Wavelet Filtering with the Mellin Transform*
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ABSTRACT
It is shown that any convolution operator in the time domain can be represented exactly as a multiplication operator in the time-scale (wavelet) domain. The Mellin transform gives a one-to-one correspondence between frequency filters (system functions) and scale filters (multiplication operators in the scale domain), subject to the convergence of the defining integrals. The usual wavelet reconstruction theorem is a special case. Applications to the denoising of random signals are proposed. It is argued that the present method is more suitable for removing the effects of atmospheric turbulence than the conventional procedures because it is ideally suited for resolving spectral power laws.

Keywords: Non-stationary processes: Time-scale representations.

1. Scale Filtering in the Wavelet Domain
Time-scale analysis of signals is similar to their time-frequency analysis, but with the frequency replaced by a scale parameter [1]. Instead of having to choose a basic window in time, one must begin with a basic wavelet $\psi(t)$. We define the wavelet family of $\psi(t)$ as the two-parameter family of functions

$$\psi_{\sigma, \tau}(t) = \psi(\sigma t - \tau) \quad \text{with } \sigma, \tau \text{ real and } \sigma \neq 0.$$  

(1)

Note that this differs from the usual convention $\psi_{s, \tau'}(t) = |s|^{-1/2}\psi((t - \tau')/s)$. As will be seen below, (1) is equivalent but formally simpler. Whereas the usual scale parameter $s$ can be interpreted as time scale ($|s| \gg 1$ means coarse scale and $0 < |s| \ll 1$ means fine scale), our parameter $\sigma = 1/s$ can be interpreted as frequency scale: Large $\sigma$ isolates high frequencies, while small $\sigma$ isolates low frequencies. Thus we may identify $\sigma$ with the frequency band passed by the wavelet $\psi_{\sigma, \tau}(t)$, and $(\sigma, \tau)$ can be regarded as a pair of time-frequency parameters, with the understanding that now “frequency” means “frequency

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scale.” That is, instead of shifting \( f \to f + f_0 \), our frequencies now multiply \( \sigma \to \sigma \sigma_0 \). The absence of the factor \(|s|^{-1/2} = |\sigma|^{1/2} \) will be seen to be unimportant and, moreover, to lead to simpler reconstruction formulas. Our parameter \( \tau \) also differs from the above \( \tau' \) by \( \tau = \sigma \tau' \). That is, \( \tau \) is a time shift in the scaled time. (First scale, then shift rather than the other way around.) The wavelet transform of a signal \( \chi(t) \) with respect to the family \( \psi_{\sigma,\tau} \) is defined as the inner product

\[
\tilde{\chi}(\sigma, \tau) \equiv \langle \psi_{\sigma,\tau}, \chi \rangle = \int_{-\infty}^{\infty} dt \, \psi(\sigma t - \tau)^* \chi(t). \tag{2}
\]

Inserting the Fourier expansion of \( \psi(\sigma t - \tau) \) gives

\[
\tilde{\chi}(\sigma, \tau) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} df \, e^{-2\pi i f(\sigma t - \tau)} \hat{\psi}(f)^* \chi(t) = \int_{-\infty}^{\infty} df \, e^{2\pi i f\tau} \hat{\psi}(f)^* \hat{\chi}(f\sigma) = \mathcal{F}^{-1} \left[ \hat{\psi}(f)^* \hat{\chi}(f\sigma) \right](\tau), \tag{3}
\]

where \( \hat{\psi}(f) \) denotes the Fourier transform of \( \psi(t) \) and \( \mathcal{F}^{-1} \) is the inverse Fourier transform operator. Thus by Plancherel’s theorem,

\[
\int_{-\infty}^{\infty} d\tau \, |\tilde{\chi}(\sigma, \tau)|^2 = \int_{-\infty}^{\infty} df \, |\hat{\psi}(f)|^2 |\hat{\chi}(f\sigma)|^2 = \frac{1}{|\sigma|} \int_{-\infty}^{\infty} df \, |\hat{\psi}(f/\sigma)|^2 |\hat{\chi}(f)|^2. \tag{4}
\]

Suppose we integrate both sides over \( \sigma \) with an arbitrary positive weighting function \( w(\sigma) \). (For reasons to become clear below, we need all scales \( \sigma \neq 0 \). For some purposes, \( \sigma > 0 \) will do; this can be arranged simply by choosing \( w(\sigma) = 0 \) for \( \sigma < 0 \).) Thus

\[
\int_{-\infty}^{\infty} d\sigma \, w(\sigma) \int_{-\infty}^{\infty} d\tau \, |\tilde{\chi}(\sigma, \tau)|^2 = \int_{-\infty}^{\infty} d\sigma \, |w(\sigma)| \int_{-\infty}^{\infty} df \, |\hat{\psi}(f/\sigma)|^2 |\hat{\chi}(f)|^2 = \int_{-\infty}^{\infty} df \, |\hat{\chi}(f)|^2 \int_{-\infty}^{\infty} d\sigma \, |w(\sigma)| |\hat{\psi}(f/\sigma)|^2, \tag{5}
\]

where the integral over \( \sigma \) is understood to exclude \( \sigma = 0 \) (integrate over \( 0 < \varepsilon \leq |\sigma| < \infty \), then take the limit \( \varepsilon \to 0 \)). Let

\[
\Psi(f) \equiv |\hat{\psi}(f)|^2
\]

denote the spectral density of \( \psi(t) \), and define

\[
W(f) \equiv \int_{-\infty}^{\infty} d\sigma \, \frac{w(\sigma)}{|\sigma|} \Psi(f/\sigma) \equiv \langle w \ast \Psi \rangle(f) = \int_{-\infty}^{\infty} d\sigma \, \Psi(\sigma) w(f/\sigma) \equiv \langle \Psi \ast w \rangle(f), \tag{6}
\]
where the second line is obtained from the first by the substitution $\sigma \to f/\sigma$, assuming that $f \neq 0$. (The “DC component” $f = 0$ gets special treatment throughout wavelet theory, as does $\sigma = 0$, and for similar reasons.) Then (5) becomes

$$\int \int d\sigma d\tau w(\sigma) |\tilde{\chi}(\sigma, \tau)|^2 = \int_{-\infty}^{\infty} df W(f) |\hat{\chi}(f)|^2. \quad (7)$$

The left-hand side in (7) is a scale-weighted norm (“energy”) of the signal, whereas the right-hand side is a frequency-weighted norm with weight function $W(f)$. (For example, Sobolev norms are defined with $W(f) = (1 + f^2)^s$ for some power $s$.) Then (7) is a kind of “Plancherel theorem” equating the two weighted norms. Like the Plancherel theorem, it can be polarized to give the analog of Parseval’s relation for two signals $\phi(t)$ and $\chi(t)$:

$$\int \int d\sigma d\tau \tilde{\phi}(\sigma, \tau)^* w(\sigma) \tilde{\chi}(\sigma, \tau) = \int_{-\infty}^{\infty} df \hat{\phi}(f)^* W(f) \hat{\chi}(f). \quad (8)$$

This shows that the convolution operator $\mathcal{W}$ with system function (or “symbol”) $W(f)$ can be expressed in the wavelet domain as

$$\mathcal{W}\chi(t) \equiv \int_{-\infty}^{\infty} df e^{2\pi ift} W(f) \hat{\chi}(f) = \int \int d\sigma d\tau \psi_{\sigma,\tau}(t) w(\sigma) \tilde{\chi}(\sigma, \tau). \quad (9)$$

Thus, to represent $\mathcal{W}$ in the wavelet domain, we need merely to replace $\hat{\chi}(f)$ with $\tilde{\chi}(\sigma, \tau)$, $W(f)$ with $w(\sigma)$, and the “basis vectors” $e^{2\pi ift}$ of the Fourier transform with the “basis vectors” $\psi_{\sigma,\tau}(t)$ of the wavelet transform!

In deriving (9) we assumed that $w(\sigma)$ (and therefore, by (6), also $W(f)$) is nonnegative. This assumption was made for technical reasons, since then all the integrands occurring in Equations (5)–(8) are nonnegative and the integrals therefore either converge absolutely or diverge to infinity, making it unnecessary to worry about conditional convergence. We now relax this assumption and allow $w(\sigma)$ (hence also $W(f)$) to be complex-valued, with the understanding that the integrals may no longer converge absolutely. Then (9) shows that operations normally performed in the frequency domain can now also be performed in the time-scale domain, provided we can find a weighting function $w(\sigma)$ which gives $W(f)$ by (6). So the first problem that must be addressed is: Given $W(f)$, solve (6) for $w(\sigma)$. We now show that this problem has a unique solution, subject to the admissibility condition that its defining integrals converge.

The key is to note that Equation (6) defines $W(f)$ as a scaling convolution of $w(\sigma)$ with $\Psi(f)$, in the sense that the usual difference variable $f - \sigma$ is replaced by the quotient $f/\sigma$ and the translation-invariant (Lebesgue) measure $d\sigma$ is replaced by the scaling-invariant measure $d\sigma/|\sigma|$. Just as convolutions are converted into products by the Fourier transform, scaling convolutions are converted into products by the Mellin transform [2]. The latter
takes a function $F(\sigma)$ of a positive variable $\sigma > 0$ into a function $\tilde{F}(p)$ of a complex variable $p$ by

$$\mathcal{M}\{F(\sigma)\}(p) = \int_0^\infty \frac{d\sigma}{\sigma} \sigma^{-p} F(\sigma) \equiv \tilde{F}(p), \quad (10)$$

and the original function is reconstructed by integrating over a contour $C$ in the complex plane of the “power parameter” $p$ going from $c - i\infty$ to $c + i\infty$, for an appropriate choice of $c$ which depends on $F(\sigma)$:

$$F(\sigma) = \frac{1}{2\pi i} \int_C dp \sigma^p \tilde{F}(p) \equiv \mathcal{M}^{-1}\{\tilde{F}(p)\}(\sigma). \quad (11)$$

To solve (6) for $w(\sigma)$ using the Mellin transform, we must separate the positive- and negative-frequency components of $W(f)$, since the Mellin transform uses only $\sigma > 0$. Let us assume that

$$\hat{\psi}(f) = 0 \text{ for } f < 0, \quad \text{hence } \Psi(f) = 0 \text{ for } f < 0. \quad (12)$$

Then the wavelet $\psi(t)$ is necessarily complex. It is an analytic signal in the sense of Gabor [3], extending analytically to the upper half of the complex time plane. (See also [1], Section 9.3.) Let $U(\sigma)$ be the unit step function

$$U(\sigma) = \begin{cases} 1 & \text{if } \sigma > 0 \\ 0 & \text{if } \sigma < 0, \end{cases}$$

and write

$$w(\sigma) = U(\sigma)w(\sigma) + U(-\sigma)w(\sigma) = w_+(\sigma) + w_-(-\sigma), \quad (13)$$

where

$$w_{\pm}(\sigma) \equiv U(\sigma)w(\pm\sigma) = 0 \text{ for } \sigma < 0. \quad (14)$$

By (12), (6) becomes (for $f \neq 0$)

$$W(f) = \int_0^\infty \frac{d\sigma}{\sigma} \Psi(\sigma) w(f/\sigma) = W_+(f) + W_-(-f), \quad (15)$$

where

$$W_{\pm}(f) = \int_0^\infty \frac{d\sigma}{\sigma} \Psi(\sigma) w_\pm(f/\sigma) = (\Psi \ast w_\pm)(f) = 0 \text{ for } f < 0. \quad (16)$$

Therefore the positive-frequency component $W_+(f)$ of $W(f)$ depends only on the positive-scale component $w_+(\sigma)$ of $w(\sigma)$ and the negative-frequency component $W_-(-f)$ depends only on the negative-scale component $w_-(-\sigma)$. This decoupling was the purpose of the assumption (12) above. Applying the Mellin transform to the scaling convolutions (16), we obtain

$$\tilde{W}_\pm(p) = \mathcal{M}^{-1}\{\Psi \ast w_\pm\}(p) = \tilde{\Psi}(p) \tilde{w}_\pm(p). \quad (17)$$
This gives a formal expression for $w_{\pm}(\sigma)$ in terms of the transform of $W_{\pm}(f)$:

$$w_{\pm}(\sigma) = \frac{1}{2\pi i} \int_C dp \, \sigma^{-p} \frac{\tilde{W}_{\pm}(p)}{\tilde{\Psi}(p)}, \quad \sigma > 0. \quad (18)$$

We give three examples. First, let $n$ be an integer and $w(\sigma) = \sigma^n$. Then

$$w_{\pm}(\sigma) = U(\sigma) (\pm \sigma)^n, \quad (19)$$

and (16) gives

$$W_{\pm}(f) = U(f) (\pm f)^n \int_0^\infty \frac{d\sigma}{\sigma} \Psi(\sigma) \sigma^{-n} = U(f) (\pm f)^n \tilde{\Psi}(n). \quad (20)$$

Hence

$$W(f) = \tilde{\Psi}(n) [U(f) f^n + U(-f) f^n] = \tilde{\Psi}(n) f^n. \quad (21)$$

That is, pure integer powers are invariant under the correspondence $w(\sigma) \leftrightarrow W(f)$ between the frequency and scale domains, except for the renormalization constant $\tilde{\Psi}(n)$. The differential operator

$$P(D) = \sum_{n=0}^N a_n D^n,$$

where $D = \frac{d}{dt}$, is represented in the frequency domain as multiplication by

$$W(f) = \sum_{n=0}^N a_n (2\pi i f)^n. \quad (22)$$

Hence, by (21), it is represented in the wavelet domain as multiplication by

$$w(\sigma) = \sum_{n=0}^N a_n (2\pi i \sigma)^n \tilde{\Psi}(n). \quad (23)$$

In order for this representation to be defined, we must have

$$0 < \tilde{\Psi}(n) \equiv \int_0^\infty \frac{d\sigma}{\sigma} \Psi(\sigma) \sigma^{-n} < \infty \quad \text{for all } n \text{ with } a_n \neq 0. \quad (24)$$

We say that the operator $P(D)$ is admissible if the condition (24) holds. In particular, the identity operator $P(D) \equiv 1$ is represented in frequency by $W(f) \equiv 1$, hence

$$w(\sigma) = \frac{1}{\tilde{\Psi}(0)}$$
and (9) becomes
\[ \chi(t) = \frac{1}{\Psi(0)} \int d\sigma d\tau \psi_{\sigma,\tau}(t) \tilde{\chi}(\sigma, \tau). \] (25)
This is the usual wavelet reconstruction formula, which inverts the wavelet transform. It shows that the identity operator is admissible if and only if
\[ \tilde{\Psi}(0) \equiv \int_0^\infty \frac{d\sigma}{\sigma} |\hat{\psi}(\sigma)|^2 < \infty, \] (26)
which is the usual admissibility condition \[1\] for the wavelet \( \psi(t) \). Indeed, (18) indicates the following general admissibility condition for convolution operators:

In order for the convolution operator with system function \( W(f) \) to be representable in the wavelet domain of the wavelet \( \psi(t) \), the Mellin transforms \( \tilde{W}_\pm(p) \) of \( W_{\pm}(f) \) must be analytic along an appropriate contour \( C \) parallel to the imaginary axis such that \( \tilde{\Psi}(p) \) is also analytic with no zeros along \( C \) and the integral (18) converges.

This example suggests that wavelet methods may be used instead of Fourier methods to solve differential equations. To make this precise, one must examine the admissibility condition of the given operator and the range of the wavelet transform, i.e., the set of all functions \( \tilde{\chi}(\sigma, \tau) \) obtained from \( \chi(t) \) as the latter is allowed to range over the space of functions of interest. (Not every function \( F(\sigma, \tau) \) of time and scale is the wavelet transform of some time signal \( \chi(t) \); \( F(\sigma, \tau) \) must satisfy a consistency condition related to the fact that the “basis vectors” \( \psi_{\sigma,\tau}(t) \) are redundant \[1\].)

Our second example is
\[ w(\sigma) = |\sigma|^p, \] (27)
where \( p \) is now any power (not necessarily an integer). Then
\[ w_{\pm}(\sigma) = U(\sigma) \sigma^p, \] (28)
and (16) gives
\[ W_{\pm}(f) = U(f) f^p \int_0^\infty \frac{d\sigma}{\sigma} \Psi(\sigma) \sigma^{-p} = U(f) f^p \tilde{\Psi}(p). \] (29)
Hence
\[ W(f) = \tilde{\Psi}(p) [U(f) f^p + U(-f) (-f)^p] = \tilde{\Psi}(p) |f|^p. \] (30)
Like the pure integer powers \( \sigma^n \) of \( \sigma \), the pure powers \( |\sigma|^p \) of \( |\sigma| \) are invariant under the correspondence \( w(\sigma) \leftrightarrow W(f) \), again with the renormalization factor \( \tilde{\Psi}(p) \). Note that this representation of the filter \( W(f) = |f|^p \) (with \( p = -\alpha \)) is implicit in \[1\], Section 9.3.

Our final example is
\[ h(\sigma) = \frac{1}{\tilde{\Psi}(0)} \left\{ \begin{array}{ll} -i & \text{if } \sigma > 0 \\ i & \text{if } \sigma < 0. \end{array} \right. \] (31)
This gives

\[ H(f) = \begin{cases} 
-i & \text{if } f > 0 \\
i & \text{if } f < 0, 
\end{cases} \]  

which is the system function of the Hilbert transform

\[ (\mathcal{H}\chi)(t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{du}{u-t} \chi(t), \]  

where \( P \) denotes the principal part of the integral. Thus

\[ (\mathcal{H}\chi)(t) = \frac{i}{\Psi(0)} \left[ \iint_{\sigma<0} d\sigma d\tau \psi_{\sigma,\tau}(t) \tilde{\chi}(\sigma,\tau) - \iint_{\sigma>0} d\sigma d\tau \psi_{\sigma,\tau}(t) \tilde{\chi}(\sigma,\tau) \right]. \]  

This representation of \( \mathcal{H} \) is implicit in Theorem 3.3 of [1].

The combination of Mellin and Fourier transforms has long been used by harmonic analysts in connection with singular integral operators, in particular to obtain estimates [4]. But although the concrete representation of convolution operators derived here would seem to be fundamental in the application of wavelet analysis to signal processing, I was unable to find its existence in the literature. A discrete version of this problem in the framework of multiresolution analysis was the subject of a recent paper by Beylkin and Törrésani [5].

2. Applications

The time-scale representation (9) can be applied, in principle, to any problem involving convolution operators. Its usefulness depends on the nature of the system function \( W(f) \) for the problem at hand. The above examples show that systems whose functions obey power laws will have simple representations. In particular, the problem of filtering out the effects of atmospheric turbulence (denoising) seems a likely candidate, since it involves asymptotic power laws. There is strong empirical evidence that denoising in the scale domain is superior to denoising in the frequency domain because it tends to smooth out the spectrum without smearing the scales, so that the exponents in the power laws are clearly resolved [6].

The above time-scale analysis was developed for one dimension (time). It can be extended to several dimensions, like space-time, and in that case it may be applied to physical wavelet representations. Physical wavelets, which were defined in [1], are localized acoustic or electromagnetic waves (i.e., solutions of the scalar wave equation or of Maxwell’s equations) which share many of the properties of the one-dimensional wavelets. They are related to one another by geometrical operations such as space-time translations, scalings, and Lorentz transformations (boosting to moving coordinate systems), and they can be used as “building blocks” to form arbitrary acoustic or electromagnetic waves. Such wavelets have recently found natural applications in sonar and radar [7]. With them,
the problem of denoising atmospheric turbulence can be approached from first principles, since the physical wavelets already fully describe the deterministic space-time behavior (dynamics) of the mean process. Progress on this will be reported elsewhere.

References

1. G. Kaiser, A Friendly Guide to Wavelets, Birkhäuser, Boston, 1994.
2. R. Courant and D. Hilbert, Methods of Mathematical Physics, Volume 1, Interscience, New York, 1953.
3. D. Gabor, Theory of communication, J. IEE 93 (1946) (III), 429–457.
4. R.R. Coifman and Y. Meyer, On commutators of singular integrals and bilinear singular integrals, Transactions of the AMS 212, 315–331 (1975).
5. G. Beylkin and B. Torrésani, Implementation of operators via filter banks, preprint, University of Colorado, October 13, 1995.
6. L. Hudgins, C.A. Friehe, and M.E. Mayer, Wavelet transforms and atmospheric turbulence, Physical Review Letters 71, 3279–3282 (1993).
7. G. Kaiser, Physical wavelets and radar, to appear in the IEEE Antennas and Propagation Magazine, February, 1996.