BF THEORY ON A BRANE

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Abstract

An alternative approach to introducing gravitational dynamics on a brane embedded in a higher dimensional spacetime is presented. The brane is treated as a boundary of a higher dimensional manifold in which the bulk action is described by a metric independent topological quantum field theory. The example of a five dimensional non-Abelian BF theory with a boundary brane is considered. A natural boundary condition is adopted chosen for consistency of the topological action despite the presence of a boundary. The resulting effective action on the brane is the action of general relativity in first order form plus terms involving the extrinsic curvature of the brane.
1 Introduction

The idea of a brane embedded in large extra dimensions as an alternative to Kaluza-Klein compactification has received considerable attention since a specific model to achieve this was proposed in \cite{1}. In this approach matter fields are confined to the brane as open strings ending on the brane while gravitational modes are confined close to the brane by the unique characteristics of the gravitational field of a brane within a five dimensional anti-deSitter spacetime. In \cite{1} the action of general relativity was taken to govern the gravitational dynamics of the five dimensional spacetime. This choice for the gravitational action was motivated as a low energy effective theory of an underlying string theory. A metric was therefore introduced from the outset. On the other hand, the idea has frequently been expressed that in string theory, or indeed in any fundamental theory of quantum gravity, the metric should really be a derived feature \cite{2}. Therefore, one is motivated to consider the possible consistency of a brane embedded in a spacetime of large extra dimensions without the initial assumption of a metric.

Consistent metric independent theories are of course known in the form of topological quantum field theories \cite{3}. Therefore, one approach to exploring the consistency of a brane embedded in a metric free spacetime of large extra dimensions would be to consider a brane embedded in a higher dimensional topological field theory. This has a collateral benefit in that there would be no gravitational modes propagating in the bulk since a topological field theory has no local dynamical modes. In such an approach several problems immediately present themselves, however. First of all, it is not clear how a brane having local dynamics could couple to a topological quantum field theory without spoiling its consistency. Secondly, there must be a natural way for general relativity to emerge on the brane from the topological theory in the bulk. That is, general relativity cannot simply be introduced on the brane in an ad hoc manner.

In approaching these problems, the consistency issue is most easily examined if the brane is considered as a boundary of a one higher dimension manifold on which is defined a specific topological field theory. For example, a 3 brane may be taken to enclose a finite four volume or the 3 brane may divide an infinite four volume into two half spaces with the other boundary at infinity. The question of consistency then becomes whether a topological theory and a boundary condition can be found in which the consistency of the topological theory in the bulk is maintained and which allows dynamics
on the boundary. Topological field theories on manifolds with boundary have been considered by a number of authors [4]-[10]. The easiest way to ensure consistency of the bulk topological theory when introducing a boundary is to simply impose a boundary condition where all fields vanish at the boundary. However, this prevents any dynamics from emerging on the boundary. Several authors have considered the possibility of other consistent boundary conditions which can also introduce local dynamics on the boundary. In the most extensively studied such approach dynamical “edge states” are introduced on the boundary in a manner chosen to cancel surface terms arising from variation of the bulk action to thereby maintain a consistent topological field theory in the bulk [6]-[9]. While very interesting for a variety of reasons, such edge states do not appear to be able to provide the dynamics of general relativity. In another approach, general relativity was introduced on the boundary of a five dimensional space and then a corresponding topological field theory was identified in the bulk [10].

In the present paper non-Abelian BF theory on a manifold with a boundary is considered with a natural boundary condition in the form of a normal boundary condition on B, chosen in [5] solely for mathematical consistency of the bulk topological field theory. The normal boundary condition is expressed in an equivalent dual formulation which takes advantage of the metric needed to define the normal and tangential forms at the boundary. As a specific model, non-Abelian BF theory in a 5D arbitrarily large metric free manifold with a boundary brane is considered. The B and F[A] fields are chosen to take values in SO(4,1). Remarkably, the normal boundary condition on B is satisfied by a relation between the B field and the soldering form induced on the boundary which leads to the first order Palatini action for general relativity on the boundary after integration over the extra dimension.

Therefore, the obstacles to a consistent embedding of a brane world in a metric independent topological field theory can be overcome in the model considered in this paper.

The paper concludes with some speculations on the possibility of branes naturally arising in topological field theories as defects in the context of topological field theory as an “intrinsically broken” gauge field theory.
2 BF Theory On A Manifold With Boundary

First the approach of [5] will be briefly reviewed before extending the approach to non-Abelian BF theory. The topological field theory considered in [5] had the following action:

\[ S = \int B \wedge dC \]  

(1)

where the integral is over an n-dimensional manifold M with boundary, B is a p form and C is an n-p-1 form. To kill the boundary term which appears in the variation of the action (1) the following natural boundary condition was chosen:

\[ B \in \Omega^n_{\text{nor}} \]  

(2)

where \( \Omega^n_{\text{nor}} \) is the space of all p-forms normal to the boundary of M, i.e., the space of all p-forms which vanish when contracted with any tangent vector of the boundary of M. A number of features of the above action were then considered in [5] which will not be needed here. However, an observation noted in [5] which will be useful here is that a metric, which is needed to distinguish tangential and normal forms to the boundary, can be used to define the Hodge dual map over M. The Hodge dual in turn maps the space of p forms tangential to the boundary to n-p forms normal to the boundary (and vice versa). Specifically:

\[ * : \Omega^p_{\text{tan}}(M) \to \Omega^{n-p}_{\text{nor}}(M) \]  

(3)

where \( \Omega^p_{\text{tan}}(M) \) is the space of p-forms tangential at the boundary and * is the Hodge dual defined over M.

The action for Abelian BF theory can be written as follows in arbitrary dimension:

\[ S = \int B \wedge dA \]  

(4)

where A is the connection of an Abelian group over M and B is an n-2 form. Therefore, Abelian BF theory is just a special case of the topological field theories considered in [5]. To extend the approach of [5] to non-Abelian BF theory we consider the general BF theory action:

\[ S = \int B \wedge F \]  

(5)
where $F[A]$ is the curvature two form of the connection $A$ of a principle bundle over $M$ associated with a non-Abelian group $G$ and $B$ is a group algebra valued $n$-2 form where $n$ is the dimension of the manifold $M$ [3]. Variation of the action results in the following equations of motion:

$$F = 0 \text{ and } DB = 0$$

(6)

where

$$DB = dB + [A, B]$$

(7)

Variation of the action also introduces a surface term on the boundary $\partial M$ and a boundary condition must be introduced to cancel this term. Following the above approach we choose the normal boundary condition (2) to kill the surface term. Because of the above dual map (3) we may equivalently write the boundary condition (2) as:

$$B \in \ast \Omega^{n-(n-2)}_{\text{tan}}$$

(8)

i.e., $B$ is restricted to the space of forms dual to the space of tangential two forms.

### 3 BF Theory On A World Brane

We next apply the above approach to consistent boundary conditions in non-Abelian BF theory to a 3 dimensional brane evolving inside an arbitrarily large 5D manifold $M$ without metric and with the bulk action described by non-Abelian BF theory. The four dimensional world brane may be considered as a boundary of the 4+1 dimensional manifold $M$ since there is no metric in the bulk and we can without loss of generality identify the two timelike directions in the boundary and bulk. Therefore, we can equivalently consider $M$ to be a five dimensional manifold with four dimensional boundary brane $\partial M$. We take $G$ to be $\text{SO}(4,1)$ (the analysis will be essentially the same for $\text{SO}(3,2)$). The 3 brane may be taken to enclose a finite four volume in $M$ or the 3 brane may divide an infinite four volume into two spaces with the other boundary taken at infinity (e.g., with trivial boundary condition with all fields vanishing at infinity). The total action is:

$$S = k \int B \wedge F$$

(9)
where $k$ is a coupling constant, with the boundary condition at the brane given by (8) to kill the surface terms on the brane, i.e.:

$$B \in \ast \Omega^2_{\text{tan}}$$

(10)

Now a natural basis for the space of tangential forms is given by the local frame fields corresponding to the induced metric on the brane. Since B also takes values in $G$ the tangent basis forms must also take values in $G=\text{SO}(4,1)$. Therefore, a tangential 2 form basis is given by the wedge product $e^A \wedge e^B$ (where $A,B$ are group indices). Therefore, a solution of (11) at the boundary brane can be written:

$$B = \ast^{(5)}(e \wedge e)$$

(11)

where $\ast^{(5)}$ refers to the 5D Hodge dual.

The total 5D action (9) can be split into two pieces $S_1$ and $S_2$ where $S_1$ is the action for a 5D volume comprising a thin region at the boundary where the boundary condition holds (e.g., brane thickness) and $S_2$ is the action in the bulk. Thus, using the above dual formulation of the boundary condition:

$$S = S_1 + S_2 = k \int \ast^{(5)}(e \wedge e) \wedge F + k \int B \wedge F$$

(12)

where both integrals are 5D volume integrals. The first term is the effective action at the brane since the second term has no dynamical modes. The first term has a similar form to 5D general relativity in first order form but is not 5D covariant since the frame fields $e$ are purely tangential. The 5D curvature $F$ at the brane can be split into normal-normal (5-5), normal-tangent (5-i) and tangent-tangent (i-i) (where $i=1-4$) components using the metric introduced at the brane to define the normal boundary condition. The 5-5 and 5-i components of $F$ get truncated by the contraction with the purely tangential frame fields $e \wedge e$ leaving only the tangent-tangent components of the 5D curvature $F$ in the first term of (12). In general the internal 5 components will not be killed, however. This can be better examined in a component form of the first term of (12):

$$S_1 = k \int d^5 x |e| e^{[i}_A e^{j]}_B F^{AB}_{i j} = k \int d^5 x |e| (e^{[i}_a e^{j]}_b F^{ab}_{i j} + 2 e^{[i}_5 e^{j]}_b F^{5b}_{i j})$$

(13)

where $|e|$ is the magnitude of the determinant of $e$, $a,b=1-4$ are group indices in an $\text{SO}(3,1)$ subgroup of $\text{SO}(4,1)$ and $F=F^{(5)}$ is the 5D curvature. The
equations of Gauss and Codazzi may now be used to express the 5D curvature in terms of the intrinsic curvature $F^{(4)}$ on the brane and the extrinsic curvature $K$ [13]:

$$F^{(5)}_{ij} = F^{(4)}_{ij} + K_i K^i - K_j K^j + K_{i;j} - K_{j;i}$$  \hspace{1cm} (14)

where internal indices have been suppressed. Inserting this in $S_1$ in equation (13) gives the action at the brane in purely 4D terms:

$$S_1 = k \int d^5 x |e| (e^{[i} e^{j]} (F^{(4)ab}_{ij} + K^a_i K^b_j - K^a_j K^b_i) + 2 e^{[i} e^{j]} (K^b_{i;j} - K^b_{j;i}))$$  \hspace{1cm} (15)

where $F^{(4)}$ is the 4D curvature but where the integration is still over the 5D volume at the brane where the boundary condition holds.

The action (15) may be reduced to a 4D action in a trivial manner if we assume all fields are independent of the fifth normal direction over the local region where the boundary condition holds so that we can integrate $S_1$ over $x^5$. The result is simply (15) integrated over the 4D volume times a constant of integration $t$ which is just the normalized thickness of the local boundary region where the boundary condition holds (for example, $t$ may be the brane thickness, where the brane is viewed as a defect as discussed below). This assumed independence of the action on $x^5$ also affects the extrinsic curvature terms. The extrinsic curvature $K$ may be written in Gaussian normal coordinates as [13]:

$$K_{ij} = -\frac{1}{2} \partial g_{ij}/\partial x^5$$  \hspace{1cm} (16)

With the assumed independence of all quantities on $x^5$ the extrinsic curvature will vanish. Equation (15) then becomes, removing the explicit reference to 4D and replacing $F$ with the more suggestive $R$:

$$S_1 = k' \int d^4 x |e'| (e^{[i} e^{j]} R^{ab}_{ij})$$  \hspace{1cm} (17)

(where $|e'|$ is the magnitude of the determinant of $e'^i_a$, and where $k' = tk$), which is just the action for 4D general relativity in the first order Palatini form.

Although the above assumptions resulting in the action (17) for 4D general relativity are reasonable, it is interesting to look closer at the action
In particular, it is interesting to note that while the equations of motion require $F^{(5)}$ to vanish, $F^{(4)}$ need not due to the extrinsic curvature terms. That is, the extrinsic curvature acts like a source for the 4D curvature. Thus, starting from a topological BF theory with only flat connections as solutions a dynamical theory with a source emerges from the boundary condition.

4 Defect Formation In Topological Field Theories

The present section provides a very qualitative discussion of how a world brane could arise as a defect in the context of a bulk topological quantum field theory. As noted in the introduction the present paper is motivated, at least in part, by the philosophy that the most basic formulation of quantum gravity should be metric free. Topological quantum field theories provide many attractive aspects for such a metric free underlying theory. Topological quantum field theories have typically not been considered as viable models for a realistic physical theory due to the lack of any realistic symmetry breaking mechanism or other mechanism for creating a lower energy theory which possesses local dynamics. The model in this paper illustrates how general relativity can emerge naturally on a boundary brane of a bulk topological field theory. However, ideally the topological field theory should also be able to explain brane formation. Since topological field theories have no local dynamics it is not clear how such a boundary brane could form in the first place, especially absent any mechanism for symmetry breaking.

Topological field theories do inherently possess aspects of spontaneous symmetry breaking, however. The characteristic feature of spontaneously broken field theories is the existence of distinct degenerate vacuum states. A specific one of these vacuum states must be chosen as the vacuum for the theory and the symmetry reflecting the label that distinguishes the vacua is broken. The same vacuum state need not be chosen at every place, however, and different domains characterized by different vacua are possible. In particular, where regions are initially causally disconnected distinct vacuum domains are expected. It is well known that such domains can give rise to domain walls and other topological defects in a variety of cosmological scenarios based on spontaneously broken field theories. [11]

Topological quantum field theories are metric independent theories, there-
Therefore:

\[ \delta S/\delta g = T^{\mu\nu} = 0 \]  

(18)

and hence have vanishing energy for all observables. [3] Also, distinct vacuum solutions are known for various theories which have been studied, which solutions are labeled by topological properties of the underlying manifold. Therefore, topological quantum field theories share the characteristic feature of degenerate vacuum solutions possessed by spontaneously broken field theories. Nonetheless, despite the presence of distinct vacuum solutions the complete theory may not truly have distinct vacuum states if the states are mixed in the quantum theory. Typically, topological field theories are studied on compact manifolds where such mixing does occur. Where a theory with degenerate solutions is defined on an infinite manifold, however, mixing does not occur and distinct degenerate vacua will be present. [12]

Therefore, a topological quantum field theory defined on an infinite manifold intrinsically does possess the characteristic features of a spontaneously broken field theory. As a result topological quantum field theories intrinsically possess the possibility of giving rise to domain walls and other defects without any further breaking. Once a defect is present the domain wall will act as a boundary to the region described by the topological field theory and, as shown above, the boundary dynamics can give rise to realistic physical theories. Therefore, based on this “intrinsic symmetry breaking” effect present in a large class of topological field theories such theories without more can potentially present models for both defect creation and dynamics on the brane. Thus in effect two stages of symmetry breaking occur without the need to introduce scalars to break the high degree of symmetry present in topological theories. Typically scalars have been considered necessary for symmetry breaking since only scalars can avoid breaking Poincare invariance at the same time. Topological field theories seem to provide an exception to this rule.

Therefore, although the model considered in the present paper is far from a realistic model of a fundamental theory, nonetheless it is suggestive of a possible road from a fundamental topological quantum field theory to realistic four dimensional dynamics.

Another suggestive feature of a topological “intrinsically broken” fundamental theory comes from an analogy with superconductivity. A domain wall dividing two regions of a topological field theory is analogous to a supercon-
ducting region of broken $U(1)$ gauge symmetry sandwiched between two unbroken normal regions. In type II superconductors flux tubes will penetrate the superconducting region and interact with each other. If a topological field theory is a candidate for a fundamental theory, perhaps with string theory or M theory as a dynamical phase, the analog of the flux tubes would be strings of topological phase coupled between parallel branes defining the edges of the domain wall. Therefore, string theory could naturally emerge as an effective field theory on a brane/defect as a broken phase of a topological field theory in the bulk.

5 Discussion

A model with a bulk topological field theory defined on a manifold with a boundary brane has been presented which provides dynamics on the boundary brane in a natural manner. Specifically, SO(4,1) BF theory with a natural boundary condition, expressed in its dual form using the induced metric at the boundary, results in the action for general relativity on the brane after integration over the extra dimension. This provides an alternative to “edge states” to introduce dynamics into a topological field theory with boundary. Also, an alternative means of suppressing gravitational modes propagating off a brane world to that in [1] is provided by this model. In principle this approach could be extended in a variety of ways, such as other groups, other topological field theory actions and higher dimensions which allow consistent introduction of a soldering form at the boundary. The soldering form requirement is reminiscent of soliton solutions of purely internal gauge theories and the boundary condition of this paper could be viewed as a consistency condition for the brane to be a TQFT defect. A qualitative discussion of possible defect formation in topological field theories was provided.

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