Note on the pinned distance problem over finite fields

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Abstract. Let $F_q$ be a finite field with odd $q$ elements. In this article, we prove that if $E \subseteq F_q^d$, $d \geq 2$, and $|E| \geq q$, then there exists a set $Y \subseteq F_q^d$ with $|Y| \sim q^d$ such that for all $y \in Y$, the number of distances between the point $y$ and the set $E$ is similar to the size of the finite field $F_q$. As a corollary, we obtain that for each set $E \subseteq F_q^d$ with $|E| \geq q$, there exists a set $Y \subseteq F_q^d$ with $|Y| \sim q^d$ so that any set $E \cup \{y\}$ with $y \in Y$ determines a positive proportion of all possible distances. An averaging argument and the pigeonhole principle play a crucial role in proving our results.

1. Introduction

Let $F_q^d$ be the $d$-dimensional vector space over the finite field $F_q$ with $q$ elements. In 2005, Iosevich and Rudnev [5] initially posed and studied an analogue of the Falconer distance problem over finite fields. They asked for the minimal exponent $\alpha > 0$ such that if $E \subseteq F_q^d$ and $|E| \geq Cq^\alpha$ for a sufficiently large constant $C > 0$, then $|\Delta(E)| \geq cq$ for some $0 \leq c \leq 1$, where $|\Delta(E)|$ denotes the cardinality of the distance set $\Delta(E)$, defined by $\Delta(E) = \{|\langle x - y \rangle| : x, y \in E\}$.

Here we recall that $||\alpha|| := \sum_{j=1}^{d} \alpha_j^2$ for $\alpha = (\alpha_1, \ldots, \alpha_d) \in F_q^d$.

By developing the discrete Fourier machinery, Iosevich and Rudnev [5] proved that $|\Delta(E)| \sim q$ whenever $|E| \geq Cq^{(d+1)/2}$. We recall that $A \ll B$ means that $A \leq CB$ for some constant $C > 0$, which is independent of $q$, and we use $A \sim B$ if $A \ll B$ and $B \ll A$. The authors in [3] showed that the exponent $(d+1)/2$ is optimal for all odd dimensions $d \geq 3$ except for the cases when $-1$ is not a square and $d = 4k - 1$ for $k \in \mathbb{N}$. However, in any other cases including even dimensions $d \geq 2$, it has been conjectured by Iosevich and Rudnev [5] that in order to have a positive proportion of all distances, the exponent $(d+1)/2$ can be improved to $d/2$.

Conjecture 1.1 (Iosevich-Rudnev’s Conjecture). Let $E \subseteq F_q^d$. Suppose that $d \geq 2$ is even or $d, q \equiv 3 \mod 4$. Then if $|E| \geq Cq^{d/2}$ for a sufficiently large constant $C > 0$, we have $|\Delta(E)| \sim q$.

Iosevich-Rudnev’s Conjecture is still open and even the threshold $(d+1)/2$ has not been improved except for two dimensions. In the case of $d = 2$ over general finite fields, the authors in [2] obtained the 4/3 exponent, which is the first result to break down the exponent $(d+1)/2$.

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This result was obtained by applying the restriction estimates for the circles on the plane. More precisely, they proved the following result with an explicit constant.

**Theorem 1.2** \([2]\). Let \(E\) be a subset of \(\mathbb{F}^2_q\) with \(|E| \geq q^{1/3}\). Then following statements hold:

1. If \(q \equiv 3 \mod 4\), then \(|\Delta(E)| \geq \frac{q}{1+\sqrt{3}}\).
2. If \(q \equiv 1 \mod 4\), then \(|\Delta(E)| \geq C_q q\), where the constant \(C_q\) is defined by
   \[
   C_q := \frac{(1 - 2q^{-1})^2}{1 + \sqrt{3} - \sqrt{3}q^{-2/3}}.
   \]

Notice that \(C_q > 0\) for all \(q \geq 3\), and \(C_q\) converges to \(\frac{1}{3}\sqrt{3}\) as \(q \to \infty\). Since a convergent sequence is bounded, we therefore choose a constant \(c > 0\), independent of \(q\), such that \(C_q \geq c > 0\).

From this observation, the following corollary is a direct consequence of Theorem 1.2.

**Corollary 1.3** \([2]\). Suppose that \(E \subseteq \mathbb{F}^2_q\) with \(|E| \geq q^{4/3}\). Then we have

\(|\Delta(E)| \sim q\).

Using a group action approach, Bennett, Hart, Iosevich, Pakianathan, and Rudnev \([1]\) provided an alternative proof for the exponent 4/3 in the above corollary.

As a strong version of the Falconer distance problem, one has studied the pinned distance problem over finite fields. Given \(E \subseteq \mathbb{F}^d_q, d \geq 2, \) and \(y \in \mathbb{F}_q\), the pinned distance set with a pin \(y\), denoted by \(\Delta_y(E)\), is defined by

\[
\Delta_y(E) = \{||x - y|| : x \in E\}.
\]

The Chapman, Erdo\'gan, Hart, Iosevich, and Koh \([2]\) showed that the exponent \((d + 1)/2\) due to Iosevich and Rudnev holds true for the pinned distance sets. More precisely they proved the following.

**Theorem 1.4** \([2]\). Let \(E \subseteq \mathbb{F}^d_q, d \geq 2\). If \(|E| \geq q^{d+1/2}\), then there exists a subset \(E'\) of \(E\) with \(|E'| \sim |E|\) so that for every \(y \in E'\), we have

\(|\Delta_y(E)| \sim q\).

As seen in the conjecture of the Falconer distance set problem, the exponent \((d + 1)/2\) cannot be improved except for the cases when \(d, q \equiv 3 \mod 4\) or \(d \geq 2\) is even. However, in those cases it have been believed that \(d/2\) can be the best possible exponent for the pinned distance sets. As partial evidence for this prediction, the 4/3 exponent result was extended to the pinned distance sets in \(\mathbb{F}^2_q\) by Hanson, Lund, and Roche-Newton \([4]\), who successfully performed the bisector energy estimate.

**Theorem 1.5** \([4]\). Let \(E \subseteq \mathbb{F}^2_q\). If \(|E| \geq q^{4/3}\), then the conclusion of Theorem 1.4 holds.

When \(q\) is prime, the exponent 4/3 have been improved to 5/4 by Murphy, Petridis, Pham, Rudnev, and Stevenson \([6]\).

**Theorem 1.6** \([6]\). Let \(q\) be prime. Then if \(E \subseteq \mathbb{F}^2_q\) with \(|E| \geq q^{5/4}\), we have

\[
\max_{y \in E} |\Delta_y(E)| \sim q.
\]

Despite researchers’ efforts, the conjectured exponent \(d/2\) has not been proven. It is unlikely that one can establish the conjecture by using the known techniques. Moreover, there is very little evidence to support that the conjecture is true.

The main purpose of this paper is not to derive an improved result on the distance problem, but to address that the probability that random sets satisfy the distance conjecture is very high.
1.1. The statement of main results. Our main theorem is as follows.

THEOREM 1.7. Let \( E \subseteq \mathbb{F}_q^d \). Then given \( a > 1 \), there exists \( Y \subseteq \mathbb{F}_q^d \) with \( |Y| \geq \frac{a-1}{a}q^d \) such that for all \( y \in Y \),

\[
|\Delta_y(E)| \geq \min \left\{ \frac{q}{2a}, \frac{|E|}{2a} \right\}.
\]

The following result is a direct consequence of Theorem 1.7.

COROLLARY 1.8. Suppose that \( E \subseteq \mathbb{F}_q^d, d \geq 2 \), with \( |E| \geq q \). Then for any \( a > 1 \), there exists \( Y \subseteq \mathbb{F}_q^d \) with \( |Y| \geq \frac{a-1}{a}q^d \) so that for all \( y \in Y \), we have

\[
|\Delta_y(E \cup \{y\})| \geq \frac{1}{2a}.
\]

PROOF. Since \( |E| \geq q \), we have

\[
\min \left\{ \frac{q}{2a}, \frac{|E|}{2a} \right\} = \frac{q}{2a}.
\]

In addition, note that for all \( y \in \mathbb{F}_q \), we have \( |\Delta(E \cup \{y\})| \geq |\Delta_y(E)| \). Hence, the statement of the corollary follows immediately from Theorem 1.7. \( \square \)

2. Proof of main result (Theorem 1.7)

We begin with the standard counting argument as in [2].

To find a lower bound of the cardinality of the \( y \)-pinned counting function \( \nu_y : \mathbb{F}_q \rightarrow \mathbb{N} \cup \{0\} \), which maps an element \( t \) in \( \mathbb{F}_q \) to the number of elements \( x \) in \( E \) such that \( ||x-y|| = t \). In other words, for \( y \in \mathbb{F}_q^d, t \in \mathbb{F}_q \), we have

\[
\nu_y(t) = \sum_{x \in E: ||x-y|| = t} 1.
\]

Since \( |E|^2 = \left( \sum_{t \in \Delta_y(E)} \nu_y(t) \right)^2 \), it follows from the Cauchy-Schwarz inequality that

\[
|\Delta_y(E)| \geq \frac{|E|^2}{\sum_{t \in \mathbb{F}_q} \nu_y^2(t)}.
\] (2.1)

2.1. Key lemmas. The average of \( \sum_{t \in \mathbb{F}_q} \nu_y^2(t) \) over \( y \) in \( \mathbb{F}_q^d \) is explicitly given as follows:

LEMMA 2.1. Let \( E \subseteq \mathbb{F}_q^d \). Then we have

\[
\frac{1}{q^d} \sum_{y \in \mathbb{F}_q^d} \sum_{t \in \mathbb{F}_q} \nu_y^2(t) = \frac{|E|^2}{q} + \frac{q-1}{q} |E|.
\]

PROOF. By the definition of the \( y \)-pinned counting function \( \nu_y(t) \), we have for each \( y \in \mathbb{F}_q \),

\[
\sum_{t \in \mathbb{F}_q} \nu_y^2(t) = \sum_{x, z \in E: ||x-y|| = ||z-y||} 1.
\]

Hence, the average of it over \( y \in \mathbb{F}_q^d \) is given as follows:

\[
\frac{1}{q^d} \sum_{y \in \mathbb{F}_q^d} \sum_{t \in \mathbb{F}_q} \nu_y^2(t) = \frac{1}{q^d} \sum_{x, z \in E: x \neq z} \sum_{y \in \mathbb{F}_q^d: ||x-y|| = ||z-y||} 1 + \frac{1}{q^d} \sum_{x, z \in E: x \neq z} \sum_{y \in \mathbb{F}_q^d: ||x-y|| = ||z-y||} 1
\]

\[
= |E| + \frac{1}{q^d} \sum_{x, z \in E: x \neq z} \sum_{y \in \mathbb{F}_q^d: ||x-y|| = ||z-y||} 1.
\] \( \square \)
Now, we notice that for \( x, z \in E \) with \( x \neq z \), we have
\[
(2.3) \quad \sum_{y \in \mathbb{F}_q^d : ||x-y|| = ||z-y||} 1 = q^{d-1}.
\]
In fact, since \( x \neq z \), the quantity \( \sum_{y \in \mathbb{F}_q^d : ||x-y|| = ||z-y||} 1 \) is the number of the elements in the hyper-plane which bisects the line segment joining \( x \) and \( z \). Alternatively we can prove this rigorously by using the finite field Fourier analysis. To see this, let \( \chi \) denote a nontrivial additive character of \( \mathbb{F}_q \). Then by the orthogonality of \( \chi \), we see that if \( x \neq z \), then
\[
\sum_{y \in \mathbb{F}_q^d : ||x-y|| = ||z-y||} 1 = q^{-1} \sum_{y \in \mathbb{F}_q^d} \sum_{s \in \mathbb{F}_q} \chi(s(||x-y|| - ||z-y||)) = q^{d-1} + q^{-1} \sum_{y \in \mathbb{F}_q^d} \sum_{s \neq 0} \chi(s(||x-y|| - ||z-y||)).
\]
Applying the orthogonality of \( \chi \) to the sum over \( y \), we see that the second term above is zero since \( \chi(s(||x-y|| - ||z-y||)) = \chi(-2s(x-z) \cdot y) \chi(s(||x|| - ||z||)) \) and \( s(x-z) \) is not a zero vector. Hence, the equation (2.3) holds.

Finally, combining the above two estimates (2.2), (2.3), we obtain the desirable estimate.

The following result can be obtained by the pigeonhole principle together with Lemma 2.1.

**Lemma 2.2.** Let \( E \subseteq \mathbb{F}_q^d \). Then for any \( a > 1 \), there exists \( Y \subseteq \mathbb{F}_q^d \) with \( |Y| \geq \frac{a-1}{a} q^d \) such that for every \( y \in Y \),
\[
\sum_{t \in \mathbb{F}_q} \nu_y^2(t) \leq \frac{a}{q} |E|^2 + \frac{a(q-1)}{q} |E|.
\]

**Proof.** Let us fix \( a > 1 \). Define
\[
Y = \left\{ y \in \mathbb{F}_q^d : \sum_{t \in \mathbb{F}_q} \nu_y^2(t) \leq \frac{a}{q} |E|^2 + \frac{a(q-1)}{q} |E| \right\}.
\]
To complete the proof, it remains to show that
\[
|Y| \geq \frac{a-1}{a} q^d.
\]
By contradiction, let us assume that
\[
(2.4) \quad |Y| < \frac{a-1}{a} q^d.
\]
It is clear that
\[
(\mathbb{F}_q^d \setminus Y) = \left\{ y \in \mathbb{F}_q^d : \sum_{t \in \mathbb{F}_q} \nu_y^2(t) > \frac{a}{q} |E|^2 + \frac{a(q-1)}{q} |E| \right\}.
\]
We also notice that for all \( y \in \mathbb{F}_q^d \),
\[
(2.6) \quad \sum_{t \in \mathbb{F}_q} \nu_y^2(t) \geq \sum_{t \in \mathbb{F}_q} \nu_y(t) = |E|.
\]
Now by Lemma 2.1 it follows that
\[
(2.7) \quad \frac{1}{q^d} \sum_{y \in \mathbb{F}_q^d} \sum_{t \in \mathbb{F}_q} \nu_y^2(t) = \frac{|E|^2}{q} + \frac{q-1}{q} |E|.
\]
However, we can also estimate it as follows. Using (2.5) and (2.6), we have

\[
\frac{1}{q^d} \sum_{y \in Y} \sum_{t \in E} \nu_y^2(t) = \frac{1}{q^d} \sum_{y \in Y} \sum_{t \in E} \nu_y^2(t) + \frac{1}{q^d} \sum_{y \in \mathbb{F}_q^d \setminus Y} \sum_{t \in E} \nu_y^2(t) > \frac{1}{q^d} \left| Y \right| |E| + \frac{1}{q^d} (q^d - |Y|) \left( \frac{a}{q} |E|^2 + \frac{a(q-1)}{q} |E| \right) \\
= \frac{a|E|^2}{q} + \frac{a(q-1)|E|}{q} + \left( \frac{|E|}{q^d} - \frac{a|E|^2}{q^d+1} - \frac{a(q-1)|E|}{q^d+1} \right) \left( a - \frac{1}{a} q^d \right).
\]

Since \( a > 1 \), in the third term above, the coefficient of \( |Y| \) is negative. Hence, we can combine the above estimate with (2.4) to deduce that

\[
\frac{1}{q^d} \sum_{y \in \mathbb{F}_q^d} \sum_{t \in E} \nu_y^2(t) > \frac{|E|^2}{q} + \frac{a(q-1)|E|}{q} + \left( \frac{|E|}{q^d} - \frac{a|E|^2}{q^d+1} - \frac{a(q-1)|E|}{q^d+1} \right) \left( a - \frac{1}{a} q^d \right)
\]

Simplifying the RHS of the above equation, we get

\[
\frac{1}{q^d} \sum_{y \in \mathbb{F}_q^d} \sum_{t \in E} \nu_y^2(t) > \frac{|E|^2}{q} + \frac{q-1}{q} |E| + \frac{a-1}{a} |E|,
\]

which contradicts the equation (2.7) since \( a > 1 \). \( \square \)

2.2. Proof of Theorem 1.7. Combining (2.1) and Lemma 2.2 we get the required result:

\[
|\Delta_y(E)| \geq \frac{|E|^2}{\frac{q}{2} |E|^2 + \frac{q(a-1)}{2a(q-1)}} \geq \min \left\{ \frac{q}{2a}, \frac{|E|}{2a(q-1)} \right\} \geq \min \left\{ \frac{q}{2a}, \frac{|E|}{2a} \right\}.
\]

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