Dynamics of \( f(R) \)-cosmologies containing Einstein static models

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Abstract
We study the dynamics of homogeneous isotropic FRW cosmologies with positive spatial curvature in \( f(R) \)-gravity, paying special attention to the existence of Einstein static models and only study forms of \( f(R) = R^n \) for which these static models have been shown to exist. We construct a compact state space and identify past and future attractors of the system and recover a previously discovered future attractor corresponding to an expanding accelerating model. We also discuss the existence of universes which have both a past and a future bounce, a phenomenon which is absent in general relativity.

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1. Introduction

After more than one hundred years, Einstein’s theory of general relativity still remains our best description of gravity and has up to now survived the scrutiny of a multitude of tests, most of which have been on solar system scales. However, in the last decade, high precision astrophysical and cosmological observations appear to suggest that general relativity might be incomplete. In particular, cosmological data indicate an underlying cosmic acceleration of the universe that cannot be recast in the framework of general relativity without resorting to additional exotic matter components (known as dark energy), which have not yet been directly observed. Several models have been proposed [1] in order to address this problem and at the moment, the one which appears to fit all available observations (supernovae Ia [2], cosmic microwave background anisotropies [3], large-scale structure formation [4], baryon oscillations [5], weak lensing [6]), turns out to be the so-called concordance model in which a tiny cosmological constant is present [7] and ordinary matter is dominated by cold dark matter (CDM). However, despite its success, the \( \Lambda \)-CDM model is affected by significant fine-tuning
problems related to the vacuum energy scale, so it is important to investigate other viable theoretical schemes.

Currently, one of the most popular alternatives to the *concordance model* is based on modifications of standard Einstein gravity. Although modifications of Einstein’s theory of gravity were already proposed in the early years after the publication of general relativity, a detailed investigation of cosmological models within this framework only got underway a few years ago. Such models became popular in the 1980s because it was shown that they naturally admit a phase of accelerated expansion which could be associated with an early universe inflationary phase [8]. The fact that the phenomenology of dark energy requires the presence of a similar phase (although only a late time one) has recently revived interest in these models. In particular, the idea that dark energy may have a geometrical origin, i.e., that there is a connection between dark energy and a non-standard behavior of gravitation on cosmological scales is currently a very hot topic of research.

Among these models the so-called extended theory of gravitation (ETG) and, in particular, *higher-order theories of gravity* (HOG) [9] have provided a number of extremely interesting results on both cosmological [10, 11, 13, 14] and astrophysical [13, 15] scales. These models are based on gravitational actions which are nonlinear in the Ricci curvature $R$ and/or contain terms involving combinations of derivatives of $R$ [16–18]. One of the nice features of these theories is that the field equations can be recast in a way that the higher-order corrections are written as an energy–momentum tensor of geometrical origin describing an ‘effective’ source term on the right-hand side of the standard Einstein field equations [10, 11]. In this curvature quintessence scenario, the cosmic acceleration can be shown to result from such a new geometrical contribution to the cosmic energy density budget, due to higher-order corrections of the Hilbert–Einstein Lagrangian.

Unfortunately the analysis of HOG is complicated by the fact that the resulting fourth-order field equations are highly nonlinear and apart from a few exact solutions [12], finding cosmological solutions by solving the field equations directly has proved to be extremely difficult. This problem has been eased somewhat by using the theory of dynamical systems which has over the last few years proved to be a powerful scheme for investigating the physical behavior of such theories (see, for example, [19–27]). In fact, studying cosmologies using the dynamical systems approach has the advantage of providing a relatively simple method for obtaining exact solutions (even if these only represent the asymptotic behavior) and obtain a (qualitative) description of the global dynamics of these models. Consequently, such an analysis allows for an efficient preliminary investigation of these theories, suggesting which background models deserve further investigation and this provides a starting point for the analysis of the growth of structure in HOG [28, 29].

A remarkable feature of the background dynamics of $f(R)$ gravity is that there are a number of classes of these theories that admit a Friedmann transient matter-dominated decelerated expansion phase, followed by one with an accelerated expansion rate [19, 30]. The first phase provides a setting during which structure formation can take place and this is followed by a smooth transition to a dark energy like era which drives the cosmological acceleration. It would therefore be of great interest if orbits could be found in the phase space of $f(R)$ models that connect such a Friedmann matter dominated phase to an accelerating phase via an Einstein static solution in a way which is indistinguishable (at least at the level of the background dynamics) to what occurs in the $\Lambda$CDM model of general relativity [31]. If a similar evolution exists in an $f(R)$ model, we would expect the associated Einstein static solution to be a saddle point (as it is in general relativity) and consequently unstable. It therefore follows that a careful examination of the existence and stability of the Einstein static model in the more general setting of modified gravity is of critical importance.
In general relativity, the issue of stability of the Einstein static model has been studied several times since the classic paper by Eddington [32] in the 1930s, where it was shown that such models are unstable with respect to homogeneous and isotropic perturbations, exactly the feature that allows a transition between a decelerated expansion era to one which is accelerating. However, later work by Harrison [33] and Gibbons [34], which extended Eddington’s work, considered generic inhomogeneous and anisotropic perturbations of an Einstein static model filled with a perfect fluid and found that provided the sound speed satisfies $c_s^2 > \frac{1}{3}$, these models are neutrally stable with respect to such perturbations. The reason for this ‘non-Newtonian’ stability stems from the fact that in general relativity, the Einstein static universe is spatially closed, and therefore has a maximum scale associated with it, which is greater than the largest physical scale of the perturbations. Since Jean’s length is a significant fraction of this maximum scale, perturbations in the fluid oscillate, rather than grow, leading to the conclusion that in general relativity at least, the Einstein static solution is neutrally stable with respect to such perturbations. This result was also found more recently by Barrow et al [35]. Also a comprehensive dynamical description including the asymptotic behavior of models in the neighborhood of Einstein static model was done in [36].

Recently [38, 39] the existence and stability of the Einstein static models in the more general setting of $f(R)$ gravity was examined. It was found that only one class of $f(R)$ theories admits an Einstein static model, and that this class is neutrally stable with respect to vector and tensor perturbations for all equations of state on all scales. Scalar perturbations are only stable on all scales if the matter fluid equation of state satisfies $c_s^2 > \frac{\sqrt{5} - 1}{6} \approx 0.21$. This result is remarkably similar to the GR case discussed above.

In this paper, we use the theory of dynamical systems and numerics to examine in detail the background evolution of this special class of $f(R)$ theories. This analysis compliments the work in [39] and provides a more complete picture of how the Einstein static solution fits into the overall structure of the phase space of solutions in $f(R)$ gravity.

The following conventions will be used in this paper: the metric signature is $(- + + +)$; Latin indices run from 0 to 3; units are used in which $c = 8\pi G = 1$.

2. The field equations

For homogeneous and isotropic spacetimes, a general form of the action for fourth-order gravity is given by

$$ A = \int d^4x \sqrt{-g} [f(R) - 2\Lambda + L_m], $$

where $L_m$ is the Lagrangian of the matter fields. The fourth-order field equations can be obtained by varying 1:

$$ G_{ab} + g_{ab} \frac{\Delta}{f'} = T^m_{ab} + T^R_{ab} = T^T_{ab}, $$

where primes denote derivatives with respect to $R$. Here $T^T_{ab}$ is the total effective energy–momentum tensor composed of the ordinary matter energy momentum tensor $T^m_{ab}$ and the correction term $T^R_{ab}$ (often referred to as the ‘curvature fluid’):

$$ T^R_{ab} = \frac{1}{f'} \left[ \frac{1}{2} g_{ab}(f - Rf') + f'_{,cd}(g^{cd}g_{ab} - g^{cd}g_{ab}) \right]. $$

As shown in [39], if we wish to keep the extra degrees of freedom in fourth-order gravity, the existence of an Einstein static universe imposes a strong constraint on the function $f(R)$, which was found to be

$$ f(R) = 2\Lambda + KR^{\frac{1}{w}(1+w)}, \quad w \neq -1. $$
Here we assume that the standard matter is a barotropic perfect fluid with equation of state
\[ p_m^m = w \rho_m^m, \] (5)
where \( \rho_m^m \) and \( p_m^m \) are the standard matter density and pressure, respectively. The effective field equations will look like those for \( R^n \)-gravity, but subject to the constraint
\[ n = \frac{3}{2}(1 + w), \] (6)
which guarantees the existence of an Einstein static solution. This means that the equation of state is fixed as a function of \( n \), or in other words: for a given value of \( n \) we can only find Einstein static solutions if the equation of state satisfies (6). As we know, in order to satisfy all the energy conditions for standard matter, \( w \) has to take values in \([-1/3, 1]\). We will therefore only consider the range \( n \in [1, 3] \), and in particular will analyze in detail the cases of dust \((n = 3/2, w = 0)\) and radiation \((n = 2, w = 1/3)\). In what follows we will use equation (6) to eliminate \( w \) from the field equations.

For the homogeneous and isotropic spacetimes, the independent field equations for \( R^n \)-gravity are (with \( n \) given by (6)):

- The Raychaudhuri equation
  \[ \Theta + \frac{1}{3} \Theta^2 - \frac{1}{2n} R - (n - 1) \frac{\dot{R}}{R} \Theta + \frac{\rho_m^m}{n R^{n-1}} = 0, \] (7)
  where \( \Theta \) is the volume expansion which defines a scale factor \( a(t) \) along the fluid flow lines via the standard relation \( \Theta = 3 \dot{a}/a \).
- The Friedmann equation
  \[ \Theta^2 + \frac{3}{2} \ddot{R} = -3(n - 1) \frac{\dot{R}}{R} \Theta + \frac{3(n - 1)}{2n} R + \frac{3 \rho_m^m}{n R^{n-1}}, \] (8)
  where \( \ddot{R} \equiv (6\kappa)/a^2 \) is the curvature of the 3-spaces and \( \kappa \in (1, 0, -1) \) denotes closed, flat and open universes, respectively.
- The trace equation
  \[ \frac{\dot{R}}{R} = \frac{1}{3} \frac{(n - 2)}{n(n - 1)} R - \frac{n - 2}{2} \frac{\dot{R}}{R} \Theta + \frac{2}{3} \frac{(2 - n)}{n(n - 1)} \rho_m^m. \] (9)
- The conservation equation for standard matter and the propagation equation for the 3-curvature are
  \[ \dot{\rho}_m^m = -\frac{2}{3} n \rho_m^m \Theta, \quad \dot{\tilde{R}} = -\frac{2}{3} \ddot{R} \Theta \] (10)
  respectively.

Combining the Raychaudhuri and Friedmann equations, we get
\[ R = 2 \dot{\Theta} + \frac{4}{3} \Theta^2 + \ddot{R}, \] (11)
which is equivalent to the general definition of the Ricci scalar in terms of the scale factor in a FRW spacetimes, given by
\[ R = 6 \left( \frac{\dot{a}}{a} + \frac{a^2}{a^2} + \frac{\kappa}{a^2} \right). \] (12)

For our analysis of the dynamical system, which will be presented in the following section, it is useful to complete the square of equation (8) and rewrite the Friedmann equation as
\[ \left( \Theta + \frac{3(n - 1)}{2} \frac{\ddot{R}}{R} \right)^2 + \frac{3}{2} \dot{R} = \frac{9(n - 1)^2}{4} \frac{\dot{R}^2}{R^2} + \frac{3(n - 1)}{2n} R + \frac{3 \rho_m^m}{n R^{n-1}}. \] (13)
We can easily see that all the additive terms on both sides of the equation above are strictly non-negative for all spacetimes with non-negative Ricci scalar $R$ and 3-curvature $\tilde{R}$. This will help us to compactify the full FRW state space for non-negative spatial curvature and Ricci scalar, and thus allow us to analyze the dynamics of the system in a compact framework.

### 3. Dynamics of non-negative curvature FRW spacetime

We now study the dynamics of FRW models with non-negative spatial curvature and Ricci scalar, which include the Einstein static universe. In order to convert the equations above into a system of autonomous first-order differential equations, we define the following set of normalized variables:

$$
x = \frac{3\dot{R}}{2RD} (n-1), \quad y = \frac{3R}{2nD^2} (n-1), \quad z = \frac{3\rho_m}{nR^{n-1}D^2},
$$

$$
K = \frac{3\tilde{R}}{2D^2}, \quad Q = \Theta D,
$$

(14)

together with the normalized time variable

$$
\tau' = \frac{d}{dt} = \frac{1}{D} \frac{d}{d\tau},
$$

(15)

Here the normalization $D$ is chosen as

$$
D = \sqrt{\left(\frac{\Theta}{3} + \frac{3(n-1)\tilde{R}}{2R} \right)^2 + \frac{3}{2} \tilde{R}}
$$

(16)

in order to compactify the variables as shown below. In terms of these variables, the Friedmann equation (13) becomes

$$
x^2 + y + z = 1, \quad y, z \geq 0,
$$

(17)

and from the definition of the normalization we get another constraint

$$
(Q + x)^2 + K = 1, \quad K \geq 0.
$$

(18)

The above constraints show that the variables must be compact and take the following ranges:

$$
x \in [-1, 1], \quad y \in [0, 1], \quad z \in [0, 1], \quad Q \in [-2, 2], \quad K \in [0, 1].
$$

(19)

Using the five compact variables together with the two constraints, we reduce the complete dynamical system to a three-dimensional one, and the cosmological equations become equivalent to the autonomous system

$$
Q' = [(3-n)x^2 - n(y - 1) - 1] \frac{Q^2}{3} + [(3-n)x^2 - n(y - 1) + 1] \frac{Qx}{3}
$$

$$
+ \frac{1}{3} \left[ x^2 - 1 + \frac{ny}{n-1} \right],
$$

$$
y' = \frac{2yx^2}{3}(3-n)(x + Q) + \frac{2xy}{3} \left[ \frac{n^2 - 2n + 2}{n-1} - ny \right] + \frac{2}{3} Qny(1-y),
$$

$$
x' = \frac{x^3}{3}(3-n)(Q + x) + \frac{x^2}{3} [n(2 - y) - 5] + \frac{Qx}{3} [n(1 - y) - 3] + \frac{1}{3} \left[ \frac{n(n-2)}{n-1} y - n + 2 \right].
$$

(20)

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3 It is important to note that this choice of variables will exclude general relativity, i.e., the case of $n = 1$. See [31] for the dynamical systems analysis of the corresponding cosmologies in GR.
4. Local analysis of the dynamical system

The complete description of all the equilibrium points of the dynamical system described by the above equations is given in table 1. We note that the points $A$ and $B$ correspond to the similarly labeled points in [40]. The line $LC$ has been labeled as such since it includes the point $C$ found in [40] in the flat limit $|Q + x| = |(2 - n)Q| \to 1$. Strictly speaking the system (20) has an additional equilibrium point, but its coordinates only lie in the state space for $n > 3$, which for us corresponds to an unrealistic equation of state $w > 1$. Note that the subscript $\epsilon$ only labels expanding ($\epsilon = +1$) and collapsing ($\epsilon = -1$) models for the point $A$. Since the other points correspond to asymptotically Minkowski spaces, the subscript merely labels the different limits from which these asymptotic solutions were obtained. The line $LC$ has an expanding branch $LC^{exp}$ with $Q > 0$, a collapsing branch $LC^{coll}$ with $Q < 0$, and in between the Einstein static model $ES$, with $Q = 0$. The stability of the equilibrium points and the line is summarized in table 2. We will now briefly discuss several interesting features of these equilibrium points:

(i) The position of the equilibrium points $N_\epsilon$ in the state space does not depend on the equation of state for the standard matter. In fact, the existence of these points is a generic feature of $R^2$ gravity, and they form a global source ($\epsilon = +1$) or sink ($\epsilon = -1$) in the state space considered here. This implies that for any equation of state, there always exists the possibility of the universe originating from a vacuum Minkowski model in the infinite past, or evolving toward a vacuum Minkowski model in the infinite future.

(ii) The position of the equilibrium points $L_\epsilon$ also does not depend on the equation of state for standard matter. However, the stability properties of these points do depend on the equation of state. For matter with high negative pressure ($-1/3 \leq w \leq -1/6$), or for very stiff matter ($2/3 \leq w \leq 1$), these points are unstable saddles. For other values of the barotropic index, these points represent a global source/sink depending on the sign of $\epsilon$.

(iii) The points $B_\epsilon$ only exist for the specific range of the barotropic index $w \in [-1/3, 2/3]$, and they are always unstable. An interesting feature of these points is, they are vacuum Minkowski for all values of barotropic index in the above mentioned range, except for

| Point | Constraints | Solution/Description |
|-------|-------------|----------------------|
| $N_\epsilon$ | $Q = 0, x = 0$ | Vacuum Minkowski |
| $L_\epsilon$ | $Q = 0, x = 0$ | Vacuum Minkowski |
| $B_\epsilon$ | $Q = 0, x = 0$ | Vacuum Minkowski if $n \neq 2$ |
| $A_\epsilon$ | $Q = 0, x = 0$ | Accelerating for $P_\epsilon < n < 2$ |
| $LC$ | $Q = 0, x = 0$ | Non-accelerating curved |

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Table 2. This table summarizes the nature of the equilibrium points. As in the previous table, we have abbreviated \( P_+ = (1 + \sqrt{3})/2 \approx 1.37 \). The physically interesting values correspond to \( n = 3/2 \) (dust) and \( n = 2 \) (radiation). Note for \( n = 3/2 \), the Einstein static point becomes a bifurcation, and its stability cannot be determined within the linear theory (see the text).

| Point | Type | Range of \( n \) |
|-------|------|-----------------|
| \( A_+ \) | Expanding | \((1, 5/4)\) \(\approx\) \(1\) \(\approx\) \(1.37\) |
| \( A_- \) | Collapsing | \((5/4, P_+)\) |
| \( B_\pm \) | Static | \((P_+, 3/2)\) |
| \( \mathcal{L}_\pm \) | Static | \((3/2, 5/2)\) |
| \( \mathcal{N}_\pm \) | Static | \((5/2, 3)\) |
| \( \mathcal{L}_\text{exp} \) | Expanding | Sink for \( Q < Q_b \) |
| \( \mathcal{L}_\text{coll} \) | Collapsing | Source for \( |Q| > Q_b \) |

\( w = 1/3 \). In a radiation dominated universe these points represent power-law solutions \( a(t) = a_0 t^{1/3} \).

(iv) The points \( A_+ \) exist for \( w \in [-1/6, 1] \) only, and they have the interesting property of exhibiting de Sitter (dS)/anti-de Sitter (AdS) like acceleration/deceleration depending on the barotropic index. In this case however, the acceleration has a power-law behavior instead of the exponential behavior of dS/AdS. These points are vacuum solutions, and the expanding point is a global sink if \( n > P_+ \approx 1.37 \) or \( w > w_+ \approx -0.09 \). These points are also a generic feature of \( R^n \) gravity, and the existence of these points implies that even with an ultra-violet correction to general relativity, one can evolve toward late time acceleration for the ranges of \( w \) given in table 1.

(v) The line \( \mathcal{L} \) is entirely an artifact of the \( n - w \) correspondence (6), and is absent in a general \( R^n \)-gravity. All the points on this line are expanding/collapsing non-accelerating solutions, containing the Einstein static solution as the only static point. For \( n \in [1, P_+] \approx [1, 1.37] \) (see table caption), the expanding branch of the line forms spiral sinks, while the collapsing branch of the line forms spiral sources. The Einstein static point in between these branches forms a center, that means it is neutrally stable for this range of \( n \). For \( n \in [P_+, 3/2] \) a bifurcation appears on the line: for \( Q \in [-Q_b, Q_b] \), the expanding (contracting) points on the line remain spiral sinks (sources), and the Einstein static point remains a center. The points with \( |Q| > Q_b \) however now are saddles. Here \( Q_b \) is the bifurcation value defined by

\[
Q_b = \sqrt{\frac{2n - 3}{(n - 1)(11 - 11n + 2n^2)}}. \tag{21}
\]

We can see that the bifurcation enters the state space from both endpoints of the lines for \( n = P_+ \). As \( n \) increases, \( Q_b \) and \( -Q_b \) move closer toward the center of the line, and for \( n = 3/2 \) the two bifurcations \( \pm Q_b \) merge at \( Q = 0 \). For equations of state stiffer than dust \((n > 3/2)\), all the points on the line are saddles, including the Einstein static model.

(vi) The state space in general contains several past and future attractors. Interestingly, we find that for any equation of state, there is no expanding past attractor. The only possible
past attractors are the asymptotically Minkowski points $\mathcal{L}_+$ and $N_+$, and the decelerating collapsing point $A_-$. For $w < 0$, the points on the collapsing non-decelerating line $L^C_{coll}$ may also be past attractors. This means that in this class of models, there is no stable big bang scenario for any realistic equation of state $w \in [-1/3, 1]$, but only possible bounce scenarios or expansion after an initial asymptotic Minkowski phase. Figure 1 shows the state space for dust ($n = 3/2$), with all the equilibrium points. There are bounce scenarios along those trajectories where the quantity $Q$ changes sign. Figure 2 shows the state space for $n = 4/3$, and we can easily see how the nature of the equilibrium points changes as we change the barotropic index. Figure 4 shows a solution which is bouncing in the past as well as in the future. These kinds of solutions are not present in general relativity.

4.1. Stability of the Einstein static model

We will now discuss the stability of the Einstein static point in more detail. The stability of the Einstein static model against general covariant, gauge-invariant linear perturbations was studied in [39]. The standard procedure of harmonic decomposition is used, employing the trace-free symmetric tensor eigenfunctions $\tilde{Q}$ of the spatial Laplace–Beltrami operator:

$$\tilde{\nabla}^2 \tilde{Q} = -\frac{k^2}{a_0^2} \tilde{Q}, \quad \dot{\tilde{Q}} = 0,$$

(22)

where the constant $a_0$ defines the lengthscale for Einstein static universe. For the spatially closed models, the spectrum of eigenvalues is discrete and given by $[33, 41]$

$$k^2 = \tilde{n}(\tilde{n} + 2),$$

(23)

where the co-moving wave number takes the values $\tilde{n} = 1, 2, 3, \ldots$. We note that the dynamical systems analysis discussed in the present paper corresponds to the homogeneous
Figure 2. The complete state space for matter with $w = -1/9$, with all the equilibrium points. The expanding part of line $LC$ now behaves as a spiral sink. Also the points $A_-$ and $A_+$ are now saddles.

Figure 3. The quasiperiodic nature of the Einstein static point for $w = -1/9$. Here the initial conditions are taken very close to the Einstein static point on the plane perpendicular to the line $LC$ containing the Einstein static point.

mode with comoving wave number $\tilde{n} = 0$ in the harmonically decomposed perturbation equations. This mode was not studied in [39], since it corresponds to a change in the background of the linear perturbation theory.
It can easily be seen from equation (53) in [39] that the scalar perturbations for $\ddot{\bar{n}} = 0$ have a growing and a decaying mode if $w > 0$. If $w < 0$ on the other hand, the density gradient has two purely oscillatory modes. These results agree with the stability of the Einstein static model in the dynamical systems context analyzed here, where we found that the Einstein static model is a saddle for $w > 0$. For $w < 0$, since all the points of the line $LC$ at one side of Einstein static are sources and other side of Einstein static are sinks, and numerical studies show that the point Einstein static becomes quasiperiodic in nature, in the sense that orbits emerge from a source, spiral around Einstein static for some time and goes to a sink. Figure 3 shows such an orbit, where the initial conditions are taken very close to the Einstein static point on the plane perpendicular to the line $LC$ containing the Einstein static point. This quasiperiodic nature reflects in the oscillatory modes of the scalar perturbations in [39].

As we can easily see from the dynamical system equations, for dust-like matter ($w = 0$) the $ES$ point is a double bifurcation point, and consequently the linear theory breaks down. The physical interpretation is as follows: from equation (53) in [39] we can easily see that for $w = 0$ and $\ddot{\bar{n}} = 0$, the linear perturbation is independent of time, i.e., it is a constant. Hence to study the actual behavior of the homogeneous perturbation for dust, we must consider higher-order corrections.

4.2. The state space for radiation

For $n = 2$ (or $w = 1/3$) the state space has some interesting properties. Though qualitatively the nature of all equilibrium points are similar to that of dust, due to vanishing of the trace of the energy–momentum tensor for standard matter, the surface defined by $x = 0$ becomes an invariant subspace, containing the global sink/source $A_s$, global saddles $B_s$ and the Einstein static point $ES$ which is also a saddle. By definition, $x = 0$ corresponds to $R = 0$, and therefore the points $A_s$ are not accelerated expansion/collapse solutions but they are instead static Minkowski points. Figure 5 shows the phase portrait of this subspace. It is interesting
It is interesting to note that in this subspace the points $B_{\epsilon}$ behave like source/sink, however in the complete state space they are saddles. All the trajectories from $A_{-}$ to $A_{+}$, and from $B_{+}$ to $B_{-}$ admit bouncing universes. Also figure 4 shows a solution for radiation-like matter linking $N_{+}$ to $L_{-}$ which has a bounce in the future.

5. Discussion and conclusions

In this work, we have studied the state space of the class of FRW models in $f(R)$-gravity. We were particularly interested in the stability of the Einstein static model, which in general only exists for the specific form of $f(R) = R^n$ with $n$ constrained to be a function of the equation of state parameter $w$ as shown in [39]. For this reason, we here considered $R^n$-gravity with the constraint $n = 3/2(1 + w)$ only.

We have identified the past and future attractors of the state space of these models. We have found that the Einstein static model is an unstable saddle point for all equations of state stiffer than dust ($w > 0$ or $n > 3/2$), and a neutrally stable center for $w \in (-1/3, 0)$, i.e. $n \in (1, 3/2)$. This is different to general relativity, where the Einstein static model is an unstable saddle for all $w \in (-1/3, 1)$. We have also numerically found bouncing orbits that link the decelerating collapsing point $A_{-}$ to the accelerating expanding point $A_{+}$ via an asymptotically Einstein static phase represented by the saddle point $ES$.

We note that the stability of the Einstein static model obtained here is in agreement with the analysis of inhomogeneous perturbations [39], provided we carefully examine the relationship between the dynamical systems analysis here and the harmonically decomposed linear perturbations described in [39]. The homogeneous mode with comoving wave number $\tilde{n} = 0$ was excluded in [39], since it corresponds to a change in the background of the linear perturbation theory. It is this mode however that corresponds to the stability properties of the
dynamical system examined here. This mode is neutrally stable if $w < 0$, and unstable with a decaying and a growing mode if $w > 0$. These results exactly match the results from the dynamical systems analysis presented in this paper.

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