Quantum Group based Modelling for the description of high temperature superconductivity and antiferromagnetism

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Abstract

Following our recent conjecture to model the phenomena of antiferromagnetism and superconductivity by quantum symmetry groups, we propose in the present note three toy models, namely, one based on $SO_q(3)$ the other two constructed with the $SO_q(4)$ and $SO_q(5)$ quantum groups. Possible motivations and rationale for these choices are outlined. One of the prime motivations underlying our proposal is the experimental observation of stripe structure [phase] in high $T_c$ superconductivity [HTSC] materials. A number of experimental techniques have recently observed that the CuO$_2$ are rather inhomogeneous, providing evidence for phase separation into a two component system, i.e. carrier-rich and carrier-poor regions. In particular, extended x-ray absorption fine structure [EXAFS] demonstrated that these domains forms stripes of undistorted and distorted local structures alternating with mesoscopic length scale comparable with coherence length in HTSC.

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The Hubbard Hamiltonian [HH] and its extensions dominate the study of strongly correlated electrons systems and the insulator metal transition [1]. One of the attractive feature of the Hubbard Model is its simplicity. It is well known that in the HH the band electrons interact via a two-body repulsive Coulomb interaction; there are no phonons in this model and neither in general are attractive interactions incorporated. With these points in mind it is not surprising that the HH was mainly used to study magnetism. In contrast superconductivity was understood mainly in light of the BCS theory, namely as an instability of the vacuum [ground-state] arising from effectively attractive interactions between electron and phonons. However Anderson [2] suggested that the superconductivity in high T,c material could arise from purely repulsive interaction. The rationale of this suggestion is grounded in the observation that superconductivity in such materials arises from the doping of an otherwise insulating state. Thus following this suggestion the electronic properties in such a high T,c superconductor material close to a insulator-metal transition must be considered. In particular the one-dimensional HH is considered to be the most simple model which can account for the main properties of strongly correlated electron systems including the metal-insulator transition. Long range antiferromagnetic order at half-filling has been reported in the numerical studies of this model [3,4]. Away from half-filling this model has been studied in [5].

In theories based on magnetic interactions [2] for modelling of HTSC, it has been assumed that the CuO₂ planes in HTSC materials are microscopically homogeneous. However, a number of experimental techniques have recently observed that the CuO₂ are rather inhomogeneous, providing evidence for phase separation into a two component system. i.e. carrier-rich and carrier-poor regions [6]. In particular, extended x-ray absorption fine structure [EXAFS] demonstrated that these domains forms stripes of undistorted and distorted local structures alternating with mesoscopic length scale comparable with coherence length
in HTSC. The neutron pair distribution function of Egami et al. [7] also provides structural
evidence for two component charge carriers. Other techniques also seem to point that be-
low a certain temperature $T^*$ the CuO$_2$ planes may have ordered stripes of carrier-rich
and carrier-poor domains [7]. The emergence of experimental evidence for inhomogeneous
structure has led to renewal of interest, in theories of HTSC which are based on alterna-
tive mechanism, such as phonon scattering, the lattice effect on high $T_c$ superconductivity
[8–10,7]. Polarized EXAFS study of optimally doped YBa$_2$Cu$_3$O$_y$ shows in-plane lattice
anomaly [8] below a characteristic temperature $T^{*′}$ which lies above $T_c$, and close to the
characteristic temperature of spin gap opening $T^*$. It is an interesting question if the in-
plane lattice anomaly is related to the charge stripe or spin-phonon interaction. We note
that it has been attempted in [11,12] to relate the spin gap observed in various experiments
such as NMR, neutron scattering and transport properties to the short-range ordering of
spin singlets.

Zhang [13] proposed a unified theory of superconductivity and antiferromagnetism
based on SO(5) symmetry and suggested that there exists an approximate global SO(5)
symmetry in the low temperature phase of the high $T_c$ cuprates. In this model one has
a five component order parameter. Three components correspond to a spin one, charge
zero particle-hole pair condensed at the center of mass momentum ($\pi, \pi$), these components

\[ \text{The following can be taken as a definition of } T^*: \quad T^* \text{ is an onset temperature of pseudogap opening in spin or charge excitation spectra.} \]

\[ T^{*′} \text{ may be defined as follows: } T^{*′} \text{ is an onset temperature of local phonon anomalies and } T^{*′} < T^*. \]

\[ \text{We note that theories of cuprates based upon a quantum critical point have also been suggested by others, see for e.g. Sachdev et al. [15].} \]
represent antiferromagnetic order in the middle of Mott insulating state. The remaining two components correspond to a spin-singlet charge $\pm 2e$ Cooper pair of orbital symmetry $d_{x^2-y^2}$ condensed in zero momentum state, these last two components are supposed to correspond to superconductivity in the doped Mott insulator. Baskaran and Anderson [14] have presented an elegant series of criticisms of the work in Ref. [13]. In view of the critique presented in Ref. [14], and the features of quantum groups we were led to our conjecture that at the simplest level and as a preliminary step, it is tempting to base a model for superconductivity and antiferromagnetism on a quantum group symmetry rather than the usual classical Lie group [24].

We now recall some useful details about quantum groups [17,16] and our simple discussion about quantum groups outlined in [24]. We caution the reader that currently there is no ‘satisfactory’ general definition of a quantum group**. However it is commonly accepted [17] that quantum groups are certain ‘well-behaved’ Hopf algebras and that the standard deformations of the enveloping Hopf algebras of semisimple Lie algebras and of coordinate Hopf algebras of the corresponding Lie groups are guiding examples. An amazing feature of quantum group theory is the unexpected connections with apparently unrelated concepts in physics and mathematics such as Lie Groups, Lie Algebras and their representations, special functions, knot theory, low-dimensional topology, operator algebras, noncommutative geometry, combinatorics, quantum inverse scattering problem, theory of integrable models, conformal and quantum field theory and perhaps others.

We consider even the modelling of HTSC materials based on quantum groups as a preliminary step, since it is quite likely that a realistic model which unifies a complex system containing antiferromagnetic and superconducting phases, may require the mathematical

**We mean in terms of rigorous mathematics
machinery currently being used for string theory, or something even beyond it. Another important motivation for our conjecture is to model Stripes. Stripes can be aptly described as being found in the unstable two-phase region between the antiferromagnetic and metallic states [18]. One of the real challenges is to formulate a theory which gives rise to the equivalent of “Fermi surface”†† whose excitation surface derives from fluctuations that are not uniform in space. Intuitively one may imagine these non-uniform fluctuations as arising out of a nonlinear sigma-like model. We further conjecture that superconductivity arises when two immiscible phases, namely a 2-D antiferromagnetic state and a 3-D metallic state, are “forced” to meet at $T_c \to \infty$. As is well-known, ordinary low temperature superconductors arise entirely out of a “metallic” like state. In contrast, High $T_c$ superconductors have relatively large $T_c$ since we cannot smoothly map two immiscible states [metallic and antiferromagnetic] together. It is the lack of this smooth mapping that is precisely responsible for the High $T_c$. The lack of smooth mappings may be modelled by the nontrivial phases which arise out of the braiding operations of quantum groups.

In lieu of our remarks on quantum groups [24] to model antiferromagnetism and superconductivity we first consider a simple toy model, namely $\text{SO}_q(3)$. Why $\text{SO}_q(3)$? To this end we recall the observation that superconductivity in HTSC materials arises from the doping of an otherwise insulating state. On the other hand it is known that the low energy effective Hamiltonian of an antiferromagnet can be described by the $\text{SO}(3)$ nonlinear $\sigma$ model [19]. There are also claims [20–22] that the $\text{SO}(3)$ nonlinear sigma model may also suffice to describe the underdoped region of the oxide superconductors. Simply put, experimentally

† † We mean, as is usual [18], that with a metallic state we can always associate a “Fermi surface” which describes the low-energy excitations. This is a finite volume surface in momentum space which arises due to all the one-particle amplitudes [18].
we know that the doping of oxide antiferromagnet leads to superconductivity for a region of the doping parameter. The actual picture is more complicated for example by the presence of stripes. To be concrete we have in mind the generalized phase diagram as seen in La-SrCuO$_4$ [see Fig. 3.26, page 98 of Ref. [23], where the following phases are enumerated: antiferromagnetic insulator, messy insulating phase, 2D strange metal, 3D metal and superconductor on the temperature versus doping diagram.]. In a simple toy model we may model the antiferromagnetic material plus the doping by SO$_q$(3), where the quantum group parameter $q$ is related to the physical doping $\delta$. In a naive scenario we may take $\delta = q - 1$, when $q = 1$ SO$_q$(3) reduces to SO(3), $\delta = 0$ i.e. zero doping and we recover the SO(3) nonlinear $\sigma$ model description for the low energy effective Hamiltonian.

There are several reasons for choosing SO$_q$(4) and SO$_q$(5) As already mentioned SO(3) is a symmetry group for antiferromagnet insulator at the level of effective Hamiltonian. On the other hand effective Hamiltonian of a superconductor may be described by a U(1) nonlinear sigma model [XY model], for example it was indicated in Ref. [25] that the metal insulator transition may be described by the XY model. It was further pointed out in Ref. [26] that superconducting transition on the underdoped side of oxides by a renormalized classical model. In addition it is worth noting that SO(3) spin rotation and U(1) phase/charge rotation are symmetries of the microscopic t-J model. One of the simplest group to embed SO(3)$\times$ U(1) is SO(5). Now keeping in mind the discussion in [24] and here we propose to model a unified group for antiferromagnetic, superconducting and other phases in HTSC materials by SO$_q$(5). Even in the considerations based on SO(5) the concepts of Hopf maps, quaternions, Yang monopole and Berry phase [27] have crept up. This leads more support to our contention that model for HTSC materials must be based on quantum group rather than the classical group in this case SO$_q$(5) instead of SO(5). A singlet-triplet model has been suggested recently by Mosvkin and Ovchinnikov [28] in light of experimental consider-
ations. This model is based on the Hamiltonian of the two-component spin liquid [28]. The Hamiltonian considered in [28] has SO(4) group symmetry. The calculations and results in [28] are encouraging in that they elaborate and give insight into static and dynamical spin properties of cuprates including paramagnetic susceptibility, nuclear resonance and inelastic magnetic neutron magnetic scattering. However a central issue as recognized in [28] and not dealt with is charge inhomogeneity and phase separation. It is tempting to replace SO(4) by SO_q(4) and see if one could account for charge inhomogeneity and phase separation.

We must also clearly state that we don’t at the moment have a magic insight to tell us which specific quantum group must be chosen to model HTSC. However we feel that the quantum groups arising from the classical orthogonal groups, i.e. SO(N) are a good and worthwhile starting point, since they naturally incorporate the symmetry group of the insulating antiferromagnetic state and are naturally rich enough to accommodate quantum liquid behaviour.

In conclusion, the quantum orthogonal groups SO_q(N) are proposed as potential candidate for modelling the theory of HTSC materials. A strong feature of quantum groups is that they unify classical Lie algebras and topology. In more general sense it is expected that quantum groups will lead to a deeper understanding of the concept of symmetry in physics.

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