Magnetohydrodynamic instabilities in a self-gravitating rotating cosmic plasma

Jyoti Turi and A P Misra

Department of Mathematics, Siksha Bhavana, Visva-Bharati (A Central University), Santiniketan-731 235, West Bengal, India

* Author to whom any correspondence should be addressed.

E-mail: jyotituri.maths@gmail.com and apmisra@visva-bharati.ac.in

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Abstract

The generation of magnetohydrodynamic (MHD) waves and their instabilities are studied in galactic gaseous rotating plasmas with the effects of the magnetic field, the self gravity, the diffusion-convection of cosmic rays as well as the gas and cosmic-ray pressures. The coupling of the Jeans, Alfvén and magnetosonic waves, and the conditions of damping or instability are studied in three different cases, namely when the propagation direction is perpendicular, parallel and oblique to the static magnetic field, and are shown to be significantly modified by the effects of the Coriolis force due to the rotation of cosmic fluids and the cosmic-ray diffusion. The coupled modes can be damped or anti-damped depending on the wave number is above or below the Jeans critical wave number that is reduced by the effects of the Coriolis force and the cosmic-ray pressure. It is found that the deviation of the axis of rotation from the direction of the static magnetic field gives rise to the coupling between the Alfvén wave and the classical Jeans mode which otherwise results into the modified slow and fast Alfvén waves as well as the modified classical Jeans modes. Furthermore, due to the effects of the cosmic rays diffusion, there appears a new wave mode (may be called the fast Jeans mode) in the intermediate frequency regimes of the slow and fast Alfvén waves, which seems to be dispersionless in the long-wavelength propagation and has a lower growth rate of instability in the high density regimes of galaxies. The dispersion properties and the instabilities of different kinds of MHD waves reported here can play pivotal roles in the formation of various galactic structures at different length scales.

1. Introduction

One of the paramount examples of the magnetohydrodynamic (MHD) instability is the Parker instability [1], which has been known to be relevant to the Galaxy evolution, i.e., possibly the formation of molecular clouds and the galactic dynamo. Such instabilities are typically characterized by the length-scales that can be shorter than the galactic radius. However, on relatively large scales, the Parker instability together with the Jeans instability [2], associated with the self-gravitational perturbations, can induce the formation of filament-like structures in the interstellar medium (ISM). It has been shown that such instability produces most likely small-scale structures instead of the giant molecular clouds [3]. Furthermore, the Parker instability may be significantly suppressed by means of fluctuating galactic magnetic fields [4] and so, other MHD instabilities should intervene to the formation of large-scale galactic structures. In this context, the generation of magnetosonic waves and the formation of structures in galactic gaseous plasmas has been studied by Bonanno et al taking into account the effects of the differential rotation, the magnetic field and the self-gravity [5]. Recent investigations indicate that the modification of the Parker instability with the effects of cosmic-ray transport can have large implications for the evolution and structure formation of galaxies [6].

On the other hand, the magnetogravitational instability [7] of space and astrophysical plasmas has been extensively studied with the effects of the Earth’s rotation, the self-gravitational force and the magnetic field [8–13]. To mention a few of them, Prajapati et al [14] have investigated the self-gravitational instability in
anisotropic rotating plasmas and showed that the effects of rotation can stabilize the system. Kossacki [15] studied the Jeans instability in viscous plasmas with the effects of the Coriolis force due to the rotation of fluids and finite electric conductivity. In other investigations, many authors have considered the MHD instabilities in anisotropic plasmas with the influences of the Hall current and finite electric and thermal conductivities [16–22]. Ren et al [23] have advanced the theory of MHD waves in quantum magnetoplasmas by considering the effects of plasma resistivity. The propagation of MHD waves was also explored in self-gravitating dusty plasmas [24–27]. The Jeans instability criteria in two-component plasmas containing ionized and neutral components has also been discussed in other related works [28–31]. Recently, Sharma et al [32] have studied the MHD instabilities in finitely conducting neutrino-coupled magnetized plasmas by the effects of the self-gravity and dissipation.

The aim of this work is to advance the theory of MHD waves and their instabilities in ISM consisting of thermal gas and cosmic rays using a two-fluid approximation. Especially, we focus on the generation of different kinds of wave modes including new type of coupling and the generation of a new wave mode, not reported before, as well as the characteristics of wave damping and instabilities that are modified by the Coriolis force due to the rotation of cosmic fluids, the self-gravitational force, the magnetic field, the thermal gas and cosmic-ray pressures as well as the cosmic-ray diffusion. The paper is organized as follows: in section 2, we describe the two-fluid MHD model for an ionized thermal gas and cosmic rays, and obtain a general dispersion relation for MHD waves. Section 3 demonstrates the characteristics of wave dispersion and damping/instability in three different cases of propagation (perpendicular, parallel and oblique to the magnetic field). Finally, the results are summarized and concluded in section 4.

2. Basic equations

We consider the propagation of MHD waves in an interstellar medium that consists of a thermal ionized gas and cosmic rays under the influences of a self-gravitational force, the magnetic force, the pressure gradient force, and the Coriolis force due to the rotation of cosmic fluids with an angular velocity \( \Omega = (0, \Omega_0 \cos \lambda, \Omega_0 \sin \lambda) \), where \( \lambda \) is the angle made by the axis of rotation with the \( y \)-axis. Here, the cosmic rays are considered to be a gas with a negligible density but a significant contribution to the pressure [5]. We consider a polytropic thermal gas, i.e., no diffusion along any direction but only the compression and rarefaction processes, which include heat transfer, can take place, and describe the evolution of cosmic rays in the diffusion approximation along the magnetic field lines as they are strongly magnetized and the diffusion transverse to the magnetic field is thus less important [33]. The plasma is assumed to be homogeneous, fully ionized, and highly conducting with a higher Reynolds number. Furthermore, we assume that the uniform magnetic field is along the \( z \)-axis, i.e., \( \mathbf{B}_0 = B_0 \hat{z} \) and ignore the effect of the centrifugal force compared to the Coriolis force. The basic equations governing the dynamics of MHD waves are

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1}
\]

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla \left( P + \frac{\mathbf{B}^2}{2\mu_0} \right) + \frac{1}{\rho \mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} - 2\Omega \times \mathbf{v} + \nabla \psi, \tag{2}
\]

\[
\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{v} - (\nabla \cdot \mathbf{v}) \mathbf{B}, \tag{3}
\]

\[
\nabla^2 \psi = -4\pi G \rho, \tag{4}
\]

together with the equations of state for the thermal gas and the diffusion-convection equation for cosmic rays [5]:

\[
\frac{\partial P_g}{\partial t} + (\mathbf{v} \cdot \nabla) P_g + \gamma_g \nabla \cdot \mathbf{v} = 0, \tag{5}
\]

\[
\frac{\partial P_c}{\partial t} + (\mathbf{v} \cdot \nabla) P_c + \gamma_c \nabla \cdot \mathbf{v} = \kappa \nabla \cdot (\nabla P_c). \tag{6}
\]

Here, \( \rho \) and \( \mathbf{v} \) are, respectively, the fluid (thermal gas) mass density and velocity; \( P = P_g + P_c \) is the total pressure in which \( P_g \) is the gas pressure and \( P_c \) that of cosmic rays; \( \mu_0 \) is the vacuum permeability, \( \psi \) is the gravitational potential, \( G \) is the gravitational constant, \( \kappa \) is the diffusion coefficient of cosmic rays along the magnetic field, and \( \nabla P_c = \mathbf{B} (\mathbf{B} \cdot \nabla P_c)/B^2 \). Also, \( \gamma_g \) and \( \gamma_c \) are the adiabatic indices corresponding to the thermal gas and cosmic rays. In the following section 3, we will obtain a general linear dispersion relation from these basic set of equations using a perturbative approach.
3. Dispersion relation

In equilibrium state, we assume that the plasma is uniform with constant density \( \rho_0 \) and zero velocity. To linearize equations (1)–(6), we split up the physical quantities into their equilibrium (with suffix 0) and perturbation (with suffix 1) parts according to \( \rho = \rho_0 + \rho_1, \mathbf{v} = 0 + \mathbf{v}_1, \mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1, \psi = \psi_0 + \psi_1, P_j = P_{0j} + P_{1j} \) (with \( j = g, c \)), and \( P = P_0 + P_1 \) with \( P_0 = P_{0g} + P_{0c} \) and \( P_1 = P_{1g} + P_{1c} \). Thus, we obtain

\[
\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0, \tag{7}
\]

\[
\frac{\partial \mathbf{v}_1}{\partial t} = -\frac{1}{\rho_0} \nabla P_1 - \frac{1}{\mu_0 \rho_0} \nabla (\mathbf{B}_0 \cdot \mathbf{B}_1) + \frac{1}{\rho_0 \mu_0} (\mathbf{B}_0 \cdot \nabla) \mathbf{B}_1
- 2 \Omega \times \mathbf{v}_1 + \nabla \psi_1, \tag{8}
\]

\[
\frac{\partial \mathbf{B}_1}{\partial t} = (\mathbf{B}_0 \cdot \nabla) \mathbf{v}_1 - (\nabla \cdot \mathbf{v}_1) \mathbf{B}_0, \tag{9}
\]

\[
\nabla^2 \psi_1 = -4\pi G \rho_1, \tag{10}
\]

\[
\frac{\partial P_{1g}}{\partial t} + \gamma_k P_{0g} \nabla \cdot \mathbf{v}_1 = 0, \tag{11}
\]

\[
\left( \frac{\partial}{\partial t} + u_0 \right) P_1 + \gamma_k P_{0g} \nabla \cdot \mathbf{v}_1 = 0, \tag{12}
\]

where the frequency (inverse time scale) of cosmic rays diffusion \( \nu_c \) is given by \([5] \nu_c B_1 = -k \nabla \cdot [\mathbf{B}_0 \times (\mathbf{B}_0 \cdot \nabla) P_1 / B_0^2] \).

Next, assuming the perturbations to vary as plane waves of the form \( \sim \exp (i \mathbf{k} \cdot \mathbf{r} - i \omega t) \) with the wave vector \( \mathbf{k} \) and wave frequency \( \omega \), we obtain from equations (7) to (12) the following general dispersion relation.

\[
\omega^2 \mathbf{v}_1 - \left( \frac{C_s^2}{\omega^2 + \nu_c^2} \right) (\mathbf{k} \cdot \mathbf{v}_1) \mathbf{k}
- \frac{k}{\mu_0 \rho_0} [(\mathbf{k} \cdot \mathbf{v}_1) B_0^2 - (\mathbf{B}_0 \cdot \mathbf{k}) (\mathbf{B}_0 \cdot \mathbf{v}_1)]
- \frac{\mathbf{B}_0 \cdot \mathbf{k} [(\mathbf{k} \cdot \mathbf{v}_1) \mathbf{B}_0 - (\mathbf{B}_0 \cdot \mathbf{k}) \mathbf{v}_1]}{\mu_0 \rho_0}
+ 2i \omega \Omega \times \mathbf{v}_1 + \frac{4\pi G \rho_0}{k^2} (\mathbf{k} \cdot \mathbf{v}_1) \mathbf{k} = 0, \tag{13}
\]

where the reduced expression of \( \nu_c \) is given by \( \nu_c = \kappa (\mathbf{k} \cdot \mathbf{B}_0)^2 / B_0^2 \). Also, \( C_s = \sqrt{\gamma_k P_{0g}/\rho_0} \) is the sound speed and \( C_A = \sqrt{\gamma_k P_{0g}/\rho_0} \) that associated with cosmic rays.

In what follows, we consider the propagation of MHD waves at an arbitrary direction with respect to the static magnetic field \( \mathbf{B}_0 \). Without loss of generality, we assume that \( \mathbf{k} = k (\hat{x} \sin \theta + \hat{z} \cos \theta) \), where \( \theta \) is the angle between \( \mathbf{k} \) and \( \mathbf{B}_0 \). Also, \( \mathbf{v}_1 = (v_{1x}, v_{1y}, v_{1z}) \) in which \( v_{1x}, v_{1y}, \) and \( v_{1z} \) denoting the perturbed velocity components along \( x, y, \) and \( z \) axes. Thus, equation (13) reduces to

\[
(\omega^2 - C_s^2 \cos^2 \theta + \omega_j^2 \sin^2 \theta)
\times \left[ (\omega^2 - k^2 V_A^2 + (\omega_j^2 - C_A^2 \cos^2 \theta) \sin^2 \theta)(\omega^2 - k^2 V_A^2 \cos^2 \theta)
- 4 \Omega_A^2 \omega^2 \sin^2 \lambda - (\omega^2 - k^2 V_A^2 \cos^2 \theta)
\times [4 \Omega_A^2 \omega^2 \cos^2 \lambda + (\omega_j^2 - C_A^2 \cos^2 \theta) \sin \theta \cos \theta] \right] = 0, \tag{14}
\]

where \( C_s^2 = C_s^2 + C_A^2 \omega^2 / (\omega + i \omega_c) \) with \( C_j^2 = C_s^2 + C_A^2 \) denoting the squared acoustic speed in absence of the cosmic rays diffusion \( \nu_c = 0 \), \( \omega_j = \sqrt{4\pi \rho_0 G} \) is the classical Jeans frequency and \( V_A = B_0 / \sqrt{\rho_0 \mu_0} \) is the Alfvén velocity.

The dispersion equation (14) is, in general, complex due to the appearance of the factor \( \omega + i \nu_c \). It can be analyzed for the identification of different kinds of MHD wave modes, their propagation characteristics as well as their stability with the effects of the self-gravitational force, the Coriolis force, and the dissipation due to the cosmic rays diffusion. Furthermore, equation (14) can be studied in three different cases of interest, namely

(i) the propagation vector is perpendicular to the magnetic field, i.e., \( \mathbf{k} \perp \mathbf{B}_0 \) with \( \theta = \pi / 2 \),
(ii) the propagation vector is parallel to the magnetic field, i.e., \( \mathbf{k} \parallel \mathbf{B}_0 \) with \( \theta = 0 \), and
(iii) the propagation direction is arbitrary, i.e., the angle \( \theta \) assumes any values in \( 0 \leq \theta \leq \pi / 2 \).

It is to be mentioned that for the sake of simplicity, we have neglected the effects of particle collision by assuming that the collision time is much longer than the time scale of oscillations. Also, in a highly conducting medium with high Reynolds number, both the effects of fluid viscosity and magnetic viscosity (plasma
registivity) on the attenuation of MHD waves can be shown to be small (Although the attenuation increases with the viscosity effects, the same decreases as the fluid conductivity increases). On the other hand, it has been shown that for large-scale structure formation, the two possible sources of dissipation due to self-interaction and gravitational coupling of cosmic fluids can operate together and the interplay between them can play an important role in determining the dynamics of cosmic fluids [34]. However, inclusion of such dissipative effects is beyond the scope of the present study.

3.1. Propagation perpendicular to the magnetic field
We consider the direction of propagation across the magnetic field, i.e., \( \mathbf{k} = k \hat{\mathbf{x}} \), \( \mathbf{B}_0 = B_0 \hat{\mathbf{z}} \) with \( \theta = \pi / 2 \) and \( \Omega = (0, \Omega_0 \cos \lambda, \Omega_0 \sin \lambda) \). In this case, the general dispersion relation (14) reduces to
\[
\omega^2 = \left( C_g^2 + V_A^2 \right) k^2 - \omega_i^2 + 4 \Omega_0^2 k^2, \tag{15}
\]
equation (15) describes the wave dispersion and instability for the Jeans–Alfvén–magnetosonic (JAM) wave which is influenced by the thermal pressure due to gas \( C_g \) of charged particles and cosmic rays \( C_r \), the cosmic rays diffusion \( \nu_r \), the external magnetic field \( B_z \), the gravitational force \( \omega_i \), and the Coriolis force \( \Omega_0 \) due to the rotation of cosmic fluids. By disregarding the effects of the cosmic rays pressure and diffusion, the Coriolis force and the external magnetic field, one can recover the classical Jeans mode in electron-ion plasmas, given by,
\[
\omega^2 = C_g^2 k^2 - \omega_i^2, \tag{16}
\]
which is known to be stable (unstable) for \( k > k_j \) \((k < k_j)\), where \( k_j = \omega_i / C_g \) is the Jeans critical wave number. Also, in absence of the gas and cosmic rays pressures (or if the contributions of these pressures are much smaller than that of the magnetic pressure) together with the gravitational force and the Coriolis force, one obtains the pure Alfvén wave with the phase velocity \( \omega / k = V_A \). Furthermore, the typical magnetosonic mode can be recovered with the phase velocity \( \omega / k = \sqrt{C_g^2 + V_A^2} \) if one neglects the cosmic rays pressure (or when the gas pressure dominates over the cosmic rays pressure), the gravitational force and the Coriolis force. Such magnetosonic (or magnetoacoustic) waves become ion–acoustic waves with velocity \( C_g \) if the gas pressure is much higher than the cosmic rays pressure and the magnetic pressure. Thus, it follows that in presence of all these aforementioned effects, a new coupled JAM mode is generated which can be damped or anti-damped due to the effects of the cosmic rays diffusion and the Coriolis force and the self-gravity.

We note that equation (15) is also, in general, complex due to the effect of the cosmic rays diffusion \( \nu_r \). Even in absence of this effect, the Jeans instability criterion for the JAM mode is significantly modified by the cosmic rays pressure and the Coriolis force. In the case of \( \nu_r = 0 \), the JAM mode is always stable for \( \omega_i < 2 \Omega_0 \) or when the influence of the self-gravitational force is negligible (i.e., \( \omega_i = 0 \)). However, for \( \omega_i > 2 \Omega_0 \) the mode can be stable or unstable according to when \( k > k_i \) or \( k < k_i \), where \( k_i \) is the modified Jeans critical wave number, given by,
\[
k_i = \left\{ \frac{\omega_i^2 - 4 \Omega_0^2}{C_g^2 + V_A^2} \right\}^{1/2}. \tag{17}
\]
Thus, the Jeans instability criterion is not only modified, but the critical wave number is also significantly reduced due to the cosmic rays pressure and the Coriolis force. It follows that the JAM mode or the gravitational mode becomes stable (unstable) in the longer (shorter) domains of wavelengths than that predicted for the classical Jeans mode [2]. This means that the effects of rotation and the cosmic rays pressure favor the stability of JAM modes in cosmic plasmas. Typically, for plasmas relevant to spiral galaxies [8] we have \( B_0 \sim 1 \text{ nT}, \) \( P_{\text{gas}} \sim 10^{-13} \text{ Nm}^{-2}, \) \( \rho_0 \sim 10^{-21} \text{ Kg/m}^3 \). So, considering \( \omega_i \sim 0.5 \Omega_0, \) \( \Omega_0 \sim 0.2 \Omega_0 \) and \( V_A / C_g \sim 0.5 \), where \( \tau = \omega_i^{-1} \) is the typical time scale of oscillations \((\approx 10^{15} \text{ s})\), the critical wavelength can be estimated as \( k_i^{-1} \sim 3.7 C_g / \omega_i \), and the maximum instability growth rate as \( \gamma_{\max} \sim \sqrt{\omega_i^2 - 4 \Omega_0^2} / \omega_i \sim 0.3 \Omega_0 \) at \( k = 0 \). The corresponding growth time of instability can then be estimated as \( \tau_{\text{growth}} \sim 10^8 \text{ yrs} \), which is comparable to the evolutionary time of diffuse interstellar clouds.

On the other hand, in presence of the cosmic rays diffusion, the JAM mode becomes unstable, i.e., it can be either damped or anti-damped. The expressions for the damping and growth rates can be obtained from the dispersion equation (15) by assuming \( \omega = \omega_i + i \gamma \); \( \gamma \ll |\omega_i| \); \( \gamma = \omega / (\omega + i \nu_r) \approx 1 \nu_r / \omega_i \), and using the conditions \( \omega_i \lesssim \Omega_0 \) and \( \omega_i > \Omega_0 \) separately. In the former case, \( \omega_i \) is real and \( \omega_i = \omega_i \), while in the latter, \( \omega_i \) is real (\( = \omega_i \)) or purely imaginary according to when \( k > k_j \) or \( k < k_j \). Here, \( \omega_i \) is a solution of equation (15) at \( \nu_r = 0 \). Thus, when \( \omega_i \lesssim \Omega_0 \), the JAM wave gets damped and the damping rate is \( \gamma \approx (1/2) C_g^2 k_j^2 \nu_r / \omega^2 \), where \( \omega \) is the the wave frequency at \( \nu_r = 0 \). However, when \( \omega_i > 2 \Omega_0 \), the instability (damping) occurs for \( k < k_j \) \((k > k_j)\). In this case, the damping rate has the same expression as above, however, the growth rate of instability can be estimated as \( \gamma \sim (\omega^2 + C_g^2 k_j^2 \nu_r / \omega^2)^{1/2} \) with \( \omega \) denoting the wave frequency at \( \nu_r = 0 \). In order to study the characteristics of the JAM mode in details we numerically solve equation (15). The results are displayed in figures 1 and 2 corresponding to the cases with \( \omega_i \lesssim 2 \Omega_0 \) and \( \omega_i > 2 \Omega_0 \) respectively. In the former,
while only the damping takes place irrespective of the values of $k$, in the latter, the damping or instability (anti-damping) can occur according to when $k k J > 1$ or $k k J < 1$.

From figure 1 it is seen that the real part of the wave frequency increases with $k$. Also, the absolute value of the damping rate increases within the domain $0 \leq k \leq 1$ and it reaches a steady state value for $k \geq 1$. This means that the wavelength is decreased to accommodate a large number of wave modes to pass through a given point per unit time. However, they die out within a short interval of time due to the higher damping rate. From the subplots (a) and (b) it is noted that while the wave frequency (real part, $\Re(\omega)$) is increased, the damping rate (imaginary part, $|\Im(\omega)|$) is significantly reduced due to a small increment of the parameters $C_g$ and $V_A$ associated with the thermal gas pressure and the static magnetic field (Compare the solid line with the dashed and dash-dotted lines). A significant increase of both the wave frequency and the damping rate also occurs by the effects of the cosmic ray pressure (Compare the solid lines with the dotted lines). On the other hand, subplots (c) and (d) of figure 1 show that the effects of the rotational frequency ($\Omega_0$) with its increased value and the Jeans frequency ($\omega_J$) with a small reduction are to enhance the wave frequency but to reduce the damping rate significantly (Compare the solid line with the dotted and dash-dotted lines). As expected, a small increase of the diffusion frequency $\nu_c$ leads to a significant enhancement of the damping rate [See the dashed lines in subplot (d)]. Thus, it may be concluded that for the propagation of JAM waves in rotating magnetoplasmas, the damping rate can be diminished by either increasing the thermal gas pressure / the magnetic field strength or decreasing the cosmic rays pressure / Jeans frequency. So, highly magnetized gaseous plasmas with dominating effects from the Coriolis force over the self-gravity can exhibit the stabilizing behaviors of JAM modes. Figure 2 shows the plots of the growth and the damping rates in the case when the contribution from the self-gravity is higher than that due to the Coriolis force, i.e., $\omega_J > 2\Omega_0$. As discussed before, the instability (damping) occurs for $k < k_J$ ($k > k_J$). The peaks of the curves appear at the critical value, $k = k_J$, at which the damping or the growth rate of the JAM mode may not be defined. However, the growth (decay) rate tends to decrease with increasing values of

**Figure 1.** The dispersion properties [subplots (a) and (c)] and the damping rates [subplots (b) and (d)] are shown for the Jean-Alfvén-magnetosonic wave [equation (15)] in the case of $\omega_J \leq 2\Omega_0$. The fixed parameter values for subplots [(a), (b)] and [(c), (d)], respectively, are $(\nu_c, 1\Omega_0, \omega_J) = (0.1, 0.2, 0.3)\omega_0$ and $(C_g, C_s, V_A) = (0.5, 0.4, 0.5)C_s$. 

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k in the interval $0 \leq k < k_J$ ($k > k_J$). The growth rate becomes minimum or the damping rate gets maximized close to this critical value. Here, for larger values of $k (> k_J)$, the damping rate reaches a steady state. Furthermore, due to a small increment of the parameters $C_g$, $C_c$, and $V_A$, associated with the gas pressure, cosmic rays pressure and the static magnetic field, the growth or damping rates are reduced together with a shift of the critical value $k_J$ towards the lower region of $k$, i.e, the domain of $k$ for the wave instability shortens while that for the damping is extended [See subplot (a)]. On the other hand, the effect of an increment of the diffusion frequency $\nu_c$ is to enhance both the growth and the damping rates even though the critical wave number $k_J$ remains the same. However, in contrast to the effects of the rotational frequency (with a small increase of which both the growth and the damping rates are significantly reduced together with a shift of the critical wave number towards lower values of $k$), the effects of the Jeans frequency are to enhance both the growth and the damping rates as well as an increase of the critical wave number. The latter results into an increase of the instability domain $0 < k < k_J$, but a decrease of the domain for damping ($k > k_J$). So, it may be concluded that when the self-gravitational force dominates over the Coriolis force due to the rotation of fluids, cosmic magnetoplasmas always exhibit instability. Such magneto-Jeans instability can have important contributions to the formation of large-scale clouds in spiral arms or galactic centers [35].

3.2. Propagation parallel to the magnetic field

We consider the propagation of MHD waves parallel to the static magnetic field, i.e., $k = k\hat{z}$, $B_0 = B_0\hat{z}$, $\theta = 0$, and $\Omega = (0, \Omega_0 \cos \lambda, \Omega_0 \sin \lambda)$. In this case, the dispersion relation (14) reduces to

$$\left(\omega^2 - \tilde{C}_J^2 k^2 + \omega_J^2\right)\left[\left(\omega^2 - k^2 V_A^2\right)^2 - 4\Omega_0^2 \omega^2 \sin^2 \lambda\right] - 4\Omega_0^2 \omega^2 \cos^2 \lambda \left(\omega^2 - k^2 V_A^2\right) = 0, \tag{18}$$

where $\tilde{C}_J^2 = C_J^2$ at $\nu_c = 0$. Equation (18) represents the dispersion relation for coupled Jeans and Alfvén wave modes, to be called Jeans–Alfvén (JA) modes that are typically modified by the cosmic-ray pressure and diffusion, as well as the Coriolis force. Interestingly, the coupling of these modes occurs due to the obliqueness of the axis of rotation with respect to the direction of the magnetic field and they become decoupled when $\Omega || B_0$ or $\lambda = \pi/2$. We note that equation (18) is, in general, complex, which can exhibit either the wave instability or damping depending on the system parameters as stated before and the wave number below or above a critical value.

Before we proceed to the general case, we first examine the nature of the roots of equation (18) in absence of the cosmic-ray diffusion ($\nu_c = 0$). To this end, we recast equation (18) as a polynomial equation in $\omega_J$, i.e.,
\[
(\omega^2)^3 + (\omega^2 - C_f^2 k^2 - 2V_A^2 k^2 - 4\Omega_0^2)(\omega^2)^2 \\
- [(\omega_f^2 - C_f^2 k^2)(2V_A^2 k^2 + 4\Omega_0^2 \sin^2 \lambda)] \omega^2 \\
- (V_A^2 k^4 + 4\Omega_0^2 \cos^2 \lambda k^2 V_A^2) \omega^2 \\
+ (\omega_f^2 - C_f^2 k^2)V_A^4 k^4 = 0. \tag{19}
\]

When the contribution of the self-gravitational force is higher than that of the Coriolis force, i.e., \(\omega_f > 2\Omega_0\), applying the Decartes' rule of signs, we note that equation (19) has a maximum of two positive roots (i.e., a maximum of two stable wave modes) and a maximum of one negative root (i.e., either one unstable mode or no unstable mode) for \(k < k_h\) and \(k > k_h\), where \(k_h = \omega_f/\sqrt{C_f^2 + V_A^2} \approx k_f\). For further insights of these modes, we express equation (19) as a quadratic equation in \(\omega^2\), i.e.,

\[
(\omega^2 - C_f^2 k^2 + \omega_f^2)(\omega^2 - k^2 V_A^2) - 4\Omega_0^2 \omega^2 = 0. \tag{20}
\]

Similarly, considering \(\omega_f \leq 2\Omega_0\), one can also conclude that there may be two stable wave modes and one unstable mode (or no unstable mode) for \(k < k_h\) and \(k > k_h\) respectively, where \(k_h = \omega_f/\sqrt{C_f^2 + V_A^2} < k_f \equiv \omega_f/C_r\).

In particular, for \(\lambda = \pi/2\), i.e., when the axis of rotation is along the static magnetic field \((\Omega \parallel B_0)\) for which the Jeans mode and the Alfvén mode get decoupled, one obtains from equation (19) the following dispersion relation for the classical Jeans mode but modified by the cosmic-ray pressure.

\[
\omega^2 - C_f^2 k^2 + \omega_f^2 = 0. \tag{21}
\]

This gives an unstable (or a stable) mode for \(k < k_f\) (or \(k > k_f\)). In this case, we obtain the dispersion relations for the fast and slow Alfvén waves modified by the Coriolis force, i.e.,

\[
\omega = \pm \Omega_0 + \sqrt{\Omega_0^2 + k^2 V_A^2}. \tag{22}
\]

While for the slow mode the cut-off frequency is zero, for the fast mode it is shifted by \(\omega = 2\Omega_0\). Both the modes are clearly stable being independent of the self-gravity effect.

On the other hand, when \(\lambda = 0\), i.e., when the axis of rotation is transverse to the direction of the magnetic field, equation (19) reduces to

\[
(\omega^2 - k^2 V_A^2) [(\omega^2 - C_f^2 k^2 + \omega_f^2)(\omega^2 - k^2 V_A^2) - 4\Omega_0^2 \omega^2] = 0. \tag{23}
\]

The first factor of equation (23), when equated to zero, gives the pure Alfvén mode having the phase velocity as the Alfvén velocity \(V_A\). The second factor gives the following reduced dispersion relation for the coupled Jeans and the Alfvén modes.

\[
(\omega^2 - C_f^2 + \omega_f^2)(\omega^2 - k^2 V_A^2) - 4\Omega_0^2 \omega^2 = 0. \tag{24}
\]

For further insights of these modes, we express equation (24) as a quadratic equation in \(\omega^2\), i.e.,

\[
(\omega^2)^2 + [\omega_f^2 - (C_f^2 + V_A^2) k^2 - 4\Omega_0^2] \omega^2 - \omega_f^2 V_A^2 k^2 = 0. \tag{25}
\]

Here, two particular cases may be of interest, namely when \(\omega_f > 2\Omega_0\) and \(\omega_f < 2\Omega_0\). In the former case, the dispersion relation (25) is rewritten as

\[
(\omega^2)^2 + (\omega_f^2 - C_f^2 k^2 - 2V_A^2 k^2) \omega^2 - \omega_f^2 V_A^2 k^2 = 0, \tag{26}
\]

where \(\omega_f^2 = |\omega_f^2 - 4\Omega_0^2|\) is the square of the reduced Jeans frequency. Solving equation (26) for \(\omega_f^2\), we obtain two different roots of \(\omega_f^2\), given by,

\[
\omega_f^{1,2} = \frac{1}{2} \left[ k^2 (C_f^2 + V_A^2) - \omega_f^2 \right] \\
\pm \sqrt{\left[ k^2 (C_f^2 + V_A^2) \right]^2 + 4k^2 V_A^4 \omega_f^4}. \tag{27}
\]

We note that the expression for \(\omega_f^{1} \) is always positive, giving a real root, while that of \(\omega_f^{2} \) is always negative (giving an imaginary root) irrespective of the values of \(k\). Thus, it follows that the fast (slow) Jeans-Alfvén mode is always stable (unstable) for \(\omega_f > 2\Omega_0\). The instability growth rate for the slow mode is

\[
\gamma = \frac{1}{\sqrt{2}} \sqrt{\left[ k^2 (C_f^2 + V_A^2) \right]^2 + 4k^2 V_A^4 \omega_f^4} \\
+ \omega_f^2 - k^2 (C_f^2 + V_A^2), \tag{28}
\]

which at the Jeans critical wave number \(k_h\) can be estimated as \(\gamma \sim (k_f V_A \omega_f)^{1/2}\). On the other hand, for \(\omega_f \lesssim 2\Omega_0\), we recast the dispersion relation (25) as

\[
(\omega^2)^2 - (\omega_f^2 + C_f^2 k^2 + V_A^2 k^2) \omega^2 - \omega_f^2 V_A^2 k^2 = 0. \tag{29}
\]

Similar to equation (26), this equation gives also fast and slow wave modes, and it can be shown that while the fast mode is always stable, the slow mode becomes unstable for any value of \(k\). In this case, the growth rate of
instability is

\[ \gamma = \frac{1}{\sqrt{2}} \left[ \sqrt{\left( \omega_{j0}^2 + k^2(C_g^2 + V_A^2) \right) F + 4k^2V_A^2\omega_j^2} - \omega_{j0} - k^2(C_g^2 + V_A^2) \right], \]  

which at the Jeans critical wave number \( k_{j} \) can be estimated as \( \gamma \sim (1/\sqrt{2})k_{j}V_A\omega_j/\omega_{j0} \lesssim \omega_j \).

Next, in a more general way and in presence of the cosmic rays diffusion, i.e., when \( \nu_c \neq 0 \), we can also similarly discuss the characteristics of different kinds of wave modes including those in the particular cases of \( \lambda = 0 \) and \( \lambda = \pi/2 \). However, instead of the classical Jeans mode as in equation (21), we can have a modified Jeans mode, given by,

\[ \omega^2 - C_g^2k^2 + \omega_j^2 = 0, \]

which may exhibit both the damping and the instability (due to the effects of \( \nu_c \)) similar to those displayed in figure 2. Also, in addition to the fast and slow Alvén waves [cf. equation (22)] there may appear a new wave mode in the intermediate frequency range, i.e., in between the frequencies of the modified slow and fast Alvén modes. In the following section 3.3, we will discuss these features in a more general situation, i.e., when the wave propagation direction and the axis of rotation are oblique with respect to the external magnetic field.

### 3.3. Propagation at an arbitrary direction

In this section, we study the characteristics of MHD waves in a more general situation, i.e., when the direction of propagation \( \theta \) is not necessarily 0 or \( \pi/2 \), but may assume any value in the interval \( 0 \leq \theta \leq \pi/2 \), and the angle \( \lambda \) between the axis of rotation and the y-axis is also arbitrary in the interval \( 0 \leq \lambda \leq \pi/2 \). Furthermore, we assume that the diffusion frequency \( \nu_c \) is not necessarily too small but may be of moderate value, i.e., \( \nu_c, \Omega_0, \omega_0 \approx \omega_j \). Figure 3 shows the dispersion curves [subplot (a)] and the instability growth rates [subplot (b)] of obliquely propagating MHD waves for two different values of \( \theta \) as in the legends. The fixed parameter values are \((C_p, C_g, V_A) = (0.5, 0.4, 0.5)C_s\) and \((\nu_c, \Omega_0, \omega_0) = (0.1, 0.2, 0.3)\omega_j\).

\[ \begin{array}{c}
\text{Figure 3. The dispersion curves [subplot (a)] and the instability growth rates [subplot (b)] of obliquely propagating MHD waves [equation (14)] are shown for different values of the angle of propagation} \theta \text{ as in the legends. The fixed parameter values are} \ (C_p, C_g, V_A) = (0.5, 0.4, 0.5)C_s \ \\
\text{and} \ (\nu_c, \Omega_0, \omega_0) = (0.1, 0.2, 0.3)\omega_0. 
\end{array} \]
In what follows, we have also studied the characteristics of the wave frequencies and the instability growth rates of oblique modes by the effects of the parameters that are associated with the gas and cosmic rays pressures ($C_g$ and $C_c$), and the magnetic field intensity ($V_A$). The results are displayed in figure 4. While subplots (a), (b) and (c) are for the dispersion curves, the corresponding growth rates are shown in subplots (d), (e) and (f). From subplots (a) and (b) we note that the thermal gas pressure and the cosmic rays pressure have the similar effects on the slow and fast oblique Alfvén waves in increasing the wave frequencies. However, while the frequency of the slow Jeans mode increases with an increasing value of $C_g$, it decreases with increasing values of $C_c$. Here, both $C_g$ and $C_c$ do not have significant impacts on the slow oblique Alfvén waves. Furthermore, while the parameter $V_A$ has no effect on the slow Jeans mode, it increases (reduces) the frequency of slow (fast) oblique Alfvén modes [See subplot (c)]. On the other hand, subplots (d), (e) and (f) show that the growth rates for all the modes are increased with increasing values of $C_g$, $C_c$ and $V_A$ except for the slow Jeans mode which exhibits stabilizing behaviors with zero growth rate. The enhancement of the growth rate is, however, noticeable by the effects of $C_g$ and $V_A$.

It is also instructive to study the influences of the parameters $\Omega_0$ and $\omega_J$ associated with the Coriolis and gravitational forces on the dispersion curves and the growth rates of instability of oblique MHD waves which are exhibited in figure 5. As is seen, the rotational frequency of cosmic fluids does not have any significant influence on the Jeans modes as expected. However, it reduces (increases) the frequency of slow (fast) oblique Alfvén modes [subplot (a)]. Subplot (b) shows that the wave frequencies for all the modes are reduced by the effects of the Jeans frequency with its increased values. On the other hand, subplots (c) and (d) show that the effects of $\Omega_0$ are to enhance the growth rate of the fast oblique Alfvén mode, but to reduce that of the fast Jeans mode. It has no
significant effect on the instability of slow Alfvén mode. However, the instability growth rates for all the modes (except for the slow Jeans mode) are significantly reduced by the effects of the Jeans frequency. It is also noted that the effect of the cosmic rays diffusion frequency \( \nu_c \) has no significant impact on the wave modes and instability growth rates except that it enhances the frequencies of the slow and fast Jeans modes. This happens as from equation (14) it is evident that \( \nu_c \) contributes only to the factors \( (\tilde{C}_s^2) \) associated with the Jeans modes.

4. Summary and conclusion

We have studied the generation of different kinds of MHD waves as well as the characteristics of the wave dispersion and the damping or growth rates of instability in rotating cosmic magnetoplasmas under the influences of the self-gravitational force, the Coriolis force, both the thermal gas and cosmic rays pressures as well as the cosmic rays diffusion. The main results can be summarized as follows:

- The Jeans, Alfvén and magnetosonic waves are shown to be unstable due to the effects of cosmic rays diffusion. They can be damped or anti-damped within the finite domains of wave numbers above or below some critical values. The latter are reduced by the effects of the Coriolis force and the cosmic-ray pressure.
- For the propagation of MHD waves perpendicular to the magnetic field, when the Jeans frequency is less than or comparable to the rotational frequency, i.e., \( \omega_J \lesssim 2\Omega_0 \), the damping rate of Jeans-Alfvén-magnetosonic...
modes can be reduced by either increasing the thermal gas pressure (or the magnetic field intensity) or decreasing the cosmic rays pressure (or the Jeans frequency). On the other hand, when \( \omega_{J} > 2\Omega_{0} \), both the damping and instability can occur, however, in two different domains of \( k \): \( k > k_{J} \) and \( k < k_{J} \) respectively, where \( k_{J} \) is the reduced Jeans critical wave number. However, the instability prevails if the self-gravity force strongly dominates over the Coriolis force.

- When the propagation direction of MHD waves is parallel to that of the magnetic field, the coupling of Jeans and Alfvén waves occurs due to the obliqueness \( (0 \leq \lambda \leq \pi/2) \) of the axis of rotation about the magnetic field. They get decoupled when \( \Omega_{\parallel}B_{0} \), i.e., \( \lambda = \pi/2 \). In the latter, when the effects of the cosmic rays diffusion is ignored, the classical Jeans mode (modified by the cosmic rays pressure) as well as the fast and slow Alfvén modes (modified by the Coriolis force) are recovered. The fast (slow) Jeans-Alfvén mode is, however, stable (unstable) when \( \lambda = 0 \).

- In a more general situation with the effects of the cosmic rays pressure and diffusion, when the propagation direction and the orientation of the axis of rotation are arbitrary, a new MHD mode, to be called fast Jeans mode, is found to exist for \( 0 \leq \theta < \pi/2 \) and \( 0 \leq \lambda \leq \pi/2 \) in the intermediate frequency ranges of the slow and fast oblique Alfvén waves. While the slow Jeans mode exhibits stabilizing behaviors, the other modes are always unstable.

To conclude, it is to be noted that the insentropic plasma equilibrium specified by the density, pressure and magnetic field inhomogeneities may alter the propagation characteristics of MHD waves significantly, especially in astrophysical environments. However, we have limited our model to those regimes where the gradient length scale of the plasma equilibrium is much larger than the wavelengths of perturbations. In these regimes, it is possible to identify the MHD wave modes as distinct modes (i.e., Alfvén, magnetosonic, fast and slow modes). However, in the case of an inhomogeneous background, the MHD modes may not necessarily be distinct. We plan to include this effect of plasma inhomogeneity on the propagation of MHD waves in our future projects. Furthermore, apart from the wave damping of MHD waves to occur, the growth rate of instability can be typically large in highly magnetized plasmas with strong gravitational effects. Such high growth rates can be sufficient for the seed perturbations to reach at the nonlinear regimes, thereby contributing to the gravitational collapse or the formation of large-scale clouds such as those in spiral arms or galactic centers.

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**Data availability statement**

The data that support the findings of this study are available upon reasonable request from the authors.

**ORCID iDs**

A P Misra \( \text{https://orcid.org/0000-0002-6167-8136} \)

**References**

[1] Parker EN 1966 Astrophys. J. 145 811
[2] Jeans JH 1902 The stability of a spherical nebula Philosophical Transactions of the Royal Society of London. Series A 199 1–53
[3] Asseo E, Cesarsky C, Lachieze-Rey M and Pellat R 1978 Astrophys. J. 225 L21
[4] Kim J, Ryu D and Jones T W 2001 Astrophys. J. 557 464
[5] Bonanno A and Urpin V 2008 Mon. Not. R. Astron. Soc. 388 1679
[6] Heintz E, Bustom C and Zweibel E G 2020 Astrophys. J. 891 157
[7] Howard L N 1962 J. Fluid Mech. 13 158
[8] Giddon J 1966 The Astrophys. J. 145 583
[9] Hamabata H 1984 and T. Namikawa J. Plasma Phys. 31 153
[10] Hamabata H and Namikawa T 1985 J. Plasma Phys. 33 437
[11] Hamabata H and Namikawa T 1985 J. Plasma Phys. 33 443
[12] Singh B and Kalra G 1986 Astrophys. J. 304 6
[13] Bora M and Nayyar N 1991 Astrophys. Space Sci. 179 313
[14] Prajapati R, Parihar A and Chhajlani R 2008 Phys. Plasmas 15 012107
[15] Kossacki K 1961 Acta Astron. 11 83 (https://adsabs.harvard.edu/full/1961AcA....11...83K)
[16] Prajapati R, Soni G and Chhajlani R 2008 Phys. Plasmas 15 062108
[17] Chhajlani R and Purohit P 1985 Beiträge aus der Plasmafisophysik 25 615
[18] Kalra G, Hosking R and Talwar S 1970 Astrophys. Space Sci. 9 34
[19] Ariel P 1970 The Physics of Fluids 13 1644
[20] Vyas M and Chhajlani R 1988 Astrophys. Space Sci. 140 89
[21] Chhajlani R and Parihar A 1993 Contrib. Plasma Phys. 33 227
[22] Bhatia P and Hazarika A R 1995 Phys. Scr. 51 775
[23] Ren H, Wu Z, Cao J and Chu P K 2009 Phys. Plasmas 16 072101
[24] Jacobs G and Shukla P K 2004 Phys. Scr. 70 262
[25] Sharma K 1982 Astrophys. Space Sci. 85 263
[26] Chhajlani R and Sanghvi R 1986 Astrophys. Space Sci. 124 33
[27] Chhajlani R and Parihar A 1994 Astrophys. J. 422 746
[28] Herrnegger F 1972 J. Plasma Phys. 8 393
[29] Bhatia P K and Gupta O 1973 Publ. Astron. Soc. Japan 25 541
[30] Chhonkar R and Bhatia P 1977 J. Plasma Phys. 18 273
[31] Vaghela D and Chhajlani R 1990 Contrib. Plasma Phys. 30 783
[32] Sharma P and Patidar A 2021 Phys. Plasmas 28 052116
[33] Giacalone J and Jokipii J R 1999 Astrophys. J. 520 204
[34] Natwariya P K and Bhatt A K 2020 Jitesh R. andPandey The European Physical Journal C 80 767
[35] Kim W-T, Ostriker E C and Stone J M 2002 Astrophys. J. 581 1080