IMPROVED SURROGATE BI-PARAMETER MAXIMUM PRINCIPLE

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Abstract. Logarithmic potentials and many other potentials satisfy maximum principle. The dyadic version of logarithmic potential can be easily introduced, it lives on dyadic tree and also satisfies maximum principle. But its analog on bi-tree does not have this property. We prove here that “on average” we can still have something like maximum principle on bi-tree.

1. Potentials and maximum principle

Let us recall some basic facts on logarithmic potential. For a compact Dirichlet regular set $E$ on the complex plane (in particular $E$ has positive logarithmic capacity $\text{cap}(E)$) there exists unique positive measure $\mu$ on $E$ such that

$$\int \log \frac{1}{|z - \zeta|} d\mu(\zeta) \equiv 1, \quad z = x + iy \in E.$$ 

And potential $U_\mu(z) := \int \log \frac{1}{|z - \zeta|} d\mu(\zeta)$ also satisfies

$$U_\mu(z) \leq 1 \quad \forall z \in \mathbb{C}.$$ 

For such a measure, its normalization $\mu/|\mu|$ is called capacitary measure of $E$ and $|\mu| = \text{cap}(E)$.

For any positive measure $\nu$ on $E$ the maximal principle holds: for

$$U_\nu(z) = \int \log \frac{1}{|z - \zeta|} d\nu(\zeta) \quad \zeta = \xi + i\eta, \quad z = x + iy$$

we have

$$U_\nu(z) \leq \sup_{z \in E} U_\nu(z), \quad \forall z \in \mathbb{C}.$$ 

However, in the area of complex analysis dealing with Hardy spaces in the poly-disc considered in [AMPS], [AHMV], [AMPVZ], [MPVZ] the following very different
potential (or its dyadic version) appears absolutely naturally and plays a vital part of poly-disc theory:

\[ V^\nu(x, y) = \int_E \log \frac{1}{|x - \xi|} \cdot \log \frac{1}{|y - \eta|} d\nu(\xi, \eta). \]

It is a bi-parameter potential. This a very unusual potential and there is no maximum principle. The reader can be familiarized with this kind of potential theory (along with the classical one) through the book [AH].

In [MPV] we built several examples that demonstrate how crucially multi-parameter potential theory is different from the usual one.

But a certain “shadow” of the maximum principle is still preserved. We used a certain surrogate maximum principle that turned out to be vital for our multi-parameter Carleson embedding theorems (≡weighted multi-parameter paraproducts theorems) in [AHMV], [AMPVZ], [MPVZ], [MPVZ1].

We feel that our method of proving the surrogate maximum principle is a certain variant of a convex optimization. But we did not manage to cast it in this language, and, instead we use trick after trick to get it.

In the current paper we improve the dyadic surrogate maximum principle found in [AMPVZ] and [MPVZ], [MPVZ1], so everything happens on a direct product of dyadic trees, we call it a bi-tree and the symbol for it is \( T^2 \).

2. Surrogate maximal principle on a bi-tree

Consider a bi-tree \( T^2 = T \times T \), where \( T \) is a finite (but unboundedly large) simple dyadic tree. Let \( \alpha = (\alpha_1, \alpha_2) \) be the notation for the node of \( T^2 \) and Hardy operator be defined

\[ I f(\alpha) := \sum_{\alpha' \geq \alpha} f(\alpha'), \]

where \( \geq \) is a natural partial order on \( T^2 \). This is “summing-up along bi-tree”. The summing-down is a formal conjugate operator:

\[ I^* f(\alpha) := \sum_{\alpha' \leq \alpha} f(\alpha'). \]

Let \( \mu \) be a non-negative function on \( T^2 \) (on a finite graph there is no difference between non-negative functions and measures). The potential \( \nabla^{\mu} \) is defined as follows

\[ \nabla^{\mu}(\alpha) := \mathbb{I}(I^* \mu)(\alpha) = \sum_{\alpha' \geq \alpha} \sum_{\gamma \leq \alpha'} \mu(\gamma). \]

\[ E_s := \{ \alpha \in T^2 : \nabla^{\mu}(\alpha) \leq s \}. \]

More notations:

\[ \nabla^{\mu}_\delta(\alpha) := \mathbb{I}(1_{E_\delta} I^* \mu)(\alpha) = \sum_{\alpha' \geq \alpha} \sum_{\gamma \leq \alpha'} \mu(\gamma). \]

\[ \mathcal{E}[\mu] := \int \nabla^{\mu} d\mu = \int_{T^2} (I^* \mu)^2. \]

\[ \mathcal{E}_\delta[\mu] := \int \nabla^{\mu}_\delta d\mu = \int_{T^2} 1_{E_\delta}(I^* \mu)^2. \]
The same operators exist obviously on the simple dyadic tree, and we call them \(I, I^*\). The potential of measure \(\mu\) on \(T\) will be denoted by \(V^\mu\). We also define

\[
V^\mu_\delta(a) := \sum_{a' \geq a, \sum_{a'' \geq a'} I^* \mu(a') \leq \delta} I^* \mu(a') .
\]

The maximum principle (at least one variant of it) on a simple tree is the following obvious inequality valid for all non-negative \(h\) on a simple tree:

(1) \(\max I h = \max_{\text{supp } h} I h\).

(This is not true in general on a bi-tree.)

Denoting \(h(a') := (I^* \mu)(a')\) we can use (1) equality to write

(2) \(I(1_{Ih \leq \delta} \cdot h) \leq \max_{Ih \leq \delta} I(1_{Ih \leq \delta} \cdot h) \leq \delta\),

and, hence, to conclude another obvious thing:

\[V^\mu_\delta \leq \delta\]

uniformly on a simple tree, and so

(3) \(\int V^\mu_\delta d\mu \leq \delta |\mu|\).

Potential \(V^\mu\) can be considered as a dyadic version of \(\int_0^1 \log \frac{1}{|x-y|} d\mu(y)\). Notice also that an obvious equality (1) gives the following variant of maximum principle for potentials \(V^\mu\):

**Proposition 2.1** (Maximum principle for potentials on a simple tree) Let \(\mu\) be a positive measure on \(T\) and \(V^\mu \leq 1\) on \(\text{supp } \mu\). Then \(V^\mu \leq 1\) everywhere on \(T\).

*Proof.* Let \(h = I^* \mu\), let \(E = \text{supp } h\). Then \(E = \{a \in T : \exists b \leq a, b \in \text{supp } \mu\}\). By definition \(E \subset \{V^\mu \leq 1\} = \{Ih \leq 1\}\). By (1) we write \(\max Ih = \max_{\text{supp } h} Ih = \max 1_E Ih, 1_E Ih \leq I(1_E h)\) (this is true for any up-set on any multi-tree), and so \(\leq I(1_{Ih \leq 1}) \leq 1\) by (2). \(\square\)

**Remark 2.1.** On a bi-tree neither this maximum principle nor (2) or (1) or (3) would be true anymore. One can find an example in Section 6 below or in [MPV]. But on average we have only slightly worse that (3). Namely, we have the following result below.

**Theorem 2.2** Let \(0 < |\mu| \leq \mathcal{E}[\mu] < \infty\). Let \(\delta > 0\) and \(\tau > 0\). Then there exists \(C_\tau < \infty\) such that \(\mathcal{E}_\delta [\mu] \leq C_\tau \delta^{1-\tau} |\mu|^{1-\tau} \mathcal{E}[\mu]^{\tau}\).

First we need a lemma.

**Lemma 1** Let \(0 < \delta \leq \lambda/6\). Then we can find function \(\varphi\) non-negative on \(T^2\) such that

1. The domain of majorization: \(\|\varphi\| \geq \frac{1}{4} \mathcal{V}^\mu_\delta\), where \(\mathcal{V}^\mu_\delta \geq 40 \lambda\);
2. Support of majorant: \(\text{supp } \varphi \subset E_{3\lambda} \setminus E_\delta\);
3. Energy estimate: \(\int_{T^2} \varphi^2 \leq A_0 \mathcal{E}_\delta [\mu]\).
Remark 2.3. In fact, instead of 1) we will prove the following statement that implies 1). Everywhere on $T^2$, we have

$$\mathbb{I} \varphi + 10\lambda \geq \frac{1}{2} V_\delta^\mu.$$ 

2.1. Discussion of Lemma 1. Lemma 4.10 of [MPVZ1] claims “almost” the same. But there are two very delicate differences. The first difference is that although in Lemma 4.10 the majorization claim 1) is present, but in a weaker form: the similar majorization happens there on the set $\{\lambda \leq V_\delta^\mu \leq 2\lambda\}$.

The fact that $40\lambda$ happens instead of $\lambda$ is immaterial, but the fact that Lemma 4.10 has also the estimate from above $V_\delta^\mu \leq 2\lambda$ on the domain of majorization makes Lemma 4.10 much weaker than Lemma 1 above.

But we should also mention that by restricting majorization condition 1) to a “smaller” set Lemma 4.10 allowed us to have much better energy estimate 3):

$$\int_{T^2} \varphi^2 \leq A_0 \delta^2 E_\delta[\mu].$$

Such an estimate in our Lemma 1 is not possible on a “larger” domain of majorization $V_\delta^\mu \geq 40\lambda$ that does not include the upper bound on potential $V_\delta^\mu$ (if we also want to keep the support claim 2)).

The attentive reader should be warned that Lemma 4.10 allows us to get an estimate of type 1) on a set $V_\delta^\mu \geq 40\lambda$ without an upper bound on potential $V_\delta^\mu$, and even with the energy estimate $\int_{T^2} \varphi^2 \leq A_0 \delta^2 E_\delta[\mu]$. But the price to pay is to throw away the support claim 2) of Lemma 1.

We are saying this to emphasize that even though Lemma 4.10 of [MPVZ1] and Lemma 1 above look “the same”, they are actually very different, and they have quite different proofs.

This also explains that on tri-tree $T^3$ we do not have so far an analog of Lemma 1 but we have the analog of Lemma 4.10, that is Lemma 4.17 of [MPVZ1].

What consequences have these subtle delicate differences between Lemmas and $T^2$ and $T^3$? Here they are. For example, Lemma 4.10 was needed to prove on bi-tree $T^2$ the estimate

$$E_\delta[\mu] \lesssim \delta^{2/3} |\mu|^{2/3} E[\mu]^{1/3}.$$ 

As we can see in Theorem 2.2 this $2/3$ can be replaced by any $1 - \tau, \tau > 0$. On $T^3$ we do not have the analog of Lemma 1 so far, and this is the reason we cannot improve the estimate of Lemma 4.20 of [MPVZ1] on $T^3$:

$$E_\delta[\mu] \lesssim \delta^{1/2} |\mu|^{1/2} E[\mu]^{1/2}.$$ 

Again the reader should compare this with absolutely trivial estimate $\mathcal{E}$ on a simple tree. We do not have any estimate of the type

$$E_\delta[\mu] \lesssim \delta^\varepsilon |\mu|^{-\varepsilon} E[\mu]^{1-\varepsilon},$$

with any positive $\varepsilon$ on $T^n, n \geq 4$. This prevents us to extend the results of [MPVZ], [MPVZ1] to higher dimension. In particular, Carleson embedding theorem on $T^n, n \geq 4$, is not known.
2.2. From Lemma 1 to Theorem 2.2. Let us see how Lemma 1 implies Theorem 2.2. By the first part of Lemma 1 we conclude:

\[
\frac{1}{4} \mathcal{E}_\delta[\mu] \leq 40\lambda|\mu| + \int \| \varphi \| d\mu = 40\lambda|\mu| + \int_{E_{3\lambda}} \varphi \| \mu \| \leq 40\lambda|\mu| + \left( \int \varphi^2 \right)^{1/2} \mathcal{E}_{3\lambda}[\mu]^{1/2}.
\]

So

\[
\mathcal{E}_\delta[\mu] \leq 2A_0^{1/2} \left( \frac{\delta}{3\lambda} \right)^{1/2} \left( \mathcal{E}_\delta[\mu] \right)^{1/2} + C_\lambda|\mu|.
\]

We solve this quadratic inequality with respect to \( x := (\mathcal{E}_\delta[\mu])^{1/2} \) to obtain

\[
\mathcal{E}_\delta[\mu] \leq 10A_0 \left( \frac{\delta}{3\lambda} \right) \mathcal{E}_{3\lambda}[\mu] + A_1\lambda|\mu|.
\]

We have for any \( \delta, \lambda > 0 \) with \( \delta \leq \frac{\lambda}{6} \):

\[
\mathcal{E}_\delta[\mu] \leq \frac{\delta}{\lambda} \mathcal{E}_{3\lambda}[\mu] + \lambda|\mu|.
\]

Let \( T > 6 \) to be specified later (it depends only on \( \tau \)). We set \( \lambda_i := T^i \delta \) for \( i \in \mathbb{Z}_+ \). If \( \mathcal{E}_\delta[\mu] \leq \lambda_1|\mu| \) then

\[
\mathcal{E}_\lambda[\mu] = \mathcal{E}_\delta[\mu]^\tau \mathcal{E}_\delta[\mu]^{1-\tau} \leq C_\tau \mathcal{E}[\mu]^{\tau}(\delta|\mu|)^{1-\tau}
\]

as \( \lambda_1 = T\delta \). This gives the desired estimate with \( C_\tau := T^{1-\tau} \).

If \( \mathcal{E}_\delta[\mu] > \lambda_1|\mu| \) then we use a stopping argument. We fix the first \( k \in \mathbb{N} \) such that i) \( \mathcal{E}_{\lambda_{i-1}}[\mu] > \lambda_k|\mu| \) but ii) \( \mathcal{E}_{\lambda_i}[\mu] \leq \lambda_{k+1}|\mu| \). The “stopping” happens as \( \mathcal{E}[\mu] \) is finite but \( \lambda_k \to \infty \).

Now, using (1) for \( 1 \leq i \leq k \) we have

\[
\mathcal{E}_{\lambda_{i-1}}[\mu] \leq \lambda_i^{-1} \mathcal{E}_{\lambda_i}[\mu] + \lambda_i|\mu| \leq T^{-1} \mathcal{E}_{\lambda_i}[\mu]
\]

For the last inequality we used i) : \( \lambda_i|\mu| = T^{-1} \lambda_{i+1}|\mu| \leq T^{-1} \mathcal{E}_{\lambda_i}[\mu] \). Next, we multiply these inequalities to get:

\[
\mathcal{E}_\delta[\mu] \leq \left( \frac{c_0}{T} \right)^k \mathcal{E}_{\lambda_k}[\mu] \leq c_0^k T\delta|\mu|,
\]

where \( c_0 \) is the constant in (1) which we can assume to be \( \geq 6 \). We choose \( T := c_0^\frac{1}{\lambda} \).

By i) we see that \( k \leq \log_T \left( \frac{A}{\delta} \right) \) where \( A = \frac{\mathcal{E}_{\mu}|\mu|}{\lambda_i} \). Thus, \( c_0^k \leq \left( \frac{A}{\delta} \right)^{\frac{1}{\lambda}} \), which gives the desired result with \( C_\tau := T = c_0^\frac{1}{\lambda} \).

In fact,

\[
\mathcal{E}_\delta[\mu] \leq T^\tau \left( \frac{\mathcal{E}[\mu]}{\delta|\mu|} \right) \delta|\mu| = T\delta^{1-\tau}|\mu|^{1-\tau} \mathcal{E}[\mu]^{\tau}.
\]

In the previous case the constant \( C_\tau \) was \( T^{1-\tau} \), but obviously this is less than \( T \). Notice also that \( \min_{\tau \in (0,1)} c_0^\frac{1}{\lambda} \delta^{1-\tau} = \delta e^{\sqrt{\log \frac{A}{\delta}}} \). Thus we have
Corollary 2.1 Let \( \mu \) be a measure on bi-tree such that \( |\mu| \leq \mathcal{E}[\mu] \). Then
\[
\mathcal{E}_\delta[\mu] \leq \delta e^{\sqrt{\log 2} \mathcal{E}[\mu]}.
\]

Remark 2.4. For \( \mu \) be such that \( |\mu| \leq \mathcal{E}[\mu] \) on a simple tree we have a trivial estimate
\[
\mathcal{E}_\delta[\mu] := \int V^\mu_d d\mu \leq \delta \mathcal{E}[\mu].
\]
It just follows from another obvious one parameter claim \( \text{[3]} \).

3. Majorization in Lemma \( \text{[1]} \)

Now we prove Lemma \( \text{[1]} \)

Proof. Consider \( m := 1_{E_3}1^\ast \mu \), and \( n := 1^\ast \mu \).

Also fix \( (\beta_0, \alpha_0) \in T^2 \) and let this node be such that
\[
\forall \beta \geq \beta_0, \alpha \geq \alpha_0, \mathcal{V}_\delta^\mu(\beta, \alpha) \geq 20\lambda.
\]

Now put
\[
\varphi = \frac{1}{\lambda(I_1 m \cdot I_2 n \cdot 1_{E_{3\lambda} \setminus E_3} + I_2 m \cdot I_1 n \cdot 1_{E_{3\lambda} \setminus E_3}).
\]

Then
\[
\mathbb{I} \varphi(\beta_0, \alpha_0) = \frac{1}{\lambda}(I_2 I_1(I_1 m \cdot I_2 n \cdot 1_{E_{3\lambda} \setminus E_3}) + I_2 I_1(I_2 m \cdot I_1 n \cdot 1_{E_{3\lambda} \setminus E_3}))(\beta_0, \alpha_0).
\]

And
\[
I_2 I_1(I_1 m \cdot I_2 n \cdot 1_{E_{3\lambda} \setminus E_3})(\beta_0, \alpha_0) = \sum_{\alpha \geq \alpha_0} I_1(I_1 m \cdot I_2 n \cdot 1_{E_{3\lambda} \setminus E_3})(\beta_0, \alpha) = \sum_{\alpha \geq \alpha_0} I_1 m(\beta_0, \alpha) \cdot I_1(N1_{\delta < I_1 N \leq 3\lambda})(\beta_0, \alpha),
\]
where \( N := I_2 n \). But
\[
I_1(N1_{\delta < I_1 N \leq 3\lambda})(\beta_0, \alpha) \geq \frac{3}{2} \lambda - 2\delta \geq \lambda,
\]
if \( \mathbb{I} n(\beta_0, \alpha) = (I_1 N)(\beta(\alpha), \alpha) > 3\lambda \). This set is non-empty by \( \text{[7]} \). We denote the set of such \( \alpha \geq \alpha_0 \) by \( L(\beta_0, \alpha_0) \). So \( (\beta_0, \alpha) \notin E_{3\lambda} \). Let \( \alpha \geq \alpha_0 \) be in \( L(\beta_0, \alpha_0) \), and \( \alpha \geq \alpha' \geq \alpha_0 \). Then \( \alpha' \in L(\beta_0, \alpha_0) \) as well. In fact, \( (I_1 N)(\beta_0, \alpha') > (I_1 N)(\beta_0, \alpha) > 3\lambda \), because \( I_1 N = \mathbb{I} n \) is monotone increasing in each variable. So the set \( L(\beta_0, \alpha_0) \) is the ray, or segment \([\alpha_0, \ell(\beta_0, \alpha_0)]\).

To check \( \text{[5]} \) let us denote by \( \beta(3\lambda) \) the place such that it is the first \( > \beta_0 \) such that \( (\beta(3\lambda), \alpha) \in E_{3\lambda} \). If it exists. If it does not exists we put \( \beta(3\lambda) := \beta_0 \). We also denote by \( \beta(\delta) \) the last \( > \beta_0 \) such that \( (\beta(\delta), \alpha) \notin E_{\delta} \). So it may happen that \( \beta(\delta) := 0 = [0, 1] \), the maximal dyadic interval.

Then we get \( \text{[6]} \) as follows
\[
I_1(N1_{\delta < I_1 N \leq 3\lambda})(\beta_0, \alpha) = \sum_{\beta(3\lambda) \leq \beta' \leq \beta(\delta)} N(\beta', \alpha) = \sum_{\beta(3\lambda) \leq \beta'} N(\beta', \alpha) - \sum_{\beta(\delta) < \beta'} N(\beta', \alpha) \geq \frac{3}{2} \lambda - 2\delta.
\]
Using (8) we continue:

\[ I_2 I_1(I_1 m \cdot I_2 n \cdot 1_{E_{2,\lambda \setminus E_{\delta}} })(\beta_0, \alpha_0) \geq \lambda \sum_{\alpha \geq \alpha_0, \alpha \in L} I_1 m(\beta_0, \alpha). \]

Symmetrically, if \( R(\beta_0, \alpha_0) \) is the set of \( \beta \) such that there exist \( \alpha(\beta) \geq \alpha_0 \) such that \( I n(\beta, \alpha(\beta)) = I_2 I_1 m(\beta, \alpha) \geq 3\lambda, \) where \( H := I_1 n, \) we have

\[ I_1 I_2 (I_2 m \cdot I_1 n \cdot 1_{E_{2,\lambda \setminus E_{\delta}} })(\beta_0, \alpha_0) \geq \lambda \sum_{\beta \geq \beta_0, \beta \in R} I_2 m(\beta, \alpha_0). \]

We also conclude as before that the set \( R(\beta_0, \alpha_0) \) is the ray, or segment \([\beta_0, r(\beta_0, \alpha_0)].\)

But we assumed in (7) that

\[ I_1 I_2 m(\beta_0, \alpha_0) = I_2 I_1 m(\beta_0, \alpha_0) = \mathcal{V}_\delta^\mu(\beta_0, \alpha_0) \geq 20\lambda. \]

This assumption allows us to see that

\[ \sum_{\alpha \geq \alpha_0, \alpha \in L(\beta_0, \alpha_0)} I_1 m(\beta_0, \alpha) + \sum_{\beta \geq \beta_0, \beta \in R(\beta_0, \alpha_0)} I_2 m(\beta, \alpha_0) \geq \mathcal{V}_\delta^\mu(\beta_0, \alpha_0) - \delta \geq \frac{19}{20} \mathcal{V}_\delta^\mu(\beta_0, \alpha_0). \]

To see (10) let us rewrite it as follows

\[ \sum_{\alpha_0 \leq \alpha \leq \ell(\beta_0, \alpha_0)} I_1 m(\beta_0, \alpha) + \sum_{\beta_0 \leq \beta \leq r(\beta_0, \alpha_0)} I_2 m(\beta, \alpha_0) \geq \mathcal{V}_\delta^\mu(\beta_0, \alpha_0) - \delta \geq \frac{19}{20} \mathcal{V}_\delta^\mu(\beta_0, \alpha_0). \]

To prove (11) let us assume first that

\[ (\ell(\beta_0, \alpha_0), r(\beta_0, \alpha_0)) \in E_{\delta}. \]

Recall that \( \mathcal{V}_\delta^\mu = I_2 I_1 m. \) Thus the first sum in (11) gives us the summation of \( I^* \mu \) over the part of \( E_{\delta} \) that is \( E_{\delta} \cap \{(\beta, \gamma) \geq (\beta_0, \alpha_0) : \alpha \leq \ell(\beta_0, \alpha_0)\}. \)

But we symmetrically have \( \mathcal{V}_\delta^\mu = I_2 I_1 m. \) Thus the second sum in (11) gives us the summation of \( I^* \mu \) over the part of \( E_{\delta} \) that is \( E_{\delta} \cap \{(\beta, \gamma) \geq (\beta_0, \alpha_0) : \beta \leq r(\beta_0, \alpha_0)\}. \)

For the sake of brevity let us denote \( \ell := \ell(\beta_0, \alpha_0), r := r(\beta_0, \alpha_0). \) The only part of the summation of \( I^* \mu \) involved in the definition of \( \mathcal{V}_\delta^\mu(\beta_0, \alpha_0), \) which is left uncovered by both sums of (11) is, therefore, \( \sum_{\beta \geq \ell, \alpha \geq r} I^* \mu(\beta, \alpha). \) But by assumption (12) and the definition of \( E_{\delta} \) this sum is at most \( \delta. \) Thus (11) is proved when (12) holds.

Now assume that (12) does not hold. In this case we are going to estimate \( \mathcal{V}_\delta^\mu(\beta_0, \alpha_0) \) from above and to come to contradiction with (9). In fact, \( \mathcal{V}_\delta^\mu(\beta_0, \alpha_0) \) is bounded by three sums:

- \( \sum_{\beta_0 \leq \beta < r, \alpha_0 \leq \alpha < \ell} 1_{E_{\delta}}(\beta, \alpha) I^* \mu(\beta, \alpha), \)
- \( \sum_{\beta_0 \leq \beta \geq \ell, \alpha_0 \leq \alpha < \ell} 1_{E_{\delta}}(\beta, \alpha) I^* \mu(\beta, \alpha), \)
- \( \sum_{\alpha_0 \leq \alpha, \beta \geq r} 1_{E_{\delta}}(\beta, \alpha) I^* \mu(\beta, \alpha). \)
The first sum vanishes as by the fact that we have negation of (12) each term of this sum is zero. In fact, let

$$(\beta, \alpha) \in [\beta_0, r] \times [\alpha_0, \ell],$$

then $(\beta, \alpha) \notin E_\delta$ because by the negation of (12) $(r, \ell) \notin E_\delta$. Thus $1_{E_\delta}(\beta, \alpha) = 0$ in the first sum.

The second sum $\leq 6\lambda$. This is because $\ell$ is the maximal element for which such a sum is $\geq 3\lambda$. This maximality, and the monotone increasing of $\alpha \to I_1m(\beta_0, \alpha)$, imply that the second sum is at most $6\lambda$. The monotone increasing of $\alpha \to I_1m(\beta_0, \alpha)$ follows from the monotone increasing of function $m = 1_{E_\delta \cap \ell} I^* \mu$ in both variables.

By the same reasoning (symmetrically) the third sum is at most $6\lambda$.

We conclude that if (12) does not hold then $V_\mu E_\delta(\beta_0, \alpha_0) \leq 12\lambda$, but this contradicts (9). We finally proved (11), and, therefore, (10), under the assumption (7) that $V_\mu E_\delta(\beta_0, \alpha_0) \geq 20\lambda$.

From (10) it follows now that

$$I \varphi(\beta_0, \alpha_0) \geq \frac{19}{20} V_\mu E_\delta(\beta_0, \alpha_0),$$

if (9) is satisfied. Obviously this gives the majorization claim in Lemma. The support condition follows by the definition of $\varphi$.

4. THE ENERGY ESTIMATION

We are left to prove the norm (energy) claim of Lemma

$$(13) \quad \int_{T^2} \varphi^2 \leq K \frac{\delta}{\lambda} E_\delta [\mu].$$

By the definition of $\varphi$,

$$\int_{T^2} \varphi^2 \leq \frac{2}{\lambda^2} \int (I_1 m)^2 \cdot (I_2 n)^2 1_{E_\delta \cap \ell} + \frac{2}{\lambda^2} \int (I_2 m)^2 \cdot (I_1 n)^2 1_{E_\delta \cap \ell}.$$

These two terms are symmetric, we will estimate the first one. To do that we need two lemmas.

**Lemma 2** Let $T$ be a finite dyadic tree, and $g, h$ be non-negative functions on $T$. Let $g$ be superadditive, and let $1h \leq \lambda$ on supp $g$. Then for any $\beta \in T$

$$I^*(gh)(\beta) = \sum_{\alpha \leq \beta} g(\alpha) h(\alpha) \leq \lambda g(\beta).$$

**Proof.** Let us prove that

$$I^*(gh)(\beta) = \sum_{\alpha \leq \beta} g(\alpha) h(\alpha) \leq g(\beta) \max_{\alpha \in T, \alpha \leq \beta} Ih(\alpha).$$

The support of $g$ is an up-set by superadditivity. Then this holds trivially if $g(\beta) = 0$, and so we need to check the claim only on the support of $g$. Let $\beta \in \text{supp } g$
and let $\beta_+, \beta_-$ be two children of $\beta$. Then by induction

\[
I^* (gh)(\beta) = g(\beta)h(\beta) + I^* (gh)(\beta_+) + I^* (gh)(\beta_-) \leq

\]

\[
g(\beta)h(\beta) + g(\beta_+) \max_{\alpha \in T, \alpha \leq \beta_+} Ih(\alpha) + g(\beta_-) \max_{\alpha \in T, \alpha \leq \beta_-} Ih(\alpha) \leq

\]

\[
g(\beta)h(\beta) + (g(\beta_+) + g(\beta_-)) \cdot \max_{\alpha \in T, \alpha \leq \beta_+} \max_{\alpha \in T, \alpha \leq \beta_-} Ih(\alpha) \leq

\]

\[
g(\beta)h(\beta) + (g(\beta_+) + g(\beta_-)) \cdot \max_{\alpha \in T, \alpha \leq \beta_+} \max_{\alpha \in T, \alpha \leq \beta_-} Ih(\alpha) =

\]

\[
g(\beta) \cdot [h(\beta) + \max_{\alpha \in T, \alpha < \beta} Ih(\alpha)] = g(\beta) \cdot \left( \max_{\alpha \in T, \alpha \leq \beta} Ih(\alpha) \right)
\]

Lemma \[2\] is proved. \[\square\]

In the next lemma the operator $I$ is an operator on any abstract space with any positive measure.

**Lemma 3** Let $I$ be an operator with positive kernel, and $f, g$ non-negative functions. Then

\[
\int (If)^* g \leq \sup_{\text{supp } g} II^*(g) \int f^2
\]

**Proof.** Let $Ih(x) = \int K(x,y)h(y)dy$, here $dy$ is just any measure. Then $I^* f(y) = \int K(x,y) f(x) dx$. Then

\[
(14) \quad \int (If)^2 g = \int f I^*(gIf) \leq \|f\|_2 \left( \int I^*(gIf)^2 (gIf) dy \right)^{1/2}.
\]

On the other hand, we can write (we use symmetry between $x$ and $x'$ in the third line)

\[
\int I^*(gIf) I^*(gIf) dy = \int dy \int dx' \int dx K(x,y) g(x) I f(x) K(x',y) g(x') I f(x') \leq
\]

\[
\frac{1}{2} \int dy \int dx' \int dx K(x,y) g(x) \left[ If(x)^2 + If(x')^2 \right] K(x',y) g(x') =
\]

\[
\int dy \int dx' \int dx K(x,y) g(x) If(x)^2 K(x',y) g(x') = \int I^* g(y) \cdot I^* (g(If)^2)(y) dy =
\]

\[
\int II^* g \cdot (If)^2 \leq \sup_{\text{supp } g} II^* g \cdot \int (If)^2 g.
\]

Now we plug this estimate into (14) and we get

\[
\int (If)^2 g \leq \left( \int f^2 \right)^{1/2} \cdot \left( \sup_{\text{supp } g} II^* g \right)^{1/2} \cdot \left( \int (If)^2 g \right)^{1/2}.
\]

This is exactly the claim of Lemma \[3\] \[\square\]

**Remark 4.1.** There is always a stronger version:

\[
\int (If)^2 g \leq \sup_{\text{supp } g} I (\text{1supp } f I^*(g)) \int f^2.
\]

In fact, we just apply Lemma \[3\] to a new operator $I\phi := I (\text{1supp } f \phi).$
Then let $f := m, I := I_1, g := (I_2 n)^2 1_{E_{3\lambda}},$ then from this lemma and this remark we conclude the following:

$$\int (I_1 m)^2 \cdot (I_2 n)^2 1_{E_{3\lambda}} \leq \sup I_1(1_{\text{supp } m}(I_1^* \rho)) \int m^2,$$

where $\rho := (I_2 n)^2 1_{E_{3\lambda}}$.

Now we apply Lemma $2$ with $g := (I_2 n)^2 1_{E_{3\lambda}}, h := (I_2 n), I := I_1$. Notice

1. $I_1 I_2 n = \mathbb{1} n = \mathbb{1}^* \mu = \mathbb{V}^\mu$.
2. So $E_{3\lambda} = \{ \mathbb{V}^\mu \leq 3 \lambda \} = \{ I_1 h \leq 3 \lambda \}$, as $h := (I_2 n)$.
3. Hence, $\text{supp } g \subset E_{3\lambda} = \{ I_1 h \leq 3 \lambda \}$.

Therefore, we are in the assumptions of Lemma $2$ with $g = (I_2 n)^2 1_{E_{3\lambda}}, h = (I_2 n)^2 1_{E_{3\lambda}}, \text{supp } g \subset E_{3\lambda} = \{ I_1 (I_2 n) \leq 3 \lambda \}$. We conclude that the following pointwise estimate holds:

$$I_1^* \rho = I_1^* ((I_2 n)^2 1_{E_{3\lambda}}) \leq 3 \lambda \cdot I_2 n.$$

**Remark 4.2.** Lemma $2$ was vital here. This is the only place we used that $I_1$ is a simple tree operator.

But $\text{supp } m \subset E_{\delta}$. So we get an estimate

$$I_1(1_{\text{supp } m}(I_1^* \rho)) \leq I_1(1_{E_{\delta}}(I_1^* \rho)) \leq 3 \lambda \cdot I_1(1_{E_{\delta}} I_2 n) \leq 3 \delta \lambda,$$

where the last estimate follows by the following simple observation. We denoted $h = I_2 n$ and we know that (see (1), (2) above with $\delta$ replacing $3 \lambda$):

$$E_{\delta} = \{ I_1 h \leq \delta \}.$$

The latter set is of course an up-set on a simple tree. Now on a simple tree (but not on a multiple tree)

$$I_1(1_{E_{\delta}} \cdot I_2 n) = I_1(1_{I_{1,h \leq \delta} \cdot h}) \leq \delta.$$

Hence, plugging (16) into (15), we conclude that

$$\int T^2 \varphi^2 \leq A_0 \frac{\delta}{3 \lambda} \int m^2,$$

but

$$\int m^2 = \int 1_{E_{\delta}} \| \mathbb{V}^\mu \|^2 = \int \| 1_{E_{\delta}} \mathbb{V}^\mu \| d\mu = \int \mathbb{V}^\mu d\mu = \mathbb{E}_{\delta}[\mu],$$

and Lemma $1$ is proved.

5. A SHORTER PROOF OF THEOREM 2.2 GIVEN LEMMA 1

**Theorem 5.1** Let $\mu$ be a measure on $T^2$. If $\mathbb{E}[\mu] \geq 2 \delta |\mu|$, then

$$\mathbb{E}_{\delta}[\mu] \lesssim \delta \exp \left( c \sqrt{\log \frac{\mathbb{E}[\mu]}{\delta |\mu|}} \right) |\mu|.$$ 

**Remark 5.2.** Corollary $2.1$ is a consequence of this result. This is immediate if one notices that function $x \rightarrow \frac{1}{2} e^{\sqrt{\log x}}$ is decreasing for $x \geq e$. 

Remark 5.3. Theorem 2.2 is the consequence of this result. In fact, for any $\tau > 0$
$$\sqrt{\log x} \leq \tau \log x + c_\tau, \quad \forall x \geq 2.$$ 

Proof of Theorem 5.1 assuming Lemma 1. By Lemma 1, we have
$$E_\delta[\mu] \leq 10\lambda|\mu| + \int \mathbb{I} \varphi \, d\mu = 10\lambda|\mu| + \int \varphi^* \mu \leq 10\lambda|\mu| + \left( \int \varphi^2 \right)^{1/2} E_{3\lambda}[\mu]^{1/2}.$$ 

So
$$E_\delta[\mu] \leq A_0^{1/2} \left( \frac{\delta}{3\lambda} \right)^{1/2} \left( E_\delta[\mu] \right)^{1/2} E_{3\lambda}[\mu]^{1/2} + 10\lambda|\mu|.$$ 

We solve this quadratic inequality with respect to $x := \left( E_\delta[\mu] \right)^{1/2}$ to obtain
(18) $$E_\delta[\mu] \leq 2A_0 \left( \frac{\delta}{3\lambda} \right) E_{3\lambda}[\mu] + 10\lambda|\mu|.$$ 

Let us denote
(19) $$A = \frac{E_\delta[\mu]}{|\mu|}.$$ 

Let $\delta_k := (4A_0)^{-k(k+1)/2}$. By induction on $k \geq 0$, we will show that
$$E_{\delta_k}[\mu] \leq (4A_0)^{k+1} \delta_k|\mu|,$$
which, together with monotonicity of $E_\delta[\mu]$, implies the conclusion of the theorem.

For $k = 0$, the claim holds trivially. We have
$$\frac{\delta_{k+1}}{\delta_k} = (4A_0)^{-(k+1)(k+2)/2+k(k+1)/2} = (4A_0)^{-k-1} \leq 1/6,$$
so that, using (18) and the inductive hypothesis, we obtain
$$E_{\delta_{k+1}}[\mu] \leq 2A_0 \frac{\delta_{k+1}}{\delta_k} E_{\delta_k}[\mu] + 4\delta_k|\mu|.$$ 

$$\leq 2A_0 \cdot \frac{\delta_{k+1}}{\delta_k} \cdot (4A_0)^{k+1} \delta_k|\mu| + 4\delta_k|\mu|$$
$$\leq \left( 2A_0 (4A_0)^{k+1} + 4 \cdot (4A_0)^{k+1} \right) \delta_{k+1}|\mu|$$
$$\leq (4A_0)^{k+2} \delta_{k+1}|\mu|.$$ 

□

6. The Lack of Maximum Principle

All measures and dyadic rectangles below will be $N$-coarse.

In this section we build another example when Carleson condition holds, but restricted energy condition fails. But the example is more complicated (and more deep) than the previous one. In it the weight $\alpha$ again has values either 1 or 0, but the support $S$ of $\alpha$ is an up-set, that is, it contains every ancestor of every rectangle in $S$.

Let $N$ below be $2^M$, $M \in \mathbb{Z}_+$. The example is based on the fact that potentials on bi-tree may not satisfy maximal principle. So we start with constructing $N$-coarse $\mu$ such that given a small $\delta > 0$
(20) $$\forall \mu \preceq \delta \quad \text{on} \quad \text{supp} \mu,$$
but with an absolute strictly positive $c$

$$\max \mathcal{V}^\mu \geq \mathcal{V}^\mu(\omega_0) \geq c \delta \log N,$$

where $\omega_0 := [0, 2^{-N}] \times [0, 2^{-N}]$.

We define a collection of rectangles

$$Q_j := [0, 2^{-2^j}] \times [0, 2^{-2^{-1}N}], \quad j = 1 \ldots M \approx \log N,$$

and we let

$$Q_j^{++} := [2^{-2^j-1}, 2^{-2^j}] \times [2^{-2^{-1}N-1}, 2^{-2^{-1}N}],
Q_j^t := [0, 2^{-2^j-1}] \times [0, 2^{-2^{-1}N}], \quad j = 1 \ldots M,$$

$$Q_j^r := Q_j \setminus Q_j^t,
Q_j^l := [0, 2^{-2^j}] \times [2^{-2^{-1}N-1}, 2^{-2^{-1}N}],
Q_j^{--} := Q_j \setminus Q_j^l$$

to be their upper right quadrants, lower halves, top halves, right halves, and lower quadrant respectively. Now we put

$$\mathcal{R} := \{ R : Q_j \subset R \text{ for some } j = 1 \ldots M \},
\alpha_Q := \chi_\mathcal{R}(Q),$$

$$\mu(\omega) := \frac{\delta}{N} \sum_{j=1}^{M} \frac{1}{|Q_j^{++}|} \chi_{Q_j^{++}}(\omega),
\mu(\omega) := \frac{\delta}{N} \sum_{j=1}^{M} \frac{1}{|Q_j^{++}|} \chi_{Q_j^{++}}(\omega),$$

$$P_j = (2^{-2^j}, 2^{-2^{-1}N}).$$

Here $|Q|$ denotes the total amount of points $\omega \in (\partial T)^2 \cap Q$, i.e. the amount of the smallest possible rectangles (of the size $2^{-2N}$) in $Q$.

Observe that on $Q_j$ the measure is basically a uniform distribution of the mass $\frac{\delta}{N}$ over the upper right quarter $Q_j^{++}$ of the rectangle $Q_j$ (and these quadrants are disjoint).

To prove (20) we fix $\omega \in Q_j^{++}$ and split $\mathcal{V}^\mu(\omega) = \mathcal{V}^\mu_{Q_j^{++}}(\omega) + \mu(Q_j^r) + \mu(Q_j^{--}) + \mathcal{V}^\mu_{Q_j^{++}}(\omega)$, where the first term sums up $\mu(Q)$ for $Q$ between $\omega$ and $Q_j^{++}$. This term obviously satisfies $\mathcal{V}^\mu_{Q_j^{++}}(\omega) \lesssim \frac{\delta}{N}$. Trivially $\mu(Q_j^r) + \mu(Q_j^{--}) \leq \frac{2\delta}{N}$. The non-trivial part is the estimate

$$\mathcal{V}^\mu_{Q_j^{++}}(\omega) \lesssim \delta.$$

To prove (25), consider the sub-interval of interval $[1, n]$ of integers. We assume that $j \in [m, m+k]$. We call by $C_j^{[m, m+k]}$ the family of dyadic rectangles containing $Q_j^{++}$ along with all $Q_i^{++}$, $i \in [m, m+k]$ (and none of the others). Notice that $C_j^{[m, m+k]}$ are not disjoint families, but this will be no problem for us as we wish to estimate $\mathcal{V}^\mu(Q_j^{++})$ from above.

Notice that, for example, $C_j^{[m, m+1]}$ are exactly the dyadic rectangles containing point $P_j$. It is easy to calculate that the number of such rectangles is

$$(2^j + 1) \cdot (2^{-1}N + 1) \lesssim N.$$
Analogously, dyadic rectangles in family $C_j^{[m,m+k]}$ have to contain points $P_m, P_{m+k}$. Therefore, each of such rectangles contains point $(2^{-2^m}, 2^{-2^{m+k}N})$. The number of such rectangles is obviously at most $\lesssim 2^{-k}N$. The number of classes $C_j^{[m,m+k]}$ is at most $k+1$.

Therefore, $\mathcal{V}^{\mu}((Q_j^{++}))$ involves at most $(k+1)2^{-k}N$ times the measure in the amount $k \cdot \frac{\delta}{N}$. Hence

$$\mathcal{V}^{\mu}((Q_j^{++})) \leq \sum_{k=1}^n k(k+1)2^{-k}N \cdot \frac{\delta}{N},$$

and (25) is proved. Inequality (20) is also proved.

We already denoted

$$\omega_0 := [0, 2^{-N}] \times [0, 2^{-N}],$$
calculate now $\mathcal{V}^{\mu}(\omega_0)$. In fact, we will estimate it from below. The fact that $C_j^{[m,m+k]}$ are not disjoint may represent the problem now because we wish estimate $\mathcal{V}^{\mu}(\omega_0)$ from below.

To be more careful for every $j$ we denote now by $c_j$ the family of dyadic rectangles containing the point $P_j$ but not containing any other point $P_i, i \neq j$. Rectangles in $c_j$ contain $Q_j^{++}$ but do not contain any of $Q_i^{++}, i \neq j$. There are $(2^j - 2^{j-1} - 1) \cdot (2^{j+1}N - 2^{j}N - 1), j = 2, \ldots, M-2$. This is at least $\frac{1}{8}N$.

But now families $c_j$ are disjoint, and rectangles of class $c_j$ contribute at least $\frac{1}{8}N \cdot \frac{\delta}{N}$ into the sum that defines $\mathcal{V}^{\mu}(\omega_0)$. We have $M-4$ such classes $c_j$, as $j = 2, \ldots, M-2$. Hence,

$$\mathcal{V}^{\mu}(\omega_0) \geq \frac{1}{8}N \cdot \frac{\delta}{N} \cdot (M-4) \geq \frac{1}{9}M \cdot \frac{\delta}{N}.$$

Choose $\delta$ to be a small absolute number $\delta_0$. Then we will have (see (20))

$$\mathcal{V}^{\mu} \leq 1, \quad \text{on supp } \mu.$$

But (26) proves also (21) as $M \asymp \log N$.

**Remark 6.1.** Notice that in this example $\mathcal{V}^{\mu} \leq 1$ on supp $\mu$, and

$$\text{cap}\{\omega : \mathcal{V}^{\mu} \geq \lambda\} \leq ce^{-2\lambda}.$$

Here capacity is the bi-tree capacity defined e. g. in [?]. So there is no maximal principle for the bi-tree potential, but the set, where the maximal principle breaks down, has small capacity.

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