Quantum loop representation for
fermions coupled to Einstein - Maxwell field

Kirill V. Krasnov

Bogolyubov Institute for Theoretical Physics, Kiev, Ukraine

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Quantization of the system comprising gravitational, fermionic and electromagnetic fields is developed in the loop representation. As a result we obtain a natural unified quantum theory. Gravitational field is treated in the framework of Ashtekar formalism; fermions are described by two Grassmann-valued fields. We define a $C^*$-algebra of configurational variables whose generators are associated with oriented loops and curves; “open” states – curves – are necessary to embrace the fermionic degrees of freedom. Quantum representation space is constructed as a space of cylindrical functionals on the spectrum of this $C^*$-algebra. Choosing the basis of “loop” states we describe the representation space as the space of oriented loops and curves; then configurational and momentum loop variables become in this basis the operators of creation and annihilation of loops and curves. The important difference of the representation constructed from the loop representation of pure gravity is that the momentum loop operators act in our case simply by joining loops in the only compatible with their orientation way, while in the case of pure gravity this action is more complicated.

I. INTRODUCTION

Recent developement in nonperturbative quantum gravity has shown that the loop representation is quite a decent tool for dealing with generally covariant field theories. This representation allowed one to find a wide class of solutions of quantum general relativity constraints [1], [2]; there has also been found an interesting interface with the knot theory [3]. The main goal of this paper is to construct the loop representation for the system which includes fermionic and two gauge fields: gravitational and electromagnetic.

The loop representation in quantum theory is based on using the so-called loop variables which are the well-known in Yang-Mills theories Wilson loop functionals. The dynamical variables of Yang-Mills theory (as well as of general relativity in the framework of Ashtekar variables) are connection field over the spatial manifold and its conjugate momentum. Wilson loop functionals form a set of gauge invariant non-local quantities built from the connection field and the “loop” approach is to regard these quantities as basic variables. This becomes a very powerful means when one considers a generally covariant field theory. In this case the lack of background structure does not allow one to construct a renormalized operator corresponding to a local classical variable – in other words a renormalization procedure for constructing such an operator turns out to be background dependent. On the other
hand, loop variables are non-local quantities; one does not need any background structure to construct a representation of these variables in terms of operators in a Hilbert space. So, when there is no background structure available for a quantization procedure, the following strategy has been proposed: it is to regard loop quantities as basic variables at the classical level and construct quantum theory representing the corresponding Poisson algebra of loop variables by a certain operator algebra.

There is also another problem which loop representation seems to be suited for – it is the problem of presence of constraints. Constraints generate symmetry transformations and because of presence of symmetries not all degrees of freedom of the Lagrange formulation are physical. The general strategy for quantizing such a system is to choose the coordinates on its phase space which have the simplest properties under the symmetry transformations and regard them as basic variables. The loop variables are just these quantities. The symmetry transformations of general relativity in the framework of Ashtekar variables are gauge transformations and spatial diffeomorphisms. As we will see, for the system including also fermionic and electromagnetic fields the symmetry group consists of two similar parts: gauge and diffeomorphism transformations. It is the advantage of using loop variables that they are gauge invariant and transform very naturally under the diffeomorphism group, namely as the geometrical objects with which they are associated. That is why their usage simplifies considerably the problem of finding solutions to the gauge and diffeomorphism constraints.

Because loop variables contain all gauge invariant information, any local gauge invariant quantity can be expressed as a limit of corresponding loop variable. This means that the Hamiltonian constraint of the theory can be written in terms of loop variables with a properly chosen limit procedure. This provides us with the Hamiltonian operator regularization method, because the operator corresponding to a classical “loop” expression becomes a well defined operator in the loop space. It has been shown by Rovelli and Smolin that there exists such a way to take a limit that no divergences will appear and there arise a well defined operator in the loop space. So the loop representation which is based on the usage of loop variables can reduce the problem of solving the Hamiltonian constraint to a simple combinatorical problem in the loop space.

As it has been stated above, we develop quantization program for the system which includes not just pure gravitational field, but also fermionic and electromagnetic fields. It was noted by Ashtekar et al that there exists a natural possibility of unification gravity with other gauge fields in the Hamiltonian framework; it is to enlarge the gauge group of pure gravity $SL(2, C)$ to a group which describes a unified gauge field. The first work along this line concerned the loop representation for such a unified theory and its connection to the knot theory. We continue the developement of loop representation for the unified theory. It turns out that enlargening of the gauge degrees of freedom, and therefore enlargening of the symmetry group, leads to some appealing features of the quantum theory in the loop representation. One of them is that the loop operators act even simpler than in the case of pure gravity: in the latter case momentum operators act with a result which includes both a loop and its inverse; this is connected to the fact that loop variables corresponding to a loop and to its inverse coincide. In the case of the unified field these two quantities become independent: the loop variables acquire orientation. The difference from the case of pure gravity is that the loop operators never change this orientation when they act in the
“loop” space. The Poisson algebra of loop variables is described solely in terms of breaking, rearranging and rejoining loops and turns out to be simpler than in the case of pure general relativity.

The example of the loop technique for fermions coupled to gravitational field was given by Morales-Técolt and Rovelli [7]. Unlike these authors, we consider the full-featured case when two independent fermionic fields are present. Fermionic fields are described by two complex Grassmann-valued spinor fields so the “loop” variables, which are mixed “gauge – fermionic” quantities, are even Grassmann algebra elements. We construct the loop representation in which the action of quantum analogs of these variables can again be described in a geometric way: we will see that fermionic operators act in the loop space as operators of creation and annihilation of curves.

The organization of this paper is as follows. In Sec. II we remind briefly the properties of our system in the Hamiltonian formulation and introduce the unified Einstein-Maxwell gauge field. In this section we also describe the Hamiltonian formulation for the fermionic system. In Sec. III we introduce the loop variables and study the Poisson algebra structure. Sec. IV is the heart of our paper: it concerns constructing the loop representation. We introduce a $C^*$-algebra of configurational variables, find a representation space in which these configurational variables become operators and, choosing a basis in this space, define “loop” operators.

II. HAMILTONIAN FORMULATION

We begin with the action for gravity and matter fields. Fix a four-manifold $M$, which is topologically a direct product $\mathbb{R} \times \Sigma$ for some three-manifold $\Sigma$. In the framework of Ashtekar variables the Lagrangian density for gravity $L_E$ is the functional of an anti-Hermitian soldering form $\sigma^a A^I$ and a self-dual connection $4A_{a A} B$ on $M$ [8]

$$L_E(\sigma, A) = G^2(\sigma) \bar{\sigma}^a A^I a^b B A^4 F_{ab} 4A^I B,$$

where $(\sigma)$ is the determinant of the inverse soldering form and $4F_{ab}$ is the curvature tensor of $4A_a$. Self-dual connection field is chosen to be of dimension $1/m$, what is convenient because allows one to regard $4A_a$ as a usual Yang-Mills field. Factor $G$ in (1) is the fundamental constant; $G$ is set to have a dimension of $1/m$ so that the action is dimensionless. Other fundamental constants are set to be $\hbar = c = 1$.

Let us note that the action functional (1) is complex; the fields $\sigma^a$ and $4A_a$ are complex dynamical variables of the complexified general relativity system.

The connection $D_a$ defined by $4A_{a A} B$ and by electromagnetic vector-potencial $a_a$ via $4D_a \lambda_A = \partial_a \lambda_A + 4A_{a A} B \lambda_B + a_a \lambda_A$ acts only on unprimed spinors. Thus, we shall take the Dirac Lagrangian density for fermionic fields $\xi^A, \eta^{A'}$ (Grassmann-valued) to be

$$L_D(\xi, \eta, \sigma, A, a) = \sqrt{2}(\sigma) [\sigma^a A^I A [\eta^{A'} 4D_a \xi^A - (4D_a \bar{\eta}^A) \eta^{A'}] + \frac{im}{\sqrt{2}} [\bar{\eta} A \xi^A - \bar{\eta}^{A'} A \xi^{A'}].$$

The Lagrangian density for electromagnetic field is

$$L_{Em}(a, \sigma) = \frac{1}{2} (\sigma) g^{ac} g^{bd} 4f_{ab} 4f_{cd},$$
where $f_{ab}$ is the curvature tensor of $a_a$; metric field $g^{ab}$ here is defined as the squared soldering form

$$g^{ab} = \sigma^a_{AA'} \sigma^{bAA'}.$$

The total action of the theory is the sum

$$S = S_E + S_D + S_{Em}.$$

In order to develop the canonical quantization program we should pass on to the Hamiltonian framework, carrying out a space+time decomposition in the action functional (see [8] for details). Then the action takes the following form

$$S = \int dt \int d^3x \left( \text{Tr} \ E^a \mathcal{L}_t A_a + \mathcal{L}_t \xi^A \bar{\pi}_A + \mathcal{L}_t \bar{\eta}^A \bar{\omega}_A + \bar{\epsilon}^a \mathcal{L}_t a_a \right)$$

- Hamiltonian constraint

- Diffeomorphism constraint

- Gauge transformations constraint

- Gauge transformations constraint (spin basis rotations)

- Gauge transformations constraint (phase rotations)

As it is common for generally covariant field theories, the Hamiltonian will be the sum of constraints. The action in this Hamiltonian form provides a canonical phase space description of the system. This section is aimed to discuss the arising Poisson structure and to introduce a set of unified coordinates on this phase space.

**A. Einstein-Maxwell unified field**

Let us for the moment restrict our consideration only to the part of the Hamiltonian describing the dynamics of the gauge fields. The last two terms in the Hamiltonian are the generators of local gauge transformations on the phase space. These transformations are: rotations of the complexified spin basis at each spatial point, which form the group $SL(2, \mathbb{C})$, and phase rotations, which form the group $U(1)$; the gauge fields lie in the corresponding Lie algebras. Therefore, the full gauge group, which is formed by all internal space symmetry transformations, is $SL(2, \mathbb{C}) \times U(1)$. From the Hamiltonian, i.e. geometric, point of view it is superfluous to distinguish the two gauge fields – the dynamical variables of the theory should be a connection on some bundle over the spatial manifold (which takes values in the Lie algebra of the gauge group) and its conjugate momentum. Thus, we should regard the two independent connection fields of initial Lagrange formulation as the two parts of one connection field – the unified Einstein-Maxwell field.

So we are to choose the new “coordinates” on the phase space of the system which will correspond to the unified gauge field. The expression for the new gauge variables is straightforward. Let us choose the new connection field to be
Then the initial Einstein and Maxwell connection fields can be expressed through \( \mathcal{A} \) as follows

\[
\mathcal{A}^{AB} = \mathcal{A}^{AB} - \frac{1}{2} (\text{Tr} \mathcal{A}) \delta_{A}^{B},
\]

\[
\mathfrak{a}^{a} = \frac{1}{2} (\text{Tr} \mathcal{A}) \delta_{A}^{B}.
\]

Having introduced the unified connection field \( \mathcal{A} \) we can define the corresponding momentum field \( \mathcal{E} \). We shall take it in the form

\[
\tilde{\mathcal{E}}^{aB}_{A} := \tilde{E}^{aB}_{A} + \frac{1}{2} \mathfrak{e}^{a} \delta_{A}^{B},
\]

so that it is the canonically conjugate to \( \mathcal{A} \):

\[
\{ \tilde{\mathcal{E}}^{a}_{CD}(x), \mathcal{A}^{AB}_{B}(y) \} = - \delta^{3}(x - y) \delta^{a}_{B} \delta^{A}_{D} \delta^{B}_{C}.
\]

Here \( \mathcal{A} \) is the pullback of \( \mathcal{A} \) to the tree-manifold \( \Sigma \). The factor \( \frac{1}{2} \) in (3) is important; it provides the correct (canonical) commutational relations between the connection field and its momentum (4). The gravitational and electromagnetic momentum fields can also be expressed through the unified field

\[
\tilde{E}^{aB}_{A} = \tilde{E}^{aB}_{A} - \frac{1}{2} \text{Tr}(\tilde{E}^{a}) \delta_{A}^{B},
\]

\[
\tilde{e}^{a} = \text{Tr}(\tilde{E}^{a})
\]

Having these relations it is just an exercise to rewrite the constraints in terms of the unified fields. The last two terms in the Hamiltonian are the Gauss law constraints for the gravitational and electromagnetic fields

\[
\langle \mathcal{A}^{a} \rangle^{B}_{A} D_{a} \tilde{E}^{aB}_{B} + \langle \mathfrak{a}^{a} \rangle \partial_{a} \tilde{e}^{a},
\]

where

\[
D_{a} = \partial_{a} + \mathcal{A}_{a}.
\]

We can express it in terms of the new Lagrange multiplier \( \langle \mathcal{A}^{a} \rangle \), so the Gauss law constraint for the unified field takes the form

\[
\frac{\delta(S_{E} + S_{Em})}{\delta(\mathcal{A}^{B}_{A} \dot{t})} = D_{a} \tilde{E}^{aB}_{B} = 0,
\]

where we have introduced

\[
D_{a} := \partial_{a} + \mathcal{A}_{a}.
\]

The part of the diffeomorphism constraint

\[
\frac{\delta(S_{E} + S_{Em})}{\delta N^{a}} = - \text{Tr}(\tilde{E}^{b} F_{ab}) - \tilde{e}^{b} f_{ab}
\]

expressed through the unified variables takes the form

\[
\frac{\delta(S_{E} + S_{Em})}{\delta N^{a}} = - \text{Tr}(\tilde{E}^{b} F_{ab}).
\]

Again, the factor \( \frac{1}{2} \) from (3) was necessary to cancel the factor 2 which appeared from the trace operation. Here we introduced the curvature field \( F_{ab} \) of the connection \( \mathcal{A}_{a} \)

\[
F_{ab} := 2 D_{[a} \mathcal{A}_{b]} = 2 \partial_{[a} A_{b]} + 2 \partial_{[a} \mathfrak{a}_{b]} + [A_{a}, A_{b}] = F_{ab} + f_{ab}.
\]
B. Fermionic part

For the Dirac action functional the space-time decomposition leads to the following expression (see [8] for details)

\[ S_D = \int dt \int_{\Sigma_t} d^3x \left\{ -i(\sigma)\{(\xi^\dagger)_A L_t \xi^A + (\bar{\eta}^\dagger)_A L_t \bar{\eta}^A\} + \mathcal{H}(\xi, \xi^\dagger, \bar{\eta}, \bar{\eta}^\dagger)\right\}, \]  

(13)

where \( \mathcal{H} \) means the Hamiltonian functional. The \( ^\dagger \)-operation here descends from the complex conjugation on the Grassmann algebra of \( SL(2, \mathbb{C}) \) spinors and satisfies the following properties:

(a) \( (a\alpha_A + b\beta_A)^\dagger = a^*\alpha_A^\dagger + b^*\beta_A^\dagger \); (b) \( (\alpha_A^\dagger)^\dagger = -\alpha_A \); (c) \( (\alpha^A)^\dagger \alpha_A \geq 0 \); (d) \( (\epsilon_{AB})^\dagger = \epsilon_{AB} \); (e) \( (\alpha_A \beta_B)^\dagger = \alpha_A^\dagger \beta_B^\dagger \), for all Grassmann fields \( \alpha_A \) and \( \beta_B \) and complex functions \( a, b \). Being Grassmann-valued, the fermionic fields anti-commute, so having rearranged them in (13) we got the different from [8] sign in square brackets. We define the momentum fields by the left variational derivatives

\[ \pihat_A := \frac{\delta S}{\delta L_t \xi^A} = i(\sigma)(\xi^\dagger)_A; \]  

(14)

\[ \omegahat_A := \frac{\delta S}{\delta L_t \bar{\eta}^A} = i(\sigma)(\bar{\eta}^\dagger)_A. \]

Then the action takes the form

\[ S_D = \int dt \int_{\Sigma_t} d^3x \left[ L_t \xi^A \pihat_A + L_t \bar{\eta}^A \omegahat_A + \mathcal{H}(\xi, \pihat, \bar{\eta}, \omegahat)\right]. \]  

(15)

The momentum fields have appeared at the right side to the configurational fields because of the usage of the left derivatives in the momentum field definition. The full Hamiltonian density for the spinor fields is

\[ \mathcal{H}(\xi, \pihat, \bar{\eta}, \omegahat) = \]  

\[ \mathcal{N} \left[ G^{-2} \bar{E}^a_B [D_a \xi^A \pihat_B + D_a \bar{\eta}^A \omegahat_B] + i m [(\sigma)^2 \bar{\eta}^A \xi^A + \pihat^A \omegahat_A] \right. \]

\[ \left. - (\mathcal{A} t)_B^A [\xi^B \pihat_A + \bar{\eta}^B \omegahat_A] - N^a [D_a \xi^A \pihat_A + D_a \bar{\eta}^A \omegahat_A]\right], \]  

(16)

where we used the “unified” Lagrange multiplier \( (\mathcal{A} t) \) (see (4)).

The equations of motion are now straightforward from the variational principle. Using the left variation which turns into zero at the initial and final time points one finds the dynamics

\[ L_t \xi^A = - \frac{\delta H}{\delta \pihat_A}; \quad L_t \bar{\eta}^A = - \frac{\delta H}{\delta \omegahat_A}; \]  

\[ L_t \pihat_A = - \frac{\delta H}{\delta \xi^A}; \quad L_t \omegahat_A = - \frac{\delta H}{\delta \bar{\eta}^A}. \]
So the evolution of any functional of the dynamical variables is given by

\[ \mathcal{L}_t f(\xi, \tilde{\pi}, \tilde{\eta}, \tilde{\omega}) = \{ H, f \}, \]

where the Poisson structure on the phase space is defined via

\[
\{ f, g \} = -\int d^3x \left[ \frac{\delta f}{\delta \xi^A} \frac{\delta g}{\delta \tilde{\pi}^A} + \frac{\delta f}{\delta \tilde{\pi}^A} \frac{\delta g}{\delta \xi^A} + \frac{\delta f}{\delta \tilde{\eta}^A} \frac{\delta g}{\delta \tilde{\omega}^A} + \frac{\delta f}{\delta \tilde{\omega}^A} \frac{\delta g}{\delta \tilde{\eta}^A} \right].
\]

(17)

All functional derivatives in this formula are left. Then one can obtain the Poisson brackets between the canonical variables

\[
\{ \xi^A(x), \xi^B(y) \} = 0; \quad \{ \tilde{\pi}^A(x), \tilde{\pi}^B(y) \} = 0;
\]

\[
\{ \tilde{\pi}^B(y), \xi^A(x) \} = -\delta^A_B \delta(x - y)
\]

and analogously for \( \tilde{\eta}, \tilde{\omega} \) fields.

**C. The Hamiltonian constraint**

The Hamiltonian constraint of the theory consists of the three parts

\[
\frac{\delta S_E}{\delta \tilde{N}} = \frac{1}{2} G^2 \text{Tr}(\tilde{E}^a \tilde{E}^b F_{ab})
\]

(20)

\[
\frac{\delta S_{Em}}{\delta \tilde{N}} = \frac{1}{32} \left( \frac{1}{G^2} \right)^4 (\sigma)^{-2} \text{Tr}(\tilde{E}^a \tilde{E}^c) \text{Tr}(\tilde{E}^b \tilde{E}^d) (e_{ab} e_{cd} + b_{ab} b_{cd})
\]

(20a)

\[
\frac{\delta S_D}{\delta \tilde{N}} = G^{-2} \tilde{E}^A_B \left( D_a \xi^A \tilde{\pi}^B + D_a \tilde{\eta}^A \tilde{\omega}^B \right) + im ((\sigma)^2 \tilde{\eta}^A \xi^A + \tilde{\pi}^A \tilde{\omega}^A),
\]

(20b)

where \( b_{ab} = 2 f_{ab} \). Having introduced the unified connection field and the corresponding conjugate momentum, we shall express the Hamiltonian constraint in terms of these fields. This gives for the Einstein part of the Hamiltonian

\[
\frac{\delta S_E}{\delta \tilde{N}} = \frac{1}{2} G^2 \mathcal{U}_{abc} \text{Tr}(\tilde{E}^a \tilde{E}^b \tilde{B}^c).
\]

(21)

Here we introduced the magnetic field \( \tilde{B}^a \) as the dual of the curvature of the unified field

\[
F_{ab} = \mathcal{U}_{abc} \tilde{B}^c;
\]

so that it has the dimension and the weight of \( \tilde{E}^a \). The tensor \( \mathcal{U}_{abc} \) is the totally antisymmetric tensor of weight \( -1 \).

The other two parts of the Hamiltonian become

\[
\frac{\delta S_{Em}}{\delta \tilde{N}} = \frac{1}{32} \left( \frac{1}{G^2} \right)^4 (\sigma)^{-2} \mathcal{U}_{abcdef} \text{Tr}(\tilde{E}^a \tilde{E}^c) \left( \text{Tr}(\tilde{E}^b \tilde{E}^d) \text{Tr}(\tilde{B}^e) \text{Tr}(\tilde{B}^f) - \text{Tr}(\tilde{B}^a \tilde{B}^b) \text{Tr}(\tilde{E}^c) \text{Tr}(\tilde{E}^f) - \text{Tr}(\tilde{E}^a \tilde{E}^b) \text{Tr}(\tilde{E}^d) \text{Tr}(\tilde{B}^c) \text{Tr}(\tilde{B}^f) \right),
\]

(21a)
\[
\frac{\delta S_D}{\delta N} = G^{-2}(\tilde{\mathcal{E}}_A^B - \frac{1}{2} \text{Tr}(\tilde{\mathcal{E}}^B)(D_a \xi^A \tilde{\pi}_B + D_a \bar{\eta}^A \tilde{\omega}_B) + i m ((\sigma)^2 \bar{\eta} \xi^A + \bar{\pi}^A \tilde{\omega}_A). \quad (21b)
\]

Let us also give here the complete (including fermionic degrees of freedom) expression for the Gauss law and diffeomorphism constraints in terms of the Einsein-Maxwell field

\[
\frac{\delta S}{\delta N^a} = - \text{Tr}(\tilde{\mathcal{E}}^b F_{ab}) - (D_a \xi^A \tilde{\pi}_A + D_a \bar{\eta}^A \tilde{\omega}_A) \quad (22)
\]

\[
\frac{\delta S}{\delta (4A_t)_B} = - (\xi^A \tilde{\pi}_B + \bar{\eta}^A \tilde{\omega}_B) + D_a \tilde{\mathcal{E}}_a^A. \quad (23)
\]

As we have seen, the Hamiltonian constraint contains the determinant of the inverse soldering form

\[
(\sigma)^2 = - \frac{1}{3\sqrt{2}} \Omega_{abc} \text{Tr}(\tilde{\sigma}^a \tilde{\sigma}^b \tilde{\sigma}^c), \quad (24)
\]

so we need its expression through the unified variables. It is given by

\[
(\sigma)^2 = \frac{1}{12 G^a} \Omega_{abc} \text{Tr}(\tilde{\mathcal{E}}^a - \frac{1}{2} \text{Tr}(\tilde{\mathcal{E}}^a))(\tilde{\mathcal{E}}^b - \frac{1}{2} \text{Tr}(\tilde{\mathcal{E}}^b))(\tilde{\mathcal{E}}^c - \frac{1}{2} \text{Tr}(\tilde{\mathcal{E}}^c)). \quad (25)
\]

This accomplishes the aim of this Section, which was to obtain all the constraints of the Hamiltonian framework expressed in terms of the unified gauge and the fermionic fields. We will conclude by pointing out that in the form (21) the Hamiltonian is not polynomial in \(\tilde{\mathcal{E}}^a\) variables because of the presence of the factor \((\sigma)^{-2}\) in the electromagnetic part; this might cause problems in constructing the corresponding quantum operator. Possible solution of the problem was proposed by Ashtekar et al [8]. Multiplying the Hamiltonian constraint by \((\sigma)^2\) one may restore its polynomial character; the Hamiltonian constraint becomes a density of weight four (therefore the corresponding Lagrange multiplier - lapse function - becomes a density of weight minus three). Another possible way to tackle this problem is discussed in [17].

III. ALGEBRA OF LOOP VARIABLES

In this Section we construct gauge invariant non-local functionals of the dynamical variables. These functionals are associated with loops, curves and ribbons so they will play a role of “loop” gauge-invariant coordinates on the system’s phase space. We will discuss the algebra of these “loop” variables with respect to the Poisson brackets. A special attention is paid to a graphical representation of the algebra obtained.

A. Configurational loop variables

The set of variables which we call configurational loop variables will play a crucial role in the quantization procedure. Let us denote the space of the unified connection fields (the space of connections on a certain \(SL(2, C) \times U(1)\) bundle over \(\Sigma\) by \(F\) and consider a Wilson loop functional on \(F\).
\[(\gamma) \equiv T_{\gamma}[A] := \text{Tr} \mathcal{P} \exp \oint_{\gamma} \mathcal{A}, \tag{26}\]

\[\gamma : [0, 1] \rightarrow \Sigma.\]
The expression under the trace operation here is the parallel transport (with the connection \(A\)) matrix \(U\)

\[U[\gamma]_A^B = \mathcal{P} \exp \int_\gamma d\tau \dot{\gamma}^a A_{aA}^B\]
taken for a closed loop, so

\[(\gamma) = \text{Tr} U[\gamma].\]

The main difference from the case of pure gravity is that

\[(\gamma^{-1})[A] \neq (\gamma)[A] \tag{27}\]
because of the presence of the additional electromagnetic part in the connection field. Since loop quantities form a set of (complex) coordinate\(^1\) on the configurational space \(\mathcal{F}/\mathcal{G}\) (we denoted by \(\mathcal{F}/\mathcal{G}\) the quotient space of \(\mathcal{F}\) with respect to the action of gauge transformations) \((27)\) means that the loop quantities \((\gamma)\) and \((\gamma^{-1})\) are independent coordinate variables. This explains why, unlike the case of pure general relativity, loop variables are associated with oriented loops. We will exploit a simple graphical representation of these quantities

![Diagram](image)

FIG. 1. Configurational loop variables for the unified field are associated with oriented loops.

Loop variables form an over-complete set of coordinate quantities in the sense that they satisfy the following identities\(^3\)

1. They are invariant under reparametrizations of loops. If \(\gamma'\) is a reparametrized loop \(\gamma'(s) = \gamma(f(s))\) then

\[(\gamma') = (\gamma).\]

2. The Mandelstam identity. For any three loops \(\alpha, \beta,\) and \(\gamma\) intersecting at a point one has

\[(\alpha)(\beta)(\gamma) = (\alpha \circ \beta)(\gamma) + (\alpha)(\beta \circ \gamma) + (\alpha \circ \gamma)(\beta) - (\alpha \circ \beta \circ \gamma) - (\alpha \circ \gamma \circ \beta).\]

\(^{1}\)in the sense that for a pair of gauge not equivalent fields \(\mathcal{A}_1\) and \(\mathcal{A}_2\) there exist such a pair of loops \(\gamma_1\) and \(\gamma_2\) that \((\gamma_1)[\mathcal{A}_1] \neq (\gamma_2)[\mathcal{A}_2]\)

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This, on the first sight cumbersome relation has replaced the simpler Mandelstam identity for the case of pure gravity [1]

\[(\alpha)(\beta) = (\alpha \circ \beta) + (\alpha \circ \beta^{-1})\]

owing to the independence of \((\gamma)\) and \((\gamma^{-1})\) variables in our case.

3. Loop variables are invariant under retracing operation

\[(\gamma \circ \eta \circ \eta^{-1}) = (\gamma),\]

where \(\circ\) means the composition of loops which intersect at a point, \(\eta^{-1}\) is the inverse of a curve \(\eta\).

As configurational loop variables involving fermionic degrees of freedom we will take certain even Grassmann algebra elements. The infinite-dimensional Grassmann algebra is generated by the anticommuting complex objects – our dynamical field variables \(\xi(x), \bar{\eta}(x), \bar{\pi}(x), \bar{\omega}(x)\). A basis of the Grassmann algebra is formed by the powers of these generators. Let us consider the following gauge-invariant even elements associated with open curves

\[(\xi|\gamma|\bar{\eta}) := \text{Tr}\{\xi U[\gamma] \bar{\eta}\} = \xi^A U[\gamma]^A B \bar{\eta}_B,\]  \hspace{1cm} (28)

which we will regard as fermionic configurational variables.

We propose a convenient notation in which any “loop” quantity is denoted by a Greek letter in parenthesis. Since ends of a curve correspond to the fermionic degrees of freedom, it is convenient to include symbols of fermionic fields in parenthesis on both sides of a loop symbol to get a symbol which describes the mixed quantity. Thus, \(\gamma\) in the above expression is the open curve with ends marked by \(\xi, \bar{\eta}\;\text{we will always put \(\xi\) at the final point of a curve and \(\eta\) will mark the initial (recall that any curve (loop) has an orientation)}\)

![FIG. 2. Fermionic variables are associated with open curves.](image)

\(^2\)This is the point where our approach differs from that of Morales-Técolt and Rovelli [1]. As the quantities involving fermionic fields they considered \((\psi|\gamma|\psi)\) (in our notations). It is one of the reasons for which we introduce the loop variables that the quantum Hamiltonian operator can be defined as a certain loop limit of an operator constructed from the basic loop operators. However, the quantities they consider as basic ones turn into zero when the corresponding curve shrinks to a point. In this sense the loop quantities quadratic in a Grassmann field can not serve as basic variables.
The quantities introduced satisfy the relations analogous to those for closed loops

1. Reparametrization invariance

\[(\xi|\gamma|\bar{\eta}) = (\xi|\gamma|\bar{\eta}).\]

2. The Mandelstam identity. Consider a curve \(\alpha\) and a point \(s\) on it. This point divides \(\alpha\) into two parts for which we use the special notation

\[\alpha = \alpha_s \circ \alpha_s.\]

Thus, \(\alpha_s\) is the part of the \(\alpha\) from the initial point to the point \(s\) and \(\alpha_s\) is the remaining part. Then for any three curves \(\alpha, \beta,\) and \(\gamma\) intersecting at a point the following identity holds

\[(\xi|\alpha|\bar{\eta})(\xi|\beta|\bar{\eta})(\xi|\gamma|\bar{\eta}) = (\xi|\alpha_s \circ \beta_s |\bar{\eta})(\xi|\gamma_s|\bar{\eta}) + (\xi|\alpha_s \circ \gamma_s |\bar{\eta})(\xi|\beta_s|\bar{\eta}) - (\xi|\gamma_s \circ \alpha_s |\bar{\eta})(\xi|\beta_s|\bar{\eta}) + (\xi|\gamma_s \circ \beta_s |\bar{\eta})(\xi|\alpha_s|\bar{\eta}),\]

or using graphical notation for \((\xi|\alpha|\bar{\eta})\)

![Image of Mandelstam identity for open curves](image)

3. The retracing identity

\[(\xi|\gamma_s \circ \alpha \circ \alpha^{-1}\circ \gamma_s|\bar{\eta}) = (\xi|\gamma|\bar{\eta}).\]

So we have introduced the gauge invariant quantities \((\gamma)\) and \((\xi|\gamma|\bar{\eta})\). The above identities imply that these quantities are associated not with loops and curves themselves but with classes of equivalence of loops and curves. In the case of pure gravity these classes are called hoops and it seems reasonable to keep this name and for equivalence classes of curves. Two loops (or curves) belong to the same equivalence class (or hoop) if they define the same loop quantities for all fields \(A, \xi, \bar{\eta}\). These hoop variables form an Abelian algebra under the Poisson brackets and will play a role of "coordinates" in the loop representation.
B. Momentum loop variables

Let us first construct the quantities which do not involve any fermionic degrees of freedom. We associate such momentum variables with piecewise strips, i.e. piecewise ribbons with ends glued. Inserting the momentum field $\mathcal{E}^a$ at points of loops, one can construct the following gauge invariant loop quantities linear in the field $\mathcal{E}$

$$ (\gamma)^a(s) := \text{Tr}\{U[\gamma/s] \mathcal{E}^a(s) U[\gamma_s]\}. $$

These quantities are almost what one needs as the momentum variables. As we have stated, they are gauge invariant but, because of their vector character, they transform under the action of diffeomorphisms somewhat complicatedly. We shall construct the other quantities which are associated with piecewise strips and which transform under diffeomorphisms as geometrical objects (i.e. the transformed quantity is also a strip quantity which is associated to another strip – a transformed one). In this paper we consider only the basic, linear in momentum field strip variables which we will again denote by a letter in parenthesis

$$ (S) := \int_S d^2 s \eta_{abc} \gamma^c(p). $$

Here $\gamma(p)$ is a loop which goes through a point $p$ on $S$, and $\eta_{abc}$ denotes the Levi-Civita tensor density on $\Sigma$. The loop family $\{\gamma(p)\}$ is supposed to cover all the strip surface (the loops $\gamma(p)$ and $\gamma(p')$ for different $p, p'$ may coincide). The quantity defined is the gauge invariant functional on the phase space associated with a strip $S$

![Fig. 4. Linear in $\mathcal{E}$ momentum variables are associated with strips.](image)

The gauge-invariant loop quantities involving fermionic momentum fields are

$$ (\xi|\gamma|\bar{\pi}) := \text{Tr}\{\xi U[\gamma] \bar{\pi}\}, $$

$$ (\bar{\omega}|\gamma|\bar{\eta}) := \text{Tr}\{\bar{\omega} U[\gamma] \bar{\eta}\}, $$

$$ (\bar{\omega}|\gamma|\pi) := \text{Tr}\{\bar{\omega} U[\gamma] \pi\}. $$

Again $\gamma$ is a curve with ends marked by the corresponding fermionic variables. The introduced quantities are represented respectively by

![Fig. 5. Fermionic momentum variables.](image)
C. Loop variables algebra

The loop variables introduced are functionals on the phase space and the Poisson algebra they generate can be computed. It is induced by the Poisson structure on the space of gauge and fermionic fields (\( (7) \) and \( (19) \) respectively). The remaining part of these section is aimed to describe the resulting algebra of loop variables in a graphical form.

The brackets of loop quantities with the momentum loop variables can be obtained by using the following useful expression for the matrix \( U \)

\[
U[\gamma]_A^B = \int ds^a \ U[\gamma/s]_A^C \ A_a(s) D \ U[\gamma]_s^D.
\]

So one gets

\[
\{ (S), (\beta) \} = (\gamma^{(S)} \circ \beta), \quad (33)
\]

where \( \gamma^{(S)} \) is the loop from the loop family covering \( S \) which intersect with the loop \( \beta \). It is tempting to represent this result as

FIG. 6. The Poisson brackets between a strip and a loop variables.

The brackets of the fermionic momentum variables \( (32) \) with the coordinate loop quantities are given by

\[
\int d^3x \ \{ (\xi|\beta(x)|\bar{\pi}(x)), (\xi|\gamma|\bar{\eta}) \} = (\xi|\beta \circ \gamma|\bar{\eta}); \quad (34)
\]

in the right side of this expression \( \beta \) is the curve from the family \( \beta(x) \) whose initial point coincide with the final point of \( \gamma \). There is the graphical representation for this expression

FIG. 7.

Computing the other brackets one gets

\[
\int d^3x \ \{ (\bar{\omega}(x)|\beta(x)|\bar{\eta}), (\xi|\gamma|\bar{\eta}) \} = (\xi|\gamma \circ \beta|\bar{\eta}); \quad (34a)
\]
\[ \int d^3x d^3y \{ (\bar{\omega}(x)|\beta(x;y)\bar{\pi}(y)), (\xi|\gamma|\bar{\eta}) \} = (\xi|\gamma \circ \beta|\bar{\pi}) + (\bar{\omega}|\beta \circ \gamma|\bar{\eta}); \quad (34b) \]

So, as we have seen, the algebra of introduced “loop” variables can be described solely in terms of geometrical objects: loops, curves and strips. Because of the natural action of the diffeomorphism transformations on the introduced variables (namely as on geometrical objects), the elements of the quotient algebra of these variables with respect to the diffeomorphisms’ action have a clear geometrical meaning. They are represented by classes of diffeomorphism equivalent curves, loops and strips. The algebraic (induced Poisson) structure on this quotient algebra is given by the same relations (33)-(34) but understanding as the relations for equivalence classes. This fact helps one to solve the problem of finding the solutions of the diffeomorphism constraint in the loop representation.

**IV. THE LOOP REPRESENTATION**

Constructing the quantum representation for our system we will mostly follow the programm of Asthekar group (for recent developments see [14]); however, the approach described below is more ”physical” because much importance is attached to the visualization of all relations.

The program of quantization of generally covariant field theories proposed in the number of publications (see [14] and references therein) uses the idea to realize the quantum representation space as the representation space of a configurational variables Abelian C*-algebra. The construction is an infinite-dimensional generalization of the standart coordinate representation in quantum mechanics: the space of coordinate representation is the representation space of the Abelian algebra of \( \hat{x} \) operators, i.e. \( \text{Span}\{ | x \rangle \} \) where \( \hat{x}|x\rangle = x|x\rangle \). The canonical realization of this space is the space \( L^2(\mathbb{R}, dx) \) of functions \( \varphi(x) \) over the spectrum of
Momentum variables are represented by derivative operators on $L^2$. The infinite-dimensional case repeats all these points: the representation space is the space of all continuous functionals on the spectrum of the configuration variables algebra and “momentum” variables naively can be represented by variational derivative operators.

In order to take advantage of the standard representation theory of $C^*$-algebras we have to define a $C^*$-algebra of configurational variables.

A. $C^*$-algebra of configurational variables

The most natural candidate for this algebra is the algebra of our configurational loop variables over complex numbers. Its elements would be complex even Grassmann numbers (i.e. it would be formed by powers of our configurational loop variables $(\gamma)$, $(\xi|\gamma|\bar{\eta})$) and we would define an involution as the $\dagger$-operator. One, however, runs into some problems on this way. First, because of the complexity of the Ashtekar connection, the parallel transport matrix $U[\gamma]$ is not unitary (it belongs to a larger group $SL(2, C) \times U(1)$); therefore, our loop variables behave somewhat complicatedly under the complex conjugation operation; for example, given any loop $\gamma$ there may not exists such a loop $\gamma'$ that $(\gamma)^\ast = (\gamma')'$, but it is still not the worst. Because of the non-unitarity of $U(\gamma)$ the elements $(\gamma)$ are not bounded so the natural sup-norm $\| \cdot \|$ does not exists on this algebra. Owing to these facts the case of complex $A_a$ (or the case of Lorentzian general relativity) remained a problem by the last time. The situation get somewhat changed after the coherent state transform had appeared [15].

Recall that the complex Ashtekar connection $A_a$ satisfies the following reality conditions

$$ A_a + A_a^\dagger = 2 \Gamma(\sigma). $$

Thus, being the complex $SL(2, C)$ connection field, $A$ bears some additional “unphysical information” and one can expect that only its “real”, $SU(2)$ part must play a role in the quantization procedure. This leads to the idea to consider quantum states $\Psi[A]$ which depend only on the connection $A_a$ and do not depend on its complex conjugate $\bar{A}_a$. Such functionals $\Psi$ are called holomorphic, so the first step to eliminate the superfluous degrees of freedom from the quantization procedure is to consider a representation of Lorentzian general relativity in the space of holomorphic functionals of the Ashtekar connection. Then, as it has been shown in [15], there exists an isomorphism (given by the coherent state transform) between this space and the space of functionals of the $SU(2)$ connection. Therefore, having built a quantum representation of the $SU(2)$ variables algebra, one will have the representation of Lorentzian genaral relativity given by the coherent state transform.

---

3 Although in the given form the reality conditions are non-polynomial in $\sigma$, there is a form in which they are polynomial.
So, according to this scheme, we have got to construct a representation of the $SU(2)$ variables algebra. For the case of pure gravity this actually has been done in \cite{14} and our aim is to show that the construction allows a natural enlargening to the case when the gauge field and the fermionic matter present.

The $C^*$-algebra of the $SU(2)$ configurational variables is described as follows. It is formed by the same “loop” quantities $(\gamma), (\xi|\gamma|\bar{\eta})$ (with the multiplication, additive and $\dagger$-operations from the Grassmann algebra). The only difference is that the parallel transport matrix $U[\gamma]$ becomes now unitary, so it has a bounded trace and there exists the sup-norm \cite{35} for the loop algebra elements $(\gamma)$. The unitarity of $U[\gamma]$ leads also to the following simple properties of our algebra generators with respect to the $\dagger$-operation:

• Operation $\dagger$ acts on the loop quantities $(\gamma)$ as simple complex conjugation and
\[ (\gamma)^* = (\gamma^{-1}). \]

• The action of $\dagger$-operation on the fermionic “loop” variables is the consequence of our momentum fields definition \cite{14} and is given by
\begin{align*}
(\sigma(x)) (\sigma(y)) (\xi(x)|\gamma|\bar{\eta}(y))\dagger &= (\bar{\omega}(y)|\gamma^{-1}|\bar{\pi}(x)), \\
(\sigma(x)) (\xi(x)|\gamma|\bar{\pi}(y))\dagger &= (\sigma(y)) (\xi(y)|\gamma^{-1}|\bar{\pi}(x)), \\
(\sigma(y)) (\bar{\omega}(x)|\gamma|\bar{\eta}(y))\dagger &= (\sigma(x)) (\bar{\omega}(y)|\gamma^{-1}|\bar{\eta}(x)).
\end{align*}
(36)

These relations are, in fact, the reality conditions which one should impose on the fermionic phase space in order to single out its real part. It is worthwhile to note that we have chosen the form in which they are non-polynomial in the $\tilde{E}$ variable (because of the presence of $(\sigma)$).

Next, let us introduce a norm which makes the algebra of “loop” elements a $C^*$-algebra. The algebra is spanned by powers of the “loop” generators; i.e. it is spanned by a set of elements which are labeled by pairs of points with curves connecting them (this correspond to a product of $(\xi|\gamma|\bar{\eta})$ generators) and by loops (this correspond to a product of different $(\gamma)$’s). We suppose that algebra elements depend no more than on a countable set of points $p_1, \ldots, p_k, \ldots, p'_1, \ldots, p'_k, \ldots$ and loops $\gamma_1, \ldots, \gamma_n, \ldots$, so arbitrary algebra element $X$ can be written as a functional of the type\cite{36}
\[ X = \Phi(\xi_{p_1}, \ldots, \xi_{p_k}, \ldots, \bar{\eta}_{p'_1}, \ldots, \bar{\eta}_{p'_k}, \ldots, U[\gamma_1], \ldots, U[\gamma_n], \ldots). \]

We call such functionals the cylindrical functionals. Function $\Phi$ is assumed to be a continuous function of all its arguments.

We shall introduce a norm on our algebra by means of the scalar product which we define in the Appendix. This scalar product has the form (see \cite{37})

\[ \text{Here k of this “loops” $\gamma$ are those connecting points } p, p' \text{ where fields } \xi, \bar{\eta} \text{ are taken.} \]
\[ \langle \Phi_1 | \Phi_2 \rangle = \int [\Phi_1(\xi, \bar{\eta})]^{\dagger} \Phi_2(\xi, \bar{\eta}) \exp \{ \sum (\xi)^{\dagger}(\xi) + \sum (\bar{\eta})^{\dagger}(\bar{\eta}) \} \, d(\xi, \xi^{\dagger}, \bar{\eta}, \bar{\eta}^{\dagger}), \]

where the integration is carried over the fermionic fields; the result does not depend on \( \xi, \eta \): it is a cylindrical functional of connection field (i.e. it depends on \( A \) as a continuous function \( f(U[\gamma_1], \ldots, U[\gamma_n]) \)). Therefore, the following function of our algebra elements exists

\[ \| X \| := \sup_{A \in u(2)} \sqrt{\langle X | X \rangle}, \quad (37) \]

which we will show has all the properties of a norm.

1. First,

\[ \| X + Y \| = \sup \sqrt{\langle X + Y | X + Y \rangle} \leq \sup(\sqrt{\langle X | X \rangle} + \sqrt{\langle Y | Y \rangle}) = \| X \| + \| Y \|. \]

2. For two elements \( X, Y \) which do not contain fermionic fields the property \( \| XY \| \leq \| X \| \cdot \| Y \| \) is obvious. When for example \( X \) is a purely fermionic while \( Y \) does not contain fermionic fields the property still holds, because then \( \langle X | X \rangle \) does not depend on the connection field, as it follows from our scalar product definition. So we have to check this property only for the case when both \( X \) and \( Y \) contain fermionic fields; we may always consider one of these elements, for instance \( X \), to be a generator, i.e. a product of the fermionic “loop” quantities. In the case when \( X = Y \) it can be explicitly checked that \( \| X^2 \| = \| X \|^2 \). For instance,

\[ \| (\xi | \gamma \bar{\eta})^2 \| = \sqrt{4} = \sqrt{2}\sqrt{2} = \| (\xi | \gamma \bar{\eta}) \|^2, \]

so we have the property required. In the opposite case, when \( Y \) does not comprise any curves from those which compose the generator \( X \), the scalar product definition gives \( \langle XY | XY \rangle = \langle X | X \rangle \langle Y | Y \rangle \) and the property is satisfied. So we have only check it for the case when \( Y = X + Y_1 \), where \( Y_1 \) does not contain any curves from those composing \( X \) (what is equivalent to \( \langle X | Y_1 \rangle = 0 \)). In this case it turns out also that \( \langle X^2 | XY_1 \rangle = 0 \), so \( \langle (X + Y_1)X | (X + Y_1)X \rangle = \langle X^2 | X^2 \rangle + \langle X | X \rangle \langle Y_1 | Y_1 \rangle \), and, because any fermionic generator has the property that \( \langle X^2 | X^2 \rangle = (\langle X | X \rangle)^2 \), we finally have that \( \langle (X + Y_1)X | (X + Y_1)X \rangle = \langle X | X \rangle (\langle X | X \rangle + \langle Y_1 | Y_1 \rangle) = \langle X | X \rangle \langle Y | Y \rangle \).

3. Next, we have to check the property \( \| XX^\dagger \| = \| X \|^2 \) which makes an algebra with involution a \( C^* \)-algebra. Because for the loop quantities this property is obvious, it should be checked only for the fermionic generators. The explicit calculation gives

\[ \| (\xi | \gamma \bar{\eta})(\xi | \gamma \bar{\eta})^\dagger \| = \sqrt{\langle (\xi | \gamma \bar{\eta})(\xi | \gamma \bar{\eta})^\dagger | (\xi | \gamma \bar{\eta})(\xi | \gamma \bar{\eta})^\dagger \rangle} = \sqrt{4} = \| (\xi | \gamma \bar{\eta}) \|^2. \]

For the elements of higher orders in the configurational fermionic quantities the required property follows from the fact that for any two such quantities \( X, Y \)

\[ \langle XY | XY \rangle = \langle X | X \rangle \langle Y | Y \rangle. \]
So the algebra of configurational “loop” variables becomes an Abelian *-algebra with norm \( \| \cdot \| \) (which satisfies the relation \( \| AA^\dagger \| = \| A \|^2 \)) and we can take a completion to obtain a C*-algebra of configurational “loop” variables.

### B. Representation space

Having the C*-algebra of configurational variables we are at the point to implement the standard representation theory. According to Gelfand an Abelian C*-algebra is isomorphic to the algebra of all continuous functions on its spectrum. Let us give the description of the spectrum of our loop variables algebra. Denote by \( \mathcal{F} \) the space of all (satisfying the certain boundary conditions) fields \( A_{\alpha A}(x), f^A(x), g^A(x) \) where \( A_{\alpha} \in u(2) \) and \( f, g \) are complex spinor fields (non-Grassmann-valued) which take values in fibres \( F \) of some \( G \)-bundle over \( \Sigma \). The corresponding space quotient by the gauge transformations will be \( \mathcal{F}/\mathcal{G} \) where \( \mathcal{G} = SU(2) \times U(1) = U(2) \). Then each point of \( \mathcal{F}/\mathcal{G} \) defines a linear homomorphism \( \omega \) from the loop variables algebra to \( C \) (a character) as follows:

\[
\omega_{A,f,g}(\gamma) := \text{Tr}(\exp \oint_\gamma A), \\
\omega_{A,f,g}(\xi|\gamma|\bar{\eta}) := \text{Tr}(f \exp \int_\gamma A g).
\]

The spectrum is the set of all characters so we have that the points of \( \mathcal{F}/\mathcal{G} \) distinguish the elements of our algebra spectrum; it is easy to show that \( \mathcal{F}/\mathcal{G} \) is dense (in Gelfand topology) in the spectrum, so we will denote the later by \( \overline{\mathcal{F}/\mathcal{G}} \). This space becomes a quantum configurational space of our theory. As in the case of pure gravity its limit points are distributions which we shall regard as generalized fields (in the sense of Dirac’s \( \delta \)-function). We will denote the generalized fields by the same symbols \( A, f, g \); so \( \overline{\mathcal{F}/\mathcal{G}} = \{ A, f, g \} \).

Thus, the space of “loop” algebra representation is the space \( \mathcal{C}^0(\overline{\mathcal{F}/\mathcal{G}}) \) of all continuous functions over \( \overline{\mathcal{F}/\mathcal{G}} \). This space, however, is too large to define integral and differential calculus on it. The construction of a smoller space, measure and differential calculus on this smoller space has been proposed by Ashtekar et al [14]. They proposed to regard the quantum configuration space of an infinite-dimensional case as the projective limit of finite-dimensional configurational spaces of gravity on floating lattices. Then the representation space becomes the space \( Cy(\overline{\mathcal{F}/\mathcal{G}}) \) of cylindrical functionals over the algebra spectrum. By a cylindrical functional on \( \overline{\mathcal{F}/\mathcal{G}} \) we understand a map \( \Psi \) of the form

\[
\Psi = \Phi(f^1, \ldots, f^k, g^1, \ldots, g^m, \mathcal{P} \exp \int_{\gamma_1} A, \ldots, \mathcal{P} \exp \int_{\gamma_n} A) \\
\Psi : (F^k \times F^m \times G^n) \to C.
\]

One can develop a calculus on \( Cy(\overline{\mathcal{F}/\mathcal{G}}) \) by means of the projective limit from the finite-dimensional configurational spaces [14]. Thus, one can introduce a measure \( \mu \) on \( Cy(\overline{\mathcal{F}/\mathcal{G}}) \); this gives rise to the Hilbert representation space \( L^2(\overline{\mathcal{F}/\mathcal{G}}, \mu) \).
In order to construct the loop representation we choose a certain basis in the representation space $Cyl(\mathcal{F}/\mathcal{G})$ so that the loop variables become in this basis simple operators which can be interpreted in terms of operators of creation and annihilation. The idea is very similar to one, which is used to define, for example, the momentum representation in quantum mechanics. One chooses the basis formed by all proper states $|p\rangle$ of the momentum operator (the corresponding wave-functions are $\sim \exp ipx$) and defines all the operators by their action on states from this basis. We introduce the basis of “loop” states which in some sense are proper states of momentum loop operators and the loop operators become the operators of creation and annihilation of loops and curves in this basis.

It is convenient to use the Dirac’s notations and denote the following functionals in our space by Dirac’s kets

$$|\alpha\rangle = \begin{cases} \text{Tr}(f \mathcal{P} \exp \int_{\alpha} A g) & \text{or} \\ \text{Tr}(\mathcal{P} \exp \int_{\alpha} A) \end{cases}$$

depending on whether $\alpha$ has ends or not; and in a similar way a “multiloop” state is

$$|\alpha, \beta\rangle = \begin{cases} \text{Tr}(f \mathcal{P} \exp \int_{\alpha} A g) \text{Tr}(f \mathcal{P} \exp \int_{\beta} A g) \\ \cdots \end{cases}, \text{etc.}$$

The order in which loops are taken to compose a multiloop state is not important.

These states form the basis in the representation space and we will call them $n$-loop states. The configurational loop variables become operators of multiplication and the Dirac’s notations allow us to express their action simply by

$$(\hat{\gamma}) |\alpha\rangle = |\alpha, \gamma\rangle$$

$$(\xi \hat{\gamma} \eta |\alpha\rangle = |\alpha, \gamma\rangle.$$

In a similar manner they act on states containing more ”loops”. We see, therefore, that if one thinks about the state $|\alpha_1, \alpha_2, \cdots, \alpha_n\rangle$ as about a state containing n ”loops”, then the action of the configurational operators consists in simply the adding of one more ”loop” to a state. It is tempting to regard these operators as “creation” operators. Thus, the basis can be obtained acting on a cyclic vector (unity functional) by the creation “loop” operators.

Let us examine the basis introduced more thoroughly. Due to the identities satisfied by Wilson functionals this basis is overcomplete. Its elements are linear dependent, so some of them may be rewritten as linear combinations of others. For example, any three-loop state may be first realized as the state with three loops intersecting at a point and then reduced to the sum of states containing two and one loop.

$\text{Although, in the absence of a scalar product in the representation space, it is not quite legitimate to call the set of states which we will introduce a basis, we use this term because there exists a scalar product in which the set of states is a basis. Note also, that this basis is not countable.}$
Thus, as it has been found by Gambini and Pullin \cite{6}, any n-loop state may be reduced to a linear combination of two- and one-loop states.

Consider then a state containing two loops and one open curve. Repeating the above procedure, we may reduce this state to a linear combination of states containing one loop, one open curve and merely open curve states

Thus, any state containing n curves and m loops may be reduced to a linear combination of states containing n curves and one loop or n-curve states (the number of ends in a state cannot be reduced). The set of irreducible elements of our basis consists of states

- n curves no loops, \( n = 0, 1, \cdots \),
- n curves one loop, \( n = 0, 1, \cdots \),
- two loops.

Having this “loop” basis in the representation space we are ready to define the action of other operators by defining it on the basis elements.
C. Momentum operators

In the last part of this Section we construct the representation of the classical momentum “loop” variables in the space considered, i.e. we build operators

\[(\hat{S}), \ (\xi | \gamma | \bar{\eta}), \ (\bar{\omega} | \gamma | \bar{\eta}), \ (\bar{\pi} | \gamma | \bar{\omega}),\]

so that their commutational relations coincide to the first degrees in \(\hbar\) with the Poisson brackets of their classical analogs. Note that we represent the Poisson brackets by commutational relations even though the variables involve Grassmann fields.

The advantage of the built representation is that we have simultaneously two equivalent descriptions of operators’ action. The first, visual one is based on a graphical representation of states and operators, i.e. on dealing with Dirac’s kets \(|\alpha\rangle\). The second description is based on representing states as functionals of generalized fields and operators act in the space of functionals. In this later one there is a naive way to define the momentum operators; one should just use the corresponding classical expressions and replace all momentum fields by the functional derivative operators. The resulting operators will act in the space of functionals of generalized fields \(\Psi[A, f, g]\)

\[
\begin{align*}
\hat{\xi}^a \rightarrow \hat{\xi}^a, \quad (\hat{\xi}^a \circ \Psi)[A] &:= i \frac{\delta}{\delta A_a} \Psi[A]; \\
\hat{\pi}_A \rightarrow \hat{\pi}_A, \quad (\hat{\pi}_A \circ \Psi)[f, g] &:= i \frac{\delta}{\delta f_A} \Psi[f, g]; \\
\hat{\omega}^A \rightarrow \hat{\omega}^A, \quad (\hat{\omega}^A \circ \Psi)[f, g] &:= i \frac{\delta}{\delta g_A} \Psi[f, g].
\end{align*}
\]

No problems arise there with operator ordering because functional derivative operators commute when they act at different space points. These construction leads to the well defined operators, whose action on the introduced basis vectors can be described graphically. Moreover, this latter graphical description can be used as an alternative definition of the “momentum” operators. We shall give the result in the both descriptions.

First, let us define the “strip” operator, which describes gauge degrees of freedom. It is represented by the following operator

\[
(\hat{\hat{S}}) \circ |\alpha\rangle := i |\gamma^S \circ \alpha\rangle,
\]

\[\text{No.}\] The “right” and “left” functional derivatives here have nothing to do with the Grassmann properties: this is a useful convention which implies the rules by which our fermionic momentum operators act on the loop states. For example, it is easy to see that \((\xi | \bar{\gamma} | \bar{\pi}) \circ (\xi | \alpha | \bar{\eta}) = (\xi | \gamma \circ \alpha | \bar{\eta})\) and \((\xi | \alpha | \bar{\eta}) \circ (\omega^A | \gamma | \bar{\eta}) = (\xi | \alpha \circ \gamma | \bar{\eta})\) (we omitted the integration over the momentum fields in this formula), so the right and left vectors over our momentum operators mean simply that the \(\omega^A\) operator glues a curve to the right side of a curve in our ordering (i.e. to the begining point) while \(\bar{\omega}\) glues a curve to the left (to the final point). This also explains the usage of \((\cdot | \gamma | \cdot)\) symbols for our loop quantities: one simply has to replace \((\bar{\pi}, \xi)\) or \((\bar{\eta}, \bar{\omega})\) pair by the composition operation \(\circ\).
where $\gamma^S$ is the loop from the loop family covering $S$ which intersects with the loop $\alpha$ (when there is no intersection of $\alpha$ with the strip the result is zero). The graphical representation of this operator is

$$
\begin{align*}
(S) & \quad \alpha \quad \gamma \quad \alpha \\
\quad & = \quad i \gamma^S \alpha
\end{align*}
$$

FIG. 12.

i.e. the operator adds the loop to a one loop state and glues these loops in the only way compatible with their orientation. As we have stated above, the “loop” states are the “proper” states of our momentum operators in the sense that the result of their action on a $n$-loop state is also a $n$-loop state (when the strip intersects more than one loop from the state the result will be the sum of $n$-loop states).

Let us define the fermionic “momentum” operators. The construction is straightforward from the form of their Poisson brackets with the configurational variables (one can also obtain the following expressions using functional derivative operators from (39)). We shall define

$$
\int d^3x \left(\xi |\hat{\gamma}(x)|\hat{\pi}(x)\right) \circ |\alpha\rangle := i |\gamma \circ \alpha\rangle.
$$

(40)

Here we integrated over the initial point of the curve $\gamma$ in order to have the same density at the right and the left sides of the expression and the curve $\gamma$ at the right side is that one from the family $\gamma(x)$ whose final point coincides with the initial point of $\alpha$. The graphical description of this operator is

$$
\begin{align*}
\gamma & \quad \alpha \\
\quad & = \quad i \gamma \circ \alpha
\end{align*}
$$

FIG. 13.

i.e. it prolonges $\alpha$ by adding the corresponding curve to the final point. This, linear in the fermionic momentum field operator does not change the number of open ends in a state. The other linear in the momentum field operator is defined in a similar way

$$
\int d^3x \left(\bar{\omega}(x)|\hat{\gamma}(x)|\bar{\eta}\right) \circ |\alpha\rangle := i |\alpha \circ \gamma\rangle;
$$

(41)
the only difference with the previous operator is that this one adds the corresponding curve to the initial point of \( \alpha \). And, finally, the operator quadratic in the fermionic momentum fields is defined as

\[
\int d^3x \, d^3y \, (\tilde{\omega}(x)|\tilde{\gamma}(x; y)|\tilde{\pi}(y)) \circ |\alpha\rangle := (i)^2 |\gamma \circ \alpha\rangle;
\]
That is, the operator glues two open ends of a curve, converting it into a loop. Acting on more complicated states this operator also reduces (by two) the number of open ends in a state. The result is the sum of states; in each of these states the operator glues together two open ends (of different kind), and the sum goes through all possible pairs of open ends.

Restoring the \( \hbar \) factor in all relations one can easily check that the above definitions really give a representation of the classical algebra, i.e. that the commutators among defined operators (scaled by the factor \( i\hbar \)) turn into their classical analogs when \( \hbar \) goes to zero.

There are few words left to be said about the solving of the diffeomorphism constraint in our approach. This is particularly simple while describing the states and operators graphically, because the elements of quantum algebra are described in terms of geometrical objects and diffeomorphism constraint is represented in quantum case by a generator of transformations of these objects. Commutational relations are also written in terms of geometrical objects and the relations similar in the form hold for all representatives of the diffeomorphism equivalent classes of objects. In order to pick up the physical states, i.e. to solve the constraint, one should find a representation of the “physical” operators which lie in the corresponding quotient algebra. It is easy to see that in the approach described this will be the representation in the space of equivalence classes of loops and curves and all “physical” operators will act on classes of diffeomorphic equivalent objects. There is also another, more rigorous approach for solving the diffeomorphism constraint, which is based on the description of our quantum operators as operators in the space of functionals; this approach is given by the averaging procedure from \[14\].

V. DISCUSSION

We have constructed the representation for our quantum system in which the classical loop variables became operators in the “loop” space. The representation we built differs from the loop representation of pure gravity in the following important points:

1. Unlike loops describing pure gravitational excitations, loops and curves of the unified theory are oriented.

2. The momentum loop operators (corresponding to the gauge as well as to the fermionic degrees of freedom) act on the “loop” states merely prolonging these “loops” in the only compatible with their orientation way.
One can propose an interesting classification of the constructed “loop” operators in terms of creation and annihilation operators. We have seen that the operators corresponding to the configurational loop variables act by adding a “loop” to a state, so it is natural to regard them as creation operators. This terminology is especially good for the operator represented by a curve because open ends of a curve in our formalism correspond to fermions. It can easily be shown that this operator actually creates a pair of “fermions” of different charge sign. Indeed, one can define the charge of a quantum state as the eigenvalue of the charge operator for this state. Classical charge is the generator of gauge transformations; it is given by the quantity

$$i Q = - \int_\Sigma d^3x \, C(x) \, (\xi^A(x)\pi_A(x) + \eta^A(x)\tilde{\omega}_A(x)),$$

where $C(x)$ is an arbitrary (real, integrable) function. One can define the quantum charge operator with the regularization procedure of point splitting. The result is a well defined operator which acts only on the open ends of curves in a state. Each final point on a curve gives $-1$ while initial points give $+1$. Thus, the result of this operator’s action on any state in our representation is zero. This means that our “fermions” are born only in pairs with their “anti-particles” and that all the states in our representation are electrically neutral.

The operators corresponding to the quantities linear in momentum fields do not change neither number of loops nor number of open ends in a state; in this sense the “loop” states are the “eiglenstates” of these operators. And finally, there are momentum operators of higher orders which change the number of “loops” in a “loop” state; they create or annihilate “loops” depending on a state they act on. This completes the description of quantum kinematics for our unified system.

The important problem which has not been discussed in this paper is the construction of a scalar product in the representation space. There are several points in our scheme where a scalar product is necessary. First, having defined the “loop” operators with respect to the “basis” of “loop” states, one should have a scalar product in which the set of “loop” states becomes a basis. Second, in order to have a Hilbert representation space we need a scalar product with respect to which the space $Cyl/F/G$ is complete. And, finally, one needs a scalar product so that the properties of our operators with respect to the Hermitian conjugation coincide with the corresponding properties of the classical quantities with respect to the $\dagger$-operation. This collection of problems for the system considered is discussed in [18].

Let us conclude by pointing out a possible physical meaning of the formalism obtained. It describes the unified theory, i.e. the gravitational and electromagnetic fields enter the formalism only in a certain combination. On the quantum level excitations of these fields are described by loops and curves so the formalism predicts that there do not exist pure gravitational or pure electromagnetic quantum excitations and these fields appear always only together.

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7 Or rather to the fermionic degrees of freedom because of the lack of interpretation in terms of particles when no background structure presents.

8 This is what one would expect from the demand of gauge invariance.
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APPENDIX:

Let us introduce a scalar product in the space of cylindrical functionals of fermionic fields. We call a functional $\Phi(\xi, \bar{\eta})$ on the Grassmann algebra cylindrical if it depends on fields $\xi, \bar{\eta}$ taken no more than in a countable set of points

$$\Phi = \Phi(\xi_{p_1}, \ldots, \xi_{p_k}, \bar{\eta}_{p_{k'}}^1, \ldots, \bar{\eta}_{p_{k'}}^{k'});$$

here we denote $\xi_p = \xi(p)$. Then, having the involution $\dagger$, we can define a scalar product of two cylindrical functionals $\Phi_1, \Phi_2$ as

$$\langle \Phi_1 | \Phi_2 \rangle := \int [\Phi_1(\xi, \bar{\eta})]^\dagger \Phi_2(\xi, \bar{\eta}) \exp \sum_p (\xi_p^\dagger) A(\xi_p) A + \sum_{p'} (\bar{\eta}_{p'}^\dagger) A(\bar{\eta}_{p'}) A$$

$$\prod_p d\xi_p^\dagger \wedge d\xi_p \wedge d\xi_p^\dagger \wedge d\xi_p \prod_{p'} d\bar{\eta}_{p'}^\dagger \wedge d\bar{\eta}_{p'} \wedge d\bar{\eta}_{p'}^\dagger \wedge d\bar{\eta}_{p'}^\dagger;$$  (A1)

$$\langle 1 | 1 \rangle := 1.$$ 

Here $\Phi^\dagger$ is to be understood as a cylindrical functional of the fields $\xi^\dagger, \bar{\eta}^\dagger$. The two sums and products are taken over the points $p, p'$ which the functionals $\Phi_1, \Phi_2$ depend on. The integral here is the integral over fermionic Grassmann variables; it acquires a sense if we define the following basic operations on the Grassmann algebra

$$\int \xi^A \xi^B d\xi \wedge d\xi := \epsilon^{AB};$$

$$\int (\xi^\dagger)_A (\xi^\dagger)_B d\xi^\dagger \wedge d\xi^\dagger := \epsilon_{AB};$$

$$\int \bar{\eta}_A \bar{\eta}_B d\bar{\eta} \wedge d\bar{\eta} := \epsilon_{AB};$$

$$\int (\bar{\eta}_\dagger)^A (\bar{\eta}_\dagger)^B d\bar{\eta}^\dagger \wedge d\bar{\eta}^\dagger := \epsilon^{AB},$$

and

$$\int d\xi \wedge d\xi := 0; \quad \int d\xi^\dagger \wedge d\xi^\dagger := 0;$$

$$\int d\bar{\eta} \wedge d\bar{\eta} := 0; \quad \int d\bar{\eta}^\dagger \wedge d\bar{\eta}^\dagger := 0.$$ 

Thus, the result of integration of any functional $\Phi$ is defined as proportional to the coefficient near the highest order non-zero term in the expansion of $\Phi$ by powers of the fermionic fields.
Then the integration in (A1) gives a complex number, so this expression really defines a scalar product in the space of cylindrical functionals.

Finally, let us illustrate the definition on the following simple examples. The explicit calculation yields

\[
\langle (\xi|\gamma|\bar{\eta})| (\xi|\gamma'|\bar{\eta}) \rangle = \begin{cases} 
0, & \gamma \neq \gamma' \\
2, & \gamma = \gamma',
\end{cases}
\]

\[
\langle (\xi|\gamma|\bar{\eta})(\xi|\gamma|\bar{\eta})^\dagger | (\xi|\gamma|\bar{\eta})(\xi|\gamma|\bar{\eta})^\dagger \rangle = 4. \tag{A2}
\]
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