On the Form of Solutions of Fuchsian differential Equations with $n$ regular singular Points

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Abstract

The form of the coefficients of power series expressions corresponding to solutions of Fuchsian differential equations (or their associated degenerated confluent forms) with $n$ regular singular points is determined by solving the corresponding $n$-term recurrence relations in full generality. Some important special cases are discussed in which the solutions coincide with special functions of mathematical physics.

Key words: Fuchsian differential equations, regular singular points, recurrence relations

1 Introduction: The Theory of Fuchsian Differential Equations

In the theory of ordinary second-order linear differential equations with variable coefficients, which are of great interest both for mathematics and theoretical physics, it may happen that the corresponding coefficients are not globally well-behaved analytic functions but instead have singularities.

In this rather common case, the behavior of solutions is usually studied in the immediate vicinity of those (typically isolated) singular points, where at least one of the coefficients of the equation diverges and it is therefore to be expected - based on the fact that the corresponding singular points of said solutions lie amongst those of their associated singular coefficients - that at least one of the linearly independent solutions strives toward infinity as well. The manner in which this occurs in detail depends very much on the nature of the singular point examined, that is, in particular, on whether the point in question is a so-called regular or irregular singular point.

A particularly important class of ordinary differential equations of the aforementioned type, which deals exclusively with the much simpler case of regular singular points, is the famous Fuchsian class of second-order linear ordinary differential equations; a class being specified by the property that its coefficients

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are (typically) complex-valued functions that have poles of at most first and second order. Famous representatives of this class are Gauß’ hypergeometric differential equation, Heun’s equation, the Lamé equation and the generalized Lamé equation [1, 22].

To explain the main characteristics of this class, one may first look at the fact that any singular ordinary homogeneous second-order linear differential equation with variable coefficients can be written in the form

\[ f'' + pf' + qf = 0, \]  

where the validity of the initial conditions \( f'|_{\xi = \xi_i} = w_0, \) \( f'|_{\xi = \xi_i} = w_1 \) is assumed in relation to a fixed singular point \( \xi_i \), the prime denotes differentiation with respect to the (typically) complex variable \( \xi \) and the coefficients \( p = p(\xi) \) and \( q = q(\xi) \) are (typically) complex-valued functions with altogether \( n \) different isolated singularities \( \xi_n \).

Given this setting, which ensures that the functions \( p(\xi) \) and \( q(\xi) \) are regular everywhere except in the immediate vicinity of each individual singular point, it becomes clear that the main problem in this context is to find a solution around a fixed singular point \( \xi_i \). This is not least because the method of successive approximation can be used to show that the system of first order differential relations \( f' = u \) and \( u' = -pu + qf \) defined by (1) has a unique regular solution everywhere except in the vicinity of those isolated points where \( p(\xi) \) and \( q(\xi) \) become singular [23]. Consequently, serious problems occur only close to the singular points \( \xi_1, \xi_2, ..., \xi_n \), so that it can be concluded that the real difficulty is to find a solution around these points.

Accordingly, if, for the purpose of solving (1) around a fixed singular point, it is assumed that \( \xi_i \) that the coefficients \( p(\xi) \) and \( q(\xi) \) can be expanded in Laurent series of the form \( p(\xi) = \sum_{k=-\infty}^{\infty} p_k (\xi - \xi_i)^k \) and \( q(\xi) = \sum_{k=-\infty}^{\infty} q_k (\xi - \xi_i)^k \) in an annulus of radius \( 0 < |\xi - \xi_i| < R \), it follows from general considerations on the theory of differential equations with variable coefficients [17, 24] that the solutions of (1) must have the form

\[ f_1(\xi) = \sum_{k=-\infty}^{\infty} u_k (\xi - \xi_i)^{k+\sigma_1}, \quad f_2(\xi) = \sum_{k=-\infty}^{\infty} v_k (\xi - \xi_i)^{k+\sigma_2} \]  

in the case that \( \sigma_1 \neq \sigma_2 \) or

\[ f_1(\xi) = \sum_{k=-\infty}^{\infty} u_k (\xi - \xi_i)^{k+\sigma_1}, \quad f_2(\xi) = \sum_{k=-\infty}^{\infty} v_k (\xi - \xi_i)^{k+\sigma_2} + aw_1 \ln(\xi - \xi_i) \]  

in the case that \( \sigma_1 = \sigma_2 \).

The necessary conditions for a point \( \xi_i \) to be a regular singular point are that \( p(\xi) \) has a pole of at most first order and \( q(\xi) \) one of at most second order, so that the expressions \( \lim_{\xi \to \xi_i} (\xi - \xi_i)p \) and \( \lim_{\xi \to \xi_i} (\xi - \xi_i)^2q \) remain finite. As a result,
differential equation (1) can be re-written in the form
\[ f'' + \frac{p_1}{\xi - \xi_i} f' + \frac{q_1}{(\xi - \xi_i)^2} f = 0. \]  

(4)

A prerequisite for this to be the case is that the complex functions \( p_1(\xi) = \sum_{j=0}^{n} p_j(\xi - \xi_i)^j \) and \( q(\xi) = \sum_{j=0}^{n} q_j(\xi - \xi_i)^j \) are holomorphic in an annulus \( K \) of radius \( R \), i.e. in the local domain \( 0 < |\xi - \xi_i| < R \), for arbitrary linear coefficients \( p_j \) and \( q_j \).

The relation above can also be written down in the form
\[ f'' + \sum_{i=1}^{n} \frac{\gamma_i}{\xi - \xi_i} f' + \frac{V}{\prod_{i=0}^{n} (\xi - \xi_i)} f = 0, \]  

(5)

where the \( \gamma_i \) are constant coefficients and \( V = V(\xi) \) is a polynomial of degree \( n - 2 \), usually called the Van Vleck polynomial.

One of the main differences to differential equations with irregular singular points, which show a stronger singular behavior, is that the coefficients of a differential equation with regular singular points can be expanded in the vicinity of a singular point \( \xi_i \) (for fixed \( i \)) in a Laurent series with a finite instead of an infinite number of negative exponents. Accordingly, one of the two linearly independent solutions must be of the form
\[ f_1(\xi) = \sum_{k=-m}^{\infty} u_k(\xi - \xi_i)^{k+\sigma_1} = (\xi - \xi_i)^{\sigma_1-m} (u_{-m} + u_{-m+1}(\xi - \xi_i) + ...) =: \sum_{k=0}^{\infty} w_k(\xi - \xi_i)^{k+\sigma_1-m} \]  

(6)

in full accordance with Fuchs’ theorem [7, 23]. Hence, it becomes clear that one of the solutions of (4) (resp. (5)) around a given singular point \( \xi = \xi_i \) for fixed \( i \) can be determined by using Frobenius’ method, i.e. by making a generalized power series ansatz of the form
\[ f_1(\xi) = \sum_{k=0}^{\infty} w_k(\xi - \xi_i)^{k+\rho_1}, \]  

(7)

where the so-called critical exponent \( \rho_1 \) must be determined by solving the indicial equation arising in the course of the calculation.

In the case of the second solution, on the other hand, due to the fact that (5) is a linear differential relation, the method of variation of parameters can be used to obtain another solution of the form \( f_2 = f_1 \int_{\xi^0}^{\xi} e^{\int_{\xi^0}^{\xi'} \frac{p(\xi'')}{(\xi' - \xi_i)^2} d\xi''} d\xi' \). This again gives an expression of the form (7), but for a different critical exponent.
$\rho_2$. However, based on the fact that the integrant of this second solution can also be expanded in a generalized Frobenius series, it is not hard to realize - in the event that $\rho_2 = \rho_1 + m$, where $m$ is some positive integer - that said solution is of the form

$$f_2(\xi) = A \sum_{k=0}^{\infty} w_k (\xi - \xi_j)^{k+\rho_1} \ln(\xi - \xi_j) + \sum_{k=0}^{\infty} v_k (\xi - \xi_j)^{k+\rho_2},$$  \hspace{1cm} (8)$$

where $A$ is an arbitrary constant. Of course, the corresponding power series have positive radii of convergence. A rescaling of the solutions $f_1(\xi)$ and $f_2(\xi)$ of (3) by a factor of $(\xi - \xi_j)^\alpha$, where $\alpha$ is some positive integer, yields again an equation of type (5).

Another related way of approaching the problem solving differential equation (5) is to bring the equation into the form

$$\prod_{i=1}^{n} (\xi - \xi_i)^{f''} + \prod_{i=1}^{n} (\xi - \xi_i) \left( \sum_{i=1}^{n} \gamma_i (\xi - \xi_i)^f \right) + V f = 0$$  \hspace{1cm} (9)$$

and to seek polynomial solutions of the resulting expression. These solutions, if they exist, are then given by what are called Heine–Stieltjes polynomials (sometimes also called Stieltjes polynomials) [10, 16, 25, 28], which form the basis for the construction of ellipsoidal harmonics and their generalizations [29]. In the case that no such polynomial solutions of (9) exist, the solution will be of the form

$$f_1(\xi) = \sum_{k=0}^{\infty} w_k \xi^{k+\rho_1}.$$  \hspace{1cm} (10)$$

In order to obtain the second possible solution with the aid of the already determined one, one may then either use ansatz (8) or possibly first seek another solution by applying the method of variation of parameters in a slightly different way, namely by trying to find a solution $\Delta = \Delta(\xi)$ of (9) for $V = 0$ and then make an ansatz of the form

$$f_2 = f_1 \Delta + G,$$  \hspace{1cm} (11)$$

where $G = G(\xi)$ can be assumed to be a generalized power series of the form $G(\xi) = \sum_{k=0}^{\infty} v_k \xi^{k+\rho_2}$. Provided that the definition $\mathcal{U} \equiv \Delta'$ is used in the present context, $\Delta(\xi)$ is then obtained by solving the first order relation

$$\mathcal{U}' + \sum_{j=1}^{n} \frac{\gamma_j}{\xi - \xi_j} \mathcal{U} = 0,$$  \hspace{1cm} (12)$$

which follows directly from (9) for the special case $V(\xi) = 0$; leading to the result

$$\Delta = C \int \mathcal{U} d\xi,$$

where $\mathcal{U} = \mathcal{U}(\xi) = e^{-\sum_{j=1}^{n} \gamma_j \ln(\xi - \xi_j)}$ and $\Delta = \Delta(\xi) = \frac{df_2}{d\xi} - f_1 \frac{df_1}{d\xi}$.

is, of course, the Wronskian determinant.
Both methods are equivalent in that they lead to exactly the same results, from which it can be concluded that the respective pairs (7) and (8) and (10) and (11) can be superimposed in such a way that they represent one and the same solution of (5); just written down in different ways. The concrete choice of one of these methods for solving Fuchsian differential equations is therefore purely a matter of taste.

Anyway, the question of the convergence of the corresponding power series expressions still remains to be clarified. For the sake of simplicity, $\xi_1 = 0$ shall be assumed at this point. Given this case, equation (4) can be re-written in the form

$$\xi^2 f'' + \xi \sum_{j=0}^{n} p_j \xi^j f' + \sum_{j=0}^{n} q_j \xi^j f = 0. \quad (13)$$

Using ansatz (7), which in the given case coincides with (10), the system of equations

\[
\begin{align*}
  w_0 f_0(\rho_1) &= 0; \\
  w_1 f_0(\rho_1 + 1) + w_0 f_1(\rho_1) &= 0; \\
  w_2 f_0(\rho_1 + 2) + w_1 f_1(\rho_1 + 1) + w_0 f_2(\rho_1) &= 0; \\
  &\vdots \\
  w_n f_0(\rho_1 + n) + w_{n-1} f_1(\rho_1 + n - 1) + \ldots + w_0 f_n(\rho_1) &= 0; \\
\end{align*}
\]

(14)
is obtained for the coefficients $w_k$, where the abbreviations $f_0(x) = x(x-1) + p_0 x + q_0$ and $f_k(x) = p_k x + q_k$ were introduced for the sake of brevity. If $R$ is the radius of convergence of the coefficients occurring in (13) and $R_1 < R$, one can use the estimates $|p_k| < \frac{m_k}{R_1}$ and $|q_k| < \frac{m_k}{R_1}$ to conclude that $|p_k| + |q_k| < \frac{m_1 + m_2}{R_1}$ must hold. With these result at hand, (14) can be used to show that $|w_n| \leq \frac{|f_1(\rho_1 + n-1)|}{|f_0(\rho_1 + n)|} |w_{n-1}| + \frac{|f_2(\rho_1 + n-2)|}{|f_0(\rho_1 + n)|} |w_{n-2}| + \ldots + \frac{|f_n(\rho_1)|}{|f_0(\rho_1 + n)|} |w_0|$, and thus $|w_n| \leq \frac{M}{R_1} |w_{n-1}| + \frac{M}{R_1} |w_{n-2}| + \ldots + \frac{M}{R_1} |w_0|$. For the first $N$ coefficients, it is always possible find a sufficiently large number $P > 1 + M$ such that one can estimate $|w_k| \leq \frac{M^k}{R_1}$ for $k = 1, 2, \ldots, N - 1$. For fixed $n \geq N$, using the fact that $P^n(P - M - 1) > M > 0$, one then obtains $|w_n| \leq \frac{M^k}{R_1} (P^{n-1} + P^{n-2} + \ldots + 1) = \frac{P^n - 1}{P - 1} \frac{M^n}{R_1}$. The estimate thus remains correct for any given value of $n$. However, since the series $\sum_{k=0}^{\infty} \frac{P^k}{R_1} \xi^k$ is absolutely convergent in an annulus of radius $|\xi| < \frac{R}{P}$, this proves that the generalized power series expression (10)
represents a solution of (4) in the immediate vicinity of \( \xi_0 = 0 \). And since analog results can be deduced in the immediate vicinity of all other singular points, one comes to the conclusion that (7) indeed represents a solution of the differential equations (4) and (5).

Having clarified this, one can take now advantage of the fact that, by analytic continuation along any path not passing through the poles of \( p(\xi) \) and \( q(\xi) \), any set of linearly independent solutions being valid around a singular point of differential equation (5) gives a new set of solutions. However, it typically occurs in this context that the extended functions obtained from the analytic continuation of a previously given set of solutions in the vicinity of an isolated singular point \( \xi = \xi_i \) are multivalued complex functions, whose value at another point \( \xi = \xi_j \) depends on the chosen curve from \( \xi_i \) to \( \xi_j \). In particular, by choosing a path around one of the singular points, it often happens that this point becomes a branch point, which has the effect that a given pair \( f_1(\xi) \) and \( f_2(\xi) \) of solutions transitions into a new pair \( \tilde{f}_1(\xi) \) and \( \tilde{f}_2(\xi) \). However, by taking advantage of the fact that pairs of solutions form a vector space, it becomes clear that there is a monodromy transformation, which, after a certain singularity has been circulated either clock- or counterclockwise, turns one pair of solutions into another. In this context, the components of the corresponding monodromy matrix are constants and its determinant must be different from zero in order to ensure that the solutions \( \tilde{f}_1(\xi) \) and \( \tilde{f}_2(\xi) \) are linearly independent. The monodromy matrices are the generators of the monodromy group, which is prescribed by means of a finite-dimensional complex linear representation of the so-called fundamental group, which is the first and simplest homotopy group \([18, 19]\).

The monodromy concept is important in this context not least because its definition reveals an important property of analytical continuations along curves between regular singular points. This can be seen if one moves alongside special paths around an isolated singularity, which all have the same start and endpoints and can be continuously deformed into one another, because in this case the analytic continuations along different curves will yield the same results at their common endpoint, which is subject to the famous monodromy theorem.

Given the concrete form of the linearly independent solutions in the vicinity of an isolated singular point, the question is how the form of the solution looks at other singular points of the equation.

Thus, in the case that an analytic continuation of a given pair of solutions in the vicinity of a fixed regular singular point can be defined, it is ensured that those pairs of solutions, which are valid in the vicinity of all other regular singular points, can be converted into each other by simple linear transformations. The analytic continuation of solutions along curves, which ‘connect’ in this way pairs of regular singular points with each other, then defines a Riemann surface, i. e. a one-dimensional complex manifold, which is a connected Hausdorff space that is endowed with an atlas of charts to the open unit disk of the complex plane (whereas the transition maps between two overlapping charts are required to be holomorphic).

Consequently, it is clear how to transform any set of solutions of differential
equation (5), which is defined in the immediate vicinity of a given singular point \( \xi_i \), to one being defined in the immediate vicinity of another singular point \( \xi_j \). In this way, the respective form of solutions can be easily calculated in the vicinity of each isolated singular point, whereas it turns out the group of symmetries is isomorphic to the Coxeter group \( D_n \) of order \( n!2^{n-1} \). The total number of possible solutions can therefore easily be given for any number of regular singular points, which proves to be extremely interesting not only from a mathematical, but also from a physical point of view, since a large number of important equations of mathematical physics happen to be Fuchsian differential equations with a (typically) small number of regular singular points.

Of course, all of this has been known for a long time. In point of fact, the theory of Fuchsian differential equations has been a fixed, fully established part of the mathematical theory of second-order linear homogeneous differential equations with variable coefficients since the late nineteenth, early twentieth century. Nevertheless, there is an important point to be addressed here: What all Fuchsian differential equations with more than three regular singular points have in common is that their solutions, which have to be of the form (7) and (8) are difficult to write down explicitly. More precisely, it is hard to write down the form of the coefficients of the corresponding generalized power series expression for any finite number of terms of the associated recurrence relation. In particular, given the case that relation (9) is used as a starting point for the analysis and ansatz (7) is used for finding a solution around a given singular point \( \xi_j \), there occurs the problem that one (typically) has to deal with a \( n - 2 \)-term recursion relation of the form

\[
w_k + n - 2 = \sum_{j=1}^{n-2} m(j)_{k+n-2}w_k + n - j - 2,
\]

where the \( m(j)_{k} = m(j)(k) \) are (not necessarily smooth or even regular) functions of \( k \). Note that, of course, this is completely in line with what is obtained from relation (14).

Except for the almost trivial case \( n = 3 \), it is often a delicate problem to write down solutions of (15). Nevertheless, as shall now be discussed, this problem can be solved by considering special distributionally-valued coefficients that can be used to determine the exact form of solutions of (9), i.e. of solutions that are given in terms of power series expressions of the form (10) and (11). The next section shall clarify this in full detail.

2 Recurrence Relations and distributionally-valued Coefficients

In the previous section, it has been pointed out that in order to find solutions of the form (7) of representatives of the class of Fuchsian differential equations given by (5), one typically has to deal with the problem of solving associated
recovery of the sequence of coefficients \( \theta \). The form of the solutions of this relation is subject to the following theorem:

**Theorem 1:** Let (10) be a solution of (9) for fixed critical exponent \( p_1 \) and (15) be the associated recurrence relation. Given the choice \( w_1 := m(1)_0 w_0 \), \( w_2 := m(1)_1 w_1 + m(2)_1 w_0 = (m(1)_1 m(1)_0 + m(2)_1) w_0 \), ..., \( w_{n-2} := m(1)_{n-2} w_{n-3} + m(2)_{n-2} w_{n-4} + ... + m(n-2)_{n-2} w_0 \), solutions of (15) take the form

\[
w_{k+n-2} = \ll m(1), m(2), ..., m(n-2) \gg_{k+n-2} w_0,
\]

where the symbol \( \ll m(1), m(2), ..., m(n-2) \gg_{k+n-2} \) can be written in terms of a multi-linear form of the type

\[
\ll m(1), m(2), ..., m(n-2) \gg_{k+n-2} := W_{a_{k+n-2}...a_0} X_{k+n-2}^{a_{k+n-2}} ... X_0^{a_0}.
\]

In this context, each \( a_j \) runs from zero to \( k + n - 2 \) and the corresponding objects \( X_j^{a_j} = X^{a_j}(j) \) have the non-zero components \( X_j^0 = \theta(j) \), \( X_j^1 = m(1)_j \), \( X_j^2 = m(2)_j \), ..., \( X_j^{k+n-2} = m(k+n-2)_j \), where \( \theta(j) := \begin{cases} 0 & \text{if } j < 0 \\ 1 & \text{if } j \geq 0 \end{cases} \) is the Heaviside step function. The object \( W_{a_{k+n-2}...a_0} \) is defined in such a way that all its components are either zero or one. The non-zero components are exactly those for which on the one hand \( \sum_{j=0}^{k+n-2} a_j = k + n - 1 \) applies and, on the other hand, all indices that take the value zero occur only as successors of those that take a value of two, all pairs of indices that take the value zero combined occur only as successors of indices with a value of three, all triples of indices that take the value zero occur only as successors of indices with a value of four and so on and so forth. This implies in particular that all \( W_{0a_{k+n-3}...a_0}, W_{00a_{k+n-4}...a_0}, ..., W_{00...0a_{n-2}...a_0} \) are zero. It is also assumed that all coefficients with negative values are zero by definition.

**Proof:** As a basis for proving that (16) is a solution of (15), it must be proven by induction that

\[
W_{a_{k+n-2}...a_0} X_{k+n-2}^{a_{k+n-2}} ... X_0^{a_0} = m(1)_{k+n-2} W_{a_{k+n-3}...a_0} X_{k+n-3}^{a_{k+n-3}} ... X_0^{a_0} + \ldots + m(2)_{k+n-2} W_{a_{k+n-4}...a_0} X_{k+n-4}^{a_{k+n-4}} ... X_0^{a_0} + \ldots + m(n-2)_{k+n-2} W_{a_{n-2}...a_0} X_{n-2}^{a_{n-2}} ... X_0^{a_0}.
\]

Considering the sequence of coefficients \( w_1 := m(1)_0 w_0, w_2 := m(1)_1 w_1 + m(2)_1 w_0 = (m(1)_1 m(1)_0 + m(2)_1) w_0, ..., w_{n-2} := m(1)_{n-2} w_{n-3} + m(2)_{n-2} w_{n-4} + ... + m(n-2)_{n-2} w_0 \) in combination with relations (16) and (17), it immediately becomes clear that \( W_{a_0} X_0^{a_0} = W_1 X_1^{a_1} = m(1)_0, W_{a_1} X_1^{a_1} X_0^{a_0} = m(1)_1 m(1)_0 + m(2)_1 = m(1)_1 W_{a_0} X_0^{a_0} + m(2)_1, W_{a_2} X_2^{a_2} X_1^{a_1} X_0^{a_0} = m(1)_2 (m(1)_1 m(1)_0 + m(2)_1) + m(2)_2 m(1)_1 m(1)_0 + m(3)_0 = m(1)_2 W_{a_1} X_1^{a_1} X_0^{a_0} + m(2)_2 W_{a_0} X_0^{a_0} + m(3)_0 \) must hold. This gives the non-zero components \( W_1, W_{11}, W_{20}, W_{111}, W_{120}, W_{201}, \ldots \). Repeating this procedure for continuously increasing numbers of \( n \), it becomes clear that the above sequence of coefficients can be combined to multi-linear form of the type \( \ll m(1), m(2), ..., m(n-2) \gg_{n-2} = W_{a_{n-2}...a_0} X_{n-2}^{a_{n-2}} ... X_0^{a_0} \), provided that the condition \( \sum_{j=0}^{n-2} a_j = n - 1 \) is met and all indices that take the
value zero occur only as successors of those that take a value of two, all pairs of indices that take the value zero combined occur only as successors of indices with a value of three, all triples of indices that take the value zero occur only as successors of indices with a value of four and so on and so forth.

The first non-trivial case to be considered is \( k = 1 \). In this case, the above relation reads

\[
W_{a_1-a_2-a_3-a_4} X_{n-1}^{a_{n-1}} X_{n-2}^{a_{n-2}} \cdots X_{0}^{a_0} = W_{1a_3-a_2-a_4} X_{n-1}^{1} X_{n-2}^{a_{n-2}} \cdots X_{0}^{a_0} + \\
+ W_{2a_3-a_2-a_4} X_{n-1}^{2} X_{n-2}^{a_{n-2}} \cdots X_{0}^{a_0} + W_{3a_3-a_2-a_4} X_{n-1}^{3} X_{n-2}^{a_{n-2}} \cdots X_{0}^{a_0} + \cdots = \\
= m_{1n-1} W_{a_3-a_2-a_4} X_{n-2}^{a_{n-2}} \cdots X_{0}^{a_0} + m_{2n-1} W_{a_3-a_2-a_4} X_{n-3}^{a_{n-3}} \cdots X_{0}^{a_0} + \\
+ m_{3n-1} W_{a_3-a_2-a_4} X_{n-4}^{a_{n-4}} \cdots X_{0}^{a_0} + \cdots \tag{19}
\]

Given the fact that \( W_{1a_3-a_2-a_4} - W_{a_3-a_2-a_4} = W_{2a_3-a_2-a_4} - W_{a_3-a_2-a_4} = \cdots = 0 \) is fulfilled in the given case, since all non-zero components of both \( W_{1a_3-a_2-a_4} \) and \( W_{a_3-a_2-a_4} \) have the same value equal to one, assertion (18) defines a distributional relation which is actually fulfilled for all possible combinations of indices \( a_{n-3} \ldots a_0 \), \( a_{n-2} \ldots a_0 \), \ldots, \( a_{n-1} \ldots a_0 \) due to the fact that \( X_j^0 = 1 \) applies for all fixed non-negative values \( j \).

The next step is \( k = 2 \), but this works completely the same if all indices with values of \( n - 1 \) and \( n - 2 \) in (19) are replaced by indices with values \( n \) and \( n - 1 \).

Finally, the consistency of the induction step \( k \to k + 1 \) can easily be verified by replacing again all indices with values of \( n - 1 \) and \( n - 2 \) in (19) by indices with values \( k + n - 1 \) and \( k + n - 2 \), so that the validity of assertion (18) follows as a direct consequence. Thus, one obtains the result

\[
W_{a_{k+n-2} \ldots a_0} X_{k+n-2}^{a_{k+n-2}} \cdots X_{0}^{a_0} = \\
= \sum_{j=1}^{n-2} m_{(j)k+n-2} W_{a_{k+n-j-2} \ldots a_0} X_{k+n-j-2}^{a_{k+n-j-2}} \cdots X_{0}^{a_0} \tag{20}
\]

After multiplication with \( w_0 \), it then becomes clear that - under the given choice for the coefficients \( w_1, w_2, \ldots, w_{n-2} \) (12) actually represents a solution to (15).

\[\square\]

The additional solution \( f_2 = f_2(\xi) \) has a related, but more complicated form. This form can be obtained by inserting ansatz (10) for a different choice of critical exponent \( \rho_2 \) into (9), which then gives a second solution of precisely the same form. However, in the case that the corresponding critical exponents differ by a positive integer, ansatz (11) can be used, which then leads to the differential relation

\[
\prod_{j=1}^{n} (\xi - \xi_j) G'' + \prod_{j=1}^{n} (\xi - \xi_j) (\sum_{j=1}^{n} \frac{\gamma_j}{\xi - \xi_j}) G' + VG + 2 \prod_{j=1}^{n} (\xi - \xi_j) \phi f_1' = 0 \tag{21}
\]
in the power series $G(\xi) = \sum_{k=0}^{\infty} v_k \xi^{k+\rho_2}$. By expanding the term $\prod_{j=1}^{n} (\xi - \xi_j) \tilde{u} f'_j$ in a power series as well, one then typically ends up with the recurrence relation of the form

$$v_{k+n-2} = \sum_{j=1}^{n-2} m(j) k+n-2 v_{k+n-j-2} + \Phi_{k+n-2},$$

(22)

where the coefficient $\Phi_k$ results from the expansion. The solutions of this equation are subject to the following theorem:

**Theorem 2:** Let (11) be a solution of (9) for fixed critical exponents $\rho_1$ and $\rho_2$ and (22) be the associated recurrence relation. Given the choice $v_1 := m(1)_0 v_{00} + \Phi_0$, $v_2 := m(1)_1 v_1 + m(2)_1 v_0 + \Phi_1 = (m(1)_1 m(1)_0 + m(2)_1) v_0 + \Phi_1 + m(1)_1 \Phi_0$, ..., $v_{n-2} := m(1)_n - 2 v_{n-3} + m(2)_n - 2 v_{n-4} + ... + m(n-2)_n - 2 v_0 + \Phi_{n-2} + m(1)_n - 2 \Phi_{n-3} + (m(1)_1 m(1)_0 + m(2)_1) \Phi_{n-4} + ...$, solutions of recurrence relation (22) take the form

$$v_{k+n-2} = \lll m(1), m(2), \ldots, m(n-2) \rrr k+n-2$$

(23)

where, assuming that $\sum_{m=0}^{k+n-j-3} a_m = k + n - j - 2$ and $\sum_{m=k+n-j-2}^{k+n-2} a_m = j + 1$ applies in the present context, the symbol $\lll m(1), m(2), \ldots, m(k+n-2) \rrr k+n-2$ can be written in terms of a multi-linear form of the type

$$\lll m(1), m(2), \ldots, m(n-2) \rrr k+n-2 = \lll m(1), m(2), \ldots, m(n-2) \rrr k+n-2 v_0 + \Phi_{k+n-2} + \sum_{j=0}^{k+n-2} W_{a_k+n-2 \ldots a_k+n-j-2} X_{k+n-2}^a \ldots X_{k+n-j-2}^a \Phi_{k+n-j-3}.$$  

(24)

**Proof:** The first non-trivial case to be considered is $k = 1$. In this case, it can be concluded that (22) is solved if and only if

$$\lll m(1), m(2), \ldots, m(n-2) \rrr n-1 = \lll m(1), m(2), \ldots, m(n-2) \rrr n-1 + \Phi_{n-1}.$$  

(25)

Using (24), the left hand side of (25) can be written down in the form

$$\lll m(1), m(2), \ldots, m(n-2) \rrr n-1 = \lll m(1), m(2), \ldots, m(n-2) \rrr n-1 v_0 + \Phi_{n-1} + \sum_{i=0}^{n-2} W_{a_n-2 \ldots a_n-i-2} X_{n-2}^a \ldots X_{n-i-2}^a \Phi_{n-i-2},$$

(26)

whereas the right hand side reads
\[
\begin{align*}
\sum_{j=1}^{n-2} m(j)_{n-1} & \ll m(1), m(2), \ldots, m(n-2) \gg n-j-1 \ v_0 + \Phi_{n-1} + \\
+ \sum_{j=1}^{n-2} m(j)_{n-1} \left\{ \Phi_{n-j-1} + \sum_{i=0}^{n-j-2} W_{a_{n-j-2} \ldots a_{n-j-i-2}} X_{n-j-i}^{a_{n-j-2}} \ldots X_{n-j-i}^{a_{n-j-1}} \Phi_{n-j-i-2} \right\}.
\end{align*}
\]

Due to the validity of (20), one thus is left to prove

\[
\begin{align*}
\sum_{i=0}^{n-2} W_{a_{n-2} \ldots a_{n-i-2}} X_{n-2}^{a_{n-2}} \ldots X_{n-i-2}^{a_{n-i-2}} \Phi_{n-i-2} & = \\
\sum_{j=1}^{n-2} m(j)_{n-1} \left\{ \Phi_{n-j-1} + \sum_{i=0}^{n-j-2} W_{a_{n-j-2} \ldots a_{n-j-i-2}} X_{n-j-i}^{a_{n-j-2}} \ldots X_{n-j-i}^{a_{n-j-1}} \Phi_{n-j-i-2} \right\}.
\end{align*}
\]

Using here the fact that this relation can be re-written in the form

\[
\begin{align*}
W_{a_{n-2} \ldots a_{n-2}} X_{n-2}^{a_{n-2}} \Phi_{n-2} + W_{a_{n-2} \ldots a_{n-3}} X_{n-3}^{a_{n-2}} X_{n-2}^{a_{n-3}} \Phi_{n-3} + \ldots + W_{a_{n-2} \ldots a_0} X_{n-2}^{a_{n-2}} \ldots X_{0}^{a_0} \Phi_0 & = \\
= \left( m(1)n-1 \Phi_{n-2} + \left( m(1)n-1 W_{a_{n-3} \ldots a_0} X_{n-3}^{a_{n-3}} + m(2)n-1 \right) \Phi_{n-3} + \ldots + \right)
\end{align*}
\]

one finds that it is sufficient to show that the condition

\[
W_{a_{n-2} \ldots a_0} X_{n-2}^{a_{n-2}} \ldots X_{0}^{a_0} = \sum_{j=1}^{n-2} m(j)n-1 W_{a_{n-j-2} \ldots a_0} X_{n-j-2}^{a_{n-j-2}} \ldots X_{0}^{a_0}
\]

is met in the present context. However, by taking into account that the left hand side can be written in the form

\[
\begin{align*}
W_{a_{n-2} \ldots a_0} X_{n-2}^{a_{n-2}} \ldots X_{0}^{a_0} & = W_{1a_{n-3} \ldots a_0} X_{n-2}^{a_{n-3}} \ldots X_{0}^{a_0} \ldots W_{20 \ldots a_0} X_{n-2}^{a_{n-4}} \ldots X_{0}^{a_0} + \ldots + W_{n-200 \ldots 0} X_{n-2}^{a_{n-4}} \ldots X_{0}^{a_0},
\end{align*}
\]

one finds that condition (29) is actually met due to the fact that $W_{1a_{n-3} \ldots a_0} - W_{a_{n-3} \ldots a_0} = W_{20 a_{n-4} \ldots a_0} X_{n-2}^{a_0} - W_{a_{n-4} \ldots a_0} = \ldots = W_{n-200 \ldots 0} X_{n-3}^{a_0} X_{n-4}^{a_0} \ldots X_{0}^{a_0} - 1 = 0$ holds by definition for all $n-j > 0$ with $3 \leq j \leq n$.

Accordingly, since one can proceed completely the same way for any given larger value of $k$, it can be concluded that ansatz (23) can be used to solve recurrence relation (22) and therefore that (11) indeed represents a solution to (9) for the given choice of coefficients.

\[
\square
\]
As an application of the present investigation of Fuchs’ mathematical framework for solving second-order linear differential equations with a regular singular points, special types of differential equations belonging to this class shall be considered next, which play an important role in both mathematics and theoretical physics. Since, however, the mathematical literature pays much attention to the discussion of these special differential equations with a small number of regular singular points anyway, the remaining part of this section will be content with giving some relevant examples without discussing them in full detail on a case-by-case basis or listing the exact structure of the associated solutions and all their relevant properties. For a more detailed treatment of the subject, one should therefore consult the relevant mathematical literature, such as for instance the books by Bateman, Slavnov and Lay or Smirnov.

The simplest non-trivial example of a Fuchsian differential equation is one with three regular singular points. However, as it turns out, any equation of this type can be transformed into Gauss’ hypergeometric differential equation, which is the reason why it is not only indisputably the most prominent, but also the only relevant representative of this class. From a physical point of view, this is certainly because it leads to some important special cases, including the Legendre equation, the Jacobi equation, the Chebyshev equation and the Gegenbauer equation; all of which possess polynomial solutions that belong to the superordinate class of Heine-Stieltjes polynomials. By performing a linear transformation, which serves as a basis of a limiting procedure by means of which it can be achieved that a singularity lying at a finite position is shifted into infinity and thus coincides with the singularity already existing there, the said differential equation can be converted into the so-called confluent hypergeometric differential equation, which leads to other important special cases such as the Bessel, Hermite and Laguerre equations. It thus becomes apparent that a large number of the special functions relevant for mathematical physics are solutions of the hypergeometric equation in one form or another.

The said differential equation can be written down in the form

\[ \xi(\xi - 1)f'' + [(a + b + 1)\xi - c]f' + abf = 0, \tag{31} \]

where \(a, b\) and \(c\) are arbitrary linear coefficients. It has three singular points at 0, 1 and \(\infty\) and critical exponents \(\rho_1 = 0\) and \(\rho_2 = 1 - c\). The first solution leads to the two-term recursion relation

\[ w_{k+1} = m_{(1)k+1} w_k, \tag{32} \]

which contains the definition \(m_{(1)k+1} := \frac{(a+k)(b+k)}{(k+1)(c+k)}\). This relation can trivially be solved by continuous iteration, which gives the result

\[ w_{k+1} = \ll m_{(1)} \gg_{k+1} w_0, \tag{33} \]

which can be re-written by using the fact that \(\ll m_{(1)} \gg_{k+1} = W_{11}...X_{k+1}^1X_1^1...X_0^1 = \prod_{j=0}^{k+1} m_{(1)j} = \frac{(a)_{k+1}(b)_{k+1}}{(k+1)!(c)_{k+1}!} \), where \((x)_k = \frac{\Gamma(x+k)}{\Gamma(k)} = \{ \begin{array}{ll} x(x+1)...(x+k-1) & \text{if } k \geq 1 \\ 1 & \text{if } k = 0 \end{array} \) is the
Pochhammer symbol. The solution found is therefore given by the hypergeometric function \( f_1(\xi) = F(a, b; c; \xi) = \sum_{k=0}^{\infty} \frac{(\alpha)(\beta)_k}{(c)_k} \frac{\xi^k}{k!} \), as it ought to be. Depending on the values for \( a, b \) or \( c \), different cases have to be distinguished. In the case that \( 1 - c \) is not a positive integer, the second solution reads \( f_2(\xi) = \xi^{1-c} F(a - c + 1, b - c + 1, 2 - c; \xi) \). In turn, given the case that it is a positive integer, one of the solutions given above loses its meaning. In this case, as well as in the special case in which \( c = 1 \), one of the solutions will be of the form (8) and thus contains a logarithmic term. And while these more complicated cases can, of course, be treated within the formalism in the same way as all those other solutions that combine to Kummer’s collection of 24 solutions of the hypergeometric differential equation in the vicinity of all regular singular points \( 0, 1 \) and \( \infty \), it seems more reasonable in the present context, in order to demonstrate the practicality of the formalism at hand, to pass over directly to Fuchsian differential equations with a larger number of regular singular points.

For this purpose, consider the case of Fuchsian differential equations with four regular singular points. Any representative of this class can be reduced to the form

\[
f'' + \left( \frac{\gamma}{\xi} + \frac{\delta}{\xi - 1} + \frac{\epsilon}{\xi - a} \right) f' + \frac{\alpha \beta \xi - q}{\xi (\xi - 1)(\xi - a)} f = 0,
\]

which is known as Heun’s differential equation. The coefficients occurring in this form must satisfy \( \alpha + \beta - \gamma - \delta - \epsilon + 1 = 0 \); the constant \( q \) is called the accessory parameter. The regular singular points of this equation lie at \( 0, 1, a \) and \( \infty \). Heun’s differential equation admits 2 solutions, usually called Heun functions, which are contained in a set of no less than 192 different solutions. One can find a lot of information on these functions, such as for instance the fact that said functions are often written down as power series expressions in Riemann’s \( P \)-symbols and thus in series of hypergeometric functions, which are solutions of Riemann’s \( P \)-differential equation. The most prominent among these solutions are certainly the polynomial ones \( H_p(a, q; \alpha, \beta, \gamma, \delta; \xi) \), generally known as Heun polynomials, which are again special types of Heine-Stieltjes polynomials. The non-polynomial solutions are often denoted by \( H_\ell(a, q; \alpha, \beta, \gamma, \delta; \xi) \) and rarely written down explicitly, which is mainly due to the fact that the coefficients have to be determined by solving a three-term recursion relation. More precisely, the situation is as follows: Since a Frobenius ansatz of the form (10) leads to the critical exponents \( \rho_1 = 0 \) and \( \rho_2 = 1 - \gamma \), it seems justified to consider - given the case that \( \gamma \) is not a positive integer and there holds \( |\xi| < 1 \) - the power series expression \( f_1(\xi) = H_\ell(a, q; \alpha, \beta, \gamma, \delta; \xi) = \sum_{k=0}^{\infty} w_k \xi^k \), which, after the result \( w_1 := \ll m_{(1)}, m_{(2)} \gg \), \( w_0 = m_{(1)} w_0 = -\frac{a}{\alpha \gamma} w_0 \) is obtained, leads to the three-term recurrence relation

\[
w_{k+2} = m_{(1)} k+2 w_{k+1} + m_{(2)} k+2 w_k,
\]

according to which

\[
m_{(1)} k+2 = m_{(1)}(a, q; \alpha, \beta, \gamma, \delta; k+2) := \frac{(k+1)(k+\gamma)(k+\alpha+1)}{a(k+2)(k+\gamma+1)} + q,
\]

and

\[
m_{(2)} k+2 = m_{(2)}(a, q; \alpha, \beta, \gamma, \delta; k+2) := -\frac{(k+\alpha)(k+\delta)}{a(k+2)(k+\gamma+1)}
\]

is assumed to
apply by definition. As a direct result, it can be concluded that the solution of this recurrence relation must be of the form (17), so that \( w_{k+2} \approx m_{(1)}, m_{(2)} \gg k+2 \) \( w_0 \); at least provided that the objects \( X_j^a = X_j^a(j) \) have the components \( X_j^0 = \theta(j) \), \( X_j^1 = m_{(1)(j)} \) and \( X_j^2 = m_{(2)(j)} \). In particular, the situation is as follows: For \( k = 0 \), relation (17) reads \( \approx m_{(1)}, m_{(2)} \gg 2 = W_{a_1 a_0} X_1^{a_1} X_0^{a_0} = W_{11} X_1^1 X_0^1 + W_{20} X_2^2 X_0^0 = m_{(1)1} m_{(1)0} + m_{(2)0} \), which coincides exactly with what is obtained from (35). For \( k = 2 \), relation (17) reads \( \approx m_{(1)}, m_{(2)} \gg 3 = W_{a_1 a_0} X_2^{a_2} X_1^{a_1} X_0^{a_0} = W_{111} X_2^2 X_1^1 X_0^0 + W_{120} X_1^2 X_2^2 X_0^0 + W_{201} X_0^2 X_1^2 X_0^1 = m_{(1)2} m_{(1)1} m_{(1)0} + m_{(1)2} m_{(2)0} + m_{(2)2} m_{(1)0} \), which also coincides with what is obtained from (35). By further iteration, one then easily verifies that solutions of the Heun equation (34) can actually be written as a power series of the form (10) with coefficients of the form (16) and (17), respectively. Additionally, one finds that the solution which corresponds to the critical exponent \( 1 - \gamma \) in the vicinity of the singular point zero reads \( f_2(\xi) = \xi^{1-\gamma} \ell(\alpha, (a d + \epsilon)(1 - \gamma) + \delta; \alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma, \delta; \xi) \).

Note that Heun’s equation and its confluent form as well as their generalizations [13, 22] have numerous applications in both mathematics and theoretical physics. This is not least because Heun functions and their associated confluent forms cannot only be used for finding solutions to the Mathieu, Whittaker-Hill and Ince equations, they can also be used to determine solutions of the Schrödinger equation for different types of potentials in different dimensions [13, 22]. Besides that, the said functions can be used to solve both the Klein-Gordon equation for real and complex scalar fields and the Dirac equation for both massless and massive Fermions in quantum field theory on different curved geometric backgrounds as well as the the Regge-Wheeler equation, the Zerelli equation the and the Teukolsky master equation [2, 3, 4, 5, 6, 8, 9, 13, 14, 17, 20, 24, 26, 27, 31]. For a more detailed overview of the applications of the Heun equation and its confluent form, see [11].

However, the methods developed in the given work can also be applied to differential equations with a higher number of regular singular points, such as, for instance, the generalized Lamé equation, which has five regular singular points or to the so-called hypergeneralized Heun equation [3], which has six such points. In fact, it may be concluded that all non-trivial solutions of equations of the Fuchsian class can be represented in the form presented in this work.

Finally, it shall be pointed out that the method for determining the coefficients of solutions of Fuchsian differential equations presented in this work also has been successfully applied to another physical problem, namely the problem of how to calculate the exact structure of a so-called profile function corresponding to the gravitational field of a massless ultrarelativistic particle on the event horizon of a charged rotating black hole [22]. This problem leads to a Fuchsian differential equation with five regular singular points, the so-called generalized Dray-’t Hooft equation, which occurs as a special case of the generalized Lamé equation. As it turns out, the coefficients of solutions to this equation are exactly of the form presented in this work.
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