Quadratic Embedding Constants of Graph Joins

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Abstract
The quadratic embedding constant (QE constant) of a graph is a new characteristic value of a graph defined through the distance matrix. We derive formulae for the QE constants of the join of two regular graphs, double graphs and certain lexicographic product graphs. Examples include complete bipartite graphs, wheel graphs, friendship graphs, completely split graph, and some graphs associated to strongly regular graphs.

Keywords Distance matrix · Double graph · Graph join · Lexicographic product graph · Quadratic embedding constant · Strongly regular graphs

Mathematics Subject Classification Primary 05C50 · Secondary 05C12 · 05C76

1 Introduction
In the study of spectral characteristics of a graph the adjacency matrix and the Laplacian matrix have played central roles, see e.g., [5, 10, 15, 39]. For further development of spectral graph theory there is growing interest in the distance matrix
keeping a profound relation to Euclidean distance geometry. For some recent works on distance spectrum see e.g., [2, 27–29]. The quadratic embedding constant (QE constant) of a graph, recently introduced by means of the distance matrix, is a new characteristic value of a graph [35]. In this paper we derive formulae for the QE constants of the join of two regular graphs, double graphs and certain lexicographic product graphs. Moreover, we give some computational examples related to strongly regular graphs. Our results support potential interest in classifying graphs in terms of the QE constants.

Let \( G = (V, E) \) be a finite connected graph (always assumed to be simple and undirected), where \( V \) is the set of vertices and \( E \) the set of edges. Let \( d(x,y) = d_G(x,y) \) be the distance between two vertices \( x,y \in V \), that is, the length of a shortest path connecting them, and \( D = [d(x,y)]_{x,y \in V} \) the distance matrix. Various realizations of a graph in a Euclidean space or in a more general metric space have been discussed from different aspects, see [17] and references cited therein, and also [18, 25, 30]. A quadratic embedding of a graph \( G = (V, E) \) in a Euclidean space \( \mathbb{R}^N \) is a map \( \varphi : V \to \mathbb{R}^N \) satisfying

\[
\| \varphi(x) - \varphi(y) \|^2 = d(x,y), \quad x,y \in V,
\]

where the left-hand side is the square of the Euclidean distance between two points \( \varphi(x) \) and \( \varphi(y) \). A graph \( G \) is called of QE class or of non-QE class according as \( G \) admits a quadratic embedding or not.

The concept of quadratic embedding traces back to the early works of Schoenberg [37, 38] (see also [40]) and has been studied considerably along with Euclidean distance geometry [1, 3, 23, 24, 26]. Schoenberg’s theorem says that a finite connected graph \( G \) is of QE class if and only if the distance matrix \( D \) is conditionally negative definite, namely, \( \langle f, Df \rangle \leq 0 \) for all \( f \in C(V) \) with \( \langle 1, f \rangle = 0 \), where \( C(V) \) is the space of \( \mathbb{R} \)-valued functions on \( V \), \( 1 \in C(V) \) the constant function taking value one, and \( \langle \cdot, \cdot \rangle \) the canonical inner product on \( C(V) \). Moreover, it is noteworthy that \( D \) is conditionally negative definite if and only if the \( Q \)-matrix \( Q = [q^d(x,y)]_{x,y \in V} \) is positive definite (allowing zero eigenvalues) for all \( 0 \leq q \leq 1 \). The last condition is essential for noncommutative harmonic analysis on free groups [7] and \( q \)-deformed spectral analysis of growing graphs [20].

A new quantitative approach was initiated in the recent paper [35]. For a finite connected graph \( G = (V, E) \) with \( |V| \geq 2 \) the quadratic embedding constant (QE constant for short) is defined by

\[
\text{QEC}(G) = \max \{ \langle f, Df \rangle ; f \in C(V), \langle f, f \rangle = 1, \langle 1, f \rangle = 0 \}. \tag{1.1}
\]

By definition, a graph on at least two vertices admits a quadratic embedding if and only if \( \text{QEC}(G) \leq 0 \). Thus, the QE constant gives rise to a criterion for graphs to be of QE class or not. Moreover, the QE constant is interesting for itself as a new numerical invariant of graphs. In the previous papers [32, 34, 35] we obtained explicit values of the QE constants of particular graphs and their estimates in relation to graph operations. More concrete examples of QE constants have been obtained in the recent papers [22, 31, 36]. We are also interested in classifying the finite
connected graphs in terms of the QE constants, see e.g., [4], where we started an attempt to characterize graphs along with the increasing sequence of the QE constants of paths.

Another interesting aspect of the QE constant is found in a relation to the minimal eigenvalue of the adjacency matrix. In fact, for a connected strongly regular graph $G$ we will derive the simple relation:

$$\text{QEC}(G) = -2 - \lambda_{\text{min}}(G),$$

see Propositions 2.3 and 5.1. Therefore, for a connected strongly regular graph $G$ the condition $\text{QEC}(G) \leq 0$ is equivalent to $\lambda_{\text{min}}(G) \geq -2$. On the other hand, graphs with smallest eigenvalue at least $-2$ have been extensively studied and strongly regular graphs with such property are classified [9, 12, 13]. In this aspect too the QE constant is expected to provide an interesting characteristic value of a graph.

Finally, as is expected from the definition (1.1) the QE constant is closely related to the distance spectrum, for the comprehensive account see [2] and references cited therein. The eigenvalues of the distance matrix $D$ are arranged as

$$\delta_1(G) > \delta_2(G) \geq \delta_3(G) \geq \cdots \geq \delta_n(G), \quad n = |V|,$$

where the first strict inequality, due to the Perron-Frobenius theorem, means that the maximal eigenvalue of $D$ is simple. In this line a remarkable relation to the QE constant is shown by the inequalities:

$$\delta_2(G) \leq \text{QEC}(G) < \delta_1(G),$$

the proof of which is by the standard min-max theorem. It is an interesting open question to characterize graphs with $\delta_2(G) = \text{QEC}(G)$.

This paper is organized as follows. In Sect. 2 we review some methods for calculating the QE constants as well as their basic properties. In Sect. 3 we derive a formula for the QE constant of the join $G_1 + G_2$ of regular graphs $G_1$ and $G_2$ (Theorem 3.1) and show typical examples. In Sect. 4 we discuss the double graph $\text{Double}(G)$ and lexicographic product graph $G \triangleright K_2$, which are subgraphs of the graph join $G + G$. We see that the formulae for their QE constants (Theorems 4.2 and 4.7) are compatible to the ones for distance spectra [21]. In Sect. 5 we examine some strongly regular graphs and their graph joins, and construct families of graphs which are not of QE class (Theorem 5.4).

2 Preliminaries

2.1 Some Basic Properties of QE Constants

As first examples, for the complete graphs and complete bipartite graphs we have

$$\text{QEC}(K_n) = -1, \quad n \geq 2,$$  \hfill (2.1)
\[
\text{QEC}(K_{m,n}) = \frac{2(mn - m - n)}{m + n}, \quad m \geq 1, \quad n \geq 1. \tag{2.2}
\]

For cycles we have

\[
\text{QEC}(C_n) = \begin{cases} 
-\left(4 \cos^2 \frac{\pi}{n}\right)^{-1}, & \text{if } n \geq 3 \text{ is odd,} \\
0, & \text{if } n \geq 4 \text{ is even.} 
\end{cases} \tag{2.3}
\]

For paths $\text{QEC}(P_n)$ can be seen [31] and a sharp estimate can be seen [32]. More examples of QE constants can be found in [4, 34, 35].

Let $G = (V, E)$ be a finite connected graph and $H = (W, F)$ a connected subgraph with $|W| \geq 2$. Let $D_G$ and $D_H$ be the distance matrices of $G$ and $H$, respectively. If $H$ is isometrically embedded in $G$, that is,

\[
d_H(x, y) = d_G(x, y), \quad x, y \in W,
\]

the distance matrix $D_H$ becomes a submatrix of $D_G$ and by definition we have

\[
\text{QEC}(H) \leq \text{QEC}(G).
\]

In particular, every finite connected graph $G = (V, E)$ with $|V| \geq 2$ fulfills

\[
-1 \leq \text{QEC}(G),
\]

since $G$ contains $K_2$ isometrically and $\text{QEC}(K_2) = -1$. Moreover, if $G = (V, E)$ is not complete, it contains $P_3$ isometrically and we have

\[
\text{QEC}(P_3) = -\frac{2}{3} \leq \text{QEC}(G).
\]

Hence $\text{QEC}(G) = -1$ if and only if $G$ is a complete graph.

### 2.2 Formulae for QE Constants

We first recall a general formula for the QE constant (1.1) for a graph $G = (V, E)$ with $V = \{1, 2, \ldots, n\}, \ n \geq 3$. Following the method of Lagrange's multipliers we consider

\[
F(f, \lambda, \mu) = f(Df) - \lambda(f, f) - 1 - \mu(1, f). \tag{2.4}
\]

Identifying $f \in C(V)$ with $[f_i] \in \mathbb{R}^n$ in such a way that $f(i) = f_i$, we regard $F(f, \lambda, \mu)$ as a function of $(f = [f_i], \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$. Let $S(D)$ be the set of all stationary points of $F(f, \lambda, \mu)$, that is, the set of $(f, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ such that

\[
\frac{\partial F}{\partial f_i} = \ldots = \frac{\partial F}{\partial f_n} = \frac{\partial F}{\partial \lambda} = \frac{\partial F}{\partial \mu} = 0. \tag{2.5}
\]

The above equations are written explicitly as follows:
\[ \frac{\partial F}{\partial f_i} = 2Df(i) - 2\lambda f(i) - \mu = 0, \quad i = 1, 2, \ldots, n, \quad (2.6) \]

or equivalently,
\[ (D - \lambda)f = \frac{\mu}{2} I, \quad (2.7) \]

and
\[ \langle f, f \rangle = 1, \quad (2.8) \]
\[ \langle 1, f \rangle = 0. \quad (2.9) \]

We then see that \( S(D) \) is the set of \((f, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \) satisfying (2.7)–(2.9).

**Proposition 2.1** ([35]) Let \( G = (V, E) \) be a finite connected graph on \( n = |V| \geq 3 \) vertices. Then we have
\[ \text{QEC}(G) = \max \Lambda(D), \]
where \( \Lambda(D) \) is the projection of \( S(D) \) onto the \( \lambda \)-axis, that is, the set of \( \lambda \in \mathbb{R} \) such that \((f, \lambda, \mu) \in S(D) \) for some \( f \in \mathbb{R}^n \) and \( \mu \in \mathbb{R} \).

The diameter of a finite connected graph \( G = (V, E) \) is defined by
\[ \text{diam}(G) = \max \{d(x, y) \mid x, y \in V\}. \]

Recall that the adjacency matrix \( A = [a(x, y)]_{x,y \in V} \) of \( G \) is defined by setting \( a(x, y) = 1 \) if \( x \) and \( y \) are adjacent and \( a(x, y) = 0 \) otherwise. If \( 1 \leq \text{diam}(G) \leq 2 \), then \( |V| \geq 2 \) and we have
\[ D = 2J - 2I - A, \quad (2.10) \]
where \( J \) is the matrix whose entries are all one, and \( I \) the identity matrix. (The symbols \( J \) and \( I \) are used without specifying their sizes.)

**Proposition 2.2** ([34]) For a finite connected graph \( G \) with \( 1 \leq \text{diam}(G) \leq 2 \) we have
\[ \text{QEC}(G) = -2 - \min \{\langle f, Af \rangle \mid f \in C(V), \langle f, f \rangle = 1, \langle 1, f \rangle = 0\}. \quad (2.11) \]

**Proposition 2.3** For a finite connected regular graph \( G \) with \( 1 \leq \text{diam}(G) \leq 2 \) we have
\[ \text{QEC}(G) = -2 - \lambda_{\min}(G), \quad (2.12) \]
where \( \lambda_{\min}(G) = \min \text{ev}(A) \) is the minimal eigenvalue of the adjacency matrix \( A \).
Proof If \( G \) is a regular connected graph of degree \( r \geq 0 \), we have \( r \in \text{ev}(A) \) and \( AI = rI \). Moreover, it is known that \( r \) is the maximal eigenvalue of \( A \) and is simple. Then, (2.12) follows from (2.11) immediately.

Obviously, \( \text{diam}(G) = 1 \) if and only if \( G \) is a complete graph \( G = K_n \) with \( n \geq 2 \). Then \( \text{QEC}(K_n) = -1 \) follows also from Proposition 2.3. Graphs with \( \text{diam}(G) = 2 \) are interesting from several points of view, e.g., [9]. We will discuss in Section 5 some topics related to strongly regular graphs.

2.3 Graph Join

For \( i = 1, 2 \) let \( G_i = (V_i, E_i) \) be a finite graph with \( V_1 \cap V_2 = \emptyset \), and \( A_i \) its adjacency matrix. Set

\[
\tilde{V} = V_1 \cup V_2,
\tilde{E} = E_1 \cup E_2 \cup \{ \{x, y\} ; x \in V_1, y \in V_2 \}.
\]

Then \( \tilde{G} = (\tilde{V}, \tilde{E}) \) becomes a graph, which is called the join of \( G_1 \) and \( G_2 \), and is denoted by

\[ \tilde{G} = G_1 + G_2. \]

With the natural arrangement of rows and columns the adjacency matrix of \( \tilde{G} = G_1 + G_2 \) is written in a block-matrix form:

\[
\tilde{A} = \begin{bmatrix}
A_1 & J \\
J & A_2
\end{bmatrix}.
\]

Note the graph join \( G_1 + G_2 \) becomes connected even when \( G_1 \) and \( G_2 \) are not connected. Moreover, we have \( 1 \leq \text{diam}(G_1 + G_2) \leq 2 \). We then see from (2.10) that the distance matrix of \( \tilde{G} = G_1 + G_2 \) is given by

\[
\tilde{D} = 2J - 2I - \tilde{A} = \begin{bmatrix}
2J - 2I - A_1 & J \\
J & 2J - 2I - A_2
\end{bmatrix}.
\] (2.13)

For the QE constant of \( \tilde{G} = G_1 + G_2 \) we need to investigate \( S(\tilde{D}) \), the set of stationary points of

\[ F(f, \lambda, \mu) = \langle f, \tilde{D}f \rangle - \lambda(\langle f, f \rangle - 1) - \mu(\langle I, f \rangle), \quad f \in C(\tilde{V}), \quad \lambda \in \mathbb{R}, \quad \mu \in \mathbb{R}. \]

According to the block-matrix notation in (2.13) we write

\[
f = \begin{bmatrix}
g \\
h
\end{bmatrix}, \quad g \in C(V_1), \quad h \in C(V_2).
\]

Then after simple algebra, we see that \( S(\tilde{D}) \) is the set of \( (g, h, \lambda, \mu) \in C(V_1) \times C(V_2) \times \mathbb{R} \times \mathbb{R} \) satisfying
\[2(A_1 + \lambda + 2)g - (2(I, g) - \mu)I = 0, \quad (2.14)\]
\[2(A_2 + \lambda + 2)h - (2(I, h) - \mu)I = 0, \quad (2.15)\]
\[\langle g, g \rangle + \langle h, h \rangle = 1, \quad (2.16)\]
\[\langle I, g \rangle = \langle I, h \rangle = 0. \quad (2.17)\]

Then as an immediate consequence of Proposition 2.1 we obtain

**Proposition 2.4** Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be finite (not necessarily connected) graphs with \( V_1 \cap V_2 = \emptyset \). Let \( S(\tilde{D}) \) be the same as above and \( \Lambda(\tilde{D}) \) the projection of \( S(\tilde{D}) \) onto the \( \lambda \)-axis, namely, the set of \( \lambda \in \mathbb{R} \) such that \((g, h, \lambda, \mu) \in S(\tilde{D}) \) for some \( g \in C(V_1), h \in C(V_2) \) and \( \mu \in \mathbb{R} \). Then we have
\[\text{QEC}(G_1 + G_2) = \max \Lambda(\tilde{D}), \quad (2.18)\]

### 3 Join of Regular Graphs

The main purpose of this section is to derive a formula of the QE constant of the join of two regular graphs. The result is stated in the following

**Theorem 3.1** For \( i = 1, 2 \) let \( G_i = (V_i, E_i) \) be a \( r_i \)-regular graph on \( n_i = |V_i| \) vertices with \( V_1 \cap V_2 = \emptyset \), where \( r_i \geq 0 \) and \( n_i \geq 1 \). Then we have
\[\text{QEC}(G_1 + G_2) = -2 + \max \left\{ -\lambda_{\text{min}}(G_1), -\lambda_{\text{min}}(G_2), \frac{2n_1n_2 - r_1n_2 - r_2n_1}{n_1 + n_2} \right\}, \quad (3.1)\]

where \( \lambda_{\text{min}}(G_i) \) is the minimal eigenvalue of the adjacency matrix of \( G_i \).

**Proof** Note that \( r_i = 0 \) is allowed. In that case \( G_i \) is an empty graph on \( n_i \) vertices. For (3.1) we will describe \( S(\tilde{D}) \) explicitly and apply Proposition 2.4.

Since \( G_1 \) is \( r_1 \)-regular, we have \( A_1 I = r_1 I \). Then, taking the inner product of (2.14) with \( I \), we obtain
\[ (\lambda + 2 - n_1 + r_1)\langle I, g \rangle = -\frac{\mu}{2} n_1. \quad (3.2)\]

Similarly, from (2.15) we obtain
\[ (\lambda + 2 - n_2 + r_2)\langle I, h \rangle = -\frac{\mu}{2} n_2, \]

and using (2.17) we have
\[ (\lambda + 2 - n_2 + r_2)\langle I, g \rangle = \frac{\mu}{2} n_2. \quad (3.3)\]

It then follows from (3.2) and (3.3) that
\{(n_1 + n_2)(\lambda + 2) - 2n_1n_2 + r_1n_2 + r_2n_1\}\langle I, g \rangle = 0.

Accordingly, we consider two cases.

(Case 1) \((n_1 + n_2)(\lambda + 2) + n_1r_2 + n_2r_1 - 2n_1n_2 = 0\). This happens when \(\lambda = \lambda^*\), where

\[
\lambda^* = \frac{2n_1n_2 - r_1n_2 - r_2n_1}{n_1 + n_2} - 2.
\] (3.4)

We set

\[
g^* = \sqrt{\frac{n_2}{n_1(n_1 + n_2)}} I, \quad h^* = -\sqrt{\frac{n_1}{n_2(n_1 + n_2)}} I,
\]

and

\[
\mu^* = \frac{2(n_1 - r_1 - n_2 + r_2)}{n_1 + n_2} \sqrt{\frac{n_1n_2}{n_1 + n_2}}.
\]

Then, simple calculation shows that (2.14)–(2.17) are satisfied by \((g^*, h^*, \lambda^*, \mu^*)\). Hence \(\lambda^* \in \Lambda(\bar{D})\).

(Case 2) \(\lambda \neq \lambda^*\) and \(\langle I, g \rangle = 0\). By (2.17) and (3.2) we obtain \(\langle I, h \rangle = \mu = 0\). Moreover, (2.14) and (2.15) become

\[
(A_1 + \lambda + 2)g = 0, \quad (3.5)
\]

\[
(A_2 + \lambda + 2)h = 0, \quad (3.6)
\]

respectively. Thus, our task is to find \((g, h, \lambda, 0)\) satisfying (3.5), (3.6), (2.16) and

\[
\langle I, g \rangle = \langle I, h \rangle = 0. \quad (3.7)
\]

It follows from (3.5) and (3.6) that \(-\lambda - 2 \in \text{ev}(A_1) \cup \text{ev}(A_2)\). In fact, if otherwise we have \(g = 0\) and \(h = 0\) which do not fulfill (2.16). Hence

\[
\Lambda(\bar{D}) \setminus \{\lambda^*\} \subset \{-\lambda - 2 \; ; \; \lambda \in \text{ev}(A_1) \cup \text{ev}(A_2)\}. \quad (3.8)
\]

Now recall that \(r_1 \in \text{ev}(A_1)\) is simple and \(A_1 I = r_1 I\). Let \(\alpha \in \text{ev}(A_1) \setminus \{r_1\}\) and take an eigenvector \(g\) such that \(\langle g, g \rangle = 1\). Since the eigenspaces with distinct eigenvalues are mutually orthogonal, we have \(\langle I, g \rangle = 0\). Then \((g, h = 0, -\alpha - 2, 0)\) satisfies (3.5), (3.6), (2.16) and (3.7). Hence \(-\alpha - 2 \in \Lambda(\bar{D})\). Similarly, if \(\alpha \in \text{ev}(A_2) \setminus \{r_2\}\), we have \(-\alpha - 2 \in \Lambda(\bar{D})\). Thus,

\[
\{-\alpha - 2 \; ; \; \alpha \in \text{ev}(A_1) \setminus \{r_1\}\} \cup \{-\alpha - 2 \; ; \; \alpha \in \text{ev}(A_2) \setminus \{r_2\}\} \subset \Lambda(\bar{D}) \quad (3.9)
\]

Since \(r_1\) is the largest eigenvalue of \(A_1\),

\[
\max\{-\alpha - 2 \; ; \; \alpha \in \text{ev}(A_1)\} = \max\{-\alpha - 2 \; ; \; \alpha \in \text{ev}(A_1) \setminus \{r_1\}\} = -\lambda_{\min}(G_1) - 2.
\]

A similar relation holds for \(r_2\) and \(A_2\). Consequently, we see from (3.8) and (3.9) that
\[
\max \Lambda(D) = \max\{-\lambda_{\min}(G_1) - 2, -\lambda_{\min}(G_2) - 2, \lambda^*\},
\]
which proves (3.1).

**Corollary 3.2** Notations and assumptions being the same as in Theorem 3.1, we have
\[
\text{QEC}(G_1 + G_2) \geq \max\{-2 - \lambda_{\min}(G_1), -2 - \lambda_{\min}(G_2)\}.
\]

**Example 3.3** (complete bipartite graphs [35]) Let \(m \geq 1\) and \(n \geq 1\). The complete bipartite graph \(K_{m,n}\) is the graph join \(K_{m,n} = \bar{K}_m + \bar{K}_n\), where \(\bar{K}_m\) and \(\bar{K}_n\) the empty graphs on \(m\) and \(n\) vertices, respectively. Obviously,
\[
\lambda_{\min}(\bar{K}_m) = 0, \quad m \geq 1.
\]
It then follows from Theorem 3.1 that
\[
\text{QEC}(G_1 + G_2) = -2 + \max\{0, 0, \frac{2mn}{m+n}\} = -2 + \frac{2mn}{m+n}.
\]
Hence
\[
\text{QEC}(K_{m,n}) = \frac{2(mn - m - n)}{m+n}, \quad m \geq 1, \quad n \geq 1.
\]

**Example 3.4** (complete split graph) Let \(m \geq 1\) and \(n \geq 1\). The graph join \(K_n + \bar{K}_m\) is called the complete split graph. Note that
\[
\lambda_{\min}(K_n) = \begin{cases} 0, & \text{for } n = 1, \\ -1, & \text{for } n \geq 2. \end{cases}
\]
For \(n = 1\) we have \(K_n + \bar{K}_m = K_1 + \bar{K}_m = K_1 + \bar{K}_m = K_{1,m}\), which is a special case of Example 3.3. Assume that \(n \geq 2\) and \(m \geq 1\). Then by Theorem 3.1 we obtain
\[
\text{QEC}(K_n + \bar{K}_m) = -2 + \max\{1, 0, \frac{2mn - (n-1)m}{m+n}\} = -2 + \frac{mn + m}{m+n} = \frac{mn - 2n}{m+n}.
\]
In particular, the tri-partite graph \(K_{1,1,m}\) being the join of \(K_2\) and \(\bar{K}_m\), we set \(n = 2\) in (3.12) to get
\[
\text{QEC}(K_{1,1,m}) = \text{QEC}(K_2 + \bar{K}_m) = \frac{m - 4}{m + 2},
\]
see also [35].
Example 3.5 (friendship graphs) Let \( n \geq 1 \). The friendship graph \( F_n = nK_2 + K_1 \), where \( nK_2 \) is the disjoint union of \( n \) copies of \( K_2 \). Note that \( nK_2 \) is a 1-regular graph on \( 2n \) vertices. Since \( \lambda_{\min}(nK_2) = \lambda_{\min}(K_2) = -1 \) and \( \lambda_{\min}(K_1) = 0 \), by Theorem 3.1 we obtain
\[
\text{QEC}(F_n) = -2 + \max \left\{ 1, 0, \frac{4n - 1}{2n + 1} \right\} = -2 + \frac{4n - 1}{2n + 1} = \frac{-3}{2n + 1}.
\]

Example 3.6 Let \( n \geq 3 \) and \( m \geq 1 \). We consider the graph joins \( C_n + K_m \) and \( C_n + \overline{K}_m \). Recall first that
\[
\lambda_{\min}(C_n) = \min \left\{ 2 \cos \frac{2\pi j}{n} ; 1 \leq j \leq n - 1 \right\} = \begin{cases} -2 + 4 \sin^2 \frac{\pi}{2n}, & \text{if } n \text{ is odd}, \\ -2, & \text{if } n \text{ is even}, \end{cases}
\]
of which verification is elementary, see e.g., [10]. Since \( \lambda_{\min}(C_n) \leq \lambda_{\min}(K_n) \) for all \( n \geq 3 \) and \( m \geq 1 \), we obtain
\[
\text{QEC}(C_n + K_m) = -2 + \max \left\{ -\lambda_{\min}(C_n), \frac{mn - 2m + n}{m + n} \right\} = \max \left\{ -2 - \lambda_{\min}(C_n), \frac{mn - 4m - n}{m + n} \right\}.
\]

Case of \( n \) being even in (3.13) is simple. In fact, after simple algebra we obtain
\[
\text{QEC}(C_{2n} + K_m) = \begin{cases} 0, & \text{if } mn - 2m - n \leq 0, \\ \frac{2mn - 4m - 2n}{m + 2n}, & \text{otherwise}, \end{cases}
\]
where \( m \geq 1 \) and \( n \geq 2 \). In case of \( n \) being odd in (3.13), we only mention the following formula:
\[
\text{QEC}(C_{2n-1} + K_m) = \max \left\{ -4 \sin^2 \frac{\pi}{2(2n - 1)}, \frac{2mn - 5m - 2n + 1}{m + 2n - 1} \right\},
\]
where \( n \geq 2 \) and \( m \geq 1 \). For \( n \geq 3 \) the wheel graph \( W_n \) on \( n + 1 \) vertices is the graph join \( W_n = C_n + K_1 \). Then after simple observation we obtain
\[
\text{QEC}(W_n) = \text{QEC}(C_n + K_1) = \begin{cases} -4 \sin^2 \frac{\pi}{2n}, & \text{if } n \text{ is odd}, \\ 0, & \text{if } n \text{ is even}. \end{cases}
\]
The above result was obtained in [34] by different calculation. In a similar and, in fact, simpler fashion we obtain
\[
\text{QEC}(C_n + \overline{K}_m) = \frac{2mn - 4m - 2n}{m + n}, \quad n \geq 3, \quad m \geq 2.
\]
In particular, \( \text{QEC}(C_n + \overline{K}_m) > 0 \) except \((m, n) = (2, 3), (3, 3), (2, 4), \) and
In fact, \( QEC(C_3 + \bar{K}_2) = -\frac{2}{5} \), \( QEC(C_3 + \bar{K}_3) = QEC(C_4 + \bar{K}_2) = 0 \).

Remark 3.7  Let \( G \) be a graph and \( A \) be the adjacency matrix. Following Cvetković [14] an eigenvalue of \( A \) is called a main eigenvalue if it has an eigenvector which is not orthogonal to \( I \). Then a graph \( G \) has exactly one main eigenvalue if and only if \( G \) is regular. In that case this main eigenvalue is equal to the regular degree. It seems interesting to study generalization of Theorem 3.1 along the idea of main and non-main eigenvalues.

4 Two Subgraphs of Graph Join

4.1 Double Graphs Double\((G)\)

Let \( G = (V,E) \) be a finite graph (not necessarily connected). We set

\[
\tilde{V} = V \times \{0, 1\} = \{ (x,i) : x \in V, i \in \{0,1\} \},
\]

\[
\tilde{E} = \{ \{(x,i),(y,j)\} : (x,i) \in \tilde{V}, (y,j) \in \tilde{V}, \{x,y\} \in E \}.
\]

Then \( (\tilde{V}, \tilde{E}) \) becomes a graph, which is called the double graph of \( G = (V,E) \) and is denoted by Double \((G)\). Clearly, Double \((G)\) is a subgraph of \( G + G \). For a review on the double graphs we refer to [33].

The double graph is understood in a slightly informal manner as follows. Divide \( \tilde{V} \) into two parts:

\[
\tilde{V} = V_0 \cup V_1,
\]

\[
V_i = \{ (x,i) : x \in V \}, \quad i = 0, 1.
\]

Then the induced subgraph spanned by \( V_i \) is isomorphic to \( G \). Namely, the double graph Double \((G)\) is a graph obtained from the union of two copies of \( G \) by adding edges between them. The new edges appear between \( (x, 0) \) and \( (y, 1) \) if and only if \( x \in V \) and \( y \in V \) are adjacent.

Example 4.1  As is seen in Fig. 1 we have

![Fig. 1 Double \((K_2)\), Double \((K_3)\) and Double \((P_3)\)
Double \((K_2) = C_4\), Double \((K_3) = K_6 \setminus 3K_2\), Double \((P_3) = K_{2,4}\).

In fact, for the complete graph \(K_n\) we have

\[
\text{Double} (K_n) = K_{2n} \setminus nK_2,
\]

where \(nK_2\) stands for the disjoint union of \(n\) edges.

We are interested in the QE constants of double graphs. As is easily verified, for a graph \(G = (V, E)\) with \(n = |V| \geq 2\) the double graph \(\text{Double} (G)\) is connected if and only if \(G\) is connected.

**Theorem 4.2** Let \(G = (V, E)\) be a finite connected graph with \(n = |V| \geq 2\). Then we have

\[
\text{QEC} (\text{Double} (G)) = 2\text{QEC} (G) + 2. \tag{4.1}
\]

**Proof** Let \(D\) be the distance matrix of \(G\). Then, with the natural arrangement of rows and columns the distance matrix \(\tilde{D}\) of the double graph \(\text{Double} (G)\) is written in the form:

\[
\tilde{D} = \begin{bmatrix} D & D + 2I \\ D + 2I & D \end{bmatrix}. \tag{4.2}
\]

According to the general argument in Sect. 2.2 we consider

\[
F(g, h, \lambda, \mu) = \left\langle \begin{bmatrix} g \\ h \end{bmatrix}, \tilde{D} \begin{bmatrix} g \\ h \end{bmatrix} \right\rangle - \lambda(\langle g, g \rangle + \langle h, h \rangle - 1) - \mu(\langle I, g \rangle + \langle I, h \rangle),
\]

where \(g \in C(V_0), h \in C(V_1), \lambda \in \mathbb{R}\) and \(\mu \in \mathbb{R}\). Let \(S(\tilde{D})\) be the set of all stationary points of \(F(g, h, \lambda, \mu)\). By Proposition 2.1 we have

\[
\text{QEC} (\text{Double} (G)) = \max \Lambda(\tilde{D}), \tag{4.3}
\]

where \(\Lambda(\tilde{D})\) is the projection of \(S(\tilde{D})\) onto the \(\lambda\)-axis, namely, the set of \(\lambda \in \mathbb{R}\) such that \((g, h, \lambda, \mu) \in S(\tilde{D})\) for some \(g \in C(V_0), h \in C(V_1)\) and \(\mu \in \mathbb{R}\).

After simple calculation we see that \(S(\tilde{D})\) is the set of solutions to the following system of equations:

\[
2Dh + 2Dg + 4g - 2\lambda h - \mu I = 0, \tag{4.4}
\]

\[
2Dh + 2Dg + 4h - 2\lambda g - \mu I = 0, \tag{4.5}
\]

\[
\langle h, h \rangle + \langle g, g \rangle = 1, \tag{4.6}
\]
\[ \langle I, h \rangle + \langle I, g \rangle = 0. \quad (4.7) \]

Taking the difference of (4.4) and (4.5), we obtain
\[ (\lambda + 2)(h - g) = 0. \]

Here we do not need to check whether \( \lambda = -2 \) belongs to \( \Lambda(\tilde{D}) \) because of (4.3) and the fact that the QE constant is always \( \geq -1 \).

Thus we consider the case of \( f = g \). Equations (4.4)–(4.7) are reduced to the following equations:
\[ (4D + 4 - 2\lambda)h = \mu I, \quad \langle h, h \rangle = \frac{1}{2}, \quad \langle I, h \rangle = 0. \quad (4.8) \]

On the other hand, in computing \( \text{QEC}(G) \) we consider the set \( S(D) \) of all \((f_1, \hat{\lambda}_1, \mu_1) \in C(V) \times \mathbb{R} \times \mathbb{R} \) satisfying
\[ (D - \hat{\lambda}_1)f_1 = \frac{\mu_1}{2} I, \quad \langle f_1, f_1 \rangle = 1, \quad \langle I, f_1 \rangle = 0. \quad (4.9) \]

We see from (4.8) and (4.9) that the correspondence
\[ h = \frac{1}{\sqrt{2}} f_1, \quad \lambda = 2\hat{\lambda}_1 + 2, \quad \mu = \sqrt{2} \mu_1, \]

gives rise to a one-to-one correspondence between \( S(D) \) and \( \{(f, f, \lambda, \mu) \in S(\tilde{D})\} \). Consequently,
\[ \text{QEC}(\text{Double}(G)) = \max \{\lambda; (g, h, \lambda, \mu) \in S(\tilde{D})\} \]
\[ = \max \{2\hat{\lambda}_1 + 2; (f_1, \hat{\lambda}_1, \mu_1) \in S(D)\} \]
\[ = 2\text{QEC}(G) + 2, \]

as desired. \( \square \)

**Corollary 4.3** Let \( G = (V, E) \) be a finite connected graph with \( n = |V| \geq 2 \) vertices. Then we have
\[ \text{QEC}(\text{Double}(G)) \geq 0. \]

Moreover, \( \text{QEC}(\text{Double}(G)) = 0 \) if and only if \( G = K_n \) is a complete graph.

**Proof** Since \( \text{QEC}(G) \geq -1 \) for any finite connected graph with \( n = |V| \geq 2 \) vertices, we have
\[ \text{QEC}(\text{Double}(G)) = 2 \text{QEC}(G) + 2 \geq 2(-1) + 2 = 0. \]

The equality occurs if and only if \( \text{QEC}(G) = -1 \), that is, \( G \) is a complete graph. \( \square \)

**Example 4.4** Since \( \text{QEC}(K_n) = -1 \) for \( n \geq 2 \), we have

\[ \text{QEC}(\text{Double}(K_n)) = 2 \text{QEC}(K_n) + 2 \geq 2(-1) + 2 = 0. \]
QEC(\text{Double }(K_n)) = 2QEC(K_n) + 2 = 0, \quad n \geq 2.

It is noted that \text{Double }(K_n) = K_{2n} \setminus nK_2. By elementary calculus we have

\begin{align*}
QEC(K_n \setminus K_2) &= \frac{-2}{n}, \\
QEC(K_n \setminus rK_2) &= 0, \quad 2 \leq r \leq \left\lceil \frac{n}{2} \right\rceil,
\end{align*}

for details see [35].

\textbf{Remark 4.5} The distance spectrum of \text{Double }(G) is related to that of \( G \) in a similar fashion as in (4.1), for details see [21].

\section*{4.2 Lexicographic Product Graphs \( G \triangleright K_2 \)}

The lexicographic product \( G \triangleright K_2 \) gives another subgraph of the graph join \( G + G \), for the general definition of the lexicographic product, see e.g., [19].

Here we introduce the lexicographic product \( G \triangleright K_2 \) in a slightly informal manner. Let \( G = (V, E) \) be a graph (not necessarily connected). The lexicographic product \( G \triangleright K_2 \) is a graph \( (\tilde{V}, \tilde{E}) \) defined by

\begin{align*}
\tilde{V} &= V \times \{0,1\} = \{(x,i) ; x \in V, i \in \{0,1\}\}, \\
\tilde{E} &= \{(x,i),(y,j) \} ; \ (x,i) \in \tilde{V}, \ (y,j) \in \tilde{V}, \ \{x,y\} \in E \text{ or } x = y, i \neq j\}.
\end{align*}

In other words, \( G \triangleright K_2 \) is obtained from \text{Double }(G) by adding new edges of the form \( \{(x,0),(x,1)\} \), \( x \in V \). Obviously, \( G \triangleright K_2 \) is connected if and only if so is \( G \).

\textbf{Example 4.6} As is shown in Fig. 2 we have

\( K_2 \triangleright K_2 = K_4 \), \quad \( K_3 \triangleright K_2 = K_6 \), \quad \( P_3 \triangleright K_2 = K_6 \setminus C_4 \).

In fact, \( K_n \triangleright K_2 = K_{2n} \).

Let \( G = (V,E) \) be a connected graph with \( n = |V| \geq 2 \) and \( D \) the distance matrix. The distance matrix \( \tilde{D} \) of the lexicographic product \( G \triangleright K_2 \) is written in the form:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{\( K_2 \triangleright K_2, K_3 \triangleright K_2 \) and \( P_3 \triangleright K_2 \)}
\end{figure}
\[D = \begin{bmatrix} D & D + I \\ D + I & D \end{bmatrix},\] (4.10)

which is similar to the distance matrix of the double graph Double(G). Then after similar argument as in the case of the double graphs we come to the following

**Theorem 4.7** Let \(G = (V, E)\) be a finite connected graph with \(n = |V| \geq 2\) vertices. Then we have

\[
\text{QEC}(G \triangleright K_2) = 2\text{QEC}(G) + 1.
\] (4.11)

**Remark 4.8** It follows from Theorem 4.7 that \(G \triangleright K_2\) is not of QE class if and only if \(\text{QEC}(G) > -1/2\). In this relation characterizing finite connected graphs \(G\) with \(\text{QEC}(G) \leq -1/2\) is an interesting question, as is mentioned in Introduction.

**Remark 4.9** The distance spectrum of \(G \triangleright K_2\) is related to that of \(G\) in a similar fashion as in (4.11), for details see [21].

### 5 Strongly Regular Graphs

A **strongly regular graph** with parameters \((n, r, e, f)\) is an \(r\)-regular graph on \(n\) vertices such that any two adjacent vertices have \(e\) common neighbors and any two non-adjacent vertices have \(f\) common neighbors. By definition the complete graph \(K_n\) and the empty graph \(\overline{K}_n\) are not strongly regular. Strongly regular graphs, tracing back to Bose [6], have been extensively studied from various points of view, for example see [10, 11, 16] as well as the large list [8].

In this section we are interested in the QE constant of a strongly regular graph. Note that for a connected strongly regular graph \(G\) we have \(n \geq 4, 2 \leq r \leq n - 2, f \geq 1\) and \(\text{diam}(G) = 2\).

**Proposition 5.1** Let \(G\) be a connected strongly regular graph with parameters \((n, r, e, f)\). Then we have

\[
\text{QEC}(G) = -2 - \lambda_{\text{min}}(G) = -2 + \frac{(f - e) + \sqrt{(f - e)^2 + 4(r - f)}}{2},
\]

where \(\lambda_{\text{min}}(G)\) is the minimal eigenvalue of the adjacency matrix of \(G\).

**Proof** The first equality follows from Proposition 2.3. For the second equality let \(A\) be the adjacency matrix of \(G\). An elementary observation leads

\[
(A - rI)(A^2 + (f - e)A + (f - r)I) = 0,
\]

from which we obtain
\[ \lambda_{\min}(G) = \frac{-(f - e) - \sqrt{(f - e)^2 + 4(r - f)}}{2}, \]

for details see e.g., [10, 15]. \hfill \Box

**Theorem 5.2** Let \( G \) be a connected strongly regular graph. Then \( \text{QEC}(G) \leq 0 \) (resp. \( \text{QEC}(G) \geq 0 \)) if and only if \( \lambda_{\min}(G) \geq -2 \) (resp. \( \lambda_{\min}(G) \leq -2 \)).

**Proof** A direct consequence from Proposition 5.1. \hfill \Box

Graphs with smallest eigenvalue at least \(-2\) have been extensively studied after the famous classification [12]. Here we focus on the following significant result.

**Proposition 5.3** ([9]) Let \( G \) be a connected regular graph on \( n \geq 1 \) vertices. If \( \lambda_{\min}(G) \geq -2 \), then either \( G \) is a complete graph \( K_n \) or \( G \) is a cycle \( C_n \) with odd \( n \geq 3 \).

**Theorem 5.4** For any connected strong regular graph \( \Gamma \) except \( C_5 \) we have

\[ \text{QEC}(\Gamma + K_m) \geq 0, \quad \text{QEC}(\Gamma + \bar{K}_m) \geq 0, \quad m \geq 1. \quad (5.1) \]

**Proof** Let \( \Gamma \) be a connected strong regular graph and suppose that \( \lambda_{\min}(\Gamma) > -2 \). It follows from Proposition 5.3 that either \( \Gamma \) is a complete graph \( K_n \) or \( \Gamma \) is a cycle \( C_n \) with odd \( n \geq 3 \). Among them only \( C_5 \) is strongly regular. Therefore, for any connected strong regular graph \( \Gamma \) except \( C_5 \) we have \( \lambda_{\min}(\Gamma) \leq -2 \). For such a graph \( \Gamma \), with the help of Corollary 3.2 we see that

\[ \text{QEC}(\Gamma + K_m) \geq -2 - \lambda_{\min}(\Gamma) \geq 0, \quad m \geq 1. \]

Similarly, \( \text{QEC}(\Gamma + \bar{K}_m) \geq 0 \) is shown. \hfill \Box

**Example 5.5** (cycle \( C_5 \)) Already studied in Example 3.6. First note that

\[ \lambda_{\min}(C_5) = -\frac{1 + \sqrt{5}}{2} > -2. \]

We have

\[ \text{QEC}(C_5 + K_1) = \text{QEC}(C_5 + K_2) = -\frac{3 - \sqrt{5}}{2} < 0, \]

\[ \text{QEC}(C_5 + K_m) = \frac{m - 5}{m + 5}, \quad m \geq 3, \]

\[ \text{QEC}(C_5 + \bar{K}_m) = \frac{6m - 10}{m + 5}, \quad m \geq 2. \]

For strongly regular graphs the following result is very interesting. For definitions and properties of the graphs in the statement, see e.g., [10, 12, 16].
Proposition 5.6 ([13]) Let $\Gamma$ be a connected, coconnected, strongly regular graph with $\lambda_{\text{min}}(\Gamma) \geq -2$. Then $\Gamma$ is either a 5-cycle $C_5$, a triangular graph $T(n)$ with $n \geq 5$, an $(n \times n)$-grid with $n \geq 3$, the Petersen graph, the Shrikhande graph, the Clebsch graph, the Schlafli graph, or one of the three Chang graphs.

Below we list the QE constants of the joins of a graph mentioned in Proposition 5.6 with $K_n$ and $\overline{K}_n$. The calculation is routine application of Theorem 3.1.

Example 5.7 (triangular graphs) The line graph of the complete graph $K_n$ is called a triangular graph and denoted by $T(n)$ where $n \geq 2$. For $n \geq 4$ the triangular graph $T(n)$ is strongly regular with parameters:

\[
\left( \frac{n(n-1)}{2}, 2(n-2), n-2, 4 \right).
\]

Moreover,

\[
ev(T(n)) = \{2(n-2)^1, (n-4)^{n-1}, (-2)^{n(n-3)/2}\},
\]

\[
\text{QEC}(T(n)) = 0.
\]

For graph joins we have

\[
\text{QEC}(T(n) + K_m) = \max \left\{ 0, \frac{(n-1)(nm-n-4m)}{n(n-1) + 2m} \right\},
\]

\[
\text{QEC}(T(n) + \overline{K}_m) = \max \left\{ 0, \frac{2(n-1)(nm-n-2m)}{n(n-1) + 2m} \right\},
\]

where $n \geq 4$ and $m \geq 1$. Note that $T(4) \cong K_{2,2,2}$ and hence $\overline{T}(4) \cong 3K_2$ is not connected.

Example 5.8 (grids) For $n \geq 2$ the Cartesian product $K_n \times K_n$ is called an $(n \times n)$-grid. The $(n \times n)$-grid is strongly regular with parameters $(n^2, 2(n-1), n-2, 2)$. Moreover,

\[
ev(K_n \times K_n) = \{2(n-1)^1, (n-2)^{2n-2}, (-2)^{n^2-2n+1}\},
\]

\[
\text{QEC}(K_n \times K_n) = 0.
\]

For graph joins we have

\[
\text{QEC}((K_n \times K_n) + K_m) = \max \left\{ 0, \frac{n(nm-n-2m)}{n^2 + m} \right\}, \quad n \geq 2, \ m \geq 1,
\]

\[
\text{QEC}((K_n \times K_n) + \overline{K}_m) = \frac{2n(nm-n-m)}{n^2 + m}, \quad n \geq 2, \ m \geq 2.
\]

Note that $K_2 \times K_2 \cong C_4 \cong K_{2,2}$ and hence $\overline{K}_2 \times \overline{K}_2 \cong 2K_2$ is not connected.

Example 5.9 (Petersen graph) The Petersen graph $\Gamma_1$ is the unique strongly regular graph with parameters $(10, 3, 0, 1)$. We have
ev(\(\Gamma_1\)) = \{3^1, 1^5, (-2)^4\}, \quad \text{QEC}(\Gamma_1) = 0.

For graph joins we have
\[
\begin{align*}
\text{QEC}(\Gamma_1 + K_1) &= \text{QEC}(\Gamma_1 + K_2) = 0, \\
\text{QEC}(\Gamma_1 + K_m) &= \frac{5m - 10}{m + 10} > 0, \quad m \geq 3, \\
\text{QEC}(\Gamma_1 + \bar{K}_m) &= \frac{15m - 20}{m + 10} > 0, \quad m \geq 2.
\end{align*}
\]

**Example 5.10** (Shrikhande graph) There are two strongly regular graphs with parameters (16, 6, 2, 2). One is \((4 \times 4)\)-grid and the other is the Shrikhande graph \(\Gamma_2\). We have
\[
ev(\Gamma_2) = \{6^1, 2^6, (-2)^9\}, \quad \text{QEC}(\Gamma_2) = 0.
\]

For graph joins we have
\[
\begin{align*}
\text{QEC}(\Gamma_2 + K_1) &= \text{QEC}(\Gamma_2 + K_2) = 0, \\
\text{QEC}(\Gamma_2 + K_m) &= \frac{8m - 16}{m + 16} > 0, \quad m \geq 3, \\
\text{QEC}(\Gamma_2 + \bar{K}_m) &= \frac{24m - 32}{m + 16} > 0, \quad m \geq 2.
\end{align*}
\]

**Example 5.11** (Clebsch graph) The (10-regular) Clebsch graph \(\Gamma_3\) is the unique strongly regular graph with parameters (16, 10, 6, 6). We have
\[
ev(\Gamma_3) = \{10^1, 2^5, (-2)^{10}\}, \quad \text{QEC}(\Gamma_3) = 0.
\]

For graph joins we have
\[
\begin{align*}
\text{QEC}(\Gamma_3 + K_m) &= 0, \quad 1 \leq m \leq 4, \\
\text{QEC}(\Gamma_3 + K_m) &= \frac{4m - 16}{m + 16} > 0, \quad m \geq 5, \\
\text{QEC}(\Gamma_3 + \bar{K}_m) &= \frac{20m - 32}{m + 16} > 0, \quad m \geq 2.
\end{align*}
\]

**Example 5.12** (Schläfli graph) The Schläfli graph \(\Gamma_4\) is the unique strongly regular graph with parameters (27, 16, 10, 8). We have
\[
ev(\Gamma_4) = \{16^1, 4^6, (-2)^{20}\}, \quad \text{QEC}(\Gamma_4) = 0.
\]

For graph joins we have
\[
QEC(\Gamma_4 + K_1) = QEC(\Gamma_4 + K_2) = QEC(\Gamma_4 + K_3) = 0,
\]
\[
QEC(\Gamma_4 + K_m) = \frac{9m - 27}{m + 27} > 0, \quad m \geq 4,
\]
\[
QEC(\Gamma_4 + \bar{K}_m) = \frac{36m - 54}{m + 27} > 0, \quad m \geq 2.
\]

**Example 5.13** (Chang graphs) There are four strongly regular graphs with parameters (28, 12, 6, 4). One is the triangular graph \( T(8) \), i.e., the line graph of \( K_8 \). Each of the other three is called a Chang graph. For a Chang graph \( \Gamma_5 \) we have

\[
ev(\Gamma_5) = \{12^1, 4^7, (-2)^{20}\}, \quad QEC(\Gamma_5) = 0.
\]

For graph joins we have

\[
QEC(\Gamma_5 + K_1) = QEC(\Gamma_5 + K_2) = 0,
\]
\[
QEC(\Gamma_5 + K_m) = \frac{14m - 28}{m + 28} > 0, \quad m \geq 3,
\]
\[
QEC(\Gamma_5 + \bar{K}_m) = \frac{42m - 56}{m + 28} > 0, \quad m \geq 2.
\]

Every strongly regular graph in the above Examples 5.7–5.13 has the minimal eigenvalue \(-2\) and hence the zero QE constant. We add some more examples with positive QE constants.

**Example 5.14** (Hoffman–Singleton graph) The Hoffman–Singleton graph \( \Gamma_6 \) is the unique strongly regular graph with parameters (50, 7, 0, 1). We have

\[
ev(\Gamma_6) = \{7^1, 2^{28}, (-3)^{21}\}, \quad QEC(\Gamma_6) = 1.
\]

For graph joins we have

\[
QEC(\Gamma_6 + K_m) = 1, \quad m = 1, 2,
\]
\[
QEC(\Gamma_6 + K_m) = \frac{41m - 50}{m + 50} > 0, \quad m \geq 3,
\]
\[
QEC(\Gamma_6 + \bar{K}_m) = \frac{91m - 100}{m + 50} > 0, \quad m \geq 2.
\]

**Example 5.15** (Higman–Sims graph) The Higman–Sims graph \( \Gamma_7 \) is the unique strongly regular graph with parameters (100, 22, 0, 6). We have

\[
ev(\Gamma_7) = \{22^1, 2^{27}, (-8)^{22}\}, \quad QEC(\Gamma_7) = 6.
\]

For graph joins we have
\( \text{QEC}(\Gamma_7 + K_m) = 6, \quad 1 \leq m \leq 9, \)
\( \text{QEC}(\Gamma_7 + K_m) = \frac{76m - 100}{m + 100} > 0, \quad m \geq 10, \)
\( \text{QEC}(\Gamma_7 + K_m) = 6, \quad 1 \leq m \leq 4, \)
\( \text{QEC}(\Gamma_7 + K_m) = \frac{176m - 200}{m + 100} > 0, \quad m \geq 5. \)

**Example 5.16** (Suzuki graph) The Suzuki graph \( \Gamma_8 \) is a rank 3 strongly regular graph with parameters \((1782, 416, 100, 96)\). We have
\[
\text{ev}(\Gamma_8) = \{416^1, 20^{780}, (-16)^{1001}\}, \quad \text{QEC}(\Gamma_8) = 14.
\]
For graph joins we have
\[
\text{QEC}(\Gamma_8 + K_m) = 14, \quad 1 \leq m \leq 19,
\]
\[
\text{QEC}(\Gamma_8 + K_m) = \frac{1364m - 1782}{m + 1782} > 0, \quad m \geq 20.
\]
\[
\text{QEC}(\Gamma_8 + K_m) = 14, \quad 1 \leq m \leq 9,
\]
\[
\text{QEC}(\Gamma_8 + K_m) = \frac{3146m - 3564}{m + 1782} > 0, \quad m \geq 10.
\]

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