Weighted Grand Lebesgue Spaces norm estimation
for Hardy - Sobolev - Poincare - Wirtinger operators.

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Abstract.

We extend the classical Hardy - Sobolev - Poincare - Wirtinger inequalities from the ordinary Lebesgue - Riesz spaces into the Grand Lebesgue ones, with exact constants evaluation.

Key words and phrases. Hardy - Sobolev - Poincare - Wirtinger inequalities, domain, distance from the boundary, upper and lower estimates and limits, Lebesgue - Riesz, dilation method, Sobolev’s and Grand Lebesgue Space norm and spaces, vector, dilatation and parameter of dilatation, Talenty’s method, gradient, convex and property set, operators, examples.

1 Definitions. Notations. Statement of problem. Previous results.

Let $D$ be certain proper own: $D \neq \mathbb{R}^n$ sub-domain of the whole space $\mathbb{R}^n$, equipped with ordinary Euclidean norm $|x|, x \in \mathbb{R}^n$, for instance open, connected,
convex, having non-trivial interior and Lipschitz boundary $\partial D$. The distance between arbitrary vector $x \in \mathbb{R}^n$ and boundary $\partial D$ will be denoted as follows

$$d(x) = d(x, \partial D) = \inf_{y \in \partial D} |x - y|.$$  

Let also $u = u(x), \ x \in D$ be arbitrary valued function belonging to the Sobolev's space $W^1_p$:

$$||u||_{W^1_p} \overset{def}{=} |\nabla u|^1_p + |u|^1_p < \infty;$$

where the usually Lebesgue - Riesz norm $|u|^1_p$ has the form

$$|u|^1_p := \left[ \int_D |u(x)|^p \ dx \right]^{1/p}, 1 \leq p < \infty;$$

the gradient in a distributional sense of the function $u$.

The following (and close) key weight Hardy - Sobolev - Poincare - Wirtinger (briefly, HSPW) inequality

$$\int_D \frac{\nabla u|^p}{\alpha(x, \partial D)} \geq \left( \frac{\alpha + p - n}{p} \right)^p \cdot \int_D \frac{|u|^p}{d^\alpha(x, \partial D)}; \ p > n - \alpha, \alpha = \text{const} < n \quad (1)$$

belongs to many authors, see e.g. [1], [2], [3], [4], [18], [19], [27]. See also [12], [20], [21], [26], [28], [29] etc.

It may be rewritten as follows. Introduce the following (Borelian) measure $\mu_\alpha = \mu_{\alpha,D}$ on the measurable subsets of the source set $D$

$$\mu_{\alpha,D}(dx) \overset{def}{=} \frac{dx}{d^\alpha(x, \partial D)} \quad (2)$$

and put

$$K(p) = K(p; \alpha, n) := \frac{p}{p + \alpha - n}, \ p > n - \alpha. \quad (3)$$

Define also the (linear) operator defined on all the functions having as a domain of definition the set $D \setminus \partial D$:

$$T[u](x) \overset{def}{=} \frac{u(x)}{d(x, \partial D)}. \quad (4)$$

Then when $p > n - \alpha$

$$||T[u]||_{L_p(D, \mu_\alpha)} \leq K(p) \cdot ||\nabla u||_{L_p(D, \mu_\alpha)}. \quad (5)$$

Herewith the ”constant” $K = K(p)$ is the best possible for arbitrary proper domain $D$:

$$\sup_{u: \ ||\nabla u||_{L_p(D, \mu_\alpha)} = 1} ||T[u]||_{L_p(D, \mu_\alpha)} = K(p), \ p > n - \alpha. \quad (6)$$
Our claim in this short preprint is to extrapolate the last estimate onto the so-called Grand Lebesgue Spaces instead the classical Lebesgue - Riesz ones, builded on the our set $D$ equipped with the measure $\mu_{\alpha,D}$.

We intent herewith to calculate the exact value of correspondent embedding constants.

The particular case of these statement was considered in [26]; see also [23] - [25].

Let us recall here for readers convenience some known definitions and facts from the theory of Grand Lebesgue Spaces (GLS) adapted exclusively to offered article.

Let the number $b$ be constants such that $n - \alpha < b \leq \infty$; and let $\psi = \psi(p) = \psi[b](p)$, $p \in (n - \alpha, b)$, be numerical valued strictly positive function not necessary to be finite in every point:

$$\inf_{p \in (n - \alpha, b)} \psi[b](p) > 0. \quad (7)$$

The set of all such a functions $\psi(\cdot)$ will be denoted by $\Psi(b) = \{ \psi = \psi(\cdot) \}$.

For instance

$$\psi_m(p) := p^{1/m}, \ m = \text{const} > 0,$$

or more generally

$$\psi_{m,\beta,L}(p) := p^{1/m} \ln^\beta(p + 1) L(\ln(p + 1)), \ \beta = \text{const}, \quad (8)$$

where $L(\cdot)$ is positive continuous slowly varying as $p \to \infty$ function.

**Definition 1.1.**

By definition, the (Banach) Grand Lebesgue Space (GLS) $G\psi = G\psi[b]$, consists on all the real (or complex) numerical valued measurable functions $f : D \to \mathbb{R}$ defined on the whole our space $\Omega$ and having a finite norm

$$||f||_{G\psi} = ||f||_{G\psi[b]} \overset{def}{=} \sup_{p \in (n - \alpha, b)} \left[ \frac{||f||_{p,D,\mu_{\alpha}}}{\psi(p)} \right]. \quad (9)$$

The function $\psi = \psi(p) = \psi[b](p)$ is named as the *generating function* for this space.

If for instance

$$\psi(p) = \psi^{(r)}(p) = 1, \ p = r; \ \psi^{(r)}(p) = +\infty, \ p \neq r,$$

where $r = \text{const} \in [1, \infty)$, $C/\infty := 0, \ C \in \mathbb{R}$, (an extremal case), then the correspondent $G\psi^{(r)}(p)$ space coincides with the classical Lebesgue - Riesz space $L_r = L_r(\Omega, \mathbb{P})$.  

3
The finiteness of some GLS $G\psi$ norm for the function $f : D \to R$ is closely related with its tail function

$$T_f(u) := \mu_\alpha \{ x : |f(x)| > u \}, \quad u \geq 0.$$ 

The GLS spaces are also closely related with the suitable exponential Orlicz ones, builded on the our measurable space $(D, \mu_\alpha)$.

See the detail investigation of these spaces in the works [5], [6] - [7], [9], [10], [8], [11], [15] - [17], [22], chapters 1,2.

## 2 Main result.

**Theorem 2.1.** Suppose that for certain function $u = u(x), \ x \in D$ its gradient belongs to some Grand Lebesgue Space $G\psi[b] :$

$$||\nabla u||_{G\psi[b]} < \infty. \quad (10)$$

Introduce an auxiliary such a function

$$\psi_K(p) := K(p) \cdot \psi(p), \quad n - \alpha < p < b.$$ 

Our proposition:

$$||T[u]||_{G\psi_K} \leq 1 \times ||\nabla u||_{G\psi}, \quad (11)$$

where the constant "1" in (11) is the best possible.

**Remark 2.1.** The generating function $\psi[b](\cdot)$ in (10) may be chosen by the natural way:

$$\psi[b](\cdot) := ||\nabla u||_p, \quad n - \alpha < p < b,$$

of course, if there exists such a value $b > n - \alpha$.

**Proof of upper bound** is very simple and is alike to one in the articles [24], [25]; as well as the proof of the lower bound. Suppose

$$||\nabla u||_{G\psi} < \infty;$$

one can assume without loss of generality

$$||\nabla u||_{G\psi} = 1.$$ 

It follows from the direct definition of the norm in GLS

$$\forall p \in (n - \alpha, b) \Rightarrow ||\nabla u||_p \leq \psi(p).$$
We apply the inequality (5)
\[ ||T[u]||_{L_p(D, \mu_\alpha)} \leq K(p) \cdot \psi(p) = \psi_K(p), \]
and on the other words
\[ ||T[u]||_{G\psi_K} \leq 1 = ||\nabla u||_{G\psi}. \]

The lower bound in (11) follows immediately from one of the results of the article [25]; see also [24], taking into account the exactness of the value \( K = K(p) \), see (6).
Q.E.D.

3 Necessary conditions for these estimations.

We will ground in this section that the "configuration" given by the estimate (1) is essentially non-improvable. Namely, suppose that there exists non-trivial constants \( a, h; G(D, a, h) \) such that for arbitrary proper non-trivial sub-domain \( D \subset \mathbb{R}^n \) and for arbitrary function \( u = u(x), x \in D \) belonging to the space \( C^0_\infty(D) \) there holds the estimate

\[ L[u] := L[u](a) \overset{def}{=} \left[ \int_D \frac{|u(x)|^q}{d^a(x, \partial D)} \, dx \right]^{1/q} \leq \]

(12)

\[ R[u] := R[u](h) \overset{def}{=} G(D, a, h) \left[ \int_D \frac{||\nabla u||^p}{d^h(x, \partial D)} \, dx \right]^{1/p} \]

(13)

for certain fixed values \( p, q \in [1, \infty) \).

**Theorem 3.1.** Suppose that the relations (12) and (13) holds true for arbitrary proper domain \( D \) as well as for certain non-zero function \( u = u(x), x \in D \) from the set \( C^0_\infty(D) \). Then

\[ a - n \frac{q}{p} = 1 + h - n \frac{p}{p}. \]

(14)

**Proof.** We will apply the so-called dilation method, belonging at first, perhaps, to G.Talenty, see the article [29], in which was considered the particular case \( a = h = 0 \).

Let us choose the domain \( D \) on the form

\[ D := R^d \otimes R^r_+, \quad n = \dim D = d + r, \quad r, d > 0. \]

Let’s agree to write instead of \( x \) the vector \( (x, y), x \in R^d, y \in R^r_+ \). Then
\[
L^q[u] = \int_{R^d} \int_{R^d_+} \frac{|u(x, y)|^q}{|y|^a} \, dx \, dy,
\]
\[
R^p[u] = \int_{R^d} \int_{R^d_+} \frac{|\nabla(x, y)|^p}{|y|^h} \, dx \, dy.
\]

One can choose the function \( u = u(x, y) \) from the set \( C^0_\infty(D) \) such that \( L(u) > 0, \ R[u] > 0; \) therefore \( 0 < L[u] \leq CR[u] < \infty. \)

Let also \( \lambda \in (0, \infty) \) be a parameter of dilatation:

\[
V_\lambda[u](x, y) \overset{\text{def}}{=} u(\lambda x, \lambda y).
\]

Obviously, \( V_\lambda[u](\cdot, \cdot) \in C^0_\infty(D). \) Therefore, it satisfies the inequality (12) - (13):

\[
L_a[V_\lambda u] \leq CR_h[V_\lambda u]. \tag{15}
\]

But

\[
\{L_a[V_\lambda u]\}^q = \lambda^{-d-r+a} L^q[u];
\]

\[
L_a[V_\lambda u] = \lambda^{(a-d-r)/q} L_a[u]
\]

and analogously

\[
R_h[V_\lambda u] = \lambda^{(p-d-r+h)/p} R_h[u],
\]

and we get to the inequality

\[
C_1 \lambda^{(a-d-r)/q} \leq C_2 \lambda^{(p-d-r+h)/p}
\]

for arbitrary positive values \( \lambda. \) Following,

\[
\frac{a-d-r}{q} = \frac{p-d-r+h}{p},
\]

which is equal to (14).

Of course, in the case when \( a \cdot h = 0 \) or \( p = q \) we get to the classical Sobolev’s inequality, see e.g. [29].

4 Concluding remarks.

Open question: Suppose that the there holds (14) and that the domain \( D \) is proper. Assume also the function \( u = u(x) \) belongs to the set \( C^0_\infty(D). \) Will be the inequality (12) - (13) true, of course, in general case, i.e. when \( p \neq q? \)
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