Orbit stability in billiards in magnetic field

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We study the stability properties of orbits in dispersing billiards in a homogeneous magnetic field by using a modified formalism based on the Bunimovich–Sinai curvature (horocycle method). We identify simple periodic orbits that can be stabilized by the magnetic field in the four-disk model and the square-lattice Lorentz gas. The stable orbits can play a key role in determining the transport properties of these models.

I. INTRODUCTION

Recent experiments on ballistic transport in mesoscopic systems [1,2,3] have raised the interest on chaotic billiards where the addition of an external magnetic field can yield simple models expected to account for certain aspects of the observed behaviour. Some of the transport anomalies found in the experiments are thought to have classical origins [4,5,6,7,8,9], which makes classical dynamics in these models worth studying. In addition, the classical orbits are essential ingredients of semiclassical calculations intended to account for these effects. A powerful formalism can be based on periodic orbits and their stability properties [10,11,12]; moreover, classical quantities characterizing the billiard and appearing in various formulae describing transport properties depend very strongly on the hyperbolic or nonhyperbolic nature of the billiard. These considerations indicate that calculating the stability of orbits and locating stable islands in phase space for these models can be of central importance.

The geometry of experiments done so far on microjunctions [1] and antidot-lattices [2,3] can be imitated by models consisting of circular scatterers (disks). The area between four disks centered on the corners of a square (four-disk model) has an adequate geometry for a cross-junction while the square-lattice Lorentz gas [3] corresponds to an antidot lattice with a periodic arrangement of scatterers. In fact, the four-disk model has been studied in connection with cross-junction experiments [1,5], and transport in the Lorentz gas in magnetic field have been investigated to explain some of the magnetoresistance phenomena in antidot-lattices [1,7]. For the above reasons, we will focus on these hard-disk billiards.

Concerning the role of stable orbits, the three-disk model in magnetic field has already been studied from the point of view of chaotic scattering, demonstrating the effect of stable islands created by the field on the global properties of the dynamics [5]. Closed billiards like the stadium in an external magnetic field have been investigated in Ref. [15] where the form of the tangent map for billiards in magnetic field has also been given. In the present paper, we propose an alternative tool, namely the extension of the horocycle method of Bunimovich
and Sinai [16] worked out originally for the calculation of Lyapunov exponents in Euclidean billiards (it has also been applied to billiards on a surface of constant negative curvature [17]). This modified method is considerably simpler to implement than calculations based on tangent maps and makes it easy for us to find stabilized orbits and follow their stability properties as a function of the field strength.

II. DISK MODELS IN MAGNETIC FIELD

We consider the motion of a charged particle of unit mass in a plane colliding with hard disks of radius \( r \) in the presence of a uniform external magnetic field \( h \) perpendicular to the plane; the velocity of the particle is fixed at \( v = 1 \). In addition we set the unit of electric charge so that the cyclotron radius is simply \( s = 1/h \) and choose the signs of the charge and the field so that the particle goes around counterclockwise along the cyclotron orbit.

The disk centers are in the vertices of a square lattice with a period \( 2d = 2 \); the whole lattice filled with disks yields the (periodic) Lorentz gas, while taking a single square only we get the four-disk model. The geometry of the system is controlled by the radii \( r \) and \( s \). Throughout the paper, we will use the cyclotron radius \( s \) measuring the strength of the magnetic field as our control parameter while we keep the disk radius constant at \( r = 1/2 \).

All the numeric values of \( s \) will refer to this case and we will comment on the dependence of our findings on \( r \) when necessary.

During the free motion the particle follows a cyclotron orbit of radius \( s \) around a given center depending on the initial conditions; a collision (considered specular) with a disk puts the particle onto another circular path with the same radius but around a different center. The dynamics as a sequence of collisions can be kept track of by following the center positions of these circles [14]. A jump in the center position always occurs when the particle collides with a disk. Thus, the sequence of the circle centers defines a discrete mapping (a Poincaré map) associated with the time-continuous dynamics. Let \((p, q)\) denote the orbit center at a given instant of time; after collision, a new center \((p', q')\) defines the particle trajectory. The Poincaré map \((p', q') = P(p, q)\) connecting these coordinates relative to the position of the disk on which the collision occurs has been given explicitly in Ref. [14]; we do not need its precise form here.

The complete dynamics can be described by subsequent applications of the map \( P \) with the proper disk center coordinates. \( p \) and \( q \) are canonically conjugate pairs, and the plane \((p, q)\) can be taken as the phase space of our system. They can also be used in conjunction with the disk centers to compute other quantities useful in our study. For example, we will need the impact angles at collisions and the lengths of the arcs between them; they are easy to obtain from the center coordinates. The tangent map of the process can be calculated as the derivative of \( P \) with respect to \( p \) and \( q \) [14], and the result is equivalent to the map given in Ref. [15].

III. LYAPUNOV EXPONENTS AND HOROCYCLES

The stability of a periodic orbit in the Poincaré section is described by its two Lyapunov exponents; due to the Hamiltonian character of the billiard models, the two exponents
add up to zero. Hyperbolic orbits have real Lyapunov exponents so the largest one is positive describing the exponential divergence of nearby trajectories, while elliptic orbits are characterized by imaginary stability exponents connected to their winding numbers. The Lyapunov exponent of the cycle can be calculated as the logarithm of the eigenvalue of its stability matrix obtained from the tangent map. The eigenvalues are negative when there is an odd number of inversion of nearby orbits during the cycle; then we take the modulus first. We are mainly interested in the evolution of stability properties of periodic orbits when changing the control parameter, but the fact that the stability matrix is usually a product of tangent map matrices, each describing one collision, makes it rather difficult to obtain considerable insight to orbit stability from the complicated form of the stability matrix elements. To avoid these difficulties, we will apply in this paper a more suitable approach, which is based on the curvature of local orbit fronts.

A. The curvature $\kappa$

Making use of the local divergence property, there is another possibility for the calculation of Lyapunov exponents. Its central quantity is the Bunimovich–Sinai curvature $\kappa$, which is the local curvature of the front (horocycle) formed around a central trajectory by nearby orbits started from a common point of origin $[13]$. Its evolution in time $\kappa(t)$ along the central orbit can be given easily: it is a smooth function with jumps due to the collisions. Without an external field, the local radius of the front grows linearly in time, so between collisions the curvature simply evolves as

$$\kappa(t) = \frac{\kappa_0}{1 + \kappa_0 t}$$

with $\kappa_0$ being the curvature at $t = 0$. In a collision, $\kappa$ jumps from its value $\kappa_-$ immediately before hitting the disk to a new value $\kappa_+$ connected by

$$\kappa_+ = \kappa_- + \frac{2}{r \cos \phi}$$

where $r$ is the disk radius and $-\pi/2 < \phi < \pi/2$ is the impact angle formed by the direction of the incoming velocity and the inward normal of the disk at the point of impact. Since this jump is always positive, starting from $\kappa_0 > 0$ (diverging orbits), the curvature remains positive for all time.

The Lyapunov exponent per collision of an orbit is given as

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \int_0^{T_N} \kappa(t) \, dt,$$

where $T_N$ is the time at the $N$th collision along the orbit. In the long-time limit, the dependence on the initial curvature $\kappa(0)$ is expected to disappear in the same way as the dependence on initial conditions in any other method for calculating Lyapunov exponents.

For a periodic orbit with $n$ collisions, the integral can be taken over one time period $T$ with a periodicity condition implied on $\kappa(t)$ so that $\kappa(T) = \kappa(0)$. It is easy to show that
there are two solutions, one positive and the other negative, for this requirement. If we denote the values of the front curvature right after the \( m \)th collision and the next one by \( \kappa_m \) and \( \kappa_{m+1} \), respectively, then it follows from Eqs. (1) and (2) that they are connected by a rational function with positive coefficients. This means that in the periodicity condition \( \kappa_n = R(\kappa_0) = \kappa_0 \) the rational function \( R \) also has this property and thus we obtain two solutions with opposite signs. Obviously, the positive value of \( \kappa_0 \) leads to the positive exponent when evaluating the integral. The corresponding eigenvalues of the stability matrix are \( \Lambda_{\pm} = (-1)^n \exp(\pm \lambda n) \), where the sign factor in front of the exponential accounts for the inversion of nearby orbits occurring at each collision.

It is important to note that the curvature of the front can also be written as a ratio by using the local Jacobi coordinate \( J \), which is the (signed) distance of a nearby orbit from the central one along the front [18]. The inverse of \( J \) multiplied by the angle between the two velocities gives just \( \kappa \). Since we set the velocity \( v = 1 \), this (small) angle is equal to the velocity component of the nearby orbit in the direction perpendicular to that of the central one, thus it can be written as the time derivative \( \dot{J} \). Therefore \( \kappa = \dot{J}/J \), and since \( \dot{J} \) and \( J \) can be considered as components of a vector in the tangent space of the dynamics, this means that \( \kappa \) is determined by the direction of that vector. This relationship also implies that the free-flight evolution of \( \kappa(t) \) satisfies the Ricatti equation:

\[
\dot{\kappa} = -K - \kappa^2 ,
\]

where the curvature term \( K \) follows from the Jacobi equation describing the separation of geodesics:

\[
\ddot{J} + KJ = 0 .
\]

In our case, \( J(t) \) is a linear function, which corresponds to \( K = 0 \) in Eqs. (4) and (5) as expected for a Euclidean billiard. We also note that the situation of crossing orbits means \( J \) changing sign while going through 0, which leads to a \( 1/t \) type singularity in \( \kappa(t) \). Such an inversion is accompanied by a sign change in the stability eigenvalue, so we may complete the sign rule for the eigenvalues: each jump (at collision) and each pole singularity in \( \kappa(t) \) contributes a factor \( -1 \) to the eigenvalues.

**B. \( \kappa \) in magnetic field**

The presence of the external field modifies the evolution of \( \kappa \) both in the free motion and at the collisions. The fact that \( \kappa(t) \) can be expressed through the Jacobi coordinate \( J(t) \) makes it easy to calculate it in a magnetic field. During the free flight, the particle moves along a circle of radius \( s \). Starting two orbits at \( t = 0 \) with parallel velocities, their distance remains constant along the direction perpendicular to the initial velocity while the tangent of the front turns with the velocity direction along the orbits. The local coordinate \( J(t) \) is the projection of the constant distance \( J_0 \) on the front direction at time \( t \): \( J(t) = J_0 \cos(t/s) \). By moving the origin of time to the point where the orbits cross, the cosine is replaced by sine; with this choice the free evolution of the curvature can be written as

\[
\kappa(t) = \frac{1}{s} \cot \frac{t}{s} .
\]
This form has exactly the same singularity \((1/t)\) approaching the points of orbit crossing as in the fieldless case. It is also periodic with a period equal to half of the total turnover time \(T_c = 2\pi s\) along the cyclotron orbit. It is important to stress that \(\kappa(t)\) can have negative values as well corresponding to the focusing effect of the magnetic field.

The jump in \(\kappa\) at collisions comes from the fact that there is a sudden change in the relative direction of two nearby orbits due to their different impact angles. In the fieldless case, the change in the relative direction is just twice the angle difference between the normal directions at the points of impact on the disk perimeter. With magnetic field, there is an additional effect due to the constant change in the velocity direction of the orbits during free flight: If the central orbit of the front reaches the disk under an impact angle \(\phi\), then another orbit in the front with a local Jacobi coordinate \(J\) will have a relative velocity direction different from \(\dot{J}\) when colliding with the disk because of the time difference \(\delta = J\tan\phi\) between the two impacts. This leads to an additional jump \(2\delta/s\) in \(\kappa\) at the collision. As a result, we obtain a modified jump formula:

\[
\kappa_+ = \kappa_- + \frac{2}{\cos\phi}\left(\frac{1}{r} + \frac{1}{s}\sin\phi\right).
\]

The parenthesized part can be interpreted as an effective curvature of the scatterer for the orbits: the field changes it from its fieldless value \(1/r\) so that even a collision with a flat wall causes a jump in \(\kappa\) in magnetic field. The field can decrease (increase) the defocusing effect of a collision on the disc at negative (positive) impact angles; the collision can even be focusing if \(s < r\). It is worth emphasizing that both the modified free evolution and the jump condition simplify, as expected, to the fieldless formulae (1) and (2) in the limit \(s \to \infty\).

In the following, we will use the normalized time variable \(\tau = t/s\) and its function \(\xi(\tau) = s\kappa(\tau s)\), which we call the curvature function of an orbit. It consists of pieces of the cotangent function \(\cot(\tau - \tau_i + \epsilon_i)\) \((i = 0, 1, 2, \ldots)\) describing the arcs of free flights and positioned by the phase shifts \(0 \leq \epsilon_i < \pi\) so that the jumps due to the collisions at \(\tau_i\) satisfy the condition (7) prescribed for \(\kappa\) (now obviously multiplied by a factor \(s\) for \(\xi\)). We will also use the notation \(\xi_i = \xi(\tau_i + 0) = \cot\epsilon_i\). For a periodic orbit of \(n\) collisions, we will assume that the periodicity condition \(\xi_0 = \xi_n\) is imposed when possible; in addition, we choose the origin of time so that the phase shift of the first piece is \(\epsilon_0 = \tau_0\), i.e., we move the initial collision when we start following the orbit to the value \(\tau_0 = \cot^{-1}\xi_0\). This way we can compare the curvature functions of periodic orbits at different values of the magnetic field.

### C. Lyapunov exponents of periodic orbits

The Lyapunov exponent \(\lambda\) of a cycle of period \(T\) with \(n\) collisions can now be obtained as

\[
n\lambda = \frac{T}{T_0} \int_{T_0}^{T_n} \xi(\tau) d\tau ,
\]

where \(T_n = T_0 + T/s\). The integral can easily be evaluated over the continuous pieces, but we can also combine the calculation of \(\lambda\) with the solution of the periodicity condition. For
this, it is enough to notice that by using elementary trigonometrics we can still write the relationship between $\xi_i$ and $\xi_{i-1}$ as a rational function:

$$
\xi_i = \cot(\epsilon_{i-1} + \alpha_i) + \Delta_i \\
= \frac{(\cot \alpha_i + \Delta_i)\xi_{i-1} + (\Delta_i \cot \alpha_i - 1)}{\xi_{i-1} + \cot \alpha_i} \\
= \frac{(\cos \alpha_i + \Delta_i \sin \alpha_i)\xi_{i-1} + (\Delta_i \cos \alpha_i - \sin \alpha_i)}{\xi_{i-1} \sin \alpha_i + \cos \alpha_i},
$$

(9)

where $\alpha_i = \tau_i - \tau_{i-1}$ is the length of arc $i$ measured in angle and $\Delta_i$ is the jump in $\xi$ at $\tau_i$. We wrote the third equality using normalized coefficients with a determinant 1. This means that the periodicity condition can also be written in the form

$$
\xi_0 = \frac{A\xi_0 + B}{C\xi_0 + D} \quad (10)
$$

with a determinant $AD - BC = 1$. This is a quadratic equation for $\xi_0$, and it is easy to check that the two eigenvalues of the matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ are just equal to the denominator in Eq. (10) taken with the two solutions for $\xi_0$. Using the connection between $\kappa$ and the Jacobi coordinate $J$, it can be shown that the matrix $M$ is equivalent to the stability matrix of the cycle obtained from the tangent map, so the Lyapunov exponents are simply $n\lambda = \log |C\xi_0 + D|$

We can immediately see that the periodicity condition can be satisfied if and only if the orbit is hyperbolic, since the two solutions of $\xi_0$ yield the two Lyapunov exponents. However, now the coefficients are not necessarily positive, thus in magnetic field we have the possibility of elliptic cycles with complex eigenvalues when Eq. (10) has no real solution. For these stable orbits, $\xi(\tau)$ is not periodic and the connection between its integral and the complex eigenvalues becomes more complicated. It could be restored by a formal extension allowing complex values for $\kappa$ and $\xi$, but there is a more plausible interpretation that we will discuss later.

**IV. ORBIT STABILIZATION IN MAGNETIC FIELD**

The positivity of $\kappa(t)$ in the fieldless case ensures that all the periodic orbits in the hard-disk billiards are hyperbolic. Small field values ($s \gg 1$) are not expected to change that: the angles $\alpha_i$ of the arcs are small, so the coefficients in Eq. (10) remain positive. Turning to a stronger field, there can be periodic orbits with wider negative parts in their curvature function leading to a decreased instability. The cycle becomes marginally stable at the field value for which the integral of $\xi(\tau)$ vanishes. Increasing the field further, we reach the region of elliptic stability with imaginary Lyapunov exponents, and at another critical field strength the cycle disappears through an inverse saddle-center bifurcation. This process, already observed in Ref. [14], can be studied best on the simplest periodic orbits of the system.
A. The clockwise orbit

Based on the form of Eq. (6), one expects that the candidates for stabilization are the orbits consisting in strong fields mostly of arcs close to a semicircle so that the corresponding pieces in the curvature function stretch almost over the whole period of the cotangent function. The simplest case is a cycle containing arcs of equal length and identical collisions ensured by the symmetry of the system. We take the four-disk model as our first example. Each disk has a symmetry point, the section of the disk perimeter with the diagonal of the square. There are two period-4 orbits connecting all these points one after the other in a clockwise or counterclockwise direction. Without magnetic field, these two orbits coincide on the configuration plane, forming a square; taking the fourfold symmetry into account, they are essentially fixed points of the symmetry-reduced version of the map $P$ acting on the arc centers.

![Diagram of the four-disk model](image.png)

FIG. 1. The clockwise (heavy line) and counterclockwise (dashed line) orbits in the four-disk model for $r = 0.5$ and $s = 1.5$; the crosses mark the disk centers.

Turning on the field, the straight lines in the cycle are replaced by arcs and the orbit pair splits up (Fig. 1). It can be checked, e.g. by direct calculation using the tangent map as in Ref. [14], that the largest Lyapunov exponent of the clockwise (counterclockwise) orbit decreases (increases) with increasing field strength. The fourfold symmetry remains, and the curvature function of the full orbit consists of four copies of the block describing free flight along an arc and the jump at the end of it (Fig. 2a); in other words, the periodicity condition can just be applied to this block due to the symmetry as if we had a period-1 cycle.
FIG. 2. The curvature function $\xi(\tau)$, plotted with vertical lines drawn from the axis $\xi = 0$, of the clockwise orbit in the hyperbolic regime (a) at cyclotron radii $s = .737 \approx s_1$ (full lines) and 1.5 (dashed lines) for a full period of four collisions; $\xi(\tau)$ of the clockwise orbit in the elliptic regime (b) at $s = 0.7$ for 5 full periods (20 collisions); the curvature function of the saddle clockwise orbit (c) at $s = 0.7$ for a full period. Notice the difference of scales in the plots. The curves of the first blocks of the two cases in Fig. 2a coincide because of the normalization of $\xi(\tau)$. In Fig. 2b, there is approximately one pole singularity for each period of four collisions due to the fact that the phase of the complex eigenvalue is close to $3\pi/4$ (cf. Eq. (15)).
In the clockwise orbit, the jump at the collision decreases with increasing field strength, while the length of the arc increases leading to negative regions appearing in $\xi(\tau)$. For a cyclotron radius $s_1$, we arrive at a situation (Fig. 2a) where the jump $\Delta$ is just equal to the minimum possible difference between two values of the cotangent function at points separated by a distance $\alpha$, the angular length of the arcs:

$$\Delta = 2 \tan \frac{\alpha}{2}.$$  

Because of the symmetry of the cotangent function, the integral of $\xi(\tau)$ over the block is zero, i.e., the orbit is marginally stable. By writing $\Delta$ and $\alpha$ as functions of $s$, Eq. (11) can be used to calculate the critical field value:

$$s_1 = \sqrt{(1 - r/\sqrt{2})^2 + (r/\sqrt{2})^2}$$  

(remember that $d = 1$). This means that the clockwise cycle reaches marginal stability, at $s_1 \approx 0.7368$ for $r = 0.5$, when the center of an arc in the cycle falls in line with the centers of the disks connected by the arc.

B. The elliptic regime

For field values $s < s_1$, we are in the elliptic regime and the periodicity condition (10) cannot be satisfied by any real value of $\xi_0$. The connection to the complex eigenvalues $e^{\pm i\phi}$ of the stability matrix could then be restored by a formal extension allowing complex values for $\kappa$ and $\xi$, but there is also a more plausible interpretation based on the dynamics unfolding from the sequence of phase shifts $\epsilon_i$ in the blocks of the curvature function. To see this in details, let us follow the clockwise orbit over a large number of repetitions and record the sequence $\epsilon_i = \cot^{-1}(\xi_i)$ ($i = 0, 1, 2, \ldots$) starting from a certain $\epsilon_0$ value. It follows from the interpretation of $\kappa$ as the ratio of the two components of a vector in tangent space that the values $\epsilon_i$ are just the angles (taken with mod $\pi$) of these vectors in the Poincaré section defined by the collisions and parametrized by $\dot{J}$ and $J$.

The connection between $\epsilon_i$ and $\epsilon_{i-1}$ can easily be obtained from the first line of Eq. (9). Since now each block has the same width $\alpha$ and jump $\Delta$ at its end, this relationship does not depend on $i$ and can be written as a one dimensional map:

$$\epsilon_i = C(\epsilon_{i-1}) = \cot^{-1}(\cot(\epsilon_{i-1} + \alpha) + \Delta)$$  

It has the topology of a circle map on the interval $[0, \pi]$, since the cotangent function is periodic by $\pi$. The sequence $\epsilon_i$ then follows from the dynamics of this circle map. Recalling that the members of this sequence are angles of vectors connected by the stability matrix of the cycle in the Poincaré section, we conclude that the circle map $C$ represents the effect of the Poincaré map $M$ restricted to the directions of vectors.

The map $C$ is essentially a shift by $\alpha$ distorted by the presence of $\Delta$. In the hyperbolic regime, where $\Delta < 2 \tan(\alpha/2)$, it has two fixed points, one stable and the other unstable, connected to the two solutions of the periodicity condition for $\kappa$. For an elliptic cycle, however, it has no fixed points, so $C$ generates a quasiperiodic sequence of the $\epsilon_i$ values, and
one can also easily check that for $0 < \epsilon < \pi - \alpha$ the difference $C(\epsilon) - \epsilon$ is bounded from below. This implies that for any starting $\epsilon_0$ there must be a certain step $j$ so that $\epsilon_j + \alpha > \pi$, i.e., the corresponding block in the curvature function will contain the pole singularity of the cotangent function (Fig. 2b).

If we look for the total angle $\Theta_n$ by which a vector is rotated under $n$ applications of the Poincaré map $M$, then we must add to the difference $\epsilon_n - \epsilon_0$ given by the map $C$ an extra $\pi$ “lost” in the circle map for each $\epsilon_i > \pi - \alpha$, i.e., for each pole singularity in $\xi(\tau)$:

$$\Theta_n = \epsilon_n - \epsilon_0 + k_n \pi,$$

(14)

where $k_n$ is the number of poles in $n$ blocks. Formally, the extra term can be considered as the contribution of the poles to the imaginary part of the integral of $\xi(\tau)$ taken over $n$ blocks. The real part of the integral, which can be evaluated by excluding small boxes centered on the poles, will fluctuate in $n$ as a consequence of the quasiperiodic dynamics in the $\epsilon_i$ values, but for large $n$ the elliptic property of the cycle ensures that it vanishes when divided by $n$. Thus we may conclude that the integral of the curvature function yields the Lyapunov exponent of a periodic orbit also in the elliptic regime in the sense that if we take it over a large number of repetitions of the orbit, then the phase of the eigenvalues $e^{\pm i\phi}$ is determined by the average number of poles per collision in the orbit:

$$\phi = \lim_{n \to \infty} \left( 1 - \frac{k_n}{n} \right) \pi,$$

(15)

where we took into account the effect of the jump in $\kappa$ at the collision. The argument above and its conclusion could obviously be generalized to cycles of longer periods too.

C. Bifurcations and pruning

At another critical field value $s_0 = 1 - r/\sqrt{2} \approx 0.6464$, the arcs become complete semicircles, and for $s < s_0$ the cycle cannot exist any more. The curvature function for $s_0$ is just the perfect cotangent function for all times: $\xi(\tau) = \cot \tau$. The jumps induced by the collisions sit right on top of the poles at $\tau_n = n\pi$; thus in the integral of $\xi(\tau)$ there will be an additional sign factor for each block. This means that approaching $s_0$ from below (decreasing field strength) the clockwise orbit is born at $s = s_0$ with the stability eigenvalue $\Lambda = 1$: the singularity in each block of $\xi(\tau)$ compensates for the sign change due to the collision.

The birth of the elliptic cycle is a usual saddle-center bifurcation as in smooth Hamiltonian systems: it is born together with a hyperbolic cycle. This orbit has the same symmetry but the length of its arcs is increasing for $s > s_0$, so they become longer than a semicircle. The corresponding block in the curvature function contains more than a full period of the cotangent function, so the periodicity condition can be met and the cycle is hyperbolic (Fig. 2c). By increasing $s$ further, the saddle orbit disappears in a way typical in hard-wall billiards: it reaches the stage where the arcs should cut through the disks to arrive at the desired symmetry points. This illustrates the main mechanism for pruning (disappearance of orbits by varying a system parameter) in our billiard models: the orbit becomes
forbidden when an arc hits a disk that is not in its “itinerary” or when a collision angle becomes flat \([20]\).

In the elliptic regime, the clockwise orbit in phase space is in the middle of a stable island formed by periodic, quasiperiodic, and chaotic trajectories as in general Hamiltonian systems. Outside the outermost KAM torus surrounding the island, there is the well-known infinite hierarchy of chains of smaller islands around longer periodic orbits, also stabilized by the field, and cantori, the remnants of broken KAM tori; we will discuss these aspects of the stabilization later in more details. It should be emphasized that the main steps of the above scenario for the stabilization of the clockwise orbit do not depend on the disk size relative to the disk spacing: the saddle-center bifurcation takes place at any value of the ratio \(r/d\), therefore the orbit stabilization occurs even at small disk sizes where the fieldless orbits are very unstable. The disk size affects, however, the range of stability: for small disk radii, it is proportional to \(r\) (the geometric interpretation given earlier for \(s_1\) illustrates this point very clearly).

In the four-disk model, the clockwise orbit is the only important example of the stabilization induced by the field. The two other candidates with a simple symmetry (orbits bouncing forth and back between two disks along the side or the diagonal of the square connecting the disk centers) have eigenvalues independent of the magnetic field: the focusing effect of the field along the arcs is exactly cancelled by the additional defocusing effect at the collisions.

V. STABLE ORBITS IN THE LORENTZ GAS

By extending the four-disk model periodically on the plane, we obtain the (square-lattice) Lorentz gas for which the four-disk model serves as a unit cell. This means, obviously, that the stable clockwise orbit of the four-disk model is present in the Lorentz gas too. However, the extension creates various other orbits that can be stabilized by the field: there are localized orbits extending over more than one unit cell as well as nonlocalized (travelling) orbits proceeding at a constant average speed.

A. Symmetric cycles

As we have seen, the focusing property of the magnetic field may lead to stabilization if the arcs building up the orbit are close to semicircles, so first we will look for orbits with this property. The symmetries of the system can help us again to find the simplest stable cycles: those with arcs of equal length. Each disk now has eight symmetry points due to the four symmetry axes, and connecting two such points by an arc generates a symmetric cycle with period 1, 2, 4, or 8 (period-1 and period-2 cycles if we take the fourfold symmetry into account). The configurations not prohibited by the pruning rules mentioned earlier are all born stable and have their ranges of elliptic stability. The most important examples are shown in Fig. 3 while Table I lists the corresponding field values \(s_0\) and \(s_1\).
FIG. 3. Symmetric stable cycles in the Lorentz gas; the cyclotron radii are $s = 1.05$ (a and b), 1.38 (c), 1.77 (d), 0.9 (e), and 1.22 (f).

TABLE I. The critical field values $s_0$ and $s_1$ of the symmetric cycles in Figs. 1, 3, and 5 (the labels in the first column refer to the figures) for disk size $r = 0.5$. The numbers are rounded to four decimal digits. The $s_1$ value for the cycle 3d is in parentheses since it cannot be reached because of the pruning.

| Cycle | $s_0$   | $s_1$   |
|-------|---------|---------|
| 1     | 0.6464  | 0.7368  |
| 3a    | 1.0000  | 1.1180  |
| 3b    | 1.0000  | 1.0607  |
| 3c    | 1.3536  | 1.3990  |
| 3d    | 1.7678  | (1.8028)|
| 3e    | 0.8265  | 0.9309  |
| 3f    | 1.1791  | 1.2543  |
| 5a    | 0.6105  | 0.6197  |
| 5b    | 0.7792  | 0.8363  |
| 5c    | 0.5586  | 0.5711  |
The simplest of these cycles is the travelling period-1 orbit that connects all the disks along a lattice axis (Fig. 3a). It is not affected by pruning except for large disk sizes \( r > 2/3 \). It is born at \( s_0 = 1 \) and becomes hyperbolic at \( s_1 = \sqrt{1+r^2} \approx 1.1180 \), i.e., when the arc center lines up with the disk centers as was also the case with the clockwise orbit. Another simple nonlocalized orbit with period 2 travels along the diagonal direction (Fig. 3b). It is also born at \( s_0 = 1 \) and is allowed for not too large disk sizes until pruning kills it at \( s = \sqrt{2} \) by a flat collision angle. The other cycles shown in Fig. 3 are localized; their symmetry-reduced periods are 1 (Figs. 3c and d) or 2 (Figs. 3e and f).

The curvature functions of the period-1 cycles evolve exactly the same way as described for the clockwise orbit. The period-2 cycles contain two different collisions, so the two jumps in \( \xi \) are different. This means that for \( s = s_1 \) the contributions of the two blocks to the Lyapunov exponent must cancel each other through a symmetric arrangement since the integral taken over one block is different from 0 (Fig. 4). It is also worth noting that while the period-1 orbits and their islands go through the evolution described for the clockwise orbit, the period-2 cycles show a slightly more complicated behaviour.

![FIG. 4. The curvature function of the 8-cycle in Fig. 3e for its symmetry-reduced period of two collisions at \( s = s_1 \).](image)

For a symmetric period-2 cycle, the \( s_1 \) values listed in Table I is specified by the criterion that the cycle is hyperbolic for \( s > s_1 \), but this is not the only crossover value between hyperbolic and elliptic stability. When increasing \( s \) from \( s_0 \) to \( s_1 \), the elliptic cycle becomes hyperbolic through a period-doubling bifurcation soon after its birth, but later it restabilizes and remains elliptic up to \( s = s_1 \) where a symmetry-breaking equal-period bifurcation (also called Rimmer bifurcation [21]) takes place. This behaviour turns out to be connected to the fact that even-period cycles has positive stability eigenvalues in weak fields.

So far we have dealt with symmetric orbits built of identical arcs, although obviously there exist many other cycles with more complicated symmetries or with no symmetry at all that are stabilized by the field after their births. We focused on the symmetric ones not only because they are the simplest ones to study but also because apparently they are the most important examples of orbit stabilization in these models in terms of the size of their ranges of stability and the phase space regions occupied by their islands. In longer cycles with arcs of different lenghts, the focusing effect of the field in longer arcs in one part of the orbit may
be put off by defocusing in other parts, so the stabilization is more difficult to achieve and pruning may kill the orbit before it could stabilize. In that respect, the symmetry reduces the effective period of the cycle and makes it more “sensitive” to the stabilizing field effects.

B. Multiple collisions

In a strong magnetic field, we can obtain new stable cycles from the symmetric configurations discussed so far by making use of the the possibility of multiple bounces on the same disk. The simplest case is when the particle bounces twice on a disk before it hits another one. The extra arcs created by the multiple bounces can make orbit types already extinct in such a strong field reappear by replacing a simple collision in the original orbit by one or more arcs with both ends on the same disk. Figure 5 shows three examples in the Lorentz gas.

![Diagram showing symmetric stable cycles with multiple bounces in the Lorentz gas.](image)

**FIG. 5.** Symmetric stable cycles with multiple bounces in the Lorentz gas; the cyclotron radii are $s = 0.615$ (a), 0.82 (b), and 0.565 (c). These cycles are connected to the regular orbits in Figs. 2a, 2b, and 1, respectively.

The stabilization mechanism for these orbits is slightly more complicated than for the “regular” orbits because now there are arcs of different lengths in the cycle. Looking at the curvature functions of these orbits (Fig. 6), we can see that at the brink of stabilization, the block corresponding to the extra arc occupies a symmetric position with respect to the singularity inside so that its contribution to the Lyapunov exponent is 0 and the values $\xi_j$ and $\xi_{j+1}$ at its ends are equal. This means that by removing the extra block we could still stick the other blocks together as if they belonged to a regular orbit with a similar symmetry: e.g., the period-3 orbit in Fig. 6b, which is a version of the period-2 travelling orbit of Fig. 3b, has two normal blocks that could form in themselves a curvature function similar to regular period-2 orbits (cf. Fig. 4).
FIG. 6. The curvature function of the cycles in Fig. 5c (a) and Fig. 5b (b) when losing stability at their respective field values $s_1$ (see Table I) for one symmetry-reduced period. In Fig. 6a, the first block is centered on the node inside as are the blocks in Fig. 2a for the clockwise orbit at $s = s_1$, while the two normal blocks in Fig. 6b could form a curvature function similar to that in Fig. 4 if the extra middle block were removed.

The curvature function also allows us to calculate easily the stability of special orbits consisting of collisions with one disk only, rotating around it (usually quasiperiodically) forever. In these orbits, each arc and each collision is equivalent so they can be treated as period-1 cycles. As we have seen, the centered position of the blocks in the curvature function automatically satisfies the periodicity condition in this case due to the special relationship between the arc length and the impact angle. This implies that the Lyapunov exponents of these orbits are always 0, i.e., they are marginally stable at any field strength where they can exist. This “insensitivity” can be connected to the fact that the islands of such orbits have smooth boundaries: they lack the complicated structure seen around the usual KAM-islands.
VI. STABLE ISLANDS AND TRANSPORT PROPERTIES

A. The effects of hard-wall pruning

In the Poincaré map, an elliptic cycle is surrounded by an island of bounded motion. The orbits inside the island visit the disks in the same sequence as the central cycle remaining inside a certain domain, both in configuration space and phase space, that they cannot escape. The island is born at $s = s_0$ together with the corresponding orbit, and at first it grows in size with decreasing field strength. It is still there at $s = s_1$ when the cycle in its center becomes hyperbolic, but this event starts a bifurcation sequence that leads to the breakup and gradual dissolution of the island in the chaotic sea. The island disappears completely at a cyclotron radius $s_2$ slightly larger than $s_1$.

In a certain way, this scenario is similar to those observed in smooth Hamiltonian systems; however, the hard-wall property, in the form of pruning, plays a key role in the evolution of the islands. This means that the size of an island at a specific field value and the details of its disappearance when changing the field may not be solely the results of the “natural” smooth evolution of its orbits as in smooth systems: sudden changes in the disk sequences of neighbouring orbits limit the room to live for an island and its neighbourhood. If the outermost orbits of the island should cut through the interior of other disks in following the disk sequence characterising the island, then they are removed from the system, and a well-behaving inner orbit becomes the actual border of the island. This pushes the edges of the chaotic sea closer to the central orbit and can also limit the width of the border region around the island where the hierarchy of small island chains and cantori can exist.

The influence of hard-wall pruning can clearly be seen in the example of the orbit in Fig. 3d, which is a “enlarged” version of the orbit of Fig. 3c connecting next-nearest neighbour disks instead of nearest neighbours. The enlarged cycle has to avoid “unnecessary” disks so it is much more sensitive to pruning. In fact, it is killed sooner by pruning, when $s$ is increased from $s_0$, than it could reach destabilization at $s = s_1$. This means that the whole island disappears well before the smooth development of its orbits could make that happen. In special cases, the onset of hard-wall pruning can even coincide with the birth of a cycle so that the island cannot develop at all. For example, the enlarged version of the travelling period-1 orbit proceeding along the diagonal line of the unit cells is born as a pair of two cycles at $s = \sqrt{2}d$ with arcs just touching another disk off that diagonal. The elliptic member of the pair, with arcs shortening when $s$ increasing, should cut through that disk which means it is forbidden together with its island; meanwhile, the hyperbolic cycle moves away from the disturbing disk so it exists for a certain field range.

B. Trappings near the islands

The universal orbit structure of smaller islands and cantori around the edge of an island can affect the “free” orbits of the chaotic sea. Although trajectories started outside the area of an island cannot step into it, they can reach its immediate vicinity, through the holes of the cantori, where they get trapped for a while until they find their way out again. This stickiness of the island leads to qualitative changes in the dynamical properties, like the decay of correlations, that are typical in nonhyperbolic systems. The nonhyperbolic effects
should be considered when applying the model to explain various transport phenomena. Its consequences from the point of view of chaotic scattering (together with the evolution of the island for the analogous triangular orbit) in the three-disk model were discussed in Ref. [14].

In the Lorentz gas, the stable orbits can affect the diffusion properties by trapping other “free” trajectories in their vicinity for a while, thus raising the possibility of anomalous diffusion [22]. If a trajectory enters the neighbourhood of the island around a travelling orbit, then the trapping means moving along with the orbits of the island at a constant average speed. On the other hand, in a trapping near a localized orbit the wandering particles become temporally localized too. (For certain ranges of the field strength, the two types of trappings can even coexist as is the case, e.g., for the orbits of Figs. 3e and 3b: it can be seen from Table I that around $s \approx 0.83$ both cycles are stable.) Whether these trappings can actually modify the character of the diffusion by forcing a super- or sublinear time dependence on the mean square displacement depends on the value of the exponent in the power law describing the distribution of trapping times. This question can be approached by using the formalism of Lévy walks as in Ref. [23]. The results indicate that a typical exponent $\gamma$ in the trapping time distributions associated with stable islands ($1 < \gamma < 2$, see Ref. [24]) leads to superlinear diffusion in the case of travelling orbits but the localized islands cannot alter the linear behaviour (although they can decrease considerably the value of the diffusion constant). This also implies that the diffusion is superlinear for the coexistence of the two trapping types too.

All this shows that from a theoretical point of view, it is important to know what kind of stable orbits (if any) are present in the system at a given value of the magnetic field. This does not necessarily mean in a particular application of the model, especially in connection with experiments, that the presence of an island is actually felt in the transport properties, since the ratio of trajectories affected by the island on the characteristic time scale of the experiment can be negligible if the island does not occupy a large area in phase space. It is worth noting that the islands associated with some of the cycles of Figs. 3 and 5 in the Lorentz gas are very small indeed, therefore they are easy to miss on a usual phase space portrait unless we know where to look for them. The detailed knowledge of the stabilization mechanism, based on the properties of the curvature function, can help us in finding them.

VII. CONCLUSIONS

In the present paper we extended the horocycle description of orbit separation to the case of billiards in magnetic field. We demonstrated that this formalism provides a convenient alternative for the usual tangent map calculations as a tool to investigate stability properties of orbits. In particular, the form of the curvature function suggests which type of orbits can be stabilized by the field, and this gives an edge to our method in searching for stable islands compared to plotting a simple phase space portrait where the small-size islands can easily be overlooked. As a simple result, we obtained that all the symmetric periodic orbits built of arcs of the same length are born stable at a field strength where the cyclotron diameter takes on the desired value. We also established the connection between the eigenvalues of elliptic orbits and the integral of the curvature function.

We identified the most prominent examples in the four-disk model and in the Lorentz gas and studied their evolutions, also discussing the consequences of the hard-wall property. The
details concerning which orbits are stabilized by the magnetic field may depend on the disk size $r/d$ in the billiard because of the interplay between smooth orbit evolution and hard-wall pruning. The longer, more complicated orbits or the enlarged versions of the simple symmetric orbits may be allowed for a small disk size to go through their “natural” smooth evolution while they may not even be present for larger disk sizes because of the pruning. For weaker field values, when $s$ is considerably larger than $d$, an elliptic orbit should avoid many other disks in order to form the long arcs necessary for stabilization, which makes their presence rather unlikely; therefore, one expects that the creation or disappearance of most orbits in weak fields take place without stabilization.

From this point of view, it is an interesting open question if there is a critical value $s_c$ of the cyclotron radius, depending on $r$, so that for $s > s_c$ all the existing orbits are unstable in the system. For chaotic scattering in the three-disk model, such a critical field apparently exists [14], so we expect a similar property in the four-disk model too, but in the Lorentz gas one cannot rule out the possibility of some unlikely cycles built of long arcs in weak field extending over many unit cells and still avoiding disrupting collisions with “wrong” disks. (This question is also connected to the existence of collision-free cycles of closed circles in weak fields.)

The idea of the curvature function and its use as a tool to study stability properties can be extended to models quite different from the ones discussed here. Thermostatted billiard models in external electric field have drawn large interest recently from the point of view of the understanding of irreversibility and transport in nonequilibrium systems [25]; in such systems one can use the equations of motion for the velocity direction to derive the time evolution of $\kappa$ explicitly. As for smooth potentials, the formulae for the time evolution of the monodromy matrix of Eckhardt and Wintgen [26] can be modified to accomodate the effects of an external magnetic field in the quantity $K$ appearing in the Jacobi equation. The latter approach is also suitable to treat models, both smooth and hard wall, with inhomogeneous fields.

Finally, it is worth noting that the curvature $\kappa$ plays a very important theoretical role in semiclassical calculations using periodic orbits and the Gutzwiller trace formula. As was shown in Ref. [12], the convergence of the method can be drastically improved by extending the dynamics by adding the tangent space to it so that the resulting evolution operator becomes multiplicative along the orbits. The addition of the tangent space dynamics means following the evolution of $\kappa$ as a passive function depending on the “real” dynamical variables. Then taking the trace of the evolution operator requires not only looking at periodic orbits but also prescribing the periodicity of $\kappa$ for the orbit, which is just the condition we used in our study of stability properties.

*Added note.* While finishing the manuscript, the author became aware of Ref. [27] dealing with the behaviour of nearby trajectories in general magnetic billiards on Riemannian surfaces and presenting similar results, obtained independently of the present paper, concerning the evolution of $\kappa$. In particular, our Eq. (8) is a solution of Eq. (50) of Ref. [27] reduced to the special case of Euclidean billiards in a homogeneous magnetic field, while Eq. (49) there, describing the jump of $\kappa$ at collisions, agrees with our Eq. (7).
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