Regularization of self inductance

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Abstract

We introduce several methods to define the self-inductance of a single loop as the regularization of divergent integrals which we obtain by applying Neumann (or Weber) formula for the mutual inductance of a pair of loops to the case when two loops are identical.

Keywords: self-inductance, Neumann’s formula, knot

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1 Introduction

Inductance is an important notion in classical electrodynamics. Suppose two closed circuits \(\Gamma_1\) and \(\Gamma_2\), mutually disjoint, carry currents \(I_1\) and \(I_2\) respectively. The magnetic field energy \(U_{12}\) of the system is given by

\[
U_{12} = I_1 I_2 L_{12},
\]

where the coefficient \(L_{12}\) is called the mutual inductance. According to Neumann formula (\[N\]), the mutual inductance can be expressed by

\[
L_N(\Gamma_1, \Gamma_2) := \frac{\mu_0}{4\pi} \int_{\Gamma_1} \int_{\Gamma_2} \frac{dx_1 \cdot dx_2}{|x_1 - x_2|},
\]

(1.1)

where \(\mu_0 = 4\pi \times 10^{-7}\) H/m is the permeability of free space. There is another expression of the mutual inductance, Weber’s formula; by

\[
L_W(\Gamma_1, \Gamma_2) := \frac{\mu_0}{4\pi} \int_{\Gamma_1} \int_{\Gamma_2} \frac{\hat{r}_{12} \cdot dx_1 (\hat{r}_{12} \cdot dx_2)}{|x_1 - x_2|},
\]

(1.2)

where \(r_{12} := x_2 - x_1\) and \(\hat{r}_{12} := r_{12}/|r_{12}|\). It is known that these two expressions are equivalent. The reader is referred to [Da] p.230 for the proof. The equivalence also follows from \[2.1\].

If we put \(\Gamma_1 = \Gamma_2\) in (1.1) (or (1.2)) the integral diverges in logarithmic order because of the contribution of a neighbourhood of the diagonal set. There have been some studies to avoid this divergence difficulty. Bueno and Assis (\[BA\]) used higher dimensional objects, i.e., a surface or a 3-dimensional wire thickened around a core loop \(\Gamma\) instead of \(\Gamma\) itself. They also showed that self-inductances obtained from Neumann formula and Weber formula are equivalent. Dengler (\[D\]) replace a neighbourhood of a point on the loop by a short straight wire segment to avoid the divergence of self-energy.

In this article, we propose a new definition of the self-inductance of a single loop by regularizing \(L_N(\Gamma, \Gamma)\) (and \(L_W(\Gamma, \Gamma)\)) without approximating the loop by higher dimensional objects

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To avoid the divergence of self-energy, we use two methods studied in the theory of generalized functions, Hadamard regularization and regularization via analytic continuation (see, for example, [GS] or [Z]). The same methods have been used to define the Möbius energy of a knot by regularizing \( \int_{\Gamma} \int_{\Gamma} |x - y|^{-2} \, dx \, dy \) in [O] and [B] respectively, and in general, the Riesz energy of a manifold \( M \) by regularizing \( \int_{M} \int_{M} |x - y|^\alpha \, dx \, dy \) in [OS2].

There is no difference between making a choice of two methods, but unlike in the result by Bueno and Assis, the difference between making a choice of Neumann and Weber formulae is equal to \( (\mu_0/2\pi) \) times the length of the loop.

We then show that our regularized self-inductance coincides with the regularization of the mutual inductance of a pair of close parallel loops as the distance between them tends to 0 (up to addition by a constant depending on the length of the loop) (Theorem 2.7). The same method has been used in [OS1] (see Subsection 3.2).

Finally we study regularized self-inductance of solenoids (i.e. coils that wind cylinders) (Theorem 2.8). If we expand it in a series in the number \( n \) of turns per unit length, the dominant term is of the order of \( n^2 \), and the coefficient does not depend on the choice of Neumann or Weber formula to start with. This coefficient, when the length \( \ell \) of the cylinder goes to infinity, is asymptotic to \( \mu_0\pi r^2 \) times \( \ell \) plus constant. Thus, by taking the asymptotics as \( n \) and \( \ell \) go to infinity, we can deduce that the dominant term of self-inductance of a solenoid is equal to \( \mu_0\pi r^2 n^2 \ell \) which fits a well-known formula, which implies the validity of our definition of regularized self-inductance.

2 Regularization of self-inductance

2.1 Regularization with a single loop

We start with explaining the idea of Hadamard regularization of a divergent integral in a general setting. Take an “\( \varepsilon \)-neighbourhood” of the set where the integrand blows up, restrict the integration to the complement of it, expand the result in a Laurent series in \( \varepsilon \) (possibly with a log term or terms with non-integer powers), and take the constant term. The constant term is called \textit{Hadamard's finite part} of the integral. The terms with negative powers (and a log term if exists) are called the \textit{counter terms}.

Let \( \Gamma \) be a smooth \( C^1 \) simple loop in \( \mathbb{R}^3 \) with length \( L \) parametrized by \( \gamma(s) \) by the arc-length.

**Theorem 2.1** The self-inductance can be regularized in the following way.

1. **Hadamard regularization** of \( L_N(\Gamma, \Gamma) \) and \( L_W(\Gamma, \Gamma) \) can be carried out as follows. Let \( \Delta_\varepsilon \) be an “\( \varepsilon \)-neighbourhood” of the diagonal set with respect to the distance in \( \mathbb{R}^3 \):

\[
\Delta_\varepsilon := \{(x_1, x_2) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x_1 - x_2| \leq \varepsilon \}.
\]

There exist the limits

\[
H_N(\Gamma) := \lim_{\varepsilon \to 0^+} \left( \frac{\mu_0}{4\pi} \int_{(\Gamma \times \Gamma) \setminus \Delta_\varepsilon} \frac{dx_1 \cdot dx_2}{|x_1 - x_2|} + \frac{\mu_0 L}{2\pi} \log \frac{2\pi}{\varepsilon} \right), \tag{2.1}
\]

\[
H_W(\Gamma) := \lim_{\varepsilon \to 0^+} \left( \frac{\mu_0}{4\pi} \int_{(\Gamma \times \Gamma) \setminus \Delta_\varepsilon} \frac{\hat{r}_{12} \cdot dx_1)(\hat{r}_{12} \cdot dx_2)}{|x_1 - x_2|} + \frac{\mu_0 L}{2\pi} \log \frac{2\pi}{\varepsilon} \right). \tag{2.2}
\]

\[\text{We assume the smoothness for the sake of simplicity. In fact, } \Gamma \text{ being } C^1 \text{ is enough.}\]
(2) Fix the loop $\Gamma$ and consider

$$F_N(z) := \frac{\mu_0}{4\pi} \int_{\Gamma \times \Gamma} |x_1 - x_2|^2 d^2x_1 \cdot d^2x_2,$$

$$F_W(z) := \frac{\mu_0}{4\pi} \int_{\Gamma \times \Gamma} |x_1 - x_2|^2 (\hat{r}_{12} \cdot dx_1)(\hat{r}_{12} \cdot dx_2)$$

as functions of a complex variable $z$. Then both $F_N(z)$ and $F_W(z)$ are well-defined and holomorphic on $\{z \in \mathbb{C} : \Re z > -1\}$. The domains of $F_N(z)$ and $F_W(z)$ can be extended to the whole complex plane $\mathbb{C}$ by analytic continuation to make meromorphic functions with possible simple poles at negative odd integers. Let them be denoted by the same symbols $F_N(z)$ and $F_W(z)$. The first residues are given by

$$\text{Res}(F_N, -1) = \text{Res}(F_W, -1) = \frac{\mu_0 \mathcal{L}}{2\pi}.$$ 

There exist the limits

$$A_N(\Gamma) := \lim_{z \to -1} \left( F_N(z) - \frac{\mu_0 \mathcal{L}}{2\pi(z + 1)} \right),$$

$$A_W(\Gamma) := \lim_{z \to -1} \left( F_W(z) - \frac{\mu_0 \mathcal{L}}{2\pi(z + 1)} \right).$$

(3) In each case starting with Neumann formula or Weber formula, there is no difference between making a choice of two methods of regularization, namely, $H_N(\Gamma) = A_N(\Gamma)$ and $H_W(\Gamma) = A_W(\Gamma)$.

(4) The difference between making a choice of Neumann formula and Weber formula is equal to $(\mu_0/2\pi)$ times the length of the loop, namely,

$$H_W(\Gamma) = H_N(\Gamma) + \frac{\mu_0 \mathcal{L}}{2\pi}.$$ 

**Definition 2.2** Let us call $H_N(\Gamma) = A_N(\Gamma)$ and $H_W(\Gamma) = A_W(\Gamma)$ the regularized self-inductances of $\Gamma$ in the sense of Neumann and Weber respectively.

The proof of Theorem 2.1 is divided in several steps.

Assertions (1), (2) and (3) follow from the argument in Section 3 of [OS2] with slight modification.

Let $t_1 x_j$ be the unit tangent vector to $\Gamma$ at $x_j$ and $\theta_j (j = 1, 2)$ be the angle between $t_1 x_j$ and $r_{12}$. We remark that the numerator of integrand of (2.2) can be expressed as

$$(\hat{r}_{12} \cdot dx_1)(\hat{r}_{12} \cdot dx_2) = \cos \theta_1 \cos \theta_2 dx_1 dx_2.$$ 

Let us give the relation between the arc-length parameter and the chord length (the distance in $\mathbb{R}^3$) of a nearby point from a fixed point on the knot explicitly.
Lemma 2.3 Let \( x_1 \) be a point on \( \Gamma \). Suppose a nearby point \( x_2 \) on \( \Gamma \) is expressed by the arc-length parameter \( s \) from \( x_1 \). Namely, \( x_1 = \gamma(s_1) \) for some \( s_1 \) and \( x_2 = \gamma(s_1 + s) \) \((-\mathcal{L}/2 < s < \mathcal{L}/2\)). Let \( t \) be the chord length (the distance in \( \mathbb{R}^3 \)) between \( x_1 \) and \( x_2 \). Then\(^2\)

\[
t = |s| \left( 1 - \frac{\kappa^2}{24} s^2 - \frac{\kappa \kappa'}{24} s^3 + O(s^4) \right), \tag{2.5}
\]

\[
s = \begin{cases}
  t + \frac{\kappa^2}{24} t^3 + \frac{\kappa \kappa'}{24} t^4 + O(t^5) & (s \geq 0), \\
  -t - \frac{\kappa^2}{24} t^3 + \frac{\kappa \kappa'}{24} t^4 + O(t^5) & (s < 0),
\end{cases} \tag{2.6}
\]

where \( \kappa \) and \( \kappa' \) are the curvature and its derivative with respect to \( s \) at point \( x_1 \), i.e., at \( s = s_1 \). Furthermore the numerators of integrands of (2.1) and (2.2) can be estimated by

\[
\frac{d}{d\gamma} \left( \frac{d}{d\gamma} \right)^{s} \frac{d}{d\gamma} \frac{d}{d\gamma} = 1 - \frac{\kappa^2}{4} s^2 + O(s^3) = 1 - \frac{\kappa^2}{4} t^2 + O(t^3). \tag{2.8}
\]

Proof. Computation using Frenet-Serret’s formula implies that \( \gamma(s_1 + s) \) can be expressed with respect to the Frenet-Serret frame at \( \gamma(s_1) \) as

\[
\xi = s - \frac{\kappa^2}{6} s^3 - \frac{\kappa \kappa'}{8} s^4 + O(s^5)
\]

\[
\eta = \frac{\kappa}{2} s^2 + \frac{\kappa'}{6} s^3 - \frac{\kappa^2 + \kappa \tau - \kappa''}{24} s^4 + O(s^5)
\]

\[
\zeta = \frac{\kappa \tau}{6} s^3 + \frac{2 \kappa' \tau + \kappa''}{24} s^4 + O(s^5),
\]

where \( \tau \) means the torsion. Then, (2.5) can be obtained by substituting the above series expansion to

\[
t = |\gamma(s_1 + s) - \gamma(s)| = \left( |\gamma(s_1 + s) - \gamma(s)| \cdot (\gamma(s_1 + s) - \gamma(s)) \right)^{\frac{1}{2}}.
\]

With this frame \( \hat{r}_{12} \) is expressed as

\[
\hat{r}_{12} = \left( 1 - \frac{\kappa^2}{8} s^2, \frac{\kappa}{2} s + \frac{\kappa'}{6} s^2, \frac{\kappa \tau}{6} s^2 \right) + O(s^3).
\]

Next lemma is needed for the proof of (2).

Let \( B_t(x) \) denote the 3-ball with center \( x \) and radius \( t \). Let \( d \) be the diameter of \( \Gamma \).

\(^2\)The last terms of (2.9), (2.10) and (2.11) are not necessary in this article. We put them here to illustrate \( \psi_{x_1}^{(4)}(0) = 0 \) in (2.10) in Lemma 2.4.
Lemma 2.4 Fix \( x_1 \) on \( \Gamma \) and put
\[
\psi_{x_1}(t) := \int_{\Gamma \cap B_\epsilon(x_1)} t_{x_1} \cdot t_{x_2} \, dx_2 \quad (0 \leq t \leq d).
\]
Then \( \psi_{x_1}(t) \) extends to a smooth function on \((-d, d)\), denoted by the same symbol, with \( \psi_{x_1}(-t) = -\psi_{x_1}(t) \). Put
\[
\varphi(t) := \psi'_{x_1}(t).
\]
Then \( \varphi^{(2j-1)}(0) = 0 \) for \( j \in \mathbb{N} \). To be precise, \( \varphi(t) \) can be expanded in a series in \( t \) as
\[
\varphi(t) = 2 - \frac{3k^2}{4} t^2 + O(t^4).
\]

Proof. The first half follows from Proposition 3.1 of \([\text{OS2}]\) by putting \( \rho(x_1, x_2) = t_{x_1} \cdot t_{x_2} \). We remark that the assertion \( \psi^{(2j)}(0) = 0 \) \( (j \in \mathbb{N}) \) can be illustrated by the fact that \( \int_0^\delta u^{2j-1} \, du = 0 \).

We prove (2.9) in what follows. The equalities (2.6) and (2.7) imply
\[
\psi_{x_1}(t) = \int_0^t \left( 1 - \frac{k^2}{2} u^2 - \frac{k k'}{2} u^3 + O(u^4) \right) \left( 1 + \frac{k^2}{8} u^2 + \frac{k k'}{6} u^3 + O(u^4) \right) \, du
\]
\[
+ \int_0^t \left( 1 - \frac{k^2}{2} u^2 + \frac{k k'}{2} u^3 + O(u^4) \right) \left( 1 - \frac{k^2}{8} u^2 + \frac{k k'}{6} u^3 + O(u^4) \right) \, du
\]
\[
= 2t - \frac{k^2}{4} t^3 + O(t^5),
\]
and hence
\[
\varphi(t) = 2 - \frac{3k^2}{4} t^2 + O(t^4),
\]
which completes the proof. \( \square \)

Finally, we give a lemma which is needed for the proof of (4) of Theorem 2.1.

Lemma 2.5 Let \( \omega_1 \) be the 1-form on \( \Gamma \) given by \( \omega_1 = dx_1 \cdot \hat{r}_{12} \). Then we have
\[
\frac{dx_1 \cdot dx_2}{|x_1 - x_2|^\alpha} = \alpha \left( \frac{\hat{r}_{12} \cdot dx_1}{|x_1 - x_2|^\alpha} \right) - d \left( \frac{\omega_1}{|x_1 - x_2|^{a-1}} \right).
\]

Proof. It can be obtained by modifying the proof of Proposition 4.11 of \([\text{OS1}]\) (cf. Proposition 5 of \([\text{BP}]\)).

Suppose \((x_1, x_2) \in \Gamma \times \Gamma \setminus \Delta\), where \( \Delta \) is the diagonal \( \Delta = \{(x, x) : x \in \mathbb{R}^3\} \) (the case when \((x_1, x_2) \in \Gamma_1 \times \Gamma_2\) can be proved in the same way). Let \( \tau \) be the angle between the two oriented planes containing the line \( \overrightarrow{x_1 x_2} \) tangent to \( \Gamma \) at \( x_1 \) and \( x_2 \) respectively.

Let \( \{e_1, e_2, e_3\} \) be an orthonormal moving frame (along \( \Gamma \)) with \( e_1 = \hat{r}_{12} \), and \( e_3 \perp T_{x_1} \Gamma \).

Let \( \omega_i = dx_1 \cdot e_i \) and \( \omega_{ij} = de_i \cdot e_j \). Then there hold
\[
\omega_2 = \sin \theta_1 \, dx_1, \quad \omega_3 = 0, \quad \omega_{12} = \cos \tau \sin \theta_2 \frac{dx_2}{|x_2 - x_1|},
\]
which implies
\[
d\omega_1 = \omega_{12} \wedge \omega_2 + \omega_{12} \wedge \omega_3 = -\cos \tau \sin \theta_1 \sin \theta_2 \frac{dx_1 \wedge dx_2}{|x_2 - x_1|}.
\]
(1) The case $\alpha \neq 1$. We have
\[
\cos \theta_1 \cos \theta_2 \frac{dx_1 \wedge dx_2}{|x_2 - x_1|^\alpha} = \frac{(\hat{\mathbf{r}}_{12} \cdot dx_1)(\hat{\mathbf{r}}_{12} \cdot dx_2)}{|x_1 - x_2|^\alpha} \\
= -d(|x_2 - x_1| \wedge \omega_1) \frac{1}{|x_2 - x_1|^\alpha} = 1 \frac{d}{\alpha - 1} \frac{1}{|x_2 - x_1|^\alpha} \wedge \omega_1 \\
= \frac{1}{\alpha - 1} \frac{1}{|x_2 - x_1|^{\alpha - 1}} \cdot |x_2 - x_1| \wedge \omega_1 \\
= \frac{1}{\alpha - 1} \cos \tau \sin \theta_1 \sin \theta_2 \frac{dx_1 \wedge dx_2}{|x_2 - x_1|^\alpha} + \frac{1}{\alpha - 1} \frac{d}{|x_2 - x_1|^{\alpha - 1}},
\]
and therefore
\[
\cos \tau \sin \theta_1 \sin \theta_2 \frac{dx_1 \wedge dx_2}{|x_2 - x_1|^\alpha} = (\alpha - 1) \cos \theta_1 \cos \theta_2 \frac{dx_1 \wedge dx_2}{|x_2 - x_1|^\alpha} - d \frac{\omega_1}{|x_2 - x_1|^{\alpha - 1}}.
\]
Since
\[
\frac{dx_1 \cdot dx_2}{|x_2 - x_1|^\alpha} = (\cos \theta_1 \cos \theta_2 + \cos \tau \sin \theta_1 \sin \theta_2) \frac{dx_1 \wedge dx_2}{|x_2 - x_1|^\alpha},
\]
(2.13) follows.

(2) The case $\alpha = 1$ follows from (2.12) and (2.13).

Proof of Theorem 2.1. We prove only in the case of Neumann formula, as the argument goes parallel for Weber formula. We drop off the coefficient $\mu_0/(4\pi)$ in the proof to make formulae shorter and simpler.

(1) Since
\[
\int_{\Gamma \times \Gamma \setminus \Delta} \frac{dx_1 \cdot dx_2}{|x_1 - x_2|} = \int_{\Gamma} \left( \int_{\Gamma \setminus \partial \Omega(x_1)} \frac{t_{x_1} \cdot t_{x_2}}{|x_1 - x_2|} \, dx_2 \right) \, dx_1
\]
the regularization process can be reduced to that of $\int_{\Gamma} |x_1 - x_2|^{-1} (t_{x_1} \cdot t_{x_2}) \, dx_2$. The assertion follows from (2.5) and (2.7) since the integrand can be estimated by
\[
\frac{t_{x_1} \cdot t_{x_2}}{|x_1 - x_2|} = \frac{1}{|s|} + O(s).
\]

(2) Since
\[
\int_{\Gamma \times \Gamma} |x_1 - x_2|^z \, dx_1 \cdot dx_2 = \int_{\Gamma} \left( \int_{\Gamma \setminus \partial \Omega(x_1)} |x_1 - x_2|^z (t_{x_1} \cdot t_{x_2}) \, dx_2 \right) \, dx_1,
\]
the regularization of $F_N(z)$ can be reduced to that of $\int_{\Gamma} |x_1 - x_2|^z (t_{x_1} \cdot t_{x_2}) \, dx_2$.

By the coarea formula (a kind of geometric version of Fubini’s theorem), we have
\[
\int_{\Gamma} |x_1 - x_2|^z (t_{x_1} \cdot t_{x_2}) \, dx_2 = \int_0^d t^z \psi_1(t) \, dt = \int_0^d t^z \varphi(t) \, dt.
\]
Put
\[ f(z) := \int_0^d t^z \varphi(t) \, dt, \]
which converges for \( \Re z > -1 \). Let \( k \) be any natural number, and consider the right hand side of
\[
\int_0^d t^z \varphi(t) \, dt = \int_1^d t^z \varphi(t) \, dt + \int_0^1 t^z \left[ \varphi(t) - \varphi(0) - \varphi'(0) t - \cdots - \frac{\varphi^{(k-1)}(0)}{(k-1)!} t^{k-1} \right] \, dt \\
+ \sum_{j=1}^k \int_0^1 \frac{\varphi^{(j-1)}(0)}{(j-1)!} t^{z+j-1} \, dt.
\]
(2.14)
The first term is a holomorphic function of \( z \). The integrand of second term can be estimated by \( t^{z+k} \), hence the integral converges for \( \Re z > -k - 1 \). Since
\[
\int_0^1 \frac{\varphi^{(j-1)}(0)}{(j-1)!} t^{z+j-1} \, dt = \frac{\varphi^{(j-1)}(0)}{(j-1)!} (z+j) (z \neq -j)
\]
f(\( z \)) is a meromorphic function on \( \Re z > -k - 1 \) possibly with simple poles at \( z = -1, \ldots, -k \)
with residues given by
\[
\text{Res}(f, -j) = \frac{\varphi^{(j-1)}(0)}{(j-1)!} \quad (j = 1, \ldots, k)
\]
(2.15)
(GS Ch.1, 3.2).

Since \( k \in \mathbb{N} \) is arbitrary as \( \varphi \) is smooth, and \( \varphi^{(2i)}(0) = 0 \) \( (i = 0, 1, 2, \ldots) \) by Lemma 2.4, this proves that \( F_N(z) \) is a meromorphic function with possible simple poles at negative odd integers. Since \( \varphi(0) = 2 \) by (2.9), the residue at \( z = -1 \) is given by
\[
\text{Res}(F_N, -1) = \frac{\mu_0}{4\pi} \int_{\Gamma} \varphi(0) \, dx_1 = \frac{\mu_0 C}{2\pi}.
\]

(3) The above argument can be paraphrased into Hadamard regularization as follows. Putting \( k = 1 \) and \( z = -1 \) in (2.14), one obtains
\[
\int_{\Gamma \setminus B_\varepsilon(x_1)} \frac{t \cdot x_1 - t \cdot x_2}{|x_1 - x_2|} \, dx_2 = \int_0^d t^{-1} \varphi(t) \, dt \\
= \int_1^d t^{-1} \varphi(t) \, dt + \int_0^1 t^{-1} \left[ \varphi(t) - \varphi(0) \right] \, dt + \varphi(0) \int_0^1 t^{-1} \, dt,
\]
(2.16)
Since
\[
\int_0^1 t^{z} \, dt = \frac{1}{z+1} \quad (z \neq -1),
\]
the residue of \( z \mapsto \int_0^1 t^z \, dt \) at \( z = -1 \) is equal to 1. Comparing (2.14) with \( k = 1 \) and (2.16), the first two terms in the right hand sides coincide since the integrals converge. The equality \( H_N(\Gamma) = A_N(\Gamma) \) follows from
\[
\lim_{z \to -1} \left( \int_0^1 t^z \, dt - \frac{1}{z+1} \right) = \lim_{\varepsilon \to 0^+} \left( \int_{\varepsilon}^1 \frac{dt}{t} + \log \varepsilon \right).
\]
Here we divide the domain of integration at 1. When \( d < 1 \) we can take any \( d_0 \) with \( 0 < d_0 < d \) instead of 1. The argument goes parallel.
For a sufficiently small \( \varepsilon > 0 \), \( B_\varepsilon(x) \cap \Gamma \) consists of a single curve segment for any \( x \in \Gamma \). We remark that the condition is satisfied if \( \varepsilon \) is smaller than the \textit{thickness} of \( \Gamma \) (see, for example, [LSDR] for the definition of thickness of a knot). Let \( x_{\varepsilon, +} \) and \( x_{\varepsilon, -} \) be the endpoints of the curve \( B_\varepsilon(x) \cap \Gamma \), where we assume \( x_{\varepsilon, +} \) is a little bit ahead of \( x \) with respect to the orientation of \( \Gamma \).

Then the boundary of \( \Gamma \times \Gamma \setminus \Delta_\varepsilon \) consists of two disjoint curves on \( \Gamma \times \Gamma \) given by

\[
\Gamma_{\varepsilon, \pm} := \{ (x, x_{\varepsilon, \pm}) : x \in \Gamma \}.
\]

Suppose \( \Gamma_{\varepsilon, \pm} \) are endowed with the same orientation as \( \Gamma \). Then the boundary of \( \Gamma \times \Gamma \setminus \Delta_\varepsilon \) is given by

\[
\partial (\Gamma \times \Gamma \setminus \Delta_\varepsilon) = \Gamma_{\varepsilon, +} \cup (-\Gamma_{\varepsilon, -}),
\]

where \(-\Gamma_{\varepsilon, -}\) is endowed with the reverse orientation. Since on \( \Gamma_{\varepsilon, +} \) and \( \Gamma_{\varepsilon, -} \)

\[
\omega_1 = dx_1 \cdot \frac{x_2 - x_1}{|x_2 - x_1|} = (1 + O(\varepsilon^2)) \, dx_1,
\]

we have

\[
\int \int_{(\Gamma \times \Gamma \setminus \Delta_\varepsilon)} \left( \frac{\hat{r}_{12} \cdot dx_1}{|x_1 - x_2|} \right) - \frac{dx_1 \cdot dx_2}{|x_1 - x_2|} = \int \int_{(\Gamma \times \Gamma \setminus \Delta_\varepsilon)} d\omega_1
\]

\[
= \int_{\Gamma_{\varepsilon, +} \cup (-\Gamma_{\varepsilon, -})} \omega_1
\]

\[
= 2L + O(\varepsilon^2).
\]

We close this subsection with giving a property and an example of the self-inductance.

**Proposition 2.6** The regularized self-inductance behaves under homothety as follows.

\[
H_N(\lambda \Gamma) = \lambda H_N(\Gamma) + \frac{\mu_0 L}{2\pi} \cdot \lambda (\log \lambda) \quad (\lambda > 0).
\]

This is immediate from the definition (2.1).

**Example 2.1** The regularized self-inductance of a unit circle \( \Gamma_o \) is given by

\[
H_N(\Gamma_o) = 2\pi \lim_{\varepsilon \to 0^+} \left( \frac{\mu_0}{4\pi} \int_{\varepsilon}^{2\pi - \varepsilon} \frac{\cos \theta}{2 \sin^2 \frac{\theta}{2}} \, d\theta + \frac{\mu_0}{2\pi} \log \varepsilon \right) = \frac{(-1 + \log 2) \mu_0}{\pi}.
\]

2.2 Regularization with parallel loops

**Theorem 2.7** Let \( \Gamma \) be a smooth\(^4\) simple loop with non-vanishing curvature. Let \( \Gamma_\delta \) be a \( \delta \)-parallel curve given by \( \Gamma_\delta = \{ x + \delta n(x) : x \in \Gamma \} \), where \( n \) is the unit principal normal vector to \( \Gamma \). Then

\[
\lim_{\delta \to 0^+} \left( L_N(\Gamma, \Gamma_\delta) + \frac{\mu_0 L}{2\pi} \log \delta \right) = H_N(\Gamma) + \frac{(\log 2) \mu_0 L}{2\pi}.
\]

\(^4\)We assume the smoothness for the sake of simplicity. In this case we need \( C^{\infty} \).
We remark that if $\delta$ is smaller than the thickness of $\Gamma$ then $\Gamma_\delta \cap \Gamma = \emptyset$.

**Proof.** We drop off the coefficient $\mu_0/(4\pi)$ in the proof to make formulae shorter and simpler. The proof of Proposition 4.18 of [OS1] goes parallel with slight modification. We estimate, fixing $x_1$ on $\Gamma$,

$$
\int_{\Gamma_\delta \cap B_\varepsilon(x_1)} \frac{t_{x_1} \cdot t_{x_2}}{|x_1 - x_2|} \, dx_2
$$

for $0 < \delta \ll \varepsilon \ll 1$.

Suppose $\Gamma$ is parametrized by the arc-length as $\Gamma = \{ \gamma(s) \}_{0 \leq s \leq \ell}$. Then $\Gamma_\delta$ can be expressed as $\Gamma_\delta = \{ \gamma_\delta(s) \}_{0 \leq s \leq \ell}$, where $\gamma_\delta(s) = \gamma(s) + \delta \kappa(s)^{-1} \gamma''(s)$. Let $x_1 = \gamma(s_1)$ and $\kappa$ be the curvature of $\Gamma$ at $x_1$. We have

$$
\gamma'(s_1) \cdot \gamma_\delta'(s_1 + s) = (1 - \kappa \delta) + O(1) \delta s + O(s^2),
$$

$$
|\gamma_\delta(s_1 + s) - \gamma(s_1)|^2 = (\delta^2 + (1 - \kappa \delta)s^2)
$$

for $0 < \delta \ll \varepsilon \ll 1$.

Let $s_1 \pm s_\pm$ be the values of parameter when $\Gamma_\delta$ passes through the sphere $\partial B_\varepsilon(x_1)$. Then

$$
s_\pm = \pm \sqrt{\frac{\varepsilon^2 - \delta^2}{1 - \kappa \delta}} + O(\varepsilon^3).
$$

The above equalities imply that [2.18] can be estimated by

$$
\int_{\sqrt{\frac{\varepsilon^2 - \delta^2}{1 - \kappa \delta}} + O(\varepsilon^3)}^{\sqrt{\frac{\varepsilon^2 - \delta^2}{1 - \kappa \delta}} + O(\varepsilon^3)} \frac{(1 - \kappa \delta) + O(1) \delta s + O(s^2)}{\sqrt{\delta^2 + (1 - \kappa \delta)s^2}} \, ds
$$

$$
= 2 \int_{0}^{\sqrt{\frac{\varepsilon^2 - \delta^2}{1 - \kappa \delta}}} \frac{1 - \kappa \delta}{\sqrt{\delta^2 + (1 - \kappa \delta)s^2}} \, ds + O(\varepsilon^2) + o(\delta)
$$

$$
= 2 \sqrt{1 - \kappa \delta} \log \left( \frac{\varepsilon + \sqrt{\varepsilon^2 - \delta^2}}{\delta} \right) + O(\varepsilon^2) + o(\delta)
$$

$$
= 2 \log 2 + 2 \log \varepsilon - 2 \log \delta + O(\varepsilon).
$$

Since $\delta \ll \varepsilon$, it follows that

$$
\int_{\Gamma} \int_{\Gamma_\delta} \frac{dx_1 \cdot dx_2}{|x_1 - x_2|} = \int_{\Gamma} \left( \int_{\Gamma_\delta \cap B_\varepsilon(x_1)} \frac{t_{x_1} \cdot t_{x_2}}{|x_1 - x_2|} \, dx_2 + \int_{\Gamma_\delta \setminus B_\varepsilon(x_1)} \frac{t_{x_1} \cdot t_{x_2}}{|x_1 - x_2|} \, dx_2 \right) \, dx_1
$$

$$
= \int_{\Gamma} \left( 2 \log 2 + 2 \log \varepsilon - 2 \log \delta + \int_{\Gamma_\delta \setminus \Gamma_\varepsilon(x_1)} \frac{t_{x_1} \cdot t_{x_2}}{|x_1 - x_2|} \, dx_2 \right) \, dx_1 + O(\varepsilon)
$$

$$
= \int_{(\Gamma \times \Gamma) \setminus \Delta_\varepsilon} \frac{dx_1 \cdot dx_2}{|x_1 - x_2|} + 2 \mathcal{L} \log \varepsilon + 2(\log 2) \mathcal{L} - 2 \mathcal{L} \log \delta + O(\varepsilon),
$$

which means

$$
\int_{\Gamma} \int_{\Gamma_\delta} \frac{dx_1 \cdot dx_2}{|x_1 - x_2|} + 2 \mathcal{L} \log \delta - 2(\log 2) \mathcal{L} = \int_{(\Gamma \times \Gamma) \setminus \Delta_\varepsilon} \frac{dx_1 \cdot dx_2}{|x_1 - x_2|} + 2 \mathcal{L} \log \varepsilon + O(\varepsilon).
$$

Taking the limit as $\varepsilon$ goes down to 0 and multiplying the both sides by $\mu_0/(4\pi)$, we obtain [2.17].

$\square$
2.3 Hadamard regularization in terms of the arc-length parameter

We introduce formulae of regularized self-inductance obtained by Hadamard regularization in terms of the arc-length parameter. They are more suitable for numerical experiments, and one of them will be used in the next subsection.

In (2.1) and (2.2) we used the chord-length i.e. the distance in \( \mathbb{R}^3 \) to define an “\( \varepsilon \)-neighbourhood” \( \Delta_\varepsilon \) of the diagonal set \( \Delta \) of \( \Gamma \times \Gamma \), which we use in Hadamard regularization. However, since it is easier to express a curve by the arc-length than by the chord-length, it is useful to have the formulae of Hadamard regularization in terms of the arc-length parametrization. In this setting, \( \Delta_\varepsilon \) should be replaced by

\[
\tilde{\Delta}_\varepsilon := \{(x_1, x_2) \in \Gamma \times \Gamma : d_\Gamma(x_1, x_2) \leq \varepsilon\},
\]

where \( d_\Gamma \) is the arc-length along \( \Gamma \). Nevertheless, the counter terms and Hadamard’s finite parts do not change, which follows from the equalities (2.5) – (2.8), namely,

\[
H_N(\Gamma) = \lim_{\varepsilon \to 0^+} \left( \frac{\mu_0}{4\pi} \iint_{(\Gamma \times \Gamma) \setminus \tilde{\Delta}_\varepsilon} \frac{dx_1 \cdot dx_2}{|x_1 - x_2|} + \frac{\mu_0 L}{2\pi} \log \varepsilon \right),
\]

\[
H_W(\Gamma) = \lim_{\varepsilon \to 0^+} \left( \frac{\mu_0}{4\pi} \iint_{(\Gamma \times \Gamma) \setminus \tilde{\Delta}_\varepsilon} \frac{\hat{\tau}_{12} \cdot dx_1)(\hat{\tau}_{12} \cdot dx_2)}{|x_1 - x_2|} + \frac{\mu_0 L}{2\pi} \log \varepsilon \right).
\]

We remark that such a phenomenon can be observed when the power of denominator in the integrand is smaller than 3, as was explained for a similar functional in Remark 2.2.1 of \( \text{[O]} \).

Next, to have faster convergence in numerical experiments, it is better to increase the number of counter terms. The series expansion in \( \varepsilon \), after dropping off the constant \( \mu_0/(4\pi) \), is given by

\[
\int \int_{(\Gamma \times \Gamma) \setminus \tilde{\Delta}_\varepsilon} \frac{dx_1 \cdot dx_2}{|x_1 - x_2|} = 2\mathcal{L} \log \frac{1}{\varepsilon} + \frac{11}{24} \left( \int_{\Gamma} \kappa(x)^2 \, dx \right) \varepsilon^2 + O(\varepsilon^4), \tag{2.19}
\]

\[
\int \int_{(\Gamma \times \Gamma) \setminus \tilde{\Delta}_\varepsilon} \frac{(\hat{\tau}_{12} \cdot dx_1)(\hat{\tau}_{12} \cdot dx_2)}{|x_1 - x_2|} = 2\mathcal{L} \log \frac{1}{\varepsilon} + \frac{5}{24} \left( \int_{\Gamma} \kappa(x)^2 \, dx \right) \varepsilon^2 + O(\varepsilon^4).
\]

Let us show that (2.19) follows from (2.5) and (2.7). Fix a point \( x_1 \). For a small positive number \( b \) and \( \varepsilon \) with \( 0 < \varepsilon < b \),

\[
\int_{\varepsilon \leq d_\Gamma(x_1, x_2) \leq b} \frac{tx_1 \cdot tx_2}{|x_1 - x_2|} \, dx_2 = \int_{\varepsilon}^{b} \left( 1 - \frac{\kappa^2}{2} s^2 - \frac{\kappa \kappa'}{2} s^3 + O(s^4) \right) \left[ 1 - \frac{\kappa^2}{24} s^2 - \frac{\kappa \kappa'}{24} s^3 + O(s^4) \right]^{-1} \, ds
\]

\[
= \int_{-\varepsilon}^{-\varepsilon} \left( 1 - \frac{\kappa^2}{2} s^2 - \frac{\kappa \kappa'}{2} s^3 + O(s^4) \right) \left[ -s \left( 1 - \frac{\kappa^2}{24} s^2 - \frac{\kappa \kappa'}{24} s^3 + O(s^4) \right) \right]^{-1} \, ds
\]

\[
= O(1) - 2\log \varepsilon + \frac{11}{24} \kappa^2 \varepsilon^2 + O(\varepsilon^4).
\]

2.4 The self-inductance of solenoids

Let us consider a solenoid \( \Gamma_n = \Gamma_{r, \ell, n} \) that winds a cylinder of radius \( r \) and length \( \ell \) with a number of turns per unit length \( n \) carrying a current \( I \). Define the regularized self-inductance of \( \Gamma \) in the sense of Neumann and of Weber by (2.1) and (2.2).
Theorem 2.8  The asymptotic of the regularized self-inductances as \( n \) goes to infinity is given by
\[
\lim_{n \to \infty} \frac{H_N(\Gamma_{r,\ell,n})}{n^2} = \lim_{n \to \infty} \frac{H_W(\Gamma_{r,\ell,n})}{n^2} = \frac{8\mu_0}{3} \left[ -r^3 + \frac{1}{8} \left( -\ell (\ell^2 - 4r^2) E \left( -\frac{4r^2}{\ell^2} \right) + \ell (\ell^2 + 4r^2) K \left( -\frac{4r^2}{\ell^2} \right) \right) \right],
\] (2.20)
where \( K(k) \) and \( E(k) \) are the complete elliptic integrals of the first and second kind respectively:
\[
E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k \sin^2(t)} \, dt, \quad K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k \sin^2(t)}} \, dt.
\]

Let (2.20) be denoted by \( L_{r,\ell} \). The asymptotic of \( L_{r,\ell} \) as \( \ell \) goes to infinity is given by
\[
\lim_{\ell \to \infty} \left( L_{r,\ell} - \mu_0 \left( \frac{\pi r^2 \ell}{2} - \frac{8}{3} r^3 \right) \right) = 0.
\] (2.21)

Thus the asymptotics of regularized self-inductance as \( n \) and \( \ell \) go to infinity is given by
\[
H_N(\Gamma_{r,\ell,n}) \sim H_W(\Gamma_{r,\ell,n}) \sim \mu_0 \pi r^2 n^2 \ell \quad (n, \ell \to \infty),
\]
as is expected.

Remark 2.9  The right hand side of (2.20) multiplied by \( n^2 \) has already appeared in [BAB] and [DO] (and in [L] according to [BAB]) as the limit of the mutual inductance of coaxial solenoids as they overlap.

Proof.  (1) The first equality of (2.20) follows from Theorem 2.1 (4) since \( \mathcal{L} = \ell \sqrt{(2\pi nr)^2 + 1} = O(n) \).

(2) The second equality of (2.20) can be proved as follows. Let \( M = M_{r,\ell} \) be a cylinder with radius \( r \) and length \( \ell \) parametrized by cylindrical coordinates;
\[
p: [0, 2\pi r] \times [-\ell/2, \ell/2] \ni (t, z) \mapsto (r \cos(t/r), r \sin(t/r), z).
\]
Let \( v_x \) be the unit tangent vector to the “meridean circle” of \( M_{r,\ell} \) through \( x \) which is given by \( v_x = p_t = (\partial p/\partial t) \). We show
\[
\lim_{n \to \infty} \frac{H_N(\Gamma_n)}{n^2} = \frac{\mu_0}{4\pi} \int_{M \times M} \frac{v_{x_1} \cdot v_{x_2}}{|x_1 - x_2|} \, dx_1 dx_2
\] (2.22)
by the following steps.

Let \( \varepsilon_0 \) be any positive number.

(i) For any positive number \( \delta \) there exists a natural number \( n_0 \) such that for any point \( x \) in \( M \) there holds
\[
\left| \int_{\Gamma_n \times \Gamma_n \setminus N_\delta(x)} \frac{dx_1 \cdot dx_2}{|x_1 - x_2|} - \int_{(M \times M) \setminus N_\delta(x)} \frac{v_{x_1} \cdot v_{x_2}}{|x_1 - x_2|} \, dx_1 dx_2 \right| < \varepsilon_0,
\]
where \( N_\delta(x) \) is a “curved square neighbourhood” of \( x = p(t, z) \) in \( M \) given by
\[
N_\delta(x) = p((t - \delta, t + \delta) \times ((z - \delta, z + \delta) \cap [-\ell/2, \ell/2])).
\]
where we assume that the coordinate $\theta$ is considered modulo $2\pi$.

(ii) There exists a positive number $\delta_2$ such that if $0 < \delta \leq \delta_2$ then for any $x$ in $M$,

$$
\iint_{(M \times M) \cap N_\delta(x)} \frac{v_{x_1} \cdot v_{x_2}}{|x_1 - x_2|} \, dx_1 dx_2 < \varepsilon_0.
$$

(iii) There is a natural number $n_1$ such that if $n \geq n_1$ then for any positive number $\delta$ with $0 < \delta \leq \delta_2$,

$$
\frac{1}{n} \left| \lim_{\varepsilon \to 0^+} \left( \int_{[-\delta, -\varepsilon] \cup [\varepsilon, \delta]} \frac{p_t(t, 0) \cdot p_t(0,0)}{|p(t,0) - p(0,0)|} \, dt + 2 \log \varepsilon \right) + 2 \sum_{i=1}^{\delta/n} \int_{-\delta}^{\delta} \frac{p_t(t, i/n) \cdot p_t(i/n,0) - p_t(0,0)}{|p(t,i/n) - p(0,0)|} \, dt \right| < 2\varepsilon_0,
$$

where $\lfloor u \rfloor$ is the floor function that gives the greatest integer less than or equal to $u$.

(iv) There is a natural number $n_2$ with $n_2 \geq n_1$ such that for any positive number $\delta$ with $0 < \delta \leq \delta_2$ and for any point $x$ in $M$,

$$
\frac{1}{n} \left| \lim_{\varepsilon \to 0^+} \left( \int_{(\Gamma_n \cap N_\delta(x)) \setminus \Delta_\varepsilon} \frac{t x \cdot t x_2}{|x_1 - x_2|} \, dx_2 + 2 \log \varepsilon \right) \right| < 3\varepsilon_0.
$$

Since the right hand side of (2.22) is given by

$$
\frac{\mu_0}{16\pi} \int_0^{2\pi} \int_0^\ell \int_0^{2\pi} \int_0^\ell \frac{\cos(\theta_1 - \theta_2)}{\sqrt{4r^2 \sin^2 \frac{\theta_1 - \theta_2}{2} + (z_1 - z_2)^2}} r^2 \, dz_1 d\theta_1 dz_2 d\theta_2,
$$

$$
= 2\mu_0 r^3 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{1 - 2 \sin^2 \theta}{\sqrt{4 \sin^2 \theta + (z_1 - z_2)^2}} \, dz_1 dz_2 d\theta,
$$

Some computation shows that it is equal to the right hand side of (2.20).

(3) The equality (2.21) follows from the following expansion;

$$
\frac{1}{3} \left( -\ell (\ell^2 - 4) E \left( -\frac{4}{\ell^2} \right) + \ell (\ell^2 + 4) K \left( -\frac{4}{\ell^2} \right) \right) = \pi \ell + \frac{\pi}{2} \cdot \frac{1}{\ell} + O \left( \frac{1}{\ell^3} \right).
$$

\[ \square \]

3 Appendix

3.1 The second residues of $F_N$ and $F_W$

We can proceed the analysis of $F_N(z)$ and $F_W$ a bit more.

**Proposition 3.1** The second residues of $F_N$ and $F_W$ defined by (2.3) and (2.4) are given by

$$
\text{Res}(F_N, -3) = -\frac{3\mu_0}{16\pi} \int \kappa(x)^2 \, dx, \quad \text{Res}(F_W, -3) = -\frac{\mu_0}{16\pi} \int \kappa(x)^2 \, dx,
$$

where $\kappa$ denotes the curvature.

**Proof.** The first equality follows from (2.9) and (2.15).

The second equality can be proved similarly using (2.8).

\[ \square \]
3.2 When the power of the denominator is 2

If we change the power of the denominator of \( \frac{1}{1} \) from 1 to 2, we obtain “average linking with random circles” ([OS1]). Let \( S(1,3) \) be the set of oriented circles in \( \mathbb{R}^3 \), \( dC \) a measure on \( S(1,3) \) which is invariant under Möbius transformations, and \( \text{lk}(C, \Gamma_i) \) the linking number of \( \Gamma \) and an oriented circle \( C \). Then we have

\[
\int_{\Gamma_1} \int_{\Gamma_2} \frac{d\mathbf{x}_1 \cdot d\mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^2} = 2 \int_{\Gamma_1} \int_{\Gamma_2} \frac{(\hat{\mathbf{r}}_{12} \cdot d\mathbf{x}_1)(\hat{\mathbf{r}}_{12} \cdot d\mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|^2} = \frac{3\pi}{4} \int_{S(1,3)} \text{lk}(C, \Gamma_1) \cdot \text{lk}(C, \Gamma_2) \, dC.
\]

When \( \Gamma_1 \) and \( \Gamma_2 \) coincide, the above integrals blow up, and they can be regularized as

\[
\lim_{\varepsilon \to 0^+} \left( \int_{(\Gamma \times \Gamma) \setminus \Delta_{\varepsilon}} \frac{d\mathbf{x}_1 \cdot d\mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^2} - \frac{2L}{\varepsilon} \right) = \lim_{\varepsilon \to 0^+} \left( \frac{3\pi}{4} \int_{S_{\varepsilon}(1,3)} \text{lk}(C, \Gamma)^2 \, dC - \frac{3\pi L}{2\varepsilon} \right),
\]

where \( S_{\varepsilon}(1,3) \) is the set of oriented circles with radius \( r \geq \varepsilon \). The reader is referred to [OS1] for the details.

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