Brakke’s inequality for the thresholding scheme

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Abstract

We continue our analysis of the thresholding scheme from the variational viewpoint and prove a conditional convergence result towards Brakke’s notion of mean curvature flow. Our proof is based on a localized version of the minimizing movements interpretation of Esedoḡlu and the second author. We apply De Giorgi’s variational interpolation to the thresholding scheme and pass to the limit in the resulting energy-dissipation inequality. The result is conditional in the sense that we assume the time-integrated energies of the approximations to converge to those of the limit.

Keywords: Mean curvature flow, thresholding, MBO, diffusion generated motion

Mathematical Subject Classification: 35A15, 65M12, 74N20

1 Introduction

The thresholding scheme is a time discretization for mean curvature flow. Its structural simplicity is intriguing to both applied and theoretical scientists. Merriman, Bence and Osher \cite{MBO} introduced the algorithm in 1992 to overcome the numerical difficulty of multiple scales in phase-field models. Their idea is based on an operator splitting for the Allen-Cahn equation, alternating between linear diffusion and thresholding. The latter replaces the fast reaction coming from the nonlinearity, i.e., the reaction-term, in the Allen-Cahn equation. We refer to Algorithm 1.1 below for a precise description of the scheme in the multi-phase setting. The convolution can be implemented efficiently on a uniform grid using the Fast Fourier Transform and the thresholding step is a simple pointwise operation. Because of its simplicity and efficiency, thresholding received a lot of attention in the last decades. Large-scale simulations \cite{Xia, MBO-2, MBO-3} demonstrate the efficiency of a slight modification of the scheme. For applications in materials science and image segmentation it is desirable to design algorithms that are efficient enough to handle large numbers of phases but flexible enough to incorporate external forces, grain-dependent and even anisotropic surface energies. Not long ago, the natural extension to the multi-phase case \cite{EO1} was generalized to arbitrary surface tensions by Esedoḡlu and the second author \cite{EO2}. In this paper, it was realized that thresholding preserves the gradient-flow structure of (multi-phase) mean-curvature flow in the sense that it can be viewed as a minimizing movements scheme for an energy that $\Gamma$-converges to the total interfacial area. This viewpoint allowed to incorporate a wide class of surface tensions including the well-known Read-Shockley formula for small-angle grain boundaries \cite{Read-Shockley}.

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The development of thresholding schemes for anisotropic motions started with the work [19] of Ishii, Pires and Souganidis. Efficient schemes were introduced by Bonnetier, Bretin and Chambolle [7], where the convolution kernels are explicit and well-behaved in Fourier space but not necessarily in real space. The recent work [10] of Elsey and Esedoḡlu is inspired by the variational viewpoint [15] and shows that not all anisotropies can be obtained when structural features such as positivity of the kernel are required. However, variants of the scheme developed by Esedoḡlu and Jacobs [14] share the same stability conditions even for more general kernels.

The rigorous asymptotic analysis of thresholding schemes started with the independent convergence proofs of Evans [16] and Barles and Georgelin [5] in the isotropic two-phase case. Since the scheme preserves the geometric comparison principle of mean curvature flow, they were able to prove convergence towards the viscosity solution of mean curvature flow. Recently, Swartz and Yip [31] proved convergence for a smooth evolution by establishing consistency and stability of the scheme, very much in the flavor of classical numerical analysis. They prove explicit bounds on the curvature and injectivity radius of the approximations and get a good understanding of the transition layer. However, also their result does not generalize to the multi-phase case immediately. In our previous work [22] we established the convergence of thresholding to a distributional formulation of multi-phase mean-curvature flow based on the assumption of convergence of the energies. In [24], Swartz and the first author applied these techniques to the case of volume-preserving mean-curvature flow and other variants.

Since the works [5, 16] are based on the comparison principle, the proofs do not apply in the multi-phase case. Our guiding principle in this work is instead the gradient-flow structure of (multi-phase) mean curvature flow. In general, a gradient-flow structure is given by an energy functional and a metric tensor, which endows the configuration space with a Riemannian structure that encodes the dissipation mechanism. A simple computation reveals this structure for mean curvature flow. If the hypersurface \( \Sigma = \Sigma(t) \) evolves smoothly by its mean curvature (here and throughout we use the time scale such that \( \frac{2}{V} = H \)) the change of area is given by

\[
2 \frac{d}{dt} |\Sigma| = - \int_\Sigma 2V \cdot H = - \int_\Sigma |H|^2, \tag{1}
\]

where \( V \) denotes the normal velocity vector and \( H \) denotes the mean curvature vector of \( \Sigma \). Although (1) does not characterize the mean curvature flow one can read off the metric tensor, the \( L^2 \)-metric \( \int_\Sigma |V|^2 \) on the space of normal vector fields, when fixing the energy to be the surface area. However, some care needs to be taken when dealing with this metric as for example the geodesic distance vanishes identically [28]. The implicit time discretization developed by Almgren, Taylor and Wang [2] and Luckhaus and Sturzenhecker [25] makes use of this gradient-flow structure. In fact, it inspired De Giorgi to define a similar implicit time discretization for abstract gradient flows which he named “minimizing movements”. His abstract scheme consists of a family of minimization problems that mimic the principle of a gradient flow moving in direction of the steepest descent in an energy landscape. The configuration \( \Sigma^n \) at time step \( n \) is obtained from its predecessor \( \Sigma^{n-1} \) by minimizing \( E(\Sigma) + \frac{h}{2n} \text{dist}^2(\Sigma, \Sigma^{n-1}) \), where \( \text{dist} \) denotes the geodesic distance induced by the Riemannian structure and \( h > 0 \) denotes the time-step size. In the case of a Euclidean configuration space, the scheme boils down to the implicit Euler scheme. In its Riemannian version, it has been used for applications in partial differential equations and for instance allowed Jordan, Kinderlehrer and the second author [20] to interpret diffusion equations as gradient flows for the entropy w.r.t. the Wasserstein distance. In view of the degeneracy in the case of mean curvature flow it is evident.
that the scheme in \cite{2,25} uses a proxy for the geodesic distance. The replacement for the distance of two boundaries $\Sigma = \partial \Omega$ and $\tilde{\Sigma} = \partial \tilde{\Omega}$ is the (non-symmetric) quantity $4 \int_{\Omega \Delta \tilde{\Omega}} d_{\tilde{\Omega}} \, dx$, where $d_{\tilde{\Omega}}$ denotes the (unsigned) distance to $\partial \tilde{\Omega}$. Chambolle \cite{9} showed that the scheme \cite{2,25} which seems academic at a first glance can be implemented rather efficiently. Recently, Bellettini and Kholmatov \cite{6} analyzed the scheme in the multi-phase case. However, neither a conditional convergence result to a distributional BV-solution, nor one to a Brakke flow are available yet.

Also Brakke’s pioneering work \cite{8} is inspired by the gradient-flow structure of mean curvature flow. His definition is similar to the one of an abstract gradient flow and characterizes solutions by the optimal dissipation of energy. Brakke measures the dissipation of energy only in terms of the mean curvature. As \cite{1} cannot characterize the solution, Brakke monitors localized versions of the surface area, which leads to a sensible notion of solution; we refer to Definition \ref{def:loc} for a precise definition in our context of sets of finite perimeter. Ilmanen \cite{18} used a phase-field version of Huisken’s monotonicity formula \cite{17} to prove the convergence of solutions to the scalar Allen-Cahn equation to Brakke’s mean curvature flow. Extending his proof to the multi-phase case is a challenging open problem. Only recently, Simon and the first author \cite{23} proved a conditional convergence result for the vector-valued Allen-Cahn equation very much in the spirit of \cite{25,22}. However, an unconditional result is not yet available. Even the construction of non-trivial global solutions to multi-phase mean-curvature flow has only been achieved recently by Tonegawa and Kim \cite{21}.

In the present work we establish the convergence of the thresholding scheme to Brakke’s motion by mean curvature. As our previous result \cite{22}, also this one is only a conditional convergence result in the sense that we assume the time-integrated energies to converge to those of the limit. Our proof is based on the observation that thresholding does not only have a global minimizing movements interpretation, but indeed solves a family of localized minimization problems. In Section 2 we state our main results, in particular Theorem \ref{thm:main}. We use De Giorgi’s variational interpolation for these localized minimization problems to derive an exact energy-dissipation relation and pass to the limit in the inequality with help of our strengthened convergence. We first recall the known results from the abstract framework of gradient flows in metric spaces (cf. Chapter 3 in \cite{4}). Then we pass to the limit $h \to 0$ in these terms with help of our strengthened convergence. It is worth pointing out that such a result is not known for the time discretization scheme \cite{2,25}.

The starting point for our analysis of thresholding schemes is the minimizing movements interpretation of Esedoğlu and the second author \cite{15}. Let us explain this interpretation with help of the example of the two-phase scheme. The combination $\chi^n = 1_{\{G_h \ast \chi^{n-1} > \frac{1}{2}\}}$ of convolution and thresholding is equivalent to minimizing $E_h(\chi) + \frac{1}{2h}d^2_h(\chi, \chi^{n-1})$, where $E_h$ is an approximation of the perimeter functional and $d_h$ is a metric. The latter serves as a proxy for the induced distance, just like $4 \int_{\Omega \Delta \tilde{\Omega}^{n-1}} d_{\tilde{\Omega}^{n-1}} \, dx$ in the minimizing movements scheme of Almgren, Taylor and Wang \cite{2}, and Luckhaus and Sturzenhecker \cite{25}. The $\Gamma$-convergence of similar functionals has been developed some time ago by Alberti and Bellettini \cite{1} and more recently by Ambrosio, De Philippis and Martinazzi \cite{3}, and was proven for the functionals $E_h$ by Miranda, Pallara, Paronetto and Preunkert \cite{29}. Esedoğlu and the second author found a simpler proof in the case of the energies $E_h$, which extends to the multi-phase case.

Let us recall the thresholding scheme and the basic notation.

**Algorithm 1.1.** Given the partition $\Omega_{n-1}^{1}, \ldots, \Omega_{n-1}^{P}$ at time $t = (n-1)h$, obtain the partition $\Omega_{n}^{1}, \ldots, \Omega_{n}^{P}$ at time $t = nh$ by the following two operations:
1. **Convolution step:** \( \phi_i := G_h * \left( \sum_{j \neq i} \sigma_{ij} \mathbf{1}_{\Omega_i^{n-1}} \right) \).

2. **Thresholding step:** \( \Omega_i^n := \{ \phi_i < \phi_j \text{ for all } j \neq i \} \).

Here and throughout the paper

\[
G_h(z) := \frac{1}{(2\pi h)^d/2} \exp \left( -\frac{|z|^2}{2h} \right)
\]

denotes the centered Gaussian of variance \( h \), which we also think of as the heat kernel at time \( \frac{h}{2} \).

We assume the matrix of surface tensions \( \sigma = (\sigma_{ij})_{i,j} \) to satisfy the obvious relations

\[
\sigma_{ij} = \sigma_{ji} > 0 \text{ for } i \neq j, \quad \sigma_{ii} = 0
\]

and the usual (strict) triangle inequality

\[
\sigma_{ij} < \sigma_{ik} + \sigma_{kj} \text{ for all pairwise different } i, j, k.
\]

Furthermore, we ask the matrix \( \sigma \) to be conditionally negative definite \( \sigma < 0 \) as a bilinear form on \((1, \ldots, 1)\)\(^\perp\). This condition can be simply spelled out as \( \xi \cdot \sigma \xi \leq -|\sigma| |\xi|^2 \) for all \( \xi \in \mathbb{R}^P \) such that \( \sum_{i=1}^P \xi_i = 0 \), where \( \sigma > 0 \) is a positive constant. The condition was introduced by Esedo˘ glu and the second author \([15]\) and guarantees the dissipation of energy. Indeed, the conditional negativity \((2)\) ensures that

\[
\frac{|\xi|^2}{|\sigma|} := -\xi \cdot \sigma \xi = -\sum_{i,j} \sigma_{ij} \xi_i \xi_j, \quad \text{for } \xi \in \mathbb{R}^P \text{ s.t. } \sum_i \xi_i = 0
\]

defines a norm \( |\cdot|_\sigma \) on the space \((1, \ldots, 1)\)\(^\perp\). For convenience we will work with periodic boundary conditions, i.e., on the flat torus \([0, \Lambda)^d\). We write \( \int dx \) short for \( \int_{[0, \Lambda)^d} dx \) and \( \int dz \) short for \( \int_{\mathbb{R}^d} dz \).

Furthermore, \( \chi^n \) given by \( \chi_i^n := \mathbf{1}_{\Omega_i^n}, \ i = 1, \ldots, P \), denotes the vector of characteristic functions of the phases \( \Omega_i^n \) at time step \( n \) and we denote its piecewise constant interpolation by

\[
\chi_h(t) := \chi^n = \left( \mathbf{1}_{\Omega_1^n}, \ldots, \mathbf{1}_{\Omega_P^n} \right) \quad \text{for } t \in [nh, (n+1)h).
\]

However, we will mostly use a nonlinear interpolation which will be introduced later. Selim Esedo˘ glu and the second author \([15]\) showed that thresholding preserves the gradient-flow structure of (multi-phase) mean curvature flow in the sense that it can be viewed as a minimizing movements scheme

\[
\chi^n = \arg \min_u \left\{ E_h(u) + \frac{1}{2h} d_h^2(u, \chi^{n-1}) \right\}, \quad \text{for } t \in [nh, (n+1)h), \quad \text{(3)}
\]

where the minimum runs over all measurable \( u: [0, \Lambda)^d \rightarrow \mathbb{R}^P \) such that \( 0 \leq u_i \leq 1, \ i = 1, \ldots, P \) and \( \sum_i u_i = 1 \) a.e.. Here the dissipation functional

\[
\frac{1}{2h} d_h^2(u, \chi) := \frac{1}{\sqrt{h}} \int |G_{h/2} * (u - \chi)|_\sigma^2 \, dx = -\frac{1}{\sqrt{h}} \int G_{h/2} * (u - \chi) \cdot \sigma G_{h/2} * (u - \chi) \, dx \quad \text{(4)}
\]
is, because of (2), the square of a metric and the energy

\[ E_h(u) := \frac{1}{\sqrt{h}} \int u \cdot \sigma G_h * u \, dx \]  \tag{5} 

is an approximation of the total interfacial area. Indeed, this functional Γ-converges to the energy

\[ E(\chi) := c_0 \sum_{i,j} \sigma_{ij} \int \frac{1}{2} (|\nabla \chi_i^i| + |\nabla \chi_j^j| - |\nabla (\chi_i + \chi_j)|) \],

defined for partitions \( \chi: [0, \Lambda]^d \to \{0,1\}^P \) s.t. \( \sum_i \chi_i = 1 \). Writing \( \Omega_i = \{ \chi_i = 1 \} \) and \( \partial^* \Omega_i \) for the reduced boundary of \( \Omega_i \), the term

\[ \int \frac{1}{2} (|\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)|) = \mathcal{H}^{d-1}(\partial^* \Omega_i \cap \partial^* \Omega_j), \]

is the measure of the interface between Phases \( i \) and \( j \), so that the energy \( E \) is indeed the total interfacial area

\[ E(\chi) = c_0 \sum_{i,j} \sigma_{ij} \mathcal{H}^{d-1}(\partial^* \Omega_i \cap \partial^* \Omega_j). \]

The constant \( c_0 \) is given by the first moment of \( G \), i.e.,

\[ c_0 = \int_0^\infty |z| G(z) \, dz = \frac{1}{\sqrt{2\pi}}. \]

The above mentioned Γ-convergence is an immediate consequence of the pointwise convergence of these functionals and the monotonicity property

\[ E_{N^2 h}(u) \leq E_h(u) \quad \text{for all} \ u: [0, \Lambda]^d \to [0,1]^P, \ \text{s.t.} \ \sum_{i=1}^P u_i = 1, \ h > 0, \ \text{and} \ N \in \mathbb{N}, \ \tag{6} \]

see \cite{15} Lemma A.2. We write \( A \lesssim B \) to express that \( A \leq CB \) for a generic constant \( C < \infty \) that only depends on the dimension \( d \), on the size \( \Lambda \) of the domain, and the matrix \( \sigma \) of surface tensions. By \( A = O(B) \) we mean the quantitative \( |A| \lesssim B \) while \( A = o(B) \) as \( h \to 0 \) means the qualitative \( \frac{A}{B} \to 0 \) as \( h \to 0 \).

2 Brakke’s inequality and main result

The main statement of this work is Theorem \( \ref{2} \) below. Assuming there was no drop of energy as \( h \to 0 \), i.e.,

\[ \int_0^T E_h(\chi^h) \, dt \to \int_0^T E(\chi) \, dt, \]  \tag{7} 

it states that the limit of the approximate solutions satisfies a \( BV \)-version of Brakke’s inequality \cite{8}.
Brakke’s inequality is a weak formulation of motion by mean curvature $2V = H$ and is motivated by the following characterization of the normal velocity. Given a smoothly evolving hypersurface $\partial \Omega(t) = \Sigma(t)$ with normal velocity vector $V$ we have

$$\frac{d}{dt} \int_{\Sigma} \zeta = \int_{\Sigma} \left( -\zeta H \cdot V + V \cdot \nabla \zeta + \partial_{t} \zeta \right)$$

for any smooth test function $\zeta \geq 0$. The converse is also true: Given a function $V : \Sigma \rightarrow \mathbb{R}$ such that $|V|$ holds for any such test function $\zeta \geq 0$ then $V$ is the normal velocity of $\Sigma$. In the pioneering work [3], Brakke uses this idea for his definition of the equation $2V = -H$ to extend the concept of motion by mean curvature to general varifolds. We recall his definition in our more restrictive setting of finite perimeter sets, which in the smooth two-phase case simplifies to the inequality

$$2 \frac{d}{dt} \int_{\Sigma} \zeta \leq \int_{\Sigma} \left( -\zeta |H|^2 + H \cdot \nabla \zeta + 2 \partial_{t} \zeta \right).$$

**Definition 2.1.** We say that the time-dependent partition $\chi : (0, T) \times [0, \Lambda)^d \rightarrow \{0, 1\}^P \in BV$ with $\sum_i \chi_i = 1$ a.e. moves by mean curvature with initial data $\chi^0 : [0, \Lambda)^d \rightarrow \{0, 1\}^P \in BV$ with $\sum_i \chi_i^0 = 1$ a.e. if there exists a $\sum_{i,j} \sigma_{ij} \frac{1}{2} (|\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)|)$ dt-measurable normal vector field $H : (0, T) \times [0, \Lambda)^d \rightarrow \mathbb{R}^d$ with

$$\sum_{i,j} \sigma_{ij} \int_0^T \left( |H|^2 + \frac{1}{2} (|\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)|) \right) dt < \infty,$$

which is the mean curvature vector of the partition in the sense that for all test vector fields $\xi \in \mathcal{C}_c^\infty((0, T) \times [0, \Lambda)^d, \mathbb{R}^d)$

$$\sum_{i,j} \sigma_{ij} \int_0^T \left( \nabla \cdot \xi - \nu_i \cdot \nabla \xi \nu_i \right) \frac{1}{2} (|\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)|) dt$$

$$= -\sum_{i,j} \sigma_{ij} \int_0^T \int H \cdot \xi \frac{1}{2} (|\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)|) dt,$$

such that for any test function $\zeta \in \mathcal{C}^\infty([0, T] \times [0, \Lambda)^d)$ with $\zeta \geq 0$ and $\zeta(\cdot, T) = 0$ we have

$$\sum_{i,j} \sigma_{ij} \int_0^T \left( -\zeta |H|^2 + H \cdot \nabla \zeta + 2 \partial_{t} \zeta \right) \frac{1}{2} (|\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)|) dt$$

$$\geq -\sum_{i,j} \sigma_{ij} \int_0^T \zeta(\cdot, 0) \frac{1}{2} (|\nabla \chi_i^0| + |\nabla \chi_j^0| - |\nabla (\chi_i^0 + \chi_j^0)|)$$

(11)

Here and throughout, $\nu_i$ denotes the measure theoretic normal of Phase $i$ characterized by the equation $\nabla \chi_i = \nu_i |\nabla \chi_i|$. Note that with this choice, $\nu_i$ points inwards.

Equation (10) encodes not only that $H$ is the mean curvature vector along the smooth part of the surface cluster but furthermore enforces the Herring angle condition along triple junctions, which comes from the integration by parts rule for smooth hypersurfaces $\Sigma$ with boundary $\Gamma$:

$$\int_{\Sigma} (\nabla \cdot \xi - \nu \cdot \nabla \xi \nu) = \int_{\Gamma} \xi \cdot b - \int_{\Sigma} H \cdot \xi,$$

$$\int_{\Sigma} (\nabla \cdot \xi - \nu \cdot \nabla \xi \nu) = \int_{\Gamma} \xi \cdot b - \int_{\Sigma} H \cdot \xi,$$
where $b$ denotes its conormal.

Equation (11) does not only encode the mean curvature flow equation via the optimal dissipation of energy but also the initial data $\chi^0$ in a weak sense.

**Theorem 2.2 (Brakke’s inequality).** Given initial data $\chi^0: [0, \Lambda)^d \to \{0, 1\}^P$ with $E(\chi^0) < \infty$ and a finite time horizon $T < \infty$, for any sequence there exists a subsequence $h \downarrow 0$ such that the approximate solutions given by Algorithm 1.1 converge to a limit $\chi: (0, T) \times [0, \Lambda)^d \to \{0, 1\}^P$ in $L^1$ and a.e. in space-time. Given the convergence assumption (7), $\chi$ evolves by mean curvature in the sense of Definition 2.1.

**Remark 2.3.** Given initial conditions $\chi^0$ with $E(\chi^0) < \infty$ the compactness in [22, Proposition 2.1] yields a subsequence such that $\chi^h \to \chi$ in $L^1$ and a.e. for a partition $\chi$ with $\text{ess sup}_t E(\chi(t)) \leq E(\chi^0)$.

This statement is similar to our result in [22]. There we proved the convergence of thresholding towards a distributional formulation of (multi-phase) mean-curvature flow, the same notion as in [25]. Under the same assumption (7) as in the present work, for any $i = 1, \ldots, P$ we constructed a $|\nabla \chi_i| \, dt$-measurable function $\tilde{V}_i: (0, T) \times [0, \Lambda)^d \to \mathbb{R}$ with

$$
\int_0^T \int \tilde{V}_i^2 |\nabla \chi_i| \, dt < \infty,
$$

which is the (scalar) normal velocity of the $i$-th phase in the sense that

$$
\int_0^T \int \partial_t \zeta \chi_i \, dx \, dt + \int \zeta(\cdot, 0) \chi_i^0 \, dx = -\int_0^T \int \zeta \tilde{V}_i |\nabla \chi_i| \, dt
$$

for all $\zeta \in C^\infty((0, T) \times [0, \Lambda)^d)$ with $\zeta(\cdot, T) = 0$, such that

$$
\sum_{i,j} \sigma_{ij} \int_0^T \int \left( \nabla \cdot \xi - \nu_i \cdot \nabla \xi \nu_i - 2 \xi \cdot \nu_i \tilde{V}_i \right) \frac{1}{2} (|\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)|) \, dt = 0 \quad (12)
$$

for all $\xi \in C^\infty_0((0, T) \times [0, \Lambda)^d, \mathbb{R}^d)$.

Without any regularity assumption, none of the two formulations is stronger in the sense that it implies the other. Nevertheless (12) requires more regularity as it is formulated for sets of finite perimeter, whereas Brakke’s inequality naturally extends to general varifolds. Finally, we note that our proof here is much softer than the result (12) proved in our earlier work [22].

### 3 De Giorgi’s variational interpolation and idea of proof

It is a well-appreciated fact that a classical gradient flow $\dot{u}(t) = -\nabla E(u(t))$ of a smooth energy functional $E$ on a Riemannian manifold can be characterized by the optimal rate of dissipation of the energy $E$ along the solution $u$:

$$
\frac{d}{dt} E(u(t)) \leq -\frac{1}{2} |\dot{u}(t)|^2 - \frac{1}{2} |\nabla E(u(t))|^2.
$$

(13)

This is the guiding principle in generalizing gradient flows to metric spaces where one replaces $|\dot{u}|$ by the metric derivative and $|\nabla E(u)|$ by some upper gradient, e.g. the local slope $|\partial E(u)|$, see [19] for a definition in our context.
As discussed in the introduction, mean curvature flow can be viewed as a gradient flow in the sense that for a smooth evolution \( \Sigma = \Sigma(t) \) the energy, which in this case is the surface area \( |\Sigma(t)| \), satisfies the inequality
\[
2 \frac{d}{dt} |\Sigma| = -\int_{\Sigma} 2 V \cdot H \leq -\frac{1}{2} \int_{\Sigma} |H|^2 - \frac{1}{2} \int_{\Sigma} |2V|^2.
\]
While in the abstract framework, the dissipation of the energy is measured w.r.t. both terms \( |\dot{u}|^2 \approx \int_{\Sigma} |2V|^2 \) and \( |\partial E(u)|^2 \approx \int_{\Sigma} |H|^2 \), Brakke measures the rate only in terms of the local slope \( \int_{\Sigma} H^2 \) but asks for the localized version \( \ref{local_energy} \), which due to \( \ref{energy_bound} \) in the simple setting of a single surface evolving smoothly by its mean curvature is precisely \( \ref{local_energy} \) with equality.

The basis of this work is the approximate version of Brakke’s inequality, Lemma \ref{local_min_self} below. In view of the minimizing movements interpretation \( \ref{minmovements} \) it should be feasible to obtain at least the global inequality
\[
2 \frac{d}{dt} |\Sigma| \leq -\int_{\Sigma} |H|^2
\]
but the localized inequality \( \ref{local_energy} \) would be still out of reach. The lemma states that thresholding does not only solve the global minimization problem \( \ref{minmovements} \) but a whole family of local minimization problems, which will allow us to establish the family of localized inequalities \( \ref{local_energy} \).

**Lemma 3.1** (Local minimality). Let \( \chi^n \) be obtained from \( \chi^{n-1} \) by one iteration of Algorithm \ref{algo} and \( \zeta \geq 0 \) an arbitrary test function. Then
\[
\chi^n = \arg \min_u \left\{ E_h(u, \chi^{n-1}; \zeta) + \frac{1}{2h} d_h^2(u, \chi^{n-1}; \zeta) \right\},
\]
where the minimum runs over all \( u : [0, \Lambda)^d \to [0, 1]^d \) with \( \sum_i u_i = 1 \) a.e.. By \( d_h(u, \chi; \zeta) \) we denote the localization of the metric \( d_h(u, \chi) \) given by
\[
\frac{1}{2h} d_h^2(u, \chi; \zeta) := \frac{1}{\sqrt{h}} \int \zeta |G_{h/2} \ast (u - \chi)|^2_{\sigma} dx,
\]
which is again a (semi-)metric on the space of all such \( u \)’s as above and in particular satisfies a triangle inequality. By \( E_h(u, \chi; \zeta) \) we denote the localized (approximate) energy incorporating the localization error in both energy and metric:
\[
E_h(u, \chi; \zeta) := \frac{1}{\sqrt{h}} \int \zeta u \cdot \sigma G_h \ast u \, dx + \frac{1}{\sqrt{h}} \int (u - \chi) \cdot \sigma [\zeta, G_h] \chi \, dx \nonumber
\]
\[
- \frac{1}{\sqrt{h}} \int (u - \chi) \cdot \sigma [\zeta, G_{h/2}] G_{h/2} \ast (u - \chi) \, dx.
\]
Here and throughout the paper
\[
[\zeta, G_h] \ast u := \zeta G_h \ast u - G_h \ast (\zeta u) \approx -\nabla \zeta \cdot h \nabla G_h \ast u
\]
denotes the commutator of the multiplication with the (smooth) function \( \zeta \) and the convolution with the kernel \( G_h \), both of which act componentwise on the vector \( u \).
Let us briefly comment on the structure of the localized energy $E_h$. First, by definition of $E_h$ we have
\[ E_h(u, u; \zeta) = \frac{1}{\sqrt{h}} \int \zeta \cdot \sigma G_h * u \, dx \quad \text{and} \quad E_h(u, \chi; 1) = E_h(u), \] cf. [3],
so that in particular we recover the minimizing movements interpretation [8] in the case $\zeta \equiv 1$.

Second, with the localization $\zeta$, the first integral in the definition of $E_h$ is an approximation of the localized total interfacial energy $c_0 \sum_{i,j} \sigma_{ij} \int_{\Sigma^1} \zeta$. We will see shortly that the second term gives rise to the transport term in Brakke’s inequality, while the last term, the commutator arising from the metric term, will be shown to be negligible in the limit $h \to 0$ for all quantities under consideration here.

Thanks to the local minimization property [14] of the thresholding scheme we can apply the abstract framework of De Giorgi, cf. Chapters 1–3 in [4], to this localized setting. As for any minimizing movements scheme, the comparison of $\chi^n$ to the previous time step $\chi^{n-1}$ in the minimization problem [14] yields an energy-dissipation inequality which serves well as an a priori estimate, but which fails to be sharp by a factor of 2. To obtain a sharp inequality we follow the ideas of De Giorgi. We introduce his variational interpolation $u^h$ of $\chi^n$ and $\chi^{n-1}$: For $t \in (0, h]$ and $n \in \mathbb{N}$ we let
\[ u^h((n-1)h + t) := \arg\min_u \left\{ E_h(u, \chi^{n-1}; \zeta) + \frac{1}{2t} d_h^2(u, \chi^{n-1}; \zeta) \right\}. \] (17)

Note that the choice of $u^h$ is not necessarily unique, but given $u^h(nh) = \chi^n$, we can choose $u^h$ to depend continuously on $t$ w.r.t. the metric $d_h$. Comparing $u^h(t)$ with $u^h(t+\delta t)$ in this minimization problem and taking the limit $\delta t \to 0$ while keeping $h$ fixed, one obtains the sharp energy-dissipation inequality along this interpolation, the following approximate version of Brakke’s inequality [11].

It is worth pointing out that opposed to the special case $t = h$, we do not have an explicit formula for the interpolations $u^h$, and there is no guarantee for $u^h \in \{0,1\}$; indeed we expect that generically $u^h \in (0,1)$.

**Corollary 3.2 (Approximate Brakke inequality).** For any test function $\zeta \geq 0$, a time-step size $h > 0$ and $T = Nh$ we have
\[
\frac{h}{2} \sum_{n=1}^{N} \left| \partial E_h(\cdot, \chi^{n-1}; \zeta) \right|^2(\chi^n) + \frac{1}{2} \int_0^T \left| \partial E_h(\cdot, u^h(t); \zeta) \right|^2(u^h(t)) \, dt \\
+ \frac{1}{h} \sum_{n=1}^{N} \left( E_h(\chi^n, \chi^{n-1}; \zeta) - E_h(\chi^n, \chi^n; \zeta) \right) \leq E_h(\chi^0, \chi^0; \zeta) - E_h(\chi^N, \chi^N; \zeta), \] (18)

where $|\partial E_h(\cdot, \chi; \zeta)|(u)$ is the “local slope” of $E_h(\cdot, \chi; \zeta)$ at $u$ defined by
\[ |\partial E_h(\cdot, \chi; \zeta)|(u) := \lim_{v \to u} \sup_{\zeta} \frac{(E_h(u, \chi; \zeta) - E_h(v, \chi; \zeta))}{d_h(u, v; \zeta)}. \] (19)

The convergence $v \to u$ is in the sense of the metric $d_h$.

Our goal is to derive Brakke’s inequality [11] from its approximate version [18], i.e., we want to relate the limits of the expressions in [18] to the terms appearing in [11]: In Propositions 3.7 we
will show that the transport term arises from the increments $\frac{1}{h} \left( E_h(\chi^n, \chi^{n-1}; \zeta) - E_h(\chi^n, \chi^n; \zeta) \right)$. Then we will derive a lim inf-inequality between the local slope of the approximate energies $E_h$ and the squared mean curvature of the limiting partition in Proposition 3.10.

While Corollary 3.2 is a mere application of the abstract theory in [4], we will now use the particular character of thresholding, i.e., the structure of the energy (16) and the metric term (15) in order to pass to the limit in the approximate Brakke inequality (18).

We start with the basic a priori estimate for the piecewise constant interpolation $\chi^h$.

**Corollary 3.3 (Energy-dissipation estimate).** Given initial conditions $\chi^0: [0, \Lambda)^d \to \{0, 1\}^P$ with finite energy $E_0 := E(\chi^0) < \infty$, a time-step size $h > 0$ and a finite time horizon $T = Nh$ we have

$$\sup_N \left( E_h(\chi^N) + h \sum_{n=1}^N \frac{d_n^2(\chi^n, \chi^{n-1})}{2h^2} \right) \leq E_0. \quad (20)$$

We recall the following proposition from [22] which will allow us to pass to the limit in the approximate Brakke inequality for the scheme. It is only for this proposition we use the convergence assumption (17).

**Proposition 3.4 (Lemma 2.8 and Proposition 3.5 in [22]).** Given $u^h \to \chi$ and $E_h(u^h) \to E(\chi)$, for any test function $\zeta \in C^\infty([0, \Lambda)^d)$ it holds

$$\frac{1}{\sqrt{h}} \int \nabla u^h : \nabla G_h * u^h \, dx = c_0 \sum_{i,j} \sigma_{ij} \int \frac{1}{2} \left( |\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)| \right), \quad (21)$$

and for any test matrix field $A \in C^\infty([0, \Lambda)^d, \mathbb{R}^{d \times d})$ we have

$$\sum_{i,j} \sigma_{ij} \frac{1}{\sqrt{h}} \int A \cdot u^h \nabla G_h * u^h \, dx = c_0 \sum_{i,j} \sigma_{ij} \int \nu_i \cdot A \nu_i \frac{1}{2} \left( |\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)| \right). \quad (22)$$

In [22] we used the above proposition to pass to the limit in the first variation of the energy

$$\delta E_h(u, \xi) := \frac{d}{ds} \bigg|_{s=0} E_h(u_s), \quad (23)$$

where the inner variations $u_s$ of $u$ along a vector field $\xi$ are given by the transport equation

$$\partial_s u_s + (\xi \cdot \nabla) u_s = 0 \quad u_s|_{s=0} = u. \quad (24)$$

**Proposition 3.5 (Proposition 3.2, Remark 3.3 and Lemma 3.4 in [22]).** Given $u: [0, \Lambda)^d \to [0, 1]^P$ with $\sum_i u_i = 1$ a.e. we have

$$\left| \delta E_h(u, \xi) - \frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int \nabla \xi : u_i (G_h Id - h\nabla^2 G_h) * u_j \, dx \right| \leq \sqrt{h} \|
abla^2 \xi\|_{\infty} E_h(u). \quad (25)$$

In particular if $u^h \to \chi \in \{0, 1\}^P$ and $E_h(u^h) \to E(\chi) < \infty$ we have

$$\delta E_h(u^h, \xi) \to c_0 \sum_{i,j} \sigma_{ij} \int \nabla \xi : (Id - \nu_i \otimes \nu_i) \frac{1}{2} \left( |\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)| \right). \quad (26)$$
**Remark 3.6.** Although the proof is contained in [22], we will recall the short argument for [23] and rephrase it in the language of commutators to introduce the reader to the notation. Note that the argument for [20] is only based on the fact that the measure on the right-hand side of [21] agrees with the trace of the right-hand side measure of [22].

In the absence of the localization, i.e., if the test function $\zeta$ is constant, the last left-hand side term $h \sum_{n=1}^{N} \frac{1}{h} (E_h(\chi^n, \chi^{n-1}; \zeta) - E_h(\chi^n, \chi^n; \zeta))$ in (18) vanishes. However, for a non-constant test function we have to pass to the limit in this extra term. Let us again restrict ourselves to the two-phase case for the following short discussion to see that formally, the behavior of this term is obvious. Expanding $\zeta$ (and ignoring the commutator in the metric term for a moment), the form of this term depends on the test fields $\zeta$ and $\xi$. Afterwards we apply the Euler-Lagrange equation $\delta(\frac{1}{2h} d_h^2(\cdot, \chi))(u, \nabla \zeta) = -\delta E_h(u, \zeta)$ of the global minimizing movements principle (3) and Proposition 3.5 to obtain the transport term in the form

$$\frac{C_0}{2} \int_0^T \int \nabla^2 \zeta: (\text{Id} - \nu \otimes \nu) |\nabla \chi| \, dt.$$

**Proposition 3.7.** Given the convergence assumption (7) and with $T = Nh$ we have

$$\lim_{h \to 0} h \sum_{n=1}^{N} \frac{1}{h} (E_h(\chi^n, \chi^{n-1}; \zeta) - E_h(\chi^n, \chi^n; \zeta)) \quad = \quad \frac{C_0}{2} \sum_{i,j} \sigma_{ij} \int_0^T \int \nabla^2 \zeta: (\text{Id} - \nu_i \otimes \nu_i) \frac{1}{2} (|\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)|) \, dt.$$

The following a priori estimate for the variational interpolation $u^h$ defined in (17) follows now very easily.

**Corollary 3.8 (A priori estimate).** Given initial conditions $\chi^0: [0, \Lambda]^d \to \{0, 1\}^P$ with finite energy $E_0 := E(\chi^0) < \infty$, a time-step size $h > 0$ and a finite time horizon $T = Nh$, if the test function $\zeta$ is strictly positive, then for the interpolation (17) we have

$$\limsup_{h \to 0} \left( \sup_{t} E_h(u^h(t)) + \int_0^T \frac{d_h^2(u^h(t), \chi^h(t))}{2h^2} \, dt \right) < \infty. \quad (28)$$
In particular, we have the following quantitative proximity of \( u^h(t) \) to \( \chi^h(t) \) in our metric:

\[
\sqrt{h} \int_0^T \left| G_{h/2} \ast \left( \frac{u^h - \chi^h}{h} \right) \right|^2 \sigma \ dt \quad \text{stays bounded as } \quad h \to 0. \quad (29)
\]

The following statement is a post-processed version of our assumption (7).

**Lemma 3.9.** Given the convergence assumption (7), for a subsequence, we also have the pointwise property

\[
E_h(\chi^h) \to E(\chi) \quad \text{a.e. in } (0, T)
\]

and furthermore for the variational interpolation \( u^h \) given by (17)

\[
E_h(u^h) \to E(\chi) \quad \text{a.e. in } (0, T). \quad (31)
\]

Moreover, Lebesgue’s dominated convergence theorem implies the integrated version

\[
\int_0^T E_h(u^h) \ dt \to \int_0^T E(\chi) \ dt. \quad (32)
\]

Additionally, the interpolations \( u^h \) converge to the same limit in \( L^1 \), i.e., \( \lim_h u^h = \lim_h \chi^h = \chi \).

In the following proposition, we probe the definition of the local slope (19) with inner variations \( u_s \). These are given by the transport equation (24) and in the simpler two-phase case we obtain

\[
\left| \partial E_h(\cdot, \chi^h; \zeta(u)) \right| \geq \frac{\delta E_h(\cdot, \chi; \zeta)(u, \xi)}{\sqrt{2\sqrt{h} \int \zeta \left( G_{h/2} \ast (\xi \cdot \nabla u) \right)^2 \ dx}}. \quad (33)
\]

Then we will find that the localization \( \zeta \) acts trivially on this term: As \( h \to 0 \), the first variation of the localized energy \( \delta E_h(\cdot, \chi; \zeta)(u, \xi) \) behaves like the first variation of the global energy in direction of the localized vector field \( \zeta \xi \), i.e., \( \delta E_h(\cdot)(u, \xi) \):

\[
\left| \delta E_h(\cdot, \chi^h; \zeta)(u, \xi) \right| \leq \zeta, \xi \ h^{1/4} \frac{d_h(u, \chi)}{h}.
\]

Similarly, it is straightforward to see that

\[
\left| \sqrt{h} \int \zeta \left( G_{h/2} \ast (\xi \cdot \nabla u) \right)^2 \ dx - \frac{1}{\sqrt{h}} \int \zeta \xi \otimes \xi: (1-u)h \nabla^2 G_h \ast u \ dx \right| \leq \zeta, \xi \ h^{1/4} E_h(u),
\]

where again the implicit constant depends on \( \zeta \) and \( \xi \). Taking the limit \( h \to 0 \), we may apply Proposition 3.4 to both terms, the numerator and the denominator. Then taking the supremum over all possible vector fields \( \xi \) we obtain the lim inf-inequality

\[
\frac{c_0}{2} \int_0^T \zeta \left| \nabla \chi \right| dt \leq \liminf_{h \to 0} \frac{1}{2} \int_0^T \left| \partial E_h(\cdot, \chi^h; \zeta) \right|^2 (u^h) dt,
\]

which provides the final ingredient for the proof of Theorem 2.2.
Proposition 3.10. Let $\zeta > 0$ be smooth, $\chi^h(t)$ the approximate solution obtained by Algorithm 1.1 and let $u^h(t)$ be either the variational interpolation (17) or the approximate solution $\chi^h(t + h)$ at time $t + h$. Given the convergence assumption (11), there exists a measurable normal vector field $H \in L^2(\sum_{i,j} \sigma_{ij} \frac{1}{2} (|\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)|) dt)$ which is the mean curvature vector of the partition $\chi$ in the sense of (11), such that

$$\frac{c_0}{2} \sum_{i,j} \sigma_{ij} \int_0^T \int \left| \frac{1}{2} (|\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)|) dt \leq \lim_{h \to 0} \int_0^T \left| \partial \mathcal{E}_h(\cdot, \chi^h; \zeta) \right|^2 (u^h) dt.$$  

(34)

4 Proofs

We first give the proofs of the main results, Theorem 2.2, Lemma 3.1 and Corollary 3.2 and then turn to the other statements which form the basis of the proof of Theorem 2.2.

Proof of Theorem 2.2 Step 1: Time-freezing for $\zeta$. We claim that it is enough to prove

$$\sum_{i,j} \sigma_{ij} \int_0^T \left( \zeta |H|^2 - H \cdot \nabla \zeta \right) \left| \frac{1}{2} (|\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)|) dt \right|_{t=0}^T$$

$$- 2 \sum_{i,j} \sigma_{ij} \left( \int_0^T \left| \frac{1}{2} (|\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)|) | \right|_{t=0}^Tight. \right.$$  

(35)

for any time-independent, strictly positive test function $\zeta = \zeta(x) > 0$ and a.e. $T$.

This is a standard approximation argument: In order to reduce (11) to (35) we fix a time-dependent test function $\zeta = \zeta(t, x) \geq 0$ and two time instances $0 \leq s < t$. It is no restriction to assume $s = 0$. Writing $t =: \tilde{T}$ for the time horizon we take a regular partition $0 = T_0 < \cdots < T_M = \tilde{T}$ of the interval $(0, \tilde{T})$ of fineness $\tau = M / \tilde{T}$. We write $\zeta_M$ for the piecewise constant interpolation of $\zeta$ plus a small perturbation $\frac{1}{M}$ so that $\zeta_M \geq \frac{1}{M} > 0$:

$$\zeta_M(t) := \zeta(T_{m-1}) + \frac{1}{M} \quad \text{if } t \in [T_{m-1}, T_m).$$

Writing $\partial^{-\tau} \zeta_M(t) := \frac{1}{\tau} (\zeta_M(t) - \zeta_M(t - \tau))$ for the discrete (backwards) time derivative we have

$$\zeta_M \to \zeta, \quad \nabla \zeta_M \to \nabla \zeta \quad \text{and} \quad \partial^{-\tau} \zeta_M \to \partial \zeta \quad \text{uniformly as } M \to \infty.$$  

(36)

Using (33) for $\zeta_M \geq \frac{1}{M} > 0$ on each interval $[T_{m-1}, T_m)$ and summing over $m$ we obtain (11).

Step 2: Proof of (35). Given a test function $\zeta = \zeta(x) > 0$, we want to prove (35) for a.e. $\tilde{T} > 0$. For simplicity, we may assume that $\tilde{T} = Nh$ is a multiple of the time step size $h$. Furthermore by (33) we may assume that $E_h(\chi^h(\tilde{T})) \to E(\chi(\tilde{T}))$. We pass to the limit in the approximate Brakke inequality (18) to prove Brakke’s inequality (35) for this time-independent test function.
By \( \sigma \) we may apply Proposition 3.10 to obtain

\[
\frac{c_0}{4} \sum_{i,j} \sigma_{ij} \int_0^T \int \zeta |H|^2 \frac{1}{2} \left( |\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)| \right) dt
\leq \liminf_{h \to 0} \frac{1}{2} \sum_{n=1}^N |\partial E_h(\cdot, \chi^{n-1}; \zeta)|^2 (\chi^n),
\]

as well as

\[
\frac{c_0}{4} \sum_{i,j} \sigma_{ij} \int_0^T \int \zeta |H|^2 \frac{1}{2} \left( |\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)| \right) dt
\leq \liminf_{h \to 0} \frac{1}{2} \int_0^T |\partial E_h(\cdot, \chi^h(t); \zeta)|^2 (u^h(t)) dt.
\]

In addition, we may apply Proposition 3.7 for the transport term and after division by the common prefactor \( c_0 \) we obtain (35).

**Proof of Lemma 3.1.** Given initial conditions \( \chi \in \{0,1\}^P \) with \( \sum_i \chi_i = 1 \) and a time-step size \( h > 0 \), one iteration of the thresholding scheme yields \( \chi^1 = \chi \{ (\sigma G_h \ast \chi)_i = \min_j (\sigma G_h \ast \chi)_j \} \). Then \( \chi^1 \) clearly minimizes

\[
2u \cdot \sigma G_h \ast \chi
\]

among all \( u \in [0,1]^P \) s.t. \( \sum_i u_i = 1 \). This expression is equal to

\[
u \cdot \sigma G_h \ast u - (u - \chi) \cdot \sigma G_h \ast (u - \chi) + u \cdot \sigma G_h \ast \chi - \chi \cdot \sigma G_h \ast u + \chi \cdot \sigma G_h \ast \chi
\]

\[
= u \cdot \sigma G_h \ast u - (u - \chi) \cdot \sigma G_h \ast (u - \chi) + (u - \chi) \cdot \sigma G_h \ast \chi - \chi \cdot \sigma G_h \ast (u - \chi) + \chi \cdot \sigma G_h \ast \chi,
\]

where the last right-hand side term is independent of \( u \) and thus irrelevant for the minimization. Multiplying with \( \zeta \geq 0 \) and integrating shows that \( \chi^1 \) minimizes

\[
\int \zeta \left[ u \cdot \sigma G_h \ast u - (u - \chi) \cdot \sigma G_h \ast (u - \chi) + (u - \chi) \cdot \sigma G_h \ast \chi - \chi \cdot \sigma G_h \ast (u - \chi) \right] dx + \text{const.}
\]

Dividing by \( \sqrt{h} \), recalling the definitions (15) and (16) of the localized distance and energy, and using the semi-group and symmetry properties of the kernel and the symmetry of \( \sigma \) yield (14).

**Corollary 3.2** is an immediate consequence of interpreting our problem from the point of view of gradient flows in metric spaces.

Given \( \chi \) and \( \zeta \), the Moreau-Yosida approximation \( E_{h,t} \) of \( E_h \) is defined by

\[
E_{h,t}(\chi; \zeta) := \min_u \left\{ E_h(u, \chi; \zeta) + \frac{1}{2t} d_h^2(u, \chi; \zeta) \right\}
\]

and furthermore we recall the (not necessarily unique) variational interpolation \( u^h(t) \) of \( \chi \) and \( \chi^1 := u^h(h) \), cf. (17).
As $t$ decreases we have a stronger penalization and thus we expect $u^h(t)$ to be “closer” to $\chi = u^h(0)$ than $\chi^1 = u^h(h)$ which justifies the name “interpolation”. Note that $E_h(u, \chi; \zeta)$ and $d_h(u, \chi; \zeta)$ are, because of the smoothing property of the kernel $G_h$, weakly continuous in $u$ and $\chi$. Furthermore, we recall that we choose $u^h(\cdot)$ in such a way that it is continuous in $t$ w.r.t. the metric $d_h$.

The following general theorem monitors the evolution of the (approximate) energy along the interpolation $u^h(t)$ in terms of the distances at different time instances measured by the metric $d_h$, and gives a lower bound in terms of the local slope $|\chi|$. Furthermore, we recall that we choose $u^h(\cdot)$ in such a way that it is continuous in $t$ w.r.t. the metric $d_h$.

Theorem 4.1 (Theorem 3.1.4 and Lemma 3.1.3 in [4]). For every $\chi: [0, T]^d \to \{0, 1\}^P$ with $\sum_i \chi_i = 1$ a.e. the map $t \mapsto E_{h,t}(\chi; \zeta)$ is locally Lipschitz in $(0, h]$ and continuous in $[0, h]$ with

$$
\frac{t}{2} |\partial E_h(\cdot, \chi; \zeta)|^2 (u^h(t)) + \frac{1}{2} \int_0^t |\partial E_h(\cdot, \chi; \zeta)|^2 (u^h(s)) \, ds \\
\leq \frac{1}{2t} d_h^2(u^h(t), \chi; \zeta) + \int_0^t \frac{d_h^2(u^h(s), \chi; \zeta)}{2s^2} \, ds = E_h(\chi, \chi; \zeta) - E_h(u^h(t), \chi; \zeta).
$$

The idea behind Theorem 4.1 is rather simple: By testing the minimality of $u^h(t)$ against $u^h(s)$ and taking $s \uparrow t$ (and similarly with reversed roles for $s \downarrow t$) one obtains $\frac{d}{dt} E_{h,t}(\chi; \zeta) = -\frac{d_h^2(u^h(t), \chi; \zeta)}{2t^2}$. Integrating this equation from $t = 0$ to $t = h$ yields the equality between the energy difference and the metric term in (37). The first inequality between the local slope and the metric term comes from the general estimate $|\partial E_h(\cdot, \chi; \zeta)| (u^h(t)) \leq \frac{d_h(u^h(t), \chi; \zeta)}{t}$, which follows from the definition of the local slope and the triangle inequality.

Proof of Corollary 3.2. We apply (37) in Theorem 4.1 with $\chi = \chi^{n-1}$ and $t = h$, and sum over $n = 1, \ldots, N$.

Now we turn to the more problem-specific statements, which use the special character of thresholding.

Proof of Corollary 3.3. The statement simply follows from testing the global minimization problem (4) for $\chi^n$ with its predecessor $\chi^{n-1}$ and summation over $n$.

Proof of Proposition 3.3. The first variation of $E_h$ at $u$ along the vector field $\xi$ defined through (24) and (26) is given by

$$
\delta E_h(u, \xi) = \frac{1}{\sqrt{h}} \int - (\xi \cdot \nabla) u \cdot \sigma G_h * u - u \cdot \sigma (G_h * ((\xi \cdot \nabla) u)) \, dx
$$

$$
= \frac{1}{\sqrt{h}} \int u \cdot \sigma (\xi \cdot \nabla) G_h * u - u \cdot \sigma (G_h * (\xi \otimes u)) \, dx
$$

$$
+ \frac{1}{\sqrt{h}} \int (\nabla \cdot \xi) u \cdot \sigma G_h * u + u \cdot \sigma G_h * ((\nabla \cdot \xi) u) \, dx.
$$
This can be compactly rewritten as

$$\delta E_h(u, \xi) = \frac{1}{\sqrt{h}} \int 2 (\nabla \cdot \xi) u \cdot \sigma G_h \ast u + u \cdot \sigma [\xi, \nabla G_h] u - u \cdot \sigma [\nabla \cdot \xi, G_h] u \, dx.$$ 

Componentwise in $$u$$, we expand the first commutator:

$$([\xi, \nabla G_h] u_i) (x) = \int (\xi(x) - \xi(x - z)) \cdot \nabla G_h(z) u_i(x - z) \, dz$$

$$= \nabla \chi(x) \cdot \left( - \frac{z}{\sqrt{h}} \otimes \frac{z}{\sqrt{h}} G_h(z) u_i(x - z) \, dz + O \left( \| \nabla^2 \xi \|_{\infty} \sqrt{h} k_h \ast u_i \right) (x) \right),$$

where we used the identity $$\nabla G(z) = -G(z)z$$ and where the kernel $$k_h$$ is given by the mask $$k(z) = |z|^2 G(z)$$ and can be controlled by a Gaussian with slightly larger variance $$k(z) \lesssim G(z/2)$$. Likewise, the second commutator can be estimated pointwise by

$$\| \nabla \cdot \xi, G_h \ast u_i \| \lesssim \| \nabla^2 \xi \|_{\infty} \sqrt{h} \tilde{k}_h \ast u_i,$$

where $$\tilde{k}_h$$ is given by the mask $$\tilde{k}(z) = |z|G(z) \lesssim G(z/2)$$. By the identity $$G(z) (Id - z \otimes z) = -\nabla^2 G(z)$$ we indeed obtain [23] with an error of order $$\| \nabla^2 \xi \|_{\infty} \sqrt{h} E_{4h}(u)$$, which by the monotonicity [16] of $$E_h$$ yields the claim.

**Proof of Proposition 3.7** We first note that by definition [16],

$$E_h(\chi^n, \chi^{n-1}; \zeta) - E_h(\chi^n, \chi^n; \zeta) = \frac{1}{\sqrt{h}} \int (\chi^n - \chi^{n-1}) \cdot \sigma [\zeta, G_h] \chi^{n-1} \, dx$$

$$- \frac{1}{\sqrt{h}} \int (\chi^n - \chi^{n-1}) \cdot \sigma [\zeta, G_{h/2}] G_{h/2} \ast (\chi^n - \chi^{n-1}) \, dx.$$ 

By the antisymmetry of the commutator (and the symmetry of $$\sigma$$), we may replace $$\chi^{n-1}$$ by $$\chi^n$$ on the right-hand side:

$$\frac{1}{\sqrt{h}} \int (\chi^n - \chi^{n-1}) \cdot \sigma [\zeta, G_h] \chi^n - (\chi^n - \chi^{n-1}) \cdot \sigma [\zeta, G_{h/2}] G_{h/2} \ast (\chi^n - \chi^{n-1}) \, dx.$$ 

Now we prove the proposition in two steps. First, we show that the first term converges to the right-hand side of the claim:

$$\lim_{h \to 0} \int_0^T \partial_i^{-h} \chi^h \cdot \sigma [\zeta, G_h] \chi^h \, dx$$

$$= c_0 \sum_{i,j} \sigma_{ij} \int_0^T \nabla^2 \zeta : (Id - \nu_i \otimes \nu_i) \frac{1}{2} (|\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)|) \, dt,$$  

(39)

where $$\partial_i^{-h} \chi^h = \chi^h - \chi^{h-1}$$ denotes the discrete backwards time derivative of $$\chi^h$$. Then we prove that the second term is negligible:

$$\lim_{h \to 0} \int_0^T \sqrt{h} \int \partial_i^{-h} \chi^h \cdot \sigma [\zeta, G_{h/2}] G_{h/2} \ast \partial_i^{-h} \chi^h \, dx \, dt = 0.$$  

(40)
Step 1: Argument for \( \| \zeta \|_{\infty} \). Expanding the commutator to second order

\[
\frac{1}{\sqrt{h}} [\zeta, G_h * v] = \sqrt{h} \nabla G_h * (-\nabla \zeta v) + \frac{\sqrt{h}}{2} (G_h \text{Id} + h \nabla^2 G_h) * (\nabla^2 \zeta v) + O (\| \nabla^3 \zeta \|_{\infty} h k_h * |v|),
\]

where the kernel \( k_h \) is given by the mask \( k(z) = |z|^3 G(z) \), we obtain for the first-order term

\[
h \sum_{n=1}^{N} \int \frac{\chi^n - \chi^{n-1}}{h} \cdot \sigma \sqrt{h} \nabla G_h * (-\nabla \zeta \chi^n) \, dx
\]

\[
= h \sum_{n=1}^{N} \frac{1}{\sqrt{h}} \int (\chi^n - \chi^{n-1}) \cdot \sigma G_h * (- (\nabla \zeta \cdot \nabla) \chi^n - \Delta \zeta (1 - \chi^n)) \, dx.
\]

Now we recognize the first variation of the (unlocalized) dissipation functional, cf. (4), on the right-hand side:

\[
\delta \left( \frac{1}{2h} d^2_h (\cdot, \chi^{n-1}) \right) (\chi^n, \zeta) = - \frac{2}{\sqrt{h}} \int (\chi^n - \chi^{n-1}) \cdot \sigma G_h * (- (\zeta \cdot \nabla) \chi^n) \, dx
\]

with \( \nabla \zeta \) playing the role of \( \xi \). Using the semi-group and symmetry properties of the kernel, the extra term involving the Laplacian of the test function can be estimated by Jensen’s inequality and the energy-dissipation estimate (20):

\[
\left| h \sum_{n=1}^{N} \frac{1}{\sqrt{h}} \int (\chi^n - \chi^{n-1}) \cdot \sigma G_h * (\Delta \zeta \chi^n) \, dx \right| \lesssim \| \Delta \zeta \|_{\infty} \left( \frac{T}{\sqrt{h}} \sum_{n=1}^{N} \frac{1}{\sqrt{h}} \int |G_{h/2} * (\chi^n - \chi^{n-1})|^2_2 \, dx \right) \leq \| \Delta \zeta \|_{\infty} T^{1/2} E_0^{1/2} h^{1/4}.
\]

Formally, the leading-order term, i.e., the first variation of the dissipation functional, converges to the transport term, which in the two-phase case is \(-c_0 \int_{\Sigma} \nu \cdot \nabla \zeta \). Since instead we want to obtain the term \(-c_0 \int_{\Sigma} H \cdot \nabla \zeta \) (in its weak form \( c_0 \int_{\Sigma} \nabla^2 \zeta \cdot (\text{Id} - \nu \otimes \nu) \)), we employ the minimizing movements interpretation (3) in form of its Euler-Lagrange equation

\[
\delta E_h (\chi^n, \xi) + \delta \left( \frac{1}{2h} d^2_h (\cdot, \chi^{n-1}) \right) (\chi^n, \xi) = 0 \quad \text{for all} \ \xi \in C^\infty ([0, \Lambda]^d, \mathbb{R}^d).
\]

We thus have

\[
h \sum_{n=1}^{N} \int \frac{\chi^n - \chi^{n-1}}{h} \cdot \sigma \sqrt{h} \nabla G_h * (\nabla \zeta \cdot \nabla \chi^n) \, dx = \frac{h}{2} \sum_{n=1}^{N} \delta E_h (\chi^n, \nabla \zeta).
\]

By the convergence of the energies (7) we may apply Proposition 5.3 in (25) and pass to the limit \( h \to 0 \) in the right-hand side:

\[
\lim_{h \downarrow 0} \frac{1}{2} \int_0^T \delta E_h (\chi^h, \nabla \zeta) \, dt
\]

\[
= \frac{c_0}{2} \sum_{i,j} \sigma_{ij} \int_0^T \nabla^2 \zeta \cdot (\text{Id} - \nu_i \otimes \nu_i) \frac{1}{2} (|\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)|) \, dt.
\]
Indeed, by Lebesgue’s dominated convergence theorem, we are allowed to interchange the order of integration in time and the limit \( h \to 0 \).

Now we conclude the argument for (39) by showing that the contributions of the second- and third-order terms in the expansion (41) are negligible in the limit \( h \to 0 \). The contribution of the second-order term is estimated as follows. We argue componentwise in \( \chi^h \), fix \( i, j \in \{1, \ldots, P\} \), \( i \neq j \) and observe that by Cauchy-Schwarz

\[
\int_0^T \int \partial_t^{-h} \chi^h \sqrt{h} \left( G_h Id + h \nabla^2 G_h \right) * \left( \nabla^2 \zeta \chi^h_j \right) \, dx \, dt 
\leq \left( \int_0^T \sqrt{h} \int \left| \left( G_h Id + h \nabla^2 G_h \right) * \partial_t^{-h} \chi^h_i \right|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T \sqrt{h} \int \left| \nabla^2 \zeta \chi^h_j \right|^2 \, dx \, dt \right)^{\frac{1}{2}}.
\]

The second right-hand side integral is bounded by \( T \Lambda^d \| \nabla^2 \zeta \|_\infty \sqrt{h} \to 0 \), while the first right-hand side integral can be estimated by

\[
\sqrt{h} \int \left| \left( G_h Id + h \nabla^2 G_h \right) * \partial_t^{-h} \chi^h_i \right|^2 \, dx \leq \sqrt{h} \int \left( G_{h/2} * \partial_t^{-h} \chi^h_i \right)^2 \, dx \quad \overset{43}{\lesssim} \quad E_h(u^0),
\]

where in the first estimate we have used the semi-group property \( G_h Id + h \nabla^2 G_h = (G_{h/2} Id + h \nabla^2 G_{h/2}) * G_{h/2} \) and the fact that the kernel \( G_{h/2} Id + h \nabla^2 G_{h/2} \) is uniformly bounded in \( L^1 \).

Since the kernel \( k_h \) is uniformly bounded in \( L^1 \), the contribution of the third-order term in (41) is controlled by

\[
\int_0^T h \left| \partial_t^{-h} \chi^h \right| \, dx \, dt = \int_0^T \int \left| \chi^h(t) - \chi^h(t-h) \right| \, dx \, dt.
\]

The following basic estimate, which is valid for any pair of characteristic functions,

\[
|\chi - \tilde{\chi}| = |\chi - \tilde{\chi}|^2 \lesssim |G_{h/2} * (\chi - \tilde{\chi})|^2 + |G_{h/2} * \chi - \chi|^2 + |G_{h/2} * \tilde{\chi} - \tilde{\chi}|^2,
\]  
and the fact that by the normalization \( \int G_{h/2}(z) \, dz = 1 \) and the pointwise estimate \( G_{h/2}(z) \lesssim G_h(z) \), for each \( i \in \{1, \ldots, P\} \), we have

\[
\frac{1}{\sqrt{h}} \int |G_{h/2} * \chi_i - \chi_i| \, dx \leq \frac{1}{\sqrt{h}} \int G_{h/2}(z) \int |\chi_i(x) - \chi_i(x-z)| \, dx \, dz \lesssim \frac{1}{\sqrt{h}} \int G_h(z) \int |\chi_i(x) - \chi_i(x-z)| \, dx \, dz = \frac{1}{\sqrt{h}} \int G_h(z) \int (1 - \chi_i)(x) \chi_i(x-z) + (1 - \chi_i)(x-z) \chi_i(x) \, dx \, dz = \frac{2}{\sqrt{h}} \sum_{1 \leq j \leq P, j \neq i} \int \chi_j G_h * \chi_j \, dx \leq \frac{1}{\min_{i \neq j} \sigma_{ij}} E_h(\chi)
\]
yield the estimate

\[
\int_0^T \int \left| \chi^h(t) - \chi^h(t-h) \right| \, dx \, dt \lesssim (1 + T) E_0 \sqrt{h} \to 0.
\]

This concludes the proof of (39).
Step 2: Argument for (40). We may argue componentwise and omit the index in the following. We expand the commutator to first order

\[
[\zeta, G_{h/2}] v = \nabla G_{h/2} \ast \left( -\frac{h}{2} \nabla \zeta v \right) + O \left( \| \nabla^2 \zeta \|_\infty h \kappa_h \ast |v| \right),
\]

where the kernel \( \kappa_h \) is given by the mask \( k(z) = |z|^2 G_{1/2}(z) \), and first consider the contribution of the first-order term to (40), namely

\[
-\frac{h}{2} \int_0^T \sqrt{h} \int \partial_{t-h} \chi \nabla G_{h/2} \ast \nabla \zeta (G_{h/2} \ast \partial_{t-h} \chi^h) \ dx \ dt.
\]

Using the antisymmetry of \( \nabla G \), the chain rule and integration by parts this is equal to

\[
\frac{h}{2} \int_0^T \sqrt{h} \int \nabla \zeta : \nabla \left( \frac{1}{2} \left( G_{h/2} \ast \partial_{t-h} \chi^h \right)^2 \right) \ dx \ dt
\]

By the energy-dissipation estimate (20) this term vanishes as \( h \to 0 \).

The contribution of the second-order term coming from the expansion (44) is controlled by

\[
\frac{h}{2} \int_0^T \sqrt{h} \int |\partial_{t-h} \chi^h| \kappa_h \ast |G_{h/2} \ast \partial_{t-h} \chi^h| \ dx \ dt \leq \int_0^T \sqrt{h} \int |G_{h/2} \ast \partial_{t-h} \chi^h| \ dx \ dt.
\]

Therefore, this term vanishes as \( h \to 0 \) by Jensen’s inequality and the energy-dissipation estimate (20).

Proof of Corollary 3.8. In contrast to the piecewise constant interpolation \( \chi_h \), the variational interpolation \( u^h \) is not given in an explicit form but only by the minimization problem (17). In particular, since in general \( u^h \) may depend on the test function \( \zeta \), we are tied to the local minimization problem (17). By (37) we have in particular

\[
E_h(u^h(\tilde{T}), u^h(\tilde{T}); \zeta) + \int_0^{\tilde{T}} \frac{d^2_h(u^h, \chi^h; \zeta)}{2h^2} \ dt \leq E_h(\chi^0, \chi^0; \zeta) - \sum_{n=1}^{N} \left( E_h(\chi^n, \chi^n; \zeta) - E_h(\chi^n, \chi^n; \zeta) \right)
\]

for any \( \tilde{T} \in [Nh, (N+1)h) \), where \( N \in \mathbb{N} \). The left-hand side is bounded from below by

\[
\inf \zeta \left( E_h(u^h(\tilde{T})) + \int_0^{\tilde{T}} \frac{d^2_h(u^h, \chi^h)}{2h^2} \ dt \right)
\]

while the right-hand side can be controlled by Proposition 3.7.

Proof of Lemma 3.9. The convergence assumption (7) together with the lim inf-inequality of the \( \Gamma \)-convergence implies the convergence \( E_h(\chi^h) \to E(\chi) \) in \( L^1(0,T) \). In order to understand the
behavior of the energies of the variational interpolations $u^h$ we compare them to the energies of the piecewise constant interpolation:

$$|E_h(u^h) - E_h(\chi^h)| = \frac{1}{\sqrt{h}} \left| \int (G_{h/2} * u^h \cdot \sigma G_{h/2} * (u^h - \chi^h) + G_{h/2} * (u^h - \chi^h) \cdot \sigma G_{h/2} * \chi^h) \, dx \right|$$

$$\lesssim \frac{1}{\sqrt{h}} \int |G_{h/2} * (u^h - \chi^h)|_\sigma \, dx$$

and by Jensen we obtain

$$\int_0^T |E_h(u^h) - E_h(\chi^h)| \, dt \lesssim T^{1/2} \frac{1}{\sqrt{h}} \left( \int_0^T \int |G_{h/2} * (u^h - \chi^h)|^2_\sigma \, dx \, dt \right)^{1/2},$$

which by (29) vanishes as $h \to 0$. That means the approximate energies converge to the same limit in $L^1(0, T)$ and therefore we obtain the $L^1$-convergence (32) and – after the possible passage to a further subsequence – the pointwise convergences (30) and (31).

The convergence of $u^h$ to $\chi = \lim \chi^h$ can now be proven by the very same argument as the one following (13). The only difference here is that the components $u^h_i$ are not characteristic functions. Then the first equality in (13) can be replaced by the inequality $|u - \chi| \leq 2|u - \chi|$ and in the chain of inequalities following (13), the equality $|\chi_i(x) - \chi_i(x-z)| = (1-\chi_i)(x) \chi_i(x-z) + (1-\chi_i)(x-z) \chi_i(x)$ can simply be replaced by the inequality $|u_i(x) - u_i(x-z)| \leq (1-u_i)(x) u_i(x-z) + (1-u_i)(x-z) u_i(x)$, which is valid for any function $u_i$ with values in $[0, 1]$. Together with (28), this leads to the estimate

$$\lim_{h \to 0} \frac{1}{\sqrt{h}} \int_0^T \left| u^h - \chi^h \right| \, dx \, dt < \infty,$$

so that indeed $u^h \to \chi$ in $L^1$.

**Proof of Proposition 3.1.** We give ourselves a test vector field $\xi$ and let the variations $u_s$ defined in (24) play the role of $v$ in the definition of the local slope (19) so that we obtain the inequality

$$|\partial E_h(\cdot, \chi^h; \zeta)(u^h)| \geq \lim_{s \to 0} \frac{(E_h(u_s^h, \chi^h; \zeta) - E_h(u_s^h, \chi^h; \zeta))}{d_s(u_s^h, u_s^h; \zeta)}.$$

As $s \to 0$ we expand the numerator in the following way

$$E_h(u_s^h, \chi^h; \zeta) = E_h(u_s^h, \chi^h; \zeta) + s \frac{d}{ds} \bigg|_{s=0} E_h(u_s^h, \chi^h; \zeta) + o(s) \quad \text{as} \quad s \to 0.$$

For the denominator we have, cf. (4),

$$\frac{1}{2h} d_s^2(u_s^h, u_s^h; \zeta) = \frac{s^2}{\sqrt{h}} \int \zeta \left| G_{h/2} * (\xi \cdot \nabla u_s^h) \right|^2_\sigma \, dx + o(s^2) \quad \text{as} \quad s \to 0.$$

Taking the limit $s \to 0$ we obtain

$$|\partial E_h(\cdot, \chi^h; \zeta)(u^h)| \geq \frac{\frac{d}{ds}\bigg|_{s=0} E_h(u_s^h, \chi^h; \zeta)}{\sqrt{2\sqrt{h}} \int \zeta \left| G_{h/2} * (\xi \cdot \nabla u_s^h) \right|^2_\sigma \, dx}.$$  (45)
Now we expand $\zeta$ and $\xi$ to analyze the leading order terms as $h \to 0$. Using (16) we compute the first variation of the localized energy $E_h(u, \chi; \zeta)$, cf. (16):

$$\frac{d}{ds} \bigg|_{s=0} E_h(u_s, \chi; \zeta)$$

$$= \frac{1}{\sqrt{h}} \int -\zeta (\xi \cdot \nabla) u \cdot \sigma G_h \ast u - \zeta u \cdot \sigma G_h \ast ((\xi \cdot \nabla) u)
- (\xi \cdot \nabla) u \cdot \sigma [\zeta, G_h] u + (\xi \cdot \nabla) u \cdot \sigma [\zeta, G_h^*] (u - \chi)
+ (\xi \cdot \nabla) u [\zeta, G_{h/2}] G_{h/2} \ast (u - \chi) - (\xi \cdot \nabla) u \cdot \sigma G_{h/2} \ast [\zeta, G_{h/2}]^*(u - \chi) \, dx.$$  

The fourth term in the sum comes from replacing $\chi$ by $u$ in the third term, while for the last term we used the antisymmetry $\int u [\zeta, G_{h/2}] v \, dx = -\int v [\zeta, G_{h/2}] u \, dx$ (and the symmetry of $\sigma$). Note that due to the symmetry of $G$ there is a cancellation between the second and third term in this sum:

$$\int -\zeta u \cdot \sigma G_h \ast ((\xi \cdot \nabla) u) - (\xi \cdot \nabla) u \cdot \sigma [\zeta, G_h^*] u \, dx = \int -\zeta (\xi \cdot \nabla) u \cdot \sigma G_h \ast u \, dx
= \int -u \cdot \sigma G_h \ast (\zeta (\xi \cdot \nabla) u) \, dx.$$  

A direct computation based on the semi-group property $G_h = G_{h/2} \ast G_{h/2}$ yields

$$[\zeta, G_h^*] v + [\zeta, G_{h/2}^*] G_{h/2} \ast v - G_{h/2} \ast [\zeta, G_{h/2}^*] v = 2 [\zeta, G_{h/2}^*] G_{h/2} \ast v$$

so that the last three terms in the first variation of $E_h$ above can be combined using once more the antisymmetry of the commutator, and we get

$$\frac{d}{ds} \bigg|_{s=0} E_h(u_s, \chi; \zeta) = \frac{1}{\sqrt{h}} \int -\zeta (\xi \cdot \nabla) u \cdot \sigma G_h \ast u - u \cdot \sigma G_h \ast (\zeta (\xi \cdot \nabla) u) \, dx$$

$$- \frac{2}{\sqrt{h}} \int G_{h/2} \ast (u - \chi) \cdot \sigma [\zeta, G_{h/2}^*] ((\xi \cdot \nabla) u) \, dx.$$  

Note that the first right-hand side integral is exactly $\delta E_h(u, \zeta \xi)$, the first variation of the energy along the “localized” vector field $\zeta \xi$, see (38). Now we plug $u = u^h$ into the above formula. Since by Lemma 3.3 $u^h \to \chi$ in $L^1$ and $E_h(u^h) \to E(\chi)$ for a.e. $t$, using Proposition 3.3 with $\zeta \xi$ playing the role of $\xi$, for a.e. $t$, along the sequence $u^h$ the first right-hand side integral of (47) converges to

$$\delta E(\chi; \zeta \xi) = \sum_{i,j} \sigma_{i,j} \int \nabla (\zeta \xi) : (Id - v_i \otimes v_i) \frac{1}{2} ((\nabla \chi_i) + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)|).$$

We now give the argument that the second integral in (47) is negligible:

$$\frac{2}{\sqrt{h}} \int G_{h/2} \ast (u - \chi) \cdot \sigma [\zeta, G_{h/2}^*] ((\xi \cdot \nabla) u) \, dx \to 0 \quad \text{in} \ L^1(0,T).$$

In view of (29) in Corollary 3.8 by Cauchy-Schwarz it is enough to prove

$$\sqrt{h} \int_0^T \int |[\zeta, G_{h/2}] (\xi \cdot \nabla u^h)|^2 \, dx \, dt \to 0$$

(49)
for all $i = 1, \ldots, P$. We fix $i$ and omit the index in the following. Rewriting the commutator

$$[\zeta, G_{h/2}^*] (\xi \cdot \nabla u^h) = \int G_{h/2}(z) (\zeta(x) - \zeta(x - z)) \xi(x - z) \cdot \nabla u^h(x - z) \, dz$$

and integrating by parts in $z$ we obtain the pointwise estimate

$$||[\zeta, G_{h/2}^*] (\xi \cdot \nabla u^h)|| \leq \left| \int \nabla G_{h/2}(z) \cdot (\zeta(x) - \zeta(x - z)) u^h(x - z) \, dz \right|$$

$$+ \left| \int G_{h/2}(z) \nabla z \cdot [(\zeta(x) - \zeta(x - z)) \xi(x - z)] u^h(x - z) \, dz \right|$$

$$\lesssim \|\nabla \zeta\|_\infty \|\|u\|_\infty + \|\xi\|_\infty \|\nabla \xi\|_\infty$$

and hence (19) holds with the rate $O((\|\nabla \zeta\|_\infty \|\xi\|_\infty + \|\|u\|_\infty \|\nabla \xi\|_\infty)^2 T \sqrt{h})$. Therefore we have proven the following convergence of the first variation of the localized energy (16):

$$\lim_{h \to 0} \int_0^T \frac{d}{ds} \left| E_h(u^h, \chi^h; \zeta) \right| ds = \lim_{h \to 0} \int_0^T \delta E_h(u^h, \zeta, \zeta) \, dt$$

$$= k_0 \sum_{i,j} \sigma_{ij} \int_0^T \int \nabla (\zeta \cdot \zeta) : (\zeta \cdot \zeta) \frac{1}{2} (|\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)|) \, dt. \quad (50)$$

With the same methods we can handle the term in the expansion of the metric term $d_h(u^h, u^h; \zeta)$:

We claim that

$$\lim_{h \to 0} \frac{2}{\sqrt{h}} \int_0^T \zeta \left| G_{h/2} * ((\xi \cdot \nabla^2 u^h)) \right|_\sigma^2 \, dx \, dt$$

$$= \lim_{h \to 0} \frac{2}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int_0^T \int \zeta (\xi \cdot \xi) \cdot (u^h_i \nabla^2 G_h) * u^h_j \, dx \, dt$$

$$= 2 k_0 \sum_{i,j} \sigma_{ij} \int_0^T \int \zeta (\xi \cdot \xi) \frac{1}{2} (|\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)|) \, dt. \quad (51)$$

To this end we plug $\nabla \cdot (\xi \cdot \nabla u^h) = \nabla \cdot (\xi u^h) - (\nabla \cdot \xi) u^h$ into the quadratic term on left-hand side and expand the square. First we note that only the term

$$2 \frac{2}{\sqrt{h}} \int \zeta \left| G_{h/2} * (\nabla \cdot (\xi u^h)) \right|_\sigma^2 \, dx = 2 \frac{2}{\sqrt{h}} \int \zeta \left| \nabla G_{h/2} * (\xi \cdot u^h) \right|_\sigma^2 \, dx$$

survives in the limit $h \to 0$. Indeed, we have

$$2 \frac{2}{\sqrt{h}} \int \zeta \left| G_{h/2} * ((\nabla \cdot (\xi u^h)) \right|_\sigma^2 \, dx \lesssim \|\nabla \xi\|_\infty \|\nabla \xi\|_\infty \frac{1}{\sqrt{h}} \int \zeta \, dx$$

and the mixed term can be estimated by Young's inequality and the boundedness of the leading-order term which we will show now. Using the antisymmetry of $\nabla G$ we have

$$2 \frac{2}{\sqrt{h}} \int \zeta \left| \nabla G_{h/2} * (\xi \cdot u^h) \right|_\sigma^2 \, dx = 2 \frac{2}{\sqrt{h}} \int u^h \cdot (\xi \cdot \nabla) G_{h/2} * (\xi u^h) \, dx.$$
We now want to commute the multiplication with $\xi$ and the outer convolution and afterwards the multiplication with $\zeta \xi$ and the inner convolution. For this we use the $L^\infty$-commutator estimate
\[ \|\xi, \nabla G_h \ast u\|_\infty \lesssim \|\nabla \xi\|_\infty \|u\|_\infty \]
for the vector fields $\xi$ and $\zeta \xi$, which implies the $L^1$-estimate
\[ \int |\nabla G_{h/2} \ast (\xi u^h_j)| \, dx \lesssim \|\nabla \xi\|_\infty + \|\xi\|_\infty \int |\nabla G_{h/2} \ast u^h_j| \, dx, \]
and the a priori estimate (28) for the last term:
\[ \lim_{h \to 0} \sup_{\xi} \int |\nabla G_{h/2} \ast u^h_j| \, dx \lesssim \lim_{h \to 0} \sup_{\xi} E_h(u^h) \]
(53)
For the first estimate in (53) we exploited $\int \nabla G(z) \, dz = 0$ as follows
\[ \int |\nabla G_{h/2} \ast u^h_j| \, dx = \int \int |\nabla G_{h/2}(z) (u^h_j(x) - u^h_j(x - z))| \, dz \, dx \]
\[ \leq \int \int |\nabla G_{h/2}(z)| \, |u^h_j(x) - u^h_j(x - z)| \, dz \, dx \]
and used the pointwise estimates $|\nabla G_{h/2}(z)| \lesssim \frac{1}{\sqrt{h}} G_h(z)$ and $|u - v| \leq (1 - u)v + u(1 - v)$ for any $u, v \in [0, 1]$. This gives indeed
\[ \int |\nabla G_{h/2} \ast u^h_j| \, dx \lesssim \frac{2}{\sqrt{h}} \int (1 - u^h_j) \, G_h \ast u^h_j \, dx \leq \frac{1}{\min_{i \neq j} \sigma_{ij}} E_h(u^h). \]

Therefore, the left-hand side of (51) is indeed to leading order given by
\[ \frac{2}{\sqrt{h}} \int \zeta (\xi \otimes \xi) : u^h \cdot \sigma (h \nabla^2 G_h) \ast u^h \, dx. \]
Then (51) follows from the convergence of the energies (cf. Lemma 3.9) and Proposition 3.4.

Using (40) for the numerator and (51) for the denominator of the right-hand side of (45) and $(\xi \cdot \nu_i)^2 \leq |\xi|^2$ along the way, we obtain by Fatou’s Lemma in $t$
\[ \liminf_{h \to 0} \int_0^T |\partial E_h(\cdot, \chi^h, \zeta)|^2(u^h) \, dt \]
\[ \geq c_0 \int_0^T \left( \sup_{\xi} \frac{\int \nabla (\zeta \xi) : (Id - \nu_i \otimes \nu_i) \sum_{i,j} \sigma_{ij} \frac{1}{2} (|\nabla \chi_i | + |\nabla \chi_j | - |\nabla (\chi_i + \chi_j) |)}{\sqrt{\zeta |\xi|^2 \sum_{i,j} \sigma_{ij} \frac{1}{2} (|\nabla \chi_i | + |\nabla \chi_j | - |\nabla (\chi_i + \chi_j) |)}} \right)^2 \, dt. \]
Applying this estimate for $u^h(h) = \chi^h(t + h)$ and $\zeta = 1$ furnishes the existence of the mean curvature vector
\[ H \in L^2 \left( \sum_{i,j} \sigma_{ij} \frac{1}{2} (|\nabla \chi_i | + |\nabla \chi_j | - |\nabla (\chi_i + \chi_j) |) \, dt, \mathbb{R}^d \right) \]
as claimed in the proposition. Turning back to the interpolation $u^h$, we obtain the desired lim inf inequality (54) for the interpolations as well.
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