NON-SIMPLE SLE CURVES ARE NOT DETERMINED BY THEIR RANGE

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ABSTRACT. We show that when observing the range of a chordal SLE\(_\kappa\) curve for \(\kappa \in (4, 8)\), it is not possible to recover the order in which the points have been visited. We also derive related results about conformal loop ensembles (CLE): (i) The loops in a CLE\(_\kappa\) for \(\kappa \in (4, 8)\) are not determined by the CLE\(_\kappa\) gasket. (ii) The continuum percolation interfaces defined in the fractal carpets of conformal loop ensembles CLE\(_\kappa\) for \(\kappa \in (8/3, 4)\) (we defined these percolation interfaces in [15], and showed there that they are SLE\(_{16/\kappa}\) curves) are not determined by the CLE\(_\kappa\) carpet that they are defined in.

1. Introduction

The Schramm-Loewner evolutions (SLE) defined by Oded Schramm [19] in 1999 are the canonical conformally invariant, non-crossing, fractal curves which connect a pair of boundary points in a simply connected planar domain, and their importance has since been highlighted in numerous settings. The SLE family is indexed by the positive real parameter \(\kappa\), and there are three different regimes of \(\kappa\) values which correspond to different sample path behavior of an SLE\(_\kappa\) curve [18]: it is a simple curve for \(\kappa \in (0, 4]\), a self-intersecting but not space-filling curve for \(\kappa \in (4, 8)\), and a space-filling curve for \(\kappa \geq 8\).

In this work, we will answer the following question: Is it possible to recover the trajectory \(\eta\) of the SLE\(_\kappa\) (i.e. the map \(t \mapsto \eta(t)\)) when one observes its range (i.e. the set of points \(\eta([0, \infty])\))? In other words, if one knows the set of points that an SLE\(_\kappa\) visited, can one recover the order in which they were traced? The answer in the regime where \(\kappa \in (4, 8)\) is the following:

**Theorem 1.1** (SLE\(_\kappa\) range does not determine the path). Fix \(\kappa \in (4, 8)\), suppose that \(\eta\) is an SLE\(_\kappa\) process from \(0\) to \(\infty\) in the upper half-plane, and consider some fixed \(T \in (0, \infty]\). Then the trajectory \(\eta|_{[0, T]}\) is almost surely not determined by its range \(\eta([0, T])\). In fact, the conditional law of the trajectory given its range is almost surely non-atomic.

We note that the answer to this question in the other regimes of \(\kappa\) values is trivial: For a simple curve one can always recover the trajectory given its range, and for a space-filling curve, the range does provide no information at all.

In the proof of this result, we will view our SLE\(_\kappa\) path as living in an ambient space with many other SLE\(_\kappa\) paths around. More specifically, we will take here this space to be a collection of loops which come from a conformal loop ensemble CLE\(_\kappa\) [22, 25] with \(\kappa \in (4, 8)\). This is in contrast to several other recent works, in which it was natural to use the Gaussian free field (GFF) as the structure in which the path is naturally embedded [23, 12, 13, 11, 11].

We will also answer two natural questions about CLE. Recall that a CLE describes the distribution of a natural random collection of loops in a simply connected domain. The law of a CLE in a simply

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connected domain is conformally invariant (and one can therefore always view it as the conformal image of a CLE defined in the unit disk) and it is described by the same parameter $\kappa$ as SLE, but with the constraint that $\kappa$ has to be in the interval $(8/3, 8)$. The loops of a $\text{CLE}_\kappa$ are $\text{SLE}_{\kappa}$-type paths. Again, there are two ranges depending on whether or not $\kappa > 4$. When $\kappa \in (8/3, 4]$, the loops are simple, disjoint and do not touch the boundary of the domain they are defined in, whereas when $\kappa \in (4, 8)$, the loops are non-simple (but non-self-crossing) and can touch each other and the boundary.

Our first result about CLE will deal with the latter case (where $\kappa \in (4, 8)$). The set of points that is not surrounded by any of the loops (in the sense that the index of all the loops around those points is 0) is a random closed set called the $\text{CLE}_\kappa$ gasket, and can be viewed as the natural conformally invariant random version of the Sierpinski gasket. Because the individual loops of the $\text{CLE}_\kappa$ touch each other and the boundary, it is a priori not clear whether one can recover the individual loops by just looking at the gasket. Indeed, it is not possible:

**Theorem 1.2** ($\text{CLE}_\kappa$ gasket does not determine $\text{CLE}_\kappa$ loops). Fix $\kappa \in (4, 8)$ and suppose that $\Gamma$ is the collection of loops in a (non-nested) $\text{CLE}_\kappa$. Then $\Gamma$ is almost surely not determined by the $\text{CLE}_\kappa$ gasket. In fact, the conditional law of $\Gamma$ given its gasket is almost surely non-atomic.

We now turn to our second result for CLE, which is focused on the case of $\text{CLE}_\kappa$ for $\kappa \in (8/3, 4)$. Recall that in this case, the loops in the $\text{CLE}_\kappa$ form a disjoint collection of simple loops that also do not touch the boundary, so that the set of points that are encircled by no loop (this set is now called the $\text{CLE}_\kappa$ carpet) can be viewed as a natural conformally invariant random version of the Sierpinski carpet. In [15], we have defined and described natural continuous percolation interfaces (CPI) within such $\text{CLE}_\kappa$ carpets, that can intuitively describe boundaries of critical percolation clusters within these random fractal sets. These interfaces turn out to be variants of $\text{SLE}_{16/\kappa}$ curves, that are coupled with the $\text{CLE}_\kappa$.

**Theorem 1.3** (Continuous percolation within $\text{CLE}_\kappa$ is random). Fix $\kappa \in (8/3, 4)$, suppose that $\Gamma$ is a $\text{CLE}_\kappa$, and that $\eta$ is an $\text{SLE}_{16/\kappa}$-type curve coupled with $\Gamma$ as a CPI in the sense of [15]. Then the range of $\eta$ (and therefore also the path) is almost surely not determined by $\Gamma$. In fact, the conditional law of $\eta$ given $\Gamma$ is almost surely non-atomic.

In fact, this statement also holds in the case of the “labeled” $\text{CLE}_\kappa$ for $\kappa \in (8/3, 4)$ which are described in [15], where for each of the $\text{CLE}_\kappa$ loops, one tosses an independent biased coin to decide whether it is open or closed for the considered percolation process that one constructs. We note that the analog of Theorem 1.3 for the labeled $\text{CLE}_4$ is known to be false (see [15]): the continuous percolation interfaces in a labeled balanced (one uses a fair coin to choose the labels) $\text{CLE}_4$ are deterministic functions of the labeled $\text{CLE}_4$ itself.

Theorem 1.3 also sheds some light about the coupling between the GFF and the $\text{CLE}_\kappa$ carpets when $\kappa \in (8/3, 4)$, as it shows that in these couplings, the GFF is not a deterministic function of the nested $\text{CLE}_\kappa$ carpets.

The rough idea of the proof of Theorem 1.2 will be to construct a measure $\mu$ which is supported on a certain set of exceptional points. These points are either intersection points between two distinct macroscopic $\text{CLE}$ loops, or double points on one single loop. In both cases, there are four different macroscopic strands that emanate from these points. We will then show that if one performs the Markov step of picking a point at random using $\mu$, and then resamples the way that the four macroscopic strands are hooked up at that point, one roughly preserves the law of the global CLE.
Figure 1. Idea of the proof of Theorem 1.2: Choosing a pivotal and resampling its state can merge loops without changing the gasket.

Figure 2. Idea of the proof of Theorem 1.1: An SLE\(\kappa\) path with \(\kappa \in (4, 8)\) and two intertwined double points. The path depends on whether it visits the plain part before the dashed part or not, but the range does not. Choosing two such double points according to some well-chosen measure \(\mu'\) and resampling the order between dashed and plain will preserve the range but not the path.

During this resampling step, the gasket is preserved, but one can merge two loops into one or split one loop into two (see Figure 1). This shows that it is not possible to identify the individual loops by just observing the gasket.

The proofs of Theorem 1.3 and of Theorem 1.1 will follow a similar idea, except that in the latter case, one will need to construct a measure \(\mu'\) on special pairs of intertwined double points on the path, and the Markov step will consist in switching simultaneously the hook-up configuration between the four strands at both points in order to preserve the range of the path (see Figure 2). This then leads to a coupling of a pair of SLE\(\kappa\) paths which have the same range but visit their common range in a different order.

The construction and non-triviality of these measures \(\mu\) and \(\mu'\) is based on the multi-scale second moment method, which has also been used in many instances in the last decades to study the Hausdorff dimensions of random fractals.
2. Preliminaries

In this section, we make some comments and review some preliminary facts before proceeding to the proofs of our main theorems. In Section 2.1, we will explain a possible approach to proving our results when $\kappa = 6$ using properties of Bernoulli percolation. In the next three subsections, we will give a brief review of the construction of CLE$_\kappa$ and some results from [15]. We will assume that the reader of this work is familiar with the definition of SLE$_\kappa$ [19] and with the so-called SLE$_\kappa(\rho)$ processes [9]. We direct the reader to [10, 29] for reviews of the former and to [15, 12] for more on the latter.

2.1. Percolation pivotal points analogy. Although it will not be used directly in our proofs, it is worthwhile to first describe a possible proof of Theorem 1.1 in the special case where $\kappa = 6$. Indeed, our proof will have some analogies with the strategy that we will now outline.

The SLE$_6$ curve is known to be the scaling limit of percolation interfaces and the way in which the discrete interface approaches the SLE$_6$ in the scaling limit is well-understood [26, 2, 30]. In particular, the double points of SLE$_6$ correspond to the scaling limit of the double points in the discrete percolation interface (see for instance [30]). These discrete double points form a subset of the set of points in the percolation configuration where a so-called four-arm event holds (two disjoint closed and two open arms touch these points in alternating circular order, and they create four percolation interface strands). Furthermore, thanks to the work of Garban, Pete and Schramm [6, 7], the way in which the counting measure on such double points approaches a natural measure on the set of double points of SLE$_6$ in the scaling limit is well-understood.

Suppose now that one considers a long percolation interface, and consider two given disjoint macroscopic domains (here the domains will be thought of as fixed, while the mesh of the lattice will tend to zero). With positive probability, it will happen that the percolation interfaces visits these two domains twice and create intertwined double points as in Figure 3. On this event, we can decide to choose at random a pair of such intertwined double points using the counting measure on such pairs, and to change the status of both of these two points simultaneously. Note that this will basically not change the range of the percolation interface, but only the order in which the three strands of the percolation interface that join the two double points are traced. This indicates that the probability of the obtained configuration is comparable to the probability of the initial one (before switching how the strands are hooked up), which in turn indicates that in the scaling limit, the SLE$_6$ curve cannot be a deterministic function of its range.

Let us note that in order to make the previous idea work, it is sufficient to use the counting measure on some “special” intertwined double points which satisfy some extra conditions. For instance, one can use points where the four strands are well-separated at some macroscopic scale. Such points are easier to work with when obtaining uniform estimates: In particular, when one conditions on the event that such a well-separated four-arm event occurs at two given points, then one can see that the conditional probability that these two points end up being intertwined on the percolation interface is bounded from below. Instead of sampling according to the counting measure along the curve, one can (up to constants in the probabilities) therefore first choose the two points at random in some uniform way in two prescribed domains, then sample the nice four-arm events in their respective macroscopic neighborhood, and then finally the percolation configuration in the remaining domains, in such a way that they hook up the arms so that the percolation interface visits these two points, and then finally the state of these two points.
Our approach to the general SLE\(\kappa\) case will have a similar flavor, although we will work directly in the continuum. The percolation configuration will be replaced by a CLE\(\kappa\) instance. We will discover the CLE\(\kappa\) first near the two points \(z, w\) using branches of the CLE\(\kappa\) exploration tree. We will also define some “nice four-arm type events” and use the conformal Markov property of SLE\(\kappa\) and CLE\(\kappa\) to control the correlations between what happens in different regions.

2.2. CLE\(\kappa\) background when \(\kappa \in (4, 8)\). We will now give a very brief review of the construction and the main properties of CLE\(\kappa\) for \(\kappa \in (4, 8)\); mind that some of the statements that we will survey here do not hold for \(\kappa \in (8/3, 4]\). These results follow fairly directly from [22] – see also [15] for a more extensive review. In the present paper, we will focus primarily on the non-nested versions of conformal loop ensembles, where each given point in the domain is almost surely surrounded by one CLE loop.

Non-nested CLE\(\kappa\) for \(\kappa \in (4, 8)\) are random collections of loops, that are the conjectural scaling limits of the collection of outermost interfaces for a critical FK model for \(q = 2 + 2\cos(8\pi/\kappa) \in (0, 4)\) with free boundary conditions on a deterministic planar lattice approximation of a simply connected domain. The special cases \(\kappa = 6\) and \(\kappa = 16/3\) have been shown to correspond respectively to the scaling limits of site percolation on the triangular lattice [26, 2] (as mentioned in the previous subsection) and of the critical Ising-FK model [27]. One can also relate the CLE\(\kappa\) with these FK-models in the framework on planar maps and Liouville quantum gravity: Combining the results of [24, 5] implies that CLE\(\kappa\) is the scaling limit (for the so-called peanosphere-topology) of the interfaces in the critical FK model for \(q = 2 + \cos(8\pi/\kappa)\) on certain types of random planar map models.

Let us now recall the definition of the CLE\(\kappa\) exploration tree from [22] – we still assume that \(\kappa \in (4, 8)\) here. The SLE\(\kappa(\kappa - 6)\) process is a variant of SLE\(\kappa\) with a special target-invariance property established in [3, 21]. Suppose that \(D \subseteq \mathbb{C}\) is a non-trivial simply connected domain and that one chooses a starting point (or root) \(x\) on \(\partial D\). The target-invariance property makes it
possible to couple \( \text{SLE}_\kappa(\kappa - 6) \) processes from this root point to all points in the domain, in such a way that any two of them (targeting \( y \) and \( y' \), say) coincide up to the first time at which their trace disconnects \( y \) from \( y' \). After \( y \) and \( y' \) are first separated, the processes then evolve conditionally independently. We note that this coupling exists precisely because of the target-invariance of these processes. This collection of \( \text{SLE}_\kappa(\kappa - 6) \) processes starting from \( x \) is called the \( \text{SLE}_\kappa(\kappa - 6) \) branching tree, rooted at \( x \).

Then, using this tree, and guided by the conjectures about discrete models, it is explained in [22] how to define a collection of loops: For each given point \( z \), the loop that surrounds \( z \) in the loop-ensemble is simply defined as the first clockwise loop that is traced around \( z \) by the branch of the exploration tree that targets \( z \). This defines the \( \text{CLE}_\kappa \), and gives rise to a number of natural conjectures [22] about this object.

Let us list a few properties of these \( \text{CLE}_\kappa \)'s for \( \kappa \in (4, 8) \) that have been derived using the imaginary geometry approach to SLE processes:

- It was conjectured in [22] that these loops are in fact continuous curves. This is now known to hold because (see [12]) \( \text{SLE}_\kappa(\rho) \) curves have been proved to be continuous when \( \rho > -2 \) (and when \( \kappa > 4 \), then \( \kappa - 6 > -2 \)).
- It was conjectured in [22] (this is very natural because this property holds in the case of discrete models) that the law of this collection of loops does not depend on the choice of the root of the exploration tree. This property does not follow trivially from the branching tree definition and setup, but it has now been established, using the reversibility properties of \( \text{SLE}_\kappa(\kappa - 6) \) for \( \kappa \in (4, 8) \) derived in [14].
- The local finiteness of \( \text{CLE}_\kappa \), i.e. that the number of loops with diameter at least \( \epsilon \) is for each \( \epsilon > 0 \) almost surely finite, was established in [11] as a consequence of the almost sure continuity of the so-called space-filling SLE.
- The \( \text{CLE}_\kappa \) is clearly a deterministic function of the exploration tree. Conversely, as explained in [22], when \( \kappa \in (4, 8) \), the exploration tree are in fact a deterministic function of the \( \text{CLE}_\kappa \) and the chosen root. This makes it possible to discover simultaneously different portions of different exploration trees starting from different roots but that are associated with a the \( \text{CLE}_\kappa \). This idea will play a key role in the present paper.

2.3. \( \text{CLE}_\kappa \) background when \( \kappa \in (8/3, 4] \) and continuum percolation construction. Let us first say some very brief words about the basic properties of \( \text{CLE}_\kappa \) when \( \kappa \in (8/3, 4] \). In the present paper, we will actually only use their construction via boundary conformal loop ensembles (BCLE) but we need to recall a few things about them first.

- Just as for \( \kappa \in (4, 8) \), it is possible to choose a boundary point and to define (see [22]) a branching \( \text{SLE}_\kappa(\kappa - 6) \) tree that in turn defines a collection of loops called \( \text{CLE}_\kappa \). There are however some important differences that are due to the fact that \( \kappa - 6 \) is not greater than \(-2\) anymore. For instance, in order to define these \( \text{SLE}_\kappa(\kappa - 6) \) processes, one needs to use side-swapping and/or Lévy compensation. So, when \( \kappa \in (8/3, 4] \), one has to choose a side-swapping parameter \( \beta \in [-1, 1] \) to define such a tree, and when \( \kappa = 4 \), one has to choose a drift parameter \( \mu \).
- The fact that the law of \( \text{CLE}_\kappa \) that is constructed in this way does not depend on the root is non-trivial, and does rely on another construction of these CLEs using the Brownian loop-soups (see [25]). Actually, the results of the paper [15] that we will recall in a few paragraphs do provide as a by-product another derivation of the root-independence of the...
CLE$_\kappa$ distribution, based on the coupling of SLE with the GFF. The fact that the law of the CLE$_\kappa$ does not also depend on the choice of $\beta$ (or $\mu$) is explained in [31]. So in summary, there is indeed just one CLE$_\kappa$ distribution for each $\kappa \in (8/3, 4]$.

- One very important feature of the branching tree construction is that it has not been proved that the exploration trees that define these CLE$_\kappa$’s can be recovered deterministically from the CLE$_\kappa$ and the root. In fact, Theorem 1.3 of the present paper will show that in the case $\kappa \in (8/3, 4)$, the CLE$_\kappa$ exploration tree is not a deterministic function of the CLE$_\kappa$ and the root (for the special case $\kappa = 4$, we refer to the discussion in [15]: The exploration defines a tree only for the balanced labeled CLE$_4$, which together with the choice of the root, does determine the exploration tree – as can be shown using the direct relationship between this labeled CLE$_4$ and the GFF).

We now recall some features from the paper [15] that will be relevant in the present paper: Suppose now that $\kappa \in (2, 4)$ and $\rho \in (-2, \kappa - 4)$. We first begin by reminding the reader of the construction of the so-called boundary conformal loop ensembles BCLE$_\kappa(\rho)$ defined in [15, Section 7]. This is a conformally invariant family of boundary-touching SLE$_\kappa$-type loops which live in a simply connected domain $D$. Despite the fact that $\kappa < 4$, its definition does follow rather closely that of non-simple CLEs that we recalled in the previous subsection. As we will here sometimes describe simultaneously some SLE processes for different values of $\kappa$, we will use in this subsection the notation $\kappa \in (2, 4)$ and $\kappa' = 16/\kappa \in (4, 8)$, as in [15].

We fix a root point $x \in \partial D$. For each $y \in \partial D$ distinct from $x$, we let $\eta_y$ be a SLE$_\kappa(\rho; \kappa - 6 - \rho)$ process starting from $x$ (with marked points at $x^-$ and $x^+$ to the infinitesimal left and right of $x$) and targeted at $y$. Note that the chosen range of $\rho$ values is precisely so that $\rho$ and $\kappa - 6 - \rho$ are both greater than $-2$. Processes of this type are also target-independent, see [3, 21]. We can therefore couple together a family of such processes $\eta_y$ which are targeted at a countable, dense subset of $\partial D$ with the property that for $y, z$ in this set, the processes $\eta_y$ and $\eta_z$ almost surely agree with each other until they first separate $y$ from $z$, and after $y$ and $z$ are separated, the two processes evolve conditionally independently. This is therefore also a branching tree rooted at $x$ as in the previous subsection, but this one targets only boundary points.

The union of all branches in this tree divides $D$ into a countable collection of connected components. The boundary of each connected component is naturally oriented by the paths of the tree which form its boundary. The collection of boundaries of subdomains that are naturally oriented clockwise are called the loops of the BCLE$_\kappa(\rho)$ and the counterclockwise ones are referred to as the false loops of the BCLE$_\kappa(\rho)$. If we want to emphasize that the loops have a clockwise orientation, we will use the notation BCLE$_{\kappa}^\circ(\rho)$. One similarly defines BCLE$_{\kappa}^\circ(\rho)$ using SLE$_\kappa(\kappa - 6 - \rho; \rho)$ in place of SLE$_\kappa(\rho; \kappa - 6 - \rho)$ and takes the loops (resp. false loops) which are traced counterclockwise (resp. clockwise). Again, it is not obvious from the construction, but it is shown in [15, Proposition 7.1] (using the reversibility of the SLE$_\kappa(\rho_1; \rho_2)$ processes with $\rho_1, \rho_2 > -2$ established in [13]) that these BCLE$_\kappa(\rho)$ do not depend on the choice of root $x$.

For $\kappa' \in (4, 8)$ and $\rho' \in (\kappa'/2 - 4, \kappa'/2 - 2)$, the BCLE$_{\kappa'}^\circ(\rho')$ and BCLE$_{\kappa'}^\circ(\rho')$ are defined in an analogous way and it follows from the reversibility of SLE$_{\kappa'}(\rho_1'; \rho_2')$ with $\rho_1', \rho_2' \geq \kappa'/2 - 4$ established in [14] that the resulting family of loops does not depend on the choice of root [15, Proposition 7.1]. Recall that $\kappa'/2 - 2$ is the threshold below which the SLE$_{\kappa'}(\rho')$ processes are boundary intersecting. The range of $\rho'$ values considered in the definition of BCLE$_{\kappa'}(\rho')$ is precisely so that $\rho' < \kappa'/2 - 2$ and $\kappa' - 6 - \rho' < \kappa'/2 - 2$ so that the loops do in fact hit the domain boundary. Note that BCLE$_{\kappa'}(0)$ is simply the collection of loops in a CLE$_{\kappa'}$ which intersect the boundary.
We now suppose again that \( \kappa \in (8/3, 4) \) (so that \( \kappa' \in (4, 6) \)). As explained in [15] Section 7.2, one can iterate BCLEs, alternating between \( \kappa \) and \( \kappa' \) loops in order to produce natural couplings of CLE\( \kappa \) and CLE\( \kappa' \). The construction proceeds as follows.

- Sample a BCLE\( \kappa'/2 \) (0) process \( \Gamma' \). Sampling these loops is the continuum analog of exploring the boundary-touching FK clusters with free boundary conditions.
- Given \( \Gamma' \), we then sample independent BCLE\( \kappa'(-\kappa/2) \) processes in each of the (clockwise) loops of \( \Gamma' \). Call the resulting collection of loops \( \Gamma \). Sampling these loops is the continuum analog of exploring the boundary touching interfaces in the corresponding Potts model with free boundary conditions.
- Iterate the exploration in the false loops of \( \Gamma' \) and \( \Gamma \).

It is shown in [15] Theorems 7.2 and 7.3 that the law of the collection of SLE\( \kappa \)-type loops thus defined is in fact a CLE\( \kappa \). Note that the proof uses the SLE commutation relations which are encoded by the GFF [12, 11].

The branch of the CLE\( \kappa' \) exploration tree in this construction is called a continuum percolation exploration (CPI) inside of the CLE\( \kappa \) carpet (see [15] Definition 2.1 as well as [15] Section 4) because it can be interpreted as a percolation interface within this CLE\( \kappa \) carpet. This CLE\( \kappa'/\)CLE\( \kappa \) coupling derived in [15] is a continuum version of the random cluster representation of the Potts model (see, e.g., [8] for a review). See also [1] for the case of the Ising model.

Note that in the BCLE\( \kappa'/\)BCLE\( \kappa \) iteration scheme described just above, the BCLE\( \kappa' \) loops are always attached to the right side of the BCLE\( \kappa' \) loops. This means that the CPI always reflects to the left whenever it hits a CLE\( \kappa \) loop (in the percolation interpretation, the interiors of the CLE\( \kappa \) loops are closed, and the CPI traces the open/closed interface, with closed to the right and open to the left of the interface).

This construction can be generalized to the setting in which each of the CLE\( \kappa \) holes is labeled either + or − independently with a given probability \( p \in (0, 1) \). Then, the CPI reflects to the left (resp. right) when it hits a loop with a + (resp. −) label. In this case the CPI is a branch in a BCLE\( \kappa'/\rho' \) exploration tree where \( \rho' \) is a function of \( p \). This provides for each choice of \( p \), a different BCLE\( \kappa'/\)BCLE\( \kappa \)-type iteration scheme that construct a CLE\( \kappa \) (see [15] for all this).

2.4. The trunk construction of SLE\( \kappa(\kappa-6) \) for \( \kappa \in (8/3, 4) \). In Section 2.3, we recalled the construction of CLE\( \kappa \) which is based on iterated BCLE’s from [15]. We will now describe the analogous procedure where one builds just one SLE\( \kappa(\kappa-6) \) process for \( \kappa \in (8/3, 4) \), which is a single branch of the SLE\( \kappa(\kappa-6) \) exploration tree.

We recall (see [15] and the references therein for additional background) that when \( \kappa \in (8/3, 4) \) so that \( \kappa - 6 < -2 \), for each choice of \( \beta \in [-1, 1] \), one can define an SLE\( ^{\beta}_{\kappa} \) \((\kappa - 6) \) process so that the following is true. The value \( p = (1 + \beta)/2 \) represents the probability that when it traces a loop, it traces it counterclockwise and the trunk passes to the left of that loop while when this process traces a loop clockwise, its trunk passes to the right of that loop. When \( \kappa = 4 \) so that \( \rho = -2 \), one has to use a symmetric side-swapping (i.e., \( \beta = 0 \) and \( p = 1/2 \) in the previous setup), but there is an additional drift-type parameter \( \mu \). This leads to the so-called SLE\( ^{\beta,\mu}_{4} \) processes. All of these processes are called conformally invariant explorations of the CLE\( \kappa \) that they construct.

We will now describe in a bit more detail the case where \( \kappa \neq 4 \) (i.e., \( \kappa \in (8/3, 4) \)) and for simplicity we will again focus on the case \( \beta = 1 \) (so that all of the loops are attached to the right side of the trunk). We suppose that we have a simply connected domain \( D \subseteq \mathbb{C} \) and fix \( x, y \in \partial D \) distinct.
We then carry out the following steps (see the right-hand side of Figure 4). Here \( \rho' = 0 \) because we are dealing with the case \( \beta = 1 \).

- Sample an SLE\(_{\kappa'}(\kappa' - 6)\) process \( \eta' \) from \( x \) to \( y \) with the force point located at \( x^+ \) (i.e., infinitesimally on the counterclockwise side of \( x \)). Note that the time-reversal of \( \eta' \) is an SLE\(_{\kappa'}(\kappa' - 6)\) from \( y \) to \( x \) with the forced point located at \( y^− \). In other words, the law of \( \eta' \) is reversible up to swapping the force point from the right to the left side. This process \( \eta' \) will end up being the trunk of an SLE\(_1^\kappa(\kappa - 6)\) from \( x \) to \( y \) that we will construct.

- We next trace the collection of CLE\(_\kappa\)-loops that are attached to the trunk \( \eta' \) using the following procedure. In each component \( U \) in the complement of the range of \( \eta' \) which is either to the right of \( \eta' \) or surrounded clockwise by \( \eta' \) we sample a collection of SLE\(_\kappa\)-type loops as follows. Let \( x_U \) (resp. \( y_U \)) be the first (resp. last) point on \( \partial U \) which is visited by \( \eta' \). We then draw a branching SLE\(_\kappa(3\kappa/2 - 6)\) process which starts from \( x_U \) and is targeted at every point on \( \partial U \) which is in the range of \( \eta' \). When viewed as targeting \( y_U \), this process is an SLE\(_\kappa(3\kappa/2 - 6)\) process with a single force point at \( x_U \); note that \( 3\kappa/2 - 6 = \kappa - 6 - (-\kappa/2) \). This process will trace a collection of SLE\(_\kappa\)-type loops. We note that in the case that \( x_U = y_U \), this collection of loops has the same law as for a BCLE\(_\kappa(-\kappa/2)\). In the case that \( x_U \neq y_U \), we will still refer to this collection of loops as a BCLE\(_\kappa(-\kappa/2)\) but with the marked boundary points \( x_U \) and \( y_U \).

- We can the construct a continuous path \( \eta \) from \( x \) to \( y \) as follows. It moves along the trunk \( \eta' \) from \( x \) to \( y \), and each time it meets one of the SLE\(_\kappa\)-type loops for the first time, it traces it counterclockwise. As explained in \([15]\), the law of \( \eta \) is that of an SLE\(_1^\kappa(\kappa - 6)\) in \( D \) from \( x \) to \( y \).

\[\text{Figure 4. Left: A collection of loops in } D \text{ created by a branching SLE}_\kappa(3\kappa/2 - 6) \text{ process from } -i \text{ to } i. \text{ The right boundary of the loops corresponds to this process targeted at } i. \text{ The loops altogether form a BCLE}_\kappa(-\kappa/2) \text{ with marked points } -i \text{ and } i. \text{ Right: An SLE}_\kappa'(\kappa' - 6) \text{ process } \eta' \text{ from in } D \text{ from } -i \text{ to } i. \text{ In each clockwise loop or component to the right of } \eta' \text{ is an independent BCLE}_\kappa(-\kappa/2) \text{ (their interior is filled). Following the BCLE}_\kappa(-\kappa/2) \text{ loops in the order they are first visited by } \eta' \text{ yields an SLE}_1^\kappa(\kappa - 6).}\]
Note that the components of $D \setminus \eta'$ which are to the right of $\eta'$ have boundary which can be decomposed into two parts: the part which is in $\partial D$ and the part which is in the range of $\eta'$. So, in the second step, the loops of $\eta$ which are in these components are drawn by independent branching $\text{SLE}_\kappa(3\kappa/2 - 6)$ processes in each component, starting from the first point on the boundary which is visited by $\eta'$. Moreover, the right boundary of these loops is given in each component by an $\text{SLE}_\kappa(3\kappa/2 - 6)$ starting from the first point on the component boundary visited by $\eta'$ and targeted at the last. It will be convenient in some places below to think of the concatenation of these $\text{SLE}_\kappa(3\kappa/2 - 6)$ processes as a single curve (which we will do) and refer to as an $\text{SLE}_\kappa(3\kappa/2 - 6)$.

We can use the same trunk and the same collection of $\text{SLE}_\kappa$-type loops to build a continuous path $\tilde{\eta}$ from $y$ to $x$. By reversibility, $\tilde{\eta}$ has the law of an $\text{SLE}^{-1}_{\kappa}(\kappa - 6)$ from $y$ to $x$ that is tracing exactly the same loops at $\eta$ (but clockwise instead of counterclockwise). Note that while the ranges of $\eta$ and of $\tilde{\eta}$ coincide and the trunk of $\tilde{\eta}$ is the time-reversal of $\eta'$, the two processes $\eta$ and $\tilde{\eta}$ are not the time-reversal of each other. Indeed, the trunk will meet each loop more than once, so that the order in which $\tilde{\eta}$ encounters the loops is not exactly the reversed order in which $\eta$ meets them. In a certain sense, this construction in fact provides a very precise description of the lack of reversibility of the $\text{SLE}_\kappa(\kappa - 6)$ processes.

### 3. Conformal invariance of $\text{CLE}_\kappa$ exploration tree hookup probabilities

The goal of the present section will be to derive a conformal invariance statement related to pairs of explorations of $\text{CLE}_\kappa$’s. In Section 3.1 we will address the case that $\kappa \in (4, 8)$, which is the one that will be an essential ingredient in the proofs of our main three theorems. For completeness and future reference, we also discuss the case that $\kappa \in (8/3, 4)$ in Section 3.2 (note that the story in the case that $\kappa = 4$ is anyway much simpler because of the connection between the GFF and $\text{SLE}_4(\rho)$ processes).

![Figure 5](https://example.com/figure5.png)  
**Figure 5.** Exploration of a $\text{CLE}_\kappa$ for $\kappa \in (4, 8)$ starting from $-i$ and from $i$, creating four branches.

#### 3.1. The case $\kappa \in (4, 8)$

Let us consider a $\text{CLE}_\kappa$ for $\kappa \in (4, 8)$ in $D$. Some of the CLE loops will hit $\partial D$, and some others will not. For each loop $\mathcal{L}$ that intersects the counterclockwise half-circle
from $-i$ to $i$, we can define its first and last intersection points $z(\mathcal{L})$ and $\overline{z}(\mathcal{L})$ on this half-circle, when one moves from $-i$ to $i$. One can then define a continuous path $\eta_1^\#$ from $-i$ to $i$ as follows: One moves counterclockwise along the half-circle, and each time one hits $z(\mathcal{L})$ for some loop $\mathcal{L}$, one attaches the clockwise loop $\mathcal{L}$ to the path, and then proceeds. We note that this defines a continuous path by the local finiteness of CLE$_\kappa$ proved in \[14\]. One can then define the subpath $\eta_1$ of this path that corresponds to its growth as seen from $i$ (loosely speaking, one cuts out all parts that are growing while hidden away from $i$). This path $\eta_1$ is the concatenation of all clockwise portions of loops $\mathcal{L}$ from $z(\mathcal{L})$ to $\overline{z}(\mathcal{L})$ that are not disconnected from $i$ and $-i$ by any other such portion. The law of the path $\eta_1$ is that of an SLE$_\kappa(\kappa - 6)$ from $-i$ to $i$ in $D$. In fact, $\eta_1$ is exactly the branch from $-i$ to $i$ of the CLE$_\kappa$ exploration tree.

One can note that after some time $t$ ($t$ can be a deterministic time or a stopping time with respect to the filtration generated by $\eta_1$), one can define $z(t)$ to be the last point on the half-circle from $-i$ to $i$ that $\eta_1$ visited before $t$. If $\eta_1(t) \neq z(t)$, then this point $z(t)$ is (by construction) equal to $z(\mathcal{L})$ where $\mathcal{L}$ is the loop that $\eta_1$ is (partially) tracing at time $t$.

One can also interchange the roles of $i$ and $-i$ and perform the backward procedure: one moves clockwise along the half-circle from $i$ to $-i$, and attaches the loops of the CLE$_\kappa$ drawn in counterclockwise manner. In this way, one defines a path $\eta_2^\#$ and a subpath $\eta_2$ (which is $\eta_2^\#$ seen as growing from $-i$). While the time-reversal of $\eta_1^\#$ is not identical to $\eta_2^\#$ because the order of the loops and the way in which they are discovered have been changed, the time-reversal of $\eta_1$ is exactly $\eta_2$ (it is described also via the concatenation of the same portions of loops $\mathcal{L}$ between $\overline{z}(\mathcal{L})$ and $z(\mathcal{L})$): The time-reversal of the SLE$_\kappa(\kappa - 6)$ $\eta_1$ from $-i$ to $i$ is the SLE$_\kappa(\kappa - 6)$ $\eta_2$ from $i$ to $-i$ (modulo the convention that the marked point is now on the other side of the path).

Let us now suppose that one has discovered $\eta_1$ up to a stopping time $t$ and that $\eta_1(t) \neq z(t)$. As we have already mentioned, the point $z(t)$ is then the starting point of the CLE loop $\mathcal{L}$ that $\eta_1(t)$ is part of. In particular, the conditional law of the rest of this loop given $\eta_1|_{[0,t]}$ is exactly an SLE$_\kappa$ from $\eta_1(t)$ to $z(t)$ in the component of $D \setminus \eta_1([0,t])$ with $z(t)$ on its boundary. We can now decide to discover part of this loop counterclockwise starting from $z(t)$. By the time-reversal symmetry of SLE$_\kappa$ \[13\], the law of this path is an SLE$_\kappa$ from $z(t)$ to $\eta_1(t)$ in the component of $D \setminus \eta_1([0,t])$ with $\eta_1(t)$ on its boundary. Let us call this path $\widetilde{\eta}_1$, and discover $\widetilde{\eta}_1$ up to some stopping time $s$ (note that the definition of $\widetilde{\eta}_1$ and the notion of stopping time depend on $\eta_1|_{[0,t]}$). Given $\eta_1$ up to time $t$ and $\widetilde{\eta}_1$ up to time $s$, the law of the missing part of $\mathcal{L}$ joining $\eta_1(t)$ to $\widetilde{\eta}_1(s)$ is just an SLE$_\kappa$ in the remaining to be discovered domain (i.e. in the connected component $D_{s,t}$ of $D \setminus (\eta_1([0,t]) \cup \widetilde{\eta}_1([0,s]))$ which has $\widetilde{\eta}_1(s)$ and $\eta_1(t)$ on its boundary) because of the reversibility of SLE.

We now define symmetrically the path $\eta_2$ up to some stopping time $t_2$, the point $\overline{z}(t_2)$ and the path $\overline{\eta}_2$ from $\overline{z}(t_2)$ to $\eta_2(t_2)$. We assume that we are in a configuration as depicted in Figure 5 where all four points $\eta_1(t_1), \overline{\eta}_2(s_2), \eta_2(t_2)$ and $\eta_2(s_2)$ are four different boundary points of the same connected component $D(t, s, t_2, s_2)$ of the remaining to be discovered domain. Typical examples of stopping times $t$, $s$, $t_2$ and $s_2$ can be the respective hitting times of a circle of radius $r$ around the origin by the respective four strands (if they do make it to that circle).

We can note that conditionally on these four branches, two possibilities arise:

- $\eta_1(t)$ and $\eta_2(t_2)$ correspond to parts of the same CLE loop. In this case, $\overline{\eta}_2$ will hook up with $\eta_2^\#$, while the path $\eta_1$ will first hook up with $\eta_2$ (and these last two paths respectively coincide with $\eta_1^\#$ and $\eta_2^\#$ up to when they hook up). We call this the one-loop event $E_1$. 


• \( \eta_1(t) \) and \( \eta_2(t_2) \) correspond to different CLE loops. In this case, \( \tilde{\eta}_1 \) will hook up with \( \eta_1^\# \) without meeting \( \tilde{\eta}_2(s_2) \), and \( \tilde{\eta}_2 \) will hook up with \( \eta_2^\# \) without meeting \( \tilde{\eta}_1(s) \). We call this the two-loop event \( E_2 \).

With the previous notation, we define \( C(t, s, t_2, s_2) \) to be the configuration given by the domain \( D(t, s, t_2, s_2) \) and the four counterclockwise ordered boundary points \( \eta_1(t), \tilde{\eta}_1(s), \tilde{\eta}_2(s_2), \eta_2(t_2) \). We are going to establish the following conformal invariance statement.

**Lemma 3.1.** If we are in a configuration as depicted in Figure 5, the conditional distribution of the remaining pieces of \( \eta_1, \tilde{\eta}_1, \tilde{\eta}_2 \) in \( D(t, s, t_2, s_2) \) until they hook up into one or two loops is a conformally invariant function of the configuration \( C(t, s, t_2, s_2) \).

In fact, in order to prove this lemma it suffices to prove the following seemingly weaker result:

**Lemma 3.2.** If we are in a configuration as depicted in Figure 5, the conditional distribution of \( E_1 \) (and therefore of \( E_2 \)) is a function \( f_\kappa(\cdot) \) of the cross-ratio between the four boundary points in the configuration \( C(t, s, t_2, s_2) \).

Indeed, conditionally on \( E_1 \), we can describe the joint conditional law of the remaining pieces of the loops containing \( \eta_1 \) and \( \tilde{\eta}_1 \) (and therefore of \( \eta_2 \) and \( \tilde{\eta}_2 \)) in \( D(t, s, t_2, s_2) \) as the bi-chordal SLE\( _\kappa \) joining the four end-points in that domain, which is characterized uniquely by the fact that conditionally on one of the two paths, the law of the other one is an ordinary SLE\( _\kappa \) in the remaining domain (see [13, Theorem 4.1]), and we know that this property is satisfied in the present case (the same argument can also be applied when one conditions on \( E_2 \)).

In fact, we can notice that this conditional distribution is in fact symmetric when one formally interchanges the roles of \( (\eta_1(t), \tilde{\eta}_1(s), \tilde{\eta}_2(s_2), \eta_2(t_2)) \) and \( (\tilde{\eta}_2(s_2), \eta_2(t_2), \eta_1(t), \tilde{\eta}_1(s)) \). It therefore follows that it is in fact sufficient to prove Lemma 3.2 in the case where we do not grow \( \tilde{\eta}_1 \) or \( \tilde{\eta}_2 \) after \( \eta_1 \) and \( \eta_2 \) have been defined (in other words, when \( s_2 = t_2 = 0 \)). Indeed, we can then decide to switch the roles of \( \eta_1, \eta_2 \) and \( \tilde{\eta}_1, \tilde{\eta}_2 \), and to continue growing \( \eta_1 \) and \( \eta_2 \) instead of growing \( \tilde{\eta}_1 \) and \( \tilde{\eta}_2 \), and by conformal invariance, the general case of Lemma 3.2 follows.

As we have already pointed out, \( \eta_1 \) (and its time-reversal \( \eta_2 \)) is an SLE\( _\kappa(\kappa - 6) \) process, so that the following result implies Lemma 3.2 (here, modulo conformal invariance, \( \eta \) plays the role of the remaining part of \( \eta_1 \), including \( \eta_2 \), while \( \eta_2 \) plays the role of the beginning of the time-reversal of \( \eta \)).

**Lemma 3.3.** Fix \( \kappa \in (4,8) \) and suppose that \( \eta \) is an SLE\( _\kappa(\kappa - 6) \) process in \( \mathbb{H} \) from 0 to \( \infty \) with force point located in \( \mathbb{R}_+ \). Let \( \eta_R \) be the time-reversal of \( \eta \). Let \( \kappa R \) be an \( \eta_R \)-stopping time, and let \( D_{\kappa R} \) be the component of \( \mathbb{H} \setminus \eta_R([0, \tau_R]) \) without on its boundary. Then the conditional law of \( \eta \) given \( \eta_R|_{[0, \tau_R]} \) viewed as a path in \( D_{\kappa R} \) is a conformally invariant function of the configuration consisting of the domain \( D_{\kappa R} \) and the four marked boundary points given by 0, the location of the force point, \( \eta_R(\tau_R) \), and \( \min(\eta_R|_{[0, \tau_R]} \cap \mathbb{R}_+) \).

**Proof of Lemma 3.3.** Note that the case \( \kappa = 6 \) is trivial. We will treat the two cases \( \kappa \in (6,8) \) and \( \kappa \in (4,6) \) separately. The distinction between these two cases reflects the change in sign of \( \kappa - 6 \). When \( \kappa \in (6,8) \), we will use the SLE/GFF coupling [12] and the reversibility of SLE\( _\kappa \) [14]. When \( \kappa \in (4,6) \) we will use the CLE setup and results from [15] as reviewed just above in Section 2.3.

The proof in both cases will make use of a variant of the resampling characterization of bi-chordal SLE. More precisely, we will use a more general version of [13, Theorem 4.1], which is proved in Appendix A that states that there is a unique law on pairs of curves \( (\eta_1, \eta_2) \) connecting a pair of
boundary points $x, y$ with $\eta_1$ to the left of $\eta_2$ so that the conditional law of $\eta_1$ given $\eta_2$ (resp. $\eta_2$ given $\eta_1$) is that of a certain SLE$_\kappa(\rho)$ type process with force points on its left (resp. right) side.

We choose again the following notation: we fix $\kappa' \in (4, 8)$ and let $\kappa = 16/\kappa' \in (2, 4)$. We first consider the case that $\kappa' \in (6, 8)$; see Figure 6 for an illustration of the argument. (The reader may find it helpful to look at [17, Figure 2.5].) We can view $\eta'$ as the counterflow line from $\infty$ to 0 of a GFF $h$ on $H$ with constant boundary conditions given by $-\lambda' + \pi \chi$ on $R$. Let $\eta$ be the flow line of $h$ with angle $\theta = 3\pi/2 - 2\lambda/\chi$ from 0 to $\infty$. Then $\eta$ is an SLE$\kappa(3\kappa/2 - 4; 2 - 3\kappa/2)$ process. (Note that $2 - 3\kappa/2 > -2$ provided $\kappa < 8/3$ and $\kappa' > 6$.) Let $\eta'_R$ be the time-reversal of $\eta'$ and let $\tau, \tau_R$ be stopping times for $\eta', \eta'_R$, respectively. Then:

- It follows from [12] that the conditional law of $\eta$ given $\eta'$ is independently that of an SLE$\kappa(\kappa - 4; 2 - 3\kappa/2)$ process in each of the components of $H \setminus \eta'$ which are to the right of $\eta'$ and whose boundary have non-empty intersection with $\partial H$.
- The conditional law of $\eta'$ given $\eta$ is that of an SLE$_{\kappa'}$ process in the component of $H \setminus \eta$ which is to the left of $\eta$. Consequently, by the reversibility of SLE$_{\kappa'}$ processes for $\kappa' \in (4, 8)$ proved in [14], it follows that the conditional law of $\eta'$ given $\eta$, $\eta'|[0, \tau]$, and $\eta'_R|[0, \tau_R]$ is that of an SLE$_{\kappa'}$ process in the remaining domain.
Figure 7. Illustration of the setup used in the proof of Lemma 3.3 in the case that $\kappa' \in (4, 6)$. Shown is an SLE$_{\kappa'}(\kappa' - 6)$ process from $-i$ to $i$ in $D$ viewed as a CPI in a CLE$_\kappa$, $\kappa = 16/\kappa' \in (8/3, 4)$, process $\Gamma$. If we condition on $\eta'$ up to a stopping time $\tau$, the time-reversal $\eta'_R$ of $\eta'$ up to an $\eta'_R$-stopping time $\tau_R$ and the loops of $\Gamma$ which touch this path segment (green loops), then the conditional law of the rest of $\eta'$ (blue path) is an SLE$_{\kappa'}(\kappa' - 6)$ in the remaining domain. Conversely, if we condition on all of $\eta'$ (red and blue paths), then the conditional law of the loops which touch $\eta'$ is given by independent BCLE$_\kappa$'s in the components of $D \setminus \eta'$ which are surrounded by the right side of $\eta'$. Thus as these conditional laws are conformally invariant, the conformal invariance of the joint law follows from the bi-chordal arguments of Appendix A.2.

Since the two conditional laws are conformally invariant given $\eta'|[0, \tau]$ and $\eta'_R|[0, \tau_R]$, it follows from the bi-chordal SLE characterization (Appendix A.1) that the joint law of $\eta$ and $\eta'$ given $\eta'|[0, \tau]$ and $\eta'_R|[0, \tau_R]$ is conformally invariant.

We next consider the case that $\kappa' \in (4, 6)$; see Figure 7 for an illustration of the argument. We let $\eta'$ be an SLE$_{\kappa'}(\kappa' - 6)$ process in $H$ from 0 to $\infty$ with force point located at 0. Let $\eta'_R$ be the time-reversal of $\eta'$ and let $\tau$ (resp. $\tau_R$) be a stopping time for $\eta'$ (resp. $\eta'_R$). We view $\eta'$ as a CPI (in the sense of [13] Definition 2.1) coupled with a CLE$_\kappa$, say $\Gamma$, in $H$. We note that then $\eta'_R$ is also a CPI coupled with $\Gamma$. The CPI property of $\eta'_R$ implies that $\eta'_R|[\tau_R, \infty)$ is a CPI associated with the CLE$_\kappa$ given by including those loops of $\Gamma$ which are contained in the complementary component of $\eta'_R([0, \tau_R])$ with 0 on its boundary. In particular, conditioned on this we have that the law of the remainder of $\eta'$ given $\eta'|[0, \tau]$ and $\eta'_R|[0, \tau_R]$ and the loops of $\Gamma$ which hit $\eta'_R|[0, \tau_R]$ is that of an SLE$_{\kappa'}(\kappa' - 6)$ in the remaining domain. The same also holds if we switch the roles of $\eta'$ and $\eta'_R$.

Summarizing, we have that:

- Given $\eta'|[0, \tau]$, $\eta'_R|[0, \tau_R]$, and the loops of $\Gamma$ which hit $\eta'_R|[0, \tau_R]$, the remainder of $\eta'$ has the law of an SLE$_{\kappa'}(\kappa' - 6)$ process in the remaining domain.
- Given $\eta'|[0, \tau]$, $\eta'_R|[0, \tau_R]$, and the loops of $\Gamma$ which hit $\eta'|[0, \tau]$, the remainder of $\eta'_R$ has the law of an SLE$_{\kappa'}(\kappa' - 6)$ process in the remaining domain.
• Given all of \( \eta' \), the conditional law of the loops of \( \Gamma \) which hit \( \eta' \) is given by a \( \text{BCLE}_\kappa^c(\kappa/2) \) in each of the complementary domains which are to the right of \( \eta' \).

We will now use the conformal invariance of these three laws to deduce from the bi-chordal characterization that the joint law of \( \eta' \) and the loops of \( \Gamma \) which hit \( \eta'\quad | \quad [0,\tau] \) and \( \eta'_R\quad | \quad [0,\tau_R] \) is conformally invariant given \( \eta'\quad | \quad [0,\tau] \) and \( \eta'_R\quad | \quad [0,\tau_R] \) in the remaining domain. As explained in Appendix \[A.2\] this implies the conformal invariance of the conditional law of \( \eta' \) given \( \eta'\quad | \quad [0,\tau] \) and \( \eta'_R\quad | \quad [0,\tau_R] \).

As explained in \[16\], building on this lemma, on Dubédat’s commutation relations \[3\] and some SLE estimates, it is actually possible to explicitly identify the hook-up probability function \( f_\kappa \) in terms of a ratio of hypergeometric functions. We conclude this subsection with the following simpler result that just states that the function \( f_\kappa \) is well-behaved (this will be sufficient for the purpose of the present paper). Here and in the sequel, we refer to the definition of the cross-ratio of a conformal rectangle to be defined on \((0,\infty)\) and equal to 1 for a conformal square such as the unit disk with the four boundary points \(1, i, -1, -i\).

**Lemma 3.4.** The function \( f_\kappa(\cdot) \) is bounded away from 0 and from 1 on any compact subset of \((0,\infty)\). The function \( f_\kappa(c) \) converges to either 0 or 1 as \( c \to 0 \) or \( c \to \infty \).

**Proof.** It for instance suffices to start from the configuration \( C(t, s, t_2, s_2) \) with cross-ratio \( c \) and to note that it is possible (with positive probability \( p_\kappa(c) \)) to let \( \eta_1 \) grow further until the new cross-ratio hits 1. Hence, we get that \( f_\kappa(c) \geq p_\kappa(c)f_\kappa(1) \). The same argument can be applied to the two-loop event.

The result about the limit of \( f_\kappa(c) \) as \( c \to 0 \) and \( c \to \infty \) follows from the fact that when one explores one strand (say \( \eta_1 \)) until then end, one will eventually discover which hook-up event holds. So, just before that moment, the conditional probability of the hook-up event will tend to 0 or 1. But by construction, the cross-ratio will tend to 0 or \( \infty \) at the same time. \( \square \)

### 3.2. The case \( \kappa \in (8/3, 4) \)

We are now going to establish the analog of Lemma 3.2 for the case \( \kappa \in (8/3, 4) \). The setup will take a slightly different form than in the case of Lemma 3.2 because we cannot use the fact that the branches of the exploration tree used to build a CLE_\kappa are deterministic functions of the CLE_\kappa (indeed, it is one of the main results of the present paper that it is not the case), so we need to first explain how we define the joint law of the two explorations. As we mentioned earlier, the content of the present subsection will not be used later in the present paper and it is included here for future reference (it is used in \[16\]). We also leave out the (easier) case \( \kappa = 4 \), as this one can be dealt with via the relation between CLE_4 and the Gaussian free field (see for instance \[16\]).

Throughout, we suppose that we have a simply connected domain \( D \subseteq \mathbb{C} \), that \( x \) and \( y \) are distinct boundary bounds, and that \( \eta \) is an SLE_\kappa(\kappa - 6) path in \( D \) from \( x \) to \( y \). We also suppose that \( \eta \) is the SLE_\kappa^{-1}(\kappa - 6) \) from \( y \) to \( x \) whose trunk and loops are the same as those of \( \eta \), as described in Section 2.4. We emphasize again that \( \eta \) is not exactly the time-reversal of \( \eta \), but that the trace of \( \eta \) and \( \eta \) coincide, that the trunk of \( \eta \) is the time-reversal of that of \( \eta \), and that \( \eta \) visits exactly the SLE_\kappa-type loops as \( \eta \).

When we explore \( \eta \), we can choose any deterministic way to parameterize it (so that its “time” is a deterministic function function of its trace). We will then use the natural filtration \( (\mathcal{F}_t) \) generated by the path. Note that (because the SLE_\kappa loops are simple while the trunk is not a simple curve), at a stopping time \( \tau \), the knowledge of the last point on the trunk that has been visited by \( \eta \) before
time \( \tau \) is contained in the \( \sigma \)-field \( \mathcal{F}_\tau \). We call this point \( X(\tau) \), and we use the same notation for \( \tilde{\eta} \) (thus defining \( \tilde{X}(\tilde{\tau}) \)).

We are now going to discover a piece of \( \eta \) and a piece of \( \tilde{\eta} \). More precisely, suppose that \( \tau \) (resp. \( \tilde{\tau} \)) is a stopping time for \( \eta \) (resp. \( \tilde{\eta} \)). We define the event

\[
A(\tau, \tilde{\tau}) := \{ \eta([0, \tau]) \cap \tilde{\eta}([0, \tilde{\tau}]) = \emptyset, \ \eta(\tau) \neq X(\tau), \ \tilde{\eta}(\tilde{\tau}) \neq \tilde{X}(\tilde{\tau}) \}.
\]

On this event, we call \( D_{\tau, \tilde{\tau}} \) the connected component of the complement of \( \eta([0, \tau]) \cup \tilde{\eta}([0, \tilde{\tau}]) \) that has \( \eta(\tau) \) on its boundary. Note that \( X(\tau) \) and \( \tilde{X}(\tilde{\tau}) \) each correspond to two prime ends in \( D_{\tau, \tilde{\tau}} \).

We will consider implicitly the one that is on the left-hand side of \( \eta(\tau) \) (resp. the right-hand side of \( \tilde{\eta}(\tilde{\tau}) \)), i.e., on the side of the trunk.

**Lemma 3.5.** The conditional probability given \( \mathcal{F}_\tau, \tilde{\mathcal{F}}_{\tilde{\tau}} \) and the event \( A = A(\tau, \tilde{\tau}) \), of the event that \( \eta(\tau) \) and \( \tilde{\eta}(\tilde{\tau}) \) are part of the same \( \text{CLE}_\kappa \) loop is a function of the cross-ratio of the four marked points \( \eta(\tau), \tilde{\eta}(\tilde{\tau}), X(\tau) \) and \( \tilde{X}(\tilde{\tau}) \) in \( D_{\tau, \tilde{\tau}} \).

**Proof.** Let \( \eta' \) denote the trunk of \( \eta \) up until first hitting \( X = X(\tau) \). Define \( \tilde{\eta}' \) to be the trunk of \( \tilde{\eta} \) up until hitting \( \tilde{X} = \tilde{X}(\tilde{\tau}) \). We let \( \eta'' \) the missing middle piece of the trunk (joining \( X \) and \( \tilde{X} \)), so that the concatenation of \( \eta', \eta'' \) and (the time-reversal of) \( \tilde{\eta}' \) together form the trunk of \( \eta \).

Let us define \( E := (\eta(t), t \leq \tau) \) (resp. \( \tilde{E} = (\tilde{\eta}(t), t \leq \tilde{\tau}) \)), so that the \( \sigma \)-field generated by \( E \) is \( \mathcal{F}_\tau \).

(There could a priori be more information in this \( \sigma \)-field than the one provided by the trace \( \eta([0, \tau]) \) – the results of the present paper will actually show that it is the case).

Let us call \( L \) the remaining part of the loop that \( \eta \) has started to trace at time \( \tau \), and let similarly \( \tilde{L} \) be the remaining part of the loop of \( \tilde{\eta} \) that \( \tilde{\eta} \) has started to trace at time \( \tilde{\tau} \). Note that \( L \) could contain \( \tilde{\eta}(\tilde{\tau}) \) if the loop explored by \( \eta \) and \( \tilde{\eta} \) at time \( \tau \) and \( \tilde{\tau} \) are the same.

Using resampling techniques, we will show that the conditional law of \( (L, \tilde{L}) \) is a conformally invariant function of the domain and the four marked points (which clearly implies the lemma).

We first note that conditionally on \( (E, L) \), the law of \( \tilde{\eta} \) up until the time at which it hits \( \eta([0, \tau]) \cup L \) is that of the beginning of an \( \text{SLE}_{\kappa}^{-1}(\kappa - 6) \) from \( y \) to \( X \) in the component of the complement of \( \eta([0, \tau]) \cup L \) with \( y \) on its boundary. If we now condition on \( F := (E, L, \tilde{E}, \tilde{L}) \) (and suppose that the event \( A \) holds), and let \( U \) denote the connected component of the complement of \( \eta([0, \tau]) \cup L \cup \tilde{\eta}([0, \tilde{\tau}]) \cup \tilde{L} \) that lies outside of \( L \) and \( L' \), and has \( X \) and \( \tilde{X} \) on its boundary (which is the one where the middle piece of the trunk will be), we see that this middle piece of the trunk (i.e., \( \tilde{\eta}' \)) will join \( X \) and \( \tilde{X} \) in this domain. In fact, its conditional distribution (given \( F \)) is that of an \( \text{SLE}_{\kappa'}(\kappa' - 6) \) from \( X \) to \( \tilde{X} \) in \( U \) (this follows directly from the conformal Markov property of the exploration mechanism: One can first discover \( E, L \), then discover \( \tilde{\eta} \) up to the time at which it hits \( L \) or finishes drawing \( \tilde{L} \), and see that \( \tilde{\eta}' \) is the trunk of an \( \text{SLE}_{\kappa}(\kappa - 6) \) in the remaining domain).

Conversely, if we condition on the entire trunk, then we can describe the law of all the \( \text{CLE}_\kappa \) loops that are attached to it. In particular, the outer boundary of the set of \( \text{CLE} \) loops of \( \Gamma \) that touch the trunk is distributed like an \( \text{SLE}_{\kappa}(3\kappa/2 - 6) \) from \( x \) to \( y \) in the collection of complementary components which lie “to the right” of the trunk (in the sense described at the end of Section 2.4 – note that there could be in fact a countable collection of such components). We can note that both \( \eta(\tau) \) and \( \tilde{\eta}(\tilde{\tau}) \) will be on this path.

In particular, if we further condition on \( (E, \tilde{E}) \), we get an \( \text{SLE}_{\kappa}(3\kappa/2 - 6) \) conditioned on part of its beginning and part of its end. Recall from [13, Theorem 6.2] that this conditional distribution is
We want to show that these two conditional distributions characterize uniquely the conditional probability that after resampling one has a configuration with two loops, is a function of \( \tau, \tilde{\tau} \).

In the present setup, this implies in particular that the conditional distribution of \((L, \tilde{L})\) has the following properties: We consider the connected components of the complement of \( \eta([0, \tau]) \cup \tilde{\eta}(0, \tilde{\tau}) \) that have \( \eta(\tau) \) or \( \tilde{\eta}(\tilde{\tau}) \) on their boundary. There are two possibilities here: Either there is just one such component (that has both points on their boundary) in which case both hook-up scenarios are possible, or there are two components (and then \( L \) and \( \tilde{L} \) must be in these two different components). Then, the conditional distribution of \((L, \tilde{L})\) is conformally invariant in the sense that:

- In the former case (with one connected component), it is a conformally invariant function of the connected component with the four marked points \( \eta(\tau), \tilde{\eta}(\tilde{\tau}), X, \tilde{X} \).
- In the latter case (with two connected components), \( L \) and \( \tilde{L} \) are conditionally independent, and the conditional law of \( L \) (resp. \( \tilde{L} \)) is a conformally invariant function of its corresponding component and the two marked boundary points \( \eta(\tau) \) and \( X \) (resp. \( \tilde{\eta}(\tilde{\tau}) \) and \( \tilde{X} \)).

So, the previous two paragraphs establish the existence and the conformal invariance of:

- The conditional distribution of \((L, \tilde{L})\) given \((E, \tilde{E}, \tilde{\eta}')\).
- The conditional distribution of \(\tilde{\eta}'\) given \((L, \tilde{L}, E, \tilde{E})\).

We are in a setup where we can apply the resampling ideas of the type described in the appendix. We want to show that these two conditional distributions characterize uniquely the conditional law of \((L, \tilde{L}, \tilde{\eta}')\) given \((E, \tilde{E})\) (and this will in particular imply the conformal invariance of this conditional distribution and the conformal invariance of the hook-up probability).

What is needed to apply the ergodicity-based argument is to see that for almost any given \((L_1, \tilde{L}_1, E, \tilde{E})\) and \((L_2, \tilde{L}_2, E, \tilde{E})\), if we sample \(\tilde{\eta}'_1\), then resample \((L_i, \tilde{L}_i)\), then resample \(\tilde{\eta}'_2\) and so on, then one can couple these two resamplings so that the two configurations coincide with positive probability after a finite number of steps.

Let us first do this in the case where both \((L_1, \tilde{L}_1)\) and \((L_2, \tilde{L}_2)\) correspond to configurations with two loops (i.e., the event that the loops being drawn by \(\eta, \tilde{\eta}\) at the times \(\tau, \tilde{\tau}\), respectively, are not the same). (In the remaining paragraphs, \((E, \tilde{E})\) are considered to be fixed). In this case, the conditional law of \(L_i\) given \(\tilde{L}_i\) for \(i = 1, 2\) is that of an SLE-\(\kappa\) process in the remaining domain and the same is likewise true for the conditional law of \(\tilde{L}_i\) given \(L_i\). Therefore by first resampling \(L_i\) given \(\tilde{L}_i\) and then \(L_i\) given \(L_i\) for \(i = 1, 2\) we can couple together the configurations \((L_1, \tilde{L}_1)\) and \((L_2, \tilde{L}_2)\) on a common probability space to agree with positive probability. On this event, we can then resample simultaneously \(\tilde{\eta}'_i\) so that the entire configurations agree with positive probability.

Now we consider the case in which \((L_1, \tilde{L}_1)\) and/or \((L_2, \tilde{L}_2)\) correspond to configurations with only one loop (i.e., the event that the loops being drawn by \(\eta, \tilde{\eta}\) at the times \(\tau, \tilde{\tau}\) are the same). It will suffice to show that in one iteration step, we can obtain for both cases a configuration with two loops (and then apply the previous argument which applies to the setting of two loops in a second iteration step). To see this, one can first see that with positive probability, if we resample \((L_i, \tilde{L}_i)\) given \(\tilde{\eta}'_i\) (after having sampled \(\tilde{\eta}'_i\)) then \(L_i\) and \(\tilde{L}_i\) will not be part of the same loop (this follows easily from the fact that SLE-\(\kappa\)(3\(\kappa)/2 - 6) does hit the boundary and from the conformal invariance of SLE-\(\kappa\)(3\(\kappa)/2 - 6) conditioned by part of its beginning and part of its end, so that the conditional probability that after resampling one has a configuration with two loops, is a function of
the crossratio of the four marked points). Therefore after this one iteration step, there is a positive chance that we obtain a configuration with two loops, as desired.

Just as in the case $\kappa \in (4, 8)$, the CLE hook-up probabilities for $\kappa \in (8/3, 4)$ are actually worked out in the paper [10], building among other things on the present Lemma 3.3 on commutation relation considerations and on some SLE estimates.

A final remark is that in Lemma 3.5, we could have in fact chosen $\tilde{\tau}$ to be a stopping time for the filtration $\sigma(E, (\tilde{\eta}(s), s \leq t))$ (i.e., one can use information about $E$ to choose the stopping time $\tilde{\tau}$) without changing the proofs. This fact can turn out be handy, as one can for instance choose $\tilde{\tau}$ to be the first time at which the cross-ratio between the four marked points reaches a certain value.

4. Proof of Theorem 1.2

This section is devoted to the proof of the fact that the CLE$_\kappa$ loops are not determined by the CLE$_\kappa$ gasket when $\kappa \in (4, 8)$. We will explain in the subsequent section what modifications in the argument of the proof of this result enable us to also establish Theorem 1.1 and Theorem 1.3. Throughout this section, $\kappa \in (4, 8)$ is fixed and all constants that will appear in the proofs can depend on our choice of $\kappa$.

4.1. Notation. When $z = (z_1, z_2, z_3, z_4)$ is a quadruple of counterclockwise ordered points on the unit circle, we denote by $P_z$ the joint law of the configuration with the four strands $\eta_1, \tilde{\eta}_1, \tilde{\eta}_2$ and $\eta_2$ that were described in the previous section, starting respectively from these four points. More precisely, this is the conformal image of the law described in Lemma 3.1.

From now on, we will actually denote these paths by $\gamma_1, \gamma_2, \gamma_3$ and $\gamma_4$, and we will also use the notation $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$. We note that $\gamma_1$ (resp. $\gamma_3$) hooks up with either $\gamma_2$ or $\gamma_4$ (and then ends at $z_2$ or $z_4$) but does not hook up with $\gamma_3$ (resp. $\gamma_1$). To prove Theorem 1.2 we will first show that, on the positive probability event that $\gamma_1$ and $\gamma_3$ intersect each other, it is not possible to determine whether $\gamma_1$ terminates at $z_2$ or $z_4$ when one just observes the union of the ranges of $\gamma_1$ and $\gamma_3$.

We let $q = (a_1, \ldots, a_4) = (-i, 1, i, -1)$ and define $T$ to be the collection all 4-tuples $z$ where for each $j$, $|z_j - a_j| < 1/100$. This implies in particular that the cross-ratio of these four points is close to 1. Hence, the results of the previous section show that there exists a positive $p_0 = p_0(\kappa)$ such that the paths $\gamma$ hookup in each of the two possible ways with $P_z$ probability at least $p_0$ for all $z \in T$.

We also denote by $\nu$ the measure on $z \in T$ obtained by sampling independently each $z_j$ uniformly on the part of the unit circle that is at distance less than 1/100 from $a_j$. The law $P_z$ is then obtained by first choosing $z$ according to $\nu$ and then sampling $P_z$.

Let us also introduce some further notation that we will use throughout this section. We denote by $U(r) = U(r, \gamma)$ the event that all four strands $\gamma_1, \ldots, \gamma_4$ reach the circle of radius $r$ around the origin, and we call $t_j(r)$ their respective hitting times of this circle. On this event $U(r)$, we then define the connected component $D_r = D_r(\gamma)$ of $D \setminus \cup_j \gamma_j([-1, r])$ that contains the origin, and the conformal transformation $\psi_r = \psi_{r, \gamma}$ from $\tilde{D}_r(\gamma)$ back onto $D$ with $\psi_{r, \gamma}(0) = 0$ and $\psi_{r, \gamma}'(0) > 0$. We then also consider the image $z(r) = \tilde{z}(r, \gamma)$ under $\psi_{r, \gamma}$ of the four endpoints $\gamma_j(t_j(r))$. The previous considerations show that conditionally on $U(r)$ and on the four strands up to the hitting times $t_j(r)$, the law of the image $\gamma'$ of the remaining to be discovered parts of the four strands under $\psi_{r, \gamma}$ is exactly $P_{z(r, \gamma)}$. □
We can finally recall again that the conditional law of $\gamma_3$ given all of $\gamma_1$ is that of an $\text{SLE}_\kappa$ process from its initial to its target point in the complement of $\gamma_1$ and that the same is true when we switch the roles of $\gamma_1$ and $\gamma_3$. These facts show that the paths do various things with positive probability because one can first sample how they are hooked up using Lemma 4.2 and then resample each of the paths given the other path one at a time. We will use this at several stages in the proof.

4.2. A priori four-arm probabilities estimates. The purpose of this subsection is to derive Lemma 4.2 which is a crude lower bound of the probability that the four strands $\gamma_1, \ldots, \gamma_4$ all get close to the origin in a fairly well-separated way. We note that our goal here is to prove in a short way a lemma that will be sufficient for our purpose, and that it is not difficult to derive stronger statements.

When $n \in \mathbb{N}$, we define $\varepsilon_n := 2^{-n}$ (we will use this notation throughout this section) and the event $E_n = U(\varepsilon_n, \gamma)$ that all four paths $\gamma_1, \ldots, \gamma_4$ reach the circle of radius $\varepsilon_n$ around the origin. Let us first point out the following fact:

**Lemma 4.1.** There exist $\alpha_0 \in (0, 2)$ and some constant $c_0 > 0$ such that $P_{\nu}[E_n] \geq c_0 \times (\varepsilon_n)^{\alpha_0}$ for all $n \geq 1$.

**Proof.** We will use here a known estimate about the set of double points of an SLE curve when $\kappa \in (4, 8)$. This estimate follows for instance from [17, Theorem 1], where the almost sure double point dimension of $\text{SLE}_\kappa$ is actually derived, but we note that it would also be possible to derive the weaker statement that we will use here in a fairly elementary way. Indeed, we will just need a rather crude lower-bound of the first moment.

Suppose that the lemma would not hold, and let us then prove that it would imply an estimate that in turn implies that almost surely, the Hausdorff dimension of the set of points that are in a certain fixed neighborhood of the origin and on $\gamma_1 \cap \gamma_3$ is almost surely equal to 0. This leads to a contradiction, because we know that this is not the case, see for instance [17, Theorem 1].

First of all, let us suppose that $|z| < r$, when $r > 0$ is very small, and consider the conformal transformation $\psi_z: D \to \overline{D}$ with $\psi_z(0) = z$ and $\psi_z'(0) > 0$. We note that $|\psi_z'|$ converges uniformly to 1 on the closed unit disk when $r \to 0$, and that $\psi_z'$ converges uniformly to 1 on the unit circle. Let $P_{\nu'}$ denote the same law as $P_{\nu}$ except that the four starting points are now chosen uniformly such that for each $j$, $|z_j - o_j| < 1/200$ (instead of 1/100), and let $U(\varepsilon_n, \gamma, z)$ denote the event that all four paths $\gamma_1, \ldots, \gamma_4$ reach the circle of radius $\varepsilon_n/2$ around $z$. Note that when $r$ is small enough, then the image under $\psi$ of the disk of radius $\varepsilon_n$ around 0 contains the disk of radius $\varepsilon_n/2$ around $z$ (for all $n$), and we can also control the image of the uniform measure on the part of the unit circle at distance smaller than 1/100 of $o_j$ under $\psi_z$ (and see that its density is everywhere larger than $(2 + c)$ times that of the uniform uniform measure on the part of the unit circle at distance smaller than 1/100 for any very small given $c$).

It then follows readily by conformal invariance that when $r$ is fixed and small enough, then for all $n$ and $|z| < r$,

$$P_{\nu'}[U(\varepsilon_n, \gamma, z)] \leq 32 P_{\nu}[E_n]$$

(where we have chosen $c$ so that $(2 + c)^4 = 32$). If we now suppose that the lemma would not hold, then it would imply that for each $\alpha \in (0, 2)$, there exists $n_k \to \infty$ such that

$$P_{\nu}[E_{n_k}] / \varepsilon_{n_k}^\alpha \to 0 \quad \text{as} \quad k \to \infty.$$
Hence,
\begin{equation}
\sup_{z:|z|<r} \mathbb{P}_\nu[U(\varepsilon_{nk}, \gamma, z)] = o(\varepsilon_{nk}^\alpha) \quad \text{as} \quad k \to \infty.
\end{equation}

Note that (4.1) implies an upper bound on the expectation of the area of the set of points in the disk of radius \( r \) around the origin that are in the \( \varepsilon_{nk}/2 \)-neighborhood of both \( \gamma_1 \) and \( \gamma_3 \), and one can conclude that, under the probability measure \( \mathbb{P}_\nu \), the Hausdorff dimension of \( \gamma_1 \cap \gamma_3 \cap \{ z : |z| < r \} \) is at most \( 2 - \alpha \). As this is true for all \( \alpha \in (0, 2) \), we conclude that this Hausdorff dimension is almost surely equal to 0, and as we have already explained, this is not the case.

For each \( \delta > 0 \), we then define the event \( F_{n,\delta} \subset E_n \) that \( \min_{i \neq j} |z_i(\varepsilon_n, \gamma) - z_j(\varepsilon_n, \gamma)| \geq \delta \). In other words, the event \( F_{n,\delta} \) says that in terms of harmonic distance from the origin, the four points of \( \z(\varepsilon_n, \gamma) \) are \( \delta \)-separated in \( D_{\varepsilon_n}(\gamma) \).

**Lemma 4.2.** There exists \( \delta_0 > 0 \) so that for infinitely many values of \( n \), \( \mathbb{P}_\nu[F_{n,\delta_0}] \geq c_0(\varepsilon_n)^{\alpha_0}/8 \).

**Proof.** Let us first note that, if we choose \( \delta_0 > 0 \) small enough, then for all \( n \), \( \mathbb{P}_\nu[E_{n+1}\setminus(E_n \setminus F_{n,\delta_0})] \leq 1/8 \). Indeed, when \( E_n \setminus F_{n,\delta_0} \) holds, then at least two of the strands of \( \gamma \) corresponding to very close points \( z_j(t_j(\varepsilon_n)) \) will be very likely to hook up without reaching the circle of radius \( \varepsilon_{n+1} \) (here, we can consider separately the following two cases: Either, one point \( z_j(t_j(\varepsilon_n)) \) is \((\delta_0)^{1/2}\)-close to only one other \( z_j'(t_j'(\varepsilon_n)) \), in which case the two corresponding strands will hook-up with a probability close to 1 because of Lemma 3.4, and will therefore have a very small probability of reaching the circle of radius \( \varepsilon_{n+1} \). Or one \( z_j'(t_j'(\varepsilon_n)) \) is \((\delta_0)^{1/2}\)-close to at least two other \( z_j(t_j(\varepsilon_n)) \), in which case, two of the three corresponding strands do necessarily hook-up and will have a very small probability of reaching the circle of radius \( \varepsilon_{n+1} \)— we leave the details to the reader).

Suppose now that for such a choice of \( \delta_0 \), and for all \( n \) greater than some \( n_0 \), \( \mathbb{P}_\nu[F_{n,\delta_0}] \leq \mathbb{P}_\nu[E_n]/8 \). Then, we would get that for all \( n > n_0 \),
\begin{equation}
\mathbb{P}_\nu[E_{n+1}] \leq \mathbb{P}_\nu[F_{n,\delta_0}] + \mathbb{P}_\nu[E_n \setminus F_{n,\delta_0}]/8 \leq \mathbb{P}_\nu[E_n]/4
\end{equation}
which would imply that \( \mathbb{P}_\nu[E_n] \) is bounded by a constant times \( 4^{-n} = (2^{-n})^2 \). But we know from the previous estimate that this is not the case, and we can therefore conclude that \( \mathbb{P}_\nu[F_{n,\delta_0}] \geq \mathbb{P}_\nu[E_n]/8 \) for infinitely many values of \( n \).

From now on, \( \delta \) will be fixed and equal to such a \( \delta_0 > 0 \), and we will just write \( F_n \) instead of \( F_{n,\delta_0} \).

We are now going to define a new event \( E_n' \) by “composition” of \( F_n \) with another event, which is an idea that we will repeatedly use. Let us first define the event \( G \) that four strands \( \gamma \) reach the circle of radius \( 1/2 \), that \( \z(1/2, \gamma) \in \mathcal{T} \) and that \( D_{1/2}(\gamma) \) is a subset of the disk of radius \( 3/4 \) around the origin.

We now say that the event \( E_n' \) holds if \( F_n \) holds, and if \( \gamma_{\varepsilon_n} \) satisfies \( G \). Then, we can note that there exists a constant \( c_1 > 0 \), so that for infinitely any values of \( n \), \( \mathbb{P}_\nu[E_n'] \geq c_1 \times (\varepsilon_n)^{\alpha_0} \). Indeed, because of Lemma 4.2, it suffices to see that \( \mathbb{P}_\nu[G] \) is bounded from below uniformly for all \( \alpha \)-separated quadruples of starting points, (which in turn implies that \( \mathbb{P}_\nu[E_n'|F_n] \) is bounded from below). This fact can then easily be checked using the absolute continuity and resampling arguments mentioned at the end of Section 4.4.

We now explain how to control the dependence of our events with respect to the initial configuration \( \z \in \mathcal{T} \). Let us consider the four strands until they reach the circle of radius \( 1/2 \) around the unit circle. We now want to argue that there exists a constant \( c_2 > 0 \), such that for each configuration
of starting points $z \in \mathcal{T}$, one can find an event $G_1 = G_1(z)$ (possibly extending the probability space in order to use extra randomness) with $P_z[G_1] = c_2$, and such that $G_1 \subset \{z(1/2, \gamma) \in \mathcal{T}\}$ and, conditionally on $G_1(z)$, the law of $z(1/2, \gamma)$ is that of four independent uniformly chosen points in the $1/100$ neighborhoods of the $o_j$’s in the unit circle. This fact can be easily worked out, for instance using the resampling property of the law of the four strands.

We now define the event $E''_n = E''_n(z)$ to hold if $G_1(z)$ holds and if $\gamma^{1/2}$ satisfies $E'_n$. For a general configuration of four strands $\gamma$, we will just say that $E''_n$ holds if $E_n(\gamma_1(0), \ldots, \gamma_4(0))$ holds. In this way, we can notice that $P_z[E''_n]$ does in fact not depend on the choice of $z \in \mathcal{T}$. It does however depend on $n$. We have just seen that for infinitely many values of $n$,

$$P_z[E''_n] \geq c_1 c_2 (\varepsilon_n)^{\alpha_0}.$$  

This shows in particular that there exist infinitely many values of $n$ so that

$$P_z[E''_n] > (10^4 \varepsilon_n)^{(\alpha_0+2)/2}.  \tag{4.2}$$

We now choose a value of $N \geq 10$ so that (4.2) holds. We will keep this $N$ fixed until the end of this section. We define also $b$, $\beta_0$ and $\beta$ so that

$$b := P_z[E''_N] = (\varepsilon_N)^{\beta_0} = (10^4 \varepsilon_N)^{\beta}$$

and note that $0 < \beta_0 < \beta < 2$.

We can also note that on the event $E''_n$, it is natural to define the domain $D''_N$ corresponding to the complementary connected component containing the origin of the four strands up to the respective times at which one sees that $E''_n$ is satisfied. Note that the conformal radius (from the origin) of $D''_N$ is in the interval $[4\varepsilon_N, \varepsilon_N/4]$ (this follows from multiplicativity of the conformal radii, Koebe’s 1/4 Theorem and from the definitions of $E''_N$ and $G_1$). When mapping $D''_N$ back to the unit disk, the remaining to be discovered strands are exactly $((\gamma^{1/2})\varepsilon_N)^{1/2}$, that we denote by $e(\gamma)$.

Finally, and this will be important for what follows, we can note that our definition of $G_1$ ensures that for any bounded function $f$ we have that $E[z[1E''_N f(e(\gamma))]]$ does not depend on $z \in \mathcal{T}$.

### 4.3. The good pivotal regions and their number.

We are now going to use the previous considerations to define further events $E^k$ for $k \geq 1$. The event $E^1$ is just the event $E''_N$ with $N$ chosen as before. Then, we define $E^2$ to be the event that $E^1$ holds and that $e(\gamma)$ also satisfies the event $E''_N$. As on the event $E''_N$, the configuration of the images of the end-points is in $\mathcal{T}$, it follows that for all $z \in \mathcal{T}$ we have that $P_z[E^2] = P_z[E^1]^2 = b^2$. We then iteratively define the decreasing family of events $E^k$ is an analogous manner for $k \geq 3$. More precisely, $E^k$ is the event that $E^1$ holds, and that $e(\gamma)$ satisfies the event $E^{k-1}$. Then, the very same argument shows that for all $z \in \mathcal{T}$,

$$P_z[E^k] = P_z[E^k] = P_z[E^1]^k = b^k = (10^4 \varepsilon_N)^{k\beta} = (\varepsilon_N)^{k\beta_0}$$

where $0 < \beta_0 < \beta < 2$.

Let us now make some comments of the shape of the connected component $D^k$ containing the origin of the complement of the four strands up to the stopping times corresponding to the event $E^k$ (for instance, $D^1 = D''_N$). Let us denote by $\rho_k$ the conformal radius of $D^k$ as viewed from the origin. It follows from our definitions of the event $E''_N$ together with Koebe’s 1/4 Theorem that:

- For all $k$, $\varepsilon_N/4 \leq \rho_{k+1}/\rho_k \leq 4\varepsilon_N$. This implies in particular that $\rho_k \leq (4\varepsilon_N)^k$.
- The boundary of $D^k$ is included in the annulus between the circles of radii $\rho_k/4$ (this is just Koebe’s 1/4 Theorem) and $10\rho_k$ around the origin (this last fact follows from the last event in the definition of $G$).
Suppose now that $u$ is a point in the unit disk. We define the Möbius transformation $\psi_u: \mathbf{D} \to \mathbf{D}$ with $\psi_u(u) = 0$ and $\psi_u'(u) > 0$. For a given configuration defined under $\mathbf{P}_\zeta$, we say that the event $E^k(u)$ holds if the image of the configuration under $\psi_u$ satisfies $E^k$.

We define $\mathcal{T}' \subset \mathcal{T}$ to be the collection all 4-tuples $\zeta$ where for each $j$, $|z_j - o_j| < 1/200$. We can note that one can then find $r_0$, so that for all $u$ with $|u| < r_0$, $\psi_u(\zeta) \in \mathcal{T}$ as soon as $\zeta \in \mathcal{T'}$. Hence, for $|u| < r_0$ and all $\zeta \in \mathcal{T}'$, $\mathbf{P}_\zeta[E^k(u)] = \mathbf{P}_{\psi_u(\zeta)}[E^k] = b^k$.

Our next goal is now to derive the following second moment bound:

**Lemma 4.3.** There exists a constant $C' > 0$ so that for all $\zeta \in \mathcal{T}'$, for all $k$ and all $u, v \in B(0, r_0)$ with $|u - v| \geq 2^{-Nk}$, we have

$$\mathbf{P}_\zeta[E^k(u) \cap E^k(v)] \leq \frac{C' \times b^{2k}}{|u - v|^{\beta}}.$$ 

**Proof.** Let us define $D^k(u)$ and $D^k(v)$ just as before, except that they correspond to the domain around $u$ and $v$ respectively (so that $\psi_u(D^k(u))$ has the same law as $D^k$, for instance). Let $K(u, v)$ denote the smallest $k$ such that $D^k(u)$ does not contain the disk of radius $16(\varepsilon N)^{-1}|u - v|$ around $u$, and $K(v, u)$ similarly (interchanging $u$ and $v$). By symmetry, it is sufficient to bound the probability of the event $E^k(u) \cap E^k(v) \cap \{K(u, v) \leq K(v, u)\}$. We are going to decompose this according to the value of $K(u, v)$.

Note that by our previous bounds on the conformal radius of $D^k$,

$$|u - v|/(4\varepsilon N) \leq \rho_{K(u, v)} \leq (4\varepsilon N)^{K(u, v)}$$

so that $(4\varepsilon N)^{K(u, v)} \geq |u - v|$. We can therefore restrict ourselves to the values $k_0$ taken by $K(u, v)$, so that

$$b^{k_0} = (100\varepsilon N)^{k_0\beta} \geq |u - v|^\beta.$$ 

Suppose that $k_0 = K(u, v) \leq K(v, u)$ and let us consider the four strands $\gamma_1, \ldots, \gamma_4$ up to the time at which the event $E^{k_0+10}(u)$ is realized. Observing these four strands, we are only missing the pieces in $D^{k_0+10}(u)$, so that can already see what happened near $v$. In particular, we can see -- modulo whether the paths $\gamma_1, \ldots, \gamma_4$ hook up in the right way near $u$ -- if $E^k(v)$ can hold or not. Furthermore, the conditional probability of the four paths making it so that $E^k(u)$ holds is bounded by a constant times $b^{k - (k_0 + 10)}$. From this, we can deduce that

$$\mathbf{P}_\zeta[E^k(u) \cap E^k(v) \cap \{k_0 = K(u, v) \leq K(v, u)\}]$$

$$\leq b^{k - k_0 - 10} \mathbf{P}[E^k(v) \cap \{k_0 = K(u, v) \leq K(v, u)\}]$$

$$\leq b^{-10} b^k |u - v|^{-\beta} \mathbf{P}[E^k(v) \cap \{k_0 = K(u, v)\}].$$

Summing over all possible values of $k_0$, and using the symmetry in $u$ and $v$, we finally get that

$$\mathbf{P}[E^k(u) \cap E^k(v)] \leq 2b^{-10} b^{2k}|u - v|^{-\beta}.\qed$$

Let $N_k = B(0, r_0) \cap (2^{-kN}\mathbf{Z}^2)$. Let us now define the number $N_k$ of points in $N_k$ such that $E_k(u)$ holds. Our previous moment bounds imply some control on the law of $N_k$ as $k \to \infty$. Recall that $\beta_0$ is the value chosen so that $b = 2^{-N\beta_0}$.

**Lemma 4.4.** There exist a constant $a > 0$ such that for all $\zeta \in \mathcal{T}'$ and all $k \geq 1$, we have that

$$\mathbf{P}_\zeta[a \leq N_k/2^{kN(2-\beta_0)} \leq 1/a] \geq a.$$
Proof. Let $X = X_k$ denote the random variable $N_k/2^kN(2-\beta_0)$. As $P_\zeta[E^k(u)] = b^k$, we have that

$$E_\zeta[X_k] = 2^{-2kN}2^{\beta_0 kN} \sum_{u\in\mathcal{N}_k} P_\zeta[E^k(u)] = 2^{-2kN} \#\mathcal{N}_k$$

which is bounded from above and from below by positive constants that are independent of $k$ and of $\zeta \in \mathcal{T}'$.

On the other hand,

$$E_\zeta[(X_k)^2] = (2^{-2kN}2^{\beta_0 kN})^2 \sum_{u,v\in\mathcal{N}_k} P_\zeta[E^k(u) \cap E^k(v)].$$

The sum when $u = v$ is again easily taken care of by the fact that $P_\zeta[E^k(u)] = b^k$, and Lemma 4.3 takes care of all the terms in the sum for $u \neq v$; we get that for some constant $C$, for all $\zeta \in \mathcal{T}'$ and all $k$,

$$E_\zeta[(X_k)^2] \leq C + C(2^{-2kN})^2 \sum_{v \neq u \in \mathcal{N}_k} |u - v|^{-\beta}$$

which is easily shown to be bounded by some explicit constant independent of $k$ because $\beta \in (0, 2)$.

This information on the first and second moments of $X_k$ then classically imply the lemma. \hfill \Box

4.4. Rerandomizing configurations in pivotal regions and conclusion of the proof. We now complete the proof of Theorem 1.2. We begin by establishing the following intermediate result.

Lemma 4.5. Let $\gamma$ be the branch of the $\text{CLE}_\kappa$ exploration tree from $-i$ to $i$ in $D$. The probability that the conditional law of $\gamma$ given the $\text{CLE}_\kappa$ gasket is not supported on a single path is strictly positive.

Proof. Let $\gamma$ be the four strands defined under the law $P_\tau$. We also let $\Gamma$ be the corresponding loop ensemble and $\Upsilon$ its gasket. We will show that, with positive probability, the conditional probability that $\gamma_1$ terminates at $z_4$ given $\Upsilon$ is in $(0, 1)$.

For each given $k$, consider the Markov chain on $(\Gamma, \gamma)$ configurations defined as follows:

- Pick a point $u \in \mathcal{N}_k$ uniformly at random,
- Check whether the event $E^k(u)$ occurs, and
- If so, resample the terminal segments of the paths in $D^k(u)$ and the rest of the $\text{CLE}_\kappa$ in $D^k(u)$.

Note that this chain preserves the joint law of $(\Gamma, \gamma)$.

We now want to use Lemma 4.4 to see that if we run the chain for $\lfloor 2^{kN(2-\beta_0)} \rfloor$ steps there is a positive chance (bounded from below uniformly w.r.t. $k$) that there exists exactly one single time during these steps at which the chain discovers a point $u \in \mathcal{N}_k$ where $E^k(u)$ occurs and switches the connections near this point.

To see this, we first notice that if we sample $\lfloor 2^{kN(2-\beta_0)} \rfloor$ times a uniformly chosen point in $\mathcal{N}_k$ for a given configuration $\Gamma$, then with a probability bounded uniformly from below (say, larger than some $a_0$), one hits exactly one point $u$ for which $E^k(u)$ holds and switches it. Then, we need to argue that once this point has been switched and we get a new configuration $\tilde{\Gamma}$, with positive probability, if we sample the remaining uniformly chosen points in $\mathcal{N}_k$ (so that we have sampled a total of $2^{kN(2-\beta_0)}$ such points), then we do not find a point $v$ for which the event $E^k(v)$ occurs for the new configuration.
To justify this, we note that the proof of Lemma 4.3 (in particular, the fact that we derived an upper bound on the conditional probability of $E^k(v)$ occurring given $E^k(u)$, regardless of how the paths hook up near $u$), we get that for every $u \in N_k$, conditionally on the event that $u$ has been picked among the $2^{kN(2-\beta_0)}$ times and on the fact that $E^k(u)$ did hold at that time, the mean number of points $v \in N_k$ such that $E^k(v)$ holds either before or after the switch near $u$ is bounded by a constant times $b^k \times \#N_k$. In particular, the conditional probability that this number of points is greater than some explicit large but fixed constant times $b^k \times \#N_k$, then the conditional probability that no further changes are made to the configuration during the remaining switching attempts is bounded uniformly from below.

Then, if $\tilde{\Gamma}$ denotes the resulting loop ensemble and $\tilde{\Upsilon}$ its gasket, the Hausdorff distance between $\Upsilon$ and $\tilde{\Upsilon}$ is at most $(8\varepsilon_N)^k$, while $\tilde{\Gamma}$ has changed more dramatically (now $\gamma_1$ hooks up with the other strand). Therefore sending $k \to \infty$ (and possibly passing to an appropriate subsequence), we get an asymptotic coupling which satisfies the desired property. \[
\square
\]

We now conclude the proof of Theorem 1.2 via a zero-one law type argument.

**Proof of Theorem 1.2.** Let $\eta$ be the branch of the CLE$_{\kappa}$ exploration tree from $-i$ to $i$ in the unit disk. By Lemma 4.5, we know that $\eta$ is not determined by the gasket $\Upsilon$ of $\Gamma$ with probability at least $p \in (0, 1]$. We can parameterize $\eta$ as seen from $i$. Since $\eta(t)$ converges almost surely to $i$ as $t \to \infty$, it is clear that for some given large $t_0$, the probability $p(t_0)$ that $\eta$ up to time $t_0$ is not determined by $\Upsilon$ is strictly positive. By scaling and conformal invariance, this probability $p(t_0)$ is independent of $t_0$.

For a given fixed $t_0$, as explained in the CLE$_{\kappa}$ description, it is possible to discover simultaneously $\eta$ up to time $t_0$ and the CLE$_{\kappa}$ loops it traces. In particular, we can trace $\eta$ up to the first time $s_0$ after $t_0$ at which it will touch the semi-circle from $-i$ to $i$ again, and leave the loop that it was tracing at time $t_0$ in order to branch towards $i$. At that time $s_0$, the conditional law of the CLE in the remaining to be explored domain with $i$ on its boundary is just the law of a CLE in this domain. It can in particular be resampled without affecting $\eta$ up to time $t_0$. Hence, after that time $s_0$, the conditional probability that future of $\eta$ is not determined by the gasket is still $p$, independently of $\eta$ up to time $t_0$. Hence, we get that $1 - p \leq (1 - p)(1 - p(t_0))$. As $p > 0$, we conclude that $p = 1$. \[
\square
\]

Note that this argument in fact can be adapted to see that the conditional law of $\eta$ given $\Upsilon$ is almost surely non-atomic. Indeed, the previous result shows that for some $\lambda < 1$, there is a positive probability $a$ that the conditional law of $\eta$ up to time $t_0$ given the $\Upsilon$ has no atom of mass greater than $\lambda$. Using the conditional independence after $s_0$, if we define $Q(x)$ to be the probability that the conditional law of $\eta$ given $\Upsilon$ has an atom of mass at least $x$, we get readily that for all $x \leq 1$,

\[Q(\lambda x) \leq (1 - a)Q(\lambda x) + a(Q(x)),\]

which (together with the fact that $Q(1) = 0$) implies that $Q(x) = 0$ for all positive $x$.

5. **Derivation of Theorem 1.1 and of Theorem 1.3**

We now explain how to adapt the previous ideas in order to derive the other two theorems stated in the introduction.
5.1. **Randomness of continuum percolation interfaces.** We first give the proof of Theorem 1.3.

*Proof of Theorem 1.3.* As mentioned in the introduction, our proof for CPIs in the case where \( p = 1 \) (i.e. where all of the CLE loops have the same label) can in fact be easily adapted to the more general setting of labeled CLE carpets. So, for simplicity, we will only include the argument in the case that \( p = 1 \).

Fix \( \kappa \in (8/3, 4) \) and let \( \kappa' = 16/\kappa \in (4, 6) \). We suppose that we have a coupling of a CLE\( ^\kappa \) process \( \Gamma' \) and a CLE\( ^\kappa \) process \( \Gamma \) as described at the end of Section 2.3. Just as in Section 3.1, we then explore part of the branch of the CLE\( ^{\kappa'} \) exploration tree from \(-i\) to \(i\), and then, starting from its most recent intersection with \( \partial D \) in the counterclockwise direction starting from \(-i\), we start drawing the currently explored loop backwards up to some stopping time. We then condition on all of the CLE\( ^\kappa \) loops which intersect the part of the CLE\( ^{\kappa'} \) exploration tree that we have observed so far. See the left side of Figure 8 for an illustration. By [15, Theorem 7.3], we know in particular that the conditional law of the CLE\( ^{\kappa'} \) exploration tree in the connected component of the remaining domain which has \( i \) on its boundary, is given by that of an independent CLE\( ^{\kappa'} \) exploration tree in the remaining domain. (In the FK-Potts analogy in a domain \( D \), one considers a monochromatic with color “A” boundary condition for Potts that one can view as a wired boundary condition for the coupled FK model, and one explores the inner boundaries that touch \( \partial D \) of the open FK-cluster up to some time – this corresponds to the exploration of the CLE\( ^{\kappa'} \) tree. Then, one attaches to its right all the clusters of “non-A” sites which therefore have A’s on their outer boundary, and notes that in the remaining domain, one has again A-monochromatic boundary conditions).

This allows us to set things up in a manner which is similar to the proof of Theorem 1.2 except we will choose the point from which we start to explore another part of the CLE\( ^{\kappa'} \) exploration tree differently. Namely, we pick a point which is on one of the aforementioned CLE\( ^\kappa \) loops and then explore a branch of the CLE\( ^{\kappa'} \) exploration tree from there and then part of the time-reversal of the branch targeted at its most recent intersection with the domain boundary on its right side, as illustrated in the right side of Figure 8.

Then we have four marked boundary points and we can proceed in this setting with the same argument as in the proof of Theorem 1.2. In particular, the argument of Theorem 1.2 implies that if the inner/outer paths create special intersection points, then we cannot tell if the inner paths hook up with each other and the outer paths with each other or if the inner and outer paths hook up. Note that the actual gasket of the CLE\( ^{\kappa'} \) is then not the same depending on the way in which the paths hook up (which is a stronger statement than just saying that the exploration path is not the same).

This implies that the percolation exploration of the CLE\( ^\kappa \) is with positive probability not determined by the CLE\( ^\kappa \) carpet. A simple zero-one argument (by looking at smaller and smaller pieces of the CPI) then implies that it is in fact the case that the percolation exploration is almost surely not determined by the CLE\( ^\kappa \) carpet and in fact the conditional law is almost surely non-atomic. □

5.2. **Randomness of the SLE curve given its range.** We now turn to Theorem 1.1. In the present section, we again assume that \( \kappa \in (4, 8) \). Let us first note the following fact, that allows us to consider an SLE\( ^{\kappa} \) \((\kappa - 6)\) instead of an SLE\( ^\kappa \).

**Lemma 5.1.** The probability that the conditional law of an SLE\( ^\kappa \) process given its entire range is non-trivial is either equal to 0 or to 1, and it is equal to the corresponding probability for an SLE\( ^{\kappa} \) \((\kappa - 6)\) process.
Proof. By [12, Proposition 7.30] the conditional law of an SLE$_\kappa(\kappa - 6)$ process given its left and right boundaries is independently that of an SLE$_\kappa(\kappa/2 - 4; \kappa/2 - 4)$ in each of the bubbles formed by the left and right boundaries. Note that there are almost surely infinitely many such bubbles, because there are infinitely many global cut points on an SLE$_\kappa(\kappa - 6)$ (this follows from the fact that the same is true for an SLE$_\kappa$, see [17] and the references therein). Note that these left and right boundaries are determined by the range of the path. Therefore, the probability that an SLE$_\kappa(\kappa - 6)$ process is determined by its range is equal to 1 if the same is true for an SLE$_\kappa(\kappa/2 - 4; \kappa/2 - 4)$, and it is equal to 0 otherwise.

But we know that the conditional law of an SLE$_\kappa$ process given its left and right boundaries is also independently that of an SLE$_\kappa(\kappa/2 - 4; \kappa/2 - 4)$ in each of the infinitely many bubbles formed by the left and right boundaries. Hence, we also have that the probability that an SLE$_\kappa(\kappa - 6)$ process is determined by its range is equal to 1 if the same is true for an SLE$_\kappa(\kappa/2 - 4; \kappa/2 - 4)$, and it is equal to 0 otherwise.

This proves the statement in the lemma. 

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. By the previous lemma, we can consider the case where $\eta$ is an SLE$_\kappa(\kappa - 6)$ process in $D$ from $-i$ to $i$ which is given as the branch of a CLE$_\kappa$ exploration tree in $D$. 

**Figure 8.** Left: The branch of the CLE$_\kappa'$ exploration tree from $-i$ to $i$ drawn up to a given time (red), as well as the time-reversal of the path targeted at the most recent intersection with the counterclockwise segment of $\partial D$ from $-i$ to $i$ (blue), just like in Section 3.1. We take the CLE$_\kappa'$ to be coupled with a CLE$_\kappa$ as a percolation in the CLE$_\kappa$ carpet; the filled loops are the CLE$_\kappa$ loops which intersect the part of the CLE$_\kappa'$ we have explored so far. The conditional law of the CLE$_\kappa$ in the unexplored region is then independently a CLE$_\kappa$ in each of the components. Right: Shown is a partial exploration of a loop which is attached to the green CLE$_\kappa$ loops. If the two new branches hook up with the previous two, then they all correspond to the same outermost CLE$_\kappa'$ loop and if they do not, then the first ones correspond to an outermost loop and the other ones to another loop, surrounded by the first loop.
As explained in the introduction, and in contrast to the proof of Theorem 1.2, we will need to resample the configuration in two different well-chosen regions rather than just at one, in order to globally preserve the range of $\eta$.

We suppose that $r$ is chosen to be very small, and we let $B_1$ and $B_2$ be the open disks of radii $r$ around $-1/2$ and around $1/2$ respectively. As in the proof of Theorem 1.2, we can control the number $N_k^1$ of good approximate pivotals in $B_1$ (resp. the number $N_k^2$ of good approximate pivotals in $B_2$) instead of the number $N_k$ of approximate pivotals in the ball of radius $r_0$ around the origin. The argument of the proof of Theorem 1.2 allows to control separately $N_k^1$ and $N_k^2$. Let us now explain how one can in fact control both simultaneously, i.e. that there exist a positive constant $a$ such that for all $z \in \mathcal{T}'$ and all $k \geq 1$,

\begin{equation}
P_z[a2^{kN(2-\beta_0)} \leq \min(N_k^1, N_k^2) \leq \max(N_k^1, N_k^2) \leq 2^{kN(2-\beta_0)}/a] \geq a.
\end{equation}

One way to proceed is to notice that the probability of the event $A$ that $\eta$ visits $B_1$, then $B_2$, then $B_1$, then $B_2$ in that order, and hits the boundary of the unit disk and itself in-between (so that the branches disconnect $B_1$ from $B_2$) as depicted in Figure 9(i) is strictly positive. We can furthermore impose on $A$ the fact that the cross-ratios of the four landing points on $B_1$ and on $B_2$ as indicated on the right of Figure 9(ii) in the complement of the paths drawn there are in $[u, 1/u]$ for some positive constant $u$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9.png}
\caption{From top left to bottom right: (i) The event $A$. (ii) The same picture without the parts near $B_1$ and $B_2$. (iii) Before resampling near $B_1$. (iv) Before resampling near $B_2$.}
\end{figure}
The arguments in Section 3.1 show that when A holds, then conditionally on the four strands up to when they hit \(B_1\) as in Figure 9(iii), the hook-up probabilities are bounded away from 0 and from 1 (this is because these four strands near \(B_1\) do correspond exactly to an exploration as in Section 3.1, except that it is started from \(-1\) and \(i\) instead of from \(-i\) and \(i\)). If this resampling has changed the hook-up configuration near \(B_1\), one can apply the same argument near \(B_2\), noting that the four strands near \(B_2\) do correspond to the “exploration” of the CLE starting from \(-1\) and \(i\), as in the fourth picture. This shows that on the event \(A'\) that there exists strands like on Figure 9(i), the joint conditional law of the configuration near \(B_1\) and of the configuration near \(B_2\) is absolutely continuous with respect to each other, and that there exists a Radon-Nikodym derivative which is uniformly bounded and bounded away from 0, which in turn allows to deduce (5.1).

Then, as in the proof of Theorem 1.2 with positive probability, if we perform the Markov step where we pick \(|2^{kN\beta_0}|\) points uniformly in \(2^{-kN}Z^2 \cap B_1\) and in \(2^{-kN}Z^2 \cap B_2\), then we will have a positive chance (bounded from below) of hitting just one pivotal in \(B_1\) and just one pivotal in \(B_2\). On this event, if one flips the configurations near both of these two points, the exploration tree path in the new CLE will have Hausdorff distance at most \(e^{-(\beta-c)k}\) from the original exploration tree path but visit its range in a different order.

Taking a (possibly subsequential) limit as \(k \to \infty\) thus yields an asymptotic coupling of two \(\text{SLE}_\kappa(\kappa - 6)\) processes which with positive probability have the same range but visit points in a different order. Hence, the probability that an \(\text{SLE}_\kappa(\kappa - 6)\) process is not determined by its range is strictly positive, which concludes the proof, by Lemma 5.1.

6. Comments

6.1. Relationship with the SLE/GFF coupling. \(\text{SLE}_\kappa\) and \(\text{CLE}_\kappa\) can be naturally coupled with an instance \(h\) of the Gaussian free field (GFF) on a simply connected domain \(D \subseteq \mathbb{C}\) with appropriately chosen boundary data (see e.g. [20, 23, 4, 12, 11, 15]). Theorem 1.1 has some consequences for the coupling of \(\text{SLE}_\kappa\) for \(\kappa \in (4, 8)\) with the GFF, Theorem 1.2 for the coupling of \(\text{CLE}_\kappa\) for \(\kappa \in (4, 8)\), and Theorem 1.3 for \(\text{CLE}_\kappa\) for \(\kappa \in (8/3, 4)\).

Let us first comment on the \(\text{SLE}_\kappa/\text{GFF}\) coupling for \(\kappa \in (4, 8)\). Suppose that \(h\) is a GFF on a simply connected domain \(D \subseteq \mathbb{C}\) with boundary data so that it may be coupled with an \(\text{SLE}_\kappa\) process \(\eta\) from one point on \(\partial D\) to another. In this coupling, the boundary data for the conditional law of \(h\) given \(\eta\) is in each component \(U\) of \(D \setminus \eta\) given by a constant plus a multiple of the argument of the derivative of the uniformizing conformal map \(\varphi: U \to \mathbb{H}\). Although the winding of \(\partial U\) is not defined in the usual sense as it is fractal, \(\arg \varphi'\) has the interpretation of being the harmonic extension of the winding of \(\partial U\) from \(\partial U\) to \(U\). In particular, there is a marked point on \(\partial U\) where \(\arg \varphi'\) makes a jump of size \(2\pi\). In terms of the path, this point corresponds to the first (equivalently last) point on \(\partial U\) visited by \(\eta\). If one observes only the range of \(\eta\) in addition to the GFF boundary data then it is in fact possible to recover the trajectory of \(\eta\) in a measurable way. This follows because \(\eta\) turns out to be a deterministic function of \(h\) [4, 12] and the values of \(h\) in the components of \(D \setminus \eta\) are conditionally independent of \(\eta\) itself given the values of \(h\) along \(\eta\). Theorem therefore implies that one cannot recover the marked points or GFF field heights by observing the range of \(\eta\) and the orientations of the loops that it makes alone.

The case of \(\text{CLE}_\kappa\) for \(\kappa \in (8/3, 8) \setminus \{4\}\) is similar to that of \(\text{SLE}_\kappa\). The reason for this is that one couples \(\text{CLE}_\kappa\) for \(\kappa \in (4, 8)\) with the GFF by coupling the whole exploration tree of \(\text{SLE}_\kappa(\kappa - 6)\) processes with the GFF [12, 11, 15]. The case that \(\kappa = 4\) is different because in this case the
conditional law of the GFF given the loops is given by a constant which is determined by the loop orientations. In particular, the loops are not marked by a special point so that there is no additional randomness involved here.

The Markov step used to prove Theorems 1.1–1.3 is also interesting to think about in the context of the GFF: While this operation only affects the small regions of the CLE picture, it does have a less localized influence for the corresponding GFF. This is because changing the manner in which the loops of a CLE are hooked up has the effect of moving the marked point along the component boundaries which in turn translates into changing the GFF heights along the loop boundaries. That is, our Markov step is a measure preserving transformation defined on GFF instances which leaves the CLE gasket fixed but makes a macroscopic change to the corresponding GFF instance because the heights are changed.

6.2. Quantum gravity perspective. It is natural to wonder whether techniques involving quantum gravity and mating of trees, as described e.g. in [5], could be used to give an alternate proof of Theorems 1.1–1.3. In this short subsection, we make some brief and informal remarks about how the operations described in this paper could potentially be understood and studied within that framework. The re-randomization procedure that we have described here also naturally fits into the quantum gravity framework developed in [5]. In particular, it is implicit in the construction of [5] that there is a quantum version of the “natural” measure on SLE_κ double points and intersections of CLE_κ loops. It is not difficult to see that if one picks a typical such point using this measure in either setting and then “zooms in,” the resulting limit is the same if one starts in either the SLE_κ double point or CLE_κ loop intersection settings. In fact, it can be described as a gluing of eight so-called quantum wedges which correspond to the four strands of path and the four regions which separate the path strands. The operation of resampling how the paths are hooked up has a natural interpretation in the quantum gravity perspective. Indeed, it is shown in [5] that an SLE_κ path or CLE_κ path for κ ∈ (4, 8) can be represented as a gluing of a pair of loop-trees which arise from a pair of independent (κ/4)-stable Lévy processes; see [5] Figure 1.6 and Figure 1.7. These loop-trees correspond to the components which are cut off on the left and right sides of the path. Regluing the paths in order to switch the direction of a pivotal point corresponds to natural operations that one can perform directly on the trees (hence Lévy processes) themselves, namely cutting the pair of trees to form new trees or grafting trees together.

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Appendix A. Bi-chordal resampling

The purpose of this appendix is to explain how the bi-chordal resampling arguments developed in [13] Section 4 can be extended to the setting in which the paths in question are allowed to intersect each other. We write this as an appendix as it can be read independently of the present paper and may also serve as a future reference for such resampling questions. First, in Section A.1 we will establish a general statement which is used in the proof of Lemma 3.3 in the case that κ ∈ (6, 8) and in Section A.2 a variant of this that is used in the proof of Lemma 3.3 in the case κ ∈ (4, 6).
A.1. Basic characterization.

**Theorem A.1.** Fix $\kappa > 0$, $\rho_1, \rho_2, \rho'_1, \rho'_2 > (-2) \lor (\kappa/2 - 4)$. Suppose that $D \subseteq \mathbb{C}$ is a Jordan domain and $x_1, x_2, y_1, y_2 \in \partial D$ are distinct and given in counterclockwise order. There is at most one probability measure on pairs of paths $\eta_L, \eta_R$ respectively connecting $x_1$ to $y_1$ and $x_2$ to $y_2$ such that the conditional law of $\eta_L$ given $\eta_R$ is independently that of an SLE$_\kappa(\rho_1; \rho_2)$ process in each of the components of $D \setminus \eta_R$ which are to the left of $\eta_R$ and the conditional law of $\eta_R$ given $\eta_L$ is independently that of an SLE$_\kappa(\rho'_1; \rho'_2)$ process in each of the components of $D \setminus \eta_L$ which are to the right of $\eta_L$.

We emphasize that we do not prove the existence of a probability measure which satisfies the hypotheses of Theorem A.1. However (and this is the case in the present paper’s CLE$_\kappa$ setting or in the imaginary geometry framework), the existence of such a measure is typically provided by ad-hoc constructions.

We note that it suffices to prove Theorem A.1 in the case that $\kappa \in (0, 4]$ because of SLE-duality considerations. More precisely, in the case that $\kappa > 4$ we can replace $\eta_L$ by its right boundary and $\eta_R$ by its left boundary because the conditional law of $\eta_L$ and $\eta_R$ in each case is known (see, e.g., [12]).

Throughout, we will make the concrete choice that $D = (0, 1) \times (0, \ell)$ for some fixed $\ell > 0$. We denote the left boundary segment $[0, i\ell]$ by $L$ and the right boundary segment by $R$. Let $K_L$ (resp. $K_R$) denote the set of all compact connected subsets of $\overline{D}$ that contain $L$ (resp. $R$). We endow $K_L, K_R$ with the Hausdorff distance between compact sets and consider them with their associated Borel $\sigma$-algebra.

We are now going to give a definition of a measurable family of elements of $K_L$ that we will call good, but we already note that if $K$ consists of the union of $L$ with a continuous curve from 0 to $i\ell$ in $\overline{D}$, then it will necessarily be such a good set.

When $K_1 \in K_L$, let us look at the connected components of $D \setminus K_1$ whose boundary contains a non-trivial interval of $R \setminus K_1$. We call $(O_j)$ this family of open sets, and we denote by $a_j$ and $b_j$ the corresponding extremities of this interval of $(R \setminus K_1) \cap \overline{O_j}$.

For each $n$, we also define the set $K_1^n$ which is obtained by taking the union of all closed dyadic squares of side-length $2^{-n}$ (i.e., with corners in $2^{-n}\mathbb{Z}^2$) that intersect $K_1$. For each $n$, the map $K_1 \mapsto K_1^n$ is easily shown to be measurable and it can take only finitely many values. Hence, any function of $K_1^n$ will be a measurable function of $K_1$. Furthermore, $K_1^n$ is also in $K_L$ and we can then define the family $(O_j^n)$ and the points $a_i^n$ and $b_i^n$ just as before.

**Definition A.2.** We say that a set $K_1$ is good (in $K_L$) if the following two conditions hold:

(i) For each $N > 0$, for each $n$, the number of $(O_j^n)$ with diameter greater than $1/N$ is finite and bounded uniformly in $n$.

(ii) For each $j$, the boundary of $O_j$ is a continuous curve.

Note that since each $O_j^n$ is contained in some $O_j$, and that each $O_j$ contains at least one $O_j^n$ of half its diameter when $n$ is sufficiently large, this implies also that for each $N$, the number of $O_j$ with diameter greater than $1/N$ is finite.

Let us give some further definitions: For each $O_j$ and each large $n$, let us denote by $O_j^n(i)$ the largest $O_j^n$ that is contained in $O_j$ (for instance, the one with largest diameter, breaking possible
ties using some deterministic rule) if it exists (and it always does provided $n$ is large enough). Let us write $\pi^n_j := a^n_{j,n(j)}$ and $\bar{b}^n_j := b^n_{j,n(j)}$ the corresponding boundary points.

A first remark is that the set of good sets is a measurable subset of $K_L$. Indeed, both (i) and (ii) can be expressed in terms (i.e., as countable unions of intersections of unions of intersections) of events involving finitely many sets $K^1_i$. This is clear for (i). To see that this is also the case for (ii), one can note that (ii) is equivalent to the fact that for each $j$, there exists parameterizations of the boundaries of $O^n_{j,n(j)}$ (which are continuous curves) such that this sequence of continuous curves is uniformly Cauchy as $n \to \infty$. This can be seen by considering the sequence of conformal maps $\phi^n_i$ which take $D$ to $O^n_{j,n(j)}$ with $-i, -1, i$ respectively taken to $\pi^n_j, (\pi^n_j + \bar{b}^n_j)/2, \bar{b}^n_j$ and noting that they are uniformly Cauchy if and only if (ii) holds.

For each given $K_1 \in K_L$, we now describe a procedure $\Phi$ to define a random element $K_2 \in K_R$: First, if $K_1$ is not good, we just set $K_2 = R$. If $K_1$ is good, then inside each of the sets $O_j$ (using the same notations as above), we then sample an independent SLE$_n(\rho'_1; \rho'_2)$ denoted by $\gamma_j$ from $a_j$ to $b_j$, with marked points that are both located at $a_j$ (one on each “side”). We then define the set $K_2 := R \cup (\cup_j \gamma_j)$. Note that this set is compact, connected and contains $R$, i.e., it is in $K_R$.

A first important observation is the following:

**Lemma A.3.** If we endow $K_L$ and $K_R$ with the Hausdorff metric, the previous procedure describes a Borel-measurable Markov kernel (in other words, the map $\Phi$ that associates to each $K_1$ the law of $K_2$ is measurable).

**Proof.** It suffices to show that the law of $K_2$ can be viewed as the weak limit as $n \to \infty$ of $\Phi(K^n_1)$ (which are therefore measurable with respect to $K_1$):

When $K_1$ is good, for each $n$, let us define $\gamma^n_j$ the corresponding SLE curves in $O^n_j$. For each $j$, condition (ii) shows that the law of $\gamma^n_{j,n(j)}$ converges to that of $\gamma_j$. Indeed, if we define the conformal map $\psi^n_j$ from $O_j$ onto $O^n_{j,n(j)}$ that maps $a_j, (a_j + b_j)/2, b_j$ onto $\pi^n_j, (\pi^n_j + \bar{b}^n_j)/2, \bar{b}^n_j$ respectively, then because $K_1$ is good, then $\psi^n_j$ extends continuously to the boundary (i.e. to the set of prime-ends) and converges uniformly to the identity map as $n \to \infty$.

Now we note that for each $\varepsilon > 0$, the number of connected components $O^n_j$ with diameter greater than $\varepsilon$ is equal to the number of connected component $O_j$ with diameter greater than $\varepsilon$ for all $n$ large enough (only the $O^n_{j,n(j)}$ will have diameter greater than $\varepsilon$ (which follows readily from condition (ii)).

From this, it follows readily that when $n$ is large enough, one can couple $K^n_2$ (defined just as $K_2$ but replacing $K_1$ by $K^n_1$) and $K_2$ so that the Hausdorff distance between the two is smaller than $2\varepsilon$ (as one just needs to control the SLE paths in the finitely many $O_j$’s). Hence, the law of $K_2$ can indeed be viewed as the weak limit of the law of $K^n_2$. \hfill $\square$

In a completely symmetric manner (with respect to the line $1/2 + i\mathbb{R}$), we can define for each $K_2 \in K_R$ a procedure to define a random set $K'_1 \in K_L$, just replacing $(\rho'_1; \rho'_2)$ with $(\rho_1; \rho_2)$. We denote by $\Psi$ the measurable Markovian kernel that associates to each $K_1 \in K_L$ the law of the random set $K'_1$ obtained iteratively by first choosing $K_2$ applying the first procedure, and then $K'_1$ (in a conditionally independent way, given $K_2$) using this second procedure. This kernel is Markovian as a composition of Markovian kernels. One then has the following result:

We now state the following proposition that immediately implies Theorem A.1.
Proposition A.4. There exists at most one probability measure $\pi$ on $K_L$ that is invariant under $\Psi$ and such that $\pi$ a.e. $K_1$ is the union of the range of a continuous curve and $L$.

This is useful in the present paper, because in the CLE/SLE settings that we work with, we are given a probability measure on $K_L$ that satisfies these properties.

Remark A.5. Stronger statements probably do hold, but the previous one is often sufficient (and it is the case in the present paper). For instance, one can probably show uniform exponential mixing (i.e., that starting from any two different configurations, the two chains started from these two configurations can be coupled so that they coincide before the $N$-th iteration step with a probability that is bounded by $\exp(-cN)$ for some constant $c$ that is independent of the initial configurations).

Recall some basic features of Markov kernels (see for instance Chapter 6 of [28]) that will be useful in the proof: The set of invariant probability measures is convex, and the extremal points in this convex set are exactly the ergodic invariant measures. As a consequence, two extremal ergodic invariant measures are either mutually singular or equal.

Proof of Proposition A.4. We are going to prove that there exists at most one probability measure $\pi$ on $K_L$ that is invariant under $\Psi$ and such that $\pi$ a.e. $K_1$ is good. Such a measure $\pi$ being necessarily a mixture of extremal invariant ergodic measures supported on good sets, it will suffice to see that there exists at most one extremal ergodic measure supported on good sets.

Let $\nu$, $\tilde{\nu}$ be two extremal ergodic measures supported on good sets. Let us choose $K_1$ and $\tilde{K}_1$ independently according to these two probability measures (on the same probability space) and then let us first apply (independently) the first step $\Phi$ of $\Psi$ to construct $K_2$ and $\tilde{K}_2$.

Note that for every good $K_1$ there exists $\delta(K_1) > 0$ such that with probability at least $\delta$, $K_2$ is a subset of the right-hand half $D_+ = (1/2, 1] \times [0, \ell]$ of the rectangle $D$ (recall that finitely many $O_j$’s have diameter at least $1/2$). Hence, with a random but positive conditional probability (given $K_1$, $\tilde{K}_1$), both $K_2$ and $\tilde{K}_2$ are subsets of $D_+$. But, then, conditionally on this event, it is possible (simply using absolute continuity of $\text{SLE}_{\infty}(\rho_1; \rho_2)$ processes defined in two different domains) to see that one can couple the second iteration step that constructs $K'_1$ and $\tilde{K}'_1$ in such a way that these two sets do coincide (and stay in the left-hand half of $D$) with positive probability. But since the laws of $K'_1$ and $\tilde{K}'_1$ are respectively equal to $\nu$ and $\tilde{\nu}$, this shows that these two measures are not singular, which implies (because extremal ergodic measures are either singular or equal) that they are equal.

A.2. Second variant. We now explain a closely related result, which will be derived using a variation of the argument used to prove Theorem A.1. Throughout, we suppose that $\kappa \in (8/3, 4)$ and $\kappa' = 16/\kappa \in (4, 6)$. As we mentioned earlier, this version is relevant for the proof of Lemma 3.3 in the case $\kappa' \in (4, 6)$.

Let us consider the same rectangle $D$. We denote by $T$ and $B$ its top and bottom sides. For $l \in (0, \ell)$, we also denote by $I_l$ the horizontal segment $[0, 1] \times \{l\}$.

When $\eta'$ is a continuous non-self-crossing and non-self-tracing path in $\overline{D}$ from $0$ to $i\ell$, we say that a connected component of $D \setminus \eta'$ is to the right of $\eta'$ if its boundary contains an open interval of $(T \cup R \cup B) \setminus \eta'$.

Suppose that we have a law on pairs $(\eta', \Gamma)$ where $\eta'$ is a continuous non-self-crossing path in $\overline{D}$ from $0$ to $i\ell$ and $\Gamma$ is a collection of loops in the components of $D \setminus \eta'$ (i.e., each loop is in the closure of one of these components) which are to the right of $\eta'$ which satisfy the following properties:
Given \( \eta' \), the conditional law of \( \Gamma \) is given independently by that of a BCLE_{\kappa}(\kappa' - 6) from 0 to \( i\ell \) with a single force point at the right-most point \( y \) of \( B \) so that no loop of \( \Gamma \) intersects both \( T \) and \( [0, y) \).

If we condition on the loops of \( \Gamma \) which intersect \( T \), then the conditional law of \( \eta' \) in the remaining domain is that of an SLE_{\kappa}(\kappa' - 6) from \( i\ell \) to 0 with a single force point at the right-most point \( y \) of \( T \) so that no loop of \( \Gamma \) intersects both \( B \) and \([i\ell, y)\).

Then, we see that the law of \( \eta' \) is invariant under two different operations:

- Sample \( \Gamma \) given \( \eta' \), keep only the loops that touch \( T \), and then resample \( \eta' \).
- Sample \( \Gamma \) given \( \eta' \), keep only the loops that touch \( B \), and then resample \( \eta' \).

Exactly as in the previous argument, one can see that these two resampling operations correspond to two Markovian kernels that we denote by \( \Psi_1 \) and \( \Psi_2 \) (one can view the paths in question as corresponding to compact sets, define good sets in a similar manner as before, and define the operation in a measurable way, and see that it coincides with the above description in the case where the set is a continuous path). Then:

**Proposition A.6.** There exists at most one probability measure \( \pi \) that is invariant under both \( \Psi_1 \) and \( \Psi_2 \) and that is supported on the set of continuous non-self-crossing curves from 0 to \( i\ell \) in \( \overline{D} \).

Most of the proof of this statement is almost identical to the previous arguments, except for the final resampling argument, that we now describe in more detail:

Suppose that \( \eta' \) and \( \tilde{\eta}' \) are two independent samples of two probability measures \( \pi \) and \( \tilde{\pi} \) that satisfy the conditions of the proposition. In order to prove that \( \pi = \tilde{\pi} \), it is sufficient to show that these two probability measure do not have disjoint support. For this we just need to show that by performing a resampling step corresponding to \( \Psi_1 \) and then a resampling step corresponding to \( \Psi_2 \), we can arrange so that the obtained paths coincide with positive probability. First, we do a resampling step corresponding to \( \Psi_1 \) as follows.

- Given \( \eta \) and \( \tilde{\eta} \), we sample \( \Gamma, \tilde{\Gamma} \) independently. Then there is a positive chance that no loops of \( \Gamma \) and \( \tilde{\Gamma} \) intersects both \( T \) and \( I_{3\ell/4} \). We call \( E' \) this event.
- We then resample \( \eta' \) and \( \tilde{\eta}' \) using the second part of the resampling step \( \Psi_1 \). We do this independently except when the event \( E' \) occurred. Let us denote \( \tau, \tilde{\tau} \) the respective hitting times of \( I_{\ell/2} \) by \( \eta' \) and \( \tilde{\eta}' \). By standard absolute continuity properties for SLE, when \( E' \) holds, we can couple \( \eta'_{|[0,\tau]} \) with \( \tilde{\eta}'_{|[0,\tilde{\tau}]} \) so that with positive probability, these two portions agree and do intersect \( \tilde{R} \) (so that they disconnect \( T \) from \( B \)) – we call \( E \) this event. So, we do couple them in this way.

We note that for any two continuous curves \( \eta' \) and \( \tilde{\eta}' \), the probability that in this resampling step corresponding to \( \Psi_1 \), we obtain that the probability of the event \( E \) is strictly positive. We then perform a second resampling step corresponding to \( \Psi_2 \): On the event \( E \), we can couple the loops of \( \Gamma, \tilde{\Gamma} \) that intersect \( B \) in such a way that they are identical (this is just because they are defined in identical domains). We can then resample \( \eta' \) and \( \tilde{\eta}' \) in the second step of \( \Psi_2 \) in such a way that they coincide.
Hence, we conclude that \((\Psi_2 \circ \Psi_1)(\pi) = \pi\) and \((\Psi_2 \circ \Psi_1)(\bar{\pi}) = \bar{\pi}\) are not mutually singular, which implies that \(\pi = \bar{\pi}\).

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