SOME INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE FOR FUNCTIONS WHOSE DERIVATIVES OF $n$-TH ORDER ARE $(\alpha, m)$-CONVEX

FENG QI, MUHAMMAD AMER LATIF, WEN-HUI LI, AND SABIR HUSSAIN

Abstract. In the paper, the authors find some new integral inequalities of Hermite-Hadamard type for functions whose derivatives of the $n$-th order are $(\alpha, m)$-convex and deduce some known results. As applications of the newly-established results, the authors also derive some inequalities involving special means of two positive real numbers.

1. Introduction

It is common knowledge in mathematical analysis that a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on an interval $I \neq \emptyset$ if
\[
(1.1) \quad f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)
\]
for all $x, y \in I$ and $\lambda \in [0, 1]$; If the inequality (1.1) reverses, then $f$ is said to be concave on $I$.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on an interval $I$ and $a, b \in I$ with $a < b$. Then
\[
(1.2) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]
This inequality is well known in the literature as Hermite-Hadamard integral inequality for convex functions. See [4, 10] and closely related references therein.

The concept of usually used convexity has been generalized by a number of mathematicians. Some of them can be recited as follows.

Definition 1.1 ([17]). Let $f : [0, b] \to \mathbb{R}$ be a function and $m \in [0, 1]$. If
\[
(1.3) \quad f(\lambda x + m(1 - \lambda)y) \leq \lambda f(x) + m(1 - \lambda)f(y)
\]
holds for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$, then we say that $f(x)$ is $m$-convex on $[0, b]$.

Definition 1.2 ([9]). Let $f : [0, b] \to \mathbb{R}$ be a function and $(\alpha, m) \in [0, 1] \times [0, 1]$. If
\[
(1.4) \quad f(\lambda x + m(1 - \lambda)y) \leq \lambda^\alpha f(x) + m(1 - \lambda^\alpha)f(y)
\]
is valid for all $x, y \in [0, b]$ and $\lambda \in (0, 1]$, then we say that $f(x)$ is $(\alpha, m)$-convex on $[0, b]$.

It is not difficult to see that when $(\alpha, m) \in \{(0, 0), (1, 0), (1, m), (1, 1), (\alpha, 1)\}$ the $(\alpha, m)$-convex function becomes the $\alpha$-star-shaped, star-shaped, $m$-convex, convex, and $\alpha$-convex functions respectively.

The famous Hermite-Hadamard inequality (1.2) has been refined or generalized by many mathematicians. Some of them can be reformulated as follows.

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Key words and phrases. Hermite-Hadamard integral inequality; convex function; $(\alpha, m)$-convex function; differentiable function; application; mean.
\textbf{Theorem 1.1} ([12, Theorem 3]). Let \( f : I^o \subset \mathbb{R} \to \mathbb{R} \) be a twice differentiable function such that \( f''(x) \) is \( \alpha \)-convex on \([a, b]\) for some fixed \( q \geq 1 \). If \( |f''(x)|^q \) is \( m \)-convex on \([a, b]\) for some fixed \( q > 1 \) and \( m \in [0, 1] \), then

\[
\left| \frac{f'(a) + f'(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)^2}{24} \left\{ \left[ \lambda^2 + (1 + \alpha)(1 - \lambda)^3 + \frac{5\alpha - 3}{4} \right] |f''(a)| \right.
\]

\[
+ \left. \left[ \lambda^4 + (2 - \alpha)\lambda^3 + \frac{1 - 3\alpha}{4} \right] |f''(b)| \right\}, \quad 0 \leq \lambda \leq \frac{1}{2}.
\]

\[
\left\{ \frac{(b-a)^2}{48} (3\lambda - 1) (|f''(a)| + |f''(b)|), \quad \frac{1}{2} \leq \lambda \leq 1. \right. \]

\textbf{Theorem 1.2} ([15, Theorem 4]). Let \( I \subseteq \mathbb{R} \) be an open interval and \( a, b \in I \) with \( a < b \), and let \( f : I \to \mathbb{R} \) be a twice differentiable mapping such that \( f''(x) \) is integrable. If \( 0 \leq \lambda \leq 1 \) and \( |f''(x)| \) is convex on \([a, b]\), then

\[
\left| (\lambda - 1) \frac{f'(a) + f'(b)}{2} - \lambda \frac{f(a) + f(b)}{2} + \int_a^b f(x) \, dx \right|
\]

\[
\leq \frac{(b-a)^2}{24} \left\{ \left[ \lambda^2 + (1 + \alpha)(1 - \lambda)^3 + \frac{5\alpha - 3}{4} \right] |f''(a)| \right.
\]

\[
+ \left. \left[ \lambda^4 + (2 - \alpha)\lambda^3 + \frac{1 - 3\alpha}{4} \right] |f''(b)| \right\}, \quad 0 \leq \lambda \leq \frac{1}{2}.
\]

\[
\left\{ \frac{(b-a)^2}{48} (3\lambda - 1) (|f''(a)| + |f''(b)|), \quad \frac{1}{2} \leq \lambda \leq 1. \right. \]

\textbf{Theorem 1.3} ([11, Theorem 3]). Let \( b^* > 0 \) and \( f : [0, b^*] \to \mathbb{R} \) be a twice differentiable function such that \( f''(x) \) is \( \alpha \)-convex on \([a, b]\) for \( (\alpha, m) \in [0, 1] \times [0, 1] \) and \( q \geq 1 \), then

\[
\left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x) \, dx \right|
\]

\[
\leq \frac{(mb-a)^2}{2} \left\{ 1 \right. \left. \right|^\frac{1}{q} \left\{ \frac{|f''(a)|^q}{(\alpha + 2)(\alpha + 3)} + m|f''(b)|^q \left[ \frac{1}{6} - \frac{1}{(\alpha + 2)(\alpha + 3)} \right] \right\} \right. \right. \}

In recent years, some other kinds of Hermite-Hadamard type inequalities were generated in [1, 2, 3, 13, 14, 16, 20, 21, 22], for example. For more systematic information, please refer to monographs [4, 10] and related references therein.

In this paper, we will establish some new inequalities of Hermite-Hadamard type for functions whose derivatives of \( n \)-th order are \((\alpha, m)\)-convex and deduce some known results in the form of corollaries.

\section{A Lemma}

For establishing new integral inequalities of Hermite-Hadamard type for functions whose derivatives of \( n \)-th order are \((\alpha, m)\)-convex, we need the following lemma.
Lemma 2.1. Let $0 < m \leq 1$ and $b > a > 0$ satisfying $a < mb$. If $f^{(n)}(x)$ for $n \in \{0\} \cup \mathbb{N}$ exists and is integrable on the closed interval $[0, b]$, then

$$
(2.1) \quad \frac{f(a) + f(mb)}{2} - \frac{1}{mb - a} \int_a^{mb} f(x) \, dx - \frac{1}{2} \sum_{k=2}^{n-1} \frac{(k-1)(mb-a)^k}{(k+1)!} f^{(k)}(a) = \frac{(mb-a)^n}{n!} \int_0^1 t^{n-1}(n-2t)f^{(n)}(ta + m(1-t)b) \, dt,
$$

where the sum above takes 0 when $n = 1$ and $n = 2$.

Proof. When $n = 1$, it is easy to deduce the identity (2.1) by performing an integration by parts in the integrals from the right side and changing the variable.

When $n = 2$, we have

$$
(2.2) \quad \frac{f(a) + f(mb)}{2} - \frac{1}{mb - a} \int_a^{mb} f(x) \, dx = \frac{(mb-a)^2}{2} \int_0^1 t(1-t)f''(ta + m(1-t)b) \, dt.
$$

This result is same as [11, Lemma 2].

When $n = 3$, the identity (2.1) is equivalent to

$$
(2.3) \quad \frac{f(a) + f(mb)}{2} - \frac{1}{mb - a} \int_a^{mb} f(x) \, dx - \frac{(mb-a)^2}{12} f''(a) = \frac{(mb-a)^3}{12} \int_0^1 t^2(3-2t)f^{(3)}(ta + m(1-t)b) \, dt,
$$

which may be derived from integrating the integral in the second line of (2.3) and utilizing the identity (2.2).

When $n \geq 4$, computing the second line in (2.1) by integration by parts yields

$$
\frac{(mb-a)^n}{n!} \int_0^1 t^{n-1}(n-2t)f^{(n)}(ta + m(1-t)b) \, dt
$$

$$
= -\frac{(n-2)(mb-a)^{n-1}}{n!} f^{(n-1)}(a) + \frac{(mb-a)^{n-1}}{(n-1)!} \int_0^1 t^{n-2}(n-1-2t)f^{(n-1)}(ta + m(1-t)b) \, dt,
$$

which is a recurrent formula

$$
S_{a,mb}(n) = -T_{a,mb}(n-1) + S_{a,mb}(n-1)
$$
on $n$, where

$$
S_{a,mb}(n) = \frac{1}{2} \frac{(mb-a)^n}{n!} \int_0^1 t^{n-1}(n-2t)f^{(n)}(ta + m(1-t)b) \, dt
$$

and

$$
T_{a,mb}(n-1) = \frac{1}{2} \frac{(n-2)(mb-a)^{n-1}}{n!} f^{(n-1)}(a)
$$

for $n \geq 4$. By mathematical induction, the proof of Lemma 2.1 is complete. □

Remark 2.1. Similar integral identities to (2.1), produced by replacing $f^{(k)}(a)$ in (2.1) by $f^{(k)}(b)$ or by $f^{(k)}(\frac{a+b}{2})$, and corresponding integral inequalities of Hermite-Hadamard type have been established in [8, 18, 19].

Remark 2.2. When $m = 1$, our Lemma 2.1 becomes [5, Lemma 2.1].
3. Inequalities of Hermite-Hadamard Type

Now we are in a position to establish some integral inequalities of Hermite-Hadamard type for functions whose derivatives of \( n \)-th order are \((\alpha, m)\)-convex.

**Theorem 3.1.** Let \((\alpha, m) \in [0, 1] \times (0, 1)\) and \( b > a > 0 \) with \( a < mb \). If \( f(x) \) is \( n \)-time differentiable on \([0, b]\) such that \( |f^{(n)}(x)| \in L([0, mb])\) and \( |f^{(n)}(x)|^p \) is \((\alpha, m)\)-convex on \([0, mb]\) for \( n \geq 2 \) and \( p \geq 1 \), then

\[
(3.1) \quad \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb - a} \int_a^{mb} f(x) \, dx \right| \leq \frac{1}{2} \sum_{k=2}^{n-1} \frac{(k-1)(mb-a)^k}{(k+1)!} f^{(k)}(a)
\]

\[
\leq \frac{1}{2} \frac{(mb-a)^n}{n!} \left( \frac{n-1}{n+1} \right)^{1-1/p} \left\{ \frac{n(n-1) + \alpha(n-2)}{(n+\alpha)(n+\alpha+1)} |f^{(n)}(a)|^p \right. \\
+ m \left[ \frac{n-1}{n+1} - \frac{n(n-1) + \alpha(n-2)}{(n+\alpha)(n+\alpha+1)} \right] |f^{(n)}(b)|^{1/p} \right\},
\]

where the sum above takes 0 when \( n = 2 \).

**Proof.** It follows from Lemma 2.1 that

\[
(3.2) \quad \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb - a} \int_a^{mb} f(x) \, dx \right| \leq \frac{1}{2} \sum_{k=2}^{n-1} \frac{(k-1)(mb-a)^k}{(k+1)!} f^{(k)}(a)
\]

\[
\leq \frac{1}{2} \frac{(mb-a)^n}{n!} \int_0^1 t^{n-1}(n-2t)|f^{(n)}(ta + m(1-t)b)| \, dt.
\]

When \( p = 1 \), since \( |f^{(n)}(x)| \) is \((\alpha, m)\)-convex, we have

\[
|f^{(n)}(ta + m(1-t)b)| \leq t^n |f^{(n)}(a)| + m(1-t^n)|f^{(n)}(b)|.
\]

Multiplying by the factor \( t^{n-1}(n-2t) \) on both sides of the above inequality and integrating with respect to \( t \in [0, 1] \) lead to

\[
\int_0^1 t^{n-1}(n-2t)|f^{(n)}(ta + m(1-t)b)| \, dt \\
\leq \int_0^1 t^{n-1}(n-2t)\left[ t^n |f^{(n)}(a)| + m(1-t^n)|f^{(n)}(b)| \right] \, dt \\
= |f^{(n)}(a)| \int_0^1 t^{n+\alpha-1}(n-2t) \, dt + m|f^{(n)}(b)| \int_0^1 t^{n-1}(n-2t)(1-t) \, dt \\
= \left( \frac{n}{n+\alpha} - \frac{2}{n+\alpha+1} \right) |f^{(n)}(a)| + m|f^{(n)}(b)| \left( \frac{n-1}{n+1} - \frac{n}{n+\alpha} + \frac{2}{n+\alpha+1} \right) \\
= \frac{n(n-1) + \alpha(n-2)}{(n+\alpha)(n+\alpha+1)} |f^{(n)}(a)| + m \left[ \frac{n-1}{n+1} - \frac{n(n-1) + \alpha(n-2)}{(n+\alpha)(n+\alpha+1)} \right] |f^{(n)}(b)|.
\]

The proof for the case \( p = 1 \) is complete.
When $p > 1$, by the well-known Hölder integral inequality, we obtain

\begin{equation}
\int_0^1 t^{n-1}(n-2t)|f^{(n)}(ta + m(1-t)b)| \, dt \leq \left[ \int_0^1 t^{n-1}(n-2t) \, dt \right]^{1-1/p} \left[ \int_0^1 t^{n-1}(n-2t)|f^{(n)}(ta + m(1-t)b)|^p \, dt \right]^{1/p}.
\end{equation}

Using the $(\alpha, m)$-convexity of $|f^{(n)}(x)|^p$ produces

\begin{equation}
\int_0^1 t^{n-1}(n-2t)|f^{(n)}(ta + m(1-t)b)|^p \, dt \leq \int_0^1 t^{n-1}(n-2t)[t^{\alpha}|f^{(n)}(a)|^p + m(1-t^\alpha)|f^{(n)}(b)|^p] \, dt
\end{equation}

\begin{equation}
= \frac{n(n-1) + \alpha(n-2)}{(n+\alpha)(n+\alpha+1)} |f^{(n)}(a)|^p + m\left[ \frac{n-1}{n+1} - \frac{n(n-1) + \alpha(n-2)}{(n+\alpha)(n+\alpha+1)} \right] |f^{(n)}(b)|^p.
\end{equation}

Substituting (3.3) and (3.4) into (3.2) yields the inequality (3.1). This completes the proof of Theorem 3.1.

\textbf{Corollary 3.1.} Under conditions of Theorem 3.1,

(1) when $m = 1$, we have

\begin{equation}
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{1}{2} \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{(k+1)!} f^{(k)}(a) \left( \frac{1}{n} \right)^{1-1/p} \left( \frac{1}{(n+\alpha)(n+\alpha+1)} \right) |f^{(n)}(b)|^p
\end{equation}

\begin{equation}
\times \left\{ \frac{n(n-1) + \alpha(n-2)}{(n+\alpha)(n+\alpha+1)} |f^{(n)}(a)|^p + \left[ \frac{n-1}{n+1} - \frac{n(n-1) + \alpha(n-2)}{(n+\alpha)(n+\alpha+1)} \right] |f^{(n)}(b)|^p \right\}^{1/p};
\end{equation}

(2) when $n = 2$, we have

\begin{equation}
\left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x) \, dx \right| \leq \frac{1}{b-a} \int_a^b f(x) \, dx \left( \frac{1}{3} \right)^{1-1/p} \left( \frac{2}{(\alpha + 2)(\alpha + 3)} \right)^{1/p} \left( \frac{1}{2} \right)^{1-1/p} \left( \frac{n}{n+1} \right)^{1-1/p} |f''(a)|^p + m \left[ \frac{1}{3} - \frac{2}{(\alpha + 2)(\alpha + 3)} \right] |f''(b)|^p
\end{equation}

\begin{equation}
\times \left\{ \frac{n(n-1) + \alpha(n-2)}{(n+\alpha)(n+\alpha+1)} |f^{(n)}(a)|^p + \left[ \frac{n-1}{n+1} - \frac{n(n-1) + \alpha(n-2)}{(n+\alpha)(n+\alpha+1)} \right] |f^{(n)}(b)|^p \right\}^{1/p};
\end{equation}

(3) when $m = \alpha = p = 1$ and $n = 2$, we have

\begin{equation}
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)^2}{24} \left( |f''(a)| + |f''(b)| \right);
\end{equation}

(4) when $m = \alpha = 1$ and $p = n = 2$, we have

\begin{equation}
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)^2}{12} \left( \frac{|f''(a)|^2 + |f''(b)|^2}{2} \right)^{1/2}.
\end{equation}

\textbf{Remark 3.1.} Under conditions of Theorem 3.1,

(1) when $n = 2$, the inequality (3.1) becomes the one (1.8) in [11, Theorem 3];

(2) when $\alpha = m = 1$, Theorem 3.1 becomes [5, Theorem 3.1].
Theorem 3.2. Let \( (a, m) \in [0, 1] \times (0, 1) \) and \( b > a > 0 \) with \( a < mb \). If \( f(x) \) is \( n \)-time differentiable on \([0, b]\) such that \( |f^{(n)}(x)| \in L([0, mb]) \) and \(|f^{(n)}(x)|^p \) is \((a, m)\)-convex on \([0, mb]\) for \( n \geq 2 \) and \( p > 1 \), then

\[
(3.5) \quad \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb - a} \int_a^{mb} f(x) \, dx - \frac{1}{2} \sum_{k=2}^{n-1} \frac{(k-1)(mb - a)^k}{(k+1)!} f^{(k)}(a) \right| \\
\leq \frac{1}{2} \frac{(mb - a)^n}{n!} \left[ \frac{n^{q+1} - (n-2)^q + 1}{2(q+1)} \right]^{1/q} \left\{ \frac{1}{p(n-1) + \alpha + 1} |f^{(n)}(a)|^p \right. \\
+ \left. \frac{ma}{p(n-1) + \alpha + 1} |f^{(n)}(b)|^p \right\}^{1/p},
\]

where the sum above takes 0 when \( n = 2 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

Proof. It follows from Lemma 2.1 that

\[
(3.6) \quad \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb - a} \int_a^{mb} f(x) \, dx - \frac{1}{2} \sum_{k=2}^{n-1} \frac{(k-1)(mb - a)^k}{(k+1)!} f^{(k)}(a) \right| \\
\leq \frac{1}{2} \frac{(mb - a)^n}{n!} \int_0^1 t^{n-1}(n - 2t)|f^{(n)}(ta + m(1-t)b)| \, dt.
\]

By the well-known Hölder integral inequality, we obtain

\[
(3.7) \quad \int_0^1 t^{n-1}(n - 2t)|f^{(n)}(ta + m(1-t)b)| \, dt \\
\leq \left[ \int_0^1 (n - 2t)^q \, dt \right]^{1/q} \left[ \int_0^1 t^{p(n-1)}|f^{(n)}(ta + m(1-t)b)|^p \, dt \right]^{1/p} \\
= \left[ \frac{n^{q+1} - (n-2)^q + 1}{2(q+1)} \right]^{1/q} \left[ \int_0^1 t^{p(n-1)}|f^{(n)}(ta + m(1-t)b)|^p \, dt \right]^{1/p}.
\]

Making use of the \((a, m)\)-convexity of \(|f^{(n)}(x)|^p\) reveals

\[
(3.8) \quad \int_0^1 t^{p(n-1)}|f^{(n)}(ta + m(1-t)b)|^p \, dt \\
\leq \int_0^1 t^{p(n-1)}|f^{(n)}(a)|^p + (m(1-t)^{\alpha})|f^{(n)}(b)|^p \, dt \\
= |f^{(n)}(a)|^p \int_0^1 t^{p(n-1)+\alpha} \, dt + m|f^{(n)}(b)|^p \int_0^1 t^{p(n-1)}(1-t^{\alpha}) \, dt \\
= \frac{|f^{(n)}(a)|^p}{p(n-1) + \alpha + 1} + \frac{ma}{p(n-1) + \alpha + 1} |f^{(n)}(b)|^p.
\]

Combining (3.7) and (3.8) with (3.6) results in the inequality (3.5). This completes the proof of Theorem 3.2. \( \square \)

Corollary 3.2. Under conditions of Theorem 3.2,
(1) when \( m = 1 \), we have
\[
\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx - \frac{1}{2} \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{(k+1)!} f^{(k)}(a)
\leq \frac{1}{2} \frac{(b-a)^n}{n!} \left[ \frac{n^{q+1} - (n-2)^{q+1}}{2(q+1)} \right]^{1/q} \left\{ \frac{1}{p(n-1) + \alpha + 1} |f^{(n)}(a)|^p + \frac{\alpha}{[p(n-1) + \alpha + 1] |f^{(n)}(b)|^p} \right\}^{1/p}.
\]

(2) when \( n = 2 \), we have
\[
\frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x) \, dx
\leq \frac{(mb-a)^2}{2} \left( \frac{1}{q+1} \right)^{1/q} \left[ \frac{1}{p+\alpha+1} |f''(a)|^p + \frac{ma}{(p+\alpha+1)(p+\alpha+1)} |f''(b)|^p \right]^{1/p};
\]

(3) when \( m = \alpha = 1 \) and \( n = 2 \), we have
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)^2}{2(p+1)!} \frac{1}{(q+1)^{1/q}} \frac{1}{(q+1)} \left[ \frac{1}{(q+1)} |f''(a)|^q + |f''(b)|^q \right]^{1/q},
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Theorem 3.3.** Let \((a, m) \in [0, 1] \times (0, 1)\) and \( b > a > 0 \) with \( a < mb \). If \( f(x) \) is \( n \)-time differentiable on \([0, b]\) such that \( |f^{(n)}(x)| \in L([0, mb]) \) and \( |f^{(n)}(x)|^p \) is \((a, m)\)-convex on \([0, mb]\) for \( n \geq 2 \) and \( p \geq 1 \), then
\[
\left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x) \, dx - \frac{1}{2} \sum_{k=2}^{n-1} \frac{(k-1)(mb-a)^k}{(k+1)!} f^{(k)}(a) \right|
\leq \frac{(n-1)^{1-1/p}}{n!} \frac{(mb-a)^n}{2} \left\{ \frac{(n-2)(pm-p+\alpha) + 2(n-1)}{(pm-p+\alpha+1)(pm-p+\alpha+2)} |f^{(n)}(a)|^p + m \left[ \frac{(n-1)(pm-2p+2)}{(pm-p+1)(pm-p+2)} - \frac{(n-2)(pm-p+\alpha) + 2(n-1)}{(pm-p+\alpha+1)(pm-p+\alpha+2)} \right] |f^{(n)}(b)|^p \right\}^{1/p},
\]
where the sum above takes 0 when \( n = 2 \).

**Proof.** Utilizing Lemma 2.1, Hölder integral inequality, and the \((a, m)\)-convexity of \( |f^{(n)}(x)|^p \) yields
\[
\left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x) \, dx - \frac{1}{2} \sum_{k=2}^{n-1} \frac{(k-1)(mb-a)^k}{(k+1)!} f^{(k)}(a) \right|
\leq \frac{1}{2} \frac{(mb-a)^n}{n!} \int_0^1 t^{n-1}(n-2t) |f^{(n)}(ta + m(1-t)b)| \, dt
\leq \frac{1}{2} \frac{(mb-a)^n}{n!} \left[ \int_0^1 (n-2t) \, dt \right]^{1-1/p}
\times \left\{ \int_0^1 t^{p(n-1)(n-2t)} \left[ |f^{(n)}(a)|^p + m(1-t^\alpha) |f^{(n)}(b)|^p \right] \, dt \right\}^{1/p}.
\]
This follows from applying the inequality (3.9) to the function $f(x) = x^r$. 

**Corollary 3.3.** Under conditions of Theorem 3.3,

1. when $m = 1$, we have

$$
\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx = \frac{1}{2} \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{(k+1)!} \left( f^{(k)}(a) - f^{(k)}(b) \right) 
\leq \frac{(n-1)^{1-1/p}}{2} \left\{ \frac{(n-2)(p-\alpha + 1)}{(p-\alpha + 1)(p-\alpha + 2)} \right\}^{1/p} + m \left\{ \frac{(n-2)(p-\alpha + 1)}{(p-\alpha + 1)(p-\alpha + 2)} \right\}^{1/p}.
$$

2. when $n = 2$, we have

$$
\frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x) \, dx \leq \frac{2}{4} \left\{ \frac{(n-1)^{1-1/p}}{2} \left\{ \frac{(p+\alpha + 1)}{(p+\alpha + 1)(p+\alpha + 2)} \right\}^{1/p} + m \left\{ \frac{(p+\alpha + 1)}{(p+\alpha + 1)(p+\alpha + 2)} \right\}^{1/p} \right\}^{1/p}.
$$

3. when $m = \alpha = 1$ and $n = 2$, we have

$$
(3.11) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)^2}{2^{2-1/p}} \left\{ \frac{(p+\alpha + 1)}{(p+\alpha + 1)(p+\alpha + 2)} \right\}^{1/p}.
$$

### 4. Applications to special means

It is well known that, for positive real numbers $\alpha$ and $\beta$ with $\alpha \neq \beta$, the quantities

$$
A(\alpha, \beta) = \frac{\alpha + \beta}{2}, \quad G(\alpha, \beta) = \sqrt{\alpha \beta}, \quad H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}},
$$

$$
I(\alpha, \beta) = \frac{1}{e} \left( \frac{\beta^\alpha}{\alpha^\beta} \right)^{1/(\beta - \alpha)}, \quad L(\alpha, \beta) = \frac{\alpha - \beta}{\ln \alpha - \ln \beta}, \quad L_r(\alpha, \beta) = \left[ \frac{\beta^{r+1} - \alpha^{r+1}}{(r+1)(\beta - \alpha)} \right]^{1/r}
$$

for $r \neq 0, -1$ are respectively called the arithmetic, geometric, harmonic, exponential, logarithmic, and generalized logarithmic means.

Basing on inequalities of Hermite-Hadamard type in the above section, we shall derive some inequalities of the above defined means as follows.

**Theorem 4.1.** Let $r \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$ and $b > a > 0$. Then, for $p, q > 1$,

$$
|A(a^r, b^r) - [L_r(a, b)]^r| \leq \frac{(b-a)^2(r-1)}{2(p+1)^{1/p}(q+2)^{1/q}} \left[ a^{(r-2)q} + \frac{b^{(r-2)q}}{q+1} \right]^{1/q},
$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

**Proof.** This follows from applying the inequality (3.9) to the function $f(x) = x^r$. 

$\square$
Theorem 4.2. Let $r \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$ and $b > a > 0$. Then, for $p \geq 1$,\hspace{1cm}
\begin{equation}
|A(a^r, b^r) - [L_r(a,b)]^r| \leq \frac{(b-a)^2 r(r-1)}{2^{1-1/p}} \left( \frac{(p+1)a^{(r-2)p} + 2b^{(r-2)p}}{(p+1)(p+2)(p+3)} \right)^{1/p}.
\end{equation}
\hspace{1cm}
Proof. This follows from applying the inequality (3.11) to the function $f(x) = x^r$. \hfill \Box

Theorem 4.3. Let $r \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$ and $b > a > 0$. Then\hspace{1cm}
\begin{equation}
|A(a^r, b^r) - [L_r(a,b)]^r| \leq \frac{(b-a)^2 r(r-1)}{24} A(a^{r-2}, b^{r-2}).
\end{equation}
\hspace{1cm}
Proof. This follows from applying the inequality (3.11) for $p = 1$ to the function $f(x) = x^r$. \hfill \Box

Theorem 4.4. Let $b > a > 0$. Then for $p, q > 1$ we have\hspace{1cm}
\begin{equation}
\left| \frac{1}{H(a,b)} - \frac{1}{L(a,b)} \right| \leq \frac{(b-a)^2}{(p+1)^{1/p}(q+2)^{1/q}} \left[ \frac{1}{a^{3p} b^{3q}} + \frac{1}{(q+1)b^{3q}} \right]^{1/q},
\end{equation}
where $\frac{1}{p} + \frac{1}{q} = 1$.\hspace{1cm}
Proof. This follows from applying the inequality (3.9) to the function $f(x) = \frac{1}{x}$. \hfill \Box

Theorem 4.5. Let $b > a > 0$. Then for $p \geq 1$ we have\hspace{1cm}
\begin{equation}
\left| \frac{1}{H(a,b)} - \frac{1}{L(a,b)} \right| \leq \frac{(b-a)^2 r(r-1)}{2^{1-1/p}[(p+2)(p+3)]^{1/p}} \left[ \frac{1}{a^{3p} b^{3q}} + \frac{2}{(p+1)b^{3p}} \right]^{1/p}.
\end{equation}
\hspace{1cm}
Proof. This follows from the inequality (3.11) to the function $f(x) = x^r$. \hfill \Box

Theorem 4.6. Let $b > a > 0$. Then we have\hspace{1cm}
\begin{equation}
\ln \frac{I(a,b)}{G(a,b)} \leq \frac{(b-a)^2}{24} A\left( \frac{1}{a^2}, \frac{1}{b^2} \right).
\end{equation}
\hspace{1cm}
Proof. This follows from applying the inequality (3.11) for $p = 1$ to the function $f(x) = - \ln x$. \hfill \Box

Remark 4.1. This paper is a combined version of the preprints [6, 7].

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References

[1] R.-F. Bai, F. Qi, and B.-Y. Xi, Hermite-Hadamard type inequalities for the $m$- and $(\alpha, m)$-logarithmically convex functions, Filomat 27 (2013), no. 1, 1–7; Available online at http://dx.doi.org/10.2298/FIL1301001B, 2
[2] S.-P. Bai, S.-H. Wang, and F. Qi, Some Hermite-Hadamard type inequalities for $n$-time differentiable $(\alpha, m)$-convex functions, J. Inequal. Appl. 2012, 2012:267, 11 pages; Available online at http://dx.doi.org/10.1186/1029-242X-2012-267, 2
[3] L. Chun and F. Qi, Integral inequalities of Hermite-Hadamard type for functions whose third derivatives are convex, J. Inequal. Appl. 2013, 2013:451, 10 pages; Available online at http://dx.doi.org/10.1186/1029-242X-2013-451, 2
[4] S. S. Dragomir and C. E. M. Pearce, Selected Topics on Hermite-Hadamard Type Inequalities and Applications, RGMIA Monographs, Victoria University, 2000; Available online at http://rgmia.org/monographs/hermite_hadamard.html, 1, 2
[5] D.-Y. Hwang, Some inequalities for $n$-time differentiable mappings and applications, Kyungpook Math. J. 43 (2003), no. 3, 335–343, 3, 5
[6] M. A. Latif and S. Hussain, New inequalities of Hermite-Hadamard type for n-time differentiable \((\alpha, m)\)-convex functions with applications to special means, RGMIA Res. Rep. Coll. 16 (2013), Art. 17, 12 pages; Available online at http://rgmia.org/v16.php. 9

[7] W.-H. Li and F. Qi, Hermite-Hadamard type inequalities of functions whose derivatives of n-th order are \((\alpha, m)\)-convex, available online at http://arxiv.org/abs/1308.2948v1. 9

[8] W.-H. Li and F. Qi, Some Hermite-Hadamard type inequalities for functions whose n-th derivatives are \((\alpha, m)\)-convex, Filomat 27 (2013), no. 8, 1575–1582; Available online at http://dx.doi.org/10.2298/FIL1308575L. 3

[9] V. G. Miheşan, A generalization of the convexity, Seminar on Functional Equations, Approx. Convex, Cluj-Napoca, 1993. (Romania) 1

[10] C. P. Niculescu and L.-E. Persson, Convex Functions and their Applications, CMS Books in Mathematics, Springer-Verlag, 2005. 1, 2

[11] M. E. Özdemir, M. Avci, and H. Kavurmaci, Hermite-Hadamard-type inequalities via \((\alpha, m)\)-convexity, Comput. Math. Appl. 61 (2011), no. 9, 2614–2620; Available online at http://dx.doi.org/10.1016/j.camwa.2011.02.053. 2, 3, 5

[12] M. E. Özdemir, M. Avci, and E. Set, On some inequalities of Hermite-Hadamard type via m-convexity, Appl. Math. Lett. 23 (2010), no. 9, 1065–1070; Available online at http://dx.doi.org/10.1016/j.aml.2010.04.037. 2

[13] F. Qi, Z.-L. Wei, and Q. Yang, Generalizations and refinements of Hermite-Hadamard’s inequality, Rocky Mountain J. Math. 35 (2005), no. 1, 235–251; Available online at http://dx.doi.org/10.1216/rmjm/1181069779. 2

[14] F. Qi and B.-Y. Xi, Some integral inequalities of Simpson type for GA-\(\epsilon\)-convex functions, Georgian Math. J. 20 (2013), no. 4, 775–788; Available online at http://dx.doi.org/10.1515/gmj-2013-0043. 2

[15] M. Z. Sarikaya and N. Aktan, On the generalization of some integral inequalities and their applications, available online at http://dx.doi.org/10.1016/j.jmcn.2011.05.026. 2

[16] Y. Shuang, Y. Wang, and F. Qi, Some inequalities of Hermite-Hadamard type for functions whose third derivatives are \((\alpha, m)\)-convex, J. Comput. Anal. Appl. 17 (2014), no. 2, 272–279. 2

[17] G. Toader, Some generalizations of the convexity, Univ. Cluj-Napoca, Cluj-Napoc. 1985, 329–338. 1

[18] S.-H. Wang and F. Qi, Inequalities of Hermite-Hadamard type for convex functions which are \(n\)-times differentiable, Math. Inequal. Appl. 17 (2014), in press. 3

[19] S.-H. Wang, B.-Y. Xi, F. Qi, Some new inequalities of Hermite-Hadamard type for \(n\)-time differentiable functions which are \(m\)-convex, Analysis (Munich) 32 (2012), no. 3, 247–262; Available online at http://dx.doi.org/10.1524/anly.2012.1167. 3

[20] B.-Y. Xi, R.-F. Bai, and F. Qi, Hermite-Hadamard type inequalities for the \(m\)- and \((\alpha, m)\)-geometrically convex functions, Aequationes Math. 84 (2012), no. 3, 261–269; Available online at http://dx.doi.org/10.1007/s00010-011-0114-x. 2

[21] B.-Y. Xi and F. Qi, Some inequalities of Hermite-Hadamard type for \(h\)-convex functions, Adv. Inequal. Appl. 2 (2013), no. 1, 1–15. 2

[22] B.-Y. Xi and F. Qi, Some integral inequalities of Hermite-Hadamard type for convex functions with applications to means, J. Funct. Spaces Appl. 2012 (2012), Article ID 980438, 14 pages; Available online at http://dx.doi.org/10.1155/2012/980438. 2

(Qi) Department of Mathematics, School of Science, Tianjin Polytechnic University, Tianjin City, 300387, China; College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China

E-mail address: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com
URL: http://qifeng618.wordpress.com

(Latif) College of Science, Department of Mathematics, University of Hail, Hail 2440, Saudi Arabia
E-mail address: m.amer.latif@hotmail.com, m.alatif@uoh.edu.sa

(Li) Department of Mathematics, School of Science, Tianjin Polytechnic University, Tianjin City, 300387, China
E-mail address: wen.hu.i11@yahoo.com, wen.hu.i1102@gmail.com

(Hussain) Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan
E-mail address: sabirhus@gmail.com