DIVERGENCE OF SPECTRAL DECOMPOSITIONS OF HILL OPERATORS WITH TWO EXPONENTIAL TERM POTENTIALS

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Abstract. We consider the Hill operator
\[ Ly = -y'' + v(x)y, \quad 0 \leq x \leq \pi, \]
subject to periodic or antiperiodic boundary conditions (bc) with potentials of the form
\[ v(x) = ae^{-2irx} + be^{2isx}, \quad a, b \neq 0, \quad r, s \in \mathbb{N}, \quad r \neq s. \]
It is shown that the system of root functions does not contain a basis in \( L^2([0, \pi], \mathbb{C}) \) if bc are periodic or if bc are antiperiodic and \( r, s \) are odd or \( r = 1 \) and \( s \geq 3 \).

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1. Introduction

We consider the Hill operators \( L = L_{Per^\pm}(v) \) with smooth \( \pi \)-periodic (complex-valued) potentials \( v \)
\[ Ly = -y'' + v(x)y, \quad 0 \leq x \leq \pi, \]
subject to periodic (\( Per^+ \)) or antiperiodic (\( Per^- \)) boundary conditions:
\[ Per^\pm : \quad y(\pi) = \pm y(0), \quad y'(\pi) = \pm y'(0). \]
See basics and details in [15].
If \( v \) is real-valued, then \( L_{Per^\pm}(v) \) is a self-adjoint operator with a discrete spectrum. The system of its normalized eigenfunctions
\[ \Phi = \{ \varphi_k : L\varphi_k = \lambda_k \varphi_k, \quad \| \varphi_k \| = 1 \} \]
is orthonormal, and the spectral decompositions
\[ f = \sum_k \langle f, \varphi_k \rangle \varphi_k \]
converge (unconditionally) in \( L^2([0, \pi]) \) for every \( f \in L^2([0, \pi]) \).
If \( v \) is a complex-valued potential the picture becomes more complicated – see [11, 12, 14, 18, 19, 20, 23, 24, 25]. In 2006 A. Makin [16, 17]
and the authors [3, Thm 71] gave the first examples of such potentials that the system of root functions for periodic or antiperiodic boundary conditions does not contain a basis in $L^2([0,\pi])$ even though there all but finitely many eigenvalues are simple.

It is well known that the spectra of the operators $L_{\text{Per}}^\pm$ are discrete, and the following localization formulas hold (see, for example, [4, Prop 1]):

\begin{equation}
\text{Sp} (L_{\text{Per}}^\pm) \subset \Pi_N \cup \bigcup_{n>N, n \in \Gamma^\pm} D_n, \quad \#\{\text{Sp} (L_{\text{Per}}^\pm) \cap D_n\} = 2,
\end{equation}

where $D_n = \{ z : |z - n^2| < 1 \}$, $\Gamma^+ = 2\mathbb{N}$, $\Gamma^- = 2\mathbb{N} - 1$, $N = N(v)$,

\begin{equation}
\Pi_N = \{ z = x + iy \in \mathbb{C} : |x| < (N + 1/2)^2, |y| < N \}.
\end{equation}

In either case the spectral block decompositions

\begin{equation}
g = S_N g + \sum_{n>N, n \in \Gamma^\pm} P_n g, \quad \forall g \in L^2([0,\pi]),
\end{equation}

where

\begin{equation}
S_N = \frac{1}{2\pi i} \int_{\partial \Pi_N} (z - L_{\text{Per}}^\pm)^{-1} dz, \quad P_n = \frac{1}{2\pi i} \int_{\partial D_n} (z - L_{\text{Per}}^\pm)^{-1} dz,
\end{equation}

converge unconditionally in $L^2([0,\pi])$. This is true even if the $\pi$-periodic potential $v$ is singular, i.e., $v \in H^{-1}_{\text{loc}}(\mathbb{R})$, as A. Savchuk and A. Shkalikov showed in [22]. An alternative proof is given in [5].

The unconditional convergence of decompositions (1.6) implies that for every set $\Delta$ (finite or infinite) of even (or odd) integers $n > N$ the sum of projections

\begin{equation}
P(\Delta) = \sum_{k \in \Delta} P_k
\end{equation}

converges unconditionally, so the projections $P(\Delta)$ are well defined and

\begin{equation}
\sup_{\Delta} \| P(\Delta) \| \leq M(v) < \infty.
\end{equation}

Invariant subspaces $E(\Delta) = \text{Ran} P(\Delta)$ have $\{ P_k, k \in \Delta \}$ as their Riesz system of projections, $\dim P_k = 2$.

Could $P_k$ be split to give a basis of root functions for $E(\Delta)$? We put the question in this way because for one and the same operator $L_{\text{Per}}^\pm(v)$ the answer could be yes and no depending on $\Delta$. For example, if

$v(x) = ae^{-10ix} + be^{10ix}$

and

\begin{equation}
\Delta_0 = \{ n \in \Gamma^\pm : n \not\equiv 0 \mod 5 \},
\end{equation}

then the system of root functions for $L_{\text{Per}}^\pm(v)$ does not contain a basis in $L^2([0,\pi])$.
then the answer is positive, but for $\Delta_1 = 5\mathbb{N}$ the answer is no if $|a| \neq |b|$, and yes if $|a| = |b|$. We explain this phenomenon in Section 4 (see Proposition 19).

In view of (1.8) and (1.9), the following holds (see Corollary 10 in [9, Section 3] for details).

**Remark 1.** If $\Delta$ is an infinite set of even (or odd) integers, then the corresponding system of periodic (or antiperiodic) root functions contains a basis of $E(\Delta)$ if and only if it contains an unconditional basis of $E(\Delta)$.

The spectra localization formula (1.4) allows us to apply the Lyapunov–Schmidt projection method (see [3, Lemma 21]) and reduce the eigenvalue equation $Ly = \lambda y$ to a series of eigenvalue equations in two-dimensional eigenspaces $E_n^0$ of the free operator. This leads to the following (see [3, Section 2.2]).

**Lemma 2.** Let $L$ be a Hill operator with a potential $v \in L^2$. Then, for large enough $n \in \mathbb{N}$, there are functionals $\alpha_n(v; z)$ and $\beta_n^\pm(v; z)$, $|z| < n$ such that a number $\lambda = n^2 + z$, $|z| < n/4$, is a periodic (for even $n$) or anti-periodic (for odd $n$) eigenvalue of $L$ if and only if $z$ is an eigenvalue of the matrix

\[
\begin{pmatrix}
\alpha_n(v; z) & \beta_n^-(v; z) \\
\beta_n^+(v; z) & \alpha_n(v; z)
\end{pmatrix}.
\]

Moreover, $\alpha_n(z; v)$ and $\beta_n^\pm(z; v)$ depend analytically on $v$ and $z$, and $z^- = \lambda^- - n^2$ and $z^+ = \lambda^+ - n^2$ are the only solutions of the equation

\[
(z - \alpha_n(v; z))^2 = \beta_n^-(v; z)\beta_n^+(v; z).
\]

The functionals $\alpha_n(v; z)$ and $\beta_n^\pm(v; z)$ are well defined for large enough $n$ by explicit expressions in terms of the Fourier coefficients of the potential (see [3, Formulas (2.16)-(2.33)] for Hill operators with $L^2$-potentials).

Here we provide formulas for $\alpha_n(v; z)$ and $\beta_n^\pm(v; z)$ using the combinatorial approach that has been developed in [2, 3] and used there to obtain the asymptotics of the spectral gaps $\gamma_n = \lambda^+_n - \lambda^-_n$ for potentials of the form $v(x) = a \cos 2x + b \cos 4x$.

For each $n \in \mathbb{N}$ a walk $x$ from $-n$ to $n$ (or from $n$ to $-n$ or from $n$ to $n$) is defined through its sequence of steps

\[
x = (x(t))_{t=1}^{\nu+1}, \quad 1 \leq \nu = \nu(x) < \infty,
\]

where $x(t) \in 2\mathbb{Z} \setminus \{0\}$, and respectively,

\[
\sum_{t=1}^{\nu+1} x(t) = 2n \quad \text{or} \quad \sum_{t=1}^{\nu+1} x(t) = -2n \quad \text{or} \quad \sum_{t=1}^{\nu+1} x(t) = 0.
\]
A walk $x$ is called *admissible* if its *vertices* $j(t) = j(t, x)$ given, respectively, by

\[ j(0) = -n \quad \text{or} \quad j(0) = +n \]  

and

\[ j(t) = -n + \sum_{i=1}^{t} x(i) \quad \text{or} \quad j(t) = n + \sum_{i=1}^{t} x(i), \quad 1 \leq t \leq \nu + 1, \]

satisfy

\[ j(t) \neq \pm n \quad \text{for} \quad 1 \leq t \leq \nu. \]

Let

\[ v = \sum_{m \in 2\mathbb{Z}} V(m)e^{imx} \]

be the Fourier expansion of the potential $v$ with respect to the system \( \{e^{imx}, \ m \in 2\mathbb{Z}\} \), and let $X_n, Y_n$ and $W_n$ be, respectively, the set of all admissible walks from $-n$ to $n$, from $n$ to $-n$ and from $n$ to $n$. For each admissible walk $x$ we set

\[ h_1(x; z) = \prod_{t=1}^{\nu} (n^2 - j(t)^2 + z)^{-1}, \quad h(x) = h_1(x) \prod_{t=1}^{\nu+1} V(x(t)); \]

then

\[ \alpha_n(z) = \sum_{x \in W_n} h(x, z), \quad \beta_n^+(z) = \sum_{x \in X_n} h(x, z), \quad \beta_n^-(z) = \sum_{x \in Y_n} h(x, z). \]

The core of our approach is analysis of asymptotic behavior of the functionals $\beta_n^\pm(z) = \beta_n^\pm(v; z)$. In particular, the following criterion (which is a slight modification of Theorem 1 in [7] or Theorem 2 in [6]) gives a constructive approach to determine the basisness properties of the root function system.

**Criterion 3.** Let $v \in L^2([0, \pi])$, and let $\Delta \subset \Gamma^+$ (or $\Delta \subset \Gamma^-$) be an infinite set of sufficiently large numbers. If $\Delta = \Delta_0 \cup \Delta_1$, where

\[ \beta_n^+(z) \equiv \beta_n^-(z) \equiv 0 \quad \text{for} \quad n \in \Delta_0, \]

\[ \beta_n^+(0) \neq 0, \quad \beta_n^-(0) \neq 0 \quad \text{for} \quad n \in \Delta_1 \]

and there is a constant $c > 0$ such that

\[ c^{-1}|\beta_n^+(0)| \leq |\beta_n^+(z)| \leq c|\beta_n^+(0)|, \quad \text{for} \quad n \in \Delta_1, \ |z| \leq 1, \]

then:

(a) for large enough $n \in \Delta$, the operator $L_{Pr^\pm}(v)$ has in the disc $D_n = \{z : |z-n^2| < 1\}$ exactly one periodic (or antiperiodic) eigenvalue.
of geometric multiplicity 2 if $n \in \Delta_0$, and exactly two simple periodic (or antiperiodic) eigenvalues if $n \in \Delta_1$;
(b) the system of root functions of $L_{Per}^\pm(v)$ contains a Riesz basis of $E(\Delta)$ if and only if
\begin{equation}
\limsup_{n \in \Delta_1} t_n(0) < \infty,
\end{equation}
where
\begin{equation}
t_n(z) = \max\{|\beta_n^-(z)|/|\beta_n^+(z)|, |\beta_n^+(z)|/|\beta_n^-(z)|\}.
\end{equation}

In the framework of this criterion one can explain practically all known cases of existence or non-existence of bases consisting of root functions of the operators $L_{Per}^\pm(v)$ for specific classes of potentials $v$. For example, the main result in [26] follows from Criterion 3.

In general form, i.e., without the restrictions (1.21) - (1.23), Criterion 3 is given in [8] in the context of 1D Dirac operators but the formulation and proof are the same in the case of Schrödinger operators (see Proposition 19 in [9]). Moreover, the same argument gives the following more general statement.

**Criterion 4.** Let $\Gamma^+ = 2\mathbb{N}$, $\Gamma^- = 2\mathbb{N} - 1$ in the case of Hill operators with $H_{per}^\pm$-potentials, and $\Gamma^+ = 2\mathbb{Z}$, $\Gamma^- = 2\mathbb{Z} - 1$ in the case of one dimensional Dirac operators with $L^2$-potentials. There exists $N_* = N_*(v)$ such that for $|n| > N_*$ the operator $L = L_{Per}^\pm(v)$ has in the disc $D_n = \{z : |z-n^2| < n/2\}$ (respectively $D_n = \{z : |z-n| < 1/2\}$) exactly two periodic (for $n \in \Gamma^+$) or antiperiodic (for $n \in \Gamma^-$) eigenvalues, counted with multiplicity. Let
\begin{equation}
\mathcal{M}^\pm = \{n \in \Gamma^\pm : n \geq N_*, \lambda_n^- \neq \lambda_n^+\}.
\end{equation}

(a) If $\Delta \subset \Gamma^\pm$ is an infinite set such that $|n| > N_*$ for $n \in \Delta$, then the system of periodic (or antiperiodic) root functions contains a Riesz basis in $E(\Delta)$ if and only if
\begin{equation}
\limsup_{n \in \Delta \cap \mathcal{M}^\pm} t_n(z_n^*) < \infty,
\end{equation}
where
\begin{equation}
z_n^* = \frac{1}{2}(\lambda_n^- + \lambda_n^+) - \lambda_n^0 \quad \text{with} \quad \lambda_n^0 = n^2 \quad \text{for Hill operators and} \quad \lambda_n^0 = n \quad \text{for Dirac operators}.
\end{equation}

(b) The system of root functions of $L_{Per}^\pm(v)$ contains a Riesz basis, (respectively, in $L^2([0, \pi])$ in the Hill case or in $L^2([0, \pi], \mathbb{C}^2)$ in the Dirac case) if and only if (1.26) holds for $\Delta = \Gamma^\pm$.

Another interesting abstract criterion of basisness is the following.
Criterion 5. The system of root functions of the operator $L_{\text{Per}}^{-1}(v)$ contains a Riesz basis in $E(\Delta)$ if only if

$$\limsup_{n \in \Delta \cap \mathcal{M}^\pm} \frac{|\lambda_n^+ - \mu_n|}{|\lambda_n^- - \lambda_n^-|} < \infty,$$

where (for large enough $n$) $\mu_n$ is the Dirichlet eigenvalue close to $n^2$.

In the case $\Delta = \Gamma^\pm$ this criterion was given (with completely different proofs) in [13] for Hill operators with $L^2$-potentials and in [9] for Hill operators with $H^{-1}_{\text{per}}$-potentials and for one-dimensional Dirac operators with $L^2$-potentials as well. The proof of the criterion in the more general case $\Delta \subset \Gamma^\pm$ is the same.

However, if one wants to apply Criterion 5 to specific potentials $v$, say $v(x) = a \cos 2x + b \cos 4x$ with $a, b \in \mathbb{C}$, it is necessary first to obtain the asymptotics of the spectral gaps $|\lambda_n^+ - \lambda_n^-|$ and deviations $|\mu_n - \lambda_n^+|$, what is by itself quite a difficult problem.

In [6, 7] we considered low degree trigonometric polynomials with nonzero coefficients $v(x)$ of the form

(i) $ae^{-2ix} + be^{2ix}$,
(ii) $ae^{-2ix} + Be^{4ix}$,
(iii) $ae^{-2ix} + Ae^{-4ix} + be^{2ix} + Be^{4ix}$.

It is shown that the system of eigenfunctions and (at most finitely many) associated functions is complete but it is not a basis in $L^2([0, \pi], \mathbb{C})$ if $|a| \neq |b|$ in the case (i), if $|A| \neq |B|$ and neither $-b^2/4B$ nor $-a^2/4A$ is an integer square in the case (iii), and it is never a basis in the case (ii) subject to periodic boundary conditions. In connection with Example (iii) see also [1, 21].

In this paper we extend the analysis of the above example (ii) to potentials of the form

$$v(x) = ae^{-2ix} + be^{2ix}, \quad a, b \neq 0, \ r, s \in \mathbb{N}, \ r \neq s.$$ 

In Section 2, Theorem 11, it is shown that the system of root functions does not contain a basis in $L^2([0, \pi], \mathbb{C})$ for periodic $bc$ or if $bc$ is antiperiodic but $r, s$ are odd.

In Section 3, the case $r = 1, \ s > 2$ any (i.e., odd or even) with antiperiodic boundary conditions is completely analyzed as well, and it is shown that the system of root functions does not contain a basis in $L^2([0, \pi], \mathbb{C})$ – see Theorem 18.

In our proofs we face series of questions related to enumerative combinatorics and diophantine equations. Their solution would dramatically extend the class of trigonometric polynomial potentials $v(x)$ for which the problem of convergence of spectral decompositions could be
resolved. In our study of potentials (iii) in [2, 4] we discover a combinatorial identity (see also [1, 21]) that could be a prototype of such results. In this connection see [10] for more comments and open problems.

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2. TWO EXPONENTIAL TERM POTENTIALS

1. Our main objects are the potentials of the form

\begin{equation}
(2.1) \quad v(x) = ae^{-2Rix} + be^{2Six}, \quad a, b \neq 0,
\end{equation}

with \( R, S \in \mathbb{N}, \ R \neq S. \) Then

\begin{equation}
(2.2) \quad R = dr, \ S = ds, \quad \text{where} \ r, s \ \text{are coprime;}
\end{equation}

they are the main parameters in what follows.

In view of (2.1), an admissible path \( x = (x(t))_{t=1}^{\nu+1} \) from \(-n\) to \(n\) gives a non-zero term \( h(x, z) \) in \( \beta^+_{n}(z) \) (see (1.20)) if and only if

\begin{equation}
(2.3) \quad x(t) \in \{-2R, 2S\}, \quad t = 1, 2, \ldots, \nu + 1.
\end{equation}

Let \( x \) be such a path, and let

\begin{equation}
(2.4) \quad \tilde{p} = \#\{t : \ x(t) = -2R\}, \quad \tilde{q} = \#\{t : \ x(t) = 2S\}.
\end{equation}

Consider

\begin{equation}
(2.5) \quad n \in \Delta := (rsd)\mathbb{N}, \quad \text{i.e.,} \ n = rsdm, \ m \in \mathbb{N};
\end{equation}

then

\begin{equation}
(2.6) \quad -2R\tilde{p} + 2S\tilde{q} = 2n, \quad s\tilde{q} = r\tilde{p} + rsm.
\end{equation}

and therefore,

\begin{equation}
(2.7) \quad \tilde{q} = rq, \quad \tilde{p} = sp \quad \text{with} \ q = p + m.
\end{equation}

Under the assumptions (2.4) - (2.7) we denote by \( X_n(p) \) the set of all admissible paths from \(-n\) to \(n\) with \( \tilde{p} = ps \) negative steps \(-2R\) and \( \tilde{q} = qr \) positive steps \(2S.\) Then \( n \in \Delta \) (see (2.5)) implies

\begin{equation}
(2.8) \quad \#X_n(0) = 1, \quad X_n(0) = \{x^*\},
\end{equation}

where

\begin{equation}
(2.9) \quad x^*(k) = 2sd, \quad j_k^* := j(k, x^*) = -n + 2sdk, \quad 1 \leq k \leq rm - 1.
\end{equation}
Therefore, for \( n = rsdm \) we have \( n^2 - (j_k^*)^2 = 4s^2d^2k(rm - k) \), which implies that

\[
(2.10) \quad h(x^*, 0) = b^{rm} \prod_{k=1}^{rm-1} \frac{1}{n^2 - (j_k^*)^2} = \frac{b^{rm}}{(4s^2d^2)^{rm-1}[(rm - 1)!]^2}.
\]

Moreover, in these notations, we have

\[
(2.11) \quad \beta_n^+(z) = \sum_{p=0}^{\infty} \sum_{x \in X_n(p)} h(x, z),
\]

where, for \( x \in X_n(p) \),

\[
(2.12) \quad h(x, z) = a^{\tilde{p}}b^{\tilde{q}}h_1(x, z), \quad h_1(x, z) = \prod_{t=1}^{\tilde{p}+\tilde{q}-1} (n^2 - j(t, x)^2 + z)^{-1}.
\]

2. Next we show that the leading term in the asymptotics of \( \beta_n^+(z) \) is determined by \( h(x^*, 0) \) only. Fix \( p \geq 1 \) and \( x \in X_n(p) \); choose a set of vertices \( j(t_k, x), \ k = 1, \ldots, rm - 1 \) so that

\[
(2.13) \quad 0 \leq \delta_k := j_k^* - j(t_k, x) < 2S = 2sd.
\]

(This is possible since the positive steps of \( x \) are equal to \( 2S \).)

We have \( h_1(x, z) = \Pi_1(z) \cdot \Pi_2(z) \), where

\[
\Pi_1(z) = \prod_{k=1}^{rm-1} (n^2 - j(t_k, x)^2 + z)^{-1}
\]

and \( \Pi_2(z) \) is the product of those factors of \( h_1(x, z) \) which are not included in \( \Pi_1(z) \). In view of (2.4) and (2.7), the number of factors in \( \Pi_2(z) \) is equal to

\[
\nu(x) - (rm - 1) = \tilde{p} + \tilde{q} - 1 - (rm - 1) = (r + s)p.
\]

For \( n \geq 2 \) and \( |z| \leq 1 \) we have

\[
|n^2 - j(t_k, x)^2 + z| \geq |n^2 - j(t_k, x)^2| - 1 \geq 2n - 2 \geq n,
\]

so the absolute value of each factor is less than \( 1/n \). Therefore,

\[
(2.14) \quad |\Pi_2(z)| \leq (1/n)^{(r+s)p}.
\]

To estimate \( \Pi_1(z) \) we need the following (compare with [7, Lemma 2]).

**Lemma 6.** If \( \{j_1, \ldots, j_K\} \subset \{j = -n + 2t, \ t = 1, \ldots, n - 1\} \), then for large enough \( n \) and \( |z| \leq 1 \)

\[
(2.15) \quad \prod_{k=1}^{K} |n^2 - j_k^2 + z|^{-1} = \left( \prod_{k=1}^{K} |n^2 - j_k^2|^{-1} \right)^{(1+\theta_n)}, \quad |\theta_n| \leq \frac{4 \log n}{n}.
\]
Proof. Indeed, we have
\[
\theta_n = \prod_{k=1}^{K} \frac{n^2 - (j_k)^2}{n^2 - (j_k)^2 + z} - 1 = e^{-w_n} - 1,
\]
where \( w_n = \sum_{k=1}^{K} \log \left(1 + \frac{z}{n^2 - (j_k)^2}\right) \). Therefore, by the inequality
\[
|\log(1 + \zeta)| \leq \sum_{k=1}^{\infty} |\zeta|^k \leq 2|\zeta| \quad \text{for} \quad |\zeta| \leq 1/2,
\]
it follows that for large enough \( n \)
\[
|w_n| \leq \sum_{k=1}^{K} \frac{2|z|}{n^2 - (j_k)^2} \leq \sum_{k=1}^{n-1} \frac{2}{n^2 - (-n + 2k)^2} = \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{k} \leq 2 \log n < \frac{1}{2}.
\]
On the other hand, if \( |w| \leq 1/2 \) then \( |e^{-w} - 1| \leq \sum_{k=1}^{\infty} |w|^k \leq 2|w| \), which implies (2.15).

Now we could estimate the product \( \Pi_1(z) \) by Lemma 6. Indeed, if \( j_k = j(t_k, x) \) then due to the choice of \( t_k \) (see (2.13)) the vertices \( j_k \) are distinct and \( -n < j_k < n \). Therefore, (2.15) implies that
\[
\Pi_1(z) = \Pi_1(0)(1 + \theta_n), \quad \text{where} \quad |\theta_n| = O\left(\frac{\log n}{n}\right).
\]

3. Next we estimate \( \Pi_1(0) \) by comparing it with \( h_1(x^*, 0) \). To this end we need the following.

Lemma 7. Let \( n, K, S \in \mathbb{N} \) and \( n \geq (K + 1)S \), and let
\[
j_k = \pm(n - 2kS), \quad 0 \leq \delta_k \leq 2(S - d), \quad k = 1, \ldots, K, \quad d \in (0, S).
\]
Then
\[
\prod_{k=1}^{K} \frac{n^2 - (j_k)^2}{n^2 - (j_k - \delta_k)^2} \leq Cn^{1-d/S}.
\]
(This lemma is a more general assertion than Lemma 12 in [7], where \( S = 2 \) and \( \delta_k = 2 \) so \( d = 1 \).)

Proof. First we consider the case \( j_k = -(n - 2kS) \), i.e., moving forward from \( -n \) to \( +n \). Then \( n - j_k = 2n - 2kS \geq 2S \), and we have
\[
\frac{n^2 - (j_k)^2}{n^2 - (j_k - \delta_k)^2} = \frac{(n + j_k)(n - j_k)}{(n + j_k - \delta_k)(n - j_k + \delta_k)} \leq \frac{n + j_k}{n + j_k - \delta_k}.
\]
If \( j_k = -n + 2Sk \), then
\[
\frac{n + j_k}{n + j_k - \delta_k} = \left( 1 - \frac{\delta_k}{2kS} \right)^{-1} \leq \left( 1 - \frac{S - d}{kS} \right)^{-1}.
\]
Therefore, the product in (2.18) does not exceed
\[
\prod_{k=1}^{K} \left( 1 - \frac{\gamma}{K+1-k} \right)^{-1} \leq Cn^\gamma \text{ where } \gamma = 1 - \frac{d}{S}, \quad C = C(\gamma).
\]

When we are moving backward from \(+n\) to \(-n\), then \( j_k = n - 2Sk \), so
\[
\frac{n + j_k}{n + j_k - \delta_k} = \left( 1 - \frac{\delta_k}{2n - 2kS} \right)^{-1} \leq \left( 1 - \frac{(S - d)}{(K + 1 - k)S} \right)^{-1}.
\]
Therefore, the product in (2.18) does not exceed
\[
\prod_{k=1}^{K} \left( 1 - \frac{\gamma}{K+1-k} \right)^{-1} \leq Cn^\gamma \text{ where } \gamma = 1 - \frac{d}{S}, \quad C = C(\gamma),
\]
which completes the proof.

4. By Lemma 6 \(|\Pi_1(z)/\Pi_1(0)| = 1 + O ((\log n)/n)\). On the other hand, applying Lemma 7 to \( \Pi_1(0)/h_1(x^*, 0) \) we obtain (since \( S = sd \))
\[
\Pi_1(0) \leq Cn^{1-1/s}h_1(x^*, 0).
\]
Together with the estimate (2.14) for \( \Pi_2 \), this leads to
\[
|h_1(x, z)/h_1(x^*, 0)| \leq Cn^{1-1/s}(1/n)^{(s+r)p}.
\]

Let us take into account the coefficients \( a, b \) of the potential. We set
\[
T = \max\{|a|, |b|\};
\]
then
\[
\frac{|h(x, z)|}{|h(x^*, 0)|} = \frac{|a|^{\tilde{p}}|b|^{\tilde{q}}|h_1(x, z)|}{|b|^{rm}h_1(x^*, 0)} \leq Cn^{1-1/s} \left( \frac{T}{n} \right)^{(s+r)p},
\]
because \( \tilde{p} + \tilde{q} - rm = (r + s)p \) due to (2.4) and (2.7).

5. The number of paths \( x \in X_n(p) \) does not exceed
\[
\#X_n(p) \leq \left( \frac{\tilde{p} + \tilde{q}}{\tilde{p}} \right).
\]
In view of (2.7),

(2.22)

\[ \#X_n(p) \leq \left( \frac{(s + r)p + rm}{sp} \right) \leq \begin{cases} \frac{1}{(sp)!} [(s + 2r)m]^p & \text{if } p \leq m, \\ 2^{(s+2r)p} & \text{if } p > m. \end{cases} \]

By (2.20) and (2.22), it follows that

(2.23)

\[ \sum_{p=1}^{\infty} \sum_{x \in X_n(p)} |h(x, z)| \leq |h(x^*, 0)|((\sigma_1 + \sigma_2)), \]

where

\[ \sigma_1 = Cn^{1 - \frac{1}{s}} \sum_{p=1}^{m} \frac{1}{(sp)!} [(s + 2r)m]^p \left( \frac{T}{n} \right)^{(s+r)p}, \]

\[ \sigma_2 = Cn^{1 - \frac{1}{s}} \sum_{p=m+1}^{\infty} 2^{(s+2r)p} \left( \frac{T}{n} \right)^{(s+r)p}. \]

Since \( n = rsdm \) we have

\[ [(s + 2r)m]^p \left( \frac{T}{n} \right)^{(s+r)p} = \left( \frac{T \cdot (s + 2r)}{rsd} \right)^{s/r} \left( \frac{T}{n} \right)^{(s+r)p} = \left( \frac{T}{n} \right)^{(s+r)p} \]

where \( T_1 = T \left( \frac{T \cdot (s + 2r)}{rsd} \right)^{s/r} \). Therefore, for \( n \geq 2T_1 + 1 \),

\[ \sigma_1 = Cn^{1 - \frac{1}{s}} \sum_{p=1}^{m} (T_1/n)^{rp} \leq 2Cn^{1 - \frac{1}{s}} (T_1/n)^{r} \leq C_1 n^{1 - r - \frac{1}{2}}, \]

where \( C_1 = C_1(r, s, T) \).

The second sum \( \sigma_2 \) is much smaller than the first one:

\[ \sigma_2 \leq C_2 \left( \frac{4T}{n} \right)^{(s+r)p} \leq C_2 n^{1 - (s+r)(m+1) - \frac{1}{s}}, \]

where \( C_2 = C_2(r, s, T) \).

In view of (2.23), the obtained estimates for \( \sigma_1 \) and \( \sigma_2 \) prove that

(2.24)

\[ \sum_{p=1}^{\infty} \sum_{x \in X(p)} |h(x, z)| \leq C(r, s, T)|h(x^*, 0)| \cdot n^{1 - r - \frac{1}{2}}, \quad |z| \leq 1. \]

Hence, the following is true.

**Lemma 8.** For large enough \( n = mdsr, \ m \in \mathbb{N} \),

(2.25)

\[ \frac{1}{2} \beta_n^+(0) \leq |\beta_n^+(z)| \leq 2 \beta_n^+(0) \]
and
\begin{equation}
\beta_n^+(0) = h(x^*, 0) \left(1 + O \left( n^{1-s^{-\frac{1}{2}}} \right) \right)
\end{equation}

with
\begin{equation}
h(x^*, 0) = 4s^2d^2 \left( \frac{b}{4s^2d^2} \right)^{rm} ((rm - 1)!)^{-2}.
\end{equation}

6. To analyze the paths \( y \in \mathcal{Y}_n \) from \( n \) to \( -n \), i.e.,
\begin{equation}
\nu + 1 \sum_{1}^{\nu+1} y(t) = -2n,
\end{equation}
we can just exchange the roles of \( R \) and \( S \) and repeat the above statements with proper adjustments. Then
\[
Y_n(0) = \{y^*\}, \quad y^*(t) = -2R, \quad 1 \leq t \leq sm - 1,
\]
and the following holds.

**Lemma 9.** For large enough \( n = mdsr \), \( m \in \mathbb{N} \),
\begin{equation}
\frac{1}{2} \beta_n^{-}(0) \leq |\beta_n^{-}(z)| \leq 2\beta_n^{-}(0)
\end{equation}
and
\begin{equation}
\beta_n^{-}(0) = h(y^*, 0) \left(1 + O \left( n^{1-s^{-\frac{1}{2}}} \right) \right)
\end{equation}

with
\begin{equation}
h(y^*, 0) = 4r^2d^2 \left( \frac{a}{4r^2d^2} \right)^{sm} ((sm - 1)!)^{-2}.
\end{equation}

**Remark 10.** If \( R = 1 \) then \( d = r = 1 \), \( S = s \), and for any \( n \) if we go backward from \( +n \) to \( -n \) it could be done without using forward steps \( +2s \). Analogues of (2.30) could be given for any \( s \) – see Section 3.5.

7. The set \( \Delta \) defined in (2.5) certainly contains infinitely many even integers because \( m \) could run over \( 2\mathbb{N} \). But if \( rsd \) is even, then \( \Delta \cap (2\mathbb{N} + 1) = \emptyset \) while \( \Delta \cap (2\mathbb{N} + 1) \) is infinite if \( rsd \) is odd, i.e., if \( R \) and \( S \) are odd. In any case, if \( R \neq S \), say \( R < S \),
\[
\min \{|\beta_n^\pm(0)/\beta_n^\mp(0)|, \; n = (rsd)m\} \leq (C_3)^m \frac{(rm - 1)!}{(sm - 1)!} \leq (C_4)^m m^{-|r-s|m}.
\]
In view of Criterion \([3]\), these observations lead to the following.

**Theorem 11.** For any potential \( v \) in (2.1) there is no basis consisting of root functions of \( L_{Per^+}(v) \). If \( R \) and \( S \) are odd, the same is true for \( L_{Per^-}(v) \).
3. Potentials $ae^{-2ix} + be^{2six}$, $s > 2$.

1. If we analyze $bc = \text{Per}^-$ in the case the potential is of the form (2.1) and one of the parameters $r, s$ in (2.2) is even then the constructions in Section 2 cannot be applied to give us a negative statement like Theorem 11. In this section we present elaborate analysis in the case $r = 1$, $s > 2$ and

$$\Delta = \{n = sm - 1, m \in \mathbb{N}\}. \tag{3.1}$$

Observe, that if $s$ is even, then $\Delta$ consist of odd numbers, and if $s$ is odd then $\Delta \cap (2\mathbb{N} - 1) \neq \emptyset$ and $\Delta \cap 2\mathbb{N} \neq \emptyset$. So, by showing that

$$\inf \{|\beta_n^{\pm}(0)|/|\beta_n^{\mp}(0)| : n \in \Delta, n \geq N(v)\} = 0$$

we would obtain by Criterion 3 that there is no basis in $L^2([0, \pi])$ consisting of root functions of $L_{\text{Per}^-}(v)$ for potentials of the form

$$v(x) = ae^{-2ix} + Be^{2six}, \ a, b \neq 0, \ s \geq 3. \tag{3.2}$$

Let us remind that Theorem 11 in Section 2 considers the operators $L_{\text{Per}^+}(v)$ for any $s$. Its claim follows from Criterion 3 because

$$\inf \{|\beta_n^{\pm}(0)|/|\beta_n^{\mp}(0)| : n \in s\mathbb{N}, n \geq N(v)\} = 0.$$ 

In the sequel we write for convenience $h_1(x)$ instead of $h_1(x, 0)$, and $h(x)$ instead of $h(x, 0)$.

2. Fix $n = sm - 1$; a path $x = (x(t))_{t=1}^{\nu+1}$ from $-n$ to $n$ gives a non-zero term $h(x, z)$ in $\beta_n^+(z)$ if and only if (compare with (2.3))

$$x(t) = -2 \quad \text{or} \quad x(t) = 2s. \tag{3.3}$$

Set

$$p = \#\{t : x(t) = -2, \ 1 \leq t \leq \nu(x) + 1\},$$

$$q = \#\{t : x(t) = 2s, \ 1 \leq t \leq \nu(x) + 1\};$$

then we have

$$2n = -2p + 2sq \Rightarrow sm - 1 = -p + sq \Rightarrow p = 1 + s(q - m). \tag{3.5}$$

We set

$$p = 1 + s\kappa, \quad q = m + \kappa \tag{3.6}$$

to satisfy (3.5), and define $X_n(\kappa)$ as the set of all admissible paths satisfying (3.3) which parameters $p$ and $q$ are given by (3.6). Then

$$\#X_n(0) = m + 1, \tag{3.7}$$
and with \( p = 1, \ q = m \) a path \( \xi^\tau \in X_n(0) \) is uniquely determined by the position \( \tau \) of its only step \(-2\). In other words, the paths in \( X_n(0) \) are given by

\[
(3.8) \quad \xi^\tau(t) = \begin{cases} 
2s, & t \neq \tau \\
-2, & t = \tau
\end{cases} \quad 1 \leq \tau, t \leq m + 1.
\]

Among them the two paths \( \xi^1 \) and \( \xi^{m+1} \) are special in the sense that \( h_1(\xi^1) = h_1(\xi^{m+1}) < 0 \), while \( h_1(\xi^\tau) > 0 \) for \( \tau = 2, \ldots, m \). More precisely, since \( j(t, \xi^1) = -n - 2 + 2s(t - 1) \), we have

\[
n^2 - j(1, \xi^1)^2 = n^2 - (-n - 2)^2 = -4(n + 1) = -4ms
\]

and

\[
n^2 - j(t+1, \xi^1)^2 = n^2 - (-n - 2 + 2st)^2 = 4s(m-t)(st-1), \quad t = 1, \ldots, m-1,
\]

so it follows that

\[
(3.9) \quad h_1(\xi^1) = \prod_{t=1}^{m} [n^2 - j(t, \xi^1)^2]^{-1} = \frac{-1}{(4s)^{m}m!} \left( \prod_{t=1}^{m-1} (st - 1) \right)^{-1}.
\]

By symmetry \( h_1(\xi^{m+1}) = h_1(\xi^1) \), so we obtain for their sum

\[
(3.10) \quad h_1(\xi^1) + h_1(\xi^{m+1}) = -H^-(m)
\]

with

\[
(3.11) \quad H^-(m) = \frac{2}{(4s)^{m}m!} \left( \prod_{t=1}^{m-1} (st - 1) \right)^{-1} = \frac{2s}{(2s)^{2m}m!} \frac{\Gamma(1 - \frac{1}{s})}{\Gamma(m - \frac{1}{s})}.
\]

For \( \xi^\tau \) with \( 2 \leq \tau \leq m \) we have

\[
(3.12) \quad j(t, \xi^\tau) = \begin{cases} 
-n + 2st, & t \leq \tau - 1, \\
-n - 2 + 2s(t - 1), & \tau \leq t \leq m.
\end{cases}
\]

By (3.1) and (3.12)

\[
n^2 - j(\xi^\tau, t)^2 = \begin{cases} 
4st[(m - t)s - 1], & 1 \leq t \leq \tau - 1, \\
4s(s(t - 1) - 1)(m - (t - 1)), & \tau \leq t \leq m,
\end{cases}
\]

which implies, for \( 2 \leq \tau \leq m \), that

\[
(3.13) \quad h_1(\xi^\tau) = \frac{1}{(4s)^{m}((\tau - 1)!(m - \tau + 1)!)} \left( \prod_{t=\tau+1}^{m-1} (st - 1) \right)^{-1} \left( \prod_{t=1}^{m-1} (st - 1) \right)^{-1}.
\]
One can easily see that the sum
\[ (3.14) \quad H^+ = H^+(m) := \sum_2^m h_1(\xi^r) \]
can be written (if we change \( \tau \) to \( \tau - 1 \)) as
\[ (3.15) \quad H^+ = \frac{1}{(4s)^m} \sum_{\tau=1}^{m-1} \frac{\prod_{t=1}^{\tau-1}(st-1)}{\tau!} \frac{\prod_{t=1}^{m-\tau-1}(st-1)}{(m-\tau)!} \left( \prod_{t=1}^{m-1}(st-1) \right)^{-2}. \]

We set \( \alpha = 1/s; \) then
\[ (3.16) \quad \alpha < 1/2 \quad (\text{so } 1 - 2\alpha > 0) \quad \text{for } s > 2. \]

Let
\[ (3.17) \quad A_\alpha(k) = \frac{\alpha \prod_{t=1}^{k-1}(t-\alpha)}{k!} = \frac{\alpha \Gamma(k-\alpha)}{\Gamma(1-\alpha)\Gamma(k+1)}, \quad k \geq 2, \]

\[ (3.18) \quad A_\alpha(0) = 0, \quad A_\alpha(1) = \alpha. \]

Then
\[ (3.19) \quad 2A_\alpha(m) \times (H^+/H^-) = \sum_{\tau=1}^{m-1} A_\alpha(\tau)A_\alpha(m-\tau), \]

and
\[ (3.20) \quad \sum_{k=0}^{\infty} A_\alpha(k)w^k = f_\alpha(w) := 1 - (1 - w)^\alpha \]
happens to be a nice generating function. The right-hand side of (3.19) is the \( m \)-th Taylor coefficient \( T_m \) of the square
\( (f_\alpha(w))^2 = (1-(1-w)^\alpha)^2 = 1 - 2(1-w)^\alpha + (1-w)^{2\alpha} = 2f_\alpha(w) - f_{2\alpha}(w), \)
so it equals
\[ T_m([f_\alpha]^2) = 2A_\alpha(m) - A_{2\alpha}(m) \]
Hence, dividing by \( 2A_\alpha(m) \) and taking into account (3.16) and (3.17), we obtain
\[ (3.21) \quad \frac{H^+}{H^-} = 1 - \frac{A_{2\alpha}(m)}{2A_\alpha(m)} = 1 - \frac{\Gamma(1-\alpha)\Gamma(m-2\alpha)}{\Gamma(m-\alpha)\Gamma(1-2\alpha)}, \quad \alpha = 1/s. \]
The Stirling formula shows that
\[ (3.22) \quad r(m) := \frac{A_{2\alpha}(m)}{2A_\alpha(m)} = \frac{\Gamma(1-\alpha)\Gamma(m-2\alpha)}{\Gamma(m-\alpha)\Gamma(1-2\alpha)} = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} \rho(m)m^{-\alpha}, \]
where \( \rho(m) \to 1 \). Therefore,

\[
(3.23) \quad \frac{H^+ - H^-}{H^+ + H^-} = \frac{r(m)}{2 - r(m)} \approx \frac{1}{2} r(m)
\]

for large enough \( m \), i.e., we proved the following.

**Lemma 12.** In the above notations,

\[
(3.24) \quad H^-(m) - H^+(m) \gtrsim m^{-1/s} (H^-(m) + H^+(m)) \quad \text{as} \quad m \to \infty.
\]

By Lemma 6, for large enough \( n \) and \( |z| \leq 1 \) we have that

\[
(3.25) \quad h_1(\xi, z) = h_1(\xi, 0)(1 + \theta(\xi, z)), \quad |\theta(\xi, z)| \leq \frac{4 \log n}{n}, \quad \xi \in X_n(0).
\]

Indeed, if \( \xi = \xi^\tau, \quad \tau = 2, \ldots, m - 1 \) then (3.25) follows directly from Lemma 6. To handle \( h_1(\xi^1, z) \), we write it in the form

\[
\frac{1}{n^2 - (-n - 2)^2 + z} \prod_{k=1}^{m-1} \frac{1}{n^2 - (-n - 2 + 2sk)^2 + z}.
\]

Then we apply Lemma 6 to the product on the right and estimate the single factor by \((-4n - 4 + z)^{-1} = -(4n + 4)^{-1}(1 + O(1/n))\). The case \( \xi = \xi^{m+1} \) is symmetric.

From (3.25) and (3.10) it follows that

\[
(3.26) \quad h_1(\xi^1, z) + h_1(\xi^{m+1}, z) = -H^-(m) \left[ 1 + O((\log n)/n) \right].
\]

On the other hand, by (3.14) and (3.25) we obtain that

\[
\sum_{\tau=2}^{m} h_1(\xi^\tau, z) = \sum_{\tau=2}^{m} h_1(\xi^\tau)(1 + \theta(\xi^\tau, z)) = H^+(m) + \Omega,
\]

where

\[
|\Omega| = \left| \sum_{\tau=2}^{m} h_1(\xi^\tau)\theta(\xi^\tau, z) \right| \leq \sum_{\tau=2}^{m} h_1(\xi^\tau) \frac{4 \log n}{n} = H^+(m) \frac{4 \log n}{n}.
\]

Thus, we have

\[
(3.27) \quad \sum_{\tau=2}^{m} h_1(\xi^\tau, z) = H^+(m)[1 + O((\log n)/n)].
\]

Now (3.26) and (3.27) give us, for \( |z| \leq 1 \), that

\[
(3.28) \quad \sum_{\xi \in X_n(0)} h_1(\xi^\tau, z) = (H^+(m) - H^-(m)) \left[ 1 + O((\log n)/n) \right].
\]
3. Next we estimate the ratio of
\[ \sum_{x \in X_n(\kappa)} |h_1(x, z)| \quad \text{and} \quad \sum_{\xi \in X_n(0)} |h_1(\xi)| = H^- + H^+. \]

Fix \( x \in X_n(\kappa), \kappa \geq 1 \), and set
\[ \tau = \min \{ m + 1, \min\{ t : x(t) = -2 \} \}. \]  

(3.29)

Let
\[ j_k^* = j(k, \xi^\tau), \quad k = 1, \ldots, m. \]

(3.30)

denote the vertices of \( \xi^\tau \).

Next we choose \( m \) vertices \( j_k = j(t_k, x) \) of \( x \) so that \( j_k \) is "close" to \( j_k^* \) as follows. If \( \tau = m \) or \( \tau = m + 1 \) we set \( j_k = j_k^* \), \( k = 1, \ldots, m \). If \( \tau < m \) we set
\[ t_k = k \quad \text{if} \quad 1 \leq k \leq \tau \]
and
\[ t_k = \min \{ t > \tau : j(t, x) > j_{k-1}^* \}, \quad \tau + 1 \leq k \leq m. \]  

(3.31) \hspace{1cm} (3.32)

Let \( J(x) := (j(t, x))_{t=1}^{\nu(x)} \) be the sequence of the vertices of \( x \). The sequence \( (j_k)_{k=1}^m = (j(t_k, x))_{t=1}^{\nu(x)} \) is a subsequence of \( J(x) \); let
\[ I(x) = (i_1, \ldots, i_\rho), \quad \rho = \nu(x) - m = (1 + s)\kappa \]
be its complementary subsequence in \( J(x) \). Consider the mapping
\[ \Phi_\kappa : X_n(\kappa) \to X_n(0) \times \mathbb{Z}^{(1+s)\kappa}, \quad \Phi_\kappa(x) = (\xi^\tau, I(x)). \]  

(3.33)

**Lemma 13.** The mapping \( \Phi_\kappa \) is injective.

**Proof.** The lemma will be proved if we show that given \( \Phi_\kappa(x) = (\xi^\tau, I(x)) \) we can restore in an unique way the path \( x \) (or equivalently, the sequence of its vertices \( J(x) \)).

In view of the construction, if \( \tau = m \) or \( \tau = m + 1 \) then
\[ J(x) = (j_1^*, \ldots, j_m^*, i_1, \ldots, i_\rho), \quad \rho = \nu(x) - m. \]

In the case \( \tau < m \) we have to find the vertices \( j_k \) and their places in \( J(x) \). By (3.31),
\[ j_k = j(k, x) = j_k^*, \quad 1 \leq k \leq \tau. \]

Consider the first term \( i_1 \) of the sequence \( I(x) \). By (3.32), there is an integer \( \mu_1 \) such that \( 0 \leq \mu_1 \leq m - \tau \) and
\[ i_1 = j_\tau^* = -2 + 2s \cdot \mu_1; \]
then \( j_k = j(k, x) = j_\tau^* + 2s(k - \tau) \) for \( \tau + 1 \leq k \leq k_1 := \tau + \mu_1 \). If \( k_1 = m \) we have \( j(t, x) = i_{t-m} \) for \( m + 1 \leq t \leq \nu(x) \), so \( J(x) \) is restored.
Otherwise, we set
\[ \tau_1 = \min \{ t : i_{t+1} - i_t \notin \{-2, 2s\}, 1 \leq t < \rho \}. \]
From (3.32) it follows that \( i_1, \ldots, i_{\tau_1} \) are successive vertices of \( x \), so we have
\[ j(t, x) = i_{t-k_1}, \quad k_1 + 1 \leq t \leq \tau_1 + k_1. \]
Moreover, there is \( \mu_2 \in \mathbb{N} \) such that
\[ i_{\tau_1+1} - i_{\tau_1} = -2 + 2s \cdot \mu_2, \]
which implies
\[ j_k = i_{\tau_1} + 2s(k - k_1), \quad k_1 + 1 \leq k \leq k_2 := k_1 + \mu_2, \]
so
\[ j(t, x) = i_{\tau_1} + 2s(t - \tau_1 - k_1), \quad \tau_1 + k_1 + 1 \leq t \leq \tau_1 + k_2. \]
In the case \( k_2 = m \) we have \( j(t, x) = i_{t-m} \) for \( m + \tau_1 + 1 \leq t \leq \nu(x) \), so \( J(x) \) is restored. Otherwise, we set
\[ \tau_2 = \min \{ t : i_{t+1} - i_t \notin \{-2, 2s\}, \tau_1 + 1 \leq t < \rho \}. \]
and continue by induction.

Fix \( x \in X_n(\kappa) \), and let \( (j_k)_{k=1}^m \) and \( \Phi(x) = (\xi^\tau, I(x)) \) be defined as above. Then
\[ h_1(x, z) = \prod_{k=1}^m (n^2 - j_k^2 + z)^{-1} \cdot \prod_{i \in I(x)} (n^2 - i^2 + z)^{-1}, \]
and by Lemma \( \ref{lem:product} \) we have
\[ \prod_{k=1}^m (n^2 - j_k^2 + z)^{-1} = \left( \prod_{k=1}^m (n^2 - j_k^2)^{-1} \right) (1 + O((\log n)/n)). \]
On the other hand, by (3.31) and (3.32), \( j_k = j_k^* \) for \( 1 \leq k \leq \tau \) and \( j_{k-1} < j_k \leq j_k^* \) for \( \tau < k \leq m \). Therefore, by Lemma \( \ref{lem:inequality} \) we obtain
\[ \frac{1}{h_1(\xi^\tau)} \prod_{k=1}^m (n^2 - j_k^2)^{-1} = \prod_{k=\tau}^m \frac{n^2 - (j_k^*)^2}{n^2 - (j_k)^2} \leq Cn^{1-\frac{1}{s}}. \]
(Since \( j_k^* = n - 2s(m + 1 - k) \), we apply Lemma \( \ref{lem:inequality} \) after changing the summation index by \( k = m + 1 - k \).) Thus, the above inequalities imply that
\[ \prod_{k=1}^m (n^2 - j_k^2 + z)^{-1} \leq C h(\xi^\tau) n^{1-\frac{1}{s}}. \]
Let $X_n(\kappa, \tau)$ be the set of all $x \in X_n(\kappa)$ such that (3.29) holds. The sets $X_n(\kappa, \tau)$, $1 \leq \tau \leq m + 1$ are disjoint, and

$$X_n(\kappa) = \bigcup_{\tau = 1}^{m+1} X_n(\kappa, \tau).$$

In view of (3.34) and (3.35) we have

$$\sum_{x \in X_n(\kappa, \tau)} |h_1(x, z)| \leq Cn^{1-\frac{1}{s}} \prod_{x \in X_n(\kappa, \tau)} |n^2 - i^2 + z|^{-1}.\quad (3.36)$$

By Lemma 12 the mapping $\Phi_\kappa$ is injective, so the sequence $I(x) = (i_1, \ldots, i_\rho)$ is uniquely determined by $x \in X_n(\kappa, \tau)$. Moreover,

$$|n^2 - i^2 + z| \geq |n^2 - i^2| - 1 \geq \frac{1}{2} |n^2 - i^2| \quad \text{for } |z| \leq 1.$$

Therefore,

$$\sum_{x \in X_n(\kappa, \tau)} \prod_{i \in I(x)} \frac{2^\rho}{|n^2 - i^2| \cdots |n^2 - i^2_\rho|}$$

$$= \left( \sum_{i \neq \pm n} 2 \right)^\rho \leq \left( \frac{C \log n}{n} \right)^\rho, \quad \rho = (s + 1)\kappa,$$

because $\sum_{i \neq \pm n} |n^2 - i^2|^{-1} \leq (C \log n) / n$ (e.g., see Lemma 10 in [4]).

Therefore, taking a sum over $\tau = 1, \ldots, m + 1$ in (3.36), we obtain the following.

**Lemma 14.** In the above notations, for $\kappa = 1, 2, \ldots,$

$$\sum_{x \in X_n(\kappa)} |h_1(x, z)| \leq Cn^{1-\frac{1}{s}} \left( \frac{C \log n}{n} \right)^{(s+1)\kappa} (H^- + H^+).\quad (3.37)$$

4. Now we are going to show that the main term of the asymptotics of $\beta_n^+(z)$, $|z| \leq 1$, is given by $H^+ - H^-$. First we prove the following.

**Lemma 15.** In the above notations, for $n = sm - 1$, we have

$$\sum_{x \in X_n \setminus X_n(0)} |h(x, z)| \leq C(a, b) \frac{\log n}{n^{1/s}} \left( \frac{\log n}{n} \right)^s (H^-(m) + H^+(m)).\quad (3.38)$$

Proof. If $x \in X_n(\kappa)$, then $\nu(x) = p + q$ with $p = 1 + sk$, and $q = m + \kappa$, so

$$h(x, z) = a^p b^q h_1(x, z) = ab^m (a^s b)^\kappa h_1(x, z).$$
By (3.37) it follows
\[ \sum_{x \in X_n(\kappa)} |h(x, z)| \leq C|a||b|^m n^{1 - \frac{\kappa}{s}} (D(n))^\mu \left( H^-(m) + H^+(m) \right), \]
with \( D(n) = |a^s b| \left( \frac{C \log n}{n} \right)^{s+1} \). For large enough \( n \) we have \( D(n) < 1/2 \), so
\[ \sum_{\kappa=1}^{\infty} \sum_{x \in X_n(\kappa)} |h(x, z)| \leq C|a||b|^m n^{1 - \frac{\kappa}{s}} D(n) \left( H^-(m) + H^+(m) \right), \]
which completes the proof.

Since
\[ \beta_n^+(z) = \sum_{x \in X_n(0)} h(x, z) + \sum_{x \in X_n \setminus X_n(0)} h(x, z), \]
Lemma 12 and Lemma 15 lead to the following.

**Proposition 16.** In the above notations, for \( n = sm - 1 \), we have
\[ (3.39) \quad \beta_n^+(z) = \beta_n^+(0) \left[ 1 + O\left( (\log n)/n \right) \right], \]
where
\[ (3.40) \quad \beta_n^+(0) = \frac{-2sab^m \Gamma^2(1 - \frac{1}{2}) \Gamma(m - \frac{2}{s})}{(2s)^{2m} m! \Gamma^2(m - \frac{1}{2}) \Gamma(1 - \frac{2}{s})} \left( 1 + O\left( (\log n)^{s+1}/n^s \right) \right). \]

**Proof.** Indeed, by (3.24) one can easily see that (3.39) follows from (3.28) and (3.38).

To prove (3.40), let us recall that
\[ \sum_{\xi \in X_n(0)} h(\xi, 0) = H^+(m) - H^-(m), \]
so (3.38) and (3.24) imply that
\[ (3.41) \quad \beta_n^+(0) = ab^m (H^+(m) - H^-(m)) \left( 1 + O\left( (\log n)^{s+1}/n^s \right) \right). \]
Therefore, (3.40) follows from (3.11) and (3.21), which completes the proof.

5. Next we estimate \( \beta_n^-(z) \) for \(|z| \leq 1\) – compare Lemma 8 – without any restriction like (2.5) or (3.1) on \( n \). For every \( n \), if \( y \) is a path from \(+n\) to \(-n\) satisfying (3.3) - (3.4), then we have \(-2n = -2p + 2sq\), i.e.,
\[ (3.42) \quad p = n + sq. \]
We define \( Y_n(q) \) as the set of all paths with parameters \( p, q \) satisfying (3.42). Then

(3.43) \[ \#Y_n(0) = 1 \]

and the only path \( \eta \in Y_n(0) \) is defined by

(3.44) \[ \eta(t) = -2, \quad 1 \leq t \leq n, \]

so its vertices are

(3.45) \[ j(t; \eta) = n - 2t, \quad 0 \leq t \leq n. \]

Therefore,

(3.46) \[ h_1(\eta) = \prod_{t=1}^{n-1} \left( n^2 - (n - 2t)^2 \right)^{-1} = \frac{1}{4^{n-1}(n-1)!^2} \]

and, due to Lemma 6

(3.47) \[ h_1(\eta, z) = \prod_{t=1}^{n-1} \left( n^2 - (n - 2t)^2 + z \right)^{-1} = h_1(\eta) \left[ 1 + O((\log n)/n) \right]. \]

If \( q \geq 1 \), then any path \( y \in Y_n(q) \) has a sub-path with \( s + 1 \) steps of the form \((2s, -2, \ldots, -2)\). Indeed, choose

(3.48) \[ t^* = \max\{t : y(t) = 2s\}; \]

then \( t^* \leq \nu(y) - s - 1 \), and

(3.49) \[ y(t) = -2, \quad t^* + 1 \leq t \leq t^* + s. \]

Now define a new path \( \tilde{y} \in Y(q-1) \) by

(3.50) \[ \tilde{y}(t) = \begin{cases} y(t), & 1 \leq t < t^*, \\ y(t+1+s), & t^* \leq t \leq \nu(y) - s. \end{cases} \]

Then

(3.51) \[ h_1(y, z) = h_1(\tilde{y}, z) \cdot \prod_{t=t^*}^{t^*+s} \left( n^2 - (n - j^2(t, y))^2 + z \right)^{-1}, \]

so

\[ |h_1(y, z)| \leq (2n)^{-qs} |h_1(\tilde{y}, z)| \quad \text{for} \quad |z| \leq 1. \]

After \( q \) such restructuring we come, in view of (3.47), to the inequality

(3.52) \[ |h_1(y, z)| \leq 2(2n)^{-qs} |h_1(\eta)|, \quad |z| \leq 1, \ n > N_1. \]
If $T = \max\{|a|, |b|\}$, then – compare (2.19) - (2.20) – for $y \in Y_n(q)$ it follows from (3.52) that

$$|h(y, z)| = |a^{n+qs}b^q|h_1(y, z)| \leq \frac{2T^{q(s+1)}n^{q(s+1)}}{2n^{q(s+1)}}|a^n h_1(\eta)| = 2|h(\eta)| \left(\frac{T}{2n}\right)^{q(s+1)}.$$ 

As in (2.21), now we can claim that

$$\#Y_n(q) \leq \left(\begin{array}{c} p + q \\ q \end{array}\right) = \left(\begin{array}{c} n + q(s + 1) \\ q \end{array}\right) \leq \left\{\begin{array}{ll} \frac{1}{q}(s + 2)n^q & \text{if } q < n, \\ 2(s+2)^q & \text{if } q \geq n. \end{array}\right.$$

Therefore, by (3.53) and (3.54) we obtain

$$\sum_{q \geq 1} \sum_{y \in Y_n(q)} |h(y, z)| \leq 2|h(\eta)|(\sigma_1 + \sigma_2),$$

where for large enough $n$

$$\sigma_1 = \sum_{q=1}^{n-1} \frac{(s + 2)^n q^q}{q!} \left(\frac{T}{2n}\right)^{q(s+1)} \leq C_1 n^{-s},$$

and

$$\sigma_2 = \sum_{q=n}^{\infty} 2q(s+2)^q \left(\frac{T}{2n}\right)^{q(s+1)} \leq 2q \left(\frac{T}{n}\right)^{n(s+1)}.$$ 

Certainly, the inequalities (3.54) - (3.57) imply

$$\sum_{Y_n \setminus \{\eta\}} |h(y, z)| \leq \frac{C}{n^s} |h(\eta)|, \quad |z| \leq 1.$$ 

**Proposition 17.** In the above notations,

$$\beta_{\eta}^-(z) = \beta_n^-(0)(1 + O((\log n)/n)),$$

where

$$\beta_n^-(0) = \frac{a^n}{4^{n-1}[(n-1)!]^2}(1 + O(1/n^s)).$$

**Proof.** Indeed, (3.60) follows from (3.46) and (3.58), and (3.59) follows from (3.46), (3.60), (3.47) and (3.58). \hfill \Box

**Theorem 18.** For any potential of the form

$$v(x) = ae^{-2ix} + be^{2ix}, \quad a, b \neq 0, \quad s \geq 3,$$

there is no basis consisting of root functions of $L_{Per}(v)$. 
Proof. In view of (3.39), (3.40) and (3.59), we may apply Criterion 3 to the set \( \Delta = \{ n = sm - 1, \ m \in \mathbb{N} \} \). By (3.40), (3.59) and the Stirling formula, we have

\[
|\beta_n^{(0)}(0)|/|\beta_n^{(0)}(0)| \leq C_1^n (m!/n!)^2 \leq C_2^m m^{2(1-s)m} \to 0, \quad n \in \Delta.
\]

Hence, Criterion 3 implies that there is no basis consisting of root functions of \( L_{\text{Per}}(v) \). \( \square \)

4. Comments

Theorems 11 and 18 claim divergence of spectral decompositions in the case of potentials of the form

\[
v(x) = ae^{-2iRx} + be^{2iSx}
\]

for many pairs \( R, S \) such that \( R \neq S \).

If \( R = S \) the picture is much simpler; it is similar to the case \( R = S = 1 \) which is analyzed in [7], see Theorem 7 in Section 3 there.

If \( R = S > 1 \), then an admissible path \( x \) from \( -n \) to \( n \) (or from \( n \) to \( -n \)) gives a nonzero term \( h(x, z) \) of \( \beta_n^{(0)}(z) \) if and only if \( x(t) = \pm 2R \).

Let \( p \) and \( q \) be, respectively, the number of steps equal to \( -2R \) and \( 2R \). Then – compare (2.1) - (2.7) –

\[
2n = -2Rp + 2Rq = 2R(p + q),
\]

so

\[
\beta_n^{-(z)}(z) = 0, \quad \beta_n^{+(z)}(z) = 0 \quad \text{if} \quad n \not\equiv 0 \mod R,
\]

Choose \( N \) so large that (1.6) holds and the claim of Lemma 2 is valid for \( n > N \). Set

\[
\Delta_0^\pm = \{ n \in \Gamma^\pm : n > N, \ n \not\equiv 0 \mod R \}
\]

and let \( E(\Delta_0^\pm) = \text{Ran} \ P_{\Delta_0^\pm} \), where \( P_\Delta \) is the projection defined by (1.8). Then, in view of (1.3), Criterion 3 implies that \( E(\Delta_0^\pm) \) (respectively \( E(\Delta_0^-) \)) has a basis consisting of periodic (antiperiodic) root functions. In particular, this holds for the set \( \Delta_0 \) defined by (1.10).

On the other hand, let us consider the set

\[
\Delta_0^\pm = \{ n \in \Gamma^\pm, \ n = Rm, \ m \geq N \}.
\]

By Criterion 3 the system of root functions of \( L_{\text{Per}} \) contains a (Riesz) basis in \( L^2([0, \pi]) \) if and only if \( E(\Delta_0^\pm) \) has a basis consisting of periodic (respectively antiperiodic) root functions.
One can show using the same argument as in [7, Section 3] (see Lemmas 3 and 4, and Propositions 5 and 6 there) that if $n \in \Delta_{\pm}^1$, then

\begin{equation}
\beta_n^+(z) = 4R^2 \left( \frac{b}{4R^2} \right)^m \frac{1}{[(m - 1)!]^2} \left( 1 + O \left( \frac{\log n}{n} \right) \right), \quad |z| \leq 1,
\end{equation}

\begin{equation}
\beta_n^-(z) = 4R^2 \left( \frac{a}{4R^2} \right)^m \frac{1}{[(m - 1)!]^2} \left( 1 + O \left( \frac{\log n}{n} \right) \right), \quad |z| \leq 1.
\end{equation}

Now Criterion 3 says when $E(\Delta_{\pm}^1)$ has a basis consisting of root functions, which leads to the following generalization of Theorem 7 in [7].

**Proposition 19.** If $R$ is even, then a root function system of the operator

\begin{equation}
L = -\frac{d^2}{dx^2} + ae^{-2iRx} + be^{2iRx},
\end{equation}

considered with antiperiodic boundary conditions, contains a Riesz basis in $L^2([0, \pi])$.

If $R$ is odd and $L$ is considered with antiperiodic boundary conditions, or $R$ is arbitrary and $L$ is considered with periodic boundary conditions, then the system of root functions of the operator $L$ contains a Riesz basis in $L^2([0, \pi])$ if and only if

\begin{equation}
|a| = |b|.
\end{equation}

**Proof.** By (4.5) and (4.6) we have

\[
\frac{\beta_n^-(z)}{\beta_n^+(z)} = \left( \frac{a}{b} \right)^n \left( 1 + O \left( \frac{\log n}{n} \right) \right), \quad n \in \Delta_{\pm}^1, \quad |z| \leq 1.
\]

Then the assertion follows from the simple observation that $\Delta_{\pm}^1 \cap 2\mathbb{N}$ is an infinite set for any $R$ but $\Delta_{\pm}^1 \cap (2\mathbb{N} - 1) = \emptyset$ if $R$ is even and $\Delta_{\pm}^1 \cap (2\mathbb{N} - 1)$ is infinite if $R$ is odd. \qed

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