New twisted quantum deformations of
$D = 4$ super-Poincaré algebra

A. Borowiec$^{1,3}$, J. Lukierski$^1$ and V.N. Tolstoy$^{1,2}$

$^1$Institute for Theoretical Physics,
University of Wroclaw, pl. Maxa Borna 9,
50–205 Wroclaw, Poland

$^2$Institute of Nuclear Physics,
Moscow State University, 119 992 Moscow, Russia

$^3$Bogoliubov Laboratory of Theoretical Physics,
Joint Institute for Nuclear Research, Dubna,
Moscow region 141980, Russia

Abstract

We show how some classical $r$-matrices for the $D = 4$ Poincaré algebra can be
supersymmetrized by an addition of part depending on odd supercharges. These $r$-
matrices for $D = 4$ super-Poincaré algebra can be presented as a sum of the so-called
subordinated $r$-matrices of super-Abelian and super-Jordanian type. Corresponding
twists describing quantum deformations are obtained in an explicit form. These twists
are the super-extensions of twists obtained in the paper [1] (arXiv:math/0712.3962v1).

1 Introduction

In the theory of fundamental interactions and field-theoretical models the following two basic
ideas were considered:
(i) Since more than thirty years one studies the supersymmetric extensions of standard sym-
metries, dynamical models and classical geometries (see e.g. [2] – [7]).
(ii) In last twenty years the idea of introducing quantum symmetries [8] - [11] and quantum
deformations of dynamical models (see e.g. [12, 13]) is gaining popularity as a way to describe
the effects related with quantization of gravity [14, 15] and the description of D-branes defined
on noncommutative space-time manifolds [16, 17].

*The talk given by third author at the International Workshop "Supersymmetries and Quantum Symme-
tries" (SQS’07) (Dubna, July 30 - August 4, 2007)
In this paper we shall investigate the superposition of these two concepts in the area of Hopf-algebraic description of relativistic symmetries. The first example of supersymmetric extension of quantum symmetries were related with the algebra \( \mathfrak{osp}(1|2) \) \([18, 19]\); further the supersymmetric extension of Drinfeld-Jimbo deformation of arbitrary Lie algebra was considered \([20, 21]\). The supersymmetric extension of \( D = 4 \) quantum relativistic symmetries was firstly obtained as the supersymmetrization of \( \kappa \)-deformed Poincare symmetries \([22] - [25]\); only quite recently new results were obtained as an extension of canonical Abelian twist for Poincare symmetries \([26, 27]\) to the case of super-Poincare symmetries (see e.g. \([28, 29, 30]\)). It should be stressed that recently there was introduced also general classification of twisted quantum deformations of semisimple Lie superalgebras \([31, 32]\). Subsequently these general considerations were applied to the quantum deformation of \( \mathfrak{osp}(1|2) \) \([33]\) which can be interpreted as \( D = 1 \) superconformal symmetry and recently to quantum deformation of \( \mathfrak{osp}(1|4) \) \([34]\) which can be seen as describing the \( D = 3 \) quantum superconformal symmetry or \( D = 4 \) quantum anti-de-Sitter supersymmetry.

The quantum deformations of relativistic supersymmetries are described by Hopf-algebraic deformations of the Poincaré superalgebra. Such quantum deformations are classified by super-Poincaré Poisson structures which are given by classical \( r \)-matrices. In the case of the Poincaré algebra complete classification of classical \( r \)-matrices, which do satisfy the homogeneous classical Yang-Baxter equation, was obtained by S. Zakrzewski in \([35]\). In the case of Poincaré superalgebra even partial classification of corresponding \( r \)-matrices has not been achieved. In this paper we undertake such a task. We shall extend supersymmetrically part of the Zakrzewski’s list by an addition to the Poincare classical \( r \)-matrices the terms containing supercharges. Moreover these extended \( r \)-matrices can be presented as a sum of the so-called subordinated \( r \)-matrices which are of super-Abelian and super-Jordanian type. This subordination structure allows to construct a sequence of quantizations described by the product of twists providing complete quantum deformation. The twists considered below are super-extensions of the twists obtained recently in \([1]\).

2 Supertwists − general considerations

Let \( r \) be a classical \( r \)-matrix of a Lie superalgebra \( \mathfrak{g} \) \((r \in \bigwedge ^2 \mathfrak{g})\) satisfying the classical Yang–Baxter equation (CYBE)

\[
[r^{12}, r^{13} + r^{23}] + [r^{13}, r^{23}] = \Omega ,
\]

where \( \Omega \) is \( \mathfrak{g} \)-invariant element, \( \Omega \in (\bigwedge ^3 \mathfrak{g})_B \). We consider two types of the classical \( r \)-matrices and corresponding twists.

a) **Graded Abelian twist.** Let the classical \( r \)-matrix \( r = r_{sA} \) has the form\(^1\)

\[
r_{sA} = \sum_{i=1}^{n} y_i \wedge x_i + \sum_{i=1}^{m} v_k \wedge u_k ,
\]

where all elements \( x_i, y_i (i = 1, \ldots, n) \) are even (bosonic) and they commute among themselves, all elements \( u_k, v_k (k = 1, \ldots, m) \) are odd (fermionic) and they anti-commute, the

\(^1\)We do not consider here situation when \( r \)-matrix contains terms of odd degree, i.e. the terms of form \( x_i \wedge u_k \), where \( \deg x_i = 0, \deg u_k = 1 \). In a case of the such mixed terms we need to introduce odd (fermionic) parameters of deformation in the \( r \)-matrix (for example see \([28]\); for general discussion see \([31]\)).
elements $x_i, y_i$ do commute with the elements $u_k, v_k$. Moreover the symbol "∧" for bosonic
and fermionic elements is defined as follows

\[ y_i \land x_i := y_i \otimes x_i - x_i \otimes y_i , \quad \text{for } \deg x_i = \deg y_i = 0 , \quad (2.3) \]

\[ v_k \land u_k := u_k \otimes u_k + u_k \otimes v_k , \quad \text{for } \deg u_k = \deg v_k = 1 . \quad (2.4) \]

The $r$-matrix (2.2) is called of super-Abelian type. The corresponding twist is given as follows

\[ F_{r,s_A} = \exp \frac{R_{s_A}}{2} = \exp \left( \frac{1}{2} \sum_{i=1}^n x_i \land y_i + \frac{1}{2} \sum_{k=1}^m u_k \land v_k \right) . \quad (2.5) \]

This twisting two-tensor $F := F_{r_A}$ satisfies the cocycle equation [36]

\[ F^{12}(\Delta \otimes \text{id})(F) = F^{23}(\text{id} \otimes \Delta)(F) , \quad (2.6) \]

and the "unital" normalization condition

\[ (\epsilon \otimes \text{id})(F) = (\text{id} \otimes \epsilon)(F) = 1 . \quad (2.7) \]

The twisting element $F$ defines the deformation of universal enveloping algebra $U(g)$ considered as a Hopf algebra. The new deformed coproducts and antipodes are given as follows

\[ \Delta^{(F)}(a) = F \Delta(a) F^{-1} , \quad S^{(F)}(a) = u S(a) u^{-1} \quad (2.8) \]

for any $a \in U(g)$, where $\Delta(a)$ is a coproduct before twisting, and

\[ u = \sum_i f_i^{(1)} S(f_i^{(2)}) \quad (2.9) \]

if $F = \sum_i f_i^{(1)} \otimes f_i^{(2)}$.

b) Extended super Jordanian twist. Let the classical $r$-matrix $r = r_{J_{n|m}}(\xi)$ has the following form\(^2\)

\[ r_{J_{n|m}}(\xi) = \xi \left( \sum_{\nu=0}^n y_\nu \land x_\nu + \sum_{k=1}^m v_k \land u_k \right) , \quad (2.10) \]

where the elements $x_\nu, y_\nu (\nu = 0, 1, \ldots, n)$ and $u_k, v_k (k = 1, 2, \ldots, m)$ satisfy the relations:

\[ [x_0, y_0] = y_0 , \quad [x_0, x_i] = t_i x_i , \quad [x_0, y_i] = (1 - t_i) y_i , \]
\[ [x_i, y_j] = \delta_{ij} y_0 , \quad [x_i, y_j] = [y_i, y_j] = 0 , \quad [y_0, x_j] = [y_0, y_j] = 0 , \]
\[ [x_0, u_k] = t_k' u_k , \quad [x_0, v_k] = (1 - t_k') v_k , \quad [y_0, u_k] = [v_0, v_k] = [y_0, v_k] = 0 , \]
\[ [x_i, u_k] = [x_i, v_k] = [y_i, u_k] = [y_i, v_k] = 0 , \]
\[ \{u_k, u_l\} = \delta_{kl} y_0 , \quad \{u_k, u_l\} = \{v_k, v_l\} = 0 , \]

\[ (i, j = 1, 2, \ldots, n), \ (k, l = 1, 2, \ldots, m), \ (t_i, t_k' \in \mathbb{C}) . \]

Such an $r$-matrix is called of super-Jordanian type and it is easy to verify that the two-tensor (2.10) indeed satisfies the homogeneous classical Yang-Baxter equation (2.1) (with $\Omega = 0$), provided the elements $x_\nu, y_\nu$

\(^2\)Here introduction of the deformation parameter $\xi$ is a matter of convenience.
(ν = 0, 1, . . . , n) and u_k, v_k (k = 1, 2, . . . , m) satisfy the relations (2.11). The corresponding twist is given as follows [31, 32]

\[
F_{r,J_{n|m}} = \exp \left( \xi \sum_{k=1}^{m} u_k \otimes v_k \ e^{-2(1-t_k)\sigma} + \xi \sum_{i=1}^{n} x_i \otimes y_i \ e^{-2(1-t_i)\sigma} \right) \exp(2x_0 \otimes \sigma), \quad (2.12)
\]

where \( \sigma := \frac{1}{2} \ln(1 + \xi y_0) \).

Remark. The bosonic part \( F_{r,J_{n|m}}(\xi) = \xi \sum_{\nu=0}^{n} y_{\nu} \wedge x_{\nu} \) of (2.10) is the classical \( r \)-matrix of Jordanian type and the corresponding twist of Jordanian type is given by the formula (2.12) for \( m = 0 \).

Let \( r \) be an arbitrary \( r \)-matrix of the superalgebra \( \mathfrak{g} \) and let \( \text{Sup}(r) \) be a support of \( r \). We recall the useful notion of subordination [1, 37].

**Definition 2.1** Let \( r_1 \) and \( r_2 \) be two arbitrary classical \( r \)-matrices. We say that \( r_2 \) is subordinated to \( r_1 \), \( r_1 \succeq r_2 \), if \( \delta_{r_1}(\text{Sup}(r_2)) = 0 \), i.e.

\[
\delta_{r_1}(x) := [x \otimes 1 + 1 \otimes x, r_1] = 0, \quad \forall x \in \text{Sup}(r_2).
\]

If \( r_1 \succeq r_2 \) then \( r = r_1 + r_2 \) is also a classical \( r \)-matrix. The subordination enables us to construct a correct sequence of quantizations. For instance, if the \( r \)-matrix of super-Jordanian type (2.10) is subordinated to the \( r \)-matrix of super-Abelian type (2.2), \( r_{sA} \succeq r_{J_{n|m}} \), then the total twist corresponding to the resulting \( r \)-matrix \( r = r_{sA} + r_{J_{n|m}} \) is given as follows

\[
F_r = F_{r_{J_{n|m}}} F_{r_{sA}}.
\]

### 3 Quantum deformations of super-Poincaré algebra

The \( D = 4 \) Poincaré algebra \( \mathcal{P}(3,1) \) is described by ten generators:

(i) six-dimensional Lorentz algebra \( \mathfrak{o}(3,1) \) with the generators \( M_i, N_i \) (i = 1, 2, 3):

\[
[M_i, M_j] = i \epsilon_{ijk} M_k, \quad [M_i, N_j] = i \epsilon_{ijk} N_k, \quad [N_i, N_j] = -i \epsilon_{ijk} M_k \quad (3.1)
\]

(ii) Abelian fourmomenta \( P_0, P_j \) (j = 1, 2, 3) with the standard commutation relations:

\[
[M_j, P_k] = i \epsilon_{jkl} P_l, \quad [M_j, P_0] = 0, \quad [N_j, P_k] = -i \delta_{jk} P_0, \quad [N_j, P_0] = -i P_j. \quad (3.2)
\]

The physical generators of the Lorentz algebra \( \mathfrak{o}(3,1), M_i, N_i \) (i = 1, 2, 3), are related with the canonical basis \( h, h', e_\pm, e'_\pm \) as follows

\[
h = i N_3, \quad e_\pm = i (N_1 \pm M_2), \quad \quad (3.4)
\]

\[
h' = i M_3, \quad e'_\pm = i (M_1 \mp N_2), \quad \quad (3.5)
\]

where the generators (3.4-3.5) satisfy the following nonvanishing commutation relations:

\[
[h, e_\pm] = \pm e_\pm, \quad [e_\pm, e_-] = 2h, \quad (3.6)
\]

\[
[h, e'_\pm] = \pm e'_\pm, \quad [h', e_\pm] = \pm e'_\pm, \quad [e_\pm, e'_\mp] = \pm 2h', \quad (3.7)
\]

\[
[h', e'_\mp] = \mp e_\pm, \quad [e'_\mp, e'_-] = -2h'. \quad (3.8)
\]

3. The support \( \text{Sup}(r) \) is a subalgebra of \( \mathfrak{g} \) generated by the elements \( \{x_i, y_i; u_k, v_k\} \) if \( r = \sum_i y_i \wedge x_i + \sum_k v_k \wedge u_k \).
Moreover one can introduce the reality structure as follows

\[ a^* = -a \quad (\forall a \in \mathfrak{o}(3,1)) \]  

(3.9)

The subalgebra generated by the four momenta \( P_0, P_j \ (j = 1, 2, 3) \) will be denoted by \( \mathbf{P} \) and we also set \( P_\pm := P_0 \pm P_3 \).

S. Zakrzewski has shown in [35] that each classical \( r \)-matrix, \( r \in \mathcal{P}(3,1) \cap \mathcal{P}(3,1) \), has the following decomposition

\[ r = a + b + c \]  

(3.10)

where \( a \in \mathbf{P} \cap \mathbf{P}, b \in \mathbf{P} \cap \mathfrak{o}(3,1), c \in \mathfrak{o}(3,1) \cap \mathfrak{o}(3,1) \) satisfy the following relations

\[ [[c, c]] = 0 \]  

(3.11)

\[ [[b, c]] = 0 \]  

(3.12)

\[ 2[[a, c]] + [[b, b]] = t \Omega \quad (t \in \mathbb{R}, \Omega \neq 0) \]  

(3.13)

\[ [[a, b]] = 0 \]  

(3.14)

and \([[-, -]]\) means the Schouten bracket. Moreover a complete list of the classical \( D = 4 \) \( r \)-matrices for the cases \( c \neq 0 \) and \( c = 0, t = 0 \) is known.\(^4\) The results are presented in the following table taken from [35]:

\[
\begin{array}{cccccc}
\gamma h' \wedge h & b & a & \# & N \\
0 & \alpha P_+ \wedge P_- + \tilde{\alpha} P_1 \wedge P_2 & 2 & 1 \\
\gamma e_+ \wedge e_+ & \beta_1 b_{P_+} + \beta_2 P_+ \wedge h' & 0 & 1 & 2 \\
& \beta_1 b_{P_+} & \alpha P_+ \wedge P_1 & 1 & 3 \\
& \gamma \beta_1 (P_1 \wedge e_+ + P_2 \wedge e'_+) & \alpha P_+ \wedge (\alpha P_1 + \alpha_2 P_2) - \gamma \beta^2 P_1 \wedge P_2 & 2 & 4 \\
\gamma (h \wedge e_+) & 0 & 0 & 1 & 5 \\
\gamma (h \wedge e_+) & 0 & 0 & 1 & 5 \\
\gamma (h \wedge e_+) & 0 & 0 & 1 & 5 \\
\end{array}
\]

\[
\begin{array}{cccccc}
\gamma h \wedge e_+ & \beta_1 b_{P_+} + \beta_2 P_2 \wedge e_+ & 0 & 1 & 6 \\
& \beta_1 b_{P_+} + \beta_2 P_+ \wedge h' & 0 & 1 & 7 \\
& \beta_1 b_{P_+} + \beta_2 P_+ \wedge e_+ & 0 & 1 & 8 \\
& P_1 \wedge (\beta_1 e_+ + \beta_2 e'_+) & \alpha P_+ \wedge P_2 & 2 & 9 \\
& \beta_1 P_+ \wedge (h + \chi e_+), \chi = 0, \pm 1 & \beta_1 (P_1 \wedge e'_+ + P_+ \wedge e_+) & \alpha_1 P_- \wedge P_1 + \alpha_2 P_+ \wedge P_2 & 2 & 10 \\
& & \beta_1 P_2 \wedge e_+ & \alpha_1 P_+ \wedge P_1 + \alpha_2 P_- \wedge P_2 & 1 & 11 \\
& & \beta_1 P_+ \wedge e_+ & \alpha_1 P_+ \wedge P_1 + \alpha_2 P_- \wedge P_2 & 1 & 11 \\
& & \beta_1 P_0 \wedge h' & \alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2 & 2 & 13 \\
& & \beta_1 P_3 \wedge h' & \alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2 & 2 & 14 \\
& & \beta_1 P_+ \wedge h' & \alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2 & 1 & 15 \\
& & \beta_1 P_1 \wedge h & \alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2 & 2 & 16 \\
& & \beta_1 P_+ \wedge h & \alpha P_1 \wedge P_2 + \alpha_1 P_+ \wedge P_1 & 1 & 17 \\
& & P_+ \wedge (\beta_1 h + \beta_2 h') & \alpha_1 P_1 \wedge P_2 & 1 & 18 \\
& & 0 & \alpha_1 P_1 \wedge P_2 & 0 & 19 \\
& & 0 & \alpha_1 P_1 \wedge P_2 & 0 & 20 \\
& & 0 & \alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2 & 1 & 21 \\
\end{array}
\]

\(^4\)Classification of the \( r \)-matrices for the case \( c = 0, t \neq 0 \) is still not complete.
where \( b_{P_+} \) and \( b_{P_2} \) are given as follows:

\[
b_{P_+} = P_1 \land e_+ - P_2 \land e'_+ + P_+ \land h , \tag{3.15}
\]

\[
b_{P_2} = 2P_1 \land h' + P_- \land e'_- - P_+ \land e'_-. \tag{3.16}
\]

The Table 1 contains 21 cases labelled by the number \( N \) in the last column. In the forth column (labelled by \#) there is indicated the number of essential parameters defining given deformation. This number is smaller than the number of parameters occurring in the Table 1. In particular, we introduced an additional parameter \( \gamma \) in the component \( a \) (in the cases 2, 3, 4, 5, 6) the parameter \( \beta_1 \) in the component \( b \) (in the cases 7–18) and the parameter \( \alpha_1 \) in the component \( a \) (in the cases 19–21)\(^5\). The reduction of the number of parameters to the essential ones is achieved by taking into account all possible automorphisms of the Poincaré algebra \( \mathcal{P}(3,1) \) (see details in [35]).

The super-Poincaré algebra \( \mathcal{P}(3,1|1) \) is generated by the algebra \( \mathcal{P}(3,1) \) and four real supercharges \( Q_\alpha \) \((\alpha = \pm 1, \pm 2)\) with the anticommutation relations

\[
\{Q^+_{\alpha}, Q^+_{\beta}\} = 0 , \quad \{Q^-_{\alpha}, Q^-_{\beta}\} = 2(\delta_{\alpha\beta} P_0 + (\sigma_j)_{\alpha\beta} P_j) , \tag{3.17}
\]

where \( \sigma_j \) \((j = 1, 2, 3)\) are the standard \( 2 \times 2 \) \( \sigma \)-matrices, and moreover

\[
[M_j, Q^\pm_\alpha] = \pm \frac{1}{2}(\sigma^\pm_j)_{\alpha\beta} Q^\pm_\beta ,
\]

\[
[N_j, Q^\pm_\alpha] = -\frac{i}{2}(\sigma^\pm_j)_{\alpha\beta} Q^\pm_\beta , \quad [P_\mu, Q^\pm_\alpha] = 0 . \tag{3.18}
\]

Here \( \sigma^+_{j} \equiv \sigma_j \) and \( \sigma^-_{j} \) denotes the complex conjugate of \( \sigma_j \).

It is clear that all classical \( r \)-matrices satisfying the homogeneous YBE are \( r \)-matrices for the super-Poincaré algebra and the twists constructed in [31, 32] can be used for the derivation of explicit super-Poincare twists. It turns out that it is possible to extend supersymmetrically the Zakrzewski’s classification for the Poincare superalgebra by an addition of terms expressed in terms of supercharges. Moreover these extended \( r \)-matrices can be presented as a sum of subordinated \( r \)-matrices which are of super-Abelian and super-Jordanian types. We consider below several examples of such superextensions and give the corresponding twists describing the quantum deformations for the first six \( r \)-matrices from the Table 1, with \( c \neq 0, t = 0 \). These twist quantizations are described by the superextensions of the twists obtained in [1].

1. The \( r \)-matrix \( r_1 \) describing the superextension of \( N = 1 \) in Table 1

\[
r_1 = \gamma h' \land h + \alpha P_+ \land P_- + \bar{\alpha} P_1 \land P_2
\]

has the following superextension:

\[
r_{1s} = \gamma h' \land h + \alpha P_+ \land P_- + \bar{\alpha} P_1 \land P_2 + \eta Q_2^+ \land Q_1^+ . \tag{3.20}
\]

The \( r \)-matrix (3.19) can be presented as a sum of two subordinated \( r \)-matrices of super-Abelian and Abelian types

\[
r_{1s} = r_{1s}' + r_{1s}'' , \quad r_{1s}' \succ r_{1s}'' ,
\]

\[
r_{1s}' = \alpha P_+ \land P_- + \bar{\alpha} P_1 \land P_2 + \eta Q_2^+ \land Q_1^+ ,
\]

\[
r_{1s}'' = \gamma h' \land h . \tag{3.21}
\]

\(^5\)In the paper by S. Zakrzewski [35] all these additional parameters are equal to 1 and the numbers in the forth column correspond to this situation.
The complete twist defining quantization is the ordered product of super-Abelian and Abelian twists

\[ F_{r_{1s}} = F_{r_{1s}}' F_{r_{2s}} = \exp(\gamma h \wedge h') \exp(\alpha P_- \wedge P_+ + \tilde{\alpha} P_2 \wedge P_1 + \eta Q_1^+ \wedge Q_2^+) \]  

(3.22)

2). The \( r \)-matrix \( r_2 \), describing the superextension of \( N = 2 \) in **Table 1**

\[ r_2 = \gamma e'_+ \wedge e_+ + \beta_1 (P_1 \wedge e_+ - P_2 \wedge e'_+ + P_+ \wedge h) + \beta_2 P_+ \wedge h' , \]  

(3.23)

admits the superextension

\[ r_{2s} = \gamma e'_+ \wedge e_+ + \beta_1 (P_1 \wedge e_+ + P_2 \wedge e'_+ + P_+ \wedge h) + \beta_2 P_+ \wedge h' + \frac{\beta_1}{2} Q_1^- \wedge Q_1^+ , \]  

(3.24)

The \( r \)-matrix \( r_{2s} \) can be presented as a sum of three subordinated \( r \)-matrices where one of them is of super-Jordanian type and two are of Abelian type

\[ r_{2s} = r_{2s}' + r_{2s}'' + r_{2s}''' , \quad r_{2s}' \succ r_{2s}'' , \quad r_{2s}'' \succ r_{2s}' , \quad r_{2s}''' \succ r_{2s}'' . \]  

(3.25)

The corresponding twisting two-tensor is given by the following formula

\[ F_{r_{2s}} = F_{r_{2s}'} F_{r_2} F_{r_{2s}''} , \]  

(3.26)

where

\[ F_{r_{2s}'} = \exp(\beta_1 (\frac{1}{2} Q_1^- \otimes Q_1^- e^{-\frac{1}{2} \sigma_+} + e_+ \otimes P_1 - e'_+ \otimes P_2)) \exp(2h \otimes \sigma_+) , \]  

\[ F_{r_2} = \exp(\gamma e_+ \wedge e'_+) , \]  

(3.27)

\[ F_{r_{2s}''} = \exp\left( \frac{\beta_2}{2} h' \wedge \sigma_+ \right) . \]  

Here and below we set \( \sigma_+ := \frac{1}{2} \ln(1 + \beta_1 P_+) . \)

It should be noted that the first formula (3.27) containing the supercharges \( Q_{1}^{\pm} \) is related with the result presented in [34]

3). The \( r \)-matrix \( r_3 \), describing the superextension of \( N = 3 \) in **Table 1**.

\[ r_3 = \gamma e'_+ \wedge e_+ + \beta_1 (P_1 \wedge e_+ - P_2 \wedge e'_+ + P_+ \wedge h) + \alpha P_+ \wedge P_1 , \]  

(3.28)

also admits the superextension in analogy to the previous case

\[ r_{3s} = \gamma e'_+ \wedge e_+ + \beta_1 (P_1 \wedge e_+ - P_2 \wedge e'_+ + P_+ \wedge h) + \alpha P_+ \wedge P_1 + \frac{\beta_1}{2} Q_1^- \wedge Q_1^+ , \]  

(3.29)

This \( r \)-matrix can be presented as a sum of three subordinated \( r \)-matrices where one of them is of super-Jordanian type and two are of Abelian type

\[ r_{3s} = r_{3s}' + r_{3s}'' + r_{3s}''' , \quad r_{3s}' \succ r_{3s}'' , \quad r_{3s}'' \succ r_{3s}' , \quad r_{3s}''' \succ r_{3s}'' . \]  

(3.30)

\[ r_{3s}' = \beta_1 (\frac{1}{2} Q_1^- \wedge Q_1^+ + P_1 \wedge (e_+ - \frac{\alpha}{\beta_1} P_+) - P_2 \wedge e'_+ + P_+ \wedge h) , \]  

\[ r_{3s}'' = \gamma e'_+ \wedge (e_+ - \frac{\alpha}{\beta_1} P_+) , \]  

\[ r_{3s}''' = \frac{\gamma \alpha}{\beta_1} e'_+ \wedge P_+ . \]
The corresponding twist is given by the following formula

\[ F_{r3} = F_{r3}' F_{r3}'' F_{r3}'', \]  

(3.31)

where

\[
F_{r3}' = \exp\left(\beta_1 \left(\frac{1}{2} Q_1^+ \otimes Q_1^- e^{-\frac{1}{2} \sigma_+} + e_+ - \frac{\alpha}{\beta_1} P_+ \right) \otimes P_1 - e'_+ \otimes P_2 \right) \exp(2h \otimes \sigma_+),
\]

\[
F_{r3}'' = \exp\left(\gamma (e_+ - \frac{\alpha}{\beta_1} P_+) \wedge e'_+\right),
\]

\[
F_{r3}''' = \exp\left(\frac{\gamma \alpha}{\beta_1^2} \sigma_+ \wedge e'_+\right).
\]

4). The \( r \)-matrix \( r_4 \), describing the superextension of \( N = 4 \) in Table 1.

\[
r_4 = \gamma(e'_+ \wedge e_+ + \beta_1 P_1 \wedge e_+ + \beta_1 P_2 \wedge e'_+ - \beta_1^2 P_1 \wedge P_2) + P_+ \wedge (\alpha_1 P_1 + \alpha_2 P_2),
\]

(3.33)

has the following superextension

\[
r_4 = \gamma(e'_+ \wedge e_+ + \beta_1 P_1 \wedge e_+ + \beta_1 P_2 \wedge e'_+ - \beta_1^2 P_1 \wedge P_2) + P_+ \wedge (\alpha_1 P_1 + \alpha_2 P_2) + \eta Q_1^+ \wedge Q_1^+,
\]

(3.34)

This \( r \)-matrix can be written as a sum of two subordinated \( r \)-matrices of super-Abelian and Abelian types

\[
r_{4s} = r_{4s}' + r_{4s}'', \quad r_{4s}' > r_{4s}'',
\]

\[
r_{4s}' = P_+ \wedge (\alpha_1 P_1 + \alpha_2 P_2) + \eta Q_1^+ \wedge Q_1^+,
\]

(3.35)

\[
r_{4s}'' = \gamma(e'_+ + \beta_1 P_1) \wedge (e_+ - \beta_1 P_2).
\]

The corresponding twist is given by the following formula

\[ F_{r4s} = F_{r4}' F_{r4s}', \]  

(3.36)

where

\[
F_{r4s}' = \exp((\alpha_1 P_1 + \alpha_2 P_2) \wedge P_+ + \eta Q_1^+ \wedge Q_1^+),
\]

\[
F_{r4s}''' = \exp(\gamma(e_+ - \beta_1 P_1) \wedge (e'_+ + \beta_1 P_2)).
\]

(3.37)

It should be noted that the parameter \( \beta_1 \) can be removed by the rescaling automorphism \( \beta_1 P_\nu \rightarrow P_\nu (\nu = 0, 1, 2, 3) \).

5). The fifth \( r \)-matrix \( r_5 \) (\( N = 5 \) in Table 1) is the \( r \)-matrix for the Lorentz algebra \( \mathfrak{o}(3, 1) \) which does not admit any superextension. The explicit formulae for the corresponding twist and algebra coproduts are calculated in [38].

6). The sixth \( r \)-matrix \( r_6 \), describing the superextension of \( N = 6 \) in Table 1

\[
r_6 = \gamma h \wedge e_+ + \beta_1 (2P_1 \wedge h' + P_- \wedge e'_+ - P_+ \wedge e'_-) + \beta_2 P_2 \wedge e_+,
\]

(3.38)

has the following superextension

\[
r_{6s} = \gamma h \wedge e_+ + \beta_1 (2P_1 \wedge h' + P_- \wedge e'_+ - P_+ \wedge e'_-) + \beta_2 P_2 \wedge e_+
\]

\[
+ \beta_2 P_2 \wedge e_+ + \frac{i\beta_1}{4} (Q_1^+ + Q_1^-) \wedge (Q^+_2 - Q^-_2),
\]

(3.39)
This $r$-matrix can be presented as a sum of three subordinated $r$-matrices

$$r_{6s} = r'_{6s} + r''_{6s} + r'''_{6s}, \quad r'_{6s} \succ r''_{6s}, \quad r''_{6s} \succ r'''_{6s},$$

$$r'_{6s} = \beta_1 \left( 2P_1 \wedge h' + P_- \wedge e'_+ - P_+ \wedge e'_- + \frac{i}{4}(Q^+_1 + Q^-_1) \wedge (Q^+_2 - Q^-_2) \right),$$

$$r''_{6s} = \gamma h \wedge e_+,$$

$$r'''_{6s} = \beta_2 P_2 \wedge e_+. \quad (3.40)$$

The $r$-matrix $r''_{6s}$ is of Jordanian type, the $r$-matrix $r'''_{6s}$ is of Abelian type but the first $r$-matrix $r'_{6s}$ is neither of super-Abelian nor of super-Jordanian type. One can check that $r$-matrix (3.39) satisfies the non-homogeneous classical Yang-Baxter equation (3.13) with $t \neq 0$. In terms of the generators $M_i, N_i (i = 1, 2, 3)$ the $r$-matrix $r'_{6s}$ has the form

$$r'_{6s} = 2\iota \beta_1 \left( P_1 \wedge M_3 - P_3 \wedge M_1 - P_0 \wedge N_2 + \frac{1}{8}(Q^+_1 + Q^-_1) \wedge (Q^+_2 - Q^-_2) \right). \quad (3.41)$$

Unfortunately the quantum deformation corresponding to the $r$-matrix (3.39) can not be described by a supertwist. It is quite plausible that the quantization of (3.41) can be obtained by particular contraction procedure from the $q$-deformation $U_q(\mathfrak{osp}(1|4))$ of $D = 4$ AdS superalgebra in analogy with the $\kappa$-deformation of super-Poincaré algebras derived in [23].

We conclude that we presented superextensions for all Zakrzewski’s classical $r$-matrices when $c \neq 0$ (see (3.10)) and we constructed the corresponding twists for the cases $t = 0$. Analogous results can be obtained for classical $r$-matrices $r_N (7 \leq N \leq 18)$ in Table 1 with $c = 0, t = 0$.

### 4 Outlook

In our paper we provided a supersymmetrization of the part of Zakrzewski’s list of Poincare algebra deformations, presented in the form of $r$-matrices ([35]; see Table 1). It should be recalled that known examples of $D = 4$ quantum deformations of supersymmetries, e.g. standard $\kappa$-deformation [23] does not belong to the considered class, because they correspond in formula (3.10) to the case $c = 0, b \neq 0$. We see therefore as important physically task the completion of the classification of quantum deformations of relativistic supersymmetries, corresponding to cases with $c = 0$ (see $7 \leq N \leq 21$ in Table 1). In particular, one can consider the supersymmetrization of twisted versions of $\kappa$-deformed Poincare algebra [39].

In this presentation we consider only the Hopf-algebraic framework of quantum-deformed $D = 4$ Poincare supersymmetries and did not study its applications. It should be recalled that in simplest case of Abelian twist there were already considered examples of deformed field-theoretic models (see e.g. [40, 41]). It should be pointed out that most of the deformations of supersymmetric field theories were made without any reference to twisting and Hopf algebra structure (see e.g. [42, 43, 44]). The next step in our investigations will be as well the results presented in Sec.3 in terms of physical super-Poincare generators, to consider their superspace realizations and finally to construct new dynamical models with quantum supersymmetries.
Acknowledgments

The paper has been supported by MNiSW grant NN202 318534 (A.B., J.L., V.N.T) and the grant RFBR-08-01-00392 (V.N.T.), and the French National Research Agency grant NT05-241455GIPM (V.N.T.).

References

[1] V.N. Tolstoy, *Twisted Quantum Deformations of Lorentz and Poincar algebras*, To appear in Proceedings of VII International Workshop ”Lie Theory and Its Applications in Physics”. Ed. V.K. Dobrev et al, Heron Press, Sofia, 2008; arXiv:0712.3962[math.QA].

[2] J. Wess and B. Zumino, *Nucl. Phys.* B 78, 1 (1974).

[3] R. Haag, J. Lopuszanski and M. Sohnius, *Nucl. Phys.* B88, 257 (1985).

[4] P. Fayet and S. Ferrera, *Phys. Rep.* 32C, 249 (1977).

[5] P. Van Nieuwenhuizen, *Phys. Rep.* 68, 189 (1981).

[6] V. Ogievetsky and E. Sokatchev, *Yad. Fiz.* 31, 264 (1980); *ibid*, 31, 821 (1980); *ibid*, 32, 862 (1980).

[7] I.L. Buchbinder and S. M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity*, IOP Publ. Bristol, 1995.

[8] V.G. Drinfeld, in *Proc. of XX-th Int. Math. Congress (1986), Berkeley (USA)*, p. 820 (1987).

[9] M. Jimbo, *Lett. Math. Phys.*, 10, 63 (1985).

[10] S.L. Woronowicz, *Publ. Res. Inst. Math. Sci.*, Kyoto, 23, 117 (1987).

[11] L.D. Faddeev, N.Yu. Reshetikhin, and L.A. Takhtadjian, *Algebra i Analiz*, 1, 178 (1989).

[12] P. Kosinski, J. Lukierski and P. Maslanka, *Phys. Rev.*, D62, 025004 (2000); arXiv:0706.3658

[13] L. Freidel, J. Kowalski-Glikman and S. Nowak,

[14] S. Doplicher, K. Fredenhagen, and J.E. Roberts, *Phys. Lett.*, B331, 39 (1994); *Commun. Math. Phys.* 172, 187 (1995).

[15] L.J. Garay, Int.J. Mod. Phys. A10, 145 (1995).

[16] C.S. Chu and P.M. Ho, *Nucl. Phys.* B568, 447 (2000).

[17] N. Seiberg and E. Witten, *JHEP*, 9909, 032 (1999); hep-th/9908142.

[18] P.P. Kulish, Kyoto preprint RIMS-615 (1988)

[19] P.P. Kulish and N.Yu. Reshetikhin, *Lett. Math. Phys.*, 18, 143 (1989).

[20] M. Chaichian, P.P. Kulish, *Phys. Lett.*, B234, 72 (1990).
[21] S.M. Khoroshkin and V.N. Tolstoy, *Comm. Math. Phys.*, **141** (3), 599 (1991).

[22] J. Lukierski, A. Nowicki, H. Ruegg, and V.N. Tolstoy, *Phys. Lett.*, **B264**, 331 (1991).

[23] J. Lukierski, A. Nowicki, J. Sobczyk, *J. Phys.* **A26**, L1109 (1993).

[24] P. Kosinski, J. Lukierski, P. Maslanka and J. Sobczyk, *J. Phys.*, **A27**, 6827 (1994); hep-th/9405076; ibid. **A28**, 2255 (1995); hep-th/9411115.

[25] P. Kosinski, J. Lukierski, P. Maslanka and J. Sobczyk, *Mod. Phys. Lett.*, **A10**, 2599 (1995); hep-th/9412144.

[26] M. Chaichian, P.P. Kulish, K. Nishijima, and A. Tureanu, *Phys. Lett.*, **B604**, 98 (2004); hep-th/0408069.

[27] J. Wess, *Proc. of 2003 Workshop in Vrnacha Banya, Serbia, Aug. 2003*; publ. Belgrad, 2004, p. 122; hep-th/0408080.

[28] Y. Kobayashi and S. Sasaki, *Int. J. Mod. Phys.*, **A20**, 7175 (2005).

[29] B.M. Zupnik, *Phys. Lett.*, **B627**, 208 (2005).

[30] R. Banerjee, Ch. Lee and S. Siwach, *Eur. Phys. J.*, **C48**, 305 (2006).

[31] V.N. Tolstoy, *Proc. of International Workshop ”Supersymmetries and Quantum Symmetries (SQS’03)”, Russia, Dubna, July, 2003*, eds: E. Ivanov and A. Pashnev, publ. JINR, Dubna, p. 242 (2004); arXiv:math/0402433v1.

[32] V.N. Tolstoy, *Nankai Tracts in Mathematics ”Differential Geometry and Physics”. Proceedings of the 23-th International Conference of Differential Geometric Methods in Theoretical Physics (Tianjin, China, 20-26 August, 2005)*. Editors: Mo-Lin Ge and Weiping Zhang. World Scientific, 2006, Vol. 10, 443-452; arXiv:math/0701079v1.

[33] A. Borowiec, J. Lukierski, V.N. Tolstoy, *Mod. Phys. Lett.*, **A18**, 1157 (2003); arXiv:hep-th/0301033.

[34] A. Borowiec, J. Lukierski, V.N. Tolstoy, *Eur. Phys. J.*, **C44**, 139 (2005); arXiv:hep-th/0412131.

[35] S. Zakrzewski, *Commun. Math. Phys.*, **187**, 285 (1997); arXiv:q-alg/9602001v1.

[36] V.G. Drinfeld, *Leningrad Math. J.*, **1**, 1419 (1990).

[37] V.N. Tolstoy, *Quantum Deformations of Relativistic Symmetries*, To appear in Proceedings of the XXII Max Born Symposium ”Quantum, Super and Twistors”, 2008; arXiv:0704.0081v1[math.QA].

[38] A. Borowiec, J. Lukierski, V.N. Tolstoy, *Eur. Phys. J.*, **C48**, 636 (2006); arXiv:hep-th/0604146; *Czech. J. Phys.*, **55**, 11 (2005); arXiv:hep-th/0510154.

[39] J. Lukierski and V. Lyakhovsky, Proc. of Conf. on Noncommutative Geometry in Mathematics and Physics, Karlstad, 2004; *Contemporary Math.*, **391**, 281 (2006); hep-th/0406155.
[40] P. Aschieri, Ch. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp and J. Wess, *Class. Quant. Grav.*, 22, 3511 (2005); hep-th/0504183.

[41] P. Aschieri, M. Dimitrijevic, F. Meyer, S. Schraml and J. Wess, *Lett. Math. Phys.*, 78, 61 (2006); hep-th/0603024.

[42] S. Ferrara and M.A. Lledo, *JHEP*, 0005, 008 (2000); hep-th/0002084.

[43] N. Seiberg, *JHEP*, 0306, 010 (2003); hep-th/0305248.

[44] S. Ferrara, M.A. Lledo and O. Macia, *JHEP*, 0309, 068 (2003); hep-th/0307039.