ and $0 < \kappa < \kappa' < 1$ be as in 3.11. For $i = 1, 2$, let $(v_0^{(i)}, w_0^{(i)}) \in \mathcal{C}^{\frac{5}{2}\theta - 2q + (\theta - 1)\kappa'}_{T} \times \mathcal{C}^{\frac{5}{2}\theta - 5 - \theta q' - \theta \kappa}_{T}$ and $X^{(i)} \in \mathcal{X}^{p, \kappa'}_{T}$. Then, there exists a positive constant $K_3$ depending only on $\kappa$ and $\kappa'$ such that the following estimate holds: For $(v^{(i)}, w^{(i)}) \in \mathcal{D}^{p, \kappa'}_{T}$, $i = 1, 2$, it holds that

\[
\|\mathcal{M}_{X^{(1)}}(v_0^{(1)}, w_0^{(1)})(v^{(1)}, w^{(1)}) - \mathcal{M}_{X^{(2)}}(v_0^{(2)}, w_0^{(2)})(v^{(2)}, w^{(2)})\|_{\mathcal{D}^{p, \kappa'}_{T}}
\leq K_3(1 + M^a) \|v_0^{(1)} - v_0^{(2)}\|_{\mathcal{C}^{\frac{5}{2}\theta - 2q + (\theta - 1)\kappa'}} + \|w_0^{(1)} - w_0^{(2)}\|_{\mathcal{C}^{\frac{5}{2}\theta - 5 - \theta q' - \theta \kappa}}^2 + K_3(1 + M^a) T^a \|G^{(1)} - G^{(2)}\|_{\mathcal{D}^{p, \kappa'}_{T}} + \|X^{(1)} - X^{(2)}\|_{\mathcal{X}^{p, \kappa'}_{T}}^2.
\]
Proposition 5.2. Let $0 < \kappa < \kappa' \ll 1$ be as in (3.1). Then, there exists a continuous mapping

$$
\tilde{T}_s : C^2_{\theta-2-\theta q+(\theta-1)\kappa'} \times C^2_{\theta-5-\theta q'-\theta \kappa} \times \mathcal{X}_1^\kappa \to (0, 1]
$$

such that the following (1) and (2) hold:

1. For every $(v_0, w_0) \in C^2_{\theta-2-\theta q+(\theta-1)\kappa'} \times C^2_{\theta-5-\theta q'-\theta \kappa}$ and $X \in \mathcal{X}_1^\kappa$, set $T_s = \tilde{T}_s(v_0, w_0, X)$. Then, the system (3.14) admits a unique solution $(v, w) \in D^\kappa_{T_s}$ and there is a positive constant $K$ depending only on $\kappa, \kappa'$, $T_s$ and $\|X\|_{\mathcal{X}_1^\kappa}$ such that

$$
\|(v, w)\|_{D^\kappa_{T_s}} \leq K \left( 1 + \|v_0\|_{C^{-\frac{3}{2}+\kappa'}} + \|w_0\|_{C^{-\frac{5}{2}-2\kappa}} \right).
$$

2. Let $\{(v^{(n)}_0, w^{(n)}_0)\}_{n=1}^\infty$ and $\{X^{(n)}\}_{n=1}^\infty$ converge to $(v_0, w_0)$ in $C^2_{\theta-2-\theta q+(\theta-1)\kappa'} \times C^2_{\theta-5-\theta q'-\theta \kappa}$ and $X$ in $\mathcal{X}_1^\kappa$, respectively. Set

$$
T_s^{(n)} = \tilde{T}_s(v^{(n)}_0, w^{(n)}_0, X^{(n)})
$$

and let $(v^{(n)}, w^{(n)})$ be a unique solution on $[0, T_s^{(n)}]$ to the system (3.14) with the initial condition $(v^{(n)}_0, w^{(n)}_0)$ driven by $X^{(n)}$. Then, for every $0 < t < T_s$, we have

$$
\lim_{n \to \infty} \|(v^{(n)}, w^{(n)}) - (v, w)\|_{D^\kappa_{T_s}} = 0.
$$

As usual, the local solution admits a prolongation up to the explosion time. Namely, for every $(v_0, w_0)$ as in Proposition 5.2 and $X \in \mathcal{X}_1^\kappa$, $T > 0$, there exists $T_{\text{exp}} \in (0, T]$ such that QGE (3.14) admits a unique solution $(v, w) \in D^\kappa_{T_{\text{exp}}}$ for every $t < T_{\text{exp}}$ and

$$
\limsup_{t \to T_{\text{exp}}} (\|v_t\|_{C^2_{\theta-2-\theta q+(\theta-1)\kappa'}} + \|w_t\|_{C^2_{\theta-5-\theta q'-\theta \kappa}}) = \infty
$$

if $T_{\text{exp}} < T$. Moreover, the function $(v_0, w_0, X) \mapsto T_{\text{exp}}$ is lower semicontinuous.
6. Relation between paracontrolled solution and classical mild solution

In this section we show that the solution of our paracontrolled QGE coincides in a suitable sense with the classical mild solution of QGE when the driver $X$ is a natural enhancement of nice (and deterministic) $\xi$.

Assume that $\xi \in C_T C^\alpha$, $X_0 \in C^\alpha$ and $Y_0, V_0 \in C^{\alpha-1}$ for some $\alpha > 2$. Set

$$X_t = P^{\frac{\alpha}{2}} X_0 + I[\xi]$$

(or equivalently, $\partial_t X_t = -((\Delta)^{\frac{\alpha}{2}} + 1) X_t + \xi$ with the initial value $X_0$). We define its natural enhancement $X = (X, V, Y, Z, W, \hat{Z}, \hat{W})$ as in Example 3.4.

**Proposition 6.1.** Let the notation be as above and assume $(v_0, w_0) \in C^\theta \times C^{-\theta}$. We denote by $(v, w)$ a unique solution of the paracontrolled QGE (3.14). Then, $u := X + Y + v + w$ solves

$$\begin{align*}
\partial_t u_t &= -((\Delta)^{\frac{\alpha}{2}} + 1) u_t + R^+ u_t \cdot \nabla u_t + \xi_t \\
&= -((\Delta)^{\frac{\alpha}{2}} + 1) u_t + R^+ u_t \cdot \nabla u_t + u_t + \xi_t, \quad \text{with } u_0 = X_0 + Y_0 + v_0 + w_0
\end{align*}$$

in the classical mild sense (locally in time), or equivalently,

$$u = P^{\frac{\alpha}{2}} u_0 + I[R^+ u \cdot \nabla u + u + \xi].$$

**Proof.** We use the quantities $T_1, T_2, \ldots, T_{10}$ defined in (4.16). It is sufficient to show that

$$v + w = P^{\frac{\alpha}{2}} (v_0 + w_0) + I[\Phi \cdot \nabla X] + I[R^+ X \cdot \nabla(Y + v + w)] + I[X + Y + v + w]$$

Indeed, adding $X + Y$ to (6.1) and setting $u = X + Y + v + w$, we obtain

$$
\begin{align*}
u &= P^{\frac{\alpha}{2}} (X_0 + Y_0 + v_0 + w_0) + I[\xi] + I[R^+ X \cdot \nabla X] \\
&+ I[R^+ (u - X) \cdot \nabla X] + I[R^+ X \cdot \nabla (u - X)] + [R^+ (u - X) \cdot \nabla(u - X)] + I[u] \\
&= P^{\frac{\alpha}{2}} (X_0 + Y_0 + v_0 + w_0) + I[R^+ u \cdot \nabla u + u + \xi] \\
&= P^{\frac{\alpha}{2}} (u_0) + I[R^+ u \cdot \nabla u + u + \xi],
\end{align*}
$$

which proves the assertion of the proposition.

Now we show (6.1). Before doing so, one should note that since $v$ satisfies the first equation of (3.14), we have $v = \text{com}(v, w) + \Phi \otimes V$ thanks to (3.10) and (3.11).

First, we calculate $\Phi \cdot \nabla X$:

$$\Phi \cdot \nabla X = \Phi(\otimes + \otimes + \otimes) \nabla X$$

$$= \Phi(\otimes) \nabla X + Z + R^+ w \circ \nabla X + R^+ \{\text{com}(v, w) + \Phi \otimes V\} \circ \nabla X$$

$$= \Phi(\otimes) \nabla X + Z + R^+ w \circ \nabla X + R^+ \text{com}(v, w) \circ \nabla X$$

$$+ \{R^+ (\Phi \otimes V) - \Phi \otimes R^+ V\} \circ \nabla X + \{R^+ (\Phi \otimes V) \otimes \nabla X + \Phi \cdot W + C(\Phi, R^+ V, \nabla X)$$

$$= T_3 + Z + T_4 + R^+ \text{com}(v, w) \circ \nabla X$$

$$+ T_6 + \Phi \cdot W + C(\Phi, R^+ V, \nabla X).$$
Next, we calculate $R^\perp X \cdot \nabla(Y + v + w)$:
\[
R^\perp X \cdot \nabla(Y + v + w) = \hat{Z} + R^\perp X \cdot \nabla \{ \text{com}(v, w) + \Phi \otimes V \} + R^\perp X \cdot \nabla w
\]
\[
= \hat{Z} + R^\perp X \cdot \nabla w + R^\perp X \cdot \nabla \text{com}(v, w) + R^\perp X \cdot \{ \nabla \Phi \otimes V \}
\]
\[
+ R^\perp X (\otimes + \otimes) \{ \Phi \otimes \nabla V \}
\]
\[
= \hat{Z} + R^\perp X \cdot \nabla w + R^\perp X \cdot \nabla \text{com}(v, w) + R^\perp X \cdot \{ \nabla \Phi \otimes V \}
\]
\[
+ R^\perp X (\otimes + \otimes) \{ \Phi \otimes \nabla V \} + \Phi \cdot \hat{W} + C(\Phi, \nabla V, R^\perp X)
\]
\[
= \hat{Z} + T_3 + R^\perp X \cdot \nabla \text{com}(v, w) + T_7 + T_8 + \Phi \cdot \hat{W} + C(\Phi, \nabla V, R^\perp X).
\]

Note that we used Lemma 2.8 for the second equality.

Combining these all, we can easily check that the right-hand side of (6.1) is equal to
\[
P^{\theta/2}(v_0 + w_0) + I[F(v, w) + G(v, w)],
\]
where $F$ and $G$ were defined in (3.12). This completes the proof. \hfill \Box

7. Probabilistic Part: Enhancement of white noise

The main purpose of this section is to prove the enhancement of white noise. In this section we do not always assume $\theta \in (7/4, 2]$ and the condition on $\theta$ will be specified in each theorem or each proposition. The time interval is basically $[0, T]$ for arbitrary $T > 0$. If $u: (-\infty, T] \to C^\alpha$ for some $\alpha \in \mathbb{R}$, then as before we write $I[u]_t = \int_{-\infty}^{t} P^{\theta/2} u_s ds, t \in (-\infty, T)$, whenever this makes sense.

Denote by $\xi$ the space-time white noise associated with $L^2(\mathbb{R} \times \mathbb{T}^2)$. Let $\chi: \mathbb{R}^2 \to \mathbb{R}$ be a smooth function with compact support such that $\chi(0) = 1$. See Section 7.1 below. We write
\[
(7.1) \quad \chi^\varepsilon(x) = \chi(\varepsilon x)
\]
for $\varepsilon > 0$ and $x \in \mathbb{R}^2$. Set
\[
(7.2) \quad \xi^\varepsilon(t, x) = \sum_{k \in \mathbb{Z}^2} \chi^\varepsilon(k) \hat{\xi}(k) e_k(x),
\]
where the random variable $\hat{\xi}(k)$ stands for the $k$th Fourier coefficient of $\xi$ (see Remark 7.2 below). Heuristically, $\chi^\varepsilon$ kills high frequencies. This is called a smooth approximation of $\xi$. We study the stationary natural enhancement $X^\varepsilon = (X^\varepsilon, V^\varepsilon, Y^\varepsilon, Z^\varepsilon, W^\varepsilon, \hat{Z}^\varepsilon, W^\varepsilon)$ as in Example 3.3.

**Theorem 7.1.** Let $\theta \in (8/5, 2]$ and $\kappa > 0$. Then, there exists a random driver $X$ such that
\[
\mathbb{E}[\|X\|_{X^\varepsilon_T}^p] < \infty, \quad \lim_{\varepsilon \searrow 0} \mathbb{E}[\|X - X^\varepsilon\|_{X^\varepsilon_T}^p] = 0
\]
for every $1 < p < \infty$. Moreover, $X$ does not depend on the cut-off function $\chi$.

Due to a few cancellations in the enhancement procedure, we do not have to do any renormalization. The proof of Theorem 7.1 is decomposed into Lemmas 7.13, 7.14, 7.15, 7.16, 7.17, 7.20, 7.21, and 7.23 below. The rest of this section is devoted to showing Theorem 7.1.

7.1. Notation and preliminaries. In this subsection we let $\theta \in (0, 2]$ and we suppose the space dimension $d$ is general. One should be careful of the difference between subindices and superindices of $k$ and $x$. 

Let $T^d = (\mathbb{R}/\mathbb{Z})^d$ denote the $d$-dimensional torus. For $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ and $x = (x^1, \ldots, x^d) \in T^d$, we set $k \cdot x = k_1 x^1 + \cdots + k_d x^d$ (modulo $\mathbb{Z}$) and $e_k(x) := \exp(2\pi \sqrt{-1} k \cdot x)$.

We set $\hat{E} := \mathbb{R} \times \mathbb{Z}^d$. An element of $\hat{E}$ is often denoted by $\mu = (\sigma, k)$, where $\sigma \in \mathbb{R}$ and $k \in \mathbb{Z}^d$.

Their norm-like quantities are defined by $|k|_w := 1 + |k|$ and $|\mu|_w = |(\sigma, k)|_w := 1 + |\sigma|^{1/2} + |k|^{1/2}$, where $|k| = \sqrt{(k_1)^2 + \cdots + (k_d)^2}$ as usual. As a measure on $\hat{E}$, the product of the Lebesgue measure and the counting measure is used, which will be denoted by $d\mu = d(\sigma, k)$.

For the partition of unity $\{\rho_j\}_{j \geq -1}$ used in the definition of paraproduct, we write

$$\psi_0(x, x') = \sum_{|j-j'| \leq 1} \rho_j(x) \rho_{j'}(x').$$ (7.3)

Let $\xi$ be a space-time white noise on $E := \mathbb{R} \times T^d$ which is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that is, $\xi : \Omega \to L^2(\mathbb{E}, dtdx)$ is a centered Gaussian random variable whose Cameron–Martin space is $L^2(\mathbb{E}, dtdx)$. (Equivalently, $\xi$ is an isonormal Gaussian process associated with $L^2(\mathbb{E}, dtdx)$.) For every $f \in L^2(\mathbb{E}, dtdx)$, $\langle \xi, f \rangle$ is a real-valued centered Gaussian random variable whose covariance is given by $\mathbb{E}(\langle \xi, f \rangle \langle \xi, g \rangle) = \langle f, g \rangle_{L^2(\mathbb{E})}$.

The orthogonal projection onto the $n$th homogeneous Wiener chaos $\mathcal{Q}_n$ is denoted by $\Pi_n : L^2(\mathbb{E}) \to \mathcal{Q}_n$ ($n \geq 0$).

For example,

$$\Pi_0[F] = \mathbb{E}[F].$$ (7.4)

For $f = f((s_1, x_1), \ldots, (s_n, x_n)) \in L^2(\mathbb{E}^n, (dtdx)^{\otimes n})$, $\mathcal{J}_n(f)$ stands for the $n$th multiple Wiener-Itô integral with respect to $\xi$. In particular, if $n = 1$, then $\mathcal{J}_1(f) = \langle \xi, f \rangle$. As is well known, $\mathcal{J}_n(f) \in \mathcal{Q}_n$ and satisfies

$$\mathbb{E}[|\mathcal{J}_n(f)|^2] \leq \|f\|^2_{L^2(\mathbb{E})}.$$ (7.5)

In an obvious way, $\mathcal{J}_n$ extends to complex-valued functions and (7.5) still holds. (For white noise and multiple Wiener-Itô integrals, see Section 1.1 [18] among others.)

**Remark 7.2.** Here, we give a simple remark on $\hat{\xi}(k), k \in \mathbb{Z}^d$. Its definition is given by $\langle \hat{\xi}(k), h \rangle = \langle \xi, h \otimes e_{-k} \rangle$ for (real-valued) $h \in L^2(\mathbb{R})$. Clearly, $\hat{\xi}(-k) = \hat{\xi}(k)$, $\mathbb{P}$-a.s. Both the real and the imaginary part of $\hat{\xi}(k)$ are a constant multiple of the time white noise. Indeed, $\hat{\xi}(k) = (\eta_1 + \sqrt{-1} \eta_2)/\sqrt{2}$ in law for every $k \neq 0$ and $\hat{\xi}(0) = \eta_1$ in law, where $\eta_1, \eta_2$ are two independent copies of the time white noise associated with $L^2(\mathbb{R})$. Their covariance is given by $\mathbb{E}[\langle \xi(k), h \rangle \langle \xi(l), \hat{h} \rangle] = \delta_{k+l,0}(h, \hat{h})_{L^2(\mathbb{R})}$ for every (real-valued) $h, \hat{h} \in L^2(\mathbb{R})$.

Then, $\hat{\xi}^e = \sum_{k \in \mathbb{Z}^d} \hat{\xi}(k)e_k$ as in (6.2) satisfies $\langle \xi^e, f \rangle = \langle \xi, (\mathcal{F}_{R_d}^{-1})^* f(t, \cdot) \rangle$ for $f \in L^2(\mathbb{R})$, where $\mathcal{F}_{R_d}$ stands for the Fourier transform on $\mathbb{R}^d$ and $*$ for the convolution with respect to the space variable. (This could be used as an alternative definition of $\xi^e$, but the series representation as in (6.2) will be used throughout this paper).

Concerning the covariance, the following formal contraction rule holds:

$$\mathbb{E}[\hat{\xi}_{s_1}(k)\hat{\xi}_{s_2}(l)] = \delta(s_1 - s_2)\delta_{k+l,0}.$$ (7.6)

Though not really mathematically rigorous, (7.6) is used very often in mathematical physic literature. We will also use it in this paper since (7.6) is quite useful and computations using this rule (at least those in the this paper) can easily be made rigorous.

Now we prove a couple of auxiliary estimates for later use.
Lemma 7.3. Let $d \geq 1$ and $\theta \in (0,2]$. Assume that $\alpha, \beta \in (0,2 + \frac{2d}{\theta})$ satisfy $\alpha + \beta > 2 + \frac{2d}{\theta}$. Then,
\[ \int_{\mathbb{E}} |\mu|^{-\alpha} |\nu - \mu|^{-\beta} d\mu \lesssim |\nu|^{-\alpha - \beta + 2 + \frac{2d}{\theta}}, \]
where the implicit constant is independent of $\nu \in \mathbb{E}$.

Proof. We mimic the proof of Lemma 9.8 [9]. For $l \in \mathbb{N}$ with $|\nu| \approx 2^l$, we have
\[ \int_{\mathbb{E}} |\mu|^{-\alpha} |\nu - \mu|^{-\beta} d\mu \lesssim \sum_{i,j \geq 0} 2^{-\alpha i - \beta j} \chi_{[l-l_0,\infty)}(\max(i,j)) \int_{\mathbb{E}} 1_{|\nu| \approx 2^l, |\nu - \mu| \approx 2^j} (\mu) d\mu, \]
where $l_0 \in \mathbb{N}$ is a fixed constant depending only on $\theta$. We use
\[ \int_{\mathbb{E}} 1_{|\nu| \approx 2^l, |\nu - \mu| \approx 2^j} (\mu) d\mu \lesssim \int_{\mathbb{E}} 1_{|\nu| \approx 2^l} (\mu) d\mu \lesssim (2^j)^{2 + \frac{2d}{\theta}} \]
and
\[ \int_{\mathbb{E}} 1_{|\nu| \approx 2^l, |\nu - \mu| \approx 2^j} (\mu) d\mu \leq \int_{\mathbb{E}} 1_{|\nu - \mu| \approx 2^j} (\mu) d\mu = \int_{\mathbb{E}} 1_{|\nu| \approx 2^l} (\mu) d\mu \lesssim (2^j)^{2 + \frac{2d}{\theta}}. \]
Hence, since we are assuming $\alpha, \beta \in (0,2 + \frac{2d}{\theta})$ and $\alpha + \beta > 2 + \frac{2d}{\theta}$, then
\[ \int_{\mathbb{E}} |\mu|^{-\alpha} |\nu - \mu|^{-\beta} d\mu \lesssim \sum_{i,j \geq 0} \chi_{[l-l_0,\infty)}(\max(i,j)) \min(2^i,2^j)^{2 + \frac{2d}{\theta}} 2^{-\alpha i - \beta j} \]
\[ \lesssim (2^j)^{-\alpha - \beta + 2 + \frac{2d}{\theta}} \approx |\nu|^{-\alpha - \beta + 2 + \frac{2d}{\theta}}. \]
Here, the summations are taken over all $(i,j)$ with the described conditions. \hfill \square

Let $\psi_0$ be a function given by (7.3).

Lemma 7.4. (i) Let $\lambda \geq 0$. Then, we have
\[ \psi_0(k,l) \lesssim \frac{|l|^{\lambda}}{|k|^{\lambda}} 1_{|k|, |l|}, \quad (k,l \in \mathbb{R}^d), \]
where the implicit constant is independent of $k,l$.
(ii) Let $\lambda \in [0,1)$ and $k = (k^1, \ldots, k^d), l = (l^1, \ldots, l^d) \in \mathbb{R}^d$. Then,
\[ \left| \frac{\partial}{\partial k^i} \psi_0(k,l) \right| \lesssim |l|^{-1+\lambda} 1_{|k|, |l|}, \quad (k,l \in \mathbb{R}^d), \]
where the implicit constant is independent of $k,l$.

Proof. Modify results in [10] Section 5 or [9] Lemma 5.10. \hfill \square

Proposition 7.5. Let $n \geq 0$ be an integer and let $\Xi$ be a $\mathcal{D}'(\mathbb{T}^d)$-valued random variable defined on a Gaussian probability space whose Fourier coefficient $\hat{\Xi}(k) = (\Xi, e_{-k})$ belongs to the (complex-valued) $n$th inhomogeneous Wiener chaos $\oplus_{i=0}^{n} \mathbb{Q}$, for every $k \in \mathbb{Z}^d$. Then, for every $s \in \mathbb{R}$ and $p \in (1, \infty)$, we have
\[ \mathbb{E}[|\Xi|^p] \lesssim \sum_{j=-1}^{\infty} 2^{j(s p + d)} \left( \sup_{x \in \mathbb{T}^d} \mathbb{E}[|\Delta_j \Xi(x)|^2] \right)^{p/2}. \]
In particular, if the right-hand side of (7.7) is finite, then $\Xi$ belongs to $\mathcal{C}^s$, a.s.
Definition 7.7. A random field $A = \{A_{t,x}\}_{(t,x) \in E}$ on $E = \mathbb{R} \times \mathbb{T}^d$ is said to have a good kernel if each $A_{t,x}$ is written as

$$A_{t,x} = J_n(f_{t,x}).$$

for some $n \geq 1$ and some $f_{t,x} \in L^2(\hat{E}^n, (dt\,dx)^n)$ satisfying the following conditions (7.11)–(7.14):

- Let $Q_0 \in L^2(\hat{E}^n, \mathcal{C})$ be such that

$$Q_0((-\sigma_1, -k_1), \ldots, (-\sigma_n, -k_n)) = \frac{Q_0(\sigma_1, k_1), \ldots, (\sigma_n, k_n))}{\langle (\sigma_i, k_i) \rangle_{i=1}^n} \in \hat{E}^n. \tag{7.11}$$

We define $Q_t \in L^2(\hat{E}^n, \mathcal{C})$ by

$$Q_t((\sigma_1, k_1), \ldots, (\sigma_n, k_n)) = e^{-2\pi \sqrt{-1} \sum_{i=1}^n (\sigma_1 + \cdots + \sigma_n) t}Q_0((\sigma_1, k_1), \ldots, (\sigma_n, k_n)). \tag{7.12}$$

The function $Q_t$ is called the $Q$-function of $A$.

- Define $H_t \in L^2(\hat{E}^n, \mathcal{C})$ so that

$$Q_t((\sigma_1, k_1), \ldots, (\sigma_n, k_n)) = \mathcal{F}_{t \text{im}}H_t((\sigma_1, k_1), \ldots, (\sigma_n, k_n)) \tag{7.13}$$

for every $(k_1, \ldots, k_n) \in (\mathbb{Z}^d)^n$. Here, $\mathcal{F}_{t \text{im}}H_t$ stands for the Fourier transform that acts on the function

$$s \mapsto H_t(s_1, \ldots, s_n) \in \mathbb{C}.$$
for a fixed \((k_1, \ldots, k_n)\). It is called the time-Fourier transform. The function \(H_t\) is called the \(H\)-function of \(A\).

- Define a real-valued function by \(f_{t,x} \in L^2(E^n, (dt dx) \otimes n)\) by

\[
(7.14) \quad f_{t,x}((s_1, y_1), \ldots, (s_n, y_n)) = \sum_{k_1, \ldots, k_n \in \mathbb{Z}^d} \prod_{i=1}^n e_{k_i}(x - y_i) H_t((s_1, k_1), \ldots, (s_n, k_n))
\]

for every \(\{(s_i, y_i)\}_{i=1}^n \in E^n\).

**Remark 7.8.** We make some comments on Definition 7.7 above.

1. Condition (7.11) is equivalent to

\[
H_t((s_1, -k_1), \ldots, (s_n, -k_n)) = H_t((s_1, k_1), \ldots, (s_n, k_n))
\]

for every \(t \in \mathbb{R}\) and \(\{(s_i, k_i)\}_{i=1}^n \in E^n\), which in turn is equivalent to that \(f_{t,x}\) is real-valued.

2. Since \(Q_0 \in L^2(E^n)\), the function

\[
Q_t((\sigma_1, \ldots, \sigma_n) \mapsto Q_t((\sigma_1, k_1), \ldots, (\sigma_n, k_n))
\]

belongs to \(L^2(\sigma_1 \cdots d\sigma_n)\) for every \(t \in \mathbb{R}, (k_1, \ldots, k_n) \in (\mathbb{Z}^d)^n\). Hence, \(J_{\text{time}}\) in (7.13) works perfectly well as an isometry. In particular,

\[
(7.15) \quad \int_{\mathbb{R}^n} ds_1 \cdots ds_n H_t((s_1, k_1), \ldots, (s_n, k_n)) = \int_{\mathbb{R}^n} ds_1 \cdots ds_n |Q_t((\sigma_1, k_1), \ldots, (\sigma_n, k_n))|^2
\]

for every \(t, (k_1, \ldots, k_n)\). If in addition \((\sigma_1, \ldots, \sigma_n) \mapsto H_t((s_1, k_1), \ldots, (s_n, k_n))\) is \(L^1(\mathbb{R}^n)\) for every \(t, k_1, \ldots, k_n\), then \(Q_t = J_{\text{time}} H_t\) is explicitly given by

\[
Q_t((\sigma_1, k_1), \ldots, (\sigma_n, k_n)) = \int_{\mathbb{R}^n} ds_1 \cdots ds_n e^{-2\pi i (\sigma_1 s_1 + \cdots + \sigma_n s_n)} H_t((s_1, k_1), \ldots, (s_n, k_n)).
\]

3. The square-integrability of \(Q_0, Q_t, H_t\) and \(f_{t,x}\) are mutually equivalent. Indeed,

\[
(7.16) \quad \|Q_0\|_{L^2(E^n)} = \|Q_t\|_{L^2(E^n)} = \|H_t\|_{L^2(E^n)} = \|f_{t,x}\|_{L^2(E^n)}, \quad (t, x) \in E.
\]

4. As a (random) function in \(x, A_{t,x}\) can be expressed as a (random) Fourier series as follows:

\[
(7.17) \quad A_{t,x} = \sum_{l \in \mathbb{Z}^d} e_{l}(x) J_{n}(g_{t,l}),
\]

where we set

\[
(7.18) \quad g_{t,l}((s_1, y_1), \ldots, (s_n, y_n)) = \sum_{k_1 + \cdots + k_n = l} \prod_{i=1}^n e_{k_i}(-y_i) H_t((s_1, k_1), \ldots, (s_n, k_n))
\]

for \(\{(s_i, y_i)\}_{i=1}^n \in E^n\). By the Plancherel theorem, \(\sum_{l \in \mathbb{Z}^d} \|g_{t,l}\|_{L^2(E^n)}^2 = \|f_{t,x}\|_{L^2(E^n)}^2\) for all \((t, x) \in E\).

Without a further condition on \(Q_0\), we can hardly prove that \(A = \{A_{t,x}\}_{(t,x) \in E}\) belongs to a suitable Banach space. A typical condition for this purpose is as follows: for almost all \(\mu = (\sigma, k) \in E\)

\[
(7.19) \quad \int_{E^n} |Q_0(\mu_1, \ldots, \mu_{n-1}, \mu - \mu_1 - \cdots - \mu_{n-1})|^2 d\mu_1 \cdots d\mu_{n-1} \leq M^2 |\mu|^2 e^{2\gamma} |k|^{2\delta}.
\]
Here, γ, δ ∈ ℜ and M > 0 are constants (a further condition on them will be imposed later). The integration is actually taken over the “hyperplane” \{ (μ_1, \ldots, μ_n) ∈ ℰ^n | μ_1 + \cdots + μ_n = μ \}. Hence, (7.19) remains the same even if any argument of Q_0 plays a special role instead of the n-th argument. When n = 1, (7.19) simply reads as

(7.20) \[ |Q_0(μ)|^2 ≤ M^2|μ|^2]\gamma|k|^{−2\delta}.

Now we present the key lemma for enhancing the white noise. This can be regarded as a fractional version of the corresponding results in [6, 10]. One should note that the condition γ > 1 is quite important. It will be understood that C^γ C^α = C^α when κ = 0.

**Lemma 7.9.** Let d ≥ 1 and θ ∈ (0, 2). Assume that A = \{A_{t,x}\}_{t,x}∈ ℰ has a good kernel in the sense of Definition 7. Let \[ Assume (7.19) for some γ > 1, δ ∈ ℜ, M > 0.

Then, we have

\[ E\left[ \|A\|_{C^α_{γ}}^{2ρ} \right] ≤ M^{2p}, \quad p ∈ [0, ∞), \quad α < \frac{θ + d}{2} + \frac{γθ}{2} + δ, \quad κ ∈ \left[ 0, \frac{γ - 1}{2} \right] \cap 1 \].

Here, the implicit constant may depend on n, p, α, γ, δ, T, d, θ. In particular, the random field A almost surely belongs to C^α_{γ}.

**Proof.** In [10] it is assumed δ > 0. But, it looks redundant. So we give a proof for the sake of completeness. We may assume that p ≥ 1. Indeed, for the case p = 0 the conclusion is trivial and so we can use the Hölder inequality to have the conclusion for all p > 0.

We see from the assumption and (7.19) that, for any κ ∈ [0, (γ - 1)/2) and j = 0, 1, \ldots,

(7.21) \[ \left\| (∑_{j=1}^{n}\sum_{k_1,...,k_1}ρ_j(k_1+\cdots+k_n)Q_0(μ_1,\ldots,μ_n)) \right\|_{L^2(ℰ^n)} ≤ \sum_{k \in ℤ^d} ρ_j(k)^2 \left( ∑_{l=1}^{n} dσ[σ]^{2κ} \right) \int_{E^{-1}} |Q_0(μ_1,\ldots,μ_n-1,μ-μ_1-\cdots-μ_n)|^2 dμ_1 \cdots dμ_{n-1} \]

\[ ≤ M^2 \sum_{k \in ℤ^d} ρ_j(k)^2 |k|^{−2\delta} \int dσ[(σ,k)]^{−2(γ−2\delta)} \]

\[ ∼ M^2 \sum_{k \in ℤ^d} ρ_j(k)^2 |k|^{−2\delta} k^{θ(1−γ+2\delta)} \sim M^2 (2^j)^{d+θ−γθ+2\delta+2\delta θ}.\]

Now we show E[\|A_{t,x}\|_{C^α_{γ}}^{2ρ}] < ∞ for every fixed t by using (7.21). Note that △_j applies to functions in the x-variable, while \( J_n \) applies to functions in the (s_i, y_i)-variables. Then, we can easily see that

(7.22) \[ \triangle_j A_{t,x} = \int_n \frac{1}{\sum_{k_1,\ldots,k_n} ρ_j \left( ∑_{i=1}^{n} k_i \right) e^{−k_i(y_i)}H_t((s_1,k_1),\ldots,(s_n,k_n))}. \]

Since \( J_n \) does not act on x, we have

\[ \triangle_j A_{t,x} = \sum_{k_1,\ldots,k_n} ρ_j \left( ∑_{i=1}^{n} k_i \right) e^{−k_i(y_i)}J_n \left( ∑_{i=1}^{n} e^{−k_i(y_i)}H_t((s_1,k_1),\ldots,(s_n,k_n)) \right) \]

thanks to equality (7.17). From this equality, (7.15), (7.12) and (7.15) (the L^2-isometric property of \( J_n \)), we have

\[ E[∥△_j A_{t,x}∥^2] ≤ \int_{R^d} ds_1 \cdots ds_n \int_{(T^d)^n} dy_1 \cdots dy_n \]
Since \( \kappa = 0 \), we have

\[
| \frac{d}{dt} E A_t |_{C_0}^{2p} \lesssim M^{2p} \sum_{j=-1}^{\infty} (2^j)^{d+\theta-\gamma\theta-2\delta}.
\]

Since \( \alpha < -\frac{(d + \theta)}{2} + \gamma\theta/2 + \delta \), the right-hand side is finite for sufficiently large \( p \). Thus,

\[
E[|A_t|_{C_0}^{2p}] \lesssim 1.
\]

Next we estimate \( E[|A|^2 \cdot [\sigma_s] \cdot [\sigma_n] \cdot [\sigma_m] \cdot [\sigma_l] \). Going through an argument which is essentially the same way as above, we see that

\[
E[|\Delta_j A_{t,x} - \Delta_j A_{t',x}|^2] \leq \int_{R^n} d\sigma_1 \cdots d\sigma_n \sum_{k_1, \ldots, k_n} \left| \rho_j \left( \frac{n}{k_1, \ldots, k_n} \sum_{i=1}^n k_i \right) \right|^2 |Q_0((\sigma_1, k_1), \ldots, (\sigma_n, k_n))|^2 \times |e^{-2\pi \sqrt{-1}(\sigma_1 + \cdots + \sigma_n)t} - e^{-2\pi \sqrt{-1}(\sigma_1 + \cdots + \sigma_n)t'}|^2 \lesssim |t - t'|^{2\kappa} \int_{R^n} d\sigma_1 \cdots d\sigma_n \sum_{k_1, \ldots, k_n} \left| \sum_{i=1}^n \sigma_i \right|^2 \left| \rho_j \left( \frac{n}{k_1, \ldots, k_n} \sum_{i=1}^n k_i \right) \right|^2 |Q_0((\sigma_1, k_1), \ldots, (\sigma_n, k_n))|^2.
\]

for \( t, t' \in [0, T] \) and \( \kappa \in [0, (\gamma - 1)/2] \cap [0, 1] \). Let

\[
\Theta = 2(\alpha - \theta\kappa)p + d + p(d + \theta - \gamma\theta - 2\delta + 2\kappa\theta).
\]

Combining this with (7.7) and (7.21), we have

(7.23) \[ E[|A_t - A_{t'}|^2 \cdot [\sigma_s] \cdot [\sigma_n] \cdot [\sigma_m] \leq M^{2p}|t - t'|^{2\kappa} \sum_{j=-1}^{\infty} (2^j)^{\Theta} \sim M^{2p}|t - t'|^{2\kappa}. \]

if \( p \) is sufficiently large since \( \alpha < -\frac{(d + \theta)}{2} + \gamma\theta/2 + \delta \). We may also assume that \( 2\kappa p > 1 \) holds. Then, the so-called Besov-Hölder embedding theorem yields

(7.24) \[ E\left[ \left( \frac{\sup_{0 \leq t < t' \leq T} \| A_t - A_{t'} \|_{C_{0}\inf} \right)^{2p} \right] \lesssim \int_{[0,T]^2} \frac{E[|A_t - A_{t'}|^2 \cdot [\sigma_s] \cdot [\sigma_n] \cdot [\sigma_m]}{|t - t'|^{1+2\kappa}} dt dt'. \]

We can find \( \alpha < \alpha' < \alpha'' \) and \( 0 \leq \kappa < \kappa' < \kappa'' \) such that \( (\alpha', \kappa') \) and \( (\alpha'', \kappa'') \) still satisfy

\[
\alpha', \alpha'' < -\frac{\theta + d}{2} + \frac{\gamma\theta}{2} + \delta, \kappa', \kappa'' \in \left[ 0, \frac{\gamma - 1}{2} \right] \cup \left[ 0, \gamma - 1 \right].
\]
and \( \alpha - \theta \kappa = \alpha' - \theta \kappa' = \alpha'' - \theta \kappa'' \). If \( p \) is sufficiently large, we see from (7.24) for \((\alpha', \kappa')\) and (7.23) for \((\alpha'', \kappa'')\) that
\[
\mathbb{E}
\left[
\sup_{0 \leq t < t' \leq T}
\frac{\|A_t - A_{t'}\|_{C^{0, \theta \kappa}}}{|t - t'|^{\alpha}}
\right]^{2p}
\lesssim
\int_{[0,T]^2}
\mathbb{E}[\|A_t - A_{t'}\|_{C^{0, \theta \kappa}}^{2p}]
\frac{|t - t'|^{1+2p\kappa''}}{|t - t'|^{1+2p\kappa'}}
dtdt'
\lesssim
\int_{[0,T]^2}
M^{2p}dtdt'
\lesssim
M^{2p}.
\]
This proves the lemma.

Next we generalize the definition of “random fields with a good kernel” in Definition 7.7. We assumed \( \|Q_0\|_{L^2(E^n)} = \|f_{t,x}\|_{L^2(E^n)} < \infty \) in Definition 7.7 in order to define \( A_{t,x} \) by (7.10). In many cases, however, this condition is too restrictive. For example, think of the Ornstein–Uhlenbeck process \( X = \int [\xi] \), which is the simplest component of our driver \( X \). As we will see in the next subsection, \( Q_0 \) for \( X \) is not \( L^2(E) \). This fact motivates our generalization.

The reason why we can generalize is as follows: The expressions (7.17)–(7.18) suggest that the condition “\( \|f_{t,x}\|_{L^2(E^n)}^2 < \infty \)” may be relaxed:

There exists \( N \in \mathbb{N} \) such that \( \|g_t^{(l)}\|_{L^2(E^n)}^2 = O(|l|^N) \) for every \( l \).

Indeed, if
\[
(7.25)
\|g_t^{(l)}\|_{L^2(E^n)}^2 = O(|l|^N)
\]
for some \( N \) as \( |l| \to \infty \), then from (7.20)
\[
\mathbb{E}[\|\mathcal{J}_n(g_t^{(l)})\|_2^2] = O(|l|^N)
\]
and
\[
(7.26)
A_t := \sum_{l \in \mathbb{Z}^d} e_l(\cdot)\mathcal{J}_n(g_t^{(l)}) \quad \text{in } \mathcal{D}' = \mathcal{D}'(\mathbb{T}^d, \mathbb{R}),
\]
defines a random distribution (i.e., \( \mathcal{D}' \)-valued random variable) for each \( t \). (This part is a general theory.)

**Remark 7.10.** At first glance, (7.17) in the classical case and (7.26) in the generalized case seem identical. However, their meanings are different. (7.17) is for an arbitrary fixed \( x \in \mathbb{T}^d \), while (7.26) is (supposed to be) an equality in \( \mathcal{D}' \) and therefore one can not substituting a fixed \( x \) into (7.26).

**Definition 7.11.** Let a measurable function \( Q_0: E^n \to \mathbb{C} \) with (7.11) be such that
\[
(\sigma_1, \ldots, \sigma_n) \mapsto Q_0((\sigma_1, k_1), \ldots, (\sigma_n, k_n))
\]
belongs to \( L^2(d\sigma_1 \cdots d\sigma_n) \) for every \( k_1, \ldots, k_n \in \mathbb{Z}^d \). Define \( Q_t, H_t, g_t^{(l)} \) by (7.12), (7.13) and (7.18), respectively.

By slightly abusing the terminology, we say that a \( \mathcal{D}' \)-valued stochastic process \( A = \{A_t\}_{t \in \mathbb{R}} \) has a **good kernel** if there exists \( Q_0 \) as above such that the right-hand side of (7.26) makes sense as a random distribution and equals \( A_t \) a.s. for every \( t \).

In the above definition, we do not claim here that every such \( Q_0 \) defines a stochastic process by (7.26). We need an extra condition to ensure it. For instance if we assume that \( Q_0: E^n \to \mathbb{C} \) with (7.11) satisfies (7.19) for some \( \gamma > 1, \delta \in \mathbb{R}, M > 0 \). Then, a calculation quite similar
to (7.21) shows that \( g_t^{(l)} \|_{L^2(E^n)} \leq \|t\|^{-2\theta(\gamma-1)} \) and that \( g_t^{(k_1+\cdots+k_n)} \|_{L^2(E^n)} \) dominates the \( L^2 \)-norm of \((\sigma_1, \ldots, \sigma_n) \mapsto Q_0(\sigma_1, k_1), \ldots, (\sigma_n, k_n)\) for any given \((k_1, \ldots, k_n)\). Therefore, under this assumption the right-hand side of (7.20) makes sense.

Going deeper in this direction, we can naturally generalize our key lemma (Lemma 7.9) as follows:

**Lemma 7.12.** Let \( d \geq 1, \theta \in (0, 2], p \geq 0, \gamma > 1, \delta \in \mathbb{R}, M > 0 \) and \( 0 < \kappa < \frac{\gamma}{d} \). Assume that \( Q_0: \hat{E}^n \to \mathbb{C} \) with (7.11) satisfies condition (7.19). Then, \( A = \{A_t\}_{t \in \mathbb{R}} \) defined by (7.20) has a good kernel in the sense of Definition 7.11. Moreover, all the conclusions of Lemma 7.9 remain true as long as

\[
\alpha < -\frac{\theta + d}{2} + \frac{\gamma\theta}{2} + \delta.
\]

**Proof.** The proof of Lemma 7.9 works without any modification. To understand this, simply observe what is actually computed in the proof of Lemma 7.9 is always \( \Delta_j A_{t,x} \), not \( A_{t,x} \) itself.

Since we have already seen that \( A_t \) is a random distribution, \( \Delta_j A_t \) is well-defined as a finite Fourier series (and hence can be evaluated at every \( x \)):

\[
\Delta_j A_{t,x} = \sum_{l \in \mathbb{Z}^d} \rho_j(l) e_l(x) J_n(g_t^{(l)}) = J_n \left( \sum_{l \in \mathbb{Z}^d} \rho_j(l) e_l(x) g_t^{(l)} \right).
\]

By the definition of \( g_t^{(l)} \), this coincides with (7.22). The rest is exactly the same as in the corresponding parts of the proof of Lemma 7.9, where the isometry of \( J_{\text{time}} \) and condition (7.19) are used, but the assumption \( \|Q_0\|_{L^2(E^n)} < \infty \) is not.

**7.2. Fractional version of Ornstein–Uhlenbeck process.** From now on we work on \( T^2 \). In this subsection we prove the convergence of the first component of the driver \( X^\varepsilon \).

What we call (the fractional version of) the Ornstein–Uhlenbeck process is the solution of the following linearized equation:

\[
\partial_t X = -(-\Delta)^{\theta/2} X - X + \xi.
\]

Its stationary solution should be given by

\[
X_t = T[\xi]_t = \int_{-\infty}^t e^{-\theta(s)\|((-\Delta)^{\theta/2} + 1)(\xi_s)|ds} = \sum_{k \in \mathbb{Z}^2} \mathbf{e}_k(\bullet) \int_{-\infty}^t h(t-s, k) \hat{\xi}_s(k) ds,
\]

where \( h : \hat{E} = \mathbb{R} \times \mathbb{Z}^2 \to \mathbb{R} \) is defined by

\[
h(t, k) := 1_{(0, \infty)}(t) e^{-((2\pi|k|)^{\theta/2} + 1)} t.
\]

At this stage, it is not completely obvious whether \( X \) makes sense. 

So, we first compute the smooth approximation \( X^\varepsilon \). Set, for \( \varepsilon \in (0, 1] \),

\[
X^\varepsilon_{t,x} = \sum_{k \in \mathbb{Z}^2} \chi^\varepsilon(k) e_k(x) \int_{-\infty}^t h(t-s, k) \xi_s(k) ds = J_1(f^\varepsilon_{t,x}),
\]

where

\[
f^\varepsilon_{t,x}(s, y) = \sum_{k \in \mathbb{Z}^2} \chi^\varepsilon(k) e_k(x-y) h(t-s, k).
\]

Thanks to the truncation by \( \chi, f^\varepsilon_{t,x} \) (the integrand in \( J_1 \)) belongs to \( L^2(E) \). Hence, the right-hand side of (7.30) is well defined.
Let \((s, k), (\sigma, k) \in \hat{E}\). Set
\[
\begin{align*}
H^X_t(s, k) &= h(t - s, k), \\
Q^X_t(\sigma, k) &= \frac{e^{-2\pi\sqrt{-1}st}}{-2\pi\sqrt{-1}\sigma + (2\pi|k|)^\theta + 1} = e^{-2\pi\sqrt{-1}st}Q^X_0(\sigma, k), \\
H^{X,\varepsilon}_t(s, k) &= \chi^\varepsilon(k)H^X_t(s, k), \quad Q^{X,\varepsilon}_t(\sigma, k) = \chi^\varepsilon(k)Q^X_t(\sigma, k).
\end{align*}
\]
Obviously, \(Q^X_0\) satisfies (7.20), the assumption of Lemma 7.12 and hence defines a Besov space-valued process with a good kernel, which immediately turns out to be \(X\). Since \(Q^X_0\) is independent of \(\chi\), so is \(X\). Similarly, \(X^\varepsilon\) has a good kernel defined from \(Q^{X,\varepsilon}_0\).

The following lemma will be a model case to many other estimates:

**Lemma 7.13.** Let \(\theta \in (0, 2]\). For every \(\alpha < \frac{\theta}{2} - 1\) and \(1 < p < \infty\), it holds that
\[
E[\|X\|_{C_T^\alpha}^p] < \infty, \quad \lim_{\varepsilon \to 0} E[\|X - X^\varepsilon\|_{C_T^\alpha}^p] = 0.
\]

**Proof.** From (7.32)
\[
|Q^{X,\varepsilon}_0(\sigma, k)|^2 \leq |Q^X_0(\sigma, k)|^2 = \frac{1}{(2\pi\sigma)^2 + \{(2\pi|k|)^\theta + 1\}^2} \lesssim |(\sigma, k)|^{-4},
\]
so that (7.20) holds with \(\gamma = 2\), \(\delta = 0\), so that our assumption \(\alpha < \frac{\theta}{2} - 1\) matches (7.21). Then, we can use Lemma 7.12 to see \(X, X^\varepsilon \in L^p(\Omega; P; C_T^\alpha)\) if \(\alpha < \frac{\theta}{2} - 1\). Next, we prove \(X^\varepsilon\) is Cauchy. Since \(X\) and \(X^\varepsilon\) have a good kernel whose \(Q\)-functions are given by \(Q^X_0\) and \(Q^{X,\varepsilon}_0\), respectively, \(X - X^\varepsilon\) also has a good kernel whose \(Q\)-function is given by \(Q^X_0 - Q^{X,\varepsilon}_0\). For every sufficiently small \(a > 0\), we have
\[
|Q^X_0(\sigma, k) - Q^{X,\varepsilon}_0(\sigma, k)|^2 \lesssim |1 - \chi(\varepsilon|k|)^2|Q^X_0(\sigma, k)|^2
\]
\[
\lesssim \varepsilon^{\frac{\theta}{2}a}|k|^{-\frac{\theta}{2}a} \lesssim \varepsilon^{\alpha a}|(\sigma, k)|_{s(4-a)}^{-}.
\]
Thus, (7.20) is fulfilled. We use Lemma 7.12 with \(\gamma = 2 - a/2 \in (1, \infty)\), \(\delta = 0\) to prove the convergence of \(X - X^\varepsilon\) in \(L^p(\Omega; P; C_T^\alpha)\) as \(\varepsilon \to 0\) if \(\alpha < \frac{\theta}{2} - 1\) since the constant \(a\) is arbitrarily small.

Next, we study \(V = I[\nabla X]\). We will check that \(V^\varepsilon = I[\partial_\varepsilon X^\varepsilon] (i = 1, 2)\) is convergent as \(\varepsilon \to 0\) if the Besov regularity is smaller than \(\frac{\theta}{2} - 2\).

From (7.30) we have
\[
(V^\varepsilon)_t(x) = \sum_{k \in \mathbb{Z}^2} \chi^\varepsilon(k)e_k(x)(2\pi\sqrt{-1}k^i) \int_{-\infty}^{\infty} duh(t - u, k) \int_{-\infty}^{\infty} h(u - s, k)\xi_s(k)ds
\]
and the \(H\)-function and the \(Q\)-function in this case are given by
\[
\begin{align*}
H^V_t(s, k) &= 2\pi\sqrt{-1}k^i \int_{-\infty}^{\infty} duh(t - u, k)h(u - s, k), \\
Q^V_t(\sigma, k) &= \frac{e^{-2\pi\sqrt{-1}st(2\pi\sqrt{-1}k^i)}}{-2\pi\sqrt{-1}\sigma + (2\pi|k|)^\theta + 1} = e^{-2\pi\sqrt{-1}st}Q^V_0(\sigma, k).
\end{align*}
\]
We set
\[
\begin{align*}
H^{V,\varepsilon}_t(s, k) &= \chi^\varepsilon(k)H^V_t(s, k), \quad Q^{V,\varepsilon}_t(\sigma, k) = \chi^\varepsilon(k)Q^V_t(\sigma, k).
\end{align*}
\]
This shows that $V_i^\varepsilon$ has a good kernel, whose $H$-function and $Q$-function are given by $H^{V_i^\varepsilon}_t$ and $Q^{V_i^\varepsilon}_t$, respectively. It will turn out that $Q^{V_i^\varepsilon}_t$ satisfies the assumption of Lemma 7.12. The corresponding process is denoted by $V_i$. Since $Q^{V_i}_t$ is independent of $\chi$, so is $V_i$ ($i = 1, 2$).

**Lemma 7.14.** Let $\theta \in (2/3, 2]$. Then for every $\alpha < \frac{4}{3}\theta - 2$, $1 < p < \infty$ and $i = 1, 2$,

$$E[\|V_i\|_{C^p_T C^\alpha}] < \infty, \quad \lim_{\varepsilon \searrow 0} E[\|V_i - V_i^\varepsilon\|_{C^p_T C^\alpha}] = 0.$$

**Proof.** We set

$$(\gamma, \delta) = \left(4 - \frac{2}{\theta}, 0\right)$$

to apply Lemma 7.12 so that our assumption $\alpha < \frac{3\theta}{2} - 2$ matches (7.27). Note that $4 - 2/\theta > 1$ if and only if $\theta > 2/3$. In this case, we have

$$|Q^V_0^{V_i^\varepsilon}(\sigma, k)|^2 \lesssim |Q^{V_i^\varepsilon}_0(\sigma, k)|^2 \lesssim \frac{|k|^2}{[(2\pi \sigma)^2 + ((2\pi |k|)^2 + 1)^2]^{\gamma/2}} \lesssim |(\sigma, k)|_{s/2}^{2\gamma}.$$

The rest is omitted since it is quite similar to the proof of Lemma 7.13.

### 7.3. Convergence of $Y := \mathcal{I}[R^k X \cdot \nabla X]$.

In this subsection we prove that

$$Y^\varepsilon := \mathcal{I}[R_2 X^\varepsilon \cdot \partial_1 X^\varepsilon] - \mathcal{I}[R_1 X^\varepsilon \cdot \partial_2 X^\varepsilon]$$

is convergent as $\varepsilon \searrow 0$ if the Besov regularity is smaller than $2\theta - 3$.

Firstly, $R_i X^\varepsilon_{i,x}$ has the following expression ($i = 1, 2$):

$$R_i X^\varepsilon_{i,x} = \sum_{k = (k^1, k^2) \in \mathbb{Z}^2} \chi^\varepsilon(k) \frac{2\pi \sqrt{-1} k^i}{2\pi |k|} 1_{k \neq 0} e_k(x) \int_{-\infty}^{\infty} h(t - s, k) \xi_s(k) ds$$

$$= \mathcal{J}_1 \left( \sum_{k \in \mathbb{Z}^2} \chi^\varepsilon(k) \frac{2\pi \sqrt{-1} k^i}{2\pi |k|} 1_{k \neq 0} e_k(x - *) h(t - *, k) \right),$$

whose $H$-function and $Q$-function are as follows: For $(s, k), (\sigma, k) \in \tilde{E}$,

$$H^{R_i X}_t(s, k) = h(t - s, k) \frac{2\pi \sqrt{-1} k^i}{2\pi |k|} 1_{k \neq 0},$$

$$Q^{R_i X}_t(s, k) = e^{-2\pi \sqrt{-1} \sigma t} \frac{2\pi \sqrt{-1} k^i}{-2\pi \sqrt{-1} \sigma + (2\pi |k|)^2 + 1} 1_{k \neq 0}. $$

$$H^{R_i X, \varepsilon}_t(s, k) = \chi^\varepsilon(k) H^{R_i X}_t(s, k), \quad Q^{R_i X, \varepsilon}_t(s, k) = \chi^\varepsilon(k) Q^{R_i X}_t(s, k).$$

Here, $h$ is given by (7.29).

Secondly, $\partial_j X^\varepsilon_{i,x}$ has the following expression ($j = 1, 2$):

$$\partial_j X^\varepsilon_{i,x} = \sum_{l = (l^1, l^2) \in \mathbb{Z}^2} \chi^\varepsilon(l) (2\pi \sqrt{-1} l^j) e_l(x) \int_{-\infty}^{\infty} h(t - s, l) \xi_s(l) ds$$

$$= \mathcal{J}_1 \left( \sum_{l \in \mathbb{Z}^2} \chi^\varepsilon(l) (2\pi \sqrt{-1} l^j) e_l(x - *) h(t - *, l) \right).$$

In this case, the $H$-function and the $Q$-function are as follows: For $(s, k), (\sigma, l) \in \tilde{E}$,

$$H^{\partial_j X}_t(s, l) = h(t - s, l) (2\pi \sqrt{-1} l^j).$$
\[ Q^1_t \chi^X(s, l) = e^{-2\pi \sqrt{-1} \sigma t} 2\pi \sqrt{-1} |l| 2\pi \sqrt{-1} \sigma + (2\pi |l|)^\theta + 1 \]
\[ H^1_t \chi^X(s, l) = \chi^X(l) H^1_t \chi^X(s, l) \]

Hence, \( \mathcal{I}[R_t X^\varepsilon \cdot \partial_j X^\varepsilon]_{t,x} \) has the following expression:
\[
\mathcal{I}[R_t X^\varepsilon \cdot \partial_j X^\varepsilon]_{t,x} = \sum_{k,l \in \mathbb{Z}^2} \chi^X(k) \chi^X(l) \mathbf{e}_{k+1}(x) \int_{-\infty}^{\infty} du_h(t - u, k + l) \]
\[
\quad \times \left( \frac{2\pi \sqrt{-1} k^i}{2\pi |k|} \right) \left( \frac{2\pi \sqrt{-1} l^j}{2\pi |l|} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u - s_1, k) h(u - s_2, l) \hat{\xi}_{s_1}(k) \hat{\xi}_{s_2}(l) ds_1 ds_2.
\]

Since the right-hand side of (7.37) above is a homogeneous polynomial in \( \xi \) of degree two, the following Wiener chaos decomposition holds:
\[
\mathcal{I}[R_t X^\varepsilon \cdot \partial_j X^\varepsilon]_{t,x} = \Pi_2(\mathcal{I}[R_t X^\varepsilon \cdot \partial_j X^\varepsilon]_{t,x}) + \Pi_0(\mathcal{I}[R_t X^\varepsilon \cdot \partial_j X^\varepsilon]_{t,x}).
\]

Now we calculate the second order term on the right-hand side of (7.37). It is easy to see that
\[
\Pi_2(\mathcal{I}[R_t X^\varepsilon \cdot \partial_j X^\varepsilon]_{t,x}) = J_2 \left( f^\varepsilon_{t,x} \right),
\]
where
\[
f^\varepsilon_{t,x}((y_1, s_1), (y_2, s_2)) = \sum_{k,l \in \mathbb{Z}^2} \mathbf{e}_k(x - y_1) \mathbf{e}_l(x - y_2) \chi^X(k) \chi^X(l) H_t((s_1, k), (s_2, l))
\]
with
\[ H_t((s_1, k), (s_2, l)) = \int_{-\infty}^{\infty} du_h(t - u, k + l) \left( \frac{2\pi \sqrt{-1} k^i}{2\pi |k|} \right) \left( \frac{2\pi \sqrt{-1} l^j}{2\pi |l|} \right) h(u - s_1, k) h(u - s_2, l)
\]
for \( (s_1, k), (s_2, l) \in \hat{E} \).

It is straightforward to check that
\[
Q^1_t((\sigma_1, k), (\sigma_2, l)) = e^{-2\pi \sqrt{-1}(\sigma_1 + \sigma_2)t} 2\pi \sqrt{-1} |l| 2\pi \sqrt{-1} \sigma + (2\pi |l|)^\theta + 1
\]
\[ \times \left( \frac{2\pi \sqrt{-1} k^i |k|}{2\pi |k|} \right) \left( \frac{2\pi \sqrt{-1} l^j}{2\pi |l|} \right) \left( \frac{2\pi \sqrt{-1} \sigma}{2\pi |\sigma|} \right) \left( \frac{2\pi \sqrt{-1} \sigma_2}{2\pi |\sigma_2|} \right) \left( \frac{2\pi \sqrt{-1} \sigma_1}{2\pi |\sigma_1|} \right) \left( \frac{2\pi \sqrt{-1} \sigma_2}{2\pi |\sigma_2|} \right).\]

These show that \( \Pi_2(\mathcal{I}[R_t X^\varepsilon \cdot \partial_j X^\varepsilon]) \) has a good kernel given by
\[ Q^1_t((\sigma_1, k), (\sigma_2, l)) = \chi^X(k) \chi^X(l) Q_t((\sigma_1, k), (\sigma_2, l)). \]

It will turn out that \( Q_0 \) above satisfies the assumption of Lemma 7.12 and hence it defines a Besov space-valued process \( \Pi_2(\mathcal{I}[R_t X \cdot \partial_j X]) \).

In fact, the zeroth order term vanishes. By the contraction rule \( \mathbf{E}[\hat{\xi}_{s_1}(k) \hat{\xi}_{s_2}(l)] = \delta(s_1 - s_2) \delta_{k+l,0} \), we see from (7.32), (7.6) and (7.37) that
\[
\Pi_0(\mathcal{I}[R_t X^\varepsilon \cdot \partial_j X^\varepsilon]_{t,x}) = \mathbf{E}[\mathcal{I}[R_t X^\varepsilon \cdot \partial_j X^\varepsilon]_{t,x}]
\]
\[ = \sum_{k \in \mathbb{Z}^2} \chi^X(k)^2 \frac{2\pi \sqrt{-1} k^i}{2\pi |k|} 1_{k \neq 0} \cdot 2\pi \sqrt{-1} (-k^j)
\]
\[ \times \int_{-\infty}^{\infty} du_h(t - u, 0) \int_{-\infty}^{\infty} h(u - s_1, k)^2 ds_1. \]

(7.40)
from which \( \Pi_0(\mathcal{I}[R^2X^t \cdot \partial_t X^t]_{t,x} - \mathcal{I}[R^1X^t \cdot \partial_2 X^t]_{t,x}) = 0 \) immediately follows. Thanks to this cancellation, we do not need any renormalization for \( Y \) given by (7.41).

We give a main result of this subsection. Set

\[
Y = \Pi_2(\mathcal{I}[R^2X \cdot \partial_t X]) - \Pi_2(\mathcal{I}[R_1X \cdot \partial_2 X]),
\]

which is independent of \( \chi \) by definition.

**Lemma 7.15.** Let \( \theta \in (3/2, 2) \). Then, for every \( 0 < \alpha < 2\theta - 3 \) and \( 1 < p < \infty \), it holds that

\[
E[\|Y\|^p_{C^\alpha C^0} + \|Y\|^p_{C^\alpha /2 C^0}] < \infty, \quad \lim_{\varepsilon \searrow 0} E[\|Y - Y^\varepsilon\|^p_{C^\alpha C^0} + \|Y - Y^\varepsilon\|^p_{C^\alpha /p C^0}] = 0.
\]

**Proof.** We prove the lemma for \( \theta \in (3/2, 2) \). The proof for the case \( \theta = 2 \) is just a slight modification.

It is sufficient to estimate \( \Pi_2(\mathcal{I}[R_iX^t \cdot \partial_i X^t]_{t,x}) \) and \( \Pi_2(\mathcal{I}[R_iX \cdot \partial_j X]_{t,x}) \) for \( i \neq j \). Let \( Q_i^\varepsilon \) be the corresponding \( Q \)-function given by (7.38) and (7.39). We use Lemma 7.12 with

\[
(\gamma, \delta) = \left( 5 - \frac{4}{\theta}, 0 \right).
\]

Let \((\sigma_1, k), (\sigma_2, l) \in E\). It is easy to see from (7.38) and Lemma 7.3 that

\[
|Q_i^\varepsilon((\sigma_1, k), (\sigma_2, l))|^2 \lesssim |Q_0((\sigma_1, k), (\sigma_2, l))|^2 \lesssim |(\sigma_1 + \sigma_2, k + l)|^{-4} |\sigma_1|^{-4} |\sigma_2|^{-4} |\varepsilon|^{-(1 - \frac{1}{\theta})}.
\]

Then, if \( \theta \in (3/2, 2) \), we can use Lemma 7.3 to obtain that

\[
\int E |Q_i^\varepsilon((\sigma_1, k), (\tau - \sigma_1, m - k))|^2 d\sigma_1(k) \lesssim |(\tau, m)|^{-2\gamma},
\]

where the implicit constant is independent of \( \varepsilon \). Now we use Lemma 7.12. The rest is similar and is omitted. \( \square \)

**7.4. Convergence of \( \tilde{W} := R^tX \otimes \nabla \mathcal{I}[\nabla X] \).** In this subsection we prove that

\[
\tilde{W}^t = R^tX^t \otimes \partial_t \mathcal{I}[\partial_t X^t] - R^tX^t \otimes \partial_2 \mathcal{I}[\partial_2 X^t] \quad (i = 1, 2)
\]
is convergent as \( \varepsilon \searrow 0 \) if the Besov regularity is smaller than \( 2\theta - 4 \).

It is straightforward to see that \( \partial_t \mathcal{I}[\partial_t X^t] \) has the following expression \((i, j = 1, 2)\):

\[
\partial_t \mathcal{I}[\partial_t X^t]_{t,x} = \sum_{l \in \mathbb{Z}^2} \chi^\varepsilon(l)e_1(x)(2\pi \sqrt{-1})^l \int_{-\infty}^{\infty} duh(t-u, l) \int_{-\infty}^{\infty} H_u^{\partial_t X}(s,l)\dot{\xi}_s(l)ds,
\]

where \( H_u^{\partial_t X} \) is the \( H \)-function for \( \partial_t X \) given by (7.36).

Hence, we can easily see from (7.31) that

\[
[R^tX^t]_{t,x} \otimes \partial_t \mathcal{I}[\partial_t X^t]_{t,x} = \sum_{k,l \in \mathbb{Z}^2} \chi^\varepsilon(k)\chi^\varepsilon(l)\psi_\varepsilon(k,l)e_{k+l}(x) \int_{-\infty}^{\infty} H_t^{R^t X}(s_1,k)\dot{\xi}_{s_1}(l)ds_1
\]

\[
\times (2\pi \sqrt{-1})^l \int_{-\infty}^{\infty} duh(t-u, l) \int_{-\infty}^{\infty} H_u^{\partial_t X}(s_2,l)\dot{\xi}_{s_2}(l)ds_2
\]

\[
= \Pi_2([R^tX^t]_{t,x} \otimes \partial_t \mathcal{I}[\partial_t X^t]_{t,x}) + \Pi_0([R^tX^t]_{t,x} \otimes \partial_t \mathcal{I}[\partial_t X^t]_{t,x}).
\]

We calculate the second order term on the right-hand side of (7.43). It holds that

\[
\Pi_2([R^tX^t]_{t,x} \otimes \partial_t \mathcal{I}[\partial_t X^t]_{t,x}) = \mathcal{J}_2\left( f_{t,x} \right),
\]
where

\[ f_{\ell,x}((s_1, y_1), (s_2, y_2)) = \sum_{k, l \in \mathbb{Z}^2} c_k (x - y_1) c_l (x - y_2) \chi^\varepsilon(k) \chi^\varepsilon(l) H_t((s_1, k), (s_2, l)) \]

with

\[ H_t((s_1, k), (s_2, l)) = \psi_0(k, l) H_{t}^{R_2 X}(s_1, k) (2\pi \sqrt{-1} t^l) \int_{-\infty}^{\infty} du h(t - u, l) H_u^{0, X}(s_2, l). \]

Recall that \( H_{t}^{R_2 X} \) is the \( H \)-function for \( R_2 X \) given by (7.34).

By taking the Fourier transform with respect to the time variables, we have

\[ Q_t((\sigma_1, k), (\sigma_2, l)) = e^{-2\pi \sqrt{-1}(\sigma_1 + \sigma_2) t} \psi_0(k, l) Q_0^{R_2 X}(\sigma_1, k) (2\pi \sqrt{-1} t^l) \]

\[ \times \frac{\sqrt{-1} k^2 k_1 1_{k \neq 0}}{-2\pi \sqrt{-1} \sigma_2 + (2\pi |l|)^\theta + 1} \]

\[ = e^{-2\pi \sqrt{-1}(\sigma_1 + \sigma_2) t} \psi_0(k, l) \frac{\sqrt{-1} k^2 k_1}{-2\pi \sqrt{-1} \sigma_1 + (2\pi |k|)^\theta + 1} \]

\[ \times \frac{2\pi \sqrt{-1} t^l}{-2\pi \sqrt{-1} \sigma_2 + (2\pi |l|)^\theta + 1} \cdot \frac{2\pi \sqrt{-1} t^l}{-2\pi \sqrt{-1} \sigma_2 + (2\pi |l|)^\theta + 1}. \]

This shows that \( \Pi_2(R_2 X^\varepsilon \otimes \partial_t \mathcal{I}[^\varepsilon \partial_t X]) \) has the following good kernel:

\[ Q_t((\sigma_1, k), (\sigma_2, l)) = \chi^\varepsilon(k) \chi^\varepsilon(l) Q_t((\sigma_1, k), (\sigma_2, l)). \]

It will turn out that \( Q_0 \) above satisfies the assumption of Lemma 7.12 and hence it defines a Besov space-valued process \( \Pi_2(R_2 X \otimes \partial_t \mathcal{I}[^\varepsilon \partial_t X]) \). The computation for \( R_1 X^\varepsilon \otimes \partial_2 \mathcal{I}[^\varepsilon \partial_t X] \) is essentially the same and \( \Pi_2(R_1 X \otimes \partial_2 \mathcal{I}[^\varepsilon \partial_t X]) \) is also well defined.

Next we show the zeroth order term vanish. Using the contraction rule (7.6) to (7.43) and the fact that \( \psi_0(k, -k) = \psi_0(k, k) = 1 \),

\[ \Pi_0(\Pi_{R_2 X^\varepsilon}|_{t,x} \otimes \partial_t \mathcal{I}[\partial_2 X^\varepsilon]|_{t,x}) = \sum_{l \in \mathbb{Z}^2} \chi^\varepsilon(l) (2\pi \sqrt{-1} t^l) \]

\[ \times \int_{-\infty}^{\infty} du h(t - u, l) \int_{-\infty}^{\infty} H_t^{R_2 X}(s, -l) H_u^{0, X}(s, l) ds \]

\[ = \sum_{l \in \mathbb{Z}^2} \chi^\varepsilon(l) (2\pi \sqrt{-1} t^l) \sqrt{-1} (-l_1^2) 1_{l \neq 0} (2\pi \sqrt{-1} t^l) \]

\[ \times \int_{-\infty}^{\infty} du h(t - u, l) \int_{-\infty}^{\infty} dsh(t - s, -l) h(u - s, l) \]

\[ = \sum_{l \in \mathbb{Z}^2} \chi^\varepsilon(l) (2\pi \sqrt{-1} t^l) \sqrt{-1} (-l_1^2) 1_{l \neq 0} \frac{2\pi \sqrt{-1} t^l}{4((2\pi |l|)^\theta + 1)^2}. \]

Hence, \( \Pi_0(\Pi_{R_2 X^\varepsilon}|_{t,x} \otimes \partial_t \mathcal{I}[\partial_2 X^\varepsilon]|_{t,x} - [R_1 X^\varepsilon]|_{t,x} \otimes \partial_2 \mathcal{I}[\partial_2 X^\varepsilon]|_{t,x}) = 0 \). Thanks to this cancellation, we do not need any renormalization for \( W \) given by (7.37) below.

Now we set

\[ \dot{W}_i = \Pi_2(R_2 X \otimes \partial_t \mathcal{I}[\partial_i X]) - \Pi_2(R_1 X \otimes \partial_2 \mathcal{I}[\partial_i X]) \]

for \( i = 1, 2 \).
Lemma 7.16. Let $\theta \in (3/2, 2]$. Then, for every $\alpha < 2\theta - 4$, $1 < p < \infty$ and $i = 1, 2$,

$$E[||\hat{W}_i||_{C^\alpha_{t,x}}] < \infty, \quad \lim_{\varepsilon \searrow 0} E[||\hat{W}_i - \hat{W}^\varepsilon_i||_{C^\alpha_{t,x}}] = 0.$$ 

Proof. We prove the lemma for $\theta \in (3/2, 2)$. The proof for the case $\theta = 2$ is just a slight modification.

Let $Q_1$ and $Q_2$ be as in (7.43) and (7.44), respectively. As we have explained, they (should) correspond to $\Pi_2(R_0 X \otimes \partial_t I[\partial_t X])$ and its smooth approximation, respectively. We let

$$\gamma = 5 - \frac{6}{\theta}, \quad \delta = 0$$

to use Lemma 7.12. (Note that $\gamma > 1$ if and only if $\theta > 3/2$.) Since $\psi_\varepsilon(k, l)$ is bounded, we see from Lemma 7.14 that

$$|Q_0^\varepsilon((\sigma_1, k), (\sigma_2, l))|^2 \lesssim |Q_0((\sigma_1, k), (\sigma_2, l))|^2 \lesssim |(\sigma_1, k)|^\alpha |(\sigma_2, l)|^{\frac{3}{2}(1 - \frac{1}{\theta})}$$

and

$$\int_{E} |Q_0((\sigma_1, k), (\tau - \sigma_1, m - k))|^2 d(\sigma_1, k) \lesssim |(\tau, m)|^{2(5 - \frac{6}{\theta})}.$$ 

\[\Box\]

7.5. Convergence of $W := R^\varepsilon I[\nabla X] \otimes \nabla X$. In this subsection we prove that

$$W_1^\varepsilon = R_1 I[\partial_t X^\varepsilon] \otimes \partial_t X^\varepsilon - R_1 I[\partial_t X^\varepsilon] \otimes \partial_2 X^\varepsilon \quad (i = 1, 2).$$ 

is convergent as $\varepsilon \searrow 0$ if the Besov regularity is smaller than $2\theta - 4$. For brevity we write $[R_1 I[\partial_t X^\varepsilon] \otimes \partial_1 X^\varepsilon]_{t,x}$ for $[R_1 I[\partial_t X^\varepsilon]]_{t,x} \otimes [\partial_1 X^\varepsilon]_{t,x}$.

It is straightforward to see that $R_1 I[\partial_t X^\varepsilon]$ has the following expression $(i, j = 1, 2)$:

$$R_1 I[\partial_t X^\varepsilon]_{t,x} = \sum_{k \in \mathbb{Z}^2} \chi^\varepsilon(k) e_k(x) \frac{2\pi \sqrt{-1}k^j}{2\pi |k|} 1_{k \neq 0}$$

$$\times \int_{-\infty}^\infty duh(t - u, k) \int_{-\infty}^\infty H_u^{\partial_t X} (s, k) \hat{\xi}_s(k) ds,$$

where $H^{\partial_t X}$ is the $H$-function for $\partial_t X$ explicitly given by (7.36).

Hence, we can easily see from (7.36) that

$$[R_2 I[\partial_t X^\varepsilon] \otimes \partial_1 X^\varepsilon]_{t,x}$$

$$= \sum_{k, l \in \mathbb{Z}^2} \chi^\varepsilon(k) \chi^\varepsilon(l) \psi_\varepsilon(k, l) e_{k+l}(x)$$

$$\times \frac{2\pi \sqrt{-1}k^j}{2\pi |k|} 1_{k \neq 0} \int_{-\infty}^\infty duh(t - u, k) \int_{-\infty}^\infty H_u^{\partial_t X} (s, k) \hat{\xi}_{s_1}(k) ds_1$$

$$\times \int_{-\infty}^\infty H_u^{\partial_1 X} (s_2, l) \hat{\xi}_{s_2}(l) ds_2$$

$$= \Pi_2([R_2 I[\partial_t X^\varepsilon] \otimes \partial_1 X^\varepsilon]_{t,x}) + \Pi_0([R_2 I[\partial_t X^\varepsilon] \otimes \partial_1 X^\varepsilon]_{t,x}).$$

Let us calculate the second order term on the right-hand side of (7.49). It holds that

$$\Pi_2([R_2 I[\partial_t X^\varepsilon] \otimes \partial_1 X^\varepsilon]_{t,x}) = \mathcal{J}_2(f_2^\varepsilon_{(t,x)}),$$

where

$$f_2^\varepsilon_{(t,x)}((s_1, y_1), (s_2, y_2)) = \sum_{k, l \in \mathbb{Z}^2} e_k(x - y_1) e_l(x - y_2) \chi^\varepsilon(k) \chi^\varepsilon(l) H_t((s_1, k), (s_2, l))$$
Proof. We let \( H_t((s_1, k), (s_2, l)) = \psi_0(k, l) \frac{2\pi \sqrt{-1} k^2}{2\pi |k|} 1_{k \neq 0} \int_{-\infty}^{\infty} \text{d}u h(t - u, k) H_u^{\partial Y_{\theta}}(s_1, k) \cdot H_u^{\partial Y_{\theta}}(s_2, l). \)

By taking the Fourier transform with respect to the time variables, we have

\[
(7.50) \quad Q_t((\sigma_1, k), (\sigma_2, l)) = e^{-2\pi \sqrt{-1} (\sigma_1 + \sigma_2) t} \psi_0(k, l) \frac{2\pi \sqrt{-1} k^2}{2\pi |k|} 1_{k \neq 0} \\
\times Q_0^{\partial X}(\sigma_1, k) - 2\pi \sqrt{-1} \sigma_1 + (2\pi |k|)^\alpha + 1 \cdot \frac{Q_0^{\partial X}(\sigma_2, l)}{2\pi |k|} 1_{k \neq 0} \\
\times \frac{2\pi \sqrt{-1} k^4}{\{ -2\pi \sqrt{-1} \sigma_1 + (2\pi |k|)^\alpha + 1 \}^2} e^{-2\pi \sqrt{-1} l^t}.
\]

Therefore, \( \Pi_2(R_2 \mathcal{I}[\partial X^\varepsilon] \cap \partial_t X^\varepsilon) \) has the following good kernel:

\[
(7.51) \quad Q_t^\varepsilon((\sigma_1, k), (\sigma_2, l)) = \chi^\varepsilon(k) \chi^\varepsilon(l) Q_t((\sigma_1, k), (\sigma_2, l)).
\]

It will turn out that \( Q_0^\varepsilon \) above satisfies the assumption of Lemma 7.12 and hence it defines a Besov space-valued process \( \Pi_2(R_2 \mathcal{I}[\partial_t X^\varepsilon] \cap \partial_t X^\varepsilon) \). The computation for \( R_1 \mathcal{I}[\partial X^\varepsilon] \cap \partial_2 X^\varepsilon \) is essentially the same and \( \Pi_2(R_1 \mathcal{I}[\partial_t X] \cap \partial_2 X) \) is also well defined.

Next we show the zeroth order term on the right-hand side of (7.49) vanish. Using \( \psi_0(k, -k) = \psi_0(k, k) = 1 \) and applying the contraction rule \( \text{E}[\tilde{\xi}_{s_1}(k) \tilde{\xi}_{s_2}(l)] = \delta(s_1 - s_2) \delta_{k+l,0} \) to (7.49),

\[
\Pi_0([R_2 \mathcal{I}[\partial_t X^\varepsilon] \cap \partial_t X^\varepsilon]_{t,x}) = \sum_{k \in \mathbb{Z}^2} \chi^\varepsilon(k) \frac{2\pi \sqrt{-1} k^2}{2\pi |k|} 1_{k \neq 0} \\
\times \int_{-\infty}^{\infty} \text{d}u h(t - u, k) \int_{-\infty}^{\infty} H_t^{\partial Y_{\theta}}(s, k) H_u^{\partial Y_{\theta}}(s, -k) \text{d}s \\
= \sum_{k \in \mathbb{Z}^2} \chi^\varepsilon(k) \frac{2\pi \sqrt{-1} k^2}{|k|} 1_{k \neq 0} (2\pi \sqrt{-1} k^4) (2\pi |k|)^\alpha + 1 \frac{2 \{ (2\pi |k|)^\alpha + 1 \}^2}{|k|^2}.
\]

Again, we have \( \Pi_0([R_2 \mathcal{I}[\partial_t X^\varepsilon] \cap \partial_t X^\varepsilon]_{t,x} - [R_1 \mathcal{I}[\partial_t X^\varepsilon] \cap \partial_2 X^\varepsilon]_{t,x} = 0. \) Due to this cancellation, we do not need any renormalization for \( W \) given by (7.52) below.

We set

\[
(7.52) \quad W_i = \Pi_2(R_2 \mathcal{I}[\partial_t X] \cap \partial_t X) - \Pi_2(R_1 \mathcal{I}[\partial_t X] \cap \partial_2 X)
\]

for \( i = 1, 2. \)

Lemma 7.17. Let \( \theta \in (3/2, 2) \). Then, for every \( \alpha < 2\theta - 4, 1 < p < \infty \) and \( i = 1, 2, \)

\( \text{E}[\|W_i\|_{C^{0,p}}^p] < \infty, \)

\( \lim_{\varepsilon \downarrow 0} \text{E}[\|W_i - W_i^\varepsilon\|_{C^{0,p}}^p] = 0. \)

Proof. We let

\( \gamma = 4 - \frac{4}{\theta}, \quad \delta = 2. \)

Let \( Q_i \) and \( Q_i^\varepsilon \) be as in (7.50) and (7.51), respectively. They (should) correspond to \( \Pi_2([R_2 \mathcal{I}[\partial_t X] \cap \partial_t X]) \) and its smooth approximation, respectively. Since we have \( \psi_0(k, l)^2 |k|^2 |l|^2 \lesssim |k|^4, \) we see that

\[
|Q_0^\varepsilon((\sigma_1, k), (\sigma_2, l))|^2 \lesssim |Q_0((\sigma_1, k), (\sigma_2, l))|^2
\]
\[ \psi_0(k, l^2) \leq |(\sigma_1, k)|^{-\delta}(|\sigma_2, l|)^{-1} \]

\[ \leq |(\sigma_1, k)|^{-\delta}(1 - \frac{1}{2})|\sigma_2, l|^2 \]

\[ = |(\sigma_1, k)|^{-\delta}|\sigma_2, l|^{-2\delta}. \]

The rest is essentially the same as in the proof of Lemma 7.16. \qed

7.6. Convergence of \( Z := R^1Y \odot \nabla X \). In this subsection we prove that

\[ Z^\varepsilon := R^1Y^\varepsilon \odot \nabla X^\varepsilon = (R^2Y^\varepsilon) \odot (\partial_1 X^\varepsilon) - (R^1Y^\varepsilon) \odot (\partial_2 X^\varepsilon) \]

is convergent as \( \varepsilon \xrightarrow{} 0 \) if the Besov regularity is smaller than \( \frac{5}{2} \theta - 5 \). We do not need any renormalization.

Let us fix notation. For \( j \in \{1, 2\} \), \( j' = 3 - j \) stands for the other element of \( \{1, 2\} \). Hence, if \( j = 1 \) (resp. \( j = 2 \)), then \( j' = 2 \) (resp. \( j' = 1 \)).

Let \( i, j \in \{1, 2\} \). By straightforward computation, we have

\[ R_i[I[R_jX^\varepsilon \cdot \partial_j X^\varepsilon]]_{t,x} = \sum_{k,l \in \mathbb{Z}^2} \chi^\varepsilon(k)\chi^\varepsilon(l)e_{k+l}(x)\frac{2\pi \sqrt{-1}(k^2 + l^2)}{2\pi |k + l|}1_{k+l \neq 0} \int_{-\infty}^{\infty} duh(t - u, k + l) \]

\[ \times \int_{-\infty}^{\infty} H_u^{R_jX}(s_1, k)\dot{\xi}_{s_1}(k)ds_1 \int_{-\infty}^{\infty} H_u^{\partial_j X}(s_2, l)\dot{\xi}_{s_2}(l)ds_2 \]

and

\[ \partial_j X^\varepsilon_{t,x} = \sum_{m \in \mathbb{Z}^2} \chi^\varepsilon(m)e_m(x) \int_{-\infty}^{\infty} H_t^{\partial_j X}(s_3, m)\dot{\xi}_{s_3}(m)ds_3. \]

Here, \( H_u^{R_jX} \) and \( H_u^{\partial_j X} \) are given by (7.34) and (7.36), respectively. For simplicity we set \( Z_{t,x}^{ij\varepsilon} := R_i[I[R_jX^\varepsilon \cdot \partial_j X^\varepsilon]]_{t,x} \odot \partial_j X^\varepsilon_{t,x} \). Then, we have

\[ Z_{t,x}^{ij\varepsilon} = \sum_{k,l,m \in \mathbb{Z}^2} \chi^\varepsilon(k)\chi^\varepsilon(l)\chi^\varepsilon(m)e_{k+l+m}(x)\psi_0(k + l, m)\frac{2\pi \sqrt{-1}(k^2 + l^2)}{2\pi |k + l|}1_{k+l \neq 0} \]

\[ \times \int_{-\infty}^{\infty} duh(t - u, k + l) \int_{-\infty}^{\infty} H_u^{R_jX}(s_1, k)\dot{\xi}_{s_1}(k)ds_1 \]

\[ \times \int_{-\infty}^{\infty} H_u^{\partial_j X}(s_2, l)\dot{\xi}_{s_2}(l)ds_2 \int_{-\infty}^{\infty} H_t^{\partial_j X}(s_3, m)\dot{\xi}_{s_3}(m)ds_3 \]

\[ = \Pi_3(Z_{t,x}^{ij\varepsilon}) + \Pi_1(Z_{t,x}^{ij\varepsilon}). \]

Let us calculate the third order term on the right-hand side of (7.54). Note that the superindex \((i, j)\) will be omitted in many places when no confusion seems likely. It is easy to see that

\[ \Pi_3(Z_{t,x}^{ij\varepsilon}) = J_3\left(f_{t,x}(\varepsilon)\right), \]

where

\[ f_{t,x}(\varepsilon)((s_1, y_1), (s_2, y_2), (s_3, y_3)) = \sum_{k,l,m \in \mathbb{Z}^2} e_k(x - y_1)e_l(x - y_2)e_m(x - y_3)\chi^\varepsilon(k)\chi^\varepsilon(l)\chi^\varepsilon(m)H_t((s_1, k), (s_2, l), (s_3, m)) \]
Proof. Using Lemma 7.9 with

\[ H_t((s_1, k), (s_2, l), (s_3, m)) = \psi_0(k + l, m) \frac{2\pi \sqrt{-1}(k^4 + l^4)}{2\pi |k + l|} 1_{k + l \neq 0} \]

\[ \times \int_{-\infty}^{\infty} duh(t - u, k + l) H_u R^X (s_1, k) H_u^\partial X (s_2, l) H_t^\partial X (s_3, m). \]

By taking the Fourier transform with respect to the time variables, we have

\[ Q_t((\sigma_1, k), (\sigma_2, l), (\sigma_3, m)) = e^{-2\pi \sqrt{-1}(\sigma_1 + \sigma_2 + \sigma_3)t} \psi_0(k + l, m) \frac{2\pi \sqrt{-1}(k^4 + l^4)}{2\pi |k + l|} 1_{k + l \neq 0} \]

\[ \times \frac{Q^R_{\sigma} (\sigma_1, k) Q^\partial_{\sigma} X (\sigma_2, l)}{-2\pi \sqrt{-1}(\sigma_1 + \sigma_2) + (2\pi |k + l|)\theta + 1} Q_{\sigma}^\partial X (\sigma_3, m). \]

Therefore, \( \Pi_3(Z^i_{t, x}) \) has a good kernel whose \( Q \)-function is given by

\[ Q_t^i((\sigma_1, k), (\sigma_2, l), (\sigma_3, m)) = \chi^\varepsilon (k) \chi^\varepsilon (l) \chi^\varepsilon (m) Q_t((\sigma_1, k), (\sigma_2, l), (\sigma_3, m)). \]

As we will see below, \( Q_0 \) above satisfies the assumption of Lemma 7.12 and hence it defines a Besov space-valued process. We denote it by \( \Pi_3(Z^{i,j}) \).

**Lemma 7.18.** Let the notation be as above and \( \theta \in (8/5, 2] \). Then, for every \( \alpha < \frac{5}{2} \theta - 5 \), \( 1 < p < \infty \) and \( i, j = 1, 2 \),

\[ E[||\Pi_3(Z^{i,j\varepsilon})||^p_{C^\alpha}] < \infty, \lim_{\varepsilon \downarrow 0} E[||\Pi_3(Z^{i,j\varepsilon}) - \Pi_3(Z^{i,j})||^p_{C^\alpha}] = 0. \]

**Proof.** Using Lemma 7.9 with \( \gamma = 6 - \frac{\theta}{\alpha} \) and \( \delta = 0 \), we prove the lemma. Note that \( \gamma > 1 \) if and only if \( \theta > 8/5 \). For simplicity, we prove the case \( \theta \in (8/5, 2) \). (The case \( \theta = 2 \) can be shown essentially in the same way.) It is easy to see from (7.55) that

\[ |Q_0((\sigma_1, k), (\sigma_2, l), (\sigma_3, m))|^2 \lesssim \psi_0(k + l, m)^2 \frac{1}{|\sigma_1 + \sigma_2, k + l|^\theta} \frac{|l|^\theta}{|\sigma_1, k|^\theta} \frac{|m|^\theta}{|\sigma_3, m|^\theta}. \]

First, take a sum of this quantity over \((\sigma_1, k)\) and \((\sigma_2, l)\) such that \((\sigma_1 + \sigma_2, k + l) = (\hat{\tau}, \hat{n})\) for a given \((\hat{\tau}, \hat{n})\). By Lemma 7.3 and the same argument as in the proof of Lemma 7.13 it is dominated by a constant multiple of

\[ \psi_0(\hat{n}, m)^2 \frac{|m|^\theta}{|\hat{n}, \hat{n}|^{2(\theta - \frac{\theta}{\alpha})} |(\sigma_3, m)|^\theta} \lesssim \psi_0(\hat{n}, m)^2 \frac{|\hat{n}|^\theta}{|\hat{n}, \hat{n}|^{2(\theta - \frac{\theta}{\alpha})} |(\sigma_3, m)|^\theta} \lesssim \frac{1}{|\hat{n}, \hat{n}|^{2(\theta - \frac{\theta}{\alpha})} |(\sigma_3, m)|^\theta}. \]

Note that, since \( \theta > 4/3 \), we can use Lemma 7.3 again. The sum over \((\hat{\tau}, \hat{n})\) and \((\sigma_3, m)\) such that \((\hat{\tau} + \sigma_3, \hat{n} + m) = (\tau, n)\) for a given \((\tau, n)\) is dominated by a constant multiple of \(|(\tau, n)|_{\alpha}^{2(\theta - \frac{\theta}{\alpha})} \).

Summing up, we have shown that

\[ \int_{E^2} |Q_0((\sigma_1, k), (\sigma_2, l), (\tau - \sigma_1 - \sigma_2, n - k - l))|^2 \, d\sigma_1 \, d\sigma_2 \, dk \, dl \lesssim |(\tau, n)|_{\alpha}^{2(\theta - \frac{\theta}{\alpha})}. \]

\[ \square \]

Next, we calculate the first order term on the right-hand side of (7.34).
Lemma 7.19. Let the notation be as above and \( \theta \in (8/5, 2] \). Then, for every \( \alpha < \frac{2}{5} \theta - 5 \) and \( 1 < p < \infty \), there exists \( \Pi_1(Z^{2,2\varepsilon} - Z^{2,1\varepsilon} - Z^{1,2\varepsilon} + Z^{1,1\varepsilon}) \) which is independent of \( \alpha, p, \varepsilon \) such that
\[
\lim_{\varepsilon \to 0} E\|\Pi_1(Z^{2,2\varepsilon} - Z^{2,1\varepsilon} - Z^{1,2\varepsilon} + Z^{1,1\varepsilon}) - \Pi_1(Z^{2,2} - Z^{2,1} - Z^{1,2} + Z^{1,1})\|_{C_T^{\alpha}} = 0.
\]

Proof. First we calculate \( \Pi_1(Z^{i,j;\varepsilon}) \). Applying the contraction rule
\[
(7.56) \quad \Pi_1(\hat{\xi}_1(k)\hat{\xi}_2(l)) = \delta(s_1 - s_2)\delta_{k+l,0}\hat{\xi}_1(m) + \delta(s_1 - s_3)\delta_{k+m,0}\hat{\xi}_2(l) + \delta(s_2 - s_3)\delta_{l+m,0}\hat{\xi}_1(k)
\]
to (7.54), we have
\[
\Pi_1(Z^{i,j;\varepsilon}) =: A_{i,x}^{i,j;\varepsilon}(1) + A_{i,x}^{i,j;\varepsilon}(2) + A_{i,x}^{i,j;\varepsilon}(3),
\]
where \( A_{i,x}^{i,j;\varepsilon}(\nu), \nu = 1, 2, 3 \), are given as follows:
\[
A_{i,x}^{i,j;\varepsilon}(1) = \sum_{m \in \Bbb Z^2} \chi^\varepsilon(m)e(t)\psi_0(0, m)\int_{-\infty}^{\infty} H_t^{\partial_j X}(s, m)\hat{\xi}_s(m)ds
\]
\[
\times \sum_{k \in \Bbb Z^2} \chi^\varepsilon(k)\int_{-\infty}^{\infty} duh(t - u, 0)\int_{-\infty}^{\infty} H_u^{R_{ij} X}(s_1, k)H_u^{\partial_j X}(s_1, -k)ds_1.
\]
By (7.46), the second factor (i.e. the sum over \( k \)) above equals \( \Pi_0(I[R_{ij} X \cdot \partial_j X]_{t,x}) \). Therefore, \( A_{i,x}^{i,j;\varepsilon}(1) = A_{i,x}^{i,j;\varepsilon}(2) = 0 \).

Similarly, we have
\[
(7.57) \quad A_{i,x}^{i,j;\varepsilon}(2) = \sum_{k,l \in \Bbb Z^2} \chi^\varepsilon(l)\chi^\varepsilon(k)e_t(x)\psi_0(k + l, -k)\frac{2\pi}{2\pi|k + l|}1_{k+l \neq 0}
\]
\[
\times \int_{-\infty}^{\infty} duh(t - u, k + l)\int_{-\infty}^{\infty} H_u^{R_{ij} X}(s_1, k)H_t^{\partial_j X}(s_1, -k)ds_1
\]
\[
\times \int_{-\infty}^{\infty} H_u^{\partial_j X}(s_2, l)\hat{\xi}_s(s)ds_2
\]
\[
= \sum_{l \in \Bbb Z^2} \chi^\varepsilon(l)e_t(x)\left[ \sum_{k \in \Bbb Z^2} \chi^\varepsilon(k)\psi_0(k + l, -k)\frac{2\pi}{2\pi|k + l|}1_{k+l \neq 0}
\]
\[
\times \left( -\frac{2\pi}{2\pi|k|}\frac{1}{2\pi|k|} \right)^{2\varepsilon} \right) \right)^{-2\varepsilon}\int_{-\infty}^{t} du\epsilon^{-((2\pi|k|)^{\varepsilon} + (2\pi|k|)^{\varepsilon} + 2)}\int_{-\infty}^{\infty} H_u^{\partial_j X}(s_2, l)\hat{\xi}_s(s)ds_1
\]
and
\[
(7.58) \quad A_{i,x}^{i,j;\varepsilon}(3) = \sum_{k,l \in \Bbb Z^2} \chi^\varepsilon(k)\chi^\varepsilon(l)e_t(x)\psi_0(k + l, -l)\frac{2\pi}{2\pi|k + l|}1_{k+l \neq 0}
\]
\[
\times \int_{-\infty}^{\infty} duh(t - u, k + l)\int_{-\infty}^{\infty} H_u^{R_{ij} X}(s_1, k)\hat{\xi}_s(s_1)ds_1
\]
\[
\times \int_{-\infty}^{\infty} H_t^{\partial_j X}(s_2, l)H_t^{\partial_j X}(s_2, -l)ds_2
\]
\[\begin{align*}
&= \sum_{k \in \mathbb{Z}^2} \chi^e(k) e_k(x) \left[ \sum_{l \in \mathbb{Z}^2} \chi^e(l)^2 \psi_o(k + l, -l) \frac{2\pi \sqrt{-1} (k^i + l^i)}{2\pi |k + l|} \mathbf{1}_{k+l \neq 0} \\
&\quad \times \left\{ \frac{-(2\pi \sqrt{-1} l^i')(2\pi \sqrt{-1} l^i)}{2(2\pi |l|)^\theta + 1} \right\} \right] \\
&\quad \times \int_{-\infty}^t \frac{du}{\epsilon^2} \left( (2\pi |k+l|)^\theta + (2\pi |l|)^\theta + 2 \right) \int_{-\infty}^\infty H_{\alpha_j} X(s, k) \xi_s (k) ds_1.
\end{align*}\]

In what follows, the superscript "i, j" in \(A_{\alpha_j}^i(x, \nu)\) (and in other corresponding quantities) will be suppressed when no confusion seems likely.

Now we calculate the \(H\)-function and the \(Q\)-function associated with \(A_{\alpha_j}^i(2)\). Let \((s_2, l) \in \tilde{E}\).

From (7.57), we see that
\[H_{\alpha_j}^i (s_2, l) = \chi^e(l) \sum_{k \in \mathbb{Z}^2} \chi^e(k)^2 \psi_o(k + l, -k) \frac{2\pi \sqrt{-1} (k^i + l^i)}{2\pi |k + l|} \mathbf{1}_{k+l \neq 0} \\
\times \int_{-\infty}^t \frac{du}{\epsilon^2} \left( (2\pi |k+l|)^\theta + (2\pi |l|)^\theta + 2 \right) \int_{-\infty}^\infty H_{\alpha_j}^i X(s_2, l)
\]
and
\[Q_{\alpha_j}^i (s_2, l) = e^{-2\pi \sqrt{-1} \sigma_2 \theta} \chi^e(l) \sum_{k \in \mathbb{Z}^2} \chi^e(k)^2 \psi_o(k + l, -k) \frac{2\pi \sqrt{-1} (k^i + l^i)}{2\pi |k + l|} \mathbf{1}_{k+l \neq 0} \\
\times \left\{ \frac{-(2\pi \sqrt{-1} l^i')(2\pi \sqrt{-1} l^i)}{4\pi |k| \{(2\pi |k|)^\theta + 1\}} \right\} \mathbf{1}_{k \neq 0} \\
\times \frac{1}{-2\pi \sqrt{-1} \sigma_2 + (2\pi |k+l|)^\theta + (2\pi |l|)^\theta + 2} \frac{2\pi \sqrt{-1} l^i'}{-2\pi \sqrt{-1} \sigma_2 + (2\pi |l|)^\theta + 1},
\]
which shows \(A_{\alpha_j}^i(2)\) has a good kernel whose \(Q\)-function is given by \(Q_{\alpha_j}^i\). We define \(Q_{\alpha_j}(\sigma_2, l)\) by formally setting \(\epsilon = 0\) (or replacing both \(\chi^e(k)\) and \(\chi^e(l)\) by 1) in the above expression of \(Q_{\alpha_j}^i(\sigma_2, l)\).

Assume that \(\theta \in (8/5, 2)\). (The case \(\theta = 2\) can be done essentially in the same way.) Take sufficiently small \(a > 0\) and set \(\gamma := 6 - \frac{8 + 2a}{\theta}\). Then, under the condition on \(\theta\), we may assume \(0 < \gamma < 2\) and \(0 < 2\theta - 3 - a < 1\). Recall that \(\psi_o(k + l, k) \lesssim |k + l|^{2\theta - 3 - a} |l|^\gamma \) by Lemma 4.4.

We can dominate \(|Q_0(\sigma_2, l)|^2\) as follows:
\[
(7.59) \quad |Q_0(\sigma_2, l)|^2 \lesssim \sum_{k \in \mathbb{Z}^2} |\psi_o(k + l, -k)| \frac{1}{|k|^{\gamma - 1}} \frac{1}{|k + l|^\theta + |k|\theta + 1} \frac{1}{|(\sigma_2, l)|_s^{-2\gamma}}^2 \\
\lesssim \sum_{k \in \mathbb{Z}^2} \min \left\{ \frac{|k + l|^{2\theta - 3 - a}}{|k|^{\gamma - 1}} \frac{1}{|k + l|^\theta + |k|\theta + 1} \frac{1}{|(\sigma_2, l)|_s^{-2\gamma}}^2, \frac{1}{|l|^{2\theta - 3 - a}} \right\} \\
\lesssim \sum_{k \in \mathbb{Z}^2} \frac{|k|^{2\theta - 3 - a}}{|l|^{2\theta - 3 - a}} \frac{1}{|(\sigma_2, l)|_s^{-2\gamma}} \lesssim |(\sigma_2, l)|_{s}^{-2\gamma}.
\]

Here, the implicit constant may depend on the constant \(a\). Since \(6 - \frac{8}{\theta} > 1\) and \(a > 0\) is arbitrarily small, we can apply Lemma [7.9] with \(\gamma = 6 - \frac{8 + 2a}{\theta}\) and \(b = 0\) to \(A_{\alpha_j}^i(2)\) to obtain the desired estimate. We can also estimate \(|Q_0(\sigma_2, l) - Q_{\alpha_j}^i(\sigma_2, l)|^2\) in the same way using that \(|1 - \chi^e(k)| \lesssim \epsilon^b |k|^b\) for every sufficiently small \(b > 0\).

The Besov space-valued process associated with this \(Q_0\) will be denoted by \(A^{i,j}(2)\).
Next we calculate the $H$-function and the $Q$-function associated with $A^\epsilon_{t,x}(3)$. Let $(s_1, k) \in E$. In a similar way as above, we see that

$$H^\epsilon_t(s_1, k) = \chi^\epsilon(k) \sum_{l \in \mathbb{Z}^2} \chi^\epsilon(l) \psi_\sigma(k + l, -l) \frac{2\pi \sqrt{-1}(k^i + l^i)}{2\pi|k + l|} \frac{1_{k+l \neq 0}}{2[(2\pi|l|)^\theta + 1]}$$

and

$$Q^\epsilon_t(\sigma_1, k) = e^{-2\pi \sqrt{-1}\sigma_1^2} \chi^\epsilon(k) \sum_{l \in \mathbb{Z}^2} \chi^\epsilon(l) \psi_\sigma(k + l, -l) \frac{2\pi \sqrt{-1}(k^i + l^i)}{2\pi|k + l|} \frac{1_{k+l \neq 0}}{2\pi|k|^2}$$

Here and in what follows, we set

$$C(l) = -\frac{(2\pi \sqrt{-1}l^i)(2\pi \sqrt{-1}l^i)}{2[(2\pi|l|)^\theta + 1]}, \quad D(k) = \frac{2\pi \sqrt{-1}k^i}{2\pi|k|^2} 1_{k \neq 0},$$

$$B_{k,l} = (2\pi|k + l|)^\theta + (2\pi|l|)^\theta + 2, \quad B'_l = 2[(2\pi|l|)^\theta + 1].$$

Thus, $A^\epsilon_{t,x}(3) = A^{i_2,\epsilon}_{t,x}(3)$ also has a good kernel and its $Q$-function is given by $Q^\epsilon(= Q^{i_2,\epsilon})$ above.

One should note here that

$$\sum_{l \in \mathbb{Z}^2} \chi^\epsilon(l) \psi_\sigma(l, -l) \frac{2\pi \sqrt{-1}l^i}{2\pi|l|} 1_{l \neq 0}$$

is symmetric in $i$ and $i'$ for fixed $j$. Therefore, if we modify the definition a little bit as

$$\tilde{Q}^\epsilon_t(\sigma_1, k) = e^{-2\pi \sqrt{-1}\sigma_1^2} \chi^\epsilon(k) \sum_{l \in \mathbb{Z}^2} \chi^\epsilon(l) \frac{C(l)}{2\pi|k|^2}$$

then we can easily see that $Q^2, 1;\epsilon - Q^2, 1;\epsilon + Q^1, 1;\epsilon = \tilde{Q}^2, 2;\epsilon - \tilde{Q}^2, 1;\epsilon + \tilde{Q}^1, 1;\epsilon$. Hence, it is enough to estimate these modified $Q$-functions. We define $\tilde{Q}^\epsilon_t(\sigma_1, k) = \tilde{Q}^{i_2,\epsilon}_t(\sigma_1, k)$ by formally setting $\epsilon = 0$ (or replacing both $\chi^\epsilon(k)$ and $\chi^\epsilon(l)$ by 1) in the above expression of $\tilde{Q}^\epsilon_t(\sigma_1, k) = \tilde{Q}^{i_2,\epsilon}_t(\sigma_1, k)$.

In the following we will prove that this $\tilde{Q}$ satisfies the assumption of Lemma 7.12 for the desired $\gamma > 1$. The Besov space-valued process associated with this $\tilde{Q}$ will be denoted by $A^{i_2,3}(3)$.
\[
+ |\psi_0(k + l, -l) - \psi_0(l, -l)| \cdot \frac{1}{-2\pi\sqrt{-1}\sigma_1 + B_{k,l}} \\
+ |\psi_0(l, -l)| \cdot \frac{1}{-2\pi\sqrt{-1}\sigma_1 + B_{k,l}} - \frac{1}{-2\pi\sqrt{-1}\sigma_1 + B_{l}}
\]

\[= J_1 + \cdots + J_4.\]

Contribution from \(J_1\) is dominated as follows: Assume that \(\theta \in (8/5, 2). \) (The case \(\theta = 2\) can be done essentially in the same way.) Take sufficiently small \(a > 0\) and set \(\gamma := 6 - \frac{5 + 2\theta}{\theta}. \) We use \(\psi_0(k + l, -l) \lesssim |k + l|^{2\theta - 3 - a}|k|^{|2\theta - 3 - a|}\) again.

\[
\left| \sum_{l \in \mathbb{Z}^2} \chi^\varepsilon(l)^2 \frac{\sqrt{-1}J_1 C(l) D(k)}{-2\pi\sqrt{-1}\sigma_1 + (2\pi|k|)^\theta + 1} \right|^2 
\lesssim \left\{ \frac{|k|}{|k + l|} 1_{k + l \neq 0} \frac{|k + l|^{2\theta - 3 - a}}{|k|^{|2\theta - 3 - a|}} \frac{|l|^{2 - \theta}}{|l|^{|2 - \theta|}} \frac{1}{|\sigma_1, k|^2} \right\}^2 |(\sigma_1, k)|^{\gamma} \lesssim |(\sigma_1, k)|^{\gamma}.
\]

Here we used that \(1/|(\sigma_1, k)|^{\gamma} \lesssim 1/|k|^{|2\theta - 3 - a|}.\)

We estimate the contribution from \(J_2. \) By straightforward calculation, we have that

\[
\left| \frac{1}{|k + l|} 1_{k + l \neq 0} - \frac{1}{|l|} 1_{l \neq 0} \right| \lesssim \frac{|l|}{|k + l||l|} \leq \frac{|k|}{|k + l|||l|}
\]

and therefore for any \(\varepsilon \in (0, 1)\)

\[
\left| \frac{1}{|k + l|} 1_{k + l \neq 0} - \frac{1}{|l|} 1_{l \neq 0} \right| \psi_0(k + l, -l) \lesssim \frac{|k|^\varepsilon}{|k + l||l|^\varepsilon}
\]

Taking \(1 - \varepsilon = 2\theta - 3 - a, \) we see that

\[
\left| \sum_{l \in \mathbb{Z}^2} \frac{\sqrt{-1}J_2 C(l) D(k)}{-2\pi\sqrt{-1}\sigma_1 + (2\pi|k|)^\theta + 1} \right|^2 
\lesssim \left\{ \sum_{l \in \mathbb{Z}^2} |l| \frac{|k|^{-2\theta + 4 + a}}{|k + l|^{|2\theta - 3 - a|}} \frac{|l|^{2 - \theta}}{|l|^{|2 - \theta|}} \frac{1}{|\sigma_1, k|^2} \right\}^2 |(\sigma_1, k)|^{\gamma} \lesssim |(\sigma_1, k)|^{\gamma}.
\]

We estimate the contribution from \(J_3. \) Due to Lemma 7.3 (ii) and the boundedness of \(\psi_0, \) we have for any \(\lambda, \varepsilon \in (0, 1)\) that

\[
|\psi_0(k + l, -l) - \psi_0(l, -l)| \lesssim |\psi_0(k + l, -l) - \psi_0(l, -l)|^\varepsilon \\
\lesssim |(k + l - l|^\varepsilon |l|^{1 - \lambda + \lambda} \lesssim \frac{|(1 - \lambda)^a + a - 2\theta + 2 > 2 \text{ holds. Then, we have}}{|k|^\varepsilon |l|^{1 - \lambda + \lambda} .
\]

We take \(1 - \varepsilon = 2\theta - 3 - a \) for sufficiently small \(a > 0\) and then we choose \(\lambda \) so small that

\[
(1 - \lambda)^a + a > 2\theta - 2 > 2 \text{ holds. Then, we have}
\]

\[
\left| \sum_{l \in \mathbb{Z}^2} \frac{\sqrt{-1}J_3 C(l) D(k)}{-2\pi\sqrt{-1}\sigma_1 + (2\pi|k|)^\theta + 1} \right|^2 
\]
Lemma 7.20. Let \( \theta \in (8/5, 2] \). Then, for every \( \alpha < 6^\theta - 5 \) and \( 1 < p < \infty \),
\[
\mathbb{E}[\|Z\|_{C_\theta^\alpha}^p] < \infty, \quad \lim_{\varepsilon \searrow 0} \mathbb{E}[\|Z - Z^\varepsilon\|_{C_\theta^\alpha}^p] = 0.
\]
7.7. Convergence of $\tilde{Z} = \nabla Y \cdot R^k X$. In this subsection we prove that

\begin{equation}
\tilde{Z}^\varepsilon := \nabla Y^\varepsilon \cdot R^k X^\varepsilon = (\partial Y^\varepsilon) \cdot (R_2 Y^\varepsilon) - (\partial Y^\varepsilon) \cdot (R_1 Y^\varepsilon) = \partial_1 I[R_2 X^\varepsilon \cdot \partial_1 X^\varepsilon - R_1 X^\varepsilon \cdot \partial_2 X^\varepsilon] \cdot (R_2 Y^\varepsilon) - \partial_2 I[R_2 X^\varepsilon \cdot \partial_1 X^\varepsilon - R_1 X^\varepsilon \cdot \partial_2 X^\varepsilon] \cdot (R_1 Y^\varepsilon)
\end{equation}

is convergent as $\varepsilon \searrow 0$ if the Besov regularity is smaller than $\frac{\theta}{2} - 5$. We do not need any renormalization. The proof is quite similar to, but somewhat simpler than the one for $Z$ in the previous subsection.

Let $i, j \in \{1, 2\}$. By straightforward computation, we have

\[ \partial_1 I[R_j X^\varepsilon \cdot \partial_j X^\varepsilon]_{t,x} = \sum_{k,l \in \mathbb{Z}^2} \chi^\varepsilon(k) \chi^\varepsilon(l) e_{k+l+1}(x) 2\pi \sqrt{-1}(k^i + l^i) \int_{-\infty}^\infty duh(t - u, k + l) \]

\[ \times \int_{-\infty}^\infty H_u^{R_j X}(s_1, k) \xi_1(s_1) ds_1 \int_{-\infty}^\infty H_u^{\partial_j X}(s_2, l) \xi_2(s_2) ds_2 \]

and

\[ R_j X^\varepsilon_{t,x} = \sum_{m \in \mathbb{Z}^2} \chi^\varepsilon(m) e_m(x) \int_{-\infty}^\infty H_t^{R_j X}(s_3, m) \xi_3(s_3) ds_3. \]

We set $\tilde{Z}^i_{t,x} := \partial_1 I[R_j X^\varepsilon \cdot \partial_j X^\varepsilon]_{t,x} \cdot R_j X^\varepsilon_{t,x}$. Then, we have

\begin{equation}
\tilde{Z}^i_{t,x} = \sum_{k,l \in \mathbb{Z}^2} \chi^\varepsilon(k) \chi^\varepsilon(l) \chi^\varepsilon(m) e_{k+l+m}(x) 2\pi \sqrt{-1}(k^i + l^i) \]

\[ \times \int_{-\infty}^\infty duh(t - u, k + l) \int_{-\infty}^\infty H_u^{R_j X}(s_1, k) \xi_1(s_1) ds_1 \]

\[ \times \int_{-\infty}^\infty H_u^{\partial_j X}(s_2, l) \xi_2(s_2) ds_2 \int_{-\infty}^\infty H_t^{R_j X}(s_3, m) \xi_3(s_3) ds_3 \]

\[ = \Pi_3(\tilde{Z}^i_{t,x}) + \Pi_1(\tilde{Z}^i_{t,x}). \]

Let us calculate the third order term on the right-hand side of (7.66). It is easy to see that

\[ \Pi_3(\tilde{Z}^i_{t,x}) = \mathcal{F}_3 \left( f^i_{t,x} \right), \]

where

\[ f^i_{t,x}((s_1, y_1), (s_2, y_2), (s_3, y_3)) = \sum_{k,l,m \in \mathbb{Z}^2} e_k(x - y_1) e_l(x - y_2) e_m(x - y_3) \chi^\varepsilon(k) \chi^\varepsilon(l) \chi^\varepsilon(m) H_1((s_1, k), (s_2, l), (s_3, m)) \]

with

\[ H_1((s_1, k), (s_2, l), (s_3, m)) = 2\pi \sqrt{-1}(k^i + l^i) \int_{-\infty}^\infty duh(t - u, k + l) H_u^{R_j X}(s_1, k) H_u^{\partial_j X}(s_2, l) H_t^{R_j X}(s_3, m). \]

By Fourier transform with respect to the time variables, we have

\begin{equation}
Q_1((\sigma_1, k), (\sigma_2, l), (\sigma_3, m)) = e^{-2\pi \sqrt{-1}(\sigma_1 + \sigma_2 + \sigma_3)l} 2\pi \sqrt{-1}(k^i + l^i) \]

\[ \times \frac{Q_0^{R_j X}(\sigma_1, k) Q_0^{\partial_j X}(\sigma_2, l)}{-2\pi \sqrt{-1}(\sigma_1 + \sigma_2) + (2\pi |k + l|)^\theta + 1} Q_0^{R_j X}(\sigma_3, m). \]
Therefore, $\Pi_\delta(\hat{Z}^i;j)\text{e}^\gamma$ has a good kernel whose $Q$-function is given by

$$Q_\delta((\sigma_1,k), (\sigma_2,l), (\sigma_3,m)) = \chi^\varepsilon(k)\chi^\varepsilon(l)\chi^\varepsilon(m)Q_\delta((\sigma_1,k), (\sigma_2,l), (\sigma_3,m)).$$

As we will see below, $Q_\delta$ above satisfies the assumption of Lemma 7.12 and hence it defines a Besov space-valued process. We denote it by $\Pi_\delta(\hat{Z}^i;j)\text{e}^\gamma$.

**Lemma 7.21.** Let the notation be as above and $\theta \in [8/5, 2]$. Then, for every $\alpha < \frac{5}{2}\theta - \delta$ and $i, j = 1, 2$, 

$$\mathbb{E}[||\Pi_\delta(\hat{Z}^i;j)||^p_{C_TC^\alpha}] < \infty, \quad \lim_{\varepsilon \searrow 0} \mathbb{E}[||\Pi_\delta(\hat{Z}^i;j)\text{e}^\gamma ||^p_{C_TC^\alpha}] = 0.$$ 

**Proof.** We let 

$$\gamma = 6 - \frac{8}{\theta}, \delta = 0$$

to use Lemma 7.9. Note that $\gamma > 1$ if and only if $\theta > 8/5$. We prove the case $\theta \in (8/5, 2)$ only. It is easy to see from (7.68) that

$$|Q_\delta((\sigma_1,k), (\sigma_2,l), (\sigma_3,m))|^2 \leq \frac{\|k + l\|^2}{\|(\sigma_1, k)\|_{\alpha}^2 \|l\|^2} \frac{1}{\|(\sigma_2, l)\|_{\alpha}^2} \frac{1}{\|(\sigma_3, m)\|_{\alpha}^2}.$$ 

By using Lemma 7.3 twice in the same way as in the proof of Lemma 7.18, we have

$$\int_{E_2} |Q_\delta((\sigma_1,k), (\sigma_2,l), (\tau - \sigma_1 - \sigma_2 - n - k - l))|^2 d(\sigma_1, k)d(\sigma_2, l) \lesssim (\tau,n)^{-2\gamma}.$$ 

Using Lemma 7.9 we prove the lemma. \hfill \square

Next, we calculate the first order term on the right-hand side of (7.67).

**Lemma 7.22.** Let the notation be as above and $\theta \in [8/5, 2]$. Then, for every $\alpha < \frac{5}{2}\theta - \delta$ and $1 < p < \infty$, there exists $\Pi_\delta(\hat{Z}^2;\hat{Z}, \hat{Z}^1;\hat{Z}, \hat{Z}, \hat{Z}) \in L^p(\Omega, \mathbb{P}; C_TC^\alpha)$ which is independent of $\alpha, p, \chi$ such that

$$\lim_{\varepsilon \searrow 0} \mathbb{E}[||\Pi_\delta(\hat{Z}^2;\hat{Z}, \hat{Z}^1;\hat{Z}, \hat{Z}, \hat{Z})\text{e}^\gamma ||_{C_TC^\alpha}] = 0.$$ 

**Proof.** First we calculate $\Pi_\delta(\hat{Z}^i;j_{\nu})\text{e}^\gamma$. Applying the contraction rule (7.58) to (7.67), we have

$$\Pi_\delta(\hat{Z}^i;j_{\nu}) =: \hat{A}_{t,x}^i;\nu(1) + \hat{A}_{t,x}^i;\nu(2) + \hat{A}_{t,x}^i;\nu(3),$$

where $\hat{A}_{t,x}^i;\nu(\nu) = 2, 3$, are given as follows: (the contribution from $\hat{A}_{t,x}^i;\nu(1)$ cancels out for the same reason as in the proof of Lemma 7.12. So we do not write it down.)

(7.69) \hspace{1cm} \hat{A}_{t,x}^i(2) = \sum_{k,l \in \mathbb{Z}^2} \chi^\varepsilon(k)\chi^\varepsilon(l)\chi^\varepsilon(2\pi\sqrt{-1}(k^i + l^i)) \times \int_{-\infty}^{\infty} duh(t - u, k + l) \int_{-\infty}^{\infty} H_u^{R_jX}(s_1, k)H_t^{R_jX}(s_1, -k)ds_1 \times \int_{-\infty}^{\infty} H_u^{R_jX}(s_2, l)ds_2 \\
= \sum_{l \in \mathbb{Z}^2} \chi^\varepsilon(l)\chi^\varepsilon(2\pi\sqrt{-1}(k^i + l^i))
\[ \times \frac{2\pi \sqrt{-1} k^j}{2|k|} 1_{k \neq 0} \cdot \frac{-2\pi \sqrt{-1} k^j}{2|k|} 1_{k \neq 0} \cdot \frac{1}{2\{2\pi|k|\}^\theta + 1} \]
\[ \times \int_{-\infty}^t du e^{-(t-u)\{(2\pi|k+l|)^\theta + (2\pi|k|)^\theta + 2\}} \int_{-\infty}^\infty H_{u}^{\partial_j X}(s_2, l) \xi_{s_2}(l) ds_2 \]

and
\[
\hat{A}^{\alpha, \varepsilon}_{t,x}(3) = \sum_{k,l \in \mathbb{Z}^2} \chi^\varepsilon(k) \chi^\varepsilon(l)^2 e_k(x) 2\pi \sqrt{-1}(k^j + l^j) \]
\[ \times \int_{-\infty}^\infty du h(t - u, k + l) \int_{-\infty}^\infty H_{u}^{R_j X}(s_1, k) \xi_{s_1}(k) ds_1 \]
\[ \times \int_{-\infty}^\infty H_{u}^{\partial_j X}(s_2, l) H_{t}^{R_j X}(s_2, -l) ds_2 \]
\[ = \sum_{k \in \mathbb{Z}^2} \chi^\varepsilon(k) e_k(x) \sum_{l \in \mathbb{Z}^2} \chi^\varepsilon(l)^2 2\pi \sqrt{-1}(k^j + l^j) \frac{(2\pi \sqrt{-1} l^j)^{2}(2\pi \sqrt{-1} l^j)}{2\pi|l| \cdot 2\{(2\pi|l|)^\theta + 1\}} 1_{l \neq 0} \]
\[ \times \int_{-\infty}^t du e^{-(t-u)\{(2\pi|k+l|)^\theta + (2\pi|k|)^\theta + 2\}} \int_{-\infty}^\infty H_{u}^{R_j X}(s_1, k) \xi_{s_1}(k) ds_1 \]
This estimate (7.13) is essentially the same as (7.14). Hence, we can apply Lemma 7.9 to $\hat{A}^\varepsilon(2) = \hat{A}^{i,j,\varepsilon}(2)$ with $\gamma = 6 - (8 + 2a)/\theta$ and $\delta = 0$.

The Besov space-valued process associated with this $Q_0$ will be denoted by $\hat{A}^{i,j}(2)$.

Next, we will calculate the $H$-function and the $Q$-function associated with the function $\hat{A}^\varepsilon_s(3)$. For any $(s_1, k), (\sigma_1, k) \in E$,

$$H^\varepsilon_s(s_1, k) = \chi^\varepsilon(k) \sum_{l \in \mathbb{Z}^2} \chi^\varepsilon(l) 2 \pi \sqrt{-1} (k^l + t^l) \left( \frac{2 \pi \sqrt{-1} l^l}{2 \pi |l|} \right) \left( \frac{2 \pi \sqrt{-1} l^l}{2 \pi |l|} \right) 1_{l \neq 0}$$

and

$$Q^\varepsilon_s(\sigma_1, k) = e^{-2 \pi \sqrt{-1} \sigma_1 t^l} \chi^\varepsilon(k) \sum_{l \in \mathbb{Z}^2} \chi^\varepsilon(l) 2 \pi \sqrt{-1} (k^l + t^l) \left( \frac{2 \pi \sqrt{-1} l^l}{2 \pi |l|} \right) \left( \frac{2 \pi \sqrt{-1} l^l}{2 \pi |l|} \right) 1_{l \neq 0}$$

As in the proof of Lemma 7.19, we modify the $Q^\varepsilon = Q^{i,j,\varepsilon}$ a little bit as follows:

$$\tilde{Q}^\varepsilon_s(\sigma_1, k) = \tilde{Q}^{i,j,\varepsilon}_s(\sigma_1, k) = e^{-2 \pi \sqrt{-1} \sigma_1 t^l} \chi^\varepsilon(k) \sum_{l \in \mathbb{Z}^2} \chi^\varepsilon(l) 2 \pi \sqrt{-1} (k^l + t^l) \left( \frac{2 \pi \sqrt{-1} l^l}{2 \pi |l|} \right) \left( \frac{2 \pi \sqrt{-1} l^l}{2 \pi |l|} \right) 1_{l \neq 0}$$

We have again $Q^{2,2,\varepsilon} - Q^{2,1,\varepsilon} - Q^{1,2,\varepsilon} + Q^{1,1,\varepsilon} = \tilde{Q}^{2,2,\varepsilon} - \tilde{Q}^{2,1,\varepsilon} - \tilde{Q}^{1,2,\varepsilon} + \tilde{Q}^{1,1,\varepsilon}$. We define $\tilde{Q}_s(\sigma_1, k) = \tilde{Q}^{i,j,\varepsilon}_s(\sigma_1, k)$ by formally setting $\varepsilon = 0$ (or replacing both $\chi^\varepsilon(k)$ and $\chi^\varepsilon(l)$ by 1) in the above expression of $\tilde{Q}_s(\sigma_1, k) = \tilde{Q}^{i,j,\varepsilon}(\sigma_1, k)$.

By the same argument as in the proof of Lemma 7.19, we can show that

$$|\tilde{Q}_0(\sigma_1, k)|^2 \leq \frac{2}{1 + 2} |\tilde{Q}_0(\sigma_1, k)|^2 \lesssim |(\sigma_1, k)|^2 \epsilon^{-6 - (8 + 2a)/\theta}$$

for every $a > 0$ small enough. (This time it is actually less cumbersome since $\psi_0$ does not appear.) By Lemma 7.9 there exists a Besov space-valued process $\hat{A}^{i,j}(3)$ which corresponds to $\tilde{Q}^{i,j}$.

By setting

$$\Pi_1(\hat{Z}^{2,2} - \hat{Z}^{2,1} - \hat{Z}^{1,2} + \hat{Z}^{1,1}) = \{\hat{A}^{2,2}(2) - \hat{A}^{2,1}(2) - \hat{A}^{1,2}(2) + \hat{A}^{1,1}(2)\} + \{\hat{A}^{\otimes,2,2}(3) - \hat{A}^{\otimes,2,1}(3) - \hat{A}^{\otimes,1,2}(3) + \hat{A}^{\otimes,1,1}(3)\},$$

we finish our proof. □

Summing up, we have shown the following Lemma in this subsection: Set

$$\hat{Z} = \sum_{k=1,3} \Pi_k(\hat{Z}^{2,2} - \hat{Z}^{2,1} - \hat{Z}^{1,2} + \hat{Z}^{1,1}).$$
For the definition of $\hat{Z}^\varepsilon := \nabla Y^\varepsilon \cdot R^\perp X^\varepsilon$, see $(7.66)$.

**Lemma 7.23.** Let $\theta \in (8/5, 2]$. Then, for every $\alpha < \frac{2}{5} \theta - 5$ and $1 < p < \infty$,

$$
\mathbf{E}[\|\hat{Z}\|_{C^\alpha_T C^\alpha}] < \infty, \quad \lim_{\varepsilon \to 0} \mathbf{E}[\|\hat{Z} - \hat{Z}^\varepsilon\|_{C^\alpha_T C^\alpha}] = 0.
$$

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