Phase Transition of the Ising Model on Fractal Lattice

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Phase transition of the Ising model is investigated on a planar lattice that has a fractal structure. On the lattice, the number of bonds that cross the border of a finite area is doubled when the linear size of the area is extended by a factor of four. The free energy and the spontaneous magnetization of the system are obtained by means of the higher-order tensor renormalization group method. The system exhibits the order-disorder phase transition, where the critical indices are different from that of the square-lattice Ising model. An exponential decay is observed in the density matrix spectrum even at the critical point. It is possible to interpret that the system is less entangled because of the fractal geometry.

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I. INTRODUCTION

Phase transition and critical phenomena have been one of the central issues in statistical analyses of condensed matter physics\textsuperscript{1}. When the second-order phase transition is observed, thermodynamic functions, such as the free energy, the internal energy, and the magnetization, show non-trivial behavior around the transition temperature $T_c$. This critical singularity reflects the absence of any scale length at $T_c$, and the power-law behavior of thermodynamic functions around the transition is explained by the concept of the renormalization group\textsuperscript{1,4–6}.

Analytic investigation of the renormalization group flow in $\varphi^4$-model shows that the Ising model exhibits a phase transition when the lattice dimension is larger than one, which is the lower critical dimension\textsuperscript{6,7}. In a certain sense, the one-dimensional Ising model shows rescaled critical phenomena around $T_c = 0$. When the lattice dimension is larger than four, which is the upper critical dimension, and provided that the system is uniform, then the Ising model on regular lattices exhibits mean-field-like critical behavior.

Compared with critical phenomena on regular lattices, much less is known on fractal lattices. Renormalization flow is investigated by Gefen et al.,\textsuperscript{8–11} where correspondence between lattice structure and the value of critical indices is not fully understood in a quantitative manner. For example, the Ising model on the Sierpinski gasket does not exhibit phase transition at any finite temperature, although the Hausdorff dimension of the lattice, $d_H = \ln 3/\ln 2 \approx 1.585$, is larger than one\textsuperscript{12,13}. The absence of the phase transition could be explained by the fact that the number of interfaces, i.e. the outgoing bonds from a finite area, does not increase when the size of the area is doubled on the gasket. A non-trivial feature of this system is that there is a logarithmic scaling behavior in the internal energy toward zero temperature\textsuperscript{14}. The effect of anisotropy has been considered recently\textsuperscript{15}.

In this article, we investigate the Ising model on the Sierpinski carpet, presence of the phase transition is proved\textsuperscript{16}, and its critical indices were roughly estimated by Monte Carlo simulations\textsuperscript{17}. It should be noted that it is not easy to collect sufficient number of data plots for finite-size scaling\textsuperscript{18} on such fractal lattices by means of Monte Carlo simulations, because of the exponential blow-up of the number of sites in a unit of fractal.

In this article, we investigate the Ising model on a planar fractal lattice, shown in Fig. 1. The lattice consists of vertices around the lattice points, which are denoted by the empty dots in the figure, where there are Ising spins. The whole lattice is constructed by recursive extension processes, where the linear size of the system increases by the factor of four in each step. If the lattice was a

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fractal_lattice.png}
\caption{Composition of the fractal lattice. Upper left: a local vertex around an Ising spin shown by the empty dot. Middle: the basic cluster which contains $N_1 = 12$ vertices. Lower right: the extended cluster which contains $N_2 = 12^2$ vertices. In each step of the system extension, the linear size of the system increases by the factor of 4, where only 12 units are linked, and where 4 units at the corners are missing, if it is compared with a 4 by 4 square cluster.}
\end{figure}
regular square one, $4 \times 4 = 16$ units are connected in the extension process, whereas only 12 units are connected on this fractal lattice; 4 units are missing in the corners. As a result, the number of sites contained in a cluster after $n$ extensions is $N_n = 12^n$, and the Hausdorff dimension of this lattice is $d_H = \ln 12 / \ln 4 \approx 1.792$. The number of outgoing bonds from a cluster is only doubled in each extension process since the sites and the bonds at each corner are missing. If we evaluate the lattice dimension from the relation

$$M = L^{d-1}$$

between the linear dimension $L$ and the number of outgoing bonds $M$, we have $d = 1.5$, since $M$ is proportional to $\sqrt{L}$ on the fractal. Remark that the value is different from $d_H \approx 1.792$

We report the critical behavior of the Ising model on the fractal lattice when the system size is large enough. Thermodynamic properties of the system are numerically studied by means of the Higher-Order Tensor Renormalization Group (HOTRG) method. The system exhibits the order-disorder phase transition, where the critical indices are different from the square lattice Ising model. In the next Section we introduce a representation of the Ising model in terms of a vertex model, which is suitable for numerical analyses by means of the HOTRG method.

In Sec. III, we show the calculated result around the transition temperature $T_c$. Conclusions are summarized in the last Section.

II. VERTEX REPRESENTATION

We introduce a representation of the Ising model as a (symmetric) 16-vertex model. The Ising interaction between two adjacent Ising spins $\sigma$ and $\sigma'$, where each one takes either +1 or −1, is represented by the diagonal Hamiltonian

$$H(\sigma, \sigma') = -J \sigma \sigma' ,$$

where $J > 0$ represents the ferromagnetic coupling. Throughout this article we assume that there is no external magnetic field. The corresponding local Boltzmann weight on the bond is given by

$$\exp \left[ -\frac{H(\sigma, \sigma')}{k_B T} \right] = \exp \left[ \frac{J}{k_B T} \sigma \sigma' \right] = e^{K \sigma \sigma'} ,$$

where $T$ is the temperature, $k_B$ is the Boltzmann constant, and we have introduced a parameter $K = J/k_B T$.

It is possible to factorize the bond weight $e^{K \sigma \sigma'}$ into two parts, by introducing an auxiliary spin $s = \pm 1$, which is often called an ‘ancilla’, and which is located between $\sigma$ and $\sigma'$. A key relation is

$$e^{K \sigma \sigma'} = \frac{1}{2 (\cosh 2K)^{1/2}} \sum_s e^{\pi s (\sigma + \sigma')} ,$$

where the r.h.s. takes the value $(\cosh 2K)^{1/2}$ when $\sigma = \sigma'$, and $(\cosh 2K)^{-1/2}$ when $\sigma \neq \sigma'$, and where Eq. (11) holds under the condition

$$e^{K} = (\cosh 2K)^{1/2} .$$

The new parameter $K$ is then expressed as follows

$$e^{K} = \sqrt{e^{2K} + e^{4K} - 1} .$$

Thus, if we introduce a factor

$$W_{\sigma s} = e^{\pi \sigma s} \left[ 2 (\cosh 2K)^{1/2} \right]^{-1/2}$$

for each division of a bond, we can rewrite the Ising interaction in the following form

$$e^{K \sigma \sigma'} = \sum_s W_{\sigma s} W_{\sigma' s} .$$

By means of the factorization in Eq. (3), we can map the square-lattice Ising model into the symmetric 16-vertex model, where the local vertex weight is defined as

$$T_{ss' s'' s'''} = \sum_{\sigma} W_{\sigma s} W_{\sigma s'} W_{\sigma' s''} W_{\sigma'' s'''} .$$

In the upper-left corner of Fig. 1 we have shown the graphical representation of the vertex weight $T_{ss' s'' s'''}$, where the open circle denotes the Ising spin $\sigma$, which is summed up. The four short bars around the Ising spin in Fig. 1 show the halves of the bonds, where there are auxiliary spins $s, s', s'', and s'''$ at the end of each short bar.

In case we consider a finite-size cluster with rectangular shape with free boundary conditions, we have to prepare a new boundary Boltzmann weight

$$P_{ss's''} = \sum_{\sigma} W_{\sigma s} W_{\sigma s'} W_{\sigma s''}$$

and a corner Boltzmann weight

$$C_{ss'} = \sum_{\sigma} W_{\sigma s} W_{\sigma s'} .$$

It should be noted that these boundary weights $P_{ss's''}$ and $C_{ss'}$ are obtained by taking partial trace for the vertex weight; we have the relations

$$P_{ss's''} = \frac{\sum_{s''} T_{ss's'' s'''} \sum_{s'''} W_{ss''} W_{ss'''} \sum_{s''} W_{ss''}}{\sum_{s''} W_{ss''} \sum_{s'''} W_{ss'''} \sum_{s''} W_{ss''}} ,$$

and

$$C_{ss'} = \frac{\sum_{s''} \sum_{s'''} W_{ss''} W_{ss'''} \sum_{s''} W_{ss''} \sum_{s'''} W_{ss'''}}{\sum_{s''} W_{ss''} \sum_{s'''} W_{ss'''} \sum_{s''} W_{ss''}} .$$
where one can neglect the denominator when one is interested in the critical singularity; the denominators just subtract a regular function from the free energy of the system. In case that one needs fixed boundary conditions, it is sufficient to avoid taking the configuration sum for Eq. (11) at the beginning. In each extension process, we join 12 local tensors by a recursive joining process of the local tensors, which is nothing but a vertex weight in Eq. (9) at the beginning. We take the configuration sum for those tensor indices in the middle of Fig. 1. In the joining process, we take the rescaling effect into account, although the amount of rescaling effect is large.

In order to simplify the numerical analysis, we choose the parameterization $J = k_B = 1$, and thus we have $K = 1/T$. In the numerical calculation by means of HOTRG, we keep $D = 24$ states at most for block spin variables. We have verified that the choice $D = 24$ is sufficient for obtaining the converged free energy in the entire temperature region. We treat the free energy per site

$$f(T) = \lim_{n \to \infty} \frac{F_n(T)}{N_n}$$

in the following thermodynamic analyses, where the r.h.s. converges already for $n \lesssim 30$.

Figure shows the temperature dependence of the specific heat per site

$$c(T) = \frac{\partial}{\partial T} u(T),$$

where $u(T)$ is the internal energy per site

$$u(T) = -T^2 \frac{\partial}{\partial T} \frac{f(T)}{T},$$

and the temperature derivatives are performed numerically. There is no singularity in $c(T)$ around its maximum. One might find a weak non-analytic behavior at $T_c \approx 1.317$, which is marked by the dotted line in the figure; the numerical derivative of $c(T)$ with respect to temperature (plotted in the inset) has a sharp peak at the critical temperature $T_c$. It is, however, difficult to determine the critical exponent $\alpha$ precisely, because of the weakness in the singularity; as shown in the figure, $c(T)$ around $T_c$ is almost linear in $T$, and therefore $\alpha$ is nearly zero.

Figure shows the spontaneous magnetization per site $m(T)$, which is obtained by inserting a $\sigma$-dependent local weight

$$\tilde{T}_\sigma = \sum_{\sigma} \sigma W_{\sigma} W_{\sigma s} W_{\sigma s'} W_{\sigma s''}$$

into the system. Since the fractal lattice is inhomogeneous, the value is weakly dependent on the location of...
the spontaneous magnetization in Fig. 1. The numerical calculation by HOTRG captures sites that are in the middle of the 12-site cluster shown.

The power-law behavior below $T_c$ is observed at $T_c \approx 1.317$. The calculated spontaneous magnetization per site $m(T)$ below $T_c$ since any tiny round-off error is sufficient for breaking the symmetry inside low-temperature ordered state. Around the transition temperature, the magnetization satisfies a power-law behavior

$$m(T) \propto |T_c - T|^{0.0137},$$

where the precision of the exponent is around 2%, which can be read out from the inset of Fig. 3 as a tiny deviation from the linear dependence (the dashed lines) in $m(T)^{1/\beta}$ near $T_c$.

As a byproduct of the numerical HOTRG calculation, we can roughly observe the entanglement spectrum, which is the distribution of the eigenvalue $\omega_i$ of the density matrix that is created for the purpose of obtaining the block spin transformation. Since the effect of environment is not considered in our implementation of the HOTRG method, the eigenvalue $\omega_i = \lambda_i^2$ is obtained as the square of the singular values $\lambda_i$ in the higher-order singular value decomposition applied to the extended tensors. Figure 4 shows $\omega_i$ at $T = T_c$ in the decreasing order. The decay is rapid, and therefore further increase of the number of block-spin state from $D = 8$ to a larger number does not significantly improve the precision in $\omega_i$; the difference in $f(T_c)$ between $D = 8$ and $D = 16$ is already of the order of $10^{-6}$. It should be noted that the eigenvalues are not distributed equidistantly in logarithmic scale; the corner double line structure is absent.

IV. CONCLUSIONS AND DISCUSSIONS

We have investigated the Ising model on the fractal lattice shown in Fig. 1 by means of the HOTRG method. The calculated specific heat $c(T)$ suggests that the model shows 2nd order phase transition. Qualitatively speaking, the presence of weak singularity in the specific heat agrees with the result of the $\epsilon$-expansion, which shows the increasing nature of the critical exponent in $c(T)$ with respect to the space dimension $d$. The calculated spontaneous magnetization $m(T)$ also supports the 2nd order phase transition with the exponent $\beta_{\text{fractal}} \approx 0.0137$, which is smaller by one order of magnitude than the critical exponent $\beta_{\text{square}} = 1/8 = 0.125$ of the square-lattice Ising model.

The fractal structure of the lattice modifies the entanglement spectrum from that on the square lattice explained by the corner double line picture. Since each corner is missing in the fractal structure in Fig. 1 short-range entanglement is almost filtered out in the process of the renormalization group transformation. This may be the reason why we do not need many degrees of freedom for the renormalized tensors. The situation is similar to the entanglement structure reported in the tensor net-
The lattice geometry of the fractal lattice can be modified in several manners. For example, one can alternate the system extension process of the fractal for the purpose of modifying the Hausdorff dimension; for every odd n the extension with 12 vertices shown in Fig. 1 is performed, and for even n the normal extension with 16 vertices on the square-lattice is performed. Alternatively, one can also modify the basic cluster, in such a manner as introducing 6 by 6 cluster where 4 corners are missing, etc. It is also worth considering three-dimensional fractal lattices on the square-lattice is performed. Alternatively, one can also modify the basic cluster, in such a manner as introducing 6 by 6 cluster where 4 corners are missing, etc. It is also worth considering three-dimensional fractal lattices, and apply the HOTRG method as it was done for the cubic lattice Ising model. These modifications do not spoil the applicability of the HOTRG method while the numerical requirement is heavier than the current research. Analyses of quantum systems on a variety of fractal lattice is another possible extensions. These further study may clarify the role of the entanglement in the universality of the phase transition in both regular and fractal lattices.

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