Elastic backbone phase transition in the Ising model

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I. INTRODUCTION

The geometrical approach to thermal systems has proved to be very fruitful, especially in the vicinity of critical points. The effectiveness of the correspondence between local and global properties has led to the study of various geometrical quantities in thermal systems, like the q-state Potts model [1,2], the two-dimensional electron gas [3], and the spin glass [4] and the modified Ising models [5,6]. Backbone and elastic backbone (EB, the set of shortest paths) of the geometrical and the Fortuin-Kasteleyen (FK) clusters are examples of such extended objects, whose fractal structure can be found in optimal paths [7] and interfaces [8], which can be processed via Schramm-Loewner evolution (SLE) [9]. Actually the criticality of the original model induces fractality of these extended objects. More precisely when the thermal model experiences a second-order phase transition, it can be equivalently described as a percolation transition of FK clusters, which are fractal structures [10]. The EB will serve here as a geometrical object that can be employed to lighten some aspects of geometrical and also FK clusters.

The EB in disordered systems is the subset of the backbone that would give the first contribution to a restoring force, when the system is elongated. The EB determines the resistance of the system under tension, whose characterization involves the determination of its fractal dimension, optimal path traces, etc. [7]. A new type of transition in classical percolation for the system under tension, whose characterization involves the determination of its fractal dimension, optimal path traces, etc. [7]. A new type of transition in classical percolation for the system under tension, whose characterization involves the determination of its fractal dimension, optimal path traces, etc. [7].

II. THE FORTUIN-KASTELEYEN (FK) REPRESENTATION OF THE ISING MODEL

The FK formulation provides a geometrical description of the q-state Potts model. The determination of these geometrical properties is of special importance in the context of critical phenomena. The FK clusters of the q-state Potts model describe the critical behavior. The q-state Potts model is defined by the following Hamiltonian:

\[ H = -K \sum_{\langle i,j \rangle} (\delta_{\sigma_i,\sigma_j} - 1) - h' \sum_i \delta_{\sigma_i,1}, \quad \sigma_i = 1, 2, \ldots, q. \]  

(1)
The FK clusters are also useful to reduce the critical slowing down, which is known as the Swendsen-Wang algorithm [18]. In this approach FK clusters are used as the objects to be updated at each Monte Carlo step. If we take $\tau$ and $\sigma$ as two independent exponents, defined by $P(n) \sim n^{-\tau} \exp[-\theta n]$ in which $P(n)$ is the cluster distribution giving the average number density of clusters of $n$ sites and $\theta \sim (T - T_c)^{\nu}$, then the standard geometrical exponents of the percolation theory are given by

$$\begin{align*}
\alpha &= 2 - \frac{\tau - 1}{\sigma}, \\
\beta &= \frac{\tau - 2}{\sigma}, \\
\gamma &= \frac{3 - \tau}{\sigma}, \\
\eta &= 2 + \frac{\tau - 3}{\sigma}, \\
v &= \frac{\tau - 1}{d\sigma}, \\
d_f &= \frac{d}{\tau - 1},
\end{align*}$$

in which $\alpha$ is the exponent of the density of clusters (determined by the divergence of its third derivative with respect to temperature), $\beta$ is the exponent of the number density of the percolating cluster, $\gamma$ is the exponent of density fluctuations, $\eta$ is the Fisher exponent (anomalous dimension in the Green function), $v$ is the exponent of correlation length, and $d_f$ is the cluster fractal dimension. Therefore some hyperscaling relations relate these exponents, the most important ones being $\alpha = 2 - vd$, $d_f = \frac{1}{2}(d + 2 - \eta) = d - \beta/v$, $2\beta + \gamma = dv$. The latter hyperscaling relation is violated for the EB transition of the percolation model [11].

The Ising model is given ($q = 2$ Potts model) by (up to an additive constant)

$$H = -J \sum_{\langle i,j \rangle} s_i s_j - h \sum_i s_i, \quad s_i = \pm 1, \quad (4)$$

in which $J = \frac{1}{4}K$, and $h = \frac{1}{2}h'$. $J > 0$ corresponds to positively correlated nearest neighbors, whereas $J < 0$ is for negatively correlated ones. The temperature $T$ controls the disorder in the system. The FK clusters are simply obtained by bond diluting the geometric spin cluster, i.e., the connected cluster comprised by the same spins. In this bond dilution, one removes the bonds between nearest neighbors with the probability $p = 1 - e^{-2J}$.

For $h = 0$ the model is well known to exhibit a nonzero spontaneous magnetization per site $M = \lim_{h \to 0} \langle \sigma_i \rangle$ at temperatures below the critical temperature $T_c$. In fact, there are two transitions in the Ising model: the magnetic (paramagnetic to ferromagnetic) transition (mentioned above) and the percolation transition (in which the FK cluster percolate and become fractal). For the 2D regular Ising model at $h = 0$ these two transitions occur simultaneously [17], although it is not the case for all versions of the Ising model, e.g., for the site-diluted Ising model [6].

### III. RESULTS

As a spanning object, the elastic backbone (EB) is a geometrical subset of the spanning cluster that contains important information about the geometry of the cluster, since it is the set of points which react first to an external tension. It defines a new type of transition in ordinary percolation [11].

In this section we present the geometrical properties of the EB of the FK clusters of the Ising model. Let us define the Ising model on the $L \times L$ tilted square lattice. We impose open (periodic) boundary conditions along vertical (horizontal) directions, respectively. Then using Monte Carlo simulations of the Ising model at $h = 0$, we generated $10^5$ Ising configurations at temperatures $T \leq T_c$ for $L = 362, 512, 724, 1024, 1448$, and $2048$. After identifying the FK clusters, the backbones and the EBs are extracted using the burning algorithm [19]. The statistics of the density of these clusters as well as the loops in the backbone are calculated.
Various fractal dimensions of EBs as functions of temperature are obtained.

Our main observation is that there is a temperature, namely, \( T_{eb} < T_c \), at which the EBs undergo a phase transition from the dilute phase to the dense one. Below this temperature the EBs are dense. At this temperature, the density of the EB exhibits strong large fluctuations, which signals a second-order phase transition. Based on these observations we propose that there are three regimes in the zero magnetic field Ising model: for \( T > T_c \) there is no spanning cluster, whereas for \( T_{eb} < T < T_c \) we are in percolation regime with dilute EBs, and for \( T \leq T_{eb} \) the EBs become dense. At \( T = T_{eb} \) the system shows critical behavior with some critical exponents which are extracted analyzing the scaling relations.

In Fig. 1 we show samples of EBs (blue) in the tilted square lattice for three cases: at \( T = T_c \) (dilute phase) [Fig. 1(a)], at \( T = T_{eb} \) (critical value) [Fig. 1(b)], and at \( T = 1.70 < T_{eb} \) (dense phase) [Fig. 1(c)]. The blue traces are simply the shortest paths from top to bottom. Periodic boundary conditions have been imposed in the horizontal direction.

The first quantity to be investigated is \( m_L(T) \equiv L^{-2}(M) \) in which \( M \) is the number of sites contained in the EB, and \( \langle \rangle \) is the ensemble average. We consider it as the order parameter in this problem. Figure 2(a) shows \( m_L \) in terms of \( T \) for various sizes \( L \), exhibiting a clear transition at some temperature, below which the EBs become dense. By tracking the behavior of \( m_L \) in terms of \( T \) and \( L \), one can extract the critical temperature \( T_{eb} \), as done in the inset. From this analysis we observe that \( T_{eb} = 1.847 \pm 0.001 \). Also one can obtain the exponent \( \beta_{eb} \), which is obtained to be \( 0.48 \pm 0.03 \) through the scaling relation

\[
m_L(T) = L^{-\beta_{eb}/\nu_{eb}}G_m(\epsilon)L^{1/\nu_{eb}},
\]

in which \( G_m(x) \) is a scaling function with \( G_m(x) \sim x^{-\beta_{eb}} \) and is analytic and finite as \( x \to 0 \) (or equivalently \( T \to T_{eb} \)), and \( \epsilon = T/T_{eb} \). We note here that since the system is anisotropic, one should calculate \( \nu_{||} \) (the exponent parallel to the time direction) and \( \nu_{\perp} \) (perpendicular exponent) separately. The relation between these anisotropic exponent and \( \nu_{eb} \) will be studied at the end of this section.

To extract \( T_{eb} \), one may need a more precise method. We have used the Binder’s cumulant:

\[
B_L = 1 - \frac{\langle m_L^4 \rangle}{3\langle m_L^2 \rangle^2},
\]

which becomes \( L \)-independent at \( T = T_{eb} \). In fact, the crossing point of two successive sizes \( L \)'s may change as \( L \) increases; i.e., the crossing points are \( L \)-dependent. In this case one can extrapolate the \( T_{eb}(L) \) to find the correct value, \( T_{eb}(\infty) \), which is done in the lower inset of Fig. 2(b). This analysis confirms the finding of Fig. 2(a); i.e., it reveals that \( T_{eb} = 1.846 \pm 0.003 \).

An important test is to examine whether the scaling relation Eq. (5) holds or not, which is necessary for a second-order transitions. We plot \( G_m(\epsilon) \equiv |\epsilon|L^{1/\nu_{eb}} \) for \( T > T_{eb} \) and \( T \leq T_{eb} \) to extract the exponents. This analysis is done in Fig. 3 in which the upper branch is for \( T \leq T_{eb} \), and the lower branch for \( T > T_{eb} \). It is seen that for large enough \( \epsilon \)'s (for which we expect \( G(\epsilon) \propto \epsilon^{\beta_{eb}} \)), the slope is \( \beta_{eb} = 0.54 \pm 0.02 \), and \( \nu_{eb} = 1.20 \pm 0.03 \). This implies that \( \beta_{eb}/\nu_{eb} = 0.47 \pm 0.03 \), which is compatible with the value found above.
The total mass of the EBs is expected to behave like \( M_{eb} = L^2 m_{eb} = L^{d_f} G_m(L^{1/\gamma_1}) \) in which \( d_f = 2 - \beta_{eb}/\nu_{eb} \), and \( G_m \) is the same function as Eq. (5). \( d_f \) is therefore obtained by a log-log plot of \( M_{eb} \) in terms of \( L \), which has been done for \( T = T_{eb} \) in Fig. 4. We have additionally plotted the same graphs for the number of loops inside the EB (circles) and backbones (inverse triangles). In the burning algorithm a loop is identified each time a site is simultaneously burned from two sites and the number of loops involving \( i \) and \( j \) is \( n - 1 \) when there are \( n \) distinct paths from \( i \) to \( j \) [19,20]. The resulting fractal dimensions are \( d_f^{(1)} = 1.53 \pm 0.03 \) and \( d_f^{(2)} = 1.99 \pm 0.01 \) respectively. We see that interestingly the fractal dimension for the number of loops inside the EBs is the same as \( d_f \), and the number of loops inside the backbone grows extensively, showing that the backbone is in the dense phase. Actually we expect this for all \( T < T_c \), since the backbones behave like the FK clusters, which are in dense phase in this regime. The analysis of the fractal dimension for the other temperatures shows that \( d_f \approx 2.0 \) for \( T < T_{eb} \) (dense phase) and \( d_f \approx 1.0 \) for \( T > T_{eb} \) (dilute phase); see the inset of Fig. 4. This confirms that the clusters are space filling for the first case and effectively one-dimensional in the dilute phase.

Given the above data, the question arises concerning the presumable singular behavior of the fluctuations of the order parameter, \( m_L \), as for any second-order phase transition. Let us define the fluctuation of the order parameter \( \chi \equiv L^\beta (M_{eb}^2 - \langle M_{eb}^2 \rangle) \), which is expected to diverge at the transition point of any continuous transition. It is additionally expected to fulfill the scaling behavior:

\[
\chi_L(\epsilon) = L^{-\gamma_0/\nu} G_\chi(\beta L^{1/\nu}) ,
\]

in which again \( G_\chi(x) \) is a scaling function with \( G_\chi(x)|_{x=\infty} \propto x^{\epsilon_0} \) and is analytic and finite as \( \epsilon \to 0 \). The analysis of this function is presented in Fig. 5. This scaling hypothesis predicts that the maximum value of \( \chi \), i.e., at the transition point behaves like \( \chi_{max} \propto L^{\gamma_1/\nu} \), and also \( \chi_L(\epsilon) \propto |\epsilon|^{-\gamma_0} \) for small enough \(|\epsilon|\). Figure 5(a) suggests that \( \gamma_{eb} = 1.00 \pm 0.01 \). If we use the above-obtained \( \gamma_{eb} (1.20 \pm 0.03) \), we find that \( \gamma_{eb} = 1.20 \pm 0.03 \). Summarizing we have presented the data collapse analysis in Fig. 5(b), which confirms that \( \gamma_{eb} = 1.2 \pm 0.1 \). The inset is also consistent with this result.

Here it is worthwhile to comment on the hyperscaling relations. As mentioned in Sec. II the exponents are not independent, and there are some hyperscaling relations between them. For example, \( d_f = 2 - \beta/\nu \), and \( 2 \beta + \gamma = d \nu \) (\( d = 2 \) here). The latter has been shown to be violated for the EB transition in percolation [11]. Here we note that \( \beta_{eb} + \frac{1}{2} \gamma_{eb} = 1.15 \pm 0.07 \), which agrees within the error bar with \( \gamma_{eb} = 1.2 \pm 0.1 \). Therefore, we conclude that the hyperscaling relation is restored in the FK clusters of the Ising model.

The set of all shortest paths leaving one point can be seen as an anisotropic object, and the corresponding critical point (the transition point) should be described by an anisotropic universality class. Recently it was suggested by Deng et al. [12] that the transition point of the EBs defined in the percolation system is in the universality class of DP. To this end, they calculated two fractal dimensions for both the EB of percolation and the backbone of DP: first, the number of occupied sites along the center line \( N_b = \langle N_{o-L/2} \rangle \) (which represents the behavior of the bulk) and the number of occupied sites at the top and bottom edges \( N_t = \frac{1}{2} (N_{o-L} + N_{o-L}) \) (representing the behavior of boundaries) in terms of system size \( L \), and,
second, the chemical distance (shortest path) exponent $d_{\text{min}}$ defined by $(l_t) \sim L^{d_{\text{min}}}$. From the similarities between the obtained fractal dimensions and the exponents of the DP $d_{\text{DP}} = 2 - \frac{\nu}{\gamma}$ characterizing the full DP, and $d_{\text{DP}, \|} = 2 - \frac{\nu}{\gamma} - \delta$ characterizing the bulk of the DP, in which the exponent $\delta$ is defined by the survival probability $P(t) \propto t^{-\delta}$, Deng et al. concluded that they are in the same universality classes. Note that $N_e$ and $N_b$ are expected to scale like $L^{d-1}$ and $L^{d-1}$ (the subtraction of exponents by one is due to the fact that we are taking one-dimensional cuts through the clusters). The above described procedure still requires some consistent (the subtraction of exponents by one is due to the fact that $d = d_{\text{fractal}} + 1$, Hou et al. [32]). For OP ($p = p_{\text{eb}}$), which is close to each other, the other exponents are drastically different. The other exponent that is relevant in characterizing the geometrical properties of the model at $T = T_c$ is the fractal dimension of the shortest path ($d_{\text{min}}$). This dependence is shown in Fig. 6(b), from which we see that $d_{\text{min}} = 1.090 \pm 0.004$. This exponent has previously been conjectured by Deng et al. to be 1.09375 [33] and numerically calculated by Hou et al. where the value 1.0940(3) was reported [34]. This value should also be compared with $d_{\text{min}}$ for the FK clusters of the Ising model.

One may be interested in calculating $\sigma$ and $\tau$. We obtain $\tau = (2d\nu - \beta)/(d\nu - \beta) = 2.29 \pm 0.02$, and $\sigma = 1/(d\nu - \beta) = 0.53 \pm 0.02$. Since our model is anisotropic, it will not be conformally invariant [38], and a Loewner transformation would map its paths to anomalous diffusion [39].

### IV. DISCUSSION AND CONCLUSION

The elastic backbone (EB) of the Ising model (in the zero magnetic field limit) has numerically been considered in

#### TABLE I. The exponents for the Ising model at $T = T_{\text{eb}}$ (rows 2, 3, and 4), for the ordinary percolation (OP) model at $p = p_{\text{eb}}$ (row 5) [11], for OP at $p = p_{\text{eb}}$, (row 6, in which the exact results can be found in Refs. [21–28], and are numerically confirmed in Refs. [29–31]), and finally for the DP model at $p = p_{\text{eb}}$. [32]. For OP and DP, although the sole values of $\beta$ and $\nu$ are different, $\beta/\nu$ (which is equal to $\beta/\nu_{\|}$) is the same. The exponents for the Ising model are considerably different from the exponents of DP and therefore define a new anisotropic universality class. Two hyperscaling relations ($d_{\text{f}} = 2 - \beta/\nu$ and $2\beta + \gamma = 2\nu$) are also reported, which are shown to be valid for the Ising model, whereas the latter is violated for OP.

| Exponent | $\beta$ | $\nu = \nu_{\|}$ | $\nu_{\perp}$ | $\gamma$ | $2 - \beta/\nu$ | $d_{\text{f}}$ | $d_{\text{e}}$ | $d_{\text{b}}$ | $d_{\text{min}}$ | $\frac{\beta}{\nu} + \frac{\gamma}{\nu}$ |
|----------|--------|----------------|-------------|--------|----------------|------------|------------|------------|------------|------------------|
| Figs. 2(a), 4, 6 | – | – | – | – | 1.52(3) | 1.53(2) | 1.71(1) | 1.52(1) | 1.090(4) | – |
| Fig. 3 | 0.54(2) | 1.20(3) | – | – | 1.53(3) | – | – | – | – | – |
| Figs. 5(a), 6(a) | – | – | 1.21(4) | 1.86(1) | 1.20(3) | – | – | – | – | 0.95(3) |
| OP ($p = p_{\text{eb}}$) | 0.50(2) | 2.00(2) | – | 1.97(5) | 1.750(3) | 1.750(3) | 1.84054(4) | 1.68102(15) | – | 0.74(1) |
| OP ($p = p_{\text{eb}}$) | $\frac{\beta}{\nu} \approx 0.14$ | $\frac{\nu}{\beta} \approx 1.33$ | – | $\frac{\nu}{\beta} \approx 2.39$ | $d_{\text{f}}$ | $\frac{\nu}{\beta} \approx 1.896$ | – | – | 1.13077(2) | 7.35–37 |
| DP ($p = p_{\text{eb}}$) | 0.277(2) | 1.0969(3) | 1.7339(3) | – | 1.747(3) | 1.765(1) | – | – | – | – |
this work. The geometrical properties of the critical models are coded in the FK clusters, which are obtained simply by dilution of the geometrical clusters of the same spin. Based on our numerical evidences we proposed that the EB of the spanning FK clusters in a system of linear size $L$ has been considered as the order parameter for this transition. Using Binder’s cumulant we found $T_{eb} = 1.846 \pm 0.003$. We have obtained $\beta$ and $\nu$ exponents using various methods, which yield consistent values. The exponents are different from both critical percolation and the percolation at $p = p_{eb}, 2 - \beta_{eb}/\nu_{eb} = 1.52 \pm 0.03$. The determination of other exponents (for example, $\gamma$ obtained from the density fluctuations, $d_f$, $d_c$, $d_h$, and $d_{min}$) reveals that the universality class of this transition is considerably different from ordinary percolation at $p = p_c$ and $p = p_{eb}$ and also the Ising model at $T = T_c$. We have characterized comprehensively exponents which seem to be in a new universality class for anisotropic systems. The parallel correlation length exponent $\nu_{||}$ and $d_{min}$ were found to be $1.86 \pm 0.01$ and $1.090 \pm 0.004$, respectively, which are different from the ones for DP $[1.7339(3)$ and $1.13077(2)$, respectively$. Importantly we have shown that two relevant hyperscaling relations hold here, one of which is violated for percolation at $p = p_{eb}$.

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