Abstract. We provide a companion to the recent Bényi-Čurgus generalization of the well-known theorems of Ceva and Menelaus, so as to characterize both the collinearity of points and the concurrence of lines determined by six points on the edges of a triangle. A companion for the generalized area formula of Routh appears, as well.

The venerable theorems of (Giovanni) Ceva and Menelaus (of Alexandria) concern points on the edge-lines of a triangle. Each point defines—and is defined by—the ratio of lengths of collinear segments joining it to two of the triangle’s vertices, and the theorems use a trio of such ratios to neatly characterize the special configurations in Figure 1.

Specifically, with $D$, $E$, $F$ on edge-lines opposite respective vertices $A$, $B$, $C$, we write

\begin{align}
\begin{aligned}
    d &:= \frac{|BD|}{|DC|} \quad e := \frac{|CE|}{|EA|} \quad f := \frac{|AF|}{|FB|}
\end{aligned}
\end{align}

Figure 1

(a) Lines through a triangle’s vertices, meeting at a common point. (Ceva) (b) Points on a triangle’s edges, lying on a common line. (Menelaus)

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Throughout, we consider segment lengths to be signed, with each of $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$—for distinct $A$, $B$, $C$—indicating the direction of a positively-signed segment on the corresponding (extended) side of the triangle. Moreover, we adopt these conventions regarding ratios of these lengths:

\begin{align}
\frac{|PP|}{|PQ|} &= 0 \quad \frac{|PQ|}{|QQ|} = \infty \quad \frac{|PX|}{|XQ|} = -1, \text{ for } X \text{ the point at infinity on } \overrightarrow{PQ}
\end{align}
and express the theorems as follows:

**Theorem 1** (Ceva). Lines $\overrightarrow{AD}$, $\overrightarrow{BE}$, $\overrightarrow{CF}$ pass through a common point if and only if

\[(2a) \quad def = 1\]

**Theorem 2** (Menelaus). Points $D$, $E$, $F$ lie on a common line if and only if

\[(2b) \quad def = -1\]

Bényi and Ćurgus [1], and this author, independently (and nearly-simultaneously) considered separate aspects of the same approach to generalizing the above —namely, doubling the number of points on the triangle’s edges— arriving at equations whose terms, in the grand Ceva-Menelaus tradition, differ only in sign.

Interestingly, the Bényi-Ćurgus result concerns *Ceva*-like elements (lines through vertices) and a *Menelaus*-like phenomenon (collinearity of points). This author’s contribution, on the other hand, concerns *Menelaus*-like elements (points on edges) and a *Ceva*-like phenomenon (concurrence of lines).

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**Figure 2**

Place points $A^+$ and $A^-$ on the edge-line opposite vertex $A$; likewise, $B^+$ and $B^-$ opposite $B$, and $C^+$ and $C^-$ opposite $C$. Define these ratios:\(^2\)

\[
\begin{align*}
  a^+ &:= \frac{|BA^+|}{|A^+C|} & b^+ &:= \frac{|CB^+|}{|B^+A|} & c^+ &:= \frac{|AC^+|}{|C^+B|} \\
  a^- &:= \frac{|CA^-|}{|A^-B|} & b^- &:= \frac{|AB^-|}{|B^-C|} & c^- &:= \frac{|BC^-|}{|C^-A|}
\end{align*}
\]

\(^2\)Observe that the superscripts emphasize an opposing directionality in the definitions of the ratios. For instance, the points in ratio $a^+$ trace the path $B-A^+-C$, with endpoints oriented in the *positive* direction; in $a^-$, the path $C-A^-B$ has endpoints oriented in the *negative* direction. Were we to define all six ratios in “matching” orientations —as was done in [1]— the resulting formulas would lose some clarity and symmetry.
**Theorem 3** (Six-Point Ceva-Menelaus Theorem).

a. Lines $\overrightarrow{B \oplus C}$, $\overrightarrow{C \oplus A}$, $\overrightarrow{A \oplus B}$ pass through a common point if and only if

$$a^+b^+c^+ + a^-b^-c^- = 1 - a^+a^- - b^+b^- - c^+c^-$$

b. (Bényi-Čurgus) Points $\overrightarrow{B \ominus C}$, $\overrightarrow{C \ominus A}$, $\overrightarrow{A \ominus B}$ lie on a common line if and only if

$$a^+b^+c^+ + a^-b^-c^- = -1 + a^+a^- + b^+b^- + c^+c^-$$

Note: Identifying $A \ominus B$, $B \ominus C$, $C \ominus A$ with $C \ominus B$, $B \ominus A$, $A \ominus C$ yields $a = b = c = 0$, so that (4a) and (4b) reduce to (2a) and (2b). The Six-Point Theorem generalizes the traditional results.

For proof, one can invoke vector techniques, as indicated with Theorem 5 below.

**Routh, too.** When Ceva’s lines fail to concur, and when Menelaus’ points fail to “colline”, they determine triangles. One might well ask how the area$^4$ of each resulting triangle compares to that of the original figure. (Edward John) Routh provided answers. (See [1].)

![Figure 3](image)

(a) Triangle from lines through vertices.  
(b) Triangle from points on edges.

**Theorem 4** (Routh’s Formulas).

a. (“Routh’s Theorem”). The triangle with (non-parallel) edge-lines $\overrightarrow{AD}$, $\overrightarrow{BE}$, $\overrightarrow{CF}$ has area

$$\frac{|\triangle ABC| \cdot (def - 1)^2}{(1 + d + de)(1 + e + ef)(1 + f + fd)}$$

b. The triangle with (finite) vertices $D$, $E$, $F$ has area

$$\frac{|\triangle ABC| \cdot def + 1}{(1 + d)(1 + e)(1 + f)}$$

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$^3$To amplify the duality with (a), we write “$P \ominus Q$” for the point of intersection of lines $\overrightarrow{PP}$ and $\overrightarrow{QQ}$.

$^4$As with length, we consider triangle area to be *signed*. Areas $|\triangle ABC|$ and $|\triangle DEF|$ agree in sign when vertex paths $A-B-C-A$ and $D-E-F-D$ trace their respective figures in the same direction; similarly for triangles defined by their edge-lines.
Observe that the numerator in each of these formulas—and, thus, the area of the triangle in question—vanishes, as it should, when (and only when) the conditions for Ceva’s or Menelaus’ theorems indicate that the triangle degenerates into a point or a line. The reader may verify that a denominator vanishes when (and only when) the triangle becomes unbounded, having non-finite vertices and parallel edges.

Bényi and Čurgus [1] specifically address the six-point generalization of Theorem 4b. At the suggestion of Mr. Čurgus, this author derived the counterpart generalization of 4a.

**Theorem 5** (Six-Point Routh Formulas).

a. The triangle with (non-parallel) edge-lines $\overrightarrow{B^+C^-}$, $\overrightarrow{C^+A^-}$, $\overrightarrow{A^+B^-}$ has area

\[
|\triangle ABC| \cdot \frac{(a^+b^-c^- + a^-b^+c^- + a^+a^- + b^+b^- + c^+c^- - 1)^2}{(1 - a^+a^- + b^- (1 + a^-)) + c^+ (1 + a^+)} \cdot (1 - b^+b^- + c^- (1 + b^-) + a^+(1 + b^+)) \cdot (1 - c^+c^- + a^- (1 + c^-) + b^+(1 + c^+))
\]

b. (Bényi-Čurgus). The triangle with (finite) vertices $\overrightarrow{B^-C^+}$, $\overrightarrow{C^-A^+}$, $\overrightarrow{A^-B^+}$ has area

\[
|\triangle ABC| \cdot \frac{a^+b^-c^- + a^-b^+c^- - a^+a^- - b^+b^- - c^+c^- + 1}{(1 + b^- + c^+)(1 + c^- + a^+)(1 + a^- + b^+)}
\]

**Proof.** Treating points as vectors, we can write

\[
A^+ = \frac{B(1 + a^+) + Ca^+}{1 + a^+}, \quad A^- = \frac{Ba^- + C(1 + a^-)}{1 + a^-}, \quad \text{etc.}
\]
to find, after a bit of tedious algebra, that the vertices of the triangles in parts (a) and (b) of the theorem have the respective forms

\begin{align}
A &= \frac{A(1-a^{-a^+}) + B(c^+ + a^-b^-) + C(b^- + a^+c^+)}{1-a^-a^+ + b^-(1 + a^-) + c^+(1 + a^+)} \\
B &= \frac{A + Bc^+ + Cb^-}{1 + c^+ + b^-}
\end{align}

The area formulas follow from a bit more —and more-tedious— algebra. Of course, since ratios of lengths of collinear segments, and of areas of coplanar triangles, are preserved under affine transformation, one could simplify this analysis somewhat by assuming, say, \(A = (0,0), B = (1,0), C = (0,1)\); even in generality, however, verification of these formulas amounts to just a few seconds’ effort from a computer algebra system.

\[\square\]

**Remarks.** We can accentuate the duality of the traditional results of Ceva and Menelaus by reciting them thusly: “Points determined by pairs of lines through the (vertex-)points of a triangle coincide if and only if ...” versus “Lines determined by pairs of points on the (edge-)lines of a triangle coincide if and only if ...”.

Taking a deep breath, we can do likewise for the parts of the Six-Point Theorem: “Points determined by pairs of lines determined by pairs of points on the (edge-)lines of a triangle coincide if and only if ...” versus “Lines determined by pairs of points determined by pairs of lines through the (vertex-)points of a triangle coincide if and only if ...”.

What formulas characterize the coincidence of lines and/or points at the next order of complexity? For that matter, what strategy for pairing lines and/or points best constitutes the next order of complexity?

**References**

[1] Bényi, Árpád, and Ćurgus, Branko. “A generalization of Routh’s triangle theorem”. The American Mathematical Monthly, Vol. 120, No. 9 (Nov., 2013), pp. 841-846. (Preprint available at arXiv:math.MG/1112.4813v2)