Quantum communication, reference frames and
gauge theory

S.J. van Enk
Bell Labs, Lucent Technologies
600-700 Mountain Ave, Murray Hill, NJ 07974

April 1, 2022

Abstract
We consider quantum communication in the case that the communicating parties not only do not share a reference frame but use imperfect quantum communication channels, in that each channel applies some fixed but unknown unitary rotation to each qubit. We discuss similarities and differences between reference frames within that quantum communication model and gauge fields in gauge theory. We generalize the concept of refbits and analyze various quantum communication protocols within the communication model.

1 Introduction

In recent years a lot of attention has been paid to the role reference frames play in quantum communication [1]–[12]. We may distinguish four different types of studies. First, many papers consider the resources needed to establish a shared reference frame [1]. For example, a spatial reference frame between Alice and Bob can be established by transmitting spin-1/2 particles, their spin encoding information about direction. The questions that have been considered in that case are whether transmitting entangled qubits (here a spin-1/2 particle is viewed as a qubit) is beneficial and how the fidelity (i.e., the overlap between Alice’s and Bob’s private reference frames) approaches unity with the number of qubits sent. Similarly one can consider synchronizing clocks by transmitting “ticking” qubits (superpositions of two quantum states with different energy) [2] as establishing a different type of reference frame.

Second, there is a connection between superselection rules and reference frames. More precisely, the lack of a reference frame leads to an effective superselection rule and, conversely, a superselection rule can be effectively lifted by setting up a shared reference frame. All this was noted long ago in [13] but in recent years the question arose as to whether superselection rules can be exploited in cryptographic protocols. After all, if an eavesdropper in a cryptographic protocol is not able to perform certain desired operations as a result
of superselection rules, it may enhance the security of that protocol. Unfortunately, the answer is negative, precisely because a superselection rule can be effectively lifted. The proof of that statement is more involved than might be apparent and can be found in [8].

Third, there are new types of resources associated with reference frames, or the lack thereof. Although in many cases, for instance in teleportation protocols, it is tacitly assumed an explicit isomorphism has been established between the Hilbert spaces for qubits used by different observers, that is in fact not always trivial. Hence the ability to establish an isomorphism between different Hilbert spaces is a useful resource that until recently stayed hidden [3,4]. In order to quantify that resource, Ref. [7] introduced the local variance in particle number, which acts as a resource when a particle-number superselection rule holds. Alternatively this quantity can be translated into a refbit [10], a unit of sharing a reference frame, in the case that observers agree on the definition of the states $|0\rangle$ and $|1\rangle$ but not on the relative phase between those two states. Thus one can quantify the partial presence of a shared reference frame. In addition, a privately shared reference frame between two observers is a useful resource that can be exploited for secret communication [9].

Fourth, there is the question whether the definition of entanglement has to be modified when restrictions exist on the allowed operations [12], such as in the presence of superselection rules [6] or in the absence of a shared reference frame [11]. After all, entanglement is only a useful concept if it can be used for things like violating Bell inequalities or teleportation. Both teleportation and Bell-inequality violations require certain measurements. If those measurements are impossible to perform because of a superselection rule it makes perfect sense to modify the definition of entanglement. If those measurements are impossible to perform without a shared reference frame, then one still can (partially) convert “useless” entanglement into useful entanglement by establishing (partially) a reference frame by using the above-mentioned refbits.

In all the above papers the assumption is made that the communicating parties, although not sharing a reference frame, do share a perfect quantum communication channel. However, it is hard to see how the communicating parties can be sure to have a perfect communication channel when they do not share a reference frame. And so we will instead assume that some fixed unitary rotation is applied to the qubit upon traversing the communication channel. This unitary rotation is assumed to be measurable, at least up to redefinitions of local reference frames.

In spite of the fact that we do not even consider decoherence in the channel (i.e., we assume no entanglement is created between the transmitted qubit and some other system) this modification is nontrivial. In particular, allowing an asymmetry between quantum communication from Alice to Bob and communication from Bob to Alice leads to geometrical considerations analogous to those encountered in gauge theories (see Section 4). For example, gauge transformations can be directly translated into changes of reference frames.

This paper is organized as follows. In Section 2 we define our communication model where multiple parties share classical and quantum channels and
possess their own private reference frames. All channels and reference frames are assumed stationary in time. In Section 3 we define three different types of observables naturally arising in the context of the communication model. The three different types are defined by the answers to two questions: is the observable measurable by only one observer or by multiple observers, and is the observable independent of reference frames?

In Section 4 we discuss analogies and differences between (Yang-Mills) gauge theory and the quantum communication model of Section 2. We also discuss gauge-invariant measurements, not related to the observables discussed in Section 3.

In Section 5 we discuss three different resources for quantum communication protocols. One resource generalizes the refbit to the present communication model, the other two resources are 2-qubit and 3-qubit entangled states. One question considered for the latter resources is when they are equivalent, i.e., when the observers can convert one type of state into another by local operations and classical communication (LOCC). Since we assume here the observers do not share reference frames, not all such states are equivalent, even when they would be equivalent in the case of shared reference frames. Moreover, we find that 2-qubit entangled states behave very differently in that respect than do 3-qubit entangled states.

Finally, in Section 6 we study the modifications of standard quantum communication protocols such as quantum data hiding, superdense coding, and quantum bit commitment. The latter is still not possible—this follows from [8]—but it is perhaps illuminating to see once more why bit commitment fails in a concrete example. In addition, the example from Section 6.3 is a bit different than standard bit commitment attempts. Section 7 concludes and summarizes.

2 Communication model

2.1 Quantum communication channels

We assume we have multiple observers with the standard names Alice, Bob, Charlie etc., with sub- and superscripts pertaining to those observers abbreviated to A, B, C, . . . . The observers communicate over two-way classical channels and separate one-way quantum channels. This is unlike in Refs. [5, 10, 16] where the quantum communication channel is being used for classical communication as well. This is meant to imply we do not consider sending a classical bit of information to be a difficult task in the present context.

Each pair of observers $k,l = A, B, C, . . . .$. thus shares two quantum communication channels characterized by two unitaries $V_{kl} \in SU(2)$ and $V_{lk} \in SU(2)$, describing the transformation of a qubit state when it is sent from $l$ to $k$ or vice versa, respectively. In general we have neither $V_{kl} = V_{lk}$ nor $V_{kl} = V_{lk}^\dagger$. The latter relation could hold if the “same” physical channel is used to communicate in both directions. For instance, in the “plug-and-play” system developed for quantum key distribution [14] an essential role is played by Faraday rotators
that ensure a photon that has traveled from Bob to Alice will reverse its (unknown) polarization rotation when traveling back to Bob over the same channel. In such a case, when one does have the relation $V_{AB} = V_{BA}^\dagger$, one could transform away the action of the channel and incorporate it into the reference frames of Alice and Bob. But here we do not make that assumption.

2.2 Reference frames

Each observer possesses his/her own local reference frame, which defines the local orientation of the Bloch sphere, i.e. what the observer means by “$|0\rangle$” and “$(|0\rangle + |1\rangle)/\sqrt{2}$”, etc., and which similarly determines what is meant by “bit-flip” and “phase-flip” operations. We assume observers have no knowledge of the other observers’ reference frames.

More precisely, then, each observer $k = A, B, C \ldots$ possesses a local reference frame that defines a local basis $\{|0_k\rangle, |1_k\rangle\}$. Of course, only relative orientations of reference frames are observable. We denote the basis transformations that specify the relative orientations of the frames of reference of observers $k$ and $l$ by $R_{kl} \in SU(2)$. We thus explicitly factor out the unobservable $U(1)$ transformations from the full group of unitaries $U(2)$. These unitary operators satisfy by definition

$$\langle a_k | R_{kl} | b_l \rangle = \delta_{ab} \text{ for } a, b = 0, 1.$$  \hfill (1)

We have the relations

$$R_{kl} R_{lm} = R_{km},$$

$$R_{kl}^\dagger = R_{lk}.$$  \hfill (2)

Given (1) we may use the notation

$$|\psi_k\rangle = R_{kl} |\psi_l\rangle,$$  \hfill (3)

to connect the descriptions of different observers for any state $|\psi\rangle$. Similarly, we can write the relation between unitary operations $U$ as applied by different observers:

$$U_{kk} = R_{kl} U_{ll} R_{lk}.$$  \hfill (4)

Depending on the physical implementation of qubits used, one may impose restrictions on the form of the operators $R_{kl}$. For instance, when using polarized photons for quantum communication the right-hand and left-hand circular polarization states can be locally defined with respect to the propagation direction of the light since Nature does distinguish between “left” and “right”. (Here we assume we can ignore the possibility that the spacetime structure of the universe is that of an unorientable manifold.) In this case, the observers could agree on choosing left-hand and right-hand circular polarization to encode $|0\rangle$ and $|1\rangle$ without having to share a reference frame. In that case the form of $R_{kl}$ would be restricted to rotations around the polar axis ($z$ axis), i.e.,

$$R_{kl} = \exp(i \phi_{kl} \sigma_z/2),$$  \hfill (5)
with $\sigma_z$ the generator of rotations around the $z$ axis and $\phi_{kl}$ the phase difference characterizing the relative rotation between the two reference frames of $k$ and $l$. Here, however, we will not make the assumption on reference frames. Note also that propagation of polarized photons through a standard optical fiber can certainly transform left-hand into right-hand polarization, and hence no restriction would be placed on the unitaries $V_{kl}$.

2.3 Time evolution

We assume that all the unitaries introduced so far are stationary. That is, both quantum channels and reference frames are assumed stationary. This description, perhaps despite appearances, does include the case where “reference frames” are clocks. What is stationary in that case is, e.g., the frequency of the clock used and what is unknown is the time-offset between different clocks.

Similarly, there may be a nontrivial time evolution for traveling qubits, but that evolution is assumed to be stationary, too. That time evolution can then in fact be easily transformed away. In most cases this leads to a simpler description. On the other hand, retaining the time evolution allows one to use coordinate-independent expressions. Typically, we will transform away the time evolution of a traveling qubit, but give covariant expressions when appropriate.

3 Observables

The unitaries $R_{kl}$ and $V_{kl}$ are not observable by themselves. In fact, one may even get the impression that $V_{kl}$ can somehow be redefined to be the identity by absorbing its effects in the definition of $R_{kl}$. If that were the case, there would be no point to this paper. That this is not so we will see in this Section 1.

3.1 Private and public observables

We will distinguish three types of observables: (i) those dependent on the reference frame(s) of other observer(s), (ii) those independent of other observers’ reference frames, but dependent on the private reference frame of the observer, and (iii) those that are independent of any reference frames.

In the context of communication protocols a reasonable working assumption is that all observers have learned everything there is to learn about the reference-frame-independent observables of type (iii). In particular, we will assume that Alice knows reference-frame-independent observables involving the channels between Bob and Charlie.

On the other hand, we will assume that the observers know nothing about quantities that depend on other observers’ reference frames. So, the latter quantities, of both types (i) and (ii), can be considered “private” variables, while the type (iii) observables are “public.” So the three types of observables

---

1This is not meant to imply the present paper is not pointless, but only that one cannot set the unitaries $V_{kl}$ equal to the identity.
may be called private reference-frame dependent (of type (i)), private reference-frame independent (of type (ii)), and public reference-frame independent (of type (iii)).

The assumption that all public observables are known to all observers identifies another resource that is being used in standard quantum communication protocols. Indeed, such a resource is used implicitly when one assumes that all quantum communication channels are perfect. Here we do not explore the issue of quantifying that resource further. This would involve assuming that observers do not know anything about the public reference-frame-independent observables, assessing how that lack of knowledge affects standard communication protocols such as teleportation, superdense coding and quantum data hiding, and quantifying how much knowledge is needed to overcome obstacles in the way of teleportation etc.

3.2 Private reference-frame-dependent observables

The first type of observables allows one observer to learn something about the reference frame chosen by another observer, relative to his/her own. For example, if Alice prepares a state $|\psi_A\rangle$ and sends it to Bob, then Bob can measure matrix elements of the form

$$\langle \phi_B | V_{BA} R_{AB} | \psi_B \rangle.$$  

(6)

This procedure must make use of a classical communication (either from Alice to Bob or vice versa) containing a classical description of the state that Alice is supposed to send. In principle this can be done by transmitting two complex coefficients $(a_0, a_1)$ describing a qubit state $a_0|0_A\rangle + a_1|1_A\rangle$. In practice though, it is sufficient to choose from only a small set of states, such as $\{ |0\rangle, |1\rangle, (|0\rangle \pm |1\rangle)/\sqrt{2}, (|0\rangle \pm i|1\rangle)/\sqrt{2} \}$, as long as the set is sufficient for doing full tomography on a qubit. Thus in order to determine matrix elements (6) of the unitary $V_{BA} R_{AB}$ only a few bits of classical information need to be sent at a time.

We can also assume Bob applies some unitary $U$, not equal to the identity, and then returns the qubit to Alice. This leads to observables of the form

$$\langle \phi_A | V_{AB} U_{BB} V_{BA} | \psi_A \rangle,$$  

(7)

which can be measured by Alice. This requires classical communication of the matrix elements $\langle a_B | U_{BB} | b_B \rangle$ of the unitary that Bob applies. Here too, the unitaries may be chosen from a small set of gates, such as the Pauli gates $X, Y, Z$ (which are up to an overall phase shift equal to the Pauli matrices $\sigma_j$ for $j = x, y, z$, except now seen as unitaries rather than as generators of unitaries).

We conclude that only a small number of bits of communication is required for observers to measure private reference-frame-dependent observables.

3.3 Private reference-frame-independent observables

The second type of observables is independent of the choices of other observers’ reference frames, and always involves qubits that travel on a closed loop starting
and finishing at one and the same observer. The simplest kind is, e.g., where Alice sends a qubit to Bob who returns it to Alice without messing with its state. This allows Alice to measure the matrix elements

\[ \langle \phi_A | V_{AB} V_{BA} | \psi_A \rangle. \]  

This observable shows the definition of \( V_{AB} \) cannot be absorbed into the definition of \( R_{AB} \) and \( R_{BA} \), it being independent of Bob’s reference frame. In particular, \( V_{AB} \) and \( V_{BA} \) cannot be set equal to the identity. The observable does depend on Alice’s reference frame. That is, although Alice could communicate classically the entries of the unitary matrix \( V \) to other observers, they cannot reconstruct the same unitary operator from that description.

Similarly, Alice can measure matrix elements like

\[ \langle \phi_A | V_{AC} V_{CB} V_{BA} | \psi_A \rangle, \]  

where the qubit commutes from Alice to Bob to Charlie before returning to Alice (and so on for multiple stops at various observers).

### 3.4 Public reference-frame independent observables

From the two observables given in the preceding subsection it is easy to construct quantities that are independent of Alice’s reference frame. Namely Alice can measure the trace of those observables:

\[ \sum_{k=0,1} \langle k_A | V_{AB} V_{BA} | k_A \rangle. \]  

and

\[ \sum_{k=0,1} \langle k_A | V_{AC} V_{CB} V_{BA} | k_A \rangle. \]  

But those quantities can in fact also be measured by Bob and Charlie, thanks to the cyclic property of the trace. Hence the name public observables.

### 4 Quantum communication as a gauge theory

#### 4.1 Holonomies and Wilson loops

The quantity appearing in Eq. (8) can be written as a holonomy of a connection, i.e. a path-ordered exponential of a gauge field \( \vec{A} \) over the path of the qubit in space

\[ V_{AB} V_{BA} = \mathcal{P} \exp \left( i \oint_{\tilde{C}} q \vec{A} \cdot d\vec{r} \right), \]  

where the integral is taken over the closed circuit \( \tilde{C} \) in space the qubit traverses from Alice to Bob and back. Here \( q \) is the “charge” of the qubit, and \( \vec{A} \) takes
its value in the Lie algebra $\text{su}(2)$. The path-ordering operator $\mathcal{P}$ is necessary to account for the fact that fields $\vec{A}$ at different positions do not necessarily commute.

In writing down a path-integral in ordinary space we explicitly transformed away the time evolution of the qubit from (12). By reintroducing that time evolution we can rewrite (12) in a covariant way: First we write

$$V_{BA} = \mathcal{P} \exp \left( i \int_{x_A}^{x_B} qA_\mu \, dx^\mu, \right)$$

(13)

in terms of a gauge field $A_\mu = (A_t, \vec{A})$, that includes a time-component now. Second we explicitly include the time evolution along an observer’s world line to write a closed spacetime integral

$$V_{AB}V_{BA} = \mathcal{P} \exp \left( i \oint_C qA_\mu \, dx^\mu, \right)$$

(14)

where the integral is over the closed contour $C$ along which the qubit travels from Alice to Bob and back, and backwards in time in Alice’s frame.

The observable Eq. (14) still depends on what the starting point of the traveling qubit is on the closed contour. That is, only Alice can measure that quantity, and the similar quantity defined for Bob is in general different, $V_{AB}V_{BA} \neq V_{BA}V_{AB}$ (and that’s why in the previous Section we called this a private observable). If we define

$$W \equiv \text{tr}(V_{AB}V_{BA}) = \text{tr}\mathcal{P} \exp \left( i \oint_C qA_\mu \, dx^\mu, \right)$$

(15)

the resulting quantity becomes independent of the starting point, because of the cyclic property of the trace. This is in fact what one calls a Wilson loop. It is measurable both by Alice and Bob, and it is a public reference-frame independent observable, in the language of the preceding Section.

4.2 Matter and field degrees of freedom

In the pretty much standard case where photon polarization is used to implement a qubit for communication, there is a reversal of roles of field and matter degrees of freedom as compared to usual gauge theories. This reversal manifests itself in several different ways:

In gauge theories $A_\mu$ always describes a field and that field acts on the quantum state of a material particle, such as a proton. But in our present model the qubit describes a field degree of freedom and $A_\mu$ describes how matter (e.g. an optical fiber) acts on the polarization degree of a photon.

In addition, in our present context the photon is the quantum object while the matter degrees of freedom, as they are treated here, are classical. In gauge theories it is the matter particle that’s always treated quantum-mechanically
while the field may be described classically. Of course, the main aim of Yang-Mills theory is to quantize that field, too. Similarly, we could attempt to quantize the gauge field $A_{\mu}$ but we leave that can of worms closed.

Finally, whereas in gauge theories the material particle is charged, and the gauge field interacts with the particle through the charge, here the qubit is "charged", even if it is implemented by a photon.

4.3 Reference frame changes as gauge transformations

Wilson loops are invariant under infinitesimal gauge transformations of the form (using the covariant form now)

$$\delta A_{\mu}^a = \frac{1}{q} \partial_{\mu} \alpha^a + f^{abc} A_{\mu}^b \alpha^c,$$

for arbitrary (infinitesimal) $\alpha^a(x_{\mu})$ where we wrote

$$A_{\mu} = A_{\mu}^a t_a,$$

with $t_a = \sigma_a/2$ the generators of SU(2) and the structure constants are given by $f^{abc} = \epsilon^{abc}$ in terms of the anti-symmetric Levi-Civita symbol. Equivalently, the gauge transformation can be written as

$$A_{\mu}(x) \mapsto U(x) \left( A_{\mu} + i \frac{1}{q} \partial_{\mu} \right) U^\dagger(x),$$

with $U(x)$ any unitary, varying smoothly with spacetime. But as we noted before, the same quantity is also independent of the choices of reference frames. On the other hand, an object like (13) is both gauge dependent and reference-frame dependent.

All this may give the impression that the gauge transformation (16) on $A_{\mu}$ can be equivalently seen as changes of reference frames. This is indeed so. To see this, first consider the transformation of matrix elements (6) under changes of reference frames $R_{AA}$ and $R_{BB}$ for Alice and Bob effected by the transformation

$$R_{AB} \mapsto R_{AA} R_{AB} R_{BB}^\dagger,$$

(19)

while leaving $R_{AB}$ invariant.

We get

$$\langle \phi_B | V_{BA} R_{AB} | \psi_B \rangle \rightarrow \langle \phi_B | V_{BA} R_{AA}^\dagger R_{AB} R_{BB} | \psi_B \rangle$$

$$= \langle \phi_B | R_{BB} V_{BA} R_{AA}^\dagger R_{AB} | \psi_B \rangle,$$

(20)

where we wrote $| \psi_B \rangle = R_{BB} | \psi_B \rangle$ for the transformed state in Bob’s frame. The change of reference frames, as far as the observable (6) is concerned, can equivalently be viewed as a transformation of $V_{AB}$:

$$V_{AB} \mapsto R_{AA} V_{AB} R_{BB}^\dagger,$$

(21)

while leaving $R_{AB}$ invariant.
Next consider the transformation of observables of the form (7):

\[
\langle \phi A | V_{AB} U_{BB} V_{BA} | \psi A \rangle \mapsto \langle \phi A | R_{AA} V_{AB} R_{BB}^\dagger U_{BB} R_{BB} V_{BA} R_{AA}^\dagger | \psi A \rangle,
\]

(22)

where we used the transformation of the unitary \( U_{BB} \)

\[
U_{BB} \mapsto U_{BB}' = R_{BB} U_{BB} R_{BB}^\dagger.
\]

(23)

And again we see that the transformation (22) can be effected by the same transformations (21) applied to both \( V_{AB} \) and \( V_{BA} \) instead of transforming \( R_{AB} \). Thus, a change of reference frame \( R_{AB} \) can be seen as a gauge transformation of \( A_\mu \) instead. The easy way to see this is that \( V_{AB} \) always occurs in combination with \( R_{BA} \).

On the other hand, it is clear that gauge-invariance and reference-frame independence are different: no gauge-dependent quantity is observable, but there are reference-frame dependent observables. Indeed, those allow observers to learn about other observers’ reference frames.

4.4 Encoding and gauge-invariant measurements

It is by now well known and appreciated that one can encode information using decoherence-free subspaces (see for example [15]), which in the context of quantum communication [5] or quantum computing [15] protects quantum information against the decohering effects of joint errors of the generic form \( U \otimes U \) acting on \( N \) qubits, where \( U \in SU(2) \) is an arbitrary and possibly unknown unitary. We use this technique here to define gauge-invariant measurements (not to be confused with the observables defined in the previous Section) that will be useful in the next Sections.

4.4.1 Two qubits

The singlet state

\[
| \beta_0 \rangle \equiv \frac{[|0\rangle|1\rangle - |1\rangle|0\rangle]}{\sqrt{2}}
\]

(24)

(written in any basis) is invariant under \( U \otimes U \) for any \( U \in SU(2) \). This invariance means nothing more or less than that the state is reference-frame independent and gauge-independent. This leads immediately to the following gauge-invariant POVM on 2 qubits

\[
E_s = | \beta_0 \rangle \langle \beta_0 |; \quad E_t = I^{(2)} - E_s,
\]

(25)

with the subscripts \( s \) and \( t \) referring to “singlet” and “triplet”, the usual names for the \( J = 0 \) and \( J = 1 \) angular momentum eigenstates of 2 spin-1/2 systems.

For later use we consider the effect on the other three Bell states of \( U \otimes U \). Defining

\[
| \beta_x \rangle \equiv \frac{[|0\rangle|0\rangle - |1\rangle|1\rangle]}{\sqrt{2}}
| \beta_y \rangle \equiv \frac{[|0\rangle|0\rangle + |1\rangle|1\rangle]}{\sqrt{2}}
| \beta_z \rangle \equiv \frac{[|0\rangle|1\rangle + |1\rangle|0\rangle]}{\sqrt{2}}
\]

(26)
we have for any \( U \in SU(2) \)

\[
U \otimes U |\beta_j\rangle = I \otimes U \sigma_j U^\dagger \sigma_j |\beta_j\rangle,
\]

(27)

for \( j = 0, x, y, z \), which explains the particular choice of subscripts \( x, y, z \) for the various Bell states in (26). It is convenient to consider infinitesimal transformations

\[
U = \exp(i \epsilon_a \sigma_a) \approx I + i \epsilon_a \sigma_a,
\]

(28)

acting on the three Bell states \(|\beta_x\rangle, |\beta_y\rangle, \text{ and } |\beta_z\rangle\). Using this infinitesimal form it is easy to see we can then rewrite (27) as

\[
U \otimes U = \exp(2i \epsilon_a t_a),
\]

(29)

where \( t_a \) are the generators for the 3-D (\( J = 1 \)) representation of \( SU(2) \)

\[
t_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad t_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}; \quad t_z = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

(30)

written in the basis \( \{ |\beta_x\rangle, |\beta_y\rangle, |\beta_z\rangle \} \). We can use the same trick, writing the repeated action of a fixed unitary in \( SU(2) \) on any number \( N \) of qubits as a direct sum of various different irreducible representations of \( SU(2) \), to find gauge-invariant POVMs for any number of qubits.

### 4.4.2 Three qubits

In the direct-sum representation of \( SU(2) \) acting identically on 3 qubits, it is well-known [15] that there are two different \( J = 1/2 \) representations and one \( J = 3/2 \) representation. One 2-D gauge-invariant subspace (labeled by \( J = 1/2, \lambda = 0 \)) is spanned by

\[
\begin{align*}
|0\rangle|1\rangle|0\rangle - |1\rangle|0\rangle|0\rangle)/\sqrt{2} \\
|1\rangle|0\rangle|1\rangle - |0\rangle|1\rangle|1\rangle)/\sqrt{2},
\end{align*}
\]

(31)

where the basis states are defined in such a way that a bit flip on all three qubits on one basis state manifestly produces the other. The other 2-D gauge-invariant subspace (labeled by \( J = 1/2, \lambda = 1 \)) is spanned by

\[
\begin{align*}
[2|0\rangle|0\rangle|1\rangle - |0\rangle|1\rangle|0\rangle - |1\rangle|0\rangle|0\rangle]/\sqrt{6} \\
[2|1\rangle|1\rangle|0\rangle - |1\rangle|0\rangle|1\rangle - |0\rangle|1\rangle|1\rangle]/\sqrt{6},
\end{align*}
\]

(32)

which can be labeled by \( (J = 1/2, \lambda = 1) \). Finally, the 4-D completely symmetric subspace of three qubits provides the single \( J = 3/2 \) representation.

All gauge-invariant POVM on three qubits can then be defined in terms of the projections onto these three different subspaces

\[
\{ E_{1/2,0}, E_{1/2,1}, E_{3/2} \}
\]

(33)

in obvious notation.
4.4.3 Four qubits

In the basis \{\ket{\beta_x}, \ket{\beta_y}, \ket{\beta_z}\} defined above we can easily write down 4-qubit states that are invariant under the joint action of \(U^\otimes 4\) for any \(U \in \text{SU}(2)\), i.e., the states with zero angular momentum. Apart from the obviously invariant state \(\ket{\phi_{0,0}} \equiv \ket{\beta_0} \overline{\ket{\beta_0}}\) (which we labeled by \(J = 0, \lambda = 0\)) we also have the invariant state

\[
\ket{\phi_{0,1}} \equiv \frac{1}{\sqrt{3}} [\ket{\beta_x} \overline{\ket{\beta_x}} - \ket{\beta_y} \overline{\ket{\beta_y}} + \ket{\beta_z} \overline{\ket{\beta_z}}]
\]

labeled \(J = 0, \lambda = 1\). Of course, any linear combination of these two states is also invariant. As an example we may rewrite the states by relabeling the 4 qubits. For instance, if we relabel qubits 1,2,3,4 to 1,3,2,4 we get

\[
\ket{\phi_{0,0}}_{1234} = \frac{1}{\sqrt{2}} [\ket{\beta_0} \overline{\ket{\beta_0}} + \ket{\beta_y} \overline{\ket{\beta_y}} - \ket{\beta_z} \overline{\ket{\beta_z}}]/\sqrt{2}
\]

and these two states are obviously invariant as well.

There are, furthermore, three different \(J = 1\) representations and the one completely symmetric \(J = 2\) representation. A specific and detailed exposition of the corresponding subspaces can be found in, for example, [15]. Here we just label them by their \(J\) quantum number and, in the case of \(J = 1\) by an additional label \(\lambda = 0, 1, 2\).

5 Communication resources

Just as in Refs. [16, 17], we can define several different types of resources to be used in quantum communication protocols.

5.1 Bipartite entangled states

We can define an ebit between two observers \(k\) and \(l\) as the resource of them sharing a maximally entangled two-qubit state of the form

\[
I_{kk} \otimes V_{kl} R_{kl} [\ket{0_k} \overline{\ket{1_l}} - \ket{1_k} \overline{\ket{0_l}}]/\sqrt{2},
\]

where \(I_{kk}\) is the identity on \(k\). This ebit can be created by observer \(k\) producing locally the singlet state and subsequently sending it to \(l\) over the channel they share. Similarly we can define an alternative ebit by reversing the roles of observers \(k\) and \(l\), and this leads to the following definition of an ebit, involving the shared state

\[
V_{kl} R_{lk} \otimes I_{ll} [\ket{0_k} \overline{\ket{1_l}} - \ket{1_k} \overline{\ket{0_l}}]/\sqrt{2}.
\]

An important question is when these two definitions are equivalent, i.e., under what conditions these two states can be converted into one another by LOCC.
(local operations and classical communication). In general, we can write for any unitary $U$

$$U_{kk} \otimes I_{ll} [\langle 0_k | 1_l \rangle - \langle 1_k | 0_l \rangle] / \sqrt{2} = I_{kk} \otimes U_{ll}^\dagger [\langle 0_k | 1_l \rangle - \langle 1_k | 0_l \rangle] / \sqrt{2} \quad (38)$$

In order to convert one $ebit$ into the other, observer $l$ would have to apply the local unitary $V_{lk} V_{dl}$. This operation can indeed be implemented by observer $l$ (though observer $k$ would not be able to implement that operation!) by the assumptions of Sections 2 and 3. The two definitions of $ebit$ are, therefore, equivalent. For the other Bell states (i.e. the triplet states) the same conclusion holds as they can all be obtained from the singlet state by locally applying $X$, $Y$, or $Z$: by observer $k$ to the state (36) and by observer $l$ to the state (37), respectively. Thus, all Bell states so defined are equivalent, just as they are in the usual case where all observers do share reference frames and have perfect quantum communication channels.

The reason that Alice would have to apply the operation $V_{AB} V_{BA}$ while converting an $ebit$ she shares with Bob is obvious: the singlet state can be transmitted without change from one observer to the next. So, one way to generate an $ebit$ is to have Alice create the singlet state and transmit both qubits to Bob who then returns one of the qubits to Alice. The other way to generate an $ebit$ is for Alice to transmit only one of her qubits to Bob. The difference between the two ways of generating an $ebit$ is that one qubit is transmitted from Alice to Bob and back instead of staying in Alice’s possession.

In spite of the above observations, we note that not all bipartite entangled states are equivalent. It is easy to write down maximally entangled states $not$ equivalent to the states considered here, such as the state

$$[\langle 0_k | 1_l \rangle - \langle 1_k | 0_l \rangle] / \sqrt{2},$$

which observers $k$ and $l$ would be happy to share, as it would allow them to measure Bell inequalities or to perform teleportation without further ado. Unfortunately for them, the state cannot be obtained without sharing a reference frame ($i.e.$ knowing $R_{kl}$).

For yet another type of Bell states $not$ equivalent to $ebits$, one can consider Bell states generated by a third observer, and distributed to two other observers. This example will be examined further below, as it serves as a resource for quantum data hiding.

### 5.2 Tripartite entangled states

In discussions about three-party entangled states attention is usually focused on two different types of states, the GHZ state and the W state. Indeed, these are the only 2 inequivalent states that are truly three-party entangled, in the case where all three parties share a reference frame. We will see here that without shared reference frames there are infinitely many inequivalent states with truly three-party entanglement. The reason is simple enough, there is no gauge-invariant three-qubit state.
Since both the W state and the GHZ state are in the \( J = 3/2 \) representation of SU(2), we really have to consider only one of these in the present context. We focus on the GHZ state for no reason in particular. A GHZ state can be prepared from 2 Bell states, shared between three parties. For instance, if Alice shares one singlet state each with Bob and Charlie, a CNOT applied by her to her two qubits and measurement of the target qubit in the standard basis \( |0_A⟩ \) or \( |1_A⟩ \) leads to a GHZ-like state, namely

\[
I_{AA} \otimes V_{BA}R_{AB} \otimes V_{CA}R_{AC} [ |1_A⟩|0_B⟩|0_C⟩ + |0_A⟩|1_B⟩|1_C⟩] / \sqrt{2},
\]  
(39)

if the result of her measurement is “0”, and

\[
I_{AA} \otimes V_{BA}R_{AB} \otimes V_{CA}R_{AC} [ |0_A⟩|0_B⟩|1_C⟩ + |0_A⟩|1_B⟩|0_C⟩] / \sqrt{2},
\]  
(40)

when the result is “1”.

In the former case Alice can simply flip her qubit to convert (39) to the state

\[
I_{AA} \otimes V_{BA}R_{AB} \otimes V_{CA}R_{AC} [ |0_A⟩|0_B⟩|0_C⟩ + |1_A⟩|1_B⟩|1_C⟩] / \sqrt{2}.
\]  
(41)

Obviously the same state is created when Alice generates locally a GHZ state \([ |0_A⟩|0_A⟩|0_A⟩ + |1_A⟩|1_A⟩|1_A⟩] / \sqrt{2} \) and subsequently sends the second qubit to Bob and the third to Charlie over the channels she shares with them. In the other case, though, the state (40) generated is genuinely different and in fact not equivalent, as either Bob or both Alice and Charlie would have to apply the bit flip operation. This way they either get

\[
I_{AA} \otimes V_{BA}R_{AB} \otimes V_{CA}R_{AC} [ |0_A⟩|0_B⟩|0_C⟩ + |1_A⟩|1_B⟩|1_C⟩] / \sqrt{2}.
\]  
(42)

or

\[
I_{AA} \otimes V_{BA}R_{AB} \otimes X_{CC}V_{CA}R_{AC}X_{CC} [ |0_A⟩|0_B⟩|0_C⟩ + |1_A⟩|1_B⟩|1_C⟩] / \sqrt{2}.
\]  
(43)

These two versions of the GHZ state are equivalent by construction.

Moreover, we can easily write down four more states corresponding to (41)–(43) but with the roles of the various observers interchanged. Thus, we have four more GHZ states that are not equivalent. Generally speaking, all those states are inequivalent because for almost all unitaries \( U^{(1)} \) and \( U^{(2)} \) the relation

\[
U^{(1)}_{AA} \otimes U^{(2)}_{BB} \otimes I_{CC} [ |0_A⟩|0_B⟩|0_C⟩ + |1_A⟩|1_B⟩|1_C⟩] / \sqrt{2}
\]

holds with \( U^{(1,2)} \) a nonlocal (entangling) unitary operation on two qubits, that cannot be factorized into a product of two local unitaries. In fact, the operation \( X \) that appears in Eqs. (42)–(43) above is nothing special (other than being related to the particular protocol used to generate a GHZ state from 3 singlet states), and when we replace it by an arbitrary unitary not equal to the identity, we get yet another inequivalent state. Thus there are three continuous sets of inequivalent GHZ states.
5.3 Refbits
Here we generalize the refbit, a unit of sharing a reference frame, as defined in [10]. A useful definition, exploited in the next Section, is as follows: a refbit is a single qubit state $|\psi_k\rangle$ in a different observer $l$’s hands, with $\psi$ chosen “optimally” for the (communication) task at hand. This definition reduces to that of Ref. [10], which described the special case where all observers agree on the physical meaning of the basis states $|0\rangle$ and $|1\rangle$ but not on the relative phase between them. In that case the sharing of an equal superposition of the basis states is always optimal and constitutes a refbit.

6 Communication protocols
6.1 Quantum data hiding
Suppose Charlie prepares one of two states $[|0_C\rangle|1_C\rangle \pm |1_C\rangle|0_C\rangle]/\sqrt{2}$, and distributes this state to Alice and Bob, who consequently end up with a state

$$V_{AC}R_{CA} \otimes V_{BC}R_{CB}[|0_A\rangle|1_B\rangle \pm |1_A\rangle|0_B\rangle]/\sqrt{2}. \quad (45)$$

Alice and Bob cannot determine which one of the two states Charlie distributed to them, as they would need to know how Charlie’s reference frame is oriented with respect to theirs. Thus Charlie manages to hide one bit of information (‘+’ or ‘−’) in the quantum state (45). The question is with what additional resources Alice and Bob are able to unlock the hidden bit.

First note that Alice and Bob can always unlock the bit if one sends the qubit to the other. This is a trivial observation if Alice, Bob and Charlie share perfect communication channels, but here one has to do a little work in order to see this and it only works for certain entangles states. For example, if Bob sends the qubit he received from Charlie to Alice, she ends up with a state

$$V_{AC}R_{CA} \otimes V_{AB}V_{BC}R_{CA}[|0_A\rangle|1_A\rangle \pm |1_A\rangle|0_A\rangle]/\sqrt{2}. \quad (46)$$

By applying the unitary (which is indeed measurable by Alice in her reference frame)

$$U = (V_{AC}V_{CA})(V_{AB}V_{BC}V_{CA})^\dagger$$

to the second qubit, she transforms (46) to

$$V_{AC}R_{CA} \otimes V_{AC}R_{CA}[|0_A\rangle|1_A\rangle \pm |1_A\rangle|0_A\rangle]/\sqrt{2}. \quad (47)$$

By performing the gauge-invariant two-outcome POVM [23] she can perfectly distinguish the two states, as one is a triplet state, the other the singlet state.

Note that this trick would not work if Charlie had hidden his bit in the two other Bell states, since those two states are both triplet states. This shows the Bell states are not created equal in this context. Namely, an ebit created by Charlie is not equivalent, for Alice and Bob, to ebits created by Alice or Bob themselves.
Another resource that would allow Alice and Bob to unlock the bit with some nonzero probability is two refbits to be provided by Charlie. For instance, suppose Charlie sends Alice and Bob a qubit each that he prepares both in the same state, say \( |0_C⟩ + |1_C⟩⟩ \). We may write the resulting state in the form

\[
V_{AC}R_{CA} \otimes V_{BC}R_{BA} \sum_{i,j} C^\pm_{ij} |\beta_i⟩_A \otimes |\beta_j⟩_B
\]

in the Bell-state bases \( \{ |\beta_i⟩_k \} \) for \( i = 0, x, y, z \) and \( k = A, B \). One finds then \( C^+_0 = 1/\sqrt{8} \) whereas \( C^-_{00} = 0 \). This implies that when both Alice and Bob perform the gauge-invariant POVM (20) and both find the result “singlet”, they have conclusively identified the ‘+’ bit. A similar conclusion holds when Charlie provides Alice and Bob with orthogonal refbits, such as when he transmits \( |0_C⟩ + |1_C⟩⟩ / \sqrt{2} \) to Alice and \( |0_C⟩ - |1_C⟩⟩ / \sqrt{2} \) to Bob. In that case one finds \( C^+_{00} = 1/\sqrt{8} \) whereas \( C^-_{00} = 0 \), so that Alice and Bob can conclusively identify the ‘−’ bit, when they both find the measurement outcome “singlet.” Thus in half of the cases Alice and Bob have a 1/8 chance to unlock the bit, so that their probability for success is \( P_{\text{success}} = 1/16 \). But do note that Charlie can decide to either give them no chance to recover the bit, or a chance of 1/8. Of course, providing Alice and Bob with infinitely many refbits allows them to always unlock the bit.

### 6.2 Superdense coding

Suppose Alice and Bob share an ebit, a state of the form (36). Now Alice performs one of 4 operations \( U^{(k)} = I_{AA}, X_{AA}, Y_{AA}, Z_{AA} \) for \( k = 1 \ldots 4 \) on her half of the Bell pair, and subsequently sends her qubit to Bob. He ends up with one of four orthogonal states

\[
V_{BA}R_{AB}U^{(k)}_{BB} \otimes V_{BA}R_{AB} \left[ |0_B⟩ |1_B⟩ - |1_B⟩ |0_B⟩ \right] / \sqrt{2},
\]

but he can only distinguish the singlet from the triplet states and nothing more. And so Alice can in fact send not more than 1 bit of information with 1 qubit this way. This protocol hardly deserves the name “superdense coding.”

Alice has to provide at least two refbits to allow Bob to receive more bits of information. The easiest way to analyze this situation is to first consider the case where Alice actually provides Bob with a maximally entangled state of 2 qubits, say \( |β⟩ \) (note a singlet state \( |β_0⟩ \) would be a useless resource here), instead of two refbits that are always in a product state. Bob can then apply, after having applied the two-qubit POVM (25) and having gotten the inconclusive “triplet” outcome, a gauge-invariant POVM on his 4 qubits. If Alice in fact applied the \( X \) operation, then Bob gets, with probability 1/3, the outcome ‘φ01’, which identifies unambiguously the correct state by excluding the possibilities \( Z \) and \( Y \). With other outcomes, Bob learns nothing more about what operation Alice applied than he already knows from his first measurement.

Now if Alice sends just two refbits, say both in the state \( |0_A⟩ \), then Bob receives an equal superposition of \( |β_z⟩ \) and \( |β_y⟩ \). What he then can distinguish
unambiguously is only the $Z$ vs. the $X$ and $Y$ operations. That is, if he gets the `$\phi_{0.1}$' outcome he knows Alice cannot have applied $Z$. The `$\phi_{0.1}$' outcome occurs with probability 1/6 if Alice in fact applied $X$, and also if in fact she applied $Y$. So Alice can send more than 1 bit of information by choosing, for example, $I$ with probability 1/2, $Z$ with probability 1/4, and $X$ or $Y$ with probability 1/4. Then in 1/2 of the cases Bob gets only 1 bit, but in 1/24 (=1/6*1/4) of the cases he gets two bits, which is 1/24th of a bit more than without refbits. This is not the optimum protocol but the easiest to explain and an improvement almost worth the name “superdense coding.” Of course, in the limit of infinitely many refbits the optimum superdense coding protocol allows Alice to send 2 bits to Bob with 1 qubit.

6.3 No bit commitment

Consider the following protocol. Alice prepares either the gauge-invariant 4-qubit state $|\phi_{0.0}\rangle$ (which we call “case 0”) or the gauge-invariant 4-qubit state $|\phi_{0.1}\rangle$ (“case 1”), both defined in Section 4.4.3. Then she sends 2 qubits to Bob over the quantum channel they share. In case 0, she sends the first and the third qubit, in case 1 she either sends the first and second qubits, or the first and third qubits (without telling him (yet)). There is only 1 useful gauge-invariant measurement Bob could perform, namely projecting onto the singlet and triplet states. From the definitions in Section 4.4.3 we read off that in case 0, Bob would get the outcome “singlet” with probability 1/4. In case 1, if Alice sends the first and third qubits, he would get that outcome with probability 3/4, whereas if she sends the first and second qubits, he never gets that outcome. So, if Alice sends the first and third qubits, he would get that outcome with probability 3/4, Bob will have a probability of 1/4 = 1/3 * 1/3 of getting the “singlet” outcome, independent of Alice’s choice between cases 0 and 1.

Thus the above procedure part can act as the first stage of a quantum bit commitment protocol, where Alice commits to a bit that Bob cannot read. The difference with usual bit-commitment schemes is that in the protocol presented above the choice “0” corresponds always to the same state. For the other choice Bob gets, as usual, a mixture of two states. Since that mixture is in fact the same as the partial density matrix of particles 1 and 3 in the state $|\phi_{0.0}\rangle$, Alice can in fact rotate one choice to the other, using just one ancilla qubit. Thus she can always cheat perfectly.

This all follows from the general case analyzed in [8]. That Alice needs an ancilla to cheat without going detected, and that in contrast without ancillas bit commitment would be possible, was shown in [18].

7 Conclusions

The simple assumption that communication channels are not perfect leads to additional complications in the theory of reference frames in quantum communication. This assumption of nonperfect quantum communication channels arises
naturally— after all how could observers be sure to share perfect channels when they do not share a reference frame— but has not been studied so far.

We defined three types of observables, private or public, and reference-frame dependent or reference-frame independent. By rewriting certain observables in terms of a gauge vector field, we showed how reference frame changes can be viewed as gauge transformations. In particular, “public reference-frame independent observables” in the quantum communication context correspond to gauge-independent observables, and one class of those observables in particular corresponds to Wilson loops.

Consideration of quantum communication resources revealed that not all bipartite maximally entangled states are equivalent to 
\textit{ebits}, and for three-party entangled states the situation is worse: for example, there are 3 continuous sets of inequivalent GHZ states, that cannot be converted into one another by local operations and classical communication. Finally, by generalizing the concept of a \textit{refbit} \cite{10} one can quantify how much of a reference frame one has to share in order to be able to implement superdense coding to some given extent or to unlock a bit in a quantum data hiding protocol.

References

[1] S. Massar and S. Popescu, Phys. Rev. Lett. 74, 1259 (1995); N. Gisin and S. Popescu, \textit{ibidem} 83, 432 (1999); A. Peres and P. Scudo, \textit{ibidem} 86, 4160 (2001); 87, 167901 (2001); E. Bagan, M. Baig, and R. Muñoz-Tapia, \textit{ibidem} 87, 257903 (2001); T. Rudolph and L. Grover, \textit{ibidem} 91, 217905 (2003).

[2] I. Chuang, Phys. Rev. Lett. 85, 2006 (2000); M. de Burgh and S.D. Bartlett, Phys. Rev. A 72, 042301 (2005).

[3] S.J. van Enk, J. Mod. Optics 48, 2049 (2001).

[4] T. Rudolph and B.C. Sanders, Phys. Rev. Lett. 87, 077903 (2001). S.J. van Enk and C.A. Fuchs, Phys. Rev. Lett. 88, 027902 (2002); Quant. Inf. Comp. 2, 151 (2002); H.M. Wiseman, Journal of Optics B, S849 (2004); S.D. Bartlett, T. Rudolph, and R.W. Spekkens, \texttt{quant-ph/0507214}.

[5] S.D. Bartlett, T. Rudolph, and R.W. Spekkens, Phys. Rev. Lett. 91, 027901 (2003).

[6] H. Wiseman and J.A. Vaccaro, Phys. Rev. Lett. 91, 097902 (2003); J.A. Vaccaro, F. Anselmi, and H. Wiseman, Int. J. Quant. Info. 1, 427 (2003).

[7] F. Verstraete, and J.I. Cirac, Phys. Rev. Lett. 91, 010404 (2003); N. Schuch, F. Verstraete, and J.I. Cirac, Phys. Rev. Lett. 92, 087904 (2004).

[8] A. Kitaev, D. Mayers, and J. Preskill, Phys. Rev. A 69, 052326 (2004).

[9] S.D. Bartlett, T. Rudolph, and R.W. Spekkens, Phys. Rev. A 70, 032307 (2004).
[10] S.J. van Enk, Phys. Rev. A 71, 032339 (2005).
[11] S.J. van Enk, Phys. Rev. A 72, 064306 (2005).
[12] S.D. Bartlett, A.C. Doherty, R.W. Spekkens, and H.M. Wiseman, Phys. Rev. A 73, 022311 (2006).
[13] Y. Aharonov and L. Susskind, Phys. Rev. 155, 1428 (1967).
[14] N. Gisin, G. Ribrody, W. Tittel, and H. Zbinden, Rev. Mod. Phys 74, 145 (2002).
[15] M.S. Byrd, D.A. Lidar, L.A. Wu, and P. Zanardi, Phys. Rev. A 71, 052301 (2004).
[16] A. Harrow, Phys. Rev. Lett. 92, 097902 (2004).
[17] B.M. Terhal, D. DiVincenzo, and D.W. Leung, Phys. Rev. Lett. 86, 5807 (2001).
[18] D.P. DiVincenzo, J.A. Smolin and B.M. Terhal, New. J. Phys. 6, 80 (2004).