Parameterized Analysis of Reconfigurable Broadcast Networks

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Abstract. Reconfigurable broadcast networks (RBN) are a model of distributed computation in which agents can broadcast messages to other agents using some underlying communication topology which can change arbitrarily over the course of executions. In this paper, we conduct parameterized analysis of RBN. We consider cubes, (infinite) sets of configurations in the form of lower and upper bounds on the number of agents in each state, and we show that we can evaluate boolean combinations over cubes and reachability sets of cubes in PSPACE. In particular, reachability from a cube to another cube is a PSPACE-complete problem. To prove the upper bound for this parameterized analysis, we prove some structural properties about the reachability sets and the symbolic graph abstraction of RBN, which might be of independent interest. We justify this claim by providing two applications of these results. First, we show that the almost-sure coverability problem is PSPACE-complete for RBN, thereby closing a complexity gap from a previous paper \cite{3}. Second, we define a computation model using RBN, à la population protocols, called RBN protocols. We characterize precisely the set of predicates that can be computed by such protocols.

Keywords: Broadcast networks · Parameterized reachability · Almost-sure coverability · Asynchronous shared-memory systems

1 Introduction

Reconfigurable broadcast networks (RBN) \cite{8,10} are a formalism for modelling distributed systems in which a set of anonymous, finite-state agents execute the same underlying protocol and broadcast messages to their neighbors according to an underlying communication topology. The communication topology is reconfigurable, meaning that the set of neighbors of an agent can change arbitrarily over the course of an execution. Parameterized verification of these networks concerns itself with proving that a given property is correct, irrespective of the number of participating agents. Dually, it can be viewed as the problem of finding an

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execution of some number of agents which violates a given property. Ever since their introduction within this context \cite{10}, RBN have been studied extensively, with various results on (parameterized) reachability and coverability \cite{8,10,3,7}, along with various extensions using probabilities and clocks \cite{5,4}.

In this paper, we first consider the cube-reachability problem for RBN, in which we are given two (possibly infinite) sets of configurations $C$ and $C'$ (called cubes), each of them defined by lower and upper bounds on the number of agents in each state, and we must decide if there is a configuration in $C$ which can reach some configuration in $C'$. The cube-reachability question covers parameterized reachability and coverability problems, and as explained in \cite{3}, also covers the parameterized reachability problem for a generalized model of RBN called RBN with leaders. Moreover, a sub-problem of cube-reachability has already been studied for RBN in \cite{8}. The authors show that this sub-problem is PSPACE-complete. One of the results in our paper is that the entire cube-reachability problem is PSPACE-complete, hence extending the sub-problem considered in \cite{8}, while still retaining the same complexity upper bound.

In fact, our main result, which we call the PSPACE Theorem, is a more general result. It subsumes the above result for cube-reachability and allows for more complex parameterized analysis of RBN. The PSPACE Theorem roughly states that any boolean combination of atoms can be evaluated in PSPACE, where an atom is a finite union of cubes or the reachability set of a finite union of cubes (i.e. $post^*$ or $pre^*$). To prove the PSPACE Theorem, we first consider the so called symbolic graph of a RBN (\cite{8}, Section 5). We prove some structural properties about these graphs, using results from \cite{8}. Next, using these structural properties, we show that the set of reachable configurations of a cube $C$ can be expressed as a finite union of cubes, each having a norm exponentially bounded in the size of the given RBN and $C$. This result then allows us to give an on-the-fly exploration algorithm for proving the PSPACE Theorem.

We believe that the PSPACE Theorem and the results leading to it that we have proven in this paper have further applications to problems concerning RBN. To justify this claim, we provide two applications. First, we show that the almost-sure coverability problem for RBN is PSPACE-complete, thereby closing a complexity gap from a previous paper (\cite{3}, Section 5.3). Second, we define a computation model using RBN, called RBN protocols, which is similar in spirit to the population protocols model \cite{12}. We characterize precisely the set of predicates that can be computed using RBN protocols. This result generalizes the corresponding result for IO protocols, which are a sub-class of population protocols that can be simulated by RBN protocols, as shown in (\cite{3}, Section 6.2).

Finally, by the reduction given in (\cite{3}, Section 4.2), our results on cube-reachability and almost-sure coverability can be transferred to another model of distributed computation called asynchronous shared memory systems (ASMS), giving a PSPACE-completeness result for both of these problems. This solves an open problem from (\cite{6}, Section 6).

To summarize, we have shown that many important parameterized problems of RBN can be solved in PSPACE, that the sub-problem of the cube-reachability
problem defined in [8] can be generalized while retaining the same upper bounds, and that the almost-sure coverability problems for RBN and ASMS are PSPACE-complete, thereby solving open problems from [36]. We believe that our other results might be of independent interest, and we provide an application by introducing RBN protocols and characterizing the set of predicates that they can compute.

The paper is organized as follows. Section 2 contains preliminaries, including the definition of RBN. Section 3 defines the symbolic graph of a RBN, and proves the properties of this graph needed to derive our main result. Section 4 contains the main result that a host of parameterized problems over cubes, including cube-reachability, is PSPACE-complete for RBN. Finally, Sections 5 and 6 give applications of our main results: Section 5 solves the complexity gap for the almost-sure coverability problem, and Section 6 introduces RBN protocols and characterizes their expressive power. Due to lack of space, full proofs of some of the results can be found in the appendix.

2 Preliminaries

The definitions and notations in this section are taken from [3].

2.1 Multisets

A multiset on a finite set $E$ is a mapping $C : E \to \mathbb{N}$, i.e. for any $e \in E$, $C(e)$ denotes the number of occurrences of element $e$ in $C$. We let $\mathcal{M}(E)$ denote the set of all multisets on $E$. Let $\langle e_1, \ldots, e_n \rangle$ denote the multiset $C$ such that $C(e) = |\{ j \mid e_j = e\}|$. We sometimes write multisets using set-like notation. For example, $\langle 2 \cdot a, b \rangle$ and $\langle a, a, b \rangle$ denote the same multiset. Given $e \in E$, we denote by $e$ the multiset consisting of one occurrence of element $e$, that is $\langle e \rangle$. Operations on $\mathbb{N}$ like addition or comparison are extended to multisets by defining them component wise on each element of $E$. Subtraction is allowed as long as each component stays non-negative. We call $|C| \overset{\text{def}}{=} \sum_{e \in E} C(e)$ the size of $C$.

2.2 Reconfigurable Broadcast Networks

Reconfigurable broadcast networks (RBN) are networks consisting of finite-state, anonymous agents and a communication topology which specifies for every pair of processes, whether or not there is a communication link between them. During a single step, a single agent can broadcast a message which is received by all of its neighbors, after which both the agent and its neighbors change their state according to some transition relation. Further, in between two steps, the communication topology can change in an arbitrary manner. For the problems that we consider in this paper, it is easier to forget the communication topology and define the semantics of an RBN directly in terms of collections of agents.
Definition 1. A reconfigurable broadcast network is a tuple $\mathcal{R} = (Q, \Sigma, \delta)$ where $Q$ is a finite set of states, $\Sigma$ is a finite alphabet and $\delta \subseteq Q \times \{!a, \, ?a \mid a \in \Sigma\} \times Q$ is the transition relation.

If $(p, !a, q)$ (resp. $(p, ?a, q)$) is a transition in $\delta$, we will denote it by $p \rightarrow a q$ (resp. $p \rightarrow a q$). A configuration $C$ of a RBN $\mathcal{R}$ is a multiset over $Q$, which intuitively counts the number of processes in each state. Given a letter $a \in \Sigma$ and two configurations $C$ and $C'$ we say that there is a step $C \xrightarrow{a} C'$ if there exists a multiset $\{t, t_1, \ldots, t_k\}$ of $\delta$ for some $k \geq 0$ satisfying the following: $t = p \rightarrow a q$, each $t_i = p_i \rightarrow a q_i$, $C \geq p + \sum_i p_i$, and $C' = C - p - \sum_i p_i + q + \sum_i q_i$. We sometimes write this as $C \xrightarrow{t + t_1 + \ldots + t_n} C'$ or $C \xrightarrow{a} C'$. Intuitively it means that a process at the state $p$ broadcasts the message $a$ and moves to $q$, and for each $1 \leq i \leq k$, there is a process at the state $p_i$ which receives this message and moves to $q_i$. We denote by $\xrightarrow{a}$ the reflexive and transitive closure of the step relation. A run is then a sequence of steps.

Let $\mathcal{R} = (Q, \Sigma, \delta)$ be an RBN. Given configurations $C$ and $C'$, we say $C'$ is reachable from $C$ if $C \xrightarrow{a} C'$. We say $C'$ is coverable from $C$ if there exists $C''$ such that $C \xrightarrow{a} C''$ and $C'' \geq C'$. The reachability problem consists of deciding, given a RBN $\mathcal{R}$ and configurations $C, C'$, whether $C'$ is reachable from $C$ in $\mathcal{R}$. The coverability problem consists of deciding, given a RBN $\mathcal{R}$ and configurations $C, C'$, whether $C'$ is coverable from $C$ in $\mathcal{R}$. Let $\mathcal{S}$ be a set of configurations. The predecessor set of $\mathcal{S}$ is $\text{pre}^*(\mathcal{S}) \overset{\text{def}}{=} \{C \mid \exists C \in \mathcal{S} : C' \xrightarrow{a} C\}$, and the successor set of $\mathcal{S}$ is $\text{post}^*(\mathcal{S}) \overset{\text{def}}{=} \{C \mid \exists C' \in \mathcal{S} : C' \xrightarrow{a} C\}$.

Example 1. Figure 1 illustrates a RBN $\mathcal{R} = (Q, \Sigma, \delta)$ with $Q = \{q_1, q_2, q_3\}$. Configuration $\{3 \cdot q_1\}$ can reach $\{2 \cdot q_1, q_3\}$ in two steps. First, a process broadcasts $a$, the two other processes receive it and move to $q_2$. Then, one of the processes in $q_2$ broadcasts $b$ and moves to $q_1$, while the other one receives $b$ and moves to $q_3$. Notice that $\{q_3\}$ is only coverable from a configuration $\{k \cdot q_1\}$ if $k \geq 3$.

2.3 Cubes and Counting Sets

Given a finite set $Q$, a cube $C$ is a subset of $\mathbb{M}(Q)$ described by a lower bound $L : Q \rightarrow \mathbb{N}$ and an upper bound $U : Q \rightarrow \mathbb{N} \cup \{\infty\}$ such that $C = \{C : L \leq C \leq U\}$.
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U}. Abusing notation, we identify the set C with the pair (L, U). Notice that since U(q) can be \( \infty \) for some state q, a cube can contain an infinite number of configurations. All the results in this paper are true irrespective of whether the constants in a given input cube are encoded in unary or binary.

A finite union of cubes \( \bigcup_{i=1}^{m} (L_i, U_i) \) is called a counting constraint and the set of configurations \( \bigcup_{i=1}^{m} C_i \) it describes is called a counting set. Notice that two different counting constraints may describe the same counting set. For example, let \( Q = \{ q \} \) and let \( (L, U) = (1, 3), (L', U') = (2, 4), (L'', U'') = (1, 4) \). The counting constraints \( (L, U) \cup (L', U') \) and \( (L'', U'') \) define the same counting set. It is easy to show (see also Proposition 2 of [12]) that counting constraints and counting sets are closed under Boolean operations.

Norms. Let \( C = (L, U) \) be a cube. Let \( \|C\|_l \) be the sum of the components of L. Let \( \|C\|_u \) be the sum of the finite components of U if there are any, and 0 otherwise. The norm of \( C \) is the maximum of \( \|C\|_l \) and \( \|C\|_u \), denoted by \( \|C\| \). We define the norm of a counting constraint \( \Gamma = \bigcup_{i=1}^{m} C_i \) as \( \|\Gamma\| = \max_{i \in [1, m]} \{ \|C_i\| \} \).

The norm of a counting set \( S \) is the smallest norm of a counting constraint representing \( S \), that is, \( \|S\| = \min_{\Gamma \subseteq \{C_i\}} \{ \|\Gamma\| \} \). Proposition 5 of [12] entails the following results for the norms of the union, intersection and complement.

**Proposition 1.** Let \( S_1, S_2 \) be counting sets. The norms of the union, intersection and complement satisfy: \( \|S_1 \cup S_2\| \leq \max\{\|S_1\|, \|S_2\|\}, \|S_1 \cap S_2\| \leq \|S_1\| + \|S_2\|,\) and \( \|\overline{S}\| \leq |Q| \cdot \|S_1\| + |Q| \).

Reachability. The reachability problem can be generalized to the cube-reachability problem which consists of deciding, given an RBN \( R \) and two cubes \( C, C' \), whether there exists configurations \( C \in C \) and \( C' \in C' \) such that \( C' \) is reachable from \( C \) in \( R \). If this is the case, we say \( C' \) is reachable from \( C \). The counting set-reachability problem asks, given an RBN \( R \) and two counting sets \( S, S' \), whether there exists cubes \( C \in S \) and \( C' \in S' \) such that \( C' \) is reachable from \( C \) in \( R \). We define cube-coverability and counting set-coverability in an analogous way.

Remark 1. In the paper [8], the authors define a sub-class of the cube-reachability problem, which is called the unbounded initial cube-reachability problem in [3]. More precisely, the sub-class considered in [8] is the following: We are given an RBN and two cubes \( C = (L, U) \) and \( C' = (L', U') \) with the special property that \( L(q) = 0 \) and \( U(q) \in \{0, \infty\} \) for every state q. We then have to decide if \( C \) can reach \( C' \). This problem was shown to be \( \text{PSPACE} \)-complete ([8], Theorem 5.5), whenever the numbers in the input are given in unary. As we shall show later in this paper, the cube-reachability problem itself is in \( \text{PSPACE} \), even when the input numbers are encoded in binary, thereby generalizing the upper bound results given in that paper.
3 Reachability sets of counting sets

In this section, we set the stage for proving the main result of this paper. This main result is given in two stages: First, we show that given a RBN with state set $Q$ and a counting set $S$, the set $\text{post}^*(S)$ is also a counting set and $\|\text{post}^*(S)\| \leq 2^p(\|S\|\cdot|Q|)$ where $p$ is some fixed polynomial. Using this, we then prove that a host of cube-parameterized problems for RBN can be solved in PSPACE.

The rest of this section is organized as follows: To prove the first result, we recall the notion of a symbolic graph of a RBN from [8]. In the symbolic graph, each node is a symbolic configuration of the RBN, which intuitively represents an infinite set of configurations in which the number of agents is fixed in some states, and arbitrarily big in the others. Next, by exploiting the special structure of the symbolic graph, we prove some properties which allow us to show that whenever two nodes in this graph are reachable, they are reachable by a path having a special structure. Finally, using these properties and the connection between symbolic configurations and configurations of the RBN, we prove the desired first result. Once we have shown the first result, we then show how the PSPACE Theorem can be obtained from it.

Throughout this section, we fix an RBN $\mathcal{R} = (Q, \Sigma, \delta)$.

3.1 Symbolic graph

In this subsection, we recall the notion of a symbolic graph of an RBN from [8]. Here, for the sake of convenience, we define it in a slightly different way, but the underlying notion is the same as [8]. Throughout this subsection and the next, we fix a number $k \in \mathbb{N}$.

The symbolic graph of index $k$ associated with the RBN $\mathcal{R}$ is an edge-labelled graph $G_k = (N, E, L)$ where $N = \mathbb{M}_k(Q) \times 2^Q$ is the set of nodes. Here $\mathbb{M}_k(Q)$ denotes the set of multisets on $Q$ of size at most $k$. $E$ is the set of edges and $L : E \rightarrow \Sigma$ is the labelling function. Each node of $G_k$ is also called a symbolic configuration. Intuitively, in each symbolic configuration $(v, S)$, the multiset $v$ (called the concrete part) is used to keep track of a fixed set of at most $k$ agents, and the subset $S$ (called the abstract part) is used to keep track of the support of the remaining agents.

Let $\theta = (v, S)$ and $\theta' = (v', S')$ be two symbolic configurations. There is an edge labelled by $a$ between $\theta$ and $\theta'$ if and only if the following is satisfied: There exists a transition $(q, !a, q') \in \delta$ such that at least one of the following two conditions holds:

- (Broadcast from $v$) There exists a multiset of transitions $\tilde{(p_1, ?a, p'_1), \ldots, (p_l, ?a, p'_l)}$ such that $v' = v - \sum_i p_i + \sum_i p'_i - q + q'$, and for each $q_s \in Q$:
  
  - If $q_s \in S' \setminus S$ then there exists $q'_s \in S$ and $(q'_s, ?a, q_s) \in R$,
  
  - If $q_s \in S \setminus S'$ then there exists $q'_s \in S'$ and $(q_s, ?a, q'_s) \in R$.

- (Broadcast from $S$) There exists a multiset of transitions $\tilde{(p_1, ?a, p'_1), \ldots, (p_l, ?a, p'_l)}$ such that $v' = v - \sum_i p_i + \sum_i p'_i$, $q \in S, q' \in S'$, and for each $q_s \in Q \setminus \{q, q'\}$:
• if \( q_s \in S' \setminus S \) then there exists \( q'_s \in S \) and \( (q'_s, a, q_s) \in R \),
• if \( q_s \in S \setminus S' \) then there exists \( q'_s \in S' \) and \( (q_s, a, q'_s) \in R \).

An edge labelled by \( a \) between \( \theta \) and \( \theta' \) is denoted by \( \theta \xrightarrow{a} \theta' \). The relation \( \sim_{G_k} \) is the reflexive and transitive closure of \( \sim_{G_k} := \cup_{a \in \Sigma} \sim_{G_k}^a \). Whenever the index \( k \) is clear, we will drop the subscript \( G_k \) from these notations.

**Remark 2.** Let \( \theta = (v, S), \theta' = (v', S') \) be two symbolic configurations. By construction, \( \theta \) can only reach \( \theta' \) if \( \|v\| = \|v'\| \).

To give an intuition behind the edges in \( G_k \), recall the intuition that in a symbolic configuration, the concrete part is used to keep track of a fixed set of at most \( k \) processes and the abstract part is used to keep track of the support of the remaining processes. The first condition for the existence of an edge asserts the following: 1) In the concrete part, some process broadcasts the message \( a \) and some subset of processes receive \( a \), 2) In the abstract part, any new state added or any old state deleted comes because of receiving \( a \). The second condition asserts exactly the same, except we now require the process broadcasting the message \( a \) to be from the abstract part.

The symbolic graph of index \( k \) can be thought of as an abstraction of the set of configurations of \( \mathcal{R} \), where only a fixed number of processes are explicitly represented and the rest are abstracted by means of their support alone. To formalize this, given a symbolic configuration \( \theta = (v, S) \), we let \( \llbracket \theta \rrbracket \) denote the following (infinite) set of configurations: \( C \in \llbracket \theta \rrbracket \) if and only if \( C(q) = v(q) \) for \( q \notin S \) and \( C(q) \geq v(q) \) for \( q \in S \).

**Example 2.** The symbolic graph \( G_0 \) of index 0 of the RBN of Example 1 is illustrated in Figure 2. At this index, the graph only keeps track of a subset \( S \subseteq Q \), and the edges correspond to broadcasts from \( S \). Consider the edges from \( \{q_1\} \). The self-loop corresponds to a broadcast of \( a \) that is not received. The edge to \( \{q_1, q_2\} \) corresponds to a broadcast of \( a \) received by at least one process.
in \( q_1 \). There is no edge from \( \{q_3\} \) because there is no broadcast transition from \( q_3 \).

We then have the following lemma, which asserts that runs between two configurations in an RBN induce corresponding runs in the symbolic graph. The proof of the lemma is easily obtained from the definition of the symbolic graph.

**Lemma 1.** Let \( C, C' \) be two configurations of \( \mathcal{R} \) such that \( C \xrightarrow{a} C' \). Then, for every \( \theta \) such that \( C \in \llbracket \theta \rrbracket \), there exists \( \theta' \) such that \( C' \in \llbracket \theta' \rrbracket \) and \( \theta \overset{a}{\Rightarrow} \theta' \).

### 3.2 Properties of the symbolic graph

In this subsection, we prove some properties of the symbolic graph (of any index \( k \)). The first two properties that we prove exhibit some structural properties on the paths of the symbolic graph. The next two properties relate paths over the symbolic graph to runs over the configurations of the given RBN. These four properties will ultimately lead us to prove our two main contributions in the next section.

**First property: Monotonicity.** Let \( k \in \mathbb{N} \) and let \( G_k \) be the symbolic graph of index \( k \) associated with \( \mathcal{R} \). The first key property of \( G_k \) is the following property, which we call **monotonicity**.

**Proposition 2.** Let \( \theta = (v, S) \) and \( \theta' = (v', S') \) be symbolic configurations of \( G_k \). Then the following are true:

- If \( Z \subseteq S \) and \( \theta \overset{a}{\Rightarrow} \theta' \), then \( (v, S) \overset{a}{\Rightarrow} (v', Z \cup S') \).
- If \( Z \subseteq Q \) and \( \theta \overset{a}{\Rightarrow} \theta' \), then \( (v, Z \cup S) \overset{a}{\Rightarrow} (v', Z \cup S') \).

**Proof.** The two points follow immediately from the definition of \( \overset{a}{\Rightarrow} \).

**Second property: Normal Form.** To state the second property, we first need a small definition.

**Definition 2.** Let \( (v_0, S_0) \overset{a}{\Rightarrow} \cdots \overset{a}{\Rightarrow} (v_m, S_m) \) a path in \( G_k \). A pair of indices \( 0 \leq i < j \leq m \) is called a bad pair if \( (S_i \setminus S_{i+1}) \cap S_j \neq \emptyset \). A path is said to be in normal form if it contains no bad pairs, i.e., for all \( 0 \leq i < m \) and any \( j > i \), \( (S_i \setminus S_{i+1}) \cap S_j = \emptyset \).

Intuitively, a path is in normal form if during each step, the states that disappear from the abstract part never reappear again. The following lemma asserts that whenever there is a path between two symbolic configurations, then there is a path between them that is in normal form.

**Lemma 2.** Let \( \theta, \theta' \) be symbolic configurations of \( G_k \) such that there is a path between \( \theta \) and \( \theta' \) of length \( m \). Then, there is a path in normal form between \( \theta \) and \( \theta' \) of length \( m \).
Proof Sketch. Let \( \theta = \theta_0 \leadsto \theta_1 \leadsto \theta_2 \leadsto \ldots \theta_{m-1} \leadsto \theta_m = \theta' \) be the path between \( \theta \) and \( \theta' \). We proceed by induction on \( m \). The claim is clearly true for \( m = 0 \). Suppose \( m > 0 \) and the claim is true for \( m - 1 \). By induction hypothesis, we can assume that the path \( \theta_0 \leadsto \theta_1 \leadsto \ldots \leadsto \theta_{m-1} \) is already in normal form.

Let each \( \theta_i = (v_i, S_i) \). Let \( l \) be the number of bad pairs in the path between \( \theta_0 \) and \( \theta_m \). If \( l = 0 \), then the path is already in normal form and we are done. Suppose \( l > 0 \) and let \( (w, w') \) be a bad pair. Since the path between \( \theta_0 \) and \( \theta_{m-1} \) is already in normal form, it has to be the case that \( w' = m \). Hence, we have \( Z := (S_w \setminus S_{w+1}) \cap S_m \neq \emptyset \).

By Proposition 2 the following is a valid path: \( (v_w, S_w) \leadsto (v_{w+1}, S_{w+1} \cup Z) \leadsto (v_{w+2}, S_{w+2} \cup Z) \ldots (v_{m-1}, S_{m-1} \cup Z) \leadsto (v_m, S_m \cup Z) = (v_m, S_m) \). Let \( \theta'_j := \theta_j \) if \( j \leq w \) and \( (v_j, S_j \cup Z) \) otherwise. Hence, we get a path \( \theta'_0 \leadsto \theta'_1 \leadsto \ldots \theta'_{m-1} \leadsto \theta'_m \).

Let each \( \theta'_i = (v'_i, S'_i) \) and let \( 0 \leq i < j \leq m - 1 \). By a case analysis on where \( i \) and \( j \) are relative to the index \( w \), we can prove that \( (S'_i \setminus S'_{i+1}) \cap S'_j = \emptyset \). Having proved this, it is then clear by construction, that this new path from \( \theta'_0 := \theta_0 \) to \( \theta'_m := \theta_m \) has at most \( l - 1 \) bad pairs only. Hence, we now have a path from \( \theta_0 \) to \( \theta_m \) such that the prefix of length \( m - 1 \) is in normal form and the number of bad pairs has been strictly reduced to \( l - 1 \). Repeatedly applying this procedure leads to a path in normal form between \( \theta_0 \) and \( \theta_m \).

Third property: Refinement. Before we state the third property, we need a small definition. Recall that, given a symbolic configuration \( \theta = (v, S) \), the set \([\theta]\) denotes the set of configurations \( C \) such that \( C(q) = v(q) \) if \( q \notin S \) and \( C(q) \geq v(q) \) otherwise. The following definition refines the set \([\theta]\).

Definition 3. Given a symbolic configuration \( \theta = (v, S) \) and a number \( N \in \mathbb{N} \), let \([\theta]_N\) denote the set of configurations \( C \) such that \( C(q) = v(q) \) if \( q \notin S \) and \( C(q) \geq v(q) + N \) otherwise. Note that \([\theta] = [\theta]_0\).

This definition along with the above two properties now enable us to prove the third property. It roughly states that if a symbolic configuration \( \theta' \) can be reached from another symbolic configuration \( \theta \), then there is a “small” \( N \) such that any configuration in \([\theta']_N\) can be reached from some configuration in \([\theta]\).

Theorem 1. Let \( \theta, \theta' \) be symbolic configurations of \( G_k \) such that \( \theta \leadsto^* \theta' \). Then there exists \( N \leq k \times (2k)^{|Q|} \times (|Q| + 1)^{|Q|+1} + 1 \) such that for all \( C' \in [\theta']_N \), there exists \( C \in [\theta] \) such that \( C \Rightarrow C' \).

Proof Sketch. Suppose \( \theta \leadsto^* \theta' \). If the length of the path is 0, then there is nothing to prove. Hence, we restrict ourselves to the case when the length of the path is bigger than 0. By Lemma 2 there is a path in normal form from from \( \theta \) to \( \theta' \) (say) \( \theta = \theta_0 \leadsto \theta_1 \leadsto \theta_2 \ldots \leadsto \theta_{m-1} \leadsto \theta_m = \theta' \) with each \( \theta_i := (v_i, S_i) \).

Let \( N_0 = 0 \) and let \( N_i = (N_{i-1} + 1) \cdot (|S_{i-1} \setminus S_i| + 1) \) for every \( 1 \leq i \leq m \). In Lemma 5.3 of [8] (more precisely in its proof, in Lemma 6 of the long version [9]), the following fact has been proved:
For every $1 \leq i \leq m$ and for every $C' \in \llbracket \theta_i \rrbracket_{N_i+1}$, there exists $C \in \llbracket \theta_i-1 \rrbracket_{N_i-1+1}$ such that $C \rightarrow C'$.

This immediately proves that for all $C' \in \llbracket \theta' \rrbracket_{N_m+1}$, there exists $C \in \llbracket \theta \rrbracket$ such that $C \rightarrow C'$. If we prove $N_m \leq k \times (2k)^{|Q|} \times (|Q| + 1)^{|Q|+1}$, then the proof of the theorem will be complete.

Notice that if $(v, \theta) \rightsquigarrow (v', S')$ is an edge in $G_k$ then $S' = \emptyset$. This fact, along with the definition of a path in normal form, allows us to easily conclude that the number of indices $i$ such that $|S_{i-1} \setminus S_i| > 0$ is at most $|Q|$. It then follows that except for at most $|Q|$ indices, each index $N_i$ is obtained from $N_i-1$ by simply adding 1 and in the remaining indices, $N_i$ is obtained from $N_i-1$ by adding 1 and then multiplying by a number which is at most $|Q| + 1$. Using this, we can deduce that the maximum value for $N_m$ is at most $(m - |Q| + 1)|Q|(|Q| + 1)^{|Q|}$. Since $m$ is itself the length of the path between $\theta_0$ and $\theta_m$, $m$ is upper bounded by the number of symbolic configurations in $G_k$ which is at most $k \times k^{|Q|} \times 2^{|Q|}$. Overall we get that $N_m \leq k \times (2k)^{|Q|} \times (|Q| + 1)^{|Q|+1}$.

**Remark 3.** A similar result was proved in Lemma 5.3 of [3], but there it was just stated that there exists an $N$ satisfying this property. Moreover from the proof of that lemma, only a doubly exponential bound on $N$ could be inferred.

**Fourth property: Compatibility.** To describe the fourth property, we need the following notion of order on configurations, relative to a given symbolic configuration.

**Definition 4.** Let $\theta = (v, S)$ be a symbolic configuration, and let $C, C'$ be two configurations of $\mathcal{R}$. We define an order $\preceq_\theta$ such that $C \preceq_\theta C'$ if and only if $C, C' \in \llbracket \theta \rrbracket$, and $\forall q \in S, C(q) \leq C'(q)$.

This definition enables us to state our next property, which we dub compatibility. It intuitively says that the order that we have defined is, in some sense, compatible with the edges of the symbolic configurations.

**Lemma 3.** Let $\theta$ be a symbolic configuration of $G_k$, and let $C, C'$ be two configurations of $\mathcal{R}$. If $C \in \llbracket \theta \rrbracket$ and $C \rightarrow C'$, then there exists a symbolic configuration $\theta'$ such that 1) $C' \in \llbracket \theta' \rrbracket$, 2) $\theta \rightsquigarrow^* \theta'$ and 3) for all $C'_1$ such that $C'_1 \preceq_\theta C'$, there exists $C_1 \in \llbracket \theta \rrbracket$ such that $C_1 \rightarrow C'_1$.

**Proof.** Let $\theta$ be a symbolic configuration and $C, C'$ be configurations such that $C \in \llbracket \theta \rrbracket$ and $C \rightarrow C'$. Let $C = C_0 \rightarrow \cdots \rightarrow C_{m-1} \rightarrow C_m = C'$ denote the run between $C$ and $C'$. We prove the property by induction on $m$. For $m = 0$, we have $C = C'$. The property is easily seen to hold with $\theta' = \theta$.

Suppose now that $m \geq 1$, and that the property holds for all $n \leq m$. By induction hypothesis, for the configuration $C_{m-1}$, there exists a symbolic configuration $\theta_{m-1}$ satisfying the property, in particular $\theta \rightsquigarrow^* \theta_{m-1}$. Since $C_{m-1} \rightarrow a C_m$ for some $a \in \Sigma$, by Lemma 1 there exists a symbolic configuration $\theta_m$ such that $C_m \in \llbracket \theta_m \rrbracket$, and $\theta_{m-1} \rightsquigarrow^a \theta_m$. Using $\theta \rightsquigarrow^* \theta_{m-1}$, we obtain that $\theta \rightsquigarrow^* \theta_m$. 


Let $\theta_{m-1} = (v_{m-1}, S_{m-1})$ and $\theta_m = (v_m, S_m)$. Let $C'_m \in [\theta_m]$ be such that $C'_m \supseteq \theta_m$ $C_m$. We will construct a configuration $C'_{m-1} \in [\theta_{m-1}]$ such that $C'_{m-1} \supseteq \theta_{m-1}$ $C_{m-1}$ and $C'_{m-1} \Rightarrow C'_m$. If we construct such a configuration, then by induction hypothesis, there is a $C_1 \in [\theta]$ such that $C_1 \Rightarrow C'_{m-1} \Rightarrow C'_m$, which will conclude the proof.

Let $C'_{m-1}(q) = C_{m-1}(q)$ for all $q \not\in S_{m-1}$. To define $C'_{m-1}$ on $S_{m-1}$, we first define a mapping $\text{pred}$ from states in $S_m$ to states of $S_{m-1} \cup \overline{S_{m-1}} = Q$ as follows. Given $q' \in S_m$:

- If $q' \in S_{m-1}$, $\text{pred}(q') = q'$;
- If $q' \not\in S_{m-1}$, by definition of edges in the symbolic graph, there exists $q \in S_{m-1}$ such that $(q, ?a, q')$ is a transition. Then $\text{pred}(q') = q$ for one (arbitrary but fixed) such $q$.

By definition, $C'_m(q) = C_m(q)$ for all $q \not\in S_m$. For all $q \in S_m$, let $n_q = C'_m(q) - C_m(q)$. Intuitively, we want to place these $n_q$ processes in the right places of $C'_m$ so that $C'_m \Rightarrow C'_m$. For all $q \in S_{m-1}$, let $C'_{m-1}(q) = C_{m-1}(q) + \sum q' \in S_q, \text{pred}(q') = q n_{q'}$. By definition, $C'_{m-1} \supseteq \theta_{m-1} C_{m-1}$. So all that remains is to prove that $C'_{m-1} \Rightarrow C'_m$.

Let $\rho = \frac{t + t_1, \ldots, t_n}{C_m}$ where $t = (p, !a, p')$ and each $t_i = (p_i, ?a, p'_i)$. If we let $S_m \setminus S_{m-1} = \{q'_1, \ldots, q'_w\}$, then by definition there is a transition $t'_i := (\text{pred}(q'_i), ?a, q'_i)$ for each $i$. Additionally, $C'_{m-1}(\text{pred}(q'_i)) \geq C_{m-1}(\text{pred}(q'_i)) + n_{q'_i}$. This allows us to do $\rho = \frac{t + t_1, \ldots, t_n, n_{q'_1}, t'_1, n_{q'_2}, t'_2, \ldots, n_{q'_w}, t'_w}{C'_m}$, which concludes the proof.

\[\square\]

### 4 The PSPACE Theorem

In this section, we prove our two main contributions. First, we show that given a cube $C$, $\text{post}^*(C)$ is a counting set of bounded size. Using this, we show our main result: any boolean combination of atoms can be evaluated in PSPACE, where an atom is a counting set or the reachability set of a counting set. We call this the PSPACE Theorem. The intuition behind the PSPACE Theorem is that the norms of the counting sets obtained by such combinations are “small”, and so we only need to examine small configurations to verify them, thus yielding a PSPACE algorithm for checking correctness. In particular, the PSPACE Theorem will show that the cube-reachability problem is in PSPACE. We fix an arbitrary RBN $R = (Q, \Sigma, \delta)$ for the rest of the section.

We start by drawing links between cubes and symbolic configurations.

- Given a symbolic configuration $\theta = (v, S)$, we let $C_{\theta}$ be the cube $(L, U)$ where $L = v$, and $U(q) = v(q)$ if $q \not\in S$ and $U(q) = \infty$ otherwise. Then $C_{\theta} = [\theta]$.
- Given a cube $C = (L, U)$, we define $\Delta_C$ to be the set of symbolic configurations $\theta = (v, S)$ with $S = \{q \mid U(q) = \infty\}$ and $L(q) \leq v(q) \leq U(q)$ if $q \not\in S$ and $v(q) = L(q)$ otherwise. Then $[\Delta_C] = C$. 


Theorem 2. A counting set with "small" norm.  

Indeed, if \( C = (L, U) \) and \((v, S) \in \Delta_C\), then \(|v| \leq |L| + |U_f|\) where \(U_f(q) = 0\) if \(U(q) = \infty\) and \(U_f(q) = U(q)\) otherwise. Since \(\|C\| = \max(|L|, |U_f|)\), we have the desired result. By Remark 2 we know that symbolic configurations in the graph of index \(2\|C\|\) can only reach symbolic configurations which are also in the graph of index \(2\|C\|\).

Lemma 4. Given a cube \(C\), the sets \(\Delta_C\) and \(\text{post}^*(\Delta_C)\) are included in the symbolic graph of index \(2\|C\|\).

There are only a finite number of symbolic configurations in the graph of a given index. Therefore \(\text{post}^*(\Delta_C)\) is a finite set of symbolic configurations \(\theta\). It follows that \([\text{post}^*(\Delta_C)]\) is the finite union of the cubes \(C_\theta\), and thus a counting set.

Unfortunately, it is in general not the case that \(\text{post}^*(\Delta_C) = [\text{post}^*(\Delta_C)]\), which would close our argument. However, we will show that for each symbolic configuration \(\theta\) in \(\text{post}^*(\Delta_C)\), there is a counting set \(S_\theta \subseteq [\theta]\) such that the finite union of these counting sets is equal to \(\text{post}^*(\Delta_C)\). This will then show our first important result, namely that the reachability set of a counting set is also a counting set with "small" norm.

Theorem 2. Let \(C\) be a cube. Then \(\text{post}^*(\Delta_C)\) is a counting set and

\[
\|\text{post}^*(\Delta_C)\| \in O((\|C\| \cdot |Q|)^{|Q|+2})
\]

The same holds for \(\text{pre}^*\) by using the given RBN with reversed transitions.

Proof. We start by defining a counting set \(M\) of configurations, which we will then prove to be equal to \(\text{post}^*(\Delta_C)\). Given a symbolic configuration \(\theta\) of \(\text{post}^*(\Delta_C)\), we define the set \(\min(\theta, C)\) to be the set of configurations \(C \in [\theta]\) such that \(C\) is minimal for the order \(\preceq_{\theta}\) over the configurations of \(\text{post}^*(\Delta_C)\), i.e.

\[
\min(\theta, C) = \min_{\preceq_{\theta}} \{C \in [\theta] \mid C \in \text{post}^*(\Delta_C)\}
\]

We can now define \(M\) to be the following set

\[
M = \bigcup_{\theta \in \text{post}^*(\Delta_C)} \bigcup_{C \in \min(\theta, C)} C_\theta^C,
\]

where \(C_\theta^C\) is the cube \(C_{(C, S)}\) for \(S\) such that \(\theta = (v, S)\). Since \(M\) is a finite union of cubes, it is a counting set.

We show that \(\text{post}^*(\Delta_C) \subseteq M\). Let \(C \in \text{post}^*(\Delta_C)\). There exists \(C_0 \in C\) such that \(C_0 \rightarrow C\), and there exists \(\theta_0 \in \Delta_C\) such that \(C_0 \in [\theta_0]\). Applying Lemma 4 we obtain the existence of \(\theta \in \text{post}^*(\theta_0) \subseteq \text{post}^*(\Delta_C)\) such that \(C \in [\theta]\). Now, there exists a configuration \(C' \in \min(\theta, C)\) such that \(C' \preceq_{\theta} C\). By definition of \(C_\theta^C\), \(C\) is in \(C_\theta^C\) and thus in \(M\).

Now we show that \(M \subseteq \text{post}^*(\Delta_C)\). Let \(C \in M\). By definition, there must be a symbolic configuration \(\theta \in \text{post}^*(\Delta_C)\) and a configuration \(C' \in \text{post}^*(\Delta_C)\) such
that \( C' \preceq_\theta C \). By the Compatibility Lemma (Lemma 5), \( C \) is in post*(\( C \)) as well.

All that remains is to bound the norm of \( M \). To do this, let \( \theta = (v, S) \in \text{post}^*(\Delta C) \) and let \( C \in \min(\theta, C) \). If we bound the norm of \( \|C_C^N\| \) by the desired quantity, then the proof will be complete. Noticing that \( \|C_C^N\| = |C| \), it suffices to bound \(|C|\) by the desired quantity, which is what we shall do now.

By Theorem 11 and Lemma 14, there exists an \( N \leq 2\|C\| \times (4\|C\|)^Q \times (|Q| + 1)^{|Q|+1} \) such that \( [\text{post}^*(\Delta C)]_N \subseteq [\text{post}^*(\|\Delta C\|)] = \text{post}^*(C) \). By definition of \( C \), there must be a smallest \( N' \) such that \( C(q) \leq v(q) + N' \) for every state \( q \). If \( N' > N \), then let \( C_N \) be the configuration given by \( C_N(q) = \min(C(q), v(q)+N) \). We get that \( C_N \in [\theta]_N \subseteq [\text{post}^*(\Delta C)]_N \subseteq \text{post}^*(C) \), and so \( C_N \preceq_\theta C \) and \( C_N \in \text{post}^*(C) \), which is a contradiction to the minimality of \( C \). Hence \( N' \leq N \) and so \(|C| \leq |v| + |Q| \cdot N \). Since \( \theta = (v, S) \) is in \text{post}^*(\Delta C) \), by Lemma 11 we have that \(|v| \leq 2\|C\| \). Substituting the upper bounds for \(|v|\) and \( N \) in the inequality \(|C| \leq |v| + |Q| \cdot N \) then gives the required upper bound for \(|C|\), thereby finishing the proof.

This result also holds for \text{pre}*(\( C \)). If \( R = (Q, \Sigma, R) \) is the given RBN, consider the “reverse” RBN \( R_r \), defined as \( R = (Q, \Sigma, R_r) \) where \( R_r \) has a transition \((q, \star a, q') \) for \( \star \in \{!, ?\} \) iff \( R_r \) has a transition \((q', \star a, q) \). Notice that \( R_r \) is still an RBN and that \text{post}^*(\( C \)) in \( R \) is equal to \text{pre}^*(\( C \)) in \( R_r \).

Recall that counting sets are closed under boolean operations. With the above theorem, plus the fact that counting sets are finite unions of cubes, we obtain the following closure result.

**Corollary 1 (Closure).** Counting sets are closed under post*, pre* and boolean operations.

We are now ready to show our main result, the \text{PSPACE} Theorem. We show that there exist \text{PSPACE} algorithms to evaluate boolean combinations over counting sets and reachability set of counting sets. This result and its proof are adapted from a similar result for population protocols in [13].

Given a counting constraint \( \Gamma \), we let \([\Gamma]\) denote the counting set described by \( \Gamma \). To state our result, we first define some “nice” expressions.

**Definition 5.** A nice expression is any expression that is constructed by the following syntax:

\[
E := \Gamma \mid \text{post}^*(\Gamma) \mid \text{pre}^*(\Gamma) \mid E \cap E \mid E \cup E \mid \overline{E}
\]

where \( \Gamma \) is any counting constraint.

If \( E \) is a nice expression, then the size of \( E \), denoted by \(|E|\), is defined as follows:

- If \( E = \Gamma \) or \text{post}^*(\( \Gamma \)) or \text{pre}^*(\( \Gamma \)), then \(|E| = 1\);
- If \( E = E_1 \cup E_2 \) or \( E = E_1 \cap E_2 \), then \(|E| = |E_1| + |E_2|\);
- If \( E = \overline{E_1} \), then \(|E| = |E_1| + 1\).
The set of configurations that is described by a nice expression $E$ can be defined in a straightforward manner, and is denoted as $[E]$. Notice that any nice expression $E$ is a counting constraint, and $[E]$ is a counting set, by the Closure Corollary [1].

**Theorem 3 (PSPACE Theorem).** Let $E$ be a nice expression and let $N$ be the maximum norm of the counting constraints appearing in $E$. Then $[E]$ is a counting set of norm at most exponential in $N, |E|$ and $|Q|$. Further, the membership and emptiness problems for $[E]$ are in PSPACE.

**Proof.** Recall that $[E]$ is a counting set, by the Closure Corollary (Corollary [1]). The exponential bounds for the norms follow immediately from Proposition [1] and Theorem [2]. The membership complexity for union, intersection and complement is easy to see. Without loss of generality it suffices to prove that membership in $\text{post}^*(\Gamma)$ is in PSPACE, where $\Gamma$ is a counting constraint.

By Savitch’s Theorem $\text{NPSPACE} = \text{PSPACE}$, so we provide a nondeterministic algorithm. Given $(C, \Gamma)$, we want to decide whether $C \in \text{post}^*(\Gamma)$. The algorithm first guesses a configuration $C_0 \in \Gamma$ of the same size as $C$, verifies that $C_0$ belongs to $\Gamma$, and then simply guesses an execution starting at $C_0$, step by step. The algorithm stops if either the configuration reached at some step is $C$, or if it has guessed more steps than the number of configurations of size $|C|$. This concludes the discussion regarding the membership complexity.

To see that checking emptiness of $E$ is in PSPACE, notice that if $E$ is nonempty, then it has an element of size at most $\|E\|$. We can guess such an element $C$ in polynomial space (by representing each coefficient in binary), and verify that $C$ is indeed in $E$ by means of the PSPACE membership algorithm. 

This result is a powerful tool which can be used to prove that a host of problems are in PSPACE for RBN. For instance, the cube-reachability problem for cubes $C$ and $C'$ is just checking if $\text{post}^*(C) \cap C'$ is empty, which by the PSPACE Theorem can be done in PSPACE. Combining this with Remark [1], we obtain the following result.

**Theorem 4.** Cube-reachability is PSPACE-complete for RBN.

By the reduction given in Section 4.2 of [3], this result also proves that cube-reachability is PSPACE-complete for asynchronous shared-memory systems (ASMS), which is another model of distributed computation where agents communicate by a shared register. Due to lack of space, we defer a discussion of this result to the appendix.

We will demonstrate further applications of the PSPACE Theorem in the next section.

5 Application 1: Almost-sure coverability

Having presented our PSPACE Theorem and the closure property for reachability sets of counting sets, we now provide two applications. For the first one, we
consider the almost-sure coverability problem for RBN. Using our new results, we prove that this problem is PSPACE-complete.

The rest of the section is as follows: We first recall the definition of the almost-sure coverability problem, give a characterization of it in terms of counting sets and then prove PSPACE-completeness. Throughout this section, we fix a RBN \( R = (Q, \Sigma, \delta) \) with two special states \( \text{init}, \text{fin} \in Q \), which will respectively be called the initial and final states.

5.1 The almost-sure coverability problem

Let \( \uparrow \text{fin} \) denote the set of all configurations \( C \) of \( R \) such that \( C(\text{fin}) \geq 1 \). For any \( k \geq 1 \), we say that the configuration \( \uparrow k \cdot \text{init} \) almost-surely covers \( \text{fin} \) if and only if \( \text{post}^*(\uparrow k \cdot \text{init}) \subseteq \text{pre}^*(\uparrow \text{fin}) \). The reason behind calling this the almost-sure coverability relation is that the definition given here is equivalent to covering the state \( \text{fin} \) from \( \uparrow k \cdot \text{init} \) with probability 1 under a probabilistic scheduler which picks agents uniformly at random at each step.

The number \( k \) is called a cut-off if one of the following is true: Either, 1) for all \( h \geq k \), the configuration \( \uparrow h \cdot \text{init} \) almost-surely covers \( \text{fin} \), in which case \( k \) is called a positive cut-off; or, 2) for all \( h \geq k \), the configuration \( \uparrow h \cdot \text{init} \) does not almost-surely cover \( \text{fin} \), in which case \( k \) is called a negative cut-off. The following was proved in Theorem 9 of [3].

**Theorem 5.** Given an RBN with two states init, fin, a cut-off always exists. Whether the cut-off is positive or negative can be decided in EXPSPACE.

Our main result of this section is that

**Theorem 6.** Deciding whether the cut-off of a given RBN is positive or negative is PSPACE-complete. Moreover, a given RBN always has a cut-off which is at most exponential in its number of states.

5.2 A characterization of almost-sure coverability

We now rewrite the definition of almost-sure coverability in terms of counting sets. Let \([\text{init}]\) be the cube such that \( L(q) = U(q) = 0 \) if \( q \neq \text{init} \) and \( L(\text{init}) = 0, U(\text{init}) = \infty \). Notice that by definition, \( \uparrow \text{fin} \) is a cube. We now consider the set of configurations defined by \( S := \text{post}^*([\text{init}]) \cap \text{pre}^*(\uparrow \text{fin}) \). By our PSPACE Theorem 3, \( S \) is a counting set such that the norm of \( S \) is at most \( 2^{p(|Q|)} \) for some fixed polynomial \( p \). We now claim the following.

**Theorem 7.** \( R \) has a positive cut-off if and only if \( S \) is finite. Moreover, \( |Q| \cdot |S| \) is an upper bound on the size of the cut-off for \( R \) and so \( R \) has a cut-off which is exponential in its number of states.

**Proof.** Let \( N \) be the norm of \( S \). Suppose \( S \) is finite. If \( C \in S \), then \( \sum_{q \in Q} C(q) \leq |Q| \cdot N \). So, if \( C \) is any configuration of size \( h > |Q| \cdot N \) such that \( C \in \text{post}^*(\uparrow h \cdot \text{init}) \) then \( C \in \text{pre}^*(\uparrow \text{fin}) \). Hence, \( |Q| \cdot N \) is a positive cut-off for \( R \).
Suppose $S$ is infinite, and let $\cup_i C_i$ be a counting constraint for $S$ whose norm is $N$. Then there must exist an index $i$ with $C_i := (L, U)$ and a state $p$ such that $U(p) = \infty$. For each $h \geq N$, consider the configuration $C_h$ given by $C_h(q) = L(q)$ if $q \neq p$ and $C_h(p) = h$. Notice that $C_h \in S$ and so $C_h \in post^*([init]) \cap pre^*(\uparrow fin)$. Hence, for every $h \geq |Q| \cdot N$, we have exhibited a configuration of size $h$, reachable from $(\cup \cdot init)$ but from which $fin$ is not coverable. Thus $N$ is a negative cut-off for $R$.

Remark 4. Notice that we have shown that if $S$ is finite, then $R$ has a positive cut-off and if $S$ is infinite, then $R$ has a negative cut-off. This gives an alternative proof of the fact that a cut-off always exists for a given RBN.

5.3 PSPACE-completeness of the almost-sure coverability problem

Because of Theorem 4 we now have the following result.

Lemma 5. Deciding whether the cut-off of a given RBN is positive or negative can be done in PSPACE.

Proof Sketch. By Theorem 7 it follows that a given RBN has a negative cut-off iff $S = post^*([init]) \cap pre^*(\uparrow fin)$ is infinite. We have already seen that $S$ is a counting set such that the norm of $S$ is at most $N := 2^p(|Q|)$ for some fixed polynomial $p$.

Let $\cup_i C_i$ be a counting constraint for $S$ which minimizes its norm and let each $C_i = (L_i, U_i)$. Hence, $L_i(q) \leq N$ for every state $q$. Further, $S$ is infinite iff there is an index $i$ and a state $q$ such that $U_i(q) = \infty$. Using these two facts, we can then show that $S$ is infinite iff there is a state $q$ and a configuration $C \in S$ such that $C(q') \leq N$ for every $q' \neq q$ and $C(q) = N + 1$.

Hence, to check if $S$ is infinite, we just have to guess a state $q$ and a configuration $C$ such that $C(q') \leq N$ for every $q' \neq q$ and $C(q) = N + 1$ and check if $C \in S$. Since guessing $C$ can be done in polynomial space (by representing every number in binary), by the PSPACE Theorem (Theorem 3), we can check if $C \in S$ in polynomial space as well, which concludes the proof of the theorem.

We also have the accompanying hardness result.

Lemma 6. Deciding whether the cut-off of a given RBN is positive or negative is PSPACE-hard.

Similar to the cube-reachability problem, our result on almost-sure coverability also applies to the related model of ASMS. This solves an open problem from [6]. For lack of space, we once again defer this discussion to the appendix.

6 Application 2: Computation by RBN

In this section we give another application of our results. We introduce a model of computation using RBN called RBN protocols. We take inspiration from the
extensively-studied model of population protocols \cite{1213}. The reader can consult the above references for more details on population protocols.

In our model, reconfigurable networks of identical, anonymous agents interact to compute a predicate $\varphi : \mathbb{N}^k \rightarrow \{0, 1\}$. We show that RBN protocols compute exactly the threshold predicates, which we will define more formally below.

### 6.1 RBN Protocols

We introduce our computation model. The notation mimics that of \cite{14}.

**Definition 6.** An RBN protocol is a tuple $P = (Q, \Sigma, \delta, I, O)$ where $(Q, \Sigma, \delta)$ is an RBN, $I = \{q_1, \ldots, q_k\}$ is a set of input states, and $O : Q \rightarrow \{0, 1\}$ is an output function.

Configurations and runs of $P$ are the same as that of the underlying RBN. A configuration $C$ is called a $0$-consensus (respectively a $1$-consensus) if $C(q) > 0$ implies $O(q) = 0$ (respectively $O(q) = 1$). For $b \in \{0, 1\}$, a $b$-consensus $C$ is stable if every configuration reachable from $C$ is also a $b$-consensus. A run $C_0 \rightarrow C_1 \rightarrow C_2 \cdots$ of $P$ is fair if it is finite and cannot be extended by any step, or if it is infinite and the following condition holds for all configurations $C, C'$: if $C \rightarrow C'$ and $C = C_i$ for infinitely many $i \geq 0$, then the step $C \rightarrow C'$ appears infinitely along the run. In other words, if a fair run reaches a configuration infinitely often, then all the configurations reachable in a step from that configuration will be reached infinitely often from it.

A fair run $C_0 \rightarrow C_1 \rightarrow \ldots$ converges to $b$ if there is $i \geq 0$ such that $C_j$ is a $b$-consensus for every $j \geq i$. For every $v \in \mathbb{N}^k$, let $C_v$ be the configuration given by $C_v(q_i) = v_i$ for every $q_i \in I$, and $C_v(q) = 0$ for every $q \in Q \setminus I$. We call $C_v$ the initial configuration for input $v$. The protocol $P$ computes the predicate $\varphi : \mathbb{N}^k \rightarrow \{0, 1\}$, if for every $v \in \mathbb{N}^k$, every fair run starting at $C_v$ converges to $\varphi(v)$.

![Fig. 3. An RBN protocol $P$.](image)

**Example 3.** Adding the dashed line transitions to the RBN of Example 1 yields the RBN protocol $P = (Q, \Sigma, \delta, I, O)$ illustrated in Figure 3. The initial state is
$q_1$, i.e. $I = \{q_1\}$, and the output function is defined such that $O(q_1) = O(q_2) = 0$ and $O(q_3) = 1$. If there is a process in $q_3$, it can “attract” the rest of the processes there using the new dashed transitions. As with the RBN of Example 1, a process can be put in $q_3$ starting from the initial configuration $\{k \cdot q_1\}$ if and only if $k \geq 3$. This RBN protocol computes the predicate $x \geq 3$: if there are less than 3 processes originally in $q_1$ then they stay in states with output 0, and if there are more, then in a fair run a process eventually enters $q_3$, and eventually the others follow, thus converging to 1.

### 6.2 Expressivity

In this section, we show that RBN protocols compute exactly the predicates definable by counting sets. A predicate $\varphi : \mathbb{N}^k \rightarrow \{0, 1\}$ is **definable by counting sets** if for every $b \in \{0, 1\}$, the sets $\{v \mid \varphi(v) = b\}$ are counting sets.

For $b \in \{0, 1\}$, define the following sets of configurations:

- Let $C_b$ be the set of $b$-consensus configurations.
- Let $ST_b$ be the set $\text{pre}^\ast (C_b)$ of stable $b$-consensuses. These are the configurations from which one can reach only $b$-consensuses.
- Let $I_b$ be the set of initial configurations $C_v$ for inputs $v$ such that $\varphi(v) = b$.

The next lemma states that every predicate computed by a protocol is definable by counting sets.

**Lemma 7.** Let $\mathcal{P}$ be a RBN protocol that computes the predicate $\varphi : \mathbb{N}^k \rightarrow \{0, 1\}$. Then for every $b \in \{0, 1\}$, the sets $I_b, C_b$ and $ST_b$ are all counting sets. This entails that $\varphi$ is definable by counting sets.

**Proof Sketch.** Fix a $b \in \{0, 1\}$. It is easy to see that $C_b$ is a cube. Unraveling the definitions of $I_b$ and $ST_b$, we can express them in terms of $C_b$ by using boolean operations and $\text{pre}^\ast$. By the Closure Corollary (Corollary 1), they are counting sets. Set $\{v \mid \varphi(v) = b\}$ is simply $I_b$ restricted to $I$, and so we are done. □

The next lemma states the converse result. It essentially uses the fact that there is a sub-class of population protocols called IO protocols which compute exactly the predicates definable by counting sets (Theorem 7 and Theorem 39 of [14]), and that IO protocols are a sub-class of RBN (Section 6.2 of [3]).

**Lemma 8.** Let $\varphi : \mathbb{N}^k \rightarrow \{0, 1\}$ be a predicate definable by counting sets. Then there exists a RBN protocol computing $\varphi$.

By Lemma 7 and Lemma 8 we get our result.

**Theorem 8.** RBN protocols compute exactly the predicates definable by counting sets.

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Lemma 1. Let $C, C'$ be two configurations of $R$ such that $C \overset{a}{\rightarrow} C'$. Then, for every $\theta$ such that $C \in \llbracket \theta \rrbracket$, there exists $\theta'$ such that $C' \in \llbracket \theta' \rrbracket$ and $\theta \sim^a \theta'$.

Proof. Let $C \overset{t+t_1, \ldots, t_n}{\rightarrow} C'$, where $t = (p, !a, p')$ and each $t_i = (p_i, ?a, p'_i)$. Let $\theta = (v, S)$ such that $C \in \llbracket \theta \rrbracket$. Let $t_1, \ldots, t_k$ be the subset of receive transitions such that $p_1, \ldots, p_k \notin S$, and let $t_{k+1}, \ldots, t_n$ be the subset of receive transitions such that $p_{k+1}, \ldots, p_n \in S$. We have two cases.

- If $p \notin S$, then we perform a “broadcast from $v$”. Let $v' = v - \sum_i p_i + \sum_{i=1}^k p'_i - p + p'$. Let $S' = S \cup \{p'_i | i \in \{k+1, n\}\}$.
- If $p \in S$, then we perform a “broadcast from $S$”. Let $v' = v - \sum_i p_i + \sum_{i=k+1}^n p'_i$. Let $S' = S \cup \{p'\} \cup \{p'_i | i \in \{k+1, n\}\}$.

Since $C \overset{t+t_1, \ldots, t_n}{\rightarrow} C'$, $C(q) > 0$ for $q \in \{p, p_1, \ldots, p_n\}$ and thus $v'$ is well-defined. Let $\theta'$ be $(v', S')$. By our definition of $v', S'$, there is an edge $\theta \sim^a \theta'$ in the symbolic graph.
Lemma 2. Let $\theta, \theta'$ be symbolic configurations of $G_k$ such that there is a path between $\theta$ and $\theta'$ of length $m$. Then, there is a path in normal form between $\theta$ and $\theta'$ of length $m$.

Proof. Let $\theta = \theta_0 \leadsto \theta_1 \leadsto \ldots \leadsto \theta_{m-1} \leadsto \theta_m = \theta'$ be the path between $\theta$ and $\theta'$. We proceed by induction on $m$. The claim is clearly true for $m = 0$. Suppose $m > 0$ and the claim is true for $m - 1$. By induction hypothesis, we can assume that the path $\theta_0 \leadsto \theta_1 \leadsto \ldots \leadsto \theta_{m-1}$ is already in normal form.

Let each $\theta_i = (v_i, S_i)$. Let $l$ be the number of bad pairs in the path between $\theta_0$ and $\theta_m$. If $l = 0$, then the path is already in normal form and we are done. Suppose $l > 0$ and let $(w, w')$ be a bad pair. Since the path between $\theta_0$ and $\theta_{m-1}$ is already in normal form, it has to be the case that $w' = m$. Hence, we have $Z := (S_w \setminus S_{w+1}) \cap S_m \neq \emptyset$.

By Proposition 2, the following is a valid path: $(v_w, S_w) \leadsto (v_{w+1}, S_{w+1} \cup Z) \leadsto (v_{w+2}, S_{w+2} \cup Z) \ldots (v_{m-1}, S_{m-1} \cup Z) \leadsto (v_m, S_m)$. Let $\theta'_j := \theta_j$ if $j \leq w$ and $(v_j, S_j \cup Z)$ otherwise. Hence, we get a path $\theta'_0 \leadsto \theta'_1 \leadsto \ldots \theta'_{m-1} \leadsto \theta'_m$.

Let each $\theta'_i = (v'_i, S'_i)$. We first claim that the path between $\theta'_0$ and $\theta'_{m-1}$ is in normal form. Indeed, suppose there exists $0 \leq i < j \leq m - 1$ such that $(S'_i \setminus S'_{i+1}) \cap S'_j \neq \emptyset$. There are four cases:

- $i \leq w$ and $j \leq w$: In this case $S'_i \setminus S'_{i+1} = S_i \setminus S_{i+1}$ and $S'_j = S_j$, and since the path between $\theta_0$ and $\theta_{m-1}$ is in normal form, this case cannot happen.
- $w < i$ and $w < j$: In this case $S'_i \setminus S'_{i+1} = S_i \setminus S_{i+1}$ and $S'_j = S_j \cup Z$. Since the path between $\theta_0$ and $\theta_{m-1}$ is in normal form, this should then imply that $(S_i \setminus S_{i+1}) \cap Z \neq \emptyset$. By definition this means that $(S_w \setminus S_{w+1}) \cap S_i \neq \emptyset$ which contradicts the fact that the path between $\theta_0$ and $\theta_{m-1}$ is in normal form.
- $i < w$ and $w < j$: Similar to the case before, this should then imply that $(S_i \setminus S_{i+1}) \cap Z \neq \emptyset$. By definition this means that $(S_i \setminus S_{i+1}) \cap S_w \neq \emptyset$ which contradicts the fact that the path between $\theta_0$ and $\theta_{m-1}$ is in normal form.
- $i = w$ and $w < j$: In this case $S'_i \setminus S'_{i+1} = S_i \setminus (S_{i+1} \cup Z)$ and $S'_j = S_j \cup Z$. This would then imply that $(S_i \setminus S_{i+1}) \cap S_j \neq \emptyset$ which contradicts the fact that the path between $\theta_0$ and $\theta_{m-1}$ is in normal form.

It is then clear by construction, that this new path from $\theta'_0 := \theta_0$ to $\theta'_m := \theta_m$ has at most $l - 1$ bad pairs only. Hence, we now have a path from $\theta_0$ to $\theta_m$ such that the prefix of length $m - 1$ is in normal form and the number of bad pairs has been strictly reduced to $l - 1$. Repeatedly applying this procedure, leads to a path in normal form between $\theta_0$ and $\theta_m$. □

Theorem 1. Let $\theta, \theta'$ be symbolic configurations of $G_k$ such that $\theta \leadsto^* \theta'$. Then there exists $N \leq k \times (2k)^{|Q|} \times (|Q| + 1)^{|Q|+1} + 1$ such that for all $C' \in \llbracket \theta' \rrbracket_N$, there exists $C \in \llbracket \theta \rrbracket$ such that $C \leadsto C'$.

Proof. Suppose $\theta \leadsto^* \theta'$. If the length of the path is 0, then there is nothing to prove. Hence, we restrict ourselves to the case when the length of the path is
bigger than 0. By Lemma 2, there is a path in normal from from θ to θ′ (say) θ = θ₀ → θ₁ → θ₂ ⋯ θₘ₋₁ → θₘ = θ′ with each θᵢ := (vᵢ, Sᵢ).

Let N₀ = 0 and let Nᵢ = (Nᵢ₋₁ + 1) · (|Sᵢ₋₁ \ Sᵢ| + 1) for every 1 ≤ i ≤ m.

In Lemma 6 of [S], the following fact has been proved:

For every 1 ≤ i ≤ m and for every C′ ∈ [θᵢ]|Nᵢ₋₁+1, there exists C ∈ [θᵢ]|Nᵢ₋₁+1 such that C → C′.

This immediately proves that for all C′ ∈ [θ′]|Nₘ₊₁, there exists C ∈ [θ] such that C → C′. If we prove Nₘ ≤ k × (2k)|Q| × (|Q| + 1)|Q|₊₁, then the proof of the theorem will be complete.

Notice that since the path between θ₀ and θₘ is in normal form, the number of indices i such that |Sᵢ₋₁ \ Sᵢ| > 0 is at most |Q|. Indeed, suppose q ∈ Sᵢ₋₁ \ Sᵢ for some i. Then by the normal form property, q /∈ Sⱼ for any j ≥ i. Hence, in the rest of the path q does not appear in the abstract part at all. By definition of the edges in the symbolic graph, if (v, θ) → (v′, S′) is an edge, then S′ = ∅. These two facts then imply that the number of indices i such that |Sᵢ₋₁ \ Sᵢ| > 0 is at most |Q|.

It then follows that except for at most |Q| indices, each index Nᵢ is obtained from Nᵢ₋₁ by simply adding 1 and in the remaining indices, Nᵢ is obtained from Nᵢ₋₁ by adding 1 and then multiplying by a number which is at most |Q| + 1. The way to maximize Nₘ by this procedure is by letting Nᵢ = Nᵢ₋₁ + 1 for every 1 ≤ i ≤ m − |Q| and then letting Nᵢ = (Nᵢ₋₁ + 1) ⋅ (|Q| + 1) for every m − |Q| < i ≤ m. This gives an upper bound of (m − |Q| + 1)|Q|(|Q| + 1)|Q| for Nₘ. Since m is itself the length of the path between θ₀ and θₘ, m is upper bounded by the number of symbolic configurations in G_k which is at most |Nₘ|² ⋅ 2|Q| ≤ k × k|Q| ⋅ 2|Q|.

Overall we get that Nₘ ≤ k × (2k)|Q| × (|Q| + 1)|Q|₊₁.

\[ \square \]

B Proofs for Section 5

Lemma 5. Deciding whether the cut-off of a given RBN is positive or negative can be done in PSPACE.

Proof. By Theorem 7, it follows that a given RBN has a negative cut-off iff \( S = post^∗([\text{init}]) \cap pre^∗(\uparrow \text{fin}) \) is finite. We have already seen that S is a counting set such that the norm of S is at most N := 2^p(|Q|) for some fixed polynomial p.

Let \( U_iC_i \) be a counting constraint for S which minimizes its norm and let each \( C_i = (L_i, U_i) \). Hence, \( L_i(q) \leq N \) for every state q. Further, S is infinite iff there is an index i and a state q such that \( U_i(q) = \infty \).

Using these two facts, we claim that S is infinite iff there is a state q and a configuration \( C \in S \) such that \( C(q') \leq N \) for every \( q' \neq q \) and \( C(q) = N + 1 \). Indeed, if S is infinite, then there is an i and a q such that \( U_i(q) = \infty \). If we let C be such that \( C(q') = L(q') \leq N \) for every \( q' \neq q \) and \( C(q) = N + 1 \) then \( C \in C_i \in S \).
For the other direction, suppose such a state $q$ and a configuration $C$ exists. Since $C \in \mathcal{S}$ we have that $C \in \mathcal{C}_i$ for some $i$. Now, since $C(q) = N + 1$ and the norm of $C_i$ is at most $N$, it must be the case that $U_i(q) = \infty$. This then proves that $\mathcal{S}$ is infinite.

Hence, to check if $\mathcal{S}$ is infinite, we just have to guess a state $q$ and a configuration $C$ such that $C(q') \leq N$ for every $q' \neq q$ and $C(q) = N + 1$ and check if $C \in \mathcal{S}$. Since guessing $C$ can be done in polynomial space (by representing every number in binary), by the PSPACE Theorem (Theorem 3), we can check if $C \in \mathcal{S}$ in polynomial space as well, which concludes the proof of the theorem.

\begin{lemma}
Deciding whether the cut-off of a given RBN is positive or negative is PSPACE-hard.
\end{lemma}

\textbf{Proof.} We reduce from the fixed-configuration almost-sure coverability problem for RBN. In this problem, we are given a RBN $\mathcal{R} = (Q, \Sigma, \delta)$, a configuration $C$ of $\mathcal{R}$ such that $2 \leq |C| \leq |Q|$ and a state $q_f \in Q$ and we are asked to decide if $C$ can almost-surely cover $q_f$, i.e., if $\text{post}^*{(C)} \subseteq \text{pre}^*(\uparrow q_f)$. This problem is PSPACE-hard and the proof is as follows: In Theorem 4 of [14], the authors give a reduction from the acceptence problem for linear-space bounded Turing machines to the problem of covering a state $q_f$ from a given initial configuration $C$ for a subclass of RBN called IO nets which have the following property: Starting from the initial configuration of the IO net, there is exactly one execution which is possible. It then follows that the covering the state $q_f$ from $C$ is equivalent to almost-surely covering $q_f$ from $C$. Since IO nets are a subclass of RBN (Section 6.2 of [3]), it follows that the fixed-configuration almost-sure coverability problem for RBN is PSPACE-hard.

We now give a reduction from the fixed-configuration almost-sure coverability problem for RBN to the problem of checking if a given RBN has a positive cut-off. Let $(\mathcal{R}, C, q_f)$ be an instance of the fixed-configuration almost-sure coverability problem for RBN such that $\mathcal{R} = (Q, \Sigma, \delta)$ and $C = \{q_1, \ldots, q_n\}$. The required reduction proceeds in three stages.

\textbf{First stage:} We construct a new RBN $\mathcal{R}_1 = (Q_1, \Sigma_1, \delta_1)$ as follows: $Q_1 = Q \times \{1, 2, \ldots, n\} \cup \{\text{fin}\}$ where fin is a new state, $\Sigma_1 = \Sigma \cup \{\@\}$ where $\@$ is a new letter and $\delta_1 = \{(p, i) \overset{!a}{\to} (q, i) : p \overset{a}{\to} q \in \delta\} \cup \{(p, i) \overset{?a}{\to} (q, i) : p \overset{?a}{\to} q \in \delta\} \cup \{(q_f, i) \overset{!a}{\to} \text{fin}\}$. For each $i$, the set $Q \times \{i\}$, will be called the $i^{th}$ copy of $\mathcal{R}$.

Intuitively, $\mathcal{R}_1$ contains $n$ copies of $\mathcal{R}$ along with a new state fin such that it is always possible to move from any copy of the state $q_f$ to the new state fin. Note that since $n = |C| \leq |Q|$, this construction takes polynomial time.

Let $D := \{(q_1, 1), (q_2, 2), \ldots, (q_n, n)\}$. It is straightforward to verify that $C$ can almost-surely cover $q_f$ in $\mathcal{R}$ iff $D$ can almost-surely cover fin in $\mathcal{R}_1$.

\textbf{Second stage:} We now construct a second RBN $\mathcal{R}_2 = (Q_2, \Sigma_2, \delta_2)$ as follows: $Q_2 = Q_1 \cup \{\text{init}\}$ where init is a new state, $\Sigma_2 = \Sigma_1 \cup \{\#\} \cup \{\$_1, \ldots, \$_n\}$.
where $\#, \$_1, \ldots, \$_n$ are $n+1$ new letters and $\delta_2$ contains all the transitions in $\delta_1$ and also the following transitions:

- Type 1 transitions: $\text{init} \xrightarrow{\#} (q_i, i)$ and for each $1 \leq i \leq n$.
- Type 2 transitions: For every $p \in Q$ and $1 \leq i \leq n$, we have the transitions $(p, i) \xrightarrow{\$_i} (p, i)$ and $(p, i) \xrightarrow{?\$_i} \text{fin}$.

By combining the Type 1 and Type 2 transitions, it is very easy to verify the following facts:

- Fact 1: If $C' \geq \langle (p, i), (q, i) \rangle^*$ for some $i$ and some $p, q$, then $C'$ can cover $\text{fin}$.
- Fact 2: If $C' \geq \langle 2 \cdot \text{init} \rangle$ or $C \geq \langle \text{init}, (p, i) \rangle$ for some $i$ and some $p$, then $C'$ can cover $\text{fin}$.
- Fact 3: $n+1$ is a positive cut-off for $R_2$.
- Fact 4: If $C' \xrightarrow{R_2} C''$ is a run such that $C'(\text{init}) = 0$ and $C'$ does not contain two processes in the same copy of $R$, then no transitions of type 1 or type 2 could have been fired along this run. Consequently, we have $C' \xrightarrow{R_1} C''$.

We now claim that

$\langle n \cdot \text{init} \rangle$ can almost-surely cover $\text{fin}$ in $R_2$ iff $D$ can almost-surely cover $\text{fin}$ in $R_1$.

Suppose $\langle n \cdot \text{init} \rangle$ can almost-surely cover $\text{fin}$ in $R_2$. We want to show that $D$ can almost-surely cover $\text{fin}$ in $R_1$. To do this, we have to show that if $D \xrightarrow{R_1} C'$, then $C'$ can cover $\text{fin}$ in $R_1$. Notice that $C'(\text{init}) = 0$ and $C'$ does not contain two processes in the same copy of $R$.

Notice that, by using the type 1 transitions, we have $\langle n \cdot \text{init} \rangle \xrightarrow{R_2} D$ and so we have $\langle n \cdot \text{init} \rangle \xrightarrow{R_2} D \xrightarrow{R_2} C'$. By assumption, this means that $C' \xrightarrow{R_2} C''$ with $C''(\text{fin}) > 0$. By Fact 4, we have $C' \xrightarrow{R_1} C''$ and so $C'$ can cover $\text{fin}$ in $R_1$.

Suppose $D$ can almost-surely cover $\text{fin}$ in $R_1$. We want to show that $\langle n \cdot \text{init} \rangle$ can almost-surely cover $\text{fin}$ in $R_2$. To do so, we have to show that if $\langle n \cdot \text{init} \rangle \xrightarrow{R_2} C'$, then $C'$ can cover $\text{fin}$ in $R_2$. By means of Fact 1 and Fact 2, it suffices to show that this is the case when $C'$ contains exactly one process in each copy of $R$. In this case, we will prove that $D \xrightarrow{R_1} C'$ and so by assumption, this means that $C'$ can cover $\text{fin}$ in $R_1$ and hence in $R_2$ as well.

All that remains to show that is that $D \xrightarrow{R_1} C'$, which is what we do now.

Consider the run $\langle n \cdot \text{init} \rangle \xrightarrow{R_2} C_1 \xrightarrow{R_2} C_2 \ldots C_k \xrightarrow{R_2} C'$. Since $C'$ has exactly one process in each copy of $R$, it must be the case that along this run, no type 2 transitions were fired, and each type 1 transition was fired exactly once, i.e., for each $1 \leq i \leq n$, the transition $\text{init} \xrightarrow{\#} (q_i, i)$ occurred exactly once along this run. Notice that if for some $j$, we have $C_j \xrightarrow{R_2} C_{j+1}$, then $C_j \xrightarrow{R_2} C_{j+1}$.
\[ C_{j+2} \text{ where } r_j \text{ is not a type 1 transition and } r_{j+1} \text{ is a type 1 transition, then} \]
\[ C_j \xrightarrow{r_{j+1} + r_{j+1}^1, \ldots, r_{j+1}^j} C'' \xrightarrow{r_j + r_j^1, \ldots, r_j^j} C_{j+2}. \]
This means that we can push all the occurrences of type 1 transitions along this run to the beginning. But then notice that after the first \( n \) steps we would have reached the configuration \( D \) from \( \{n \cdot \text{init}\} \). This means that \( D \xrightarrow{\cdot} C' \) and by Fact 4, we have \( D \xrightarrow{\cdot} C' \), which finishes the proof.

Notice that by Fact 3, we have actually shown the following

**Fact 5:** \( D \) can almost-surely cover \( \text{fin} \) in \( \mathcal{R}_1 \) iff \( \{n \cdot \text{init}\} \) can almost-surely cover \( \text{fin} \) in \( \mathcal{R}_2 \) iff \( n \) is a positive cut-off for \( \mathcal{R}_2 \).

**Third stage:** We now construct our final RBN \( \mathcal{R}_3 = (Q_3, \Sigma_3, \delta_3) \) as follows: \( Q_3 = Q_2 \cup \{s_1, s_2, \ldots, s_n\} \) where \( s_1, \ldots, s_n \) are \( n \) new states, \( \Sigma_3 = \Sigma_2 \cup \{a_1, \ldots, a_n, b\} \) where \( a_1, \ldots, a_n, b \) are \( n + 1 \) new letters and \( \delta_3 \) contains all the transitions in \( \delta_2 \) and also the following transitions:

- Type 3 transitions: For each \( i \in \{1, \ldots, n\} \), we have \((q_i, i) \xrightarrow{l_{ai}} (q_i, i)\) and \( s_{i-1} \xrightarrow{?_{ai}} s_i \). (Here and in the sequel, \( s_0 \) is taken to be \( \text{init} \)).
- Type 4 transitions: For each \( i \in \{1, \ldots, n - 1\} \), we have \( s_i \xrightarrow{b_i} \text{init} \).

Since \( s_n \) is a sink state which does not broadcast anything and since the only way to reach \( s_n \) is through \( s_{n-1} \), it is easy to verify the following:

**Fact 6:** Suppose \( C' \xrightarrow{\cdot} C'' \) such that \( C'(s_n) = i_n \) and \( C''(s_n) = j_n \). Then \( i_n \leq j_n \) and \( C' - i_n \cdot s_n \xrightarrow{\cdot} C'' - j_n \cdot s_n + (j_n - i_n) \cdot s_{n-1} \xrightarrow{\cdot} C'' - j_n \cdot s_n + (j_n - i_n) \cdot \text{init} \).

We now claim that

There is a positive cut-off for \( \mathcal{R}_3 \) iff \( n \) is a positive cut-off for \( \mathcal{R}_2 \).

Suppose \( n \) is a positive cut-off for \( \mathcal{R}_2 \). We claim that \( n \) is also a positive cut-off for \( \mathcal{R}_3 \). To show this, we have to prove that for all \( h \geq n \), if \( \{h \cdot \text{init}\} \xrightarrow{\cdot} C' \), then \( C' \) can cover \( \text{fin} \) in \( \mathcal{R}_3 \). Let \( C'(s_l) = i_l \) for every \( 1 \leq l \leq n \) and let \( i = \sum_{1 \leq l \leq n} i_l \).

We consider two cases:

- **Case 1:** \( C'(s_n) = 0 \). Let \( \tilde{C} = C' - (\sum_{1 \leq l \leq n} i_l \cdot s_l) + i \cdot \text{init} \). Notice that \( \tilde{C} \) can be reached from \( \{h \cdot \text{init}\} \) in \( \mathcal{R}_2 \) - Simply use the same run from \( \{h \cdot \text{init}\} \) to \( C' \), but remove all the type 3 and type 4 transitions. By assumption then, \( \tilde{C} \) can cover \( \text{fin} \) in \( \mathcal{R}_2 \) and so in \( \mathcal{R}_3 \) as well.
  Further, notice that \( C' \) can reach \( \tilde{C} \) in \( \mathcal{R}_3 \) means of type 4 transitions. This means that \( C' \) can also cover \( \text{fin} \) in \( \mathcal{R}_3 \).

- **Case 2:** \( C'(s_n) > 0 \). Let \( h' = h - i_n \). By Fact 6, we have \( \{h' \cdot \text{init}\} \xrightarrow{\cdot} C' - i_n \cdot s_n \). If we show that \( h' \geq n \), then we can apply the same argument as Case 1 to finish this case as well. Indeed, for a process to reach the state
First we show that there are 0 processes in states \( q \) counting set. Let \( \phi \) computes \( I \). \( I \) puts an arbitrary number of processes in initial states of \( I \cap R \). By Fact 5, it suffices to show that \( D \) can almost-surely cover \( \text{fin} \) in \( R_1 \). To show this, we need to show that if \( D \xrightarrow{\text{fin}} C' \) then \( C' \) can cover \( \text{fin} \) in \( R_1 \).

Let \( N \) be the positive cut-off for \( R_3 \) and let \( h \geq \max\{n, N\} \). By assumption \( \langle h \cdot \text{init} \rangle \) can almost-surely cover \( \text{fin} \) in \( R_3 \). By using Type 1 and Type 3 transitions, it is easy to see that \( \langle h \cdot \text{init} \rangle \xrightarrow{\text{fin}} D + (h - n) \cdot s_n \) and so \( \langle h \cdot \text{init} \rangle \xrightarrow{\text{fin}} C' + (h - n) \cdot s_n \). By assumption, \( C' + (h - n) \cdot s_n \) can cover \( \text{fin} \) in \( R_3 \) and so we have a run \( C' + (h - n) \cdot s_n \xrightarrow{\text{fin}} C'' \) with \( C''(\text{fin}) > 0 \). By induction on the run, it is easy to prove that along this run, all the configurations \( C'' \) satisfy \( C''(s_n) = h - n \) and \( C''(s_i) = 0 \) for every \( 0 \leq i \leq n - 1 \). Hence, by Fact 6, we have \( C' \xrightarrow{\text{fin}} C'' - (h - n) \cdot s_n \). Notice that along this run, there is no possibility of firing any transition of type 1, 2 or 4. Further, if a transition of type 3, i.e., a transition of the form \((q_i, i) \xrightarrow{1a_i} (q_i, i)\) is fired, then there could have been no process which received that message. It follows that transitions of type 3 do not change the configuration along this run. Hence, we can assume that no transitions belonging to type 3 are fired along this run. This then implies that \( C' \xrightarrow{\text{fin}} C'' - (h - n) \cdot s_n \) and so \( C' \) can cover \( \text{fin} \) in \( R_1 \).

This chain of constructions then proves the desired result.

C Proof for section [6]

Lemma 7. Let \( P \) be a RBN protocol that computes the predicate \( \varphi : \mathbb{N}^k \rightarrow \{0, 1\} \). Then for every \( b \in \{0, 1\} \), the sets \( I_b, C_b \) and \( ST_b \) are all counting sets. This entails that \( \varphi \) is definable by counting sets.

Proof. Let \( P = (Q, \Sigma, \delta, I, O) \) be a RBN protocol computing \( \varphi \) and let \( b \in \{0, 1\} \). First we show that \( ST_b \) is a counting set. The set \( C_b \) is equal to the cube such that there are 0 processes in states \( q \) with \( O(q) = 1 - b \) (i.e. an upper and a lower bound of 0), and an arbitrary number of processes elsewhere (i.e. an upper bound of \( \infty \) and a lower bound of 0). By the Closure Corollary \( \square \) \( ST_b = \text{pre}^*(C_b) \) is a counting set.

Let \( I \) be the counting set of initial configurations defined by the cube which puts an arbitrary number of processes in initial states of \( I \), and 0 elsewhere. The set \( I \cap \text{pre}^*(\text{pre}^*(ST_b)) \) is the set of initial configurations from which all runs of \( P \) converge to \( b \). By the Closure Corollary \( \square \) it is a counting set. Since \( P \) computes \( \varphi \), by definition \( I_b = I \cap \text{pre}^*(\text{pre}^*(ST_b)) \). The set \( \{v \mid \varphi(v) = b\} \) is equal to \( I_b \) restricted to the initial states \( I \), and so we are done. \( \square \)
Lemma 8. Let \( \varphi : \mathbb{N}^k \rightarrow \{0, 1\} \) be a predicate definable by counting sets. Then there exists a RBN protocol computing \( \varphi \).

Proof. To prove this, we first need the notion of an immediate observation net. An immediate observation (IO) net is a tuple \( N = (Q, \delta) \) where \( Q \) is a finite set of states and \( \delta \subseteq Q \times Q \times Q \) is the transition relation. A configuration of \( N \) is a multiset over \( Q \), and there is a step between two configurations \( C, C' \) if there exists \( (p, q, p') \in \delta \) such that \( C \geq_L p, q \) and \( C' = C - p + p' \).

In Section 6.2 of [3], it is shown that RBN can simulate IO nets. More specifically, given an IO net \( N = (Q, \delta) \), Section 6.2 of [3] shows that we can compute in polynomial time, a RBN \( R = (Q, \Sigma, \delta') \) with the same set of states such that \( C \xrightarrow{\delta'} C' \) in \( N \) if and only if \( C \xrightarrow{\delta} C' \) in \( R \) for any two configurations \( C, C' \). This implies that for any two subsets of configurations \( C, C' \), \( \text{post}^*(C) \subseteq \text{pre}^*(C') \) in \( N \) if and only if \( \text{post}^*(C) \subseteq \text{pre}^*(C') \) in \( R \).

Given an IO net \( N = (Q, \delta) \), a subset \( I \subseteq Q \) and an output function \( O : Q \rightarrow \{0, 1\} \), the tuple \( (N, I, O) \) defines an immediate observation (IO) population protocol, a subclass of population protocols introduced in [2]. Similar to the definition of RBN protocols, we can define the notion of an IO protocol computing a predicate and Theorem 7 and Theorem 39 of [2] shows that IO population protocols compute exactly the predicates definable by counting sets.

Proposition 2.12 of [13] entails that an IO protocol computes a predicate \( \varphi \) if and only if \( \text{post}^*(I_b) \subseteq \text{pre}^*(ST_b) \) for every \( b \in \{0, 1\} \). This is also true for RBN. Indeed Proposition 2.12 of [13] states the above result for “well-behaved generalized protocols” (Definition 2.1 of [13]). Fix an arbitrary RBN protocol \( P \). By definition 2.1 of [13], it is a generalized protocol by setting \( \text{Conf} \) to be the set of configurations of \( P \), \( \Sigma \) to be its set of states, and \( \text{Step} \) to be the step relation of the underlying RBN. By definition 2.8 of [13], it is well-behaved, i.e., every fair execution eventually ends up in a bottom strongly connected component of the reachability graph. This is because the number of processes does not change along a run, so the reachability graph from any configuration is finite.

Since RBN can simulate IO nets, it follows that RBN protocols can compute any predicate computable by IO protocols and this concludes the proof.

D Asynchronous shared-memory systems

We now consider another model of distributed computation called asynchronous shared-memory systems (ASMS) [15,11]. Here, we have a set of finite-state, anonymous agents which can communicate by means of a single shared register, i.e., agents can either write a value to the register or read the value currently written on the register. The definitions and notations in this section are taken from [3].

Definition 7. An asynchronous shared-memory system (ASMS) is a tuple \( P = (Q, \Sigma, \delta) \) where \( Q \) is a finite set of states, \( \Sigma \) is a finite alphabet, and \( \delta \subseteq Q \times \)
\( \{ R, W \} \times \Sigma \times Q \) is the set of transitions. Here \( R \) stands for read, and \( W \) stands for write.

We use \( p \xrightarrow{R(d)} q \) (resp. \( p \xrightarrow{W(d)} q \)) to denote that \((p, R, d, q) \in \delta \) (resp. \((p, W, d, q) \in \delta \)). A configuration \( C \) of an ASMS is a multiset over \( Q \cup \Sigma \) such that \( \sum_{d \in \Sigma} C(d) = 1 \), i.e., \( C \) contains exactly one element from the set \( \Sigma \). Hence, we sometimes denote a configuration \( C \) as \((M, d)\) where \( M \) is a multiset over \( Q \) (which counts the number of processes in each state) and \( d \in \Sigma \) (which denotes the content of the shared register). The value \( d \) will be denoted by \( \text{data}(C) \).

A step between configurations \( C = (M, d) \) and \( C' = (M', d') \) exists if there is \( t = (p, \text{op}, d'', q) \in \delta \) such that \( M(p) > 0 \), \( M' = M - p + q \) and either \( \text{op} = R \) and \( d = d' = d'' \) or \( \text{op} = W \) and \( d' = d'' \). If such a step exists, we denote it by \( C \xrightarrow{t} C' \) and we let \( \ast \) denote the reflexive transitive closure of the step relation.

A run is then a sequence of steps.

A cube \( C = (L, U) \) of an ASMS \( P = (Q, \Sigma, \delta) \) is defined to be a cube over \( Q \cup \Sigma \) satisfying the following property: There exists \( d \in \Sigma \) such that \( L(d) = U(d) = 1 \) and \( L(d') = U(d') = 0 \) for every other \( d' \in \Sigma \). Hence, we sometimes denote a cube \( C \) as \((L, U, d)\) where \((L, U)\) is a cube over \( Q \) and \( d \in \Sigma \). Membership of a configuration \( C \) in a cube \( C \) is then defined in a straightforward manner.

The cube-reachability problem for ASMS is then to decide, given \( P \) and two cubes \( C, C' \), whether \( C \) can reach \( C' \), i.e., whether there are configurations \( C \in C, C' \in C' \) such that \( C \xrightarrow{\ast} C' \). In Section 4 of \cite{3}, it was shown that the cube-reachability problems for RBN and ASMS are polynomial-time equivalent. By Theorem 4, we get

**Theorem 9.** The cube-reachability problem for ASMS is PSPACE-complete.

**The almost-sure coverability problem**

Similar to RBN, we can define the almost-sure coverability problem for ASMS. Let \( P = (Q, \Sigma, \delta) \) be an ASMS with two special states \( \text{init} \) and \( \text{fin} \) and a special initial letter \# \( \in \Sigma \). Let \( \uparrow \text{fin} \) denote the set of all configurations \( C \) with \( C(\text{fin}) \geq 1 \). For any \( k \geq 1 \), we say that the configuration \((\{k \cdot \text{init}\}, \#)\) almost-surely covers \( \text{fin} \) iff \( \text{post}^\ast(\{k \cdot \text{init}\}, \#) \subseteq \text{pre}^\ast(\uparrow \text{fin}) \). Now, similar to RBN, it is easy to define the notion of a cut-off for ASMS. The following fact is known (Theorem 3 of \cite{6}).

**Theorem 10.** Given an ASMS \( P \) with two state \( \text{init}, \text{fin} \) and a letter \#, a cut-off always exists. Whether the cut-off is positive or negative can be decided in \( \text{EXPSPACE} \) and \( \text{PSPACE} \)-hard.

The main result of this subsection is that

**Theorem 11.** Deciding whether the cut-off of a given ASMS is positive or negative is in PSPACE-complete.
Note that it suffices only to prove the upper bound, since the lower bound is already known. Let \( P = (Q, \Sigma, \delta) \) be a fixed ASMS with \( \text{init}, \text{fin} \in Q \) and \( \# \in \Sigma \). Let \( ([\text{init}], \#) \) denote the cube which has an arbitrary number of agents in the state \( \text{init} \) and 0 elsewhere. Similar to the model of RBN, we first show that,

**Theorem 12.** \( P \) has a positive cut-off iff \( S := \text{post}^*(([\text{init}], \#)) \cap \text{pre}^*(\uparrow \text{fin}) \) is finite.

**Proof.** Suppose \( S \) is finite. Let \( N \) be the largest value appearing in any of the configurations of \( S \). It is easy to see that if \( h > |Q| \cdot N \), then any configuration of size \( h \) does not belong in \( S \). It follows that if \( h > |Q| \cdot N \) and \( C \in \text{post}^*((h \cdot \text{init}, \#)) \) then \( C \in \text{pre}^*(\uparrow \text{fin}) \) and so we have a positive cut-off.

Suppose \( S \) is infinite. Then there must be an infinite set of configurations which belong to \( \text{post}^*(([\text{init}], \#)) \) but not in \( \text{pre}^*(\uparrow \text{fin}) \). This means that for infinitely many numbers \( h \), there is a configuration \( C_h \in \text{post}^*((h \cdot \text{init}, \#)) \) but \( C_h \notin \text{pre}^*(\uparrow \text{fin}) \). This then implies that \( P \) cannot have a positive cut-off.

Hence, checking whether \( P \) has a positive cut-off is equivalent to deciding if \( S \) is finite. We shall now show that this is decidable in \( \text{PSPACE} \). To show this, we recall the connection established between RBN and ASMS in Section 4 of [3].

Given an ASMS \( P = (Q, \Sigma, \delta) \), in section 4.2 of [3], it is shown that in polynomial time we can come up with an RBN \( R = (Q \cup \Sigma \cup Q', \Sigma', \delta') \) which has a copy of \( Q \) and \( \Sigma \) as its states and which has the following properties:

- A good configuration of \( R \) is a configuration \( C \) such that \( \sum_{a \in \Sigma} C(a) = 1 \) and \( C(q) = 0 \) if \( q \notin Q \cup \Sigma \). Notice that there is a natural bijection between configurations of \( P \) and good configurations of \( R \).
- A configuration \( C \) of \( P \) can reach a configuration \( C' \) of \( P \) iff \( \hat{C} \) can reach \( \hat{C'} \) in \( R \), where \( \hat{C} \) and \( \hat{C'} \) are the corresponding good configurations of \( C \) and \( C' \) respectively.

Let \( I \) denote the set of all good configurations of \( R \) and let \( C \) denote the set of all good configurations of \( R \) which puts an arbitrary number of agents in \( \text{init} \), exactly one agent in \( \# \) and zero agents elsewhere.

It then follows that the set \( S := \text{post}^*(([\text{init}], \#)) \cap \text{pre}^*(\uparrow \text{fin}) \) over \( P \) is finite iff the set \( S' := (\text{post}^*(C) \cap I) \cap \text{pre}^*(\uparrow \text{fin}) \cap I \) over \( R \) is finite. Similar to the proof of Lemma [4], we can decide if this set is finite or not in \( \text{PSPACE} \). This gives the required \( \text{PSPACE} \) upper bound.