CHARACTERIZATION OF SOLUTIONS FOR BICOOPERATIVE GAMES
BY USING REPRESENTATION THEORY

Masaki Saito
Osaka University

Yoshihumi Kusunoki
Osaka Prefecture University

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Abstract When analyzing solutions for bicooperative games as well as classical cooperative games, traditional approaches regard both of games and payoff vectors as linear spaces, but in this paper, we present another approach to analyze solutions from the viewpoint of representation of the symmetric group. First, we regard the space of games as a representation of the symmetric group. Then, by using tools of representation theory, we obtain a decomposition of the space and specify useful subrepresentations. Exploiting this decomposition, we show an explicit formula of linear symmetric solutions. Additionally, we also show expressions of linear symmetric solutions restricted by parts of the axioms of the Shapley value for bicooperative games.

Keywords: Game theory, bicooperative game, solution, representation theory, symmetric group

1. Introduction
Achieving cooperation and sharing the resulting benefits are central issues in any form of organization, particularly in economic environments. The primary concern of cooperative game theory is how to deal with these issues. The most classical model is the concept of games in characteristic form. When side-payments are possible among players, this results in TU (Transferable Utility) games. A TU game is defined by giving a profit value for each coalition of players. Cooperative game theory has been studied on how to divide the profit achieved by the grand coalition or other realized coalitions. Such an allocation method is called solution in cooperative game theory. There are two main types of solutions. One is to give an allocation as set-valued, and the other is to give it as one-point. The Shapley and Banzhaf values are well-known one-point solutions. They are special cases of probabilistic values. On the other hand, the core, Weber set and selectope are included in set-valued solutions.

In a TU game, the players have only two options: to participate or not to participate in a coalition. Now then, do the players who don’t participate have no influence on the worth of the coalition when it is formed? Obviously, there is the possibility of some of the players who don’t participate in the coalition to operate against the actions of the participants. This leads to insufficiency of the classical model. For instance, we consider a group of players who develop an economic activity, and suppose that a change of the activity is proposed to the players by an internal or external motivation. This situation may be modeled in the following manner [4]. Let \( N \) be the set of all players. We consider a pair \((S,T)\) of disjoint subsets of \( N \). Players in \( S \) accept the proposal and agree with the change. On the other hand, those in \( T \) disagree with it. Finally, the other players in \( N \setminus (S \cup T) \) think that they do not receive any profit from the proposal, but they are not willing to take objection to
the change managed by \( S \). For each pair \((S, T)\), we can model the profit (or cost) derived from the change of the activity by a characteristic function \( b(S, T) \). Such a game \( b \) is called bicooperative game.

Bilbao has proposed bicooperative games in \([2]\), and discussed their combinatorial properties. Grabisch and Labreuche \([12]\) have independently studied bicooperative games as the cooperative games corresponding to bi-capacities, which are generalized capacities to evaluate scores in bipolar scales. Before those studies, a special class of bicooperative games, called ternary voting games, was proposed in \([9]\). Ternary voting games are simple voting games with abstention.

Various solutions have been proposed for bicooperative games. Bilbao et al. \([4, 5]\) have proposed and axiomatized a Shapley value and probabilistic values for bicooperative games. Moreover, Bilbao et al. \([3, 6]\) have proposed core, Weber set and selectope, and studied their relations. Grabisch and Labreuche \([12]\) and Labreuche and Grabisch \([15]\) have also proposed and axiomatized Shapley values. The Shapley values of \([4, 12, 15]\) are different in definitions of payoff vectors and efficiency. Recent results include Domènech et al. \([8]\) and Borkotokey et al. \([7]\).

In this paper, following the axiomatic characterization of the Shapley value for bicooperative games that was studied in \([4]\), we show more general characterization results. We study characterization of linear symmetric solutions in the light of elementary representation theory. Representation theory is a branch of algebra that studies algebraic structures adding an action of a group on a vector space. When applying representation theory in this paper, roughly speaking, the linear axiom leads to the introduction of a vector space and the symmetry axiom leads to the introduction of the action of the symmetric group on the vector space. Taking advantage of this approach, we obtain an axiomatic system of the Shapley value, which is slightly different from that of \([4]\). Additionally, we show explicit formulas of linear symmetric solutions with the efficiency and/or null axioms \([4, 15]\), which do not appear in the previous studies.

There are not many studies that apply representation theory to cooperative game theory. Kleinberg and Weiss \([14]\) construct a direct-sum decomposition of the null space of Shapley value by using the representation theory of the symmetric group. Hernández-Lamoneda et al. \([13]\) provide a direct-sum decomposition of the space of TU games under the symmetric group and derive some applications involving characterizations of classes of solutions. Sánchez-Pérez \([16]\) provides a direct-sum decomposition of the space of games with externals and provides some applications.

The main goal of this paper is to provide a viewpoint of the space of bicooperative games as a representation of the symmetric group, and to show this representation-theoretical approach can be a generic method of characterizing linear symmetric solutions.

Let us briefly outline the contents of this paper. In Section 2, we explain basic representation theory and introduce bicooperative games. In Section 3, a decomposition for the space of bicooperative games is introduced. In Section 4, we then present a couple of applications of this decomposition by giving characterizations of linear symmetric solutions. Section 5 concludes this paper. Long proofs are relegated to Appendix. We emphasize that the proofs in Appendix are long due to notations and many calculations, but contain little technical discussion.
2. Preliminaries
2.1. Representation theory of symmetric group

Representation theory is a fundamental tool in this paper. We shall introduce basic facts about representation theory. For more details, see textbooks of Adkins and Weintraub [1] and Fulton and Harris [10].

Let $F$ be a field and $M$ be an $F$-vector space. Let $GL(M)$ be the set of all invertible linear functions. Note that $GL(M)$ is a group with respect to function composition.

**Definition 2.1.** Let $M$ be an $F$-vector space, $H$ be a group and $\rho: H \to GL(M)$ be a group homomorphism, i.e., $\rho(h_1 h_2) = \rho(h_1) \rho(h_2)$ for all $h_1, h_2 \in H$. Then $(M, \rho)$ is called $F$-representation of $H$.

Let $S_n$ be the symmetric group of degree $n$. Hereinafter, we only consider the case that $M$ is finite-dimensional, $F = \mathbb{R}$ and $H = S_n$, and focus on $\mathbb{R}$-representation of $S_n$, but most of the definitions and theorems introduced in this section are valid in the case of an arbitrary $F$-representation of $H$. For simplicity, we simply refer to $\mathbb{R}$-representation of $S_n$ as representation.

Similar to linear spaces, a direct sum can be defined for two representations.

**Definition 2.2.** Let $(M_1, \rho_1)$ and $(M_2, \rho_2)$ be two representations. Then the direct sum of $(M_1, \rho_1)$ and $(M_2, \rho_2)$ is the representation $(M_1 \oplus M_2, \rho_1 \oplus \rho_2)$, where $M_1 \oplus M_2$ is the direct sum of $M_1$ and $M_2$ as vector spaces and $\rho_1 \oplus \rho_2: S_n \to GL(M_1 \oplus M_2)$ is defined as $[(\rho_1 \oplus \rho_2)(\sigma)](m_1 + m_2) = [\rho_1(\sigma)](m_1) \oplus [\rho_2(\sigma)](m_2)$.

Let $(M, \rho)$ be a representation. A vector subspace $N \subset M$ is $S_n$-invariant if $[\rho(\sigma)](N) \subset N$ for all $\sigma \in S_n$. A subrepresentation of $(M, \rho)$ is a representation $(N, \rho_N)$ such that $N$ is $S_n$-invariant and $\rho_N: S_n \to GL(N)$ is defined as $\rho_N(\sigma) = [\rho(\sigma)](N)$ where $[\rho(\sigma)](N)$ is the restriction of $\rho(\sigma)$ to $N$. $(M, \rho)$ is irreducible if it does not contain a nontrivial subrepresentation, that is, $\{0\} \neq \rho(\sigma)$ for any $\sigma \in S_n$.

**Example 2.1.** Let $N = \{1, 2, \ldots, n\}$ be a finite nonempty set. We define $\rho: S_n \to GL(\mathbb{R}^N)$ as $\rho(\sigma)(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Then $\rho$ is a homomorphism since

$$
(\rho(\sigma_1) \circ \rho(\sigma_2))(x_1, \ldots, x_n) = \rho(\sigma_1)(x_{\sigma_2(1)}, \ldots, x_{\sigma_2(n)})
= (x_{\sigma_1\sigma_2(1)}, \ldots, x_{\sigma_1\sigma_2(n)})
= \rho(\sigma_1\sigma_2)(x_1, \ldots, x_n)
$$

for $\sigma_1, \sigma_2 \in S_n$ and $x \in \mathbb{R}^N$. Thus $(\mathbb{R}^N, \rho)$ is a representation. We set

$$
U = \{1\} \quad \text{and} \quad V = U^\perp = \left\{ z \in \mathbb{R}^N \mid \sum_{i \in N} z_i = 0 \right\}
$$

where $1 = (1, 1, \ldots, 1) \in \mathbb{R}^N$. Note that $U$ and $V$ are $S_n$-invariant and $\mathbb{R}^N = U \oplus V$. According to [13, Lemma 7 in Appendix], $(U, \rho_U)$ and $(V, \rho_V)$ are irreducible.

**Definition 2.3.** Let $(M_1, \rho_1)$ and $(M_2, \rho_2)$ be two representations. A linear map $f: M_1 \to M_2$ is an $S_n$-equivariant map if $f \circ \rho_1(\sigma) = \rho_2(\sigma) \circ f$, i.e., $f([\rho_1(\sigma)](x)) = [\rho_2(\sigma)](f(x))$ for every $x \in M_1, \sigma \in S_n$. An invertible $S_n$-equivariant map is called an $S_n$-equivariant isomorphism. If there exists an $S_n$-equivariant isomorphism between $(M_1, \rho_1)$ and $(M_2, \rho_2)$, it is said that $(M_1, \rho_1)$ and $(M_2, \rho_2)$ are isomorphic, denoted by $(M_1, \rho_1) \cong (M_2, \rho_2)$.

$S_n$-equivariant maps preserve structures of two representations. Two isomorphic representations are essentially equivalent as representations.

The following theorem is a very powerful tool.

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Theorem 2.1. Let \((M_1, \rho_1)\) and \((M_2, \rho_2)\) be irreducible representations and \(\phi: M_1 \rightarrow M_2\) be an \(S_n\)-equivariant map. Then,

(i) Either \(\phi\) is an isomorphism, or \(\phi = 0\).

(ii) If \((M_1, \rho_1) \cong (M_2, \rho_2)\) and \((M_1, \rho_1) \cong (U, \rho_U)\) or \((M_1, \rho_1) \cong (V, \rho_V)\), then \(\phi\) is unique up to multiplication by a scalar \(c \in \mathbb{R}\), that is, if \(\psi\) is an isomorphism between \((M_1, \rho_1)\) and \((M_2, \rho_2)\), there exists \(c \in \mathbb{R}\) such that \(\phi = c\psi\).

Definition 2.4. Let \((M_1, \rho_1)\) and \((M_2, \rho_2)\) be representations. The multiplicity \(k\) of \((M_1, \rho_1)\) in \((M_2, \rho_2)\) is the largest nonnegative integer with the property that \(k(M_1, \rho_1) = (M_1, \rho_1) \oplus \cdots \oplus (M_1, \rho_1)\) is isomorphic to a subrepresentation of \((M_2, \rho_2)\).

It is known that any representation of symmetric group is a direct sum of irreducible representations. Combining this fact with Theorem 2.1(i) gives the following proposition:

Proposition 2.1. For any representation \((M, \rho)\), there is a decomposition

\[
(M, \rho) = a_1(M_1, \rho_1) \oplus \cdots \oplus a_k(M_k, \rho_k)
\]  

(2.1)

where \((M_1, \rho_1), \ldots, (M_k, \rho_k)\) are distinct irreducible representations from each other. The decomposition of \((M, \rho)\) into a direct sum of \(k\) factors is unique, as are the occurrence of \((M_1, \rho_1), \ldots, (M_k, \rho_k)\) and their multiplicities \(a_1, \ldots, a_k\).

We shall introduce character of representations. Character theory is remarkably effective in decomposing representations.

Definition 2.5. Let \((M, \rho)\) be a representation, and let \(B\) be a basis of \(M\). The character of \((M, \rho)\) is the function \(\chi_{(M, \rho)}: S_n \rightarrow \mathbb{R}\) defined by

\[
\chi_{(M, \rho)}(\sigma) = \text{Tr}(\rho(\sigma)|_B).
\]

Tr denotes the trace of a matrix or a linear transformation and \([\rho(\sigma)]_B\) denotes the matrix representation of \(\rho(\sigma)\) on the basis \(B\). This is independent of the choice of the basis \(B\). Also, since \(\text{Tr}(P \rho(\sigma) P^{-1}) = \text{Tr}(\rho(\sigma))\) if \(P\) is a regular matrix, the characters of two isomorphic representations are equal. Note that \(\chi_{(M_1, \rho_1) \oplus (M_2, \rho_2)}(\sigma) = \chi_{(M_1, \rho_1)}(\sigma) + \chi_{(M_2, \rho_2)}(\sigma)\).

For two characters \(\chi_{(V_1, \rho_1)}, \chi_{(V_2, \rho_2)}\), we define an inner product by

\[
\langle \chi_{(V_1, \rho_1)}, \chi_{(V_2, \rho_2)} \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_{(V_1, \rho_1)}(\sigma^{-1}) \chi_{(V_2, \rho_2)}(\sigma).
\]

Given a representation \((M, \rho)\), this inner product helps us compute the multiplicities of \((U, \rho_U)\) and \((V, \rho_V)\).

Proposition 2.2. The multiplicities \(a_U\) of \((U, \rho_U)\) and \(a_V\) of \((V, \rho_V)\) in (2.1) are computed by the inner product of \(\chi_{(M, \rho)}\) with \(\chi_{(U, \rho_U)}\) or \(\chi_{(V, \rho_V)}\), i.e.,

\[
a_U = \langle \chi_{(M, \rho)}, \chi_{(U, \rho_U)} \rangle,
\]

\[
a_V = \langle \chi_{(M, \rho)}, \chi_{(V, \rho_V)} \rangle.
\]

Theorem 2.1 and Proposition 2.2 are specialized in our situations, that is, \(F = \mathbb{R}\) and \(H = S_n\). For more details, see Appendix.
2.2. Bicooperative games

Let \( N = \{1, 2, \ldots, n\} \) be a fixed nonempty finite set, and let \( 2^N = \{ S \mid S \subset N \} \) and \( 3^N = \{ (S, T) \in 2^N \times 2^N \mid S \cap T = \emptyset \} \).

**Definition 2.6.** A bicooperative game is a pair \((N, b)\) with \( N \) a finite set and \( b \) a function \( b: 3^N \to \mathbb{R} \) with \( b(\emptyset, \emptyset) = 0 \).

For each \((S, T) \in 3^N\), the worth \( b(S, T) \) can be interpreted as the gain (whenever \( b(S, T) > 0 \)) or loss (whenever \( b(S, T) < 0 \)) that \( S \) can achieve when \( T \) is the opposer coalition and \( N \setminus (S \cup T) \) is the neutral coalition. The pair \((\emptyset, N)\) represents the situation if all the players object to the change under consideration in this game and \((N, \emptyset)\) represents the situation where all the players wish the change.

Let \( G = \left\{ b \in \mathbb{R}^{3^N} \mid b(\emptyset, \emptyset) = 0 \right\} \). We refer to an element of \( G \) as a bicooperative game.

Given \( b_1, b_2 \in G \) and \( c \in \mathbb{R} \), we define the sum \( b_1 + b_2 \) and the product \( cb_1 \), in \( G \), in the usual form, i.e.,

\[
(b_1 + b_2)(S, T) = b_1(S, T) + b_2(S, T) \quad \text{and} \quad (cb_1)(S, T) = cb_1(S, T),
\]

respectively. It is easy to verify that \( G \) is an \( \mathbb{R} \)-vector space with these operations.

A *solution* is a function \( \phi: G \to \mathbb{R}^N \). If \( \phi \) is a solution and \( b \in G \), then we can interpret \( \phi_i(b) \) as the utility payoff which player \( i \) should expect from the game \( b \). We call an element of \( \mathbb{R}^N \) payoff vector.

Now, the symmetric group \( S_n \) acts on \( G \) in the natural way; i.e., for \( \sigma \in S_n \), \( \sigma \cdot b \in G \) is defined as

\[
[\sigma \cdot b](S, T) = b(\sigma S, \sigma T) \quad \text{for all} \quad (S, T) \in 3^N,
\]

where \( \sigma S = \{ \sigma(i) \mid i \in S \} \) for \( S \subset N \).

And also, \( S_n \) acts on the space of payoff vectors, \( \mathbb{R}^N \);

\[
\sigma(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).
\]

Note that these actions define representations into \( G \) or \( \mathbb{R}^N \), i.e., \((G, \rho_1)\) and \((\mathbb{R}^N, \rho_2)\) are representations, where \( \rho_1: S_n \to \text{GL}(G) \) is defined as \( [\rho_1(\sigma)](b) = \sigma \cdot b \) and \( \rho_2: S_n \to \text{GL}(\mathbb{R}^N) \) is defined as \( [\rho_2(\sigma)](x) = \sigma \cdot x \).

Next, we define the usual linearity and symmetry axioms which are often cited in the cooperative game theoretic framework.

**Axiom 2.1** (Linearity). The solution \( \phi \) is linear if

\[
\phi(\alpha b_1 + \beta b_2) = \alpha \phi(b_1) + \beta \phi(b_2)
\]

for all \( \alpha, \beta \in \mathbb{R} \), and \( b_1, b_2 \in G \).

Linearity axiom states that if several games are combined linearly then the solutions of the games shall be combined in the same way to obtain the solution of the resulting game.

**Axiom 2.2** (Symmetry). The solution \( \phi \) is said to be symmetric if

\[
\phi(\sigma \cdot b) = \sigma \cdot \phi(b) \tag{2.2}
\]

for all \( \sigma \in S_n \) and \( b \in G \).

Symmetry axiom states that the rule for computing the share does not depend on the labeling of the players.

Note that a solution \( \phi \) is linear and symmetric if and only if \( \phi \) is a \( S_n \)-equivariant map since (2.2) is equivalent to \( \phi \circ \rho_1(\sigma) = \rho_2(\sigma) \circ \phi \) for all \( \sigma \in S_n \). Thus linear and symmetric
solutions for bicooperative games can be analyzed by using tools of representation theory. Hereinafter, we simply denote \((G, \rho_1)\) and \((\mathbf{R}^N, \rho_2)\) by \(G\) and \(\mathbf{R}^N\), respectively. Also, if \(X\) is \(S_n\)-invariant subspace of \(G\) or \(\mathbf{R}^N\) (subrepresentation of \(G\) or \(\mathbf{R}^N\)), we simply denote \((X, \rho_X)\) by \(X\).

3. Decomposition of \(G\)

We begin with the decomposition of \(\mathbf{R}^N\) into irreducible representations, which has been already done in Example 2.1.

**Proposition 3.1.** The decomposition of \(\mathbf{R}^N\), under \(S_n\), into irreducible subrepresentations is:

\[
\mathbf{R}^N = U \oplus V.
\]

**Proof.** See Hernández-Lamoneda et al. [13] (Lemma 7 in Appendix). \(\square\)

Note that a nontrivial irreducible subrepresentation of \(\mathbf{R}^N\) is either \(U\) or \(V\) by Proposition 3.1 and uniqueness of decomposition (Proposition 2.1). Combining Theorem 2.1 with Proposition 3.1, we have the next corollary.

**Corollary 3.1.** Let \(\phi: G \to \mathbf{R}^N\) be a symmetric solution and \(M \subset G\) be an irreducible subrepresentation. Then \(\phi(M)\) is equal to one of \(U, V\) and \(\{0\}\).

**Proof.** Let \(\phi_M: M \to \mathbf{R}^N\) be the restriction of \(\phi\) to \(M\). Note that \(\phi_M\) is also \(S_n\)-equivariant and \(\phi(M) = \phi_M(M)\). If \(\phi_M = 0\) or \(M = \{0\}\), then \(\phi(M) = \{0\}\). Otherwise, by Theorem 2.1(i), \(\phi_M\) is an isomorphism between \(M\) and \(\phi_M(M)\), that is, \(\phi_M(M)\) is a nontrivial irreducible subrepresentation of \(\mathbf{R}^N\). Then \(\phi_M(M)\) must be equal to either \(U\) or \(V\) since \(\phi_M(M)\) is a nontrivial irreducible subrepresentation of \(\mathbf{R}^N\) and a nontrivial irreducible subrepresentation of \(\mathbf{R}^N\) is either \(U\) or \(V\). Therefore, \(\phi(M)\) is equal to one of \(U, V\) and \(\{0\}\). \(\square\)

From Proposition 2.1, there is a decomposition of \(G\). By Theorem 2.1(i), when we discuss linear symmetric solutions for \(G\), it is enough to consider the direct sum of the irreducible subrepresentations of \(G\) that are isomorphic to \(U\) or \(V\), since the complement of those subrepresentations is mapped into \(\{0\}\). Therefore, we concentrate on identifying such subrepresentations. To do this, character theory plays an important role.

First we decompose \(G\) in a convenient way. Let

\[
C_n = \{ (s, t) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq s + t \leq n \land s, t \geq 0 \}.
\]

For each \((s, t) \in C_n\), define the subspace of games

\[
G^s_t = \{ b \in G \mid b(S, T) = 0 \text{ if } |S| \neq s \text{ or } |T| \neq t \}.
\]

The game space \(G\) (as a linear space) is decomposed into the direct sum of \(G^s_t\) for all \((s, t) \in C_n\) since \(\forall(s, t) \in C_n \forall(s', t') \in C_n ((s, t) \neq (s', t') \rightarrow G^s_t \cap G^s_{t'} = \{0\})\) and \(\forall(S, T) \in 3^N \setminus \{((\emptyset, \emptyset)) \exists(s, t) \in C_n (|S| = s \land |T| = t)\} \). Moreover, each \(G^s_t\) is \(S_n\)-invariant, that is, it is a subrepresentation. Hence, \(G\) (as a representation) is decomposed as follows.

\[
G = \bigoplus_{(s, t) \in C_n} G^s_t.
\]

Note that \(G^0_0 \cong G^0_n \cong U\) since \(f(b) = b(N, \emptyset)\) is an isomorphism between \(G^0_0 \cong U\) and \(f(b) = b(\emptyset, N)\) is an isomorphism between \(G^0_n \cong U\). Thus we shall identify subrepresentations of \(G^s_t\) that are isomorphic to \(U\) or \(V\) for \((s, t) \in D_n = C_n \setminus \{(n, 0), (0, n)\} \).
The following games play an important role in describing the decomposition of representation of games. For each \((s, t) \in C_n\), define \(u_t^* \in G_t^*\) as follows:

\[
 u_t^*(S, T) = \begin{cases} 
 1 & |S| = s \land |T| = t, \\
 0 & \text{otherwise.}
\end{cases}
\]

For each \((s, t) \in C_n\) and each \(x \in \mathbb{R}^N\), define the game \(x^{(s, t, 0)} \in G_t^*\) as follows:

\[
x^{(s, t, 0)}(S, T) = \begin{cases} 
 \sum_{i \in S} x_i & |S| = s \land |T| = t, \\
 0 & \text{otherwise.}
\end{cases}
\]

Also, define the game \(x^{(s, t, 1)} \in G_t^*\) as follows:

\[
x^{(s, t, 1)}(S, T) = \begin{cases} 
 \sum_{i \in T} x_i & |S| = s \land |T| = t, \\
 0 & \text{otherwise.}
\end{cases}
\]

We define the functions \(L_{s,t,0} : \mathbb{R}^N \to G_t^*\) as \(x \mapsto x^{(s, t, 0)}\) and \(L_{s,t,1} : \mathbb{R}^N \to G_t^*\) as \(x \mapsto x^{(s, t, 1)}\). \(L_{s,t,0}\) and \(L_{s,t,1}\) help us find subrepresentations of \(G_t^*\) which are isomorphic to \(U\) or \(V\).

**Lemma 3.1.**

1. For each \((s, t) \in D_n\) such that \(1 \leq s \leq n - 1\), the function \(L_{s,t,0}\) is one-to-one \(S_n^*-\)equivariant map.
2. For each \((s, t) \in D_n\) such that \(1 \leq t \leq n - 1\), the function \(L_{s,t,1}\) is one-to-one \(S_n^*-\)equivariant map.

**Proof.** We check only \(L_{s,t,0}\). It is clear that each \(L_{s,t,0}\) is an \(\mathbb{R}\)-linear transformation. To show each \(L_{s,t,0}\) is one-to-one, it suffices to show that if \(L_{s,t,0}(x) = 0\) then \(x = 0\). Since \(1 \leq s \leq n - 1\), we can choose \(S \subseteq N\) such that \(|S| = s\) and \(i \in S\) and \(j \notin S\) for arbitrary \(i \neq j\). Since \(L_{s,t,0}(x) = 0\), we have \(\sum_{k \in S} x_k = 0\) and \(\sum_{k \in S \cup \{j\} \setminus \{i\}} x_k = 0\). The two equations implies \(x_i = x_j\). Since \(x_i = x_j\) for arbitrary \(i \neq j\), we have \(x_1 = x_2 = \cdots = x_n\). Thus \(\sum_{k \in \{1, 2, \ldots, n\}} x_k = s \cdot x_1 = 0\) and we have \(x = 0\).

The rest is to show that \(\sigma \circ L_{s,t,0} = L_{s,t,0} \circ \sigma\) for all \(\sigma \in S_n\). Let \(x \in \mathbb{R}^N\) and \(\sigma \in S_n\) be arbitrary.

\[
(L_{s,t,0}(\sigma \cdot x))(S, T) = \begin{cases} 
 \sum_{i \in S} x_{\sigma(i)} & |S| = s \land |T| = t, \\
 0 & \text{otherwise}
\end{cases} = \begin{cases} 
 \sum_{\sigma^{-1}(i) \in S} x_i & |S| = s \land |T| = t, \\
 0 & \text{otherwise.}
\end{cases}
\]

On the other hand,

\[
(\sigma \cdot L_{s,t,0}(x))(S, T) = (L_{s,t,0}(x))(\sigma(S), \sigma(T)) = \begin{cases} 
 \sum_{i \in \sigma(S)} x_i & |\sigma(S)| = s \land |\sigma(T)| = t, \\
 0 & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
 \sum_{\sigma^{-1}(i) \in S} x_i & |S| = s \land |T| = t, \\
 0 & \text{otherwise.}
\end{cases}
\]

\[\square\]

**Remark 3.1.** For the sake of later discussion (especially Theorem 3.1), we organize images of \(U\) and \(V\) by mappings \(L_{s,t,0}\) and \(L_{s,t,1}\). By Lemma 3.1, for each \((s, t) \in D_n\) such that \(1 \leq s \leq n - 1\), if \(M \subseteq \mathbb{R}^N\) is a subrepresentation, then \(L_{s,t,0}\) is an isomorphism between \(M\) and \(L_{s,t,0}(M)\), that is, \(M \cong L_{s,t,0}(M)\). This is also true for \(L_{s,t,1}\). In light of this, we organize images of \(U\) and \(V\) by mappings \(L_{s,t,0}\) and \(L_{s,t,1}\).
1. Note that \( L_{n,0,0}(U) = \langle u_0 \rangle \) and \( L_{0,n,1}(U) = \langle u_n \rangle \).

For \((s, t) \in C_n, \) \[
\begin{aligned}
L_{s,t,0}(U) &= L_{s,t,1}(U) = \langle u_s \rangle \cong U & \text{if } s \neq 0 \land t \neq 0, \\
L_{s,0,0}(U) &= \langle u_s \rangle \cong U, L_{s,0,1}(U) = \{0\} & \text{if } s \neq 0 \land t = 0, \\
L_{0,t,0}(U) &= \{0\}, L_{0,t,1}(U) = \langle u_t \rangle \cong U & \text{if } s = 0 \land t \neq 0.
\end{aligned}
\]

2. For \((s, t) \in D_n, \) \[
\begin{aligned}
L_{s,t,0}(V) &= V \cong L_{s,t,1}(V) & \text{if } s \neq 0 \land t \neq 0, \\
L_{s,0,0}(V) &= V, L_{s,0,1}(V) = \{0\} & \text{if } s \neq 0 \land t = 0, \\
L_{0,t,0}(V) &= \{0\}, L_{0,t,1}(V) \cong V & \text{if } s = 0 \land t \neq 0.
\end{aligned}
\]

3. In particular, when \( s + t = n, \) we have \( L_{s,t,0}(V) = L_{s,t,1}(V) \) since
\[
\begin{aligned}
x^{(s,t,0)}(S, T) &= \begin{cases} 
\sum_{i \in S} x_i & |S| = s \land |T| = t \\
0 & \text{otherwise}
\end{cases} \\
&= \begin{cases} 
-\sum_{i \in T} x_i & |S| = s \land |T| = t \\
0 & \text{otherwise}
\end{cases} = -x^{(s,t,1)}(S, T).
\end{aligned}
\]

We set
\[
\begin{aligned}
D_{n}^1 &= \{ (s, t) \in D_n \mid s + t = n \} = \{ (k, n - k) \mid k = 1, \ldots, n - 1 \}, \\
D_{n}^2 &= \{ (s, t) \in D_n \mid t = 0 \} = \{ (s, 0) \mid s = 1, \ldots, n - 1 \}, \\
D_{n}^3 &= \{ (s, t) \in D_n \mid s = 0 \} = \{ (0, t) \mid t = 1, \ldots, n - 1 \}, \\
D_{n}^4 &= D_n \setminus (D_{n}^1 \cup D_{n}^2 \cup D_{n}^3).
\end{aligned}
\]

Note that \( \{ D_{n}^1, D_{n}^2, D_{n}^3, D_{n}^4 \} \) is a partition of \( D_n. \)

The following theorem states about identifying irreducible representations of \( G_t^s \) that are isomorphic to \( U \) or \( V \) for \((s, t) \in D_n. \)

**Theorem 3.1.** For each \((s, t) \in D_n, \)
\[
G_t^s = U_t^s \oplus V_t^s \oplus W_t^s,
\]
where \( U_t^s = \langle u_t \rangle \cong U, \)
\[
V_t^s = \begin{cases} 
\overline{V}_t^s (\cong V_t^s) & (s, t) \in D_{n}^1, \\
V_t^s & (s, t) \in D_{n}^2, \\
\overline{V}_t^s \oplus V_t^s & (s, t) \in D_{n}^3,
\end{cases}
\]
in which \( \overline{V}_t^s = L_{s,t,0}(V) = \{ x^{(s,t,0)} \mid x \in V \} \cong V \) and \( V_t^s = L_{s,t,1}(V) = \{ x^{(s,t,1)} \mid x \in V \} \cong V; \) and \( W_t^s \) is a subrepresentation that does not contain any summands isomorphic to either \( U \) nor \( V. \)

**Proof.** See Appendix. \( \square \)

**Remark 3.2.** Theorem 3.1 does not quite give us a decomposition of \( G_t^s \) into irreducible subrepresentations. The subrepresentation \( U_t^s \) is irreducible and \( V_t^s \) can be a direct sum of irreducible subrepresentations. Whereas \( W_t^s \) may or may not be irreducible, but as we shall see the exact nature of this subrepresentation plays no role in the study of linear symmetric solutions, since it lies in the kernel of any such solution.
Let $U_G = \bigoplus_{(s,t) \in C_n} U_{t,s}^*$, $V_G = \bigoplus_{(s,t) \in D_n} V_{t,s}^*$, $W_G = \bigoplus_{(s,t) \in D_n} W_{t,s}^*$, then $G = U_G \oplus V_G \oplus W_G$. In other words, for any $b \in G$, there exist unique $a_{s,t} \in \mathbb{R}$, $y_s \in V$, $z_{s,t,0}, z_{s,t,1} \in V$ and $w \in W_G$ such that

$$b = \sum_{(s,t) \in C_n} a_{s,t} u_t^* + \sum_{(s,t) \in D_n^1} y_s^{(s,t,0)} + \sum_{(s,t) \in D_n^2} z_{s,t,0}^{(s,t,0)} + \sum_{(s,t) \in D_n^3} z_{s,t,1}^{(s,t,1)} + w\quad(3.1)$$

The next result provides a good example of how the decomposition of $G$ can be used to gain information about linear symmetric solutions.

**Corollary 3.2.** If $\phi: G \to \mathbb{R}^N$ is a linear symmetric solution, then $\phi(b) = 0$ for every $b \in W_G$.

**Proof.** Let $\phi: G = U_G \oplus V_G \oplus W_G \to \mathbb{R}^N = U \oplus V$ be a linear symmetric solution (equivalently, $S_n$-equivariant map). Suppose $Z \subset W_G$ is an irreducible subrepresentation in the decomposition of $W_G$ (even while we do not know the decomposition of $W_G$ as a sum of irreducible subrepresentations, the existence of such a decomposition is ensured by Proposition 2.1). Then $\phi(Z)$ is equal to $U$ or $V$ or $\{0\}$ by Corollary 3.1. But $Z$ is not equivalent to $U$ nor $V$ by Theorem 3.1, so $\phi(Z) = \{0\}$. Since this is true for every irreducible subrepresentation $Z$ of $W_G$, $\phi$ is zero on all of $W_G$. \hfill \Box

From the decomposition of $G$, given a game $b \in G$, we may decompose it into the form of (3.1). This decomposition fits very well to the study of the image of $b$ under any linear symmetric solution.

**Theorem 3.2.** Any linear symmetric solution

$$\phi: G = U_G \oplus V_G \oplus W_G \to \mathbb{R}^N = U \oplus V$$

satisfies (a) $\phi(U_G) \subset U$; (b) $\phi(V_G) \subset V$. Moreover,

- for each $(s,t) \in C_n$, there is a constant $\alpha(s,t) \in \mathbb{R}$ such that
  $$\phi(u_t^*) = \alpha(s,t) \cdot 1 \in U,$$

- for each $(s,t) \in D_n$, there are constants $\beta(s,t,0), \beta(s,t,1) \in \mathbb{R}$ such that, for every $z \in V$,
  $$\phi(z^{(s,t,0)}) = \beta(s,t,0) \cdot z \in V, \quad \phi(z^{(s,t,1)}) = \beta(s,t,1) \cdot z \in V.$$

**Proof.** Let $\phi: G \to \mathbb{R}^N$ be a linear symmetric solution. Let $(s,t) \in D_n$ be fixed. $\phi(V_t^*)$ is equal to $U$ or $V$ or $\{0\}$ by Corollary 3.1, but $\phi(V_t^*)$ is never equal to $U$ since $V_t^* \cong V \neq U$. Thus $\phi(V_t^*) \subset V$. We can show $\phi(V_t^*) \subset V$ in the same way. Therefore $\phi(V_t^*) \subset V$ since $\phi(V_t^*) \subset V$ and $\phi(V_t^*) \subset V$ for all $(s,t) \in D_n$. We can show $\phi(U_G) \subset U$ in the same way.

We shall show $\phi(z^{(s,t,0)}) = \beta(s,t,0) \cdot z \in V$. Let $(s,t) \in D_n$ be fixed. In the same way as before, $\phi(V_t^*)$ is equal to $V$ or $\{0\}$. If $\phi(V_t^*) = \{0\}$ then $\phi(z^{(s,t,0)}) = 0 = (0, z) \in V$ for every $z \in V$. If $\phi(V_t^*) = V$, that is, $\phi_{V_t^*}: V_t^* \to V$, the restriction of $\phi$ to $V_t^*$, is an isomorphism, then there exists $\beta(s,t,0) \in \mathbb{R}$ such that

$$\phi(z^{(s,t,0)}) = \phi_{V_t^*}(z^{(s,t,0)}) = \beta(s,t,0) \cdot L_{s,t,0}^{-1}(z^{(s,t,0)}) = \beta(s,t,0) \cdot z \in V$$

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for every $z \in V$ by Theorem 2.1(ii), where $L_{s,t,0}^{-1}: V_t^s \to V$ is the inverse function of $L_{s,t,0}$.

Note that the inverse function of an isomorphism is also an isomorphism. Thus, for each $(s, t) \in D_n$, there is a constant $\beta(s, t, 0) \in \mathbb{R}$ such that, for every $z \in V$,

$$
\phi(z^{(s,t,0)}) = \beta(s, t, 0) \cdot z \in V.
$$

We can also show the cases of $\phi(u_t^s) = \alpha(s, t) \cdot 1 \in U$ and $\phi(z^{(s,t,1)}) = \beta(s, t, 1) \cdot z \in V$ in the same way. \hfill \Box

For many purposes it suffices to use merely the existence of the decomposition of the game $b \in G$, without having to worry about the precise value of each component. Nevertheless it will be useful to have it. Thus we give a formula for computing it.

Hereinafter, we the following abbreviation:

$$
b_0^b(s,t) = \sum_{(S,T)\in 3^N \mid |S|=s,|T|=t,S \ni i} b(S,T), \quad b_1^b(s,t) = \sum_{(S,T)\in 3^N \mid |S|=s,|T|=t,T \ni i} b(S,T),
$$

$$
b_{-1}(s,t) = \sum_{(S,T)\in 3^N \mid |S|=s,|T|=t,S \ni \not{i}} b(S,T), \quad b(s,t) = \sum_{(S,T)\in 3^N \mid |S|=s,|T|=t} b(S,T).
$$

(3.2)

**Theorem 3.3.** Let $b \in G$. The components of the right-hand side of (3.1) are determined as follows:

1. $a_{s,t}$ is the average of the values $b(S,T)$ with $S$ containing $s$ players and $T$ containing $t$ players:

$$
a_{s,t} = \frac{s!t!(n-s-t)!}{n!} b(s,t).
$$

2. For every $(k, n-k) \in D_n^1$, $y_k \in V$ is given by:

$$
(y_k)_i = \frac{(k-1)!(n-k-1)!}{n(n-2)!} \left( (n-k)b_0^b(k,n-k) - kb_1^b(k,n-k) \right),
$$

and for every $(s,t) \in D_n^2 \cup D_n^4$, $z_{s,t,0}$ is given by:

$$
(z_{s,t,0})_i = \frac{(s-1)!(n-s-t-1)!}{n(n-2)!} \left( (n-s-t)b_0^b(s,t) - sb_{-1}(s,t) \right),
$$

and for every $(s,t) \in D_n^3 \cup D_n^4$, $z_{s,t,1}$ is given by:

$$
(z_{s,t,1})_i = \frac{s!(t-1)!(n-s-t-1)!}{n(n-2)!} \left( (n-s-t)b_1^b(s,t) - tb_{-1}(s,t) \right).
$$

3. Moreover, $w$ may be computed as “the rest”, i.e.,

$$
w = b - \sum_{(s,t) \in C_n} a_{s,t} u_t^s - \sum_{k=1}^{n-1} y_k^{(k,n-k,0)} - \sum_{(s,t) \in D_n^2 \cup D_n^4} z_{s,t,0}^{(s,t,0)} - \sum_{(s,t) \in D_n^3 \cup D_n^4} z_{s,t,1}^{(s,t,1)} \in W_G.
$$

**Proof.** See Appendix. \hfill \Box
4. Applications

Now, we show that we can easily obtain characterizations of solutions by using the decomposition of a game given by Theorem 3.3 in conjunction with Theorem 3.2. We start by providing a characterization of all linear symmetric solutions.

**Theorem 4.1.** Let $\phi : G \rightarrow \mathbb{R}^N$ be a solution. The following three conditions are equivalent.

1. $\phi$ satisfies linearity and symmetry.
2. There exist real coefficients
   \[
   \{ \gamma(s, t) \mid (s, t) \in C_n \} \cup \{ \delta(s) \mid (s, t) \in D_1 \} \\
   \cup \{ \zeta(s, t, 0) \mid (s, t) \in D_2^0 \cup D_2^1 \} \cup \{ \zeta(s, t, 1) \mid (s, t) \in D_3^0 \cup D_3^1 \}
   \]
   such that for each $i \in N$
   \[
   \phi_i(b) = \sum_{(s, t) \in C_n} \gamma(s, t) b(s, t) + \sum_{k=1}^{n-1} \delta(k) ((n - k)b_i^0(k, n - k) - kb_i^1(k, n - k)) \\
   + \sum_{(s, t) \in D_2^0 \cup D_2^1} \zeta(s, t, 0) ((n - s - t)b_i^0(s, t) - sb_{-i}(s, t)) \\
   + \sum_{(s, t) \in D_3^0 \cup D_3^1} \zeta(s, t, 1) ((n - s - t)b_i^1(s, t) - tb_{-i}(s, t)).
   \]

3. There exist real coefficients
   \[
   \{ p(s, t) \mid (s, t) \in C_n, s > 0 \} \cup \{ q(s, t) \mid (s, t) \in C_n, t > 0 \} \cup \{ r(s, t) \mid (s, t) \in C_n, s + t < n \}
   \]
   such that for each $i \in N$
   \[
   \phi_i(b) = \sum_{(s, t) \in C_n, s > 0} p(s, t) b_i^0(s, t) + \sum_{(s, t) \in C_n, t > 0} q(s, t) b_i^1(s, t) - \sum_{(s, t) \in C_n, s + t < n} r(s, t) b_{-i}(s, t).
   \]

**Proof.** See Appendix.

Theorem 4.1 shows that any linear symmetric solution $\phi_i(b)$ for player $i \in N$ only depends on $b_i^0$, $b_i^1$ and $b_{-i}$. Since the solutions only depend on the cardinalities of coalitions, the dimension of the space of solutions is largely reduced compared with that of $G$. Additionally, from the expression, we can see that $\phi_i(b)$ is unchanged when positions of others $h, j \in N \setminus \{ i \}$ are replaced. This is a consequence of the symmetry axiom.

The next corollary shows the specific dimension of the space of linear symmetric solutions. It is obtained by counting the parameters of the expressions of Theorem 4.1.

**Corollary 4.1.** The space of all linear and symmetric solutions on $G$ has dimension $\frac{3}{2}n^2 + \frac{3}{2}n - 1$.

**Proof.** Count the number of parameters $\gamma(s, t), \delta(k), \zeta(s, t, 0), \zeta(s, t, 1)$ in (4.1).

\[
|C_n| + |D_1^n| + |D_2^n| + |D_3^n| + 2|D_4^n| = \left( \frac{n+2}{2} \right) - 1 + \left( n-1 + n-1 + n-1 + 2 \left( \frac{n+2}{2} \right) - 3 - 3(n-1) \right)
\]
Thus, by Theorem 3.3, for any $s,t \in X$, we have $\varphi_{s,t}(N,\emptyset) = \varphi_{s,t}(\emptyset,N)$. To evaluate power of players, it may not be necessary.

Let $\alpha \in \mathbb{R}^n$ be a linear symmetric solution. Then, $\alpha$ is efficient, if and only if $\sum_{i \in N} \alpha_{i} = 0$. By Theorem 3.2, there exists $\alpha \in \mathbb{R}^n$ such that $\phi(u_t^*) = \alpha(s,t) \cdot 1$. Thus $\sum_{i \in N} \alpha_{i} = \alpha(s,t)n$. Since $\phi$ is efficient,

$$
\alpha(s,t) = \begin{cases} 
0 & (s,t) \in D_n, \\
n & (s,t) = (n,0), \\ 
\frac{1}{n} & (s,t) = (0,n).
\end{cases}
$$

(Axiom 4.1) (Efficiency). A solution $\phi$ is efficient if $\sum_{i \in N} \phi_i(b) = b(N,\emptyset) - b(\emptyset, N)$ for all $b \in G$. The efficiency axiom states players should divide the value $b(N,\emptyset) - b(\emptyset, N)$ among them, where $b(N,\emptyset) - b(\emptyset, N)$ is the difference between the profits when the players change their activities and when they remain.

Proposition 4.1. Let $\phi: G \rightarrow \mathbb{R}^N$ be a linear symmetric solution. Then $\phi$ is efficient iff for every $(s,t) \in C_n$, $\phi(u_t^*) = \alpha(s,t) \cdot 1$ where

$$
\alpha(s,t) = \begin{cases} 
0 & (s,t) \in D_n, \\
n & (s,t) = (n,0), \\ 
\frac{1}{n} & (s,t) = (0,n).
\end{cases}
$$

Proof. $(\Rightarrow)$: Suppose $\phi$ is efficient. Let $(s,t) \in C_n$. By Theorem 3.2, there exists $\alpha(s,t)$ such that $\phi(u_t^*) = \alpha(s,t) \cdot 1$. Thus $\sum_{i \in N} \phi_i(u_t^*) = \alpha(s,t)n$. Since $\phi$ is efficient,

$$
\alpha(s,t)n = u_t^*(N,\emptyset) - u_t^*(\emptyset,N) = \begin{cases} 
0 & (s,t) \in D_n, \\
n & (s,t) = (n,0), \\ 
-1 & (s,t) = (0,n).
\end{cases}
$$

$(\Leftarrow)$: Note that every $b \in V_G$ trivially satisfies $\sum_{i \in N} \phi_i(b) = 0$ since (by Theorem 3.2) $\phi(V_G) \subseteq V$, and every $b \in W_G$ also satisfies $\sum_{i \in N} \phi_i(b) = 0$ since $\phi(b) = 0$ by Corollary 3.2. Thus, by Theorem 3.3, for any $b \in G$,

$$
\sum_{i \in N} \phi_i(b) = \sum_{i \in N} \sum_{(s,t) \in C_n} a_{s,t} \phi_i(u_t^*) = \sum_{i \in N} \left( \frac{a_{n,0}}{n} - \frac{a_{0,n}}{n} \right) = b(N,\emptyset) - b(\emptyset, N).
$$
Taking advantage of the decomposition of Theorem 3.3 and Theorem 3.2, we can easily see the consequence of the efficiency axiom. Now, an immediate application is to provide a characterization of all linear, symmetric and efficient solutions.

**Theorem 4.2.** Let $\phi: G \to \mathbb{R}^N$ be a solution. The following three conditions are equivalent.

1. $\phi$ satisfies linearity, symmetry and efficiency.
2. There exist real coefficients
   \[
   \{ \delta(s) \mid (s, t) \in D_{n}^1 \} \cup \{ \zeta(s, t, 0) \mid (s, t) \in D_{n}^2 \cup D_{n}^4 \} \cup \{ \zeta(s, t, 1) \mid (s, t) \in D_{n}^3 \cup D_{n}^4 \} .
   \]
   such that each $i \in N$
   \[
   \phi_i(b) = \frac{b_0^i(n, 0) - b_1^i(0, n)}{n} + \sum_{k=1}^{n-1} \delta(k) \left( (n - k)b_0^i(k, n - k) - kb_1^i(k, n - k) \right) \\
   + \sum_{(s, t) \in D_{n}^2 \cup D_{n}^4} \zeta(s, t, 0) \left( (n - s - t)b_0^i(s, t) - sb_{-1}(s, t) \right) \\
   + \sum_{(s, t) \in D_{n}^3 \cup D_{n}^4} \zeta(s, t, 1) \left( (n - s - t)b_1^i(s, t) - tb_{-1}(s, t) \right). 
   \] (4.4)
3. $\phi$ is of the form of (4.2) under the constraint
   \[
   \begin{cases}
   p(n, 0) = \frac{1}{n}, \\
   q(0, n) = -\frac{1}{n}, \\
   sp(s, t) + tq(s, t) - (n - s - t)r(s, t) = 0 \quad (s, t) \in D_n,
   \end{cases}
   \] (4.5)
   where we assume $p(0, k) = q(k, 0) = r(k, n - k) = 0$ for $k = 1, \ldots, n - 1$.

**Proof.** See Appendix. \hfill \square

Interpretation of expressions of $\phi$ in this theorem may be difficult, however, we can easily see that (4.5) is sufficient for the efficiency axiom from the fact that:

\[
\sum_{i \in N} \phi_i(b) = \sum_{(s, t) \in C_n, s > 0} p(s, t) \sum_{i \in N} b_0^i(s, t) + \sum_{(s, t) \in C_n, t > 0} q(s, t) \sum_{i \in N} b_1^i(s, t) \\
- \sum_{(s, t) \in C_n, s + t < n} r(s, t) \sum_{i \in N} b_{-1}(s, t) \\
= \sum_{(s, t) \in C_n, s > 0} p(s, t) \sum_{i \in N} \sum_{(S, T) \in 3^N, |S| = s, |T| = t, S \ni i} b(S, T) \\
+ \sum_{(s, t) \in C_n, t > 0} q(s, t) \sum_{i \in N} \sum_{(S, T) \in 3^N, |S| = s, |T| = t, T \ni i} b(S, T) \\
- \sum_{(s, t) \in C_n, s + t < n} r(s, t) \sum_{i \in N} \sum_{(S, T) \in 3^N, |S| = s, |T| = t, T \ni i} b(S, T) \\
= \sum_{(s, t) \in C_n, s > 0} p(s, t) \sum_{(S, T) \in 3^N, |S| = s, |T| = t} \sum_{i \in S} b(S, T)
\]
The next corollary also shows the dimension of the space of linear, symmetric and efficient solutions.

**Corollary 4.2.** The space of all linear, symmetric and efficient solutions on $G$ has dimension $n^2 - 1$.

**Proof.** Count the number of parameters $\delta(s)$, $\zeta(s,t,0)$, $\zeta(s,t,1)$ in Theorem 4.2.

Next, we take particular attention to the null axiom.

**Axiom 4.2** (Nullity). If a player $i$ is null for a bicooperative game $b \in G$, then $\phi_i(b) = 0$, where a player $i$ is null for the game $b$ if $b(S \cup \{i\}, T) = b(S, T) = b(S, T \cup \{i\})$ for all $(S, T) \in 3^N \setminus \{i\}$.

The null axiom states a player who does not contribute to any coalition should gain nothing.

**Theorem 4.3.** A solution $\phi: G \to \mathbb{R}^N$ satisfies linearity, symmetry and null axioms iff there exist real coefficients $\{p(s,t) \mid (s,t) \in C_n, s > 0\} \cup \{q(s,t) \mid (s,t) \in C_n, t > 0\}$ such that for each $i \in N$

$$\phi_i(b) = \sum_{(s,t) \in C_n, s+t<n} \left( p(s + 1, t) (b_i^0(s + 1, t) - b_{-i}(s, t)) 
+ q(s, t + 1) (b_i^1(s, t + 1) - b_{-i}(s, t)) \right), \quad (4.6)$$

equivalently,

$$\phi_i(b) = \sum_{(S,T) \in 3^N \setminus \{i\}} \left( p(s + 1, t) \left( b(S \cup \{i\}, T) - b(S, T) \right) 
+ q(s, t + 1) \left( b(S, T \cup \{i\}) - b(S, T) \right) \right), \quad (4.7)$$

where $s = |S|$ and $t = |T|$.

**Proof.** See Appendix.
As we can see (4.7), the value $\phi_i(b)$ of player $i \in N$ is a weighted sum of marginal contributions of $i$ for both of agreement and disagreement with the proposal considered in game $b$. Each weight for agreement $p(s+1,t)$ (resp. disagreement $q(s,t+1)$) only depends on the cardinalities of the coalition $S$ (resp. $T$) that $i$ will enter and the opposite coalition $T$ (resp. $S$).

The next corollary also shows the dimension of the space of linear, symmetric and nullity solutions.

**Corollary 4.3.** The space of all linear, symmetric and nullity solutions on $G$ has dimension $n^2 + n$

**Proof.** Count the number of parameters $p(s,t), q(s,t)$ in Theorem 4.3.

Combining the above results, we show the linear and symmetric solutions with efficiency and null axioms.

**Theorem 4.4.** A solution $\phi: G \to \mathbb{R}^N$ satisfies linearity, symmetry, efficiency and null axioms iff it is of the form of (4.7) under the constraints

$$
\begin{aligned}
& p(n,0) = \frac{1}{n}, \\
& q(0,n) = -\frac{1}{n}, \\
& sp(s,t) + tq(s,t) - (n - s - t)(p(s+1,t) + q(s,t+1)) = 0 \quad (s,t) \in D_n,
\end{aligned}
$$

where we assume $p(k,0), p(0,k), p(k+1,n-k), q(k+1,n-k)$ are 0 for all $k = 1, \ldots, n-1$.

**Proof.** It is immediately obtained by combining Theorems 4.2 and 4.3.

Finally, we consider the Shapley value of bicooperative games. Contrary to classical cooperative games [11], in the case of bicooperative games, linearity, symmetry, efficiency and nullity axioms do not specify a unique solution. Hence, Bilbao et al. [4] proposed an additional axiom.

**Axiom 4.3** (Structural axiom). For every $(S,T) \in 3^N \setminus \{(\emptyset, \emptyset)\}$ such that $j \in S$, it holds

$$
- \frac{\phi_j(C_{ST})}{\phi_i(C_{ST \cup \{i\}})} = \frac{(n+s-1-t)!/2^{s-1}}{(n+s-t-1)!/2^s} = 2,
$$

and for every $(S,T) \in 3^N \setminus \{(\emptyset, \emptyset)\}$ such that $h \in T$, it holds

$$
- \frac{\phi_h(C_{ST})}{\phi_i(C_{S \cup \{i\},T})} = \frac{(n+t-1-s)!/2^{t-1}}{(n+t-s-1)!/2^t} = 2,
$$

where $C_{ST}$ is a bicooperative game such that

$$
C_{ST}(K,L) = \begin{cases} 
1 & (K,L) = (S,T), \\
0 & \text{otherwise.}
\end{cases}
$$

The constants appearing in the above definition come from the numbers of chains or paths from the top or bottom of coalition pairs (i.e., $(N,\emptyset)$ or $(\emptyset,N)$) to some coalition pairs related to $S$, $T$, $h$, $i$ and $j$ (e.g., $(S,T)$ and $(S,T \cup \{i\})$). For the details of the structural axiom, see [4].

Following the same discussion of [4], we can characterize the Shapley value for bicooperative games.
Corollary 4.4. A solution \( \phi: G \to \mathbb{R}^N \) satisfies linearity, symmetry, efficiency, null and structural axioms iff it is the Shapley value that is the form of (4.7) with

\[
p(s, t) = \frac{(n + s - t - 1)!((n + t - s)!(2n)!}{2^{n-s-t+1}},
\]

\[
q(s, t) = -\frac{(n + s - t - 1)!((n + s - t)!}{(2n)!}2^{n-s-t+1}.
\]

Proof. See Appendix. \qed

Since we use the null axiom instead of the dummy axiom, this axiomatic system is slightly different from that of [4].

5. Conclusion
In this paper we study linear symmetric solutions for bicooperative games using the elementary representation theory of symmetric group. This paper has two main objectives.

The first objective is to provide a viewpoint of the space of bicooperative games as a representation. Traditional approaches regard the game space and the payoff vector space as linear spaces, but in this paper, we consider them as representations of \( S_n \). This idea reflects linearity and symmetry all together in the game space and the payoff vector space. By doing this, we can consider linear symmetric solutions as \( S_n \)-equivariant maps. And we characterize all linear symmetric solutions by tools of representation theory. Additionally, we have imposed the efficiency and/or null axioms on the linear symmetric solutions, and provided their characterizations. Moreover, we have characterized the Shapley value for bicooperative games by imposing the structural axiom. In this paper we have ended up looking for linear and symmetric solutions, but more results may be obtained by representation theory.

The second objective is to advertise representation theory as a natural tool for research in cooperative game theory. This representation-theoretic approach was introduced by Hernández-Lamoneda et al. [13] and Sánchez-Pérez [16], the former applied for usual cooperative games, the latter applied for games in partition function form. In this paper we showed that this approach is also applicable to bicooperative games. This reinforces the statement that this approach is generally applicable to various frameworks in cooperative game theory. If one considers an original framework of cooperative games and need to get linear and symmetric solutions for games in that framework, the approach in this paper might be applicable.

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Appendix
We use the abbreviations (3.2). Additionally, we set \( (3^N)_{s}^{s} = \{(S, T) \in 3^N \mid |S| = s, |T| = t \} \) for \( (s, t) \in C_n \) and abbreviate \( \sum_{(S, T) \in (3^N)_{s}^{s}} b(S, T) \) to \( \sum_{(s, t)} \sum_{(k, n-k)} b(S, T) \). For example, \( \sum_{S \ni i} b(S, T) \) means

\[
\sum_{(S, T) \in (3^N)_{s}^{s}} b(S, T).
\]

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Notes on Theorem 2.1 and Proposition 2.2

In finite-dimensional representation theory of finite groups, Schur’s Lemma is the following claim.

**Fact 5.1 (Schur’s Lemma).** Let \((M_1, \rho_1)\) and \((M_2, \rho_2)\) be irreducible \(F\)-representations of \(H\) and \(\phi: M_1 \to M_2\) an \(H\)-equivariant map. Then

(i) Either \(\phi\) is an isomorphism, or \(\phi = 0\).

(ii) If \((M_1, \rho_1) \cong (M_2, \rho_2)\) and \(F\) is algebraically closed, then \(\phi\) is unique up to multiplication by a scalar \(\lambda \in F\).

Theorem 2.1 is very similar to Schur’s Lemma, but in our situation, that is, when \(F = \mathbb{R}\) and \(H = S_n\), Theorem 2.1(ii) does not immediately follow from Schur’s Lemma(ii) since \(\mathbb{R}\) is not algebraically closed. To justify Theorem 2.1(ii), the following fact plays an important role.

**Fact 5.2.** [13, Corollary 7 in Appendix] Let \((M_1, \rho_1)\) be a real irreducible representation, such that its complexification \((M_1, \rho_1)^C = (M_1, \rho_1) \oplus i(M_1, \rho_1)\) is also irreducible (as a complex representation). Let \((M_2, \rho_2)\) be a real irreducible representation. If \(\phi: M_1 \to M_2\) is equivariant, then \(\phi\) is unique up to multiplication by a real scalar.

Fact 5.2 tells us that if \((M, \rho)\) satisfies (i) \((M, \rho)\) is irreducible as a real representation and (ii) \((M, \rho)^C\) is irreducible as a complex representation, then Schur’s Lemma(ii) holds for \((M, \rho)\). According to [13], \(U\) and \(V\) satisfy (i) and (ii).

**Fact 5.3.** [13, Lemma 7 in Appendix] The decomposition of \(\mathbb{R}^N\), under \(S_n\), into irreducible subrepresentations is

\[
\mathbb{R}^N = U \oplus V.
\]

Remark 10 in Appendix of [13] tells us that Fact 5.3 holds in \(\mathbb{C}\).

**Fact 5.4.** [13] The decomposition of \(\mathbb{C}^N\), under \(S_n\), into irreducible subrepresentations is

\[
\mathbb{C}^N = U^C \oplus V^C.
\]

By the above discussion, we have shown Theorem 2.1 since \(U\) and \(V\) satisfy (i) and (ii). Note that Theorem 2.1(ii) does not apply to representations which are not isomorphic to \(U\) nor \(V\).

Next, we focus on Proposition 2.2. In general, there are orthogonality relations between irreducible characters with this inner product.

**Fact 5.5.** Let \((M_1, \rho_1)\) and \((M_2, \rho_2)\) be irreducible \(F\)-representations of \(H\). Then

(i) If \((M_1, \rho_1) \not\cong (M_2, \rho_2)\), \(\langle \chi_{(M_1, \rho_1)}, \chi_{(M_2, \rho_2)} \rangle = 0\).

(ii) If \((M_1, \rho_1) \cong (M_2, \rho_2)\) and \(F\) is algebraically closed, \(\langle \chi_{(M_1, \rho_1)}, \chi_{(M_2, \rho_2)} \rangle = 1\).

Because Fact 5.5(ii) depends on Schur’s Lemma (ii), Fact 5.5(ii) assumes that \(F\) is algebraically closed. However, by Theorem 2.1, \(\langle \chi_{(M_1, \rho_1)}, \chi_{(M_2, \rho_2)} \rangle = 1\) if \((M_1, \rho_1) \cong (U, \rho_U)\) or \((M_1, \rho_1) \cong (V, \rho_V)\). This fact and Proposition 2.1 lead Proposition 2.2.

The reader may feel that Theorem 2.1(ii) and Proposition 2.2 are a little poor since they are limited to \(U\) or \(V\). However, the first time that we face representations which are not isomorphic to \(U\) nor \(V\) is Corollary 3.1, and this corollary shows that such representations vanish when we think about linear and symmetric solutions. Thus, Theorem 2.1 and Proposition 2.2 are sufficient to analyze linear and symmetric solutions.
Proof of Theorem 3.1
First, we remark about the representation $G^*_t$ and $\chi_{G^*_t}(\sigma)$ for $(s, t) \in C_n$. For $(K, L) \in \binom{3^N}{t}$, we define a game $C_{(K, L)} \in G^*_t$ as

$$C_{(K, L)}(S, T) = \begin{cases} 1 & \text{if } (S, T) = (K, L), \\ 0 & \text{otherwise}. \end{cases}$$

Then $B = \{C_{(K, L)} \}_{(K, L) \in \binom{3^N}{t}}$ is a basis of $G^*_t$. Then $[\rho_{G^*_t}(\sigma)]_B = (a_{(S, T), (S', T')}) \in \mathbb{R}^{(3^N)_t \times (3^N)_t}$ where

$$a_{(S, T), (S', T')}(S', T') = \begin{cases} 1 & \sigma(S) = S' \land \sigma(T) = T' \\ 0 & \text{otherwise}. \end{cases}$$

Thus $\chi_{G^*_t}(\sigma) = \text{Tr}([\rho_{G^*_t}(\sigma)]_B)$ is the number of $(S, T) \in \binom{3^N}{t}$ such that $(\sigma(S) = S) \land (\sigma(T) = T)$. Note that $\chi_{G^*_t}(\sigma) = \chi_{G^*_t}(\sigma^{-1})$ for arbitrary $\sigma \in S_n$ since $\sigma(S) = S$ is equivalent to $\sigma^{-1}(S) = S$ for $S \subseteq N$.

Now, we get into the proof of Theorem 3.1. By Proposition 2.2, $\langle \chi_{G^*_t}, \chi_U \rangle$ and $\langle \chi_{G^*_t}, \chi_V \rangle$ are the numbers of subrepresentations isomorphic to $U$ and $V$ within $G^*_t$, respectively.

We start by computing the multiplicity of $U$ in $G^*_t$. Note that $\chi_U(\sigma) = 1$ for all $\sigma \in S_n$ since $U$ is 1-dimensional and $\rho_U(\sigma)$ is identity for all $\sigma \in S_n$.

$$\langle \chi_{G^*_t}, \chi_U \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_{G^*_t}(\sigma) \chi_U(\sigma) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_{G^*_t}(\sigma).$$

Define

$$\{\sigma\}_{(S, T)} = \begin{cases} 1 & \sigma(S) = S \land \sigma(T) = T, \\ 0 & \text{otherwise}. \end{cases}$$

Then, $\chi_{G^*_t}(\sigma) = \sum_{(s, t)} \{\sigma\}_{(S, T)}$ and so,

$$\langle \chi_{G^*_t}, \chi_U \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \sum_{(s, t)} \{\sigma\}_{(S, T)} = \frac{1}{n!} \sum_{(s, t)} \sum_{\sigma \in S_n} \{\sigma\}_{(S, T)}.$$

Here, we have $\sum_{\sigma \in S_n} \{\sigma\}_{(S, T)} = |\{\sigma \in S_n \mid \sigma(S) = S \land \sigma(T) = T\}| = |S|! |T|! (n - |S| - |T|)!$. Thus,

$$\langle \chi_{G^*_t}, \chi_U \rangle = \frac{1}{n!} \sum_{(s, t)} |S|! |T|! (n - |S| - |T|)! = \frac{1}{n!} \cdot s! (n - s - t)! \sum_{(s, t)} 1 = \frac{s! (n - s - t)!}{n!} \binom{n}{s} \binom{n - s}{t} = \frac{s! (n - s - t)!}{n!} \cdot \frac{n!}{s!(n - s)!} \cdot \frac{(n - s)!}{t!(n - s - t)!} = 1.$$

Now, we compute the multiplicity of $V$ in $G^*_t$. Since $\mathbb{R}^N = U \oplus V$, then $\chi_{\mathbb{R}^N} = \chi_U + \chi_V \implies \langle \chi_{G^*_t}, \chi_{\mathbb{R}^N} \rangle = \langle \chi_{G^*_t}, \chi_U \rangle + \langle \chi_{G^*_t}, \chi_V \rangle \implies \langle \chi_{G^*_t}, \chi_V \rangle = \langle \chi_{G^*_t}, \chi_{\mathbb{R}^N} \rangle - 1$.

Notice that $G^*_0 \cong \mathbb{R}^N$ since $f : G^*_0 \rightarrow \mathbb{R}^N$ defined as $f_i(b) = b(\{i\}, \emptyset)$ is an isomorphism.

Let us compute

$$\langle \chi_{G^*_t}, \chi_{\mathbb{R}^N} \rangle = \langle \chi_{G^*_t}, \chi_{G^*_0} \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_{G^*_t}(\sigma) \chi_{G^*_0}(\sigma)$$

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\[
\frac{1}{n!} \sum_{\sigma \in S_n} \sum_{(s,t)} \{\sigma\}_{(s,t)} \{\sigma\}_{(i,\emptyset)} = \frac{1}{n!} \sum_{i \in \mathbb{N}} \sum_{\sigma \in S_n} \{\sigma\}_{(s,t)} \{\sigma\}_{(i,\emptyset)}.
\]

Here, we have

\[
\sum_{\sigma \in S_n} \{\sigma\}_{(s,t)} \{\sigma\}_{(i,\emptyset)} = \left\{ \begin{array}{ll}
(|S| - 1)! |T|!(n - |S| - |T|)! & i \in S, \\
|S|!(|T| - 1)! (n - |S| - |T|)! & i \in T, \\
|S|! |T|!(n - |S| - |T|)! & i \notin S \cup T.
\end{array} \right.
\]

For \((s, t) \in D^1_n\),

\[
\left\langle \chi_{G^*_i}, \chi_{G^*_b} \right\rangle = \frac{1}{n!} \sum_{i \in \mathbb{N}} \sum_{\sigma \in S_n} \{\sigma\}_{(s,t)} \{\sigma\}_{(i,\emptyset)}
\]

\[
= \frac{1}{n!} \sum_{i \in \mathbb{N}} \sum_{\sigma \in S_n} \{\sigma\}_{(s,t)} \{\sigma\}_{(i,\emptyset)} + \sum_{i \in T} \sum_{\sigma \in S_n} \{\sigma\}_{(s,t)} \{\sigma\}_{(i,\emptyset)}
\]

\[
= \frac{1}{n!} \sum_{i \in \mathbb{N}} \sum_{\sigma \in S_n} \{\sigma\}_{(s,t)} \{\sigma\}_{(i,\emptyset)} + \frac{2s!t!(n - s - t)!}{n!} \sum_{1}^{(s,t)}
\]

\[
= \frac{2s!t!(n - s - t)!}{n!} \cdot \binom{n}{s} \binom{n - s}{t} = 2.
\]

Similarly, \(\left\langle \chi_{G^*_b}, \chi_{G^*_i} \right\rangle = \left\langle \chi_{G^*_i}, \chi_{G^*_b} \right\rangle = 2.\)

For \((s, t) \in D^4_n\),

\[
\left\langle \chi_{G^*_i}, \chi_{G^*_b} \right\rangle = \frac{1}{n!} \sum_{i \in \mathbb{N}} \sum_{\sigma \in S_n} \{\sigma\}_{(s,t)} \{\sigma\}_{(i,\emptyset)}
\]

\[
= \frac{1}{n!} \sum_{i \in \mathbb{N}} \sum_{\sigma \in S_n} \{\sigma\}_{(s,t)} \{\sigma\}_{(i,\emptyset)} + \sum_{i \in T} \sum_{\sigma \in S_n} \{\sigma\}_{(s,t)} \{\sigma\}_{(i,\emptyset)}
\]

\[
+ \sum_{i \notin S \cup T} \sum_{\sigma \in S_n} \{\sigma\}_{(s,t)} \{\sigma\}_{(i,\emptyset)}
\]

\[
= \frac{1}{n!} \sum_{i \in \mathbb{N}} \sum_{\sigma \in S_n} \{\sigma\}_{(s,t)} \{\sigma\}_{(i,\emptyset)} + \frac{3s!t!(n - s - t)!}{n!} \sum_{1}^{(s,t)}
\]

\[
= \frac{3s!t!(n - s - t)!}{n!} \cdot \binom{n}{s} \binom{n - s}{t} = 3.
\]
Therefore, \[
\langle \chi G^*_t \cdot \chi V \rangle = \langle \chi G^*_t, \chi \mathbb{R}^N \rangle - 1 = \begin{cases} 1 & (s, t) \in D^1_n \setminus D^4_n, \\ 2 & (s, t) \in D^4_n. \end{cases}
\]

The next task is to identify summands that are isomorphic to \( U \) or \( V \) inside \( G^*_t \). For that end, recall the functions \( L_{s,t,0}: \mathbb{R}^N \to G^*_t \) and \( L_{s,t,1}: \mathbb{R}^N \to G^*_t \). Since these functions are one-to-one \( S_n \)-equivariant maps, \( L_{s,t,0} \) and \( L_{s,t,1} \) are isomorphisms between \( U^*_t \) and \( U \) for \( (s, t) \in C_n \). \( L_{s,t,1} \) is an isomorphism between \( \nabla_t^s \) and \( \chi V \) for \( (s, t) \in D^1_n \setminus D^3_n \), \( L_{s,t,1} \) is an isomorphism between \( V_t^s \) and \( \chi V \) for \( (s, t) \in D^1_n \setminus D^3_n \). Thus, inside \( G^*_t \), we have the images of
\[
\begin{align*}
U^*_t & \quad \text{and} \quad \nabla_t^s (\equiv V^s) \quad \text{for} \quad (s, t) \in D^1_n, \\
U_t^s & \quad \text{and} \quad \nabla_t^s \quad \text{for} \quad (s, t) \in D^2_n, \\
U^*_t & \quad \text{and} \quad V_t^s \quad \text{for} \quad (s, t) \in D^3_n, \\
U_t^s, V_t^s & \quad \text{for} \quad (s, t) \in D^4_n.
\end{align*}
\]

These are all subrepresentations that are isomorphic to \( U \) or \( V \). \( \square \)

**Proof of Theorem 3.3**

We start by computing the projection of \( b \) onto \( U_G \). For each \( (s, t) \in C_n \), we define \( M^{s,t}: G \to U \) as
\[
M^{s,t}(b) = b(s, t) \cdot 1.
\]
For each \( (s, t) \in C_n \), it is clear that \( M^{s,t}(u^s_t) = M^{s,t}(z_{p,q,0}) = M^{s,t}(z_{p,q,1}) = 0 \) for any \( (p, q) \in C_n \setminus \{(s, t)\} \) and \( z \in V \), and \( M^{s,t}(z_{s,t,0}) = M^{s,t}(z_{s,t,1}) = 0 \) for any \( z \in V \). And since \( M^{s,t} \) is \( S_n \)-equivariant, Corollary 3.2 implies \( M^{s,t}(w) = 0 \) for any \( w \in W_G \). Since any \( b \in G \) is uniquely decomposed into the form of (3.1),
\[
M^{s,t}(b) = M^{s,t}(a_{s,t}u^s_t) \implies b(s, t) \cdot 1 = a_{s,t}u^s_t(s, t) \cdot 1 = a_{s,t} \sum (s,t) 1 \cdot 1 = \frac{n!}{s!(n-s-t)!} a_{s,t} \cdot 1
\]
Therefore \( a_{s,t} = \frac{s!!(n-s-t)!}{n!}b(s, t) \).

Now, we shall compute \( y_s \in V \) for \( (s, t) \in D^1_n \), \( z_{s,t,0} \in V \) for \( (s, t) \in D^2_n \cup D^1_n \), \( z_{s,t,1} \in V \) for \( (s, t) \in D^3_n \cup D^1_n \). For each \( (s, t) \in C_n \), we define \( f^{s,t,0}: G \to \mathbb{R}^N \) and \( f^{s,t,1}: G \to \mathbb{R}^N \) as
\[
f^{s,t,0}(b) = b^0_t(s, t), \quad f^{s,t,1}(b) = b^1_t(s, t).
\]
\( f^{s,t,0} \) is an \( S_n \)-equivariant map since
\[
f^{s,t,0}(\sigma \cdot b) = \sum_{S \supseteq i} b(\sigma S, \sigma T) = \sum_{S \sigma(i)} b(S, T) = b^0_{\sigma(i)}(s, t) = f^{s,t,0}(b).
\]
And \( f^{s,t,1} \) is also an \( S_n \)-equivariant map. Let \( z \in V \). Note that if \( s \neq p \) or \( t \neq q \) then \( f^{s,t,0}(z_{p,q,0}) = f^{s,t,0}(z_{p,q,1}) = f^{s,t,1}(z_{p,q,0}) = f^{s,t,1}(z_{p,q,1}) = 0 \) since \( z_{p,q,0}(S, T) = z_{p,q,1}(S, T) = 0 \) for all \( (S, T) \in (3^N)_t \). Whereas, for \( (s, t) \in D_n \setminus D^3_n \), and for \( i \in N \),
\[
f^{s,t,0}(z_{(s,t)}) = \sum_{S \supseteq i} \sum_{j \in S} z_j = \sum_{j \in N} \{((S, T) \in (3^N)_t \mid S \supseteq i, j)\} z_j
\]
\[ \begin{align*}
         & = \{ (S, T) \in (3^N)^s \mid S \ni i \} z_i + \sum_{j \in N \setminus \{i\}} \{ (S, T) \in (3^N)^s \mid S \ni j \} z_j \\
         & = \left( n - 1 \right) \left( n - s \right) \left( t \right) z_i + \left( n - 2 \right) \left( n - s \right) \left( t \right) \sum_{j \in N \setminus \{i\}} z_j \\
         & = \left( n - 1 \right) \left( n - s \right) \left( t \right) z_i - \left( n - 2 \right) \left( n - s \right) \left( t \right) z_i \\
         & = \left( \left( n - 1 \right) - \left( n - 2 \right) \right) \left( n - s \right) \left( t \right) z_i \\
         & = \left( n - 2 \right) \left( n - s \right) \left( t \right) z_i = \frac{(n-2)! (n-s)}{(s-1)! (n-s-t)} z_i,
\end{align*} \]

and
\[
 f^{s,t,0}_i(z^{(s,t,1)}) = \sum_{S \ni i} \sum_{t \in T} z_j = \sum_{j \in N} \{ (S, T) \in (3^N)^s \mid S \ni i, T \ni j \} z_j
\]
\[
 = \begin{cases} 
  \left( n - 2 \right) \left( n - s - 1 \right) \sum_{j \in N \setminus \{i\}} z_j & (s, t) \in D^1_n \cup D^4_n, \\
  0 & (s, t) \in D^2_n.
\end{cases}
\]

For \((s, t) \in D^2_n\), \(f^{s,t,0}_i(z^{(s,t,1)}) = 0\) holds since \( \{ (S, T) \in (3^N)^s \mid S \ni i, T \ni j \} = 0 \) when \(t = 0\). Here, we have
\[
 f^{s,t,0}_i(z^{(s,t,0)}) = -\frac{(n-2)!}{(s-1)!(t-1)!(n-s-t)!} z_i \text{ for } (s, t) \in D^1_n \cup D^4_n.
\]
We remark that \( f^{s,t,0}(z^{(s,t,0)}), f^{s,t,0}(z^{(s,t,1)}) \in V \).

Similarly, for \((s, t) \in D^1_n \setminus D^2_n\), we have
\[
 f^{s,t,1}_i(z^{(s,t,1)}) = \frac{(n-2)! (n-t)}{s!(t-1)!(n-s-t)!} z_i,
\]
and
\[
 f^{s,t,1}_i(z^{(s,t,0)}) = \begin{cases} 
  \frac{(n-2)!}{(s-1)!(t-1)!(n-s-t)!} z_i & (s, t) \in D^1_n \cup D^4_n, \\
  0 & (s, t) \in D^3_n.
\end{cases}
\]

Let \(p : \mathbb{R}^N \to V\) be the projection of \(\mathbb{R}^N\) onto \(V\), i.e.,
\[
p_i(x) = x_i - \frac{1}{n} \sum_{j=1}^n x_j.
\]
This projection is equivariant, sends \(U\) to zero and is an identity on \(V\).

Next, define \(L^{s,t,0} : G \to V\) as \(L^{s,t,0} = p \circ f^{s,t,0}\) and \(L^{s,t,1} : G \to V\) as \(L^{s,t,1} = p \circ f^{s,t,1}\). Recall that any \(b \in G\) is uniquely decomposed into the form of (3.1).

Case (i): \((s, t) \in D^1_n\)

Note that \(s + t = n\). By applying \(L^{s,t,0}\) to both sides of (3.1), we obtain
\[
 L^{s,t,0}(b) = \frac{(n-2)!}{(s-1)!(t-1)!(n-s-t)!} y_s = \frac{(n-2)!}{(s-1)!(t-1)!} y_s.
\]
because \( f^{s,t,0}(z^{(p,q,0)}) = f^{s,t,0}(z^{(p,q,1)}) = 0 \) for \((s,t) \neq (p,q)\), \( f^{s,t,0}(U_G) \subset U \) and \( f^{s,t,0}(W_G) = \{0\} \). Consider the right-hand side of the above equation.

\[
L_{s,t,0} = p_i(f^{s,t,0}(b)) = b_0^i(s,t) - \frac{1}{n} \sum_{l \in \mathbb{N}} b_0^l(s,t) = b_0^i(s,t) - \frac{s}{n} b(s,t) = \frac{t}{n} b_0^i(s,t) - \frac{s}{n} b_1^i(s,t).
\]

The third equality holds since

\[
\sum_{l \in \mathbb{N}} b_0^l(s,t) = \sum_{l \in \mathbb{N}} \sum_{(S,T) \in (3^n)^*_l, S \neq l} b(S,T) = \sum_{(S,T) \in (3^n)^*_l} |\{l \in N \mid l \in S\}| b(S,T) = s b(s,t)
\]

and the last equality comes from \( b(s,t) = b_0^i(s,t) + b_1^i(s,t) \). Thus

\[
(y_s)_i = \frac{(s - 1)! (t - 1)!}{n(n-2)!} \left( tb_1^i(s,t) - sb_1^i(s,t) \right).
\]

**Case (ii):** \((s,t) \in D_2^n\)

Note that \( t = 0 \). By applying \( L^{s,t,0} \) to both sides of (3.1), we obtain

\[
L^{s,t,0}(b) = \frac{(n-2)! (n-s)}{(s-1)! (n-s-t)!} z_{s,t,0} = \frac{(n-2)!}{(s-1)! (n-s-1)!} z_{s,t,0},
\]

because \( f^{s,t,0}(z^{(p,q,0)}) = f^{s,t,0}(z^{(p,q,1)}) = 0 \) for \((s,t) \neq (p,q)\), \( f^{s,t,0}(U_G) \subset U \) and \( f^{s,t,0}(W_G) = \{0\} \). Moreover,

\[
L_{s,t,0} = b_0^i(s,t) - \frac{s}{n} \sum_{l \in \mathbb{N}} b(s,t) = \frac{n-s}{n} b_0^i(s,t) - \frac{s}{n} b_{-i}(s,t).
\]

Thus, we have

\[
(z_{s,t,0})_i = \frac{(s - 1)! (n-s-1)!}{n(n-2)!} ((n-s)b_0^i(s,t) - sb_{-i}(s,t)).
\]

**Case (iii):** \((s,t) \in D_3^n\)

Similarly as Case (ii),

\[
(z_{s,t,1})_i = \frac{(t-1)! (n-t-1)!}{n(n-2)!} ((n-t)b_1^i(s,t) - tb_{-i}(0,t)).
\]

**Case (iv):** \((s,t) \in D_4^n\)

Observe that

\[
L^{s,t,0}(b) = \frac{(n-2)! (n-s)}{(s-1)! (n-s-t)!} z_{s,t,0} - \frac{(n-2)!}{(s-1)! (n-s-1)!} z_{s,t,1},
\]

\[
L^{s,t,1}(b) = \frac{(n-2)! (n-t)}{s! (t-1)! (n-s-t)!} z_{s,t,1} - \frac{(n-2)!}{(s-1)! (n-s-1)!} z_{s,t,0},
\]

because \( f^{s,t,0}(z^{(p,q,0)}) = f^{s,t,0}(z^{(p,q,1)}) = f^{s,t,1}(z^{(p,q,0)}) = f^{s,t,1}(z^{(p,q,1)}) = 0 \) for \((s,t) \neq (p,q)\), \( f^{s,t,0}(U_G) \subset U \) and \( f^{s,t,1}(W_G) = \{0\} \) for \((s,t) \neq (p,q)\). Moreover,

\[
L_{s,t,0} = b_0^i(s,t) - \frac{s}{n} b_0^i(s,t) - \frac{s}{n} b_{-i}(s,t).
\]
Solving the system of these equations, we obtain

\[
(z_{s,t,0})_i = \frac{(s-1)!(n-s-t-1)!}{n(n-2)!} \left( (n-s-t)b_i^0(s,t) - sb_{-i}(s,t) \right),
\]

\[
(z_{s,t,1})_i = \frac{s!(t-1)!(n-s-t-1)!}{n(n-2)!} \left( (n-s-t)b_i^1(s,t) - tb_{-i}(s,t) \right).
\]

\[\Box\]

**Proof of Theorem 4.1**

Let \( \phi : G \to \mathbb{R}^N \) be a solution.

(1) \(\Rightarrow\) (2):

From the decomposition (3.1) and the linearity of \( \phi \), we obtain the following expression:

\[
\phi(b) = \sum_{(s,t) \in C_n} a_{s,t} \phi(u_i^s) + \sum_{k=1}^{n-1} \phi(y_k^{(k,n-k,0)}) + \sum_{(s,t) \in D_n^3} \phi(z_{s,t,0}) + \sum_{(s,t) \in D_n^4} \phi(z_{s,t,1}) + \phi(w).
\]

Now, \( \phi(w) = 0 \) and Theorem 3.2 implies

\[
\phi(b) = \sum_{(s,t) \in C_n} a_{s,t} \alpha(s,t) \mathbf{1} + \sum_{k=1}^{n-1} \beta(k,n-k,0)y_k + \sum_{(s,t) \in D_n^3} \beta(s,t,0)z_{s,t,0} + \sum_{(s,t) \in D_n^4} \beta(s,t,1)z_{s,t,1},
\]

for some constants \( \alpha(s,t), \beta(s,t,0), \beta(s,t,1) \). Hence, for each \( i \in N \),

\[
\phi_i(b) = \sum_{(s,t) \in C_n} a_{s,t} \alpha(s,t) + \sum_{k=1}^{n-1} \beta(k,n-k,0)(y_k)_i + \sum_{(s,t) \in D_n^3} \beta(s,t,0)(z_{s,t,0})_i + \sum_{(s,t) \in D_n^4} \beta(s,t,1)(z_{s,t,1})_i.
\]

By the representation of Theorem 3.3, we have

\[
\sum_{(s,t) \in C_n} a_{s,t} \alpha(s,t) = \sum_{(s,t) \in C_n} \frac{s!(n-s-t)!}{n!} \alpha(s,t)b(s,t),
\]

\[
\sum_{k=1}^{n-1} \beta(k,n-k,0)(y_k)_i = \sum_{k=1}^{n-1} \beta(k,n-k,0) \frac{(k-1)!(n-k-1)!}{n(n-2)!} \left( (n-k)b_i^0(k,n-k) - kb_i^1(k,n-k) \right).
\]
\[
\sum_{(s,t) \in D_n^2 \cup D_n^4} \beta(s, t, 0)(z_{s,t,0})_i \\
= \sum_{(s,t) \in D_n^2 \cup D_n^4} \frac{(s - 1)!((n - s - t - 1))}{n(n - 2)!} \beta(s, t, 0) \left( (n - s - t)b_i^0(s, t) - sb_{-i}(s, t) \right), \\
\]

\[
\sum_{(s,t) \in D_n^2 \cup D_n^4} \beta(s, t, 1)(z_{s,t,1})_i \\
= \sum_{(s,t) \in D_n^2 \cup D_n^4} \frac{s!(t - 1)!((n - s - t - 1))}{n(n - 2)!} \beta(s, t, 1) \left( (n - s - t)b_i^0(s, t) - tb_{-i}(s, t) \right). \\
\]

We set
\[
\gamma(s, t) = \frac{s!(n - s - t)!}{n!} \alpha(s, t), \quad \delta(k) = \frac{(k - 1)!(n - k - 1)!}{n(n - 2)!} \beta(k, n - k, 0), \\
\zeta(s, t, 0) = \frac{(s - 1)!((n - s - t - 1))}{n(n - 2)!} \beta(s, t, 0), \quad \zeta(s, t, 1) = \frac{s!(t - 1)!((n - s - t - 1))}{n(n - 2)!} \beta(s, t, 1). \\
\]

Then, we obtain (4.1).

(2) \Rightarrow (3):

Let \( \phi \) be of the form of (4.1). Note that for all \( i \in N, b(s, t) = b_i^0(s, t) + b_i^1(s, t) + b_{-i}(s, t) \) for \( (s, t) \in C_n \), \( b_i^0(s, t) = 0 \) for \( (s, t) \in D_n^3 \), \( b_i^1(s, t) = 0 \) for \( (s, t) \in D_n^2 \) and \( b_{-i}(s, t) = 0 \) for \( (s, t) \in D_n^1 \). Then for each \( b \in G \) and \( i \in N \)

\[
\phi_i(b) = \left( \sum_{(s,t) \in C_n \setminus D_n^2} \gamma(s, t)b_i^0(s, t) + \sum_{(s,t) \in D_n^1} \delta(s)tb_i^0(s, t) \\
+ \sum_{(s,t) \in D_n^2 \cup D_n^4} \zeta(s, t, 0)(n - s - t)b_i^0(s, t) \right) \\
+ \left( \sum_{(s,t) \in C_n \setminus D_n^2} \gamma(s, t)b_i^1(s, t) - \sum_{(s,t) \in D_n^1} \delta(s)b_i^1(s, t) \\
+ \sum_{(s,t) \in D_n^2 \cup D_n^4} \zeta(s, t, 1)(n - s - t)b_i^0(s, t) \right) \\
- \left( - \sum_{(s,t) \in D_n \setminus D_n^1} \gamma(s, t)b_{-i}(s, t) + \sum_{(s,t) \in D_n^2 \cup D_n^4} \zeta(s, t, 0)b_{-i}(s, t) \\
+ \sum_{(s,t) \in D_n^2 \cup D_n^4} \zeta(s, t, 1)b_{-i}(s, t) \right) \\
= \left( \gamma(n, 0)b_i^0(n, 0) + \sum_{(s,t) \in D_n^1} \gamma(s, t) + \delta(s)t \right) b_i^0(s, t) \\
+ \sum_{(s,t) \in D_n^2 \cup D_n^4} \gamma(s, t) + \zeta(s, t, 0)(n - s - t) b_i^0(s, t) \right)
\]
+ \left( \gamma(0, n) b_1^1(0, n) + \sum_{(s, t) \in D^1_n} (\gamma(s, t) - \delta(s) s) b_1^1(s, t) \right) \\
+ \sum_{(s, t) \in D^2_n \cup D^4_n} (\gamma(s, t) + \zeta(s, t, 1)(n - s - t)) b_1^1(s, t) \\
- \sum_{(s, t) \in D_n \setminus D^3_n} (-\gamma(s, t) + \zeta(s, t, 0)s + \zeta(s, t, 1)t) b_{-1}(s, t).

Then we have (4.2) by setting \( p(s, t), q(s, t) \) and \( r(s, t) \) as follows: for \( (s, t) = (n, 0) \) and \( (s, t) \in D_n \setminus D^3_n \),

\[
p(s, t) = \begin{cases} 
\gamma(s, t) & (s, t) = (n, 0), \\
\gamma(s, t) + \delta(s)t & (s, t) \in D^1_n, \\
\gamma(s, t) + \zeta(s, t, 0)(n - s - t) & (s, t) \in D^2_n \cup D^4_n,
\end{cases}
\]

for \( (s, t) = (0, n) \) and \( (s, t) \in D_n \setminus D^2_n \),

\[
q(s, t) = \begin{cases} 
\gamma(s, t) & (s, t) = (0, n), \\
\gamma(s, t) - \delta(s)s & (s, t) \in D^1_n, \\
\gamma(s, t) + \zeta(s, t, 1)(n - s - t) & (s, t) \in D^3_n \cup D^4_n,
\end{cases}
\]

for \( (s, t) \in D_n \setminus D^1_n \),

\[
r(s, t) = -\gamma(s, t) + \zeta(s, t, 0)s + \zeta(s, t, 1)t.
\]

(3) \Rightarrow (1): Linearity is straightforward and symmetry holds since for each \( i \in N \)

\[
\phi_i(\sigma \cdot b) = \sum_{(s, t) \in C_n, s > 0} p(s, t) \sum_{S \ni i} b(\sigma S, \sigma T) + \sum_{(s, t) \in C_n, t > 0} q(s, t) \sum_{T \ni i} b(\sigma S, \sigma T) \\
- \sum_{(s, t) \in C_n, s + t < n} r(s, t) \sum_{S \ni i} b(\sigma S, \sigma T)
\]

\[
= \sum_{(s, t) \in C_n, s > 0} p(s, t) \sum_{S \ni \sigma(i)} b(S, T) + \sum_{(s, t) \in C_n, t > 0} q(s, t) \sum_{T \ni \sigma(i)} b(S, T) \\
- \sum_{(s, t) \in C_n, s + t < n} r(s, t) \sum_{S, T \ni \sigma(i)} b(S, T)
\]

\[
= \sum_{(s, t) \in C_n, s > 0} p(s, t) b_{\sigma(i)}^0(s, t) + \sum_{(s, t) \in C_n, t > 0} q(s, t) b_{\sigma(i)}^1(s, t) \\
- \sum_{(s, t) \in C_n, s + t < n} r(s, t) b_{-\sigma(i)}(s, t)
\]

\[
= \phi_{\sigma(i)}(b).
\]
Solutions for games by representation

Proof of Theorem 4.2
Let $\phi : G \to \mathbb{R}^N$ be a solution.

(1) $\Rightarrow$ (2):
The same proof of (1) $\Rightarrow$ (2) in Theorem 4.1 but $\alpha(s, t)$ are determined by (4.3).

(2) $\Rightarrow$ (3):
The same proof of (2) $\Rightarrow$ (3) in Theorem 4.1 but $\gamma(s, t)$ are determined by $\alpha(s, t)$ and (4.3).

(3) $\Rightarrow$ (1):
Theorem 4.1 (3) $\Rightarrow$ (1) implies linearity and symmetry. Efficiency is shown in the paragraph below Theorem 4.2.

Proof of Theorem 4.3
Let $\phi$ be of the form of (4.7). Then we can show linearity and symmetry by the same way of the proof of Theorem 4.1 (3) $\Rightarrow$ (1). If $i \in N$ is null for $b \in G$, then

$$
\phi_i(b) = \sum_{(S,T) \in \mathbb{R}^N \setminus \{i\}} \left( p(s + 1, t)(b(S \cup \{i\}, T) - b(S, T)) + q(s, t + 1)(b(S, T \cup \{i\}) - b(S, T)) \right)
$$

$$
= \sum_{(S,T) \in \mathbb{R}^N \setminus \{i\}} \left( p(s + 1, t)(b(S, T) - b(S, T)) + q(s, t + 1)(b(S, T) - b(S, T)) \right)
$$

$$
= 0.
$$

We show that (4.2) with the null axiom must be the first form of (4.6).

For every $(K, L) \in 3^N \setminus \{\{0, 0\}\}$, we define games $D_{K,L} \in G$ as follows: for $(S, T) \in 3^N \setminus \{i\},$

$$
D_{K,L}(S, T) = \begin{cases} 1 & (S, T) = (K, L) \\ 0 & (S, T) \neq (K, L) \end{cases},
$$

and

$$
D_{K,L}(S \cup \{i\}, T) = D_{K,L}(S, T), \quad D_{K,L}(S, T \cup \{i\}) = D_{K,L}(S, T).
$$

By the null axiom, we have $\phi_i(D_{K,L}) = 0$. Let $k = |K|$ and $l = |L|$. Additionally, from (4.2), we necessarily have the following equation:

$$
\phi_i(D_{K,L}) = p(k + 1, l) + q(k, l + 1) - r(k, l) = 0
$$

for any $k, l$ such that $k + l < n$. We reformulate (4.2) as follows:

$$
\phi_i(b) = \sum_{(s,t) \in \mathbb{C}_n, s+t<n} (p(s + 1, t)b^0_i(s + 1, t) + q(s, t + 1)b^1_i(s, t + 1) - r(s, t)b_{-i}(s, t)).
$$

Combining it with the above equation, we obtain (4.6). Equivalence between (4.6) and (4.7) holds because

$$
\sum_{(s,t) \in \mathbb{C}_n, s+t<n} (p(s + 1, t)(b^0_i(s + 1, t) - b_{-i}(s, t)) + q(s, t + 1)(b^1_i(s, t + 1) - b_{-i}(s, t)))
$$

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\[
\begin{align*}
&= \sum_{(s,t) \in C_n, s+t < n} \left( p(s+1, t) \sum_{(S,T) \in 3^{N \setminus \{i\}}, |S|=s, |T|=t} (b(S \cup \{i\}, T) - b(S, T)) \\
&\quad + q(s, t+1) \sum_{(S,T) \in 3^{N \setminus \{i\}}, |S|=s, |T|=t} (b(S, T \cup \{i\}) - b(S, T)) \right) \\
&= \sum_{(s,t) \in C_{n-1}} \sum_{(S,T) \in 3^{N \setminus \{i\}}, |S|=s, |T|=t} (p(s+1, t)(b(S \cup \{i\}, T) - b(S, T)) \\
&\quad + q(s, t+1)(b(S, T \cup \{i\}) - b(S, T))) \\
&= \sum_{(S,T) \in 3^{N \setminus \{i\}}} (p(s+1, t)(b(S \cup \{i\}, T) - b(S, T)) + q(s, t+1)(b(S, T \cup \{i\}) - b(S, T))).
\end{align*}
\]

\[\square\]

**Proof of Corollary 4.4**

First, we remark that the axiomatic characterization of the Shapley value in ([4], pp. 112–114, Theorem 7) is composed of linearity, symmetry, efficiency, dummy and structural axioms. In this corollary, the dummy axiom is replaced with the null axiom. The dummy axiom (see [4]) implies the null axiom. Therefore, the result of [4] entails that the Shapley value satisfies linearity, symmetry, efficiency, null and structural axioms. Hence, we only show the sufficiency. This proof of the sufficiency is essentially the same as that of [4], despite the difference of dummy and null axioms.

If a solution \( \phi \) satisfies linearity, symmetry, efficiency, null axioms, by Theorem 4.4, it is expressed as follows: for \( i \in N \)

\[
\phi_i(b) = \sum_{(S,T) \in 3^{N \setminus \{i\}}} \left( p(s+1, t)(b(S \cup \{i\}, T) - b(S, T)) \\
+ q(s, t+1)(b(S, T \cup \{i\}) - b(S, T)) \right),
\]

where

\[
\begin{align*}
|p(n,0) = 1/n, \\
q(0,n) = -1/n, \\
sp(s,0) - (n-s)(p(s+1,0) + q(s,1)) = 0, & \quad 0 < s < n, \\
tq(0,t) - (n-t)(p(1,t) + q(0,t+1)) = 0, & \quad 0 < t < n, \\
sp(s,t) + tq(s,t) = 0, & \quad s+t = n, s,t > 0, \\
sp(s,t) + tq(s,t) - (n-s-t)(p(s+1,t) + q(s,t+1)) = 0, & \quad s + t < n, s,t > 0.
\end{align*}
\]

Furthermore, when \( \phi \) satisfies the structural axiom, we have

\[
\phi_j(C_{S,T}) = -2\phi_i(C_{S,T\cup\{i\}}) \implies p(s, t) = -2q(s, t+1), \tag{5.2}
\]

for \( (S, T) \in 3^{N \setminus \{i\}} \setminus \{(\emptyset, \emptyset)\} \) such that \( j \in S \), i.e. \((s, t) \in C_n\) such that \( s + t < n \) and \( s > 0 \), and

\[
\phi_k(C_{S,T}) = -2\phi_i(C_{S\cup\{i\},T}) \implies q(s, t) = -2p(s+1, t), \tag{5.3}
\]
for \((S, T) \in 3^N \setminus \{(0, 0)\}\) such that \(h \in T\), i.e. \((s, t) \in C_n\) such that \(s + t < n\) and \(t > 0\). Combining (5.1)–(5.3), we obtain

\[
p(s, t) = \begin{cases} 
\frac{1}{n} & \text{if } s = n, t = 0, \\
\frac{n-s}{n+s}2p(s+1, 0) & \text{if } 0 < s < n, t = 0, \\
t^1 & \text{if } s + t = n, s, t > 0, \\
\frac{s}{2}^12p(s, t-1) & \text{if } s + t = n, s, t > 0, \\
\frac{n-s+t}{n+s-t}2p(s+1, t) & \text{if } s + t < n, s, t > 0.
\end{cases}
\]

These recurrence relations uniquely determine \(p(s, t)\) for all \((s, t) \in C_n\) such that \(s > 0\), since each \(p(s, t)\) is recursively determined by \(p(s+1, t)\) when \(s + t < n\) or \(p(s, t-1)\) when \(s + t = n\), and there is a boundary condition \(p(n, 0) = 1/n\). We can check the following function satisfies the recurrence relations:

\[
p(s, t) = \frac{(n+s-t-1)!\ (n+t-s)!\ (2^n-s-t+1)}{(2n)!}, \text{ for } (s, t) \in C_n, \ s > 0.
\]

Similarly, the recurrence relations for \(q(s, t)\) is obtained as follows:

\[
q(s, t) = \begin{cases} 
\frac{-1}{n} & \text{if } s = 0, t = n, \\
\frac{n-t}{n+t}2p(0, t+1) & \text{if } s = 0, 0 < t < n, \\
\frac{s}{1}^12p(s-1, t) & \text{if } s + t = n, s, t > 0, \\
\frac{n+s-t}{n+s+t}2q(s, t+1) & \text{if } s + t < n, s, t > 0.
\end{cases}
\]

The function that is determined by these recurrence relations is obtained as follows:

\[
q(s, t) = -\frac{(n+t-s-1)!\ (n+s-t)!\ (2^n-t-s+1)}{(2n)!}, \text{ for } (s, t) \in C_n, t > 0.
\]

\[\square\]

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Yoshifumi Kusunoki
Osaka Prefecture University
Gakuen-cho 1-1, Naka-ku, Sakai, Osaka 599-8531, Japan
E-mail: yoshifumi.kusunoki@kis.osakafu-u.ac.jp

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