Invitation in Crowdsourcing Contests

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Abstract

In a crowdsourcing contest, a requester holding a task posts it to a crowd. People in the crowd then compete with each other to win the rewards. Although in real life, a crowd is usually networked and people influence each other via social ties, existing crowdsourcing contest theories do not aim to answer how interpersonal relationships influence peoples’ incentives and behaviors, and thereby affect the crowdsourcing performance. In this work, we novelty take peoples’ social ties as a key factor in the modeling and designing of agents’ incentives for crowdsourcing contests. We then establish a new contest mechanism by which the requester can impel agents to invite their neighbours to contribute to the task. The mechanism has a simple rule and is very easy for agents to play. According to our equilibrium analysis, in the Bayesian Nash equilibrium agents’ behaviors show a vast diversity, capturing that besides the intrinsic ability, the social ties among agents also play a central role for decision-making. After that, we design an effective algorithm to automatically compute the Bayesian Nash equilibrium of the invitation crowdsourcing contest and further adapt it to large graphs. Both theoretical and empirical results show that, the invitation crowdsourcing contest can substantially enlarge the number of contributors, whereby the requester can obtain significantly better solutions without a large advertisement expenditure.

1 Introduction

1.1 Background

Crowdsourcing is the action of an organisation outsourcing its task to an undefined and generally large network of people using an open call for participation [Howe et al. 2006]. Contest has proven to be a very successful approach in many platforms for crowdsourcing. A contest is a social and economic interactions where a group of agents exert costly and irretrievable effort in order to win prizes and the principal needs to make decision of judiciously awarding prizes among these agents based on their performances. It can be modeled as a normal-form game in which players are agents, strategies are effort, and payoffs are the expected utilities. The applications of contest design have been many and varied in both human society and multi-agent systems, ranging from political campaigns, sports, advertising, school admission, labor contract, R&D competition to crowd-sourcing. The development of contest design theory has been especially advanced over recent years in artificial intelligence, fueled by the needs of various applications of Internet online services.

In a crowdsourcing contest a requester posts a task on a platform and announces a monetary reward that he is willing to pay for a winning solution. Participants submit solutions to the platform and the requester chooses the best solution and awards the prize. For example, in the world’s largest competitive software development portal TopCoder.com, software development phases are realised through contests in which the crowd developers compete with each other for best innovation. Popular projects on GitHub also can provide bounty to elicit a better code from GitHub users. The Netflix Prize contest awarded one million dollars to an outstanding team that created the most accurate

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algorithm for predicting user ratings. Social recruitment on LinkedIn or Recruiter.com also can be seen as a crowdsourcing contest where the recruiters try to acquire the staffs with the highest ability from a crowd.

1.2 Our Contribution

The crowdsourcing contest is essentially about how to elicit efforts from a crowd of people and how to allocate reward among them. Although in real life, a crowd is usually networked and people influence other people via social ties, existing crowdsourcing contest platforms and theories do not formally consider how interpersonal relationships influence peoples' incentives and decisions, and thereby affect the overall performance of the crowdsourcing. Without considering the social relations, incentive design in crowdsourcing contests is inadequate. This is because an individual can exert her influence on a crowdsourcing not only by contributing her own solution, but also via her relations with other people. For instance, it is commonly seen that people share with their friends information about job opportunities, investment opportunities, challenges, contests or money-making tasks. However, the state-of-the-art crowdsourcing contests do not aim to answer why people share information with others. They do not treat peoples’ social influences as a key factor in behavioral reasoning and incentive design. By the current contest models, the requester can only convene some people via her own effort. Talents with high ability may not know the task and will not contribute, which results in a limited quality of the crowdsourcing.

We propose a novel crowdsourcing contest model taking into consideration that people may invite their friends to contribute to the task. To improve the effort elicitation, in this work, we novelly introduce social networks into contest mechanism design. Our model is essentially a Bayesian game on a social graph where each node’s behavior is two dimensional, one is whether to contribute her ability and the other is which friends to invite. This is a multi-dimensional contest model and is different from the traditional crowdsourcing contests where the requester can only call together agents via her own effort.

Based on this invitation model, we establish an invitation mechanism for crowdsourcing contests by which a requester can impel agents to invite their friends to contribute. The mechanism has three steps: (i) agents register in the platform and they can invite friends to register; (ii) the requester exposes the invitation graph to the agents and agents can contribute their efforts; (iii) the requester chooses the winners and allocates the rewards. Since the contest mechanism allows agents to observe the social graph, it empowers the agent to make deliberative decisions taking into consideration of social relations. We find that in equilibrium, regarding the contribution dimension, each agent uses a simple threshold strategy to decide whether to contribute her effort or not. Regarding the invitation dimension, each agent prefers to invite all her neighbours, which is in a sharp contrast to conventional contest.

With the invitation model and the mechanism, the crowdsourcing contest principal can convene as many people as possible but without a large amount of advertisement expenditure. Thus the hidden valuable people in the social network get the chance of participation even though they are not known directly by the principal. This may elicit more participants and better performance, and increase the profit of the principal. We give a detailed reasoning of the Bayesian Nash equilibria under these new mechanism, and also conduct an extensive experiment of them on Facebook network datasets. Both theoretical and empirical results show that, agents’ social invitations substantially enlarge the number of contributors, whereby the requester can obtain significantly better solutions without a large expenditure on advertising.

1.3 Related Works

The works on contest design can be traced back T[lulock [1967] and Krueger [1974] which studied the rent-seeking problems. Another pioneer work Lazear and Rosen [1981] showed the rank-order tournaments could serve as an efficient scheme to elicit effort from strategic agents. Optimal contest design which aims at eliciting better submissions dates back to Glazer and Hassin [1988], and
recently has been investigated in a wide variety of settings in both economics and artificial intelligence. For instance, Dasgupta and Nti [1998] and Fu and Wu [2020] focused on the effects of contest success functions; Che and Gale [2003] and Olszewski and Siegel [2020] studied the prize structures; Ghosh and Hummel [2012] and Ghosh and Kleinberg [2016] found the optimal contest under different output-skill-effort functions and maximizing goals; Fu and Lu [2012] studied the multi-stage contests. With the concept of crowdsourcing burgeoning with internet, many works embed contest researches into the crowdsourcing scenario, such as prize structures [Archak and Sundararajan, 2009, Cavallo and Jain, 2012, Luo et al., 2015, Chawla et al., 2019], multiple contests on crowdsourcing platforms [DiPalantino and Vojnovic, 2009], crowdsourcing effectiveness under certain schemes [Gao et al., 2012, Cavallo and Jain, 2013]. A comprehensive survey of these works can be found in Brabham [2013] and Segev [2020].

Very recently, researchers proposed the simple contest [Ghosh and Kleinberg, 2016, Levy et al., 2017, Sarne and Lepioshkin, 2017]. The simple contest mechanisms capture the scenarios where a contributor’s output quality is a fixed level if an agent decides to participate the contest, either for she would do her best or for she cannot control the quality. For example, in a job interview, all the candidates would do their best and the performances only depend on their abilities. Or in a best-staff contest of your institution, where only the works of previous years are admitted, no one can improve her quality by making instant effort. In a simple contest, the submission quality is intrinsic and cannot be adjusted. Thus a simple agent only need to decide on participating or not. The previous works on simple contest all deal with a fixed set of agents. There are several models of simple contest having been discussed, including models of single or multiple winners [Ghosh and Kleinberg, 2016], certain or uncertain performance evaluation [Habani et al., 2019], parallel or sequential competing order [Levy et al., 2019] and so on. In this work, we also follow the basic ideas of simple contest design but design a game and mechanisms with a more realistic setting: the domain of players is expanded from a fixed set to all agents in the social network, where the contest designer (principal) can only reach a portion of the whole crowd.

There have been some works concentrating on the query scenario, which is a branch of the crowdsourcing problem. Kleinberg and Raghavan [2005] first formulated a model for query propagation on a random network and investigated it from a game-theoretic view. The following works such as Arcaute et al. [2007], Dikshit and Yadati [2009] and Jin et al. [2011] extended this work to more general conditions. The famous 2009 DARPA Network Challenge witnessed the outstanding achievements of this series of work [Pickard et al., 2011]. Some up-to-date works including Wang et al. [2018] and Shen et al. [2019] has extended the scenario from query to contest, where the propagation of crowdsourcing through the social ties plays an important role. However, these works consider little about agents’ incentive design and equilibrium reasoning with the social interactions. Besides the query incentive network problems, there have been some very recent works on mechanism design in social networks [Li et al., 2017] Zhao et al. [2018] Jeong and Lee [2020] Kawasaki et al. [2020]. As far as we know, all these works concentrate on introducing social networks into typical auction problems which are quite different from contest design and these works do not treat graph based BNE analysis as their central problem. A comprehensive survey on these works can be seen in Guo and Hao [2021].

2 Invitation Contest Model

2.1 Invitation Contest as a Bayesian Game on Graph

Consider a crowd of people forming a set N. The social network of these people is a graph \( G = (N, E) \) where the nodes represent people and E is social relationships. Two nodes can reach each other directly if there is an edge linking them, but the structure of the whole graph is unobservable by any agent. Assume there is a requester \( p \in N \), who has a task in some professional field and wants to obtain high-quality solutions. \( p \) holds a contest to find high-ability people as well as to acquire solutions. Without agents’ invitations, \( p \) can only call together a small portion of people in N which she can directly influence (either by advertisement or notification). But agents’ invitations can further
bring in more potential participants.

Any agent can participate in the contest with her own effort, or invite her neighbours, or both, or neither. We assume the participation will incur a cost $c$ but invitation is costless. According to the specific mechanism rule, an agent could be rewarded for providing a good solution to the task, while she could also be rewarded for inviting a high-ability agent. Agents’ utilities is the difference between reward and cost. Assume the solution quality of each agent is just her ability in this field and is private information. Let the ability of agent $i$ be a value from a continuous interval, i.e., $q_i \in [l, u]$. Assume the abilities of all agents are i.i.d with probability density function (pdf) $f(q)$ and cumulative distribution function (cdf) $F(q)$. We assume that $c > 0$, $l \geq 0$, and $f$, $F$ and $c$ are all public information.

Each agent’s decision is two dimensional: one is whether to contribute her ability, the other is which friends to invite. We consider scenarios that agents either contribute with full ability or do not contribute at all [Ghosh and Kleinberg, 2016; Priel et al., 2018]. A strategy of $i$ is $s_i : q_i \to (a_i, c_i) \in \{q_i, 0\} \times \mathcal{P}(E_i)$, where $i$’s action $a_i = q_i$ if $i$ contributes her ability and $a_i = 0$ otherwise. $E_i$ is the neighbour set of $i$, with powerset $\mathcal{P}(E_i)$. The set of agents invited by $i$ is denoted as $c_i$, which could be any subset of $E_i$. Note that an agent is invited does not necessarily mean she must contribute. Also, an agent inviting some friends does not necessarily mean she does not contribute.

The strategy profile from all agents is $\vec{s} = (s_1, \ldots, s_n)$. Profile $\vec{s}$ is a Bayesian Nash equilibrium if no agent can benefit by unilaterally deviating from her present strategy $s_i$.

With agents’ invitations, the task information propagates in the social network $\mathcal{G}$, and agents who receive invitations form a subgraph $\mathcal{H} = (\mathcal{U} \cup \{p\}, E') \subseteq \mathcal{G}$, where the invited agent set $\mathcal{U} \subseteq \mathcal{N}$. We call it the invitation graph. Now based on the crowdsourcing contest and the invitation model, we define some notions which will be used for the analysis in the following sections. First define a ranking of agents’ contributions.

**Definition 1.** (Outperform) Agent $i$ outperforms agent $j$, if $i$ and $j$ both contribute and qualities $q_i > q_j$, or if $i$ contributes but $j$ does not.

With agents’ invitations taken into consideration, besides the ability, an agent’s position in the network also plays a crucial role. We use the following notion to depict the relative position of any two agents in the invitation graph.

**Definition 2.** (Lead) we say $i$ leads $j$ (formally, $i \triangleright j$), if $j$ cannot know the task when $i$ doesn’t invite any neighbour.

That is, $i$ is a cut vertex in the invitation graph and if she does not propagate the task information, then all the propagation routes from the requester $p$ to $j$ are broken, resulting in that $j$ can not know the task and will not contribute.

Because there are many pairs of nodes where there is no leading relation between them, thus the leading relation is a partial ordering. Denote by $\mathcal{U}$ the set of all agents who have been invited to the task, by $D_i$ the set of all agents led by $i$, and by $C_i$ the set of all agents leading $i$. Note that both $D_i$ and $C_i$ exclude $i$ herself.

### 2.2 Invitation Graph and Order Tree

The invitation graph is a connected graph constructed with invited agents to be the nodes and invitations to be the edges. As Definition 1 and 2 show, if an agent $j$ is led by another agent $i$, then every path from the requester node $p$ to the agent node $j$ passes the agent node $i$ in the invitation graph. The leading relation is asymmetric and transitive. That is, if $i$ leads $j$, then $j$ does not lead $i$; if $i$ leads $j$ and $j$ leads $k$, then $i$ leads $k$. Therefore, when $i$ leads $j$, we have $i \in C_j$, $j \in D_i$ and $D_j \subset D_i$. Moreover, we have the following proposition about leading relation.

**Proposition 1.** For an arbitrary agent $i$ with more than one leader (i.e. $|C_i| > 1$), for any $j, k \in C_i (j \neq k)$, there must be that $j$ leads $k$, or $k$ leads $j$ (i.e. $j \in C_k$ or $k \in C_j$).
Proof. Because both $j$ and $k$ are $i$’s leaders, every path from the requester $p$ to $i$ passes both $j$ and $k$. According to the definition of leading, if $j$ does not lead $k$, there is at least one path $p_{jk}$ from the requester $p$ to $k$ that doesn’t pass $j$. The premise that $j$ leads $i$ requires that $j$ must be on every path from $p$ to $i$. Then in this case, every path from $k$ to $i$ must pass $j$. Otherwise, there will exist one path $p_{ki}$ not passing $j$, making a path $p_{jk} + p_{ki}$ from $p$ to $i$ that doesn’t pass $j$. This contradicts the premise that $j$ is on every path from $p$ to $i$. Therefore, when $j$ leads $i$ but does not lead $k$, every path from $k$ to $i$ must pass $j$.

Now we know that, if $j$ does not lead $k$, every path from $k$ to $i$ must pass $j$. This also means every path $p_{ji}$ from $j$ to $i$ doesn’t pass $k$. Suppose there is a path $p_{ji}$ from $p$ to $j$ that doesn’t pass $k$. Then there must be a path $p_{ji} + p_{ji}$ from $p$ to $i$ that doesn’t pass $k$, which contradicts the fact that $k$ leads $i$. Hence, if $j$ does not lead $k$, every path from $p$ to $j$ must pass $k$, which means $k$ leads $j$. □

From Proposition 1, we know that for any agent $i$ with $C_i \neq \emptyset$, there’s a leader sequence $\Phi_{pi} = (c^1_i, c^2_i, \ldots, c^{|C_i|}_i)$ between $p$ and $i$, where every one in this sequence leads all her latter ones, which can be notated as: $c^j_i \in C_{c^j_i}$ for any $1 \leq j < k \leq |C_i|$. Also, we know that $\forall 1 < j \leq |C_i|, (c^1_i, \ldots, c^{j-1}_i)$ is the leader sequence of $j$.

With the above features, we can form a rooted tree to describe the leading relations in an invitation graph. In this tree, the root node is the requester and other nodes are the invited agents. For any agent $i$, if $C_i = \emptyset$, then her parent node is the requester; if $|C_i| = 1$, then her parent node is her unique leader; if $|C_i| \geq 2$, then her parent node is the last one in her leader sequence (i.e., $c^{|C_i|}_i$). We call this tree by Order Tree of the invitation graph. In the order tree, every node in the path from the requester $p$ to $i$ is $i$’s leader, and all the nodes in the sub-tree taking $i$ as the root form the set $D_i$.

From the above analysis, we derive the following proposition.

**Proposition 2.** The leading/led relations among all agents in any given invitation graph $\mathcal{H}$ can be described by an order tree $T$. In an order tree, the root is the requester $p$, and other nodes are invited agents. $C_i$ forms the path from $p$ to $i$.

Given the social network of a crowd, there could be different invitation graphs depending on different strategy profiles of all agents. But with a specific invitation graph, the order tree is unique. We show an example of the above notions in Figure 2 and Example 1.

**Example 1.** (1) Let Fig. 2(a) be the social network, where agents $a$ and $d$ didn’t invite any one while other agents invited all their neighbours. (2) Then the invitation graph is in Fig. 2(b). From this invitation graph, we have $U = \{a, b, c, d, e, f\}$, $C_b = \emptyset$, $D_b = \{e, d, e, f\}$, $C_e = \{b\}$ and $D_e = \{f\}$. (3) The unique order tree generated from this invitation graph is shown in Fig. 2(c).

In the following section, based on the above model of invitation contest, we propose a novel contest mechanisms taking into consideration of both the agents’ invitations and their contributions.

## 3 Collective Invitation Mechanism

In this section, we design a crowdsourcing contest where all agents in the crowd prefer to invite their neighbours. We call it collective invitation mechanism (CIM). After characterization of the
3.1 Contest Mechanism Rules

The collective invitation mechanism (CIM) has three phases.

- **Registration phase**: Agents register in the contest and they can invite their friends to register.
- **Competition phase**: At the beginning of this phase, the requester publishes the whole invitation graph to all registered agents. Then the registered agents decide whether to contribute their abilities or not.
- **Award phase**: Each agent who outperforms *all the agents not led by herself* is awarded with a prize $M$. Agents who do not contribute have no reward.

Note that CIM does not publish any information in the registration phase, but it publishes the invitation graph at the beginning of the competition phase. CIM may select out multiple high ability agents and award each of them. The allocation rule of the award phase is in Algorithm 1.

**Algorithm 1**: Allocation rule in award phase of CIM

Input: invited agents $U$, invitation graph $H$, agents’ contribution qualities $q_i, \forall i \in U$

Output: a set $W$ of winners

1. initialize $W = \emptyset$
2. for every $i \in U$ do
3.   if $i$ has contributed then
4.     compute $D_i$ according to $H$;
5.     if $\forall j \in (U - D_i - \{i\}), i \succ j$ then
6.       $W \leftarrow W + \{i\}$;
7. return $W$.

For ease of notation, denote $P_i = U - D_i - \{i\}$. According to line-5 in Algorithm 1 only agents in $P_i$ can be competitors of $i$, but the set $P_i$ is not affected by $i$’s own invitation. If $i$ outperforms all $j$ in $P_i$, she will be selected out as a winner. Throughout this section, $P_i$ will be used as a key factor for analyzing agents’ behaviors. What is the intuition of designing such an award rule? Recall that by definition, $D_i$ is the set of agents who can join the contest only when $i$ invites her neighbours. If a mechanism allows a competition between $i$ and agents from $D_i$, then $i$ will not invite anyone. Therefore, ensuring any agent $i$ not to compete with her $D_i$ can provide her with incentives to invite. From the later analysis, we will further know that the setting achieves far more than this.

According to the algorithm, there are two basic properties of CIM. Firstly, if there is a unique best contributor, then she wins prize $M$. Secondly, the winners other than the best contributor must
be her leaders. This is because for any winner \( i \) who is not the best contributor, the best contributor can not be in \( P_i \), otherwise \( i \) can not be a winner. Note that CIM does not provide reward to a non-contributor, which means the utility of a non-contributor is always 0. Thus an agent would choose to contribute if the expected utility for contributing is greater than 0.

### 3.2 Agent’s Strategy and Utility

Based on the above description of CIM, now we start to analyze agents’ behaviors under CIM. First we will show that under CIM, only two kinds of strategies will be preferred by agents.

Consider an agent \( i \) in a given invitation graph who has intrinsic contribution quality \( q_i \). Given any strategy of another agent \( j \), the probability that \( i \) outperforms \( j \) when contributing is \( P(a_j < q_i) \). This value is non-decreasing and continuous in \( q_i \). The expected utility for \( i \) to contribute is

\[
\pi_i(q_i) = M \cdot \prod_{j \in P_i} P(a_j < q_i) - c,
\]

which is also non-decreasing and continuous in \( q_i \). Using this utility as well as the continuity and monotonicity of \( P \), we can obtain the following theorem.

**Theorem 1.** Under CIM, the Bayesian Nash equilibrium is a threshold equilibrium, in which each agent contributes only if her ability surpasses her own threshold.

**Proof.** First consider a case that \( i \) contributes with the upper-bound ability \( q_i = u \). No matter what \( j \) does, it is always that \( P(a_j < u) = 1 \). Thus Eq. 1 degenerates to

\[
\pi_i(u) = M - c > 0,
\]

while the utility for \( i \) not to contribute is 0. This means agents who have a close-to-upper-bound ability always contribute.

Next consider a case that agent \( i \) contributes with the lower-bound ability \( q_i = l \). With this lowest ability, \( i \) can outperform \( j \) only when \( j \) does not contribute, which happens with the probability \( P(a_j = 0) < 1 \). Now Eq. 1 becomes

\[
\pi_i(l) = M \cdot \prod_{j \in P_i} P(a_j < l) - c.
\]

The sign of this utility is not determined. (1) If \( \pi_i(l) \leq 0 \), since \( \pi_i(q_i) \) is non-decreasing and continuous in \( q_i \), and we already know \( \pi_i(u) > 0 \), there must be a value \( r_i \in [l, u] \) such that \( \pi_i(r_i) = 0 \). And \( i \) contributes only when her ability \( q_i > r_i \). (2) When \( \pi_i(l) > 0 \), since \( q_i \geq l \) and \( \pi_i(q_i) \) is non-decreasing and continuous in \( q_i \), \( i \) will always contribute. We call such an \( i \) unconditional contributor under the second condition. Unconditional contributors is a special group of agents in CIM. Appendix A.3 discusses more about them. For the unconditional contributors, we can simply set their thresholds identically as the lower-bound, i.e., \( r_i = l \). This does not change any results. \( \square \)

The result of this theorem is intuitive. If an agent \( i \) does not contribute, then her utility is 0. If she contributes, her utility \( \pi(q_i) \) is a non-decreasing and continuous function of her ability \( q_i \). Specifically, if she has the highest ability \( q_i = u \), then \( \pi \) is positive. The non-decreasing monotonicity of \( \pi \) ensures that there is a threshold value \( r_i \geq l \) such that \( \pi(q_i) \geq 0 \) only if \( q_i \geq r_i \). Theorem 1 says that we only need to focus on the threshold strategies. Note that the thresholds for different agents in CIM are not necessarily identical.

With this theorem and Definition 1 we can further know that, the probability for \( i \) to outperform \( j \) by contributing is \( P(q_j \leq \max\{q_i, r_j\}) = F_{q_j}(\max\{q_i, r_j\}) \). Therefore under the i.i.d assumption, Eq. 1 can be reformulated into

\[
\pi_i(q_i) = M \cdot \prod_{j \in P_i} F_{q_j}(\max\{q_i, r_j\}) - c.
\]

This formula is an agent’s utility regarding others’ thresholds. In the next subsection, it will be used as a key component for analysing agents’ invitation incentives.
3.3 Behavior in Equilibrium

Theorem 1 is of great importance. By narrowing the strategy space to threshold strategies, we can further capture agents’ invitations and contributions in detail. The following lemma shows that the threshold $r_i$ of any agent $i$ has an upper-bound $r_i \leq F^{-1} \left( \frac{|P|}{\sqrt{M}} \right)$.

**Lemma 1.** For any agent $i$ under CIM, $i$’s threshold $r_i \leq F^{-1} \left( \frac{|P|}{\sqrt{M}} \right)$, where $P_i = U - D_i - \{i\}$.

**Proof.** Prove by contradiction. Let $s_i = F^{-1} \left( \frac{|P|}{\sqrt{M}} \right)$. Suppose that there exists an agent $i$ where $r_i > s_i$. Then, $q_i = s_i + \delta < r_i$ where $\delta > 0$, the expected utility for $i$ to contribute is

$$
\pi_i(q_i) = M \cdot \prod_{j \in P_i} F_{q_j}(\max\{r_j, q_i\}) - c
\geq M \cdot \prod_{j \in P_i} F'_{q_j}(q_i) - c
> M \cdot \prod_{j \in P_i} F'_{q_j}(s_i) - c
= M \cdot \prod_{j \in P_i} \frac{|P|}{\sqrt{M}} - c = 0.
$$

Thus $i$ will contribute with $q_i < r_i$. This contradicts that $r_i$ is the threshold.

Note the only variable is $|P_i|$, which is the number of $i$’s competitors under CIM, the group of agents whom are not affected by $i$’s invitation. This means the upper-bound of an agent’s threshold is independent from her invitation decision. Moreover, if two agents have leading relation, then the following lemma can be used to compare their thresholds.

**Lemma 2.** In the Bayesian Nash equilibrium of CIM, if agent $i$ leads agent $j$, then $j$ has higher threshold than $i$, that is, $j \in D_i \Rightarrow r_j > r_i$.

**Proof.** Suppose two agents $i$ and $j$, where $i$ leads $j$, then $j \in D_i$, and by definition, we know that $D_j \subset D_i$, thus $P_j \subset P_i$. First consider $j$. Since $r_j$ is the threshold of $j$, if the quality of $j$ is $q_j = r_j$, then her utility of contributing is

$$
\pi_j(q_j = r_j) = M \cdot \prod_{k \in P_j} F_{q_k}(\max\{r_k, r_j\}) - c \geq 0.
$$

The product in this equality can be factorized as

$$
\prod_{k_1 \in P_i} F_{r_{k_1}}(\max\{r_{k_1}, r_j\}) \cdot \prod_{k_2 \in P_j, k_2 \notin P_i} F_{r_{k_2}}(\max\{r_{k_2}, r_j\}). \quad (3)
$$

Now consider $i$. If $i$’s ability is $q_i = r_j$, then her utility of contributing is

$$
\pi_i(q_i = r_j) = M \cdot \prod_{k_1 \in P_i} F_{r_{k_1}}(\max\{r_{k_1}, r_j\}) - c,
$$

where the product is exactly the same as the first term in Eq. (1). Since the second term in Eq. (3) is a probability less than 1, we have $\pi_i(q_i = r_j) > \pi_j(q_j = r_j) \geq 0$. The utility of $i$ can only be 0 if she does not contribute. This means if $q_i = r_j$, $i$ will contribute. Therefore, $r_i < r_j$.

Specifically, agents not leading anyone must be leaf nodes in the order tree. The following definition formally defines the leaf type nodes.

**Definition 3.** (leaf type) An agent $i$ is of leaf type if leading no one (i.e. $D_i = \emptyset$).
The agents of leaf type lead nobody. Note that a leaf type agent in the order tree does not necessarily mean she has no friends. It only means this agent is not a cut vertex in the invitation graph. For an agent of leaf type, unilaterally altering the invitation does not change the group of invited agents. In fact, they are all the leaf nodes in the order tree. Proposition\textsuperscript{3} proves that all leaf nodes have an identical threshold.

**Proposition 3.** In the equilibrium under CIM, every agent who is a leaf node in the order tree of the invitation graph has an identical threshold \( r_z = F^{-1}\left(\frac{|\mathcal{U}| - \sqrt{\frac{c}{M}}}{|\mathcal{U}|}\right) \), which is the highest among all agents’ thresholds.

**Proof.** Assume all leaf agents except \( i \) use the threshold \( r_z \) in equilibrium. If \( i \) is also a leaf agent, when \( i \) contributes with \( q_i \), the expected utility is

\[
\pi_i(q_i) = \mathcal{M} \cdot \prod_{z \in \mathcal{Z}, z \neq i} F_{q_z}(\max\{r_z, q_i\}) \cdot \prod_{k \notin \mathcal{Z}} F_{q_k}(\max\{r_k, q_i\}) - c.
\]

Here \( \mathcal{Z} \) is the set of all leaf agents. For any agent \( k \notin \mathcal{Z}, |\mathcal{D}_k| \geq 1 \) and \( |\mathcal{P}_k| \leq |\mathcal{U}| - 2 \). Thus according to Lemma\textsuperscript{1} and that \( F(x) \) is monotonically increasing, we know

\[
r_k \leq F^{-1}\left(\frac{|\mathcal{P}_k|}{\sqrt{|\mathcal{U}|}}\right) \leq F^{-1}\left(\frac{|\mathcal{U}| - 1}{\sqrt{|\mathcal{U}|}}\right) = r_z.
\]

If \( q_i = r_z - \delta \) where \( \delta \to 0^+ \), then \( i \)'s expected utility for contributing is

\[
\pi_i(q_i) = \mathcal{M} \cdot \prod_{z \in \mathcal{Z}, z \neq i} F_{q_z}(r_z) \cdot \prod_{k \notin \mathcal{Z}} F_{q_k}(q_i) - c
\]

\[
< \mathcal{M} \cdot \prod_{z \in \mathcal{Z}, z \neq i} F_{q_z}(r_z) \cdot \prod_{k \notin \mathcal{Z}} F_{q_k}(r_z) - c = \mathcal{M} \cdot F(r_z)^{|\mathcal{U}| - 1} - c = 0.
\]

This means \( r_i > r_z - \delta \) where \( \delta \to 0^+ \). On the other hand, according to Lemma\textsuperscript{1} we have \( r_i \leq r_z \).

Now we know that since \( |\mathcal{P}_z| = |\mathcal{U}| - 1 \) for all leaf nodes, \( r_z \) is larger than any non-leaf agent’s threshold. Note when we consider the symmetric equilibrium, this conclusion won’t change even if every agent is leaf agent.

**Remark 1.** In the Bayesian Nash equilibrium of CIM, (i) An agent contributes only if her ability surpasses her threshold, (ii) As long as some agents are not leaves, there is no common threshold. (iii) Leaf nodes have the highest threshold.

These three points are straightforward from the above results. Especially, from point (iii), it is apparent that under CIM, no agent wants to be a leaf node, this may incentivize each agent to try her best to invite her neighbours and to make herself leading some nodes. The following Lemma formally proved this.

**Lemma 3.** Under CIM, for any agent, she can decrease her threshold by inviting her neighbours.

**Proof.** Consider an arbitrary agent \( i \). Given the other agents’ invitations, when \( i \) invites no one, she becomes a leaf node. Suppose in this case the whole agent set is \( \mathcal{U} \) and \( i \)'s threshold is \( r_i \). When \( i \) invites some (or all) of her neighbours, suppose the whole agent set is \( \mathcal{U}' \) and her threshold is \( r_i' \). It is straightforward that \( \mathcal{U} \subseteq \mathcal{U}' \).

When \( i \) invites no one, \( i \) is a leaf node in the order tree. By Proposition\textsuperscript{3} we have \( r_i = F^{-1}\left(\frac{|\mathcal{U}| - \sqrt{\frac{c}{M}}}{|\mathcal{U}|}\right) \) and \( F(r_i')^{|\mathcal{U}'| - 1} = \frac{c}{M} \). Note that \( r_i \) is also the upper-bound threshold shown in Lemma\textsuperscript{1} When \( i \) invites some neighbours, if \( \mathcal{U} = \mathcal{U}' \), \( i \) is still a leaf node and \( r_i' = r_i \). Otherwise, \( \mathcal{U} \subset \mathcal{U}' \). Under this condition, some agents in \( \mathcal{P}_i \) could still be leaf nodes. According to Proposition
Considering all these cases, inviting friends is better off for every agent under CIM.

Proof. By Lemma \( 3 \), \( i \)'s ability when she invites is equal lower than that when she does not invite, i.e., \( r_i' \leq r_i \). Given ability \( q_i \), there are three cases of the ordering of \( q_i, r_i \), and \( r_i' \).

1) \( q_i \leq r_i' \leq r_i \). In this case, \( i \) won’t contribute either she makes invitation or not. Then \( \pi_i(U', q_i) = \pi_i(U, q_i) = 0 \), where \( U' \) is the agent set when \( i \) invites, and \( U \) is that when \( i \) does not invite.

2) \( r_i' < q_i \leq r_i \). In this case, if she doesn’t make any invitation in the registration phase, \( i \) won’t contribute in the competition phase, and then her utility is 0; if she invites in the registration phase, \( i \) will contribute in the competition phase and then obtain utility greater than 0. Therefore, \( \pi_i(U', q_i) > \pi_i(U, q_i) \).

3) \( r_i' \leq q_i < r_i \). In this case, \( i \) will contribute in the competition phase no matter whether she has invited or not. \( r_i \) is \( i \)'s threshold when she is a leaf node, thus by the threshold upper-bound in Lemma \( 1 \) for any \( k \in P_i, r_k \leq r_i \). Thus, \( r_k < q_i \), and

\[
\pi_i(U', q_i) = M \cdot \prod_{k \in P_i} F_{q_k}(\max\{r_k', q_i\}) - c \\
\geq M \cdot \prod_{k \in P_i} F_{q_k}(q_i) - c \\
= M \cdot \prod_{k \in P_i} F_{q_k}(\max\{r_k, q_i\}) - c = \pi_i(U, q_i).
\]

Considering all the these cases, inviting friends is better off for every agent under CIM.

From Theorem \( 3 \) we know that CIM provides a good incentive such that in the Bayesian Nash equilibrium, every agent would invite some, if not all, of her neighbours.

Now we derive an expression of an arbitrary agent’s threshold. For an arbitrary \( i \), if her ability just meets her threshold, i.e., \( q_i = r_i \), her utility will be 0. Then Eq. \( 3 \) can be rewritten as

\[
\pi_i(r_i) = M \cdot \prod_{j \in J_i} F_{q_j}(r_j) \cdot \prod_{j \in Q_i} F_{q_j}(r_i) - c = 0,
\]
where $J_i$ denotes the set of $i$’s competitors whose thresholds are larger than $r_i$, and $Q_i$ denotes the set of $i$’s competitors whose thresholds are not larger than $r_i$. Under i.i.d. assumption of agents’ abilities, all $F$ are identical for different $j$. Then we rewrite Eq.(4) into

$$F(r_i)^{|Q_i|} = \frac{c}{M} \prod_{j \in J_i} F(r_j).$$

This formula together with Proposition[3] directly leads to the following theorem.

**Theorem 3 (Contribution Threshold).** In the Bayesian Nash equilibrium of CIM, the contribution threshold of any leaf agent $z$ is

$$r_z = F^{-1} \left( \frac{|U|-1}{c} \right),$$

while that of any non-leaf agent $i$ is

$$r_i = F^{-1} \left( \sqrt{\frac{c}{M} / \prod_{j \in J_i} F(r_j)} \right).$$

The special case that $|Q_i| = 0$ has been handled in Appendix A.1. Recall that $J_i$ is the set of competitors whose thresholds are greater than $i$’s. Theorem[3] tells us that, to decide $r_i$, we only need to know $r_j$ of each $j \in J_i$, as well as the size of $Q_i$. For this purpose, we can devise an algorithm to traverse all nodes and recursively compute each $r_i$ by leveraging Theorem[3].

This section has shown how agents make invitation decisions in the registration phase and also gives the formula of their contribution thresholds. But still, we have not see the detailed invitation behavior and what are the contributions from agents in equilibrium. In the next section, we will propose a computational method to characterize the agents’ equilibrium behaviors under CIM.

### 4 Computation of Bayesian Nash Equilibrium

In this section, we first show that it is possible to rank agents’ contribution thresholds even without knowing the values of these thresholds. The key idea is to find another computable quantity which is monotone in $r_i$. Since this mediator quantity of an agent has a one-to-one correspondence relation with the agent’s contribution threshold, we will first rank the thresholds by ranking agents’ mediator quantities. Then we can design an algorithm to compute the values of the agents’ thresholds in equilibrium by using this ranking and the contribution threshold’s formula in Eq.(6). These thresholds finally determines the equilibrium and the performance of CIM.

#### 4.1 Monotonicity of Contribution Thresholds

In the competition phase of CIM, for an agent $i$, if all the agents led by her (i.e. $D_i$) do not have enough ability to contribute, we say $i$ is a leader of mediocre (lom). Denote the probability for $i$ to be a lom as $p_i^{lom}$, which is calculated as

$$p_i^{lom} = P(\forall k \in D_i, q_k < r_k) = \prod_{k \in D_i} F(q_k(r_k)).$$

Recall that $F(q_k(r_k))$ is the probability that $k$’s ability is below her threshold $r_k$, thus the product is the probability no agent in $D_i$ can contribute. The intuition behind the probability $p_i^{lom}$ is: if agent $i$ has a small $p_i^{lom}$, it is more likely that $i$’s invitation brings in some contributors, thus scaring off the competitors of $i$ (i.e. agents in $P_i$). When $i$’s competitors are scared off, $i$’s winning probability will increase, and $i$ can have a positive expected utility when contributing with a smaller $q_i$. This is why the agent with small $p_i^{lom}$ has low threshold. Based on this intuition, we first prove the following Lemmas[4] and [5]
Lemma 4. For any two agents $i$ and $j$ who don’t have leading relation (i.e. $i \not\sim j$ and $j \not\sim i$), if $p_{i}^{\text{nom}} < p_{j}^{\text{nom}}$, then $r_{i} < r_{j}$.

Proof. Because $i$ and $j$ don’t have leading relation, we have $i \notin D_{j}, j \notin D_{i}$, and $D_{i} \cap D_{j} = \emptyset$. Denote the set of common competitors of $i$ and $j$ by $P_{ij} = P_{i} \cap P_{j}$. Moreover, we know that $U = D_{i} + P_{i} + \{i\} = D_{j} + P_{j} + \{j\}$, the relationships within these sets are shown in Figure 3.

We have that $P_{ij} = P_{i} - D_{j} - \{j\} = P_{j} - D_{i} - \{i\}$.

![Figure 3: Illustration of $i$ and $j$ who don’t have any leading relation.](image)

Assume that $r_{i} \geq r_{j}$ in equilibrium, according to Lemma 2, we have $\forall g \in D_{i}, r_{g} > r_{i} \geq r_{j}$. According to Lemma 6 in Appendix A.1, when $r_{i} \geq r_{j}$, only $j$ could be an unconditional contributor. Therefore, the expected utility of $i$ when contributing with $q_{i} = r_{i}$ is

$$\pi_{i}(q_{i} = r_{i}) = M \prod_{h \in D_{j}} F_{q_{h}}(\max\{r_{h}, r_{i}\}) \cdot \prod_{k \in P_{ij}} F_{q_{k}}(\max\{r_{k}, r_{i}\}) \cdot F_{q_{j}}(\max\{r_{j}, r_{i}\}) - c, \quad (8)$$

which equals 0. On the other hand, the expected utility of $j$ when contributing with $q_{j} = r_{j}$ is

$$\pi_{j}(q_{j} = r_{j}) = M \prod_{g \in D_{i}} F_{q_{g}}(\max\{r_{g}, r_{j}\}) \cdot \prod_{k \in P_{ij}} F_{q_{k}}(\max\{r_{k}, r_{j}\}) \cdot F_{q_{j}}(\max\{r_{j}, r_{j}\}) - c, \quad (9)$$

which is non-negative. Since $r_{g} > r_{i} \geq r_{j}, \forall g \in D_{i}, F_{q_{g}}(\max\{r_{g}, r_{j}\}) = F_{q_{g}}(r_{g})$, and under the i.i.d. assumption, $F_{q_{j}}(\max\{r_{j}, r_{j}\}) = F_{q_{j}}(\max\{r_{j}, r_{j}\}) = F(r_{j}).$ Combining Eqs. (8) and (9), we have

$$\prod_{h \in D_{j}} F_{q_{h}}(\max\{r_{h}, r_{i}\}) \cdot \prod_{k \in P_{ij}} F_{q_{k}}(\max\{r_{k}, r_{i}\}) \leq \prod_{g \in D_{i}} F_{q_{g}}(r_{g}) \cdot \prod_{k \in P_{ij}} F_{q_{k}}(\max\{r_{k}, r_{j}\}). \quad (10)$$

Since $r_{i} \geq r_{j}$, it is easy to know $F_{q_{h}}(\max\{r_{h}, r_{i}\}) \geq F_{q_{h}}(\max\{r_{h}, r_{j}\})$, thus

$$\prod_{k \in P_{ij}} F_{q_{k}}(\max\{r_{k}, r_{i}\}) \geq \prod_{k \in P_{ij}} F_{q_{k}}(\max\{r_{k}, r_{j}\}).$$

Combining with Eq. (10), we further have $\prod_{h \in D_{j}} F_{q_{h}}(\max\{r_{h}, r_{i}\}) \leq \prod_{g \in D_{i}} F_{q_{g}}(r_{g})$. Together with a simply known inequality that $\prod_{h \in D_{j}} F_{q_{h}}(r_{h}) \leq \prod_{h \in D_{j}} F_{q_{h}}(\max\{r_{h}, r_{i}\})$, we have

$$\prod_{h \in D_{j}} F_{q_{h}}(r_{h}) \leq \prod_{g \in D_{i}} F_{q_{g}}(r_{g}). \quad (11)$$

Note that, Eq. (11) is the conclusion under the assumption that $r_{i} \geq r_{j}$ for $i$ and $j$ who don’t have leading relation, which means, $r_{i} \geq r_{j} \Rightarrow \prod_{g \in D_{i}} F_{q_{g}}(r_{g}) \geq \prod_{h \in D_{j}} F_{q_{h}}(r_{h}),$ then by contrapositive we have $\prod_{g \in D_{i}} F_{q_{g}}(r_{g}) < \prod_{h \in D_{j}} F_{q_{h}}(r_{h}) \Rightarrow r_{i} < r_{j},$ which according to the definition of $p_{i}^{\text{nom}}$ is equivalent to that $p_{i}^{\text{nom}} < p_{j}^{\text{nom}} \Rightarrow r_{i} < r_{j}$. □

Lemma 5. In equilibrium, for any two agents $i$ and $j$, if $p_{i}^{\text{nom}} = p_{j}^{\text{nom}}$, then $r_{i} = r_{j}$. 

![](image)
Proof. According to Lemma 2, we know $\forall y \in \mathcal{D}_i, r_y > r_i$ and $\forall h \in \mathcal{D}_j, r_h > r_j$. If $i$ leads $j$, then $(\{i\} + \mathcal{D}_j) \subseteq \mathcal{D}_i$. This relation leads to

$$\prod_{g \in \mathcal{D}_i} F_{q_g}(r_g) \leq \prod_{h \in \mathcal{D}_j} F_{q_h}(r_h) \cdot F_{q_i}(r_i) < \prod_{h \in \mathcal{D}_j} F_{q_h}(r_h).$$

By contrapositive, if $\prod_{g \in \mathcal{D}_i} F_{q_g}(r_g) = \prod_{h \in \mathcal{D}_j} F_{q_h}(r_h)$, there is no leading relation between $i$ and $j$. Recall that by definition, $p_{i \text{om}} = \prod_{g \in \mathcal{D}_i} F_{q_g}(r_g)$. This means, given $p_{i \text{om}} = p_{j \text{om}}$, $i$ and $j$ don’t have leading relation. Then $\tilde{\mathcal{P}} = \mathcal{D}_i + \mathcal{P}_j + \{i\}$ and $\mathcal{P} = \mathcal{D}_j + \mathcal{P}_j + \{j\}$ as Figure 3 shows. In equilibrium, the expected utility for $j$ to contribute with $q_j = r_i$ is

$$\pi_j(q_j = r_i) = M \cdot \prod_{g \in \mathcal{D}_i} F_{q_g}(\max\{r_g, r_i\}) \cdot \prod_{k \in \mathcal{P}_j} F_{q_k}(\max\{r_k, r_i\}) \cdot F_{q_i}(\max\{r_i, r_i\}) - c$$

$$= M \cdot \prod_{g \in \mathcal{D}_i} F_{q_g}(r_g) \cdot \prod_{k \in \mathcal{P}_j} F_{q_k}(\max\{r_k, r_i\}) \cdot F_{q_i}(r_i) - c$$

$$= M \cdot \prod_{h \in \mathcal{D}_j} F_{q_h}(r_h) \cdot \prod_{k \in \mathcal{P}_j} F_{q_k}(\max\{r_k, r_i\}) \cdot F_{q_i}(r_i) - c. \quad (12)$$

In equilibrium, $i$’s expected utility of contributing with $q_i = r_i$ must be non-negative, that is

$$\pi_i(q_i = r_i) = M \cdot \prod_{h \in \mathcal{D}_j} F_{q_h}(\max\{r_h, r_i\}) \cdot \prod_{k \in \mathcal{P}_j} F_{q_k}(\max\{r_k, r_i\}) \cdot F_{q_i}(\max\{r_i, r_i\}) - c \geq 0.$$

Since $F_{q_h}(r_h) \leq F_{q_h}(\max\{r_h, r_i\})$, it is easy to see $\prod_{h \in \mathcal{D}_j} F_{q_h}(r_h) \leq \prod_{h \in \mathcal{D}_j} F_{q_h}(\max\{r_h, r_i\})$. Moreover, $F_{q_i}(r_i) \leq F_{q_i}(\max\{r_i, r_i\})$ under the i.i.d. assumption. Therefore, the above equation is reformulated into

$$\pi_i(q_i = r_i) \geq M \cdot \prod_{h \in \mathcal{D}_j} F_{q_h}(r_h) \cdot \prod_{k \in \mathcal{P}_j} F_{q_k}(\max\{r_k, r_i\}) \cdot F_{q_i}(r_i) - c.$$

The right-hand side is identical to that in Eq. (12). Thus we have

$$\pi_i(q_i = r_i) \geq \pi_j(q_j = r_i). \quad (13)$$

According to Lemma 6 in the Appendix A.1, if $i$ is an unconditional contributor, she is the unique agent who has the lowest threshold $l$, and $\pi_i(q_i = r_i) > 0$. Under this condition, $r_j > r_i = l$, which leads to $\pi_j(q_j = r_i) < 0 < \pi_i(q_i = r_i)$. This satisfies the relation in Eq. (13). If $i$ is not an unconditional contributor, then $\pi_j(q_j = r_i) \leq \pi_i(q_i = r_i) = 0$, so $r_j \geq r_i$. In conclusion, when $p_{i \text{om}} = p_{j \text{om}}$, there is that $r_j \geq r_i$.

With the same process on reversed $i$ and $j$, we can derive that when $p_{i \text{om}} = p_{j \text{om}}$, there is that $r_i \geq r_j$. Therefore, there must be that $r_i = r_j$ if $p_{i \text{om}} = p_{j \text{om}}$.

From Lemma 2, Lemma 4 and Lemma 5, it’s easy to derive that, in any equilibrium under CIM, $\forall i, j \in \mathcal{U}(i \neq j)$,

$$r_i = r_j \iff p_{i \text{om}} = p_{j \text{om}},$$

$$r_i < r_j \iff p_{i \text{om}} < p_{j \text{om}}. \quad (14)$$

Thus we can directly have the following Proposition 4 which serves as a corner stone for identifying the equilibrium under CIM.

**Proposition 4 (Monotonicity of Threshold).** In the Bayesian Nash equilibrium of CIM, for any two agents $i$ and $j$, $r_i < r_j$ iff $p_{i \text{om}} < p_{j \text{om}}$.

The significance of Proposition 4 is, it shows that if we want to rank all agents’ thresholds, we just need to rank the value of all $p_{i \text{om}}$, which is more easily accessible than $r_i$. 

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4.2 Algorithm for Equilibrium Computation

Based on Theorem 3 and the fact that leaf nodes have the highest threshold, we design Algorithm 2 to compute the Bayesian Nash equilibrium (BNE) of CIM. It has four components. The first one is for initialization (line [1] to [6]). Then, the second component calculates agents’ probabilities of being a lom (line [8] to [11]). After that, by sorting these probabilities, the third component identifies the top agents who have the highest threshold (line [12] to [13]). At last, the fourth component calculates the thresholds for these top agents (line [14] to [16]). The algorithm traverses the whole order tree and stops after obtaining all agents’ thresholds. The following work flow explains this algorithm in detail.

Algorithm 2: Computing BNE of CIM by Tree Search

| Input: | invited agents $U$, invitation graph $H$ |
| Output: | thresholds $\vec{r} = (r_1, \ldots, r_i, \ldots, r_{|U|})$ |
| 1 | initialize flag $\phi_i = 0$ and $r_i = \text{nil}$ for all $i \in U$; |
| 2 | find leaf node set $Z$; |
| 3 | store $D_i$ and $P_i$ for all $i \in U$; |
| 4 | initialize qualified competitor set $A = Z$; |
| 5 | for each leaf node $i \in Z$ do |
| 6 | assign $r_i$ with value in Eq.(5); |
| 7 | while $A \neq U$ do |
| 8 | for each $i$ whose $\phi_i = 0$ and $r_i = \text{nil}$ do |
| 9 | if $r_k \neq \text{nil}$ for all $k \in D_i$ then |
| 10 | compute $i$’s probability $p_{i\text{lom}}$ by Eq.(7); |
| 11 | set $i$’s flag to $\phi_i = 1$; |
| 12 | rank all $i$ whose $\phi_i = 1$ by descending $p_{i\text{lom}}$; |
| 13 | move all top $i$ into temp set $T$; |
| 14 | for all $i \in T$ do |
| 15 | use $J_i = A \cap P_i$ to compute $r_i$ by Eq.(6); |
| 16 | reset $\phi_i = 0$; |
| 17 | update $A = A + T$; then reset $T = \emptyset$; |
| 18 | return $\vec{r}$. |

1. (line [1] to [6]) This is the initialization. With the all registered agents $U$ and the invitation graph $H$, it is easy to compute the leaf node set $Z$, and $D_i$ and $P_i$ for each agent $i$. It is also shown that the threshold of leaf nodes $Z$ can be directly computed by Eq.(5), this is done in the loop.

2. (line [8] to [11]) According to Eq. (7), this component calculates probabilities $p_{i\text{lom}}$ for every $i$ whose child nodes’ thresholds $r_k$ are all known. The reason for this calculation is that, Proposition 4 shows we can sort thresholds $r_i$ via sorting probabilities $p_{i\text{lom}}$. Line [8] and [9] scan over all agents whose threshold are not obtained (i.e. $r_i = \text{nil}$) and can be obtained (i.e. $r_k \neq \text{nil}$ for all $k \in D_i$).

3. (line [12] to [13]) Given all the agents who have a known $p_{i\text{lom}}$ and an unknown $r_i$, this component ranks these agents and selects out top agents who has the highest $p_{i\text{lom}}$. The reason for computing this is Theorem 3 finds that $r_i$ is dictated only by $r_j$ for $j \in J_i$. This $J_i$ is obtained by updating $A$ in line [13] and [17].

4. (line [14] to [16]) With the top-ranked agents given by the Component [8] this component computes the thresholds for these top-ranked agents. Note that in each iteration of the while loop, there is a new set of top-ranked agents $T$. With a new set $T$ in each iteration, set $A$ gradually expands until all agents’ thresholds are calculated.
By finding the agents with the highest \( p^{l_{\text{oms}}} \) from the ones with \( \phi_i = 1 \) in line 12 and 13, Algorithm 2 can find the agents who have the highest threshold among all agents whose thresholds are not obtained. This is because, for an agent \( j \) whose threshold is not obtained (i.e. \( r_j = \text{nil} \)), if \( \phi_j = 0 \), then \( j \) must lead at least one agent \( i \) for whom \( r_i = \text{nil} \) and \( \phi_i = 1 \), and by Lemma 2 \( r_j < r_i \). Starting from the leaf nodes in set \( \mathbf{Z} \) who have been proved to have the highest thresholds, Algorithm 2 keeps finding the agents who have the highest threshold among all agents whose thresholds are not obtained, then moving them to the temp set \( \mathbf{T} \), calculating their thresholds, and moving them to the set \( \mathbf{A} \). This is exactly the process of traversing the order tree from the leaf nodes. By this mean, the equilibrium thresholds are calculated in descending order. Therefore, for any agent \( i \) who is chosen into the temp set \( \mathbf{T} \), \( \mathbf{J} \), must be the subset of \( \mathbf{A} \), and \( \mathbf{Q} \), has no common element with \( \mathbf{A} \). Note that, for the unconditional contributor \( i \) whose \( |\mathbf{Q}_i| = 0 \) shown in A.1, Eq.6 cannot be solved. However, the unconditional contributor can be easily identified because she must be the unique last one whose threshold needs to be decided, and her threshold is \( l \) in our setting.

With Proposition 3, Theorem 3, and the characterization of Algorithm 2, now we summarize the theoretical result for the computation of BNE under CIM.

**Theorem 4.** Under CIM, it is a Bayesian Nash equilibrium that every agent invites some or all of her neighbours and their contribution thresholds are computed by Algorithm 2.

Using this algorithm, we randomly generate 1000 different invitation graphs and check whether it is profitable for an agent to unilaterally deviate from inviting all her neighbours. We have the following observations.

**Remark 2.** Given all the other agents invite all their neighbours, if an agent unilaterally deviates from ‘inviting all’, then (i) the threshold of the deviator does not decrease; (ii) For the agents whose thresholds were higher than that of the deviator, their thresholds decrease; (iii) For the agents whose threshold were less equal than that of the deviater, their thresholds decrease; (iv) For the agents whose threshold were higher than that of the deviater, their thresholds decrease; (v) For the agents whose threshold were less equal than that of the deviator, their thresholds are still less equal than the new threshold of the deviator.

Moreover, the utility of the deviator \( i \) contributing with \( q_i > r_i \) before her deviation is

\[
\pi_i(q_i) = M \cdot \prod_{j \in \mathbf{P}, r_j > r_i} F_{q_j}(\max\{r_j, q_i\}) \cdot \prod_{k \in \mathbf{P}, r_k \leq r_i} F_{q_k}(q_k) - c.
\]

The simulating results show that once \( i \) deviates from ‘inviting all’, \( r_i' \geq r_i \), \( \forall j, r_j' < r_j \) and \( \forall k, r_k' \leq r_k' \). Thus, if \( q_i \leq r_i' \), \( i \) won’t contribute after deviation. Otherwise,

\[
\pi'_i(q_i) = M \cdot \prod_{j \in \mathbf{P}, r_j > r_i} F_{q_j}(\max\{r_j', q_i\}) \cdot \prod_{k \in \mathbf{P}, r_k \leq r_i} F_{q_k}(q_k) - c.
\]

Since \( \forall j, r_j' < r_j \), it is easy to know \( \pi'_i(q_i) \leq \pi_i(q_i) \), which takes an equal sign only when \( q_i = \max\{q_j\} \). This means deviating from ‘inviting all’ decreases \( i \)’s expected utility. Therefore, we obtain the following corollary.

**Corollary 1.** For any focal agent \( i \), when all the other agents invites all their neighbours, ‘inviting all her neighbours’ is a best response for \( i \). That is, every agent invites all her neighbours should be a Bayesian Nash equilibrium of CIM.

### 4.3 Efficient Computation on Large Graphs

Though in the previous subsection, it has been proved that every agent makes invitation in equilibrium under CIM, to decide which subset of neighbours an arbitrary agent invites is not easy. Among various of invitation actions, ‘inviting all neighbours’ is the most simple one that results in a good crowdsourcing performance. In this subsection, we analyze that to what extent each agent wants to invite her friend and whether it is possible that ‘all agents invite all neighbours’ in the Bayesian Nash equilibrium. We design a method to implement Algorithm 2 on large graphs.
Recall in Eq. (14), the expected utility for \( i \) to contribute with ability \( q_i \) is
\[
\pi_i(q_i) = \mathcal{M} \cdot \prod_{j \in P_i} F_{q_j}(\max\{r_j, q_i\}) - c, \tag{15}
\]
which depends on the value of equilibrium thresholds of \( i \)'s competitors. Thus how the competitors’ thresholds change is a key point. By definition of Bayesian Nash equilibrium, we only need to verify that ‘inviting all is the best response when all other agents invite all’. This means we need to examine how an agent’s expected utility changes when she changes her invitation from ‘inviting all’ to ‘inviting some’, which requires verification on every agent’s whole action space. This is obviously with high time complexity, and we need to simplify this procedure.

Note that, if different invitation graphs have identical order tree structure, then these different graphs have identical input for Algorithm 2 thus we have the following proposition.

**Proposition 5.** If two invitation graphs \( \mathcal{H} \) and \( \mathcal{H}' \) can generate an identical order tree \( T \), and agent \( i \) in \( \mathcal{H} \) and agent \( i' \) in \( \mathcal{H}' \) correspond to a same node in \( T \), then these two agents \( i \) and \( i' \) have the same contribution threshold in equilibrium.

Assume all the other agents are inviting all their neighbours. Consider the focal agent \( i \) changes her invitation action from ‘inviting all’ to ‘inviting some’. By the above proposition, the equilibrium will be affected only when \( i \)'s change affects the generated order tree. Given that every other agent has invited all her neighbours, \( i \)'s invitation only affects the agents led by herself, while the leading relations among the other agents will not change at all. When \( i \) changes her invitation from ‘inviting all’ to ‘inviting some’, some agents who are led by \( i \) may be excluded from the contest, while all \( i \)'s competitors are still in the contest.

We give the next definition and some propositions, which can effectively simplify the verification.

**Definition 4.** (Type of Agents) Two agents \( i \) and \( j \) are of the same type if the sub-trees root from them have the same structure. Formally, \( i \) and \( j \) are of the same type if (1) they are both leaf nodes, or (2) there exists a bijective relation \( \mathcal{R}_{i,j} : \mathcal{D}^{ir} \rightarrow \mathcal{D}^{jr} \) where for every \( (d_{ir}^i, d_{jr}^j) \in \mathcal{R}_{i,j}, d_{ir}^i \) and \( d_{jr}^j \) are of the same type.

We give an example to illustrate the above definition.

**Example 2.** Consider an order tree shown in Figure 4. Within all the nodes except \( p \) in it, \( j, f, g, h, k \) and \( d \) are leaf nodes, so they are of the same type. For \( e, b \) and \( i \), the sub-trees taking them as roots have the same structure, so they are of the same type. Formally, \( \mathcal{D}^{er} = \{j\}, \mathcal{D}^{br} = \{g\}, \mathcal{D}^{ir} = \{k\} \). There exists a relation \( \mathcal{R}_{e,b} = \{(j,g)\}, \) where \( j \) and \( g \) are of the same type, so \( e \) and \( b \) are of the same type; similar for \( e \) and \( i \), \( b \) and \( i \). Therefore, \( e, b \) and \( i \) are of the same type. For \( a \) and \( c \), the sub-trees taking them as roots have the same structure, so they are of the same type too. Formally, \( \mathcal{D}^{ar} = \{e, f\}, \mathcal{D}^{cr} = \{b, i\} \). There exists a relation \( \mathcal{R}_{a,c} = \{(e, i), (f, h)\}, \) where \( e \) and \( i \) are of the same type, and \( f \) and \( h \) are of the same type, so \( a \) and \( c \) are of the same type.

![Figure 4: A order tree \( T \) to illustrate Definition 4](image)

With Definition 4, the agents in an order tree can be categorized into different classes, each capturing a type. Generally, for two agents \( i \) and \( j \) of the same type, if there is that \( r_{d_{ir}^i} = r_{d_{jr}^j} \) for every
\[(d_{ij}^i, d_{ij}^j) \in \mathcal{R}_{i,j}, \text{ then by Eq.(14), } p_{ij}^{\text{com}} = p_{ij}^{\text{com}}, \text{ that is } \prod_{h \in D_{d_{ij}^i}} F_{q_h}(r_h) = \prod_{k \in D_{d_{ij}^j}} F_{q_k}(r_k).\]

Since \(r_{d_{ij}^i} = r_{d_{ij}^j}\) and \(\mathcal{R}_{i,j}\) is bijective by Definition 4, we can do the following calculation:

\[
\prod_{d_i \in D_i} F_{q_{d_i}}(r_{d_i}) = \prod_{d_j \in D_j} F_{q_{d_j}}(r_{d_j}) \Rightarrow r_i = r_j \text{ by Lemma 5.}
\]

With the same process that we inductively define the agent type, we can have the following proposition.

**Theorem 5.** In the Bayesian Nash equilibrium of CIM, agents of the same type have identical contribution threshold.

With the above Theorem 5, we directly have the following corollary for general conditions.

**Corollary 2.** Given all agents invite all their neighbours, if one agent of a type has no incentive to deviate from ‘inviting all’, then every agent of this type has no incentive to deviate from ‘inviting all’.

With Theorem 5 and Corollary 2, we can further simplify Algorithm 2 by only calculating threshold of one agent in each type, rather than calculating threshold of every agent. This significantly reduce the computation complexity of the Bayesian Nash equilibrium of CIM. This makes it possible to compute the BNE of CIM even for large social networks.

5 Experiments

5.1 Invitation Contests Outperform Traditional Contests

We implement our contest mechanisms on a well-known Facebook network dataset which has 4039 nodes and 88234 edges (see McAuley and Leskovec [2012]). For each single task, each node’s ability \(q_i\) is sampled from the exponential distribution \(E(\lambda = 1)\). This sampling well captures that people’s abilities for different tasks may show large diversity. We evaluate the overall performance of our mechanisms over many different tasks. The experiment results are shown in Figure 5.

Each column is for a fixed requester with a certain degree. The first row is for mechanism with no invitations (abbreviated as MN). The second rows is for Collective Invitation Mechanism (CIM). Each subfigure is a statistic result of 1000 different tasks. The \(x\)-axis is the bins for the highest quality obtained in each task, while the \(y\)-axis is the proportion of the bins in 1000 tasks. Thus each colored chart depicts a distribution of the best solution within 1000 tasks.

It is observed that the best solution elicited under MN is dependent on the requester’s degree. When the requester has more neighbours in the social network, the quality of the best solution is higher, and vice versa. CIM has significantly better performance in eliciting good solutions comparing with MN. The frequency of no-contributor cases is increasing with the requester’s degree under MN, but it is always 0 under CIM.

1. The best solutions under CIM are much better than that under MN. This can be seen by comparing each row in a fixed column. MN can never surpass CIM, even for the requester who has very large degree. This is because the invitation mechanisms provide strong incentives for both inviting and contributing.

2. The performance of CIM is robust against the requesters’ social resource diversity. This can be seen by comparing the columns in the second and the third row. The best solution distribution...
does not change too much for different requesters. This robustness makes CIM not only work well for companies or big organizations, but also have advantage for requesters who have limited social resources. This is especially suitable for websites like Github or StackExchange where the majority is normal users who have small degree.

3. In all the subfigures, the black bars above \( x = 0 \) depict how the chances are that no agent contributes under this mechanism. We can see MN suffers from this failure chances. In contrast, CIM totally solves this problem, it always has contributors, which makes its performance stable.

With these experiments, it is observed that in all the experiments under CIM, we find that every agent always fully invites all her neighbours. This is the strong evidence that CIM has performance beyond Theorem \( \text{(2)} \) it actually provide very strong incentive for agents to invite all neighbours. Moreover, in 98.6\% of all the 1000 tasks, only one winner was selected under CIM, this effectively controls the budget under it. Moreover, in all task experiments under CIM, the requester only needs to award at most two winners. This means CIM is not only very effective and robust in the sense that it elicits solutions with significantly higher qualities; it is also very efficient since the requester’s expenditure is controlled to a low level.

### 5.2 Population Dynamics in Invitation Contests

By the notion highest quality, we mean the quality of the best solution among all agents’ submitted solutions. In this part, we first show that, given a number \( n \) of agents with i.i.d. abilities, regardless of any underlying social network structure or crowdsourcing mechanism, to what extent the highest quality could be. That is, given a number \( n \) of agents, we theoretically find the upperbound of the performance of any crowdfunding contests.

Recall that \( q_i \) is an agent’s ability and \( q_i (\forall i) \) are i.i.d. variables, with a common PDF \( f \). \( q_i \) has an upperbound \( u \) and a lowerbound \( l \). Consider a group of \( n \) agents with an arbitrary social network structure. The highest ability among the \( n \) agents is the maximum value of \( n \) random variables \( q_i (1 \leq i \leq n) \), which is just the largest order statistic \( q_{(n)} \). The CDF of \( q_{(n)} \) is

\[
\hat{F}_{q_{(n)}}(x) = P(q_{(n)} \leq x) = P(\forall i, q_i \leq x) = \prod_{i \in [n]} P(q_i \leq x) = \prod_{i \in [n]} F_{q_i}(x) = F(x)^n.
\]

Thus the PDF of \( q_{(n)} \) is

\[
\hat{f}(x) = \frac{d\hat{F}(x)}{dx} = \frac{dF(x)^n}{dx} = n \cdot F(x)^{n-1} \cdot f(x).
\]

And the expected value of \( q_{(n)} \) is

\[
E(q_{(n)}) = \int_l^u x \cdot \hat{f}(x)dx = \int_l^u n \cdot x \cdot F(x)^{n-1} \cdot f(x)dx.
\] 

Figure 5: Overall performances of invitation mechanisms over 1000 different tasks.
This is the expectation of the highest ability in a crowd of \( n \) agents. It is also the upperbound of the expected highest quality obtainable in any crowdsourcing contest. Traditional crowdsourcing contest cannot reach this upperbound performance, since the number of agents in them cannot reach value \( n \) without agents’ invitations. If the requester can only call together a very small portion of the \( n \) agents, their performance could even be much less than this upperbound value. In the next part, we will experimentally see how CIM will perform comparing with this performance upperbound.

To challenge CIM, we choose a node with a very small degree (\( d = 5 \)) to be the task requester. This means, if there is no agents’ invitations (i.e., by traditional crowdsourcing mechanism), the task posted by the requester can only acquire 5 participants. To objectively simulate an expected performance and compare it with the upperbound \( E(q(n)) \) in Eq.(16), we sample the abilities of agents in this same Facebook network 500 times according to a same distribution. This means, we simulate totally 500 different crowdsourcing tasks in a same social network. The results are shown in Figure 6, where the \( x \)-axis represents the agent number in the invitation graph and the maximum number is 4038; the \( y \)-axis is the highest quality. The red solid curve represents the average highest qualities in the 500 simulations, while the red shadow alongside this solid curve shows the standard deviation (SD) of the qualities in 500 different tasks. The blue dashed line represents the theoretical upperbound of the highest ability of any crowdsourcing contest. It has been show in Eq.(16).

As Figure 6 shows, under CIM, the participant population dynamically expands. At the very beginning of the crowdsourcing, only the requester’s 5 neighbours participate, and the highest quality is pretty low. As the population increases from tens of people to nearly one thousand of people, the highest quality increases rapidly. After that, the highest quality grows steadily with the population size, until the population size stops growing. The performance of CIM is below the theoretical upperbound. This is because, agents in CIM have different thresholds, and when the agent with the highest ability does not contribute, there might be other agents with lower abilities who contribute. For this reason, the CIM in experiments cannot reach the upperbound. Note that traditional contest mechanism without invitation confront a problem that there exists a chance that the highest ability agent will not contribute if it is below the common contribution threshold, and the requester obtains no contribution under this occasion. This drawback of traditional contest has been shown as the black bars in Figure 5. The standard deviations of CIM (depicted by the red shadow) is not very small. However, our mechanisms are designed not only for a single contest, but more suitable for the platforms and websites which have a large amount of users and many contest. Thus in expectation, our mechanism can provide stable and high performance.
6 Conclusions

Existing crowdsourcing contest models only consider the agents who directly link to the requester, while a large amount of agents in the social network who do not know the task could not participate. We extend contest design theory to social network environments and introduce new contest mechanism CIM whereby agents are impelled to invite their friends. CIM incentivizes all agents to make some invitation, and in simulation it shows an even stronger incentive that all agents refer all their neighbours. Our new invitation model and contest mechanism capture the commonly seen scenarios that an agent can exert influence on the crowdsourced task not only by her own ability but also via her social influence. Comparing with traditional crowdsourcing contest methods, our new mechanism CIM can discover high ability agents who hide deeply in the social network, which significantly improves the crowdsourcing quality. Our work is the first attempt to clarify agents’ incentives to invite their friends in a crowdsourced task. Future researches could be the extensions of invitation contests mechanisms under various settings, such as different output functions and contest success functions. Another future direction is to investigate the change of the principal’s marginal profit when more agents are invited into the contest. Extended research could also be the invitation contests under other kinds of settings, such as sequential contest, contest with uncertain performance and so on.

References

Jeff Howe et al. The rise of crowdsourcing. *Wired magazine*, 14(6):1–4, 2006.

Gordon Tullock. The welfare costs of tariffs, monopolies, and theft. *Economic Inquiry*, 5(3):224–232, 1967.

Anne O Krueger. The political economy of the rent-seeking society. *The American economic review*, 64(3):291–303, 1974.

Edward P Lazear and Sherwin Rosen. Rank-order tournaments as optimum labor contracts. *Journal of political Economy*, 89(5):841–864, 1981.

Amihai Glazer and Refael Hassin. Optimal contests. *Economic Inquiry*, 26(1):133–143, 1988.

Ani Dasgupta and Kofi O Nti. Designing an optimal contest. *European Journal of Political Economy*, 14(4):587–603, 1998.

Qiang Fu and Zenan Wu. On the optimal design of biased contests. *Theoretical Economics*, 15(4):1435–1470, 2020.

Yeon-Koo Che and Ian Gale. Optimal design of research contests. *American Economic Review*, 93(3):646–671, 2003.

Wojciech Olszewski and Ron Siegel. Performance-maximizing large contests. *Theoretical Economics*, 15(1):57–88, 2020.

Arpita Ghosh and Patrick Hummel. Implementing optimal outcomes in social computing: a game-theoretic approach. In *Proceedings of the 21st international conference on World Wide Web*, pages 539–548, 2012.

Arpita Ghosh and Robert Kleinberg. Optimal contest design for simple agents. *ACM Transactions on Economics and Computation (TEAC)*, 4(4):1–41, 2016.

Qiang Fu and Jingfeng Lu. The optimal multi-stage contest. *Economic Theory*, 51(2):351–382, 2012.

Nikolay Archak and Arun Sundararajan. Optimal design of crowdsourcing contests. *ICIS 2009 proceedings*, page 200, 2009.
Ruggiero Cavallo and Shaili Jain. Efficient crowdsourcing contests. In AAMAS, pages 677–686, 2012.

Tie Luo, Salil S Kanhere, Sajal K Das, and Hwee-Pink Tan. Incentive mechanism design for heterogeneous crowdsourcing using all-pay contests. IEEE transactions on mobile computing, 15(9):2234–2246, 2015.

Shuchi Chawla, Jason D Hartline, and Balasubramanian Sivan. Optimal crowdsourcing contests. Games and Economic Behavior, 113:80–96, 2019.

Dominic DiPalantino and Milan Vojnovic. Crowdsourcing and all-pay auctions. In Proceedings of the 10th ACM conference on Electronic commerce, pages 119–128, 2009.

Xi Alice Gao, Yoram Bachrach, Peter Key, and Thore Graepel. Quality expectation-variance trade-offs in crowdsourcing contests. In Twenty-Sixth AAAI Conference on Artificial Intelligence, 2012.

Ruggiero Cavallo and Shaili Jain. Winner-take-all crowdsourcing contests with stochastic production. In First AAAI Conference on Human Computation and Crowdsourcing, 2013.

Daren C Brabham. Crowdsourcing, 2013.

Ella Segev. Crowdsourcing contests. European Journal of Operational Research, 281(2):241–255, 2020.

Priel Levy, David Sarne, and Igor Rochlin. Contest design with uncertain performance and costly participation. In Proceedings of the 26th International Joint Conference on Artificial Intelligence, pages 302–309, 2017.

David Sarne and Michael Lepioshkin. Effective prize structure for simple crowdsourcing contests with participation costs. In Proceedings of the AAAI Conference on Human Computation and Crowdsourcing, 2017.

Michal Habani, Priel Levy, and David Sarne. Contest manipulation for improved performance. In Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems, pages 2000–2002, 2019.

Priel Levy, David Sarne, and Yonatan Aumann. Selective information disclosure in contests. In Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems, pages 2093–2095, 2019.

Jon Kleinberg and Prabhakar Raghavan. Query incentive networks. In 46th Annual IEEE Symposium on Foundations of Computer Science (FOCS’05), pages 132–141. IEEE, 2005.

Esteban Arcaute, Adam Kirsch, Ravi Kumar, David Liben-Nowell, and Sergei Vassilvitskii. On threshold behavior in query incentive networks. In Proceedings of the 8th ACM conference on Electronic commerce, pages 66–74, 2007.

Devansh Dikshit and Narahari Yadati. Truthful and quality conscious query incentive networks. In International Workshop on Internet and Network Economics, pages 386–397. Springer, 2009.

Xin Jin, Kuang Xu, Victor OK Li, and Yu-Kwong Kwok. Discovering multiple resource holders in query-incentive networks. In 2011 IEEE consumer communications and networking conference (CCNC), pages 1000–1004. IEEE, 2011.

Galen Pickard, Wei Pan, Iyad Rahwan, Manuel Cebrian, Riley Crane, Anmol Madan, and Alex Pentland. Time-critical social mobilization. Science, 334(6055):509–512, 2011.

Yufeng Wang, Wei Dai, Qun Jin, and Jianhua Ma. Bcinet: A biased contest-based crowdsourcing incentive mechanism through exploiting social networks. IEEE Transactions on Systems, Man, and Cybernetics: Systems, 50(8):2926–2937, 2018.
Wen Shen, Yang Feng, and Cristina V Lopes. Multi-winner contests for strategic diffusion in social networks. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 33, pages 6154–6162, 2019.

Bin Li, Dong Hao, Dengji Zhao, and Tao Zhou. Mechanism design in social networks. In *Proceedings of the AAAI Conference on Artificial Intelligence*, 2017.

Dengji Zhao, Bin Li, Junping Xu, Dong Hao, and Nicholas R Jennings. Selling multiple items via social networks. In *Proceedings of the 17th International Conference on Autonomous Agents and MultiAgent Systems*, pages 68–76, 2018.

Seungwon Eugene Jeong and Joosung Lee. The groupwise-pivotal referral mechanism: Core-selecting referral strategy-proof mechanism. *Available at SSRN: https://ssrn.com/abstract=3574093*, 2020.

Takehiro Kawasaki, Nathanaël Barrot, Seiji Takanashi, Taiki Todo, and Makoto Yokoo. Strategy-proof and non-wasteful multi-unit auction via social network. In *Proceedings of the AAAI Conference on Artificial Intelligence*, pages 2062–2069, 2020.

Yuhang Guo and Dong Hao. Emerging methods of auction design in social networks. In Zhi-Hua Zhou, editor, *Proceedings of the Thirtieth International Joint Conference on Artificial Intelligence, IJCAI-21*, pages 4434–4441. International Joint Conferences on Artificial Intelligence Organization, 8 2021. Survey Track.

Levy Priel, Sarne David, and Yonatan Aumann. Tractable (simple) contests. In *Proc. of IJCAI*, pages 361–367, 2018.

Julian J McAuley and Jure Leskovec. Learning to discover social circles in ego networks. In *NIPS*, volume 2012, pages 548–56. Citeseer, 2012.
Appendix

A  Supplements for Collective Invitation Mechanism (CIM)

A.1  Discussion about the unconditional contributors

As is discussed previously, there are two cases for an arbitrary agent $i$ in equilibrium: (1) $\pi_i(l) \leq 0$; (2) $\pi_i(l) > 0$. In the first case, $i$ has an equilibrium threshold $r_i \in [l, u]$, where $\pi_i(q_i = r_i) = 0$, and $i$ contributes only when $q_i > r_i$. In the second condition, we artificially set $r_i = l$, which does not change the analysis, because under this condition, $i$ will always contribute, and

- From $i$’s perspective, the probability that she outperforms any other $j$ is $P(q_j < q_i \lor q_j < r_j) = F_{q_j}(\max\{q_j, r_j\})$, which is not affected by the value of $r_i$;
- From any other $j$’s perspective, the probability that $j$ outperforms $i$ is $P(q_i < q_j) = F_{q_i}(q_j)$.

Since $q_j \geq l$, this value exactly equals $F_{q_i}(\max\{q_j, r_i\})$.

Therefore, setting $r_i = l$ exerts the same influence on every agent, thus does not affect the equilibrium of the game. It is worth noting that, for such agent $i$ whose $\pi_i(l) > 0$, under the setting that $r_i = l$, $\pi_i(r_i) > 0$, which is different from the other kind of agent $j$, for whom $\pi_j(r_j) = 0$. Given the above facts, we conclude that, for an arbitrary agent $i$,

- If $r_i > l$, then $\pi_i(r_i) = 0$;
- If $r_i = l$, then $\pi_i(r_i) \geq 0$.

$i$ might be an unconditional contributor for whom $\pi_i(r_i) > 0$ only if $r_i = l$. We have one thing for sure that, such an $i$ has the lowest threshold among all agents in equilibrium. Now we illustrate a lemma to see some properties about the lowest threshold.

**Lemma 6.** If there is a unique agent $i$ who has the lowest threshold among all agents, then this lowest threshold $r_{\text{min}} = r_i = l$. If there are multiple agents $i, \cdots, j$ who have the lowest thresholds among all agents, then the lowest threshold $r_{\text{min}} = r_i = \cdots = r_j > l$.

**Proof.** When there is a unique agent $i$ with $r_{\text{min}}$ in equilibrium, denote the second lowest threshold by $r_{\text{se}}$. We have that $\forall j \neq i, r_j \geq r_{\text{se}} > r_{\text{min}}$. On the one hand, when $i$ contributes with $q_i = r_{\text{se}} > r_{\text{min}}$, her expected utility should be positive, that is

$$\pi_i(q_i = r_{\text{se}}) = M \cdot \prod_{j \in P_i} F_{q_j}(\max\{r_j, r_{\text{se}}\}) - c > 0.$$  

On the other hand, when $i$ contributes with $q_i = l \leq r_{\text{min}} < r_{\text{se}}$, the expected utility is

$$\pi_i(q_i = l) = M \cdot \prod_{j \in P_i} F_{q_j}(\max\{r_j, l\}) - c$$

$$= M \cdot \prod_{j \in P_i} F_{q_j}(r_j) - c$$

$$= M \cdot \prod_{j \in P_i} F_{q_j}(\max\{r_j, r_{\text{se}}\}) - c$$

$$= \pi_i(q_i = r_{\text{se}}) > 0.$$  

This indicates that $i$ is an unconditional contributor who will contribute with any ability and for whom $r_i = l$. Every other agent is not an unconditional contributor since they have thresholds higher than $l$.

When there are multiple agents $i, \cdots, j$ with $r_{\text{min}}$ in equilibrium, $r_i = r_j = r_{\text{min}}$. Now we consider the leading relation between $i$ and $j$. We know $U = \{i\} + D_i + P_i$. Thus $j \in D_i$ or $j \in P_i$.
If \( j \in D_i \), which means, \( i \) leads \( j \), then \( i \in P_j \);

• If \( j \notin D_i \), then \( j \in P_i \).

We summarize the above as \( i \in P_j \) or \( j \in P_i \).

Lemma 6 gives a clear illustration about the unconditional contributors, which we concludes as follows:

1. there is at most one unconditional contributor in each contest;

2. only for this unconditional contributor (denote by \( i \)), \( r_i = l \) and \( \pi_i(r_i) > 0 \);

3. for every other agent \( j \), \( r_j > l \) and \( \pi_j(r_j) = 0 \);

4. the unconditional contributor does not necessarily exist in every contest.

We can further tell that, within one contest under CIM, for an arbitrary agent \( i \),

• \( r_i > l \) if and only if \( \pi_i(r_i) = 0 \);

• \( r_i = l \) if and only if \( \pi_i(r_i) > 0 \), and such an \( i \) is the unique unconditional contributor in this contest.

Note that, for the unconditional contributor \( i \), all other agents have higher thresholds than her, which is exactly the condition that \( |Q_i| = 0 \) which will be discussed in Proposition 3.

B Supplements for Experiments

B.1 Reducing the computation complexity for CIM equilibrium verification

Example 3. Consider three similar order trees in Figure 7: (a), \( i \) and \( j \) are of the same type; in (b), \( i \) doesn’t invite \( d_{i1} \); in (c), \( j \) doesn’t invite \( d_{j1} \).

Figure 7: Three similar order trees. In (a), \( i \) and \( j \) are of the same type; in (b), \( i \) doesn’t invite \( d_{i1} \); in (c), \( j \) doesn’t invite \( d_{j1} \).
With Lemmas 5 and 2, it is easy to know the equilibrium thresholds $r_i = r_j$ in (a), $r'_i > r'_j$ in (b), and $r''_i < r''_j$ in (c). It is worth noting that, if we run Algorithm 2 on both (b) and (c), the programs execute in a symmetric way until $r'_i$ is determined in (b), or $r''_j$ is determined in (c).

To be specific, the agent numbers in (b) and (c) are equal. At the beginning of both programs, $h'$, $d_{i3}$, $d_{j3}$ in (b) and $h''$, $d'_{i3}$, $d'_{j3}$ in (c) are all leaf agents with identical threshold. Then in the first loop of both programs, $d'_{i2}$, $d'_{j2}$ in (b) and $d''_{i2}$, $d''_{j2}$ in (c) are of the same type with identical threshold. In the next loop, $i'$ in (b) and $j''$ in (c) are of the same type with identical threshold. That is to say, $r'_i = r''_j$.

Considering Eq. (15), we know for any $q > r'_i = r''_j$, the expected utility for $i$ to contribute with $q$ in (b) equals the expected utility for $j$ to contribute with the same $q$ in (c) (i.e. $\pi'_i(q) = \pi''_j(q)$). And for any $q \leq r'_i = r''_j$, neither $i'$ nor $j''$ contributes. The former facts show that, if $i$ has no incentive to deviate from (a) to (b), then $j$ has no incentive to deviate from (a) to (c) either.

Take the case in Figure 7(a) as an example to illustrate the above common changes.

**Example 4.** Suppose Figure 7(a) is the order tree when all agents invite all their neighbours. Under this condition, $i$ leads $d_{i1}$, $d_{i2}$ and $d_{i3}$; the direct descendants are $d_{i1}$ and $d_{i2}$.

There are three possible conditions if $i$ deviates from ‘inviting all’: (1) $d_{i1}$ is excluded and $i$ leads $d_{i2}$ and $d_{i3}$ (as Figure 8(d) shows); (2) $d_{i2}$ is excluded, thus $d_{i3}$ also being excluded and $i$ leads $d_{i1}$ only (as Figure 8(e) shows); (3) $d_{i1}$, $d_{i2}$ and $d_{i3}$ are all excluded (as Figure 8(f) shows).

We ran the equilibrium verification on the order tree in Figure 7(a) by calculating the equilibrium thresholds on the order trees in Figure 7(a) and Figure 8(d-f). Figure 9(a) shows these equilibrium thresholds when the prize $M = 1$, the contributing cost $c = 0.1$, and agents’ qualities are independently and identically distributed on the exponential distribution $E(\lambda = 1)$.

From Figure 9(a), we know that when $i$ invites all her neighbours, among all $i$’s competitors, $d_{j1}, d_{j2}, d_{j3}, h$ have higher thresholds than $i$; $k, j$ have thresholds equal to or lower than $i$. After $i$’s deviating, $i$’s threshold increases, thresholds of $d_{j1}, d_{j2}, d_{j3}, h$ decrease, and thresholds of $k, j$ are still equal lower than that of $i$. 

![Figure 8: The three different order trees when $i$ in Figure 7(a) deviates from ‘inviting all’](image)
Figure 9: In the case shown in Figs. 7(a), i’s different invitations may lead to different order trees shown in 8(d-f). (a) and (b) illustrate the equilibrium information one these four order trees under the assumption that $\mathcal{M} = 1$, $c = 0.1$ and $q \sim E(\lambda = 1)$. 

(a) Thresholds of $i$ and her competitors under 4 equilibria

(b) $i$’s expected utility under different intrinsic quality