A DICHOTOMY FOR MEASURES OF MAXIMAL ENTROPY NEAR TIME-ONE MAPS OF TRANSITIVE ANOSOV FLOWS

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ABSTRACT. We show that time-one maps of transitive Anosov flows of compact manifolds are accumulated by diffeomorphisms robustly satisfying the following dichotomy: either all of the measures of maximal entropy are non-hyperbolic, or there are exactly two ergodic measures of maximal entropy, one with a positive central exponent and the other with a negative central exponent.

We establish this dichotomy for certain partially hyperbolic diffeomorphisms isotopic to the identity whenever both of their strong foliations are minimal. Our proof builds on the approach developed by Margulis for Anosov flows where he constructs suitable families of measures on the dynamical foliations.

1. Introduction

In his pioneering work [27], Margulis studied measures of maximal entropy of geodesic flows in order to count closed geodesics for manifolds with variable negative curvature. More precisely, he constructed a family of measures \( \{m_x\}_{x \in M} \) such that for all \( x \in M \) the measure \( m_x \) is carried by the unstable manifold at \( x \), and for all \( t \in \mathbb{R} \) we have

\[
(\varphi_t)_* m_x = e^{-t \cdot h_{\varphi}(x)} m_{\varphi^t x}.
\]

He then built an invariant probability measure which was observed to be a measure of maximal entropy and is now called the Bowen-Margulis measure. It was then proved to be the unique measure of maximal entropy. We refer to Ledrappier [25] for an introduction.

In this paper, we will extend Margulis’ construction to a class of partially hyperbolic maps and obtain a striking dichotomy.

**Theorem 1.1.** If \( \varphi^t \) is a transitive Anosov flow on a compact manifold \( M \), then there is an open set \( U \) in \( \text{Diff}^1(M) \) which contains \( \varphi^1 \) in its closure such that for any \( f \in U \cap \text{Diff}^2(M) \) we have the following dichotomy:

1. either all the measures of maximal entropy have zero central Lyapunov exponents, or
2. there are exactly two ergodic measures of maximal entropy where one has a positive central exponent and the other has a negative central exponent, and both measures are Bernoulli.

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Related results. These results are part of a larger program to understand properties of entropy beyond uniform hyperbolicity. In that classical setting, say for a transitive Anosov diffeomorphism, there is a unique measure of maximal entropy (MME). Even though there are a number of significant results beyond the hyperbolic setting [29, 9, 10] there are still many fundamental open questions beyond the uniformly hyperbolic setting. Partially hyperbolic diffeomorphisms with one-dimensional center have been studied as the “next nontrivial class”. A MME always exists in this setting by entropy expansivity (see [12, 15, 26]). Its uniqueness is a more delicate question.

Uniqueness of the MME has been shown for certain systems that are derived from Anosov, a subclass introduced by Mañé, first for specific constructions, then in greater and greater generality [7, 13, 38, 16, 8].

The partially hyperbolic diffeomorphisms with a center foliation into circles form another subclass with a more subtle behavior. Assuming accessibility, [31] has established the following dichotomy:

– either the dynamics is isometric in the center direction and there exists a unique MME which is nonhyperbolic;
– there are multiple hyperbolic MMEs.

Strategy of proof. We introduce a new subclass of partially hyperbolic diffeomorphisms with one-dimensional center which we call flow type. They are isotopic to the identity and the fundamental examples are the perturbations of time-one maps of Anosov flows. Our main result is Theorem 3.9: the above dichotomy holds for partially hyperbolic flow type diffeomorphisms whose strong foliations are both minimal.

The uniqueness of the MME for a given sign of the central exponent (say nonpositive) follows from a variant of Margulis’ approach. Namely, we first build a family of measures on the center-unstable leaves. Then we construct measures on unstable leaves, which we call Margulis $u$-conditionals. This is more difficult for maps than for flows.

We then use the entropy with respect to the unstable foliation as introduced by Ledrappier and Young [22] and an argument of Ledrappier [24] to show that, when its central exponent is nonpositive, a measure maximizes the entropy if and only if its disintegration along the unstable leaves is given by the Margulis conditionals.

A Hopf argument shows that if there is a MME with negative central exponent, then any MME with nonpositive central exponent must coincide with it. The symmetry between positive and negative central exponents in the hyperbolic case follows from the one-dimensionality of the central leaves: we associate to any measure with, say negative central exponent, an isomorphic one with nonnegative central exponent.

A hyperbolic ergodic MME is isomorphic to a Bernoulli shift times a circular permutation, according to a general result by Ben Ovadia [30]. The triviality of the permutation follows by considering iterates. This concludes the proof of Theorem 3.9.

Finally, to prove Theorem 1.1 we establish Theorem 3.10, i.e., we find open sets of partially hyperbolic flow type diffeomorphisms with both strong foliations minimal near any time-one map of a transitive Anosov flow. We first show that such time-one maps are robustly flow type partially hyperbolic diffeomorphisms. Then
Bonatti and Díaz [3] provide a perturbation ensuring robust transitivity. Lastly, by a further perturbation following Bonatti, Díaz, and Urés [4] we can ensure robust minimality of both strong foliations. Theorem 1.1 now follows from Theorem 3.9.

The use of Margulis conditionals. The construction of Margulis has given rise to a large body of work, mainly devoted to the estimation of the number of periodic orbits, sometimes beyond the uniformly hyperbolic setting [20]. We refer to Sharp’s survey in [28], the long awaited publication of Margulis’ thesis. The works of Hamenstädt [17] and Hasselblatt [18] that give a geometric description of the Margulis conditionals \( \{ m_x \}_{x \in M} \) are perhaps closer to our concerns.

While this work was being written, we learned that a different but related approach has been developed in [11]. This approach can deal with equilibrium measures (i.e., generalizations of measures of maximal entropy taking into account a weight function) but requires non-expansion along the center. Separately, Jiagang Yang has told us that he also has results on the MMEs for the same type of diffeomorphisms as we consider.

Comments. Let us first note that part of our results could be obtained from symbolic dynamics, using generalizations of ideas going back to the classical works of Sinai, Ruelle, and Bowen (see, e.g., [36, 5, 34]). More precisely, existence of at most one MME with, say, positive central exponent can be deduced from [9, Section 1.6] since, in the terminology of this work, our minimality assumption implies that there is a unique homoclinic class of measures with a given sign of the central exponent. However, the dichotomy does not seem to follow from this approach which is blind to nonhyperbolic measures.

Second, one usually expects that results such as ours can be extended to \( C^{1+\alpha} \) smoothness, for any \( 0 < \alpha < 1 \), and generalized to equilibrium measures with respect to Hölder-continuous potentials (although uniqueness holds for generic potentials [34]).

Questions. Our techniques demand a very strong form of irreducibility and the flow type property is somewhat technical. Hence we ask:

**Question 1.** In Theorem 3.9, can one replace minimality of both strong foliations by minimality of just one or by robust transitivity? Can one replace flow type by isotopic to the identity?

In the volume-preserving setting there is a rigidity result [2]. We think that some version of it may hold for MMEs in the dissipative setting.

**Question 2.** In the setting of Theorem 3.9, is the hyperbolic case open and dense? When the MME is nonhyperbolic, does this imply that the diffeomorphism is the time one map of a flow? does it at least exclude the existence of hyperbolic periodic points?

Though we will identify the disintegrations of nonhyperbolic MMEs along both strong foliations, their analysis remains incomplete:

**Question 3.** Consider a partially hyperbolic diffeomorphism \( f \) with flow type and with minimality of both strong foliations. Can its disintegration along the center be atomic like in the hyperbolic case? Can there be more than one nonhyperbolic MME? Are nonhyperbolic MMEs Bernoulli?
We prove that the hyperbolic MMEs are Bernoulli, hence strongly mixing. One can try to establish some speed (see [39] for a related result).

**Question 4.** If $\mu$ is a hyperbolic MME for a flow type diffeomorphism $f$ with minimality of both strong foliations, does it satisfy exponential decay of correlations for Hölder-continuous functions, i.e., for any Hölder-continuous functions $u, v : M \rightarrow \mathbb{R}$, does there exist a number $\kappa < 1$ such that:

$$\int_M u \circ f^n.v \, d\mu - \int_M u \, d\mu \int_M v \, d\mu = O(\kappa^n)?$$

For Anosov flows, the topological entropy can obviously be changed by perturbations whereas it is locally constant for Anosov diffeomorphisms. What is the situation for the maps we consider?

**Question 5.** Consider a flow type diffeomorphism whose strong foliations are robustly both minimal. Is it true that the volume growth of each strong leaf is equal to the topological entropy? Does the topological entropy have a homological interpretation? Can an arbitrarily small perturbation make the topological entropy locally constant as a function of the diffeomorphism?

2. **Background**

In this section we review concepts of partial hyperbolicity, Lyapunov exponents, and disintegration of measures.

2.1. **Partial hyperbolicity.** For a diffeomorphism $f : M \rightarrow M$ of a compact manifold to itself recall the norm and conorm with respect to a subspace of $V \subset T_x M$ for some $x \in M$: $\|Df|V\| := \max\{\|Tf(v)\| : v \in V, \|v\| = 1\}$ and $\text{conorm}(Df|V) := \min\{\|Tf(v)\| : v \in V, \|v\| = 1\}$.

A splitting $E \oplus F$ is dominated if it is nontrivial, invariant, and if there is some $N \geq 1$ such that, for all $x \in M$:

$$\|Df^N|E_x\| < \frac{1}{2} \text{conorm}(Df^N|F_x).$$

**Definition 2.1.** A diffeomorphism is (strongly) partially hyperbolic if there is an invariant splitting of the tangent bundle: $TM = E^s \oplus E^c \oplus E^u$ such that $E^s \oplus (E^c \oplus E^u)$ and $(E^s \oplus E^c) \oplus E^u$ are dominated, $E^s$ is uniformly contracted, and $E^u$ is uniformly expanded.

The stable and unstable bundles $E^s$ and $E^u$ of a partially hyperbolic diffeomorphism are always uniquely integrable into stable and unstable foliations, respectively, denoted by $F^s$ and $F^u$. The bundles $E^c$, $E^{cs} := E^s \oplus E^c$, and $E^{cu} := E^c \oplus E^u$ fail to be integrable for some strongly partially hyperbolic diffeomorphisms.

**Definition 2.2.** A strongly partially hyperbolic diffeomorphism $f$ is dynamically coherent if there exists invariant foliations $F^{cs}$ and $F^{cu}$ that are tangent to the $E^{cs}$ and $E^{cu}$ bundles respectively. In this case there is a center foliation $F^c$ given by $F^c(x) = F^{cs}(x) \cap F^{cu}(x)$ for $x \in M$. 

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1This is sometimes called pointwise domination, see [1].
We refer to [6] for various other definitions of dynamical coherence and their relationships.

For a dynamically coherent diffeomorphism $f$ each leaf of $F^{cs}$ is subfoliated by the leaves of $F^{c}$ and the leaves of $F^{s}$. A similar statement holds for the center-unstable foliation. Then for any points $p, q \in M$ where $q \in F^{s}(p)$ there is a neighborhood $U_{p}$ of $p$ in the leaf $F^{s}(p)$ and a homeomorphism $h^{s}_{p,q} : U_{p} \rightarrow F^{c}(q)$ such that

$$h^{s}_{p,q}(x) \in F^{s}(x) \cap F^{loc}_{c}(q).$$

The map $h^{s}_{p,q}$ is the (local) stable holonomy map. We can similarly define the unstable holonomy map.

2.2. Center Lyapunov exponents. For a strongly partially hyperbolic diffeomorphism $f : M \rightarrow M$ a real number $\chi$ is a center Lyapunov exponent at $x \in M$ if there exists a nonzero vector $v \in E^{c}_{c}$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \| Df^{n}(v) \| = \chi.$$ 

If $\dim E^{c} = 1$, then the limit above only depends on $x$ and exists $m$-almost everywhere for any $f$-invariant Borel probability measure $m$. For $m$ an ergodic $f$-invariant Borel probability measure the limit takes on a single value for $m$-almost every $x \in M$.

2.3. Disintegration of a measure. Let $X$ be a Polish space and $\mu$ be a finite Borel measure on $X$. Let $\mathcal{P}$ be a partition of $X$ into measurable sets. Let $\hat{\mu}$ be the induced measure on the $\sigma$-algebra generated by $\mathcal{P}$. A system of conditional measures of $\mu$ with respect to $\mathcal{P}$ is a family $\{\mu_{P}\}_{P \in \mathcal{P}}$ of probability measures on $X$ such that

(1) $\mu_{P}(P) = 1$ for $\mu$-almost every $P \in \mathcal{P}$, and
(2) given any continuous function $\psi : X \rightarrow \mathbb{R}$, the function $P \rightarrow \int \psi d\mu_{P}$ is integrable, and

$$\int_{X} \psi d\mu = \int_{\mathcal{P}} \left( \int \psi d\mu_{P} \right) d\hat{\mu}(P).$$

Rokhlin [32, 33] proved that if $\mathcal{P}$ is a measurable partition, then the disintegration always exists and is essentially unique.

We will consider partitions given by foliations of a manifold. If a foliation has a positive measure set of noncompact leaves, then the result of Rokhlin does not immediately apply. However, one can extend the result of Rokhlin by disintegrating into measures defined up to scaling (see Avila, Viana, and Wilkinson [2]).

Let $M$ be a manifold where $\dim(M) \geq 2$ and $m$ be a locally finite measure on $M$. Let $\mathcal{B}$ be a small foliation box. Then Rokhlin’s result implies there is a disintegration $\{m_{x}^{B} : x \in \mathcal{B}\}$ of the restriction of $m$ to the foliation box into conditional probability measures along the local leaves of the foliation, i.e., the connected components $F_{B}(x)$ of $F(x) \cap B$ for $x \in \mathcal{B}$. From [2, Lemma 3.2] we know that if $\mathcal{B}$ and $\mathcal{B}'$ are foliation boxes and $m$-almost any $x \in \mathcal{B} \cap \mathcal{B}'$, then the restriction of $m_{x}^{B}$ and $m_{x}^{B'}$ coincide up to a constant factor.

We then know that for $m$-almost every $x \in M$ there is a projective measure $m_{x}$ (i.e., defined up to some scaling possibly depending on $x$) such that $m_{x}(M \setminus F(x)) = 0$. Furthermore, the function $x \mapsto m_{x}$ is constant along the leaves of $F$, and the conditional probabilities $m_{x}^{B}$ along the local leaves of any foliation box $\mathcal{B}$ coincide
almost everywhere with the normalized restriction of the \( m_x \) to the local leaves of \( B \).

Finally, we note that if the foliation \( \mathcal{F} \) is fixed by some diffeomorphism (i.e., \( f(\mathcal{F}(x)) = \mathcal{F}(x) \)) without fixed points, one can replace the projective measures by true measures using the global normalization: \( m_x([x, f(x)]) = 1 \) for all \( x \in M \).

2.4. Continuous systems of measures. We will work with families of measures carried by the leaves of the dynamical foliations up to a union of exceptional leaves.

**Definition 2.3.** Given a foliation \( \mathcal{F} \) of some manifold \( M \) and some \( \mathcal{F} \)-saturated subset \( M_1 \subset M \) a continuous system of measures on \( \mathcal{F}|_{M_1} \) is a family \( \{ m_x \}_{x \in M_1} \) such that:

(i) for all \( x \in M_1 \), \( m_x \) is a Radon measure on \( \mathcal{F}(x) \);

(ii) for all \( x, y \in M_1 \), \( m_x = m_y \) if \( \mathcal{F}(x) = \mathcal{F}(y) \);

(iii) \( M \) is covered by foliation charts \( B \) such that: \( x \mapsto m_x(\phi|_{\mathcal{F}(x)}) \) is continuous on \( M_1 \) for any \( \phi \in C_c(B) \).

The Radon property (i) means that each \( m_x \) is a Borel measure and is finite on compact subsets of the leaf \( \mathcal{F}(x) \) (here, and elsewhere, we consider the intrinsic topology on each leaf).

If \( \{ \mu_x \}_{x \in M} \) is the disintegration of some probability measure \( \mu \) along a foliation \( \mathcal{F} \) as defined in the previous definition and if \( \{ m_x \}_{x \in M_1} \) is a continuous system of measures on \( \mathcal{F}|_{M_1} \), we will say that they coincide if \( \mu(M_1) = 1 \) and for \( \mu \)-a.e. \( x \in M_1 \), \( \mu_x \) and \( m_x \) are proportional.

**Definition 2.4.** Assume that \( \mathcal{F} \) is a foliation which is invariant under some diffeomorphism \( f : M \to M \), i.e., for all \( x \in M \), \( f(\mathcal{F}(x)) = \mathcal{F}(f(x)) \). Let \( M_1 \subset M \) be \( \mathcal{F} \)-saturated. A continuous system of measures \( \{ m_x \}_{x \in M_1} \) on \( \mathcal{F}|_{M_1} \) is dilated if there is some number \( D > 1 \) such that for all \( x \in M_1 \cap f^{-1}(M_1) \):

\[
(1) \quad f_* m_x = D^{-1} m_{f(x)}.
\]

\( D \) is called the **dilation factor**. We call the family \( \{ m_x \}_{x \in M_1} \) a Margulis system on \( \mathcal{F} \) and the measures \( m_x \) the **Margulis \( \mathcal{F} \)-conditionals**.

Our construction (following Margulis) relies on properties of the holonomy between foliations defined as follows:

**Definition 2.5.** Let \( \mathcal{F}_1, \mathcal{F}_2 \) be foliations which are invariant under some diffeomorphism \( f \in \text{Diff}^1(M) \). Let \( M_1 \) be an \( \mathcal{F}_1 \)-saturated subset of \( M \). Assume that \( \{ m_x \}_{x \in M_1} \) is a Margulis system of measures on \( \mathcal{F}_1|_{M_1} \) and that \( \mathcal{F}_2 \) is transverse to \( \mathcal{F}_1 \). The system \( \{ m_x \}_{x \in M_1} \) is **invariant**, respectively **quasi-invariant**, along \( \mathcal{F}_2 \) if, for all \( \mathcal{F}_2 \)-holonomies \( h : U \to V \) with \( U, V \) contained in \( \mathcal{F}_1 \)-leaves included in \( M_1 \):

\[
(2) \quad h_* (m_x|U) = m_{h(x)}|V \text{ for any } x \in U,
\]

respectively:

\[
(3) \quad h_* (m_x|U) \text{ and } m_{h(x)}|V \text{ are equivalent for any } x \in U.
\]

**Remark 2.6.** The quasi-invariance in (3) can be characterized by the absolute continuity of the holonomies along \( \mathcal{F}_2 \) with respect to a class of transversal measures defined by the Margulis system on \( \mathcal{F}_1 \).
Though an arbitrary continuous system of measures along the strong unstable foliation does not need to correspond to the disintegration of any invariant probability measure, those we construct in this paper will (see Proposition 5.1.)

3. Main Results

This section collects our main results. Our techniques deal with the following type of diffeomorphisms. For convenience, we fix some Riemannian structure on the compact manifold $M$.

**Definition 3.1.** A diffeomorphism $f : M \to M$ has flow type if:

(I) **Partial hyperbolicity:** $f$ is strongly partially hyperbolic with splitting $TM = E^s \oplus E^c \oplus E^u$ and $\dim E^c = 1$;

(II) **Dynamical coherence:** there are invariant foliations $F^{cs}$ and $F^{cu}$ tangent to $E^s \oplus E^c$ and $E^c \oplus E^u$;

Let $F^c$ be the foliation whose leaves are the connected components of the intersections $F^{cs}(x) \cap F^{cu}(x)$, $x \in M$.

(III) **Center leaves:** The center foliation $F^c$ is oriented and has at least one compact leaf.

Let $F : \mathbb{R} \times M \to M$ be the continuous flow along $F^c$ with unit positive speed.

(IV) **Flow like dynamics:** there is a continuous $\tau : M \to \mathbb{R}$ such that, for all $x \in M$, $f(x) = F(x, \tau(x))$ and $\tau(x) > 0$.

Following Margulis, we build special measures on most strong stable and strong unstable leaves. Let $M^u := M \setminus C^u$ where $C^u$ is the union of the unstable leaves that intersect some compact center leaf. Define $M^s$ and $C^s$ likewise.

**Theorem 3.2.** Let $f$ be a $C^2$ diffeomorphism $f$ with flow type and minimal stable and unstable foliations on a compact manifold $M$. Then there is a continuous system of measures $\{m^u_x\}_{x \in M^u}$ on $F^u$ such that:

1. each $m^u_x$ is atomless, locally finite, with full support;
2. $\{m^u_x\}_{x \in M^u}$ is a Margulis system along $F^u$ with dilation factor $D_u > 1$;
3. the system $\{m^u_x\}_{x \in M^u}$ is $cs$-quasi-invariant;
4. $M^u$ is dense with full measure for any ergodic measure with positive entropy.

**Addendum 3.3.** In the setting of the previous theorem, there is a unique system of measures $\{m^u_x\}_{x \in M^u}$ satisfying the above items (1) and (2). Moreover its dilation factor is $D_u = \exp h_{\text{top}}(f)$.

We call the system of measures $\{m^u_x\}_{x \in M^u}$ the unstable Margulis system and the measures $m^u_x$ the unstable Margulis conditionals.

We will build such a system $\{m^u_x\}_{x \in M^u}$ in Section 4, show its uniqueness and compute its dilation factor $D_u$ in Section 5.

**Remarks 3.4.**

1. The above theorem and addendum, applied to $f^{-1}$, defines a stable Margulis system $\{m^s_x\}_{x \in M}$ with dilation factor $D_s = D_u^{-1} = \exp(-h_{\text{top}}(f)) < 1$.
2. The $C^2$ smoothness assumption is only used by Theorem 4.3 to establish absolute continuity of the $s$-holonomy but it is probably enough to assume $C^{1+\alpha}$ smoothness. We do not know how to deal with $C^1$ smoothness.
Using tools from Ledrappier and Young [22], we prove the following result in Section 5.

**Theorem 3.5.** Let \( f \) be a \( C^2 \) diffeomorphism with flow type and minimal stable and unstable foliations on a compact manifold \( M \). Let \( \mu \) be an ergodic MME.

If \( \lambda^c(\mu) \leq 0 \), then the disintegration of \( \mu \) along \( \mathcal{F}^u \) is given by the unstable Margulis system \( \{ m_x^u \}_{x \in \mathcal{M}^u} \) from Theorem 3.2. In particular, the measure \( \mu \) has full support.

The above applied to \( f^{-1} \) shows that an ergodic MME with \( \lambda^c(\mu) \geq 0 \) has disintegration along \( \mathcal{F}^s \) given by \( \{ m_x^s \}_{x \in \mathcal{M}^s} \). In particular, any MME has full support.

**Remark 3.6.** The above theorem gives more information in the non-hyperbolic case. Indeed, if \( \mu \) is an ergodic measure of maximal entropy with \( \lambda^c(\mu) = 0 \), then the disintegrations, along both strong foliations \( \mathcal{F}^u \) and \( \mathcal{F}^s \), are given by the corresponding Margulis systems from Theorem 3.2.

The dichotomy will follow from two results about hyperbolic measures of maximal entropy. The first is a uniqueness result, based on the Hopf argument.

**Proposition 3.7.** Let \( f \) be a \( C^2 \) diffeomorphism with flow type and minimal stable and unstable strong foliations on a compact manifold \( M \). Let \( \mu \) be some ergodic MME. If \( \mu \) is hyperbolic, say \( \lambda^c(\mu) < 0 \), then there is no other ergodic MME \( \nu \) with \( \lambda^c(\nu) \leq 0 \).

The second result is a symmetry argument, using the one-dimensional center leaves. It builds so-called twin measures (see [31, 14]).

**Proposition 3.8.** Let \( f \) be a \( C^1 \) diffeomorphism of a compact manifold \( M \). Let \( \mathcal{F}^c \) be an orientable one-dimensional foliation (with continuously varying \( C^1 \) leaves). Assume that, for all \( x \in M \), \( f \) maps \( \mathcal{F}^c(x) \) to itself in an orientation-preserving way. Let \( \mu \in \mathcal{P}_{\text{erg}}(f) \) satisfy:

1. its Lyapunov exponent along \( \mathcal{F}^c \) is \( \lambda_c(\mu) < 0 \);
2. for \( \mu \)-a.e. \( x \in M \), the following set is relatively compact in the intrinsic topology of \( \mathcal{F}^c(x) \):
   \[
   W^c(x) := \{ y \in \mathcal{F}^c(x) : \limsup_{n \to \infty} \frac{1}{n} \log d(f^n x, f^n y) < 0 \};
   \]
   (3) and for \( \mu \)-a.e. \( x \in M \), the leaf \( \mathcal{F}^c(x) \) is noncompact and contains no fixed point.

Then there is another invariant probability measure \( \nu \) which is isomorphic to \( \mu \) and with exponent \( \lambda_c(\nu) \geq 0 \).

Finally, we state the abstract version of our main result:

**Theorem 3.9.** For any \( C^2 \) diffeomorphism \( f \) with flow type and minimal stable and unstable foliations on a compact manifold \( M \), we have the following dichotomy:

1. either all the measures of maximal entropy have zero central Lyapunov exponents, or
2. there are exactly two ergodic measures of maximal entropy where one has a positive central exponent and one has a negative central exponent, and both are Bernoulli.
The next theorem shows that there is an abundance of diffeomorphisms satisfying the above assumptions. It follows from properties of perturbations of time-one maps of transitive Anosov flows established in [3] and [4], as discussed in Section 6.

**Theorem 3.10.** If \( \varphi^t \) is a transitive Anosov flow on a compact manifold \( M \), then for all \( T \neq 0 \) there exists a \( C^1 \) open set \( \mathcal{U} \) in \( \text{Diff}^1(M) \) such that \( \varphi^T \) belongs to the \( C^1 \)-closure of \( \mathcal{U} \) and every \( f \in \mathcal{U} \cap \text{Diff}^2(M) \) has flow type with both stable and unstable foliations minimal.

4. Building Margulis Systems of Measures

In this section, we consider flow type diffeomorphisms whose strong foliations are both minimal. To begin with, we follow Margulis’ construction of a system of measures on the \( cu \)-leaves that are invariant under stable holonomies. We then deduce from this a system of \( u \)-conditionals that are quasi-invariant under center-stable holonomies. This proves Theorem 3.2 except for the uniqueness of the Margulis \( u \)-system and the equality \( D_u = \exp h_{\text{top}}(f) \) that will be deduced in Section 5 from the analysis of MMEs.

4.1. The \( cu \)-conditionals. We following Margulis’ construction.

**Proposition 4.1.** Let \( f \in \text{Diff}^2(M) \) on a compact manifold \( M \) with a dominated splitting \( E^s \oplus E^{cu} \) with \( E^s \) uniformly contracted. Assume that:

1. there are foliations \( \mathcal{F}^{cu} \) and \( \mathcal{F}^s \) which are tangent to, resp. \( E^{cu} \) and \( E^s \);
2. \( \mathcal{F}^s \) is minimal.

Then there is a Margulis system \( \{m^{cu}_x\}_{x \in M} \) on \( \mathcal{F}^{cu} \) which is invariant under \( \mathcal{F}^s \)-holonomy and such that each \( m^{cu}_x \) is atomless, Radon, and fully supported on \( \mathcal{F}^{cu}(x) \).

We introduce some convenient definitions. Let \( \sigma \in \{c, cu, cs, s, u\} \). We denote by \( \lambda^\sigma \) the intrinsic Riemannian volume on each \( \sigma \)-leaf: for any subset \( E \) of a \( \sigma \)-leaf, \( \lambda^\sigma(E) \) is its volume with respect to the Riemannian structure on the leaf. We denote by \( d_\sigma \) the distance defined on each leaf by the induced Riemannian structure and defining the intrinsic topology.

The \( \sigma \)-balls are \( B^\sigma(x, r) := \{y \in \mathcal{F}^\sigma(x) : d_\sigma(y, x) < r\} \). For a subset \( A \) of such a leaf, we set \( B^\sigma(A, r) := \bigcup_{x \in A} B^\sigma(x, r) \). A \( \sigma \)-subset is a \( d_\sigma \)-bounded subset of a \( \sigma \)-leaf. A \( \sigma \)-test function is a nonnegative function \( \psi : M \to \mathbb{R} \) such that \( \{x \in M : \psi(x) \neq 0\} \) is a \( \sigma \)-subset and the restriction of \( \psi \) to

\[ \text{supp}(\psi) := \{x \in M : \psi(x) \neq 0\} \]

is continuous. We write \( \psi > 0 \) if \( \psi \geq 0 \) and \( \{\psi > 0\} \) has non-empty interior in the intrinsic topology. We denote by \( \mathcal{T}^\sigma \) the collection of all \( \sigma \)-test functions.

Given a \( \sigma \)-holonomy \( h : A \to B \) its size is \( \sup_{x \in A} d_\sigma(x, h(x)) \), and the two subsets \( A, B \) are called equivalent along \( \mathcal{F}^\sigma \) through \( h \) provided \( h(A) = B \). We say that they are \( \epsilon \)-equivalent if the holonomy has size at most \( \epsilon \). Two functions \( \psi, \phi \) are \( \epsilon \)-equivalent along \( \mathcal{F}^\sigma \) if their supports are equivalent through a \( \sigma \)-holonomy \( h \) with size at most \( \epsilon \) and satisfying \( \phi = \psi \circ h \). Two submanifolds \( N', N'' \) are \( t \)-transverse if they are transverse and if for every \( x \in N' \cap N'' \), the angle between any two nonzero vectors in \( T_x N' \) and \( T_x N'' \) is at least \( t \).
Following Margulis, we consider functionals $\lambda : T^c u \to \mathbb{R}$. Note that $\lambda^c u$ is one such functional. The map $f$ acts on them by:

$$\forall \psi \in T^c u \quad f(\lambda)(\psi) := \lambda(\psi \circ f^{-1}).$$

A key class of such functionals are $\ell_n := f^n(\lambda^c u)$ for any $n \in \mathbb{N}$. That is, for any $\phi \in T^c u$,

$$\ell_n(\phi) := \int \phi \circ f^{-n} \, d\lambda^c u.$$

To normalize, we fix some $\phi_1 \in T^c u$ with $\phi_1 > 0$. Considering the topology of pointwise convergence (i.e., working in $\mathbb{R} T^c u$ with the product topology), let $L$ be the closure of the following set:

$$L_1 := \{ \lambda = \sum_{i=1}^{n} c_i \ell_{t_i} : n \in \mathbb{N}^*, t_1, \ldots, t_n \in \mathbb{N}, c_1, \ldots, c_n > 0 \text{ with } \lambda(\phi_1) = 1 \}.$$

We will use the following covering numbers. For $K$, a $cu$-subset, and $\rho > 0$, we denote by $r_{cu}(K, \rho)$ the smallest integer $k \geq 0$ such that there are $x_1, \ldots, x_k \in K$ with $K \subset \bigcup_{i=1}^{k} B_{cu}(x_i, \rho)$.

**Proposition 4.2.** There exist $\Lambda \in L$ and $D_u > 0$ such that

$$f(\Lambda) = D_u \cdot \Lambda$$

and, for some positive numbers $C, R$, for any $\phi \in T^c u$:

(a) $\Lambda(\phi) \leq C r^c u(\sup \phi, R) \| \phi \|_\infty$;

(b) if $\phi > 0$, then $\Lambda(\phi) > 0$; and

(c) if $\psi \in T^c u$ is s-equivalent to $\phi$, then $\Lambda(\phi) = \Lambda(\psi)$.

To prove this, we will show that $L$ is a convex and compact set and then apply the Schauder-Tychonoff fixed point Theorem to a normalized action of $f$ on $L$.

We will relate the iterations of different $cu$-test functions by using the invariance under s-holonomy and especially the following theorem (see, e.g., [1, Theorem C]).

For convenience let $E(t) = (e^t - 1)^+$. Observe that for $t \geq \log 2$, $E(t) \geq \frac{1}{2} e^t$ and $E(kt) \leq E(t)^{2k}$.

**Theorem 4.3.** Let $f$ be a $C^2$ diffeomorphism on a compact Riemannian manifold $M$. Assume that there is a dominated splitting $E^s \oplus E^c u$ with $E^s$ uniformly contracting. Fix $t > 0$ and let $A_1, A_2$ be two submanifolds $t$-transverse to $E^s$ and s-equivalent through $h : A_1 \to A_2$. Then $h$ is absolutely continuous.

More precisely, writing $\lambda_1, \lambda_2$ for the Riemannian volume on $A_1, A_2$, the measures $h_* \lambda_1$ and $\lambda_2$ are equivalent and there are constants $C < \infty$ and $\alpha > 0$ depending only on $f$, $E^s$, and $t$ such that, letting $\epsilon$ be the size of the holonomy $h$:

$$\left| \frac{d h_* \lambda_1}{d \lambda_2} - 1 \right| \leq C(e^\alpha + E(\epsilon)).$$

The second term $E(\epsilon)$ simply ensures that the above bound holds even for large $\epsilon > 0$. 
We need two additional lemmas. The first one will give a uniform bound on the volume growth in the center unstable leaves.

**Lemma 4.4.** For any open, non-empty cu-subset $A$, there are constants $C_A < \infty$ and $r_A > 0$ such that,

\[
\forall x \in M \forall n \geq 0 \lambda^{cu}(f^n B^{cu}(x, r_A)) \leq C_A \lambda^{cu}(f^n A).
\]

**Proof.** Fix $r_1 > 0$ so small that $A_1 := A \setminus B^{cu}(\partial A, r_1)$ is not empty. The minimality implies that any $x$ is $s$-equivalent to some point in $A_1$. The continuity of the foliation $F$ and its transversality to $M$ yield $r_x > 0$ and $R_x < \infty$ such that, for any $y \in B(x, r_x)$, $B^{cu}(y, r_x)$ is $(R_x, s)$-equivalent to a subset of $A$. The compactness of $M$ yields $r_A > 0$, $R_A < \infty$ such that, for any point $x \in M$, $B^{cu}(x, r_A)$ is $(R_A, s)$-equivalent to a subset of $A$.

Since $F$ is contracted, there are numbers $C < \infty$ and $\kappa < 1$ such that the set $f^n(B^{cu}(x, r_A))$ is $(C\kappa^n R_A, s)$-equivalent to a subset of $f^n(A)$. Theorem 4.3 proves the claim. \qed

**Corollary 4.5.** For any $\phi \in T^{cu}$ with $\phi > 0$, there are numbers $C(\phi) < \infty$ and $R(\phi) > 0$ such that for any $\psi \in T^{cu}$ and any $n \geq 0$:

\[
\int \psi \circ f^{-n} d\lambda^{cu} \leq C(\phi) r^{cu}(\supp \psi, R(\phi)) \|\psi\|_{\infty} \int \psi \circ f^{-n} d\lambda^{cu}.
\]

**Proof.** The left hand side is bounded by $\|\psi\|_{\infty} \cdot \lambda^{cu}(f^n(\supp \psi))$. Fix some $0 < t < \sup \phi$. The previous lemma with $A := \{\phi > t\}$ yields $r_A > 0$ and $C_A < \infty$. Since $\supp(\phi)$ is compact in its cu-leaf $F(A)$, there are $x_1, \ldots, x_N \in F(A)$ with $N = r^{cu}(\supp \psi, r_A)$ such that $\supp(\psi) \subset \bigcup_{i=1}^N B^{cu}(x_i, r_A)$. Now

\[
\lambda^{cu}(f^n(B^{cu}(x_i, r_A))) \leq C_A \lambda^{cu}(f^n A) \leq C_A \lambda^{cu}(\phi \circ f^{-n})/t.
\]

Summing over the cover of $\supp \psi$, the claim follows with $C(\phi) := C_A/t$ and $R(\phi) := r_A$. \qed

The next lemma establishes approximate holonomy invariance.

**Lemma 4.6.** There are numbers $C < \infty$ and $0 < \rho < 1$ with the following properties. Let $\psi, \phi, \in T^{cu}$ be $(s, \Delta)$-equivalent for some $\Delta < \infty$.

First, if $\psi > 0$, then, for all $n \geq 0$:

\[
\forall \lambda \in L \quad |\lambda(\psi \circ f^{-n}) - \lambda(\phi \circ f^{-n})| \leq C \rho^n(\Delta^\alpha + E(C\Delta))\lambda(\psi \circ f^{-n}).
\]

Second, for any $\phi_1, \in T^{cu}$ with $\phi_1 > 0$, there are numbers $C(\phi_1)$ and $R(\phi_1)$ > 0 such that, for any $n \geq 0$ we have

\[
|\ell_n(\psi) - \ell_n(\phi)| \leq C(\phi_1) r(\supp(\psi), R(\phi_1)) \|\psi\|_{\infty} \rho^n(\Delta^\alpha + E(C\Delta))\ell_n(\phi_1).
\]

**Proof.** Since $\phi$ and $\psi$ are $\Delta$-equivalent, there is $h : \supp(\psi) \to \supp(\phi)$ with size at most $\Delta$ such that $\psi = \phi \circ h$. Since $F$ is uniformly contracted, $\phi \circ f^{-n}$ and $\psi \circ f^{-n}$ are $C\kappa^n \Delta$-equivalent through $h_n := f^n \circ h \circ f^{-n}$ for some $C < \infty$ and
\( \kappa < 1 \). Theorem 4.3 yields:
\[
|\ell_n(\phi) - \ell_n(\psi)| = \left| \int \phi \circ f^{-n} d\lambda_{f^n}^\psi - \int \psi \circ f^{-n} d\lambda_{f^n}^\phi \right| \\
= \left| \int \phi \circ f^{-n} \circ h_n \frac{d(h_n)_\ast\lambda_{f^n}^\psi}{d\lambda_{f^n}^\phi} - \psi \circ f^{-n} d\lambda_{f^n}^\phi \right| \\
\leq C((\Delta \kappa)^\alpha + E(C\Delta \kappa)) \int |\psi \circ f^{-n}| d\lambda_{f^n}^\phi \\
\leq C(\Delta \kappa^\alpha + E(C\Delta \kappa)) \ell_n(|\psi|).
\]

Let \( \lambda \in \mathbb{L}_1 \) so that \( \lambda = \sum_{i=1}^{I} c_i \ell_{t_i} \). Using the identity \( \ell_t(\phi \circ f^{-n}) = \ell_n(\phi \circ f^{-t}) \), we get:
\[
|\lambda(\phi \circ f^{-n}) - \lambda(\psi \circ f^{-n})| \leq C(\Delta \kappa^\alpha + E(C\Delta \kappa)) \lambda(\psi \circ f^{-n}),
\]
proving eq. (8) for all \( \lambda \in \mathbb{L}_1 \) by continuity.

Let \( \phi_1 \in \mathcal{T}^c \) with \( \phi_1 > 0 \). Applying Corollary 4.3, we obtain \( \mathcal{C}(\phi_1) \) and \( \mathcal{R}(\phi_1) > 0 \) such that eq. (7) holds.

**Proof of Proposition 4.4**. We prove the first two claims (a) and (b) for arbitrary \( \lambda \in \mathbb{L} \).

**Step 1**: Claim (a): \( \exists R > 0 \ \forall \lambda \in \mathcal{L} \ \lambda(\psi) \leq C r^\mathcal{C}e(\text{supp} \psi, R) \|\psi\|_{\infty} \).

Corollary 4.3 for \( \phi = \phi_1 \) yields numbers \( C < \infty \) and \( R > 0 \) such that for any \( \psi \in \mathcal{T}^c \), \( \forall n \geq 0 \ \ell_n(\psi) \leq C r^\mathcal{C}e(\text{supp} \psi, R) \|\psi\|_{\infty} \ell_n(\phi_1) \). Therefore, for any \( \lambda \in \mathbb{L}_1 \):
\[
(8) \quad \lambda(\psi) = \sum_{i=1}^{I} c_i \ell_{t_i}(\psi) \leq \sum_{i=1}^{I} c_i C r^\mathcal{C}e(\text{supp} \psi, R) \|\psi\|_{\infty} \ell_{t_i}(\phi_1) \\
= C r^\mathcal{C}e(\text{supp} \psi, R) \|\psi\|_{\infty} \lambda(\phi_1).
\]

This proves the claim since \( \lambda(\phi_1) = 1 \) for \( \lambda \in \mathbb{L}_1 \). It extends to the closure \( \mathbb{L} \), concluding Step 1.

Note that eq. (8) implies that \( \mathbb{L}_1 \) is a subset of the compact set
\[
\prod_{\psi \in \mathcal{T}^c} [0, C \cdot r^\mathcal{C}e(\text{supp} \psi, R) \|\psi\|].
\]

Hence its closure \( \mathbb{L} \) is compact.

**Step 2**: Claim (b): \( \forall \psi \in \mathcal{T}^c \) if \( \psi > 0 \) there is \( C(\psi) > 0 \) s.t. \( \forall \lambda \in \mathbb{L}, \ \lambda(\psi) \geq C(\psi) \).

We assume that \( \phi > 0 \) and apply again Corollary 4.3 exchanging \( \psi \) and \( \phi_1 \). We get new numbers \( C' \) and \( R' \) defined by \( \phi \). Setting \( C_1 := C' r^\mathcal{C}e(\text{supp} \phi_1, R') \), we have \( \ell_n(\phi_1) \leq C_1 \ell_n(\psi) \). That is, \( \ell_n(\psi) \geq (1/C_1) \ell_n(\phi_1) \) so that, for any \( \lambda \in \mathbb{L}_1 \):
\[
(9) \quad \lambda(\psi) \geq \frac{1}{C_1} \lambda(\phi_1) = \frac{1}{C_1}.
\]

This again extends to \( \mathbb{L} \), concluding Step 2 with the lower bound \( 1/C_1 \).

**Step 3**: Existence of \( \Lambda \in \mathbb{L} \) with \( f(\Lambda) = D \cdot \Lambda \)

We now build the functional \( \Lambda \) as a fixed point of the map
\[
\tilde{f} : \mathbb{L} \to \mathbb{L}, \ \lambda \mapsto \frac{\lambda(\psi \circ f^{-1})}{\lambda(\phi_1 \circ f^{-1})}.
\]
We claim that $\tilde{f}$ is well-defined and continuous from $\mathbb{L}$ to $\mathbb{R}^{T^\infty}$. Indeed, the map $\lambda \mapsto \lambda(\phi \circ f^{-1})$ from $\mathbb{L}$ to $\mathbb{R}$ is well-defined since $\phi \circ f^{-1} \in T^\text{cu}$, is obviously continuous, and is positive by Step 2. Note that $\lambda(\cdot \circ f^{-1}) : T^\text{cu} \to \mathbb{R}$ is well-defined and $\lambda \mapsto \lambda(\cdot \circ f^{-1})$ is continuous from $\mathcal{L}$ to $\mathbb{R}^{T^\infty}$. The claim is proved.

Finally, it is obvious that $\tilde{f}(L_1) = L_1$, hence $\tilde{f} : \mathbb{L} \to \mathbb{L}$ is a well-defined continuous map. Since $\mathbb{L}$ is a convex, compact subset of the locally convex linear space $\mathbb{R}^{T^\infty}$, the Schauder-Tychonoff Theorem applies and yields $\Lambda \in \mathbb{L}$ with $\tilde{f}(\Lambda) = \Lambda$.

Therefore, $f(\Lambda) = D \cdot \Lambda$ with $D = \lambda(\phi_1 \circ f^{-1}) > 0$.

**Step 4:** Claim (c): s-holonomy invariance of $\Lambda$

Let $\phi, \psi \in T^\text{cu}$. Writing $\psi = \max(\psi,0) - \max(-\psi,0)$ and likewise for $\phi$, we can assume $\phi, \psi > 0$. Assume that $\psi$ and $\phi$ are $s$-equivalent. By compactness of their support, they are $(s, \Delta)$-equivalent for some $\Delta < 0$ and therefore $\phi \circ f^{-n}$ and $\psi \circ f^{-n}$ are $(s, C\kappa^n \Delta)$-equivalent with $0 < \kappa < 1$. Using the dilation and the approximate holonomy invariance eq. (6) from Lemma 4.6, we get that, for any $\epsilon > 0$, for large enough $n \geq 0$:

$$|\Lambda(\phi) - \Lambda(\psi)| = D^{-n} |\Lambda(\phi \circ f^{-n}) - \Lambda(\psi \circ f^{-n})|$$

$$\leq D^{-n} \epsilon \cdot \Lambda(\psi \circ f^{-n}) = \epsilon \cdot \Lambda(\psi)$$

As $\epsilon > 0$ is arbitrarily small this implies $\Lambda(\phi) = \Lambda(\psi)$, i.e., Claim (c). \hfill $\square$

We deduce a Margulis system of $cu$-measures from the functional $\Lambda$.

**Proof of Proposition 4.4.** Proposition 4.2 yields a functional $\Lambda$ on $T^\text{cu}$, which contains $C_c(F^{cu}(x))$ for all $x \in M$. Note that $\Lambda(C_c(F^{cu}(x)))$ is linear and positive (because this holds for all $\lambda \in L_1$ and extends by continuity to $L$). Hence, Riesz’s Representation Theorem gives a measure $m_x$ on $F^{cu}(x)$, for each $x \in M$, by setting:

$$\forall \phi \in C_c(F^{cu}(x)), \quad m_x(\phi) = \Lambda(\phi).$$

The local finiteness, full support, and $s$-invariance of each $m_x$ follows from properties (a), (b), (d) of the functional $\Lambda$ from Proposition 4.2.

We deduce that each $m^\text{cu}_x$ is atomless from the holonomy invariance. Assume by contradiction that there is $y \in F^{cu}(x)$ with $m^\text{cu}_x(\{y\}) > 0$. Consider the stable leaf $F^s(y)$. By assumption it is dense in $M$, hence by transversality in $F^{cu}(x)$. By $s$-invariance, $m^\text{cu}_x$ must have a dense set of atoms $z \in F^{cu}(x)$, all of which have measure $m^\text{cu}_x(\{y\}) > 0$. This contradicts the finiteness of $m^\text{cu}_x$ on compact sets.

We finally deduce the continuity from the holonomy invariance. As $F^{cu}$ and $F^s$ are transverse, for any $x_0 \in M$, there is a neighborhood $B$ of $x_0$ and a continuous map $h_0 : B \times F^{cu}(x_0) \to B$ with $\{h_0(x,y)\} = F^{cu}_{\text{loc}}(x) \cap F^s_{\text{loc}}(y)$ (in particular, $h_0(x_0,y) = y$). Let $\phi \in C(M)$. By holonomy invariance, $m^\text{cu}_x(\phi|_{F^{cu}_B(x)}) = m^\text{cu}_{x_0}(\phi \circ h_0(x_0,\cdot))$. Hence,

$$|m^\text{cu}_x(\phi|_{F^{cu}_B(x)}) - m^\text{cu}_{x_0}(\phi|_{F^{cu}_B(x_0)})| \leq \int_{F^{cu}_{\text{loc}}(x_0)} |\phi \circ h_0(x,\cdot) - \phi| \, dm^\text{cu}_{x_0}$$

which converges to 0 as $x$ goes to $x_0$. This is the continuity property. \hfill $\square$

The next lemma establishes that the constant $D_u$ is larger than 1.

**Lemma 4.7.** Let $f \in \text{Diff}^2(M)$ have flow type with both strong foliations minimal. Let $\{m^\text{cu}_x\}_{x \in M}$ be a Margulis cu-system. The dilation of the Margulis system on $F^{cu}$ satisfies: $D_u > 1$. 
Propositions 5.5 and 5.8).

for the equality \( D \) the family of measures \( \{ m_x \} \).

Here \( t \) maps for small enough \( I \) curves proving \( D \) cu-mu to disjoint subsets of the cu-leaf. This ensures that \( m_x \) inherits the \( \sigma \)-additivity of \( m_x \).

In our setting there is no flow commuting with the dynamics and we have to proceed differently. To keep the equivariance, we will replace \( \phi^t(x) \) by the \( c \)-segment \( I_c(x) \) “between \( x \) and \( f(x) \)”. Such center curves cannot be assumed to be arbitrarily short, so even restricting \( x \) to a small subset of a single \( u \)-leaf, the curves \( I_c(x) \) might intersect and destroy the additivity property. It turns out that this problem only occurs on \( C_u \), i.e., the union of the cu-leaves containing a compact center leaf. This is why we will build \( u \)-measures \( m_x \) only for \( x \in M \setminus C_u \).

Preparations. We recall or establish some useful properties of the dynamical foliations and of the cu-system built in Section 4. In particular, it allows us to disregard the closed center leaves.

Remark 4.8. There can be at most countably many compact center leaves, since each one is normally hyperbolic (see [19] Theorem 4.1(b)).

Lemma 4.9. Let \( f \in \text{Diff}(M) \) have flow type. If a center leaf meets some unstable leaf in more than one point, then it is contained in the cu-leaf of a compact center leaf.

Proof. To simplify notations a bit, we prove the symmetric statement involving stable and center stable manifolds.

Let \( x, y \) be two distinct points with \( y \in F^s(x) \cap F^c(x) \) and \( x \leq y \) along the center leaf (recall that \( f \) being flow type, it maps each center leaf to itself preserving some orientation). Let \( L \) be the center segment \( [x, y]_c \). We are going to show that \( f^n(L) \) converges to a closed center leaf \( \gamma \) as a compact subset with respect to the intrinsic distance in \( F^c(x) \). The existence of \( \gamma \) in \( F^c(x) \) will prove the lemma.

Proof. By assumption (III), there is a compact center leaf \( F^c(x) \subset M \). It is contained in the cu-leaf \( F^{cu}(x) \). Since \( F^c(x) \) is a topological attractor for the restriction of \( f \) to the invariant set \( F^{cu}(x) \), there is a relatively compact neighborhood \( U \) of \( F^c(x) \) in \( F^{cu}(x) \) such that \( U \setminus f^{-1}(U) \) has non empty interior. As \( m_x^{cu} \) has full support in \( F^{cu}(x) \), it follows that:

\[
D_u^{-1}m_x^{cu}(U) = m_x^{cu}(f^{-1}U) < m_x^{cu}(U),
\]

proving \( D_u > 1 \). \( \square \)

4.2. Building the \( u \)-conditionals. We complete the proof of Theorem 3.2 (except for the equality \( D_u = e^{b_{top}(f)} \) and the uniqueness of the Margulis systems, see Propositions 5.5 and 5.8).

We start with the previously built Margulis cu-system \( \{ m_x^{cu} \}_{x \in M} \) and define the family of measures \( \{ m_x^u \}_{x \in M} \) by extending subsets of \( u \)-leaves to subsets of cu-leaves along the center foliation. For flows, Margulis used the formula

\[
m_x^u(A) = m_x^{cu} \left( \bigcup_{0 \leq t < t_0} \phi^t(A) \right).
\]

Here \( t_0 > 0 \) is chosen small so that disjoint subsets of the same \( u \)-leaf correspond to disjoint subsets of the cu-leaf. This ensures that \( m_x^u \) inherits the \( \sigma \)-additivity of \( m_x^{cu} \).

Alternatively, one could consider a covering of \( M \) where all center leaves are noncompact. Note also that this problem could be altogether avoided by restricting to perturbation of time \( t \) maps for small enough \( t \), instead of taking \( t = 1 \).
There are two cases, depending on the position of \( y \) with respect to \( f(x) \). In the first case, we have:
\[
(10) \quad x \leq f(x) \leq y \leq f(y)
\]
hence the decompositions (the union are disjoint up to one point):
\[
L = [x, y]_c = [x, f(x)]_c \cup [f(x), y]_c
\]
\[
f(L) = [f(x), f(y)]_c = [f(x), y]_c \cup [y, f(y)]_c.
\]
For large enough \( k \geq 0 \), the points \( f^k(x) \) and \( f^k(y) \) are arbitrarily close while the length of \( [f^k(x), f(f^k(x))]_c \) is bounded independently of \( k \). We can assume that this holds already for \( k = 0 \). Thus there is a \( s \)-holonomy \( h : [x, f(x)]_c \to [y, f(y)]_c \) such that \( h(x) = y \).

Now define the map \( \psi : L \to f(L) \) by \( \psi([x, f(x)]_c) = h \) and \( \psi([f(x), y]_c) = Id \).
Observe that it is a bijection such that \( \psi(x) \in G_{loc}^\delta(x) \) and that there is a finite bound \( \delta := \sup_{z \in L} d(z, \psi(z)) \) on the Hausdorff distance \( d_H(L, f(L)) \).
This implies that, for any \( n \geq 0 \),
\[
d_H(f^n(L), f^{n+1}(L)) \leq \sup_{z \in L} d(f^n(z), f^n(\psi(z)))
\]
decays exponentially fast since \( F^s \) is uniformly contracted. It follows that
\[
\sum_{n \geq 0} d_H(f^n(L), f^{n+1}(L)) < \infty
\]
and so \( f^n(L) \) converges to some nonempty compact \( \gamma \) in the complete space of all nonempty compact subsets of \( F^{cu}(x) \). Obviously \( \gamma \) is a closed center leaf. This proves the lemma in the case (10) holds. In all other cases,
\[
x \leq y < f(x) \leq f(y).
\]
As above, perhaps after replacing \( x, y \) by some forward iterates, we find a \( s \)-holonomy \( h : [x, f(x)]_c \to [y, f(y)]_c \) such that \( h(x) = y \). By transversality of \( F^c \) and \( F^s \), the length of \( [z, h(z)]_c \) is lower bounded for \( z \in M \). Hence \( h^N(x) < f(x) \leq h^{N+1}(x) \) for some \( N \geq 1 \), and one has:
\[
L = [x, h^{-N}(f(x))]_c \cup [h^{-N}(f(x)), y]_c
\]
\[
f(L) = [f(x), h^N(y)]_c \cup [h^N(y), f(y)]_c.
\]
and the following maps are \( s \)-holonomies:
\[
h^N : [h^{-N}(f(x)), y]_c \to [f(x), h^N(y)]_c
\]
\[
h^{N+1} : [x, h^{-N}(f(x))]_c \to [h^N(y), f(y)]_c
\]
(since \( h(f(x)) = f(h(x)) = f(y) \)).
As before, we conclude that \( f^n(L) \) converges to a compact center leaf in \( F^{cu}(x) \).

Center-unstable leaves have a simple structure:

**Lemma 4.10.** Let \( f \in \text{Diff}^1(M) \) be a flow type diffeomorphism. For any \( x \in M \),
\[
F^{cu}(x) = \bigcup_{y \in F^c(x)} F^u(y).
\]
In particular, \( C^u = \bigcup_\gamma F^c(\gamma) \) where \( \gamma \) ranges over the countably many compact center leaves.
This is [35 Prop. 2.9]. Note that the statement there assume that \( f \) is a perturbation of some Anosov flow, but inspection of the proof shows that flow type is sufficient. Now we can show that the exceptional set \( C^u \) will not obstruct our construction:

**Lemma 4.11.** The following holds:

1. For any \( x \in M \), any compact center leaf has zero \( m^c_x \)-measure.
2. For any \( \mu \in \mathbb{P}_{\text{erg}}(f) \) not carried by a compact center leaf, \( \mu(C^u) = 0 \).
3. For any \( x \in M \), \( m_x^c(F^c(x) \cap C^s) = 0 \).

**Proof.** The first point is immediate consequence of the center leaves being fixed so that \( m_x^c(\gamma) = D_u \cdot m_x^c(\gamma) \) but \( D_u \neq 1 \) and \( m_x^c(\gamma) < \infty \) (by the Radon property).

We prove the second point by contradiction. Let \( \mu \in \mathbb{P}_{\text{erg}}(f) \) be such that \( \mu(C^u) > 0 \). The previous lemma implies that \( \mu \) gives positive measure to some \( F^s(\gamma) \) for some compact center leaf \( \gamma \). By ergodicity, we have \( \mu(F^s(\gamma)) = 1 \).

We turn to the third point. By the previous lemma, it is enough to prove the claim with \( C^s \) replaced by \( F^c(\gamma) \), where \( \gamma \) is an arbitrary compact center leaf. Since \( \gamma \subset F^c(\gamma) \cap F^u(\gamma) \), we can find a countable cover
\[
\{y \in F^c_u(x) : y \in F^c(\gamma)\} \subset \bigcup_{i \geq 1} h_i^c(\gamma \cap U_i)
\]
where \( h_i^c : U_i \to F^c_u(x) \), \( i \geq 1 \), are \( s \)-holonomies with \( U_i \subset F^c_u(\gamma) \). Now, by \( s \)-invariance of the \( cu \)-system and point 1, each term on the right hand side has zero \( m^c_x \)-measure. \( \square \)

The following property will show that the strong Margulis conditionals are themselves atomless.

**Lemma 4.12.** Let \( f \) be a diffeomorphism on a compact manifold. Assume it has flow type and admits a Margulis system of \( cu \)-measures \( \{m^c_x\}_{x \in M} \) with dilation factor \( D_u > 1 \). Then, for each \( x \in M \), \( m^c_x(F^c(x)) = 0 \).

**Proof.** We proceed by contradiction, assuming that \( m^c_x(F^c(x)) > 0 \) for some \( x \in M \). By \( \sigma \)-additivity and the dilation, we must have \( m^c_x([x, f(x)]) > 0 \). Since \( M \) is compact, one can find \( n_k \to \infty \) such that \( f^{n_k}(x) \) converges. Call the limit \( y \in M \).

By property (IV) from the definition of flow type, the lengths of the segments \( [f^n x, f^{n+1} x] \) are uniformly bounded for all \( n \geq 0 \).

For \( k \) large enough, \( [f^{n_k} x, f^{n_k+1} x] \) is close to \( [y, f(y)] \), so that this entire segment \( [f^{n_k} x, f^{n_k+1} x] \) projects to \( F^c(y) \) by a local stable holonomy. The projection is a segment \( I_k \) with uniformly bounded length and
\[
m_y^c(I_k) = m_x^c([f^{n_k} x, f^{n_k+1} x]) = D_{u}^{n_k} m_x^c([x, f(x)] \to \infty.
\]
Since \( I_k \to [y, f(y)] \), this contradicts the local finiteness of \( m_y^c \). \( \square \)

**Proof of Theorem 3.2** (except for the uniqueness and the dilation factor). We begin by defining the measures \( m^u_x \) along the unstable foliation. For each \( x \in M^u \) and Borel subset \( A \subset F^u(x) \) we let
\[
m^u_x(A) = m_x^c(\hat{A}) \text{ with } \hat{A} := \bigcup_{y \in A} [x, f(x)]. \]
This is well-defined since \( \hat{A} \) is measurable when \( A \) is. Consider a \( u \)-leaf \( \mathcal{F}^u(x) \) for some \( x \in M^u \). Let \( A_1, A_2, \ldots \) be pairwise disjoint measurable sets. By Lemma 4.9 the extensions \( \hat{A}_1, \hat{A}_2, \ldots \) are also pairwise disjoint. Hence,

\[
m^u_x(\bigsqcup_n A_n) = m^u_x(\bigsqcup_n \hat{A}_n) = m^u_x(\hat{A}_n) = \sum_n m^u_x(A_n),
\]

proving the \( \sigma \)-additivity. Obviously, \( m^u_x(\emptyset) = 0 \). Thus \( m^u_x \) is a measure.

\( C^u \) is a countable union of submanifolds with positive codimension. Thus Baire theorem shows that \( M^u \) is dense. Lemma 4.11 shows that it has zero measure for any ergodic measure with positive entropy. Item 3 is proven.

Observe that relative compactness, respectively nonempty interior, hold for \( \hat{A} \) if it holds for \( A \) yielding that \( m^u_x \) is Radon and fully supported. Lemma 4.12 shows that each \( m^u_x \) is atomless. Item 4 follows.

To prove that \( \{m^u_x(x) : x \in M \} \) is a Margulis system, observe that this holds for \( \{m^u_x(x) : x \in M \} \) by Proposition 4.11. The continuity of \( m^u_x \) follows. To determine its dilation, note that \( f(\hat{A}) = f(\hat{A}) \), since \( f([x, f(x)]_c) = f([x, f^2(x)]_c) \) so that:

\[
m^u_x(f(A)) = m^u_x(f(\hat{A})) = m^u_x(f(\hat{A})) = D_u m^u_x(\hat{A}) = D_u m^u_x(A).
\]

Item 5 is proved.

We turn to the quasi-invariance under \( cs \)-holonomies. Since \( E^c \) and \( E^s \) are transverse, we see that:

**Lemma 4.13.** Let \( x, y \in M \) satisfy \( \mathcal{F}^c(x) = \mathcal{F}^c(y) \) with \( d_{cs}(x, y) \) small. Then there exists a \( s \)-holonomy \( h^s : U \rightarrow V \) with \( U \) a neighborhood of \([x, f(x)]_c\) s.t.

\[
h^s([x, f(x)]_c) \subset (f^{-1}(y), f^2(y))_c.
\]

Let us prove the \( cs \)-quasi-invariance. Since this is a local property, we can assume that \( A \subset \mathcal{F}^u_\delta(x) \) for \( x \in M^u \) with \( \delta > 0 \) small so that the \( cs \)-holonomy extends to \( h^{cs}(A) = B \). By construction and the Margulis property,

\[
m^y_{\delta}(f^{-1}(B) \cup \hat{B} \cup f(\hat{B})) = (D^{-1}_u + 1 + D_u)m^u_{\delta}(B).
\]

By the previous lemma, \( h^s(\hat{A}) \subset f^{-1}(\hat{B}) \cup \hat{B} \cup f(\hat{B}) \), so the \( s \)-invariance of the \( cu \)-conditionals yields:

\[
m^u_x(A) = m^u_x(\hat{A}) = m^u_x(h^s(\hat{A})) \leq (D^{-1}_u + 1 + D_u)m^u_{\delta}(B).
\]

We have shown \( m^u_x \ll (h^{cs})^{-1}(m^u_{\delta}) \) with Radon-Nikodym derivative bounded by \( D^{-1}_u + 1 + D_u \). Exchanging the roles of \( x \) and \( y \) we obtain the equivalence of the measures. This finishes the proof of item 6 and of Theorem 3.2 except for the uniqueness of \( \{m^u_x(x) : x \in M^u \} \) and the equality \( D_u = e^{h_{\text{top}}(f)} \).

By definition, an arbitrary holonomy can be written as the composition of holonomies with small sizes. Hence we get:

**Corollary 4.14.** For any \( \delta > 0 \) there exists \( C(\delta) > 1 \) such that if \( A_1 \) and \( A_2 \) inside \( \mathcal{F}^u \) are \( \delta, cs \)-equivalent, then

\[
\frac{1}{C(\delta)} < \frac{m^u(A_1)}{m^u(A_2)} < C(\delta).
\]
5. Dichotomy for flow type diffeomorphisms

This section is devoted to the proof of our main result on the MMEs of flow-type diffeomorphisms (Theorem 5.3).

We first build an invariant probability measure $\mu^{cu\otimes s}$ from the $cu$- and $s$-Margulis conditionals. We deduce that dilation factors of these conditionals are inverse of each other: $D_u D_u^t = 1$. We then show that any MME with nonnegative central exponent has $u$-disintegrations given by the $u$-Margulis conditionals by using the entropy along the unstable foliation as introduced by Ledrappier and Young [22] and a classical convexity argument of Ledrappier (see [25] for a pedagogical exposition).

A Hopf argument now gives the uniqueness of the MME with nonpositive center exponent. We finish the proof of the dichotomy by building twin measures and considering $f^{-1}$ instead of $f$. We conclude this section by showing some additional properties: uniqueness of the Margulis systems, properties on the sign of the central exponent of $\lambda_c(\mu^{cu\otimes u})$, and the Bernoulli property of hyperbolic MMEs.

5.1. Quasi-product measures. Given a diffeomorphism $f \in \text{Diff}(M)$ with flow type and minimal strong foliations, we build an invariant probability measure from the Margulis systems $\{m^c_x\}_{x \in M}$ and $\{m^s_x\}_{x \in M \setminus C^s}$ as those provided by Proposition 4.11 and Theorem 5.2 applied to the inverse $f^{-1}$.

**Proposition 5.1.** Let $f \in \text{Diff}^1(M)$ have flow type with minimal strong foliations. Assume that there are Margulis systems:

- $\{m^c_x\}_{x \in M}$ on $\mathcal{F}^c$ with dilation $D^u$ which is $s$-invariant;
- $\{m^s_x\}_{x \in M \setminus C^s}$ on $\mathcal{F}^s$ with dilation $D^s$ which is $cu$-quasi-invariant;
- $C^s$ satisfies Lemma 4.11(3).

Then there is an invariant Borel probability measure $\mu^{cu\otimes s}$ such that near each $y \in M$, its conditional measures along $\mathcal{F}^s$ are given by $\{m^s_x\}_{x \in M \setminus C^s}$ with $m^c_y$ as quotient measure. Moreover, $D_u = D_u^{-1}$.

**Proof.** We first define local measures $\{m_p\}_{p \in M}$. Let $p \in M$. Fix $W^s_{loc}(p)$ and $W^c_{loc}(p)$ be sufficiently small neighborhoods of $p$ in $W^s_{loc}(p)$ and $W^c_{loc}(p)$. Let $U_p$ be the image of $W^s_{loc}(p) \times W^c_{loc}(p)$ by the local product map $(x, y) \mapsto W^s_{loc}(x) \cap W^c_{loc}(y)$, a local homeomorphism by transversality of the foliations. We define a measure $m_p$ on $U_p$ according to the following formula:

$$m_p = \int_{W^c_{loc}(p)} m^s_y dm^c_p(y)$$

as we now explain. Fix $\phi$ a continuous function with $\text{supp} \phi \subset U_p$. Let $\alpha^p_\phi : W^c_{loc}(p) \to \mathbb{R}$ be defined by

$$\alpha^p_\phi(y) := m^c_y(1_{W^s_{loc}(y)} \cdot \phi).$$

Note that $\alpha^p_\phi$ is defined and continuous on $\mathcal{F}^c_{loc}(p) \setminus C^s$. By Lemma 4.11(3), this set has full $m^c_{\phi}$-measure in $\mathcal{F}^c_{loc}(p)$. Since $U_p$ is small, Corollary 4.13 shows that the measurable function $\alpha^p_\phi$ is bounded hence integrable for the Radon measure $m^c_{\phi}$. Thus one can give the following meaning to eq. (11):

$$m_p(\phi) := \int_{\mathcal{F}^c_{loc}(p)} \alpha^p_\phi(y) dm^c(y).$$
Note that these measures are finite, positive Borel measures defined for every \( p \in M \).

**Claim 5.2.** The previously defined measures \( m_p, p \in M \), with support contained in some neighborhood \( U_p \), satisfy the following compatibility condition:

\[
(13) \quad \text{If } p, q \in M \text{ and } \phi \in C(U_p \cap U_q), \text{ then } m_p(\phi) = m_q(\phi).
\]

To prove this claim, note that, since \( U_p \) and \( U_q \) are small (and may be assumed to intersect) for every \( x \in U_p \cap U_q \), \( W_{loc}^s(x) \) intersects \( \tilde{W}_{loc}^s(p) \), resp. \( \tilde{W}_{loc}^u(q) \), at exactly one point \( y \), resp. \( z \). Note that \( z = h(y) \) for some given \( s \)-holonomy \( h : \tilde{W}_{loc}^s(p) \to \tilde{W}_{loc}^u(q) \). Additionally, \( \tilde{W}_{loc}^s(y) = W_{loc}^s(x) \cap U_p \) and \( \tilde{W}_{loc}^u(z) = W_{loc}^u(x) \cap U_q \). Since \( \phi \in C(U_p \cap U_q) \), it follows that

\[
\alpha_p^q(y) = \alpha_p^q(h(y)).
\]

Since \( \{m^u_x\}_{x \in M} \) is invariant under \( s \)-holonomy:

\[
m_p(\phi) = \int_{\tilde{W}_{loc}^s(p)} \alpha_p^q(y) \, dm^u_p(y) = \int_{\tilde{W}_{loc}^s(p)} \alpha_p^q(h(y)) \, dm^u_p(y)
\]

\[
= \int_{\tilde{W}_{loc}^s(q)} \alpha_p^q(z) \, dm^u_p(z) = m_q(\phi),
\]

proving the claim.

We now define a finite Borel measure \( m \) on \( M \) by picking a partition of unity

\[ 1 = \chi_1 + \cdots + \chi_r \]

1

subordinated to a finite cover \( U_1, \ldots, U_r \) determined by points \( p_1, \ldots, p_r \) and setting:

\[ m(\phi) = m_{p_1}(\phi \chi_1) + \cdots + m_{p_r}(\phi \chi_r). \]

Finally, we set \( \mu^{cu \otimes s} = m(M)^{-1} \cdot m \). Observe it is a Borel probability measure on \( M \). It is locally finite, hence finite on the compact set \( M \).

Observe that \( m \) (and \( \mu^{cu \otimes s} \)) does not depend on the choice of the partition of unity. Indeed, if \( 1 = \chi_1 + \cdots + \chi_{r'} \) is another partition of unity subordinated to some finite cover defined by points \( p_1', \ldots, p_{r'} \), then, for every \( \phi \in C(M) \),

\[
m_{p_1}(\phi \chi_1) + \cdots + m_{p_{r'}}(\phi \chi_{r'}) = \sum_{i=1}^{r} \sum_{j=1}^{r'} m_{p_i}(\phi \chi_i \chi_{j}') = \sum_{i=1}^{r} \sum_{j=1}^{r'} m_{p_i'}(\phi \chi_i \chi_j')
\]

\[
= m_{p_1'}(\phi \chi_1) + \cdots + m_{p_{r'}}(\phi \chi_{r'}).
\]

The local formula in eq. (12), valid in any open set \( U_p \), implies that the disintegration of \( \mu^{cu \otimes s} \) wrt to any partition subordinate to \( F^s \) is given by the Margulis \( s \)-conditionals \( m^s_x \) for a.e. \( x \in M \).

To show \( f \)-invariance, observe that for any measurable subset \( A \) of \( U_p \cap f^{-1}(U_q) \) with \( p, q \in M \),

\[
m(f(A)) = m_q(f(A)) = \int_{\tilde{W}_{loc}^s(q)} m^s_q(f(A)) \, dm^u_q(y)
\]

\[
= D_s \int_{\tilde{W}_{loc}^s(q)} m^s_{f^{-1}y}(A) \, dm^u_q(y) = D_s \int_{f^{-1}(\tilde{W}_{loc}^s(q))} m^s_{f^{-1}y}(A) \, df^{-1}_{-1}(m^u_q)(x)
\]

\[
= D_s D_{-1} m_p(A) = D_s D_{-1} m(A).
\]

The same holds for any measurable subset of \( M \). The case \( A = M \) implies that \( D_s = D_u \). Hence \( m \) and therefore \( \mu^{cu \otimes s} \) are \( f \)-invariant. \( \square \)
5.2. Identification of the conditionals. We show that any measure of maximal entropy has a disintegration along s- or u-conditionals given by the Margulis systems and that their dilation factor is given by the topological entropy.

Let $\mathcal{F}$ be an invariant foliation. Assume that $\mathcal{F}$ admits a generating increasing measurable partition $\xi$ subordinated to $\mathcal{F}$. Let $\{\mu_{\xi(x)}\}_{x \in M}$ be the corresponding disintegration. The entropy with respect to $\mathcal{F}$ is defined \[23\] as:

$$h(f, \mu, \mathcal{F}) = -\int \log \mu_{\xi(x)}(f^{-1}\xi(fx)) d\mu.$$ 

It does not depend on the choice of $\xi$ \[23\]. It was also shown that uniformly expanding foliations or more generally Pesin unstable foliations, admit such generating, increasing partitions. The next result follows from Theorem C′ in \[22\] and items (i)-(iii) after the statement of the theorem.

**Proposition 5.3** (Ledrappier-Young). Let $f \in \text{Diff}^2(M)$ be partially hyperbolic with strong unstable foliation $\mathcal{F}^u$. For any $\mu \in \mathbb{P}_{\text{erg}}(f)$,

$$h(f, \mu, \mathcal{F}^u) \leq h(f, \mu).$$

If all center Lyapunov exponents of $\mu$ are nonpositive, then the above inequality is an equality.

Indeed, as $\mu$ has non-positive center exponents there exists $\lambda > 0$ such that the corresponding measurable foliation

$$\mathcal{W}^u(x) := \left\{ y \in M : \limsup_{n \to \infty} \frac{1}{n} \log d(f^{-n}(x), f^{-n}(y)) < -\lambda \right\}.$$ 

coincides a.e. with $\mathcal{F}^u$.

Using the above result we are able to show the following proposition which is an extension of Ledrappier’s argument in \[21\] (see also \[25\]). The next two results prove Theorem 3.5 and complete the proof of Theorem 3.2.

**Proposition 5.4.** Let $f \in \text{Diff}^2(M)$ be a diffeomorphism with a Margulis u-system $\{m^u_x\}_{x \in M \setminus \mathcal{C}^u}$ with dilation factor $D^u > 1$. If $\mu \in \mathbb{P}_{\text{erg}}(f)$ with $\lambda^c(\mu) \leq 0$, then $h(f, \mu) \leq \log D^u$. Moreover, there is equality if and only if the disintegration of $\mu$ along $\mathcal{F}^u$ is given by $m^u_x$, $\mu$-a.e.

**Proof.** First suppose that $\mu$ gives full measure to $\mathcal{C}^u$. Lemma \[4.11\] implies that $h(f, \mu) = 0$ so $h(f, \mu) < \log D^u$. Now suppose that $\mathcal{C}^u$ has zero measure. Define a normalized family of measures adapted to the partition $\xi$:

$$m_x(A) := \frac{m^u_x(A \cap \xi(x))}{m^u_x(\xi(x))}.$$ 

Observe that above ratio is well defined, as $m^u_x$ is fully supported and $\xi(x)$ contains an open set. The dilation property of Margulis measures yields:

$$m_x((f^{-1}\xi)(x)) = D^u m^u_x(\xi(fx)) \frac{m^u_x(\xi(x))}{m^u_x(\xi(fx))}.$$ 

Following Ledrappier, observe that $g(x) := -\log m_x((f^{-1}\xi)(x)) \geq 0$ so that, by the pointwise ergodic theorem, we know $\lim_{n \to \infty} \frac{1}{n} S_n g(x)$ (possibly $+\infty$) exists almost everywhere. To identify this limit, observe that it is also a limit in probability. Taking the logarithm of the previous identity, we see that $g(x) = h(fx) - h(x) +
\( \log D_u \) for a measurable function \( h \). Therefore the limit in probability and therefore almost everywhere is the constant \( \log D_u \). Thus \( g \) is integrable with:

\[
- \int \log m_x((f^{-1}\xi)(x)) d\mu = \log D_u.
\]

Now recall that \( h(f, \mu, \mathcal{F}^u) = - \int \log \mu_{\xi(x)}((f^{-1}\xi)(x)) d\mu \) and so,

\[
- \int \log \mu_{\xi(x)}((f^{-1}\xi)(x)) d\mu \leq - \int \log m_x((f^{-1}\xi)(x)) d\mu = \log(D_u).
\]

The inequality comes from Jensen’s inequality and the (strict) concavity of the logarithm. The case of equality for Jensen’s inequality yields that this is an equality if and only if \( \mu_{\xi(x)}((f^{-1}\xi)(x)) = m_x((f^{-1}\xi)(x)) \) for \( \mu \)-a.e. \( x \in M \). Replacing \( \xi \) by \( f^{-n}\xi \), we obtain that

\[
\mu_{f^{-n}\xi(x)}(f^{-n-1}\xi(x)) = \frac{m_x((f^{-n-1}\xi)(x))}{m_x((f^{-n}\xi)(x))}
\]

so

\[
\mu_{\xi(x)}((f^{-n-1}\xi)(x)) = \prod_{k=0}^n \frac{\mu_{\xi(x)}((f^{-k-1}\xi)(x))}{\mu_{\xi(x)}((f^{-k}\xi)(x))} = \prod_{k=0}^n \mu_{(f^{-k}\xi)(x)}((f^{-k-1}\xi)(x))
\]

\[
= \prod_{k=0}^n \frac{m_x((f^{-k-1}\xi)(x))}{m_x((f^{-k}\xi)(x))} = \frac{m_x((f^{-n-1}\xi)(x))}{m_x((f^{-n}\xi)(x))}.
\]

Since \( \xi \) is generating and increasing, the disintegration of \( \mu \) along \( \xi \) is given by the Margulis \( u \)-conditionals as claimed. \( \Box \)

The next result identifies the dilation factor.

**Proposition 5.5.** Let \( f \) be of flow type with minimal strong foliations and \( D_u = D_s^{-1} \) as in Proposition 5.1 then \( D_u = \exp(h_{\text{top}}(\emptyset)) \).

**Proof.** Let \( \mu \) be an ergodic measure for \( f \). If \( \lambda^c(\mu) \leq 0 \) then by Proposition 6.4 we know that \( h(f, \mu) \leq \log D_u \). On the other hand, if \( \lambda^c(\mu) > 0 \), then for \( f^{-1} \) we see that \( h(f, \mu) = h(f^{-1}, \mu) \leq \log D_{f^{-1}} = \log D_u \). Hence, for any ergodic measure we have \( h(f, \mu) \leq \log D_u \) and by the Variational Principle we have \( h_{\text{top}}(f) \leq \log D_u \).

To finish the proof, it is enough to show that, \( \mu^{u \otimes s} \) being the measure constructed in Proposition 5.1

\[
h(\mu^{u \otimes s}) \geq \log(D).
\]

Since the conditional measures of \( \mu^{u \otimes s} \) along \( \mathcal{F}^s \) are given by the Margulis system \( \{m^s\}_{x \in M \setminus C^s} \), the previous proposition shows that:

\[
h(\mu^{u \otimes s}, f^{-1}, \mathcal{F}^s) = - \log(D_s).
\]

Note that \( \mathcal{F}^s \) is the strong unstable foliation of \( f^{-1} \). By Proposition 5.4

\[
h(\mu^{u \otimes s}, f) = h(\mu^{u \otimes s}, f^{-1}) \geq - \log(D_s) = \log(D_u).
\]

\( \Box \)

**Proof of Theorem 3.3** Let \( f \) be a \( C^2 \) diffeomorphism of a compact manifold \( M \). Assume that \( f \) has flow type and minimal strong foliations. The dilation factor \( D_u \) is greater than 1 by Lemma 4.7. By Proposition 5.5 \( D_u = D_s^{-1} = \exp h_{\text{top}}(f) \).
Therefore Proposition 5.4 shows that any ergodic MME $\mu$ with a nonpositive central exponent has unstable conditionals given by the unstable Margulis system \( \{ m^{u}_x \}_{x \in M^u} \).

Finally to see that $\mu$ has full support, recall that each $m^{u}_x$ has full support in $\mathcal{F}^u(x)$ which is dense in $M$, since $\mathcal{F}^u$ is minimal. \( \square \)

5.3. Hyperbolic MMEs. In this section we assume the existence of some hyperbolic MME. We build its twin measure, which is a MME with opposite central exponent. We get a uniqueness result from a version of Hopf’s argument.

**Proof of Proposition 5.5.** By assumption (1), the exponent along $\mathcal{F}^c(x)$ is negative for almost every $x$. Hence an easy version of the Pesin stable manifold theorem shows that $\mathcal{W}^c(x)$, the Pesin stable manifold of $x$ in the center foliation, the intersection of the Pesin stable manifold of $x$ and the center leaf $\mathcal{F}^c(x)$, is an open curve $\mathcal{W}^c(x) \subset \mathcal{F}^c(x)$ for $\mu$-a.e. $x \in M$. By item (2), this curve is bounded. Thus, the following is well-defined $\mu$-almost everywhere (recall that $\mathcal{F}^c(x)$ is oriented):

$$\beta : M \to M, \quad x \mapsto \sup \mathcal{W}^c(x).$$

Note that $\beta$ is measurable and satisfies $\beta \circ f = f \circ \beta$ and $\beta(x) \in \mathcal{F}^c(x)$ for all $x \in M$.

**Claim 5.6.** There is a measurable subset $Z \subset M$ with $\mu(Z) = 1$ such that the restriction $\beta|Z$ is injective.

We now finish the proof of the proposition by assuming the above claim. We will prove the claim below. Let $\nu := \beta_* (\mu)$. Note that the claim implies that $\beta$ is a measure-preserving conjugacy between $(f, \mu)$ and $(f, \nu)$. If $\lambda_* (\nu)$ was negative, Pesin theory would contradict that for $\mu$-a.e. $x \in M$, $\beta(x)$ is on the boundary of some $\mathcal{W}^u(y)$. Thus $\lambda_* (\nu) \geq 0$ and the proposition is established. \( \square \)

To prepare the proof of the claim, recall from Section 2.3 one can find a measurable disintegration of $\mu$ along the foliation $\mathcal{F}^c$ into (projective) Radon measures $\{ \mu^c_x \}_{x \in M}$. Since $f(x) \in \mathcal{F}^c(x)$ with $x \neq f(x)$ for $\mu$-a.e. $x$ one can define Radon measures $\{ \mu^c_x \}_{x \in M}$ by using the invariant normalization $\mu^c_x(y, f(y)) = 1$ for $\mu$-a.e. $x \in M$ and for all $y \in \mathcal{F}^c(x)$. We first show:

$$\text{(14)} \quad \text{for } \mu\text{-a.e. } x \in M \quad \mu^c_x(\mathcal{W}^c(x)) = c_x \delta_x \quad \text{for some } c_x > 0.$$

For each $\epsilon > 0$, let $\mathcal{W}^{c,\epsilon}(x) := \{ y \in \mathcal{W}^c(x) : d_{c}(y, \partial \mathcal{W}^c(x)) > \epsilon \}$ where $d_{c}(\cdot, \cdot)$ is the induced distance on the center foliation. Note that for any $\delta > 0$, there are a set $S$ of positive $\mu$-measure of points $x$ and some arbitrarily small $\epsilon > 0$ such that

$$\forall x \in S \quad \mu^c_x(\mathcal{W}^c(x)) \geq (1 - \delta) \mu^c_x(\mathcal{W}^c(x)).$$

The last measure is positive since $\mu$-a.e. point $x$ belongs to the support of $\mu^c_x$. Now, for $\mu$-a.e. $x$, for all large $n \geq 0$,

$$f^n(\mathcal{W}^{c,\epsilon}(f^{-n}x)) \subset \{ y \in \mathcal{W}^c(x) : d_{c}(y, x) < \epsilon \}.$$

The ergodicity of $\mu$ implies that $f^{-n}x \in S$ for some arbitrarily large integers $n$. Hence, by invariance of $\mu$:

$$\mu^c_x(\{ y \in \mathcal{W}^c(x) : d_{c}(y, x) < \epsilon \}) \geq \mu^c_{f^{-n}x}(\mathcal{W}^{c,\epsilon}(f^{-n}x))$$

$$\geq (1 - \delta) \mu^c_{f^{-n}x}(\mathcal{W}^c(f^{-n}x)) = (1 - \delta) \mu^c_x(\mathcal{W}^c(x)).$$

Since $\epsilon, \delta > 0$ were arbitrarily small, eq. (14) follows.
Proof of Claim 5.6. To prove the claim, let \( Z \) be the set of \( x \in M \) for which \( \mu^c_x \) satisfies eq. (14). This is a measurable set with full measure \( \mu \). Now, let \( x, y \in Z \). If \( \beta(x) = \beta(y) \) then \( W^c(x) = W^c(y) \). In particular \( \mu^c_x|W^c(x) = \mu^c_y|W^c(y) \). By eq. (14), this implies \( x = y \), concluding the proof of the claim. \( \square \)

We now turn to the Hopf argument.

Proof of Proposition 5.7. Let \( \mu, \nu \) be ergodic MME with \( \lambda_<(\mu) < 0 \) and \( \lambda_<(\nu) \leq 0 \). Proposition 5.4 implies that their conditional measures along unstable foliations are both given by the \( w \)-Margulis system \( \{m^u_y\}_{y \in M \cap C^u} \). Let \( B_\mu := \{x \in M : \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^kx} \to \mu \} \) be the ergodic basin of \( \mu \) (the convergence is in the weak star topology as \( n \to +\infty \)). By ergodicity \( \mu(M \setminus B_\mu) = 0 \) and so \( m^u_\mu(M \setminus B_\mu) = 0 \) for \( \mu \)-a.e. \( x \). Similarly, letting \( B_\nu \) be the ergodic basin of \( \nu \), for \( \nu \)-a.e. \( y \) we have \( m^u_\nu(M \setminus B_\nu) = 0 \).

As the center Lyapunov exponent of \( \mu \) is negative, for \( \mu \)-a.e. \( y \in B_\mu \), \( m^u_\mu(M \setminus B_\mu) = 0 \) and there is a subset \( K \subset \mathcal{F}^u(y) \cap B_\mu \) with \( m^u_\mu(K) > 0 \) and such that the size of the Pesin local stable manifolds \( \mathcal{W}^s_{loc}(z) \) of points \( z \in K \) is uniformly bounded from below.

Pick \( x \in B_\nu \) with \( m^u_\nu(M \setminus B_\nu) = 0 \). The density of \( \mathcal{F}^u(x) \) implies that there is a local \( cs \)-holonomy \( h : K \to \mathcal{F}^u(x) \) (perhaps after replacing \( K \) with a smaller subset). This holonomy is absolutely continuous from \( (K, m^u_\nu) \) to \( (\mathcal{F}^u(x), m^u_y) \), hence:

\[
m^u_\nu(h(K)) > 0.
\]

By the choice of \( K \), \( h(z) \) belongs to the Pesin stable manifold of \( z \). Since the ergodic basin is saturated by stable manifolds, \( h(K) \subset B_\mu \) and therefore \( m^u_\nu(B_\mu) > 0 \). As \( m^u_\nu(M \setminus B_\nu) = 0 \), we conclude that \( B_\nu \cap B_\mu \neq \emptyset \) and consequently \( \mu = \nu \). \( \square \)

We will use the following consequence of the above proof.

Remark 5.7. Let \( f \) be a flow type \( C^2 \) diffeomorphism with minimal strong foliations. Whenever \( \mu \) is an ergodic MME with \( \lambda_<(\mu) < 0 \), it is the only invariant probability measure whose \( u \)-conditionals are given by an unstable Margulis system.

Proof of Theorem 5.8. Let \( f \) be a \( C^2 \) diffeomorphism with flow type and minimal strong foliations. Observe that since \( f \) is partially hyperbolic with a one-dimensional center, there must exist an ergodic MME \( \mu \) \cite{12} \cite{13} \cite{23}. For instance, assume that \( \lambda^c(\mu) < 0 \) (if not, use \( f^{-1} \)).

Proposition 5.7 shows that there is no other ergodic MME with nonpositive central exponent. Now Proposition 5.3 gives some ergodic MME \( \nu \) with \( \lambda^c(\nu) \geq 0 \). In particular \( \nu \neq \mu \). The previous uniqueness shows that the case \( \lambda^c(\nu) = 0 \) cannot occur so \( \lambda^c(\nu) > 0 \). We see that there are exactly two ergodic MME. The dichotomy is proved.

It remains to prove that \( \mu \) is Bernoulli. The symbolic dynamics of \cite{30} implies that \( (f, \mu) \) is isomorphic to the product of a Bernoulli measure with a circular permutation of some order \( p \geq 1 \). It follows that \( \mu = (\mu_1 + \cdots + \mu_p)/p \) where the measures \( \mu_1, \ldots, \mu_p \) are distinct ergodic MMEs for \( f^p \) such that \( \lambda^c(f^p, \mu_k) = p \cdot \lambda^c(f, \mu) \) does not depend on \( k \) and is not zero. Observe that \( f^p \) like \( f \) has flow type with minimal strong foliations. Hence, the previous uniqueness result applies to \( f^p \) showing that \( p = 1 \), i.e., \( (f, \mu) \) is Bernoulli.

The theorem is established. \( \square \)
5.4. **Uniqueness of the \( u \)-conditionals.** The following together with Proposition 5.5 proves Addendum 3.3.

**Proposition 5.8.** Let \( f \) be as in Theorem 3.9. Then the stable and unstable Margulis systems are unique. Moreover:

In the nonhyperbolic case\(^3\), these Margulis systems give the disintegration of all MMEs along both unstable and stable foliations.

In the hyperbolic case\(^4\), the center-stable and center-unstable Margulis systems are also unique.

Moreover, the quasi-product measures defined from the Margulis systems have center exponents with the expected signs:

\[
\lambda_c(\mu^{cs} \otimes u) \leq 0 \quad \text{and} \quad \lambda_c(\mu^{cu} \otimes s) \geq 0.
\]

**Proof.** In the non-hyperbolic case, take any ergodic measure of maximal entropy. As it has zero central exponent, the Ledrappier argument (Proposition 5.4) shows that its conditional measures along strong foliations should coincide with the unique stable and unstable Margulis systems \( \{m^s_x\}_{x \in M \setminus C} \) and \( \{m^u_x\}_{x \in M \setminus C^u} \).

Now consider the hyperbolic case. There is one ergodic MME \( \mu_- \) with \( \lambda_c(\mu_-) < 0 \). It follows from Proposition 5.7 that \( \mu^{cs} \otimes u = \mu_- \), hence \( \lambda_c(\mu^{cs} \otimes u) < 0 \). Similarly, for \( \mu_+ \) one has \( \lambda_c(\mu^{cu} \otimes s) > 0 \).

Let us prove that there exists a unique (up to normalization) system of Margulis measures \( \{m^{cs}_x\} \).

Suppose that there exists another Margulis \( cs \) system called \( \{\hat{m}^{cs}_x\} \). Consider the quasi-product probability locally defined as

\[
\nu := \int m^u(y) d\hat{m}^{cs}_x(y).
\]

It has \( u \)-conditionals \( m^u_y \). By Remark 5.7, we have \( \nu = \mu^{cs} \otimes u \) and consequently \( \hat{m}^{cs}_x = m^{cs}_x \) for \( \nu \)-almost every \( x \). Now the continuity of Margulis families together with the full support of \( \nu \) imply that the last equality holds for every \( x \in M \). A similar argument shows uniqueness of the Margulis family \( \{m^{cu}_x\} \). \( \square \)

6. **Perturbing time-one maps to get flow type with minimal strong foliations**

We prove Theorem 3.10. Let \( \varphi^t : M \to M \) be a topologically transitive Anosov flow on a compact manifold and let \( T > 0 \). We find an open set of diffeomorphisms with flow type and minimal strong foliations accumulating on \( \varphi^T \). We first use structural stability results, mainly from \[19\], to show that flow type holds for all \( C^1 \) close to \( \varphi^T \). We then use the minimality of the strong foliations.

**Proposition 6.1.** Let \( \varphi^T : M \to M \) be the time \( T > 0 \) map of an Anosov flow. Then there is a \( C^1 \)-neighborhood \( \mathcal{V} \) of \( \varphi^T \) such that \( f \in \mathcal{V} \) has flow type.

**Proof.** We must prove (I), (II), (III), and (IV) for all diffeomorphisms \( C^1 \)-close to \( \varphi^T \). Observe that these properties are well-known for \( \varphi^T \) itself. Let us see that they hold for all \( C^1 \)-close diffeomorphisms using the structural stability theory in \[19\].

Partial hyperbolicity with the center subbundle of a given dimension is well-known to be robust. Since the center foliation \( \mathcal{F}^c \) of \( \varphi^T \) is the partition into the

---

\(^3\)The nonhyperbolic case is the first alternative of Theorem 3.9.

\(^4\)The hyperbolic case is the second alternative of Theorem 3.9.
orbits of the flows is smooth, it is plaque expansive \[19\] (7.2)]. Therefore \((\varphi^T,F^c)\) is structurally stable \[19\] (7.1) and therefore dynamical coherence (II) holds robustly. Indeed, this theorem yields a center foliation \(F^c_g\) for \(g\), whereas its proof, especially \[19\] thm. (6.8), gives stable and unstable manifolds of center leaves that coincides with \(F^c_{gu}\) and \(F^c_{gs}\).

The center foliation of \(\varphi^T\) is obviously oriented by the vector field. The flow being expansive it has at most countably many closed orbits. The flow having the specification property it has (infinitely many) closed orbits. The structural stability of \((\varphi^T,F^c)\) implies that (III) holds robustly.

To establish (IV), we need to refer to the proof of the structural stability of \((\varphi^T,F^c)\) \[19\] Thm. (7.1) and especially of \[19\] Thm. (6.8). We use terminology and notations from \[19\] chap. 6, 7. On page 107 of \[19\], it is shown that the perturbed diffeomorphism \(f'\) has a center foliation \(F'\) whose every leaf is close to the corresponding leaf of \(F\) in the sense that \(F'((h(x))\) is represented by section of the formal normal bundle to \(F(x)\) which is close to zero when \(f'\) is close to \(f = \varphi^T\).

Since the lifts \(i^*f, i^*f'\) of \(f, f'\) to this formal bundle are close to each other, we see that \(f'(x) = F'((\tau(x), x))\) for some function \(\tau : M \to \mathbb{R}\) with \(\sup |\tau - T| < \epsilon\) where \(\epsilon > 0\) can be taken arbitrarily small by assuming \(f'\) to be close enough to \(f = \varphi^T\).

The previous reasoning shows that the minimum of the lengths of the closed loops for \(F'\) is close to that of \(F\). In particular it can be assumed to be larger than \(3\epsilon\).

If \(\tau\) is not continuous, then there are two sequences of points \(x_n, y_n \in M\) that converge to the same limit \(z\) and such that \(s := \lim_{n \to \infty} \tau(x_n)\) and \(t := \lim_{n \to \infty} \tau(y_n)\) exist and are distinct. This implies that \(F^s(z) = F^t(z)\) with \(|t - s| < 2\epsilon\), i.e., \(F'\) contains a loop of length less than \(2\epsilon\). The contradiction proves the continuity of \(\tau\). \(\square\)

**Remark 6.2.** Note that flow type property together with plaque expansivity is a \(C^1\)-open property. We do not know if this is true for the flow type property itself.

It remains to check the minimality of both strong foliations. Let \(T\) be the set of time \(T\) maps of topologically transitive Anosov flows on some compact manifold. We need to find an open set \(U\) of diffeomorphisms whose strong stable and strong unstable foliations are both minimal and such that \(T \subset \overline{U}\). We proceed in two steps.

**Step 1.** Any diffeomorphism in \(T\) can be \(C^1\)-approximated by robustly transitive diffeomorphisms.

This follows from the work of Bonatti and Diaz in \[8\].

**Step 2.** Any diffeomorphism in \(T\) has a \(C^1\)-neighborhood \(V\) with the following property. Any robustly transitive diffeomorphism in \(V\) can be \(C^1\)-approximated by diffeomorphisms whose strong stable and strong unstable foliations are both minimal.

This follows from the work of Bonatti, Díaz and Ures \[4\] (it is in fact a simpler situation since in our setting hyperbolic periodic points are dense and contained in invariant compact leaves).

This completes the proof of Theorem 3.10.
Remark 6.3. A recent work of Ures, Viana, and Yang \cite{ures-viana-yang} extends \cite{bonatti-diaz-ures} to a larger class of systems among the $C^\infty$ volume preserving diffeomorphisms. More precisely, they prove that in a neighborhood of the time one map of the geodesic flow of any hyperbolic surface, there is an open and dense set of diffeomorphisms with both strong foliations minimal.

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