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Abstract

The soliton formation is considered in the Nambu-Jona-Lasinio model with local four quark interaction and various schemes to regularize the energy contribution of the polarized vacuum. No additional constraints are admitted in order to stabilize the soliton. While solitons are unstable in the proper–time regularized version the three momentum cut–off regularization apparently is more appropriate. Using a semi–classical approach multi-quark solitons obtained from that scheme are discussed. However, no self–consistent non–trivial unit baryon number configuration has been found. We also study a renormalizable extension of the model. In this case no stable multi–quark solitons are obtained within the semi–classical approach.

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1 Introduction

The strong interaction dynamics of mesons and baryons is described by quantum chromodynamics (QCD). It is assumed that this theory explains hadronic phenomena in terms of the quark and gluon degrees of freedom. At low energies, comparable with the low lying hadron masses, QCD exhibits a non–perturbative behavior. This circumstance renders the analytic study of the theory rather difficult. Nevertheless, a qualitative description of important aspects of QCD is possible on the basis of effective theories which have in common with QCD the symmetries of the quark–flavor dynamics, in particular the pattern of spontaneous chiral symmetry breaking. In this respect the Nambu–Jona–Lasinio (NJL) model [1] has been very successful in the description of the meson sector [2, 3].

It is therefore natural to investigate the soliton solutions of the model in order to describe baryons according to the $1/N_C$ expansion. In this context the important issue to be addressed concerns the stability of these soliton configurations. To be specific we consider the NJL model with a local four quark interaction and various regularization prescriptions for the contribution of the polarized vacuum to the mass of the soliton. This is motivated by the common belief that its stability strongly depends on the imposed regularization procedures. For example, in refs [4, 5] the description of baryons as solitons with three valence quarks derived from the NJL model with proper–time regularization [6] is reviewed. For this description to work the scalar and pseudoscalar fields $\sigma$ and $\pi$ must artificially be stabilized by the so–called chiral circle condition $\sigma^2 + \pi^2 = \text{const.}$. However, it is known that without this condition the soliton using proper–time regularization is not stable [7, 8]. Here, we wish to define the stable soliton as a non–trivial mean-field solution, which is stable by itself once the regularization prescription has been fixed. In particular we do not allow for additional constraints which artificially stabilize the field configuration but do not directly follow from the model. This criterion is apparently not met by the just mentioned soliton.

Another example is the application of the proper–time regularization to the imaginary (anomalous) part of the effective action as well [9]. As a consequence the baryon number is no longer quantized and may assume non–integer values. The soliton can then be stabilized by constraining the regularized baryon number e.g. to unity. As this stabilization mechanism originates from a particular regularization prescription the question of whether or not it fulfills our above criterion remains a matter of taste. Alternatively one might add further interaction terms to the model Lagrangian. In that respect the inclusion of a Higgs–type meson self–interaction, i.e. $(\sigma^2 + \pi^2 - \text{const.})^2$, has been shown to render the soliton stable [10].

Not to be misinterpreted, we do not regard these additional constraints as un-

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3This term is conditionally finite and hence does not require regularization. Its regularization might spoil the anomaly structure of the model.
reasonable. The chiral circle condition as suggested by Skyrme type models, which have this condition implemented by definition, may certainly be in accordance with a reasonable description of single baryons as solitons. The baryon number constraint to integer values is obvious, and the forth–order self–interaction is unavoidable in sensibly renormalizable models. An additional forth–order self–interaction may also be motivated by higher multi–quark interactions following from QCD when gluons are integrated out. The point, however, is that these additional constraints do not directly follow from the NJL model together with an adopted regularization procedure.

Recently it has been shown, using the Thomas–Fermi method, that the NJL model with three dimensional momentum cut–off regularization of the Dirac sea, apparently accounts for the existence of stable solitons, provided the number of valence quarks is large enough. Needless to emphasize that the phenomenology of multi–quark or multi–baryon systems requires the $\sigma$ and $\pi$ fields to be treated independently. We compute the critical size of this soliton below which it ceases to be stable. This size is determined by the interplay between the bulk and surface energy densities. For the proper description of the surface the lowest order gradient corrections to the Thomas–Fermi method in the Wigner–Kirkwood expansion are taken into account. These investigations are described in section 2 and are completed in section 3 with a self-consistent calculation for unit baryon number imposing the corresponding regularization scheme on the single particle Dirac Hamiltonian.

Finally, in section 4, we briefly discuss a renormalizable extended version of the NJL model whose solitons are yet unstable, apparently for the same reason as those in the proper–time regularized version. Our conclusions are summarized in section 5.

2 Thomas–Fermi method and Wigner–Kirkwood expansion

Our starting point is the semi–bosonized version of the NJL model where the scalar and pseudoscalar fields $\sigma$ and $\pi$ have been introduced through a Hubbard–Stratonovich transformation to eliminate the original four–fermion interaction

$$ L_{NJL} = \bar{q} [i\gamma^\mu \partial_\mu - (\sigma + i\gamma_5 \tau \cdot \pi)] q - \frac{\sigma^2 + \pi^2}{2G} . $$

We do not consider explicit chiral symmetry breaking. In the vacuum sector the scalar field is identical to the constituent quark mass $m$. In order to prevent the model from becoming a trivial theory of non–interacting mesons a regulator has to be retained or alternatively the necessary counterterms have to be added to the Lagrangian. In the subsequent section we will study the three–momentum cut–off in more detail because this regulator suppresses the high momentum components, which in the proper–time regularization scheme cause the soliton to shrink to a point–like singularity. For that reason we present the following expressions in...
that particular scheme, the generalization to other regularization schemes is straightforward. The three–momentum cut–off \( \Lambda \) is related to the four–fermion coupling \( G \) and the constituent quark mass \( m \) via the gap–equation

\[
\frac{1}{G} = 4N_C \int_{|p| \leq \Lambda} \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{p^2 + m^2}}.
\]  

(2)

In the chiral limit the pion decay constant obtained from

\[
f_\pi^2 = N_C m^2 \int_{|p| \leq \Lambda} \frac{d^3 p}{(2\pi)^3} \frac{1}{(p^2 + m^2)^{3/2}} ,
\]  

(3)

will later be used to determine the value of the constituent mass \( m \) as a function of the cut–off \( \Lambda \).

The semi–classical, relativistic particle and energy densities for fermions in the background of a scalar field are derived within the Wigner–Kirkwood (WK) expansion up to order \( \hbar^2 \) [13, 14, 15]

\[
\rho = \rho^{(0)} + \rho^{(2)}, \quad \mathcal{E} = \mathcal{E}^{(meson)} + \mathcal{E}^{(0)} + \mathcal{E}^{(2)}. \]  

(4)

The leading order (Thomas–Fermi method) was already investigated in Ref. [12] and the next to leading order includes gradient corrections

\[
\rho^{(0)} = -4N_C \int_{p_F \leq |p| \leq \Lambda} \frac{d^3 p}{(2\pi)^3} \rho^{(2)} = \frac{N_C}{6} \int_{p_F \leq |p| \leq \Lambda} \frac{d^3 p}{(2\pi)^3} \frac{1}{p^4} \left[ (1 - \frac{3\sigma^2}{p^2})(\nabla \sigma)^2 - 2\sigma \Delta \sigma \right] \]

\[
\mathcal{E}^{(meson)} = \frac{\sigma^2}{2G} \quad \mathcal{E}^{(0)} = -4N_C \int_{p_F \leq |p| \leq \Lambda} \frac{d^3 p}{(2\pi)^3} \sqrt{p^2 + \sigma^2} \quad \mathcal{E}^{(2)} = \frac{N_C}{6} \int_{p_F \leq |p| \leq \Lambda} \frac{d^3 p}{(2\pi)^3} \frac{\sqrt{p^2 + \sigma^2}}{p^4} \left[ (1 - \frac{3\sigma^2}{p^2})(\nabla \sigma)^2 - 2\sigma \Delta \sigma \right].
\]  

(5)

Note that the scalar field is space–dependent but does not depend on the momenta. For the valence contribution the three–momenta were integrated from zero to the Fermi momentum \( p_F \). To this the contribution of the polarized vacuum has been added. It is obtained from the same integrals, however, with the opposite sign and the upper bound changed to the cut–off \( \Lambda \). This not only explains the integration bounds in eq. (4) but also removes the infra–red singularities dwelling in the individual contributions to the gradient terms.
In principle pseudoscalar fields may be introduced in a chirally symmetric way by the replacements
\[ \sigma^2 \rightarrow \sigma^2 + \pi^2, \quad (\nabla \sigma)^2 \rightarrow (\nabla \sigma)^2 + (\nabla \pi)^2 \] and
\[ \sigma \Delta \sigma \rightarrow \sigma \Delta \sigma + \pi \cdot \Delta \pi. \]
We do not give the explicit expressions because within the semi–classical approach the pseudoscalar fields are not excited. The densities
\[ \bar{\rho} = \rho - \rho^{(\text{vac})}, \quad \rho^{(\text{vac})} = -4N_C \int_{|p| \leq \Lambda} \frac{d^3p}{(2\pi)^3}, \]
\[ \bar{\mathcal{E}} = \mathcal{E} - \mathcal{E}^{(\text{vac})}, \quad \mathcal{E}^{(\text{vac})} = \frac{m^2}{2G} - 4N_C \int_{|p| \leq \Lambda} \frac{d^3p}{(2\pi)^3} \sqrt{p^2 + m^2} \]
determine the quark number and the energy measured relative to the vacuum. All further applications follow from the variation of the functional
\[ \delta \int d^3r (\bar{\mathcal{E}} - \lambda \bar{\rho}) = 0, \]
where the Lagrange multiplier \( \lambda \) (chemical potential) has been introduced to fix the quark number \( A \). This constraint should not be confused with the one employed in [9]. Here it is due to the semi–classical treatment and serves solely to determine the Fermi momentum \( p_F \). In a microscopical calculation as e.g. the one presented in subsection 3.3 no such constraint is required.

Subsequently we apply this method to hadronic matter and to finite hadronic systems.

3 Three dimensional momentum cut–off regularization

Apparently the three–momentum cut–off regularization for the vacuum contributions is a good candidate to yield stable solitons. It successfully suppresses the high momentum components contained in the infinitely high and narrow peak of the scalar field at the origin which arises e.g. in the proper–time regularization scheme [7, 8]. In the following subsections we apply the Thomas–Fermi method together with its gradient corrections to hadronic matter and to finite hadronic systems. Finally, in subsection 3.3, we present a fully self–consistent calculation for unit baryon number.

3.1 Hadronic matter

For hadronic matter the gradient terms in (5) do not contribute, only the leading order in the WK expansion which corresponds to the Thomas–Fermi method survives. The variation (3) with respect to the Fermi momentum \( p_F \) and the scalar field \( \sigma \), respectively, leads then to simple expressions for the particle number and the energy discussed already in [12] in connection with the chiral phase transition. The
binding energy per quark $\bar{\mathcal{E}}/\rho - m$ is plotted in Fig. 1 of that reference for various ratios of the cut–off in units of the constituent mass. The maximal binding energy (minima of those curves) is always reached in the symmetric phase, $\sigma \equiv 0$, at

$$\bar{\mathcal{E}}/\rho - m = p_F - m$$

(8)

$$p_F^4 = \frac{3}{2} \left[ \Lambda \sqrt{\Lambda^2 + m^2 (2\Lambda^2 - m^2)} + m^4 \ln \frac{\Lambda + \sqrt{\Lambda^2 + m^2}}{m} - 2\Lambda^4 \right],$$

which in units of the free constituent quark mass $m$ is the function of $\Lambda/m$ depicted

![Figure 1: Binding energy per quark for hadronic matter in dependence of the cut–off $\Lambda$. Solitons are obtained for $\Lambda/m \leq 1.80$.](image)

in our Fig. 1. From this figure we notice that hadronic matter becomes bound for $\Lambda/m \leq 1.80$, for larger ratios no solitons exist. Since for finite systems the gradient terms (cf. next subsection) introduce surface repulsion, the latter statement is quite general. With decreasing ratio $\Lambda/m$ we expect to find solitons also for a smaller number of quarks. For what follows we choose $\Lambda/m = 1.7, 1.6$ and $1.5$ which corresponds to a NJL coupling $G = 1.60, 1.86$ and $2.18$ according to the gap-equation (2). All dimensionalfull quantities will be expressed in terms of the free constituent mass $m$. This scale may be fixed by the empirical value of the pion decay constant $f_\pi \simeq 90$MeV in the chiral limit using (3). The corresponding model parameters and the results for the binding energy per quark in hadronic matter are listed in Table 1. The last row refers to an unphysically low value for the cut–off $\Lambda/m = 1.0$ with strong coupling $G = 6.17m^2$ leading to a considerable increase of the binding energy.
Table 1: Parameters of the model for various ratios of the cut–off divided by the constituent mass. The pion decay constant is kept fixed at $f_\pi = 90\text{MeV}$. The last two columns give the quark condensate and the binding energy per quark in hadronic matter.

| $\Lambda/m$ | $m$ [MeV] | $\Lambda$ [MeV] | $G$ [GeV$^{-2}$] | $(-<\bar{q}q>)^\frac{1}{2}$[MeV] | $\bar{E}/\bar{\rho} - m$ [MeV] |
|-------------|-----------|-----------------|-----------------|------------------|------------------|
| 1.7         | 348       | 592             | 13.2            | 298              | -5.8             |
| 1.6         | 365       | 583             | 14.0            | 297              | -12.5            |
| 1.5         | 383       | 575             | 14.8            | 296              | -20.6            |
| ...         |           |                 |                 |                  |                  |
| 1.0         | 553       | 553             | 20.2            | 301              | -101.8           |

per quark. We provided this parameter set for later reference in subsection 3.3, where we discuss the three quark soliton as obtained in a self-consistent calculation.

In the following subsection we are taking the next to leading terms in the WK expansion into account in order to study the soliton formation in finite systems.

### 3.2 Finite hadronic systems

Here we consider the second order terms in the WK expansion (5), whose gradient terms contribute a repulsive mesonic kinetic energy corresponding to the surface repulsion of the soliton. Since this semi–classical expansion is designed for large particle numbers, spin–isospin uncorrelated quark states are assumed. Consequently the pseudoscalar field enters these expressions at least quadratically and the variation (7) makes this field vanish identically. Note that the non–zero value of the scalar field in the vacuum provides the driving term for $\sigma(r)$. For symmetry reasons it is a radial function.

The variation (7) with respect to the Fermi momentum $p_F(r)$, which is now a radial function as well, leads to

$$
\epsilon_F = \sqrt{p_F^2 + \sigma^2} = \lambda \quad r \leq R \quad \text{‘inside’}
$$

$$
(1 - 3\sigma^2/p_F^2)(\nabla \sigma)^2 - 2\sigma \Delta \sigma = 24p_F^4 \quad r > R \quad \text{‘outside’}. \quad (9)
$$

The “radius” $R$ divides coordinate space into an interior region, where the Fermi energy is fixed by the chemical potential as in the hadronic matter case, and into an outer region, where in the Thomas–Fermi approximation $p_F = 0$ when the gradients are neglected. The presence of these gradient terms prevents $p_F$ and hence the quark density from becoming exactly zero in the outer region. Instead all these quantities will obtain a smooth exponential tail.
In the interior region a non–linear differential equation follows from variation (7) with respect to \( \sigma(r) \). For completeness we give this equation explicitly

\[
\frac{\sigma}{G} + \frac{N_C \sigma}{\pi^2} \left[ \epsilon_F p_F - \epsilon_{\Lambda} \Lambda - \sigma^2 \ell n \frac{p_F + \epsilon_F}{\Lambda + \epsilon_{\Lambda}} \right] \\
+ \frac{N_C}{12\pi^2} \left[ 2 \left( -\frac{\epsilon_F}{p_F} + \frac{\epsilon_{\Lambda}}{\Lambda} \right) (1 - \frac{\Lambda^2}{\Lambda^2}) - \frac{3}{\Lambda} \epsilon_F + 3 \ell n \frac{p_F + \epsilon_F}{\Lambda + \epsilon_{\Lambda}} \right] \Delta \sigma \\
+ \left( \frac{\epsilon_{\Lambda}}{\Lambda} \left( 4 - \frac{\Lambda^2}{\Lambda^2} \right) - \frac{\epsilon_F}{p_F} (2 + \frac{\epsilon_{\Lambda}^2}{p_F^2}) - 2 \frac{\sigma^4}{\epsilon_{\Lambda} \Lambda^3} + 2 \frac{\epsilon_F \sigma^2}{\epsilon_{\Lambda} \Lambda^3} \right) \frac{(\nabla \sigma)^2}{\sigma} = 0 ,
\]

where we have introduced the abbreviation \( \epsilon_{\Lambda} = \sqrt{\Lambda^2 + \sigma^2} \). This second order differential equation, subject to appropriate boundary conditions, is then solved for a given chemical potential \( \lambda = \epsilon_F(R) = \sqrt{p_F^2(R) + \sigma^2(R)} \) related to a corresponding quark number \( A \). In practice this is achieved by fixing the radius parameter \( R \).

In principle, a similar differential equation has to be solved in the outside region together with (9), resulting in an additional set of two coupled non–linear second order differential equations for the functions \( \sigma(r) \) and \( p_F(r) \). Fortunately for large distances the asymptotical solution is known analytically

\[
\sigma(r) \mathop{\rightarrow}\limits^{r \to \infty} m - \text{const.} \cdot e^{-2mr}/r , \\
\rho(r) - \rho^{(\text{vac})} \mathop{\rightarrow}\limits^{r \to \infty} \frac{2N_C}{\pi^2} p_F^3(r) , \\
\epsilon(r) - \epsilon^{(\text{vac})} \mathop{\rightarrow}\limits^{r \to \infty} \frac{2N_C}{\pi^2} mp_F^3(r) .
\]

The constant appearing in \( \sigma(r) \) is determined by the boundary conditions at \( r = R \). We assume these solutions to be valid in the entire outer region, which is justified by the observation that at the radius \( R \) the profile \( \sigma(R) \) is already close to its asymptotical value \( m \) (e.g. for \( \Lambda/m = 1.5 \) with \( A = 12 \) quarks we have \( \sigma(R) = 0.943m \) at \( R = 3.12 \text{ m}^{-1} \), cf. solid curve in Fig.3). Using a logarithmic scale the resulting binding energy per quark \( \bar{\epsilon}/\bar{\rho} - m \) is plotted in Fig.2 as a function of the quark number \( A \) for various values of the three–momentum cut–off. The binding is always weaker than that of hadronic matter which is slowly approached with increasing particle number. The surface effect increases rapidly with decreasing particle number such that there exists a critical number of quarks for which the soliton ceases to exist.

This number sensitively depends on the ratio \( \Lambda/m \) which controls the coupling constant \( G \) and the strength of binding in the NJL model with three–momentum cut–off. This dependence is shown in more detail in Fig.3 where the quark number is again plotted logarithmically. For \( \Lambda/m > 1.8 \), as we have seen above, quark matter becomes unbound. With decreasing ratio then also solitons with a smaller number of quarks become bound, e.g. for \( \Lambda/m = 1.5 \) we may expect solitons with \( A \gtrsim 10 \).
Figure 2: Binding energy per quark as a function of the number of quarks $A$. The solid, dashed and dotted lines corresponds to different values of the cut–off $\Lambda/m = 1.5, 1.6$ and 1.7 respectively. The corresponding infinite matter binding energies (Table 1) are $-0.054, -0.034$ and $-0.017\, m$.

Figure 3: Critical number of quarks in dependence of the cut–off $\Lambda/m$ as obtained in the semi–classical approximation. Above the critical curve soliton formation is possible. The dashes indicate that this result should not be trusted for small quark numbers.
quarks to exist. However, we should be cautious in trusting our results for too small quark numbers. Already for \( A = 12 \) with \( \Lambda/m = 1.5 \) (solid curve in Fig.2) the second order WK contribution to particle number and energy respectively, amounts to 20\% and we may expect an error of about 5\% from neglecting the higher orders in this expansion. In particular, for the case of special interest, \( A = 3 \), a hedgehog solution with non–vanishing pseudoscalar field is required. This should alter the results considerably. Finally in Fig.4 we show the calculated profiles for \( \Lambda/m = 1.5 \)

![Figure 4: Profiles \( \sigma \) for \( \Lambda/m = 1.5 \) as function of the radius in inverse quark masses. The solid, dashed and dotted lines represent \( A = 12, 60 \) and 120 quarks.](image)

for various quark numbers. For large particle numbers (\( A = 60, 120 \) etc.) the profiles differ only by their radii. Inside they are essentially zero and outside they approach the free constituent mass, both regions being smoothly connected at the surface of the soliton. For small particle numbers (\( A = 12 \), solid line) the attraction is no longer strong enough to drive the scalar field into the interior region to zero, instead the profile starts in the origin at a finite value.

### 3.3 Self–consistent calculation for \( A = 3 \)

Here follows a short description of searching for a self-consistent configuration with unit baryon number in the three momentum cut–off regularization scheme. For this purpose we consider the following energy functional

\[
\mathcal{E}[\sigma, \pi] = \frac{N_c}{2} \epsilon_{\text{val}} (1 - \text{sign}(\epsilon_{\text{val}})) - \frac{N_c}{2} \sum_{\mu} \epsilon_{\mu}(\Lambda) + \frac{1}{2G} \int d^3r \left( \sigma^2 + \pi^2 - m^2 \right). \tag{12}
\]
In contrast to other covariant regularization schemes which regularize the functional trace, which arises when integrating out the fermions, like e.g. proper–time \cite{4,5}, the three momentum cut–off scheme is already applied at an earlier stage, namely at the level of the single particle Dirac Hamiltonian which is projected onto the corresponding subspace in momentum space. That is, the single particle energies $\epsilon_\mu(\Lambda)$ are eigenvalues of

$$h_\Lambda = P_\Lambda (\alpha \cdot p + \beta [\sigma + i\gamma_5 \tau \cdot \pi]) P_\Lambda,$$ (13)

where the projection operator acts on the spinorial wave–functions,

$$P_\Lambda \Psi_\mu = \Psi_\mu \left\{ \begin{array}{ll} 1 & \text{for } \langle \mu | p^2 | \mu \rangle \leq \Lambda^2 \\ 0 & \text{for } \langle \mu | p^2 | \mu \rangle > \Lambda^2 \end{array} \right..$$ (14)

Technically this projection is accomplished by diagonalizing the unprojected Dirac Hamiltonian in a basis consisting of free Dirac spinors with momentum less than the cut–off $\Lambda$. That is, the Fock space for quarks on which the baryon is built, is truncated by the cut–off \cite{12}. We will comment on the special role of the valence quark level later. The baryon number is defined as the asymmetry in the spectrum of $h_\Lambda$. Apparently the just defined regularization yields unit baryon number for the functional (12).

A self–consistent soliton solution would correspond to a local minimum of the energy functional (12). In contrast to the multi–quark systems discussed above we expect a strong coupling to isospin and hence the pions for a single baryon. Of course, this driving role of the pions is a well–known feature of the soliton picture for individual baryons where the pion fields assume a hedgehog form. We therefore choose the static ans"atze

$$\sigma(r) = \sigma(r) \quad \text{and} \quad \pi(r) = \hat{r} \pi(r)$$ (15)

which introduces the two radial profile functions $\sigma(r)$ and $\pi(r)$. These profile functions obey the self–consistency conditions

$$\sigma = - \frac{N_C}{4\pi} G \int d\Omega \left\{ \Psi^\dagger_{\text{val}} \beta \Psi_{\text{val}} - \frac{1}{2} \sum_\mu \Psi^\dagger_{\mu} \beta \Psi_\mu \right\}$$ (16)

$$\pi = - \frac{N_C}{4\pi} G \int d\Omega \left\{ \Psi^\dagger_{\text{val}} i \beta \gamma_5 \tau \cdot \hat{r} \Psi_{\text{val}} - \frac{1}{2} \sum_\mu \Psi^\dagger_{\mu} i \beta \gamma_5 \tau \cdot \hat{r} \Psi_\mu \right\}$$ (17)

with $\Psi_\mu$ being the eigenfunctions of the projected Dirac Hamiltonian (13). Suitably for the spherical configuration (15) the free basis to diagonalize $h_\Lambda$ is discretized by requiring appropriate boundary conditions for the quark wave–function at a finite but large radius, $D$. Of course, the existence of the continuum limit, $D \to \infty$ is
crucial. States with momenta larger than \( \Lambda \) are omitted from that basis. In addition the basis states are labeled by the conserved grand spin quantum number with the valence level possessing grand spin zero. For more details on these technical aspects we refer to the literature \([18,19]\).

The search for a self–consistent solution starts with choosing a trial configuration for the profile functions \( \sigma(r) \) and \( \pi(r) \). The resulting eigenstates of \( h_\Lambda \) are subsequently employed to update these profile functions according to the equations of motion \( (16,17) \). Iteration of this procedure eventually yields a stable configuration. Unfortunately we have been unsuccessful in finding a non–trivial configuration for the model parameters of table 1 when taking the valence quark level to be the lowest eigenstate of the projected Hamiltonian. This includes the extreme case \( \Lambda = m \), which supposedly yields strong binding (cf. table 1). This means that during the iteration the meson profiles tend to the vacuum configuration \( \sigma(r) = m \) and \( \pi(r) = 0 \). It should be noted that much care had to be taken in gaining this result as there appeared to be spurious finite size effects which faked a non–trivial solution. Although the typical extension of a soliton solution is expected to be of the order of 1fm we were enforced to take \( D = 25 \text{fm} \) or larger to avoid these finite size effects.

On the other hand it should be noted that taking the valence level as resulting from diagonalizing the projected Hamiltonian \( h_\Lambda \) corresponds to also regularizing the valence level which is an un–common approach\(^4\). We have therefore also studied the case where the valence quark contribution to the equations of motion stems from diagonalizing the un–projected Hamiltonian. In that case something peculiar happens. Upon iteration the system tends to approach the un–stable (peaked) configuration observed in the proper–time scheme. This configuration in particular is characterized by a negative valence quark eigenenergy \( \epsilon_{\text{val}} \). However, as \( \epsilon_{\text{val}} \) turns negative, it is considered part of the distorted vacuum and must be taken from the projected Hamiltonian, which, as discussed above, does not yield stable solutions. Hence in the iterative approach the configuration fluctuates about a point where \( \epsilon_{\text{val}} \) flips sign\(^5\). Apparently that is at best a saddle point solution of the equations of motion and cannot be considered a real solution.

4 Renormalizable extension of the NJL model

As already discussed in connection with the proper–time regularization scheme, the scalar field may develop an infinitely high and narrow peak at the origin, which

\(^4\)In the proper–time scheme, for example, this would correspond to a non–integer baryon number and also destabilize the soliton. In the case of the three–momentum cut–off, however, demanding unit baryon number does not prohibit the regularization of the valence level.

\(^5\)We have also considered the inclusion of a finite ‘chemical potential’ \(-m < \mu < m\) against which we measure the single quark energies. In that case the configuration oscillates about the point \( \epsilon_{\text{val}} = \mu \).
finally destroys the soliton. It is intuitively clear that any regularization scheme which does not suppress the high momentum components contained in that peak will encounter this problem. Here, we want to shed some light on how this instability occurs. The Wigner–Kirkwood expansion (section 2) proves suitable to give a simple explanation of the phenomenon.

For definiteness we consider a non–trivial and renormalizable version of the NJL model, where the necessary counterterms, namely the mesonic kinetic energies and quartic mesonic interactions, are added to the Lagrangian (1). Here we give only a brief description of the renormalization procedure with particular emphasis on the soliton sector, for the details we refer to [16]. For simplicity we consider the scalar field only, pseudo-scalar fields may be added straightforwardly

\[
\mathcal{L} = \bar{q}(i\gamma^\mu \partial_\mu - \sigma)q + \frac{f_0^2}{2} \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2G_0} \sigma^2 - \frac{\lambda_0}{2} \sigma^4 .
\]  

(18)

This amounts to a linear sigma model coupled to quarks which is known to be renormalizable [17]. According to the renormalization prescription [16] the bare couplings indicated with subscripts zero are replaced by the renormalized (finite) parameters \(g_\sigma\) and \(\mu_\sigma\). The quadratic and logarithmic divergencies (denoted by \(I_{\text{quad}}\) and \(I_{\text{log}}\) in [16]) stemming from the quark loop are isolated and expressed as momentum integrals,

\[
\mathcal{A} = -iN_C \text{Tr} \log (i\gamma^\mu \partial_\mu - \sigma) + \int d^4 x \mathcal{L}_M
\]

with

\[
\mathcal{L}_M = \frac{1}{2} \left[ \frac{1}{g_\sigma^2} + 4iN_C \int \frac{d^4 p}{(2\pi)^4} \left( \frac{p^2 + \frac{1}{2} m^2}{(p^2 - m^2)^2} \right) \partial_\mu \sigma \partial^\mu \sigma \right] \\
+ \left[ \frac{\mu_\sigma^2}{4g_\sigma^2} - 4iN_C \int \frac{d^4 p}{(2\pi)^4} \left( \frac{p^2 - 2m^2}{(p^2 - m^2)^2} \right) \sigma^2 \right] \\
- \frac{1}{2} \left[ \frac{\mu_\sigma^2}{4g_\sigma^2 m^2} + 4iN_C \int \frac{d^4 p}{(2\pi)^4} \left( \frac{1}{(p^2 - m^2)^2} \right) \sigma^4 \right].
\]  

(19)

Note that the scalar field used here \(\sigma = m + g_\sigma \sigma'\) is connected to \(\sigma'\) defined in [16] and that the parameters are related \(\mu_\sigma^2 = 4m^2 (1 + N C g_\sigma^2 / 6\pi^2)\) such that the Nambu relation for the sigma mass \(m_\sigma = 2m\) holds. In principle the action (19) is well suited to study the soliton in the renormalized extension of the NJL model. This is in particular the case because the counterterms, which are completely fixed in the meson sector, also render finite the fermion determinant in the soliton background. However, this is technically quite involved because identical regularization schemes have to be employed for the functional trace and the counterterms. In a numerical treatment this is technically quite complicated and beyond the scope of the present paper.

\[\text{Cf. ref [20] for a suitable path to approach this problem.}\]
Instead we would like to gain some first insight by employing the much simpler semi–classical method described in section 2. It is noticed that the infinities of the extra terms contained in \((19)\) just cancel those appearing in \(E^{(0)}\) and \(E^{(2)}\) (eq.\((5)\)) when \(\Lambda \to \infty\) (\(\rho^{(0)}\) and \(\rho^{(2)}\) are finite in that limit). This is not obvious, apparently the WK expansion reproduces all singularities of the exact model correctly. Stated otherwise, the WK expansion allows us to carry out the renormalization program analytically and avoiding a numerical regularization of the fermion determinant.

As the resulting action is a perfectly finite functional of the form \((4)\) we can proceed analogously to subsection 3.1. We find that quark matter (symmetric phase with \(\sigma = 0\)) is in principle bound for \(g_2^\sigma > 24\pi^2/N_C\). For finite systems the variation in the interior of the soliton leads to a non–linear differential equation similar to \((10)\). However, it turns out that in contrast to \((10)\) this differential equation possesses no stable solutions. Using a relaxation method for the solution of the non–linear differential equation with appropriate boundary conditions, the development of the infinitely high and narrow peak in the scalar profile at the origin is observed quite similar as reported in the proper–time regularization scheme \([7, 8]\). The kinetic part of the functional \((7)\) inside the soliton \((\lambda = \epsilon_F = \sqrt{p_F^2 + \sigma^2})\) as it follows from \((5)\),

\[
\frac{1}{2} \left[ \frac{1}{g_2^\sigma} + \frac{N_C}{6\pi^2} \left( \frac{\epsilon_F}{p_F} - 3\ln \frac{p_F + \epsilon_F}{m} \right) \right] (\nabla \sigma)^2,
\]

explains the cause of this result. Close to the center of the soliton, where \(p_F\) increases towards \(\epsilon_F < \sim m\) (\(\sigma \to 0\)), the second term changes sign and eventually exceeds the constant and positive contribution from the first term. As a consequence the total kinetic energy becomes negative. Exactly this happened in all considered cases quite independently from the size of the soliton. Therefore, within the WK expansion, it is obviously the kinetic energy being no longer positive definite which causes the instability of the soliton. Of course, this is equivalent to the observation of an infinitely narrow and high peak in the scalar profile.

5 Conclusions

In this short note we have reported on the investigation of the soliton formation in the NJL model with local four quark interaction and various regularization procedures. Our interest was focused on solitons in the absence of additional constraints, which are commonly imposed in order to achieve stabilization.

According to the investigations presented here the following picture emerges. Whether or not stable solitons may be found in the NJL model depends on the chosen regularization scheme. First of all there are the more sophisticated regularization schemes as e.g. Schwinger’s proper–time regularization, which do not limit the high momentum components contained in the Dirac sea. Solitons in these schemes are
unstable because the scalar field develops an infinitely high and narrow peak at the origin. Using the WK expansion we were able to show that within a renormalizable extension of the NJL model the instability is related to the mesonic kinetic energy density not being positive definite.

Apparently the three–momentum cut–off regularization scheme circumvents the above addressed problem by successfully suppressing the high momentum components contained in the sharp peak of the scalar field. This regularization scheme is therefore particularly appropriate to study the soliton formation in the unconstrained NJL model. In detail our findings are:

- Hadronic matter is bound for $\Lambda/m \leq 1.80$. Since finite systems add surface repulsion no solitons exist for larger ratios.

- In order to describe finite size effects the lowest order gradient corrections to the Thomas–Fermi method in the WK expansion have been considered. We find that the binding energy per quark is continuously weakened with decreasing particle number until the soliton ceases to exist. This behavior, caused by the repulsive kinetic terms active in the surface of the soliton, reveals a peculiarity of the NJL model, which consequently prefers the formation of large clusters of hadronic matter.

- The critical size, or equivalently the critical quark number of the soliton obviously depends sensitively on the ratio $\Lambda/m$. We have computed the limiting function above which soliton formation takes place. Because of the semi–classical approximation employed, this result should not be trusted for small quark numbers.

- An extensive search for a self-consistent configuration with unit baryon number was performed. No solitons were found, not even with an unphysically low cut–off $\Lambda/m = 1$ which should provide optimal attraction. In all cases the iterative procedure ran into the trivial configuration. Allowing the distinct valence quark not to undergo regularization, the sharp peak in the scalar field reappeared.

Concludingly we have to state that although we do find solitons in the three–momentum regularized NJL model without further constraints, the results are unsatisfactory in various respects, in particular of course for the unit baryon number case. It seems that the NJL model together with an adopted regularization procedure alone cannot account for a reliable description of individual baryons as solitons. However, this may not necessarily be bad news for model builders. As discussed, the stabilizing extensions of the model can well be motivated by phenomenological considerations.
Finally, we would like to further add a few comments concerning the stability. Within the semi-classical approach the multi-quark solitons discussed in section 3 are stable against variations of the scalar field. Independently we checked that the infinite system is at least locally stable also when spatially dependent pseudo–scalar fields are allowed. However, for finite systems, in particular of course for very small particle numbers, we may not exclude that in a self-consistent calculation pseudo–scalar fields leading to instabilities are excited. For the three–quark system we did not find a non–trivial solution. We may conjecture that the quartic meson self–interaction inherent in the NJL model is too weak in order to stabilize this soliton.

On the other hand, with the chiral circle condition imposed, the proper-time regularized NJL soliton seems to be stable. Fluctuations in the lowest channels (grand spin) were searched and no instabilities detected [21]. It is natural to assume that this remains still valid when the chiral circle condition is softened and replaced by an additional quartic meson self–interaction. Thus, in both cases the hedgehog presumably represents the stable minimum configuration.

For other versions of the model as e.g. its non-local extensions [22, 23] the latter statement is not at all obvious. Although there the hedgehog solution is reported to be stable with respect to monopole deformations it is not excluded that this soliton leaks through other normal modes.

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