Spin Susceptibility and Gap Structure of the Fractional-Statistics Gas

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Abstract

This paper establishes and tests procedures which can determine the electron energy gap of the high-temperature superconductors using the $t-J$ model with spinon and holon quasiparticles obeying fractional statistics. A simpler problem with similar physics, the spin susceptibility spectrum of the spin 1/2 fractional-statistics gas, is studied. Interactions with the density oscillations of the system substantially decrease the spin gap to a value of $(0.2 \pm 0.2) \hbar \omega_c$, much less than the mean-field value of $\hbar \omega_c$. The lower few Landau levels remain visible, though broadened and shifted, in the spin susceptibility. As a check of the methods, the single-particle Green’s function of the non-interacting Bose gas viewed in the fermionic representation, as computed by the same approximation scheme, agrees well with the exact results. The same mechanism would reduce the gap of the $t-J$ model without eliminating it.

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I. INTRODUCTION

In this paper, we investigate the excitation spectrum of the spin $1/2$ fractional-statistics gas, focusing upon the minimum energy required to flip a spin. We develop techniques enabling the computation of the energy gap of the cuprate superconductors in terms of the $t$–$J$ model possessing spinon and holon quasiparticles obeying fractional statistics [1]. Using that framework, Tikofsky, Laughlin, and Zou [2] determined the optical conductivity and other quantities in good agreement with exact diagonalizations of the $t$–$J$ Hamiltonian for small systems. However, their computed electron energy gap is roughly five times larger than the experimental limit [3] for the cuprates. In addition, their linear response functions displayed unreasonable sharp structure, which had to be smoothed out in an arbitrary fashion. The spin-one excitation spectrum of the fractional-statistics gas provides insight into the electron gap of the more complicated $t$–$J$ model. Specifically, an electron hole in the $t$–$J$ model consists of a composite of two particles, a spinon and a holon, each obeying fractional statistics, just as the spin-one excitation of the fractional-statistics gas is a composite of a spin-up particle and a spin-down hole. As will be demonstrated later, interactions with phonons, the density oscillations of the fractional-statistics gas, broaden the spin susceptibility spectrum, thereby lowering the spin gap, the minimum spin-one excitation energy. As a possible comparison, Girvin and collaborators [4] found the exact ground state of the spin $1/2$ fractional-statistics Hamiltonian with an attractive delta-function interaction. However, they neglected the broadening of the spin susceptibility spectrum and consequently obtained an overly large spin gap.

Let us define the system being studied. The non-interacting fractional statistics gas [3] with spin corresponds to the spin $S$ fermionic Hamiltonian,

$$H_{\text{anyon}} = \sum_{i=1}^{N} \frac{1}{2m} \left| P_i + \frac{e}{c} A_i \right|^2,$$  \hspace{1cm} (1.1)

where the gauge field $A_i$ has the value

$$A_i = \sum_{j \neq i}^{N} A_{ij} = (1-\nu) \frac{\hbar c}{e} \frac{e}{z} \times \sum_{j \neq i}^{N} \frac{r_i - r_j}{|r_i - r_j|^2} \left(1 - e^{-\alpha |r_i - r_j|^2}\right).$$  \hspace{1cm} (1.2)
and \( \mathbf{r}_i \) denotes the two-dimensional vector locating the \( i \)th particle. In the problem being studied, the particles have spin \( S = \frac{1}{2} \) and statistics \( \nu = \frac{1}{2} \). The cutoff of the gauge field \( \mathbf{A} \) at small distances prevents a weak spin instability investigated by Béran and Laughlin \[3\]. Setting the cutoff parameter \( \alpha \) equal to \( 10^4 \) avoids the Hartree-Fock instability described in that paper with a negligible effect upon the subsequent spin gap calculations. In the lattice system to which the fractional-statistics formalism will be applied, the gauge force would be cut off at a distance of one lattice spacing.

The spin susceptibility, defined for an unpolarized spin \( \frac{1}{2} \) system to be

\[
\chi(\mathbf{r}_1 t_1 | \mathbf{r}_2 t_2) = -i < \Phi | T\{\sigma^z(\mathbf{r}_1 t_1) \sigma^z(\mathbf{r}_2 t_2)\} | \Phi >
\]

\[
= -2i < \Phi | T\{\sigma^-(\mathbf{r}_1 t_1) \sigma^+(\mathbf{r}_2 t_2)\} | \Phi > ,
\]

(1.3)
determines the spin gap of the fractional-statistics gas. The wavefunction \( |\Phi> \) is the ground state, and the operator time dependence follows the Heisenberg picture,

\[
\hat{O}(t) = e^{(i/\hbar)\hat{H}t} \hat{O} e^{-(i/\hbar)\hat{H}t}.
\]

(1.4)
The \( z \)-component of spin is

\[
\sigma^z(\mathbf{r}) = \Psi^\dagger_\uparrow(\mathbf{r})\Psi_\uparrow(\mathbf{r}) - \Psi^\dagger_\downarrow(\mathbf{r})\Psi_\downarrow(\mathbf{r}),
\]

(1.5)
while the spin-raising and spin-lowering operators are defined by

\[
\sigma^+(\mathbf{r}) = \Psi^\dagger_\uparrow(\mathbf{r})\Psi_\downarrow(\mathbf{r})
\]

(1.6)
and

\[
\sigma^-(\mathbf{r}) = \Psi^\dagger_\downarrow(\mathbf{r})\Psi_\uparrow(\mathbf{r}) ,
\]

(1.7)
where \( \Psi^\dagger_\sigma(\mathbf{r}) \) and \( \Psi_\sigma(\mathbf{r}) \) are fermionic operators creating or destroying a spin \( \sigma \) particle at the position \( \mathbf{r} \). The Fourier transform of the susceptibility,

\[
\chi(\mathbf{Q}, \omega) = L^2 \int \int e^{-i(\mathbf{Q} \cdot \mathbf{r} - \omega t)} \chi(\mathbf{r} t | 0 0) dt d\mathbf{r} ,
\]

(1.8)
determines the spin gap $\Delta$, the smallest positive frequency $\omega$ for which $\text{Im} \chi(Q, \omega)$ is non-zero at some momentum $Q$.

Finding the spin gap at the mean-field level is straightforward. The system may be approximated by the mean-field Hartree Hamiltonian $H_0$, \[ H_0 = \sum_{i=1}^{N} \frac{1}{2m} \left| P_i + \frac{e}{c} \vec{A}_i \right|^2, \tag{1.9} \]
corresponding to particles in a constant magnetic field

$$\vec{A} = \frac{1}{2} \vec{B} \times \vec{r}, \tag{1.10}$$

where

$$\vec{B} = (1 - \nu) \frac{\hbar c}{e} \bar{\rho} \hat{z}, \tag{1.11}$$

and $\bar{\rho}$ is the average particle density of all spins. The resulting energy spectrum consists of sharp Landau levels, with a gap between Landau levels of the cyclotron energy,

$$\hbar \omega_c = \frac{\hbar e B}{mc} = 2\pi(1 - \nu) \frac{\hbar^2}{m \bar{\rho}} \tag{1.12}$$
as displayed in Fig. 1. For spin $\frac{1}{2}$ semions, the lowest Landau level is occupied with particles of both spins, and the other levels are empty. Thus, the spin-one excitation with the smallest energy consists of removing a spin-down particle from Landau level 0 and adding a spin-up particle to Landau level 1, implying a mean-field spin gap of $\hbar \omega_c$. For any momentum $Q$, the imaginary part of the spin susceptibility $\text{Im} \chi(Q, \omega)$ consists of delta functions at the frequencies $\omega$ equal to multiples of the cyclotron energy $\omega_c$. The mean-field spin susceptibility is represented as the thick vertical lines in Fig. 2, in which, for $Q = 0.6/a_0$, the height of each line is proportional to the relative strength of its delta function. For smaller $Q$, the delta function at $\omega = \omega_c$ dominates the others to an even greater extent.

Our final result for the spin susceptibility of the fractional-statistics gas differs substantially from the mean-field value. Fig. 2 displays the imaginary part of the spin susceptibility, calculated in Section IV, as a function of the frequency $\omega$ for various values of the momentum.
Q. For any momentum, the lowest energy excitation has an energy Δ of $0.32 \hbar \omega_c$. Actually, the calculation has rather large error bars, as discussed in section V, and we only assert that the spin gap is less than $0.4 \hbar \omega_c$. A numerical approach is needed to determine the spin gap more precisely. In any case, the spin gap, if it exists, is substantially less than the mean-field value of $\hbar \omega_c$. The susceptibility spectrum has broadened considerably, though half of the spectral weight lies within $0.01 \omega_c$ of the spin gap Δ. At larger momenta, a definite peak appears at the frequency $\omega = 1.2 \omega_c$.

Interactions with phonons, the density oscillations of the system, broaden the spin susceptibility spectrum into the form displayed in Fig. 2. Consider the effect of the phonons upon the single-particle density of states of the unoccupied Landau levels $n \geq 1$. As illustrated in Fig. 3(a), a particle in Landau level 2 may emit a phonon and decay into Landau level 1, resulting in the broadening of Landau level 2. Also, assuming Landau level 1 is broadened, as in Fig. 3(b), a particle in that Landau level may make an intralevel transition to a lower energy state within the Landau level and emit a phonon, in turn broadening Landau level 1 self-consistently. This broadening extends the particle spectrum to lower energies. Similarly, the intralevel transitions within Landau level 0 broaden the hole spectrum, so that it reaches higher energies. Thus, the energy gap separating the particle and hole states decreases by an amount of order one, since the theory contains no small parameters. The phonon interaction between the particle and the hole lowers the spin gap further.

Conceptually, a flipped spin modifies its local environment to diminish the overall excitation energy, the spin gap. In other words, the lowest energy state $|q> \rangle$ with spin one and momentum $q$ may be approximated by density waves in addition to spin waves,

$$|q> \approx \left[ \sigma_q^+ - \sum_k h_k \rho_k \sigma_{q-k}^+ \right] |\Phi> \right). \tag{1.13}$$

Here $|\Phi> \rangle$ is the ground state, $\sigma_q^+$ is the Fourier transform of the spin-raising operator $\sigma^+(r)$ defined in Eq. (1.9),

$$\sigma_q^+ = \int dr \sigma^+(r) e^{-iqr}. \tag{1.14}$$
and the density operator \( \rho_q \) is defined in Eq. (1.21). The coefficients \( h_k \) must be determined, and we have left out terms involving more than one phonon. In real space, the excited state \(|q>\) may be written,

\[
|q> \approx \sum_i e^{-i q \cdot r_i} \sigma_i^+ \left[ Z - \sum_{j \neq i} h(r_j - r_i) \right], 
\]

(1.15)

where \( h(r) \) is the Fourier transform of \( h_k \),

\[
h(r) = L^2 \int \frac{dk}{(2\pi)^2} h_k e^{ik \cdot r},
\]

(1.16)

\( \sigma_i^+ \) is the spin-raising operator on particle \( i \), and \( Z \) has the value,

\[
Z = 1 - L^2 \int \frac{dk}{(2\pi)^2} h_k.
\]

(1.17)

The other particles are repelled from the flipped spin present in a higher Landau level. At the level of a Hartree approximation, the reduced density near the flipped spin leads to a reduced mean gauge field and a reduced cyclotron energy \( \hbar \omega_c \), and thus a reduced excitation energy. In conclusion, the single-mode approximation of the spin-one excitation made by Girvin et al., [4]

\[
|q_{SMA}> = \sigma_q^+ |\Phi>,
\]

(1.18)

describes the lowest lying spin-one excitation poorly.

Let us define certain conventions used throughout this paper. We shall often use units in which the cyclotron energy \( \hbar \omega_c \), defined in Eq. (1.12), and the magnetic length \( a_0 \),

\[
a_0 = \sqrt{\frac{\hbar c/e}{2\pi B}} = [2\pi(1-\nu)\bar{p}]^{-1/2}
\]

(1.19)

both equal one. Let us define the density operator \( \rho(r) \),

\[
\rho(r) = \sum_{i=1}^{N} \delta(r - r_i),
\]

(1.20)

along with its Fourier transform \( \rho_q \),

\[
\rho_q = \sum_{i=1}^{N} e^{-i q \cdot r_i}.
\]

(1.21)
The mean-field current operator \( j(r) \), \( j(r) = \frac{1}{2m} \sum_{i=1}^{N} \left\{ \mathbf{p}_i + \frac{e}{c} \mathbf{A}_i, \delta(\mathbf{r} - \mathbf{r}_i) \right\} \), has a Fourier transform \( j_q \),

\[
j_q = \frac{1}{2m} \sum_{i=1}^{N} \left\{ \mathbf{p}_i + \frac{e}{c} \mathbf{A}_i, e^{-iq \cdot \mathbf{r}_i} \right\},
\]

and a transverse component,

\[
j_q^T = j_q \cdot (\hat{z} \times \hat{q}),
\]

where \( \mathbf{A} \) represents the constant magnetic field of Eq. (1.10).

II. EFFECTIVE INTERACTION

Interactions with phonons, the density oscillations of the fractional-statistics system, broaden the spin-susceptibility spectrum and lower the spin gap. To analyze the system, we shall replace the Hamiltonian \( \mathcal{H}_{\text{anyon}} \), Eq. (1.1), with an effective Hamiltonian consisting of particles in a magnetic field coupled to phonons. The phonon system accurately describes the spin-one excitation spectrum of the fractional-statistics gas. Perturbative techniques will then determine the spin susceptibility and spin gap.

The effective Hamiltonian \( \mathcal{H}_{\text{eff}} \) modelling the fractional statistics gas has the form

\[
\mathcal{H}_{\text{eff}} = \sum_{i=1}^{N} \frac{1}{2m} |\mathbf{p}_i + \frac{e}{c} \mathbf{A}_i|^2 + \sum_{q} \frac{1}{2} \hbar \omega_q \left( a_q^\dagger a_q + a_q a_q^\dagger \right) + \sum_{q} \frac{\alpha_q}{\sqrt{2\omega_q}} j_q^T \left( a_q^\dagger + a_q \right). \tag{2.1}
\]

The operators \( a_q^\dagger \) and \( a_q \) create and destroy a phonon of momentum \( \hbar \mathbf{q} \) and energy \( \hbar \omega_q \). The particles lie in the constant magnetic field \( \mathbf{A} \) given by Eq. (1.10), arising from the mean-field Hamiltonian \( \mathcal{H}_0 \), Eq. (1.9). The mean-field transverse-current operator \( j_q^T \), defined in Eq. (1.24), couples a particle to a phonon with coupling strength \( \alpha_q \). As in the mean-field Hamiltonian \( \mathcal{H}_0 \), the particle density corresponds to filling the lowest Landau level due to the magnetic field \( \mathbf{A} \) with particles of both spins. The phonon dispersion \( \omega_q \) and coupling \( \alpha_q \), along with the justification of the effective Hamiltonian \( \mathcal{H}_{\text{eff}} \), are given below.
Let us first approximate the part of the Hamiltonian left out of the mean-field treatment as a two-body interaction $H_1$, neglecting the three-body terms which have a limited effect upon the spin gap. The perturbation Hamiltonian $H_1$, defined by subtracting the mean-field Hamiltonian, Eq. (1.1), from the full fractional-statistics Hamiltonian, Eq. (1.1), in the limit of large $\alpha$, may be written,

$$H_1 = H_{\text{anyon}} - H_0$$

$$= \frac{e}{c} \sum_{q \neq 0} V(q) \cdot j_{-q} \rho_q - \frac{1}{2m} \left( \frac{e}{c} \right)^2 \sum_{q \neq 0} \sum_{p \neq 0} V(q) \cdot V(p) \rho_{p-q} \rho_q \rho_{-p} .$$  \hspace{1cm} (2.2)

The potential $V(q)$ is defined to be,

$$V(q) = (1 - \nu) \frac{1}{iL^2} \frac{\hbar c}{eq} (\hat{z} \times \hat{q}) ,$$  \hspace{1cm} (2.3)

and the density operator $\rho_q$ and the mean-field current operator $j_q$ are defined in Eqs. (1.21) and (1.23). The second term of $H_1$, the three-body interaction, is largest for $p$ equal to $q$, where the density operator $\rho_{p-q}$ reduces to the total particle number $N$. Removing the terms with $p$ not equal to $q$ leaves the two-body interaction Hamiltonian,

$$H_{1-\text{body}} = \frac{e}{c} \sum_{q \neq 0} V(q) \cdot j_{-q} \rho_q - \frac{N}{2m} \left( \frac{e}{c} \right)^2 \sum_{q \neq 0} |V(q)|^2 \rho_q \rho_{-q}$$

$$= \frac{1}{2} \left( \rho_{-q} j^x_{-q} j^y_{-q} \right) \mathcal{V}^0(q) \begin{pmatrix} \rho_q \\ j^x_q \\ j^y_q \end{pmatrix} ,$$  \hspace{1cm} (2.4)

where, for $q$ in the $\hat{x}$ direction, the bare interaction $\mathcal{V}^0$ has the form,

$$\mathcal{V}^0(q) = \frac{12\pi}{2 L^2 q^2} \begin{pmatrix} 1 & 0 & iq \\ 0 & 0 & 0 \\ -iq & 0 & 0 \end{pmatrix} .$$  \hspace{1cm} (2.5)

In the preceding equation and the remainder of this paper, the cyclotron energy $\hbar \omega_c$, Eq. (1.12), and the magnetic length $a_0$, Eq. (1.19), are both set equal to one.
Various higher order processes modify the effective interaction between any two particles, in a fashion similar to the screening of the Coulomb interaction in an electron gas. Diagrammatically, the effective interaction is the sum of a bare interaction and a polarization bubble attached to two bare interactions,

\[ \mathcal{V}(q, \omega) = \mathcal{V}_0(q) + \mathcal{V}_0(q) \mathcal{D}(q, \omega) \mathcal{V}_0(q) . \]  

(2.6)

The full polarization bubble \( \mathcal{D} \) is the correlation function

\[
\mathcal{D}(q,\omega) = L^2 \int \frac{1}{i\hbar} \int_0^\infty dt \, dr \, e^{-i(q \cdot r - \omega t)} e^{-\eta t} \langle \Phi | \left(T\{\rho(r,t),\rho(0,0)\} \, T\{\rho(r,t),j^x(0,0)\} \, T\{\rho(r,t),j^y(0,0)\} \right) \langle \Phi | \left(T\{j^y(r,t),\rho(0,0)\} \, T\{j^y(r,t),j^x(0,0)\} \, T\{j^y(r,t),j^y(0,0)\} \right)\rangle .
\]

(2.7)

The density operator \( \rho(r) \) and the mean-field current operator \( j(r) \) are given by Eqs. (1.20) and (1.22), the time dependence of the operators follows the definition of the Heisenberg representation, Eq. (1.4), and the operators are time ordered. Eq. (2.6) for the effective interaction may be expressed in terms of the proper polarization \( \mathcal{D}^P \), the Feynman diagrams in the polarization which remain connected after removing any single interaction line, in the form,

\[ \mathcal{V}(q, \omega) = \mathcal{V}_0(q) + \mathcal{V}_0(q) \mathcal{D}^P(q, \omega) \mathcal{V}_0(q) , \]  

(2.8)

as displayed in Fig. 4(a). The bare interaction \( \mathcal{V}_0 \), Eq. (2.5), is denoted by a thin wavy line, the effective interaction \( \mathcal{V} \) is denoted by the thick wavy line, and the proper polarization \( \mathcal{D}^P \) is denoted by the black square.

Let us calculate the effective interaction using the Random Phase Approximation. In the Random Phase Approximation (RPA), the proper polarization \( \mathcal{D}^P \) in Eq. (2.6) is estimated by the bare polarization \( \mathcal{D}^0 \) of the mean-field Hamiltonian \( \mathcal{H}_0 \), Eq. (1.9). As displayed diagrammatically in Fig. 4(b),
\[ V_{\text{RPA}}(q, \omega) = V^0(q) + V^0(q)D^0(q, \omega)V_{\text{RPA}}(q, \omega), \]  

(2.9)

where the straight lines represent single-particle propagators of the Hamiltonian \( H_0 \). The bare polarization \( D^0 \) is found by replacing \( |\Phi> \) with \( |\Phi_0> \), the ground state of \( H_0 \), in Eq. (2.7) and using the Hamiltonian \( H_0 \) in the Heisenberg representation, Eq. (1.4). The calculation of the bare polarization of a spinless system in [7] and [8] is modified for spin \( \frac{1}{2} \) semions to have the value,

\[ D^0(q, \omega) = 2 \frac{L^2}{2\pi} \begin{pmatrix} q^2 \Sigma_0 & q\omega \Sigma_0 & -iq \Sigma_1 \\ q\omega \Sigma_0 & \omega^2 \Sigma_0 - 1 & -i\omega \Sigma_1 \\ iq \Sigma_1 & i\omega \Sigma_1 & \Sigma_2 \end{pmatrix}, \]  

(2.10)

where

\[ \Sigma_j(q, \omega) \equiv \sum_{n=1}^{\infty} \frac{e^{-b}b^{n-1}}{(n-1)!(\omega^2 - (n-i\eta)^2)^2}(n-b)^j \]  

(2.11)

and \( b \) represents \( \frac{1}{2}q^2 \). The factor of two in Eq. (2.10) arises from the spin degrees of freedom. Substituting Eqs. (2.5) and (2.10) into Eq. (2.9) results in the RPA interaction,

\[ V_{\text{RPA}}(q, \omega) = \frac{1}{2\pi} \frac{2\pi}{L^2q^2D} \begin{pmatrix} 1 + \Sigma_2 & 0 & iq(1 + \Sigma_1) \\ 0 & 0 & 0 \\ -iq(1 + \Sigma_1) & 0 & q^2 \Sigma_0 \end{pmatrix}, \]  

(2.12)

where

\[ D = (1 + \Sigma_1)^2 - \Sigma_0(1 + \Sigma_2). \]  

(2.13)

For fixed \( q \), the RPA interaction \( V_{\text{RPA}}(q, \omega) \) contains poles at discrete values of \( \omega \), with a spacing of approximately one between poles. Fig. \[\text{F}3\] graphs the imaginary part of the current-current component of \( V_{\text{RPA}} \) for three values of \( q \). The solid line in each plot has been broadened with a small value of \( \eta \) for visibility. However, that discrete Landau level structure is not physical, and would disappear in a more accurate calculation of the proper polarization \( D^P \). The effective interaction at finite \( q \) will be broadened over a range of energies, resembling the dashed line in Figs. \[\text{F}3\](b) and \[\text{F}3\](c).
Nevertheless, we will approximate the dynamic effective interaction as a sharp mode for each momentum \( q \) in the current-current channel, disregarding both the broadening of the effective interaction and the effect of the density-density and density-current channels. For \( q < 2 \), the lowest energy singularity in the current-current RPA interaction dominates the other singularities in the three interaction channels, while for \( q > 2 \) the dynamic interaction is too weak to have a significant effect. The effective interaction then has the form,

\[
\mathcal{V}^{\text{eff}}(q, \omega) = \mathcal{V}^0(q) + \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & \frac{|\alpha_q|^2}{\omega^2-(\omega_q-i\eta)^2} & 0
\end{pmatrix}.
\] (2.14)

The excitation, a phonon, has an energy \( \hbar \omega_q \) and a transverse-current coupling \( \alpha_q \).

Let us determine the coefficients \( \alpha_q \) and \( \omega_q \) appearing in Eq. (2.14) for the effective interaction \( \mathcal{V}^{\text{eff}} \). We will define a quantity \( A_q \) equal to the area under the curves in Fig. 5,

\[
A_q \equiv \int_0^\infty \left[-\frac{1}{\pi} \text{Im} \mathcal{V}_{JJ}^{\text{RPA}}(q, \omega)\right] d\omega.
\] (2.15)

Then \( \omega_q \) is defined as the weighted average value of \( \omega \),

\[
\omega_q \equiv \frac{1}{A_q} \int_0^\infty \omega \left[-\frac{1}{\pi} \text{Im} \mathcal{V}_{JJ}^{\text{RPA}}(q, \omega)\right] d\omega,
\] (2.16)

and the coupling strength \( \alpha_q \) obeys

\[
|\alpha_q|^2 \equiv 2 A_q \omega_q.
\] (2.17)

Substituting Eq. (2.3) into Eq. (2.8) demonstrates that

\[
\mathcal{V}_{JJ} = \left(\frac{\pi}{L^2}\right)^2 \frac{1}{q^2} \mathcal{D}_{\rho\rho},
\] (2.18)

which, along with the f-sum rule [10], reduces Eqs. (2.16) and (2.17) to

\[
|\alpha_q|^2 = \frac{\pi}{L^2}.
\] (2.19)

Numerical calculations confirm this constant value of \( |\alpha_q|^2 \), and fit \( \omega_q \) to the formula,
\[ \omega_q^{(\text{formula})} = \left[ (v_s q)^4 + \left( \frac{q^2}{2} \right)^4 \right]^{1/4}, \quad (2.20) \]

with a sound velocity \( v_s \) of 1. Both \( \omega_q \) and \( \omega_q^{(\text{formula})} \) are graphed in Fig. 3.

Actually, the Random Phase Approximation does not determine the speed of sound \( v_s \) sufficiently accurately. Together, Hartree-Fock and correlation effects [5,7,9] lower the sound speed of the spinless fractional-statistics gas by approximately 9%. The sound speed of the spin \( \frac{1}{2} \) fractional-statistics gas would presumably shift by a comparable amount. Inserting \( v_s = 0.9 \) into Eq. (2.20) has a sizable effect upon the resulting spin gap, which we included in the error bars of the spin gap calculation in Section V.

The static interaction \( \mathcal{V}^0 \) in the effective interaction \( \mathcal{V}^{\text{eff}} \), Eq. (2.14), does not affect the spin gap. When calculating the spin susceptibility diagrammatically, as in Fig. 4, each of the phonon interaction lines is short-ranged. Dampening the interaction \( \mathcal{H}_{\text{anyon}} - \mathcal{H}_0 \) for large particle separations does not affect those Feynman diagrams. By Kohn’s theorem [11], the remaining short-ranged interactions between particles in a magnetic field do not change the excitation energy at small momenta from the value without an interaction, the cyclotron energy \( \hbar \omega_c \). Thus, the static interaction \( \mathcal{V}^0 \) does not change the \( Q \to 0 \) spin gap and may be omitted. Béran and Laughlin [6] demonstrated that with a cutoff parameter \( \alpha \) in the fractional-statistics Hamiltonian \( \mathcal{H}_{\text{anyon}} \), Eq. (1.1), corresponding to a reasonable doping of a lattice model, the static interaction \( \mathcal{V}^0 \) does not create a roton to lower the spin gap at finite \( Q \). Thus, the fractional-statistics system may be described as the mean-field Hamiltonian \( \mathcal{H}_0 \), Eq. (1.9), and a dynamic transverse-current interaction, the second term in Eq. (2.14), all of which reduces to the effective Hamiltonian \( \mathcal{H}_{\text{eff}} \), Eq. (2.1).

In summary, we have obtained a simpler system \( \mathcal{H}_{\text{eff}} \), Eq. (2.1), which possesses the same spin gap as the fractional-statistics gas \( \mathcal{H}_{\text{anyon}} \), Eq. (1.1). The effective Hamiltonian consists of particles in a magnetic field coupled transversely to phonons. The phonon dispersion \( \omega_q \) is given by Eq. (2.20), and the particle-phonon coupling \( \alpha_q \) is given by Eq. (2.19). In the remainder of this paper, we will investigate the effective Hamiltonian \( \mathcal{H}_{\text{eff}} \) to find the spin gap.
III. SINGLE-PARTICLE DENSITY OF STATES

Let us compute the single-particle density of states of the fractional-statistics gas. The density of states allows a crude estimate of the spin gap, since interactions between a particle and a hole present in a spin-one excitation lower the spin gap further. However, the single-particle computation clearly illustrates the mechanisms of broadening and gap lowering present in the spin susceptibility spectrum. Also, the formal methods used to compute the spin susceptibility in section IV resemble the techniques used to compute the density of states. We will start with the effective Hamiltonian $H_{\text{eff}}$, Eq. (2.1), consisting of particles in a magnetic field coupled to phonons. As displayed in Fig. 3, a particle in any Landau level may decay into either a lower Landau level or a lower energy state in the same Landau level, emitting a phonon. In this section, we will find the extent of the spectrum broadening induced by that decay process.

The density of states depends upon the single-particle Green’s function of the fractional statistics gas,

$$G(r_1t_1|\mathbf{r}_2t_2\sigma_2) = -i < \Phi | T \{ \Psi_{\sigma_1}(\mathbf{r}_1, t_1) \Psi_{\sigma_2}^{\dagger}(\mathbf{r}_2, t_2) \} | \Phi > ,$$

(3.1)

where $|\Phi >$ is the ground state. The operators $\Psi_{\sigma}^{\dagger}(\mathbf{r}, t)$ and $\Psi_{\sigma}(\mathbf{r}, t)$ create and destroy a spin $\sigma$ particle at position $\mathbf{r}$ and time $t$ in the Heisenberg representation, Eq. (1.4). The particle may emit and later absorb a virtual phonon, giving the Green’s function a complex self-energy which broadens the spectrum and lowers the spin gap. As illustrated in Fig. 8, we will solve Dyson’s equation for the exchange of a single phonon in the effective Hamiltonian $H_{\text{eff}}$, Eq. (2.1), to find the spin gap.

Let us first find the Green’s function corresponding to the bare Hamiltonian $H_0$, Eq. (1.9), in the absence of phonons. A constant magnetic field in the symmetric gauge possesses the Landau level orbitals $\varphi_{nk}$ [5],

$$\varphi_{nk}(z) = \frac{1}{\sqrt{2\pi 2^n n! k!}} \left( \frac{1}{2} z - 2 \frac{\partial}{\partial z} \right)^n \left( \frac{1}{2} z^* - 2 \frac{\partial}{\partial z^*} \right)^k e^{-|z|^2/4} ,$$

(3.2)

with Landau level energies,
The complex number \( z = x + iy \) refers to the position, and the natural units \( \hbar \omega_c \), Eq. (1.12), and \( a_0 \), Eq. (1.19), are set equal to one. As drawn in Fig. 1 Landau level 0 is occupied with particles of both spins, while the higher Landau levels are empty. The bare Green’s function may be written in the Landau level basis as,

\[
G^0(\mathbf{r}_1 t_1 \sigma_1 | \mathbf{r}_2 t_2 \sigma_2) = \delta_{\sigma_1 \sigma_2} \frac{1}{2\pi} \int \sum_{nk} \sum_{n'k'} <n'k' | G^0(E) | nk> \varphi_{n'k'}(\mathbf{r}_1) \varphi^*_{nk}(\mathbf{r}_2) e^{iE(t_2-t_1)} dE ,
\]

(3.4)

where

\[
<n'k' | G^0(E) | nk> = \delta_{nn'}\delta_{kk'} \frac{1}{E - (n + 1/2) + i\eta_n} ,
\]

(3.5)

and the infinitesimal \( \eta_n \) is positive for levels \( n \geq 1 \) and negative for level \( n = 0 \).

The Green’s function of the effective Hamiltonian \( \mathcal{H}_{\text{eff}} \), Eq. (2.1), may be found by solving Dyson’s equation for the exchange of a single phonon. This process is illustrated in Fig. 8, where the thin straight line is the bare Green’s function \( G^0 \), the thick straight line is the full Green’s function \( G \), and the thick wavy line is the phonon propagator. We included the diagrams in which, besides the phonons, only a single particle or a single hole is present at any point in time. For instance, the exchange graph in Fig. 8 where \( n \geq 1 \) and \( m = 0 \) has been neglected. This approximation is similar to the Tamm-Dancoff approximation [12] in nuclear physics. We have also left out diagrams with crossed phonon interaction lines. The relevant particle quantum states \( |nk> \), in addition to the phonons, consist of Landau level 0 filled with particles of both spins, with an extra particle in the orbital \( \varphi_{nk} \) for \( n \geq 1 \), or a hole in the orbital \( \varphi_{0k} \) for \( n = 0 \).

Dyson’s equation involves the matrix elements of the transverse-current operator \( j^T_q \), Eq. (1.24), between the Landau level states \( |nk> \). Summing over the intermediate states \( |mp> \) within a Landau level \( m \) and averaging over the direction \( \theta \) of the phonon momentum \( \mathbf{q} \) combines those matrix elements into a simple expression. After performing those operations, we find,
\[- \frac{1}{2\pi} \int_0^{2\pi} \sum_p \langle n' k' | j_q^T | m p \rangle < m p | j_q^T | n k \rangle \, d\theta = \delta_{mm'}\delta_{kk'}|\mathcal{M}_{nm}(q)|^2, \quad (3.6)\]

The effective matrix element $\mathcal{M}_{nm}(q)$, which is independent of the index $k$, has the value

$$\mathcal{M}_{nm}(q) = \sqrt{\frac{m!}{2n!}} L_{n-m}^{(n-m-1/2)} \left[ nL_m^{n-m-1}(b) - bL_m^{n-m+1}(b) \right] e^{-b/2}, \quad (3.7)$$

where $L$ denotes an associated Laguerre polynomial,

$$L_{\alpha n}^\alpha(b) = \frac{1}{n!} e^{b} b^{\alpha} \frac{d^n}{db^n} \left( e^{-b} b^{n+\alpha} \right), \quad (3.8)$$

and $b$ stands for $\frac{1}{2}q^2$.

We may now write down Dyson’s equation for the single-particle propagator arising from the exchange of a single phonon. The Green’s function will be computed in the Landau level basis,

$$G(r_1t_1\sigma_1|r_2t_2\sigma_2) = \delta_{\sigma_1\sigma_2} \frac{1}{2\pi} \int \sum_{nk} \sum_{n'k'} <n'k'|G(E)|nk> \varphi_{n'k'}(r_1)\varphi^*_n(r_2) e^{iE(t_2-t_1)} \, dE. \quad (3.9)$$

Because of Eq. (3.6), the full Green’s function $G(E)$ is diagonal in the basis of Landau level orbitals,

$$<n'k'|G(E)|nk> = \delta_{nm'}\delta_{kk'} G_n(E). \quad (3.10)$$

The Landau level Green’s function $G_n$ has the standard form,

$$G_n(E) = \frac{1}{E - (n+1/2) - \Sigma_n(E) + i\eta}, \quad (3.11)$$

with a self-energy,

$$\Sigma_n(E) = iL^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{dq}{(2\pi)^2} \sum_m \frac{|\mathcal{M}_{nm}(q)|^2 |\alpha_q|^2}{\omega^2 - (\omega_q - i\eta)^2} G_m(E + \omega). \quad (3.12)$$

The coupling $|\alpha_q|^2$ is given by Eq. (2.19), and the phonon energy $\omega_q$ is approximated by Eq. (2.20). When $n$ is an unoccupied Landau level, $n \geq 1$, the index $m$ is summed over the levels $m \geq 1$, and when $n = 0$, only $m = 0$ is present in the sum.

The analytic structure of the Landau level Green’s function $G_m(E)$ allows a simplification of the self-energy expression, Eq. (3.12). When Landau level $m$ is unoccupied, each of the
poles of $G_m(E + \omega)$ lies in the lower half of the complex plane of $\omega$, while the phonon propagator has a pole on each side of the real axis, as indicated in Fig. 9. The $\omega$ integral may be evaluated by closing the contour upwards, so that for $n \geq 1$,

$$
\Sigma_n(E) = L^2 \int \frac{d\mathbf{q}}{(2\pi)^2} \sum_{m=1}^{\infty} |M_{nm}(q)|^2 \frac{|\alpha_q|^2}{2\omega_q} G_m(E - \omega_q) \cdot
$$

(3.13)

Landau level 0 has a similar self-energy expression,

$$
\Sigma_0(E) = L^2 \int \frac{d\mathbf{q}}{(2\pi)^2} |M_{00}(q)|^2 \frac{|\alpha_q|^2}{2\omega_q} G_0(E + \omega_q) \cdot
$$

(3.14)

Even though the resulting density of states is physically incorrect, it provides a reasonable estimate of the spin gap. We solved Eqs. (3.11), (3.13), and (3.14) for $G_n(E)$. In Fig. 10, we display the density of states $D_n(E)$ within the lowest four Landau levels,

$$
D_n(E) = \bar{\rho} \left| \frac{1}{\pi} \text{Im} G_n(E) \right| \cdot
$$

(3.15)

Our calculation left out the effect of the static interaction, $V_0$ in Eq. (2.14), which changes the density of states greatly. The static interaction causes a gap between the occupied and unoccupied states which diverges logarithmically with the size of the sample. However, due to Kohn’s theorem, as argued in Section II, the static interaction does not affect the spin gap, the quantity of physical interest. Thus, except for leaving out the particle-hole and multiple-phonon interactions, the resulting spin gap remains valid.

The resulting broadening of the spectrum significantly reduces the gap energy relative to the mean-field gap displayed in Fig. 1. The broadened unoccupied states in Fig. 10 extend down to a minimum particle energy of

$$
E_{\text{min},p} = \left( \epsilon_1^{(0)} - 0.48 \right) \hbar\omega_c = 1.02 \hbar\omega_c \cdot
$$

(3.16)

and the occupied states reach a maximum hole energy of

$$
E_{\text{max},h} = \left( \epsilon_0^{(0)} + 0.05 \right) \hbar\omega_c = 0.55 \hbar\omega_c \cdot
$$

(3.17)

If the particle-hole interactions are neglected, the lowest lying spin-one state then has an energy,
\[ \Delta^{\text{no p-h}} = E_{\text{min,p}} - E_{\text{max,h}} = 0.47 \hbar \omega_c. \] \hfill (3.18)

Both the dynamic interaction between the particle and the hole and multiple phonon interactions will lower the spin gap further.

Finally, let us examine the overall distribution of the single-particle spectrum. The density of states of Landau level 1 diverges as the energy approaches \( E_{\text{min,p}} \) from above, so that roughly 60% of that level’s states are within \( 0.01 \hbar \omega_c \) of \( E_{\text{min,p}} \), while the tail extending to higher energies contains the rest of the states. Similarly, 90% of Landau level 0 lies very close to \( E_{\text{max,h}} \), with the tail containing the remaining spectral weight extending to lower energies. Landau level 2 contains a strong peak at an energy somewhat below \( \epsilon_2^{(0)} \), in addition to some broadened structure at both higher and lower energies. Landau level 3 is more broadened than the lower Landau levels, though a peak remains visible. Despite the broadening, the average energy of each Landau level,

\[ \langle E_n \rangle \equiv \frac{1}{\rho} \int_{-\infty}^{\infty} E D_n(E) \, dE, \] \hfill (3.19)

maintains the unperturbed value \( \epsilon_n^{(0)} \).

IV. SPIN SUSCEPTIBILITY AND GAP

Let us compute the spin susceptibility of the fractional-statistics gas, and consequently its gap. We shall find the retarded spin-one particle-hole propagator,

\[ \mathcal{F}(r_1 r_4 t_1 | r_2 r_3 t_2) = -i \theta(t_1 - t_2) \langle \Phi | \Psi_\uparrow(r_1, t_1) \Psi_\uparrow(r_4, t_1) \Psi_\downarrow(r_2, t_2) \Psi_\downarrow(r_3, t_2) | \Phi \rangle. \] \hfill (4.1)

The particle-hole pair produced in the intermediate state may emit and then subsequently absorb a phonon. The phonons arise in the effective Hamiltonian \( \mathcal{H}_{\text{eff}} \), Eq. (2.1), which as argued in Section II has the same spin gap as the fractional-statistics gas. The single-particle self-energy, \( \Sigma_n(E) \) in Eq. (3.13), accounts for the case when the particle both emits and absorbs a phonon, and the single-hole self-energy, \( \Sigma_0(E) \) in Eq. (3.14), accounts for the case when the hole both emits and absorbs a phonon. However, the full particle-hole propagator
\( \mathcal{F} \) is needed to include the sizable interaction between the particle and the hole, where one emits and the other absorbs a phonon. Consequently, the spin one spectrum exhibits more broadening and gap lowering than does a convolution of the single-particle Green’s functions computed earlier. The spin susceptibility and spin gap are readily expressed in terms of the pair propagator.

As in the section III, we have only included processes in which a single particle-hole pair propagates forwards in time, emitting and absorbing phonons. The Random Phase Approximation (RPA) Feynman graphs, which for the spin 0 particle-hole propagator exhibit the corrections to the Hartree-Fock ground state most strongly, as a reversed time-ordering [7], do not appear in the spin 1 propagator. Thus, the Tamm-Dancoff approximation [12] should be reliable. Graphs containing crossed phonon interaction lines have again been excluded.

A magnetoexciton basis [13], describing a particle and a hole in a magnetic field in terms of eigenstates of both momentum and the bare Hamiltonian \( H_0 \), proves to be convenient in studying the particle-hole propagator. The magnetoexciton particle-hole wavefunction has the form,

\[
\psi_{n\alpha}(z_1, z_2) = \frac{(-1)^n}{L\sqrt{2\pi 2^n n!}} \left( 2 \frac{\partial}{\partial z_1^*} - \frac{1}{2} z_1 \right)^n e^{-\frac{1}{4} |z_1|^2 + |z_2|^2 + |z_\alpha|^2} e^{\frac{1}{4} |z_1^* z_2 + z_1^* z_\alpha - z_2 z_\alpha^*|^2} \ ,
\]

where \( z = x + iy \) is a position written as a complex number. The spin-up particle lies in Landau level \( n \geq 1 \), the spin-down hole lies in Landau level 0, and the pair has momentum \( Q \), represented as a complex number \( z_\alpha = iQ_x - Q_y \). The corresponding quantum state \( |n \ Q \rangle \) is an energy eigenstate of the mean-field Hamiltonian \( H_0 \), Eq. (1.9), with energy

\[
\epsilon_{nQ}^{(0)} = n \ .
\]

The bare particle-hole Green’s function \( \mathcal{F}^0 \) of the Hamiltonian \( H_0 \) may be expressed in the magnetoexciton basis,

\[
\mathcal{F}^0(r_1 r_4 t_1 | r_2 r_3 t_2) = \frac{1}{2\pi} \int \sum_{n, Q} \sum_{n', Q'} \frac{dE}{E} \left( \langle n' Q' | \mathcal{F}^0(E) | n \ Q \rangle \right) \times \psi^*_{n' \alpha'}(r_1, r_4) \psi_{n\alpha}(r_2, r_3) e^{iE(t_2 - t_1)} \ ,
\]
where

\[\langle n' Q' | F^0(E) | n Q \rangle = \delta_{nn'} \delta QQ' \frac{1}{E - n + i\eta}. \quad (4.5)\]

The phonon coupling appearing in the particle-hole propagator depends upon the transverse-current matrix element between two magnetoexcitons. The transverse-current operator \( j^T_q \), defined in Eq. (1.24), may act on either the particle or the hole, and both couplings must be included. The coupling between the particle and a phonon has a matrix element,

\[\langle m Q - q | j^T(P) | n Q \rangle = \int \int dr_1 dr_2 \lim_{r'_1 \to r_1} e^{-iqr_1} \left[ \frac{1}{2}(P_1 - P_1') + \hat{A}(r_1) \right] \cdot \hat{e}_q \psi^*_{m\beta}(r_1', r_2) \psi_{n\alpha}(r_1, r_2), \quad (4.6)\]

while the coupling between the hole and a phonon has a matrix element,

\[\langle m Q - q | j^T(H) | n Q \rangle = -\int \int dr_1 dr_2 \lim_{r'_2 \to r_2} e^{-iqr_2} \left[ \frac{1}{2}(P_2 - P_2') - \hat{A}(r_2) \right] \cdot \hat{e}_q \psi^*_{m\beta}(r_1, r_2) \psi_{n\alpha}(r_1, r_2'). \quad (4.7)\]

The mean-field \( \hat{A}(r) \) is specified by Eq. (1.10), \( \hat{e}_q \) is the transverse direction \( \hat{z} \times \hat{q} \), and \( z_\beta = i(Q_x - q_x) - (Q_y - q_y) \). Since the hole has the opposite charge, its interaction with both the phonons and the magnetic field has the opposite sign. The resulting matrix elements have the values,

\[\langle m Q - q | j^T(P) | n Q \rangle = e^{-(|z_\alpha|^2 + |z_\beta|^2)/4} e^{z_\alpha^*/z_\alpha/2} \times \frac{i}{|z_\gamma|} \sqrt{\frac{m!}{n!}} \left( -\frac{z_\gamma}{\sqrt{2}} \right)^{n-m} \left[ n L_{m-1}^{n-m}(b) - b L_{m}^{n-m-1}(b) \right], \quad (4.8)\]

and

\[\langle m Q - q | j^T(H) | n Q \rangle = \delta_{mn} e^{-(|z_\alpha|^2 + |z_\beta|^2)/4} e^{z_\alpha^*/z_\alpha/2} \frac{i}{|z_\gamma|} [-b], \quad (4.9)\]

where \( z_\gamma = i(q_x - q_y) \), \( b \) denotes \( \frac{1}{2}q^2 \), and the Laguerre polynomials \( L \) are defined in Eq. (3.8). The transverse-current matrix element is the sum of the particle and the hole components,
\[ \langle m \mathbf{Q} - q | j_q^T | n \mathbf{Q} \rangle = \langle m \mathbf{Q} - q | j_q^{(P)} | n \mathbf{Q} \rangle + \langle m \mathbf{Q} - q | j_q^{(H)} | n \mathbf{Q} \rangle \]
\[ = \sqrt{\frac{m!}{n!} \frac{i}{q} \left( \frac{-\varphi}{\sqrt{2}} \right)^{n-m}} e^{-b/2} \times \left\{ e^{(i/2)(q \times \mathbf{Q}) \cdot \hat{z}} \left[ n L_m^{n-m-1}(b) - b L_m^{n-m+1}(b) \right] - \delta_{nm} e^{-(i/2)(q \times \mathbf{Q}) \cdot \hat{z}} b \right\}. \] (4.10)

Since small momenta dominate the interaction, we may neglect the phases proportional to \( q \times \mathbf{Q} \) in Eq. (4.10), approximating the current matrix element \( \langle m \mathbf{Q} - q | j_q^T | n \mathbf{Q} \rangle \) by a term \( M_{nm}'(q) \) independent of \( \mathbf{Q} \),
\[ M_{nm}'(q) = \sqrt{\frac{m!}{n!} \frac{i}{q} \left( \frac{-\varphi}{\sqrt{2}} \right)^{n-m}} \left[ n L_m^{n-m-1}(b) - b L_m^{n-m+1}(b) - \delta_{nm} b \right] e^{-b/2}, \] (4.11)
which for \( n \neq m \) has the same magnitude as \( M_{nm}(q) \), defined in Eq. (3.6).

The Hamiltonian \( \mathcal{H}_{\text{eff}} \), Eq. (2.1), restricted to a single particle-hole pair being present, may be treated exactly as a discrete quantum state coupled to phonons, allowing the diagrammatic computation of the particle-hole propagator \( \mathcal{F} \), Eq. (4.1), and the spin gap. Since momentum is conserved, consider states in which the system has total momentum \( \mathbf{Q} \). The particle’s Landau level \( n \) and the set of phonons present specify a basis for the single-pair configurations. A system which may occupy one of a discrete set of quantum states \( |n> \), where \( n \) is a positive integer, interacting with phonons has an isomorphic Hilbert space. If the corresponding matrix elements are the same, the spectrum of the discrete system agrees with the spectrum of \( \mathcal{H}_{\text{int}} \) constrained to the restricted Hilbert space. Thus, \( \mathcal{H}_{\text{eff}} \) with a single particle-hole pair may be transformed into the discrete system,
\[ \mathcal{H}_{\text{discrete}} = \sum_{n=1}^{\infty} n \Psi_n^\dagger \Psi_n + \sum_q \frac{1}{2} \hbar \omega_q \left( a_q^\dagger a_q + a_q a_q^\dagger \right) \]
\[ + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_q \frac{\alpha_q}{\sqrt{2 \omega_q}} M_{nm}'(q) \Psi_m^\dagger \Psi_n \left( a_q^\dagger - a_{-q} \right). \] (4.12)

The operators \( \Psi_n^\dagger \) and \( \Psi_n \) create and destroy a particle in the discrete state \( |n> \).

We found the particle-hole propagator by solving Dyson’s equation for the exchange of a single phonon in \( \mathcal{H}_{\text{discrete}}, \) as illustrated in Fig. 11. In effect, this method includes the particle exchange, hole exchange, and particle-hole ladder Feynman diagrams. The bare particle-hole
propagator $\mathcal{F}^0$, defined in Eq. (4.4), is represented by a thin dashed line, while the phonon propagator is denoted by a thick wavy line. The full pair propagator $\mathcal{F}$, represented by a thick dashed line, may be examined in the magnetoexciton basis,

$$
\mathcal{F}(\mathbf{r}_1\mathbf{r}_4 \mid \mathbf{r}_2\mathbf{r}_3) = \frac{1}{2\pi} \int \sum_{n\, Q} \sum_{n'\, Q'} <n'\, Q' \mid \mathcal{F}(E) \mid n\, Q> \times \psi_{n'\, \alpha'}(\mathbf{r}_1, \mathbf{r}_4) \psi_{n\, \alpha}(\mathbf{r}_2, \mathbf{r}_3) e^{iE(t_2-t_1)} dE.
$$

(4.13)

Due to the phases of $\mathcal{M}_{nm}(q)$ in Eq. (4.11), the integral over the direction of the phonon momentum $q$ cancels in Dyson’s equation whenever $n \neq n'$, keeping the propagator diagonal.

The resulting Dyson’s equation of the particle-hole Green’s function has a form similar to Eqs. (3.10) through (3.12) for the single-particle Green’s function. The particle-hole propagator remains both diagonal and independent of momentum in the magnetoexciton basis,

$$
<n'\, Q' \mid \mathcal{F}(E) \mid n\, Q> = \delta_{nn'} \delta_{QQ'} \mathcal{F}_n(E).
$$

(4.14)

The magnetoexciton propagator $\mathcal{F}_n(E)$ for particle Landau level $n$ obeys the equations,

$$
\mathcal{F}_n(E) = \frac{1}{E - n - \Sigma_n(E) + i\eta},
$$

(4.15)

with a self-energy,

$$
\Sigma_n(E) = iL^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d\mathbf{q}}{(2\pi)^2} \sum_{m=1}^{\infty} \frac{|\mathcal{M}_{nm}(q)|^2 |\alpha_q|^2}{\omega^2 - (\omega_q - i\eta)^2} \mathcal{F}_m(E + \omega).
$$

(4.16)

Performing the integral over $\omega$, using the analytic properties of $\mathcal{F}$ identical to the analytic properties of $G$ leading to Eq. (3.13), simplifies the self-energy to

$$
\Sigma_n(E) = L^2 \int \frac{d\mathbf{q}}{(2\pi)^2} \sum_{m=1}^{\infty} |\mathcal{M}_{nm}(q)|^2 |\alpha_q|^2 \frac{1}{2\omega_q} \mathcal{F}_m(E - \omega_q).
$$

(4.17)

Eqs. (4.13) and (4.17) may be solved for $\mathcal{F}_n(E)$.

Let us compute the spin susceptibility $\chi(q,\omega)$, as defined in Eq. (1.3), and thus find the spin gap. Setting $\mathbf{r}_1$ equal to $\mathbf{r}_4$ and $\mathbf{r}_2$ equal to $\mathbf{r}_3$ in the particle-hole propagator $\mathcal{F}$, Eq. (4.13), and Fourier transforming yields the spin susceptibility. In terms of the magnetoexciton basis,
\[ \chi(Q, \omega) = \frac{2}{L^2} \sum_{n=1}^{\infty} |<0| \rho_Q | n Q>|^2 \mathcal{F}_n(\omega), \]  

where \(<0| \rho_Q | n Q>| is the density matrix element between the magnetoexciton \(|n Q>| and the ground state,

\[ <0| \rho_Q | n Q>| = \int d\mathbf{r}_1 \psi_0^0(\mathbf{r}_1, \mathbf{r}_1) e^{-i\mathbf{q} \cdot \mathbf{r}_1} = \frac{L}{\sqrt{2\pi n!}} \left( \frac{-i\mathbf{z}_\alpha}{\sqrt{2}} \right)^n e^{-b/2}, \]  

where \(b = \frac{1}{2}Q^2\) and \(z_\alpha = iQ_x - Q_y\). The resulting spin susceptibility is graphed in Fig. 2 and discussed in the introduction. This diagrammatic approach finds a spin gap \(\Delta = 0.32 \hbar \omega_c\).

The next section will apply a different computational technique to estimate the error bars of the spin gap calculation.

V. SECOND-ORDER PERTURBATION THEORY

The spin gap may also be investigated with second-order perturbation theory. We can thereby estimate the uncertainty arising from leaving out the Feynman diagrams containing crossed phonon lines. Since the coupling constant \(\alpha_q\) is \(\sqrt{\pi}\) in the natural units of the problem, processes involving multiple phonons have a noticeable effect, lowering the spin gap further. In fact, the lower spin gap energy given by second-order perturbation theory appears to be more accurate than the value arising from summing diagrams, as discussed at the end of Section VI. However, second-order perturbation theory cannot determine the full spin susceptibility spectrum, but only the spin gap.

Let us compute the spin gap, the lowest energy spin-one state, using second-order perturbation theory. For a system with a ground state \(|0>| and excited states \(|l>| possessing energies \(E_l^{(0)}\) according to a bare Hamiltonian \(\mathcal{H}_0\), an interaction \(\mathcal{H}_1\) shifts the ground state energy to the value

\[ E^{(2)} = E_0^{(0)} - \sum_l \frac{|<l| \mathcal{H}_1 |0>|^2}{E_l^{(0)} - E_0^{(0)}}. \]  

We will find the energy shift of the spin 1 particle-hole state \(|1 Q>|, which serves as the ground state \(|0>| in Eq. (5.1). The magnetoexciton \(|1 Q>|, defined by the wavefunction in
Eq. (4.2), has a spin-up particle in Landau level 1, a spin-down hole in Landau level 0, and total momentum $Q$. The relevant excited states $|l>$ consist of the phonon of momentum $q$ and the magnetoexciton $|m Q-q>$. 

Inserting these states and the interaction of $H_{\text{eff}}$, Eq. (2.1), into Eq. (5.1) leads to the spin gap, 

$$\Delta^{(2)} = \left(\epsilon^{(0)}_1 - \epsilon^{(0)}_0\right) - L^2 \int \frac{d\mathbf{q}}{(2\pi)^2} \frac{|\alpha_q|^2}{2\omega_q} \sum_{m=1}^{\infty} \frac{|<m Q-q| j^T_q |1 Q>|^2}{\epsilon^{(0)}_m - \epsilon^{(0)}_1 + \omega_q}. \quad (5.2)$$

Replacing the current matrix element in Eq. (5.2) with $M'_{m1}(q)$, as defined in Eq. (4.11), and using the values in Eq. (3.3) for $\epsilon^{(0)}_n$ yields the lowest energy of a spin-one excitation,

$$\Delta^{(2)} = 1 - L^2 \int \frac{d\mathbf{q}}{(2\pi)^2} \frac{|\alpha_q|^2}{2\omega_q} \sum_{m=1}^{\infty} \frac{|M'_{m1}(q)|^2}{m - 1 + \omega_q} = 1 - 0.83. \quad (5.3)$$

Thus, at the level of second-order perturbation theory, the phonon interaction reduces the spin gap to $0.17 \hbar\omega_c$, less than 20% of its mean-field value, compared to a spin gap of $0.32 \hbar\omega_c$ obtained by diagrammatic methods in Section IV.

Altering the sound speed $v_s$ in the phonon dispersion, Eq. (2.21), changes the energy gap significantly, increasing the error bars on the spin gap calculation. For instance, a 10% decrease in the sound speed to $v_s = 0.9$ reduces the second-order perturbation theory energy gap calculated in Eq. (5.3) to

$$\Delta^{(2)} v_s=0.9 = 1 - 0.96. \quad (5.4)$$

Since such a sound speed is plausible [9], we cannot set a lower bound on the size of the spin gap, or even prove that a spin gap exists. Still, the spin gap definitely lies in the range

$$\Delta = (0.2 \pm 0.2) \hbar\omega_c, \quad (5.5)$$

much less than the mean-field value of $\hbar\omega_c$.

VI. BOSON TEST CASE

We will compute the Green’s function of the non-interacting spinless bose gas in order to test the computational procedure used to find the spin gap of the spin $\frac{1}{2}$ fractional-
statistics gas. The same gauge transformation changing particles with fractional statistics into fermions obeying $\mathcal{H}_{\text{anyon}}$, Eq. (1.1), will also transform a gas of non-interacting bosons,

$$\mathcal{H}_{\text{bose}} = \sum_{j=1}^{N} \frac{|\mathbf{P}_j|^2}{2m},$$  \hspace{1cm} (6.1)

into $\mathcal{H}_{\text{anyon}}$ with a statistical factor $\nu = 0$, and with the cutoff parameter $\alpha \to \infty$. Since the bose energy eigenstates are known exactly, the bose Green’s function may also be computed exactly. Agreement between the exact and approximate bose Green’s functions demonstrates the validity of the semion calculation, because bose statistics involve a larger deviation from non-interacting fermions than do fractional statistics.

We shall compute the part of the zero-distance bose Green’s function,

$$G(E) = -i \int_{-\infty}^{\infty} e^{iEt} <\Phi | T\{\Psi(0, t) \Psi^\dagger(0, 0)\} | \Phi > dt ,$$  \hspace{1cm} (6.2)

arising from the occupied states, where $|\Phi >$ is the ground state. In the fermionic representation, $\Psi^\dagger(\mathbf{r})$ and $\Psi(\mathbf{r})$ are local fermionic operators creating or destroying a fermion at position $\mathbf{r}$. In the bosonic representation, however, those operators also multiply the wavefunction by a phase depending upon the positions of all the other particles, effectively creating or destroying a vortex. Thus, the Green’s function defined in Eq. (6.2) differs substantially from the traditional bose propagator. In terms of the excited states $|l >$ with energies $\epsilon_l$, the Green’s function has the value,

$$G(E) = \sum_l \left\{ \frac{|<l | \Psi^\dagger(0) | \Phi >|^2}{E - \epsilon_l + i\eta} + \frac{|<l | \Psi(0) | \Phi >|^2}{E + \epsilon_l - i\eta} \right\} .$$  \hspace{1cm} (6.3)

The imaginary part of the Green’s function corresponding to the occupied states is then,

$$\frac{1}{\pi} \Im G(E) = \sum_l |<l | \Psi(0) | \Phi >|^2 \delta(E + \epsilon_l) ,$$  \hspace{1cm} (6.4)

valid for $E < 0$, which are the only energies we shall consider.

The Green’s function calculated with the approximation techniques used to find the spin gap of the fractional-statistics gas agrees fairly well with the exact boson spectrum, as displayed in Fig. 12. We are using units in which $a_0$, Eq. (1.19), and $\omega_c$, Eq. (1.12),
defined with \( \nu = 0 \), are both set equal to one. The exact bose Green’s function, the solid line in Fig. 12, was found by substituting the exact bose wavefunctions and energies into Eq. (6.4). The approximate bose Green’s function, the dashed line in Fig. 12, was found by first determining the bose effective Hamiltonian in the form of \( H_{\text{eff}} \), Eq. (2.1). We then considered, in addition to the single-phonon exchange processes considered in Section 11, processes involving arbitrary numbers of crossed phonon interaction lines, such as the diagram in Fig. 13. The effect of such crossed phonon graphs in the fractional-statistics gas is discussed at the end of this section. The exact and approximate Green’s functions possess a similar overall structure. In addition, the highest energy hole states in the two calculations differ in energy by only \( 0.06 \hbar \omega_c \), demonstrating the reliability of the calculational technique used to compute the spin gap of the fractional-statistics gas.

Let us compute the bose Green’s function exactly. The system consists of \( N \) bosons confined to a region of size \( L^2 \), with an average density,

\[
\bar{\rho} = \frac{N}{L^2} = \frac{1}{2\pi}.
\]

The gauge transformation from \( H_{\text{bose}} \), Eq. (6.1), to \( H_{\text{anyon}} \), Eq. (1.1), multiplies the bose wavefunction by a phase depending upon the direction of the separation vector between each pair of particles, but leaves the excitation energies unchanged. In the fermionic representation, the \( N \) particle ground state \( |\Phi\rangle \) has the wavefunction,

\[
|\Phi\rangle = \frac{1}{L^N} \prod_{i<j}^N \left( \frac{z_i - z_j}{|z_i - z_j|} \right),
\]

where \( z = x + iy \) expresses a position as a complex number. The \( N-1 \) particle excited states \( |k_1, \ldots, k_{N-1}\rangle \) consist of plane waves, with energies

\[
\epsilon_{k_1,\ldots,k_{N-1}} = \sum_{j=1}^{N-1} \frac{1}{2} k_j^2
\]

and wavefunctions

\[
|k_1, \ldots, k_{N-1}\rangle = \prod_{i<j}^N \left( \frac{z_i - z_j}{|z_i - z_j|} \right) N^{(N-1)!} \sum_{\tau} e^{i[k_1 \cdot r_\tau(1) + \ldots + k_{N-1} \cdot r_{\tau(N-1)}]} \]

\[25\]
summed over permutations $\tau$. The normalization $\mathcal{N}$,

$$
\mathcal{N} = \frac{1}{L^{N-1} \sqrt{(N-1)! n_1! \cdots n_M!}},
$$

depends upon the momenta $k_j$ with an occupancy $n_j$ greater than one.

The bose Green’s function may then be calculated from the energy eigenstates. The matrix element in Eq. (6.4) coupling the ground state $|\Phi>\,$, Eq. (6.6), to the excited state $|k_1,\ldots,k_{N-1}>\,$, Eq. (6.8), has the magnitude,

$$
|<k_1,\ldots,k_{N-1} \mid \Psi(0) \mid \Phi>|^2 = \frac{\mathcal{N}^2}{L^{2N}} \prod_{j=1}^{N-1} \left( \frac{2\pi}{k_j^2} \right)^2.
$$

Substituting these matrix elements and the excitation energies of Eq. (6.7) into Eq. (6.4) leads to,

$$
\frac{1}{\pi} \text{Im} G(E) = \frac{1}{L^{2N}} (2\pi)^{2(N-1)} \sum_{\{k_1,\ldots,k_{N-1}\}} \frac{\mathcal{N}^2}{k_1^4 \cdots k_{N-1}^4} \delta \left( E + \sum_{j=1}^{N-1} \frac{1}{2} |k_j|^2 \right) = \frac{1}{L^{2N}} \int_{k_1^4 \cdots k_{N-1}^4} \delta \left( E + \sum_{j=1}^{N-1} \frac{1}{2} |k_j|^2 \right) \, dk_1 \cdots dk_{N-1}.
$$

Changing the sum into an integral generates a combinatoric factor cancelling out a similar term in the normalization $\mathcal{N}$. To avoid an infrared divergence, we will cut off the momentum at a value $k_0$, requiring a normalization of $\text{Im} G(E)$ consistent with the density $\bar{\rho}$,

$$
\bar{\rho} = \int_{-\infty}^{0} \frac{1}{\pi} \text{Im} G(E) \, dE = \frac{1}{L^{2N}} \int_{k_j>k_0} \frac{1}{k_1^4 \cdots k_{N-1}^4} \, dk_1 \cdots dk_{N-1},
$$

so that

$$
k_0 = \frac{\sqrt{\pi}}{L}.
$$

With the cutoff, one can take the Fourier transform of $\text{Im} G(E)$,

$$
\int_{-\infty}^{0} e^{iEt} \frac{1}{\pi} \text{Im} G(E) \, dE = \frac{1}{L^{2N}} \left( \int_{k_j>k_0} \frac{1}{k_1^4} e^{\frac{i}{2}tk^2} \, dk \right)^{N-1}
\begin{align*}
&\quad = \bar{\rho} \left( \left( \int_{\frac{1}{2}k_0^2}^{\infty} \frac{1}{\epsilon^2} e^{it\epsilon} \, d\epsilon \right) / \left( \int_{\frac{1}{2}k_0^2}^{\infty} \frac{1}{\epsilon^2} \, d\epsilon \right) \right)^{N-1} \\
&\quad = \bar{\rho} \left\{ \left[ \frac{1}{2}k_0^2 - it(\gamma - 1) - it \ln \left( -it \frac{1}{2}k_0^2 \right) \right] \frac{1}{2}k_0^2 \right\}^{N-1}.
\end{align*}
$$
Transforming back to $E$ yields the exact hole Green’s function,

$$
\frac{1}{\pi} \text{Im} G_{\text{exact}}^\text{exact}(E) = \frac{\bar{\rho}}{2\pi} \int_{-\infty}^{\infty} e^{(i/4)t[\ln|t|+1]} e^{-(\pi/8)|t|} e^{-i(E-e_{0}^{\text{exact}})t} \, dt,
$$

where

$$
e_{0}^{\text{exact}} = -\frac{1}{2} \ln(R) - \frac{1}{4} [\ln(2) - \gamma],
$$

with $L^2 = \pi R^2$ and $\gamma$ denoting Euler’s constant. The exact Green’s function is graphed as the solid line in Fig. 12.

Let us now compute the hole propagator of the bosons with a technique similar to that used in Section III for semions, in order to test our approximation method in comparison with the exact result $G_{\text{exact}}^\text{exact}(E)$, Eq. (6.15). Some changes are necessary in the bose case. In particular, the collective mode of the bose gas has a quadratic dispersion,

$$
\omega_q = \frac{1}{2} q^2.
$$

(6.17)

Even though the RPA calculation of Section II for a bose gas obtains a linear dispersion, we will use the quadratic dispersion, which would arise if all Feynman diagrams were included. However, the effective interaction $\mathcal{V}$ between two bosons in the fermionic representation will still be approximated by Eq. (2.3), which was used in Section II to compute the effective interaction of the fractional-statistics gas. In that formula, the bare potential $\mathcal{V}^0$ for bosons is twice as large as the semion case value, Eq. (2.5). We will again approximate the effective interaction as the bare potential $\mathcal{V}^0$ added to a sharp mode at each momentum $q$ in the transverse current-transverse current channel, as in Eq. (2.14). The polarization bubble $\mathcal{D}$ of the bose gas, defined in Eq. (2.7), satisfies the sum-rule argument in Section II, resulting in a coupling strength of

$$
|\alpha_q|^2 = \frac{2\pi}{L^2}.
$$

(6.18)

The static interaction $\mathcal{V}^0$ moves each Landau level $n$ from the bare energy $\epsilon_n^{(0)}$, Eq. (3.3), to its Hartree-Fock energy $\epsilon_n^{\text{HF}}$, which for Landau level 0 has the value
\[ \epsilon_{0}^{\text{HF}} = -\frac{1}{2} \ln (R) + \frac{1}{4} [\ln (2) - \gamma] \] (6.19)

for a sample of size \( L^2 = \pi R^2 \). In summary, the resulting system is described by the effective Hamiltonian \( \mathcal{H}_{\text{eff}} \), Eq. (2.1), with the phonon energy \( \omega_q \) given by Eq. (6.17) and the coupling \( \alpha_q \) given by Eq. (6.18). The energies corresponding to Landau level 0 should be shifted by the amount \( \epsilon_{0}^{\text{HF}} - \epsilon_{0}^{(0)} \).

Because the phonons have a quadratic dispersion, processes with crossed phonon lines, such as those in Fig. 13, make a significant contribution to the bose spectrum. The solution to Dyson’s equation for the exchange of a single phonon, Eqs. (3.11) and (3.14) does not determine the propagator accurately. Instead, we will sum the Feynman diagrams with any number of crossed phonon lines. In the last paragraph of this section, we will discuss how such multi-phonon graphs affect the fractional-statistics gas. Since level 0 is the only occupied Landau level, we shall only consider graphs where the hole lies in Landau level 0 in the initial, final, and all intermediate states.

To find the bose Green’s function, a single Landau level interacting with phonons will be treated as a single quantum state interacting with phonons. The effective Hamiltonian \( \mathcal{H}_{\text{eff}} \), Eq. (2.1), representing the bose gas in the fermionic representation, Eq. (1.1), will in turn be transformed into the discrete state Hamiltonian

\[ \mathcal{H}_{\text{single}} = \sum_{q} \frac{1}{2} \omega_q \left( a_{q} a_{q}^{\dagger} + a_{q}^{\dagger} a_{q} \right) + \sum_{q} \lambda_q \Psi^{\dagger} \Psi \left( a_{q} + a_{-q}^{\dagger} \right), \] (6.20)

similar to the Hamiltonian \( \mathcal{H}_{\text{discrete}} \), Eq. (4.12), utilized to find the particle-hole propagator of the fractional-statistics gas in Section IV. The operators \( \Psi^{\dagger} \) and \( \Psi \) create and destroy a fermion which may occupy a single quantum state, \( a_{q}^{\dagger} \) and \( a_{q} \) create and destroy a phonon, and the fermion-phonon coupling \( \lambda_q \) has the value

\[ |\lambda_q|^2 = \frac{\left|\alpha_q\right|^2}{2 \omega_q} |\mathcal{M}_{00}(q)|^2 = \frac{\pi}{2 L^2} e^{-\frac{1}{2} r^2}, \] (6.21)

where \( \omega_q \) and \( \alpha_q \) have the bosonic values of Eqs. (6.17) and (6.18) and \( \mathcal{M} \) is defined in Eq. (3.6). The Green’s function of \( \mathcal{H}_{\text{single}} \),
\[ G^{\text{single}}(t_1, t_2) = -i < \Phi \mid T\{\Psi(t_1) \Psi^\dagger(t_2)\} \mid \Phi > = \int_{-\infty}^{\infty} G(E) e^{iE(t_2-t_1)} \, dE , \] (6.22)

where \( |\Phi> \) is the vacuum state, may be converted into an approximate zero-distance bose Green’s function defined in Eq. (6.2),

\[ \text{Im} \, G^{\text{approx}}(E) = -\bar{\rho} \text{Im} \, G^{\text{single}}(\epsilon_{0} - E) . \] (6.23)

The differences between a discrete quantum state and a hole in any orbital of a filled Landau level account for the changes in sign and normalization. For processes involving the exchange of a single phonon, the self-energy expression for \( H^{\text{single}} \) agrees with the single-hole self-energy formula Eq. (3.12) with \( n \) and \( m \) both set to Landau level 0. For the process involving two crossed phonon lines, as in Fig. 13, the spatial integral due to the Hamiltonian \( H^{\text{eff}} \), Eq. (2.1), reduces to the sum of the matrix elements of transverse-current operator \( j^T_q \), Eq. (1.24), between the Landau level orbitals in Eq. (3.2),

\[ \sum_{pp'p''} < 0 k' \mid j^T_{-q'} \mid 0 p>< 0 p' \mid j^T_{-q} \mid 0 p''>< 0 p'' \mid j^T_q \mid 0 k> = \delta_{kk'} e^{i(q' \times q) \cdot \hat{z}} |M_{00}(q)|^2 |M_{00}(q')|^2 , \] (6.24)

Since small momentum phonons have the dominant effect, the phase \( e^{i(q' \times q) \cdot \hat{z}} \) may be neglected. Then the two-phonon Feynman diagrams for both \( H^{\text{eff}} \) and \( H^{\text{single}} \) have the same self-energy expression,

\[ \Sigma(E)_{(2 \text{ phonon})} = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d\mathbf{q}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int \frac{d\mathbf{q}'}{(2\pi)^2} \times |M_{00}(q)|^2 |M_{00}(q')|^2 \left( \frac{|\alpha_p|^2}{(\omega + i\eta)^2 - \omega_q^2 (\omega' + i\eta)^2 - \omega_{q'}^2} \right) \times G_0(E + \omega) G_0(E + \omega + \omega') G_0(E + \omega') . \] (6.25)

This agreement may be generalized to Feynman diagrams with any number of crossed phonon lines. Thus, we may study \( H^{\text{single}} \) to find the hole Green’s function of the bose gas in the fermionic representation.
The Hamiltonian $H_{\text{single}}$, Eq. (6.20), whose Green’s function is readily converted into the zero-distance Bose green’s function, may be solved exactly by transforming the phonons, treated as harmonic oscillators, into the conjugate operators $x_q$ and $P_q$ at each momentum $q$. The phonon creation and annihilation operators $a_q^\dagger$ and $a_q$ are the standard linear combinations of $x_q$ and $P_q$,

$$a_q = \frac{1}{\sqrt{2\omega_q}} \left( \sqrt{K_q} x_q + \frac{iP_q}{\sqrt{m_q}} \right),$$

(6.26)

and

$$a_q^\dagger = \frac{1}{\sqrt{2\omega_q}} \left( \sqrt{K_q} x_q - \frac{iP_q}{\sqrt{m_q}} \right),$$

(6.27)

where $K_q$ and $m_q$ satisfy

$$\omega_q = \sqrt{\frac{K_q}{m_q}}.$$  \hspace{1cm} (6.28)

The Hamiltonian $H_{\text{single}}$ may then be reexpressed as,

$$H_{\text{single}} = \sum_q \left[ \frac{1}{2} K_q x_q^2 + \frac{P_q^2}{2m_q} \right] + \sum_q \lambda_q \sqrt{\frac{2K_q}{\omega_q}} x_q \Psi^\dagger \Psi$$

$$= \sum_q \left[ \frac{1}{2} K_q \left( x_q + \lambda_q \sqrt{\frac{2}{K_q \omega_q}} \Psi^\dagger \Psi \right)^2 \right] + \Delta E \Psi^\dagger \Psi.$$ \hspace{1cm} (6.29)

When the discrete quantum state is occupied, its interaction shifts the harmonic oscillators displacements and lowers the total energy by

$$\Delta E = -\sum_q \frac{\lambda_q^2}{\omega_q}.$$ \hspace{1cm} (6.30)

The vacuum state is then

$$|\Phi> = \prod_q e^{-\frac{1}{2} (x_q/d_q)^2},$$ \hspace{1cm} (6.31)

with a length scale

$$d_q = (K_q m_q)^{-1/4}.$$ \hspace{1cm} (6.32)

A relevant excited state contains the discrete state and $n_q$ phonons of each momentum $q$, with the shifted harmonic oscillator wavefunction
\[ |n_q\rangle = \Psi^\dagger \prod_q \left[ 2^n q_n! \sqrt{\pi d_q} \right]^{-1/2} H_{n_q} \left( \frac{1}{d_q} \left[ x_q + \lambda q \sqrt{\frac{2}{K_q \omega_q}} \right] \right) e^{-\frac{1}{2}(x_q/d_q)^2}, \quad (6.33) \]

where the \( H_n \) are Hermite polynomials, and an energy,

\[ \epsilon_{n_q} = \Delta E + \sum_q n_q \omega_q. \quad (6.34) \]

The excited states \( |n_q\rangle \), Eq. (6.33), and energies \( \epsilon_{n_q} \), Eq. (6.34), determine the imaginary part of the Green’s function, defined in Eq. (6.22),

\[ -\frac{1}{\pi} \text{Im} G^{\text{single}}(E) = \sum_{\{n_q\}} |<n_q| \Psi^\dagger |\Phi\rangle|^2 \delta(E - \epsilon_{n_q}). \quad (6.35) \]

The overlap between \( \Psi^\dagger |\Phi\rangle \) and the excited state \( |n_q\rangle \) has a magnitude,

\[ |<n_q| \Psi^\dagger |\Phi\rangle|^2 = \prod_q \frac{\epsilon_{n_q}^{n_q}}{n_q!} e^{-c_q}, \quad (6.36) \]

where \( c_q \) stands for,

\[ c_q = \left( \frac{\lambda q}{\omega_q} \right)^2 = \frac{2\pi}{L^2 q^4} e^{-\frac{1}{2} q^2}. \quad (6.37) \]

The energy shift \( \Delta E \) in Eq. (6.30) diverges due to phonons of small momentum. However, the state \( \Psi^\dagger |\Phi\rangle \) has an expected energy of

\[ <H_{\text{single}}> = \Delta E + \sum_q <n_q> \omega_q = 0, \quad (6.38) \]

where \( <O> \) is shorthand for \( <\Phi|O|\Psi\Psi^\dagger \Phi> \). Since on average the excited phonons cancel the energy shift, the energy of a state depends upon the fluctuations,

\[ \epsilon_{n_q} = \sum_q (n_q - <n_q>) \omega_q. \quad (6.39) \]

Substituting Eq. (6.30) and Eq. (6.39) into Eq. (6.35) and taking the Fourier transform leads to

\[ \int_{-\infty}^{\infty} e^{-iEt} \left( -\frac{1}{\pi} \text{Im} G^{\text{single}}(E) \right) = \sum_{\{n_q\}} \prod_q \frac{\epsilon_{n_q}^{n_q}}{n_q!} e^{-c_q} e^{-i(n_q-c_q)\omega_q t} \]

\[ = \exp \left\{ \sum_{q} c_q \left( e^{-it\omega_q} - 1 + it\omega_q \right) \right\} \]

\[ = e^{(1/4) [(1+it) \ln(1+it) - it]} \quad (6.40) \]
where the imaginary part of the logarithm lies in the range,
\[-\frac{\pi}{2} < \text{Im} \left( \ln (1 + it) \right) < \frac{\pi}{2} . \tag{6.41}\]

After Fourier transforming back to energy dependence and shifting the \( t \) contour in the complex plane, the Green’s function of \( H_{\text{single}} \) becomes,
\[-\frac{1}{\pi} \text{Im} G_{\text{single}}^{\text{approx}}(E) = \frac{1}{2\pi} e^{1/4 - E} \int_{-\infty}^{\infty} e^{(i/4)t[\ln |t| - 1]} e^{-\left(\pi/8\right) |t|} e^{iEt} dt . \tag{6.42}\]

The approximate zero-distance Green’s function for the hole states, after applying Eq. (6.23),
\[
\frac{1}{\pi} \text{Im} G_{\text{approx}}(E) = \frac{\bar{D}}{2\pi} e^{1/4 + (E - \epsilon_{0}^{\text{HF}})} \int_{-\infty}^{\infty} e^{(i/4)t[\ln |t| - 1]} e^{-\left(\pi/8\right) |t|} e^{-i(E - \epsilon_{0}^{\text{HF}})t} dt , \tag{6.43}\]
is plotted in Fig. 12 as a dashed line, and displays good agreement with the exact spectrum \( G_{\text{exact}}(E) \), Eq. (6.15).

The calculation of the spin gap in Section IV left out the Feynman diagrams with crossed phonon lines. The method used in this section to deal with those graphs fails when Landau level excitations are involved. Although the higher Landau states affect the spin susceptibility, interactions involving only Landau levels 0 and 1 cause a majority of the shift of the spin gap. If the Landau levels \( n \geq 2 \) are neglected, the preceding analysis of the fractional-statistics lowers the gap by the amount \( \Delta E \) in Eq. (6.30). Then, since \( \Delta E \) is simply the second-order perturbation theory result found in Section V, second-order perturbation theory gives the same spin gap as does including all the Feynman diagrams, for the Landau levels causing most of the gap’s shift. Thus, the spin gap found by second-order perturbation theory is more accurate than the diagrammatic calculation in Section IV.

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FIGURES

FIG. 1. Illustration of the mean-field density of states of the spin $\frac{1}{2}$ fractional-statistics gas. Landau level 0 is occupied with particles of both spins.

FIG. 2. Imaginary part of the spin susceptibility $\chi(q, \omega)$, defined in Eq. (1.3), at three momenta: (a) $Q = 0.2/a_0$, (b) $Q = 0.4/a_0$, and (c) $Q = 0.6/a_0$, as calculated using Eq. (4.18). The vertical lines (d) display the delta-function mean-field susceptibility at $Q = 0.6/a_0$.

FIG. 3. Illustration of decay processes causing broadening of the single-particle spectrum: (a) interlevel transitions and (b) intralevel transitions. Each decay process emits a phonon.

FIG. 4. Diagrammatic representation of the effective interaction: (a) exact expression in terms of the proper polarization bubble $D^P$, as in Eq. (2.8), and (b) RPA approximate expression in terms of the bare polarization bubble $D^0$, as in Eq. (2.3).

FIG. 5. Imaginary part of $V_{JJ}$, the transverse current-transverse current component of the RPA effective interaction given in Eq. (2.12), at three momenta: (a) $q = 0.5/a_0$, (b) $q = 1/a_0$, and (c) $q = 4/a_0$. The solid lines are broadened with a small $\eta$ for visibility, while the dashed lines of (b) and (c) are broadened with a larger $\eta$ to resemble the full effective interaction. The collective mode energy $\omega_q$ is defined in Eq. (2.16) to be the average value of $\omega$, and the coupling $|\alpha_q|^2$ defined in Eq. (2.17) is proportional to the integrated interaction strength at a given $q$.

FIG. 6. Collective mode dispersion $\omega_q$. The solid line is the averaged RPA interaction defined by Eq. (2.16), and the dashed line is the approximation formula given in Eq. (2.20).

FIG. 7. Typical Feynman diagram involving the static interaction, the $V^0$ of Eq. (2.14), which as discussed at the end of Section II does not affect the spin gap.

FIG. 8. Illustration of Dyson’s equation due to phonon exchange for the single-particle Green’s function $G$, as expressed in Eqs. (3.9) to (3.12). The labels on the particle lines refer to the Landau level orbitals given in Eq. (3.2).
FIG. 9. Contour used to evaluate the $\omega$ integral in the self-energy expression $\Sigma_n(E)$, Eq. (3.12), for Landau levels $n \geq 1$ and $m \geq 1$.

FIG. 10. Density of states $D_n(E)$, as defined in Eq. (3.15), for Landau levels 0 to 3. The bare Landau level energies $\epsilon_n^{(0)}$ given in Eq. (3.3) are indicated on the top of the graph. Neglecting the particle-hole interactions, the spin gap is the difference between the maximum hole energy $E_{\text{max},h}$ and the minimum particle energy $E_{\text{min},p}$. As discussed following Eq. (3.15), the calculation leaves out the divergent gap separating the occupied and unoccupied states.

FIG. 11. Diagrammatic sum used to compute the spin one particle-hole propagator $F$, defined in Eq. (4.1). The particle-hole pair representation is transformed into a discrete system obeying $\mathcal{H}_{\text{discrete}}$, Eq. (4.12). The indices on the dashed lines refer to either the particle Landau level in the magnetoexciton wavefunctions given in Eq. (4.2) or the states of $\mathcal{H}_{\text{discrete}}$.

FIG. 12. Green’s function $G$, defined in Eq. (6.2), for the holes states of the non-interacting bose gas in the fermionic representation. The solid line denotes the exact spectrum $G^{\text{exact}}(E)$, given by Eq. (6.13), and the dashed line denotes the approximate spectrum $G^{\text{approx}}(E)$, given by Eq. (6.43). The energy $E = 0$ on the graph corresponds to $\epsilon_0^{\text{HF}}$, the Hartree-Fock energy of Landau Level 0 given in Eq. (6.19).

FIG. 13. Illustration of a process involving two crossed phonon interaction lines included in the bose gas calculation, as expressed in Eq. (6.25). The hole propagators are labeled by the Landau level orbitals defined in Eq. (3.2).
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