STEF DOMAINS WITH EXOTIC CONTACT BOUNDARIES.

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ABSTRACT. We construct infinitely many Stein domains almost symplectomorphic to any given one, whose boundaries are all in the same almost contactomorphism class but are non-contactomorphic pairwise. We introduce a new invariant for asymptotically dynamically convex contact manifolds, the positive idempotent index, which is used to distinguish these contact structures.

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1. Introduction

This paper is mainly concerned with the problem of distinguishing contact boundaries of Stein manifolds, which has a long history. Y. Eliashberg \cite{Eliashberg91} constructed an exotic contact structure representing the standard almost contact structure on $S^{4k+1}$, and I. Ustilovsky \cite{Ustilovsky99} proved that every almost contact class on $S^{4k+1}$ has infinitely many different contact structures. M. McLean \cite{McLean07} has shown that there are infinitely many exotic Stein structures $\mathbb{C}^n_k$ on $\mathbb{C}^n$, $n \geq 4$. Using flexible Weinstein structures, O. Lazarev \cite{Lazarev16} proved that any contact manifold admitting an almost Weinstein filling admits infinitely many exotic contact structures with flexible fillings. The aim of this paper is to prove the following theorem:

**Theorem 1.1.** For any Stein domain $V^{2n}$, $n > 4$, there are infinitely many Stein domains $(V_i)_{i \in \mathbb{N}}$ such that:

- $V_i$ is almost symplectomorphic to $V$ for each $i \in \mathbb{N}$,
- the contact boundaries of $V_i$ are in the same almost contact class as $\partial V$.
- $\partial V_i$ are mutually non-contactomorphic.

1.1. Sketch of the proof. The contact structure on $V_i$ in Theorem 1.1 is asymptotically dynamically convex, which will be defined in Section 3.1. To distinguish them, we introduce an invariant called the *positive idempotent group* $I_+(Y)$ for every contact manifold $(Y, \xi)$, see Section 3.2. The definition of positive idempotent group $I_+(Y)$ depends on the filling $W$: it is well defined when $SH_+(W) \neq 0$ for some Liouville filling $W$, and $I_+(Y)$ is independent of filling when $(Y, \xi)$ is an ADC contact manifold. In the case when $I_+(Y)$ is finite, we can define the *positive idempotent index* $i(Y) := |I_+(Y)|$ (see Section 3.2). The main theorem 1.1 is based on the following theorem, which will be proved in Section 6:

**Theorem 1.2.** There exists infinitely many Weinstein domains $(W^{2n}_k, \lambda_k, \psi_k)$, $n > 4$ such that

- $(Y_k, \xi_k) := (\partial W_k, \lambda_k)$ is asymptotically dynamically convex and in the same almost contact class as $(S^{2n-1}, \xi_{std})$,
- $W_k$ is almost symplectomorphic to $(B^{2n}, \omega_{std})$,
- $I_+(Y_k)$ is finite for all $k$, and $i(Y_k) \neq i(Y_l)$ for $k \neq l$.

**Remark 1.3.** In \cite{McLean07}, for $n > 4$, M. McLean constructed infinitely many exotic Stein structures $\mathbb{C}^n_k$ on $\mathbb{C}^n$, which are Weinstein homotopic to $(W_k, \lambda_k, \psi_k)$. So the boundaries of $\mathbb{C}^n_k$ are all ADC, and non-contactomorphic.

We will need the fact that any almost Weinstein domain admits a flexible Weinstein structure in the same almost symplectic class (See Section 2.7). Moreover, if a contact manifold admits a flexible filling, then it is asymptotically dynamically convex, as stated in the following lemma:

**Lemma 1.4** (Corollary 4.1 \cite{Lazarev16}). If $(Y^{2n-1}, \xi)$, $n \geq 3$, has a flexible filling, then $(Y, \xi)$ is asymptotically dynamically convex.

**Proof of Theorem 1.1.** For any Stein domain $V$, by the Weinstein existence $h-$ principle (see Section 2.7), there is a flexible Weinstein domain $W$ that is almost symplectomorphic to $V$. Let $V_i = W_2 W_i$ be the boundary connected sum of $W$ and $W_i$. Since $W_i$ is almost symplectomorphic to $(B^{2n}, \omega_{std})$, $V_i$ is almost symplectomorphic to $V$. The boundary connect sum is equivalent to attaching a Weinstein 1-handle, so $V_i$ is a Weinstein domain. Theorem 2.7
allows us to deform a Weinstein structure into a Stein structure, which is still denoted by \( V_i \). Meanwhile,
\[
\partial V_i = \partial W \partial W_i = \partial W_i Y_i
\]
is the connect sum. So \( \partial V_i \) is in the same almost contact class as \( \partial W \) since \((Y_i, \xi_i)\) is almost contactomorphic to \((S^{2n-1}, \xi_{std})\).

So we only need to verify the third condition in Theorem 1.1, which follows from the following proposition:

**Proposition 1.5.** Let \( V_i \) be defined as above, then the contact structure on the boundary is asymptotically dynamically convex. Moreover, the positive idempotent group \( I_+(\partial V_i) \cong I_+(Y_i) \).

We will defer the proof till Subsection 3.3.

\( \square \)

### 1.2. Conventions and notation.

(See Section 2 for detailed definitions.)

Let \( \lambda \) be a Liouville 1-form on a Liouville manifold \( W \).

\[
d\lambda(\cdot, J\cdot) = g_J \quad (\text{Riemannian metric}),
\]

\[
d\lambda(X_H, \cdot) = -dH, \quad X_H = J\nabla H \quad (\text{Hamiltonian vector field}),
\]

\[
\mathcal{L} \hat{W} := C^\infty(S^1, \hat{W}), \quad S^1 = \mathbb{R}/\mathbb{Z} \quad (\text{loop space}),
\]

\[
A_H : \mathcal{L} \hat{W} \to \mathbb{R}, \quad A_H(x) := \int_{S^1} x^*\lambda - \int_{S^1} H(t, x(t)) dt \quad (\text{action}),
\]

\[
\nabla A_H(x) = -J(x)(\dot{x} - X_H(t, x)) \quad (L^2\text{-gradient}),
\]

\[
u : \mathbb{R} \to \mathcal{L} W, \quad \partial_s u = \nabla A_H(u(s, \cdot)) \quad (\text{gradient line})
\]

\[
\iff \partial_s u + J(u)(\partial_t u - X_H(t, u)) = 0 \quad (\text{Floer equation}), \tag{1.1}
\]

\[
P(H) := \text{Crit}(A_H) = \{1\text{-periodic orbits of the Hamiltonian vector field } X_H \},
\]

For each \( h \in H_1(W) \), \( P^h(H) := \text{Crit}_h(A_H) = \{x \in P(H)[[x] = h \in [S^1 \to W] \} \)

\[
\mathcal{M}(x_-, x_+; H, J) = \{u : \mathbb{R} \times S^1 \to W \mid \partial_s u = \nabla A_H(u(s, \cdot)), u(\pm \infty, \cdot) = x_\pm \}/\mathbb{R}
\]

(moduli space of Floer trajectories connecting \( x_\pm \in P(H) \)),

\[
\mathcal{M}_h(x_-, x_+; H, J) = \{u : \mathbb{R} \times S^1 \to W \mid \partial_s u = \nabla A_H(u(s, \cdot)), u(\pm \infty, \cdot) = x_\pm \in P^h(H) \}/\mathbb{R}
\]

(moduli space of Floer trajectories connecting \( x_\pm \in P^h(H) \)),

\[
\dim \mathcal{M}(x_-, x_+; H, J) = \mu_{CZ}(x_+) - \mu_{CZ}(x_-) - 1,
\]

\[
A_H(x_+) - A_H(x_-) = \int_{\mathbb{R} \times S^1} |\partial_s u|^2 ds dt = \int_{\mathbb{R} \times S^1} u^*(d\lambda - dH \wedge dt).
\]

Here the formula expressing the dimension of the moduli space in terms of Conley-Zehnder indices is to be understood with respect to a symplectic trivialization of \( u^*TW \).

Let \( \mathbb{K} \) be a field and \( a < b \) with \( a, b \notin \text{Spec}(\partial W, \alpha) \). We define the filtered Floer chain groups with coefficients in \( \mathbb{K} \) by

\[
SC_{<b}^s(H) := \bigoplus_{x \in P(H), A_H(x) < b} \mathbb{K} \cdot x, \quad SC_{(a,b)}^s(H) = SC_{<b}^s(H)/SC_{<a}^s(H),
\]
with the differential \( d : SC_*^{a,b}(H) \to SC_*^{a,b}(H) \) given by
\[
dx_+ = \sum_{x \in \mu_{CZ}(x_+) - 1} \#M(x_-, x_+; H, J) \cdot x_-
\]
Here \( \# \) denotes the signed count of points with respect to suitable orientations. We think of the cylinder \( \mathbb{R} \times S^1 \) as the twice punctured Riemann sphere, with the positive puncture at \( +\infty \) as incoming, and the negative puncture at \( -\infty \) as outgoing. This terminology makes reference to the corresponding asymptote being an input, respectively an output for the Floer differential. Note that the differential decreases both the action \( A_H \) and the Conley-Zehnder index. The filtered Floer homology is now defined as
\[
SH_*^{a,b}(H) = \ker d/\text{im } d.
\]
Note that for \( a < b < c \) the short exact sequence
\[
0 \to SC_*^{a,b}(H) \to SC_*^{a,c}(H) \to SC_*^{b,c}(H) \to 0
\]
induces a tautological exact triangle
\[
SH_*^{a,b}(H) \to SH_*^{a,c}(H) \to SH_*^{b,c}(H) \to SH_*^{a,b}(H)[-1].
\] (1.2)

**Remark.** We will suppress the field \( \mathbb{K} \) from the notation. As noted in the Introduction, the definition can also be given with coefficients in a commutative ring. In this paper, \( \mathbb{K} = \mathbb{Z}_2 \).

**Notation.** Let \( a = (a_0, a_1, \cdots , a_n) \) be an \((n+1)\)-tuple of integers \( a_i > 1, z := (z_0, z_1, \cdots , z_n) \in \mathbb{C}^{n+1} \), and set \( f(z) := z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n} \), and let \( B(s) \) to be the closed ball of radius \( s \).
\[
V_a(t) := \{ (z_0, z_1, \cdots , z_n) \in \mathbb{C}^{n+1} | f(z) = t \}.
\]
We will often suppress \( a \) from the notation. Let \( X_t^s = V(t) \cap B(s) \).

and let \( \beta \in C^\infty(\mathbb{R}) \) be a smooth monotone decreasing cut-off function with \( \beta(x) = 1, x \leq \frac{1}{4} \) and \( \beta(x) = 0, x \geq \frac{3}{4} \).
\[
U_a(\epsilon) := \{ z \in \mathbb{C}^{n+1} | z_0^{a_0} + \cdots + z_n^{a_n} = \epsilon \cdot \beta(||z||^2) \}.
\]
Likewise \( a \) will often be suppressed. Moreover let \( W_\epsilon^s = U(\epsilon) \cap B(s) \).

2. **Background**

2.1. **Symplectic and contact structures.** A symplectic manifold \((M, \omega)\) is a smooth \(2n\)-dimensional manifold \( M \) together with a nondegenerate, closed 2-form \( \omega \). A function \( H \in C^\infty(M) \) on a symplectic manifold \((M, \omega)\) is called Hamiltonian. We define its Hamiltonian vector field \( X_H \) via
\[
dH = -\iota_{X_H} \omega = -\omega(X_H, \cdot) = \omega(\cdot, X_H).
\]

A contact manifold \( \Sigma \) is a smooth \((2n-1)\)-dimensional manifold together with a completely non-integrable smooth hyperplane distribution \( \xi \in T\Sigma \). The distribution is called a contact structure. It can be locally defined as \( \xi = \ker \alpha \) for some local 1-form \( \alpha \) such that \( \alpha \wedge (d\alpha)^{n-1} \neq 0 \) pointwise. If \( \alpha \) is globally defined, then \( \alpha \) is called a contact form. We will always assume \( \alpha \) is globally defined. Under this assumption \( \alpha \wedge (d\alpha)^{n-1} \) gives rise to a volume form and hence
\( \Sigma \) is orientable. Once an orientation is chosen we require that \( \alpha \wedge (d\alpha)^{n-1} > 0 \). Associated with a contact form \( \alpha \) one has a Reeb vector field \( R \), uniquely defined by the equations
\[
\iota_R(\alpha) = 0,  \\
\iota_R(d\alpha) = 1.
\]
Clearly \( R \) is transverse to \( \xi \). If we have two different forms \( \alpha, \alpha' \) which define the same contact structure, then we can find a nowhere vanishing function \( f \) such that \( \alpha' = f \cdot \alpha \). Indeed, \( f = \alpha'(R) \). The flow of a Reeb vector field is called Reeb flow, and closed trajectories of Reeb flow are called the Reeb orbits. The action of a Reeb orbit \( \gamma \) is defined as
\[
A(\gamma) := \int_{S^1} \gamma^* \alpha
\]
Note that \( A(\gamma) \) is always positive and equals the period of \( \gamma \). The spectrum \( \text{spec}(\Sigma, \alpha) \) is the set of actions of all Reeb orbits of \( \alpha \). We will need the following definition for Reeb trajectories which is part of a closed Reeb orbit.

**Definition 2.2.** \( \gamma : [0, T] \rightarrow X \) is called a fractional Reeb orbit for contact manifold \((X, \xi)\) if there is a closed Reeb orbit \( \gamma_0 \) of \((X, \xi)\) such that \( \gamma(t) = \gamma_0(t), t \in [0, T] \).

We say that a Reeb orbit \( \gamma \) of \( \alpha \) is non-degenerate if the linearized Reeb flow along \( \gamma \) from \( \xi_p \) to itself for some \( p \in \gamma \) has no eigenvalue 1. Moreover we say that a contact form is non-degenerate if all Reeb orbits of \( \alpha \) are non-degenerate. We can always assume a contact form is non-degenerate after a \( C^0 \)-small perturbation, since a generic contact form is non-degenerate. Notice that when \( \alpha \) is non-degenerate, \( \text{spec}(\Sigma, \alpha) \) is a discrete subspace of \( \mathbb{R}^+ \).

2.2. Liouville and Weinstein domains. A Liouville domain is a pair \((W^{2n}, \lambda)\) such that
- \( W^{2n} \) is a compact manifold with boundary,
- \( d\lambda \) is a symplectic form on \( W \),
- the Liouville field \( \lambda_X \), defined by \( i_X d\lambda = \lambda \), is outward transverse along \( \partial W \).

Let \( \alpha := \lambda|_{\partial W} \) be a contact one-form on \( \partial W \). The negative flow of \( X \) gives rise to a collar:
\[
\phi : (1 - \epsilon, 1] \times \partial W \rightarrow W,  \\
\phi^* \lambda = r\alpha,  \\
\phi^* X = r\partial_r.
\]

We can attach an infinite cone to it, which is called the completion of \((W, \lambda)\):
\[
\hat{W} = W \cup_{\partial W} ([1, \infty) \times \partial W),  \\
\hat{\lambda}|_W = \lambda  \\
\hat{\lambda}|([1, \infty) \times \partial W) = r\alpha,  \\
\hat{X}|([0, \infty) \times \partial W) = r\partial_r,  \\
\hat{\omega} = d\hat{\lambda}.
\]

A Liouville isomorphism between domains \( W_0, W_1 \) is a diffeomorphism \( \psi : \hat{W}_0 \rightarrow \hat{W}_1 \) satisfying \( \psi^* \hat{\lambda}_1 = \hat{\lambda}_0 + df \), for some \( f \) compactly supported. We also say that \( \hat{W}_0 \) and \( \hat{W}_1 \) are Liouville isomorphic. Clearly \( \psi \) is compatible with the Liouville flow at infinity.

**Definition 2.2.** A Liouville domain \((W, \lambda)\) is called \( G \)-equivariant if a group \( G \) acts on \( W \) and \( \lambda \) is \( G \)-invariant, i.e, \( g^* \lambda = \lambda, \forall g \in G \). A diffeomorphism \( f \) between two \( G \)-equivariant Liouville domains is called \( G \)-equivariant if the following diagram commutes, for all \( g \in G \):
\[
\begin{array}{ccc}
(W_1, \lambda_1) & \xrightarrow{f} & (W_0, \lambda_0) \\
\downarrow{g^*} & & \downarrow{g^*} \\
(W_1, \lambda_1) & \xrightarrow{f} & (W_0, \lambda_0)
\end{array}
\]
Remark 2.3. A manifold $M$ is called $G$-equivariant if $G$ acts on it.

Proposition 2.4 (Proposition 11.8 [CE12]). Let $W$ be a compact symplectic manifold with contact type boundary and $\lambda_t, t \in [0, 1]$ be a homotopy of Liouville forms on $W$. Then there exits a diffeomorphism of the completions $f : \hat{W}_0 \to \hat{W}_1$ such that $f^*\hat{\lambda}_1 = \hat{\lambda}_0 = dg$ where $g$ is a compactly supported function.

We have an immediate corollary for Proposition 2.4:

Corollary 2.5. Let $(\lambda_t)_{0 \leq t \leq 1}$ be a family of ($G$-equivariant) Liouville structures on $W$. Then all the $(W, \lambda_t)$ ($(W, \hat{\lambda}_t)$) are mutually ($G$-equivariantly) Liouville isomorphic.

A Weinstein domain is a triple $(W^{2n}, \lambda, \phi)$ such that

- $(W, \lambda)$ is a Liouville domain,
- $\phi : W \to \mathbb{R}$ is an exhausting Morse function with $\partial W$ being a regular level set,
- $X_\lambda$ is a gradient-like vector field for $\phi$.

Since $W$ is compact and $\phi$ is an exhausting Morse function with $\partial W$ as a regular level set, $\phi$ has finitely many critical points. Liouville and Weinstein cobordisms are defined similarly. If a contact manifold $(Y, \xi)$ is contactomorphic to $\partial(W, \lambda)$, then we say that $(W, \lambda)$ is a Liouville or Weinstein filling of $(Y, \xi)$.

Definition 2.6. A Stein manifold $(M, J, \phi)$ is a complex manifold $(M, J)$ with an exhausting plurisubharmonic function $\phi : M \to \mathbb{R}$. A manifold of the form $\phi^{-1}((-\infty, c])$ is called a Stein domain, where $c$ is a regular value of $\phi$.

We also have the following famous theorem by Eliashberg:

Theorem 2.7 (Theorem 1.1 [CE12]). Given a Weinstein structure $\mathcal{M} = (\omega, X, \phi)$ on $V$, there exists a Stein structure $(J, \phi)$ on $V$ such that $\mathcal{M}(J, \phi)$ is Weinstein homotopic to $\mathcal{M}$ with fixed $\phi$.

2.3. Symplectic homology. This section is mainly taken out from [Laz16]. The convention used here agrees with [CO18].

2.3.1. Admissible Hamiltonians and almost complex structures. Let $\mathcal{H}_{std}(W)$ denote the class of admissible Hamiltonians, which are functions on $\hat{W}$ defined up to smooth approximation as follows:

- $H^s \equiv 0$ in $W$,
- $H^s$ is linear in $r$ with slope $s \in \text{Spec}(Y, \alpha)$ in $\hat{W} \setminus W = Y \times [1, \infty)$. We will often suppress $s$.

To be precise, $H$ is a $C^2$-small Morse function in $W$ and $H = h(r)$ in $\hat{W} \setminus W$ for some function $h$ that is increasing convex in a small collar $(Y \times [1, 1 + \epsilon], r\alpha)$ of $Y$ and linear with slope $s$ outside this collar.

For $H \in \mathcal{H}_{std}(W)$, recall the Hamiltonian vector field $X_H$ is defined by the condition $d\lambda(\cdot, X_H) = dH$. The time-1 orbits of $X_H$ are called the Hamiltonian orbits of $H$ and fall into two categories depending on their location in $\hat{W}$:

- In $W$, the only Hamiltonian orbits are constants corresponding to critical points of $H|_W$.
- In $\hat{W} \setminus W$, we have $X_H = h'(r)R_\alpha$, where $R_\alpha$ is the Reeb vector field of $(Y, \alpha)$. So all Hamiltonian orbits lie on level sets of $r$ and come in $S^1$-families corresponding to reparametrizations of some Reeb orbit of $\alpha$ with period $h'(r)$.
Since the slope $s$ of $H$ at infinity is not in $\text{Spec}(Y, \alpha)$, all nonconstant Hamiltonian orbits lie in a small neighborhood of $Y$ in $\hat{W}$. After a $C^2$-small time-dependent perturbation of $H$, the orbits become non-degenerate, i.e. the linearized Hamiltonian flow from $T_pW$ to $T_pW$, for some $p$ in the Hamiltonian orbit, does not have 1 as an eigenvalue. These non-degenerate orbits also lie in a neighborhood of $W$ and so their number is finite. In fact, under this perturbation, each $S^1$-family of Hamiltonian orbits degenerates into two Hamiltonian orbits.

We say that an almost complex structure $J$ is cylindrical on the symplectization $(Y \times (0, \infty), r\alpha)$ if it preserves $\xi = \ker \alpha$, $J|_{\xi}$ is independent of $r$, is compatible with $d(r\alpha)|_{\xi}$, and $J(r\partial_\nu) = R_\nu$. Let $\mathcal{J}_{\text{std}}(W)$ denote the class of admissible almost complex structures $J$ on $\hat{W}$ which satisfy

- $J$ is compatible with $\omega$ on $\hat{W}$
- $J$ is cylindrical on $\hat{W}\setminus W = (Y \times [1, \infty), r\alpha)$.

2.3.2. Floer complex. For $H \in \mathcal{H}_{\text{std}}(W), J \in \mathcal{J}_{\text{std}}(W)$, the Floer complex $SC(W, \lambda, H, J)$ is generated as a free abelian group by Hamiltonian orbits of $H$. In this paper we need to consider all Hamiltonian orbits, as opposed to only the contractible ones, see [Wen].

First, let’s fix a reference loop

$$l_h : S^1 \to W$$

with $[l_h] = h \in H_1(W, \mathbb{Z})$. Denote by $\mathcal{P}^h(H)$ the set of all 1–periodic orbits of $X_Ht$ in the homology class $h$.

For a fixed reference class $h$, we will often write the chain complex generated as a free abelian group by orbits in $\mathcal{P}^h(H)$ as $SC^h(H, J)$ when we do not need to specify $(W, \lambda)$. We will suppress $h$ when it causes no confusion.

The differential is given by counts of Floer trajectories. In particular, for two Hamiltonian orbits $x_-, x_+ \in H$, let $\hat{\mathcal{M}}(x_-, x_+; H, J)$ be the moduli space of smooth maps $u : \mathbb{R} \times S^1 \to \hat{W}$ such that

$$\lim_{s \to \pm \infty} u(s, \cdot) = x_{\pm}$$

and $u$ satisfies Floer’s equation

$$\partial_su + J(\partial_tu - X_H) = 0.$$ 

Here $s, t$ denotes the $\mathbb{R}, S^1$ coordinates on $\mathbb{R} \times S^1$ respectively. Since the Floer equation is $\mathbb{R}$-invariant, there is a free $\mathbb{R}$-action on $\hat{\mathcal{M}}(x_-, x_+; H, J)$ for $x_- \neq x_+$. Let $\mathcal{M}(x_-, x_+; H, J)$ be the quotient by this $\mathbb{R}$-action, i.e. $\hat{\mathcal{M}}(x_-, x_+; H, J)/\mathbb{R}$. After a small time-dependent perturbation of $(H, J), \mathcal{M}(x_-, x_+; H, J)$ is a smooth finite-dimensional manifold.

A maximal principle ensures that Floer trajectories do not escape to infinity in $\hat{W}$. For the following, let $V \subset (W, \lambda_W)$ be a Liouville subdomain, i.e. $(V, \lambda_W|_V)$ is a Liouville domain. Then $(Z, \alpha_Z) = \partial(V, \lambda)$ is a contact manifold. Since $V$ is a Liouville subdomain, there is a collar of $Z$ in $W$ that is symplectomorphic to $(Z \times [1, 1 + \delta], d(\alpha_Z))$ for some small $\delta$.

Lemma 2.8. [AS10] Consider $H : \hat{W} \to \mathbb{R}$ such that $H = h(r)$ is increasing near $Z$, where $r$ is the cylindrical coordinate and $J \in \mathcal{J}_{\text{std}}(W)$ is cylindrical near $Z$. If both asymptotic orbits of a $(H, J)$-Floer trajectory $u : \mathbb{R} \times S^1 \to \hat{W}$ are contained in $V$, then $u$ is contained in $V$.

Applying this result to $V = W$, we can proceed as if $W$ were closed and conclude by the Gromov-Floer compactness theorem that $\mathcal{M}(x_-, x_+; H, J)$ has a codimension one compactification. This implies that $\mathcal{M}_h(x_-, x_+; H, J)$, the zero-dimensional component of $\mathcal{M}(x_-, x_+; H, J)$, has finitely many elements and the map $d : SC(H, J) \to SC(H, J)$, defined by

$$dx_+ = \sum_{x_-} \#\mathcal{M}_h(x_-, x_+; H, J)x_-$$
on generators and extended to $\text{SC}(H, J)$ by linearity, is a differential. Here $\#\mathcal{M}_h(x_-, x_+; H, J)$ denotes the mod 2 count of elements of $\mathcal{M}_h(x_-, x_+; H, J)$. We have that $(\text{SC}(H, J), d)$ is a chain complex. Note that the underlying vector space $\text{SC}(H, J)$ depends only on $H$ while the differential $d$ depends on both $H$ and $J$. The resulting homology $HF(H, J)$ is independent of $J$ and compactly supported deformations of $H$.

If $c_1(W, \omega) = 0$, as will always be the case in this paper, $HF(H, J)$ has a $\mathbb{Z}$-grading. More precisely, if $c_1(W, \omega) = 0$, the canonical line bundle of $(W, \omega)$ is trivial. After choosing a global trivialization of this bundle, we can assign an integer called the Conley-Zehnder index $\mu_{\text{CZ}}(x)$ to each Hamiltonian orbit $x$; see Subsection 2.6. For a general orbit $x$, $\mu_{\text{CZ}}(x)$ depends on the choice of trivialization of the canonical bundle. For a Hamiltonian orbit corresponding to a critical point $p$ of the Morse function $H|_W$, the Conley-Zehnder index coincides with $n - \text{Ind}(p)$, where $\text{Ind}(p)$ is the Morse index of $H|_W$ at $p$.

Furthermore, $\dim \mathcal{M}(x, y; H, J) = \mu_{\text{CZ}}(y) - \mu_{\text{CZ}}(x) - 1$ so the differential, which counts the zero-dimensional components of $\mathcal{M}(x, y; H, J)$, decreases the degree by one.

2.3.3. Continuation map. Although $HF(H, J)$ is independent of $J$ and compactly supported deformations of $H$, $HF(H, J)$ does depend on the slope of $H$ at infinity and so is not an invariant of $W$. In particular, $HF(H, J)$ only sees Reeb orbits of period less than the slope of $H$ at infinity. To incorporate all Reeb orbits, we need to consider Hamiltonians with arbitrarily large slope. More formally, this can be done by considering continuation maps between $\text{SC}(H, J)$ for different $H$. Given $H_-, H_+ \in \mathcal{H}_{\text{std}}(W)$, let $H_s \in \mathcal{H}_{\text{std}}(W), s \in \mathbb{R}$, be a family of Hamiltonians such that $H_s = H_-$ for $s < 0$ and $H_s = H_+$ for $s \gg 0$. Similarly, let $J_s \in \mathcal{J}_{\text{std}}(W)$ interpolate between $J_-, J_+$. For Hamiltonian orbits $x_-, x_+$ of $H_-, H_+$ respectively, let $\mathcal{M}(x_-, x_+; H_s, J_s)$ be the moduli space of parametrized Floer trajectories, i.e. maps $u : \mathbb{R} \times S^1 \to \widetilde{W}$

$$\partial_s u + J_s(\partial_t u - X_{H_s}) = 0$$

To ensure that parametrized Floer trajectories do not escape to infinity, we again need to use a maximal principle. For this principle to hold, it is crucial that the homotopy of Hamiltonian functions is decreasing, i.e. $\partial H_s/\partial s \leq 0$. If $J_s$ is $s$-independent, we use the following parametrized version of ‘no escape’ Lemma 2.8, which is proven in Proposition 3.1.10 of [Gut15]. If $J_s$ does depend on $s$ and $V = W$, then we use the maximal principle from [Sei06]. We state both in the following lemma.

**Lemma 2.9.** [Gut15], [Sei06] Consider a decreasing homotopy $H_s : \widetilde{W} \to \mathbb{R}$ such that $H_s = h_s(t)$ is increasing in $t$ near $Z = \partial V$ and $H_s|_Z$ is $s$-independent; let $J \in \mathcal{J}_{\text{std}}(W)$ be cylindrical near $Z$. If $u : \mathbb{R} \times S^1 \to \widetilde{W}$ is a $(H_s, J)$-Floer trajectory with both asymptotes in $V$, then $u$ is contained in $V$. If $V = W$, the same claim also holds for a homotopy $J_s \in J_{\text{std}}(W)$ that is cylindrical near $Z$.

By applying the second part of Lemma 2.9, we can proceed as if $W$ were closed and conclude that $\mathcal{M}(x_-, x_+; H_s, J_s)$ has a codimension one compactification. Then the continuation map $\phi_{H_s, J_s} : \text{SC}(H_+, J_+) \to \text{SC}(H_-, J_-)$ defined by

$$\phi_{H_s, J_s}(x_+) = \sum_{x_-} \#\mathcal{M}_h(x_-, x_+; H_s, J_s)x_-$$

on generators and extended to $\text{SC}(H_+, J_+)$ by linearity is a chain map. Up to chain homotopy, this map is independent of $J_s$ and $H_s$. Note that there is no $\mathbb{R}$-action since the parametrized Floer equation is not $\mathbb{R}$-invariant. As a result, $\phi_{H_s, J_s}$ is degree-preserving. Finally, we define
symplectic homology as the direct limit

\[ SH(W, \lambda) := \lim_{\rightarrow} HF(H, J). \]

The direct limit is taken over continuation maps \( \phi_{H_0, J_0} : HF(H_0, J_0) \to HF(H_1, J_1) \) on homology. One key property is that \( SH(W, \lambda) \) depends only on the symplectomorphism type of \((\hat{W}, d\lambda)\) [Sei06].

2.4. Positive symplectic homology. Positive symplectic homology \( SH^+(W) \) is defined using the action functional. For a small time-dependent perturbation of \( H \in \mathcal{H}_{std}(W) \), the action functional \( A_H : C^\infty(S^1, \hat{W}) \to \mathbb{R} \) is

\[ A_H(x) := \int_{S^1} x^* \lambda - \int_{S^1} H(x(t)) dt. \]

Under our conventions, the Floer equation is the positive gradient flow of the action functional and so action increases along Floer trajectories, i.e. if \( u \in \mathcal{M}(x_-, x_+) \) is a non-constant Floer trajectory, then \( A_H(x_+) > A_H(x_-) \). Let \( SC^{<a}(H, J) \) be generated by orbits of action less than \( a \). Since action increases along Floer trajectories, the differential decreases action and hence \( SC^{<a}(H, J) \) is a subcomplex of \( SC(H, J) \); we define \( SC^{>a}(H, J) \) to be the quotient complex \( SC(H, J)/SC^{<a}(H, J) \).

For \( H \in \mathcal{H}_{std}(W) \), the constant orbits corresponding to Morse critical points \( x \in W \) have action \(-H(x)\). The non-constant orbits that correspond to Reeb orbits have action close to the action of the corresponding Reeb orbit, which is positive. In fact, for sufficiently small \( \epsilon \), \( SC^{<\epsilon}(H, J) \) corresponds to the Morse complex of \(-H|_W\) with a grading shift. More precisely, \( H_k(SC^{<\epsilon}(H, J)) \cong H^{n-k}(W; \mathbb{Z}) \). Define \( SC^+(H, J) \) to be the quotient complex \( SC(H, J)/SC^{<\epsilon}(H, J) \) and let \( HF^+(H, J) \) be the resulting homology. Using a direct limit construction, we can also define \( HF^+(W) \). More precisely, suppose \( H_s \) is a decreasing homotopy such that \( H_s = H_- \) for \( s < 0 \) and \( H_s = H_+ \) for \( s > 0 \). Then the continuation Floer trajectories are also action increasing and hence there is an induced chain map \( \phi_{H_s, J_s}^+ : SC^+(H_+, J_+) \to SC^+(H_-, J_-) \). As with \( SH(W) \), we define \( SH^+(W) \) by

\[ SH^+(W, \lambda) := \lim_{\rightarrow} HF^+(H, J). \]

The direct limit is taken over the continuation maps \( \phi_{H_0, J_0}^+ : HF^+(H_0, J_0) \to HF^+(H_1, J_1) \) on homology.

Like \( SH(W) \), \( SH^+(W) \) depends only on the symplectomorphism type of \((\hat{W}, d\lambda)\). Note that as a vector space, \( SC^+(H, J) \) essentially depends only on \((Y, \alpha)\) and not on the interior \((W, \lambda)\). This is because \( SC^+(H, J) \) is generated by non-constant Hamiltonian orbits, which live in the cylindrical end of \( W \) and correspond to Reeb orbits of \((Y, \alpha)\). On the other hand, the differential for \( SC^+(H, J) \) may depend on the filling \( W \) of \((Y, \alpha)\) since Floer trajectories between non-constant orbits may go into the filling, so different Liouville fillings of \((Y, \xi)\) might have different \( SH^+ \).

The chain-level short exact sequence

\[ 0 \to SC^{<\epsilon}(H, J) \to SC(H, J) \to SC^+(H, J) \to 0 \]

induces the ‘tautological’ long exact sequence in homology

\[ \cdots \to H^{n-k}(W; \mathbb{Z}) \to SH_k(W, \lambda) \to SH^+_k(W, \lambda) \to H^{n-k+1}(W; \mathbb{Z}) \to \cdots. \]
2.5. Summary of the TQFT structure on $SH_*(W)$. This is taken out of chapter 6 in [Rit13]. For a detailed construction, see chapter 16 of [Rit13]. Note that both the grading and action functional differ from ours by a negative sign, and our homology $SH_*(W^{2n})$ is cohomology $SH^*(W^{2n})$ in [Rit13]. We summarize here the TQFT structure. Suppose we are given:

(1) a Riemann surface $(S,j)$ with $p + q$ punctures, with fixed complex structure $j$;

(2) ends: a cylindrical parametrization $s + it$ near each puncture, with $j\partial_a = \partial_t$;

(3) $p \geq 1$ of the punctures are negative (i.e, we converge to the puncture as $s \to -\infty$), they are indexed by $a = 1, \ldots, p$;

(4) $q \geq 0$ of the punctures are positive (i.e, we converge to the puncture as $s \to +\infty$), they are indexed by $b = 1, \ldots, q$;

(5) weights: constants $A_a, B_b > 0$ satisfying $\sum A_a - \sum B_b \geq 0$;

(6) a 1-form $\beta$ on $S$ with $d\beta \leq 0$, and on the ends $\beta = A_a dt$, $\beta = B_b dt$ for large $|s|$.

**Remark 2.10.** Negative/positive parametrizations are modelled on $(-\infty, 0] \times S^1$ and $[0, \infty) \times S^1$, respectively. In (6), $d\beta \leq 0$ means $d\beta(v, jv) \leq 0$ for all $v \in TS$. By Stokes’ theorem, $\sum A_a - \sum B_b = -\int_S d\beta \geq 0$. This forces $p \geq 1$ and (5). Subject to this inequality, such $\beta$ exists. See Lemma 16.1 [Rit13].

Fix a Hamiltonian $H : \tilde{W} \to \mathbb{R}$ linear at infinity with $H \geq 0$ (required in Section 16.3 [Rit13]), this defines $X = X_H$. Fix an almost complex structure $J$ on $W$ of contact type at infinity.

The moduli space $M(x_a; y_b; S, \beta)$ of Floer solutions consists of smooth maps $u : S \to \tilde{W}$ such that $du - X \otimes \beta$ is $(j, J)$-holomorphic, and $u$ converges on the ends to 1-orbits $x_a, y_b$ of $A_a H, B_b H$ which we call the asymptotics.

After a small generic $S$-dependent perturbation $J_z$ of $J$, $M(x_a; y_b; S, \beta)$ is a smooth manifold. One can ensure that on the ends $J_z$ does not depend on $z = s + it \in S$ for $|s| \gg 0$. Just as for Floer continuations maps (2.3.3), a maximum principle and an a priori energy estimate $E(u) = \sum A_B H(y_b) - \sum A_a H(x_a)$ holds, so the $M(x_a; y_b; S, \beta)$ have compactifications by broken Floer solutions: Floer trajectories for $A_a H, B_b H$ can break off at the respective ends. When gradings are defined (2.6),

$$\dim M(x_a; y_b; S, \beta) = - \sum \mu_{CZ}(x_a) + \sum \mu_{CZ}(y_b) + n \chi(S)$$

$$= - \sum \mu_{CZ}(y_b) - \sum \mu_{CZ}(x_a) + n(2 - 2g - p - q).$$

(2.1) (2.2)

Define $\psi_S : \otimes_{a=1}^q SC_*(B_b H) \to \otimes_{a=1}^p SC_*(A_a H)$ on generators by counting isolated Floer solutions

$$\psi_S(y_1 \otimes \cdots \otimes y_q) = \sum_{u \in M_0(x_a; y_b; S, \beta)} \epsilon_u x_1 \otimes \cdots \otimes x_p,$$

where $\epsilon_u \in \{ \pm 1 \}$ are orientation signs (In this paper we use $\mathbb{Z}_2$ coefficients, so these signs don’t matter. In general, see Section 17 of [Rit13]). Then extend $\psi_S$ linearly.

The $\psi_S$ are chain maps. On homology,

$$\psi_S : \otimes_{a=1}^q SH_*(B_b H) \to \otimes_{a=1}^p SH_*(A_a H)$$

is independent of the choices $(\beta, j, J)$ relative to the ends. Taking direct limits, we get induced maps:

$$\psi_S : SH_*(W)^{\otimes q} \to SH_*(W)^{\otimes p} \quad (p \geq 1, q \geq 0).$$

So $SH_*(W)$ has a unit $\psi_C(1)$. 

2.5.1. The product. The pair of pants surface $P$ (Figure 1) defines the product
\[ \psi_P : SH_i(W) \otimes SH_j(W) \to SH_{i+j}(W), \ x \cdot y = \psi_P(x,y), \]
which is graded-commutative and associative.

**Remark 2.11.** The pair of pants product also respects the action filtration. As mentioned in [Ueb15] and in Section 16.3 of [Rit13], we have
\[ A_{2H}(x_3) \leq A_H(x_1) + A_H(x_2). \]
Hence the product restricts to a map
\[ SH_{[a',b]}(W) \times SH_{[a'^{\prime},b']} (W) \to SH_{[\max\{a',a^{\prime}+b\},b^{\prime}]} (W), \]
where on the right hand side it is necessary to divide out all generators with action less than $\max\{a',a^{\prime}+b\}$ to make the map well defined. So one does not get a product on the whole positive symplectic homology, but we can define maps:
\[ SH_{[\delta,b]}(W) \times SH_{[\delta,b]} (W) \to SH_{[b^{\prime},2\delta]}(W), \]

2.5.2. The unit. Let $C = \mathbb{C}$ with $p = 1$, $q = 0$. The end is parametrized by $(-\infty, 0] \times S^1$ via $s + it \mapsto e^{-2\pi(s+it)}$. On this end, $\beta = f(s)dt$ with $f'(s) \leq 0$, $f(s) = 1$ for $s \leq -2$ and $f(s) = 0$ for $s \geq -1$. Extend by $\beta = 0$ away from the end (See Figure 2). Thus we get a map $\psi_C : \mathbb{K} \to SH_*(H)$.

**Definition 2.12.** Let $e_H = \psi_C(1) \in SH_*(H)$. We can define $e = \lim_{\to} e_H \in SH_n(W)$.

**Theorem 2.13** (Theorem 6.1 [Rit13]). $e$ is the unit for the product on $SH_*(W)$.

**Proof.** By the gluing illustrated in the Figure 3, $\psi_P(e, \cdot) = \psi_P\#C(\cdot) = \psi_Z(\cdot) = \text{id}$. □

**Remark 2.14.** For “gluing = compositions” results, see Theorems 16.10, 16.12, 16.14 in [Rit13]. Before taking direct limits, the above is the continuation map
\[ SH_*(H) \xrightarrow{e_H \otimes} SH_*(H) \otimes SH_* (2H) \xrightarrow{\psi_P} SH_*(2H). \]
\[ \beta = 0 \]
\[ \beta = f(s)dt \]
\[ \beta = dt \]

Floer’s equation for \( H \)

\[ \beta = \frac{df(s)}{dt} \]

\[ J \)-holomorphic since \( 0 \cdot H = 0 \]

Floer’s continuation equation \( f(s) \cdot H \)

**Figure 2.** A cap \( C \), and its interpretation as a continuation cylinder.

**Figure 3.** Unit for pair of pants product

**Lemma 2.15** (Lemma 6.2 [Rit13]). \( e_H \) is a count of the isolated finite energy Floer continuation solutions \( u : \mathbb{R} \times S^1 \rightarrow \widehat{W} \) for the homotopy \( f(s)H \) from \( H \) to 0.

**Lemma 2.16** (Lemma 6.3 [Rit13]). For \( H \) as in Section 2.3.1, \( e_H = \) sum of the local minima of \( H \).

**Theorem 2.17** (Theorem 6.4 [Rit13]). \( e = \lim e_H \) is the image of 1 under \( c_* : H_*(W) \rightarrow SH_*(W) \), and \( e_H = c_{*,H}(1) \) where \( c_{*,H} : H_*(\hat{M}) \cong SH_*(H) \rightarrow SH_*(W) \) is the inclusion map.

2.5.3. The TQFT structure on \( SH_*(W) \) is compatible with the grading by \( H_1(W) \). We can grade \( SC_*(H) = \bigoplus_{h \in H_1(W)} SC^h_*(H) \) by the homology classes \( h \in H_1(W) \) of the generators. The Floer differential preserves the \( H_1 \) grading, and so do Floer operations on a cylinder and a cap. The pair of pants product respects this grading as follows: \( \psi_S : SH^{h_1}_*(W) \otimes SH^{h_2}_*(W) \rightarrow SH^{h_1+h_2}_*(W) \). We can also grade \( SH_*(W) = \bigoplus_{h} SH^h_*(M) \) by the free homotopy classes \( h \in [S^1, M] \) of the generators. The TQFT operations for genus zero surfaces are compatible with the grading (the equation above holds after replacing \( \sum \) by concatenation of free loops).

**Remark 2.18.** Let \( SH^0_*(W) \) denote the summand corresponding to the contractible loops. Considering only contractible loops determines a TQFT with operations \( \psi_S : SH^{0}_*(W)^{\otimes q} \rightarrow SH^0_*(W)^{\otimes p} \) \((p \geq 1, q \geq 0)\). Also \( c_* : H_*(W) \rightarrow SH^0_*(W) \subset SH_*(W) \) naturally lands in \( SH^0_*(W) \).
2.5.4. Viterbo Functoriality. For Liouville subdomains \( W \subset \hat{M} \), Viterbo [Vit99] constructed a restriction map \( SH_*(M) \rightarrow SH_*(W) \) and McLean [McL07] proved that it is a ring homomorphism.

**Theorem 2.19** ([McL07] [CO18]). Let \( W \) and \( V \) be compact symplectic manifolds with contact type boundary and assume that the Conley-Zehnder index is well-defined on \( W \). If \( V \) is obtained from \( W \) by attaching to \( \partial W \times [0,1] \) a subcritical symplectic handle \( H_k^n, k < n \), then it holds that

\[
SH_*(V,\mathbb{Z}_2) \cong SH_*(W,\mathbb{Z}_2)
\]

as rings.

**Remark 2.20.** A.Ritter proved a stronger statement in Theorem 9.5 of [Rit13].

2.6. Conley-Zehnder index. In this section we discuss Conley-Zehnder index as in Fauck [Fau16]. To define \( \mu_{CZ} \), let \( Sp(2n) \) denote the group of \( 2n \times 2n \) symplectic matrices. We will discuss a generalization, called the Robbin-Salamon index as follows: any smooth path \( \Psi : [a,b] \rightarrow Sp(2n) \) satisfies an ordinary differential equation

\[
\Psi'(t) = J_0 S(t) \Psi(t), \quad \Psi(a) \in Sp(2n),
\]

Where \( t \rightarrow S(t) = S(t)^T \) is a smooth path of symmetric matrices and \( J_0 \) is the standard almost complex structure. We say \( t \in [a,b] \) is called a crossing if \( \det(id - \Psi(t)) = 0 \). The crossing form at time \( t \) is a quadratic form \( \Gamma(\Psi,t) \) defined for \( v \in \ker(id - \Psi(t)) \) by

\[
\Gamma(\Psi,t)v = <v, S(t)v>
\]

A crossing \( t \) is called regular if \( \Gamma(\Psi,t) \) is non-degenerate. For a path with only regular crossings, the Robbin-Salamon index is defined by

\[
\mu_{CZ}(\Psi,a,b) := \frac{1}{2} \text{sign} \Gamma(\Psi,a) + \sum_{a < t < b} \text{sign} \Gamma(\Psi,t) + \frac{1}{2} \text{sign} \Gamma(\Psi,b)
\]

where the sum runs all over crossings \( t \in (a,b) \), and \( \text{sign}(M) \) denotes the signature of the matrix \( M \), which equals the number of positive eigenvalues minus the number of negative eigenvalues. Here we use \( \mu_{CZ} \) to denote the Robbin-Salamon index. The fact that the Robbin-Salamon index coincides with Conley-Zehnder index when \( \det(id - \Psi(b)) \neq 0 \) sort of justifies this abuse of notation.

We have the following properties for \( \mu_{CZ} \):

- **(Naturality)** For any path \( \Phi : [a,b] \rightarrow Sp(2n) \), \( \mu_{CZ}(\Phi \Phi^{-1}) = \mu_{CZ}(\Psi) \)
- **(Homotopy)** \( \mu_{CZ}(\Psi_s) \) is constant for any homotopy \( \Psi_s \) with fixed endpoints.
- **(Product)** If \( Sp(2n) \oplus Sp(2n') \) is identified with a subgroup of \( Sp(2(n+n')) \) in the natural way, then \( \mu_{CZ}(\Psi \oplus \Psi') = \mu_{CZ}(\Psi) + \mu_{CZ}(\Psi') \).

The homotopy property allows us to define \( \mu_{CZ}(\Psi,a,b) \) also for paths with non-regular crossings, given that having regular crossings is a \( C^\infty \) generic property among paths with fixed endpoints.

**Remark 2.21** ([Lemma 59 [Fau16]]). Let \( \Psi_1, \Psi_2, \Psi_3 : [0,T] \rightarrow Sp(2) \) be the following paths:

\[
\Psi_1(t) = e^{it}, \quad \Psi_2(t) = e^{-it}, \quad \Psi_3(t) = \text{diag}(e^{f(t)},e^{-f(t)}), f \in C^1(\mathbb{R}).
\]
Then, their Conley-Zehnder indices are given as follows:

\[
\mu_{CZ}(\Psi_1) = \begin{bmatrix} T \\ 2\pi \end{bmatrix} + \begin{bmatrix} T \\ 2\pi \end{bmatrix},
\]

\[
\mu_{CZ}(\Psi_2) = \begin{bmatrix} -T \\ 2\pi \end{bmatrix} + \begin{bmatrix} -T \\ 2\pi \end{bmatrix} = -\mu_{CZ}(\Psi_1),
\]

\[
\mu_{CZ}(\Psi_3) = 0.
\]

**Trivialization.** Suppose we have a symplectic manifold \((M,\omega)\) with \(c_1(M) = 0\) and \(J\) is an \(\omega\)-compatible almost complex structure. Then the anti-canonical bundle of \(M\) is the highest exterior power of \((TM,J)\), i.e., \(\kappa^* = \wedge^n(TM,J)\). The canonical bundle \(\kappa_J\) is the dual of \(\kappa^*_J\).

In the same manner, we can define the canonical bundle of a contact manifold \((C,\xi)\) with a choice of one form \(\alpha\) and \(d\alpha\)-compatible almost complex structure on \(\xi\).

A **trivialization of the canonical bundle** is a bundle isomorphism \(\Phi : \kappa_J \to M \times \mathbb{C}\). A **trivialization** of \((\gamma^*TM,J)\) (where \(\gamma\) is a loop in \(M\)) is a bundle isomorphism \(\Psi : \gamma^*TM \to S^1 \times \mathbb{C}^n\). Such a trivialization has a one-to-one correspondence (up to homotopy) with the trivialization of \(\gamma^*\kappa^*_J\) and hence the trivialization of the canonical bundle via:

\[
\det_{\mathbb{C}}(\Psi) : \wedge^n(\gamma^*TM) = \gamma^*\kappa^*_J \to S^1 \times \mathbb{C}.
\]

For a 1-periodic Hamiltonian orbit \(x\), we fix a trivialization of \((x^*TM,J)\) along \(x\) as:

\[
\Psi : x^*TM \to S^1 \times \mathbb{C}^n
\]

Suppose \(\psi\) is the Hamiltonian flow and \(d\psi_t : TM|_{x(0)} \to TM|_{x(t)}\) is its linearization, then define

\[
M_t(x) := \Psi_t \circ d\psi_t \circ \Psi^{-1}_0
\]

The Conley-Zehnder index of \(x\) is defined as \(\mu_{CZ}(x) := \mu_{CZ}(M_t(x))\). Similarly, we can define the Conley-Zehnder index of a Reeb orbit. In particular, let \((W,\lambda)\) be a Liouville domain and \((C := \partial W,\xi := \ker \lambda|_C)\) its boundary. We have \(TM|_C = \xi \oplus < X_{\text{Reeb}} > \oplus < X >\), where \(X_{\text{Reeb}}, X\) are a Reeb vector field and a Liouville vector field, respectively. Since \(< X_{\text{Reeb}} > = J < X >\), we can identify \(< X_{\text{Reeb}} > \oplus < X >\) with \(\mathbb{C}\), i.e., \(\gamma^*TM = \gamma^*\xi \oplus \mathbb{C}\), where \(\gamma\) is a Reeb orbit. Due to the fact that Reeb flow preserves \(X_{\text{Reeb}}\) and extends to the symplectization, we have

\[
M_{t,M}(\gamma) = M_{t,\xi}(\gamma) \oplus \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

where \(M_{t,M}(\gamma)\) is the symplectic matrix associated to the linearization of the Reeb flow with respect to a trivialization of \(TM|_{\gamma}\), i.e., \(M_{t,\xi}(\gamma)\) is defined in the same manner. The product property of Conley-Zehnder index implies that \(\mu_{CZ}(M_{t,M}(\gamma)) = \mu_{CZ}(M_{t,\xi}(\gamma))\). Hence we will not specify which index we are referring to in the rest of this paper.

Now consider a \(G\)-equivariant Liouville domain \((W,\lambda)\). Suppose the group action is free and \(|G| < \infty\). Then we have that the quotient map

\[
\pi_G : (W,\lambda) \to (W/G,\lambda)
\]

is a finite covering map. Each Reeb orbit \(\gamma\) in \(\partial(W/G)\) then lifts to a fractional orbit \(\bar{\gamma}\) in \(\partial W\). That is, \(\bar{\gamma}(t) = \gamma_0(t), t \in [0,T]\) for some closed Reeb orbit \(\gamma_0\) in \(\partial W\). In particular, we can choose \(\gamma_0\) with period of \(|G| \cdot T\). If we choose a \(G\)-equivariant trivialization for the canonical bundle \(\kappa_W\), then such trivialization descends down to \(\kappa_{W/G}\). Equivalently, if we
choose $G$-equivariant trivialization of $\xi|_{\gamma_0}$, and $M_{t,\xi}(\gamma_0)$ is the matrix of the linearized map, then we have for some $M_G \in \text{Sp}(2n, \mathbb{R})$,

$$M_G \cdot M_{t,\xi}(\gamma_0) = M_{t+T,\xi}(\gamma_0).$$

where $M_G$ satisfies $M_G^{[G]} = M_{[G];\xi}(\gamma_0)$ is a constant matrix, which only depends on the homotopy class of our $G$-equivariant trivialization. In particular, $M_{T,\xi}(\gamma_0) = M_G$, so $\mu_{CZ}(M_{t,\xi}(\gamma_0)), t \in [0, T]$ is well defined since the Conley-Zehnder index is constant for any homotopy with fixed endpoints. We can therefore define the Conley-Zehnder index of such a fractional Reeb orbit of $\gamma$ to be the Conley-Zehnder index of $M_{t,\xi}(\gamma_0), t \in [0, T]$.

As a consequence, we have

$$\mu_{CZ}(\gamma) = \mu_{CZ}(\tilde{\gamma}).$$

Lemma 2.22. Let $(\mathbb{R} \times S^1, d(rd\theta))$ be the symplectization of $(S^1, \theta)$. Choose the canonical trivialization of $T(\mathbb{R} \times S^1) = T\mathbb{R} \times TS^1$, then all fractional Reeb orbits of $(S^1, \theta)$ have Conley-Zehnder index (Robbin-Salamon index) zero, with respect to any cyclic group action rotating the cylinder.

Proof. Since Reeb flow preserves $(\partial_r, \partial_\theta)$, so the matrix for linearized return map is

$$M(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$ 

Therefore, the Conley-Zehnder index is zero. □

The following lemma gives a formula for the Reeb vector field in terms of the Hamiltonian and Liouville vector field.

Lemma 2.23. Let $(W, \lambda)$ be a Liouville manifold. Suppose $H$ is a function on $W$ with $0$ as its regular value and the Liouville vector field $X$ is transverse to the $0$-level set. Then $\Sigma := H^{-1}(0), \lambda)$ is a contact manifold whose Reeb vector field is given by $X_{\text{Reeb}} = \frac{X_H}{X(H)}$, where $X_H$ is the Hamiltonian vector field of $H$.

Proof. $(\Sigma, \lambda)$ is well known to be contact. We only need to prove the latter part of the lemma. Since

$$\iota_{X_H}d\lambda|_\Sigma = -dH|_\Sigma = 0$$

and

$$\iota_{X_H}d\lambda = \iota_{X_H}(X, X_H) = dH(X) = X(H),$$

it follows $X_{\text{Reeb}} = \frac{X_H}{X(H)}$. □

Lemma 2.24 (Lemma 5.20 [McL16]). Let $(C, \xi)$ be a contact manifold with associated contact form $\alpha$ and let $h : \mathbb{R} \to \mathbb{R}$ be a function with $h' > 0, h'' > 0$ and $h'(0) = 1$. Let $\tilde{C} := C \times \mathbb{R}$ be the symplectization of $C$ with symplectic form $d(e^r \alpha)$ where $r$ parameterizes $\mathbb{R}$. Let $\gamma(t)$ be a Reeb orbit of $\alpha$ of period $L$ with a choice of trivialization of the symplectic vector bundle $\oplus_{j=1}^{N} TM$ along this orbit. This choice of trivialization induces a choice of trivialization of $\gamma^* \oplus_{j=1}^{N} \xi$ in a natural way. Then the Hamiltonian $Lh(e^r)$ has a 1 periodic orbit $x$ equal to $\gamma(Lt)$ inside $C \times \{0\} = C$ and its Conley-Zehnder index is equal to $\mu_{CZ}(\gamma) + \frac{1}{2}$.

Remark 2.25. Notice that the Hamiltonian vector field in [McL16] differs from ours by a minus sign. We have

$$M_{t,M}(x) = M_{t,\xi}(\gamma) \oplus \begin{bmatrix} 1 & 0 \\ \frac{h''}{h'}t & 1 \end{bmatrix},$$
for some constant $a > 0$.

If instead, $h'' < 0$, then the index equals $\mu_{CZ}(\gamma) - \frac{1}{2}$. And if $h' < 0$, then the Hamiltonian orbit goes in the opposite direction of the Reeb orbit, and the index differs by a minus sign.

We will conclude this subsection with a lemma relating Morse index of critical point with Conley-Zehnder index of the corresponding constant Hamiltonian orbit.

**Lemma 2.26.** If $S$ is an invertible symmetric matrix with $||S|| < 2\pi$ and $\Psi(t) = \exp(tJ_0S)$, then

$$\mu_{CZ}(\Psi) = n - \text{Ind}(S)$$

where $\text{Ind}(S)$ is the number of negative eigenvalues of $S$.

**Corollary 2.27** (Corollary 7.2.2 [AD14]). Let $W$ be a symplectic manifold of dimension $2n$, let $H : W \to \mathbb{R}$ be a Hamiltonian and $x$ be a critical point of $H$. We assume that $H$ is $C^2$-small (in this case, we can choose a Darboux chart centered at $x$ such that the usual norm $||\text{Hess}_x(H)|| < 2\pi$). Then the Conley-Zehnder index $\mu_{CZ}(x)$ of $x$ as a periodic solution of the Hamiltonian system and its Morse index $\text{Ind}(x)$ as a critical point of the function $H$ are connected by

$$\mu_{CZ}(x) = n - \text{Ind}(x).$$

2.7. **Weinstein handle attachment and contact surgery.**

2.7.1. **Contact surgery.** This section is already included in Chapter 6 of [Gei08]. We will highlight the parts which should be paid attention to in this paper, namely, the trivialization of the conformal symplectic normal bundle.

**Definition 2.28.** Let $(M, \xi)$ be a contact manifold. A submanifold $L$ of $(M, \xi)$ is called an isotropic submanifold if $T_pL \subset \xi_p$ for all point $p \in L$.

Let $L \subset (M, \xi = \ker \alpha)$ be an isotropic submanifold in a contact manifold with cooriented contact structure. Let $(TL)^\perp \subset \xi_L$ be the subbundle of $\xi_L$ that is symplectically orthogonal to $TL$ with respect to the symplectic bundle structure $d\alpha|_\xi$. The conformal structure of this bundle does not depend on the choice of contact form and therefore $(TL)^\perp$ is determined by $\xi$. The fact $L$ is isotropic implies that $TL \subset (TL)^\perp$. So we have the following definition,

**Definition 2.29.** The quotient bundle

$$CSN_M(L) := (TL)^\perp/TL$$

with the conformal symplectic structure induced by $d\alpha$ is called the conformal symplectic normal bundle of $L$ in $M$.

So we have

$$\xi|_L = \xi|_L/(TL)^\perp \oplus (TL)^\perp/TL \oplus TL = TL \oplus \xi|_L/(TL)^\perp \oplus CSN_M(L).$$

Let $J : \xi \to \xi$ be a complex bundle structure on $\xi$ compatible with the symplectic structure given by $d\alpha$. Then the bundle $\xi|_L/(TL)^\perp$ is isomorphic to $J(TL)$. So the contact structure has the following natural splitting on the isotropic submanifold:

**Lemma 2.30.**

$$\xi|_L = TL \oplus J(TL) \oplus CSN_M(L)$$
Therefore, if we fix a trivialization of $TL \oplus J(TL)$, then the trivialization of $CSN_M(L)$ is determined by the trivialization of $\xi|_L$. Now we can state the contact surgery theorem:

**Theorem 2.31** (Theorem 6.2.5 [Gei08]). Let $\Lambda^{k-1}$ be an isotropic sphere in a contact manifold $(M, \xi = \text{ker} a)$ with a trivialization of the conformal symplectic normal bundle $CSN_M(\Lambda^{k-1})$. Then there is a symplectic cobordism from $(M, \xi)$ to the manifold $M'$ obtained from $M$ by surgery along $\Lambda^{k-1}$ with the natural framing. In particular, the surgered manifold $M'$ carries a contact structure that coincides with the one on $M$ away from the surgery region.

**Remark 2.32.** The resulting contact structure on $M'$ is uniquely determined up to isotopy by the isotopic isotropy class of $\Lambda^{k-1}$ and the homotopy class of the trivialization of $CSN_M(\Lambda^{k-1})$.

2.7.2. Weinstein handlebodies. For the purposes of this paper, we need to attach a handle to a Weinstein domain. We will follow Section 13 in [CE06]. The standard handle of index $k$ will be the bidisk in $\mathbb{C}^n$:

$$\left\{ \sum_{j=1}^{k} x_j^2 \leq (1 + \varepsilon)^2, \sum_{j=1}^{k} y_j^2 + \sum_{j=k+1}^{n} |z_j|^2 \leq \varepsilon^2 \right\},$$

where $z_j = x_j + iy_j, j = 1, 2, \ldots, n$, are the complex coordinates in $\mathbb{C}^n$. In particular, the handle $H$ carries the standard complex structure $i$, along with the standard symplectic structure $\omega_{std}$. The symplectic form $\omega_{std}$ on $H$ admits a hyperbolic Liouville field

$$X_{std} = \sum_{j=1}^{k} \left( x_j \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial y_j} \right) + \frac{1}{2} \sum_{l=k+1}^{n} \left( x_l \frac{\partial}{\partial x_l} + y_l \frac{\partial}{\partial y_l} \right).$$

Let us denote by $\xi^-$ the contact structure $\alpha_{std}|_{\partial^+ H} = 0$ defined on $\partial^+ H$ by the Liouville form $\alpha_{std} = \iota_{X_{std}} \omega_{std}$, where $\partial^+ H := \partial D^k \times \mathbb{D}^{2n-k}$ is the lower boundary. Notice that the bundle $\xi^-|_{\Lambda^{k-1}}$ canonically splits as $TA^{k-1} \oplus J(TA^{k-1}) \oplus \varepsilon^{n-k}$, where $\varepsilon^{n-k}$ is a trivial $(n-k)$-dimensional complex bundle. We will denote by $\sigma_{\Lambda}$ the isomorphism

$$TA^{k-1} \oplus J(TA^{k-1}) \oplus \varepsilon^{n-k} \cong \xi^-|_{\Lambda}.$$

Suppose we are given a real $k$-dimensional bundle $E$, a complex $n$-dimensional bundle $\tau, n \geq k$, and injective totally real homomorphism $\phi : E \rightarrow \tau$. Then $\phi$ canonically extends to a complex homomorphism $\phi \otimes \mathbb{C} : E \otimes \mathbb{C} \rightarrow \tau$. If $n > k$ and $\phi \otimes \mathbb{C}$ extends to a fiberwise complex isomorphism $\Phi : E \otimes \mathbb{C} \cong \varepsilon^{n-k}$ then $\Phi$ is called a saturation of $E$ covering $\phi$. When $n = k$ the saturation is unique.

Let $(V, \omega, X, \phi)$ be a Weinstein manifold, $p$ a critical point of index $k$ of the function $\phi$, $a < b = \phi(p)$ a regular value of $\phi$. Denote $W := \{ \phi \leq a \}$. Suppose that the stable manifold of $p$ intersects $V \setminus Int W$ along a disc $D^k$, and let $\Lambda^{k-1} = \partial D^k$ be the attaching sphere. The inclusion $TA^{k-1} \hookrightarrow \xi$ extends canonically to an injective complex homomorphism $TA^{k-1} \oplus J(TA^{k-1}) \hookrightarrow \xi$, while the inclusion $TD^k \hookrightarrow TV$ extends to an injective complex homomorphism $TD^k \oplus J(TD^k) \hookrightarrow TV$. There exists a homotopically unique complex trivialization of the conformal symplectic normal bundle $CSN_{\partial W}(\Lambda^{k-1})$ in $\xi$ which extends to $D^k$ as a trivialization of the conformal symplectic normal bundle to $D^k$ in $TV$. This trivialization provides a canonical isomorphism $\Phi_{D^k} : TA \oplus J(TA) \oplus \varepsilon^{n-k} \rightarrow \xi|_{\Lambda^{k-1}}$, and we will call this the canonical saturation of the inclusion $\Lambda^{k-1} \hookrightarrow \partial W$.

We have the following theorem on attaching a handle to a Weinstein domain:
Theorem 2.33 (Prop13.11 [CE06], [W+91]). Let \( (W, \omega, X, \phi) \) be a 2n–dimensional Weinstein domain with boundary \( \partial W \) and \( \xi \) the induced contact structure \( \{ \alpha|_{\partial W} = 0 \} \) on \( W \) defined by the Liouville form \( \alpha = i_X \omega \). Let \( h : \Lambda \to \partial W \) be an isotropic embedding of the \( (k-1) \)-sphere \( \Lambda \). The saturation \( \Phi : T\Lambda \oplus J(T\Lambda) \oplus \mathbb{R}^{n-k} \to \xi \) covers the differential \( dh : TA \to \xi \). Then there exists a Weinstein domain \( (\tilde{W}, \tilde{\omega}, \tilde{X}, \tilde{\phi}) \) such that \( W \subset \text{Int}\tilde{W} \), and

(i) \( (\tilde{\omega}, \tilde{X}, \tilde{\phi})|_W = (\omega, X, \phi) \);
(ii) the function \( \tilde{\phi}|_{\text{Int}\tilde{W}} \) has a unique critical point \( p \) of index \( k \).
(iii) the stable disc \( D \) of the critical point \( p \) is attached to \( \partial W \) along the sphere \( h(\Lambda) \), and

the canonical saturation \( \Phi_D \) coincides with \( \Phi \).

Given any two Weinstein extensions \( (W_0, \omega_0, X_0, \phi_0) \) and \( (W_1, \omega_1, X_1, \phi_1) \) of \( (W, \omega, X, \phi) \) which satisfy properties (i)-(iii), there exists a diffeomorphism \( g \) fixed on \( W \) such that \( g : W_0 \to W_1 \) satisfying \( (\omega_0, X_0, \phi_0) \) and \( (g^*\omega_1, g^*X_1, g^*\phi_1) \) are homotopic in the class of Weinstein structures which satisfy (i)-(iii). In particular, the completion of these two Weinstein domains are symplectomorphic via a symplectomorphism fixed on \( W \).

We say that the Weinstein domain \( (\tilde{W}, \tilde{\omega}, \tilde{X}, \tilde{\phi}) \) is obtained from \( (W, \omega, X, \phi) \) by attaching a handle of index \( k \) along an isotropic sphere \( h(\Lambda) \to \partial W \) with the given trivialization \( \Phi \).

Definition 2.34. A Weinstein domain \( (W^{2n}, \lambda, \phi) \) is flexible if there exist regular values \( c_1, \ldots, c_{k} \) of \( \phi \) such that \( c_1 < \min \phi < c_2 < \cdots < c_{k-1} < \max \phi < c_k \) and for all \( i = 1, \ldots, k-1 \), \( \{ c_i \leq \phi \leq c_{i+1} \} \) is a Weinstein cobordism with a single critical point \( p \) whose attaching sphere \( \Delta_p \) is either subcritical or a loose Legendrian in \( (Y^{\omega(c_i)}, \lambda|_{Y^{\omega(c_i)}}) \).

Flexible Weinstein cobordisms are defined similarly. Also, a Weinstein handle attachment or contact surgery is called flexible if the attaching Legendrian is loose. So any flexible Weinstein domain can be constructed by iteratively attaching subcritical or flexible handles to \( (B^{2n}, \omega_{\text{std}}) \). A Weinstein domain that is Weinstein homotopic to a Weinstein domain satisfying Definition 2.34 will also be called flexible. Loose Legendrians have dimension at least 2 so if \( (Y_+, \xi_+) \) is the result of flexible contact surgery on \( (Y_-, \xi_-) \), then by Proposition 2.38 \( c_1(Y_+) \) vanishes if and only if \( c_1(Y_-) \) does. Finally, we note that subcritical domains are automatically flexible.

Since they are built using loose Legendrians, which satisfy an h-principle, flexible Weinstein domains also satisfy an h-principle [CE12]. Again, the h-principle has an existence and uniqueness part:

- any almost Weinstein domain admits a flexible Weinstein structure in the same almost symplectic class
- any two flexible Weinstein domains that are almost symplectomorphic are Weinstein homotopic (and hence have exact symplectomorphic completions and contactomorphic boundaries).

2.7.3. Formal structures. There are also formal versions of symplectic, Weinstein, and contact structures that depend on just the underlying algebraic topological data. For example, an almost symplectic structure \( (W, J) \) on \( W \) is an almost complex structure \( J \) on \( W \); this is equivalent to having a non-degenerate (but not necessarily closed) 2-form on \( W \). An almost symplectomorphism between two almost symplectic manifolds \( (W_1, J_1), (W_2, J_2) \) is a diffeomorphism \( \phi : W_1 \to W_2 \) such that \( \phi^*J_2 \) can be deformed to \( J_1 \) through almost complex structures on \( W_1 \). Equivalently, it also means that there is a family of non-degenerate 2-forms \( \omega_t \) interpolating between \( \omega_1 \) and \( \omega_2 \). We have the following lemma:

Lemma 2.35. \( (B^{2n}, \omega), \omega^n > 0 \) is almost symplectomorphic to \( (B^{2n}, \omega_{\text{std}}) \).
Proof. We only need to find a family of non-degenerate 2–forms connecting $\omega$ and $\omega_{\text{std}}$. Consider the bundle $\Omega^2(B^{2n}) \to B^{2n}$. Since $B^{2n}$ is contractible, so we only need to consider 2–forms on $\mathbb{R}^{2n}$. Clearly, the space of non-degenerate forms satisfying $\omega^n > 0$ on $\mathbb{R}^{2n}$ is path-connected. \hfill \square

Remark 2.36. Suppose $W \# B^{2n}$ is the boundary connected sum of $(W,\lambda)$ and $(B^{2n},\omega_{\text{std}})$, then $(W \# B^{2n},\lambda')$ is almost symplectomorphic to $(W,\lambda)$. Indeed, we can perform the connected sum in a Darboux chart; a standard argument shows that the almost symplectic class remains the same.

An almost Weinstein domain is a triple $(W,J,\phi)$, where $(W,J)$ is a compact almost symplectic manifold with boundary and $\phi$ is a Morse function on $W$ with no critical points of index greater than $n$ and maximal level set $\partial W$. An almost contact structure $(Y,J)$ on $Y$ is an almost complex structure $J$ on the stabilized tangent bundle $TY \oplus e^1$ of $Y$. Therefore an almost symplectic domain $(W,J)$ has almost contact boundary $(\partial W,J|_{\partial W})$; it is an almost symplectic filling of this almost contact manifold. Therefore a family of almost symplectic structures give rise to a family of almost contact structures on the boundary:

Lemma 2.37. Almost symplectomorphic Liouville domains have almost contactomorphic boundaries.

Note that any symplectic, Weinstein, or contact structure can also be viewed as an almost symplectic, Weinstein, or contact structure by considering just the underlying algebraic topological data.

Note that the first Chern class $c_1(J)$ is an invariant of almost symplectic, almost Weinstein, or almost contact structures. In this paper, we will often need to assume that $c_1(J)$ vanishes. The following proposition, which will be used several times in this paper, shows that the vanishing of $c_1(Y,J)$ is often preserved under contact surgery and furthermore implies the vanishing of $c_1(W,J)$.

Proposition 2.38 (Proposition 2.1 [Laz16]). Let $(W^{2n},J),n \geq 3$, be an almost Weinstein cobordism between $\partial_- W = (Y_-,J_-)$ and $\partial_+ W = (Y_+,J_+)$. If $H^2(W,Y_-) = 0$, the following are equivalent:

- $c_1(J_-) = 0, c_1(J_+) = 0$
- $c_1(J) = 0$.

If $\partial_- W = \emptyset$, the vanishing of $c_1(J_+)$ and $c_1(J)$ are equivalent.

Proof. Let $i_\pm : Y_\pm \hookrightarrow W$ be inclusions. Then $i_\pm^* c_1(J) = c_1(J_\pm)$ so the vanishing of $c_1(J)$ implies the vanishing of $c_1(J_-)$ and $c_1(J_+)$. To prove the converse, consider the cohomology long exact sequences of the pairs $(W,Y_-)$ and $(W,Y_+)$:

$$H^2(W,Y_\pm;\mathbb{Z}) \to H^2(W;\mathbb{Z}) \xrightarrow{i_\pm^*} H^2(Y_\pm;\mathbb{Z}).$$

By assumption, $H^2(W,Y_-;\mathbb{Z})$ vanishes and hence $i_-^*$ is injective. By Poincaré-Lefschetz duality, $H^2(W,Y_+;\mathbb{Z}) \cong H_{2n-2}(W,Y_-;\mathbb{Z})$. Since $2n-2 \geq n+1$ for $n \geq 3$ and $W$ is a Weinstein cobordism, $H_{2n-2}(W,Y_-;\mathbb{Z})$ vanishes and hence $i_+^*$ is also injective. Then if either $c_1(J_-) = i_-^* c_1(J)$ or $c_1(J_+) = i_+^* c_1(J)$ vanish, so does $c_1(J)$.

If $\partial_- W = \emptyset$, we just need the vanishing of $H^2(W,Y_+;\mathbb{Z})$, which holds for $n \geq 3$. \hfill \square

2.8. Morse-Bott case. The results of this section largely come from M.McLean [McL16].
Definition 2.39. A Morse-Bott family of Reeb orbits of \((C, \alpha)\) of period \(T\) is a closed sub-manifold \(B \subset C\) where \(B\) is a union of closed Reeb orbits of period \(T\), if \(\psi : C \to C\) is the Reeb flow then \(\ker(D\psi_T)|_B = TB\).

We are interested in indices of Reeb orbits and so from now on we assume that we work with a fixed trivialization of a fixed power of the canonical bundle of \((C, \alpha)\).

Note that the Conley-Zehnder index of the period \(T\) orbits starting in \(B_T\) are all the same because \(B_T\) is path connected. Hence we define the Conley-Zehnder index of \(B_T\), \(\mu_{CZ}(B_T)\), to be the Conley-Zehnder index of one of its period \(T\) Reeb orbits.

We can define an index closely related to the Conley-Zehnder index, called lower SFT index, \(lSFT(\gamma)\), as follows:

\[
lSFT(\gamma) := \mu_{CZ}(\gamma) - \frac{1}{2} \dim \ker(D\psi_T|_\xi - \id) + (n - 3)
\]

Similarly, we have the following definition:

Definition 2.40. Let \(K\) be a Hamiltonian on a symplectic manifold \((X, \omega_X)\) and \(B\) is a set of fixed points of its time \(T\) flow. We say that \(B\) is isolated if any such fixed point near \(B\) is contained in \(B\). If \(B\) is a path connected topological space and we have fixed a symplectic trivialization of \(TX\). Then every such Hamiltonian orbit has the same Conley-Zehnder index and we will write \(\mu_{CZ}(B, K)\) for the Conley-Zehnder index. The set \(B\) is said to be Morse-Bott if \(B\) is a submanifold and \(\ker(D\psi_T^V - \id) = TB\) along \(B\) where \(\psi_T^V : X \to X\) is the \(T\) Hamiltonian flow of \(K\).

The following lemma is a technical lemma which relates the index of Reeb orbits in a contact hypersurface (which is a regular level set of a Hamiltonian) and the index of the corresponding Hamiltonian orbits.

Lemma 2.41 (Lemma 5.22 [McL16]). Let \((W, \omega_W)\) be a symplectic manifold with a choice of symplectic trivialization of \(TW\) along a basis of \(H^1(W, \mathbb{Z})\), which is torsion free. Let \(\theta_W\) be a 1-form satisfying \(d\theta_W = \omega_W\), and \(K\) be an Hamiltonian with the property that \(b := \iota_{X_{\theta_W}} dK > 0\). This means \(C_r := K^{-1}(r)\) is a contact manifold with contact form \(\alpha_r := \theta_W|_{C_r}\). Let \(B \subset W\) be a connected submanifold transverse to \(C_r\) for each \(r\) so that \(B_0 := C_r \cap B\) is a Morse-Bott submanifold of the contact manifold \((C_r, \alpha_r)\) of period \(L_r\), where \(L_r\) smoothly depends on \(r\). Suppose that \(b = L_0\) along \(B_0\) and that \(db(V) > \frac{d(L_0)}{dr}|_{r=0}\) along \(B_0\), where \(V\) is a vector field tangent to \(B\) satisfying \(dK(V) = 1\). Then \(B_0\) is Morse-Bott for \(K\) and \(\mu_{CZ}(B_0, K) = \mu_{CZ}(B_0, \alpha_0) + \frac{1}{2}\).

Remark 2.42. Our sign convention is different from McLean’s in [McL16] since we use \(\omega(\cdot, X_H) = dH\). So the condition on \(b := \iota_{X_{\theta_W}} dK > 0\) differs by a minus sign. If \(b \neq L_0\) along \(B_0\), we have to either rescale \(b\) or \(L_t\).

Remark 2.43. In light of lemma 2.22, we have \(\mu_{CZ}(\gamma, r^2) = \frac{1}{2}\), where \(\gamma\) is the Morse-Bott manifold of Hamiltonian orbits.

3. Asymptotically dynamically convex contact structures

We will define (strongly) asymptotically dynamically convex contact structure first. Then we introduce a new invariant called positive idempotent group base on \(SH_*(W)\). It does not depend on the filling for ADC contact structure and therefore can be seen as a contact invariant. The proof will be deferred to Section 7. In subSection 3.3, we show that (strongly) ADC property is preserved under subcritical contact surgery.
3.1. Reeb orbits and asymptotically dynamically convex contact structures. Let's take a moment to look at the degree of Reeb orbits, which is essential to the definition of ADC contact structures. For any contact manifold $(\Sigma, \alpha)$ with $c_1(\Sigma, \xi)$, the canonical line bundle of $\xi$ is trivial, as will always be the case in this paper. After choosing a global trivialization of this bundle, we can assign an integer to each Reeb orbit $\gamma$ of $(\Sigma, \alpha)$-the reduced Conley-Zehnder index:

$$|\gamma| := \mu_{CZ}(\gamma) + n - 3.$$ 

For a general Reeb orbits, $|\gamma|$ depends on the choice of trivialization of canonical bundle. However, if the Reeb orbit $|\gamma|$ is contractible in $\Sigma$, then the grading does not depend on the trivialization. We will consider both the contractible and non-contractible Reeb orbits in this paper.

Let $\mathcal{P}_s^D(\Sigma, \alpha)$ be the set of Reeb orbits $\gamma$ of $(\Sigma, \alpha)$ and have $A(\gamma) < D$, where $\Phi$ is a specific trivialization of the canonical bundle. In a similar manner, we can define $\mathcal{P}_s^{D/s}(\Sigma, \alpha)$ to be the set of contractible Reeb orbits $\gamma$ of $(\Sigma, \alpha)$ and have $A(\gamma) < D$. Here we drop the subscription of $\Phi$ since the degree of contractible Reeb orbits does not depend on the choice of trivialization.

**Lemma 3.1** (Proposition 3.1 [Laz16]). For any $D, s > 0$, $\mathcal{P}_s^{D}(Y, s\alpha)$ and $\mathcal{P}_s^{D/s}(Y, \alpha)$ are in grading-preserving bijection.

**Proof.** Note that $R_{s\alpha} = \frac{1}{s} R_\alpha$. So if $\gamma_{s\alpha} : [0, T] \to Y$ is a Reeb trajectory of $\alpha$ with action $T$, then $\gamma_{s\alpha} = \gamma_\alpha \circ m_{s^{-1}} : [0, sT] \to Y$ is a Reeb trajectory of $s\alpha$ with action $sT$; here $m_{s^{-1}} : [0, sT] \to [0, T]$ is multiplication by $\frac{1}{s}$. The map $\gamma_\alpha \to \gamma_{s\alpha}$ is a bijection between the set of Reeb orbits. If $T < D/s$, then $sT < D$ and so it is a bijection between $\mathcal{P}_s^{D/s}(Y, \alpha)$ and $\mathcal{P}_s^{D}(Y, s\alpha)$. This bijection is grading-preserving since the Conley-Zehnder index of a Reeb orbit is determined by the linearized Reeb flow on the trivialized contact planes $\xi$ but does not depend on the speed of the flow. \hfill \Box

We will also need the following notation. If $\alpha_1, \alpha_2$ are contact forms for $\xi$, then there exists a unique $f : Y \to \mathbb{R}^+$ such that $\alpha_2 = f \alpha_1$. We write $\alpha_2 > \alpha_1, \alpha_2 \geq \alpha_1$ if $f > 1, f \geq 1$ respectively. Note that if $\alpha_2 > \alpha_1, \alpha_2 \geq \alpha_1$, then for any diffeomorphism $\Psi : Y' \to Y$, we have $\Psi^* \alpha_2 > \Psi^* \alpha_1, \Psi^* \alpha_2 \geq \Psi^* \alpha_1$, respectively.

**Definition 3.2.** A contact manifold $(\Sigma, \xi)$ is asymptotically dynamically convex (strongly asymptotically dynamically convex with respect to $\Phi$) if there exists a sequence of non-increasing contact forms $\alpha_1 \geq \alpha_2 \geq \alpha_3 \cdots$ for $\xi$ and increasing positive numbers $D_1 < D_2 < D_3 \cdots$ going to infinity such that all elements of $\mathcal{P}_0^{D_k}(\Sigma, \alpha_k)$ ($\mathcal{P}_s^{D_k}(\Sigma, \alpha_k)$) have positive lower SFT index.

**Remark 3.3.** Since we define the strongly ADC property (with respect to $\Phi$) using lower SFT index. So the contact form $\alpha_1$ in the definition doesn’t have to be non-degenerate. It is an immediate corollary of lemma 3.4.

**Lemma 3.4** (Lemma 4.10 [McL16]). Let $\gamma$ be any Reeb orbit of $\alpha$ of period $T$ and define $K := \dim \ker(D\psi_T|\xi(\gamma(0)) - id)$. Fix some Riemannian metric on $C$. There is a constant $\delta > 0$ and a neighborhood $N$ of $\gamma(0)$ so that for any contact form $\alpha_1$ with $|\alpha - \alpha_1|_{C^2} < \delta$ and any Reeb orbit $\gamma_1$ of $\alpha_1$ starting in $N$ of period in $[T - \delta, T + \delta]$ we have $\mu_{CZ}(\gamma_1) \in [\mu_{CZ}(\gamma) - \frac{K}{T}, \mu_{CZ}(\gamma) + \frac{K}{T}]$. 


3.2. Positive idempotent group $I_+$. Now we consider a strongly asymptotically dynamically convex contact manifold $(\Sigma, \xi, \Phi)$ with Liouville filling $(W, \lambda)$. We have the following result due to Lazarev:

**Theorem 3.5** (Proposition 3.8 [Laz16]). If $(\Sigma, \xi, \Phi)$ is a strongly asymptotically dynamically convex contact structure, then all Liouville fillings of $(\Sigma, \xi, \Phi)$ have isomorphic $SH^+$. 

3.2.1. Definition of positive idempotent group $I_+$. We also want to define the ring structure. However as in Remark 2.11, we can not define a product on $SH^+$. Having said that, we can use the pair-of-pants product on $SH_*(W)$ to define an invariant for $SH^+_*$ which is independent of the Liouville filling.

First, let’s recall the tautological short exact sequence:

$$0 \to SC^{<e}_*(W) \to SC_*(W) \to SC_*(W)/SC^{<e}_*(W) \to 0$$

We have long exact sequence:

$$\cdots \to SH^{<e}_*(W) \to SH_*(W) \to SH^+_*(W) \to SH^{<e}_{*-1}(W) \to \cdots$$

(3.1)

We also have $H^{n-*}(W, H) \cong SH^{<e}_*(W, H)$ since the admissible Hamiltonian $H$ is $C^2$ small in $W$. Therefore we can replace $SH^{<e}_*$ terms in equation 3.1 by $H^{n-*}$, in particular, we have

$$\cdots \to H^0(W) \to SH_n(W) \to SH^+_n(W) \to H^1(W) \to \cdots$$

(3.2)

In fact, the map $H^0(W) \to SH_n(W)$ in equation 3.2 is a ring homomorphism, see Appendix A of [CO18]. Suppose $SH_* (W) \neq 0$, then $1_W$ does not maps to the unit in $SH_*(W)$, where $1_W$ is the unit of $H^0(W)$, by Theorem 2.17 (also see Lemma A.3 of [CO18]). Therefore $H^0(W) \to SH_n(W)$ is injective, and we will regard it as a subring of $SH_n(W)$. We can thus identify elements in $SH_n(W)/H^0(W)$ with elements in $SH^+_n(W)$. In particular, $SH_n(W)/H^0(W) \cong SH^+_n(W)$ if $H^1(W) = 0$.

Now let’s consider a subgroup of $SH_n(W)$ as follows:

$$I(W) := \{ \alpha \in SH_n(W) \mid \alpha^2 - \alpha \in H^0(W) \}.$$  

(3.3)

Define positive idempotent group $I_+(W)$ by

$$I_+(W) := I(W)/H^0(W).$$

By the previous analysis, we can regard $I_+(W)$ as a subgroup of $SH^+_n(W)$. In the case $I_+(W)$ is finite, we can further define positive idempotent index $i(W) := |I_+(W)|$.

3.2.2. Properties of $I_+$. Since $H^0(W, \mathbb{Z}_2) \cong \mathbb{Z}_2$, $I_+(W, \mathbb{Z}_2)$ is determined by $I(W, \mathbb{Z}_2)$. Recall that $SH_*(W, \mathbb{Z}_2)$ has a $H_1(W, \mathbb{Z})$ grading. The first observation is that elements in $H^0(W)$ have $H_1$ grading zero. Indeed, it’s true for all elements in $I(W)$. Suppose $R$ is an algebra over $\mathbb{Z}_2$ which is graded by a finitely generated torsion-free abelian group $K$. This means that as a vector space, $R = \bigoplus_{k \in K} R_k$ with the property that if $a \in R_{k_1}, b \in R_{k_2}$ then $ab \in R_{k_1 + k_2}$. Define $I_0(R) := \{0, 1\}$ and $I(R) := \{x \in R \mid x^2 - x \in I_0\}$.

**Lemma 3.6** (Lemma 7.6 [McL07]). If $a \in I(R)$ then $a \in R_e$ where $e$ is identity of group $K$.

**Proof.** We argue by contradiction. Suppose we have $a = a_{k_1} + \cdots + a_{k_n}$ where $k_1 \in K$ and $a_{k_1} \in R_{k_1}$. Then $a^2 = a_{k_1}^2 + \cdots + a_{k_n}^2$. Since $K$ is torsion free, there is a group homomorphism $p : K \to \mathbb{Z}$ such that $p(k_1) \neq 0$. This map actually gives $R$ a $\mathbb{Z}$ grading. Let $b$ be an element in $R$, then it can be uniquely written as $b = b_1 + \cdots + b_k$ where $b_k$ are non-zero elements of $R$ with grading $d_k \in \mathbb{Z}$. We can define a function $f$ as follows:

$$f(b) := \min\{|d_i| \neq 0\}$$
Note that $f$ is well-defined only if at least one of the $d'_s$ is non-zero. And when it is well-defined, $f(b+1) = f(b)$ because $1 \in R_e$ and has grading 0. The assumption $p(k_1) \neq 0$ implies that $f(a)$ is well defined and positive. On the other hand, we have $a \in I(R)$, which means $a^2 = a$ or $a^2 = a + 1$. Either way, it implies $f(a^2) = f(a)$, which contradicts the fact that $f(a^2) \geq 2f(a)$. \hfill $\square$

**Corollary 3.7.** Any element in $I(W)$ is null-homologous in $H_1(W, \mathbb{Z})$, given that $H_1(W, \mathbb{Z})$ is torsion-free.

Therefore, we can refine our definition of $I(W)$ to be

$$I(W) := \{ \alpha \in SH^0_n(W) \mid \alpha^2 - \alpha \in H^0(W) \},$$

where $SH^0_n(W)$ is generated by all null-homologous Reeb orbits. In the case of strongly asymptotically dynamically convex contact manifolds (with respect to certain framing $\Phi$), we have the following Proposition:

**Proposition 3.8.** If $(\Sigma, \xi, \Phi)$ is a strongly asymptotically dynamically convex contact structure with a filling $W$ such that $SH_*(W) \neq 0$, then all Liouville filling of $(\Sigma, \xi, \Phi)$ have isomorphic positive idempotent group $I_+$.

The proof will be deferred to Section 7.

3.3. Effect of contact surgery.

**Theorem 3.9** (Theorem 3.15 [Laz16], [Yau04]). If $(Y_1^{2n-1}, \xi_1)$, $n \geq 2$, is an asymptotically dynamically convex contact structure and $(Y_2, \xi_2)$ is the result of index $k \neq 0$ subcritical contact surgery on $(Y_1, \xi_1)$, then $(Y_2, \xi_2)$ is also asymptotically dynamically convex.

Now we are in the position to prove Proposition 1.5.

**Proof of Proposition 1.5.** Recall that $W$ is a flexible Weinstein domain, so $SH_*(W) = 0$ (see [BEE12]). By Lemma 1.4, $\partial W$ is asymptotically dynamically convex and so is $Y_i$ by assumption. Moreover, $\partial V_i$ is obtained by attaching a Weinstein 1-handle to asymptotically dynamically convex contact manifold, therefore it is asymptotically dynamically convex by Theorem 3.9. A well known fact is that subcritical surgery does not change symplectic homology as a ring, see Corollary 2.19. Thus,

$$SH_*(V_i) \cong SH_*(W) \oplus SH_*(W) = SH_*(W),$$

as rings. By the definition of positive idempotent group, we have $I_+(V_i) \cong I_+(W_i)$, thence $I_+(\partial V_i) \cong I_+(Y_i)$. \hfill $\square$

**Theorem 3.10** ([Laz16], [Yau04]). Let $(\Sigma_1, \xi_1)$ be a strongly asymptotically dynamically convex contact structure with respect to $\Phi$, and $(\alpha_k, D_k)$ as in Definition 3.2 and $(\Sigma_2, \xi_2)$ be the result of index 2 contact surgery on $\Lambda^1 \subset \Sigma_1$ which extends trivialization $\Phi$ to the handle. Then $(\Sigma_2, \xi_2)$ is also strongly asymptotically dynamically convex with respect to $\Phi$.

**Remark 3.11.** Since the trivialization $\Phi$ of the canonical bundle of $(\Sigma_1, \xi_1)$ extends to the attaching handle, so by abuse of notation, the trivialization of canonical bundle of $(\Sigma_2, \xi_2)$ which is obtained by extending $\Phi$ to the attaching handle is still denoted by $\Phi$.

**Proposition 3.12** (Proposition 5.5 [Laz16], [Yau04]). Let $\Lambda^{k-1} \subset (\Sigma_1^{2n-1}, \alpha_1)$, $n > 1$, be an isotropic sphere with $k < n$. For any $D > 0$ and integer $i > 0$, there exists $\epsilon = \epsilon(D,i) > 0$ such that if $(\Sigma_2, \alpha_2)$ is the result of contact surgery on $U^*(\Lambda, \alpha)$ with respect to the trivialization $\Phi$, then there is a grading preserving bijection between $\mathcal{P}_D^\Phi(\Sigma_2, \alpha_2)$ and $\mathcal{P}_D^\Phi(\Sigma_1, \alpha_1) \cup \{ \gamma^1, \cdots, \gamma^l \}$ where $|\gamma^i| = 2n - k + 2i$. 


Remark 3.13. The proof largely follows [Laz16] proposition 5.5, with only minor changes regarding the non-contractible Reeb orbits. The difference in the Strongly ADC case is that we need to choose the trivialization to define the Conley-Zehnder index.

Proof of Proposition 3.12. As explained in [Yau04], the surgery belt sphere $S^{2n-k-1}$ contains a contact sphere $(S^{2n-2k-1},\xi_{std})$. After taking appropriate sequence of contact forms on $(\Sigma_2,\xi_2)$, the Reeb orbits of $(\Sigma_2,\xi_2)$ correspond to the old Reed orbits of $(\Sigma_1,\xi_1)$, plus the new orbits of $(S^{2n-2k-1},\xi_{std})$ inside the belt sphere, up to action $D$. The correspondence is natural since the surgery extends the trivialization. And the latter corresponds to the iterations $\gamma^1,\cdots,\gamma^l$ of a single Reeb orbit $\gamma$, see [Yau04]. Moreover, $\mu_{CZ}(\gamma_i) = n-k-1+2i$ and therefore $|\gamma_i| = 2n-k-4+2i$. Meanwhile, by shrinking the handle, the action can be made arbitrarily small and therefore we can ensure that arbitrarily large iterations of $\gamma$ have action less than $D$.

For $\Lambda \subset \Sigma$, choose a Riemannian metric on $\Lambda$. Then let $U^c(\Lambda) \subset (J^1(\Lambda),\alpha_{std})$ be $\{|y| < \epsilon,|z| < \epsilon\}$, where we use the metric on $\Lambda$ to define $|y|$.

**Proposition 3.14** (Proposition 6.7 [Laz16]). Let $\alpha_1 > \alpha_2$ be contact forms for $(\Sigma,\xi)$ and let $\Lambda \subset (\Sigma,\xi)$ be an isotropic submanifold with trivial symplectic conormal bundle. Then for any sufficiently small $\delta_1,\delta_2$, there exists a contactomorphism $h$ of $(\Sigma,\xi)$ such that
- $h$ is supported in $U^c(\Lambda,\alpha_1),h|_{\Lambda} = Id$, and $h^*\alpha_2 < 4\alpha_1$
- $h^*\alpha_2|_{U^c(\Lambda,\alpha_1)} = c\alpha_1|_{U^c(\Lambda,\alpha_1)}$ for some constant $c$ (depending on $\delta_1,\delta_2$)
- $h(U^{\delta_1}(\Lambda,\alpha_1)) \subset U^{\delta_2}(\Lambda,\alpha_2)$.

**Proposition 3.15** (Remark 6.5 [Laz16]). Let $\Lambda \subset (\Sigma_{2n-1},\xi_1), n > 2$ be an isotropic sphere and $(\Sigma_2,\xi_2)$, the result of contact surgery on $\Lambda$ which extends the chosen trivialization $\Phi$ of the canonical bundle. Suppose $(\Sigma_1,\xi_1)$ is a strongly asymptotically dynamically convex contact structure with respect to the trivialization $\Phi$ and has $(\alpha_k,D_k)$ as in Definition 3.2. If $\alpha_k|_{U^c(\Lambda,\alpha_1)} = c_k\alpha_1|_{U^c(\Lambda,\alpha_1)}$ for some constants $c,c_k$, then $(\Sigma_2,\xi_2)$ is also strongly asymptotically dynamically convex with respect to the trivialization $\Phi$.

Proof of Theorem 3.10. Now we will proceed exactly as Lazarev did, keeping in mind that we are dealing with the strongly ADC property. We can apply Proposition 3.14 so that the conditions of Proposition 3.15 are satisfied.

4. Brieskorn Manifolds

4.1. Definition of Brieskorn manifolds. Let $a = (a_0, a_1, \cdots, a_n)$ be an $(n+1)$-tuple of integers $a_i > 1$, $z := (z_0, z_1, \cdots, z_n) \in \mathbb{C}^{n+1}$, and set $f(z) := z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n}$, we define Brieskorn Variety as

$$V_a(t) := \{(z_0, z_1, \cdots, z_n) \in \mathbb{C}^{n+1}| f(z) = t\}. \quad (4.1)$$

We will often suppress $a$ when it causes no confusion, and define $X_t^z = V(t) \cap B(s)$.

Further, with $S^{2n+1}$ denoting the unit sphere in $\mathbb{C}^{n+1}$, we define the Brieskorn Manifold as the intersection of Brieskorn Variety $V_a(0)$ with the unit sphere:

$$\Sigma(a) := V_a(0) \cap S^{2n+1}.$$ 

**Lemma 4.1** (Lemma 96 [Fau16], Lemma 7.1.1 [Gei08]). $\Sigma(a)$ and $V(a,t), t \neq 0$ are smooth manifolds.
Remark 4.4. By Theorem 4.2 we find the Jacobian matrix

\[ D(f, \rho) = \begin{bmatrix}
    a_0 z_0^{a_0-1} & \cdots & a_n z_n^{a_n-1} & 0 & \cdots & 0 \\
    0 & \cdots & 0 & a_0 \bar{z}_0^{a_0-1} & \cdots & a_n \bar{z}_n^{a_n-1}
\end{bmatrix} \]

For \( z \neq 0 \) the first two rows of \( D(f, \rho) \) are linearly independent, which implies that \( \epsilon \neq 0 \) is a regular value of \( f \). If \( z \) is a point where this matrix has rank smaller than 3, there exists a non-zero complex number \( \lambda \) such that \( \bar{z}_k = \lambda a_k z_k^{a_k-1} \) for all \( k \) and hence

\[ \sum_{k=0}^{n} \frac{z_k \bar{z}_k}{a_k} = \lambda \sum_{k=0}^{n} z_k = \lambda \cdot f(z) \]

This equality is incompatible with the conditions \( \rho(z) = 1 \) and \( f(z) = 0 \) for a point \( z \in \Sigma(a) \).

\[ \square \]

4.2. Topology of Brieskorn manifolds. Next, we give some topological facts about Brieskorn manifolds without proof.

Proposition 4.2 (Theorem 5.2 [Mil16]). A Brieskorn manifold \( \Sigma(a)^{2n-1} \) is \((n-2)\)-connected.

We associate to each tuple \( a = (a_0, a_1, \ldots, a_n) \) the following graph \( \Gamma_a \):

- \( \Gamma_a \) has \( n + 1 \) vertices labeled \( a_0, a_1, \ldots, a_n \).
- \( \Gamma_a \) contains an edge between \( a_j, a_k \) iff \( \gcd(a_j, a_k) > 1 \).

Theorem 4.3 ([Bri66]). If \( n \geq 3 \), then \( \Sigma(a) \) is homeomorphic to a sphere if and only if the graph \( \Gamma_a \) satisfies one of the following two conditions:

- The graph \( \Gamma_a \) has at least two isolated points.
- The graph \( \Gamma_a \) has an isolated point and a connected component \( K \) with an odd number of points such that for \( a_i, a_j \in K, i \neq j \), one has \( \gcd(a_i, a_j) = 2 \).

Remark 4.4. By Theorem 4.3, we have the family \( \Sigma(2, 2, 2, p_1, \ldots, p_n) \) of Brieskorn spheres, where \( p_i \)'s are distinct odd prime numbers.

4.3. Trivialization and Conley-Zehnder index. Let us consider on \( \mathbb{C}^{n+1} \) the following Hermitian form given by

\[ < \xi, \zeta >_a := \frac{1}{2} \sum_{k=0}^{n} a_k \xi_k \bar{\zeta}_k. \]

It defines a symplectic 2-form

\[ \omega_a := \frac{i}{4} \sum_{k=0}^{n} a_k dz_k \wedge d\bar{z}_k. \]

Notice that \( Y_{\lambda}(z) := \frac{z}{\bar{z}} \) is a Liouville vector field for \( \omega_a \), with the corresponding 1-form

\[ \lambda_a := \omega_a(Y_{\lambda}, \cdot) = \frac{i}{8} \sum_{k=0}^{n} a_k (z_k d\bar{z}_k - \bar{z}_k dz_k). \]
**Proposition 4.5** (Proposition 97 [Fau16], [LM76]). The restriction \( \alpha_a := \lambda_a|_\Sigma \) is a contact form on \( \Sigma(a) \) with Reeb vector field \( R_a \) given by

\[
R_a = 4i(z_0/a_0, z_1/a_1, \ldots, z_n/a_n).
\]

**Proof.** The gradient of \( f \) with respect to \( < \cdot, \cdot >_a \) is given by

\[
\nabla_a f := 2(\bar{z}_0 a_0^{-1}, \bar{z}_1 a_1^{-1}, \ldots, \bar{z}_n a_n^{-1}).
\]

The Liouville vector field \( Y_V \) of the restricted 1-form \( \lambda_a|_{V_a(0)} \) with respect to the restricted symplectic form \( \omega_a|_{V_a(0)} \) is given by

\[
Y_V := Y_\lambda - \frac{\nabla_a f}{||\nabla_a f||^2_a} \cdot \nabla_a f.
\]

Note that \( TV_a(t) = \ker df = \ker <\nabla_a f, \cdot >_a \), which shows that \( Y_V \in TV_a(0) \). Furthermore, we have for any \( \xi \in TV_a(0) \),

\[
\omega_a(Y_V, \xi) = \omega_a(Y_\lambda, \xi) - \frac{\nabla_a f, Y_\lambda >_a}{||\nabla_a f||^2_a} \omega_a(\nabla_a f, \xi) = \lambda_a(\xi) + \frac{\nabla_a f, Y_\lambda >_a}{||\nabla_a f||^2_a} \cdot 2f(\bar{z}) = 0.
\]

So this indicates that \( Y_V \) is the Liouville vector field for the pair \( (\omega_a|_{V_a(0)}, \lambda_a|_{V_a(0)}) \). Now notice that \( d\rho = \sum_{k=0}^n \bar{z}_k d\bar{z}_k + z_k dz_k \) and we have

\[
d\rho(Y_V) = \sum_{k=0}^n \frac{z_k \bar{z}_k}{2} - \frac{\nabla_a f, Y_\lambda >_a}{||\nabla_a f||^2_a} \sum_{k=0}^n 2\bar{z}_k \cdot \bar{z}_k a_k^{-1} = \frac{\rho(z)}{2} - \frac{\nabla_a f, Y_\lambda >_a}{||\nabla_a f||^2_a} \cdot 2f(z) = \frac{1}{2}.
\]

since \( \rho(z) = 1 \) and \( f(z) = 0 \). It follows that \( Y_V \) points out of the unit sphere and hence out of \( \Sigma(a) \) in \( V_a(0) \). It follows that \( \Sigma(a) \) is a contact hypersurface in \( V_a(0) \). Now we are going to check that \( R_a \) is the Reeb vector field of \( \alpha_a \). For any \( z \in \Sigma(a) \), we have

\[
< R_a, \nabla_a f >_a = 4i \sum_{k=0}^n z_k a_k = 4i f(z) = 0,
\]

\[
d\rho(R_a) = \sum_{k=0}^n z_k (-4i) \bar{z}_k + \bar{z}_k 4iz_k = 0
\]

The two equations above shows that \( R_a \) is a tangent vector. We also have

\[
\alpha_a(R_a) = \lambda_a(R_a) = \frac{i}{8} \sum_{k=0}^n a_k (\frac{4i}{a_k} \bar{z}_k - \bar{z}_k \frac{4i}{a_k} z_k) = \rho(z) = 1,
\]

\[
\iota_{R_a} d\alpha_a = \frac{i}{4} \sum_{k=0}^n (4i \frac{z_k}{a_k} d\bar{z}_k - (-4i) \frac{\bar{z}_k}{a_k}) = -\sum_{k=0}^n (z_k d\bar{z}_k + \bar{z}_k dz_k) = -d\rho.
\]

The latter form is zero for vectors in \( T\Sigma(a) \), therefore, \( R_a \) is the Reeb vector field. \( \square \)

**Proposition 4.6** (Corollary 98 [Fau16]). The symplectic complement \( \xi_a^\perp \) with respect to \( \omega_a \) of the contact structure \( \xi_a := \ker \alpha_a \) inside \( \mathbb{C}^{n+1} \) is symplectically trivialized by the following 4 vector fields:

- \( X_1 := \frac{\nabla_a f}{||\nabla_a f||_a} \)
- \( Y_1 := i \cdot X_1 \)
- \( X_2 := Y_V \)
- \( Y_2 := R_a \).
Proposition 4.7 \((\mathbb{1})\)

The latter equation comes from the proof of proposition

Meanwhile we have

\[
\omega_a(X_1, Y_1) = 1, \quad \omega_a(Y_1, X_2) = \lambda_a(R_a) = 1.
\]

The latter equation comes from the proof of proposition 4.5. \(\square\)

The Reeb vector field \(R_a = 4i(z_{a_0} - z_{a_1}, \ldots, z_{a_n})\) generates the following flow:

\[
\psi_a^t(z) = (e^{\frac{4it}{\pi}} \cdot z_0, \ldots, e^{\frac{4it}{\pi}} \cdot z_n)
\]

The submanifolds \(\Sigma_T\) of period \(T \in \pi\mathbb{Z}/2\) are given by

\[
\Sigma_T = \left\{ z \in \Sigma(a) \left| z_k = 0, \text{ if } T \frac{a_k}{a} \notin \pi\mathbb{Z}/2 \right. \right\}
\]

\(\Sigma_T\) is not empty if and only if the relation \(T \frac{a_k}{a} \notin \pi\mathbb{Z}/2\) is satisfied by at least 2 different \(k\), as \(z \in \Sigma(a)\) has at least 2 non-zero entries. Note that \(\Sigma_T\) is the intersection \(\Sigma(a) \cap V(a, T)\), where \(V(a, T)\) denotes the complex linear subspace

\[
V(a, T) := \left\{ z \in \mathbb{C}^{n+1} \left| z_k = 0, \text{ if } T \frac{a_k}{a} \notin \frac{\pi}{2} \right. \right\}
\]

whose complex dimension is given by

\[
\dim \mathbb{C} V(a, T) := \left| \left\{ k \left| 0 \leq k \leq n, \frac{T}{a_k} \in \frac{\pi}{2} \right. \right\} \right|
\]

where \(|S|\) denotes the cardinality of the set \(S\). We notice that \(\Sigma_T\) is therefore isomorphic to the Brieskorn manifold \(\Sigma(a(T))\), where

\[
a(T) = (a_0, \ldots, \hat{a}_i, \ldots, a_n)
\]

is a subset of \(a\). Here \(\hat{a}_i\) means the term \(a_i\) is omitted, when \(T \frac{a_i}{a} \notin \frac{\pi}{2}\). The differential of \(\phi_a\) at time \(t\) is given by

\[
D\psi_a^t = \text{diag}(e^{4it/a_0}, \ldots, e^{4it/a_n})
\]

It follows that

\[
\ker(D\psi_a^t |_{T_z \Sigma(a)}) = T_z \Sigma(a) \cap V(a, T) = T_z \Sigma_T
\]

Therefore \(\Sigma_T\) is Morse-Bott submanifold.

The calculation of the indices of all closed Reeb orbits can be found in various literatures, see [KvK16], [Ust99]. We conclude this subsection with the following proposition:

**Proposition 4.7** ([KvK16], [Fau16]). Let \(\gamma \in \Sigma(a)\) be a fractional Reeb of period \(t\). We have

\[
\mu_CZ(\gamma) = \sum_{k=0}^{n} \left( \left[ \frac{2t}{a_k\pi} \right] + \left[ \frac{2t}{a_k\pi} \right] \right) - \left( \left[ \frac{2t}{\pi} \right] + \left[ \frac{2t}{\pi} \right] \right)
\]

**Proof.** First we notice that the indices are canonically defined when \(n \geq 4\), by Proposition 4.2. Recall the Reeb vector field in Proposition 4.5, \(R_a = 4i(z_{a_0} - z_{a_1}, \ldots, z_{a_n})\). The associated Reeb flow is

\[
\psi_a^t(z) = (e^{\frac{4it}{\pi}} \cdot z_0, \ldots, e^{\frac{4it}{\pi}} \cdot z_n).
\]
We regard them as flow on $\mathbb{C}^{n+1}$ as opposed to $\Sigma(a)$. This perspective gives us the advantage of calculating the indices directly on $\mathbb{C}^{n+1}$. If we take the standard trivialization of $T\mathbb{C}^{n+1}$, then the linearized return map is

$$D\psi_t^a = \text{diag}(e^{4it/a_0}, \cdots, e^{4it/a_n}) =: \Psi_t.$$ 

By Proposition 4.6, we have the trivialization of the symplectic complement $\xi^\bot_a$. The linearized return map of the flow on $\xi^\bot_a$ gives:

- $D\psi_t^a(X_1(z)) = e^{4it} \cdot X_1(\psi_t^a(z))$,
- $D\psi_t^a(Y_1(z)) = e^{4it} \cdot Y_1(\psi_t^a(z))$,
- $D\psi_t^a(X_2(z)) = X_2(\psi_t^a(z))$,
- $D\psi_t^a(Y_2(z)) = Y_2(\psi_t^a(z))$.

It follows that the linearized return map of $D\psi_t^a$ on $\xi^\bot_a$ under the prescribed trivialization are the diagonal matrices:

$$\Psi_t^2 := \begin{bmatrix} e^{4it} & 0 \\ 0 & 1 \end{bmatrix}.$$ 

A trivialization of $\xi_a$ along the Reeb orbit gives us the linearization of $\Psi_t^1$ of $\psi_t^a$ on $\xi_a$. Any trivialization of $\xi_a$ and $\xi^\bot_a$ combined gives rise to a trivialization of $T\mathbb{C}^{n+1}$, which is homotopic to the standard one. Therefore by the product property of the Conley-Zehnder index and using remark 2.21, we find that

$$\mu_{CZ}(\gamma) = \mu_{CZ}(\Psi_1) = \mu_{CZ}(\Psi) - \mu_{CZ}(\Psi_2)$$

$$= \sum_{k=0}^n \left( \left\lfloor \frac{2t}{a_k \pi} \right\rfloor + \left\lceil \frac{2t}{a_k \pi} \right\rceil \right) - \left( \left\lfloor \frac{2t}{\pi} \right\rfloor + \left\lceil \frac{2t}{\pi} \right\rceil \right).$$

□

**Lemma 4.8.** Let $a = (a_0, a_1, a_2, \cdots, a_n)$, where $a_k$’s are positive integers, and $\sum \frac{1}{a_k} \geq 1$. Then the following function $f_a : \mathbb{R}_+ \to \mathbb{Z}$,

$$f_a(x) = \sum_{k=0}^n \left( \left\lfloor \frac{x}{a_k} \right\rfloor + \left\lceil \frac{x}{a_k} \right\rceil \right) - \left( \left\lfloor \frac{x}{\pi} \right\rfloor + \left\lceil \frac{x}{\pi} \rceil \right)$$

has a minimum, denoted by $m(a)$. In particular, if $a = (2, 2, 2, p_1, \cdots, p_n)$, then $m(a) \geq 2$, where $p_k$’s are distinct odd prime numbers, $n \geq 2$.

**Proof.** We notice that $2x - 1 < \lfloor x \rfloor + \lceil x \rceil < 2x + 1$, we have

$$f_a(x) > 2 \left( \sum_{k=0}^n \frac{1}{a_k} - 1 \right) x - n - 1 \geq -n - 1,$$

which proves the first part. For the second part, we have

$$f_a(x) = 3 \left( \left\lfloor \frac{x}{2} \right\rfloor + \left\lceil \frac{x}{2} \rceil \right) + \sum_{k=1}^n \left( \left\lfloor \frac{x}{p_k} \right\rfloor + \left\lceil \frac{x}{p_k} \rceil \right) - (\lfloor x \rfloor + \lceil x \rceil)$$
Note that \( f_a(x + 2) \geq f_a(x) + 2 \), so the minimum is obtained in \( x \in (0, 2] \). On this interval, we have \( f_a(x) = 3 \left( \left\lfloor \frac{x}{2} \right\rfloor + \left\lceil \frac{x}{2} \right\rceil \right) + n - \left( \left\lfloor x \right\rfloor + \left\lceil x \right\rceil \right) \), which is

\[
\begin{align*}
  f_a(x) &= \begin{cases} 
    2 + n & x \in (0, 1), \\
    1 + n & x = 1, \\
    n & x \in (1, 2), \\
    4 + n & x = 2.
  \end{cases}
\end{align*}
\]

hence our conclusion. \( \square \)

5. Exotic contact manifolds

5.1. Liouville domains admitting group actions. We need to find a Liouville domain \((W, \lambda)\) with contact manifold \((\Sigma(a))\) as its boundary. While \(V_a(0)\) has a singularity at origin, \(V_a(\epsilon)\) is smooth. So we will follow Alexander Fauck’s approach to overcome this by constructing an interpolation between \(V_a(0)\) and \(V_a(\epsilon)\). First, we choose a smooth monotone decreasing cut-off function \(\beta \in C^\infty(\mathbb{R})\) with \(\beta(x) = 1, x \leq \frac{1}{4}\) and \(\beta(x) = 0, x \geq \frac{3}{4}\). Then we define (we will often omit \(a\))

\[
U_a(\epsilon) := \{ z \in \mathbb{C}^{n+1} | z_0^{a_0} + \cdots + z_n^{a_n} = \epsilon \cdot \beta(||z||^2) \}.
\]

Let

\[
W_s^\epsilon := U_s \cap B(s)
\]

we have

**Proposition 5.1** (Proposition 99 [Fau16]). For sufficient small \(\epsilon, (X^1_\epsilon, \lambda)\) is a Liouville domain with boundary \((\Sigma(a), \alpha_a)\) and vanishing first Chern class.

Moreover, we have a cyclic group

\[
C(L) := \{ e^{2\pi ki \frac{1}{L}} | k \in \mathbb{Z} \} = \langle \zeta \rangle
\]

acting on \((\mathbb{C}^{n+1})^*\), which is generated by :

\[
\zeta_* : (\mathbb{C}^{n+1})^* \rightarrow (\mathbb{C}^{n+1})^* \\
(z_0, z_1, \cdots, z_n) \mapsto (z_0 \zeta^{b_0}, z_1 \zeta^{b_1}, \cdots, z_n \zeta^{b_n})
\]

where \( L := \text{lcm } a_j, b_j := L/a_j, \zeta := e^{2\pi i / L} \). We can easily see that the 1-form \(\lambda_a\) is \(C(L)\)-invariant.

We can restrict this group action to the subsets of \((\mathbb{C}^{n+1})^*\) mentioned above and obtain a \(C(L)\)-action on the manifolds \(X^s_\epsilon\) and \(W^s_\epsilon\). By definition, \(X^{1/2}_\epsilon = U(\epsilon) \cap B(1/2) = V(\epsilon) \cap B(1/2) = W^{1/2}_\epsilon\). We have the following proposition:

**Proposition 5.2.** For sufficiently small \(\epsilon > 0\), there is a \(C(L)\)-equivariant isotopy between the following pairs of Liouville domains:

- \(X^1_\epsilon\) and \(X^{1/2}_\epsilon\),
- \(W^1_\epsilon\) and \(W^{1/2}_\epsilon\).
Proof. We only give a proof for the existence of a $C(L)$-equivariant isotopy between $W^1_t$ and $W^{1/2}_t$. We can prove the same results for $X^1_t$ and $X^{1/2}_t$ verbatim. Consider the function $\rho(z) = ||z||^2$ on $V_t$. If for sufficiently small $\epsilon$, the critical values of $\rho$ are less that $1/4$, then we are done, by lemma 5.3. Indeed, we have $f_\epsilon(z) := f(z) - \epsilon \cdot \beta(||z||^2)$ on $\mathbb{C}^{n+1}$ and its differential is given by

$$Df_\epsilon = Df - \epsilon \cdot \beta'(||z||^2) \cdot D\rho$$

so the map

$$(f_\epsilon, \rho) : \mathbb{C}^{n+1} \to \mathbb{C} \times \mathbb{R}$$

has Jacobian matrix $(Df - \epsilon \cdot \beta'(||z||^2) \cdot D\rho, D\rho)$, which has the same rank as $(Df, D\rho)$. So by the same argument in the proof of lemma 4.1, if $z$ is a point where the Jacobian is not full rank, then we have for some complex number $\lambda$, $z_k = \lambda a_k z_k^{-1}$ for all $k$. For $||z|| \geq 1/2$, we have $|z_k| \geq \frac{1}{2}$ for some $k_0$, so

$$|z_{k_0}| = |\lambda| \cdot a_{k_0} \cdot |z_{k_0}| \geq |z_{k_0}|$$

i.e.,

$$|\lambda| = \frac{|z_{k_0}|^{2-a_{k_0}}}{a_{k_0}} \leq \frac{(2\sqrt{n})^{a_{k_0}-2}}{a_{k_0}} \leq C(a) \quad (5.1)$$

where $C(a) := \max_{0 \leq k \leq n} \left\{ \frac{(2\sqrt{n})^{a_{k_0}-2}}{a_{k_0}} \right\}$ only depends on $a$ and $n$. Meanwhile, we have

$$\sum_{k=0}^{n} \frac{z_k \bar{z}_k}{a_k} = \lambda \sum_{k=0}^{n} \frac{z_k a_k}{a_k} = \lambda \cdot f(z) = \lambda \cdot \epsilon \beta(||z||^2) \quad (5.2)$$

Combining equations (5.2) and (5.1), we have

$$\sum_{k=0}^{n} \frac{z_k \bar{z}_k}{a_k} = \lambda \cdot \epsilon \beta(||z||^2) \leq \epsilon \cdot C(a) \quad (5.3)$$

On the other,

$$\sum_{k=0}^{n} \frac{z_k \bar{z}_k}{a_k} \geq \frac{1}{\max\{a_j\}} \sum_{k=0}^{n} z_k \bar{z}_k \geq \frac{1}{\max\{a_j\}} \cdot ||z||^2 = \frac{1}{4 \max\{a_j\}} \quad (5.4)$$

Equations (5.4) and (5.3) cannot hold for sufficiently small $\epsilon$ at the same time, and therefore the function $\rho$ has no critical points in $||z|| \geq 1/2$, hence all critical values are less than $1/4$.

Lemma 5.3 (Theorem 2.2.2 [Nic11]). Suppose $G$ acts on a manifold $M$ and $f$ is a $G$-invariant exhausting function on $M$. Moreover, assume that no critical value of $f$ is contained in $[a, b] \subset \mathbb{R}$, then there is a $G$-equivariant isotopy $\Phi_t$ between the sublevel sets $M^a := f^{-1}((-\infty, a])$ and $M^b := f^{-1}((-\infty, b])$, and $\Phi_t$ coincides with Id outside a compact set.

Proof. Since there are no critical values of $f$ in $[a, b]$ and the sublevel sets are compact, we deduce that there exists $\epsilon > 0$ such that

$$\{a - \epsilon < f < b + \epsilon \} \subset M \setminus \text{Crit}(f)$$

First we fix a gradient-like $G$-invariant vector field $Y$ and construct a compactly supported $G$-equivariant smooth function

$$g : M \to [0, \infty)$$
such that
\[ g(x) = \begin{cases} \frac{1}{|Yf|}, & a \leq f(x) \leq b, \\ 0, & f(x) \notin (a - \epsilon, b + \epsilon). \end{cases} \]

We can now construct a $G$-invariant vector field $X := -gY$ on $M$ and we denote by
\[ \Phi : \mathbb{R} \times M \to M, \quad (t, x) \to \Phi_t(x) \]
the flow generated by $X$. Clearly the flow commutes with the group action, so $\Phi_t$ is $G$-equivariant. If $u(t)$ is an integral curve of $X$, then differentiating $f$ along $u(t)$ in the region \{\(a \leq f \leq b\)} and get
\[ \frac{df}{dt} = Xf = -\frac{1}{Yf} Yf = -1 \]
This implies
\[ \Phi_{b-a}(M_a) = M_b \]
and $\Phi_t$ is identity outside the region \{\(a - \epsilon < f < b + \epsilon\).

**Remark 5.4.** By proposition 5.1, $(X^1_\epsilon, \Phi^1_\epsilon \lambda)$ is a family of $C(L)$–equivariant Liouville structures. Then by corollary 2.5, we have $(X^1_\epsilon, \lambda)$ is $C(L)$–equivariant Liouville isomorphic to $(X^1_\epsilon, \lambda)$. By the same token, $(W^1_\epsilon, \lambda)$ is $C(L)$–equivariant Liouville isomorphic to $(W^1_\epsilon, \lambda)$ and therefore to $(X^1_\epsilon, \lambda)$.

Let $\phi_t(z) := \frac{1}{8} \sum_{j=0}^n c_j(t) |z_j|^2$, where $c_j(t)$ is a linear interpolation such that $c_j(0) = 1, c_j(1) = a_j$. It’s easy to check that $\phi_t$ is plurisubharmonic on $V_a(\epsilon)$. Indeed, $\phi_t$ is $i$-convex on $\mathbb{C}^{n+1}$ since $\Delta \phi_t > 0$ and $V_a(\epsilon)$ is a smooth complex submanifold. So $(V_a(\epsilon), i, \phi_t)$ are $C(L)$–equivariant Stein manifolds.

Since $X^1_\epsilon = \phi_0^{-1}((-\infty, 1/8])$, $(X^1_\epsilon, i, \phi_0)$ is a $C(L)$–equivariant Stein domain. Seen as a Liouville domain, $(X^1_\epsilon, -d^C \phi_0)$ is $C(L)$–equivariant Liouville isomorphic to $(X^1_\epsilon, \lambda)$ as follows:

**Proposition 5.5.** There is a $C(L)$-equivariant Liouville homotopy between $(X^1_\epsilon, -d^C \phi_0)$ and $(X^1_\epsilon, \lambda)$, for sufficiently small $\epsilon$.

**Proof.** Notice that for $\lambda = -d^C \phi_1$, it suffices to prove the critical points of $\phi_t$ are contained in a compact set \{\(|z| \leq 1/3\)} then $\nabla \phi_t, \phi_t$ will be transversal to the boundary, and $-d^C \phi_t$ will be a family of $C(L)$–equivariant Liouville structures on $X^1_\epsilon$, so we can conclude the result by corollary 2.5. In the following we are going to prove that all critical points satisfy \(|z| \leq 1/3\).

Consider the map
\[ (f, \phi_t) : \mathbb{C}^{n+1} \to \mathbb{C} \times \mathbb{R} \]
Its Jacobian matrix is
\[
D(f, \rho) = \begin{bmatrix}
0 & \cdots & 0 \\
0 & \cdots & 0 \\
\frac{1}{8} c_0(t) z_0 & \cdots & \frac{1}{8} c_n(t) z_n
\end{bmatrix} - \begin{bmatrix}
\frac{z_0}{8} & \cdots & \frac{z_n}{8}
\end{bmatrix}
\]
If $z$ is a point where this matrix has rank smaller than 3, there exists a non-zero complex number $\lambda \in \mathbb{C}$ such that
\[
\frac{z_k}{8} = \lambda a_k z_k^{a_k - 1}
\]
for all $k$ and
\[
\sum_{k=0}^n \frac{c_k(t) z_k z_k^*}{8 a_k} = \lambda \sum_{k=0}^n a_k z_k = \lambda \cdot f(z) = \lambda \cdot \epsilon
\] (5.5)
For $||z|| > \frac{1}{3}$, we have $|z_r| > \frac{1}{3(n+1)}$ for some $0 \leq r \leq n$. So we have

$$\frac{c_r(t)}{8} \cdot |z_r| = |\lambda| \cdot a_r \cdot |z_r^{a_r-1}|$$

i.e.,

$$|\lambda| = \frac{c_r(t)}{8a_r |z_r|^{a_r-2}} < \frac{(3(n+1))a_r-2}{8} \leq C$$

where $C = \max_{0 \leq i \leq n} \frac{(3(n+1))a_i-2}{8}$, only depends on $a$. On one hand, we have

$$\sum_{k=0}^{n} \frac{c_k(t)z_k \bar{z}_k}{8a_k} = \lambda \sum_{k=0}^{n} z_k^{a_k} = \lambda \cdot f(z) = \lambda \cdot \epsilon < C \cdot \epsilon. \quad (5.6)$$

On the other hand, we have

$$\sum_{k=0}^{n} \frac{c_k(t)z_k \bar{z}_k}{8a_k} \geq \sum_{k=0}^{n} \frac{|z_k|^2}{8a_k} \geq \frac{1}{72} \max_{0 \leq i \leq n} \{a_i\}. \quad (5.7)$$

So for $\epsilon$ small enough, equations (5.6) and (5.7) cannot both hold, which implies the critical points of $\phi_t$ is contained in $\{||z|| \leq 1/3\}$. □

**Remark 5.6.** Since we have $\phi_0(z) = \frac{||z||^2}{8}, \nabla \phi_0 \phi_0 = \sum_{i=0}^{n} (z_i \partial \bar{z}_i + \bar{z}_i \partial z_i)/2$ is complete in $\mathbb{C}^{n+1}$.

Therefore $\phi_0$ is a completely exhausting function on $V_a(\epsilon)$. By the proof of Proposition 5.5, all critical points of $\phi_0$ are in the interior of $X^1$. It follows that $V_a(\epsilon)$ is the completion of $X^1$ by matching the corresponding trajectories of the Liouville fields.

### 5.2. Topology of manifolds $M_0$ and $M_1$

Now we have a $C(L)$-equivariant Stein manifold $(\mathbb{C}^*, i, (\log |z|)^2/2)$ where the $C(L)$-action is multiplication. The map

$$\mathbb{R} \times (\mathbb{R}/2\pi \mathbb{Z}) \longrightarrow \mathbb{C}^*, \quad (r, \theta) \mapsto e^{r+\theta i}$$

gives rise to polar coordinates form of the same Stein manifold $(\mathbb{R} \times S^1, j, r^2/2)$ and the Liouville vector is $r \partial_r$, which is complete.

Now we consider the product of Stein manifolds $(\mathbb{C}^*, i, (\log |z|)^2/2)$ and $(V_a(\epsilon), i, \phi_0)$. It has a free $C(L)$ action as follows:

$$(z_0, z_1, \ldots, z_n) \mapsto (z_0 \zeta^{b_0}, z_1 \zeta^{b_1}, \ldots, z_n \zeta^{b_n}, \eta),$$

where $b_i = L/a_i, \zeta \in C(L)$. The product function $\phi := (\log |z|)^2/2 + \phi_0$ is a completely exhausting $J$-convex Morse function, and the product Stein manifold is of finite type. By abuse of the notation, we use $\phi$ to denote the function on the quotient manifold as well, and $M_0 := \{\phi \leq C\}$ is a Stein domain, where $C$ is greater than all critical values of $\phi$. Hence the completion $\tilde{M}_0(\tilde{a}) = (V_a(\epsilon) \times \mathbb{C}^*) / C(L)$ since $\phi$ is complete. Oftentimes we will suppress $a$.

If we consider the Weinstein structure instead, the Weinstein domain can be cut out in other ways, as stated in the following lemma:

**Lemma 5.7.** Suppose $(W, \lambda, \phi)$ is a finite type Weinstein manifold. Let $\psi : W \rightarrow \mathbb{R}$ be an exhausting Morse function. Suppose $X_\lambda$ is nondegenerate and gradient-like for $\psi$ outside $\{\psi \leq 0\}$. Then $\{\psi \leq 0\}$ together with $\lambda$ is Liouville homotopic to a Weinstein domain $W_1 := \{\phi \leq K\}$, for $K$ sufficiently large.
Proof. Let $K$ satisfy
\[ \{ \psi \leq 0 \} \subset W_1 \subset W_2 := \{ \psi \leq C \} \]
for some large enough $C$ (conditions will be evident along the line of proof). Notice that $\{ \psi \leq 0 \}$ is Liouville homotopic to $W_2$. Fix a smooth function $\rho$ (it can be constructed on the level sets of $\phi$) such that
- $\rho = 1$ in $W_1$, $\rho = 0$ outside $W_2$.
- $X_\lambda(\rho) \leq 0$

Let $M := \max_{p \in W_2 \setminus W_1} (\phi - \psi)$. Now consider the function $f = \rho \phi + (1 - \rho)(\psi + M)$. We will show that $f$ is Morse and $X_\lambda$ is gradient-like for $f$. We only need to verify $X$ is gradient-like in $W_2 \setminus W_1$. We have
\[ X_\lambda(f) = \rho X_\lambda(\phi) + (1 - \rho)X_\lambda(\psi) + (\phi - \psi - M)(X_\lambda(\rho)) \geq \rho X_\lambda(\phi) + (1 - \rho)X_\lambda(\psi) > 0 \]
So $X_\lambda$ is gradient-like for $f$ and $f$ doesn’t have new critical points outside $W_1$. Because $f|_{W_1} = \phi|_{W_1}$, $f$ is Morse. Hence $(\lambda, f)$ is also a Weinstein structure on $W_2$, and a linear interpolation between $f$ and $\phi$ gives rise to a family of Weinstein structures. In particular, it gives rise to a Liouville homotopy.

In fact, we have an explicit form for the topology of $M_0$. The following quotient map
\[ \pi : V_\alpha(c) \times \mathbb{C}^* \to \mathbb{C}^{n+1} \setminus V_\alpha(0) \]
\[ (z_0, z_1, \cdots, z_n, t) \mapsto (z_0 t^{b_0}, z_1 t^{b_1}, \cdots, z_n t^{b_n}) \]
coincides with $C(L)$—action.

Therefore $\tilde{M_0}(a)$ (hence $M_0$) is diffeomorphic $\mathbb{C}^{n+1} \setminus V_\alpha(0)$). In particular, when $\Sigma(a)$ is a sphere, $V_\alpha(0)$ is homeomorphic to $\mathbb{R}^{2n}$ since it’s a cone over a sphere. So $M_0$ is homeomorphic to $\mathbb{R}^{2n+2} \setminus \mathbb{R}^{2n}$. We have the following proposition about $M_0$:

**Proposition 5.8.** Let $M_0(a)$, $n \geq 3$ be the manifold defined above, with $\Sigma(a)$ a sphere. Then $\pi_1(M_0) = \mathbb{Z}, H_i(M_0) = 0, i \geq 2$.

**Proof.** It suffices to prove the results for $\mathbb{C}^{n+1} \setminus V_\alpha(0)$. We have a deformation retraction
\[ r : \mathbb{C}^{n+1} \setminus V_\alpha(0) \to S^{2n+1} \setminus \Sigma(a), \]
and we have the Milnor fibration:
\[ S^{2n+1} \setminus \Sigma(a) \to S^1 \]
\[ (z_0, z_1, \cdots, z_n) \mapsto \frac{f(z)}{||f(z)||} \]
The fibers are homotopic to a bouquet of $n$–spheres, which is simply connected since $n \geq 3$, the long exact sequence gives us $\pi_1(M_0) = \mathbb{Z}$. Meanwhile, $M_0$ is homeomorphic to $\mathbb{R}^{2n+2} \setminus \mathbb{R}^{2n}$, so we have $H_i(M_0) = 0, i \geq 2$.

**Proposition 5.9.** Let $M_0$ be a manifold with $\pi_1(M_0) = \mathbb{Z}, H_i(M_0) = 0, i \geq 2$. Suppose $\gamma$ is a generator for $\pi_1(M_0)$ and $M_1$ is the result of attaching a 2-handle along $\gamma$. Then $M_1$ is contractible.

**Proof.** The attaching 2-handle kills the generator $[\gamma]$ so $\pi_1(M_1) = 0$. Meanwhile, $H_i(M_0) = 0, i \geq 2$ implies $H_k(M_1) = 0, k \geq 3$ since attaching a 2-handle does not change higher homology. Let’s denote the 2-handle by $H$. We have the Mayer-Vietoris sequence:
\[ \cdots \to H_2(H) \oplus H_2(M_0) \to H_2(M_1) \to H_1(M_0 \cap H) \xrightarrow{i_*} H_1(H) \oplus H_1(M_0) \to H_1(M_1) \to \cdots \]
Here $[\gamma]$ is the generator of both $H_1(M_0 \cap H)$ and $H_1(M_0)$, so $i_*$ is isomorphism. Hence we have

$$\cdots \to 0 \to H_2(M_1) \to \mathbb{Z} \xrightarrow{\sim} \mathbb{Z} \to 0 \to \cdots$$

So $H_2(M_1) = 0$. Then since $M_1$ is simply connected, we have $\pi_k(M_1) = 0$ by the Hurewicz Theorem. It follows that $M_1$ is contractible by Whitehead’s Theorem. □

**Lemma 5.10** (Corollary 2.30 [McL07]). Let $M$ be a contractible Stein manifold of finite type. If $n := \dim \mathbb{C} M \geq 3$ then $M$ is diffeomorphic to $\mathbb{C}^n$.

**Proof.** The proof is taken out of McLean [McL07]. Let $(J, \phi)$ be the Stein structure associated with $M$. We can also assume that $\phi$ is a Morse function. For $R$ large enough, the domain $M_R := \{ \phi < R \}$ is diffeomorphic to the whole of $M$ as $M$ is of finite type. It suffices to show that the boundary of $M_R := \{ \phi \leq R \}$ is simply connected, then the result follows from the $h$–cobordism theorem. The function $\psi := R - \phi$ only has critical points of index $\geq n \geq 3$ because the function only has critical points of index $\leq n$. So $M_R$ can be reconstructed by attaching handles of index $\geq 3$ with the help of a Morse function $\psi$. This does not change the fundamental group, hence $\partial M_R$ is simply connected because $M_R$ is. □

5.2.1. **Handle attachment and trivialization.** Now we need to fix a trivialization of the canonical bundle $\kappa_{\widetilde{M}_0}$ of $(\mathcal{T}\widetilde{M}_0, J)$. Since we have the $C(L)$–equivariant quotient map $V_a(\epsilon) \times \mathbb{C}^* \to \widetilde{M}_0$, it suffices to fix $C(L)$–trivializations on both $V_a(\epsilon)$ and $\mathbb{C}^*$ since

$$T(V_a(\epsilon) \times \mathbb{C}^*) = TV_a(\epsilon) \times TC^*$$

Notice that the trivialization of the symplectic complement in Proposition 4.6 is $C(L)$–equivariant, and the standard trivialization of $TC^{n+1}$ is also $C(L)$–equivariant, therefore a $C(L)$–equivariant trivialization of $TV_a(\epsilon)$ exists. Since $V_a(\epsilon)$ is simply connected, the trivialization of $TV_a(\epsilon)$ is homotopically unique. We will take the natural trivialization of $TC^* \to \mathbb{C}^* \times \mathbb{C}$, which determines the trivialization $\Phi$ of $T(V_a(\epsilon) \times \mathbb{C}^*)$. We will also fix $\Phi$ for the rest of this paper, which will be crucial in two places:

- Determining the framing for the Weinstein 2-handle attachment in Proposition 5.11.
- Determining the trivialization for the calculation of Conley-Zehnder index in Proposition 5.24.

**Proposition 5.11.** There is a contractible Weinstein domain $(M_1, \omega_1, X_1, \psi_1)$ obtained from the Weinstein domain $(M_0, -dd^C \phi, \nabla \phi, \phi)$ by attaching a 2-handle such that the canonical saturation (see subSection 2.7.2) coincides with the trivialization $\Phi$.

**Proof.** If we can find an isotropic circle in $M_0$ which generates the fundamental group, then by Theorem 2.33, we can attach a Weinstein handle in such a way that the trivialization of the contact structure extends to the Weinstein handle body. The existence of such an isotropic circle is guaranteed by the $h$–principle in lemma 5.13, which states a subcritical embedding can be perturbed into an isotropic embedding. □

**Remark 5.12.** By Proposition 5.9, $M_1$ is contractible.

Let $M$ be a contact manifold of dimension $2n + 1$ and $V$ a smooth manifold of subcritical dimension, i.e. $\dim V \leq n$. Let $\text{Mono}^{\text{emb}}$ be the space of monomorphisms $TV \to TM$ which cover embeddings $V \to M$, and $\text{Mono}_{\text{isot}}^{\text{emb}}$ its subspace which consists of isotropic monomorphisms $F : TV \to TM$. Let $\text{Mono}_{\text{isot}}^{\text{emb}}$ be the space of homotopies

$$\text{Mono}_{\text{isot}}^{\text{emb}} = \{ F_t, t \in [0, 1] | F_t \in \text{Mono}^{\text{emb}}, F_0 = df_0, F_1 \in \text{Mono}_{\text{isot}}^{\text{emb}} \}.$$
The space $\text{Emb} \text{isot}$ of isotropic embeddings $V \rightarrow M$ can be viewed as a subspace of $\text{Mono} \text{emb}_{\text{isot}}$. Indeed, we can associate to $f \in \text{Emb} \text{isot}$ the homotopy $F_t \equiv df, t \in [0, 1]$, in $\text{Mono} \text{emb}_{\text{isot}}$.

**Lemma 5.13** (Proposition 12.4.1 [EM02]). The inclusion

$$\text{Emb} \text{isot} \hookrightarrow \text{Mono} \text{emb}_{\text{isot}}$$

is a homotopy equivalence.

The above $h-$principle also holds in the relative and $\mathcal{C}^0-$dense forms.

**Remark 5.14.** By Theorem 2.7, $(M_1, \omega_1, X_1, \psi_1)$ is homotopic to a Stein domain through Weinstein structures. We denote the Stein structure by the same notation $(M_1, J_1, \phi_1)$.

In light of proposition 5.8, $\pi_1(M_0) = \mathbb{Z}$ (when $\Sigma(a)$ is a sphere). Suppose $\gamma$ is the generator of $\pi_1(M_0)$ and let $(M_1, \omega_1, X_1, \psi_1)$ be the Weinstein domain obtained by attaching a Weinstein 2-handle to $M_0$. It is contractible by proposition 5.9. As remark 5.14 shows above, $(M_1, \omega_1, X_1, \phi_1)$ admits a Stein structure. As a matter of fact, since $(M_0, J_0, \phi_0)$ is of finite type, so is $(M_1, J_1, \psi_1)$. Then by lemma 5.10, we have the following:

**Lemma 5.15.** $M_1$ is diffeomorphic to $B^{2n}$.

5.3. The Weinstein domain $M_0$. We notice $(V_\alpha(\epsilon), -d^C \phi_0) = (\hat{X}_1^1, -d^C \phi_0)$ while $(X_1^1, -d^C \phi_0)$ is $C(L)-$equivariant Liouville homotopic to $(W^1_\epsilon, \lambda)$, and in light of lemma 5.7, we can cut out a domain in $(\hat{W}^1_\epsilon \times \mathbb{C}^*, \lambda_0 := \lambda + rd\theta)$ by different functions.

First of all, we need the following technical proposition.

**Proposition 5.16.** Let $(M, \lambda)$ be a $G$-equivariant Liouville domain, and $R$ be the coordinate for its cylindrical end. Assume $\phi$ is a $G$-equivariant Morse function on $M$ such that $X(\phi) < 0$ near the boundary of $M$. Then for any $\epsilon > 0$, there exists $\delta_1 \gg \delta_2 > 0$ and a $G$-equivariant Morse function $f$ (see Figure 4) such that:

- $||1 - f||_{C^2} < \epsilon$ in the region $M \setminus \{R > 1 - \delta_1 + 2\delta_2\}$.
- $f$ and $\phi$ have same set of critical points, and the Morse indices are the same.
- $f$ satisfies the equation
  $$(\frac{f}{a})^2 + (\frac{R - (1 - \delta_1)}{\delta_1})^6 = 1$$
  on the region $1 - \delta_1 + \delta_2 < R \leq 1, 0 < a < 1$.

**Proof.** First, we can fix the canonical collar of the boundary using the negative Liouville flow

$$\iota : (1 - \epsilon_1, 1] \times \partial M \rightarrow M$$

$$\iota^* \lambda = R\lambda, \quad \iota^* X = R\partial_R$$

where $\epsilon_1 > 0$ is sufficiently small, so that $X(\phi) < 0$ in the canonical collar and $R$ is the cylindrical coordinate. Notice that $R$ is $G$-equivariant and so is any function in $R$.

Now let’s fix a sufficiently small $\epsilon_1 > \delta_1 \gg \delta_2 \gg \epsilon_2 > 0$ (the exact constraints on $\delta_1, \delta_2, \epsilon_2$ will be clear along the proof), and an increasing bump function $\rho$ such that $\rho(R) = 1$ for $R \geq 1$ and $\rho(R) = 0$ for $R \leq 0$. Let $\hat{\rho}(R) := \rho(\frac{R - (1 - \delta_1)}{\delta_2})$ Then we have

$$||\hat{\rho}||_{C^2} \leq \frac{1}{\delta_2}||\rho||_{C^2}$$

(5.9)
Define a bump function $\hat{\rho}$ on $M$ to be $\rho(\hat{R})$ on its canonical collar and extended by 0. Apparently $\hat{\rho}$ is $G$-equivariant. Let $h > 0$ be a function on $[1 - \delta_1, 1] \times \partial M$ satisfying the conditions:

$$\left( \frac{h}{1 - \epsilon_2} \right)^2 + \left( \frac{R - (1 - \delta_1)}{\delta_1} \right)^6 = 1$$

Then $h$ can be extended to a smooth function on $M$. Without loss of generality, we can assume $\|1 - \phi\|_{C^2} < \epsilon_2$. Otherwise we can simply replace $\phi$ by $1 + c\phi$ for $c > 0$ sufficiently small. We claim the function

$$f = \phi \cdot (1 - \hat{\rho}) + h\hat{\rho}$$

satisfies all conditions in this proposition. Firstly, $h$ is well-defined and $G$-equivariant, and since $f$ coincides with $h$ on the region $\{ R \geq 1 - \delta_1 + \delta_2 \}$, equation 5.8 is satisfied.

Secondly, we only need to show that $f$ has no critical points in the region $\{ 1 - \delta_1 \leq R \leq 1 - \delta_1 + 2\delta_2 \}$, for which we have

$$\partial_{R}(f) = h'\hat{\rho} + h\hat{\rho}' + (1 - \hat{\rho})\partial_{R}(\phi) - \phi\hat{\rho}' = (h - \phi)\hat{\rho} + h'\hat{\rho} + (1 - \hat{\rho})\partial_{R}(\phi) < 0$$

since $R\partial_{R}(\phi) = X(\phi) < 0$ and $h \leq 1 - \epsilon_2 \leq \phi$. Therefore $f$ has no critical point in the canonical collar. Since outside the canonical collar $f \equiv \phi$, the second condition follows.

Now we show that $f$ also satisfies the first condition. In the region $M \setminus \{ R > 1 - \delta_1 \}$, we have $f \equiv \phi$, so we only need to check the region $\{ 1 - \delta_1 \leq R \leq 1 - \delta_1 + 2\delta_2 \}$, where

\[ f \text{ only depends on } R, \ f'(R) < 0, (f(R)^2)' < 0 \]

\[ \|1 - f\|_{C^2} < \epsilon \]

\[ f(R) = (1 - \epsilon_2)\sqrt{1 - \left( \frac{R - (1 - \delta_1)}{\delta_1} \right)^6} \]

\[ 1 - \delta_1 + 2\delta_2 \]
\[ \|f - 1\|_\mathcal{E}^2 = \|\phi - 1\|_\mathcal{E}^2 + (h - \phi)\hat{\rho}\|_\mathcal{E}^2 \]
\[ \leq \|\phi - 1\|_\mathcal{E}^2 + \|\phi - h\|_\mathcal{E}^2 \]
\[ \leq \varepsilon_2 + 2\|\phi - 1\|_\mathcal{E}^2 \cdot \|\rho\|_\mathcal{E}^2 \]
\[ \leq \varepsilon_2 + \frac{1}{\delta_2^2} (\|\phi - 1\|_\mathcal{E}^2 + \|1 - h\|_\mathcal{E}^2) \cdot \|\rho\|_\mathcal{E}^2 \]
\[ \leq \varepsilon_2 + \frac{1}{\delta_2^2} (\|1 - h\|_\mathcal{E}^2) \cdot \|\rho\|_\mathcal{E}^2 \]

The Taylor expansion of \(1 - h\) at \(R = 1 - \delta_1\) is:
\[ 1 - h((1 - \delta_1) + t) = \varepsilon_2 + Ct^6 + o(t^{11}), \quad C = \frac{1 - \varepsilon_2}{2\delta_1^6} \]

Therefore \(\|1 - h\|_\mathcal{E}^2 \leq \varepsilon_2 + C_1\delta_2^6\), for \(t < 2\delta_2 \ll \delta_1\), where \(C_1 = C_1(\delta_1)\). Thus we have
\[ \|1 - f\|_\mathcal{E}^2 \leq \varepsilon_2 + \frac{2\varepsilon_2 + C_1\delta_2^4}{\delta_2^2} < \varepsilon \]

The last inequality holds as long as \(\varepsilon_2 \leq \delta_2^2\) and \(\delta_2 \ll \delta_1\).

**Lemma 5.17.** Let \(f, \rho\) be defined as above, \(g := \hat{\rho}(\delta_2 - R)\). Then \(g \cdot X_f\) is \(\mathcal{C}^1\) small, where \(X_f\) is the Hamiltonian vector field of \(f\) with respect to \(d\lambda_0\).

**Proof.** We only need to prove this in \(\{1 - \delta_1 + \delta_2 < R < 1 - \delta_1 + 2\delta_2\}\). Notice that
\[ \|X_f\|_\mathcal{E}^1 \leq \|1 - f\|_\mathcal{E}^2 < K\delta_2^2, \]
where \(K\) is independent of \(\delta_2\). Meanwhile, we have
\[ \|g \cdot X_f\|_\mathcal{E}^1 \leq |g| \cdot \|X_f\| + |dg| \cdot \|X_f\| + |g| \cdot \|dX_f\| \]
\[ \leq ((|g| + |dg|) \cdot (\|X_f\| + \|dX_f\|)) \]
\[ \leq \|g\|_\mathcal{E}^1 \cdot \|X_f\|_\mathcal{E}^1 \]
\[ \leq \frac{\|\rho\|_\mathcal{E}^1}{\delta_2} \cdot K\delta_2^2 \]
\[ \leq K'\delta_2 \]

Suppose \(G\) is a finite group and \(M\) is a \(G\)-manifold. Let \(\mathcal{M}^G(M, \mathbb{R})\) denote the set of \(G\)-equivariant Morse functions on \(M\) and \(\mathcal{C}(M, \mathbb{R})\) the set of smooth functions.

**Lemma 5.18 (Density Lemma 4.8 [Was69]).** \(\mathcal{M}^G(M, \mathbb{R})\) is dense in \(\mathcal{C}(M, \mathbb{R})\) with respect to the \(C^k\) topology.

**Remark 5.19.** Note that \((W^\lambda_\varepsilon_1, \lambda)\) is \(G\)-equivariantly Liouville isomorphic to \((X^\lambda_\varepsilon_1, -d^\varepsilon\phi_0)\). Since \(\phi_0\) is \(\varepsilon\)-convex on \(X^\lambda_\varepsilon_1\) (and we can perturb it into a \(G\)-equivariant Morse function if necessary), the index of \(-\phi_0\) is at least \(n\) (half of the dimension of a Stein Manifold). Therefore we can find such function \(\phi'\) on \(W^\lambda_\varepsilon_1\) as well.

Apply proposition 5.16 to \((W^\lambda_\varepsilon_1, \lambda)\), with \(\phi'\) as in remark 5.19. Then consider the function \(F\) on the product Liouville manifold \((W^\lambda_\varepsilon_1 \times \mathbb{R} 	imes S^1, \lambda_0 := \lambda + \varepsilon_1 d\theta)\) defined as:
$F : \tilde{W}_1^L \times \mathbb{R} \times S^1 \to \mathbb{R}$,  
\begin{align}
(p, (r, \theta)) & \mapsto r^2 - f(p)^2 \quad \text{for} \quad p \in W^1_\epsilon \\
((q, R), (r, \theta)) & \mapsto r^2 - a^2 \left(1 - \left(\frac{R - (1 - \delta_1)}{\delta_1}\right)^6\right) \quad \text{for} \quad (q, R) \in \partial W^1_\epsilon \times (1 - \delta_1 + 2\delta_2, \infty). 
\end{align}

where $a = 1 - \epsilon_2$. It is easy to check that $F$ is a smooth $C(L)$-equivariant function on $\tilde{W}_1^L \times \mathbb{R} \times S^1$, and 0 is a regular value. Furthermore, the following lemma shows that the Liouville vector field $Y := Y_\lambda + r\partial_r$ (where $Y_\lambda$ is the Liouville field on $(W_1^L, \lambda)$) is gradient-like for $F$ on $\{F \geq 0\}$.

**Lemma 5.20.** The Liouville vector field $Y$ of $(\tilde{W}_1^L \times \mathbb{R} \times S^1, \lambda + r d\theta)$ is gradient-like for $F$ outside $W_0 := \{F \leq 0\}$.

**Proof.** We will verify the statement on the regions $\{R > 1 - \delta_1 + \delta_2\}$ and $\{R > 1 - \delta_1 + 2\delta_2\}$ separately. In the region $\{R > 1 - \delta_1 + \delta_2\}$, $Y = r\partial_r + R\partial_R$ with $F = r^2 - a^2 \left(1 - \left(\frac{R - (1 - \delta_1)}{\delta_1}\right)^6\right)$, the claim is trivial. In the region $\{R > 1 - \delta_1 + 2\delta_2\}$, we have $Y = r\partial_r + Y_\lambda$. Notice that this region is a product $W' \times (\mathbb{R} \times S^1)$, where $W' = W^1_\epsilon \setminus \{R > 1 - \delta_1 + 2\delta_2\}$ is $W^1_\epsilon$ attached with a cylindrical cobordism. Now,

$$Y(F) = (r\partial_r + Y_\lambda)(r^2 - f^2) = 2(r^2 - fY_\lambda(f))$$  \hspace{1cm} (5.13)

Since $1 - f$ is $C^2$ small, the coordinate $r$ is nonzero in the region $W_0^c \cap \{R > 1 - \delta\}^c$, and $W'$ is compact, we have $2(r^2 - fY_\lambda(f)) > 0$. The conclusion follows. \hfill \Box

**Remark 5.21.** Note that $\{F \leq 0\}$ is a $C(L)$-equivariant Liouville domain, and $C(L)$ acts freely on it. The quotient domain is Liouville homotopic to the Stein domain $(M_0, J, \phi)$, by Lemma 5.7. Since the properties of interest are invariant under Liouville isomorphism, we will also denote the quotient domain $\{(F \leq 0), \lambda_0\}/C(L)$ by $(M_0, \lambda_0)$.

**Remark 5.22.** The region $(U := \{R > 1/2\}^c \cap \{|r| \leq 1/2\}, \lambda)$ is a Liouville domain with corners. We can smooth out the corner with a $C^\infty$-small perturbation. By abuse of notation, the boundary of this Liouville domain is denoted by $M = \{R > 1/2\} \times \{|r| \leq 1/2\} \cup \{|R > 1/2\} \times \{|r| = 1/2\}$, with Liouville vector field $Y = R\partial_R + r\partial_r$. The time 1 flow of $Y$ sends $M$ to a new boundary $\{R = e/2\} \times \{|r| \leq e/2\} \cup \{|R > e/2\} \times \{|r| = e/2\}$, that is, $U \cup M \times [0, 1] = \{R > e/2\} \cap \{|r| \leq e/2\}$. It’s easy to check that $U \subset \{F \leq 0\} \subset U \cup M \times [0, 1]$.

### 5.4. Strongly ADC property of $M_0$

In this subsection, we will prove that the contact boundary of $(M_0, \lambda_0)$ (as in remark 5.21) with respect to the trivialization $\Phi$ is strongly asymptotically dynamically convex. Let us first state what the framing is. Since $(\tilde{W}_1^L \times \mathbb{R} \times S^1, \lambda_0)$ is a product, it suffices to choose the $G$-equivariant trivialization on both components, since it descends naturally to the quotient $(W_0, \lambda_0)$ (see subSection 5.2.1). We denote the boundary of $(M_0, \lambda_0)$ by $(\Sigma_0, \lambda_0)$.

**Theorem 5.23.** Let $F$ be the function of Lemma 5.20. Then $(\Sigma_0, \lambda_0)$ has the strongly ADC property with respect to a trivialization $\Phi$, provided $a$ satisfies the conditions $n \geq 3$ and $m(a) \geq 2$. 
Figure 5. Lower SFT index of Fractional Reeb orbits in $\Sigma$.

Proposition 5.24. For any $K > 0$, there exists a $C(L)$-equivariant function $F$ as defined in 5.10 on the Liouville domain $(\hat{W}_\epsilon^1 \times \mathbb{R} \times S^1, \lambda_0)$ with a chosen trivialization $\Phi$ such that

1. $\Sigma := \{ F = 0 \}$ is a regular level set and the Liouville vector field $Y$ points outwards along $\Sigma$.
2. The quotient $\Sigma_0 := \Sigma/G$ satisfies that all elements of $\mathcal{P}_{\phi}^{<K}(\Sigma_0, \lambda_0)$ have lower SFT index at least $\min\{m(a) - 3/2, n - 5/2\}$.

Proof. We will show that by choosing a proper $C^2$-small function $f$ as in Proposition 5.16, the corresponding function $F$ satisfies the required conditions. The first condition is satisfied by the construction of $F$, as proved in Lemma 5.20, we only need to show the second condition is also satisfied. Recall the quotient map

$$\pi : \Sigma \to \Sigma_0$$

is an $L$-sheeted covering map. Therefore, the Reeb orbits in $\Sigma_0$ lift to fractional Reeb orbits in $\Sigma$. To be precise, if $\gamma(t), t \in [0, T]$ is a Reeb orbit in $\Sigma_0$, then the $L$-fold Reeb orbit $\gamma(t), t \in [0, qT]$ can be lifted to a Reeb orbit $\tilde{\gamma}(t), t \in [0, LT]$ in $\Sigma$. It follows that the index of $\gamma$ in $\Sigma_0$ can be calculated through the index of $\tilde{\gamma}$ in $\Sigma_0$. We will proceed by investigating the Reeb orbits in three regions:

(a) $\Sigma \cap \{ 1 > R > 1 - \delta_1 + \delta_2 \}$, where $\tilde{\gamma}$ has constant $r, R$ coordinates.
(b) $\Sigma \cap \{ 1 = R \}$, where $\gamma$ is contractible, and $\gamma$ lifts to closed Reeb orbit $\tilde{\gamma}$ in $\Sigma$.
(c) $\Sigma \cap \{ R > 1 - \delta_1 + 2\delta_2 \}$, where $\tilde{\gamma}$ has constant coordinate in the $W^1_\epsilon$ component.

We will show that all elements of $\mathcal{P}_{\phi}^{<K}(\Sigma_0, \lambda_0)$ (see Figure 5) can be lifted to fractional Reeb orbits either entirely contained in part (a), (b) or (c) and

(a) orbits in part (a) have lower SFT index at least $m(a) - 3/2$;
(b) orbits in part (b) have lower SFT index at least $m(a) - 3/2$;
(c) orbits in part (c) have lower SFT index at least $n - 5/2$. 
First, in region (a), by lemma 2.23, we have
\[ X_{\text{Reeb}} = \frac{X_F}{Y(F)} = \frac{2r \partial_{\theta} - 2f X_f}{2r^2 - 2f Y(\lambda)} = \frac{2r \partial_{\theta} - 2f f' J \partial_R}{2r^2 - 2f f'} . \]

So \( X_{\text{Reeb}} \) has no \( \partial_R \) component, and therefore the Reeb flow in the region (a) has constant \( R \) coordinate. So any Reeb orbits \( \gamma \) intersecting \( \{ R > 1 - \delta_1 + \delta_2 \} \) remains entirely in region (a). Let us begin the proof with a lemma:

**Lemma 5.25.** With \( W_0 \) and \( F \) defined as in Lemma 5.20, the conditions in Lemma 2.41 are satisfied.

**Proof.** We will verify the conditions in three cases:

a. in the region \( W_0 \setminus \{ R > 1 - \delta_1 + 2\delta_2 \} \), where \( \| 1 - f \|_{C^2} < \epsilon \);

b. in the region \( \{ 1 - \delta_1 + \delta_2 < R < 1 \} \), where \( Y = r \partial_r + R \partial_R \), and \( f = f(R) \).

c. in the region \( R = 1, r = 0 \).

First of all, note that \( b = dF(Y) = 2r^2 - 2f Y(\lambda) > 0 \) by Lemma 5.20.

Case (a): Let \( p \) be a critical point of \( f \) and define
\[ A := \{(p, \sqrt{f(p)}^2 + t, \theta) \in W_0^1 \times \mathbb{R} \times S^1 | t \in (-\epsilon_0, \epsilon_0)\} \]
Define \( C_t := F^{-1}(t) \), which is transverse to \( A \) as \( \partial_r \) is transverse to it. Let
\[ A_t := C_t \cap A = \{(p, \sqrt{f(p)}^2 + t, \theta) \in W_0^1 \times \mathbb{R} \times S^1 \} \]
and \( L_t = \sqrt{f(p)^2} + t, b/L_0 = 2f(p) \). We can rescale \( L_t \) by \( 2f(p) \), and with \( V = \frac{\partial}{\partial r} \), we have
\[ \frac{dV}{dt} = 2 > 2f(p) \frac{dL_t}{dt} \Big|_{t=0} = 1 \]

Case(b): Let \( B \) be a Morse-Bott manifold of the Brieskorn manifold \( (\Sigma(a), \lambda) \), and \( g(R) = -f^2(R) \). Then \( F(t) = r(t)^2 + g(R(t)) \). We have the following:
\[ dF = 2r dr + g'(R) dR, \quad b = dF(Y) = 2r^2 + R g'(R), \]
\[ X_{\text{Reeb}} = X_F/Y(F) = (2r \partial_{\theta} + g'(R) J \partial_R)/b \]

Now, define for any constant \( a > 0 \) \((-1/a)\) is the slope of tangent line of \( F \) at \((r, R)\),
\[ A(a) := \{(q, R(t), \theta, r(t)) \in B \times (-1-\delta_1 + \delta_2, 1) \times S^1 \times \mathbb{R} | r^2 - f(R)^2 = t, r = ag'(R), t \in (-\epsilon_0, \epsilon_0)\} \]
Again let \( C_t := F^{-1}(t) \), which is transverse to \( A(a) \), and
\[ A_t := C_t \cap A = \{(q, R(t), \theta, r(t)) \} \]
Then \( A(t) \) is a Morse-Bott manifold in \( C_t \). Since
\[ L_t = b/2r(t) = r + \frac{Rg'(R)}{2r} = r + \frac{R}{2a}, \quad b/L_0 = 2r(0) \]
\[ 2r \frac{dL_t}{dt} \Big|_{t=0} = 2rr' + \frac{rr'}{a} \]
and on the other hand, \( V = r' \partial_r + R \partial_R, \quad db = 4rdr + (g'(R) + R g''(R)) dR \), we have that
\[ \frac{dV}{dt} = 4rr' + R^2 g'(R) + RR' g''(R) \geq 2rr' + \frac{rr'}{a} = 2r \frac{dL_t}{dt} \Big|_{t=0} \]
since \( r' > 0, R' > 0, g''(R) > 0 \).

Case (c): Let \( g = -f^2(R), B \) defined as above, define
\[ A := \{(q, R(t), \theta, 0) \in B \times \mathbb{R} \times S^1 \times \mathbb{R} | g(R) = t, t \in (-\epsilon_0, \epsilon_0)\} \]
Once more, \( C_t := F^{-1}(t) \), which is transverse to \( A \) as \( \partial_R \) is transverse to it, and
\[ A_t := C_t \cap A = \{ (q, R(t), \theta, 0) \in B \times \mathbb{R} \times S^1 \times \mathbb{R} \} \]
are pseudo Morse-Bott manifolds. Moreover,
\[ dF = g'(R)dR, \quad b = dF(Y) = Rg'(R), \quad L_t = R(t), \quad b/L_0 = g'(1) \]
Here,
\[ g'(1) \frac{dL_t}{dt} \bigg|_{t=0} = g'(R)R'|_{R=1} = \frac{d}{dt}(g(R(t)))|_{t=0} = 1 \]
and with \( V = \frac{\partial g}{\partial g(R)} \), we have
\[ db(V) = 1 + \frac{Rg''(R)}{g'(R)} > 1 = g'(1) \frac{dL_t}{dt} \bigg|_{t=0}, \]
\[ \square \]

Now let us compute the index of the Reeb orbits in region (a). Any Reeb orbit \( \gamma \) can be lifted to a fractional Reeb orbit \( \tilde{\gamma} \) in the region \( \Sigma \cap \{ R > 1 - \delta_1 + \delta_2 \} \), where \( F = r^2 - f(R)^2 \). The Reeb orbit can be written as \( \tilde{\gamma} = (\gamma_1, \gamma_2) \), where \( \gamma_1, \gamma_2 \) are fractional Reeb orbits of \( (\Sigma(a), R\lambda) \) and \( (S^1, r\theta) \), for fixed \( r, R \), so by Lemma 5.25,
\[ \mu_{CZ}(\gamma, F) = \mu_{CZ}(\gamma, \lambda_0) + \frac{1}{2}. \]
Meanwhile,
\[ \mu_{CZ}(\gamma, F) = \mu_{CZ}(\gamma_1, -f(R)^2) + \mu_{CZ}(\gamma_2, r^2) \]
follows the product property of Conley-Zehnder index. Note that
\[ (-f(R)^2)' > 0, (-f(R)^2)'' > 0, \]
therefore we have
\[ \mu_{CZ}(\gamma_1, -f(R)^2) = \mu_{CZ}(\gamma_1, c_1\lambda) + \frac{1}{2} \]
Moreover, by Remark 2.43,
\[ \mu_{CZ}(\gamma_2, r^2) = \frac{1}{2} \]
Notice \( \gamma_1 \) is a fractional Reeb orbits on the Brieskorn manifold \( (\Sigma(a), R\lambda) \), which has the same index as \( (\Sigma(a), \lambda) \). By Lemma 4.7 and Lemma 4.8, we then have \( \mu_{CZ}(\gamma_1) = m(a) \). Putting all equations together:
\[ lSFT(\gamma) = \mu_{CZ}(\gamma, \lambda_0) - \frac{1}{2} \dim B + (n + 1) - 3 \]
\[ \geq (\mu_{CZ}(\gamma, F) - \frac{1}{2}) - n + (n + 1) - 3 \]
\[ \geq (\mu_{CZ}(\gamma_1, \lambda) + \frac{1}{2}) + \mu_{CZ}(\gamma_2, r^2) - \frac{5}{2} \]
\[ \geq m(a) + \frac{1}{2} + \frac{1}{2} - \frac{5}{2} \]
\[ = m(a) - 3/2. \]

For region (b), the claim will be proved in Lemma 5.27.

Now suppose \( \gamma_0(t), t \in [0, T], T < K \) is a Reeb orbit in \( \Sigma_0 \) and can be lifted to a fractional Reeb orbit in region (c). Then the \( L\)–fold Reeb orbit \( \gamma(t) := \gamma_0(t), t \in [0, LT] \) can be lifted to a closed Reeb orbit \( \gamma(t) \) in this region. Let \( g(R) \) be a smooth function defined in
Lemma 5.17. So \( g(R) = 1 \) for \( R < 1 - \delta_1 + \delta_2 \) and \( g(R) = 0 \) for \( R > 1 - \delta_1 + 2\delta_2 \). By abuse of notation, \( g \) can be regarded as a function on \( \Sigma \cap \{ 1 - \delta_1 + \delta_2 < R < 1 - \delta_1 + 2\delta_2 \} \). We extend \( g \) to \( \Sigma \) by a constant. Now define a new vector field \( X = g \cdot X_{\text{Reeb}} \). Let \( X_W \) be the projection of \( X \) to \( W^1 \), i.e.

\[
X_W = g \cdot \frac{-X_f}{f - Y_\lambda(f)} = \frac{-1}{f - Y_\lambda(f)} \cdot gX_f
\]

Since \( (1 - f) \) is \( \mathcal{C}^2 \)-small, \( \| \frac{1}{f - Y_\lambda(f)} \|_{\mathcal{C}^1} < 2 \). By Lemma 5.17, \( X_W \) is \( \mathcal{C}^1 \)-small. Then by Corollary 5.31, for \( f \) sufficiently \( \mathcal{C}^2 \)-small, any periodic orbit of period less than \( LK \) is a constant orbit, and therefore corresponds to a critical point of \( f \). We claim that any such Reeb orbit \( \gamma_0 \) has lower SFT index at least \( n - 5/2 \), which will be proved in Proposition 5.29.

**Remark 5.26.** Reeb orbits in \( W_0 \) can be graded by its \( H_1 \) class. Let have a closer look at the Reeb orbits with \( H_1 \) grading 0. In the proof of Proposition 5.24, the Reeb orbits in region (a) and (c) are never null-homologous.

**Lemma 5.27.** Any Reeb orbit \( \gamma \) in region (b) is contractible in \( \Sigma_0 \), and its lower SFT index is at least \( m(a) - 3/2 \).

*Proof.* In region (b), \( X_{\text{Reeb}} = J\partial_R \). In fact, \( \Sigma \cap \{ 1 = R \} = \Sigma(a) \times S^1 \). The Reeb flow is stationary on \( S^1 \) and coincides with the Reeb flow on the Brieskorn manifold \( \Sigma(a) \). Therefore, any Reeb orbit \( \gamma \) is contractible. Suppose \( \tilde{\gamma} \) is a lift of \( \gamma \). Let \( B \subset \Sigma(a) \) be a Morse-Bott manifold for \( (\Sigma(a), \lambda) \). In light of Lemma 5.25, we have

\[
\mu_{\text{CZ}}(B \times S^1, F) = \mu_{\text{CZ}}(B \times S^1, \lambda_0) + \frac{1}{2}
\]

By the product property of Conley-Zehnder index,

\[
\mu_{\text{CZ}}(B \times S^1, F) = \mu_{\text{CZ}}(B, -f^2(R)) + \mu_{\text{CZ}}(S^1, r^2) = \mu_{\text{CZ}}(B, -f^2(R)) + \frac{1}{2}
\]

On the other hand,

\[
\mu_{\text{CZ}}(B, -f^2(R)) = \mu_{\text{CZ}}(B, \lambda) + \frac{1}{2} \geq m(a) + \frac{1}{2}
\]

So we conclude that

\[
\mu_{\text{CZ}}(B \times S^1, \lambda_0) = \mu_{\text{CZ}}(B \times S^1, F) - \frac{1}{2} \geq m(a) + \frac{1}{2}
\]

and

\[
\text{LSFT}(\gamma) = \mu_{\text{CZ}}(\gamma) - \frac{1}{2} \dim \ker(D_{\gamma(0)} \psi_T - \text{id}) + (n + 1 - 3)
\]

\[
= \mu_{\text{CZ}}(B \times S^1, \lambda_0) - \frac{1}{2} (\dim B + 1) + (n + 1 - 3)
\]

\[
\geq m(a) + 1/2 - n + n - 2 = m(a) - 3/2.
\]

**Remark 5.28.** Let \( MB(p) \) be the Morse-Bott manifold of return time \( \frac{p \pi}{2} \) in the Brieskorn manifold, then \( MB(p) \times S^1/C(L) \) is a Morse-Bott manifold in \( \Sigma_0 \). Conversely, any Morse-Bott manifold of contractible Reeb orbits in \( \Sigma_0 \) can be lifted to \( \Sigma \). By Lemma 5.27 and Remark 5.26, the contractible Morse-Bott manifolds in \( \Sigma_0 \) can be lifted to \( \Sigma(a) \times S^1 \subset \Sigma \).
Indeed, each Reeb orbit in $\Sigma_0$ has $L$ different lifts in $\Sigma$. In terms of Morse-Bott manifolds of contractible Reeb orbits, we have a one-to-one correspondence:
\[ \pi : \Sigma(a) \times S^1 \to (\Sigma(a) \times S^1)/C(L) \subset \Sigma_0 \]
\[ MB(p) \times S^1 \mapsto (MB(p) \times S^1)/C(L). \]
The group action is trivial on the first factor, therefore
\[ (MB(P) \times S^1)/C(L) = MB(p) \times (S^1/C(L)) \cong MB(P) \times S^1 \]
and
\[ \mu_{CZ}(MB(p) \times S^1, \lambda_0) = \mu_{CZ}(MB(p), \lambda) + \frac{1}{2}. \]

**Proposition 5.29.** As defined in Proposition 5.24, Reeb orbit $\gamma_0$ has lower SFT index at least $n - 5/2$.

**Proof.** The $L$-fold of Reeb orbit $\gamma(t)$ can be lifted to a Reeb orbit $\tilde{\gamma}(t)$ in Region (b). Its $W^1_\epsilon$ component is a critical point $p$ of $f$. Since the conditions of Lemma 2.41 are satisfied,
\[ \mu_{CZ}(B_0, \lambda_0) + \frac{1}{2} = \mu_{CZ}(B_0, F). \]
Everything descends down to the quotient $M_0$. We will use the same notations for the quotient. We have the Hamiltonian orbit $\gamma_0 = (p, \gamma_2)$, where $p$ is a constant orbit in $W^1_\epsilon$ while $\gamma_2$ is an orbit in $\mathbb{R} \times S^1$. The index is
\[ \mu_{CZ}(B_0, F) = \mu_{CZ}(p, -f^2) + \mu_{CZ}(\gamma_2, r^2) = \mu_{CZ}(p, -f^2) + \frac{1}{2} \]
Since $f(p) \neq 0$, $\text{Ind}_p(f^2) = \text{Ind}_p(f)$, hence
\[ \mu_{CZ}(p, -f^2) = \text{Ind}_p(f^2) - n = \text{Ind}_p(f) - n \]
by Corollary 2.27. Since indices of critical points of $f$ is at least $n$, so $\mu_{CZ}(p) \geq 0$ (see remark 5.19). Thus lower SFT index
\[ lSFT(\gamma) = \mu_{CZ}(B_0, \lambda_0) - \frac{1}{2} \dim B_0 + (n + 1) - 3 \]
\[ = \mu_{CZ}(B_0, F) - \frac{1}{2} - \frac{1}{2} + (n + 1) - 3 \]
\[ \geq 0 + \frac{1}{2} + n - 3 = n - 5/2 \]
where the Morse-Bott manifold $B_0 = S^1$. \(\square\)

**Proof of Theorem 5.23.** Recall the definition of a strongly ADC contact manifold: there exists a sequence of non-increasing contact forms $\alpha_i$ and increasing positive numbers $D_i$ going to infinity such that all elements of $\mathcal{P}_{<D_i}^< (\Sigma, \alpha_i)$ have positive lower SFT index.

In light of Proposition 5.24, for any $K_i = K^i (K$ is a fixed large number, the explicit conditions will be clear later in this proof), there exists a $C(L)$-equivariant function $F_i$ such that all elements of $\mathcal{P}_{<K_i}(\Sigma_i, \lambda_0|\Sigma_i)$ have positive lower SFT index (since $\min \{m(a) - 3/2, n - 5/2\} > 0$), where $\Sigma_i := F_i^{-1}(0)/C(L)$ is the boundary of the quotient manifold.

By Remark 5.22, we notice that conditions of Corollary 5.34 are satisfied, so there exists a contactomorphism $f_i : \Sigma_0 \to \Sigma_{i+1}$ and a constant $C$ independent of $F_i$, such that
\[ \frac{1}{C} \cdot \lambda_0|\Sigma_0 < f_i^*(\lambda_0|\Sigma_i) < C \cdot \lambda_0|\Sigma_i. \]
So the non-increasing contact forms $\alpha_i$ can be defined as $\alpha_i = \frac{1}{C_i} f_i^*(\lambda_0|_{\Sigma_i}) < \alpha_{i-1}$, and $D_i := K_i/C_i$, which goes to infinity as long as $K > C$. Then $\mathcal{P}^{< \lambda_i}(\Sigma_0, \alpha_i) = \mathcal{P}^{< \lambda_i}(\Sigma_i, \lambda_0|_{\Sigma_i})$, which shows that all elements have positive lower SFT index. □

We follow the idea of F.laudenbach in the proof of the following lemma.

**Lemma 5.30** (Proposition 6.1.5 [AD14], [Lbdm04]). Let $X$ be a vector field on $\mathbb{R}^{2n}$. If $||dX||_{L^2} < \frac{2\pi}{L}$, the only periodic orbits with period less than $L$ are constant orbits.

**Proof.** Consider the solution $u(t)$ of period $T \leq L$ and take its Fourier expansion as well as $\dot{u}, \ddot{u}$.

$$u(t) = \sum_k c_k(u)e^{2k\pi i t/T}, \quad \dot{u}(t) = \sum_k \frac{2k\pi i}{T} c_k(u)e^{2k\pi i t/T}$$

So by Parseval’s identity, we have

$$||\ddot{u}||_{L^2} = \sum_k \frac{4\pi^2}{T^2} |c_k(\dot{u})|^2 \geq \sum_{k \neq 0} \frac{4\pi^2}{T^2} |c_k(\ddot{u})|^2 = \frac{4\pi^2}{T^2} ||\ddot{u}(t)||_{L^2}^2$$

since $c_0(\ddot{u}) = 0$. Hence,

$$||\ddot{u}||_{L^2} \geq \frac{2\pi}{T} ||\ddot{u}(t)||_{L^2}$$

On the other hand, since $\ddot{u} = (dX)(\dot{u})$, $||dX||_{L^2} < \frac{2\pi}{L}$, so

$$||\ddot{u}||_{L^2} < \frac{2\pi}{L} ||\ddot{u}(t)||_{L^2} \leq \frac{2\pi}{T} ||\ddot{u}(t)||_{L^2}$$

if $\dot{u} \neq 0$. Therefore $u(t)$ is a constant orbit. □

**Corollary 5.31** ([Lbdm04]). If $M$ is a compact manifold with boundary and $X$ is a vector field which vanishes in the neighborhood of the boundary. Then for any $L > 0$, the flow generated by $X$ has no nonconstant periodic orbit with period less than $L$ for sufficiently $C^1$-small $X$.

**Proof.** First we get rid of the boundary by doubling $M$ (glue $M$ with itself along the boundary). Now that $X$ can be smoothly extended since it vanishes in a neighborhood of the boundary. Now consider the new closed manifold $\hat{M}$. Let us fix a finite collection of compact atlas $K_i$. Since $X$ is $C^1$-small, every closed orbit with bounded period $T$ to the flow of $X$ has a small diameter ($D \leq ||X||_{uniform} \cdot L$), which implies the entire orbit remains in one of the charts $K_i$. The $C^1$ norm is equivalent to the Euclidean norm so the lemma above applies. □

**Lemma 5.32.** Let $(U, \lambda)$ be a Liouville domain, $(\hat{U}, \hat{\lambda})$ its completion, and $\Sigma_1 := \partial U$ be the contact boundary. Suppose we have a Liouville domain $(V, \bar{\lambda})$ such that $U \subset V \subset U \cup \Sigma_1 \times [0, M]$. Then there is a contactomorphism $\Psi$

$$\Psi : (\Sigma_1, \lambda_2 = \hat{\lambda}|_{\Sigma_1}) \to (\Sigma_2 := \partial V, \lambda_2 = \bar{\lambda}|_{\Sigma_2}),$$

such that $\lambda_1 \leq \Psi^* \lambda_2 \leq e^M \lambda_1$.

**Proof.** Since

$$U \subset V \subset \Sigma_1 \times [0, M]$$

let $\psi$ be the flow generated by the Liouville vector field and $t(p)$ be the time when the flow starting at $p \in \Sigma_1$ reaches $\Sigma_2$, i.e, $\psi_{t(p)}(p) \in \Sigma_2$. Then $M \geq t(p) \geq 0$. Let $\rho$ be a function
on $\tilde{U}$ supported on $(-\varepsilon, M + 1) \times \Sigma_1$, such that $\rho((r, p)) \equiv t(p)$ on the region $[-0, M] \times \Sigma_1$.

Now consider the vector field $Y := \rho \cdot \partial_r$ and we denote by

$$\Psi : \mathbb{R} \times \tilde{U} \to \tilde{U}, \quad (t, p) \mapsto \Psi_t(p)$$

the flow generated by $Y$. Clearly we have $\Psi_1(\Sigma_1) = \Sigma_2$ and $\Psi^*\lambda_1 = e^{t(p)}\lambda_2$. The now conclusion follows.

**Remark 5.33.** If the Liouville domains in the Lemma above are $G$-equivariant, then there is $G$-equivariant contactomorphism satisfying the above statement.

**Corollary 5.34.** Let $(U, \lambda)$ be a Liouville domain, suppose we have two Liouville domains $V_1, V_2$ with $\Sigma_1 = \partial V_1, \Sigma_2 = \partial V_2$ such that $U \subset V_i \subset U \cup \partial U \times [0, M]$. Then there exists a contactomorphism $f$ and a constant $C$ independent of $V_i$, such that

$$\frac{1}{C} \cdot \lambda|_{\Sigma_1} < f^*\lambda|_{\Sigma_2} < C \cdot \lambda|_{\Sigma_1}.$$ 

6. Finiteness of positive idempotent group $I_+(\Sigma_0)$

We are going to show that the positive idempotent group $I_+(\Sigma_0)$ is finite. Let’s recall the definitions: for any filling $W$ of $\Sigma_0$ such that $SH_*(W) \neq 0$, we have

$$I(W) = \{ \alpha \in SH^0_\alpha(W) \mid \alpha^2 - \alpha \in H^0(W) \}$$

and $I_+(W) = I(W)/H^0(W)$, hence it suffices to prove $I(W)$ is a finite group. Indeed, for the Liouville filling $(M_0, \lambda_0)$ as in remark 5.21, $SH^0_k(M_0, \mathbb{Z}_2)$ is finite. We begin by introducing a spectral sequence which converges to $SH^0_k(M_0, \mathbb{Z}_2)$:

**Theorem 6.1** (Theorem 5.4 [KvK16]). Let $(W, \omega = d\lambda)$ be a Liouville domain satisfying the assumptions:

1. The Reeb flow on $\partial W$ is periodic with minimal periods $T_1 \cdot \frac{\pi}{2}, T_2 \cdot \frac{\pi}{2}, \ldots, T_k \cdot \frac{\pi}{2}$, where $T_k \cdot \frac{\pi}{2}$ is the common period, i.e. the period of a principal orbit. We assume that all $T_k$ are integers.
2. The restriction of the tangent bundle to the symplectization to $\partial W, T(\mathbb{R} \times \partial W)|_{\partial W}$, is trivial as a symplectic vector bundle, $c_1(W) = 0$ and we have a choice of the trivialization of the canonical bundle.
3. There is a compatible complex structure $J$ for $(\xi := \ker \lambda_{\partial W}, d\lambda_{\partial W})$ such that for every periodic Reeb orbit $\gamma$ the linearized Reeb flow is complex linear with respect to some unitary trivialization of $(\xi, J, da)$ along $\gamma$.

For each positive integer $p$ define $C(p)$ to be the set of Morse-Bott manifolds with return time $p$, and for each Morse-Bott manifolds $\Sigma \in C(p)$ put

$$\Delta(\Sigma) = \mu_{CZ}(\Sigma) - \frac{1}{2} \dim \Sigma/S^1,$$

where the Robbin-Salamon index is computed for a symplectic path defined on $[0, p]$. Then there is a spectral sequence converging to $SH(W; R)$, whose $E^1$-page is given by

$$E^1_{pq} = \begin{cases} \bigoplus_{\Sigma \in C(p)} H_{p+q-\Delta(\Sigma)}(\Sigma; R) & p > 0 \\ H_{q+p}(W, \partial W; R) & p = 0 \\ 0 & p < 0. \end{cases}$$
Remark 6.2. The above spectral sequence respects the $H_1$ grading. Therefore, to compute $SH^0_n(M_0)$, we only need to focus on the Morse-Bott manifolds of null-homologous Reeb orbits.

Lemma 6.3. $SH^0_k(M_0, \mathbb{Z}_2)$ is finite for all $k$.

Proof. Note that it suffices to find all the Morse-Bott manifolds. By Remark 5.28, the first page of the spectral sequence which converges to $SH^0_k(M_0, \mathbb{Z}_2)$ is

$$E^1_{pq} = \begin{cases} \bigoplus H_{p+q-\Delta(MB(p))(MB(p) \times S^1; \mathbb{Z}_2)} & p > 0 \\ H_{q+n}(M_0, \partial M_0; \mathbb{Z}_2) & p = 0 \\ 0 & p < 0. \end{cases}$$

The finiteness of $SH^0_k(M_0, \mathbb{Z}_2)$ follows from the following two facts: first, there are only finitely many Morse-Bott manifolds $MB(p)$ satisfying $\Delta(MB(p)) = k$, i.e.

$$k = \mu_{CZ}(MB(p) \times S^1) - \frac{1}{2}(\dim(MB(p) \times S^1)/S^1) = f_a(p) - \frac{1}{2}(\dim MB(p) - 1).$$

The above equation can only be satisfied by finitely many $p \in \frac{1}{2}Z$, and for any $p$ there is at most one Morse-Bott manifold with return time $p\pi/2$ in the Brieskorn manifold $\Sigma(a)$. Secondly,

$$H_*(MB(p) \times S^1; \mathbb{Z}_2) = 0 \quad \text{if } * < 0 \text{ or } * > 2n.$$

and $H_*(MB(p) \times S^1; \mathbb{Z}_2)$ is finite dimensional for $0 \leq * \leq 2n$. Therefore $SH^0_k(M_0, \mathbb{Z}_2)$ is finite for each $k$, since the dimension of $\bigoplus_{p+q=k} E^1_{pq}$ is finite for each $k$.

□

Now we are going to prove that $SH^0_k(M_0, \mathbb{Z}_2) \neq 0$, where $a$ is defined as in Remark 4.4.

Lemma 6.4. For $a = (2, 2, 2, \cdots, p_k)$, we have $SH^0_k(M_0, \mathbb{Z}_2) \neq 0$, where $k + 3 = n, p_i$s are sufficiently large primes.

Proof. It suffices to prove that $SH^0_{n-1}(M_0, \mathbb{Z}_2) \neq 0$. To that end we will focus on the total degree $p + q = n - 2, n - 1, n$ in the spectral sequence above. First of all, for $p = 0$, we have

$$E^1_{0q} = H_{n+q}(M_0, \partial M_0; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & q = n \\ 0, & q \neq n. \end{cases}$$

On the other hand,

$$\Delta(MB(p) \times S^1) = \mu_{CZ}(MB(p) \times S^1) - \frac{1}{2}(\dim(MB(p) \times S^1)/S^1) = f_a(p) - \frac{1}{2}(\dim MB(p) - 1)$$

where $p\pi/2, p \in \mathbb{Z}$ is the period. Meanwhile,

$$f_a(p) = 3\left(\left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{p}{2} \right\rfloor \right) + \sum \left(\left\lfloor \frac{p}{p_i} \right\rfloor + \left\lfloor \frac{p}{p_i} \right\rfloor \right) - \left(\left\lfloor \frac{p}{3} \right\rfloor + \left\lfloor \frac{p}{3} \right\rfloor \right) \geq 3p + k - 2p = p + n - 3$$

so $\Delta(MB(p) \times S^1) \geq p - 4$. Thus, if we require $p_i > 2n$, then the only Morse-Bott manifolds could possibly contribute to total degree $p + q \leq n$ is $MB(p) \times S^1$, $p = 2l$ for some $0 < l \in \mathbb{Z}$ (see [KvK16]). Now that $p = 2l, l < n$, we have $MB(p) = \Sigma(2, 2, 2) \cong \mathbb{R}P^3$.

$$H_i(\mathbb{R}P^3 \times S^1) = \begin{cases} \mathbb{Z}_2, & i = 0, 4 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & i = 1, 2, 3 \\ 0, & \text{otherwise} \end{cases}$$
In this case,
\[ \Delta(MB(2l) \times S^1) = 6l + n - 3 - 4l - 1 = 2l + n - 4 = p + n - 4. \]
So for \( l > 2 \), \( \Delta(MB(2l) \times S^1) > n \). For \( l = 1, 2 \), we have (see Figure 6)
\[ E^1_{pq}(M_0, \mathbb{Z}_2) = H^q_{-(n-4)}(\mathbb{R}P^3 \times S^1; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & q = n - 4, n \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & q = n - 3, n - 2, n - 1 \\ 0, & \text{otherwise}. \end{cases} \]
Hence, \( E^k_{2,n-3}(M_0, \mathbb{Z}_2) \neq 0 \) stabilizes at the second page, so \( SH^0_{n-1}(M_0, \mathbb{Z}_2) \neq 0 \). It follows that \( SH^0_n(M_0, \mathbb{Z}_2) \neq 0 \). In particular, \( SH^0_n(M_0, \mathbb{Z}_2) \neq 0 \), since the unit lives in degree \( n \).

\[ \square \]

Remark 6.5. Lemma 6.4 shows that \( I_+ (\Sigma_0(a)) \) is well-defined since \( SH_*(M_0) \neq 0 \). Furthermore, \( I_+ (\Sigma_0(a)) \) is a finite group.

Now we are ready to prove Theorem 1.2: first, we will take \( a = (2, 2, 2, p_1, \ldots, p_k) \). Recall
\[ U_a(\epsilon) = \{ z \in \mathbb{C}^{n+1} | z_0^{a_0} + \cdots + z_n^{a_n} = \epsilon \cdot \beta(||z||^2) \}, \]
and
\[ W^1_\epsilon = U_\epsilon \cap B(1), \quad \partial W^1_\epsilon = \Sigma(a). \]
Let \( (M_0(a), \lambda_0) \) be defined as in Remark 5.21. Then we have the following facts:
(1) \( (M_0, \lambda_0) \) is strongly ADC;
(2) \( SH^0_n(M_0) \neq 0 \) and is finitely dimensional.
The first claim is true due to Proposition 5.23. We only need to check the condition that \( m(a) \geq 3 \), which in turn is the result of Lemma 4.8. The second claim is proved in Lemma 6.3.
On the other hand, \( \Sigma(a) \) is a sphere, because \( a \) has at least two prime numbers, which are isolated nodes in \( \Gamma_a \) (Remark 4.4). Liouville vector field \( Y_\lambda \) is gradient-like (Lemma 5.20) for the function \( F \) which we used to cut out the Weinstein domain. Therefore, it is Liouville homotopic to Stein domain \( (M_0, J, \phi) \) (Remark 5.21). \( \pi_1(M_0) = \mathbb{Z} \) since \( M_0 \) is diffeomorphic to \( \mathbb{C}^{n+1} \setminus V_0(0) \) (Proposition 5.8). Let \( \gamma \) be an isotropic circle generating \( \pi_1(M_0) \). Such \( \gamma \)
exists by the $h$–principle (Lemma 5.13). Let $M_1$ be Weinstein manifold obtained from $M_0$ by attaching a Weinstein 2-handle with respect to the trivialization $\Phi$ (Proposition 5.11). $M_1$ is of finite type because $M_0$ is. Furthermore, attaching 2-handle along $\gamma$ kills the fundamental group. Therefore, $M_1$ is contractible (Lemma 5.15). Any contractible Stein domain of finite type is diffeomorphic to $B^{2n}$. Therefore we have the following facts about $(M_1, \lambda_1, \psi_1)$:

1. $(M_1, d\lambda_1)$ is almost symplectomorphic to $(B^{2n}, \omega_{std})$;
2. $(\partial M_1, \lambda_1)$ is asymptotically dynamically convex;
3. $\text{SH}_*(M_1) \cong \text{SH}_*(M_0)$ as rings.

Note that $(d\lambda_1)^n > 0$ holds on region away from the attaching area, since the Liouville form remains the same as $\lambda_0$, so the first claim follows from Lemma 2.35. The second statement is true because subcritical surgery preserves the ADC property, by Theorem 3.10. The third statement is due to the fact that subcritical surgery doesn’t change the ring structure of symplectic homology, see Theorem 2.19. Let’s define

$$(W_k, \lambda_k, \psi_k) := (M_1, \lambda_1, \psi_1) \sharp (M_1, \lambda_1, \psi_1) \sharp \cdots \sharp (M_1, \lambda_1, \psi_1)$$

$k$



to be the boundary connected sum of $n$ copies of $(M_1, \lambda_1, \psi_1)$. Now we are going to prove that $(W_k, \lambda_k, \psi_k)$ satisfies all conditions in Theorem 1.2

**proof of Theorem 1.2.** The first two conditions are evident due to the construction of $W_k$. We only need to check the third condition. By Theorem 2.19, we have

$$\text{SH}_n(W_k) \cong \underbrace{\text{SH}_n(M_1) \oplus \cdots \oplus \text{SH}_n(M_1)}_{k}$$

so we have

$$I(W_k) \cong \underbrace{I(M_1) \oplus \cdots \oplus I(M_1)}_{k}.$$ 

Since $\text{SH}_*(M_1) \neq 0$ and is finite dimensional, $\{0, 1_{M_1}\} \subset I(M_1)$. We therefore have $2 \leq |I(M_1)| < \infty$, so $|I(W_k)| = |I(M_1)|^k$ are mutually distinct. Therefore, $i(\partial W_k) \neq i(\partial W_l)$ for $k \neq l$. 

**7. $I_+$ is an invariant of ADC contact manifolds**

We will follow Lazarev’s approach. First we will introduce the procedure called stretching-the-neck. Here we use the notation in [Laz16].

Let $V \subset \tilde{W}$ be a Liouville subdomain with contact boundary $(Z, \alpha_Z)$. Consider a collar of $Z$ in $V$ symplectomorphic to $(Z \times [1 - \delta, 1], d(t\alpha_Z))$ for small $\delta$. Let $J \in \mathcal{J}_{std}(W)$ be cylindrical in $Z \times [1 - \delta, 1]$ and set $J' := J|_{Z \times [1 - \delta, 1]}$. For $0 < R < 1 - \delta$, we extend $J'$ to a cylindrical almost almost structure on $Z \times [R, 1]$, which we also call $J'$. Now let $f_R$ be any diffeomorphism $[R, 1] \to [1 - \delta, 1]$ whose derivative equals 1 near the boundary. We can define $J_R \in \mathcal{J}_{std}(W)$ to be $(\text{Id} \times f_R)_* J'$ on $Z \times [1 - \delta, 1]$ and $J$ outside $Z \times [1 - \delta, 1]$. It is smooth because of the derivative condition on $f_R$. If $J_s \in \mathcal{J}_{std}(W)$ is a homotopy that is cylindrical and $s$-independent in $Z \times [1 - \delta, 1]$, then we can apply the same construction to obtain a homotopy $J_{R,s}$.

Let $(H_s, J_s)$ be a homotopy with $(H_s, J_s) = (H_-, J_-)$ for $s \ll 0$ and $(H_s, J_s) = (H_+, J_+)$ for $s \gg 0$, and $H_s \equiv 0$ in $Z \times [1 - \delta, 1] \subset V$. Furthermore, let $x_+, x_-$ be the Hamiltonian orbits of $H_+, H_-$ respectively in the source and target of the maps induced by $(H_s, J_s)$. 


Proposition 7.1 (Proposition 3.10 [Laz16]). Suppose that \((Z, \alpha)\) is strongly ADC with respect to the trivialization \(\Phi\) and all elements of \(\mathcal{P}_{\Phi}^{<D}(Z, \alpha)\) have positive degree. If \(A_{H_+}(x_+) - A_{H_-}(x_-) < D\), then there exists \(R_0 \in (0, 1 - \delta)\) such that for any \(R \leq R_0\), all rigid \((H_s, J_{R,s})\)-Floer trajectories are contained in \(\hat{W} \setminus V\).

Remark 7.2. If \(H_s\) is independent of \(s\), then the Floer trajectories define the differential; if \(H_s\) is an decreasing homotopy, then \((H_s, J_{R,s})\)-Floer trajectories define the continuation map.

In P.Uebele’s paper [Ueb15], the pair-of-pants product is defined for ”index-positive” contact manifold, where the symplectic homology used is actually Rabinowitz-Floer homology. Though as the paper points out, the ring structure is not well defined on \(SH^+\). However, at the chain level, if the pair-of-pants product is asymptotic to Hamiltonian orbits of positive action, then by the stretching-the-neck technique, we can prove the pair-of-pants does not enter the interior of the Liouville filling. In [Ueb15], this is proved for index-positive contact manifold. However, it is not true for Strongly ADC contact manifold in general. That being said, the pair-of-pants does not enter the interior of filling when indices of Hamiltonian orbits of the asymptotes are large enough. To be precise, we have the following:

Proposition 7.3. Suppose that \((Z, \alpha)\) is strongly ADC with respect to the trivialization \(\Phi\) and all elements of \(\mathcal{P}_{\Phi}^{<D}(Z, \alpha)\) have positive reduced Conley-Zehnder index. Furthermore, let \(A_H(x_i) < D/2(i = 1, 2, 3)\) be non-constant Hamiltonian orbits such that \(\mu_{CZ}(x_1) + \mu_{CZ}(x_2) - \mu_{CZ}(x_3) = n\) and \(\mu_{CZ}(x_i) \geq n, i = 1, 2\), then there exists \(R_0 \in (0, 1 - \delta)\) such that for any \(R \leq R_0\), all pair-of-pants products are contained in \(\hat{W} \setminus V\).
Proof. The proof is a combination of proposition 3.10 of [Laz16] and lemma 3.12 of [Ueb15]. First of all, we have to rule out the breaking as in Figure 7 (Similarly with $x_1$ and $x_2$ exchanged).

Suppose we have the breaking as in Figure 7, then the top level have positive dimension, and we have (see lemma 3.10 of [Ueb15])

$$\mu_{CZ}(x_1) - \mu_{CZ}(x_3) - \mu_{CZ}(\gamma_1) - n + 3 \geq 0$$

and

$$\mu_{CZ}(x_2) - \mu_{CZ}(\gamma_2) + 3 \geq 0$$

Then, since

$$\mu_{CZ}(x_1) + \mu_{CZ}(x_2) - \mu_{CZ}(x_3) = n$$

these conditions are reduced to

$$\mu_{CZ}(\gamma_1) \leq 3 - \mu_{CZ}(x_2) \quad \text{and} \quad \mu_{CZ}(\gamma_2) \leq 3 + \mu_{CZ}(x_2)$$

In particular,

$$\mu_{CZ}(\gamma_1) \leq 3 - n.$$  

Meanwhile, the Floer energy of the top level would be

$$0 \leq E = A_H(x_1) - A_H(x_3) - A(\gamma_1).$$

So

$$0 < A(\gamma_1) \leq A_H(x_1) - A_H(x_3) < D$$

We have $\gamma_1 \in \mathcal{P}^{<D}(Z, \alpha)$, which implies $\mu_{CZ}(\gamma_1) > 3 - n$, which is a contradiction. Now we know that the top of the Floer building is connected, so we can proceed as in the proof of proposition 3.10 of [Laz16]. We prove this by contradiction. Suppose the pair-of-pants product breaks after neck-stretching and $\gamma_k$ be the Reeb orbits in the top of Floer building as in [Laz16]. The virtual dimension of the moduli space of the top Floer building is

$$|x_1| + |x_2| - |x_3| - \sum |\gamma_k| < 0.$$  

Contradiction. □

Now if we further require that the admissible Hamiltonian $H$ has a unique minimum (which is always possible and compatible with our requirements on admissible Hamiltonians), then the Floer chain complex $SC_*(W, H, J) = O_*(W, H, J) \oplus C_*(W, H, J)$, where $O_*(W, H, J)$ is generated by all non-constant Hamiltonian orbits and $C_*(W, H, J)$ is generated by all constant Hamiltonian orbits (critical points of $H$). Since $H$ is $C^2$ small in $W$, the action of the critical points is small, and the Floer differential $d$ coincides with the Morse boundary operator $d_1$.

We therefore have $(SC_*^{<\delta}(W, H, J), d) = (C_*(W, H, J), d_1)$ and $SC^+_* (W, H, J) = O_*(W, H, J)$.

For degree reasons, $C_n(W, H, J) = \mathbb{Z}_2 < p >$, where $p$ is the unique minimum of $H$. Note that $d(p) = d_1(p) = 0$ and we have the fact that $d(x + p) = 0$ implies $d(x) = d(p) = 0$.

Proof of Proposition 3.8. As shown in Figure 8, let $W, V$ be two different Liouville fillings for a strongly ADC contact manifold $(\Sigma, \lambda)$. Suppose $H^D_W, H^D_V$ are Hamiltonians (as in SubSection 2.3.1) whose slopes at infinity are $D \notin \text{Spec}(\Sigma, \lambda)$. We can further assume that they have unique minima which are denoted by $p, q$ respectively. Note that any element $x \in O_*(W, H_W, J_W)$ has action $A_{H_W}(x) < D$. As shown above, $O_*(W, H_W, J_W) = SC^+_*(W, H_W, J_W)$.

After neck-stretching, we can assume that

$$(H_W, J_W)|_{\Sigma \times [R, \infty)} \equiv (H_V, J_V)|_{\Sigma \times [R, \infty)}$$
So we have $O_*(W, H_W, J_W) = O_*(W, H_V, J_V)$. Proposition 7.1 shows that Floer cylinders with asymptotes in $SC^+_n(W, H_W, J_W)$ are entirely contained in $\Sigma \times [R, \infty)$. Therefore Floer differentials of $O_*(W, H_W, J_W)$ and $O_*(W, H_V, J_V)$ coincide. We will suppress $W$ and $V$ in the notation and denote them by $(O_*(H, J), \partial)((O_*^{<K}(H, J), \partial) if it is filtered above by action $K).$ we have the pair-of-pants product $\otimes_W$ on

$$SC^{D/2}_n(W, H^D, J) = O^{<D/2}_n(H^D, J) \oplus C^{<D/2}_n = O^{<D/2}_n(H^D, J) \oplus \mathbb{Z}_2 < p >$$
defined as
\[
SC_n^{< D/2}(W, H^D, J) \otimes SC_n^{< D/2}(W, H^D, J) \rightarrow SC_n^{< D}(W, H^D, J)
\]
(7.1)
\[
(x, y) \mapsto x \otimes_W y.
\]
(7.2)

By Proposition 7.3, \( \otimes_W \) coincides with \( \otimes_V \) on components in \( O_n^{< D}(H^D, J) \), that is, for \( x, y \in O_n^{< D/2}(H^D, J) \), \( x \otimes_W y = z + \delta_W(x, y) \), where \( z \in O_n^{< D/2}(H^D, J) \) and \( \delta_W(x, y) \) is closed in \( SC_n(H, J) \). Likewise, we have \( x \otimes_V y = z + \delta_V(x, y) \), where \( z \in O_n^{< D}(H^D, J) \) and \( \delta_V(x, y) \) is closed in \( SC_n^{-}(H, J) \). Now for any \( \alpha \in I^{< D/2}(W, H_W, J_W) \subset SH_n^{< D/2}(W, H_W, J_W) \), we have
\[
\alpha = [x + \epsilon p]_W = [x]_W + \epsilon[p]_W = [x]_W + \epsilon H_W
\]
where \( x \in O_n^{D/2}(H, J) \), \( \epsilon = 0 \) or 1. \( x \otimes_W x = z + \delta_W(x, x) \) implies
\[
\alpha^2 - \alpha = [x]_W^2 + \epsilon^2 H_W - [x]_W - \epsilon H_W = [z - x + \delta(x, x)]_W = [z - x]_W + [\delta(x, x)]_W
\]
So \( \alpha \in I^{< D/2}(W, H_W, J_W) \) is equivalent to
\[
[z - x]_W + [\delta(x, x)]_W \in H^0(W).
\]
But since \( [\delta(x, x)]_W \in H^0(W) \), \( \alpha \in I^{< D/2}(W, H_W, J_W) \) is equivalent to \( [z - x]_W \in H^0(W) \). Thence for \( x \in O_n^{< D/2}(H, J) \), \( \partial(x) = 0 \) (\( \partial \) is Floer differential on \( O_n^{< D/2}(H, J) \)),
\[
[x]_W^+ \in I_+(W, H_W, J_W) \iff [z - x]_W^+ \in SH_n^{+, < D/2}(H, J)
\]
where \( [y]_W^+ \) stands for the equivalence class of \( y \in O_+(H, J) \) in \( SH_n^{+, < D/2}(W, H_W, J_W) \). We can prove the same results for \( V \) similarly. Therefore we have an isomorphism between \( I_n^{< D/2}(W, H_W, J_W) \) and \( I_n^{< D/2}(V, H_V, J_V) \):
\[
[x]_W^+ \mapsto [x]_V^+.
\]
Since \( SH_n^{+, < D/2}(W, H_W, J_W) \) and \( SH_n^{+, < D/2}(V, H_V, J_V) \) can be defined by \( (\Sigma \times [R, \infty], H, J) \) as the Floer cylinder never enters the interior. Therefore we have the identity
\[
SH_n^{+, < D/2}(W, H_W, J_W) \cong H_*(O_n^{< D/2}(H, J), \partial) \cong SH_n^{+, < D/2}(V, H_V, J_V),
\]
the inclusion map \( SH_n^{+, < D/2}(W, H_W, J_W) \rightarrow SH_n^{+(W, H_W, J_W)} \) commute with the above isomorphism,
\[
I_n^{< D/2}(W, H_W, J_W) \rightarrow I_n^{< D/2}(V, H_V, J_V) \rightarrow \text{Id}
\]
and we can therefore take the direct limit with respect to \( H_W \). Since we already know \( SH_n^{+(W)} \) is isomorphic to \( SH_n^{+(V)} \) by Theorem 3.5, it follows that \( I_+(W) \cong I_+(V) \).

\[\square\]

**Remark 7.4.** We can also proceed exactly as in proof of proposition 3.8 in [Laz16]. The key point is to use the *essential complex* as defined in that proof.
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