Minimizing volatility increases large risks

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Abstract: We introduce a faithful representation of the heavy tail multivariate
distribution of asset returns, as parsimonious as the Gaussian framework. Using
calculation techniques of functional integration and Feynman diagrams borrowed from
particle physics, we characterize precisely, through its cumulants of high order, the
distribution of wealth variations of a portfolio composed of an arbitrary mixture of
assets. The portfolio which minimizes the variance, i.e. the relatively “small” risks,
often increases larger risks as measured by higher normalized cumulants and by the
Value-at-risk.
Finance is all about risks and risk is usually quantified by the volatility. As is now well recognized, due to the presence of heavy tails and long-range correlations, the volatility is only an imperfect measure of risk. In principle, the risk associated with a given portfolio is fully embedded in the multivariate distribution of the returns of these assets. Practically, dealing with this multivariate distribution is a formidable task for both its specification (important for scenario simulations), for portfolio optimization and for the control of risks. Until now, simpler one-dimensional measures of risks have been developed, for instance in terms of the Value-at-risk. However, they suffer from their reliance on a stable and accurate determination of the covariance matrix of returns, which is problematic in the presence of heavy tails and of time-varying volatilities and correlations. A variety of methods have been also proposed that are however all limited in their domain of application.

Here, we focus our attention upon the “fat tail” problem, having in mind that a large part of the time-varying volatilities and correlations may result from their unstable determination precisely due to the presence of non-gaussian effects. Generalization of our “fractal” covariance matrix approach described below in the spirit of GARCH models is straightforward and will be described elsewhere.

To address the “fat tail” problem, we present three important innovations. First, we develop a new method that provides an approximate but faithful representation of the full multivariable “fat tail” distribution of asset returns. Second, we adapt theoretical tools from theoretical physics to calculate precisely the distribution of returns of the full portfolio. Third, we compare different portfolio optimization procedures and show that minimizing the variance is not optimal as it may often increase large risks. We provide the relevant tools for better optimization suitable to a given risk aversion.

1 “Fractal” Covariance Approximation

Consider two heterogeneous assets, such as the US index SP500 and the Swiss Franc (CHF), both quoted in US dollars. The empirical joint bivariate distribution of their daily annualized returns

\[ r_i(t) = 250 \ln \frac{s_i(t + 1)}{s_i(t)} \]

is plotted in Fig.1 for the time interval from Jan. 1971 to Oct. 1998. \( s_i(t) \) is the price at time \( t \) valued in US dollars, where \( i = 1 \) for the SP500 and \( i = 2 \) for the CHF. The contour lines define the probability confidence level: 95% of the events fall inside the domain limited by the outer line. Thus, there is a 5% probability to observe events falling outside. The other confidence levels of 90%, 50% and 10% are similarly defined. Fig. 1 also shows the marginal distributions for the SP500 and the CHF in US$. The abcissa axis are the same as for the bivariate representation so that the projection from the bivariate to the monovariate distributions is highlighted. The ordinate of the marginal distributions uses a logarithmic scale: a linear plot then qualifies an exponential distribution. One can observe that, while the distributions are not far from an exponential, they exhibit a slightly upward curvature in the tails indicating a slightly more heavy tail than the exponential.
1.1 Contracting maps as a new quantification of departure from Gaussian

Let us call $F_1(r)$ and $F_2(r)$ their cumulative marginal distributions, giving the probability that the return be less than $r$. Let us introduce the transformation $r_1 \rightarrow y_1$ and $r_2 \rightarrow y_2$ which transforms $F_1(r_1)$ and $F_2(r_2)$ into Gaussian distributions with unit variance. By the conservation of probabilities, this reads

$$F(r_1, r_2) = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{y_1 y_2}{\sqrt{2}} \right) \right],$$

where erf($y$) is the error function. We can rewrite it in order to make explicit the nonlinear transformation from the $r$ variables to the $y$ variables:

$$y_1(r_1, r_2) = \sqrt{2} \text{erf}^{-1} \left( 2 F_1(r_1, r_2) - 1 \right)$$

where erf$^{-1}$ is the inverse of the error function. The Gaussian $y$ variables, together with the nonlinear transformation (3), embody fully and with no approximation the heavy tail nature of the marginal distributions.

As an analytic illustration, consider stretched exponential (Weibull) distributions of the form

$$P(r) \equiv \frac{dF}{dr} = \frac{c}{2\sqrt{\pi}} |r|^{\frac{5}{2} - 1} e^{-|r|^c r_0^c}. \tag{4}$$

This function provides a reasonable fit to the distributions $F_1(r_1)$ and $F_2(r_2)$, especially in the tails as shown in Fig. 1, with $A_1 = 4500$, $c_1 = 0.7$, $r_{01} = 0.79$ and $A_2 = 700$, $c_2 = 1.1$, $r_{02} = 2.13$. It is clear that the tails are much “fatter” than for a Gaussian.

In this case, the change of variable (3) can be written, using a slight change of normalization, as

$$y_i(t) = \text{sign}(r_i(t)) \left| r_i(t) \right|^{\frac{c_i}{2}}. \tag{5}$$

The distribution of $y_i$ is Gaussian with a variance now equal to $V_{ii} = (r_{ii})^{c_i}$.

We stress that the transformation (3) is exact and valid for any distribution. It will be used in the simulations. In contrast, expression (5) is exact only for Weibull distributions. It is found to provide a good approximation of the tails of the return distributions. It is used below to present the novel theoretical approach.

Fig. 2 shows $y_1$ as a function of $r_1$ and $y_2$ as a function of $r_2$ from the data and the comparison with (3) using the same parameters as above and shown in Fig. 1. The negative returns have been folded back to the positive quadrant. The fits with expression (3) shown in Fig. 2 are good for the large values, while deviations for small returns indicate that the departure from a Gaussian is less strong in the center of the distributions. This plot provides a novel quantification of the departure from a Gaussian. The downward curvatures result from the fact that the tails of the distribution are “fatter” than a Gaussian: the $r \rightarrow y$ transformation is thus a contracting map.
1.2 Optimal multivariate distributions for “fat tails”

To put our next step into its relevant context, we recall that multivariate Gaussian distributions have played and still play a key role, not only because they are convenient to use, but also because they are optimal in an information theoretical sense: with the only prior information of the covariance matrix, they contain the least possible assumptions, in other words they are the most likely representation of the data. As already pointed out, they are however inconsistent with the presence of heavy tails and non-normal dependence. In this light, we can now capitalize upon the transformation (3) and include the information on the heavy tails to better characterize the multivariate asset return distributions. In this goal, the dependence between the assets is characterized by the covariance matrix $V$ of the transformed Gaussian variables $y$’s:

$$V = \langle Y Y^T \rangle - \langle Y \rangle \langle Y^T \rangle,$$  

where $\langle Y \rangle$ denotes the expectation of $Y$ and $Y$ is the unicolumn matrix with elements $y_1$ and $y_2$. This generalizes straightforwardly for a larger number $N$ of assets. For stretched exponential variables for which the relation (5) holds, the definition (6) leads to the covariance elements

$$V_{ij} = \langle \left( \frac{\text{sign}(r_i)}{|r_i|^{c/2}} \right) \left( \frac{\text{sign}(r_j)}{|r_j|^{c/2}} \right) \rangle - \langle \left( \frac{\text{sign}(r_i)}{|r_i|^{c/2}} \right) \rangle \langle \left( \frac{\text{sign}(r_j)}{|r_j|^{c/2}} \right) \rangle$$  

(7)

Note that the essential information on the sign of the returns is kept while a fractional power of their amplitudes is taken, hence the term “fractal” covariance matrix (that we keep even for the general case to refer to the contracting nature of the $r \rightarrow y$ mapping). $V_{ij}$ has a faster convergence rate for sparse data and is better behaved statistically than the usual covariance matrix since it is less sensitive to large fluctuations due to the small power $c/2$. As a test, we have verified that the normalized correlation coefficient $\rho_V \equiv \frac{V_{12}}{\sqrt{V_{11}V_{22}}}$ for the covariance matrix for the $y_1$ and $y_2$ variables is significantly more stable than the usual correlation coefficient $\rho_r \equiv \frac{r_{12}}{\sqrt{r_{11}r_{22}}}$ for the covariance matrix of the returns $r_1$ and $r_2$, as a function of time in running windows of various sizes. The introduction of ARCH models and their generalizations has been motivated by the observed non-stationarity of the usual covariance matrix [2]. The improved stability of $\rho_V$ suggests that this non-stationarity results in part from the inadequacy of the covariance matrix to provide an efficient characterization of the asset risk profiles, resulting from the presence of “fat tails”. Our new approach directly addresses this problem.

Conditionned only on the measurement (6) of the “fractal” covariance matrix $V$, the most likely representation of the time series becomes the usual Gaussian multivariate distribution in terms of the $y$ variables:

$$\hat{P}(Y) = (2\pi)^{-N/2} |V|^{-1/2} \exp \left( -\frac{1}{2} \left( Y^T - \langle Y^T \rangle \right) V^{-1} \left( Y - \langle Y \rangle \right) \right),$$  

(8)

where $|V|$ is the determinant of $V$. We stress that this parameterization is fundamentally different from the usual Gaussian approximation on the price returns $r$. To get the implied multivariate distribution $P(R)$ in terms of the return variables $R^T = \{r_1, r_2\}$, we use the identity $P(R) = \hat{P}(Y) \frac{dY}{dR}$, where $\frac{dY}{dR}$ is the jacobian of the
transformation from $R \rightarrow Y$:

$$P(R) = |V|^{-1/2} \exp \left( -\frac{1}{2} (Y^T - \langle Y^T \rangle) (V^{-1} - I) (Y - \langle Y \rangle) \right) \prod_{j=1}^{N} \frac{dF_j(r_j)}{dr_j}, \quad (9)$$

where $V$ is again the covariance matrix for $Y$ (i.e. the “fractal” covariance matrix for $R$) and $I$ is identity matrix. Changing the normalization as in the change of variable (5) leads to the same form (9) except for the identity matrix $I$ being changed into the diagonal matrix of elements $V_{ii} = (r_0^c)$. This representation is exact for arbitrary uncorrelated variables, in which case $V = I$. It is also exact for a Gaussian distribution modified by monotonic one-dimensional variable transformations for any number of variables, or equivalently by multiplication by a non-negative separable function. This method has recently been independently introduced in the context of multivariate distributions of particle physics data [3].

Fig. 3 presents the bivariate distribution $\hat{P}(Y)$ obtained from Fig. 1 using the transformation (3) as well as the corresponding Gaussian marginal distributions. The contour lines are defined as in Fig. 1. Note their smooth elliptic shape that contrast with the diamond shape shown in Fig. 1. The principal axis of the ellipses are almost perfectly along the $y_1, y_2$ axis, a signature of the weak “fractal” correlation between the SP500 and the CHF. In the limit of absence of correlation, the ratio of the small over large principal axis is equal to $\sqrt{\frac{V_{11}}{V_{22}}}$.

As a simple and efficient “goodness of fit” test for the reliability of this representation (9), we have studied the fraction of events (points) shown in Fig. 3 within an ellipse of equation $\chi^2 = (Y^T - \langle Y^T \rangle) V^{-1} (Y - \langle Y \rangle)$ as a function of the $\chi^2$ density $(1/2) e^{-\chi^2/2}$ for two degrees of freedom. We observe a very straight bisector line which qualifies the multivariate Gaussian representation (9). Varying $\chi^2$ from 0 to 1 spans the distribution from the small most probable returns to the large least probable returns.

## 2 Characterization of portfolios

### 2.1 Empirical investigation

We can now capitalize upon the rather good stationarity properties of the representation of the bivariate distributions provided by (9) and use this information to optimize portfolios and characterize risks. Consider a portfolio investing a fixed fraction $p$ of its wealth $W$ in the SP500 and the remaining fraction $1 - p$ in the CHF. Using the historical time series, we construct numerically the time series $W(t)$ from the recursion

$$W(t + 1) = pW(t)s_1(t) + (1 - p)W(t)s_2(t) \quad (10)$$

which ensures that $p$ is fixed. The annualized daily return $r_W$ of $W(t)$ is defined by $r_W(t) = 250 \ln \frac{W(t+1)}{W(t)}$. Fig. 4 shows the dependence as a function of $p$ of the variance

$$C_2 \equiv \langle (r_W - \langle r_W \rangle)^2 \rangle \quad (11)$$
and of the kurtosis
\[ \kappa \equiv \frac{C_4}{C_2^2} = \frac{\langle (r_W - \langle r_W \rangle)^4 \rangle}{\langle (r_W - \langle r_W \rangle)^2 \rangle^2} - 3, \] (12)

of the daily portfolio returns. The kurtosis quantifies the deviation from a Gaussian distribution and provides a measure for the degree of “fatness” of the tails, i.e. a measure of the “large” risks. Taking into account only the variance and the kurtosis and neglecting all higher order cumulants, a distribution can be approximated by the following expression valid for small kurtosis \[7\]
\[ P(r_W) \simeq \exp\left[ -\frac{(r_W - \langle r_W \rangle)^2}{2C_2} \left( 1 - \frac{5\kappa (r_W - \langle r_W \rangle)^2}{12C_2} \right) \right]. \] (13)

The negative sign of the correction proportional to \( \kappa \) means that large deviations are more probable than extrapolated from the Gaussian approximation. For a typical fluctuation \(|S - \langle S\rangle| \sim \sqrt{C_2}\), the relative size of the correction in the exponential is \(\frac{5\kappa}{12}\). For the large values of \( \kappa \) found below this approximation (13) break down and the deviation from a Gaussian is much more dramatic.

As seen in Fig. 4, the variance has a well-defined quadratic minimum at \( p_V = 0.375 \). The kurtosis has a S-shape with two local minima at \( p_{\kappa_2} = -0.405 \) (absolute minimum) and \( p_{\kappa_1} = 0.125 \) (local minimum). The table gives the corresponding variance \( C_2 \) and kurtosis \( \kappa \) for these three portfolios and for the benchmark \( p_B = 0.5 \).

| \( p \) | \( C_2 \) | \( \kappa \) | \( r_\ell \) | \( c \) | VaR (20 days) | VaR (10 years) |
|---|---|---|---|---|---|---|
| \( p_B = 0.5 \) | 2.42 | 19.9 | 1.0 | 0.75 | -3.77 | -19.4 |
| \( p_V = 0.375 \) | 2.28 | 9.53 | 1.77 | 1.09 | -4.41 | -13.6 |
| \( p_{\kappa_1} = 0.125 \) | 2.85 | 4.20 | 3.44 | 1.73 | -6.12 | -12.4 |
| \( p_{\kappa_2} = -0.405 \) | 7.77 | 3.92 | 4.39 | 1.35 | -9.19 | -22.8 |

Table: \( p \) (resp. \( 1 - p \)) is the weight in value invested in the SP500 (resp. CHF). \( C_2 \) (resp. \( \kappa \)) is the variance (resp. kurtosis) of the distribution of returns of the portfolios. \( r_\ell \) and \( c \) are the scale and exponent of the Weibull fit to their tail. The last two columns report the calculated Value-at-Risk at the 95% and 99.96% confidence levels.

The conclusion of this analysis is striking: the portfolio with \( p_{\kappa_1} = 0.125 \) has a variance only 25% higher than that of the minimum variance portfolio while its kurtosis is smaller than half that of the minimum variance portfolio. It is thus possible to construct a portfolio which has about the same degree of “small” risks (as measured by the variance) while having significantly smaller “large” risks than would give the standard “mean-variance” portfolio approach \[4\].

This result can also be interpreted in a way that highlights the danger of standard practice: minimizing “small” risks as quantified by the variance may increase (here more than double) the “large” risks. In trouble times of large volatility fluctuations, it is particularly important to recognize this fact. Fig. 5 further exemplifies this phenomenon by plotting the cumulative distributions \( F(r_W) \) for the four portfolios...
in an inverse axis representation, corresponding to the so-called Zipf or rank-ordering plot: this representation of the nth largest value as a function of its rank n emphasizes the information in the tail of the distribution. We can collapse the tails of the distributions of the four portfolios by choosing suitable pairs of parameters c and r_ℓ for each portfolio distribution and by plotting \((r_W/r_ℓ)^c\) as a function of \(\ln n\): this collapse is the signature that all the tails are approximately of the same functional form (4) and that we have correctly identified the values of the parameters. The table lists the values of c and r_ℓ that best fit the tail of each portfolio return distribution.

The portfolio with \(p = 0.125\) provides the best compromise with a low variance and a low kurtosis: not surprisingly, the exponent c of its tail is the largest corresponding to the faster asymptotic decay (thinnest tail).

2.2 Theoretical formulation

We now present briefly how these stylized facts can be rationalized by a systematic theory based on the representation (9). Up to a very good approximation, it is harmless and much simpler to replace the returns \(r_i(t)\) defined in (1) by \((s_i(t+1) - s_i(t))/s_i(t)\) and, over reasonable large time intervals (e.g. a year), neglect the variation the denominator in comparison to the variation of the numerator \(\delta s_i(t) \equiv s_i(t+1) - s_i(t)\). The daily wealth variation at time t of a portfolio of N assets reads

\[
\delta W(t) = \sum_{i=1}^{N} p_i \delta s_i(t),
\]

where \(p_i\) is again the weight in value of the \(i\)th asset in the portfolio. We normalize the weights \(\sum_{i=1}^{N} p_i = 1\). Our strategy is to express the \(\delta s_i(t)\) variables as a function of the \(y_i(t)\) using (3) and calculate directly the distribution \(P(\delta W)\) of the daily portfolio wealth variations. We stress that \(P(\delta W)\) embodies completely all possible information on risks and in particular embodies the usual volatility and VAR measures. We illustrate the procedure for the case of Weibull distributions for which (3) reduces to (5):

\[
\delta W(t) = \sum_{i=1}^{N} p_i \text{sign}(y_i) |y_i|^{2c_i}.
\]

The formal expression for \(P(\delta W)\) is

\[
P(\delta W) = C \prod_{i=1}^{N} \left( \int dy_i \right) e^{-\frac{1}{2} Y^T V^{-1} Y} \delta \left( \delta W(t) - \sum_{i=1}^{N} p_i \text{sign}(y_i) |y_i|^{\frac{2}{c_i}} \right).\]

In order to simplify the notation, we assume that the average price variations are zero. It is easy to reintroduce non-zero average returns in the formalism. Taking the Fourier transform of (16), we get

\[
\hat{P}(k) = \frac{1}{(2\pi)^{N/2} \det V^{1/2}} \prod_{i=1}^{N} \left( \int du_i \right) e^{-\frac{1}{2} Y^T V^{-1} Y + ik \sum_{i=1}^{N} p_i y_i^{q_i}}.
\]

where \(c_i = 2/q_i\). We only show the expression (17) for the case where \(q_i\) are integers and odd such that the “interaction” terms sign\((y_i) |y_i|^{\frac{2}{q_i}}\) simplify into \(y_i^{q_i}\). Note that
the case \( q = 3 \) corresponding to an exponent \( c = 2/3 \) is realistic empirically for the SP500 data. Our results below holds for general \( q \)'s. Expression (17) bears strong resemblance with quantities that appear in field theories of particle physics and we have used the relevant “technology” to evaluate it.

For \( q = 1 \), i.e. \( c = 2 \), the change of variable (5) is linear, all integrals are gaussian which yields the standard result that the distribution \( P(\delta W) \) is Gaussian with a variance

\[
C_2 = p^T V p . \tag{18}
\]

This retrieves the results covered by the standard Markovitz’s theory [4] at the basis of the CAPM [5].

Consider now the more general “heavy tail” case of arbitrary \( q > 1 \), i.e. \( c = 2/q < 2 \). For uncorrelated assets, \( V \) is diagonal and the multiple integral becomes the product of one-dimensional integrals. We have shown [6] that cumulants of \( P(W) \) of all orders can be calculated exactly:

\[
C_{2n}(q) = \sum_i C(n, q_i) (p_i^2 v_{ii})^n , \tag{19}
\]

where \( C(n, q) \) is a function of \( n \) and \( q \) [6]. We have \( C(1, q) = (2^q/\sqrt{\pi})\Gamma(q + 1/2) \) and \( C(2, q) = (2^q/\sqrt{\pi}) \Gamma(2q + 1/2) - (3 \, 2^q/\pi) [\Gamma(q + 1/2)]^2 \), where \( \Gamma \) is the Gamma function. In this diagonal case, the \( q \)th power of the variance of \( y_i \) is equal to the variance \( v_{ii} \) of the \( i \)th asset daily price variation \( \delta s_i \), leading to \( (V_{ii})^q = v_{ii} \). We stress that this expression (14) is valid even when \( q \) is real and the interaction term is \( \propto \text{sign}(y_i)|y_i|^q \) and thus applies to arbitrary Weibull exponential distributions. Odd cumulants are vanishing due to our restriction to distribution with zero mean.

It is well-known that, conditioned on mild regularity conditions, the knowledge of all cumulants uniquely determines the distribution function \( P(\delta W) \). We have thus been able to characterize fully in this case all aspects of risks associated to a given portfolio. Recall that the cumulant \( C_2 \) is the variance of the portfolio wealth variation distribution. The normalized fourth cumulant \( \kappa \equiv C_4/C_2^2 \) is its kurtosis. As already mentioned, it is zero for a Gaussian distribution and provides a standard measure of departure from Gaussian. Higher order cumulants quantify the deviation from a Gaussian further in the tail of the distribution.

We have also been able to calculate the cumulants for the correlated case. The calculation is significantly more involved and uses a systematic Feynman diagrammatic procedure [6, 7] that has been invented in quantum electrodynamics [8]. The results and corresponding empirical tests will be given in [6].

This completes our brief summary of our complete analytical determination of the distribution of the portfolio wealth variation for multivariate correlated fat tail multivariate distributions. Our technique can be extended to more general asset distributions of the form \( P(r) = e^{-f(r)} \), as long as \( f(r) \to +\infty \) for \( |r| \to +\infty \) no slower than a power law with positive exponent. This condition covers all cases of practical interest.

We now use these analytical results to generalize our empirical finding that minimizing “small” risks as quantified by the variance often increases significantly the “large” risks.
3 Risk quantification

3.1 Optimal portfolios

To keep the presentation simple, we consider the uncorrelated diagonal case (19). Being presented the full spectrum of cumulants that quantify all possible measures of risks, we now determined two “optimal” portfolios.

- The first portfolio \( P_V \) has the smaller variance. The corresponding asset weights are found to be:

\[
p_1 v_{11} = p_2 v_{22} = \ldots = p_N v_{NN} = \frac{1}{\sum_i v_{ii}},
\]

where \( v_{ii} \) is the variance of the \( i \)th asset. The assets contribute to this portfolio in value inversely proportional to their variance.

- The second portfolio \( P_K \) has simultaneously the smallest kurtosis \( C_4 \) and smallest higher normalized cumulants \( \lambda_{2m} \equiv \frac{C_{2m}}{(C_2)^m} \) for \( m > 2 \). The corresponding asset weights are:

\[
p_1 v_{11}^{1/2} = p_2 v_{22}^{1/2} = \ldots = p_N v_{NN}^{1/2} = \frac{1}{\sum_i v_{ii}^{1/2}}.
\]

Since the normalized cumulants \( \lambda_{2m} \) with \( m \geq 2 \) measure the deviation from a Gaussian in the tail, \( P_K \) minimizes the large risks.

3.2 Small versus large risk optimization

The asset weights given by (21) do not minimize the portfolio variance but do correspond to the smallest possible large risks. Reciprocally, the asset weights given by (20) that minimize the portfolio variance, increase the large risks. We state two results among several others that we have obtained that generalize this observation.

Let us denote

\[
X_i \equiv \frac{1}{\sum_{j=1}^N \frac{1}{v_{jj}^{1/2}}}
\]

the relative inverse risk brought by asset \( i \). Let us also call \( \lambda_{2m}^{(K)} \) (resp. \( \lambda_{2m}^{(V)} \)) the normalized cumulant of order \( 2m \) of the portfolio \( P_K \) (resp. \( P_V \)). Then,

\[
\frac{\lambda_{2m}^{(K)}}{\lambda_{2m}^{(V)}} = \frac{1}{N^{m-1}} \left( \frac{\sum_i X_i^2}{{\left( \sum_j X_j^2 \right)^m}} \right).
\]

We thus find that \( \lambda_{2m}^{(K)} \) is always smaller or equal to \( \lambda_{2m}^{(V)} \) for \( m \geq 2 \) for all possible values of \( X_i \)’s. The equality occurs only for all \( X_i \)’s being equal to \( 1/N \), i.e. for assets with identical variances. This demonstrates that the weights that minimize the
variance increase the higher normalized cumulants. It also interesting to compare the portfolio $P_K$ with the benchmark portfolio $P_{1/N}$ defined by $p_1 = p_2 = \ldots = p_N = 1/N$. We find

$$\text{ratio} \equiv \frac{\lambda_4^{(V)}}{\lambda_4^{(1/N)}} = \frac{\left(\sum_i \frac{1}{v_{ii}}\right)^2 \left(\sum_j v_{jj}\right)^2}{\left(\sum _k \frac{1}{v_{kk}}\right)^2 \left(\sum_l v_{ll}^2\right)}, \quad (23)$$

Notice that changing all variances $v_{ii}$ into their inverse change the ratio of kurtosis into its inverse. This implies that, if we find a set of $v_{ii}$‘s for which the ratio of kurtosis is smaller than one, then the set of the inverses $1/v_{ii}$’s gives a ratio of kurtosis larger than one. This proves that there are many situations for which minimizing the variance of the portfolio may either increase its kurtosis and therefore its large risks as compared to that of the benchmark.

### 3.3 Empirical test

#### 3.3.1 Kurtosis

Fig. 6 compares the dependence of the empirical kurtosis shown in Fig. 4 to the prediction obtained from Eq. (19) of the theory. We use the result for uncorrelated assets as the coefficient of correlations are small $\rho_v \approx \rho_V \approx 0.03$. We have checked that taking into account the non-zero value of $\rho$ does not change significantly the results.

We show six theoretical curves for all the combinations of the values $c_1 = 0.7$, $c_1 = 0.8$ and $c_2 = 1.05$, $c_2 = 1.1$ and $c_2 = 1.15$. For relatively large positive (‘long’) and negative (‘short’) weights $p$ of the SP500, the kurtosis $\kappa$ is mostly sensitive to the estimation of the exponent $c_1$ of the SP500 return distribution, because the SP500 has the fatest tail (smallest exponent $c$). For small values of $p$, the reverse is true and the portfolio kurtosis is mostly sensitive to the exponent $c_2$ of the CHF return distribution. The empirical determination shown in Fig. 4 is replotted as circles. In the domain of $p$ with reasonable variance and kurtosis, we find a quite good agreement for $c_1 = 0.75, c_2 = 1.15$. The other theoretical curves provide the range of uncertainty in the kurtosis estimation coming from measurement errors in the exponents $c$. The main point here is that the theory adequately identifies the set of portfolios which have small kurtosis and thus small ‘large risks’ and still reasonable variance (‘small risk’). We stress the importance of such precise analytical quantification to increase the robustess of risk estimators: historical data becomes notoriously unreliable for medium and large risks for lack of suitable statistics.

#### 3.3.2 Value-at-Risk

As a final test, we show how the different portfolios perform with respect to the Value-at-Risk (VaR) at different confidence levels. Recall that the VaR determines the probability of a portfolio of assets losing a certain amount in a given time period due to adverse market conditions with a particular level of confidence $C_L$. For instance, a VaR-measure of one million dollars at the $C_L = 95\%$ level of confidence implies that total portfolio losses would not exceed one million dollars more than
1 − \( C_L = 5\% \) of the time (i.e. typically one day in twenty) over a given holding period. In essence, VaR provides a measure of extreme events that occur in the lower tail of the portfolio’s return distribution.

We have estimated the VaR for each of the four portfolios both from historical data and from the stretched exponential model. For each weight, we constructed the distribution of returns \( P(r_W) \) obtained from (10) and estimated directly the VaR such that the fraction of negative returns smaller than VaR is \( 1 - C_L \). Mathematically, this corresponds to determine the return \( r_W \) such that \( F(r_W) = 1 - C_L \). The corresponding VaRs at the \( C_L = 95\% \) level are given in the table. This confidence level corresponds to a typical maximum daily loss encountered once every 20 days.

An independent estimation was performed by using the fits of the distributions of \( r_W \) by stretched exponentials, with the values of \( c \) and \( r_\ell \) reported in the table. That the portfolio distributions can still be considered of this form in their tail is validated by an “extreme deviation” theorem [10]. Then, the VaR is solution of

\[
C_L = \frac{1}{2\alpha} \left[ 1 + \text{erf} \left( \frac{(\text{VaR}/r_\ell)^{\frac{1}{c}}}{\sqrt{2}} \right) \right],
\]

which has to be solved with respect to \( \text{VaR} \). The additional multiplicative factor \( \alpha \approx 10 \) accounts for the empirical fact that the stretched exponential is valid only in the tail of the distribution. \( \alpha \) has been calibrated for one confidence level and checked to remain approximately the same for the others.

This calibration allows us to predict the VaR at higher confidence levels, i.e. for daily losses that can typically occur over longer period of times than 20 days. For relatively low confidence interval like \( C_L = 95\% \), we find that the VaR for the Variance portfolio \( P_V \; p = 0.375 \) is significantly smaller than that for the kurtosis portfolio \( P_K \) with \( p = 0.125 \). But since the exponent \( c \) of \( P_K \) is larger than that of \( P_V \), the tail of \( P_K \) is bounded to become thinner and the VaR of the kurtosis portfolio \( P_K \) is bound to become smaller than that of the variance portfolio \( P_V \) at high confidence levels. We calculate that the cross-over occurs approximately at a confidence level of 99.93% corresponding to a typical largest daily loss of about 12% occurring once every five years. For larger time horizon, the kurtosis portfolio becomes better, having a smaller VaR. We show the VaR at the confidence level of 99.96% corresponding to the decadal daily shock, i.e. to the typical largest loss seen once every ten years. As expected, the kurtosis portfolio has the smallest VaR.
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FIGURE CAPTIONS

Fig. 1: Bivariate distribution of the daily annualized returns of the SP500 US index 1 and of the CHF 2 (in US $) for the time interval from Jan. 1971 to Oct. 1998. Half of the data is represented for clarity of the figure. The contour lines define the probability confidence level of 95% (outer line), 90%, 50% and 10%. The upper and left diagrams show the projected marginal distributions for the SP500 and the CHF in US$ and their fit to (3). The parameters of the fit are $A_1 = 4500$, $c_1 = 0.7$, $r_{01} = 0.79$ and $A_2 = 700$, $c_2 = 1.1$, $r_{02} = 2.13$.

Fig. 2: Dependence of the gaussian variables defined by (3) as a function of the return for the SP500 and CHF data shown in Fig. 1. The negative returns have been folded back onto the positive quadrant. The continuous lines are given by (5) with $c_1 = 0.7$ and $c_2 = 1.1$ respectively for the SP500 1 and CHF 2.

Fig. 3: Bivariate distribution $\hat{P}(Y)$ obtained from Fig. 1 using the transformation (3). The contour lines are defined as in Fig. 1. The upper and left diagrams show the corresponding projected marginal distributions, which are gaussian by construction of the change of variable (3). Both are fitted by the continuous line of equation $P_{1,2} = 150 \exp(-|y_{1,2}|^2/2)$.

Fig. 4: Empirical dependence as a function of $p$ of the variance $C_2$ and of the kurtosis $\kappa$ of the distribution of returns $r_W(t) = 250 \ln \frac{W(t+1)}{W(t)}$ of a portfolio with a fraction $p$ (resp. 1-$p$) in value invested in the SP500 index (resp. in the CHF), whose total value is given by (10). The variance has a well-defined quadratic minimum at $p_V = 0.375$. The kurtosis has a S-shape with two local minima at $p_{\kappa 2} = -0.405$ (absolute minimum) and $p_{\kappa 1} = 0.125$ (local minimum). The table gives the corresponding variance $c_2$ and kurtosis $\kappa$ for these three portfolios and for the benchmark $p_B = 0.5$.

Fig. 5: Rescaling of the distributions $P(r_W)$ of returns $r_W$ obtained from the four portfolios studied in the table. The rescaling uses for the ordinate the reduced variable $(r_W/r_e)^c$ where the exponent $c$ and the characteristic return scale $r_e$ have been determined by a direct fit to each portfolio return distributions. The abcissa is the rank $n$ of the $n$th largest value plotted along the ordinate. This rank-ordering plot, which is the same as a cumulative plot, but with reversed axis, emphasizes the information contained in the tail. The symbols correspond to: + : $p = -0.405$; o : $p = 0.125$; * : $p = 0.375$; x : $p = 0.5$. The straight line has equation $7.50 - 1.16 \ln n$.

Fig. 6: Comparison of the empirical kurtosis (circles) shown in Fig. 4 with the prediction obtained from Eq. (19) of the theory. The six theoretical curves correspond to all combinations of pairs of values $c_1 = 0.7, c_1 = 0.9$ and $c_2 = 1.15$ (solid line); $c_2 = 1.1$ (dotted-dashed line) and $c_2 = 1.05$ (dotted line).
$c_1 = 0.8$

$\kappa$

Weight $p$

$c_1 = 0.7$