Comparison of a general series expansion method and the homotopy analysis method

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January 26, 2013

Abstract
A simple analytic tool namely the general series expansion method is proposed to find the solutions for nonlinear differential equations. By choosing a set of suitable basis functions \( \{e_n(t, t_0)\}_{n=0}^{\infty} \) such that the solution to the equation can be expressed by \( u(t) = \sum_{n=0}^{\infty} c_n e_n(t, t_0) \). In general, \( t_0 \) can control and adjust the convergence region of the series solution such that our method has the same effect as the homotopy analysis method proposed by Liao, but our method is more simple and clear. As a result, we show that the secret parameter \( h \) in the homotopy analysis methods can be explained by using our parameter \( t_0 \). Therefore, our method reveals a key secret in the homotopy analysis method. For the purpose of comparison with the homotopy analysis method, a typical example is studied in detail.

Keywords: general series expansion method; the homotopy analysis method; nonlinear differential equation

PACS: 02.30.Mv, 02.60.Cb, 02.30.Hq, 02.30.Jr

1 Introduction
For a given nonlinear differential equation with initial and boundary conditions, it is well known that we can use the Taylor series expansion at the initial point to find its solution. On the other hand, according to the initial and boundary conditions, if we choose a set of suitable basis functions \( \{e_n(t, t_0)\}_{n=0}^{\infty} \) to expand
the solution, we can obtain more efficient solution. For example, for a vibrate problem, we can choose periodic functions such as sine or cosine functions as the basis functions. Fourier series or transformation method is a typical example. In general, the convergence region of the series solution is relative small. How to enlarge the convergence region of the solution? The homotopy analysis method proposed by Liao[1] provides a way to solve the problem. We describe the Liao’s method as follows. Consider the nonlinear differential equation

\[ Lu + N(u) = 0, \]

with some initial or boundary conditions, where \( L \) is a linear operator, and \( N \) is a nonlinear operator. The key step is to construct the following homotopy

\[ (1 - p)[Lu - Lu_0] + ph[Lu + N(u)] = 0. \]

When \( p \) changes from 0 to 1, the solution of Eq.(2) changes from \( u_0 \) to the solution of Eq.(1). Concretely, take the solution of Eq.(2) in the following series form

\[ u(t, p) = \sum_{n=0}^{+\infty} u_n(t)p^n, \]

where \( u_n(t) \) are unknown functions to be determined, for \( n = 0, 1, \ldots \). Substituting \( u(t, p) \) into Eq.(2) and setting the coefficients of \( p^n \) to zero yields the system of the differential equations of \( u_n(t) \)'s. Solving the system gives every \( u_n \). In finally, taking \( p = 1 \) gives the solution \( u(t) = u(t, 1) \) of Eq.(1). In a series of papers[2-10] and the book[11], liao developed and applied his method to deal with a lot of nonlinear problems. According to different choice of \( u_0(t) \), the final form of solution of Eq.(2) is given by

\[ u(t) = \lim_{m \to \infty} \sum_{n=0}^{m} \mu_{m,n}(h)e_n(t), \]

where \( \{e_n(t)\}_{n=0}^{+\infty} \) is a set of functions and the approaching function \( \mu_{m,n}(h) \) satisfies \( \lim_{m \to \infty} \mu_{m,n}(h) = 1 \). For example, if we take \( e_n(t) = t^n \) for \( n = 0, 1, 2, \ldots \), we obtain the so-called generalized Taylor series solution. Liao points out that the generalized Taylor series provides a way to control and adjust the convergence region through an auxiliary parameter \( h \) such that the homotopy analysis method is particularly suitable for problems with strong nonlinearity[1,11].

It is easy to see that the auxiliary parameter \( h \) is the key to the homotopy analysis method. However, the mathematical meaning of the parameter \( h \) is unknown so that the homotopy analysis method is still wrapped in a secret veil. This is an open problem in the theory of the homotopy analysis method. Thus a natural aim is to solve it.

In the present paper, we propose a general series expansion method for nonlinear differential equations. By comparing with the homotopy analysis method,
we give the answer of the problem on the mathematical meaning of \( h \) in the homotopy analysis method. In order to understand the key of my idea, we first prove that the so-called generalized Newton binomial series at the initial point is just the usual Newton binomial series expansion at another point \( t_0 \). For other basis functions, we also introduce an auxiliary point \( t_0 \) whose mathematical meaning is clear. Indeed, we find that the point \( t_0 \) can control and adjust the region of the corresponding series solution. Thus our method includes two key steps: one is the choice of the basis function, another is the choice of the point \( t_0 \). These two simple choices can provide us with an efficient way to expand the solution and control its convergence region. By the comparison of these two methods, we uncovers some secret aspects of the homotopy analysis method. Of course, as a simple but efficient method, the general series expansion method can be directly used to solve nonlinear equations. We will give other applications in the future.

2 On the generalized Newton binomial theorem

In order to understand easily the key of our idea, we discuss the Liao’s generalized Newton binomial theorem. For real number \( \alpha (\alpha \neq 0, 1, 2, 3, \cdots) \), the generalized Newton binomial theorem is given by

\[
(1 + t)^\alpha = \lim_{m \to \infty} \sum_{n=0}^{m} \frac{\alpha^{m-n}(h)}{n!} t^n,
\]

whose convergence region is

\[
-1 < t < \frac{2}{|h|} - 1, (-2 < h < 0),
\]

where

\[
\mu_{\alpha}^{m,n}(h) = (-h)^{n-\alpha} \sum_{j=0}^{m-n} (-1)^j \binom{\alpha - n}{j} (1 + h)^j,
\]

and

\[
\binom{\alpha}{n} = \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!}.
\]

The approaching function \( \mu_{\alpha}^{m,n}(h) \) satisfies \( \lim_{m \to \infty} \mu_{m,n}(h) = 1 \) when \( n \geq 1 \). Liao[11] points out that the generalized Newton binomial theorem provides a way to control and adjust the convergence region through an auxiliary parameter \( h \). We next point out that the generalized Newton binomial theorem is exactly the usual Newton binomial expansion at another point \( t_0 = -1 - \frac{1}{h} \). Indeed, when \( t_0 = -1 - \frac{1}{h} \), the Taylor expansion at the point \( t_0 \) is given by

\[
(1 + t)^\alpha = (1 + t_0)^\alpha \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{1}{(1 + t_0)^n} (t - t_0)^n,
\]
\[ \lim_{m \to \infty} \sum_{k=0}^{m} \mu_{\alpha}^{m,k}(h) \binom{\alpha}{k} t^k, \quad (9) \]

where
\[ \mu_{\alpha}^{m,k}(h) = (-h)^{k-\alpha} \sum_{j=0}^{m-k} \binom{\alpha-k}{j} (h+1)^j (-1)^j. \quad (10) \]

Correspondingly, the convergence region \(|t - t_0| < |1 + t_0|\) of the binomial expansion at the point \(t_0 = -1 - \frac{1}{h}\) is just (6). This result reveals clearly the secret of the auxiliary parameter \(h\) and the essence of the generalized Newton binomial theorem. More details can be found in Ref. [12]. Furthermore, we can prove that the generalized Taylor series is essentially the usual Taylor expansion at another point [13].

### 3 General series expansion method

For a given differential equation of function \(f(t)\)
\[ N(f, f', \cdots) = 0, \quad (11) \]
with some original or boundary conditions at the point \(t = 0\), e.g., \(f(0) = 0, f'(0) = 0, f(+\infty) = 0\). Firstly, we choose a set of base functions \(e_0(t, t_0), e_1(t, t_0), \cdots\). Then we expand the solution as a series,
\[ f(t) = \sum_{n=0}^{+\infty} a_n e_n(t, t_0). \quad (12) \]

Substituting the series solution into equation (11) and using the original and boundary conditions, we can determine the values of the parameters \(a_n\) for \(n = 0, 1, \cdots\).

On the other hand, we can expand every \(e_n(t, t_0)\) as
\[ e_n(t, t_0) = \sum_{m=0}^{+\infty} b_{nm}(t_0) e_m(t), (n = 0, 1, \cdots), \quad (13) \]
where
\[ e_m(t) = e_m(t, 0). \quad (14) \]

Then we have
\[ f(t) = \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} a_n b_{nm}(t_0) e_m(t), \quad (15) \]
Therefore there is a relationship between the point \(t_0\) and the auxiliary parameter \(h\) such that
\[ f(t) = \lim_{m \to +\infty} \sum_{n=1}^{m} \mu_{\alpha}^{m,n}(h) c_n e_n(t), \quad (16) \]
where $c_n$ satisfies
\[ f(t) = \sum_{n=1}^{+\infty} c_n e_n(t). \] (17)

**Remark 1.** Comparing with the usual series expansion methods such as power series method whose expanding point is in general the original point, an important different point of our proposed general series method is that the expanding point of our series is a motive point $t_0$ which can control the convergence region.

**Remark 2.** Ma and Fuchssteiner proposed a powerful approach for finding exact solutions to nonlinear differential equations [14]. The crucial idea is to expand solutions of given differential equations as functions of solutions of solvable differential equations, in particular, polynomial and rational functions. A more systematical theory on decompositions and transformations is presented very recently in Refs.[15] and [16]. The resulting theory unifies many existing approaches to exact solutions such as the tanh-function methods, the homogeneous balance method, the exp-function method and the Jacobi elliptic function method.

**Remark 3.** The introduction of the parameter $h$ in the homotopy analysis method is to control the convergence region. In order to increase the speed of convergence of the series solution, Liao used the Pade approximation to do it (see Ref.[11] for details). Therefore, we can also use Pade approximation to increase the speed of convergence of the general series solution.

### 4 A typical example

In the chapter 2 of the book [11], Liao studies a typical example which reads
\[ V'(t) = 1 - V^2(t), \quad V(0) = 0, \] (18)
whose exact solution is given by
\[ V(t) = \tanh(t). \] (19)

Liao constructs the following homotopy
\[ (1 - q)(\frac{\partial(\Phi(t; q) - V_0(t))}{\partial t}) = qh(\frac{\partial \Phi(t; q)}{\partial t} + \Phi^2(t; q) - 1) \] (20)
where $q \in [0, 1]$. Then we have $\Phi(t; 0) = V_0(t), \Phi(t; 1) = V(t)$. We take
\[ \Phi(t; q) = V_0(t) + \sum_{m=1}^{+\infty} V_m(t)q^m, \] (21)
where $V_m(t)$ are the functions to be determined. Substituting the solution into the homotopy equation yields a polynomial of $q$. Setting the coefficients of the polynomial gives a series of linear differential equations which solution $V_m(t)$
can be obtained. In final, we take \( q = 1 \) to give the solution needed. According to the rule of solution expression, by taking \( V_0(t) = t \), Liao obtains a generalized Taylor series solution

\[
V(t) = \lim_{m \to \infty} \sum_{n=0}^{m} \mu_0^{m,n}(h)\alpha_{2n+1}t^{2n+1},
\]

(22)

where

\[
\mu_0^{m,n}(h) = (-h)^n \sum_{k=0}^{m-n} \binom{n-1+k}{k}(1+h)^k,
\]

(23)

and \( \alpha_{2n+1} \) are the coefficients of the Taylor expansion of \( \tanh(t) \) at the point \( t = 0 \),

\[
\tanh(t) = \sum_{n=0}^{+\infty} \alpha_{2n+1}t^{2n+1},
\]

(24)

whose convergence radius is \( \rho_0 \approx \frac{3}{2} \) (indeed, we have \( \alpha_{2n+1} = \frac{2^{2k+2}(2^{2k+2}-1)}{(2k+2)!}B_{2k+2} \) where \( B_{2k+2} \) are Bernoulli numbers). Liao gives the convergence region of the generalized Taylor series (22)

\[
0 \leq t < \rho_0\sqrt{\frac{2}{|h|} - 1},
\]

(25)

with \(-2 < h < 0\). In the paper [13], we have studied this case in detail and showed that the generalized Taylor series was just the usual Taylor series at another point. In the present paper, I study another set of basis functions. As the same as Liao [11], take the set of basis functions

\[
e_n(t) = e^{-nt}, (n = 0, 1, 2, \cdots).
\]

(26)

Liao gives the \( m \)-th order approximate solution

\[
V(t) \approx 1 + 2 \sum_{n=1}^{m} \{( -1)^n \exp(-2nt)\mu_0^{m,n}(h)\} - \exp(-t)\left\{1 + \frac{h}{2} + \frac{h}{2}\exp(-2t)\right\}^m.
\]

(27)

When \( h = -2 \), the solution becomes

\[
V(t) \approx 1 + 2 \sum_{n=1}^{m} \{( -1)^n \exp(-2nt)\} + (-1)^{m+1}\exp(-(2m+1)),
\]

(28)

which converges to the exact solution in the region \( 0 \leq t < +\infty \). However, if we expand directly the exact solution using the basis functions \( \{e_n(t)\}_{0}^{+\infty} \), we have

\[
V(t) = 1 + 2 \sum_{n=1}^{+\infty} \{( -1)^n \exp(-2nt)\},
\]

(29)
which converges to the exact solution in the region $0 < t < +\infty$, and does not converge at the point $t = 0$. This is a serious shortcoming for solution (29). Although the Liao’s solution (27) can avoid this weakness, its accurate convergence region can’t be given. Our method can overcome these two difficulties.

We next solve the equation (18) using our general series expansion method at the point $t_0$ to give the same results with Liao’s. We take the basis functions $e_n(t, t_0) = \{\exp(-2t) - t_0\}^n$ and expand the solution as

$$V(t) = \sum_{n=1}^{+\infty} a_n \{\exp(-2t) - t_0\}^n,$$

and substitute it into Eq.(18) to give

$$a_0 = \frac{1 - t_0}{1 + t_0}, a_n = 2(-1)^n \frac{1}{(1 + t_0)^{n+1}}, n = 1, 2, \cdots.$$

Therefore the series solution is given by

$$V(t) = \sum_{n=1}^{+\infty} a_n \{\exp(-2t) - t_0\}^n,$$

whose convergence region is

$$\frac{-1}{2} \ln(2t_0 + 1) < t < +\infty,$$

which can be obtained by computing $|a_{n+1}/a_n (\exp(-2t) - t_0)| < 1$ from (30). When $t_0 > 0$ such as $t_0 = 1$, the convergence region consists of the region $0 < t < +\infty$. Thus $t_0$ can control and adjust the convergence region. This shows that our method is more simple and efficient in some degrees than the homotopy analysis method.

Take $t_0 = -1 - \frac{1}{h}$, and denote

$$\mu^{m,n}(h) = \sum_{k=n}^{m} \frac{k}{n} \frac{t_0^{k-n}}{(1 + t_0)^{k+1}} = (-h)^{n+1} \sum_{k=n}^{m} \frac{k}{n} (h + 1)^{k-n}.$$

Therefore the solution can be written as

$$V(t) = 1 + \lim_{m \to \infty} 2 \sum_{n=1}^{m} \mu^{m,n}(h)(-1)^n \exp(-2nt),$$

whose convergence region is

$$\frac{-1}{2} \ln(-1 - \frac{2}{h}) < t < +\infty.$$

This result once again shows that the role of our parameter $t_0$ is the same as the Liao’s parameter $h$. 

7
5 Conclusion

We propose a general series expansion method to find the solutions to nonlinear differential equations. Because the expanding point of the corresponding series solution is a motive point, the method has a freedom to control the convergence region. Through detailed analysis of a typical example, we show that we can use the general series expansion to give the same result obtained by the homotopy analysis method. As a result, we reveal the secret of the auxiliary parameter $h$ which is the key of the homotopy analysis method. In some degrees our method is more simple and efficient than the homotopy analysis method.

Acknowledgements. Thanks to referees for their valuable suggestions and calling our attention to existing papers [14] and [16]. The first author (CS Liu) would like to thank Prof. Wen-Xiu Ma for his helpful discussion.

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