Strong convergence of a full discretization for stochastic wave equation with polynomial nonlinearity and additive noise

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Abstract

In this paper, we propose a full discretization for $d$-dimensional stochastic wave equation with both polynomial nonlinearity and additive noise, which is based on the spectral Galerkin method in spatial direction and splitting averaged vector field method in temporal direction. Uniform bounds for high order derivatives of the continuous and the full discrete problem are obtained by constructing and analyzing Lyapunov functionals, which are crucial to derive the strong convergence rate of the proposed scheme. Furthermore, we show the exponential integrability properties of both the exact and numerical solutions, which are another key gradients to analyze the approximate error, due to the averaged energy preserving property of both the spatial and full discretization. Based on these regularity estimates and exponential integrability properties, the strong convergence order in both spatial and temporal direction are obtained. Numerical experiments are presented to verify these theoretical results.

AMS subject classification: 60H08, 60H35, 65C30.

Key Words: stochastic wave equation, polynomial nonlinearity, strong convergence, spectral Galerkin method, exponentially integrability, Lyapunov functionals

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1 Introduction

As a kind of commonly observed physical phenomena, the wave motions are usually described by the stochastic partial differential equations (SPDE) of hyperbolic type. The main objective of this paper is to investigate the spectral Galerkin splitting averaged vector fields approximations for the following stochastic wave equation with polynomial nonlinearity, driven by an additive noise

\[
\begin{aligned}
&du(t) = v(t)dt, & \text{in } \mathcal{O} \times (0, T], \\
dv(t) = \Lambda u(t)dt - f(u(t))dt + dW(t), & \text{in } \mathcal{O} \times (0, T], \\
u(0) = u_0, & \text{in } \mathcal{O},
\end{aligned}
\]

where \( \mathcal{O} = (0, 1)^d \) with \( d = 1, 2, 3 \), \( T \in (0, \infty) \) and the initial values \( u_0, v_0 : \mathcal{O} \to \mathbb{R} \) are random variables. Furthermore, we impose that \( \Lambda = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} \) is the Dirichlet Laplace operator. Here, the nonlinear term \( f(u) = c_\rho u^\rho + \cdots + c_1 u + c_0 \) is assumed to be polynomial with odd degree \( \rho \leq 3 \) and \( c_\rho > 0 \). Throughout this paper, \( W \) is an \( L^2 := L^2(\mathcal{O}; \mathbb{R}) \)-valued \( \mathbb{Q} \)-Wiener process with respect to a filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P}) \), i.e., there exists an orthonormal basis \( \{e_k\}_{k \in \mathbb{N}_+} \) of \( L^2 \) and a sequence of mutually independent real-valued Brownian motions \( \{\beta_k\}_{k \in \mathbb{N}_+} \) such that \( W(x, t) = \sum_{k \in \mathbb{N}_+} \mathbb{Q}^{1/2} e_k(x) \beta_k(t) \) with \( \mathbb{Q} \) being a symmetric, positive definite and finite trace operator.

The stochastic wave equation with Lipschitz and regular coefficients has been systematically investigated both theoretically and numerically (see for example [1,6,15–17,19,20] and references therein). However, in many practical applications the drift coefficient \( f \) of the stochastic wave equation (1) has a specific structure, e.g., polynomial, but fails to satisfy a global Lipschitz condition, while in the vast majority of research articles concern the numerical approximation of SPDEs a global Lipschitz assumption is used. We are only aware that in the literature [3,18,22] and [21] the well-posedness and the full discrete scheme without a global Lipschitz assumption are obtained. [5] proves the existence and uniqueness of local and global solutions of stochastic wave equation in \( \mathbb{R}^d \) with \( d \leq 3 \), for which the nonlinear terms are polynomial. [18] investigates the existence and uniqueness of the solution for a class of stochastic wave equations in two space-dimensions containing a nonlinearity of polynomial type. [22] gives the necessary and sufficient conditions for the existence of a function-valued solution of nonlinear stochastic wave equation. Since exact solutions are rarely known, one needs approximations of solutions of such SPDEs. [21] presents some nonstandard partial-implicit methods preserving the energy functional for 1-dimensional stochastic wave equation with cubic power law perturbed by \( \mathbb{Q} \)-Wiener process. However, as far as we know, there are no known results about the strong convergence analysis for the numerical approximation of stochastic wave equation with non-globally Lipschitz coefficients. We emphasize that our main goal is to present the strong convergence analysis of a full discrete numerical method for the stochastic wave equations (1).

The main topic of this work is to propose and analyze the following full discretization (see (15)), in which the splitting averaged vector field (AVF) method is applied in temporal
direction and the spectral Galerkin method is used in spatial direction for (1),

\begin{align*}
v^N_{m+1} &= u^N_m + h \frac{v^N_m + \tilde{v}^N_m}{2}, \\
\tilde{v}^N_{m+1} &= v^N_m + h \Lambda_N \left( u^N_m + v^N_m \right) - h P^N \left( \int_0^1 f(u^N_m + \theta (u^N_{m+1} - u^N_m))d\theta \right), \\
v^N_{m+1} &= \tilde{v}^N_{m+1} + P_N \delta W_m,
\end{align*}

where \( N \) is the dimension of the spectral Galerkin projection space, \( h = T/M \) with \( M \in \mathbb{N}^+ \) is the the time step-size, \( m \in \mathbb{N}_M := \{0, 1, \cdots, M\} \). Moreover, \( \Lambda_N = P^N \Lambda \) is defined by (4), and the increment \( \delta W_m := W(t_{m+1}) - W(t_m) \) of the Wiener process is defined by (15), respectively. By denoting \( H^\beta = \tilde{H}^\beta \times \tilde{H}^{\beta-1} \), \( \beta \in \mathbb{R} \) with \( \tilde{H}^\beta := D\left((-\Lambda)^{\frac{\beta}{2}}\right) \) with Dirichlet boundary condition being the interpolation space, the strong convergence rate of the above full discrete scheme for stochastic wave equation is stated in the following theorem.

**Theorem 1.1.** Assume that \( d = 1 \), \( \beta \geq 1 \) or that \( d = 2 \), \( \beta = 2 \) and in addition suppose that \((u_0, v_0)^T \in H^\beta\), \( \|(-\Lambda)^{\frac{\beta-1}{2}}\|_{L^2} < \infty \). Let \( \gamma = \min(\beta, 2) \). Then there exists a positive constant \( C \) such that

\[
\sup_{m \in \mathbb{Z}_M} \mathbb{E} \left[ \left( \|u(t_m) - u^N_m\|_{L^2}^2 + \|v(t_m) - v^N_m\|_{L^2}^2 \right)^p \right] \leq C \left( h^{\gamma p} + \lambda^{-\beta p}_N \right),
\]

where \( C \) depends on \( p, T, u_0, v_0 \) and operator \( Q \), but is independent of \( h \) and \( N \).

We would like to mention that, proving the error estimate in Theorem 1.1 rigorously is rather demanding, confronted with the difficulty brought by the superlinearly growing of the drift term \( f(u) \). This difficulty leads that the classical method to derive the strong convergence rate, that is, taking expectation first and then applying Gronwall inequality, is not available. To overcome this difficulty, we take expectation first and then apply the Gronwall inequality, which needs us to show the exponential integrability property of both the semidiscretization and full discrete scheme. The exponential integrability property is of vital importance to obtain the strong convergence, and we refer to [11,12] for a detailed discussion of exponential integrability for stochastic evolution equations with non-monotone coefficients. In this paper, we introduce the Lyaponouv energy function and get the \( H^1 \)-regularity of the semidiscretization. Combining \( H^1 \)-regularity and then applying Sobolev–Gagliard–Nirenberg inequality and the exponential integrability lemma, we get the exponential integrability property of spatial semidiscretization. With the spatial discretization, we split it into a deterministic system and a stochastic system whose exact solution is known, and apply the AVF method to discretize the deterministic system. Due to the Lyaponouv energy preserving property of the AVF method, we derive the exponential integrability property of the proposed full discretization based on some Sobolev–Gagliard–Nirenberg inequality. The first benefit of the exponential integrability property is the \( H^\beta \)-regularity of both the spectral Galerkin semidiscretization and full discretization for the case that \( \beta \geq 1, d = 1 \) and that \( \beta = 2, d = 2, 3 \). The uniform bounds with high regularities and Hölder continuity of solutions for both
the continuous and the semi-discrete problems are crucial to derive the strong convergence of the proposed full discrete scheme with certain order. Furthermore, it follows from the error estimate between the semigroup and rational approximation, the Hölder continuity of the semidiscretization in $L^p(\Omega; H^r)$-norm and the exponential integrability property of both the exact and numerical solution that the numerical solution of (15) converges to the exact solution of (1) in strong sense, which is stated in Theorem 1.1. To the best of our knowledge, this is the first result about both the exponential integrability and strong convergence rate for the full discretization of stochastic wave equation (1).

The rest of this paper is organized as follows. Section 2 presents an abstract formulation of (1) for the analysis of stochastic wave equation, and some properties of the semigroup are also considered. In Section 3, the regularity estimate and exponential integrability property of the mild solution of the spectral Galerkin semidiscretization are studied. The analysis of strong convergence for the spectral Galerkin semidiscretization is also given. Section 4 is devoted to obtaining a full discretization and give $L^p(\Omega)$ error estimates of the proposed scheme. Finally, numerical experiments are carried out in Section 5 to verify our theoretical results.

2 Preliminary and frame work

In this section, we set forth an abstract formulation of (1) for the analysis of stochastic wave equation, and some properties of the semigroup generated by the coefficient operator are also considered. Throughout this paper, the constants $C$ may be different from line to line. When it is necessary to indicate the dependence on some parameters, we will use the notation $C(\cdot)$.

Denote by $X = (u, v)^\top$, thus we can rewrite (1) as the following abstract form in infinite-dimensional space

$$
dX(t) = AX(t)dt + F(X(t))dt + G(t)dW(t), \quad t \in (0, T],$$
$$X(0) = X_0,$$

(2)

where

$$X_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ \Lambda & 0 \end{bmatrix}, \quad F(X(t)) = \begin{bmatrix} 0 \\ f(u(t)) \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Here and below we denote $I$ by the identity operator. We assume that the eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ and the corresponding eigenfunctions $e_i$ of the operator $-\Lambda$, i.e., with

$$-\Lambda e_i = \lambda_i e_i, \quad i = 1, 2, \cdots,$$

form an orthonormal basis in $L^2(\Omega)$. Define the interpolation space $H^r := \mathcal{D}((-\Lambda)^{1/2})$ for $r \in \mathbb{R}$ equipped with the inner product

$$\langle x, y \rangle_{H^r} = \langle (-\Lambda)^{1/2} x, (-\Lambda)^{1/2} y \rangle_{L^2} = \sum_{i=1}^{\infty} \lambda_i^r \langle x, e_i \rangle_{L^2} \langle y, e_i \rangle_{L^2}$$

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and the corresponding norm $\|x\|_r := ((x,x)_{\mathbb{H}^r})^{1/2}$. Furthermore, we introduce the product space

$$\mathbb{H}^r := \mathbb{H}^r \times \mathbb{H}^{r-1}, \quad r \in \mathbb{R}$$

endowed with the inner product

$$\langle X_1, X_2 \rangle_{\mathbb{H}^r} = \langle x_1, x_2 \rangle_{\mathbb{H}^r} + \langle y_1, y_2 \rangle_{\mathbb{H}^{r-1}}, \quad X_1 = (x_1, y_1)^\top, \quad X_2 = (x_2, y_2)^\top$$

for any $X_1 = (x_1, y_1)^\top$, $X_2 = (x_2, y_2)^\top$, and the corresponding norm

$$\|X\|_{\mathbb{H}^r} := ((X,X)_{\mathbb{H}^r})^{1/2} = (\|u\|^2_r + \|v\|^2_{r-1})^{1/2}.$$  

Moreover, we define the domain of operator $A$ by

$$\mathcal{D}(A) = \left\{ X \in \mathbb{H} : AX = \begin{bmatrix} u \\ \Lambda u \end{bmatrix} \in \mathbb{H} = L^2 \times \mathbb{H}^{-1} \right\} = \mathbb{H}^1 \times L^2,$$

then the operator $A$ thus generates a $C_0$-semigroup $E(t)$, $t \geq 0$ on the Hilbert space $\mathbb{H}^1$, given by

$$E(t) = \exp(tA) = \begin{bmatrix} C(t) & (-\Lambda)^{-\frac{3}{2}}S(t) \\ -(-\Lambda)^{\frac{3}{2}}S(t) & C(t) \end{bmatrix},$$

where $C(t) = \cos(t(-\Lambda)^{\frac{1}{2}})$ and $S(t) = \sin(t(-\Lambda)^{\frac{1}{2}})$ are the cosine and sine operators, respectively.

The following two lemmas concern with the temporal Hölder continuity of the sine and cosine operators, which have been discussed, for example, in [1].

**Lemma 2.1.** For all $r \in [0,1]$, there exists a positive constant $C := C(r)$ such that

$$\| (S(t) - S(s))(-\Lambda)^{-\frac{3}{2}} \|_{L(L^2)} \leq C(t-s)^r, \quad \| (C(t) - C(s))(-\Lambda)^{-\frac{3}{2}} \|_{L(L^2)} \leq C(t-s)^r$$

and

$$\| (E(t) - E(s))X \| \leq C(t-s)^r \| X \|_{\mathbb{H}^r}$$

for all $t \geq s \geq 0$.

**Lemma 2.2.** $C(t)$ and $S(t)$ satisfy a trigonometric identity in the sense that

$$\| S(t)x \|_{L^2}^2 + \| C(t)x \|_{L^2}^2 = \| x \|_{L^2}^2, \quad x \in L^2.$$

Based on the above trigonometric identity, we have $\|E(t)\|_{L(\mathbb{H})} \leq 1.$
3 Spatial semidiscretization

This section is devoted to constructing the numerical approximation for (1) or (2) in spatial direction by using a spectral Galerkin method. And we present the existence and uniqueness and give the regularity analysis for the solution to the spectral Galerkin semi-discretization method, including the uniform boundedness of the solution in $L^p(\Omega; H^\beta)$-norm and Hölder continuity of the solution in $L^p(\Omega; \dot{H})$-norm. While, we give the strong convergence rate of the spectral Galerkin approximation.

3.1 Spectral Galerkin approximation

In this subsection, we consider the spectral Galerkin approximation for stochastic wave equation. The Galerkin method is always used to discretize stochastic partial differential equation in spatial direction (see for example [4,10] and references therein). For the considered equation and $N \in \mathbb{N}^+$, we define a finite dimensional subspace $U_N$ of $L_2$ spanned by $\{e_1, e_2, \cdots, e_N\}$, and the projection operator $P_N : \dot{H}^r \rightarrow U_N$ by

$$P_N \zeta = \sum_{i=1}^{N} \langle \zeta, e_i \rangle_{L^2} e_i, \quad \forall \ \zeta \in \dot{H}^r, \quad r \geq -1. \quad (3)$$

The definition of $P_N$ immediately implies

$$\|P_N\|_{L^2} \leq 1, \quad \Delta P_N \zeta = P_N \Delta \zeta, \quad \forall \ \zeta \in \dot{H}^r.$$

Now we define $\Lambda_N : U_N \rightarrow U_N$ by

$$\Lambda_N \zeta = \Lambda P_N \zeta = P_N \Lambda \zeta = -\sum_{i=1}^{N} \lambda_i \langle \zeta, e_i \rangle_{L^2} e_i, \quad \forall \ \zeta \in U_N. \quad (4)$$

By denoting $X^N = (u^N, v^N)^\top$, the spectral Galerkin method for (2) yields

$$dX^N(t) = A_N X^N(t) dt + F_N(X^N(t)) dt + G_N(t) dW(t), \quad t \in (0, T],$$

$$X^N(0) = X^N_0,$$  

where

$$X^N_0 = \begin{bmatrix} u^N_0 \\ v^N_0 \end{bmatrix}, \quad A_N = \begin{bmatrix} 0 & I \\ \Lambda_N & 0 \end{bmatrix}, \quad F_N(X^N) = \begin{bmatrix} 0 \\ P_N (f(u^N)) \end{bmatrix}, \quad G_N(t) = \begin{bmatrix} 0 \\ P_N \end{bmatrix}.$$

Similarly, the discrete operator $A_N$ generates a $C_0$-semigroup $E_N(t), \ t \geq 0$, given by

$$E_N(t) = \exp(tA_N) = \begin{bmatrix} C_N(t) & (-\Lambda_N)^{-\frac{1}{2}} S_N(t) \\ -(-\Lambda_N)^{\frac{1}{2}} S_N(t) & C_N(t) \end{bmatrix},$$

where $C_N(t) = \cos(t(-\Lambda_N)^{\frac{1}{2}})$ and $S_N(t) = \sin(t(-\Lambda_N)^{\frac{1}{2}})$ are the discrete cosine and sine operators defined in $U_N$, respectively. It can be verified straightforwardly that

$$C_N(t)P_N \zeta = C(t)P_N \zeta = P_N C(t) \zeta, \quad S_N(t)P_N \zeta = S(t)P_N \zeta = P_N S(t) \zeta$$

for any $\zeta \in \dot{H}^r, \ r \geq -1.$
3.2 Regularities of the solution of spatial semidiscretization

Now we show the existence and uniqueness of the mild solution of (5) under certain conditions on the operator \( Q \) and initial data \( X_0 \). Moreover, thanks to the Lyaponouv function we present a priori estimation on \( \|X_N(t)\|_{L^2(\Omega;H^1)} \) in Lemma 3.1.

Before we begin, we denote \( \frac{\delta E}{\delta u} = f(u) \). Then we have that

\[
 a_1 \|u\|_{L^4}^4 - b_1 \leq F(u) \leq a_2 \|u\|_{L^4}^4 + b_2 \tag{6}
\]

for some positive constants \( a_1, a_2, b_1, b_2 \). Similar to [5], applying Itô formula to the Lyaponouv function \( V^1(u_N, v_N) = \frac{1}{2}\|u_N\|_{H^1}^2 + \frac{1}{2}\|v_N\|_{L^2}^2 + F(u_N) + C_1, C_1 \geq b_1 \), we have the following lemma.

**Lemma 3.1.** Assume that \( X_0 \in H^1 \), and \( Q \in L_2(L^2) \). Then the spectral Galerkin semidiscretization (5) has a unique mild solution given by

\[
 X_N(t) = E_N(t)X_0^N + \int_0^t E_N(t-s)F(X_N(s))ds + \int_0^t E_N(t-s)G_NdW(s) \tag{7}
\]

for any \( t \in [0, T] \). Moreover, there exists a positive constant \( C := C(T, Q) \in (0, \infty) \) such that

\[
 \|X_N(t)\|_{L^2(\Omega;H^1)} \leq C(\|X_0\|_{L^2(\Omega;H^1)} + 1). \tag{8}
\]

**Remark 3.1.** Applying Itô formula to \( (V_1(u_N, v_N))^p, p \geq 2 \), we also can get \( \mathbb{E}\|u_N(t)\|_{H^1}^p \leq C(T), \mathbb{E}\|v_N(t)\|_{L^2}^p \leq C(T) \), whose proof are similar to the above one.

To investigate the strong convergence order of semidiscretization (5), we show the following exponential integrability property of the semidiscrete numerical solution \( X_N \) based on the exponential Lemma in [7, Corollary 2.4]. For the application of exponential integrability, we also refer to [2,8–10,13,14] and references therein.

**Lemma 3.2.** Assume that \( X_0 \in H^1 \), and \( Q \in L_2(L^2) \). Then there exist a positive constant \( C := C(X_0, T) \) and a constant \( \alpha \geq \frac{1}{2}\text{Tr}(Q) \) such that the mild solution (7) satisfies

\[
 \mathbb{E}\left[ \exp\left( \int_0^T \frac{V_1(u_N(s), v_N(s))}{C'} \exp(\alpha s)ds \right) \right] \leq C, \quad \forall C' > T. \tag{9}
\]

Here and throughout this paper \( \mathbb{E}[\cdot] \) denotes the expectation operator.

**Proof.** To obtain the estimate, we define the linear operator

\[
 G_{\mathbb{F}^N}(V_1) := \langle DV_1, \mathbb{F}^N \rangle_{L^2} + \frac{1}{2}\text{Tr}(G_NG_N^*(D^2V_1)) = \frac{1}{2}\text{Tr}(P^NQP^N).
\]
Then we get
\[
G_{\mathcal{F},N,G_N}(V_1) + \frac{1}{2\exp(\alpha t)}\|G_N^*(\nabla V_1)\|^2
\leq \frac{1}{2} \text{Tr}(Q) + \frac{1}{2\exp(\alpha t)} \sum_{i=1}^{\infty} (v^N, Q^i e_i)^2_{L^2} \leq \frac{1}{2} \text{Tr}(Q) + \frac{1}{2\exp(\alpha t)}\|v^N\|^2_{L^2} \text{Tr}(Q)
\]
\leq \frac{1}{2} \text{Tr}(Q) + \frac{1}{2\exp(\alpha t)}V_1(u^N, v^N)\text{Tr}(Q).
\]

Let \(\bar{U} = -\frac{1}{2} \text{Tr}(Q), \alpha \geq \frac{1}{2} \text{Tr}(Q)\). According to exponential integrability lemma in [7, Corollary 2.4], we have
\[
\mathbb{E} \left[ \exp \left( \frac{V_1(u^N(t), v^N(t))}{\exp(\alpha t)} + \int_0^t \bar{U}(s, X_s) \, ds \right) \right] \leq \mathbb{E}(\exp(V_1(u_0, v_0))).
\]

The Jensen’s inequality then yields
\[
\mathbb{E} \left[ \exp \left( \int_0^T \frac{V_1(u^N(s), v^N(s))}{C' \exp(\alpha s)} \, ds \right) \right] = \mathbb{E} \left[ \exp \left( \frac{1}{T} \int_0^T \frac{TV_1(u^N(s), v^N(s))}{C' \exp(\alpha s)} \, ds \right) \right]
\leq \mathbb{E} \left[ \frac{1}{T} \int_0^T \exp \left( \frac{TV_1(u^N(s), v^N(s))}{C' \exp(\alpha s)} \right) \, ds \right]
\leq C \left[ \frac{1}{T} \int_0^T \mathbb{E}(\exp(V_1(u_0, v_0))) \int_0^s \exp(-\alpha u) \, du \, ds \right] \leq C\mathbb{E}(\exp(V_1(u_0, v_0))),
\]
which implies the estimate [9].

\[\square\]

**Corollary 3.1.** Let \(d = 1, 2\). For any positive constant \(c\), it holds that
\[
\mathbb{E} \left( \exp \left( \int_0^T c \|u^N(s)\|^2_{L^6} \, ds \right) \right) < \infty.
\]

**Proof.** By using the Jensen’s inequality, Young’s inequality and the Gagliardo–Nirenberg inequality \(\|u\|_{L^6} \leq C\|\nabla u\|_{L^2}^a \|u\|_{L^2}^{1-a}\), where \(a = \frac{1}{3}\) if \(d = 1\) and \(a = \frac{2}{3}\) if \(d = 2\), we have that
\[
\mathbb{E} \left( \exp \left( \int_0^T c \|u^N(s)\|^2_{L^6} \, ds \right) \right) \leq C(T) \sup_{s \in [0,T]} \mathbb{E} \left( \exp(cT \|u^N(s)\|^2_{L^6}) \right)
\leq C(T) \sup_{t \in [0,T]} \mathbb{E} \left( \exp \left( \frac{cT \|\nabla u^N(t)\|^2_{L^2}}{C' \exp(\alpha t)} \right) \exp(\|u^N(t)\|^2_{L^2} \exp(\alpha T)) \right),
\]
with \(C' > cT\). The Hölder and Young’s inequalities imply that for any sufficiently small \(\epsilon > 0\),
\[
\mathbb{E} \left( \exp \left( \int_0^T c \|u^N(s)\|^2_{L^6} \, ds \right) \right)
\leq C(T) \sup_{t \in [0,T]} \mathbb{E} \left( \exp \left( \frac{\|\nabla u^N(t)\|^2_{L^2}}{\exp(\alpha t)} \right) \exp(\epsilon \|u^N(t)\|^4_{L^4} + C(\epsilon, \alpha, T)) \right).
\]

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Choosing $\epsilon = \frac{1}{\exp(\alpha t)}$, together with Lemma 3.2,

$$
\mathbb{E} \left( \exp \left( \int_0^T c \| u^N(s) \|_{L^2}^2 ds \right) \right) \leq C(\alpha, T),
$$

which completes the proof. \qed

**Remark 3.2.** When $d = 1$, using the Gagliardo–Nirenberg inequality $\| u \|_{L^\infty} \leq \| \nabla u \|^{\frac{1}{2}} \| u \|^{\frac{1}{2}}$, the solution (7) also satisfies

$$
\mathbb{E} \left( \exp \left( \int_0^T \| u^N(s) \|_{L^\infty}^2 ds \right) \right) < \infty.
$$

**Proposition 3.1.** Let $p \geq 1$, $d = 1$, $\beta \geq 1$, $X_0 \in \mathbb{H}^\beta$, $\| (\Lambda)^{\frac{\beta-1}{2}} Q \|_{L^2(L^2)} < \infty$. Then the mild solution $X^N$ of (5) satisfies

$$
\| X^N \|_{L^p(\Omega; \mathbb{H}^\beta)} \leq C(X_0, T, p).
$$

**Proof.** Due to (7) and the regularity of the stochastic convolution, it suffices to estimate $\| \int_0^t E_N(t-s)\mathbb{P}(X^N(s))ds \|_{\mathbb{H}^\beta}$. We note

$$
E_N(t-s)\mathbb{P}(X^N(s)) = \left[ (-\Lambda)^{-\frac{1}{2}} S(t)f(u^N(s)) \right] C(t) f(X^N(s)),
$$

thus we only need to estimate $\int_0^t \| (-\Lambda)^{\frac{\beta-1}{2}} f(u^N(s)) \|_{L^2} ds$. The Sobolev embedding theorem leads to

$$
\int_0^t \| (-\Lambda)^{\frac{\beta-1}{2}} f(u^N(s)) \|_{L^2} ds \leq C \int_0^t \left( 1 + \| u^N \|_{L^\infty}^2 \right) \| u^N(s) \|_{\mathbb{H}^{\beta-1}} ds
$$

$$
\leq C \int_0^t \left( 1 + \| u^N \|_{L^2}^2 \right) \| u^N(s) \|_{\mathbb{H}^{\beta-1}} ds,
$$

which, together with Lemma 3.1, show the desired result for the case that $\beta \in [1, 2)$. For the case that $\beta \in [n, n+1)$, $n \in \mathbb{N}^+$, we can complete the proof through induction arguments. \qed

Now we show the higher regularity of the discrete solution of (5) in the following proposition.

**Proposition 3.2.** Assume that $X_0 \in \mathbb{H}^2$ and $\| (\Lambda)^{\frac{\beta-1}{2}} Q \|_{L^2(L^2)} < \infty$ for $\beta \geq 2$. Then for any $p \geq 2$, there exists a positive constant $C := C(p, T, X_0, Q)$ such that

$$
\| X^N(t) \|_{L^p(\Omega; \mathbb{H}^2)} \leq C(\| X^N(0) \|_{L^p(\Omega; \mathbb{H}^2)} + 1).
$$

**Proof.** We only present the proof for $p = 2$ here, since the proof for general $p > 2$ is similar. To get the $\mathbb{H}^2$-regularity of the mild solution, we introduce another Lyapounov function

$$
V_2(u^N, v^N) = \frac{1}{2} \| \nabla^2 u^N \|_{L^2}^2 + \frac{1}{2} \| \nabla v^N \|_{L^2}^2 + \frac{1}{2} \langle (\Lambda) u^N, f(u^N) \rangle_{L^2}.
$$
By applying Itô formula formally to $V_2$, it yields
\[
dV_2 = \langle \nabla^2 u^N, \nabla^2 v^N \rangle_{L^2} dt + \langle \nabla v^N, \nabla(\Lambda u^N) \rangle_{L^2} dt + \frac{1}{2} \text{Tr} \left( (\nabla P_N Q^{\frac{3}{2}})(\nabla P_N Q^{\frac{3}{2}})^* \right) dt
\]
\[
+ \langle \nabla v^N, -\nabla (P_N f(u^N)) \rangle dt + \nabla P_N dW(t)_{L^2}
\]
\[
+ \frac{1}{2} \langle (\Delta u^N, f(u^N)) \rangle_{L^2} dt + \frac{1}{2} \langle (\Delta u^N, f'(u^N)v^N) \rangle_{L^2} dt
\]
\[
= \frac{1}{2} \langle \Delta u^N, f(u^N) \rangle_{L^2} dt + \frac{1}{2} \langle \nabla v^N, \nabla P_N dW(t) \rangle_{L^2} + \frac{1}{2} \text{Tr} \left( (\nabla P_N Q^{\frac{3}{2}})(\nabla P_N Q^{\frac{3}{2}})^* \right) dt
\]
\[
+ \frac{1}{2} \langle \nabla u^N, f'(u^N)\nabla v^N \rangle_{L^2} dt + \frac{1}{2} \langle \nabla u^N, f''(u^N)\nabla u^N v^N \rangle_{L^2} dt
\]
\[
= I_1 dt + \langle \nabla v^N, \nabla P_N dW(t) \rangle_{L^2} + \frac{1}{2} \text{Tr} \left( (\nabla P_N Q^{\frac{3}{2}})(\nabla P_N Q^{\frac{3}{2}})^* \right) dt,
\]
due to the commutativity between $\Lambda$ and $P_N$, with $I_1 = \frac{1}{2} \langle \nabla u^N, f''(u^N)\nabla u^N v^N \rangle_{L^2} dt$. Thus, by the Hölder inequality and the Gagliardo–Nirenberg inequality $\|\nabla u\|_{L^4} \leq C\|\Delta u\|^{\alpha^2} \|\nabla u\|^{1-\alpha^2}$, $\alpha = \frac{d}{4}$, we have
\[
I_1 \leq C\|\nabla u^N\|_{L^4}^2 (1 + \|u^N\|_{L^{\infty}}) \|v^N\|_{L^2}^2
\]
\[
\leq C\|\Delta u^N\|_{L^2}^2 \|u^N\|_{L^6}^{2-2\alpha} (1 + \|u^N\|_{L^{\infty}}) \|v^N\|_{L^2}^2.
\]
By further applying the Gagliardo–Nirenberg inequality $\|u\|_{L^{\infty}} \leq C\|\Delta u\|_{L^2}^{\alpha'} \|u\|_{L^6}^{1-\alpha'}$, $\alpha' = (\frac{2}{d} - \frac{1}{3})^{-1}\frac{1}{6}$ and using the Young’s inequality, we get for $d \leq 2$,
\[
I_1 \leq C\|\Delta u^N\|_{L^2}^2 \|u^N\|_{L^6}^{2-2\alpha} (1 + \|\Delta u^N\|_{L^2}^{\alpha'} \|u^N\|_{L^6}^{1-\alpha'}) \|v^N\|_{L^2}^2
\]
\[
\leq C(1 + (\|u^N\|_{L^2}^2 \|u^N\|_{H^{\frac{1}{6}}}^{1-\alpha'} \|v^N\|_{L^2})^m + \|\Delta u^N\|_{L^2}^2)
\]
with some positive number $m$, and for $d = 3$,
\[
I_1 \leq C(1 + \|u^N\|_{L^2}^2 \|u^N\|_{H^{\frac{1}{6}}}^2 \|v^N\|_{L^2}^2) (1 + \|\Delta u^N\|_{L^2}^2)
\]
\[
\leq C(1 + \|\Delta u^N\|_{L^2}^2) + C(\epsilon \|v^N\|_{L^2}^2 + \epsilon \|u^N\|_{H^{\frac{1}{6}}}^2 + \epsilon \|u^N\|_{L^2}^2 + C(\epsilon) \|\Delta u^N\|_{L^2}^2
\]
with sufficiently small $\epsilon > 0$. Using Cauchy–Schwarz inequality, Young’s inequality and the fact $H^{\frac{1}{6}} \hookrightarrow L^6$, we deduce that
\[
|\langle (-\Lambda)u^N, (u^N)^3 \rangle_{L^2}| \leq \|(-\Lambda)u^N\|_{L^2} \|(u^N)^3\|_{L^2} \leq \frac{1}{2}\|\Delta u^N\|_{L^2}^1 + \frac{1}{2}\|u^N\|_{L^6}^6
\]
\[
\leq \frac{1}{2}\|\Delta u^N\|_{L^2}^2 + C\|u^N\|_{H^{\frac{1}{6}}}^6.
\]
The above inequality means $V_2(u^N, v^N) \geq \frac{1}{4}\|\Delta u^N\|_{L^2}^2 - \frac{C}{4}\|u^N\|_{H^{\frac{1}{6}}}^6$, which deduces that for $d \leq 2$,
\[
dV_2 \leq C(4V_2 + C\|u^N\|_{H^{\frac{1}{6}}}^6) dt + C(1 + (\|u^N\|_{L^2}^{2-2\alpha} \|u^N\|_{H^{\frac{1}{6}}}^{1-\alpha'} \|v^N\|_{L^2}^m) dt
\]
\[
+ \langle \nabla v^N, \nabla P_N dW(t) \rangle_{L^2} + \frac{1}{2} \text{Tr} \left( (\nabla P_N Q^{\frac{3}{2}})(\nabla P_N Q^{\frac{3}{2}})^* \right) dt.
\]
Taking the expectation on both sides and applying Gronwall’s inequality, we have

\[ EV_2(u^N(t), v^N(t)) \leq C \exp(Ct) \left( V_2(u^N(0), v^N(0)) + \frac{1}{2} \text{Tr} \left( \left( \nabla P_N \mathbf{Q}^\frac{1}{2} \right) \left( \nabla P_N \mathbf{Q}^\frac{1}{2} \right)^* \right) t \right. \]

\[ + \left. \int_0^t E \left( 1 + \left( \|u^N\|_{L^2}^{2-2\alpha} \|u^N\|_{\dot{H}^1}^{1-a'} \|v^N\|_{L^2}^m \right) ds \right) \right) \]

which, combined with Lemmas 3.1 and 3.2, show the desired result in the case that \( d \leq 2 \). For \( d = 3 \), we have that

\[ dV_2 \leq C \left( \epsilon \|v^N\|_{L^2}^2 + \epsilon \|u^N\|_{H^1}^2 + \epsilon \|u^N\|_{L^4}^4 + C(\epsilon) \right) \left( 4V_2 + C\|u^N\|_{H^1}^6 \right) dt \]

\[ + \left( \nabla v^N, \nabla P_N dW(t) \right)_{L^2} + \frac{1}{2} \text{Tr} \left( \left( \nabla P_N \mathbf{Q}^\frac{1}{2} \right) \left( \nabla P_N \mathbf{Q}^\frac{1}{2} \right)^* \right) dt. \]

Gronwall’s inequality yields that

\[ V_2(u^N(t), v^N(t)) \leq C \int_0^t \exp(\epsilon \|v^N\|_{L^2}^2 + \epsilon \|u^N\|_{H^1}^2 + \epsilon \|u^N\|_{L^4}^4) ds V_2(u_0^N, v_0^N) \]

\[ + C \int_0^t \exp(\epsilon \|v^N\|_{L^2}^2 + \epsilon \|u^N\|_{H^1}^2 + \epsilon \|u^N\|_{L^4}^4) ds \int_0^T \|u^N\|_{H^1}^6 ds \]

\[ + C \int_0^t \exp(\epsilon \|v^N\|_{L^2}^2 + \epsilon \|u^N\|_{H^1}^2 + \epsilon \|u^N\|_{L^4}^4) ds \int_0^T \left| \left( \nabla v^N, \nabla P_N dW(t) \right)_{L^2} \right| \]

\[ + C \int_0^t \exp(\epsilon \|v^N\|_{L^2}^2 + \epsilon \|u^N\|_{H^1}^2 + \epsilon \|u^N\|_{L^4}^4) ds \int_0^T \left( \left( \nabla P_N \mathbf{Q}^\frac{1}{2} \right) \left( \nabla P_N \mathbf{Q}^\frac{1}{2} \right)^* \right) ds. \]

Taking expectation, using the Hölder and Burkholder–Davis–Gundy inequalities, and Lemmas 3.1 and 3.2, we get

\[ EV_2(u^N(t), v^N(t)) \]

\[ \leq C(T, X_0, \mathbf{Q}) + C \int_0^t \mathbb{E} \left[ \exp(\epsilon \|v^N\|_{L^2}^2 + \epsilon \|u^N\|_{H^1}^2 + \epsilon \|u^N\|_{L^4}^4) ds \right. \]

\[ \cdot \left. \left| \int_0^T \left( \nabla v^N, \nabla P_N dW(t) \right)_{L^2} \right| \right] \]

\[ \leq C(T, X_0, \mathbf{Q}) + C \sqrt{\mathbb{E} \left[ \left| \int_0^T \left( \nabla v^N, \nabla P_N dW(t) \right)_{L^2} \right|^2 \right]} \]

\[ \leq C(T, X_0, \mathbf{Q}, \epsilon) + \epsilon \sup_{t \in [0, T]} \mathbb{E} \left[ \|\nabla v^N\|_{L^2}^2 \right], \]

which completes the proof in the case that \( d = 3 \). \( \square \)

Next we derive Hölder continuity in time for the solution \( u^N \) and \( X^N \) with respect to the spatial \( L^p(\Omega; \dot{H}) \)-norm. Both results play a key role in our error analysis in Section 4.
Lemma 3.3. Assume that conditions in the Lemma 3.1 hold. Then there exists a positive constant $C$ depending on $X_0, p, T$ such that for each $0 \leq s \leq t \leq T$,
\[
\| u^N(t) - u^N(s) \|_{L^p(\Omega; \mathbb{H})} \leq C|t - s|,
\]
\[
\| X^N(t) - X^N(s) \|_{L^p(\Omega; \mathbb{H})} \leq C|t - s|^\frac{1}{2}.
\]

Proof. From (5), we have
\[
\begin{align*}
&u^N(t) - u^N(s) = (C_N(t) - C_N(s))P_N(u_0) + (-\Lambda_N)^{-\frac{1}{2}}(S_N(t) - S_N(s))P_N(v_0) \\
&\quad + \int_0^s (-\Lambda_N)^{-\frac{1}{2}}(S_N(t - r) - S_N(s - r))P_N(f(u^N))dr \\
&\quad + \int_s^t (-\Lambda_N)^{-\frac{1}{2}}S_N(t - r)P_N(f(u^N))dr \\
&\quad + \int_0^s (-\Lambda_N)^{-\frac{1}{2}}(S_N(t - r) - S_N(s - r))P_N(f(u^N))dr \\
&\quad + \int_t^T (-\Lambda_N)^{-\frac{1}{2}}S_N(t - r)P_N(f(u^N))dr.
\end{align*}
\]

Therefore, using the properties of $C_N(t)$ and $S_N(t)$ (see Lemma 2.1) and the Burkholder–Davis–Gundy inequality,
\[
\begin{align*}
\| u^N(t) - u^N(s) \|_{L^p(\Omega; \mathbb{H})} &\leq C(t - s) \left( \| u_0 \|_{L^p(\Omega; \mathbb{H})} + \| v_0 \|_{L^p(\Omega; \mathbb{H})} \right) \\
&\quad + C \int_0^s (t - s) \| f(u^N) \|_{L^p(\Omega; \mathbb{H})}ds + C \int_s^t \| f(u^N) \|_{L^p(\Omega; \mathbb{H})}ds \\
&\quad + \left( \int_0^s \| (-\Lambda_N)^{-\frac{1}{2}}(S_N(t - r) - S_N(s - r))P_N \|_{L^2(\Omega; \mathbb{H})}^2dr \right)^{\frac{1}{2}} \\
&\quad + \left( \int_s^t \| (-\Lambda_N)^{-\frac{1}{2}}S_N(t - r)P_N \|_{L^2(\Omega; \mathbb{H})}^2dr \right)^{\frac{1}{2}} \\
&\quad \leq C|t - s| \left( 1 + \| u_0 \|_{L^p(\Omega; \mathbb{H})} + \| v_0 \|_{L^p(\Omega; \mathbb{H})} + \sup_{0 \leq t \leq T} \| u^N \|_{L^p(\Omega; \mathbb{H})}^3 \right) \leq C|t - s|,
\end{align*}
\]
which is the claim for $u^N$. For $X^N$, the proof is similar. The main difference lies on
\[
\mathbb{E} \left[ \int_s^t A^{-\frac{1}{2}}C(t - r)dW(r) \right]^{2p} \leq C\mathbb{E} \left[ \int_s^t (\Lambda)^{-\frac{1}{2}}C(t - r)\| dW(r) \|_{L^2}^{2p} \right] \leq C(t - s)^p.
\]

\[
\square
\]

3.3 Strong convergence of the spatial semi-discretization

Based on the Lemma 3.2, we prove that the discrete solution $X_N$ (7) converges to that of (2) in strong sense and the strong convergence order of the semi-discretization (5) is stated in the following theorem.
Theorem 3.1. Assume that $d = 1$, $\beta \geq 1$ or that $d = 2$, $\beta = 2$ and in addition suppose that $X_0 \in \mathbb{H}^\beta$, $\|(-\Lambda)^{\beta-\frac{1}{2}}\|_{L^2_0} < \infty$. Then stochastic wave equation $[2]$ admits a unique mild solution given by

$$X(t) = E(t)X_0 + \int_0^t E(t-s)\mathbb{E}(X(s))ds + \int_0^t E(t-s)GdW(s),$$

such that

$$\sup_{0 \leq t \leq T} \|X_N - X\|_{L^p(\Omega;\mathbb{H})} = O(\lambda_N^{-\frac{\beta}{2}}).$$

**Proof.** For sake of simplicity, we consider the strong convergence of $u_N$ whose proof is similar to $X^N$ as example.

**Step 1: Existence and uniqueness of the mild solution $X(t)$.** We claim that $\{u_N\}_{N \in \mathbb{N}}$ is a Cauchy sequence in $C([0,T], L^2)$. Notice that

$$u^N(t) - u^M(t) = (u^N(t) - P^N u^M(t)) + ((P^N - I)u^M(t)),$$

where $N, M \in \mathbb{N}_+$, and

$$u^N(t) = C_N(t)u^N_0 + (-\Lambda)^{-\frac{1}{2}} S_N(t)u^N_0 - \int_0^t (-\Lambda)^{-\frac{1}{2}} S(t-s)P_Nf(u^N)ds$$

$$+ \int_0^t (-\Lambda)^{-\frac{1}{2}} S(t-s)P_NdW(s),$$

$$u^M(t) = C_M(t)u^M_0 + (-\Lambda)^{-\frac{1}{2}} S_M(t)u^M_0 - \int_0^t (-\Lambda)^{-\frac{1}{2}} S(t-s)P_Mf(u^M)ds$$

$$+ \int_0^t (-\Lambda)^{-\frac{1}{2}} S(t-s)P_MdW(s).$$

Without loss of generality, it may be assumed that $M > N$. According to the expression of both $u^N$ and $u^M$, using the definition of $P_N$, we have

$$\|(P^N - I)u^M(t)\|^2_{L^2} = \sum_{i=N+1}^{\infty} \lambda_i^{-\beta} \langle u^M(t), \lambda_i^{\frac{\beta}{2}} e_i \rangle^2_{L^2} \leq \lambda_N^{-\beta} \|u^M\|^2_{\mathbb{H}^\beta}$$

with $\beta \geq 1$. With respect to the term $u^N(t) - P^N u^M(t)$, we have

$$u^N(t) - P^N u^M(t) = \int_0^t (-\Lambda)^{-\frac{1}{2}} S(t-s)P^N (f(u^M) - f(u^N)) ds.$$

From the Sobolev embedding $L^\frac{d}{2} \hookrightarrow \mathbb{H}^{-1}$, and using the Hölder inequality,

$$\|u^N(t) - P^N u^M(t)\|_{L^2} \leq \int_0^t \left\|(-\Lambda)^{-\frac{1}{2}} S(t-s)P^N (f(u^M) - f(u^N)) \right\|_{L^2} ds$$

$$\leq C \int_0^t (1 + \|u^N\|^2_{L^6} + \|u^M\|^2_{L^6})(\|u^N(t) - P^N u^M(t)\|_{L^2} + \|(P^N - I)u^M(t)\|_{L^2}) ds,$$
which implies
\[
\|u^N(t) - P^N u^M(t)\|_{L^2} \\
\leq C\lambda_N^{-\beta} \exp \left( \int_0^T \left( \|u^N\|_{L^6}^2 + \|u^M\|_{L^6}^2 \right) ds \right) \int_0^t \left( 1 + \|u^N\|_{L^6}^2 + \|u^M\|_{L^6}^2 \right) \|u^M\|_{L^6} \ ds
\]
due to Gronwall’s inequality. Taking \(p\)th moment and then using Hölder inequality, Young’s inequality, and Corollary 3.1 we obtain that
\[
\|u^N(t) - u^M(t)\|_{L^p(\Omega; L^2)} \\
\leq C\lambda_N^{-\frac{\beta}{p}} \left\| \exp \left( \int_0^T \left( \|u^N\|_{L^6}^2 + \|u^M\|_{L^6}^2 \right) ds \right) \right\|_{L^{2p}(\Omega; \mathbb{R})} \\
\cdot \left\| \int_0^T \left( 1 + \|u^N\|_{L^6}^2 + \|u^M\|_{L^6}^2 \right) \|u^M\|_{L^6} ds \right\|_{L^{2p}(\Omega; \mathbb{R})},
\]
which leads to
\[
\sup_{0 \leq t \leq T} \|u^N(t) - u^M(t)\|_{L^p(\Omega; L^2)} \leq C\lambda_N^{-\frac{\beta}{p}}.
\]
Similarly, we can prove that \(\{v^M\}_{M \in \mathbb{N}}\) is a Cauchy sequence in \(\mathbb{H}^{-1}\), which means that \(\{X^M\}_{M \in \mathbb{N}}\) is a Cauchy sequence in \(\mathbb{H}\).

Denote \(X = (u, v)^\top \in \mathbb{H}\) the limit of \(\{X^M\}_{M \in \mathbb{N}}\). Due to Propositions 3.1 and Fatou’s Lemma, we also have that \(\mathbb{E}[\|X\|_{\mathbb{H}^1}^p] \leq C(T, p, X_0, Q)\). By the Gagliardo–Nirenberg inequality and the boundedness of \(X\) and \(X^N\) in \(\mathbb{H}^1\), we get that \(X^N\) also converges to \(X\) in \(L^p\). Then Fatou’s lemma leads that for any \(c > 0\),
\[
\mathbb{E} \left( \exp \left( \int_0^T c \|u(s)\|_{L^6}^2 ds \right) \right) = \mathbb{E} \left( \exp \left( \int_0^T c \lim_{N \to \infty} \|u^N(s)\|_{L^6}^2 ds \right) \right) \\
\leq \lim_{N \to \infty} \mathbb{E} \left( \exp \left( \int_0^T c \|u^N(s)\|_{L^6}^2 ds \right) \right) < \infty.
\]

**Step 2: Strong convergence order.** To derive the strong convergence rate, we need to prove that
\[
X(t) = E(t)X_0 + \int_0^t E(t-s)F(X(s))ds + \int_0^t E(t-s)GdW(s)
\]
(12)
for any \(t \in [0, T]\). We take the convergence of \(\{u^N\}_{N \in \mathbb{N}}\) as example for convenience, that is, we need to show \(u\) satisfies
\[
u(t) = C(t)u_0 + (-\Lambda)^{(-\frac{1}{2})}S(t)v_0 - \int_0^t (-\Lambda)^{(-\frac{1}{2})}S(t-s)f(u)ds \\
+ \int_0^t (-\Lambda)^{(-\frac{1}{2})}S(t-s)dW(s).
\]
To this end, we show that the mild form of the exact solution \( u^N \) is convergent to that of \( u \). The assumption on \( X \) yields that
\[
\|C(t)(I - P^N)u_0\| + \|(-\Lambda)^{-\frac{1}{2}} S(t)(I - P^N)v_0\| \leq C\lambda_N^{-\frac{3}{2}}(\|u_0\|_{H^3} + \|v_0\|_{H^{\beta-1}}).
\]
Based on the Sobolev embedding theorem \( L^\frac{6}{5} \hookrightarrow H^{-1} \), we have
\[
\int_0^t \|(-\Lambda)^{-\frac{1}{2}} S(t-s)P^N (f(u) - f(u^N)) \|_{L^p(\Omega;L^2)} ds
\]
\[
\leq C \int_0^t (1 + \|u\|_{L^6}^2 + \|u^N\|_{L^{5}}^2)\|u - u^N\|_{L^{2p}(\Omega;L^2)} ds \leq C\lambda_N^{-\frac{\beta}{2}}.
\]
For the stochastic term, by the Burkholder–Davis–Gundy inequality we obtain that for \( p \geq 2 \),
\[
\| \int_0^t (-\Lambda)^{-\frac{1}{2}} S(t-s)(I - P^N) dW(s) \|_{L^p(\Omega;L^2)}
\]
\[
\leq \sqrt{\int_0^t \|(-\Lambda)^{-\frac{1}{2}} S(t-s)(-\Lambda)^{-\frac{\beta}{2}} (I - P^N)(-\Lambda)^{\frac{\beta}{2} - \frac{1}{2}}\|_{L^p(\Omega;\mathcal{L}_d^2)}^2 ds}
\]
\[
\leq C\lambda_N^{-\frac{3}{2} - \frac{\beta}{2}}.
\]
Then summing up all the above estimates and using the Hölder inequality finish the proof.

\[\square\]

**Remark 3.3.** Under the same conditions of Theorem 3.1, one could also get the following stronger result,
\[
\|X_N - X\|_{L^2(\Omega;C([0,T];H))} = O(\lambda_N^{-\frac{\beta}{2}}).
\]

The main difference of the proof compared to that of Theorem 3.1 lies on dealing with the stochastic term. For instance, we use the properties of trigonometric functions and Burkholder–Davis–Gundy inequality and get that for \( p \geq 2 \),
\[
\| \int_0^t (-\Lambda)^{-\frac{1}{2}} S(t-s)(I - P^N) dW(s) \|_{L^p(\Omega;C([0,T];L^2))}
\]
\[
\leq C\| \int_0^t (-\Lambda)^{-\frac{1}{2}} C(s)(I - P^N) dW(s) \|_{L^p(\Omega;C([0,T];L^2))}
\]
\[
+ C\| \int_0^t (-\Lambda)^{-\frac{1}{2}} S(s)(I - P^N) dW(s) \|_{L^p(\Omega;C([0,T];L^2))}
\]
\[
\leq C\lambda_N^{-\frac{3}{2} - \frac{\beta}{2}}.
\]

**Remark 3.4.** For the case that \( d = 3 \), \( \{u^M\}_{M \in \mathbb{N}} \) is only strongly convergent and possesses pathwise convergence rate, since the mild solution \( \{X\} \) of the spectral Galerkin approximation in this case does not have exponential integrability in \( L^6 \).

**Remark 3.5.** We prove that the mild solution \( X \) of (1) satisfies \( \|X\|_{L^p(\Omega;\mathbb{H}^\beta)} \leq C(X_0, T, p) \) if \( p \geq 1 \), \( d = 1 \), \( \beta \geq 1 \), \( X_0 \in \mathbb{H}^\beta \), \( \|(-\Lambda)^{\frac{\beta-1}{2}} \|_{\mathcal{L}_d^2} < \infty \). Moreover, the mild solution satisfies \( \|X\|_{L^p(\Omega;\mathbb{H}^\beta)} \leq C(X_0, T, p) \), when \( d = 2, 3 \), \( X_0 \in \mathbb{H}^\beta \), \( \|(-\Lambda)^{\frac{1}{2}} \|_{\mathcal{L}_d^2} < \infty \).
4 Strong convergence of the temporal discretization

In this section, we propose the full discretization for stochastic wave equation (1) by applying splitting AVF method to (5) and, finally, we will state a convergence theorem for this full discretization. For any $T > 0$, we partition the time domain $[0, T]$ uniformly with nodes $t_m = mh$, $m = 0, 1, \cdots, M$, and $Mh = T$.

We first decompose (1) into a deterministic differential equation

$$
\begin{align*}
du^{N,D}(t) &= v^{N,D}(t)dt, \quad dv^{N,D}(t) = \Lambda_N u^{N,D}(t)dt - P_N(f(u^{N,D}(t)))dt, \\
u^{N,D}(t_m) &= u^N(t_m), \quad v^{N,D}(t_m) = v^N(t_m),
\end{align*}
$$

and a stochastic system

$$
\begin{align*}
du^{N,S}(t) &= 0, \quad dv^{N,S}(t) = P_NdW(t), \\
u^{N,S}(t_m) &= u^{N,D}(t_{m+1}), \quad v^{N,S}(t_m) = v^{N,D}(t_{m+1}),
\end{align*}
$$

respectively. Further by using the AVF scheme to discretize (13) and the analytical solution of (14), we have the splitting AVF scheme

$$
\begin{align*}
u^N_{m+1} &= u^N_m + h\bar{v}^N_{m+\frac{1}{2}}, \\
\bar{v}^N_{m+\frac{1}{2}} &= v^N_m + h\Lambda_N u^N_{m+\frac{1}{2}} - hP_N\left(\int_0^1 f(u^N_m + \theta(u^N_{m+1} - u^N_m))d\theta\right), \\
v^N_{m+1} &= \bar{v}^N_{m+1} + P_N\delta W_m,
\end{align*}
$$

where $\bar{v}^N_{m+\frac{1}{2}} = \frac{1}{2}(\bar{v}^N_{m+1} + v^N_m)$ and the increment $\delta W_m$ is given by

$$
\delta W_m := W(t_{m+1}) - W(t_m) = \sum_{k=1}^{\infty} (\beta_k(t_{m+1}) - \beta_k(t_m)) Q^2 e_k.
$$

Denote

$$
\mathbb{A}(t) = \begin{pmatrix} I & \frac{1}{2}I \\ \Lambda_N \frac{1}{2} I & I \end{pmatrix}, \quad \mathbb{B}(t) = \begin{pmatrix} I & -\frac{1}{2}I \\ -\Lambda_N \frac{1}{2} I & I \end{pmatrix}
$$

and $\mathbb{M}(t) = I - \Lambda_N \frac{Q^2}{4}$, then we have

$$
\mathbb{B}^{-1}(t)\mathbb{A}(t) = \begin{pmatrix} \mathbb{M}^{-1}(t) & 0 \\ 0 & \mathbb{M}^{-1}(t) \end{pmatrix}, \quad \mathbb{A}^2(t) = \begin{pmatrix} 2\mathbb{M}^{-1}(t) - I & \mathbb{M}^{-1}(t)t \\ \mathbb{M}^{-1}(t)\Lambda_N t & 2\mathbb{M}^{-1}(t) - I \end{pmatrix}.
$$

This formula leads that (15) can be rewritten as

$$
\begin{align*}
\begin{pmatrix} u^N_{m+1} \\ v^N_{m+1} \end{pmatrix} &= \mathbb{B}^{-1}(h)\mathbb{A}(h) \begin{pmatrix} u^N_m \\ v^N_m \end{pmatrix} \\
&\quad + \mathbb{B}^{-1}(h) \begin{pmatrix} 0 \\ hP_N\left(\int_0^1 f(u^N_m + \theta(u^N_{m+1} - u^N_m))d\theta\right) \end{pmatrix} + \begin{pmatrix} 0 \\ P_N(\delta W_m) \end{pmatrix}.
\end{align*}
$$

To study the strong convergence of the proposed full discretization, we first give some estimates of the matrix $\mathbb{B}^{-1}\mathbb{A}$.
Lemma 4.1. Consider any $r \geq 0$ and $t \geq 0$, then for any $w \in \mathbb{H}^r$, one has
\[
\|B^{-1}(t)A(t)w\|_{\mathbb{H}^r} = \|w\|_{\mathbb{H}^r},
\]
following from [3, Theorem 3], we gave the following lemma which is used to give the error estimate for [15].

**Lemma 4.2.** For any $r \geq 0$ and $h \geq 0$, there exists a positive constant $C := C(r)$ such that
\[
\|\left(E_N(h) - B^{-1}(h)A(h)\right)w\|_{\mathbb{H}^{r+2}} \leq C|h|\|w\|_{\mathbb{H}^{r+2}},
\]
\[
\|\left(E_N(h) - B^{-1}(h)\right)w\|_{\mathbb{H}^{r+1}} \leq C h \|w\|_{\mathbb{H}^{r+1}},
\]
for any $w \in \mathbb{H}^{r+2}$.

To obtain the strong convergence order of full discretization [15], we show a priori estimate and the exponential integrability property of discrete solution of [15].

**Proposition 4.1.** The splitting process $X^N$ is uniquely solvable, and for any $t \in T$ and $p > 1$, there exists a positive constant $C'$ such that
\[
\sup_{m \in \mathbb{N}_M} \mathbb{E}(V_1^p((u_m^N, v_m^N))) \leq C',
\]
where $V_1(u_m^N, v_m^N) = \frac{1}{2}u_m^N\|_{L^2}^2 + \frac{1}{2}\|v_m^N\|_{L^2}^2 + F(u_m^N)$.  

**Proof.** Fixing $t \in T_m$ with $m \in \mathbb{N}_M$, we have
\[
V_1(u_{m+1}^N, v_{m+1}^N) = V_1(u_m^N, v_m^N),
\]
due to equations [15]. Applying Itô formula to equations [14], we obtain
\[
dV_1(u_{m,s}^N, v_{m,s}^N) = \langle v_{m,s}^N, P_N dW(t) \rangle_{L^2} + \frac{1}{2} \text{Tr} \left( (P_N Q^\frac{1}{2})(P_N Q^\frac{1}{2})^* \right) dt,
\]
which implies that
\[
V_1(u_{m+1}^N, v_{m+1}^N) = V_1(u_m^N, v_m^N) + \int_{t_m}^t \langle v_{m,s}^N, P_N dW(s) \rangle_{L^2} + \frac{1}{2} \text{Tr} \left( (P_N Q^\frac{1}{2})(P_N Q^\frac{1}{2})^* \right) ds.
\]
In a similar manner, by applying Itô formula to $V_1^p(u_{m,s}^N, v_{m,s}^N)$, for any $p \geq 1$, we obtain
\[
V_1^p(u_{m,s}^N, v_{m,s}^N) = V_1^p(u_{m,s}^N, v_{m,s}^N) + \frac{p}{2} \int_{t_m}^t V_1^{p-1}(u_{m,s}^N, v_{m,s}^N) \text{Tr} \left( (P_N Q^\frac{1}{2})(P_N Q^\frac{1}{2})^* \right) ds \\
+ \frac{p}{2} \int_{t_m}^t V_1^{p-1}(u_{m,s}^N, v_{m,s}^N) \langle v_{m,s}^N, P_N dW(t) \rangle_{L^2} \\
+ \frac{p(p-1)}{2} \sum_{i=1}^N \int_{t_m}^t V_1^{p-2}(u_{m,s}^N, v_{m,s}^N) \langle v_{m,s}^N Q^\frac{1}{2} e_i, v_{m,s}^N Q^\frac{1}{2} e_i \rangle_{L^2} ds.
\]
We note that, for the proposed full discrete scheme (15), \((u^N(t_m), v^N(t_m))\) takes values at \((u^N_{m+1}, \bar{v}^N_{m+1})\). Taking the expectation on both sides of the above equation, and using Hölder inequality,

\[
\mathbb{E}(V^p(u^N_1(t), v^N_1(t))) \leq \mathbb{E}(V^p(u^N_1(u^N_{m+1}, \bar{v}^N_{m+1})) + C \int_{t_m}^t (1 + \mathbb{E}(V^p(u^N_1(s), v^N_1(s))))ds,
\]

which leads to

\[
\mathbb{E}(V^p(u^N_{m+1}, \bar{v}^N_{m+1})) \leq \exp(C) \left( \mathbb{E}(V^p(u^N_1(t_m), v^N(t_m))) + C \right),
\]

and therefore

\[
\sup_{m \in \mathbb{N}_m} \mathbb{E}(V^p((u^N_{m+1}, \bar{v}^N_{m+1}))) \leq \exp(C) \mathbb{E}(V^p(u^N_0, v^N_0)) + \exp(C),
\]

which implies the estimate (18). \(\square\)

**Remark 4.1.** From the above proof of Proposition 4.1 it can be seen that

\[
\mathbb{E}(V_1(u^N_m, v^N_m(t))) = \mathbb{E}(V_1(u^N_0, v^N_0)) + \int_{0}^{t_m} \frac{1}{2} \text{Tr} \left( (P_N Q^\frac{1}{2} ) (P_N Q^\frac{1}{2})^* \right) ds.
\]

This means that the proposed scheme preserves the energy evolution law of the original system (1).

In order to get the strong convergence of the temporal discretization, we also need the exponential integrability property of \(\{u^N_i\}_{1 \leq i \leq M}\). Similar to the proof of Lemma 3.2, we demonstrate exponential integrability property of \(u^N\) in the following lemma.

**Lemma 4.3.** Let \(d = 1, 2\) and \(\|Q^\frac{1}{2}\|_{L_2(L^2)} < \infty\). Then there exists a positive constant \(C := C(X_0, Q, T)\) such that

\[
\mathbb{E} \left( \exp \left( c h \sum_{i=1}^{m} \|u^N_i\|_{L^2}^2 \right) \right) \leq C,
\]

for any constant \(c > 0\).

**Proof.** We note that the proposed scheme satisfies

\[
V_1(u^N_m, \bar{v}^N_m) = V_1(u^N_{m+1}, \bar{v}^N_{m+1}),
\]

where \(V_1(u^N_m, \bar{v}^N_m) = \frac{1}{2} \|u^N_m\|^2_{H^1} + \frac{1}{2} \|\bar{v}^N_m\|^2_{L^2} + F(u^N_m) + C\). Similar to the proof in Lemma 3.2, we have that for \(t \in [t_m, t_{m+1}]\)

\[
\mathbb{E} \left( \exp \left( \frac{V_1(u^N_i, v^N_i)}{\exp(\alpha t)} + \int_{t_m}^{t} \bar{U}(s, X_s) ds \right) \right) \leq \mathbb{E} \left( \exp \left( \frac{V_1(u^N_i, v^N_i)}{\exp(\alpha t)} \right) \right),
\]

where \(\bar{U} = -\frac{1}{2} \text{Tr}(Q)\), and \(\alpha > \frac{1}{2} \text{Tr}(Q)\).

Finally, based on the iterative arguments, the desired estimate (19) follows from Jensen’s inequality, and Young’s inequality. The proof is completed. \(\square\)
Theorem 4.1. Assume that \( d = 1, \beta \geq 1 \) or that \( d = 2, \beta = 2 \) and in addition suppose that \( X_0 \in \mathbb{H}^\beta, \| (-\Lambda)^{\beta-1} \|_{L^2_0} < \infty \). Let \( \gamma = \min(\beta, 2) \). Then there exists a positive constant \( C := C(X_0, Q, T) \) such that

\[
\sup_{m \in \mathbb{Z}_M} \mathbb{E} \left[ \| X^N(t_m) - X^N_m \|^{2p} \right] \leq C h^{\gamma p}.
\]

(20)

Proof. The numerical approximation given by equation (16) can be rewritten as

\[
X^N_m = (\mathbb{B}^{-1}(h)\mathbb{A}(h))^m X^N_0 + \sum_{j=0}^{m-1} (\mathbb{B}^{-1}(h)\mathbb{A}(h))^{(m-1-j)} \left( \mathbb{P}_N(\delta W_j) \right)
\]

(21)

Then the mild solution (7) can be expressed as

\[
X^N(t_m) = E(t_m) X^N_0 + \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} E(t_m - s) \mathbb{F}_N(X^N(s)) ds
\]

(22)

Let \( \varepsilon_m = X^N(t_m) - X^N_m \) for \( m = 0, \ldots, M \). Subtracting (21) from (22), we get

\[
\varepsilon_m = (E(t_m) - (\mathbb{B}^{-1}(h)\mathbb{A}(h))^m) X^N_0
\]

\[
+ \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left( E(t_m - s) - (\mathbb{B}^{-1}(h)\mathbb{A}(h))^{(m-1-j)} \right) \mathbb{P}_N(\delta W(s))
\]

\[
+ \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left( E(t_m - s) \left( \mathbb{P}_N(f(u^N(s))) \right) - (\mathbb{B}^{-1}(h)\mathbb{A}(h))^{(m-1-j)}
\]

\[
\cdot (\mathbb{B}^{-1}(h) \left( \mathbb{P}_N(f(u^N + \theta(u^N_{j+1} - u^N_j)) \right) ds,
\]

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which yields
\[ \|\alpha_m\|_{H} \leq \left\| \left( E(t_m) - \left( B^{-1}(h)a(h) \right)^m \right) X_0^N \right\|_H \]
\[ + \left\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left( E(t_m - s) - \left( B^{-1}(h)a(h) \right)^{(m-1-j)} \right) \begin{pmatrix} 0 \\ P_N(f(u^N(s))) \end{pmatrix} ds \right\|_H \]
\[ + \left\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left( E(t_m - s) - \left( B^{-1}(h)a(h) \right)^{(m-1-j)} \right) \begin{pmatrix} 0 \\ P_N(f(u^N(s))) \end{pmatrix} ds \right\|_H \]
\[ \leq C h^{\frac{d}{2}} \|X_0^N\|_{H^0} + Err^1_m + Err^2_m. \]

We decompose the term \( Err^2_m \) in the strong sense into several parts as following
\[ Err^2_m \leq \left\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left( E(t_m - s) - E(t_{m-1} - s)B^{-1}(h) \right) \begin{pmatrix} 0 \\ P_N(f(u^N(s))) \end{pmatrix} ds \right\|_H \]
\[ + \left\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} E(t_{m-1} - s)B^{-1}(h) \begin{pmatrix} 0 \\ P_N(f(u^N(s))) \end{pmatrix} - \left( B^{-1}(h)a(h) \right)^{(m-1-j)} \right. \]
\[ \left. \cdot B^{-1}(h) \begin{pmatrix} 0 \\ P_N(\int_0^1 f(u^N_j + \theta(u^N_j + u^N_j) - u^N_j) d\theta) \end{pmatrix} ds \right\|_H \]
\[ = : I + II. \]

Using Hölder inequality, Young’s inequality and (17), we have
\[ I \leq \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left\| \left( E(t_m - s) - E(t_{m-1} - s)B^{-1}(h) \right) \begin{pmatrix} 0 \\ P_N(f(u^N(s))) \end{pmatrix} \right\|_H ds \]
\[ \leq \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left\| \left( E(h) - B^{-1}(h) \right) \begin{pmatrix} 0 \\ P_N(f(u^N(s))) \end{pmatrix} \right\|_H ds \]
\[ \leq C h \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \|f(u^N(s))\|_{L^2} ds. \]
For the term II, denoting \( \left\lfloor \frac{s}{h} \right\rfloor \) the integer part of \( \frac{s}{h} \), we obtain

\[
II \leq \left\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} E(t_{m-1} - s)B^{-1}(h) \left( P_N(f(u^N(s)) - f(u^N(t_{[\frac{s}{h}]^N}))) \right) ds \right\|_H
\]

\[
+ \left\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} E(t_{m-1} - s)B^{-1}(h) \left( P_N(f(u^N(t_{[\frac{s}{h}]^N}))) - f(u^N(t_{[\frac{s}{h}]^N})) \right) ds \right\|_H
\]

\[
+ \left\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} E(t_{m-1} - s)B^{-1}(h) \left( P_N(f(u^N(t_{[\frac{s}{h}]^N}))) - f(u^N(t_{[\frac{s}{h}]^N}) + \theta(u^N_{[\frac{s}{h}]^N} - u^N_{[\frac{s}{h}]^N}))d\theta \right) ds \right\|_H
\]

\[= : II_1 + II_2 + II_3 + II_4. \]

Now we estimate \( II_i, i = 1, 2, 3, 4 \), separately. Namely, we have

\[
II_1 \leq \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left\| B^{-1}(h) \left( P_N(f(u^N(s)) - f(u^N(t_{[\frac{s}{h}]^N}))) \right) \right\|_H ds
\]

\[
\leq \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left\| [M^{-1} h P_N(f(u^N(s)) - f(u^N(t_{[\frac{s}{h}]^N})))] \right\|_{L^2} ds
\]

\[+ \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left\| M^{-1} P_N(f(u^N(s)) - f(u^N(t_{[\frac{s}{h}]^N}))) \right\|_{H^{-1}} ds. \]

By the means of the inequality \( \left| \frac{2h}{4 + \lambda \cdot h^2} \right|^2 \leq C\lambda_i^{-1} \), for every \( i = 1, \ldots, N \),

\[
II_1 \leq C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left\| M^{-1} P_N(f(u^N(s)) - f(u^N(t_{[\frac{s}{h}]^N}))) \right\|_{H^{-1}} ds
\]

\[\leq C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left\| (u^N(s))^2 + (u^N(t_{[\frac{s}{h}]^N}))^2 + u^N(s)u^N(t_{[\frac{s}{h}]^N}))(u^N(s) - u^N(t_{[\frac{s}{h}]^N})) \right\|_{H^{-1}} ds
\]

\[+ C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left\| (u^N(s) + u^N(t_{[\frac{s}{h}]^N}))(u^N(s) - u^N(t_{[\frac{s}{h}]^N})) \right\|_{H^{-1}} ds
\]

\[+ C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left\| u^N(s) - u^N(t_{[\frac{s}{h}]^N}) \right\|_{H^{-1}} ds. \]
Based on Young’s inequality, Sobolev embedding inequality and Hölder inequality,

\[
\Pi_1 \leq C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (\|u^N(s)\|_L^2 + \|u^N(t_{\bar{s}_j})\|_L^2)\|u^N(s) - u^N(t_{\bar{s}_j})\|_L^2 ds
\]

\[
+ C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left(\|u^N(s)\|_L^6 + \|u^N(t_{\bar{s}_j})\|_L^6\right)\|u^N(s) - u^N(t_{\bar{s}_j})\|_L^2 ds
\]

\[
+ C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left(\|u^N(s) - u^N(t_{\bar{s}_j})\|_L^2\right) ds
\]

\[
\leq C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (\|u^N(s)\|_L^2 + \|u^N(t_{\bar{s}_j})\|_L^2 + 1)\|u^N(s) - u^N(t_{\bar{s}_j})\|_L^2 ds.
\]

Similarly, the term \(\Pi_2\) satisfies

\[
\Pi_2 \leq C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left(\|u^N(t_{\bar{s}_j})\|_L^2 + \|u^N(t_{\bar{s}_j})\|_L^2 + 1\right)\|u^N(t_{\bar{s}_j}) - u^N(t_{\bar{s}_j})\|_L^2 ds.
\]

For the term \(\Pi_3\), based on Taylor expansion, we have

\[
\Pi_3 \leq \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \int_0^1 \|f(u_j^N) - f(u_j^N + \theta(u_{j+1}^N - u_j^N))\|_{H^{-1}} d\theta ds
\]

\[
\leq \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \int_0^1 \|f'(u_j^N + \theta' u_{j+1}^N)\theta(u_{j+1}^N - u_j^N))\|_{H^{-1}} d\theta ds
\]

with \(\theta' \in (0, 1)\). Then \(u_{j+1}^N - u_j^N = \frac{h}{2} (v_{j+1}^N + v_j^N)\) and Sobolev embedding inequality yield that

\[
\Pi_3 \leq C h^2 \sum_{j=0}^{m-1} (\|u_j^N\|_{L^6}^2 + \|u_{j+1}^N\|_{L^6}^2 + 1)(\|v_{j+1}^N\|_{L^2} + \|v_j^N\|_{L^2}).
\]
With respect to the term $\Pi_4$, we have

$$\Pi_4 \leq \left\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (E(t_{m-1} - s) - E(t_{m-1} - t_j)) \mathbb{B}^{-1}(h) \right\|_H$$

$$+ \left\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (E(t_{m-1} - t_j) - (\mathbb{B}^{-1}(h)\mathbb{A}(h))^{(m-1-j)}) \mathbb{B}^{-1}(h) \right\|_H$$

$$\leq C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (s - t_j) \|f(u_j^N + \theta(u_{j+1}^N - u_j^N))\|_{L^2} ds$$

$$+ Ch^\beta \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \int_0^1 \|f(u_j^N + \theta(u_{j+1}^N - u_j^N))\|_{\mathbb{H}^{\beta-1}} d\theta ds.$$

For the case that $d = 1, \beta \geq 1$, the Sobolev embedding theorem leads to

$$\int_{t_j}^{t_{j+1}} \int_0^1 \|f(u_j^N + \theta(u_{j+1}^N - u_j^N))\|_{\mathbb{H}^{\beta-1}} d\theta ds$$

$$\leq C \int_{t_j}^{t_{j+1}} \int_0^1 (1 + \|u_j^N + \theta(u_{j+1}^N - u_j^N)\|_{L^\infty}) \|u_j^N + \theta(u_{j+1}^N - u_j^N)\|_{\mathbb{H}^{\beta-1}} d\theta ds$$

$$\leq C \int_{t_j}^{t_{j+1}} (1 + \|u_j^N\|_{\mathbb{H}^{\frac{1}{2}}} + \|u_{j+1}^N\|_{\mathbb{H}^{\frac{1}{2}}}) \|u_j^N + \theta(u_{j+1}^N - u_j^N)\|_{\mathbb{H}^{\beta-1}} + \|u_j^N + \theta(u_{j+1}^N - u_j^N)\|_{\mathbb{H}^{\beta-1}} d\theta ds$$

$$= Ch(1 + \|u_j^N\|_{\mathbb{H}^{\frac{1}{2}}} + \|u_{j+1}^N\|_{\mathbb{H}^{\frac{1}{2}}}) (\|u_j^N\|_{\mathbb{H}^{\beta-1}} + \|u_{j+1}^N\|_{\mathbb{H}^{\beta-1}})$$

$$\leq Ch(1 + \|u_j^N\|_{\mathbb{H}^{\frac{1}{2}}} + \|u_{j+1}^N\|_{\mathbb{H}^{\frac{1}{2}}}) (\|u_j^N\|_{\mathbb{H}^{\frac{3}{2}}} + \|u_{j+1}^N\|_{\mathbb{H}^{\frac{3}{2}}}).$$

For the case that $d = 2, \beta = 2$, using the Sobolev embedding theorem $\mathbb{H}^{1+} \hookrightarrow L^\infty$, we have

$$\int_{t_j}^{t_{j+1}} \int_0^1 \|f(u_j^N + \theta(u_{j+1}^N - u_j^N))\|_{\mathbb{H}^{\beta-1}} d\theta ds$$

$$\leq Ch(1 + \|u_j^N\|_{\mathbb{H}^{\frac{3}{2}}} + \|u_{j+1}^N\|_{\mathbb{H}^{\frac{3}{2}}}) (\|u_j^N\|_{\mathbb{H}^{1}} + \|u_{j+1}^N\|_{\mathbb{H}^{1}}).$$

Combining all the estimates together, we have

$$\Pi_4 \leq Ch^\beta h \sum_{j=0}^{m-1} (1 + \|u_j^N\|_{\mathbb{H}^{\frac{3}{2}}} + \|u_{j+1}^N\|_{\mathbb{H}^{\frac{3}{2}}}) (\|u_j^N\|_{\mathbb{H}^{\beta-1}} + \|u_{j+1}^N\|_{\mathbb{H}^{\beta-1}}).$$
Combining those together, we have
\[
\|\varepsilon_m\|_{\mathbb{H}} \leq \sum_{j=0}^{m-1} \Phi_j \|\varepsilon_j\|_{\mathbb{H}} + \sum_{j=0}^{m-1} \psi_j + Err^1_m + C(h^{q/2} \|X^N_0\|_{\mathbb{H}^\beta}),
\]
where for \( j = 0, 1, \ldots, (m - 1) \), \( \Phi_j = Ch(\|u^N(t_j)\|_{L^6}^2 + \|u^N_j\|_{L^6}^2 + 1) \), and
\[
\psi_j = Ch \int_{t_j}^{t_{j+1}} \| (f(u^N(s))) \|_{L^2} ds \\
+ Ch^{p/2} h(1 + \|u^N_j\|_{\mathbb{H}^1}^2 + \|u^N_{j+1}\|_{\mathbb{H}^1}^2)(\|u^N_j\|_{\mathbb{H}^{\beta-1}} + \|u^N_{j+1}\|_{\mathbb{H}^{\beta-1}}) \\
+ C \int_{t_j}^{t_{j+1}} (\|u^N(s)\|_{L^6}^2 + \|u^N(t_{\frac{s}{h}})\|_{L^6}^2 + 1)\|u^N(s) - u^N(t_{\frac{s}{h}})\|_{L^2} ds \\
+ Ch^{p}(\|u^N_j\|_{L^6}^2 + \|u^N_{j+1}\|_{L^6}^2 + 1)(\|\dot{u}^N_j\|_{L^2} + \|\dot{u}^N_j\|_{L^2}).
\]
By the Gronwall’s inequality, we have
\[
\|\varepsilon_m\|_{\mathbb{H}} \leq \left( Ch^{q/2} \|X^N_0\|_{\mathbb{H}^\beta} + Err^1_m + \sum_{j=0}^{m-1} \psi_j \right) \exp \left( \sum_{j=0}^{m-1} \Phi_j \right).
\]
Taking expectation and using Hölder inequality, we have
\[
\mathbb{E}\|\varepsilon_m\|_{\mathbb{H}}^{2p} \leq \left[ \mathbb{E} \left( Ch^{q/2} \|X^N_0\|_{\mathbb{H}^\beta} + Err^1_m + \sum_{j=0}^{m-1} \psi_j \right)^{4p} \right]^{1/2} \left[ \mathbb{E} \exp \left( 4p \sum_{j=0}^{m-1} \Phi_j \right) \right]^{1/2}.
\]
According to the exponential integrability of \( u^N \) and \( u^N_m \), the above inequality becomes
\[
\mathbb{E}\|\varepsilon_m\|_{\mathbb{H}}^{2p} \leq C \left[ \mathbb{E} \left( Ch^{q/2} \|X^N_0\|_{\mathbb{H}^\beta} + Err^1_m + \sum_{j=0}^{m-1} \psi_j \right)^{4p} \right]^{1/2} \\
\leq Ch^{\beta p} \|X^N_0\|_{\mathbb{H}^\beta}^{2p} + C \left[ \mathbb{E}(Err^1_m)^{4p} \right]^{1/2} + C \left[ \mathbb{E} \left( \sum_{j=0}^{m-1} \psi_j \right)^{4p} \right]^{1/2} \\
\leq Ch^{\beta p} \|X^N_0\|_{\mathbb{H}^\beta}^{2p} + C \left[ \mathbb{E}(Err^1_m)^{4p} \right]^{1/2} + Cm^{2p-1/2} \left[ \sum_{j=0}^{m-1} \mathbb{E}\psi_j^{4p} \right]^{1/2}.
\]
Thanks to Burkholder–Davis–Gundy inequality and properties of the semigroup (see Lemmas
We then have Theorem 1.1, which states the strong convergence of the above numerical scheme and continuity of the \( \mathbb{E} \). According to the Hölder inequality, the a prior estimates of \( 2.1 \) and \( 4.2 \) are

\[
\mathbb{E}(Err_{m}^{1})^{4p} = \mathbb{E} \left\| \sum_{j=0}^{m-1} \int_{t_{j}}^{t_{j+1}} (E(t_{m} - s) - (B^{-1}(h)A(h))^{(m-1-j)}) G_{N}dW(s) \right\|^{4p}_{\mathbb{H}} \\
\leq \mathbb{E} \left\| \int_{0}^{t_{m}} (E(t_{m} - s) - (B^{-1}(h)A(h))^{(m-1-[\frac{s}{h}]}) G_{N}dW(s) \right\|^{4p}_{\mathbb{H}} \\
\leq \left[ \int_{0}^{t_{m}} \left\| (E(t_{m} - s) - (B^{-1}(h)A(h))^{(m-1-[\frac{s}{h}]}) G_{N}Q_{s}^{1/2} \right\|^{2}_{\mathbb{H}} \right]^{2p}.
\]

Then

\[
\mathbb{E}(Err_{m}^{1})^{4p} \leq C \sup_{0 \leq s \leq t_{m}} \left\| (E(t_{m} - s) - E(t_{m-1} - t_{[\frac{s}{h}]}) G_{N}Q_{s}^{1/2} \right\|^{4p}_{\mathbb{H}} \\
+ C \sup_{0 \leq s \leq t_{m}} \left\| (E(t_{m-1} - t_{[\frac{s}{h}]}) - (B^{-1}(h)A(h))^{(m-1-[\frac{s}{h}]}) G_{N}Q_{s}^{1/2} \right\|^{4p}_{\mathbb{H}} \\
\leq C(h^{2\beta p} + h^{4p}),
\]

which leads to

\[
\mathbb{E}\| \varepsilon_{m} \|^{2p}_{\mathbb{H}} \leq C h^{\beta p} \| X_{0}^{N} \|^{2p}_{\mathbb{H}} + C(h^{\beta p} + h^{2p}) + C m^{2p-\frac{1}{2}} \left[ \sum_{j=0}^{m-1} \mathbb{E} \psi_{j}^{4p} \right]^{\frac{1}{2}}.
\]

According to the Hölder inequality, the a prior estimates of \( u^{N} \) and \( u_{m}^{N} \) and the Hölder continuity of \( u^{N} \), we obtain

\[
\mathbb{E} \psi_{j}^{4p} \leq C h^{4p} \mathbb{E} \left( \int_{t_{j}}^{t_{j+1}} \| (f(u^{N}(s))) \|_{L^{2}} ds \right)^{4p} \\
+ C h^{2\beta p} h^{4p} \mathbb{E} \left( \left( 1 + \| u_{j}^{N} \|_{L^{2}}^{2} + \| u_{j+1}^{N} \|_{L^{2}}^{2} \right) \left( \| u_{j}^{N} \|_{H^{\beta - 1}}^{2} + \| u_{j+1}^{N} \|_{H^{\beta - 1}}^{2} \right) \right)^{4p} \\
+ C \mathbb{E} \left( \int_{t_{j}}^{t_{j+1}} \left( \| u^{N}(s) \|_{L^{2}}^{2} + \| u^{N}(t_{[\frac{s}{h}]}) \|_{L^{2}}^{2} + 1 \right) \| u^{N}(s) - u^{N}(t_{[\frac{s}{h}]}) \|_{L^{2}} ds \right)^{4p} \\
+ C h^{8p} \mathbb{E} \left( \left( \| u_{j}^{N} \|_{L^{2}}^{2} + \| u_{j+1}^{N} \|_{L^{2}}^{2} + 1 \right) \left( \| v_{j}^{N} \|_{L^{2}} + \| u_{j}^{N} \|_{L^{2}} \right) \right)^{4p} \\
\leq C(h^{8p} + h^{4p+2\beta p} + h^{8p} + h^{8p}) \leq C h^{8p} + C h^{4p+2\beta p}.
\]

This yields that

\[
\mathbb{E}\| \varepsilon_{m} \|^{2p}_{\mathbb{H}} \leq C h^{\beta p} \| X_{0}^{N} \|^{2p}_{\mathbb{H}} + C(h^{\beta p} + h^{2p}) + C m^{2p} h^{2p+\beta p} + C m^{2p} h^{4p} \leq C h^{\gamma p},
\]

which completes the proof. \( \square \)

Combining both results will then give the desired estimate, we are able to present our main Theorem 1.1 which states the strong convergence of the above numerical scheme and also provides a rate for this strong convergence.
5 Numerical experiments

This section illustrates numerically the main results of this paper by considering the following 2-dimensional stochastic wave equation with cubic nonlinearity

\[ du = v dt, \]
\[ dv = [u_{xx} + u_{yy}] dt - u^3 dt + dW, \quad \text{in } O \times (0, T), \]
\[ u(0) = 0, \quad v(0) = 1, \quad \text{in } O \]

with \( O = (0, 1) \times (0, 1), T = 1 \) and homogenous Dirichlet boundary condition.

In the sequel, we choose the orthonormal basis \( \{e_{k,l}\}_{k,l \in \mathbb{N}} \) and the the corresponding eigenvalues \( \{\eta_{k,l}\}_{k,l \in \mathbb{N}} \) of as

\[ e_{k,l} = 2 \sin(k\pi x) \sin(l\pi y), \quad \eta_{k,l} = \frac{1}{k^3 + l^3}. \]

Now let us start with tests on the convergence rates. First of all, we consider the spatial convergence rate of the full discrete scheme (15). In order to verify the theoretical analysis results, we choose the parameter \( \beta = 3/2, 2, \text{ and } 5/2 \), respectively. The left figure in Fig. 1 displays the spatial approximation errors \( \sup_{0 \leq t \leq T} \|X_N - X\|_{L^p(O;H)} \) against \( N \) on a log-log scale with \( N = 2^s, \ s = 4, 5, \cdots, 9 \). It can be observed that the errors decrease at a slope of order \( \beta \) as \( N \) increases, which is consistent with our previous theoretical result (see Theorem 3.1) on the spatial convergence order. Note that for the temporal discretization we used here the proposed scheme (15) at a sufficiently small time step-size \( h = 2^{-10} \). In addition, \( N = 2^{11} \) is used to compute the exact solution.

![Figure 1: Strong convergence order in space and time for the full discrete scheme (15) applied to stochastic wave equation (23). Left: spatial errors for the full discrete scheme; Right: temporal errors for the full discrete scheme.](image)

We now fix \( N = 100 \) and investigate the strong convergence order in temporal direction of the proposed full discrete scheme (15) by using various step-sizes \( h = 2^{-r}, \ r = 2, 3, \cdots, 7 \).
Again, the “exact” solution is approximated by the method \((15)\) with a very small time step-size \(h = 2^{-12}\). The right figure in Fig. \([\text{1}]\) presents the strong approximation errors of the proposed scheme \((15)\) in temporal direction. It can be seen that this numerical performance coincides with the theoretical assertion (see Theorem \([\text{1.1}]\)).

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