Differential Inequalities and Univalent Functions

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Abstract—Let \( M \) be the class of analytic functions in the unit disk \( D \) with the normalization \( f(0) = f'(0) - 1 = 0 \), and satisfying the condition

\[
\left| z^2 \left( \frac{z}{f(z)} \right)'' + f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 \right| \leq 1, \quad z \in D.
\]

Functions in \( M \) are known to be univalent in \( D \). In this paper, it is shown that the harmonic mean of two functions in \( M \) are closed, that is, it belongs again to \( M \). This result also holds for other related classes of normalized univalent functions. A number of new examples of functions in \( M \) are shown to be starlike in \( D \). However we conjecture that functions in \( M \) are not necessarily starlike, as apparently supported by other examples.

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1. INTRODUCTION

Let \( H \) denote the family of analytic functions in the open unit disk \( D := \{ z \in \mathbb{C} : |z| < 1 \} \), and \( A \) its subclass of normalized functions \( f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \). Further, let \( S \) denote the subclass of \( A \) consisting of functions \( f \) univalent in \( D \). Denote by \( S^* \) and \( C \) respectively the subclasses of \( S \) consisting of starlike and convex functions. Functions \( f \in S^* \) map \( D \) onto starlike domains with respect to the origin, while \( f \in C \) whenever \( f(D) \) is a convex domain. Analytically, \( f \in S^* \) if \( \text{Re} \left( zf'(z)/f(z) \right) > 0 \), while \( f \in C \) if \( \text{Re} (1 + zf''(z)/f'(z)) > 0 \).

Investigations into particular subclasses of \( A \) continued to be of recent interest. These include the class \( U \) consisting of functions \( f \in A \) satisfying

\[
\left| f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 \right| \leq 1, \quad z \in D,
\]

as well as the class \( P \) of functions \( f \in A \) with

\[
\left| \frac{z}{f(z)} \right|'' \leq 2, \quad z \in D.
\]
The strict inclusion $\mathcal{P} \subseteq \mathcal{U} \subseteq \mathcal{S}$ holds within these classes (see [2, 5, 14] for a proof). There are several generalizations [7] of this result. For recent investigations on $\mathcal{U}$ and its generalization, we refer to [11–13] and the references therein.

In this paper, the phrase \( f \in \mathcal{U} \) (respectively, \( f \in \mathcal{P} \)) in \(|z| < r\) means that the defining inequality holds in \(|z| < r\) instead of the full disk \(|z| < 1\). We also follow this standard convention for other classes. In [8] and [9], the authors discussed the classes $\mathcal{M}$ and $\mathcal{N}$ of functions from $\mathcal{A}$ satisfying respectively the differential inequality

\[
|M_f(z)| \leq 1 \quad \text{and} \quad |N_f(z)| \leq 1, \quad z \in \mathbb{D},
\]

where

\[
M_f(z) = z^2 \left( \frac{z}{f(z)} \right)'' + f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 \quad \text{and} \quad N_f(z) = -z^3 \left( \frac{z}{f(z)} \right)'' + f'(z) \left( \frac{z}{f(z)} \right)^2 - 1.
\]

These classes are also closely related to the class $\mathcal{U}$ in the sense of the strict inclusions $\mathcal{N} \subseteq \mathcal{M} \subseteq \mathcal{P} \subseteq \mathcal{U}$. A slightly general version of this result is given in [1].

In [10], Obradović and Ponnusamy discussed “harmonic mean” of two univalent analytic functions. These are functions $F$ of the form

\[
F(z) = \frac{2f(z)g(z)}{f(z) + g(z)}, \tag{1}
\]

or equivalently,

\[
\frac{1}{F(z)} - \frac{1}{z} = \frac{1}{2} \left[ \left( \frac{1}{f(z)} - \frac{1}{z} \right) \right] + \left( \frac{1}{g(z)} - \frac{1}{z} \right), \tag{2}
\]

where $f, g \in \mathcal{S}$. In particular, the authors in [10] determined the radius of univalency of $F$, and proposed the following two conjectures.

**Conjecture 1.** (a) The function $F$ defined by (1) is not necessarily univalent in $\mathbb{D}$ whenever $f,g \in \mathcal{S}$ such that \(((f(z) + g(z))/z) \neq 0 \in \mathbb{D}$$.$

(b) The function $F$ defined by (1) is univalent in $\mathbb{D}$ whenever $f,g \in \mathcal{C}$ such that \(((f(z) + g(z))/z) \neq 0 \in \mathbb{D}$$.$

The authors in [10] showed that whenever $f,g \in \mathcal{U}$, then the function $F$ defined by (1) belongs to $\mathcal{U}$ in the disk $|z| < \sqrt{(\sqrt{5}-1)/2} \approx 0.78615$.

While Conjecture 1 remains open, the aim of this paper is to show that Conjecture 1 (a) does not hold when the class $\mathcal{S}$ is replaced by $\mathcal{U}$. Indeed, it does not hold true even for the classes $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{P}$.

The second objective of the paper is to consider several examples in examining starlikeness of functions in the classes $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{P}$. We conclude with a conjecture that functions in the class $\mathcal{M}$ are not necessarily starlike in $\mathbb{D}$.

## 2. ON THE HARMONIC MEAN OF UNIVALENT FUNCTIONS

**Theorem 1.** Let $f,g \in \mathcal{U}$ satisfy \([f(z) + g(z)]/z \neq 0 \text{ for } z \in \mathbb{D}$. Then the function $F$ given by (1) also belongs to the class $\mathcal{U}$.

**Proof.** From (2), it readily follows from the triangle inequality that the function $F$ satisfies

\[
\left| F'(z) \left( \frac{z}{F(z)} \right)^2 - 1 \right| = \left| -z^2 \left( \frac{1}{f(z)} - \frac{1}{z} \right) \right| \leq \frac{1}{2} \left| -z^2 \left( \frac{1}{f(z)} - \frac{1}{z} \right) \right| + \frac{1}{2} \left| -z^2 \left( \frac{1}{g(z)} - \frac{1}{z} \right) \right| \leq \frac{1}{2} \left| f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 \right| + \frac{1}{2} \left| g'(z) \left( \frac{z}{g(z)} \right)^2 - 1 \right| < 1.
\]

Thus $F \in \mathcal{U}$. \(\Box\)

Moreover, we see that Theorem 1 holds true if the class $\mathcal{U}$ is replaced by the class $\mathcal{M}$.

**Theorem 2.** Suppose $f,g \in \mathcal{M}$ satisfy \([f(z) + g(z)]/z \neq 0 \text{ for } z \in \mathbb{D}$. Then the function $F$ given by (1) also belongs to the class $\mathcal{M}$.
Proof. Now
\[ f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 = -z^2 \left( \frac{1}{f(z)} - \frac{1}{z} \right). \]

Using this equality, it follows that
\[
\mathcal{M}_f(z) = z^2 \left[ \left( \frac{z}{f(z)} \right)'' - \left( \frac{1}{f(z)} - \frac{1}{z} \right)'' \right] = z^2 \left[ \left( \frac{z}{f(z)} \right)' - \frac{1}{f(z) + 1} \right]'
\]
\[
= z^2 \left[ \left( \frac{1}{f(z)} + \frac{1}{z} \right)' \right] = z^2 \left[ \left( \frac{1}{f(z)} - \frac{1}{z} \right)'' + z^2 \left( \frac{1}{f(z)} - \frac{1}{z} \right)' \right].
\]

In view of (2), this means that \( \mathcal{M}_f(z) = \frac{1}{2} (\mathcal{M}_f(z) + \mathcal{M}_g(z)) \), and use of the triangle inequality yields the desired result. \( \square \)

**Theorem 3.** Let \( f, g \in \mathcal{N} \) satisfy \( |f(z) + g(z)|/z \neq 0 \) for \( z \in \mathbb{D} \). Then the function \( F \) given by (1) also belongs to the class \( \mathcal{N} \).

**Proof.** As in the proof of Theorem 2, we see that
\[
\mathcal{N}_f(z) = -z^2 \left[ \left( \frac{z}{f(z)} \right)'' + \left( \frac{1}{f(z)} - \frac{1}{z} \right)'' \right]
\]
\[
= -z^2 \left[ \left( \frac{1}{f(z)} - \frac{1}{z} f'(z) \left( \frac{z}{f(z)} \right)^2 \right)'' + \left( \frac{1}{f(z)} - \frac{1}{z} \right)'' \right]
\]
\[
= -z^4 \left( \frac{1}{f(z)} - \frac{1}{z} \right)' - 3z^3 \left( \frac{1}{f(z)} - \frac{1}{z} \right)' - z^2 \left( \frac{1}{f(z)} - \frac{1}{z} \right)'.
\]

Thus relation (2) gives \( \mathcal{N}_F(z) = \frac{1}{2} (\mathcal{N}_f(z) + \mathcal{N}_g(z)) \), and the proof of theorem readily follows. \( \square \)

Finally, it is also readily shown that the above theorem holds true for the class \( \mathcal{P} \).

### 3. Examples and a Conjecture

It is known that functions in the class \( \mathcal{U} \) are not necessarily starlike. There are a number of examples displaying functions in \( \mathcal{U} \) that are not starlike in \( \mathbb{D} \), see for instance [6]. However, is \( \mathcal{M} \subset \mathcal{S}^* \)? This section discusses the latter problem.

**Example 3.** To present a one-parameter family of functions in \( \mathcal{M} \) that are also starlike, consider the function \( f \) given by \( z/f(z) = 1 + (1 - \alpha)z + \alpha z^m \), where \( \alpha \in (0, 1) \) and \( m \in \mathbb{N} \setminus \{1\} = \{2, 3, \ldots\} \) are such that \( \alpha(m - 1)^2 = 1 \). Then \( z/f(z) \neq 0 \) in \( \mathbb{D} \) and
\[
\sum_{k=2}^{\infty} (k - 1)^2 |b_k| = (m - 1)^2 \alpha = 1,
\]
and therefore, \( f \in \mathcal{M} \).

Next, we show that \( f \) is starlike whenever \( m > 1 \) is an odd integer. Now, a simple calculation shows
\[
\frac{zf'(z)}{f(z)} = \frac{1 - \alpha(m - 1)z^m}{1 + (1 - \alpha)z + \alpha z^m}
\]

With \( z = e^{i\theta} \), then
\[
\frac{e^{i\theta} f'(e^{i\theta})}{f(e^{i\theta})} = \frac{A(\theta) + iB(\theta)}{|1 + (1 - \alpha)e^{i\theta} + \alpha e^{im\theta}|^2}.
\]

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where
\[ A(\theta) = 1 + (1 - \alpha) \cos \theta - \alpha (m-2) \cos (m\theta) - \alpha (1 - \alpha)(m-1) \cos (m-1)\theta - \alpha^2 (m-1). \]

Note that \( A(\theta) = A(-\theta). \) As \( \alpha = 1/(m-1)^2, \) the expression for \( A(\theta) \) reduces to
\[ A(\theta) = 1 - \frac{1}{(m-1)^3} - \frac{m(m-2)}{(m-1)^2} D(\theta), \quad \text{where} \quad D(\theta) = -\cos \theta + \frac{1}{m} \cos (m\theta) + \frac{\cos (m-1)\theta}{m-1}. \]

To show starlikeness, that is, \( f \in S^*, \) it suffices to show that \( A(\theta) \geq 0 \) for \( 0 \leq \theta \leq \pi. \) First we prove the assertion for the case \( m = 3, \) while the general case is obtained separately. Setting \( m = 3, \) \( A(\theta) \) reduces to
\[ A(\theta) = \frac{7}{8} - \frac{3}{4} \left[ -\cos \theta + \frac{1}{3} \cos 3\theta + \frac{1}{2} \cos 2\theta \right], \]
and from the identities \( \cos 2\theta = 2\cos^2 \theta - 1 \) and \( \cos 3\theta = 4\cos^3 \theta - 3\cos \theta, \)
\[ A(\theta) = \frac{1}{4} (5 + 6 \cos \theta - 4 \cos^3 \theta - 3 \cos^2 \theta) = \frac{1}{4} (1 + \cos \theta)^2 (5 - 4 \cos \theta), \]
which shows that \( A(\theta) \geq 0. \) Thus, the function \( f_3(z) \) given by
\[ f_3(z) = \frac{z}{1 + \frac{3}{4} z + \frac{1}{4} z^2} = \frac{4z}{(1+z)(4-z+z^2)}, \]
is starlike in \( \mathbb{D}. \)

Next, we proceed to prove starlikeness for the general case. This requires more computations. First,
\[ D'(\theta) = \sin \theta - \sin (m\theta) - \sin (m-1)\theta = \sin \theta - 2\sin \left( \frac{2m-1}{2} \right) \cos \frac{\theta}{2} \]
\[ = 2\cos \frac{\theta}{2} \left[ \sin \frac{\theta}{2} \sin (\frac{2m-1}{2}) \right] = 4 \cos \frac{\theta}{2} \cos \frac{m\theta}{2} \sin \left( \frac{m-1}{2} \right). \]
We need to show that \( A(\theta) \geq 0 \) for \( 0 \leq \theta \leq \pi. \) It is convenient to set \( m = 2n+1, n \geq 2 \) so that
\[ D'(\theta) = 4 \cos \frac{\theta}{2} \cos \frac{(2n+1)\theta}{2} \sin n\theta, \quad n \geq 2, \]
where \( D(\theta) \) takes the form
\[ D(\theta) = -\cos \theta + \frac{1}{2n+1} \cos (2n+1)\theta + \frac{1}{2n} \cos (2n\theta). \]
Clearly, \( D'(\theta) = 0 \) for \( \theta = 0, \pi, \) and the critical points of \( D(\theta) \) in the open interval \( (0, \pi) \) are given by
\[ \begin{cases} 
\theta_j = \frac{(2j-1)\pi}{2n+1} & \text{for } j = 1, 2, \ldots, n, \\
\theta'_j = \frac{j\pi}{n} & \text{for } j = 1, 2, \ldots, n-1,
\end{cases} \]
for \( n \geq 2. \) Moreover, for each \( n \geq 2, \)
\[ \begin{cases} 
\cos \frac{(2n+1)\theta}{2} > 0 & \text{for } 0 < \theta < \theta_1, \\
(-1)^j \cos \frac{(2n+1)\theta}{2} > 0 & \text{for } \theta_j < \theta < \theta_{j+1} \quad \text{and for } j = 1, 2, \ldots, n, \\
(-1)^{j-1} \sin n\theta > 0 & \text{for } \theta'_j < \theta < \theta'_{j+1} \quad \text{and for } j = 1, 2, \ldots, n.
\end{cases} \]
In view of the above inequalities and after a careful scrutiny, it follows that
\[ D'(\theta) \begin{cases} 
= 0 & \text{for } \theta = 0, \theta_j, \theta'_j \quad \text{for } j = 1, 2, \ldots, n, \\
> 0 & \text{for } \theta \in (0, \theta_1) \cup (\theta'_1, \theta_{j+1}) \text{ for } j = 1, 2, \ldots, n-1, \\
< 0 & \text{for } \theta \in (\theta_j, \theta'_j) \quad \text{for } j = 1, 2, \ldots, n.
\end{cases} \]

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where \(0 < \theta_1 < \theta'_1 < \theta_2 < \cdots < \theta_j < \theta'_j < \theta_{j+1} < \cdots < \theta_n < \theta'_n = \pi\). Therefore,

\[
D(\theta) \leq \max \{ D(0), D(\theta_j), D(\theta'_j) : j = 1, \ldots, n \}.
\]

Since

\[
D(0) = -1 + \frac{1}{2n+1} + \frac{1}{2n} = -\frac{2n}{2n+1} + \frac{1}{2n}, \quad D(\pi) = 1 - \frac{1}{2n+1} + \frac{1}{2n} = \frac{2n}{2n+1} + \frac{1}{2n} > 0,
\]

then \(D(0) \leq D(\pi)\). Moreover,

\[
D(\theta_j) = -\cos \theta_j + \frac{1}{2n+1} \cos(2j-1)\pi + \frac{1}{2n} \cos(2n+1-1)\theta_j
\]

\[
= -\cos \theta_j - \frac{1}{2n+1} \cos \theta_j - \left(\frac{2n+1}{2n}\right) \cos \theta_j - \frac{1}{2n+1},
\]

and

\[
D(\theta'_j) = -\cos \theta'_j + \frac{1}{2n+1} \cos(2n+1)\pi + \frac{1}{2n} \cos(2j\pi)
\]

\[
= -\left(1 - \frac{1}{2n+1}\right) \cos \theta'_j + \frac{1}{2n+1} \cos \theta'_j + \frac{1}{2n}.
\]

We deduce that \(D(\theta_j) \leq D(\pi)\) and \(D(\theta'_j) \leq D(\pi)\) holds for each \(j = 1, 2, \ldots, n\). Thus, \(D(\theta) \leq D(\pi)\) for \(\theta \in [0, \pi]\). This observation shows that

\[
A(\theta) \geq A(\pi) = 1 - \frac{1}{8n^3} - \frac{(2n+1)(2n-1)}{4n^2} \left(\frac{2n}{2n+1} + \frac{1}{2n}\right) = 0 \text{ for } \theta \in [0, \pi].
\]

Hence \(\text{Re}(e^{i\theta} f'(e^{i\theta})/f(e^{i\theta})) \geq 0\), which implies that \(f\) is starlike in \(\mathbb{D}\). Summarizing, for each \(n \geq 1\), the function \(f_n\) given by

\[
\frac{z}{f_n(z)} = 1 + \left(1 - \frac{1}{4n^2}\right) z + \frac{1}{4n^2} z^{2n+1},
\]

belongs \(\mathcal{M}\), and \(f_n\) is starlike in \(\mathbb{D}\).

**Example 4.** Consider

\[
f(z) = \frac{z}{\phi(z)}, \quad \phi(z) = 1 + \left(1 - \frac{\zeta(5)}{\zeta(3)}\right) z + \frac{1}{\zeta(3)} \sum_{n=2}^{\infty} \frac{z^n}{(n-1)^3}.
\]

We may rewrite \(\phi\) as

\[
\phi(z) = 1 + \left(1 - \frac{\zeta(5)}{\zeta(3)}\right) z + \frac{1}{\zeta(3)} \sum_{n=2}^{\infty} \frac{z^n}{(n-1)^3}.
\]

It is a simple exercise to see that \(\phi(z) \neq 0\) in \(\mathbb{D}\) and \(f \in \mathcal{M}\). The Mathematica software is used to display the image of the unit disk under \(f\) as shown in Figure 1. It apparently displays that \(f(\mathbb{D})\) is a starlike domain.

**Example 5.** It is illustrative to present a general example showing that functions in \(\mathcal{U}\) do not necessarily belong to \(\mathcal{S}^*\). For \(n \geq 3\), consider the function

\[
f_n(z) = \frac{z}{1 + ibz + (1/(n-1))e^{2i\beta} z^n}.
\]

For \(|b| \leq (n-2)/(n-1)\) and \(\beta\) a real number, then

\[
\text{Re}\left(\frac{z}{f_n(z)}\right) > 1 - |b| - \frac{1}{n-1} \geq 0,
\]
and

\[
\left| \left( \frac{z}{f_n(z)} \right)^2 f_n'(z) - 1 \right| = |e^{2i\beta} z^n| < 1 \quad \text{for} \quad z \in \mathbb{D},
\]

so that \( f_n \in U \) for each \( n \geq 3 \). On the other hand, \( f_n \) is not in \( S^* \) when \( 0 < b \leq (n - 2)/(n - 1) \) and \( 0 < \beta < \arctan(b(n - 1)/(n - 2)) \). This follows on account that

\[
\text{Re} \left( \frac{zf_n'(z)}{f_n(z)} \right) = \left| \frac{(2(n - 2)/((n - 1)) \sin \beta - 2b \cos \beta \sin \beta}{1 + ib + (e^{2i\beta}/((n - 1)))^2} \right| < 0.
\]

**Example 6.** Consider the function \( f \) defined by \( z/f(z) = 1 + (1 - \alpha)z + \alpha z^m \), where \( \alpha \in (0, 1) \) and \( m \geq 3 \) is an odd integer such that \( \alpha m(m - 1) = 2 \). Then \( z/f(z) \neq 0 \) in \( \mathbb{D} \) and

\[
\left| \left( \frac{z}{f(z)} \right)^m \right| = |\alpha m(m - 1)z^{m-2}| < \alpha m(m - 1) = 2,
\]

and therefore, \( f \in \mathcal{P} \). As in Example 3,

\[
\text{Re} \left( \frac{e^{i\theta} f'(e^{i\theta})}{f(e^{i\theta})} \right) = \frac{A(\theta)}{|1 + (1 - \alpha)e^{i\theta} + \alpha e^{im\theta}|^2},
\]

where

\[
A(\theta) = 1 + (1 - \alpha) \cos \theta - \alpha(m - 2) \cos(m\theta) - \alpha(1 - \alpha)(m - 1) \cos(m - 1) \theta - \alpha^2(m - 1).
\]

Substituting \( \alpha = 2/(m(m - 1)) \) and \( m = 2n + 1 \) (\( n \geq 1 \)), the last expression for \( A(\theta) \) reduces to

\[
A(\theta) = 1 - \frac{2}{n(2n + 1)^2} + \frac{2n - 1}{n(2n + 1)} D(\theta),
\]

where

\[
D(\theta) = (n + 1) \cos \theta - \cos(2n + 1)\theta - \frac{2(n + 1)}{2n + 1} \cos 2n\theta.
\]
To prove that $f$ is not starlike in $\mathbb{D}$, it suffices to show that $A(\theta) < 0$ for some $\theta \in (-\pi, \pi)$. In the case of $m = 3$ (i.e. $n = 1$), it is a simple exercise to see that

$$A(\theta) = \frac{1}{9}(1 + \cos \theta)(11 + 4 \cos \theta - 12 \cos^2 \theta),$$

which is clearly negative for $\theta$ near $\pi$. Indeed, substituting $\cos \theta = -8/9$ or $\theta_0 = 6\pi/7$, it can be verified that $A(\theta) \approx -55/2187 < 0$, and $A(\theta_0) \approx -0.25811 < 0$. Thus, the function

$$f_3(z) = \frac{z}{1 + \frac{2}{3}z + \frac{1}{3}z^3} = \frac{3z}{(1 + z)(3 - z + z^2)}$$

belongs to $\mathcal{P}\setminus\mathcal{S}^*$.

To do away the problem for some other values of $n$, we proceed as follows. Set

$$\theta = \frac{2(2n + 1)\pi}{4n + 3} \quad \text{and} \quad \phi = \frac{\pi}{2(4n + 3)}$$

so that $\phi = (\pi - \theta)/2$. Then $\cos \theta = -\cos 2\phi = 2\sin^2 \phi - 1$, $\cos(2n + 1)\theta = -\cos 2(2n + 1)\phi = -\sin \phi$, and $\cos 2n\theta = \cos 4n\phi = \sin 3\phi = 3\sin \phi - 4\sin^3 \phi$. Thus, $A(\theta)$ given by (3) can be simplified leading to

$$A(\theta) = 1 - \frac{2(n(2n + 1)^2)}{n(2n + 1) + 2}\left[\frac{2(n + 1)\sin^2 \phi - \frac{4n + 5}{2n + 1}\sin \phi + \frac{8(n + 1)}{2n + 1}\sin^3 \phi}{2n(2n + 1)^2}ight].$$

It is seen from the computer algebra system Mathematica that $A(\theta) < 0$ for $n = 1, 2, \ldots, 15$. For easy reference, Table I lists the values of $A(\theta)$ for $n = 1, 2, \ldots, 14$.

Thus, we conclude that the above procedure helps us to show that for each $n \in \{1, 2, \ldots, 14\}$, the function $f_n$ given by

$$\frac{z}{f_n(z)} = 1 + \left(1 - \frac{1}{n(2n + 1)}\right)z + \frac{1}{n(2n + 1)}z^{2n+1}$$

is not starlike in $\mathbb{D}$. By a minor modification in the choice of $\theta$, one can show that $f_n$ is not starlike for some $n \geq 15$ although it is not clear whether $f_n$ is starlike for larger values of $n$.

The ideas and the motivations behind the above examples lead to the following

**Conjecture.** The class $\mathcal{M}$ is not contained in $\mathcal{S}^*$.

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