Beyond the Dirac phase factor

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Abstract

We report on previously overlooked solutions of the usual gauge transformation equations that exhibit a new form of nonlocal quantal behavior with the well-known Relativistic Causality of classical fields affecting directly the phases of wavefunctions. The new nonlocalities compete with Aharonov-Bohm behaviors and they provide: a correction to a number of erroneous results in the literature, a new interpretation of semiclassical observations and further extensions to delocalized states, a natural remedy of earlier “paradoxes”, and a new formulation in the study of time-dependent slit-experiments.

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I. INTRODUCTION

The Dirac phase factor — with a phase containing integrals over potentials (of the general form \( \int A \cdot dx' - c \int \phi dt' \)) — is the standard and widely used solution of the usual gauge transformation equations of Electrodynamics (with \( A \) and \( \phi \) vector and scalar potentials respectively). In a quantum mechanical context, it connects wavefunctions of two systems (with different potentials) that experience the same classical fields, i.e. either systems that are gauge-equivalent (a trivial case with no physical consequences), or systems that exhibit phenomena of the Aharonov-Bohm type (magnetic or electric) — and then this Dirac phase has nontrivial observable consequences. However, it has not been realized that the gauge transformation equations, viewed in a more general context, can have more general solutions than simple Dirac phases, and these lead to wavefunction-phase-nonlocalities that have been widely overlooked and that seem to have important physical consequences. In this paper we will briefly demonstrate these generalized solutions and will present cases (and closed analytical results for the wavefunction-phases) that actually connect (or map) two quantal systems that are neither physically equivalent nor of the usual Aharonov-Bohm type. We will also explore the consequences of the new (nonlocal) contributions (that appear in the wavefunction-phases) and will see that they are numerous and important; they are also of a different type in static and in time-dependent configurations (and in the latter cases they seem to lead to Relativistically causal behaviors, that apparently resolve earlier “paradoxes” arising in the literature from the use of standard Dirac phase factors).

Let us first remind the reader of a property that is more general than usually realized: the solutions \( \Psi(r, t) \) of the \( t \)-dependent Schrödinger (or Dirac) equation (SE) for a quantum particle of charge \( q \) that moves (as a test particle) in two distinct sets of (predetermined and classical) vector and scalar potentials \( (A_1, \phi_1) \) and \( (A_2, \phi_2) \), that are generally spatially- and temporally-dependent [and such that, at the spacetime point of observation \( (r, t) \), the magnetic and electric fields are the same in the two systems], are formally connected through

\[
\Psi_2(r, t) = e^{i \frac{q}{\hbar} \Lambda(r, t)} \Psi_1(r, t),
\]

with the function \( \Lambda(r, t) \) required to satisfy
\[
\n\nabla \Lambda(r, t) = \mathbf{A}_2(r, t) - \mathbf{A}_1(r, t) \quad \text{and} \quad -\frac{1}{c} \frac{\partial \Lambda(r, t)}{\partial t} = \phi_2(r, t) - \phi_1(r, t). \tag{2}
\]

The above property can be immediately proven by substituting each \(\Psi_i\) into its corresponding \((i)\)th time-dependent SE (namely with the set of potentials \((\mathbf{A}_i(r, t), \phi_i(r, t))\)): one can then easily see that (1) and (2) guarantee that both SEs are indeed satisfied together (after cancellation of a global phase factor in system 2). [In addition, the equality of all classical fields at the observation point, namely \(\mathbf{B}_2(r, t) = \nabla \times \mathbf{A}_2(r, t) = \nabla \times \mathbf{A}_1(r, t) = \mathbf{B}_1(r, t)\) for the magnetic fields (MFs) and \(\mathbf{E}_2(r, t) = -\nabla \phi_2(r, t) - \frac{1}{c} \frac{\partial \mathbf{A}_2(r, t)}{\partial t} = -\nabla \phi_1(r, t) - \frac{1}{c} \frac{\partial \mathbf{A}_1(r, t)}{\partial t} = \mathbf{E}_1(r, t)\) for the electric fields (EFs), is obviously consistent with all equations (2) — provided, at least, that \(\Lambda(r, t)\) is such that interchanges of partial derivatives with respect to all spatial and temporal variables (at the point \((r, t)\)) are allowed].

The above fact is of course well-known within the framework of the theory of quantum mechanical gauge transformations (the usual case being with \(\mathbf{A}_1 = \phi_1 = 0\), hence a mapping from a system with no potentials); but in that framework, these transformations are supposed to connect (or map) two physically equivalent systems (more rigorously, this being true for ordinary gauge transformations, in which case the function \(\Lambda(r, t)\), the so-called gauge function, is unique (single-valued) in spacetime coordinates). In a formally similar manner, the above argument is also often used in the context of the so-called “singular gauge transformations”, where \(\Lambda\) is multiple-valued, but the above equality of classical fields is still imposed (at the observation point, that always lies in a physically accessible region); then the above simple phase mapping (at all points of the physically accessible spacetime region, that experience equal fields) leads to the standard phenomena of the Aharonov-Bohm (AB) type, reviewed below, where unequal fields in physically-inaccessible regions have observable consequences. However, we should keep in mind that that above property ((1) and (2) taken together) can be more generally valid — and, as already stated, we will present cases (and closed analytical results for the appropriate phase function \(\Lambda(r, t)\)) that actually connect (or map) two systems (in the sense of (1)) that are neither physically equivalent nor exhibiting the usual AB behaviors. And naturally, because of the above provision of field equalities at the observation point, it will turn out that any nonequivalence of the two systems will involve remote regions of spacetime, namely regions that do not contain the observation point \((r, t)\) (and in which regions, as we shall see, the classical fields experienced by the particle may be different in the two systems).
Returning to the standard cases, usual Λ’s are given in terms of Dirac phases, namely integrals over potentials. I.e. in static cases, and if, for simplicity, we start from system 1 being completely free of potentials (A1 = φ1 = 0), the wavefunctions of the particle in system 2 (moving only in a static vector potential A(r)) will acquire an extra phase with an appropriate “gauge function” Λ(r) that must satisfy \( \nabla \Lambda = A \). The standard (and widely-used) solution of this is the line integral \( \Lambda = \Lambda_0 + \int_{r_0}^{r} A(r') dr' \) (which, by considering two paths encircling an enclosed inaccessible magnetic flux, formally leads to the well-known magnetic AB effect[1]). It should however be stressed that the above is only true if \( \nabla \Lambda = A \) is valid for all points r of the region where the particle moves, i.e. if the particle in system 2 moves (as a narrow wavepacket) always outside MFs (\( \nabla \times A = 0 \) everywhere). Similarly, if the particle in system 2 moves only in a spatially uniform scalar potential \( \phi(t) \), the appropriate Λ must satisfy \( -\frac{1}{c} \frac{\partial \Lambda(t)}{\partial t} = \phi(t) \), the standard solution being \( \Lambda(t) = \Lambda(0) - c \int_{t_0}^{t} \phi(t')dt' \) that gives the extra phase acquired by system 2 (this result formally leading to the electric AB effect[1] by applying it to two equipotential regions, such as two metallic cages held in distinct time-dependent scalar potentials). Once again however it should be stressed that the above is only true if \( -\frac{1}{c} \frac{\partial \Lambda(t)}{\partial t} = \phi(t) \) is valid at all times t of interest, i.e. if the particle in system 2 moves (as a narrow wavepacket) always outside EFs (\( E = -\nabla \phi - \frac{1}{c} \frac{\partial A}{\partial t} = 0 \) at all times). (In the electric AB setup, the above is ensured by the fact that t lies in an interval of a finite duration T for which the potentials are turned on, in combination with the narrowness of the wavepacket; and the assumption is that, during T, the particle has vanishing probability of being at the edges of the cage where the potential starts having a spatial dependence. The reader is referred to Appendix B of Peshkin[2] that demonstrates the intricacies of the electric AB effect, to which we return with an important comment at the end of this paper).

For potentials more general than in the above cases, (and if, for notational simplicity, we restrict our attention to only one spatial variable x) it is usually stated that the general gauge function that connects (through a phase factor \( e^{i\frac{\phi}{\hbar c} \Lambda(x,t)} \)) the wavefunctions of a quantum system with no potentials to those in a general set (A, φ) is the obvious combination (and a natural extension) of the above two forms, namely

\[
\Lambda(x,t) = \Lambda(x_0,t_0) + \int_{x_0}^{x} A(x',t) \cdot dx' - c \int_{t_0}^{t} \phi(x,t')dt',
\]

(3)
which, however, is generally incorrect for $x$ and $t$ uncorrelated variables: it does not generally satisfy the standard system (2) (viewed as a system of partial differential equations (PDEs)), namely

$$\nabla \Lambda = A \quad \text{and} \quad -\frac{1}{c} \frac{\partial \Lambda}{\partial t} = \phi. \quad (4)$$

Indeed: (i) When the $\nabla$ operator acts on eq. (3), it gives the correct $A(x, t)$ from the 1st term, but it also gives some annoying additional nonzero quantity from the 2nd term (that survives because of the $x$-dependence of $\phi$); hence it invalidates the first of the basic system (4). (ii) Similarly, when the $-\frac{1}{c} \frac{\partial}{\partial t}$ operator acts on eq. (3), it gives the correct $\phi(x, t)$ from the 2nd term, but it also gives some annoying additional nonzero quantity from the 1st term (that survives because of the $t$-dependence of $A$); hence it invalidates the second of the basic system (4). It is only when $A$ is $t$-independent, and $\phi$ is spatially-independent, that eq. (3) is correct. It is also interesting to note that the line integrals appearing in (3) do not form a path (in spacetime) that connects the initial to the final point (see below). [An alternative form that is also given in the literature is again eq. (3), but with the variables that are not integrated over implicitly assumed to belong to the initial point (hence a $t_0$ replaces $t$ in $A$, and an $x_0$ replaces $x$ in $\phi$). However, one can see again that the system (4) is not satisfied (the above differential operators, when acted on $\Lambda$, give $A(x, t_0)$ and $\phi(x_0, t)$, hence not the values of the potentials at the point of observation $(x, t)$ as they should), this not being an acceptable solution either. And in this case also there is no spacetime-path connecting the initial $(x_0, t_0)$ to the final point $(x, t)$ either, as the reader can easily verify]. What is the problem here, or, better put, what is the deeper reason for the above inconsistency? The short answer is the uncritical use of Dirac phase factors that come from path-integral treatments (where $x$ and $t$ are not uncorrelated variables, but actually correlated to produce a path $x(t)$). The general inadequacy of (3) was actually one of the main points that has motivated this work. By looking for the most general form of $\Lambda$ that solves the basic system of PDEs we have recently found generalized results that actually correct eq. (3) in 2 ways: through the proper appearance of $x_0$ and $t_0$ (as in eq. (5) and eq. (6) of next Section) – which happens to give a path-sense (that connects the initial to the final point) in either of the two solutions (see Fig.1), being therefore consistent with Feynman’s path integral result in the special case of narrow-wavepacket states – but most importantly, through the additional presence of novel nonlocal terms that had so far
These generalized results are the \textbf{exact} solutions of the system (4) but, even most importantly, the formulation (and methodology of solution) that produces them, if applied to $\Lambda(x, y)$ (in the 2-D static case) and also to $\Lambda(x, y, t)$ (in the full dynamical 2-D case), leads to the exact (nontrivial) forms of the phase function $\Lambda$ that, apart from satisfying (in all cases) the system (4), seems to also have far reaching consequences for the wavefunction-phases in the Schrödinger picture (the most important being their causal behavior).

Summarizing, we will see in this paper that the full form of a general $\Lambda$ goes beyond the usual Dirac phases: apart from integrals over potentials, it also generally contains terms of classical fields that act \textit{nonlocally} (in spacetime) on the solutions of the $t$-dependent SE. As a result, the phases of wavefunctions in the Schrödinger picture are affected nonlocally by MFs and EFs — nonlocal contributions that have apparently escaped from path-integral approaches. We will then focus on two types of application of the new formulation: (i) Application to particles passing through static MFs or EFs will lead to cancellations of AB phases at the observation point; these cancellations will be linked to behaviors at the semiclassical level (to early experimental observations by Werner & Brill or to recent reports of Batelaan & Tonomura) but will be shown to be far more general (valid not only for narrow wavepackets but also for completely delocalized quantum states). By using them we will provide a new interpretation of semiclassical results and we will point out a number of sign errors in popular reports in the literature: we will clearly show that semiclassical phase-differences picked up by classical trajectories (deflected by fields) are \textit{opposite} (and \textit{not} equal, as usually stated or implied) to the corresponding “AB phase” (due to the flux enclosed by the same trajectories). (ii) Application to $t$-dependent situations will provide a remedy for a number of misconceptions (on improper use of simple Dirac phase factors) propagating in the literature (Feynman, Erlichson and others), and will lead to nontrivially extended phases that contain an AB part and a nonlocal field-part: their competition will be shown to recover Relativistic Causality in earlier “paradoxes” (such as the van Kampen thought-experiment) and will provide a fully quantitative formulation of Peshkin’s qualitative discussion (on expected causal behavior) in the electric AB effect (discussion that was also based on a simple Dirac phase factor). The temporal nonlocalities found in this work demonstrate in part \textit{a causal propagation of phases of quantum wavefunctions in the Schrödinger picture} (through the well-known causal propagation of fields), something that may open a new
and direct way for addressing $t$-dependent double-slit experiments and the associated causal issues.

II. 1-D DYNAMIC CASE

Let us first consider 1-D cases and find the proper $\Lambda(x, t)$ that takes us from (maps) a system in a set $(A_1, \phi_1)$ to a set $(A_2, \phi_2)$. As already emphasized, we must assume that at the point $(x, t)$ of observation we have equal EFs, i.e. $-\frac{\partial \phi_2}{\partial x} - \frac{1}{c} \frac{\partial A_2}{\partial t} = -\frac{\partial \phi_1}{\partial x} - \frac{1}{c} \frac{\partial A_1}{\partial t}$, but we will not exclude the possibility of the two systems passing through different EFs in other regions of spacetime (that do not contain the observation point). In fact, this possibility will come out naturally from a careful solution of the basic PDEs, namely $\frac{\partial \Lambda}{\partial x} = A$, $-\frac{1}{c} \frac{\partial \Lambda}{\partial t} = \phi$. This system is underdetermined in the sense that we only have knowledge of $\Lambda$ at an initial point $(x_0, t_0)$ and with no further boundary conditions (hence multiplicities of solutions being generally expected, see below). By following a careful procedure of integrations we finally obtain 2 distinct solutions (depending on which eq. we integrate first): the first solution is

$$\Lambda(x, t) = \Lambda(x_0, t_0) + \int_{x_0}^{x} A(x', t) dx' - c \int_{t_0}^{t} \phi(x_0, t') dt' + \left\{ c \int_{t_0}^{t} dt' \int_{x_0}^{x} E(x', t') + g(x) \right\} + \tau(t_0)$$

with $g(x)$ required to be chosen so that the quantity $\left\{ c \int_{x_0}^{x} E + g(x) \right\}$ is indep. of $x$, and the second solution is

$$\Lambda(x, t) = \Lambda(x_0, t_0) + \int_{x_0}^{x} A(x', t_0) dx' - c \int_{t_0}^{t} \phi(x, t') dt' + \left\{ -c \int_{x_0}^{x} dx' \int_{t_0}^{t} E(x', t') + \hat{g}(t) \right\} + \chi(x_0)$$

with $\hat{g}(t)$ to be chosen in such a way that $\left\{ -c \int_{t_0}^{t} E + \hat{g}(t) \right\}$ is indep. of $t$. We can directly verify that (5) or (6) are indeed solutions of the basic PDEs. [For (5) we have (even for $E(x', t') \neq 0$): $\frac{\partial \Lambda(x, t)}{\partial x} = A(x, t)$ satisfied trivially (because $\{..\}$ is indep. of $x$), and $-\frac{1}{c} \frac{\partial \Lambda(x, t)}{\partial t} = -\frac{1}{c} \int_{x_0}^{x} \frac{\partial A(x', t)}{\partial t} dx' + \phi(x_0, t) - \int_{x_0}^{x} E(x', t) dx'$, and then with the substitu-
tion $-\frac{1}{c} \frac{\partial A(x',t)}{\partial t} = \frac{\partial \phi(x',t)}{\partial x'} + E(x', t)$ we obtain $-\frac{1}{c} \frac{\partial \Lambda(x,t)}{\partial t} = \int_{x_0}^{x} \frac{\partial \phi(x',t)}{\partial x'} dx' + \int_{x_0}^{x} E(x',t) dx' + \phi(x_0,t) - \int_{x_0}^{x} E(x',t) dx'$. Since the 2nd and 4th terms cancel each other, and the 1st term is $\int_{x_0}^{x} \frac{\partial \phi(x',t)}{\partial x'} dx' = \phi(x,t) - \phi(x_0,t)$ we obtain $-\frac{1}{c} \frac{\partial \Lambda(x,t)}{\partial t} = \phi(x,t)$. ✓ We have directly shown therefore that the basic system of PDEs is indeed satisfied by our generalized solution (5) even for any nonzero $E(x',t')$ (in regions $(x',t') \neq (x,t)$). (Note however that at the point of observation $E(x,t) = 0$, signifying the essential fact that the fields in the two systems are identical (recall that $E = E_2 - E_1$ at the point of observation $(x,t)$). It can similarly be shown that (6) is also a solution. In (5) and (6) the placement of $x_0$ and $t_0$ gives a “path-sense” to the line integrals in each solution (each path consisting of 2 perpendicular line segments connecting $(x_0,t_0)$ to $(x,t)$, with solution (5) having a clockwise and solution (6) a counter-clockwise sense, see red and green arrow paths in Fig.1); this way a natural rectangle is formed, within which the enclosed “electric fluxes” in spacetime appear to be crucial (showing up as nonlocal contributions of the EFs-difference from regions $(x',t')$ of space and time that are remote to the observation point $(x,t)$). These nonlocal terms in $\Lambda$ have a direct effect on the wfs’ phases at $(x,t)$. The actual manner in which this happens is determined by the functions $g(x)$ or $\hat{g}(t)$—these must be chosen in such a way that they satisfy their respective conditions. In Fig.1a we show an extended vertical striped-$E$-distribution (the case of a 1-D capacitor that is arbitrarily charged for all time), where, for $x$ located outside (and on the right of) the capacitor, the simplest proper choices are $g(x) = 0$ and $\hat{g}(t) = +c \int_{x}^{t} E$ (since the quantity $\int_{t}^{x} E$ is already indep. of $x$ (a displacement of the $(x,t)$-corner of the rectangle to the right does not change the enclosed “electric flux” – hence the choice of $g(x) = 0$) but is not a constant; this enclosed flux depends on $t$ (since it does change with a displacement of the $(x,t)$-corner upwards) - hence the choice of $\hat{g}(t)$ above). These choices then of $g(x)$ and $\hat{g}(t)$ lead (through (5) and (6)) to new (generalized) solutions for this particular field-configuration. We then note that the difference of the two solutions (5) and (6) is zero (the flux determined by the potential-integrals is exactly cancelled by the nonlocal term of EFs), a cancellation effect that is important and that will be generalized below. For other shapes of $E$ the choices of $g(x)$ and $\hat{g}(t)$ will be different: for
an extended horizontal strip (the case of a nonzero EF in all space that has a finite duration \( T \), proper choices (for observation instant \( t > T \)) are \( \hat{g}(t) = 0 \) and \( g(x) = -c \int_{t_0}^{t} \int E \) (since the electric flux enclosed in the “observation rectangle” now depends on \( x \), but not on \( t \)) – or a more involved example would correspond to a triangular shape (see Fig.1b for the corresponding magnetic case to be discussed later), where the enclosed flux depends on both \( x \) and \( t \) (but can be shown to be separable, see next Section). As for the last constant terms \( \tau(t_0) \) and \( \chi(x_0) \) (what we will call “multiplicities”), these are only present when \( \Lambda \) is expected to be multivalued, i.e. in cases of motion in multiple-connected spacetimes, and are then related to the fluxes in the inaccessible regions: in the electric AB setup, the prototype of multiple-connectivity in spacetime, it turns out\([3]\) that \( \tau(t_0) = -\chi(x_0) = \) enclosed “electric flux”, and if these values are substituted in \((5)\) and \((6)\) they cancel out the new nonlocal terms and lead to the usual electric AB result. In simple-connected spacetimes, it can be rigorously shown\([3]\) that solutions \((5)\) and \((6)\) are equal (with \( g(x) \) being equal to the \( t \)-indep. bracket of \((6)\), and \( \hat{g}(t) \) being equal to the \( x \)-indep. bracket of \((5)\)), the nonlocal terms having therefore the tendency to exactly cancel the “AB terms” (this being true for arbitrary shapes and analytical form of \( E(x,t) \)).

III. 2-D STATIC CASE

The same method applied to static 2-D cases (now for the system of PDEs \( \frac{\partial \Lambda}{\partial x} = A_x, \frac{\partial \Lambda}{\partial y} = A_y \)) finally gives 2 general solutions\([3]\): the first is

\[
\Lambda(x, y) = \Lambda(x_0, y_0) + \int_{x_0}^{x} A_x(x', y)dx' + \int_{y_0}^{y} A_y(x_0, y')dy' + \left\{ \int_{y_0}^{y} \int_{x_0}^{x} B(x', y') + g(x) \right\} + f(y_0)
\]

with \( g(x) \) such that \( \left\{ \int_{y_0}^{y} \int B + g(x) \right\} \) : indep. of \( x \), and the second is

\[
\Lambda(x, y) = \Lambda(x_0, y_0) + \int_{x_0}^{x} A_x(x', y_0)dx' + \int_{y_0}^{y} A_y(x, y')dy' + \left\{ -\int_{x_0}^{x} \int_{y_0}^{y} B(x', y') + h(y) \right\} + \hat{h}(x_0)
\]
with \( h(y) \) such that \( \left\{ - \int_x^y B + h(y) \right\} : \text{indep. of } y \). These results apply to cases where the particle goes through different perpendicular MFs (recall \( B = B_2 - B_1 \) in spatial regions remote to the observation point \((x, y)\)). One can again show that the 2 solutions are equal for simple-connected space, and for multiple-connectivity the values of multiplicities \( f(y_0) \) and \( \hat{h}(x_0) \) cancel out the nonlocalities and reduce the above to the usual result of mere \( A \)-integrals along the 2 paths (i.e. two simple Dirac phases). For striped \( B \)-distributions, functions \( g(x) \) and \( h(y) \) must be chosen in ways compatible with their above conditions (as in the earlier \((x, t)\)-cases); by then taking the difference of (7) and (8) we obtain that the “AB phase” (originating from the closed line integral of \( A \)'s) is exactly cancelled by the nonlocal term of MFs. This is reminiscent of the cancellation of phases observed in the early experiments of Werner & Brill\(^4\) for particles passing through a MF (a cancellation between the “AB phase” and the semiclassical phase picked up by the trajectories), and our method seems to provide a natural explanation: as our results are general (and for delocalized states in simple-connected space they basically demonstrate the uniqueness of \( \Lambda \)), they are also valid and applicable to states that describe wavepackets in classical motion, as \textit{was} the case in Werner & Brill’s work.

The above cancellations can then be understood as a compatibility between the AB fringe-displacement and the trajectory-deflection due to the Lorentz force (i.e. the semiclassical phase picked up due to the optical path difference of the two deflected trajectories \textit{exactly cancels} (is \textit{opposite in sign} from) the AB phase picked up by the same trajectories due to the flux that they enclose). This opposite sign seems to have been rather unnoticed: In Feynman’s Fig.15-8\(^5\), or in Felsager’s Fig.2.16\(^6\), classical trajectories are deflected after passing through a strip of a MF placed on the right of a double-slit apparatus. Both authors determine the semiclassical phase picked up by the deflected trajectories and find it consistent with the AB phase. One can see on closer inspection, however, that the two phases actually \textit{have opposite signs} (see our own Fig.2 and the discussion that follows below, where this is proved in detail). Similarly, in the very recent review of Batelaan & Tonomura\(^7\), their Fig.2 shows wavefronts associated to deflected classical trajectories where it is stated that “the phase shift calculated in terms of the Lorentz force is the same as that predicted by the AB effect in terms of the vector potential”. Once more, however, it turns out that the sign of the classical phase-difference is really opposite to the sign of the AB phase (see
proof below). The phases are not equal as stated by the authors. And it turns out that even “electric analogs” of the above cases also demonstrate this opposite-sign relationship (see proofs further below). All the above examples can be viewed as a manifestation of the cancellations that have been found in the present work for general quantum states (but in those examples they are just special cases for wavepacket-states in classical motion).

Let us give a brief elementary proof of the above claimed opposite sign-relationships: Indeed, in our Fig.2, the “AB phase” due to the flux enclosed between the two classical trajectories (of a particle of charge $q$) is

$$
\Delta \varphi^{AB} = 2\pi \frac{q}{e} \frac{\Phi}{\Phi_0},
$$

(9)

with $\Phi_0 = \frac{h}{e}$ the flux quantum, and $\Phi \approx BDd$ the enclosed flux between the two trajectories (for small trajectory-deflections), with the deflection originating from the presence of the magnetic strip $B$ and the associated Lorentz forces. On the other hand, the semiclassical phase difference between the same 2 classical trajectories is $\Delta \varphi^{semi} = 2\pi \lambda \Delta l$, with $\lambda = \frac{h}{mv}$ being the de Broglie wavelength (and $v$ being the speed of the particle, taken almost constant (as usually done) due to the small deflections), and with $\Delta l$ being $\Delta l \approx d \sin \theta \approx d \frac{V}{L} (x_c$ being the (displaced) position of the central fringe on the screen). We have therefore

$$
\Delta \varphi^{semi} = \frac{2\pi}{\lambda} \frac{x_c}{L},
$$

(10)

Now, the Lorentz force (exerted only during the passage through the thin magnetic strip, hence only during a time interval $\Delta t = \frac{Wv}{v}$) has a component parallel to the screen (let us call it $x$-component) that is given by

$$
F_x = \frac{q}{c} (v \times B)_x = -\frac{q}{c} vB = -\frac{BWq}{\frac{Wv}{v}} = -\frac{BWq}{c\Delta t},
$$

(11)

which shows that there is a change of kinematic momentum (parallel to the screen) equal to $-\frac{BWq}{c}$, or, equivalently, a change of parallel speed

$$
\Delta v_x = -\frac{BWq}{mc},
$$

(12)

which is the speed of the central fringe’s motion (i.e. its displacement over time along the screen). Although this has been caused by the presence of the thin deflecting magnetic strip, this displacement is occurring uniformly during a time interval $t = \frac{L}{v}$, and this time interval must satisfy
\[ \Delta v_x = \frac{x_c}{t} \]  \hspace{1cm} (13)

(as, for small displacements, the wps travel most of the time in uniform motion, i.e. \( \Delta t \ll t \)). We therefore have that the central fringe displacement must be \( x_c = \Delta v_x t = -\frac{BWqL}{mc}v \), and noting that \( mv = \frac{\hbar}{\lambda} \), we finally have

\[ x_c = -\frac{BWqL\lambda}{hc}. \]  \hspace{1cm} (14)

By substituting (14) into (10), the lengths \( L \) and \( \lambda \) cancel out, and we finally have \( \Delta \varphi^{semi} = -2\pi \frac{q}{e} \frac{BWd}{e} \), which with \( \frac{hc}{e} = \Phi_0 \) the flux quantum, and \( BWd \approx \Phi \) the enclosed flux (always for small trajectory-deflections) gives (through comparison with (9)) our final proof that

\[ \Delta \varphi^{semi} = -2\pi \frac{q}{e} \frac{\Phi}{\Phi_0} = -\Delta \varphi^{AB}. \]  \hspace{1cm} (15)

The “electric analog” of the above exercise is also outlined below, now with a homogeneous EF (pointing downwards everywhere in space, but switched on for only a finite duration \( T \)) on the right of a double-slit apparatus (see our Fig.3): In this case the electric Lorentz force \( qE \) is exerted on the trajectories only during the small time interval \( \Delta t = T \), which we take to be much shorter \( (T \ll t) \) than the time of travel \( t = \frac{L}{v} \) (we now have a thin electric strip in time rather than the thin magnetic strip in space that we had earlier). The electric type of AB phase is now

\[ \Delta \varphi^{AB} = -2\pi \frac{q}{e} cT \frac{\Delta V}{\Phi_0}, \]  \hspace{1cm} (16)

with \( \Delta V \) being the electric potential difference between the two trajectories, hence \( \Delta V \approx Ed \) (again for small trajectory-deflections). On the other hand, the semiclassical phase difference between the two trajectories is again given by (10), but the position \( x_c \) of the central fringe must now be determined by the EF force \( qE \): The change of kinematic momentum (always parallel to the screen) is now \( qET \), hence the analog of (12) is now

\[ \Delta v_x = \frac{qET}{m}, \]  \hspace{1cm} (17)

which if combined with (13) (that is obviously valid in this case as well, again for small deflections, due to the \( \Delta t = T \ll t \)), and always with \( t = \frac{L}{v} \), gives that the central fringe
displacement must be  

\[ x_c = \Delta v_x t = \frac{qETL}{mv}, \]

and using again  

\[ mv = \frac{L}{\lambda}, \]

we finally have the following analog of (14)

\[ x_c = \frac{qELL\lambda}{\hbar}. \tag{18} \]

By substituting (18) into (10), the lengths  \( L \) and  \( \lambda \) again cancel out, and we finally have

\[ \Delta \varphi_{semi} = 2\pi qETL\lambda \frac{\lambda}{\hbar} = 2\pi qEEdT \frac{L\lambda}{\hbar}, \]

which with \( \frac{L\lambda}{\hbar} = \Phi_0 \) the flux quantum, and through comparison with (16) leads once again to our final proof that

\[ \Delta \varphi_{semi} = -\Delta \varphi^{AB}. \tag{19} \]

We note therefore that even in the electric case, the semiclassical phase difference (between two trajectories) picked up due to the Lorentz force (exerted on them) is once again opposite to the electric AB phase picked up by the same trajectories (due to the electric flux that they enclose).

We should point out once again, however, that although the above elementary considerations apply to semiclassical motion of narrow wavepackets, in this paper we have given a more general understanding of the above opposite sign-relationships that applies to general (even completely delocalized) states, and that originates from our generalized Werner & Brill cancellations.

In a slightly different vein, the cancellations that we found above give an explanation of why certain classical arguments (invoking the past  \( t \)-dependent history of an experimental setup) seem to be successful in giving at the end an explanation of AB effects (namely a phase consistent with that of a static AB configuration). However, there is again an opposite sign that seems to have been largely unnoticed in such arguments as well (i.e. in Silverman\[8\], where in his eq.(1.34) there should be an extra minus sign).

Finally, on other shapes of  \( B \), see Fig.1b for an example of a homogeneous  \( B \) distributed in a triangular shape (now the part of the magnetic flux contained inside the “observation rectangle” depending on both  \( x \) and  \( y \)). It turns out that this flux can be written as a sum of separate  \( x \)- and  \( y \)-contributions, and for an equilateral triangle of side  \( a \) we obtain that proper functions (for the solutions (7) and (8)) are

\[ g(x) = B \left( -\sqrt{3}ax - \frac{\sqrt{3}}{2}x^2 \right) + \frac{\sqrt{3}}{4}a^2 \]

and \[ h(y) = B \left( ay - \frac{y^2}{2\sqrt{3}} \right) - \frac{\sqrt{3}}{4}a^2 \]. These, if substituted in (7) and (8), lead to new and nontrivial nonlocal solutions (or, correspondingly, to nonlocal phases of wavefunctions).
In cases of circularly shaped distributions (when the enclosed flux is not separable) it is advantageous to solve the PDEs directly in polar coordinates (for corresponding results see [3])—while for general shapes, one may need to first transform to an appropriate coordinate system, and only then apply the above methodology (i.e. strategy, for solving the resulting PDEs).

IV. FULL ($x,y,t$)-CASE

Finally, for the $t$-dependent 2-D case we have to solve
\[
\frac{\partial \Lambda}{\partial x} = A_x, \quad \frac{\partial \Lambda}{\partial y} = A_y, \quad -\frac{1}{c} \frac{\partial \Lambda}{\partial t} = \phi,
\]
in order to see how the solutions combine the spatial and temporal nonlocal effects found above.

We now have $3! = 6$ alternative routes to follow for integrating the system and, at the end, 12 different results are derived, where the $t$-propagation of $B$ and of $E_x$ and $E_y$ in all space is nontrivially important. By leaving out all the long details [3] we merely show one solution, where only $B(\ldots,t_0)$ appears (the $t$-dependence of $B$ having already been incorporated in the behavior of $E_x$ and $E_y$ through Faraday’s law), namely

\[
\Lambda(x, y, t) = \Lambda(x_0, y_0, t_0) + \int_{x_0}^{x} A_x(x', y_0, t) \, dx' + \int_{y_0}^{y} A_y(x, y', t) \, dy' - \int_{x_0}^{x} dx' \int_{y_0}^{y} dy' B(x', y', t_0) + G(y, t_0) -
\]

\[-c \int_{t_0}^{t} \phi(x_0, y_0, t') \, dt' + c \int_{x_0}^{x} dt' \int_{x_0}^{x} dx' E_x(x', y_0, t') + c \int_{y_0}^{y} dt' \int_{y_0}^{y} dy' E_y(x, y', t') + F(x, y) + f(x_0, t_0),
\]

with conditions:
\[
\left\{ G - \int B(x', y', t_0) \right\} : \text{indep. of } y, \quad \left\{ F + c \int E_x(x', y', t') \right\} : \text{indep. of } x,
\]
and
\[
\left\{ F + c \int E_y(x, y', t') \right\} : \text{indep. of } y. \quad \text{In the above, } f \text{ accounts for possible multiplicities at } t_0. \quad \text{This solution, together with its spatial “dual”}
\]
[now with $\int_{x_0}^{x} A_x(x', y, t) \, dx' + \int_{y_0}^{y} A_y(x_0, y', t) \, dy'$ replacing the above $A$-terms, and with $c \int_{t_0}^{t} \int_{x_0}^{x} dx' E_x(x', y, t') + c \int_{t_0}^{t} \int_{y_0}^{y} dy' E_y(x, y', t')$ replacing the above $E$-terms, and with $G(y, t_0)$ being replaced by a $\hat{G}(x, t_0)$ that must satisfy:
\[
\left\{ \hat{G} + \int B(x', y', t_0) \right\} : \text{indep.}
\]
of $x$, are both crucial for the discussion of the thought-experiment that follows: In [9] van Kampen considered a magnetic AB setup, but with an inaccessible magnetic flux that is $t$-dependent: he envisaged turning on the flux very late, or equivalently, observing the interference of the two wavepackets on a distant screen very early, earlier than the time it takes light to travel the distance to the screen (i.e. $t < \frac{L}{c}$), hence using the (instantaneous nature of the) AB phase to transmit information (on the presence of a confined flux somewhere in space) superluminally. Indeed, the AB phase at any $t$ is determined by differences of $\frac{q}{\hbar c} \Lambda (r, t)$ with $\Lambda (r, t) \sim \int_{r_0}^{r} A (r', t).dr'$ (basically a special case of (3)). However, if we use, instead, our results above (that contain the additional nonlocal terms), it turns out that, for a spatially-confined flux $\Phi (t)$ and for $t < \frac{L}{c}$, functions $G$, $\dot{G}$ and $F$ can all be taken zero (their conditions are all satisfied), the point being that at instant $t$, the $E$-field has not yet reached the spatial point $(x, y)$ of the screen — a generalization of the striped cases that we saw earlier but now to the case of 3 spatio-temporal variables (with now the spatial point $(x, y)$ being outside the light-cone defined by $t$ (see Fig.4)); as the electric flux is independent of the upper limits $x$ and $t$, this construction rigorously gives $F = 0$. Moreover, the AB multiplicities (at $t_0$) lead to cancellation of the $B$-terms (always at $t_0$), with the final result (after subtraction of the 2 solutions) being

$$\Delta \Lambda (x, y, t) = \oint A (r', t).dr' + c \int_{t_0}^{t} dt' \oint E (r', t').dr'$$

(20)

which, with $\oint A (r', t).dr' = \Phi (t)$ the instantaneous enclosed magnetic flux and with the help of Faraday’s law $\oint E (r', t').dr' = -\frac{1}{c} \frac{d\Phi (t')}{dt'}$, gives

$$\Delta \Lambda (x, y, t) = \Phi (t) - (\Phi (t) - \Phi (t_0)) = \Phi (t_0).$$

(21)

Although $\Delta \Lambda$ is generally $t$-dependent, we obtain the intuitive (causal) result that, for $t < \frac{L}{c}$ (i.e. if the physical information has not yet reached the screen), the phase-difference turns out to be $t$-independent, and leads to the magnetic Aharonov-Bohm phase that we would observe at $t_0$. The new nonlocal terms have conspired in such a way as to exactly cancel the Causality-violating AB phase (that would be proportional to the instantaneous $\Phi (t)$). This gives a resolution of the van Kampen “paradox” within a canonical formulation, without using any vague electric AB argument (as there is no multiple-connectivity in $(x, t)$-plane).
An additional physical element is that, for the above cancellation, it is not only the $E$-fields but also the $t$-propagation of the $B$-fields (the full “radiation field”) that plays a role\(^3\).

Use of the other 10 solutions can also address bound-state analogs (in $t$-driven 1-D nanorings) or even “electric” analogs of the van Kampen case: In Peshkin’s review\(^2\), on the electric AB effect, the author correctly states “One cannot wait for the electron to pass and only later switch on the field to cause a physical effect”. Although Peshkin uses his eq.(B.5) and (B.6) (based on \(^3\)), he carefully states that it is not the full solution; actually, if we view it as an ansatz, then it is understandable why he needs to enforce a condition (his eq.(B.8), and later (B.9)) on the EF outside the cages (in order for certain (annoying) terms (resulting from a minimal substitution due to the incorrect ansatz) to vanish and for (B.5) to be a solution). But then he notes that the extra condition cannot always be satisfied (hence (B.5) is not really the solution for all times), drawing from this the above qualitatively correct conclusion on Causality. As it turns out, our treatment gives exactly what Peshkin describes in words (with the total “radiation field” outside the cages being once again crucial in recovering Causality), but in a direct and fully quantitative manner, and with no ansatz based on an incorrect form. We should also point out that improper uses of simple Dirac phases appear often in the literature: even in Feynman\(^5\) it is stated that the simple phase factor $\int^x A \cdot dr' - c \int^t \phi dt'$ is valid even for dynamic fields; this is also explicitly stated in Erlichson’s review\(^10\) – Silverman\(^8\) being the only report with a careful wording about $\int^3$ being only restrictively valid (for $t$-indep. $A$ and $r$-indep. $\phi$), although even there the nonlocal terms have been missed.

At the level of the basic Lagrangian $L(r, v, t) = \frac{1}{2}mv^2 + e v \cdot A(r, t) - q\phi(r, t)$ there are no fields present, and the view holds in the literature\(^11\) that EFs or MFs cannot contribute directly to the phase. This view originates from the path-integral treatments widely used (where the Lagrangian determines directly the phases of Propagators), but, nevertheless, our canonical treatment shows that fields do contribute nonlocally, and they are actually crucial in recovering Relativistic Causality. Moreover, path-integral discussions\(^12\) of the van Kampen case use wave (retarded)-solutions for $A$ (hence in Lorenz gauge) and are incomplete; our results take advantage of the retardation of fields $E$ and $B$ (true in any gauge), and not of potentials. In addition, Troudet\(^12\) correctly states that his path-integral treatment is good for not highly-delocalized states in space, and that in case of delocalization the proper treatment “would be much more complicated, and would require a
much more complete analysis”. Such an analysis has actually been provided in the present work. It should be added that the van Kampen “paradox” seems to be still thought of as remarkable[13]. The present work has provided a natural and general resolution, and most importantly, through nonlocal (and Relativistically causal) propagation of wavefunction-phases.

On a broader significance of the new solutions we conclude that a causal behavior may exist at the level of quantum mechanical phases, enforced by the nonlocal terms (through the well-known causal behavior of fields). The nonlocal terms found in this work at the level of $\Lambda$ reflect a causal propagation of wavefunction-phases in the Schrödinger picture (at least a part of them, the one containing the fields, that competes with the AB types of phases containing the potentials). This nonlocality and Causality of quantum phases is an entirely new concept (given the local nature but also the nonrelativistic character of the SE) and deserves to be further explored. Possible immediate applications would be in $t$-dependent slit-experiments recently discussed using a completely different method (with modular variables in the Heisenberg picture)[14]. It has been recently noted[15] that Physics cannot currently predict how we dynamically go from the single-slit diffraction to the double-slit diffraction pattern (whether it is in a gradual and causal manner or not). Application of our nonlocal terms to such questions (i.e. by introducing scalar potentials on the slits in a $t$-dependent way) provides a completely new formulation for addressing causal issues of this type. Finally, one can always wonder what the consequences of these new nonlocalities would be, if these were included in other systems of High-Energy or Condensed Matter Physics with a gauge structure; alternatively, it is worth noting that, if $E$’s were substituted by gravitational fields and $B$’s by Coriolis force fields arising in non-inertial frames of reference, the above nonlocalities (and their apparent causal nature) could possibly have an interesting story to tell about quantum mechanical phase behavior in a Relativistic/Gravitational framework.

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FIGURE LEGENDS

**Fig. 1.** (Color online): Examples of field-configurations (in simple-connected spacetime) where the nonlocal terms are nonzero: (a) a strip in 1+1 spacetime, where the electric flux enclosed in the “observation rectangle” is dependent on $t$ but independent of $x$; (b) a triangular distribution in 2-D space, where the part of the magnetic flux inside the “observation rectangle” depends on both $x$ and $y$. The appropriate choices for the corresponding functions $g(x)$ and $\hat{g}(t)$ for case (a), or $g(x)$ and $h(y)$ for case (b), are given in the text.

**Figure 2.** (Color online): The standard double-slit apparatus with an additional strip of a perpendicular magnetic field $B$ of width $W$ placed between the slit-region and the observation screen. The deflection shown is for a negative charge $q$ (and in the text it is assumed small, due to $W << L$).

**Figure 3.** (Color online): The analog of Fig.2 (again for a negative charge $q$) but with an additional electric field parallel to the observation screen that is turned on for a time interval $T$ (with $T << t$, and $t$ the time of travel).

**Figure 4.** (Color online): The analog of paths of Fig.1 but now in 2+1 spacetime for the van Kampen thought-experiment, when the instant of observation $t$ is so short that the physical information has not yet reached the spatial point of observation $(x, y)$. The two solutions (that, for wavepackets, have to be subtracted in order to give the phase difference at $(x, y, t)$) are described in the text, and are here characterized through their electric field $E$-line-integral behavior: “electric field path (I)” (the red-arrow route) denotes the “dual” solution, and “electric field path (II)” (the green-arrow route) denotes the “primary” solution given in the beginning of Section IV.