Poisson Hypothesis for information networks
(A study in non-linear Markov processes)

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Abstract

In this paper we prove the Poisson Hypothesis for the limiting behavior of the large queueing systems in some simple cases. We show in particular that the corresponding dynamical systems, defined by the non-linear Markov processes, have a line of fixed points which are global attractors. To do this we derive the corresponding non-linear integral equation and we explore its self-averaging properties.

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1 Introduction

The Poisson Hypothesis deals with large queueing systems. For general systems one can not compute exactly the quantities of interest, so various approximations are used in practice. The Poisson Hypothesis was formulated first by L. Kleinrock in [K]. It is the statement that certain approximation becomes exact in the appropriate limit. It concerns the following situation.
Suppose we have a large network of servers, through which customers are travelling, being served at different nodes of the network. If the node is busy, the customers wait in the queue. Customers are entering into the systems via some nods, and the external flows of customers from the outside are Poissonian. The service time at each node is random, with some fixed distribution, depending on the node. We are interested in the stationary distribution $\pi_N$ at a given node $\mathcal{N}$: what is the distribution of the queue at it, what is the average waiting time, etc. If the service time distributions are different from the Poisson distribution, then the distribution $\pi_N$ in general can not be computed. The recipe of the Poisson Hypothesis for approximate computation of $\pi_N$ is the following:

- consider the total flow $F$ of customers to the node $\mathcal{N}$. (In general, $F$ is not Poissonian, of course.) Replace $F$ with a constant rate Poisson flow $P$, the rate being equal to the average rate of $F$. Compute the stationary distribution $\hat{\pi}_N$ at $\mathcal{N}$, corresponding to the inflow $P$. (This is an easy computation.) The claim is that $\hat{\pi}_N \approx \pi_N$.

The Poisson Hypothesis is supposed to give a good estimate if the internal flow to every node $\mathcal{N}$ is a sum of flows from many other nodes, and each of these flows constitute only a small fraction of the total flow to $\mathcal{N}$.

Clearly, the Poisson Hypothesis can not be literally true. It can hopefully hold only after some kind of “thermodynamic” limit is taken.

In the present paper we prove the Poisson Hypothesis for the information networks in some simple cases. Namely, we will consider the following closed queueing network. Let there be $M$ servers and $N$ customers to be served. The distribution of the service time is given by some fixed random variable $\eta$. Upon being served, the customer chooses one of $M$ servers with probability $\frac{1}{M}$, and goes for the service there. Then in the limit $M, N \to \infty$, with $\frac{M}{N} \to \rho$, the Poisson Hypothesis holds, under certain general restrictions on $\eta$.

An important step in this problem was made in the paper [KR1]. Namely, it was shown there that the above mentioned flow $F$ converges in our limit to a Poisson random process with some rate function $\lambda(t)$. If one would be able to show additionally that $\lambda(t) \to c = const$ as $t \to \infty$, that will be sufficient to establish the Hypothesis. However, the technique of [KR1] was not enough to prove the relaxation property $\lambda(t) \to c$. It was proven there that the situation at a given single server is described by the so-called non-linear Markov process $\mu_t$ with Poissonian input with rate $\lambda(t)$, and the (non-Poissonian)
output with the same rate. Another way of saying this is that corresponding non-linear Markov process defines some complicated dynamical system, and the problem was to study its invariant measures. Namely, this system has one parameter family of fixed points, and the question is about whether it has other invariant measures.

In the present paper we complete the picture, showing that the above relaxation $\lambda(t) \to c$ indeed takes place, and so $\mu_t \to \mu_c$, where $\mu_c$ is the stationary distribution of the stationary Markov process with the Poisson input, corresponding to constant rate $\lambda(t) = c$. In the language of dynamical systems we show that there are no other invariant measures except these defined by the fixed points.

The central discovery of the present paper, which seems to be the key to the solution of the problem, is that, roughly speaking, the function $\lambda(t)$ has to satisfy the following non-linear equation:

$$\lambda(t) = \left[\lambda(\cdot) \ast q_{\lambda(\cdot),t}(\cdot)\right](t).$$  \hspace{1cm} (1)

Here $\ast$ stays for convolution: for two functions $a(t), b(t)$ it is defined as

$$[a(\cdot) \ast b(\cdot)](t) = \int a(t-x)b(x) \, dx,$$

while $q_{\lambda(\cdot),t}(\cdot)$ is a one-parameter family of probability densities with $t$ real, which depends also in an implicit way on the unknown function $\lambda(\cdot)$. We call (1) the self-averaging property. The present paper consists therefore of two parts: we prove that indeed the self-averaging relation holds, and we prove then that it implies relaxation.

It is amazing that the relation (1) depends crucially on some purely combinatorial statement concerning certain problem of the placement of the rods on the line $\mathbb{R}^1$, see Section 6.

To fix the terminology, we remind the reader here what we mean by the non-linear Markov process (see [M1], [M2]). We do this for the simplest case of discrete time Markov chains, taking values in a finite set $S$, $|S| = k$. In such a case the set of states of this Markov chain is a simplex $\Delta_k$ of all probability measures on $S$, $\Delta_k = \{\mu = (p_1, \ldots, p_k) : p_i \geq 0, p_1 + \ldots + p_k = 1\}$, while the Markov evolution defines a map $P : \Delta_k \to \Delta_k$. In the case of usual Markov chain $P$ is affine, and this is why we will call it linear chain. In this case the matrix of transition probabilities coincides with $P$. If $P$ is non-linear, we will call such a process a non-linear Markov chain. It is defined
by a family of transition probability matrices $P_\mu$, $\mu \in \Delta_k$, so that matrix element $P_\mu (i, j)$ is a probability of going from $i$ to $j$ in one step, starting in the state $\mu$. The (non-linear) map $P$ is then defined by $P (\mu) = \mu P_\mu$.

The ergodic properties of the linear Markov chains are settled by the Perron-Frobenius theorem. In particular, if the linear map $P$ is such that the image $P (\Delta_k)$ belongs to the interior $\text{Int} (\Delta_k)$ of $\Delta_k$, then there is precisely one point $\mu \in \text{Int} (\Delta_k)$, such that $P (\mu) = \mu$, and for every $\nu \in \Delta_k$ we have $P^n (\nu) \to \mu$ as $n \to \infty$.

In case $P$ is non-linear, we are dealing with more or less arbitrary dynamical system, and the question about stationary states of the chain or about measures on $\Delta_k$ invariant under $P$ can not be settled in general.

Therefore it is natural to ask about the specific features of our dynamical system, which permit us to find all its invariant measures. We explain this in the following subsection.

**Dynamical systems aspect.** Here we will use the notation of the paper, though in fact the situation of the paper is more complicated; in particular the underlying space is not a manifold, but a space of all measures over some non-compact set.

Let $M$ be a manifold, supplied with the following structures:

- for every point $\mu \in M$ and every $\lambda > 0$ a tangent vector $X (\mu, \lambda)$ at $\mu$ is defined,
- a function $b : M \to \mathbb{R}^+$ is fixed.

We want to study the dynamical system

$$\frac{d}{dt} \mu (t) = X (\mu (t), b (\mu (t))) . \quad (2)$$

Its flow conserves another given function, $N : M \to \mathbb{R}^+$, and we want to prove that our dynamical system has one-parameter family of fixed points - each corresponding to one value of $N$ - and no other invariant measures.

We have the following extra properties of our dynamical system:

Let $\lambda (t) > 0$; consider the differential equation

$$\frac{d}{dt} \mu (t) = X (\mu (t), \lambda (t)) , \ t \geq 0 , \quad (3)$$

with $\mu (0) = \nu$. We denote the solution to it by $\mu_{\nu, \lambda (\cdot)} (t)$ . We know that
for every $c > 0$ and every initial data $\nu$, the solution $\mu_{\nu,\lambda(t)}(t)$ to (3) converges to some stationary point $\nu_c \in M$,

$$\mu_{\nu,\lambda(t)}(t) \to \nu_c, \text{ provided } \lambda(t) \to c \text{ as } t \to \infty,$$

$$(4)$$

- for the function $N$ we have

$$\frac{d}{dt} N \left( \mu_{\nu,\lambda(t)}(t) \right) = \lambda(t) - b \left( \mu_{\nu,\lambda(t)}(t) \right).$$

In particular, for every trajectory $\hat{\mu}_\nu(t)$ of (2) (where $\hat{\mu}_\nu(0) = \nu$) we have $N(\hat{\mu}_\nu(t)) = N(\nu)$. Also, $N(\nu_c)$ is continuous and increasing in $c$;

- for every $\nu, \lambda(\cdot)$ and every $t > 0$ there exists a probability density $q_{\nu,\lambda,t}(x), \ x \geq 0$, such that

$$b \left( \mu_{\nu,\lambda(t)}(t) \right) = (\lambda * q_{\nu,\lambda,t})(t),$$

where

$$(\lambda * q_{\nu,\lambda,t})(y) = \int_{x \geq 0} q_{\nu,\lambda,t}(x) \lambda(y - x) \ dx.$$

The family $q_{\nu,\lambda,t}(x)$ satisfies:

$$\int_0^1 q_{\nu,\lambda,t}(x) \ dx = 1 \text{ for all } \nu, \lambda, t,$$

and

$$\inf_{\nu,\lambda,t} \inf_{x \in [0,1]} q_{\nu,\lambda,t}(x) > 0.$$

Then for every initial state $\nu$

$$\hat{\mu}_\nu(t) \to \nu_c,$$

$$(5)$$

where $c$ satisfies $N(\nu_c) = N(\nu)$.

Our statement follows from the fact that the self-averaging property,

$$f(t) = (f * q_t)(t),$$

where $c$ satisfies $N(\nu_c) = N(\nu)$. 
with \( q_t(\cdot) \) being a family of probability densities on \([0, 1]\), implies that \( f(t) \to \text{const} \) as \( t \to \infty \), so (5) follows from (4).

We feel that the relation (1) is an important feature of the subject we are interested in. Therefore in the present paper we study it and the related questions in some generality.

i) We start with the equation

\[
   f(t) = [f(\cdot) * q_t(\cdot)](t).
\]

Here we suppose that \( q_t(\cdot) \) is just some one-parameter family of probability densities (without functional dependence), so (1) is a special case of (6). On the other hand, we suppose additionally that all the distributions \( q_t(\cdot) \) are supported by some finite interval. We establish relaxation in this case.

ii) We then do the same for the case of distributions \( q_t(\cdot) \) with unbounded support.

iii) Last, we treat the true problem, where in addition to the infinite support, an extra parameter \( \mu \) appears and an extra perturbation is added to convolution term in (6):

\[
   \lambda(t) = (1 - \varepsilon_{\lambda,\mu}(t)) [\lambda(\cdot) * q_{\lambda,\mu,t}(\cdot)](t) + \varepsilon_{\lambda,\mu}(t) Q_{\lambda,\mu}(t).
\]

Here the parameter \( \varepsilon_{\lambda,\mu}(t) \) is small: \( \varepsilon_{\lambda,\mu}(t) \to 0 \) as \( t \to \infty \), the term \( Q_{\lambda,\mu}(t) \) is uniformly bounded, while the meaning of \( \mu \) will be explained later.

As we proceed from i) to iii), we will have to assume more about the class of distributions \( \{q_{\cdot}\} \), for which the self-averaging implies relaxation.

We finish this introduction by a brief discussion of the previous work on the subject, and their methods.

As we said before, part of the proof of the Poissonian Hypothesis – the so called Weak Poissonian Hypothesis – was obtained in (KR1). By proving that the Markov semigroups describing the Markov processes for finite \( M, N \), converge, after factorization by the symmetry group of the model, to the semigroup, describing the non-linear Markov process, the authors have proven that the limit flows to each node are independent Poisson flows with the same rate function \( \lambda(t) \). This statement is fairly general, and can be generalized to other models with the same kind of the symmetry – the so-called mean-field models. The general theory – see, for example, (L) – implies then, that all the limit points of the stationary measures of the Markov processes with finite \( M, N \) are invariant measures of the limiting non-linear Markov process. The remaining step – the proof that the limiting dynamical system
has no other attractors except the one-parameter family of the fixed points – is done in the present paper. Apriori this fact is not at all clear, and one can construct natural examples of the systems with many complicated attractors, which are reflected in the complex behavior of the Markov processes with finite $M, N$. However, the self-averaging property, explained above, rules out such a possibility. It seems that the self-averaging property can also be generalized to other mean-field models.

The Poisson Hypothesis was fully established in a pioneer paper [St] for a special case when the service time is non-random. This is a much simpler case, and the methods of the paper can not be extended to our situation. They are sufficient for a simpler case of the Poissonian service times, which case was studied in [KR2].

The paper [DKV] deals with another mean-field model, describing some open queueing network. Though the Poisson Hypothesis does not hold for it, the spirit of the main statement there is the same as in the present paper: the limiting dynamical system has precisely one global attractor, which corresponds to the fixed point.

One of specific feature of the method of the paper [DKV], as well as related paper [DF], is that the Markov processes have countable sets of values. So one can in principle use monotonicity arguments and stochastic domination. In our situation the phase space is (one-dimensional) real manifold, and this technique does not seem to be applicable.

The importance of the Poisson Hypothesis as the central problem of the theory of large queueing systems was emphasized, among others, by Roland Dobrushin [D] and Alexander Borovkov [B].

2 Notation

In this section we will fix the notation for the non-linear Markov process, which takes place at a given server in the above described limit.

Server. It is defined by specifying the distribution of the random serving time $\eta$, i.e. by the function

$$F(t) = \Pr\{\text{serving time } \eta \geq t\}.$$  

We suppose that $\eta$ is such that:

- the density function $p(t)$ of $\eta$ is positive on $t \geq 0$ and uniformly bounded from above;
• $p(t)$ satisfies the following strong Lipschitz condition: for some $C < \infty$, some positive function $u(t)$ and for all $t \geq 0$

$$|p(t + \Delta t) - p(t)| \leq C p(t) |\Delta t|,$$  

(8)

provided $t + \Delta t > 0$ and $|\Delta t| < u(t)$;

• introducing the random variables

$$\eta \mid \tau = (\eta - \tau \mid \eta > \tau), \tau \geq 0,$$

we suppose that for some $\delta > 0$, $M_\delta < \infty$

$$\mathbb{E} \left( \eta \mid \tau \right)^{2+\delta} < M_\delta$$  

(9)

for all $\tau$; therefore, for the conditional expectations we have

$$\mathbb{E} \left( \eta \mid \tau \right) < \bar{C}$$  

(10)

uniformly in $\tau \geq 0$;

• Without loss of generality we can suppose that

$$\mathbb{E}(\eta) = 1.$$  

(11)

In what follows, the function $p(\cdot)$ will be fixed.

**Configurations.** By a configuration of a server at a given time moment $t$ we mean the following data:

• The number $n \geq 0$ of customers waiting to be served. The customer who is served at $t$, is included in the total amount $n$. Therefore by definition, the length of the queue is $n - 1$ for $n \geq 1$, and $0$ for $n = 0$.

• The duration $\tau$ of the elapsed service time of the customer under the service at the moment $t$.

Therefore the set of all configurations $\Omega$ is the set of all pairs $(n, \tau)$, with an integer $n > 0$ and a real $\tau > 0$, plus the point $0$, describing the
situation of the server being idle. For a configuration \( \omega = (n, \tau) \in \Omega \) we define \( N(\omega) = n \). We put \( N(0) = 0 \).

**States.** By a state of the system we mean a probability measure \( \mu \) on \( \Omega \). We denote by \( M(\Omega) \) the set of all states on \( \Omega \).

**Observables.** There are some natural random variables associated with our system. One is the queue length, \( N_\mu = N_\mu(\omega) \). We denote by \( N(\mu) \) the mean queue length in the state \( \mu \):

\[
N(\mu) = \mathbb{E}(N_\mu) \equiv \langle N_\mu(\omega) \rangle_\mu,
\]

and we introduce the subsets \( M_q(\Omega) \subset M(\Omega) \), \( q \geq 0 \) by

\[
M_q(\Omega) = \{ \mu \in M(\Omega) : N(\mu) = q \}.
\]

Another one is the expected service time

\[
S(\omega) = \begin{cases}
0 & \text{for } \omega = 0, \\
(n - 1) \mathbb{E}(\eta) + \mathbb{E}\left(\eta \bigg| \tau\right) & \text{for } \omega = (n, \tau), \text{ with } n > 0.
\end{cases}
\]

Again, we define

\[
S(\mu) = \langle S(\omega) \rangle_\mu.
\]

Clearly,

\[
S(\mu) \leq \bar{C} N(\mu). \tag{12}
\]

**Input flow.** Suppose that a function \( \lambda(t) \geq 0 \) is given. We suppose that the input flow to our server is a Poisson process with rate function \( \lambda(t) \), which means in particular that the probabilities \( P_k(t, s) \) of the events that \( k \) new customers arrive during the time interval \([t, s]\) satisfy

\[
P_k(t, t + \Delta t) = \begin{cases}
\lambda(t) \Delta t + o(\Delta t) & \text{for } k = 1, \\
1 - \lambda(t) \Delta t + o(\Delta t) & \text{for } k = 0, \\
o(\Delta t) & \text{for } k > 1,
\end{cases}
\]
as \( \Delta t \to 0 \), while for non-intersecting time segments \([t_1, s_1], [t_2, s_2]\) the flows are independent.

**Output flow.** Suppose the initial state \( \nu = \mu(-T), T > 0 \), as well as the rate function \( \lambda(t) \), with \( \lambda(t) = 0 \) for \( t \leq -T \), of the input flow are given. Then the system evolves in time, and its state at the moment \( t \) is given by the measure

\[
\mu(t) = \mu_{\nu, \lambda(\cdot)}(t).
\]
In particular, the probabilities $Q_k (t, s) = Q_k (t, s; \nu, \lambda (\cdot), p (\cdot))$ of the events that $k$ customers have finished their service during the time interval $[t, s]$ are defined. We suppose that the customer, once served, leaves the system.

The resulting random point process $Q (\cdot, \cdot)$ need not, of course, be Poissonian. However we still can define its rate function $b (t)$ as the one satisfying

$$Q_k (t, t + \Delta t) = \begin{cases} b (t) \Delta t + o (\Delta t) & \text{for } k = 1, \\ 1 - b (t) \Delta t + o (\Delta t) & \text{for } k = 0, \\ o (\Delta t) & \text{for } k > 1, \end{cases}$$

as $\Delta t \to 0$. The rate function $b (\cdot)$ of the output flow is determined once the initial state $\nu = \mu (0)$ and the rate function $\lambda (\cdot)$ of the input flow are given. Therefore the following (non-linear) operator $A$ is well defined:

$$b (\cdot) = A (\nu, \lambda (\cdot), -T).$$

We will call the general situation, described by the triple $\nu, \lambda (\cdot), b (\cdot) = A (\nu, \lambda (\cdot), -T)$, as a General Flow Process (GFP).

The following is known about the operator $A$, see [KRI]:

- For every initial state $\nu$ the equation
  $$A (\nu, \lambda (\cdot)) \equiv A (\nu, \lambda (\cdot), 0) = \lambda (\cdot)$$
  has exactly one solution $\lambda (\cdot) = \lambda_\nu (\cdot)$. Then the evolving state $\mu_{\nu, \lambda_\nu (\cdot)} (t)$ is what is called the non-linear Markov process, which we will abbreviate as NMP.

- The non-linear Markov process has the following conservation property: for all $t$
  $$N \left( \mu_{\nu, \lambda_\nu (\cdot)} (t) \right) = N (\nu)$$
  (because “the rates of the input flow and the output flow coincide”). So one can say that the spaces $\mathcal{M}_q (\Omega)$ are invariant under non-linear Markov evolutions.

- All the functions $\lambda_\nu (\cdot)$ are bounded:
  $$\lambda_\nu (t) \leq C = C (\eta) \quad (13)$$
  uniformly in $\nu$ and $t$. 

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For every constant $c \in [0, 1)$ there exists the initial state $\nu_c$, such that

$$A(\nu_c, c) = c. \quad (14)$$

(Here we identify the constant $c$ with the function taking just one value $c$ everywhere.) Moreover, this measure $\nu_c$ is a stationary state: $\mu_{\nu_c, c}(t) = \nu_c$ for all $t > 0$. The function $c \mapsto N(\nu_c)$ is continuous increasing, with $N(\nu_0) = 0$, $N(\nu_c) \uparrow \infty$ as $c \to 1$.

The non-linear Markov process $\mu_{\nu, \lambda(\cdot)}(t)$ is the main object of the present paper. Therefore we will give below another definition of this process, via jump rates of transitions during the infinitesimal time, $\Delta t$. So suppose that our process is in the state $\mu \in \mathcal{M}(\Omega)$, and assumes the value $\omega = (n, \tau) \in \Omega$. During the time increment $\Delta t$ the following two transitions can happen with probabilities of order of $\Delta t$:

- the customer under the service will finish it and will leave the server, so the value $(n, \tau)$ will become $(n - 1, \varsigma)$, with $\varsigma \leq \Delta t$. The probability of this event is $c_1 \Delta t + o(\Delta t)$, where
  $$c_1 = c_1(\omega) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{\int_{\tau}^{\tau + \Delta t} p(x) \, dx}{\int_{\tau}^{\infty} p(x) \, dx};$$

- a new customer will arrive to the server, so the value $(n, \tau)$ will become $(n + 1, \tau + \Delta t)$. The probability of this event is given by
  $$c_2 \Delta t + o(\Delta t),$$
  where
  $$c_2 = c_2(\mu) = \mathbb{E}_\mu(c_1(\omega)).$$

In words, the input rate is the average rate of the output in the state $\mu$.

It is curious to note that while for the general nonlinear continuous time Markov processes its rates depend both on the value and on the state of the process, in our case the rate $c_1$ depends only on the value, while the rate $c_2$ – only on the state of the process.
3 More facts from [KR1]

Consider the following continuous time Markov process $\mathcal{M}$. Let there be $M$ servers and $N$ customers. The serving time is $\eta$. The configuration of the system consists of specifying the numbers of customers $n_i$, $i = 1, ..., M$, waiting at each server, plus the duration $\tau_i$ of the service time for every customer under service. Therefore it is a point in

$$\Theta_{M,N} = \left\{ (\omega_1, ..., \omega_M) \in \prod_{i=1}^{M} \Omega_i : n_1 + ... + n_M = N \right\}.$$

Upon being served, the customer goes to one of $M$ servers with equal probability $1/M$, and is there the last in the queue.

The permutation group $S_M$ acts on $\Theta_{M,N}$, leaving the transition probabilities invariant. Therefore we can consider the factor-process. Its values are (unordered) finite subsets of $\Omega$. It can be equivalently described as a measure

$$\nu = \frac{1}{M} \sum_{i=1}^{M} \delta_{(n_i, \tau_i)}.$$ 

We can identify such measures with the configurations of the symmetrized factor-process. Note that

$$\langle n \rangle_\nu = \frac{N}{M}.$$ 

So if we introduce the notation $\mathcal{M}_q(\Omega) \subset \mathcal{M}(\Omega)$ for the measures $\mu$ on $\Omega$ for which $\langle n \rangle_\mu = q$, then we have that $\nu \in \mathcal{M}_{\frac{N}{M}}(\Omega)$. We also introduce the notation $\mathcal{M}_{\frac{N}{M}}(\Omega) \subset \mathcal{M}(\Omega)$ for the family of atomic measures, such that each atom has a weight $\frac{k}{M}$ for some integer $k$.

A state of our Markov process is a probability measure on the set of configurations, i.e. an element of $\mathcal{M}(\mathcal{M}(\Omega))$. If the initial state of the process is supported by $\mathcal{M}_q(\Omega)$, then at any positive time it is still the element of $\mathcal{M}(\mathcal{M}_q(\Omega))$. A natural embedding $\mathcal{M}(\Omega) \subset \mathcal{M}(\mathcal{M}(\Omega))$, which to each configuration $\nu \in \mathcal{M}(\Omega)$ corresponds the atomic measure $\delta_\nu$, will be denoted by $\delta$.

For $\mu_0 = \delta_\nu \in \mathcal{M}\left(\mathcal{M}_{\frac{N}{M}}(\Omega)\right)$ to be the initial state of our Markov process, we denote by $\mu_t$ the evolution of this state. Clearly, in general $\mu_t \notin \delta(\mathcal{M}(\Omega))$ for positive $t$. This process is ergodic. We denote by $\pi_{M,N}$ the stationary measure of this process.
Let now \( \kappa \in \mathcal{M}_q(\Omega) \) be some measure, let the sequences of integers \( N_j, M_j \to \infty \) be such that \( \frac{N_j}{M_j} \to q \), and let the measures \( \nu^j \in \mathcal{M}_{\frac{N_j}{M_j}, M_j}(\Omega) \) be such that \( \nu_j \to \kappa \) weakly. Consider the Markov processes \( \mu^j_t \in \mathcal{M}_{\frac{N_j}{M_j}, M_j}(\Omega) \), corresponding to the initial conditions \( \delta_{\nu^j} \).

As we just said, in general \( \mu^j_t \notin \delta_{\mathcal{M}_{\frac{N_j}{M_j}, M_j}(\Omega)} \) for any \( j \), once \( t > 0 \). However, for the limit \( \mu_t = \lim_{j \to \infty} \mu^j_t \) we have that \( \mu_t \in \mathcal{M}(\mathcal{M}_q(\Omega)) \), and moreover \( \mu_t \in \delta(\mathcal{M}_q(\Omega)) \), so we can say that the random evolutions \( \mu^j_t \) tend to the non-random evolution \( \kappa_t \equiv \mu_t \) as \( N_j, M_j \to \infty \).

Therefore we have a dynamical system

\[
\mathcal{T}_t : \mathcal{M}_q(\Omega) \to \mathcal{M}_q(\Omega).
\]

This dynamical system \( \mu_t \) is nothing else but the non-linear Markov process, mentioned above.

Another way of obtaining the same dynamical system is to look on the behavior of a given server. Here instead of taking the symmetrization of the initial process \( \mathfrak{M} \) on \( \Theta_{M,N} \), we have to consider its projection on the first coordinate, \( \Omega_1 \), say. To make the correspondence with the above, we have to take for the initial state of this process a measure \( \tilde{\nu}^j \) on \( \Theta_{M,N} \), which is \( S_M \)-invariant, and which symmetrization is the initial state \( \nu^j \) of the preceding paragraph. The projection of \( \mathfrak{M} \) on \( \Omega_1 \) would not be, of course, a Markov process. However, it becomes the very same non-linear Markov process \( \mu_t \) in the above limit \( N_j, M_j \to \infty \).

We can generalize further, and study the projection of \( \mathfrak{M} \) to a finite product, \( \prod_{j=1}^R \Omega_j \). Then in the limit \( N_j, M_j \to \infty \) this projection converges to a process on \( \prod_{j=1}^R \Omega_j \), which factors into the product of \( R \) independent copies of the same non-linear Markov process \( \mu_t \). This statement is known as the “propagation of chaos” property.

The main result of the present paper is that for every \( q \) the dynamical system (15) has exactly one fixed point \( \nu_c, c = c(q) \), and that it is globally attractive. In particular that means that \( \pi_{N_j, M_j} \to \nu_c \), provided \( \frac{N_j}{M_j} \to q \) as \( j \to \infty \) and \( c = c(q) \).
4 Main result

Our goal is to show the following:

**Theorem 1** For every initial state \( \nu \) the solution \( \lambda_\nu (\cdot) \) of the equation

\[
A (\nu, \lambda (\cdot)) = \lambda (\cdot)
\]

has the **relaxation** property:

\[
\lambda_\nu (t) \to c \text{ as } t \to \infty,
\]

where the constant \( c \) satisfies

\[
E_{\nu} (N (\omega)) = E_{\nu_c} (N (\omega)).
\]

Moreover, \( \mu_{\nu,\lambda_\nu (\cdot)} (t) \to \nu_c \) weakly, as \( t \to \infty \).

A special case of the above theorem is the following

**Proposition 2** Let \( T > 0 \) be some time moment, and suppose that the function \( \lambda (\cdot) \) satisfies

\[
A (0, \lambda (\cdot), -T) = b (\cdot)
\]

with

\[
\lambda (t) = b (t) \text{ for all } t \geq 0.
\]

Let also

\[
\int_{-T}^{0} \lambda (t) \, dt \leq C < \infty.
\]

Then for some \( c \geq 0 \)

\[
\lambda (t) \to c \text{ as } t \to \infty.
\]

Our theorem follows from the Proposition 2 immediately in the special case when the initial state \( \nu \) is of the form \( \nu = \mu_{0,\lambda (\cdot)} (t) \) for some \( \lambda \) and some \( t > 0 \). These initial states are easier to handle, so we treat them separately.

The heuristics behind the Proposition 2 is the following. One expects that if

\[
b (\cdot) = A (\nu, \lambda (\cdot)),
\]

then the function \( b \) for large times is “closer to a constant” than the function \( \lambda \). More precisely, if \( t \) belongs to some segment \([T_1, T_2], \) with \( T_1 \gg 1 \), then the
dependence of $b(t)$ on $\nu$ is very weak, so $b$ is determined mainly by $\lambda$. One then argues that under that assumption $\sup_{t \in [T_1, T_2]} b(t)$ should be strictly less than $\sup_{t \in [T_1, T_2]} \lambda(t)$. Indeed, one can visualize the random configuration of the exit moments $y_i$-s as being obtained from the input flow configuration of $x_i$-s by making it *sparser*. Namely, we have to consider a sequence $\eta_i$ of i.i.d. random variables, having the same distribution as $\eta$, and then to move the particles $x_i$ to the right, positioning them at locations $y_i$, so that in the result

$$y_{i+1} - y_i \geq \eta_i$$

for all $i$-s, see (29), (30) below for more details. However this is a very rough idea, since some particles need not be moved, due to the fact that (19) may hold even prior to the sparsening step, in which case it will happen that $y_{i+1} = x_{i+1}$, while $y_i > x_i$, and so the configuration becomes locally denser. (And if $\lambda$ is a constant, then $b$ is this same constant, so again the above argument is not literally true.)

To be more precise, we will show the following **self-averaging property**. Let the functions $\lambda(\cdot)$ and $b(\cdot)$ are related by

$$b(\cdot) = A(0, \lambda(\cdot), -T).$$

One of the main points of the following would be to show that for every $x$ one can find a probability density $q_{\lambda,x}(t)$, vanishing for $t \leq 0$, such that

$$b(x) = [\lambda * q_{\lambda,x}](x).$$

(20)

We then will show that this self-averaging property of the system implies (18), provided we know in advance certain regularity properties of the family $\{q_{\lambda,x}\}$. Note that apriori the condition (20) is not evident at all for our FIFO system: one has to rule out the situation that, say, the input rate function $\lambda$ is uniformly bounded from above by 1, while the output rate $b$ is occasionally reaching the level 2; this is clearly inconsistent with (20).

In general case, when

$$b(\cdot) = A(\mu, \lambda(\cdot)),$$

we have

$$b(x) = (1 - \varepsilon_{\lambda,\mu}(x)) [\lambda * q_{\lambda,\mu,x}](x) + \varepsilon_{\lambda,\mu}(x) Q_{\lambda,\mu}(x),$$

(21)
where $\varepsilon_{\lambda,\mu}(x) > 0$, $\varepsilon_{\lambda,\mu}(x) \to 0$ as $x \to \infty$, while $Q_{\lambda,\mu}(x)$ is a bounded term, see Section 8 for details.

To get the above mentioned regularity property we will need few preparatory lemmas.

**Lemma 3** Let $\mu_{\nu,\lambda,\nu}(\cdot)$ be NMP, with $N\left(\mu_{\nu,\lambda,\nu}(t)\right) = N(\nu) = q$. Then there exists a time moment $T = T(q)$ and $\varepsilon = \varepsilon(q) > 0$, such that for all $t > T$

$$\langle \omega = 0 \rangle_{\mu_{\nu,\lambda,\nu}(t)} > \varepsilon. \quad (22)$$

In words, the probability of observing the system $\mu_{\nu,\lambda,\nu}(t)$ to be in the idle state is uniformly positive, after some time $T(q)$.

**Proof.** Let the initial state $\kappa$ of the NMP be such that $N(\kappa) = q$. Then by (10), (12), we have $S(\kappa) \leq Cq$. Consider now the GFP, started in $\kappa$ and having zero input flow, i.e. $\lambda \equiv 0$. We denote it by $\mu_{\kappa,0}(t)$. Consider the probability $\langle N(\omega) > 0 \rangle_{\mu_{\kappa,0}(t)}$ that such a system is still occupied at the moment $t$. Then clearly

$$S(\kappa) \geq t \langle N(\omega) > 0 \rangle_{\mu_{\kappa,0}(t)}.$$ 

Therefore

$$\langle \omega = 0 \rangle_{\mu_{\kappa,0}(t)} \geq 1 - \frac{Cq}{t}.$$ 

In particular, if we put $T = 2Cq$, then for all $t \geq T$

$$\langle \omega = 0 \rangle_{\mu_{\kappa,0}(t)} \geq \frac{1}{2}.$$ 

Consider now the NMP started at $\kappa$. Let us introduce the event

$$E_{\kappa}(t) = \begin{cases} \text{in the Poisson random flow, defined by the rate} \\ \lambda_{\kappa}(\cdot), \text{ no customer arrives before time } t. \end{cases}$$

Then

$$\langle \omega = 0 \rangle_{\mu_{\kappa,\lambda_{\kappa}(T)}} \geq \Pr (E_{T}) \langle \omega = 0 \bigg| E_{\kappa}(T) \rangle \mu_{\kappa,\lambda_{\kappa}(T)}(T) \quad (23)$$

$$= \Pr (E_{\kappa}(T)) \langle \omega = 0 \rangle_{\mu_{\kappa,0}(T)} \geq \frac{1}{2} \Pr (E_{\kappa}(T)),$$
so we need an estimate on the probability \( \Pr (E_\kappa (T)) \). This is easy, because of (13):

\[
\Pr (E_\kappa (T)) = \exp \left\{ - \int_0^T \lambda_\kappa (t) \, dt \right\} \geq \exp \left\{ -T C (\eta) \right\}.
\] (24)

That proves our statement with \( T = 2 \bar{C}q \) and \( \varepsilon = \exp \left\{ -T C (\eta) \right\} /2 \), though thus far only for \( t = T \).

But in fact we are done! Indeed, for an arbitrary \( t > T \) let us take the state \( \kappa = \kappa_{t-T} = \mu_{\nu, \lambda_\nu} (t-T) \) of the process \( \mu_{\nu, \lambda_\nu} (\cdot) \) as the initial state of a new NMP, \( \mu_{\kappa, \lambda_\kappa} (\cdot) \). Then for every \( \tau > T - t \) we have \( \mu_{\nu, \lambda_\nu} (\tau) = \mu_{\kappa, \lambda_\kappa} (\tau - (t-T)) \), so in particular \( \mu_{\nu, \lambda_\nu} (t) = \mu_{\kappa, \lambda_\kappa} (T) \). Since \( \kappa_{t-T} \in M_q (\Omega) \), we can apply (23), (24) and thus complete the proof.

**Lemma 4** Let \( \mu_{\nu, \lambda_\nu} (\cdot) \) be NMP, with \( N (\mu_{\nu, \lambda_\nu} (t)) = N (\nu) = q \). Then there exists a time moment \( T' = T' (q) \) and \( \varepsilon' = \varepsilon' (q) > 0 \), such that for all \( T \geq T' \)

\[
\int_0^T \lambda_\nu (t) \, dt < T (1 - \varepsilon').
\] (25)

**Proof.** A configuration \( \chi \) of our process in the segment \([0, T']\) consists from

i) the initial configuration \((n, \tau)\), drawn from the distribution \( \mu \);

ii) the random set \( 0 < x_1 < \ldots < x_m < T' \), which is a realization of the Poisson random field defined by the rate function \( \lambda_\nu \) (restricted to the segment \([0, T']\)), independent of \((n, \tau)\);

iii) one realization \( \eta_1 \) of the conditional random variable \( (\eta - \tau \mid \eta > \tau) \) and \( n + m - 1 \) independent realizations \( \eta_k, k = 2, \ldots, n + m \) of the random variable \( \eta \). We denote by \( \mathbb{P}_{\mu \otimes \lambda_\nu \otimes \eta} (d\chi) \) the corresponding (product) distribution.

Let \( \bar{A} (\chi) \subset [0, \infty) \) be the set on the real line, covered by the rods of \( \chi \) after the resolution of conflicts. Let \( B (\chi) = [0, \infty) \setminus \bar{A} (\chi) \). Finally, let \( A (\chi) \subset \bar{A} (\chi) \) be the set covered only by the last \( m \) rods, while \( C (\chi) = \bar{A} (\chi) \setminus A (\chi) \). A moment thought shows that

\[
\mathbb{E}_\chi \left( \int_0^\infty I_{A(\chi)} (x) \, dx \right) = \int \mes \{ A (\chi) \} \mathbb{P}_{\mu \otimes \lambda_\nu \otimes \eta} (d\chi) = \int_0^{T'} \lambda_\nu (t) \, dt.
\] (26)
Also
\[ \int_0^{T'} \left( I_{\bar{A}(x)}(x) + I_{B(x)}(x) \right) \, dx \equiv T'. \]

Evidently,
\[ E_x(I_{B(x)}(x)) = \Pr \{ \text{the system is idle at the moment } x \} . \]

From the previous lemma we know that \( E_x(I_{B(x)}(x)) > \varepsilon \) for all \( x \) large enough. Therefore
\[ E_x \left( \int_0^{T'} I_{\bar{A}(x)}(x) \, dx \right) < T' (1 - \varepsilon/2) \quad (27) \]
once \( T' \) is large enough. Finally,
\[
\Bigg| E_x \left( \int_0^{T'} I_{\bar{A}(x)}(x) \, dx \right) - E_x \left( \int_0^{\infty} I_{\bar{A}(x)}(x) \, dx \right) \Bigg|
= \Bigg| E_x \left( \int_0^{\infty} I_{C(x)}(x) \, dx \right) - E_x \left( \int_{T'}^{\infty} I_{\bar{A}(x)}(x) \, dx \right) \Bigg| \quad (28)
\]

Note that each of the last two expectations is the mean occupation time of our system when it is initially in the states \( \mu_{\nu,\lambda}(0) = \nu \) and \( \mu_{\nu,\lambda}(T') \), while no extra input flows are present. Since \( N(\nu) = N(\mu_{\nu,\lambda}(T')) = q \), the difference between the expectations of these occupation times does not exceed \( 2\bar{C} \), see (10). This, together with (26-28) proves our statement to hold for \( T' \) large, with \( \varepsilon' = \varepsilon/4 \).

We finish this section with a statement about the regularity of the exit flow.

**Lemma 5** Let the function \( p(t) \) satisfies the strong Lipschitz condition \( \Box \): for some \( C \)
\[ |p(t + \Delta t) - p(t)| \leq C p(t) \Delta t. \]
Then the function \( b(t) \) is Lipschitz.

**Proof.** Let \( t \) be fixed. The idea of the proof is to correspond to every elementary event, which contribute to the output rate \( b(t) \), the elementary event, contributing to \( b(t + \Delta t) \), by enlarging by \( \Delta t \) the service time of the customer, whose service ends at the moment \( t \). This correspondence,
however, does not “cover” all the events, contributing to \( b(t + \Delta t) \). Namely, the elementary events not covered by the above correspondence, are precisely those, for which the customer, whose service terminated at \( t + \Delta t \), started his service after the moment \( t \).

Let us first estimate the probability \( \pi(t, \Delta t) \) of the event \( \Pi(t, \Delta t) \) that some customer started to be served after the moment \( t \), and was served before \( t + \Delta t \). Consider an elementary event, contributing to \( \Pi(t, \Delta t) \). It is some configuration \( (\vec{x}_1, \ldots, \vec{x}_n; \vec{l}_1, \ldots, \vec{l}_n) \), where a certain rod \( \vec{l}_k \) satisfies \( \vec{l}_k \leq \Delta t \).

Comparing the collection of events \( \{ (\vec{x}_1, \ldots, \vec{x}_n; \vec{l}_1, \ldots, \vec{l}_{k-1}, l_k, \ldots, \vec{l}_n) : l_k \leq \Delta t \} \) with the collection \( \{ (\vec{x}_1, \ldots, \vec{x}_n; \vec{l}_1, \ldots, \vec{l}_{k-1}, l_k, \ldots, \vec{l}_n) : l_k > \Delta t \} \) (Peierls transformation), we conclude that

\[
\pi(t, \Delta t) \leq \frac{\int_0^{\Delta t} p(t) \, dt}{\int_0^\infty p(t) \, dt} \leq C_p \Delta t
\]

for some \( C_p < \infty \).

Denote by \( \zeta(t) \) the random moment of the beginning of the service of the client, who happens to be the last one started to be served before \( t \). Then one can define the rate \( \gamma_t(x) \) for all \( x < t \) by

\[
\gamma_t(x) = \lim_{\Delta x \to 0} \frac{\Pr \{ \zeta(t) \in [x, x + \Delta x] \}}{\Delta x}.
\]

Then

\[
b(t) = \int_0^\infty \gamma_t(t - x) p(x) \, dx.
\]

Clearly,

\[
b(t + \Delta t) = \int_0^\infty \gamma_t(t - x) p(x + \Delta t) \, dx + \pi(t, \Delta t).
\]

Therefore

\[
|b(t + \Delta t) - b(t)| \leq C_p \Delta t + \int_0^\infty \gamma_t(t - x) |p(x + \Delta t) - p(x)| \, dx
\]

\[
\leq C_p \Delta t + C \Delta t \int_0^\infty \gamma_t(t - x) p(x) \, dx
\]

\[
= C_p \Delta t + C \Delta t b(t).
\]

Since \( b(\cdot) \) is uniformly bounded, the proof follows.
5 The self-averaging relation

Here we will derive a formula, expressing the function \( b(\cdot) = A(0, \lambda(\cdot)) \) in terms of the functions \( \lambda(\cdot) \) and \( p(\cdot) \). This will be the needed self-averaging relation (20).

**Theorem 6** Let the functions \( b(\cdot) \) and \( \lambda(\cdot) \) are related by

\[
b(\cdot) = A(0, \lambda(\cdot)).
\]

Then there exists a family of probability densities \( q_{\lambda,x}(t) \), such that

\[
b(x) = \int_0^\infty \lambda(x-t) q_{\lambda,x}(t) \, dt.
\]

**Proof.** To see this we first introduce some new notions.

Let \( l_1, \ldots, l_n > 0 \) be a collection of positive real numbers, which we will interpret as the lengths of hard rods, placed in \( \mathbb{R}^1 \). A configuration of rods can be then given by specifying, say, their left-ends, \( x_1 < x_2 < \ldots < x_n \), so that the rod \( l_i \) occupies the segment \([x_i, x_i + l_i] \). This configuration will be denoted by \( \sigma_n(x_1, \ldots, x_n;l_1, \ldots, l_n) \).

In case some of the rods are intersecting over a nondegenerate segments, we say that such a configuration has conflicts. By a resolution of conflicts we call another configuration of the rods \( l_1, \ldots, l_n \), where these rods have the following set \( z_1 < z_2 < \ldots < z_n \) of the left-ends:

it is defined inductively by

\[
z_1 = x_1,
\]

and

\[
z_i = \max \{z_{i-1} + l_{i-1}, x_i\}. \tag{29}
\]

We will denote by \( y-s \) the corresponding set of the right-ends:

\[
y_i = z_i + l_i. \tag{30}
\]

Any configuration with no conflicts, and in particular any configuration obtained by resolution of the conflicting one, will be called an r-configuration. The operation of resolving the conflict will be denoted by \( R \), so

\[
\sigma_n(z_1, \ldots, z_n;l_1, \ldots, l_n) = R\sigma_n(x_1, \ldots, x_n;l_1, \ldots, l_n).
\]
For any configuration $\sigma$ of rods we will denote by $Y(\sigma)$ the set of their right-ends. So, in our notations

$$(y_1, ..., y_n) = Y(R\sigma_n(x_1, ..., x_n; l_1, ..., l_n)).$$

Suppose now that the lengths $l_1, ..., l_n$, as well as the locations $x_1, ..., x_{n-1}$ and $y$ are specified. We define the values $X(y) \equiv X(y \mid x_1, ..., x_{n-1}; l_1, ..., l_n)$ as the solutions of the equation

$$y \in Y(R\sigma_n(x_1, ..., x_{n-1}, X(y); l_1, ..., l_n)).$$

Note that for the general position data $(x_1, ..., x_{n-1}; l_1, ..., l_n)$ the function $X(y \mid x_1, ..., x_{n-1}; l_1, ..., l_n)$ is not defined for some $y$-s of positive measure, while for some other $y$-s it is multivalued, having several (finitely many) branches, provided $n \geq 2$. (The case $n = 1$ is simple: $X(y \mid l_1) = y - l_1$.)

Now we can write the desired formula:

$$b(y) = \exp \left\{-I_\lambda(y)\right\} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \times$$

$$\times \int_0^\infty \cdots \int_0^\infty \left[ \int_0^y \cdots \int_0^y \lambda \left( X(y \mid x_1, ..., x_{n-1}; l_1, ..., l_n) \right) \prod_{i=1}^{n-1} \lambda(x_i) \, dx_i \right] \prod_{i=1}^n p(l_i) \, dl_i,$$

where

$$I_\lambda(y) = \int_0^y \lambda(x) \, dx.$$ 

The integral in (32) should be understood as follows: the range of integration coincides with the domain where the function $X(y \mid x_1, ..., x_{n-1}; l_1, ..., l_n)$ is defined, while over the domains where the function $X$ is multivalued one should integrate each branch separately and then take the sum of integrals.

In words, the meaning of the relation (32) is the following: for every realization $x_1, ..., x_{n-1}$ of the Poisson random field and every realization $l_1, ..., l_n$ of the sequence of the service times, we look for time moments $X = X(y \mid x_1, ..., x_{n-1}; l_1, ..., l_n)$, at which the $l_n$-customer has to arrive, so as to ensure that at the moment $y$ some (other) customer will exit, after being served. In some cases such moments might not exist, while in other
cases there might be more than one such moment. If $X_i$ are these moments, we then have to add all the rate values, $\lambda(X_i)$, and to integrate the sum $\sum_i \lambda(X_i)$ over all $n$ and all $x_1, ..., x_{n-1}; l_1, ..., l_n$, thus getting the exit rate $b(y)$.

The first term ($n = 1$) in (32) is by definition the convolution,

$$b_1(y) = \int_0^y \lambda(y - l) \, p(l) \, dl.$$  \hspace{1cm} (33)

Since $p(l) \geq 0$ and

$$\int_0^y p(l) \, dl \leq 1, \hspace{1cm} (34)$$

we have indeed that $b_1(y) < \sup_{x \leq y} \lambda(x)$ in case when, say, the maxima of $\lambda$ are isolated, or when $\lambda$ is not a constant and the support of the distribution $p$ is the full semiaxis $\{l > 0\}$. We want to show that in some sense the same is true for all the functions $b_n$, defined as

$$b_n(y) = \int \left[ \int \lambda(X(y \mid x_1, ..., x_{n-1}; l_1, ..., l_n)) \prod_{i=1}^{n-1} \left( \frac{\lambda(x_i)}{I_{\lambda}(y)} \right) dx_i \right] \prod_{i=1}^n p(l_i) \, dl_i.$$  \hspace{1cm} (35)

The crucial step will be the analog of (33), (34) for all $n > 1$, that is that

$$b_n(y) = \int_0^y \lambda(y - l) \, p_n(l) \, dl,$$

with $p_n(l) \geq 0$, $\int_0^y p_n(l) \, dl \nearrow 1$ for $y \to \infty$. This turns out to be quite an involved combinatorial statement.

Note that, evidently, the measure $\prod_{i=1}^n p(l_i) \, dl_i$ is invariant under the coordinate permutations in $\mathbb{R}^n$; therefore we can rewrite the expression (35) for the function $b_n(y)$ as

$$b_n(y) = \int \left[ \int \frac{1}{n!} \lambda(\bar{X}(y \mid x_1, ..., x_{n-1}; \{l_1, ..., l_n\})) \prod_{i=1}^{n-1} \left( \frac{\lambda(x_i)}{I_{\lambda}(y)} \right) dx_i \right] \prod_{i=1}^n p(l_i) \, dl_i,$$

where the following notations and conventions are used:

- the (multivalued) function $\bar{X}(y \mid x_1, ..., x_{n-1}; \{l_1, ..., l_n\})$ by definition assigns to every $y$ the union of the sets of solutions $X(y)$ of all the equations

$$y \in Y(R\sigma_n(x_1, ..., x_{n-1}, X(y); l_{\pi(1)}, ..., l_{\pi(n)})).$$  \hspace{1cm} (37)
with \( \pi \) running over all the permutation group \( S_n \) (the notation \( \{l_1, ..., l_n\} \) stresses the fact that the function \( \bar{X} \) does not depend on the order of \( l_i \)-s);

- the entries of the set \( \bar{X} \left( y \left| x_1, ..., x_{n-1}; \{l_1, ..., l_n\} \right. \right) \) have to be counted with multiplicities, which for a given \( x \in \bar{X} \left( y \left| x_1, ..., x_{n-1}; \{l_1, ..., l_n\} \right. \right) \) is by definition the number of equations \( 37 \) with different \( \pi \)-s, to which \( x \) is a solution;

- the integration in \( 36 \) of the multivalued function means that each sheet should be integrated and the results added. Moreover, each sheet has to be taken as many times as its multiplicity is.

Since each contribution \( \lambda \left( X \left( y \left| x_1, ..., x_{n-1}; l_1, ..., l_n \right. \right) \right) \) to \( 35 \) appears \( n! \) times in \( 36 \), we have to divide by \( n! \).

We repeat that while for some \( x \)-s, \( \pi \)-s and \( l \)-s the equation \( 37 \) might have no solutions, for other data it can have more than one solution. Clearly, the set \( \bar{X} \left( y \left| x_1, ..., x_{n-1}; \{l_1, ..., l_n\} \right. \right) \), for almost every data \( x_1, ..., x_{n-1} \), can have no other entries than those of the form

\[
x_{A,y,l_i} = y - \sum_{i \in A \subset \{1,2,..,n\}} l_i,
\]

where \( A \) runs over all nonempty subsets of \( \{1, 2, ..., n\} \) (i.e. at most \( 2^n - 1 \) different entries). So the function \( \bar{X} \left( y \left| x_1, ..., x_{n-1}; \{l_1, ..., l_n\} \right. \right) \), as a function of \( x_1, ..., x_{n-1} \), has to be piece-wise constant. It is not ruled out apriori that for some data the set \( \bar{X} \left( y \left| x_1, ..., x_{n-1}; \{l_1, ..., l_n\} \right. \right) \) can be empty. This is not, however, the case. Moreover, as the Theorem 7 below states,

- the number of elements in the set \( \bar{X} \left( y \left| x_1, ..., x_{n-1}; \{l_1, ..., l_n\} \right. \right) \), counted with multiplicities, is precisely \( n! \) for almost every value of the arguments.
Therefore we have for the inner integral in (36):

\[
\int \frac{1}{n!} \lambda \left( \bar{X}(y \mid x_1, ..., x_{n-1}; \{l_1, ..., l_n\}) \right) \prod_{i=1}^{n-1} \left( \frac{\lambda(x_i)}{I_\lambda(y)} dx_i \right)
\]

\[= \int \frac{1}{n!} \sum_{A \subset \{1,2,\ldots,n\}, A \neq \emptyset} k(A, y, x_1, ..., x_{n-1}; \{l_1, ..., l_n\}) \lambda \left(x_{A,y,\{l_i\}}\right) \prod_{i=1}^{n-1} \left( \frac{\lambda(x_i)}{I_\lambda(y)} dx_i \right),
\]

where the integer \(k(A, y, x_1, ..., x_{n-1}; \{l_1, ..., l_n\})\) is the multiplicity of the value \(x_{A,y,\{l_i\}}\) of the function \(\bar{X}\) at the point \((y, x_1, ..., x_{n-1}; \{l_1, ..., l_n\})\). Since

\[\sum_{A \subset \{1,2,\ldots,n\}, A \neq \emptyset} k(A, y, x_1, ..., x_{n-1}; \{l_1, ..., l_n\}) = n!
\]
a.e., we have

\[
\int \frac{1}{n!} \lambda \left( \bar{X}(y \mid x_1, ..., x_{n-1}; \{l_1, ..., l_n\}) \right) \prod_{i=1}^{n-1} \left( \frac{\lambda(x_i)}{I_\lambda(y)} dx_i \right)
\]

\[= \sum_{A \subset \{1,2,\ldots,n\}, A \neq \emptyset} q_{\lambda,y} \left(A \mid \{l_1, ..., l_n\}\right) \lambda \left(x_{A,y,\{l_i\}}\right),
\]

where

\[q_{\lambda,y} \left(A \mid \{l_1, ..., l_n\}\right) = \int \frac{1}{n!} k(A, y, x_1, ..., x_{n-1}; \{l_1, ..., l_n\}) \prod_{i=1}^{n-1} \left( \frac{\lambda(x_i)}{I_\lambda(y)} dx_i \right),\]

so

\[0 \leq q_{\lambda,y} \left(A \mid \{l_1, ..., l_n\}\right) \leq 1, \text{ with } \sum_{A \subset \{1,2,\ldots,n\}, A \neq \emptyset} q_{\lambda,y} \left(A \mid \{l_1, ..., l_n\}\right) = 1,
\]

since the measures \(\frac{\lambda(x_i)}{I_\lambda(y)} dx_i\) are probability measures on \([0, y]\). (Note that the functions \(k(A, y, x_1, ..., x_{n-1}; \{l_1, ..., l_n\})\) do depend on the variables \(x_1, ..., x_{n-1};\)
hence the measures $q_{\lambda,y} \left( \cdot \mid \{l_1, \ldots, l_n\} \right)$ indeed depend on $\lambda, y$.) Therefore, for the function $b_n (y)$ we obtain a sort of a convolution expression:

$$b_n (y) = \int \sum_{A \subset \{1,2, \ldots, n\}, \, A \neq \emptyset} \left[ q_{\lambda,y} \left( A \mid \{l_1, \ldots, l_n\} \right) \lambda \left( x_{A,y,\{l_i\}} \right) \right] \prod_{i=1}^{n} p (l_i) \, dl_i. \quad (40)$$

Be it the case that the probability measure $q_{\lambda,y} \left( \cdot \mid \{l_1, \ldots, l_n\} \right)$ is concentrated on just one subset $A = \{1,2, \ldots, n\}$, we would obtain the usual convolution

$$b_n (y) = \int \lambda \left( y - l_1 - \ldots - l_n \right) \prod_{i=1}^{n} p (l_i) \, dl_i = \lambda \ast p * \ast \cdots \ast p (y).$$

Here the situation is more subtle, and in (40) we have a stochastic mixture of convolutions with random number of summands.

Taking into account the relations (32), (35), (40), the result can be summarized as follows. Let $\nu \equiv \nu_{\lambda,y}$ be the integer valued random variable with the distribution

$$\Pr \{ \nu = n \} = \exp \left\{ -I_{\lambda} (y) \right\} \left[ I_{\lambda} (y) \right]^n / n!, \quad n = 0,1,2, \ldots,$$

and $\eta_1, \eta_2, \ldots$ be the i.i.d. random serving times. Consider the random function $\xi_{\lambda,y} = \xi_{\lambda,y} ( \nu_{\lambda,y}, \eta_1, \eta_2, \ldots )$, such that its conditional distribution under condition that the realization $\nu_{\lambda,y}, \eta_1, \eta_2, \ldots$ is given, is supported by the finite set

$$L (\nu_{\lambda,y}; \eta_1, \eta_2, \ldots) = \left\{ \sum_{i \in A} \eta_i : A \subset \{1,2, \ldots, \nu_{\lambda,y}+1\}, \, A \neq \emptyset \right\} \subset \mathbb{R}^1,$$

and is given by

$$\Pr \left\{ \xi_{\lambda,y} = \sum_{i \in A} \eta_i \mid \nu_{\lambda,y}, \eta_1, \eta_2, \ldots \right\} = q_{\lambda,y} \left( A \right) \left( \eta_1, \ldots, \eta_{\nu_{\lambda,y}+1} \right)$$

(see (38)). Then the following holds:

$$b (y) = \mathbb{E} \left( \lambda (y - \xi_{\lambda,y}) \right).$$

This is precisely the relation (20), with $q_{\lambda,y}$ being the distribution of $\xi_{\lambda,y}$. ■
6 Combinatorics of the rod placements

In this section we will prove the Theorem A which was used in the previous section. We will use the notation of the previous section, introduced at its beginning, up to relation (31).

By a cluster of the r-configuration \(\sigma_n (z_1, ..., z_n; l_1, ..., l_n)\) we call any maximal subsequence \(z_i < z_{i+1} < ... < z_j\) such that \(z_j = z_i + l_i + l_{i+1} + ... + l_{j-1}\). If \(z_i < z_{i+1} < ... < z_j\) is a cluster of an r-configuration, then the point \(z_i\) will be called the root of the cluster, while the point \(z_j\) will be called the head of the cluster. Note that for a general position configuration \(\sigma_n (x_1, ..., x_n; l_1, ..., l_n)\) the point \(z_i\) is a root of a cluster of the corresponding r-configuration if and only if \(z_i = x_i\). The segment \([z_i, z_j + l_j]\) will be called the body of the cluster \(z_i < z_{i+1} < ... < z_j\), and the point \(z_j + l_j\) will be called the end of the cluster.

The notation \(\sigma_n (x_1, ..., x_n; l_1, ..., l_n) \cup \sigma_1 (X, L)\) has the obvious meaning of adding an extra rod of the length \(L\) at the location \(X\). Note though, that in general

\[
R [\sigma_n (x_1, ..., x_n; l_1, ..., l_n) \cup \sigma_1 (X, L)] \neq R [R \sigma_n (x_1, ..., x_n; l_1, ..., l_n) \cup \sigma_1 (X, L)].
\]

It is however the case, if the point \(X\) is outside the union of all bodies of clusters of \(R \sigma_n (x_1, ..., x_n; l_1, ..., l_n)\). This will be used later.

In what follows we will need a marked point in \(\mathbb{R}^1\). For all our purposes it is convenient to chose the origin, \(0 \in \mathbb{R}^1\), as such a point.

We will say that the resolution of conflicts in the configuration \(\sigma_n (x_1, ..., x_n; l_1, ..., l_n)\) results in a hit of the origin, iff for some \(k\) we have

\[
y_k \equiv z_k + l_k = 0.
\] (41)

Such a hit will be called an \(x_r\)-hit, iff the cluster of the point \(z_k\) has its root at \(z_r = x_r\). (Necessarily, we have that \(r \leq k\).) An \(x_r\)-hit will be called an \((x_r, x_k)\)-hit, if (41) holds.

Now we are ready to formulate our problem. Let \(n\) be an integer, and \(\lambda_1 < \lambda_2 < ... < \lambda_n\) be a fixed set of positive lengths of rods. Let \(x_1 < x_2 < ... < x_{n-1}\) be a set of \((n - 1)\) left-ends. We want to compute the number \(N (x_1, x_2, ..., x_{n-1}; \lambda_1, \lambda_2, ..., \lambda_n)\), which is defined as follows. For any permutation \(\pi\) of \(n\) elements and for any \(X \in \mathbb{R}^1\), \(X \neq x_1, x_2, ..., x_{n-1}\) we can consider the configuration \(\sigma_{n-1} (x_1, ..., x_{n-1}; \lambda_{\pi(1)}, ..., \lambda_{\pi(n-1)}) \cup \sigma_1 (X, \lambda_{\pi(n)})\) of rods, when the rods \(l_i = \lambda_{\pi(i)}\) are placed at \(x_i, i = 1, ..., n - 1\), while the free rod \(l_n = \lambda_{\pi(n)}\) is placed at \(X\). Given \(\pi\), we count the number
$N_{\pi} (x_1, ..., x_{n-1}; \lambda_1, ..., \lambda_n)$ of different locations $X$, such that the corresponding r-configuration $R \left[ \sigma_{n-1} (x_1, ..., x_{n-1}; \lambda_{\pi(1)}, ..., \lambda_{\pi(n-1)}) \cup \sigma_1 (X, \lambda_{\pi(n)}) \right]$ has a hit, and moreover this hit is an $X$-hit. (In certain cases one cannot produce an $X$-hit by putting the rod $l_n = \lambda_{\pi(n)}$ anywhere on $\mathbb{R}^1$; then $N_{\pi} (x_1, ..., x_{n-1}; \lambda_1, ..., \lambda_n) = 0$. In certain other cases there are more than one possibility to place the free rod so as to produce an $X$-hit.) Then we define

$$N (x_1, ..., x_{n-1}; \lambda_1, ..., \lambda_n) = \sum_{\pi \in S_n} N_{\pi} (x_1, ..., x_{n-1}; \lambda_1, ..., \lambda_n).$$

**Theorem 7** For almost every $x_1, ..., x_{n-1}$ and $\lambda_1, ..., \lambda_n$,

$$N (x_1, ..., x_{n-1}; \lambda_1, ..., \lambda_n) = n!$$

**Proof.** Let us explain why the result is plausible. Let the set $x_1, ..., x_{n-1}$ be given. Then we can choose the positive numbers $\lambda_1, ..., \lambda_n$ so small that for any $\pi$ the configuration

$$\sigma_{n-1} (x_1, ..., x_{n-1}; \lambda_{\pi(1)}, ..., \lambda_{\pi(n-1)}) \cup \sigma_1 (X = -\lambda_{\pi(n)}, \lambda_{\pi(n)})$$

has no conflicts, while no other choice of $X$ results in a hit. Therefore in our case $N_{\pi} (x_1, ..., x_{n-1}; \lambda_1, ..., \lambda_n) = 1$ for every $\pi$, so indeed $N (x_1, ..., x_{n-1}; \lambda_1, ..., \lambda_n) = n!$.

Now we explain why our result is non-trivial. To see it, take $n = 2$, $x_1 = -3$, $\lambda_1 = 1$, $\lambda_2 = 10$. Then

$$N_{12} (x_1; \lambda_1, \lambda_2) = 2$$

– one can place the rod 10 at $-10$ or at $-11$. On the other hand,

$$N_{21} (x_1; \lambda_1, \lambda_2) = 0$$

– the rod 10, placed at $-3$, blocks the origin from being hit. Still, $2 + 0 = 2!$. Note that this example is a general position one.

We will derive our theorem from its special case, explained in the first paragraph of the present proof. The idea of computing $N (x_1, ..., x_{n-1}; \lambda_1, ..., \lambda_n)$ for a general data is to decrease one by one the numbers $\lambda_1 < \lambda_2 < ... < \lambda_n$, starting from the smallest one, to the values very small, keeping track on the quantities $N_{\pi} (x_1, ..., x_{n-1}; \lambda_1, ..., \lambda_n)$. During this evolution some of these will jump, but the total sum $N (x_1, ..., x_{n-1}; \lambda_1, ..., \lambda_n)$ would stay unchanged, as we will show. That will prove our theorem.
We begin by presenting a simple formula for the number $N_\pi(x_1, \ldots, x_{n-1}; \lambda_1, \ldots, \lambda_n)$.

Consider the rod configuration $R$ \[(\sigma_{n-1}(x_1, \ldots, x_{n-1}; \lambda_{\pi(1)}, \ldots, \lambda_{\pi(n-1)}) \]. which will be abbreviated as $R_\pi(\lambda_1, \ldots, \lambda_n) \equiv R_\pi(\lambda)$. Let us compute the quantity $S_\pi(x_1, \ldots, x_{n-1}; \lambda_1, \ldots, \lambda_n)$, which is the number of points $y_i \in Y(R_\pi(\lambda_1, \ldots, \lambda_n))$, falling into the segment $[-\lambda_{\pi(n)}, 0]$. Then

$$N_\pi(x_1, \ldots, x_{n-1}; \lambda_1, \ldots, \lambda_n) = \begin{cases} S_\pi(x_1, \ldots, x_{n-1}; \lambda_1, \ldots, \lambda_n) & \text{if the point } -\lambda_{\pi(n)} \\
S_\pi(x_1, \ldots, x_{n-1}; \lambda_1, \ldots, \lambda_n) + 1 & \text{ otherwise.} \end{cases}$$

(42)

Indeed, for every $y_i$, falling inside $[-\lambda_{\pi(n)}, 0]$, there is a position $X_i(z_1, \ldots, z_{n-1}, y_1, \ldots, y_{n-1}) < 0$, such that once the free rod $\lambda_{\pi(n)}$ is placed there, the site $y_i$ is pushed to the right and hits the origin. In case the point $-\lambda_{\pi(n)}$ is outside all clusters of $R_\pi(\lambda_1, \ldots, \lambda_n)$, placing the free rod $\lambda_{\pi(n)}$ at $X_0 = -\lambda_{\pi(n)}$ produces an extra hit.

Now let $\Delta > 0$ be such that

$$\lambda_1 < \lambda_2 < \ldots < \lambda_{i-1} < \lambda_i - \Delta < \lambda_i < \lambda_i + \Delta < \lambda_{i+1} < \ldots < \lambda_n,$$

for $i = 1, \ldots, n$, and some of the functions $N_\pi$ exhibit jumps in $\lambda_i$ as it goes down from $\lambda_i + \Delta$ to $\lambda_i - \Delta$. We denote by $\lambda(\delta)$ the vector $\lambda_1, \ldots, \lambda_i + \delta, \ldots, \lambda_n$. We suppose that $\Delta$ is small enough, so that for any $\pi$ the difference

$$|N_\pi(x_1, \ldots, x_{n-1}; \lambda(\Delta)) - N_\pi(x_1, \ldots, x_{n-1}; \lambda(-\Delta))|$$

is at most one. Moreover, we want $\Delta$ to be so small that on the segment $\lambda \in [\lambda_i - \Delta, \lambda_i + \Delta]$ there is precisely one point, say $\lambda_i$, at which some of the functions $N_\pi(x_1, \ldots, x_{n-1}; \lambda)$ do jump. (In general, there will be several permutations $\pi$, for which such a jump will happen at $\lambda = \lambda_i$. Indeed, if we observe an $(X, x_k)$-hit in our rod configuration with $l_i = \lambda_{\pi(i)}$, while we have that $x_1 < x_2 < \ldots x_{s-1} < X < x_s < \ldots < x_k < \ldots < x_{n-1}$, then in some cases we will have an $(X, x_k)$-hit for every rearrangement of the rods $l_s, \ldots, l_k$, i.e. for all permutations of the form $\pi \circ \rho$, where $\rho$ permutes the elements $s, \ldots, k$, leaving the other fixed.)

Let us begin with the case when

$$N_\pi(x_1, \ldots, x_{n-1}; \lambda(\Delta)) - N_\pi(x_1, \ldots, x_{n-1}; \lambda(-\Delta)) = 1.$$  

(43)
That means that $N_\pi (x_1, \ldots, x_{n-1}; \lambda (\Delta)) \geq 1$. So the intersection $Y (R_\pi (\lambda (\Delta))) \cap \left[ -\lambda (\Delta)_{\pi(n)} ; 0 \right] \neq \emptyset$. Let $y_k (\lambda (\Delta) , \pi) < \ldots < y_r (\lambda (\Delta) , \pi)$ are all the points of this intersection. The relation (43) implies via (42) that the point $y_k (\lambda (\delta) , \pi)$ leaves the segment $\left[ -\lambda (\Delta)_{\pi(n)} ; 0 \right]$ as $\delta$ passes the zero value:

$$y_k (\lambda (\delta) , \pi) > -\lambda (\delta)_{\pi(n)} \quad \text{for} \quad \delta > 0,$$

$$y_k (\lambda (0) , \pi) = -\lambda (0)_{\pi(n)},$$

$$y_k (\lambda (\delta) , \pi) < -\lambda (\delta)_{\pi(n)} \quad \text{for} \quad \delta < 0.$$  

Moreover, the point $y_k (\lambda (\delta) , \pi)$ is not the end of the cluster. Therefore $y_k (\lambda (\delta) , \pi) = z_{k+1} (\lambda (\delta) , \pi)$. We now claim that if we assign the rod $\lambda (\delta)_{\pi(n)}$ to $x_{k+1}$, and will take for the free rod the rod $\lambda (\delta)_{\pi(k+1)}$, then for the corresponding permutation the opposite to (43) happens:

$$N_{\pi(n \leftrightarrow k+1)} (x_1, \ldots, x_{n-1}; \lambda (\Delta)) - N_{\pi(n \leftrightarrow k+1)} (x_1, \ldots, x_{n-1}; \lambda (-\Delta)) = -1. \quad (47)$$

Here we denote by $\pi (n \leftrightarrow k + 1)$ the permutation which is the composition of the transposition $n \leftrightarrow k + 1$, followed by $\pi$. Indeed, after the above reassignment and the resolution of conflicts, the rod $\lambda (\delta)_{\pi(n)}$ will be positioned at the point $y_k (\lambda (\delta) , \pi)$. The relations (44) – (46) then tell us, that during the $\delta$-evolution the right endpoint of this rod will move from the positive semiaxis to the negative one, thus adding one unit to the value $S_{\pi(n \leftrightarrow k+1)} (x_1, \ldots, x_{n-1}; \lambda (\Delta))$.

The above construction corresponds to every permutation $\pi$, satisfying (43), another permutation, $\pi' = \Phi (\pi)$, which satisfy (47). We will be done if we show that $\Phi$ is one to one. We prove this by constructing the inverse of $\Phi$.

So let $\pi'$ be such that

$$N_{\pi'} (x_1, \ldots, x_{n-1}; \lambda (\Delta)) - N_{\pi'} (x_1, \ldots, x_{n-1}; \lambda (-\Delta)) = -1.$$

According to the above that means that the intersection $Y (R_{\pi'} (\lambda (\Delta))) \cap \left( 0, +\infty \right) \neq \emptyset$. Let $y_{k'} (\lambda (\Delta) , \pi') < \ldots < y_{r'} (\lambda (\Delta) , \pi')$ are all the points of this intersection. The relation (43) implies via (42) that the point $y_{k'} (\lambda (\delta) , \pi')$ moves from the positive semiaxis to the negative one as $\delta$ passes the zero value:

$$y_{k'} (\lambda (\delta) , \pi') > 0 \quad \text{for} \quad \delta > 0,$$

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\[ y_{k'}(\lambda(0), \pi') = 0, \]
\[ y_{k'}(\lambda(\delta), \pi') < 0 \text{ for } \delta < 0. \]

But that precisely means that the point \( y_{k'-1}(\lambda(\delta), \pi') \) is inside the segment \([-\lambda(\delta)\pi'(k'), 0]\) for \( \delta = \Delta \), and outside it for \( \delta = -\Delta \). So if we assign the free rod \( \lambda(\delta)\pi'(n) \) to the point \( x_{k'} \), making the rod \( \lambda(\delta)\pi'(k') \) free, then we construct the permutation \( \pi'' = \Phi'(\pi') \), for which \( (43) \) holds.

The statement that \( \Phi' \) is inverse to \( \Phi \) is straightforward. ■

Below we will need a version of the above theorem, which follows. Let \( T, L \)
be positive reals, \( L < T \). Let again \( n \) be an integer, and \( \lambda_1 < \lambda_2 < \ldots < \lambda_n \)
be a fixed set of positive lengths of rods. Let \( -T < x_1 < x_2 < \ldots < x_{n-1} < 0 \) be a set of \((n-1)\) left-ends. We want to compute the number
\( N(-T, x_1, x_2, \ldots, x_{n-1}; L, \lambda_1, \lambda_2, \ldots, \lambda_n) \), which is defined as follows. For any permutation \( \pi \) of \( n \) elements and for any \( X \in (-T, 0) \), \( X \neq x_1, x_2, \ldots, x_{n-1} \) we can consider the configuration \( \sigma_n(-T, x_1, \ldots, x_{n-1}; L, \lambda_{\pi(1)}, \ldots, \lambda_{\pi(n-1)}) \cup \sigma_1(X, \lambda_{\pi(n)}) \)
of rods, when the rod \( L \) is placed at \(-T\), the rods \( l_i = \lambda_{\pi(i)} \) are placed at \( x_i, i = 1, \ldots, n - 1 \), while the free rod \( l_n = \lambda_{\pi(n)} \) is placed at \( X, -T < X < 0 \).

Given \( \pi \), we count the number \( \tilde{N}_\pi(-T, x_1, x_2, \ldots, x_{n-1}; L, \lambda_1, \lambda_2, \ldots, \lambda_n) \) of different locations \( X \), such that the corresponding r-configuration
\( R[\sigma_n(-T, x_1, \ldots, x_{n-1}; L, \lambda_{\pi(1)}, \ldots, \lambda_{\pi(n-1)}) \cup \sigma_1(X, \lambda_{\pi(n)})] \) has a hit, and moreover this hit is an \( X \)-hit. Then we define
\[ \tilde{N}(-T, x_1, x_2, \ldots, x_{n-1}; L, \lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{\pi \in S_n} \tilde{N}_\pi(-T, x_1, x_2, \ldots, x_{n-1}; L, \lambda_1, \lambda_2, \ldots, \lambda_n). \]

**Theorem 8** Suppose that
\[ L + \lambda_1 + \lambda_2 + \ldots + \lambda_n < T. \] (48)

Then \( \tilde{N}(-T, x_1, x_2, \ldots, x_{n-1}; L, \lambda_1, \lambda_2, \ldots, \lambda_n) = n! \) for almost every \( x_1, \ldots, x_{n-1} \) and \( \lambda_1, \ldots, \lambda_n \).

The theorem \( 8 \) differs from the theorem \( 7 \) by the presence of the additional rod \( L \), which is placed at \(-T\), and by the restriction that all points \( X, x_1, x_2, \ldots, x_{n-1} \) has to be within the segment \((-T, 0)\). Therefore the rod \( L \)
will not move under the resolution of conflicts. Without the restriction \( (48) \)
the statement of the theorem is not valid, as it is easy to see.

**Proof.** Let the numbers \( 0 < \epsilon_1 < \ldots < \epsilon_{n-1} \) be so small that the sum \( \epsilon_1 + \ldots + \epsilon_{n-1} \) is less than any of the numbers \( |\delta_0(T-L) + \delta_1\lambda_1 + \delta_2\lambda_2 + \ldots + \delta_n\lambda_n| \),
where $\delta_i$ are taking any of three values $-1, 0, 1$, with the only restriction that not all of them vanish simultaneously. Let us replace the configuration $x_1, x_2, ..., x_{n-1}$ by the configuration $x'_1, x'_2, ..., x'_{n-1}$, where

$$x'_i = \begin{cases} L - T + \varepsilon_i & \text{if } x_i < L - T, \\ x_i & \text{otherwise.} \end{cases}$$

Let $k$ be the largest integer for which $x'_i > x_i$. (The meaning of the configuration $x'_1, x'_2, ..., x'_{n-1}$ is the following: were all $\varepsilon_i$ zeroes, it is the result of resolving the first conflict, between the first rod $L$ and the rods intersecting it, which have to be pushed to the right-hand end of $L$. We use positive $\varepsilon$-s in order to have all the point $x'_i$ different.) By the previous theorem we know that $N (x'_1, x'_2, ..., x'_{n-1}; \lambda_1, ..., \lambda_n) = n!$. Let the location $X$ be such that for some permutation $\pi$ the corresponding r-configuration

$$R [\sigma_{n-1} (x'_1, ..., x'_{n-1}; \lambda_{\pi(1)}, ..., \lambda_{\pi(n-1)}) \cup \sigma_1 (X, \lambda_{\pi(n)})]$$

has an X-hit. The condition [15] implies that the cluster of the r-configuration

$$R [\sigma_{n-1} (x'_1, ..., x'_{n-1}; \lambda_{\pi(1)}, ..., \lambda_{\pi(n-1)}) \cup \sigma_1 (X, \lambda_{\pi(n)})]$$

rooted at $X$, does not contain any of the points $z'_1 = z'_1, z'_2, ..., z'_k$ (see (29) for the notation), so $X > L - T$, and the r-configuration

$$R [\sigma_n (\lambda, \pi, L, \lambda_{\pi(1)}, ..., \lambda_{\pi(n-1)}) \cup \sigma_1 (X, \lambda_{\pi(n)})]$$

has an X-hit as well. Therefore

$$\tilde{N} (\lambda, x_1, x_2, ..., x_{n-1}; L, \lambda_1, \lambda_2, ..., \lambda_n) \geq n!.$$.

On the other hand, if the r-configuration

$$R [\sigma_{n} (\lambda, x_1, ..., x_{n-1}; L, \lambda_{\pi(1)}, ..., \lambda_{\pi(n-1)}) \cup \sigma_1 (X, \lambda_{\pi(n)})]$$

has an X-hit, then by the same reasoning $X$ has to be to the right of the location $L - T$, and moreover the cluster of this configuration, rooted at $X$, does not contain any of the points $z_1 = -T, z_2 = -T + L, ..., z_{k+1}$; therefore the r-configuration

$$R [\sigma_{n-1} (x'_1, ..., x'_{n-1}; \lambda_{\pi(1)}, ..., \lambda_{\pi(n-1)}) \cup \sigma_1 (X, \lambda_{\pi(n)})]$$

has an X-hit. Hence

$$\tilde{N} (\lambda, x_1, x_2, ..., x_{n-1}; L, \lambda_1, \lambda_2, ..., \lambda_n) \leq n!,$$

and the proof follows.

\section{Estimates on dissipators}

For the future use we have to estimate the densities $q_{\lambda, x} (t)$, entering into the relation $b (x) = [\lambda * q_{\lambda, x}] (x)$.

\textbf{Lemma 9}

$$q_{\lambda, y} (t) \geq p (t) \Pr \{ \text{server is idle at the moment } y - t \}. \quad (49)$$
Proof. We will obtain this estimate by invoking the initial relation (52) for $b$:

$$b(y) = \exp \{-I_\lambda(y)\} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int_0^\infty \cdots \int_0^\infty$$

$$\left[ \frac{y}{\lambda} \left( x_1, \ldots, x_{n-1}; l_1, \ldots, l_n \right) \prod_{i=1}^{n-1} \lambda(x_i) \ dx_i \right] \prod_{i=1}^{n} p(l_i) \ dl_i$$

Namely, the contribution to the value $q_{\lambda,y}(t)$ comes from all realizations $(x_1, \ldots, x_{n-1}; l_1, \ldots, l_n)$, for which $y-t \in X(y \mid x_1, \ldots, x_{n-1}; l_1, \ldots, l_n)$. Among such realizations let us pick the following class: the rod $l_n = t$, while the rods of the configuration $R\sigma_n(x_1, \ldots, x_{n-1}; l_1, \ldots, l_{n-1})$ does not cover the point $y-t$. Let us denote the indicator of the complement to the union of rods forming the set $R\sigma_n(x_1, \ldots, x_{n-1}; l_1, \ldots, l_{n-1})$ by $I_{x,1}$. Then we have

$$q_{\lambda,y}(t) \geq p(t) \exp \{-I_\lambda(y)\} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \times \int_0^\infty \cdots \int_0^\infty$$

$$\left[ \frac{y}{\lambda} \left( x_1, \ldots, x_{n-1}; l_1, \ldots, l_n \right) \prod_{i=1}^{n-1} \lambda(x_i) \ dx_i \right] \prod_{i=1}^{n} p(l_i) \ dl_i$$

$$= p(t) \Pr\{\text{server is idle at the moment } y-t\}.$$ 

Next we establish the upper bound on $q_{\lambda,y}$.

Lemma 10

$$q_{\lambda,y}(t) \leq \sum_{n=1}^{\infty} p^n(t) \Pr\{N_{i}^{\lambda,y} \geq n-1\}, \quad (50)$$

where $N_{i}^{\lambda,y}$ is the random number of $\lambda$-Poisson points in the segment $[y-t, y]$.

In particular, for $t \leq C$ there exists a constant $\tilde{C} = \tilde{C}(C, p)$, such that for all $\lambda, y$

$$q_{\lambda,y}(t) \leq \tilde{C}. \quad (51)$$
Proof. As above, we have

\[ q_{\lambda,y}(t) = \exp \left\{ -I_{\lambda}(y) \right\} \sum_{n=1}^{\infty} \prod_{i=1}^{n-1} \int_{0}^{\infty} \int_{x_1}^{y} \cdots \int_{x_{n-2}}^{y} I_{x,1}(y-t) \left( \sum_{j} p(\beta_j) \right) \lambda(x_i) \, dx_i \prod_{i=1}^{n-1} p(l_i) \, dl_i, \]

where the values \( \beta_j \) are all solutions \( \beta \) of the equation

\[ X \left( y \mid x_1, \ldots, x_{n-1}; l_1, \ldots, l_{n-1}, \beta \right) = y - t, \quad (52) \]

and where we integrate only over the set \( 0 < x_1 < \ldots < x_{n-1} < y \), so we do not have the factorials any more. Note that the equation (52) has solutions only if the configuration \( (x_1, \ldots, x_{n-1}; l_1, \ldots, l_{n-1}) \) satisfies the condition \( I_{x,1}(y-t) = 1 \); in that case the solutions do not depend on all these \( x_i, l_i \), for which \( x_i < y - t \). Let us define the values \( \beta_j \) outside the event \( I_{x,1}(y-t) = 1 \) as the solutions of

\[ X \left( y \mid x_1, \ldots, x_{n-1}; \tilde{l}_1, \ldots, \tilde{l}_{n-1}, \beta \right) = y - t, \]

where

\[ \tilde{l}_i = \begin{cases} 0 & \text{if } x_i < y - t, \\ l_i & \text{otherwise}. \end{cases} \]
Replacing the indicator by 1, we have

\[ q_{\lambda,y}(t) \leq \exp\{-I_\lambda(y)\} \sum_{n=1}^{\infty} \left[ \int_0^\infty \cdots \int_0^\infty \left( \sum_{j=1}^{n-1} \prod_{i=1}^{j} \lambda(x_i) \, dx_i \right) \prod_{i=1}^{n-1} p(l_i) \, dl_i \right] \]

\[ = \exp\{-I_\lambda(y) + I_\lambda(y-t)\} \sum_{n=1}^{\infty} \left[ \int_0^\infty \cdots \int_0^\infty \left( \sum_{j=1}^{n-1} \prod_{i=1}^{j} \lambda(x_i) \, dx_i \right) \prod_{i=1}^{n-1} p(l_i) \, dl_i \right], \tag{53} \]

where the x-integration is now taken over \( y-t < x_1 < \ldots < x_{n-1} < y \). Let us enumerate \( \beta_j \) in decreasing order. Then clearly \( \beta_1 = t \), and the corresponding term in (53) equals \( p(t) \). The next solution, \( \beta_2 \), of the equation (52), exists only if \( l_1 < t \) and \( x_1 < y - l_1 \). Then \( \beta_2 = t - l_1 \). The second term is therefore

\[ \exp\{-I_\lambda(y) + I_\lambda(y-t)\} \sum_{n=2}^{\infty} \int_0^\infty \cdots \int_0^\infty \left[ \int_{x_2}^{y} \cdots \int_{x_{n-2}}^{y} \left[ \int_0^{y-l_1} \lambda(x_1) \, dx_1 \right] p(t-l_1) p(l_1) \, dl_1 \prod_{i=2}^{n-1} \lambda(x_i) \, dx_i \right] \prod_{i=2}^{n-1} p(l_i) \, dl_i \]

\[ \leq \exp\{-I_\lambda(y) + I_\lambda(y-t)\} \sum_{n=2}^{\infty} \int_0^\infty \cdots \int_0^\infty \left[ \int_{x_2}^{y} \cdots \int_{x_{n-2}}^{y} \left[ \int_0^{y-t} \lambda(x_1) \, dx_1 \right] p(y-t-l_1) p(l_1) \, dl_1 \prod_{i=2}^{n-1} \lambda(x_i) \, dx_i \right] \prod_{i=2}^{n-1} p(l_i) \, dl_i \]

\[ = p^2(t) \Pr\{\text{the } \lambda\text{-Poisson field has at least 1 point in the segment } [y-t, y]\}. \]
So by induction we arrive to the bound (50):

\[ q_{\lambda,y}(t) \leq \sum_{n=1}^{\infty} p^* n(t) \Pr\{ N_{\lambda,y}^{(t)} \geq n - 1 \} \equiv Q_{\lambda,y}(t), \]

where \( N_{\lambda,y}^{(t)} \) is the random number of \( \lambda \)-Poisson points in the segment \([y - t, y]\).

To see (51), we use a rough form of (50):

\[ q_{\lambda,y}(t) \leq \sum_{n=1}^{\infty} p^* n(t). \]

(54)

Let \( A = \sup_t p(t) \). Then it is immediate from (54) that for all \( t \leq C \)

\[ q_{\lambda,y}(t) \leq A \left( 1 + \sum_{n=1}^{\infty} \Pr\{ \eta_1 + \ldots + \eta_n \leq C \} \right), \]

where \( \eta_i \) are i.i.d. random variables, distributed as \( \eta \). But the probabilities \( \Pr\{ \eta_1 + \ldots + \eta_n \leq C \} \) decay exponentially in \( n \). ■

We will need the compactness estimate on the distributions \( q_{\lambda,y}(t) \). We will obtain them using the estimate (50). As the following statement show, the estimate (50) is rather rough; we believe that all the moments of the distribution \( q_{\lambda,y}(t) \) of order less than \( 1 + \delta \) are finite.

**Lemma 11** Suppose that \( \lambda \) is such that for some \( T' \) and \( \varepsilon' > 0 \) and for all \( T \geq T' \)

\[ \int_0^T \lambda_v(t) \, dt < T (1 - \varepsilon') \]

(see (25)). Then for any \( b < \frac{\delta}{2} \)

\[ \int_0^{\infty} t^b q_{\lambda,y}(t) \, dt < C(\lambda, b) < \infty, \]

(55)

where \( C(\lambda, b) \) depends on \( \lambda \) only via \( T' \) and \( \varepsilon' \).

**Proof of Lemma 11** We are going to use the simple estimate: for every random variable \( \zeta \) and every \( \nu > 0 \)

\[ \bar{Q}(T) \equiv \Pr\{ \zeta > T \} \leq T^{-\nu} \mathbb{E}\{|\zeta|^\nu\}. \]

(56)
We also will need an estimate on \( \int_A \int_0^\infty t^a \tilde{q}(t) \, dt \), \( a < \kappa \), where \( \tilde{q} \) is the density of \( \zeta \). We have:

\[
\int_A \int_0^\infty t^a \tilde{q}(t) \, dt = - \int_A t^a \, d\left( \tilde{Q}(t) \right)
= A^a \tilde{Q}(A) + a \int_A t^{a-1} \tilde{Q}(t) \, dt.
\]

To apply (56) to (50) we will use the Dharmadhikari-Yogdeo estimate (see, e.g. [P], p.79): if \( \xi_i \) are independent centered random variables, then

\[
\mathbb{E}\left( |\xi_1 + ... + \xi_n|^{2+\delta} \right) \leq R n^{\delta/2} \sum_{i=1}^n \mathbb{E}\left( |\xi_i|^{2+\delta} \right).
\]

Here \( R = R(\delta) \) is some universal constant. Introducing \( \xi_i = \eta_i - 1 \) (see (11)), we have (see (9))

\[
Q_n(t) \equiv \Pr\{\eta_1 + ... + \eta_n > t\} = \Pr\{\xi_1 + ... + \xi_n > t - n\} \leq RM_{\delta}(t - n)^{-(2+\delta)} n^{1+\delta/2}.
\]

To proceed, we use (50) to write

\[
\int_0^\infty t^b q_{\lambda,y}(t) \, dt \leq \int_0^\infty t^b Q_{\lambda,y}(t) \, dt
= \sum_{n=1}^\infty \left[ \int_0^\infty t^b p^{*n}(t) \Pr\{N_{t}^{\lambda,y} \geq n - 1\} \, dt \right].
\]

Note that \( \mathbb{E}\left( N_{t}^{\lambda,y} \right) \leq (1 - \alpha) t \) once \( t \) is large enough. The first step is to write

\[
\int_0^\infty t^b p^{*n}(t) \Pr\{N_{t}^{\lambda,y} \geq n - 1\} \, dt
\]

\[
\leq \int_0^{n(1+\frac{\alpha}{2})} t^b p^{*n}(t) \Pr\{N_{t}^{\lambda,y} \geq n - 1\} \, dt + \int_{n(1+\frac{\alpha}{2})}^\infty t^b p^{*n}(t) \, dt.
\]
Now, using (56) and (57), we have

\[ \int_{n(1+\frac{\alpha}{2})}^{\infty} t^b p^* (t) \, dt \leq \left[ n \left( 1 + \frac{\alpha}{2} \right) \right]^b R M_{\delta} \left( \frac{\alpha}{2} n \right)^{-(2+\delta)} n^{1+\delta/2} \]

\[ + b R M_{\delta} n^{1+\delta/2} \int_{n(1+\frac{\alpha}{2})}^{\infty} t^{b-1} (t-n)^{-(2+\delta)} \, dt \]

\[ \leq C n^{b-1-\delta/2}, \]

where \( C = C (\alpha, \delta, M_{\delta}) \).

The first term in (59) is negligible. To see that, we first observe:

**Lemma 12** Let \( 0 < \nu < 1 \), and \( N^{\nu} \) be a Poisson random variable:

\[ \Pr \{ N^{\nu} = k \} = e^{-\nu n} (\nu n)^k \]

Then

\[ \Pr \{ N^{\nu} \geq n \} \leq \frac{1}{1-\nu} e^{-(1-\nu)^2 n}, \]

provided \( n \) is large enough.

**Proof.** Note first of all, that if \( \chi > 0 \) and \( n > \chi \), then

\[ e^{-\chi} \sum_{k \geq n} \frac{\chi^k}{k!} \leq e^{-\chi} \sum_{k \geq 0} \left( \frac{\chi}{n+1} \right)^k \]

\[ = e^{-\chi} \frac{\chi^n}{n!} \frac{1}{1 - \frac{\chi}{n+1}}. \]

In our case we thus have

\[ \sum_{k \geq n} \Pr \{ N^{\nu} = k \} \leq e^{-\nu n} (\nu n)^n \frac{1}{n!} \frac{1}{1-\nu}. \]

By Stirling, for \( n \) large

\[ \sum_{k \geq n} \Pr \{ N^{\nu} = k \} \leq \frac{1}{1-\nu} e^{-\nu n} \frac{\nu^n n^n}{n! e^{-n}} \]

\[ = \frac{1}{1-\nu} e^{(1-\nu + \ln \nu) n} \]

\[ \leq \frac{1}{1-\nu} e^{-(1-\nu)^2 n}. \]

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Therefore,

\[
\int_0^{n(1+\frac{\delta}{2})} t^b p^n(t) \Pr\{N_{\lambda,y}^n \geq n-1\} \, dt \\
\leq \frac{1}{\alpha} e^{-\frac{\alpha^2}{2} n} \int_0^{n(1+\frac{\delta}{2})} t^b p^n(t) \, dt \\
\leq \frac{1}{\alpha} e^{-\frac{\alpha^2}{2} n} \left[ n \left( 1 + \frac{\alpha}{2} \right) \right]^b.
\]

Hence, the moment \( \int_0^\infty t^b q_{\lambda,y}(t) \, dt \) is finite as soon as the series \( \sum_n n^{b-1-\delta/2} \) converges, which happens when \( b < \frac{\delta}{2} \). That proves Lemma 11. ■

8 The self-averaging relation: general case

Here we derive a formula, expressing the function \( b(\cdot) = A(\mu, \lambda(\cdot)) \) in terms of the functions \( \lambda(\cdot), p(\cdot) \) and the measure \( \mu \). This will be the needed self-averaging relation (21). We remind the reader that \( \mu \) is a probability measure on the set of pairs \( \{(n, \tau)\} \cup \emptyset \).

**Theorem 13** Let \( N(\mu) = q \), and the rate function \( \lambda(\cdot) \) satisfies the conclusions of the Lemma 4 and the relation (25):

\[
\int_0^T \lambda_\nu(t) \, dt < T (1 - \varepsilon') \quad \text{for all } T \geq T' > 0.
\]

Then there exists the family of probability densities \( q_{\lambda,\mu,x}(\cdot), x > 0 \), and the functionals \( \varepsilon_{\lambda,\mu}(x) \) and \( Q_{\lambda,\mu}(x) \), such that

\[
b(x) = (1 - \varepsilon_{\lambda,\mu}(x)) \left[ \lambda * q_{\lambda,\mu,x}(x) \right] + \varepsilon_{\lambda,\mu}(x) Q_{\lambda,\mu}(x) \quad \text{. (60)}
\]

Moreover,

\[
\varepsilon_{\lambda,\mu}(x) \to 0 \quad \text{as } x \to \infty, \quad \text{(61)}
\]

while \( Q_{\lambda,\mu}(x) \leq C \), uniformly in \( \lambda, \mu \) and \( x \), once \( q, T', \varepsilon' \) are fixed.

**Proof.** We start by defining the functional \( \varepsilon_{\lambda,\mu}(x) \). Note that the description of the realization of our process up to the moment \( x \) consists of the following data:
i) the initial configuration \((n, \tau)\), drawn from the distribution \(\mu\);

ii) the random set \(0 < x_1 < \ldots < x_m < x\) (with random number \(m\) of points), which is a realization of the Poisson random field defined by the rate function \(\lambda\) (restricted to the segment \([0, x]\)), independent of \((n, \tau)\);

iii) one realization \(\eta_1\) of the conditional random variable \(\left(\eta - \tau \mid \eta > \tau\right)\) and \(n + m - 1\) independent realizations \(\eta_k, k = 2, \ldots, n + m\) of the random variable \(\eta\). We denote by \(P_{\mu \otimes \lambda \otimes \eta}\) the corresponding (product) distribution. The difference \(1 - \varepsilon_{\lambda, \mu}(x)\) is then just the \(P_{\mu \otimes \lambda \otimes \eta}\)-probability of the event

\[
\sum_{k=1}^{n+m} \eta_k < x.
\]

(If \(n = 0\), then by definition we put \(\tau = 0\); we put also \(\sum_{i}^{0} \equiv 0\).)

The meaning of the decomposition (60) can be explained now: the first term corresponds to the exit flow computed over those realizations where the relation (62) holds, while the second term represents the rest of the flow.

Let us show (61), that is that

\[
\Pr\left\{ \sum_{k=1}^{n+m} \eta_k > x \right\} \to 0 \text{ as } x \to \infty.
\]

To do this, we introduce two independent random variables:

\[ S_{\mu} = \sum_{k=1}^{n} \eta_k, \quad S_{\lambda} = \sum_{k=n+1}^{n+m} \eta_k. \]

Then for every \(\alpha \in (0, 1)\) we have

\[
\Pr\left\{ \sum_{k=1}^{n+m} \eta_k > x \right\} = \Pr\{S_{\mu} + S_{\lambda} > x\} \leq \Pr\{S_{\mu} > \alpha x\} + \Pr\{S_{\lambda} > (1 - \alpha) x\}.
\]

Indeed, if \(S_{\mu} + S_{\lambda} > x\), then either \(S_{\mu} > \alpha x\), or else \(S_{\lambda} > (1 - \alpha) x\). Since \(E(S_{\mu}) \leq C + q\), the probability \(\Pr\{S_{\mu} > \alpha x\}\) goes to zero for every \(\alpha\) positive, as \(x \to \infty\), uniformly in \(\mu\). For the second term we have

\[
\Pr\{S_{\lambda} > (1 - \alpha) x\} = \sum_{m=1}^{\infty} \left(\int_{(1-\alpha)x}^{\infty} p^{*m}(t) \, dt\right) \Pr\{N^{\lambda,x} = m\}.
\]

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Here $N^{\lambda,x}$ is the random number of points in the realization of the Poisson field in $[0, x]$. Note that $\mathbb{E}(N^{\lambda,x}) < x(1 - \varepsilon')$ once $x > T'$. Therefore we can apply the same argument which was used in the proof of Lemma $11$ when showing that the integral $\int_T^\infty Q_{\lambda,y}(t) \to 0$ as $T \to \infty$, see (50). It implies that $\Pr\{S_{\lambda} > (1 - \alpha)x\} \to 0$ once $\alpha$ is small enough, uniformly in $\lambda$.

Next we define the distributions $q_{\lambda,\mu,x}$. They are constructed from the random field of the rods $\{\eta_k, k = 1, \ldots, n + m\}$, defined above, placed at locations $\{0, \ldots, 0, x_1, \ldots, x_m\}$, via the procedure of resolution of conflicts, defined in the previous section. To do it we first introduce the rate $b_L(x)$ to be the exit rate of the conditional service process under the conditions that

$$\sum_{k=1}^{n} \eta_k = L, \quad \sum_{n+1}^{n+m} \eta_k < x - L. \quad (63)$$

We claim that for some probability distributions $q_{\lambda,L,x}$ we have

$$b_L(x) = [\lambda * q_{\lambda,L,x}](x).$$

The distribution $q_{\lambda,\mu,x}$ is then obtained by integration:

$$q_{\lambda,\mu,x} = \int q_{\lambda,L,x} \mu \otimes \lambda \otimes \eta \left( \sum_{k=1}^{n} \eta_k \in dL \right).$$

(The random variable $\sum_{k=1}^{n} \eta_k$ is of course independent of the Poisson $\lambda$-field.) The output rate $b_L(x)$ corresponds to the situation when we have customers arriving at the moments $0, x_1, \ldots, x_m$, which have serving times $L, \eta_{n+1}, \ldots, \eta_{n+m}$, and which satisfy the relation

$$L + \sum_{n+1}^{n+m} \eta_k < x.$$

So we have to repeat the construction of the Section 5 in the present situation. Few steps require some comments. The transition from the relation (55) to (56) uses the fact that for any $s$ the measure $\prod_{i=1}^{s} p(l_i) \, dl_i$ is invariant under the coordinate permutations $S_s$ in $\mathbb{R}^s$. But the same $S_m$ symmetry evidently holds for the conditional distribution of the random vector $\{(\eta_k, k = n + 1, \ldots, n + m) \mid \sum_{n+1}^{n+m} \eta_k < x - L\}$, since both the unconditional distribution and the distribution of the condition are $S_m$-invariant.
The next crucial step was the relation (39), stating that the functions \( q_{\lambda,y} \) are probability distributions. It was based on the theorem 7. The situation at hand is somewhat more delicate, since the rods we are dealing now with, are of two kinds: the first one has a non-random length \( L \), produced by the initial state \( \mu \), while others are situated at the Poissonian locations \( \{ x_i \} \), defined by the rate function \( \lambda \). However, under condition \( \sum_{n+1}^{n+m} \eta_k < x - L \) the needed combinatorial statement (about \( m! \)) still holds, and is the content of the theorem 8. These remarks allow one to carry over the construction of the previous section, and so to establish the existence of the probability densities \( q_{\lambda,L,x} \), and thus also \( q_{\lambda,\mu,x} \). The upper and lower estimates on \( q_{\lambda,\mu,x} \) are obtained in the same way as were the estimates for \( q_{\lambda,x} \) in the preceding section.

The function \( Q_{\lambda,\mu}(x) \) is the rate of exit flow of our process, conditioned by the event

\[
\sum_{1}^{n+m} \eta_k \geq x.
\]

The boundedness of the \( Q_{\lambda,\mu}(x) \) follows from the following property of the service time distribution \( p(x) \): for every \( x, \tau, x > \tau > 0, 1 > t > 0 \)

\[
\frac{p(x)}{p(x + t)} \leq C', \quad \frac{p(x - \tau \mid \eta > \tau)}{p(x - \tau + t \mid \eta > \tau)} \leq C'.
\]

The relation (64) follows easily from the condition (8). To explain the boundedness, consider the elementary event

\[
(n, \tau) \times \{ x_1, ..., x_m : 0 < x_1 < ... < x_m < x \} \times \{ \eta_1, ..., \eta_{n+m} \},
\]

which contributes to the output flow inside the segment \( [x, x + \Delta x] \), which flow is accounted by the second term of (60). That means that our rod configuration produces after resolution of conflicts a hit inside \( [x, x + \Delta x] \), and also that

\[
\sum_{1}^{n+m} \eta_k > x.
\]

In the notation of the Section 6 it means that after resolution of conflicts the endpoint \( y_k \) of some (shifted) rod fits within \( [x, x + \Delta x] \), for some \( k \in \).
\{1, \ldots, n + m\}. Let \( k \) be the smallest such index. But then the elementary events
\[(n, \tau) \times \{x_1, \ldots, x_m : 0 < x_1 < \ldots < x_m < x\} \times \{\eta_1, \ldots, \eta_{k-1}, \eta_k + t, \eta_{k+1}, \ldots, \eta_{n+m}\},\]
with any \( t \in (\Delta x, 1) \), do not contribute to the output flow inside the segment \([x, x + \Delta x]\), while still satisfying (65). Therefore, due to (64), the probability that the customer would finish his service during the period \([x, x + \Delta x]\), is of the order of \( \Delta x \), and, moreover,
\[
Q_{\lambda, \mu} (x) \leq \frac{1}{C'}. 
\]

Let now \( M \in \mathcal{M}_q (\Omega) \) be some invariant measure of the dynamical system (15). Then \( \mathcal{M} \)-almost every state \( \tilde{\mu}_0 \in \mathcal{M}_q (\Omega) \) belongs to the family \( \{\tilde{\mu}_t : -\infty < t < +\infty\} \), such that for all \( \tau > 0 \), all \( t \)
\[
T_{\tau} (\tilde{\mu}_t) = \tilde{\mu}_{t+\tau}.
\]
Let us fix one such family \( \{\tilde{\mu}_t\} \). Then the function \( \lambda (t), -\infty < t < +\infty \), which for every \( -\infty < \tau < +\infty \) satisfies on \([\tau, +\infty)\) the equation
\[
\lambda (\cdot) = A (\tilde{\mu}_\tau, \lambda (\cdot), \tau),
\]
is well defined. Then, according to the equation (60), for every \( \tau, -\infty < \tau < +\infty \), and for all \( x \geq \tau \)
\[
\lambda (x) = (1 - \varepsilon_{\lambda, \tilde{\mu}_\tau} (x)) \left[\lambda * q_{\lambda, \tilde{\mu}_\tau, x}\right] (x) + \varepsilon_{\lambda, \tilde{\mu}_\tau} (x) Q_{\lambda, \tilde{\mu}_\tau} (x). \tag{66}
\]
One would like to pass here to the limit \( \tau \to -\infty \). According to (61), for every \( x \) we have \( \varepsilon_{\lambda, \tilde{\mu}_\tau} (x) \to 0 \) as \( \tau \to -\infty \). Moreover, it is not difficult to show that in the same limit \( q_{\lambda, \tilde{\mu}_\tau, x} (\cdot) \to q_{\lambda, x} (\cdot) \). So the following equation holds for \( \lambda : \)
\[
\lambda (x) = [\lambda * q_{\lambda, x}] (x), \quad -\infty < x < +\infty. \tag{67}
\]
By the methods developed below one can show that every bounded solution of (67) is a constant. Since, however, we are proving a stronger statement, that the dynamical system \( T_{\tau} \) has one fixed point on each \( \mathcal{M}_q (\Omega) \), which is, moreover, globally attractive, we will not provide the details.

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9 Self-averaging $\Rightarrow$ relaxation: a warm-up

Before presenting the general proof that self-averaging implies relaxation, we consider the following simpler system: we have infinitely many servers, with service time $\eta$, its distribution given by the probability density $p$. As the customer comes, he chooses any free server, and is served, leaving the system afterwards. The inflow is Poissonian, given by the rate function $f(x)$. If we impose the condition that the customers are coming at the rate they are living the system, we get the non-linear Markov process. The self-averaging relation (20) in such a case simplifies to

$$b(x) = [f \ast p](x).$$

Lemma 14 Let $p(x)$ be some smooth probability density, $\text{supp } p = [0, 1]$. Let $f$ be a positive bounded function on $\mathbb{R}^1$, with $f(x) \leq C$ for $x \in [1-, 0)$. Suppose that

$$f \ast p(x) = f(x) \quad \text{for all } x \geq 0.$$  \hspace{1cm} (68)

Then $f(x) \to c$ as $x \to \infty$, for some $c > 0$.

Proof. Let us first show that for every function $\varphi \geq 0$ on $\mathbb{R}^1$, $\varphi = 0$ for $x$ outside the segment $[1-, 0)$ there exists a function $f_\varphi$ on $\mathbb{R}^1$, satisfying (68), which coincides with $\varphi$ on $[1-, 0)$. To construct such a function consider the sequence $f_n$ of functions, defined inductively by

$$f_0 = \varphi,$$

$$f_{n+1}(x) = \begin{cases} \varphi(x) & \text{for } x < 0, \\ [f_n \ast p](x) & \text{for } x > 0. \end{cases}$$

A straightforward check shows that the sequence $f_n(x)$ is non-decreasing for every $x$, and that $f_n(x) \leq C$ for all $n$ and $x$. Therefore the function $f_\varphi(x) = \lim_{n \to \infty} f_n(x)$ is defined. Clearly, it satisfies (68). This function is given by the formula

$$f_\varphi(x) = \begin{cases} [\varphi \ast p](x) + \left( \varphi \ast p \bigg|_{\{x \geq 0\}} \right) \ast \left( \sum_{n=1}^{\infty} p^n \right) (x) & \text{for } x > 0, \\ \varphi(x) & \text{for } x < 0. \end{cases}$$  \hspace{1cm} (69)

(Note that for every $x$ the sequence $p^n(x)$ decays exponentially, so the last expression is well-defined.)
Let us show that the function $f$, satisfying (68), is uniquely defined by its restriction to $[1−, 0)$. Indeed, let $g$ be another such function. Then $h(x) = f(x) - g(x)$ is bounded in absolute value and satisfies

\[ h(x) = 0 \text{ for } x \leq 0, \]
\[ h * p(x) = h(x) \text{ for all } x \geq 0. \]

But then $h * p^n(x) = h(x)$ for all $x$ and for all $n$. Since for every $A$ we have $\int_0^A p^n(x) \, dx \to 0$, as $n \to \infty$, it follows that $h \equiv 0$.

Let us show that the function

\[ s(x) = \sum_{n=1}^{\infty} p^*(x) \]

goes to the limit as $x \to \infty$; together with (69) it would imply our statement. We will compute that limit, $S(\eta)$. For that we will use the local limit theorem approximation for the convolutions $p^n(x)$. Let $m$ be the mean value of $\eta$, and $v$ be its variance. Denote by $q_{M,V}(\cdot)$ the density of the normal distribution with the mean $M$ and the variance $V$. Then easy calculus computations tell us that

\[ S(\eta) = \lim_{N \to \infty} \sum_{k \geq N/2} q_{km,kv}(Nm) = 2 \lim_{N \to \infty} \sum_{k \geq N} q_{km,kv}(Nm) \]
\[ = 2 \lim_{N \to \infty} \sum_{k \geq N} \frac{1}{\sqrt{2\pi kv}} e^{-(Nm-km)^2/2kv} = \frac{1}{m}. \]

Therefore the limit

\[ \lim_{x \to \infty} f_\varphi(x) = \frac{1}{m} \int_0^{+\infty} [\varphi * p](x) \, dx. \]

In a special case when

\[ \varphi(x) = 1 + x \text{ on } [-1, 0], \]

and $p(x)$ is the uniform distribution on a segment $[0, 1]$, there is another formula for $f_\varphi$:

\[ f_\varphi(x) = 1 + x - \sum_{0 \leq k < x} \frac{(-1)^k}{2k!} (x-k)^k e^{x-k}. \]
(We got it together with Prof. O. Ogieveckij.) It satisfies the equation:

\[ f(x) = \int_{x-1}^{x} f(x) \, dx, \quad x \geq 0 \]  

(70)

with the initial data

\[ f(x) = 1 + x \]

on the segment \([-1, 0]\). Note that at \(x = 0\) it has a jump, from 1 to \(\frac{1}{2}\). It becomes more and more smooth; in the non-integer points it is analytic, but at the integer point \(n\) it has \(n - 1\) derivatives.

It is bounded, and it goes to \(\frac{2}{3}\) as \(x \to \infty\), since \(m = \frac{1}{2}\) and

\[ \int_{0}^{1} \left[ (1 + x) \big|_{[-1,0]} * p \right](x) \, dx = \frac{1}{3}. \]

But to see analytically that this series defines a bounded function, and, moreover,

\[ 1 + x - \sum_{0 \leq k < x} \frac{(-1)^k}{2k!} (x - k)^k e^{x-k} \to \frac{2}{3} \text{ as } x \to \infty \]

seems to be quite hard. So, it looks amazing, that the above arguments give relatively simple proof of this convergence, which proof is probabilistic!

10 Self-averaging \(\Rightarrow\) relaxation: probabilistic proof?

As we know, any function \(\lambda\), defined for \(x < 0\), and vanishing for \(x < -T\), can be uniquely extended to \(x \geq 0\) in such a way that the relation

\[ A(0, \lambda(\cdot), -T) = b(\cdot) \]

holds with \(b(x) = \lambda(x)\) for \(x \geq 0\). Therefore for every \(x \geq 0\) we have

\[ \lambda(x) = [\lambda \ast q_{\lambda,x}](x), \]  

(71)

where \(q_{\lambda,x}(\cdot)\) is a probability density supported by the semiaxis \(\{y \geq 0\}\), and which is defined only by the restriction \(\lambda \big|_{\{y \leq x\}}\). Our goal is to show that

(71) implies that \(\lambda(x)\) relaxes to some constant \(c\) as \(x \to \infty\).
Since the distributions \( q_{\lambda,x} \) depend on \( \lambda (\cdot) \) in a very complicated way, we have to treat a more general statement. Suppose a family of probability densities \( q_x (\cdot) \), supported by the semiaxis \( \{ y \geq 0 \} \), is given, where \( x \geq 0 \). Let \( f (x) \) be a non-negative function, defined on \( \mathbb{R}^1 \), such that

\[
f (x) \leq C \text{ for } x < 0, \\
f (x) = [f \ast q_x] (x) \text{ for } x \geq 0.
\]

(72)

One would like to show that

\[
\lim_{x \to \infty} f (x) = c,
\]

(73)

for some \( c \geq 0 \). That will imply the relaxation needed.

Motivated by the analysis of the previous section, we will study the equation (72) by considering the corresponding inhomogeneous Markov random walk. Unfortunately, the relation (73) does not follow from (72) in general, and the reasons are probabilistic! Before explaining it let us “solve” (72).

So, let the family \( \{ q_x, x \geq 0 \} \) be given; we solve (72) for \( f \), given its restriction \( f \big|_{\{ x < 0 \}} \). We do this in close analogy with the previous section, see (69). We put

\[
f_0 (x) = \begin{cases} 
  f (x) & \text{for } x < 0 \\
  0 & \text{for } x \geq 0.
\end{cases}
\]

We define

\[
f_{n+1} (x) = \begin{cases} 
  f (x) & \text{for } x < 0, \\
  [f_n \ast q_x] (x) & \text{for } x \geq 0.
\end{cases}
\]

Then for every \( x \) the sequence \( f_n (x) \) is increasing, and the function \( f (x) = \lim_{n \to \infty} f_n (x) \) solves (72).

To proceed, it is convenient to rewrite the function \( f \) in a different way. We define

\[
g_1 (x) = \begin{cases} 
  [f_0 \ast q_x] (x) & \text{for } x \geq 0, \\
  0 & \text{for } x < 0,
\end{cases}
\]

\[
g_{n+1} (x) = [g_n \ast q_x] (x).
\]

Then for \( x \geq 0 \) we have

\[
f (x) = \sum_{n \geq 1} g_n (x).
\]
Now we will write the formula for \( g_n \) in terms of convolution. To simplify the exposition we consider the case \( n = 5 \), say.

\[
g_5(x) = [g_4 \ast q_x](x)
\]

\[
= \int g_4(x-u)q_x(u)\,du
\]

\[
= \int g_3(x-u-v)q_{x-u}(v)q_x(u)\,dudv
\]  

\[
= \int g_2(x-u-v-w)q_{x-u-v}(w)q_x(u)\,dudvdw
\]

\[
= \int g_1(x-t)q_{x-u-v-w}(t-u-v-w)q_x(u)\,dudvdwdt.
\]  

Motivated by the last line we will introduce now for every \( x > 0 \) the following family of d.d.d. (dependent, differently distributed) positive random variables \( \chi_1, \chi_2, \chi_3, \ldots \):

- the distribution of \( \chi_1 \) is given by the density \( q_x(\cdot) \),
- the conditional distribution of \( \chi_2 \) under condition \( \chi_1 \) is given by the density \( q_{x-\chi_1}(\cdot) \),
- the conditional distribution of \( \chi_3 \) under condition \( \chi_1, \chi_2 \) is given by the density \( q_{x-\chi_1-\chi_2}(\cdot) \),
- etc. ...

To make this definition complete, we put \( q_x(\cdot) \) to be the uniform distribution on \([0,1]\) for all negative \( x \)-s; note that this extension does not contribute to (74), since the support of all the functions \( g_i \) is the positive semiaxis.

Hence we are led naturally to consider for every \( x > 0 \) the sums \( \theta_i = \chi_1 + \chi_2 + \ldots + \chi_i \); if we denote by \( p_x^{(i)} \) the probability density of \( \theta_i \), then we have, by (74):

\[
g_{n+1}(x) = [g_1 \ast p_x^{(n)}](x).
\]

Summarizing, we have for \( x > 0 \):

\[
f(x) = g_1(x) + \sum_{n \geq 2} [g_1 \ast p_x^{(n)}](x).
\]

The distributions \( p^{(i)}_x \) are describing the following non-stationary Markov chain: at every point \( y \in \mathbb{R}^1 \) we are given a probability distribution \( q_y(\cdot) \), which has to be interpreted as the transition probability to make a move once in \( y \). So we start at some \( x \), and we make a (random) move \(-\chi_1\), where \( \chi_1 \)
is distributed according to $q_x(\cdot)$. Arriving thus to $x - \chi_1$, we make a second move $-\chi_2$, where $\chi_2$ is distributed according to $q_{x-\chi_1}(\cdot)$, and so on. Clearly, the local limit theorem for this chain would imply (73).

We have to note, however, that the relation between the validity of the local limit theorem for this Markov chain and the validity of the relation (73) is more complicated. First of all, the CLT for $\theta_i$ might not hold, notwithstanding that the family $q_y(\cdot)$ have very nice compactness properties. To give one example, consider the family of probability densities $u_x(t), x \in \mathbb{R}^1$, where all $u_x(\cdot)$ have for their support the segment $[0, 1]$, and satisfy there $0 < c < u_x(t) < C < \infty$, uniformly in $x$ and $t$. We define now

$$q_x(t) = u_x(t - \{x\}),$$

where $\{\cdot\}$ stays for the fractional part. Then all $q_y(\cdot)$-s have their supports within the segment $[0, 2]$. But the random variables $\theta_i$ do not have CLT behavior! Indeed, the random variable $\theta_i$, is localized in the segment $[\lfloor x \rfloor - i, \lfloor x \rfloor - i + 1]$, where $\lfloor \cdot \rfloor$ denotes the integer part. Nevertheless, for this example it can be shown that the relation (73) still holds, and that involves certain statement of the type of Perron-Frobenius theorem for our Markov chain. Further modification of this example, when

$$q_x(t) = u_x(t - \{x\} - 1),$$

results in the Markov chain with two classes, and in this case both the CLT and the relation (73) fail.

We conjecture here that the CLT theorem for the sums $\theta_i$ holds, if the family $q_x(\cdot)$ of transition densities has the following additional property:

- For some $k, K, 0 < k < K < \infty$,

$$k \leq \frac{q_{x_1}(t)}{q_{x_2}(t)} \leq K,$$  \hspace{1cm} (75)

provided at least one of the values $q_{x_i}(t)$ is positive.

The condition (75) is reminiscent of the positivity of ergodicity coefficient condition, introduced by Dobrushin [D] in his study of the limit theorems for the non-stationary Markov chains.

In what follows we will take another road, and we get the relaxation property by analytical methods, which seems in our case to be simpler. But we still use probability theory, though not the CLT. It would be interesting to obtain the desired result by proving the corresponding limit theorem.
11 Self-averaging \( \Rightarrow \) relaxation: finite range case

In this section we prove the relaxation for the solution of the equation (72) in the finite range case.

**Theorem 15** Suppose that

\[
0 \leq f (x) \leq C \quad \text{for } x < 0, \\
f (x) = [f * q_x] (x) \quad \text{for } x \geq 0,
\]

while the following conditions on the family \( q_x \) hold: for some \( T \)

\[
\int_0^T q_x (t) \, dt = 1 
\]

for all \( x \), and

\[
C \geq q_x (t) \geq \kappa (t) > 0 
\]

(76)

for \( 0 \leq t \leq T \), with continuous positive \( \kappa (t) \). Then the limit exists:

\[
\lim_{x \to \infty} f (x) = c \geq 0.
\]

The property (76) holds for the NMP, as follows from the relations (49) and (22).

**Proof.** i) We know that the function \( f \) is continuous and bounded, \( 0 \leq f \leq C \). So if there exists a value \( X \) such that \( f \) is monotone for \( x \geq X \), then the function \( f \) has to be constant for \( x \geq X + T \), and we are done. So we are left with the case when the function \( f \) has infinitely many points of local maxima and local minima, which go to \( \infty \).

ii) Given a local maximum, \( x_0 \), we will construct now a sequence \( x_i \) of local maximums, \( i = 0, -1, -2, ..., -n = -n (f, x_0) \) such that

- \( x_0 > x_{-1} > x_{-2} > ... \),
- \( x_i - x_{i-1} < 2T, \ x_i - x_{i-2} \geq T \) for all \( i \),
- \( 0 < x_{-n} < 2T \),
- \( f (x_{i-1}) \geq f (x) \) for any \( x_{i-1} \leq x \), and \( f (x_{i-1}) > f (x_i) \),
• for every \( x \in [x_{i-1}, x_i - T] \) we have \( f(x) \geq f(x_i) \) (of course if the segment is non-empty).

The construction is the following. Let \( x_0 \) be some point of local maxima. Since

\[
  f(x_0) = \int_0^T f(x_0 - t) q_{x_0}(t) \, dt,
\]

we have \( f(x_0) < F(x_0) \equiv \sup \{ f(x) : x \in [x_0 - T, x_0] \} \), unless \( f \) is a constant on \([x_0 - T, x_0] \), in which case we are done. Let

\[
y = \inf \{ x \in [x_0 - T, x_0] : f(x) = F(x_0) \equiv \sup \{ f(x) : x \in [x_0 - T, x_0] \} \} .
\]

If \( y > x_0 - T \), or if \( y = x_0 - T \) and is a local maximum, we define \( x_{-1} = y \).

In the opposite case we have that the point \( x_0 - T \) is not a local maximum of the function \( f \) on the segment \([x_0 - 2T, x_0 - T] \). We then consider two cases. In the first one we suppose that the function \( f \) on the segment \([x_0 - 2T, x_0 - T] \) takes values below \( \bar{F} = \frac{f(x_0) + f(x_0 - T)}{2} \). Let \([y, x_0 - T] \subset [x_0 - 2T, x_0 - T] \) be the largest segment for which the inequality \( f(x) \geq \bar{F} \) holds for every \( x \in [y, x_0 - T] \). We define \( x_{-1} \) to be the leftmost point of maximum of \( f \) in \([y, x_0 - T] \). In the opposite case we consider the set \( S = \{ x \in [x_0 - 2T, x_0 - T] : f(x) \geq f(x_0 - T) \} \). It contains other points besides \( x_0 - T \). However, it can not contain all the segment \([x_0 - 2T, x_0 - T] \).

Since \( f \) is not a constant on \([x_0 - 2T, x_0 - T] \), \( \sup S f > f(x_0 - T) \). Let \( z \in (x_0 - 2T, x_0 - T) \) be such that \( f(z) < f(x_0 - T) \). Let \( S_1 = S \cap [z, x_0 - T] \).

We necessarily have that \( \sup S_1 f > f(x_0 - T) \) as well. We define \( x_{-1} \) to be any point in \( S_1 \) where \( f(x_{-1}) = \sup S_1 f \). Clearly, \( x_{-1} \) is a local maxima of \( f \), while \( x_0 - x_{-1} < 2T \).

We proceed to define the sequences \( x_i \) by induction, \( i = 0, -1, -2, ... \), until we arrive to a first value below \( 2T \), where we stop.

iii) In the same way, starting from a local minima \( y_0 \), we can construct a sequence \( y_i \) of local minima, such that

• \( y_0 > y_{-1} > y_{-2} > ... \),

• \( y_i - y_{i-1} < 2T, \ y_i - y_{i-2} \geq T \) for all \( i \),

• \( 0 < y_{-n} < 2T, \)

• \( f(y_{i-1}) \leq f(x) \) for any \( y_{i-1} \leq x \), and \( f(y_{i-1}) < f(y_i) \),

• for every \( x \in [y_{i-1}, y_i - T] \) we have \( f(x) \leq f(y_i) \) (if the segment is non-empty).
We can suppose additionally that \( x_0 \geq y_0 \geq x_{-1} \).

iv) Note that the (finite) sequence \( x_i \) do depend on the initial local minima \( x_0 \), which was used for the starter. The bigger \( x_0 \) is, the longer the sequence \( x_i \) is. So let us introduce the sequence \( x_0^{(N)} \) of such starters, and we suppose that \( x_0^{(N)} \geq N \). In that way we will obtain the family \( x_i^{(N)} \) of sequences of local maximums of \( f \), \( i = 0, -1, \ldots, -n \left( f, x_0^{(N)} \right) \), with \( n \left( f, x_0^{(N)} \right) \to \infty \) as \( N \to \infty \). (Of course, in well may happen that for different \( N \)-s the corresponding sequences share many common terms.)

Denote by \( M \) the limit \( \liminf_{N \to \infty} f \left( x_0^{(N)} \right) \). In the same way we can introduce the limit \( m = \limsup_{N \to \infty} f \left( y_0^{(N)} \right) \). Clearly, \( M \geq m \), and if we can show that \( M = m \), then we are done. So we suppose that \( M - m > 0 \), and we will bring that to contradiction.

v) Let us fix \( \varepsilon > 0 \), \( \varepsilon < \frac{M-m}{10} \), which is possible if \( M - m > 0 \). Then one can choose \( N \) so large, that at least 99% of terms of the sequence \( f \left( x_i^{(N)} \right) - f \left( x_{i-1}^{(N)} \right) \) are less than \( \frac{\varepsilon^2}{2} \). We will fix that value of \( N \), and we will omit \( N \) from our notation. Therefore without loss of generality we can assume that for some \( i \) (in fact, for many) we have \( f \left( x_i \right) < f \left( x_i \right) + \varepsilon^2 \) for all \( x \in \left[ x_i - T, x_i \right] \). Therefore for the set \( K = \{ x \in \left[ x_{i-1} - T, x_i \right] : f \left( x \right) > f \left( x_i \right) - \varepsilon \} \) we have:

\[
\int_{K-(x_i-T)} q_{x_i}(t) \, dt > 1 - \varepsilon. \tag{77}
\]

Hence, for its Lebesgue measure we have

\[
\mes \{ K \} \geq \frac{1 - \varepsilon}{C}.
\]

Consider now the corresponding sequence of minima, \( \{ y_k \} \), and the segments \( [y_k - T, y_k] \). We claim that the set \( K \) has to belong to the union of these segments. That would be evident if the segments in question were covering the corresponding region without any holes. However, that is not necessarily the case, and there can be holes between the segments, since in general the differences \( y_i - y_{i-1} \) can be bigger than \( T \). Yet, this does not present a problem, since by construction the function \( f \) is smaller than \( m \) outside the union of the segments \( [y_k - T, y_k] \), which implies that the set \( K \) indeed is covered by these segments. Since \( \text{diam} \ (K) \leq T \), there exists \( k = k \left( K \right) \), such that \( K \subset [y_{k-1} - T, y_{k-1}] \cup [y_k - T, y_k] \cup [y_{k+1} - T, y_{k+1}] \).
Without loss of generality we can assume the set \( K \) “fits into \([y_k - T, y_k]\)”, in the sense that
\[
\text{mes} \{ K \cap [y_k - T, y_k]\} \geq \frac{\text{mes} \{ K \}}{3} \geq \frac{1 - \varepsilon}{3C},
\]
while we have \( f(x) > f(y_k) - \varepsilon^2 \) for all \( x \in [y_k - T, y_k] \). So we have
\[
\int_{\{ K \cap [y_k - T, y_k]\} - (y_k - T)} q_{y_k}(t) \, dt \geq \bar{\kappa} \left( \frac{1 - \varepsilon}{3C} \right), \tag{78}
\]
where we define the function \( \bar{\kappa} (\alpha) \) by
\[
\bar{\kappa} (\alpha) = \inf_{A \subset [0,T]: \text{mes} \{ A \} \geq \alpha} \int_A \kappa(t) \, dt.
\]
By construction, the set \( K \cap [y_k - T, y_k] \) is disjoint from the set \( L \subset [y_k - T, y_k] \), which is defined by \( L = \{ x \in [y_k - T, y_k] : f(x) < f(y_k) + \varepsilon \} \). Since
\[
f(y_k) = \int_0^T f(y_k - t) \, q_{y_k}(t) \, dt,
\]
we have similar to (77) that
\[
\int_{L - (y_k - T)} q_{y_k}(t) \, dt > 1 - \varepsilon. \tag{79}
\]
But since \( q_{y_k}(t) \, dt \) is a probability measure, we should have that
\[
\bar{\kappa} \left( \frac{1 - \varepsilon}{3C} \right) + 1 - \varepsilon \leq 1,
\]
because of (78), (79). This, however, fails once \( \varepsilon \) is small enough. \( \blacksquare \)

12 Self-averaging \( \implies \) relaxation: infinite range case

We return to the equation (72), \( f(x) = [f * q_x](x) \). Now we will not suppose that the measures \( q_x \) have finite support. Instead we suppose that the family
$q_x$ has the following compactness property: for every $\varepsilon > 0$ there exists a value $K(\varepsilon)$, such that
\[
\int_0^{K(\varepsilon)} q_x(t) \, dt \geq 1 - \varepsilon
\] (80)
uniformly in $x$. We will also suppose that for every $T$ the (monotone continuous) function
\[
F_T(\delta) = \inf_{x} \inf_{D \subset [0,T]} \int_D q_x(t) \, dt
\] (81)
is positive once $\delta > 0$. Finally we assume that the family $q_x$ is such that the function $f$, with solves (72), is Lipschitz, with Lipschitz constant $L = L(\{q_x\})$. As we know from the Sections 4 and 7, these conditions are indeed satisfied in the specific case of the non-linear Markov process and the equation (71).

12.1 Approaching stationary point

**Lemma 16** i) Let $M = \limsup_{x \to \infty} f(x)$. Then for every $T$ and every $\varepsilon$ given there exists some value $K_1$, such that
\[
\inf_{x \in [K_1, K_1 + T]} f(x) \geq M - \varepsilon.
\]

ii) Let $m = \liminf_{x \to \infty} f(x)$. Then for every $T$ and every $\varepsilon$ given there exists some value $K_2$, such that
\[
\sup_{x \in [K_2, K_2 + T]} f(x) \leq m + \varepsilon.
\]
Moreover, the conclusions of the lemma remains valid if the function $f$ satisfies a weaker equation (see (82))
\[
f(x) = (1 - \varepsilon(x)) \left[ f \ast q_x \right](x) + \varepsilon(x) Q(x),
\] (82)
with $\varepsilon(x) \to 0$ as $x \to \infty$ and $Q(\cdot) \leq C$.

**Proof.** i) Let $\delta > 0$. Then there exists a value $S > 0$, such that for all $x > S$ we have $f(x) < M + \delta$, and $\varepsilon(x) Q(x) < \frac{\delta}{2}$. Further, there exists a value $R > S$, such that for all $y \geq R$
\[
\int_{R-S}^\infty q_y(t) \, dt < \delta,
\]
see (80). Finally, there exists a point \( y > R + T \), such that \( f(y) > M - \frac{\delta}{2} \).

Due to the equation (82) we have

\[
f(y) = (1 - \varepsilon(y)) \left[ \int_0^{y-S} f(y - t) q_y(t) \, dt + \int_{y-S}^\infty f(y - t) q_y(t) \, dt \right] + \varepsilon(y) Q(y).
\]

Let \( \Delta > 0 \), and \( A = \{ x \in [y - T, y] : f(x) < M - \Delta \} \), while \( a = \int_A q_y(t) \, dt \).

We want to show that the measure \( a \) has to be small for small \( \delta \). Splitting the first integral into two, according to whether the point \( y - t \) is in \( A \) or not, we have

\[
M - \delta < a (M - \Delta) + (1 - a - \delta) (M + \delta) + \delta C,
\]

so

\[
a < \frac{\delta C + 2 - M}{\Delta},
\]

which goes to zero with \( \delta \), provided \( \Delta \) is fixed. Therefore

\[
\text{mes} \{ A \} \leq F_T^{-1} \left( \frac{\delta C + 2 - M}{\Delta} \right),
\]

(see (81)). Since \( F_T^{-1}(u) \to 0 \) as \( u \to 0 \), that proves that for any given \( \Delta \) the Lebesgue measure \( \text{mes} \{ A \} \to 0 \) as \( \delta \to 0 \). Since the function \( f \) is Lipschitz, we conclude that \( \inf_{x \in [y - T, y]} f(x) \geq M - \Delta - \mathcal{L} \text{mes} \{ A \} \geq M - 2\Delta \), provided \( \delta \) is small enough. Taking \( \Delta = \varepsilon/2 \) finishes the proof.

\( ii \) Let \( \delta > 0 \). Then there exists a value \( S > 0 \), such that for all \( x > S \) we have \( f(x) > m - \delta \). Again, take \( R > S \), such that for all \( y \geq R \)

\[
\int_0^{R-S} q_y(t) \, dt > 1 - \delta.
\]

Finally, there exists a point \( y > R + T \), such that \( f(y) < m + \delta \). Due to the equation (82) we have

\[
f(y) > (1 - \varepsilon) \left[ \int_0^{y-S} f(y - t) q_y(t) \, dt + \int_{y-S}^\infty f(y - t) q_y(t) \, dt \right], \tag{83}
\]

where \( \varepsilon \) can be supposed arbitrarily small. Let \( \Delta > 0 \), and

\( A = \{ t \in [0, T] : f(y - t) > m + \Delta \} \), while \( a = \int_A q_y(t) \, dt \). We want to show that the measure \( a \) has to be small for small \( \delta \). Splitting the first integral
into two, according to whether the point $y - t$ is in $A$ or not, and disregarding
the second one, we have

$$m + \delta > (1 - \kappa) [a (m + \Delta) + (1 - \delta - a) (m - \delta)].$$

For $\kappa$ so small that $\kappa [a (m + \Delta) + (1 - a - \delta) (m - \delta)] < \delta$, we have

$$m + 2\delta > a (m + \Delta) + (1 - a - \delta) (m - \delta),$$

so

$$a < \delta \frac{m + 3}{\Delta},$$

which goes to zero with $\delta$, provided $\Delta$ is fixed. Therefore

$$\text{mes}\{A\} \leq F_{-1}^{-1} \left( \delta \frac{m + 3}{\Delta} \right),$$

and the rest of the argument coincides with that of the part $i$).

\section{Absorbing by stationary point}

We now will show that if $f$ satisfies (72), then the property $\inf_{x \in [K, K + T]} f (x) \geq M - \varepsilon$ implies that for all $x > K + T$

$$f (x) > M - \varepsilon - c (T), \quad (84)$$

with $c (T) \to 0$ as $T \to \infty$. That clearly implies relaxation. (Note that we do not claim that (57) holds for the solutions of (82)). We will show it under the extra assumption that the distribution $p (\cdot)$ has finite moment of some order above 4. This assumption, as well as (87) below, will be used only throughout the rest of the present subsection.

Using the linearity of (72), we will rewrite our problem slightly, in order to simplify the notation.

Let the function $f \geq 0$ satisfies $f (x) = [f * q_x] (x)$ for $x > 0$, and

i) $f (x) > 1$ for $x \in [-T, 0]$,

ii) for some $\beta > 1$ and $B < \infty$ and for every $x$ we have

$$\int_0^\infty t^\beta q_x (t) \, dx \leq B, \quad (85)$$

55
compare with (55). We want to derive from that data that for some \( c(T) > 0 \), \( c(T) \to 0 \) as \( T \to \infty \)

\[
f(x) > 1 - c(T)
\]
for all \( x > 0 \).

Denote by

\[
g_0(x) = \begin{cases} 
1 & x \in [-T,0] \\
0 & x \notin [-T,0]
\end{cases}
\]

Since \( f \geq g \), we have \( f(x) \geq g_1(x) = [g_0 \ast q_x](x) \) for \( x \geq 0 \). We define \( g_1(x) = g_0(x) \) for \( x < 0 \). Iterating, we have \( f(x) \geq g_n(x) \), where

\[
g_n(x) = \begin{cases} 
g_0(x) & x < 0 \\
g_{n-1} \ast q_x(x) & x \geq 0
\end{cases}
\]

Hence, \( f(x) \geq g_\infty(x) \). The function \( g_\infty(x) \) has the following probabilistic interpretation: we have a Markov chain on \( \mathbb{R}^1 \), where transition from the point \( x \) is governed by transition densities \( q_x \) to make the step (to the left), (and which steps to the left are defined in an arbitrary way for \( x \leq 0 \)); then the value \( g_\infty(x) \) for \( x > 0 \) is the probability that starting from \( x \) we will visit the interval \([-T,0]\). The question now is about the lower bound on \( g_\infty(x) \) over all possible \( q_x \) from our class.

So let us take \( x > 0 \), and let start the Markov chain \( X_n \) from \( x \), (i.e. \( X_0 = x \)), which goes to the left, and which makes a transition from \( y \) to \( y - t \) with the probability \( q_y(t) \, dt \). We need to know the probability of the event

\[
\mathbb{P}_x \{ \text{there exists } n \text{ such that } X_n \in [-T,0] \}.
\]

In other words, we want to know the probability of \( X \) visiting \([-T,0]\). We would like to show that

\[
\mathbb{P}_x \{ X, \text{ visits } [-T,0] \} \geq \gamma(\beta, B, T) \tag{86}
\]

with

\[
\gamma(\beta, B, T) \to 1 \text{ as } T \to \infty
\]

uniformly over the families \( q_x \) from our class.

Note, however, that in general such an estimate does not hold. For example, the process \( X \) can well stay positive for all times. The more interesting example where the process goes to \( -\infty \), follows, so we will need further restrictions on the family \( q_x \).
Example. Let $T$ be given. We will construct the family $q^T_x$ from our class (85), such that for the corresponding process $X^T$

\[ \mathbb{P}_x \{ X^T \text{ visits } [0,T] \} = 0. \]

We define $q^T_x(t)$ for $x \in (k, k+1]$ with integer $k \neq 0$ to be any distribution localized in the segment $[k-1, k]$ (the uniform distribution on $[k-1, k]$ is OK). For $x \in \left(\frac{k}{2}, \frac{k+1}{2}\right]$, $k = 1, 2, \ldots$, it is defined by

\[
q^T_x(t) = \begin{cases} 
e^{-t} & \text{if } t > T + 1 \\ 2^{k+1} \left(1 - \int_{T+1}^{\infty} e^{-t} dt\right) & \text{if } t \in \left[x - \frac{1}{2^n}, x - \frac{1}{2^{n+1}}\right] \\ 0 & \text{otherwise.} \end{cases}
\]

For $x \leq 0$ it is defined in an arbitrary way. ■

The mechanism of violating the relation (86) is that the time the process $X^T$ can spend in the segment $[0,1]$ is unbounded in $T$. As the following theorem shows, this feature is the only obstruction for the statement desired to hold.

**Theorem 17** Consider the Markov chain $X \cdot$ defined above via the transition densities $q_x(t)$. Suppose that condition (85) holds, and that in addition these densities are uniformly bounded in the vicinity of the origin: for all real $x$ and all $t$ in the segment $[0,1]$, say,

\[ q_x(t) \leq C. \tag{87} \]

Then for some $\gamma(\beta, B, C, T) \to 1$ as $T \to \infty$ we have:

\[ \mathbb{P}_x \{ X. \text{ visits } [-T,0] \} \geq \gamma(\beta, B, C, T). \]

The condition (87) in the case of NMP follows easily from the estimate (50), see Lemma 10.

**Proof.** We will estimate the probability of the complementary event:

\[ \mathbb{P}_x \{ X. \text{ misses } [-T,0] \}
\]

\[ = \sum_{k=0}^{\infty} \int_0^{x} \left[ \int_{y+T}^{\infty} q_y(t) \, dt \right] P_k(x, dy). \]

Here $P_k(x, dy)$ is the probability distribution of the chain $X$ after $k$ steps, and the expression $\left[ \int_{y+T}^{\infty} q_y(t) \, dt \right] P_k(x, dy)$ is the probability that the chain
X. arrives after \( k \) steps to the location \( y \), and then makes a jump over the segment \([-T, 0]\). So we have

\[
\mathbb{P}_x \{X. \text{ misses } [-T, 0]\} \\
\leq \int_0^x B(y + T)^{-\beta} \sum_{k=0}^{\infty} P_k(x, dy) \\
\leq \sum_{n=0}^{x+1} B(n + T)^{-\beta} \sum_{k=0}^{\infty} P_k(x, [n, n + 1]),
\]

where \( P_k(x, [n, n + 1]) \) is the probability of the event \( X_k \in [n, n + 1] \), and where in the second line we are using the following simple estimate:

\[
\int_{r}^{\infty} q_y(t) \, dt = r^{-\beta} \int_{r}^{\infty} t^{\beta} q_y(t) \, dt \leq r^{-\beta} \int_{0}^{\infty} t^{\beta} q_y(t) \, dt.
\]

Now,

\[
\sum_{k=0}^{\infty} \mathbb{P}_x \{X_k \in [n, n + 1]\} \\
= \sum_{k=0}^{\infty} \sum_{l < k} \mathbb{P}_x \{X_k \in [n, n + 1], X_l > n + 1, X_{l+1} \in [n, n + 1]\} \\
= \sum_{l=0}^{\infty} \mathbb{P}_x \{X_l > n + 1, X_{l+1} \in [n, n + 1]\} \\
\times \sum_{k>0} \mathbb{P}_x \left\{X_{l+k} \in [n, n + 1] \mid X_l > n + 1, X_{l+1} \in [n, n + 1]\right\}.
\]

Let now the random variables \( \zeta_i \) be i.i.d., uniformly distributed in the segment \([0, \frac{1}{C}]\), where \( C \) is the same as in (86). Then is easy to see that

\[
\mathbb{P}_x \left\{X_{l+k} \in [n, n + 1] \mid X_l > n + 1, X_{l+1} \in [n, n + 1]\right\} \leq \Pr \{\zeta_1 + \ldots + \zeta_k \leq 1\}.
\]

Since the last probability decays exponentially in \( k \), while \( \sum_{l=0}^{\infty} \mathbb{P}_x \{X_l > n + 1, X_{l+1} \in [n, n + 1]\} = 1 \), we conclude that

\[
\sum_{k=0}^{\infty} \mathbb{P}_x \{X_k \in [n, n + 1]\} \leq K(C).
\]

Since the series \( \sum n^{-\beta} \) converges for \( \beta > 1 \), the proof follows. \( \blacksquare \)
13 Self-averaging $\Rightarrow$ relaxation: noisy case

In this section we prove the relaxation for the NMP with general initial condition, i.e. for the solution of the equation

$$\lambda (x) = (1 - \varepsilon_{\lambda,\mu} (x)) [\lambda \ast q_{\lambda,\mu} (x)] + \varepsilon_{\lambda,\mu} (x) Q_{\lambda,\mu} (x)$$

see (60). We are not able to prove it in the generality of the previous Sections. Below we will use all the specific features of the NMP, and in particular we will use the comparison between different NMP-s and GMP-s, corresponding to various initial states and input rates. The comparison mentioned is based on the coupling arguments.

13.1 Coupling

Definition 18 Let $\mu_1, \mu_2$ be two states on $\Omega$. We call the state $\mu_1$ to be **higher** than $\mu_2$, $\mu_1 \succcurlyeq \mu_2$, if there exists a coupling $P [d\omega_1, d\omega_2]$ between the states $\mu_1, \mu_2$, with the property:

$$P [ (\Omega \times \Omega)^\succcurlyeq ] = 1,$$

where

$$(\Omega \times \Omega)^\succcurlyeq = \{ (n_1, \tau_1), (n_2, \tau_2) \in \Omega \times \Omega : n_1 \geq n_2 \}.$$

Clearly, if $\mu_1 \succcurlyeq \mu_2$, then $N (\mu_1) \geq N (\mu_2)$.

Definition 19 Let $\mu_1, \mu_2$ be two states on $\Omega$. We call the state $\mu_1$ to be **taller** than $\mu_2$, $\mu_1 \succ \mu_2$, if there exists a coupling $P [d\omega_1, d\omega_2]$ between the states $\mu_1, \mu_2$, with the property:

$$P [ (\Omega \times \Omega)^\succ ] = 1,$$

where

$$(\Omega \times \Omega)^\succ = \{ (n_1, \tau_1), (n_2, \tau_2) \in \Omega \times \Omega : \tau_1 = \tau_2, n_1 \geq n_2 \text{ or } \omega_2 = 0 \}.$$

Lemma 20 Let $\mu_1 (0), \mu_2 (0)$ be two initial states on $\Omega$ at $t = 0$, and $\lambda_1 (t), \lambda_2 (t), t \geq 0$ be two Poisson densities of the input flows. The service time distribution is the same $\eta$ as before. Let $\mu_i (t)$ be two corresponding GFP-s, with $\mu_i (0) = \mu (0)$. Suppose that $\mu_1 (0) \succ \mu_2 (0)$, and that $\lambda_1 (t) \geq \lambda_2 (t)$. Then $\mu_1 (t) \succ \mu_2 (t)$, so in particular

$$N (\mu_1 (t)) \geq N (\mu_2 (t)).$$
Proof. To see this let us construct the coupling between the processes \( \mu_i(t) \). Let us color the customers arriving according to the \( \lambda_2(t) \) flow as red. We also assign the red color to the customers which were present at time \( t = 0 \) from the initial state \( \mu_2(0) \). Let \( \gamma(t) = \lambda_1(t) - \lambda_2(t) \), and consider \( \gamma(t) \) as the extra input flow of blue customers (with independent service times). We also add blue customers at time \( t = 0 \), which are needed to complete the state \( \mu_2(0) \) up to \( \mu_1(0) \). Then the total (color blind) flow coincides with \( \lambda_1 \) flow, while the total (color blind) process coincides with \( \mu_1(t) \).

The service rule for the two-colored process is color blind: all the customers are served in order of their arrival time. We claim now that along every coupled trajectory \((\omega_1(t), \omega_2(t))\) we have \( R(\omega_1(t)) \geq R(\omega_2(t)) \), where \( R(\cdot) \) is the number of red customers at the moment \( t \), waiting to be served. That evidently will prove our statement.

Clearly, the number \( R(\omega(t)) \) is the difference,

\[
R(\omega(t)) = A(\omega(t)) - S(\omega(t)),
\]

where \( A(\omega(t)) \) is the total number of red customers, having arrived before \( t \), while \( S(\omega(t)) \) is the total number of red customers, who left the system before \( t \). Clearly, \( A(\omega_1(t)) = A(\omega_2(t)) \). Let us show that \( S(\omega_1(t)) \leq S(\omega_2(t)) \).

This is easy to see once one visualizes the procedure of resolving the rod conflicts, which corresponds to our service rule, for the two-colored rod case. Namely, one has first to put all the red rods, and resolve all their conflicts by shifting some of them to the right accordingly. The number of thus obtained rods to the left of the point \( t \) is the number \( S(\omega_2(t)) \). Clearly, if one adds some blue rods to the red ones, then each red rod would be shifted to the right by at least the same amount as without the blue rods. As a result, every red rod would either stay where it was, or move to the right, so indeed \( S(\omega_1(t)) \leq S(\omega_2(t)) \).

13.2 Compactness

Consider a General Flow Process \( \mu(t) \) with initial state \( \mu(0) = \nu \) at \( T = 0 \) and the input rate \( \lambda(t) \equiv c < 1 \). This is an ergodic process, so the weak limit

\[
\lim_{t \to \infty} \mu_{\nu,c}(t) = \nu_c
\]

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exists and does not depend on the initial state $\nu$. We would like to show that if $N(\nu) < \infty$, then also
\[
\lim_{t \to \infty} N(\mu_{\nu,c}(t)) = N(\nu_c)
\]
(see (14)). This turns out to be somewhat delicate problem, because the speed of the convergence $\mu_{\nu,c}(t) \to \nu_c$ is only linear in time, and it can happen that for every $\delta > 0$ the moment $E_\nu(n(\omega)^{1+\delta})$ does not exist, which property persists in time, and the moments $E_{\mu_{\nu,c}(t)}(n(\omega)^{1+\delta})$ are infinite for every $t$.

### 13.2.1 Compactification

With every point $\omega = (n, \tau) \in \Omega$, $n > 0$, there is associated the random variable $\eta \big|_\tau = (\eta - \tau \mid \eta > \tau)$. Consider now the following queueing problem: the customers are arriving at positive Poissonian times with the rate $\lambda(t) \equiv c$, while service times are independent and $\eta$-distributed. In addition, at moment 0 there is a customer with service time distributed according to $\eta \big|_\tau$, and $n-1 \eta$-distributed customers. Then the expected size of queue at the moment $t$ is precisely $N(\mu_{\delta,\nu,c}(t))$. In general, $N(\mu_{\nu,c}(t)) = E_{\mu_{\nu,c}(t)}(N(\mu_{\delta,\nu,c}(t)))$. We want to study the dependence of $N(\mu_{\nu,c}(t))$ on $\nu$. We abbreviate it by $N(\nu, t)$.

In general, the family $\eta \big|_\tau$ is not weakly compact. In order to prove our statement we have to generalize it, including $\left\{ \eta \big|_\tau , \tau \geq 0 \right\}$ into a compact family. The generalization is as follows. Note that the random variables $\eta \big|_\tau$ have the property that $E\left(\eta \big|_\tau\right)^{2+\delta} \leq M_\delta$. Consider now the set $\tilde{N}$ of all positive random variables with this property: $\zeta \in \tilde{N} \iff E(\zeta)^{2+\delta} \leq M_\delta$. This set $\tilde{N}$ already is weakly compact, due to Prokhorov theorem. We denote by $\mathcal{N} \subset \tilde{N}$ the closure of the family $\left\{ \eta \big|_\tau , \tau \geq 0 \right\}$ in $\tilde{N}$. We now extend our configuration space $\Omega$, to consist of pairs $\omega = (n, \zeta)$, $\zeta \in \mathcal{N}$. The initial state will then be a measure $\nu$ on $\Omega$, i.e. on the set of pairs $(n, \zeta)$, $n \geq 1$, plus the single point $n = 0$. We will use the old notation for all the extended objects.

Unfortunately, the function $N(\nu, t)$ is not continuous on $\mathcal{M}(\Omega)$, due to the fact that $\Omega$ is (still) not compact, in the $n$-direction. This obstruction
would be removed once one is contented to restrict the function \(N(\nu, t)\) to \(\mathcal{M}(\Omega_N) \subset \mathcal{M}(\Omega)\), where \(\Omega_N = \{\omega = (n, \zeta) \in \Omega : n \leq N\}\). Then it is enough to check that \(N(\nu, t)\) is continuous on \(\Omega \subset \mathcal{M}(\Omega)\), where the imbedding \(\Omega \subset \mathcal{M}(\Omega)\) is via \((n, \zeta) \sim \delta_{(n, \zeta)}\). The function we are dealing with is then the following:

Let \(x > 0\), and \(N((n, x), t)\) be the expected size of the queue at the moment \(t\), if

- the customers are arriving at positive Poissonian times with the rate \(\lambda(t) \equiv c\),
- the service times are independent and \(\eta\)-distributed,
- at moment 0 there is a customer with non-random service time, which equals \(x\), together with \(n-1\) \(\eta\)-distributed customers, waiting in the queue.

Now,

\[N((n, \zeta), t) \equiv N(\delta_{(n, \zeta)}, t) = \mathbb{E}(N((n, \zeta), t)).\]

Since the function \(N((n, x), t)\), though continuous, has infinite support in the \(x\)-variable for \(n, t\) fixed, the continuity of \(N((n, \zeta), t)\) in \(\zeta\) (in weak topology) needs to be checked. However, \(N((n, x), t) = ct + n\) for all \(x > t\), and that makes the check trivial.

In general case

\[N(\nu, t) = \mathbb{E}_\nu(\mathbb{E}_\zeta(N((n, \zeta), t))).\]

Now, since \(\mathbb{E}_\zeta(N((n, \zeta), t))\) is a continuous function on \(\{1, ..., N\} \times \mathcal{N} \times \mathbb{R}_+\), the continuity of \(N(\nu, t)\) on \(\mathcal{M}(\Omega_N)\) follows from compactness of \(\{1, ..., N\} \times \mathcal{N}\).

Therefore we have obtained the following conditional statement:

**Lemma 21** Suppose for some \(\ell \geq 0\) the convergence \(N(\nu, t) \to \ell\) holds for every \(\nu \in \mathcal{M}(\Omega)\), as \(t \to \infty\). Then the convergence is uniform on every \(\mathcal{M}(\Omega_N)\).
13.2.2 Convergence

Lemma 22 For every $n, \tau$ we have

$$\lim_{t \to \infty} N \left( \mu_{\delta(n,\tau),c}(t) \right) = N(\nu_c)$$

(though, of course, not uniform in $n, \tau$).

Proof. Since $\mu_{\delta(n,\tau),c}(t) \to \nu_c$ weakly, $\limsup_{t \to \infty} N \left( \mu_{\delta(n,\tau),c}(t) \right) \geq N(\nu_c)$.

To prove the opposite inequality we need to produce a uniform upper bound on the family $\mu_{\delta(n,\tau),c}(t)$, in order to have its uniform integrability. By this we mean the following property: for every $\kappa > 0$ there exists a value $N_\kappa$ such that for all $t$

$$\mathbb{E}_{\mu_{\delta(n,\tau),c}(t)} \left( N(\omega) I_{N(\omega) \geq N_\kappa} \right) < \kappa,$$

where $I$ stands for the indicator. If it were possible to find an $\varepsilon$ such that $\nu_c \geq \varepsilon \delta(n,\tau) + (1-\varepsilon) \delta_0$, then we would be done, since we then have that $N \left( \mu_{\delta(n,\tau),c}(t) \right) \leq N(\nu_c)$ for all $t$. However this is not the case, since the measure $\nu_c$ has no atoms. Therefore we have to pass to the imbedded Markov chain, as it is done in [S], sect. 5.1.

Consider the service process started in the configuration $(n, \tau)$, with the customer arrival rate $\lambda \equiv c$. Let $\xi_i$ be the number of customers in the system right after the (random) moment $t_i$, when the $i$-th customer was served. We put $\xi_0 = n, t_0 = 0$. Then

$$\xi_{i+1} = \max \{0, \xi_i - 1 + \theta_i\},$$

where $\theta_i$ is the number of customers which came to the system during the $i$-th service session. Then the random variables $\theta_0, \theta_1, \theta_2, \ldots$ are independent, with $\theta_1, \theta_2, \ldots$ identically distributed. The Markov chain $\xi_i$ is ergodic. It is stationary, except for the first step. We denote by $\pi$ its stationary distribution, and by $\pi^{(n,\tau)}$ the distribution of the variable $\xi_1$. We claim that it is enough for our purposes to study this chain. Indeed, if we define the process

$$\tilde{\omega}(t) = \xi_i + 1 \text{ for } t \in (t_{i-1}, t_i],$$

and $\tilde{\mu}(t)$ be its distribution, then we clearly have $\tilde{\mu}(t) \succ \mu(t)$. 63
Let $\pi_1, \pi_2$ be two probability distributions on $\mathbb{N} = \{0, 1, 2, \ldots\}$. As above, we will say that $\pi_1 \succcurlyeq \pi_2$ if there is a coupling $\Pi$ of $\pi_1, \pi_2$, supported by $(\mathbb{N} \times \mathbb{N})^\succcurlyeq = \{(n_1, n_2) \in \mathbb{N} \times \mathbb{N} : n_1 \geq n_2\}$. If $\xi^1_i, \xi^2_i$ are two stationary Markov chains, corresponding to two initial distributions $\pi_1, \pi_2$ at the moment $i = 1$, and if $\pi^1 \succcurlyeq \pi^2$, then also $\xi^1_i \succcurlyeq \xi^2_i$.

Let us show now that for some $\varepsilon > 0$ we have

$$\pi \succcurlyeq \varepsilon \pi^{(n, \tau)} + (1 - \varepsilon) \delta_0.$$  

(91)

This is almost evident. Indeed, in case $t_i - t_{i-1} > \tau$ let us consider the random number $\theta_i^\tau$ of customers arriving to the server during the initial portion $\tau$ of time of the $i$-th service session. Then the event $\{t_i - t_{i-1} > \tau, \theta_i^\tau \geq n\}$ happens with positive probability, while conditioning by this event we have that the conditional distribution of $\xi_i$ is higher than $\pi^{(n, \tau)}$. The relation (91) then follows, since the stationary measure can be computed by averaging over the trajectories.

From Lemma 21 we then know that the convergence in (90) is uniform over $\{(n, \tau) : n \leq N\}$ for every $N$.

The next statement will allow us to treat the unbounded component.

**Lemma 23** There exist $N_0 = N_0(c)$ and $T = T(c)$, such that the following holds:

Let $\nu = \delta_{(n, \tau)}$ denote the initial state, concentrated on the configuration with $n \geq N_0$ customers, with the first one being already served for a time $\tau$. Then

$$N\left(\mu_{\delta_{(n, \tau)}, c}(t)\right) \leq n,$$

(92)

for every $\tau \geq 0$ and every $t \geq T$.

**Proof.** We start with presenting our choice of $N_0$ and $T$. Namely, we take $N_0$ to be any integer bigger than $N(\nu_c)$, while $T$ is defined by the property that for every $\tau$ and every $t \geq T$

$$N\left(\mu_{\delta_{(N_0, \tau)}, c}(t)\right) < N_0.$$

(The existence of $T$ follows from the uniformity of the convergence $N\left(\mu_{\delta_{(N_0, \tau)}, c}(t)\right) \rightarrow N(\nu_c)$ in $\tau$.)

We claim now that for any $n > N_0$, any $\tau$ and any $t \geq T$

$$N\left(\mu_{\delta_{(n, \tau)}, c}(t)\right) < n.$$
To see this let us consider the following auxiliary service discipline: we start in the state \((n, \tau)\), the input rate is \(\lambda \equiv c\), but the server serves the customers only if the queue has more than \(n - N_0\) clients; otherwise the server remains idle. Let us denote the resulting states of the corresponding process by \(\tilde{\mu}_{\delta(n,\tau),c}(t)\). Evidently, \(N\left(\tilde{\mu}_{\delta(n,\tau),c}(t)\right) = N\left(\mu_{\delta(N_0,\tau),c}(t)\right) + n - N_0\). Therefore, for any \(\tau\) and any \(t \geq T\)

\[
N\left(\tilde{\mu}_{\delta(n,\tau),c}(t)\right) < n.
\]

On the other hand, the processes \(\tilde{\mu}_{\delta(n,\tau),c}(\cdot)\) and \(\mu_{\delta(n,\tau),c}(\cdot)\) can be coupled in such a way that with probability one \(N(\tilde{\omega}) \geq N(\omega)\) at all times, so

\[
N\left(\tilde{\mu}_{\delta(n,\tau),c}(t)\right) \leq N\left(\tilde{\mu}_{\delta(n,\tau),c}(t)\right).
\]

Together, the three last lemmas imply the relation (89):

**Lemma 24** For all initial states \(\nu\) with \(N(\nu) < \infty\)

\[
\lim_{t \to \infty} N(\mu_{\nu,c}(t)) = N(\nu_c).
\]

**Proof.** Let \(N_\varepsilon\) be the smallest integer \(k\), satisfying

- \(k > N_0\) (see Lemma 23),
- \(N(\nu^\uparrow) < \varepsilon\), where we denote by \(\nu^\uparrow\) the measure obtained from \(\nu\) by restricting it to the set \(\Omega^\uparrow = \{(n, \tau) : n > k\}\).

Likewise, we define the measure \(\nu^\downarrow\) by \(\nu^\downarrow = \nu - \nu^\uparrow\).

Let us write

\[
\mu_{\nu,c}(t) = \int d\nu(n, \tau) \mu_{\delta(n,\tau),c}(t).
\]

Then

\[
N(\mu_{\nu,c}(t)) = N\left(\int d\nu^\downarrow(n, \tau) \mu_{\delta(n,\tau),c}(t)\right) + N\left(\int d\nu^\uparrow(n, \tau) \mu_{\delta(n,\tau),c}(t)\right).
\]

From compactness we know that

\[
N\left(\int d\nu^\downarrow(n, \tau) \mu_{\delta(n,\tau),c}(t)\right) \to \nu^\downarrow(\Omega) N(\nu_c)\quad\text{as } t \to \infty.
\]
And the relation \( (92) \) tells us that for \( t > T(c) \)

\[
N \left( \int d\nu \uparrow (n, \tau) \mu_{\delta(n, \tau), c} (t) \right) \leq N(\nu^\uparrow).
\]

Hence for all \( t \) large enough

\[
(1 - 2\varepsilon) N(\nu_c) \leq N(\mu_{\nu, c}(t)) \leq N(\nu_c) + \varepsilon.
\]

\[ \blacksquare \]

13.3 End of the proof in noisy case

Let \( \mu_{\nu, \lambda_{\nu}}(\cdot) (t) \) be the non-linear Markov process with the initial state \( \nu \), having finite mean queue, \( N(\nu) < \infty \). We will show that the function \( \lambda(t) \equiv \lambda_{\nu}(t) \) goes to a limit as \( t \to \infty \). The idea is the following:

Suppose \( m = \lim \inf_{t \to \infty} \lambda(t) < \lim \sup_{t \to \infty} \lambda(t) = M. \) As we already know, for every \( T \) and every \( \varepsilon > 0 \) there exist some values \( K_1, K_2 \) such that

\[
\sup_{x \in [K_1, K_1 + T]} \lambda(x) \leq m + \varepsilon, \tag{93}
\]

while

\[
\inf_{x \in [K_2, K_2 + T]} \lambda(x) \geq M - \varepsilon. \tag{94}
\]

We want to bring this to contradiction, arguing as follows:

- First of all, we note that the mean queue, \( N(\mu_{\nu, \lambda_{\nu}}(\cdot)(t)) \) does not change in time, staying equal to the initial value \( N(\nu) \). On the other hand

- We can compare the state \( \mu_{\nu, \lambda_{\nu}}(K_1 + T) \) with the state \( \mu_{\mu_{\nu, \lambda_{\nu}}(\cdot)(K_1), m+\varepsilon} (T) \). Due to \( (93) \), the latter is higher, so

\[
N \left( \mu_{\mu_{\nu, \lambda_{\nu}}(K_1), m+\varepsilon} (T) \right) \geq N(\nu). \tag{95}
\]

By the same reasoning,

\[
N \left( \mu_{\mu_{\nu, \lambda_{\nu}}(K_2), M-\varepsilon} (T) \right) \leq N(\nu). \tag{96}
\]
We then claim that once $T$ is large enough, the state $\mu_{\nu,\lambda_{\nu}}(K_1,m+\epsilon)(T)$ is close to the equilibrium $\nu_{m+\epsilon}$, and moreover

$$N \left( \mu_{\nu,\lambda_{\nu}}(K_1,m+\epsilon)(T) \right) \leq N \left( \nu_{m+\epsilon} \right) + \epsilon'. \quad (97)$$

By the same reasoning,

$$N \left( \mu_{\nu,\lambda_{\nu}}(K_2,M-\epsilon)(T) \right) \geq N \left( \nu_{M-\epsilon} \right) - \epsilon''. \quad (98)$$

Since $N \left( \nu_{M-\epsilon} \right) > N \left( \nu_{m+\epsilon} \right)$ once $\epsilon$ is small, the relations (97) and (98) are inconsistent once $\epsilon'$ and $\epsilon''$ are also small enough.

We need to prove the relations (97) and (98). It turns out that the relation (98) is much easier. Indeed, to show it, we can compare the state $\mu_{\nu,\lambda_{\nu}}(K_2,M-\epsilon)(T)$ with the state $\mu_0,M-\epsilon)(T)$. The latter is evidently lower –

$$N \left( \mu_{\nu,\lambda_{\nu}}(K_2,M-\epsilon)(T) \right) \geq N \left( \mu_0,M-\epsilon)(T) \right),$$

– and as soon as $T$ is large enough, $\mu_0,M-\epsilon)(T)$ is close to $\nu_{M-\epsilon}$. Since $\mu_0,M-\epsilon)(T)$ is also lower than $\nu_{M-\epsilon}$,

$$N \left( \mu_0,M-\epsilon)(T) \right) \leq N \left( \nu_{M-\epsilon} \right). \quad (99)$$

Since $\mu_0,M-\epsilon)(T) \to \nu_{M-\epsilon}$ as $T \to \infty$, (99) implies that $N \left( \mu_0,M-\epsilon)(T) \right) \to N \left( \nu_{M-\epsilon} \right)$, which proves (98).

In the above proof the important step was to replace the state $\mu_{\nu,\lambda_{\nu}}(K_2)$ with a lower state $0$, which is in fact the lowest. Turning to (97), we see that this step can not be mimicked there, since there is no highest state! So, to proceed, we need some a priori upper bound on the state $\mu_{\nu,\lambda_{\nu}}(K_1)$.

**Lemma 25** Let $\nu$ be an arbitrary initial state, with $N(\nu) < \infty$. Then there exist $\bar{c}(\nu) < 1$ and $T < \infty$, such that for every $t > T$

$$\lambda_{\nu}(t) < \bar{c}(\nu).$$

**Proof.** The statement of the lemma is equivalent to the fact that $M = \limsup_{t \to \infty} \lambda(t) < 1$. So suppose the opposite, that $M \geq 1$. As we then know from Lemma 16 for every $T$ and every $\epsilon > 0$ we can find a segment
\([K, K + T]\), such that \(\lambda_\nu (t) > 1 - \varepsilon\) for all \(t \in [K, K + T]\). This, however, contradicts to the statement (25) of Lemma 4. ■

So without loss of generality we can assume that the initial state \(\nu\) is such that \(N (\nu) < \infty\), while \(\lambda_\nu (t) < \bar{c} < 1\) for all \(t > 0\). Clearly, the state \(\mu_{\nu, \lambda_\nu} (t)\) is dominated by \(\mu_{\nu, \bar{c}} (t)\). From the previous section we know that \(N (\mu_{\nu, \bar{c}} (t)) \to N (\nu_{\bar{c}})\) as \(t \to \infty\). Moreover, for any \(\varepsilon > 0\) fixed we know from Lemma 23 that there exist a level \(N (\bar{c}, \varepsilon)\) and a time \(T (\bar{c})\), such that for all \(t > T (\bar{c})\) in the state \(\mu_{\nu, \bar{c}} (t)\) we have:

\[
\sum_{n > N (\bar{c}, \varepsilon)} n \Pr \{N (\omega) = n\} < \varepsilon. \tag{100}
\]

Again we may assume that \(T (\bar{c}) = 0\).

Define now the time duration \(\tilde{T} = \tilde{T} (N (\bar{c}, \varepsilon), m + \varepsilon)\) as such that for every state \(\tilde{\nu}\) on \(\Omega\), supported by configurations \(\{(n, \tau) : n \leq N (\bar{c}, \varepsilon)\}\), and every \(t > \tilde{T}\)

\[
N (\mu_{\tilde{\nu}, m+\varepsilon} (t)) \leq N (\nu_{m+\varepsilon}) + \varepsilon. \tag{101}
\]

(The existence of \(\tilde{T}\) follows from the compactness, as was explained in the preceding section.) As we know from Lemma 10 there exist a moment \(K = K (\tilde{T})\), such that \(\sup_{t \in [K, K + \tilde{T}]} \lambda_\nu (t) \leq m + \varepsilon\). We claim that at the moment \(K + \tilde{T}\) the state \(\mu_{\nu, \lambda_\nu} (K + \tilde{T})\) is not much higher than \(\nu_{m+\varepsilon}\), so in particular

\[
N (\mu_{\nu, \lambda_\nu} (K + \tilde{T})) \leq N (\nu_{m+\varepsilon}) + 2\varepsilon.
\]

Indeed, let us write

\[
\mu_{\nu, \lambda_\nu} (K)
\]

\[
= \mu_{\nu, \lambda_\nu} (K) \big|_{\{(n, \tau) : n \leq N (\bar{c}, \varepsilon)\}} + \mu_{\nu, \lambda_\nu} (K) \big|_{\{(n, \tau) : n > N (\bar{c}, \varepsilon)\}}
\]

\[
\equiv \kappa_1 + \kappa_2.
\]

Then

\[
\mu_{\nu, \lambda_\nu} (K + \tilde{T}) = \mu_{\kappa_1, \tilde{\lambda} (\cdot)} (\tilde{T}) + \mu_{\kappa_2, \tilde{\lambda} (\cdot)} (\tilde{T}),
\]

where \(\tilde{\lambda} (t) = \lambda_\nu (K + t)\). Then for the first summand the relation (101) holds, since the state \(\kappa_1\) relaxes after time \(\tilde{T}\) under “higher dynamics” with the rate \(m + \varepsilon \geq \tilde{\lambda}\), so is very close to \(\kappa_1 (\Omega) \nu_{m+\varepsilon}\). For the second summand the relation (100) holds, since \(\tilde{\lambda} (t) < \bar{c}\). That proves the relation (97).
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