A GENERALIZED GROBMAN-HARTMAN THEOREM

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Abstract. We prove that any generalized hyperbolic operator on any Banach space is structurally stable. As a consequence, we obtain a generalization of the classical Grobman-Hartman theorem.

1. Introduction

Let $E$ be a metric space. Recall that two continuous maps $\varphi, \psi : E \to E$ are topologically conjugate if there is a homeomorphism $h : E \to E$ such that $h \circ \varphi = \psi \circ h$. Given a Banach space $X$, an invertible bounded linear operator $T$ on $X$ is said to be structurally stable if there exists $\varepsilon > 0$ such that $T + \varphi$ is topologically conjugate to $T$ whenever $\varphi : X \to X$ is a Lipschitz map with norm $\|\varphi\|_\infty = \sup_{x \in X} \|\varphi(x)\| \leq \varepsilon$ and Lipschitz constant $\text{Lip}(\varphi) = \sup_{x \neq y} \frac{\|\varphi(x) - \varphi(y)\|}{\|x - y\|} \leq \varepsilon$. Structural stability is a fundamental notion in the area of dynamical systems. It was introduced by Andronov and Pontrjagin [1] for a certain class of differentiable flows on the plane. Nowadays, there are many variations of this notion in different contexts. In the definition of structural stability it is usual to consider $C^1$ perturbations with small $C^1$ norm. Here we are following Pugh [15], where it is allowed more general perturbations, namely Lipschitz perturbations with small Lipschitz norm. We refer the reader to [11, 16, 17] for nice expositions about structural stability.

Another fundamental notion in the area of dynamical systems is that of hyperbolicity. Recall that a bounded linear operator $T$ on a complex Banach space $X$ is said to be hyperbolic if its spectrum $\sigma(T)$ does not intersect the unit circle $\mathbb{T}$ in the complex plane. In the case of real Banach spaces, it is required that $\sigma(T_{\mathbb{C}}) \cap \mathbb{T} = \emptyset$, where $T_{\mathbb{C}}$ denotes the complexification of $T$. It is well-known that $T$ is hyperbolic if and only if there are an equivalent norm $\|\cdot\|$ on $X$ and a splitting $X = X_s \oplus X_u$, $T = T_s \oplus T_u$ (the hyperbolic splitting of $T$), where $X_s$ and $X_u$ are closed $T$-invariant subspaces of $X$ (the stable and the unstable subspaces for $T$), $T_s = T|_{X_s}$ is a proper contraction (i.e., $\|T_s\| < 1$), $T_u = T|_{X_u}$ is invertible and is a proper dilation (i.e., $\|T_u^{-1}\| < 1$), and the identification of $X$ with the product $X_s \times X_u$ identifies $\|\cdot\|$ with the max norm on the product.

Let us recall the following classical result from the 1960’s.

Theorem A. Every invertible hyperbolic operator on a Banach space is structurally stable.

This result was originally obtained by Hartman [9] for operators on finite-dimensional euclidean spaces. The general case was independently obtained by Palis [14] and Pugh [15], motivated by an argument in Moser [13].

It is natural to ask: Does the converse of Theorem A hold? It was soon realized that the answer is “yes” in the finite-dimensional setting. Indeed, the 1972 paper [16] by Robbin already contains a proof of this fact. However, the full question was answered only very
recently by Bernardes and Messaoudi [4]. In fact, in this paper it was characterized
the invertible weighted shifts on the spaces $\ell_p(\mathbb{Z})$ ($1 \leq p < \infty$) and $c_0(\mathbb{Z})$ that have
the shadowing property and it was proved that all of them are structurally stable; as
a consequence, examples of structurally stable operators that are not hyperbolic were
obtained, answering the above question in the negative. More precisely, the following
result is contained in [4].

**Theorem B.** Let $Y = \ell_p(\mathbb{Z})$ ($1 \leq p < \infty$) or $Y = c_0(\mathbb{Z})$. Let $w = (w_n)_{n \in \mathbb{Z}}$ be a bounded
sequence of scalars with $\inf_{n \in \mathbb{Z}} |w_n| > 0$ and consider the bilateral weighted backward shift
$B_w : (x_n)_{n \in \mathbb{Z}} \in Y \mapsto (w_{n+1}x_{n+1})_{n \in \mathbb{Z}} \in Y$.

If
\[
\lim_{n \to \infty} \sup_{k \in \mathbb{N}} |w_{-k}w_{-k-1} \cdots w_{-k-n}|^{\frac{1}{n}} < 1 \quad \text{and} \quad \lim_{n \to \infty} \inf_{k \in \mathbb{N}} |w_kw_{k+1} \cdots w_{k+n}|^{\frac{1}{n}} > 1,
\]
then $B_w$ is structurally stable and not hyperbolic.

In the present work we will obtain a result that unifies Theorems A and B. In order
to be more precise, recall that an invertible bounded linear operator $T$ on $X$ is said to
be generalized hyperbolic if we can write
\[
X = M \oplus N,
\]
where $M$ and $N$ are closed subspaces of $X$ such that $T(M) \subset M$, $T^{-1}(N) \subset N$,
\[
\sigma(T|_M) \subset \mathbb{D} \quad \text{and} \quad \sigma(T^{-1}|_N) \subset \mathbb{D},
\]
where $\mathbb{D}$ denotes the open unit disc in the complex plane. This class of operators was
introduced by Bernardes et al. [3], where it was proved that each element of this class
has the shadowing property. But the terminology "generalized hyperbolic" was given by
Cirilo et al. [5], where it was proved that this class is open in the space of all invertible
bounded linear operators. It is clear that this class contains the invertible hyperbolic
operators. It also contains the invertible weighted shifts from Theorem B. In order to
see this, it is enough to consider
\[
M = \{(x_n)_{n \in \mathbb{Z}} \in Y : x_n = 0 \text{ for all } n > 0\},
\]
\[
N = \{(x_n)_{n \in \mathbb{Z}} \in Y : x_n = 0 \text{ for all } n \leq 0\},
\]
and to observe that the spectral radius formula shows that the estimates in (1) give the
inclusions in (3). We will prove in Section 2 that every generalized hyperbolic operator
on a Banach space is structurally stable, which unifies Theorems A and B. The class
of generalized hyperbolic operators contains all the structurally stable operators that
are known up to now. It is an open problem whether or not every structurally stable
operator lies in this class.

The classical Grobman-Hartman theorem asserts that if $p$ is a hyperbolic fixed point
of a $C^1$ diffeomorphism $F$ on a Banach space $X$, then there is a neighborhood of $p$
where $F$ is topologically conjugate to its derivative at $p$. This linearization theorem was
independently obtained by Grobman [7] (announced in [6]) and Hartman [9,10] in the
finite-dimensional setting. The extension to Banach spaces is due independently to Palis
[14] and Pugh [15]. The Grobman-Hartman theorem plays a major role in the areas of
dynamical systems and differential equations. We refer the reader to [8,11,12,17] for
more details on this important theorem and its applications.
In Section 3 we will obtain a generalization of the Grobman-Hartman theorem by showing that we can replace the hyperbolicity hypothesis on the fixed point by generalized hyperbolicity. Moreover, we will prove that, even in this more general case, the homeomorphism conjugating the map and its derivative at the fixed point can be chosen to be $\theta$-Hölder (for suitable values of $\theta$) near the fixed point.

2. Generalized hyperbolic operators are structurally stable

Given Banach spaces $X$ and $Y$, we denote by $U_b(X; Y)$ the Banach space of all bounded uniformly continuous maps $\varphi : X \to Y$ endowed with the supremum norm. In the case $X = Y$, we write $U_b(X)$ instead of $U_b(X; X)$.

Our goal in this section is to prove that generalized hyperbolicity implies structural stability. Actually, we will obtain a formally stronger property, namely: strong structural stability. Recall that an invertible bounded linear operator $T$ on a Banach space $X$ is said to be strongly structurally stable if for every $\gamma > 0$ there exists $\varepsilon > 0$ such that the following property holds: for any Lipschitz map $\varphi \in U_b(X)$ with $\|\varphi\|_{\infty} \leq \varepsilon$ and $\text{Lip}(\varphi) \leq \varepsilon$, there is a homeomorphism $h : X \to X$ such that $h \circ T = (T + \varphi) \circ h$ and $\|h - I\|_{\infty} \leq \gamma$. So, it is now required that the homeomorphism $h$ conjugating $T$ and $T + \varphi$ is close to the identity operator. Although this notion is formally stronger than structural stability, it is still an open problem whether or not these two notions are equivalent.

**Theorem 1.** Every generalized hyperbolic operator on a Banach space is strongly structurally stable.

**Remark 2.** An important difference between the proofs of Theorem 1 and Theorem A is that in case where the operator is generalized hyperbolic and not hyperbolic, the conjugation $H$ is not unique and we have to choose $H = I_d + h$ where $h$ belongs to an adequate space of functions.

**Proof.** Let $T$ be a generalized hyperbolic operator on a Banach space $X$ and let $M$ and $N$ be as in (2) and (3). Let

$$P_M : X \to M \quad \text{and} \quad P_N : X \to N$$

be the projections associated to the decomposition of $X$ given by (2), and put

$$d = \max\{\|P_M\|, \|P_N\|\}.$$

By (3) and the spectral radius formula, there are constants $c \geq 1$ and $0 < t < 1$ such that

(4) $\|T^n y\| \leq c t^n \|y\|$ and $\|T^{-n} z\| \leq c t^n \|z\|$ whenever $n \in \mathbb{N}_0$, $y \in M$ and $z \in N$.

Consider the closed subspace $Y = M + T^{-1}(N)$ of $X$. In order to prove that $T$ is strongly structurally stable, we fix $0 < \gamma < 1$ and put

$$\varepsilon = \frac{\gamma (1 - t)}{cd(1 + t)}.$$

Let $\beta \in U_b(X)$ be a Lipschitz map with $\|\beta\|_{\infty} \leq \varepsilon$ and $\text{Lip}(\beta) \leq \varepsilon$. Put $S = T + \beta$. We have to find a homeomorphism $H : X \to X$ such that $H \circ T = S \circ h$ and $\|H - I\|_{\infty} \leq \gamma$. Actually, our $H$ will be a uniform homeomorphism of the form $H = I + h$, where $h \in U_b(X; Y)$ and $\|h\|_{\infty} \leq \gamma$. We divide the remaining of the proof in five steps.

**Step 1.** For any uniform homeomorphism $R : X \to X$, the bounded linear map

$$\Psi : \varphi \in U_b(X; Y) \mapsto \varphi \circ R - T \circ \varphi \in U_b(X)$$
is bijective. Moreover, its inverse is given by

\[ \Psi^{-1}(\alpha)(x) = \sum_{k=0}^{\infty} T^k P_M(\alpha(R^{k-1}x)) - \sum_{k=1}^{\infty} T^{-k} P_N(\alpha(R^k x)). \]

In particular,

\[ \|\Psi^{-1}(\alpha)\|_{\infty} \leq \frac{c d (1 + t)}{1 - t} \|\alpha\|_{\infty}. \]

Indeed, fix \( \alpha \in U_b(X) \) and suppose that \( \varphi \in U_b(X; Y) \) satisfies \( \Psi(\varphi) = \alpha \), that is,

\[ \varphi(Rx) - T(\varphi(x)) = \alpha(x) \quad \text{for all } x \in X. \]

Then, a simple induction argument shows that

\[ \varphi(R^n x) = T^n(\varphi(x)) + \sum_{k=1}^{n} T^{n-k}(\alpha(R^{k-1}x)) \quad \text{for all } n \in \mathbb{N}. \]

By applying \( T^{-n} \) to both sides of the above equality, we obtain

\[ \varphi(x) = T^{-n}(\varphi(R^n x)) - \sum_{k=1}^{n} T^{-k}(\alpha(R^{k-1}x)) = y_n(x) + z_n(x), \]

where

\[ y_n(x) = T^{-n} P_M(\varphi(R^n x)) - \sum_{k=1}^{n} T^{-k} P_M(\alpha(R^{k-1}x)), \]

\[ z_n(x) = T^{-n} P_N(\varphi(R^n x)) - \sum_{k=1}^{n} T^{-k} P_N(\alpha(R^{k-1}x)). \]

It is clear that \( z_n(x) \in T^{-1}(N) \) for all \( n \in \mathbb{N} \) and \( x \in X \). We claim that \( y_n(x) \in M \) for all \( n \) and \( x \). For this purpose, write

\[ y_n(x) = a_n(x) + b_n(x) \quad \text{with } a_n(x) \in M \text{ and } b_n(x) \in N. \]

Consider the case \( n = 1 \). We have that

\[ T(y_1(x)) = P_M(\varphi(Rx)) - P_M(\alpha(x)) \in M, \]

and so \( T(b_1(x)) = T(y_1(x)) - T(a_1(x)) \in M \) as well. Since \( \varphi(x) = a_1(x) + (b_1(x) + z_1(x)) \), \( a_1(x) \in M \) and \( b_1(x) + z_1(x) \in N \), the fact that \( \varphi(x) \in Y \) implies that \( b_1(x) \) must belong to the set \( T^{-1}(M) \cap T^{-1}(N) = \{0\} \), that is, \( y_1(x) \in M \). Now, suppose that for a certain \( n \geq 1 \), we have that \( y_n(x) \in M \) for all \( x \). Then,

\[ T(y_{n+1}(x)) = y_n(Rx) - P_M(\alpha(x)) \in M. \]

By arguing as above, we conclude that \( b_{n+1}(x) = 0 \), that is, \( y_{n+1}(x) \in M \). By induction, our claim is proved. Thus, \( 9 \) gives \( P_N(\varphi(x)) = z_n(x) \) for all \( n \in \mathbb{N} \). Since \( T^{-n} P_N(\varphi(R^n x)) \to 0 \) as \( n \to \infty \), we obtain

\[ P_N(\varphi(x)) = -\sum_{k=1}^{\infty} T^{-k} P_N(\alpha(R^{k-1}x)) \quad (x \in X). \]

Now, if we apply \( 8 \) with \( R^{-n} x \) in place of \( x \), we get

\[ \varphi(x) = T^n(\varphi(R^{-n} x)) + \sum_{k=0}^{n-1} T^k(\alpha(R^{-k-1}x)) = y'_n(x) + z'_n(x), \]
where
\[ y'_n(x) = T^n P_M(\varphi(R^{-n}x)) + \sum_{k=0}^{n-1} T^k P_M(\alpha(R^{-k-1}x)), \]
\[ z'_n(x) = T^n P_N(\varphi(R^{-n}x)) + \sum_{k=0}^{n-1} T^k P_N(\alpha(R^{-k-1}x)). \]

It is clear that \( y'_n(x) \in M \) for all \( n \in \mathbb{N} \) and \( x \in X \). We claim that \( z'_n(x) \in T^{-1}(N) \) for all \( n \) and \( x \). Indeed, consider the case \( n = 1 \). Since \( \varphi(R^{-1}x) \in Y \), we can write \( \varphi(R^{-1}x) = a + b \) with \( a \in M \) and \( b \in T^{-1}(N) \). Hence,
\[ z'_1(x) = TP_N(\varphi(R^{-1}x)) + P_N(\alpha(R^{-1}x)) = Tb + P_N(\alpha(R^{-1}x)) \in N, \]
which implies that \( z'_1(x) \in T^{-1}(N) \), because \( \varphi(x) \in Y \). Now, suppose that for a certain \( n \geq 1 \), we have that \( z'_n(x) \in T^{-1}(N) \) for all \( x \). Then,
\[ z'_{n+1}(x) = T(z'_n(x)) + P_N(\alpha(R^{-1}x)) \in N, \]
which implies that \( z'_{n+1}(x) \in T^{-1}(N) \), since \( \varphi(x) \in Y \). This proves our second claim. Hence, \((11)\) gives \( P_M(\varphi(x)) = y'_n(x) \) for all \( n \in \mathbb{N} \). Since \( T^n P_M(\varphi(R^{-n}x)) \to 0 \) as \( n \to \infty \), we obtain
\[ P_M(\varphi(x)) = \sum_{k=0}^{\infty} T^k P_M(\alpha(R^{-k-1}x)) \quad (x \in X). \]

By \((10)\) and \((12)\), \( \varphi \) must be unique. On the other hand, the estimates
\[ \sum_{k=1}^{\infty} \| T^{-k} P_N(\alpha(R^{-k-1}x)) \| \leq \frac{cd}{1-t} \| \alpha \|_{\infty}, \] \[ \sum_{k=0}^{\infty} \| T^k P_M(\alpha(R^{-k-1}x)) \| \leq \frac{cd}{1-t} \| \alpha \|_{\infty} \]
show that the series in \((10)\) and \((12)\) converge absolutely and uniformly on \( X \). Therefore, if we define \( \varphi : X \to Y \) by means of equations \((10)\) and \((12)\), we obtain a map \( \varphi \in U_b(X;Y) \). Moreover, an easy computation shows that \((7)\) holds, that is, \( \Psi(\varphi) = \alpha \). This shows that \( \Psi \) is bijective and that \( \Psi^{-1} \) is given by \((5)\). Finally, the estimate \((6)\) follows immediately from the estimates \((13)\) and \((14)\), which completes the proof of Step 1.

**Step 2.** There is a unique \( h \in U_b(X;Y) \) such that the uniformly continuous map \( H = I + h : X \to X \) satisfies
\[ H \circ T = S \circ H. \]
Moreover,
\[ \| H - I \|_{\infty} = \| h \|_{\infty} \leq \gamma. \]

We apply Step 1 with \( R = T \) to obtain the linear isomorphism
\[ \Psi_1 : \varphi \in U_b(X;Y) \mapsto \varphi \circ T - T \circ \varphi \in U_b(X). \]
Since \((15)\) is equivalent to \( h = \Psi_1^{-1}(\beta \circ (I + h)) \), we have that \( h \in U_b(X;Y) \) has the desired property if and only if it is a fixed point of the map
\[ \Phi_1 : \varphi \in U_b(X;Y) \mapsto \Psi_1^{-1}(\beta \circ (I + \varphi)) \in U_b(X;Y). \]
Since $\Phi_1$ is Lipschitz with
\[
\text{Lip}(\Phi_1) \leq \|\Psi_1^{-1}\| \text{Lip}(\beta) \leq \frac{cd(1+t)}{1-t} \varepsilon < 1
\]
(where we have used (6) and our choice of $\varepsilon$), the existence and uniqueness of $h$ follows from Banach’s contraction principle. Moreover,
\[
\|h\|_\infty = \|\Phi_1(h)\|_\infty \leq \|\Psi_1^{-1}\| \|\beta\|_\infty \leq \frac{cd(1+t)}{1-t} \varepsilon = \gamma,
\]
which gives (16).

**Step 3.** There is a unique $h' \in U_b(X;Y)$ such that the uniformly continuous map $H' = I + h' : X \to X$ satisfies
\[
(17) \quad H' \circ S = T \circ H' \circ S.
\]
We apply Step 1 with $R = S$ to obtain the linear isomorphism
\[
\Psi_2 : \varphi \in U_b(X;Y) \mapsto \varphi \circ S - T \circ \varphi \in U_b(X).
\]
In this case, a simple computation shows that (17) is equivalent to $\Psi_2(h') = -\beta$. Thus, $h' = \Psi_2^{-1}(-\beta)$ is the only solution.

**Step 4.** $H' \circ H = I$.

By (15) and (17),
\[
(18) \quad H' \circ H \circ T = T \circ H' \circ H.
\]
Since $H' \circ H = I + u$ with $u \in U_b(X;Y)$, (18) gives $\Psi_1(u) = 0$, and so $u = 0$.

**Step 5.** $H \circ H' = I$.

By (15) and (17),
\[
(19) \quad H \circ H' \circ S = S \circ H \circ H'.
\]
Write $H \circ H' = I + v$ with $v \in U_b(X;Y)$. A simple computation shows that (19) is equivalent to $v = \Psi_2^{-1}(\beta \circ (I + v) - \beta)$, that is, $v$ is a fixed point of the map
\[
\Phi_2 : \varphi \in U_b(X;Y) \mapsto \Psi_2^{-1}((\beta \circ (I + \varphi)) - \beta) \in U_b(X;Y).
\]
As before, $\Phi_2$ is a contraction, and so $\Phi_2$ has $v$ as its unique fixed point. However, now we have $\Phi_2(0) = 0$. Thus, $v = 0$, as was to be shown.

Finally, Steps 4 and 5 show that $H$ is a uniform homeomorphism, completing the proof that $T$ is strongly structurally stable. \qed

3. A generalized Grobman-Hartman theorem

Let $X$ be a Banach space and $F : X \to X$ be a differentiable map. Suppose that $p$ is a fixed point of $F$. We say that $p$ is a generalized hyperbolic fixed point of $F$ if the derivative $DF_p$ of $F$ at $p$ is a generalized hyperbolic operator on $X$, that is, there is a splitting
\[
X = E_p^- \oplus E_p^+,
\]
where $E_p^-$ and $E_p^+$ are closed subspaces of $X$ with $DF_p(E_p^+) \subset E_p^+$, $(DF_p)^{-1}(E_p^-) \subset E_p^-$,
\[
\sigma(DF_p|_{E_p^+}) \subset \mathbb{D} \quad \text{and} \quad \sigma((DF_p)^{-1}|_{E_p^-}) \subset \mathbb{D}.
\]
Let $U$ and $V$ be open subsets of $X$ and let $\theta > 0$. Recall that a homeomorphism $H : U \to V$ is said to be $\theta$-Hölder if there is a constant $c > 0$ such that
\[
\|H(x) - H(x')\| \leq c \|x - x'\|^\theta \text{ for all } x, x' \in U
\]
and
\[
\|H^{-1}(y) - H^{-1}(y')\| \leq c \|y - y'\|^\theta \text{ for all } y, y' \in V.
\]
As a consequence of Theorem 1, we will now obtain a generalization of the Grobman-Hartman theorem to the case of generalized hyperbolic fixed points of $C^1$ diffeomorphisms on Banach spaces. We will also show that the linearization can be chosen to be $\theta$-Hölder near the fixed point, provided $\theta > 0$ is small enough.

**Theorem 3.** Let $X$ be a Banach space and $F : X \to X$ be a $C^1$ diffeomorphism. If $p$ is a generalized hyperbolic fixed point of $F$, then $F$ is topologically conjugated to $DF_p$ near $p$, that is, there exist a homeomorphism $H : X \to X$ and an open neighborhood $U$ of $p$ in $X$ such that
\[
H \circ F = DF_p \circ H \text{ on } U.
\]
Moreover, for $\theta > 0$ small enough, we have that $U$ and $H$ can be chosen so that $H : U \to H(U)$ is a $\theta$-Hölder homeomorphism.

**Proof.** We first assume that $p = 0$. Put $T = DF_0$ and $\alpha = F - T$. We have that $\alpha(0) = F(0) = 0$ and $D\alpha_0 = 0$. By Theorem 1 and the hypothesis on the fixed point $p$, the operator $T$ is structurally stable. Hence, there exists $0 < \varepsilon < \|T^{-1}\|^{-1}$ such that $T + \varphi$ is topologically conjugate to $T$ whenever $\varphi : X \to X$ is a Lipschitz map with $\|\varphi\|_{\infty} \leq \varepsilon$ and $\text{Lip}(\varphi) \leq \varepsilon$. Let $U$ be an open neighborhood of $0$ in $X$ such that $\text{Lip}(\alpha|_{U}) < \frac{\varepsilon}{2}$. By a classical result (see, for example, Lemma 2 of [14]), there exists a bounded Lipschitz map $\beta : X \to X$ such that $\beta|_{U} = \alpha|_{U}$, $\|\beta\|_{\infty} \leq \varepsilon$ and $\text{Lip}(\beta) \leq \varepsilon$. Let $H : X \to X$ be a homeomorphism such that $H \circ (T + \beta) = T \circ H$. Then,
\[
H \circ F = DF_0 \circ H \text{ on } U,
\]
which proves the first assertion in Theorem 3. In order to prove the second assertion, let $M = E_0^+, N = E_0^-, Y = M + T^{-1}(N)$, $S = T + \beta$ and
\[
\Psi : \varphi \in U_b(X; Y) \mapsto \varphi \circ S - T \circ \varphi \in U_b(X).
\]
By Step 3 in the proof of Theorem 1, we can assume that $H$ has the form $H = I + h$, where $h \in U_b(X; Y)$ is given by $h = \Psi^{-1}(-\beta)$. Hence, by (5),
\[
h(x) = -\sum_{k=0}^{\infty} T^k P_M(\beta(S^{-k-1}x)) + \sum_{k=1}^{\infty} T^{-k} P_N(\beta(S^{-k-1}x)).
\]
By renorming $X$, if necessary, we may assume that
\[
\|T|_M\| < 1 \text{ and } \|T^{-1}|_N\| < 1.
\]
We assume $M \neq \{0\}$ and $N \neq \{0\}$, leaving the other cases to the reader. Let
\[
0 < \theta < \min \left\{ -\frac{\ln \|T^{-1}|_N\|}{\ln \|T\|}, -\frac{\ln \|T|_M\|}{\ln \|T^{-1}\|} \right\} \leq 1.
\]
Then
\[
\max \left\{ \|T|_M\| \|T^{-1}\|^\theta, \|T^{-1}|_N\| \|T\|^\theta \right\} < 1.
\]
Since $\beta : X \to X$ is a bounded Lipschitz map and $\theta \in (0, 1]$, $\beta$ is $\theta$-Hölder. More precisely, since $\|\beta\|_{\infty} \leq \varepsilon$ and $\text{Lip}(\beta) \leq \varepsilon$, we have that
\[
\|\beta(x) - \beta(y)\| \leq 2\varepsilon \|x - y\|^\theta \text{ for all } x, y \in X.
\]
On the other hand,

\[(24)\quad \|S^nx - S^ny\| \leq (\|T\| + \varepsilon)^n \|x - y\| \quad \text{for all } x, y \in X.\]

Since \(S^{-1}x = T^{-1}x - T^{-1}(\beta(S^{-1}x))\), we deduce that

\[\|S^{-1}x - S^{-1}y\| \leq \frac{\|T^{-1}\|}{1 - \|T^{-1}\|\varepsilon} \|x - y\| = (\|T^{-1}\| + \varepsilon s)\|x - y\|,\]

where \(s = \|T^{-1}\|^2/(1 - \|T^{-1}\|\varepsilon) > 0\). Thus,

\[(25)\quad \|S^{-n}x - S^{-n}y\| \leq (\|T^{-1}\| + \varepsilon s)^n \|x - y\| \quad \text{for all } x, y \in X.\]

Now, by using (21), (23), (24) and (25), we obtain

\[
\|h(x) - h(y)\| \leq \sum_{k=0}^{\infty} \|T^k P_M(\beta(S^{-k-1}x) - \beta(S^{-k-1}y))\| \\
+ \sum_{k=1}^{\infty} \|T^{-k} P_N(\beta(S^{k-1}x) - \beta(S^{k-1}y))\| \\
\leq C\|x - y\|^\theta,
\]

where

\[
C = 2\varepsilon\|P_M\| \sum_{k=0}^{\infty} \|T\|_M \|k(\|T^{-1}\| + \varepsilon s)^{(k+1)\theta} + 2\varepsilon\|P_N\| \sum_{k=1}^{\infty} \|T^{-1}\|_N \|k(\|T\| + \varepsilon)^{(k-1)\theta}.\]

It follows from (22) that \(C\) is a finite constant provided we choose \(\varepsilon > 0\) small enough. Hence, the map \(h : X \to Y\) is \(\theta\)-Hölder. We know from the proof of Theorem 1 that \(H^{-1} = I + h'\), where \(h'\) is of the same type as \(h\). Thus, the map \(h' : X \to Y\) is also \(\theta\)-Hölder. By choosing \(U\) so that both \(U\) and \(V = H(U)\) have diameters < 1, we conclude that \(H : U \to V\) is a \(\theta\)-Hölder homeomorphism.

Now, suppose that \(p \neq 0\) and consider the \(C^1\) diffeomorphism \(G : X \to X\) defined by

\[G(x) = F(x + p) - p \quad \text{for all } x \in X.\]

Since \(G(0) = 0\) and \(DG_0 = DF_p\), there exist a homeomorphism \(K : X \to X\) and an open neighborhood \(V\) of 0 in \(X\) such that

\[K \circ G = DG_0 \circ K \quad \text{on } V.\]

Consider the open neighborhood \(U = p + V\) of \(p\) in \(X\) and the homeomorphism \(H : X \to X\) given by \(H(y) = K(y - p)\) for all \(y \in X\). Then,

\[H \circ F = DF_p \circ H \quad \text{on } U.\]

Moreover, \(H\) can be chosen to be \(\theta\)-Hölder on \(U\) for \(\theta > 0\) small enough.

\[\square\]

**Remark 4.** The fact that the linearization can be chosen to be locally \(\theta\)-Hölder for small enough \(\theta\) was proved in the case of a hyperbolic fixed point in [2].

We close this work by proposing the following open problem: Does every infinite-dimensional (separable) Banach space support a nonhyperbolic (strongly) structurally stable operator?
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