Fine Properties of Charge Transfer Models

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January 31, 2022

Abstract

We prove $L^1 \to L^\infty$ estimates for charge transfer Hamiltonians $H(t)$ in $\mathbb{R}^n$ for $n \geq 3$, followed by a discussion on $W^{\kappa,p} \to W^{\kappa,p}$ estimates for the same model, where $2 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then, geometric methods are developed to establish the time boundedness of the $H^\kappa$ norm for the evolution of charge transfer operators and asymptotic completeness of the Hamiltonian $H(t)$ in the $H^\kappa$ norm, where $\kappa$ is any positive integer.

1 Introduction

This paper is devoted to the study of the model corresponding to the time-dependent charge transfer Hamiltonian

$$H(t) = -\frac{1}{2}\Delta + \sum_{j=1}^{m} V_j(x - \vec{v}_j t)$$

with rapidly decaying smooth stationary potentials $V_j(x)$ and a set of mutually distinct constant velocities $\vec{v}_j$. First we focus on the $L^1 \to L^\infty$ dispersive estimate for the solutions of the time-dependent problem

$$\frac{1}{i}\partial_t \psi + H(t)\psi = 0$$

associated with a charge transfer Hamiltonian $H(t)$. This kind of dispersive estimate has been studied intensively during the past twenty years. The starting point is the well-known $L^p$ estimates for the free Schrödinger equation ($H_0 = -\frac{1}{2}\Delta$) on $\mathbb{R}^n$:

$$\|e^{itH_0}f\|_{L^p} \leq C_p |t|^{-n(\frac{1}{2} - \frac{1}{p})}\|f\|_{L^{p'}},$$

with $+\infty \geq p \geq 2, \frac{1}{p} + \frac{1}{p'} = 1$.

Writing the evolution as a convolution operator, the estimate is straight forward. One application is that they imply the Strichartz estimates

$$\|e^{itH_0}f\|_{L^r_tL^q_x} \leq C_q\|f\|_{L^2}, \quad 2 \leq r, q \leq \infty, \quad \frac{n}{r} + \frac{2}{q} = \frac{n}{2}, \quad n \geq 3 \quad [GV, KT]$$

*The paper is for the author’s candidacy of Ph.D. thesis at Caltech. The author feels deeply grateful to his advisor Dr. Wilhelm Schlag. His influence can be seen in every page. The author also wants to thank Rowan Killip for his very helpful comments.
Such estimates play a fundamental role, among other things in the theory of nonlinear dispersive equations. The extension of such theories motivated the efforts to establish the $L^p$ decay estimates for the general time independent Schrödinger operators of the type $H = -\frac{1}{2}\Delta + V(x)$. In this case there may be bound states, i.e., $L^2$ eigenfunctions of $H$. Under the evolution $e^{-itH}$, such bound states are merely multiplied by oscillating factors and thus do not disperse. So we need to project away any bound state and the estimates should take the form
\begin{equation}
\|e^{-itH}P_c(H)\psi_0\|_{L^p} \leq C_p |t|^{-\frac{n}{2}-\frac{1}{p}}\|\psi_0\|_{L^p'} \quad p \geq 2, \quad \frac{1}{p} + \frac{1}{p'} = 1, \tag{1.1}
\end{equation}

\begin{equation}
\|e^{-itH}P_c(H)f\|_{L^q_t L^r_x} \leq C_q \|f\|_{L^2} \quad \text{for} \quad 2 \leq q \leq \infty, \quad \frac{n}{r} + \frac{2}{q} = \frac{n}{2}, \quad n \geq 3, \tag{1.2}
\end{equation}

where $P_c(H)$ is the projection onto the continuous part of the spectrum of the self-adjoint operator $H$.

Before (1.1) and (1.2) were established, Rauch [R], Jensen and Kato [JK], and Jensen [J] proved the dispersion of $e^{-itH}P_c(H)$ in weighted $L^2$ spaces. Rauch required exponential decay of the potential, whereas Jensen and Kato assumed polynomial decay of a certain rate. Because the $L^2$ norm is preserved by the evolution, we need only prove (1.1) for the case where $p = \infty$ by interpolation. There are several approaches to the proof of (1.1). In [Ya], Yajima proved that the wave operators are bounded on the Sobolev spaces $W^{\kappa,p}$ and (1.1) is a consequence of the intertwining property of the wave operators. In [JSS], the evolution operator is expanded by Duhamel’s formula and its cancellation property was explored. Their proof splits into two parts: a “high energy” estimate, which holds for all potentials, and a “low energy” estimate where some spectral property of $H$ is assumed. This approach is generalized in [RSS1] to prove a “weak version” of the dispersive estimate of $H(t)$ (see inequality (1.13)). The first goal of this paper is to extend this idea further and prove the dispersive estimate of $H(t)$. Recently, M. Goldberg and W. Schlag ([GS]) proved (1.1) with much less restrictive conditions on the potential in $\mathbb{R}^1$ and $\mathbb{R}^3$. Their method is expected to give (1.1) in all dimensions.

We proceed by defining the charge transfer model and specifying our basic assumption.

**Definition 1.1.** By a charge transfer model we mean a Schrödinger equation
\begin{equation}
\frac{1}{i}\partial_t \psi - \frac{1}{2}\Delta \psi + \sum_{\kappa=1}^{m} V_\kappa(x - \vec{v}_\kappa t)\psi = 0 \quad \psi|_{t=0} = \psi_0, \quad x \in \mathbb{R}^n, \tag{1.3}
\end{equation}

where $\vec{v}_\kappa$ are distinct vectors in $\mathbb{R}^n$, $n \geq 3$, and the real potentials $V_\kappa$ are such that for every $1 \leq \kappa \leq m$,

1. $V_\kappa$ is time independent and has compact support (or fast decay), $V_\kappa, \nabla V_\kappa \in L^\infty$.

2. 0 is neither an eigenvalue nor a resonance of the operators

$$H_\kappa = -\frac{1}{2}\Delta + V_\kappa(x).$$

Recall that $\psi$ is a resonance if it is a distributional solution of the equation $H_\kappa \psi = 0$ which belongs to the space $L^2(\langle x \rangle^{-\sigma} \, dx)$ for any $\sigma > \frac{1}{2}$, but not for $\sigma = 0$. 

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The conditions in the above definition is always assumed when we prove and apply the dispersive estimates, i.e. Theorem 1.7 and Theorem 1.8. The conditions are not for optimal, but for convenience. This definition is standard, see [Gr1], [Ya2]. The Schrödinger group \( e^{-itH_\kappa} \) is known to satisfy the decay estimates (see Journé, Soffer, Sogge [JSS] and Yajima [Ya1])

\[
\|e^{-itH_\kappa} P_c(H_\kappa) \psi_0 \|_{L^\infty} \lesssim |t|^{-\frac{n}{2}} \| \psi_0 \|_{L^1}
\]

for \( n \geq 3 \) under various conditions on the potential. Here \( P_c(H_\kappa) \) is the spectral projection onto the continuous spectrum of \( H_\kappa \) and \( \lesssim \) denotes bounds involving multiplicative constants independent of \( \psi_0 \) and \( t \). For \( n = 3 \), [GS] proved (1.4) under the assumption that \( |V_\kappa(x)| \leq C(1 + |x|)^{-\beta} \), for some \( \beta > 3 \). For \( n > 3 \), [1.4] holds [JSS] under the additional assumption: \( \mathcal{F}(V_\kappa) \in L^1 \). Yajima [Ya1] proved (1.4) with slightly weaker conditions than [JSS]. We shall assume that \( \beta > \psi \).

To establish similar dispersive estimates for time dependent Schrödinger equations is more involved. Heuristically, we can’t project away the bounded states as they are moving in different directions. Rodnianski and Schlag [RS] proved dispersive estimate for small time dependent potentials. In this paper, we will focus on the charge transfer model.

An indispensable tool in the study of the charge transfer model are the Galilean transforms

\[
\phi_{\vec{e},y}(t) = e^{-\frac{y^2}{2t}} e^{-ix \cdot \vec{e}} e^{i(y + t\vec{v}) \cdot \vec{p}},
\]

cf. [Gr1], where \( \vec{p} = -i \vec{\nabla} \). Under \( \phi_{\vec{e},y}(t) \), the Schrödinger equation transforms as follows:

\[
\phi_{\vec{e},y}(t) e^{it \frac{\hat{X}}{2}} = e^{it \frac{\hat{X}}{2}} \phi_{\vec{e},y}(0)
\]

and moreover, with \( H = -\frac{1}{2} \Delta + V \),

\[

\psi(t) := \phi_{\vec{e},y}(t)^{-1} e^{-itH} \phi_{\vec{e},y}(0) \phi_0, \quad \phi_{\vec{e},y}(t)^{-1} = e^{-iy \vec{v}} \phi_{0, -\vec{e}, y}(t)
\]

solves

\[
\frac{1}{i} \partial_t \psi - \frac{1}{2} \Delta \psi + V(\cdot - t\vec{v} - y) \psi = 0
\]

\[
\psi|_{t=0} = \phi_0.
\]

Since in our case always \( y = 0 \), we set \( \phi_{\vec{v}}(t) := \phi_{\vec{e},0}(t) \). Note that the transformations \( \phi_{\vec{e},y}(t) \) are isometries on all \( L^p \) spaces and \( \phi_{\vec{v}_1}(t)^{-1} = \phi_{-\vec{v}_1}(t) \) because of (1.17). In the following, we shall assume that the number of potentials is \( m = 2 \) and that the velocities are \( \vec{v}_1 = 0, \vec{v}_2 = (1, 0, \ldots 0) = \vec{e}_1 \). The arguments generalize easily to \( m \geq 3 \).

We now introduce the appropriate analog to project away bounded states for the problem

\[
\frac{1}{i} \partial_t \psi - \frac{1}{2} \Delta \psi + V_1 \psi + V_2 (\cdot - t\vec{e}_1) \psi = 0
\]

\[
\psi|_{t=0} = \psi_0
\]

with compactly supported potentials \( V_1, V_2 \). Let \( u_1, \ldots, u_m \) and \( w_1, \ldots, w_\ell \) be the normalized bound states of \( H_1 \) and \( H_2 \) corresponding to the negative eigenvalues \( \lambda_1, \ldots, \lambda_m \) and \( \mu_1, \ldots, \mu_\ell \), respectively (recall that we are assuming that 0 is not an eigenvalue). We denote by \( P_0(H_1) \) and
$P_0(H_2)$ the corresponding projections onto the bound states of $H_1$ and $H_2$, respectively, and let $P_c(H_\kappa) = \text{Id} - P_0(H_\kappa)$, $\kappa = 1, 2$. The projections $P_0(H_{1,2})$ have the form

$$P_0(H_1) = \sum_{i=1}^{m} \langle \cdot, u_i \rangle u_i, \quad P_0(H_2) = \sum_{j=1}^{\ell} \langle \cdot, w_j \rangle w_j.$$ 

The following orthogonality condition in the context of the charge transfer Hamiltonian (1.9) was introduced in [RSS1].

**Definition 1.2.** Let $U(t)\psi_0 = \psi(t,x)$ be the solutions of (1.9). We say that $\psi_0$ (or also $\psi(t,\cdot)$) is asymptotically orthogonal to the bound states of $H_1$ and $H_2$ if

$$\|P_0(H_1)U(t)\psi_0\|_{L^2} + \|P_0(H_2,t)U(t)\psi_0\|_{L^2} \to 0 \text{ as } t \to +\infty.$$ 

Here

$$P_0(H_2,t) := g_{-\overrightarrow{c_1}}(t)P_0(H_2)g_{\overrightarrow{c_1}}(t)$$

for all times $t$.

**Remark 1.3.** From Corollary 2.8 $\|U(t)\psi_0\|_{L^p} \leq C_t \|\psi_0\|_{L^{p'}}$, we know that $U(t)\psi_0 \in L^p$ is well-defined for $\psi_0 \in L^{p'}$. As the bound states $u_i, w_j$ are exponentially decaying at infinity, Definition 1.2 makes sense for any initial data $\psi_0 \in L^{p'}$ for $p' \in [1, 2]$.

**Remark 1.4.** Clearly, $P_0(H_2,t)$ is again an orthogonal projection for every $t$. It gives the projection onto the bound states of $H_2$ that have been translated to the position of the potential $V_2(\cdot - t\overrightarrow{c_1})$. Equivalently, one can think of it as translating the solution of (1.9) from that position to the origin, projecting onto the bound states of $H_2$, and then translating back.

**Remark 1.5.** From Proposition 3.1 of [RSS1], the decay rate of (1.10) is actually exponential. More precisely, the following holds:

$$\|P_0(H_1)U(t)\psi_0\|_{L^2} + \|P_0(H_2,t)U(t)\psi_0\|_{L^2} \lesssim e^{-\alpha t} \|\psi_0\|_{L^2},$$ 

for some $\alpha > 0$.

**Remark 1.6.** It is clear that all $\psi_0$ that satisfy (1.10) form a closed subspace. This subspace coincides with the space of scattering states for the charge transfer problem. The latter is well-defined by Graf’s asymptotic completeness result [Gr].

We can only expect the dispersive estimate for (1.9) for the initial data satisfying Definition 1.2 just as we have to project away the bound states for (1.4). Rodnianski, Schlag, Soffer [RSS1] established the following estimate

$$\|U(t)\psi_0\|_{L^2 + L^\infty} \lesssim \langle t \rangle^{-\frac{3}{4}} \|\psi_0\|_{L^1 \cap L^2},$$

with initial data $\psi_0 \in L^1 \cap L^2$ satisfying (1.10), where $U(t)$ is the evolution of the charge transfer model and $\langle t \rangle = (1 + t^2)^{\frac{1}{2}}$. By definition, $\|f\|_{L^2 + L^\infty} := \inf_{h+g} \{\|h\|_{L^2} + \|g\|_{L^\infty}\}$ and $\|f\|_{L^1 \cap L^2} = \inf_{h+g} \{\|h\|_{L^1 \cap L^2} + \|g\|_{L^\infty}\}$.
\[ \|f\|_{L^1} + \|f\|_{L^2} \quad (1.13) \] has important application to the asymptotic stability and asymptotic completeness for the small perturbation of non-colliding solitons for NLS (\textbf{RSS2}).

\textbf{RSS1} decomposes the evolution into different channels according to each potential. Every channel splits into a large velocity part and a low velocity part. For the large velocity part, they employed Kato’s smoothing estimate; for the low velocity part, a propagation estimate is used. In this paper, we will combine the methods from \textbf{JSS} and \textbf{RSS1} and obtain the following:

\textbf{Theorem 1.7.} Consider the charge transfer model as in Definition 1.1 with two potentials, cf. (1.3). Assume \( \hat{V}_1, \hat{V}_2 \in L^1(\mathbb{R}^n) \). Let \( U(t) \) denote the propagator of (1.9). Then for any initial data \( \psi_0 \in L^1 \), which is asymptotically orthogonal to the bound states of \( H_1 \) and \( H_2 \) in the sense of Definition 1.2, one has the decay estimates

\[ \|U(t)\psi_0\|_{L^\infty} \lesssim |t|^{-\frac{\beta}{2}} \|\psi_0\|_{L^1}. \quad (1.14) \]

An analogous statement holds for any number of potentials, i.e., with arbitrary \( m \) in (1.3).

Inspection of the argument in the following sections shows that it applies to exponentially decaying potentials, say. But also sufficiently fast power decay at infinity is allowed. We shall prove (1.14) by means of a bootstrap argument. More precisely, we prove that the bootstrap assumption

\[ \|U(t)\psi_0\|_{L^\infty} \leq C_0 |t|^{-\frac{\beta}{2}} \|\psi_0\|_{L^1} \quad \text{for all } 0 \leq t \leq T \quad (1.15) \]

implies that

\[ \|U(t)\psi_0\|_{L^\infty} \leq \frac{C_0}{2} |t|^{-\frac{\beta}{2}} \|\psi_0\|_{L^1} \quad \text{for all } 0 \leq t \leq T. \quad (1.16) \]

Here \( T \) is any fixed large constant and (1.15) holds for \( C_0 \), some sufficiently large constant because of Corollary 2.3. \( C_0 \) may depend on \( T \) at the beginning. The above implication (1.15) \( \Rightarrow \) (1.16) holds as long as \( C_0 \) is larger than some universal constant independent of the time \( T \). Thus iterating (1.15) \( \Rightarrow \) (1.16) then yields a constant that does not depend on \( T \). The theorem follows by letting \( T \to +\infty \).

As the \( L^2 \) norm of \( U(t)\psi_0 \) remains constant, by interpolation, the following holds:

\[ \|U(t)\psi_0\|_{L^p} \lesssim C_p |t|^{-n\left(\frac{1}{p} - \frac{1}{p'}\right)} \|\psi_0\|_{L^{p'}} \quad p \geq 2, \quad \frac{1}{p} + \frac{1}{p'} = 1. \quad (1.17) \]

Our next theorem is about the decay estimates of \( \partial^\alpha U(t)\psi_0 \), where \( \alpha = (\alpha_1, \cdots, \alpha_n) \) is an \( n \)-tuple of nonnegative integers and \( \partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \). We write \( |\alpha| = \alpha_1 + \cdots + \alpha_n \).

\textbf{Theorem 1.8.} Let \( U(t) \) denote the propagator of the equation (1.9). Assume (1.14) holds for \( H_1 \) and \( H_2 \). Let \( V_j \in C^\kappa_0 \) for \( \kappa \) is a positive integer and \( j = 1, 2 \). Moreover, assume that for \( \forall |\beta| \leq \kappa \) and \( j = 1, 2 \), \( \partial^\beta V_j \in L^1(\mathbb{R}^n) \). Then for any initial data \( \psi_0 \in W^{\kappa,3'} \), which is asymptotically orthogonal to the bound states of \( H_j \) \( (j = 1, 2) \) in the sense of Definition 1.2, one has the decay estimates

\[ \|U(t)\psi_0\|_{W^{\kappa,p'}} \lesssim |t|^{-n\left(\frac{1}{p} - \frac{1}{p'}\right)} \|\psi_0\|_{W^{\kappa,p'}}. \quad (1.18) \]

where \( 2 \leq p < \infty \) and \( \frac{1}{p} + \frac{1}{p'} = 1 \).
Remark 1.9. It suffices to prove Theorem 1.8 for \( p > \frac{2n}{n-2} \), because interpolating with Theorem 1.10 which holds under the assumption of Theorem 1.8, we derive Theorem [?] for any \( p \in [2, +\infty] \). \( p > \frac{2n}{n-2} \) guarantees that \( \int_1^\infty |t|^{-n(\frac{1}{2} - \frac{1}{p})} < \infty \). We need to exclude the case \( p = \infty \), since part of our proof relies on singular integrals and we do not know whether or not (1.18) holds for \( p = \infty \).

The second part of this paper is motivated by Graf [Gr]. Graf proved energy boundedness for \( U(t, s) \) by a geometric method, where \( U(t, s) \) is the solution operator corresponding to the time-dependent Schrödinger equation

\[
\frac{1}{t} \partial_t \psi - \frac{\Delta}{2} \psi + \sum_{j=1}^m V_j(x - \vec{v}_j t) \psi = 0,
\]

i.e., \( \psi(t, \cdot) = U(t, s)\psi_0 \). [Gr] proved that \( \|U(t, s)\psi_0\|_{L^1} \) is bounded as \( t \to \infty \) provided that the initial data \( \psi_0 \in H^1(\mathbb{R}^n) \), \( n \geq 1 \). For the higher degree Sobolev norms, J. Bourgain [Bo] proved the following for the general time dependent Hamiltonian \( H(t) \):

Suppose the time dependent potential \( V(x, t) \) is bounded, real and \( \sup_{t \in \mathbb{R}} |V(x, t)| \) is compactly supported. Moreover, for any \( n \)-tuple \( \alpha \)

\[
\sup_{t \in \mathbb{R}} \|D_x^\alpha V(t)\|_{L^\infty} < C_\alpha.
\]

Then for \( \forall \epsilon > 0 \) and \( \kappa > 0 \),

\[
\|U(t, 0)\psi_0\|_{H^\kappa} \leq C_{\epsilon, \kappa} |t|^\epsilon \|\psi_0\|_{H^\kappa} \quad \text{for all } t.
\]

An example (Bo) is given to show that we can not remove the \( |t|^{\epsilon} \) growth for general time dependent potentials. From this paper, it is shown that (1.20) does hold with \( \epsilon = 0 \) for the case of the charge transfer Hamiltonian. More precisely, in Section 4, the time-boundedness of \( \|U(t, s)\psi_0\|_{H^\kappa} \) for Charge Transfer Models is established by the same geometric method as in [Gr] for any real number \( \kappa \). We write \( [\cdot] \) as the least integer no less than \( \cdot \). The precise statement is as follows:

**Theorem 1.10.** Let \( U(t, s) \) be the evolution operator for (1.19), and let \( \kappa \in \mathbb{R} \) and the dimension \( n \geq 1 \). Furthermore, suppose \( V_j \in C_0^{[\kappa]}(\mathbb{R}^n) \), \( (j = 1, 2, \ldots, m) \), i.e. \( V_j \) has derivatives up to degree \( [\kappa] \), which are all continuous and compactly supported. Then for \( \forall t, s \in \mathbb{R} \)

\[
\|U(t, s)\psi_0\|_{H^\kappa} \leq C_\kappa \|\psi_0\|_{H^\kappa},
\]

where \( C_\kappa \) depends on \( \kappa \) and the potentials \( V_j \).

**Remark 1.11.** By interpolation, it clearly suffices to consider the case where \( \kappa \) is an integer. By duality, it suffices to prove the case where \( \kappa \) is a positive integer. Indeed, assuming \( \kappa < 0 \), due to the fact that \( U(t, s) \) is unitary on \( L^2(\mathbb{R}^n) \), we have

\[
\|U(t, s)\psi_0\|_{H^\kappa} = \sup_{\|\phi\|_{H^{-\kappa}} = 1} \langle U(t, s)\psi_0, \phi \rangle_{L^2} = \sup_{\|\phi\|_{H^{-\kappa}} = 1} \langle \psi_0, U(s, t)\phi \rangle_{L^2} \leq C_{-\kappa} \|\psi_0\|_{H^\kappa}.
\]
No assumption is made on the spectra of the subsystems \( H_j \). The assumption of compact support of \( V_j \) is for convenience only and the proof works for sufficiently fast polynomial decay at infinity without essential change. Suppose all assumptions of both Theorem 1.8 and Theorem 1.10 hold, then by interpolation, the estimate (1.18) holds for \( 2 \) at infinity without essential change. Suppose all assumptions of both Theorem 1.8 and Theorem 1.10 hold, then by interpolation, the estimate (1.18) holds for \( 2 < p < \infty \).

**Remark 1.12.** It follows from Duhamel’s formula and Gronwall’s inequality, that

\[
(1.21) \quad \|U(t,s)\psi_0\|_{H^s} \leq C(I)\|\psi_0\|_{H^s} \quad t, s \in I,
\]

for any compact interval \( I \). Therefore, it suffices to prove Theorem 1.10 when \( |t| \) or \( |s| \) is large.

As an important consequence, we apply Theorem 1.8 and Theorem 1.10 to obtain the following asymptotic completeness for the charge transfer model in the \( H^\kappa \) sense:

**Theorem 1.13.** Let \( u_1, \ldots, u_m \) and \( w_1, \ldots, w_\ell \) be the eigenfunctions of \( H_1 = -\frac{\Delta}{2} + V_1(x) \) and \( H_2 = -\frac{\Delta}{2} + V_2(x) \), respectively, corresponding to the negative eigenvalues \( \lambda_1, \ldots, \lambda_m \) and \( \mu_1, \ldots, \mu_\ell \). Assume that \( V_j \in C_n^{n+2\kappa+2}(\mathbb{R}^n) \), \( n \geq 3 \), \( j = 1, 2 \), and that 0 is neither an eigenvalue nor a resonance of \( H_1, H_2 \), where \( \kappa \) is a nonnegative integer. Then for any initial data \( \psi_0 \in H^2 \), the solution \( U(t)\psi_0 \) of the charge transfer problem (1.9) can be written in the form

\[
U(t)\psi_0 = \sum_{r=1}^{m} A_r e^{-it\lambda_r} u_r + \sum_{k=1}^{\ell} B_k e^{-it\mu_k} g_{-\varepsilon_1}(t) w_k + e^{-it\frac{\Delta}{2}} \phi_0 + R(t),
\]

for some choice of the constants \( A_r, B_k \) and the function \( \phi_0 \in H^\kappa \). The remainder term \( R(t) \) satisfies the estimate

\[
\|R(t)\|_{H^\kappa} \to 0, \quad \text{as} \ t \to \infty.
\]

**Remark 1.14.** The above theorem holds for \( m \) potentials. We are not aiming to give the optimal regularity condition on the potentials. The theorem is equivalent to claiming that \( H^\kappa(\mathbb{R}^n) \) is the sum of the ranges of the wave operators \( \Omega_l^- \), \( l = 0, 1, 2 \) defined in Section 6.1. [Gr] proved that the ranges of the wave operators are orthogonal to each other in the \( L^2 \) sense. Therefore, \( H^\kappa(\mathbb{R}^n) \) again is a direct sum of \( \Omega_l^- \) \( (H^\kappa) \).

## 2 Cancellation Lemma

The first ingredient of our proof is the notion of cancellation. In this section, \( U(t) \) will denote the evolution operator of (1.9) or (1.3). It is clear from their proofs that the following lemmas also hold for general time dependent Hamiltonian \( H_0 + V(t) \).

**Lemma 2.1.**

\[
(2.1) \quad \sup_{-\infty < t < \infty} \|e^{it\Delta} V e^{-it\Delta}\|_{p \to p} \leq \|\hat{V}\|_1,
\]

where \( p \in [1, \infty] \) and \( \cdot \|_{p \to p} \) means the operator norm from \( L^p \) to \( L^p \). For the proof of the lemma, just notice that equation (1.6) implies \( [e^{it\Delta} e^{ic\xi} e^{-it\Delta} f](x) = g_{-\varepsilon}(2t)f(x) = e^{-it|\zeta|^2} e^{ix\zeta} f(x - 2\zeta t) \).
Lemma 2.2. Suppose $t, s \in \mathbb{R}$, then we have the following:

\begin{equation}
\sup_{r \in \mathbb{R}} \| e^{-i(t-s)H_0} V(r) U(s) \|_{1 \to \infty} < |t|^{-\frac{\gamma}{2}} C M e^{M|s|},
\end{equation}

where $M = \max_{r \in \mathbb{R}} \| \hat{V}(r) \|_1 < \infty$.

Proof. Let’s write $\Psi(t, s) := \sup_{r \in \mathbb{R}} \| e^{-i(t-s)H_0} V(r) U(s) \|_{1 \to \infty}$. Without loss of generality, we suppose that $s > 0$. By Duhamel’s formula,

\[ e^{-i(t-s)H_0} V(r) U(s) = e^{-i(t-s)H_0} V(r) \{ e^{-i s H_0} - i \int_0^s e^{-i(s-\tau)H_0} V(\tau) U(\tau) d\tau \}, \]

it follows that

\[ \| e^{-i(t-s)H_0} V(r) U(s) \|_{1 \to \infty} \]
\[ \leq \| e^{-i(t-s)H_0} V(r) e^{-i s H_0} \|_{1 \to \infty} + \int_0^s \| e^{-i(t-s)H_0} V(r) e^{-i(s-\tau)H_0} V(\tau) U(\tau) \|_{1 \to \infty} d\tau \]
\[ \leq C \| \hat{V}(r) \|_1 |t|^{-\frac{\gamma}{2}} + \| \hat{V}(r) \|_1 \int_0^s \| e^{-i(t-\tau)H_0} V(\tau) U(\tau) \|_{1 \to \infty} d\tau \]
\[ \leq C M |t|^{-\frac{\gamma}{2}} + M \int_0^s \Psi(t, \tau) d\tau. \]

Taking the supremum over $r$, we get $\Psi(t, s) \leq C M |t|^{-\frac{\gamma}{2}} + M \int_0^s \Psi(t, \tau) d\tau$. By Gronwall’s inequality,

\[ \Psi(t, s) \leq C M |t|^{-\frac{\gamma}{2}} e^{M s}. \]

Note that the lemma still holds with other constants $C$ and $M$ on the right-hand side if we replace $V(r)$ with $V_j(r)$ or replace $U(s)$ with another evolution, say $e^{-i s H_j}$. Another observation is that the lemma can be generalized to the following by the same proof:

\begin{equation}
\sup_{r \in \mathbb{R}} \| e^{-i(t-s)H_0} V(r) U(s) \psi_0 \|_{p'} \lesssim |t|^{-\gamma} M e^{M s} \| \psi_0 \|_p
\end{equation}

where $\gamma = n(\frac{1}{2} - \frac{1}{p})$ and $2 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. This will be useful in Section 4.

Corollary 2.3. Suppose $U(t)$ is the evolution operator of \textcolor{red}{1.2} or \textcolor{red}{1.3}. Assume $t > 0$, then

\begin{equation}
\| U(t) \|_{p' \to p} \lesssim t^{-n(\frac{1}{2} - \frac{1}{p})} e^{M t} \frac{1}{p} + \frac{1}{p'} = 1, 2 \leq p \leq \infty
\end{equation}

Proof. By Duhamel’s formula, $U(t) = e^{-i t H_0} - i \int_0^t e^{-i(t-\tau)H_0} V(\tau) U(\tau) d\tau$. Write $\gamma = n(\frac{1}{2} - \frac{1}{p})$, then by Lemma 2.2 we have

\[ \| U(t) \|_{p' \to p} \leq C t^{-\gamma} + \int_0^t \Psi(t, \tau) d\tau \leq C t^{-\gamma} + \int_0^t C t^{-\gamma} M e^{M \tau} d\tau \leq C t^{-\gamma} e^{M t}. \]

\[ \square \]
From the corollary, the bootstrap assumption (1.15) holds for any time $T$ if we take $C_0 = Ce^{MT}$.

**Lemma 2.4.** Suppose $m \geq 1$ and $\varepsilon > 0$. If $u_1, u_2, \ldots, u_m$ are either all positive or all negative, satisfying $|\sum_{j=1}^m u_j| > \varepsilon$, then there exists a constant $C = C(m, \varepsilon)$ such that

\begin{equation}
\| \prod_{j=1}^{m-1} (e^{iu_j H_0} V(s_j)) e^{iu_m H_0} \|_{1 \to \infty} \leq CM^{m-1} \prod_{j=1}^m (u_j)^{-\frac{\varepsilon}{2}}
\end{equation}

(2.5)

\begin{equation}
\| \prod_{j=1}^{m-1} (e^{iu_j H_0} V(s_j)) U(u_m) \|_{1 \to \infty} \leq CM^{m-1} \prod_{j=1}^m (u_j)^{-\frac{\varepsilon}{2}} e^{Mu_m}
\end{equation}

(2.6)

where $s_j$ is any real number and $M = \sup_{s \in \mathbb{R}} (\|V(s)\|_1 + \|\hat{V}(s)\|_1)$.

**Proof.** The first inequality is from [JSS]. Assume that $u_1, u_2, \ldots, u_m$ are all positive without loss of generality. We apply the dispersive estimate for $e^{iu_j H_0}$ repeatedly and the left-hand side is dominated by $CM^{m-1} \prod_{j=1}^m u_j^{-\frac{\varepsilon}{2}}$, which is dominated by the right-hand side up to a constant, provided each $u_j > \varepsilon$. If some $u_j \leq \varepsilon$, it is inefficient to use a dispersive estimate for $e^{iu_j H_0}$. Instead, we apply the cancellation lemma 2.1 and obtain

\[ e^{iu_j H_0} V(s_j) e^{iu_{j+1} H_0} = \int e^{iu_j H_0} e^{i\xi \zeta} e^{-iu_j H_0} V(s_j)(\zeta) d\zeta e^{i(u_{j+1} + u_j) H_0}. \]

where $e^{iu_j H_0} e^{i\xi \zeta} e^{-iu_j H_0}$ is the Galilean transform $g_{-\xi}(-u_j)$ according to (1.6). If again $u_j + u_{j+1} < \varepsilon$, we can repeat this procedure until $u_{j-l} + \cdots + u_j + \cdots + u_{j+k} > \varepsilon$ which always happens because $|\sum_{j=1}^m u_j| > \varepsilon$. Then we apply the dispersive estimate to obtain the inequality.

We sketch the proof of the second equation. When $m = 1$, it is just (2.4) provided that $u_m > \varepsilon$. When $m = 2$, if $u_1 > \frac{\varepsilon}{2}$ and $u_2 > \frac{\varepsilon}{2}$,

\[ \| (e^{iu_1 H_0} V(s_1)) U(u_2) \|_{1 \to \infty} \lesssim |u_1|^{-\frac{\varepsilon}{2}} \|V(s_1) U(u_2)\|_{1 \to 1} \]

\[ \lesssim |u_1|^{-\frac{\varepsilon}{2}} \|U(u_2)\|_{1 \to \infty} \]

\[ \lesssim |u_1|^{-\frac{\varepsilon}{2}} |u_2|^{-\frac{\varepsilon}{2}} e^{Mu_2} \lesssim \langle u_1 \rangle^{-\frac{\varepsilon}{2}} \langle u_2 \rangle^{-\frac{\varepsilon}{2}} e^{Mu_2} \]

If $u_1 \leq \frac{\varepsilon}{2}$ or $u_2 \leq \frac{\varepsilon}{2}$, we apply Lemma 2.2.

\[ \| e^{iu_1 H_0} V(s_1) U(u_2) \|_{1 \to \infty} \lesssim (|u_1| + |u_2|)^{-\frac{\varepsilon}{2}} e^{Mu_2} \lesssim \langle u_1 \rangle^{-\frac{\varepsilon}{2}} \langle u_2 \rangle^{-\frac{\varepsilon}{2}} e^{Mu_2} \]

The case where $m > 2$ follows exactly as the first inequality using Lemma 2.1.

**3 Proof of the decay estimates**

Theorem 1.7 will be proved in this section by a bootstrap argument. By Corollary 2.3, we can assume that $t$ is large enough in Theorem 1.7. More precisely, $t$ will be bigger than any constant appearing in our estimate, except the bootstrap constant $C_0$ in (1.8). By assumption, $H_1, H_2$ can only admit finitely many negative eigenvalues. Let $\alpha > 0$ satisfy: $-\alpha$ is bigger than any eigenvalue
of $H_1, H_2$. For technical reasons, we will assume that the initial data $\psi$ belong to $L^1 \cap L^2$ and employ the following bootstrap argument:

Specifically, we will show that

$$
(3.1) \quad \|U(t)\psi_0\|_{L^\infty} \leq C_0 |t|^{-\frac{n}{2}} (\|\psi_0\|_{L^1} + e^{-\frac{nT}{2}} \|\psi_0\|_{L^2}) \quad \text{for all} \quad 0 \leq t \leq T,
$$

implies that

$$
(3.2) \quad \|U(t)\psi_0\|_{L^\infty} \leq \frac{C_0}{2} |t|^{-\frac{n}{2}} (\|\psi_0\|_{L^1} + e^{-\frac{nT}{2}} \|\psi_0\|_{L^2}) \quad \text{for all} \quad 0 \leq t \leq T,
$$

provided that $\frac{C_0}{2}$ remains larger than some constant that does not depend on $T$. The logic here is that for arbitrary but fixed $T$, the assumption \eqref{3.1} can be made to hold for some $C_0$ depending on $T$, because of Corollary \ref{2.3}. Iterating the implication \eqref{3.1} $\implies$ \eqref{3.2} yields a constant that does not depend on $T$. So we can let $T \to +\infty$ to eliminate $\|\psi_0\|_{L^2}$ on the right-hand side. Since $L^1 \cap L^2$ is dense in $L^1$ and $U(t)$ is a linear operator, we get the dispersive estimate \eqref{1.14} for any initial data $\psi_0 \in L^1$. To simplify the notation, we write $\|\psi_0\|_{L^1} + e^{-\frac{nT}{2}} \|\psi_0\|_{L^2}$ as $\|\psi_0\|^{(T)}_L$ or $\|\psi_0\|$.

We proceed by expanding $U(t)$ via Duhamel’s formula with respect to the free evolution $H_0$:

$$
(3.3) \quad U(t)\phi_0 = e^{-itH_0}\phi_0 - i \int_0^t e^{-i(t-s)H_0}V(s)U(s)\psi_0 \, ds
$$

$$
= e^{-itH_0}\psi_0 - i \int_0^t e^{-i(t-s)H_0}V(s)e^{-isH_0}\psi_0 \, ds
$$

$$
- \int_0^t \int_s^t e^{-i(t-s)H_0}V(s)e^{-i(s-\tau)H_0}V(\tau)U(\tau)\psi_0 \, d\tau \, ds
$$

$$
(3.4)
$$

Note that $\|e^{-itH_0}\psi_0\|_\infty \lesssim |t|^{-\frac{n}{2}} \|\psi_0\|_1$. For the second term in \eqref{3.4}, we divide the integration interval $(0, t)$ into three pieces and handle them by means of the cancellation lemma. Firstly,

$$
\| \int_0^1 e^{-i(t-s)H_0}V(s)e^{-isH_0}\psi_0 \, ds \|_\infty \lesssim |t|^{-\frac{n}{2}} \sup_s \|e^{isH_0}V(s)e^{-isH_0}\|_{1 \to 1} \|\psi_0\|_1 \lesssim |t|^{-\frac{n}{2}} \|\psi_0\|_1.
$$

Similarly, we have

$$
\| \int_{t-1}^t e^{-i(t-s)H_0}V(s)e^{-isH_0}\psi_0 \, ds \|_\infty \lesssim |t|^{-\frac{n}{2}} \sup_s \|e^{isH_0}V(s)e^{-isH_0}\|_{1 \to 1} \|\psi_0\|_1 \lesssim |t|^{-\frac{n}{2}} \|\psi_0\|_1.
$$

The third piece is

$$
\| \int_{t-1}^1 e^{-i(t-s)H_0}V(s)e^{-isH_0}\psi_0 \, ds \|_\infty \lesssim \int_{t-1}^1 |t-s|^{-\frac{n}{2}} \sup_s \|V(s)\|_1 |s|^{-\frac{n}{2}} \, ds \|\psi_0\|_1 \lesssim |t|^{-\frac{n}{2}} \|\psi_0\|_1,
$$

where we observed that

$$
(3.5) \quad \int_{t-1}^1 |t-s|^{-\frac{n}{2}} |s|^{-\frac{n}{2}} \, ds \lesssim t^{-\frac{n}{2}} \quad \text{given} \quad n \geq 3.
$$

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The third term in (3.4) is

\[
\int_0^t ds \int_0^s d\tau \, e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V(\tau) U(\tau) \psi_0.
\]

We will decompose the domain of integration \( \int_0^t ds \int_0^s d\tau \) into several pieces and treat each piece separately. We fix \( A > 0 \) as a large constant and \( \epsilon > 0 \) as a small constant. Write \( \min\{s, A\} = s \wedge A \). Then Lemma 2.4 and (3.5) implies that

\[
\| \int_0^t ds \int_0^{s \wedge A} d\tau \, e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V(\tau) U(\tau) d\tau ds \|_{1 \to \infty} \leq \int_0^t ds \int_0^{s \wedge A} d\tau \, \langle t-s \rangle^{-\frac{n}{2}} \langle s-\tau \rangle^{-\frac{n}{2}} \langle -\frac{n}{2} \rangle^A M \\
\leq t^{-\frac{n}{2}}.
\]

By \( \| \cdot \|_{1 \to \infty} \), we mean the operator norm from \( L^1 \) to \( L^\infty \). However when we apply the bootstrap assumption, \( \| \psi_0 \|_{L^1} \) has to be modified to \( \| \psi_0 \|_{L^1} + e^{-\frac{M}{2\epsilon}} \| \psi_0 \|_{L^2} := \| \psi_0 \|_{\epsilon} \).

An application of Lemma 2.4 and the bootstrap assumption show that

\[
\| \int_{t - \epsilon}^t ds \int_{t - \epsilon}^s d\tau \, e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V(\tau) U(\tau) d\tau ds \psi_0 \|_{L^\infty} \leq \int_{t - \epsilon}^t ds \int_{t - \epsilon}^s d\tau \, \| e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} \|_{1 \to \infty} \| V(\tau) U(\tau) \psi_0 \|_1 d\tau ds \\
\leq \int_{t - \epsilon}^t d\tau \int_\tau^t ds |t-\tau|^{-\frac{n}{2}} \max_{\tau \in (t-\epsilon, t)} \| U(\tau) \psi_0 \|_{\infty}.
\]

If \( n = 3 \), then the above is dominated by

\[
\leq \int_{t - \epsilon}^t d\tau |t-\tau|^{-\frac{3}{2}} C_0 t^{-\frac{3}{2}} \| \psi_0 \| \leq \sqrt{\epsilon} C_0 t^{-\frac{3}{2}} \| \psi_0 \|.
\]

Taking \( \epsilon \) small enough, the above term can be dominated by \( \frac{1}{100} C_0 t^{-\frac{3}{2}} \| \psi_0 \| \). When \( n > 3 \), we need to expand \( U(t) \) further to remove the singularity of \( |t-\tau|^{-\frac{n}{2}} \) at \( \tau = t \), (see [JSS] Section 2 for details). The following is another piece of (3.6):

\[
\| \int_A^{t-A} \int_A^{s} e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V(\tau) U(\tau) \psi_0 d\tau ds \|_{L^\infty} \leq \frac{1}{100} C_0 \| \psi_0 \| t^{-\frac{n}{2}}.
\]
where \( \kappa_A < \int_A^{+\infty} ds(s)^{-\frac{\alpha}{2}} \to 0 \) as \( A \to \infty \). Lemma 2.4 and the bootstrap assumption are applied in turn in the above. The last line of above inequality holds provided that \( A \) is large enough. By Corollary 2.3, we can assume \( t >> A \). Similarly, the following piece in (3.6) also requires that \( A \) is large:

\[
\sum_{j=1}^m \int_{t-A}^t ds \int_{s-A}^{s-(t-\epsilon)} d\tau (t-s)^{-\frac{\alpha}{2}} (s-\tau)^{-\frac{\alpha}{2}} \| U(\tau)\psi_0 \|_\infty \leq \frac{1}{100} C_0 \| \psi_0 \| t^{-\frac{\alpha}{2}}
\]

So By what remains in (3.6) is

\[
(3.7) \quad \sum_{j=1}^m \int_{t-A}^t ds \int_{s-A}^{s-(t-\epsilon)} d\tau e^{-i(t-s)H_0} V(s)e^{-i(s-\tau)H_0} V_j(\cdot-\tau \vec{v}_1) U(\tau)\psi_0
\]

For the term containing \( V_j \) in (3.7), \( U(\tau) \) will be expanded with respect to \( H_j \) by Duhamel’s formula. Abusing notation, we will write \( V_j(\cdot-\tau \vec{v}_1) \) as \( \tilde{V}(\tau) \). In the following, we only deal with the term containing \( \tilde{V} \) which will be decomposed into two parts by \( U(\tau) = P_b(H_1,\tau) U(\tau) + P_c(H_1,\tau) U(\tau) \).

### 3.1 Bound States

**Proposition 3.1.** Let \( \psi(t,x) = (U(t)\psi_0)(x) \) be a solution of (1.9) which is asymptotically orthogonal to the bound states of \( H_j, j = 1,2 \) in the sense of Definition 1.2. Provided the bootstrap assumption \( 3.1 \), we have for any \( t \in (0,T) \)

\[
(3.8) \quad \| P_b(H_1,t) U(t)\psi_0 \|_\infty \lesssim C_0 e^{-\frac{\alpha t}{2}} t^{-\frac{\alpha}{2}} (\| \psi_0 \|_{L^1} + e^{-\frac{\alpha t}{2}} \| \psi_0 \|_{L^2}),
\]

where \( C_0 \) is the constant in the bootstrap assumption.

**Proof.** Let \( \tilde{U}(t) := g(\psi_1(t) U(t) \) and \( \phi(t) = \tilde{U}(t)\psi_0 \). Then \( \phi(t) \) solves

\[
\frac{1}{t} \partial_t \phi - \frac{1}{2} \Delta \phi + V(\cdot + t \vec{v}_1) \phi = 0,
\]

\[
\phi|_{t=0}(x) = ((g(\psi_1(0)\psi_0))(x),
\]

Then \( \| P_b(H_1,t) U(t)\psi_0 \|_\infty = \| P_b(H_1) \tilde{U}(t)\phi_0 \|_\infty \) so without loss of generality, we can assume that \( \vec{v}_1 \) is the zero vector. Suppose that the bound states of \( H_1 \) are \( u_1, u_2, \ldots, u_i \) and we decompose

\[
(3.10) \quad U(t)\psi_0 = \sum_{i=1}^l a_i(t) u_i + \psi_1(t,x)
\]

with respect to \( H_1 \) so that \( P_c(H_1)\psi_1 = \psi_1 \) and \( P_b(H_1)\psi_1 = 0 \). By the asymptotic orthogonality assumption,

\[
\sum_{i=1}^l |a_i(t)|^2 \to 0 \text{ as } t \to \infty.
\]
Substituting (3.10) into (1.9) yields

\[
\frac{1}{i} \partial_t \psi_1 - \frac{1}{2} \Delta \psi_1 + V_1 \psi_1 + V_2 (\cdot - t e_1) \psi_1 + \sum_{j=1}^l \left[ \frac{1}{i} \tilde{a}_j(t) u_j - \frac{1}{2} \Delta u_j a_j(t) + V_1 u_j a_j(t) + V_2 (\cdot - t e_1) u_j a_j(t) \right] = 0.
\]

(3.11)

Since \( P_c(H_1) \psi_1 = \psi_1 \), we have

\[
(- \frac{1}{2} \Delta + V_1) \psi_1 = H_1 \psi_1 = P_c(H_1) H_1 \psi_1, \quad \partial_t \psi_1 = P_c(H_1) \partial_t \psi_1.
\]

In particular,

\[
P_b(H_1) \left( \frac{1}{i} \partial_t \psi_1 - \frac{1}{2} \Delta \psi_1 + V_1 \psi_1 \right) = 0.
\]

Thus taking an inner product of the equation (3.11) with \( u_\kappa \) and using the fact that \( \langle u_\kappa, u_j \rangle = \delta_{j\kappa} \) as well as the identity

\[
- \frac{1}{2} \Delta u_j + V_1 u_j = \lambda_j u_j,
\]

we obtain the ODE

\[
\frac{1}{i} \dot{a}_\kappa(t) + \lambda_\kappa a_\kappa(t) + \langle V_2 (\cdot - t e_1) \psi_1, u_\kappa \rangle + \sum_{j=1}^m a_j(t) \langle V_2 (\cdot - t e_1) u_j, u_\kappa \rangle = 0
\]

for each \( a_\kappa \) with the condition that

\[
a_\kappa(t) \to 0 \text{ as } t \to +\infty.
\]

Recall that \( u_\kappa \) is an eigenfunction of \( H_1 = - \frac{1}{2} \Delta + V_1 \) with eigenvalue \( \lambda_\kappa < 0 \). It is well-known (see e.g. Agmon [Ag]) that such eigenfunctions are exponentially localized, i.e.,

\[
\int_{\mathbb{R}^n} e^{2\alpha |x|} |u_\kappa(x)|^2 \, dx \leq C = C(V_1, n) < \infty \text{ for some positive } \alpha.
\]

(3.12)

Therefore, the assumption that \( V_2 \) has compact support implies

\[
\| V_2 (\cdot - t e_1) u_\kappa \|_2 \lesssim e^{-\alpha t} \text{ for all } t \geq 0.
\]

(3.13)

The implicit constant in (3.13) depends on the size of the support of \( V_2 \) and \( \| V_2 \|_\infty \).

By the bootstrap assumption, \( f_\kappa(t) := \langle V_2 (\cdot - t e_1) \psi_1, u_\kappa \rangle \) satisfies

\[
|f_\kappa(t)| \lesssim \| \psi_1 \|_\infty \| V_2 (\cdot - t e_1) u_\kappa \|_1 \lesssim e^{-\alpha t} \| \psi_1 \|_\infty
\]

\[
\lesssim e^{-\alpha t} \| (\text{Id} - P_b(H_1)) U(t) \psi_0 \|_\infty
\]

\[
\lesssim e^{-\alpha t} t^{-\frac{n}{4}} C_0(\| \psi_0 \|_{L^1} + e^{-\frac{\alpha t}{2}} \| \psi_0 \|_{L^2}) + e^{-\alpha t} \sum_{i=1}^l |a_i(t)| \| u_i \|_\infty,
\]

(3.14)
where \( t \in (0, T) \). Notice that (3.14) fails for \( t > T \) because the bootstrap assumption only applies to \( 0 < t < T \). Instead, we have the following for \( t > T \):

\[
|f_\kappa(t)| < \|V_2(\cdot - te_1)u_\kappa\|_2\|\psi_1\|_2 \lesssim e^{-\alpha t}\|\psi_0\|_2.
\]

(3.15)

In view of (3.14), \( a_\kappa \) solves the equation

\[
\frac{1}{t} \ddot{a}_\kappa(t) + \lambda_\kappa a_\kappa(t) + \sum_{j=1}^m a_j(t) C_{\kappa j}(t) + f_\kappa(t) = 0
\]

where \( C_{\kappa j}(t) = C_{\kappa j}(t) = \langle V_2(\cdot - te_1)u_j, u_\kappa \rangle \). By (3.13), \( \max_{j, \kappa} |C_{\kappa j}(t)| \lesssim e^{-\alpha t} \). Solving (3.16) explicitly, we obtain

\[
\vec{a}(t) = ie^{-i \int_0^t B(s) ds} \int_t^\infty e^{i \int_0^\tau B(\tau) d\tau} \vec{f}(s) ds,
\]

where \( B_{\kappa j}(t) = \lambda_j \delta_{\kappa j} + C_{\kappa j}(t) \).

By (3.14), (3.15) and the unitarity of \( e^{i \int_0^\tau B(\tau) d\tau} \), we conclude that

\[
|\vec{a}(t)| \lesssim \int_t^T \int_0^t |\vec{f}(s)| ds + \int_t^T e^{-\alpha s} C_0 \|\psi_0\|_2 + \int_t^\infty \int_T e^{-\alpha s} \sum_{j=1}^l |a_j(s)| \|u_i\|_\infty ds + \int_T^\infty e^{-\alpha s} \|\psi_0\|_L^2.
\]

Choose a large constant \( t_0 > 0 \) such that for all \( t_1 > t_0 \), the following holds:

\[
\int_{t_1}^T e^{-\alpha s} \sum_{j=1}^l |a_j(s)| \|u_i\|_\infty ds \leq \frac{1}{2} \sup_{t_1 < t < T} |\vec{a}(t)|,
\]

then

\[
\sup_{t_1 < t < T} |\vec{a}(t)| \lesssim e^{-\alpha t_1 t_{1}^{-\frac{\gamma}{2}} C_0 \|\psi_0\|_2 + e^{-\alpha T} \|\psi_0\|_L^2} \lesssim e^{-\frac{\alpha t}{M} t_{1}^{-\frac{\gamma}{2}} C_0 \|\psi_0\|_2}
\]

\[
\square
\]

Remark 3.2. In the above proof, if we change (3.14) into the following:

\[
|f_\kappa(t)| \lesssim \|\psi_1(t)\|_p \|V_2(\cdot - te_1)u_\kappa\|_{p'} \lesssim e^{-\alpha t} \|\psi_1(t)\|_p
\]

\[
\lesssim e^{-\alpha t} \| (\operatorname{Id} - P_\gamma(H_1)) U(t) \psi_0 \|_p
\]

\[
\lesssim e^{-\alpha t} \gamma C_0 (\|\psi_0\|_{p'} + e^{-\alpha T} \|\psi_0\|_L^2) + e^{-\alpha t} \sum_{i=1}^l |a_i(t)| \|u_i\|_p
\]

where \( \gamma = n(\frac{1}{2} - \frac{1}{p}) > 1 \), and follow the same arguments, we see that for large \( t \),

\[
\|P_\gamma(H_1, t) U(t) \psi_0\|_p \lesssim t^{-\gamma} C_0 e^{-\frac{\alpha T}{M}} (\|\psi_0\|_{p'} + e^{-\alpha T} \|\psi_0\|_L^2).
\]
If the potential $V_1$ is smooth enough, it is known (see e.g. [Ag]) that the bound state $u_j$ of $H_1$ is differentiable. Moreover, its derivatives decay exponentially at infinity. Thus,

$$\|\partial P_b(H_1, t)U(t)\psi_0\|_p \leq \sum_{i=1}^l |a_i(t)||\partial u_i|_p \lesssim t^{-\gamma}C_0e^{-\frac{\alpha T}{2}}(\|\psi_0\|_{\infty} + e^{-\frac{\alpha T}{2}}\|\psi_0\|_{L^2}).$$

In addition, the above claims hold with $H_1$ replaced by $H_j$, $j = 2, \ldots, m$. These results will be used to prove Theorem 1.8 in Section 4. □

With Proposition 5.1, the $P_b(H_1, \tau)U(\tau)$ part of (3.7) can be estimated by the following:

$$\int_{t-A}^{t} ds \int_{s-A}^{s+e(t-\epsilon)} d\tau \|e^{-i(\tau-s)H_0V(s)}e^{-i(s-\tau)H_0V_1(\tau)}P_b(H_1, \tau)U(\tau)\psi_0\|_{\infty}$$

$$\lesssim \int_{t-A}^{t} ds \int_{s-A}^{s+e(t-\epsilon)} d\tau \langle t-s \rangle^{-\frac{n}{2}} \langle s-\tau \rangle^{-\frac{n}{2}} \|V_1(\tau)P_b(H_1, \tau)U(\tau)\psi_0\|_{1}$$

$$\lesssim A^2 \sup_{\tau \in (t-2A, t)} \|P_b(H_1, \tau)U(\tau)\psi_0\|_{\infty}$$

$$< \frac{C_0}{100} t^{-\frac{n}{2}} (\|\psi_0\|_{L^1} + e^{-\frac{\alpha T}{2}} \|\psi_0\|_{L^2}).$$

For the $P_c(H_1, \tau)U(\tau)$ part of (3.7), we need to apply Duhamel’s formula again and expand (3.7) further with respect to $H_1$. We assume that $\bar{v}_1 = 0$ and $m = 2$ to simplify our notation. Specifically, we plug the following

$$P_c(H_1, \tau)U(\tau) = P_c(H_1)U(\tau) = P_c(H_1)e^{-i\tau H_1} - iP_c(H_1) \int_0^\tau e^{-i(\tau-r)H_1}V_2(r)U(r) \, dr$$

into (3.7). For the term containing $P_c(H_1)e^{-i\tau H_1}$, we apply the dispersive decay for $P_c(H_1)e^{-i\tau H_1}$:

$$\int_{t-A}^{t} ds \int_{s-A}^{s+e(t-\epsilon)} d\tau \|e^{-i(\tau-s)H_0V(s)}e^{-i(s-\tau)H_0V_1(\tau)}P_c(H_1)e^{-i\tau H_1}\psi_0\|_{\infty}$$

$$\lesssim \int_{t-A}^{t} ds \int_{s-A}^{s+e(t-\epsilon)} d\tau \langle t-s \rangle^{-n/2} \langle s-\tau \rangle^{-n/2} \|\psi_0\|_{1} \lesssim t^{-n/2} \|\psi_0\|_{1}.$$
To simplify the notation, we will write $A_1$ as $A$. Our goal is to estimate each term in (3.20). The second term of (3.20) is estimated as follows:

\[
\int_{t-A}^t ds \int_{s-A}^{s+(t-\epsilon)} d\tau \int_\delta^\tau d\tau' \|e^{-i(t-s)H_0}V(s)e^{-i(s-\tau)H_0}V_1(e^{-i(\tau-r)H_1}V_2(r)U(r)\psi_0\|_\infty \\
\lesssim \int_{t-A}^t ds \int_{s-A}^{s+(t-\epsilon)} d\tau \int_\delta^\tau d\tau' \langle t-s \rangle^{-n/2} \langle s-\tau \rangle^{-n/2} \langle \tau-r \rangle^{-n/2} C_0 \|\psi_0\|_1 \\
\lesssim t^{-n/2} \|\psi_0\|_1.
\]

The implicit constant above depends on $A, \delta$.

The third term of (3.20) is estimated as follows:

\[
\int_{t-A}^t ds \int_{s-A}^{s+(t-\epsilon)} d\tau \int_\delta^\tau d\tau' \|e^{-i(t-s)H_0}V(s)e^{-i(s-\tau)H_0}V_1(e^{-i(\tau-r)H_1}V_2(r)U(r)\psi_0\|_\infty \\
\lesssim \int_{t-A}^t ds \int_{s-A}^{s+(t-\epsilon)} d\tau \int_\delta^\tau d\tau' \langle t-s \rangle^{-n/2} \langle s-\tau \rangle^{-n/2} \langle \tau-r \rangle^{-n/2} \|U(r)\psi_0\|_\infty \\
\lesssim \int_{t-A}^t ds \int_{s-A}^{s+(t-\epsilon)} d\tau \int_\delta^\tau d\tau' \langle t-s \rangle^{-n/2} \langle s-\tau \rangle^{-n/2} \langle \tau-r \rangle^{-n/2} C_0 \|\psi_0\|_1 \\
\lesssim t^{-n/2} C_0 \kappa_A \|\psi_0\| \leq \frac{1}{100} C_0 t^{-n/2} \|\psi_0\|,
\]

where $\kappa_A \to 0$ as $A \to \infty$. So the above inequality holds for large enough $A$.

For the fourth term in (3.20), we have:

\[
\int_{t-A}^t ds \int_{s-A}^{s+(t-\epsilon)} d\tau \int_{\tau-\delta}^\tau d\tau' \|e^{-i(t-s)H_0}V(s)e^{-i(s-\tau)H_0}V_1(e^{-i(\tau-r)H_1}V_2(r)U(r)\psi_0\|_\infty \\
\lesssim \int_{t-A}^t ds \int_{s-A}^{s+(t-\epsilon)} d\tau \int_{\tau-\delta}^\tau d\tau' \langle t-s \rangle^{-n/2} \langle s-\tau \rangle^{-n/2} \|V_1(e^{-i(\tau-r)H_1}V_2(r)U(r)\psi_0\|_1 \\
\lesssim \int_{t-A}^t ds \int_{s-A}^{s+(t-\epsilon)} d\tau \int_{\tau-\delta}^\tau d\tau' \langle t-s \rangle^{-n/2} \langle s-\tau \rangle^{-n/2} \|U(r)\psi_0\|_\infty \\
\lesssim \int_{t-A}^t ds \int_{s-A}^{s+(t-\epsilon)} d\tau \int_{\tau-\delta}^\tau d\tau' \langle t-s \rangle^{-n/2} \langle s-\tau \rangle^{-n/2} C_0 \|\psi_0\| \\
\lesssim t^{-n/2} C_0 \kappa_\delta \|\psi_0\| \leq \frac{1}{100} C_0 t^{-n/2} \|\psi_0\|,
\]

where $\kappa_\delta \to 0$ as $\delta \to 0$. So the above inequality holds for $\delta$ small enough.

For the $\int_0^\delta d\tau$ part of (3.20), we expand

\[
e^{-i(\tau-r)H_1} = e^{-i(\tau-r)H_0} - i \int_0^{t-r} e^{-i(\tau-r-\beta)H_1}V_1 e^{-i\beta H_0} d\beta.
\]

Here we put $H_0$ after $H_1$ in the integral because we want $H_0$ to appear immediately before $U(r)$ and apply Lemma 2.4. Substitute this expansion into the $\int_0^\delta d\tau$ part of (3.20) and we get two
either case, we can conclude that

\begin{equation}
\int_{t-A}^{t} ds \int_{s-A}^{s+(t-\epsilon)} d\tau \int_{0}^{\delta} dr e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V_1(\tau) P_c(H_1) e^{-i(\tau-r)H_0} V_2(r) U(r) \psi_0
\end{equation}

Notice that \( P_c(H_1) = \text{Id} - P_0(H_1) \), and because \( \|P_0(H_1)\|_{p \to p} \) is bounded, \( \|P_c(H_1)\|_{p \to p} \) is bounded as well. Therefore the \( L^\infty \) norm of (3.21) is estimated as follows:

\[ \lesssim \int_{t-A}^{t} ds \int_{s-A}^{s+(t-\epsilon)} d\tau \int_{0}^{\delta} dr (t-s)^{-n/2} (s-\tau)^{-n/2} (\tau-r)^{-n/2} e^{Mr} \|\psi_0\|_1 \lesssim t^{-n/2} \|\psi_0\|_1. \]

The second term of the integral part of (3.20) after substitution is

\begin{equation}
\int_{t-A}^{t} ds \int_{s-A}^{s+(t-\epsilon)} d\tau \int_{0}^{\delta} dr e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V_1(\tau) P_c(H_1) \int_{0}^{\tau-r} e^{-i(\tau-r-\beta)H_1} V_1 e^{-i\beta H_0} d\beta V_2(r) U(r) \psi_0
\end{equation}

Decompose \( \int_{\tau-r}^{\tau} d\beta \) so we can rewrite (3.22) as \( J_1 + J_2 + J_3 \), where \( J_1, J_2 \) and \( J_3 \) correspond to \( \int_{\tau-r}^{\delta} d\beta, \int_{\tau-r}^{\tau-r-1} d\beta \) and \( \int_{\tau-r-1}^{\tau-r} d\beta \) respectively.

We proceed to estimate \( J_1 \) as follows:

\[ \int_{0}^{\delta} dr \int_{0}^{\delta} d\beta \|P_c(H_1) e^{-i(\tau-r-\beta)H_1} V_1 e^{-i\beta H_0} V_2(r) U(r) \psi_0\|_\infty \]

\[ \lesssim \int_{0}^{\delta} dr \int_{0}^{\delta} d\beta (\tau - r - \beta)^{-n/2} e^{-i\beta H_0} V_2(r) U(r) \psi_0\|_\infty \]

\[ \lesssim (\tau)^{-n/2} \int_{0}^{\delta} dr \int_{0}^{\delta} d\beta (\beta + r)^{-n/2} e^{Mr} \|\psi_0\|_1. \]

In the above expression, when \( n = 3 \), \( \int_{0}^{\delta} dr \int_{0}^{\delta} d\beta (\beta + r)^{-n/2} e^{Mr} \) is integrable. When \( n > 3 \), we need to further expand \( e^{-i(\tau-r-\beta)H_1} \) to remove the singularity of \( (\beta + r)^{-n/2} \) at \( \beta + r = 0 \). In either case, we can conclude that \( \|J_1\|_\infty \lesssim t^{-n/2} \|\psi_0\|_1. \)

For \( J_2 \), our estimate is the following:
\[
\int_0^\delta \int_0^{\tau-r-1} d\beta \| P_c(H_1) e^{-i(\tau-r-\beta)H_1} V_1 e^{-i\beta H_0} V_2(r) U(r) \psi_0 \|_\infty \\
\lesssim \int_0^\delta \int_0^{\tau-r-1} d\beta \| P_c(H_1) e^{-i(\tau-r-\beta)H_1} V_1 e^{-i\beta H_0} V_2(r) U(r) \psi_0 \|_1 \\
\lesssim \int_0^\delta \int_0^{\tau-r-1} d\beta \| P_c(H_1) e^{-i(\tau-r-\beta)H_1} V_1 e^{-i\beta H_0} V_2(r) U(r) \psi_0 \|_\infty \\
\lesssim \int_0^\delta \int_0^{\tau-r-1} d\beta \| P_c(H_1) e^{-i(\tau-r-\beta)H_1} V_1 e^{-i\beta H_0} V_2(r) U(r) \psi_0 \|_1 \\
\lesssim \int_0^\delta \int_0^{\tau-r-1} d\beta \| P_c(H_1) e^{-i(\tau-r-\beta)H_1} V_1 e^{-i\beta H_0} V_2(r) U(r) \psi_0 \|_1 \\
\lesssim \tau^{-n/2} \| \psi_0 \|_1.
\]

The implicit constant above depends on \( \delta \) and is independent of \( t \) and \( \psi_0 \). Plugging the above estimate into \( J_1 \), we derive that \( \| J_2 \|_\infty \lesssim t^{-n/2} \| \psi_0 \|_1 \).

To estimate \( J_3 \), we notice that

\[
\| \int_0^\tau d\beta V_1(\tau) P_c(H_1) e^{-i(\tau-r-\beta)H_1} V_1 e^{-i\beta H_0} V_2(r) U(r) \psi_0 \|_1 \\
\lesssim \int_0^\tau d\beta \| V_1(\tau) \|_2 \| P_c(H_1) e^{-i(\tau-r-\beta)H_1} V_1 e^{-i\beta H_0} V_2(r) U(r) \psi_0 \|_2 \\
\lesssim \int_0^\tau d\beta \| V_1(\tau) \|_2 \| e^{-i\beta H_0} V_2(r) U(r) \psi_0 \|_\infty \\
\lesssim \int_0^\tau d\beta \| V_1(\tau) \|_2 \| e^{-i\beta H_0} V_2(r) U(r) \psi_0 \|_1.
\]

Observe that \( r \) is small and \( \beta \simeq \tau \). Plugging the above estimate into \( J_3 \), we derive that \( \| J_3 \|_\infty \lesssim t^{-\frac{n}{2}} \| \psi_0 \|_1 \). Thus, we finished the estimate of the \( \int_0^\delta \) \( dr \) part of (3.20).

### 3.2 Low and high velocity estimates

So far we have estimated four parts of (3.20). This subsection is devoted to deriving the estimate of the \( \int_{\tau-A}^{\tau-\delta} \) \( dr \) part of (3.20), which will be decomposed as follows:

\[
\int_{\tau-A}^{\tau-\delta} ds \int_{s-A}^{s+\lambda(t-\epsilon)} d\tau \int_{t-A}^{\tau-\delta} d\tau \int_{s-A}^{s+\lambda(t-\epsilon)} \, dr e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V_1 P_c(H_1) e^{-i(\tau-r)H_1} V_2(r) U(r) \psi_0 \\
= \int_{t-A}^{\tau-\delta} ds \int_{s-A}^{s+\lambda(t-\epsilon)} d\tau \int_{t-A}^{\tau-\delta} d\tau \int_{s-A}^{s+\lambda(t-\epsilon)} \, dr e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} \\
V_1 P_c(H_1) e^{-i(\tau-r)H_1} (F(|\vec{p}| \geq N) + F(|\vec{p}| \leq N)) V_2(r) U(r) \psi_0 \\
= J_{\text{high}} + J_{\text{low}}.
\]
$F(|\vec{p}| \leq N)$ and $F(|\vec{p}| \geq N)$ denote smooth projections onto the frequencies $|\vec{p}| \leq N$ and $|\vec{p}| \geq N$, respectively. For the low velocity part $J_{low}$, firstly, $(t - s) + (s - \tau) \geq \epsilon$ and Lemma 3.3 imply

$$\|e^{i(t-s)H_0} V(s)e^{i(s-\tau)H_0}\|_{1 \rightarrow \infty} \lesssim (t - s)^{\frac{\alpha}{2}}(s - \tau)^{-\frac{n}{2}}.$$  

Secondly, we need the following proposition (see [RSS31] for its proof):

**Proposition 3.3.** Let $\chi_\tau$ be a smooth cut of $B(0, \tau \delta)$, where $\delta$ is a small constant depending only on $\vec{e}_1$ and $B(0, \tau \delta)$ is a ball in $\mathbb{R}^n$ centered at 0 with radius $\tau \delta$. Let $A, N$ be large positive constants and $A, N << r$ then

$$\sup_{0 < \tau - r \leq A} \|\chi_\tau e^{-i(\tau - r)H_1} P_c(H_1) F(|\vec{p}| \leq N)V_2(\cdot - r\vec{e}_1)\|_{L^2 \rightarrow L^2} \leq \frac{AN}{\delta t}.$$  

The idea behind Proposition 3.3 can be explained as the following:

The support of $V_2(\cdot - r\vec{e}_1)$ is contained in $B(r\vec{e}_1, R)$. Here $R$ is the size of the support of $V_2$. The operator $e^{-i(\tau - r)H_1} P_c(H_1) F(|\vec{p}| \leq N)$ “propagate” $B(r\vec{e}_1, R)$ into $B(0, \tau \delta)$ only if $(\tau - r)N \geq \text{dist}(B(r\vec{e}_1, R), B(0, \tau \delta))$ according to the classical picture. However if $|\tau - r| < A$, $r \ll A, N$, $(\tau - r)N \ll \text{dist}(B(r\vec{e}_1, R), B(0, \tau \delta))$.

To apply this proposition to $J_{low}$, note that $\chi_\tau V_1 = V_1$. Let $\chi_2$ be a smooth cut of the support of $V_2$ and $f$ be any function in $L^\infty(\mathbb{R}^n)$. Then it follows from Proposition 3.3 that

$$\|V_1 P_c(H_1)e^{-i(\tau-r)H_1} F(|\vec{p}| \leq N)V_2(r)f\|_1 \leq \|V_1 \chi_\tau P_c(H_1)e^{-i(\tau-r)H_1} F(|\vec{p}| \leq N)V_2(r)\chi_2(\cdot - r\vec{e}_1)f\|_1 \leq \|V_1\|_2\|\chi_\tau P_c(H_1)e^{-i(\tau-r)H_1} F(|\vec{p}| \leq N)V_2(r)\chi_2\|_2\|f\|_\infty \lesssim \frac{ANM^2}{\delta t} \|f\|_\infty.$$  

Combining the above estimate with (3.23) and noting $A, M, N \ll t$, we conclude

$$\|J_{low}\|_\infty \lesssim \int_{t-A}^t ds \int_{s-A}^{s+(t-s)} d\tau \int_{\tau-A}^{\tau-\delta} dr \langle t - s \rangle^{-\alpha/2}(s - \tau)^{-\alpha/2}\frac{ANM^2}{\delta t} \|U(r)\|_\infty \|\psi_0\| \leq \frac{C_0}{100} t^{-\alpha/2} \|\psi_0\|.$$  

From the above estimate for $J_{low}$, it is worth remarking that the purpose of the multiple expansions by Duhamel’s formula is to prepare a cushion (the potentials $V_1$ and $V_2$) to apply the $L^2 \rightarrow L^2$ estimate (Prop 3.3) between $L^1 \rightarrow L^\infty$ estimates.

For the high velocity part $J_{high}$, we shall further expand $U(r)$ with respect to $H_0$, followed by a commutator argument. By Duhamel’s formula

$$U(r) = e^{-irH_0} - i \int_0^r e^{-i(r - \alpha)H_0} V(\alpha)U(\alpha) d\alpha,$$

we write $J_{high} = J_{high,1} - iJ_{high,2}$, where

$$J_{high,1} = \int_{t-A}^t ds \int_{s-A}^{s+(t-s)} d\tau \int_{\tau-A}^{\tau-\delta} dr e^{-i(t-s)H_0} V(s)e^{-i(s-\tau)H_0} V_1 P_c(H_1) e^{-i(\tau-r)H_1} F(|\vec{p}| \geq N)V_2(r)e^{-irH_0}\psi_0.$$
\[ J_{\text{high,2}} = \int_{t-A}^{t} ds \int_{s-A}^{s+(t-\epsilon)} d\tau \int_{\tau-A}^{\tau-\delta} dr e^{-i(t-s)H_0} V(s)e^{-i(s-\tau)H_0} V_1 P_c(H_1) e^{-i(t-\tau)H_1} F(|\vec{p}| \geq N). \]

\[
\cdot V_2(r) \int_{0}^{r} e^{-i(t-\alpha)H_0} V(\alpha) U(\alpha) \psi_0 d\alpha.
\]

The decay of \( J_{\text{high,1}} \) will come easily from \( e^{-irH_0} \). Indeed, we apply Lemma 2.4 to \( e^{-i(t-s)H_0} V(s)e^{-i(s-\tau)H_0} \) as in (3.23) and notice that

\[
J_{\text{high,1}} \leq \| J_{\text{high,1}} \|_{L^2 \rightarrow L^2} \leq 1.
\]

Then it is clear that \( \| J_{\text{high,1}} \|_{L^\infty} \) is dominated by

\[
\int_{t-A}^{t} ds \int_{s-A}^{s+(t-\epsilon)} d\tau \int_{\tau-A}^{\tau-\delta} dr (t-s)^{-n/2}(s-\tau)^{-n/2}(r)^{-n/2} \| \psi_0 \|_1 \lesssim t^{-n/2} \| \psi_0 \|_1.
\]

\( J_{\text{high,2}} \) will be decomposed into three parts \( J_{\text{high,2}}^1, J_{\text{high,2}}^2 \) and \( J_{\text{high,2}}^3 \), corresponding to \( \int_B^r d\alpha \), \( \int_{0}^{B} d\alpha \) and \( \int_{r-B}^{0} d\alpha \) respectively, where \( B > 0 \) is a large constant to be specified.

For \( J_{\text{high,2}}^1 \), the decay comes from \( e^{-i(t-\alpha)H_0} \). Indeed, it follows from Lemma 2.2 and \( 0 < \alpha < B \) that

\[
\| e^{-i(t-\alpha)H_0} V(\alpha) U(\alpha) \|_{L^1 \rightarrow L^\infty} \lesssim r^{-n/2} e^{M\alpha} \lesssim (r)^{-n/2}
\]

Hence, it follows from (3.23), (3.24) and the above inequality that \( \| J_{\text{high,2}}^1 \|_{L^\infty} \) is dominated by

\[
\int_{t-A}^{t} ds \int_{s-A}^{s+(t-\epsilon)} d\tau \int_{\tau-A}^{\tau-\delta} dr (t-s)^{-n/2}(s-\tau)^{-n/2}(r)^{-n/2} \| \psi_0 \|_1 \lesssim (t)^{-n/2} \| \psi_0 \|_1.
\]

\( J_{\text{high,2}}^2 \) will be estimated by an application of the bootstrap assumption and the smallness comes from choosing \( B \) sufficiently large. Indeed, it follows from (3.23), (3.24), Lemma 2.4 and the bootstrap assumption that

\[
\| J_{\text{high,2}}^2 \|_{L^\infty} \lesssim \int_{t-A}^{t} ds \int_{s-A}^{s+(t-\epsilon)} d\tau \int_{\tau-A}^{\tau-\delta} dr (t-s)^{-n/2}(s-\tau)^{-n/2}.
\]

\[
\cdot \int_{B}^{r-B} (r-\alpha)^{-n/2} \alpha^{-n/2} d\alpha C_0 \| \psi_0 \|
\]

\[
\lesssim \int_{t-A}^{t} ds \int_{s-A}^{s+(t-\epsilon)} d\tau \int_{\tau-A}^{\tau-\delta} dr (t-s)^{-n/2}(s-\tau)^{-n/2}(r)^{-n/2} \kappa_B C_0 \| \psi_0 \|
\]

\[
\lesssim \frac{1}{100} C_0 t^{-n/2} \| \psi_0 \|.
\]

In the above inequality, \( B \) is chosen sufficiently large, because \( \kappa_B = \int_{B}^{\infty} \alpha^{-n/2} d\alpha \rightarrow 0 \) when \( B \rightarrow \infty \).
The decay of $J_{\text{high,2}}^3$ can only come from $U(\alpha)$. As usual we need to generate the smallness \( \frac{1}{100} \) for the bootstrap assumption. Here the smallness \( \frac{1}{100} \) comes from the high velocity and a commutator argument. Write $F(|\vec{p}| \geq N)V_2(r) = [F(|\vec{p}| \geq N), V_2(r)] + V_2(r)F(|\vec{p}| \geq N)$ and correspondingly, we decompose $J_{\text{high,2}}^3 = J_{\text{high,2}}^{3.1} + J_{\text{high,2}}^{3.2}$. That is to say $J_{\text{high,2}}^{3.1}$ and $J_{\text{high,2}}^{3.2}$ are just $J_{\text{high,2}}^3$ with $F(|\vec{p}| \geq N)V_2(r)$ replaced by $[F(|\vec{p}| \geq N), V_2(r)]$ and $V_2(r)F(|\vec{p}| \geq N)$.

Specifically, the smallness \( \frac{1}{100} \) for $J_{\text{high,2}}^{3.1}$ comes from the following standard fact, namely

\begin{equation}
\| [F(|\vec{p}| \leq N), V_2] \|_{L^2} \lesssim N^{-1} \| \nabla V_2 \|_{\infty}.
\end{equation}

To see this, write $F(|\vec{p}| \leq N)f = [\hat{\eta}(\xi/N)\hat{f}(\xi)]' \psi$ with some smooth bump function $\eta$. Hence the kernel $K$ of $[F(|\vec{p}| \leq N), V_2]$ is

\begin{equation*}
K(x, y) = N^n \eta(N(x - y))(V_2(y) - V_2(x)),
\end{equation*}

and follows from Schur’s test and $\sup_x \| K(x, \cdot) \|_{L^1} = \sup_y \| K(\cdot, y) \|_{L^1} \lesssim N^{-1} \| \nabla V_2 \|_{\infty}$.

It follows from (3.23), $\| P_c(H_1)e^{-i(t - r)H_1} \|_{2 \rightarrow 2} \leq 1$, (3.25) and the bootstrap assumption that

\begin{equation*}
\| J_{\text{high,2}}^{3.1} \|_{\infty} \lesssim \int_t^t ds \int_{s - A}^{s + (t - \epsilon)} d\tau \int_{t - \delta}^{\tau - \delta} dr (t - s)(s - \tau)^{-n/2} \frac{\| V_1 \|_{2} \| \nabla V_2 \|_{\infty}}{N} \int_{r - B}^{r} \| e^{-i(r - \alpha)H_0} V(\alpha)U(\alpha)\psi_0 \|_{2} d\alpha
\end{equation*}

\begin{equation*}
\lesssim \frac{1}{N} \sup_{t - 3A < \alpha < t} \| U(t)\psi_0 \|_{\infty} \lesssim \frac{C_0}{N} t^{-n/2} \| \psi_0 \| \leq \frac{C_0}{100} t^{-n/2} \| \psi_0 \|,
\end{equation*}

where $\frac{1}{N}$ is chosen sufficiently small to dominate the implicit constant in "\( \lesssim \)" which only depends on $n, V, \vec{v}_2$ and $\epsilon, \delta, A, B$.

The smallness for $J_{\text{high,2}}^{3.2}$ comes from the following version of Kato’s $\frac{1}{2}$-smoothing estimate:

\begin{equation}
\| \int_{\alpha}^{\alpha + B} \chi_2(\cdot - r\vec{v}_2)F(|\vec{p}| \geq N)e^{-i(r - \alpha)H_0} \|_{2} \lesssim \frac{BR}{\sqrt{N}},
\end{equation}

where $\chi_2(\cdot)$ is a smooth cut around the support of $V_2$ and $R$ is radius of the support of $\chi_2$. The implicit constant only depends on $n, V_2$. We refer to Section 3.5 for its proof and further references.

Now observe that the region of integration $\int_{T - A}^{T - \delta} d\tau \int_{T - \tau}^{T - \tau - \delta} d\alpha (t - s)(s - \tau)^{-n/2} \frac{BR}{\sqrt{N}} \| U(\alpha)\psi_0 \|_{\infty} \lesssim \frac{C_0}{100} t^{-n/2} \| \psi_0 \|,

where $\frac{1}{\sqrt{N}}$ is chosen to be sufficiently small to dominate the implicit constant which only depends on $n, V, \vec{v}_2$ and $\epsilon, \delta, A, B$. Therefore, we conclude that (3.41) implies (3.24), from which Theorem 1 follows.
4 Decay estimate of the derivatives of \( U(t) \)

In this section we prove Theorem 1.8 by induction on \( \kappa \) by following the same scheme of the proof of Theorem 1.7. The first step is to set up the cancellation lemma for \( \partial U(t)\psi_0 \).

**Lemma 4.1.** Let \( \kappa \) be a nonnegative integer. Assume \( \sup_{0 \leq \beta \leq \kappa} \sup_{r \in \mathbb{R}} \| \partial^\beta V(r) \|_{L^1} < M \). Let \( \alpha \) be a nonnegative integer \( n \)-tuple with \( |\alpha| = \kappa \). Suppose \( U(t) \) is the evolution operator of (1.9) as before. Then

\[
(4.1) \quad \sup_{r \in \mathbb{R}} \| e^{-i(t-s)H_0} V(r) \partial^\alpha U(s) \psi_0 \|_p < |t|^{-\gamma} Me^{(\kappa+1)Ms} \| \psi_0 \|_{W^{\kappa,p'}}
\]

where \( \gamma = n(\frac{1}{2} - \frac{1}{p}) \) and \( 2 \leq p < \infty, \frac{1}{p} + \frac{1}{p'} = 1 \).

**Proof.** Write the left-hand side of (4.1) := \( \Psi(t,s) \). When \( \kappa = 0 \), (4.1) is just the inequality (2.3). Note that the inequality (2.3) holds with \( V \) replaced by its derivative \( \partial^\beta V \), as long as \( \partial^\beta V \) lies in \( L^1(\mathbb{R}^n) \). Assume \( \kappa = 1 \) and apply Duhamel’s formula:

\[
\| e^{-i(t-s)H_0} V(r) \partial U(s) \psi_0 \|_p \\
\leq \| e^{-i(t-s)H_0} V(r) \partial e^{-isH_0} \psi_0 \|_p + \int_0^s \| e^{-i(t-s)H_0} V(r) e^{-i(s-t)H_0} \partial V(\tau) U(\tau) \psi_0 \|_p d\tau \\
\leq C \| V(\tau) \|_1 t^{-\gamma} \| \partial \psi_0 \|_{p'} + \| V(\tau) \|_1 \int_0^s \| e^{-i(t-t')H_0} (\partial V)(\tau) U(\tau) \psi_0 \|_p d\tau \\
+ \| V(\tau) \|_1 \int_0^s \| e^{-i(t-t')H_0} V(\tau) \partial U(\tau) \psi_0 \|_p d\tau \\
\leq CMt^{-\gamma} \| \psi_0 \|_{W^{1,p'}} + M \int_0^s t^{-\gamma} e^M d\tau \| \psi_0 \|_{p'} + M \int_0^s \Psi(t,\tau) d\tau \\
\leq CMt^{-\gamma} e^{Ms} \| \psi_0 \|_{W^{1,p'}} + M \int_0^s \Psi(t,\tau) d\tau.
\]

Taking supremum over \( r \), we get \( \Psi(t,s) \leq CMt^{-\gamma} e^{Ms} \| \psi_0 \|_{W^{1,p'}} + M \int_0^s \Psi(t,\tau) d\tau \). By Gronwall’s inequality, \( \Psi(t,s) \leq CMt^{-\gamma} e^{2Ms} \).

For \( \kappa > 1 \), the above argument goes through by induction, provided that the Fourier transform of the derivatives up to degree \( \kappa \) of \( V(r) \) are uniformly bounded in \( L^1(\mathbb{R}^n) \).

The following is an analog of Corollary 2.3:

**Corollary 4.2.** With the same notations and assumptions as in Lemma 4.1, we have

\[
(4.2) \quad \| U(t) \psi_0 \|_{W^{\kappa,p'}} \lesssim t^{-\gamma} e^{(1+\kappa)Ms} \| \psi_0 \|_{W^{\kappa,p'}}
\]

**Proof.** By Duhamel’s formula, Lemma 4.1 and the fact that \( \partial \) commutes with \( e^{-itH_0} \), we have the following estimate:

\[
\| \partial^\alpha U(t) \psi_0 \|_p \lesssim \| e^{-itH_0} \partial^\alpha \psi_0 \|_p + \sum_{\beta \leq \alpha} \int_0^t \| e^{-i(t-\tau)H_0} (\partial^\beta V)(\tau) \partial^{\alpha-\beta} U(\tau) \psi_0 \|_p d\tau \\
\lesssim t^{-\gamma} \| \psi_0 \|_{W^{\kappa,p'}} + \sum_{\beta \leq \alpha} \int_0^t t^{-\gamma} e^{(\beta+1)M\tau} d\tau \| \psi_0 \|_{W^{\kappa,p'}} \\
\leq Ct^{-\gamma} e^{(\kappa+1)Mt} \| \psi_0 \|_{W^{\kappa,p'}}.
\]
Similarly, the following lemma generalizes Lemma 2.4.

**Lemma 4.3.** Let $\alpha$ be an $n$-tuple with $|\alpha| = \kappa$ and $U(t)$ be the evolution operator of (1.9). For each $m \geq 1$ and $\epsilon > 0$, $u_1, u_2, \ldots, u_m$ are all in either $\mathbb{R}_+$ or $\mathbb{R}_-$, satisfying $|\sum_{j=1}^m u_j| > \epsilon$, then there exists constant $C = C(m, \epsilon, \kappa, p)$ such that

\[
\| \prod_{j=1}^{m-1} (e^{iu_jH_0}V(s_j))\partial^\alpha U(u_m)\psi_0\|_p \leq CM^{m-1} \prod_{j=1}^m \|u_j\|^{-\gamma} e^{(\kappa+1)Mu_m} \|\psi_0\|_{W^{\kappa, p}},
\]

where $s_j$ is any real number, $\frac{2n}{n-2} < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and

\[
M = \Sigma_{0 \leq \beta \leq \alpha} \sup_{s \in \mathbb{R}}(\|\partial^\beta V(s)\|_1 + \|\partial^\beta V(s)\|_1).
\]

Using Lemma 4.1 and Corollary 4.2, the proof of Lemma 4.3 is exactly the same as that of Lemma 2.4.

We only prove Theorem 1.8 for the case $\kappa = 1, 2$. The case $\kappa > 2$ can be proved by induction. Specifically, we prove the following implication:

For any fixed sufficiently large time $T$,

\[
\|U(t)\psi_0\|_{W^{\kappa, p}} \leq C_0 |t|^{-\gamma}(\|\psi_0\|_{W^{\kappa, p}} + e^{-\frac{\alpha T}{2}} \|\psi_0\|_{L^2}) \quad \text{for} \quad 0 \leq t \leq T, \kappa = 1, 2
\]

implies that

\[
\|U(t)\psi_0\|_{W^{\kappa, p}} \leq \frac{C_0}{2} |t|^{-\gamma}(\|\psi_0\|_{W^{\kappa, p}} + e^{-\frac{\alpha T}{2}} \|\psi_0\|_{L^2}) \quad \text{for} \quad 0 \leq t \leq T, \kappa = 1, 2
\]

provided that $\frac{2n}{n-2}$ remains larger than some constant that does not depend on $T$. The assumption (1.4) can be made to hold for some $C_0$ depending on $T$, because of Corollary 4.2. Letting $T \to +\infty$ to eliminate $\|\psi_0\|_{L^2}$, Theorem 1.8 follows from the iteration of the above implication.

We will first prove (1.3) for $\kappa = 1$. For technical reasons (see (1.15)), we need the above bootstrap assumption (1.4) for $\kappa + 2$. To simplify the notation, we write $\partial^\alpha = \partial$ and

$\|\psi_0\|_{W^{1, p}} + e^{-\frac{\alpha T}{2}} \|\psi_0\|_{L^2} := \|\psi_0\|_{(1, p')}$. 

With these cancellation lemmas for $\partial U(t)\psi_0$, the proof of Theorem 1.8 follows the scheme of that of Theorem 1.7. The difference is that now we need to commute $\partial_x$ with operators such as $e^{itH_0}$, $V$ and $e^{itH_1}$ to apply the cancellation lemma and the bootstrap assumption.

We proceed by expanding $U(t)$ with Duhamel’s formula:

\[
\partial U(t)\psi_0 = \partial e^{-itH_0}\psi_0 - i \int_0^t \partial e^{-i(t-s)H_0} V(s)U(s)\psi_0 \, ds
\]

\[
= \partial e^{-itH_0}\psi_0 - i \int_0^t e^{-i(t-s)H_0}(\partial V)(s)U(s)\psi_0 \, ds
\]

\[
- i \int_0^t e^{-i(t-s)H_0} V(s)\partial U(s)\psi_0 \, ds.
\]

(4.6)
Notice that \[ \{\partial, V\} = (\partial V) \cdot \] is a multiplication operator, which can be viewed as another potential and Theorem 1.7 can be applied to the second term of (4.6). This idea has appeared in the proof of Lemma 4.1. Specifically, it follows from the proof of Theorem 1.7 and an interpolation with the \( L^\gamma \) conservation of \( U(t) \) that

\[
\| \int_0^t e^{-i(t-s)H_0} V(s) U(s) \psi_0 \, ds \|_p \lesssim t^{-\gamma} \| \psi_0 \|_{p'}.
\]

By assumption, \( \partial V_j \) satisfies the regularity and smoothness conditions for \( V_j \) in Theorem 1.7 and we conclude that

\[
\| \int_0^t e^{-i(t-s)H_0} (\partial V)(s) U(s) \psi_0 \, ds \|_p \lesssim t^{-\gamma} \| \psi_0 \|_{p'}.
\]

We expand the last term of (4.6) by Duhamel’s formula just as in Section 3 and perform the same decomposition. With the cancellation lemma for \( \partial U(t) \) and Remark 3.2, the last term (4.6) is reduced to the following:

\[(4.7) \quad \sum_{j=1}^{2} \int_{t-A}^{t} ds \int_{s-A}^{s+(t-c)} d\tau e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V_j(x - \tau \vec{v}_j) \partial P_c(H_1, \tau) U(\tau) \psi_0.\]

Before we proceed, we observe that our assumptions guarantee

\[(4.8) \quad \| P_c(H_1)e^{-itH_1} \psi_0 \|_{L^q} \leq C_q |t|^{-\gamma} \| \psi_0 \|_{L^{q'}}.\]

This implies

\[
\| H_1 P_c(H_1)e^{-itH_1} \psi_0 \|_{L^q} = \| P_c(H_1)e^{-itH_1} H_1 \psi_0 \|_{L^q}
\leq C_q |t|^{-\gamma} \| H_1 \psi_0 \|_{L^{q'}} \leq C_q |t|^{-\gamma} \| \psi_0 \|_{W^{2,q'}}.
\]

As \( V_1 \in L^\infty(\mathbb{R}^n) \) and double Riesz transforms is bounded on \( L^q(\mathbb{R}^n) \) \( 1 < q < +\infty \), the above inequality in the case of \( 1 < q < +\infty \), implies that

\[(4.9) \quad \| P_c(H_1)e^{-itH_1} \psi_0 \|_{W^{2,q}} \lesssim |t|^{-\gamma} \| \psi_0 \|_{W^{2,q'}}.\]

Interpolating between (4.8) and (4.9) (Theorem 6.4.5 [BL]), we conclude that

\[(4.10) \quad \| P_c(H_1)e^{-itH_1} \psi_0 \|_{W^{1,q}} \leq C_q |t|^{-\gamma} \| \psi_0 \|_{W^{1,q'}}.\]

where \( 2 \leq q < \infty \), \( \frac{1}{q} + \frac{1}{q'} = 1 \) and \( \gamma = n(\frac{1}{2} - \frac{1}{q}) \). Because double Riesz transforms are unbounded on \( L^\infty(\mathbb{R}^n) \), we exclude \( p = \infty \) in Theorem 1.8.

We write \( P_c(H_1)U(\tau) = P_c(H_1)e^{-itH_1} - iP_c(H_1) \int_0^\tau e^{-i(\tau-x)H_1} V_2(r)U(r) \, dr \) and (4.7) is broken into two terms.
It follows from (4.10), among other things that the first term of (4.7), which contains \( P_c(H_1)e^{-i\tau H_1} \) is dominated by \( |t|^{-\gamma}\|\psi_0\|_{W^{1,p}} \).

The second term of (4.7) is decomposed as follows:

\[
(4.11) \quad \int_0^\tau \frac{dr}{\delta} \delta > 0 \text{ is chosen sufficiently small.}
\]

We estimate each term in (4.11) with similar methods as that for (3.20). Because of (4.10), the terms containing \( \int_A^\tau c \) can be estimated exactly as that there is no derivative before \( P(H_1) \), and we omit the details here. Again by (4.10) with \( q = 2 \), the term containing \( \int_0^{\tau - \delta} dr \) in (4.11) is estimated as follows:

\[
\int_{t-A}^t ds \int_{s-A}^{s+(t-c)} d\tau \int_{\tau-\delta}^{\tau} \left\| e^{i(t-s)H_0}V(s)e^{i(s-\tau)H_0}V_1(\tau)\partial e^{-i(\tau-r)H_1}P_c(H_1) V_2(r) U(\tau)\psi_0 \right\|_p \\
\lesssim \sup_{t-2A < \tau < t} \int_{\tau-\delta}^{\tau} \left\| \partial P_c(H_1)e^{-i(\tau-r)H_1} V_2(r) U(\tau)\psi_0 \right\|_2 \\
\lesssim \sup_{t-2A < \tau < t} \int_{\tau-\delta}^{\tau} \left\| V_2(r) U(\tau)\psi_0 \right\|_{W^{1,2}} \\
\lesssim \sup_{t-2A < \tau < t} \int_{\tau-\delta}^{\tau} \left\| U(\tau)\psi_0 \right\|_{W^{1,p}} \\
\lesssim t^{-\gamma} C_0 \|\psi_0\|_{(1,p')} \leq \frac{C_0}{100} t^{-\gamma} \|\psi_0\|_{(1,p')}
\]

Here \( \delta > 0 \) is chosen sufficiently small.

The \( \int_0^{\delta} dr \) term in (4.11) is expanded by Duhamel’s formula:

\[
e^{-i(\tau-r)H_1} = e^{-i(\tau-r)H_0} - i \int_0^{\tau-r} e^{-i(\tau-r-\beta)H_1} V_1 e^{-i\beta H_0} d\beta.
\]

Plugging the above expression into the \( \int_0^{\delta} dr \) term, we get two terms. The first one containing \( e^{-i(\tau-r)H_0} \) is

\[
(4.12) \quad \int_{t-A}^t ds \int_{s-A}^{s+(t-c)} d\tau \int_0^{\delta} d\tau e^{-i(t-s)H_0}V(s)e^{-i(s-\tau)H_0}V_1(\tau)\partial P_c(H_1)e^{-i(\tau-r)H_0} V_2(r)U(\tau)\psi_0.
\]

Since \( P_c(H_1) = Id - P_b(H_1) \) and \( P_b(H_1) \) is a bounded operator from \( L^p \) to \( L^p \), \( P_c(H_1) \) is bounded from \( L^p \) to \( L^p \). It follows from Lemma (4.1) \( 0 < r < \delta \), and the Leibnitz rule that

\[
\| H_1 P_c(H_1)e^{-i(\tau-r)H_0} V_2(r) U(\tau)\psi_0 \|_p = \| P_c(H_1)H_1 e^{-i(\tau-r)H_0} V_2(r) U(\tau)\psi_0 \|_p \\
\leq C \| H_1 e^{-i(\tau-r)H_0} V_2(r) U(\tau)\psi_0 \|_p \\
\leq C \| V_1 \|_\infty \| e^{-i(\tau-r)H_0} V_2(r) U(\tau)\psi_0 \|_p + \| e^{-i(\tau-r)H_0} \Delta V_2(r) U(\tau)\psi_0 \|_p \\
\leq C \tau^{-\gamma} \|\psi_0\|_{W^{2,p'}}.
\]
Since $H_1 = H_0 + V_1$ and $V_1$ is bounded, we see that
\[
\| \Delta P_c(H_1)e^{-i(\tau-t)H_0}V_2(r)U(r)\psi_0 \|_p \leq C\tau^{-\gamma}\|\psi_0\|_{W^{2,p}}.
\]
Because the double Riesz transforms are bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, it follows that
\[
\| P_c(H_1)e^{-i(\tau-t)H_0}V_2(r)U(r)\psi_0 \|_{W^{2,p}} \leq C\tau^{-\gamma}\|\psi_0\|_{W^{2,p}}.
\]

Therefore, by complex interpolation, we see
\[
\| P_c(H_1)e^{-i(\tau-t)H_0}V_2(r)U(r)\psi_0 \|_{W^{1,p}} \leq C\tau^{-\gamma}\|\psi_0\|_{W^{1,p}}.
\]
which implies that $\|P_c\|_{W^{1,p}} \lesssim_t t^{-\gamma}\|\psi_0\|_{W^{1,p}}$.

For the term containing $\int_0^\delta dr \int_0^{\tau-t} d\beta$, we perform the exact same decomposition as in (4.10) and each step there goes through provided (4.13) and (1.12).

The term containing $\int_{\tau-A}^\tau ds \int_{s-A}^s ds$ in (4.11) is
\[
\int_{\tau-A}^\tau \int_{s-A}^s ds \int_{\tau-t}^{\tau-t} d\tau \|e^{-i(t-s)H_0}V(s)e^{-i(s-r)H_0}V_i\partial P_c(H_1)e^{-i(\tau-t)H_1}V_2(r)U(r)\psi_0 \|_p.
\]

The proof of Theorem 1.7 showed that $\forall \epsilon > 0$, the following holds:
\[
\| V_1 P_c(H_1)e^{-i(\tau-t)H_1}V_2(r)U(r)\psi_0 \|_\infty < \epsilon C_0 t^{-\frac{\beta}{2}}\|\psi_0\|_1,
\]
given $t$ sufficiently large. Going through the proof, we see that the same argument also shows
\[
\| V_1 P_c(H_1)e^{-i(\tau-t)H_1}V_2(r)U(r)\psi_0 \|_p < \epsilon C_0 t^{-\gamma}\|\psi_0\|_{p'}.
\]

Furthermore the above inequality holds if $V_1$ or $V_2$ is replaced by its derivative. Another observation is that, given our new cancellation lemma for $\partial U(r)\psi_0$,
\[
\| V_1 P_c(H_1)e^{-i(\tau-t)H_1}V_2(r)\partial \beta U(r)\psi_0 \|_p < \epsilon C_0 t^{-\gamma}\|\psi_0\|_{(1,\beta,2,p')}.
\]

Indeed, to prove the above inequality, we decompose the left-hand side into a high velocity part and a low velocity part. Each part generates the small constant $\epsilon$ for the same reason as in Section 3.3. The same argument with the bootstrap assumption (4.10) implies:
\[
\| V_1 H_1 P_c(H_1)e^{-i(\tau-t)H_1}V_2(r)U(r)\psi_0 \|_p
\]
\[
= \| V_1 P_c(H_1)e^{-i(\tau-t)H_1}H_1V_2(r)U(r)\psi_0 \|_p \lesssim \epsilon C_0 t^{-\gamma}\|\psi_0\|_{(2,p')}.
\]

It follows from the above inequality and an elementary calculation that
\[
\| V_1 P_c(H_1)e^{-i(\tau-t)H_1}V_2(r)U(r)\psi_0 \|_{W^{2,p}} \lesssim \epsilon C_0 t^{-\gamma}\|\psi_0\|_{(2,p')}.
\]

Hence, by complex interpolation, for $\forall \epsilon > 0$,
\[
\| V_1 P_c(H_1)e^{-i(\tau-t)H_1}V_2(r)U(r)\psi_0 \|_{W^{1,p}} \lesssim \epsilon C_0 t^{-\gamma}\|\psi_0\|_{(1,p')}.
\]
given \( t \) sufficiently large. This implies that \( \|f_{1.14}\|_{W^{1,p}} \) can be estimated by \( \frac{1}{1+p}C_0 t^{-\gamma}\|\psi_0\|_{1,p'} \).

Therefore, we proved (1.5) for \( \kappa = 1 \). The same procedure also proves (1.5) for \( \kappa = 2 \). Thus, we finish the bootstrap argument and conclude that

\[
\|U(t)\psi_0\|_{W^{\kappa,p}} \lesssim \|\psi_0\|_{W^{(\kappa,p')}},
\]

by letting \( T \to \infty \). The proof for \( \kappa > 2 \) is similar by induction. Thus we have proved Theorem 1.8.

5 Boundedness of the Sobolev norm of \( U(t)\psi_0 \)

The goal of this section is to prove Theorem 1.10 when \( \kappa \) is a positive integer. The intuition comes from the case \( \kappa = 1 \) ([Gr]). To bound the kinetic energy (the \( H^1 \) norm), we look at the observable \( K(t) = \frac{1}{2}(p - \vec{v})^2 + \sum_{l=1}^m V_l(t) \). \( \langle K(t) \rangle \) will decrease if the particle is far away from any potential, since the observable \( (p - \vec{v})^2 \) decreases like \( t^{-2} \) for the free motion (the Pseudo-conformal identity). If the particle is close to the center of potential \( V_l \), then \( \frac{\vec{v}}{t} \approx \vec{v}_l \) and \( \langle K(t) \rangle \approx (\frac{1}{2}(p - \vec{v})^2 + V_l(x - \vec{v}(t))) \), which clearly is the total energy of this one potential stationary subsystem up to a Galilean transform. To carry this boundedness from \( \langle K(t) \rangle \) to \( \langle pt^2 \rangle \), we need to replace the vector field \( \vec{v} \) by \( \nu(x,t) \), such that \( \nu(x,t) \) is uniformly bounded and is equal to \( \vec{v}_l \) in an increasingly big neighborhood of \( x = \vec{v}_l t \).

Vigorously, consider a smooth, uniformly bounded vector field

\[
\nu(x,t) : \mathbb{R}^n \times (-\infty, -T] \cup [T, +\infty) \to \mathbb{R}^n
\]

and let

\[
K_0(t) = \frac{1}{2}(p - \nu(x,t))^2 + \sum_{l=1}^m V_l(t),
\]

where \( T \) is a large positive constant, \( p = (p_1, \cdots, p_n) \) and \( p_j = -i\frac{\partial}{\partial x_j} \). Note \( p^2 = H_0 \) and \( \frac{1}{2}(p - \nu(x,t))^2 \) is a well-defined self-adjoint positive operator.

In [Gr], Graf constructed \( \nu(x,t) \) and proved \( \|U(t,s)\psi_0\|_{H^1} \) is bounded as \( t \to \infty \) by bounding \( \frac{d}{dt}\langle K_0(t) \rangle \) from above by a time-integrable function, where \( \langle K_0(t) \rangle = (U(t,s)\psi_0, K_0(t)U(t,s)\psi_0)_{L^2} \).

We write \( (f,g) \) as the inner product of \( f, g \) in the \( L^2(\mathbb{R}) \) sense.

To prove Theorem 1.10, we need to define the proper analog of \( K_0(t) \) suitable to the \( H^\kappa \) norm of \( U(t,s)\psi_0 \) to match the intuition given by the classical system. Fortunately the following observable works:

\[
K(t) = \sum_{l=1}^m (\frac{1}{2}(p - \nu(x,t))^2 + V_l(t))^\kappa - (m - 1)(\frac{1}{2}(p - \nu(x,t))^2)^\kappa.
\]

Notice that \( K(t) = K_0(t) \) if \( \kappa = 1 \). Because \( \nu(x,t) \) and its derivatives are bounded uniformly in space time and \( V_j \in C^1(\mathbb{R}^n) \), we have the following, writing \( \langle K(t,s) \rangle = (U(t,s)\psi_0, K(t)U(t,s)\psi_0)_{L^2} \):

\[
\|U(t,s)\psi_0\|_{H^\kappa}^2 \lesssim \langle K(t,s) \rangle + \|U(t,s)\psi_0\|_{H^{\kappa-1}}^2; \quad \langle K(t,s) \rangle \lesssim \|U(t,s)\psi_0\|_{H^\kappa}^2.
\]

By induction on \( \kappa \), it suffices to show \( \langle K(t,s) \rangle \) is bounded uniformly in \( t \) and \( s \).

Expand \( K(t) \) as polynomial of \( \frac{1}{2}(p - \nu(x,t))^2 \). Though \( \frac{1}{2}(p - \nu(x,t))^2 \) and \( V_l(t) \) do not commute with each other, viewing \( K(t) \) as a differential operator, the term of highest degree is
It is convenient to describe \( \nu \) holds.

Choosing \( T > 0 \) large enough such that \( K(t,s) \) is always nonnegative. The other part containing the low degree terms can be dominated by \( \|U(t,s)\psi_0\|^2_{L^{\infty}} \). By the induction hypothesis, it follows that \( \langle K(t,s) \rangle \) is bounded from below. To bound \( \langle K(t,s) \rangle \) from above, it suffices to show that for \( t > T \)

\[
\frac{d}{dt} \langle K(t,s) \rangle \leq -t^{-(1+\delta)} C(\langle K(t,s) \rangle + \|U(t,s)\psi_0\|^2_{L^{\infty}}).
\]

For \( t < -T \), the opposite of the above inequality should hold:

\[
\frac{d}{dt} \langle K(t,s) \rangle \geq t^{-(1+\delta)} C(\langle K(t,s) \rangle + \|U(t,s)\psi_0\|^2_{L^{\infty}}).
\]

First let’s consider \( t > T \), integrating (5.1),

\[
\langle K(t_2,s) \rangle - \langle K(t_1,s) \rangle \leq C \int_{t_1}^{t_2} t^{-(1+\delta)} \langle K(t,s) \rangle dt + C \sup_{t,s} \|U(t,s)\psi_0\|^2_{L^{\infty}}.
\]

Choosing \( T > 0 \) large enough such that \( C \int_T^{\infty} t^{-1-\delta} dt < \frac{1}{2} \), then

\[
\langle K(t_2,s) \rangle \leq \langle K(t_1,s) \rangle + C \sup_{t,s} \|U(t,s)\psi_0\|^2_{L^{\infty}} + \frac{1}{2} \max_{t_1 < t < t_2} \langle K(t,s) \rangle
\]

\[
\max_{t_1 < t < t_2} \langle K(t,s) \rangle \leq 2 \langle K(t_1,s) \rangle + C \|\psi_0\|^2_{L^{\infty}}
\]

Letting \( t_2 \to +\infty \) and \( t_1 = T \), it follows that \( \max_{t > T} \langle K(t,s) \rangle < C \langle K(T,s) \rangle + C \|\psi_0\|^2_{L^{\infty}} \). Hence \( \langle K(t,s) \rangle \leq C_T \|\psi_0\|^2_{L^{\infty}} \) for \( t > T \) and \( s \in [-T,T] \). For \( t < -T \), we integrate (5.2) to bound \( \langle K(t,s) \rangle \) from above and Theorem 1.10 follows in this case by the same argument given that (5.2) holds.

Before we proceed to prove (5.1) and (5.2), let’s specify some properties of the vector field \( \nu(x,t) \).

It is convenient to describe \( \nu(x,t) \) in the rescaled coordinates \( y = \frac{x}{\gamma} \). Let \( u_0 = 2 \max_{1 \leq l \leq m} |\vec{v}_l| \).

When \( |y| > u_0 \), \( \nu(x,t) = u_0 \frac{y}{|y|} \). When \( y \in B_l \), we specify \( \nu(x,t) = \vec{v}_l \), where \( B_l \) is a fixed ball centered at \( \vec{v}_l \). We suppose that \( B_l(l = 1, \cdots , m) \) lie in the big ball \( B_0 \) centered at the origin with radius \( u_0 \) and that they are disjoint from each other. When \( y \in B_0 - \bigcup_{l=1}^m B_l \), we specify \( \nu(x,t) = y \).

To make the vector field smooth, we modify and smooth the vector field in the scale of \( |t|^{1-\gamma} \), where \( \gamma \) is a small positive number. In the rescaled coordinates \( y \), the scale is \( |t|^{-\gamma} \). Specifically, consider

\[
\omega(s, \alpha) = s \varphi\left(\frac{u_0 - s}{\alpha}\right) + u_0(1 - \varphi\left(\frac{u_0 - s}{\alpha}\right)),
\]

where \( \varphi \in C^\infty(\mathbb{R}) \) with \( \varphi' \geq 0 \) and

\[
\varphi(x) = 0 \text{ for } x \leq 0 \quad \text{and} \quad \varphi(x) = 1 \text{ for } x > 1.
\]
Then writing $y = \frac{x}{t}$, we define
\[ \omega^{(0)}(x,t) = \omega(|y|, |t|^{-\gamma}) \frac{y}{|y|} \quad \text{and} \quad \omega^{(\ell)}(x,t) = -(y - \vec{v}t) \varphi(2 - |t|^{\frac{1}{\gamma}}|y - \vec{v}t|), \]
where $\ell = 1, 2, \cdots, m$. Finally, $\nu(x,t) := \sum_{\ell=0}^{m} \omega^{(\ell)}$

The properties of the vector field $\nu(x,t)$ that concern us are listed as follows:

1. $\nu$ is bounded in space time. The $k$-th space derivatives of $\nu$ uniformly decay as $|t|^{-k(1-\gamma)}$ as $t \to \infty$.

2. $(\nu_{i,j})_{n \times n}$ as a matrix is symmetric and positive semi-definite when $t > 0$, negative semi-definite when $t < 0$, where $\nu_i$ is the $i$-th component of vector $\nu$ and the indices following a comma stand for partial derivatives in space. As $\nu_{k,j} = \nu_{j,k}$, $p_k - \nu_k$ and $p_j - \nu_j$ commute with each other, i.e. $[p_k - \nu_k, p_j - \nu_j] = 0$.

3. $\|\nu_{i,j}\nu_j + \frac{d\nu_i}{dt}\|_\infty \leq C|t|^{-(1+\delta)}$. Here we make the choice $1 + \delta = \min\{1 + \gamma, 2 - 2\gamma\} > 1$. Summation over double indices is understood.

These properties can be shown by a direct calculation ([Gr]). Now we are going to prove (5.1) and (5.2) and proceed by observing that

\[ i \frac{\partial}{\partial t} U(t,s) \psi_0 = H(t)U(t,s) \psi_0 \quad \text{and} \quad -i \frac{\partial}{\partial s} U(t,s) \psi_0 = U(t,s)H(s) \psi_0. \]

It follows from the above that $\frac{d}{dt} \langle K(t,s) \rangle = (U(t,s) \psi_0, (i[H(t), K(t)] + \frac{dK}{dt})U(t,s) \psi_0)$. A straightforward calculation shows:

\begin{align*}
\frac{\partial K}{\partial t} &= \sum_{l=1}^{m} \sum_{k=0}^{n-1} \frac{1}{2} (p - \nu(x,t))^2 + V_l(t) \frac{d}{dt} \left( \frac{1}{2} (p - \nu)^2 + V_l(t) \right) \frac{1}{2} (p - \nu)^2 + V_l(t) \kappa^{-1-k} \\
&\quad - (m - 1) \sum_{k=0}^{n-1} \frac{1}{2} (p - \nu)^2 \frac{d}{dt} 2 \left( p - \nu(x,t) \right)^2 \left( \frac{1}{2} (p - \nu)^2 \right) \kappa^{-1-k} := J_1 + J_2,
\end{align*}

and the commutator

\[ [H(t), K(t)] = \frac{1}{2} p^2 + \sum_{l=1}^{m} V_l(t), \sum_{l=1}^{m} \left( \frac{1}{2} (p - \nu)^2 + V_l(t) \right)^\kappa - (m - 1) \left( \frac{1}{2} (p - \nu)^2 \right)^\kappa \]
\[ \frac{\partial}{\partial t} \sum_{i=1}^{m} \frac{\nu_{i,j} \nu_{j,i} + \frac{\partial \nu_i}{\partial t} + \frac{1}{4} \nu_{i,j,j} + (\nu - \bar{v}) \cdot \nabla V_i}{ \sum_{l,j} \nu_{l,j} \nu_{j,l} + \frac{\partial \nu_l}{\partial t} } \leq C |t|^{-(1+\delta)} \]

and that the \( L_\infty \) norm of derivatives of these terms decay even faster because each space derivative gains a factor \(|t|^{\delta-1}\). Moreover \( \sum_{i=1}^{m} (\nu - \bar{v}_l) \cdot \nabla V_i \) vanishes as \( |t| > T \) is sufficiently large, since \( \nu - \bar{v}_l \) vanishes on an increasing neighborhood of \( x = t\bar{v}_l \), which will eventually contain the support of \( \nabla V_i \).

Plugging the expression of \( M_1 \) into expression (5.5), we claim the decaying terms listed in equation (5.7) only produce time integrable term. We calculate the term containing \( \frac{1}{2} p_i (\nu_{i,j} \nu_{j,i} + \frac{\partial \nu_i}{\partial t}) \) as an example to illustrate this point:

\[ \|(U(t,s)\psi_0, \frac{1}{2} (p - \nu)^2 + V_i(t))^k \frac{1}{2} p_i (\nu_{i,j} \nu_{j,i} + \frac{\partial \nu_i}{\partial t}) (\frac{1}{2} (p - \nu)^2 + V_i(t))^{\kappa-1-k} U(t,s)\psi_0) \]

\[ = |(\frac{1}{2} p_i (\nu_{i,j} \nu_{j,i} + \frac{\partial \nu_i}{\partial t}) (\frac{1}{2} (p - \nu)^2 + V_i(t))^{\kappa-1-k} U(t,s)\psi_0)| \]

\[ = |(\frac{1}{2} p_i (\nu_{i,j} \nu_{j,i} + \frac{\partial \nu_i}{\partial t}) (\frac{1}{2} (p - \nu)^2 + V_i(t))^{\kappa-1-k} U(t,s)\psi_0)|. \]
If $2k + 1 = \kappa$ or $2k + 2 = \kappa$, (5.8) can be dominated by

\[
C|t|^{-1-\delta}||p_i \left( \frac{1}{2} (p - \nu) + V_i(t) \right) U(t, s) \psi_0||_{L^2} \leq C|t|^{-1-\delta}||U(t, s) \psi_0||^2_{H^\kappa} \\
\leq C|t|^{-1-\delta} (\langle K(t, s) \rangle + ||U(t, s) \psi_0||^2_{H^{\kappa-1}}). 
\]

If $\kappa \neq 2k + 1$ or $2k + 2$, first consider $2k + 2 < \kappa$ and $\kappa = 2d + 1$, an odd integer. We need to commute $\nu_{i,j} V_j + \frac{\partial \nu_i}{\partial t}$ with $(\frac{1}{2} (p - \nu)^2 + V_i)^{d-k}$. Specifically, we claim that

\[
(\frac{1}{2} (p - \nu)^2 + V_i(t))^{d-k} (\nu_{i,j} V_j + \frac{\partial \nu_i}{\partial t}) (\frac{1}{2} (p - \nu)^2 + V_i(t))^{k}
\]

is an differential operator of degree $2d + 1$, whose coefficients are of magnitude $t^{-1-\delta}$. This is clear because $\nu_{i,j} V_j + \frac{\partial \nu_i}{\partial t}$ and its derivatives decay at least as $|t|^{-1-\delta}$. Hence, (5.8) is dominated by $C|t|^{-1-\delta} (\langle K(t, s) \rangle + ||U(t, s) \psi_0||^2_{H^{\kappa-1}})$. In the case that $2k + 2 < \kappa$ and $\kappa = 2d$ or $2k + 1 > \kappa$, (5.8) is dominated by $C|t|^{-1-\delta} (\langle K(t, s) \rangle + ||U(t, s) \psi_0||^2_{H^{\kappa-1}})$ due to the same reason.

Therefore, it remains to estimate the following in expression (5.3):

\[
(5.9) \quad \sum_{l=1}^{m} \sum_{k=0}^{\kappa-1} \left( \frac{1}{2} (p - \nu)^2 + V_l(t) \right)^k \left( i \left( \sum_{j \neq l} V_j \right) \frac{1}{2} (p - \nu)^2 + V_l \right) + A \left( \frac{1}{2} (p - \nu)^2 + V_l(t) \right)^{\kappa-1-k}.
\]

Observe that for given time $t$, $\nu(x, t)$ is a constant vector on a ball centered at $t \hat{v}_l$ with radius growing linearly in $|t|$ approximately. So as long as $|t|$ is large, $\nu(x, t)$ will be constant on the support of $V_l(t)$. This implies that $\nu_{j,i}, \nu_{i,j}$ both vanish on the support of $V_l(t)$. Hence it follows from $A = -(p_i - \nu_i) \frac{\nu_{i,j} + \nu_{j,i}}{2} (p_j - \nu_j)$ that $AV_l = 0$ and $V_l A = 0$. Moreover, for $j \neq l$, $V_j(t), V_l(t)$ have disjoint supports given that $t$ is large. So the expression (5.3) is reduced to the following:

\[
(5.10) \quad \sum_{l=1}^{m} \sum_{k=0}^{\kappa-1} \left( \frac{1}{2} (p - \nu)^2 \right)^k \left( i \left( \sum_{j \neq l} V_j \right) \frac{1}{2} (p - \nu)^2 \right) + A \left( \frac{1}{2} (p - \nu)^2 \right)^{\kappa-1-k}
\]

\[
(5.11) \quad = \sum_{k=0}^{\kappa-1} \left( \frac{1}{2} (p - \nu)^2 \right)^k (mA + (m - 1) i \left( \sum_{j} V_j \right) \frac{1}{2} (p - \nu)^2) \left( \frac{1}{2} (p - \nu)^2 \right)^{\kappa-1-k}
\]

Secondly, we consider

\[
J_2 + iJ_4 + iJ_6 = -(m-1) \sum_{k=0}^{\kappa-1} \frac{1}{2} (p - \nu)^2 \left( i \frac{1}{2} p^2 + \sum_{j=1}^{m} V_j(t) \right) \left( \frac{1}{2} (p - \nu)^2 \right) + \frac{d}{dt} \frac{1}{2} (p - \nu)^2 \left( \frac{1}{2} (p - \nu)^2 \right)^{\kappa-1-k}.
\]

Setting all potentials $V_l = 0$ in (5.6), we see that

\[
i \left( \frac{1}{2} p^2, \frac{1}{2} (p - \nu)^2 \right) + \frac{d}{dt} \frac{1}{2} (p - \nu)^2 = A + \text{time integrable terms},
\]

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where the time integrable terms equal to
\[
A - \frac{1}{2}p_i(\nu_{i,j}\nu_j + \partial\nu_{i,j}/\partial t) - \frac{1}{2}(\nu_{i,j}\nu_j + \partial\nu_{i,j}/\partial t)p_i + \nu_i(\nu_{i,j}\nu_j + \partial\nu_{i,j}/\partial t) + \frac{1}{4}\nu_{i,ijj}
\]
and can be estimated exactly as those in \(J_1 + iJ_3 + iJ_5\). We are left to estimate in \(J_2 + iJ_4 + iJ_6\):

\[
-(m - 1)\sum_{k=0}^{\kappa-1}\left(\frac{1}{2}(p - \nu)^2\right)^k(A + i\sum_{j=1}^{m}V_j(t), \frac{1}{2}(p - \nu)^2\right)((\frac{1}{2}(p - \nu)^2)^{\kappa-1-k}
\]

Now adding (5.11) and (5.13) together, we see that \(\langle i[H(t),K(t)] + \partial K/\partial t \rangle \) is simplified as some time integrable terms plus the following:

\[
\sum_{k=0}^{\kappa-1}\left(\frac{1}{2}(p - \nu)^2\right)^kA\left(\frac{1}{2}(p - \nu)^2\right)^{\kappa-1-k},
\]

which is a differential operator of degree \(2\kappa\).

First we observe that \([p_k - \nu_k, p_j - \nu_j] = 0\) and \((p_k - \nu_k)^{\nu_{i,j} + \nu_{j,i}} = \nu_{i,j}^2(p_k - \nu_k) + \nu_{j,i}^2(p_k - \nu_k)\). Second, \(\nu_{i,j,k} + \nu_{j,i,k}\) and its derivatives decay at least as fast as \(|t|^{-1-\delta}\) when \(t \to \infty\) and thus is integrable in time. Hence if we commute \(A\) with \((p - \nu)^2\) or \(p_j - \nu_j\), the commutator is time integrable.

If \(\kappa = 2d + 1\), an odd integer, then

\[
\left(\frac{1}{2}(p - \nu)^2\right)^kA\left(\frac{1}{2}(p - \nu)^2\right)^{\kappa-1-k} = \left(\frac{1}{2}(p - \nu)^2\right)^dA\left(\frac{1}{2}(p - \nu)^2\right)^d + \text{time-integrable terms}.
\]

The first summand is negative (positive) definite when \(t > 0\) (\(t < 0\)).

If \(\kappa = 2d\), an even integer, then \((\frac{1}{2}(p - \nu)^2)^kA(\frac{1}{2}(p - \nu)^2)^{\kappa-1-k} = \frac{1}{2}(p - \nu)^2)^{\kappa-1-k} = \frac{1}{2}(p - \nu)^2)^{\kappa-1-k} + \text{time-integrable terms}\). Again the first summand is negative (positive) definite if \(t > 0\) (\(t < 0\)).

Hence, we have written \(\frac{\partial}{\partial t}\langle K(t,s) \rangle\) as a sum of a negative (positive if \(t < 0\)) term and other time-integrable terms. More precisely, the time-integrable terms decay at least as fast as \(|t|^{-1-\delta}\). Therefore, we have proved (5.11) for \(t > T\) and (5.2) for \(t < -T\).

Finally, we deal with the case where \(|t| < T, s > T\) by time reversal. Write \(r = s - t\) and \(\bar{U}(r,s) = U(s - r,s), \bar{H}(r) = H(s - r)\). Then we have \(i\partial_r\bar{U}(r,s) = -\bar{H}(r)\bar{U}(r,s)\). Define the corresponding observable:

\[
\bar{K}(r) = \sum_{l=1}^{m}\left(\frac{1}{2}(p + \nu(x, s - r))^2 + V_l(x - s\bar{v}_l + \bar{v}(r))\right)^k - (m - 1)\left(\frac{1}{2}(p + \nu(x, s - r))^2\right)^k.
\]

It can be shown that \(\bar{U}(r,s)\) is a bounded operator from \(H^k\) to itself by the same argument with \(U(t,s)\) replaced by \(\bar{U}(r,s)\). The case of \(|t| < T, s < -T\) is similar.
6 Asymptotic completeness in Sobolev spaces

Recall that we are considering \( V_1 \). \( V_1 \) is stationary (we denote its velocity as \( \vec{e}_0 = 0 \)) and \( V_2 \) is moving with velocity \( \vec{e}_1 \). There are two approaches to prove Theorem 1.13. Graf (\cite{Gr}) proved the asymptotic completeness for the charge transfer model in the \( L^2 \) sense by proving a RAGE theorem. Our first option to prove Theorem 1.13 is to generalize Graf’s idea. We find that this approach works, provided the fact that each individual subsytem (i.e. \( p^2 + V_l \)) is asymptotically complete in the \( H^\kappa \) sense. However the only direct way to prove this fact, as we know, is by the dispersive estimate. The good point of this approach is that it requires less restrictive condition on the potentials and the spectrum of the individual subsystem, given that nontrivial fact. Our second option to prove Theorem 1.13 is to apply the dispersive estimate (Theorem 1.8) directly. To illustrate both of these ideas, the following proof is somehow a combination of these two options. Specifically, we follow \( \text{Gr} \) to prove the existence of the wave operators and then apply Theorem 1.8 to prove Theorem 1.13.

6.1 Existence of wave operators

The well-known wave operators are defined as following:

\[
\Omega^-_0 (s) = s - \lim_{t \to +\infty} U(s,t)e^{-i(t-s)H_0},
\]

\[
\Omega^-_1 (s) = s - \lim_{t \to +\infty} U(s,t)e^{-i(t-s)H_1}P_b(H_1),
\]

\[
\Omega^-_2 (s) = s - \lim_{t \to +\infty} U(s,t)\, \mathfrak{g}_-\vec{e}_1(t)\, e^{-i(t-s)H_2}P_b(H_2)\, \mathfrak{g}_\vec{e}_1(s).
\]

**Theorem 6.1.** Under the assumption of Theorem 1.10 the above wave operators exist in the space \( H^\kappa \). More precisely, for \( l = 0, 1, 2 \) and \( \forall \psi_0 \in H^\kappa \), the limits converges in the \( H^\kappa \) sense and \( \Omega^-_l (s)\psi_0 \) lies in \( H^\kappa (\mathbb{R}^n) \).

**Remark 6.2.** The above theorem can be proved by Cook’s method together with Theorem 1.8 and Theorem 1.10 if we are willing to impose more regularity on the potentials and the spectrum condition. The following proof originated in \( \text{Gr} \), which we believe, requires the least conditions on the system.

We present some preliminary facts before we proceed:

**Lemma 6.3.** Let \( g \in C^\infty_0 (\mathbb{R}^n) \) and \( \nu > 0 \). Suppose

1. \( g(p) = 0 \) for \( |p| \geq \nu \) and fix \( \alpha > 1 \). Then for \( R > 0, t > 0 \) and any \( N > 0 \),

\[
\| F(|x| > \alpha(R + \nu t))e^{-i\frac{p^2}{2}t} g(p) F(|x| < R)\psi \|_{H^\kappa} \leq C_{N,\kappa} (R + \nu t)^{-N} \| \psi \|_{L^2}.
\]

2. \( g(p) = 0 \) for \( |p| \leq \nu \) and \( \nu_0 > 0, 0 < \alpha < 1 \). Then for \( t > 0 \) and any \( N > 0 \),

\[
\| F(|x| > \alpha(\nu - \nu_0) t)e^{-i\frac{p^2}{2}t} g(p) F(|x| < \nu_0 t)\psi \|_{H^\kappa} \leq C_{N,\kappa} t^{-N} \| \psi \|_{L^2}.
\]
These estimates are fairly common for \( \kappa = 0 \) and may be proved by the stationary phase methods (e.g. [En], Lemma (6.3)). For the case \( \kappa \geq 1 \), it follows from a commutator argument and the fact that the derivative on the left-hand side can be absorbed into Lemma 6.4.

**Lemma 6.4.** Let \( g \in C^0_0(\mathbb{R}) \) and \( v > 0 \). Suppose \( g(e) = 0 \) for \( e \geq v^2/2 \) and fix \( \alpha > 1 \). Then for \( l = 1, 2, R > 0 \) and \( t \geq 0 \), we have

\[
(6.1) \quad \| F(|x| > \alpha(R + vt))e^{-iH_0(t)}g(H_1)F(|x| < R)\psi(x)\|_{H^\kappa} \leq C_{N, \kappa}(R + vt)^{-\epsilon}\| \psi \|_{L^2}.
\]

When \( \kappa = 0 \), the lemma is just Lemma 4.2 of [Gr]. For \( \kappa \geq 1 \), the left-hand side of (6.1) is dominated up to a constant by

\[
\|(H_1 + M)\tilde{g}e^{-iH_0(t)}g(H_1)F(|x| < R)\psi(x)\|_{L^2(|x| > \alpha(R + vt))}.
\]

where \( M \) is chosen so large that \( H_1 + M \) is a positive operator. If we define \( \tilde{g}(H_1) = (H_1 + M)^{-\frac{1}{2}}g(H_1) \), then \( \tilde{g} \in C^0_0(\mathbb{R}) \). The above is of the form of \( \kappa = 0 \) and the lemma follows from the case \( \kappa = 0 \).

**Lemma 6.5.**

1. Let \( 0 < v_0 < v \) and \( g \in C^0_0(\mathbb{R}^n) \) with \( g(p) = 0 \) for \( \{|p| < v\} \cup \{|p - e_1| < v\} \).

   Then for any \( s \in \mathbb{R} \),

   \[
   \lim_{t_1 \to +\infty_{t_2 > t_1}} \sup \| (U(t_2, t_1) - e^{-iH_0(t_2-t_1)})e^{-iH_0(t_1-s)}g(p) \prod_{l=0}^1 F(|x - e_l s| < v_0(t_1 - s)) \|_{L^2 \to H^\kappa} = 0.
   \]

2. Let \( v_0, v > 0 \) with \( v_0 + v < |e_1| \) and \( g \in C^0_0(\mathbb{R}) \) with \( g(p) = 0 \) for \( p > v^2/2 \). Then

   \[
   \lim_{t_1 \to +\infty_{t_2 > t_1}} \sup \| (U(t_2, t_1) - e^{-iH_1(t_2-t_1)})g(H_1)F(|x| < v_0 t_1) \|_{L^2 \to H^\kappa} = 0.
   \]

For \( \kappa = 0 \), the lemma was proved in [Gr]. We will follow the approach there to prove the case \( \kappa > 0 \).

**Proof.** Part (1): Take \( \alpha < \alpha_1 < 1 \) and let \( f \in C^0_0(\mathbb{R}^n) \) with \( f(y) = 0 \) if \( |y - e_l| > \alpha(v - v_0) \) for both \( l = 0, 1 \). Since \( \alpha t < \alpha_1(t - s) \), we have \(|f(x/t)| \leq |f(x/t)| \sum_{l=0}^1 F(|x - e_l t| < \alpha_1(v - v_0)(t - s))\) for \( t \) large enough.

\[
\| f(\frac{x}{t})e^{-iH_0(t-s)}g(p) \prod_{l=0}^1 F(|x - e_l s| < v_0(t - s)) \|_{L^2 \to H^\kappa} \]

\[
\leq \sum_{|\beta| < \kappa} \sum_{l=0}^1 \| \partial^\beta f \|_{L^\infty} \| F(|x - e_l t| < \alpha_1(v - v_0)(t - s)) \|
\]

\[
(6.2) \quad \leq C \sum_{|\beta| < \kappa} \sum_{l=0}^1 \| F(|x| < \alpha_1(v - v_0)(t - s))e^{-iH_0(t-s)}g^\beta(p + e_l)F(|x| < v_0(t - s)) \|
\]

\[
\leq C(t - s)^{-N}
\]

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where $g^\beta(p) = \sum_{|\beta+\gamma|=\kappa} p^\gamma g(p)$. The above inequality follows by commuting the derivative through $f(x/t)$, by applying Galilean transform to the second expression, and by Lemma 6.3. By (6.2) and Theorem 1.10, it suffices to show

$$\sup_{t_2 > t_1} \| (U(t_2, t_1)(1 - f(x/t_1)) - (1 - f(x/t_2)) e^{-iH_0(t_2 - t_1)} e^{-iH_0(t_1 - s)} g(p) \|
\cdot \prod_{l=0}^1 F(|x - \vec{e}_l s| < v_0(t_1 - s)) \|_{L^2 \to H^\kappa} \to 0.$$  

Substituting

$$(U(t_2, t_1)(1 - f(x/t_1)) - (1 - f(x/t_2)) e^{-iH_0(t_2 - t_1)}) = \int_{t_1}^{t_2} \frac{d}{dt} (U(t_2, t)(1 - f(x/t)) e^{-iH_0(t - t_1)}) dt$$

into (6.3), it follows from Theorem 1.10 that the left-hand side of (6.3) is dominated by

$$\int_{t_1}^{+\infty} dt \| [iH(t)(1 - f(x/t)) - i(1 - f(x/t)) H_0 - \frac{\partial}{\partial t} f(x/t)] e^{-iH_0(t - s)} g(p) \| \prod_{l=0}^1 F(|x - \vec{e}_l s| < v_0(t_1 - s)) \|_{L^2 \to H^\kappa}.$$  

The expression within the square brackets consists of (1)-(3) which are estimated as follows:

1. Suppose $t$ is sufficiently large, then $V(t)(1 - f(x/t)) = 0$, because $V$ is compactly supported, where we take $f(y) = 1$ for $|y - \vec{e}_1| < \alpha(v - v_0)/2$.
2. $H_0 f(x/t) - f(x/t) H_0 = -\frac{1}{2} t^{-2}(\Delta f)(x/t) - i t^{-1} (\nabla f)(x/t) p$ and
3. $\frac{\partial}{\partial t} f(x/t) = -t^{-1} (x/t)(\nabla f)(x/t)$

are treated using (6.2).

Part (2): Choose $\alpha > 1$ and $v_1$ with $\alpha(v + v_0) < v_1 < |e_1|$ and let $f \in C_0^\infty(\mathbb{R}^n)$ with $f(y) = 1$ for $|y| < \alpha(v + v_0)$ and $f(y) = 0$ for $|y| > v_1$. We first claim that

$$\lim_{t_1 \to +\infty} \sup_{t_2 > t_1} \| (1 - f(x/t)) e^{-iH_1(t - t_1)} g(H_1) F(|x| < v_0 t_1) \|_{L^2 \to H^\kappa} = 0$$

Since $1 - f(x/t)$ is supported in $|x| > \alpha(v + v_0) t > \alpha(v_0 t_1 + v(t - t_1))$, it follows from Lemma 6.4 that

$$\| F(|x| > \alpha(v_0 t_1 + v(t - t_1)) e^{-iH_1(t - t_1)} g(H_1) F(|x| < v_0 t_1) \|_{L^2 \to H^\kappa} \leq C_{N, \kappa}(v_0 t_1 + v(t - t_1))^{-\epsilon}.$$  

Now by Theorem 1.10 and

$$(U(t_2, t_1)f(x/t_1) - f(x/t_2) e^{-iH_1(t_2 - t_1)}) = \int_{t_1}^{t_2} \frac{d}{dt} (U(t_2, t)f(x/t) e^{-iH_1(t - t_1)}) dt,$$

it suffices to estimate
\[
\sup_{t_2 > t_1} \| (U(t_2, t_1) f(x/t_1) - f(x/t_2)e^{-iH_1(t_2-t_1)})g(H_1)F(|x| < v_0 t_1) \|_{L^2 \to H^s}
\]
\[
\leq \int_{t_1}^{+\infty} dt \| [iH(t)f(x/t) - if(x/t)H_1 + \frac{\partial}{\partial t}f(x/t)]e^{-iH_1(t-t_1)}g(H_1)F(|x| < v_0 t_1) \|_{L^2 \to H^s},
\]

As in Part (1), a discussion of terms (a)-(d) in the square brackets now follows:

(a) \( V_1(x)f(x/t) - f(x/t)V_1(x) = 0 \)

(b) \( V_2(x)\bar{f}(x/t) = 0 \) if \( t \) is large enough because \( V_2 \) is compactly supported and \( f(y) = 0 \) for \( |y| > v_1 \) and \( |e_1| > v_1 \).

(c) \( [H_0, f(x/t)] = \frac{1}{2} t^{-2} \Delta f(x/t) - \frac{i}{2} \bar{\nabla} f(x/t) \). Since \( V_1 \in C_0^\infty \), we can take \( M \) large enough, so that the corresponding term can be dominated by

\[
\| (M + H_1)^{\frac{1}{2}}(\frac{1}{2} t^{-2} \Delta f(x/t) - \frac{i}{2} \bar{\nabla} f(x/t)) e^{-iH_1(t-t_1)}g(H_1)F(|x| < v_0 t_1) \|_{L^2 \to L^2}.
\]

Commute \( (M + H_1)^{\frac{1}{2}} \) through \( (\frac{1}{2} t^{-2} \Delta f(x/t) + \frac{i}{2} \bar{\nabla} f(x/t) \bar{\nabla}) \) and the commutators generated will decay at least as fast as \( t^{-2} \), hence they are time-integrable. Note \( \| (p^2 + 1)\frac{1}{2} g(H_1) \|_{L^2 \to L^2} < C_\sigma \).

The only term that does not decay as fast as \( t^{-2} \) is

\[
\| (M + H_1)^{\frac{1}{2}}(\frac{i}{2} \bar{\nabla} f(x/t)) e^{-iH_1(t-t_1)}(M + H_1)^{\frac{1}{2}+1}g(H_1)F(|x| < v_0 t_1) \|_{L^2 \to L^2},
\]

which is integrable, due to the fact that \( (M + H_1)^{-1}p \) is a bounded operator from \( L^2 \) to \( L^2 \) and due to \( (b,5) \) (with \( g(H_1) \) replaced by \( (M + H_1)^{\frac{1}{2}+1}g(H_1) \)), and due to the support property of \( \nabla f(x/t) \).

(d) \( \frac{\partial}{\partial t}f(x/t) = \frac{\partial}{\partial t}f(x/t) t^{-1} \), which can be treated as part (c), using \( (b,5) \) with \( f(x) \) replaced by \( xf(x) \).

**proof of Theorem 6.1.** Since \( U(s,t)e^{-i(t-s)H_0} \) and \( U(s,t)e^{-i(t-s)H_1} \) are uniformly bounded operators from \( H^s \) to \( H^s \), it suffices to prove the existence of the strong limits \( \Omega_0^s(s) \) and \( \Omega_1^s(s) \) on a dense set \( D \):

\[
D = \{ g(p)f(x)\psi : g \in C_0^\infty(\mathbb{R}^n \setminus \{0, e_1\}), f \in C_0^\infty(\mathbb{R}^n), \psi \in L^2(\mathbb{R}^n) \}.
\]

\( g(p) \) satisfies the hypothesis of Lemma 5.5 Part (1), with a suitable \( v > 0 \). Take \( 0 < v_0 < v \) and note that

\[
\prod_{l=1}^{2} F(|x - \vec{e}_l| < v_0(t_1 - s)) f(x) = f(x)
\]

for \( t_1 \) big enough. For \( t_2 > t_1 \), it follows from Theorem 1.10 that

\[
\| U(s, t_1)e^{-iH_0(t_1-s)} - U(s, t_2)e^{-iH_0(t_2-s)} \|_{H^s} \leq \| U(t_2, t_1) - e^{-iH_0(t_2-t_1)} e^{-iH_0(t_1-s)} \|_{L^2 \to H^s} \| f(x) \|_{L^2}.
\]
Hence Lemma 6.5 implies that \( U(s, t)e^{-iH_0(t-s)}g(p)f(x)\psi \) is Cauchy sequence in \( H^\kappa(\mathbb{R}^n) \) as \( t \to +\infty \), which is equivalent to the existence of \( \Omega_1^{-}(s) \).

We will only show the existence of \( \Omega_1^{-}(s) \). The existence of \( \Omega_2^{-}(s) \) follows from the same argument up to a Galilean transform \( (\mathcal{G}_t) \). Since the eigenfunctions of \( H_1 \) span the range of \( P_b(H_1) \), it suffices to prove convergence on the eigenfunctions \( \psi : H_1\psi = E\psi \). Due to our assumptions on the potentials, the positive eigenvalues are excluded. Thus for any \( v > 0 \), we can find a suitable \( g \) as in Lemma 6.5 Part (2) with \( g(H_1)P(H_1) = P(H_1) \). More precisely, we take \( v, v_0 > 0 \) with \( v + v_0 < |e_1| \). For \( t_2 \geq t_1 \),

\[
\| (U(s, t_1)e^{-iH_1(t_1-s)}P(H_1) - U(s, t_2)e^{-iH_1(t_2-s)})\psi \|_{H^\kappa} = \|U(s, t_2)(U(t_2, t_1) - e^{iH_1(t_2-t_1)})e^{-iH_1(t_1-s)}g(H_1)(F(|x| < v_0t_1) + F(|x| > v_0t_1))\psi \|_{H^\kappa} \lesssim \|U(t_2, t_1) - e^{iH_1(t_2-t_1)}e^{-iH_1(t_1-s)}g(H_1)F(|x| < v_0t_1)|_{L^2 \to H^\kappa} \|\psi\|_{L^2} + \|F(|x| > v_0t_1)\psi \|_{H^\kappa},
\]

since \( U(s, t), H_1(s) \) and \( g(H_1) \) are bounded operators on \( H^\kappa(\mathbb{R}^n) \) with a uniform bound. Lemma 6.5 part (2) and the fact that \( \|F(|x| > v_0t_1)\psi \|_{H^\kappa} \to 0 \) when \( t_1 \to +\infty \) imply that \( U(s, t)e^{-iH_1(t^r-s)}P(H_1)\psi \) is a Cauchy sequence in \( H^\kappa \).

### 6.2 Asymptotic completeness

In this section we will apply Theorem 1.8 and 1.10 to prove Theorem 1.13. For the case \( \kappa = 0 \), we refer the reader to [RSS1].

**Proof of Theorem 1.13.** First let us assume that \( \psi_0 \in W^{\kappa,2} \cap W^{\kappa,p'} \) for some \( 1 < p' < \frac{2n}{2+n} \). Decompose

\[
\psi(t) := U(t)\psi_0 = P_b(H_1)U(t)\psi_0 + P_b(H_2, t)U(t)\psi_0 + R(t).
\]

By construction, we clearly have

\[
(6.6) \quad P_b(H_2, t)U(t)\psi_0 + R(t) \in \text{Ran}(P_c(H_1)),
\]

\[
P_b(H_1)U(t)\psi_0 + R(t) \in \text{Ran}(P_c(H_2, t)).
\]

We further write

\[
P_b(H_1)U(t)\psi_0 = \sum_{r=1}^{m} e^{-i\lambda_r t}a_r(t)u_r(x)
\]

for some choice of unknown functions \( a_r(t) \). Due to the smoothness of the potentials, \( u_r \) belongs to \( H^\kappa(\mathbb{R}^n) \). It follows from (6.6) that, similar to (5.1),

\[
\dot{a}_r + i \langle V_2(\cdot - t\bar{e}_1)\psi(t), u_r \rangle = 0 \quad \text{for all} \quad 1 \leq r \leq m.
\]

The exponential localization of \( u_r \) implies that \( |\langle V_2(\cdot - t\bar{e}_1)\psi(t), u_r \rangle| \lesssim e^{-\alpha t} \). Therefore, \( a_r(t) \) has a limit, writing \( \lim_{t \to +\infty} a_r(t) = A_r \), and

\[
(6.7) \quad \left\| P_b(H_1)U(t)\psi_0 - \sum_{r=1}^{m} A_re^{-i\lambda_r t}u_r \right\|_{H^\kappa} \to 0, \quad t \to +\infty.
\]
We next define the functions \( v_r = \lim_{t \to +\infty} U(t)^{-1} e^{-i\lambda_r t} u_r \). The existence of \( v_r \) and \( v_r \in H^\kappa \) is guaranteed by Theorem 6.1. By Theorem 1.10, we have

\[
(6.8) \quad \left\| U(t) \left( \sum_{r=1}^m A_r v_r \right) - \sum_{r=1}^m A_r e^{-i\lambda_r t} u_r \right\|_{H^\kappa} \to 0, \quad t \to +\infty.
\]

We then infer from (6.7) that

\[
(6.9) \quad \left\| U(t) \left( \sum_{r=1}^m A_r v_r \right) - P_b(H_1) U(t) \psi_0 \right\|_{H^\kappa} \to 0, \quad t \to +\infty.
\]

The above arguments apply to \( P_b(H_2, t) U(t) \psi_0 \) in a similar fashion. More precisely, we write

\[
U(t) \psi_0 = P_b(H_2, t) U(t) \psi_0 + \Gamma(t) = g_{-\vec{e}_i}(t) P_b(H_2) g_{\vec{e}_1}(t) U(t) \psi_0 + \Gamma(t).
\]

Therefore,

\[
(6.10) \quad g_{\vec{e}_1}(t) U(t) \psi_0 = P_b(H_2) g_{\vec{e}_1}(t) U(t) \psi_0 + g_{\vec{e}_1}(t) \Gamma(t)
\]

Recall that the function \( \tilde{\psi}(t) = g_{\vec{e}_1}(t) U(t) \psi_0 \) is a solution of the problem

\[
(6.11) \quad \frac{1}{i} \partial_t \tilde{\psi} - \frac{\Delta}{2} \tilde{\psi} + V_2(x) \tilde{\psi} + V_1(x + t \vec{e}_1) \tilde{\psi} = 0, \quad \tilde{\psi} \big|_{t=0} = g_{\vec{e}_1}(0) \psi_0.
\]

According to (6.10), \( \tilde{\psi}(t) = P_b(H_2) \tilde{\psi}(t) + \Gamma_1(t) \), where \( \Gamma_1(t) = g_{\vec{e}_1}(t) \Gamma(t) \). In particular,

\[
\Gamma_1(t) \in \text{Ran}(P_c(H_2)).
\]

Decompose

\[
P_b(H_2) \tilde{\psi}(t) = \sum_{s=1}^{\ell} b_s(t) e^{-i\mu_s t} w_s
\]

for some choice of unknown functions \( b_s(t) \). Again due to the smoothness of the potentials, \( w_s \in H^\kappa(\mathbb{R}^n) \). After substituting the decomposition in (6.11) we obtain the equations

\[
\dot{b}_s(t) + i \langle V_1(\cdot + t \vec{e}_1) \tilde{\psi}, w_s \rangle = 0 \quad \text{for all} \quad 1 \leq s \leq \ell.
\]

Using exponential localization of \( w_s \) we conclude the existence of the limit \( b_s(t) \to B_s \) as \( t \to +\infty \). Thus \( \left\| P_b(H_2) \tilde{\psi}(t) - \sum_{s=1}^{\ell} B_s e^{-i\mu_s t} w_s \right\|_{H^\kappa} \to 0, \quad t \to \infty \). Equivalently, after applying \( g_{-\vec{e}_1}(t) \), we have

\[
(6.12) \quad \left\| P_b(H_2, t) U(t) \psi_0 - \sum_{s=1}^{\ell} B_s e^{-i\mu_s t} g_{-\vec{e}_1}(t) w_s \right\|_{H^\kappa} \to 0.
\]

Now Theorem 6.1 allows us to define

\[
\omega_s := \Omega_2 w_s = s - \lim_{t \to +\infty} U(t)^{-1} g_{-\vec{e}_1}(t) e^{-itH_2} P_b(H_2) w_s \in H^\kappa.
\]
Moreover,

\[(6.13) \quad \left\| U(t) \left( \sum_{s=1}^{\ell} B_s \omega_s \right) - \sum_{s=1}^{\ell} B_s e^{-i\mu_s t} g_{-\varepsilon}(t) w_s \right\|_{H^\kappa} \to 0, \quad t \to +\infty.\]

It then follows from \((6.12)\) that

\[(6.14) \quad \| P_b(H_2, t) U(t) \psi_0 - U(t) \left( \sum_{s=1}^{\ell} B_s \omega_s \right) \|_{H^\kappa} \to 0, \quad t \to +\infty.\]

We now define the function

\[(6.15) \quad \phi := \psi_0 - \sum_{r=1}^{m} A_r v_r - \sum_{s=1}^{\ell} B_s \omega_s,\]

which will lead to the initial data \(\phi_0\) for the free channel. We have that

\[P_b(H_1) U(t) \phi = P_b(H_1) U(t) \psi_0 - P_b(H_1) U(t) \left( \sum_{r=1}^{m} A_r v_r \right) - P_b(H_1) U(t) \left( \sum_{s=1}^{\ell} B_s \omega_s \right).\]

It follows from \((6.9)\) and the identity \(P_b^2(H_1) = P_b(H_1)\) that

\[(6.16) \quad \left\| P_b(H_1) U(t) \psi_0 - P_b(H_1) U(t) \left( \sum_{r=1}^{m} A_r v_r \right) \right\|_{H^\kappa} \to 0 \quad \text{as} \quad t \to +\infty.\]

Furthermore,

\[(6.17) \quad P_b(H_1) \sum_{s=1}^{\ell} B_s e^{-i\mu_s t} g_{-\varepsilon}(t) w_j = \sum_{r=1}^{m} \sum_{s=1}^{\ell} B_s e^{-i\mu_s t} \langle g_{-\varepsilon}(t) w_j, u_r \rangle u_r \to 0\]

in the \(H^\kappa\) sense as \(t \to +\infty\), due to the exponential localization of the eigenfunctions \(u_r\). We infer from \((6.16), (6.13),\) and \((6.17)\) that \(\| P_b(H_1) U(t) \phi \|_{H^\kappa} \to 0\). Similarly, \(\| P_b(H_2, t) U(t) \phi \|_{H^\kappa} \to 0\). Thus, \(U(t)\) is asymptotically orthogonal to the bound states of \(H_1\) and \(H_2\). \(V_j \in C_0^{n+2k+2}\) implies that \((1 + |\xi|)^{\kappa+1+2k} \hat{V}_j(\xi) \in L^2(\mathbb{R}^n)\). So \((1 + |\xi|)^{\kappa} \hat{V}_j(\xi) \in L^1(\mathbb{R}^n)\). Therefore, according to Theorem \ref{thm:orth}, \(U(t)\) satisfies the estimate

\[(6.18) \quad \| U(t) \phi \|_{W^{\kappa,p}} \lesssim |t|^{-n(\frac{1}{2} - \frac{1}{p})} \| \phi \|_{W^{\kappa,p'}}\]

where \(\frac{2m}{n-2} < p < +\infty\). In order to be able to apply the estimate \((6.18)\), one needs to verify that \(\phi \in W^{\kappa,p'}\). By assumption, \(\psi_0 \in W^{\kappa,p'}\). Thus it remains to check \(v_r \in W^{\kappa,p'}, r = 1, \ldots, m\) and \(\omega_s \in W^{\kappa,p'}, s = 1, \ldots, \ell\), which is guaranteed by Lemma \ref{lem:estimate} below. Assuming this lemma for the moment, we now consider the expression

\[e^{-it\frac{\Delta}{2}} U(t) \phi = \phi - i \int_0^t e^{-is\frac{\Delta}{2}} \left( V_1(x) + V_2(x - s \varepsilon) \right) U(s) \phi ds.\]
Writing $\frac{2\nu}{p-2} = r$, we have the following estimate:

$$
\int_{t}^{+\infty} \|e^{-is\frac{\Delta}{2}} (V_1(x) + V_2(x - se\gamma_1)) U(s)\phi\|_{H^r} ds
\lesssim (\|V_1\|_{W^{2,r}} + \|V_2\|_{W^{2,r}}) \int_{t}^{+\infty} \|U(s)\phi\|_{W^{2,r}} ds
\lesssim \int_{t}^{+\infty} |s|^{-n\left(\frac{1}{2} - \frac{1}{p}\right)} \|\phi\|_{W^{2,r}} (\|V_1\|_{W^{2,r}} + \|V_2\|_{W^{2,r}}) \to 0, \quad \text{as } t \to +\infty.
$$

Here we note that $-n\left(\frac{1}{2} - \frac{1}{p}\right) < -1$. This allows us to show the existence of the limit

$$
\phi_0 := \lim_{t \to \infty} e^{it\frac{\Delta}{2}} U(t)\phi \in H^r.
$$

It follows that

(6.19) \hspace{1cm} \|U(t)\phi - e^{-it\frac{\Delta}{2}} \phi_0\|_{H^r} \to 0, \quad t \to +\infty.

Combining (6.8), (6.13), (6.15), and (6.19) we infer that

$$
\left\|U(t)\psi_0 - \sum_{r=1}^{m} A_r e^{-i\lambda_r t} u_r - \sum_{s=1}^{\ell} B_s e^{-i\mu_s t} g_{-\varepsilon_1}(t) w_s - e^{-it\frac{\Delta}{2}} \phi_0 \right\|_{H^r} \to 0, \quad \text{as } t \to +\infty,
$$

as claimed. Because $W^{\kappa,2} \cap W^{\kappa,p'}$ is dense in $W^{\kappa,2}$, for any $\psi_0 \in W^{\kappa,2}$, there is a sequence $\psi_l \in W^{\kappa,2} \cap W^{\kappa,p'}$ converging to $\psi_0$ in the $W^{\kappa,2}$ norm. Then for each $\psi_l$, we have the following decomposition:

$$
U(t)\psi_l = \sum_{r=1}^{m} A_r e^{-i\lambda_r t} u_r + \sum_{s=1}^{\ell} B_s e^{-i\mu_s t} g_{-\varepsilon_1}(t) w_s + e^{-it\frac{\Delta}{2}} \phi_l + \mathcal{R}_l(t),
$$

It follows from Theorem 1.10 that $\psi_l = \sum_{r=1}^{m} A_r \Omega_1^{-} u_r + \sum_{k=1}^{\ell} B_k \Omega_2^{-} w_k + \Omega_0^{-} \phi_l$.

Since the ranges of $\Omega_{0,1,2}$ are orthogonal to each other in $L^2(\mathbb{R}^n)$ (Gr), the fact that $\psi_l$ converges as $l \to +\infty$, implies that each component in the above equation converges. Hence, $\lim_{l \to +\infty} A_r^l = A_r^0$, $\lim_{l \to +\infty} B_k^l = B_k^0$. These imply that $\Omega_0^{-} \phi_l$ converges in $H^r$, since all other terms in the above identity converges in $H^r$. Write $\lim_{l \to +\infty} \Omega_0^{-} \phi_l = f_0 \in H^r$.

By the asymptotic completeness theorem for $L^2$ (Gr), there are $\phi_0 \in L^2$ such that the following holds:

$$
\psi_0 = \sum_{r=1}^{m} A_r^0 \Omega_1^- u_r + \sum_{k=1}^{\ell} B_k^0 \Omega_2^- w_k + \Omega_0^- \phi_0
$$

in the $L^2$ sense. This implies that $f_0 = \Omega_0^- \phi_0$.

Then by the definition of the wave operator, $U(0, t) e^{-it\mathcal{H}_0} \phi_0 - f_0 \to 0$ as $t \to +\infty$ in $L^2(\mathbb{R}^n)$. Since $U(t, 0)$ and $e^{it\mathcal{H}_0}$ are uniformly bounded operators on $H^r(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$, we see that $\phi_0 = \lim_{t \to +\infty} e^{it\mathcal{H}_0} U(t, 0) f_0$ in $L^2(\mathbb{R}^n)$. We claim that this implies $\phi_0 \in H^r(\mathbb{R}^n)$. It suffices to prove the following:
Assume \(g_n\) is a sequence in \(H^\kappa(\mathbb{R}^n)\) and \(\|g_n\|_{H^\kappa} < 1\). Moreover, \(g_n\) converges to \(g\) in the \(L^2\) norm. Then \(g\) lies in \(H^\kappa(\mathbb{R}^n)\).

To see this, note that on Fourier side, \(H^\kappa(\mathbb{R}^n)\) is just a weighted \(L^2(\mathbb{R}^n)\) space. More precisely, \(\|g_n - \hat{g}\|_{L^2(\mathbb{R}^n)} \to 0\) implies that for the ball \(B_R\) with radius \(R\), centered at the origin,

\[
\|(1 + |\xi|^2)^{\hat{2}}(\hat{g_n}(\xi) - \hat{g}(\xi))\|_{L^2(B_R)} \to 0 \quad \text{as} \quad n \to +\infty.
\]

This implies that \(\|(1 + |\xi|^2)^{\hat{2}}\hat{g}(\xi)\|_{L^2(B_R)}\) is uniformly bounded by \(\sup\|g_n\|_{H^\kappa} \leq 1\). Let \(R \to +\infty\), we see that \(\|g\|_{H^\kappa} \leq 1\).

Now it is clear that the following decomposition holds in the space \(H^\kappa\) for any \(\psi_0 \in H^\kappa\):

\[
\psi_0 = \sum_{r=1}^{m} A^0_r \Omega^+_1 u_r + \sum_{k=1}^{l} B^0_k \Omega^+_2 w_k + \Omega^-_0 \phi_0.
\]

To complete the proof of Theorem 1.13, it remains to prove the following lemma:

**Lemma 6.6.** Assume that the potentials \(V_1(x), V_2 \in C^{n+2\kappa+2}_{0}(\mathbb{R}^n)\). Let \(U(t)\) be the evolution operator of (1.9) and \(\Omega_{1,2}\) the wave operators corresponding to \(U(t)\), as defined at the beginning of this section. Then for all \(f \in L^2(\mathbb{R}^n)\), \(\Omega_{1,2} f\) lies in \(W^{\kappa,p'}\), where \(1 < p' < \frac{2n}{m+2}\).

**Proof.** The proof is essentially contained in [RSS1] Section 4. For the reader’s convenience, we present the details here. Without loss of generality we only consider the wave operator \(\Omega^+_1\). For an arbitrary \(L^2\) function \(f\)

\[
\Omega^+_1 f = \sum_{r=1}^{m} f_r \lim_{t \to +\infty} U(t)^{-1} e^{-it H_1} u_r,
\]

where \(P_b(H_1)f = \sum_{r=1}^{m} f_r u_r\) for some constants \(f_r\). It follows from Duhamel’s formula that

\[
U(t)^{-1} e^{-it H_1} u_r = u_r + i \int_0^t U(s)^{-1} V_2(\cdot - s e_1^1)e^{-i s H_1} u_r \, ds
\]

\[
= u_r + i \int_0^t U(s)^{-1} V_2(\cdot - s e_1^1)e^{-i \lambda_r s} u_r \, ds,
\]

(6.20)

since \(u_r\) is an eigenfunction of \(H_1\) corresponding to an eigenvalue \(\lambda_r\). The function \(u_r\) is exponentially localized in \(L^2\) together with its \(n + 2\) derivatives \(^1\)

\[
\sum_{0 \leq |\gamma| \leq n+2} \int_{\mathbb{R}^n} e^{2\alpha |x|} |\partial_x^\gamma u_r(x)|^2 \, dx \leq C
\]

for some positive constant \(\alpha\) appearing in (3.12). This implies that the function

\[
G_r(s, x) := e^{-i \lambda_r s} V_2(x - s e_1^1) u_r(x)
\]

has the property that for any \(k \geq 0\) and multi-index \(\gamma, 0 \leq |\gamma| \leq n + 2\)

\[
\|\langle s \rangle^k \partial_x^\gamma G_r(s, \cdot)\|_{L^2_s} \leq c(r, |\gamma|, k) \langle s \rangle^{-3j_0 - 2 - \kappa}.
\]

\(^1\)The localization of higher derivatives of \(u_r\) follows from the localization of \(u_r\) stated in (3.12) and the equation \(-\Delta u_r + V_1(x) u_r = \lambda_r u_r\) with potential \(V_1(x)\) which is bounded together with all its derivatives of order \(\leq (n + 2)\).
By Hölder’s inequality, we have, writing \( q = \frac{2p'}{2-p'} \),

\[
\tag{6.21}
\| \partial_x^j U(-s)G(s,x) \|_{L^{p'}} \leq \| \langle x \rangle^{-j_0} \|_{L^q} \| \langle x \rangle^j \partial_x^j U(-s)G(s,x) \|_{L^2}. 
\]

Take \( j_0 = \left\lceil \frac{2p'}{2p'} n \right\rceil + 1 > \frac{n}{q} \), then \( \| \langle x \rangle^{-j_0} \|_{L^q} < +\infty \).

To prove the desired conclusion, it would then suffice to show that for any \( |\gamma| = \kappa \), there exists a positive constant \( k \) such that for any function \( g(x) \)

\[
\tag{6.22}
\| \langle x \rangle^j \partial^\gamma U(t)g \|_{L^2} \lesssim \langle t \rangle^{3j_0 + \kappa} \sum_{|\beta| \leq j_0 + \kappa} \| \langle x \rangle^k \partial_x^\beta g \|_{L^2}, \quad \forall t \geq 0. 
\]

We note that the estimates of the type (6.22) for problems with time independent potentials are well-known. They have been proved in the paper by Hunziker [H]. In the time-dependent case the argument is essentially the same. More precisely, define the functions

\[
\Phi_{j,|\gamma|}(t) := \sum_{j'=0}^j \sum_{|\gamma'|=0} \| \langle x \rangle^{j'} \partial^\gamma U(t)g \|_{L^2}
\]

for any index \( j \geq 0 \) and any multi-index \( \gamma \). Using equation (1.9) we obtain that

\[
\frac{d}{dt} \| \langle x \rangle^j \partial_x^\gamma U(t)g \|_{L^2}^2 = i(-1)^{|\gamma|} \| \left( \frac{\Delta}{2} - V(t,x), \partial^\gamma \langle x \rangle^{2j} \partial_x^\gamma \right) U(t)g, U(t)g \|_{L^2}.
\]

Computing the commutator we obtain the recurrence relation

\[
\Phi_{j,|\gamma|}(t) \lesssim \Phi_{j,|\gamma|}(0) + \langle t \rangle^2 \sum_{|\gamma| \leq 2|\gamma|} \left\| \frac{\langle x \rangle}{\langle t \rangle} \partial_x^\gamma V \right\|_{L^\infty_{t,x}} \sup_{0 \leq \tau \leq t} \Phi_{j-1,|\gamma|+1}(|\gamma|+1) \leq
\]

\[
C(V) \left( \sum_{k=0}^{j-1} \langle t \rangle^{2k} \Phi_{j-k,|\gamma|+k}(0) + \langle t \rangle^{2j} \Phi_{0,|\gamma|+j}(|\gamma|+j) \right),
\]

where \( C(V) \) is a constant depending on

\[
\tag{6.23}
\sum_{|\gamma| \leq 2(|\gamma|+j-1)} \left\| \frac{\langle x \rangle}{\langle t \rangle} \partial_x^\gamma V \right\|_{L^\infty_{t,x}}
\]

In addition, differentiating the equation (1.9) \( |\gamma| + j \) times with respect to \( x \) and using the standard \( L^2 \) estimate, we have

\[
\Phi_{0,|\gamma|+j}(\tau) \leq C(V)(1 + |\tau|^{3|\gamma|+j})\Phi_{0,|\gamma|+j}(0)
\]

Therefore,

\[
\Phi_{j,|\gamma|}(t) \leq C(V)(1 + |t|^{3j+|\gamma|})\Phi_{j,|\gamma|+j}(0).
\]

Now setting \( j = j_0 \) and \( |\gamma| = \kappa \) we obtain the desired estimate (6.22) with \( k = j_0 \). Observe that the assumption \( V_1, V_2 \in C_0^{n+2k+2}(\mathbb{R}^n) \) controls the constant \( C(V) \) in (6.23) for the potential \( V(t,x) = V_1(x) + V_2(x - t\vec{e}_1) \).

\[\square\]
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