Automorphic Inflation

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Abstract

A framework of inflation is formulated based on automorphic forms. In this setting the inflaton multiplet takes values in a curved target space constructed from a reductive group $G$ and an arithmetic subgroup $\Gamma$. The dynamics of inflationary models is essentially determined by the choice of the groups $(G, \Gamma)$ and a form $\Phi$. Automorphic inflation provides a natural structure in which the shift symmetry of large field inflation arises as one of generating elements of the arithmetic group $\Gamma$. The model of $j$—inflation is discussed as an example of modular inflation associated to the group $SL(2)$.

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1 Introduction

Symmetries have been useful as guides to the dynamics of fundamental theories for more than a century. The most dramatic examples involve continuous groups, but discrete groups have been of importance as well. The purpose of this paper is to formulate a field theoretic framework based on the combination of both types of groups and an associated function space. The continuous group $G$ is taken to be reductive and the discrete group $\Gamma$ is assumed to be an arithmetic subgroup $\Gamma \subset G$. Given a pair $(G, \Gamma)$ the dynamics is determined by an automorphic form $\Phi$, defined relative to $\Gamma$ as a function on $G$. This group function descends to a form $f_\Phi$ on the target space $X$ of the inflaton multiplet $\phi^I$, hence induces automorphic potentials $V(f_\Phi(\phi^I))$. The nontrivial target space metric $G_{IJ}$ of the kinetic term is induced by the Lie algebra structure $\mathfrak{g}$ of the group $G$. This set-up thus leads to a highly structured field theory space $\mathcal{F}(G, \Gamma)$.

A natural application of automorphic field theories arises in the context of inflation. Estimates of the effects of higher dimension operators expected to appear in the UV-completion of inflationary models suggest that these operators can make model-specific predictions unstable. This is particularly pronounced in the framework of large field
inflation, where the inflaton varies over an energy range that is super-Planckian. Unless these operators have very small coefficients higher dimension corrections will have dramatic effects on the parameters. An early discussion of these issues in the context of chaotic inflation [1] can be found in ref. [2]. A device often postulated to avoid such corrections is the existence of an inflaton shift symmetry $\phi \mapsto \phi + s$. Historically, the first model to incorporate this idea is natural inflation [3], but many modifications and extensions have been introduced in the intervening two decades, including models that aim at realizations of this symmetry in UV-complete theories [4]. Reviews of some of this work can be found in [5, 6]. This idea has received renewed attention following the possibility of a sizeable tensor ratio [7]. If a major component of this signal is of primordial origin this implies that the inflationary scale is quite high, not too far from that of GUT models [8].

In the context of the shift symmetry it is natural to ask whether it is part of a larger group that operates on the inflaton space. The existence of such a group would provide a systematic framework in which different types of invariant potentials could be classified. In the present paper such a program is initiated by formulating inflation in terms automorphic forms and their associated automorphic functions. The shift symmetry will be part of arithmetic group $\Gamma \subset G$ which characterizes the space of forms.

While classical modular forms are ubiquitous in physics, having found applications in many different fields, higher rank automorphic forms have only relatively recently entered physics. Examples involve Eisenstein series in effective actions, Siegel modular forms in the context of black hole entropy, and $GL(n)$-automorphic forms in the construction of automorphic spacetime. In the following the range of automorphic physics is extended to a cosmological context.

## 2 Automorphic inflation framework

In the present section the framework of automorphic inflation is outlined, in particular the form of the potential in terms of automorphic forms and the structure of the metric
$G_{IJ}$ of the kinetic term. The concept of automorphic forms was introduced under this name by Klein in the late 19th century [9]. In its original formulation the focus was on discrete subgroups $\Gamma$ of the modular group $\text{SL}(2, \mathbb{R})$, a framework that is too narrow for general multi-field inflation. In the following a distinction is therefore made between automorphic forms associated to higher rank groups, and modular forms, where the latter will be viewed as objects associated to $\text{SL}(2, \mathbb{R})$ (or $\text{GL}(2, \mathbb{R})$).

### 2.1 Automorphic actions

In the simplest case the concept of automorphic inflation can be formulated in the context of multi-field theories defined by an action of the form

$$
\mathcal{A}_{\text{aut}} = \int d^4 x \sqrt{-g} \left( \frac{M_{Pl}^2}{2} R - \frac{1}{2} G_{IJ} g^{\mu \nu} \partial_\mu \phi^I \partial_\nu \phi^J - V(\phi^I) \right),
$$

(1)

where the spacetime metric $g_{\mu \nu}$ is taken to be of signature $(-, +, +, +)$. The basic building blocks include potentials $V(\phi^I)$ induced by automorphic forms $f : X \rightarrow \mathbb{C}$ on the inflaton field space $X$ with coordinates $\tau = (\tau^1, ..., \tau^N)$ as

$$
V(f(\tau^I)) = L^4 F(f(\tau^I)).
$$

(2)

Here $L^4$ is an energy scale, and $F(\tau^I)$ is a dimensionless function of the inflaton multiplet expressed in terms of dimensionless variables $\tau = \phi/\mu$, where $\mu$ is a second free energy scale. The potential function $F$ should be a real function $F(f, \overline{f})$, a simple class of examples given by $F(f, \overline{f}) = (f \overline{f})^p$ for arbitrary exponents $p$. Inflationary models $\mathcal{I}$ of this type are thus characterized by a number of choices $\mathcal{I} = \mathcal{I}(G, \Gamma, F, \Phi, L, \mu)$. The space $X$ is to be taken as a bounded domain in $\mathbb{C}^n$ whose structure is constrained by the nature of the inflaton $\phi^I$. These spaces have a nontrivial geometry encoded in the metric $G_{IJ}, I, J = 1, ..., n$, which will be described in more detail further below after the nature of the space $X$ has been made more explicit.

Automorphic inflation can be formulated in more general context for theories of the form $P(\mathbb{K}, V)$, where $\mathbb{K} = G_{IJ} g^{\mu \nu} \partial_\mu \phi^I \partial_\nu \phi^J$, and $P$ is some general, not necessarily polynomial, function. Examples of this type include extension of DBI inflation [10] to the multi-field context [11].
2.2 Automorphic forms associated to reductive groups

As noted already, the generalization of the notion of automorphic forms has made the concept less precise and no standard language has emerged. In the following automorphic forms are taken to be defined on higher rank groups and distinguished from modular forms. For a physically motivated discussion it is best to focus first on the inflaton space $X$ as a symmetric bounded domain that arises via the action of an algebraic reductive group $G$ as

$$X = G/K \cdot A,$$

where $K$ is a maximal compact subgroup and $A$ is the split component. For an arithmetic subgroup $\Gamma \subset G$ the transformation behavior of the function $f$ on $X$ is determined by an automorphy factor $J(\gamma, x)$ associated to $\gamma \in \Gamma$. This function $J : G \times X \rightarrow \mathbb{C}$ is a 1-cocycle which satisfies the relation

$$J(gh, x) = J(g, hx) J(h, x)$$

for $g, h \in G$ and $x \in X$. A $J$-automorphic form $f(x)$ is induced by a certain type of group function $\Phi(g)$ as

$$f(x) = J(g, x_0) \Phi(g),$$

where $x = g \cdot x_0$ and $x_0$ is the base point of the maximal compact subgroup $K \subset G$. In order to obtain the standard transformation behavior as

$$f(\gamma x) = \epsilon(\gamma) J(\gamma, x) f(x),$$

where $\gamma \in \Gamma \subset \Gamma$ and $x \in X$, the functions $\Phi$ have to satisfy a number of constraints as follows.

The first condition specifies the behavior of the functions $\Phi$ with respect to the action of an arithmetic subgroup $\Gamma \subset G$ on the forms $\Phi$, which in general is allowed to transform with a character $\epsilon$ as $\Phi(\gamma g) = \epsilon(\gamma) \Phi(g)$, for $\gamma \in \Gamma$. The second condition restricts the behavior of $\Phi$ with respect to the action of the maximal compact subgroup $K \subset G$ by requiring that the forms span a finite dimensional space under this action, i.e. $\dim(\Phi(gk))_{k \in K} < \infty$. A final transformation behavior is imposed with respect to
the center $Z(G)$ of the group $G$. This is encoded in terms of a central character $\omega$ on $Z$ as $\Phi(ga) = \omega(a)\Phi(g)$. A fourth condition, structurally different in type from the transformation constraints above, generalizes the eigenvalue constraints of classical modular forms, either of holomorphic or of Maaβ type. In the higher rank case this extension is implemented via the universal enveloping algebra $U(\mathfrak{g})$ associated to the Lie algebra $\mathfrak{g}$ of the group $G$, by requiring that for automorphic forms the space of functions generated by the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ is finite dimensional. This can be formulated more explicitly in terms of differential constraints by mapping $U(\mathfrak{g})$ into the algebra of left invariant differential operators. This correspondence is generated by the action of the Lie algebra on groups functions $\Phi$ as

$$D_V\Phi(g) = \frac{d}{dt}\Phi(ge^{tv}) \bigg|_{t=0}, \quad V \in \mathfrak{g}. \quad (7)$$

While meromorphic forms are of importance, finiteness results have been proven by imposing a convergence constraint selects an important subclass of function via the condition that there exists a constant $C$ and an integer $n$ such that $|\Phi(g)| \leq C||g||^n$. These include in particular the cusp forms. More details concerning the conceptual framework of automorphic forms can be found in the reviews of ref. [12].

The notion of automorphic inflation just defined in terms of the group theoretic framework associates different models to each reductive group $G$ and $\Gamma \subset G$ via the choice of an automorphic form $\Phi$ and the function $F$. Once a symmetric bounded domain has been chosen, together with an arithmetic discrete subgroup $\Gamma \subset G$, the space of automorphic forms is finite dimensional [13, 14]. This implies that modulo the function $F$ each choice $(G, \Gamma)$ leads to a finite dimensional theory space.

### 2.3 The kinetic term of automorphic inflation

The kinetic term of automorphic field theory is characterized by a nontrivial metric on the target space $X = G/K$ of the inflaton multiplet. This metric is induced in terms of the adjoint representation of the Lie algebra $\mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$, defined as $\text{ad}_V(W) = [V, W]$, via the Cartan-Killing form $B(X, Y) = \text{tr} \text{ad}_X\text{ad}_Y$ on $\mathfrak{g}$ of $G$, which is isomorphic to
the tangent space $T_eG$ at the identity element $e$. The inner product on $T_eG$ can be transported to other tangent space $T_gG$ by the differential $dL_g$ of the left translation map $L_g$. The pullback via $L_g^{-1}$ can be used to define the inner product on $T_gG$ as

$$\langle V, W \rangle_g = \langle dL_g^{-1}V, dL_g^{-1}W \rangle_e,$$

for $V, W \in T_gG$. The associated metric descends to the quotient $X = G/K$.

The coordinate form $G_{IJ}$ of the metric on $X$ can be obtained explicitly from the Iwawasa decomposition $G = NAK$ of $G$, defined as a refinement of the Cartan decomposition $g = k \oplus p$, where $k$ is the Lie algebra of the maximal compact subgroup $K$. By choosing a maximal abelian subspace $a$ of $p$ the decomposition $p = a \oplus n$ leads to factors $NAK$ of the group $G$, where $N$ is nilpotent and $A$ is abelian. The map from $G$ to $X$ can be made explicit by choosing the maximal compact subgroup $K$ to be the isotropy group of a point $x_0 \in X$, leading to $gx_0 = nakx_0 = nax_0$.

In the remainder of this paper this framework will be illustrated in the context of modular inflation, with $j$-inflation as a particular example.

## 3 Modular inflation

The simplest theories of automorphic inflation can be formulated in the context of classical modular functions and forms, which are functions understood to be defined relative to arithmetic subgroups of the modular group $SL(2, \mathbb{Z})$, such as the Hecke congruence subgroups $\Gamma_0(N)$ of level $N$, the principal congruence subgroups $\Gamma(N)$, or other similar groups. The reductive group here is $G = SL(2, \mathbb{R})$, and the maximal compact group $K = SO(2, \mathbb{R})$ leads to the domain $X$ which can be viewed as by the upper halfplane

$$\mathcal{H} = SL(2, \mathbb{R})/SO(2, \mathbb{R}).$$

Modular forms $\Phi$ defined as group function on $SL(2, \mathbb{R})$ are constrained by $\Gamma$–equivariance to admit a character $\epsilon_N$ such that

$$\Phi(\gamma g) = \epsilon_N(\gamma)\Phi(g),$$

(10)
where $\epsilon_N$ can be the trivial character.

The $K-$finiteness condition takes a more precise form in the modular case because the compact subgroup is abelian, hence its irreducible representations are one-dimensional. The resulting character determines the weight $w$ of the form via

$$\Phi(k_\theta g) = e^{i w \theta} \Phi(g),$$

where $k_\theta \in \text{SO}(2, \mathbb{R})$ is a rotation by $\theta$. The center of PSL(2, $\mathbb{R}$) is trivial, hence the third constraint is empty.

The differential constraint also simplifies and can be made more explicit because the rank of SL(2, $\mathbb{R}$) is one, hence the center of the universal enveloping algebra is one-dimensional. An element of $U(g)$ which is always in the center is the Casimir element $C = h_{ij} X_i X_j$, where $h_{ij}$ is the inverse Killing metric $h_{i j}$, defined with respect to a basis $\{X_i\} \subset g$ as

$$h_{ij} = \text{tr } \text{ad} X_i \text{ad} X_j.$$  \hfill (12)

Hence for SL(2, $\mathbb{R}$) the center $Z(g)$ is a polynomial algebra in the single generator $C$. Mapping $C$ to a differential operator $\Delta_C$ therefore leads to an operator which is a multiple of the Laplace-Beltrami operator

$$\Delta_g = \frac{1}{\sqrt{g}} \partial_i g^{-1} \sqrt{g} \partial_j.$$  \hfill (13)

The eigenform constraint thus distinguishes between holomorphic and Maass forms in terms of the structure of the eigenvalue.

Modular forms $f(\tau)$ on the upper halfplane $\mathcal{H} = G/K$, characterized by their weight $w$, level $N$ and character $\epsilon$, are obtained from group functions $\Phi$ on SL(2, $\mathbb{R}$) by considering the base point $x_0 = i$ of the maximal compact subgroup $K$ as

$$f(\tau) = J(g, i) \Phi(g),$$

where $\tau = g \cdot i$. For elements $\gamma$ in the discrete subgroup $\Gamma(N)$ of level $N$ the automorphy factor is

$$J(\gamma, \tau) = (c\tau + d)^w, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N),$$

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and the transformation behavior is given by

\[
f(\gamma \tau) = \epsilon_N(\gamma)(c\tau + d)^w f(\tau), \quad \gamma \tau = \frac{a\tau + b}{c\tau + d}
\]  

(16)

Modular functions can be obtained by considering quotients \( f(\tau) = g(\tau)/h(\tau) \) of modular forms \( g, h \) of the same weight with respect to the same discrete groups. Such functions then induce inflationary potentials of the type \( V(f) = L^4 F(f, \overline{f}) \). The inflaton doublet \( \phi = (\phi^1, \phi^2) \) is parametrized by \( \phi = \mu \tau \) and the target space is equipped with the hyperbolic metric

\[
ds^2 = \frac{d\tau d\overline{\tau}}{(\text{Im } \tau)^2}.
\]

(17)
i.e. the metric of the kinetic term is conformally flat with

\[
G_{IJ} = \left( \frac{\mu}{\phi^2} \right)^2 \delta_{IJ}.
\]

(18)

The general action reduces to

\[
A_{\text{mod}} = \int d^4x \sqrt{-g} \left( \frac{M_{\text{Pl}}^2}{2} R - G_{\tau \overline{\tau}} g^{\mu \nu} \partial_\mu \tau \partial_\nu \overline{\tau} - V(\tau, \overline{\tau}) \right).
\]

(19)

This set-up of modular inflation provides an extensive framework in which finite dimensional Hilbert spaces of modular forms can be used to construct modular functions by considering quotients \( f/g \) of modular forms \( f, g \) of the same weight. In this way modular functions can be constructed in terms of forms that arise naturally in a variety of different contexts.

4 \( j \)-Inflation

In this final section the framework of modular inflation is exemplified by considering an inflaton potential determined by a modular function with respect to the full modular group. The space of classical modular forms with respect to \( \text{SL}(2, \mathbb{Z}) \) is in principle completely known since it is spanned by only two modular forms,

\[
M_4(\text{SL}(2, \mathbb{Z})) = \langle E_4, E_6 \rangle,
\]

(20)
where \( E_4 \) and \( E_6 \) are the Eisenstein series of weight 4 and 6. Eisenstein series of weight \( w \) can be defined in terms of the divisor function \( \sigma_m(n) = \sum_{d\mid n} d^m \) as

\[
E_w = 1 - \frac{2w}{B_w} \sum_n \sigma_{w-1}(n) q^n, \tag{21}
\]

where \( q = e^{2\pi i \tau}, \tau \in \mathcal{H} \), and \( B_w \) are the Bernoulli numbers, which are related via Euler’s formula to the Riemann zeta function as

\[
\zeta(w) = -\frac{(2\pi i)^w B_w}{2w!}, \tag{22}
\]

where \( w > 0 \) is even. The forms \( E_4 \) and \( E_6 \) are the unique elements of \( M_4(\text{SL}(2, \mathbb{Z})) \) and \( M_6(\text{SL}(2, \mathbb{Z})) \) respectively, up to normalization.

Holomorphic modular functions can be constructed from modular forms by considering quotients such that the denominator form is non-vanishing in the upper halfplane. A venerable modular form with this property is the discriminant function \( \Delta \in S_{12}(\text{SL}(2, \mathbb{Z})) \), the unique cusp form of weight twelve, up to normalization, defined by

\[
\Delta(\tau) = \eta(\tau)^{24}, \tag{23}
\]

where \( \eta(\tau) \) is the Dedekind function

\[
\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n). \tag{24}
\]

The form \( \Delta \) arises in many different contexts, for example the partition function of the bosonic string, or as a geometric object, but is here considered as a building block of modular functions.

Since \( \Delta \) does not vanish on \( \mathcal{H} \), modular functions without poles in \( \mathcal{H} \) can be obtained by considering numerators of weight twelve. One of these is \( E_4^3 \), leading to the elliptic modular function

\[
j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)}, \tag{25}
\]

which is perhaps the most prominent modular function, going back to Kronecker’s Jugendtraum, in particular his work on the class numbers of imaginary quadratic fields \([15]\), and Hermite’s work on the quintic equation \([16]\). Both \( E_4 \) and \( \Delta \) are modular forms...
with respect to the full modular group, hence $j$ is a holomorphic function invariant under $\text{SL}(2, \mathbb{Z})$.

Given the function $j$ one can consider $j$-inflationary models defined by potentials $V = L^4 F(j, \overline{j})$. The simplest cases are obtained by setting e.g. $F(f, \overline{f}) = (f \overline{f})^p$. In the following the focus will be on the model with $p = 1$. As a first step toward a phenomenological analysis it is most convenient to consider the slow-roll approximation since this allows an analytic discussion of many aspects of the model, leaving a more precise numerical analysis of the exact dynamics to a more detailed discussion. Important parameters emphasized by the WMAP and Planck analyses are given by the scalar spectral index $n_s$, defined in terms of the scalar power spectrum $P_s(k)$ as

$$n_s = 1 + \frac{d \ln P_s}{d \ln k},$$

and the tensor-to-scalar ratio $r$ defined via the tensor power spectrum $P_t$ as

$$r = \frac{P_t}{P_s}.$$

Similar to $n_s$ one considers the tensor power spectral index defined as

$$n_t = \frac{d \ln P_t}{d \ln k}.$$

The power spectrum of the curvature perturbation can be written in multi-field inflation in terms of the number of $e$-foldings $N$ as [17]

$$P_s = \left( \frac{H}{2\pi} \right)^2 G^{IJ} \frac{\partial N}{\partial \phi^I} \frac{\partial N}{\partial \phi^J},$$

while the tensor power spectrum was determined by Starobinsky [18] to be given by the Hubble parameter

$$P_t = \frac{2}{\pi^2} \left( \frac{H}{M_{\text{Pl}}} \right)^2.$$

For the inflationary model based on the $j$-function the parameters $n_s$ and $r$ can be computed analytically in the slow roll approximation in terms of the dimensionless parameters $\epsilon_I$ and $\eta_{IJ}$, defined as

$$\epsilon_I = M_{\text{Pl}} \frac{\partial_I V}{V}, \quad \eta_{IJ} = M_{\text{Pl}}^2 \frac{\nabla_I \nabla_J V}{V}.$$
where $\partial I = \partial V/\partial \phi^I$, $\nabla_I$ denotes the covariant derivative in field space, and $M_{Pl} = 1/\sqrt{8\pi G}$ is the reduced Planck mass. If the $\epsilon_I$ are sufficiently small the universe accelerates, leading to inflation.

In multi-field inflation the spectral indices can be obtained in the slow-roll approximation as

\[ n_s = 1 - 3\epsilon_I\epsilon^I + 2\eta_{IJ}\epsilon^I\epsilon^J \]  

(32)

and

\[ n_t = -\epsilon_I\epsilon^I, \]  

(33)

while the tensor-to-scalar ratio takes the form $r = -8n_t$.

Here the indices are raised and lowered with the field space metric $G_{IJ}$ and its inverse. For $I = 1$ these reduce to the known single-field results $n_s = 1 - 6\epsilon_V + 2\eta_V$ and $r = 16\epsilon_V$ where $\epsilon_V = \frac{1}{2}\epsilon_I\epsilon^I$ and $\eta_V \equiv M_{Pl}^2 V''/V = \eta_{11}$.

The observables $P_s(k_p)$, $n_s$ and $r$ can be expressed in the case of $j$-inflation in terms of the modular forms $E_4, E_6$ as well as the quasi-modular form $E_2$. The height $L$ of the potential does not enter the scalar spectral index $n_s$ and the tensor ratio $r$, leading to expressions $n_s = n_s(\mu, E_2, E_4, E_6)$ and $r = r(\mu, E_2, E_4, E_6)$. With these explicit results the spectral index and the tensor ratio can be evaluated in terms of the inflaton variable $\tau_s = \phi_s/\mu$ at horizon crossing in dependence of the scale $\mu$. Values of the dimensionless inflaton $\tau$ in the neighborhood of the zero of $E_6$ lead to $(P_s(k_p), n_s, r)$-parameters that are consistent with the observational results reported by the Planck collaboration in ref. [19].

It is similarly possible to constrain further parameters of these models for which WMAP and Planck have provided limits. Among these are the running $\alpha_s$ of the spectral index, in the presence of which the scalar power spectrum is parametrized as [19]

\[ P_s(k) = P_s(k_p) \left( \frac{k}{k_p} \right)^{n_s - 1 + \gamma_s}, \]  

(34)

with

\[ \gamma_s = \frac{\alpha_s}{2} \ln \frac{k}{k_p}, \quad \alpha_s = \frac{dn_s}{d\ln k}, \]  

(35)
as well as the non-Gaussianity parameter $f_{NL}$ of the bispectrum $B(\vec{k}_1, \vec{k}_2, \vec{k}_3)$ and the parameters $\tau_{NL}$ and $g_{NL}$ of the trispectrum $T(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4)$. These parameters can be expressed in terms of the Eisenstein series $E_w$, $w = 2, 4, 6$ as well, leading again to analytic expressions, and the results can be compared with the limits imposed by the Planck collaboration [20]. These results involve higher derivative features of the potential and their more involved analysis, including the effects of the isocurvature perturbations, typically encoded in the transfer functions, is left to a more extensive future discussion.

The model of $j$-inflation arises within class of functions comprised of genus zero modular functions. Within the framework of modular inflation these provide a reservoir of different models that lend themselves to a similar analysis. It is also possible to consider modular inflation based on forms of higher level by reducing the symmetry group to some congruence subgroup of level $N$.

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