COMPLETE INTERPOLATING SEQUENCES FOR SMALL FOCK SPACES.

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ABSTRACT. We give a characterization of complete interpolating sequences for the Fock spaces $F_p^\phi$, $1 \leq p < \infty$, where $\phi(z) = \alpha (\log^+ |z|)^2$, $\alpha > 0$. Our results are analogous to the classical Kadets-Ingham’s $1/4$–Theorem on perturbation of Riesz bases of complex exponentials, and they answer a question asked by A. Baranov, A. Dumont, A. Hartmann and K. Kellay in [4, page 31].

1. Introduction and main results

Let $\alpha > 0$ and $\phi(z) = \alpha (\log^+ |z|)^2$, $z \in \mathbb{C}$. The associated Fock spaces $F_p^\phi$, $1 \leq p < \infty$, are the following

$$F_p^\phi := \left\{ f \in \text{Hol}(\mathbb{C}) : \|f\|_{p,\phi}^p := \int_{\mathbb{C}} |f(z)e^{-\phi(z)}|^p \ dm(z) < \infty \right\}$$

and

$$F_\infty^\phi := \left\{ f \in \text{Hol}(\mathbb{C}) : \|f\|_{\infty,\phi} := \sup_{z \in \mathbb{C}} |f(z)e^{-\phi(z)}| < \infty \right\},$$

where $dm$ stands for the area Lebesgue measure in the complex plane $\mathbb{C}$. The Fock space $F_p^\phi$ endowed with the above norm is a Banach space for every $1 \leq p \leq \infty$.

In order to define interpolating and sampling sets for $F_p^\phi$, $1 \leq p < \infty$, we recall that the point evaluation functional $L_z : F_p^\phi \rightarrow f(z) \in \mathbb{C}$, for fixed $z \in \mathbb{C}$, is a bounded linear map. Its norm $\|L_z\|_{F_p^\phi \rightarrow \mathbb{C}}$ satisfies

$$\frac{1}{C}(1 + |z|)^{-2/p} e^{\phi(z)} \leq \|L_z\|_{F_p^\phi \rightarrow \mathbb{C}} \leq C(1 + |z|)^{-2/p} e^{\phi(z)},$$

for some constant $C \geq 1$, (see Lemma 2.1).

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Let $\Lambda \subset \mathbb{C}$ be a countable set. The sequence $\Lambda$ is called *sampling* for $\mathcal{F}_p^\varphi$, $1 \leq p < \infty$, if there exists a constant $C \geq 1$ such that

$$\frac{1}{C} \|f\|_{p,\varphi} \leq \|f\|_{p,\varphi,\Lambda} \leq C \|f\|_{p,\varphi},$$

for all $f \in \mathcal{F}_p^\varphi$, where

$$\|f\|_{p,\varphi,\Lambda} := \sum_{\lambda \in \Lambda} (1 + |\lambda|)^2 |f(\lambda)|^p e^{-p\varphi(\lambda)}.$$

Similarly, we say that a sequence $\Lambda$ is *sampling* for $\mathcal{F}_\infty^\varphi$ whenever there exists a constant $C > 0$ such that

$$\|f\|_{\infty,\varphi} \leq C \|f\|_{\infty,\varphi,\Lambda},$$

for every $f \in \mathcal{F}_\infty^\varphi$, where

$$\|f\|_{\infty,\varphi,\Lambda} := \sup_{\lambda \in \Lambda} |f(\lambda)| e^{-\varphi(\lambda)}.$$

The set $\Lambda \subset \mathbb{C}$ is said to constitute an *interpolating sequence* for $\mathcal{F}_p^\varphi$, $1 \leq p \leq \infty$, if for every sequence $v = (v_\lambda)_{\lambda \in \Lambda}$ that satisfies $\|v\|_{p,\varphi,\Lambda} < \infty$, there exists a function $f \in \mathcal{F}_p^\varphi$ such that $f(\lambda) = v_\lambda$, for every $\lambda \in \Lambda$. It is called a *complete interpolating sequence* for $\mathcal{F}_p^\varphi$, $1 \leq p \leq \infty$, whenever it is simultaneously sampling and interpolating for $\mathcal{F}_p^\varphi$. Finally, for the Hilbert case $p = 2$, standard arguments ensure that a sequence $\Lambda$ is a complete interpolating set for $\mathcal{F}_2^\varphi$ if and only if the system of normalized reproducing kernels $\{k_\lambda\}_{\lambda \in \Lambda}$ is a Riesz basis for $\mathcal{F}_2^\varphi$, that means $\{k_\lambda\}_{\lambda \in \Lambda}$ is a linear isomorphic image of an orthonormal basis for $\mathcal{F}_2^\varphi$.

The problem of existence of complete interpolating sequences for the Fock spaces has occupied many authors. Recall that K. Seip in [19] and K. Seip and R. Wallstén in [21], proved that there exists no complete interpolating set for the classical Fock spaces $\mathcal{F}_p^\varphi$ ($\varphi(z) = \alpha |z|^2$, $\alpha > 0$), see also [6]. The absence of such sequences was obtained by J. Ortega-Cerdà and K. Seip in [18] for Fock spaces with weights $\varphi$ satisfying $\frac{1}{C} \leq \Delta \varphi(z) \leq C$, for some constant $C \geq 1$. Later on, for the subharmonic weights $\varphi(z)$ whose associated Riesz measures are doubling, the nonexistence of complete interpolating sets was established by N. Marco, X. Massaneda and J. Ortega-Cerdà in [16]. A. Borichev, R. Dhuez and K. Kellay in [7] checked that large Fock spaces (those associated with weighted $\varphi(z)$ growing more rapidly than $|z|^2$ and satisfying some natural regularity conditions) have no complete interpolating set. Furthermore, S. Brekke and K. Seip in [10] and A. Borichev, A. Hartmann, K. Kellay, and X. Massaneda in [8] showed that the spaces $\mathcal{F}_p^\varphi$ possess no multiple complete interpolating sequences.

On the other hand, for $\varphi(z) = \text{Const.} \log |z|$, the associated Fock spaces are of finite dimension and obviously every sequence $\Lambda \subset \mathbb{C}$ of distinct points that satisfies $\text{Cardinal}(\Lambda) = \dim(\mathcal{F}_p^\varphi)$ is a complete interpolating set for $\mathcal{F}_p^\varphi$. In [9], A. Borichev and Yu. Lyubarskii provided Riesz bases of normalized reproducing kernels for $\mathcal{F}_2^\varphi$, where
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\( \varphi(z) = (\log^+ |z|)^{\beta}; 1 < \beta \leq 2 \). They proved also that \( \mathcal{F}_\varphi^2 \) possesses such bases if and only if \( \varphi(x) \) grows at most as \( (\log(x))^2 \). More recently, A. Baranov, Yu. Belov and A. Borichev in [2] described the radial Hilbert Fock spaces which have Riesz bases of normalized reproducing kernels and are (or are not) isomorphic to de Branges spaces, see also [3]. In [4], A. Baranov, A. Dumont, A. Hartmann and K. Kellay gave a complete description of complete interpolating sequences for \( \mathcal{F}_\varphi^p \), when \( p \in \{2, \infty\} \) and \( \varphi(z) = \alpha (\log^+ |z|)^2 \). Riesz bases of normalized reproducing kernels for \( \mathcal{F}_\varphi^2 \), where \( \varphi = (\log^+ |z|)^{\beta}; 1 < \beta < 2 \), are characterized recently by K. Kellay and Y. Omari in [13].

The central result of this paper is a characterization of complete interpolating sequences for \( \mathcal{F}_\varphi^p \), \( 1 \leq p < \infty \) and \( \varphi(z) = \alpha (\log^+ |z|)^2 \), where \( \alpha > 0 \). An interesting feature is that for our case the endpoint of the scale \( p = 1 \) also admits complete interpolating sequences. This happens because the discrete Hilbert transform in sparse sequences is bounded even for \( p = 1 \), see [5] for more details. Our results answer a question asked by the authors in [4]. In fact, we employ the ideas and methods suggested in [4] and we develop additional techniques in order to cover the remaining values \( 1 \leq p < \infty \). Finally, we directly treat the problem in the space \( \mathcal{F}_\varphi^p \) without using a complex interpolating theorem.

Before stating our main results, we give some necessary definitions and notations. In order to be more precise, let \( \varphi(z) = \alpha (\log^+ |z|)^2 \), where \( \alpha > 0 \), we set
\[
\rho(z) := (\Delta \varphi(z))^{-1/2} = |z|/\sqrt{2\alpha}.
\]
We associate with the function \( \rho \) the "distance" as follows:
\[
d_\rho(z, w) := \frac{|z - w|}{1 + \min \{\rho(z), \rho(w)\}}, \quad z, w \in \mathbb{C}.
\]
(1.2)

Now, let \( \Lambda \) be a sequence of complex numbers. We say that \( \Lambda \) is \( d_\rho - \text{separated} \), if there exists \( \delta > 0 \), such that
\[
\inf \{d_\rho(\lambda, \lambda') : \lambda \neq \lambda', \lambda, \lambda' \in \Lambda\} \geq \delta.
\]
It is not difficult to see that a sequence \( \Lambda \) is \( d_\rho - \text{separated} \) if and only if there exists a constant \( c \in (0, 1) \) such that the discs \( \{D(\lambda, cp(\lambda))\} \) are pairwise disjoint. In what follows, for \( \Gamma = \{\gamma_n\} \) a sequence of complex numbers ordered in such a way that \( \{|\gamma_n|\} \) is nondecreasing, we denote by \( \Gamma \cup \{*\} \) the sequence consisting of \( \Gamma \) union any point from \( \mathbb{C} \setminus \Gamma \), and \( \Gamma \setminus \{*\} \) the resulting sequence by removing any one point from \( \Gamma \). Also, for a given real sequence \( (\delta_n) \) we use the following notations
\[
\nabla_N := \inf_n \frac{1}{N} \sum_{k=n+1}^{n+N} \delta_k, \quad \Delta_N := \sup_n \frac{1}{N} \sum_{k=n+1}^{n+N} \delta_k,
\]
where \( N \) is a positive integer. For \( 1 \leq p < \infty \), we denote by \( q \) its Hölder conjugate exponent \( (1/p + 1/q = 1) \). The main results of this work are the following theorems.
Theorem 1. Let $\alpha > 0$, $\varphi(r) = \alpha (\log^+ r)^2$, $\Lambda = \{\lambda_n = e^{\frac{\lambda_n}{\alpha}} : n \geq 0\}$ and $\Gamma = \{\gamma_n : n \geq 0\} \subset \mathbb{C}$ such that $|\gamma_n| \leq |\gamma_{n+1}|$, we write $\gamma_n = \lambda_n e^{i\delta_n} e^{i\theta_n}$, where $\delta_n, \theta_n \in \mathbb{R}$. Then

(A) $\Gamma \setminus \{\ast\}$ is a complete interpolating sequence for $\mathcal{F}_\varphi^p$, $1 \leq p < 4/3$, if and only if the following three assertions hold:

1. $\Gamma$ is $d_p$-separated,
2. $(\delta_n)$ is a bounded real sequence,
3. there exists an integer $N \geq 1$ such that

\[
-\left(\frac{1}{4\alpha} + \frac{1}{q\alpha}\right) < \nabla_N \leq \Delta_N < \frac{1}{4\alpha} - \frac{1}{q\alpha}. \tag{1.3}
\]

(B) $\Gamma$ is a complete interpolating set for $\mathcal{F}_\varphi^p$, $4/3 < p < 4$, if and only if (a) and (b) are verified and

\[
\frac{1}{4\alpha} - \frac{1}{q\alpha} < \nabla_N \leq \Delta_N < \frac{1}{p\alpha} - \frac{1}{4\alpha}, \tag{1.4}
\]

for some integer $N \geq 1$.

(C) $\Gamma \cup \{\ast\}$ is a complete interpolating sequence for $\mathcal{F}_\varphi^p$, $p > 4$, if and only if (a) and (b) are satisfied and for some positive integer $N$ we have

\[
\frac{1}{p\alpha} - \frac{1}{4\alpha} < \nabla_N \leq \Delta_N < \frac{1}{p\alpha} + \frac{1}{4\alpha}. \tag{1.5}
\]

For $p \in \{4/3, 4\}$, we have the following

Theorem 2. Let $\alpha > 0$, $\varphi(r) = \alpha (\log^+ r)^2$ and $\Lambda = \{\lambda_n = e^{\frac{\lambda_n}{\alpha}} : n \geq 0\}$. Let $\Gamma = \{\gamma_n : n \geq 0\} \subset \mathbb{C}$ such that $|\gamma_n| \leq |\gamma_{n+1}|$, we write $\gamma_n = \lambda_n e^{i\delta_n} e^{i\theta_n}$, where $\delta_n, \theta_n \in \mathbb{R}$. The sequences $\{e^{-\frac{1}{4\alpha}} \Gamma \} \setminus \{\ast\}$ and $e^{-\frac{1}{\pi}} \Gamma$ are complete interpolating sets for $\mathcal{F}_\varphi^{4/3}$ and $\mathcal{F}_\varphi^4$, respectively, if and only if the conditions (a) and (b) hold and

\[
-\frac{1}{4\alpha} < \nabla_N \leq \Delta_N < \frac{1}{4\alpha}. \tag{1.6}
\]

for some integer $N \geq 1$.

Note that the cases $p = 2, \infty$ were treated in [4]. If $N = 1$, $p = 2$, the condition (1.4) in Theorem 1 becomes $\sup_{n \geq 0} |\delta_n| < 1/(4\alpha)$, which is analogue to the well known Kadets-Ingham’s $1/4$—Theorem, see [12, 20]. This latter is related to a stability problem of Riesz bases of normalized reproducing kernels for the Paley-Weiner spaces $PW_\alpha^2$, for more details on this problem we refer to [11, 14] and references therein. Also, for an arbitrary integer $N \geq 1$, Theorem 1 appears like Avdonin’s Theorem (see, e.g. [1, 4]). For $1 < p < \infty$, Theorems 1 and 2 give similar results to those proved by K. Seip and Yu. Lyubarski in [15] about complete interpolating results for $PW_\alpha^p$, $1 < p < \infty$, and those proved by J. Marzo and K. Seip in [17] for the space of polynomials equipped with the $L^p$—norm.
We can summarize Theorem 1 and Theorem 2 by the following table, let $\Gamma = \{\gamma_n\}_{n \geq 0}$ be a sequence of $\mathbb{C}$ and write $\gamma_n = \lambda_n e^{\delta_n} e^{i\theta_n}$, we have

| $p$ | Complete interpolating sequence | $\nabla_N \leq \Delta_N$ |
|-----|-------------------------------|--------------------------|
| $1 \leq p < \frac{4}{3}$ | $\Gamma \setminus \{\ast\}$ | $-\left(\frac{1}{4\alpha} + \frac{1}{q\alpha}\right) < \nabla_N \leq \Delta_N < \frac{1}{4\alpha} - \frac{1}{q\alpha}$ |
| $p = \frac{4}{3}$ | $\left(e^{-\frac{1}{4\alpha}}\Gamma\right) \setminus \{\ast\}$ | $-\frac{1}{4\alpha} < \nabla_N \leq \Delta_N < \frac{1}{4\alpha}$ |
| $\frac{4}{3} < p < 4$ | $\Gamma$ | $\frac{1}{4\alpha} - \frac{1}{q\alpha} < \nabla_N \leq \Delta_N < \frac{1}{p\alpha} - \frac{1}{4\alpha}$ |
| $p = 4$ | $e^{-\frac{1}{4\alpha}}\Gamma$ | $-\frac{1}{4\alpha} < \nabla_N \leq \Delta_N < \frac{1}{4\alpha}$ |
| $p > 4$ | $\Gamma \cup \{\ast\}$ | $\frac{1}{p\alpha} - \frac{1}{4\alpha} < \nabla_N \leq \Delta_N < \frac{1}{4\alpha} + \frac{1}{p\alpha}$ |

The paper is organized as follows: In the next section, we prove some technical lemmas and preliminary results needed in the proofs of the main theorems. Section 3 is devoted to prove Theorem 2.6 (Section 2). Theorems 1 and 2 will be proved in the last section.

We end this introduction with some words on notation. Throughout this paper, the notation $U(z) \lesssim V(z)$ for $z$ in some set $\Omega$ means that the ratio $U(z)/V(z)$ of the two positive functions $U(z)$ and $V(z)$ is bounded from above by a positive constant independent of $z$ in $\Omega$. We write $U(z) \asymp V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$ hold simultaneously.

### 2. Key Lemmas and Preliminary Results

In this section, we will prove some key lemmas and secondary results. In fact, we give the necessary estimates of some infinite products and we also prove some preliminary results about $d_\rho$—separated sequences. Throughout the rest of this paper, we assume that $\alpha = 1$ and consequently $\varphi(r) = (\log^+ r)^2$. We begin by estimating the norm of the point evaluation functional.

**Lemma 2.1.** Let $z \in \mathbb{C}$ be fixed. The point evaluation functional $L_z : f \in \mathcal{F}_p^\varphi \mapsto f(z) \in \mathbb{C}$ is a bounded linear map and its norm satisfies the following estimate

$$\|L_z\|_{\mathcal{F}_p^\varphi \to \mathbb{C}} \asymp (1 + |z|)^{-2/p} e^{\varphi(z)}.$$  

**Proof.** Let $f \in \mathcal{F}_p^\varphi$ and set $F(r) = \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi}$. First we write

$$\|f\|_{\mathcal{F}_p^\varphi} \asymp \int_1^\infty F(r) e^{-p\varphi(r)} r dr = \int_0^\infty F(e^t) e^{-pt^2 + 2t} dt.$$
Let $z \in \mathbb{C}$ and write $u := \log |z|$. Let $m \in \mathbb{N}$ be the integer that satisfies $m \leq u < m + 1$ and $0 < \varepsilon < 1$ sufficiently small, we have

\[
\|f\|_{p,\varphi}^p \gtrsim \int_{u-\varepsilon}^{u+\varepsilon} F(e^t)e^{-pt^2+2t}dt \\
\times e^{-pu^2+2u+2pmu} \int_{u-\varepsilon}^{u+\varepsilon} F(e^t)e^{-2pmt}dt \\
= e^{-pu^2+2u+2pmu} \int_{e^{u-\varepsilon}}^{e^{u+\varepsilon}} F(t)e^{-2pmt} \frac{dt}{t} \\
= e^{-pu^2+2u+2pmu} \int_{A(z,e^{-\varepsilon},e^{\varepsilon})} |f(\xi)|^{p}/|\xi|^{2pm+2} dm(\xi),
\]

where $A(z,e^{-\varepsilon},e^{\varepsilon}) := \{ \xi \in \mathbb{C} : |\xi|e^{-\varepsilon} \leq |\xi| \leq |\xi|e^{\varepsilon} \}$. Applying now mean’s theorem to the function $|f(\xi)|^{p}/|\xi|^{2pm+2}$ in a disk $D(z,\delta|z|)$, for some $\delta > 0$ small enough so that $D(z,\delta|z|) \subset A(z,e^{-\varepsilon},e^{\varepsilon})$. We obtain

\[
|f(z)|^p/|z|^{2pm+2} \leq \frac{C}{|z|^2} \int_{D(z,\delta|z|)} |f(\xi)|^{p}/|\xi|^{2pm+2} dm(\xi) \\
\leq \frac{C}{|z|^2} \int_{A(z,e^{-\varepsilon},e^{\varepsilon})} |f(\xi)|^{p}/|\xi|^{2pm+2} dm(\xi).
\]

Thus,

\[
|f(z)|^pe^{-pu^2+2u} \lesssim e^{-pu^2+2u+2mpu} \int_{A(z,e^{-\varepsilon},e^{\varepsilon})} |f(\xi)|^{p}/|\xi|^{2pm+2} dm(\xi) \lesssim \|f\|_{p,\varphi}^p.
\]

For the reverse estimate, let $u = \log |z|$ and let $n$ be the integer that satisfies $n \leq 2u < n + 1$. Consider next the function $f(\xi) = \xi^n$. We have

\[
\|f\|_{p,\alpha}^p = \int_{\mathbb{C}} |\xi|^n e^{-p\alpha(\xi)} dm(\xi) \asymp \int_1^\infty t^n e^{-p(\log t)^2}dt \\
= \int_0^\infty e^{(pn+2)t-pt^2}dt = e^{\frac{n}{2}(n+2/p)^2} \int_{-(n+2/p)}^\infty e^{-pt^2}dt \\
\lesssim e^{(pn+2)u-pu^2}. \tag{2.1}
\]

Since $|f(z)| = e^{nu}$, we obtain $\|L_z\|_{p,\varphi} \geq e^{u^2-\frac{2}{p}u}$ and this completes the proof. \hfill $\square$

Now, for a given sequence $\Lambda$ of $\mathbb{C}$, $G_\Lambda$ is the following infinite product, whenever it converges, associated with $\Lambda$

\[
G_\Lambda(z) := \prod_{\lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right), \quad z \in \mathbb{C}. \tag{2.2}
\]

In what follows, we consider the sequences

\[
\Lambda_l := \left\{\lambda_n := e^{\frac{1+n}{2}}e^{i\theta_n} : n \geq l\right\}, \tag{2.3}
\]
where \( l \in \{-1, 0, 1\} \) and \( \theta_n \) are arbitrary real numbers. The notation \( \text{dist}(z, \Lambda) \) stands for the Euclidean distance between \( z \) and \( \Lambda \). The following lemma describes some basic properties of the infinite product associated with these sequences.

**Lemma 2.2.** Let \( l \in \{-1, 0, 1\} \), the infinite product \( G_{\Lambda_l} \) converges uniformly on compact sets of \( \mathbb{C} \) and hence it defines an entire function. We have the following estimates

\[
|G_{\Lambda_l}(z)| e^{-\varphi(z)} \leq \frac{\text{dist}(z, \Lambda_l)}{1 + |z|^{3/2+l}}, \quad z \in \mathbb{C},
\]

and

\[
|G'_{\Lambda_l}(\lambda)| e^{-\varphi(\lambda)} \leq \frac{1}{|\lambda|^{3/2+l}}, \quad \lambda \in \Lambda_l.
\]

**Proof.** The estimate (2.5) can be obtained from (2.4) by letting tend \( z \) to \( \lambda \). For the proof of (2.4) we refer to [9, Lemma 2.6]. \( \square \)

The following lemma can be obtained easily from the proof of [4, Theorem 1.1]. We include the proof for completeness.

**Lemma 2.3.** Let \( l \in \{-1, 0, 1\} \), \( N \geq 1 \), and \( \Lambda_l = \{\lambda_n : \ n \geq l\} \). Let \( \Gamma_l = \{\gamma_n : \ n \geq l\} \) and write \( \gamma_n = |\lambda_n|e^{\delta_n}e^{i\theta_n} \). Assume that the conditions (a) and (b) are satisfied, then the infinite product \( G_{\Gamma_l} \) satisfies the estimate

\[
\frac{\text{dist}(z, \Gamma_l)}{1 + |z|^{3/2+l+2\delta}} \lesssim |G_{\Gamma_l}(z)| e^{-\varphi(z)} \lesssim \frac{\text{dist}(z, \Gamma_l)}{1 + |z|^{3/2+l+2\eta}}, \quad z \in \mathbb{C},
\]

where \( \delta = \Delta_N \) and \( \eta = \nabla_N \). Also, for every \( \gamma_n \in \Gamma \) we have

\[
\frac{e^{\varphi(\gamma_n)}}{1 + |\gamma_n|^{3/2+l+2\delta}} \lesssim |G'_{\Gamma_l}(\gamma_n)| \lesssim \frac{e^{\varphi(\gamma_n)}}{1 + |\gamma_n|^{3/2+l+2\eta}}.
\]

**Proof.** Let \( |z| = e^t \), such that \( |\gamma_{n-1}| \leq |z| \leq |\gamma_n| \), and assume that \( \text{dist}(z, \Gamma) = |z - \gamma_{n-1}| \). Take \( m \) the integer number that satisfies

\[
\frac{m}{2} - \frac{1}{4} \leq t < \frac{m}{2} + \frac{1}{4}.
\]

First, remark that \( |m - n| \) is uniformly bounded in \( |z| \). Now, we have

\[
\log |G_{\Gamma_l}(z)| = \sum_{t \leq k < n-1} \log \left| \frac{z}{\gamma_k} \right| + \log \left| 1 - \frac{z}{\gamma_{n-1}} \right| + O(1)
\]

\[
= \sum_{k=l}^{n-1} \log \left| \frac{z}{\gamma_k} \right| + \log \frac{\text{dist}(z, \Gamma_k)}{|z|} + O(1)
\]

\[
= \sum_{k=l}^{m-1} \left( \log |z| - \frac{k + 1}{2} - \delta_k \right) + \log \frac{\text{dist}(z, \Gamma_k)}{|z|} + O(1)
\]

\[
= (m - l) \log |z| - \frac{m(m + 1)}{4} - \sum_{k=0}^{m-1} \delta_k + \log \frac{\text{dist}(z, \Gamma_l)}{|z|} + O(1). \quad (2.6)
\]
For a fixed $N \geq 1$, we write $m = sN + r$, for some $0 \leq r < N$. On the one hand, we get
\[ \sum_{k=0}^{m-1} \delta_k = \sum_{k=0}^{s-1} \sum_{j=kN}^{kN+N-1} \delta_j + O(1) \leq \Delta_N sN + O(1) = 2\Delta_N \log |z| + O(1). \quad (2.7) \]
On the other hand we have
\[ \sum_{k=0}^{m-1} \delta_k = \sum_{k=0}^{s-1} \sum_{j=kN}^{kN+N-1} \delta_j + O(1) \geq \nabla_N sN + O(1) = 2\nabla_N \log |z| + O(1). \quad (2.8) \]
The result follows immediately by combining (2.6), (2.7) and (2.8).

We will need the following classical lemma for which we include the proof for completeness.

**Lemma 2.4.** Let $\nu$ be a real parameter and $0 < p < \infty$. Let $\Lambda \subset \mathbb{C}$ be a finite union of $d_\rho$-separated sequences. The integral
\[ \int_{\mathbb{C}} \left( \frac{\text{dist}(z, \Lambda)}{1 + |z|^{3/2+\nu}} \right)^p dm(z) \quad (2.9) \]
converges if and only if $\nu > 2/p - 1/2$.

**Proof.** Firstly, for every $z \in \mathbb{C}$ we have $\text{dist}(z, \Lambda) \lesssim 1 + |z|$. On the other hand, since the sequence $\Lambda$ is a finite union of $d_\rho$-separated sequences then the estimate $\text{dist}(z, \Lambda) \gtrsim 1 + |z|$ holds for every $z \in \{ w \in \mathbb{C} : d_\rho(w, \Lambda) \geq \delta \}$, for some small positive real number $\delta$ and where
\[ d_\rho(w, \Lambda) = \inf \{ d_\rho(w, \lambda) : \lambda \in \Lambda \}. \]
The remaining part of the integral can be obtained by a simple argument of subharmonicity of the function $z \mapsto \frac{1}{(1 + |z|)^{1/2+\nu}}$. Therefore, the convergence of the integral in (2.9) is equivalent to the convergence of the integral
\[ \int_{\mathbb{C}} \left( \frac{1}{(1 + |z|)^{1/2+\nu}} \right)^p dm(z). \]
This ends the proof.

A countable set $\Lambda$ of the complex plane is called a **uniqueness set** for $\mathcal{F}_p^\varphi$ whenever every function in $\mathcal{F}_p^\varphi$ vanishing on $\Lambda$ is identically zero. $\Lambda$ is said to be a **zero set** for $\mathcal{F}_p^\varphi$ if there exists $f \in \mathcal{F}_p^\varphi \setminus \{0\}$ such that $\Lambda$ is exactly the zero set of $f$, counting multiplicities. We say that $\Lambda$ is a set of **uniqueness of zero excess** or **exact uniqueness set** for $\mathcal{F}_p^\varphi$ whenever it is a set of uniqueness for $\mathcal{F}_p^\varphi$ and when we remove any point of $\Lambda$ we obtain a zero set for $\mathcal{F}_p^\varphi$. A direct consequence of the above lemma is the following corollary

**Corollary 1.** Let $\Lambda_l$ be the sequence defined as above. We have
1. The sequence $\Lambda_{-1}$ is a uniqueness set of zero excess for $\mathcal{F}_p^\varphi$ if and only if $p \in (4, \infty]$.
2. The set $\Lambda_0$ is a uniqueness sequence of zero excess for $\mathcal{F}_p^\varphi$ if and only if $4/3 < p < 4$.
3. Let $1 \leq p < 4/3$. The sequence $\Lambda_1$ is a uniqueness set of zero excess for $\mathcal{F}_p^\varphi$. 
**Proof.** (1) Assume that there exists \( f \in \mathcal{F}_\varphi^p \) vanishing on \( \Lambda_{-1} \), taking into account Hadamard’s factorization theorem, we can write \( f = hG_{\Lambda_{-1}} \), for an entire function \( h \). By Lemma 2.2 we have

\[
|f(z)| = |h(z)G_{\Lambda_{-1}}(z)| \asymp \frac{\text{dist}(z, \Lambda_{-1})}{(1 + |z|)^{1/2}} |h(z)|e^{\varphi(z)}, \quad z \in \mathbb{C}. \tag{2.10}
\]

The last estimate and the fact that \( f \in \mathcal{F}_\varphi^p \) imply with Lemma 2.1

\[
|h(z)| \text{dist}(z, \Lambda_{-1}) \lesssim (1 + |z|)^{1/2 - 2/p}, \quad z \in \mathbb{C}. \tag{2.11}
\]

Using the fact that \( \text{dist}(z, \Lambda_{-1}) \asymp 1 + |z| \) for every \( z \in \mathcal{A} := \{ w \in \mathbb{C} : \text{dist}(w, \Lambda_{-1}) \geq \delta|w| \} \), for some \( \delta \in (0, 1) \), this implies that \( |h(z)| \lesssim (1 + |z|)^{-1/2 - 2/p} \), for every \( z \in \mathcal{A} \) and therefore \( h = 0 \). Hence \( \Lambda_{-1} \) is a uniqueness set for \( \mathcal{F}_\varphi^p \), for every \( p \geq 1 \). Now, fix \( \lambda \in \Lambda_{-1} \), the sequence \( \Lambda' = \Lambda_{-1} \setminus \{ \lambda \} \) is a zero set for \( \mathcal{F}_\varphi^p \), whenever \( p > 4 \). Indeed, by Lemmas 2.2 and 2.4 we have

\[
\int_{\mathbb{C}} \left| \frac{G_{\Lambda_{-1}}(z)}{z - \lambda} e^{-\varphi(z)} \right|^p \, dm(z) \asymp \int_{\mathbb{C}} \left( \frac{\text{dist}(z, \Lambda_{-1})}{(1 + |z|)^{3/2}} \right)^p \, dm(z) < \infty.
\]

Thus, the function \( \frac{G_{\Lambda_{-1}}(z)}{z - \lambda} \) belongs to \( \mathcal{F}_\varphi^p \) and vanishes exactly on \( \Lambda \setminus \{ \lambda \} \).

To prove the converse, it is sufficient to show that \( \Lambda' = \Lambda_{-1} \setminus \{ \lambda \} \) is a uniqueness set for \( \mathcal{F}_\varphi^p \), for every \( p \leq 4 \). Let \( f \in \mathcal{F}_\varphi^p \) be a given function that vanishes in \( \Lambda' \). Hadamard’s factorization theorem ensures the existence of an entire function \( h \) such that \( f = h \frac{G_{\Lambda_{-1}}}{z - \lambda} \). Lemma 2.2 implies that

\[
|f(z)| = \left| h(z) \frac{G_{\Lambda_{-1}}(z)}{z - \lambda} \right| \asymp \frac{\text{dist}(z, \Lambda')}{(1 + |z|)^{3/2}} |h(z)|e^{\varphi(z)}, \quad z \in \mathbb{C}. \tag{2.12}
\]

Since \( f \in \mathcal{F}_\varphi^p \), we obtain

\[
|h(z)| \text{dist}(z, \Lambda') \lesssim (1 + |z|)^{3/2 - 2/p}, \quad z \in \mathbb{C}.
\]

The same arguments as above imply that \( h \) is a polynomial of degree at most \( 3/2 - 2/p - 1 = 1/2 - 2/p \leq 0 \), i.e. \( h \) is a constant if \( p = 4 \) and the zero function if \( 1 \leq p < 4 \). Thus \( \Lambda' \) is a uniqueness set for \( \mathcal{F}_\varphi^p \), for every \( 1 \leq p < 4 \). For \( p = 4 \), since \( f = c \frac{G_{\Lambda_{-1}}}{z - \lambda} \in \mathcal{F}_\varphi^p \), the identity (2.12) implies that

\[
\int_{\mathbb{C}} \left( \frac{\text{dist}(z, \Lambda')}{(1 + |z|)^{3/2}} \right)^p \, dm(z) \asymp \int_{\mathbb{C}} |f(z)|^p e^{-p\varphi(z)} \, dm(z) < \infty. \tag{2.13}
\]

This contradicts Lemma 2.4 and hence \( \Lambda' \) is a uniqueness set for \( \mathcal{F}_\varphi^p \), for every \( p \leq 4 \). This completes the proof.

The proofs of (2) and (3) in the corollary are completely the same as that of (1), we use the estimates of the functions \( G_{\Lambda_i} \) in Lemma 2.2. The proof is complete. \( \square \)

The following lemma is a crucial tool in the proof of our results
Lemma 2.5. Let \( \Lambda \) be a sequence of complex numbers and let \( 1 \leq p < \infty \). The following are equivalent:

1. There exists a constant \( C > 0 \) such that
   \[
   \sum_{\lambda \in \Lambda} (1 + |\lambda|)^2 |f(\lambda)e^{-\varphi(\lambda)}|^p \leq C\|f\|_{p,\varphi}^p,
   \]
   for every \( f \in \mathcal{F}_\varphi^p \).

2. \( \Lambda \) is a finite union of \( d_\rho \)-separated sequences.

Proof. The proof is similar to the classical one given for [1, Lemma 2.6]. \( \square \)

The description of complete interpolating sequences for \( \mathcal{F}_\varphi^p \) is obtained by comparing them to the sequence \( \Lambda_l = \{ e^{\frac{n+1}{2\beta}} e^{i\theta_n} \}_{n \geq l} \), this leads us to the following result which will play an important role in the proof of the main theorems. The proof will be given in the next section.

Theorem 2.6. We have the following

1. The sequences \( \Lambda_{-1}, \Lambda_0 \) and \( \Lambda_1 \) are complete interpolating sets for \( \mathcal{F}_\varphi^p \), for every \( 4 < p < \infty \), \( \frac{4}{3} < p < 4 \) and \( 1 \leq p < \frac{4}{3} \) respectively.

2. The sets \( e^{-\frac{1}{4}} \Lambda_1 \) and \( e^{-\frac{1}{4}} \Lambda_0 \) are complete interpolating sequences for \( \mathcal{F}_\varphi^p \), when \( p = \frac{4}{3} \) and \( p = 4 \) respectively.

As an immediate consequence of Theorem 2.6, the quantity
\[
\|f\|_{p,\varphi,\Lambda} := \left( \sum_{\lambda \in \Lambda} |\lambda|^2 |f(\lambda)|^p e^{-p\varphi(\lambda)} \right)^{1/p}, \quad f \in \mathcal{F}_\varphi^p,
\]
where \( \Lambda \) is the corresponding sequence to \( \mathcal{F}_\varphi^p \), defines a norm equivalent to that of \( \mathcal{F}_\varphi^p \).

To make the proof of Theorem 2.6 clearer, we single out the next technical ingredient as a lemma.

Lemma 2.7. Let \( \Lambda = \Lambda_l \), for some \( l \in \{-1, 0, 1\} \) and \( \lambda \in \Lambda \). Let \( \beta > 2 \) be fixed, we have
\[
I_{\beta,\lambda} := \int_{\mathbb{C}} \frac{\text{dist}(z, \Lambda)}{|z - \lambda|(1 + |z|)^{\beta}} \ dm(z) \asymp |\lambda|^{2-\beta}.
\]
Also if \( \nu < 1 \), we have
\[
(1 + |z|)^{-\nu} \sum_{\lambda \in \Lambda} |\lambda|^\nu \frac{\text{dist}(z, \Lambda)}{|z - \lambda|} \asymp 1.
\]

Proof. First, we write
\[
I_{\beta,\lambda} = \int_{|\lambda|/2 \leq |z| \leq 2|\lambda|} \frac{\text{dist}(z, \Lambda)}{|z - \lambda|(1 + |z|)^{\beta}} \ dm(z) + \int_{\mathbb{C}\setminus\{|\lambda|/2 \leq |z| \leq 2|\lambda|\}} \frac{\text{dist}(z, \Lambda)}{|z - \lambda|(1 + |z|)^{\beta}} \ dm(z) \]
\[
= I_1 + I_2.
\]
For $|\lambda|/2 \leq |z| \leq 2|\lambda|$, we have $\text{dist}(z, \Lambda) \asymp |z - \lambda|$ and hence

$$I_1 \asymp \int_{|\lambda|/2}^{2|\lambda|} \frac{rdr}{(1 + r)^\beta} \lesssim |\lambda|^{2-\beta}.$$ 

On the other hand, using the fact that $|z - \lambda| \asymp |\lambda|$, $\text{dist}(z, \Lambda) \leq |z|$ when $|z| \leq |\lambda|/2$ and $\text{dist}(z, \Lambda) \leq |z - \lambda|$ for $|z| \geq 2|\lambda|$, we get

$$I_2 \leq \frac{1}{|\lambda|} \int_{|\lambda|/2}^{2|\lambda|} \frac{r^2dr}{(1 + r)^\beta} + \int_{2|\lambda|}^{\infty} \frac{rdr}{(1 + r)^\beta} \lesssim |\lambda|^{2-\beta}.$$ 

This ensures the desired result in (2.16). Now, to prove the estimate (2.17), we write

$$\sum_{\lambda \in \Lambda} |\lambda|^\nu \frac{\text{dist}(z, \Lambda)}{|z - \lambda|} = \left( \sum_{|\lambda| \leq |z|/2} + \sum_{|z|/2 \leq |\lambda| \leq 2|z|} + \sum_{|\lambda| \geq 2|z|} \right) |\lambda|^\nu \frac{\text{dist}(z, \Lambda)}{|z - \lambda|}$$

$$= J_1 + J_2 + J_3.$$ 

As in the proof of the first estimate, we have

$$J_1 + J_3 \lesssim \sum_{|\lambda| \leq |z|/2} |\lambda|^\nu + (1 + |z|) \sum_{|\lambda| \geq 2|z|} |\lambda|^{\nu-1} \asymp (1 + |z|)^\nu,$$

and also

$$J_2 \asymp \sum_{|z|/2 \leq |\lambda| \leq 2|z|} |\lambda|^\nu \asymp (1 + |z|)^\nu.$$ 

This completes the proof. \qed

### 3. Proof of Theorem 2.6

This section is devoted to prove Theorem 2.6. First, note that a sequence $\Lambda$ of complex numbers is a complete interpolating set for $\mathcal{F}_\varphi^p$ if and only if the following associated operator

$$T_\Lambda : \mathcal{F}_\varphi^p \to \mathcal{L}^p$$

$$f \mapsto \left( (1 + |\lambda|)^{2/p} f(\lambda) e^{-\varphi(\lambda)} \right)_{\lambda \in \Lambda}$$

is bounded and invertible.

(1) Fix $p \in [1, \infty) \setminus \{4/3, 4\}$ and let $l \in \{-1, 0, 1\}$ be the integer index of the corresponding sequence to $p$ as in Theorem 2.6.

First, since the sequence $\Lambda_l$ is $d_p$-separated, as stated in Lemma 2.5, there exists $C > 0$ such that

$$\|f\|_{p, \varphi, \Lambda_l}^p := \sum_{\lambda \in \Lambda_l} (1 + |\lambda|^2) |f(\lambda)e^{-\varphi(\lambda)}|^p \leq C \|f\|_{p, \varphi}^p,$$

for every $f \in \mathcal{F}_\varphi^p$ and hence $T_{\Lambda_l}$ is bounded. Also, if $f \in \mathcal{F}_\varphi^p$ is an element of the kernel of $T_{\Lambda_l}$, then $f$ must vanish on $\Lambda$. According to Corollary 1 the sequence $\Lambda_l$ is a uniqueness set for $\mathcal{F}_\varphi^p$, therefore $f$ must be identically zero and consequently $T_{\Lambda_l}$ is
one-to-one from $\mathcal{F}_p$ to $\ell^p$.

Now, let us prove that $T_{\Lambda_i}$ is onto. For that purpose, let $a = (a_\lambda)_{\lambda \in \Lambda_i} \in \ell^p$ and consider the function defined by

$$f(a)(z) = \sum_{\lambda \in \Lambda_i} a_\lambda (1 + |\lambda|)^{-2/p} e^{p(\lambda)} \frac{G_{\Lambda_i}(z)}{G'_{\Lambda_i}(\lambda)(z - \lambda)}, \quad z \in \mathbb{C}, \quad (3.1)$$

where $G_{\Lambda_i}$ is the infinite product associated with $\Lambda_i$. The series defining $f_a$ converges uniformly on every compact set of $\mathbb{C}$ and hence $f_a$ is an entire function satisfying $T_{\Lambda_i}f_a = a$. Indeed, let $r > 0$ and $|z| \leq r$, using (2.5) in Lemma 2.2 we have for every $|\lambda| \geq 2r$

$$\left| a_\lambda (1 + |\lambda|)^{-2/p} e^{p(\lambda)} \frac{G_{\Lambda_i}(z)}{G'_{\Lambda_i}(\lambda)(z - \lambda)} \right| \lesssim C(r)(1 + |\lambda|)^{1/2l-2/p}. \quad (3.2)$$

This ensures the uniform convergence of the series in compact sets of the complex plane when $l = -1$ for every $p$, when $l = 0$ for every $p < 4$, when $l = 1$ for all $p < 4/3$.

To complete the proof, it suffices to prove that $f_a \in \mathcal{F}_p$. Indeed, according to the estimates of the function $G_{\Lambda_i}$ given in Lemma 2.2, we obtain

$$\left| f_a(z) e^{-p(z)} \right|^p \leq \left( \sum_{\lambda \in \Lambda_i} |a_\lambda| (1 + |\lambda|)^{-2/p} e^{p(\lambda)} \left| \frac{G_{\Lambda_i}(z)}{G'_{\Lambda_i}(\lambda)(z - \lambda)} \right| e^{-p(z)} \right)^p \times \left( \sum_{\lambda \in \Lambda_i} |a_\lambda| |\lambda|^{3/2l-2/p} \frac{\text{dist}(z, \Lambda_i)}{|z - \lambda|(1 + |z|)^{3/2l}} \right)^p. \quad (3.3)$$

From Lemma 2.7, we have

$$\sum_{\lambda \in \Lambda_i} |\lambda|^{3/2l-2/p} \frac{\text{dist}(z, \Lambda_i)}{|z - \lambda|(1 + |z|)^{3/2l-2/p}} \lesssim 1, \quad z \in \mathbb{C}. \quad (3.4)$$

Applying now Jensen’s inequality, the identities (3.3) and (3.4) ensure that

$$\left| f_a(z) e^{-p(z)} \right|^p \lesssim \left( \sum_{\lambda \in \Lambda_i} \frac{|a_\lambda|}{(1 + |z|)^{2/p}|\lambda|^{3/2l-2/p}} \frac{\text{dist}(z, \Lambda_i)}{|z - \lambda|(1 + |z|)^{3/2l-2/p}} \right)^p \lesssim \left( \sum_{\lambda \in \Lambda_i} \frac{|a_\lambda|^p}{(1 + |z|)^{2}} |\lambda|^{3/2l-2/p} \frac{\text{dist}(z, \Lambda_i)}{|z - \lambda|(1 + |z|)^{3/2l-2/p}} \right)^p. \quad (3.5)$$

for every $z \in \mathbb{C}$. Integrating both sides of the last inequality with respect to the Lebesgue measure $dm(z)$, we get

$$\|f_a\|_{p,\varphi}^p \lesssim \sum_{\lambda \in \Lambda_i} |a_\lambda|^p |\lambda|^{3/2l-2/p} \int_{\mathbb{C}} \frac{\text{dist}(z, \Lambda_i)}{|z - \lambda|(1 + |z|)^{2+3/2l-2/p}} dm(z). \quad (3.6)$$
Setting $\beta = 2 + 3/2 + l - 2/p$, we deduce from Lemma 2.7 that the integral in the last inequality is controlled by a constant times $|\lambda|^{2-\beta}$ when $l = -1$ and $p > 4$, when $l = 0$ and $p > 4/3$ or when $l = 1$ and $p \geq 1$. Consequently,

$$
\|f_a\|_{p,\varphi}^p \lesssim \sum_{\lambda \in \Lambda_l} |a_{\lambda}|^p |\lambda|^{3/2 + l - 2/p} |\lambda|^{2(2+3/2 + l - 2/p)} = \|a\|_p^p < \infty.
$$

The proof is complete.

(2) Firstly, recall that we assume $\alpha = 1$. Now, for $l \in \{0, 1\}$ let $p = 4/(2l + 1)$. As in the previous case, the $d_p$-separation of the sequence $\Sigma = e^{-1/4} \Lambda_l$ affirms that $T_\Sigma$ is bounded from $\mathcal{F}_\varphi^p$ to $\ell^p$. On the other hand, if $G_\Sigma$ is the entire function associated with $\Sigma$, then we have

$$
G_\Sigma(z) = \prod_{\lambda \in \Lambda_l} \left(1 - \frac{z}{e^{-1/4}\lambda}\right) = G_{\Lambda_l}(e^{1/4}z), \quad z \in \mathbb{C}. \quad (3.7)
$$

Using the estimates of $G_{\Lambda_l}$ given in Lemma 2.2, we obtain

$$
|G_\Sigma(z)| \asymp e^{\varphi(e^{1/4}z)} \frac{\text{dist}(e^{1/4}z, \Lambda_l)}{(1 + |z|)^{3/2 + l}}
\asymp e^{(\log|z|+1/2)^2} \frac{\text{dist}(z, \Sigma)}{(1 + |z|)^{3/2 + l}}
\asymp e^{\varphi(z)} \frac{\text{dist}(z, \Sigma)}{(1 + |z|)^{l+1}}, \quad (3.8)
$$

for every $z \in \mathbb{C}$. Analogously, we have

$$
|G'_\Sigma(\sigma)| \asymp \frac{e^{\varphi(\sigma)}}{(1 + |\sigma|)^{l+1}}, \quad \sigma \in \Sigma. \quad (3.9)
$$

Following the same steps of the proof of Corollary 1, one can easily show that $\Sigma$ is a uniqueness set for $\mathcal{F}_\varphi^p$. This implies that $T_\Sigma$ is a one-to-one operator from $\mathcal{F}_\varphi^p$ to $\ell^p$. Now, and in order to prove that $T_\Sigma$ is onto, take $a = (a_\sigma)_{\sigma \in \Sigma}$ in $\ell^p$ and consider the function

$$
f_a(z) = \sum_{\sigma \in \Sigma} a_{\sigma} (1 + |\sigma|)^{-2/p} e^{\varphi(\sigma)} \frac{G_\Sigma(z)}{G'_\Sigma(\sigma)(z - \sigma)}, \quad z \in \mathbb{C}. \quad (3.10)
$$

Now, fix $r > 0$. According to the estimate in (3.9), we have for every $|z| \leq r$ and every $|\sigma| \geq 2r$

$$
\left|a_{\sigma} (1 + |\sigma|)^{-2/p} e^{\varphi(\sigma)} \frac{G_\Sigma(z)}{G'_\Sigma(\sigma)(z - \sigma)}\right| \lesssim C(r)(1 + |\sigma|)^{l-2/p}. \quad (3.11)
$$

Since $p = 4/(2l + 1)$ we obtain $l - 2/p = -1/2$. This ensures the convergence of the series in (3.10) uniformly in compact sets of $\mathbb{C}$. Using the same arguments of the previous proof one can prove that the function $f_a$ belongs to $\mathcal{F}_\varphi^p$ and satisfies
(1 + |σ|)^{2/p} f_a(σ) e^{-φ(σ)} = a_σ, for every σ ∈ Σ, i.e. \( T_Σ f_a = a \). This completes the proof.

4. PROOF OF THEOREMS 1 AND 2

In this section we prove Theorems 1 and 2. First, we prove that our conditions are sufficient in both situations and next that they are necessary. The proofs of our results appear generally like that of [4, Theorem 1.1], we will focus on the parts when the two proofs differ. In what follows we use the notation \( Γ_l = \{ γ_n : n ≥ l \} \), where \( l \) is an integer.

Now, Let \( Γ = \{ γ_n : n ≥ 0 \} \) be a countable set of \( \mathbb{C} \). Without loss of generality, we can suppose that \( Γ \cup \{ * \} = Γ \cup \{ γ_0 \} \) is a countable set of \( \mathbb{C} \) such that \( |γ_1| < |γ_0| \). Also \( Γ = Γ_0 \) and \( Γ \setminus \{ * \} = Γ \setminus \{ γ_0 \} = Γ_1 \).

4.1. Sufficient Conditions.

(1) As a first point we state sufficient conditions of Theorem 1. To this end, fix \( p \in [1, ∞) \setminus \{ 4/3, 4 \} \) and let \( l \in \{ -1, 0, 1 \} \) be the integer index of the corresponding sequence as mentioned above.

Now, since the sequence \( Γ_l \) is \( d_ρ \)-separated, \( T_{Γ_l} \) maps \( F_ϕ^p \) boundedly on \( ℓ^p \). On the other hand, take \( f ∈ F_ϕ^p \) such that \( T_{Γ_l} f = 0 \), then \( f \) must vanish on \( Γ_l \). Using Hadamard’s factorization theorem we can write \( f = h G_{Γ_l} \), for some entire function \( h \). According to Lemmas 2.3 and 2.1 the following estimate holds

\[
\frac{|h(z)| \text{dist}(z, Γ_l)}{1 + |z|^{3/2 + 2δ + l}} \lesssim |h(z)| G_{Γ_l(z)} e^{-φ(z)} = |f(z)| e^{-φ(z)} \lesssim (1 + |z|)^{-2/p}, \quad z ∈ \mathbb{C}.
\]

(4.1)

Since \( \text{dist}(z, Γ_l) = 1 + |z| \) in \( C \setminus \bigcup_{γ ∈ Γ_l} D(γ, β|γ|) \), for some \( β ∈ (0, 1) \), \( h \) is a polynomial. As \( f ∈ F_ϕ^p \), using the two first lines in (4.1) we get

\[
\int_C \left( \frac{|h(z)| \text{dist}(z, Γ_l)}{1 + |z|^{3/2 + 2δ + l}} \right)^p \, dm(z) \lesssim \int_C |f(z)|^p e^{-pφ(z)} \, dm(z) < ∞.
\]

Taking into account Lemma 2.4 we get \( ν = 2δ + l > 2/p - 1/2 \), i.e. \( δ > 1/p - 1/4 - l/2 \). Now, the conditions required in Theorem 1 in the different ranges of \( p \) depending on \( l \) allow us to conclude that \( h \) is the zero function and hence \( f \) is identically zero too.

That is, \( T_{Γ_l} \) is a bounded one-to-one operator from \( F_ϕ^p \) to \( ℓ^p \). To prove that \( T_{Γ_l} \) is onto, let \( a = (a_n)_{n ≥ l} \) in \( ℓ^p \) and consider the function

\[
g_a(z) = \sum_{n ≥ l} a_n (1 + |γ_n|)^{-2/p} e^{-φ(γ_n)} \frac{G_{Γ_l}(z)}{G_{Γ_l}(γ_n)(z - γ_n)}, \quad z ∈ \mathbb{C}.
\]

(4.2)
The last series converges uniformly on every compact set of the complex plane. Indeed, let \( r > 0 \) and \( |z| \leq r \). By (2.5) of Lemma 2.2 we have for every \( |\gamma| \geq 2r \)

\[
|a_n(1 + |\gamma|)^{-2/p} e^{\varphi(\gamma)}| \frac{G_{\Gamma_{l}}(z)}{G'_{\Gamma_{l}}(\gamma_n)(\gamma_n - \gamma)} \leq C(r) |\gamma|^{1/2 + l + 2\delta - 2/p}.
\]

(4.3)

Again, the condition on \( \delta = \Delta_N \) (taking account of the relation between \( p \) and the corresponding \( l \)) ensures that \( g_a \) is an entire function satisfying \((1 + |\gamma|)^{2/p} g_a(\gamma) e^{-\varphi(\gamma)} = a_n\), for every \( n \geq l \). It remains to check that \( g_a \in \mathcal{F}_p^\phi \). To this end, since \( \Lambda_l \) is a complete interpolating sequence for \( \mathcal{F}_p^\phi \) (see Theorem 2.6), we immediately obtain

\[
\|g_a\|_{\ell^p,\varphi}^p \approx \sum_{m \geq l} \left| \lambda_m \right|^2 \left| g_a(\lambda_m) e^{-\varphi(\lambda_m)} \right|^p
\]

\[
= \sum_{m \geq l} \left| \lambda_m \right|^2 \sum_{n \geq l} a_n(1 + |\gamma|)^{-2/p} e^{\varphi(\gamma)} \frac{G_{\Gamma_{l}}(\lambda_m)}{G'_{\Gamma_{l}}(\gamma_n)(\lambda_m - \gamma_n)} e^{-\varphi(\lambda_m)} \left| g_a(\lambda_m) \right|^p
\]

\[
= \sum_{m \geq l} \sum_{n \geq l} a_n \left( \frac{|\lambda_m|}{1 + |\gamma|} \right)^{2/p} e^{\varphi(\gamma_n) - \varphi(\lambda_m)} \frac{G_{\Gamma_{l}}(\lambda_m)}{G'_{\Gamma_{l}}(\gamma_n)(\lambda_m - \gamma_n)} \left| g_a(\lambda_m) \right|^p.
\]

It follows from this identity that the functions \((g_a)_{a \in \ell^p}\) belong to \( \mathcal{F}_p^\phi \) if and only if the matrix \( A_p = (A_{p,n,m})_{n,m \geq l} \) maps \( \ell^p \) continuously onto itself, where

\[
A_{p,n,m} := \left( \frac{|\lambda_m|}{1 + |\gamma_n|} \right)^{2/p} e^{\varphi(\gamma_n) - \varphi(\lambda_m)} \frac{G_{\Gamma_{l}}(\lambda_m)}{G'_{\Gamma_{l}}(\gamma_n)(\lambda_m - \gamma_n)},
\]

(4.4)

for every \( n, m \geq l \). On the other hand, note that there exists a polynomial \( P_l \) if \( l = 0, 1 \) (or a fraction if \( l = -1 \)) of degree \(|l|\) satisfying \( G_{\Gamma_{0}}(z) = G_{\Gamma_{l}}(z) P_l(z) \). Hence we have \( G'_{\Gamma_{0}}(\gamma_n) = G'_{\Gamma_{l}}(\gamma_n) P_l(\gamma_n) \), for every \( n \geq 0 \). Using this simple fact, we obtain

\[
|A_{p,n,m}| \asymp \left| \frac{\lambda_m}{\gamma_n} \right|^{2/p-l} e^{\varphi(\gamma_n) - \varphi(\lambda_m)} \frac{G_{\Gamma_{0}}(\lambda_m)}{G'_{\Gamma_{0}}(\gamma_n)(\lambda_m - \gamma_n)} \left| P_l(\gamma_n) \right|.
\]

(4.5)

According to the two first lines in (2.6) we have

\[
|G_{\Gamma_{0}}(\lambda_m)| \asymp \text{dist}(\lambda_m, \Gamma_0) \prod_{0 \leq k \leq m-1} \left| \frac{\lambda_m}{\gamma_k} \right|,
\]

(4.6)

and

\[
|G'_{\Gamma_{0}}(\gamma_n)| \asymp \frac{1}{|\gamma_n|} \prod_{0 \leq k \leq n-1} \left| \frac{\gamma_n}{\gamma_k} \right|.
\]

(4.7)
Consequently,

\[ |A_{p,n,m}| \asymp \frac{\lambda_m^{2/p-l-1}}{\gamma_n} e^{\varphi(\gamma_n)-\varphi(\lambda_m)} \text{dist}(\lambda_m,\Gamma_0) \left( \prod_{0 \leq k \leq m-1} \left| \frac{\lambda_m}{\gamma_k} \right| \right) \left( \prod_{0 \leq k \leq n-1} \left| \frac{\gamma_n}{\gamma_k} \right| \right)^{-1} \]

where

\[ \alpha(n, m) = \frac{m(m+1)}{2} - n \left( \frac{n+1}{2} + \delta_n \right) + \left( \frac{n+1}{2} + \delta_n \right)^2 - \left( \frac{m+1}{2} \right)^2 \]

\[ + \left( \frac{1}{p} - \frac{l+1}{2} \right) (m-n) + \sum_{k=1}^{n-1} \left( \frac{k+1}{2} + \delta_k \right) - \sum_{k=l}^{m-1} \left( \frac{k+1}{2} + \delta_k \right) \]

\[ = \left( \frac{1}{p} - \frac{1}{4} - \frac{l+1}{2} \right) (m-n) + \sum_{k=l}^{n-1} \delta_k - \sum_{k=l}^{m-1} \delta_k. \]

Since the sequence \((\delta_n)\) is bounded, there exists an integer \(M\) such that \(|\gamma_n - \lambda_m| \asymp \lambda_m\)

if \(m > n + M\), \(|\gamma_n - \lambda_m| \asymp |\gamma_n|\) if \(n > m + M\), and \(|\gamma_n| \asymp \lambda_m\) for every \(|n-m| \leq M\).

Also, since our spaces are rotation invariant we can assume that \(\text{dist}(\lambda_m, \Gamma_0) \asymp |\lambda_m|\).

Otherwise, we may replace if necessary \(\lambda_m\) by \(\lambda_m e^{i\theta_m}\), for an appropriate \(\theta_m \in \mathbb{R}\).

Consequently,

\(\star\) if \(|n-m| \leq M\) we get \(|A_{p,n,m}| \lesssim 1\).

\(\star\) if \(m > n + M\) we then obtain

\[ |A_{p,n,m}| \asymp \exp \left[ - \left( \frac{1}{4} - \frac{1}{p} - \frac{l+1}{2} \right) |m-n| - \sum_{k=n}^{m-1} \delta_k \right] \]

\[ \lesssim \exp \left[ - \left( \frac{1}{4} - \frac{1}{p} + \frac{l+1}{2} + \nabla_N \right) |m-n| \right]. \]

\(\star\) if \(n > m + M\) we obtain

\[ |A_{p,n,m}| \asymp \exp \left[ - \left( \frac{1}{4} + \frac{1}{p} - \frac{l+1}{2} \right) |m-n| + \sum_{k=m}^{n-1} \delta_k \right] \]

\[ \lesssim \exp \left[ - \left( \frac{1}{4} + \frac{1}{p} - \frac{l+1}{2} - \Delta_N \right) |m-n| \right]. \]

Thus,

\(\bullet\) If \(l = -1\) and \(p > 4\), then (1.5) ensures that \(\nabla_N > 1/p - 1/4\) and \(\Delta_N < 1/4 + 1/p\), for some \(N \in \mathbb{N}\), so that \(c := 1/4 - 1/p + \nabla_N > 0\) and \(c' := 1/4 + 1/p - \Delta_N > 0\) and hence

\[ |A_{p,n,m}| \lesssim \begin{cases} 
  e^{-(1/4 - 1/p + \nabla_N)|n-m|} = e^{-c|n-m|}, & m \geq n; \\
  e^{-(1/4 + 1/p - \Delta_N)|n-m|} = e^{-c'|n-m|}, & n \geq m. 
\end{cases} \]
• If \( l = 0 \) and \( 4/3 < p < 4 \), then (1.4) implies that \( 1/4 - 1/q < \nabla_N \leq \Delta_N < 1/p - 1/4 \), so that \( c := 1/q - 1/4 + \nabla_N > 0 \) and \( c' := 1/p - 1/4 - \Delta_N > 0 \), and consequently
\[
|A_{p,n,m}| \lesssim \begin{cases} 
  e^{-(1/q-1/4+\nabla_N)[n-m]} = e^{-c[n-m]}, & m \geq n; \\
  e^{-(1/p-1/4-\Delta_N)[n-m]} = e^{-c'[n-m]}, & n \geq m.
\end{cases}
\]

• The remaining case is \( l = 1 \) and \( 1 \leq p < 4/3 \), the condition required in (1.3) implies that \( c := 1/q + 1/4 + \nabla_N > 0 \) and \( c' := 1/4 - 1/q - \Delta_N > 0 \). Thus,
\[
|A_{p,n,m}| \lesssim \begin{cases} 
  e^{-(1/q+1/4+\nabla_N)[n-m]} = e^{-c[n-m]}, & m \geq n; \\
  e^{-(1/4-1/q-\Delta_N)[n-m]} = e^{-c'[n-m]}, & n \geq m.
\end{cases}
\]

This shows that the matrix \( A_p \) is bounded in the different cases, which ends the proof of the sufficient condition.

(2) Let \( l \in \{0, 1\}, \ p = 4/(2l + 1) \). Denote by \( \hat{G}_l = e^{-1/4} \Gamma_l = \{\nu_n : n \geq l\} \) and \( \Sigma = e^{-1/4} \Lambda_l = \{\sigma_n : n \geq l\} \). First since \( \hat{G}_l \) is \( d_p \)-separated, \( T_{\hat{G}_l} \) is bounded from \( \mathcal{F}_p^\varphi \) to \( \ell^p \) (see Lemma 2.5). On the other hand, remark that \( G_{\hat{G}_l}(z) = G_{\Gamma_l}(e^{1/4}z) \), for every \( z \in \mathbb{C} \). According to this remark and to Lemma 2.3, \( G_{\hat{G}_l} \) satisfies the estimates
\[
\frac{\text{dist}(z, \hat{G}_l)}{1 + |z|^{1+l+2\delta}} \lesssim \left| G_{\hat{G}_l}(z) \right| e^{-\varphi(z)} \lesssim \frac{\text{dist}(z, \hat{G}_l)}{1 + |z|^{1+l+2\eta}}, \quad z \in \mathbb{C},
\]
where \( \delta = \Delta_N \) and \( \eta = \nabla_N \). Using the above estimates and following the same steps of the proof in (1), one can conclude easily that \( T_{\hat{G}_l} \) is one-to-one. Now, to check that \( T_{\hat{G}_l} \) is onto we associate with each \( a = (a_n)_{n \geq l} \in \ell^p \) the following function
\[
G_a(z) = \sum_{n \geq l} a_n (1 + |\nu_n|)^{-2/p} e^{\varphi(\nu_n)} \frac{G_{\hat{G}_l}(z)}{G_{\hat{G}_l}'(\nu_n)(z - \nu_n)}, \quad z \in \mathbb{C}.
\]
Again, since \( G_{\hat{G}_l}(z) = G_{\Gamma_l}(e^{1/4}z) \), for every \( z \in \mathbb{C} \), Lemma 2.3 implies that
\[
\frac{e^{\varphi(\nu_n)}}{1 + |\nu_n|^{1+l+2\delta}} \lesssim \left| G_{\hat{G}_l}'(\nu_n) \right| \lesssim \frac{e^{\varphi(\nu_n)}}{1 + |\nu_n|^{1+l+2\eta}}, \quad \nu_n \in \hat{G}_l.
\]
Thus, for every \( r > 0 \) and every \( |z| \leq r \) we have
\[
\left| a_n (1 + |\nu_n|)^{-2/p} e^{\varphi(\nu_n)} \frac{G_{\hat{G}_l}(z)}{G_{\hat{G}_l}'(\nu_n)(z - \nu_n)} \right| \lesssim C(r)(1 + |\nu_n|)^{-2/p+l+2\delta}, \quad |\nu_n| \geq 2r.
\]
Since \( p = 4/(2l+1) \) we obtain \( \beta := -2/p+l+2\delta = 2\delta - 1/2 \). The condition on \( \delta = \Delta_N \) implies that the above series converges uniformly on compact sets of \( \mathbb{C} \) and hence \( g_a \) is holomorphic in \( \mathbb{C} \) and satisfies \( T_{\hat{G}_l} g_a = a \). To prove that \( g_a \in \mathcal{F}_p^\varphi \) we use the fact
that $\Sigma$ is a complete interpolating sequence for $F^p_\phi$ (see Theorem 2.6), and we obtain
\[
\|g_a\|_{p,\phi} \asymp \sum_{m \geq t} |\sigma_m|^2 |g_a(\sigma_m)|^p e^{-p\varphi(\sigma_m)}
\]
\[
= \sum_{m \geq t} \left( \sum_{n \geq t} a_n \left( \frac{|\sigma_m|}{1 + |\nu_n|} \right)^{2/p} e^{\varphi(\nu_n)-\varphi(\sigma_m)} \frac{G_{\hat{f}_l}(\sigma_m)}{G_{\hat{f}_l}'(\nu_n)(\sigma_m - \nu_n)} \right)^p.
\]

Once more, the functions $(g_a)_{a \in \ell^p}$ belong to $F^p_\phi$ if and only if the matrix $A_p = (A_{p,n,m})_{n,m \geq t}$ acts continuously on $\ell^p$, where
\[
|A_{p,n,m}| = \left| \frac{\sigma_m}{\nu_n} \right|^{2/p} e^{\varphi(\nu_n)-\varphi(\sigma_m)} \left| \frac{G_{\hat{f}_l}(\sigma_m)}{G_{\hat{f}_l}'(\nu_n)(\sigma_m - \nu_n)} \right|.
\]

Again, note that there exists a polynomial $P_l$ of degree $l \in \{0, 1\}$ satisfying $G_{\Gamma_0}(e^{1/4}\gamma) = G_{\hat{f}_l}^\prime(z)P_l(z)$ and hence we have $G_{\Gamma_0}'(\gamma) = G_{\hat{f}_l}'(\nu)P_l(\nu)$ for every $\gamma = e^{1/4}\nu \in \Gamma_l$. Using this simple remark and observing that $|\lambda_m| \asymp |\sigma_m|$ and $|\nu_n| \asymp |\gamma_n|$, we get

\[
|A_{p,n,m}| \asymp \left| \frac{\lambda_m}{\gamma_n} \right|^{2/p} e^{\varphi(\gamma_n)-\varphi(\lambda_m)} \left| P_l(\gamma_n)/P_l(\lambda_m) \right| \left| \frac{G_{\Gamma_0}(\lambda_m)}{G_{\Gamma_0}'(\gamma_n)(\lambda_m - \gamma_n)} \right|.
\]

Since $p = 4/(2l + 1)$. From the calculations of the previous proof we have

\[
|A_{2,n,m}| \asymp \exp \left( -\frac{|m-n|}{4} \pm \sum_{k=m+1}^n \delta_k \right),
\]

where the sum in the last identity takes the positive sign if $m \leq n$ and the negative sign if $m \geq n$. This implies the desired result.

4.2. Necessary conditions. Suppose that $\Gamma$ is a complete interpolating sequence for $F^p_\phi$.

(a) Since $\Gamma$ is an interpolating sequence, classical arguments show that $\Gamma$ is $d_\rho$-separated, see [4, Lemma 2.1 and Corollary 2.3].

(b) Let $\Lambda = \{\lambda_n = e^{\frac{n+1}{2}} : n \geq l\}$ and let $\Gamma = \{\gamma_n : n \geq l\}$ be a complete interpolating sequence for $F^p_\phi$ such that $|\gamma_n| \leq |\gamma_{n+1}|$. We write $|\gamma_n| = \lambda_n e^{\delta_n}$, and we will prove that $(\delta_n)$ is a bounded sequence.
First, for every $\gamma \in \Gamma$ there exists a unique function $f_\gamma \in \mathcal{F}_p^\Gamma$ which is a solution of the interpolating problem $f_\gamma(\gamma) = 1$ and $f_\gamma|_{\Gamma \setminus \{\gamma\}} = 0$. The zero set of $f_\gamma$ is exactly $\Gamma \setminus \{\gamma\}$. Indeed, if not, this contradicts the fact that $\Gamma$ is a uniqueness set for $\mathcal{F}_p^\Gamma$. The Hadamard factorization theorem ensures that $f_\gamma(z) = e \frac{G(z)}{G(z-\gamma)}$ for some constant $e \in \mathbb{C}$, and $G_\Gamma$ is the infinite product associated with $\Gamma$. Now, since $f_\gamma(\gamma) = 1$, we directly get $f_\gamma = \frac{G(z)}{G(z-\gamma)} \in \mathcal{F}_p^\Gamma$. Consequently, Lemma 2.1 implies that

$$|f_\gamma(z)| \lesssim \|f_\gamma\|_{p,\varphi} \frac{e^{\varphi(z)}}{1 + |z|^{2/p}}, \quad z \in \mathbb{C}. \quad (4.11)$$

On the other hand, the sequence $\Gamma$ is a complete interpolating set for $\mathcal{F}_p^\Gamma$. Hence

$$\|f_\gamma\|_{p,\varphi} \simeq \sum_{\gamma' \in \Gamma} |\gamma'|^2 \left| f_\gamma(\gamma') e^{-\varphi(\gamma')} \right|^p \quad = |\gamma|^2 e^{-p\varphi(\gamma)}, \quad \gamma \in \Gamma. \quad (4.12)$$

Now, assume that $(\delta_n)$ contains a sub-sequence $(\delta_{n_k})$ which tends to infinity. Thus, for every $k$ there exists $m_k$ such that $d_\rho(\gamma_{n_k}, \lambda_{m_k}) \lesssim 1$ and $|n_k - m_k| \to \infty$. Since the sequence $\Gamma$ is $d_\rho$-separated, a similar reasoning as for (4.6) and (4.7) gives

$$|G_\Gamma(\lambda_{m_k})| \asymp \frac{|\lambda_{m_k} - \gamma_{n_k}|}{\lambda_{m_k}} \prod_{j=1}^{n_k-1} \frac{\lambda_{m_k}}{|\gamma_j|}, \quad (4.13)$$

$$|G'_\Gamma(\gamma_{m_k})| \asymp \frac{1}{|\gamma_{n_k}|} \prod_{j=1}^{n_k-1} \frac{|\gamma_{n_k}|}{|\gamma_j|}. \quad (4.14)$$

Combining (4.11), (4.12), (4.13), and (4.14) we obtain

$$\left| \frac{\lambda_{m_k}}{\gamma_{n_k}} \right|^{n_k} \asymp |f_{\gamma_{n_k}}(\lambda_{m_k})| \lesssim e^{\varphi(\lambda_{m_k})-\varphi(\gamma_{n_k})}. \quad (4.15)$$

Without loss of generality we may suppose that $\delta_{n_k} \to \infty$, we assume also that $|\gamma_{n_k}| \geq e^{2\lambda_{m_k}}$. Otherwise, we replace $m_k$ by $m_k - m'_k$ for an adequate integer $m'_k$ (the case $\delta_{n_k} \to -\infty$ is similar). By our assumption $d_\rho(\gamma_{n_k}, \lambda_{m_k}) \lesssim 1$ and $|\gamma_{n_k}| = |\lambda_{m_k}|$, there exists a sequence $(\eta_{n_k,m_k}) \subset [1,2]$ such that $\frac{1+m_{k} \nu_{n_k}}{2} + \delta_{n_k} = \frac{1+m_{k} \nu_{n_k}}{2} + \eta_{n_k,m_k}$, i.e. $n_k + 2\delta_{n_k} = m_k + 2\eta_{n_k,m_k}$. By simple calculations we get

$$A := n_k \left( \frac{m_k - n_k}{2} - \delta_{n_k} \right) - \left( \frac{m_k + 1}{2} \right)^2 + \left( \frac{n_k + 1}{2} + \delta_{n_k} \right)^2$$

$$= \delta_{n_k}^2 + \delta_{n_k} + \frac{(n_k - m_k)(m_k - n_k + 2)}{2}$$

$$= \delta_{n_k}^2 + \delta_{n_k} + (\eta_{n_k,m_k} - \delta_{n_k})(\delta_{n_k} - \eta_{n_k,m_k} + 1)$$

$$= 2\delta_{n_k} \eta_{n_k,m_k} + O(1).$$
Passing to logarithms in (4.15) we reach a contradiction to the above inequality.

Simple modifications in this proof ensure the result for the cases \( p = 4/3 \) and 4 in Theorem 2.

\((c)\) For \( p \in \{4/3, 4\} \), assume that for every \( N \geq 1 \) we have

\[
\sup_n \frac{1}{N} \left| \sum_{k=n+1}^{n+N} \delta_k \right| = \frac{1}{4} + \varepsilon_N,
\]

for some nonnegative sequence \((\varepsilon_N)\). By the supremum property, for every \( N \geq 1 \) there exists an integer \( n_N \) such that

\[
\left| \sum_{k=n_N+1}^{n_N+N} \delta_k \right| \geq N \left( \frac{1}{4} + \varepsilon_N \right) - 1.
\]

Also by definition, for every \( K \leq N \)

\[
\left| \sum_{k=n_N+1}^{n_N+K} \delta_k \right| \leq K \left( \frac{1}{4} + \varepsilon_K \right).
\]

It follows that

\[
\left| \sum_{k=n_N+K+1}^{n_N+N} \delta_k \right| \geq \frac{N - K}{4} + N\varepsilon_N - K\varepsilon_K - 1.
\]

Since \((\delta_n)\) is bounded, we have proved in (4.10) that

\[
|A_{p,n,m}| \asymp |A_{2,n,m}| \asymp \exp \left( -\frac{|m - n|}{4} \pm \sum_{k=m+1}^{n} \delta_k \right), \quad n, m \geq 0,
\]

where the sum in the last identity takes the positive sign if \( m \leq n \) and the negative sign if \( m \geq n \).

Now, if the sequence \((N\varepsilon_N)\) contains a subsequence which tends to infinity, then for \( N \) in this subsequence and fixed \( K \) we get that the sum \(|A_{p,n_N+K,n_N+N}| + |A_{p,n_N+N,n_N+K}|\) is unbounded. Hence the matrix \( A_p = (A_{p,n,m}) \) cannot define a bounded operator in \( \ell^p \).

Suppose now that the sequence \((N\varepsilon_N)\) is bounded. Let \( N \geq 1 \), if \( \sum_{k=n_N+1}^{n_N+N} \delta_k > 0 \) then for every \( 0 < K < N \) we have

\[
\sum_{k=n_N+K+1}^{n_N+N} \delta_k \geq \frac{N - K}{4} + N\varepsilon_N - K\varepsilon_K - 1.
\]
Hence, \(|A_{p,n} + n, n + K + 1| \geq 1\). Also, if \(\sum_{k=n+1}^{n+N} \delta_k < 0\) then for every \(0 < K < N\) we have
\[
\sum_{k=n+1}^{n+N-K} \delta_k \leq -\frac{N-K}{4} + O(1).
\]
Consequently, \(|A_{p,n+1, n + N - K}| \geq 1\). In both cases, the matrix \(A_p = (A_{p,n,m})_{n,m}\) contains an increasing number of entries in one line which are bounded away from zero, and hence it cannot define a bounded operator in \(\ell^p\). This completes the proof of Theorem 2.

Let us turn to Theorem 1 and suppose first that \(4/3 < p < 4\). Assume also that
\[
\Delta_N = \frac{1}{p} - \frac{1}{4} + \varepsilon_N \quad \text{or} \quad \nabla_N = \frac{1}{4} - \frac{1}{q} - \eta_N,
\]
where either \((\varepsilon_N)\) or \((\eta_N)\) is a nonnegative real sequence. By the supremum and the infimum properties, for every \(N \geq 1\) there exist some integers \(n_N\) and \(m_N\) such that
\[
\sum_{k=n_N+1}^{n+N} \delta_k \geq N \left(\frac{1}{p} - \frac{1}{4} + \varepsilon_N\right) - 1, \quad \text{or} \quad \sum_{k=m_N+1}^{m+N} \delta_k \leq N \left(\frac{1}{4} - \frac{1}{q} - \eta_N\right) + 1.
\]
Also by definition, for every \(K \leq N\)
\[
\sum_{k=n_N+1}^{n+N} \delta_k \leq K \left(\frac{1}{p} - \frac{1}{4} + \varepsilon_K\right), \quad \text{or} \quad \sum_{k=m_N+1}^{m+N} \delta_k \geq K \left(\frac{1}{4} - \frac{1}{q} - \varepsilon_K\right) + 1.
\]
It follows that
\[
\sum_{k=n+N+1}^{n+N+K} \delta_k \geq \left(\frac{1}{p} - \frac{1}{4}\right) (N-K) + N\varepsilon_N - K\varepsilon_K - 1,
\]
and also
\[
\sum_{k=m+N+1}^{m+N+K} \delta_k \leq \left(\frac{1}{4} - \frac{1}{q}\right) (N-K) - N\eta_N + K\eta_K + 1.
\]
Recall that, since the sequence \((\delta_n)\) is bounded we proved that
\[
|A_{p,n,m}| \asymp \begin{cases} 
\exp \left[-\left(\frac{1}{q} - \frac{1}{4}\right)|m-n| - \sum_{k=n}^{m-1} \delta_k\right], & m \geq n; \\
\exp \left[-\left(\frac{1}{p} - \frac{1}{4}\right)|m-n| + \sum_{k=m}^{n-1} \delta_k\right], & n \geq m.
\end{cases} \quad (4.17)
\]
Now, suppose that the sequence \((N\varepsilon_N)\) (or \((N\eta_N)\)) contains a subsequence tending to infinity. This implies
\[
|A_{p,n+N+1, n+1}| \asymp \exp(N\varepsilon_N), \quad \text{or} \quad |A_{p,n+1, n+N+1}| \asymp \exp(N\eta_N),
\]
and the matrix $A_p = (A_p,n,m)$ cannot map $\ell^p$ continuously into itself.

Suppose now that the sequence $(N\varepsilon_N)$ is bounded, as in the previous proof we obtain that $|A_{p,n,N+N,n+N+K+1}| \leq 1$ for every $1 \leq K < N$. Analogously, if $(N\eta_N)$ is bounded we get again that $|A_{p,n,N+1,n+N-N-K}| \leq 1$ for every $1 \leq K < N$. In both situations, the matrix $A_p = (A_p,n,m)$ contains an increasing number of coefficients in one line which are bounded away from zero and hence $A_p$ cannot define a bounded operator in $\ell^p$. This completes the proof for the case $4/3 < p < 4$.

Remark that
• For $p > 4$, we have

$$|A_{p,n,m}| \lesssim \begin{cases} \exp \left[ -\left( \frac{1}{4} - \frac{1}{p} \right) |m - n| - \sum_{k=n}^{m-1} \delta_k \right], & m \geq n; \\ \exp \left[ -\left( \frac{1}{4} + \frac{1}{p} \right) |m - n| + \sum_{k=m}^{n-1} \delta_k \right], & n \geq m. \end{cases}$$

(4.18)

• For $1 \leq p < \frac{4}{3}$, we have

$$|A_{p,n,m}| \lesssim \begin{cases} \exp \left[ -\left( \frac{1}{4} - \frac{1}{p} \right) |m - n| - \sum_{k=n}^{m-1} \delta_k \right], & m \geq n; \\ \exp \left[ -\left( \frac{1}{4} - \frac{1}{p} \right) |m - n| + \sum_{k=m}^{n-1} \delta_k \right], & n \geq m. \end{cases}$$

(4.19)

Using these estimates and following the same steps of the proof of the case $4/3 < p < 4$ one can check easily that the matrix $(A_{p,n,m})_{n,m}$ cannot map boundedly $\ell^p$ into itself, whenever the condition on $\Delta_N$ (or on $\nabla_N$) is not satisfied, and hence $T^{-1}_\Gamma$ is not bounded too. This ends the proof of Theorem 1.

5. Final Remarks

In this section we give some final remarks about the results obtained in Theorems 1 and 2. In fact, roughly speaking, in Theorem 1 if $p$ increases to 4, then B) implies that $\Gamma = \{\gamma_n := \lambda_n e^{\delta_n} e^{i\theta_n}\}_{n \geq 0}$ is a complete interpolating set for $\mathcal{F}^\delta$ if and only if (a) and (b) are satisfied and for some $N \geq 1$ we have $-\frac{1}{2} < \nabla_N(\Gamma) \leq \Delta_N(\Gamma) < 0$. $\nabla_N(\Gamma)$ and $\Delta_N(\Gamma)$ are the corresponding means to the sequence $\Gamma$. Now, if $\Gamma = e^{-1/4}\Sigma := \{\sigma_n := e^{-1/4} \lambda_n e^{s_n} e^{i\psi_n}\}_{n \geq 0}$, this is equivalent to $\lambda_n e^{\delta_n} e^{i\theta_n} = \gamma_n = \sigma_n = e^{-1/4} \lambda_n e^{s_n} e^{i\psi_n}$, i.e. $\delta_n = s_n - 1/4$, for every $n \geq 0$. Simple calculations show that

$$\nabla_N(\Gamma) = \nabla_N(\Sigma) - \frac{1}{4}, \text{ and } \Delta_N(\Gamma) = \Delta_N(\Sigma) - \frac{1}{4}.$$ 

Thus,

$$-\frac{1}{2} < \nabla_N(\Gamma) \leq \Delta_N(\Gamma) < 0 \iff -\frac{1}{4} < \nabla_N(\Sigma) \leq \Delta_N(\Sigma) < \frac{1}{4},$$

and hence the result recovered from B) in Theorem 1, by tending $p$ to 4, is equivalent to that given in Theorem 2.
On the other hand, if $p$ decreases to $4$, then $C)$ ensures that $\Gamma_{-1} = \{\gamma_n := \lambda_n e^{\delta_n e^{i\theta_n}}\}_{n \geq -1}$ is a complete interpolating sequence for $F^4_p$ if and only if (a) and (b) are verified and $0 < \nabla_N(\Gamma) \leq \Delta_N(\Gamma) < \frac{1}{2}$, for some $N \geq 1$. Now, let $\Gamma_{-1} = e^{-1/4}\Sigma := \{e^{-1/4}\lambda_n e^{s_n e^{i\psi_n}}\}_{n \geq 0}$. As above we obtain

$$0 < \nabla_N(\Gamma) \leq \Delta_N(\Gamma) < \frac{1}{2} \iff -\frac{1}{4} < \nabla_N(\Sigma) \leq \Delta_N(\Sigma) < \frac{1}{4},$$

and hence the result obtained from $C)$ in Theorem 1 is equivalent to that proved in Theorem 2. The case $p = 4/3$ is also the same. This clarify the nonexistence of a certain discontinuity in the obtained results.

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References

[1] S. A. Avdonin. On the question of Riesz bases of exponential functions in $L^2$. Vestn. Leningr. Univ., Mat. Meh. Astron., 3(13):5–12, 1974.
[2] A. Baranov, Yu. Belov, and A. Borichev. Fock type spaces with Riesz bases of reproducing kernels and de Branges spaces. Stud. Math., 236:127–142, 2017.
[3] A. Baranov, Yu. Belov, and A. Borichev. Spectral synthesis in de Branges spaces. Geom. Funct. Anal., 25(2):417–452, 2015.
[4] A. Baranov, A. Dumont, A. Hartmann, and K. Kellay. Sampling, interpolation and Riesz bases in small Fock spaces. J. Math. Pures Appl., 103(6):1358–1389, 2015.
[5] Yu. Belov, T. Y. Mengestie, and K. Seip. Discrete Hilbert transforms on sparse sequences. Proc. Lond. Math. Soc., 103(1):73–105, 2011.
[6] B. Berndtsson, and J. Ortega-Cerdà, On interpolation and sampling in Hilbert spaces of analytic functions, J. Reine Angew. Math., 464:109–128, 1995.
[7] A. Borichev, R. Dhuez, and K. Kellay. Sampling and interpolation in large Bergman and Fock spaces. J. Funct. Anal., 242(2):563–606, 2007.
[8] A. Borichev, A. Hartmann, K. Kellay, and X. Massaneda. Geometric conditions for multiple sampling and interpolation in the Fock space. Adv. Math., 304:1262–1295, 2017.
[9] A. Borichev, and Yu. Lyubarskii. Riesz bases of reproducing kernels in Fock-type spaces. J. Inst. Math. Jussieu, 9(3):449–461, 2010.
[10] S. Brekke, and K. Seip. Density theorems for sampling and interpolation in the Bargmann-Fock space III. Math. Scand., 73(1):112–126, 1993.
[11] S.V. Hruščëv, N.K. Nikol’skii, and B.S. Pavlov. Unconditional bases of exponentials and of reproducing kernels. In Complex analysis and spectral theory, 214–335, Springer, 1981.
[12] M. Kadets. The exact value of the Paley-Wiener constant. Sov. Math. Dokl., 5, 559–561, 1964.
[13] K. Kellay, and Y. Omari. Riesz bases of reproducing kernels in small Fock spaces. preprint hal-01918516, 2018.
[14] Yu. Lyubarskii, and K. Seip. Weighted Paley-Wiener spaces. J. Amer. Math. Soc., 15(4):979–1006, 2002.
[15] Yu. Lyubarskii, and K. Seip. Complete interpolating sequences for Paley-Wiener spaces and Muckenhoupt’s ($A_p$) condition. Rev. Mat. Iberoam., 13(2):361–376, 1997.
[16] N. Marco, X. Massaneda, and J. Ortega-Cerdà. Interpolating and sampling sequences for entire functions. Geom. Funct. Anal., 13(4):862–914, 2003.
[17] J. Marzo, and K. Seip. The Kadets 1/4 Theorem for polynomials. *Math. Scand.*, 104(2):311–318, 2009.
[18] J. Ortega-Cerdà, and K. Seip. Beurling-type density theorems for weighted $L^p$ spaces of entire functions. *J. Anal. Math.* 75:247–266, 1998.
[19] K. Seip. Density theorems for sampling and interpolation in the Bargmann-Fock space. *Bull. Amer. Math. Soc.*, 26(2):322–328, 1992.
[20] K. Seip. Interpolating and Sampling in Spaces of Analytic Functions, *Amer. Math. Soc.*, Providence., 2004.
[21] K. Seip, and R. Wallstén. Density theorems for sampling and interpolation in the Bargmann-Fock space. II. *J. Reine Angew. Math.*, 429:107–113, 1992.

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