FINITE-TIME BLOW-UP OF A NON-LOCAL STOCHASTIC PARABOLIC PROBLEM

NIKOS I. KAVALLARIS AND YUBIN YAN

Abstract. In the current work we investigate the behaviour of a non-local stochastic parabolic problem. We first prove the local-in-time existence and uniqueness of a weak solution. Then we check the extendability in time of the weak solution. In particular, the rest of the paper is devoted to the investigation of the conditions under which finite-time blow-up occurs. We first prove that noise term induced finite-time blow up takes place when the stochastic term is rather big independently of the size of the non-local term. Afterwards, non-local-term induced finite-time blow occurs when the nonlinearity satisfies some specific growth conditions. In this case some fundamental results like maximum principle and Hopf’s type lemma in the context of SPDEs are first provided and later Kaplan’s eigenfunction method adjusted to the our non-local SPDE is employed.

1. Introduction

In the current work we consider the following non-local stochastic parabolic problem

\[
\frac{\partial u}{\partial t} = \Delta u + F(u) + \sigma(u) \partial_t W(x,t), \quad (x,t) \in D_T := D \times (0,T),
\]

\[
u(x,t) = 0, \quad (x,t) \in \partial D \times (0,T),
\]

\[
u(x,0) = \xi(x), \quad x \in D,
\]

where \( T > 0 \) and \( D \) is a bounded subset of \( \mathbb{R}^d, \ d \geq 1 \), with smooth boundary. Here the non-local term \( F(u) \) is defined by

\[
F(u) := \frac{\lambda f(u)}{\left( \int_D f(u) \, dx \right)^q}, \quad q > 0,
\]

for some positive constant \( \lambda \) where \( f(u), \sigma(u) \) are assumed to be locally Lipschitz and strictly positive functions. Moreover \( \partial_t W(x,t) \) denotes by convention the formal (time) derivative of the Wiener random process \( W(x,t) \) in a complete probability space \( (\Omega, \mathcal{F}_t, \mathbb{P}) \) with filtration \( (\mathcal{F}_t)_{t \in [0,T]} \) which is defined more rigorously in the following section. Here \( \xi \) is a \( \mathcal{F}_0 \)-random variable in some suitable Hilbert spaces introduced later.

The motivation of studying problem (1.1) – (1.3) is that this kind of non-local stochastic problems arise in the mathematical modelling of a variety of phenomena coming from
industrial applications (e.g. Ohmic heating in food sterilization \cite{20, 21, 28} and shear banding formation in high strain metals \cite{2, 3, 15}), biology (e.g. chemotaxis phenomenon \cite{18, 30}), statistical mechanics \cite{19} and so on. The presence of the multiplicative noise term $\sigma(u) \partial_t W(x, t)$ is natural when one considers noisy control systems, see \cite{4}. In particular, it represents the existence of external perturbations or a lack of knowledge of certain physical parameters which is actually quite often the case for this kind of systems. For a detailed construction of a mathematical model of the form (1.1) - (1.3) arising in shear banding formation in metals see \cite{15}.

A solution of (1.1)-(1.3) should be understood as a $H^{-1}$ valued stochastic process $u : [0, T] \times \Omega \to H = L^2(D)$, for $T > 0$. Then questions like local existence and uniqueness of a solution of (1.1)-(1.3) arise. The regularity in the PDE sense with respect to space and time of such a solution is also a very interesting issue to be addressed. In particular, for proving the occurrence of finite-time blow-up arising by the presence of the non-local term we need at least a $C^1$ - spatial regularity result, which only quite recently was obtained for the general case of quasilinear SPDEs in \cite{9}.

The current work mainly focuses on the phenomenon of finite-time blow-up, which in the probabilistic sense means that the expectation of the solution becomes infinitely big in finite time. Such a singular behaviour is intriguing not only from mathematical point of view but it is also very interesting from applications point of view since quite often is associated with a destructive behaviour of the described physical and biological systems. Thus the investigation of the conditions under which a finite-time blow-up occurs becomes vital and is the main aim of the present work.

Finite-time blow-up has been studied extensively in the context of deterministic parabolic PDEs. In particular, for the non-local problem

$$\frac{\partial u}{\partial t} = \Delta u + \frac{\lambda f(u)}{\left( \int_D f(u) \, dx \right)^q}, \quad (x, t) \in D_T, \quad q > 0,$$

$$u(x, t) = 0, \quad (x, t) \in \partial D \times (0, T),$$

$$u(x, 0) = \xi(x), \quad x \in D,$$

finite-time blow-up, i.e. the occurrence of $T < \infty$ such that

$$\limsup_{t \to T} ||u(\cdot, t)||_\infty = \infty,$$

was investigated in \cite{3, 16, 17, 18, 20, 21, 28}. More precisely, the authors in \cite{16} proved the occurrence of finite-time blow-up either for big values of the control parameter $\lambda$ or for big enough initial data $\xi(x)$, provided that $f(s)$ is a positive, increasing, convex function for any $s \in \mathbb{R}$ satisfying the following conditions

$$[f^{1-q}(s)]'' \geq 0 \quad \text{for} \quad s \in \mathbb{R} \quad \text{and} \quad \int_b^\infty \frac{ds}{f^{1-q}(s)} < \infty, \quad \text{for any} \quad b \in \mathbb{R}.$$
On the other hand, only a few results exist in the literature associated with the finite-time blow-up (which is rigorously defined in section 4) of the semilinear local SPDE

$$\frac{\partial u}{\partial t} = \Delta u + F(u) + \sigma(u) \partial_t W(x, t), \quad (x, t) \in D_T,$$

$$u(x, t) = 0, \quad (x, t) \in \partial D \times (0, T),$$

$$u(x, 0) = \xi(x), \quad x \in D,$$

see for example [7, 8] where the finite-time blow-up with respect to $L^p$-norms, $p > 1$, is investigated. Nevertheless, according to our knowledge there are no any results regarding the finite-time blow-up of the non-local problem (1.1)-(1.3), hence the current paper initiates an investigation on the blow-up behaviour of some stochastic non-local problems associated with a variety of real world applications.

The structure of the paper is as follows. Section 2 contains an introduction into the function spaces and the noise terms used throughout the manuscript. In Section 3 we study the local-in-time existence and uniqueness of solutions of problem (1.1)-(1.3). Section 4 is devoted to the derivation of finite-time blow-up results for (1.1)-(1.3). In particular, in subsection 4.1 we prove the occurrence of finite-time blow-up in the case the stochastic term is big enough. On the other hand, the case of non-local term induced finite-time blow-up is studied in subsection 4.2, where also some auxiliary results like maximum principle and Hopf’s lemma for parabolic SPDEs are provided. Indeed, it is the first time that Hopf’s lemma is stated and proven in the context of SPDEs and it is used obtain an estimate of solution $u$ near the boundary $\partial \Omega$ by applying the moving plane method.

2. Functional Setting

In the following we define some function spaces needed in our further analysis. Let $C^{\bar{\alpha}, \bar{\beta}}([0, T] \times \overline{D}), 0 < \bar{\alpha} \leq 1, \quad 0 < \bar{\beta} \leq 1$ denote Hölder spaces equipped with the norm

$$\|g\|_{C^{\bar{\alpha}, \bar{\beta}}} = \sup_{(t,x)} |g(t, x)| + \sup_{(t,x) \neq (s,y)} \frac{|g(t, x) - g(s, y)|}{|t - s|^{\bar{\alpha}} + |x - y|^{\bar{\beta}}}.$$

With usual modifications, we can also consider the case for $\bar{\alpha}, \bar{\beta} \geq 1$. Note that it holds

$$C^\alpha([0, T]; C^{\beta}(\overline{D})) \nsubseteq C^{\bar{\alpha}, \bar{\beta}}([0, T] \times \overline{D}),$$

and therefore we have to distinguish these two spaces.

Also for each $t > 0$ and for all real numbers $p, r \geq 1$ we define the space

$$L^{p,r}(D \times [0, t]) = \left\{ h : \|h\|_{p,r,t} := \left( \int_0^t \|h(\cdot, s)\|_p^r ds \right)^{1/r} < \infty \right\},$$

where $\| \cdot \|_p$ denotes the $L^p(D)$ norm. In the limiting case $p = r = \infty$ we define

$$L^{\infty,\infty}(D \times [0, t]) = \left\{ h : \|h\|_{\infty,\infty,t} := \text{ess sup}_{s \in [0,t]} \|h(\cdot, s)\|_{\infty} < \infty \right\}.$$
For some parameter $\theta \in [0, 1)$ and for space dimension $d = 1, 2$

$$\Gamma_\theta^* = \left\{ (p, r) \in [1, \infty]^2 : \frac{2^*}{2^* - 2p} + \frac{1}{r} = 1 - \theta \right\},$$

where $2^*$ may be any number in $(2, +\infty)$ if $d = 2$ and $\frac{2^*}{2^* - 2} = 1$ if $d = 1$. While for $d \geq 3$

$$\Gamma_\theta^* = \left\{ (p, r) \in [1, \infty]^2 : d \frac{2}{2p} + \frac{1}{r} = 1 - \theta \right\}.$$

In each case we define the functional space

$$L^*_\theta = \sum_{(p, r) \in \Gamma^*_\theta} L^{p, r}(D \times [0, t])$$

endowed with the norm

$$||h||_{L^*_\theta, t} = \inf \left\{ \sum_{i=1}^n ||h_i||_{p_i, r_i, t} : h = \sum_{i=1}^n h_i, h_i \in L^{p_i, r_i}(D \times [0, t]), (p_i, r_i) \in \Gamma^*_\theta, i = 1, \ldots, n; n \geq 1 \right\}.$$

For more details on this function space see the Appendix in [10].

Let, for any $p > 1, r \geq 0$

$$H_r^p(D) = \left\{ h : ||h||_{H_r^p(D)} = \inf \{ ||g||_{H_r^p(\mathbb{R}^d)}, g|_D = h \} < \infty \right\},$$

where, the so called Bessel potential space, is defined as

$$H_r^p(\mathbb{R}^d) = \left\{ h : (I - \Delta)^{r/2} h \in L^p(\mathbb{R}^d) \right\},$$

for

$$(I - \Delta)^{r/2} h := \mathcal{F}^{-1} \left( (1 + |\xi|^2)^{r/2} \hat{h} \right).$$

Here $\hat{h}$ denotes the Fourier transform of $h$, i.e., $\hat{h} = \mathcal{F}(h)$, and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. The choice of this scale of function spaces is more natural for our method than the Sobolev spaces $W_r^p(D)$.

Let $1 < r \leq 2$ then we say that a Banach space $X$ is $r$-smooth if the modulus of smoothness [26]

$$\rho_{||\cdot||}(t) = \sup \left\{ \frac{1}{2} \left( ||x + ty|| + ||x - ty|| \right) - 1 : ||x|| = ||y|| = 1, \forall x, y \in X \right\},$$

satisfies $\rho_{||\cdot||}(t) \leq C t^r$, for all $t > 0$, see [26].

Let $K$ be a separable Hilbert space and $X$ be a 2-smooth Banach space. Let us denote $\gamma(K, X)$ the space of the $\gamma$-radonifying operators from $K$ to a 2-smooth Banach space $X$. Here a bounded linear operator $\Psi : H \to X$ is called radonifying, and we denote $\Psi \in \gamma(K, X)$, if the series

$$\sum_{j=1}^\infty \beta_j \Psi(x_j)$$
converges in $L^2(\Omega, X)$, for any sequence $\{\beta_j\}_{j=1}^\infty$ of independent Gaussian real-valued random variables on the probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ and any orthonormal basis $\{\chi_j(x)\}_{j=1}^\infty$ of $K$.

We recall that the Bessel potential spaces $H^p_r(D), p \geq 2, r > 0$ belong to the class of 2-smooth Banach spaces since they are isomorphic to $L^r(0,1)$ according to [29, Theorem 4.9.3] and hence they are well suited for the stochastic Itô integration (see [5] for the precise construction of the stochastic integral).

Let us introduce the noises considered in this work. The $Q$-Wiener process $\{W(x,t) : t \geq 0\}$ is defined as a $U$-valued process, where $Q \in L(U)$ is non-negative definite and symmetric and has an orthonormal basis $\chi_j(x) \in U, j = 1, 2, 3, \ldots$ of eigenfunctions with corresponding eigenvalues $\gamma_j \geq 0, j = 1, 2, 3, \ldots$ such that $\text{Tr}(Q) = \sum_{j=1}^{\infty} \gamma_j < \infty$. (i.e., $Q$ is of trace class). It is well-known that $W(x,t)$ is a $Q$-Wiener process if and only if

$$W(x,t) = \sum_{j=1}^{\infty} \gamma_j^{1/2} \chi_j(x) \beta_j(t), \quad \text{almost surely (a.s.)},$$

where $\beta_j(t)$ are independent and identically distributed $\mathcal{F}_t$-Brownian motions and the series converges in $L^2(\Omega, U)$. We point out that the eigenfunctions $\{\chi_j(x)\}_{j=1}^\infty$ may be different from the eigenfunctions $\{\phi_j(x)\}_{j=1}^\infty$ of the elliptic operator $A = -\Delta$. Note that the trace class operator $Q$ is also a Hilbert-Schmidt operator.

For each Hilbert-Schmidt operator $Q$ on $H = L^2(D)$, there exists a kernel $q(x,y)$ such that, [25 Definition 1.64]

$$(Qu)(x) := \int_D q(x,y)u(y) \, dy, \quad \text{for any } x \in D, u \in H,$$

and

$$\|Q\|_{HS} = \|q\|_{L^2(D \times D)}.$$ 

The kernel $q(x,y)$ is also called the covariance function of the $Q$-Wiener process $W(x,t)$ and $\| \cdot \|_{HS}$ denotes the Hilbert-Schmidt norm.

For the purposes of subsection 4.2, we consider the following infinite-dimensional spatial independent Wiener process in $U = \mathbb{R}^\infty$ defined by

$$W(t) = \sum_{j=1}^{\infty} \beta_j(t) e_j,$$

where $e_j = (0, \ldots, 0, 1, 0, \ldots) \in \mathbb{R}^\infty$ are the orthonormal basis in $\mathbb{R}^\infty$. We note that for each of the above Wiener process there hold,

$$\mathbb{E}[(W,g)_U] = 0, \quad \text{for any } g \in U,$$

where $\mathbb{E}$ stands for the expectation.
We can interpret (1.1)-(1.3) as an abstract Itô equation in a proper Hilbert space which is actually a more appropriate formulation for the presentation of most of our results. We set $H = L^2(D)$ the Hilbert space equipped with an inner product and norm denoted by $(\cdot, \cdot)_H$ and $|| \cdot ||_H$ respectively and define

$$F : H \to H, \quad F(u)(x) = \frac{f(u(x))}{(\int_D f(u(x)) \, dx)^{\frac{q}{2}}}, \quad \text{for } x \in D,$$

and

$$G : H \to H, \quad G(u)(x) = b(u(x)), \quad \text{for } x \in D,$$

for all $u \in H$. Here $f, b : \mathbb{R} \to \mathbb{R}$ are local Lipschitz continuous functions, i.e. for any $s_0 \in \mathbb{R}$ there exist $\delta > 0$ and $C_f, C_b > 0$ such that for any $s_1, s_2 \in \{ s \in \mathbb{R} : |s - s_0| < \delta \}$ there holds

$$|f(s_1) - f(s_2)| \leq C_f|s_1 - s_2| \quad \text{and} \quad |b(s_1) - b(s_2)| \leq C_b|s_1 - s_2|.$$  

Furthermore, if $U$ is a separable Hilbert space with norm $\| \cdot \|_U$ and inner product $(\cdot, \cdot)_U$ then we define

$$\sigma : H \to L(U, H), \quad (\sigma(u)(w))(x) = (G(u))(x)w(x),$$

for every $x \in D, \ u \in H$ and $w \in U$, where $L(U, H)$ denotes the set of bounded operators from $U \to H$.

The problem (1.1)-(1.3) can be considered as an Itô equation in the Hilbert space $H = L^2(D)$ and is rewritten by suppressing the dependence on space as

$$du_t = [-Au_t + F(u_t)] \, dt + \sigma(u_t) \, dW_t, \quad 0 < t < T, \quad (3.3)$$

$$u_0 = \xi, \quad (3.4)$$

where $u_t = u(\cdot, t)$ can be interpreted as a predictable $H$-valued stochastic process and the linear operator $A = -\Delta : D(A) \subset H \to H$ is self-adjoint, positive definite with compact inverse. Moreover, $-A$ is the generator of an analytic semigroup $E(t) = e^{-tA}$ on $H$.

We now introduce the following two different notions of solution (3.3)-(3.4), see [25, Definitions 10.18 and 10.19].

**Definition 3.1.** A predictable $H$-valued stochastic process $\{u_t : t \in [0, T]\}$ is called a strong solution of (3.3)-(3.4) if for any $t \in [0, T]$,

$$u_t = \xi + \int_0^t [-Au_s + F(u_s)] \, ds + \int_0^t \sigma(u_s) \, dW_s, \quad (3.5)$$

where the last integral in (3.5) is a stochastic integral which is well defined, see Theorem 2.4 in [6].
Definition 3.2. A predictable $H$-valued stochastic process $\{u_t : t \in [0,T]\}$ is called a weak solution of (3.3)-(3.4) if for any $v \in \mathcal{D}(A)$, $t \in [0,T]$,

$$
(u_t, v) = (\xi, v) + \int_0^t \left[ -(u_s, Av) + (F(u_s), v) \right] ds + \int_0^t (\sigma(u_s) dW_s, v). \tag{3.6}
$$

The weak (variational) formulation (3.6) can be easily obtained by the integral formulation (3.5) by testing it with any $v \in \mathcal{D}(A)$, and is more appropriate for our study on finite-time blow-up.

It is also known that the weak solution of (3.3)-(3.4) is the mild solution of (3.3)-(3.4), that is,

$$
u(t) = E(t)\xi + \int_0^t E(t-s)F(u(s)) ds + \int_0^t E(t-s)\sigma(u(s)) dW(s),
$$

where $E(t) = e^{-tA}$ is the analytic semigroup generated by $-A$, see [25].

Before proceeding with the local existence result we prove a lemma which will be used frequently throughout this section.

Lemma 3.3. Assume that $f$ satisfies condition (3.2) and that it is bounded below by a positive constant, i.e. $f(s) \geq m > 0$, $s \in \mathbb{R}$. Then operator $F$ defined by (3.1), satisfies a locally Lipschitz condition. In particular, for any $u_0 \in H$ there exist $\delta > 0$ and $C_F > 0$ such that for any $u_1, u_2 \in B_{u_0, \delta} = \{u \in H : ||u - u_0||_\infty < \delta\}$ there holds

$$
||F(u_1) - F(u_2)||_H \leq C_F ||u_1 - u_2||_H. \tag{3.7}
$$

Proof. We have

$$
|F(u_1)(x) - F(u_2)(x)| = \left| \frac{f(u_1(x))}{(\int_D f(u_1(x)) dx)^{q}} - \frac{f(u_2(x))}{(\int_D f(u_2(x)) dx)^{q}} \right|

\leq \frac{|f(u_1(x)) - f(u_2(x))|}{(\int_D f(u_1(x)) dx)^{q}}

+ \frac{|f(u_2(x))|}{(\int_D f(u_1(x)) dx)^{q}} \left| \left( \int_D f(u_1(x)) dx \right)^{q} - \left( \int_D f(u_2(x)) dx \right)^{q} \right|

\leq C_F (m|D|)^{-q} |u_1(x) - u_2(x)|

\leq C_F (m|D|)^{-q} \left| \int_D (f(u_1(x)))^{q} - \left( \int_D f(u_2(x)) dx \right)^{q} \right|. \tag{3.8}
$$
Note also that, via mean value theorem and (3.2) we obtain
\[
\left| \left( \int_D f(u_1(x)) \, dx \right)^q - \left( \int_D f(u_2(x)) \, dx \right)^q \right|
\]
\[
= q \left( \int_D f(\bar{u}(x)) \, dx \right)^{q-1} \left| \int_D f(u_1(x)) \, dx - \int_D f(u_2(x)) \, dx \right|
\]
\[
\leq q \left( \int_D f(\bar{u}(x)) \, dx \right)^{q-1} \int_D |f(u_1(x)) - f(u_2(x))| \, dx
\]
\[
\leq C_f q \left( \int_D f(\bar{u}(x)) \, dx \right)^{q-1} \int_D |u_1(x) - u_2(x)| \, dx
\]
\[
\leq \tilde{C}_f \int_D |u_1(x) - u_2(x)| \, dx,
\]
where \( \bar{u}(x) \) is a value between \( u_1(x) \), \( u_2(x) \). Note that if \( 0 < q < 1 \) then \( \tilde{C}_f = C_f q(m|D|)^{q-1} \), otherwise if \( q \geq 1 \) then we take \( \tilde{C}_f = C_f q(M|D|)^{q-1} \) where \( M = \sup_{x \in D} \{ f(u(x)) \} \).

Combining now (3.8) and (3.9) and using Hölder’s inequality we finally derive
\[
\|F(u_1) - F(u_2)\|_H \leq C_F \|u_1 - u_2\|_H, \quad \text{whenever} \quad u_1, u_2 \in B_{u_0, \delta}.
\]

The proof of Lemma 3.3 is now complete. \( \Box \)

Let us first ensure the local-in-time existence of a weak solution to (3.3)-(3.4).

**Theorem 3.4.** (Local Existence) Assume that \( \xi \) is a \( \mathcal{F}_0 \)-random variable in \( H \) with \( \xi \in L^2(\Omega; L^\infty(D)) \) and (3.7) holds. Assume also that \( \sigma : H \to HS(U, H) \) is a locally Lipschitz continuous mapping, i.e.
\[
\|\sigma(u_1) - \sigma(u_2)\|_{HS(U, H)} \leq C_\sigma \|u_1 - u_2\|_H.
\]

Here \( HS(U, H) \) is defined by
\[
HS(U, H) = \left\{ \phi \in L(U, H) : \|\phi\|^2_{HS(U, H)} = \sum_{k=1}^{\infty} \|\phi(\chi_k)\|^2_H < \infty \right\},
\]
where \( \{\chi_k\}_{k=1}^\infty \subset U \) is an orthonormal basis in \( U \). Then there exists \( T > 0 \) such that (3.3)-(3.4) has a unique solution \( u \in L^2((0, T); L^\infty(D) \cap H^1_2(D)) \cap L^\infty((0, T); L^2(D)) \) which is actually a weak solution. Moreover \( u \) admits \( L^2(D) \)-continuous trajectories and satisfies the following estimates, with some suitable positive constant \( C = C(\sigma, F) \),
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|u_t\|^2_{L^2(D)} \right] + \mathbb{E} \left[ \int_0^T \|\nabla u_t\|^2_{L^2(D)} \, dt \right] \leq C \mathbb{E} [\|\xi\|^2_{L^2(D)}]. \tag{3.11}
\]

Thus
\[
u \in L^2\left( \Omega; C([0, T]; L^2(D)) \right) \cap L^2\left( \Omega; L^2((0, T); L^\infty(D) \cap H^1_2(D)) \right)
\]
and moreover
\[ u \in L^p(\Omega; L^\infty((0, T); L^p(D))), \quad \text{for} \quad p \geq 2. \]

To prove Theorem 3.4, we need the following version of Itô Lemma in a Hilbert space.

**Lemma 3.5** ([6]). Assume that \( F \) and \( \sigma \) satisfies (3.7) and (3.10) respectively. Assume that \( \xi \) is \( \mathcal{F}_0 \)-random variable in \( H \). Further assume that \( u \) satisfies the Itô process
\[ du_t = (-Au_t + F(u_t)) \, dt + \sigma(u_t) \, dW_t, \quad u_0 = \xi. \]
Let \( \psi : H \to \mathbb{R} \) be a \( C^2(H, \mathbb{R}) \) functional. Then we have
\[ d\psi(u_t) = \psi'(u_t) \left[ (-Au_t + F(u_t)) \, dt + \sigma(u_t) \, dW_t \right] + \frac{1}{2} \text{Tr} \left( \sigma(u_t)^* \psi''(u_t) \sigma(u_t) \right) \, dW_t. \]

**Proof of Theorem 3.4** We follow the idea of the proof of [10, Theorem 3] where the quasilinear stochastic parabolic case was investigated by the energy method. However, here we consider the stochastic semilinear parabolic case and use the semigroup approach instead.

Denote
\[ S_T = \left\{ u : u \in L^2(\Omega \times [0, T]; L^\infty(D) \cap H^1_0(D)) \right\}, \]
equipped with the norm
\[ \| u_t \|^2_{\gamma, \delta} := \mathbb{E} \left[ \int_0^T e^{-\gamma t} \left( \| u_t \|^2_{L^2(D)} + \delta \| \nabla u_t \|^2_{L^2(D)} \right) \, dt \right], \]
where \( H^1_0(D) = \{ v : v \in H^1_2(D), \, v|_{\partial D} = 0 \}. \)

It is clear that \( \| \cdot \|_{\gamma, \delta} \) is equivalent to \( \| \cdot \|_{S_T} \), where
\[ \| u_t \|^2_{S_T} := \mathbb{E} \left[ \int_0^T \| u_t \|^2_{H^1_0(D)} \, dt \right]. \]

Define the map \( \mathcal{M} \) by
\[ \mathcal{M}(u_t) := E(t) \xi + \int_0^t E(t-s)F(u_s) \, ds + \int_0^t E(t-s)\sigma(u_s) \, dW_s, \]
where \( E(t) \) is the semigroup generated by \(-A\).

In the following we employ Banach’s fixed point theorem to prove existence and uniqueness of the equation \( \mathcal{M}(u_t) = u_t \) in \( S_T \).

**Step 1:** Show that \( \mathcal{M} : S_T \to S_T. \)
For any \( u_t \in S_T \), we need to show \( \mathcal{M}(u_t) \in S_T \), i.e.,
\[ \| \mathcal{M}(u_t) \|^2_{S_T} = \mathbb{E} \left[ \int_0^T \| u_t \|^2_{H^1_0(D)} \, dt \right] < \infty, \]
which follows by the assumptions on \( \xi, F \) and \( \sigma \).
Step 2: We show that there are positive constants $\gamma, \delta$ and $0 < \kappa < 1$ such that
\[
\| \mathcal{M}(u_t) - \mathcal{M}(v_t) \|_{\gamma, \delta} \leq \kappa \| u_t - v_t \|_{\gamma, \delta},
\]
where $\kappa = \kappa(F, \sigma)$ depends on $F$ and $\sigma$.

In fact, we have
\[
\tilde{u}_t := \mathcal{M}(u_t) - \mathcal{M}(v_t) = \int_0^t E(t-s) \left( F(u_s) - F(v_s) \right) \, ds
+ \int_0^t E(t-s) \left( \sigma(u_s) - \sigma(v_s) \right) \, dW_s,
\]
which implies that
\[
d\tilde{u}_t + A\tilde{u}_t \, dt = [F(u_t) - F(v_t)] \, dt + [\sigma(u_t) - \sigma(v_t)] \, dW_t.
\]
Let $w_t = \tilde{u}_t e^{-\frac{\gamma t}{2}}$, then $w_t$ satisfies
\[
dw_t + Aw_t \, dt = -\frac{\gamma}{2} w_t \, dt + [F(u_t) - F(v_t)] e^{-\frac{\gamma t}{2}} \, dt + [\sigma(u_t) - \sigma(v_t)] e^{-\frac{\gamma t}{2}} \, dW_t.
\]
By virtue of Itô formula, see Lemma 3.3 with $\varphi(w_t) = \|w_t\|^2$, $w_t = \tilde{u}_t e^{-\gamma t}$, we deduce
\[
e^{-\gamma T} \| \tilde{u}_t \|_{L^2(D)}^2 + 2 \int_0^T e^{-\gamma t} \| \nabla \tilde{u}_t \|_{L^2(D)}^2 \, dt
+ \gamma \int_0^T e^{-\gamma t} \| \tilde{u}_t \|_{L^2(D)}^2 \, dt + 2 \int_0^T e^{-\gamma t} \langle \tilde{u}_t, F(u_t) - F(v_t) \rangle \, dt
+ \int_0^T e^{-\gamma t} \| \sigma(u_t) - \sigma(v_t) \|^2_{H^1(U; H)} \, dt.
\]
Now for any small $\epsilon > 0$, by using Young’s inequality we obtain some constant $C_\epsilon$ depending on $\epsilon$ such that
\[
2 \int_0^T e^{-\gamma t} \langle \tilde{u}_t, F(u_t) - F(v_t) \rangle \, dt
\leq \epsilon \int_0^T e^{-\gamma t} \| F(u_t) - F(v_t) \|^2_{L^2(D)} \, dt + C_\epsilon \int_0^T e^{-\gamma t} \| \tilde{u}_t \|_{L^2(D)}^2 \, dt
\leq \epsilon C_F \int_0^T e^{-\gamma t} \| u_t - v_t \|^2_{L^2(D)} \, dt + C_\epsilon \int_0^T e^{-\gamma t} \| \tilde{u}_t \|_{L^2(D)}^2 \, dt,
\]
taking also into account that $F$ satisfies a locally Lipschitz condition with constant $C_F$, by Lemma 3.3.

Moreover due to the assumption satisfied by $\sigma$ we have
\[
\int_0^T e^{-\gamma t} \| \sigma(u_t) - \sigma(v_t) \|^2_{H^1(U; H)} \, dt \leq C_\sigma \int_0^T e^{-\gamma t} \| u_t - v_t \|^2_{L^2(D)} \, dt.
\]
Thus we derive with expectation
\[
\gamma \mathbb{E} \left[ \int_0^T e^{-\gamma t} \| \bar{u}_t \|^2_{L^2(D)} \, dt \right] - C_\epsilon \mathbb{E} \left[ \int_0^T e^{-\gamma t} \| \bar{u}_t \|^2_{L^2(D)} \, dt \right] + 2 \mathbb{E} \left[ \int_0^T e^{-\gamma t} \| \nabla \bar{u}_t \|^2_{L^2(D)} \, dt \right] \\
\leq \epsilon C_F \mathbb{E} \left[ \int_0^T e^{-\gamma t} \| u_t - v_t \|^2_{L^2(D)} \, dt \right] + C_\sigma \mathbb{E} \left[ \int_0^T e^{-\gamma t} \| u_t - v_t \|^2_{L^2(D)} \, dt \right],
\]
or equivalently
\[
\mathbb{E} \left[ \int_0^T e^{-\gamma t} \| \bar{u}_t \|^2_{L^2(D)} \, dt \right] + \frac{2}{\gamma - C_\epsilon} \mathbb{E} \left[ \int_0^T e^{-\gamma t} \| \nabla \bar{u}_t \|^2_{L^2(D)} \, dt \right] \\
\leq \epsilon \frac{C_F + C_\sigma}{\gamma - C_\epsilon} \mathbb{E} \left[ \int_0^T e^{-\gamma t} \| u_t - v_t \|^2_{L^2(D)} \, dt \right].
\]

Choosing \( \gamma \) sufficiently large and suitable \( \epsilon > 0 \) such that \( 0 < \frac{\epsilon C_F + C_\sigma}{\gamma - C_\epsilon} < \kappa < 1 \) and taking \( \delta = \frac{2}{\gamma - C_\epsilon} \), we have
\[
\mathbb{E} \left[ \int_0^T e^{-\gamma t} \left( \| \bar{u}_t \|^2_{L^2(D)} + \delta \| \nabla \bar{u}_t \|^2_{L^2(D)} \right) \, dt \right] \\
\leq \kappa \mathbb{E} \left[ \int_0^T e^{-\gamma t} \left( \| u_t - v_t \|^2_{L^2(D)} + \delta \| u_t - v_t \|^2_{L^2(D)} \right) \, dt \right],
\]
that is
\[
\| \mathcal{M}(u_t) - \mathcal{M}(v_t) \|_{\gamma, \delta} \leq \kappa \| u - v \|_{\gamma, \delta}, \quad 0 < \kappa < 1.
\]
Then by Banach’s fixed point theorem, there exists a unique solution of problem (3.3)–(3.4) \( u \in \mathcal{S}_T \). Finally, the estimates (3.11) can be obtained by following a similar argument as in the proof of Theorem 3 in [10].

Step 3: To show \( u \in L^p(\Omega; L^\infty((0, T); L^p(D))) \), \( p \geq 2 \), we can employ again Banach’s fixed point theorem as above. In particular, the fact that \( b : \mathbb{R} \to \mathbb{R} \) satisfies a local Lipschitz condition combined with Lemma 3.3 actually implies that
\[
\| F(u_t) - F(v_t) \|_{L^p(D)} \leq C_F \| u_t - v_t \|_{L^p(D)},
\]
and
\[
\| \sigma(u_t) - \sigma(v_t) \|_{\gamma(U; L^p(D))} \leq C_\sigma \| u_t - v_t \|_{L^p(D)}.
\]
Thus the same reasoning exploited in steps 1 and 2 can be used to obtain the desired result. The proof of the theorem is complete. \( \square \)

To obtain higher spatial regularity we now assume that \( (\sigma) \) \( \sigma : H \to \gamma(U; H^r_p(D)) \) satisfies the linear growth condition, with \( p \geq 2 \), i.e.
\[
\| \sigma(u) \|_{\gamma(U; H^r_p(D))} \leq C(1 + \| u \|_{H^r_p(D)}), \quad 0 \leq r \leq 1.
\]
The following regularity result follows the same reasoning with Theorem 5.1 in [9]. For reader’s convenience we give a proof below adjusted to our circumstances.
Theorem 3.6. Let $p \geq 2$. Assume that $(3.3)-(3.4)$ has a weak solution $u \in L^2(\Omega; C([0, T]; L^2(D))) \cap L^2(\Omega; L^2((0, T); L^\infty(\Omega) \cap H^1_2(D))) \cap L^p(\Omega; L^\infty((0, T); L^p(D)))$.

Further assume that condition $(\sigma)$ holds and that $f$ satisfies $(3.2)$. If $\xi \in L^m(\Omega; C^{1+l}(\overline{D}))$, for $m \geq 2$, $l > 0$, then for all $\alpha \in (0, 1/2)$, there exists $\beta > 0$ such that

$$u \in L^m(\Omega; C^{\alpha,1+\beta}([0, T] \times \overline{D})), \quad m \geq 2. \quad (3.13)$$

Proof. We first show that, there exists $\eta > 0$ such that

$$u \in L^m(\Omega; C^\eta([0, T] \times \overline{D})), \quad m \geq 2. \quad (3.14)$$

Set $u = y + z$, where $z$ solves the following linear SPDE problem

$$dz_t = -A z_t dt + \sigma(u_t) \, dW_t, \quad 0 < t < T,$$

$$z_0 = 0,$$

and $y$ is the unique solution of the linear PDE problem

$$\partial_t y_t = -Ay_t + F(u_t), \quad 0 < t < T,$$

$$y_0 = \xi.$$

Step 1. Hölder regularity of $z$. By Theorem 3.4, the weak solution $u$ of $(3.3)-(3.4)$ belongs to $L^m(\Omega; L^m((0, T); L^m(D)))$, $m \geq 2$. The assumption $(\sigma)$ with $r = 0$ implies that $\sigma(u)$ belongs to $L^m(\Omega; L^m((0, T); \gamma(U, L^m(D))))$. Hence we can derive the Hölder’s regularity for the stochastic integral

$$z_t = \int_0^t E(t - s) \sigma(u_s) \, dW_s. \quad (3.15)$$

More precisely, taking also into account the linear growth of $\sigma$, we have, by using factorization method, see [5, Corollary 3.5],

$$\mathbb{E} \left[ \|z\|_{C^\gamma([0, T]; H^\delta_m(D))}^m \right] \leq C \left( 1 + \mathbb{E} \|u\|_{L^m((0, T); L^m(D))}^m \right),$$

where $\gamma \in [0, \frac{1}{2} - \frac{1}{m} - \frac{\delta}{2})$, $\delta \in (0, 1 - \frac{2}{m})$, $m > 2$. Assume that $m \geq 3$, then $\delta = \frac{1}{6}, \gamma = \frac{1}{12}$ satisfy the conditions above for any $m \geq 3$. Choose $m \geq m_0, m_0 = 7d$, where $d$ is the spatial dimension. Also take $\alpha = \delta - \frac{d}{m_0}$, then by Sobolev’s embedding theorem, we have $H^\delta_m(D) \hookrightarrow C^\alpha(D)$,

since $\delta - \frac{d}{m} > \delta - \frac{d}{m_0} = \alpha$. Thus for any $m \geq m_0$,

$$\mathbb{E} \left[ \|z\|_{C^\gamma([0, T]; C^\alpha(D))}^m \right] \leq C \left( 1 + \mathbb{E} \left[ \|u\|_{L^m((0, T); L^m(D))}^m \right] \right) < \infty.$$

For $m \in [2, m_0)$, we have

$$\mathbb{E} \left[ \|z\|_{C^\gamma([0, T]; C^\alpha(D))}^m \right] \leq \left( \mathbb{E} \left[ \|z\|_{C^\gamma([0, T]; C^\alpha(D))}^{m_0} \right] \right)^{m/m_0} < \infty.$$
Thus for any $m \geq 2$, we have
\[ \mathbb{E} \left[ \|z\|_{C^\gamma([0,T];C^{\alpha}(D))}^m \right] < \infty. \] (3.16)

Step 2. Hölder regularity of $y$. Due to Lemma 3.3 the functional $F$ satisfies a locally Lipschitz condition and hence the following estimate is valid
\[ \mathbb{E} \left[ \|F(u)\|_{L^r((0,T);L^r(D))}^r \right] \leq C \left( 1 + \mathbb{E} \left[ \|u\|_{L^r((0,T);L^r(D))}^r \right] \right) < \infty, \quad r \geq 2. \]

By choosing $r \geq r_0$ such that $\frac{2+d}{r} < \frac{1}{2}$, we have by classical parabolic PDE theory (see Theorems 10.1 and 7.1 in [23]),
\[ \|y\|_{C^{\alpha/2,\alpha}(0,T) \times \overline{D}} \leq C \left( 1 + \|\xi\|_{C^\eta(\overline{D})} \right) \left( 1 + \|F(u)\|_{L^r((0,T);L^r(D))}^{2d+1} \right), \quad r \geq r_0, \]
for some $\alpha > 0$, or if $2(2d+1)m < r$,
\[ \|y\|_{C^{\alpha/2,\alpha}(0,T) \times \overline{D}}^m \leq C \left( 1 + \|\xi\|_{C^\eta(\overline{D})}^{2m} \right) \left( 1 + \|F(u)\|_{L^r((0,T);L^r(D))}^r \right). \]

Since $r$ is arbitrary in $[r_0, \infty)$, we get
\[ \mathbb{E} \left[ \|y\|_{C^{\alpha/2,\alpha}(0,T) \times \overline{D}}^m \right] < \infty, \quad \text{for any} \quad m \in [2, \infty). \] (3.17)

Choose now $\eta = \min\{\frac{\alpha}{2}, \gamma, \lambda\} > 0$, then taking into account (3.16) and (3.17) we derive
\[ u \in C^\eta([0,T];C^\eta(\overline{D})) \subset C^\eta([0,T] \times \overline{D}). \]

Step 3. Higher spatial Hölder regularity of $z$. Now with the estimate (3.14) in hand we conclude that $u$ belongs to $L^m(\Omega;L^m((0,T);H^k_m(D)))$ for $k < \eta < 1/2$, hence by the assumption $(\sigma)$, we have $\sigma(u) \in L^m(\Omega;L^m((0,T);\gamma(U,H^k_m(D))))$. Using again the factorization method [5, Corollary 3.5], we obtain
\[ \mathbb{E} \left[ \|z\|_{C^{\gamma}(0,T);H^{\delta+k}(D)}^m \right] \leq C \left( 1 + \mathbb{E} \left[ \|u\|_{L^m((0,T);H^k_m(D))}^m \right] \right), \]
where $\gamma \in [0,1] - \frac{1}{m} - \frac{\delta}{m}$, $\delta \in (0,1 - \frac{2}{m})$, $m > 2$. In the sequel we assume $m \geq m_0 := (d+4)/k$. Thus $\delta = 1 - 3/m_0$ and $\gamma = 1/(4m_0)$ satisfy the conditions above uniformly in $m \geq m_0$. Choosing $\theta := k + 1 - \frac{d+4}{m_0} > 1$ and using the Sobolev embedding
\[ H^\delta_m(D) \hookrightarrow C^\theta(D), \quad \theta = k + \delta - d/m_0, \]
we finally deduce, for some $0 < \gamma < 1/2$,
\[ \mathbb{E} \left[ \|z\|_{C^{\gamma}(0,T);C^{\theta}(D)}^m \right] \leq C \left( 1 + \mathbb{E} \left[ \|u\|_{L^m((0,T);H^k_m(D))}^m \right] \right), \quad m \geq 2, \]
i.e.
\[ z \in L^m(\Omega;C^{\gamma,\theta}([0,T] \times \overline{D})). \] (3.18)

Step 4. Higher spatial Hölder regularity of $y$. Taking estimate (3.14) as starting point and using Schauder’s theory for deterministic parabolic PDEs [24, Theorem 6.48] as well as the linear growth condition on $F$ we derive
\[ \|y\|_{C^{(1+\alpha)/2,1+\alpha}(0,T) \times \overline{D}} \leq C \left( 1 + \|\xi\|_{C^{1+\alpha}(\overline{D})} + \|F(u)\|_{L^r((0,T);L^r(D))}^r \right). \]
for \( r \geq 2 \) large enough, hence
\[
y \in L^m \left( \Omega; C^{(1+\alpha)/2,1+\alpha}([0,T] \times \overline{D}) \right), \quad m \geq 2,
\]
which combined with (3.18) implies
\[
u \in L^m \left( \Omega; C^{\gamma,1+\beta_1}([0,T] \times \overline{D}) \right),
\]
with \( \beta_1 = \min\{\theta - 1, \alpha\} \).

**Step 5. Time regularity.** For any \( \gamma \in (0,1/2) \), due to (3.19), it suffices to improve only the time regularity of \( z \). By following the same arguments employed in step 1 for the stochastic integral and using estimate (3.14) we deduce
\[
E \left[ \|z\|^{m}_{C^{\gamma}([0,T];H^{1+k}_{\alpha}(D))} \right] < \infty,
\]
which, via the Sobolev embedding \( H^{1+k}_{\alpha}(D) \hookrightarrow C^{1+\beta}(D), \beta < k \), implies that
\[
z \in L^m \left( \Omega; C^{\gamma,1+\beta}([0,T] \times \overline{D}) \right), \quad m \geq 2.
\]
Combining now the above estimate with (3.19) we obtain the desired regularity for \( u \). The proof of Theorem 3.6 is complete. \( \square \)

**Remark 3.7.** For the purposes of the current work the spatial regularity provided by Theorem 3.6 is sufficient. Nevertheless under the assumption that \( F \) is bounded, which is guaranteed due to (3.2) and (4.10), the spatial regularity for the solution \( u \) of (3.3)-(3.4) can be improved. In particular in that case for all \( \alpha \in (0,1/2) \) there exists \( \beta > 0 \) such that
\[
u \in L^m \left( \Omega; C^{\alpha,2+\beta}([0,T] \times \overline{D}) \right), \quad m \geq 2,
\]
provided also that the initial data are smoother \( \xi \), i.e. \( \xi \in L^2 \left( \Omega; C^{2+l}(\overline{D}) \right) \). Indeed, we can increase the spatial regularity of \( u \) as long as we consider smoother initial data \( \xi \) and smoother drift terms \( F \). For more details see Propositions 5.2 and 5.3 in [9].

**Remark 3.8.** If we consider initial data \( \xi(x) \geq 0 \) a.s. then our local solution \( u(x,t) \) is also a.s. positive by application of the comparison principle, see [10].

4. **Finite-time Blow-up**

In this section we investigate under which circumstances the solution of problem (3.3)-(3.4) blows up in finite time. In subsection 4.1 we focus on the impact of the noise term in the occurrence of finite-time blow-up and we actually prove that finite-time blow-up takes place once the noise term dominates. On the other hand, subsection 4.2 concentrates on the case of non-local term induced finite-time blow-up.

We first give a definition of finite-time blow-up for the stochastic process solving problem (3.3)-(3.4).
Definition 4.1. The solution $u(x,t)$ of problem (3.3)-(3.4) blows up in finite time if there exists $0 \leq t^* < \infty$ such that
\[
\limsup_{t \to t^*} \mathbb{E} [||u_t||_p] = \infty,
\]
for some $1 \leq p \leq \infty$.

4.1. Noise term induced finite-time blow-up. In this subsection we investigate the occurrence of finite-time blow-up which is due to the contribution by the noise term. For that purpose we make the following assumptions:

(S1) The correlation function $q(x,y)$ is continuous and positive for any $x,y \in \bar{D}$ and satisfies
\[
\int_D \int_D q(x,y)w(x)w(y) \, dx \, dy \geq q_1 \int_D w^2(x) \, dx
\]
for any positive $w \in H$ and for some $q_1 > 0$. This actually means that the correlation function behaves as a steep Gaussian function.

(S2) $\sigma(s)$ is convex function and there also exists a positive, strictly increasing, convex and superlinear function $G(s)$ such that
\[
\sigma^2(s) \geq 2 G(s^2) \quad \text{for} \quad s \geq 0 \quad \text{and} \quad \int_0^\infty \frac{ds}{G(s)} < \infty.
\]

Let $(\lambda_1, \phi_1(x))$ be the first eigenpair of the operator $A = -\Delta : \mathcal{D}(A) \subset H \rightarrow H$, i.e.
\[
A\phi_1 = \lambda_1 \phi_1, \quad x \in D, \quad \phi_1(x) = 0, \quad x \in \partial D.
\]
It is known that $\phi_1$ has a constant sign on $D$ so we can take $\phi_1 \geq 0$ on $\bar{D}$ and consider the normalized such that
\[
\int_D \phi_1 \, dx = 1. \tag{4.1}
\]

Now following the approach of [7] we obtain the following.

Theorem 4.2. Suppose that (3.3)-(3.4) has a (unique) local-in-time solution $u$ whose existence is guaranteed by Theorem 3.4. Assume also that conditions (S1) and (S2) hold then $u$ blows up in finite time provided that $\xi \in L^2(\Omega; L^2(D))$, $\xi(x) \geq 0$ a.s. and
\[
\theta(0) = \theta_0 = \mathbb{E} \left[ \left( \int_D \xi(x) \phi_1(x) \, dx \right)^2 \right] > \gamma
\]
where $\gamma$ is the largest root of the equation $\beta(s) := 2 \tilde{q}_1 G(s) - 2\lambda_1 s$ and $\tilde{q}_1$ is some positive constant.

Proof. Define $\hat{u}(t)$ as
\[
\hat{u}(t) := \int_D u_t \phi_1 \, dx,
\]
then taking $v = \phi_1$ as a test function into (3.6) we derive
\begin{align*}
\hat{u}(t) &= \int_D u_t \phi_1 \, dx = \int_D \xi \phi_1 \, dx - \int_0^t \int_D u_s A \phi_1 \, dx \, ds \\
&\quad + \lambda \int_0^t \int_D \frac{f(u_s) \phi_1}{\int_D f(u_s) \, dx} q \, dx \, ds + \int_0^t \int_D \sigma(u_s) \phi_1 \, dx \, dW_s \\
&= \int_D \xi \phi_1 \, dx - \lambda_1 \int_0^t \int_D u_s \phi_1 \, dx \, ds \\
&\quad + \lambda \int_0^t \int_D \frac{f(u_s) \phi_1}{\int_D f(u_s) \, dx} q \, dx \, ds + \int_0^t \int_D \sigma(u_s) \phi_1 \, dx \, dW_s. \quad (4.2)
\end{align*}

Using now Itô’s formula, i.e. Lemma 3.5, for $\psi(u) = u^2$ and taking also into account (4.2) we obtain
\begin{align*}
\hat{u}^2(t) &= \left( \int_D \xi(x) \phi_1(x) \, dx \right)^2 - 2\lambda_1 \int_0^t \hat{u}^2(s) \, ds \\
&\quad + 2\int_0^t \int_D \hat{u}^2(s) \frac{f(u_s) \phi_1(x)}{\int_D f(u_s) \, dx} q \, dx \, ds + 2\int_0^t \int_D \hat{u}(s) \sigma(u_s) \phi_1(x) \, dx \, dW_s \\
&\quad + \int_0^t \int_D \int_D q(x,y) \phi_1(x) \phi_1(y) \sigma^2(u_s) \, dx \, dy \, ds. \quad (4.3)
\end{align*}

Set $\theta(t) = \mathbb{E}[\hat{u}^2(t)]$ then by taking the expectation into (4.3) and interchanging the order of expectation and integration by virtue of Fubini’s theorem, it reads
\begin{align*}
\theta(t) &= \mathbb{E}\left[ \left( \int_D \xi(x) \phi_1(x) \, dx \right)^2 \right] - 2\lambda_1 \int_0^t \theta(s) \, ds \\
&\quad + 2\mathbb{E}\left[ \int_0^t \int_D \hat{u}(s) \frac{f(u_s) \phi_1(x)}{\int_D f(u_s) \, dx} q \, dx \, ds \right] \\
&\quad + \mathbb{E}\left[ \int_0^t \int_D \int_D q(x,y) \phi_1(x) \phi_1(y) \sigma^2(u_s) \, dx \, dy \, ds \right] \quad (4.4)
\end{align*}

using also (2.3). Alternatively (4.4) can be written in differential form as follows
\begin{align*}
\frac{d\theta}{dt} &= -2\lambda_1 \theta(t) + 2\mathbb{E}\left[ \hat{u}(t) \int_D \frac{f(u_t) \phi_1(x)}{\int_D f(u_t) \, dx} q \, dx \, ds \right] \\
&\quad + \mathbb{E}\left[ \int_D \int_D q(x,y) \phi_1(x) \phi_1(y) \sigma^2(u_t) \, dx \, dy \, ds \right], \quad t > 0 \quad (4.5)
\end{align*}

with initial condition
\begin{align*}
\theta(0) = \theta_0 = \mathbb{E}\left[ \left( \int_D \xi(x) \phi_1(x) \, dx \right)^2 \right].
\end{align*}
Using assumptions \((S_1)\) and \((S_2)\) together with \((4.1)\), Jensen’s and Hölder’s inequalities we can estimate the third term into \((4.7)\) as follows

\[
E \left[ \int_D \int_D q(x,y) \phi_1(x) \phi_1(y) \sigma^2(u_t) \, dx \, dy \, ds \right] 
\geq q_1 E \left[ \int_D \phi_1(x)^2 \sigma^2(u_t) \, dx \right] 
\geq \tilde{q}_1 E \left[ \int_D \phi_1(x) \sigma(u_t) \, dx \right]^2 
\geq 2 \tilde{q}_1 G(\theta(t)) \tag{4.6}
\]

and thus

\[
\frac{d\theta}{dt} \geq -2\lambda_1 \theta(t) + 2 \tilde{q}_1 G(\theta(t)) := \beta(\theta(t)), \quad t > 0 \tag{4.7}
\]

\[
\theta(0) = E \left[ (\xi, \phi)_H^2 \right] \tag{4.8}
\]

taking also into account that the second term in \((4.7)\) is positive, see also Remark 3.8.

Let now \(\gamma\) be the largest root of the equation \(\beta(s) = 0\) then \(\beta(s) > 0\) for any \(s > \gamma\) if \(\gamma > 0\). Otherwise \(\beta(s) > 0\) for any \(s > 0\). Therefore if we take \(\theta(0) > \gamma\) then \((4.7)-(4.8)\) implies that

\[
t \leq \int_{\theta_0}^{\infty} \frac{ds}{\beta(s)} \leq \frac{1}{N} \int_{\theta_0}^{\infty} \frac{ds}{G(s)} < \infty
\]

for some positive constant \(N\), hence \(\theta(t) \to \infty\) as \(t \to T^*\) where

\[
T^* \leq \int_{\theta_0}^{\infty} \frac{ds}{G(s)} < \infty.
\]

The latter by virtue of Hölder’s inequality implies that \(E[||u_t||_2] \to \infty\) as \(t \to t^* \leq T^*\) and this completes the proof of the Theorem. \(\square\)

**Remark 4.3.** The result of Theorem 4.2 when \(f(s) = e^s\) and \(q > 1\) is a complementary result of Theorems 4.1 and 4.2 in [4] where it is proven that when \(\sigma(s) = 0\), i.e. for the deterministic case, then only a global-in-time solution exist. Indeed, the novelty of Theorem 4.2 is that a dominant noise can change dramatically the behaviour of the solution and lead to finite-time blow-up. Moreover Theorem 4.2 ensures the occurrence of finite-time blow-up in the case \(f(s) = e^s, q = 1\), for any dimension \(d > 2\) a result that was only conjectured for the deterministic case in [18] and only proven for \(d = 2\). In the latter case problem (1.1)-(1.3) is stochastic perturbation of a problem which describes the biological phenomenon of chemotaxis and so the occurrence of finite-time blow-up describes the aggregation of a biological population.

4.2. **Non-local term induced finite-time blow-up.** In this section, we investigate the occurrence of finite-time blow-up that is induced by the presence of the non-local term. For that purpose we try to adjust the approach introduced in [16] for the deterministic case to stochastic problem (1.1)-(1.3). According to the approach in [16] the proof of the finite-time blow-up requires the validity of a critical estimate of the solution close to
the spatial boundary which is actually a byproduct of the maximum principle. Therefore in the following we concentrate on proving some auxiliary results are coming from the maximum principle of SPDES.

For the purposes of the current section we shall consider the noise \(W(t)\) defined by (2.1). Moreover the diffusion coefficients \(\sigma\) is then defined as in [13]

\[
\sigma(u) : \mathbb{R}^\infty \to H,
\]

\[
\sigma(u)h = \sum_{j=1}^{\infty} \sigma_j(u(\cdot))(e_j, h), \quad \forall h \in \mathbb{R}^\infty,
\]

where the functions \(\sigma_j : \mathbb{R} \to \mathbb{R}, j = 1, 2, 3, \ldots\) satisfy the following linear growth condition

\[
\sum_{j=1}^{\infty} |\sigma_j(\xi)|^2 \leq C(1 + |\xi|^2), \quad \xi \in \mathbb{R}. \tag{4.9}
\]

Under the condition (4.9), we may show that \(\sigma(u) \in \gamma(\mathbb{R}^\infty, H)\) for any \(u \in H\)

\[
\|\sigma(u)\|_{\gamma(\mathbb{R}^\infty; H)} \leq C(1 + \|u\|_H^2),
\]

see also [13].

4.2.1. Maximum Principle and Hopf’s Lemma. Maximum principle and comparison methods are well established in the context of parabolic PDEs and they are actually essential tools for the qualitative study of these equations. In particular, moving plane method, [22], can be employed to derive symmetry results as well as provide control of the behaviour of the solution of semilinear parabolic PDE near the boundary of the spatial domain.

On the other hand, only recently maximum principle together with comparison results were established in the context of quasilinear and semilinear parabolic SPDEs, see for example [10] [11]. Nevertheless, according to our knowledge, the literature of SPDEs lacks a Hopf’s maximum principle result. But such a result, according to the approach introduced in [16], is essential for the proofs of Theorems 4.10 and 4.11. Besides, a Hopf’s maximum principle has its own importance in the context of the study of parabolic SPDEs therefore in the following we state and prove such a result for parabolic SPDEs.

For reader’s convenience we first give a required definition as well as we recall Hopf’s maximum principle for parabolic PDEs, see also [12] [27].

**Definition 4.4.** Let \(P_0 = (x_0, t_0)\) be a point on the boundary of \(D_T\). If there exists a closed ball \(B\) centered at \((\bar{x}, \bar{t})\) such that

\[
B \subset \overline{D_T}, \quad B \cap \partial D_T = \{P_0\}, \quad \bar{x} \neq x_0,
\]

then we say that \(P_0\) has the inside strong sphere property.
**Theorem 4.5.** Let \( \mathcal{H} \) denote the heat operator \( \mathcal{H}(u) := \Delta u - \frac{\partial u}{\partial t} \). Assume that \( \mathcal{H}(u) \geq 0 \) and \( \max u = M \) is attained at \( P_0 \in \partial D_T \), i.e. \( u(P_0) = M \), where \( P_0 \) has the inside strong sphere property. Further assume that for some neighbourhood \( \mathcal{V} \) of \( P_0 \),

\[
\forall x \in D_T \cap \mathcal{V},
\]

Then we have

\[
\frac{\partial u}{\partial \nu}(P_0) < 0,
\]

where \( \nu(P_0) = \nu \) is the outer normal direction at \( P_0 \).

In the following we will need a version of maximum principle associated with the semi-linear SPDE problem under consideration. In particular we have:

**Theorem 4.6.** (Maximum Principle) Let \( V(x,t; \hat{\xi}) \) be the solution of the following SPDE

\[
dV_t = (-AV_t + G(V_t)) dt + \chi(V_t) dW_t, \quad 0 < t < T, \ T > 0
\]

\[
V_0 = \hat{\xi} \geq 0 \ a.s.,
\]

satisfying homogeneous Dirichlet boundary conditions on \( \partial \Omega \). Assume further that \( G : H \to H \) and \( \chi : H \to H \) are Lipschitz continuous functions. If \( M = M_T := \max_{x \in \partial D_T} V(x,t) \) then

\[
V(x,t) \leq M, \quad \text{for any } (x,t) \in \overline{D_T}.
\]

**Proof.** For any \( 0 < t < T \), set \( M_t = \max_{x \in \partial D_t} V(x,t) \) where \( D_t := D \times (0,t) \). Note that

\[
z(x,t) = V(x,t) - M_t
\]

satisfies

\[
dz_t = (-A z_t + \hat{G}(z_t)) dt + \hat{\chi}(z_t) dW_t, \quad 0 < t < T,
\]

\[
z_0 = \hat{\xi} - ||\hat{\xi}||_{\infty},
\]

where \( \hat{G}(z_t) := G(z_t + M_t) - G(M_t) \) and \( \hat{\chi}(z_t) := \chi(z_t + M_t) - \chi(M_t) \), since \( M_t \) solves the following SDE

\[
dM_t = G(M_t) dt + \chi(M_t) dW_t, \quad 0 < t < T,
\]

\[
M_0 = ||\hat{\xi}||_{\infty}.
\]

Then by virtue of Theorem 18 in [10] we get

\[
E \left[ ||z^+||_{\infty,\infty; t}^2 \right] \leq s(t) E \left[ ||\hat{\xi} - M_0||_{\infty}^2 + \left( ||\hat{G}(0)||_{x,0}^2 + ||\hat{\chi}(0)||_{x,0}^2 \right)^2 \right],
\]

where \( s(t) \) is a constant depends only on \( t \). Since \( \hat{G}(0) = \hat{\chi}(0) \equiv 0 \), we finally derive that

\[
E \left( ||z^+||_{\infty,\infty; t}^2 \right) = 0,
\]

where \( z^+ \) stands for the positive part of \( z \).

The latter implies that \( V(x,t) \leq M_t \leq M_T \) for any \( (x,t) \in \overline{D_T} \) and this completes the proof of the Theorem. \( \square \)
Now we are ready to prove Hopf’s maximum principle for a general semilinear parabolic SPDE. We first define the stochastic operator
\[ \mathcal{P}(u_t) := -du_t + (-Au_t + G(u_t)) dt + \chi(u_t) dW_t, \]
for any stochastic process \( u : [0, T] \times \Omega \to H = L^2(D) \). Then the following holds:

**Theorem 4.7. (Hopf’s Lemma)** Let \( u \) satisfy the following problem
\[
\begin{align*}
\frac{\partial u}{\partial t} &= -Au + G(u) + \chi(u) \partial_t W(t), \quad (x, t) \in D_T \\
u(x, t) &= 0, \quad (x, t) \in \partial D \times (0, T), \\
u(x, 0) &= \xi(x), \quad x \in D,
\end{align*}
\]
or equivalently the Itô problem
\[ \mathcal{P}(u_t) = 0, \quad 0 < t < T, \quad u_0 = \xi, \]
where \( G : H \to H \) and \( \chi : H \to H \) are also considered to be Lipschitz continuous functions and the initial condition \( \xi(x) \) is positive a.s. Assume that \( \max_{D_T} u = M_T = M \) is attained at \( P_0 = (x_0, t_0) \in \partial D_T \), where \( P_0 \) has the inside strong sphere property, and there is some neighbourhood \( \mathcal{V} \) of \( P_0 \), such that
\[ u < M, \quad \text{in} \quad D_T \cap \mathcal{V}. \]

Then
\[ \frac{\partial u}{\partial \nu}(P_0) < 0. \]

**Proof.** Since \( P_0 \) has the inside strong sphere property we can construct a closed ball \( B \) with center \((\bar{x}, \bar{t})\) such that
\[ B \subset D_T, \quad B \cap \partial D_T = \{P_0\}, \quad \bar{x} \neq x_0, \]
i.e., the ball \( B \) is tangent to \( \partial D_T \) at point \( P_0 \).

Without loss of generality we may assume that the interior of \( B \) lies in \( D_T \cap \mathcal{V} \) and denote the boundary of \( B \) by \( S \). Let \( \pi \) be a hyperplane which divides the \((x, t)\)-plane into two half-planes \( \pi^- \) and \( \pi^+ \) such that \( (\bar{x}, \bar{t}) \in \pi^- \) and \( (x_0, t_0) \in \pi^+ \).

Since \( \bar{x} \neq x_0 \), we can choose \( \pi \) such that \( B^+ = \pi^+ \cap B \) is not empty and such that \( |\bar{x} - x| > \text{const} > 0 \), for any \((x, t) \in B^+. \) The boundary of \( B^+ \) consists of one part \( C_1 \) lying on \( S \) and another part \( C_2 \) lying on \( \pi \).

Introduce the function
\[ h(x, t) = e^{-\alpha(\|x-\bar{x}\|^2+(t-\bar{t})^2)} - e^{-\alpha R^2}, \quad (x, t) \in \overline{D_T}, \]
where \( R \) is the radius of \( S \). We then have
\[
\begin{align*}
h &= 0 \quad \text{on} \ C_1, \\
h &\geq 0 \quad \text{on} \ B^+, \\
\mathcal{H}(h) > 0 \quad \text{in} \ B^+ \text{ for } \alpha \text{ sufficiently large,}
\end{align*}
\]
(recalling that $\mathcal{H}$ stands for the heat operator).

Set

$$\Theta = u + \delta h, \quad \text{for} \quad \delta > 0,$$

then by choosing $\delta$ sufficiently small we obtain

$$\Theta < M \quad \text{on} \quad C_2,$$

$$\Theta(P) = u(P) < M \quad \text{on} \quad C_1, \quad \text{if} \quad P \neq P_0,$$

$$\Theta(P_0) = u(P_0) = M.$$

Moreover, since $h(x,t)$ solves the heat equation, i.e. $dh(t) = -Ah(t)\, dt$, then $\Theta$ satisfies

$$d\Theta(t) = -A\Theta(t) + G(\Theta(t) - \delta h(t))\, dt + \chi(\Theta(t) - \delta h(t))\, dW(t), \quad 0 < t < T,$$

with $\Theta(0) = \xi(x) - e^{-\alpha(|x-\bar{x}|^2 + R)}$ and $G_\delta, \chi_\delta$ defined as

$$G_\delta(\Theta(t)) := G(\Theta(t) - \delta h(t)), \quad \chi_\delta(\Theta(t)) := \chi(\Theta(t) - \delta h(t)).$$

By the assumptions on functions $G$ and $\chi$ and by virtue of Theorem 4.6, we deduce

$$\Theta(x,t) \leq \max_{\partial D_T^+} \Theta = \Theta(P_0) = M, \quad \text{for any} \quad (x,t) \in B^+.$$

Thus, since by (3.13) $\Theta$ is regular enough, we have

$$\frac{\partial \Theta}{\partial \nu}(P_0) \leq 0,$$

or equivalently

$$\frac{\partial u}{\partial \nu} + \delta \frac{\partial h}{\partial \nu} \leq 0, \quad \text{at} \quad P_0.$$

However, by virtue of Theorem 4.5 there holds $\frac{\partial h}{\partial \nu} < 0$ at $P_0$, thereupon we finally obtain

$$\frac{\partial u}{\partial \nu}(P_0) < 0,$$

and the proof of the Theorem 4.7 is complete. \hfill \Box

Remark 4.8. Theorem 4.7 can be generalized to the case of a general second order elliptic operator $\mathcal{L}$.

4.2.2. Estimates near the boundary. In order to tackle the difficulties arising from the presence of the non-local term $K(t) = (\int_D f(u)\, dx)^{-q}$, we need to estimate the contribution of $u(x,t)$ near the boundary. For that purpose we will use the moving plane method as in [16]. Although most of the applied arguments are quite standard in the context of deterministic PDEs, since it is the first time that these ideas are employed for SPDEs a detailed proof is provided.
Lemma 4.9. Let \( u(x,t) \) be the solution of \((3.3)-(3.4)\). Assume further that \( \xi \in L^2(\Omega; L^\infty(D)) \), \( \xi \geq 0 \) a.s. in \( D \) and \( f \) is an increasing function. If \( D \subset \mathbb{R}^d \), \( d \geq 1 \), is convex, there exists \( D_0 \subset D \) such that
\[
\int_D f(u_t) \, dx \leq (\ell + 1) \int_{D_0} f(u_t) \, dx, \quad 0 \leq t < T,
\]
for some positive integer \( \ell \).

Proof. For any \( y \in \partial D \) we define the hyperplane
\[
\mathcal{T}(\mu, y) := \{ x \in \mathbb{R}^d : (x, \nu(y))_d = \mu \},
\]
where \((\cdot, \cdot)_d\) stands for the inner product in \( \mathbb{R}^d \).

We can find \( \mu_0 \) such that \( \mathcal{T}(\mu_0, y) \) coincides with the tangent hyperplane to \( D \) at \( y \) and \( y \in \mathcal{T}(\mu_0, y) \cap \overline{D} \) (when \( D \) is strictly convex then \( \mathcal{T}(\mu_0, y) \cap \overline{D} = \{ y \} \)).

Since \( D \) is a bounded set there exists \( \mu_1 < \mu_0 \) such that \( \mathcal{T}(\mu, y) \cap \overline{D} = \emptyset \) for \( \mu > \mu_0 \) and \( \mu < \mu_0 - \mu_1 \).

We define
\[
\Sigma(\mu, y) := \{ x \in D : \mu < (x, \nu(y))_d < \mu_0 \},
\]
while by \( \Sigma'(\mu, y) \) we denote the reflection of \( \Sigma(\mu, y) \) across \( \mathcal{T}(\mu, y) \). Now using the convexity of \( D \) we can choose \( \bar{\mu} \) sufficiently close to \( \mu_0 \) so that \( \Sigma'(\bar{\mu}, y) \subset D \).

Applying Theorem 4.7 we derive for any \( y \in \partial D \)
\[
\frac{\partial u(y,t)}{\partial \nu} = (\nabla u(y,t), \nu(y))_d < 0, \quad \text{for any } t \geq t_0 > 0.
\]

By the spatial regularity of \( u \), see \((3.13)\), we can find a neighbourhood of \( y \), say \( \mathcal{N}_y \), such that
\[
\frac{\partial u(x,t_0)}{\partial \nu} = (\nabla u(x,t_0), \nu(y))_d < 0, \quad \text{for any } x \in \mathcal{N}_y.
\]

We consider now a coordinate system centered at \( y \) and defined by \((y; \nu(y), \mathcal{T}(\mu_0, y))\) such that every \( x \in \mathbb{R}^d \) is expressed as \( x = (x_\nu, x_\mathcal{T}) \), where \( x_\nu \) is the component in the direction of \( \nu(y) \) while \( x_\mathcal{T} \) stands for the component in the direction of the hyperplane \( \mathcal{T}(\mu_0, y) \).

Let us define the cylinder \( C_\delta(y) = \{ y \in \mathbb{R}^d : |x_\nu| < \delta, |x_\mathcal{T}| < \delta \} \). We may pick \( \delta > 0 \) small enough so that the reflection of \( \overline{C_\delta(y)} \cap \mathcal{T}(\bar{\mu}, y) \) across \( \mathcal{T}(\bar{\mu}, y) \), denoted by \( C'_\delta(y) \), is compact in \( D \).

Set \( K_y = \mathcal{T}(\mu_0, y) \cap \overline{D} \), then \( K_y \) is a compact convex set and \( K_y = \bigcap_{\mu < \mu_0} \Sigma(\mu, y) \). Every \( \hat{y} \in K_y \) has the same exterior normal \( \nu(y) \). Then we can define an open neighbourhood of \( \hat{y} \) of the shape \( C_\delta(\hat{y}) \) and on which \( (\nabla u(\hat{y}, t_0), \nu(y))_d < 0 \). Moreover, \( K_y \subset \bigcup_{\hat{y}} C_\delta(\hat{y}) \) and since \( K_y \) is compact we can extract a finite cover of \( C_\delta(\hat{y}) \), say \( B = \bigcup_{i=1}^n C_\delta(\hat{y}_i) \) which contains \( K_y \), for some positive integer \( n = n(y) \).

Since \( D \) is convex we can find \( \mu < \mu_0 \) such that \( \Sigma(\mu, y) \subset B \) and \( \Sigma'(\rho_0, y) \subset D \), \( \Sigma(\rho_0, y) \cup \Sigma'(\rho_0, y) \subset B \) for \( \rho_0 = \frac{\mu + \mu_0}{2} \). (Note that if \( D \) is strictly convex then the above construction is unnecessary).
We now set \( z(x, t) = z(x_y, x_{\mathcal{F}}, t) = u(2\rho_0 - x_y, x_{\mathcal{F}}, t) \) for \( x \in \Sigma(\rho_0, y) \); actually \( z \) is the reflection of \( u \) across \( T(\rho_0, y) \). Then \( z \) satisfies
\[
dz(\cdot, t) = \left[-A z(\cdot, t) + K(t) e^{z(\cdot, t)}\right] dt + \sigma(z(\cdot, t)) dW(t), \quad \text{on} \quad \Sigma(\rho_0, y) \times (0, T_{\text{max}}),
\]
\[
z \geq u \geq 0 \quad \text{on} \quad \partial \Gamma_1 = (\partial D \cap \Sigma(\rho_0, y)) \times (0, T_{\text{max}}),
\]
\[
z_t = u_t \quad \text{on} \quad \partial \Gamma_2 = (D \cap \mathcal{F}(\rho_0, y)) \times (0, T_{\text{max}}).
\]
Thus \( z \) and \( u \) satisfy the same SPDE on \( \Sigma(\rho_0, y) \times (0, T_{\text{max}}) \) while \( z \geq u \) on \( \partial \Gamma_1 \cup \partial \Gamma_2 \) and \( z(\cdot, t_0) \geq u(\cdot, t_0) \) on \( \Sigma(\rho_0, y) \), hence by the comparison principle, see [10], we deduce that \( z \geq u \) almost surely (a.s.) on \( \Sigma(\rho_0, y) \times (0, T_{\text{max}}) \).

Note that \( \Sigma(\rho_0, y) \) contains an open set of the type \( C_\delta(y) \cap D \) and if we choose \( \delta < \mu_0 - \rho_0 \) then the reflection of \( C_\delta(y) \cap D \) across \( \mathcal{F}(\rho_0, y) \) has a compact closure in \( D \). We can repeat the above construction for any \( y \in \partial D \) and the collection of all cylinders \( \{C_\delta(y)\}_{y \in \partial D} \) makes up an open cover of \( \partial D \), and we can extract a finite subcover denoted by \( C_\delta(y_1), ..., C_\delta(y_\ell) \) such that \( \partial D \subseteq C_\delta(y_1) \cup ... \cup C_\delta(y_\ell) \).

Set \( D_0 = D \setminus \bigcup_{i=1}^\ell C_\delta(y_i) \), then \( \overline{D_0} \subset D \) and we derive
\[
\int_D u \, dx \quad \leq \quad \int_{D_0} u \, dx + \sum_{i=1}^\ell \int_{C_\delta(y_i) \cap D} u \, dx \quad \leq \quad \int_{D_0} u \, dx + \sum_{i=1}^\ell \int_{C_\delta(y_i) \cap D} z \, dx
\]
\[
\leq \quad \int_{D_0} u \, dx + \sum_{i=1}^\ell \int_{C_\delta(y_i)} z \, dx = \int_{D_0} u \, dx + \sum_{i=1}^\ell \int_{C_\delta(y_i)} u \, dx
\]
\[
\leq \quad \int_{D_0} u \, dx + \ell \int_{D_0} u \, dx \leq (\ell + 1) \int_{D_0} u \, dx,
\]

taking also into account that \( u \leq z \) on \( C_\delta(y_i) \cap D \) and \( u = z \) on \( C_\delta(y_i) \) by reflection.

Now since \( f(s) \) is increasing we also deduce
\[
\int_D f(u_t) \, dx \leq (\ell + 1) \int_{D_0} f(u_t) \, dx,
\]
and the proof of Lemma is now complete. \( \square \)

4.2.3. Blow-up results. Furthermore \( f(s) \) is assumed to be increasing and convex, i.e.,
\[
f'(s), \ f''(s) \geq 0 \quad \text{for} \quad s \in \mathbb{R}, \quad (4.10)
\]
whereas \( f^{1-q}(s) \) is considered to be convex and superlinear, i.e.
\[
[f^{1-q}(s)]'' \geq 0 \quad \text{for} \quad s \in \mathbb{R} \quad \text{and} \quad \int_b^\infty \frac{ds}{f^{1-q}(s)} < \infty, \quad \text{for any} \quad b \in \mathbb{R}. \quad (4.11)
\]

We first prove that finite-time blow-up occurs when the parameter \( \lambda \) is large enough.
Theorem 4.10. Suppose that \((3.3)-(3.4)\) has a (unique) local-in-time solution \(u\) whose existence is guaranteed by Theorem 3.4. Assume also that \(f(s)\) satisfies conditions \((4.10)-(4.11)\) then \(u\) blows up in finite time for sufficiently large values of the parameter \(\lambda\), provided that \(\xi \in L^2(\Omega; L^2(D))\), \(\xi(x) \geq 0\) a.s.

Proof. Let us define \(\hat{u}(t)\) as in Theorem 4.2. Now taking the expectation over (4.2) we have

\[
E[\hat{u}(t)] = E\left[\int_D \xi \phi_1 \, dx\right] - \lambda_1 E\left[\int_0^t \int_D u_s \phi_1 \, dx \, ds\right] + \lambda E\left[\int_0^t \int_D \frac{f(u_s) \phi_1}{\int_D f(u_s) \, dx} \, dx \, ds\right] \quad (4.12)
\]

taking also into account that

\[
E\left[\int_0^t \int_D \sigma(u_s) \phi_1 \, dx \, dW_s\right] = 0,
\]
due to the fact that \(W\) is a Wiener process.

Set \(\Psi(t) = E[\hat{u}(t)]\), then by using again of Fubini’s theorem, we deduce

\[
\Psi(t) = \Psi_0 - \lambda_1 \int_0^t \Psi(s) \, ds + \lambda E\left[\int_0^t \int_D \frac{f(u_s) \phi_1}{\int_D f(u_s) \, dx} \, dx \, ds\right], \quad (4.13)
\]

where \(\Psi_0 = E[\langle \xi, \phi_1 \rangle_H]\), or equivalently the initial value problem

\[
\frac{d\Psi}{dt} = -\lambda_1 \Psi(t) + \lambda E\left[K(t) \int_D f(u_t) \phi_1 \, dx\right], \quad t > 0, \quad \Psi(0) = \Psi_0. \quad (4.14)
\]

By Lemma 4.9 we can construct \(D_0 \subset D\) with \(\overline{D_0} \subset D\) such that

\[
\int_D f(u_t) \, dx \leq (\ell + 1) \int_{D_0} f(u_t) \, dx,
\]

for some \(k \in \mathbb{N}\). Let \(m = \inf_{x \in D_0} \phi_1(x)\), then since \(\overline{D_0} \subset D\) we have \(m > 0\). Hence

\[
\int_D f(u_t) \, dx \leq \frac{\ell + 1}{m} \int_{D_0} f(u_t) \phi_1 \, dx \leq \frac{\ell + 1}{m} \int_D f(u_t) \phi_1 \, dx,
\]

and so

\[
K(t) = \left(\int_D f(u_t) \, dx\right)^{-q} \geq R \left(\int_D f(u_t) \phi_1 \, dx\right)^{-q}, \quad (4.15)
\]

for

\[
R = \left(\frac{m}{\ell + 1}\right)^q. \quad (4.16)
\]
Therefore by virtue of (4.15) and applying Jensen’s inequality twice, since both \( f(s) \) and \( f^{1-q}(s) \) are convex functions, see (4.10) and (4.11), we deduce
\[
\mathbb{E} \left[ K(t) \int_D f(u_t) \phi_1 \, dx \right] \geq \mathbb{E} \left[ R \left( \int_D f(u_t) \phi_1 \, dx \right)^{1-q} \right] \\
\geq R f^{1-q} \left[ \mathbb{E}[\hat{u}(t)] \right] = R f^{1-q} \left( \Psi(t) \right).
\] (4.17)

By (4.14) and (4.17) we deduce the differential inequality
\[
\frac{d\Psi(t)}{dt} \geq \lambda_1 \Psi(t) + R f^{1-q} (\Psi(t)), \quad t > 0,
\]
with initial condition \( \Psi(0) = \Psi_0 \).

Define
\[
0 < B := \sup_{s > \Psi(0)} s / f^{1-q}(s),
\]
then due to (4.11) we have that \( B < \infty \), and so choosing \( \lambda > \lambda_1 B / R \), we deduce
\[
t \leq \int_{\Psi(0)}^{\Psi(t)} \frac{ds}{\lambda R f^{1-q}(s) - \lambda_1 s} \leq \frac{1}{\Lambda} \int_{\Psi(0)}^{\Psi(t)} \frac{ds}{f^{1-q}(s)} < \frac{1}{\Lambda} \int_{\Psi(0)}^{\infty} \frac{ds}{f^{1-q}(s)} < \infty,
\]
for
\[
0 < \Lambda \leq \lambda R - \lambda_1 B < \infty.
\] (4.18)

Thus \( \Psi(t) \) blows up in finite time, i.e. \( \Psi(t) \to \infty \) as \( t \to T^* \) where \( T^* \) is estimated as
\[
T^* \leq \int_{\Psi(0)}^{\infty} \frac{ds}{\lambda R f^{1-q}(s) - \lambda_1 s} \leq \frac{1}{\Lambda} \int_{\Psi(0)}^{\infty} \frac{ds}{f^{1-q}(s)} < \infty.
\] (4.19)

Indeed, since by Theorem 3.6 \( u_t \) is bounded in \( D \) and due to (4.11) we have
\[
\Psi(t) = \mathbb{E} \left[ \int_D u_t \phi_1(x) \, dx \right] \leq \mathbb{E} \left[ \|u_t\|_\infty \right],
\]
and thus \( \mathbb{E} \left[ \|u_t\|_\infty \right] \to \infty \) as \( t \to t^* - \leq T^* \). The proof of the Theorem is complete. \( \square \)

In the following we prove that finite time blow-up occurs for large enough initial data as well.

**Theorem 4.11.** Suppose that the assumptions of Theorem 4.10 hold true. Assume also that
\[
\mathbb{E} \left[ \int_D \xi \phi_1 \, dx \right] > \zeta,
\] (4.20)
where \( \zeta = \zeta(\lambda) \) is the largest root of the equation
\[
\alpha(s) := \lambda R f^{1-q}(s) - \lambda_1 s = 0,
\]
and \( R \) is the constant given by (4.16). Then the solution \( u \) of (3.3) - (3.4) blows up in finite time.
**Proof.** Following the same steps as in the proof of Theorem 4.10 we obtain that $\Psi(t) = \mathbb{E}\left[ \int_D u_t \phi_1 \, dx \right]$ satisfies the differential inequality

$$
\frac{d\Psi(t)}{dt} \geq -\lambda_1 \Psi(t) + \lambda R f^{1-q}(\Psi(t)) = \alpha(\Psi(t)), \quad t > 0,
$$

with $\Psi(0) = \Psi_0 := \mathbb{E}\left[ \int_D \xi \phi_1 \, dx \right]$.

Let $\zeta = \zeta(\lambda)$ be the largest root of the equation $\alpha(s) = 0$. Then by choosing $\Psi_0 > \zeta$ we deduce

$$
\int_{\Psi_0}^{\infty} \frac{ds}{\alpha(s)} \leq \frac{1}{\Lambda_1} \int_{\Psi_0}^{\infty} \frac{ds}{f^{1-q}(s)} < \infty,
$$

for some positive constant $\Lambda_1$. But the above relation guarantees that $\Psi(t)$ blows up in finite time $T^* < \infty$, where

$$
T^* \leq \frac{1}{\Lambda_1} \int_{\Psi_0}^{\infty} \frac{ds}{f^{1-q}(s)} < \infty,
$$

which, similarly to Theorem 4.10 implies that $\mathbb{E}[\|u_t\|_\infty] \to \infty$ as $t \to t^* - \leq T^*$.

**Remark 4.12.** Theorems 4.10 and 4.11 both imply explosion of the mean $L^q$-norm for any $q \geq 1$ as well. Indeed, since $\phi_1$ is bounded and continuous on $D$ by applying Hölder’s inequality for each $q \geq 1$ we derive

$$
\Psi(t) \leq C_q \mathbb{E}\left[ \left( \int_D |u|^q \, dx \right)^{1/q} \right],
$$

for $C_q = (\int_D |\phi_1|^r \, dx)^{1/r}$ with $r = q/(q-1)$, which actually yields that the mean $L^q$-norm explodes in finite time $T_q < t^*$.

**Acknowledgement.** We would like to thank Dr. M. Hofmanová for letting us know about the paper [9]. We would also like to thank Prof. S. Larson for his stimulating comments which helped in improving the presentation of our results.

**References**

[1] J. W. Bebernes & A. A. Lacey *Global existence and finite-time blow-up for a class of nonlocal parabolic problems*, Adv. Differential Equations, Volume 2, (1997), 927–953.

[2] J. W. Bebernes & P. Talaga, *Non-local problems modelling shear banding*, Comm. Appl. Nonlinear Anal. 3 (1996), 79–103.

[3] J. W. Bebernes, C. Li & P. Talaga, *Single-point blow-up for non-local parabolic problems*, Physica D 134 (1999), 48–60.

[4] D. Blömker, *Amplitude Equations for Stochastic Partial Differential Equations*, Interdisciplinary Mathematical Sciences-Vol. 3, World Scientific Publishing Co. Inc., 2007.

[5] Z. Brzeźniak, *On Stochastic convolution in Banach spaces and applications*, Stochastics and Stochastic Reports 61(1997), 245-295.

[6] P-L Chow, *Stochastic Partial Differential Equations*, Chapman and Hall/CRC, 2007.

[7] P-L Chow, *Explosive solutions of stochastic reaction-diffusion equations in mean $L^p$–norm*, J. Diff. Equations, 250 (2011), 2567–2580.
[8] P.-L. Chow, *Unbounded positive solutions of nonlinear parabolic Itô equations*, Comm. Stoch. Anal. 3 (2009), 211–222.

[9] A. Debussche, S. De Moor & M. Hofmanova, *A regularity result for quasilinear stochastic partial differential equations of parabolic type*, SIAM J. Math. Anal., Vol 47 (2015) 1590-1614, DOI:10.1137/130950549.

[10] L. Denis, A. Matoussi & L. Stoica, *Maximum principle for quasi-linear SPDE’s on a bounded domain without regularity assumptions*, Stoch. Proc. Appl., Vol 123, (2013), 1104–1137.

[11] L. Denis, A. Matoussi & L. Stoica, *Maximum principle and comparison theorem for quasilinear stochastic PDE’s*, Electr. Jour. Probability, Vol. 14 (2009), 500–530.

[12] A. Friedman, *Partial Differential Equations of Parabolic Type*, 1983, Prentice-Hall Inc.

[13] M. Hofanova, *Strong solutions of semilinear stochastic partial differential equations*, Nonlinear Differ. Equ. Appl. 20 (2013), 757–778.

[14] S. Kaplan, *On the growth of solutions of quasilinear parabolic equations*, Comm. Pure Appl. Math 16 (1963), 327–330.

[15] N.I. Kavallaris, *Explosive solutions of a stochastic non-local reaction-diffusion equation arising in shear band formation*, to appear in Math. Meth. Appl. Sciences.

[16] N. I. Kavallaris & D. E. Tzanetis, *On the blow-up of a non-local parabolic problem*, Appl. Math. Letters 19, (2006), 921–925.

[17] N.I. Kavallaris & T. Nadzieja, *On the blow-up of the non-local thermistor problem*, Proc. Edinburgh. Math. Soc. 50 (2007).

[18] N.I. Kavallaris & T. Suzuki, *On the finite-time blow-up of a non-local parabolic equation describing chemotaxis*, Diff. Int. Equations 20, (2007), 293–308.

[19] A. Krzywicki & T. Nadzieja, *Some results concerning the Poisson–Boltzmann equation*, Zastosowania Mat. (Appl. Math. (Warsaw)) 21 (1991), 265–272.

[20] A. A. Lacey, *Thermal runaway in a non-local problem modelling Ohmic heating. Part I: Model derivation and some special cases*, Euro. J. Appl. Math. 6 (1995), 127–144.

[21] A. A. Lacey, *Thermal runaway in a non-local problem modelling Ohmic heating. Part II: General proof of blow–up and asymptotics of runaway*, Euro. J. Appl. Math. 6 (1995), 201–224.

[22] A.A. Lacey & D.E. Tzanetis, *Global unbounded solutions to a parabolic equation*, J. Diff. Equations, 101, (1993), 80-102.

[23] O.A. Ladyzhenskaya, V. A. Solonnikov & N.N. Ural’ceva, *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs 23, Amer. Math. Soc., Providence, R. I., 1968.

[24] G.M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific Publishing Co. Inc., River Edge, NJ, 1996.

[25] G. J. Lord, C. E. Powell, and T. Shardlow, *An Introduction to Computational Stochastic PDEs*, Cambridge University Press, Cambridge, UK, 2014.

[26] J. M. A. M. van Neerven & J. Zhu, *A maximal inequality for stochastic convolutions in 2-smooth Banach spaces*, Electr. Comm. Probability 16 (2011), 689–705.

[27] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, 2nd Edition 1983, Springer-Verlag.

[28] D. E. Tzanetis, *Blow-up of radially symmetric solutions of a non-local problem modelling ohmic heating*, Electron. J. Diff. Eqns. 11 (2002), 1–26.

[29] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, 2nd Edition 1995, Johann Ambrosius Barth, Heidelberg.

[30] G. Wolansky, *A critical parabolic estimate and application to non-local equations arising in chemotaxis*, Appl. Anal. 66 (1997), 291–321.
Department of Mathematics, University of Chester, Thornton Science Park Pool Lane, Ince, Chester CH2 4NU, UK

E-mail address: n.kavallaris@chester.ac.uk

Department of Mathematics, University of Chester, Thornton Science Park Pool Lane, Ince, Chester CH2 4NU, UK

E-mail address: y.yan@chester.ac.uk