NONLINEAR VORTEX STRUCTURES IN OBLIQUELY ROTATING STRATIFIED FLUIDS DRIVEN BY SMALL SCALE NON HELICAL FORCES

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Abstract

In this paper, we study a new type of large-scale instability in obliquely rotating stratified fluids with small scale non-helical turbulence. The small-scale turbulence is generated by the external force with zero helicity and low Reynolds number. The theory uses the method of multiscale asymptotic developments. The nonlinear equations for large scale motions are obtained in the third order of the perturbation theory. In this paper, we consider the linear instability and the stationary nonlinear modes. We obtain solutions in the form of nonlinear Beltrami waves and localized vortex structures as kinks of new type.

1 Introduction

The problem of generation of large-scale vortex motions and formation of stationary nonlinear vortex structures in turbulent media is very important. The characteristic scales of the velocity fields for these structures are much bigger than the characteristic scales of turbulent motions or waves which engender their appearance. The study of the generation mechanism of large-scale vortex structures (LSVS) by small-scale turbulence has not only scientific but also application interest. For instance, tornados, cyclones and anticyclones represent the particular cases of LSVS. They also play an important role in the dynamics of Earth’s atmosphere, since they determine the global transport of air masses and are responsible for the weather and climate on our planet [1-3]. The study of the mechanisms of LSVS generation is important for a number of astrophysical problems, such as the origin of the Great Spot of Jupiter, Venus super rotation, vortex structures in solar prominences and so on [4-8]. The generation of LSVS in atmospheres and bowels of the planets (or other space objects) is essentially due to thermal phenomena occurring under the influence of internal or external sources of the thermal energy.
Articles \[9\]-[11] present the theory of convective vortex dynamo. It follows from this theory that the small-scale helical turbulence leads to a large-scale instability, which engenders the formation of one convective cell considered as a huge vortex of type of tropical cyclone. Number of numerical \[12\] and laboratory \[13\] experiments confirm this. The theory of the convective vortex dynamo was further developed in \[14\], \[15\], applying the method of multiscale asymptotic developments. This method, unlike the functional techniques \[16\], \[17\] used in \[9\]-[11] allows to determine strictly the principal order of perturbations theory in which the instability arises. The method of multiscale asymptotic developments to describe the generation of LSVS in reflective non invariant turbulence was first used in \[18\]. The small Reynolds number is a parameter of the asymptotic developments. Nonlinear stabilization of the convective large-scale instability discussed in \[14\], \[15\] leads to the formation of helical vortex solitons or kinks of a new type. The generation of helical turbulence in natural conditions is usually associated with the influence of the Coriolis force on turbulent motion \[19\], \[20\]. Obviously, the question arises of the possibility of generation of large-scale vortex field in a rotating medium under influence of small-scale force with zero helicity. An example of the generation of LSVS in a rotating incompressible fluid was found in \[21\]. It was also shown that the development of large-scale instability in an inclined rotating fluid generates the nonlinear large-scale helical vortex structures or localized Beltrami vortices with the internal helical structure.

In this paper we give a generalization of the $\alpha$ – effect found in \[21\] for the case of temperature- stratified fluid. As a result of this generalization, we obtain the large-scale instability which generates the LSVS.

The organization of this paper is as follows. In the Section 2 we obtain the averaged hydrodynamic equations in the Boussinesq approximation in an obliquely rotating fluid for the large-scale fields using the method of multiscale asymptotic developments. The technical aspect of this question is described in detail in Appendix I. The correlation functions in the averaged equations are expressed in terms of the small-scale fields in the zero-order approximation with respect to $R$. In Appendix II in order to obtain the averaged equations in closed form, we find solutions for small-scale fields of zero order approximation. In Appendix III we calculate the Reynolds stresses using these solutions. In Sec. 2 we obtain a closed system of the nonlinear equations for large-scale vortex fields (vortex dynamo). In Section 3 we consider the generation of small large-scale vortex perturbations, which arises from instability of the type $\alpha$ – effect. Also the conditions of the appearance of this instability are determined depending on the effects of rotation and stratification of the medium. In Section 4, an analytic and numerical analysis of nonlinear equations in the steady state is performed, which shows the existence of nonlinear Beltrami waves and localized vortex structures in the form of kinks. The results obtained in the work can be applied to the numerous geophysical and astrophysical problems.

## 2 Equations for the large-scale fields

Let us consider a system of equations for the perturbations of velocity $\tilde{v}$, temperature $T$, pressure $P$ in the Boussinesq approximation with the constant temperature gradient $\nabla T$ for a rotating coordinate system:

$$
\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} = \nu \Delta v_i - \frac{1}{\rho} \frac{\partial P}{\partial x_i} + 2\varepsilon_{ijk}v_j\Omega_k + ge_i\beta T + F_0^i
$$

(1)
In general case, the angular velocity $\vec{\Omega}$ is inclined to the plane $(X,Y)$ where the external force $\vec{F}_0,_{\perp}$ is located.

\begin{equation}
\frac{\partial T}{\partial t} + v_k \frac{\partial T}{\partial x_k} - Ae_k v_k = \chi \Delta T
\end{equation}

\begin{equation}
\frac{\partial v_i}{\partial x_i} = 0
\end{equation}

The system of equations (1)-(3) describes the evolution of perturbations on the background of the basic equilibrium state $T(z), \rho(z)$, which is set by the constant temperature gradient $\nabla T = -A\vec{e}(A > 0)$ (heated from below) and by the hydrostatic pressure:

$\nabla P = \rho \vec{g} - \rho \left[ \vec{\Omega} \times \left( \vec{\Omega} \times \vec{r} \right) \right]$, where $\vec{r}$ is the radius-vector of the fluid element. We consider the angular velocity of rotation $\vec{\Omega}$ as constant (solid rotation) and inclined to the plane $(X,Y)$, as shown in Fig. 1, i.e. for the Cartesian geometry problem: $\vec{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$.

Where $\vec{e} = (0,0,1)$ is unit vector in the direction of the axis $Z$, $\vec{g}$ is the gravity directed vertically downwards $\vec{g} = (0,0,-g)$, $\beta$ is the coefficient of thermal expansion. The equation (1) includes the external force $\vec{F}_0$, that models the excitation source in the environment of small scale and high frequency fluctuations of the velocity field with small Reynolds number. Unlike the previous studies \[14, \[15\] we consider here the non-helical external force with the following properties:

\begin{equation}
div \vec{F}_0 = 0, \quad \vec{F}_0,rot \vec{F}_0 = 0, \quad rot \vec{F}_0 \neq 0, \quad \vec{F}_0 = f_0 \vec{F}_0 \left( \frac{x}{\lambda_0}; \frac{t}{t_0} \right) \end{equation}

Where $\lambda_0$ is the characteristic scale, $t_0$ is the characteristic time, $f_0$ is the characteristic amplitude. The external force is given in the plane $(X,Y)$, where the perpendicular projection of the angular velocity is located. We choose the external force in rotating coordinate system in the following form:

$F_0^z = 0, \quad \vec{F}_0,_{\perp} = f_0 \left( i \cos \phi_2 + j \cos \phi_1 \right), \quad \phi_1 = k_1 x - \omega_0 t, \quad \phi_2 = k_2 x - \omega_0 t.$

\begin{equation}
k_1 = k_0 (1,0,0), \quad k_2 = k_0 (0,1,0). \end{equation}

Obviously, this non-helical external force satisfies all the conditions (1). Let us now use in equations (1)-(3) the dimensionless variables. For convenience, we will designate these variables
like dimensional variables:

\[
\begin{align*}
\vec{x} &\to \frac{\vec{x}}{\lambda_0}, \quad t \to \frac{t}{t_0}, \quad \vec{v} \to \frac{\vec{v}}{v_0}, \quad \vec{F}_0 \to \frac{\vec{F}_0}{f_0}, \quad P \to \frac{P}{\rho \rho_0}, \\
t_0 &= \frac{\lambda_0^2}{v}, \quad p_0 = \frac{v_0 \nu}{\lambda_0}, \quad f_0 = \frac{v_0 \nu}{\lambda_0^2}, \quad T \to \frac{T}{\lambda_0 A}
\end{align*}
\]

(6)

Then, in dimensionless variables the equation (1)-(3) takes the following form:

\[
\begin{align*}
\frac{\partial v_i}{\partial t} + R v_k \frac{\partial v_i}{\partial x_k} &= \Delta v_i - \frac{\partial P}{\partial x_i} + \varepsilon_{ijk} v_j D_k + \varepsilon_i T \bar{Ra} T + F_0^i \\
\frac{\partial T}{\partial t} + R v_k \frac{\partial T}{\partial x_k} &= P r^{-1} \Delta T \\
\frac{\partial v_i}{\partial x_i} &= 0
\end{align*}
\]

(7)-(9)

Here, \( \bar{Ra} = \frac{Ra}{P r} \), \( Ra = \frac{g \beta A}{\nu \chi} \lambda_0^4 \) is the Rayleigh number for the scale \( \lambda_0 \), \( P r = \frac{\nu}{\chi} \) is the Prandtl number, \( D_i = \frac{\Omega_0 \lambda_0^2}{v} \) is the dimensionless parameter of rotation for the scale \( \lambda_0 \) related with the Taylor number \( T \bar{a}_i = D_i^2 \) which is the characteristic of the influence of Coriolis force on viscous forces.

We will consider the Reynolds number \( R = \frac{v_0 t_0}{\lambda_0} \ll 1 \) on the scale \( \lambda_0 \) as a small parameter of an asymptotic development. Concerning the parameters \( D_i \) and \( Ra \), we do not choose any range of values for the moment. Let us examine the following formulation of the problem. We consider the external force as small and of high frequency. This force leads to the small scale fluctuations in velocity and temperature fields on the equilibrium background. After averaging, these quickly oscillating fluctuations vanish. Nevertheless, due to the small nonlinear interactions in some orders of perturbation theory, nonzero terms can occur after averaging. This means that they are not oscillatory and are of large scale. From a formal point of view, these terms are secular, i.e., they create the conditions for the solvability of large scale asymptotic development. So the purpose of this paper is to find and study the solvability equations, i.e., the equations for large scale perturbations. The method of asymptotic equations is well presented in works [7], [14], [15]. In accordance with these papers we introduce spatial and temporal derivatives in equations (7)-(9) in the form of asymptotic expansions:

\[
\frac{\partial}{\partial t} \to \partial_t + R^4 \partial_T, \quad \frac{\partial}{\partial x_i} \to \partial_i + R^2 \nabla_i
\]

(10)

where \( \partial_t \) and \( \partial_i \) denote derivatives with respect to fast variables \( x_0 = (\vec{x}_0, t_0) \) and \( \nabla_i, \partial_T \) derivatives with respect to slow variable \( X = (\vec{X}, T) \). Variables \( x_0 \) and \( X \) can be called small-scale and large-scale variables. To construct the nonlinear theory, the variables \( \vec{v}, \vec{T}, P \) are presented in the form of asymptotic series:

\[
\vec{v}(\vec{x}, t) = \frac{1}{R} \vec{W}_{-1}(X) + \vec{v}_0(x_0) + R \vec{v}_1 + R^2 \vec{v}_2 + R^3 \vec{v}_3 + \cdots
\]
Figure 2: Relation of Cartesian projections of rotation parameter $\vec{D}$ (or angular velocity of rotation $\vec{\Omega}$) with their projections in a spherical coordinate system.

\[
T(\vec{x}, t) = \frac{1}{R} T_{-1} (X) + T_0 (x_0) + R T_1 + R^2 T_2 + R^3 T_3 + \cdots
\]  

\[
P(\vec{x}, t) = \frac{1}{R^3} P_{-3} + \frac{1}{R^2} P_{-2} + \frac{1}{R} P_{-1} + P_0 + R (P_1 + \overrightarrow{P}_1 (X)) + R^2 P_2 + R^3 P_3 + \cdots
\]

Substituting developments (10)-(11) into the initial equations (7)-(9) and then gathering together the terms of the same order, we obtain the equations of the multi-scale asymptotic development and write down the obtained equations up to order $R^3$ including. The algebraic structure of the asymptotic development of equations (7)-(9) in various orders of $R$ is given in Appendix I. It is also shown that in order $R^3$ we get the main secular equation or equation for the large-scale fields:

\[
\partial_T W^i_{-1} - \Delta W^i_{-1} + \nabla_k \left( \frac{\epsilon^k_{\overrightarrow{v}_0} T_0}{v_0^k} \right) = -\nabla_i \overrightarrow{P}_1
\]

Equations (12)-(13) with secular equations are obtained in Appendix I:

\[
-\nabla_i P_{-3} + \widetilde{Rae}_i T_{-1} + \epsilon_{ijk} W_j D_k = 0
\]

\[
W^k_{-1} \nabla_k W^i_{-1} = -\nabla_i P_{-1}
\]

\[
W^k_{-1} \nabla_k T_{-1} = 0
\]

\[
\nabla_i W^i_{-1} = 0
\]

\[
W^z_{-1} = 0
\]

These equations are satisfied by choosing the following geometry for the velocity field:

\[
\tilde{W}_{-1} = (W^x_{-1} (Z), W^y_{-1} (Z), 0), \quad T_{-1} = T_{-1} (Z), \quad P_{-1} = \text{const}
\]
In the frame of this quasi-two-dimensional approximation, we assume that the large-scale derivative over \( Z \) is much more than others derivatives, i.e.,

\[
\nabla_Z \equiv \frac{\partial}{\partial Z} \gg \frac{\partial}{\partial X}, \frac{\partial}{\partial Y}
\]

Then the system of equations (12)-(13) is simplified and takes the following form:

\[
\begin{align*}
\partial_t W_1 - \nabla_Z^2 W_1 + \nabla_Z \left( \overline{v_0' v_0'} \right) &= 0, \quad W_{z1} = W_1 \\
\partial_t W_2 - \nabla_Z^2 W_2 + \nabla_Z \left( \overline{v_0' v_0'} \right) &= 0, \quad W_{y2} = W_2 \\
\partial_t T_{z1} - Pr^{-1} \Delta T_{z1} + \nabla_Z \left( \overline{v_0^z T_0} \right) &= 0
\end{align*}
\]

Equations (15)-(16) describe the evolution of large-scale eddy fields \( \vec{W} \). In order to obtain the final closed form of equations (15)-(16) we have to calculate the Reynolds stresses \( \nabla_k \left( \overline{v_0' v_0'} \right) \). This shows that we need to find solutions for the small-scale velocity field \( \vec{v}_0 \). Appendix II contains a detailed technique to calculate the velocity field in a rotating stratified medium. Further, in Appendix III solutions for small-scale velocity field \( \vec{v}_0 \) are used to find the Reynolds stresses. Then equations (15)-(16) take a closed form:

\[
\begin{align*}
(\partial_t - \nabla_Z^2)\tilde{W}_1 &= \frac{f_0^2}{2} D_2 \nabla_Z \left[ \frac{1 + \tilde{W}_2^2 - Ra}{(1 + \tilde{W}_2^2)((1 + \tilde{W}_2^2)^2 + 2(D_2^2 - Ra)(1 - \tilde{W}_2^2) + (D_2^2 - Ra)^2)} \right] \\
(\partial_t - \nabla_Z^2)\tilde{W}_2 &= -\frac{f_0^2}{2} D_1 \nabla_Z \left[ \frac{1 + \tilde{W}_1^2 - Ra}{(1 + \tilde{W}_1^2)((1 + \tilde{W}_1^2)^2 + 2(D_1^2 - Ra)(1 - \tilde{W}_1^2) + (D_1^2 - Ra)^2)} \right]
\end{align*}
\]

To simplify the equations, we use here the designations: \( \tilde{W}_1 = 1 - W_1, \tilde{W}_2 = 1 - W_2 \). Thus, in this section we obtain the closed equations (18)-(19), which will be called the equations of nonlinear vortex dynamo in obliquely rotating stratified fluids with small scale non-helical force. If the rotation effect disappears (\( \Omega = 0 \)), then the usual diffusion dissipation of large-scale fields occurs. In the limit of a homogeneous fluid the equation (18), (19) coincide with the results of [21]. We consider first the stability of small perturbations of fields (linear theory) and then examine the question of the possible existence of stationary structures.

### 3 Large-scale instability

Equations (18)-(19) describe the nonlinear dynamics of large scale disturbances of the vortex field \( \vec{W} = (W_1, W_2) \). Therefore it is interesting to clarify the question of the stability of small perturbations of the field \( \vec{W} \). Then for small values \( (W_1, W_2) \) the equations (18)-(19) are linearized and can be reduced to the following system of linear equations:

\[
\begin{align*}
\begin{cases}
\partial_t W_1 - \nabla_Z^2 W_1 - \alpha_2 \nabla Z W_2 = 0 \\
\partial_t W_2 - \nabla_Z^2 W_2 + \alpha_1 \nabla Z W_1 = 0
\end{cases}
\end{align*}
\]
where we have introduced the following designations for the coefficients:

\[
\alpha_1 = f_0^2 D_1 \left[ \frac{(D_1^2 - Ra - 2)(2 - Ra) + Ra(4 + (D_1^2 - Ra)^2)}{(4 + (D_1^2 - Ra)^2)^2} \right]
\]

\[
\alpha_2 = f_0^2 D_2 \left[ \frac{(D_2^2 - Ra - 2)(2 - Ra) + Ra(4 + (D_2^2 - Ra)^2)}{(4 + (D_2^2 - Ra)^2)^2} \right]
\]

It is clear that equations (20) are similar to the equations for the vortex dynamo [9]-[15].

To study the large-scale instability described by the system of equations (20), we choose perturbations in the form of plane waves with wave vector \( \vec{K} \parallel OZ \), i.e.

\[
W_{1,2} = A_{W_{1,2}} \exp(-i\omega t + iKZ)
\]

Substituting (22) into the system of equations (20) we get the dispersion equation:

\[
(-i\omega + K^2)^2 - \alpha_1\alpha_2K^2 = 0
\]

From equation (23) we find the increment of instability:

\[
\Gamma = Im\omega = \pm \sqrt{\alpha_1\alpha_2}K - K^2
\]

Solutions (24) show the existence of instabilities for large-scale vortical perturbations when \( \alpha_1\alpha_2 > 0 \). If \( \alpha_1\alpha_2 < 0 \), damped oscillations arise with frequency \( \omega_0 = \sqrt{\alpha_1\alpha_2}K \) instead of the instabilities. The coefficients \( \alpha_1, \alpha_2 \) give a positive feedback loop between the components of the velocity. It should be noted that in the linear theory, the coefficients \( \alpha_1, \alpha_2 \) do not depend on the amplitudes of the fields and depend only on the rotation parameters \( D_{1,2} \), Rayleigh number \( Ra \) and amplitude of the external force \( f_0 \). Let us analyze the dependence of these
coefficients on the dimensionless parameters assuming, for simplicity, that the dimensionless amplitude of the external force $f_0$ is equal $f_0 = 10$. In the coefficients $\alpha_1$, $\alpha_2$, instead of the Cartesian projections $D_1$ and $D_2$ it is convenient to use their projections in the spherical coordinate system $(D, \phi, \theta)$ (see Fig. 2). The coordinate surface $D = const$ is a sphere, $\theta$ is latitude $\theta \in [0, \pi]$, $\phi$ is longitude $\phi \in [0, 2\pi]$.

We analyze the dependence of the amplification coefficients $\alpha_1$, $\alpha_2$ on the effects of rotation and stratification, assuming for simplicity that $D_1 = D_2$, which corresponds to a fixed value of longitude $\phi = \pi/4 + \pi n$, where $n = 0, 1, 2...k$, $k$ is integer. In this case, the amplification coefficients of the vortex perturbations are, respectively, equal to:

$$\alpha = \alpha_1 = \alpha_2 = f_0^2 \sqrt{2} D \sin \theta \left[ \frac{4(D^2 \sin^2 \theta - 2Ra - 4)(2 - Ra) + 2Ra(16 + (D^2 \sin^2 \theta - 2Ra)^2)}{(16 + (D^2 \sin^2 \theta - 2Ra)^2)^2} \right]$$

We can see from this equation that at the poles ($\theta = 0$, $\theta = \pi$) the generation of vortex perturbations is not effective because $\alpha \to 0$. The dependence of $\alpha$ coefficient on the stratification parameter of fluid (Rayleigh number $Ra$) at a fixed latitude $\theta = \pi/2$ and the number $D = 2$ is presented in the left part of Fig. 3. Also it shows the case of a homogeneous medium $Ra = 0$, where the generation of large-scale vortex perturbations is caused by the external non-helical small-scale force and the Coriolis force [21]. Fig. 3 shows that the presence of temperature stratification ($Ra \neq 0$) can engender a significant increase the coefficient $\alpha$. Consequently we have faster generation of large-scale vortex perturbations than in homogeneous medium. This effect manifests especially with numbers $Ra \to 2$. Further, with the increasing of Rayleigh numbers, the value of coefficient $\alpha$ decreases. It is also interesting to find out the impact of rotation effect on the amplification coefficients $\alpha$. For these purposes, we take the value of the Rayleigh number $Ra = 2$ at $\theta = \pi/2$. For this case the functional dependency $\alpha(D)$ is shown.
in the right part of Fig. 3. This shows that for some parameter of $D$ the coefficient $\alpha$ reaches its maximum value $\alpha_{max}$. Then with the increasing of $D$ the coefficient $\alpha$ tends gradually to zero i.e. the suppression of $\alpha$ – effect occurs. A similar phenomenon was described in [19], [20].

The left part of Fig. 4 shows the plot of the joint effect of rotation and stratification in the plane $(D, Ra)$. Here the instability area is highlighted in gray. The maximum increment of instability $\Gamma_{max} = \frac{\alpha_{max}}{4}$ is reached on the wave number $K_{max} = \frac{\sqrt{\alpha_{max}}}{2}$. The plot of the function $\Gamma$ from the wave number $K$ (see right part of Fig. 4) has a standard form of $\alpha$ – effect. Thus, the development of large-scale instability in rotating stratified atmosphere generates the large-scale helical circularly polarized vortices of Beltramy type.

4 Stationary nonlinear vortex structures

It is obvious that with the increase of amplitude, nonlinear terms decrease and the instability becomes saturated. As a result the nonlinear vortex structures appear. In order to find these structures let us examine the stationary case of equations (18)-(19) and integrate once with respect to $Z$. For the sake of simplicity we assume that $D_1 = D_2$ and $\theta = \pi/2$. Consequently we get a system of nonlinear equations of the following form:

$$\frac{d\tilde{W}_1}{dZ} = -f_0^2 D \sqrt{2} \frac{1 + \tilde{W}_2^2 - Ra}{(1 + \tilde{W}_2^2)(4(1 + \tilde{W}_2^2)^2 + 4(D^2 - 2Ra)(1 - \tilde{W}_2^2) + (D^2 - 2Ra)^2)} + C_1 \quad (25)$$

$$\frac{d\tilde{W}_2}{dZ} = f_0^2 D \sqrt{2} \frac{1 + \tilde{W}_1^2 - Ra}{(1 + \tilde{W}_1^2)(4(1 + \tilde{W}_1^2)^2 + 4(D^2 - 2Ra)(1 - \tilde{W}_1^2) + (D^2 - 2Ra)^2)} + C_2 \quad (26)$$

Here, $C_1$, $C_2$ are arbitrary constants of integration. It should be noted that the dynamic system of equations (25)-(26) is conservative, and hence is Hamiltonian. It’s easy to find it we
Figure 6: On the left a nonlinear helical wave, which corresponds to a closed trajectory on the phase plane; on the right a localized nonlinear vortex structure (kink), which corresponds to the separatrix on the phase plane ($C_1 = -1$, $C_2 = 1$, $D = Ra = 2$)

write down the equations (25)-(26) in the Hamiltonian form:

\[
\begin{align*}
\frac{d\tilde{W}_1}{dZ} &= -\frac{\partial H}{\partial \tilde{W}_2}, \\
\frac{d\tilde{W}_2}{dZ} &= \frac{\partial H}{\partial \tilde{W}_1}
\end{align*}
\]

where the Hamiltonian has the form:

\[
H = H_1(\tilde{W}_1) + H_2(\tilde{W}_2) + C_2\tilde{W}_1 - C_1\tilde{W}_2
\]  
(27)

The functions $H_{1,2}$ are respectively equal to:

\[
H_{1,2} = f_0^2 D \sqrt{2} \int \frac{(1 + \tilde{W}_{1,2}^2 - Ra)d\tilde{W}_{1,2}}{(1 + \tilde{W}_{1,2}^2)(4(1 + \tilde{W}_{1,2}^2)^2 + 4(D^2 - 2Ra)(1 - \tilde{W}_{1,2}^2) + (D^2 - 2Ra)^2)}
\]

Let us put $D = Ra = 2$ and $f_0 = 10$. Then we can calculate the Hamiltonian (27):

\[
H = -\frac{25}{2} \sqrt{2} \left( \frac{\tilde{W}_1(\tilde{W}_1^2 + 3)}{(\tilde{W}_1^2 + 1)^2} + \frac{\tilde{W}_2(\tilde{W}_2^2 + 3)}{(\tilde{W}_2^2 + 1)^2} + \arctg\tilde{W}_1 + \arctg\tilde{W}_2 \right) + C_2\tilde{W}_1 - C_1\tilde{W}_2
\]

Since the equations (25)-(26) are Hamiltonian, only fixed points of two types: elliptic and hyperbolic can be observed in a phase space. This can be checked if we carry out a qualitative analysis of the system of equations (25)-(26). Linearizing the right-hand sides of equations (25)-(26) in the neighborhood of fixed points, we establish their type and construct a phase portrait. As a result of the analysis, we find the appearance of four fixed points, two of hyperbolic and two of elliptic type. Phase portrait of a dynamical system of equations (25)-(26) for the parameters $C_1 = -1$, $C_2 = 1$, $D = Ra = 2$ and $f_0 = 10$ is shown in Fig. 5. The phase portrait allows us to describe qualitatively the possible stationary solutions. The most interesting localized solutions
correspond to the phase portrait trajectories, which connect the stationary (singular) points on the phase plane. Fig. 5 presents closed trajectories on the phase plane around the elliptic points and separatrices which connect the hyperbolic points. Closed trajectories correspond to nonlinear periodic solutions or nonlinear waves. The separatrices correspond to a localized vortex structures of the kinks type (see Fig. 6).

5 Conclusion

In this paper we obtain a new type of large-scale instability generated by vertical temperature gradient and small-scale force with zero helicity $\vec{F}_0 \cdot \text{rot} \vec{F}_0 = 0$ in an inclined rotating fluid. This force supports small-scale fluctuations in the fluid and models the effect of small-scale turbulence with the Reynolds number $R \ll 1$. It is assumed that the external force is in the plane $(X, Y)$. The force of gravity is directed vertically downwards along the axis $OZ$. Using the method of multiscale asymptotic expansions we get in our work the closed system of equations for large scale perturbations of velocity. For small amplitudes, this system of equations describes the instability, which is called the hydrodynamic $\alpha$–effect, since there is a positive feedback between the components of the velocity. It is shown that the instability arises only in the case, when the angular velocity vector of rotation deviates from the axis $OZ$. Joint effect of rotation and stratification of the medium (temperature heated from below) leads to a substantial enhancement of large-scale vortex perturbations, unlike the case of a homogeneous medium [21]. This phenomenon appears especially when the parameters of medium $D \to 2$, $Ra \to 2$ (see Fig. 3). In this case there is a maximum generation of small-scale helical motions due to the action of the Coriolis force and the inhomogeneity of the medium temperature. The rapid growth of vortex perturbations contributes to the increasing role of nonlinear effects and the consequent saturation of the instability. The study of stationary state by numerical method with parameters $D = Ra = 2$ shows the existence of two types of solutions: nonlinear waves and kinks. These solutions are similar to those found in homogeneous obliquely rotating fluid [21].

Appendix I. Multiscale asymptotic developments

Let us find the algebraic structure of the asymptotic development in various orders of $R$, starting from the lowest one. In order of $R^{-3}$ there is only one equation:

$$\partial_t P_{-3} = 0 \Rightarrow P_{-3} = P_{-3}(X)$$

(28)

In order $R^{-2}$ appears the equations:

$$\partial_t P_{-2} = 0 \quad \Rightarrow \quad P_{-2} = P_{-2}(X)$$

(29)

In order $R^{-1}$, we obtain more complicated system of equations:

$$\partial_t W_{-1}^i + W_{-1}^j \partial_j W_{-1}^i = -\partial_t P_{-1} - \nabla_i P_{-3} + \partial_k^2 W_{-1}^i + \varepsilon_{ijk} W_j D_k e_k + \tilde{Rae}_i T_{-1}$$

(30)

$$\partial_t T_{-1} - Pr^{-1} \partial_k^2 T_{-1} = -W_{-1}^k \partial_k T_{-1} + W_{-1}^z$$

(31)
$$\partial_i W^i_{-1} = 0 \quad (32)$$

The averaging of equations (30)-(32) over the fast variables give the following secular equations:

$$- \nabla_i P_{-1} + \tilde{\text{Rae}}_i T_{-1} + \varepsilon_{ijk} W_j D_k = 0 \quad (33)$$

$$W^z_{-1} = 0 \quad (34)$$

In zero order in $R^0$ we have the equations:

$$\partial_t v^i_0 + W^k_{-1} \partial_k v^i_0 + v^k_0 \partial_k W^i_{-1} = -\partial_i P_0 - \nabla_i P_{-2} + \partial^2_k v^i_0 + \varepsilon_{ijk} v^j_0 D_k + \tilde{\text{Rae}}_i T_0 + F^i_0 \quad (35)$$

$$\partial_t T_0 - Pr^{-1} \partial^2_k T_0 = -W^k_{-1} \partial_k T_0 - \partial_k (v^k_0 T_{-1}) + v^z_0 \quad (36)$$

$$\partial_i v^i_0 = 0 \quad (37)$$

These equations give one secular equation:

$$\nabla P_{-2} = 0 \quad \Rightarrow \quad P_{-2} = \text{const} \quad (38)$$

Let us consider the equations of the first approximation $R^1$:

$$\partial_t v^i_1 + W^k_{-1} \partial_k v^i_1 + v^k_0 \partial_k v^i_0 + v^k_1 \partial_k W^i_{-1} + W^k_{-1} \nabla_k W^i_{-1} = -\nabla_i P_{-1} - \partial_i \left( P_1 + \overline{P}_1 \right) + \partial^2_k v^i_1 + 2\partial_k \nabla_k W^i_{-1} + \text{Rae}_i T_1 + \varepsilon_{ijk} v^j_1 D_k \quad (39)$$

$$\partial_t T_1 - Pr^{-1} \partial^2_k T_1 - Pr^{-1} 2\partial_k \nabla_k T_{-1} = -W^k_{-1} \partial_k T_1 - W^k_{-1} \nabla_k T_{-1} - v^k_0 \partial_k T_0 - v^k_1 \partial_k T_{-1} + v^z_1 \quad (40)$$

$$\partial_i v^i_1 + \nabla_i W^i_{-1} = 0 \quad (41)$$

The secular equations follow from this system of equations:

$$W^k_{-1} \nabla_k W^i_{-1} = -\nabla_i P_{-1} \quad (42)$$

$$W^k_{-1} \nabla_k T_{-1} = 0 \quad (43)$$

$$\nabla_i W^i_{-1} = 0 \quad (44)$$

The secular equation (42)-(44) are satisfied by choosing the following geometry:

$$\vec{W}_{-1} = \left( W_{-1}^x (Z), W_{-1}^y (Z), 0 \right), T_{-1} = T_{-1} (Z), P_{-1} = \text{const} \quad (45)$$

In the second order $R^2$, we obtain the equations:
\[ \partial_t v^i_2 + W^{-1}_k \partial_k v^i_2 + v^k_0 \partial_k v^i_1 + W^{-1}_k \nabla_k v^i_0 + v^k_0 \nabla_k W^i_1 + v^i_1 \partial_k v^i_0 + v^k_2 \partial_k W^i_1 = -\nabla_i P_2 - \nabla_i P_0 + \]
\[ + \partial^2_k v^i_2 + 2 \partial_k \nabla_k v^i_0 + \widetilde{Rae}_i T_2 + \varepsilon_{ijk} v^j_2 D_k \]  
(46)

\[ \partial_i T_2 - Pr^{-1} \partial^2_k T_2 - Pr^{-1} 2 \partial_k \nabla_k T_0 = -W^{-1}_k \partial_k T_2 - W^{-1}_k \nabla_k T_0 - v^k_0 \partial_k T_1 - v^k_2 \nabla_k T_1 - \]
\[ - v^i_1 \partial_k T_0 - v^i_2 \partial_k T_1 + v^i_0 + \]
(47)
\[ \partial_i v^i_2 + \nabla_i v^i_0 = 0 \]  
(48)

It is easy to see that there are no secular terms in this order. Let us consider now the most important order $R^3$. In this order we obtain the equations:

\[ \partial_t v^i_3 + \partial_t W^i_1 + W^{-1}_k \partial_k v^i_3 + v^k_0 \partial_k v^i_2 + W^{-1}_k \nabla_k v^i_1 + v^k_0 \nabla_k v^i_0 + v^i_1 \partial_k v^i_1 + v^i_1 \nabla_k W^i_1 + \]
\[ + v^k_2 \partial_k v^i_0 + v^k_2 \partial_k W^i_1 = -\partial_i P_3 - \nabla_i (P_1 + \overline{P_1}) + \partial^2_k v^i_3 + 2 \partial_k \nabla_k v^i_1 + \Delta W^i_1 + \widetilde{Rae}_i T_3 + \]
\[ + \varepsilon_{ijk} v^j_3 D_k \]  
(49)

\[ \partial_i T_3 + \partial_i T_{-1} - Pr^{-1} \partial^2_k T_3 - Pr^{-1} 2 \partial_k \nabla_k T_1 - Pr^{-1} \Delta T_{-1} = -W^{-1}_k \partial_k T_3 - W^{-1}_k \nabla_k T_1 - \]
\[ - v^k_0 \partial_k T_2 - v^k_0 \nabla_k T_0 - v^1_k \nabla_k T_1 - v^i_1 \nabla_k T_{-1} - v^i_2 \partial_k T_0 - v^i_2 \partial_k T_{-1} + v^i_0 + \]
(50)
\[ \partial_i v^i_3 + \nabla_i v^i_1 = 0 \]  
(51)

After averaging this system of equations over the fast variables, we obtain the main system of secular equations to describe the evolution of large-scale perturbations:

\[ \partial_T W^i_1 - \Delta W^i_1 + \nabla_k \left( \overline{W^i_0 v^i_0} \right) = -\nabla_i \overline{P_1} \]  
(52)

\[ \partial_T T_{-1} - Pr^{-1} \Delta T_{-1} = -\nabla_k \left( \overline{W^i_0 T_0} \right) \]  
(53)

**Appendix II. Small-scale fields in the zero order in $R$**

In Appendix I, we obtain the equations in the zero order in $R$, which can be written in the following form:

\[ \widehat{D}_W v^i_0 = -\partial_i P_0 + \widetilde{Rae}_i T_0 + \varepsilon_{ijk} v^j_0 D_k + F^i_0 \]  
(54)

\[ \widehat{D}_0 T_0 = e_k v^k_0 \]  
(55)
\[ \partial_i v^i_0 = 0 \]  
(56)
where we introduce the designations for operators:

\[ \hat{D}_W = \partial_t - \partial_k^2 + W_{-1}^r \partial_k, \hat{D}_\theta = \partial_t - P r^{-1} \partial^2 + W_{-1}^r \partial_k \]

Small-scale oscillations of temperature are easily found from the equation (55)

\[ T_0 = \frac{v_0^z}{\hat{D}_\theta} \]  \( (57) \)

Let us substitute (57) into (54) and using the condition of solenoidality of fields \( \vec{v}_0, \vec{F}_0 \), we can find the pressure \( P_0 \):

\[ P_0 = \hat{P}_1 u_0 + \hat{P}_2 v_0 + \hat{P}_3 w_0 \]  \( (58) \)

Here we introduced the designation for operators

\[ \hat{P}_1 = \frac{D_2 \partial_z - D_3 \partial_y}{\partial^2}, \quad \hat{P}_2 = \frac{D_3 \partial_x - D_1 \partial_z}{\partial^2}, \quad \hat{P}_3 = \frac{D_1 \partial_y - D_2 \partial_x}{\partial^2} + \frac{\sim Ra}{\hat{D}_\theta} \frac{\partial \partial_z}{\partial \partial^2} \]

and velocities: \( u_0^x = u_0, v_0^y = v_0, w_0^z = w_0 \). Using the representation (58), we can eliminate pressure from the equations (54) and obtain the system of equations for velocity fields of the zero approximation:

\[ \left( \hat{D}_W + \hat{p}_{1x} \right) u_0 + \left( \hat{p}_{2x} - D_3 \right) v_0 + \left( \hat{p}_{3x} + D_2 \right) w_0 = F_0^x \]

\[ \left( D_3 + \hat{p}_{1y} \right) u_0 + \left( \hat{D}_W + \hat{p}_{2y} \right) v_0 + \left( \hat{p}_{3y} - D_1 \right) w_0 = F_0^y \]  \( (59) \)

\[ \left( \hat{p}_{1z} - D_2 \right) u_0 + \left( \hat{p}_{2z} + D_1 \right) v_0 + \left( \hat{D}_W - \frac{\sim Ra}{\hat{D}_\theta} + \hat{p}_{3z} \right) w_0 = 0 \]

The components of the tensor have the following form:

\[ \hat{p}_{1x} = \frac{D_2 \partial_x \partial_z - D_3 \partial_y \partial_y}{\partial^2}, \hat{p}_{2x} = \frac{D_3 \partial_x \partial_z - D_1 \partial_z \partial_z}{\partial^2}, \hat{p}_{3x} = \frac{D_1 \partial_y \partial_y - D_2 \partial_x \partial_x}{\partial^2} + \frac{\sim Ra}{\hat{D}_\theta} \frac{\partial \partial_z}{\partial \partial^2} \]

\[ \hat{p}_{1y} = \frac{D_2 \partial_y \partial_z - D_3 \partial_y \partial_y}{\partial^2}, \hat{p}_{2y} = \frac{D_3 \partial_y \partial_z - D_1 \partial_z \partial_z}{\partial^2}, \hat{p}_{3y} = \frac{D_1 \partial_y \partial_y - D_2 \partial_x \partial_x}{\partial^2} + \frac{\sim Ra}{\hat{D}_\theta} \frac{\partial \partial_z}{\partial \partial^2} \]

\[ \hat{p}_{1z} = \frac{D_2 \partial_z \partial_z - D_3 \partial_z \partial_y}{\partial^2}, \hat{p}_{2z} = \frac{D_3 \partial_z \partial_z - D_1 \partial_x \partial_x}{\partial^2}, \hat{p}_{3z} = \frac{D_1 \partial_z \partial_y - D_2 \partial_z \partial_x}{\partial^2} + \frac{\sim Ra}{\hat{D}_\theta} \frac{\partial \partial_z}{\partial \partial^2} \]  \( (60) \)

The solution for equations system (59) can be found in accordance with Cramer’s rule:

\[ u_0 = \frac{1}{\Delta} \left\{ \left[ \left( \hat{D}_W + \hat{p}_{2y} \right) \left( \hat{D}_W - \frac{\sim Ra}{\hat{D}_\theta} + \hat{p}_{3z} \right) - \left( \hat{p}_{2z} + D_1 \right) \left( \hat{p}_{3y} - D_1 \right) \right] \right\} F_0^x + \]

\[ + \left\{ \left( \hat{p}_{3x} + D_2 \right) \left( \hat{p}_{2z} + D_1 \right) - \left( \hat{p}_{2x} - D_3 \right) \left( \hat{D}_W - \frac{\sim Ra}{\hat{D}_\theta} + \hat{p}_{3z} \right) \right\} F_0^y \]  \( (61) \)
\[ v_0 = \frac{1}{\Delta} \left\{ \left[ (\bar{D}_W + \hat{p}_1 x) \left( \bar{D}_W - \frac{Ra}{D_\theta} + \hat{p}_3 x \right) - (\hat{p}_3 x + D_2)(\hat{p}_1 z - D_2) \right] F_0^w + \right. \\
+ \left. \left[ (\hat{p}_3 y - D_1)(\hat{p}_1 z - D_2) - (D_3 + \hat{p}_1 y) \left( \bar{D}_W - \frac{Ra}{D_\theta} + \hat{p}_3 z \right) \right] F_0^x \right\} \] (62)

\[ w_0 = \frac{1}{\Delta} \left\{ \left[ (D_3 + \hat{p}_1 y)(\hat{p}_2 z + D_1) - (\bar{D}_W + \hat{p}_2 y)(\hat{p}_1 z - D_2) \right] F_0^\prime + \right. \\
+ \left. \left[ (\hat{p}_2 x - D_3)(\hat{p}_1 z - D_2) - (\bar{D}_W + \hat{p}_1 x)(\hat{p}_2 z + D_1) \right] F_0^\prime \right\} \] (63)

Here \( \Delta \) is the determinant of the system of equations (59):

\[
\Delta = \left( \bar{D}_W + \hat{p}_1 x \right) \left( \bar{D}_W + \hat{p}_2 y \right) \left( \bar{D}_W - \frac{Ra}{D_\theta} + \hat{p}_3 z \right) + \\
+ (D_3 + \hat{p}_1 y)(\hat{p}_2 z + D_1)(\hat{p}_3 x + D_2) + (\hat{p}_2 x - D_3)(\hat{p}_3 y - D_1)(\hat{p}_1 z - D_2) - \\
- (\hat{p}_3 x + D_2)(\bar{D}_W + \hat{p}_2 y)(\hat{p}_1 z - D_2) - \\
- (\hat{p}_2 z + D_1)(\hat{p}_3 y - D_1)(\bar{D}_W + \hat{p}_1 x) - \\
- (D_3 + \hat{p}_1 y)(\hat{p}_2 x - D_3)(\bar{D}_W - \frac{Ra}{D_\theta} + \hat{p}_3 z) \] (64)

In order to calculate the expressions (61)-(64) we present the external force (6) in complex form:

\[ \bar{F}_0 = \frac{7}{2} f_0 e^{i\phi_1} + \frac{7}{2} f_0 e^{i\phi_2} + k.c. \] (65)

Then all operators in formulae (61)-(64) act from the left on their eigen function. In particular:

\[ \bar{D}_{W,H} e^{i\phi_1} = e^{i\phi_1} \bar{D}_{W,\theta} (\bar{k}_1, -\omega_0), \quad \bar{D}_{W,\theta} e^{i\phi_2} = e^{i\phi_2} \bar{D}_{W,\theta} (\bar{k}_2, -\omega_0), \]

\[ \Delta e^{i\phi_1} = e^{i\phi_1} \Delta (\bar{k}_1, -\omega_0), \quad \Delta e^{i\phi_2} = e^{i\phi_2} \Delta (\bar{k}_2, -\omega_0) \] (66)

To simplify the formulae, let us choose \( k_0 = 1, \omega_0 = 1 \) and introduce new designations:

\[ \bar{D}_W (\bar{k}_1, -\omega_0) = \bar{D}_{W_1} = 1 - i (1 - W_1), \quad \bar{D}_W (\bar{k}_2, -\omega_0) = \bar{D}_{W_2} = 1 - i (1 - W_2) \] (67)

\[ \bar{D}_{\theta} (\bar{k}_1, -\omega_0) = \bar{D}_{\theta_1} = P r^{-1} - i (1 - W_1), \quad \bar{D}_{\theta} (\bar{k}_2, -\omega_0) = \bar{D}_{\theta_2} = P r^{-1} - i (1 - W_2) \]
The velocity field of zero approximation has the following form:

\[ u_0 = \frac{f_0}{2} \frac{\hat{A}_2^*}{\hat{A}_2^* \hat{D}^*_W + D_2^*} e^{i \phi_2} + c.c. = u_{03} + u_{04} \]  \hspace{1cm} (68)

\[ v_0 = \frac{f_0}{2} \frac{\hat{A}_1^*}{\hat{A}_1^* \hat{D}^*_W + D_1^*} e^{i \phi_1} + c.c. = v_{01} + v_{02} \]  \hspace{1cm} (69)

\[ w_0 = -\frac{f_0}{2} \frac{D_1}{\hat{A}_1^* \hat{D}^*_W + D_1^*} e^{i \phi_1} + \frac{f_0}{2} \frac{D_2}{\hat{A}_2^* \hat{D}^*_W + D_2^*} e^{i \phi_2} + c.c. = w_{01} + w_{02} + w_{03} + w_{04} \]  \hspace{1cm} (70)

where

\[ \hat{A}_{1,2}^* = \hat{D}_{W,1,2}^* - \frac{\tilde{R} \alpha}{\tilde{D}_{W,1,2}^*} \]  \hspace{1cm} (71)

Components of velocity satisfy the following relations:

\[ w_{02} = (w_{01})^*, \quad w_{04} = (w_{03})^*, \quad v_{02} = (v_{01})^*, \quad v_{04} = (v_{03})^*, \quad u_{02} = (u_{01})^*, \quad u_{04} = (u_{03})^*. \]

Appendix III. Calculation of the Reynolds stresses

To close the equations (15)-(16) we have to calculate the Reynolds stresses \( T^{ik} = v^i_0 v^k_0 \) or rather its components:

\[ T^{31} = \overline{w_0 u_0} = \overline{w_{01}(u_{01})^*} + (w_{01})^* u_{01} + (w_{03})^* u_{03} + (w_{03})^* u_{03} \]  \hspace{1cm} (72)

\[ T^{32} = \overline{w_0 v_0} = \overline{w_{01}(v_{01})^*} + (w_{01})^* v_{01} + (w_{03})^* v_{03} + (w_{03})^* v_{03} \]  \hspace{1cm} (73)

Substituting the solutions for the small-scale velocity fields (68)-(70) obtained in Appendix II, into the equations (72)-(73), we can find the following expression for the correlators:

\[ T^{31} = \frac{f_0^2}{4} \frac{D_2(\hat{A}_2 + \hat{A}_2^*)}{\hat{A}_2 \hat{D}_W + D_2^*} \]  \hspace{1cm} (74)

\[ T^{32} = -\frac{f_0^2}{4} \frac{D_1(\hat{A}_1 + \hat{A}_1^*)}{\hat{A}_1 \hat{D}_W + D_1^*} \]  \hspace{1cm} (75)

Then with the definition of the operators (67) and (71), we write down the series of useful relations for the calculation of \( T^{31} \) and \( T^{32} \):

\[ \left| \hat{D}_{W,1,2} \right|^2 = \hat{D}_{W,1,2} \hat{D}_{W,1,2}^* = 1 + \tilde{W}_{1,2}^2, \quad \left| \hat{D}_{\theta,1,2} \right|^2 = \hat{D}_{\theta,1,2} \hat{D}_{\theta,1,2}^* = P R^{-2} + \tilde{W}_{1,2}^2. \]
\[ |\hat{A}_{1,2}|^2 = \hat{A}_{1,2}\hat{A}^*_{1,2} = 1 + \tilde{\tilde{W}}_{1,2}^2 - 2Ra \frac{1 - Pr\tilde{\tilde{W}}_{1,2}^2}{1 + Pr^2\tilde{\tilde{W}}_{1,2}^2} + \frac{Ra^2}{1 + Pr^2\tilde{\tilde{W}}_{1,2}^2}, \]

\[ \hat{D}_{W_{1,2}}\hat{D}_{\theta_{1,2}} + \hat{D}^*_{W_{1,2}}\hat{D}^*_{\theta_{1,2}} = 2( Pr^{-1} - \tilde{\tilde{W}}_{1,2}^2 ), \]

\[ \hat{D}_{W_{1,2}}\hat{A}_{1,2} + \hat{D}^*_{W_{1,2}}\hat{A}^*_{1,2} = 2(1 - \tilde{\tilde{W}}_{1,2}^2) - 2Ra \frac{1 + Pr\tilde{\tilde{W}}_{1,2}^2}{1 + Pr^2\tilde{\tilde{W}}_{1,2}^2} \]

Here we apply the following designations: \( \tilde{\tilde{W}}_1 = 1 - W_1, \tilde{\tilde{W}}_2 = 1 - W_2 \). Using these relations, we can obtain the following expressions:

\[ \hat{A}_{1,2} + \hat{A}^*_{1,2} = 2 \left( 1 - \frac{Ra}{1 + Pr^2\tilde{\tilde{W}}_{1,2}^2} \right), \]

(76)

\[ |\hat{D}_{W_{1,2}}\hat{A}_{1,2} + D_{1,2}^2|^2 = \left( 1 + \tilde{\tilde{W}}_{1,2}^2 \right)^2 + Ra \frac{1 + \tilde{\tilde{W}}_{1,2}^2}{1 + Pr^2\tilde{\tilde{W}}_{1,2}^2} + 2D_{1,2}^2(1 - \tilde{\tilde{W}}_{1,2}^2) + \]

\[ +D_{1,2}^4 - 2Ra \frac{2 + \tilde{\tilde{W}}_{1,2}^4}{1 + Pr^2\tilde{\tilde{W}}_{1,2}^2} - Pr\tilde{\tilde{W}}_{1,2}^4 - (1 + Pr\tilde{\tilde{W}}_{1,2}^2)(1 - D_{1,2}^2) \]

Substituting (76) in (74)-(75) we can find expressions for the Reynolds stresses in general form. For instance, for the atmosphere the Prandtl number is approximately equal to one \( Pr = 1 \). In this case, the expressions for the components of Reynolds stresses are simplified:

\[ T^{31} = \frac{f_0^2}{2} D_2 \frac{(1 + \tilde{\tilde{W}}_{2}^2 - Ra)}{(1 + \tilde{\tilde{W}}_{2}^2)((1 + \tilde{\tilde{W}}_{2}^2)^2 + 2(D_{2}^2 - Ra)(1 - \tilde{\tilde{W}}_{2}^2) + (D_{2}^2 - Ra)^2)} \]

(77)

\[ T^{32} = -\frac{f_0^2}{2} D_1 \frac{(1 + \tilde{\tilde{W}}_{1}^2 - Ra)}{(1 + \tilde{\tilde{W}}_{1}^2)((1 + \tilde{\tilde{W}}_{1}^2)^2 + 2(D_{1}^2 - Ra)(1 - \tilde{\tilde{W}}_{1}^2) + (D_{1}^2 - Ra)^2)} \]

(78)

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