KOSZUL COMPLEXES AND FULLY FAITHFUL INTEGRAL FUNCTORS

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ABSTRACT. We characterise those objects in the derived category of a scheme which are a sheaf supported on a closed subscheme in terms of Koszul complexes. This is applied to generalise to arbitrary schemes the fully faithfulness criteria of an integral functor.

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Introduction

Let $X, Y$ be two proper schemes over a field $k$ and let

$$\Phi : D^b_c(X) \rightarrow D^b_c(Y)$$

be an integral functor between their derived categories of complexes of quasi-coherent modules with bounded and coherent cohomology. Let $K^\bullet \in D^b_c(X \times Y)$ be the kernel of $\Phi$. We want to characterise those kernels $K^\bullet$ such that $\Phi$ is a fully faithful. This was solved in [1] for smooth projective schemes over a field of zero characteristic. For Gorenstein schemes and zero characteristic it was solved in [3]. For Cohen-Macaulay schemes and arbitrary characteristic it was solved in [2]. Here we remove the Cohen-Macaulay hypothesis and reproduce the fully faithfulness criteria of [2] for arbitrary schemes. The point is to replace the locally complete intersection zero-cycles of [2] by Koszul complexes associated to a system of parameters. These Koszul complexes allow to characterise, for an arbitrary scheme $X$, those objects in $D^b_c(X)$ consisting of a sheaf supported on a closed subscheme (Propositions 1.7 and 1.9). This is the main ingredient for the fully faithfulness criteria.

Date: February 2, 2008.

2000 Mathematics Subject Classification. Primary: 18E30; Secondary: 14F05, 14J27, 14E30, 13D22, 14M05.

Key words and phrases. Geometric integral functors, Fourier-Mukai, fully faithful, equivalence of categories.

Work supported by research projects MTM2006-04779 (MEC) and SA001A07 (JCYL).
Acknowledgements. I would like to thank Leovigildo Alonso, who suggested to me the use of Koszul complexes to deal with the general (non Cohen-Macaulay) case.

1. Koszul complexes, depth and support

We introduce Koszul complexes and use them to characterize those objects in the derived category consisting of a sheaf supported on a closed subscheme.

1.1. System of parameters. Koszul complex. Let $\mathcal{O}$ be a noetherian local ring of dimension $n$ and maximal ideal $m$. Let $x$ be the closed point.

Definition 1.1. A sequence $f = \{f_1, \ldots, f_n\}$ of $n$ elements in $m$ is called a system of parameters of $\mathcal{O}$ if $\mathcal{O}/(f_1, \ldots, f_n)$ is a zero dimensional ring. In other words, $(f_1, \ldots, f_n)$ is a $m$-primary ideal. We shall also denote $\mathcal{O}/f = \mathcal{O}/(f_1, \ldots, f_n)$.

It is a basic fact of dimension theory that there always exists a system of parameters. In fact, for any $m$-primary ideal $I$, there exist $f_1, \ldots, f_n$ in $I$ which are a system of parameters of $\mathcal{O}$.

We shall denote by $\text{Kos}^*(f)$ the Koszul complex associated to a system of parameters $f$. That is, if we denote $L = \mathcal{O}^{\oplus n}$ and $\omega: L \to \mathcal{O}$ the morphism given by $f_1, \ldots, f_n$, then the Koszul complex is $\wedge^i L$ in degree $-i$ and the differential $\wedge^i L \to \wedge^{i-1} L$ is the inner contraction with $\omega$. It is immediate to see that $\text{Hom}^*(\text{Kos}^*(f), \mathcal{O}) \simeq \text{Kos}^*(f)[-n]$.

The cohomology modules $H^i(\text{Kos}^*(f))$ are supported at $x$ (indeed they are annihilated by $(f_1, \ldots, f_n)$). Moreover $H^0(\text{Kos}^*(f)) = \mathcal{O}/f$ and $H^i(\text{Kos}^*(f)) = 0$ for $i > 0$ and $i < -n$.

For any complex $\mathcal{M}^*$ of $\mathcal{O}$-modules, we shall denote

$$\text{Tor}^0_\mathcal{O}(\text{Kos}^*(f), \mathcal{M}^*) = H^{-i}(\text{Kos}^*(f) \otimes \mathcal{M}^*)$$

$$\text{Ext}^0_\mathcal{O}(\text{Kos}^*(f), \mathcal{M}^*) = H^i(\text{Hom}^\mathcal{O}_\mathcal{O}(\text{Kos}^*(f), \mathcal{M}^*))$$

From the isomorphism $\text{Hom}^*(\text{Kos}^*(f), \mathcal{O}) \simeq \text{Kos}^*(f)[-n]$ it follows easily that

(1.1) $$\text{Ext}^0_\mathcal{O}(\text{Kos}^*(f), \mathcal{M}^*) \simeq \text{Tor}^0_{n-i}(\text{Kos}^*(f), \mathcal{M}^*)$$

1.2. Depth. Singularity set. The depth of an $\mathcal{O}$-module $M$, $\text{depth}(M)$, is the first integer $i$ such that either:

- $\text{Ext}^i(\mathcal{O}/m, M) \neq 0$ or
- $H^i_x(\text{Spec} \mathcal{O}, M) \neq 0$ or
- $\text{Ext}^i_{\mathcal{O}}(N, M) \neq 0$ for some non zero finite $\mathcal{O}$-module $N$ supported at $x$ or
- $\text{Ext}^i_{\mathcal{O}}(N, M) \neq 0$ for any non zero finite $\mathcal{O}$-module $N$ supported at $x$.

Lemma 1.2. The depth of $M$ is the first integer $i$ such that either:

- $\text{Ext}^i_{\mathcal{O}}(\text{Kos}^*(f), M) \neq 0$ for some system of parameters $f$ of $\mathcal{O}$ or
- $\text{Ext}^i_{\mathcal{O}}(\text{Kos}^*(f), M) \neq 0$ for every system of parameters $f$ of $\mathcal{O}$.

Proof. It is an easy consequence of the spectral sequence

$$E_{2}^{p,q} = \text{Ext}^{p}(H^{-q}(\text{Kos}^*(f), M)) \implies E_{\infty}^{p+q} = \text{Ext}^{p+q}(\text{Kos}^*(f), M)$$

Indeed, let $d = \text{depth}(M)$, $f$ a system of parameters of $\mathcal{O}$ and $r$ the first integer such that $\text{Ext}^i_{\mathcal{O}}(\text{Kos}^*(f), M) \neq 0$. Let us see that $d = r$. Since $\text{Ext}^d_{\mathcal{O}}(H^0(\text{Kos}^*(f)), M) \neq 0$, one obtains, by the spectral sequence, that $\text{Ext}^d_{\mathcal{O}}(\text{Kos}^*(f), M) \neq 0$. Hence $d \geq r$.\[\square\]
Assume that \( r \neq d \). Then \( \text{Hom}^{r-i}(H^{-i}(\text{Kos}^*(f)), M) = 0 \) for any \( i \geq 0 \), because \( H^{-i}(\text{Kos}^*(f)) \) is supported at \( x \) and \( r - i < d \). From the exact triangles

\[
\text{Kos}^*(f)_{\leq -i} \rightarrow \text{Kos}^*(f)_{\leq -i} \rightarrow H^{-i}(\text{Kos}^*(f)[i])
\]

and taking into account that \( \text{Hom}^r(\text{Kos}^*(f)_{\leq 0}, M) = \text{Hom}^r(\text{Kos}^*(f), M) \neq 0 \) one obtains that \( \text{Hom}^r(\text{Kos}^*(f)_{\leq -i}, M) = 0 \) for any \( i \geq 0 \). This is absurd because \( \text{Kos}^*(f)_{\leq -i} = 0 \) for \( i >> 0 \).

\[
\square
\]

Let \( \mathcal{F} \) be a coherent sheaf on a scheme \( X \) of dimension \( n \). We write \( n_x \) for the dimension of the local ring \( \mathcal{O}_x \) of \( X \) at a point \( x \in X \), \( \mathcal{F}_x \) for the stalk of \( \mathcal{F} \) at \( x \) and \( k(x) \) for the residual field of \( x \). \( \mathcal{F}_x \) is a \( \mathcal{O}_x \)-module. The integer number \( \text{codepth}(\mathcal{F}_x) = n_x - \text{depth}(\mathcal{F}_x) \) is called the codepth of \( \mathcal{F} \) at \( x \). For any integer \( m \in \mathbb{Z} \), the \( m \)-th singularity set of \( \mathcal{F} \) is defined to be

\[
S_m(\mathcal{F}) = \{ x \in X \mid \text{codepth}(\mathcal{F}_x) \geq n - m \}.
\]

Then, if \( X \) is equidimensional, a closed point \( x \) is in \( S_m(\mathcal{F}) \) if and only if \( \text{depth}(\mathcal{F}_x) \leq m \).

Since \( \text{depth}(\mathcal{F}_x) \) is the first integer \( i \) such that either

- \( \text{Ext}^i_{\mathcal{O}_x}(k(x), \mathcal{F}_x) \neq 0 \) or
- \( H^i_x(\mathcal{F}_x) \neq 0 \) or
- \( \text{Ext}^i_{\mathcal{O}_x}(\text{Kos}^*(f_x), \mathcal{F}_x) \neq 0 \) for some system of parameters \( f_x \) of \( \mathcal{O}_x \) or
- \( \text{Ext}^i_{\mathcal{O}_x}(\text{Kos}^*(f_x), \mathcal{F}_x) \neq 0 \) for every system of parameters \( f_x \) of \( \mathcal{O}_x \)

we have alternative descriptions of \( S_m(\mathcal{F}) \):

\[
S_m(\mathcal{F}) = \{ x \in X \mid H^i_x(\text{Spec} \mathcal{O}_{X,x}, \mathcal{F}_x) \neq 0 \text{ for some } i \leq m + n_x - n \}
\]

\[
= \{ x \in X \mid \text{Ext}^i_{\mathcal{O}_x}(\text{Kos}^*(f_x), \mathcal{F}_x) \neq 0 \text{ for some } i \leq m + n_x - n \}
\]

and some system of parameters \( f_x \) of \( \mathcal{O}_{X,x} \)

\[
= \{ x \in X \mid \text{Ext}^i_{\mathcal{O}_x}(\text{Kos}^*(f_x), \mathcal{F}_x) \neq 0 \text{ for some } i \leq m + n_x - n \}
\]

and any system of parameters \( f_x \) of \( \mathcal{O}_{X,x} \)

(1.2)

**Lemma 1.3.** [3 Lemma 1.10]. If \( X \) is smooth, then the \( m \)-th singularity set of \( \mathcal{F} \) can be described as

\[
S_m(\mathcal{F}) = \bigcup_{p \geq n - m} \{ x \in X \mid \text{Tor}^p_{\mathcal{O}_x}(k(x), \mathcal{F}_x) \neq 0 \},
\]

where \( k(x) \) is the residue field of \( \mathcal{O}_x \).

In the singular case, this characterization of \( S_m(\mathcal{F}) \) is not true. There is a similar interpretation for Cohen-Macaulay schemes replacing \( k(x) \) by \( \mathcal{O}_{Z_x} \) where \( Z_x \) is a locally complete intersection zero cycle supported on \( x \) (see [2 Lemma 3.5]). Now, for arbitrary schemes, the analogous interpretation is the following.

**Lemma 1.4.** The \( m \)-th singularity set \( S_m(\mathcal{F}) \) can be described as

\[
S_m(\mathcal{F}) = \{ x \in X \mid \text{there is an integer } i \geq n - m \text{ with } \text{Tor}^i_{\mathcal{O}_x}(\text{Kos}^*(f_x), \mathcal{F}_x) \neq 0 \}
\]

for any system of parameters \( f_x \) of \( \mathcal{O}_{X,x} \).

**Proof.** It follows from (1.1) and (1.2). \( \square \)

**Proposition 1.5.** [3 Prop 1.13]. Let \( X \) be an equidimensional scheme of dimension \( n \) and \( \mathcal{F} \) a coherent sheaf on \( X \).
(1) $S_m(\mathcal{F})$ is a closed subscheme of $X$ and codim $S_m(\mathcal{F}) \geq n - m$.
(2) If $Z$ is an irreducible component of the support of $\mathcal{F}$ and $c$ is the codimension of $Z$ in $X$, then codim $S_{n-c}(\mathcal{F}) = c$ and $Z$ is also an irreducible component of $S_{n-c}(\mathcal{F})$.

**Corollary 1.6.** [3 Cor. 1.14]. Let $X$ be a scheme and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $h: Y \hookrightarrow X$ be an irreducible component of the support of $\mathcal{F}$ and $c$ the codimension of $Y$ in $X$. There is a non-empty open subset $U$ of $Y$ such that for any $x \in U$ and any system of parameters $f_x$ of $\mathcal{O}_{X,x}$, one has

$$\text{Tor}_{i}^{\mathcal{O}_x}(\text{Kos}^*(f_x), \mathcal{F}_x) \neq 0$$

$$\text{Tor}_{i+1}^{\mathcal{O}_x}(\text{Kos}^*(f_x), \mathcal{F}_x) = 0,$$

for every $i > 0$.

**Proof.** By Lemma [1.3] the locus of the points that verify the conditions is $U = Y \cap (S_{n-c}(\mathcal{F}) - S_{n-c-1}(\mathcal{F}))$, which is open in $Y$ by Proposition [1.5]. Proving that $U$ is not empty is a local question, and we can then assume that $Y$ is the support of $\mathcal{F}$. Now $Y = S_{n-c}(\mathcal{F})$ by (2) of Proposition [1.5] and $U = S_{n-c}(\mathcal{F}) - S_{n-c-1}(\mathcal{F})$ is non-empty because the codimension of $S_{n-c-1}(\mathcal{F})$ in $X$ is greater or equal than $c + 1$ again by Proposition [1.5].

For any scheme $X$ we denote by $D(X)$ the derived category of complexes of quasi-coherent $\mathcal{O}_X$-modules and by $D_c^b(X)$ the faithful subcategory consisting of those complexes with bounded and coherent cohomology sheaves.

The following proposition characterises objects of the derived category supported on a closed subscheme.

**Proposition 1.7.** [1 Prop. 1.5][3 Prop. 1.15]. Let $j: Y \hookrightarrow X$ be a closed immersion of codimension $d$ of irreducible schemes and $\mathcal{K}^\bullet$ an object of $D_c^b(X)$. Assume that

1. If $x \in X - Y$ is a closed point, then there exists a system of parameters $f_x$ of $\mathcal{O}_x$ such that $\text{Tor}_i^{\mathcal{O}_x}(\text{Kos}^*(f_x), \mathcal{K}^\bullet_x) = 0$ for every $i$.  
2. If $x \in Y$ is a closed point, then there exists a system of parameters $f_x$ of $\mathcal{O}_x$ such that $\text{Tor}_i^{\mathcal{O}_x}(\text{Kos}^*(f_x), \mathcal{K}^\bullet_x) = 0$ when either $i < 0$ or $i > d$.

Then there is a sheaf $\mathcal{K}$ on $X$ whose topological support is contained in $Y$ and such that $\mathcal{K}^\bullet \simeq \mathcal{K}$ in $D_c^b(X)$. Moreover, this topological support coincides with $Y$ unless $\mathcal{K}^\bullet = 0$.

**Proof.** We just reproduce the proof of [3 Prop. 1.15], with the corresponding changes. Let us write $\mathcal{H}^q = \mathcal{H}^q(\mathcal{K}^\bullet)$. For every system of parameters $f_x$ of $\mathcal{O}_x$ there is a spectral sequence

$$E_2^{p,q} = \text{Tor}_p^{\mathcal{O}_x}(\text{Kos}^*(f_x), \mathcal{H}^q_x) \implies E_\infty^{p,q} = \text{Tor}_p^{\mathcal{O}_x}(\text{Kos}^*(f_x), \mathcal{K}^\bullet_x)$$

Let $q_0$ be the maximum of the $q$'s with $\mathcal{H}^q \neq 0$. If $x \in \text{supp}(\mathcal{H}^{q_0})$, one has that $\text{Tor}_p^{\mathcal{O}_x}(\text{Kos}^*(f_x), \mathcal{H}^{q_0}_x) \simeq \mathcal{H}^0(\text{Kos}^*(f_x)) \otimes_{\mathcal{O}_x} \mathcal{H}^{q_0}_x \neq 0$ for every system of parameters $f_x$ of $\mathcal{O}_x$. A nonzero element in $\text{Tor}_p^{\mathcal{O}_x}(\text{Kos}^*(f_x), \mathcal{H}^{q_0}_x)$ survives up to infinity in the spectral sequence. Since there is a system of parameters $f_x$ of $\mathcal{O}_x$ such that $E_\infty^{q} = \text{Tor}_p^{\mathcal{O}_x}(\text{Kos}^*(f_x), \mathcal{K}^\bullet) = 0$ for every $q > 0$ by hypothesis, one has $q_0 \leq 0$. A similar argument shows that the topological support of all the sheaves $\mathcal{H}^q$ is contained in $Y$; assume that this is not true and let us consider the maximum $q_1$ of the $q$'s such that $\mathcal{H}^{q_1}_x \neq 0$ for a certain point $x \in X - Y$; then $\text{Tor}_p^{\mathcal{O}_x}(\text{Kos}^*(f_x), \mathcal{H}^{q_1}_x) \neq 0$ and a nonzero
element in $\text{Tor}_0^{O_x}(\text{Kos}^i(f_x), \mathcal{H}_x^{q_{i+1}})$ survives up to infinity in the spectral sequence, which is impossible since $\text{Tor}_0^{O_x}(\text{Kos}^i(f_x), \mathcal{K}^i) = 0$ for every $i$.

Let $q_2 \leq q_0$ be the minimum of the $q$’s with $\mathcal{H}_x^{q} \neq 0$. We know that $\mathcal{H}_x^{q_2}$ is topologically supported on a closed subset of $Y$. Take a component $Y' \subseteq Y$ of the support. If $c \geq d$ is the codimension of $Y'$, then there is a non-empty open subset $U$ of $Y'$ such that $\text{Tor}_c^{O_x}(\text{Kos}^i(f_x), \mathcal{H}_x^{q_{i+1}}) \neq 0$ for any closed point $x \in U$ and any system of parameters $f_x$ of $\mathcal{O}_x$, by Corollary 1.6. Elements in $\text{Tor}_c^{O_x}(\text{Kos}^i(f_x), \mathcal{H}_x^{q_{i+1}})$ would be killed in the spectral sequence by $\text{Tor}_p^{O_x}(\text{Kos}^i(f_x), \mathcal{H}_x^{q_{i+1}})$ with $p \geq c + 2$. By Lemma 1.8, the set
\[
\{ x \in X \mid \text{Tor}_c^{O_x}(\text{Kos}^i(f_x), \mathcal{H}_x^{q_{i+1}}) \neq 0 \text{ for some } i \geq c + 2 \text{ and any parameters } f_x \text{ of } \mathcal{O}_x \}
\]
is equal to $S_{n-(c+2)}(\mathcal{H}_x^{q_{i+1}})$ and then has codimension greater or equal than $c + 2$ by Proposition 1.8. Thus there is a point $x \in Y'$ such that any nonzero element in $\text{Tor}_c^{O_x}(\text{Kos}^i(f_x), \mathcal{H}_x^{q_{i+1}})$ survives up to the infinity in the spectral sequence. Therefore, $\text{Tor}_c^{O_x}(\text{Kos}^i(f_x), \mathcal{K}^i) \neq 0$ for any system of parameters $f_x$ of $\mathcal{O}_x$. Thus $c - q_2 \leq d$ which leads to $q_2 \geq c - d \geq 0$ and then $q_2 = q_0 = 0$. So $\mathcal{K}^i = \mathcal{H}_x^0$ in $D^b(\mathcal{X})$ and the topological support of $\mathcal{K} = \mathcal{H}_x^0$ is contained in $Y$. Actually, if $\mathcal{K}^i \neq 0$, then this support is the whole of $Y$: if this was not true, since $Y$ is irreducible, the support would have a component $Y' \subseteq Y$ of codimension $c > d$ and one could find, reasoning as above, a non-empty subset $U$ of $Y'$ such that $\text{Tor}_c^{O_x}(\text{Kos}^i(f_x), \mathcal{K}^i) \neq 0$ for all $x \in U$ and any system of parameters $f_x$ of $\mathcal{O}_x$. This would imply that $c \leq d$, which is impossible.

Assume now that $X$ is separated. Let $x$ be a closed point of $X$ and $\phi_x : \text{Spec } \mathcal{O}_x \to X$ the natural morphism. Let $f_x$ be a system of parameters of $\mathcal{O}_x$. We shall still denote by $\text{Kos}^i(f_x)$ the direct image by $\phi_x$ of the Koszul complex $\text{Kos}^i(f_x)$. Let $U$ be an affine open subset containing $x$. Then $\phi_x$ is the composition of $\phi_x' : \text{Spec } \mathcal{O}_x \to U$ with the open embedding $i_U : U \hookrightarrow X$. Since $X$ is separated, $i_U$ is an affine morphism, and then $\phi_{x*} \simeq R\phi_{x*}$.

One has that

**Lemma 1.8.** For any $\mathcal{K}^i \in D(X)$ one has
\[
\text{Hom}_{D(X)}^i(\text{Kos}^i(f_x), \mathcal{K}^i) \simeq \text{Ext}_{\mathcal{O}_x}^i(\text{Kos}^i(f_x), \mathcal{K}^i)
\]

**Proof.** Let $C$ be the cone of $\phi_x^* \phi_{x*}^i \mathcal{K}^i$. It is clear that $x \notin \text{supp}(C)$. On the other hand $\phi_{x*}^i \text{Kos}^i(f_x)$ is supported at $x$. Then $\text{Hom}^i(\phi_{x*} \text{Kos}^i(f_x), C) = 0$ and
\[
\text{Hom}_{D(X)}^i(\phi_{x*} \text{Kos}^i(f_x), \mathcal{K}^i) \simeq \text{Hom}_{D(X)}^i(\phi_{x*} \text{Kos}^i(f_x), \phi_x^* \phi_{x*}^i \mathcal{K}^i)
\]
and one concludes because $\phi_x^* \phi_{x*} \text{Kos}^i(f_x) \simeq \text{Kos}^i(f_x)$.

Taking into account the equation (1.10), Proposition 1.7 may be reformulated as follows:

**Proposition 1.9.** Let $j : Y \hookrightarrow X$ be a closed immersion of codimension $d$ of irreducible schemes of dimensions $m$ and $n$ respectively, and let $\mathcal{K}^i$ be an object of $D^b_0(X)$. Assume that for any closed point $x \in X$ there is a system of parameters $f_x$ of $\mathcal{O}_x$ such that
\[
\text{Hom}_{D(X)}^i(\text{Kos}^i(f_x), \mathcal{K}^i) = 0,
\]
unless $x \in Y$ and $m \leq i \leq n$. Then there is a sheaf $\mathcal{K}$ on $X$ whose topological support is contained in $Y$ and such that $\mathcal{K}^i \simeq \mathcal{K}$ in $D^b_0(X)$. Moreover, the topological support is $Y$ unless $\mathcal{K}^i = 0$. \qed
1.2.1. Spanning classes.

Lemma 1.10. For each closed point \( x \in X \) choose a system of parameters \( f_x \) of \( \mathcal{O}_x \). The set

\[
\Omega = \{ \text{Kos}(f_x) \text{ for all closed points } x \in X \}
\]

is a spanning class for \( D^b_c(X) \).

Proof. Take a non-zero object \( \mathcal{E}^\bullet \) in \( D^b_c(X) \). Let \( q_0 \) be the maximum of the \( q \)'s such that \( \mathcal{H}^p(\mathcal{E}^\bullet) \neq 0 \), \( x \) a closed point of the support of \( \mathcal{H}^p(\mathcal{E}^\bullet) \) and \( -l \) the minimum of the \( p \)'s such that \( \mathcal{H}^p(\text{Kos}(f_x)) \neq 0 \).

Then

\[
\text{Hom}_{D(X)}^{-(l-q_0)}(\mathcal{E}^\bullet, \text{Kos}(f_x)) \simeq \text{Hom}_{\mathcal{O}_X}(\mathcal{H}^{q_0}(\mathcal{E}^\bullet), \mathcal{H}^{-l}(\text{Kos}(f_x)))
\]

\[
\simeq \text{Hom}_{\mathcal{O}_Y}(\mathcal{H}^{q_0}(\mathcal{E}^\bullet)_x, \mathcal{H}^{-l}(\text{Kos}(f_x))) \neq 0.
\]

On the other hand, by Proposition 1.9 with \( Y = \emptyset \), if \( \text{Hom}_{D(X)}^i(\text{Kos}(f_x), \mathcal{E}^\bullet) = 0 \) for every \( i \) and every \( x \), then \( \mathcal{E}^\bullet = 0 \). \( \square \)

2. Fully faithful Integral functors

In this section scheme means a separated scheme of finite type over an algebraically closed field \( k \).

Let \( X \) and \( Y \) be proper schemes, \( \mathcal{K}^\bullet \) an object in \( D^b_c(X \times Y) \) and

\[
\Phi_{X \rightarrow Y}^\mathcal{K}^\bullet: D(X) \rightarrow D(Y)
\]

the integral functor associated to \( \mathcal{K}^\bullet \). If \( X \) is projective and \( \mathcal{K}^\bullet \) has finite homological dimension over both \( X \) and \( Y \), then \( \Phi_{X \rightarrow Y}^\mathcal{K}^\bullet \) maps \( D^b_c(X) \) to \( D^b_c(Y) \) and it has an integral right adjoint (see [2, Def. 2.1], [2, Prop. 2.7] and [2, Prop. 2.9]).

The notion of strong simplicity is the following.

Definition 2.1. An object \( \mathcal{K}^\bullet \) in \( D^b_c(X \times Y) \) is strongly simple over \( X \) if it satisfies the following conditions:

1. For every closed point \( x \in X \) there is a system of parameters \( f_x \) of \( \mathcal{O}_x \) such that

\[
\text{Hom}^i_{D(Y)}(\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\text{Kos}(f_x), \Phi_{X \rightarrow Y}^\mathcal{K}^\bullet(\mathcal{K}(x)_x)), 0) = 0
\]

unless \( x_1 = x_2 \) and \( 0 \leq i \leq \dim X \).

2. \( \text{Hom}_{D(Y)}^0(\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{K}(x)), \Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{K}(x))) = k \) for every closed point \( x \in X \).

\( \triangle \)

Theorem 2.2. Let \( X \) and \( Y \) be proper schemes over an algebraically closed field of characteristic zero, and let \( \mathcal{K}^\bullet \) be an object in \( D^b_c(X \times Y) \) of finite homological dimension over both \( X \) and \( Y \). Assume also that \( X \) is projective and integral. Then the functor \( \Phi_{X \rightarrow Y}^\mathcal{K}^\bullet: D^b_c(X) \rightarrow D^b_c(Y) \) is fully faithful if and only if the kernel \( \mathcal{K}^\bullet \) is strongly simple over \( X \).

Proof. The same proof as [2, Thm. 3.6] works, replacing the use of Proposition 3.1 of [2] by its analogous result (Proposition 1.9). \( \square \)

Definition 2.3. An object \( \mathcal{K}^\bullet \) of \( D^b_c(X \times Y) \) satisfies the orthonormality conditions over \( X \) if it has the following properties:
(1) For every closed point \( x \in X \) there is a system of parameters \( f_x \) of \( \mathcal{O}_x \) such that
\[
\text{Hom}^i_{D(Y)}(\Phi^{K^*}_{X-Y}(\text{Kos}(f_x), \Phi^{K^*}_{X-Y}(\mathbb{k}(x_2))) = 0
\]
unless \( x_1 = x_2 \) and \( 0 \leq i \leq \dim X \).

(2) There exists a closed point \( x \) such that at least one of the following conditions is fulfilled:
\[
(2.1) \text{Hom}^0_{D(Y)}(\Phi^{K^*}_{X-Y}(\mathcal{O}_X), \Phi^{K^*}_{X-Y}(\mathbb{k}(x))) \simeq k.
\]
\[
(2.2) \text{Hom}^0_{D(Y)}(\Phi^{K^*}_{X-Y}(\text{Kos}(f_x)), \Phi^{K^*}_{X-Y}(\mathbb{k}(x))) \simeq k \text{ for any system of parameters } f_x \text{ of } \mathcal{O}_x.
\]
\[
(2.2^*) \text{Hom}^0_{D(Y)}(\Phi^{K^*}_{X-Y}(\mathcal{O}_x/f_x), \Phi^{K^*}_{X-Y}(\mathbb{k}(x))) \simeq k \text{ for any system of parameters } f_x \text{ of } \mathcal{O}_x.
\]
\[
(2.3) 1 \leq \dim \text{Hom}^0_{D(Y)}(\Phi^{K^*}_{X-Y}(\text{Kos}(f_x)), \Phi^{K^*}_{X-Y}(\mathcal{O}_x/f_x)) \leq l(\mathcal{O}_x/f_x) \text{ for any system of parameters } f_x \text{ of } \mathcal{O}_x, \text{ where } l(\mathcal{O}_x/f_x) \text{ is the length of } \mathcal{O}_x/f_x.
\]
\[
(2.3^*) 1 \leq \dim \text{Hom}^0_{D(Y)}(\Phi^{K^*}_{X-Y}(\mathcal{O}_x/f_x), \Phi^{K^*}_{X-Y}(\mathcal{O}_x/f_x)) \leq l(\mathcal{O}_x/f_x) \text{ for any system of parameters } f_x \text{ of } \mathcal{O}_x.
\]


\[\square\]

**Theorem 2.4.** Let \( X \) and \( Y \) be proper schemes over an algebraically closed field of arbitrary characteristic, and let \( K^* \) be an object in \( D_c^b(X \times Y) \) of finite homological dimension over both \( X \) and \( Y \). Assume also that \( X \) is projective, Cohen-Macaulay, equidimensional and connected. Then the functor \( \Phi^{K^*}_{X-Y} : D_c^b(X) \to D_c^b(Y) \) is fully faithful if and only if the kernel \( K^* \) satisfy the orthonormality conditions over \( X \) (Definition 2.3).

**Proof.** The proof is essentially the same as [2, Thm. 3.8]. We give the details.

The direct is immediate. Let us see the converse. Let us denote \( \Phi = \Phi^{K^*}_{X-Y} \). One knows that \( \Phi \) has a right adjoint \( H \) and that \( H \circ \Phi \simeq \Phi^M_{X-Y} \). Using condition (1) of Definition 2.3 one sees that \( \mathcal{M} \) is a sheaf whose support is contained in the diagonal and \( \pi_{1*} \mathcal{M} \) is locally free. Since \( X \) is connected, we can consider the rank \( r \) of \( \pi_{1*} \mathcal{M} \), which is nonzero by condition (2) of Definition 2.3, thus the support of \( \mathcal{M} \) is the diagonal. To conclude, we have only to prove that \( r = 1 \).

Since \( \mathcal{M} \) is a sheaf topologically supported on the diagonal and \( \pi_{1*} \mathcal{M} \) is locally free, it follows that if \( \mathcal{F} \) is a sheaf, then \( \Phi^M_{X-Y}(\mathcal{F}) \) is also a sheaf.

Now assume that \( K^* \) satisfies (2.1) of Definition 2.3. Then
\[
\text{Hom}^0_{D(X)}(\mathcal{O}_X, \Phi^M_{X-Y}(\mathbb{k}(x))) \simeq \text{Hom}^0_{D(Y)}(\Phi^{K^*}_{X-Y}(\mathcal{O}_X), \Phi^{K^*}_{X-Y}(\mathbb{k}(x))) \simeq k.
\]

Hence \( \Phi^M_{X-Y}(\mathbb{k}(x)) \simeq \mathbb{k}(x) \); that is, \( j_x^* \mathcal{M} \simeq \mathbb{k}(x) \), where \( j_x : \{x\} \hookrightarrow X \) is the inclusion, and \( r = 1 \).

If \( K^* \) satisfies (2.2) of Definition 2.3 then
\[
\text{Hom}_{\mathcal{O}_Y}(\mathcal{O}_x/f_x, j_x^* \mathcal{M}) \simeq \text{Hom}^0_{D(X)}(\text{Kos}(f_x), j_x^* \mathcal{M})
\]
\[
\simeq \text{Hom}^0_{D(X)}(\text{Kos}(f_x), \Phi^M_{X-Y}(\mathbb{k}(x)))
\]
\[
\simeq \text{Hom}^0_{D(Y)}(\Phi^{K^*}_{X-Y}(\text{Kos}(f_x)), \Phi^{K^*}_{X-Y}(\mathbb{k}(x))) \simeq k
\]

for any system of parameters \( f_x \) of \( \mathcal{O}_x \). Hence \( j_x^* \mathcal{M} \simeq \mathbb{k}(x) \) and \( r = 1 \).
(2.2*) is equivalent to (2.2), because
\[ \text{Hom}_D^0(\Phi^*_{X,Y}(\text{Kos}^*(f_x)), \Phi^*_{X,Y}(k(x))) \simeq \text{Hom}_D^0(X,\Phi^*_M(k(x))) \]
\[ \simeq \text{Hom}_D^0(O_x/f_x, \Phi^*_M(k(x))) \]
\[ \simeq \text{Hom}_D^0(D(Y), (\Phi^*_{X,Y}(O_x/f_x), (\Phi^*_{X,Y}(k(x)))) \]
where the second isomorphism is due to the fact that $\Phi^*_M(k(x))$ is a sheaf and to $H^0(\text{Kos}^*(f_x)) = 0$.

Finally, assume that $K^*$ satisfies (2.3) of Definition 2.3 (which is equivalent to (2.3*) by similar arguments), and let us prove that then condition (2.2*) of Definition 2.3 holds as well.

We already know that if $F$ is a sheaf supported at a point $x$, then $\phi(F) = \Phi^*_M(F)$ is also a sheaf supported at $x$. Moreover $\phi$ is exact and it has a left adjoint $G^0$ (see the proof of [2, Thm. 3.8]). Let us denote $B = O_x/f_x$.

First notice that
\[ \text{Hom}_D^0(D(Y), (\Phi^*_{X,Y}(B), \Phi^*_M(B))) \simeq \text{Hom}_{O_X}(B, \Phi^*_M(B)) \simeq \text{Hom}_{O_X}(G^0(B), B) \]
Hence, condition (2.3*) means that
\[ (*) \quad 1 \leq \text{dim}\text{Hom}_{O_X}(G^0(B), B) \leq l(B). \]

Analogously, condition (2.2*) means that $\text{Hom}_{O_X}(G^0(B), k(x)) \simeq k$.

Using the exactness of $\phi$, one proves by induction on the length $l(F)$ that the unit map $F \rightarrow \phi(F)$ is injective for any sheaf $F$ supported on $x$. It follows easily (see the proof of [2, Thm. 3.8]) for details) that the morphism $G^0(F) \rightarrow F$ is an epimorphism.

In particular $\eta: G^0(B) \rightarrow B$ is surjective, and $\dim \text{Hom}_{O_X}(G^0(B), B) \geq l(B)$. By $(*)$, $\dim \text{Hom}_{O_X}(G^0(B), B) = l(B)$. Now the proof follows as in [2, Thm. 3.8]: Let $j: \text{Spec } B \hookrightarrow X$ be the inclusion. The exact sequence of $B$-modules
\[ 0 \rightarrow N \rightarrow j^*G^0(B) \xrightarrow{j^*(\eta)} B \rightarrow 0 \]
splits, so that
\[ 0 \rightarrow \text{Hom}_B(B, B) \rightarrow \text{Hom}_B(j^*G^0(B), B) \rightarrow \text{Hom}_B(N, B) \rightarrow 0 \]
is an exact sequence. Then, $\text{Hom}_B(N, B) = 0$ because the two first terms have the same dimension. Let us see that this implies $N = 0$. If $k(x) \rightarrow B$ is a nonzero, and then injective, morphism, we have $\text{Hom}_B(N, k(x)) = 0$ so that $N = 0$ by Nakayama’s lemma. In conclusion, $j^*G^0(B) \simeq B$, and then $\text{Hom}_{O_X}(G^0(B), k(x)) \simeq k$. 

\begin{thebibliography}{9}

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