Dynamical Critical Phenomena and Large Scale Structure of the Universe:

the Power Spectrum for Density Fluctuations.

J. F. Barbero G.*, A. Dominguez*, T. Goldman+ and J. Pérez–Mercader*

*Laboratorio de Astrofísica Espacial y Física Fundamental
Apartado 50727
28080 Madrid

+Theoretical Division, Los Alamos National Laboratory
Los Alamos, New Mexico 87545

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Abstract

As is well known, structure formation in the Universe at times after decoupling can be described by hydrodynamic equations. These are shown here to be equivalent to a generalization of the stochastic Kardar–Parisi–Zhang equation with time–dependent viscosity in epochs of dissipation. As a consequence of the Dynamical Critical Scaling induced by noise and fluctuations, these equations describe the fractal behavior (with a scale dependent fractal dimension) observed at the smaller scales for the galaxy–to–galaxy correlation function and also the Harrison–Zel’dovich spectrum at decoupling. By a Renormalization Group calculation of the two–point correlation function between galaxies in the presence of (i) the expansion of the Universe and (ii) non–equilibrium, we can account, from first principles, for the main features of the observed shape of the power spectrum.

Subject headings: Gravitation; Cosmology; Critical Phenomena.
Contemporary cosmology is riddled with problems like the Problems of the Cosmological Constant, Dark Matter and the Age of the Universe. In addition, we have no dynamical understanding of why the two–point correlation function for galaxies, a key point of contact between theory and observation, has the observed behavior. It is known from observations that the galaxy–to–galaxy correlation function, $\xi_{OBS}(r)$, is well fit by a power law of the form

$$\xi_{OBS}(r) \propto r^{-\gamma},$$

where $r$ is the comoving separation between the galaxies and $\gamma$ is determined from catalogs to be between 1.5 and 1.8 at distances of the order of the megaparsec. Nevertheless, at the epoch of decoupling, we know from COBE data [1] that the correlation function goes like $r^{-(4.2\pm0.3)}$. How is this so? How can this deviation from an integer in the power law exponent be accounted for? Why is there an evolution in the exponent of the power law? Can this fact be established from some generic physics?

In this letter we will compute the galaxy–to–galaxy correlation function from the hydrodynamics that describes the formation of structure in the Universe and will present an answer to the above questions.

We generalize previous results obtained by Berera and Fang in Ref. [2] and, independently by the authors of Ref. [3]; we also compute the power spectrum of density perturbations. In the first case we have generalized their calculation to include the effects of self–gravity, expansion and, simultaneously, non–equilibrium; we have also generalized their asymptotic calculation to the full range of distances, from COBE and well into the realm of the galaxies. In the latter case, we have included deviations from equilibrium and the expansion of the Universe also in a self–consistent way, and we have extended their calculation back into the decoupling era. We will not need to introduce any new physics: our conclusions follow solely from a straightforward (albeit non–naive) analysis of the hydrodynamic equations and the extension to the Dynamical Renormalization Group and Dynamical Critical Phenomena of
techniques familiar from Condensed Matter and Elementary Particle Physics.

As argued in Refs. [2] and [3], if the power law behavior of the two point correlation function for galaxies is due to some form of critical phenomena, it follows that in the realm of the galaxies there must exist some kind of fluctuations which should account for the observed behavior of $\xi(r)$. We model them by means of power law correlated noise.

The suitably averaged value of the 2–point correlation function for the density contrast, $\delta(\vec{r},t)$, written in comoving coordinates, is identified in phenomenology with $\xi_{OBS}(r)$ (see, e. g., references [4], [5] and [6]). We will study the scaling behavior of the contrast–contrast correlation function.

Under the assumption\textsuperscript{1} of irrotational peculiar velocity $\vec{u}$ and peculiar acceleration $\vec{w}$, it is straightforward to check that the hydrodynamic equations arising from the application of Newtonian considerations to structure formation in the Universe, can be written in comoving coordinates in terms of a velocity potential $\psi$ and the gravitational potential $\phi$ due to the contrast as (Ref. [7], [2], [6])

\begin{align*}
\frac{\partial}{\partial t} \psi + H \psi - \frac{1}{2a} (\nabla \psi)^2 - \frac{1}{a} \phi + \frac{1}{a} \tilde{f} \left[ \rho_b \{ 1 + (4\pi G a^2 \rho_b)^{-1} \nabla^2 \phi \} \right] &= 0 \quad (1) \\
\frac{\partial}{\partial t} \phi + H \phi - 4\pi G a \rho_b \psi + F &= 0 \quad (2) \\
\nabla^2 F &= -\frac{1}{a} \nabla \cdot [(\nabla \psi)(\nabla^2 \phi)] \quad (3) \\
\vec{u} &= -\nabla \psi \quad (4) \\
\vec{w} &= -a^{-1} \nabla \phi \quad (5) \\
\nabla^2 \phi &= 4\pi G a^2 \rho_b \delta(\vec{r},t) \quad (6)
\end{align*}

\textsuperscript{1}The rotational components of the velocity decouple at a quicker rate than their non–rotational counterparts, and thus for late enough times it is always possible to justify this assumption.
\[ \rho(\vec{r}, t) = \rho_b [1 + \delta(\vec{r}, t)]. \] (7)

Here, \( a(t) \) is the scale parameter of the homogeneous cosmological background, and the function \( \tilde{F}(\vec{r}, t) \) originates in the gauge freedom associated with the irrotational characters of \( \vec{a} \) and \( \vec{w} \). \( H \) is the Hubble parameter. The function \( \tilde{f} \) (also known as minus the specific enthalpy) is determined by the equation of state assumed for the matter whose clustering is described by the above equations: \( p = f(\rho), \tilde{f}'(x) = -f'(x)/x \). We know however that for dust \( (p = 0) \), the Zel’dovich approximation, \( \vec{w} = F(t)\vec{u} \), with

\[ \dot{F} = 4\pi G\rho_b - H F - F^2 \] (8)

works very well during the early non-linear regime. One can (and we will) assume that this approximation is also valid when pressure is included and provided that the pressure is small (cf. Ref. [7]).

Under the above assumptions we have

\[ \phi(\vec{x}, t) = 4\pi G a \rho_b \frac{D(t)}{D(t)} \psi(\vec{x}, t) \] (9)

\[ \delta(\vec{x}, t) = \frac{1}{a} \frac{D(t)}{D(t)} \nabla^2 \psi(\vec{x}, t) \] (10)

where \( D(t) \) is the growing mode component of the density perturbation (Refs. [4], [5] and [6]). The velocity potential satisfies

\[ \frac{\partial}{\partial t} \psi + \frac{\dot{a}}{a} \psi - \frac{1}{2a} (\nabla \psi)^2 = 4\pi G \rho_b \frac{D(t)}{D(t)} \psi + \frac{1}{a} c_s^2 \log \left(1 + \frac{1}{a} \frac{D(t)}{D(t)} \nabla^2 \psi\right), \] (11)

and expanding the logarithm to lowest order, we get an equation of the form

\[ \frac{\partial}{\partial t} \psi = f_1(t) \nabla^2 \psi + f_2(t)(\nabla \psi)^2 + f_3(t) \psi, \] (12)

where the \( f_i(t) \) are determined by the background geometry as

\[ f_1(t) = \frac{c_s^2}{a^2(t) \dot{D}(t)} \] (13)
\[
\frac{f_2(t)}{2a(t)} = 1
\] (14)

\[
f_3(t) = 4\pi G\rho_b(t) \frac{D(t)}{\dot{D}(t)} - \frac{\dot{a}(t)}{a(t)}
\] (15)

This is a generalization with time–dependent coefficients (and a mass–like term) of the Kardar–Parisi–Zhang (KPZ) equation

\[
\frac{\partial}{\partial t} h = \nu \nabla^2 h + \frac{1}{2} \lambda (\nabla h)^2 + \eta(\vec{x}, t),
\] (16)

which plays a central rôle in surface growth phenomena, and whose scaling behavior in the IR (large distance, small \(k\)) and UV (short distance, large \(k\)) regimes are well understood in terms of the correlation properties of the noise or in the absence of noise, the correlation properties of the initial conditions (e.g., Ref. [8]).

By a series of changes of variables and rescalings, one can rewrite equation (12) as

\[
\frac{\partial}{\partial \tau} H(\vec{x}, \tau) = \nabla^2 H(\vec{x}, \tau) + (\nabla H(\vec{x}, \tau))^2 + \frac{\partial A(\tau)/\partial \tau}{A(\tau)} H(\vec{x}, \tau) + \eta(\vec{x}, \tau)
\] (17)

with \(A(\tau) = \frac{f_2(\tau)}{f_1(\tau)} \exp \int_{\tau_0}^{\tau} \frac{f_3(\tau')}{f_1(\tau')} d\tau'\) and the noise term is \(\eta = (f_2/f_1^2)\bar{\eta}\). The quantity \(H(\vec{x}, \tau)\) is related to \(\psi(\vec{x}, t)\) by \(H(\vec{x}, \tau) = \int_\tau^t d\tau' f_1(\tau') = \int_\tau^t d\tau' f_1(\tau') \psi(\vec{x}, \tau)\). The first term in the rhs represents the smoothing effect of diffusion and the second term is due to non–equilibrium effects. The third term describes the effects due to the time–dependence and matter content of the background geometry (it contains effects from the expansion of the Universe and self–gravity); it behaves, in the linear approximation and in a variant of “conformal–time”, as a mass term, and therefore introduces a natural correlation length into the problem. Finally, the noise term models the various fluctuations that can appear during epochs in the evolution of large scale structure in the Universe.

It is clearly seen from Eq. (17) that the scaling properties of \(H(\vec{x}, \tau)\) depend on two key features of the equation: (a) the characteristics of the noise and/or (b) the specific features of the background geometry.
The power spectrum $P(k; t)$ is the Fourier transform of the two–point correlation function for the density contrast (see [4], [5] and [6]). We get

$$P(k; t) = \Phi(t)k^2 \bar{q}^2 \langle H(\vec{k}, \tau) \ H(\vec{q}, \tau) \rangle_{\text{classical}}\big|_{\vec{q} = -\vec{k}}$$

(18)

where the two–point function for $H(k, \tau)$ does not include the effects of non–linearities nor higher order effects due to fluctuations, i. e., it is computed from the classical theory, and $\Phi^{1/2}(t) = \frac{1}{a(t)} \frac{D(t)}{\partial D(t)/\partial t} \frac{f_1(t)}{f_2(t)}$ comes from the changes of variables needed to transform from Eq. (1) to Eq. (17).

The renormalized two–point correlation function $\langle H(\vec{k}, \tau)H(\vec{q}, \tau') \rangle$ obeys a Callan–Symanzik equation, Ref. [9], the solution of which contains all its scale dependence, and $P(k; t)$ can be written as

$$P(k; t)_{\text{Full}} = \Phi(t)k^2 \bar{q}^2 \langle H(\vec{k}, \tau) \ H(\vec{q}, \tau) \rangle_{\text{Improved}}$$

(19)

where $\langle H(\vec{k}, \tau)H(\vec{q}, \tau) \rangle_{\text{Improved}}$ is the solution to the Callan–Symanzik equation. As is done within the context of the Renormalization Group (RG), this object is computed by inserting into the free two–point correlation function for $\langle H(\vec{k}, \tau)H(\vec{q}, \tau) \rangle$ the values of the couplings obtained by solving their RG equations.

We will now restrict ourselves to the following cosmological and noise scenario: flat FRW cosmologies where the noise is arbitrarily power–law correlated both in space and in time [4]. In flat FRW it is straightforward to see that $\Phi_{FRW}(t) = (81/4)c_s^4 t_0^{8/3} t^{4/3}$.

For noise with properties given by

$$\langle \eta(\vec{k}, \omega) \rangle = 0$$

(20)

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2In what follows and to simplify the writing, we will write down $H(\vec{x}, \tau)$ instead of $H(\vec{x}, \tau = \int^\tau d\tau' f_1(\tau'))$.

3Obtained from Eq. (14) by setting $\lambda = 0$.

4A cosmology with constant $A$ is flat FRW.
\[ \langle \eta(\vec{k}, \omega) \eta(\vec{q}, \Omega) \rangle = 2 \left[ D_0 + D_\theta k^{-2\rho} \left| \frac{\omega}{\omega_0} \right|^{-2\theta} \right] \delta^3(\vec{k} + \vec{q}) \delta(\omega + \Omega) \] (21)
a straightforward calculation gives (here for convergence of some integrals, \(-1/2 < \theta < 1/2\))
\[ \langle H(\vec{k}, \tau) H(\vec{q}, \tau) \rangle_{\text{Improved}} = \frac{\nu^2(k)}{2\pi K_3 \lambda^2(k)} \left\{ \frac{U_0(k)}{k^2} + \frac{U_\theta(k) \sec(\pi \theta)}{k^2(1 + 2\theta + \rho)} \right\} \delta^3(\vec{k} + \vec{q}) \] (22)
where we have introduced \( U_0 \equiv \frac{D_0 \lambda^2}{\rho^2} K_3, \ U_\theta \equiv \frac{D_\theta \lambda^2}{\rho^2} K_3, \) and \( K_3 \equiv \frac{1}{2\pi^2}. \)

The running coupling constants \( \nu(k), \lambda(k), U_0(k) \) and \( U_\theta(k) \) obey the following renormalization group equations (Ref. [10]),
\[ -\mu \frac{d\nu}{d\mu} = \nu \left[ -\frac{1}{12} U_0 + \frac{2\rho - 1}{12} U_\theta (1 + 2\theta) \sec(\pi \theta) \right] \] (23)
\[ -\mu \frac{d\lambda}{d\mu} = \frac{1}{3} \lambda \theta U_\theta (1 + 2\theta) \sec(\pi \theta) \] (24)
\[ -\mu \frac{dU_\theta}{d\mu} = (4\theta - 1 + 2\rho) U_\theta + \frac{3 + 2\theta}{12} U_0 U_\theta + \frac{3 + 10\theta - 6\rho - 4\theta \rho}{12} (1 + 2\theta) \sec(\pi \theta) U_\theta^2 \] (25)
\[ -\mu \frac{dU_0}{d\mu} = -U_0 + \frac{1}{2} U_0^2 + \frac{9 - 6\rho + 8\theta}{12} (1 + 2\theta) \sec(\pi \theta) U_0 U_\theta + \frac{1}{4} U_\theta^2 (1 + 4\theta) \sec(2\pi \theta). \] (26)

The parameter \( \mu \) has dimensions of momentum. These equations have several fixed points, and the asymptotic behavior of the solutions depends on the values of the noise parameters \( \theta \) and \( \rho \) (see Fig. 1 for a plot and definitions of the parameter regions.) The fixed point structure in the region of the \((U_\theta - U_0)\) plane where the correlations are positive is shown in Figure 1.

From COBE observations we know that in the IR (small \( k \)-regime) the power spectrum is Harrison–Zel’dovich; similarly at “large” momentum, the power spectrum is also scale invariant but with a different exponent. Therefore the boundary condition on the improved two point function is that for \( k \to 0 \) the power spectrum be Harrison–Zel’dovich, i. e., \( \lim_{k \to 0} P_{\text{Full}}(k; t) \sim k. \) This translates into a condition that must be satisfied by the two noise exponents
\[ 2 - 4\theta - 2\rho = 1; \] (27)
the subsequent evolution of $P(k)$ is controlled by the RG according to the above equations. The results of integrating the RG equations in (23) – (26) with a typical set of initial conditions compatible with COBE data are shown in Fig. 2. They reproduce both the general form of the power spectrum and its main observational features. However, as one evolves into shorter distances (larger momenta) higher order terms in the expansion of the logarithm in the hydrodynamic equation must be included since they become *relevant* at these shorter scales; the behavior displayed by the power spectrum in the calculation presented here, indicates that the evolution when higher order terms are included will also be in the direction of the observational evidence, since the largest momenta at which our calculation can be trusted is *also* consistent with the behavior inferred from catalogs of galaxies.

We have shown that the hydrodynamics of a fluid of galaxies interacting through gravity can be studied using scaling techniques based on the dynamical renormalization group. We have taken into consideration all the effects due to self–gravity, expansion of the Universe and non–equilibrium present in the “fluid of galaxies”. We have applied these ideas to the calculation of the 2–point correlation function for the density contrast, and have found that (i) its scaling behavior depends on the background geometry and the noise and/or initial conditions for the density contrast, but (ii) can be computed and (iii) comparison of our results with observations shows excellent agreement. In summary, we have seen that the power spectrum can be viewed as evidence of dynamical critical behavior in the Universe. In fact, because of this critical behavior, once the power spectrum at decoupling is known to be Harrison–Zel’dovich what happens at smaller scales is fairly insensitive to small deviations from this initial condition, since criticality implies that the system will eventually be attracted to one of its fixed points irrespective of the *details* of the physics.

Although we have demonstrated the feasibility of our approach for the simplest background (flat FRW), it is clear that more general and complete cases can be similarly treated. Furthermore, these considerations lead to a very interesting view of the large scale structure of the Universe, where all kinds of new phenomena and behaviors can now be described; these
include pattern formation, roughening transitions, nucleation, defect generation, ecological–like behavior for galactic and other many body gravitational systems, etc..

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FIG. 1. Noise spectrum and fixed points for $U_0$ and $U_{\theta}$. The fixed point structure for these RG equations depends on the noise spectrum. There exist five distinct regions in noise parameter space leading to three un–related behaviors of the RG equations. In the upper panel we display the noise “space” and in the lower panels the corresponding map of $U_0$ and $U_{\theta}$ fixed points together with their IR or UV characters. Different choices in the $\rho-\theta$ plane lead to qualitatively different solutions to the RGEs for $U_0$ and $U_{\theta}$. In the region where correlations are positive, different behaviors can be found as shown in this figure. The difference between I and V (or the set III, IV, VI and VII) appears in the region where $U_{\theta} < 0$ and where the RGEs have different types of critical points.
FIG. 2. The predicted power spectrum for the contrast as a function of the momentum scale and some typical observational data. The data plotted in the figure are taken from the CfA–101 and CfA–130 catalogues (Ref. [10]). The steep fall–off in our predicted curve past its maximum indicates the need to include higher order terms in the expansion of the logarithm in Equation (11) which, as explained in the text, become relevant at shorter distances.