Generalized Fibonacci numbers and extreme value laws for the Rényi map

N.B-S. Boer*  A.E. Sterk*

April 14, 2020

Abstract

In this paper we prove an extreme value law for a stochastic process obtained by iterating the Rényi map. Haiman (2018) derived a recursion formula for the Lebesgue measure of threshold exceedance sets. We show how this recursion formula is related to a rescaled version of the $k$-generalized Fibonacci sequence. For the latter sequence we derive a Binet formula which leads to a closed-form expression for the probability of partial maxima of the stochastic process. The proof of the extreme value law is completed by deriving sharp bounds for the dominant root of the characteristic polynomial of our Fibonacci sequence.

Contents

1 Introduction 2
2 The relation with generalized Fibonacci numbers 3
3 The Binet formula 7
4 Exponentially growing sequences 10
5 Proof of the extreme value law 11

*Bernoulli Institute for Mathematics, Computer Science, and Artificial Intelligence, University of Groningen, PO Box 407, 9700 AK Groningen, The Netherlands. E-mail: n.b.boer@student.rug.nl, a.e.sterk@rug.nl. Corresponding author: A.E. Sterk.
1 Introduction

Extreme value theory for a sequence of i.i.d. random variables \((X_i)\) studies the asymptotic distribution of the partial maximum

\[
M_n = \max(X_0, \ldots, X_{n-1})
\]

as \(n \to \infty\). Since the distribution of \(M_n\) has a degenerate limit it is necessary to consider a rescaling. Under appropriate conditions there exist sequences \(a_n > 0\) and \(b_n \in \mathbb{R}\) for which the limiting distribution of \(a_n(M_n - b_n)\) is nondegenerate. As an elementary example, assume that the variables \(X_i \sim U(0, 1)\) are independent. Then with \(a_n = n\) and \(b_n = 1\) it follows for \(\lambda > 0\) that

\[
\lim_{n \to \infty} \mathbb{P}(a_n(M_n - b_n) \leq -\lambda) = \lim_{n \to \infty} \mathbb{P}
\left(M_n \leq 1 - \frac{\lambda}{n}\right) = \lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}. \tag{2}
\]

More generally it can be proven that extreme value distributions for i.i.d. random variables are either a Weibull, Gumbel, or Fréchet distribution \([6, 7, 17]\). For extensions of extreme value theory to dependent random variables, see \([11]\).

In the last twenty years, the applicability of extreme value theory has been extended to the setting of deterministic dynamical systems such as iterated maps or differential equations. This development has important applications in the context of climate modelling. Typically, the approach is to show that two mixing conditions are satisfied which guarantee that an extreme value law for a time series generated by a dynamical system can be obtained as if it were an i.i.d. stochastic process. An example of this approach applied to the tent map can be found in \([5]\). For more details on the subject of extremes in dynamical systems the interested reader is referred to the recent monograph \([13]\) and the extensive list of references therein.

In this paper we consider the Rényi map \([16]\) given by

\[
f : [0, 1) \to [0, 1), \quad f(x) = \beta x \mod 1,
\]

where \(\beta > 1\). This map is an active topic of study within the field of dynamical systems and ergodic theory. In the special case \(\beta = 2\) the map \(f\) is also known as the doubling map, which is an archetypical example of a chaotic dynamical system of which the properties can be studied in great detail \([2]\). Other applications involve the study of random number generators \([11]\) or dynamical systems with holes in their state space \([10]\).

From now on we will assume that \(\beta \geq 2\) is an integer. In this case the Lebesgue measure is an invariant probability measure of the map \(f\):

**Lemma 1.1.** If \(X\) is a random variable such that \(X \sim U(0, 1)\), then \(f(X) \sim U(0, 1)\).

**Proof.** For \(u \in [0, 1]\) we have that \(\mathbb{P}(X \in [0, u)) = u\). This gives

\[
\mathbb{P}(f(X) \in [0, u)) = \mathbb{P}(X \in f^{-1}([0, u)) = \sum_{k=1}^{\beta} \mathbb{P}
\left(X \in \left[\frac{k-1}{\beta}, \frac{k-1+u}{\beta}\right]\right) = u,
\]

which implies that \(f(X) \sim U(0, 1)\). \qed
For non-integer values of $\beta > 1$ the map $f$ can have an invariant measure that is different from the Lebesgue measure; for the case $\beta = (\sqrt{5} + 1)/2$ see [3, 16].

Consider the stochastic process $(X_i)$ defined by $X_{i+1} = f(X_i)$, where $X_0 \sim U(0, 1)$. Lemma 1.1 implies that the variables $X_i$ are identically distributed, but they are no longer independent. Let $M_n$ be the partial maximum as defined in (1). Haiman [9] proved the following result:

**Theorem 1.2.** For fixed $\lambda > 0$ and the sequence $n_k = \lfloor \beta^k \lambda \rfloor$ it follows that

$$\lim_{k \to \infty} P(M_{n_k} \leq 1 - \beta^{-k}) = e^{-\frac{\beta - 1}{\beta \lambda}}.$$ 

Note that for $\lambda \in \mathbb{N}$ we have $P(M_{n_k} \leq 1 - \beta^{-k}) = P(\beta^k \lambda(M_{\beta^k \lambda} - 1) \leq -\lambda)$. Therefore, the result of Theorem 1.2 is in spirit similar to the example in (2), albeit that a subsequence of $M_n$ is considered. The fact that the limit is not equal to $e^{-\lambda}$ has a particular statistical interpretation. The coefficient $(\beta - 1)/\beta$ in the exponential is called the extremal index and measures the degree of clustering in extremes arising as a consequence of the dependence between the variables $X_i$; see [11, 13] for more details.

The aim of this paper is to give an alternative proof for Theorem 1.2 which relies on asymptotic properties of a rescaled version of the $k$-generalized Fibonacci numbers.

## 2 The relation with generalized Fibonacci numbers

In this section we fix the numbers $k \in \mathbb{N}$ and $u = \beta^{-k}$. For any integer $i \geq 0$ we define the set

$$E_i = \{x \in [0, 1) : f^i(x) > 1 - u\},$$

where the dependence on $k$ is suppressed in the notation for convenience. Then

$$P(M_n \leq 1 - u) = 1 - B_n \text{ where } B_n = \text{Leb} \left( \bigcup_{i=0}^{n-1} E_i \right),$$

where Leb denotes the Lebesgue measure. Based on self-similarity arguments Haiman [9] derived the following recursion formula:

$$B_n = (n-1) \frac{\beta - 1}{\beta} u + u \text{ if } 1 \leq n \leq k + 1,$$

$$B_{n+1} = B_n + \frac{\beta - 1}{\beta} u(1 - B_{n-k}) \text{ if } n \geq k + 1. \quad (3)$$

The same idea was used earlier by Haiman to study extreme value laws for the tent map [8]. For $n \in \mathbb{Z}$ we define the following numbers:

$$F_n = \begin{cases} 0 & \text{if } n < 1, \\ 1 & \text{if } n = 1, \\ \frac{B_n - B_{n-1}}{u/\beta^{n-1}} & \text{if } n > 1. \end{cases} \quad (5)$$
These numbers have the following geometric meaning. Note that the sets $E_i$ can be written as a union of $\beta^i$ intervals:

$$E_i = \bigcup_{j=1}^{\beta^i} \left[ \frac{j - u}{\beta^i}, \frac{j}{\beta^i} \right), \quad i \geq 0.$$ 

For $n \geq 2$ the number $F_n$ equals the number of subintervals of the set $E_{n-1}$ which need to be added to $E_0 \cup \cdots \cup E_{n-2}$ in order to obtain $E_0 \cup \cdots \cup E_{n-1}$. The figure clearly shows that $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, and $F_6 = 8$ which are the starting numbers of the Fibonacci sequence.

**Lemma 2.1.** For any $k, n \in \mathbb{N}$ it follows that

$$\mathbb{P}(M_n \leq 1 - \beta^{-k}) = \frac{\beta^{1-n-k}}{\beta-1} F_{n+k+1}.$$ 

**Proof.** For $n \geq k + 2$ equation (1) gives

$$F_n = \frac{B_n - B_{n-1}}{u/\beta^{n-1}} = (\beta - 1)\beta^{n-2}(1 - B_{n-1}),$$ 

or, equivalently,

$$B_{n-k-1} = 1 - \frac{\beta^{2-n}}{\beta-1} F_n.$$ 

The proof is completed by substituting $n$ for $n - k - 1$. 

The following result provides the connection between the sequence $(B_n)$ and generalizations of the Fibonacci numbers. In particular, for $\beta = 2$ the sequence $(F_n)$ is the well-known $k$-generalized Fibonacci sequence.
Lemma 2.2. The following statements are equivalent:

(i) Equations (3) and (4) hold;

(ii) The sequence \((F_n)\) defined in (5) satisfies

\[
F_n = \begin{cases} 
0 & \text{if } n < 1, \\
1 & \text{if } n = 1, \\
(\beta - 1)(F_{n-1} + F_{n-2} + \cdots + F_{n-k}) & \text{if } n \geq 2.
\end{cases}
\]

In particular, \(F_n = (\beta - 1)\beta^{n-2}\) for \(2 \leq n \leq k + 1\).

Proof. Assume that statement (i) holds. By definition \(F_1 = 1\) and for \(2 \leq n \leq k + 1\) equation (3) implies that

\[
F_n = \frac{B_n - B_{n-1}}{u/\beta^{n-1}} = \frac{\beta^{n-1}}{u} \left[ (n - 1)\frac{\beta - 1}{\beta} u + u \right] - \left( n - 2 \right) \frac{\beta - 1}{\beta} u + u = (\beta - 1)\beta^{n-2}.
\]

We proceed with induction on \(n\). For any \(n \geq k + 1\) equation (4) gives

\[
F_{n+1} = \frac{B_{n+1} - B_n}{u/\beta^n} = (\beta - 1)\beta^{n-1}(1 - B_{n-k}).
\]

In particular, for \(n = k + 1\) we have

\[
F_{k+2} = (\beta - 1)\beta^k(1 - B_1)
= (\beta - 1)(\beta^k - 1) = (\beta - 1)^2 \sum_{i=1}^{k} \beta^{k-i} = (\beta - 1) \sum_{i=1}^{k} F_{k+2-i}.
\]

Assume that for some \(n \geq k + 1\) it follows that

\[
F_{n+1} = (\beta - 1) \sum_{i=1}^{k} F_{n+1-i}.
\]

First using equation (5) and then equation (4) twice gives

\[
F_{n+2} = (\beta - 1)\beta^n(1 - B_{n-k+1})
= (\beta - 1)\beta^n(1 - B_{n-k}) - (\beta - 1)^2 \beta^{n-k-1}(1 - B_{n-2k})
= \beta F_{n+1} - (\beta - 1)F_{n-k+1},
\]

where the last equality follows from (7). Finally, the induction hypothesis implies that

\[
F_{n+2} = (\beta - 1)F_{n+1} + F_{n+1} - (\beta - 1)F_{n-k+1}
= (\beta - 1)F_{n+1} + (\beta - 1) \sum_{i=1}^{k} F_{n+1-i} - (\beta - 1)F_{n-k+1}
= (\beta - 1) \sum_{i=1}^{k} F_{n+2-i}.
\]
Hence, statement (ii) follows.

Conversely, assume that statement (ii) holds. In particular, \( F_n = (\beta - 1)^{n-2} \) for \( 2 \leq n \leq k + 1 \) so that by equation (5) it follows that
\[
B_n = B_{n-1} + \frac{u}{\beta^{n-1}}F_n = B_{n-1} + \frac{\beta - 1}{\beta}u.
\]

Equation (3) now follows by recalling that \( B_1 = u \).

We proceed by strong induction on \( n \). We have
\[
F_{k+2} = (\beta - 1) \sum_{i=1}^{k} F_{k+2-i} = (\beta - 1) \sum_{i=1}^{k} (\beta - 1)^{k-i} = (\beta - 1)(\beta^k - 1).
\]

Recalling that \( B_1 = u = \beta^{-k} \), equation (5) implies that
\[
B_{k+2} = B_{k+1} + \frac{u}{\beta^{k+1}}F_{k+2} = B_{k+1} + \frac{u}{\beta^{k+1}}(\beta - 1)(\beta^k - 1) = B_{k+1} + \frac{\beta - 1}{\beta}u(1 - B_1),
\]
which shows that equation (4) holds for \( n = k + 1 \). Assume that there exists \( m \in \mathbb{N} \) such that (4) holds for all \( k + 1 \leq n \leq m \). Observe that
\[
F_{m+2} = (\beta - 1) \sum_{i=1}^{k} F_{m+2-i}
\]
\[
= (\beta - 1) \left( F_{m+1} - F_{m+1-k} + \sum_{i=1}^{k} F_{m+1-i} \right)
\]
\[
= (\beta - 1) \left( F_{m+1} - F_{m+1-k} + \frac{F_{m+1}}{\beta - 1} \right)
\]
\[
= \beta F_{m+1} - (\beta - 1)F_{m+1-k}.
\]

Therefore,
\[
B_{m+2} - B_{m+1} = \frac{u}{\beta^{m+1}}F_{m+2}
\]
\[
= \frac{u}{\beta^{m+1}}(\beta F_{m+1} - (\beta - 1)F_{m+1-k})
\]
\[
= B_{m+1} + \frac{\beta - 1}{\beta^{k+1}}(B_{m+1-k} - B_{m-k}).
\]

The induction hypothesis gives
\[
B_{m+2} - B_{m+1} = \frac{\beta - 1}{\beta}u(1 - B_{m-k}) - \frac{(\beta - 1)^2}{\beta^{k+2}}u(1 - B_{m-2k})
\]
\[
= \frac{\beta - 1}{\beta}u \left( 1 - \left( B_{m-k} + \frac{\beta - 1}{\beta}u(1 - B_{m-2k}) \right) \right)
\]
\[
= \frac{\beta - 1}{\beta}u(1 - B_{m-k+1}).
\]

Hence, statement (i) follows. \( \square \)
3 The Binet formula

Let the sequence \((F_n)\) be as defined in (6), where \(\beta \geq 2\) is assumed to be an integer. In this section we will derive a closed-form expression for \(F_n\) as a function of \(n\) along the lines of Spickerman and Joyner [18] and Dresden and Du [4]. Levesque [12] derived a closed-form expression for sequences of the form (6) in which each term is multiplied with a different factor. Another interesting paper by Wolfram [19] considers explicit formulas for the \(k\)-generalized Fibonacci sequence with arbitrary starting values, but we will not pursue those ideas here.

The characteristic polynomial of the sequence \((F_n)\) is given by

\[
p_k(x) = x^k - (\beta - 1) \sum_{i=0}^{k-1} x^i.
\]

(8)

The following result concerns properties of the roots of this polynomial. The proof closely follows Miller [15]. For alternative proofs for the special case \(\beta = 2\), see [14, 19].

Lemma 3.1. Let \(k \geq 2\) and \(\beta \geq 2\) be integers. Then

(i) the polynomial \(p_k\) has a real root \(1 < r_{k,1} < \beta\);

(ii) the remaining roots \(r_{k,2}, \ldots, r_{k,k}\) of \(p_k\) lie within the unit circle of the complex plane;

(iii) the roots of \(p_k\) are simple.

Proof. (i) Descartes’ rule of signs implies that \(p_k\) has exactly one positive root \(r_{k,1}\). Since

\[
p_k(1) = 1 - k(\beta - 1) < 0 \quad \text{and} \quad p_k(\beta) = 1
\]

the Intermediate Value Theorem implies the existence of a root \(1 < r_{k,1} < \beta\).

(ii) Define the polynomial

\[
q_k(x) = (x - 1)p_k(x) = x^{k+1} - \beta x^k + \beta - 1,
\]

and make the following observations:

(O1) if \(x > r_{k,1}\), then \(p_k(x) > 0\), and if \(0 < x < r_{k,1}\), then \(p_k(x) < 0\);

(O2) if \(x > r_{k,1}\), then \(q_k(x) > 0\), and if \(1 < x < r_{k,1}\), then \(q_k(x) < 0\).

Note that \(p_k\) has no root \(r\) such that \(|r| > r_{k,1}\). Indeed, if such a root exists, then \(p(|r|) = 0\) so that the triangle inequality implies that \(p(|r|) \leq 0\), which contradicts observation (O1).

In addition, \(p_k\) has no root \(r\) with \(1 < |r| < r_{k,1}\). Indeed, if such a root exists, then \(q_k(r) = (r - 1)p_k(r) = 0\) so that \(\beta r^k = r^{k+1} + \beta - 1\). The triangle inequality gives \(q_k(|r|) \geq 0\), which contradicts observation (O2).
Finally, $p_k$ has no root $r$ with either $|r| = 1$ or $|r| = r_{k,1}$ but $r \neq r_{k,1}$. Indeed, if such a root exists, then $q_k(r) = (r - 1)p_k(r) = 0$, which implies $\beta r^k = r^{k+1} + \beta - 1$ and 

$$\beta |r|^k = |r^{k+1} + \beta - 1| \leq |r|^{k+1} + \beta - 1.$$  

(9)

If the inequality in (9) is strict, then $q_k(|r|) > 0$. Since $q_k(1) = 0$ and $q_k(r_{k,1}) = 0$ it then follows that $|r| \neq 1$ and $|r| \neq r_{k,1}$. If the inequality in (9) is an equality, then $r^{k+1}$ must be real. Since $q_k(r) = 0$, it follows that $r^k = ((\beta - 1) + r^{k+1})/\beta$ is real as well and hence $r$ itself is real. An application of Descartes’ rule of signs to $q_k$ implies that when $k$ is even $p_k$ has one negative root, and when $k$ is odd $p_k$ has no negative root. If $k$ is even, then $p_k(0) = -(\beta - 1)$ and $p_k(-1) = 1$. By the Intermediate Value Theorem it follows that $-1 < r < 0$. We conclude that no root of $p_k$, except $r_{k,1}$ itself, has absolute value 1 or $r_{k,1}$.

(iii) If $p_k$ has a multiple root, then so has $q_k$. In that case, there exists $r$ such that $q_k(r) = q_k'(r) = 0$. Note that $q_k'(r) = 0$ implies that $r = 0$ or $r = \beta k/(k + 1)$. Clearly, $r = 0$ is not a root of $q_k$. By the Rational Root Theorem it follows that the only rational roots of $q_k$ can be integers that divide $\beta - 1$. Hence, $r = \beta k/(k + 1)$ is not a root of $q_k$ either. We conclude that $q_k$, and thus $p_k$, cannot have multiple roots. 

The proof of the following result closely follows the method of Spickerman and Joyner [18] and then uses a rewriting step as in Dresden and Du [4].

**Lemma 3.2.** The sequence $(F_n)$ as defined in (6) is given by the following Binet formula:

$$F_n = \sum_{j=1}^{k} \frac{r_{k,j} - 1}{\beta + (k + 1)(r_{k,j} - \beta)} r_{k,j}^{n-1},$$

where $r_{k,1}, \ldots, r_{k,k}$ are the roots of the polynomial defined in (8).

**Proof.** The generating function of the sequence $(F_n)$ is given by

$$G(x) = \sum_{n=0}^{\infty} F_{n+1} x^n.$$ 

The equation

$$\sum_{n=k}^{\infty} \left( F_{n+1} - (\beta - 1) \sum_{i=1}^{k} F_{n+1-i} \right) x^n = 0$$

leads to

$$G(x) = \sum_{n=0}^{k-1} F_{n+1} x^n - (\beta - 1) \sum_{i=1}^{k-1} \sum_{n=0}^{k-i-1} F_{n+1} x^n + (\beta - 1) G(x) \sum_{i=1}^{k} x^i.$$ 

Finally, using that $F_1 = 1$ and $F_n = (\beta - 1)\beta^{n-2}$ for $2 \leq n \leq k - 1$ implies that

$$G(x) = \frac{1}{1 - (\beta - 1) \sum_{i=1}^{k} x^i}.$$
Note that $1/r$ is a root of the denominator of $G$ if and only if $r$ is a root of the characteristic polynomial $p_k$. By Lemma 3.1 part (iii) we can expand the generating function in terms of partial fractions as follows:

$$G(x) = \sum_{j=1}^{k} \frac{c_j}{x - 1/r_{k,j}},$$

where the coefficients are given by

$$c_j = \lim_{x \to 1/r_{k,j}} \frac{x - 1/r_{k,j}}{1 - (\beta - 1)\sum_{i=1}^{k} x^i} = -\frac{1}{(\beta - 1)\sum_{i=1}^{k} i(1/r_{k,j})^{i-1}}.$$

Observe that

$$\left(1 - \frac{1}{r_{k,j}}\right)\sum_{i=1}^{k} i \left(\frac{1}{r_{k,j}}\right)^{i-1} = \sum_{i=1}^{k} \left[i \left(\frac{1}{r_{k,j}}\right)^{i-1} - (i+1) \left(\frac{1}{r_{k,j}}\right)^i\right] + \sum_{i=1}^{k} \left(\frac{1}{r_{k,j}}\right)^i$$

$$= 1 - (k+1) \frac{1}{r_{k,j}} + \frac{1}{\beta - 1}.$$

This results in

$$c_j = -\frac{1 - 1/r_{k,j}}{\beta - (\beta - 1)(k+1)/r_{k,j}}.$$

Since $r_{k,j}^{k+1} - \beta r_{k,j}^k + \beta - 1 = (r_{k,j} - 1)p(r_{k,j}) = 0$ it follows that $r_{k,j} - \beta = (1 - \beta)/r_{k,j}$ so that

$$c_j = -\frac{1 - 1/r_{k,j}}{\beta + (k+1)(r_{k,j} - \beta)}.$$

Finally, we have that

$$G(x) = \sum_{j=1}^{k} c_j \left(-r_{k,j}\sum_{n=0}^{\infty} r_{k,j}^n x^n\right) = \sum_{n=0}^{\infty} \left(-\sum_{j=1}^{k} c_j r_{k,j}^{n+1}\right) x^n.$$

Substituting the values for the coefficients completes the proof.

For the special case $\beta = 2$ Dresden and Du [4] go one step further and derive the following simplified Binet formula:

$$F_{n} = \left|\frac{r_j - 1}{\beta + (k+1)(r_j - \beta)} \cdot \frac{1}{r_j^{n-1}} + \frac{1}{2}\right| \text{ for } n \geq k - 2.$$

We expect that this formula can be proven for all integers $\beta > 1$ for $n$ sufficiently large, where the lower bound on $n$ may depend on both $\beta$ and $k$. However, we will not pursue this question in this paper.
4 Exponentially growing sequences

In preparation to the proof of Theorem 1.2 we will prove two facts on sequences that exhibit exponential growth. The first result is a variation on a well-known limit:

Lemma 4.1. If \((a_k)\) is a sequence such that \(\lim_{k \to \infty} ka_k = c\), then

\[
\lim_{k \to \infty} (1 - a_k)^k = e^{-c}.
\]

Proof. Let \(\varepsilon > 0\) be arbitrary. Then there exists \(N \in \mathbb{N}\) such that \(|ka_k - c| \leq \varepsilon\), or, equivalently,

\[
\left(1 - \frac{c + \varepsilon}{k}\right)^k \leq (1 - a_k)^k \leq \left(1 - \frac{c - \varepsilon}{k}\right)^k
\]

for all \(k \geq N\). Hence, we obtain

\[
e^{-c+\varepsilon} \leq \liminf_{k \to \infty} (1 - a_k)^k \leq \limsup_{k \to \infty} (1 - a_k)^k \leq e^{-(c-\varepsilon)}.
\]

Since \(\varepsilon > 0\) is arbitrary, the result follows.

The next result provides sufficient conditions under which the difference of two exponentially increasing sequences grows at a linear rate:

Lemma 4.2. If \(a > 1\) and \((b_k)\) is a positive sequence such that \(\lim_{k \to \infty} a^k b_k = c\), then

\[
\lim_{k \to \infty} \frac{a^k - (a - b_k)^k}{k} = \frac{c}{a}.
\]

Proof. The algebraic identity

\[
a^k - y^k = (x - y) \sum_{i=0}^{k-1} x^{k-1-i} y^i
\]

leads to

\[
\frac{a^k - (a - b_k)^k}{k} = \frac{a^k b_k}{a} \cdot S_k \quad \text{where} \quad S_k = \frac{1}{k} \sum_{i=0}^{k-1} \left(1 - \frac{b_k}{a}\right)^i.
\]

It suffices to show that \(\lim_{k \to \infty} S_k = 1\). To that end, note that the assumption implies that \(\lim_{k \to \infty} b_k = 0\) so that \(-1 < -b_k/a < 0\) for \(k\) sufficiently large. Bernoulli’s inequality gives

\[
1 - i \frac{b_k}{a} \leq \left(1 - \frac{b_k}{a}\right)^i < 1,
\]

which implies that

\[
1 - \frac{k - 1}{2} \cdot \frac{b_k}{a} < S_k < 1
\]

for \(k\) sufficiently large. Moreover, the assumption implies that \(\lim_{k \to \infty} kb_k = 0\). An application of the Squeeze Theorem completes the proof. \(\square\)
5 Proof of the extreme value law

Let $\lambda > 0$ and define $n_k = \lceil \beta^k \lambda \rceil$. Combining Lemma 2.1 and 3.2 gives

$$P(M_{n_k} \leq 1 - \beta^{-k}) = \frac{\beta}{\beta - 1} \sum_{i=1}^{k} a_i(k) \quad \text{where} \quad a_i(k) = \frac{r_{k,i} - 1}{\beta + (k+1)(r_{k,i} - \beta)} \left( \frac{r_{k,i}}{\beta} \right)^{n_k + k}.$$  

where $r_{k,i}$ are the roots of $p_k$. Recall that $r_{k,1}$ is the unique root in the interval $(1, \beta)$, and that $|r_{k,i}| < 1$ for $i = 2, \ldots, k$. In the remainder of this section Theorem 1.2 will be proven by a careful analysis of the asymptotic behaviour of the dominant root $r_{k,1}$.

We define the following numbers:

$$r_{k,\text{min}} = \beta - \frac{\beta - 1}{\beta^k - 1} (1 + \beta^{-k/2}) \quad \text{and} \quad r_{k,\text{max}} = \beta - \frac{\beta - 1}{\beta^k - 1}.$$

The number $r_{k,\text{max}}$ is obtained by applying a single iteration of Newton’s method to $p_k$ using the starting point $x = \beta$. The number $r_{k,\text{min}}$ is a correction of $r_{k,\text{max}}$ with an exponentially decreasing factor.

**Lemma 5.1.** For $\beta \geq 2$ integer and $k \in \mathbb{N}$ sufficiently large it follows that

(i) $p_k(r_{k,\text{max}}) > 0$;

(ii) $p_k(r_{k,\text{min}}) < 0$;

(iii) $r_{k,\text{min}} < r_{k,1} < r_{k,\text{max}}$.

**Proof.** (i) For $x \neq 1$ we have

$$p_k(x) = x^k - (\beta - 1) \sum_{i=0}^{k-1} x^i = x^k - (\beta - 1) \frac{1 - x^k}{1 - x} = \frac{1}{1 - x} ((\beta - x)x^k - (\beta - 1)).$$

In particular, for $k \geq 2$ it follows that

$$p(r_{k,\text{max}}) = \frac{1}{\beta^k - 2} \left[ \beta^k - \left( \beta - \frac{\beta - 1}{\beta^k - 1} \right)^k \right] - 1.$$

It suffices to show that the expression between brackets is positive for $k$ sufficiently large. Lemma 4.2 gives

$$\lim_{k \to \infty} \frac{1}{k} \left( \beta^k - \left( \beta - \frac{\beta - 1}{\beta^k - 1} \right)^k \right) = \frac{\beta - 1}{\beta}.$$

Hence, for $k$ sufficiently large it follows that

$$\beta^k - \left( \beta - \frac{\beta - 1}{\beta^k - 1} \right)^k - 1 \geq \frac{\beta - 1}{2\beta} k - 1,$$

and the right-hand side is positive for $k > 2\beta/((\beta - 1)$.
(ii) Similar to the proof of part (i) it follows that

\[
p_k(r_{k,\min}) = \frac{1}{2 + \beta^{-k/2} - \beta} \left[ (\beta - \frac{\beta - 1}{\beta^k - 1} (1 + \beta^{-k/2})^k (1 + \beta^{-k/2}) - \beta^k + 1 \right].
\]

It suffices to show that the expression between brackets is positive for \( k \) sufficiently large. Lemma 4.2 gives

\[
\lim_{k \to \infty} \frac{1}{k} \left( \beta^k - \left( \beta - \frac{\beta - 1}{\beta^k - 1} (1 + \beta^{-k/2})^k \right) \right) = \frac{\beta - 1}{\beta}.
\]

Hence, for \( k \) sufficiently large it follows that

\[
\beta^k - \left( \beta - \frac{\beta - 1}{\beta^k - 1} (1 + \beta^{-k/2})^k \right) \leq k.
\]

This gives

\[
\left( \beta - \frac{\beta - 1}{\beta^k - 1} (1 + \beta^{-k/2}) \right)^k (1 + \beta^{-k/2}) - \beta^k + 1
\]

\[
= \beta^{k/2} + 1 - (1 + \beta^{-k/2}) \left( \beta - \frac{\beta - 1}{\beta^k - 1} (1 + \beta^{-k/2})^k \right)
\]

\[
\geq \beta^{k/2} + 1 - (1 + \beta^{-k/2})k,
\]

and the right-hand side is positive for \( k \) sufficiently large.

(iii) By the Intermediate Value Theorem there exists a point \( c \in (r_{k,\min}, r_{k,\max}) \) such that \( p_k(c) = 0 \). Note that \( c > 1 \) for \( k \) sufficiently large. Since \( r_{k,1} \) is the only zero of \( p_k \) which lies outside the unit circle it follows that \( c = r_{k,1} \).

In the particular, for \( \beta = 2 \) the previous result improves the bound \( 2(1 - 2^{-k}) < r_{1,k} < 2 \) derived by Wolfram [19].

**Lemma 5.2.** We have that

\[
\lim_{k \to \infty} a_1(k) = \frac{\beta - 1}{\beta} e^{-\frac{2}{\beta} \lambda}.
\]

**Proof.** From Lemma 5.1 it follows for sufficiently large \( k \) that

\[
\beta - \frac{\beta - 1}{\beta^k - 1} (1 + \beta^{-k/2}) < r_{k,1} < \beta - \frac{\beta - 1}{\beta^k - 1}.
\]

In particular, this implies

\[
\lim_{k \to \infty} r_{k,1} = \beta \quad \text{and} \quad \lim_{k \to \infty} (k + 1)(r_{k,1} - \beta) = 0
\]

so that

\[
\lim_{k \to \infty} \frac{r_{k,1} - 1}{\beta + (k + 1)(r_{k,1} - \beta)} = \frac{\beta - 1}{\beta}.
\]
Define the sequences
\[ a_k = \frac{\beta - 1}{\beta^{k+1} - \beta} (1 + \beta^{-k/2}) \quad \text{and} \quad b_k = \frac{\beta - 1}{\beta^{k+1} - \beta}. \]
The inequality \( \beta^k \lambda - 1 \leq n_k \leq \beta^k \lambda \) combined with (10) implies that
\[ (1 - a_k)^{\beta^k \lambda - 1 + k} \leq (1 - b_k)^{\beta^k \lambda + k}. \] (12)

By Lemma 4.1 it follows that
\[ \lim_{k \to \infty} (1 - b_k)^{\beta^k \lambda + k} = e^{-\frac{\beta - 1}{\beta} \lambda} \quad \text{and} \quad \lim_{k \to \infty} (1 - b_k)^k = 1, \]
which implies that
\[ \lim_{k \to \infty} (1 - b_k)^{\beta^k \lambda + k} = e^{-\frac{\beta - 1}{\beta} \lambda}. \]

A similar result holds for the sequence \( (a_k) \). Hence, (11) together with the Squeeze Theorem applied to (12) completes the proof.

**Lemma 5.3.** For \( k \) sufficiently large we have that
\[ |a_i(k)| < \frac{2}{|\beta + (k + 1)(1 - \beta)|} \cdot \frac{1}{\beta^{n_k + k}} \quad \text{for} \quad i = 2, \ldots, k. \]

**Proof.** Using that \( |r_{k,i}| < 1 \) for \( i = 2, \ldots, k \) gives
\[ |a_i(k)| = \frac{|r_{k,i} - 1|}{|\beta + (k + 1)(r_{k,i} - \beta)|} \cdot \left( \frac{|r_{k,i}|}{\beta} \right)^{n_k + k} < \frac{2}{|\beta + (k + 1)(r_{k,i} - \beta)|} \cdot \frac{1}{\beta^{n_k + k}}. \]

For \( z \in \mathbb{C} \) we consider the function
\[ f(z) = \beta + (k + 1)(z - \beta). \]
Writing \( z = x + iy \) gives
\[ |f(z)|^2 = (\beta + (k + 1)(x - \beta))^2 + (k + 1)^2 y^2 \geq (\beta + (k + 1)(x - \beta))^2. \]
The right-hand side attains its minimum value at \( x_k = \beta - \beta / (k + 1) \), and for \( k \) sufficiently large it follows that \( x_k > 1 \). Using that \( \text{Re} r_{k,i} \in (-1, 1) \) implies that
\[ |f(r_{k,i})| \geq |\beta + (k + 1)(1 - \beta)|. \]

This completes the proof. \( \square \)

From Lemma 5.3 it follows for \( k \) sufficiently large that
\[ \left| \sum_{i=2}^{k} a_i(k) \right| \leq \sum_{i=2}^{k} |a_i(k)| \leq \frac{2(k - 1)}{|\beta + (k + 1)(1 - \beta)|} \cdot \frac{1}{\beta^{n_k + k}}, \]
so that Lemma 5.2 implies that
\[ \lim_{k \to \infty} \mathbb{P}(M_{n_k} \leq 1 - \beta^{-k}) = \lim_{k \to \infty} \frac{\beta}{\beta - 1} \sum_{i=1}^{k} a_i(k) = \lim_{k \to \infty} \frac{\beta}{\beta - 1} a_1(k) = e^{-\frac{\beta - 1}{\beta} \lambda}, \]
whereby Theorem 1.2 has been proven.
References

[1] T. Addabbo, M. Alioto, M. Fort, A. Pasini, S. Rocchi, and V. Vignoli. A class of maximum-period nonlinear congruential generators derived from the Rényi chaotic map. *IEEE Transactions on Circuits and Systems*, 54:816–828, 2007.

[2] H.W. Broer and F. Takens. *Dynamical Systems and Chaos*, volume 172 of *Applied Mathematical Sciences*. Springer, 2011.

[3] G.H. Choe. *Computational Ergodic Theory*. Springer, 2005.

[4] G.P.B. Dresden and Z. Du. A simplified Binet formula for the $k$-generalized Fibonacci numbers. *Journal of Integer Sequences*, 17:Article 14.4.7, 2014.

[5] A.C.M. Freitas. Statistics of the maximum of the tent map. *Chaos, Solitons and Fractals*, 42:604–608, 2006.

[6] J. Galambos. *The Asymptotic Theory of Extreme Order Statistics*. Wiley, 1978.

[7] L. de Haan and A.F. Ferreira. *Extreme Value Theory. An Introduction*. Springer, 2006.

[8] G. Haiman. Extreme values of the tent map process. *Statistics and Probability Letters*, 65:451–456, 2003.

[9] G. Haiman. Level hitting probabilities and extremal indexes for some particular dynamical systems. *Methodology and Computing in Applied Probability*, 20:553–562, 2018.

[10] C. Kalle, D. Kong, N. Langeveld, and W. Li. The $\beta$-transformation with a hole at 0. *Ergodic Theory and Dynamical Systems*, in press, 2019.

[11] M.R. Leadbetter, G. Lindgren, and H. Rootzén. *Extremes and Related Properties of Random Sequences and Processes*. Springer-Verlag, 1980.

[12] C. Levesque. On $m$-th order linear recurrences. *The Fibonacci Quarterly*, 23:290–293, 1985.

[13] V. Lucarini, D. Faranda, A.C.M. Freitas, J.M. Freitas, M.P. Holland, T. Kuna, M. Nicol, M. Todd, and S. Vaienti. *Extremes and Recurrence in Dynamical Systems*. Wiley, 2016.

[14] E.P. Miles, Jr. Generalized Fibonacci numbers and associated matrices. *The American Mathematical Monthly*, 67:745–752, 1960.

[15] M.D. Miller. On generalized Fibonacci numbers. *The American Mathematical Monthly*, 78:1108–1109, 1971.

[16] A. Rényi. Representations for real numbers and their ergodic properties. *Acta Mathematica Academiae Scientiarum Hungaricae*, 8:477–493, 1957.
[17] S.I. Resnick. *Extreme Values, Regular Variation, and Point Processes*, volume 4 of *Applied Probability*. Springer-Verlag, New York, 1987.

[18] W.R. Spickerman and R.N. Joyner. Binet’s formula for the recursive sequence of order \( k \). *The Fibonacci Quarterly*, 22:327–331, 1984.

[19] D.A. Wolfram. Solving generalized Fibonacci recurrences. *The Fibonacci Quarterly*, 36:129–145, 1998.