Pointwise Semi-Slant Warped Product Submanifold in a Lorentzian Paracosymplectic Manifold

S. K. Srivastava and A. Sharma

Abstract. Recently Yüksel et. al. [28] shows that there doesn’t exist any proper semi-slant warped product submanifolds in a Lorentzian paracosymplectic manifold. In the present article, we first define and give preparatory lemmas for a new generalize class of semi-slant submanifolds called pointwise semi-slant submanifolds in a Lorentzian paracosymplectic manifold, and then we ensure by presenting some existence results and a non-trivial characterization theorem that there exist a pointwise semi-slant warped product submanifolds in a Lorentzian paracosymplectic manifold counter to warped product semi-slant submanifolds in a Lorentzian paracosymplectic manifold.

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1. Introduction

The premise of Lorentzian almost paracontact manifold (introduced by K. Matsumoto [17]) and warped product submanifolds one of the most effective generalization of pseudo-Riemannian products (initiated by Bishop-O’Neill, B [3]), has recognized various significant contributions in Lorentzian geometry (or pseudo-Riemannian geometry), and has been successfully employed in different models of space-time, general relativity and black holes (c.f., [4, 10, 15, 19]). Because

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of its numerous application to mathematical physics, several researcher found interest and studied the geometry of Lorentzian almost paracontact manifold and warped product submanifold in different settings (see; [1, 9, 11, 18, 20, 26]).

On the other hand, the concept of pointwise slant submanifold was introduced by Chen-Garay [8] as the natural generalization of slant submanifolds [6]. Such submanifolds were earlier studied by Etayo [12] with the name quasi-slant submanifold in almost Hermitian manifolds. Later on, Sahin [24] continued the study of pointwise slant submanifold by presenting a new class of submanifolds called warped product pointwise semi-slant submanifolds in Kählerian manifolds. Recently, Park [22, 23] and Balgeshir [2] extended the notion of pointwise slant, pointwise semi-slant submanifolds and pointwise almost h-semi-slant submanifolds along with its warped products aspects in almost contact and quaternionic Hermitian settings. Motivated by the works of these, in this research we introduced the pointwise semi-slant submanifolds in Lorentzian almost paracontact manifolds which can be considered as the generalization of slant, pointwise slant, semi-invariant, semi-slant submanifolds and investigate the warped aspects for such submanifold.

The organization of article is as follows. In Sect. 2, we recall some basic informations about Lorentzian paracosymplectic manifold. Subsect. 2.1, 2.2 and 2.3, includes some basic formulas, definitions of warped product submanifold, pointwise slant submanifold and some characterization results for such submanifolds. Sect. 3, deals with the construction of pointwise semi-slant submanifold along with the necessary and sufficient conditions for the distributions allied to the characterization of a pointwise semi-slant submanifold to be involutive and totally geodesic foliation. In Sect. 4, we first define pointwise semi-slant warped product submanifold $M$, and then give existence and nonexistence results for such warped product submanifolds. We also, obtain a characterization theorem for warped product submanifold of the form $M_T \times_f M_\theta$ with $\xi \in \Gamma(M_T)$ where, $M_T$ and $M_\theta$ are invariant and pointwise proper slant submanifolds on $M$, respectively and $f$ is a non-constant positive smooth function in a Lorentzian paracosymplectic manifold.

2. Preliminaries

Let $\tilde{M}^{2m+1}$ be a $2m + 1$-dimensional $C^\infty$ manifold. Then $\tilde{M}^{2m+1}$ is said to have an almost paracontact structure $(\phi, \xi, \eta)$, if there exist on $\tilde{M}^{2m+1}$ a tensor field $\phi$ of type $(1, 1)$, a smooth vector field $\xi$, and a 1-form $\eta$ satisfying

\begin{align}
\phi^2 &= I + \eta \otimes \xi, \quad \eta(\xi) = -1 \\
\phi \xi &= 0, \quad \eta \circ \phi = 0 \quad \text{and} \quad \text{rank}(\phi) = 2m.
\end{align}

where $I$ is the identity transformation. If the manifold $\tilde{M}^{2m+1}$ has an almost paracontact structure $(\phi, \xi, \eta)$ and admits a Lorentzian metric $g$ of type $(0, 2)$
on $\bar{M}^{2m+1}$ such that
\[ g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \] (2.3)
where signature of $g$ is necessarily $(1, 2m)$ or $(2m, 1)$ for any vector fields $X$ and $Y$; then the quadruple $(\phi, \xi, \eta, g)$ is called an Lorentzian almost paracontact structure and the manifold $\bar{M}^{2m+1}$ equipped with Lorentzian almost paracontact structure is called an Lorentzian almost paracontact manifold $\bar{M}^{2m+1}(\phi, \xi, \eta, g)$. The Lorentzian metric $g$ makes $\xi$ a timelike unit vector field, that is, $g(\xi, \xi) = -1$ (see, [17, 18]). With respect to $g$, $\eta$ is metrically dual to $\xi$, that is $g(X, \xi) = \eta(X)$.

In light of Eqs. (2.1), (2.2) and (2.3), we deduce that
\[ g(\phi X, Y) = g(X, \phi Y), \] (2.4)
for any $X, Y \in \Gamma(T\bar{M})$. Here $\Gamma(T\bar{M}^{2m+1})$ is the tangent bundle of $\bar{M}^{2m+1}$.

Finally, the fundamental 2-form $\Phi$ on $\bar{M}^{2m+1}$ is given by
\[ g(X, \phi Y) = \Phi(X, Y). \] (2.5)

Moreover,
\[ (\bar{\nabla}_Z \Phi)(X, Y) = g((\bar{\nabla}_Z \phi)X, Y) = (\bar{\nabla}_Z \Phi)(Y, X), \] (2.6)
for any $X, Y, Z \in \Gamma(TM)$, $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}^{2m+1}(\phi, \xi, \eta, g)$.

**Definition 2.1.** A Lorentzian almost paracontact manifold $\bar{M}^{2m+1}(\phi, \xi, \eta, g)$ is called [20, 28] Lorentzian paracosymplectic $\bar{M}^{2m+1}$, if the forms $\eta$ and $\Phi$ are parallel with respect to the Levi-Civita connection $\bar{\nabla}$ on $\bar{M}^{2m+1}(\phi, \xi, \eta, g)$, i.e.,
\[ \bar{\nabla}\eta = 0 \quad \text{and} \quad \bar{\nabla}\Phi = 0 \] (2.7)
for any $X, Y \in \Gamma(TM)$.

From the direct consequence of above definition, Eq. (2.2) and covariant differentiation formula, we have the following result;

**Lemma 2.2.** On a Lorentzian paracosymplectic manifold $\bar{M}^{2m+1}$ such that the structure vector field $\xi \in \Gamma(TM)$, we have
\[ \bar{\nabla}_X \xi = 0, \] (2.8)
for any $X \in \Gamma(TM)$.

**2.1. Geometry of submanifolds**

Let $M$ be a real submanifold immersed in a Lorentzian paracosymplectic manifold $\bar{M}^{2m+1}$, we denote by the same symbol $g$ the induced metric on $M$. In this article, we assume that $g$ is non-degenerate (in the sense of [11, 19]). Thus, each tangent space $T_p(M)$, for every $p \in M$, is a non-degenerate subspace of $T_p(\bar{M})$ such that $T_p(\bar{M}) = T_p(M) \oplus T_p(M)^\perp$, where $T_p(M)^\perp$ denotes the normal space of $M$. If $\Gamma(TM^\perp)$ indicate the set of vector fields normal to $M$ and $\Gamma(TM)$ the
sections of tangent bundle $TM$ of $M$, then the Gauss-Weingarten formulas are given by, respectively,
\begin{align*}
\bar{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\
\bar{\nabla}_X \zeta &= -A_\zeta X + \nabla_X^\perp \zeta,
\end{align*}
for any $X, Y \in \Gamma(TM)$ and $\zeta \in \Gamma(TM^\perp)$, where $\nabla$ is the induced connection, $\nabla^\perp$ is the normal connection on the normal bundle $\Gamma(TM^\perp)$, $h$ is the second fundamental form, and the shape operator $A_\zeta$ associated with the normal section $\zeta$ is given in [7] by
\begin{equation}
g(A_\zeta X, Y) = g(h(X, Y), \zeta).
\end{equation}
If we write, for all $X \in \Gamma(TM)$ and $\zeta \in \Gamma(TM^\perp)$ that
\begin{align*}
\phi X &= tX + nX, \\
\phi \zeta &= t'\zeta + n'\zeta,
\end{align*}
where $tX$ (resp., $nX$) is tangential (resp., normal) part of $\phi X$ and $t'\zeta$ (resp., $n'\zeta$) is tangential (resp., normal) part of $\phi \zeta$. Then the submanifold $M$ is said to be \textit{invariant} if $n$ is identically zero and \textit{anti-invariant} if $t$ is identically zero. From Eqs. (2.4) and (2.12), we obtain for all $X \in \Gamma(TM)$ that
\begin{equation}
g(X, tY) = g(tX, Y).
\end{equation}
A distribution $D$ on a submanifold $M$ is said to be \cite{10, 11}
\begin{itemize}
  \item \textit{totally geodesic} if its second fundamental form vanishes identically.
  \item \textit{umbilical} in the direction of a normal vector field $\zeta$ on $M$, if $A_\zeta = \lambda Id$, for certain function $\lambda$ on $M$; here $\zeta$ is called a umbilical section.
  \item \textit{totally umbilical} if $M$ is umbilical with respect to every (local) normal vector field.
  \item \textit{involutive} if, for all $X, Y \in D, [X, Y] \in D$.
\end{itemize}
Now we have an important results by virtue of Lemma 2.2 and Eq. (2.11),

\textbf{Lemma 2.3.} If $M$ is a isometrically immersed submanifold in a Lorentzian paracospymplectic manifold $\tilde{M}^{2m+1}$ such that the structure vector field $\xi \in \Gamma(TM)$, then
\begin{equation}
\nabla_X \xi = \nabla_\xi X = \nabla_\xi \xi = 0 \quad \text{and} \quad h(X, \xi) = 0,
\end{equation}
\begin{equation}
A_\zeta \xi = 0 \quad \text{and} \quad A_\zeta X \perp \xi
\end{equation}
for any $X \in \Gamma(TM)$ and $\zeta \in \Gamma(TM^\perp)$.
2.2. Warped product submanifolds

Let \((B, g_B)\) and \((F, g_F)\) be two pseudo-Riemannian manifolds and \(f\) be a positive smooth function on \(B\). Consider the product manifold \(B \times F\) with canonical projections
\[
\pi : B \times F \to B \quad \text{and} \quad \sigma : B \times F \to F.
\]
(2.15)
Then the manifold \(M = B \times_f F\) is said to be warped product if it is equipped with the following warped metric
\[
g(X, Y) = g_B(\pi_*(X), \pi_*(Y)) + (f \circ \pi)^2 g_F(\sigma_*(X), \sigma_*(Y))
\]
(2.16)
for all \(X, Y \in \Gamma(TM)\) and ‘\(*\)’ stands for derivation map, or equivalently,
\[
g = g_B + f^2 g_F.
\]
(2.17)
The function \(f\) is called the warping function and a warped product manifold \(M\) is said to be trivial if \(f\) is constant. In view of simplicity, we will determine a vector field \(X\) on \(B\) with its lift \(\bar{X}\) and a vector field \(Z\) on \(F\) with its lift \(\bar{Z}\) on \(M = B \times_f F\) [3].

**Proposition 2.4.** [3] For \(X, Y \in \Gamma(TB)\) and \(Z, W \in \Gamma(TF)\), we obtain on warped product manifold \(M = B \times_f F\) that
\[
\begin{align*}
(i) \quad & \nabla_X Y \in \Gamma(TB), \\
(ii) \quad & \nabla_X Z = \nabla_Z X = \left(\frac{Xf}{f}\right) Z, \\
(iii) \quad & \nabla_Z W = -\frac{g(Z, W)}{f} \nabla f,
\end{align*}
\]
where \(\nabla\) denotes the Levi-Civita connection on \(M\) and \(\nabla f\) is the gradient of \(f\) defined by \(g(\nabla f, X) = Xf\).

**Remark 2.5.** It is also important to note that for a warped product \(M = B \times_f F\); \(B\) is totally geodesic and \(F\) is totally umbilical in \(M\) [3].

Now, we prove an important results for later use;

**Theorem 2.6.** Let \(\tilde{M}^{2m+1}\) be a Lorentzian paracosymplectic manifold. Then there doesn’t exist any non-trivial warped product submanifolds \(M = B \times_f F\) of a paracosymplectic manifold such that \(\xi \in \Gamma(TF)\).

**Proof.** In light of Lemma 2.3 and Proposition 2.4, we obtain for any non-degenerate vector fields \(X \in \Gamma(TB)\) and \(Z \in \Gamma(TF)\) that \(X(\ln f)Z = 0\). This implies that \(f\) is constant function, since \(X, Z\) are non-degenerate vector fields in \(M\). This completes the proof of the theorem. \(\square\)

**Lemma 2.7.** If \(M = B \times_f F\) is a non-trivial warped product submanifold of a Lorentzian paracosymplectic manifold \(\tilde{M}^{2m+1}\) with \(\xi \in \Gamma(TB)\), then
\[
\xi(\ln f)X = 0,
\]
(2.18)
for any non-null vector field \(X \in \Gamma(TF)\).
Proof. The proof of the lemma can be directly achieved by virtue of Lemma 2.3 and Proposition 2.4. □

2.3. Pointwise slant submanifolds
Following the notion of pointwise slant immersion in [2, 8]. We define

Definition 2.8. A submanifold $M$ of a Lorentzian almost paracontact manifold $\bar{M}^{2m+1}(\phi, \xi, \eta, g)$ is said to be pointwise slant if at each given point $p \in M$, the slant angle or Wirtinger angle $\theta(X)$ between $\phi(X)$ and the space $T_pM$ is independent of the choice of the non-zero vector $X \in \Gamma(TM)$ linearly independent of $\xi$. In this case, the angle $\theta$ can be viewed as a function on $M$, which is called the slant function of the pointwise slant submanifold.

Remark 2.9. A point $p$ in a pointwise slant submanifold is called a totally real point if its slant function $\theta$ satisfies $\cos \theta = 0$ at $p$. Similarly, a point $p$ is called a complex point if its slant function satisfies $\sin \theta = 0$ at $p$. A pointwise slant submanifold $M$ of Lorentzian almost paracontact manifold $\bar{M}$ is said to be
- totally real if every point of $M$ is a totally real point.
- pointwise proper slant if it contains no totally real points.

If we denote the orthogonal distribution to $\xi \in \Gamma(TM)$ by $\mathcal{D}$ then the orthogonal direct decomposition is given as follows:

$$TM = \mathcal{D} \oplus \{\xi\},$$

where, span of the characteristic vector field $\xi$ generates the 1-dimensional distribution $\{\xi\}$ on $M$.

Furthermore, we give the following useful characterization of pointwise slant submanifolds in Lorentzian almost paracontact manifolds:

Proposition 2.10. Let $M$ be a submanifold in a Lorentzian almost paracontact manifold $\bar{M}^{2m+1}(\phi, \xi, \eta, g)$ such that $\xi \in \Gamma(TM)$. Then $M$ is pointwise slant if and only if $t^2 = \cos^2 \theta (I + \eta \otimes \xi)$ for some real-valued function $\theta$ defined on the tangent bundle $TM$ of $M$.

Proof. The proof of the proposition is similar to the proof of Lemma 2.1 of [8] for Hermitian ambient. □

The following corollaries are straightforward consequences of the above result:

Corollary 2.11. Let $\mathcal{D}_\theta$ be a distribution on $M$. Then $\mathcal{D}_\theta$ is pointwise slant if and only if there exists a function $\theta$ such that $(tP_\theta)^2Z = \cos^2 \theta Z$ for $Z \in \Gamma(\mathcal{D}_\theta)$, where $P_\theta$ denotes the orthogonal projection on $\mathcal{D}_\theta$.

Corollary 2.12. If $M$ is a pointwise slant submanifold and $\mathcal{D}_\theta$ a pointwise slant distribution on $M$ such that $\xi \in \Gamma(TM)$, then

$$g(tZ, tW) = \cos^2 \theta \{\eta(Z)\eta(W) + g(Z, W)\},$$  \hspace{1cm} (2.19)

$$g(nZ, nW) = \sin^2 \theta \{\eta(Z)\eta(W) + g(Z, W)\},$$  \hspace{1cm} (2.20)
for any $Z, W \in \Gamma(TM)$.

3. Pointwise semi-slant submanifolds

Analogous to [24] in this section, we define and study pointwise semi-slant submanifolds in a Lorentzian almost paracontact manifold $\bar{M}^{2m+1}$. We also, derive important results and deduce the geometry of leaves of the involutive distributions involved with the definition of such submanifolds.

Definition 3.1. Let $M$ a real submanifold of a Lorentzian almost paracontact manifold $\bar{M}^{2m+1}(\phi, \xi, \eta, g)$. Then we say that $M$ is a pointwise semi-slant submanifold, if it is furnished with the pair of complimentary distribution $(\mathcal{D}_T, \mathcal{D}_\theta)$ satisfying the conditions:

(i) $TM = \mathcal{D}_T \oplus \mathcal{D}_\theta \oplus \{\xi\}$,
(ii) the distribution $\mathcal{D}_T$ is invariant under $\phi$, i.e., $\phi(\mathcal{D}_T) \subseteq \mathcal{D}_T$ and
(iii) the distribution $\mathcal{D}_\theta$ is pointwise slant distribution with slant function $\theta$.

A pointwise semi-slant submanifold is proper if $\mathcal{D}_T \neq \{0\}$ and $\theta$ is not a constant. Furthermore, we say a pointwise semi-slant submanifold mixed geodesic if the second fundamental form $h$ of $M$ satisfies $h(\mathcal{D}_T, \mathcal{D}_\theta) = 0$.

In particular, we have the following:

(i). If $\mathcal{D}_T = \{0\}$ and $\theta = \pi/2$, then $M$ is an anti-invariant submanifold [1, 28].
(ii). If $\mathcal{D}_\theta = \{0\}$, then $M$ is an invariant submanifold [1, 28].
(iii). If $\mathcal{D}_T = \{0\}$ and $\mathcal{D}_\theta \neq \{0\}$ with $\theta$ globally constant such that $\theta \in (0, \pi/2)$, then $M$ is a proper slant submanifold [1].
(iv.) If $\mathcal{D}_T \neq \{0\}$ and $\mathcal{D}_\theta \neq \{0\}$ such that slant angle $\theta = \pi/2$, then $M$ is a semi-invariant submanifold [26].
(v). If $\mathcal{D}_T \neq \{0\}$ and $\mathcal{D}_\theta \neq \{0\}$ such that slant angle $\theta$ satisfies that $\theta \in (0, \pi/2)$ is independent of point and vector fields on $M$, then $M$ is a proper semi-slant submanifold [5, 28].
(vi). If $\mathcal{D}_T = \{0\}$ and $\theta$ is a slant function, then $M$ is a pointwise slant submanifold [2].

Let us consider that $M$ be a pointwise semi-slant submanifold of a Lorentzian paracosymplectic manifold $\bar{M}^{2m+1}$. If $\mathcal{P}_T$ and $\mathcal{P}_\theta$ denoted the projections on the distributions $\mathcal{D}_T$ and $\mathcal{D}_\theta$, respectively. Then we can write for any $Z \in \Gamma(TM)$ that

$$Z = \mathcal{P}_T Z + \mathcal{P}_\theta Z + \eta(Z)\xi.$$  \hspace{1cm} (3.1)

Previous equation by operating $\phi$ and Eqs. (2.2), (2.12), becomes $\phi Z = t\mathcal{P}_T Z + t\mathcal{P}_\theta Z + n\mathcal{P}_\theta Z$. Thus, from previous expression, we conclude that $t\mathcal{P}_T Z \in \Gamma(\mathcal{D}_T)$, $n\mathcal{P}_T X = 0$, and $t\mathcal{P}_\theta X \in \Gamma(\mathcal{D}_\theta)$. Using Eq. (2.12) and above expressions in Eq. (3.1), we deduce that $tZ = t\mathcal{P}_T Z + t\mathcal{P}_\theta Z$, $nZ = n\mathcal{P}_\theta Z$, for
any \( Z \in \Gamma(TM) \). Since, \( \mathfrak{D}_\theta \) is pointwise slant distribution, by the consequences of Corollary 2.11, we obtain that
\[
t^2 Z = (\cos^2 \theta)Z, \tag{3.2}
\]
for any \( Z \in \Gamma(\mathfrak{D}_\theta) \) and some real-valued function \( \theta \) defined on \( M \).

Now, by virtue of above construction, we have the following characterization result for pointwise semi-slant submanifold:

**Lemma 3.2.** If \( M \) is a proper pointwise semi-slant submanifold of a Lorentzian paracosymplectic manifold \( \bar{M}^{2m+1} \) such that \( \xi \in \Gamma(TM) \), then
\[
g(tZ, tW) = \cos^2 \theta g(\phi Z, \phi W) \tag{3.3}
\]
\[
g(nZ, nW) = \sin^2 \theta g(\phi Z, \phi W) \tag{3.4}
\]
for all \( Z, W \in \Gamma(\mathfrak{D}_\theta) \).

**Proof.** From Eq. (2.12), we can write \( g(tZ, tW) = g(\phi Z - nZ, tW) \). Hence \( g(tZ, tW) = g(Z, \phi tW) \). Using Eqs. (2.3) and (3.2), we obtain Eq. (3.3). Using Eq. (3.3) we get Eq. (3.4). \(\square\)

Next, we will find the necessary and sufficient conditions for involutive and foliation of distributions associated with pointwise semi-slant submanifold of a Lorentzian paracosymplectic manifold.

**Lemma 3.3.** If \( M \) is a proper pointwise semi-slant submanifold of a Lorentzian paracosymplectic manifold \( \bar{M}^{2m+1} \). Then a necessary and sufficient condition for the distribution \( \mathfrak{D}_T \oplus \{\xi\} \) to be involutive is that the second fundamental form \( h \) of \( M \) satisfies \( h(X, tY) = h(tX, Y) \), for any \( X, Y \in \Gamma(\mathfrak{D}_T \oplus \{\xi\}) \) and \( Z \in \Gamma(\mathfrak{D}_\theta) \).

**Proof.** In general it is not hard to see that, \( g([X, Y], Z) = g(\nabla_X Y - \nabla_Y X, Z) \) for any \( X, Y, Z \in \Gamma(TM) \). Above expression by the use of Eq. (2.3) and Lemma 2.3, reduced to
\[
g([X, Y], Z) = g(\phi \nabla_X Y, \phi Z) - g(\phi \nabla_Y X, \phi Z). \tag{3.5}
\]
Using Eqs. (2.7) and (2.12) in Eq. (3.5), we obtain for any \( X, Y \in \Gamma(\mathfrak{D}_T \oplus \{\xi\}) \) and \( Z \in \Gamma(\mathfrak{D}_\theta) \) that
\[
g([X, Y], Z) = g(\phi \nabla_X Y, tZ) + g(\nabla_X tY, nZ) - g(\phi \nabla_Y X, tZ) - g(\nabla_Y tX, nZ). \tag{3.6}
\]
Employing Eq. (2.4), (2.9) and (2.12) in Eq. (3.6), we achieve that
\[
g([X, Y], Z) = g(\nabla_X Y, t^2 Z + ntZ) + g(h(X, tY), nZ)
- g(\phi \nabla_Y X, t^2 Z + ntZ) - g(h(Y, tX), nZ). \tag{3.7}
\]
Using the fact that $h$ is symmetric and Eqs. (2.9), (3.2) in equation (3.6), we derive that
\[
\sin^2 \theta g([X, Y], Z) = g(h(X, tY), nZ) - g(h(tX, Y), nZ).
\]  
(3.8)

Thus, from (3.8), we conclude that $[X, Y] \in \Gamma(D_T \oplus \{\xi\})$ if and only if $h(X, tY) = h(tX, Y)$. Since, $M$ is a proper pointwise semi-slant submanifold and $X, Y, Z$ are non-null vector fields. This completes the proof of the lemma. □

**Lemma 3.4.** If $M$ is a proper pointwise semi-slant submanifold of a Lorentzian paracosymplectic manifold $\bar{M}^{2m+1}$. Then a necessary and sufficient condition for the distribution $D_T \oplus \{\xi\}$ defines a totally geodesic foliation is that metric $g$ in $M$ satisfies $g(A_{ntZ}W, X) = -g(A_{nZ}W, X)$, for any $X, Y \in \Gamma(D_T \oplus \{\xi\})$ and $Z \in \Gamma(D_T)$.

**Proof.** For any $X, Y \in \Gamma(D_T \oplus \{\xi\})$ and $Z \in \Gamma(D_T)$, we have from Gauss formula that $g(\nabla_X Y, Z) = g(\nabla_X Y, Z)$. Employing Eqs. (2.3), (2.7), (2.11)-(2.13) and Lemma 2.3 in above expression, we obtain that
\[
g(\nabla_X Y, Z) = g(\nabla_X Y, t^2 Z) + g(h(X, Y), ntZ) + g(h(X, tY), nZ).
\]  
(3.9)

Using Eq. (3.2) in equation Eq. (3.9), we arrive at
\[
g(\nabla_X Y, Z) = \cos^2(\theta)g(\nabla_X Y, Z) + g(h(X, Y), ntZ) + g(h(X, tY), nZ).
\]  
(3.10)

From above equation, we conclude that
\[
\sin^2 \theta g(\nabla_X Y, Z) = g(h(X, Y), ntZ) + g(h(X, tY), nZ)
\]  
(3.11)

Thus, from (3.11), we deduce that $\nabla_X Y \in \Gamma(D_T)$ if and only if $g(h(X, Y), ntZ) + g(h(X, tY), nZ) = 0$. Since, $M$ is a proper pointwise semi-slant submanifold and $X, Y, Z$ are non-null vector fields. This completes the proof of the lemma. □

**Lemma 3.5.** If $M$ is a proper pointwise semi-slant submanifold of a Lorentzian paracosymplectic manifold $\bar{M}^{2m+1}$ with $\xi \in \Gamma(TM)$. Then the pointwise slant distribution $\mathcal{D}_\theta$ is involutive if and only if the metric $g$ on $M$ satisfies $g(A_{ntZ}W, X) = g(A_{nZ}W - A_{ntW}X, Z)$, for any $X \in \Gamma(D_T)$ and $Z, W \in \Gamma(D_T)$.

**Proof.** The proof of this lemma can be achieved by following same steps as used in proving Lemma 3.3. □

**Lemma 3.6.** If $M$ is a proper pointwise semi-slant submanifold of a Lorentzian paracosymplectic manifold $\bar{M}^{2m+1}$ such that $\xi \in \Gamma(TM)$. Then the pointwise slant distribution $\mathcal{D}_\theta$ defines a totally geodesic foliation if and only if the metric $g$ on $M$ satisfies $g(A_{ntW}X, Z) = -g(A_{nW}X, Z)$, for any $X \in \Gamma(D_T)$ and $Z, W \in \Gamma(D_T)$.

**Proof.** The proof of the lemma follow same steps as in Lemma 3.4. □
4. Pointwise semi-slant warped product submanifolds

In this section, we first define pointwise semi-slant warped product submanifolds $M$, and then examine the existence or non existence results and also derive characterization theorem of such submanifolds in a Lorentzian paracosymplectic manifold $\tilde{M}^{2m+1}$ with the structure vector field $\xi$ tangent to $M$.

**Definition 4.1.** A pointwise semi-slant submanifold $M$ of a Lorentzian almost paracosymplectic manifold $\tilde{M}^{2m+1}(\phi, \xi, \eta, g)$ is called a pointwise semi-slant warped product if it is a warped product of the form: $M_T \times_f M_\Theta$ or $M_\Theta \times_f M_T$, where $M_T$ (resp., $M_\Theta$) is invariant (resp., pointwise proper slant) integral submanifolds of $\mathcal{D}_T$ (resp., $\mathcal{D}_\Theta$) on $M$ and $f$ is a non-constant positive smooth function on the first factor. If the warping function $f$ is constant then a pointwise semi-slant warped product submanifold is said to be a pointwise semi-slant product or trivial product.

From the direct consequence of Theorem 2.6, we have the following results for warped product submanifolds when $\xi$ is tangent to second factor;

**Proposition 4.2.** There doesn’t exist a non-trivial pointwise semi-slant warped product submanifold of the form $M = M_T \times_f M_\Theta$ of a Lorentzian paracosymplectic manifold $\tilde{M}^{2m+1}$ such that the structure vector $\xi$ is tangent to $M_\Theta$.

**Proposition 4.3.** There doesn’t exist a non-trivial pointwise semi-slant warped product submanifold of the form $M = M_\Theta \times_f M_T$ of a Lorentzian paracosymplectic manifold $\tilde{M}^{2m+1}$ such that the structure vector $\xi$ is tangent to $M_T$.

Now, we prove an important results for warped product submanifolds when $\xi$ is tangent to first factor;

**Theorem 4.4.** Let $\tilde{M}^{2m+1}$ be a Lorentzian paracosymplectic manifold. Then there does not exist non-trivial pointwise semi-slant warped product submanifold $M = M_\Theta \times_f M_T$ of $\tilde{M}^{2m+1}$ such that $\xi$ is tangent to $M_\Theta$.

*Proof.* We have from Eq. (2.9), that $g(\nabla_X Z, Y) = g(\bar{\nabla}_X Z, Y)$, for any $X, Y \in \Gamma(M_T)$ and $Z \in \Gamma(M_\Theta)$. Employing Eqs. (2.3), (2.4), (2.7), (2.12) and Lemma 2.3 in right hand side of above expression, we obtain that

$$g(\nabla_X Z, Y) = g(\bar{\nabla}_X t^2 Z, Y) + g(\bar{\nabla}_X nt Z, Y) + g(\bar{\nabla}_X n Z, \phi Y). \quad (4.1)$$

Using Eqs. (2.10), (2.11), (3.2) and the fact $g(Z, Y) = 0$ in Eq. (4.1), we obtain that

$$g(\nabla_X Z, Y) = \cos^2 \theta g(\nabla_X Z, Y) - g(h(X, Y), nt Z) - g(h(X, \phi Y), n Z). \quad (4.2)$$

Applying Eqs.(2.9) and (4.2) in above equation, we conclude that

$$\sin^2 \theta g(\nabla_X Z, Y) = -g(h(X, Y), nt Z) - g(h(X, \phi Y), n Z). \quad (4.3)$$
Interchanging $X$ and $Y$ in Eq. (4.3), we get
\[ \sin^2 \theta g(\nabla Y Z, X) = -g(h(X, Y), ntZ) - g(h(Y, \phi X), nZ). \] (4.4)

From Eqs. (4.3), (4.4) and Proposition 2.4, we achieve that
\[ g(h(X, \phi Y), nZ) = g(h(Y, \phi X), nZ). \] (4.5)

On the other hand, by the use of Eqs. (2.3), (2.7)-(2.12) and Lemma 2.3, we arrive at
\[ g(h(X, \phi Y), nZ) = -g(\nabla_X Z, Y) + g(\nabla_X tZ, \phi Y), \] (4.6)

for any $X, Y \in \Gamma(M_T)$ and $Z \in \Gamma(M_\theta)$. Now, from Eq. (4.6), we conclude that the Eq. (4.5) hold if and only if $g(\nabla_X tZ, \phi Y) = 0$. Moreover, by using Proposition 2.4 and replacing $Z$ by $tZ$, $X$ by $\phi X$ in above expression, we derive that $t^2 Z(\ln f)g(\phi X, \phi Y) = 0$. Hence, previous expression in light of Eqs. (2.3), (3.2) and fact that $\eta(X)\eta(Y) = 0$ reduced to, $\cos^2 \theta Z(\ln f)g(X, Y) = 0$. Thus $f$ is constant. Since, $M_\theta$ is pointwise proper slant submanifold and $X, Y, Z$ are non-null vector fields. This completes the proof of the proposition. \(\square\)

\textbf{Remark 4.5.} It is no hard to conclude that the Theorem 5.2 in [28] for $\theta$ globally constant and Theorem 4.1 in [27] for $\theta = \pi/2$ can be treat as the particular cases of the Theorem 4.4.

Next, we have an important lemma for later use

\textbf{Lemma 4.6.} If $M = M_T \times_f M_\theta$ is a non-trivial pointwise semi-slant warped product submanifold in a Lorentzian paracosymplectic manifold $\tilde{M}^{2m+1}$, then
\begin{enumerate}
  \item[(a)] $g(h(X, Z), ntW) = -tX(\ln f)g(tW, Z) + X(\ln f) \cos^2 \theta g(Z, W),$
  \item[(b)] $g(h(tX, Z), nW) = -X(\ln f)g(W, Z) + tX(\ln f)g(Z, tW),$
  \item[(c)] $g(h(X, W), ntZ) = -tX(\ln f)g(W, tZ) + X(\ln f) \cos^2 \theta g(Z, W),$
  \item[(d)] $g(h(tX, W), nZ) = -X(\ln f)g(W, Z) + tX(\ln f)g(tZ, W),$
\end{enumerate}

for all $X \in (\mathcal{D}_T \oplus \{\xi\})$ and $Z, W \in \Gamma(\mathcal{D}_\theta)$.

\textbf{Proof.} From Eqs. (2.4), (2.7), (2.12) and Gauss formulas, we attain that
\[ g(h(X, W), nZ) = g(\nabla_X tW, Z) + g(\nabla_X nW, Z) - g(\nabla_X W, tZ). \] (4.7)

Employing Proposition 2.4 and Eq. (2.14) in Eq. (4.7), we arrive at
\[ g(h(X, W), nZ) = g(\nabla_X nW, Z). \]

By using Eq. (2.10) in right hand side of previous expression, we derive that
\[ g(A_n Z W, X) = -g(A_{n W} Z, X). \] (4.8)

Moreover, Eq. (4.8), by replacing $W$ by $tW$ and Eq. (2.10) becomes
\[ g(h(tW, X), nZ) = -g(h(Z, X), ntW). \]
Applying Eqs. (2.9), (2.10), (2.12) and the fact that structure is Lorentzian paracosymplectic in above expression, we obtain that
\[ g(h(Z, X), ntW) = -g(\nabla_{tW}tX, Z) + g(\nabla_{tW}X, tZ). \] (4.9)
Using Proposition 2.4, Eq. (3.3) and the fact that \( \xi \) is orthogonal to \( Z, W \) in Eq. (4.9), we achieve the formula-(a). Thus, replacing \( W \) by \( tW \) and using Eq. (3.2) in (4.9), we get
\[ g(h(Z, X), nW) = -tX(lnf)g(W, Z) + X(lnf)g(Z, tW). \] (4.10)
Now for formula-(b), we first replace \( X \) by \( \phi X \) in Eq. (4.10), and then in light of Eqs. (2.3), (2.4), (2.7), (2.11) and fact that \( \eta(Z) = 0 \) we achieve the desired. On the other hand, Using Eq. (2.10) and interchanging \( Z \) by \( tZ \) in Eq. (4.8), we deduce that
\[ g(h(W, X), ntZ) = -g(h(tZ, X), nW) \] (4.11)
Hence, formula-(c) and formula-(d) can be attained with the help of Eq. (4.11) and by following similar steps as used to prove formula-(a) and formula-(b). This completes the proof of lemma. \( \square \)

Now, we prove an important result as the characterization for pointwise semi-slant warped product submanifold in a Lorentzian paracosymplectic manifold.

**Theorem 4.7.** Let \( M \rightarrow \bar{M}^{2m+1} \) be an isometric immersion of a submanifold \( M \) into a Lorentzian paracosymplectic manifold \( \bar{M}^{2m+1} \). Then a necessary and sufficient condition for \( M \) to be locally non-trivial pointwise semi-slant warped product submanifold \( M_T \times_f M_\theta \) is that the shape operator of \( M \) satisfies
\[ A_{ntW}X + A_{nW}tX = (\cos^2 \theta - 1)X(\nu)W, \] (4.12)
\[ \forall X \in \Gamma(\mathcal{D}_T + \{\xi\}), W \in \Gamma(\mathcal{D}_\theta) \] and for some function \( \nu \) on \( M \) such that \( Z(\nu) = 0 \), \( Z \in \Gamma(\mathcal{D}_\theta) \).

**Proof.** Let \( M \) be a non-trivial pointwise semi-slant warped product submanifold of \( \bar{M}^{2m+1} \). Then clearly from formula-(a) and formula-(b) of lemma 4.6, we obtain Eq. (4.12). Since \( f \) is a function on \( M_T \), setting \( \mu = \ln f \) implies that \( Z(\mu) = 0 \). Conversely, consider that \( M \) is a pointwise semi-slant submanifold of \( \bar{M}^{2m+1} \) such that Eq. (4.12) satisfied. By taking inner product of Eq. (4.12) with \( X \) and from Lemma 3.4, we conclude that the integral manifold \( M_T \) of \( \mathcal{D}_T + \{\xi\} \) defines a totally geodesic foliation in \( M \). Then by Lemma 3.5, the distribution \( \mathcal{D}_\theta \) is involutive if and only if
\[ g(A_{nW}Z - A_nZW, tX) = g(A_{ntZ}W - A_{ntW}Z, X), \]
for all \( X \in \mathcal{D} \) and \( Z, W \in \mathcal{D}_\theta \). Above equation in view of equation (2.11) and fact that \( h \), is symmetric can be rearranged as;
\[ g(A_{ntW}X + A_{nW}tX, Z) = g(A_{ntZ}X + g(A_nZtX, W), \] (4.13)
for all \(X \in \Gamma(\mathcal{D}_T)\) and \(Z, W \in \Gamma(\mathcal{D}_\theta)\). Employing formula-(c) and formula-(d) of Lemma 4.6 and Eqs. (2.9), (2.11) in right hand side of (4.13), we achieve that
\[
g(A_{ntZ}X + A_{nZ}tX, W) = \sin^2 \theta g(\nabla_W Z, X). \tag{4.14}
\]
Now, taking inner product of Eq. (4.12) with \(Z\), we find that
\[
g(A_{ntW}X + A_{nW}tX, Z) = (\cos^2 \theta - 1)g(X(\nu)W, Z). \tag{4.15}
\]
From Eqs. (4.13), (4.14) and (4.15), we attain that
\[
g(\nabla_W Z, X) = (\cot^2 \theta - \csc^2 \theta)X(\nu)g(Z, W), \tag{4.16}
\]
where, \(\nu = \ln f\). Hence, from Eq.(4.16), we conclude that the integrable manifold of \(\mathcal{D}_\theta\) is totally umbilical submanifold in \(M\) and its mean curvature is non-zero and \(Z(\nu) = 0\) for all \(Z \in \Gamma(\mathcal{D}_\theta)\). Thus, from [13], we can say that \(M\) is a locally non-trivial pointwise semi-slant warped product submanifold of \(M^{2m+1}\). This completes the proof of the theorem. □

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