Dynamics of hot random quantum spin chains: from anyons to Heisenberg spins

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We argue that the dynamics of the random-bond Heisenberg spin chain are ergodic at infinite temperature, in contrast to the many-body localized behavior seen in its random-field counterpart. First, we show that excited-state real-space renormalization group (RSRG-X) techniques suffer from a fatal breakdown of perturbation theory due to the proliferation of large effective spins that grow without bound. We repair this problem by deforming the SU(2) symmetry of the Heisenberg chain to its ‘anyonic’ version, SU(2)k, where the growth of effective spins is truncated at spin S = k/2. This enables us to construct a self-consistent RSRG-X scheme that is particularly simple at infinite temperature. Solving the flow equations, we compute the excited-state entanglement and show that it crosses over from volume-law to logarithmic scaling at a length scale ξk ∼ eπk3. This reveals that (a) anyon chains have random-singlet-like excited states for any finite k; and (b) ergodicity is restored in the Heisenberg limit k → ∞.

Quantum spin systems are central to condensed matter physics, underpinning aspects of the field as diverse as the theory of quantum critical phenomena [1] to the role of topology [2] and symmetry [3] in delineating zero-temperature phases of matter. Much is therefore known about their ground states and low-lying spectra. More recently, spurred in part by the ability to experimentally probe such systems in the absence of external sources of equilibration [4], there has been growing interest in understanding whether they can thermalize in isolation [5, 6] — an issue dictated by the behavior of excited states at finite energy density. Although most isolated quantum many-body systems ‘self-thermalize’, acting as their own heat bath, a handful of examples – typically one-dimensional systems with quenched randomness [7–14] – instead exhibit non-ergodic dynamics and quantum glassiness. These athermal phases violate many standard tenets of statistical mechanics, often with remarkable consequences, such as ordering below the lower critical dimension [15, 16] and quantum coherent dynamics at infinite effective temperature [17], lending added impetus to their study.

In this paper, we focus on the infinite-temperature dynamics of the random-bond Heisenberg model, described by the Hamiltonian

$$H = \sum_i J_i S_i \cdot S_{i+1},$$

(1)

where the $J_i$ are independent random variables, and we restrict to $d = 1$ throughout. We begin with spins $S_i = 1/2$, but as we shall see, a consistent finite-temperature treatment requires allowing $S_i$ to take any value. The Heisenberg chain with uniform couplings but strongly random fields is many-body localized (MBL) and hence non-ergodic [9]; what about the random-bond case?

In order to investigate the dynamics and address the question of thermalization, we focus on the spatial dependence of entanglement entropy in excited eigenstates of (1). Specifically, we define a measure of the entanglement entropy of a subsystem $A$ of size $L$ at effective temperature $T$,

$$S_A(T) = \frac{\sum_a e^{-E_a/T} S_A(\langle \Psi_a \rangle)}{\sum_a e^{-E_a/T}},$$

(2)

though we expect any suitable average over choice of sub-region, disorder configuration, or eigenstates $|\Psi_a\rangle$ to give similar characterizations. Here, $S_A(\langle \Psi_a \rangle) = -Tr[\rho_A \log \rho_A]$ is the entanglement entropy of subsystem $A$ in state $|\Psi_a\rangle$ where $\rho_A = Tr_{\bar{A}}[\rho_A(\langle \Psi_a \rangle)/\langle \Psi_a \rangle]$ is the corresponding reduced density matrix of $A$ obtained by tracing over its complement, $\bar{A}$. Eigenstate thermalization requires that $S_A(T)$ scale with the volume of $A$, $S_A(T) \sim L$. Sub-volume scaling indicates non-ergodic dynamics implying the system does not self-thermalize in isolation.

The ground state properties of such random-bond spin chains have been exhaustively studied using a real-space renormalization group (RSRG) procedure [18–22] that decimates couplings in a hierarchical fashion in which strong bonds are eliminated before weaker ones, ‘fusing’ nearest-neighbor spins into singlets or single effective spins. In the case of purely antiferromagnetic bonds, $J_i > 0$, this yields a flow of the bond strength distribution towards infinite randomness. The resulting low-energy behavior is governed by the so-called random singlet fixed point [21], which corresponds to logarithmic scaling of $S_A(T \to 0)$, when the sum in (2) is restricted to the ground state. Our strategy here is to adapt RSRG techniques to calculate $S_A(T)$ in the opposite limit of $T \to \infty$, to shed light on the dynamics of (1).

To this end, it is instructive to recall that ground state RSRG has also been applied to other systems with lower symmetry, including the random transverse-field Ising model [20]. Ref. [14] considered this simpler prob-
lem, and generalized the ground state RSRG technique to study the entire many-body spectrum by observing that at each step in the decimation procedure, it is possible to project into the excited-state manifold(s) rather than to the ground state. This yields a choice of $M$ possible decimations for each bond (for instance, $M = 2$ for the random transverse-field Ising model [13]). Formally, following each choice leads to a rapidly branching 'spectral tree' of possible decimation paths, yielding $M^{n_b}$ approximate eigenstates once all the $n_b$ bonds have been decimated. The exponentially difficult task of constructing all these possible states can be circumvented by Monte Carlo sampling the spectral tree [13] when performing thermal averages. As the RG proceeds, the characteristic energy gap shrinks, and therefore the remaining spins dominate dynamics on increasingly long time scales. This connects excited-state RSRG (RSRG-X) to a related approach where the decimation is implemented directly on the dynamics by 'integrating out' fast spins [10].

There is a prima facie obstacle to directly applying RSRG-X to the random-bond Heisenberg model [1]: namely, that due to the $SU(2)$ symmetry, one of the outcomes of decimating a pair of spins-1/2 coupled by a strong bond is that an effective spin-1 'superspin' is generated. Following the RSRG philosophy, we should compute effective couplings between this new effective spin-1 and its neighbors. As we began with a set of spin-1/2 objects, the Heisenberg Hamiltonian is not self-similar under the RG scheme. We address this complication by considering the on-site spin as an additional random variable, and tracking the RG flow of the spin distribution, in addition to that of the bond strengths (see also [23] [24]). The most general decimation step now involves a strong bond between two arbitrary spins $S_1, S_2$ that fuse to yield a new spin $S \in \{|S_1 - S_2|, \ldots, S_1 + S_2\}$. Consider the Heisenberg Hamiltonian for four successive spins $S_L, S_1, S_2, S_R$, given by $H = J_L S_L \cdot S_1 + J_L S_2 \cdot S_R + J_S S_1 \cdot S_R$, where we assume $J_S \gg J_L, J_R$. Except in the case when two identical spins fuse to a singlet, the effective coupling is obtained by first-order perturbation theory and is $H_{\text{eff}} = \tilde{J}_L S_L \cdot S + \tilde{J}_R S \cdot S_R$, with $\tilde{J}_{L,R} = \frac{S(S+1)\pm S_1(S_1+1)\mp S_2(S_2+1)}{2S(S+1)} J_{L,R}$ (Fig. 1(a)). In the case when $S_1 = S_2$ and they form a singlet (Fig. 1(b)), a second-order calculation gives $H_{\text{eff}} = J_{\text{eff}} S_L \cdot S_R$, with $J_{\text{eff}} = \frac{S(S+1)}{2} J_{L,R}$. That the resulting RSRG-X procedure is fatally flawed is especially clear at infinite temperature, where we can sample the tree at random, picking each of the fusion outcomes $S$ at each decimation step with a probability proportional to its degeneracy, i.e., $2S + 1$. The resulting flow is straightforward to determine numerically, but we sketch the intuition first. Under the random decimation, we will certainly generate some fraction of large spins. Qualitatively, the formation of increasingly large spins apparently leads to a breakdown of the RG, since a large spin leads to a large effective coupling, contrary to the perturbative justification given for the decimation. However, it is possible that the flow is such that the combination of the spin and the bare couplings that give the effective coupling remains small.

To clarify the situation, we simulated the infinite-temperature RG procedure on a $10^3$-site chain, holding the length fixed by replacing decimated spin(s). The extra spin(s) are added at a distance $L/2$ from the strong bond to avoid any accidental correlations between them and the bond being decimated. The new spins and their couplings are chosen randomly from the spins in the previous distribution, excluding those on or adjacent to the strong bond being fused. For large systems, one expects that these are representative of the true distributions so that this process approximates sampling the latter. If spin $S_1$ is added from the previous RG step, we also make one of its bonds $J_{L,R}$ to roughly preserve the correlations between spin size and bond strength.

Fig. 1(c) shows the evolution of mean spin size, $\bar{S}$, and error in neglecting higher-order contributions to $J_{\text{eff}}$, $\tilde{J}_{L,R}$ over the course of $10^3$ RG steps. Here we define the error for a given RG step as $J_{\text{err}} J_{L,R}$ when two spins are fused to a singlet or non-vanishing spin respectively. The data clearly show that the unbounded growth of spin size produces a breakdown in the validity of the RSRG-X scheme. It is natural to expect a resulting delocalized thermal phase, since the RG breakdown is caused by a proliferation of resonant bonds (with $J_{L,R} \sim J_S$) that can coherently transport energy.

To gain further insight into the fate of the high-temperature Heisenberg chain, consider running the RSRG-X procedure until it just before it begins to break down. This results in an effective Heisenberg model with a broad distribution of large superspins with weak
residual couplings. These large spins behave essentially classically, since the quantum mechanical uncertainty in spin orientation for a spin-$S$ object is suppressed as $1/S$. Conventional wisdom holds that one dimensional spin systems with local interactions (particularly those with isotropic continuous spins) cannot form a glassy state with non-ergodic dynamics [25], further supporting the hypothesis that the random-bond Heisenberg chain thermalizes at high energy density.

More generally, we expect the spectrum of any system that fails to thermalize because of disorder to be characterized by a set of approximately local conserved quantities [26] — either exponentially localized for MBL systems or algebraically so in critically localized ones [30]. For infinitely random critical systems, the local integrability will be captured by RSRG flows towards infinite-randomness fixed points [10] — so the breakdown of RSRG-X points to ergodicity.

From the preceding discussion, it is clear that the unbounded growth of spins under RSRG-X leads inevitably to the breakdown of perturbation theory. It is tempting to control this growth by simply imposing a cutoff so that spins above a certain $S_{\text{max}}$ are discarded. However, it is straightforward to see that this leads to inconsistencies in the ‘fusion rules’ for angular momentum addition — for instance, it leads to non-associative results in combining triplets of spins. Instead, we may truncate $SU(2)$ in a controlled way by “deforming” the group to its quantum version, $SU(2)_k$. The irreducible representations of $SU(2)_k$ are labeled by their “spin” $j = 0, \frac{1}{2}, 1, \ldots, \frac{k}{2}$, and obey modified fusion rules: $j_1 \otimes j_2 = \sum_{j} \delta(j-|j_1-j_2|) \oplus \cdots \oplus \delta(j_1+j_2,k-(j_1+j_2))$. Such $SU(2)_k$ algebras arise naturally in different contexts, but the one most relevant for our purposes is in describing the fusion of anyonic quasiparticles proposed as a basis for topological quantum computation [31]. Consider a one-dimensional chain of such ‘anyons’ [32] with random couplings [33] [35], each with topological charge $S = \frac{1}{2}$. The Hilbert space $\mathcal{H}$ of the chain is then spanned by the basis states $|\ldots j_{i-1} j_i \ldots \rangle$ where $j_{i-1}, j_i \in \{0, \frac{1}{2}, 1, \ldots, \frac{k}{2}\}$ and $j_i$ is contained in the fusion product of $j_{i-1}$ with $S = \frac{1}{2}$. Owing to the truncated fusion rules, $\mathcal{H}$ for $N$ anyons cannot be written as a tensor product of local Hilbert spaces, in contrast to conventional $SU(2)$ spins.

As in the Heisenberg case, it is necessary to work with generalizations of this truncated spin-$\frac{1}{2}$ chain that allow the spin $S_i$ on each site to take any value in $\{\frac{1}{2}, 1, \ldots, \frac{k}{2}\}$, and consider configurations in which $j_i \in j_{i-1} \otimes S_i$. We work with the Hamiltonian introduced in Ref. [36]

$$H = \sum_i J_i \hat{Q}_i = \sum_i \sum_{S \in S_i \otimes S_{i+1}} A_i(S) \hat{P}_S$$

where $\hat{P}_S$ is the projector onto the fusion channel $S$ in the fusion $S_i \otimes S_{i+1}$, that can be expressed in the basis introduced above using “$F$-moves”. The different fusion channels are weighted by $A_i(S) = \frac{1}{2} (|S|^2 + (S+1)^2 - (|S_i+1 - S_i||S_i+1 + S_i|)^2) - (|S_i+1 - S_i||S_i+1 + S_i|)^2$, where \{x\} = $\sin\left(\frac{\pi x}{\gamma N}\right) / \sin\left(\frac{\pi}{\gamma N}\right)$. Eq. (1) recovers the Heisenberg chain as $k \to \infty$, since $S_1 \cdot S_2 = \frac{1}{2} \sum_{S \in S_1 \otimes S_2} [S(S+1) - S_1(S_1+1) - S_2(S_2+1)] \hat{P}_S$. Crucially, Hamiltonian [3] is invariant under RSRG-X transformations. Since the maximum spin size is bounded by $\frac{k}{2}$, for strong enough initial disorder the RSRG-X scheme flows to strong randomness and becomes asymptotically exact. We will analyze the properties of this fixed point at infinite temperature $T = \infty$ for finite $k$, and then take the Heisenberg limit $k \to \infty$.

Starting from [3], we can derive the RSRG-X decimation rules explicitly and construct the spectral tree of eigenstates [37]. At infinite temperature a drastic simplification occurs: eigenstates are equiprobable, so we can construct an average RSRG-X eigenstate by decimating bonds hierarchically and choosing a fusion outcome at each node of the spectral tree with a probability proportional to the number of branches descending from this node, given by the “quantum dimension” $d_S \equiv \{2S+1\}_k$ (analogous to the $2S+1$ degeneracy of a Heisenberg spin $S$). Crucially, the RG flow of the spin distribution decouples from that of the bonds, enabling a simple computation of eigenstate entanglement. This is in marked contrast to the Monte Carlo sampling required at finite temperature, which entails performing all decimations (involving both spin and coupling distributions) leading to a single eigenstate in order to implement the Metropolis algorithm [13].
Let \( \ell \) be the RG depth, defined via \( \ell \equiv \ln \frac{N_i}{N_f} \) where \( N_i \) and \( N_f \) are the number of sites before and after the RG decimations, respectively. By examining the distribution of undecimated (super-)spins, \( \mathcal{P}(S, \ell) \), and change in total number of spins upon fusing two spins either into a superspin or a singlet \([38]\), we find that at RG depth \( \ell \),

\[
\frac{d\mathcal{P}(S)}{d\ell} = \frac{1}{1 + \Pi_0(\ell)} \left[ K_S(\ell) - \mathcal{P}(S, \ell)(1 - \Pi_0(\ell)) \right], \tag{4}
\]

where \( K_S(\ell) = \sum_{S_1, S_2 \neq 0} \frac{d\mathcal{P}(S_1)\mathcal{P}(S_2)}{dS_1 dS_2} \delta_{S_1 \otimes S_2} \) is the weighted probability of generating spin \( S \) \([38]\), and \( \Pi_0 = K_0 + k_{2/2} \) is the probability of generating a singlet (note that \( k/2 \) is also an \( SU(2)_k \) singlet). The spin distribution \( \mathcal{P}^*(S) \) at the fixed point is thus given by \( K_S^* = \mathcal{P}^*(S)(1 - K_0^* - K_{2/2}^*) \). Solving this, we find that \( \mathcal{P}^*(S) = \frac{dS^*_1}{\sum_{S' \neq 0} dS'^*_2} \). For large \( k \), the fixed point spin distribution has the scaling form \( \mathcal{P}^*(S) = \frac{1}{L^{dS}} f_k(S) \), with \( f_k(x) = 2\sin^2 \pi x \). Intuitively, the flow can be thought of as ‘annealing’ a dense gas of anyons \([39]\).

We also define the singlet participation ratio \( r_s(\ell) \), i.e. the probability that an original (‘UV scale’) spin resides in a singlet at depth \( \ell \). For \( r_s(\ell) \approx 1 \), all spins of various sizes are essentially in singlets. As \( r_s \) depends both on the probability of generating a singlet at depth \( \ell \) and the probability that the spin is not already in a singlet \([38]\),

\[
\frac{dr_s}{d\ell} = \Pi_0(\ell)(1 - r_s(\ell)), \tag{5}
\]

These flow equations permit straightforward solution for any initial spin distribution (see Fig. 2). The spin distribution \( \mathcal{P}(S) \) flows very quickly to its fixed-point value \( \mathcal{P}^*(S) \) (indicated by the plateau in \( \Pi_0 \)). Subsequently, \( r_s \) grows towards its fixed-point value. In the plateau, \( \Pi_0(\ell) \approx \Pi_0^* \) where \( (\Pi_0^*)^{-1} \approx \alpha k^3 \) for large \( k \), with the universal pre-factor \( \alpha = (4\pi^2)^{-1} \). The singlet participation ratio near the fixed point is therefore given by \( r_s(\ell) \approx 1 - e^{-\Pi_0^*} \) so that all UV spins are in singlets when \( \Pi_0^* \gg 1 \).

The random sampling RSRG-X procedure outlined above naturally lends itself to a computation of the \( T \to \infty \) entanglement entropy defined in \([2]\), averaged over disorder. Consider an interval \( A \) of the chain, of length \( L \). For \( \Pi_0^* \gg 1 \), or equivalently \( L \gg \xi_k \approx e^{\alpha k^3} \) with \( \alpha = (4\pi^2)^{-1} \), almost all the original UV spins inside \( A \) reside in singlets. Region \( A \) is entangled with the rest of the system solely by singlets that cross its boundary. The RG structure guarantees that a constant number of boundary-crossing singlets is added at each \( \ell \). Thus, up to a constant prefactor, \( S_A \) is given by counting singlet entanglement up to scale \( \ell = \ln L \), when all spins in \( A \) have been decimated \([40]\):

\[
S_A \approx \frac{1}{L \gg \xi_k} \ln L \sum_{S \neq 0} \frac{\mathcal{P}(S, \ell) \mathcal{P}(k_1/2 - S, \ell)}{dS_1 dS_2} \ln dS. \tag{6}
\]

The integrand in \([6]\) quickly saturates to its constant finite fixed point value, so that the excited-state entanglement scales as \( S_A \sim \ln L \) in this regime.

On the other hand, if \( L \ll \xi_k \) the RG shrinks interval \( A \) to a single site at depth \( \ell = \ln L \ll (K_0^*)^{-1} \) so that the entire entanglement interval is likely contained within a large superspin. Here, the physical picture is one of weakly coupled large spins. If we (loosely) think of large quantum dimensions as ‘classical’ behavior, then we expect thermalization in this regime.

In other words, the system looks thermal for \( L \ll \xi_k \), but for large intervals \( L \gg \xi_k \), the excited-state entanglement entropy scales logarithmically, characteristic of random-singlet like ‘critical points’ at zero \([41]\) or finite \([42]\) energy density \([42]\).

In the Heisenberg limit as \( k \to \infty \), we see that the crossover scale \( \xi_k \) diverges exponentially, so that at the Heisenberg point, intervals of arbitrarily large size remain in the volume-law scaling regime and the system is ergodic. This is in accord with the intuitive picture that the breakdown of perturbative RG indicates ergodicity, but now we have recovered this from an RG approach that remains controlled at any finite \( k \) as the Heisenberg limit is approached, and breaks down precisely at \( k = \infty \) when the spins are allowed to grow without bound. Although both the dynamical RG equations of \([10]\) and the \( O(3) \) limit of the discrete-symmetry spin-1 chain studied in \([14]\) hint at such a conclusion, the present approach is the first that goes beyond those intuitive but informal pictures by presenting a controlled calculation of entanglement scaling. This allows us to state with confidence that the random-bond Heisenberg model is ergodic at infinite temperature. Furthermore, the RSRG equations allow us to study the entire line of fixed points and corresponding scaling limits that control the infinite-temperature dynamics of \( SU(2)_k \) spin chains at any \( k \). These fixed points control the dynamics of Ising and likely also three-state Potts models that can be mapped to the \( k = 2 \) and \( k = 4 \) chains, respectively \([37]\).
In closing, we comment on the extension of these ideas to finite temperature. For finite $k$ there likely exists a critical disorder strength below which $SU(2)_k$ chains thermalize at $T = \infty$, yet transition into a non-ergodic random singlet phase at lower temperature. For Heisenberg chains, however, we expect thermal behavior at all $T \neq 0$, regardless of disorder strength, due to the following argument. The finite temperature RSRG-X can be approximately split into two stages. First, for RG energy scale $\Omega \gg T$, essentially all decimations yield singlets as in ground state RSRG, resulting in a renormalized chain of predominantly single-spin operators at scale $\Omega \approx T$. The remaining flow for $\Omega \ll T$ has all couplings much weaker than temperature and should essentially follow the $T \to \infty$ behavior described above. Hence, for Heisenberg chains we expect finite temperature to simply postpone the inevitable thermalization to a longer length scale. We leave a detailed analysis of the finite temperature behavior for future work.

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[42] In principle, we do not explicitly rule out a more complicated crossover with a regime having entanglement intermediate between thermal volume law, $S_A \sim L$, and critical-like athermal scaling $S_A \sim \ln L$. 
Supplemental Material

RENORMALIZATION GROUP EQUATIONS AT INFINITE TEMPERATURE

In this section, we provide a detailed derivation of the renormalization group flow equations at infinite temperature.

Probability distribution of remaining spins

We start with $N$ spins of various sizes and perform $n$ RG steps. Each step follows a fusion channel with a probability

$$ p^S_{S_1 S_2} = \frac{d_S}{\sum_{S \in S_1 \otimes S_2} d_S} \delta_{S \in S_1 \otimes S_2} = \frac{d_S}{d_{S_1} d_{S_2}} \delta_{S \in S_1 \otimes S_2}, \tag{1} $$

where $d_S = \{2S + 1\} = \frac{\sin((2S+1)\pi)}{\sin(\pi)}$ is the quantum dimension of anyon $S$ and we use the identity $\sum_{S \in S_1 \otimes S_2} d_S = d_{S_1} d_{S_2}$. The change in the number of spins of size $S$ is then

$$ \Delta N(S) \approx n \sum_{S_1, S_2} \mathcal{P}(S_1) \mathcal{P}(S_2) \left[-(\delta_{S, S_1} + \delta_{S, S_2}) + p^S_{S_1 S_2}\right], $$

$$ = n \left[-2\mathcal{P}(S) + \sum_{S_1, S_2} p^S_{S_1 S_2} \mathcal{P}(S_1) \mathcal{P}(S_2)\right], \tag{2} $$

where $\mathcal{P}(S) = N(S)/N$ is the fraction of spins of size $S$ before implementing the RG steps. The new probability distribution on spins after these $n$ RG steps is then

$$ \mathcal{P}'(S) = \frac{\sum_{S \neq 0, \frac{1}{2}} N(S) + \Delta N(S)}{N} \approx \frac{N(S)}{N} + \frac{\Delta N(S)}{N} - \frac{N(S)}{N} \sum_{S' \neq 0, \frac{1}{2}} \frac{\Delta N(S')}{N}, $$

$$ = \mathcal{P}(S) + \frac{\Delta N(S)}{N} - \mathcal{P}(S) \sum_{S' \neq 0, \frac{1}{2}} \frac{\Delta N(S')}{N}. \tag{3} $$

i.e. the probability distribution changes by

$$ \frac{\Delta \mathcal{P}(S)}{n/N} = \left[-2\mathcal{P}(S) + \sum_{S_1, S_2} p^S_{S_1 S_2} \mathcal{P}(S_1) \mathcal{P}(S_2)\right] - \mathcal{P}(S) \sum_{S' \neq 0, \frac{1}{2}} \left[-2\mathcal{P}(S') + \sum_{S_1, S_2} p^S_{S_1 S_2} \mathcal{P}(S_1) \mathcal{P}(S_2)\right], $$

$$ = \sum_{S_1, S_2} p^S_{S_1 S_2} \mathcal{P}(S_1) \mathcal{P}(S_2) - \mathcal{P}(S) \sum_{S_1, S_2, S' \neq 0, \frac{1}{2}} p^S_{S_1 S_2} \mathcal{P}(S_1) \mathcal{P}(S_2). \tag{4} $$

Now,

$$ \sum_{S' \neq 0, \frac{1}{2}} p^S_{S_1 S_2} = \frac{1}{d_{S_1} d_{S_2}} \sum_{S' \neq 0, \frac{1}{2}, S' \in S_1 \otimes S_2} d_{S'} = \frac{1}{d_{S_1} d_{S_2}} \left(d_{S_1} d_{S_2} - d_0 \delta_{S_1 S_2} - d_\frac{1}{2} \delta_{S_1 + S_2 = \frac{1}{2}}\right) = 1 - \frac{\delta_{S_1 S_2}}{d_{S_1}} - \frac{\delta_{\frac{1}{2} - S_1, S_2}}{d_{S_1} d_{\frac{1}{2} - S_1}} \tag{5} $$

where we used the fact that summing over $S' \neq 0$ includes all the $d_{S_1} d_{S_2}$ possible fusion channels unless one of the channels is a singlet (i.e., 0 or $k/2$) which is only possible when $S_1 = S_2$ (or $S_2 = \frac{k}{2} - S_1$), and the quantum dimension of the singlet channels is $d_0 = d_{\frac{1}{2}} = 1$. Using (5) in (4), we find

$$ \frac{\Delta \mathcal{P}(S)}{n/N} = \sum_{S_1, S_2} p^S_{S_1 S_2} \mathcal{P}(S_1) \mathcal{P}(S_2) - \mathcal{P}(S) \sum_{S_1, S_2} \mathcal{P}(S_1) \mathcal{P}(S_2) \left(1 - \frac{\delta_{S_1, S_2}}{d_{S_1}^2} - \frac{\delta_{\frac{1}{2} - S_1, S_2}}{d_{S_1} d_{\frac{1}{2} - S_1}}\right), $$

$$ = \sum_{S_1, S_2} p^S_{S_1 S_2} \mathcal{P}(S_1) \mathcal{P}(S_2) - \mathcal{P}(S) \left[1 - \sum_{S' \neq 0, \frac{1}{2}} \frac{\mathcal{P}(S')^2}{d_{S'}^2} - \sum_{S' \neq 0, \frac{1}{2}} \frac{\mathcal{P}(S') \mathcal{P}(k/2 - S')}{d_{S'} d_{\frac{1}{2} - S'}}, \tag{6} \right] $$
Therefore, \( \Delta \ell = -\ln \frac{N + \Delta N}{N} \approx -\frac{\Delta N}{N} \),

where, proceeding analogously to the previous calculation,

\[
\Delta N = \sum_{S \neq 0} \Delta N(S) = n \left[ -2 + 1 - \sum_{S' \neq 0, \frac{k}{2}} \frac{\mathcal{P}(S')^2}{d_S^2} - \sum_{S' \neq 0, \frac{k}{2}} \frac{\mathcal{P}(S') \mathcal{P}(\frac{k}{2} - S')}{d_S d_{\frac{k}{2} - S'}} \right] = -n \left[ 1 + \sum_{S' \neq 0, \frac{k}{2}} \frac{\mathcal{P}(S')^2}{d_S^2} + \sum_{S' \neq 0, \frac{k}{2}} \frac{\mathcal{P}(S') \mathcal{P}(\frac{k}{2} - S')}{d_S d_{\frac{k}{2} - S'}} \right].
\]

Therefore,

\[
\Delta \ell = \frac{n}{N} \left[ 1 + \sum_{S \neq 0} \frac{\mathcal{P}(S)^2}{d_S^2} + \sum_{S' \neq 0, \frac{k}{2}} \frac{\mathcal{P}(S') \mathcal{P}(\frac{k}{2} - S')}{d_S d_{\frac{k}{2} - S'}} \right].
\]

Assembling these expressions, we find

\[
\frac{d\mathcal{P}(S)}{d\ell} = \lim_{n \to 0} \frac{\Delta \mathcal{P}(S)}{\Delta \ell} = \frac{\sum_{S_1, S_2 \in S_1 \otimes S_2} d_{S_1} d_{S_2} \mathcal{P}(S_1) \mathcal{P}(S_2) - \mathcal{P}(S) \left[ 1 - \sum_{S' \neq 0, \frac{k}{2}} \frac{\mathcal{P}(S')^2}{d_S^2} - \sum_{S' \neq 0, \frac{k}{2}} \frac{\mathcal{P}(S') \mathcal{P}(\frac{k}{2} - S')}{d_S d_{\frac{k}{2} - S'}} \right]}{1 + \sum_{S' \neq 0, \frac{k}{2}} \frac{\mathcal{P}(S')^2}{d_S^2} + \sum_{S' \neq 0, \frac{k}{2}} \frac{\mathcal{P}(S') \mathcal{P}(\frac{k}{2} - S')}{d_S d_{\frac{k}{2} - S'}}} \right]
= \frac{1}{1 + K_0(\ell) + K_{\frac{k}{2}}(\ell)} \left[ K_S(\ell) - \mathcal{P}(S) \left( 1 - K_0(\ell) - K_{\frac{k}{2}}(\ell) \right) \right]
= \frac{1}{1 + \Pi_0(\ell)} \left[ K_S(\ell) - \mathcal{P}(S) (1 - \Pi_0(\ell)) \right].
\]

as claimed in the main text.

**Singlet Participation Ratio**

We now compute the flow of the singlet participation ratio \( r_s(\ell) \), defined as the probability that a microscopic (UV scale) spin resides in a singlet at RG depth \( \ell \). When this becomes \( \mathcal{O}(1) \), we are in the singlet dominated regime. To this end, we define the following quantities. Let \( N_0 \) be the initial number of spins and \( N(\ell) \) be the remaining number of super-spins after running the RG for length \( \ell \), and let \( \bar{s}(\ell) \) be the average number of microscopic spins that make up a surviving “super-spin” at scale \( \ell \). After change in RG scale \( \Delta \ell \), the change in the number of spins participating in singlets is given by

\[
\Delta r_s(\ell) N_0 = \frac{1}{2} \times \left[ N_0 e^{-\ell} \Delta \ell \right] \times \Pi_0(\ell) \times \left( 2\bar{s}(\ell) \right),
\]

from which we find that

\[
\frac{dr_s}{d\ell} = e^{-\ell} \Pi_0(\ell) \bar{s}(\ell).
\]

On the other hand, at depth \( \ell \), there are \( N_0 e^{-\ell} \) spins remaining, which are formed from \( (1 - r_s(\ell)) N_0 \) UV spins. Therefore, the typical number of microscopic spins making up each surviving super-spin is:

\[
\bar{s}(\ell) = (1 - r_s(\ell)) e^{\ell},
\]
Combining the above gives

\[ \frac{dr_s}{d\ell} = \Pi_0(\ell) (1 - r_s(\ell)). \] (14)

**Scaling of the crossover scale \( \xi_k \)**

The \( k \)-dependence of the crossover scale \( \xi_k \) is a direct consequence of the universal scaling of the spin distribution \( P^*(S) = \frac{1}{2\pi} f(2S/k) \) at the fixed point, with \( f(x) = 2\sin^2 \pi x \). To see this, recall that \( \xi_k \) is defined as the scale at which the singlet participation ratio \( r_s(\ell) \) becomes \( O(1) \). Since \( r_s(\ell) \approx 1 - e^{-\Pi_0} \) near the fixed point, we have \( \ln \xi_k = (\Pi_0^*)^{-1} \). Meanwhile, \( \Pi_0^* \) for large \( k \) scales as

\[ (\Pi_0^*)^{-1} = \left( \sum_{S \neq 0, \frac{1}{2}} \frac{P^*(S)^2}{d_S^2} + \sum_{S \neq 0, \frac{1}{2}} \frac{P^*(S)P^*(\frac{k}{2} - S)}{d_S d_{\frac{k}{2} - S}} \right)^{-1} = \frac{1}{2} \sum_{S > 0} d_S^2, \]

\[ \sim \frac{k^3}{2\pi} \int_0^1 dx \sin^2(\pi x) \sim \frac{k^3}{4\pi^2}. \] (15)

where we used the fact that the fixed point is reflection-symmetric about \( k/2 \). Therefore,

\[ \xi_k \sim \exp(\alpha k^3), \quad \text{with} \quad \alpha = \frac{1}{4\pi^2} \approx 0.0255. \] (16)

The universal prefactor \( \alpha \) is completely characterized by the spin distribution \( P^* \) at the fixed point.

Note that \( \Pi_0(\ell) \) saturates to its fixed point value \( \Pi_0^* \) with a length scale \( L_k = e^{\Pi_0^*} \) that is much smaller than \( \xi_k \) (so that the expression \( r_s(\ell) \approx 1 - e^{-\Pi_0^*} \) becomes very quickly justified). However, the important characteristic scale for the entanglement crossover (and for thermalization) is \( \xi_k \gg L_k \), which controls whether typical UV spins are involved in a singlet (in which case the excited entanglement scales logarithmically), or in a large, almost classical superspin (in which case we expect volume law scaling of the entanglement). Both length scales \( \xi_k \) and \( L_k \) are plotted in the left panel of Fig. 4.
Although the RG procedure for the Heisenberg chain quickly breaks down as higher spins are generated, it can still qualitatively reproduce the many-body spectrum of a strongly disordered short chain, where only relatively small spins are generated. This is shown in the right panel of Fig. 4 for random antiferromagnetic couplings. However, we emphasize that contrary to the RG flow that targets the ground state of a random antiferromagnetic chain, this RG scheme becomes increasingly inconsistent as the number of RG steps grows in larger systems.