HIGHER ORDER OSCILLATION AND UNIFORM DISTRIBUTION

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ABSTRACT. It is known that the M"obius function in number theory is higher order oscillating. In this paper we show that there is another kind of higher order oscillating sequences in the form $(e^{2\pi i \alpha \beta^n g(\beta)})_{n \in \mathbb{N}}$, for a non-decreasing twice differentiable function $g$ with a mild condition. This follows the result we prove in this paper that for a fixed non-zero real number $\alpha$ and almost all real numbers $\beta > 1$ (alternatively, for a fixed real number $\beta > 1$ and almost all real numbers $\alpha$) and for all real polynomials $Q(x)$, sequences $(\alpha \beta^n g(\beta) + Q(n))_{n \in \mathbb{N}}$ are uniformly distributed modulo 1.

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1. Introduction

We denote by $\mathbb{N}$ the set of positive integers. Suppose $c = (c_n)_{n \in \mathbb{N}}$, is a sequence of complex numbers. In \cite{2} (see also \cite{4}), an oscillating sequence is defined for the purpose of the study of Sarnak’s conjecture (see \cite{8, 9}) which is stated as the M"obius function is linearly disjoint from all zero entropy flows. Let us recall the definition of an oscillating sequence.
**Definition 1 (Oscillation).** The sequence \( c = (c_n)_{n \in \mathbb{N}} \) is said to be oscillating if

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} c_n e^{2\pi i nt} = 0, \quad \forall \ 0 \leq t < 1.
\]  

(1)

with the technical condition

\[
\sum_{n=1}^{N} |c_n|^\lambda = O(N) \quad \text{for some} \quad \lambda > 1.
\]  

(2)

Recall that the Möbius function \( \mu(n) \) is, by definition, \( \mu(n) = 1 \) if \( n = 1 \); \( \mu(n) = (-1)^r \) if \( n = p_1 \cdots p_r \) for \( r \) distinct prime numbers \( p_i \); \( \mu(n) = 0 \) if \( p^2 \mid n \) for some prime number \( p \). The Möbius sequence \( u = (\mu(n))_{n \in \mathbb{N}} \) is the one generated by the Möbius function. Due to Davenport’s theorem [1], the Möbius sequence is oscillating.

We proved in [2] that any oscillating sequence is linearly disjoint from all minimally mean attractable (MMA) and minimally mean-L-stable (MMLS) flows. In the same paper, we further proved that flows defined by all \( p \)-adic polynomials of integral coefficients, all \( p \)-adic rational maps with good reduction, all automorphisms of the 2-torus with zero topological entropy, all diagonalizable affine maps of the 2-torus with zero topological entropy, all orientation-preserving circle homeomorphisms are MMA and MMLS. Furthermore, in [4], we proved that flows defined by all continuous interval maps with zero topological entropy are MMA and MMLS. Therefore, we confirmed Sarnak’s conjecture for these flows which form a large class of zero topological entropy flows. However, it is also shown in [2] Example 7, only the oscillation property is not enough for the study of Sarnak’s conjecture. We need a higher order oscillation condition in the study of Sarnak’s conjecture. We gave a definition of a higher order oscillating sequence in [5].

**Definition 2 (Higher order oscillation).** We call the sequence \( c = (c_n)_{n \in \mathbb{N}} \) a higher order oscillating sequence of order \( m \geq 2 \) if

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} c_n e^{2\pi i P(n)} = 0
\]  

(3)

for every real polynomial \( P \) of degree \( \leq m \) with also the technical condition (2).

We prove in [5] that any higher order oscillating sequence of order \( d \) is linearly disjoint from all affine distal flows on the \( d \)-torus for all \( d \geq 2 \). One consequence of this result is that any higher order oscillating sequence of order 2 is linearly disjoint from all affine flows on the 2-torus with zero topological entropy. Thanks to Hua’s result [3], the Möbius sequence \( u \) is an example of a higher order
oscillating sequence of order $m$ for all $m \geq 2$ (see also [7, Lemma 2.1] and [10]). Combining this with our main result in [5], we reconfirmed in [5] that Sarnak’s conjecture for all affine flows on the 2-torus with zero topological entropy and for all affine distal flows on the $d$-torus for all $d > 2$ by using a much simpler method. Then we have the following interesting question.

**Question 1.** Is there another kind of higher order oscillation sequences as defined in Definition 2 except for the one generated by an arithmetic function like the Möbius function?

We study this question in this paper.

## 2. Statement of the main result

For a real number $x$, let $[x]$ denote the integer part of $x$, that is, the greatest integer $\leq x$; let

$$\{x\} = x - [x]$$

be the fractional part of $x$, or the residue of $x$ modulo 1.

**Definition 3** (Uniform distribution). We say a sequence $x = (x_n)_{n \in \mathbb{N}}$ of real numbers is uniformly distributed modulo 1 (abbreviated u.d. mod 1) if for any $0 \leq a < b \leq 1$, we have

$$\lim_{N \to \infty} \frac{\#(\{n \in [1, N] \mid \{x_n\} \in [a, b]\})}{N} = b - a.$$

We state three results in the u.d. mod 1 theory.

**Theorem A** (The Weyl criterion). The sequence $x = (x_n)_{n \in \mathbb{N}}$ is u.d. mod 1 if and only if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0 \quad \text{for all integers } h \neq 0.$$

**Theorem B** (Koksma’s theorem). Let $(y_n(x))_{n \in \mathbb{N}}$ be a sequence of real valued $C^1$ functions defined on an interval $[a, b]$. Suppose $y_m'(x) - y_n'(x)$ is monotone on $[a, b]$ for any two integers $m \neq n$ and suppose

$$\inf_{m \neq n, x \in [a, b]} |y_m'(x) - y_n'(x)| > 0.$$

Then for almost all $x \in [a, b]$, the sequence $y = (y_n(x))_{n \in \mathbb{N}}$ is u.d. mod 1.
Theorem C (Van der Corput’s theorem). Let $x = (x_n)_{n \in \mathbb{N}}$ be a sequence of real numbers. If for every positive integer $h$ the sequence

$$d_h x = (x_{n+h} - x_n)_{n \in \mathbb{N}}$$

is u. d. mod 1,

then $x$ itself is u. d. mod 1.

The reader who is interested in these three theorems can find proofs in [6, Theorem 2.1, Theorem 3.1, and Theorem 4.3].

For the convenience of notation, we understand that an empty sum is 0 and an empty product is one, i.e.,

$$\sum_{j=1}^{k} (\cdots) = 0 \quad \text{and} \quad \prod_{j=1}^{k} (\cdots) = 1 \quad \text{when} \quad k = 0.$$

In this notation, we have

$$y_n(x) = x^n = x^n \prod_{j=1}^{k} (\cdots) \quad \text{and} \quad y_n(x) \sum_{j=1}^{k} (\cdots) = 0 \quad \text{when} \quad k = 0.$$

Take a non empty interval $I$ in the real line $\mathbb{R}$, which can be closed, open or semi-open. Let $C^k_+(I)$ be the space of all positive real valued $k$-times continuously differentiable functions on an interval $I$, whose $i$-th derivative is non-negative for $i \leq k$. Then it is closed under addition and multiplication, that is, if

$$f, g \in C^k_+(I),$$

then

$$f + g, \quad fg \in C^k_+(I).$$

In what follows, we often use this closure property of $C^k_+(I)$. Let $\mathbb{R}[x]$ denote the space of all real polynomials. The main result we prove in this paper is

Theorem 1 (Main theorem). Let us take a function $g \in C^2_+((1, \infty))$. Then, for a fixed real number $\alpha \neq 0$ and almost all real numbers $\beta > 1$ (alternatively, for a fixed real number $\beta > 1$ and almost all real numbers $\alpha$) and for all real polynomials $Q \in \mathbb{R}[x]$, sequences

$$\left(\alpha \beta^n g(\beta) + Q(n)\right)_{n \in \mathbb{N}} \quad \text{are u. d. mod 1.} \quad (4)$$
As a countable union of exceptional null sets is null, we clearly have

\begin{equation}
\left( \alpha \beta^n g_i(\beta) + Q(n) \right)_{n \in \mathbb{N}}, \quad i \in \mathbb{N}, \quad \text{are u. d. mod 1.}
\end{equation}

Since \( x^n - 1 \in \mathcal{C}_+^2((1, \infty)) \) for all \( n \geq 1 \), we have that for any \( g \in \mathcal{C}_+^2((1, \infty)) \) and any integer \( l \geq 0 \) and \( h_j \in \mathbb{N}, 1 \leq j \leq l \),

\begin{equation}
g(x) \prod_{j=1}^{l} (x^{h_j} - 1) \in \mathcal{C}_+^2((1, \infty)).
\end{equation}

Applying this countable family to Corollary 1, we obtain:

**Theorem 2** (Equivalent statement). Given any \( g \in \mathcal{C}_+^2((1, \infty)) \). Then, for a fixed real number \( \alpha \neq 0 \) and almost all real numbers \( \beta > 1 \) (alternatively, for a fixed real number \( \beta > 1 \) and almost all real numbers \( \alpha \)), for all real polynomials \( Q \in \mathbb{R}[x] \), sequences

\begin{equation}
\left( \alpha \beta^n g_i(\beta) + Q(n) \right)_{n \in \mathbb{N}}, \quad n \in \mathbb{N}, \quad \text{are u. d. mod 1.}
\end{equation}

On the other hand, when \( l = 0 \), the product is empty, Theorem 2 is reduced to Theorem 1. So our Theorem 1 and Theorem 2 are equivalent. Our proof is based on the formulation of Theorem 2 which is already an interesting point in this paper. We will give the proof in §3.

Theorem 1 combined with Theorem A answers the question (Question 1) affirmatively.

**Corollary 2** (Main corollary). Given any \( g \in \mathcal{C}_+^2((1, \infty)) \). Then, for a fixed real number \( \alpha \neq 0 \) and almost all real numbers \( \beta > 1 \) (alternatively, for a fixed real number \( \beta > 1 \) and almost all real numbers \( \alpha \)), sequences

\begin{equation}
c = \left( e^{2\pi i \alpha \beta^n g(\beta)} \right)
\end{equation}

are higher order oscillating sequences of order \( m \) for all \( m \geq 2 \).

**Remark 1.** In particular, taking a constant function \( g \equiv 1 \), we have

\[ c = \left( e^{2\pi i \alpha \beta^n} \right)_{n \in \mathbb{N}}. \]
3. Proof of the main theorem

We start the proof for the case that \( \alpha \neq 0 \) is fixed and figure out the exceptional set for \( \beta > 1 \). We may assume that \( \alpha > 0 \) and fixed it from now on. Take \( g \in \mathcal{C}_+^2((1, \infty)) \). For \( n \in \mathbb{N} \) and an integer \( l \geq 0 \) and for a \( l \)-tuple \( (h_1, \ldots, h_l) \in \mathbb{N}^l \), define a function

\[
y_n(x) = \alpha g(x) x^n \prod_{j=1}^{l} (x^{h_j} - 1).
\]

Remember that when \( l = 0 \), \( y_n(x) = \alpha x^n g(x) \).

For \( n > m \), we have

\[
y'_n(x) - y'_m(x) = \alpha g(x) \left( nx^{n-1} - mx^{m-1} \prod_{j=1}^{k} (x^{h_j} - 1) + (x^n - x^m) \sum_{j=1}^{k} h_j x^{h_j-1} \prod_{i \neq j} (x^{h_i} - 1) \right) + \alpha g'(x) (x^n - x^m) \prod_{j=1}^{l} (x^{h_j} - 1).
\]

Since \( g', g'' \geq 0 \) and since

\[
nx^{n-1} - mx^{m-1} = nx^{n-m} - m
\]

and

\[
nx^{n-m} - m \geq n - m \geq 1 \quad \text{for} \quad x \geq 1,
\]

we see that every term in the last expression are in \( \mathcal{C}_+^1([1, \eta]) \) for any \( \eta > 1 \). By the closure property of \( \mathcal{C}_+^1([1, \eta]) \), we have that

\[
y'_n(x) - y'_m(x) \in \mathcal{C}_+^1([1, \eta]) \quad \forall n > m \in \mathbb{N}.
\]

In particular, this imply that \( y'_n - y'_m \) is increasing for \( n > m \in \mathbb{N} \). Furthermore, if \( x > a > 1 \), then \( x^h - 1 \geq a - 1 \) for any \( h \in \mathbb{N} \), we see that there is a constant \( L > 0 \) such that

\[
|y'_n(x) - y'_m(x)| \geq L \quad \forall n > m \in \mathbb{N}, \quad \forall a \leq x \leq \eta.
\]

Inequalities (7) and (8) say that the sequence of real valued \( \mathcal{C}^1 \) functions

\[
\left( y_n(x) \right)_{n \in \mathbb{N}}
\]

satisfies all hypothesizes of Theorem B.
Theorem B implies that for almost all $x$ in
\[
[(2^k + 1)/2^k, (2^{k-1} + 1)/2^{k-1}] \text{ or } [k, k + 1] \text{ for } k \geq 2,
\]
the sequence
\[
(\alpha y_n(x))_{n \in \mathbb{N}} \text{ is u.d. mod 1.}
\]
Further, this implies that for almost all
\[
x \in (1, \infty) = \bigcup_{k=2}^{\infty} \left[ \frac{2^k + 1}{2^k}, \frac{2^{k-1} + 1}{2^{k-1}} \right] \bigcup \bigcup_{k=2}^{\infty} [k, k + 1]
\]
the sequence $\left(\alpha y_n(x)\right)_{n \in \mathbb{N}}$ is u.d. mod 1.

Let
\[
A_{(h_1, h_2, \ldots, h_l)} = \left\{ \beta > 1 \mid \left(\alpha \beta^n g(\beta) \prod_{j=1}^{l} (\beta^{h_j} - 1)\right)_{n \in \mathbb{N}} \text{ is not u.d. mod 1} \right\}.
\]
Then it has one dimensional Lebesgue measure zero. By the above convention, we include the case $l = 0$ as well.

Since the set
\[
U = \bigcup_{l=0}^{\infty} \{(h_1, \ldots, h_l) \mid h_j \in \mathbb{N}\}
\]
is countable, the one dimensional Lebesgue measure of
\[
\bigcup_{(h_1, \ldots, h_l) \in U} A_{(h_1, \ldots, h_l)}
\]
is zero, too.

For the fix a real number $\alpha \neq 0$ in the theorem, take a real number
\[
\beta \in (1, \infty) \setminus \bigcup_{(h_1, \ldots, h_l) \in U} A_{(h_1, \ldots, h_l)}.
\]
This says that the sequence
\[
\left(\alpha \beta^n g(\beta) \prod_{j=1}^{l} (\beta^{h_j} - 1)\right)_{n \in \mathbb{N}} \text{ is u.d. mod 1}
\]
for all integers $l \geq 0$ and all $l$-tuple $(h_1, \ldots, h_l) \in \mathbb{N}^l$. 
Define statements $P(k)$ for $k = 0, 1, \ldots$ as follows.

$P(k):$ For any integer $l \geq 0,$
\[(h_1, \ldots, h_l) \in \mathbb{N}^l \quad \text{and} \quad t_i \in \mathbb{R} \quad (i = 0, 1, 2, \ldots, k),\]
the sequence
\[
\left(\alpha \beta^n g(\beta) \prod_{j=1}^{l} (\beta^{h_j} - 1) + \sum_{i=0}^{k} t_i n^i\right)_{n \in \mathbb{N}}\]

is u. d mod 1.

We claim that $P(k)$ holds for every integer $k \geq 0.$ We prove the claim by induction.

By our choice of $\alpha$ and $\beta,$ we know that $P(0)$ holds. Assume $P(k - 1)$ holds for $k \geq 1.$ Let
\[
x_n = \alpha \beta^n g(\beta) \prod_{j=1}^{l} (\beta^{h_j} - 1) + \sum_{i=0}^{k} t_i n^i.
\]
Then
\[
x_{n+h} = \alpha \beta^{n+h} g(\beta) \prod_{j=1}^{l} (\beta^{h_j} - 1) + \sum_{i=0}^{k} t_i (n+h)^i.
\]

Consider the difference appeared in Theorem C:
\[
x_{n+h} - x_n = \alpha \beta^n (\beta^h - 1) g(\beta) \prod_{j=1}^{l} (\beta^{h_j} - 1) + \sum_{i=0}^{k-1} T_i n^i
\]
\[
= \alpha \beta^n g(\beta) \prod_{j=1}^{l+1} (\beta^{h_j} - 1) + \sum_{i=0}^{k-1} T_i n^i
\]
with $h_{l+1} = h$ and $T_i = -t_i + \sum_{j=i}^{k} t_j \binom{j}{i} h^{j-i}.$

Since $P(k - 1)$ is valid, the resulting sequence is u. d. mod 1 for all $h \in \mathbb{N}.$ Now Theorem C implies that $P(k)$ holds too. We proved the claim. Therefore we completed the proof of Theorem 2.

The proof for the case that $\beta > 1$ is fixed and to obtain the exceptional set for $\alpha,$ is similar and easier. Let $g$ be any positive function on $(1, \infty).$ For $n \in \mathbb{N}$ and an integer $l \geq 0$ and for a $l$-tuple $(h_1, \ldots, h_l) \in \mathbb{N}^l,$ define
\[
y_n(x) = x g(\beta) \beta^n \prod_{j=1}^{l} (\beta^{h_j} - 1).
\]
Then

\[ y'_n(x) - y'_m(x) = g(\beta)(\beta^n - \beta^m) \prod_{j=1}^{l} (\beta^{h_j} - 1) \]

is a positive constant for \( n > m \) and satisfies the condition of Theorem B under the similar dissection of the interval \((1, \infty)\). The rest of the proof is the same.

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