Transport and localization in quantum walks on a random hierarchy of barriers

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Abstract
We study transport within a spatially heterogeneous one-dimensional quantum walk with a combination of hierarchical and random barriers. Recent renormalization group calculations for a spatially disordered quantum walk with a regular hierarchy of barriers alone have shown a gradual decrease in transport but no localization for increasing (but finite) barrier sizes. In turn, it is well-known that extensive random disorder in the spatial barriers is sufficient to localize a quantum walk on the line. Here we show that adding only a sparse (sub-extensive) amount of randomness to a hierarchy of barriers is sufficient to induce localization such that transport ceases. Our numerical results suggest the existence of a localization transition for a combination of both, the strength of the regular barrier hierarchy at large enough randomness as well as the increasing randomness at sufficiently strong barriers in the hierarchy.

Keywords: transport, localization, quantum walk, ultrametric

(Some figures may appear in colour only in the online journal)

1. Introduction

The dynamics of discrete-time quantum walks (DTQW), in particular, their scaling in absorption and localization phenomena, is distinctly quantum and not observed for classical walks. While localization raises the specter of many-body phenomena observed in the tight-binding model (which is akin to a continuous-time quantum walk), there is a form of localization behavior in DTQW [1, 2] that is distinct in that it can arise entirely without disorder, in otherwise perfectly homogeneous systems, for single-particle processes. Our main focus here will be on localization that entirely stops transport [3], not just for a fraction of the wave function.

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The ability to design of quantum walks with various controllable features (here, the strength of spatial heterogeneity and randomness) has motivated an expanding use of the concept. The ever increasing number of experiments with quantum walks, discrete or continuous in time, not only indicates the growth in technical facility to control such processes [4–16], it also demonstrates the intense interest in quantum walks for their myriad of applications in quantum information processing [3, 17–33], such as in algorithms for quantum search, optimization, and linear algebra [34–37]. The corresponding classical walk problem, although far less rich in phenomenology, has nonetheless been explored in meticulous detail over the last century [38–42], due to its fundamental importance to diffusive transport as well as to randomized algorithms. In an age dominated by synthetic nanotechnology appearing in everyday devices, all forms of quantum transport are bound to attain similar importance. The basic construction of quantum walks allows for many options that could significantly impact algorithmic performance, exemplified by the internal degrees of freedom in coin space of DTQW, essential to reach Grover-search efficiency in 2D [43–45]. It is important to assess the robustness of the expected algorithmic efficiency over such an array of choices, as well as to exploit these options to control and optimize it.

The real-space renormalization group (RG) was designed as a method to categorize the behavior of entire families of statistical process into universality classes [46]. Based on prior applications of RG to percolation on hierarchical networks embedded in the 1D-line (in collaboration with Bob Ziff) [47, 48], we have extended these methods to DTQW in heterogeneous environments with location-dependent transition operators [49]. The disorder there was hierarchical with a regular progression. While the strength of this hierarchy systematically reduced transport, it did not result in any localization. In contrast, in reference [3] it was shown with transfer matrix methods that even small amounts of randomness at every site of a DTQW on a line can lead to localization. Here, we show that adding a sparse amount of randomness—not on every site but merely at every level of the hierarchy—produces an interesting set of localization transitions by varying a combination of both, the barrier strength and the degree of randomness. These numerical studies will pave the way to more precise RG calculations in the future.

A beautiful example of the connection between localization and transport in DTQW is provided by the following observation, first mentioned in reference [51] and illustrated in figure 1. Those simulations concern a homogeneous DTQW on a dual Sierpinski gasket (DSG) with an absorbing wall (acting as an egress at one of its three corners). For an initial wave function spread uniformly across the network, no localization occurs and its entire weight gets absorbed at the egress rapidly, similar to any classical random walk. When DTQW is initiated while positioned at the opposite corners, however, the weight gets absorbed with a probability that decreases as an (as of yet undetermined) power of the distance between corner and egress. Consequently, an increasing fraction of the walk’s weight must become localized ever-farther from the starting sites. Similar localization phenomena of quantum walks in perfectly ordered lattices have been studied extensively [4, 32, 52–55]. However, in those cases, localization is very sharp—simply exponential—and relatively easily understood, as we have shown [2]. In contrast, the broad localization on DSG is highly non-trivial and ultimately consumes the entire walk, i.e., the arrival probability at the wall vanishes, for diverging separations.

Our discussion is organized as follows: in section 2, we start with a review of some of the fundamentals about DTQW, outline the question about asymptotic scaling in spread (i.e., transport) we will be concerned with, and we will recount the results for the original 1D quantum ultra-walk without randomness from reference [49]. In section 3, we will introduce that model extended by randomness, describe the methods we are using to determine localization, and...
discuss our results. In section 4, we conclude with a summary of our results and provide an outlook on future work.

2. Discrete-time quantum walks

2.1. Evolution equation for a walk

Our walks are governed by the discrete-time evolution equation [38] for the state of the system,

$$|\Psi(t+1)\rangle = \mathcal{U} |\Psi(t)\rangle$$

(1)

with propagator $\mathcal{U}$. This propagator is a stochastic operator for a classical, dissipative random walk. But in the quantum case it is unitary and, thus, reversible. Then, in the discrete $N$-dimensional site-basis $|x\rangle$ of some network, the PDF is given by $\rho(x, t) = \psi_{x,t}^2 = \langle x |\Psi(t)\rangle$ for random walks, or by $\rho(x, t) = |\psi_{x,t}|^2$ for quantum walks. A discrete Laplace-transform (or
‘generating function’) \[38\] of the site amplitudes

\[ \psi_x(z) = \sum_{t=0}^{\infty} \psi_{x,t} z^t \]  

has all its poles—and hence those for \( \varphi(x, z) \)—located right on the unit-circle in the complex-\( z \) plane \[56\].

On the 1D-line. The propagator in equation (1) is

\[ U = \sum_x \{A_x |x + 1\rangle \langle x| + B_x |x - 1\rangle \langle x| + M_x |x\rangle \langle x| \} \]  

for nearest-neighbor transitions. While the norm of \( \rho \) for random walks merely requires conservation of probability for the hopping coefficients, \( A_x + B_x + M_x = 1 \), for quantum walks it demands unitary propagation, \( I = U\dagger U \). The rules \[45\] then impose the conditions

\[ I_r = A_{x-1}^\dagger A_x \quad \text{and} \quad 0 = A_{x-1}^\dagger B_x + M_x^\dagger M_{x+1} = A_{x-1}^\dagger B_{x+1}. \]  

This algebra requires at least \( r = 2 \)-dimensional matrices, and it is customary \[57, 58\] to employ the most general unitary coin-matrix,

\[ C = \begin{pmatrix} \sin \theta, & e^{i\chi} \cos \theta, & -e^{i(\chi+\vartheta)} \sin \theta \end{pmatrix}. \]  

We thus define \( U = S(\overline{C} \otimes I_N) \) with shift \( S \) using matrices \( S^{(A,B,M)} \) for transfer in a direction either out of a site or back to itself, i.e., \( A = S^A \overline{C}, B = S^B \overline{C}, \) and \( M = S^M \overline{C} \) with \( S^A + S^B + S^M = I_r \), where \( C = C_x \) may be heterogeneous via \( x \)-dependent parameters. The quantum-coin entangles all \( r \) components of \( \psi_{x,t} \) and the shift-matrices facilitate the subsequent transitions to neighboring sites. For \( r = 2 \), there are no self-loops \( (S^M = 0, M = 0) \) and we shift upper (lower) components of each \( \psi_{x,t} \) to the right (left) using projectors \( S^A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) and \( S^B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \).

Alternatively, these coin-degrees of freedom could be replaced by a ‘staggered’ walk without coin \[59, 60\], for which schemes equivalent to the following RG can be developed. Finally, similar considerations apply for walks on networks, such as a fractal, except that the propagator \( U \) in equation (3) is modified to reflect the respective Laplacian.

2.2. Asymptotic scaling for walks

For random walks, the probability density \( \rho(\vec{x}, t) \) to detect a walk at time \( t \) at site \( \vec{x} \), a distance \( x = |\vec{x}| \) from its origin, obeys the scaling collapse with the scaling variable \( x/t^{1/d_w} \),

\[ \rho(\vec{x}, t) \sim t^{-d_w} f \left( \frac{x}{t^{1/d_w}} \right), \]  

where \( d_w \) is the walk-dimension and \( d_f \) is the fractal dimension of the network \[42\]. On a translationally invariant lattice of any spatial dimension \( d(= d_f) \), it is easy to show that the walk is always purely ‘diffusive’, \( d_w = 2 \), with a Gaussian scaling function \( f \), which is the content of many classic textbooks on random walks and diffusion \[39, 41\]. The scaling in equation (5) still holds when translational invariance is broken or the network is fractal (i.e., \( d_f \) is non-integer). Such ‘anomalous’ diffusion with \( d_w \neq 2 \) may arise in many transport processes \[38, 42, 61\]. For quantum walks, the only previously known value for a finite walk dimension is that for ordinary lattices \[62\], where equation (5) generically holds with \( d_w = 1 \), indicating a ‘ballistic’ spreading of the quantum walk from its origin. This value has been obtained for
various versions of one- and higher-dimensional quantum walks, for instance, with so-called weak-limit theorems [58, 62–65].

In recent work, we have developed RG for DTQW with a coin [51, 56, 66, 67]. It expands the analytic tools to understand quantum walks, since it works for networks that lack translational symmetries. Our RG provides principally similar results as in equation (5) in terms of the asymptotic scaling variable $x/t^{1/d_w}$ (or pseudo-velocity [68]), whose existence allows to collapse all data for the probability density $\rho(\vec{x}, t)$, aside from oscillatory contributions (’weak limit’). While algebraically laborious, we have developed a simple scheme to obtain RG-flow equations for unitary evolution equations [69]. Abstracting from those results, we have conjectured that the fundamental quantum walk dimension $d_w$ for a homogeneous walk always is half of that for the corresponding random walk [67],

$$d_w^Q = \frac{1}{2} d_w^R. \quad (6)$$

It is not clear how spatial inhomogeneity affects the relation between classical and quantum walks. The ability to explore a given geometry much faster than diffusion is essential for the effectiveness of quantum search algorithms [34, 70]. In fact, using equation (6) and the Alexander–Orbach relation [71], $d_w^Q d_s = 2 d_w$, we have shown [45] that the ability to attain Grover’s limit in quantum search on a homogeneous network is determined by its spectral dimension $d_s$.

2.3. Quantum ultra-walk

Walks and transport efficiency in disordered environments have been of significant interest, exemplified by the Sinai model [72]. There have been a few approaches to understand quantum walks with disorder, either through spatially [3, 32, 64, 73, 74] or temporally [75] varying coins. Even less is known about the impact of heterogeneous environments on quantum search efficiency. However, exact quantum models similar to Sinai’s for asymptotic scaling of the displacement in random environments are hard to find. For random walks, models of ‘ultra-diffusion’ with a hierarchy of ultra-metric barriers [76–79] have been proposed to study slow relaxation and aging and have been solved with RG that allow to interpolate between regular diffusion ($d_w = 2$) via anomalous sub-diffusion to the full disorder limit ($d_w \to \infty$). Even spectral properties of the tight-binding model have been explored [80].

As a hierarchical model of spatial inhomogeneity, we have considered position-dependent coins, equation (4), in such a way that all sites of odd index $x$ share the same coin, and so do all sites that are divisible by 2, $i \geq 0$, so that sites of the same value of $i$ have an identical coin, $C_i$, with hierarchy index $i = i(x)$ based on the (unique) binary decomposition of any integer $x(\neq 0)$ [49]:

$$x = 2^i(2j + 1), \quad (i \geq 0, -\infty < j < \infty). \quad (7)$$

Setting uniformly $\vartheta = \chi = 0$ in equation (4) but choosing

$$\theta_i = \theta_0 \epsilon^i, \quad (0 < \epsilon \leq 1) \quad (8)$$

with $\theta_0 = \frac{\pi}{4}$, the sequence of such coins becomes ever more reflective for a walker trying to transition through the respective site. Thus, the walker gets confined in a tree-like ultra-metric set of domains with vastly varying timescales for exit. Two neighboring domains at level $i$ form a larger domain at level $i + 1$, and so on, from which an ultra-metric hierarchy emerges, as depicted in figure 2.
The master equation (1) in Laplace-space with $U$ in equation (3) becomes $\overline{\psi}_x = zM_x \overline{\psi}_x + zA_x \overline{\psi}_{x-1} + zB_x \overline{\psi}_{x+1}$. For simplicity, we merely consider initial conditions (IC) localized at the origin, $\psi_{x=0} = \delta_{x,0} \psi_{IC}$ and define the coin at $x = 0$ simply to be the identity matrix. For the RG in reference [49], we recursively eliminated $\overline{\psi}_i$ for all sites for which $x$ is an odd number ($i = 0$), then set $x \to x/2 (i \to i - 1)$ for the remaining sites, and repeat, step-by-step for $k = 0, 1, 2, \ldots$. In each step, we successively eliminate all sites within an entire hierarchy $i$, each with an identical coin $C_i$, starting at $k = 0$ with the ‘raw’ hopping operator $A_i(0) = zA_{x(0,0)}$, $B_i(0) = zB_{x(0,0)}$, and $M_i(0) = zM_{x(0,0)} \equiv 0$. After each step, the master equation becomes self-similar in form by identifying the renormalized hopping operators $A_i^{(k)}$, $B_i^{(k)}$, $M_i^{(k)}$ for all $i > 0$ as

$$A_{i-1}^{(k+1)} = A_i^{(k)} \left[ I - M_i^{(k)} \right]^{-1} A_i^{(k)},$$

$$B_{i-1}^{(k+1)} = B_i^{(k)} \left[ I - M_i^{(k)} \right]^{-1} B_i^{(k)},$$

$$M_{i-1}^{(k+1)} = M_i^{(k)} + A_i^{(k)} \left[ I - M_i^{(k)} \right]^{-1} B_i^{(k)} + B_i^{(k)} \left[ I - M_i^{(k)} \right]^{-1} A_i^{(k)}.$$  

(9) (10) (11)

Amazingly, we can entirely eliminate the hierarchy-index $i$: if we define the $k$th renormalized shift matrices via $\{A,B,M\}_k^{(k)} = S_k^{\{A,B,M\}} C_{i+1}^{(k)}$, these satisfy the recursions:

$$S_k^{(A,B)} = S_k^{(A,B)} \left[ C_k^{-1} - S_k^M \right]^{-1} S_k^{(A,B)},$$

$$S_k^{(A,B)} = S_k^M + S_k^A \left[ C_k^{-1} - S_k^M \right]^{-1} S_k^B + S_k^B \left[ C_k^{-1} - S_k^M \right]^{-1} S_k^B,$$

(12) (13) (14)

which instead now have an explicit $k$-dependence via the inverse coins $C_k^{-1}$ of the $k$th hierarchy.

As a specific physical situation for such a setting, reference [49] considered a walk between two absorbing walls of separation $N = 2^l + 1$, equidistant from the starting site $x = 2^{l-1}$. As
the wall-sites \( x = 0 \) and \( x = 2^l \) a fully absorbing, there is no flow reflecting out of those sites such that at the end of \( l - 1 \) RG-steps, for either wall it is

\[
\bar{\psi}_{\{0,2\}} = S_{i_l - 1}^{(A,B)} (C_{i_l - 1} - S_{i_l - 1}^{M})^{-1} \psi_{IC}.
\]

(15)

In reference [49], the RG calculation yielded for the quantum ultra-walk the walk dimension, as defined in equation (5):

\[
d_{QW} = 1 + \frac{1}{2} \log_2\left(1 + \epsilon^{-2}\right),
\]

(16)

which grows without bound for decreasing \( \epsilon \), i.e., for increasing barrier heights transports diminishes from ballistic \( (d_{QW} = 1 \text{ for } \epsilon = 1) \) to extremely sub-diffusive for \( \epsilon \rightarrow 0 \). However, it was found numerically that, eventually, the entire weight of the wave function gets absorbed at arbitrarily distant walls for any finite value of \( \epsilon \).

3. Quantum walks with randomness

Reference [3] has investigated the behavior of DTQW on a 1D-line with coin parameters having extensive randomness, i.e., a different value for one of the variables \( \theta, \vartheta, \text{or } \chi \) in equation (4) on every site \( x \) (while the others were held fixed throughout). Unlike for the walk with regular hierarchical variation of \( \theta \) according to equation (8) that we have described in section 2.3, it was shown there that such extensive randomness leads to a finite localization length for the walk at any level of randomness. For example, for \( \vartheta = \chi = 0 \) in equation (4) reference [3] selected at each site \( x \) a random angle \( \theta_x \) from a uniform probability distribution,

\[
P_W(\theta) = 1/(2W), \quad \frac{\pi}{4} - W \leq \theta \leq \frac{\pi}{4} + W,
\]

(17)

with a adjustable disorder strength \( 0 \leq W \leq \pi \) and centered such that each \( C_x \) becomes a Hadamard coin for vanishing randomness, \( W \rightarrow 0 \). It was found numerically that the DTQW localized for any non-zero value of \( W \) considered there.

3.1. Quantum ultra-walk model with sub-extensive randomness

Clearly, overlaying this form of extensive randomness with the hierarchical barriers analyzed in section 2.3, e.g. by replacing in equation (8) the constant \( \theta_0 \) with a random variable \( \theta_i \) at each site \( x \) in addition to the barrier strengths \( \epsilon^{-i} \), merely enhances the already observed localization (see figure 4 below). In turn, it appears that the question regarding which level of sub-extensive randomness might be required to induce a localization transition is of some interest and can be conveniently studied in the context of such a one-dimensional quantum walk. To this end, specifically, we propose a simple two-parameter family of DTQW on the 1D-line with a combination of regular hierarchical barriers described by \( \epsilon \) in combination with hierarchical randomness, manifested by choosing random angles \( \theta_i \) from the \( W \)-controlled distribution in equation (17) merely for each level \( i = i(\lambda) \), as defined in equation (7). Indeed, we find evidence for localization transitions both for nontrivial values of \( \epsilon \) for \( W > 0 \) as well as of \( W \) for \( \epsilon < 1 \), while there is no transition either for \( \epsilon = 1 \) at any value of \( W \) or for \( W = 0 \) at any \( 0 < \epsilon \leq 1 \) (the case considered in reference [49]). In light of the ability to treat this system with RG, see section 2.3, this will open the door for high-precision calculations via numerical iteration and disorder-averaging of the (exact) RG-recursion equations in the future.
3.2. Methods

Our means to determine the existence of those transitions in this study are very simple: for given \( \epsilon \) and \( W \), we generate multiple instances of placing random angles \( \theta_i \) drawn from \( P_W \) in equation (17) into the coins on all sites \( x \) (up to \( |x| \leq L \) with \( L = 2^{16} \)) matching equation (7) with that \( i \) and any \( |j| \leq L/2^{i+1} \). We note that each instance involves at most a sub-extensive, \( O(\log L) \) choice of random angles \( \theta_i \) that could ever be experienced by the walker while reference [3] employed an extensive, \( O(L) \) selection of random angles \( \theta_i \). For each instance, we evolve DTQW initiated at \( x = 0 \) for \( t_{\text{max}} = L = 2^{16} \) time-steps to measure its mean-squared displacement, \( \sigma(t)^2 \), which is the variance of \( \rho(x, t) \) defined in section 2.1. It immediately follows from equation (5) that the root mean square displacement for large times \( t \) scales as

\[
\sigma(t) \sim t^{1/d_w},
\]

from which we can extract the walk dimension \( d_w \) asymptotically. In figure 3, we illustrate this procedure by reproducing the walk dimensions \( d_w \) in equation (16) for various values of \( \epsilon \) in the pure quantum ultra-walk without randomness \( (W = 0) \). For any finite value of \( d_w \) there is still extensive transport occurring, while \( d_w = \infty \), or \( 1/d_w = 0 \), would indicate localization. In the quantum ultra-walk, there is no localization for any \( \epsilon > 0 \), not even for any fraction of the wave function [49].

Since we are interested in the asymptotic behavior for large times and distances for the walk, we instead will plot our data for \( \sigma(t) \) in form of an extrapolation plot. To this end, we convert \( \sigma \sim A t^{1/d_w} \) in equation (18) into

\[
\frac{\log \sigma(t)}{\log t} \sim \frac{1}{d_w} + \frac{\log A}{\log t}.
\]
Figure 4. Extrapolation plot for the root mean square displacement $\sigma(t)$ for a quantum walk with extensive randomness at various strengths $W$. Both for (a) $\epsilon = 1$ and (b) $\epsilon = 0.6$, any disorder with $W > 0$ is sufficient to drive $1/d_w$ to zero asymptotically, as obtained by linear extrapolation (dashed lines) of each set of data at the intercept with the $Y$-axis. However, as the barriers in the system increase with decreasing $\epsilon$, the walker becomes more readily localized. Error bars are obtained from averaging over 25 instances for $\epsilon = 1$ and from averaging over 20 instances for $\epsilon = 0.6$.

When plotted with $X = 1/\log t$ versus $Y = \frac{\log \sigma(t)}{\log t}$, asymptotically, equation (19) describes a line from which we can read off $1/d_w$ approximately at the intercept $X = 0$, i.e., $t = \infty$. We first illustrate this technique in figure 4 for simulations employing extensive randomness which recap the findings of reference [3] (for $\epsilon = 1$) and show that localization only gets stronger for higher barriers ($\epsilon = 0.6$). Clearly, in all cases with $W > 0$, the extrapolations indicate a
Figure 5. Extrapolation plots for the quantum ultra-walk model with various strengths $W$ of sub-extensive randomness at a fixed barrier strength of (a) $\epsilon = 1$, (b) $\epsilon = 0.8$, and (c) $\epsilon = 0.6$. For $\epsilon = 1$, the case of a homogeneous quantum walk, the addition of sub-extensive randomness of any strength $W$ is insufficient to bring $1/d_w$ even anywhere near to vanishing. For lower $\epsilon$, i.e., exponentially increasing barriers, some level of such randomness $W > 0$ does prove sufficient to induce an apparent localization transition where there was none without ($W = 0$). Note that for lower $\epsilon$ (i.e., stronger barriers), the same amount of randomness suppresses transport more. Error bars were obtained by averaging over 50 instances.
vanishing of $1/d_w$, making $\sigma(t)$ bounded for large times $t$ as evidence that the walk remains localized.

### 3.3. Results

In the following, we apply the methods developed in section 3.2 to the model of a quantum ultra-walk with sub-extensive randomness introduced in section 3.1. In figure 5, we summarize our data of the walk simulations for various values in the $(\epsilon, W)$-plane. In particular, as shown in figure 5(a), for the otherwise homogeneous (barrier-free) 1D quantum walk obtained for $\epsilon = 1$, the addition of merely sub-extensive randomness placed hierarchically on the lattice is insufficient to localize the walk for any value of $W$. Even for angles chosen entirely randomly ($W = \pi$) the walk at most becomes mildly sub-diffusive and $1/d_w$ remains far from zero. This is in stark contrast with the corresponding case of extensive randomness [3] shown in figure 4(a). However, in concert with the hierarchy of barrier that emerges for $\epsilon = 0.8$ and $\epsilon = 0.6$, as shown in figures 5(b) and (c), such a sub-extensive amount of randomness proves sufficient to induce localization. In fact, it appears that for each of those fixed values of $\epsilon < 1$, there is a transition at a finite value of $W$, although it would be possible for that transition to be at $W = 0$.

We can summarize our results by sketching out a tentative phase diagram for the localization transition in the plane formed by the set of parameters $(\epsilon, W)$. In figure 6, we illustrate the emerging scenario, highlighting the region of localization $(1/d_w = 0$, in red) from that of transport $(1/d_w > 0$, in green) with a phase boundary between them estimated from our data. First, the earlier discussion in section 2.3 concerning the quantum ultra-walk without any randomness shows that the entire line $(\epsilon, W = 0)$ is not localized but that the line $(\epsilon = 0, W)$ for any $W$ certainly is. Thus, the phase boundary must pass the origin $(\epsilon = 0, W = 0)$. While we cannot entirely exclude the possibility that that boundary remains at $W = 0$ also for most $\epsilon < 1$, it is conceivable from figures 5(b) and (c) that for intermediate barrier strengths $\epsilon$ there...
is a transition at similarly intermediate values $W$, as the two interior marks at $\epsilon = 0.6$ and $\epsilon = 0.8$ insinuate. The argument for this scenario is strengthened by the fact that the point $(\epsilon = 1, W = \pi)$, yielding $1/d_{w} \approx 0.3-0.4$ from figure 5(a), appears far from localization and, thus, from the phase boundary. Hence, unless there is a discontinuous jump in $d_{w}$, that phase boundary seems to reach full randomness ($W = \pi$) at some value of $\epsilon_c$ that might be close to, but is well bounded-away from $\epsilon = 1$, as indicated in figure 6. Then, either the phase boundary varies gradually between the origin and that point, as shown, or discontinuously jumps at $\epsilon_c$ from $W = 0$ to $W = \pi$.

4. Conclusions

Based on the recent solution of the quantum ultra-walk [49], a 1D DTQW that evolves through a spatially heterogeneous (albeit not random) environment characterized by a hierarchy of progressively diverging barriers (in the form of increasingly reflective quantum coins), we have extended this model by introducing a certain, sub-extensive amount of spatial randomness. The effect of extensive randomness, with random coin parameters on every site $x$, on a 1D DTQW has been well-studied and shown to lead to localization for even the smallest amount of randomness [3]. Our new model permitted us to explore the impact of sub-extensive randomness on localization as well as to trace out an interesting phase diagram with regions of localization and transport separated by a phase boundary, generated by the interplay of such randomness with the hierarchy of barriers. We have shown that, while each ineffective by themselves, such sparse randomness in combination with even mildly escalating barriers readily induces localization. While not yet resolved in great detail, the available evidence already suggests a few interesting features exhibited within that parameter space, with localization transitions occurring at either a non-trivial finite randomness or finite barrier strength, or both. Aside from the myriad applications of highly controllable DTQW in quantum algorithms [21, 22], this model provides also a simple example with the potential for such a complex phase diagram. It would be interesting to supplement these studies with a similar continuous-time quantum walk [81]. While likely equivalent in most aspects [82], we have favored a discrete-time formulation here, since the extra coin-space allows for richer designs in the walk dynamics.

In future work, we intend to employ RG to gain a more precise description of this diagram. While it is straightforward to obtain the exact RG-recursions from equation (12) that could be evolved numerically, replacing the regular progression of hierarchical barriers solved in reference [49] with a random sequence makes the prospect of receiving analytical results (even asymptotically) very daunting. Since contributions to the fixed points of the RG seem to arise from many poles [56] throughout the complex-$z$ plane in Laplace space, originating in equation (2), even a numerical evolution of the RG-recursions does not yield insights easily. Alternatively, these questions should be more readily accessible using numerically the transfer matrix method of reference [3].

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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References

[1] Inui N, Konno N and Segawa E 2005 Phys. Rev. E 72 056112
[2] Falkner S and Boettcher S 2014 Phys. Rev. A 90 012307
[3] Vakulchyk I, Fistul M V, Qin P and Flach S 2017 Phys. Rev. B 96 144204
[4] Crespi A, Osellame R, Ramponi R, Giovannetti V, Fazio R, Sansoni L, De Nicola F, Sciarrino F and Mataloni P 2013 Nat. Photon. 7 322
[5] Grossman J M, Ciampini D, D’Arcy M, Helmerson K, Lett P D, Phillips W D, Vaziri A and Rolston S L 2004 The 35th Meeting of the Division of Atomic, Molecular and Optical Physics DAMOP04 (Tucson, AZ)
[6] Karski M, Förster L, Choi J-M, Steffen A, Alt W, Meschede D and Widera A 2009 Science 325 174
[7] Manouchehri K and Wang J 2014 Physical Implementation of Quantum Walks (Berlin: Springer)
[8] Peruzzo A et al 2010 Science 325 174
[9] Preiss P M, Ma R, Tai M E, Lukin A, Rispoli M, Lahini Y, Islam R and Greiner M 2015 Science 347 1229
[10] Qiang X, Loke T, Montanaro A, Aungskunsiri K, Zhou X, O’Brien J L, Wang J B and Matthews J C F 2016 Nat. Commun. 7 11511
[11] Ryan C A, Lafortore M, Boileau J C and Laflamme R 2005 Phys. Rev. A 72 062317
[12] Sansoni L, Sciarrino F, Vallone G, Mataloni P, Crespi A, Ramponi R and Osellame R 2012 Phys. Rev. Lett. 108 010502
[13] Schreiber A, Gábris A, Rohde P P, Laiho K, Štefaňák M, Potoček V, Hamilton C, Jex I and Silberhorn C 2012 Science 336 55
[14] Schreiber A, Cassemiro K R, Potoček V, Gábris A, Jex I and Silberhorn C 2011 Phys. Rev. Lett. 106 180403
[15] Tang H et al 2018 Sci. Adv. 4 3174
[16] Ramasesh V V, Flurin E, Rudner M, Siddiqi I and Yao N Y 2017 Phys. Rev. Lett. 118 130501
[17] Allés B, Gündüz S and Gündüz Y 2012 Quantum Inf. Process. 11 211–27
[18] Ashbóth J K and Obuse H 2013 Phys. Rev. B 88 121406
[19] Chakraborty S, Novo L, Ambainis A and Omar Y 2016 Phys. Rev. Lett. 116 100501
[20] Childs A M 2009 Phys. Rev. Lett. 102 180501
[21] Childs A M, Gosset D and Webb Z 2013 Science 339 791
[22] Goyal S K and Chandrashekar C M 2010 J. Phys. A: Math. Theor. 43 235303
[23] Kurzyńska P and Wójcik A 2013 Phys. Rev. Lett. 110 200404
[24] Lovett N B, Cooper S, Everitt M, Trevers M and Kendon V 2010 Phys. Rev. A 81 042330
[25] Manouchehri K and Wang J B 2008 J. Phys. A: Math. Theor. 41 065304
[26] Miyake S, Asboth J K, Nishimura Y and Kawakami N 2015 Phys. Rev. B 92 045424
[27] Oliveira A C, Portugal R and Donangelo R 2006 Phys. Rev. A 74 012312
[28] Rudinger K, Gamble J K, Bach E, Friesen M, Joyn R and Coppersmith S N 2013 J. Comput. Theor. Nano 10 1653
[32] Shikano Y and Katsura H 2010 Phys. Rev. E 82 031122
[33] Stefanak M, Barnett S M, Kollar B, Kiss T and Jex I 2011 New J. Phys. 13 033029
[34] Ambainis A 2007 SIAM J. Comput. 37 210
[35] Childs A M 2010 Commun. Math. Phys. 294 581
[36] Harrow A W, Hassidim A and Lloyd S 2009 Phys. Rev. Lett. 103 150502
[37] Wiebe N, Braun D and Lloyd S 2012 Phys. Rev. Lett. 109 050505
[38] Redner S 2001 A Guide to First-Passage Processes (Cambridge: Cambridge University Press)
[39] Weiss G H 1994 Aspects and Applications of the Random Walk (Amsterdam: North-Holland)
[40] Hughes B D 1996 Random Walks and Random Environments (Oxford: Oxford University Press)
[41] Feller W 1966 An Introduction to Probability Theory and its Applications vol 1 (New York: Wiley)
[42] Havlin S and Ben-Avraham D 1987 Adv. Phys. 36 695
[43] Ambainis A 2003 Int. J. Quantum Inf. 1 507
[44] Ambainis A, Kempe J and Rivosh A 2005 Proc. 16th Annual ACM-SIAM Symp. Discrete Algorithms, SODA ’05 (Philadelphia, PA: SIAM) pp 1099–108
[45] Boettcher S, Li S, Fernandes T D and Portugal R 2018 Phys. Rev. A 98 012320
[46] Plischke M and Bergersen B 1994 Equilibrium Statistical Physics 2nd edn (Singapore: World Scientific)
[47] Boettcher S, Cook J L and Ziff R M 2009 Phys. Rev. E 80 041115
[48] Boettcher S, Singh V and Ziff R M 2012 Nat. Commun. 3 787
[49] Boettcher S 2020 Phys. Rev. Res. 2 023411
[50] Grover L K 1997 Phys. Rev. Lett. 79 325
[51] Boettcher S, Falkner S and Portugal R 2014 Phys. Rev. A 90 032324
[52] Inui N, Konishi Y and Konno N 2004 Phys. Rev. A 69 052323
[53] Inui N and Konno N 2005 Physica A 353 333
[54] Joye A 2012 Quantum Inf. Process. 11 1251
[55] Ide Y, Konno N, Segawa E and Xu X-P 2014 Entropy 16 1501
[56] Boettcher S, Li S and Portugal R 2017 J. Phys. A: Math. Theor. 50 125302
[57] Portugal R 2013 Quantum Walks and Search Algorithms (Berlin: Springer)
[58] Venegas-Andraca S E 2012 Quantum Inf. Process. 11 1015
[59] Portugal R, Boettcher S and Falkner S 2015 Phys. Rev. A 91 052319
[60] Portugal R 2016 Quantum Inf. Process. 15 1387
[61] Bouchaud J-P and Georges A 1990 Phys. Rep. 195 127
[62] Grimmett G, Janson S and Scudo P F 2004 Phys. Rev. E 69 026119
[63] Konno N 2002 Quantum Inf. Process. 1 345
[64] Segawa E and Konno N 2008 Int. J. Quantum Inf. 6 1231
[65] Konno N 2008 Quantum Potential Theory (Lecture Notes in Mathematics vol 1954) ed U Franz and M Schürrmann (Berlin: Springer) pp 309–452
[66] Boettcher S, Falkner S and Portugal R 2013 J. Phys.: Conf. Ser. 473 012018
[67] Boettcher S, Falkner S and Portugal R 2015 Phys. Rev. A 91 052330
[68] Konno N 2005 J. Math. Soc. Japan 57 1179
[69] Boettcher S and Li S 2018 Phys. Rev. A 97 012309
[70] Childs A M, Cleve R, Deotto E, Farhi E, Gutmann S and Spielman D A 2003 Proc. 35th Annual ACM Symp. Theory of Computing STOC ’03 (New York: ACM) pp 59–68
[71] Alexander S and Orbach R 1982 J. Phys. Lett. 43 625
[72] Sinai Y G 1982 Theory Probab. Appl. 27 256
[73] Brun T A, Carteret H A and Ambainis A 2003 Phys. Rev. A 67 052317
[74] Konno N 2009 Quantum Inf. Process. 8 387
[75] Ribeiro P, Millman P and Mosseri R 2004 Phys. Rev. Lett. 93 190503
[76] Tietel S and Domany E 1985 Phys. Rev. Lett. 55 2176
[77] Ogilski A T and Stein D L 1985 Phys. Rev. Lett. 55 1634
[78] Huberman B A and Kerzberg M 1985 J. Phys. A: Math. Gen. 18 L331
[79] Maritan A and Stella A 1986 J. Phys. A: Math. Gen. 19 L269
[80] Ceccatto H A, Keirstead W P and Huberman B A 1987 Phys. Rev. A 36 5509
[81] Li S and Boettcher S 2017 Phys. Rev. A 95 032301
[82] Strauch F W 2006 Phys. Rev. A 74 030301
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