THE TATE CONJECTURE FOR K3 SURFACES IN ODD CHARACTERISTIC

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ABSTRACT. We show that the classical Kuga-Satake construction gives rise, away from characteristic 2, to an open immersion from the moduli of primitively polarized K3 surfaces (of any fixed degree) to a certain normal integral model for a Shimura variety of orthogonal type. This allows us to attach to every polarized K3 surface in odd characteristic an abelian variety such that divisors on the surface can be identified with certain endomorphisms of the attached abelian variety. Using a result of Kisin, we can then prove the Tate conjecture for K3 surfaces over finitely generated fields of odd characteristic. We also show that the moduli stack of primitively polarized K3 surfaces of degree $2d$ is quasi-projective and, when $d$ is not divisible by $p^2$, is geometrically irreducible in characteristic $p$. We indicate how the same method applies to prove the Tate conjecture for co-dimension 2 cycles on cubic fourfolds.

INTRODUCTION

The goal of this paper is to prove:

**Theorem 1.** Let $X$ be a K3 surface over a finitely generated field $k$ of characteristic not equal to 2. Then the Tate conjecture holds for $X$.

That is, for any prime $\ell$ invertible in $k$, the $\ell$-adic Chern class map

$$\text{Pic}(X) \otimes \mathbb{Q}_\ell \rightarrow H^2_{\text{ét}}(X_{k_{\text{sep}}}, \mathbb{Q}_\ell(1))^\Gamma$$

is an isomorphism. Here, $k_{\text{sep}}$ is a separable closure of $k$ and $\Gamma = \text{Gal}(k_{\text{sep}}/k)$ is the associated absolute Galois group.

Work of Lieblich-Maulik-Snowden [LMS12] shows that Theorem 1 implies:

**Theorem 2.** There are only finitely many isomorphism classes of K3 surfaces over a finite field of odd characteristic.

The following cases of Theorem 1 are already known:

1. When the field $k$ is of characteristic 0: cf. [Tat94, Theorem 5.6(a)] or [And96].
2. When $k$ is finite of characteristic at least 5: This is due to Nygaard and Nygaard-Ogus [Nyg83, NO85] for K3 surfaces of finite height, and Maulik [Mau12] and Charles [Cha12] for supersingular K3 surfaces. Maulik’s work utilizes the case of elliptic K3 surfaces, due to Artin-Swinnerton-Dyer [ASD73], but Charles’s is independent of it, being an application of a general result for reductions of holomorphic symplectic varieties.

The main contribution of this article is an unconditional proof of the conjecture in odd characteristic. Our methods are independent of the results above, but owe a substantial spiritual debt to the proof in characteristic 0, which combines the classical Kuga-Satake construction with Deligne’s theory of absolute Hodge cycles and Faltings’s isogeny theorem.

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1The result of [Nyg83] for ordinary K3 surfaces does not appear to have any restriction on the characteristic.
Heuristic for the proof. Here is a heuristic argument for the Tate conjecture, inspired by ideas of Kisin [Kis]:

For simplicity, we assume that \((X, \xi)\) is a K3 surface over a finite field \(k\). We can attach to it the motive \(p^2(X, \xi)\) (in the sense of motives defined by algebraic correspondences up to numerical equivalence; cf. [Jan92]) whose realizations are expected to be the various primitive cohomology groups of \((X, \xi)\). Let \(I\) be the algebraic \(\mathbb{Q}\)-group of units in the semi-simple algebra \(\text{End}(p^2(X, \xi))\): it is a reductive group over \(\mathbb{Q}\). Conjecturally, for each prime \(\ell \neq p\), \(I_{\mathbb{Q}_\ell}\) acts faithfully and \(\Gamma\)-equivariantly on \(PH_2^p(X, \xi(1))\), and the Chern class map

\[
\text{Pic}(X) \otimes \mathbb{Q}_\ell \supset \langle \xi \rangle^\perp \otimes \mathbb{Q}_\ell \xrightarrow{\text{ch}} PH_2^p(X, \xi(1))^\Gamma
\]

is \(I_{\mathbb{Q}_\ell}\)-equivariant.

Let \(I_\ell \subset \text{SO}(PH_2^p(X, \xi(1)))\) be the \(\mathbb{Q}_\ell\)-sub-group consisting of \(\Gamma\)-equivariant automorphisms. It is the commutator of the Frobenius automorphism, which is known to be semi-simple [Del72], and is thus a reductive group over \(\mathbb{Q}_\ell\). The target of the Chern class map is an irreducible representation of \(I_\ell\). So, if we knew that \(I_{\mathbb{Q}_\ell} \cong I_\ell\) (which would be implied by the general Tate conjecture), then it would follow that the Chern class map is an isomorphism (assuming \(\langle \xi \rangle^\perp\) is non-zero, which should be true after changing scalars to a quadratic extension of \(k\)). In fact, it is enough to do this for one prime \(\ell\), and so we can assume that \(I_\ell\) is a split reductive group over \(\mathbb{Q}_\ell\). In this case, a group theoretic lemma [Kis] shows that it is sufficient to prove the compactness of the space \(I(\mathbb{Q}_\ell)/I_\ell(\mathbb{Q}_\ell)\). In turn, it is sufficient to show that, for a suitable compact open \(U_\ell \subset I_\ell(\mathbb{Q}_\ell)\), the double coset space

\[
I(\mathbb{Q}) \setminus I_\ell(\mathbb{Q}_\ell)/U_\ell
\]

is finite. If one had a good notion of an \(\ell\)-power isogeny of polarized K3 surfaces, one could try to do this by identifying this double coset space with a sub-set of the ‘\(\ell\)-power isogeny class’ of \((X, \xi)\) defined over \(k\).

Kuga-Satake construction. In practice, we translate all the unproven assumptions above, via the Kuga-Satake construction, into assertions about abelian varieties. Results of Kisin [Kis] on integral canonical models of Shimura varieties then allow us to prove these assertions easily. Recall that in characteristic 0 the Kuga-Satake construction attaches to every polarized K3 surface \((X, \xi)\) an abelian variety \(A\) such that the primitive cohomology group \(PH_2^p(X, \xi)\) embeds within \(\text{End}(H^1(A))\) as a sub-Hodge structure. We extend this construction to odd characteristic. One can do this as in [Del72], by lifting to characteristic 0, applying the Kuga-Satake construction, and taking its reduction. The crystalline compatibility (up to isogeny) of such a construction is shown in [Ogu84] § 7. We make two improvements to this: First, we show the crystalline compatibility on an integral level. Second, we show that the Kuga-Satake construction sees enough geometry to allow us to view divisors on the K3 surface \(X\) as endomorphisms of \(A\). In particular, we can reduce the Tate conjecture for \(X\) to a version of Tate’s theorem on endomorphisms of \(A\). The reader is directed to [1.15] in the body of the paper for a precise version of the following result:

**Theorem 3.** Given any field \(k\) of odd characteristic \(p\) and a polarized K3 surface \((X, \xi)\) over \(k\), there exists a finite separable extension \(k'/k\) and an abelian variety \(A\) over \(k'\), the Kuga-Satake abelian variety such that the \(\mathbb{Z}_\ell\) and crystalline realizations of the primitive cohomology \(PH_2^p(X, \xi)\) embed naturally within those of \(H^1(A) \otimes H^1(A^\vee)\). Moreover, there is a canonical inclusion

\[
\text{Pic}(X_{k'}) \supset \langle \xi \rangle^\perp \hookrightarrow \text{End}(A)
\]

compatible, via the cycle class maps, with the corresponding embeddings of cohomology groups.
Moduli of K3 surfaces and the period map. It is crucial for our strategy to be able to work with families of K3 surfaces. For this, we interpret the Kuga-Satake correspondence over $\mathbb{C}$ as a period map

$$\iota_{\mathbb{C}} : M_{2d,K,\mathbb{C}}^0 \to Sh_{K,\mathbb{C}},$$

where $M_{2d,K}^0$ is the moduli space (over $\mathbb{Q}$) of degree $2d$ primitively polarized K3 surfaces with additional level structure, and $Sh_K$ is the associated orthogonal Shimura variety over $\mathbb{Q}$. This map is known to be an open immersion by the global Torelli theorem for K3 surfaces.

Results of Rizov [Riz] show that the period map descends over $\mathbb{Q}$:

$$\iota_{\mathbb{Q}} : M_{2d,K}^0 \to Sh_K.$$

The following theorem can be viewed as a positive characteristic analogue of the Torelli theorem for K3 surfaces. It is the key technical result of this paper:

**Theorem 4.** There exists a regular integral model $S_{pr}^F$ for $Sh_K$ over $\mathbb{Z}(p)$ such that $\iota_{\mathbb{Q}}$ extends to an open immersion

$$\iota_{\mathbb{Z}(p)} : M_{2d,K,\mathbb{Z}(p)}^0 \to S_{pr}^F.$$

When $p \nmid d$, this construction of the map is essentially due to Rizov [Riz10]; cf. also [Man12, §5]. With the same condition on $p$, a construction by Vasiu can be found in [Vas]. From Theorem 4 we immediately get:

**Theorem 5.** For any prime $p > 2$, the moduli stack $M_{2d,F}^0$ is quasi-projective. Moreover, the canonical bundle $\omega$ of the universal K3 surface is ample over $M_{2d,F}^0$. If $p^2 \nmid d$, then $M_{2d,F}^0$ is geometrically irreducible.

This result (apart from the irreducibility) was proven in [Man12, §5] for $p \geq 5$ with $p \nmid d$. For the irreducibility, we have to control the connected components of $S_{pr}^F$. When $p^2 \nmid d$, we can do this using the theory of toroidal compactifications from [MP13a, MP13b].

**Further remarks.** In characteristic 0, it is known that the period map is surjective, once extended to the moduli of quasi-polarized K3 surfaces. We expect the same assertion to hold in characteristic $p$. This question is intimately related to the existence of a Nerón-Ogg-Shafarevich type criterion for the good reduction of K3 surfaces over discrete valuation fields of characteristic $p$. Such a criterion is available in characteristic 0 [Kul77, PP81], and for certain K3 surfaces in finite characteristic [Mat].

There also remains the question of extending these results to characteristic 2. A major hindrance is the lack of a good theory of integral models of orthogonal Shimura varieties over 2-adic rings of integers; cf. [MP13a, 4.6.5]. Once such a theory is available, it should be straightforward to extend the ideas here to the situation where $2 \nmid d$, though highly 2-divisible $d$ are likely to present new difficulties.

The Kuga-Satake construction has appeared in many other contexts in characteristic 0: cf. [Voi86, Rap72, And96, Lyo12]. It is likely that the methods of this paper will permit us to extend the construction into positive characteristic in these cases as well, enabling us to also prove the Tate conjecture in these contexts. Certainly, for cubic fourfolds, the Torelli theorem from [Voi86] allows us to apply our methods in rather straightforward fashion, and we indicate this briefly in (4.26); cf. also [Lev01] and [Cha12, Corollary 6].

**Tour of contents.** We begin in Section 1 with a review of the theory of motives attached to absolute Hodge cycles, since this gives us a very powerful framework in which to place the Kuga-Satake correspondence. In particular, it permits us to show its compatibility with cohomological realizations in a rather natural way.
Section 2 is a quick review of the theory of moduli of \((\text{quasi-})\)polarized K3 surfaces, and in Section 3 we review what we need from \([\text{MP}13]\) about Shimura varieties of Spin and orthogonal type, their integral models, as well as the compactifications of these models. This is the most technically involved part of this paper, and the main result is \((3.19)\), which gives us our good integral models.

Section 4 is the heart of the paper. We use results from the preceding sections to extend the Kuga-Satake map over \(\mathbb{Z}_{(p)}\) and prove Theorems 4 \& 8. We also show how Theorem 3 implies Theorem 1. Finally, we sketch a version of our results that applies to cubic fourfolds and prove the Tate conjecture for them.

**Notational conventions.** For any prime \(\ell\), \(\nu_\ell\) will be the \(\ell\)-adic valuation satisfying \(\nu_\ell(\ell) = 1\). \(\mathbb{A}_f\) will denote the ring of finite ad\`eles over \(\mathbb{Q}\), and \(\mathbb{Z} \subset \mathbb{A}_f\) will be the pro-finite completion of \(\mathbb{Z}\). Given a rational prime \(p\), \(\mathbb{A}^p\) will denote the ring of prime-to-\(p\) finite ad\`eles; that is, the restricted product \(\prod_{p \notin \mathbb{P}} \mathbb{Q}_p\). Moreover, \(\mathbb{P}^p \subset \mathbb{A}^p\) will be the closure of \(\mathbb{Z}\). Given a perfect field \(k\) of finite characteristic, \(W(k)\) will denote its ring of Witt vectors, and \(\sigma : W(k) \to W(k)\) will be the canonical lift of the Frobenius automorphism of \(k\). For any group \(G\), \(\hat{G}\) will denote the locally constant \(\acute{e}tale\) sheaf (over a base that will be clear from context) with values in \(G\).

**Acknowledgements.** We thank Bhargav Bhatt, Anand Deopurkar, Mark Kisin, Davesh Maulik, George Pappas, Peter Scholze and Junecue Suh for helpful comments and conversations. This work was partially supported by NSF Postdoctoral Research Fellowship DMS-1204165.

### Contents

1. Motives
2. Shimura varieties
3. The Kuga-Satake period map over \(\mathbb{Z} \left[\frac{1}{7}\right]\)

1. **Motives**

Throughout this section (and only here), all fields will be assumed to be embeddable in \(\mathbb{C}\), and all varieties will be smooth, projective. Our main reference for this section is \([\text{DMOSS}2]\).

1.1. Fix a field \(k\) with an algebraic closure \(\overline{k}\). For any variety \(X\) over \(k\), let \(\overline{X}\) denote the \(\overline{k}\)-variety \(X \otimes_k \overline{k}\). Given an embedding \(\sigma : k \to \mathbb{C}\), let \(\sigma X\) denote the \(\mathbb{C}\)-variety \(X \otimes_{k,\sigma} \mathbb{C}\).

For any pair of integers \(d, m \in \mathbb{Z}_{\geq 0}\), let \(H^d_{\text{h}f}(X)(m)\) be the \(m\)-twisted degree \(d\) \(\acute{e}tale\) cohomology of \(\overline{X}\) with coefficients in \(\mathbb{A}_f(m)\): here, \(\mathbb{A}_f(1) = H^2_{\text{h}f}(\mathbb{P}^1_k)\), and \(\mathbb{A}_f(2) = \mathbb{A}_f(-1) \otimes^\mathbb{L} \mathbb{A}_f(-1)\).

Similarly, let \(H^d_{\text{dR}}(X)(m)\) denote the \(m\)-twisted degree \(d\) de Rham cohomology of \(X\) over \(k\): as a filtered \(k\)-vector space, it is the tensor product \(H^d_{\text{dR}}(X) \otimes_k \mathbb{Q}(-1)^{\otimes -m}\), where \(k(-1) = H^2_{\text{dR}}(\mathbb{P}^1_k)\). Finally, if \(\sigma : k \to \mathbb{C}\), let \(H^d_{\text{dR}}(X)(m) = H^d_{\text{dR}}(\sigma X)(m)\), where \(H^d_{\text{dR}}(\sigma X)\) is the degree \(d\) singular or Betti cohomology of \(\sigma X\) with coefficients in \(\mathbb{Q}\): as a rational Hodge structure, it is the tensor product \(H^d_{\text{dR}}(X) \otimes \mathbb{Q}(-1)^{\otimes -m}\), where \(\mathbb{Q}(-1) = H^2_{\text{dR}}(\mathbb{P}^1_k)\). More generally, for any \(r, s, m \in \mathbb{Z}_{\geq 0}\), set

\[
\begin{align*}
H^r_{\text{h}f}(X)(m) &= (H^r_{\text{h}f}(X) \otimes H^s_{\text{h}f}(X)^\vee)(m); \\
H^r_{\text{dR}}(X)(m) &= (H^r_{\text{dR}}(X) \otimes H^s_{\text{dR}}(X)^\vee)(m).
\end{align*}
\]
Given an embedding $\sigma : k \rightarrow \mathbb{C}$, and an extension $\overline{\sigma} : \overline{k} \rightarrow \mathbb{C}$ of $\sigma$, we have natural isomorphisms

$$H^r_{\text{et}}(X)(m) \cong H^r_{\text{et}}(\sigma X)(m);$$

$$H^r_{\text{dR}}(X)(m) \otimes_{k,\sigma} \mathbb{C} \cong H^r_{\text{dR}}(\sigma X)(m).$$

This gives us a comparison isomorphism

$$\gamma_\sigma : H^r_{\text{et}}(X)(m) \otimes \mathbb{Q} (h_f \times \mathbb{C}) \cong H^r_{\text{et}}(X)(m) \times H^r_{\text{dR}}(X)(m) \otimes_{k,\sigma} \mathbb{C}.$$ 

**Definition 1.2.** An element $s = (s_{h_f}, s_{dR}) \in H^r_{\text{et}}(X)(m) \times H^r_{\text{dR}}(X)(m)$ is rational with respect to an embedding $\overline{\sigma} : \overline{k} \rightarrow \mathbb{C}$ if $\gamma_\sigma^{-1}(s)$ lies in $H^r_{\text{et}}(X)(m)$. It is Hodge with respect to $\overline{\sigma}$ if it is rational with respect to $\overline{\sigma}$, and if $s_{\text{dR}} \in F^0 H^r_{\text{dR}}(X)(m)$. It is absolutely Hodge if it is rational with respect to every embedding $\sigma : k \rightarrow \mathbb{C}$. This last notion does not depend on the choice of algebraic closure $\overline{k}$.

We denote the space of absolute Hodge cycles in $H^r_{\text{et}}(X)(m) \times H^r_{\text{dR}}(X)(m)$ by $\text{AH}^{r,s}(X(m))$. Note that the space can be non-zero only when $2m = r - s$. We will write $\text{AH}^r(X(m))$ for $\text{AH}^{r,0}(X(m))$.

**Proposition 1.3.**

1. Let $\text{CH}^d(X)$ be the Chow group of co-dimension $d$ cycles on $X$; then there is a natural cycle class map

$$\text{cl} : \text{CH}^d(X) \rightarrow \text{AH}^{2d}(X(d)).$$

2. For every $r, s, m \in \mathbb{Z}_{\geq 0}$, $\text{AH}^{r,s}(X(m))$ is a finite dimensional $\mathbb{Q}$-vector space.

3. The natural map

$$\text{AH}^{r,s}(X(m)) \rightarrow \text{AH}^{r,s}(\overline{X}(m))^{\text{Aut}(\overline{X}/k)}$$

is an isomorphism. In fact, $s \in \text{AH}^{r,s}(\overline{X}(m))$ belongs to $\text{AH}^{r,s}(X(m))$ if and only if, for some (hence any) prime $\ell$, its $\ell$-adic realization $s_{\ell}$ is $\text{Aut}(\overline{k}/k)$-invariant.

4. If $L \supset k$ is an algebraically closed field, then, for any embedding $\overline{k} \rightarrow L$ of extensions of $k$, the natural map

$$\text{AH}^{r,s}(\overline{X}(m)) \rightarrow \text{AH}^{r,s}((X \otimes_k L)(m))$$

is an isomorphism. In particular, given an embedding $\sigma : \overline{k} \rightarrow \mathbb{C}$, we can identify $\text{AH}^{r,s}(\overline{X}(m))$ with the space of $(0,0)$-tensors in $H^r_{\sigma}(X)(m)$.

**Proof.** The first assertion is clear and the second follows from the finite dimensionality of Betti cohomology. As for the third, the first part is immediate from the definition, and the second follows, since the map $\text{AH}^{r,s}(\overline{X}(m)) \rightarrow H^r_{\text{et}}(\overline{X})(m)$ is injective and $\text{Gal}(\overline{k}/k)$-equivariant. The last assertion follows from [DMOSS2, I.2.9].

1.4. Let us now briefly recall the construction of the $\mathbb{Q}$-linear neutral Tannakian category $\text{Mot}_{\text{AH}}(k)$ of motives over $k$ for absolute Hodge cycles. We first consider the $\mathbb{Q}$-linear category whose objects are $h(X)$, where $X$ is a (smooth, projective) $k$-variety, and $h(X)$ is a formal symbol attached to it. We then decree that, for two $k$-varieties $X, Y$, $\text{Hom}(h(X), h(Y)) = \text{AH}^{\text{dim}X}(\langle X \times Y \rangle(\text{dim}X))$, with composition given by cup-product. This category has a monoidal structure given by $h(X) \otimes h(Y) = h(X \times Y)$ and an additive structure given by $h(X) \oplus h(Y) = h(X \sqcup Y)$.

Then we consider the category $\text{Mot}^{+}_{\text{AH}}(k)$ of pairs $(h(X), \pi)$, where $\pi \in \text{End}(h(X))$ is an idempotent; here,

$$\text{Hom}((h(X), \pi), (h(Y), \varpi)) = \varpi \text{Hom}(h(X), h(Y)) \pi \subset \text{Hom}(h(X), h(Y)).$$
This is a \(\mathbb{Q}\)-linear tensor category, and there are natural realization functors \(\omega_? : \text{Mot}_\text{AH}(k) \rightarrow \mathbb{Q}_?\), for \(? = \ell, d\operatorname{R}, \sigma\); here \(\mathbb{Q}_\text{DR} = k\) and \(\mathbb{Q}_\sigma = \mathbb{Q}\). For each variety \(X\), the Künneth decomposition on \(X \times X\) allows us to attach to each \(d \in \mathbb{Z}_{\geq 0}\), an object \(h^d(X)\) in \(\text{Mot}_\text{AH}^+(k)\) such that \(\omega_?(h^d(X)) = H^d(X)\), for \(? = \ell, d\operatorname{R}, \sigma\). If \(H \in \text{CH}^1(X)\) is a hyperplane section, then the Lefschetz decomposition gives us an object \(p^d(X)\) in \(\text{Mot}_\text{AH}^+(k)\) such that \(\omega_?(p^d(X)) = \text{PH}^d(X)\), the primitive cohomology group associated with \(H\); cf. \cite{DMOSS2} II.6.7.

In particular, we have the Lefschetz object \(L = h^2(\mathbb{P}^1)\) in \(\text{Mot}_\text{AH}^+(k)\). We obtain \(\text{Mot}_\text{AH}(k)\) by formally inverting \(L\). That is, its objects are pairs \((M,n)\), where \(M \in \text{Mot}_\text{AH}^+(k)\) and \(n \in \mathbb{Z}\), and morphisms are given by:

\[
\text{Hom}((M_1,n_1),(M_2,n_2)) = \text{Hom}(M_1 \otimes L^{N-n_1},M_2 \otimes L^{N-n_2}),
\]

where \(N\) is any integer such that \(N \geq n_1, n_2\). Moreover,

\[(M_1,n_1) \otimes (M_2,n_2) = (M_1 \otimes M_2,n_1 + n_2).
\]

We will denote the object \((M,n)\) by \(M(n)\). For \(? = \ell, d\operatorname{R}, \sigma\), and \(M \in \text{Mot}_\text{AH}(k)\), we will write \(M_?\) for the realization \(\omega_?(M)\), especially when we want to call attention to additional structure: that of a Galois-module, filtered vector space, or Hodge structure, respectively.

The semi-simplicity of this category rests on the existence, for each variety \(X\) and each \(d \in \mathbb{Z}_{\geq 0}\), of a perfect, polarization pairing (cf. \cite{DMOSS2} II.6.2):

\[
h^d(X) \otimes h^d(X) \rightarrow L^{\otimes d}.
\]

In particular, we can identify \(h^d(X)^\vee = h^d(X)(-d)\). Note that, to obtain a Tannakian structure on \(\text{Mot}_\text{AH}(k)\), one needs to modify the natural commutativity constraint as in \cite{Pan94} p. 470. We will refer to objects in this category simply as motives from now on.

**Theorem 1.5.**

1. \(\text{Mot}_\text{AH}(k)\) is a neutral \(\mathbb{Q}\)-linear Tannakian category, and for \(? = \ell, d\operatorname{R}, \sigma\), the natural realization functor \(\omega_?\) is a fiber functor.

2. For any extension \(L/k\), there is a natural, faithful functor of Tannakian categories compatible with fiber functors:

\[
\omega \otimes_k : \text{Mot}_\text{AH}(k) \rightarrow \text{Mot}_\text{AH}(L).
\]

If \(k\) is algebraically closed in \(L\), then this functor is also full. In general, for motives \(M,N \in \text{Mot}_\text{AH}(k)\), a map \(f : M \otimes_k L \rightarrow N \otimes_k L\) is defined over \(k\) if and only if, for some prime \(\ell\), its \(\ell\)-adic realization \(f_\ell\) commutes with \(\text{Aut}(L/k)\).

**Proof.** Cf. \cite{DMOSS2} II.6.7. \(\square\)

The following is the main result of \cite{DMOSS2} Ch. I.

**Theorem 1.6** (Deligne). Let \(X\) be a product of abelian varieties. If \(s \in H^{r,s}_\text{DR}(X)(m) \times H^{r,s}_\text{DR}(X)(m)\) is Hodge with respect to one embedding \(\sigma : \overline{k} \hookrightarrow \mathbb{C}\), then it is absolutely Hodge. \(\square\)

**Corollary 1.7.** Let \(\text{Mot}_\text{Ab}(k) \subseteq \text{Mot}_\text{AH}(k)\) be the full Tannakian sub-category generated by the motives attached to abelian varieties. Let \(\text{Hdg}_\mathbb{Q}\) be the Tannakian category of \(\mathbb{Q}\)-Hodge structures. Then, for any embedding \(\sigma : k \hookrightarrow \mathbb{C}\), the functor

\[
\text{Mot}_\text{Ab}(k) \rightarrow \text{Hdg}_\mathbb{Q}
\]

\[
M \mapsto M_\sigma
\]

is faithful. If \(k\) is algebraically closed, then it is in fact fully faithful. \(\square\)
1.8. We will need a mildly refined notion of a motive: Let $R \subset \mathbb{Q}$ be a sub-ring. A **motive with $R$-structure** or an **$R$-motive** is a motive $M$ equipped with an $\text{Aut}(\overline{k}/k)$-stable $R \otimes \mathbb{Z}$-lattice $M_R \subset M_{\hat{k}}$. For example, if $R = \mathbb{Z}$, then $M_R$ is a $\mathbb{Z}$-lattice; and, if $R = \mathbb{Z}(p)$, then giving $M_R$ amounts to giving a $\text{Aut}(\overline{k}/k)$-stable $\mathbb{Z}_p$-lattice $M_{\mathbb{Z}_p} \subset M_p$.

A morphism $f : (M, M_R) \to (N, N_R)$ of $R$-motives is a map $f : M \to N$ of motives such that the $k_f$-realization $f_{k_f}$ carries $M_R$ into $N_R$.

Suppose that $M_R = (M, M_{\hat{R}})$ is an $R$-motive. For any embedding $\sigma : k \hookrightarrow \mathbb{C}$, this also gives us a canonical $R$-lattice $M_{R, \sigma} \subset M_{\sigma}$ obtained as follows. Choose an extension $\overline{\sigma} : \overline{k} \hookrightarrow \mathbb{C}$ of $\sigma$. This gives us a comparison isomorphism

$$M_{\sigma} \otimes k_f \xrightarrow{\sim} M_{\hat{k}}.$$

We now take $M_{R, \sigma}$ to be the intersection of the pre-image of $M_{\hat{R}}$ with $M_{\sigma}$. Since $M_{\hat{R}}$ is $\text{Aut}(\overline{k}/k)$-stable, this does not depend on the choice of $\overline{\sigma}$. Clearly, for any map $f : M_R \to N_R$ of $R$-motives, the Betti realization $f_{\sigma}$ respects the $R$-lattice $M_{R, \sigma}$.

Given an $R$-motive $M$, and a prime $p$ not invertible in $R$, we will write $M_p$ for its associated $\mathbb{Z}_p$-representation of $\text{Aut}(\overline{k}/k)$, and, for any $\sigma : k \hookrightarrow \mathbb{C}$, we will write $M_{\sigma}$ for the associated $R$-Hodge structure. Note that, for each variety, $X$, $h(X)$ has a natural $R$-structure. In particular, the Lefschetz motive $L$ underlies a natural $R$-motive which we will continue to denote by $L$.

For any $R$-motive $M$, write $\text{AH}(M)$ for the $R$-module of **cycles on $M$**: This is the space of maps $\text{Hom}(1, M)$, where $1$ is the identity object; that is $1 = h(\text{pt})$ with its natural $R$-structure. If $R \hookrightarrow R'$ is an inclusion of sub-rings of $\mathbb{Q}$, then there is a natural functor $\_ \otimes_R R'$ from $R$-motives to $R'$-motives such that

$$\text{AH}(M) \otimes_R R' = \text{AH}(M \otimes_R R'),$$

for any $R$-motive $M$.

**Definition 1.9.** An $R$-motive $M$ is **pure of weight** $d$, for some $d \in \mathbb{Z}$, if, for one (hence all) $\sigma : k \hookrightarrow \mathbb{C}$, $M_{\sigma}$ is a pure Hodge structure of weight $d$. A **polarization** on an $R$-motive $M$ that is pure of weight $d$ is a pairing

$$\psi : M \otimes M \to L(-d)$$

such that, for any $\sigma : k \hookrightarrow \mathbb{C}$, $\psi$ induces a polarization of the $\mathbb{Q}$-Hodge structure $M_{\sigma} \otimes_R \mathbb{Q}$.

1.10. One problem with absolute Hodge cycles is that they do not have an analogue in positive characteristic. We will deal with this in somewhat ad hoc fashion. We now assume that the field $k$ is equipped with a discrete valuation $\nu : k^\times \to \mathbb{Z}$ such that the residue field $k(\nu)$ is perfect of characteristic $p > 0$. Let $k_\nu$ be the completion of $k$ along $\nu$, and let $\mathcal{O}_\nu$ be its ring of integers. Let $B_{\text{dR}}$ be Fontaine’s ring of de Rham periods for $k_\nu$. For any smooth projective variety over $k$, and for $d \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}$, we have the de Rham comparison isomorphism:

$$\gamma_{\text{dR}} : H^d_p(X)(m) \otimes_{k_\nu} B_{\text{dR}} \xrightarrow{\sim} H^d_{\text{dR}}(X/k) \otimes_k B_{\text{dR}}.$$

**Definition 1.11.** A cycle $s \in \text{AH}^{r,s}(X(m))$ is **de Rham** (with respect to $\nu$), if

$$\gamma_{\text{dR}}(s \otimes 1) = s_{\text{dR}} \otimes 1.$$

Write $\text{AD}^{r,s}(X(m))$ for the space of de Rham cycles in $\text{AH}^{r,s}(X(m))$.

Let $\text{Mot}_{\text{AD}, \nu}(k)$ be the category defined exactly as in [1.4], except that we only allow absolutely de Rham cycles as morphisms. It is easy to see that this is a sub-category of $\text{Mot}_{\text{AH}}(k)$. The analogue of [DMOSS2, II.6.2] holds in this setting, so $\text{Mot}_{\text{AD}, \nu}(k)$ is semi-simple and in fact Tannakian.
Theorem 1.12 (Blasius-Wintenberger). Let \( \text{Mot}_{\text{Ab}, \nu} (k) \) be the Tannakian sub-category of \( \text{Mot}_{\text{AD}, \nu} (k) \) generated by the motives attached to abelian varieties. Then the natural functor

\[
\text{Mot}_{\text{Ab}, \nu} (k) \to \text{Mot}_{\text{Ab}} (k)
\]

is an equivalence of categories.

Proof. This reduces to showing that every (absolutely) Hodge cycle on an abelian variety is de Rham, which is the main result of [Bla94]. □

1.13. We will now work with pairs \((X, \mathfrak{X})\), where \(X\) is a \(k\)-variety and \(\mathfrak{X}\) is a smooth proper \(\mathcal{O}_\nu\)-scheme equipped with an identification \(\mathfrak{X} \otimes_{\mathcal{O}_\nu} k(\nu) = X \otimes_k k(\nu)\). Write \(\mathfrak{X}_0\) for the special fiber \(\mathfrak{X} \otimes_{\mathcal{O}_\nu} k(\nu)\). Set \(W = W(k(\nu))\); then the crystalline cohomology \(H^d_{\text{cris}}(\mathfrak{X}_0/W)\) is an \(F\)-crystal over \(W\).

Let \(W(-1) = H^2_{\text{cris}}(\mathbb{P}^1_{k(\nu)}/W)\), and let \(W(1)\) be its dual; note that \(W(1) \left[ \frac{1}{p} \right] \) has the structure of an \(F\)-isocrystal over \(W \left[ \frac{1}{p} \right]\), but that \(W(1)\) is not \(F\)-stable. Let \(B_{\text{cris}}\) be Fontaine’s ring of crystalline periods for \(k_\nu\). For \(d \in \mathbb{Z}_{\geq 0}\) and \(m \in \mathbb{Z}\), we have natural comparison isomorphisms:

\[
\gamma_{B-O} : H^d_{\text{cris}}(\mathfrak{X}_0/W)(m) \otimes_W k_\nu \xrightarrow{\sim} H^d_{\text{dR}}(X)(m) \otimes_k k_\nu;
\]

\[
\gamma_{\text{cris}} : H^d_{p}(X)(m) \otimes \mathbb{Q}_p B_{\text{cris}} \xrightarrow{\sim} H^d_{\text{cris}}(\mathfrak{X}_0/W)(m) \otimes_W B_{\text{cris}}.
\]

These isomorphisms are compatibale in the sense that

\[
\gamma_{B-O} \circ (\gamma_{\text{cris}} \otimes 1) = \gamma_{\text{dR}}.
\]

Definition 1.14. A cycle \(s \in \text{AH}^{r,s}(X(m))\) is Tate (with respect to \(\mathfrak{X}\) and \(\nu\)), if

\[
\gamma_{B-O}^{-1}(s_{\text{dR}} \otimes 1) \in H^d_{\text{cris}}(\mathfrak{X}_0/W)(m) \otimes_W k(\nu)
\]

is an \(F\)-invariant element of \(H^d_{\text{cris}}(\mathfrak{X}_0/W)(m) \left[ \frac{1}{p} \right]\). We will denote this \(F\)-invariant element by \(s_{\text{cris}}\); it is the crystalline realization of \(s\).

We say that \(s\) is crystalline (with respect to \(\mathfrak{X}\) and \(\nu\)) if it is Tate, and if \(\gamma_{\text{cris}}(s_{p} \otimes 1) = s_{\text{cris}} \otimes 1\).

Denote the space of crystalline cycles in \(\text{AH}^{r,s}(X(m))\) by \(\text{AC}^{r,s}(X(m))\). Since the comparison isomorphisms are compatible with cycle classes and Poincaré duality, we see that algebraic cycle classes are crystalline. Similar statements hold for the Kümeth and Lefschetz decompositions.

Lemma 1.15. The notion of being Tate or crystalline does not depend on the choice of model \(\mathfrak{X}\). In fact, for any \(r, s, m\), we have \(\text{AC}^{r,s}(X(m)) = \text{AD}^{r,s}(X(m))\). Moreover, the \(F\)-isocrystal \(H^d_{\text{cris}}(\mathfrak{X}_0/W) \left[ \frac{1}{p} \right]\) is also independent of the choice of model \(\mathfrak{X}\).

Proof. Since \(\gamma_{\text{dR}}\) is compatible with \(\gamma_{B-O}\) and \(\gamma_{\text{cris}}\), and since \(\gamma_{\text{cris}}(s_{p} \otimes 1)\) is always \(F\)-invariant, \(s\) is crystalline if and only if \(\gamma_{\text{dR}}(s_{p} \otimes 1) = s_{\text{dR}} \otimes 1\); that is, if and only if \(s\) is de Rham. From this, the first two assertions are immediate.

For the second, we now only have to note that \(H^d_{\text{cris}}(\mathfrak{X}_0/W) \left[ \frac{1}{p} \right]\) is identified with the \(\text{Gal}(\kappa_\nu/k_\nu)\)-invariants of \(H^d_{p}(X) \otimes \mathbb{Q}_p B_{\text{cris}}\). □

From (1.12), we now have:

Corollary 1.16. If \(X\) is an abelian variety over \(k\) with good reduction at \(\nu\), then we have:

\[
\text{AC}^{r,s}(X(m)) = \text{AD}^{r,s}(X(m)) = \text{AH}^{r,s}(X(m)).
\]
1.17. Let $\text{Mot}_{AC,ν}(k) \subset \text{Mot}_{AH}(k)$ be the sub-category whose objects are triples $(h(X), π, m)$, where $X$ is a variety over $k$ with good reduction at $ν$, $π ∈ AC^{\dim X}(X × X)$ is an idempotent, and $m ∈ Z$. Morphisms are given as before, except that we restrict ourselves to absolutely crystalline cycles. Just like $\text{Mot}_{AD,ν}(k)$, $\text{Mot}_{AC,ν}(k)$ is also Tannakian. Note that, by (1.15) above, any object $M$ of $\text{Mot}_{AC,ν}$ has a canonical crystalline realization $M_{\text{cris}}$ that is an $F$-isocrystal over $W[\frac{1}{p}]$ and is equipped with a natural isomorphism of $k_ν$-vector spaces

$$M_{\text{cris}} \otimes_{W[\frac{1}{p}]} k_ν \xrightarrow{\sim} M_{\text{DR}} \otimes_k k_ν.$$ 

The next result follows easily from (1.16):

**Proposition 1.18.** Let $\text{Mot}^o_{Ab,ν}(k)$ (resp. $\text{Mot}^o_{Ab,ν,cris}(k)$) be the full sub-category of $\text{Mot}_{AH}(k)$ (resp. $\text{Mot}^o_{AC,ν}(k)$) generated by the motives attached to abelian varieties with good reduction at $ν$. Then the natural functor

$$\text{Mot}^o_{Ab,ν,cris}(k) \rightarrow \text{Mot}^o_{Ab}(k)$$

is fully faithful and its essential image is $\text{Mot}^o_{Ab,ν}(k)$.

\[\square\]

2. Moduli of K3 surfaces

Our main references for this section are [Riz06, Riz12, Man12, Ogu79].

2.1. A K3 surface over a scheme $S$ is an algebraic space $f : X \rightarrow S$ over $S$ that is proper, smooth and whose geometric fibers are K3 surfaces. A polarization (resp. a quasi-polarization) of a K3 surface $X → S$ is a section $ξ ∈ \text{Pic}(X/S)(S)$ whose fiber at each geometric point $s → S$ is a polarization (resp. a quasi-polarization); that is, the class of an ample (resp. big and nef) line bundle, of the K3 surface $X_s$ over $k(s)$. There is an intersection pairing on $\text{Pic}(X/S)$ with values in the locally constant sheaf $\mathbb{Z}$; the degree $\deg(ξ) ∈ H^1(S, \mathbb{Z})$ of a (quasi-)polarization $ξ$ is the value of its pairing with itself. The restriction of $\deg(ξ)$ to any connected component of $S$ is a non-zero positive integer. A section $ξ$ of $\text{Pic}(X/S)$ is primitive if, for all geometric points $s → S$, $ξ(s)$ is primitive; that is, $ξ(s)$ is not a non-trivial multiple of any element of $\text{Pic}(X_s)$.

Fix an integer $d ∈ \mathbb{Z}_{>0}$, and let $M_{2d}$ (resp. $M^o_{2d}$) be the moduli problem over $\mathbb{Z}[\frac{1}{2}]$ that assigns to every $\mathbb{Z}[\frac{1}{2}]$-scheme $S$ the groupoid of tuples $(f : X → S, ξ)$, where $X → S$ is a K3 surface and $ξ$ is a primitive quasi-polarization (resp. polarization) of $X$ with $\deg(ξ) = 2d$.

**Proposition 2.2.** The natural map $M^o_{2d} → M_{2d}$ is an open immersion of Deligne-Mumford stacks of finite type over $\mathbb{Z}$, fiber-by-fiber dense. Moreover, $M^o_{2d}$ is separated.

**Proof.** Everything except the fiber-by-fiber density of the image of the map can be found in [Riz06, 4.3.3] and [Man12, Proposition 2.1]. Showing the claimed density amounts to seeing that any quasi-polarized K3 surface $(X_0, ξ_0)$ over a field $k$ admits a deformation $(X, ξ)$ such that $ξ$ is an ample class. Indeed, let $D_0$ be a divisor on $X_0$ with class $ξ_0$. Then $3D_0$ determines a base-point free map $X_0 → \mathbb{P}^N$ whose image is a surface with isolated ordinary double-point singularities. The pre-images of the singularities are $(-2)$-rational curves on $X_0$. If a deformation $(X, ξ)$ of $(X_0, ξ_0)$ is not polarized, then one of these $(-2)$-curves must also permit a deformation to $X$. It is easy to check using the Riemann-Roch formula that deforming a $(-2)$-curve on a K3 surface is equivalent to deforming its divisor class, and so [LO12, Theorem A.7] shows that the deformation locus of a $(-2)$-curve in the versal deformation space of $(X_0, ξ_0)$ has co-dimension 1. This implies in turn that the locus where the versal deformation is not

\[\text{big} \] equals being the tensor product of an ample line bundle with an effective one.
polarized is a union of co-dimension 1 sub-spaces, and so finishes the proof of the proposition. Notice that the proof shows that the complement of $M_{2d}^\perp$ in $M_{2d}$ is flat over $\mathbb{Z}[\frac{1}{d}]$ and has pure co-dimension 1.

2.3. Let $(f : X \to M_{2d}, \xi)$ be the universal object over $M_{2d}$. For any prime $\ell$, the second relative étale cohomology $H_2^\ell(X)$ of $X$ over $M_{2d, \mathbb{Z}[\frac{1}{d}]}$ with coefficients in $\mathbb{Z}_\ell$ is a lisse $\mathbb{Z}_\ell$-sheaf of rank 22 equipped with a perfect, symmetric Poincaré pairing $\langle \_ , _ \rangle : H_\ell^2 \times H_\ell^2 \to \mathbb{Z}_\ell(-2)$.

We will actually be equipping $H_\ell^2$ with the negative of the conventional pairing. In characteristic 0, this means that we are viewing the Betti cohomology groups of K3 surfaces as being quadratic spaces of signature $(19+, 3-)$.

The $\ell$-adic Chern class $\text{ch}(\xi)$ of $\xi$ is a global section of the Tate twist $H_\ell^2(1)$ that satisfies $\langle \text{ch}(\xi), \text{ch}(\xi) \rangle = -2d$. We set $P_\ell^2 = \langle \text{ch}(\xi) \rangle^\perp(-1) \subset H_\ell^2$.

This is a lisse $\mathbb{Z}_\ell$-sheaf over $M_{2d, \mathbb{Z}[\frac{1}{d}]}$ of rank 21 and it inherits a symmetric $\mathbb{Z}_\ell(-2)$-valued pairing $\langle \_ , _ \rangle$, which is perfect if $\ell \nmid d$.

2.4. There is also the second relative de Rham cohomology $H^{2\cdot}_{\text{dR}}(X)$ over $M_{2d}$. This is a vector bundle with flat connection of rank 22 equipped with a Hodge filtration $F^\bullet H^{2\cdot}_{\text{dR}}$ satisfying Griffiths transversality. It is also equipped with a perfect, horizontal, symmetric pairing $\langle \_ , _ \rangle$ into $\mathcal{O}_{M_{2d}}$. The filtration then is of the form $0 = F^0 H^{2\cdot}_{\text{dR}} \subset F^1 H^{2\cdot}_{\text{dR}} \subset F^2 H^{2\cdot}_{\text{dR}} = (F^2 H^{2\cdot}_{\text{dR}})\perp \subset F^0 H^{2\cdot}_{\text{dR}} = H^{2\cdot}_{\text{dR}}$.

determined by the isotropic line $F^2 H^{2\cdot}_{\text{dR}}$. The de Rham Chern class $\text{ch}_{\text{dR}}(\xi)$ attached to $\xi$ is a horizontal global section of $F^1 H^{2\cdot}_{\text{dR}}$ satisfying $\langle \text{ch}_{\text{dR}}(\xi), \text{ch}_{\text{dR}}(\xi) \rangle = -2d$. Again, we set $F^2 H^{2\cdot}_{\text{dR}} = \langle \text{ch}_{\text{dR}}(\xi) \rangle^\perp \subset H^{2\cdot}_{\text{dR}}$.

This is a vector sub-bundle of $H^{2\cdot}_{\text{dR}}$ of rank 21, and it inherits the connection, the filtration and the symmetric pairing from $H^{2\cdot}_{\text{dR}}$.

For any prime $p$, over $M_{2d, \mathbb{F}_p}$, the induced vector bundle $H^{2\cdot}_{\text{dR}, \mathbb{F}_p}$ is equipped with an decreasing, horizontal filtration $F^\bullet_{\text{con}} H^{2\cdot}_{\text{dR}, \mathbb{F}_p}$ called the conjugate filtration (cf. [Ogu79] §1 for this and the rest of the discussion in this paragraph). Suppose that $k$ is an algebraically closed field over $\mathbb{F}_p$ and we have a map $s : \text{Spec} k \to M_{2d}$. We say that $s$ is superspecial if the fiber of $\text{ch}_{\text{dR}}(\xi)$ in $F^1 H^{2\cdot}_{\text{dR}, s}$ lies in $F^2 H^{2\cdot}_{\text{dR}, s}$. In this case, we have $F^2 H^{2\cdot}_{\text{dR}, s} = F^2_{\text{con}} H^{2\cdot}_{\text{dR}, s}$.

We say that $s$ is ordinary if $X_s$ is ordinary; that is, if $F^2 H^{2\cdot}_{\text{dR}, s} \cap F^2_{\text{con}} H^{2\cdot}_{\text{dR}, s} = 0$.

2.5. Let $k$ be a perfect field of characteristic $p$, let $W = W(k)$, and let $R$ be a formally smooth complete local $W$-algebra equipped with an augmentation map $j : R \to W$. We can arrange an identification $R = W[[t_1, \ldots, t_d]]$ in such a way that $j$ is simply the map carrying each formal variable $t_i$ to 0. Equip $R$ with a Frobenius lift $\varphi : R \to R$ with $\varphi|_{W} = \sigma$, and $\varphi(t_i) = t_i^p$, for $i = 1, \ldots, d$. A filtered $F$-crystal over $R$ is a tuple $(M, F, \text{Fil}^* M)$, where:

- $M$ is a free $R$-module.
- $F : \varphi^* M \to M$ is an isomorphism of $R_0$-modules.
- $\text{Fil}^* M$ is a decreasing, exhaustive, separated filtration of $M$ by direct summands.
If \((M, F, \text{Fil}^\bullet M)\) is a filtered \(F\)-crystal over \(R\), then we will call \(M_Q\) a \textit{filtered} \(F\)-isocrystal over \(R_Q\).

The filtered \(F\)-crystal is \textit{strongly divisible} if:

\[
F\left(\varphi^*(\sum \text{Fil}^i F M)\right) = M.
\]

**Lemma 2.6.** Let \(M = (M, F, \text{Fil}^\bullet M)\) be a filtered \(F\)-crystal over \(R\). For each integer \(i\), set

\[
M^i = \{ m \in M : F(m) \in p^i M \}.
\]

Then the following statements are equivalent:

1. \(M\) is strongly divisible.
2. For each \(i\), \(M^i = \sum_{j \leq i} \text{Fil}^{i-j} F M\).
3. For each \(i\), \(\text{Fil}^i M \subseteq M^i \subseteq \text{Fil}^i M + pM\).

**Proof.** The equivalence of (1) and (2) is immediate, and the equivalence of (2) and (3) is \cite{Ogu78} Remark 3.9. □

**Proposition 2.7.** Let \(R_0 = R \otimes \mathbb{F}_p\) and let \(X/R\) be a smooth proper scheme with fiber \(X_0\) over \(R_0\). Suppose that, for every point \(s \in \text{Spec} R_0\), the Hodge spectral sequence for \(X_s\) over \(k(s)\) degenerates at \(E_1\), and that \(H^i_{\text{dR}}(X/R)\) is a free \(R\)-module. Then, for any \(d \in \mathbb{Z}_{\geq 0}\) with \(d < p\), the relative de Rham cohomology \(H^i_{\text{dR}}(X/R)\) is a strongly divisible filtered \(F\)-crystal over \(R\).

**Proof.** Following \cite{210} and \cite{BO78} 8.26, it is enough to show that, for each \(i \in \mathbb{Z}_{\geq 0}\):

\[
F(\text{Fil}^i H^i_{\text{dR}}(X/R)) \subseteq p^i H^i_{\text{dR}}(X/R),
\]

where \(\text{Fil}^i H^i_{\text{dR}}(X/R)\) is the Hodge filtration. To show this inclusion, it is enough to show it for every \(W\)-valued point of \(R\). But then it follows from \cite{La80} Prop. 5.2. □

**Corollary 2.8.** If \((X, \xi)\) is a primitively quasi-polarized K3 surface over \(R\), then the primitive de Rham cohomology

\[
PH^2_{\text{dR}}(X/R) = \langle \text{ch}_{\text{dR}}(\xi) \rangle^\perp \subseteq H^2_{\text{dR}}(X/R)
\]

is a strongly divisible filtered \(F\)-crystal over \(R\).

**Proof.** We see from \cite{217} that \(H^2_{\text{dR}}(X/R)\) is strongly divisible. Since \(\text{ch}_{\text{dR}}(\xi)\) generates a direct summand of \(F^1 H^2_{\text{dR}}(X/R)\) and satisfies \(F(\text{ch}_{\text{dR}}(\xi)) = p \text{ch}_{\text{dR}}(\xi)\), it is not hard to see that its orthogonal complement is also strongly divisible. □

We will have use for some definitions and results due to Vasiu and Zink \cite{VZ10}.

**Definition 2.9.** A \(\mathbb{Z}_{(p)}\)-scheme \(X\) is \textit{healthy regular} if it is regular, faithfully flat over \(\mathbb{Z}_{(p)}\), and if, for every open sub-scheme \(U \subset X\) containing \(X_Q\) and all generic points of \(X_{\mathbb{F}_p}\), every abelian scheme over \(U\) extends uniquely to an abelian scheme over \(X\).

A local \(\mathbb{Z}_{(p)}\)-algebra \(R\) with maximal ideal \(m\) is \textit{quasi-healthy regular} if it is regular, faithfully flat over \(\mathbb{Z}_{(p)}\), and if every abelian scheme over \(\text{Spec} R \setminus \{m\}\) extends uniquely to an abelian scheme over \(\text{Spec} R\).

**Theorem 2.10** (Vasiu-Zink). Let \(R\) be a regular local \(\mathbb{Z}_{(p)}\)-algebra with algebraically closed residue field \(k\), of dimension at least 2, which admits a surjection

\[
R \twoheadrightarrow W(k)[T_1, T_2]/(p - h),
\]

where \(h \notin (p, T_1^p - 1, T_2^p - 1)\). Then \(R\) is quasi-healthy regular. In particular, if \(R\) is a smooth \(\mathbb{Z}_{(p)}\)-algebra, then \(R\) is quasi-healthy regular.

**Proof.** This is \cite{VZ10} Theorem 3. □
We can encapsulate the deformation theory of K3 surfaces in the following

**Theorem 2.11.** Let $X_0$ be a K3 surface over a perfect field $k$ of characteristic $p > 0$. Then:

1. The deformation functor $\text{Def}_{X_0}$ for $X_0$ is pro-representable and formally smooth of dimension 20 over $W(k)$.
2. For any class $\xi_0 \in \text{Pic}(X_0)$, the deformation functor $\text{Def}_{(X_0,\xi_0)}$ for the pair $(X_0,\xi_0)$ is pro-represented by a flat, formal sub-scheme of $\text{Def}_{X_0}$ defined by a single equation.
3. If $\xi_0$ is primitive, then $\text{ch}_{\text{DR}}(\xi_0) \neq 0$, and $\text{Def}_{(X_0,\xi_0)}$ is formally smooth, unless $\text{ch}_{\text{DR}}(\xi_0)$ lies in $F^2H^2_{\text{DR}}(X_0/k)$. In particular, $\text{Def}_{(X_0,\xi_0)}$ is formally smooth whenever $X_0$ is ordinary.
4. If $\xi_0$ is primitive and $\text{ch}_{\text{DR}}(\xi_0)$ lies in $F^2H^2_{\text{DR}}(X_0/k)$, then $\nu_p(\deg(\xi_0)) = 1$, and $\text{Def}_{(X_0,\xi_0)}$ is quasi-healthy regular.

**Proof.** ([1] and [2]) are due to Deligne; cf. [Del81, 1.2,1.5]. ([3]) can be found in [Ogu79, 2.2].

For [4], set $W = W(k)$, and let $W_0$ be its fraction field. Let $R$ be the formally smooth $W$-algebra pro-representing $\text{Def}_{X_0}$. Choose any map $R \to W$: this gives rise to a formal lift $X/W$ of $X_0$. Let $H = H^2_{\text{DR}}(X/W)$ be the de Rham cohomology of $X$. Via the identification of $H$ with the crystalline cohomology of $X_0$, if $\sigma : W \to W$ is the Frobenius lift, we have a Frobenius map:

$$F : \sigma^*H \to H.$$ 

Let $F^\bullet H \subset H$ be the Hodge filtration on $H$. Then the strong divisibility of the filtered $F$-crystal $H^2_{\text{DR}}(X/W)$ ([28]) shows that we have:

$$(2.11.1) \quad F\left(\sigma^*(p^{-2}F^2H + p^{-1}F^1H + H)\right) = H.$$ 

Let $f := \text{ch}_{\text{cris}}(\xi_0) \in H$ be the crystalline Chern class of $\xi_0$. By our hypothesis in [4], we can write

$$f = f_1 + pf_2,$$

where $f_1$ is a generator for $F^2H$, and $f_2 \in H$.

We have:

$$pf_1 + p^2f_2 = pf = F(f) = F(f_1) + pf(f_2).$$

Here, for any $v \in H$, we write $F(v)$ for $F(\sigma^*v)$.

Since $f_1 \in F^2H$, necessarily $F(f_1) \in p^2H$, and we see that we have:

$$F(f_2) = \frac{1}{p}(pf_1 + p^2f_2 - F(f_1)) \in H \setminus pH.$$ 

So ([2.11.1]) implies that $f_2$ does not lie in $pH + F^1H$. In other words, the image of $f_2$ in $\text{gr}_p^0H$ is a generator. This shows that $f_1 \cdot f_2$ is a unit, and so

$$\deg(\xi_0) = f \cdot f = 2p(f_1 \cdot f_2)$$

is not divisible by $p^2$.

Now, Ogus [Ogu79, 2.2] shows that the deformation ring for $\text{Def}_{(X_0,\xi_0)}$ is isomorphic to

$$W[[t_1, \ldots, t_{10}, u_1, \ldots, u_{10}]]/(\sum_i t_iu_i - \deg(\xi_0)).$$

So it follows from Vasiu and Zink’s criterion ([2.10]) that this ring is quasi-healthy regular. □

One of the ingredients in the proof of Deligne cited above is a description of the tangent space of the deformation functor, which we extract here for later reference.
Lemma 2.12. In the situation of (2) above, we have a canonical identification

\[ \text{Def}(X_0, \xi_0)(k[\epsilon]) = \left\{ \text{Isotropic lines } L \subset PH_{\text{dR}}^2(X_0/k) \otimes k[\epsilon] \text{ lifting } F^2H_{\text{dR}}^2(X_0/k) \right\}. \]

Here, as usual \( PH_{\text{dR}}^2(X_0/k) = \langle \text{ch}_{\text{dR}}(\xi_0) \rangle^\perp \subset H_{\text{dR}}^2(X_0/k) \).

Proof. This is standard. We only note that, under this identification, each deformation \((X, \xi)\) over \(k[\epsilon]\) is mapped to the isotropic line

\[ F^2H_{\text{dR}}^2(X/k) \subset H_{\text{dR}}^2(X/k) = H_{\text{dR}}^2(X_0/k) \otimes_k k[\epsilon]. \]

□

Corollary 2.13. Let \( r \) be the product of primes \( \ell > 2 \) such that \( \ell \mid d \), but \( \ell^2 \nmid d \).

1. \( M_{2d, Z}[\frac{1}{r}] \) is smooth over \( Z \left[ \frac{1}{p} \right] \) of relative dimension 19.
2. If \( p \mid r \), then the singular locus of \( M_{2d, F_p} \) is at most 0-dimensional, and lies within the superspecial locus.
3. All mixed characteristic complete local rings of \( M_{2d, Z}[\frac{1}{r}] \) of dimension at least 2 are quasi-healthy regular.

Proof. (1) is an immediate consequence of (3) and (4) of 2.11.

For (2), we first note that the singular points of \( M_{2d, F_p} \) are all superspecial and that their complete local rings are quasi-healthy regular, by loc. cit.. The assertion is now a consequence of the fact that there are no non-trivial infinitesimal families of quasi-polarized superspecial K3 surfaces (cf [Ogu79] Remark 2.7).

For (3), we only need to worry about the complete local rings of \( M_{2d, Z}[\frac{1}{r}] \) at points valued in fields of characteristic \( p \mid r \). By (2), the completions at the non-closed such points are formally smooth and hence quasi-healthy regular. The completions at the closed such points are quasi-healthy regular, as we have already observed.

\[ \square \]

2.14. We will now define moduli spaces of K3 surfaces with level structure, following [Riz06, §4]. Let \( U \) be the hyperbolic lattice over \( Z \) of rank 2; let \( L \) be the unimodular lattice \( U^\oplus 3 \oplus E_8^\oplus 2 \).

Choose a basis \( e, f \) for (say) the first copy of \( U \) in \( L \). Set

\[ L_d = \langle e - df \rangle^\perp \subset L \]

This is a quadratic lattice over \( Z \) of discriminant 2\( d \); let \( L_d^\perp \subset V_d := L_d \otimes_k \) be its dual lattice. Set \( G_d = \text{SO}(V_d) \): it is a semi-simple algebraic group over \( Q \).

Let \( K \subset G_d(\mathbb{A}_f) \) be a compact open sub-group that stabilizes \( L_d \) and acts trivially on \( L_d^\perp/L_d \). The maximal such sub-group is called the discriminant kernel of \( L_d \). These compact opens are called admissible in [Riz06]. Strictly speaking, Rizov’s definition of admissibility is the following: First, note that \( G_d \) can be viewed as the sub-group of isometries of \( V \) that fix \( e - df \). Now, a compact open sub-group \( K \subset G_d(\mathbb{A}_f) \) is admissible if every element of \( K \), viewed as an isometry of \( V \), stabilizes \( L_d \). That this is equivalent to our definition is shown in [MPT13, 2.2].

We will now fix an admissible compact open \( K \subset G_d(\mathbb{A}_f) \) such that \( K_p \subset G_d(\mathbb{Q}_p) \) is the discriminant kernel of \( L_d \).

Over \( M_{2d, Z(p)} \), the relative \( \ell \)-adic cohomology sheaves \( H^2_{\mathbb{Z}_p} \), for \( \ell \neq p \), can be put together to get the \( \mathbb{Z}_p \)-sheaf \( H^2_{\mathbb{Z}_p} = \prod_{\ell \neq p} H^2_{\mathbb{Z}_p} \). Then the Chern classes of \( \xi \) can also be put together to get the Chern class \( \text{ch}_{\mathbb{Z}_p}(\xi) \) in \( H^2_{\mathbb{Z}_p}(1) \). Let \( P \) be the étale sheaf over \( M_{2d, Z(p)} \), whose sections over any scheme \( T \to M_{2d, Z(p)} \) are given by

\[ P(T) = \{ \text{Isometries } \eta: L \otimes \mathbb{Z}_p^P \cong H^2_{\mathbb{Z}_p, T}(1) \text{ with } \eta(e - df) = \text{ch}_{\mathbb{Z}_p}(\xi) \} \]
This has a natural right action via pre-composition by the constant sheaf of groups $K^p$. A section $[\eta] \in H^0(T, I^p/K^p)$ is called a $K^p$-level structure over $T$.

We define $M_{2d,K}(Z_{(p)})$ to be the relative moduli problem over $M_{2d,Z_{(p)}}$ that attaches to $T \to M_{2d,Z_{(p)}}$ the set of $K^p$-level structures over $T$.

**Proposition 2.15.** $M_{2d,K}(Z_{(p)})$ is finite and étale over $M_{2d,Z_{(p)}}$. For $K^p$ small enough, it is an algebraic space over $\mathbb{Z}_{(p)}$. It is healthy regular, and, unless $\nu_p(d) = 1$, it is smooth over $\mathbb{Z}_{(p)}$.

**Proof.** Both finiteness and étaleness are clear from the definition. As for the second assertion, the key point is to show that a quasi-polarized K3 surface with $K^p$-level structure has trivial automorphism group. This is shown in [Mau12, 2.8], which is based on [Riz00b, 6.2.2].

The last assertion follows from (2.13). \qed

### 3. Shimura varieties

Our main reference for this section will be [MP13b].

3.1. We maintain the notation of (2.14). Since $V_d$ has signature $(19+, 2-)$, we can attach to it a Shimura datum $(G_d, X_d)$, where $X_d$ is the space of negative definite planes in $V_d$. The reflex field of this Shimura datum is $\mathbb{Q}$ [MP13b, 3.2], and so to any neat $\mathbb{Q}$ compact open sub-group $K \subset G_d(K_f)$ we can attach the (canonical model of the) Shimura variety $\text{Sh}_{d,K} := \text{Sh}_{d,K}(G_d, X_d)$: it is a quasi-projective variety over $\mathbb{Q}$ whose $\mathbb{C}$-points can be identified with the double coset space

$$G_d(\mathbb{Q}) \backslash (X_d \times G_d(K_f)/K).$$

We will also need Shimura varieties attached to $\text{GSpin}$ groups. Let $\varpi : G_{d,sp} \to G_d$ be the $\text{GSpin}$ central cover [MP13b, 8.1]. Then $(G_{d,sp}, X_d)$ is again a Shimura datum with reflex field $\mathbb{Q}$, and so, given a neat compact open $K_{sp} \subset G_{d,sp}$, we have the attached Shimura variety over $\mathbb{Q}$, $\text{Sh}_{d,K_{sp}} := \text{Sh}_{d,K_{sp}}(G_{d,sp}, X_d)$, whose $\mathbb{C}$-points have a description analogous to the one for $\text{Sh}_{d,K}$. Moreover, if $K = \varpi(K_{sp})$, we have a natural map of $\mathbb{Q}$-varieties

$$\text{Sh}_{d,K_{sp}} \to \text{Sh}_{d,K}$$

that is a Galois cover with Galois group $\Delta(K_{sp}) = \mathbb{A}^\times_f/\mathbb{Q}^>0(K_{sp} \cap \mathbb{A}^\times_f)$. Here, $\mathbb{A}^\times_f$ is viewed as a sub-group of $G_{d,sp}(\mathbb{A}_f)$ via the central embedding $\mathbb{G}_m \hookrightarrow G_{d,sp}$.

3.2. The Shimura varieties $\text{Sh}_{d,K_{sp}}$ and $\text{Sh}_{d,K}$ are carriers of natural families of motives. For simplicity, we will now restrict ourselves to the case where $K$ is admissible and is such that $K_{sp} \subset G_d(\mathbb{Q}_p)$ is the discriminant kernel of $L_{d,\mathbb{Z}_p}$ (2.14). Moreover, we will assume that $K_{sp,p} \subset G_{d,sp}(\mathbb{Q}_p)$ is the pre-image of $K_p$. We set $G_{d,sp}(\mathbb{Z}_{(p)}) = G_{d,sp}(\mathbb{Q}) \cap K_{sp}$; $G_d(\mathbb{Z}_{(p)})$ is defined analogously. Then we have [MP13b 3.3]:

$$\text{Sh}_{d,K_{sp}}(\mathbb{C}) = G_{d,sp}(\mathbb{Z}_{(p)})\backslash (X_d \times G_{d,sp}(\mathbb{A}_f)/K_{sp}).$$

Therefore, every representation $U$ of the discrete group $G_{d,sp}(\mathbb{Z}_{(p)})$ gives rise to a local system $U_B$ on $\text{Sh}_{d,K_{sp},\mathbb{C}}$ in a natural way. A similar statement holds for representations of $G_d(\mathbb{Z}_{(p)})$: they give rise to local systems on $\text{Sh}_{d,K,\mathbb{C}}$.

Fix an isotropic line $F^1V_d \subset V_d$; this corresponds to a maximal parabolic sub-group $P_{sp} \subset G_{d,sp}$ that stabilizes the filtration

$$0 = F^2V_d \subset F^1V_d \subset F^0V_d = (F^1V_d)^\perp \subset F^{-1}V_d = V_d.$$

Every algebraic $\mathbb{Q}$-representation $W$ of $P_{sp}$ gives rise to a vector bundle $W_\mathbb{C}$ on $\text{Sh}_{d,K_{sp},\mathbb{C}}$. Moreover, if this representation is the restriction to $P_{sp}$ of an algebraic representation $W$ of $G_d$, then the vector bundle is actually equipped with a canonical integrable connection, and its

\[3\text{Recall that } K \text{ is neat if, for every } g \in G_d(\mathbb{A}_f), \text{ the discrete group } G(\mathbb{Q}) \cap gKg^{-1} \text{ is torsion-free.}\]
(analytic) horizontal sections give rise to the local system $W_{B,C}$ attached to the corresponding discrete representation of $G_{d,sp}(\mathbb{Z}_{(p)})$. In this case, we will write $W_{dR,C}$ for the attached vector bundle with connection.

Finally, the vector bundle $W_C$ (called an automorphic vector bundle) has a canonical descent $W$ over $\text{Sh}_{d,K_{sp}}$. If it arises from a representation of $G_{d,sp}$, the integrable connection also descends, and we will write $W_{dR}$ for the flat vector bundle obtained over $\text{Sh}_{d,K_{sp}}$. For all this, cf. [M100] §III.2. Again, similar statements hold for representations of the image $P \subset G_d$ of $P_{sp}$ and vector bundles on $\text{Sh}_{d,K}$.

One way to summarize the previous two paragraphs is the following: There exists a canonical automorphic $G_{d,sp}$-torsor $\mathcal{F}_{sp}$ over $\text{Sh}_{d,K_{sp}}$ equipped with an integrable connection. This torsor has a reduction of structure group (though one that does not respect the connection) to a $P_{sp}$-torsor $\mathcal{F}_{sp,p}$. Similar statements hold for $\text{Sh}_{d,K}$.

In particular, all this applies to the representation $L_{d,\mathbb{Z}_{(p)}}(G_{d,\mathbb{Z}_{(p)}})$: it gives rise to a $\mathbb{Z}_{(p)}$-local system $L_B$ over $\text{Sh}_{d,K,C}$. Since it underlies the algebraic $\mathbb{Q}$-representation $V_d$ of $G_d$, there is an attached vector bundle with integrable connection $V_{dR}$ over $\text{Sh}_{d,K}$ such that $L_{B,C} = V_{B,C}$ is the corresponding local system. We can do the same for the representation $L\eta$ attached to the dual lattice. Moreover, the filtration $F^*V$, being stable under $P$, gives rise to the Hodge filtration $F^* V_{dR}$ on $V_{dR}$. Together with $L_B$, this determines a variation of $\mathbb{Z}_{(p)}$-Hodge structures on $\text{Sh}_{d,K,C}$.

Let $H_{(p)} = C(L_{d,\mathbb{Z}_{(p)}})$ be the Clifford algebra of $L_{d,\mathbb{Z}_{(p)}}$, and set $H = H_{(p),\mathbb{Q}} = C(V_d)$. We will view $H$ as a representation of $G_{d,sp}$ via left translation. This gives us a $\mathbb{Z}_{(p)}$-local system $H_{(p),B}$ over $\text{Sh}_{d,K_{sp},C}$ and a vector bundle with integrable connection $H_{dR}$ over $\text{Sh}_{d,K_{sp}}$ corresponding to the local system $H_{B,C}$. Note that the right translation action of $C(L_{d,\mathbb{Z}_{(p)}})$ on $H_{(p)}$ is $G_{d,sp}$-equivariant, and so descends to an action of all the associated sheaves. If we set $F^1 H \subset H$ to be the image of any generator of $F^1 V_d$, we obtain a filtration

$$0 = F^2 H \subset F^1 H \subset F^0 H = H$$

stabilized by $P$. In turn, it gives us the Hodge filtration $F^* H_{dR}$ on $H_{dR}$. Again, this determines a variation of Hodge structures on $\text{Sh}_{d,K_{sp},C}$. Note that $H$ has a natural $\mathbb{Z}/2\mathbb{Z}$-grading arising from that on $C(V_d)$.

3.3. There exists a $G_d$-equivariant idempotent projector $\pi : \text{End}_{C(V_d)}(H) \to \text{End}_{C(V_d)}(H)$ whose image is precisely $V_d$ acting on $H$ via left translations. For $? = B, dR, p, \ell$, set $H^{?(1,1)} = H^? \otimes H^\ell$. It is shown in [M100] §3 that $\pi$ gives rise to an idempotent projector of variations of $\mathbb{Q}$-Hodge structures $\pi_B : H_B^{\otimes(1,1)} \to H_B^{\otimes(1,1)}$, whose image is canonically identified with $V_B$. The attached de Rham projector $\pi_{dR}$ is defined over $\text{Sh}_{d,K_{sp}}$ and identifies $V_{dR}$ with a filtered integrable sub-bundle of $H_{dR}^{\otimes(1,1)}$. Similar statements hold for the induced $\ell$-adic and $p$-adic projectors. The inclusions obtained $V_? \subset H_?^{\otimes(1,1)}$ are precisely the ones arising from the inclusion of $G_d$-representations $V_d \subset H^{\otimes(1,1)} = \text{End}(H)$ arising from the action of $V_d$ on $H$ via left translations. In particular, since this inclusion carries $L_{d,\mathbb{Z}_{(p)}}$ into $H_{(p)}^{\otimes(1,1)}$, we obtain an inclusion $L_p \subset H_p^{\otimes(1,1)}$.

3.4. We will now show that the various sheaves described above attached to $L_{d,\mathbb{Z}_{(p)}}$ and $H_{(p)}$ are realizations of families of motives.

There is a natural reduced trace pairing on the Clifford algebra $C(V_d)$ that induces a perfect pairing $\langle \cdot, \cdot \rangle$ on $H$ with values in $\mathbb{Q}$. For an appropriately chosen $\delta \in C(L_{d,\mathbb{Z}_{(p)}}) \cap C(V_d)^\times$, the pairing $\psi(x, y) = \langle x, \delta y \rangle$ (here, $^\times$ is the canonical anti-involution on $C(V_d)$) is symplectic and induces an embedding of Shimura data

$$(G_{d,sp}, X_d) \hookrightarrow (\text{GSp}(H, \psi), S^\pm).$$
Here, $S^\pm$ is the union of the Siegel half spaces attached to $(H, \psi)$; cf. [MP13b] 3.4. In turn, as explained in (3.9) of loc. cit. and using the notation there as well, this gives us an abelian scheme (up to prime-to-$p$ isogeny; cf. [MP13b] 3.6 for the precise meaning and notation involved.)

\[ f : A^{KS} \to Sh_{d,K_p}, \]

equipped with a map

\[ C(L_d, Z_{(p)}) \to \text{End}(A^{KS})_{(p)}. \]

Now, the local system $R^1 f_*^{an}Z_{(p)}$ is canonically and $C(L_d, Z_{(p)})$-equivariantly with $H_B$, and the relative de Rham cohomology $H^{dR}(A^{KS}/Sh_{d,K_p})$ is compatibly and canonically identified with $H^{dR}$ as a filtered vector bundle with flat connection. We can also now show that the $\mathbb{Z}_p$-sheaf $H_{(p),B,Z_p}$ and the $\mathbb{Q}_\ell$-sheaves $H_{B,Q_\ell}$ (for $\ell \neq p$) descend canonically to sheaves $H_p$ and $H_\ell$, respectively, over $Sh_{d,K_p}$ (use the $p$-adic and $\ell$-adic relative cohomologies of $A^{KS}$).

Using (1.3), (1.7) and the discussion in (3.3), this shows that, for every point $s : \text{Spec} \ F \to Sh_{d,K}$, valued in a field $F$ embeddable in $\mathbb{C}$, the tuple $(V_{dR,s}, \mathcal{V}_{T,s}, L_{p,s})$ consists of realizations of a $Z_{(p)}$-motives $L_s$ over $F$. Moreover, suppose that, for a finite extension $F'/F$, there exists a lift $s_{sp} : \text{Spec} \ F' \to Sh_{d,K_p}$. Let $H_{(p),s_{sp}}$ be the $Z_{(p)}$-motivic corresponding to the degree 1 cohomology of $A^{KS}_{sp}$. Then $L_{sp}$ is identified with a sub-motive of $H^{\otimes (1,1)}_{(p),s_{sp}}$ (in fact, the motive $H^{\otimes (1,1)}_{(p),s_{sp}}$ arises from a motive $H_{(p),s}$ defined over $F$, and the inclusion of motives is also defined over $F$). The group of cycles $AH(H_{(p),s_{sp}}^{\otimes (1,1)})$ on $H_{(p),s_{sp}}$ is naturally identified with $\text{End}(A^{KS}_{sp})_{(p)}$. This means that we can identify $AH(L_{sp})$ with a sub-group of endomorphisms of $A^{KS}_{sp}$. These are called the special endomorphisms of $A^{KS}_{sp}$; cf. [MP13b] §4, and we will write $L(A^{KS}_{sp})$ for the $Z_{(p)}$-module consisting of them.

### 3.5. One can use the family of $Z_{(p)}$-motives $L$ to give a moduli theoretic description of $Sh_{d,K}$.

This is a special case of the results of [Mil94] §3.

Consider the pro-Shimura variety

\[ Sh_{d,K_p} = \lim_{\longrightarrow} Sh_{d,K_p, K_p}. \]

Over it, we have a canonical trivializing isometry

\[ \beta : V_{d,K_p} \cong \sim V_{K_p}. \]

Suppose that $f : T \to Sh_{d,K_p,\mathbb{C}}$ is a map of smooth complex analytic varieties. Then we can attach to it the polarized variation of $Z_{(p)}$-Hodge structures $(f^* L_B, F^* f^* V_{dR,\mathbb{C}})$ and the trivialization $f^* \beta$ of $f^* V_{K_p} = f^* L_B \otimes K_p$. This gives us (cf. [Mil94] 3.10):

**Proposition 3.6.** The above process gives us a canonical identification between the set of maps of analytic varieties $T \to Sh_{d,K_p,\mathbb{C}}$ and the set of isomorphism classes of tuples

\[ (U, F^* (U \otimes Z_{(p)} \mathcal{O}_T), \beta), \]

where:

- $(U, F^* (U \otimes Z_{(p)} \mathcal{O}_T))$ is a polarized variation of $Z_{(p)}$-Hodge structures over $T$ of weight 0.
- $u \in H^0(T, \det(U_{K_p}))$ is a trivialization.
- $\beta : V_{d,K_p} \cong \sim U_{K_p}$ is a trivializing isometry.

**If** $T$ is a smooth algebraic variety over $\mathbb{C}$, **then** we can also identify the set of maps of varieties $T \to Sh_{d,K_p,\mathbb{C}}$ with the same set of isomorphism classes of tuples.
3.7. The above story extends integrally over \( \mathbb{Z}_{(p)} \). For this, choose a perfect\(^4\) positive definite quadratic lattice \((\Lambda, q)\) over \( \mathbb{Z}_{(p)} \) of rank 2 such that \( 2d \) is represented primitively by \( q \); that is, there exists a unimodular element \( v_d \in \Lambda \) such that \( q(v_d) = 2d \). It is always possible to find such a lattice. Then we find that there is an isometric embedding

\[
L_{d, \mathbb{Z}_{(p)}} \hookrightarrow \tilde{L} := \Lambda \oplus (U^{\oplus 2} \oplus E_8^{\oplus 2}) \mathbb{Z}_{(p)}
\]
carrying \( e + df \) to \( v_d \in \Lambda \). Note that \( \tilde{L} \) is a perfect quadratic lattice over \( \mathbb{Z}_{(p)} \). Set \( \tilde{V} = \tilde{L}_Q \), \( \tilde{G}_{sp,(p)} = G\text{Spin}(\tilde{L}), \tilde{G}_{sp} = G\text{Spin}(\tilde{V}), \tilde{G}_{(p)} = SO(\tilde{L}), \) and \( \tilde{G} = SO(\tilde{V}) \). Since \( \tilde{V} \) has signature \((20+, 2-)\), just as above, we can attach Shimura data \((\tilde{G}_{sp}, \tilde{X})\) and \((\tilde{G}, \tilde{X})\) to it. Write \( \tilde{H}_{(p)} \) for the representation of \( \tilde{G}_{sp,(p)} \) on \( C(L) \) by left translation; and set \( \tilde{H} = \tilde{H}_{(p), Q} \).

Note that we have natural embeddings \( G_{d, sp} \hookrightarrow \tilde{G}_{sp} \) and \( G_d \hookrightarrow \tilde{G} \) that induce embeddings of the corresponding Shimura data.

Fix a compact open \( K_{sp} \subset \tilde{G}_{sp}(A_f) \) whose \( p \)-part is \( \tilde{G}_{sp,(p)}(\mathbb{Z}_p) \), the stabilizer of \( C(L)_{\mathbb{Z}_p} \) (this is a hyperspecial compact open of \( \tilde{G}_{sp}(\mathbb{Q}_p) \)), and is such that \( K_{sp} = K_{sp} \cap G_{d, sp}(A_f) \). Let \( \tilde{K} \subset \tilde{G}(A_f) \) be the image of \( K_{sp} \). This gives us a diagram of Shimura varieties, where all the maps are finite and unramified, and the vertical maps are also étale:

\[
\begin{array}{ccc}
\text{Sh}_{d, K_{sp}} & \longrightarrow & \text{Sh}_{K_{sp}} := \text{Sh}_{K_{sp}}(\tilde{G}_{sp}, \tilde{X}) \\
\downarrow & & \downarrow \\
\text{Sh}_{d, K} & \longrightarrow & \text{Sh}_K := \text{Sh}_K(\tilde{G}, \tilde{X}).
\end{array}
\]

(3.7.1)

By making \( \tilde{K}_{sp} \) small enough, we can ensure that both horizontal maps are closed immersions; cf. \([\text{MP13a}, 4.1.5]\). We will assume that this is the case from now on.

3.8. It follows from the main theorem of \([\text{Kis10}]\) that the map \( \text{Sh}_{K_{sp}} \rightarrow \text{Sh}_{\tilde{K}_{sp}} \) extends to a finite étale map \( \mathcal{S}_{\tilde{K}_{sp}} \rightarrow \mathcal{S}_{\tilde{K}} \) of smooth integral canonical models over \( \mathbb{Z}_{(p)} \). In fact, the whole square \([3.7.1]\) has a nice integral model over \( \mathbb{Z}_{(p)} \). This essentially follows from the theory of \([\text{MP13b}]\), but we will give an overview of the main points.

To begin, it follows from \([\text{Kis10}]\) (cf. \([\text{MP13b}, 3.16]\)) that the filtered vector bundle with flat connection \( V_{dr} \) over \( \text{Sh}_K \) attached to the representation \( V \) extends naturally to a filtered vector bundle with flat connection \( \tilde{L}_{dr} \) over the integral model \( \mathcal{S}_{\tilde{K}} \), attached to the representation \( \tilde{L} \) of \( \tilde{G}_{(p)} \). Strictly speaking, the extension is shown to exist over \( \mathcal{S}_{\tilde{K}_{sp}} \), but one gets our statement by finite flat descent.

Moreover, the Kuga-Satake abelian scheme \( A_{KS} \) over \( \text{Sh}_{K_{sp}} \), along with its \( C(L) \)-action, also extends uniquely over \( \mathcal{S}_{\tilde{K}_{sp}} \). If we denote its relative first de Rham cohomology by \( \tilde{H}_{(p), dr} \), then the vector bundle \( \tilde{H}_{(p), dr}^{\otimes (1,1)} \) descends over \( \mathcal{S}_{\tilde{K}} \) (using finite flat descent again, since this assertion is true over the generic fiber), and \( \tilde{L}_{dr} \) embeds as a direct summand of \( \tilde{H}_{(p), dr}^{\otimes (1,1)} \). In fact, there exists a horizontal, idempotent projector \( \pi \) on \( \tilde{H}_{(p), dr}^{\otimes (1,1)} \) whose image is \( \tilde{L}_{dr} \). By general considerations \([\text{MP13b}, 3.22]\), \( \tilde{L}_{dr} \) underlies a crystal of vector bundles \( L_{cris} \) over the crystalline site \( (\mathcal{S}_{\tilde{K}_{sp}}/\mathbb{Z}_p)_{cris} \). Let \( H_{cris} \) be the first relative cohomology of the structure sheaf under the natural map of crystalline sites

\[
(\mathbb{A}_{sp}^2/\mathbb{Z}_p)_{cris} \rightarrow (\mathcal{S}_{\tilde{K}_{sp}}/\mathbb{Z}_p)_{cris}. \]

---

\(^4\)By this we mean that the \( \mathbb{Z}_{(p)} \)-valued pairing induced by \( q \) is perfect.
Then $\widetilde{H}_{\text{cris}}$ is a crystal of vector bundles over $(\mathcal{F}_{K_{sp}},\mathbb{Z}_p)$ and the crystal $\widetilde{H}_{\text{cris}}^\otimes(1,1)$ descends over $(\mathcal{F}_{K_{sp}},\mathbb{Z}_p)$. Moreover, the embedding of de Rham realizations mentioned in the previous paragraph gives us an embedding of crystals $\tilde{L}_{\text{cris}} \subset \widetilde{H}_{\text{cris}}^\otimes(1,1)$.

3.9. If $s : \text{Spec } k \to \mathcal{F}_{K_{sp}}$ is a point valued in a field $k$ of characteristic $p$, we can now define a good notion of special endomorphism of $\widetilde{A}_s^{KS}$: an endomorphism of $\widetilde{A}_s^{KS}$ is special if its crystalline realization is a section of $\tilde{L}_{\text{cris}}$. In fact, given any morphism of schemes $T \to \mathcal{F}_{K_{sp}}$, one can define the notion of a special endomorphism of $\widetilde{A}_T^{KS}$ [MP13b §5] that specializes to the notions described here for characteristic $p$ and in [3.4] for characteristic 0 points. We will write $L(\widetilde{A}_s^{KS})$ for the space of special endomorphisms of $\widetilde{A}_s^{KS}$.

We can now define a finite, unramified scheme $Z_{K_{sp}}(v_d) \to \mathcal{F}_{K_{sp}}$ as in [MP13b 6.3]. Over $\mathcal{F}_{K_{sp}}$, we have a canonical $K_{sp}$-level structure $[\eta]$ on the $A_p$-cohomology $H_{\lambda_p}^1$ (cf. loc. cit. for the precise definition). Over each closed point of $s \to \mathcal{F}_{K_{sp}}$, this level structure provides us with a Galois-stable $K_{sp}$-orbit $[\eta_s]$ of trivializations of the $A_p$-cohomology of $\widetilde{A}_s^{KS}$ compatible with the various additional structures. For any $\mathcal{F}_{K_{sp}}$-scheme, $Z_{K_{sp}}(v_d)$ now parameterizes pairs $(f, [\eta_f])$, where $f \in L(\widetilde{A}_s^{KS})$ and, for every closed point $s \to T$, $[\eta_f]$ induces a Galois-stable $K_{sp}$-orbit $[\eta_{f,s}]$ within $[\eta]$ such that $\eta_{f,s}(v_d) = f_s.\lambda_p^v$. Here $f_s.\lambda_p^v \in \check{V}_{\lambda_p^v, \eta}$ is the étale realization of the special endomorphism $f_s \in L(\widetilde{A}_s^{KS})$.

Lemma 3.10.

1. The tangent space of $Z_{K_{sp}}(v_d)$ at any point $s$ valued in a perfect field $k(s)$ of characteristic $p$, corresponding to a special endomorphism $f$ of $\widetilde{A}_s^{KS}$, can be identified with the vector space

$$\left\{ \text{Isotropic lines in } \tilde{L}_{\text{dR}, s}[\hat{e}] \text{ lifting } F^1 \tilde{L}_{\text{dR}, s} \text{ and perpendicular to } f_{\text{dR}} \right\}.$$  

Here, $f_{\text{dR}} \in \tilde{L}_{\text{dR}, s}$ is the de Rham realization of $f$.

2. For every point $s : \text{Spec } k \to Z_{K_{sp}}(v_d)$, equipping $\widetilde{A}_s^{KS}$ with a special endomorphism $f$ of degree $2d$, the complete local ring $R_f := \hat{\mathcal{O}}_{Z_{K_{sp}}(v_d), s}$ is a quotient of $R := \hat{\mathcal{O}}_{\mathcal{F}_{K_{sp}}, s}$. It pro-represents the deformation problem for the endomorphism $f$ over $R'$ and is defined by a single equation, not divisible by $p$.

Proof. (1) follows easily from Grothendieck-Messing theory; cf. [MP13b 5.11].

For (2), that the complete local rings of $Z_{K_{sp}}(v_d)$ are deformation rings of the described kind is immediate from the definition. That the complete local rings are defined by a single equation indivisible by $p$ follows from [MP13b 5.13], which is essentially a transliteration of the proofs of [Del81 1.5.1.6].

Let $Z_{K_{sp}}^{pr}(v_d) \subset Z_{K_{sp}}(v_d)$ be the open locus where the de Rham realization of the universal special endomorphism $f$ does not vanish.

Lemma 3.11. $Z_{K_{sp}}^{pr}(v_d)$ contains $Z_{K_{sp}}(v_d)$ and is healthy regular with normal special fiber. In fact, unless $\nu_p(d) = 1$, $Z_{K_{sp}}^{pr}(v_d)$ is smooth. If $\nu_p(d) \leq 1$, then

$$Z_{K_{sp}}^{pr}(v_d) = Z_{K_{sp}}(v_d).$$

Proof. If $\nu_p(d) \leq 1$, then the lattice $L_d$ is maximal in the sense of [MP13b §2], and (6.15) of loc. cit. shows that $Z_{K_{sp}}^{pr}(v_d) = Z_{K_{sp}}(v_d)$ is healthy regular with normal special fiber. If $\nu_p(d) \geq 2$, then, for any point $s : \text{Spec } k \to Z_{K_{sp}}^{pr}(v_d)$, giving rise to a special endomorphism $f$ of $\widetilde{A}_s^{KS}$, an argument similar to that in the proof of [2.11] shows that the de Rham realization $f_{\text{dR}}$ has
non-zero image in \(gr^0_F \widetilde{L}_{dR,s}\). Now (3.10) shows that the tangent space of \(Z_{K_{sp}}^\text{pr}(v_d)\) at \(s\) has dimension 19, and so \(Z_{K_{sp}}^\text{pr}(v_d)\) is smooth at \(s\).

3.12. It is shown in [MP13b, 6.5] that we have:

\[
Z_{K_{sp}}(v_d)_{\overline{\mathbb{Q}}} = \bigsqcup_{[(v,g)]} \text{Sh}_{[(v,g)]}.
\]

Here, \([(v,g)]\) ranges over the images in \(\widetilde{G}(\mathbb{Z}(p))\backslash (\tilde{L} \times \widetilde{G}(\mathbb{A}_f^p)/\mathbb{K}^p)\) of pairs \((v,g)\) that satisfy \(g(v_d) = v \in \widetilde{V}_K\). \(\text{Sh}_{[(v,g)]}\) is a connected component of the GSpin Shimura variety attached to the quadratic space \(\langle v \rangle^\perp\) and level subgroup \(g\mathbb{K}g^{-1} \cap G_{v,sp}(\mathbb{A}_f)\). Here, \(G_v \subset \widetilde{G}\) is the stabilizer of \(v\), and \(G_{v,sp} \subset \widetilde{G}_{sp}\) is its pre-image. So \(\text{Sh}_{d,K_{sp}}\) is an open and closed sub-scheme of \(Z_{K_{sp}}(v_d)_{\overline{\mathbb{Q}}}\); in fact, we have:

\[
\text{Sh}_{d,K_{sp},\overline{\mathbb{Q}}} = \bigsqcup_{[(v,d,g)]} \text{Sh}_{[(v,d,g)]},
\]

where \(g \in G_{d,sp}(\mathbb{A}_f^p)\).

**Proposition 3.13.** Let \(\mathcal{I}_{d,K_{sp}}\) be the normalization of the Zariski closure of \(\text{Sh}_{d,K_{sp}}\) in \(Z_{K_{sp}}(v_d)\), and let \(\mathcal{I}_{d,K_{sp}}^\text{pr}\) be the Zariski closure of \(\text{Sh}_{d,K_{sp}}\) in \(Z_{K_{sp}}^\text{pr}(v_d)\). Then:

1. \(\mathcal{I}_{d,K_{sp}}^\text{pr}\) is an open sub-scheme of \(\mathcal{I}_{d,K_{sp}}\).
2. \(\mathcal{I}_{d,K_{sp}}\) is a healthy regular scheme over \(\mathbb{Z}(p)\) with geometrically normal, lci special fiber.
   It is smooth unless \(\nu_p(d) = 1\).
3. If \(\nu_p(d) \leq 1\), \(\mathcal{I}_{d,K_{sp}}^\text{pr} = \mathcal{I}_{d,K_{sp}}\).
4. Any map \(T \to \mathcal{I}_{d,K_{sp}}\) with \(T\) smooth over \(\mathbb{Z}(p)\) factors through \(\mathcal{I}_{d,K_{sp}}^\text{pr}\).

**Proof.** Assertions (1), (2) and (3) are immediate from (3.11), and (4) follows from [MP13b, 6.7].

**Lemma 3.14.** The endomorphism sheaf (say, over the big étale site) \(\text{End}(\tilde{A}_{KS})_{(p)}\) of endomorphisms of the abelian scheme (up to prime-to-\(p\) isogeny) \(\tilde{A}_{KS}\) has a natural descent over \(\mathcal{I}_{K}\).

**Proof.** Set

\[
\Delta(\tilde{K}_{sp}) := \mathbb{A}_f^\times / \mathbb{Q} \cap \tilde{K}_{sp} \cap \mathbb{A}_f^\times = \mathbb{A}_f^\times / \mathbb{Q}(\tilde{K}_{sp} \cap \mathbb{A}_f^\times).
\]

This is a finite group, and is in fact the Galois group of the étale cover \(\mathcal{I}_{K_{sp}} \to \mathcal{I}_{K}\). Given \([z] \in \Delta(\tilde{K}_{sp})\) attached to an element \(z \in \mathbb{A}_f^\times\), its action on \(\mathcal{I}_{K_{sp}}\) carries \((\tilde{A}_{KS}, [\eta_{KS}])\) to \((\tilde{A}_{KS}, [\eta_{KS}^z])\). Here, \([\eta_{KS}]\) is the canonical \(\tilde{K}_{sp}\)-level structure on \(\tilde{A}_{KS}\) (cf. [MP13b, 3.12]).

On the other hand, viewing \(z\) as a tuple \((z_\ell)_{\ell \neq p}\), we can construct the element \(u(z) = \prod_{\ell \neq p} \ell^{\nu_{\ell}(z)} \in \mathbb{Z}(p)\). This acts on \(\tilde{A}_{KS}\) by multiplication and induces an isomorphism of pairs

\[
\tilde{A}_{KS}, [\eta_{KS}^z] \cong \tilde{A}_{KS}, [\eta_{KS}]\).
\]

Conjugation by \(u(z)\) now produces a canonical identification between \(\text{End}(\tilde{A}_{KS})_{(p)}\) and \(\text{End}(\tilde{A}_{KS})_{(p)}\). This gives us the descent datum that allows us to descend the sheaf over \(\mathcal{I}_{K}\).
3.15. It follows from (3.14) that, given a map \( T \to \mathcal{S}_K \), even if the map does not factor through \( \mathcal{S}_{K_{sp}} \), and even if there is no canonical abelian scheme \( A_T^{KS} \) attached to \( T \), it still makes sense to refer to the endomorphism algebra \( \mathcal{E}(T) := \text{End}(A_T^{KS})_{(p)} \). Moreover, the associated sheaf of algebras \( \mathcal{E} \) comes equipped with canonical realization functors into \( \mathcal{H}_p^{\otimes(1,1)} \) over the generic fiber, and into \( \mathcal{H}^{\otimes(1,1)}_{\text{cris}} \) over the special fiber. Since the family of motives \( \mathcal{L} \), including its crystalline realization over the special fiber, is defined over \( \mathcal{S}_K \), we can speak of the space of ‘special endomorphisms’ \( \mathcal{L}(T) \subset \mathcal{E}(T) \), whose realizations at every closed point land in \( \mathcal{L}_\nu \subset \mathcal{H}^{\otimes(1,1)} \), where \( \nu = p \), for a point in the generic fiber, and \( \nu = \text{cris} \), for a point in the special fiber. If \( T \) is in fact an \( \mathcal{S}_{K_{sp}} \)-scheme, then we will have \( \mathcal{L}(T) = L(A_T^{KS}) \).

3.16. For each prime \( \ell \neq p \), \( \det(\mathcal{V}_\ell) = \Lambda^2 \tilde{V}_\ell \) is a locally constant \( \ell \)-adic sheaf on \( \mathcal{S}_K \) of constant rank 1. To see this, we note that it is the \( \ell \)-adic sheaf attached to the determinant character \( K_\ell \to \text{GL}(\det(\tilde{V}))(\mathbb{Q}_\ell) = \mathbb{Q}_\ell^\times \). But of course, the determinant is trivial on \( K_\ell \subset \tilde{G}(\mathbb{Q}_\ell) \).

Fix a basis element \( \mathcal{e} \in \det(\tilde{V}) \); for each prime \( \ell \neq p \), this induces a trivialization \( \mathcal{e}_\ell : \mathbb{Q}_{\ell^2} \isom to \det(\tilde{V}_\ell) \). Let \( \mathcal{e}_f \) be the induced trivialization of the \( \mathbb{A}_f^p \)-sheaf \( \mathcal{V}_{\tilde{f}} \). Let \( \tilde{I} \) be the functor such that, for every \( \mathcal{S}_K \) scheme \( T \), we have:

\[
\tilde{I}(T) = \left\{ \text{Isometries } \mathcal{V}_{\tilde{f}} \mapsto \mathcal{V}_{\tilde{f},T} \text{ such that } \mathcal{e}(\mathcal{e}_\ell) = \mathcal{e}_f \right\}.
\]

We can treat this as a sheaf, say in the pro-\( \acute{e} \text{tale} \) site over \( \mathcal{S}_K \). As such, it has a natural right action by the locally constant sheaf of groups attached to \( \tilde{G}(\mathbb{A}_f^p) \). A \( \mathbb{A}_f^p \)-level structure over \( T \) is a section \( [\eta] \in H^0(T, \tilde{I}/\mathbb{K}^p) \).

If \( f \in \tilde{I}(T) \), then we can define a sub-sheaf \( I_f \subset \tilde{I}_T \), where, for any \( T \)-scheme \( T' \):

\[
I_f(T') = \{ \eta_f \in \tilde{I}(T') : \eta_f(v_d) = f \}.
\]

We define \( Z_K(v_d) \) to be the scheme over \( \mathcal{S}_K \) such that, for any \( T \to \mathcal{S}_K \), we have:

\[
Z_K(v_d)(T) = \{ (f, [\eta_f]) : f \in \tilde{I}(T); [\eta_f] \text{ a } \mathbb{A}_f^p \text{-level structure on } (T, f) \}.
\]

Here, a \( \mathbb{A}_f^p \)-level structure on \( (T, f) \) is a section of \( I_f/K^p \) over \( T \).

**Lemma 3.17.** The composition \( \mathcal{S}_{d,K_{sp}} \to \mathcal{S}_{K_{sp}} \to \mathcal{S}_K \) factors through a surjective finite étale map \( \mathcal{S}_{d,K_{sp}} \to \mathcal{S}_{d,K} \), whose generic fiber is canonically identified with \( \text{Sh}_{d,K_{sp}} \to \text{Sh}_{d,K} \). Similarly, the restriction to \( \mathcal{S}_{d,K_{sp}}^{pr} \) factors through a surjective finite étale map \( \mathcal{S}_{d,K_{sp}}^{pr} \to \mathcal{S}_{d,K}^{pr} \).

Moreover, the finite group

\[
\Delta(K_{sp}) := \mathbb{A}_f^p/\mathbb{Q}_p > 0(K_{sp} \cap \mathbb{A}_f^p) = \mathbb{A}_f^p/\mathbb{Z}_p > 0(K_{sp} \cap \mathbb{A}_f^p)
\]

acts naturally on \( \mathcal{S}_{d,K_{sp}} \) (resp. \( \mathcal{S}_{d,K_{sp}}^{pr} \)) via prime-to-\( p \) Hecke correspondences, and \( \mathcal{S}_{d,K} \) (resp. \( \mathcal{S}_{d,K}^{pr} \)) is the quotient by this action.

**Proof.** One checks easily from the definitions that \( Z_K(v_d) \) is finite and unramified over \( \mathcal{S}_K \), equipped with a natural surjective finite étale map \( Z_{K_{sp}}(v_d) \to Z_K(v_d) \). Moreover, working over \( \mathbb{C} \), it can be checked that the composition \( \text{Sh}_{d,K_{sp}} \to Z_{K_{sp}}(v_d)_{\mathbb{Q}} \to Z_K(v_d)_{\mathbb{Q}} \) factors through an open and closed embedding \( \text{Sh}_{d,K} \to Z_K(v_d)_{\mathbb{Q}} \). We can now take \( \mathcal{S}_{d,K} \) to be the normalization of the Zariski closure of \( \text{Sh}_{d,K} \) in \( Z_K(v_d) \).

Also, define \( Z_{K_{sp}}^{pr}(v_d) \subset Z_K(v_d) \) to be the image of \( Z_{K_{sp}}(v_d) \), and \( \mathcal{S}_{d,K}^{pr} \) to be the Zariski closure of \( \text{Sh}_{d,K} \) in \( Z_{K_{sp}}^{pr}(v_d) \).

The assertion about the action of \( \Delta(K_{sp}) \) can now be checked over the generic fiber, and hence over \( \mathbb{C} \), where it is evident from the complex uniformizations of the Shimura varieties. \( \Box \)
Definition 3.18. A pro-scheme $X$ over $\mathbb{Z}_p$ satisfies the **extension property** if, for any healthy regular $\mathbb{Z}_p$-scheme $S$, any map $S \otimes \mathbb{Q} \to X$ extends to a map $S \to X$. We say that it satisfies the **restricted extension property** if the same condition holds for all smooth $\mathbb{Z}_p$-schemes $S$.

If $X$ is a healthy regular pro-scheme with the extension property, then we will say that $X$ is an **integral canonical model** for $X_{\mathbb{Q}}$. If $X$ is a smooth $\mathbb{Z}_p$-pro-scheme with the restricted extension property, we will say that $X$ is a **smooth integral canonical model** for $X_{\mathbb{Q}}$.

Proposition 3.19. Set

$$\mathcal{I}_{d,K_p} = \varprojlim_{K_p \in \mathcal{G}_d(K_p)} \mathcal{I}_{d,K_p,K_p}.$$  

Then $\mathcal{I}_{d,K_p}$ has the extension property, and if $\nu_p(d) \leq 1$, it is an integral canonical model for $\text{Sh}_{d,K_p}$. Define $\mathcal{I}_{d,K_p}^{pr}$ analogously. Then, for $\nu_p(d) \geq 2$, $\mathcal{I}_{d,K_p}$ is a smooth integral canonical model for $\text{Sh}_{d,K_p}$.

Proof. Given (3.10) and (3.13), it is enough to show that $\mathcal{I}_{d,K_p}$ has the extension property.

We first consider $\mathcal{I}_{d,K_p^{pr}}$: This will be the inverse limit

$$\varprojlim_{K_p \in \mathcal{G}_d(K_p)} \mathcal{I}_{d,K_p^{pr},K_p}.$$  

It follows from the argument in [MP13b, 6.18] that $\mathcal{I}_{d,K_p^{pr}}$ has the extension property, and now one argues as in the proof of [Moo98, 3.21.4] to show that the finite étale quotient $\mathcal{I}_{d,K_p}$ also has the extension property. Note that we need the following easy fact: A finite étale cover of a healthy regular scheme is again healthy regular; indeed, this follows from fppf descent for abelian schemes. □

3.20. We will now work over the integral model $\mathcal{I}_{d,K_p}^{pr}$ for $\text{Sh}_{d,K_p}$. Since this model is healthy regular, one sees from the argument in [MP13b, 6.26] that the Kuga-Satake abelian scheme $A^{KS} \to \text{Sh}_{d,K_p}$ extends uniquely to an abelian scheme over $\mathcal{I}_{d,K_p}^{pr}$. We will denote this abelian scheme again by $A^{KS}$, and we will from now on write $A^{KS}$ for its generic fiber over $\text{Sh}_{d,K_p}$. In particular, the first relative de Rham cohomology of $A^{KS}$ gives us a natural extension $H^{(p)} \to \mathcal{I}_{d,K_p}^{pr}$ of the filtered vector bundle with connection $H_{dR}$ over $\text{Sh}_{d,K_p}$.

Moreover, the crystalline cohomology of $A^{KS}_p$ gives rise to a crystal of vector bundles $H_{\text{cris}}$ over $\mathcal{I}_{d,K_p}^{pr}$.

Now, the de Rham realization $f_{dR}$ of the tautological special endomorphism $f$ of $\tilde{A}^{KS}$ over $\mathcal{I}_{d,K_p}^{pr}$ gives rise to a canonical trivial rank 1 sub-bundle $(f_{dR}) \subset \tilde{L}_{dR,\mathcal{I}_{d,K_p}^{pr}}$. Set

$$L_{dR} = (f_{dR})^\perp \subset \tilde{L}_{dR,\mathcal{I}_{d,K_p}^{pr}}.$$  

The following lemma is now immediate from (3.10) (1):

Lemma 3.21. The tangent space of $\mathcal{I}_{d,K}^{pr}$ at any point $s$ valued in a perfect field $k(s)$ of characteristic $p$ can be identified with the vector space

$$\left\{ \text{Isotropic lines in } L_{dR,s}[\epsilon] \text{ lifting } F^1L_{dR,s} \right\}.$$  

Similarly, the crystalline realization $f_{\text{cris}}$ of $f$ gives rise to canonical rank 1 sub-crystal $(f_{\text{cris}}) \subset \tilde{L}_{\text{cris}}$ over $(\mathcal{I}_{d,K_p}^{pr},\mathbb{F}_p/\mathbb{Z}_p)_{\text{cris}}$. Set

$$L_{\text{cris}} = (f_{\text{cris}})^\perp \subset \tilde{L}_{\text{cris}}.$$
3.22. Suppose that we are given a point \( s : \text{Spec} \, k \to \mathcal{X}_{d, K_p}^\text{pr} \) valued in a perfect field \( k \). Set \( W = W(k) \), and let \( \mathcal{C}_p \) be the \( p \)-adic completion of an algebraic closure of \( W_Q \), and let \( \mathcal{O}_{\mathcal{C}_p} \) be its ring of integers. For each \( n \in \mathbb{Z} \), let \( \mathcal{O}_{\mathcal{C}_p}(-n) \) (resp. \( \mathcal{C}_p(-n) \)) be the rank 1 free module over \( \mathcal{O}_{\mathcal{C}_p} \) (resp. \( \mathcal{C}_p \)) on which \( \Gamma = \text{Gal}(\overline{W_Q}/W_Q) \) acts semi-linearly via the \( n \)-th-power of the \( p \)-adic cyclotomic character.

Let \( \tilde{s} : \text{Spec} \, W \to \mathcal{X}_{d, K_p}^\text{pr} \) be any lift of \( s \), and let \( \tilde{s}_W \) be the induced \( W_Q \)-valued point. Then, by (1.18), \( L_{\tilde{s}_W} \) is an object in \( \text{Mot}_{\AC, \nu, \text{cris}}(W_Q) \), where \( \nu \) is the \( p \)-adic valuation on \( W_Q \).

In particular, we obtain a canonical \( \Gamma \) and \( F \)-equivariant crystalline comparison isomorphism:

\[
(3.22.1) \quad L_{\tilde{s}_W} \otimes B_{\text{cris}} \xrightarrow{\sim} L_{\text{cris}, s} \otimes B_{\text{cris}}.
\]

This in turn induces a \( \Gamma \)-equivariant filtration preserving de Rham comparison isomorphism

\[
(3.22.2) \quad L_{\tilde{s}_W} \otimes B_{\text{dR}} \xrightarrow{\sim} L_{\text{dR}, s} \otimes B_{\text{dR}}.
\]

3.23. From [MP13b, 6.29] we find that there is a canonical embedding of crystals

\[
L_{\text{cris}} \subset H_{\text{cris}}^{\otimes(1,1)}.
\]

This allows us to define a good notion of a special endomorphism of \( A_{KS}^\text{cr} \) even in characteristic \( p \) exactly as in the definition for \( \tilde{A}_{KS}^\text{cr} \) in [KM]. For any point \( s \) of \( \mathcal{X}_{d, K_p}^\text{pr} \), denote by \( L(A_{s, KS}) \) the space of special endomorphisms of \( A_{s, KS}^\text{cr} \).

**Lemma 3.24.** Suppose that we are given \( s : \text{Spec} \, k \to \mathcal{X}_{d, K_p}^\text{pr} \) valued in a perfect field \( k \) of characteristic \( p \) and a non-zero special endomorphism \( f_0 \in L(A_{s, KS}^\text{cr}) \). Let \( R_{f_0} \) be the quotient of the complete local ring of \( \mathcal{X}_{d, K_p}^\text{pr} \) that pro-represents the deformation functor for \( f_0 \). Then at least one irreducible component of \( \text{Spec} \, R_{f_0} \) is flat over \( \mathbb{Z}_p \).

**Proof.** Cf. [MP13b, 6.32]. \( \square \)

**Lemma 3.25.** Let \( \omega_{KS} \) be the canonical bundle for \( A_{KS}^\text{cr} \) over \( \mathcal{X}_{d, K_p}^\text{pr} \). Then there exists an isomorphism of line bundles over \( \mathcal{X}_{d, K_p}^\text{pr} \):

\[
\omega_{KS} \otimes^4 \xrightarrow{\sim} ((F^1 L_{dR})(-1))^\otimes 21.
\]

In particular, \( F^1 L_{dR} \) is a relatively ample line bundle for \( \mathcal{X}_{d, K_p} \) over \( \mathbb{Z}(p) \).

**Proof.** Let \( \tilde{\omega}_{KS} \) be the canonical bundle for \( \tilde{A}_{KS}^\text{cr} \) over \( \mathcal{X}_{K_p}^\text{pr} \). It follows from our construction and [MP13b, 3.24] that there is a canonical isomorphism of line bundles over \( \mathcal{X}_{K_p}^\text{pr} \):

\[
\tilde{\omega}_{KS} \otimes^2 \xrightarrow{\sim} ((F^1 \tilde{L}_{dR})(-1))^\otimes 21.
\]

In particular, over \( \mathcal{X}_{d, K_p}^\text{pr} \), we obtain a canonical isomorphism:

\[
\omega_{KS} \otimes^2 |_{\mathcal{X}_{d, K_p}^\text{pr}} \xrightarrow{\sim} ((F^1 L_{dR})(-1))^\otimes 21.
\]

We have:

\[
\omega_{KS} = \det(F^1 H_{(p), dR}) ; \quad \tilde{\omega}_{KS} = \det(F^1 \tilde{H}_{(p), dR}).
\]

We also have [MP13b, 6.29]:

\[
F^1 \tilde{H}_{(p), dR} \otimes^\text{pr} \mathcal{X}_{d, K_p}^\text{pr} = F^1 H_{(p), dR} \otimes^{C(L_d)} C(\tilde{L}).
\]

It is therefore enough to observe that \( C(\tilde{L}) \) is a free module over \( C(L_d) \mathbb{Z}_p \) of rank 2 [MP13b, 1.2]. \( \square \)
3.26. Assume that $p^2 \nmid d$. We will now study the connected components of $\mathcal{S}_{d,K,\overline{\mathbb{F}}_p}$. The first step towards this is to find a good compactification for this space:

**Proposition 3.27.** $\mathcal{S}_{d,K}$ admits a regular compactification $\overline{\mathcal{S}}_{d,K}$ over $\mathbb{Z}(p)$ such that the complement $\mathcal{D}_{d,K}$ of $\mathcal{S}_{d,K}$ in $\overline{\mathcal{S}}_{d,K}$ is a relative Cartier divisor over $\mathbb{Z}(p)$. Moreover, $\mathcal{D}_{d,K}$ admits an open neighborhood in $\overline{\mathcal{S}}_{d,K}$ that is smooth over $\mathbb{Z}(p)$.

**Proof.** We will only give a sketch of the proof.

We already know that $\overline{\mathcal{S}}_{d,K_{sp}}$ has a regular compactification $\overline{\mathcal{S}}_{d,K_{sp}}$ such that the boundary $\mathcal{D}_{d,K_{sp}}$ is a relative Cartier divisor over $\mathbb{Z}(p)$ [MP13b 8.11]. One also sees from the description in loc. cit. that the compactification can be constructed to be smooth over $\mathbb{Z}(p)$ in a neighborhood of $\mathcal{D}_{d,K_{sp}}$.

Let us be more precise: At the boundary, $\overline{\mathcal{S}}_{d,K_{sp}}$ has a stratification parameterized by equivalence classes of pairs $(\Phi_{sp}, \sigma)$. Here, $\Phi_{sp}$ is a tuple $(M, X^+, g_{sp}, \sigma)$, where $M \subset V_d$ is an isotropic sub-space (so either a plane or a line), $X^+ \subset X_d$ is a connected component, $g_{sp} \in G_{d,sp}(\mathbb{A}_f)$; moreover, $\sigma \subset U(M)(\mathbb{R})$ is a rational polyhedral cone. Here, $P(M) \subset G_{d,sp}$ is the parabolic subgroup stabilizing $M$, and $U(M) \subset P(M)$ is the center of its unipotent radical. Two pairs $(\Phi_{sp}', \sigma') = (M_r, X^+_r, g_{sp, r}, \sigma_r)$ ($r = 1, 2$) are equivalent if there exists $\gamma_{sp} \in G_{d,sp}(\mathbb{Q})$ such that $\gamma_{sp}$ carries the tuple $(M_1, X^+_1, \sigma_1)$ onto the tuple $(M_2, X^+_2, \sigma_2)$, and if $\gamma_{sp} g_{sp,1}$ and $g_{sp,2}$ are in the same double coset of $\text{(3.27.1)}$ $Q(M_2)(\mathbb{A}_f) \backslash G_{d,sp}(\mathbb{A}_f) / K_{sp}$.

Here, $Q(M_2) \subset P(M_2)$ is a certain normal sub-group described in [MP13b 8.2,8.4]. To each tuple $\Phi_{sp}$, we can attach a tower:

$$\Xi_{\Phi_{sp}} \rightarrow C_{\Phi_{sp}} \rightarrow \mathcal{S}_{\Phi_{sp}}$$

where $\mathcal{S}_{\Phi_{sp}}$ is the integral canonical model of a either 0 or 1-dimensional Shimura variety (depending on whether $M$ has dimension 1 or 2, respectively); $C_{\Phi_{sp}}$ is an abelian scheme over $\mathcal{S}_{\Phi_{sp}}$, and $\Xi_{\Phi_{sp}}$ is a torus over $C_{\Phi_{sp}}$ under a torus $E_{\phi_{sp}}$, whose co-character group is a lattice in $U(M)(\mathbb{R})$.

In particular, to each $\sigma \subset U(M)(\mathbb{R})$, we can attach a twisted torus embedding $\Xi_{\Phi_{sp}} \hookrightarrow \Xi_{\phi_{sp}}(\sigma)$.

The stratum $Z_{[(\Phi_{sp}, \sigma)]}$ attached to the equivalence class of $(\Phi_{sp}, \sigma)$ admits an étale neighborhood that is isomorphic to an étale neighborhood of $\Xi_{\Phi_{sp}}(\sigma)$.

There are many choices of compactifications, depending on choices of cone decomposition data for $(G_{d,sp}, X, K_{sp})$ (cf. [MP13a 4.2.14]). We can choose this decomposition data to be smooth; that is, such that the attached toric scheme $E_{\phi_{sp}}(\sigma)$ is smooth, for all cones $\sigma$ in the decomposition. Then, from the above description of the strata, it follows that the attached compactification $\overline{\mathcal{S}}_{d,K_{sp}}$ will be smooth in a neighborhood of $\mathcal{D}_{d,K_{sp}}$.

We can also choose the data so that the action of the finite group $\Delta(K_{sp})$ on $\mathcal{S}_{d,K_{sp}}$ extends to an action on $\overline{\mathcal{S}}_{d,K_{sp}}$. Essentially, we have to choose the cone decomposition for $(G_{d,sp}, X, K_{sp})$ to be one that is appropriately induced from one for $(G_d, X, K)$; cf. [MP13a 4.2.20]. We now take $\overline{\mathcal{S}}_{d,K}$ to be the quotient under this finite group action.

The group action is actually free near the boundary. The key point here is that, for each isotropic space $M \subset V_d$, $Q(M)$ maps isomorphically into $G_d$; this can be checked from the explicit descriptions in loc. cit.

The theory of Pink in [Pin90] shows that the generic fiber $\overline{\mathcal{S}}_{d,K}$ of $\overline{\mathcal{S}}_{d,K}$ admits a stratification parameterized by equivalence classes of pairs $(\Phi, \sigma) = (M, X^+, g, \sigma)$ exactly as above, except that $g \in G_d(\mathbb{A}_f)$ and instead of $\text{(3.27.1)}$, we consider double cosets in $Q(M_2)(\mathbb{A}_f) \backslash G_d(\mathbb{A}_f) / K$.

---

5In loc. cit., this ‘induction’ is only shown for closed immersions, but the idea is easily adapted to our situation, where we have a finite central cover instead.
The action of $\Delta(K_{sp})$ on $\mathcal{F}_{d,K_{sp}}$ has underlying it an action on the equivalence classes $[\Phi_{sp}]$, the action simply being translation of the adèlic part. This can be checked over the generic fiber; cf. [Pin90]. From this, one finds that, for any fixed $[\Phi]$ for $G_d$, $\Delta(K_{sp})$ acts simply transitively on the set of equivalence classes $[\Phi_{sp}]$ mapping onto $[\Phi]$. Using the description of the strata in [MP13b, 8.13], one can now show that each stratum attached to an equivalence class $[(\Phi_{sp}, \sigma)]$ has an étale neighborhood mapping isomorphically into an étale neighborhood of $\mathcal{F}_{d,K}$.

The required properties of $\mathcal{F}_{d,K}$ are now immediate. □

**Corollary 3.28.** Suppose that $\nu_p(d) \leq 1$. Then the geometrically connected components of $\text{Sh}_{d,K}$ and of $\mathcal{F}_{d,K,\mathcal{F}_p}$ are in natural bijection.

**Proof.** $\mathcal{F}_{d,K}$ is a proper, regular $\mathbb{Z}(p)$-scheme with normal geometric fibers containing $\mathcal{F}_{d,K}$ as a fiber-by-fiber dense open sub-scheme. Therefore, it is sufficient to prove that the geometrically connected components of the generic and special fibers of $\mathcal{F}_{d,K}$ are in natural bijection. This follows from Zariski’s connectedness theorem; cf. [DM69, 4.17]. □

**Remark 3.29.** This result on connected components should be valid for arbitrary $d$ with $\mathcal{F}_{d,K}$ replaced by $\mathcal{F}^p_{d,K}$. We intend to return to this question in future work. This would imply that $M^0_{2d,\mathcal{F}_p}$ is always geometrically irreducible, without any condition on $d$ (cf. 4.16).

4. **The Kuga-Satake period map over $\mathbb{Z} \left[ \frac{1}{2} \right]$**

4.1. Set

$$M_{2d,K_p,\mathbb{Z}(p)} = \lim_{\substack{\longrightarrow \cr K^p \subset G_d(K^p)}} M_{2d,K_p,K^p,\mathbb{Z}(p)}.$$  

This is a pro-algebraic space over $\mathbb{Z}(p)$, equipped with a natural $G_d(K^p)$-action.

The global Torelli theorem for K3 surfaces over $\mathbb{C}$ is summarized by the following

**Theorem 4.2.** There is a natural period map

$$i_{KS,\mathbb{C}} : M_{2d,K_p,\mathbb{C}} \to \text{Sh}_{d,K_p,\mathbb{C}}.$$  

It is étale, $G_d(K^p)$-equivariant, and restricts to an open immersion on $M^0_{2d,K_p,\mathbb{C}}$.

**Proof.** Over $M^0_{2d,K_p,\mathbb{C}}$, we have the weight 0 polarized variation $\mathbb{Z}(p)$-Hodge structures $(P^2_{B}(1) \otimes \mathbb{Z}(p), F^* P^2_{dR,\mathbb{C}})$. Moreover, by definition of level structures, there exists over the same space a canonical isometry

$$\beta : V_{d,K^p} \cong P^2_{K^p}(1).$$

So the map $i_{KS,\mathbb{C}}$ is the map attached to $(P^2_{B}(1) \otimes \mathbb{Z}(p), F^* P^2_{dR,\mathbb{C}}, \beta)$ by the bijection in [Riz0].

Cf. also [Riz0 Prop. 2.5] for the construction on $M^0_{2d,K_p,\mathbb{C}}$, and [Mau12 5.7] for its extension over $M_{2d,K_p,\mathbb{C}}$. The étaleness, as noted in the cited references, follows from the local Torelli theorem for K3 surfaces. The fact that the restriction to the polarized locus is an open immersion is essentially the global Torelli theorem, for which there are many proofs in the literature; including the original one of Pjateckii-Šapiro-Šafarević [PSS71], a generalization to Kähler surfaces independently by Burns-Rapoport [BR75] and Looijenga-Peters [LP80], as well as another by Friedman [Fri84] for algebraic K3s. For a good summary and yet another proof, cf. [Huy12]. Our adèlic formulation can be found in [Riz0 Prop. 2.10]. □

**Proposition 4.3.** For every point $s \in M_{2d,K_p,\mathbb{Q}}(\mathbb{C})$, there is a canonical isomorphism of $\mathbb{Z}(p)$-motives:

$$L_{i_{KS,\mathbb{C}}(s)}(-1) \cong P^2_{s} \otimes \mathbb{Z}(p).$$

⁶Recall our sign conventions from [23].
Proof. This is shown as in the proof of [DMOSS2 II.6.26(d)]. Here are some more details: To begin, from the very construction of \( t_{KS,C} \) there exists a canonical isometry

\[
\eta_B : t_{KS,C}^* L_B \to \mathbb{P}^2_B \otimes \mathbb{Z}_{(p)}
\]

of polarized variations of \( \mathbb{Z}_{(p)} \)-Hodge structures over \( M_{2d,K,p,C}^{an} \). We can view this as a section of the variation of \( \mathbb{Z}_{(p)} \)-Hodge structures (\( t_{KS,C}^* H_B^{(1,1)} \otimes P_B^2(1) \)). After replacing \( M_{2d,K,p,C}^{an} \) by a finite étale cover \( T \), we can view \( H_B^{(1,1)} \) as the relative cohomology sheaf of a family of abelian varieties.

As in loc. cit., we can show by hand that \( \eta_{B,s} \) is absolutely Hodge when \( \mathcal{X}_s \) is a Kummer K3. Now we can appeal to Principle B of [DMOSS2 Ch. I], which states that a horizontal Hodge cycle (on a family of smooth projective varieties over a smooth connected variety) that is absolutely Hodge at one point is absolutely Hodge everywhere. To apply this, we have to show that every connected component of \( T \) contains a Kummer point. Since \( M_{2d,C} \) is irreducible (cf. [11]), it suffices to exhibit a single Kummer surface over \( \mathbb{C} \) equipped with a primitive quasi-polarization of degree \( 2d \).

Let \( A \) be an abelian surface over \( \mathbb{C} \) equipped with a polarization \( \lambda \) of degree \( 2d \). Then the Kummer surface \( X \) attached to \( A \) is constructed as follows: One takes the blow-up \( \tilde{A} \) of the 2-torsion in \( A \), and then quotients \( \tilde{A} \) by the action of the canonical lift \( \iota \) of the involution \( [-1] \) on \( A \) given by multiplication by \(-1\). Any polarization on \( A \) gives rise to an ample class \( \lambda \in \text{NS}(A) \) and the pull-back of \( 2\lambda = \lambda + [-1] \lambda \) over \( A \) descends to a quasi-polarization \( \xi \in \text{NS}(X) \). Moreover, if the polarization is of degree \( d^2 \), then by Riemann-Roch [Mum70 III.16], \( \lambda \) has self-intersection \( 2d \), and, since \( \tilde{A} \to X \) is a degree 2 map of smooth surfaces, \( \xi \) has self-intersection \( 2d \) as well.

So, to finish, we have to construct an abelian surface \( A \) with a primitive polarization of degree \( d^2 \). For this, take \( A = E \times E \), with \( E \) an elliptic curve, and the polarization to be the endomorphism \( f \times (f \circ [d]) \), where \( f : E \to E' \) is the canonical polarization of \( E \). \( \square \)

Corollary 4.4 (Rizov). \( t_{KS,C} \) descends to a map

\[
t_{KS,Q} : M_{2d,K,p,Q} \to \text{Sh}_{d,K,p}.
\]

Proof. This is essentially [Riz 3.16] (cf. also [Mau12 5.7]). Rizov shows that the map descends over \( \mathbb{Q} \) by proving the existence of a dense set of ‘CM points’, for which the reciprocity law is compatible with Shimura-Taniyanma reciprocity for CM points on the canonical model \( \text{Sh}_{d,K} \).

But we will provide a different proof, using the theory of motives for absolute Hodge cycles. Indeed, it is enough to see that, for every \( \sigma \in \text{Aut}(\mathbb{C}) \), \( t_{KS,C} \circ \sigma = \sigma \circ t_{KS,C} \). For this, from [3.6], it is enough to see that both maps induce the same tuples (up to isomorphism) over \( M_{2d,K,C} \).

This is easy to deduce from the following consequence of [1.3]: For every \( s \in M_{2d,K,p,C} \), there is a canonical isomorphism of \( \mathbb{Z}_{(p)} \)-Hodge structures:

\[
L_{\sigma(t_{KS,C}(s)(s))}(-1) \cong P_{\sigma(s)}^2 \otimes \mathbb{Z}_{(p)} \cong L_{t_{KS,C}(s)(s)}(-1).
\]

\( \square \)

4.5. Fix \( K^p \subset G_d(\mathbb{A}_f^p) \) small enough so that \( K = K^p K_p \) is neat. Then \( t_{KS,Q} \) induces a map \( M_{2d,K,Q} \to \text{Sh}_{d,K} \), which we will again denote by the same symbol.

For the sake of convenience, given any sheaf \( F \) over \( \text{Sh}_{d,K} \) (with respect to any of the natural Grothendieck topologies), we will denote its pull-back along \( t_{KS,C} \) again by the same letter \( F \). This will apply in particular to the various realizations of the family of \( \mathbb{Z}_{(p)} \)-motives \( L \).

Over \( M_{2d,K,Q} \), we also have the family of \( \mathbb{Z} \)-motives \( P^2 \) attached to the primitive cohomology of the universal family of quasi-polarized K3 surfaces. In fact, the family of \( \mathbb{Z}_{(p)} \)-motives, \( P^2 \otimes \mathbb{Z}_{(p)} \) can be identified with \( L(-1) \), as we will see shortly.
Via the de Rham comparison isomorphism, \( \eta_B \) gives rise to a canonical isometry of polarized filtered vector bundles with flat connection:

\[
\eta_{\text{dR}, \mathbb{C}} : V_{\text{dR}, \mathbb{C}}(-1) \xrightarrow{\sim} P_{\text{dR}, \mathbb{C}}^2.
\]

That this isometry is algebraic follows from [Del70] and the fact that both flat bundles have regular singularities along the boundary divisor in a suitable compactification of \( \mathcal{M}^p_{2d,K,\mathbb{C}} \).

Via Artin’s comparison isomorphisms, we also obtain compatible isometries of polarized local systems on \( \mathcal{M}^p_{2d,K,\mathbb{C}} \):

\[
\eta_{\ell} : V_{\ell}(-1) \xrightarrow{\sim} P_{\ell}^2 \otimes \mathbb{Q}_{\ell}, \text{ for } \ell \neq p
\]

\[
\eta_p : L_p(-1) \xrightarrow{\sim} P_p^2.
\]

**Proposition 4.6.**

1. For each prime \( \ell \), the isometry \( \eta_{\ell} \) is defined over \( \mathcal{M}^p_{2d,K,\mathbb{Q}} \).
2. The isomorphism \( \eta_{\text{dR}, \mathbb{C}} \) descends to an isometry

\[
\eta_{\text{dR}, \mathbb{Q}} : V_{\text{dR}}(-1) \xrightarrow{\sim} P_{\text{dR}, \mathbb{Q}}^2
\]

of filtered polarized vector bundles with flat connection over \( \mathcal{M}^p_{2d,K,\mathbb{Q}} \).
3. For every point \( s : \text{Spec } F \to \mathcal{M}^p_{2d,K,\mathbb{Q}} \), there is a canonical isometry of \( \mathbb{Z}(p) \)-motives

\[
L_s(-1) \xrightarrow{\sim} P_s^2 \otimes \mathbb{Z}(p).
\]

In particular, \( P^2_s \) is a motive in \( \text{Mot}_{\text{Ab}}(F) \) with \( \mathbb{Z}(p) \)-structure.
4. If \( \nu : F \to \mathbb{Z} \) is a discrete valuation on \( F \), then the isomorphism \( L_s(-1) \xrightarrow{\sim} P_s^2 \otimes \mathbb{Z}(p) \)

is a map of motives in \( \text{Mot}_{\text{Ad},v}(F) \) with \( \mathbb{Z}(p) \)-structure (cf. [1.10]).

**Proof.** Over \( \mathcal{M}^p_{2d,K,\mathbb{Q}} \), we have canonical trivializations

\[
V_{d,K} - P_{d,K}^2
\]

By construction, over \( \mathcal{M}^p_{2d,K,\mathbb{C}} \), the composition

\[
V_{d,K}(1) \xrightarrow{\eta_{\text{dR}, \mathbb{Q}}} P_{d,K}^2
\]

is the identity, which certainly descends over \( \mathcal{M}^p_{2d,K,\mathbb{Q}} \). This shows (1), for \( \ell \neq p \).

By (3), given a point \( s \in \mathcal{M}^p_{2d,K,\mathbb{C}}(\mathbb{C}) \), the isometry of Hodge structures

\[
\eta_{B,s} : L_{B,s}(-1) \xrightarrow{\sim} P_{B,s}^2 \otimes \mathbb{Z}(p)
\]

is absolutely Hodge. If we are now given a point \( s \in \mathcal{M}^p_{2d,K,\mathbb{Q}}(F) \), where \( F \) is a field of characteristic 0 that is embeddable in \( \mathbb{C} \), using (1) for \( \ell \neq p \), (1.5) and the previous paragraph, we find that there exists a unique isometry of polarized \( \mathbb{Z}(p) \)-motives

\[
\eta_s : L_s(-1) \xrightarrow{\sim} P_s^2 \otimes \mathbb{Z}(p)
\]

such that, for any embedding \( \tau : F \hookrightarrow \mathbb{C} \), it induces the realizations \( \eta_{B,\tau(s)} \), \( \eta_{\ell,\tau(s)} \) and \( \eta_{\text{dR},\tau(s)} \). This shows (3).

Applying this to the generic points of \( \mathcal{M}^p_{2d,K,\mathbb{Q}} \), we get (1) for \( \ell = p \), as well as (2).

(4) now follows from the argument used for the proof of [Bla94 3.1(3)]; cf. also the proof of [1.19].

By (3.19), we have the normal \( \mathbb{Z}(p) \)-model \( \mathcal{Y}_{d,K} \) for \( \text{Sh}_{d,K} \) that enjoys the extension property (3.18).
**Proposition 4.7.** \( \iota_{\mathbb{Q}}^{KS} \) extends to a \( G_d(\mathbb{A}_f^p) \)-equivariant map

\[
i_{\mathbb{Q}}^{KS} : M_{2d,K_p,\mathbb{Z}(p)} \rightarrow \mathcal{I}_{d,K_p}.
\]

In fact, the map factors through the healthy regular sub-scheme \( \mathcal{I}_{d,K_p}^{pr} \subset \mathcal{I}_{d,K_p} \).

**Proof.** The existence of \( \iota^{KS} \) is simple: We know from (2.13) that \( M_{2d,K_p,\mathbb{Z}(p)} \) is healthy regular, and so we only have to invoke the extension property of \( \mathcal{I}_{d,K_p} \). It follows from (2.13) and (3.13) that \( \iota^{KS} \) factors through \( \mathcal{I}_{d,K_p}^{pr} \).

\[\square\]

The main result of this section is:

**Theorem 4.8.** The map \( \iota^{KS} \) is étale.

We will need a few preliminaries before we can prove (4.8), the main input being (4.13) below. The proof will appear right below that of loc. cit.

4.9. Fix any \( K^p \subset G(\mathbb{A}_f^p) \) small enough such that \( M_{2d,K,Z(p)} \) is represented by a smooth algebraic space. The following result, which exhibits the integral crystalline nature of the Kuga-Satake construction is the key to the proof of (4.8); cf [Mau 12, 6.8] for an essentially equivalent statement with stronger hypotheses on \( d \) and \( p \).

For any smooth point \( s \in M_{2d,K,\mathbb{Q}_p}(\mathbb{F}_p) \), the restrictions of \( L_{dR}(1) \) and \( P^2_{dR} \) to \( R = \hat{\mathcal{O}}_{M_{2d,K,Z(p)}},s \) give rise to filtered \( F \)-crystals over \( R \) in the language of (2.5). We will denote these filtered \( F \)-crystals by \( L_{dR,R}(1) \) and \( P^2_{dR,R} \), respectively. Set \( W = W(\mathbb{F}_p) \), and choose a lift \( j : R \rightarrow W \). The reductions of \( L_{dR,R}(1) \) to \( P^2_{dR,R} \) along \( j \) will be denoted \( L_{dR,W}(1) \) and \( P^2_{dR,W} \), respectively. We already know that there is an isomorphism of filtered free \( R_\mathbb{Q} \)-modules:

\[
\eta_{dR,R_\mathbb{Q}} : V_{dR,R_\mathbb{Q}}(1) \rightarrow P^2_{dR,R_\mathbb{Q}}.
\]

**Lemma 4.10.** \( \eta_{dR,R_\mathbb{Q}} \) is an isomorphism of filtered \( F \)-isocrystals over \( R_\mathbb{Q} \).

**Proof.** Let \( R_{\mathbb{Q}}^{an} \) be the ring of functions on the rigid analytic space over \( W_\mathbb{Q} \) attached to \( \text{Spf } R \). Then, by [Kat 73, 3.1], there exist unique \( F \)-equivariant, horizontal isomorphisms that reduce to the identity along \( j^* \):

\[
L_{dR,W}(1) \otimes_W R_{\mathbb{Q}}^{an} \rightarrow L_{dR,R_\mathbb{Q}}(1);
\]

\[
P^2_{dR,W} \otimes_W R_{\mathbb{Q}}^{an} \rightarrow P^2_{dR,R_\mathbb{Q}}.
\]

Here, we equip the left hand sides with the constant connection \( 1 \otimes d \) and the constant \( F \)-structures induced from the ones on \( L_{dR,W}(1) \) and \( P^2_{dR,W} \).

Since \( \eta_{dR,R_\mathbb{Q}} \) is horizontal for the connection, it now suffices to check that the induced isomorphism

\[
\eta_{dR,W_\mathbb{Q}} : L_{dR,W_\mathbb{Q}}(1) \rightarrow P^2_{dR,W_\mathbb{Q}}
\]

is a map of \( F \)-isocrystals. This is shown in [Ogu 84, § 7], but we can provide a different proof with the technology of Section 4.

Let \( \overline{W_\mathbb{Q}} \) be an algebraic closure of \( W_\mathbb{Q} \). Then we have comparison isomorphisms:

\[
L_{p,\overline{W_\mathbb{Q}}} \otimes B_{\text{cris}} \rightarrow L_{dR,W} \otimes B_{\text{cris}};
\]

\[
P^2_{p,\overline{W_\mathbb{Q}}} \otimes B_{\text{cris}} \rightarrow P^2_{dR,W} \otimes B_{\text{cris}}.
\]

The first isomorphism is (3.22.1).

We also have a natural isomorphism of \( \text{Gal}(\overline{W_\mathbb{Q}}/W_\mathbb{Q}) \)-representations:

\[
\eta_{p,\overline{W_\mathbb{Q}}} : L_{p,\overline{W_\mathbb{Q}}}(1) \rightarrow P^2_{p,\overline{W_\mathbb{Q}}}
\]
arising from an isomorphism of motives $L_{W_q}(-1) \cong P_{W_q}^2 \otimes \mathbb{Z}_{(p)}$. It now follows from \[10.6\] that $\eta_{\text{dR},W_q}$ is exactly the map obtained from $\eta_{\text{dR},W_q}$ via the crystalline comparison isomorphisms. In particular, it is $F$-equivariant. \[\square\]

**Lemma 4.11.** With the notation as above, suppose that the K3 surface attached to $s$ is ordinary. Then $\eta_{\text{dR},R_q}$ carries $L_{\text{dR},R(-1)}$ onto $P_{\text{dR},R}^2$.

**Proof.** By the argument in [Man12, 6.15], it is enough to show that, for every map $j : R \to W$ attached to a lift $\tilde{s} : \text{Spec} W \to \mathcal{M}_{2d,K,\mathbb{Z}_{(p)}}$ of $s$, the induced map $\eta_{\text{dR},W_q}$ carries $L_{\text{dR},W}(-1)$ onto $P_{\text{dR},W}^2$.

First, by the Dieudonné-Manin classification [Man02] (cf. also [Kat73, 2.1]), $L_{\text{dR},W}(-1)$ (resp. $P_{\text{dR},W}^2$) admits a canonical largest $F$-stable direct summand $L_{\text{dR},W,0}(-1)$ (resp. $P_{\text{dR},W,0}^2$) to which $F$ restricts to an isomorphism (this is the slope 0 part). In fact, this sub-$F$-crystal must be of rank 1. It suffices to check this for $P_{\text{dR},W}^2(-1)$, for which cf. [Ogu01, p. 327].

Let $U_\bullet L_{\text{dR},W}(-1)$ be the three-step ascending filtration on $L_{\text{dR},W}(-1)$ determined by

$$U_0 L_{\text{dR},W}(-1) = L_{\text{dR},W,0}(-1) ; U_1 L_{\text{dR},W} = L_{\text{dR},W,0}(-1).$$

Analogously define an ascending filtration $U_\bullet L_{\text{dR},W}^2$ on $P_{\text{dR},W}^2$. These are the canonical slope filtrations and are in particular preserved by the $F$-equivariant map $\eta_{\text{dR},W_q}$ after changing scalars to $W_{Q_q}$. Let $C_p$ be the completion of $\mathbb{T}_Q$ and let $\mathcal{O}_{C_p}$ be its ring of integers. It can now be deduced from [BK86, 9.6] that there are ascending $\Gamma$-stable filtrations $U_\bullet L_{p,\mathbb{T}_Q}(-1)$ and $U_\bullet P_{p,\mathbb{T}_Q}^2$ that satisfy the following conditions:

1. The crystalline comparison isomorphisms (for both $L$ and $P^2$) respect the $U$-filtrations on either side (in fact, one can define the $U$-filtrations on the étale side to be the unique ones that satisfy this property).

2. The canonical $\Gamma$-equivariant isomorphism

$$\eta_{p,\mathbb{T}_Q} : L_{p,\mathbb{T}_Q}(-1) \cong P_{p,\mathbb{T}_Q}^2$$

respects $U$-filtrations.

3. For each $n \in \mathbb{Z}$, we have $\Gamma$-equivariant isomorphisms compatible with the comparison isomorphisms:

\begin{align*}
\text{(4.11.1)} & \quad \text{gr}_U^1 L_{p,\mathbb{T}_Q}(-1) \otimes \mathcal{O}_{C_p} \cong \text{gr}_U^1 L_{\text{dR},W}(-1) \otimes \mathcal{O}_{C_p}(-n); \\
\text{(4.11.2)} & \quad \text{gr}_U^1 P_{p,\mathbb{T}_Q}^2 \otimes \mathcal{O}_{C_p} \cong \text{gr}_U^1 P_{\text{dR},W}^2 \otimes \mathcal{O}_{C_p}(-n).
\end{align*}

Now, $\text{gr}_U^1 \eta_{\text{dR},W_q}$ has to be compatible with $\text{gr}_U^1 \eta_{p,\mathbb{T}_Q}$ under the isomorphisms in \[4.11.1\], and $\eta_{p,\mathbb{T}_Q}$ carries $L_{p,\mathbb{T}_Q}(-1)$ onto $P_{p,\mathbb{T}_Q}^2$. Therefore, since $\mathcal{O}_{C_p}$ is faithfully flat over $W$, we find that $\text{gr}_U^1 \eta_{\text{dR},W_q}$ must carry $U_0 L_{\text{dR},W}(-1)$ onto $U_0 P_{\text{dR},W}^2$.

By the strong divisibility of $L_{\text{dR},W}(-1)$ (cf. [MPT13, 3.22]), we must have

$$U_0 L_{\text{dR},W}(-1) \cap F^1 L_{\text{dR},W}(-1) = 0.$$ 

Since $U_0 L_{\text{dR},W}(-1)$ is an isotropic line (this can be seen, for example, from the fact that $F(f) \circ F(f) = p^2(f \circ f)$, for any $f \in L_{\text{dR},W}(-1)$), we obtain a splitting of $U_\bullet L_{\text{dR},W}(-1)$:

$$L_{\text{dR},W}(-1) = F^2 L_{\text{dR},W}(-1) \oplus (F^2 L_{\text{dR},W}(-1) \oplus U_0 L_{\text{dR},W}(-1))^{1} \oplus U_0 L_{\text{dR},W}(-1).$$

We similarly define a splitting for $U_\bullet P_{\text{dR},W}^2$, and the construction shows that these splittings are compatible with $\eta_{\text{dR},W_q}$. Therefore, $\eta_{\text{dR},W_q}$ must indeed carry $L_{\text{dR},W}(-1)$ onto $P_{\text{dR},W}^2$. \[\square\]

We immediately obtain:
Corollary 4.12. If $s$ is an ordinary point, then there is an open neighborhood $U$ of $s$ in $M_{2d,K,Z_p}$ such that $\eta_{dR,U_0}$ carries $L_{dR,U}(-1)$ onto $P^2_{dR,U}$. □

Proposition 4.13. The isometry

$$\eta_{dR,Q} : V_{dR}(-1) \xrightarrow{\sim} P^2_{dR|M_{2d,K,Q}}$$

extends to an isometry (necessarily unique)

$$\eta_{dR} : L_{dR}(-1) \xrightarrow{\sim} P^2_{dR}$$

of filtered vector bundles over $M_{2d,K,Z(p)}$ with integrable connection.

Proof. It is enough to extend $\eta_{dR,Q}$ as a map of vector bundles, since the other requirements can be checked after inverting $p$. Since the ordinary locus of $M_{2d,K,F_p}$ is dense [Ogu79, 2.9], [4.12] shows that there exists an open sub-scheme $U \subset M_{2d,K,R_p}$ of co-dimension 2 such that $\eta_{dR,U_0}$ carries $L_{dR,U}(-1)$ onto $P^2_{dR,U}$. But $M_{2d,K,W}$ is a normal algebraic space, and restriction of vector bundles over a normal algebraic space to the complement of a codimension 2 closed sub-space is a fully faithful operation. Indeed, this is a local statement and is well-known for normal schemes. □

Let $T \to M_{2d,K,F_p}$ be an étale map with $T$ a scheme. Then one can also consider the crystalline realization $P^2_{\text{cris},T}$ of the primitive cohomology of the universal family $X_T \to T$: This will be a crystal of vector bundles over $(T/Z_p)_{\text{cris}}$. At the same time, one also has the crystal $L_{\text{cris},T}(-1)$ over $(T/Z_p)_{\text{cris}}$. In fact, both these crystals have the additional structure of an $F$-crystal. That is, if $\text{Fr}_T : T \to T$ is the absolute Frobenius on $T$, then we have natural maps $\text{Fr}_T^* P^2_{\text{cris}} \to P^2_{\text{cris}}$ and $\text{Fr}_T^* L_{\text{cris}}(-1) \to L_{\text{cris}}(-1)$.

Corollary 4.14. $\eta_{dR}$ induces a canonical isomorphism of $F$-crystals

$$L_{\text{cris},T}(-1) \xrightarrow{\sim} P^2_{\text{cris},T}.$$ 

Proof. If $T$ is smooth (this is always the case unless $\nu_p(d) = 1$), then this follows from [4.13] and [4.10]. Indeed, working locally if necessary, we can assume that $T$ lifts to a smooth map $\tilde{T} \to M_{2d,K,Z_p}$. Now, one can use the classical equivalence of categories between crystals on $T$ and vector bundles over $\tilde{T}$ with integrable connections.

Suppose now that $\nu_p(d) = 1$ and that $T$ is not smooth. Then, according to [2.13], $T$ has at worst isolated singular points with quadratic singularities. Let $T^\text{sm} \subset T$ be the smooth locus. The result now follows from the fact that restriction of crystals of vector bundles from $(T/Z_p)_{\text{cris}}$ to $(T^\text{sm}/Z_p)_{\text{cris}}$ is a fully faithful operation. □

Proof of (4.8). It is enough to show that, for every closed point $s \in M_{2d,K,Z_p}(\overline{\mathbb{F}}_p)$, the induced map of complete local $\mathbb{Z}_p$-algebras

$$\hat{\mathcal{O}}_{M_{2d,K,p},s} \xrightarrow{\sim} \hat{\mathcal{O}}_{M_{2d,K,p}}$$

is an isomorphism. For simplicity denote this map by $R \to R'$. Both $R$ and $R'$ are complete local Noetherian domains of the same dimension, namely 19, so it is enough to show that the induced map of tangent spaces $t_R \to t_{R'}$ is an isomorphism. But, by (5.21), (2.12) and (4.13), both $t_R$ and $t_{R'}$ can be canonically identified with the space

$$\left\{ \text{Isotropic lines } L \subset P^{2}_{dR,s} \otimes \overline{\mathbb{F}}_p[\epsilon] \text{ lifting } F^2 P^{2}_{dR,s} \right\}.$$ 

Under these identifications, the map on tangent spaces is simply the identity. This can be checked, for example, by lifting to characteristic 0. □
We can now prove the theorems stated in the introduction quite easily.

\textbf{Corollary 4.15 (Theorem 4).} The restriction of \( i^{KS} \) to \( \mathcal{M}_{d,K,\overline{\mathbb{Q}}(p)}^{2d} \) is an open immersion.

\textit{Proof.} By [LMBO00 16.5], there exists a finite \( \mathcal{S}_{d,K}^{pr} \)-scheme \( \mathcal{S} \) and a factoring as below, where the top arrow is a dense open immersion.

\[
\begin{array}{c}
\mathcal{M}_{2d,K,\overline{\mathbb{Q}}(p)}^{2d} \\
\downarrow i^{KS} \\
\mathcal{S}_{d,K}^{pr}
\end{array} \quad \begin{array}{c}
\xrightarrow{\sim} \mathcal{S} \\
\downarrow \\
\mathcal{S}_{d,K}^{pr}
\end{array}
\]

Since \( \mathcal{S}_{d,K}^{pr} \) is a normal scheme, and since \( i^{KS} \) restricted to \( \mathcal{M}_{2d,K,\overline{\mathbb{Q}}}^{2d} \) is an open immersion, it follows that \( \mathcal{S} \rightarrow \mathcal{S}_{d,K}^{pr} \) is in fact an isomorphism onto a union of connected components of \( \mathcal{S}_{d,K}^{pr} \). So the restriction of \( i^{KS} \) to \( \mathcal{M}_{2d,K,\overline{\mathbb{Q}}(p)}^{2d} \) is an open immersion. \( \square \)

\textbf{Corollary 4.16 (Theorem 5).} \( \mathcal{M}_{2d,K,\mathbb{F}_p}^{2d} \) is a quasi-projective Deligne-Mumford stack over \( \mathbb{F}_p \). Moreover, the Hodge bundle \( \omega = F^2 H^{2d}_{dR,\mathbb{F}_p} \) is ample over \( \mathcal{M}_{2d,K,\mathbb{F}_p}^{2d} \). If \( \nu_p(d) \leq 1 \), then \( \mathcal{M}_{2d,K,\mathbb{F}_p}^{2d} \) is geometrically irreducible.

\textit{Proof.} The quasi-projectivity is immediate from (4.15) and the quasi-projectivity of \( \mathcal{S}_{d,K}^{pr} \mathbb{F}_p \).

To show ampleness of \( \omega \) it suffices by (4.13) to show the ampleness of \( F^2 L_{dR,\mathbb{F}_p} \). This was shown in (3.26).

Suppose now that \( \nu_p(d) \leq 1 \). Then \( \mathcal{S}_{d,K}^{pr} = \mathcal{S}_{d,K} \), so it follows from (4.15) and (3.28) that every geometrically connected component of \( \mathcal{M}_{2d,K,\mathbb{F}_p}^{2d} \) is the specialization of a geometrically connected component of \( \mathcal{M}_{2d,K,\overline{\mathbb{Q}}}^{2d} \). It is now enough to show that \( \mathcal{M}_{2d,K} \) is an irreducible stack, which is well-known; cf. for example [BHPVdV04 Ch. VIII]. \( \square \)

4.17. Before we present the next result, we make the following remark: \textit{A priori} \( A^{KS} \) is only defined as an abelian scheme over \( \mathcal{S}_{K^{sep},p}^{pr} \) up to prime-to-\( p \) isogeny. However, it comes equipped with a \( K^{sep}_{d,K} \)-level structure: That is, a \( K^{sep}_{d,K} \)-orbit of trivializations \( H^{\alpha_{\ell}}_{2d,K} \) over the pro-\( \acute{e} \text{tale} \) cover \( \mathcal{S}_{K^{sep},p}^{pr} \). Set \( H_\ell = C(L_\ell) \); then the image of \( H_{\overline{\mathbb{F}}_p} \) under any of these trivializations is independent of the choice of trivialization, and defines a canonical \( \overline{\mathbb{F}}_p \)-sub-sheaf \( H_{\overline{\mathbb{F}}_p} \subset H^{\alpha_{\ell}}_{2d,K} \).

In particular, it pins down \( A^{KS} \) as an honest abelian scheme within its prime-to-\( p \) isogeny class; Namely, that whose \( \overline{\mathbb{F}}_p \)-cohomology can be identified with \( H_{\overline{\mathbb{F}}_p} \). It also gives us a sheaf of \( \mathbb{Z} \)-algebras \( \text{End}(A^{KS}) \subset \text{End}(A^{KS}_{K^{sep}}) \) consisting of endomorphisms that stabilize \( H_{\overline{\mathbb{F}}_p} \subset H^{\alpha_{\ell}}_{2d,K} \).

\textbf{Theorem 4.18.} Given any field \( k \) of odd characteristic \( p \) and a polarized \( K3 \) surface \( (X, \xi) \) over \( k \), there exists a finite separable extension \( k'/k \) and an abelian variety \( A \) over \( k' \), the \textbf{Kuga-Satake} abelian variety, which enjoys the following properties:

1. Fix a separable closure \( k^{sep} \) of \( k' \). For every prime \( \ell \neq p \), there exists an isomorphism of \( \mathbb{Z}_\ell \)-modules:
\[
H^1_{\text{et}}(A_{k^{sep}}, \mathbb{Z}_{\ell}) \xrightarrow{\sim} C\left(PH^2_{\text{et}}(X_{k^{sep}}, \mathbb{Z}_{\ell}(1))\right).
\]

Here, the right hand side denotes the Clifford algebra attached to the quadratic lattice \( PH^2_{\text{et}}(X_{k^{sep}}, \mathbb{Z}_{\ell}(1)) \). Moreover, let \( k^{p} \) be a perfect closure of \( k' \); then there exists an isomorphism of \( W(k^{p}) \)-modules:
\[
H^1_{\text{cris}}(A_{k^{p}}/W(k^{p})) \xrightarrow{\sim} C\left(PH^2_{\text{cris}}(X_{k^{p}}/W(k^{p}))(1)\right).
\]
(2) There is an associative $\mathbb{Z}$-algebra $C$ equipped with a map $C \to \text{End}(A)$ such that, for all primes $\ell \neq p$,

$$C \otimes \mathbb{Z}_\ell \subset \text{End} \left( H^1_{\text{ét}} \left( A_{k^{\text{sep}}, \mathbb{Z}_\ell} \right) \right)$$

is Galois-equivariantly identified with $C \left( \text{PH}^2(X_{k^{\text{sep}}, \mathbb{Z}_\ell(1)}) \right)$ acting on itself by right translation via the isomorphism in (1). Similarly, $C \otimes W(k^p)$ is $F$-equivariantly identified with $C \left( \text{PH}^2_{\text{cris}}(X_{k^p}/W(k^p))(1) \right)$.

(3) The action of $C \left( \text{PH}^2_{\text{cris}}(X_{k^{\text{sep}}, \mathbb{Z}_\ell(1)}) \right)$ on itself by left translations induces, via (1), a Galois-equivariant embedding

$$\text{PH}^2_{\text{cris}}(X_{k^{\text{sep}}, \mathbb{Z}_\ell(1)}) \subset \text{End}_{C \otimes \mathbb{Q}_\ell} \left( H^1_{\text{ét}} \left( A_{k^{\text{sep}}, \mathbb{Z}_\ell} \right) \right).$$

Similarly, there is an $F$-equivariant embedding

$$\text{PH}^2_{\text{cris}}(X_{k^p}/W(k^p))(1) \subset \text{End}_{C \otimes W(k^p)} \left( H^1_{\text{cris}}(A_{k^p}/W(k^p)) \right).$$

(4) Let $L(A) \subset \text{End}(A)$ be the sub-space of endomorphisms whose cohomological realizations lie in the image of $\text{PH}^2_{\text{cris}}(X_{k^{\text{sep}}, \mathbb{Z}_\ell(1)})$ for all $\ell \neq p$, as well as in the image of $\text{PH}^2_{\text{cris}}(X_{k^p}/W(k^p))(1)$. Then there is a natural identification

$$\text{Pic}(X_{k^p}) \supset \langle \xi \rangle^\perp \xrightarrow{\cong} L(A)$$

compatible with all cohomological realizations.

Proof. After replacing $k$ by a finite separable extension if necessary we can assume that $(X, \xi)$ admits a $K^p$-level structure for some $K^p \subset G_d(A^p)$ and that is of the form $\pi(K_{p,s}^p)$ for some compact open $K_{p,s}^p \subset G_{d,s}(k^p)$. In particular, it gives rise to a point $s \in M_{2d, K, \mathbb{Z}(p)}(k)$. After a further separable change of scalars if necessary, we can assume that $s$ lifts to a $k$-valued point of $\mathscr{S}_{d, K_{p,s}}$ to which we can attach the Kuga-Satake abelian variety $A_{K_{p,s}}^{\text{KS}}$ with properties (1), (2), and (3). The integral crystalline compatibility here follows from (4.11).

It still remains to show (1). For this we observe that, given a special endomorphism $f \in L(A^{\text{KS}})$, the deformation space of the triple $(X, \xi, f)$ admits a flat component. Indeed, by (1.8), we can identify the complete local ring of $M_{2d, K, \mathbb{Z}(p)}$ at $s$ with that of $\mathscr{S}_{K_{p,s}}$, and so the claim follows from (7.24).

We see therefore that there exists a lift $(\tilde{X}, \tilde{\xi}, \tilde{f})$ over a characteristic 0 field $F$ attached to a point $\tilde{s} \in M_{2d, K}(F)$ lifting $s$. Here we have:

$$\text{Pic}(\tilde{X}) \supset \langle \tilde{\xi} \rangle^\perp \xrightarrow{\text{Lefschetz (1.1)}} \text{AH}(\mathcal{P}_F^2) \cong \text{AH}(\mathcal{L}_s) \cap \text{End}(A^{\text{KS}}) = L(A^{\text{KS}}) \cap \text{End}(A_{K_{p,s}}^{\text{KS}}).$$

See (1.8) and (3.4) for the notation.

This shows that we have an inclusion $L(A^{\text{KS}}) \cap \text{End}(A^{\text{KS}}) \hookrightarrow \langle \xi \rangle^\perp$ compatible with cohomological realizations.

Similarly, given a class $\eta \in \langle \xi \rangle^\perp \subset \text{Pic}(X)$, the deformation space of $(X, \xi, \eta)$ again admits a flat component. Repeating the same argument as above gives us an inclusion going the other way, and so finishes the proof. \hfill \Box

Remark 4.19. (1), (2), (3) above. In the literature (cf. for example [Del72], [And90]), one usually finds the even Clifford algebra in place of the full Clifford algebra that we have chosen to use. As in [Cha12, 3.3], we do this to ensure that the statement in (3) above is not too unwieldy.

With appropriate changes, the results of the theorem can be phrased so that they hold globally for any $\mathbb{Z}(p)$-scheme $T$; cf. (4.20).
Remark 4.20. In fact, one can show more. For every map $T \to M_{2d,K,\mathbb{Z}(p)}$, let $\tilde{L}(T)$ be as in the notation of (3.14). Define the $\mathbb{Z}$-lattice $L_\mathbb{Z}(T) \subset \tilde{L}(T)$ to consist of all the elements whose $\mathbb{A}_p^\flat$-realization lands in $H^{\otimes(1,1)}_{\mathbb{A}_p^\flat}$.

Then we have a canonical identification:

$$L_\mathbb{Z}(T) = \langle \xi \rangle^\perp_T \subset \text{Pic}(\mathcal{X}_T/T).$$

Indeed, the functors $T \mapsto L_\mathbb{Z}(T)$ and $T \mapsto \langle \xi \rangle^\perp_T$ are both represented by unramified schemes over $M_{2d,K,\mathbb{Z}(p)}$ that are locally of finite type. Moreover, it is easy to deduce from the above argument that, for any field $k$, there is a canonical bijection between their $k$-valued points. In addition, one sees using deformation theory that, given a $k$-valued point of either scheme, the complete local ring at that point is canonically isomorphic to the complete local ring at the associated $k$-valued point of the other scheme. Using this and Artin approximation, one can glue together a canonical isomorphism from one scheme to the other.

Remark 4.21. Notice that we did not need the full force of the étaleness of $i^{KS}$ in the proof above. All we needed was for the deformation space of a special endomorphism to admit a flat component. This weaker condition might still be checkable in situations where the Kuga-Satake period map is not expected to be étale, such as in the context of the Catanese-Ciliberto surfaces considered in [Lyo12].

4.22. In [Riz10 4.2], Rizov shows that, when $p \nmid d$, the Kuga-Satake construction is compatible with the theory of canonical lifts for ordinary varieties. This continues to hold in our more general situation. Suppose that $(X_0,\xi_0)$ is a polarized K3 surface over a perfect field $k$ of characteristic $p$, and suppose that $X_0$ is ordinary. Let $(X,\xi)$ be the canonical lift (cf. loc. cit.) of $(X_0,\xi_0)$ over $W(k)$. After replacing $k$ by a finite extension, if necessary, we can assume that there is a Kuga-Satake abelian variety $A_0$ over $k$ attached to $(X_0,\xi_0)$, as in Theorem 3. There is also an algebraizable deformation $A$ of $A_0$ over $W(k)$ attached to the canonical lift $(X,\xi)$.

**Proposition 4.23.** $A_0$ is ordinary and $A$ is its canonical lift.

**Proof.** The proof of [Riz10 4.2.2] goes through verbatim. We recall it here briefly for the convenience of the reader. That $A_0$ is ordinary was already observed in the course of the proof of (4.11). Via Serre-Tate co-ordinates, the deformation space of $A_0$ is naturally identified with a formal torus $\mathfrak{X}$ over $W(k)$. Nygaard has shown in [Nyg83 2.7] that in this situation $A$ has to be isogenous to the canonical lift, implying that it corresponds to a torsion point of $\mathfrak{X}$. However, the only torsion point of $\mathfrak{X}$ defined over $W(k)$ is the identity, which corresponds to the canonical lift of $A_0$. □

4.24. We indicate how Theorem 3 (or, rather, its proof) implies the Tate conjecture:

**Proof of Theorem 4.** It is enough now to show the following: Suppose that we have a point $s \in M_{2d,K,\mathbb{Z}(p)}(k)$ giving rise to the Kuga-Satake abelian variety $A^{KS}_s$ (after replacing $k$ by a finite separable extension, we can assume that $A^{KS}_s$ is defined over $k$); then for any $\ell \neq p$ the map

$$L(A^{KS}_s) \otimes \mathbb{Q}_\ell \to V^{\Gamma}_{\ell,s} = P^2_{\ell,s}(1)^\Gamma.$$

is an isomorphism.

If $k$ is finite, this is exactly what is shown in Theorem 7.4 (all references in the remainder of the proof will be to [MPIT18 §7].) It requires a certain $\ell$-independence hypothesis, Assumption 7.2, which is valid in our case because of the obviously motivic origin of $V^{\Gamma}_{\ell,s} = P^2_{\ell,s}(1)$; cf. Remark 7.3.

For infinite $k$, the result follows from Corollary 7.12.
Strictly speaking, the quoted results apply directly only when \( p^2 \nmid d \), but their proofs go through even in our situation; cf. Remark 7.14.

Note also that to prove the Tate conjecture we only needed the inclusion \( L(A^K_{\text{KS}}) \subset \text{Pic}(X)_{\mathbb{Z}(p)} \).

4.25. We quickly sketch how the above ideas apply to cubic fourfolds. Let \((+1)\) (resp. \((-1)\)) be the self-dual odd positive (resp. negative) \( \mathbb{Z} \)-lattice of rank 1, and set
\[
\tilde{M} = (+1)^{\oplus 21} \oplus (-1)^{\oplus 2}.
\]
This is a self-dual lattice of signature \((21,2)\).

Let \( M_0 \) be the even rank 2 \( \mathbb{Z} \)-lattice equipped with the bi-linear form represented by the matrix \( \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \). Let \( M \) be the quadratic \( \mathbb{Z} \)-lattice:
\[
M = E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus M_0.
\]

This is a signature \((20,2)\) lattice that is maximal in the sense of [MP13b] and is self-dual over \( \mathbb{Z}[\frac{1}{2}] \). It is shown in [Has00, 2.1.2] that there exists \( m \in \tilde{M} \) with \( m \cdot m = 3 \) such that \( M \) is isometric to \( m \perp \subset \tilde{M} \).

Let \( K_M \subset \text{SO}(\tilde{M})(\mathbb{A}_f) \) be the largest compact sub-group that stabilizes \( \tilde{M} \) and acts trivially on \( m \) (this is the discriminant kernel); then \( K_M \) is in fact a compact open sub-group of \( \text{SO}(M)(\mathbb{A}_f) \). For every prime \( p \neq 2 \), and every \( K \subset K_M \) small enough with \( K_p = K_{M,p} \), just as in Section 3 (but even simpler, since we do not have to consider non-maximal lattices), the theory of [MP13b] now gives us an orthogonal Shimura variety \( \text{Sh}_K \) of the Shimura variety attached to \( M \) and the level sub-group \( K \) over \( \mathbb{Q} \), and an integral canonical healthy regular model \( \mathcal{S}_K \) over \( \mathbb{Z}(p) \) (it is smooth over \( \mathbb{Z}(p) \) if \( p \neq 3 \)).

Let \( \mathcal{C}F \) be the moduli stack of cubic fourfolds over \( \mathbb{Z} \). Just as we did for K3 surfaces in [2] we can define a notion of \( K^p \)-level structure for cubic fourfolds over \( \mathbb{Z}(p) \) using the lattice \( \tilde{M} \) and the distinguished element \( m \in \tilde{M} \). This gives us a finite étale cover \( \mathcal{C}F_{\mathbb{Z}(p)} \) of \( \mathcal{C}F(\mathbb{Z}(p)) \). Over \( \mathbb{C} \), we have a Kuga-Satake map \( \mathcal{C}F_{\mathbb{C},\mathbb{C}} \to \text{Sh}_{\mathbb{C}} \) constructed using primitive degree-4 cohomology.

Using the fact that this map is given via an absolutely Hodge correspondence [And96, § 6], just as in [1.4], we can descend the Kuga-Satake map over \( \mathbb{Q} \): \( \mathcal{C}F_{\mathbb{Q}} \to \text{Sh}_{\mathbb{K}} \). Then, since \( \mathcal{C}F_{\mathbb{Z}(p)} \) is smooth over \( \mathbb{Z}(p) \), we can extend it over \( \mathbb{Z}(p) \) using the extension property of \( \mathcal{S}_K \) (cf. [4.7]): \( \mathcal{C}F_{\mathbb{Z}(p)} \to \mathcal{S}_K \). We now have:

**Theorem 4.26.**

1. The period map \( \mathcal{C}F_{\mathbb{Z}(p)} \to \mathcal{S}_K \) is an open immersion.

2. Given any cubic fourfold \( X \) over a field \( k \) of odd characteristic, there exists a finite separable extension \( k'/k \) and an abelian variety \( A \) over \( k' \) such that the numbered assertions of [4.18] hold with \( PH^2 \) replaced by \( PH^4 \) and \( \text{Pic}(X_{k'}) \) replaced by \( \text{CH}^2(X_{k'}) \).

3. The Tate conjecture for cubic fourfolds holds in co-dimension 2 over fields of odd characteristic. That is, given a cubic fourfold \( X \) over a finitely generated field \( k \) of odd characteristic with absolute Galois group \( \Gamma = \text{Gal}(k^{\text{sep}}/k) \), the \( \ell \)-adic cycle class map \( \text{CH}^2(X) \otimes \mathbb{Q}_\ell \to H^4_{\text{et}}(X_{k^{\text{sep}}}, \mathbb{Q}_\ell(2))^\Gamma \) is an isomorphism for all \( \ell \neq p \).

\[\text{This is proven in loc. cit. via a monodromy argument, but one should also be able to prove it via Deligne’s Principle B and working with amenable points in the moduli space, much as we did with Kummer points in [1.3].}\]
(4) $\mathcal{CF}_{\mathcal{K},p}$ is geometrically irreducible for every $p > 2$.

**Sketch of proof.** If we look back at the strategy used for K3 surfaces, we see that the main step is to show that the period map

$$\mathcal{CF}_{\mathcal{K},\mathbb{Z}(p)} \to \mathcal{S}_{\mathcal{K}}$$

is étale. Indeed, once we know this, the Torelli theorem for cubic fourfolds [Vol86] will imply that the map is an open immersion. The remaining statements are proven just as for K3 surfaces. For the Tate conjecture, we reduce to the Hodge conjecture for co-dimension 2 cycles on cubic fourfolds [Voi86] Appendix 2] or [Zuc77]. This plays the same role for cubic fourfolds as Lefschetz (1,1) did for K3 surfaces. For irreducibility, we have to know that $\mathcal{S}_{\mathcal{K},F_p}$ is normal and that each of its geometrically connected components is the specialization of a geometrically connected component of $\text{Sh}_K$. This follows from the argument used in [3.28].

To prove étaleness, we note that $\mathcal{CF}_{\mathcal{K},\mathbb{Z}(p)}$ is smooth and that the tangent space at any point $s : \text{Spec } k \to \mathcal{CF}_{\mathcal{K},\mathbb{Z}(p)}$ attached to a cubic fourfold $X/k$ is given by:

$$\text{Def}_X(k[\epsilon]) = \left\{ \text{Isotropic lines } L \subset P^4_{\text{dr}}(X_0/k) \otimes k[\epsilon] \text{ lifting } F^4H^4_{\text{dr}}(X_0/k) \right\}.$$ 

This is shown in [Lev01, § 3]. So, just as in the proof of (4.8), it is enough to prove the integral crystalline compatibility of the Kuga-Satake construction. We do this using the same strategy: prove it directly for ordinary cubic fourfolds as in (4.11) and then propagate it everywhere using the density of ordinary points as in (4.13).

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