A topos-theoretic proof of Shelah’s eventual categoricity conjecture

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Abstract elementary classes

An abstract elementary class (AEC) $K$ is a category equivalent to an accessible category with directed colimits whose morphisms are monomorphisms, that admits an embedding $F: K \to A$ into a finitely accessible category preserving directed colimits and monomorphisms which is, in addition:

1. Full on isomorphisms: for every isomorphism $h: F(A) \to F(B)$ there is an isomorphism $m: A \to B$ with $F(m) = h$.
2. Nearly full: for every commutative triangle $F(A) \to F(B) \to F(C)$ $h: F(f) \to F(g)$ there is $m: A \to C$ with $F(m) = h$. 
Abstract elementary classes

Definition

An abstract elementary class (AEC) $\mathcal{K}$ is a category equivalent to an accessible category with directed colimits whose morphisms are monomorphisms, that admits an embedding $F : \mathcal{K} \to \mathcal{A}$ into a finitely accessible category preserving directed colimits and monomorphisms which is, in addition:

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2. Nearly full: for every commutative triangle:

$$
\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(C) \\
\downarrow{h} & & \downarrow{F(g)} \\
F(B) & \xrightarrow{F(m)} & F(C)
\end{array}
$$

there is $m : A \to C$ with $F(m) = h$. 
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   $\begin{array}{ccc}
   F(A) & \xrightarrow{F(f)} & F(B) \\
   h \downarrow & \nearrow & \downarrow F(g) \\
   F(C) & &
   \end{array}$

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Abstract elementary classes

FACT (Beke-Rosicky 2012): An AEC automatically admits an iso-full, nearly full embedding $E: K \to \text{Emb}(\Sigma)$ (for some signature $\Sigma$) preserving directed colimits.

It also has eventually a Łowenheim-Skolem number $\lambda$: for every substructure $i: A \to F(B)$ there is $E(f): F(C) \to F(B)$ such that $i$ factors through $E(f)$ and $|E(C)| \leq |A| + \lambda$.

FACT (Lieberman-Rosicky-Vasey 2019): For each object $A$ of an AEC, $|E(A)|$ coincides with its internal size $|A|$ defined as follows.

If $r(A)$ is the least regular cardinal $\lambda$ such that $A$ is $\lambda$-presentable, then:

- $|A| = \kappa$ if $r(A) = \kappa$;
- $|A| = r(A)$ if $r(A)$ is limit.
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$$|A| = \begin{cases} \kappa & \text{if } r(A) = \kappa \\ r(A) & \text{if } r(A) \text{ is limit} \end{cases}$$
Abstract elementary classes

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$$|A| = \begin{cases} \kappa & \text{if } r(A) = \kappa \\ r(A) + r(A) & \text{if } r(A) \text{ is limit} \end{cases}$$
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$$|A| = \begin{cases} \kappa & \text{if } r(A) = \kappa^+ \\ r(A) & \text{if } r(A) \text{ is limit} \end{cases}$$
Shelah’s conjecture

Conjecture (ZFC) For every AEC there is a cardinal $\kappa$ such that if the AEC is categorical in some $\lambda > \kappa$ then it is categorical in every $\lambda' > \kappa$.

There is a proof of the conjecture (Shelah-Vasey 2019) assuming GCH and large cardinals. In fact, the use of large cardinals is to guarantee that the AEC will eventually satisfy the amalgamation property: $\mathcal{N} \rightarrow \mathcal{M} \rightarrow \mathcal{N}'$.

We will give a short topos-theoretic proof of the conjecture assuming GCH and that the AEC satisfies amalgamation.

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A topos-theoretic proof of Shelah’s eventual categoricity conjecture

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Conjecture

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\[
\begin{array}{c}
N \\ \\
\downarrow \\
M \\
\end{array}
\quad \quad \quad
\begin{array}{c}
\rightarrow \\
A \\
\uparrow \\
N' \\
\end{array}
\quad \quad \quad
\begin{array}{c}
\rightarrow \\
M \\
\end{array}
\]
Shelah’s conjecture

Conjecture

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\[
\begin{array}{c}
N \longrightarrow A \\
\uparrow & \quad \uparrow \\
M \longrightarrow N' \\
\end{array}
\]

We will give a short topos-theoretic proof of the conjecture assuming \(GCH\) and that the AEC satisfies amalgamation.
Given an accessible category $A$ with $\kappa$-directed colimits, its $\kappa$-Scott topos $S\kappa(A)$ is the full subcategory of the presheaf $\mathbf{Set}^A$ given by those functors $F : A \to \mathbf{Set}$ preserving $\kappa$-directed colimits.

Given a topos $E$, its category of $\kappa$-points $\text{pt}\kappa(E)$ (i.e., geometric morphisms to $\mathbf{Set}$ whose inverse images preserve all $\kappa$-small limits) is an accessible category with $\kappa$-directed colimits.

**Theorem (Henry-Di Liberti)** There is a $(2,1)$-adjunction: $S\kappa : \text{Acc} \to \kappa\text{-Top}$ given by the Scott functor $S\kappa$ and the functor $\text{pt}\kappa$. 

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The Scott adjunction

Given an accessible category $A$ with $\kappa$-directed colimits, its $\kappa$-Scott topos $S_\kappa(A)$ is the full subcategory of the presheaf $Set^A$ given by those functors $F : A \to Set$ preserving $\kappa$-directed colimits.
The Scott adjunction

Given an accessible category $A$ with $\kappa$-directed colimits, its $\kappa$-Scott topos $S_\kappa(A)$ is the full subcategory of the presheaf $\mathcal{S}et^A$ given by those functors $F : A \to \mathcal{S}et$ preserving $\kappa$-directed colimits.

Given a topos $\mathcal{E}$, its category of $\kappa$-points $pt_\kappa(\mathcal{E})$ (i.e., geometric morphisms to $\mathcal{S}et$ whose inverse images preserve all $\kappa$-small limits) is an accessible category with $\kappa$-directed colimits.
The Scott adjunction

Given an accessible category $A$ with $\kappa$-directed colimits, its $\kappa$-Scott topos $S_\kappa(A)$ is the full subcategory of the presheaf $\mathcal{S}et^A$ given by those functors $F : A \to \mathcal{S}et$ preserving $\kappa$-directed colimits.

Given a topos $\mathcal{E}$, its category of $\kappa$-points $pt_\kappa(\mathcal{E})$ (i.e., geometric morphisms to $\mathcal{S}et$ whose inverse images preserve all $\kappa$-small limits) is an accessible category with $\kappa$-directed colimits.

**Theorem**

*(Henry-Di Liberti) There is a (2-)adjunction:*

\[ S : \text{Acc}_\kappa \quad \text{Top}_\kappa : pt_\kappa \]

between the category of accessible categories with $\kappa$-directed colimits and the category of $\kappa$-exact localization of presheaf toposes given by the Scott functor $S_\kappa$ and the functor $pt_\kappa$.  

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Proof idea

Let $K$ be an AEC with amalgamation that is categorical in some successor $\lambda > \kappa$.

A model $M$ is $\mu + \sigma$-saturated if for every morphism $N \rightarrow N'$ between models of size $\mu$, every morphism $N \rightarrow M$ can be extended.

Consider the following diagram of toposes and inverse images of geometric morphisms given by restriction:

\[
\begin{array}{ccc}
S_{\kappa + \kappa} & \xrightarrow{K \geq \kappa} & S_{\lambda + \lambda} \\
S_{\kappa + \kappa} & \xrightarrow{Sat_{\kappa + \kappa}} & S_{\lambda + \lambda} \\
\end{array}
\]

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A model $M$ is $\mu^+$-saturated if for every morphism $N \to N'$ between models of size $\mu$, every morphism $N \to M$ can be extended:
Proof idea

Let $\mathcal{K}$ be an AEC with amalgamation that is categorical in some successor $\lambda > \kappa$.

A model $M$ is $\mu^+$-saturated if for every morphism $N \to N'$ between models of size $\mu$, every morphism $N \to M$ can be extended:

\[
\begin{array}{ccc}
N' & \Downarrow & M \\
\uparrow & & \\
N & \rightarrow & M
\end{array}
\]

Consider the following diagram of toposes and inverse images of geometric morphisms given by restriction:
Proof idea

Let $\mathcal{K}$ be an AEC with amalgamation that is categorical in some successor $\lambda > \kappa$.

A model $M$ is $\mu^+$-saturated if for every morphism $N \to N'$ between models of size $\mu$, every morphism $N \to M$ can be extended:

\[
\begin{array}{ccc}
N' & \xrightarrow{\text{dashed}} & M \\
\uparrow & & \downarrow \\
N & \xrightarrow{} & M
\end{array}
\]

Consider the following diagram of toposes and inverse images of geometric morphisms given by restriction:

\[
\begin{array}{ccc}
S_{\kappa^+}(\mathcal{K}_{\geq \kappa^+}) & \xrightarrow{} & S_\lambda(\mathcal{K}_{\geq \lambda}) \\
\downarrow & & \downarrow \\
S_{\kappa^+}(Sat_{\kappa^+}(\mathcal{K})) & \xrightarrow{} & S_\lambda(Sat_{\lambda}(\mathcal{K}))
\end{array}
\]
Proof idea

Let $\mathcal{K}$ be an AEC with amalgamation that is categorical in some successor $\lambda > \kappa$.

A model $M$ is $\mu^+$-saturated if for every morphism $N \to N'$ between models of size $\mu$, every morphism $N \to M$ can be extended:

$$
\begin{array}{ccc}
N' & \to & M \\
\uparrow & & \downarrow \\
N & \to & M \\
\end{array}
$$

Consider the following diagram of toposes and inverse images of geometric morphisms given by restriction:

$$
\begin{array}{ccc}
S_{\kappa^+}(\mathcal{K}_{\geq \kappa^+}) & \to & S_{\lambda}(\mathcal{K}_{\geq \lambda}) \\
\downarrow & & \downarrow \\
S_{\kappa^+}(Sat_{\kappa^+}(\mathcal{K})) & \to & S_{\lambda}(Sat_{\lambda}(\mathcal{K})) \\
\end{array}
$$
FACT (Rosicky 1997): $\mathcal{K}_{\geq \lambda}$ coincides with $Sat_\lambda(\mathcal{K})$. Therefore the right morphism is an isomorphism.
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The AEC $\mathcal{K}$, as any accessible category, can be axiomatized up to equivalence by basic sentences through a theory $T$ in $\mathcal{L}_{\mu^+,\mu}$, and can have models not just in $\text{Set}$ but in any $\mu$-coherent category (i.e., in a category with enough $\text{Set}$-like properties).
FACT (Rosicky 1997): $\mathcal{K}_{\geq \lambda}$ coincides with $Sat_{\lambda}(\mathcal{K})$. Therefore the right morphism is an isomorphism.

The AEC $\mathcal{K}$, as any accessible category, can be axiomatized up to equivalence by basic sentences through a theory $\mathbb{T}$ in $L_{\mu^+,\mu}$, and can have models not just in $Set$ but in any $\mu$-coherent category (i.e., in a category with enough $Set$-like properties).

The syntactic category $C_{\mathbb{T}}$ is defined through the following universal property:
FACT (Rosicky 1997): $\mathcal{K}_{\geq \lambda}$ coincides with $Sat_\lambda(\mathcal{K})$. Therefore the right morphism is an isomorphism.

The AEC $\mathcal{K}$, as any accessible category, can be axiomatized up to equivalence by basic sentences through a theory $\mathbb{T}$ in $\mathcal{L}_{\mu^+},\mu$, and can have models not just in $Set$ but in any $\mu$-coherent category (i.e., in a category with enough $Set$-like properties). The syntactic category $\mathcal{C}_\mathbb{T}$ is defined through the following universal property:

\[
\begin{align*}
\mathcal{C}_\mathbb{T} \xrightarrow{\mu-\text{coherent}} & \mathcal{D} \\
\uparrow & \uparrow \\
M_0 \xleftarrow{} & M
\end{align*}
\]

Then $\mathcal{K}_{\geq \kappa^+}$ is axiomatized by basic sentences through $\mathbb{T}_{\kappa^+}$. 
Proof idea

The $\kappa^+$-classifying topos of $T_{\kappa^+}$ (Espíndola 2017), $\text{Set}[T_{\kappa^+}]_{\kappa^+}$ is defined through the following universal property:

$$\text{Set}[T_{\kappa^+}]_{\kappa^+}$$

$\kappa^+$-small limit preserving
The $\kappa^+$-classifying topos of $\mathbb{T}_\kappa^+$ (Espindola 2017), $\mathcal{S}et[\mathbb{T}_\kappa^+]_{\kappa^+}$ is defined through the following universal property:

\[ \mathcal{C}_T \rightarrow \mathcal{S}et[\mathbb{T}_\kappa^+]_{\kappa^+} \]

\[ \mathcal{E} \leftarrow \mathcal{S}et[\mathbb{T}_\kappa^+]_{\kappa^+} \]

\[ \kappa^+ \text{-small limit preserving} \]
Proof idea

Suppose $\text{Sat}_\kappa + (K)$ is axiomatizable by $T_{\text{sat}}\kappa +$. Then the morphism $C_{T\kappa +} \rightarrow C_{T_{\text{sat}}\kappa +}$ induces a morphism between the corresponding $\kappa +$-classifying toposes $f^*: \text{Set} \left[ T_{\kappa +} \right] \rightarrow \text{Set} \left[ T_{\text{sat}\kappa +} \right]$.

We now want to deduce from the fact that the right morphism is an isomorphism that $f^*$ is an equivalence, i.e., every model of size $\kappa +$ is $\kappa +$-saturated. Then $K$ is $\kappa +$-categorical since there is a unique such model (Rosicky 1997).
Proof idea

Suppose $Sat_{\kappa^+}(\mathcal{K})$ is axiomatizable by $\mathbb{T}_{\kappa^+}^{sat}$. Then the morphism $\mathcal{C}_{\mathbb{T}_{\kappa^+}} \to \mathcal{C}_{\mathbb{T}_{\kappa^+}^{sat}}$ induces a morphism between the corresponding $\kappa^+$-classifying toposes $f^* : Set[\mathbb{T}_{\kappa^+}]_{\kappa^+} \to Set[\mathbb{T}_{\kappa^+}^{sat}]_{\kappa^+}$. 

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Suppose $Sat_{\kappa^+}(\mathcal{K})$ is axiomatizable by $T_{\kappa^+}^{sat}$. Then the morphism $C_{T_{\kappa^+}} \to C_{T_{\kappa^+}^{sat}}$ induces a morphism between the corresponding $\kappa^+$-classifying toposes $f^* : \text{Set}[T_{\kappa^+}]_{\kappa^+} \to \text{Set}[T_{\kappa^+}^{sat}]_{\kappa^+}$. Then:

\[
\begin{array}{c}
\text{Set}[T_{\kappa^+}]_{\kappa^+} \\
\downarrow f^* \\
\text{Set}[T_{\kappa^+}^{sat}]_{\kappa^+}
\end{array}
\xrightarrow{\eta_{\text{Set}[T_{\kappa^+}]_{\kappa^+}}} \xrightarrow{\eta_{\text{Set}[T_{\kappa^+}^{sat}]_{\kappa^+}}} \xrightarrow{S_{\kappa^+}(\mathcal{K}_{\geq \kappa^+})} \xrightarrow{S_{\lambda}(\mathcal{K}_{\geq \lambda})}
\]

We now want to deduce from the fact that the right morphism is an isomorphism that $f^*$ is an equivalence, i.e., every model of size $\kappa^+$ is $\kappa^+$-saturated. Then $K$ is $\kappa^+$-categorical since there is a unique such model (Rosicky 1997).
Proof idea

Suppose $\text{Sat}_{\kappa^+}(\mathcal{K})$ is axiomatizable by $\mathbb{T}^{\text{sat}}_{\kappa^+}$. Then the morphism $C_{\mathbb{T}^{\text{sat}}_{\kappa^+}} \to C_{\mathbb{T}^{\text{sat}}_{\kappa^+}}$ induces a morphism between the corresponding $\kappa^+$-classifying toposes $f^*: \text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} \to \text{Set}[\mathbb{T}^{\text{sat}}_{\kappa^+}]_{\kappa^+}$. Then:

$$
\begin{align*}
\text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} & \xrightarrow{\eta^*_\text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+}} S_{\kappa^+}(\mathcal{K}_{\geq \kappa^+}) \xrightarrow{\eta^*_\text{Set}[\mathbb{T}^{\text{sat}}_{\kappa^+}]_{\kappa^+}} S_{\lambda}(\mathcal{K}_{\geq \lambda}) \\
f^* & \downarrow \\
\text{Set}[\mathbb{T}^{\text{sat}}_{\kappa^+}]_{\kappa^+} & \xrightarrow{\eta^*_\text{Set}[\mathbb{T}^{\text{sat}}_{\kappa^+}]_{\kappa^+}} S_{\kappa^+}(\text{Sat}_{\kappa^+}(\mathcal{K})) \xrightarrow{\eta^*_\text{Set}[\mathbb{T}^{\text{sat}}_{\kappa^+}]_{\kappa^+}} S_{\lambda}(\text{Sat}_{\lambda}(\mathcal{K}))
\end{align*}
$$

We now want to deduce from the fact that the right morphism is an isomorphism that $f^*$ is an equivalence, i.e., every model of size $\kappa^+$ is $\kappa^+$-saturated. Then $\mathcal{K}$ is $\kappa^+$-categorical since there is a unique such model (Rosicky 1997).
Proof idea

We will prove that in fact $\text{Sat}^\kappa + (K^\text{op})$ can be axiomatized and, moreover, if $\tau_D$ is the dense (alternatively, atomic) Grothendieck topology on $K^\text{op}$:

$$\text{Set}^{[\text{Sat}^\kappa + \kappa]} \cong \text{Sh}(K^\text{op}, \tau_D)$$

This is based on the following:

Theorem

Let $T^\kappa$ axiomatize $K \geq \kappa$. Then the $\kappa^+$-classifying topos of $T^\kappa$ is equivalent to the presheaf topos $\text{Set}^{K^\kappa}$. Moreover, the canonical embedding of the syntactic category is given by (note that $K^\kappa \ni M$: $	ext{CT}^\kappa \to \text{Set}$):

$$\text{CT}^\kappa \text{Set}^{K^\kappa} X \mapsto M(X)$$
Proof idea

We will prove that in fact $Sat_{\kappa^+}(\mathcal{K})$ can be axiomatized and, moreover, if $\tau_D$ is the dense (alternatively, atomic) Grothendieck topology on $K_{\kappa}^{op}$:

$$Set[\mathbb{T}_{\kappa^+}]_{\kappa^+} \cong Sh(K_{\kappa}^{op}, \tau_D)$$
Proof idea

We will prove that in fact $\text{Sat}_{\kappa^+}(\mathcal{K})$ can be axiomatized and, moreover, if $\tau_D$ is the dense (alternatively, atomic) Grothendieck topology on $\mathcal{K}^{\text{op}}_{\kappa}$:

$$\text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} \cong \text{Sh}(\mathcal{K}^{\text{op}}_{\kappa}, \tau_D)$$

This is based in the following:

*Theorem*

Let $\mathbb{T}_\kappa$ axiomatize $\mathcal{K}_{\geq \kappa}$. Then the $\kappa^+$-classifying topos of $\mathbb{T}_\kappa$ is equivalent to the presheaf topos $\text{Set}^{\mathcal{K}_\kappa}$. Moreover, the canonical embedding of the syntactic category is given by (note that $\mathcal{K}_\kappa \ni M : \mathcal{C}_{\mathbb{T}_\kappa} \to \text{Set}$):

$$\begin{array}{ccc}
\mathcal{C}_{\mathbb{T}_\kappa} & \xrightarrow{\text{ev}} & \text{Set}^{\mathcal{K}_\kappa} \\
X & \xrightarrow{} & M \mapsto M(X)
\end{array}$$
Proof idea

Proof.

Every model of $\mathcal{T}$ is a $\kappa^+$-filtered colimit of models in $\mathcal{K}_{\kappa}$.

We have:

$$C_{\mathcal{T} \kappa} \cong \text{lim}_{\to} M_i \cong \text{ev}_{\text{lim}_{\to} M_i}$$

This proves the universal property when $E = \text{Set}$.

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Proof idea

Proof.
Every model of $\mathbb{T}$ is a $\kappa^+$-filtered colimit of models in $\mathcal{K}_\kappa$. We have:

$$
\mathcal{C}_{\mathbb{T}_\kappa} \cong \mathcal{S}_{\mathcal{K}_\kappa} \xrightarrow{\sim} \operatorname{colim}_{\mathcal{K}_\kappa \rightarrow} M_i \xrightarrow{\operatorname{ev}} \operatorname{colim}_{\mathcal{K}_\kappa \rightarrow} M_i
$$

This proves the universal property when $\mathcal{E} = \mathcal{S}_{\mathcal{K}_\kappa}$. 

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Proof.

Every model of $\mathbb{T}$ is a $\kappa^+$-filtered colimit of models in $\mathcal{K}_\kappa$. We have:

$$\mathcal{C}_{\mathbb{T}_\kappa} \xrightarrow{ev} \text{Set}^{\mathcal{K}_\kappa}$$

This proves the universal property when $E = \text{Set}$. 
**Proof.**

Every model of $\mathbb{T}$ is a $\kappa^+$-filtered colimit of models in $\mathcal{K}_\kappa$. We have:

\[
\begin{align*}
\text{Set} & \xrightarrow{\text{ev}} \text{Set}^{\mathcal{K}_\kappa} \\
\mathcal{C}_{\mathbb{T}_\kappa} & \xrightarrow{\text{ev}} \text{Set}^{\mathcal{K}_\kappa} \\
M \cong \lim_{\to} M_i & \xrightarrow{\text{ev}_{M_i}} \text{Set}^{\mathcal{K}_\kappa}
\end{align*}
\]

This proves the universal property when $\mathcal{E} = \text{Set}$. 
Proof idea

Let now $E$ be the $\kappa^+$-classifying topos of $T_{\kappa}$. Then it has enough $\kappa^+$-points by the infinitary Deligne completeness theorems (Espíndola 2017).

We have:

\[
C_{T_{\kappa}}S \overset{\sim}{=} \text{ev} \left( \text{ev} \left( \text{Set}_{K_{\kappa}}E \overset{\sim}{=} \text{Set} \right) \right)
\]

Now every object $F$ in $\text{Set}_{K_{\kappa}}$ can be written as $F \overset{\sim}{=} \lim_{\to} \left[ M_i, - \right] \text{Mod}_{\lambda}(T)$, i.e.:

\[
\lim_{\to} \left[ \lim_{\to} \left[ \phi_{ij}, - \right] \right] C_{T_{\kappa}}, - \overset{\sim}{=} \lim_{\to} \left[ \lim_{\leftarrow} \phi_{ij} \right] \text{ev} \left( \text{mod}_{\lambda}(T) \right) \overset{\sim}{=} \lim_{\to} \left[ \lim_{\leftarrow} \phi_{ij} \right] C_{T_{\kappa}}, -
\]

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We have:

\[
\begin{array}{ccc}
\mathcal{C}_{\mathbb{T}_\kappa} & \xrightarrow{ev} & Set^{K_\kappa} \\
N & \downarrow & \\
\mathcal{E} & \xrightarrow{G} & \\
E & \downarrow & \\
Set^I & \xrightarrow{ev_i} & Set
\end{array}
\]
Proof idea

Let now $\mathcal{E}$ be the $\kappa^+$-classifying topos of $\mathbb{T}_\kappa$. Then it has enough $\kappa^+$-points by the infinitary Deligne completeness theorems (Espindola 2017). We have:

Now every object $F$ in $\text{Set}^{\mathcal{K}_\kappa}$ can be written as $F \cong \lim_i [M_i, -]_{\text{Mod}_{\lambda}(\mathbb{T})}$, i.e.:

$$\lim_i \lim_j [\phi_{ij}, -]_{\text{C}_\mathbb{T}, -]_{\text{Mod}_{\lambda}(\mathbb{T})} \cong \lim_i \lim_j [[\phi_{ij}, -]_{\text{C}_\mathbb{T}, -]}_{\text{Mod}_{\lambda}(\mathbb{T})} \cong \lim_i \lim_j \text{ev}(\phi_{ij})$$
Proof idea

Since $G$ preserves colimits and $\kappa$-small limits, we have:
$$G(F) \cong \lim_{\to i} \lim_{\leftarrow j} G \circ ev(\phi_{ij})$$
and hence $G$ is completely determined by its values on $ev(C_T(\kappa))$, which land in $E$. Since $E$ also preserves colimits and $\kappa$-small limits, the whole $G$ lands in $E$.

This proves the universal property when $E$ is $\kappa$-classifying topos of $T$. Since this later satisfies the same universal property, we must have $E \cong \Set_{K_\kappa}$. This finishes the proof.
Proof idea

Since $G$ preserves colimits and $\kappa^+$-small limits, we have:

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This proves the universal property when $E$ is $\kappa^+$-classifying topos of $T_\kappa$. Since this later satisfies the same universal property, we must have $E \cong Set^{K_\kappa}$. This finishes the proof.
Proof idea

Coming back to $\text{Sat}_\kappa(K)$, we have the following situation:

$$C \circ \text{Sat}_\kappa \supseteq \text{Set}_{\text{ev}}(\kappa^+) \supseteq \text{Set}_{\text{Sh}}(\text{Top}_\kappa, \tau D) \supseteq C \circ \text{Sh}(\kappa^+) \supseteq \text{Set}_{\text{FACT}}(e.g. \text{Johnstone's Elephant}):$$

The embedding $\text{Sh}(\text{Top}_\kappa, \tau D)$ factors through $\circ \text{ev}$ if and only if $\circ \text{ev}$ is dense ($\circ \text{ev}(0) = 0$, or alternatively $\circ \text{ev}(C) \neq 0$ for $C \neq 0$).
Coming back to $Sat_{\kappa^+}(\mathcal{K})$, we have the following situation:
Proof idea

Coming back to $\text{Sat}_{\kappa^+}(\mathcal{K})$, we have the following situation:

\[
\begin{align*}
C_{\mathbb{T}_{\kappa}} & \xrightarrow{ev} \text{Set}^{\mathcal{K}_{\kappa}} \\
\downarrow & \downarrow & & \downarrow \\
C_{\mathbb{T}_{\kappa^+}} & \to \text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} & \text{Set} \\
\downarrow & \downarrow & & \\
C_{\mathbb{T}_{\kappa^+}^{\text{sat}}} & \to \text{Sh}(K_{\kappa}^{\text{op}}, \tau_D)
\end{align*}
\]
Proof idea

Coming back to $\text{Sat}_{\kappa^+}(\mathcal{K})$, we have the following situation:

\[
\begin{align*}
C_{\mathbb{T}_\kappa} & \xrightarrow{\text{ev}} \text{Set}^{\mathcal{K}_\kappa} \\
\Downarrow & \quad \Downarrow \quad \Downarrow \\
C_{\mathbb{T}_{\kappa^+}} & \xrightarrow{\text{ev}} \text{Set}^{\mathbb{T}_{\kappa^+}}_{\kappa^+} \\
\Downarrow & \quad \Downarrow \\
C_{\mathbb{T}_{\kappa^+}^{\text{sat}}} & \xrightarrow{\text{Sh}(K_{\kappa}^{\text{op}}, \tau_D)} \text{Sh}(K_{\kappa}^{\text{op}}, \tau_D)
\end{align*}
\]

FACT (e.g. Johnstone’s Elephant): The embedding $\text{Sh}(K_{\kappa}^{\text{op}}, \tau_D) \hookrightarrow \text{Set}^{\mathcal{K}_{\kappa}}$ factors through $f_*$ if and only if $f_*$ is dense ($f_*(0) = 0$, or alternatively $f^*(C) \neq 0$ for $C \neq 0$).
Proof idea

If $M$ is a saturated model of size $\kappa^+$, for every $p: N \to N'$ in $K_{\kappa}$, each $N' \to M$ extends to some $p': N' \to M$. This is the same as saying that $M: \text{Set}_{K_{\kappa}} \to \text{Set}$ maps $p^*:[N',-] \to [N,-]$ to an epimorphism, since:

$$\lim_{\to} \text{ev}_{N_i}[N,-] = \lim_{\to} [N, \lim_{\to} N_i] \cong [N, M]$$

It follows that $M$ factors through $\text{Sh}(K_{\kappa}^{\text{op}}, \tau_D)$ and that $\text{Sh}(K_{\kappa}^{\text{op}}, \tau_D)$ is the $\kappa^+$-classifying topos of $T_{\text{sat}}^\kappa$. 

FACT (Rosicky 1997): $\kappa^+$-saturated models exist. This can also be seen topos-theoretically by noticing that $\text{Sh}(K_{\kappa}^{\text{op}}, \tau_D)$ has enough $\kappa^+$-points (Espindola 2017). The uniqueness of $\kappa^+$-saturated models of size $\kappa^+$ can be seen as well by noticing that $\text{Sh}(K_{\kappa}^{\text{op}}, \tau_D)$ is two-valued and Boolean (Barr-Makkai 1987+ Espindola 2017).
Proof idea

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$$\lim_{\to} i_{\text{ev}} N_i([N, -]) = \lim_{\to} i_{\text{ev}} N_i \cong [N, M].$$

It follows that $M$ factors through a $\text{Sh}(K^{\text{op}}_\kappa, \tau_D)$ and that $\text{Sh}(K^{\text{op}}_\kappa, \tau_D)$ is the $\kappa^+$-classifying topos of $T_{\text{sat}}^{\kappa^+}$.

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If $M$ is a saturated model of size $\kappa^+$, for every $p : N \to N'$ in $\mathcal{K}_\kappa$, each $N \to M$ extends to some $p' : N' \to M$. This is the same as saying that $M : \text{Set}^{\mathcal{K}_\kappa} \to \text{Set}$ maps $p^* : [N', -] \to [N, -]$ to an epimorphism, since:

$$\lim_i ev_{N_i}([N, -]) = \lim_i [N, N_i] \cong [N, \lim_i N_i] \cong [N, M]$$
If $M$ is a saturated model of size $\kappa^+$, for every $p : N \to N'$ in $\mathcal{K}_\kappa$, each $N \to M$ extends to some $p' : N' \to M$. This is the same as saying that $M : \text{Set}^{\mathcal{K}_\kappa} \to \text{Set}$ maps $p^* : [N', -] \to [N, -]$ to an epimorphism, since:

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It follows that $M$ factors through $a : \text{Set}^{\mathcal{K}_\kappa} \to \text{Sh}(\mathcal{K}_\kappa^{\text{op}}, \tau_D)$ and that $\text{Sh}(\mathcal{K}_\kappa^{\text{op}}, \tau_D)$ is the $\kappa^+$-classifying topos of $\mathbb{T}^{\text{sat}}_{\kappa^+}$.
Proof idea

If $M$ is a saturated model of size $\kappa^+$, for every $p : N \to N'$ in $\mathcal{K}_\kappa$, each $N \to M$ extends to some $p' : N' \to M$. This is the same as saying that $M : \text{Set}^{\mathcal{K}_\kappa} \to \text{Set}$ maps $p^* : [N', -] \to [N, -]$ to an epimorphism, since:

$$\lim_{i \to} \text{ev}_{N_i}([N, -]) = \lim_{i \to}[N, N_i] \cong [N, \lim_{i \to} N_i] \cong [N, M]$$

It follows that $M$ factors through $a : \text{Set}^{\mathcal{K}_\kappa} \to S\text{h}(\mathcal{K}_\kappa^{\text{op}}, \tau_D)$ and that $S\text{h}(\mathcal{K}_\kappa^{\text{op}}, \tau_D)$ is the $\kappa^+$-classifying topos of $\mathbb{T}_{\kappa^+}^{\text{sat}}$.

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Wrapping up
Wrapping up

\[
\begin{array}{cccc}
\text{Set}[[\mathbb{T}_{\kappa^+}]_{\kappa^+}} & \xrightarrow{\eta^*_\text{Set}[[\mathbb{T}_{\kappa^+}]_{\kappa^+}} & S_{\kappa^+}(\mathcal{K}_{\geq \kappa^+}) & \xrightarrow{\mathcal{R}} S_{\lambda}(\mathcal{K}_{\geq \lambda}) \\
\downarrow f^* & & \downarrow & \\
\text{Sh}(K_{\kappa}^\text{op}, \tau_D) & \xrightarrow{\eta^*_\text{Sh}(K_{\kappa}^\text{op}, \tau_D)} & S_{\kappa^+}(\text{Sat}_{\kappa^+}(\mathcal{K})) & \xrightarrow{\mathcal{R}} S_{\lambda}(\text{Sat}_{\lambda}(\mathcal{K}))
\end{array}
\]
To prove that $\text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} \cong \text{Sh}(\text{Mod}_{\kappa}(\mathbb{T})^\text{op}, \tau_D)$ we show that the embedding $\text{Sh}(\text{Mod}_{\kappa}(\mathbb{T})^\text{op}, \tau_D) \hookrightarrow \text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+}$ is an isomorphism, for which we in turn show that any basic sequent valid in $\text{Sh}(\text{Mod}_{\kappa}(\mathbb{T})^\text{op}, \tau_D)$ will also be valid in $\text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+}$.
To prove that $\text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} \cong \text{Sh}(\text{Mod}_\kappa(\mathbb{T})^{op}, \tau_D)$ we show that the embedding $\text{Sh}(\text{Mod}_\kappa(\mathbb{T})^{op}, \tau_D) \hookrightarrow \text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+}$ is an isomorphism, for which we in turn show that any basic sequent valid in $\text{Sh}(\text{Mod}_\kappa(\mathbb{T})^{op}, \tau_D)$ will also be valid in $\text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+}$.

Inspection of the diagram shows that any such basic sequent $\forall x(\phi \rightarrow \psi)$ valid in $\text{Sh}(\text{Mod}_\kappa(\mathbb{T})^{op}, \tau_D)$ is also valid in $S_\lambda(\mathcal{K}_{\geq \lambda})$, and hence in the unique model of size $\lambda$. [Christian Espíndola (Brno, MUNI)]
Wrapping up

Consider the presheaf category $\mathbf{Set}_{K \geq \kappa + \lambda}$. The interpretation of the sentence $\forall x (\phi \rightarrow \psi)$ corresponds to a subobject $S \hookrightarrow 1$.

**FACT (Kripke-Joyal semantics):** $S = 0$ if and only if for every morphism $M \rightarrow N$ in $K \geq \kappa + \lambda$, there is a morphism $N \rightarrow N'$ with $N' \not\equiv \forall x (\phi \rightarrow \psi)$.

We conclude that $S \neq 0$.

Assume now that $K$ is $\kappa$-categorical. Then $\mathbf{Set}_K$, and hence $\mathbf{Set}[T \kappa + \kappa]$ is two-valued. Thus the interpretation of $\forall x (\phi \rightarrow \psi)$ in $\mathbf{Set}[T \kappa + \kappa]$ corresponds to a subobject $T$ that is either 0 or 1. It is hence enough to prove it is not 0.

This is the last missing piece, which is proven through an infinitary generalization of a completeness theorem of Joyal:
Wrapping up

Consider the presheaf category $\text{Set}^{\mathcal{K}_{\geq \kappa^+, \leq \lambda}}$. The interpretation of the sentence $\forall x (\phi \rightarrow \psi)$ corresponds to a subobject $S \rightarrow 1$. 

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We conclude that $S \neq 0$.

Assume now that $\mathcal{K}$ is $\kappa$-categorical. Then $\text{Set}^{\mathcal{T}_{\kappa^+, \kappa^+}}$ is two-valued. Thus the interpretation of $\forall x (\phi \rightarrow \psi)$ in $\text{Set}^{\mathcal{T}_{\kappa^+, \kappa^+}}$ corresponds to a subobject $T$ that is either 0 or 1. It is hence enough to prove it is not 0.

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Assume now that $\mathcal{K}$ is $\kappa$-categorical. Then $\text{Set}^{\mathcal{K}_{\kappa}}$, and hence $\text{Set}^{[T_{\kappa^+}]_{\kappa^+}}$, is two-valued. Thus the interpretation of $\forall x(\phi \to \psi)$ in $\text{Set}^{[T_{\kappa^+}]_{\kappa^+}}$ corresponds to a subobject $T$ that is either 0 or 1. It is hence enough to prove it is not 0. This is the last missing piece, which is proven through an infinitary generalization of a completeness theorem of Joyal:
Wrapping up
Wrapping up

Theorem

*The evaluation functor:*

\[ \text{ev} : \text{Set}[^\mathbb{T}_{\kappa+}]_{\kappa+} \rightarrow \text{Set}[^\mathcal{K}_{\geq \kappa+}, \leq \lambda] \]

*preserves the interpretation of the sentence* \( \forall x (\phi \rightarrow \psi) \).
Wrapping up

Theorem

The evaluation functor:

\[ \text{ev} : \text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} \rightarrow \text{Set}^{\mathcal{K}_{\geq \kappa^+}, \leq \lambda} \]

preserves the interpretation of the sentence \( \forall x (\phi \rightarrow \psi) \).

Proof.

\[
\begin{array}{ccc}
\mathcal{C}(\mathbb{T}_{\kappa^+})_{\lambda} & \xrightarrow{\gamma'} & \text{Set}[\mathbb{T}_{\kappa^+}]_{\lambda^+} \\
\text{C}_{\mathbb{T}_{\kappa^+}} & \xrightarrow{\gamma} & \text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} \\
g & & g^*
\end{array}
\]
Wrapping up

It is enough to prove that the interpretation of $\forall x (\phi \to \psi)$ is preserved by the canonical morphism $g^*: \text{Set}[T^\kappa +] \to \text{Set}[T^\kappa +]$, since this latter is the $\lambda +$-classifying topos of $T^\kappa +$, and an entirely analogous proof to a previous theorem shows that this must be the presheaf topos $\text{Set}_{\geq \kappa +, \leq \lambda +}$. This follows immediately since $g$ preserves the interpretation of $\forall x (\phi \to \psi)$ (by the syntactic construction of the syntactic categories), and a theorem of Butz and Johnstone (1998) proves that the interpretation of $\forall x (\phi \to \psi)$ is preserved by $Y$ and $Y'$. This completes the proof.
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It is enough to prove that the interpretation of $\forall x (\phi \rightarrow \psi)$ is preserved by the canonical morphism $g^* : \text{Set}[\mathbb{T}_{\kappa^+}]_{\kappa^+} \rightarrow \text{Set}[\mathbb{T}_{\kappa^+}]_{\lambda^+}$, since this latter is the $\lambda^+$-classifying topos of $\mathbb{T}_{\kappa^+}$, and an entirely analogous proof to a previous theorem shows that this must be the presheaf topos $\text{Set}^{\mathcal{K} \geq \kappa^+, \leq \lambda}$. This follows immediately since $g^*$ preserves the interpretation of $\forall x (\phi \rightarrow \psi)$ (by the syntactic construction of the syntactic categories), and a theorem of Butz and Johnstone (1998) proves that the interpretation of $\forall x (\phi \rightarrow \psi)$ is preserved by $Y$ and $Y'$. This completes the proof.
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Wrapping up

We conclude that categoricity in \( \kappa \) and \( \lambda \) implies categoricity in \( \kappa^+ \). Repeating the argument we conclude categoricity in \( \kappa^{++} \), and so on. For a limit \( \mu \), we simply consider the diagram:

\[
\text{Set}\left[ T_{\kappa^+} \right] \text{Sh}(Kop_{\kappa^+}, \tau_D) \to \text{Set}\left[ T_{\kappa^{++}} \right] \text{Sh}(Kop_{\kappa^{++}}, \tau_D) \to \ldots
\]

This also serves for the case in which \( \lambda \) is limit.
We conclude that categoricity in $\kappa$ and $\lambda$ implies categoricity in $\kappa^+$. Repeating the argument we conclude categoricity in $\kappa^{++}$, and so on. For a limit $\mu$, we simply consider the diagram:

\[
\begin{align*}
\xymatrix{
\text{Set}[[\mathbb{T}_{\kappa^+}]_{\kappa^+} \ar[r] & \text{Set}[[\mathbb{T}_{\kappa^{++}}]_{\kappa^{++}} \ar[r] & \cdots \ar[r] & \text{Set}[[\mathbb{T}_{\mu}]_{\mu} \\
\text{Sh}(K_{\kappa}^{\text{op}}, \tau_D) \ar[r] \ar[d] & \text{Sh}(K_{\kappa^+}^{\text{op}}, \tau_D) \ar[r] \ar[d] & \cdots \ar[r] & \mathcal{E} 
}
\end{align*}
\]
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$$
\begin{array}{cccccc}
Set[\mathbb{T}_{\kappa^+}]_{\kappa^+} & \longrightarrow & Set[\mathbb{T}_{\kappa^{++}}]_{\kappa^{++}} & \longrightarrow & \cdots & \longrightarrow & Set[\mathbb{T}_\mu]_\mu \\
\downarrow & & \downarrow & & & \downarrow & \\
Sh(K^\text{op}_\kappa, \tau_D) & \longrightarrow & Sh(K^\text{op}_{\kappa^+}, \tau_D) & \longrightarrow & \cdots & \longrightarrow & \mathcal{E}
\end{array}
$$

This also serves for the case in which $\lambda$ is limit.
Wrapping up

Proof of Shelah's eventual categoricity conjecture.

FACT (Hanf numbers): For every AEC $K$ there is a cardinal $\kappa$ such that if $K$ is categorical in some $\lambda > \kappa$, it is categorical in unboundedly many cardinals.

Since categoricity in a pair of cardinals implies categoricity in all cardinals in between, we conclude that there is a tail of cardinals where $K$ is categorical. QED

Remark

Assume $GCH$. Let $K$ be an accessible category with all morphisms monomorphisms, directed bounds and amalgamation. Then the same proof outlined also proves that there is a cardinal $\kappa$ such that if $K$ is $\lambda$-categorical for some $\lambda \upmodels \kappa$ (i.e., it has only one object of internal size $\lambda$ up to isomorphism) then it is $\lambda'$-categorical for every $\lambda' \upmodels \kappa$. The hypotheses on $K$ can be spared assuming instead a proper class of strongly compact cardinals.
Wrapping up

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Christian Espíndola (Brno, MUNI)
A topos-theoretic proof of Shelah’s eventual categoricity conjecture
April 30th, 2020 21 / 22
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Thank you!