SEP ARA TING INV ARIANTS FOR 2 × 2 MA TRICES
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Abstract. A minimal separating set is found for the algebra of matrix invariants of
several 2 × 2 matrices over an infinite field of arbitrary characteristic.

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1. INTRODUCTION

1.1. Definitions. All vector spaces, algebras, and modules are over an infinite field \( F \)
of characteristic \( \text{char} F \neq 2 \), unless otherwise stated. By an algebra we always mean an
associative algebra.

To define the algebras of matrix invariants, we consider the polynomial algebra
\( R = R_{n,d} \equiv \mathbb{F}[x_i(k) \mid 1 \leq i, j \leq n, 1 \leq k \leq d] \)
together with \( n \times n \) generic matrices
\[
X_k = \begin{pmatrix}
x_{11}(k) & \cdots & x_{1n}(k) \\
\vdots & \ddots & \vdots \\
x_{n1}(k) & \cdots & x_{nn}(k)
\end{pmatrix}.
\]
Denote by \( \sigma_t(A) \) the \( t^{\text{th}} \) coefficient of the characteristic polynomial \( \chi_A \) of \( A \). As an example,
\( \text{tr}(A) = \sigma_1(A) \) and \( \det(A) = \sigma_n(A) \). The action of the general linear group \( GL(n) \) over \( R \)
is defined by the formula: \( g \cdot x_i(k) = (g^{-1} X_k g)_{ij} \), where \( (A)_{ij} \) stands for the \( (i,j)^{\text{th}} \) entry
of a matrix \( A \). The set of all elements of \( R \) that are stable with the respect to the given
action is called the algebra of matrix invariants \( R^{GL(n)} \) and this algebra is generated by
\( \sigma_t(b) \), where \( 1 \leq t \leq n \) and \( b \) ranges over all monomials in the generic matrices matrices
\( X_1, \ldots, X_d \) (see \([31], [26], [5]\)). Note that in characteristic zero case the algebra \( R^{GL(n)} \) is
generated by \( \text{tr}(b) \), where \( b \) as above. The ideal of relations between the generators of
\( R^{GL(n)} \) was described in \([28, 26, 34]\).

Denote by \( H = M(n) \oplus \cdots \oplus M(n) \) the direct sum of \( d \) copies of the space \( M(n) \) of all
matrices \( n \times n \) over \( \mathbb{F} \). The elements of \( R \) can be interpreted as polynomial functions from \( H \)
to \( \mathbb{F} \) as follows: \( x_i(k) \) sends \( u = (A_1, \ldots, A_d) \in H \) to \( (A_k)_{i,j} \). For a monomial \( c \in R \) denote
by \( \deg c \) its degree and by \( \text{md} c \) its multidegree, i.e., \( \text{md} c = (t_1, \ldots, t_d) \), where \( t_k \) is the
total degree of the monomial \( c \) in \( x_i(k) \), \( 1 \leq i, j \leq n \), and \( \deg c = t_1 + \cdots + t_d \). Similarly
we denote the degree and multidegree of a \( \mathbb{N} \)-homogeneous (\( \mathbb{N}^d \)-homogeneous, respectively)

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polynomial of $R$, where $\mathbb{N}$ stands for non-negative integers. Since $\text{deg} \sigma_i(X_{i_1} \cdots X_{i_k}) = ts$, the algebra $R^{GL(n)}$ has $\mathbb{N}$-grading by degrees and $\mathbb{N}^d$-grading by multidegrees.

In 2002 Derksen and Kemper [1] introduced the notion of separating invariants as a weaker concept than generating invariants. Given a subset $S$ of $R^{GL(n)}$, we say that elements $u, v$ of $H$ are separated by $S$ if exists an invariant $f \in S$ with $f(u) \neq f(v)$. If $u, v \in H$ are separated by $R^{GL(n)}$, then we simply say that they are separated. A subset $S \subset R^{GL(n)}$ of the invariant ring is called separating if for any $u, v$ from $H$ that are separated we have that they are separated by $S$. A subset $S \subset R^{GL(n)}$ is called $0$-separating if for any $u \in H$ such that $u$ and 0 are separated we have that $u$ and 0 are separated by $S$.

The main result of this paper is a description of a minimal (by inclusion) separating set for the algebra of matrix $GL(2)$-invariants for any $d$.

**Theorem 1.1.** The following set is a minimal separating set for the algebra of matrix invariants $R^{GL(2)}$ for every $d \geq 1$:

$$
\begin{align*}
\text{tr}(X_i), & \det(X_i), \ 1 \leq i \leq d, \\
\text{tr}(X_iX_j), & \ 1 \leq i < j \leq d, \\
\text{tr}(X_iX_jX_k), & \ 1 \leq i < j < k \leq d.
\end{align*}
$$

1.2. The known results for matrix invariants. A minimal generating set for the algebra of matrix invariants $R^{GL(2)}$ is known, namely:

$$
\begin{align*}
\text{tr}(X_i), \det(X_i), \ 1 \leq i \leq d, \text{tr}(X_{i_1} \cdots X_{i_k}), \ 1 \leq i_1 < \cdots < i_k \leq d,
\end{align*}
$$

where $k = 2, 3$ in case $\text{char} \mathbb{F} \neq 2$ and $k > 0$ in case $\text{char} \mathbb{F} = 2$ (see [27, 3]). It is easy to see that the set

$$
\begin{align*}
\text{tr}(X_i), \det(X_i), \ 1 \leq i \leq d, \sum_{i+j=k, i<j} \text{tr}(X_iX_j), 3 \leq k \leq 2d - 1
\end{align*}
$$

is a minimal (by inclusion) 0-separating set for $R^{GL(2)}$ (see also [33, 3]). By Hilbert Theorem, the algebra of invariants is a finitely generated module over the subalgebra generated by a 0-separated set. This result can be applied to construct a system of parameters (i.e. an algebraically independent set such that the algebra of invariants is finitely generated module over it) of the algebra of invariants. As an example, for $R^{GL(3)}_{3,3}$ a minimal 0-separating set is constructed, which is also a system of parameters (see [16]). Similar results are also known for $R^{GL(3)}_{3,2}$ and $R^{GL(4)}_{4,2}$ (see [32, 16]).

1.3. The known general results. The algebra of matrix invariants is a partial case of more general construction of an algebra of invariants. Namely, consider a linear algebraic group $G$ with a regular action over a finite dimensional vector space $V$. Extend this action to the action of $G$ over the coordinate ring $\mathbb{F}[V]$ by the natural way: $(g \cdot f)(v) = f(g^{-1}v)$ for all $g \in G$, $f \in \mathbb{F}[V]$ and $v \in V$. Then the algebra of invariants is the following set: $\mathbb{F}[V]^G = \{f \in \mathbb{F}[V] \mid g \cdot f = f\}$. It is well-known that there always exists a finite separating set (see [11], Theorem 2.3.15).

In [2] Domokos established that for a reductive group $G$ and $G$-modules $V, W$ a separating set $S$ for $\mathbb{F}[W \oplus V^m]^G$ can be obtained by the extension of any separating set
S_0 for F[W ⊕ V^{m_0}]^G, where m_0 = \dim V + 1 \leq m. Namely, this extension is defined as follows: a function f ∈ F[W ⊕ V^{m_0}] is send to f \circ \pi_{r_1,\ldots,r_{m_0}} : W \oplus V^m \to F, where 1 \leq r_1 < \cdots < r_{m_0} \leq m and \pi_{r_1,\ldots,r_{m_0}} : W \oplus V^m \to W \oplus V^{m_0} is the projection map sending (w, v_1, \ldots, v_m) to (w, v_{r_1}, \ldots, v_{r_{m_0}}). Note that m_0 does not depend on m. A similar result is not valid for sets of generators for matrix invariants (see Section 1.2).

For a linear algebraic group G denote by d_G ∈ \mathbb{N} \cup \{+\infty\} a minimal constant such that for each G-module V as above the invariants of F[V]^G are separated by elements of degree less or equal to d_G. Kohls and Kraft [14] proved that d_G is finite if and only if the group G is finite. Separating invariants for the finite groups were considered in [4, 6, 7, 8, 9, 10, 11, 12, 13, 15, 25, 29, 30].

2. Notations

This section contains some trivial remarks. If for A, B ∈ M(n) there exists g ∈ GL(n) such that gAg^{-1} = B, then we write A ∼ B. Denote by E_{ij} the matrix such that the (i, j)th entry is equal to one and the rest of entries are zeros. The diagonal matrix with elements a_1, \ldots, a_n we denote by diag(a_1, \ldots, a_n). The proof of the next lemma is straightforward.

**Lemma 2.1.** Assume that the field F is algebraically closed and A_1, A_2 ∈ M(2), where \[A_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}.\] Then

(a) for A_1 = diag(\alpha_1, \beta_1) there exists g ∈ GL(2) such that gA_1g^{-1} = diag(\beta_1, \alpha_1) and \[gA_2g^{-1} = \begin{pmatrix} \delta_2 & \gamma_2 \\ \beta_2 & \alpha_2 \end{pmatrix};\]

(b) if \gamma_2 ≠ 0 or \alpha_2 ≠ \delta_2 and \[A_1 = \begin{pmatrix} \alpha_1 & 1 \\ 0 & \alpha_1 \end{pmatrix},\]

then there exists g ∈ GL(2) such that gA_1g^{-1} = A_1 and \[gA_2g^{-1} = \begin{pmatrix} c & 0 \\ \gamma_2 & d \end{pmatrix}\]

for some c, d.

3. The case of three matrices

Denote the set from the formulation of Theorem 1.1 by S(d).

**Lemma 3.1.** If d ≤ 3, then the set S(d) is a minimal separating set for R^{GL(2)}.

**Proof.** Assume that d ∈ \{1, 2, 3\}. Since in this case S(d) is a (minimal) generating set for R^{GL(2)} (see Section 1.2), we have that S(d) is a separating set. It remains to show that S(d) is minimal.

Assume that d = 1. Then tr(X_1) does not separate matrices diag(1, -1), 0 and det(X_1) does not separate matrices diag(1, 0) and 0.
Assume that $d = 2$. Obviously, it is enough to prove that the set $S(2)$ without $\text{tr}(X_1X_2)$ is not separating. Consider $u = (E_{12}, E_{21}) \in H$. Then $\text{tr}(X_1X_2)$ separates $u$ and 0, but the rest of elements of $S(2)$ do not separate them.

Assume that $d = 3$. Obviously, it is enough to prove that the set $S(3)$ without $\text{tr}(X_1X_2X_3)$ is not separating. Consider $u = (E_{11}, E_{21}, E_{12})$ and $v = (E_{22}, E_{21}, E_{12})$ from $H$. Then $\text{tr}(X_1X_2X_3)$ separates $u$ and $v$, but the rest of elements of $S(3)$ do not separate them. \hfill \square

Note that Section 1.2 implies that $S(1)$ and $S(2)$ are minimal as 0-separating sets but $S(3)$ is not a minimal as 0-separating set.

4. THE CASE OF FOUR MATRICES

It is easy to see that if the assertion of Proposition 4.1 (see below) is valid over the algebraic closure of the field $\mathbb{F}$, then it is also valid over the field $\overline{\mathbb{F}}$. Therefore, in this section we assume that the field $\mathbb{F}$ is algebraically closed.

**Proposition 4.1.** Assume that $d = 4$. Consider $u = (A_1, A_2, A_3, A_4)$ and $v = (B_1, B_2, B_3, B_4)$ from $H$ such that for every $f \in S(4)$ we have $f(u) = f(v)$. Then for $h = \text{tr}(X_1 \cdots X_4)$ we have $h(u) = h(v)$.

We split the proof of the proposition into several lemmas. By the formulation of the proposition,

(T1) $\text{tr}(A_i) = \text{tr}(B_i)$, $1 \leq i \leq 4$;

(D1) $\det(A_i) = \det(B_i)$, $1 \leq i \leq 4$;

(Tij) $\text{tr}(A_iA_j) = \text{tr}(B_iB_j)$, $1 \leq i < j \leq 4$;

(Tijk) $\text{tr}(A_iA_jA_k) = \text{tr}(B_iB_jB_k)$, $1 \leq i < j < k \leq 4$.

We have to show that

(Q) $\text{tr}(A_1 \cdots A_4) = \text{tr}(B_1 \cdots B_4)$.

If (T) $f = h$ is one of the above equalities, then we write $T$ for $f - h$. As an example, $T_1 = \text{tr}(A_1) - \text{tr}(B_1)$.

We denote the entries of the matrices $A_1, \ldots, A_4$ as follows:

$$A_1 = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_5 & a_6 \\ a_7 & a_8 \end{pmatrix}, \quad A_3 = \begin{pmatrix} a_9 & a_{10} \\ a_{11} & a_{12} \end{pmatrix}, \quad A_4 = \begin{pmatrix} a_{13} & a_{14} \\ a_{15} & a_{16} \end{pmatrix}.$$

Similarly, substituting $a_i \to b_i$ for all $i$ we denote entries of the matrices $B_1, \ldots, B_4$ by $b_1, \ldots, b_{16}$.

**Remark 4.2.** (1) Since elements of $S(4)$ are invariants with the respect to the action of $GL(2)$ over $H$ diagonally by conjugation and the field is algebraically closed, we can assume that either $A_1 = \text{diag}(\alpha, \beta)$ or

$$A_1 = \begin{pmatrix} \gamma & 1 \\ 0 & \gamma \end{pmatrix}$$

for some $\alpha, \beta, \gamma$ from $\mathbb{F}$. 
(2) By part (b) of Lemma 4.1, we can assume that in the second case of part (1) we have that either \((A_2)_{12} = 0\) or \((A_2)_{21} = 0\) and \((A_2)_{11} = (A_2)_{22}\).

**Remark 4.3.** Denote by \(G(d)\) the minimal generating set from Section 1.2; in particular, \(G(3) = S(3)\). Consider \(u, v \in H\) such that \(u, v\) are not separated by elements of \(G(d)\). Then \(u\) and \(v\) are not separated by any invariant \(f\) of degree \(d\), since \(f\) is a polynomial in elements of \(G(d)\).

Remark 4.3 immediately implies the next remark:

**Remark 4.4.** Assume that \(d = 4\) and \(u, v \in H\) are not separated by elements of \(S(4)\). Then \(u, v\) are not separated by invariants \(\text{tr}(X_1 X_2 X_3 X_k)\) for any pairwise different \(1 \leq i, j, k \leq 4\). If we also have that \(u, v\) are not separated by \(\text{tr}(X_1 \cdots X_4)\), then they are not separated by the invariant \(\text{tr}(X_{\sigma(1)} \cdots X_{\sigma(4)})\) for any permutation \(\sigma \in S_4\).

**Lemma 4.5.** Assume that the condition of Proposition 4.1 holds and \(A_1 = 0\). Then the equality \((Q)\) holds.

**Proof.** By part (1) of Remark 4.2 we can assume that \(a_7 = 0\). Similarly, we can assume that \(b_3 = 0\). Considering equalities \((T_1)–(T_4)\) we obtain that

\[
\begin{align*}
b_4 &= -b_1, \quad b_8 = a_5 + a_8 - b_5, \\
b_{12} &= a_9 + a_{12} - b_9, \quad b_{16} = a_{13} + a_{16} - b_{13},
\end{align*}
\]

respectively. Equality \((D_1)\) implies that \(b_1 = 0\). In the case of \(b_2 = 0\), we have that \((Q)\) holds. Then without loss of generality we can assume that \(b_2 \neq 0\). Equalities \((T_{12}), (T_{13})\) and \((T_{14})\), respectively, imply that \(b_7 = b_{11} = b_{15} = 0\), respectively. Then \((Q)\) holds. \(\square\)

**Lemma 4.6.** Assume that the condition of Proposition 4.1 holds, \(A_1\) is scalar and \(B_1\) is diagonal. Then the equality \((Q)\) holds.

**Proof.** We have \(A_1 = \text{diag}(a_1, a_1)\) and \(B_1 = \text{diag}(b_1, b_1)\). Considering equalities \((T_1)–(T_4)\) we obtain that

\[
\begin{align*}
b_4 &= 2a_1 - b_1, \quad b_8 = a_5 + a_8 - b_5, \\
b_{12} &= a_9 + a_{12} - b_9, \quad b_{16} = a_{13} + a_{16} - b_{13},
\end{align*}
\]

respectively. Hence \((D_1)\) implies \(b_1 = a_1\) and the matrix \(B_1 = A_1\) is scalar. The equality \(Q = a_1 T_{234}\) concludes the proof. \(\square\)

**Lemma 4.7.** Assume that the condition of Proposition 4.1 holds and \(A_1, B_1\) are diagonal. Then the equality \((Q)\) holds.

**Proof.** Applying Lemma 4.6 we can assume that \(A_1\) and \(B_1\) are not scalars. Denote \(A_1 = \text{diag}(a_1, a_4)\), \(B_1 = \text{diag}(b_1, b_4)\), where \(a_1 \neq a_4\) and \(b_1 \neq b_4\). Equalities \((T_1)–(T_4)\) imply that

\[
\begin{align*}
b_4 &= a_1 + a_4 - b_1, \quad b_8 = a_5 + a_8 - b_5, \\
b_{12} &= a_9 + a_{12} - b_9, \quad b_{16} = a_{13} + a_{16} - b_{13},
\end{align*}
\]

respectively. We consider several possibilities for entries of the matrices.
(1) Assume \( b_1 = a_1 \). Since \( a_1 \neq a_4 \), equalities \((T_{12}), (T_{13}), (T_{14})\) imply that
\[
b_5 = a_5, \quad b_9 = a_9 \quad \text{and} \quad b_{13} = a_{13},
\]
respectively.

(1.1) Let \( b_7 = 0 \). It follows from equality \((D_2)\) that \( a_6 = 0 \) or \( a_7 = 0 \). Since \( A_1 \) is diagonal, we can apply part (a) of Lemma 2.1 (see also part (1) of Remark 4.2) here we use the reference to part (1) of Remark 4.2 just to show how to apply part (a) of Lemma 2.1. Namely, all elements of \( S \) how to apply part (a) of Lemma 2.1. Therefore, all elements of \( S \), then we can apply Lemma 2.1 to the pair \((A_1, A_2)\) and assume that \( a_7 = 0 \). Since \( a_1 \neq a_4 \) the following equality holds:
\[
Q = \frac{1}{a_1 - a_4} \left( (a_1 a_5 - a_4 a_8) T_{134} - a_1 a_4 (a_5 - a_8) T_{34} \right) + a_{13} T_{123} + a_{12} T_{124}.
\]
Hence \((Q)\) holds.

(1.2) Let \( b_7 \neq 0 \). Thus equalities \((T_{23}), (T_{24}), (D_2)\), respectively, imply that
\[
\begin{align*}
    b_{10} &= (a_6 a_{11} + a_7 a_{10} - b_6 b_{11})/b_7, \\
    b_{14} &= (a_6 a_{15} + a_7 a_{14} - b_6 b_{15})/b_7, \\
    b_6 &= a_6 a_7/b_7,
\end{align*}
\]
respectively. In case \( a_6 = 0 \) the equality
\[
Q = a_4 T_{234} + a_5 T_{134} - a_4 a_5 T_{34}
\]
completes the proof. Thus we assume that \( a_6 \neq 0 \). Since \( a_1 \neq a_4 \), equalities \((T_{123})\) and \((T_{124})\), respectively, imply that
\[
    a_{11} = a_7 b_{11}/b_7 \quad \text{and} \quad a_{15} = a_7 b_{15}/b_7,
\]
respectively. Thus \((Q)\) holds.

(2) Assume \( b_1 \neq a_1 \). Then it follows from \((D_1)\) that \( b_1 = a_4 \). Since \( a_1 \neq a_4 \), we equalities \((T_{12}), (T_{13})\) and \((T_{14})\), respectively, imply that
\[
    b_5 = a_8, \quad b_9 = a_{12} \quad \text{and} \quad b_{13} = a_{16},
\]
respectively.

(2.1) Let \( b_7 = 0 \). It follows from equality \((D_2)\) that \( a_6 = 0 \) or \( a_7 = 0 \). Since \( A_1 \) is diagonal, we can apply part (a) of Lemma 2.1 to the pair \((A_1, A_2)\) and assume that \( a_6 = 0 \). Thus
\[
Q = a_4 T_{234} + a_5 T_{134} - a_4 a_5 T_{34},
\]
i.e., \((Q)\) holds.

(2.2) Let \( b_7 \neq 0 \). Then equalities \((D_2), (T_{23})\) and \((T_{24})\), respectively, imply that
\[
\begin{align*}
    b_6 &= a_6 a_7/b_7, \\
    b_{10} &= (a_6 a_{11} + a_7 a_{10} - b_6 b_{11})/b_7, \\
    b_{14} &= (a_6 a_{15} + a_7 a_{14} - b_6 b_{15})/b_7,
\end{align*}
\]
respectively. If \( a_7 = 0 \), then he equality
\[
Q = a_4 T_{234} + a_5 T_{134} - a_4 a_5 T_{34},
\]
completes the proof. Thus we assume that \( a_7 \neq 0 \). Since \( a_1 \neq a_4 \), the following equalities follow from \((T_{123})\) and \((T_{124})\), respectively:

\[
a_{10} = a_6 b_{11}/b_7 \quad \text{and} \quad a_{14} = a_6 b_{15}/b_7.
\]

Thus the equality \((Q)\) is valid.

□

**Lemma 4.8.** Assume that the condition of Proposition 4.1 holds,

\[
A_1 = \begin{pmatrix} a_1 & 1 \\ 0 & a_1 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} b_1 & 0 \\ 0 & b_4 \end{pmatrix}.
\]

Then the equality \((Q)\) holds.

**Proof.** Applying part (b) of Lemma 2.1 to the pair \((A_1, A_2)\) we can assume that we have one of the following two cases:

(a) \( a_6 = 0 \);

(b) \( a_6 \neq 0, a_7 = 0, a_8 = a_5 \).

Equalities \((T_1)\)–\((T_4)\) imply that in both case

\[
b_4 = 2a_1 - b_1, \quad b_8 = a_5 + a_8 - b_5,
\]

\[
b_{12} = a_9 + a_{12} - b_9, \quad b_{16} = a_{13} + a_{16} - b_{13},
\]

respectively. Therefore, the equality \( b_1 = a_1 \) follows from \((D_1)\). Applying equalities \((T_{12})\), \((T_{13})\), \((T_{14})\) we obtain that \( a_7 = a_{11} = a_{15} = 0 \). In case (a) the matrix \( A_2 \) is diagonal and Lemma 4.7 together with Remark 4.4 concludes the proof.

Assume that case (b) holds. Since \( B_1 \) is scalar, applying part (1) of Remark 4.2 to \( B_2 \), we can assume that \( b_7 = 0 \). The same reasoning as above imply that the entries of “new” matrices \( B_2, B_3, B_4 \) satisfy the same relations as the entries of “old” matrices \( B_2, B_3, B_4 \). By equality \((D_2)\) we have \( b_5 = a_5 \). Hence

\[
Q = a_1 a_5 T_{34} + a_1 b_{13} T_{23} + a_1 (a_9 + a_{12} - b_9) T_{24},
\]

i.e., \((Q)\) holds.

□

**Lemma 4.9.** Assume that the condition of Proposition 4.1 holds for

\[
A_1 = \begin{pmatrix} a_1 & 1 \\ 0 & a_1 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} b_1 & 1 \\ 0 & b_1 \end{pmatrix},
\]

and \( \text{char} \, \mathbb{F} = 2 \). Then the equality \((Q)\) holds.

**Proof.** By Lemma 4.8 together with Remark 4.4 we can assume that for any \( i \) the matrix \( A_i \) is not diagonal and for any \( j \) the matrix \( B_j \) is not diagonal.

Applying part (b) of Lemma 2.1 to the pair \((A_1, A_2)\) we can assume that one of the following case holds:

1. \( a_6 = 0 \),
2. \( a_6 \neq 0, a_7 = 0, a_8 = a_5 \).
In both cases equalities \((T_2)–(T_4)\) imply that
\[
b_8 = a_5 + a_8 - b_5, \\
b_{12} = a_9 + a_{12} - b_9, \\
b_{16} = a_{13} + a_{16} - b_{13},
\]
respectively. Since \(\text{char } \mathbb{F} = 2\), then the equality \(b_1 = a_1\) follows from \((D_1)\). Applying equalities \((T_{12})\), \((T_{13})\) and \((T_{14})\), respectively, we obtain
\[
b_7 = a_7, \\
b_{11} = a_{11}, \\
b_{15} = a_{15},
\]
respectively.

Assume that we have case (1), i.e., \(a_6 = 0\). Since the matrix \(A_2\) is not diagonal, then \(a_7 \neq 0\). Then equalities \((T_{23})\) and \((T_{24})\), respectively, imply that
\[
b_{10} = \frac{1}{a_7}(a_5b_9 + a_8(b_9 - a_9) + a_9b_5 - a_{11}b_6 + a_{12}(b_5 - a_5) - 2b_5b_9) + a_{10}, \\
b_{14} = \frac{1}{a_7}(a_5b_{13} + a_8(b_{13} - a_9) + a_{13}b_5 - a_{15}b_6 + a_{16}(b_5 - a_5) - 2b_{13}b_5) + a_{14},
\]
respectively. Therefore, it follows from \((T_{123})\) and \((T_{124})\) that
\[
b_9 = \frac{1}{a_7}a_11(b_5 - a_5) + a_9, \\
b_{13} = \frac{1}{a_7}a_{15}(b_5 - a_5) + a_{13},
\]
respectively. If follows from equality \((D_2)\) that \(b_6 = (a_5 - b_5)(-a_8 + b_5))/a_7\). Thus equality \((Q)\) is valid.

Assume that we have case (2). Equality \((D_2)\) implies that \(b_5 = a_5\).

Assume that \(a_6 \neq b_6\). Then equalities \((T_{23})\) and \((T_{24})\) imply that \(a_{11} = 0\) and \(a_{15} = 0\), respectively. The equality \(Q = a_1a_5T_{34}\), i.e., \((Q)\) is valid.

Finally, in case \(a_6 = b_6\) the equality
\[
Q = (a_5 + a_1a_6)T_{134} - a_7^2a_6T_{34}
\]
concludes the proof. \(\square\)

**Proof of Proposition 4.1.** If \(\text{char } \mathbb{F} \) is not two, then the fact that \(S(4)\) is a generating set for \(R_{GL(2)}\) concludes the proof.

Assume that \(\text{char } \mathbb{F} = 2\). By Lemma 4.3, we can assume that for every \(i\) the matrix \(A_i\) is non-zero as well as the matrix \(B_i\). Lemmas 4.7 4.8 4.9 together with part (1) of Remark 4.2 conclude the proof. \(\square\)

5. The proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1. Assume \(d \geq 4\). Since \(S(3)\) is a minimal separating set in case \(d = 3\) (see Lemma 3.1), we have that \(S(d)\) does not contain a proper subset which is separating. On the other hand, in case \(\text{char } \mathbb{F} \neq 2\) the set \(S(d)\) generates the algebra \(R_{GL(2)}\), thus \(S(d)\) is separating.

Assume that \(\text{char } \mathbb{F} = 2\) and \(u = (A_1, \ldots, A_d), v = (B_1, \ldots, B_d)\) are not separated by \(S(d)\). Proposition 4.1 together with the description of the generating set for \(R_{GL(2)}\) from Section 1.2 implies that \(u, v\) are not separated by any invariant of degree four. This fact allows us to apply Proposition 4.1 to \((A_1, A_2, A_3, A_4A_5)\) and \((B_1, B_2, B_3, B_4B_5)\) and obtain that \(u, v\) are not separated by \(\text{tr}(X_1 \cdots X_5)\). The description of the generating set for \(R_{GL(2)}\) implies that \(u, v\) are not separated by any invariant of degree five. Repeating
this reasoning we obtain that $u, v$ are not separated by any invariant of degree $d$. Thus $u, v$ are not separated. Hence the set $S(d)$ is separating and the theorem is proven.

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