$K \to \pi \nu \bar{\nu}$: A Model Independent Analysis and Supersymmetry

Andrzej J. Buras$^a$, Andrea Romanino$^{a,b}$ and Luca Silvestrini$^a$

$^a$Technische Universität München, Physik Department
D-85748 Garching, Germany

$^b$Scuola Normale Superiore and INFN, sezione di Pisa
I-56126 Pisa, Italy

Abstract

We present a model independent analysis of new-physics contributions to the rare decays $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$. We parameterize the effects of new physics in these decays by two parameters: $r_K$ and the phase $\theta_K$, with $r_K = 1$ and $\theta_K = 0$ in the Standard Model. We show how these parameters can be extracted from future data together with the relevant CKM parameters, in particular the angle $\beta$ of the unitarity triangle. To this end CP asymmetries in $B \to \psi K_S$ and $B \to \pi^+ \pi^-$ as well as the ratio $|V_{ub}/V_{cb}|$ have to be also considered. This analysis offers simultaneously some insight in a possible violation of a “golden relation” between $K \to \pi \nu \bar{\nu}$ decays and the CP asymmetry in $B \to \psi K_S$ in the Standard Model pointed out some time ago. We illustrate these ideas by considering a general class of supersymmetric models. We find that in the “constrained” MSSM, in which $\theta_K = 0$, the measurements of $\text{Br}(K^+ \to \pi^+ \nu \bar{\nu})$ and $\text{Br}(K_L \to \pi^0 \nu \bar{\nu})$ directly determine the angle $\beta$. Moreover, the “golden relation” remains unaffected. On the other hand, in general SUSY models with unbroken R-parity the present experimental constraints still allow for substantial deviations from $r_K = 1$ and $\theta_K = 0$. Typically $0.5 < r_K < 1.3$ and $-25^0 < \theta_K < 25^0$. Consequently, in these models the violation of the “golden relation” is possible and values for $\text{Br}(K^+ \to \pi^+ \nu \bar{\nu})$ and $\text{Br}(K_L \to \pi^0 \nu \bar{\nu})$ departing from the Standard Model expectations by factors 2–3 cannot be excluded. Simultaneously, the extraction of the “true” angle $\beta$ from $K \to \pi \nu \bar{\nu}$ is not possible without additional information from other decays. Our conclusions differ in certain aspects from the ones reached in previous analyses. In particular, we stress the possible importance of left-right flavour-violating mass insertions that were not considered before.
1 Introduction

Among the CP asymmetries in $B$ decays, the CP asymmetry in $B_d \to \psi K_S$ \cite{1} is unique as it is the only one which in the Standard Model measures directly one angle of the unitarity triangle, the angle $\beta$, without any hadronic uncertainties.

On the other hand, it has been pointed out in \cite{2} that a similar role among the $K$ decays is played by $K_L \to \pi^0 \nu \bar{\nu}$ and $K^+ \to \pi^+ \nu \bar{\nu}$. Indeed, taken together, these two decays offer a very clean determination of $\sin 2\beta$ by measuring their branching ratios only \cite{2}:

$$\sin 2\beta = \frac{2r_s}{1 + r_s^2},$$  \hspace{1cm} (1)

where

$$r_s \equiv r_s(B_1, B_2) = \cot \beta = \frac{\sqrt{\sigma(B_1 - B_2) - P_c}}{\sqrt{B_2}}.$$  \hspace{1cm} (2)

Here $\sigma = 1/(1 - \lambda^2/2)^2$ with $\lambda = 0.22$,

$$B_1 = \frac{\text{Br}(K^+ \to \pi^+ \nu \bar{\nu})}{4.11 \cdot 10^{-11}}, \quad B_2 = \frac{\text{Br}(K_L \to \pi^0 \nu \bar{\nu})}{1.80 \cdot 10^{-10}},$$  \hspace{1cm} (3)

and $P_c = 0.40 \pm 0.06$ represents the internal charm contribution to the amplitude $A(K^+ \to \pi^+ \nu \bar{\nu})$ which is known including next-to-leading QCD corrections \cite{3}. The error in $P_c$ is dominated by the renormalization scale uncertainties and the value of $m_c$.

Using the well-known expression for $\sin 2\beta$ in terms of the time-integrated CP asymmetry $a_{\psi K_S}$ in $B_d \to \psi K_S$ decay, one finds an interesting connection between rare $K$ decays and $B$ physics \cite{2}:

$$\frac{2r_s(B_1, B_2)}{1 + r_s^2(B_1, B_2)} = -a_{\psi K_S} \frac{1 + x_d^2}{x_d},$$  \hspace{1cm} (4)

or equivalently

$$[\sin 2\beta]_{\pi^0 \nu \bar{\nu}} = [\sin 2\beta]_{\psi K_S},$$  \hspace{1cm} (5)

which must be satisfied in the Standard Model. Here $x_d$ measures the $B_d^0 - \bar{B}_d^0$ mixing. As stressed in \cite{2}, this “golden relation” involves, except for $P_c$, only directly measurable quantities. Due to very small theoretical uncertainties in (4), this relation is particularly suited for tests of CP violation in the Standard Model and offers a powerful tool to probe the physics beyond it. These points have been recently reemphasized in \cite{4, 5}.

The present status of the Standard Model predictions for $K \to \pi \nu \bar{\nu}$ can be summarized by \cite{3}:

$$\text{Br}(K^+ \to \pi^+ \nu \bar{\nu}) = (9.1 \pm 3.8) \cdot 10^{-11}, \quad \text{Br}(K_L \to \pi^0 \nu \bar{\nu}) = (2.8 \pm 1.7) \cdot 10^{-11},$$  \hspace{1cm} (6)
where the errors are dominated by the uncertainties in the CKM parameters. On the experimental side, the first event for $K^+ \to \pi^+ \nu \bar{\nu}$ has been recently observed by the BNL787 collaboration \[7\], giving

$$\text{Br}(K^+ \to \pi^+ \nu \bar{\nu}) = (4.2^{+0.7}_{-3.5}) \cdot 10^{-10},$$

in the ball park of the Standard Model expectations. The most recent upper bound on $\text{Br}(K_L \to \pi^0 \nu \bar{\nu})$ from FNAL-E799 is $1.8 \cdot 10^{-6}$. A new proposal at Brookhaven, AGS2000 \[8\], expects to reach the single event sensitivity $2 \cdot 10^{-12}$, allowing a 10% measurement of $\text{Br}(K_L \to \pi^0 \nu \bar{\nu})$. With a similar accuracy for $\text{Br}(K^+ \to \pi^+ \nu \bar{\nu})$ a measurement of $\sin 2\beta$ through (1) with an error of $\pm 0.05$ is possible \[2\]. This is comparable with the expected accuracy for $\sin 2\beta$ from $a_{\psi K_S}$ in the first years of the next decade at $B$ factories.

The purpose of this paper is to investigate how the relation (4) could be affected by new physics beyond the Standard Model. In particular we will consider a large class of supersymmetric models. This will also allow us to reach some general conclusions on supersymmetry effects in $K_L \to \pi^0 \nu \bar{\nu}$ and $K^+ \to \pi^+ \nu \bar{\nu}$.

Our paper is organized as follows. In section 2 we briefly recall the derivation of the formulae above in the manner useful for generalizations. In section 3 a model independent analysis of new-physics effects in $K \to \pi \nu \bar{\nu}$ decays is presented. In order to obtain some insight in the violation of the relation (4), this analysis is then combined with the model independent analysis of the unitarity triangle presented in \[9\]. In section 4, $K_L \to \pi^0 \nu \bar{\nu}$ and $K^+ \to \pi^+ \nu \bar{\nu}$ are analyzed in a large class of supersymmetric models. This analysis gives some insight in the violation of the relation (4) in these models. There we compare our results with a recent analysis of Nir and Worah \[10\]. Our conclusions differ in certain aspects from the ones reached by these authors. Section 5 summarizes the main results of our paper.

## 2 The Case of the Standard Model

It is instructive to recall first how (4) is derived. To this end let us consider the effective Hamiltonian for $K \to \pi \nu \bar{\nu}$:

$$H_{\text{eff}}^{\text{SM}} = \frac{G_F}{\sqrt{2}} \frac{2\alpha}{\pi \sin^2 \Theta_W} \sum_{i=e,\mu,\tau} \left( \lambda_i X_{NL}^i + \lambda_i X(x_i) \right) \bar{s}_L \gamma^\mu d_L \bar{\nu}_L \gamma_\mu \nu_L + \text{H.c.},$$

which originates in $Z$-penguin and box diagrams with internal charm ($X_{NL}$) and top quark ($X(x_i)$) exchanges. Here $\lambda_i = V_{is}^* V_{id}$ and $x_i = m_i^2/M_W^2$. The dependence on the charged lepton mass resulting from the relevant box-graph is negligible for the top contribution. In the charm sector this is the case only for the electron and the muon but not for the $\tau$-lepton.
Now $K_L \to \pi^0 \nu \bar{\nu}$ is purely CP violating in the Standard Model with CP violation proceeding dominantly in the decay amplitude $[11]$. Let us then introduce the quantity

$$F = \frac{1}{\lambda^5} [\lambda_e X_{NL} + \lambda_t X(x_t)],$$

where

$$X_{NL} = \frac{2}{3} X^e_{NL} + \frac{1}{3} X^\tau_{NL} \equiv \lambda^4 P_c,$$

and

$$X(x_t) = \eta_X \frac{x_t}{8} \left[ \frac{x_t + 2}{x_t - 1} + \frac{3x_t - 6}{(x_t - 1)^2} \ln x_t \right],$$

with $\eta_X = 0.985$ being the QCD correction calculated in $[3]$. Expressing the matrix elements $\langle \pi^0 | \bar{s}_L \gamma^\mu d_L | K_L \rangle$ and $\langle \pi^+ | \bar{s}_L \gamma^\mu d_L | K^+ \rangle$ through $\text{Br}(K^+ \to \pi^0 e^+ \nu)$ one finds:

$$\text{Br}(K^+ \to \pi^+ \nu \bar{\nu}) = \kappa_+ \cdot [(\text{Re}F)^2 + (\text{Im}F)^2]$$

and

$$\text{Br}(K_L \to \pi^0 \nu \bar{\nu}) = \kappa_L \cdot (\text{Im}F)^2.$$  

Here

$$\kappa_+ = r_{K^+} \frac{3\alpha^2 \text{Br}(K^+ \to \pi^0 e^+ \nu)}{2\pi^2 \sin^4 \Theta_W} \lambda^8 = 4.11 \cdot 10^{-11}$$

and

$$\kappa_L = \frac{r_{K_L} \tau(K_L)}{r_{K^+} \tau(K^+)^2} \kappa_+ = 1.80 \cdot 10^{-10},$$

where we have used

$$\alpha = \frac{1}{129} \quad \text{and} \quad \sin^2 \Theta_W = 0.23 \quad \text{and} \quad \text{Br}(K^+ \to \pi^0 e^+ \nu) = 4.82 \cdot 10^{-2}.$$  

Finally, $r_{K^+} = 0.901$ and $r_{K_L} = 0.944$ summarize isospin breaking corrections in relating $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$ to $K^+ \to \pi^0 e^+ \nu$ respectively $[12]$. The known potential ambiguity in the value for $\sin^2 \Theta_W$ can be reduced by including two-loop electroweak contributions to $K \to \pi \nu \bar{\nu}$ $[13]$. With $\sin^2 \Theta_W = 0.23$, these two-loop electroweak corrections are estimated to be negligible: at most $1 - 2\%$.

Given $\text{Br}(K^+ \to \pi^+ \nu \bar{\nu})$ and $\text{Br}(K_L \to \pi^0 \nu \bar{\nu})$ one finds:

$$\text{Re}F = -\varepsilon_1 \sqrt{B_1 - B_2}, \quad \text{Im}F = \varepsilon_2 \sqrt{B_2},$$

with the reduced branching ratios defined in $[3]$. $\varepsilon_i$ being equal to $\pm$ indicate the four-fold discrete ambiguity which must be resolved by using other decays.

In the Standard Model, the present knowledge of the CKM matrix implies $\varepsilon_1 = +$ and $\varepsilon_2 = +$. Furthermore $X_{NL}$ and $X(x_t)$ are real and the contribution proportional
to \( \text{Im} \lambda_c = - \text{Im} \lambda_t \) can be safely neglected in view of \( X_{NL}/X(x_t) \sim O(10^{-3}) \). Using the Wolfenstein parameterization \([14]\) we have
\[
\lambda_c = - \lambda, \quad \lambda_t = - A^2 \lambda^5 R_t e^{-i \beta}
\]
and consequently
\[
\text{Re} F = - [P_c + A^2 R_t X(x_t) \cos \beta]
\]
and
\[
\text{Im} F = A^2 R_t X(x_t) \sin \beta.
\]
Moreover one finds
\[
\eta = R_t \sin \beta = \frac{\sqrt{B_2}}{A^2 X(x_t)},
\]
where \( \eta \) is the height of the unitarity triangle and \( R_t \) is the length of one of its sides as shown in fig. 1. Consequently, the measurements of \( \text{Br}(K^+ \to \pi^+ \nu \bar{\nu}) \) and \( \text{Br}(K_L \to \pi^0 \nu \bar{\nu}) \) allow the construction of the unitarity triangle. Inserting \((19)\) and \((20)\) into \((17)\) one derives \((2)\) with \( \sigma = 1 \). The corrections \( O(\lambda^2) \) in \((2)\) follow from the improvement \([15]\) of the Wolfenstein parameterization. In order to simplify our presentation of the effects of new physics we will neglect these corrections. They can be included in a straightforward manner if necessary.

It should be stressed that \((12)\), \((13)\) and \((17)\) are rather general as they are valid in any extension of the Standard Model, provided

- the new physics contributions to the tree level decay \( K^+ \to \pi^0 e^+ \nu \) and
- the contributions of operators different from \( \bar{s}_L \gamma^\mu d_L \bar{\nu}_L \gamma_\mu \nu_L \)

can be neglected. Consequently, they are useful for generalizations of this discussion to models in which \( F \) includes additional contributions not present in the Standard Model.
The neglect of new-physics contributions to $K^+ \to \pi^0 e^+ \nu$ is very well justified in all known extensions of the Standard Model. On the other hand, as pointed out in [4], if lepton number is violated also operators $\bar{s}_L \gamma^\mu d_L \bar{\nu}_L \gamma_\mu \nu^L_L$ may contribute. In this case the final state $\pi^0 \nu \bar{\nu}$ is not necessarily a CP eigenstate and CP conserving contributions to $K_L \to \pi^0 \nu \bar{\nu}$ can in principle be substantial. In our opinion such a situation is rather unlikely and will not be considered in this paper.

3 A Model Independent Analysis

New physics can modify the effective Hamiltonian in (8) through new box diagram and penguin diagram contributions involving new particles such as charged Higgs, charginos, stops etc. In the case of $K_L \to \pi^0 \nu \bar{\nu}$ also new contributions to $K^0 - \bar{K}^0$ mixing should in principle be considered. However, the smallness of $\varepsilon_K = \mathcal{O}(10^{-3})$ implies that the $K^0 - \bar{K}^0$ mixing effects should be safely negligible in $K_L \to \pi^0 \nu \bar{\nu}$ independent of the model considered as long as $\text{Br}(K_L \to \pi^0 \nu \bar{\nu}) > \mathcal{O}(10^{-13})$. Since the Standard Model prediction amounts to $\text{Br}(K_L \to \pi^0 \nu \bar{\nu}) = (2.8 \pm 1.7) \cdot 10^{-11}$ [6] we expect that this inequality is satisfied also in a large class of its generalizations. Consequently the only new physics contributions relevant for $K_L \to \pi^0 \nu \bar{\nu}$ are the ones which modify the Hamiltonian in (8). Since this Hamiltonian governs both $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$, the discussion of new physics effects is considerably simplified compared to the case in which the $K^0 - \bar{K}^0$ mixing had to be considered. This situation is opposite to the one encountered in the CP asymmetry $a_{\psi K_S}$, where one expects the dominant new contributions from new phases in $B_d^0 - \bar{B}_d^0$ mixing with negligible contributions from new phases in the decay amplitude.

In order to simplify the analysis we will assume that:

- the unitarity of the three-generation CKM matrix is maintained;
- the new-physics contributions in semi-leptonic tree-level $B$ decays used to determine $|V_{cb}|$ or $A$ as well as $|V_{ub}/V_{cb}|$ can be neglected.

Then the new-physics contributions to $K \to \pi \nu \bar{\nu}$ can be described quite generally by two new parameters $r_K$ and $\theta_K$ defined by

$$X_{\text{new}} = r_K e^{-i\theta_K} X(x_t).$$

(22)

Here $X_{\text{new}}$ summarizes all contributions to $\mathcal{H}_{\text{eff}}$ except for the Standard Model contribution with internal charm quarks. In a given model $r_K$ and $\theta_K$ are generally complicated functions of masses of new particles and of new couplings. They can also carry additional $m_t$ dependence resulting from loop diagrams in which simultaneously the top quark and new particles are exchanged. Well-known examples of such a situation are box and Z-penguin diagrams with top and charged Higgs exchanges present in two Higgs doublet models and models based on supersymmetry.
In principle we could also modify the first term in (8). This, however, would unnecessarily complicate our discussion. Any new-physics effects on the term proportional to $\lambda_c$ can be included in the second term without the loss of generality. Yet one has to remember that $\lambda_c \neq \lambda_t$ and any new physics contribution proportional to $\lambda_c$ will not only modify $r_K$ but also give a contribution to $\theta_K$ even if this new contribution does not carry any new phase.

In this context, it should also be remarked that at least in the two Higgs doublet model the loop diagrams with charm and charged Higgs exchanges give negligible contribution to $K^+ \rightarrow \pi^+\nu\bar{\nu}$ because of very small charm-$H^\pm$ Yukawa couplings.

With $X(x_t)$ replaced by $X_{\text{new}}$ we have

$$\text{Re} F = -[P_e + A^2 R_t r_K X(x_t) \cos(\beta + \theta_K)] (23)$$

and

$$\text{Im} F = A^2 R_t r_K X(x_t) \sin(\beta + \theta_K), \quad (24)$$

which implies

$$\sin 2(\beta + \theta_K) = \frac{2r_s}{1 + r_s^2},$$

$$\text{Br}(K^+ \rightarrow \pi^+\nu\bar{\nu}) = \kappa_+ \cdot \left[ P_e^2 + A^4 R_t^2 r_K^2 X^2(x_t) + 2P_e A^2 R_t r_K X(x_t) \cos(\beta + \theta_K) \right] (26)$$

and

$$\text{Br}(K_L \rightarrow \pi^0\nu\bar{\nu}) = \kappa_L \cdot A^4 R_t^2 r_K^2 X^2(x_t) \sin^2(\beta + \theta_K), \quad (27)$$

where $r_s$ is given by

$$r_s = \frac{\varepsilon_1 \sqrt{B_1 - B_2} - P_e}{\varepsilon_2 \sqrt{B_2}}. \quad (28)$$

For $\varepsilon_1 = +$, $\varepsilon_2 = +$ and $\sigma = 1$, $r_s$ reduces to (2). Finally,

$$\varepsilon_2 \frac{\sqrt{B_2}}{A^2 X(x_t)} = R_t r_K \sin(\beta + \theta_K) \equiv \eta_t. \quad (29)$$

In order to simplify the discussion, we will assume that $\varepsilon_1 = +$ and $\varepsilon_2 = +$ as in the Standard Model.

We make the following observations:

- If the new-physics contributions are governed by the CKM matrix and no contributions proportional to $\lambda_c$ as well as no new complex couplings are present ($\theta_K = 0$) then, even with $r_K \neq 1$, $\sin 2\beta$ can be directly found by measuring $\text{Br}(K^+ \rightarrow \pi^+\nu\bar{\nu})$ and $\text{Br}(K_L \rightarrow \pi^0\nu\bar{\nu})$ only. It does not require the knowledge of $r_K$ which depends on unknown masses of new particles and new couplings. Examples of such a situation are the two Higgs doublet model and the simplest supersymmetric models such as the “constrained” Minimal Supersymmetric Standard Model (MSSM), to be specified in the following section.
Yet even in such situation the numerical value of $\beta$ extracted using eq. (25) may generally differ from the value one would find using the Standard Model analysis of the unitarity triangle, which is based on $B^0_{d,s} - \bar{B}^0_{d,s}$ mixings and $\varepsilon_K$. Indeed, new-physics contributions to the latter quantities may require a modified value of $\beta$, which as we have seen can be directly extracted from eq. (25) if $\theta_K = 0$.

In general, however, $\theta_K \neq 0$ and as seen in (25) $\text{Br}(K^+ \to \pi^+ \nu \bar{\nu})$ and $\text{Br}(K_L \to \pi^0 \nu \bar{\nu})$ allow to extract only $(\beta + \theta_K)$ instead of $\beta$ as in the case of the Standard Model. Once $(\beta + \theta_K)$ has been determined, one can use (26) or (29) to find the product $R_tr_K$.

We also observe that the ratio on the left-hand side of (24) no longer measures $\eta$ as in (21) but a new quantity $\eta_f$.

Thus in the presence of new physics the measurements of $\text{Br}(K^+ \to \pi^+ \nu \bar{\nu})$ and $\text{Br}(K_L \to \pi^0 \nu \bar{\nu})$ imply a “fake” unitarity triangle shown in fig. 2 in which $\beta$ and $\eta$ are replaced by $(\beta + \theta_K)$ and $\eta_f$ respectively. This implies that the side $(R_b)_f$ in this fake triangle will generally differ from $R_b$ defined as

$$R_b = \frac{1}{\lambda} \left| \frac{V_{ub}}{V_{cb}} \right|. \quad (30)$$

In order to extract $\beta$, $\theta_K$, $R_t$ and $r_K$ additional input is necessary.

Let us first remark that in order to see whether $\theta_K \neq 0$ and $r_K \neq 1$ one can simply check whether $(R_b)_f$ in fig. 2 agrees with $R_b$ in (20) which is expected to be unaffected by new-physics contributions. Similarly one can use the standard analysis of the unitarity triangle [3] which involves the Standard Model expressions for $\varepsilon_K$, $B^0_d - \bar{B}^0_d$ mixing and $|V_{ub}/V_{cb}|$. This analysis gives the values of $\beta$ and $R_t$, albeit with substantial uncertainties.

Figure 2: The “fake” unitarity triangle from $K \to \pi \nu \bar{\nu}$.
Inserting these values into (26) and (27) and comparing with the measured values of $\text{Br}(K^+ \rightarrow \pi^+ \nu \bar{\nu})$ and $\text{Br}(K_L \rightarrow \pi^0 \nu \bar{\nu})$, may give the first hint whether $r_K \neq 1$ and $\theta_K \neq 0$ are indeed required.

Now the new physics will generally contribute also to $\varepsilon_K$ and $B^0_d - \bar{B}^0_d$ mixing. Consequently in order to extract the values of $\theta_K$ and $r_K$ a more refined analysis is required. To this end one can use the model independent construction of the unitarity triangle of Grossman, Nir and Worah [9] which involves the CP asymmetries $a_{\psi K_S}$ and $a_{\pi \pi}$ in $B_d \rightarrow \psi K_S$ and $B_d \rightarrow \pi^+ \pi^-$ decays respectively. This analysis is based on the following plausible assumption:

- the $\bar{b} \rightarrow \bar{c}c\bar{s}$ and $\bar{b} \rightarrow \bar{u}u\bar{d}$ decays for $a_{\psi K_S}$ and $a_{\pi \pi}$ respectively are mediated by Standard Model tree level diagrams with negligible contributions from new physics in decay amplitudes. This implies that the only impact of new physics on $a_{\psi K_S}$ and $a_{\pi \pi}$ will come through $B^0_d - \bar{B}^0_d$ mixing.

The remaining three assumptions involving $K^0 - \bar{K}^0$ mixing, the dominance of the Standard Model contribution in the determination of $|V_{ub}/V_{cb}|$ and the unitarity of the three generation CKM matrix, made in [9], are the same as we have made in connection with $K \rightarrow \pi \nu \bar{\nu}$.

A weak point of the analysis of [9] is the use of the asymmetry $a_{\pi \pi}$ to extract the angle $\alpha$. It is well known that this extraction is affected by the “QCD penguin pollution”. The recent CLEO results for penguin dominated decays indicate that this pollution could be substantial [16]. The most popular strategy to deal with this “penguin problem” is the isospin analysis of Gronau and London [17]. It requires however the measurement of $\text{Br}(B^0_d \rightarrow \pi^0 \pi^0)$ which is expected to be below $10^{-6}$: a very difficult experimental task. For this reason several, rather involved, strategies [18] have been proposed which avoid the use of $B^0_d \rightarrow \pi^0 \pi^0$ in conjunction with the asymmetry $a_{\pi \pi}$. It is to be seen which of these methods will eventually allow us to measure $\alpha$ with a respectable precision. In what follows we will assume that all these problems will be overcome and $\alpha$ will be measured through $a_{\pi \pi}$ one day.

Under these circumstances the new-physics contributions in the analysis of [9] can be described analogously to (22) by two parameters $r_d$ and $\theta_d$, defined by

$$ (r_de^{i\theta_d})^2 \equiv \frac{\langle B^0_d | H^\text{full}_{\text{eff}}(\Delta B = 2) | \bar{B}^0_d \rangle}{\langle B^0_d | H^\text{SM}_{\text{eff}}(\Delta B = 2) | \bar{B}^0_d \rangle} $$

where $H^\text{full}_{\text{eff}}(\Delta B = 2)$ and $H^\text{SM}_{\text{eff}}(\Delta B = 2)$ denote the full Hamiltonian (including the new and Standard Model contributions) and the Standard Model Hamiltonian respectively.

With new-physics contributions to $B^0_d - \bar{B}^0_d$ mixing the asymmetry $a_{\psi K_S}$ does no longer measure $\sin 2\beta$ but $\sin 2(\beta + \theta_d)$ and the relation [3] is not satisfied if $\theta_K \neq \theta_d$:

$$ [\sin 2(\beta + \theta_K)]_{\pi \nu \bar{\nu}} \neq [\sin 2(\beta + \theta_d)]_{\psi K_S} $$
Figure 3: $\sin 2(\beta + \theta_K)$ as a function of $\text{Br}(K^+ \to \pi^+ \nu \bar{\nu})$ for $\text{Br}(K_L \to \pi^0 \nu \bar{\nu}) = 1 \cdot 10^{-11}$, $3 \cdot 10^{-11}$ and $1 \cdot 10^{-10}$ and two choices of $(\varepsilon_1, \varepsilon_2)$. The results for $(+, -)$ and $(-, +)$ are obtained by reversing the sign of $\sin 2(\beta + \theta_K)$ in these plots.

Since $\theta_K$ originates in new contributions to the decay amplitude $K \to \pi \nu \bar{\nu}$ and $\theta_d$ in new contributions to the $B^0_d - \bar{B}^0_d$ mixing, it is very likely that $\theta_K \neq \theta_d$.

Now as demonstrated in [9], the knowledge of $a_{\psi K_S}, a_{\pi \pi}$ and $R_b$ allows to extract in a model independent manner $\beta, R_t$ and $\theta_d$ and consequently the true unitarity triangle. In order to find $r_d$ also the $B^0_d - \bar{B}^0_d$ mixing parameter $x_d$ subject to hadronic uncertainties has to be considered. Inserting $R_t$ and $\beta$ into (25)–(29) allows then a model independent extraction of $\theta_K$ and $r_K$ from $K \to \pi \nu \bar{\nu}$. This is also evident from fig. 2. This in turn gives more insight in the violation of the relation (5) as anticipated in (3 2).

Having extracted the values of $r_d, \theta_d, r_K$ and $\theta_K$ in a model independent manner, one can then compare these values with explicit model calculations of these quantities. Such a comparison can either exclude certain models or put bounds on their parameters.

Clearly in a general case one would have to take care of discrete ambiguities represented by $\varepsilon_{1,2}$ in (28) and (29). Analogous discrete ambiguities are also present in the analysis of ref. [3]. In order to resolve these ambiguities, additional experimental information will be needed. In fig. 3 we plot $\sin 2(\beta + \theta_K)$ as a function of $\text{Br}(K^+ \to \pi^+ \nu \bar{\nu})$ for different values of $\text{Br}(K_L \to \pi^0 \nu \bar{\nu})$ and the choices $(+, +)$ and $(-, -)$ for $(\varepsilon_1, \varepsilon_2)$. The results for $(+, -)$ and $(-, +)$ can be obtained by reversing the sign of $\sin 2(\beta + \theta_K)$ in the left and right plot in fig. 3 respectively. The black square is in the ballpark of the central values expected in the Standard Model. We observe that even a modest accuracy for $\sin 2(\beta + \theta_K)$ should be sufficient to distinguish between various choices for $(\varepsilon_1, \varepsilon_2)$. 
4 $K \to \pi \nu \bar{\nu}$ and Supersymmetry

We will now study the effective Hamiltonian for $s \to d \nu \bar{\nu}$ transitions in the framework of a generalized SUSY extension of the Standard Model with minimal field content and unbroken R-parity.

Supersymmetric contributions to the process $K^+ \to \pi^+ \nu \bar{\nu}$ were already considered in refs. [19]-[23] and, very recently, in ref. [10], which also analyzed $K_L \to \pi^0 \nu \bar{\nu}$. With the exception of refs. [10, 23], the phenomenological analyses have been done prior to the top quark discovery and using much less stringent constraints on SUSY masses than those presently available. This makes the comparison of our numerical results with these papers very difficult. Moreover, most of the previous studies were performed in the framework of the so-called “constrained” MSSM, in which universality of the soft SUSY breaking terms is assumed. In this case, all SUSY contributions to FCNC processes are still proportional to the CKM mixing angles. Consequently, due to the heaviness of the superpartners, they are generally expected to be small in comparison with the Standard Model contributions. Furthermore, in the case of the “constrained” MSSM, the only non-negligible SUSY contribution is proportional to $\lambda$, and therefore no new phase $\theta_K$ is introduced in eq. (24). As already discussed in sec. 3, in this case it is still possible to extract $\sin 2\beta$ from $K \to \pi \nu \bar{\nu}$ decays, since eq. (1) remains unaffected.

However, it should be noted that the “constrained” MSSM is based on very strong assumptions which do not find at present a strong theoretical motivation. It is therefore useful to consider a more general definition of the MSSM, in which the assumptions about the universality of the soft SUSY breaking terms are relaxed. In this “unconstrained” version of the MSSM new sources of flavour and CP violation are present in the mass matrices of sfermions, and in general large contributions to FCNC and CP violating processes are expected in this case. To be able to study the contributions to $s \to d \nu \bar{\nu}$ transitions in such a generalized extension of the Standard Model, and to take properly into account the constraints coming from measured low-energy FCNC and CP-violating processes, one needs some kind of a model-independent parameterization of the flavour- and CP-violating quantities in SUSY. Such a parameterization has been formulated in the framework of the mass-insertion approximation [24], and will be discussed in detail below.

4.1 Computation of SUSY Contributions

There are several one-loop SUSY contributions to $s \to d \nu \bar{\nu}$ transitions, according to the kind of diagram (penguin or box) and to the virtual particles running in the loop: i) charged Higgses and up-type quarks (see fig. [4]), ii) charginos and up-type squarks (and charged sleptons in box diagrams, see fig. [5]), iii) gluinos and down-type squarks (see fig. [6], iv) neutralinos and down type squarks (and sneutrinos in box diagrams, see fig. [7]).
Figure 4: Feynman diagrams for the charged-Higgs contribution to $s \to d \nu \bar{\nu}$ transitions.
Figure 5: Feynman diagrams for the chargino contribution to $s \rightarrow d\nu \bar{\nu}$ transitions.
Figure 6: Feynman diagrams for the gluino contribution to $s \rightarrow d\nu\bar{\nu}$ transitions.

Figure 7: Feynman diagrams for the neutralino contribution to $s \rightarrow d\nu\bar{\nu}$ transitions.
First of all we note that, due to the left-handed chirality of the neutrinos and due to Lorenz invariance, the quark fields entering in dimension-six operators must have the same chirality. Therefore, there are only two possible dimension-six local operators for $s \rightarrow d\bar{\nu}\bar{\nu}$ transitions: $\bar{s}_L\gamma^\mu d_L\bar{\nu}_L\gamma^\mu\nu_L$ and $\bar{s}_R\gamma^\mu d_R\bar{\nu}_L\gamma^\mu\nu_L$.

Let us consider first gluino-mediated transitions. As already remarked in ref. [21], these contributions are negligible. First of all, there are no gluino-mediated box diagrams. Concerning the penguin diagrams, it is clear that if the $Z$ coupling to squarks were proportional to the photon one, the amplitude would be suppressed at least by a factor $q/M_Z$ with respect to the Standard Model one, since by gauge invariance there is no contribution to the effective $sd\gamma$ vertex of zeroth order in the external momenta (we denote by $q$ the momentum of the $Z$ boson). In fact, gauge invariance constrains the effective $sd\gamma$ vertex to be of the form $F_1\bar{s}_L(q^2\gamma^\mu - q^\mu \gamma_5)d_L + F_2\bar{s}_R\sigma_{\mu\nu}q^\nu d_L$. This is indeed the case if the squarks running in the loop have fixed “chirality”. To avoid the $q/M_Z$ suppression, one has to perform a double chirality flip in the down-type squark propagator, which is once again highly suppressed under the plausible assumption that the left-right mixing of squarks of the first two generations is negligible.$^1$

Let us consider next the operators that the contributions in figs. 4, 5 and 7 could generate in the effective theory. “RL” operators as $\bar{s}_R\sigma_{\mu\nu}q^\nu d_L\bar{\nu}_L\gamma^\mu\nu_L$ are suppressed by $q/M_Z$ and either by a quark mass or a chirality flip in the down-type squark propagator. “RR” operators as $\bar{s}_R\gamma^\mu d_R\bar{\nu}_L\gamma^\mu\nu_L$ can only be generated by a neutralino exchange through U(1) gauge couplings and turn out to be negligible. We are left with “LL” operators as $\bar{s}_L\gamma^\mu d_L\bar{\nu}_L\gamma^\mu\nu_L$. In a general supersymmetric extension of the Standard Model lepton flavour violation can be present due to a misalignment between lepton and slepton mass eigenstates. Therefore operator involving different neutrino flavours can be generated. However, the experimental limits on $\mu \rightarrow e\gamma$, $\tau \rightarrow \mu\gamma$ and other lepton flavour violating processes make their contribution to $K \rightarrow \pi\bar{\nu}\nu$ negligible. The operators we will consider are therefore the same as in the Standard Model. As in that case, we could have a dependence of the coefficient on the neutrino flavour involved, here due to a possible non-degeneracy of the sleptons running in the boxes. For our purposes, it is sufficient to use a “mean” value for the slepton masses of a given charge, without distinguishing among the three neutrinos, as done for the top contribution in the Standard Model.

Our effective Hamiltonian for $K \rightarrow \pi\bar{\nu}\nu$ is therefore given by eq. (8) with $X(x_t)$ replaced by

$$X_{\text{new}} = X(x_t) + X_H(x_{tH}) + C_\chi + C_N,$$

where $x_{tH} = m_t^2/m_{H^{\pm}}^2$ and the function $X_H(x_{tH})$ corresponding to the charged Higgs contribution is given in the Appendix. $C_\chi$ and $C_N$ denote respectively the chargino and neutralino contributions.

$^1$This is also the reason why the $\sin^2\Theta_W$ part of the $Z$ coupling does not contribute to the total amplitude.
While the flavour structure of the Standard Model and charged Higgs contributions involves just the CKM matrix, $C_N$ and $C_X$ depend on the transition matrices between quark and squark mass eigenstates, whose determination requires the diagonalization of the $6 \times 6$ squark mass matrices. In the so-called super-CKM basis, in which the flavour structure of the quark–squark–gaugino vertices is the same as in the quark–quark–gauge boson vertices, the squark mass matrices are given by:

$$
M_D^2 = \begin{pmatrix}
(m_D^2)_{LL} & (m_D^2)_{LR} \\
(m_D^2)_{LR}^\dagger & (m_D^2)_{RR}
\end{pmatrix},
M_U^2 = \begin{pmatrix}
(m_U^2)_{LL} & (m_U^2)_{LR} \\
(m_U^2)_{LR}^\dagger & (m_U^2)_{RR}
\end{pmatrix},
$$

where

\[
\begin{align*}
(m_U^2)_{LL} &= m_{\tilde u_L} + (m_u)^2 + \frac{M_Z^2}{6} \left( 3 - 4 \sin^2 \Theta_W \right) \cos 2 \beta \mathbf{1} \\
(m_U^2)_{LR} &= -\mu m_u \cot \beta - \frac{\tan \beta}{\sqrt{2}} A_u \\
(m_U^2)_{RR} &= m_{\tilde u_R} + (m_u)^2 + \frac{2}{3} M_Z^2 \sin^2 \Theta_W \cos 2 \beta \mathbf{1} \\
(m_D^2)_{LL} &= m_{\tilde d_L} + (m_d)^2 - \frac{M_Z^2}{6} \left( 3 - 2 \sin^2 \Theta_W \right) \cos 2 \beta \mathbf{1} \\
(m_D^2)_{LR} &= -\mu m_d \tan \beta - \frac{\tan \beta}{\sqrt{2}} A_d \\
(m_D^2)_{RR} &= m_{\tilde d_R} + (m_d)^2 - \frac{1}{3} M_Z^2 \sin^2 \Theta_W \cos 2 \beta \mathbf{1}.
\end{align*}
\]

We denote by $m_{\tilde u_L}$, $m_{\tilde u_R}$, $m_{\tilde d_L}$, and $m_{\tilde d_R}$ the soft-breaking masses for squarks and by $A_u$ and $A_d$ the soft-breaking trilinear couplings for up and down squarks in this basis, at the electroweak scale. The quark mass matrices are diagonal in this basis:

\[
\begin{align*}
m_u &= \text{diag} \left( m_u, m_c, m_t \right), \\
m_d &= \text{diag} \left( m_d, m_s, m_b \right).
\end{align*}
\]

In order to obtain simple analytic formulae in a generic model, it is useful not to perform the exact diagonalization of the squark mass matrices, but instead use the mass-insertion approximation. Since we are interested in $s \rightarrow d$ transitions, we switch to a basis in which the $d_L^i - \tilde d_L^i - N_n$ and $d_L^i - \tilde u_L^i - \chi_n$ couplings are flavour diagonal, and the flavour change in the left-handed sector is exhibited by the non-diagonality of the sfermion propagators. Denoting by $\Delta$ the off-diagonal terms in the sfermion mass matrices, the sfermion propagators can be expanded as a series in terms of $\delta = \Delta/\tilde m^2$, where $\tilde m$ is the average sfermion mass. As long as $\Delta$ is significantly smaller than $\tilde m^2$, we can just take the first term of this expansion and compute any given process in terms of these $\delta$’s.\footnote{Care must be taken in the truncation of the expansion at the first order in those models where there is a strong hierarchy between different mass insertions. For example, in the constrained MSSM, in the notations of eq. (34), $(m_D^2)_{d_L s_L}/\tilde m^2 \sim (m_D^2)_{d_L b_L} (m_D^2)_{b_L s_L}/\tilde m^4$.} This
is equivalent to a first order diagonalization of the squark mass matrices around their diagonal part.

The basis we have chosen is obtained starting from the super-CKM basis by rotating the left-handed up-type squark fields:

$$\tilde{u}_L \rightarrow V^\dagger \tilde{u}_L,$$

(37)

where $V$ is the CKM matrix. In our basis, the up-squark mass matrix is given in terms of eqs. (34) and (35) by

$$M^2_U = \left( \begin{array}{cc} (m^2_U)_{dLdL} & (m^2_U)_{dLuR} \\ (m^2_U)_{uRdL} & (m^2_U)_{uRuR} \end{array} \right),$$

(38)

with the notation

$$(m^2_U)_{dLdL} = V^\dagger (m^2_U)_{dLdL} V$$

$$(m^2_U)_{dLuR} = V^\dagger (m^2_U)_{dLuR}$$

$$(m^2_U)_{uRdL} = (m^2_U)_{uRdL} V^\dagger (m^2_U)_{uRdL}$$

$$(m^2_U)_{uRuR} = (m^2_U)_{uRuR}.$$

(39)

In the above formulae, $u$ and $d$ represent the three-generation space: $u$ stands for $u, c, t$ and analogously for $d$.

In principle one should diagonalize exactly the up-squark mass matrix in the $\tilde{t}_L - \tilde{t}_R$ sector before performing the mass-insertion expansion. However, this would make the equations more cumbersome without adding any new feature in the results, and therefore we will stick to the basis above. It is a straightforward exercise to include this exact diagonalization in our formulae.

The chargino ($C_\chi$) and neutralino ($C_N$) contributions are given by

$$C_\chi = X^0_\chi + X^{LL} R^U_{sLdL} + X^{LR} R^U_{sLtR} + X^{LR^*} R^U_{tRdL},$$

$$C_N = X_N R^D_{sLdL},$$

(40)

where the functions $X^i_\chi$ and $X_N$, which depend on SUSY masses and respectively on chargino and neutralino mixing angles, are given in the Appendix. The $R$ parameters are defined in terms of the mass insertions in the following way:

$$R^D_{sLdL} = \frac{(m^2_D)_{sLdL}}{V^*_{ts} V_{td} m^2_d}$$

$$R^U_{sLdL} = \frac{(m^2_U)_{sLdL}}{V^*_{ts} V_{td} m^2_d}$$

$$R^U_{sLtR} = \frac{(m^2_U)_{sLtR}}{V^*_{ts} m^2_{aL} m_t}$$

$$R^U_{tRdL} = \frac{(m^2_U)_{tRdL}}{V_{td} m_t m^2_{aL}},$$

(41)

where $m^2_d$ and $m^2_{aL}$ are respectively the squared masses of the down- and up-type squarks of the first two generations.

Eq. (40) shows explicitly the dependence of the various contributions on the off-diagonal entries of the squark mass matrices, that are completely unknown apart from
upper limits on their real and imaginary parts, and are strongly model dependent. The dependence on masses and other supersymmetric parameters is on the other hand contained in the $X$ functions, whose order of magnitude is rather model independent and can be read from a scatter plot for random values of the relevant SUSY parameters (see below).

The model dependence is therefore in this way almost completely contained in the $R$ parameters. We have chosen the normalization of the $R$’s in such a manner that their absolute value in the constrained MSSM is expected to be of order one or smaller. Indeed, a rough estimate gives

$$\left( R^D_{sLdL} \right)_{\text{MSSM}} \sim \left( R^U_{sLdL} \right)_{\text{MSSM}} \sim -\frac{3}{(4\pi)^2} Y_t^2 \log \frac{M_0^2}{M_Z^2}$$

(42)

and

$$\left( R^U_{sLR} \right)_{\text{MSSM}} \sim \left( R^U_{tRdL} \right)_{\text{MSSM}} \sim -\frac{A_t + \mu \cot \beta}{m_{\tilde{u}_L}},$$

(43)

where $Y_t$ is the top Yukawa coupling, $Y_t A_t$ is the (3,3) element of the trilinear scalar coupling in the up sector and $M_0$ is the universality scale. For $M_0 = \mathcal{O}(M_{\text{GUT}})$, $\frac{3}{(4\pi)^2} \log \frac{M_0^2}{M_Z^2} Y_t^2 = \mathcal{O}(1)$ and therefore $\left( R^D_{sLdL} \right)_{\text{MSSM}} \sim \left( R^U_{sLdL} \right)_{\text{MSSM}} \sim 1$. Experimental limits on the $R$ parameters are discussed below and summarized in table 1.

### 4.2 Phenomenological analysis

We are now ready to write a formula for $r_K e^{-i\theta_K}$:

$$r_K e^{-i\theta_K} = 1 + \frac{X_H}{X} + \frac{X^0}{X} + \frac{X^{LL}}{X} R^U_{sLdL} + \frac{X^{LR}}{X} R^U_{sLR} + \frac{X^{LR^*}}{X} R^U_{tRdL} + \frac{X_N}{X} R^D_{sLdL}$$

(44)
Table 2: Ranges for the SUSY parameters used in the scatter plots. Too light charginos are rejected.

| quantity | range |
|----------|-------|
| $\tan \beta$ | $2 \leftrightarrow 6$ |
| $\mu$ | $(-300 \leftrightarrow 300)$ GeV |
| $M_2$ | $(100 \leftrightarrow 300)$ GeV |
| $m_{\tilde{t}_R}$ | $(150 \leftrightarrow 300)$ GeV |
| $m_{\tilde{q}_L}$ | $(200 \leftrightarrow 500)$ GeV |
| $m_{\tilde{L}_L}$ | $(100 \leftrightarrow 300)$ GeV |

where $X \equiv X(x_t)$. The amount of SUSY corrections to $K \rightarrow \pi \nu \bar{\nu}$ branching ratios and $\sin 2\beta$ depends on how much $r_K e^{-i\theta_K}$ deviates from unity. This depends on course on the size of the mass insertions, that is largely unknown.

### 4.2.1 Results in the MSSM

In figures 8, 9, 10 and 11 the ratios $X^{\chi}_i/X$ and $X_N/X$ are reported as scatter plots for random values of SUSY parameters in the ranges given in table 4. The ratio $X_H/X$ is given separately in fig. 12 as a function of $m_H$ for $\tan \beta = 2$ (it scales as $1/\tan^2 \beta$).

Whereas figs. 8 and 12 are general in the sense that their contributions to $r_K e^{-i\theta_K}$ do not depend on the values of mass insertions, the other ones have to be multiplied by the corresponding $R$ parameter to get the contribution to $r_K e^{-i\theta_K}$. On the other hand, an upper limit for the contribution of the “constrained” MSSM, and of any other SUSY extension of the SM in which all the $R$ parameters are $O(1)$, can be directly read from these figures. Therefore in this quite large class of models the neutralino contributions are negligible. The only possible non-negligible contributions come from the charged Higgs exchange, in the “corner” of parameter space in which the Higgs mass is near its lower limit and $\tan \beta$ is very close to one, and from the term $X^{0}_\chi$ in chargino exchange. It is interesting to note that these two contributions have opposite signs and therefore tend to cancel each other, making the effects very small. This cancellation is particularly effective if one takes into account the constraints coming from $b \rightarrow s\gamma$, which imply a correlation between the mass of charginos and charged Higgses. We are therefore able to conclude that in this class of SUSY models in which the $R$ parameters are $O(1)$, the effects on $r_K$ are certainly smaller than 10%. Moreover, if the $R$ parameters are real, as it is the case in the “constrained” MSSM, we have $\theta_K = 0$.

At this point, we would like to comment on the smallness of the chargino and neutralino contributions, assuming the absence of anomalously large mass insertions. In the
limit of large squark masses, the box diagrams are suppressed by $M_Z^2/m_{	ilde{q}}^2$ relative to penguin diagrams because the $1/M_Z^2$ coming from the $Z$ propagator in the penguin diagrams is replaced by a $1/m_{	ilde{q}}^2$ coming from the additional superparticle propagator. One could then think that the penguin diagrams might still give contributions not suppressed in that limit. Actually this is not the case, once again because of gauge invariance, which requires the presence of at least one gaugino-higgsino flip in the chargino/neutralino line and an overall $M_Z^2/m_{	ilde{q}}^2$ suppression as in the box-diagram case.

\subsection{4.2.2 Results in a general SUSY extension of the Standard Model}

We now turn to the discussion of the effects which can be obtained in a general SUSY extension of the SM. In this case, we do not assume any particular model, but instead let the $R$ parameters vary within the range allowed by the available experimental constraints.

Our numerical analysis uses the constraints on off-diagonal mass insertions found in ref. \cite{25}. To obtain these constraints, it has been imposed that the gluino-mediated contributions to $K^0 - \bar{K}^0$, $D^0 - \bar{D}^0$ and $B_d^0 - \bar{B}_d^0$ mixing, as well as to the decay $b \rightarrow s\gamma$, proportional to each single off-diagonal mass insertion, do not exceed the experimental value. Barring accidental cancellations, this analysis gives the largest possible value for the $R$ parameters and consequently for the quantities as $\theta_K$ or $r_K$ considered in this paper. On the other hand, in order to simplify our analysis, we have kept the CKM parameters at their central values (see table \ref{tab:ckm}). In principle, also these parameters should be varied in the allowed ranges and this variation should be correlated with the variation of the
Figure 9: The ratio $|X^LL_X|/X$ as a function of $|\mu|$.

Figure 10: The ratio $|X^LR_X|/X$ as a function of $|\mu|$.
Figure 11: The ratio $|X_N|/X$ as a function of $|\mu|$.

Figure 12: The ratio $X_H/X$ as a function of $m_H$ (GeV), for $\tan\beta = 2$ (it scales as $1/\tan^2\beta$).
| quantity | value                      |
|----------|---------------------------|
| $V_{td}$ | $0.0076 - 0.0029 i$       |
| $V_{ts}$ | $-0.039 - 0.0006 i$       |

Table 3: Values of the relevant CKM matrix elements used in the numerical analysis.

$R$ parameters in such a manner that the experimental data on all the quantities listed above are reproduced. Such a very involved numerical analysis is beyond the scope of this paper. Consequently, the numerical results presented for this general SUSY extension of the Standard Model should be considered only as order-of-magnitude estimates.

We now discuss in more detail the constraints on the off-diagonal mass insertions which are relevant in our case. From eqs. (30) and (31), we see that the mass insertions involved in our expressions are $(m_D^2)_{s_Ld_L}$, $(m_{\ell}^2)_{s_Ld_L}$, $(m_{\ell}^2)_{s_Lt_R}$ and $(m_{\ell}^2)_{t_Rd_L}$.

Let us start with $(m_D^2)_{s_Ld_L}$. From ref. [24] we immediately get, for degenerate squarks and gluinos,

$$
\text{Re} \left( \frac{m_D^2}{m_{d_L}^2} \right) < 0.04 \frac{m_{\tilde{d}_L}}{500 \text{ GeV}},
$$

$$
\text{Im} \left( \frac{m_D^2}{m_{d_L}^2} \right) < 0.003 \frac{m_{\tilde{d}_L}}{500 \text{ GeV}},
$$

which implies the upper limit on $R_{s_Ld_L}^D$ reported in table [1].

We now turn to the constraint on $(m_{\ell}^2)_{s_Ld_L}$. First of all, following ref. [26], we note that SU(2) invariance of the soft-breaking scalar mass matrices implies the following relation:

$$
m_{d_L}^2 = V^\dagger m_{u_L}^2 V.
$$

On the other hand, we see from eqs. (38) and (39) that

$$
(m_{\ell}^2)_{s_Ld_L} = \left[ V^\dagger (m_{\ell}^2)_{LL} V \right]_{21} = \left[ V^\dagger \left( m_{s_L}^2 + (m_{d_L}^2)^2 + \frac{M_2^2}{6} \left( 3 - 4 \sin^2 \Theta_W \cos 2\beta \right) \right) V \right]_{21}
$$

$$
= \left[ m_{d_L}^2 + V^\dagger (m_{d_L}^2)^2 V \right]_{21},
$$

where we have used eqs. (35) and (46). Therefore, we see that, using eq. (45), we can set the following constraint:

$$
\text{Re} \left( \frac{m_{\ell}^2}{m_{u_L}^2} \right) < 0.04 \frac{m_{\tilde{u}_L}}{500 \text{ GeV}} + \frac{\text{Re} V_{td}^* m_t^2 V_{ld}}{m_{u_L}^2},
$$

$$
\text{Im} \left( \frac{m_{\ell}^2}{m_{u_L}^2} \right) < 0.003 \frac{m_{\tilde{u}_L}}{500 \text{ GeV}} + \frac{\text{Im} V_{td}^* m_t^2 V_{ld}}{m_{u_L}^2}.
$$

23
This point was overlooked in ref. [10], in which the authors did not use eq. (41) to set a constraint on \((m_U^2)_{sL,dL}\). Instead, they expanded the product \(V^\dagger (m_U^2)_{LL} V\) and used the available constraints on \((m_U^2)_{LL}\). This yields a weaker constraint on \((m_U^2)_{sL,dL}\), roughly a factor of four larger than the one we obtain. We will return to this point below.

Concerning the LR and RL mass insertions, things are much more involved. In this case, we cannot use the SU(2) symmetry to relate the mass terms in the up sector to the ones in the down sector, and the limits in the up sector obtained studying gluino-mediated processes are available only for \((m_U^2)_{c_{A,R}}\), with \(A, B = L, R\). On the other hand, we need to know \((m_U^2)_{t_{R,dL}}\) and \((m_U^2)_{sL,t_R}\). These mass matrix elements could be constrained by studying chargino contributions to \(K^0 - \bar{K}^0\) mixing, \(B_d - \bar{B}_d\) mixing and \(b \to s \gamma\) in the mass-insertion approximation. Unfortunately, there is no such an analysis available in the literature, except for \(b \to s \gamma\) [29], and it would go beyond the scope of this paper to discuss these constraints in detail. However, an order-of-magnitude estimate of the limits on these matrix elements can be easily obtained in the following way.

Let us consider \(K^0 - \bar{K}^0\) mixing. The gluino contribution to the coefficient of the \(s_L \bar{s}_L d_L \gamma\mu s_L d_L\) operator can be written as

\[
\frac{\alpha_s^2}{M_g^2} C_{\tilde{g}} \times \left( \frac{(m_D^2)_{sL,dL}}{m^2} \right)^2,
\]

where \(C_{\tilde{g}}\) is a dimensionless function of the average squark and gluino masses, while the chargino contribution includes the following terms:

\[
\frac{\alpha_W^2}{M_X^2} C_{\chi} \left[ \frac{(m_U^2)_{sL,t_R} Y_V t_d + V_{tR}^* Y_t (m_U^2)_{t_{R,dL}}}{m^2} \right]^2.
\]

Here \(C_{\chi}\) is again a dimensionless function of the average squark and chargino masses and of the chargino mixing matrices. We now assume that \(\alpha_s^2/M_g^2 C_{\tilde{g}} \sim \alpha_W^2/M_X^2 C_{\chi}\), which is appropriate to get an order-of-magnitude estimate of the constraints coming from chargino exchange. Comparing the chargino and gluino contributions and using the constraint reported in ref. [25], we obtain, barring accidental cancellations and interference effects,

\[
\frac{(m_U^2)_{sL,t_R} Y_V t_d}{m^2} \sim \frac{V_{ts}^* Y_t (m_U^2)_{t_{R,dL}}}{m^2} \sim \frac{(m_D^2)_{sL,dL}}{m^2} < 0.04\frac{\bar{m}}{500\text{GeV}}.
\]

The above equation shows that, due to the presence of the CKM matrix elements in the relevant couplings, essentially no constraint can be derived on the \((m_U^2)_{t_{R,dL}}\) and \((m_U^2)_{t_{R,sL}}\) entries from \(K^0 - \bar{K}^0\) mixing. Analogously, from \(B_d - \bar{B}_d\) mixing one gets

\[
\frac{(m_U^2)_{bL,t_R} Y_V t_d}{m^2} \sim \frac{V_{tb}^* Y_t (m_U^2)_{t_{R,dL}}}{m^2} \sim \frac{(m_D^2)_{bL,dL}}{m^2} < 0.1\frac{\bar{m}}{500\text{GeV}}.
\]

Therefore, we can set the following approximate limit:

\[
\left| \frac{(m_D^2)_{t_{R,dL}}}{m^2} \right| < 0.1\frac{\bar{m}}{500\text{GeV}}.
\]
Constraints coming from chargino exchange in $b \to s\gamma$ have been studied in ref. [26]. They report the following constraint:

$$\left| \frac{(m_U^2)_{tR,sL}}{\tilde{m}^2} \right| < 3 \left( \frac{\tilde{m}{}_{500 \text{ GeV}}}{} \right)^2.$$  \hfill (54)

Constraints on the relevant left-right mass insertions can be also obtained from requiring the absence in the potential of charge and colour breaking minima or unbounded from below directions. From ref. [27] we get:

$$\left| (m_U^2)_{tR,sL} \right|, \left| (m_U^2)_{tR,dL} \right| < m_t \sqrt{2\tilde{m}^2 + m_2^2}, \hfill (55)$$

where $m_2$ is the coefficient of the $|H_2|^2$ term in the superpotential. Taking for example $m_2 \simeq \tilde{m}$, one sees that the FCNC constraint on $(m_U^2)_{tR,dL}$ in eq. (53) is always stronger than the one in eq. (54) in the range ofsquark masses we consider. On the other hand, the constraint on $(m_U^2)_{tR,sL}$ in eq. (55) becomes tighter than the FCNC one in eq. (54) for squark masses larger than about 300 GeV.

The maximum values for the neutralino and chargino contributions consistent with present constraints can be estimated multiplying the scatter plots in figs. 8-11 by the upper limits on the relevant $R$ parameters reported in table 1.

To provide an order-of-magnitude estimate of the possible size of SUSY contributions to $r_K$ and $\theta_K$ in this general extension of the Standard Model, we report in the scatter plots in figs. 13 and 14 the sum of chargino and neutralino contributions obtained by varying the $R$ parameters between zero and the upper limits listed in table 1. The remaining SUSY parameters are varied in the ranges reported in table 2.

We observe that typically

$$0.5 < r_K < 1.3 \quad \text{and} \quad -25^0 < \theta_K < +25^0,$$ \hfill (56)

although values outside these ranges, even if less probable, cannot be excluded. Inspecting eqs. (25)-(27) we find that departures from the Standard Model expectations for the branching ratios in (6) by factors 2–3 are certainly possible. Furthermore, in view of substantial values of $\theta_K$, vanishing or even negative values of $\sin 2(\beta + \theta_K)$ cannot be excluded. We recall that the standard analysis of the unitarity triangle gives roughly $\beta \sim 20^0 \pm 8^0$.

Finally we would like to compare our results with those of ref. [10]. In that paper, $K \to \pi\nu\bar{\nu}$ decays have been considered in various specific supersymmetric models based on exact universality, alignment, etc. It has been found that in these special models SUSY effects were rather small and the extraction of $\beta$ from $K \to \pi\nu\bar{\nu}$ in the presence of

\footnote{We thank Y. Nir and M. Worah for pointing out to us these constraints, which were overlooked in the first version of this work.}
SUSY was possible. On the other hand, the authors of ref. [10] stressed that in generic supersymmetric models large SUSY effects are possible. We agree with this statement, yet we disagree with the manner this conclusion has been reached. Nir and Worah attribute these possibly large effects to flavour violation coming from left-left mass insertions. As we have demonstrated in [18], a constraint coming from gluino exchange in $K^0\rightarrow \bar{K}^0$ mixing, not considered in [10], excludes such large effects from these insertions. On the other hand we find that flavour violation in left-right mass insertions, which was not considered in ref. [10], can give large contributions to $K\rightarrow \pi\nu\bar{\nu}$, even when available constraints are taken into account. Another question is related to the contribution of box diagrams with chargino exchanges which have not been considered in ref. [10]. In our opinion the neglect of these contributions to $X^{\mu}_{\chi}$ cannot be justified.

4.3 SUSY effects in $B \rightarrow \psi K_s$ decays

We have seen in Sec. 4.2 that, while in the “constrained” MSSM we still expect to be able to measure $\sin 2\beta$ from $K \rightarrow \pi\nu\bar{\nu}$ decays with no additional uncertainty with respect to the Standard Model case, in a general SUSY model large contributions to $\theta_K$ are possible and therefore the extraction of the true angle $\beta$ from $K \rightarrow \pi\nu\bar{\nu}$ becomes impossible. Now we would like to briefly comment on SUSY effects in the measurement of $\sin 2\beta$ from CP asymmetries in $B \rightarrow \psi K_s$ decays. To this end we use the parameterization (31).

Let us consider first the case of the “constrained” MSSM. In this model, no new
Figure 14: The value of $r_K$ obtained by varying the $R$ parameters between zero and the upper limits given in table 3 as a function of $|\mu|$ (see the text for details).

phase is present ($\theta_d = 0$) and therefore the CP asymmetry $a_{\psi K}$ still measures $\sin 2\beta$. Consequently, in the “constrained” MSSM the relations (4) and (5) are expected to hold. However, as we already explained in Section 3, the true value of $\sin 2\beta$ that one can obtain from $a_{\psi K_s}$ and $K \to \pi \nu \bar{\nu}$ will generally differ from the one obtained using the Standard Model analysis of the unitarity triangle, which is based on $B^{0}_{d,s} - \bar{B}^{0}_{d,s}$ mixings and $\varepsilon_K$ [26, 28].

If we consider instead a general SUSY model, then large contributions to $B^{0}_{d} - \bar{B}^{0}_{d}$ mixing and, as we have seen, to $K \to \pi \nu \bar{\nu}$ decays are possible [4, 23]. Moreover, the flavour-changing mass insertions that enter in these processes are to a large extent uncorrelated. This means that large violations of the relations (4) and (5) are possible in this case.

5 Summary and Conclusions

In this paper we have presented a model independent analysis of new-physics contributions to the very clean rare decays $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$. We have illustrated this analysis by considering a large class of supersymmetric models. In particular, we have investigated whether the relation (4) between $K \to \pi \nu \bar{\nu}$ decays and the CP asymmetry $a_{\psi K_s}$ proposed in [4] could be violated in these models.

The model independent analysis of $K \to \pi \nu \bar{\nu}$ decays can be formulated in terms of
two parameters $r_K$ and $\theta_K$ in analogy to the parameters $r_d$ and $\theta_d$ introduced previously in the literature in connection with $B_d^0 - \bar{B}_d^0$ mixing. We have demonstrated in Section 3 how the parameters $r_K$ and $\theta_K$ can be extracted from future data on $\text{Br}(K^+ \to \pi^+ \nu\bar{\nu})$ and $\text{Br}(K_L \to \pi^0 \nu\bar{\nu})$. To this end the model independent analysis of the unitarity triangle of ref. [4] has to be simultaneously invoked.

Analyzing $K \to \pi\nu\bar{\nu}$ in a large class of supersymmetric models by means of the mass-insertion approximation of ref. [24] we arrive at the following conclusions:

- In the “constrained” MSSM, $\theta_d = \theta_K = 0$, implying that the relation (4) is well satisfied in this supersymmetric scenario. However, the value of $\sin 2\beta$ extracted from $K \to \pi\nu\bar{\nu}$ and $a_{\psi K_S}$ might differ from the one obtained using the Standard Model analysis of the unitarity triangle.

- On the other hand, we have demonstrated explicitly that in a more general class of supersymmetric models large deviations from the Standard Model values $\theta_K = 0$ and $r_K = 1$ are possible. Typically $0.5 < r_K < 1.3$ and $-25^0 < \theta_K < +25^0$ implying that substantial violations of the relation (4) and a departure from the Standard Model expectations for $\text{Br}(K^+ \to \pi^+ \nu\bar{\nu})$ and $\text{Br}(K_L \to \pi^0 \nu\bar{\nu})$ by factors 2–3 are certainly possible. Simultaneously, the extraction of the CKM parameters, in particular of the angle $\beta$, from $K \to \pi\nu\bar{\nu}$ alone is no longer possible and a more complicated analysis involving other decays is necessary, as described in Section 3.

Moreover, the authors of ref. [10] state that flavour violation in the mass matrices of left-handed up-type squarks could generate contributions to $K \to \pi\nu\bar{\nu}$ as large as the Standard Model ones. However, as we already noticed in Section 4.2.2, there is a constraint coming from gluino exchange in $K^0 - \bar{K}^0$ mixing which was not considered in ref. [10]; taking this constraint into account, the left-left contribution turns out to be small, about 10% of the Standard Model one. Furthermore, the contribution to $\theta_K$ is even smaller. This can be immediately seen multiplying the results in figure 9 by the upper limit for $R_{sLdL}$ reported in table 1.

On the other hand, we find that flavour violation in the left-right mass matrices for up-type squarks, which was neglected in ref. [10], can give contributions to $K \to \pi\nu\bar{\nu}$ that are much larger, even when the available constraints are taken into account. Thus, large SUSY effects in $r_K$ and especially in $\theta_K$ are mostly due to left-right flavour-changing mass insertions, not to left-left ones.

Independently of what kind of new physics is realized in nature, it is clear that the theoretically clean $K \to \pi\nu\bar{\nu}$ decays in conjunction with the very clean CP asymmetry $a_{\psi K_S}$ have great potentiality to discover new physics and to show us what this new physics is. In particular, a substantial violation of the golden relation (4) would not only require
a generalization of the Standard Model but would also tell us that it cannot be the “constrained” MSSM.

Acknowledgments

We wish to thank M. Misiak for carefully reading the manuscript. This work has been supported by the German Bundesministerium für Bildung and Forschung under contract 06 TM 874 and DFG Project Li 519/2-2. L.S. acknowledges the partial support of Fondazione Angelo della Riccia, Firenze, Italy. The work of A.R. was partially supported by the TMR Network “Physics beyond the Standard Model” under EEC contract No. ERBFMRX-CT960090.

Appendix

We collect here the expressions for various functions which appeared in Sect. 4. The formulas in terms of mass insertions that are shown below are obtained from the general ones by using the following observation. Consider a $n \times n$ Hermitian matrix $A = A^0 + A^1$ with $A^0 = \text{diag}(a^0_1, \ldots, a^0_n)$. If the unitary matrix $U$ diagonalizes $A$ by $A = U^\dagger \text{diag}(a^0_1, \ldots, a^0_n)U$ and $f$ is an arbitrary function, at the first order in $A^1$ we have

$$U^\dagger_{ik} f(a_k) U_{kj} = \delta_{ij} f(a^0_i) + A^1_{ij} F(a^0_i, a^0_j),$$

(57)

where

$$F(x,y) = \frac{f(x) - f(y)}{x - y}.$$

(58)

If the matrix $A$ has degenerate eigenvalues, the limit for $x \to y$ must be performed in eq. (58) and one obtains the derivative of $f(x)$. In our case $A$ is the up or down squark mass matrix, $A^0$ is its diagonal part and $f$ is the appropriate loop function. We are performing an expansion at the lowest order in $A^1$. Note that in this way no $O(1)$ uncertainties associated with the use of a “mean mass” for squarks of different generations are present. Therefore, we can take into account the effects of a light stop.

We consider the first two generations to be almost degenerate in each squark sector. Each term in eqs. (B3) and (H10) has a contribution from penguin and box diagrams, except for $X_H$ where the box diagram contribution can be safely neglected.

$$X_H(x) = -\frac{m_t^2}{4M_W^2 \tan^2 \beta} x \left( -\frac{\log x}{(x-1)^2} + \frac{1}{x-1} \right)$$

$$(X_\chi^0)_{\text{Pen}} = -\frac{m_t^2}{8 \sin^2 \beta M_W^2} \left[ k \left( \frac{M^2_{\chi_n}}{m^2_{\tilde{t}_R}}, \frac{M^2_{\chi_m}}{m^2_{\tilde{t}_R}} \right) H_{h^\pm n}^T H^*_{nW}, H_{W^+ m}^T H^*_{m h^\pm} \right]$$
We are using the following expressions for the chargino and neutralino mass matrices:

\[
M_X = \begin{pmatrix}
\frac{M_2}{\sqrt{2}M_W \sin \beta} & \frac{\sqrt{2}M_W \sin \beta}{\mu} \\
\frac{\sqrt{2}M_W \cos \beta}{\mu} & \frac{M_2}{\mu}
\end{pmatrix},
\]

(60)

\[
M_N = \begin{pmatrix}
M_1 & 0 & -M_Z \sin \Theta_W \cos \beta & M_Z \sin \Theta_W \sin \beta \\
0 & M_2 & M_Z \cos \Theta_W \cos \beta & -M_Z \cos \Theta_W \sin \beta \\
-M_Z \sin \Theta_W \cos \beta & M_Z \cos \Theta_W \cos \beta & 0 & -\mu \\
M_Z \sin \Theta_W \sin \beta & -M_Z \cos \Theta_W \sin \beta & -\mu & 0
\end{pmatrix}.
\]

In the chargino mass matrix rows 1 and 2 correspond respectively to the \( W^- \) and \( h_d^- \) current eigenstates, and columns 1 and 2 correspond respectively to the \( W^+ \) and \( h_u^+ \)
current eigenstates. In the neutralino mass matrix rows and columns 1, 2, 3 and 4 correspond respectively to the $B$, $W_3$, $h_d^0$ and $h_u^0$ current eigenstates.

The mass eigenstates are obtained by diagonalizing the mass matrices, using

$$M_\chi = K^T M_\chi^D H, \quad M_N = U^T M_N^D U,$$

(61)

with $H$, $K$, $U$ unitary matrices and $M_\chi^D = \text{diag}(M_\chi_{n=1,2}, M_\chi_{n>0})$ and $M_N^D = \text{diag}(M_N_{n=1,...,4}, M_N_{n>0})$.

In the neutralino sector we used the notation

$$U_n(x) \equiv t_3(x)U_{nW_3} + \tan \Theta_W y(x)U_{nB},$$

(62)

where $x$ is a generic particle, $t_3(x)$ its third isospin component, $y(x)$ is its hypercharge, and $U$ is the neutralino mixing matrix defined in eq. (61).

Finally, the loop functions are:

$$j(x) = \frac{x \log x}{x - 1} \quad j(x, y) = \frac{j(x) - j(y)}{x - y} \quad j(x, y, z) = \frac{j(x, z) - j(y, z)}{x - y}$$

$$j(x, y, z, t) = \frac{j(x, z, t) - j(y, z, t)}{x - y}$$

$$k(x) = \frac{x^2 \log x}{x - 1} \quad k(x, y) = \frac{k(x) - k(y)}{x - y} \quad k(x, y, z) = \frac{k(x, z) - k(y, z)}{x - y}$$

$$k(x, y, z, t) = \frac{k(x, z, t) - k(y, z, t)}{x - y}. \quad (63)$$

References

[1] I.I.Y. Bigi and A.I. Sanda, Nucl. Phys. B193 (1981) 85;

[2] G. Buchalla and A.J. Buras, Phys. Lett. B333 (1994) 221; Phys. Rev. D54 (1996) 6782.

[3] G. Buchalla and A.J. Buras, Nucl. Phys. B398 (1993) 285; B400 (1993) 225; B412 (1994) 106.

[4] Y. Grossman and Y. Nir, Phys. Lett. B398 (1997) 163;

[5] Y. Grossman, Y. Nir and R. Rattazzi, [hep-ph/9701231] to appear in the review volume "Heavy Flavours II", eds. A.J. Buras and M. Lindner (World Scientific, Singapore); Y. Nir, [hep-ph/9709301].

[6] A.J. Buras and R. Fleischer, [hep-ph/9704376] to appear in the review volume "Heavy Flavours II", eds. A.J. Buras and M. Lindner (World Scientific, Singapore); A.J. Buras, [hep-ph/9711217].

31
[7] S. Adler et al., hep-ex/9708031.

[8] L. Littenberg and J. Sandweiss, eds., AGS2000, Experiments for the 21st Century, BNL 52512.

[9] Y. Grossman, Y. Nir and M.P. Worah, Phys. Lett. B407 (1997) 307.

[10] Y. Nir and M.P. Worah, hep-ph/9711215.

[11] L.S. Littenberg, Phys. Rev. D39 (1989) 3322.

[12] W. Marciano and Z. Parsa, Phys. Rev. D53 (1996) 1.

[13] G. Buchalla and A.J. Buras, hep-ph/9707243.

[14] L. Wolfenstein, Phys. Rev. Lett. 51 (1983) 1945.

[15] A.J. Buras, M.E. Lautenbacher and G. Ostermaier, Phys. Rev. D50 (1994) 3433.

[16] M. Ciuchini, E. Franco, G. Martinelli and L. Silvestrini, Nucl. Phys. B501 (1997) 271.

[17] M. Gronau and D. London, Phys. Rev. Lett. 65 (1990) 3381.

[18] A. Snyder and H.R. Quinn, Phys. Rev. D48 (1993) 2139; A.J. Buras and R. Fleischer, Phys. Lett. B360 (1995) 138; J.P. Silva and L. Wolfenstein, Phys. Rev. D49 (1995) 1151; A.S. Dighe, M. Gronau and J. Rosner, Phys. Rev. D54 (1996) 3309; R. Fleischer and T. Mannel, Phys. Lett. B397 (1997) 269; C.S. Kim, D. London and T. Yoshikawa, hep-ph/9708350.

[19] S. Bertolini and A. Masiero, Phys. Lett. B174 (1986) 343.

[20] G. Giudice, Z. Phys. C 34 (1987) 57.

[21] S. Bertolini, F. Borzumati, A. Masiero and G. Ridolfi, Nucl. Phys. B353 (1991) 591.

[22] I. Bigi and F. Gabbiani, Nucl. Phys. B367 (1991) 3.

[23] G. Couture and H. König, Z. Phys. C 69 (1995) 167.

[24] L.J. Hall, V.A. Kostelecky and S. Raby, Nucl. Phys. B267 (1986) 415.

[25] F. Gabbiani, E. Gabrielli, A. Masiero and L. Silvestrini, Nucl. Phys. B477 (1996) 321.

[26] M. Misiak, S. Pokorski and J. Rosiek, Preprint IFT 3/97, hep-ph/9703442, to appear in the review volume "Heavy Flavours II", eds. A.J. Buras and M. Lindner (World Scientific, Singapore).
[27] J.A. Casas and S. Dimopoulos, Phys. Lett. B387 (1996) 107.

[28] G.C. Branco, G.C. Cho, Y. Kizukuri and N. Oshimo, Phys. Lett. B337 (1994) 316; Nucl. Phys. B449 (1995) 483; G.C. Branco, W. Grimus and L. Lavoura, Phys. Lett. B380 (1996) 119; A. Brignole, F. Feruglio and F. Zwirner, Z. Phys. C 71 (1996) 679.