UNEXPECTED LOCAL EXTREMA
FOR THE SENDOV CONJECTURE

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Abstract. Let $S(n)$ be the set of all polynomials of degree $n$ with all roots in the unit disk, and define $d(P)$ to be the maximum of the distances from each of the roots of a polynomial $P$ to that root’s nearest critical point. In this notation, Sendov’s conjecture asserts that $d(P) \leq 1$ for every $P \in S(n)$.

Define $P \in S(n)$ to be locally extremal if $d(P) \geq d(Q)$ for all nearby $Q \in S(n)$, and note that maximizing $d(P)$ over all locally extremal polynomials $P$ would settle the Sendov conjecture.

Prior to now, the only polynomials known to be locally extremal were of the form $P(z) = c(z^n + a)$ with $|a| = 1$. In this paper, we determine sufficient conditions for real polynomials of the form

$$P(z) = \int_0^z (w-a)^{n-3}(w^2 + bw + c) \, dw \quad \text{with} \quad 0 < \beta < 1$$

to be locally extremal, and we use these conditions to find locally extremal polynomials of this form of degrees $n = 8, 9, 12, 13, 14, 15, 19, 20, \text{and} 26$.

1. Introduction

In 1958, Sendov conjectured that if a polynomial (with complex coefficients) has all its roots in the unit disk, then within one unit of each of its roots lies a root of its derivative. This conjecture has yet to be settled, although it has been the subject of more than 80 papers over the intervening years, and so has been verified for many special cases. These have been documented by Sendov [8], Schmeisser [6], Sheil-Small [9, Chapter 6] and Rahman and Schmeisser [5, Section 7.3].

Let $n \geq 2$ be an integer and let $S(n)$ be the set of all complex polynomials of degree $n$ with all roots in the unit disk. For a polynomial $P$ with roots $z_1, \ldots, z_n$ and critical points $\zeta_1, \ldots, \zeta_{n-1}$, define

$$d(P) = \max_{1 \leq i \leq n} \left\{ \min_{1 \leq j \leq n-1} |z_i - \zeta_j| \right\}.$$

If $P \in S(n)$, then the Gauss-Lucas Theorem [5, Theorem 2.1.1] implies that each $d(P) \leq 2$, and Sendov’s conjecture asserts that each $d(P) \leq 1$.

We will say that a polynomial $P$ is expected if it is of the form $P(z) = c(z^n + a)$ with $|a| = 1$. In 1972, Phelps and Rodriguez defined a polynomial $P \in S(n)$ to be extremal if $d(P) = \sup\{d(Q) : Q \in S(n)\}$, and conjectured [4] after Theorem 5] that extremal polynomials are all expected. Since any expected polynomial $P$ has $d(P) = 1$, this conjecture implies Sendov’s conjecture. The Phelps-Rodriguez...
conjecture has also been verified for a number of special cases, as documented by Rahman and Schmeisser [5, Section 7.3].

Define an $\varepsilon$-neighborhood of $P \in S(n)$ to be the set of all polynomials $Q \in S(n)$ whose roots are within $\varepsilon$ of the roots of $P$ (in the sense that the roots of $Q$ can be paired with the roots of $P$ so that in each pair, the difference of the roots has a modulus less than $\varepsilon$). Define a polynomial $P \in S(n)$ to be locally extremal if $d(P) \geq d(Q)$ for all $Q$ in some $\varepsilon$-neighborhood of $P$, and note that maximizing $d(P)$ over all locally extremal polynomials $P$ would settle the Sendov conjecture.

The expected polynomials are all locally extremal, as was demonstrated by Vaja and Zaharescu [10] and by Miller [3, Theorem 3]. Given this, it is tempting to approach Sendov’s conjecture by trying to show that all locally extremal polynomials must be expected. Indeed, Schmieder has made several such attempts [7], although Borcea has shown [2, section 1] that each contains a technical flaw.

In this paper, we prove

**Theorem 1.** For each $n \in \{8, 9, 12, 13, 14, 15, 19, 20, 26\}$, there are locally extremal polynomials $P \in S(n)$ of the form

$$P(z) = \int_{\beta}^{z} (w - a)^{n-3}(w^2 + bw + c) \, dw \quad \text{with } 0 < \beta < 1.$$  

Note that polynomials of this form would not be expected, as each has a root $\beta$ that is not on the unit circle. Thus Theorem 1 implies that there are locally extremal polynomials that are not expected.

Theorem 1 has several important consequences. First, it shows that the technical flaws in Schmieder’s approach cannot be patched. Second, the structure of these polynomials (in particular, the multiple critical point) identifies circumstances that potential variational proofs of Sendov’s conjecture will need to address. Finally, finding these polynomials is a significant new step in identifying all local extrema.

In section 2, we list eight properties that are sufficient for a polynomial to be locally extremal, and in sections 3 and 4 we verify that these properties suffice. In section 5, we describe how to construct polynomials that satisfy all these properties, and in section 6 we list the resulting polynomials, thereby verifying Theorem 1.

**2. Properties**

In section 4 we prove that for a real polynomial $P$ of degree $n \geq 5$ to be locally extremal, it suffices for it to satisfy 8 properties, beginning with the following.

**A:** All roots of $P$ lie in the closed unit disk.

**B:** All roots of $P$ that are on the unit circle are simple.

**C:** $P$ has a root at $\beta$, with $0 < \beta < 1$.

**D:** All critical points of $P$ lie on a circle of positive radius centered at $\beta$.

**E:** $P$ has a real critical point $a < \beta$ of order $n - 3$.

Let $z_1, \ldots, z_n$ be the roots of $P$, numbered so that $z_1, \ldots, z_m$ are on the unit circle. Let $\zeta_1, \ldots, \zeta_{n-1}$ be the critical points of $P$, numbered (as allowed by property [E]) so that $\zeta_j = a$ for $j \geq 3$.

Note that property [E] implies that $P$ has a simple root at $\beta$. Our next property is

**F:** For each $z_i \neq \beta$, we have $\min_{1 \leq j \leq n-1} |z_i - \zeta_j| < \min_{1 \leq j \leq n-1} |\beta - \zeta_j| < 1$.  

To examine the effects of changing $\beta$ and the $\zeta_j$ by small amounts, denoted by $\Delta \beta$ (which we will require to be real) and $\Delta \zeta_j$, we will use the following notation.

$$E_k = -\Re \left[ \frac{\Delta \zeta_j - \Delta \beta}{\zeta_j - \beta} \right]$$
for $k = 1$ and $k = 2$,

$$F_k = 0$$
for $k = 1$ and $k = 2$,

$$E_3 = \frac{\Re \left[ \sum_{j=3}^{n-1} \Delta \zeta_j \right] - (n - 3) \Delta \beta}{\beta - a},$$
and

$$F_3 = -\frac{\sum_{j=3}^{n-1} (\Im \Delta \zeta_j)^2}{2(\beta - a)^2}.$$  

(2.1)

Since (up to a constant multiple) $P'(z) = \prod_{j=1}^{n-1} (z - \zeta_j)$ and by property $\text{C}$ we have $P(z) = \int_{\beta}^{z} P'(w) \, dw$, then the roots of $P$ are functions of $\beta$ and the $\zeta_j$. By property $\text{B}$ the roots of $P$ that are on the unit circle (being simple) are differentiable functions of $\beta$ and the $\zeta_j$.

Recall that $z_1, \ldots, z_m$ are on the unit circle. For $i = 1, \ldots, m$ define

$$E_{i+3} = \Re \left[ \frac{1}{z_i} \frac{\partial z_i}{\partial \beta} \right] \Delta \beta + \sum_{j=1}^{2} \Re \left[ \frac{1}{z_i} \frac{\partial z_i}{\partial \zeta_j} \right] + \Re \left[ \frac{1}{z_i} \frac{\partial z_i}{\partial \zeta_3} \sum_{j=3}^{n-1} \Delta \zeta_j \right]$$
and

$$F_{i+3} = -\Re \left[ \frac{1}{2z_i} P'(z_i) \int_{\beta}^{z_i} \frac{P'(w) \, dw}{(w - a)^2} \right] \sum_{j=3}^{n-1} (\Im \Delta \zeta_j)^2.$$  

Note that for a fixed polynomial $P$, the $m + 3$ expressions $E_k$ are all linear in the 7 real “variables” $\Delta \beta$, $\Re[\Delta \zeta_1]$, $\Im[\Delta \zeta_1]$, $\Re[\Delta \zeta_2]$, $\Im[\Delta \zeta_2]$, $\Re[\sum_{j=3}^{n-1} \Delta \zeta_j]$ and $\Im[\sum_{j=3}^{n-1} \Delta \zeta_j]$, and that the $m + 3$ expressions $F_k$ are all constant multiples of the real “variable” $\sum_{j=3}^{n-1} (\Im \Delta \zeta_j)^2$. Our final two properties are

$\text{G}$: There are constants $c_k > 0$ (depending on $P$, but independent of the 8 “variables”) so that the sums $\sum_{k=1}^{m+3} c_k E_k = 0$ and $\sum_{k=1}^{m+3} c_k F_k = \sum_{j=3}^{n-1} (\Im \Delta \zeta_j)^2$.

$\text{H}$: The coefficient matrix of the system $\{E_k = 0 : k = 1, \ldots, m + 3\}$ in our 7 “variables” is of rank 7.

### 3. Preliminary Calculations

We will show that a real polynomial $P$ of degree $n \geq 5$ satisfying properties $\text{A}, \text{H}$ is locally extremal, as follows.

Define $r$ to be the radius of the circle in property $\text{D}$ and recall that (up to a constant multiple) $P(z) = \prod_{j=1}^{n-1} (w - \zeta_j) \, dw$. From our properties $\text{A}$ and $\text{D}$ we know that $P$ has all its roots in the closed unit disk and all its critical points on a circle of radius $r > 0$ centered at $\beta$. Define the $n$-tuple $(\Delta \beta, \Delta \zeta_1, \ldots, \Delta \zeta_{n-1}) \in C^n$ to be an improvement of $P$ if $\Delta \beta$ is real and if the polynomial

$$\int_{\beta + \Delta \beta}^{z} \prod_{j=1}^{n-1} [w - (\zeta_j + \Delta \zeta_j)] \, dw$$

(3.1)

has all its roots in the closed unit disk and all its critical points strictly outside the circle of radius $r$ centered at $\beta + \Delta \beta$. By property $\text{F}$ we know that $r < 1$, so there is at least one improvement, namely $(1 - \beta, -\zeta_1, \ldots, -\zeta_{n-1})$. 
For an improvement $I = (\Delta \beta, \Delta \zeta_1, \ldots, \Delta \zeta_{n-1})$, define

$$||I|| = \left( |\Delta \beta|^2 + \sum_{j=1}^{n-1} |\Delta \zeta_j|^2 \right)^{1/2}. $$

Note that $||I|| > 0$, for if $||I|| = 0$, then the critical points of $S$ would be on (and thus not strictly outside) the circle of radius $r$ centered at $\beta + \Delta \beta$.

Take any $\epsilon$ with $0 < \epsilon < \beta/2$ and recall our definition of $\epsilon$-neighborhood from section [1]. The critical points of $P$ are continuous functions of the roots of $P$, so there is a $\delta > 0$ such that for all polynomials $Q$ in a $\delta$-neighborhood of $P$, the roots and critical points of $Q$ are within $\epsilon$ of the roots and critical points of $P$. By Property [F] we know that $d(P) = r$, so if $P$ is not locally extremal, then there is a polynomial $\tilde{P} \in S(n)$ in the $\delta$-neighborhood of $P$ with $d(\tilde{P}) > r$. Let $\beta$ be the root of $\tilde{P}$ paired with $\beta$ (so $|\tilde{\beta} - \beta| < \epsilon$), and for $j = 1, \ldots, n-1$, let $\hat{\zeta}_j$ be the critical point of $\tilde{P}$ paired with $\zeta_j$ (so $|\hat{\zeta}_j - \zeta_j| < \epsilon$). Note that $|\hat{\beta}| \geq |\beta| - |\beta - \beta| \geq \beta/2 > 0$, define $u = |\beta|/|\beta|$ and note that $|u - 1| = |\hat{\beta} - \hat{\beta}|/|\hat{\beta}| \leq 2\epsilon/|\beta/2|$. Since $|u| = 1$, then the transformation $z \rightarrow uz$ is a rotation about the origin, so $I = (u\beta - \beta, u\hat{\zeta}_1 - \zeta_1, \ldots, u\hat{\zeta}_{n-1} - \zeta_{n-1})$ is an improvement of $P$. Note that each $|u\hat{\zeta}_j - \zeta_j| \leq |u||\hat{\zeta}_j - \zeta_j| + |\zeta_j|(|u - 1| \leq C\epsilon$ for some constant $C$ (and similarly for $|u\beta - \beta|$), so $||I||$ can be made arbitrarily small.

We have just seen that if $P$ is not locally extremal then there are improvements $I$ of $P$ with $||I||$ arbitrarily small. This means that we can show that $P$ is locally extremal by proving

**Theorem 2.** If $P$ is a real polynomial of degree $n \geq 5$ that satisfies properties [A]-[H] then there is a constant $A > 0$ so that every improvement $I$ of $P$ has $||I|| \geq A$.

We begin with some preliminary calculations. For $t > 0$, we’ll say that a (real or complex) quantity is $O(t)$ if its modulus is bounded by a constant multiple of $t$. Given this, we have

**Lemma 3.** If $z = O(t)$ then

\begin{align*}
(1) \quad |1 + z| &= 1 + \Re[z] + O(t^2), \quad \text{and} \\
(2) \quad |1 + z| &= 1 + \Re[z] + (1/2)(\Im[z])^2 + O(t^3). 
\end{align*}

**Proof.** Note that $\Re[z] = O(t)$ and $|z|^2 = O(t^2)$. The results follow from the equality $|1 + z| = (1 + 2\Re[z] + |z|^2)^{1/2}$, the Taylor series $(1 + s)^{1/2} = 1 + s/2 - s^2/8 + O(s^3)$ and the substitution $s = 2\Re[z] + |z|^2$. \qed

We will be examining the relationships between the roots and critical points of $P$. We calculate the partial derivatives of these relationships with
Lemma 4. Write $P'(z) = \prod_{j=1}^{n-1}(z - \zeta_j)$ and suppose that $z_i$ is a simple root of $P(z) = \int_1^\infty P'(w) \, dw$. Then

1. $z_i$ is an analytic function of $\beta, \zeta_1, \ldots, \zeta_{n-1}$,
2. $\frac{\partial z}{\partial \beta} = \frac{P'(\beta)}{P'(z_i)}$, 
3. $\frac{\partial z_i}{\partial \zeta_j} = \frac{1}{P'(z_i)} \int_{\beta}^{z_i} \frac{P'(w)}{w - \zeta_j} \, dw$, 
4. $\frac{\partial^2 z_i}{\partial \zeta_j \partial \zeta_k} = \frac{2}{z_i - \zeta_j} \frac{\partial z_i}{\partial \zeta_j} - \frac{P''(z_i)}{P'(z_i)} \left( \frac{\partial z_i}{\partial \zeta_j} \right)^2$ and 
5. $\frac{\partial^2 z_i}{\partial \zeta_j \partial \zeta_k} = \frac{1}{z_i - \zeta_k} \frac{\partial z_i}{\partial \zeta_j} + \frac{1}{z_i - \zeta_j} \frac{\partial z_i}{\partial \zeta_k} - \frac{P''(z_i)}{P'(z_i)} \frac{\partial z_i}{\partial \zeta_j} \frac{\partial z_i}{\partial \zeta_k} - \frac{1}{P'(z_i)} \int_{\beta}^{z_i} \frac{P'(w) \, dw}{(w - \zeta_j)(w - \zeta_k)}$ for $j \neq k$.

Proof. We may assume that $z_i \neq \beta$ (else the results would be trivially true.) Proofs of parts 1–3 can be found in [1, Lemmas 2.1 and 2.3]. Part 4 can be established by writing $P'(z) = (z - \zeta_j)Q(z)$ (with $Q$ independent of $\zeta_j$) and calculating the second partial derivative (with respect to $\zeta_j$) of

$$0 = \int_{\beta}^{z_i} w Q(w) \, dw - \zeta_j \int_{\beta}^{z_i} Q(w) \, dw.$$

Part 5 can likewise be established by writing $P'(z) = (z - \zeta_j)(z - \zeta_k)Q(z)$ and calculating the mixed second partial derivative. □

4. Proof of Theorem 2

Recall that $P$ is a real polynomial of degree $n \geq 5$ that satisfies properties [A] [H]. We will prove Theorem 2 in two stages, first by estimating the roots and critical points of $P$ with linear approximations, and then by improving our estimates with quadratic approximations.

We begin by examining the conclusions that can be drawn from linear approximations to the roots and critical points of $P$ with

Proposition 5. If $I = (\Delta \beta, \Delta \zeta_1, \ldots, \Delta \zeta_{n-1})$ is an improvement of $P$ and $\Delta t = ||I||$, then $\Delta \beta = O(\Delta t^2)$, $\Delta \zeta_j = O(\Delta t^2)$ for $j \leq 2$, $\Re[\Delta \zeta_j] = O(\Delta t^2)$ for $j \geq 3$ and $\sum_{j=3}^{n-1} \Delta \zeta_j = O(\Delta t^2)$.

Proof. From the definition of $\Delta t$ we know that $\Delta \beta = O(\Delta t)$ and that each $\Delta \zeta_j = O(\Delta t)$. Since $I$ is an improvement, then each $|\zeta_j - (\zeta_j - \beta)| > r$, so $|1 + (\Delta \zeta_j - \Delta \beta)/\zeta_j - \beta| > r/|\zeta_j - \beta| = 1$, so using part 1 of Lemma 3 gives us that each

$$-\Re\left[ \frac{\Delta \zeta_j - \Delta \beta}{\zeta_j - \beta} \right] \leq O(\Delta t^2).$$

Recalling the expressions $E_k$ defined in (2.1), this gives us two inequalities $E_1 \leq O(\Delta t^2)$ and $E_2 \leq O(\Delta t^2)$. Adding (4.1) for $j = 3, \ldots, n - 1$ and recalling that $\zeta_j = a$ is real for $j \geq 3$ gives us a third inequality $E_3 \leq O(\Delta t^2)$. 


By part 1 of Lemma 2, the roots of our improved polynomial (4.1) that originate from the simple roots $z_i$ of $P$ are analytic functions of $\beta, \zeta_1, \ldots, \zeta_{n-1}$, so each such root is of the form $z_i + \Delta z_i$, with

$$
\Delta z_i = \frac{\partial z_i}{\partial \beta} \Delta \beta + \sum_{j=1}^{2} \frac{\partial z_i}{\partial \zeta_j} \Delta \zeta_j + \sum_{j=3}^{n-1} \frac{\partial z_i}{\partial \zeta_j} \Delta \zeta_j + O(\Delta t)^2.
$$

Note that each $\Delta z_i = O(\Delta t)$. Since $I$ is an improvement, then each $|z_i + \Delta z_i| \leq 1$. If $|z_i| = 1$, then $|1 + \Delta z_i/z_i| \leq 1/|z_i| = 1$ so using part 1 of Lemma 3 gives us $\Re[\Delta z_i/z_i] \leq O(\Delta t)^2$ and thus we have inequalities $E_{i+3} \leq O(\Delta t)^2$ for $i = 1, \ldots, m$.

Thus we consider the system $\{E_k = O(\Delta t)^2 : k = 1, \ldots, m+3\}$. By property $H$ the coefficient matrix of this system is of rank 7, and so solving this system shows that the values of our 7 “variables” are all $O(\Delta t)^2$. Thus we can conclude that $\Delta \beta = O(\Delta t)^2$, that $\Delta \zeta_j = O(\Delta t)^2$ for $j \leq 2$, and that $\sum_{j=3}^{n-1} \Delta \zeta_j = O(\Delta t)^2$.

Suppose that $j \geq 3$ and note that $\sum_{k=3}^{n-1} \Re[\Delta \zeta_k] = O(\Delta t)^2$. Since by property $H$ we know that $\zeta_j - \beta = a - \beta < 0$, and since $\Delta \beta = O(\Delta t)^2$, then (4.1) implies that each $\Re[\Delta \zeta_j] \leq O(\Delta t)^2$, so each

$$
\Re[\Delta \zeta_j] = -\sum_{j=3}^{n-1} \Re[\Delta \zeta_k] + O(\Delta t)^2 \geq O(\Delta t)^2
$$

and so each $\Re[\Delta \zeta_j] = O(\Delta t)^2$. This finishes the proof of Proposition 5.

At this point, for $j \geq 3$ we know only that each $\Delta \zeta_j = O(\Delta t)$. We can improve this estimate by looking at quadratic approximations to the roots and critical points of $P$ with

**Proposition 6.** If $I = (\Delta \beta, \Delta \zeta_1, \ldots, \Delta \zeta_{n-1})$ is an improvement of $P$ and $\Delta t = ||I||$, then each $\Delta \zeta_j = O(\Delta t)^{3/2}$.

**Proof.** Note that the hypotheses of Proposition 5 are satisfied, so we may use all of its conclusions. In particular, we know that $\Delta \zeta_j = O(\Delta t)^2$ for $j \leq 2$ and that $\Re[\Delta \zeta_j] = O(\Delta t)^2$ for $j \geq 3$, so to verify Proposition 6 we need only show that $\Im[\Delta \zeta_j] = O(\Delta t)^{3/2}$ for $j \geq 3$. We will do this by repeating the calculations of Proposition 5 but working now to $O(\Delta t)^3$.

Since $I$ is an improvement, then each $|1 + (\Delta \zeta_j - \Delta \beta)/(\zeta_j - \beta)| > 1$, so using part 2 of Lemma 3 gives us

$$
(4.2) \quad -\Re \left[ \frac{\Delta \zeta_j - \Delta \beta}{\zeta_j - \beta} \right] - \frac{1}{2} \left( \Im \left[ \frac{\Delta \zeta_j - \Delta \beta}{\zeta_j - \beta} \right] \right)^2 \leq O(\Delta t)^3.
$$

Define inequalities 1 and 2 by evaluating (4.2) for $j = 1$ and $j = 2$ respectively. By Proposition 5 for $j \leq 2$ we have $\Delta \zeta_j - \Delta \beta = O(\Delta t)^2$, so inequalities 1 and 2 are $E_k + F_k \leq O(\Delta t)^3$ for $k = 1, 2$. 

$$
E_i = (-1/c_i) \sum_{i \neq k=1}^{m+3} c_k E_k \geq O(\Delta t)^2
$$

so each $E_i = O(\Delta t)^2$.
Define inequality 3 to be the sum of \(\frac{1}{2}\) for \(j = 3, \ldots, n - 1\). Now each \(\zeta_j - \beta = -(\beta - a)\) is real for \(j \geq 3\), and each \(\Im[\Delta \zeta_j - \Delta \beta] = \Im[\Delta \zeta_j]\), so inequality 3 can be written as \(E_3 + F_3 \leq O(\Delta t)^3\).

Each root of our improved polynomial is of the form \(z_i + \Delta z_i\). A quadratic approximation to \(\Delta z_i\) includes terms of the form \(\Delta \beta \Delta \zeta_j = O(\Delta t)^3\) and (for \(j \leq 2\) or \(k \leq 2\)) \(\Delta \zeta_j \Delta \zeta_k = O(\Delta t)^3\), which are absorbed into the \(O(\Delta t)^3\) when we write

\[
\Delta z_i = \frac{\partial z_i}{\partial \beta} \Delta \beta + \sum_{j=1}^{2} \frac{\partial z_i}{\partial \zeta_j} \Delta \zeta_j + \frac{1}{2} \sum_{j=3}^{n-1} \sum_{k=3}^{n-1} \frac{\partial^2 z_i}{\partial \zeta_j \partial \zeta_k} \Delta \zeta_j \Delta \zeta_k + O(\Delta t)^3.
\]

Note that Proposition 5 implies that each \(\Delta z_i = O(\Delta t)^2\). Now \(\zeta_j = a\) for \(j \geq 3\), so Lemma 4 shows that for \(j \geq 3\) and \(k \geq 3\) we have

\[
\frac{\partial^2 z_i}{\partial \zeta_j \partial \zeta_k} - \frac{\partial^2 z_i}{\partial \zeta_3 \partial \zeta_4} = \begin{cases} 
\frac{1}{P'(z_i)} \int_{\beta}^{z_i} \frac{P'(w) \, dw}{(w-a)^2} & \text{if } j = k \\
0 & \text{if } j \neq k.
\end{cases}
\]

From Proposition 5 we know that

\[
\sum_{j=3}^{n-1} \sum_{k=3}^{n-1} \Delta \zeta_j \Delta \zeta_k = \left( \sum_{j=3}^{n-1} \Delta \zeta_j \right)^2 = O(\Delta t)^4
\]

and that \((\Delta \zeta_j)^2 = -2(\Im(\Delta \zeta_j))^2 + O(\Delta t)^3\) for \(j \geq 3\), so

\[
\sum_{j=3}^{n-1} \sum_{k=3}^{n-1} \frac{\partial^2 z_i}{\partial \zeta_j \partial \zeta_k} \Delta \zeta_j \Delta \zeta_k = \sum_{j=3}^{n-1} \sum_{k=3}^{n-1} \left( \frac{\partial^2 z_i}{\partial \zeta_j \partial \zeta_k} - \frac{\partial^2 z_i}{\partial \zeta_3 \partial \zeta_4} \right) \Delta \zeta_j \Delta \zeta_k + O(\Delta t)^4 = -\frac{1}{P'(z_i)} \int_{\beta}^{z_i} \frac{P'(w) \, dw}{(w-a)^2} \sum_{j=3}^{n-1} (\Im(\Delta \zeta_j))^2 + O(\Delta t)^3.
\]

Recall that each \(|z_i + \Delta z_i| \leq 1\) and that each \(\Delta z_i = O(\Delta t)^2\). If \(|z_i| = 1\), then \(|1 + \Delta z_i/z_i| \leq 1\), so using part 1 of Lemma 3 gives us \(\Re[\Delta z_i/z_i] \leq O(\Delta t)^4\) and so \(E_{i+3} + F_{i+3} \leq O(\Delta t)^3\) for \(i = 1, \ldots, m\).

Thus we have \(E_k + F_k \leq O(\Delta t)^3\) for \(k = 1, \ldots, m + 3\). From property 4 we have positive constants \(c_k\) so that \(\sum_{j=3}^{n-1} (\Im(\Delta \zeta_j))^2 = \sum_{k=3}^{n+3} c_k (E_k + F_k) \leq O(\Delta t)^3\) and thus \(\Im[\Delta \zeta_j] = O(\Delta t)^{3/2}\) for \(j \geq 3\), which completes the proof of Proposition 6. \(\square\)

Using our estimates from Propositions 5 and 6 we can now write the

**Proof of Theorem 2.** Let \(I = (\Delta \beta, \Delta \zeta_1, \ldots, \Delta \zeta_{n-1})\) be any improvement of \(P\), and let \(\Delta t = ||I||\). From Propositions 5 and 6 we know that \(\Delta \beta = O(\Delta t)^2\) and that each \(\Delta \zeta_j = O(\Delta t)^{3/2}\), so

\[
\Delta t = ||I|| = \left( |\Delta \beta|^2 + \sum_{j=1}^{n-1} |\Delta \zeta_j|^2 \right)^{1/2} = O(\Delta t)^{3/2}.
\]

Thus there is a constant \(K > 0\) so that \(\Delta t \leq K(\Delta t)^{3/2}\), so \(||I|| = \Delta t \geq 1/K^2\). \(\square\)
5. Calculations

Recall that Theorem 2 implies that any real polynomial of degree \( n \geq 5 \) that satisfies Properties A–H is locally extremal. In this section, we show how to find locally extremal polynomials by describing how to construct polynomials that satisfy our properties A–H. Maple code for these calculations can be found in the source files for this paper, located at http://arxiv.org/abs/math/0505424 [v3].

Recall that \( P \) is to be a real polynomial of degree \( n \geq 5 \). By property E we must have \( P'(z) = (z-a)^{n-3}(z^2+bz+c) \) for some real numbers \( a, b \) and \( c \).

By property C, we know that \( P(z) = \int_{\beta} z P'(w) \, dw \), so the coefficients of \( P \) are polynomials in \( \{\beta, a, b, c\} \).

Applying property D to the critical points of \( P \) (the real number \( a \) and the complex roots of \( z^2+bz+c \)), we get our first equation \( \beta^2 + b\beta + c = (\beta - a)^2 \).

Note that property G implies that the rows of the linear system \( \{E_k = 0 : k = 1, \ldots, m + 3\} \) are linearly dependent. Since property H states that the coefficient matrix of this system is of rank 7, this means that there must be at least 8 equations in the system, so \( m \geq 5 \) and thus we must look for polynomials with at least 5 roots on the unit circle.

If \( n \) is odd, we will look for polynomials with three pairs of complex conjugate roots on the unit circle. Each such pair will be the roots of a quadratic of the form \( z^2 + d_i z + 1 \), so the remainders upon dividing \( P \) by each of these three quadratics are linear polynomials with both coefficients equal to 0. This generates an additional six equations in the seven variables \( \{\beta, a, b, c, d_1, d_2, d_3\} \). Thus we have a nonlinear system of 7 equations in 7 unknowns.

If \( n \) is even, we will look for polynomials with two pairs of complex conjugate roots on the unit circle and a root at \(-1\). The conjugate roots generate an additional 4 equations in the six variables \( \{\beta, a, b, c, d_1, d_2\} \) (as above), and the equation \( P(-1) = 0 \) generates a sixth equation. Thus we have a nonlinear system of 6 equations in 6 unknowns.

Thus in either case we get a nonlinear system of equations, with the same number of equations as unknowns, so we can try to solve this system. Note that there may be more than one solution, so we will need to choose the “correct” one. For each such solution, we verify properties A–H by numerically checking the following assertions.

Property A: The maximum modulus of the roots of \( P \) is equal to 1 (to the accuracy calculated.)

Property B: The minimum distance between any two roots of \( P \) is greater than 0.1. (This shows that all roots of \( P \) are simple.)

Property C: We have 0.7 < \( \beta \) < 0.9. (We know that \( P(\beta) = 0 \), since \( P(z) = \int_{\beta} z P'(w) \, dw \) by construction.)

Property D: The distances between \( \beta \) and the critical points of \( P \) are all equal (to the accuracy calculated), and this common distance is greater than 0.9.

Property F: The quantity \( \beta - a > 0.9 \). (Note that the critical point \( a \) is real and of order \( n-3 \) by construction.)

Property E: If we define

\[
R = \max \left\{ \min_{1 \leq j \leq n-1} |z_i - \zeta_j| : z_i \neq \beta \right\} \quad \text{and} \quad r = \min_{1 \leq j \leq n-1} |\beta - \zeta_j|,
\]

then \( r < 0.97 \) and \( r - R > 0.02 \).
Property G: The linear system given by the sums has a solution in which every $c_k > 0.3$.

Property H: The seventh largest singular value of the coefficient matrix is greater than 0.04.

Once we have verified properties [A–H] for a specific polynomial $P$, we know by Theorem 2 that $P$ is locally extremal and we are done.

6. Proof of Theorem 1

For the values of $\beta$ and $P'(z)$ given below, one can verify that the polynomials $P(z) = \int_{\beta}^{z} P'(w) \, dw$ satisfy Properties [A–H] and thus by Theorem 2 are locally extremal. (Details of these computations may be found with the Maple code referenced above.)

For $n = 8$, we take $\beta = 0.7290857513$ and

$$P'(z) = (z + 0.203540979)5(z^2 - 0.5410836525z + 0.7327296666).$$

For $n = 9$, we take $\beta = 0.7145672829$ and

$$P'(z) = (z + 0.2157115753)6(z^2 - 0.8021671918z + 0.9280147829).$$

For $n = 12$, we take $\beta = 0.8403619619$ and

$$P'(z) = (z + 0.1155828545)5(z^2 - 0.4090272613z + 0.5513532168).$$

For $n = 13$, we take $\beta = 0.8275325585$ and

$$P'(z) = (z + 0.1246203379)10(z^2 - 0.5415308686z + 0.6699194279).$$

For $n = 14$, we take $\beta = 0.8158105092$ and

$$P'(z) = (z + 0.1304708647)11(z^2 - 0.6885970233z + 0.7916663399).$$

For $n = 15$, we take $\beta = 0.7999767588$ and

$$P'(z) = (z + 0.1400336168)12(z^2 - 0.8389864647z + 0.9148263642).$$

For $n = 19$, we take $\beta = 0.8684432238$ and

$$P'(z) = (z + 0.0923361850)10(z^2 - 0.6503807257z + 0.7337221736).$$

For $n = 20$, we take $\beta = 0.8570396874$ and

$$P'(z) = (z + 0.0982636528)17(z^2 - 0.7563752823z + 0.8263310816).$$

For $n = 26$, we take $\beta = 0.8817716692$ and

$$P'(z) = (z + 0.0797127446)23(z^2 - 0.7969496845z + 0.8496586550).$$

These verifications complete the proof of Theorem 1.

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