Waveguides with combined Dirichlet and Robin boundary conditions

P. Freitas\textsuperscript{1} and D. Krejčiřík\textsuperscript{2}

\textsuperscript{1} Department of Mathematics, Faculdade de Motricidade Humana (TU Lisbon) and Group of Mathematical Physics of the University of Lisbon, Complexo Interdisciplinar, Av. Prof. Gama Pinto 2, P-1649-003 Lisboa, Portugal
E-mail: freitas@cii.fc.ul.pt

\textsuperscript{2} Department of Theoretical Physics, Nuclear Physics Institute, Academy of Sciences, 250 68 Řež near Prague, Czech Republic
E-mail: krejcirik@ujf.cas.cz

Abstract

We consider the Laplacian in a curved two-dimensional strip of constant width squeezed between two curves, subject to Dirichlet boundary conditions on one of the curves and variable Robin boundary conditions on the other. We prove that, for certain types of Robin boundary conditions, the spectral threshold of the Laplacian is estimated from below by the lowest eigenvalue of the Laplacian in a Dirichlet-Robin annulus determined by the geometry of the strip. Moreover, we show that an appropriate combination of the geometric setting and boundary conditions leads to a Hardy-type inequality in infinite strips. As an application, we derive certain stability of the spectrum for the Laplacian in Dirichlet-Neumann strips along a class of curves of sign-changing curvature, improving in this way an initial result of Dittrich and Krčz \cite{9}.

MSC 2000: 35P15; 58J50; 81Q10.

Keywords: Dirichlet and Robin boundary conditions; Eigenvalues in strips and annuli; Hardy inequality; Laplacian; Waveguides.
1 Introduction

The Laplacian in an unbounded tubular region $\Omega$ has been extensively studied as a reasonable model for the Hamiltonian of electronic transport in long and thin semiconductor structures called quantum waveguides. We refer to \cite{10, 29} for the physical background and references. In this model, it is more natural to consider Dirichlet boundary conditions on $\partial \Omega$ corresponding to a large chemical potential barrier (cf \cite{17, 20, 10}).

However, Neumann boundary conditions or a combination of Dirichlet and Neumann boundary conditions have been also investigated. We refer to \cite{27, 28} for the former and to \cite{9, 30, 25} for the latter. Moreover, these types of boundary conditions are relevant to other physical systems (cf \cite{13, 8, 21}).

Although we are not aware of any work in the literature where more general boundary conditions have been considered in the case of quantum waveguides, it is possible to think also of Robin boundary conditions as modelling impenetrable walls of $\Omega$ in the sense that there is no probability current through the boundary. Furthermore, Robin boundary conditions may in principle be relevant for different types of interphase in a solid.

Moreover, the interplay between boundary conditions, geometry and spectral properties is an interesting mathematical problem in itself. To illustrate this, let us recall that it has been known for more than a decade that the curved geometry of an unbounded planar strip of uniform width may produce eigenvalues below the essential spectrum. We refer to the pioneering work \cite{17} of Exner and Řeka and the sequence of papers \cite{20, 32, 10, 25, 4} for the existence results under rather simple and general geometric conditions.

However, it has not been noticed until the recent letter \cite{9} of Dittrich and Kříž that the existence of eigenvalues in fact depends heavily on the geometrical setting provided the uniform Dirichlet boundary conditions are replaced by a combination of Dirichlet and Neumann ones. In particular, the discrete spectrum may be eliminated provided the Dirichlet-Neumann strip is “curved appropriately”, i.e., the Neumann boundary condition is imposed on the “locally shorter” boundary curve.

Recently, it has also been shown that the discrete spectrum may be eliminated by adding a local magnetic field perpendicular to a planar Dirichlet strip \cite{11, 8}, by embedding the strip into a curved surface \cite{24} or by twisting a three-dimensional Dirichlet tube of non-circular cross-section \cite{12}.

The aim of the present paper is to examine further the interplay between boundary conditions, geometry and spectral properties in the case of $\Omega$ being a planar strip with a combination of Dirichlet and (variable) Robin boundary conditions on $\partial \Omega$. Our main result is a lower bound to the spectral threshold of the Laplacian in a (bounded or unbounded) Dirichlet-Robin strip. This enables us to prove quite easily non-existence results about the discrete spectrum for certain waveguides, and generalize in this way the results of Dittrich and Kříž \cite{9}. Moreover, we show that certain combinations of boundary conditions and geometry lead to Hardy-type inequalities for the Laplacian in unbounded strips. These inequalities are new in the theory of quantum waveguides with
combined boundary conditions. As an application, we further extend the class of Dirichlet-Neumann strips with empty discrete spectrum.

2 Scope of the paper

In this section we precise the problem we deal with in the present paper and state our main results.

2.1 The model

Given an open interval \( I \subseteq \mathbb{R} \) (bounded or unbounded), let \( \Gamma \equiv (\Gamma^1, \Gamma^2) : I \to \mathbb{R}^2 \) be a unit-speed \( C^2 \)-smooth plane curve. We assume that \( \Gamma \) is an embedding. The function \( N = (-\dot{\Gamma}^2, \dot{\Gamma}^1) \) defines a unit normal vector field along \( \Gamma \) and the couple \((\dot{\Gamma}, N)\) gives a distinguished Frenet frame (cf [23, Chap. 1]). The curvature of \( \Gamma \) is defined through the Serret-Frenet formulae by \( \kappa := \det(\dot{\Gamma}, \ddot{\Gamma}) \); it is a continuous function of the arc-length parameter. We assume that \( \kappa \) is bounded. It is worth noticing that the curve \( \Gamma \) is fully determined (except for its position and orientation in the plane) by the curvature function \( \kappa \) alone (cf [26, Sec. II.20]).

Let \( a \) be a given positive number. We define the mapping

\[
\mathcal{L} : I \times [-a, a] \to \mathbb{R}^2 : \{(s, t) \mapsto \Gamma(s) + N(s)t\}
\]

and make the hypotheses that

\[
\|\kappa\|_{\infty} a < 1 \quad \text{and} \quad \mathcal{L} \quad \text{is injective.}
\]

Then the image

\[
\Omega := \mathcal{L}(I \times (-a, a))
\]

has a geometrical meaning of an open non-self-intersecting strip, contained between the parallel curves

\[
\Gamma_{\pm} := \mathcal{L}(I \times \{\pm a\})
\]

at the distance \( a \) from \( \Gamma \), and, if \( \partial I \) is not empty, the straight lines \( L_- := \mathcal{L}\{\inf I\} \times (-a, a) \) and \( L_+ := \mathcal{L}\{\sup I\} \times (-a, a) \). The geometry is set in such a way that \( \kappa > 0 \) implies that the parallel curve \( \Gamma_+ \) is locally shorter than \( \Gamma_- \), and vice versa. We refer to [12, App. A] for a sufficient condition ensuring the validity of the second hypothesis in (2).

Given a bounded continuous function \( \tilde{\alpha} : \Gamma_+ \to \mathbb{R} \), let \(-\Delta_{\kappa, \alpha}\) denote the (non-negative) Laplacian on \( L^2(\Omega) \), subject to uniform Dirichlet boundary conditions on the parallel curve \( \Gamma_- \), uniform Neumann boundary conditions on \( L_- \cup L_+ \) (i.e. none if \( \partial I \) is empty) and the Robin boundary conditions of the form

\[
\frac{\partial u}{\partial N} + \tilde{\alpha} u = 0 \quad \text{on} \quad \Gamma_+,
\]

where \( u \in D(-\Delta_{\kappa, \alpha}) \). Hereafter we shall rather use \( \alpha := \tilde{\alpha}(\mathcal{L}(\cdot, a)) \), a function on \( I \). Notice that the choice \( \alpha = 0 \) corresponds to uniform Neumann boundary conditions.
conditions on $\Gamma_+$ and $\alpha \to +\infty$ approaches uniform Dirichlet boundary conditions on $\Gamma_+$; for this reason, we shall sometimes use "$\alpha = +\infty$" to refer to the latter. The Laplacian $-\Delta_{\kappa,\alpha}$ is properly defined in Section 3 below by means of a quadratic-form approach.

### 2.2 A lower bound to the spectral threshold

If the curvature $\kappa$ is a constant function, then the image $\Omega$ can be identified with a segment of an annulus or a straight strip. We prove that, in certain situations, this constant geometry minimizes the spectrum of $-\Delta_{\kappa,\alpha}$, within all admissible functions $\kappa$ and $\alpha$ considered as parameters.

More precisely, let us denote by $D(r)$ the open disc of radius $r > 0$ and let $A(r_1, r_2) := D(r_2) \setminus D(r_1)$ be an annulus of radii $r_2 > r_1 > 0$. Abusing the notation for $\kappa$ and $\alpha$ slightly, we introduce a function $\lambda : (-a, a) \times \mathbb{R} \to \mathbb{R}$ by means of the following definition:

**Definition 1.** Given two real numbers $\alpha$ and $\kappa$, with $\kappa$ in $(-1/a, 1/a)$, we denote by $\lambda(\kappa, \alpha)$ the spectral threshold of the Laplacian on

$$A_\kappa := \begin{cases} 
A(|\kappa|^{-1} - a, |\kappa|^{-1} + a) & \text{if } \kappa \neq 0, \\
\mathbb{R} \times (-a, a) & \text{if } \kappa = 0,
\end{cases}$$

subject to uniform Dirichlet boundary condition on

$$\begin{align*}
\partial D(\kappa^{-1} + a) & \quad \text{if } \kappa > 0, \\
\mathbb{R} \times \{-a\} & \quad \text{if } \kappa = 0, \\
\partial D(|\kappa|^{-1} - a) & \quad \text{if } \kappa < 0,
\end{align*}$$

and uniform Robin boundary conditions of the type (4) (with $\alpha$ constant and $N$ being the outward unit normal on $\partial A_\kappa$) on the other connected part of the boundary.

The most general result of the present paper reads as follows:

**Theorem 1.** Given a positive number $a$ and a bounded continuous function $\kappa$, let $\Omega$ be the strip defined by (3) with (1) and satisfying (2). Let $\alpha$ be a bounded continuous function. Then

$$\inf \sigma(-\Delta_{\kappa,\alpha}) \geq \lambda(\inf \kappa, \inf \alpha) \quad \text{provided } \kappa \leq 0 \text{ or } \alpha \leq 0. \quad (5)$$

The lower bound $\lambda(\kappa, \alpha)$ as a function of curvature $\kappa$ for certain values of $\alpha$ is depicted in Figure 1. We prove the following properties which are important for (5):

**Theorem 2.** $\lambda$ satisfies the following properties:

(i) $\forall \kappa \in (-1/a, 1/a), \quad \alpha \mapsto \lambda(\kappa, \alpha) : \mathbb{R} \to \mathbb{R}$ is continuous and increasing,

(ii) $\forall \alpha \in \mathbb{R}, \quad \kappa \mapsto \lambda(\kappa, \alpha) : (-1/a, 1/a) \to \mathbb{R}$ is continuous,
Figure 1: Dependence of the lower bound $\lambda(\kappa, \alpha)$ on the curvature $\kappa$ for $a = 1$ and different values of $\alpha$. (All curves meet at $\kappa = 1/a$, the small gap for the curve with $\alpha = +\infty$ is due to a numerical inaccuracy.)

(iii) $\forall \alpha \in \mathbb{R}$, $\kappa \mapsto \lambda(\kappa, \alpha) : (-1/a, 0] \to \mathbb{R}$ is increasing,

(iv) $\forall \alpha \in (-\infty, 0]$, $\kappa \mapsto \lambda(\kappa, \alpha) : (-1/a, 1/a) \to \mathbb{R}$ is increasing,

(v) $\forall \alpha \in \mathbb{R}$, \[ \lim_{\kappa \to -1/a} \lambda(\kappa, \alpha) = \nu(\alpha), \quad \lim_{\kappa \to 1/a} \lambda(\kappa, \alpha) = \nu(+\infty), \]

where $\nu(\alpha)$, with $\alpha \in \mathbb{R} \cup \{+\infty\}$, denotes the first eigenvalue of the Laplacian in the disc $D(2a)$, subject to uniform Robin boundary conditions of the type (4) if $\alpha \in \mathbb{R}$ (with $\alpha$ constant and $N$ being the outward unit normal on $\partial D(2a)$) or uniform Dirichlet boundary conditions if $\alpha = +\infty$.

Of course, $\nu(+\infty) = J_{0,1}^2/(2a)^2$, where $J_{0,1}$ denotes the first zero of the Bessel function $J_0$, and $\nu(0) = 0$.

Theorem 1 is a natural continuation of efforts to estimate the spectral threshold in curved Dirichlet tubes [2, 14]. More specifically, in the recent article [14], Exner and the present authors established a lower bound of the type (3) for the case $\alpha = +\infty$, i.e. for pure Dirichlet strips (the results in that paper are more general in the sense that the tubes considered there were multi-dimensional and of arbitrary cross-section). Namely,

\[ \inf \sigma(-\Delta_{\kappa,+\infty}) \geq \lambda(||\kappa||_{\infty}, +\infty), \]

where $\lambda(\kappa, +\infty)$ is the spectral threshold of the Dirichlet Laplacian in $A_{\kappa}$. It is also established in [14] that $\kappa \mapsto \lambda(\kappa, +\infty)$ is an even function, decreasing on $[0,1/a)$, reaching its maximum $\pi^2/(2a)^2$ for $\kappa = 0$ (a straight strip) and approaching its infimum $\nu(+\infty)$ as $\kappa \to 1/a$ (a disc). The style and the main
idea \(i.e.\) the intermediate lower bound \((\mathbf{14})\) below) of the present paper are similar to that of \([14]\). However, we have to use different techniques to establish the properties of \(\lambda\) (Theorem \(2\)), and consequently \((\mathbf{5})\).

### 2.3 A Hardy inequality in infinite strips

Theorem \(1\) is optimal in the sense that the lower bound \((\mathbf{3})\) is achieved by a strip (along a curve of constant curvature). On the other hand, since the minimizer is bounded if the curvature is non-trivial, a better lower bound is expected to hold for unbounded strips. Indeed, in certain unbounded situations, we prove that the lower bound of Theorem \(1\) can be improved by a Hardy-type inequality.

Let us therefore consider the infinite case \(I = \mathbb{R}\) in this subsection. Let \(\alpha_0\) be a given real number. If \(\kappa\) is equal to zero identically \(i.e.\) \(\Omega\) is a straight strip and \(\alpha\) is equal to \(\alpha_0\) identically, it is easy to see that

\[
\sigma(-\Delta_{0,\alpha_0}) = \sigma_{\text{ess}}(-\Delta_{0,\alpha_0}) = \left[\lambda(0,\alpha_0), \infty\right).
\]

Although the results below hold under more general conditions about vanishing of \(\kappa\) and the difference \(\alpha - \alpha_0\) at infinity \(i.e.\) Section \(\mathbf{7}\) below), for simplicity, we restrict ourselves to strips which are deformed only locally in the sense that \(\kappa\) and \(\alpha - \alpha_0\) have compact support. Under these hypotheses, it is easy to verify that the essential spectrum is preserved:

\[
\sigma_{\text{ess}}(-\Delta_{\kappa,\alpha}) = \left[\lambda(0,\alpha_0), \infty\right).
\]

A harder problem is to decide whether this interval exhausts the spectrum of \(-\Delta_{\kappa,\alpha}\), or whether there exists discrete eigenvalues below \(\lambda(0,\alpha_0)\).

On the one hand, Dittrich and Krříž \([9]\) showed that the curvature which is negative in a suitable sense creates eigenvalues below the threshold \(\lambda(0,0)\) in the uniform Dirichlet-Neumann case \(i.e.\) in the case \(\alpha = 0\) identically. For instance, using, in analogy to \([9]\), a modification of the “generalized eigenfunction” of \(-\Delta_{0,\alpha_0}\) corresponding to \(\lambda(0,\alpha_0)\) as a test function, it is straightforward to extend a result of \([9]\) to the case of uniform Robin boundary conditions:

**Proposition 1.** \(\text{Let } I = \mathbb{R}.\) If \(\alpha(s) = \alpha_0\) for all \(s \in \mathbb{R}\) and \(\int_{\mathbb{R}} \kappa(s) \, ds < 0\), then

\[
\inf \sigma(-\Delta_{\kappa,\alpha_0}) < \lambda(0,\alpha_0).
\]

In particular, Proposition \(\mathbf{1}\) together with \((\mathbf{7})\) implies that the discrete spectrum of \(-\Delta_{\kappa,\alpha_0}\) exists if the strip is appropriately curved and asymptotically straight. Notice also that the discrete spectrum may be created by variable \(\alpha\) even if \(\Omega\) is straight \(i.e.\) \([15, 16]\) for this type of results in a similar model.

On the other hand, Dittrich and Krříž \([9]\) showed that the spectrum of \(-\Delta_{\kappa,0}\) coincides with the interval \((\mathbf{7})\) with \(\alpha_0 = 0\) provided the curvature \(\kappa\) is non-negative and of compact support. More precisely, they proved that

\[
\inf \sigma(-\Delta_{\kappa,0}) \geq \lambda(0,0) \quad \text{provided } \kappa \geq 0,
\]
which implies the result in view of (7). Of course, not only the lower bound (8) is contained in our Theorem 1, but the latter also generalizes the former to variable Robin boundary conditions:

**Corollary 1.** Let $I = \mathbb{R}$ and assume that $\kappa$ and $\alpha - \alpha_0$ have compact support. Under the hypotheses of Theorem 1,

$$\sigma(-\Delta_{\kappa, \alpha}) = \sigma_{\text{ess}}(-\Delta_{\kappa, \alpha}) = [\lambda(0, \alpha_0), \infty) \quad \text{if} \quad \kappa \geq 0, \quad \alpha_0 \leq \alpha \leq 0.$$

Apart from this generalization, Theorem 1 provides an alternative and, we believe, more elegant, proof of (8). Indeed, the proof of Dittrich and Kríž in [9] is very technical, based on a decomposition of $-\Delta_{\kappa, 0}$ into an orthonormal basis and an analysis of solutions of Bessel-type to an associated ordinary differential operator, while the proof of Theorem 1 does not require any explicit solutions whatsoever.

Furthermore, we obtain a stronger result, namely, that a Hardy-type inequality actually holds true in positively curved Dirichlet-Robin strips:

**Theorem 3.** Let $I = \mathbb{R}$. Given a positive number $a$ and a bounded continuous function $\kappa$, let $\Omega$ be the strip defined by (3) with (1) and satisfying (2). Let $\alpha$ be a bounded continuous function such that $\alpha_0 \leq \alpha \leq 0$. Assume that $\kappa$ is non-negative and that either one of $\kappa$ or $\alpha - \alpha_0$ is not identically equal to zero. Then, for any $s_0$ such that $\kappa(s_0) > 0$ or $\alpha(s_0) > \alpha_0$, we have

$$-\Delta_{\kappa, \alpha} \geq \lambda(0, \alpha_0) + \frac{c}{(\rho \circ L^{-1})^2}$$

in the sense of quadratic forms (cf (30) below). Here $c$ is a positive constant which depends on $s_0$, $a$, $\kappa$, and $\alpha$, $\rho(s, t) := \sqrt{1 + (s - s_0)^2}$ and $L$ is given by (1).

It is possible to find an explicit lower bound for the constant $c$; we give an estimate in (29) below.

Theorem 3 implies that the presence of a positive curvature or of suitable Robin boundary conditions represents a repulsive interaction in the sense that there is no spectrum below $\lambda(0, \alpha_0)$ for all small potential-type perturbations having a sufficiently fast decay at infinity. This provides certain stability of the spectrum of the type established in Corollary 1.

Moreover, in the uniform Dirichlet-Neumann case, we use Theorem 3 to show that the spectrum is stable even if $\kappa$ is allowed to be negative:

**Corollary 2.** Given a positive number $a$ and a bounded continuous function $\kappa$ of compact support, let $\Omega$ be the strip defined by (3) with (1) and satisfying (2). Assume that

$$|\kappa_-| \leq \varepsilon \quad \text{with} \quad \varepsilon \geq 0,$$

while $\kappa_+$ is independent of $\varepsilon$ and not identically equal to zero. Then there exists a positive number $\varepsilon_0$ such that for all $\varepsilon \leq \varepsilon_0$ we have

$$\sigma(-\Delta_{\kappa, 0}) = \sigma_{\text{ess}}(-\Delta_{\kappa, 0}) = [\lambda(0, 0), \infty).$$

Here $\varepsilon_0$ depends on $a$, $\kappa_+$ and $I$. 7
Corollary 2 follows as a consequence of (7) and the Hardy inequality (32) below. This generalizes a result of [9] to strips with sign-changing curvature.

2.4 Contents

The present paper is organized as follows. In the following section we introduce the Laplacian $-\Delta_{\kappa,\alpha}$ in the curved strip $\Omega$ by means of its associated quadratic form and express it in curvilinear coordinates defined by (1). We obtain in this way an operator of the Laplace-Beltrami form in the straight strip $I \times (-a,a)$.

In Section 4 we show that the structure of the Laplace-Beltrami operator leads quite easily to a “variable” lower bound (14), expressed in terms of the function $\lambda$ of Definition 1. We call this lower bound “intermediate” since this and Theorem 2 imply Theorem 1 at once.

In Section 5 we prove Theorem 2 using a combination of a number of techniques, such as the minimax principle, the maximum principle, perturbation theory, etc.

Section 6 is devoted to infinite strips, namely, to the proofs of Theorem 3 and its Corollary 2. The former is based on an improved intermediate lower bound, Theorem 2 and the classical one-dimensional Hardy inequality.

In the closing section we discuss possible extensions and refer to some open problems.

3 The Laplacian

The Laplacian $-\Delta_{\kappa,\alpha}$ is properly defined as follows. We introduce on the Hilbert space $L^2(\Omega)$ the quadratic form $Q_{\kappa,\alpha}$ defined by

$$Q_{\kappa,\alpha}[u] := \int_{\Omega} |\nabla u(x)|^2 \, dx + \int_{\Gamma_+} \tilde{\alpha}(\sigma) |u(\sigma)|^2 \, d\sigma,$$

$$u \in D(Q_{\kappa,\alpha}) := \{ u \in W^{1,2}(\Omega) \mid u(\sigma) = 0 \text{ for a.e. } \sigma \in \Gamma_- \},$$

where $u(\sigma)$ with $\sigma \in \Gamma_+ \cup \Gamma_-$ is understood as the trace of the function $u$ on that part of the boundary $\partial \Omega$ (cf Remark 1 below). The associated sesquilinear form is symmetric, densely defined, closed and bounded from below (the latter is not obvious unless $\tilde{\alpha} \geq 0$, but it follows from the results (14) and (17) below). Consequently, $Q_{\kappa,\alpha}$ gives rise (cf [22, Sec. VI.2]) to a unique self-adjoint bounded-from-below operator which we denote by $-\Delta_{\kappa,\alpha}$. It can be verified that $-\Delta_{\kappa,\alpha}$ acts as the classical Laplacian with the boundary conditions described in Section 1 provided $\Gamma$ is sufficiently regular.

It follows from assumptions (2) that $\mathcal{L} : I \times (-a,a) \to \Omega : \{(s,t) \mapsto \mathcal{L}(s,t)\}$ is a $C^1$-diffeomorphism. Consequently, $\Omega$ can be identified with the Riemannian manifold $I \times (-a,a)$ equipped with the metric $G_{ij} := (\partial_i \mathcal{L}) \cdot (\partial_j \mathcal{L})$, where $i,j \in \{1,2\}$ and the dot denotes the scalar product in $\mathbb{R}^2$. Employing the
Frenet formula $\dot{N} = -\kappa \dot{\Gamma}$, one easily finds that $(G_{ij}) = \text{diag}(g_\kappa^2, 1)$, where

$$g_\kappa(s, t) := 1 - \kappa(s) t$$

(11)
is the Jacobian of $\mathcal{L}$.

It follows that $g_\kappa(s, t) \, ds \, dt$ is the area element of the strip, $L^2(\Omega)$ can be identified with the Hilbert space

$$L^2(I \times (-a, a), g_\kappa(s, t) \, ds \, dt)$$

(12)

and $-\Delta_{\kappa, \alpha}$ is unitarily equivalent to the operator $H_{\kappa,\alpha}$ on (12) associated with the quadratic form

$$h_{\kappa, \alpha}[\psi] := \|g_\kappa^{-1} \partial_1 \psi\|_\kappa^2 + \|\partial_2 \psi\|_\kappa^2 + \int_\mathbb{R} \alpha(s) |\psi(s, a)|^2 g_\kappa(s, a) \, ds ,$$

$$\psi \in D(h_{\kappa, \alpha}) := \{ \psi \in W^{1,2}(\mathbb{R} \times (-a, a)) \mid \psi(s, -a) = 0 \text{ for a.e. } s \in \mathbb{R} \} .$$

(13)

Here $\| \cdot \|_\kappa$ stands for the norm in (12) and $\psi(s, \pm a)$ means the trace of the function $\psi$ on the part of the boundary $I \times \{ \pm a \}$ (cf Remark 1 below). In fact, if the curve $\Gamma$ is sufficiently smooth, then $H_{\kappa, \alpha}$ acts as the Laplace-Beltrami operator $-G^{-1/2} \partial_i G^{1/2} G^{ij} \partial_j$, where $(G^{ij}) = (G_{ij})^{-1}$ and $G := \det(G_{ij})$, but we will not use this fact. Finally, let us notice that the first assumption of (2) yields

$$0 < 1 - \|\kappa\|_\infty a \leq g_\kappa(s, t) \leq 1 + \|\kappa\|_\infty a < 2$$

uniformly in $(s, t) \in I \times (-a, a)$, and that is actually why we can indeed write $W^{1,2}(I \times (-a, a))$ instead of $W^{1,2}(I \times (-a, a), g_\kappa(s, t) \, ds \, dt)$ in (13).

**Remark 1.** The traces of $\psi \in W^{1,2}(I \times (-a, a))$ on the boundary of the strip $I \times (-a, a)$ are well defined and square integrable (cf [1]). In particular, the boundary integral appearing in (13) is finite (recall that $\alpha$ is assumed to be bounded). To ensure that the traces and the boundary integral appearing in (10) are well defined too, it is sufficient to notice that one can construct traces of $u \in W^{1,2}(\Omega)$ to $\Gamma_+ \cup \Gamma_-$ by means of the diffeomorphism $\mathcal{L}$, the trace operator for the straight strip $I \times (-a, a)$ and inverses of the boundary mappings $\mathcal{L}(\cdot, \{ \pm a \})$. The latter exists due to the second hypothesis in (2), which is in fact a bit stronger than an analogous assumption in the uniform Dirichlet case [10, 14] (there it is enough to assume that $\mathcal{L} \mid I \times (-a, a)$ is injective). In this context, one should point out that the approach used by Daners in [6] makes it possible to deal with Robin boundary conditions with positive $\alpha$ on arbitrary bounded domains, without using traces.

### 4 An intermediate lower bound

In this section, we derive the central lower bound of the present paper, i.e. inequality (14) below, and explain its connection with Definition 1.
Neglecting in (13) the “longitudinal kinetic energy”, i.e. the term $\|g^{-1}_{\kappa}\partial_1 \psi\|_\kappa$ in the expression for $h_{\kappa,\alpha}[\psi]$, and using Fubini’s theorem, one immediately gets
\[
\inf_{s \in I} \sigma(H_{\kappa,\alpha}) \geq \inf_{s \in I} \lambda(\kappa(s), \alpha(s)),
\] (14)
where $\lambda(\kappa, \alpha)$ denotes the first eigenvalue of the self-adjoint one-dimensional operator $B_{\kappa,\alpha}$ on $H_{\kappa} := L^2((-a, a), (1 - \kappa t)dt)$ associated with the quadratic form
\[
b_{\kappa,\alpha}[\psi] := \int_{-a}^a |\psi'(t)|^2 (1 - \kappa t)dt + \alpha |\psi(a)|^2 (1 - \kappa a),
\] (15)
\[
\psi \in D(b_{\kappa,\alpha}) := \{ \psi \in W^{1,2}((-a, a)) \mid \psi(-a) = 0 \}.
\]
With a slight abuse of notation, we denote by $\kappa \in (-1/a, 1/a)$ and $\alpha \in \mathbb{R}$ given constants now. One easily verifies that
\[
(B_{\kappa,\alpha} \psi)(t) = -\psi''(t) + \frac{\kappa}{1 - \kappa t} \psi'(t),
\]
\[
\psi \in D(B_{\kappa,\alpha}) = \{ \psi \in W^{2,2}((-a, a)) \mid \psi(-a) = 0 & \psi'(a) + \alpha \psi(a) = 0 \}.
\] (16)
Note that the values of $\psi$ and $\psi'$ at the boundary points of $(-a, a)$ are well defined due to the Sobolev embedding theorem.

$B_{\kappa,\alpha}$ is clearly a positive operator for $\alpha \geq 0$. Furthermore, using the elementary inequality $|\psi(a)|^2 \leq \varepsilon \int_{-a}^a |\psi'(t)|^2 dt + \varepsilon^{-1} \int_{-a}^a |\psi(t)|^2 dt$ with $\varepsilon > 0$, it can be easily shown that
\[
\lambda(\kappa, \alpha) \geq -\alpha^2 \frac{(1 + |\kappa| a)^2}{(1 - |\kappa| a)^2},
\] (17)
i.e., $B_{\kappa,\alpha}$ is bounded from below in any case. This and (14) prove that $H_{\kappa,\alpha}$ (and therefore $-\Delta_{\kappa,\alpha}$) is bounded from below a fortiori.

Using coordinates analogous to (1) and the circular (respectively straight) symmetry, it is easy to see that $B_{\kappa,\alpha}$ is nothing else than the “radial” (respectively “transversal”) part of the Laplacian on $L^2(A_\kappa)$ if $\kappa \neq 0$ (respectively $\kappa = 0$) in Definition 1. (We refer to [14, Lemma 4.1] for more details on the partial wave decomposition in the case $\alpha = +\infty$.) This shows that the geometric Definition 1 of $\lambda$ and the definition via (15) are in fact equivalent.

In view of (14), it remains to establish the monotonicity properties of $\lambda$ stated in Theorem 2 in order to prove Theorem 1. This will be done in the next section.

5 Dirichlet-Robin annuli

Using standard arguments (cf. [19, Sec. 8.12]), one easily shows that $\lambda(\kappa, \alpha)$, as the lowest eigenvalue of $B_{\kappa,\alpha}$, is simple and has a positive eigenfunction. We denote the latter by $\psi_{\kappa,\alpha}$ and normalize it to have unit norm in the Hilbert space $H_\kappa$. 

10
5.1 Dependence on $\alpha$

The first property of Theorem 2 follows directly from the variational definition of $\lambda(\kappa, \alpha)$. In detail, using $\psi_{\kappa, \alpha} + \delta$ with any $\delta > 0$ as a test function for $\lambda(\kappa, \alpha)$, we get

$$\lambda(\kappa, \alpha) \leq \lambda(\kappa, \alpha + \delta) - \delta \psi_{\kappa, \alpha + \delta}(a)^2 (1 - \kappa a) < \lambda(\kappa, \alpha + \delta),$$

(18)
i.e. $\alpha \mapsto \lambda(\kappa, \alpha)$ is increasing. Note that the strict monotonicity is a consequence of the fact that $\psi_{\kappa, \alpha} + \delta \in D(B_{\kappa, \alpha + \delta})$; indeed, $\psi_{\kappa, \alpha + \delta}(a) = 0$ would imply that $\psi'_{\kappa, \alpha + \delta}(a) = 0$ also, giving a contradiction. Using now $\psi_{\kappa, \alpha}$ as a test function for $\lambda(\kappa, \alpha + \delta)$, we get

$$\lambda(\kappa, \alpha + \delta) \leq \lambda(\kappa, \alpha) + \delta \psi_{\kappa, \alpha}(a)^2 (1 - \kappa a) \xrightarrow{\delta \to 0} \lambda(\kappa, \alpha),$$

(19)

which, together with (18), gives the continuity of $\lambda$ in the second variable.

5.2 Dependence on $\kappa$

Not all of the other properties of Theorem 2 are so obvious from the variational definition of $\lambda(\kappa, \alpha)$ via $B_{\kappa, \alpha}$ because the Hilbert space $\mathcal{H}_\kappa$ depends on $\kappa$. To overcome this, we introduce the unitary transformation $U_\kappa : \mathcal{H}_\kappa \rightarrow \mathcal{H}_0 : \{ \psi \mapsto (1 - \kappa t)^{\frac{1}{2}} \psi \}$

(20)

and the unitarily equivalent operator $\hat{B}_{\kappa, \alpha} := U_\kappa B_{\kappa, \alpha} U_\kappa^{-1}$ associated with the transformed form $\hat{b}_{\kappa, \alpha}[\cdot] := b_{\kappa, \alpha}[U_\kappa^{-1} \cdot]$. Given any $\phi \in D(\hat{b}_{\kappa, \alpha})$, we insert $\psi = U_\kappa^{-1} \phi$ into (15), integrate by parts and finds

$$\hat{b}_{\kappa, \alpha}[\phi] = \int_{-a}^{a} |\phi'(t)|^2 dt - \int_{-a}^{a} \frac{\kappa^2}{4(1 - \kappa t)^2} |\phi(t)|^2 dt + \left( \alpha + \frac{\kappa}{2(1 - \kappa a)} \right) |\phi(a)|^2.$$

(21)

We also verify that

$$(\hat{B}_{\kappa, \alpha} \phi)(t) = -\phi''(t) - \frac{\kappa^2}{4(1 - \kappa t)^2} \phi(t),$$

$\phi \in D(\hat{B}_{\kappa, \alpha}) = \left\{ \phi \in W^{2,2}((-a, a)) \mid \phi(-a) = 0 \right\}$

(22)

& $\phi'(a) + \left( \alpha + \frac{\kappa}{2(1 - \kappa a)} \right) \phi(a) = 0$.

It is important to notice that while $D(B_{\kappa, \alpha})$ is not invariant under $U_\kappa$, one still has $D(\hat{b}_{\kappa, \alpha}) = D(b_{\kappa, \alpha})$.

5.2.1 Continuity

Following Sec. VII. 4], $\kappa \mapsto \hat{b}_{\kappa, \alpha}$ forms a holomorphic family of forms of type (a) and $\kappa \mapsto \hat{B}_{\kappa, \alpha}$ forms a self-adjoint holomorphic family of operators of
type (B). In particular, $\kappa \mapsto \lambda(\kappa, \alpha)$ is continuous, which proves (ii) of Theorem 2. Moreover, denoting by $\phi_{\kappa, \alpha} := U_\kappa \psi_{\kappa, \alpha}$ the eigenfunction of $\hat{B}_{\kappa, \alpha}$ corresponding to $\lambda(\kappa, \alpha)$, we get that $\kappa \mapsto \phi_{\kappa, \alpha}$ is continuous in the norm of $\mathcal{H}_0$. In view of (20), it then follows that also $\kappa \mapsto \psi_{\kappa, \alpha}$ is continuous in the norm of $\mathcal{H}_0$.

5.2.2 Monotonicity

Since the function $f : \kappa \mapsto \kappa^2$ is increasing on $(-1/a, 1/a)$ for any $t \in [-a, a]$, one easily verifies Theorem 2.(iii) by means of the variational definition of $\lambda(\kappa, \alpha)$ via $\hat{B}_{\kappa, \alpha}$ and an argument similar to that used in Section 5.1.

However, the above argument fails to prove (iv) of Theorem 2 because $-f^2$ is decreasing on $(0, 1/a)$, so that one gets an interplay between the increasing boundary term and decreasing potential in (21) for positive curvatures. Therefore we come back to the initial operator (16) and calculate the derivative of $\kappa \mapsto \lambda(\kappa, \alpha)$:

**Lemma 1.** $\forall \kappa \in (-1/a, 1/a), \forall \alpha \in \mathbb{R}$,

$$\frac{\partial \lambda}{\partial \kappa}(\kappa, \alpha) = \int_{-a}^{a} \frac{\psi_{\kappa, \alpha}(t) \psi'_{\kappa, \alpha}(t)}{1 - \kappa t} dt.$$  \hspace{1cm} (23)

**Proof.** Throughout this proof, we omit the dependence of $\lambda$ and the corresponding eigenfunction on $\alpha$.

We write the eigenvalue equation for $B_{\kappa, \alpha}$ with $\psi_{\kappa}$ and $\lambda(\kappa)$ as

$$- \left[ \psi'_{\kappa}(t) (1 - \kappa t) \right]' = \lambda(\kappa) \psi_{\kappa}(t) (1 - \kappa t)$$  \hspace{1cm} (24)

and consider the analogous equation at $\kappa + \delta$, with $\delta \in \mathbb{R} \setminus \{0\}$ so small that $|\kappa + \delta| a$ is less than 1. Multiplying (24) by $\psi_{\kappa+\delta}$, integrating by parts, combining the result with the result coming from analogous manipulations applied to the problem at $\kappa + \delta$, dividing by $\delta$, integrating by parts once more and using the eigenvalue equation for $B_{\kappa, \alpha}$, we arrive at

$$\frac{\lambda(\kappa + \delta) - \lambda(\kappa)}{\delta} \int_{-a}^{a} \psi_{\kappa}(t) \psi_{\kappa+\delta}(t) (1 - \kappa t) dt$$

$$= \lambda(\kappa + \delta) \int_{-a}^{a} \psi_{\kappa}(t) \psi_{\kappa+\delta}(t) t dt$$

$$- \int_{-a}^{a} \psi'_{\kappa}(t) \psi'_{\kappa+\delta}(t) t dt - \alpha a \psi_{\kappa}(a) \psi_{\kappa+\delta}(a)$$

$$= [\lambda(\kappa + \delta) - \lambda(\kappa)] \int_{-a}^{a} \psi_{\kappa}(t) \psi_{\kappa+\delta}(t) t dt + \int_{-a}^{a} \psi'_{\kappa}(t) \psi_{\kappa+\delta}(t) dt.$$  \hspace{1cm} (25)

Letting $\delta$ go to zero yields the desired result by means of the continuity of $\kappa \mapsto \lambda(\kappa)$ and $\kappa \mapsto \psi_{\kappa}$ established in Section 5.2.1. \hfill $\square$

Lemma 1 yields (iv) of Theorem 2 whenever the integral on the right hand side of (23) is positive. In particular, this is the case when $\psi'_{\kappa, \alpha}$ is non-negative:
Lemma 2. \( \forall \kappa \in (-1/a, 1/a), \forall \alpha \in (-\infty, 0], \)
\[ t \mapsto \psi_{\kappa, \alpha}(t) : (-a, a) \to \mathbb{R} \text{ is increasing.} \]

Proof. Throughout this proof, we omit the dependence of \( \lambda \) and the corresponding eigenfunction on \( \kappa \) and \( \alpha \).

Since \( \psi \) is a positive eigenfunction and \( \psi(-a) = 0 \), respectively \( \psi'(a) = -\alpha \psi(a) \), we know that \( \psi'(-a) > 0 \), respectively \( \psi'(a) \geq 0 \). Recall also that \( \psi(a) > 0 \). We claim that \( \psi' > 0 \) on \((-a, a)\).

Case \( \lambda < 0 \). The eigenvalue problem for (16) implies that if \( \psi'(t) = 0 \) for some \( t \in (-a, a) \), then \( \psi''(t) > 0 \), i.e. \( \psi \) has a local minimum at \( t \). Consequently, if there exists a \( t_1 \in (-a, a) \) such that \( \psi'(t_1) = 0 \), then, since \( \psi'(a) > 0 \), there must also be a \( t_2 \in (-a, t_1) \) such that \( \psi \) has a local maximum at \( t_2 \), a contradiction.

Case \( \lambda > 0 \). The eigenvalue problem for (16) implies that if \( \psi'(t) = 0 \) for some \( t \in (-a, a) \), then \( \psi''(t) < 0 \), i.e. \( \psi \) has a local maximum at \( t \). Consequently, if there exists a \( t_1 \in (-a, a) \) such that \( \psi'(t_1) = 0 \), then, since \( \psi'(a) \geq 0 \), there must also be a \( t_2 \in (t_1, a) \) such that \( \psi''(t_2) = 0 \) and \( \psi' < 0 \) on \((t_1, t_2)\), i.e. \( \psi \) does not have a local maximum at \( t_2 \), a contradiction.

Case \( \lambda = 0 \). Integrating (24), we get \( \psi'(t) = -\alpha \frac{\kappa}{a} \psi(a) > 0 \) for all \( t \in [-a, a] \) (the equality would imply a trivial eigenfunction).

5.2.3 Boundary values

Using the geometrical meaning of \( \lambda(\kappa, \alpha) \) (cf. Definition 1) and since \( A_{\kappa} \) converges (e.g., in the sense of metrical convergence [11]) to the disc \( D(2a) \) with the central point removed as \( |\kappa| \to 1/a \), the limits in Theorem 2 (v) are natural to expect. We prove each of them separately.

The negative limit The limit value for \( \lambda(\kappa, \alpha) \) as \( \varepsilon \equiv -(\kappa^{-1} + a) \to 0 \) follows from Flucher’s paper [18], where an approximation formula for eigenvalues in domains with spherical holes is found. The only difference is the fact that in our case the boundary of the domain also changes as \( \varepsilon \) goes to zero. We overcome this complication by transforming the eigenvalue problem for the Laplacian on \( A_{\kappa} \) into

\[
\begin{aligned}
-\Delta u &= \lambda_{\varepsilon}(\alpha_{\varepsilon})u & \text{in} & \quad A(\varepsilon(2a + \varepsilon)^{-1}, 1), \\
 u &= 0 & \text{on} & \quad \partial D(\varepsilon(2a + \varepsilon)^{-1}), \\
 \frac{\partial u}{\partial N} + \alpha_{\varepsilon} u &= 0 & \text{on} & \quad \partial D(1),
\end{aligned}
\]

(25)

where \( \lambda_{\varepsilon}(\alpha_{\varepsilon}) \equiv (2a + \varepsilon)^2 \lambda(-(a + \varepsilon)^{-1}, \alpha), \alpha_{\varepsilon} \equiv (2a + \varepsilon)\alpha \) and \( N \) is the outward unit normal on \( \partial D(1) \). By the minimax principle,

\[
\lambda_{\varepsilon}(\alpha_{-(\text{sgn } \alpha) \varepsilon}) \leq \lambda_{\varepsilon}(\alpha_{\varepsilon}) \leq \lambda_{\varepsilon}(\alpha_{(\text{sgn } \alpha) \varepsilon})
\]

for any fixed \( \varepsilon_0 \in (\varepsilon, 2a) \), where \( \lambda_{\varepsilon}(\alpha_{\pm \varepsilon_0}) \) denotes the eigenvalue of the problem (25) with \( \alpha_{\varepsilon} \) being replaced by \( \alpha_{\pm \varepsilon_0} \). Then it is clear that \( \lambda_{\varepsilon}(\alpha_{\varepsilon}) \to \lambda_{\varepsilon}(\alpha_{\varepsilon}) \quad as \quad \varepsilon \to 0 \).
(2a)²ν(α) as ε → 0 because it is true for λε(α±ε₀) by [18] and ε₀ can be chosen arbitrarily small.

The positive limit If α > 0, the limit value for λ(κ, α) as κ−1 → a could be derived by means of a paper by Dancer and Daners, [5], where they study domain perturbations for elliptic equations subject to Robin boundary conditions. However, since they restrict to positive α and we do not know about a similar perturbation result for α < 0, we establish the limit value by rather elementary considerations.

Assuming κ ≠ 0, the eigenvalue problem for Bκ,α is explicitly solvable in terms of the Bessel functions J₀ and Y₀ (cf. [34, Chap. 7]) and the eigenvalue λ(κ, α) is then determined as the smallest (in absolute value) zero λ of the implicit equation

\[ J₀(√λ(1 + κα)/κ) \left[ √λY₁(√λ(1 − κα)/κ) + αJ₀(√λ(1 − κα)/κ) \right] = Y₀(√λ(1 + κα)/κ) \left[ √λJ₁(√λ(1 − κα)/κ) + αY₀(√λ(1 − κα)/κ) \right]. \]  

(26)

Although the case λ(κ, α) = 0 should be treated separately, a formal asymptotic expansion of (26) around √λ = 0 also gives the correct condition for a zero eigenvalue:

\[ λ(κ, α) = 0 \iff κ = α(1 − κa) log \frac{1}{1 + κa}. \]  

(27)

In particular, the condition yields that for any α < −1/(2a) there always exists κ₀ ∈ (0, 1/a) such that λ(κ₀, α) = 0. This and the properties (i), (ii) and (iv) of Theorem 1 imply that lim_{κ→1/a} λ(κ, α) > 0 for any α ∈ ℝ. We also know that the limit is bounded because λ(κ, α) < λ(κ, +∞) by the minimax principle and λ(κ, +∞) tends to the first eigenvalue of the Dirichlet Laplacian in the disc D(2a), i.e. ν(+∞) = j_{20}²/(2a)², as κ → 1/a by known convergence theorems (cf one of [31, 33, 7]). Applying the limit to (26), we get a bounded value on the right hand side, while the left hand side admits the asymptotic expansion

\[ -√λ 2a J₀(√λ(1 − κa)(2κ))^{-1} + O(κ^{-1} − a). \]

That is, √λ 2a necessarily converges to the first zero of the Bessel function J₀ as κ → 1/a.

6 Infinite strips

Let I = ℝ throughout this section. The proof of Theorem 3 is based on the following two lemmata.

Firstly, Theorem 2 implies:

Lemma 3. Assume the hypotheses of Theorem 3. Then the function µ : ℝ → ℝ defined by

\[ s \mapsto µ(s) := λ(κ(s), α(s)) − λ(0, α₀) \]

is continuous, non-zero and non-negative.
Hereafter we shall use the same notation $\mu$ for the function $\mu \otimes 1$ on $\mathbb{R} \times (-a, a)$.

Secondly, we shall need the following Hardy-type inequality for a Schrödinger operator in a strip with the potential being a characteristic function:

**Lemma 4.** For any $\psi \in W^{1,2}(\mathbb{R} \times (-a, a))$,

$$\int_{\mathbb{R} \times (-a,a)} \rho^{-2} |\psi|^2 \leq 16 \int_{\mathbb{R} \times (-a,a)} |\partial_1 \psi|^2 + \left(2 + \frac{64}{|J|^2}\right) \int_{J \times (-a,a)} |\psi|^2,$$

where $\rho(s,t) := \sqrt{1 + (s - s_0)^2}$, $J$ is any bounded subinterval of $\mathbb{R}$ and $s_0$ is the mid-point of $J$.

This lemma can be established quite easily by means of the classical one-dimensional Hardy inequality $\int_{\mathbb{R}} x^{-2} |v(x)|^2 \, dx \leq 4 \int_{\mathbb{R}} |v'(x)|^2 \, dx$ valid for any $v \in W^{1,2}(\mathbb{R})$ with $v(0) = 0$ and Fubini’s theorem; we refer the reader to [12, Sec. 3.3] or [24, proof of Lem. 2] for more details.

### 6.1 Proof of Theorem 3

Let $\psi$ belong to the dense subspace of $D(h_{\kappa,\alpha})$ given by $C^\infty$-smooth functions on $\mathbb{R} \times (-a, a)$ which vanish in a neighbourhood of $\mathbb{R} \times \{-a\}$ and which are restrictions of functions from $C^\infty_0(\mathbb{R}^2)$. Assume the hypotheses of Theorem 3 so that the conclusions of Lemma 3 hold. Let $J$ be any closed subinterval of $\mathbb{R}$ on which $\mu$ defined in Lemma 3 is positive.

The first step is to come back to the intermediate lower bound (14); we also use the definition of $\lambda$ via (15), but we do not neglect the “longitudinal kinetic energy”:

$$h_{\kappa,\alpha}[\psi] - \lambda(0, \alpha_0) \|\psi\|_\kappa^2 \geq \|g^{-1}_{\kappa} \partial_1 \psi\|_\kappa^2 + \|\mu^{1/2} \psi\|_\kappa^2$$

$$\geq \|g^{-1}_{\kappa} \partial_1 \psi\|_\kappa^2 + \epsilon (1 - \|\kappa\|_{\infty} a) \min_j \mu \int_{J \times (-a,a)} |\psi|^2.$$

Here $\epsilon \in (0, 1]$ is arbitrary for the time being. Applying Lemma 4 to the last integral, we arrive at

$$h_{\kappa,\alpha}[\psi] - \lambda(0, \alpha_0) \|\psi\|_\kappa^2 \geq \left( \frac{1}{1 + \|\kappa\|_{\infty} a} - \frac{16 \epsilon (1 - \|\kappa\|_{\infty} a) \min_j \mu}{2 + 64/|J|^2} \right) \int_{\mathbb{R} \times (-a,a)} |\partial_1 \psi|^2$$

$$+ \frac{\epsilon (1 - \|\kappa\|_{\infty} a) \min_j \mu}{2 + 64/|J|^2} \int_{\mathbb{R} \times (-a,a)} \rho^{-2} |\psi|^2.$$

Choosing now $\epsilon$ as the minimum between 1 and the value such that the first term on the right hand side of the last estimate vanishes, we finally get

$$h_{\kappa,\alpha}[\psi] - \lambda(0, \alpha_0) \|\psi\|_\kappa^2 \geq c \|\rho^{-1} \psi\|_\kappa^2$$

(28)
with
\[ c := \min \left\{ \frac{(1 - \|\kappa\|_{\infty} a) \min J \mu}{(2 + 64/J^2) (1 + \|\kappa\|_{\infty} a)} \right\}. \]  

(29)

In view of Section 3, we conclude that (28) is equivalent to
\[ Q_{\kappa,\alpha}[u] - \lambda(0, \alpha_0) \|u\|^2_{L^2(\Omega)} \geq c \|(\rho \circ \mathcal{L})^{-1} u\|^2_{L^2(\Omega)} \]
for all \( u \in D(Q_{\kappa,\alpha}) \), which is the exact meaning of (9).

6.2 Proof of Corollary 2

Let \( \psi \) be as in the previous section. The present proof is based on an algebraic comparison of \( h_{\kappa,0}[\psi] - \lambda(0,0) \|\psi\|^2_\kappa \) with \( h_{\kappa+,0}[\psi] - \lambda(0,0) \|\psi\|^2_{\kappa+} \) and a usage of (28).

For every \((s,t) \in \mathbb{R} \times (-a,a)\), we have
\[ 1 - f_\varepsilon(s) \leq \frac{g_\kappa(s,t)}{g_{\kappa+}(s,t)} \leq 1 + f_\varepsilon(s) \quad \text{with} \quad f_\varepsilon(s) := \frac{\varepsilon a \chi_I(s)}{1 - \|\kappa+\|_{\infty} a}, \]
where \( \chi_I \) denotes the characteristic function of the set \( I \times (-a,a) \). Hereafter we assume \( \varepsilon \leq (1 - \|\kappa+\|_{\infty} a) / (2a) \) so that the lower bound is greater or equal to 1/2. Using the same notation \( f_\varepsilon \) for the functions \( f_\varepsilon \otimes 1 \) on \( \mathbb{R} \times (-a,a) \), we have
\[ h_{\kappa,0}[\psi] - \lambda(0,0) \|\psi\|^2_\kappa \geq \int_{\mathbb{R} \times (-a,a)} (1 + f_\varepsilon)^{-1} g_{\kappa+}^{-1} |\partial_1 \psi|^2 \]
\[ + \int_{\mathbb{R} \times (-a,a)} ds \int_{-a}^a dt g_{\kappa+}(s,t) (|\partial_2 \psi(s,t)|^2 - \lambda(0,0) |\psi(s,t)|^2) \]
\[ - \lambda(0,0) \int_{\mathbb{R} \times (-a,a)} 2 f_\varepsilon g_{\kappa+} |\psi|^2. \]
Recalling the definition of \( \lambda \) via (15) and Lemma 3, it is clear that the term in the second line after the inequality sign is non-negative. Consequently,
\[ h_{\kappa,0}[\psi] - \lambda(0,0) \|\psi\|^2_\kappa \geq \frac{1}{2} \left( h_{\kappa+,0}[\psi] - \lambda(0,0) \|\psi\|^2_{\kappa+} \right) \]
\[ - \lambda(0,0) \int_{\mathbb{R} \times (-a,a)} 2 f_\varepsilon g_{\kappa+} |\psi|^2. \]

Using (28) with \( \alpha \) being equal to 0, with \( \kappa \) being replaced by \( \kappa+ \) and with \( s_0 \) being from the support of \( \kappa+ \), we finally obtain
\[ h_{\kappa,0}[\psi] - \lambda(0,0) \|\psi\|^2_\kappa \geq \|w^{1/2} \psi\|^2_\kappa, \]
(31)
where
\[ w(s, t) := \frac{c/4}{1 + (s - s_0)^2} - \lambda(0, 0) \frac{\varepsilon a \chi(s)}{1 - \|\kappa\|_\infty a} \]
is positive for all sufficiently small \( \varepsilon \). Equivalently,
\[ -\Delta_{\kappa, 0} \geq \lambda(0, 0) + w \circ L^{-1} \tag{32} \]
in the sense of quadratic forms on \( L^2(\Omega) \). This concludes the proof of Corollary 2.

### 7 Remarks and open questions

It follows immediately from the minimax principle that the lower bound of Theorem 1 also applies to other boundary conditions imposed on \( L_\pm \), e.g., Dirichlet, periodic, certain Robin, etc.

Of course, it is also possible to impose Robin boundary conditions on \( \Gamma_- \) instead of Dirichlet. Then the lower bound of the type (14) still holds and the problem is translated to the study of properties of the first eigenvalue in a Robin-Robin annulus. The techniques of the present paper will also apply to certain values of the parameters in such a case. However, we refrained from doing so to keep the statement of results as simple as possible.

It follows from Theorem 2 that \( \nu(\alpha) \) gives a uniform lower bound to the spectral threshold of \( -\Delta_{\kappa, \alpha} \) provided \( \alpha \leq 0 \) or \( \kappa \leq 0 \). We conjecture this to be always the case, but were not able to prove it in general. In this context, it would be desirable to prove that \( \kappa \rightarrow \lambda(\kappa, \alpha) \) does not possess local minima for any \( \alpha \in \mathbb{R} \).

We proved the fact that \( \kappa \rightarrow \lambda(\kappa, \alpha) \) is increasing on \((0, 1/a)\) only for non-positive \( \alpha \). It is clear from the limiting Dirichlet problem (cf. [14]) that this property will not hold for large positive \( \alpha \). However, formula (23) suggests that this is still true for small values of \( \alpha \). Numerical results show (cf Figure 1) that the critical value is approximately 0.78 for \( a = 1 \).

Proposition 1 contains just one example of sufficient condition which guarantees the existence of discrete eigenvalues in infinite curved strips. Further results can be obtained in the spirit of [9, 25]. An open question is, e.g., whether the discrete spectrum exists for certain strips with \( \kappa > 0 \) and \( \alpha > 0 \). Let us recall that this is always the case for \( \alpha = +\infty \).

For simplicity, we assumed that \( \kappa \) and \( \alpha - \alpha_0 \) had compact support when we considered infinite strips. However, the claim of Corollary 1 holds whenever the essential spectrum (7) is preserved, and this might be checked under much less restrictive conditions about the decay of \( \kappa \) and \( \alpha - \alpha_0 \) at infinity. For instance, modifying the approach of [25], it should be enough just to require that the limits at infinity are equal to zero. In fact, Theorem 3 holds without any condition about the decay of \( \kappa \) and \( \alpha - \alpha_0 \) at infinity, but it is of interest
only in the case the essential spectrum does not start above $\lambda(0, \alpha_0)$. In any case, a fast decay of curvature at infinity is needed to prove Corollary 2, namely, $\kappa(s) = \mathcal{O}(s^{-2})$ as $|s| \to \infty$. This quadratic decay is related to the decay of the Hardy weight in Theorem 3 which is typical for Hardy inequalities involving the Laplacian, and cannot be therefore improved by the present method.

Under suitable global geometric conditions about the reference curve $\Gamma$, the intrinsic distance $|s - s_0|$ which appears in the function $\rho$ of Theorem 3 can be estimated by an exterior one. For instance, if $\Gamma$ is an embedded unit-speed curve with compactly supported curvature, then it is easy to see that there exists a positive number $\delta$ such that

$$\forall s, s' \in \mathbb{R}, \quad \delta |s - s'| \leq |\Gamma(s) - \Gamma(s')| \leq |s - s'|.$$

Corollary 2 extends the class of strips from [9] with empty discrete spectrum. An open question is to decide whether an analogous result holds for other $\alpha$ satisfying $\alpha_0 \leq \alpha \leq 0$.

Acknowledgements

This work was partially supported by FCT (Portugal) through projects POCI/-MAT/60863/2004 (POCI2010) and SFRH/BPD/11457/2002. The second author (D.K.) was also supported by the Czech Academy of Sciences and its Grant Agency within the projects IRP AV0Z10480505 and A100480501, and by the project LC06002 of the Ministry of Education, Youth and Sports of the Czech Republic.

References

[1] R. A. Adams, *Sobolev spaces*, Academic Press, New York, 1975.
[2] M. S. Ashbaugh and P. Exner, *Lower bounds to bound state energies in bent tubes*, Phys. Lett. A 150 (1990), no. 3, 4, 183–186.
[3] D. Borisov, T. Ekholm, and H. Kovarík, *Spectrum of the magnetic Schrödinger operator in a waveguide with combined boundary conditions*, Ann. H. Poincaré 6 (2005), 327–342.
[4] B. Chenaud, P. Duclos, P. Freitas, and D. Krejčiřík, *Geometrically induced discrete spectrum in curved tubes*, Differential Geom. Appl. 23 (2005), no. 2, 95–105.
[5] E. N. Dancer and D. Daners, *Domain perturbation for elliptic equations subject to Robin boundary conditions*, J. Differential Equations 138 (1997), no. 1, 86–132.
[6] D. Daners, *Robin boundary value problems on arbitrary domains*, Trans. Amer. Math. Soc. 352 (2000), no. 9, 4207–4236.
[7] _______. Dirichlet problems on varying domains, J. Differential Equations 188 (2003), 591–624.

[8] E. B. Davies and L. Parnovski, Trapped modes in acoustic waveguides, Q. Jl Mech. Appl. Math. 51 (1998), 477–492.

[9] J. Dittrich and J. Kríž, Curved planar quantum wires with Dirichlet and Neumann boundary conditions, J. Phys. A 35 (2002), L269–275.

[10] P. Duclos and P. Exner, Curvature-induced bound states in quantum waveguides in two and three dimensions, Rev. Math. Phys. 7 (1995), 73–102.

[11] T. Ekholm and H. Kovářík, Stability of the magnetic Schrödinger operator in a waveguide, Commun. in Partial Differential Equations 30 (2005), no. 4, 539–565.

[12] T. Ekholm, H. Kovářík, and D. Krejčiřík, A Hardy inequality in twisted waveguides, Arch. Rat. Mech. Anal., to appear; preprint on math-ph/0512050 (2005).

[13] D. V. Evans, M. Levitin, and D. Vassiliev, Existence theorems for trapped modes, J. Fluid Mech. 261 (1994), 21–31.

[14] P. Exner, P. Freitas, and D. Krejčiřík, A lower bound to the spectral threshold in curved tubes, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 460 (2004), no. 2052, 3457–3467.

[15] P. Exner and D. Krejčiřík, Quantum waveguides with a lateral semitransparent barrier: Spectral and scattering properties, J. Phys. A 32 (1999), 4475–4494.

[16] _______. Waveguides coupled through a semitransparent barrier: A Birman-Schwinger analysis, Rev. Math. Phys. 13 (2001), no. 3, 307–334.

[17] P. Exner and P. Šeba, Bound states in curved quantum waveguides, J. Math. Phys. 30 (1989), 2574–2580.

[18] M. Flucher, Approximation of Dirichlet eigenvalues on domains with small holes, J. Math. Anal. Appl. 193 (1995), no. 1, 169–199.

[19] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, Berlin, 1983.

[20] J. Goldstone and R. L. Jaffe, Bound states in twisting tubes, Phys. Rev. B 45 (1992), 14100–14107.

[21] E. R. Johnson, M. Levitin, and L. Parnovski, Existence of eigenvalues of a linear operator pencil in a curved waveguide – localized shelf waves on a curved coast, SIAM J. Math. Anal. 37 (2006), no. 5, 1465-1481.
[22] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1966.

[23] W. Klingenberg, *A course in differential geometry*, Springer-Verlag, New York, 1978.

[24] D. Krejčířík, *Hardy inequalities for strips on ruled surfaces*, J. Inequal. Appl. 2006 (2006), Article ID 46409, 10 pages.

[25] D. Krejčířík and J. Kríž, *On the spectrum of curved quantum waveguides*, Publ. RIMS, Kyoto University 41 (2005), no. 3, 757–791.

[26] E. Kreyzig, *Differential geometry*, University of Toronto Press, Toronto, 1959.

[27] P. Kuchment and H. Zeng, *Convergence of spectra of mesoscopic systems collapsing onto a graph*, J. Math. Anal. Appl. 258 (2001), 671–700.

[28] ______, *Asymptotics of spectra of Neumann Laplacians in thin domains*, Advances in differential equations and mathematical physics (Birmingham, AL, 2002), Contemp. Math., vol. 327, Amer. Math. Soc., Providence, RI, 2003, pp. 199–213.

[29] J. T. Londergan, J. P. Carini, and D. P. Murdock, *Binding and scattering in two-dimensional systems*, LNP, vol. m60, Springer, Berlin, 1999.

[30] O. Olendski and L. Mikhailovska, *Localized-mode evolution in a curved planar waveguide with combined Dirichlet and Neumann boundary conditions*, Phys. Rev. E 67 (2003), art. 056625.

[31] J. Rauch and M. Taylor, *Potential and scattering theory on wildly perturbed domains*, J. Funct. Anal. 18 (1975), 27–59.

[32] W. Renger and W. Bulla, *Existence of bound states in quantum waveguides under weak conditions*, Lett. Math. Phys. 35 (1995), 1–12.

[33] P. Stollmann, *A convergence theorem for Dirichlet forms with applications to boundary value problems with varying domains*, Math. Z. 219 (1995), 275–287.

[34] Z. X. Wang and D. R. Guo, *Special functions*, World Scientific Publishing Co. Inc., Teaneck, NJ, 1989.

**List of Figures**

1. Dependence of the lower bound \( \lambda(\kappa, \alpha) \) on the curvature \( \kappa \) for \( a = 1 \) and different values of \( \alpha \). (All curves meet at \( \kappa = 1/a \), the small gap for the curve with \( \alpha = +\infty \) is due to a numerical inaccuracy.) 

20