Possible Minkowskian Language in Two-level Systems

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Abstract

One hundred years ago, in 1908, Hermann Minkowski completed his proof that Maxwell’s equations are covariant under Lorentz transformations. During this process, he introduced a four-dimensional space called the Minkowskian space. In 1949, P. A. M. Dirac showed the Minkowskian space can be handled with the light-cone coordinate system with squeeze transformations. While the squeeze is one of the fundamental mathematical operations in optical sciences, it could serve useful purposes in two-level systems. Some possibilities are considered in this report. It is shown possible to cross the light-cone boundary in optical and two-level systems while it is not possible in Einstein’s theory of relativity.

1 Introduction

Hermann Minkowski was born in Lithuania. I am very happy to talk about him in the country where he was born. He was born in 1864 near the city of Kaunas. He then studied at Albertina University in of Königsberg. Before 1945, Königsberg was the capital city of East Prussia, located in the Baltic wedge between Lithuania and Poland. After 1945, Königsberg became a Russian city of Kaliningrad serving as a Soviet naval base.

Albertina University, often called the University of Königsberg, had a very strong academic tradition. Immanuel Kant was a professor at this university. After the total destruction during the second world war, Soviets
started reconstructing the university as the University of Kaliningrad. This university is now called Immanuel Kant State University.

During the 19th century, the University of Königsberg had many mathematicians interested in physics. They formulated Maxwell’s equations in the form we use these days. Minkowski studied Maxwell’s equations there. Even after he left Königsberg, he continued his research in Maxwell’s equations. He was a professor at the University of Zurich while Einstein was a student there. There he found out that the Maxwell system is covariant under Lorentz transformations. Even though Minkowski published his result in 1908, his interest in Lorentz transformations should have influenced Einstein who completed his special theory of relativity in 1905.

Is this the end of the Lorentz completion of the Maxwell system? Yes, in classical theory of electromagnetism, but No, in the quantum world. This problem has a stormy history and was not settled until 1990 [1]. The question is whether the electromagnetic field from Maxwell’s equations can describe photons as Lorentz-covariant particles. I hope to give a review of this subject at another occasion.

Minkowski also made a strong impact on modern physics through his Minkowskian space and his geometry. He introduced space-time system where

\[ x^2 + y^2 + z^2 - t^2 \]  

remains invariant under Lorentz transformations. If the transformation is made along the \( z \) direction, we are dealing with the system where

\[ z^2 - t^2 = \text{constant}. \]  

This expression can be written as

\[ uv = \text{constant}, \]  

where

\[ u = z + t, \quad v = z - t. \]  

This means that the Lorentz transformation performs a squeeze transformation on the \( u \) and \( v \) variables

\[ \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} e^{\eta/2}u \\ e^{-\eta/2}v \end{pmatrix}. \]

This is a squeeze transformation, where one coordinate is extended while the other is contracted [2].
If we make a $45^\circ$ rotation of this coordinate system, we end up with $z = (u + v)/2$ and $t = (u - v)/2$, and the transformation is

$$\begin{pmatrix} z \\ t \end{pmatrix} = \begin{pmatrix} \cosh(\eta/2) & \sinh(\eta/2) \\ \sinh(\eta/2) & \cosh(\eta/2) \end{pmatrix},$$

which is a more familiar expression for the Lorentz transformation. Indeed, the Lorentz transformation is a squeeze transformation.

While the Lorentz transformation is known to be only for Einstein’s relativity, squeeze transformations are everywhere in optical and engineering sciences.

In Sec. 2 I review the papers I have published with Sibel Baskal and Elena Georgieva on ray optics [3, 4, 5, 6]. In Sec. 3 I discuss possible squeeze effects in two-level problems, using the same mathematical formalism developed for ray optics.

## 2 Squeeze Transformations in Ray Optics

In recent years, I have been working with Sibel Baskal and Elena Georgieva on ray optics [3, 4, 5, 6]. More specifically, we have been interested in beam transfer matrices of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with unit determinant or $AD - BC = 1$, and the matrix elements are real numbers. This matrix is often called the $ABCD$ matrix in the literature.

We have been particularly interested in the $ABCD$ matrix applicable to periodic systems. For this purpose, we have shown that this matrix can be written as a similarity transformation one of the following four matrices [4, 6].

$$\begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}, \quad \begin{pmatrix} \cosh \mu & \sinh \mu \\ \sinh \mu & \cosh \mu \end{pmatrix}, \quad \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}.$$  

We shall use the notation $W$ collectively for these matrices, and we write them as $W(\phi), W(\mu), W(\alpha), W(\beta)$ respectively. Then, according to the property of similarity transformation,

$$(W(\phi))^N = W(N\phi), \quad (W(\mu))^N = W(N\mu),$$

$$(W(\alpha))^N = W(N\alpha), \quad (W(\beta))^N = W(N\beta).$$
Depending on where the cyclic process begins, the similarity transformation matrix can be as simple as

\[ S = \begin{pmatrix} e^{-\eta/2} & 0 \\ 0 & e^{\eta/2} \end{pmatrix}. \] (10)

This is a squeeze matrix which expands one coordinate while contracting the other.

We are now allowed to write the ABCD matrix of Eq.(7) as

\[ M = SWS^{-1}. \] (11)

Thus

\[ M^N = \left( SWS^{-1} \right)^N = SW^N S^{-1}. \] (12)

We can then use Eq.(9) to compute this quantity.

The ABCD matrix can now be written as

\[ M = \begin{pmatrix} \cos \phi & -e^{-\eta} \sin \phi \\ e^{\eta} \sin \phi & \cos \phi \end{pmatrix} \] (13)

as the similarity transformation for the first matrix of Eq.(8), and we can write similar expressions for the remaining matrices.

Using the expression of Eq.(12), we have been able to deal with the cyclic problems in laser cavities and multilayer optics.

### 3 Possible Squeeze Effects in Two-level Systems

Historically, our computation started with the languages the nature speaks. We have decimal system because each human being has ten fingers. Chinese invented the abacus based on the language these fingers speak. Abacuses were used extensively in China, Russia, and many other countries until the end of the 20th century.

When French army developed long-range guns, the artillery men had to perform speedy calculations. They noticed that two bars can perform additions. Those with logarithmic scales can perform multiplications. The computer based on this principle is called the slide rule. Slide rules played essential roles in engineering applications until hand-held calculators became popular in the 1970s.
John Vincent Atanasoff and John von Neumann noticed vacuum tubes can speak flip-flop language and started computing machines based on binary mathematics. This became the electronic computers we use every day.

Feynman later noticed that the electron spin can also speak a flip-flop language, but he noted also that the system has an additional degree of freedom, leading to the idea of much more powerful computer than the present electronic system can provide [7].

We are quite familiar with the quantum mechanics of an electron in a constant magnetic field. We also know the electron system can be described by the spin matrices of the form

\[
S_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

We are quite familiar with these matrices. They are all Hermitian and generate rotations in the three-dimensional space. They satisfy the set of commutation relations

\[
[S_i, S_j] = i \epsilon_{ijk} S_k.
\]

This simple mathematical system is the language of nature for quantum computing [7]. We note that there are three additional matrices which will complete the set of commutation relations which generate the group of two-by-two matrices isomorphic to the Lorentz group, applicable to the four-dimensional Minkowskian space. They are

\[
K_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad K_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]

While the matrices of Eq.(14) generate rotations, the above three matrices generate squeeze transformations.

Let us consider a spin system with a constant magnetic field along the \( y \) direction. The Hamiltonian is then

\[
\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\]

The physics of this system is simple enough, but if we add a dissipative field along the \( x \) direction with the form

\[
g \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},
\]
the total Hamiltonian becomes

\[ H = h \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + g \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i \begin{pmatrix} 0 & -h + g \\ h + g & 0 \end{pmatrix}. \] (19)

Then the transition matrix becomes

\[ M = \exp \left( \begin{pmatrix} 0 & -h + g \\ h + g & 0 \end{pmatrix} \right) t. \] (20)

If \( g = h \), this expression becomes

\[ M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \] (21)

and

\[ M = \begin{pmatrix} 1 & -(2h)t \\ 0 & 1 \end{pmatrix}, \] (22)

if \( g = -h \).

Otherwise, we propose to compute this quantity by making a Taylor expansion. If \( h > g \), we can parametrize the Hamiltonian as

\[ i \left( h^2 - g^2 \right)^{1/2} \begin{pmatrix} 0 & -\sqrt{(h - g)/(h + g)} \\ \sqrt{(h + g)/(h - g)} & 0 \end{pmatrix}, \] (23)

which can be written as

\[ i\omega \begin{pmatrix} 0 & -e^{-\eta} \\ e^\eta & 0 \end{pmatrix}, \] (24)

with

\[ \omega = \left( h^2 - g^2 \right)^{1/2}, \quad e^{-\eta} = \left( \frac{h - g}{h + g} \right)^{1/2}. \] (25)

Thus, the transition matrix of Eq.(20) takes the form

\[ \exp \left[ \omega \begin{pmatrix} 0 & -e^{-\eta} \\ e^\eta & 0 \end{pmatrix} t \right]. \] (26)

Let us note that the matrix

\[ \begin{pmatrix} 0 & -e^{-\eta} \\ e^\eta & 0 \end{pmatrix} \] (27)
can be written as a similarity transformation
\[
\begin{pmatrix}
e^{-\eta/2} & 0 & 0 \\
0 & e^{\eta/2} & 0 \\
0 & 0 & e^{-\eta/2}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & -1 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
e^{\eta/2} & 0 & 0 \\
0 & e^{-\eta/2} & 0 \\
0 & 0 & e^{\eta/2}
\end{pmatrix}
\] (28)

Thus, we can write
\[
\left[
\begin{pmatrix}
0 & -e^{-\eta} \\
e^{\eta} & 0
\end{pmatrix}
\right]^N = \begin{pmatrix}
e^{-\eta/2} & 0 & 0 \\
0 & e^{\eta/2} & 0 \\
0 & 0 & e^{-\eta/2}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & -1 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}^N
\begin{pmatrix}
e^{\eta/2} & 0 & 0 \\
0 & e^{-\eta/2} & 0 \\
0 & 0 & e^{\eta/2}
\end{pmatrix}
\] (29)

It is therefore possible to compute the transition matrix of Eq.(20) using
Taylor expansion, and the result is
\[
\begin{pmatrix}
\cos(\omega t) & -e^{-\eta}\sin(\omega t) \\
e^{\eta}\sin(\omega t) & \cos(\omega t)
\end{pmatrix}
\]. (30)

If \( g \) is greater than \( h \), the off-diagonal elements have the same sign in
Eq.(20), and we can carry out a similar calculation. The result is
\[
\begin{pmatrix}
\cosh(\lambda t) & e^{-\eta}\sinh(\lambda t) \\
e^{\eta}\sinh(\lambda t) & \cosh(\lambda t)
\end{pmatrix}
\], (31)
with
\[
\lambda = \left( g^2 - h^2 \right)^{1/2}, \quad e^{-\eta} = \left( \frac{g - h}{h + g} \right)^{1/2}
\]. (32)

In this way, we obtain a complete analytic set of solutions for this mixed
problem, namely the rotation due to a constant magnetic field along the \( y \)
direction, and a squeeze effect along the \( x \) direction.

What significance does this result have from the computational point of
view? Historically, the Minkowskian space was introduced in connection with
special relativity, where the Minkowskian scalar of Eq.(1) remains constant.
Thus, were used to be firmly committed to the world where the light-cone
cannot be crossed. This becomes translated into the language that a circle
and cannot be translated into a hyperbola analytically. This is an old problem
existing since the ancient Greek period.

In this paper, we noted that this barrier can be crossed. We can go
to the hyperbolic geometry of Eq.(31) from the circular geometry given in
Eq.(30), by adjusting the physical parameters \( h \) and \( g \). It was noted earlier
the same transition is possible in multilayer optics [6].

This is precisely the mathematical language optical sciences and two-level
system can speak, Einstein’s special language cannot.
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