Circular chromatic index of graphs of maximum degree 3

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Abstract

This paper proves that if $G$ is a graph (parallel edges allowed) of maximum degree 3, then
\[ \chi'_c(G) \leq \frac{11}{3} \]
provided that $G$ does not contain $H_1$ or $H_2$ as a subgraph, where $H_1$ and $H_2$ are
obtained by subdividing one edge of $K_3^2$ (the graph with three parallel edges between two vertices)
and $K_4$, respectively. As $\chi'_c(H_1) = \chi'_c(H_2) = 4$, our result implies that there is no graph
$G$ with $11/3 < \chi'_c(G) < 4$. It also implies that if $G$ is a 2-edge connected cubic graph, then
$\chi'(G) \leq 11/3$.

1 Introduction

Graphs considered in this paper may have parallel edges but no loops. Given a graph $G = (V, E)$, and
positive integers $p \geq q$, a $(p, q)$-coloring of $G$ is a mapping $f : V \rightarrow \{0, 1, \ldots, p-1\}$ such that for every
edge $e = xy$ of $G$, $q \leq |f(x) - f(y)| \leq p - q$. The \textit{circular chromatic number} $\chi_c(G)$ of $G$ is defined as
\[ \chi_c(G) = \inf \{ p/q : G \text{ has a } (p, q)\text{-coloring} \}. \]

It is known [4, 6] that for any graph $G$, the infimum in the definition is always attained and
\[ \chi(G) - 1 < \chi_c(G) \leq \chi(G). \]

For a graph $G = (V, E)$, the \textit{line graph} $L(G)$ of $G$ has vertex set $E$, in which $e_1 \sim e_2$, if $e_1$ and $e_2$ have
an end vertex in common. The \textit{circular chromatic index} $\chi'_c(G)$ of $G$ is defined as
\[ \chi'_c(G) = \chi_c(L(G)). \]

Recall that the \textit{chromatic index} $\chi'(G)$ of $G$ is defined as $\chi'(G) = \chi(L(G))$. So we have
\[ \chi'(G) - 1 < \chi'_c(G) \leq \chi'(G). \]

If $G$ is connected and $\Delta(G) = 2$, then $G$ is either a cycle or a path. This implies that either $\chi'_c(G) = 2$
or $\chi'_c(G) = 2 + \frac{1}{k}$ for some positive integer $k$. Since graphs $G$ with $\Delta(G) \geq 3$ have $\chi'_c(G) \geq 3$, ‘most’
of the rational numbers in the interval $(2, 3)$ are not the circular chromatic index of any graph. The
following question was asked in [4]:

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Question 1.1 For which rational \( r \geq 3 \), there is a graph \( G \) with circular chromatic index \( r \)? In particular, is it true that for any rational \( r \geq 3 \), there is a graph \( G \) with \( \chi'_c(G) = r \)?

If \( 3 < \chi'_c(G) < 4 \), then \( G \) has maximum degree 3. It is well-known that the Four Color Theorem is equivalent to the statement that every 2-edge connected cubic planar graph \( G \) has \( \chi'_c(G) = 3 \). For nonplanar 2-edge connected cubic graphs, Jaeger [2] (see also page 197 of [3]) proposed the following conjecture (Petersen Coloring Conjecture):

Conjecture 1.2 If \( G \) is a 2-edge connected cubic graph, then one can color the edges of \( G \), using the edges of the Petersen graph as colors, in such a way that any three mutually adjacent edges of \( G \) are colored by three edges that are mutually adjacent in the Petersen graph.

Since the Petersen graph has circular chromatic index 11/3, Conjecture 1.2 would imply that every 2-edge connected cubic graph \( G \) has \( \chi'_c(G) \leq 11/3 \). The following two open problems are proposed in [6]:

Question 1.3 Prove that if \( G \) is a 2-edge connected cubic planar graph, then \( \chi'_c(G) < 4 \), without using the Four Color Theorem.

Question 1.4 Are there any 2-edge connected cubic graph \( G \) with \( \chi'_c(G) = 4 \)?

This paper proves the following result:

Theorem 1.5 Let \( H_1 \) and \( H_2 \) be the graphs as shown in Figure 1. If \( G \) is graph of maximum degree 3 and \( G \) does not contain \( H_1 \) or \( H_2 \) as a subgraph, then \( \chi'_c(G) \leq 11/3 \).

It is easy to verify that \( \chi'_c(H_1) = \chi'_c(H_2) = 4 \). Since graphs \( G \) with \( \Delta(G) \geq 4 \) have \( \chi'_c(G) \geq 4 \), we have the following corollary:

Corollary 1.6 There is no graph \( G \) with \( 11/3 < \chi'_c(G) < 4 \).

Corollary 1.6 answers the second part of Question 1.3 in the negative.

To prove Theorem 1.5, it suffices to consider 2-edge connected graphs. Indeed, if a graph \( G \) is not 2-edge connected, say \( e \) is a cut edge of \( G \), then either \( e \) is a hanging edge, i.e., incident to a degree 1 vertex, or \( e \) is a cut vertex in \( L(G) \). In the latter case, \( \chi'_c(L(G)) = \max\{\chi'_c(B) : B \) is a block of \( L(G)\}\). If \( e \) is a hanging edge of \( G \), then \( e \) has degree at most 2 in \( L(G) \), and hence any \((11,3)\)-coloring of \( L(G) - e \) can be extended to a \((11,3)\)-coloring of \( L(G) \). In the remainder of this paper, we assume that \( G \) is 2-edge connected and hence has minimum degree at least 2. It is easy to see that if \( G \) is 2-edge connected and has maximum degree at most 3, then \( G \) cannot contain \( H_1 \) or \( H_2 \) as a proper subgraph.

Therefore Theorem 1.5 is equivalent to the following:

Theorem 1.7 Suppose \( G \) is 2-edge connected and has maximum degree 3. If \( G \neq H_1, H_2 \), then \( \chi'_c(G) \leq 11/3 \).

Theorem 1.7 implies the following corollary, which answers Questions 1.3 and 1.4.

Corollary 1.8 The circular edge chromatic number of every 2-edge connected cubic graph \( G \) is less than or equal to 11/3.
2  Cubic chromatic index

The remainder of the paper is devoted to the proof of Theorem 4. In this section, we consider triangle free cubic graphs. First we prove a lemma which is needed in our proof.

Suppose $c$ is a $k$-coloring of a graph $G = (V, E)$ with colors $0, 1, \ldots, k - 1$. If $xy$ is an edge of $G$ and $c(y) = c(x) + 1 \pmod{k}$, then we say $xy$ is a tight arc with respect to $c$. Let $A$ be the set of tight arcs, and let $D_c(G) = (V, A)$, which is a directed graph with vertex set $V$. It is known \cite{1, 6} that if there is a $k$-coloring $c$ of $G$ for which $D_c(G)$ is acyclic, then $\chi_c(G) < k$. The following lemma is a strengthening of this result.

Lemma 2.1 Let $c$ be a $k$-coloring of a graph $G$ with colors $0, 1, \ldots, k - 1$, where $k > 2$. If $D_c(G)$ is acyclic and each directed path of $D_c(G)$ contains at most $n$ vertices of color $k - 1$, then $\chi_c(G) \leq \frac{k - 1}{n + 1}$.

Proof. Let $p = k(n + 1) - 1$ and $q = n + 1$. It suffices to give an $(p, q)$-coloring for $G$. For each vertex $v$ of $G$, let $l(v)$ be the maximum number of vertices with color $k - 1$ on a directed path of $D_c(G)$ which ends in $v$, without considering $v$ itself. We claim that the coloring $c'$ defined as

$$c'(v) = (c(v)q + l(v)) \mod p$$

is a proper $(p, q)$-coloring of $G$. Consider two adjacent vertices $u$ and $v$. If $2 \leq |c(u) - c(v)| \leq k - 2$, then since both $l(u)$ and $l(v)$ are less than $q$, we have $q \leq |c'(u) - c'(v)| \leq p - q$. If $c(u) - c(v) = 1$, then $vu$ is a tight arc and hence $l(u) \geq l(v)$. So we have $q \leq |c'(u) - c'(v)| \leq p - q$. Finally, if $c(u) = 0$ and $c(v) = k - 1$, then $vu$ is a tight arc and $l(u) \geq l(v) + 1$. Again we have $q \leq |c'(u) - c'(v)| \leq p - q$.

Suppose $c$ is a $k$-edge coloring of $G$ and $e = xy$ is an edge of $G$. The two arcs $xy'$ and $y'x$ are called arcs corresponding to $e$. We say an arc $xy'$ is unblocked with respect to $c$, if there is a directed walk $W = (e_1, e_2, \ldots, e_n, e, e_1', e_2', \ldots, e_n')$ in $D_c(L(G))$ such that (i) $c'(e_1) = c'(e_n') = k - 1$, and (ii) $e_n = x'x$ and $e_1' = yy'$. The arc $xy'$ is blocked with respect to $c$ if no such directed walk exists. An edge $e = xy$ is said to be blocked in the direction $x \rightarrow y$ with respect to $c$, if the arc $xy'$ is blocked. An edge $e = xy$ is completely blocked with respect to $c$, if both arcs $xy'$ and $y'x$ are blocked. Given a partial $k$-edge coloring $c'$ of $G$ (i.e., $c'$ colors a subset of edges of $G$), we say an arc $xy'$ is unblocked with respect to $c'$, if $c'$ can be extended to a $k$-edge coloring $c$ of $G$ such that $xy'$ is unblocked with respect to $c$. If no such extension exists, then we say $xy'$ is blocked with respect to $c'$. Similarly, we say an edge $e$ is completely blocked with respect to $c'$, if both arcs $xy'$ and $y'x$ are blocked with respect to $c'$.

Theorem 2.2 If $G$ is a cubic graph of girth at least 4 and has a perfect matching, then $\chi_c'(G) \leq 11/3$.

Proof. By Lemma 2.1 it suffices to prove that there exists a 4-edge coloring $\phi$ of $G$ such that $D_\phi(L(G))$ is acyclic and each directed path of $D_\phi(L(G))$ contains at most two vertices (i.e., two edges of $G$) which are colored by 3.

Let $M$ be a perfect matching of $G$. Then $G - M$ is a collection of cycles. A 4-edge coloring of $G$ is called a valid coloring with respect to $M$, if the following hold:

- All the $M$-edges (an edge in $M$ is called an $M$-edge) are colored by color 0.

- The edges of any even cycle $C$ of $G - M$ are colored by colors 1 and 2.

- The edges of any odd cycle $C$ of $G - M$ are colored by colors 1 and 2, except one edge which is colored by color 3.

Let $c'$ be a partial 4-edge coloring of $G$ which can be extended to a valid 4-edge coloring of $G$ with respect to $M$. We are interested in the blocked directions of the $M$-edges with respect to $c'$. Suppose $e = xy$ is an $M$-edge, and $C$ and $C'$ (not necessarily different) are cycles of $G - M$ such that $x \in V(C)$ and $y \in V(C')$. If $xy'$ is an unblocked arc with respect to $c'$, then we say $xy'$ is an input of $C'$ and an output of $C$ with respect to $c'$.
Let $C$ be a cycle of $G - M$, and let $c_C$ be the partial edge coloring of $G$ which is the restriction of a valid coloring $c$ to $M \cup C$. If $C$ is an even cycle, then it is easy to see that every edge $e \in M$ incident to $C$ is completely blocked with respect to $c_C$. If $C$ is an odd cycle of $G - M$, then Figure 2 shows the blocked directions of the $M$-edges incident to $C$ with respect to $c_C$.

In Figure 2, a thick edge indicates an $M$-edge. An arrow on an $M$-edge indicates a blocked direction of that edge. An $M$-edge with opposite arrows is completely blocked. Since $G$ has girth at least 4, the four vertices $v_1, v_2, v_3, v_4$ as indicated in Figure 2 are distinct. Note that an $M$-edge $e$ incident to $C$ is completely blocked with respect to $c_C$, unless $e$ is incident to one of the vertices $v_1, v_2, v_3, v_4$, which are the vertices on a path whose edges are colored by colors 1, 2, 3. So there are at most 4 $M$-edges incident to $C$ that are not completely blocked. An $M$-edge incident to $C$ could be a chord of $C$. If an $M$-edge $e$ incident to $v_1, v_2, v_3, v_4$ is a chord of $C$, then $e$ could be completely blocked. We will discuss this case later in more detail. If an $M$-edge $e$ incident to $C$ is not completely blocked with respect to $c_C$, then exactly one direction of $e$ is blocked.

For a valid 4-edge coloring $c$ of $G$, let $\phi(c)$ be the total number of not completely blocked $M$-edges. Let $\psi(c)$ be the number of not completely blocked $M$-edges that are chords of cycles of $G - M$.

**Claim 2.3** Suppose $c$ is a valid 4-edge coloring of $G$ (with respect to a perfect matching $M$). If $G - M$ has a cycle $C$ which has an input as well as an output, then there is a valid 4-edge coloring $c^*$ of $G$ for which $\phi(c^*) + \psi(c^*) < \phi(c) + \psi(c)$.

**Proof.** Assume $C$ is a cycle of $G - M$ which has an input as well as an output with respect to a valid 4-edge coloring $c$. Then $C$ is an odd cycle and the $M$-edges incident to $C$ contributes at least 2 to the summation $\phi(c) + \psi(c)$. We shall construct a valid 4-edge coloring $c^*$ of $G$ such that each $M$-edge not incident to $C$ contributes the same amount to $\phi(c^*) + \psi(c^*)$ and $\phi(c) + \psi(c)$. However, the $M$-edges incident to $C$ contributes at most 1 to the summation $\phi(c^*) + \psi(c^*)$.

Uncolor the edges of $C$ to obtain a partial 4-edge coloring $c'$ of $G$. The valid 4-edge coloring we shall construct is an extension of $c'$. It is obvious that for any valid 4-edge coloring $c'$ of $G$ which is an extension of $c'$, each $M$-edge not incident to $C$ contributes the same amount to $\phi(c^*) + \psi(c^*)$ and $\phi(c) + \psi(c)$. So we only need to make sure that the $M$-edges incident to $C$ contribute at most 1 to the summation $\phi(c^*) + \psi(c^*)$.

First we consider the case that $C$ has no chord. As each $M$-edge $e$ incident to $C$ is incident to another cycle of $G - M$, at least one direction of $e$ is blocked with respect to $c'$. Since $C$ is an odd cycle and $C$ has an input and an output with respect to $c$, it is easy to see that there are four consecutive vertices $v_1, v_2, v_3, v_4$ of $C$ such that with respect to the partial edge coloring $c'$, the $M$-edges incident to $v_1, v_2$ have a common blocked direction (i.e., either both are blocked in the direction towards $C$ or both are blocked in the direction away from $C$), and the $M$-edges incident to $v_3, v_4$ have an opposite
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block direction. Depending on which directions of the four edges are blocked, there are four cases as depicted in Figure 3.

![Figure 3: The blocked directions of M-edges incident to the uncolored cycle C of G − M](image)

We use the following convention to interpret Figure 3 and the figures in the remaining of the paper: An M-edge without an arrow could be completely blocked, or blocked in one direction, or unblocked in both directions. An M-edge with one arrow means that the indicated direction of that edge is blocked, but the other direction of that edge could be blocked or unblocked. An M-edge with a pair of opposite arrows means that edge is completely blocked.

Consider the case indicated in Figure 3 (a) and 3 (b). We extend $c'$ to a valid 4-edge coloring $c^*$ of $G$ by letting $c^*(e_1) = 3, c^*(e_2) = 2, c^*(e_3) = 1$ (the other edges of $C$ are colored by 1 and 2 alternately). It is easy to verify that in the case indicated in Figure 3(a), $e_7$ is the only edge which is probably not completely blocked with respect to $c^*$. In Figure 3(b), $e_6$ is the only edge which is probably not completely blocked. Thus the M-edges incident to $C$ contributes at most 1 to the summation $\phi(c^*) + \psi(c^*)$.

For the cases in Figure 3(c) and 3(d), let $c^*(e_1) = 1, c^*(e_2) = 2, c^*(e_3) = 3$. Then the M-edges incident to $C$ contributes at most 1 to the summation $\phi(c^*) + \psi(c^*)$.

Next we consider the case that $C$ has a chord.

Since $C$ is an odd cycle, there is an M-edge incident to $C$ which is not a chord of $C$. So there is a vertex $v_2$ of $C$ which is incident to a chord of $C$ and a neighbour $v_1$ of $v_2$ in $C$ is not incident to a chord of $C$. Let $v_3, v_4$ be the vertices of $C$ following $v_1, v_2$ (as shown in Figure 4).

Assume the M-edges incident to $v_3, v_4$ are not chords of $C$ and have a common blocked direction, as shown in Figure 4(a) or 4(b). In the case as shown in Figure 4(a), extend $c'$ to $c^*$ by letting $c^*(e_1) = 1, c^*(e_2) = 2, c^*(e_3) = 3$ (and color the other edges of $C$ alternately by colors 1 and 2). In the case as shown in Figure 4(b), extend $c'$ to $c^*$ by letting $c^*(e_1) = 3, c^*(e_2) = 2, c^*(e_3) = 1$. In any case, it is easy to verify that all the chords of $C$ are completely blocked, and there is at most one M-edge incident to $C$ which is not completely blocked.

Assume the M-edges incident to $v_3, v_4$ have opposite blocked directions or at least one of the M-edges incident to $v_3, v_4$ is a chord of $C$. Then depending on which direction of the M-edge incident to $v_1$ is blocked (with respect to $c'$), we color the edges as in Figure 5.

In each of the colorings, it is straightforward to verify that the M-edges incident to $C$ contribute at
most 1 to the summation \( \phi(c^*) + \psi(c^*) \). This completes the proof of Claim 2.3.

Now we choose a valid 4-edge coloring \( c \) of \( G \) such that \( \phi(c) + \psi(c) \) is minimum. By Claim 2.3, no cycle \( C \) of \( G - M \) has an input and an output. Since each cycle \( C \) of \( G - M \) contains at most one edge of color 3, it follows that every directed path of \( D_c(L(G)) \) contains at most 2 vertices (i.e., edges of \( G \)) with color 3. By Lemma 2.1, \( \chi_c'(L(G)) = \chi'_c(G) \leq \frac{11}{3} \).

**Corollary 2.4** If \( G \) is a 2-edge connected graph of maximum degree 3 and has girth at least 4, then \( \chi'_c(G) \leq \frac{11}{3} \).

**Proof.** If \( G \) is cubic, then by Petersen Theorem, \( G \) has a perfect matching. Otherwise, take the disjoint union of two copies of \( G \), say \( G \) and \( G' \). For each degree 2 vertex \( x \) of \( G \), connect \( x \) to the corresponding vertex \( x' \) in \( G' \) by an edge. The resulting graph \( G'' \) is cubic (as \( G \) has minimum degree 2) and is either 2-edge connected (if \( G \) has at least two degree 2 vertices), or has exactly one cut edge. In any case \( G'' \) has a perfect matching (see for example [5], page 124) and has girth at least 4. Hence \( \chi'_c(G'') \leq \frac{11}{3} \) by Theorem 2.2.

## 3 Proof of Theorem 1.7

We prove Theorem 1.7 by induction on the number of edges. If \( |E(G)| = 3 \), then it is equal to \( K_3^3 \), and has circular chromatic index 3. Assume \( |E(G)| \geq 4 \) and \( G \neq H_1, H_2 \). If \( G \) has girth at least 4, then the conclusion follows from Theorem 2.2. Thus we assume that \( G \) has a pair of parallel edges or has a triangle.

**Case I:** Suppose there is a pair of parallel edges between \( u \) and \( v \). Since \( G \) is 2-edge connected and \( G \neq H_1 \), we conclude that \( u \) is connected to another vertex \( u' \), \( v \) is connected to another vertex \( v' \), and \( u' \neq v' \). Let \( G \bowtie uv \) be the graph obtained from \( G \) by deleting the two vertices \( u \) and \( v \) from \( G \) and adding an edge between \( u'v' \). Note that this new edge may cause a multiple edge between \( u' \) and \( v' \). If \( G \bowtie uv \notin \{H_1, H_2\} \), then by induction hypothesis, \( \chi'_c(G \bowtie uv) \leq \frac{11}{3} \). Figure 6(a) illustrates that
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(a): 

(b): 

(c): 

Figure 6: (a), (b), and (c) show that how a (11/3)-edge coloring of the new graph leads to a (11, 3)-edge coloring of the previous one: (a): In the (11, 3)-edge coloring of the main graph $b = (a + 3) \mod 11$ and $c = (a + 6) \mod 11$, (b): contracting a triangle with three vertices of degree 3, (c): after contracting a triangle with one vertex of degree 2, we can always find a color $c$ to complete the (11, 3)-coloring of the old graph.

A (11, 3)-coloring of $L(G \odot uv)$ can be ‘extended’ to a (11, 3)-coloring of $L(G)$. If $G \odot uv \in \{H_1, H_2\}$, then $G$ is one of the graphs illustrated in Figure 7 or Figure 8 where a (7, 2)-coloring of $L(G)$ is given.

**Case II:** Suppose $G$ has a triangle $uvw$. Since $G$ is 2-edge connected and $G \neq H_1$, there are no multiple edges in this triangle. Let $G \odot uvw$ be the graph obtained from $G$ by contracting the triangle $uvw$ in $G$ to a new vertex. If $G \odot uvw \notin \{H_1, H_2\}$, then by induction hypothesis, $\chi'_c(G \odot uvw) \leq 11/3$.

Figure 8(b,c) illustrates that a (11, 3)-coloring of $L(G \odot uvw)$ can be ‘extended’ to a (11, 3)-coloring of $L(G)$. If $G \odot uvw \in \{H_1, H_2\}$, then $G$ is one of the graphs illustrated in Figure 8 or Figure 8 where a (7, 2)-coloring of $L(G)$ is given. So in any case, $\chi'_c(G) \leq 11/3$. This completes the proof of Theorem 1.7.

Based on the result in this paper, we propose the following conjecture:

**Conjecture 3.1** For any integer $k \geq 2$, there is an $\epsilon > 0$ such that the open interval $(k - \epsilon, k)$ is a gap for circular chromatic index of graphs, i.e., no graph $G$ has $k - \epsilon < \chi'_c(G) < k$.

If Conjecture 3.1 is true, then let $\epsilon_k$ be the largest real number for which $(k - \epsilon_k, k)$ is a gap for the circular chromatic index of graphs. The next problem would be to determine the value of $\epsilon_k$. For
Figure 8: The graphs that can be converted to $H_2$ by the “⊙” operation. For each graph a $(7, 2)$-edge coloring is given.

$k = 2, 3, 4$, Conjecture 3.1 is true and we know that $\epsilon_2 = 1, \epsilon_3 = 1/2$ and $\epsilon_4 = 1/3$. So a natural guess for $\epsilon_k$ is that $\epsilon_k = 1/(k - 1)$. However, at present time, support for such a conjecture is still weak. For $k \geq 4$, we do not have natural candidate graphs $G$ with $\chi'_c(G) = k - 1/(k - 1)$.

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