WEDDERBURN DECOMPOSITION AND IDEMPOTENTS OF SOME FINITE METACYCLIC GROUP ALGEBRAS

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ABSTRACT. In this article, we show explicitly the Wedderburn decomposition of the metacyclic group algebra $\mathbb{F}_q G$, where $G$ has a cyclic subgroup of index 2 and $\gcd(|G|, q) = 1$. We also construct the complete set of central and left idempotents of these group algebras.

1. Introduction

Let $\mathbb{F}_q$ be a finite field with $q$ elements, $G$ be a finite group with $n$ elements, where $\gcd(q, n) = 1$, and $\mathbb{F}_q G$ be the group algebra of $G$ over $\mathbb{F}_q$. Since $q$ and $n$ are coprimes, it follows from Maschke’s Theorem that $\mathbb{F}_q G$ is semisimple and, as a consequence of the Wedderburn-Artin Theorem, $\mathbb{F}_q G$ is isomorphic to a direct sum of matrix algebras over division rings. In addition, by Wedderburn’s Little Theorem, it is known that finite divisions rings are actually fields that are in our case, a finite extensions of $\mathbb{F}_q$, i.e. there exists an isomorphism $\rho$ such that

\[ \mathbb{F}_q G \cong M_{l_1}(\mathbb{F}_{q^{m_1}}) \oplus M_{l_2}(\mathbb{F}_{q^{m_2}}) \oplus \cdots \oplus M_{l_t}(\mathbb{F}_{q^{m_t}}) \]

where $l_1, \ldots, l_t, m_1, \ldots, m_t$ are appropriate positive integers such that $\sum_{j=1}^t l_j m_j = |G|$.

The explicit description of the primitive idempotents and Wedderburn decomposition of $\mathbb{F}_q G$ is an important problem in group algebras. In addition, determining ideals of $\mathbb{F}_q G$ is important in coding theory, because these ideals can be seen as subspaces of the vector field $\mathbb{F}_q^n$ that has additional algebraic properties. For instance, irreducible cyclic codes are ideals of group algebra $\mathbb{F}_q G$ that has $t$ central irreducible idempotents, each one of the form

\[ e_i = \rho^{-1}(0 \oplus \cdots \oplus 0 \oplus I_i \oplus 0 \cdots \oplus 0), \]

where $I_i$ represents the identity matrix of the component $M_{l_i}(\mathbb{F}_{q^{m_i}})$. Then, the isomorphism $\rho$ determines explicitly each central irreducible idempotent.

In the case when we consider the field $\mathbb{Q}$ instead of $\mathbb{F}_q$, the calculus of central idempotents and Wedderburn decomposition is widely studied; the classical method to calculate the primitive central idempotents of group algebras depends on computing the character group table. Other method is shown in [15], where Jespers, Leal and Paques describe the central irreducible idempotents, when $G$ is a nilpotent group, using the structure of its subgroups without employing the characters of the group. Generalizations and improvements of this method can be found in [18], where the authors provide information about the Wedderburn decomposition of $\mathbb{Q} G$. This computational method is also used in [2] to compute the Wedderburn decomposition and the central primitive idempotents of a finite semisimple group algebra $KG$, where $G$ is an abelian-by-supersolvable group $G$ and $K$ is a finite field.

The structure of $KG$ when $G = D_{2n}$ is the dihedral group with $2n$ elements is well known for $K = \mathbb{Q}$ (see [7]) and for $K = \mathbb{F}_q$ where $\gcd(2n, q) = 1$ (see [4]). In [10], Dutra, Ferraz and Polcino Milies impose conditions over $q$ and $n$ in order for $\mathbb{F}_q D_{2n}$ to have the same number of irreducible components as that of $\mathbb{Q} D_{2n}$. This result is generalized in [11], where Ferraz, Goodaire and Polcino Milies find, for some families of groups, conditions on $q$ and $G$ in order for $\mathbb{F}_q G$ to have the minimum number of simple components.

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In this article, we show explicitly the Wedderburn decomposition of the group algebra \( \mathbb{F}_qG \) where \( G \) has a cyclic subgroup of index 2. We consider two possible group’s families: The split metacyclic group that has a presentation of the form \( G = \langle x, y \mid x^n = 1, y^2 = xy \rangle \) and the non-split metacyclic group that has a presentation of the form \( G = \langle x, y \mid x^{2n} = 1, y^2 = x^n, xy = yx \rangle \). In each case, we also find the complete set of central orthogonal idempotents and, in addition, we decompose each central idempotent, when possible, in sum of non-central orthogonal idempotents.

2. Preliminaries

Throughout this article, \( \mathbb{F}_q \) and \( \overline{\mathbb{F}_q} \) denote a finite fields with \( q \) elements and its algebraic closure respectively, \( \text{ord}(a) \) is the order of \( a \) in the cyclic group \( \mathbb{F}_q^\ast \) and for each \( c \) and \( d \) positive integers such that \( \gcd(c, d) = 1 \), \( \text{ord}_c(d) \) denotes the order of \( d \) in the multiplicative group \( \mathbb{Z}_c^\ast \). For each \( f(x) \in \mathbb{F}_q[x] \) such that \( f(0) \neq 0 \), \( \text{ord}(f) \) is the least positive integer \( n \) such that \( f(x) \) divide \( x^n - 1 \). For each prime number \( p \), the function \( \nu_p \) is the \( p \)-valuation, i.e. for each integer \( c, \nu_p(c) \) is the highest exponent \( v \) such that \( p^v \) divides \( c \). In addition, for each positive integer \( k \), \( \Phi_k(x) \) denotes the \( k \)-th cyclotomic polynomial.

The decomposition into simple components of the group algebra \( \mathbb{F}_qG \), when \( G = C_n \) is a cyclic group with \( n \) elements and \( \gcd(n, q) = 1 \) is a well known result and it can be seen as direct consequence of the Chinese Remainder Theorem. Indeed

\[
\mathbb{F}_qC_n \cong \frac{\mathbb{F}_q[x]}{\langle x^n - 1 \rangle} \cong \sum_{d|n} \frac{\mathbb{F}_q[x]}{\langle \Phi_d(x) \rangle}. \tag{2.1}
\]

In addition, since \( \Phi_d(x) \) splits into \( \frac{\Phi(d)}{\text{ord}_d(q)} \) irreducible factors in \( \mathbb{F}_q[x] \), each ring \( \frac{\mathbb{F}_q[x]}{\langle \Phi_d(x) \rangle} \) can be decompose into \( \frac{\Phi(d)}{\text{ord}_d(q)} \) copies of the field \( \mathbb{F}_q(\xi_d) \), where \( \xi_d \) is a root of \( \Phi_d(x) \). In general, we have the following result for abelian group.

**Theorem 2.1** (Perlis-Walker’s Theorem). Let \( G \) be an abelian group with \( n \) elements and \( K \) be a field such that \( \text{char}(K) \nmid n \). Then

\[
KG \cong \bigoplus_{d|n} a_dK(\xi_d),
\]

where \( \xi_d \) is a \( d \)-th primitive root of unit, \( a_d = \frac{n_d}{[K(\xi_d): K]} \) and \( n_d \) is the number of elements of order \( d \) in \( G \).

We observe that the decomposition of \( \mathbb{F}_qC_n \) depends fundamentally on the factorization of \( x^n - 1 \) in \( \mathbb{F}_q[x] \). For the groups that we consider in this paper, that factorization will also be essential as well as the \( s \)-self-involutive propriety. In fact, for each polynomial \( f(x) \), we construct the so-called \( s \)-involutive of \( f(x) \), that it is one possible generalization of the notion of reciprocal polynomial.

**Definition 2.2.** Let \( g(x) = (x - a_1)(x - a_2) \cdots (x - a_k) \) be a polynomial over \( \mathbb{F}_q[x] \) and \( s \in \mathbb{Z} \) such that \( s^2 \equiv 1 \) (mod \( \text{ord}(g) \)). Let us denote \( g^s(x) \), the \( s \)-involutive of \( g(x) \), defined as \( g^s(x) := (x - a_1^s)(x - a_2^s) \cdots (x - a_k^s) \). In the case when \( g \) and \( g^s \) have the same roots in the decomposition field, the polynomial \( g \) is called \( s \)-self-involutive. The particular case when \( s = -1 \), the \( s \)-involutive of \( g(x) \) is the classical reciprocal of \( g(x) \).

The following result provides some properties on the \( p \)-adic valuation on numbers of the form \( a^k - 1 \) with \( a \equiv 1 \) (mod \( p \)), that we will using to determine the relation between the finite fields \( \mathbb{F}_q(\xi_d) \) and \( \mathbb{F}_q(\xi_d, \xi_d^{-1}) \). This result is attributed to E. Lucas and R. D. Carmichael [8].

**Lemma 2.3.** [8] Proposition 1) Let \( p \) be a prime and \( \nu_p \) be the \( p \)-valuation. The following hold: 1) if \( p \) is an odd prime that divides \( a - 1 \), then \( \nu_p(a^k - 1) = \nu_p(a - 1) + \nu_p(k) \);
Lemma 2.4. Let \( f(x) \in \mathbb{F}_q[x] \) be a monic irreducible \( s \)-self-involutive factor of \( x^n - 1 \), where \( s^2 \equiv 1 \pmod{n} \) and \( \alpha \) be a root of \( f(x) \), then

\[
[\mathbb{F}_q(\alpha) : \mathbb{F}_q(\alpha + \alpha^s, \alpha^{s+1})] = 2.
\]

Proof: Since \( f \) is an irreducible polynomial, then \( f \in \mathbb{F}_q[x] \) is the minimal polynomial of \( \alpha \) over \( \mathbb{F}_q \). Therefore \( \alpha \in \mathbb{F}_{q^m} \), where \( m \) is the degree of \( f \) and \( m \) is minimal with this property. In addition, from the fact that \( f \) is \( s \)-self-involutive, it follows that \( \alpha^s \) is also a root of \( f(x) \). Thus \( \alpha \) and \( \alpha^s \) are conjugated and there exists \( 1 \leq u \leq m - 1 \) such that \( \alpha^s = \alpha^{q_u} \), or equivalently, \( s \equiv q_u \pmod{\text{ord}(\alpha)} \). Now, we observe that

\[
(\alpha + \alpha^s)^{q_u} = \alpha^{q_u} + (\alpha^{q_u})^s = \alpha^s + \alpha^{s^2} = \alpha^s + \alpha,
\]

that implied that \( \alpha^s + \alpha \in \mathbb{F}_{q^u} \). The same way

\[
(\alpha^{s+1})^{q_u} = \alpha^{s^2 + s} = \alpha^{s+1}
\]

and hence \( \alpha^{s+1} \in \mathbb{F}_{q^u} \). It follows that

\[
[\mathbb{F}_q(\alpha + \alpha^s, \alpha^{s+1}) \subseteq \mathbb{F}_{q^u} \cap \mathbb{F}_{q^m} = \mathbb{F}_{q^\text{gcd}(u,m)} \subseteq \mathbb{F}_{q^m} = \mathbb{F}_q(\alpha),
\]

in particular we have that \([\mathbb{F}_q(\alpha) : \mathbb{F}_q(\alpha + \alpha^s, \alpha^{s+1})] \geq 2 \). On the other hand, \( \alpha \) is root of the polynomial \( x^2 - (\alpha + \alpha^s)x + \alpha^{s+1} \), and this polynomial has its coefficients in \( \mathbb{F}_q(\alpha + \alpha^s, \alpha^{s+1}) \), therefore \([\mathbb{F}_q(\alpha) : \mathbb{F}_q(\alpha + \alpha^s, \alpha^{s+1})] \leq 2 \). From these inequalities we conclude that \([\mathbb{F}_q(\alpha) : \mathbb{F}_q(\alpha + \alpha^s, \alpha^{s+1})] = 2 \). \( \square \)

The following lemma let us understand how the towel of fields of the form \( \mathbb{F}_q(\alpha^{2^k}) \) grown, when we change the values of \( k \).

Lemma 2.5. Let \( \alpha \in \overline{\mathbb{F}_q} \) be a \( 2n \)-th primitive root of the unit, where \( n \) is even and \( q \) be a power of a prime such that \( q \equiv 3 \pmod{4} \). Then

(i) \([\mathbb{F}_q(\alpha^{2^{2(n-1)}}) : \mathbb{F}_q(\alpha^{2^{2(n)}})] = 2 \) and \([\mathbb{F}_q(\alpha^{2^{2(n)}}) : \mathbb{F}_q(\alpha^{2^{2(n)+1}})] = 1 \).

(ii) If \( \nu_2(n) \leq \nu_2(q + 1) \), then

\[
[\mathbb{F}_q(\alpha^{2^j}) : \mathbb{F}_q(\alpha^{2^{j+1}})] = 1 \quad \text{for all} \quad 0 \leq j \leq \nu_2(n) - 2.
\]

(iii) If \( \nu_2(n) > \nu_2(q + 1) \), then

\[
[\mathbb{F}_q(\alpha^{2^j}) : \mathbb{F}_q(\alpha^{2^{j+1}})] = \begin{cases} 
2 & \text{if} \quad 0 \leq j < \nu_2(n) - \nu_2(q + 1) \\
1 & \text{if} \quad \nu_2(n) - \nu_2(q + 1) \leq j \leq \nu_2(n) - 2.
\end{cases}
\]

Proof: Let us denote by \( k = [\mathbb{F}_q(\alpha) : \mathbb{F}_q] \), i.e., \( k \) is the degree of the minimal polynomial of \( \alpha \) over \( \mathbb{F}_q \). It is known that \( k = \text{ord}_{2n}q \), i.e., the smallest positive integer \( l \) such that \( q^l \equiv 1 \pmod{2n} \), in particular, we have that

\[
2 \leq \nu_2(2n) \leq \nu_2(q^{k} - 1).
\]

Since \( q \equiv 3 \pmod{4} \), that inequality implies that \( k \) is even. For each \( 0 \leq j \leq \nu_2(n) \) we denote \( k_j = [\mathbb{F}_q(\alpha^{2^j}) : \mathbb{F}_q] \). It is clear that \([\mathbb{F}_q(\alpha^{2^{j-1}}) : \mathbb{F}_q(\alpha^{2^j})] = 1 \) or \( 2 \), and hence either \( k_{j+1} = k_j \) or \( k_{j+1} = 2k_j \). By definition, \( k_j \) is the smallest positive integer such that the relation \( \alpha^{2^{j+1}} = \alpha^{2^j} \) is satisfied, or equivalently \( \alpha^{2^{j+1}} = \alpha^{2^j} \). In particular we have

\[
2^j(q^{k_j} - 1) \equiv 0 \pmod{2^{2(n)+1}}
\]
and consequently
\[ \nu_2(q^{kj}) - 1 \geq \nu_2(n) + 1 - j, \quad \text{for each } 0 \leq j \leq \nu_2(n). \] (2.2)

We observe that in the case when \( j = \nu_2(n) \), it follows that \( \nu_2(q^{k\nu_2(n)}) - 1 \geq 1 \) and, from the minimality of \( k_{\nu_2(n)} \), we conclude that \( k_{\nu_2(n)} \) is odd and \( \nu_2(q^{k\nu_2(n)}) = 1 \). Furthermore, \([\mathbb{F}_q(\alpha) : \mathbb{F}_q]\) is even, \([\mathbb{F}_q(\alpha^{2^{\nu_2(n)+1}}) : \mathbb{F}_q]\) is odd and consequently \( k_{\nu_2(n)} = k_{\nu_2(n)+1} \). Putting \( j = \nu_2(n) - 1 \) in Inequality (2.2) we obtain
\[ \nu_2(q^{k\nu_2(n)-1}) - 1 \geq 2, \]

hence \( k_{\nu_2(n)-1} \) is even and therefore \( k_{\nu_2(n)-1} = 2k_{\nu_2(n)} \), i.e.
\[ [\mathbb{F}_q(\alpha^{2^{\nu_2(n)-1}}) : \mathbb{F}_q(\alpha^{2^{\nu_2(n)}})] = 2. \]

The following diagram shows, partially, the degrees of intermediary field extensions generated by the powers \( \alpha^{2^i} \) of \( \alpha \).

\[ \mathbb{F}_q(\alpha) \xrightarrow{\cdot \alpha} \mathbb{F}_q(\alpha^2) \xrightarrow{\cdot \alpha^2} \cdots \xrightarrow{\cdot \alpha^{2^{\nu_2(n)-1}}} \mathbb{F}_q(\alpha^{2^{\nu_2(n)-1}}) \xrightarrow{\cdot \alpha^{2^{\nu_2(n)}}} \mathbb{F}_q(\alpha^{2^{\nu_2(n)}}) \xrightarrow{\cdot \alpha^{2^{\nu_2(n)+1}}} \cdots \]

We still need to analyze the fields on the top of this tower. If \( 0 \leq j < \nu_2(n) \), then \( k_j \) is even. Rewriting \( k_j = 2k'_j \) and by Lemma 2.3 we obtain
\[ \nu_2(q^{k'_j}) - 1 = 1 + \nu_2(q + 1) + \nu_2(k'_j), \]

thus (2.2) is equivalent to
\[ \nu_2(k'_j) \geq \nu_2(n) - \nu_2(q + 1) - j, \] (2.3)

where \( \nu_2(k'_j) \) is the smallest non-negative integer satisfying this inequality. We remark that (2.3) is trivial in the case when the right side of the inequality is less or equal to zero. From here we need to consider two cases:

1) If \( \nu_2(n) \leq \nu_2(q + 1) \) it follows that \( \nu_2(k'_j) = 0 \) and therefore \( \nu_2(k_j) = 1 \) for every \( 0 \leq j \leq \nu_2(n) - 2 \). Thus
\[ k_j = k_{j+1} \quad \text{and} \quad [\mathbb{F}_q(\alpha^{2^j}) : \mathbb{F}_q(\alpha^{2^{j+1}})] = 1 \quad \text{for every} \quad 0 \leq j \leq \nu_2(n) - 2. \]

2) If \( \nu_2(n) > \nu_2(q + 1) \), we have to consider two possibilities:

2.1) If \( j \geq \nu_2(n) - \nu_2(q + 1) \), it follows that \( \nu_2(k'_j) = 1 \) and therefore \( k_j = k_{j+1} \) for every \( j \) such that \( \nu_2(n) - \nu_2(q + 1) \leq j \leq \nu_2(n) - 2 \).
2.2) If \( j < \nu_2(n) - \nu_2(q + 1) \) then \( \nu_2(k'_j) = \nu_2(n) - \nu_2(q + 1) - j \), hence
\[
\nu_2(k'_{j-1}) = \nu_2(n) - \nu_2(q + 1) - j + 1 \quad \text{and} \quad \nu_2(k'_j) = \nu_2(k'_{j-1}) + 1,
\]
consequently \( [\mathbb{F}_q(\alpha^{2^{j-1}}) : \mathbb{F}_q(\alpha^{2^j})] = 2 \) for every \( 1 \leq j < \nu_2(n) - \nu_2(q + 1) \). In particular, \( [\mathbb{F}_q(\alpha) : \mathbb{F}_q(\alpha^2)] = 2 \). \( \Box \)

**Corollary 2.6.** Let \( q \) be power of a prime such that \( q \equiv 3 \pmod{4} \) and \( f_i \) be an irreducible \( s \)-self-involutive factor of \( x^n - 1 \in \mathbb{F}_q[x] \). Let us denote by \( \xi_i \) some root of \( f_i \).

1) If \( \nu_2(n) > \nu_2(q + 1) \) then
\[
\mathbb{F}_q(\xi_i + \xi_i^s, \xi_i^{s+1}) = \mathbb{F}_q(\xi_i^2).
\]
2) If \( \nu_2(n) \leq \nu_2(q + 1) \) then
\[
\mathbb{F}_q(\xi_i + \xi_i^s, \xi_i^{s+1}) = \mathbb{F}_q(\xi_i^{2x^{2}(n)}).
\]

**Proof:** The proof follows from previous lemma and Lemma 2.4. \( \Box \)

### 3. Group algebra of split metacyclic group

Throughout this section, \( G \) is a non-abelian group with the following presentation
\[
G = \langle x, y \mid x^n = 1 = y^2, xy = yx^s \rangle. \tag{3.1}
\]
The polynomial \( x^n - 1 \in \mathbb{F}_q[x] \) splits into monic irreducible factors as:
\[
x^n - 1 = f_1f_2\cdots f_{r}f_{r+1}^s\cdots f_{r+2}^s\cdots f_{r+t}^s, \tag{3.2}
\]
where \( f_1 = x - 1, f_2 = x + 1 \) if \( n \) is even, \( f_j^s = f_j \) for each \( 2 \leq j \leq r \), where \( r \) is the number of \( s \)-self-involutive factors and \( 2t \) is the number of non \( s \)-self-involutive factors in the decomposition.

**Theorem 3.1.** Let \( G \) be a metacyclic group defined by the relation (3.1), where \( s^2 \equiv 1 \pmod{n} \).
If \( d := \gcd(n, s - 1) \), then
\[
\mathbb{F}_qG \cong \bigoplus_{l | d} 2 \cdot \frac{\phi(l)}{\text{ord}_l q} \mathbb{F}_q(\theta_l) \oplus \bigoplus_{1 \leq i \leq r} A_i \oplus \bigoplus_{r+1 \leq i \leq r+t} B_i,
\]
where \( \theta_l \) is a \( l \)-th primitive root of the unit and
\[
\begin{align*}
A_i & \cong M_2(\mathbb{F}_q(\xi_i + \xi_i^s, \xi_i^{s+1})) \\
B_i & \cong M_2(\mathbb{F}_q(\xi_i))
\end{align*}
\]
with \( \xi_i \) is a root of the polynomial \( f_i(x) \).

**Proof:** Let \( H \) be a subgroup of \( G \) defined as \( H = \langle z, y \rangle \), where \( z = x^{n/d} \). Since
\[
zy = x^{n/d}y = y(x^{n/d})^s = yx^{n(s-1)/d}x^{n/d} = yx^{n/d} = yz,
\]
then \( H \) is an abelian group isomorphic to the abelianization of \( G \). Let \( \psi \) be the group homomorphism defined by the generators of \( G \) as
\[
\psi : \ G \rightarrow H \\
x \mapsto z \\
y \mapsto y.
\]
This homomorphism can be extended to a homomorphism of group algebras
\[
\Psi : \mathbb{F}_qG \rightarrow \mathbb{F}_qH,
\]
that is surjective and then $F_q H$ is a non-simple component of $F_q G$. Now, using that $H = \langle y \rangle \times \langle z \rangle$ and $F_q \langle y \rangle \cong F_q \oplus F_q$, by (2.1) we have that
\[
F_q H = F_q (\langle y \rangle \times \langle z \rangle) \cong (F_q \oplus F_q) \langle z \rangle
\cong F_q C_d \oplus F_q C_d \cong \bigoplus_{i \in \mathcal{D}} 2 \cdot \frac{\phi(l)}{ord_l q} F_q (\theta_i),
\]
(3.3)
where the last isomorphism follows from Perlis-Walker’s Theorem (Theorem 2.1).

This isomorphism can be shown explicitly, considering for each irreducible factor $g(z)$ of $z^n - 1$ in $F_q [z]$, the algebra-homomorphism determined by the generators as
\[
\psi_g : F_q H \longrightarrow F_q[\frac{z}{g(z)}] \\
z \longmapsto (\zeta, \zeta) \\
y \longmapsto (1, -1)
\]

We note that $F_q[\frac{z}{g(z)}] \cong F_q (\theta_q)$, where $\theta_q$ is a root of $g$ and therefore $\theta_q$ is a $d$-th root of unit, not necessarily primitive. The isomorphism is the direct sum of the homomorphisms constructed in the previous way.

In the other hand, following the notation of (3.2), let $f_i(x)$ be an irreducible factor of $\frac{z^n - 1}{z^i - 1}$ and $\xi_i$ a root of $f_i$. Let us define $\tau_i$ the homomorphism determining by
\[
\tau_i : F_q G \longrightarrow M_2(F_q (\xi_i)) \\
x \longmapsto \left( \begin{array}{cc} \xi_i & 0 \\ 0 & \xi_i^* \end{array} \right) \\
y \longmapsto \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)
\]

By direct calculation it is easy to verify that $(\tau_i(x))^n = I$ and $\tau_i(x)\tau_i(y) = \tau_i(y)\tau_i(x)^s$ and then $\tau_i$ is a well defined homomorphism. These homomorphisms are not necessarily surjective. Indeed, for each $1 \leq i \leq \tau$, let us define $Z_i = \left( \begin{array}{cc} 1 & -\xi_i \\ 1 & -\xi_i^* \end{array} \right)$ and $\sigma_i$ be the map given by the conjugation determined by $Z_i$, i.e.
\[
\sigma_i : M_2(F_q (\xi_i)) \longrightarrow M_2(F_q (\xi_i)) \\
X \longmapsto Z_i^{-1} X Z_i
\]

(3.4)

Composing that homomorphism with $\tau_i$, we obtain a new homomorphism that satisfies the relations
\[
\sigma_i \circ \tau_i(x) = \left( \begin{array}{cc} 0 & \xi_i^{s+1} \\ -1 & \xi_i \end{array} \right) \quad \text{and} \quad \sigma_i \circ \tau_i(y) = \left( \begin{array}{cc} 1 & -\xi_i + \xi_i^* \\ 0 & -1 \end{array} \right),
\]
consequently the image of the map $\sigma_i \circ \tau_i$ is contained in $F_q (\xi_i + \xi_i^*, \xi_i^{s+1}) \subseteq F_q (\xi_i)$. It follows that, for each $1 < i \leq \tau$
\[
\dim_{F_q}(Im(\tau_i)) = \dim_{F_q}(Im(\sigma_i \circ \tau_i)) \leq 4 \dim_{F_q}(F_q (\xi_i + \xi_i^*, \xi_i^{s+1})),
\]
(3.5)
and
\[
\dim_{F_q}(Im(\tau_i)) \leq 4 \dim_{F_q}(F_q (\xi_i)).
\]
(3.6)
in the case when $r + 1 \leq i \leq r + t$.

From the homomorphisms previously defined, let us define $\tau$ the $F_q$-algebras homomorphism
\[
\bigoplus_{g|\langle x^d - 1 \rangle} \psi_g \oplus \bigoplus_{f|\langle x^n - 1 \rangle} \tau_i
\]

We claim that $\tau$ is injective. Indeed, let $u = P(x) + Q(x)y \in F_q G$ be an element in $Ker(\tau)$, where $P(x)$ and $Q(x)$ are polynomials of degree less or equal to $n - 1$. Since $\tau(u) = 0$, it follows
that $\psi_g(u) = 0$ for all $g(x)|(x^d - 1)$ and $\tau_i(u) = 0$ for each $f_i | \frac{x^n - 1}{x^d - 1}$. In the first case we have that

$$\psi_g(u) = \psi_g(P(x) + Q(x)y) = (P(\theta_g) + Q(\theta_g), P(\theta_g) - Q(\theta_g)) = (0, 0),$$

where $\theta_g$ is a root of $g(x)$. Whereas that the characteristic of $\mathbb{F}_q$ is different that 2, it follows that $P(x)$ and $Q(x)$ are zero when we evaluate these polynomials at the roots of $x^d - 1$. Besides that, for each $f_i | \frac{x^n - 1}{x^d - 1}$, we have that

$$\tau_i(u) = \left(\begin{array}{cc} P(\xi_i) & Q(\xi_i) \\ Q(\xi_i) & P(\xi_i) \end{array}\right) = \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right)$$

$$P(\xi_i) = P(\xi_i) = 0 \quad \text{and} \quad Q(\xi_i) = Q(\xi_i) = 0,$$

therefore $P(x)$ and $Q(x)$ are divisible by the polynomials $f_i(x)$ and $f_i^∗(x)$ and then also divisible by $\frac{x^n - 1}{x^d - 1}$. From these two results we obtain that $P(x)$ and $Q(x)$ are divisible by $x^n - 1$ and since the degree of these polynomials are less that $n$, we conclude that $P(x)$ and $Q(x)$ are the null polynomial. In conclusion, the homomorphism

$$\rho : F_qG \rightarrow \bigoplus_{l|d} 2 \cdot \frac{\phi(l)}{ord_q} \mathbb{F}_q(\theta_l) \oplus \bigoplus_{i=1}^r A_i \oplus \bigoplus_{i=r+1}^{r+t} B_i$$

defined by

$$\begin{cases}
\psi_g, & \text{if } g \text{ is an irreducible factor of } x^d - 1 \\
\sigma_i \circ \tau_i, & \text{if } 1 \leq i \leq r \text{ and } f_i \nmid (x^d - 1) \\
\tau_i, & \text{if } r + 1 \leq i \leq r + t \text{ and } f_i \nmid (x^d - 1)
\end{cases}$$

is an injective map. Finally by Lemma 2.4 and (3.5) we have that

$$2n \leq \dim_{\mathbb{F}_q} \left(\bigoplus_{l|d} 2 \cdot \frac{\phi(l)}{ord_q} \mathbb{F}_q(\theta_l) \oplus \bigoplus_{i=1}^r A_i \oplus \bigoplus_{i=r+1}^{r+t} B_i\right)$$

$$\leq 2d + 4 \sum_{i=1}^r \dim_{\mathbb{F}_q}(\mathbb{F}_q(\xi_i + \xi_i^s, \xi_i^{s+1})) + 4 \sum_{i=r+1}^{r+t} \dim_{\mathbb{F}_q}(\mathbb{F}_q(\xi_i))$$

$$= 2d + 2r \sum_{i=1}^r \deg(f_i) + 2 \sum_{i=r+1}^{r+t} \deg(f_i) = 2 \left(d + \deg \left(\frac{x^n - 1}{x^d - 1}\right)\right) = 2n,$$

consequently the homomorphism is also surjective. □

### 4. Group algebra of non-split metacyclic group

Throughout this section, $G$ is a group with the following presentation

$$G = \langle x, y \mid x^{2n} = 1, y^2 = x^n, xy = yx^s\rangle,$$

(4.1)

where $s^2 \equiv 1 \pmod{2n}$, and the polynomial $x^{2n} - 1 \in \mathbb{F}_q[x]$ splits into monic irreducible factors as:

$$x^{2n} - 1 = f_1 f_2 \cdots f_r f_{r+1} f_{r+2} \cdots f_{r+t},$$

(4.2)

where $f_1 = x - 1$, $f_2 = x + 1$ and $f_j^s = f_j$ for each $2 \leq j \leq r$, where $r$ is the number of $s$-self-involutive factors and $2t$ is the number of non $s$-self-involutive factors in the decomposition. We observe that if $s = -1$ then $G$ is the generalized quaternion group $Q_n$. It knows that $Q_n$ has some irreducible components that are isomorphic to non-commutative division rings. In contrast, in $\mathbb{F}_qQ_n$ this type of component does not appear, because Wedderburn’s Little Theorem (Theorem 2.55 in [13]) guarantees that every finite division ring is a field.

We observe that $[x, y] = x^{-1} y^{-1} xy = x^{s-1}$ then the commutator group $[G, G]$ is generated by $x^d$ where $d = \gcd(2n, s - 1)$. Therefore, the abelianization of $G$ is isomorphic to a subgroup $H$ of $G$ determined by

$$G_{ab} \cong H := \langle z, y \mid z^d = 1, \quad y^2 = z^{d/2}, \quad zy = yz\rangle,$$

(4.3)
where $z = x^{2n/d}$. As in the previous section, let us define the group homomorphism

$$
\psi : G \longrightarrow H
$$

\[
\begin{align*}
  x & \mapsto z \\
  y & \mapsto y,
\end{align*}
\]

that is surjective and can be extend to a group-algebra homomorphism $\Psi : \mathbb{F}_q G \longrightarrow \mathbb{F}_q H$ and hence $\mathbb{F}_q G \cong \mathbb{F}_q H \oplus \text{Ker}(\Psi)$. Thus, we need to understand the structure of $\mathbb{F}_q H$ and $\text{Ker}(\Psi)$. In following lemma, we show explicitly the components of the abelian part.

**Lemma 4.1.** Let $G$ be a group with presentation (4.1) and $H$ be a subgroup of $G$ that is isomorphic to the abelianization of $G$ defined in (4.3).

1) If $s \equiv 1 \pmod{4}$ then $\mathbb{F}_q H \cong \bigoplus_{l | d} 2 \cdot \frac{\phi(l)}{ord_q} \mathbb{F}_q(\theta_l)$.

2) If $s \equiv 3 \pmod{4}$ and $q \equiv 1 \pmod{4}$ then $\mathbb{F}_q H \cong \bigoplus_{l | d/2} 4 \cdot \frac{\phi(l)}{ord_q} \mathbb{F}_q(\theta_l)$.

3) If $s \equiv 3 \pmod{4}$ and $q \equiv 3 \pmod{4}$ then $\mathbb{F}_q H \cong \bigoplus_{l | d/2} 2 \cdot \frac{\phi(l)}{ord_q} \mathbb{F}_q(\theta_l) \oplus \frac{\phi(l)}{ord_q} \mathbb{F}_q(\theta_{2l})$.

**Proof:** It follows from Representation Theorem for Abelian Groups that $H$ is a direct product of cyclic groups. In order to show explicitly the direct product, we consider three cases:

**Case 1)** $s \equiv 1 \pmod{4}$: Since $d = \text{gcd}(2n, s - 1)$, then $\frac{d}{2}$ is even and $H$ has the following presentation:

$$
H = \langle z, w \mid z^d = 1, w^2 = 1 \rangle \cong C_d \times C_2 \quad \text{where} \quad w = yz^{d/4}.
$$

From Perlis-Walker’s Theorem we have that

$$
\mathbb{F}_q H \cong (\mathbb{F}_q \oplus \mathbb{F}_q)C_d \cong \mathbb{F}_q C_d \oplus \mathbb{F}_q C_2 \cong \bigoplus_{l | d} 2 \cdot \frac{\phi(l)}{ord_q} \mathbb{F}_q(\theta_l).
$$

This isomorphism can be presented explicitly as

$$
\psi_1 : \mathbb{F}_q H \longrightarrow \mathbb{F}_q(\theta_1) \oplus \mathbb{F}_q(\theta_1)
$$

\[
\begin{align*}
  z & \mapsto (\theta_1, \theta_1) \\
  w & \mapsto (1, -1)
\end{align*}
\]

where, for each irreducible factor of $x^d - 1$, we choose some root $\theta_l$ of order $l$. There exist exactly $\frac{\phi(l)}{ord_q}$ factors of that type. From this isomorphism we can construct a surjective homomorphism $\mathbb{F}_q G \rightarrow \mathbb{F}_q(\theta_1) \oplus \mathbb{F}_q(\theta_1)$, which by notation abuse we also denote by $\psi_1$, defined from the generators of $G$ as

$$
\psi_1 : \mathbb{F}_q G \longrightarrow \mathbb{F}_q(\theta_1) \oplus \mathbb{F}_q(\theta_1)
$$

\[
\begin{align*}
  x & \mapsto (\theta_1, \theta_1) \\
  y & \mapsto (\theta_1^{d/4}, -\theta_1^{d/4})
\end{align*}
\]

**Case 2)** $s \equiv 3 \pmod{4}$ and $q \equiv 1 \pmod{4}$: In this case, $d$ is even but not divisible by 4. We claim that $H$ can be represented as

$$
H = \langle t, y \mid t^{d/2} = 1, y^4 = 1 \rangle, \quad \text{where} \quad t = z^2.
$$

In order to prove that affirmation, it is enough to show that $z$ can be written as an expression that depends of $t$ and $y$, indeed

$$
z = z^{d+1} = z^{d/2} z^{d/2+1} = y^2 t^{d+2}.
$$

Thus $H \cong C_{d/2} \times C_4$ and

$$
\mathbb{F}_q H \cong \mathbb{F}_q(C_4 \times C_{d/2}) \cong (\mathbb{F}_q C_4)C_{d/2} \cong (\mathbb{F}_q \oplus \mathbb{F}_q \oplus \mathbb{F}_q \oplus \mathbb{F}_q)C_{d/2} \cong \mathbb{F}_q C_{d/2} \oplus \mathbb{F}_q C_{d/2} \oplus \mathbb{F}_q C_{d/2} \oplus \mathbb{F}_q C_{d/2}.
$$
Again by Perlis Walker’s Theorem it follows that
\[ F_qC_{d/2} \oplus F_qC_{d/2} \oplus F_qC_{d/2} \oplus F_qC_{d/2} \cong \bigoplus_{l|d/2} 4 \cdot \frac{\phi(l)}{\text{ord}_q} \mathbb{F}_q(\theta_l), \]
where the isomorphism can be explicitly determined by
\[ \psi_l : F_q H \rightarrow F_q(\theta_l) \oplus F_q(\theta_l) \oplus F_q(\theta_l) \oplus F_q(\theta_l) \]
\[ t \mapsto (\theta_l, \theta_l, \theta_l, \theta_l) \]
\[ y \mapsto (1, -1, \beta, -\beta) \]
where, for each irreducible factor of \( x^{d/2} - 1 \), we select an arbitrary root \( \theta_l \) of order \( l \) and \( \beta \in F_q \) satisfies \( \beta^2 = -1 \).

The same way, this isomorphism generates a surjective homomorphism \( F_q G \rightarrow F_q(\theta_l) \oplus F_q(\theta_l) \), defined by
\[ \psi_l : F_q G \rightarrow F_q(\theta_l) \oplus F_q(\theta_l) \oplus F_q(\theta_l) \oplus F_q(\theta_l) \]
\[ t \mapsto (\theta_l^{\frac{d+2}{4}}, \theta_l^{\frac{d+2}{4}}, -\theta_l^{\frac{d+2}{4}}, -\theta_l^{\frac{d+2}{4}}) \]
\[ y \mapsto (1, -1, \beta, -\beta). \]

Case 3) \( s \equiv 3 \pmod{4} \) and \( q \equiv 3 \pmod{4} \): As at before case, \( H \) can be represented as
\[ H = \langle t, y \mid t^{d/2} = 1, y^4 = 1 \rangle, \text{ where } t = z^2. \]
Since \( d/2 \) is odd, then \( F_q(\theta_l) \) is an extension of odd degree for any \( \frac{d}{2} \)-th root of unit \( \theta_l \), so \(-1\) is not a square in \( F_q(\theta_l) \), consequently
\[ F_q(C_4 \times C_{d/2}) \cong (F_qC_4)C_2 \cong (F_q \oplus F_q \oplus F_{q^2})C_{d/2} \cong F_qC_{d/2} \oplus F_qC_{d/2} \oplus F_{q^2}C_{d/2}. \]
\[ \cong \bigoplus_{l|d/2} 2 \cdot \frac{\phi(l)}{\text{ord}_q} \mathbb{F}_q(\theta_l) \oplus \frac{\phi(l)}{\text{ord}_q} \mathbb{F}_q(\theta_{2l}) \]
where the isomorphism can be written explicitly as
\[ \psi_l : F_q H \rightarrow F_q(\theta_l) \oplus F_q(\theta_l) \oplus F_q(\theta_l, \beta) \]
\[ t \mapsto (\theta_l, \theta_l, \theta_l) \]
\[ y \mapsto (1, -1, \beta) \]
where \( \theta_l \) is a root of \( x^{d/2} - 1 \) of order \( l \), \( \beta \in F_{q^2} \) is a square root of \(-1\). Therefore using previous homomorphism we construct a homomorphism with domain \( F_q G \) determined by the generators of \( G \) as
\[ \psi_l : F_q G \rightarrow F_q(\theta_l) \oplus F_q(\theta_l) \oplus F_q(\theta_l, \beta) \]
\[ x \mapsto (\theta_l^{\frac{d+2}{4}}, \theta_l^{\frac{d+2}{4}}, -\theta_l^{\frac{d+2}{4}}) \]
\[ y \mapsto (1, -1, \beta). \]

The components previously found correspond to the simple abelian components of the group algebra \( F_q G \). Thus, we now need to analyze the simple non-abelian components of this group algebra, that we know from the Wedderburn-Artin Theorem, is direct sum of components that are isomorphic to matrix algebras over some finite extension of \( F_q \). We will construct homomorphisms from \( F_q G \) to each of these components.

**Theorem 4.2.** Let \( G \) be a metacyclic group with presentation
\[ G = \langle x, y \mid x^{2n} = 1, y^2 = x^n, xy = yx^s \rangle \]
with \( s^2 \equiv 1 \pmod{2n} \). Let \( d = \gcd(2n, s-1) \) and \( H \) be the subgroup \( G \) defined in Lemma \( \ref{lem:metacyclic-group}. \)
Then \( F_q G \cong F_q H \oplus \mathcal{L} \), where \( \mathcal{L} \) can be written as sum of simple components of the form:
\[ \mathcal{L} \cong \bigoplus_{1 \leq i \leq r} A_i \oplus \bigoplus_{1 \leq i \leq r+1} B_i, \]
where \( A_i \cong M_2(F_q(\xi_i + \xi_i^s, \xi_i^{s+1})) \), \( B_i \cong M_2(F_q(\xi_i)) \) and \( \xi_i \) is a root of \( f_i(x) \).
Proof: Since \( \frac{1+x^n}{2} \) and \( \frac{1-x^n}{2} \) are central orthogonal idempotents of \( \mathbb{F}_qG \), we have that
\[
\mathbb{F}_qG \cong \mathbb{F}_qG \left( \frac{1+x^n}{2} \right) \oplus \mathbb{F}_qG \left( \frac{1-x^n}{2} \right).
\]
In addition, in \( \mathbb{F}_q(G(\frac{1+x^n}{2})) \) the element \( \overline{x} \) satisfies that \( \overline{x}^n = 1 \), hence this component is isomorphic to the group algebra generated by the group of Theorem \[2.1\]. Therefore, we only need to determine the non-abelian components of \( \mathbb{F}_qG(\frac{1-x^n}{2}) \) and these components are generated by the irreducible factors of \( x^n + 1 \) such that do not divide \( x^d - 1 \).

Let \( f_i \) be an irreducible factor of \( x^n + 1 \) such that \( \gcd(f_i, x^d - 1) = 1 \) and \( \xi_i \) be any root of \( f_i \). Let consider the natural homomorphism generated by the representation of \( G \) associated to \( \xi_i \), i.e.
\[
\omega_i : \mathbb{F}_qG \rightarrow M_2(\mathbb{F}_q(\xi_i)),
\]
which is an automorphism such that
\[
\eta_i : M_2(\mathbb{F}_q(\xi_i)) \rightarrow M_2(\mathbb{F}_q(\xi_i, \beta))
\]
is an automorphism such that
\[
\eta_i \circ \omega_i(y) = \begin{pmatrix} -\beta & 0 \\ 0 & \beta \end{pmatrix}
\]
and
\[
\eta_i \circ \omega_i(x) = \begin{pmatrix} 0 & \beta \\ \xi_i^{s+1} & \xi_i^s \end{pmatrix}
\]
thus the image of the generators are in \( \mathbb{F}_q(\xi_i + \xi_i^s, \xi_i^{s+1}, \beta) \).

We claim that, in the case that \( q \equiv 1 \pmod{4} \) or \( q \equiv 3 \pmod{4} \) and \( \nu_2(n) > \nu_2(q + 1) \), the field \( \mathbb{F}_q(\xi_i + \xi_i^s, \xi_i^{s+1}, \beta) \) is a proper subfield of \( \mathbb{F}_q(\xi_i) \). Indeed by Lemma \[2.4\] we have that
\[
(\mathbb{F}_q(\xi_i) : \mathbb{F}_q(\xi_i + \xi_i^s, \xi_i^{s+1})) = 2.
\]
In addition, if \( q \equiv 1 \pmod{4} \) then \( \beta \in \mathbb{F}_q \), or if \( q \equiv 3 \pmod{4} \) and \( \nu_2(n) > \nu_2(q + 1) \) it follows from Lemma \[2.5\] that \( [\mathbb{F}_q(\xi_i) : \mathbb{F}_q] = l \) is divisible by 4. Therefore \( [\mathbb{F}_q(\xi_i + \xi_i^s, \xi_i^{s+1}) : \mathbb{F}_q] = \frac{l}{2} \) is even and \( \beta \in \mathbb{F}_q(\xi_i + \xi_i^s, \xi_i^{s+1}) \). In any of these cases, \( \eta_i \circ \omega_i \) is an homomorphism such that the image is contained in \( M_2(\mathbb{F}_q(\xi_i + \xi_i^s, \xi_i^{s+1})) \).

The last case to consider is when \( q \equiv 3 \pmod{4} \) and \( \nu_2(n) \leq \nu_2(q + 1) \). Let \( f_i \) be a \( s \)-self-involutive irreducible factor of \( x^n + 1 \) such that \( f_i \) does not divide \( x^d - 1 \) and \( \xi_i \) be a root of \( f_i \). By Lemma \[2.5\] we know that
\[
[\mathbb{F}_q(\xi_i) : \mathbb{F}_q] = 2l \quad \text{with } l \text{ odd and } [\mathbb{F}_q(\xi_i^{\nu_2(n)+1}) : \mathbb{F}_q] = l.
\]
In addition, using the fact that \( f_i \) is \( s \)-self-involutive we also have that
\[
[\mathbb{F}_q(\xi_i) : \mathbb{F}_q(\xi_i + \xi_i^s, \xi_i^{s+1})] = 2.
\]
Thus
\[
\mathbb{F}_q = \mathbb{F}_q(\xi_i^{\nu_2(n)+1}) = \mathbb{F}_q(\xi_i + \xi_i^s, \xi_i^{s+1})
\]
and since
\[
(\xi_i^{s+1})^{(q-1)} = 1 \quad \text{and} \quad \nu_2(q^l - 1) = 1
\]
it follows that \( \nu_2(n) \leq \nu_2(s + 1) \).
Now we can suppose without loss of generality that \( \nu_2(s + 1) \leq \nu_2(q + 1) + 1 \), because if necessary we can change \( s \) by \( s + 2n \) and every conditions remain valid. In this case, there exists \( \theta_i \in \mathbb{F}_q(\xi_i) = \mathbb{F}_{q^2} \) such that
\[
\theta_i^{\nu_2(s + 1) - \nu_2(n)} = \xi_i
\]
and this element has order
\[
2^{\nu_2(s + 1) - \nu_2(n)} m = 2^{\nu_2(s + 1) + 1} m \quad \text{with} \quad m \text{ odd}.
\]
Therefore \( \theta \) is an element of \( \mathbb{F}_q(\xi_i) \) that is not a square in this field. Moreover, from
\[
s + 1 \equiv 0 \pmod{2^\nu_2(s + 1)} \quad \text{and} \quad q^l + 1 \equiv 0 \pmod{2^\nu_2(s + 1)}
\]
we have
\[
s \equiv q^l \pmod{2^\nu_2(s + 1)} \quad \text{and} \quad s \equiv q^l \pmod{m 2^\nu_2(s + 1)}.
\]
Consequently \( \theta_i \) is a root of a \( s \)-self-involutive polynomial and
\[
[\mathbb{F}_q(\theta_i) : \mathbb{F}_q(\theta_i + \theta_i^s, \theta_i^{s+1})] = 2.
\]
Let \( a, b \) be elements of \( \overline{\mathbb{F}}_q \) such that \( a^2 = -\theta_i \) and \( b^2 = \theta_i^s \) and \( Z_i \) be the matrix defined as
\[
Z_i = \begin{pmatrix} a & b \\ -\xi_i a & -\xi_i b \end{pmatrix}.
\]
Using this matrix, we can defined a conjugation over the image of \( \omega_i \) determined by the generators
\[
Z_i \cdot \begin{pmatrix} \xi_i & 0 \\ 0 & \xi_i^s \end{pmatrix} \cdot Z_i^{-1} = \begin{pmatrix} 0 & -1 \\ \xi_i^{s+1} & \xi_i + \xi_i^s \end{pmatrix}
\]
and
\[
Z_i \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot Z_i^{-1} = \frac{1}{ab} \begin{pmatrix} \theta_i \xi_i - (\theta_i \xi_i)^s & \theta_i - \theta_i^s \\ \xi_i - \xi_i^s & (\xi_i^2 \theta_i)^s - \xi_i^2 \theta_i \\ \xi_i^2 - \xi_i^s & \xi_i \xi_i^s - (\theta_i \xi_i)^s \\ \xi_i^2 - \xi_i^s & \xi_i^2 - \xi_i^s \end{pmatrix}.
\]
Clearly the matrix in (4.4) has coefficients in
\[
\mathbb{F}_q(\theta_i + \theta_i^s, \theta_i^{s+1}) = \mathbb{F}_q(\xi_i + \xi_i^s, \xi_i^s) = \mathbb{F}_q(\xi_i^{2^{\nu_2(n) + 1}}).
\]
We claim that the matrix in (4.3) also has coefficients in this field. Since \( (ab)^2 = -\theta_i^{s+1} \), in order to prove that \( ab \in \mathbb{F}_q^{2^l} \) it is enough to show that \( -\theta_i^{s+1} \) is a square in that field. Indeed
\[
(-\theta_i^{s+1})^2 = \frac{1}{2} = -\theta_i^{(s+1)(2^l-1)} \quad \text{and} \quad \nu_2 \left( (s + 1) \left( \frac{q^l - 1}{2} \right) \right) = \nu_2(s + 1).
\]
By construction \( \theta_i \) has order \( 2^{\nu_2(s + 1) + 1} m \), thus \( \theta_i^{(s+1)(2^l-1)} \neq 1 \). Since \( -1 \) is not a square, therefore \( -\theta_i^{s+1} \) is a square in \( \mathbb{F}_q(\theta_i + \theta_i^s, \theta_i^{s+1}) \). Finally, each one of matrix entries is of the form
\[
G_{c,d}(w, z) := \pm \frac{w^c - z^c}{w^d - z^d},
\]
where \( w = \theta_i \), \( z = \theta_i^s \) and \( c \) and \( d \) are appropriate integers. Since \( G_{c,d}(x, y) = G_{c,d}(y, x) \) by the Fundamental Theorem of Symmetric Functions (see [14, Theorem 2.20]), there exists \( G_{c,d} \in \mathbb{F}_q(x, y) \) quotient of polynomials such that \( G_{c,d}(x, y) = G_{c,d}(x + y, xy) \) and this identity proves that each entry of the matrix is in the field \( \mathbb{F}_q(\theta_i + \theta_i^s, \theta_i^{s+1}) \).

To finish the proof, let us consider the homomorphism \( \Lambda : \mathbb{F}_qG \to \mathbb{F}_qH \oplus \mathcal{L} \) defined as the direct sum of the homomorphism \( \mathbb{F}_qG \to \mathbb{F}_qH \) found in Lemma 4.1 with the homomorphism \( \mathbb{F}_qG \to \mathcal{L} \) found in Theorem 1.2 i.e.,
\[
\begin{cases}
\eta_i \circ \omega_i & \text{if} \quad 1 \leq i \leq r \quad \text{and} \quad f_i \nmid x^d - 1 \\
\omega_i & \text{if} \quad r + 1 \leq i \leq r + t \quad \text{and} \quad f_i \nmid x^d - 1
\end{cases}
\]
Following the same steps of the proof of Theorem 3.1, it is easy to prove that if \( u = P(x) + Q(x)y \) is in the kernel of \( \Lambda \), then \( P(x) \) and \( Q(x) \) are divisible by \( x^{2n} - 1 \), therefore \( P(x) \equiv 0 \) e \( Q(x) \equiv 0 \) and consequently \( \Lambda \) is injection. Calculating the dimension of \( \mathbb{F}_q H \oplus \mathcal{L} \) as a vector space over \( \mathbb{F}_q \), we have

\[
4n \leq \dim_{\mathbb{F}_q} \left( \mathbb{F}_q H \oplus \bigoplus_{1 \leq i \leq r} A_i \oplus \bigoplus_{r+1 \leq i \leq r+l} B_i \right)
\]

\[
\leq 4d + 4 \cdot \sum_{i=1}^{r+t} \dim_{\mathbb{F}_q} (\mathbb{F}_q(\xi_i + \xi_i^s, \xi_i^{s+1})) + 4 \cdot \sum_{i=r+1}^{r+t} \dim_{\mathbb{F}_q} (\mathbb{F}_q(\xi_i))
\]

\[
= 4d + 4 \cdot \sum_{i=1}^{r+t} \deg(f_i) + 4 \cdot \sum_{i=r+1}^{r+t} \deg(f_i)
\]

\[
= 4 \left(d + \deg \left( \frac{x^n + 1}{x^d - 1} \right) \right) = 4n,
\]

which is the dimension of \( \mathbb{F}_q G \) over \( \mathbb{F}_q \), hence it proves that \( \Lambda \) is an isomorphism. \( \square \)

**Remark 4.3.** In order to determine the number of idempotents of \( \mathbb{F}_q G \) we do not need to know the explicit factorization of the polynomial \( x^{2n} - 1 \) over \( \mathbb{F}_q[x] \), actually we only need to know how many simple component have \( \mathbb{F}_q G \) and to determine this number we observe that the polynomial \( x^{2n} - 1 \) can be factorize of the following form

\[
x^{2n} - 1 = \prod_{m \mid 2n} \Phi_m(x),
\]

where \( \Phi_m(x) \) is the cyclotomic polynomial of order \( m \) and, from Theorem 2.47 in [13], \( \Phi_m(x) \) splits into \( \phi(m)/d \) irreducible factors of degree \( d = \ord_{m|q} \). We observe that the s-self-involutive propriety depends only of the degree, specifically one factor of order \( m \) and degree \( d \) is s-self-involutive if and only if \( s \equiv q^{d/2} \mod m \).

### 5. Central Primitive Idempotents

In this section, we show explicitly an expression for every central primitive idempotent of the group algebra \( \mathbb{F}_q G \), where \( G \) is a group that have presentation of the forms (3.1) and (4.1). Before we show the expression of the idempotents, we present the form of the idempotents when the group is cyclic.

**Theorem 5.1.** [4] Theorem 2.1] Let \( \mathbb{F}_q \) be a finite field with \( q \) elements and \( n \in \mathbb{N}^* \) such that \( \gcd(q, n) = 1 \), then each primitive idempotent of \( \mathbb{F}_q C_n \) is of the form:

\[
e_{d,j}(x) = \frac{x^n - 1}{f_{d,j}(x)} h_{d,j}(x),
\]

where \( f_{d,j} \) is an irreducible factor of the cyclotomic polynomial \( \Phi_d(x) \), with \( d \) a divisor of \( n \) and \( h_{d,j}(x) \in \mathbb{F}_q[x] \) the polynomial of degree \( \deg(h_{d,j}(x)) = \ord_{q|d} q \) that is the inverse of \( \frac{x^n - 1}{f_{d,j}(x)} \) in the field \( \mathbb{F}_q[x]/(f_{d,j}) \).

We note that if we know the explicit form of the polynomial \( f_{d,j} \), then the polynomial \( h_{d,j} \) can be calculate using the Extended Euclidean Algorithm for polynomials.

**Lemma 5.2** ([4], Lemma 2.1). Let \( \mathcal{I} \subseteq \mathbb{F}_q[x]/(x^n - 1) \) be an ideal generated by the monic polynomial \( g \) that is a divisor of \( x^n - 1 \) and let \( f = \frac{x^n - 1}{g} \). Then the idempotent of \( \mathcal{I} \) is

\[
e_f := - \left( \left( \frac{f^*}{g} \right)^* \right) \cdot \frac{x^n - 1}{f}
\]
where \( f'(x) \) and \( f^*(x) \) denote respectively the formal derivation and the reciprocal of the polynomial \( f(x) \).

**Theorem 5.3.** Let \( G \) be a metacyclic group with the following presentation

\[
G = \langle x, y \mid x^n = 1 = y^2, y^{-1}xy = x^s \rangle
\]

where \( s^2 \equiv 1 \pmod{n} \) and let \( d = \gcd(n, s - 1) \). The central idempotents of the group algebra \( \mathbb{F}_q G \) are of the following form:

1. For each irreducible divisor \( f_j \) of \( x^d - 1 \), there are two primitive idempotents of the form \( \frac{1+y}{2} e_{f_j} \) and \( \frac{1-y}{2} e_{f_j} \).
2. For each \( s \)-self-involutive irreducible factor \( f_j \) of \( \frac{x^{n-1}}{x^d - 1} \), there exists one primitive idempotent of the form \( e_{f_j} \).
3. For each pair of non \( s \)-self-involutive irreducible factors \( f_j, f_j^* \) of \( \frac{x^{n-1}}{x^d - 1} \), there exists one primitive idempotents of the form \( e_{f_j} + e_{f_j^*} \).

**Proof:** From Theorem \[3.1\] we know that the homomorphism

\[
\rho : \mathbb{F}_q G \rightarrow \bigoplus_{\gamma|d} 2 \cdot \frac{\phi(\gamma)}{\text{ord}_q} \mathbb{F}_q(\theta_{\gamma}) \oplus \bigoplus_{1 \leq i \leq r} A_i \oplus \bigoplus_{1 \leq i \leq r+1} B_i
\]

is injective.

Firstly we consider the idempotents of the non-abelian components, then the image of a central primitive idempotent is an element of the form \( (0, 0, \ldots, I, 0, \ldots, 0) \), i.e. the image is zero on each component, except in one where the image is the identity \( I \) of that component, i.e. for each primitive idempotent \( e \) there exists \( i \) such that \( \rho_j(e) = \delta_{i,j} I_j \) for all \( j \), where \( I_j \) is the identity of the component \( j \) and \( \delta_{i,j} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases} \)

Let \( e = P(x) + Q(x)y \) be a presentation of \( e \), where \( P(x) \) and \( Q(x) \) are polynomials in \( \mathbb{F}_q[x] \) of degree less or equal to \( n - 1 \). Since \( \rho(e) = I \), we have that \( Q(\alpha_i) = 0 \) for every \( \alpha_i \) root of \( \frac{x^n - 1}{x^d - 1} \), hence \( Q(x) \) is divisible by this polynomial. We claim that \( Q(x) \) is also divisible by \( x^d - 1 \) and then it is null polynomial. At the abelian components we have that \( P(\theta_j) + Q(\theta_j) = 0 \) and \( P(\theta_j) - Q(\theta_j) = 0 \) for any \( \theta_j \) root of \( x^d - 1 \), therefore \( Q(x) \) is also divisible by \( x^d - 1 \).

Moreover, \( P(x) = 1 \) when we evaluate at the roots of the polynomials \( f_j \) and \( f_j^* \) and zero when we evaluate in any other root of \( x^n - 1 \). From the injectivity of \( \rho \) and Lemma \[5.2\] it follows that the unique polynomial of degree less or equal to \( n - 1 \) that satisfies that property is \( e_{f_j} \) when \( f_j = f_j^* \), and \( e_{f_j} + e_{f_j^*} \) when \( f_j \neq f_j^* \).

Finally, if \( f_i \mid (x^d - 1) \) then the image of \( \rho_j(e_{f_i}) = (1, 1) \) is not a primitive idempotent. Therefore, that this idempotent can be decomposed in two central primitive idempotents, \( \frac{1+y}{2} e_{f_j} \) and \( \frac{1-y}{2} e_{f_j} \), such that \( \rho_j \left( \frac{1+y}{2} e_{f_j} \right) = (1, 0) \) and \( \rho_j \left( \frac{1-y}{2} e_{f_j} \right) = (0, 1) \).

**Remark 5.4.** In the group determined by \( \{4, 1\} \), the non-abelian idempotents are of the same form as those found in previous theorem. Now, in order to describe the abelian idempotents, we need to consider some cases.

Let \( f \) be an irreducible factor of \( x^d - 1 \in \mathbb{F}_q[x] \). The idempotent \( e_\sigma(x) \) is a non-primitive central idempotent. The decomposition of these idempotents depend to the classes of \( s \) and \( q \) modulo 4.

Using the notation of Lemma \[4.1\], we have the following cases:

**Case 1.** \( s \equiv 1 \pmod{4} \): Since \( H \cong C_d \times C_2 \), then

\[
e_{f}(x) = e_{f}(x) \left( \frac{1+w}{2} \right) + e_{f}(x) \left( \frac{1-w}{2} \right) \quad \text{where } w = x^4 y.
\]
In this case \( e_f(x) \left( \frac{1 + \frac{q}{2} x}{2} \right) \) and \( e_f(x) \left( \frac{1 - \frac{q}{2} x}{2} \right) \) are central primitive idempotents.

Case 2. \( s \equiv 3 \pmod{4} \) \( e \equiv 1 \pmod{4} \): In this case \( H \cong C_{d/2} \times C_4 \) and the idempotent \( e_f(x) \) can be decomposed as

\[
e_f(x) = e_f(x) \left( \frac{1 + y + y^2 + y^3}{4} \right) + e_f(x) \left( \frac{1 - y + y^2 - y^3}{4} \right)
+ e_f(x) \left( \frac{1 + \beta y - y^2 - \beta y^3}{4} \right) + e_f(x) \left( \frac{1 - \beta y - y^2 + \beta y^3}{4} \right)
\]

where \( \beta \) is an element of \( \mathbb{F}_q \) such that \( \beta^2 = -1 \). Using the fact that \( x^n = y^2 \), we can rewrite the expression above as

\[
e_f(x) = e_f(x) \left( \frac{1 + x^n}{2} \right) \left( \frac{1 + y}{2} \right) + e_f(x) \left( \frac{1 + x^n}{2} \right) \left( \frac{1 - y}{2} \right)
+ e_f(x) \left( \frac{1 - x^n}{2} \right) \left( \frac{1 + \beta y}{2} \right) + e_f(x) \left( \frac{1 - x^n}{2} \right) \left( \frac{1 - \beta y}{2} \right).
\]

Some of these terms can be zero. Indeed,

- If \( f \) divides \( x^n - 1 \) then \( e_f(x) = e_f(x) \left( \frac{1 + y}{2} \right) + e_f(x) \left( \frac{1 - y}{2} \right) \).
- If \( f \) divides \( x^n + 1 \) then \( e_f(x) = e_f(x) \left( \frac{1 + \beta y}{2} \right) + e_f(x) \left( \frac{1 - \beta y}{2} \right) \).

Case 3. \( s \equiv 3 \pmod{4} \) and \( q \equiv 3 \pmod{4} \): As in the previous case, we have that \( H \cong C_{d/2} \times C_4 \) but does not exist \( \beta \in \mathbb{F}_q \) such that \( \beta^2 = -1 \). Therefore \( e_f(x) \) can decomposed as

\[
e_f(x) = e_f(x) \left( \frac{1 + y + y^2 + y^3}{4} \right) + e_f(x) \left( \frac{1 - y + y^2 - y^3}{4} \right) + e_f(x) \left( \frac{1 - y^2}{2} \right)
\]

Again using the relation \( x^n = y^2 \) we can simplify this expression:

- If \( f \) divides \( x^n - 1 \) then \( e_f(x) = e_f(x) \left( \frac{1 + y}{2} \right) + e_f(x) \left( \frac{1 - y}{2} \right) \).
- If \( f \) divides \( x^n + 1 \) then \( e_f(x) \) is primitive.

6. **Non-central Idempotents**

Unlike what happens with central primitive idempotents, non-central idempotents are not necessarily unique. In this section, we show a complete family of non-central idempotents finding, when possible, a decomposition of each central idempotent. Moreover, we only need to find the idempotents generate by the factor of \( x^n - 1 \) that are \( s \)-self-involutive, because in the case when \( f \) is not \( s \)-self-involutive, we know that \( e_f \) and \( e_{f*} \) are non-central primitive idempotents.

**Theorem 6.1.** Let \( G \) be a group with the following presentation

\[ G = \langle x, y \mid x^n = 1, y^2 = 1, xy = yx^s \rangle, \]

where \( s^2 \equiv 1 \pmod{n} \) and \( d = \gcd(n, s - 1) \). If \( f(x) \) is a \( s \)-self-involutive irreducible factor of \( x^n - 1 \) and \( \ell(x) \) is a polynomial such that \( (x^{s-1} - 1) \ell(x) \equiv 1 \pmod{f(x)} \) then

\[
e_{f,1} = -\frac{1}{n} \left( (f^*)' \cdot \frac{x^n - 1}{f(x)} \cdot [\ell(x)(1 - y) + 1] \right) \quad \text{and} \quad e_{f,2} = -\frac{1}{n} \left( (f^*)' \cdot \frac{x^n - 1}{f(x)} \cdot [\ell(x)(1 - y)] \right)
\]

are non-central primitive idempotents.

**Proof:** We observe that

\[
\gcd(x^{s-1} - 1, f(x)) = \gcd(\gcd(x^{s-1} - 1, x^n - 1), f(x)) = \gcd(x^d - 1, f(x)) = 1
\]
and this relation ensures that there exists a polynomial $\ell(x)$ that is the inverse of $x^{s-1} - 1$ modulo $f(x)$.

Let $u = P(x) + Q(x)y$ be a non-central idempotent such that its image in the component generated by $f$ for the isomorphism defined in Theorem 3.1 is of the form \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) and the image at the other components are zero. If $\xi$ is a root of the $s$-self-involutive polynomial $f(x)$, using (6.3) we have that

\[
\sigma(u) = \left( \begin{array}{cc} 1 & -\xi \\ 1 & -\xi^s \end{array} \right)^{-1} \left( \begin{array}{cc} P(\xi) & Q(\xi) \\ Q(\xi^s) & P(\xi^s) \end{array} \right) \left( \begin{array}{cc} 1 & -\xi \\ 1 & -\xi^s \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right),
\]

thus

\[
\left( \begin{array}{cc} P(\xi) & Q(\xi) \\ Q(\xi^s) & P(\xi^s) \end{array} \right) = \frac{1}{\xi - \xi^s} \left( \begin{array}{cc} 1 & -\xi \\ 1 & -\xi^s \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{cc} -\xi^s & \xi \\ -1 & 1 \end{array} \right)
\]

and this way we get

\[
P(\xi) = \frac{-\xi^s}{\xi - \xi^s} = -\frac{\xi^{s-1}}{1 - \xi^{s-1}}, \quad P(\xi^s) = \frac{\xi}{\xi - \xi^s} = \frac{1}{1 - \xi^{s-1}} \quad \text{(6.1)}
\]

\[
Q(\xi) = \frac{\xi}{\xi - \xi^s} = \frac{1}{1 - \xi^{s-1}}, \quad Q(\xi^s) = -\frac{\xi^s}{\xi - \xi^s} = \frac{-\xi}{1 - \xi^{s-1}} \quad \text{(6.2)}
\]

We observe that equations in (6.1) are equivalents. Indeed

\[
P(\xi^s) = P(\xi^{ord(\xi)/2}) = (P(\xi))^{ord(\xi)/2} = \left( \frac{-\xi^s}{\xi - \xi^s} \right)^{\frac{q^{ord(\xi)/2}}{ord(\xi)/2}} = \frac{\xi^{q^{ord(\xi)/2}}}{\xi^{q^{ord(\xi)/2}} - \xi^{q^{ord(\xi)/2}}}
\]

\[
= \frac{-\xi^s}{\xi - \xi^s} = -\frac{\xi}{\xi^s - \xi} = \frac{\xi}{\xi^s - \xi}.
\]

The same way equations in (6.2) are equivalents. In addition, the image of $u$ at the other components are zero, it follows that

\[
P(\theta) = 0 \quad \text{and} \quad Q(\theta) = 0 \quad \text{for every} \quad \theta \text{ root of} \quad x^n - 1 \quad \text{for} \quad f(x).
\]

Denoting by $g(x)$ the polynomial $\frac{x^n-1}{f(x)}$, the relation above implies that $g(x)$ is a divisor of $P(x)$ and $Q(x)$. Let us suppose that $P(x) = h(x)g(x)$, then from (6.1) it follows that the polynomial $(x^{s-1} - 1)P(x) - x^{s-1}$ is zero when we evaluate at $x = \xi$, and therefore the irreducible polynomial $f(x)$ is a factor of this polynomial, i.e.

\[
(x^{s-1} - 1)P(x) \equiv x^{s-1} \pmod{f(x)}
\]

which is equivalent to

\[
(x^{s-1} - 1)h(x)g(x) \equiv x^{s-1} \pmod{f(x)}, \quad \text{(6.3)}
\]

On the other hand, by Lemma 5.2 we know that $-\frac{1}{n}((f^*)')^* \cdot \frac{x^n-1}{f(x)} \equiv 1 \pmod{f(x)}$, hence multiplying (6.3) by $-\frac{1}{n}((f^*)')^*$ we obtain

\[
(x^{s-1} - 1)h(x) \equiv -\frac{1}{n}((f^*)')^*x^{s-1} \pmod{f(x)},
\]

and multiplying the previous equation by $\ell(x)$ it follows that

\[
h(x) = -\frac{1}{n}((f^*)')^*x^{s-1}\ell(x) \equiv -\frac{1}{n}((f^*)')^*(\ell(x) + 1) \pmod{f(x)},
\]

hence

\[
P(x) \equiv -\frac{1}{n}((f^*)')^* \cdot \frac{x^n-1}{f(x)} \cdot (\ell(x) + 1) \pmod{x^n - 1}.
\]

Following the same procedure for the polynomial $Q(x)$, we conclude that

\[
Q(x) \equiv \frac{1}{n}((f^*)')^* \cdot \frac{x^n-1}{f(x)} \cdot \ell(x) \pmod{x^n - 1},
\]
and from these two relations we concluded that
\[ e_{f,1} = -\frac{1}{n} \left( (f^*)^* \right) : \frac{x^n - 1}{f(x)} \cdot \ell(x)(1 - y) + 1 ] \]
is a non-central primitive idempotent. To finish, we observe that the orthogonal complement of \( e_{f,1} \) can be easy calculating as
\[ e_{f,2} = e_f - e_{f,1} = -\frac{1}{n} \left( (f^*)^* \right) : \frac{x^n - 1}{f(x)} \cdot \ell(x)(1 - y). \]

The following theorem describes partially a family of non-central idempotents of the group algebra \( \mathbb{F}_q G \), where \( G \) is the group with presentation given by (1.1). The same way as before, using the decomposition
\[ \mathbb{F}_q G \cong \mathbb{F}_q G \left( \frac{1 + x^n}{2} \right) \oplus \mathbb{F}_q G \left( \frac{1 - x^n}{2} \right), \]
we have that the idempotents of the first component have been described in previous theorem, so we are going show the form of the idempotents of the second component.

**Theorem 6.2.** Let \( G \) be a group with the following presentation
\[ G = \langle x, y \mid x^{2n} = 1, y^2 = x^n, xy = yx^s \rangle \]
where \( s^2 \equiv 1 \pmod{2n} \) and \( d = \gcd(2n, s - 1) \). Let \( f(x) \) be a s-self-involutive irreducible factor of \( x^n + 1 \) that does not divide \( x^d - 1 \) and \( \ell(x) \) is a polynomial such that \( (x^{s-1} - 1)\ell(x) \equiv 1 \pmod{f(x)} \). Then a pair of orthogonal non-central primitive idempotents \( e_{f,1} \) and \( e_{f,2} \), such that \( e_f = e_{f,1} + e_{f,2} \), are described in the following cases

- If \( q \equiv 1 \pmod{4} \), then
  \[ e_{f,1} = -\frac{1}{2n} \left( (f^*)^* \right) : \frac{x^{2n} - 1}{f(x)} \cdot (\ell(x) + 1)[1 + \beta y] \]
  and
  \[ e_{f,2} = \frac{1}{2n} \left( (f^*)^* \right) : \frac{x^{2n} - 1}{f(x)} \cdot [h(x) + \beta(\ell(x) + 1)y] \]
  where \( \beta^2 = -1 \).
- If \( q \equiv 3 \pmod{4}, \nu_2(n) > \nu_2(q + 1) \) and \( s \equiv 1 \pmod{4} \), then
  \[ e_{f,1} = -\frac{1}{2n} \left( (f^*)^* \right) : \frac{x^{2n} - 1}{f(x)} \cdot (\ell(x) + 1)(1 + x^{n/2}y) \]
  and
  \[ e_{f,2} = \frac{1}{2n} \left( (f^*)^* \right) : \frac{x^{2n} - 1}{f(x)} \cdot [\ell(x) + x^{n/2}(\ell(x) + 1)y)]. \]
- If \( q \equiv 3 \pmod{4}, \nu_2(n) \leq \nu_2(q + 1) \), then
  \[ e_{f,1} = -\frac{1}{2n} \left( (f^*)^* \right) : \frac{x^{2n} - 1}{f(x)} \cdot \ell(x)f'(x)(-x^s + y). \]

**Proof:** The proof is essentially the same to previous theorem. Indeed, if \( f(x) \) is an irreducible factor of \( x^n + 1 \) does not divide \( x^d - 1 \) and \( u = P(x) + Q(x)y \) is a non-central primitive idempotent such that the projection on the component generated by \( f \) is a matrix \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \), we have that
\[
\begin{pmatrix} -\xi^s & \beta \\ \beta^* & 1 \end{pmatrix}^{-1} \begin{pmatrix} P(\xi) & Q(\xi) \\ Q(\xi^*) & P(\xi^*) \end{pmatrix} \begin{pmatrix} -\xi^s & \beta \\ \beta^* & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]
where \( \xi \) is any root of \( f \) and \( \beta \in \mathbb{F}_q \) such that \( \beta^2 = -1 \). These relations are equivalent to

\[
P(\xi) = \frac{-\xi^{s-1}}{1 - \xi^{s-1}}, \quad P(\xi^s) = \frac{1}{1 - \xi^{s-1}} \tag{6.4}
\]

\[
Q(\xi) = \frac{-\beta \xi^{s-1}}{1 - \xi^{s-1}}, \quad Q(\xi^s) = \frac{\beta}{1 - \xi^{s-1}} \tag{6.5}
\]

We note that in the case when \( q \equiv 1 \pmod{4} \), the element \( \beta \) is in \( \mathbb{F}_q \). It follows that \( \xi \) and \( \xi^s \) are roots of the polynomials

\[
(x^{s-1} - 1)P(x) - x^{s-1} \quad \text{and} \quad (x^{s-1} - 1)Q(x) - \beta x^{s-1}.
\]

From this point, the proof follows exactly with the same steps as the proof of previous theorem.

In the case when \( q \equiv 3 \pmod{4} \), \(-1\) is not a square in \( \mathbb{F}_q \) and therefore \( \beta \notin \mathbb{F}_q \). Nevertheless using that \( \xi^n = -1 \) and \( n \) is even, we have that \( \beta = \xi^{n/2} \) and therefore

\[
Q(\xi) = \frac{-\xi^{n/2 + s}}{\xi - \xi^s} \quad \text{and} \quad Q(\xi^s) = \frac{\xi^{n/2 + s}}{\xi - \xi^s}.
\]

These two equations are equivalent because

\[
Q(\xi^s) = Q(\xi)^{\nu(\xi)/2} = -\xi^{\frac{n}{2} + s^2} \frac{\xi^{\frac{n}{2} + s}}{\xi^s - \xi^s} = \frac{\xi^{n/2}}{1 - \xi^{s-1}},
\]

where in the last identity we use that \( \xi^{n/2} \) is a quartic root of unite and \( s \equiv 1 \pmod{4} \). Thus \( \xi \) is a root of the polynomials

\[
(x^{s-1} - 1)P(x) - x^{s-1} \quad \text{and} \quad (x^{s-1} - 1)Q(x) - x^{n/2 + s - 1}.
\]

If \( q \equiv 3 \pmod{4} \) and \( \nu_2(n) \leq \nu_2(q + 1) \), let \( u = P(x) + Q(x)y \) is a non-central primitive idempotent such that de projection on the component generated by \( f \) is a matrix \(
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\);

we have that

\[
\left( \begin{array}{cc}
a & b \\
-\xi a & -\xi^s b
\end{array} \right)^{-1} \left( \begin{array}{cc}
P(\xi) & Q(\xi) \\
Q(\xi) & P(\xi^s)
\end{array} \right) \left( \begin{array}{cc}
a & b \\
-\xi a & -\xi^s b
\end{array} \right) = \left( \begin{array}{cc}
1 & 0 \\
0 & 0
\end{array} \right),
\]

where \( \xi \) is a root of \( f(x) \) and \( a^2 = -\theta, b^2 = \theta^s \) according to the Theorem 4.2. These relations are equivalent to

\[
P(\xi) = \frac{-\xi^{s-1}}{1 - \xi^{s-1}}, \quad P(\xi^s) = \frac{1}{1 - \xi^{s-1}} \tag{6.7}
\]

\[
Q(\xi) = \frac{1}{\xi - \xi^s}, \quad Q(\xi^s) = \frac{\xi^s}{1 - \xi^{s-1}}. \tag{6.8}
\]

It follows that \( \xi \) and \( \xi^s \) are roots of the polynomials

\[
(x^{s-1} - 1)P(x) - x^{s-1} \quad \text{and} \quad (x^s - x)Q(x) + 1.
\]

Let \( P(x) \) and \( Q(x) \) such that

\[
P(x) = -\frac{x^{2n} - 1}{2n f(x)} h(x) \quad \text{and} \quad Q(x) = -\frac{x^{2n} - 1}{2n f(x)} m(x)
\]

where \( h(x) \) is the polynomial

\[
(1 - x^{s-1})h(x) \equiv x^s f'(x) \pmod{f(x)} \quad \text{and} \quad (1 - x^s)m(x) \equiv -f'(x) \pmod{f(x)}
\]

Observe that the previous relation implies that \( h(x) \equiv -x^s m(x) \pmod{f(x)} \).

At this point, the proof follows exactly as the proof of previous theorem.
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