WEIGHTED ONE-LEVEL DENSITY OF LOW-LYING ZEROS OF DIRICHLET $L$-FUNCTIONS

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Abstract. In this paper, we compute the one-level density of low-lying zeros of Dirichlet $L$-functions in a family weighted by special values of Dirichlet $L$-functions at a fixed $s \in [1/2, 1)$. We verify both Fazzari’s conjecture and the first author’s conjecture on the weighted one-level density for our family of $L$-functions.

1. Introduction

The Riemann zeta function $\zeta(s)$ is among the most interesting objects in number theory for its close relation to the distribution of prime numbers, which are fundamental objects in number theory. In particular, zeros of $\zeta(s)$ detect prime numbers and the study of zeros of $\zeta(s)$ is inevitable in order to understand the distribution of prime numbers. For example, the famous Riemann hypothesis, which asserts that $\text{Re}(\rho) = 1/2$ should hold for all zeros $\rho$ of $\zeta(s)$ lying in the critical strip $0 < \text{Re}(s) < 1$, called non-trivial zeros of $\zeta(s)$, is among the most important unsolved problems in mathematics. One interpretation of the Riemann hypothesis links non-trivial zeros of $\zeta(s)$ to eigenvalues of a certain operator. More precisely, it is said that the Riemann hypothesis is implied by the Hilbert-Pólya conjecture, which asserts the existence of a determinant expression of $\zeta(s)$ using a Hamiltonian $H$ in the form of $\zeta(1/2+it) = \det(t\text{id} - H)$. In 1973, Montgomery [12] gave evidence of the Hilbert-Pólya conjecture, and later Odlyzko [13] gave supporting numerical data. The so-called Montgomery-Odlyzko law predicts that non-trivial zeros of $\zeta(s)$ are distributed like eigenvalues of random Hermitian matrices in the Gaussian Unitary Ensemble. At the end of the twentieth century, Katz and Sarnak [6, 7] shed light on a family of $L$-functions instead of an individual $L$-function in order to relate zeros of $L$-functions to eigenvalues of random matrices. The Katz-Sarnak philosophy (or called the density conjecture) predicts that the distribution of low-lying zeros of $L$-functions in a family is similar to that of the eigenvalues of random matrices, and that the family of $L$-functions has one of five symmetry types $U$, $Sp$, $SO$(even), $SO$(odd) and $O$ (unitary, symplectic, even orthogonal, odd orthogonal and orthogonal) in accordance with the density of low-lying zeros of $L$-functions in the family. Shortly thereafter, Iwaniec, Luo and Sarnak [5] confirmed the density conjecture for the case of automorphic $L$-functions attached to elliptic modular forms and their symmetric square lifts.

Motivated by the work of Knightly and Reno [10] in 2019, the one-level density for a family of $L$-functions weighted by $L$-values has been studied in the context of random matrix theory. Knightly and Reno [10] discovered the phenomenon that the symmetry

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type of a family of automorphic $L$-functions attached to elliptic modular forms changes \textit{from orthogonal to symplectic} due to the weight factors of central $L$-values, by comparing \cite{10} with the usual one-level density \cite{5}. Their work was inspired by Kowalski, Saha and Tsimerman \cite{11}, who treated the one-level density for a family of spinor $L$-functions attached to Siegel modular forms of degree 2 weighted by Bessel periods, identical to central $L$-values by \cite{3}. We notice by comparing the weighted one-level density \cite{11} and the usual one-level density \cite{8, 9} that the symmetry type of the family of those spinor $L$-functions changes \textit{from orthogonal to symplectic}. Recently, the first author \cite{16} observed a new phenomenon occurring in a family of symmetric square $L$-functions attached to Hilbert modular forms. He found that its symmetry type changes \textit{from symplectic to a new type of density function which does not occur in random matrix theory as Katz and Sarnak predicted}. Afterwards, Fazzari \cite{2} conjectured that for a family of $L$-functions whose symmetry type is unitary, symplectic or even orthogonal, the one-level density for the family weighted by central $L$-values should coincide with the density of eigenvalues of random matrices weighted by their characteristic polynomials. He gave evidence of his conjecture by studying the following three families of $L$-functions under the generalized Riemann hypothesis and the Ratios Conjecture: $\{\zeta(s+ia)\}_{a \in \mathbb{R}}, \{L(s, \chi_d)\}_d$ and $\{L(s, \Delta \otimes \chi_d)\}_d$. Here, $d$ is a fundamental discriminant, $\chi_d$ is the primitive quadratic Dirichlet character corresponding to the quadratic field $\mathbb{Q}(\sqrt{d})$ by class field theory, and $\Delta$ is the delta function (a cusp form of weight 12 and level 1). The first author’s result \cite{16} on the change of the symmetry type provides new evidence of Fazzari’s conjecture.

In this paper, we consider the one-level density for a family of Dirichlet $L$-functions weighted by special values of Dirichlet $L$-functions at $s \in [1/2, 1)$. We are interested to see whether the symmetry type of the family of Dirichlet $L$-functions under consideration changes. For a prime number $q$, let $\mathcal{F}_q$ be the set of all non-principal Dirichlet characters modulo $q$. We consider the one-level density of low-lying zeros of the non-completed Dirichlet $L$-function $L(s, \chi)$ attached to a Dirichlet character $\chi \in \mathcal{F}_q$ defined as

$$D(\chi, \phi) := \sum_{\rho \in 1/2 + i\gamma} \phi \left( \frac{\log q}{2\pi} \gamma \right),$$

where a test function $\phi$ is a Schwartz function on $\mathbb{R}$ such that the support supp$(\hat{\phi})$ of the Fourier transform $\hat{\phi}(\xi) := \int_{-\infty}^{\infty} \phi(x)e^{-2\pi i \xi x}dx$ of $\phi$ is compact, and $\rho$ runs over the set of all zeros of $L(s, \chi)$ in the critical strip $0 < \text{Re}(s) < 1$ counted with multiplicity. We remark that $\phi$ is naturally extended to a Paley-Wiener function on $\mathbb{C}$. With this extension, the assumption of the generalized Riemann hypothesis is not necessary and we emphasize that we do not assume the generalized Riemann hypothesis throughout this paper. Thus $\gamma$ in the summation of $D(\chi, \phi)$ is not necessarily real.

We consider the weighted one-level density of low-lying zeros of $L(s, \chi)$ for $\chi \in \mathcal{F}_q$. The usual one-level density for $\mathcal{F}_q$ has been given by Hughes and Rudnick \cite[Theorem 3.1]{3} as follows.

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Theorem 1.1 (Hughes and Rudnick, 2003). For a Schwartz function $\phi$ on $\mathbb{R}$ such that $\text{supp}(\hat{\phi}) \subset [-2, 2]$, we have

$$\frac{1}{\# F_q} \sum_{\chi \in F_q} D(\chi, \phi) = \int_{-\infty}^{\infty} \phi(x) W_U(x) dx + O\left( \frac{1}{\log q} \right), \quad q \to \infty$$

with

$$W_U(x) = 1,$$

where $q$ tends to infinity in the set of prime numbers. Thus, the symmetry type of the family $\bigcup_q \{ L(s, \chi) \mid \chi \in F_q \}$ is unitary.

We consider the one-level density for $F_q$ weighted by the square of the absolute values of central $L$-values. We note that, by the asymptotic behavior of the second moment $\sum_{\chi \in F_q} |L(1/2, \chi)|^2 \sim \frac{(q-1)^2}{q} \log q, \quad q \to \infty$ of Dirichlet $L$-functions due to Paley [14, Theorem II], the second moment of Dirichlet $L$-functions is non-zero for any sufficiently large prime number $q$. Our weighted one-level density result is stated as follows.

Theorem 1.2. Let $\phi$ be a Schwartz function on $\mathbb{R}$ such that $\text{supp}(\hat{\phi}) \subset [-1/3, 1/3]$. Then, we have

$$\sum_{\chi \in F_q} |L(1/2, \chi)|^2 = \frac{(q-1)^2}{q} \log q, \quad q \to \infty$$

of Dirichlet $L$-functions due to Paley [14, Theorem II], the second moment of Dirichlet $L$-functions is non-zero for any sufficiently large prime number $q$. Our weighted one-level density result is stated as follows.

Theorem 1.4. Take any $s \in (1/2, 1)$. Let $\phi$ be a Schwartz function on $\mathbb{R}$ such that $\text{supp}(\hat{\phi}) \subset [-2s/3, 2s/3]$. Then we have

$$\sum_{\chi \in F_q} |L(s, \chi)|^2 = q \zeta(2s), \quad q \to \infty.$$
Thus the change of symmetry type does not occur when $1/2 < s < 1$.

**Remark 1.5.** By Theorems 1.2 and 1.4 the weighted one-level density by special values $|L(s, \chi)|^2$ at $s \in [1/2, 1)$ causes the change of symmetry type if and only if $s = 1/2$. This supports the weighted density conjecture by the first author [16 Conjecture 1.3].

The key idea for proving the asymptotic behavior of the weighted one-level density is the use of Selberg's formula of the twisted second moment of Dirichlet $L$-functions with complex parameters $s$ and $s'$ [15]. Selberg's formula is a substitute of the explicit Jacquet-Zagier type trace formula by the first author and Tsuzuki [17], as we see that such a parametrized trace formula was used in [16] for analysis of the weighted one-level density for symmetric power $L$-functions attached to Hilbert modular forms. The computation in Proposition 2.3 is essentially the same as that in [16, Theorem 2.6]. Both computations include the derivation at $s = 1/2$ for deducing the main term and the second main term. Asymptotics using Selberg’s formula is explained in §3. We give proofs of Theorems 1.2 and 1.4 in §4.

**Remark 1.6.** We can extend Theorems 1.2 and 1.4 to the case of general $q$, that is, $q$ does not necessarily need to be a prime number. In that case, in place of Theorem 2.1 we need to use the more general form of Selberg's formula as stated in [15 pp. 6–7].

2. Twisted second moment of Dirichlet $L$-functions

In this section, we use the asymptotic formula for the twisted second moment of Dirichlet $L$-functions due to Selberg [15, Theorem 1] to deduce the asymptotic formula suitable for our purpose. Recall that we consider only the case when the modulus $q$ of Dirichlet characters is a prime number.

**Theorem 2.1** (Selberg, 1946). Let $q$ be a prime number. Let $m$ and $n$ be positive integers such that $m$ and $n$ are coprime to each other and to $q$. For any $(s, s') \in \mathbb{C}^2$ such that $\sigma = \text{Re}(s) \in (0, 1)$ and $\sigma' = \text{Re}(s') \in (0, 1)$, we have

$$\sum_{\chi \in \mathcal{F}_q} L(s, \chi)L(s', \overline{\chi})\chi(m)\overline{\chi(n)} = \frac{q - 1}{m^{s}s^{n}s'}\zeta(s + s') + \frac{(q - 1)q^{1-s-s'}(2\pi)^{s+s'-1}}{m^{1-s}n^{1-s'}}\Gamma(1-s)\Gamma(1-s')\frac{\pi}{(s-s')}\cos \left(\frac{\pi}{2}(s-s')\right) \zeta(2-s-s') + \mathcal{O}\left(\frac{|ss'|}{\sigma\sigma'(1-\sigma)(1-\sigma')}\left(mq^{1-\sigma} + nq^{1-\sigma'} + mnq^{1-\sigma-\sigma'}\right)\right), \quad q \to \infty,$$

where the implied constant is independent of $m, n, q, s$ and $s'$. On the right-hand side, the value at $(s, s')$ with $s + s' = 1$ is understood as the limit when $s + s' \to 1$.

**Proposition 2.2.** Let $q$ be a prime number and $m$ a positive integer coprime to $q$. For any fixed $s \in (1/2, 1)$, we have

$$\sum_{\chi \in \mathcal{F}_q} \frac{1}{|L(s, \chi)|^2} \sum_{\chi \in \mathcal{F}_q} |L(s, \chi)|^2\chi(m) = m^{-s} + \mathcal{O}_s(m^{s-1}q^{1-2s}) + \mathcal{O}_s(mq^{-s}), \quad q \to \infty,$$

(2.1)
where the implied constant is independent of $m$ and $q$.

**Proof.** Restricting Selberg’s formula in Theorem 2.1 to the case $s = s' \in (1/2, 1)$ and $n = 1$, we obtain

$$
\sum_{\chi \in \mathcal{F}_q} |L(s, \chi)|^2 \chi(m) = \frac{q-1}{m^s} \zeta(2s) + \frac{(q - 1)q^{1-2s}}{m^{1-s}} \frac{(2\pi)^{2s-1}}{\pi} \Gamma(1-s)^2 \zeta(2-2s) + O_q(mq^{1-s}).
$$

(2.2)

Let $F_m$ denote the main term of the right-hand side above and $E_m$ its error term. Then the left-hand side of (2.1) is equal to

$$
\frac{F_m + E_m}{F_1 + E_1} = \frac{F_m}{F_1} + \frac{E_m - \frac{F_m}{F_1} E_1}{F_1 + E_1}
$$

(see [10 Proposition 3.1] and [16 Corollary 2.9]). The first term on the right-hand side is then evaluated as

$$
\frac{F_m}{F_1} = \frac{1}{m^s} + \frac{q^{-1} \zeta(2s) (2\pi)^{2s-1} \Gamma(1-s)^2 \zeta(2s)}{1 + q^{1-2s} \zeta(2s)^2 \zeta(2s)} = m^{-s} + O_q \left( \frac{1}{m^{1-s} q^{2s-1}} \right).
$$

Finally, we estimate the second term of the right-hand side of (2.3) as

$$
\frac{E_m - \frac{F_m}{F_1} E_1}{F_1 + E_1} \ll_q \frac{mq^{1-s} + (m^{-s} + m^{s-1} q^{1-2s}) q^{1-s}}{q} \ll mq^{-s}.
$$

This completes the proof. □

**Proposition 2.3.** Let $q$ be a prime number and $m$ a positive integer coprime to $q$. Then we have

$$
\sum_{\chi \in \mathcal{F}_q} |L(1/2, \chi)|^2 \sum_{\chi \in \mathcal{F}_q} |L(1/2, \chi)|^2 \chi(m) = m^{-1/2} - m^{-1/2} \frac{\log m}{\log q} + O \left( \frac{m^{-1/2}}{\log q} \left( 1 + \frac{\log m}{\log q} \right) \right) + O \left( \frac{mq^{-1/2}}{\log q} \right), \quad q \to \infty,
$$

where the implied constant is independent of $m$ and $q$.

**Proof.** When $s' = 1/2$ and $n = 1$, the main term of Selberg’s formula in Theorem 2.1 is given by

$$
\frac{q-1}{m^{1/2}} \zeta(s+1/2) + (q - 1)q^{1/2-s} m^{s-1} \frac{(2\pi)^{s-1/2}}{\sqrt{\pi}} \Gamma(1-s) \cos \left( \frac{\pi}{2} (s - 1/2) \right) \zeta(3/2 - s).
$$

(2.4)

Utilizing

$$
\zeta(s+1/2) = \frac{1}{s-1/2} + \gamma + O(s-1/2), \quad s \to 1/2,
$$

$$
\zeta(3/2 - s) = \frac{-1}{s-1/2} + \gamma + O(s-1/2), \quad s \to 1/2,
$$

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where \( \gamma \) is the Euler-Mascheroni constant, and \( \cos\left(\frac{\pi}{2}(s - 1/2)\right) = 1 + \mathcal{O}((s - 1/2)^2) \) as \( s \to 1/2 \), we can rewrite the term (2.3) as

\[
\left\{ (q - 1)m^{1/2} - (q - 1)q^{1/2-s}m^{s-1}\frac{(2\pi)^{s-1/2}}{\sqrt{\pi}}\Gamma(1-s) \right\} \frac{1}{s - 1/2}
\]  

(2.5)

\[ + (q - 1)m^{-1/2}\gamma + (q - 1)q^{1/2-s}m^{s-1}\frac{(2\pi)^{s-1/2}}{\sqrt{\pi}}\Gamma(1-s)\gamma \]

(2.6)

Applying L'Hôpital's rule, we see that the term (2.5) tends to

\[ m^{-1/2}(q - 1)(\log q - \log m - \gamma - \log 8\pi) \]

as \( s \to 1/2 \), where we use \( \frac{\Gamma'(1/2)}{\Gamma(1/2)} = -\gamma - 2\log 2 \). Summing this and the value \( 2m^{-1/2}(q-1)\gamma \) of (2.6) at \( s = 1/2 \), and recalling the error term in Theorem 2.1, we obtain

\[
\sum_{\chi \in \mathcal{F}_q} |L(1/2, \chi)|^2 \chi(m) = \frac{q - 1}{m^{1/2}}(\log q - \log m + \gamma - \log 8\pi) + \mathcal{O}(mq^{1/2}).
\]

Let \( F_m \) denote the main term of the right-hand side above and \( E_m \) its error term. As (2.3), we have

\[
\frac{1}{\sum_{\chi \in \mathcal{F}_q} |L(1/2, \chi)|^2} \sum_{\chi \in \mathcal{F}_q} |L(1/2, \chi)|^2 \chi(m) = \frac{F_m + E_m}{F_1 + E_1} = \frac{F_m}{F_1} + \frac{E_m - F_m E_1}{F_1 + E_1},
\]

where \( \frac{E_m}{F_1} \) equals

\[
m^{-1/2} \frac{\log q - \log m + \gamma - \log 8\pi}{\log q + \gamma - \log 8\pi} = m^{-1/2} \frac{1 - \frac{\log m}{\log q} + \frac{\gamma - \log 8\pi}{\log q}}{1 + \frac{\gamma - \log 8\pi}{\log q}}
\]

\[ = m^{-1/2} \left( 1 - \frac{\log m}{\log q} \right) + \mathcal{O}\left( m^{-1/2} \left( 1 + \frac{\log m}{\log q} \right) \right). \]

Furthermore, we have

\[
\frac{E_m - F_m E_1}{F_1 + E_1} \ll \frac{mq^{1/2} + m^{-1/2}(1 + \frac{\log m}{\log q})q^{1/2}}{q \log q} \ll \frac{m}{q^{1/2} \log q}
\]

and the proof is complete. \( \square \)

### 3. Weighted one-level density

In this section, we prove Theorems 1.2 and 1.4. Let \( \phi \) be a Paley-Wiener function on \( \mathbb{C} \). Then for any \( \chi \in \mathcal{F}_q \) with \( q \) being a prime number, the explicit formula for \( L(s, \chi) \) à la Weil (see [4, (2.2)]) yields

\[
D(\chi, \phi) = \hat{\phi}(0) - \frac{1}{\log q} \sum_p (\chi(p) + \overline{\chi(p)}) \hat{\phi}\left(\frac{\log p}{\log q}\right) \frac{\log p}{p^{1/2}}
\]

\[ - \frac{1}{\log q} \sum_p (\chi(p)^2 + \chi(p)^2) \hat{\phi}\left(\frac{2 \log p}{\log q}\right) \frac{\log p}{p} + \mathcal{O}\left(\frac{1}{\log q}\right), \quad q \to \infty,
\]
where \( p \) runs over the set of prime numbers. For any \( s \in [1/2, 1) \), set
\[
E_q(s; \phi) := \frac{1}{\log q} \sum_{\chi \in \mathcal{F}_q} |L(s, \chi)|^2 \sum_{\chi \in \mathcal{F}_q} |L(s, \chi)|^2 D(\chi, \phi).
\]

**Proposition 3.1.** Fix \( s \in (1/2, 1) \). Let \( \phi \) be a Schwartz function on \( \mathbb{R} \) such that \( \text{supp}(\hat{\phi}) \subset [-2s/3, 2s/3] \). Then we have
\[
E_q(s; \phi) = \hat{\phi}(0) + O_s\left(\frac{1}{\log q}\right), \quad q \to \infty.
\]

**Proof.** Suppose \( \text{supp}(\hat{\phi}) \subset [-\alpha, \alpha] \), where \( \alpha > 0 \) is suitably chosen later. We recall the expression
\[
E_q(s; \phi) = \hat{\phi}(0) - M_q^{(1)}(s) - M_q^{(2)}(s) + O\left(\frac{1}{\log q}\right),
\]
where we put
\[
M_q^{(k)}(s) := \frac{1}{\log q} \sum_{p} \frac{1}{\log q} \sum_{\chi \in \mathcal{F}_q} |L(s, \chi)|^2 \sum_{\chi \in \mathcal{F}_q} |L(s, \chi)|^2 \left(\chi(p^k) + \overline{\chi(p^k)}\right) \hat{\phi}\left(\frac{k \log p}{\log q}\right) \frac{\log p}{p^{k/2}}
\]
for \( k = 1, 2 \). By Proposition 2.2, we have
\[
M_q^{(k)}(s) = 2 \left(\frac{1}{p^{k+s}} + O_s(p^{k(s-1)}q^{1-2s}) + O_s(p^k q^{-s})\right) \hat{\phi}\left(\frac{k \log p}{\log q}\right) \frac{\log p}{p^{k/2} \log q},
\]
where we use \( \sum_{\chi \in \mathcal{F}_q} |L(s, \chi)|^2 = \sum_{\chi \in \mathcal{F}_q} |L(s, \chi)|^2 \chi(p^k) \). Since we assume \( s > 1/2 \), the contribution of the term with \( \frac{1}{p^{s}} \) is
\[
\frac{2}{\log q} \sum_{p} \hat{\phi}\left(\frac{k \log p}{\log q}\right) \frac{\log p}{p^{k(s+1/2)}} \ll \frac{1}{\log q} \sum_{p} \frac{\log p}{p^{s+1/2}} \ll \frac{1}{\log q}.
\]
Furthermore, noting \( \text{supp}(\hat{\phi}) \subset [-\alpha, \alpha] \), the term \( O_s(p^k q^{-s}) \) is estimated as
\[
\frac{2}{\log q} \sum_{p} p^k q^{-s} \hat{\phi}\left(\frac{k \log p}{\log q}\right) \frac{\log p}{p^{k/2}} \ll \frac{q^{-s}}{\log q} \sum_{p \leq q^{\alpha/k}} p^{k/2} \log p.
\]
Here, by the prime number theorem and partial summation, we have
\[
\sum_{p \leq x} p^a \log p = O(x^{a+1}), \quad x \to \infty
\]
for any fixed \( a > -1 \) (see [10, (3.4)]). As a consequence, the contribution of \( O_s(p^k q^{-s}) \) is bounded by
\[
\frac{q^{-s}}{\log q} \times (q^{\alpha/k})^{k/2+1} \ll \frac{q^{-s+3\alpha/2}}{\log q}
\]
up to constant multiple. This is absorbed into \( O\left(\frac{1}{\log q}\right) \) when \( \alpha \) is taken so that \(-s+3\alpha/2 \leq 0\), i.e., \( \alpha \leq 2s/3 \). Similarly, the contribution of \( O_s(p^{k(s-1)}q^{1-2s}) \) is bounded by
\[
\frac{1}{\log q} \sum_{p} p^{k(s-3/2)} q^{1-2s} \hat{\phi}\left(\frac{k \log p}{\log q}\right) \log p \ll \frac{q^{1-2s}}{\log q} \sum_{p \leq q^{\alpha(k/3)}} p^{k(s-3/2)} \log p \ll \frac{q^{1-2s}}{\log q} \times q^{\alpha(s-1/2)},
\]
which is again absorbed into \( O\left(\frac{1}{\log q}\right) \) as long as \( 1 - 2s + \alpha(s - 1/2) \leq 0 \), i.e., \( \alpha \leq 2 \). Hence \( M_q^{(k)}(s) \) for \( k = 1, 2 \) are both bounded by the error and the proof is done. \( \square \)

Theorem 1.4 immediately follows from Proposition 3.1.

Next we consider \( \mathcal{E}_q(1/2; \phi) \).

**Proposition 3.2.** Let \( \phi \) be a Schwartz function on \( \mathbb{R} \) such that \( \text{supp}(\hat{\phi}) \subset [-1/3, 1/3] \). Then we have

\[
\mathcal{E}_q(1/2; \phi) = \hat{\phi}(0) - \phi(0) \rightleftharpoons q \to \infty.
\]

**Proof.** Suppose \( \text{supp}(\hat{\phi}) \subset [-\alpha, \alpha] \), where \( \alpha > 0 \) is suitably chosen later. We write

\[
\mathcal{E}_q(1/2; \phi) = \hat{\phi}(0) + O\left(\frac{1}{\log q}\right) - M_q^{(1)} - M_q^{(2)},
\]

where we set

\[
M_q^{(k)} := \frac{1}{\log q} \sum_p \frac{1}{2} \sum_{\chi \in \mathcal{F}_q} |L(1/2, \chi)|^2 \sum_{\chi \in \mathcal{F}_q} \left|L(1/2, \chi)^2(\chi(p^k) + \chi(p^k))\right| \hat{\phi}\left(\frac{k \log p}{\log q}\right) \frac{\log p}{p^{k/2}}
\]

for \( k = 1, 2 \). As in the proof of Proposition 3.1, with the aid of Proposition 2.3, the sum \( M_q^{(k)} \) is evaluated as

\[
M_q^{(k)} = \sum_p \frac{2}{p^{k/2}} \left(1 - \frac{k \log p}{\log q} + O\left(\frac{1}{\log q}\right) \left(1 + \frac{\log p}{\log q}\right)\right) \hat{\phi}\left(\frac{k \log p}{\log q}\right) \frac{\log p}{p^{k/2} \log q}
\]

(3.2)

+ \( O\left(\frac{q^{-1/2}}{(\log q)^2} \sum_p p^{k/2} \hat{\phi}\left(\frac{k \log p}{\log q}\right) \log p\right) \)

(3.3)

Invoking the two asymptotic formulas

\[
\sum_p \hat{\phi}\left(\frac{\log p}{\log q}\right) \frac{\log p}{p \log q} = \frac{1}{2} \phi(0) + O\left(\frac{1}{\log q}\right), \quad q \to \infty,
\]

\[
\sum_p \hat{\phi}\left(\frac{\log p}{\log q}\right) \frac{(\log p)^2}{p(\log q)^2} = \frac{1}{2} \int_{-\infty}^{\infty} \hat{\phi}(x) |x| dx + O\left(\frac{1}{\log q}\right), \quad q \to \infty
\]

(see the proof of [16, Proposition 3.2]), the term (3.2) for \( k = 1 \) is evaluated as

\[
\phi(0) - \int_{-\infty}^{\infty} \hat{\phi}(x) |x| dx + O\left(\frac{1}{\log q}\right).
\]

The term (3.2) for \( k = 2 \) is bounded by

\[
\frac{1}{\log q} \sum_p \frac{(\log p)^2}{p^2} \ll \frac{1}{\log q}.
\]
The error term (3.3) for \( k = 1, 2 \) is estimated in the following way. Noting \( \text{supp}(\hat{\phi}) \subset [-\alpha, \alpha] \) and (3.1), the error term (3.3) for \( k = 1, 2 \) is bounded by

\[
\frac{q^{-1/2}}{(\log q)^2} \sum_{p \leq q^{1/2}} p^{k/2} \log p \ll \frac{q^{-1/2}}{(\log q)^2} \times q^{(\alpha/k)(k/2+1)} \leq q^{(3\alpha/2)-1/2}.
\]

Therefore it suffices to take \( \alpha \) so that \( 3\alpha/2 - 1/2 \leq 0 \), i.e., \( \alpha \leq 1/3 \) and we are done. \( \square \)

A direct computation shows that the density function of the main term of Proposition 3.2 is equal to \( W_1^1(x) \). We state this fact as a proposition and provide the proof for convenience of the reader.

**Proposition 3.3.** Let \( \phi \) be a Schwartz function on \( \mathbb{R} \) such that \( \text{supp}(\hat{\phi}) \subset [-1, 1] \). Then we have

\[
\hat{\phi}(0) - \phi(0) + \int_{-\infty}^{\infty} \hat{\phi}(x)|x|dx = \int_{-\infty}^{\infty} \phi(x)W_1^1(x)dx = \int_{-\infty}^{\infty} \phi(x) \left(1 - \frac{\sin^2(\pi x)}{(\pi x)^2}\right)dx.
\]

**Proof.** Noting the assumption \( \text{supp}(\hat{\phi}) \subset [-1, 1] \), an elementary calculation using Fourier analysis yields

\[
\hat{\phi}(0) = \int_{-\infty}^{\infty} \phi(x)dx,
\]

\[
\phi(0) = \int_{-\infty}^{\infty} \hat{\phi}(x)dx = \int_{-\infty}^{\infty} \hat{\phi}(x)\eta(x)dx = \int_{-\infty}^{\infty} \phi(x)\hat{\eta}(x)dx,
\]

\[
\int_{-\infty}^{\infty} \hat{\phi}(x)|x|dx = \int_{-\infty}^{\infty} \hat{\phi}(x)|\eta(x)|dx = \int_{-\infty}^{\infty} \phi(x)|\hat{\eta}(x)|dx,
\]

where \( \eta(x) = 1, 1/2 \) or 0 if \( |x| < 1, |x| = 1 \) or \( |x| > 1 \), respectively. We see \( \hat{\eta}(x) = 2 \frac{\sin(2\pi x)}{2\pi x} \) by a direct computation and furthermore, we have

\[
|\hat{\eta}(\xi)| = \int_{-1}^{1} |x|e^{2\pi i x \xi}dx = \int_{-1}^{0} (-x)e^{2\pi i x \xi}dx + \int_{0}^{1} xe^{2\pi i x \xi}dx = 2 \frac{\sin(2\pi \xi)}{2\pi \xi} - \frac{\sin^2(\pi \xi)}{(\pi \xi)^2}.
\]

Hence, the left-hand side of the assertion is transformed into

\[
\int_{-\infty}^{\infty} \phi(x) \left(1 - \frac{2\sin(2\pi x)}{2\pi x} + \left(\frac{2\sin(2\pi x)}{2\pi x} - \frac{\sin^2(\pi x)}{(\pi x)^2}\right)\right)dx.
\]

This completes the proof. \( \square \)

Theorem 1.2 follows from Propositions 3.2 and 3.3.

**Remark 3.4.** If the error term of Selberg’s formula of the twisted second moment of Dirichlet \( L \)-functions in Theorem 2.4 is improved to \( O(m^a q^{-b}) \) for some \( a > 0 \) and \( b > 0 \), then the assumption on the size of \( \text{supp}(\hat{\phi}) \) in Theorem 1.2 can be weakened to \( \text{supp}(\hat{\phi}) \subset [-\alpha, \alpha] \). On the other hand, the assumption of the size of \( \text{supp}(\hat{\phi}) \) in Theorem 1.4 can be weakened only up to \( \text{supp}(\hat{\phi}) \subset [-2, 2] \) even if the error term of Selberg’s formula is improved to \( O(m^a q^{-b}) \) for some \( a > 0 \) and any sufficiently large \( b > 0 \). This is because the main term of Selberg’s formula yields the error term \( O_s(m^{-1} q^{1-2s}) \) in Proposition 2.4 which requires the assumption \( \text{supp}(\hat{\phi}) \subset [-2, 2] \) as in the last part of the proof of Proposition 3.1.
Remark 3.5. Selberg’s result (Theorem 2.1) has been improved by many authors for the case of central point, that is when $s = s' = 1/2$. The latest and currently best known result is the following result by Bettin [1, Corollary 2]:

$$\sum_{\chi \in \mathcal{F}_q} |L(1/2, \chi)|^2 \chi(m) \overline{\chi(n)} = \frac{q - 1}{(mn)^{1/2}} \left( \log \frac{q}{mn} + \gamma - \log 8\pi \right) + O \left( (m + n)^{1/2} q^{1/2} \log q \right),$$

where $q$ is a prime number, $m$ and $n$ are positive integers such that $m$ and $n$ are coprime to each other and such that $q \geq 4mn$. Using the above result, we have

$$\sum_{\chi \in \mathcal{F}_q} |L(1/2, \chi)|^2 \chi(m) \overline{\chi(n)} = \frac{1}{\sum_{\chi \in \mathcal{F}_q} |L(1/2, \chi)|^2} \sum_{\chi \in \mathcal{F}_q} |L(1/2, \chi)|^2 \chi(m)$$

$$= m^{-1/2} - m^{-1/2} \log m \log q + O \left( m^{-1/2} \log q \left( 1 + \frac{\log m}{\log q} \right) \right) + O \left( m^{1/2} q^{-1/2} \right), \quad q \to \infty.$$

This is the case when $a$ and $b$ in Remark 3.4 are $1/2$, and thus the support condition in Theorem 7.4 can be improved from $[-1/3, 1/3]$ to $[-1/2, 1/2]$.

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