Quenched noise and over-active sites in sandpile dynamics

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Abstract. – The dynamics of sandpile models are mapped to discrete interface equations. We study in detail the Bak-Tang-Wiesenfeld model, a stochastic model with random thresholds, and the Manna model. These sandpile models are, respectively, discretizations of the Edwards-Wilkinson equation with columnar, point-like and correlated quenched noise, with the constraint that the interface velocity is either zero or one. The constraint, embedded in the sandpile rules, gives rise to another noise component. Studies of this term for the Bak-Tang-Wiesenfeld model reveal long-range on-site correlations and that with open boundary conditions there is no spatial translational invariance.

Simple cellular automata have been under intensive study during the last decade. The aim has been to explain many of the power-laws that can be often seen in systems in nature. A particularly famous example is the paradigmatic Bak-Tang-Wiesenfeld (BTW) sandpile model \(^{[1]}\), which exhibits both spatial and temporal criticality using the language of statistical mechanics. Many variants of sandpile models have been developed, while the BTW model still occupies a central position due to its deceptive simplicity and complicated behavior \(^{[2]}\).

The central ingredients of a sandpile model are rules for redistributing grains and for driving the system by grain addition if it is stable. This gives rise to two time scales, slow and fast, and usually to diffusive grain dynamics. Grains are removed from unstable sites and distributed to neighbors. Those lost through the open boundaries are compensated by adding grains randomly, but on a slow time scale compared to the time that individual avalanches or periods of activity take. In the asymptotic self-organized critical (SOC) steady state power-laws emerge in the avalanche statistics. The scenario is reminiscent of another problem in non-equilibrium statistical mechanics: driven interfaces which couple a random environment to an elastic object. At a critical value of the drive force the velocity of the interface becomes non-zero, and exactly at this depinning transition critical behavior ensues. The time and length scales are renormalized, with associated critical exponents \(^{[3,4]}\).

In this Letter we demonstrate a connection between interfaces and many sandpile models. We construct a mapping of the BTW model, and two stochastic models (“rice-pile” models
The equation. The important point is the construction of the local “force” at site \(x\). The mapping establishes the upper critical dimension to linear interface model class and may be discussed with well-known scaling and renormalization and Manna models. The rice-pile model is exactly a discrete version of the (random field) linear interface model class and may be discussed with well-known scaling and renormalization arguments \([3, 4]\). The mapping establishes the upper critical dimension to \(d_u = 4\) for all the models. As a novel effect, the noise coming from the velocity constraint demonstrates a lack of translational invariance in open systems.

The sandpile models are as follows: each site \(x\) of a hyper-cubic lattice of size \(L^d\) has \(z(x, t)\) grains. When \(z(x, t)\) exceeds a critical threshold \(z_c(x)\), the site is active and topples. Grains are removed from \(x\) and given to the nearest neighbors (nn). This in turn may cause some nn’s to topple and so on. If there are no active sites in the system, one grain is added to a randomly chosen site, \(z(x, t) \to z(x, t) + 1\). The time and number of topplings till the system again contains no active sites define an avalanche and its internal lifetime. The system is usually open such that grains which topple out of the system are lost (in \(d = 1\): \(z_0 \equiv z_{L+1} \equiv 0\)). The specific toppling rule distinguishes between the models. The BTW model has \(z_c\) equal to a constant, \(z_c = 2d-1\) and the toppling rule: \(z(x, t + 1) = z(x, t) - 2d, \ z(y, t + 1) = z(y, t) + 1\), where \(y\) denotes all the \(2d\) nn’s of the site \(x\). The rice pile model has the same rule but with \(z_c(x)\) randomly chosen after each toppling from a probability distribution. In the Manna model, two grains are given to two randomly chosen neighbors with \(z_c = 1\). We consider here two kinds of ensembles. The SOC one is defined above via the drive and the open boundaries. Another possibility is to use periodic boundary conditions, and prepare the system at a certain average ‘energy’ \(\langle z(x, 0) \rangle\) in which case the critical state is reached at a certain energy \(\langle z \rangle_c \equiv L\).

The mapping of the dynamics of sandpile models \([4]\), begins with the definition of an interface or memory field \(H(x, t)\). It counts topplings at site \(x\) up to time \(t\). The dynamics of \(H\) defined in this way reads

\[
H(x, t+1) = \left\{ \begin{array}{ll}
H(x, t) + 1, & f(x, t) > 0 \\
H(x, t), & f(x, t) \leq 0
\end{array} \right. \rightarrow \frac{\Delta H}{\Delta t} = \theta (f(x, t)), \tag{1}
\]

where we have rewritten the equation of motion for \(H\) in the form of a discrete interface equation. The important point is the construction of the local “force” \(f(x, t) = z(x, t) - z_c(x)\) at site \(x\). We can express \(f(x, t)\) in terms of the grains added to site \(x\), \(n^{in}_x\), and removed from \(x\), \(n^{out}_x\), as \(f(x, t) = n^{in}_x - n^{out}_x - z_c(x)\), since \(z(x, t) = n^{in}_x - n^{out}_x\).

For the BTW and rice-pile models \(n^{in}_x\) and \(n^{out}_x\) can be derived from the local height field \(H(x, t)\) and an external force term \(F(x, t)\). Here, \(F\) counts the number of grains added to site \(x\) by the external drive up to time \(t\). Thus for SOC drive \(F(x, t)\) increases on the slow time scale and does not change during avalanches, i.e. it acts as a (columnar) quenched noise. The construction of \(f\) using \(F\) works for any drive (e.g. continuous, uniform). Observing that \(n^{out}_x\) is simply \(n^{out}_x = 2dH(x, t)\) and \(n^{in}_x = \sum_{x_{nn}} H(x_{nn}, t) + F(x, t)\), where \(x_{nn}\) denotes the
2d nn’s of \( x \), one arrives at
\[
f(x, t) = \nabla^2 H + F(x, t) - z_c(x, H), \]
where \( \nabla^2 H \) is the discrete Laplacian \( [1,2] \). The threshold \( z_c(x, H) \) depends on \( H \) for the rice-pile model and reveals that the random thresholds correspond to a quenched force field which is random in \( x \) and \( H \), and acts on the interface. For the BTW model \( z_c \) is constant.

For the Manna model we have to incorporate the randomness in the distribution rule. We use a projection technique by writing \( n_{x}^{in} \) as the average incoming flux \( \bar{n}_{x}^{in} \) plus a fluctuating part \( \delta n_{x}^{in} \) and arrive at
\[
f(x, t) = 1/d \nabla^2 H + F(x, t) - z_c(x, H) + \tau(x, H), \tag{2}\]
with the resulting noise term
\[
\tau(x, H) = \delta n_{x}^{in} \equiv n_{x}^{in} - \bar{n}_{x}^{in} = n_{x}^{in} - 1/d \sum_{x_{nn}} H(x_{nn}, t). \tag{3}\]

Each nn-toppling contributes \( 1/d \) to \( \bar{n}_{x}^{in} \) as the sum over the nn’s is the expected number of grains from neighboring sites due to their topplings. When site \( x \) topples, one uses the above definition of \( \tau(x, H) \) in terms of the fluxes \( (n^{in}, \bar{n}^{in}) \) to evaluate it at site \( x \) and height \( H \). The projection trick, used to construct the quenched noise \( \tau \), means that the randomness in the toppling rule is matched to an equivalent noise field \( \tau \) so that the interface equation for \( H \) reproduces exactly the behavior of the Manna sandpile. It can also be used in the case of other models, where the effect of a toppling is random. Note that \( \tau(x, H) \) is a conserving noise since the random toppling rule for the Manna model conserves the number of grains (except at open boundaries).

The step-function, \( \theta(f) \), in Equation (1) forces it so that the interface does not move backwards and such that the velocity \( v \equiv \Delta H/\Delta t \) is either 0 or 1 \( [3] \). This means that sandpiles are equivalent to cellular automaton models of interface depinning, with this velocity constraint. We next map this constraint into an effective noise term, denoted \( \sigma \), in the interface equation, that can be used to discuss the possible differences of ‘real’ depinning models and those that arise from sandpile models via the mapping. Consider the toppling example in Fig. 4. On the avalanche timescale \( f < 0 \) at site \( x \). As function of time, \( f \) increases until at time \( t - 1 \) several neighbors topple resulting in \( f > 0 \), so that site \( x \) will topple at time \( t \) and \( \Delta H/\Delta t = 1 \).

The sandpile rules result in an effective force \( f' \equiv 1 \) that acts on the interface \( H \) at the time of toppling: \( \Delta H/\Delta t \equiv f'\theta(f) = f'\theta(f') \). The relation between \( f \) and \( f' \) for each toppling at \( x \) (constant \( H \)) reads thus
\[
f'(x, t) = f(x, t) + \sigma(x, H) \quad \rightarrow \quad \sigma(x, H) = 1 + z_c(x, H) - z(x, t^{*}) \tag{4}\]
where \( t^{*} \) is the time at which site \( x \) topples such that \( \sigma(x, H) \) by this construction is a quenched random variable. It is computed from the difference between \( f \) and \( f' \) when \( x \) topples. Notice that \( f' \) and \( f \) are by definition time-dependent variables, since they change as grains are moved, or the Laplacian changes. They also contain a quenched force component, as is seen from Eq. (3). The easiest way to study the \( \sigma \)-noise is to determine it in a simulation using Equation (4). This construction of \( \sigma \) is similar to that of the \( \tau \)-noise term for the Manna-model and other models for which the projection trick can be used. The trick maps the difference of the expected value of the local force at toppling \( f(x, t^{*}) = z_{x} - z_{c} \) and the true one to \( \tau \). Here the difference of \( f'(x, t^{*}) \equiv 1 \) and \( f \) maps to \( \sigma \), a quenched variable. Such differences arise in the Manna model due to randomness in grain movement, and in the case of the \( \sigma \)-noise from the effect of the step-function.
The point in the noise variables \( \sigma(x, H) \) and \( \tau(x, H) \) is that given these disorder fields, the interface equation exactly reproduces the history or dynamics of a sandpile “run”. Thus we can study the interface model as such, and try to infer the properties of the original sandpile from its behavior. Also, the \( \sigma \)-noise allows one to study the effects of the peculiar discretization \((dH/dt = \theta(f)) \) explicitly, since \( \Delta H/\Delta t = f'/\theta(f') \) can be interpreted as the discretization of the continuum equation \( \partial H/\partial t = f' \). Combining Eqs. (1) and (4) we write the discretized interface equation as

\[
\frac{\Delta H}{\Delta t} = \nu \nabla^2 H + \eta(x, H) + F(x, t) + \sigma(x, H).
\]

Here the diffusion constant \( \nu \) is unity for the BTW and ricepile models and \( 1/d \) for the Manna model, and the quenched noise \( \eta(x, H) = -z_c(x, H) + \tau(x, H) \). Equation (4) is the central difference discretization of a continuum diffusion equation with quenched noise, called the linear interface model (LIM) or the quenched Edwards-Wilkinson equation \( [3,5] \). Equation (4) contains two main ingredients. First, the Laplacian character of these sandpile models which is such that the differences in accumulated topplings map exactly to an elastic force. In the interface language once the force increases sufficiently to overcome the pinning force, \( \nabla^2 H + \eta(x, H) + F(x, t) \), the interface moves by one step. For open boundary sandpiles, the right interface boundary condition is \( H = 0 \) which is to be imposed at “extra sites” \((x = 0, x = L + 1 \) for a system of size \( L \) in 1d). In the SOC steady-state the Laplacian increases, because of the roughly parabolic shape for the the toppling profile \( \tilde{H}(x) \) (see \( [3,5,20] \)). This is compensated by the ever-increasing \( \langle F(t) \rangle \), or the addition of grains by the SOC drive.

Second, the randomness in the sandpile rules map into noise variables \( \{F, \eta\} \), as do the details of the dynamics \( \{\sigma\} \). Equation (4) allows thus to make conclusions about the universality classes of models based on the noise terms and their relevance. The \( F \)-term in Eq. (4) is columnar: it integrates the deposited grains and is constant during avalanches. The \( \eta(x, H) \) term in the rice-pile and Manna models explicitly depends on \( H \). For the rice-pile \( \eta(x, H) \equiv -z_c(x, H) \), trivially. The associated LIM has point-disorder since \( z_c(x, H) \) is delta-correlated in \( x \) and \( H \). The LIM corresponding to the Manna model has a noise field \( \eta(x, H) \) which is point-like and correlated: in the \( H \)-direction because of random-walk like increments in \( \eta(x, H) \) and in the \( x \)-direction because of the short-ranged grain conservation (if a nn of \( x \) gained a grain when \( H(x, t) \) increased other nn’s did not) \( [12] \).

For non-SOC (periodic or open) boundary conditions the LIM has a depinning transition at a critical force \( F_c \). At the transition, the scaling exponents of the LIM with columnar, point-like, and, depending on the details, correlated disorder differ (see \( [3,5] \) for the effect of noise correlations). This means that avalanches have different spatial and temporal properties since the critical exponents like the roughness exponent \( \chi \) of the LIM depend on the noise. In particular, the BTW model is in a different universality class from the others \( [3,5] \) as it has no \( \eta \)-noise. The LIM is invariant to forces that are static in the \( H \)-direction \( [3,5] \) which makes \( \eta \) a relevant perturbation. For the Manna-model, we conjecture that it may be in the point-disorder LIM class despite the correlations in \( \eta \) \( [4] \). This prediction seems to be shown to be true in \( d = 2 \) in Ref. \( [15] \). The upper critical dimension of the LIM is \( d_u = 4 \) for all these kinds of noise because of the Laplacian in Eq. (4).

These conclusions are based on the continuum limit of the LIM. However, sandpile models are discretized versions thereof, with the additional \( \sigma \)-noise term in the equation. In the point-disorder LIM, various numerical approaches indicate that the \( \sigma \) is irrelevant for the rice-pile model with periodic boundary conditions \( [3,5] \). This can be understood by considering the sum of the \( \eta \) and \( \sigma \)-noise terms: the velocity constraint just renormalizes the \( \eta \)-field since
the velocity would in any case be of the order of unity, or, the actual value of \( \sigma \) depends on \( z(x, H) \).

For the other scenarios one has to answer the question, whether the \( \sigma \)-noise changes the universality class of the model at hand. This would mean simply that the sandpile model can be a “bad discretization” of a continuum interface equation so that the \( \sigma \)-noise changes the properties of the continuum model. Notice that the way \( \sigma(x, H) \) becomes non-zero (in these models \( \sigma \leq 0 \)) depends on the model. In the Manna-model a site with \( z_x = 1 \) can also get two grains from the same nn leading to \( \sigma \neq 0 \). This is point-like disorder since the decision to give two grains has no correlations with any earlier events. Thus the Manna model may be in the rice-pile universality class as noted above. Like in the rice-pile model, the \( \sigma \)-noise should not have any strong correlations due to the randomness in the avalanches.

In the BTW and rice-pile models the \( \sigma \)-term arises (see again Fig. 1) from the simultaneous topplings of nn’s of site \( j \). Thus in \( d > 2 \) the role of the noise is diminished, and in particular it should not affect the upper critical dimension. Notice that the \( \sigma \)-noise changes off the critical point (since more of the neighbors are likely to be active at the same time), and may thus play a role in systems off criticality like in the ‘fixed density’ (or energy) ensemble corresponding to a constant total force \( [14] \). The BTW model has also the Abelian property \( [2] \) and thus the \( \sigma \)-field of any particular sample depends on the exact order of the topplings.

The \( \sigma \)-noise is next studied as such to elucidate the role of the SOC boundary conditions in the interface equation (open boundaries for the sandpile). We look at (mostly) the BTW and the other models by numerical simulations in 2d in the normal SOC sandpile ensemble using parallel dynamics for all active sites at each time step, as one would do in an interface model. Quantities of interest are the average of \( \sigma \), and its spatial dependence on \( x \). The \( \sigma \)-field is constructed from the relation \( [3] \).

First we look at the finite size dependence of the probability to have a non-zero \( \sigma \), \( P_L(\sigma < 0) \). It can be studied as averaged over dissipating (grains leave the system), and non-dissipating avalanches. For all these \( P_L \) increases with the system size \( L \). The finite size scaling Ansatz \( P_L \sim \langle P \rangle_{\infty} + aL^{-b} \), fitted to the data \( [14] \), gives correction exponents that are close to \( b = 2/3 \) for bulk and dissipating avalanches, and \(-1\) for all avalanches; \( a \) is negative in all cases. The asymptotic BTW values are \( \langle P \rangle \simeq 0.081 \) for all and \( \langle P \rangle_d \simeq 0.121 \) for dissipating avalanches. Thus the dissipating avalanches have typically stronger \( \sigma \)-noise: the interface tries to move faster. They also contribute significantly into \( \langle \sigma \rangle \) since the total number of topplings is dominated by such avalanches. Therefore fluctuations of the \( \sigma \)-field may be related to the fluctuations in the average grain number, \( \langle z_x \rangle \simeq \langle z_x \rangle_c \). For the Manna and ricepile models \( \langle \sigma \rangle \) also decreases with \( L \) \( [12] \).

We checked the spatial correlations in \( \sigma(x, H) \) by computing the two-point noise-noise correlation function \( C(\delta H) = \langle \sigma(x, H + \delta H)\sigma(x, H) \rangle - \langle \sigma^2 \rangle \), at a fixed \( x = \text{const} \) \( [17] \). Figure 2 demonstrates that at the center and at the edges of the pile the noise decays exponentially. The decay is slower in the center the decay lengths being in general proportional to \( L \). In the ‘bulk’ (see the second and third curve from left) we see superposed on that behavior periodic oscillations. Thus there is no spatial translational invariance \( [18, 20] \) since the correlations depend on \( x \). The details of the correlations of \( \sigma \) in the BTW model may be related to observations of multiscaling in the BTW model \( [21] \) and manifest the columnar character of the \( F \)-noise. In contrast, for the Manna/rice-pile models the correlations in \( \sigma \) decay rapidly.

The average \( \langle \sigma(x) \rangle \) turns out to be non-uniform in \( x \). For instance, at the boundary the noise is weaker since sites can receive grains only from \( 2d - 1 \) nn’s. The non-uniform \( \sigma \) implies simply that the interface tries to move faster in the center of the pile. Next we investigate whether \( \sigma(x, H) \neq 0 \) if site \( x \) does not topple any more during an avalanche. In other words, is the velocity constraint related to the stopping properties of avalanches? We
look thus at the $\sigma$-noise at the elastic pinning paths of an interface model \cite{22}. The answer is relevant for the existence of translational invariance as well. Figure 3 shows a data collapse of $P(\sigma(x,y) \neq 0)$ along the cut $y = L/2$, $1 \leq x \leq L/2$, scaled with $P(x \simeq L/2)$, with the pinning-path constraint, i.e. the fraction of all topplings with $\sigma < 0$ such that the toppling is the last at the site during an avalanche. The probability increases at the boundaries with system size $L$. This shows that the standard BTW sandpile has no spatial translational invariance. This is true (see the inset) also for the Manna and rice-pile models. Their pinning paths are determined by the configuration prior to the avalanche and the (point) disorder the avalanche encounters. In the BTW case the paths are set by the original force/grain configuration as in a columnar-noise LIM, but not by the $\sigma$-noise field itself as the Abelian character of the BTW also implies (the choice of the toppling order does not change the final configuration). It would be interesting to compare the BTW result to the wave picture of the BTW model \cite{2, 23, 24}. The lack of translational invariance is caused by the open boundary conditions that also imply a parabolic shape for the interface and suggests that there is no simple scaling as one would expect for a normal interface model, in which case a relation $l^D \sim l^{d+\chi}$ would be valid for the avalanche size vs. its linear dimension $l$. At most, one should have $D_s \leq d + \chi$ for the cut-off dimension of the probability distribution of the avalanche sizes, since the open sandpile avalanches can of course not be ‘over-critical’ with any effective $\chi' > \chi$. Note that this observation seems to be true for any of the three models.

In conclusion, sandpiles can be mapped to driven interfaces, by describing the dynamics with various types of quenched noise. The rice-pile model is equivalent to the random field linear interface model. The BTW model has long-range on-site correlations that can be studied via the $\sigma$-noise or the restriction $v \leq 1$. It is because of the deposition noise $F(x,t)$ a columnar-noise LIM albeit with the velocity limitation. The Manna model turns out to have “correlated point-disorder” and is thus in a different universality class from the BTW model. The projection technique used for the Manna toppling dynamics can be applied to e.g. the Zhang model, models with bulk dissipation of grains, and the Olami-Feder-Christensen model \cite{25–27} and thus further extensions of our work are certainly possible. The discussion of the $\sigma$-term in the interface equation for sandpiles makes it clear that studying the noise provides a new tool for elucidating sandpile behavior and the role of various boundary conditions in sandpiles. The fundamental ingredients in sandpiles are slow drive and fast dissipation. With open boundaries these combine to make the pile spatially non-uniform, while the rules chosen are reflected in the scaling of the self-organized critical state.

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REFERENCES

[1] Bak P., Tang C. and Wiesenfeld K., Phys. Rev. Lett., 59 (1987) 381; Phys. Rev. A, 38 (1988) 364; Bak P., How Nature Works (Copernicus, New York) 1996.
[2] Dhar D., Physica A, 263 (1999) 4 and references therein.
[3] Narayan O. and Fisher D. S., Phys. Rev. B, 48 (1993) 7030.
[4] Nattermann T. et al., J. Phys. (France) II, 2 (1992) 1483; Leschhorn H. et al., Ann. Physik, 7 (1997) 1.
[5] Leschhorn H., Physica A, 195 (1993) 324.
[6] Paczuski P. and Boettcher S., Phys. Rev. Lett., 77 (1996) 111.
[7] Amaral L. A. N. and Lauritsen K. B., Phys. Rev. E, 54 (1996) R4512; ibid., 56 (1997) 231.
Fig. 1 – Rescaling of the force \( f \) and an example of how the \( \sigma \)-noise ensues (three grains added simultaneously).

[8] Manna S. S., J. Phys. A, 24 (1992) L363.
[9] For related earlier work see [6], Narayan O. and Middleton A. A., Phys. Rev. B, 49 (1994) 244, and Cule D. and Hwa T., Phys. Rev. B, 57 (1998) 8235.
[10] Dickman R., Vespignani A., and Zapperi S., Phys. Rev. E, 57 (1998) 5095; Vespignani A. et al., Phys. Rev. Lett., 81 (1998) 5676.
[11] Lauritsen K. B. and Alava M. J., preprint cond-mat/9903349.
[12] Alava M. J. and Lauritsen K. B., in preparation.
[13] Milshtein E. et al., Phys. Rev. E, 58 (1998) 303.
[14] Chessa A. et al., Phys. Rev. E, 59 (1998) R12.
[15] Vespignani A. et al., Phys. Rev. E, 62 (2000) 4564.
[16] Typically \( 10^7 \) avalanches were collected for \( L = 64, 128, 256, 512 \) for the FSS data.
[17] The values of \( \sigma \) with \( H = \text{const} \) become uncorrelated with increasing \( H \) since the interface is not flat.
[18] Drossel B., Phys. Rev. E, 61 (2000) R2168.
[19] Kttarev D. V. et al., Phys. Rev. E, 61 (2000) 81.
[20] Barrat A., Vespignani, A. and Zapperi S., Phys. Rev. Lett., 83 (1999) 1962.
[21] Tebaldi C., Demenech, M. and Stella A., Phys. Rev. Lett., 83 (1999) 3952.
[22] Makse H. A. et al., Europhys. Lett., 41 (1998) 251.
[23] Priezzhev V. B., Ivashkevich E. V., and Kttarev D. V., Phys. Rev. Lett., 76 (1996) 2093; Kttarev D. V. and Priezzhev V. B., Phys. Rev. E, 58 (1998) 2883.
[24] Paczuski M. and Boettcher S., Phys. Rev. E, 57 (1997) R3745.
[25] Zhang Y. C., Phys. Rev. Lett., 63 (1989) 470.
[26] Olami Z. et al., Phys. Rev. Lett., 68 (1992) 1244.
[27] Vespignani A. and Zapperi S., Phys. Rev. Lett., 78 (1997) 4793; Phys. Rev. E, 57 (1998) 6345.
Fig. 2 – Plots of $C(\Delta h)$ for BTW model noise at locations ranging from the edge to the center (left to right) for $L = 64$. Inset: $C(\Delta h)$ in the center for $L = 64, 128, 256, 512$ (bottom to top).

Fig. 3 – $P(\sigma(x) < 0)$ vs. $x$ scaled with $P(x = L/2)$ for various $L$ in the BTW model ($L = 32$ (circles), 64 (triangles), 128 (squares)). Inset: for $L = 64$ the same for BTW, Manna, ricepile (triangles, circles, crosses).