PSEUDODIFFERENTIAL OPERATORS ON MIXED-NORM
α-MODULATION SPACES

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Abstract. Mixed-norm α-modulation spaces were introduced recently by Cleanthous and Georgiadis [Trans. Amer. Math. Soc. 373 (2020), no. 5, 3323-3356]. The mixed-norm spaces $M^{s,\alpha}_{\vec{p},q}(\mathbb{R}^n)$, $\alpha \in [0, 1]$, form a family of smoothness spaces that contain the mixed-norm Besov spaces as special cases. In this paper we prove that a pseudodifferential operator $\sigma(x, D)$ with symbol in the Hörmander class $S^b_\rho$ extends to a bounded operator $\sigma(x, D): M^{s,\alpha}_{\vec{p},q}(\mathbb{R}^n) \to M^{s-b,\alpha}_{\vec{p},q}(\mathbb{R}^n)$ provided $0 < \alpha \leq \rho \leq 1$, $\vec{p} \in (0, \infty)^n$, and $0 < q < \infty$. The result extends the known result that pseudodifferential operators with symbol in the class $S^1_1$ maps the mixed-norm Besov space $B^{s,\alpha}_{\vec{p},q}(\mathbb{R}^n)$ into $B^{s-b,\alpha}_{\vec{p},q}(\mathbb{R}^n)$.

1. Introduction

The α-modulation spaces is a family of smoothness spaces that contains the Besov spaces and the modulation spaces, introduced by Feichtinger [17], as special cases. In the non-mixed-norm setting, the α-modulation spaces were introduced by Gröbner [21]. Gröbner used the general framework of decomposition type Banach spaces considered by Feichtinger and Gröbner in [16, 18] to build the α-modulation spaces. The parameter α determines a specific type of decomposition of the frequency space $\mathbb{R}^n$ used to define the space $M^{s,\alpha}_{\vec{p},q}(\mathbb{R}^n)$. The α-modulation spaces contain the Besov spaces and the modulation spaces, introduced by Feichtinger [17], as special cases. The choice $\alpha = 0$ corresponds to the classical modulation spaces $M^{s}_{\vec{p},q}(\mathbb{R}^n)$, and $\alpha = 1$ corresponds to the Besov scale of spaces.

The applicability of α-modulation spaces to the study of pseudo-differential operators comes rather natural, and in fact the family of coverings used to construct the α-modulation spaces was considered independently by Päivärinta and Somersalo in [29]. Päivärinta and Somersalo used the partitions to extend the Calderón-Vaillancourt boundedness result for pseudodifferential operators to local Hardy spaces.

Recently, function spaces in anisotropic and mixed-norm settings have attached considerable interest, see for example [14, 11, 12, 19, 20, 25] and reference therein. This is in part driven by advances in the study of partial and pseudodifferential operators, where there is a natural desire to be able to better model and analyse anisotropic phenomena. In particular, pseudo-differential operators on mixed-norm Besov spaces have been studied in [14, 15].

In this paper we study pseudodifferential operators on the family of mixed-norm α-modulation spaces introduced recently by Cleanthous and Georgiadis [13]. In particular, in Section 3.2 we will study pseudodifferential operators induced by symbols in the Hörmander class $S^b_\rho(\mathbb{R}^n \times \mathbb{R}^n)$ on mixed-norm α-modulation spaces. For such symbols we prove in Theorem 3.3 that

$$\sigma(x, D): M^{s,\alpha}_{\vec{p},q}(\mathbb{R}^n) \to M^{s-b,\alpha}_{\vec{p},q}(\mathbb{R}^n)$$

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provided \(0 < \alpha \leq \rho \leq 1\), \(\vec{p} \in (0, \infty)^n\), and \(0 < q < \infty\). The case \(\rho = \alpha = 1\) recovers a known result that symbols in \(S^0_1\) acts boundedly on the Besov spaces, see \cite{19}, but it is interesting to note that it is known that we have a strict inclusion \(S^0_1 \subset S^\rho_\rho\) for \(0 < \rho < 1\), so a larger class of operators is covered by allowing values of \(\alpha < 1\). This supports the claim that mixed-norm \(\alpha\)-modulation spaces are well adapted for symbols in \(S^\rho_\rho(\mathbb{R}^n \times \mathbb{R}^n)\).

In the non-mixed-norm case, pseudodifferential operators on \(\alpha\)-modulation spaces have been considered in \cite{7,9,10,28}. Pseudodifferential operators on modulation spaces were first studied by Tachizawa \cite{31}, and later by a number of authors, see e.g. \cite{5,6,22,23,27,32,33}. In the mixed-norm setting, pseudodifferential operators on Besov and Triebel-Lizorkin spaces have been studied in \cite{15,19}.

The structure of the paper is as follows. In Section \(2\) we introduce mixed-norm Lebesgues and \(\alpha\)-modulation spaces based on a so-called bounded admissible partition of unity (BAPU) adapted to the mixed-norm setting. Section \(2\) also introduces the maximal function estimates that will be needed to prove the main result. In Section \(3\) give a precise definition of the class \(S^\rho_\rho(\mathbb{R}^n \times \mathbb{R}^n)\) and the associated pseudo-differential operators. We provide some boundedness results for multiplier operators on \(\alpha\)-modulation spaces, and then proceed to prove the main result, Theorem \(5.3\).

## 2. Mixed-norm Spaces

In this section we introduce the mixed-norm Lebesgue spaces along with some needed maximal function estimates. Then we introduce mixed-norm \(\alpha\)-modulation spaces based on so-called bounded admissible partition of unity adapted to the mixed-norm setting.

### 2.1. Mixed-norm Lebesgue Spaces

Let \(\vec{p} = (p_1, \ldots, p_n) \in (0, \infty)^n\) and \(f : \mathbb{R}^n \to \mathbb{C}\). We say that \(f \in L_{\vec{p}} = L_{\vec{p}}(\mathbb{R}^n)\) provided

\[
\|f\|_{\vec{p}} := \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} |f(x_1, \ldots, x_n)|^{p_1} dx_1\right)^{\frac{p_1}{p_2}} \cdots dx_n\right)^{\frac{1}{\vec{p}}}, \quad < \infty,
\]

with the standard modification when \(p_j = \infty\) for some \(j = 1, \ldots, n\). The quasi-norm \(\| \cdot \|_{\vec{p}}\) is a norm when \(\min(p_1, \ldots, p_n) \geq 1\) and turns \((L_{\vec{p}}, \| \cdot \|_{\vec{p}})\) into a Banach space. Note that when \(\vec{p} = (p, \ldots, p)\), then \(L_{\vec{p}}\) coincides with \(L_p\). For additional properties of \(L_{\vec{p}}\), see e.g. \cite{1,3,15}.

### 2.2. Maximal operators

The maximal operator will be central to most of the estimates considered in this paper. Let \(1 \leq k \leq n\). We define

\[
M_k f(x) := \sup_{I_k \in I_{\vec{p}}} \frac{1}{|I_k|} \int_I |f(x_1, \ldots, y_k, \ldots, x_n)| dy_k, \quad f \in L^1_{\text{loc}}(\mathbb{R}^n),
\]

where \(I_{\vec{p}}^k\) is the set of all intervals \(I\) in \(\mathbb{R}^n\) containing \(x_k\).

We will use extensively the following iterated maximal function:

\[
M_{\vec{p}} f(x) := \left( M_n \left( \cdots \left( M_1 f(x) \right)^{\theta} \cdots \right)^{\frac{1}{\theta}}(x), \quad \theta > 0, \quad x \in \mathbb{R}^n.
\]

**Remark 2.1.** If \(Q\) is a rectangle \(Q = I_1 \times \cdots \times I_n\), it follows easily that for every locally integrable \(f\)

\[
\int_Q |f(y)| dy \leq |Q| M_1 f(x) = |Q| M_{\vec{p}}^\theta f^{1/\theta}(x), \quad \theta > 0, \quad x \in \mathbb{R}^n.
\]

We shall need the following mixed-norm version of the maximal inequality, see \cite{2,25}:

If \(\vec{p} = (p_1, \ldots, p_n) \in (0, \infty)^n\) and \(0 < \theta < \min(p_1, \ldots, p_n)\) then there exists a constant \(C\) such that

\[
\|M_{\vec{p}} f\|_{L_{\vec{p}}(\mathbb{R}^n)} \leq C \|f\|_{L_{\vec{p}}(\mathbb{R}^n)}
\]
An important related estimate is a Peetre maximal function estimate, which will be one of our main tools in the sequel. For \( \tilde{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n \), \( \tilde{b} = (b_1, \ldots, b_n) \in (0, \infty)^n \), consider the corresponding rectangle \( R = [a_1 - b_1, a_1 + b_1] \times \cdots \times [a_n - b_n, a_n + b_n] \), which will be denoted \( R(\tilde{a}, \tilde{b}) \).

For every \( \theta > 0 \), there exists a constant \( c = c_\theta > 0 \), such that for every \( R > 0 \) and \( f \) with \( \text{supp}(\hat{f}) \subset c_f + R[-2, 2]^n \), see [19],

\[
\sup_{y \in \mathbb{R}^n} |f(y)|/(R(x - y))^{n/\theta} \leq c\mathcal{M}_\theta f(x), \quad x \in \mathbb{R}^n.
\]

In particular, the constant \( c \) is independent of the point \( c_f \in \mathbb{R}^n \).

2.3. Mixed-norm Modulation spaces. In this section we recall the definition of mixed-norm \( \alpha \)-modulation spaces as introduced by Cleanthous and Georgiadis [13]. The \( \alpha \)-modulation spaces form a family of smoothness spaces that contain modulation and Besov spaces as special “extremal” cases. The spaces are defined by a parameter \( \alpha \), belonging to the interval \([0, 1]\). This parameter determines a segmentation of the frequency domain from which the spaces are built.

**Definition 2.2.** A countable collection \( Q \) of subsets \( Q \subset \mathbb{R}^n \) is called an admissible covering of \( \mathbb{R}^n \) if

i. \( \mathbb{R}^n = \bigcup_{Q \in Q} Q \)

ii. There exists \( n_0 < \infty \) such that \( \# \{ Q' \in Q : Q \cap Q' \neq \emptyset \} \leq n_0 \) for all \( Q \in Q \).

An admissible covering is called an \( \alpha \)-covering, \( 0 \leq \alpha \leq 1 \), of \( \mathbb{R}^n \) if

iii. \( |Q| \asymp |x|^{\alpha d} \) (uniformly) for all \( x \in Q \) and for all \( Q \in Q \),

iv. There exists a constant \( K < \infty \) such that

\[
\sup_{Q \in Q} \frac{R_Q}{r_Q} \leq K,
\]

where \( r_Q := \sup \{ r \in [0, \infty) : \exists c_r \in \mathbb{R}^n : B(c_r, r) \subseteq Q \} \) and \( R_Q := \inf \{ r \in (0, \infty) : \exists c_r \in \mathbb{R}^n : B(c_r, r) \supseteq Q \} \), where \( B(x, r) \) denotes the Euclidean ball in \( \mathbb{R}^n \) centered at \( x \) with radius \( r \).

We will need a mixed-norm bounded admissible partition of unity adapted to \( \alpha \)-coverings. We let \( \mathcal{F}(f)(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi} \, dx, \quad f \in L_1(\mathbb{R}^n) \), denote the Fourier transform, and let \( \hat{f}(\xi) = \mathcal{F}(f)(\xi) \).

**Definition 2.3.** Let \( Q \) be an \( \alpha \)-covering of \( \mathbb{R}^n \). A corresponding mixed-norm bounded admissible partition of unity (\( \tilde{p} \)-BAPU) \( \{ \psi_Q \}_{Q \in Q} \) is a family of functions satisfying

- \( \text{supp}(\psi_Q) \subset Q \)
- \( \sum_{Q \in Q} \psi_Q(\xi) = 1 \)
- \( \sup_{Q \in Q} |Q|^{-1} \| \chi_Q \|_{L_p} \| F^{-1} \psi_Q \|_{L_p} < \infty \),

where \( \tilde{p}_j : = \min \{ 1, p_1, \ldots, p_j \} \) for \( j = 1, 2, \ldots, n \) and \( \tilde{p} := (\tilde{p}_1, \ldots, \tilde{p}_n) \).

The results in Sections 3 and 5.2 rely on the known fact that it is possible to construct a smooth \( \tilde{p} \)-BAPU with certain “nice” properties. We summarise the needed properties in the following proposition proved in [13], see also [8]. Let \( \langle x \rangle := (1 + |x|^2)^{1/2} \) for \( x \in \mathbb{R}^n \).

**Proposition 2.4.** For \( \alpha \in [0, 1) \), there exists an \( \alpha \)-covering of \( \mathbb{R}^n \) with a corresponding \( \tilde{p} \)-BAPU \( \{ \psi_k \}_{k \in \mathbb{Z}^n \setminus \{0\}} \subset \mathcal{S}(\mathbb{R}^n) \) satisfying:

i. \( \xi_k \in Q_k, \quad k \in \mathbb{Z}^n \setminus \{0\} \), where \( \xi_k := k \langle k \rangle^{\alpha/(1 - \alpha)} \).

ii. The following estimate holds,

\[
|\partial^\beta \psi_k(\xi)| \leq C_\beta \langle \xi \rangle^{-|\beta| \alpha}, \quad \xi \in \mathbb{R}^n,
\]

for every multi-index \( \beta \) with \( C_\beta \) independent of \( k \in \mathbb{Z}^n \setminus \{0\} \).
iii. Define \( \tilde{\psi}_k(\xi) = \psi_k((\xi_k^n \xi + \xi_k) \). Then for every \( \beta \in \mathbb{N}^d \) there exists a constant \( C_{\beta} \) independent of \( k \in \mathbb{Z}^n \setminus \{0\} \) such that
\[
|\partial_{\xi}^\beta \tilde{\psi}_k(\xi)| \leq C_{\beta} \chi_{B(0, r)}(\xi).
\]

iv. Define \( \mu_k(\xi) = \psi_k(a_k \xi) \), where \( a_k := \langle \xi_k \rangle \). Then for every \( m \in \mathbb{N} \) there exists a constant \( C_m \) independent of \( k \) such that
\[
|\mu_k(y)| \leq C_m a_k^{(m-n)(1-\alpha)} y^{-m}, \quad y \in \mathbb{R}^n.
\]

**Remark 2.5.** A closer inspection of the construction presented in [13] reveals that the BAPU is in fact \( \tilde{p} \)-independent and only depends on \( \alpha \) through the geometry of the \( \alpha \)-covering.

The case \( \alpha = 1 \), corresponding to a dyadic-covering, is not included in Proposition 2.4, but it is known that \( \tilde{p} \)-BAPU can easily be constructed for dyadic coverings, see e.g. [14]. Using \( \tilde{p} \)-BAPUs, it is now possible to introduce the family of mixed-norm \( \alpha \)-modulation spaces.

**Definition 2.6.** Let \( \alpha \in [0, 1], s \in \mathbb{R}, \tilde{p} \in (0, \infty)^n, q \in (0, \infty] \), and let \( Q \) be an \( \alpha \)-covering with associated \( \tilde{p} \)-BAPU \( \{\psi_k\}_{k \in \mathbb{Z}^n \setminus \{0\}} \) of the type considered in Proposition 2.4. Then we define the mixed-norm \( \alpha \)-modulation space, \( M^{s,\alpha}_{\tilde{p},q}(\mathbb{R}^n) \), as the set of tempered distributions \( f \in \mathcal{S}'(\mathbb{R}^n) \) satisfying
\[
\|f\|_{M^{s,\alpha}_{\tilde{p},q}} := \left( \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \langle \xi_k \rangle^{qs} \|F^{-1}(\psi_k \mathcal{F} f)\|_{L_{\tilde{p}}(\mathbb{R}^n)}^q \right)^{1/q} < \infty,
\]
with \( \{\xi_k\}_{k \in \mathbb{Z}^n \setminus \{0\}} \) defined as in Proposition 2.4. For \( q = \infty \), we change the sum to \( \sup_{k \in \mathbb{Z}^n \setminus \{0\}} \).

It is proved in [13] that the definition of \( M^{s,\alpha}_{\tilde{p},q}(\mathbb{R}^n) \) is independent of the \( \alpha \)-covering and of the BAPU, see also [18] for the case of general decomposition space.

3. Pseudodifferential Operators on Mixed-Norm \( \alpha \)-Modulation Spaces

We now turn to the main focus of this article, the study of pseudodifferential operators on mixed-norm \( \alpha \)-modulation spaces. We will state and prove our main result later in this section, but let us first recall the Hörmander class \( \mathcal{H}^{s,\alpha}_{\tilde{p},q}(\mathbb{R}^n) \), for \( b \in \mathbb{R} \) and \( 0 \leq \rho \leq 1 \), which is the family of functions \( \sigma \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) satisfying
\[
|\sigma|^{(b)}_{N,M} := \max_{|\alpha| \leq N, |\beta| \leq M, x, \xi \in \mathbb{R}^n} \langle \xi \rangle^{\rho s} \langle x \rangle^{\beta} e^{\gamma / |x|} |x|^{-\gamma} < \infty,
\]
for all \( M, N \in \mathbb{N} \).

The class \( \mathcal{H}^{b}_{\tilde{p}}(\mathbb{R}^n \times \mathbb{R}^n) \) has been studied in details in e.g. [26]. For \( \rho < 1 \), we have a strict inclusion \( \mathcal{H}^{b}_{\tilde{p}}(\mathbb{R}^n \times \mathbb{R}^n) \subset \mathcal{H}^{0}_{\tilde{p}}(\mathbb{R}^n \times \mathbb{R}^n) \). An example of a symbol \( \sigma \in \mathcal{H}^{0}_{1/2}(\mathbb{R} \times \mathbb{R}) \setminus \mathcal{H}^{0}_{1/2}(\mathbb{R} \times \mathbb{R}) \) is the symbol associated with the convolution kernel \( K(x) = e^{i|\gamma x|} |x|^{-\gamma}, \gamma > 0 \). It can be shown that \( K(\xi) \in \mathcal{H}^{1/2-3/4}_{1/2}(\mathbb{R}^2) \), see [30] Chap. VII.

We define the pseudodifferential operator \( T_\sigma \) induced by \( \sigma \in \mathcal{H}^{b}_{\tilde{p}} \) by
\[
T_\sigma f(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \sigma(x, \xi) \hat{f}(\xi) e^{ix \cdot \xi} d\xi, \quad \text{for every } x \in \mathbb{R}^n, \quad f \in \mathcal{S}(\mathbb{R}^n),
\]
where \( \hat{f} \) is the Fourier transform of the test function \( f \in \mathcal{S}(\mathbb{R}^n) \). We let \( \text{Op} S^{b}_{\tilde{p}} \) denote the family of all operators induced by \( S^{b}_{\tilde{p}} \). Whenever convenient, we will also use the notation \( \sigma(x, D) := T_\sigma \).

An important property of \( S^{b}_{\tilde{p}} \), which we will rely on in the sequel, is the following composition result, see e.g. [20] Chap. 5,
Proposition 3.1. Let \( \sigma_1 \) and \( \sigma_2 \) be symbols belonging to \( S^0_{\rho_1}(\mathbb{R}^n \times \mathbb{R}^n) \) and \( S^0_{\rho_2}(\mathbb{R}^n \times \mathbb{R}^n) \), respectively, for some \( b_1, b_2 \in \mathbb{R} \). Then there is a symbol \( \sigma \in S^0_{\rho_1+b_2}(\mathbb{R}^n \times \mathbb{R}^n) \) so that \( T_\sigma = T_{\sigma_1} T_{\sigma_2} \). Moreover, \( \sigma \) is bounded by \( \langle \cdot \rangle \). Notice that for the Besov spaces we have the lifting property, \( \langle \cdot \rangle^{b_1} \).}

3.1. Fourier multipliers. Let us first briefly special class of pseudodifferential operators, namely Fourier multipliers where the symbol \( \sigma \) is \( x \) independent. Fourier multipliers have been studied in [13] and and we will just summarize the most crucial results for our study, where we will mainly rely on the Bessel potential operator. The Bessel potential \( J^b := (I - \Delta)^b/2 \) is defined by \( J^b f(\xi) = (\xi^2)^b \hat{f}(\xi) \). It is well known that \( \langle \cdot \rangle^b \) is bounded, so in particular \( \langle \cdot \rangle^b \) is \( S^1 \). It also known that for the Besov spaces we have the lifting property, \( J^b B^s_p(\mathbb{R}^n) = B^{s-b}_{p,q}(\mathbb{R}^n) \), see e.g. [19], and it was proven in [13] that \( J^b \) has exactly the same lifting property when considered on \( M^s_{\rho,q}(\mathbb{R}^n) \), \( 0 \leq \alpha \leq 1 \).

Proposition 3.2. Let \( \alpha \in [0, 1], s, \rho \in (0, \infty)^n \), \( q \in (0, \infty) \). Suppose \( b \in \mathbb{R} \) and let \( J^b = (1 - \Delta)^b/2 \). Then we have \( J^b M^{s,\alpha}_{\rho,q}(\mathbb{R}^n) = M^{s-b,\alpha}_{\rho,q}(\mathbb{R}^n) \), in the sense that \( \|f\|_{M^{s,\alpha}_{\rho,q}} \leq \|J^b f\|_{M^{s-b,\alpha}_{\rho,q}} \) for all \( f \in M^{s,\alpha}_{\rho,q}(\mathbb{R}^n) \).

3.2. Boundedness of pseudodifferential operators. We can now state and prove our main result, which we believe will provide a compelling case for the use of mixed-norm \( \alpha \)-modulation spaces with \( \alpha < 1 \) as the symbol classes \( S^0_{\rho} \) are increasing in size with \( \rho \) decreasing.

Theorem 3.3. Suppose \( b \in \mathbb{R} \), \( \alpha \in (0, 1], s, \rho \in (0, \infty)^n \), \( \alpha \leq \rho \leq 1 \), \( s, \rho \in (0, \infty)^n \), and \( q \in (0, \infty) \). Then

\[
\sigma(x, D) : M^{s,\alpha}_{\rho,q}(\mathbb{R}^n) \to M^{s-b,\alpha}_{\rho,q}(\mathbb{R}^n).
\]

Moreover, there exist \( L, N > 0 \) (depending on \( s, \rho, q, \) and \( \rho \)) such that the operator norm is bounded by \( C|\sigma|^{(b)}_{L,N} \), with \( C \) a constant.

Let us consider an example before we turn to the proof the result.

Example 3.4. Consider the symbol associated with the convolution kernel \( K(x) = e^{i|x||x|^{-\gamma}}, \gamma > 0, x \in \mathbb{R}^2 \). As mentioned earlier, \( K(\xi) \in S^\gamma_{1/2-3/4}(\mathbb{R}^2) \). Hence, by Theorem 3.3, \( K(x, D) : M^{s,1/2}_{\rho,q}(\mathbb{R}^2) \to M^{s-\gamma/2+3/4,1/2}_{\rho,q}(\mathbb{R}^2) \), for \( s \in \mathbb{R}, \rho \in (0, \infty)^n \), and \( q \in (0, \infty) \).

Let us now turn to the proof of Theorem 3.3. In the Besov space case, \( \alpha = 1 \) [i.e., \( M^{s,1}_{\rho,q}(\mathbb{R}^n) = B^s_{\rho,q}(\mathbb{R}^n) \)], the proof was given by Georgiadis and the author in [19]. We will therefore only consider the case \( \alpha \in (0, 1) \) below.

Proof of Theorem 3.3. Calling on Propositions 3.1 and 3.2 we have \( J^{-a} M^{s,\alpha}_{\rho,q} = M^{s+a,\alpha}_{\rho,q} \), \( \sigma(x, D) J^a \in \text{Op} S^{s+a}_{\rho,q} \), and \( J^a \sigma(x, D) \in \text{Op} S^{s+a}_{\rho,q} \) when \( \sigma \in S^0_{\rho} \), from which it follows that it is no restriction to assume that \( s \) is large and \( b = 0 \). Moreover, it suffices to prove that \( \|\sigma(x, D) f\|_{M^{s,\alpha}_{\rho,q}} \leq |C|\|f\|_{M^{s,\alpha}_{\rho,q}} \) for \( f \in S(\mathbb{R}^n) \) since \( S(\mathbb{R}^n) \) is dense in \( M^{s,\alpha}_{\rho,q}(\mathbb{R}^n) \), see [13].

Fix \( f \in S(\mathbb{R}^n) \). First we estimate the \( L^p(\mathbb{R}^n) \)-norm of \( \psi_k(D) \sigma(x, D) f \). Notice that for any \( g \in S(\mathbb{R}^n) \),

\[
|\psi_k(D) g(x)| = (2\pi)^{-d/2} \int_{\mathbb{R}^n} e^{ix \cdot y} \psi_k(y) \hat{g}(y) dy = (2\pi)^{-d/2} \int_{\mathbb{R}^n} \hat{\psi}_k(y) g(x + y) dy.
\]
Letting $\sigma^\gamma_{\eta}(x, \xi) := \partial^\gamma_{x} \partial^\alpha_{\xi} \sigma(x, \xi)$ and $\sigma^\gamma := \sigma^\gamma_{0}$, we obtain
\[
\sigma(x + y, D) f(x + y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x+y) \cdot \xi} \sigma(x+y, \xi) \hat{f}(\xi) \, d\xi
\]
\[
= (2\pi)^{-n/2} \sum_{|\gamma| \leq K-1} \frac{y^n}{\gamma!} \int_{\mathbb{R}^n} e^{i(x+y) \cdot \xi} \sigma^\gamma(x, \xi) \hat{f}(\xi) \, d\xi
\]
\[
+ (2\pi)^{-n/2} \sum_{|\gamma| = K} K^{n/\gamma!} \int_{\mathbb{R}^n} e^{i(x+y) \cdot \xi} \int_{0}^{1} (1 - \tau)^{K-1} \sigma^\gamma(x + \tau y, \xi) \hat{f}(\xi) \, d\tau \, d\xi
\]
\[
(3.4) \quad := T(x, y) + R(x, y),
\]
where we have expanded $\sigma(\cdot + y, \xi)$ in a Taylor series centered at $x$. We choose the order $K$ such that $K\alpha > s + 2(1 - \alpha)(1 + n)/r$, where $r := \min\{1, q, p_1, \ldots, p_n\}$. Using (3.4) in (3.3), we obtain
\[
(3.5) \quad \psi_k(D) \sigma(x, D) f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{\psi}_k(y) T(x, y) \, dy + (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{\psi}_k(y) R(x, y) \, dy.
\]
We estimate each of the two terms separately. First we consider the term with $T(x, y)$. We have,
\[
\int_{\mathbb{R}^n} \hat{\psi}_k(y) T(x, y) \, dy = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{\psi}_k(y) \sum_{|\gamma| \leq K-1} \frac{y^n}{\gamma!} \int_{\mathbb{R}^n} e^{i(x+y) \cdot \xi} \sigma^\gamma(x, \xi) \hat{f}(\xi) \, d\xi \, dy
\]
\[
= (2\pi)^{-n/2} \sum_{|\gamma| \leq K-1} \frac{1}{\gamma!} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma^\gamma(x, \xi) \hat{f}(\xi) \, d\xi \cdot \hat{\psi}_k(y) \, dy \, d\xi
\]
\[
(3.6) \quad = \sum_{|\gamma| \leq K-1} \frac{1}{\gamma!} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma^\gamma(x, \xi) \partial^\gamma \psi_k(\xi) \hat{f}(\xi) \, d\xi.
\]
Define $\Psi_k := \sum_{k'} \psi_{k'}$, where the sum is taken over all $k' \in \mathbb{Z}^n \setminus \{0\}$ with $\text{supp}(\psi_{k'}) \neq \emptyset$. Using the fact that $\Psi_k(\xi) = 1$ on $\text{supp}(\psi_k)$, and the relation $(\hat{f} \hat{g})^\nu = f \ast g$, we obtain for $\theta > 0$,
\[
\int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma^\gamma(x, \xi) \partial^\gamma \psi_k(\xi) \hat{f}(\xi) \, d\xi = \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma^\gamma(x, \xi) \partial^\gamma \psi_k(\xi) \hat{f}(\xi) \, d\xi
\]
\[
\leq \int_{\mathbb{R}^n} |(\sigma^\gamma(x, \cdot) \partial^\gamma \psi_k)(y)| \|\Psi_k(D) f(x - y)\| \, dy
\]
\[
\leq \int_{\mathbb{R}^n} \sup_{z \in \mathbb{R}^n} |(\sigma^\gamma(z, \cdot) \partial^\gamma \psi_k)(y)| \|\Psi_k(D) f(x - y)\| \, dy
\]
\[
= \int_{\mathbb{R}^n} \sup_{z \in \mathbb{R}^n} |(\sigma^\gamma(z, \cdot) \partial^\gamma \psi_k)(y)| \langle |\xi_k|^{\alpha} \rangle^{n/\theta} \|\Psi_k(D) f(x - y)\| \, dy,
\]
where $\xi_k = k|k|^{\alpha/(1-\alpha)}$. Using the estimate (b) from Lemma 3.5 below, and the Peetre maximal function estimate (2.6), we conclude that
\[
(3.7) \quad \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma^\gamma(x, \xi) \partial^\gamma \psi_k(\xi) \hat{f}(\xi) \, d\xi \leq C|\sigma|_{L^\infty}^{(0)} \cdot M_{\theta}(\Psi_k(D) f)(x),
\]
with $C < \infty$ independent of $k$ and $f$, provided we choose $L > n(1 + 1/\theta)$. Hence, we may also conclude that
\[
(3.8) \quad \|\int_{\mathbb{R}^n} \hat{\psi}_k(y) T(\cdot, y) \, dy\|_{L^p(\mathbb{R}^n)} \leq C|\sigma|_{L^\infty}^{(0)} \|\Psi_k(D) f\|_{L^p(\mathbb{R}^n)}.
\]
provided $0 < \theta < \min\{p_1, \ldots, p_n\}$. In particular, we may choose $L > n(1 + 1/r)$ to ensure that (3.8) holds.

We turn to the second term in (3.3). Let $\mu_k(\xi) = \psi_k(a_k \xi)$, where $a_k := (k|k|^{\alpha/(1-\alpha)})$. First notice that

$$\int_{\mathbb{R}^n} \hat{\psi}_k(y) R(x,y) \, dy = \int_{\mathbb{R}^n} \hat{\mu}_k(y) R(x,a_k^{-1}y) \, dy.$$ 

We have,

$$\left| \sum_{|\gamma|=K} a_k^{-K|\gamma|} \int_{\mathbb{R}^n} y^\gamma \hat{\mu}_k(y) \int_{\mathbb{R}^n} e^{i(x+a_k^{-1}y) \cdot \xi} \int_0^1 (1 - \tau)^{-K-1} \sigma^\gamma(x + a_k^{-1} \tau y, \xi) \hat{f}(\xi) \, d\tau \, d\xi \, dy \right| \leq C a_k^{-K} \sum_{|\gamma|=K} \int_{\mathbb{R}^n} \langle y \rangle^{-n-1} \sup_{z \in \mathbb{R}^n} \frac{|\sigma^\gamma(z,D)f(x+a_k^{-1}y)|}{\langle y \rangle^{n/\theta}} \, dy$$

Using Lemma 3.5 with $m = K + n + (1 + n)/r$, we obtain the following estimate for the right-hand side for $0 < \theta \leq r$, using that $0 < r \leq 1$,

$$C' a_k^{-K} \sum_{|\gamma|=K} \int_{\mathbb{R}^n} \langle y \rangle^{-n-1} \sup_{z \in \mathbb{R}^n} \frac{|\sigma^\gamma(z,D)f(x+a_k^{-1}y)|}{\langle y \rangle^{n/\theta}} \, dy$$

where $\tilde{K} = K\alpha - (1 + n)(1 - \alpha)/r \geq s + \frac{n+1}{q}(1 - \alpha)$, since $\alpha_k \geq 1$. Now,

$$\left( \sum_{k \in \mathbb{Z}^n \setminus \{0\}} a_k^{s-K} \left\| \int_{\mathbb{R}^n} \hat{\psi}_k(y) R(x,y) \, dy \right\|_{L^q(dx)}^{q} \right)^{1/q} \leq \left\{ C \sum_{k \in \mathbb{Z}^n \setminus \{0\}} a_k^{s-K} \left( \sum_{|\gamma|=K} \left\| \sup_{\eta, z \in \mathbb{R}^n} \frac{|\sigma^\gamma(z,D)f(x+\eta)|}{\langle \eta \rangle^{n/\theta}} \right\|_{L^q(dx)} \right)^{q} \right\}^{1/q}.$$

We notice that $L^q := C \sum_{k \in \mathbb{Z}^n \setminus \{0\}} a_k^{(s-K)q} \leq C \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|^{-n-1}$ is finite. Based on this observation, recalling that $r = \min\{1, q, p_1, \ldots, p_n\}$, we use the equivalence of $\ell^r$-norms on finite dimensional spaces to estimate the right-hand side by,

$$L \left( \sum_{|\gamma|=K} \left\| \sup_{\eta, z \in \mathbb{R}^n} \frac{|\sigma^\gamma(z,D)f(x+\eta)|}{\langle \eta \rangle^{n/\theta}} \right\|_{L^q(dx)} \right)^{1/r} \leq L \left( \sum_{|\gamma|=K} \left\| \sup_{\eta, z \in \mathbb{R}^n} \frac{|\sigma^\gamma(z,D)\psi_k(D)f(x+\eta)|}{\langle \eta \rangle^{n/\theta}} \right\|_{L^q(dx)} \right)^{1/r} \leq L \left( \sum_{|\gamma|=K} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \left\| \sup_{\eta, z \in \mathbb{R}^n} \frac{|\sigma^\gamma(z,D)\psi_k(D)f(x+\eta)|}{\langle \eta \rangle^{n/\theta}} \right\|_{L^q(dx)} \right)^{1/r}.$$
We now focus on the individual term $A_k := |[\sigma^\gamma(z, D)\psi_k(D)f](x + \eta)|$. Put $f_k(x) := |\Psi_k(D)f|(x)$, with $\Psi_k$ defined as above. We have

$$A_k = \left| \int_{\mathbb{R}^n} (\sigma^\gamma(z, \xi)\psi_k(\xi))^\vee(x + \eta - y)f_k(y) \, dy \right|$$

$$\leq \int_{\mathbb{R}^n} |(\sigma^\gamma(z, \xi)\psi_k(\xi))^\vee(x + \eta)| \, |f_k(y)| \, dy$$

$$\leq \sup_{u \in \mathbb{R}^n} \frac{|f_k(u)|}{(x - u)^{n/\theta}} \int_{\mathbb{R}^n} |(\sigma^\gamma(z, \xi)\psi_k(\xi))^\vee(x + \eta - y)| \, |x - y|^{n/\theta} \, dy.$$ 

Now, $(x - y)^{n/\theta} \leq c(x - y + \eta)^{n/\theta}(\eta)^{n/\theta}$, so

$$\sup_{x, \eta \in \mathbb{R}^n} \frac{A_k}{\langle \eta \rangle^{n/\theta}} \leq C \sup_{x, \eta \in \mathbb{R}^n} \frac{|f_k(x - \eta)|}{\langle \eta \rangle^{n/\theta}} \sup_{x, \eta \in \mathbb{R}^n} \int_{\mathbb{R}^n} |(\sigma^\gamma(z, \xi)\psi_k(\xi))^\vee(u)| \, (\eta)^{n/\theta} \, du$$

$$\leq C' \sup_{x, \eta \in \mathbb{R}^n} \frac{|f_k(x - \eta)|}{\langle \eta \rangle^{n/\theta}} |\sigma|^{(0)}_{L, K},$$

provided $L > n + n/\theta$, where we have used Lemma 3.5. Hence,

$$\sum_{|\gamma| = K} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \left\| \sup_{\eta, z \in \mathbb{R}^n} \frac{|\sigma^\gamma(z, D)\psi_k(D)f|(x + \eta)|}{\langle \eta \rangle^{n/\theta}} \right\|_{L_p(dx)}$$

$$= \sum_{|\gamma| = K} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} a_{k}^{nr/\theta} \left\| \sup_{\eta, z \in \mathbb{R}^n} \frac{|\sigma^\gamma(z, D)\psi_k(D)f|(x + \eta)|}{a_{k}^{n/\theta} \langle \eta \rangle^{n/\theta}} \right\|_{L_p(dx)}$$

$$\leq C'(|\sigma|^{(0)}_{L, K})^r \sum_{k \in \mathbb{Z}^n \setminus \{0\}} a_{k}^{nr/\theta} \left\| \sup_{\eta \in \mathbb{R}^n} \frac{|f_k(x - \eta)|}{\langle a_k \eta \rangle^{n/\theta}} \right\|_{L_p(dx)}.$$ 

We now use the Peetre maximal estimate,

$$\sup_{z \in \mathbb{R}^n} \frac{|f_k(x - z)|}{\langle a_k z \rangle^{n/\theta}} \leq C M_\theta(f_k)(x),$$

and we may apply the $L_p$-norms, using the maximal inequality, to obtain

$$\left\| \sup_{z \in \mathbb{R}^n} \frac{|f_k(x - z)|}{\langle a_k z \rangle^{n/\theta}} \right\|_{L_p(\mathbb{R}^n)} \leq C'\|f_k\|_{L_p(\mathbb{R}^n)}^r,$$

provided $0 < \theta < \min\{p_1, \ldots, p_n\}$. Putting these estimates together yields,

$$\sum_{|\gamma| = K} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \left\| \sup_{\eta, z \in \mathbb{R}^n} \frac{|\sigma^\gamma(z, D)\psi_k(D)f|(x + \eta)|}{\langle \eta \rangle^{n/\theta}} \right\|_{L_p(dx)}$$

$$\leq C'(|\sigma|^{(0)}_{L, K})^r \sum_{k \in \mathbb{Z}^n \setminus \{0\}} a_{k}^{nr/\theta} \left\| \Psi_k(D)f \right\|_{L_p(\mathbb{R}^n)},$$

provided $L > n + n/\theta$, and consequently

$$\left( \sum_{k \in \mathbb{Z}^n \setminus \{0\}} a_{k}^{eq} \left\| \int_{\mathbb{R}^n} \psi_k(y)R(x, y) \, dy \right\|_{L_p}^q \right)^{1/q} \leq C' |\sigma|^{(0)}_{L, K} \left( \sum_{k \in \mathbb{Z}^n \setminus \{0\}} a_{k}^{nr/\theta} \left\| \Psi_k(D)f \right\|_{L_p(\mathbb{R}^n)} \right)^{1/r}.$$

$$= C' |\sigma|^{(0)}_{L, K} \left( \sum_{k \in \mathbb{Z}^n \setminus \{0\}} a_{k}^{nr\theta-sr} a_{k}^{sr} \left\| \Psi_k(D)f \right\|_{L_p(\mathbb{R}^n)} \right)^{1/r},$$

$$\leq C' |\sigma|^{(0)}_{L, K} \left( \sum_{k \in \mathbb{Z}^n \setminus \{0\}} a_{k}^{sr} \left\| \Psi_k(D)f \right\|_{L_p(\mathbb{R}^n)} \right)^{1/r},$$

$$\leq C' |\sigma|^{(0)}_{L, K} \left( \sum_{k \in \mathbb{Z}^n \setminus \{0\}} a_{k}^{sr} \left\| \Psi_k(D)f \right\|_{L_p(\mathbb{R}^n)} \right)^{1/r}.$$
where for the last estimate, we used Hölder’s inequality with parameters $q/r$ and $q/(q - r)$ and the fact that $s > n(1 + \theta)/r$, where we also notice that $n(1 + \theta)/r < 3n/r$ since $0 < \theta < 2$. Finally, we can put all the estimates together to close the case $b = 0$ and $s > 3n/r$. We have, with $L > n + n/r$,

$$
\|\sigma(x, D)f\|_{M_{F,q}^{\alpha}}\leq C\left\{\|a_k^\alpha\| \int_{\mathbb{R}^n} \hat{\psi}_k(y) T(x, y) \, dy \right\}_{L_p(\mathbb{R}^n)} + \|a_k^\alpha\| \int_{\mathbb{R}^n} \hat{\psi}_k(y) R(x, y) \, dy \right\}_{L_p(\mathbb{R}^n)}
$$

and

$$
\|\sigma_j K(\alpha, D)\|_{M_{F,q}^{\alpha}} \leq C' \left[\|a_k^\alpha\| \|\psi_k(D)\|_{L_p(\mathbb{R}^n)}\right] \left[\|\sigma_j\|_{L_p(\mathbb{R}^n)}\right] \|\sigma_j\|_{L_p(\mathbb{R}^n)} \|\psi_k(D)\|_{L_p(\mathbb{R}^n)}
$$

This concludes the proof of the theorem. □

The following technical lemma has been used in the proof of Theorem 3.3.

**Lemma 3.5.** Let $\alpha \in [0, 1)$ and let $\{\psi_k\}_{k \in \mathbb{Z}^n \setminus \{0\}}$ be the $\tilde{p}$-BAPU from Proposition 2.4, depending only on $\alpha$. Suppose $\sigma \in S^0_{\rho, 0}$, $\alpha \leq \rho \leq 1$. Then for $|\gamma| \leq K$ and $m \geq 0$, we have

(a) For $|\gamma|, \nu \leq K$ and $J \in \mathbb{N}$ there exists a constant $C := C(K, J)$ such that

$$
M(x) := \sup_{z \in \mathbb{R}^n} \left| (\partial^\gamma_x \sigma(z, \cdot) \partial^\nu_x \psi_k)^\nu(x) \right| \leq C |\sigma|_{J, K} \langle k \rangle^{\alpha(1 - \gamma)} \langle k \rangle^{|\alpha| - 1} dx \leq C |\sigma|^0_{J, K}, \quad k \in \mathbb{N},
$$

for any $M \in \mathbb{N}$ satisfying $M > m + n$.

(b) For $|\gamma|, \nu \leq K$ and $m \geq 0$ there exists a constant $C' := C'(K, m)$, such that

$$
I := \int_{\mathbb{R}^n} \sup_{z \in \mathbb{R}^n} \left| (\partial^\gamma_x \sigma(z, \cdot) \partial^\nu_x \psi_k)^\nu(x) \right| \langle k \rangle^{|\alpha(1 - \gamma)} \langle k \rangle^{|\alpha| - 1} dx \leq C' |\sigma|^0_{J, K}, \quad k \in \mathbb{N},
$$

for any $M \in \mathbb{N}$ satisfying $M > m + n$.

**Proof.** First we prove (a). Let $\sigma_x^\gamma(x, \xi) := \partial^\gamma_x \partial^\nu_x \sigma(x, \xi)$ and $\sigma^\alpha := \sigma^0_x$. We have the equality

$$
M(x) = \sup_{z \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma^\gamma(z, \xi) \partial^\nu_x \psi_k(T_k \xi) \, d\xi \right|.
$$

Let $T_k = |\xi_k\|^\alpha + \xi_k$, where $\xi_k = k |\xi\|^\alpha(1 - \alpha)$. Then a substitution yields

$$
M(x) = |\xi_k|^\alpha \sup_{z \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{it^\nu \xi_k \cdot x} \sigma^\gamma(z, T_k \xi) \partial^\nu_x \psi_k(T_k \xi) \, d\xi \right|.
$$

Fix $J > 1$. We use the well-known estimate $|\langle x \rangle^J |\hat{\psi}(x)| \leq C_J \sum_{|\beta| \leq J} \|\partial^\beta g\|_{L_1}$, for some finite constant $C_J$. We apply the estimate to (3.9) to obtain

$$
M(|\xi_k|^\alpha x) \leq C |\xi_k|^\alpha \sup_{z \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} |\partial^\beta \left[ \sigma^\gamma(z, T_k \xi) \partial^\nu_x \psi_k(T_k \xi) \right] \right| \, d\xi (x)^{-J},
$$

which by Leibniz’s rule provides the bound

$$
M(|\xi_k|^\alpha x) \leq C' \sum_{|\beta| \leq J} |\xi_k|^\alpha \sum_{0 \leq \sigma \leq \beta} \sup_{z \in \mathbb{R}^n} \left| \partial^\beta \left[ \sigma^\gamma(z, T_k \xi) \right] \right| |\partial^\beta \partial^\nu_x \psi_k(T_k \xi) \right| \, d\xi (x)^{-J}.
$$

From Proposition 2.4, we have

$$
|\partial^\beta \partial^\nu_x \psi_k(T_k \xi) \right| \leq C \chi Q(\xi),
$$

where $Q(\xi)$ is a polynomial in $|\xi|$. This completes the proof. □
with $C := C(\nu, \beta, \eta)$. We also notice that for $\xi \in Q$,
\begin{equation}
\left| \partial^\gamma_\xi (\sigma^\gamma (z, T_k \xi)) \right| \leq \left| \sigma^{(0)}_{|\eta|, K} (|\xi_k|^{\alpha} \xi + \xi_k)^{-\rho|\eta|} \right| \leq C \left| \sigma^{(0)}_{|\eta|, K} (\xi_k)^{-\rho|\eta|} \right| .
\end{equation}

Now, by assumption $\alpha \leq \rho$, so using the estimates (3.11) and (3.12) in (3.10), we obtain
\[ M(|\xi_k|^{-\alpha} x) \leq C''' |\xi_k|^{\alpha} \sum_{|\beta| \leq J} |\sigma^{(0)}_{|\eta|, K} \int_{\mathbb{R}^n} \chi_Q (\xi) d\xi (x) x^{-J} \leq C''' |\xi_k|^{\alpha} \cdot |\sigma^{(0)}_{|\eta|, K} (x)^{-J} ,
\]
which proves (a), since $|\xi_k| = |k|^{1/(1-\alpha)}$.

Let us turn to (b). Pick $J > m + n$ in (a). We have
\[
I = \int_{\mathbb{R}^n} M(x)(|k|^{\alpha/(1-\alpha)} x)^m \, dx 
\leq C' |\sigma^{(0)}_{|\eta|, K} |k|^{\alpha/(1-\alpha)} \int_{\mathbb{R}^n} \langle |k|^{\alpha/(1-\alpha)} x \rangle^{-J} \langle |k|^{\alpha/(1-\alpha)} x \rangle^m \, dx 
= C' |\sigma^{(0)}_{|\eta|, K} \int_{\mathbb{R}^n} \langle x \rangle^{-J} \langle x \rangle^m \, dx 
\leq \tilde{C} |\sigma^{(0)}_{|\eta|, K} ,
\]
where we made a change of variable in the integral and used $J > n + m$, which of course implies that $m - J < -n$. This concludes the proof. \hfill \Box

4. HYPOELLIPTIC PSEUDODIFFERENTIAL OPERATORS

In this final section we consider an application of the result in the previous section to hypoelliptic pseudodifferential operators based on standard machinery, see e.g. [24]. Let us introduce some notation. Let
\[
S^\infty_{\rho} := \bigcup_{b \in \mathbb{R}} S^b_{\rho}, \quad \text{and} \quad S^{-\infty}_{\rho} := \bigcap_{b \in \mathbb{R}} S^b_{\rho}.
\]
Assume that $b_0, b \in \mathbb{R}$ such that $b_0 \leq b$. An element $\sigma \in S^b_{\rho}(\mathbb{R}^n \times \mathbb{R}^n)$ is called hypoelliptic with parameters $b_0$ and $b$ if there are positive constants $c$ and $\alpha$ such that
\[
a(\xi)^{b_0} \leq |\sigma(x, \xi)|, \quad \langle \xi \rangle \geq c,
\]
and
\[
|\partial^\gamma_\xi \partial^\delta_\xi \sigma(x, \xi)| \leq C_{\alpha, \beta} |\sigma(x, \xi)| \langle \xi \rangle^{-\rho|\alpha|}, \quad \langle \xi \rangle \geq c.
\]
Let $H S^b_{\rho, \alpha}(\mathbb{R}^n \times \mathbb{R}^n)$ the family of all such symbols. The following result is well-know, see [24] Theorem 22.1.3.

**Theorem 4.1.** Suppose $\sigma \in H S^b_{\rho, \alpha}$, with $0 < \rho \leq 1$. Then there exists $\tau \in H S^{b_0, -b}_{\rho}$ such that $I - \sigma(x, D) \tau(x, D)$ and $I - \tau(x, D) \sigma(x, D)$ are both in $\text{Op}(S^\infty_{\rho})$.

Let $M_{\rho, q}^{s, \alpha}(\mathbb{R}^n) = \cup_{s \in \mathbb{R}} M_{\rho, q}^{s, \alpha}(\mathbb{R}^n)$. Using Theorem 4.1 and the result from the previous section we have

**Theorem 4.2.** Suppose $\sigma \in H S^b_{\rho, \alpha}$, with $\rho \geq \alpha > 0$, and $f \in M_{\rho, q}^{-\infty, \alpha}(\mathbb{R}^n)$. If $\sigma(\cdot, D)f \in M_{\rho, q}^{s, \alpha}(\mathbb{R}^n)$ for some $s \in \mathbb{R}$, then $f \in M_{\rho, q}^{s+b, \alpha}(\mathbb{R}^n)$.

**Proof.** Let $S = \sigma(\cdot, D)$, and let $T = \tau(\cdot, D)$ be as in Theorem 4.1. Notice that $f = T(Sf) + (I - TS)f$. By Theorem 4.1 $T$ maps $M_{\rho, q}^{s, \alpha}(\mathbb{R}^n)$ to $M_{\rho, q}^{s+b, \alpha}(\mathbb{R}^n)$ and $(I - TS)$ maps $M_{\rho, q}^{-\infty, \alpha}(\mathbb{R}^n)$ to $M_{\rho, q}^{s+b, \alpha}(\mathbb{R}^n)$. \hfill \Box

The following example will conclude the paper.
Example 4.3. Consider the heat operator \( L \) given by
\[
L(u) := \frac{\partial u}{\partial t} - \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2}.
\]
The symbol of \( L \) is given by
\[
l(\tau, \xi) = (i\tau + |\xi|^2), \quad (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n,
\]
and one can easily verify that \( l \in HS^{2,1}_1 \). We consider an approximate inverse \( P \) to \( L \) with symbol
\[
a(\tau, \xi) = (i\tau + |\xi|^2)^{-1} \eta(\tau, \xi), \quad (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n,
\]
where \( \eta \) is a smooth cut-off function that vanishes near the origin and equals 1 for large \((\tau, \xi)\). It is easy to check that \( a \in HS^{-1,-2}_{1,1}(\mathbb{R}^{n+1}) \). Hence, if \( u \in M^{-\infty,\alpha}_{\vec{p},q}(\mathbb{R}^{n+1}), \vec{p} \in (0, \infty)^n, 0 < q < \infty, \alpha \in (0, 1] \), and \( P(u) \in M^{s,\alpha}_{\vec{p},q}(\mathbb{R}^{n+1}) \), then \( u \in M^{s-1,\alpha}_{\vec{p},q}(\mathbb{R}^{n+1}) \).

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