The Defect Functor of a Homomorphism and Direct Unions

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Abstract We will study commuting properties of the defect functor $\text{Def}_\beta = \text{CokerHom}_\mathcal{C}(\beta, -)$ associate to a homomorphism $\beta$ in a finitely presented category. As an application, we characterize objects $M$ such that $\text{Ext}^1_\mathcal{C}(M, -)$ commutes with direct unions (i.e. direct limits of monomorphisms), assuming that $\mathcal{C}$ has a generator which is a direct sum of finitely presented projective objects.

Keywords Defect functor · Coherent functor · Direct limit · Direct union · Ext-functor

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1 Introduction

Commuting properties of some canonical functors defined on some categories play important roles in the study of various mathematical objects. For instance, finitely presented objects in a category with directed colimits are defined by the condition that the induced

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covariant Hom-functor commutes with all directed colimits. In the case of module categories the equivalence between the property used in this definition and the classical notion of finitely presented module was proved by Lenzing in [26]. In that paper it is also proved that there are strong connections between commuting properties of covariant Hom-functors and commuting properties of tensor product functors with respect to direct products. These connections were extended to the associated derived functors in [9] and [12]. Moreover, Drinfeld proposed in [15] to use flat Mittag-Leffler modules in order to construct a theory for infinite dimensional vector bundles. Recent progresses in this directions were obtained in [8], [17] and [18]. Auslander introduced in [6] the class of coherent functors, and W. Crawley-Boevey characterized (in the case of module categories) these functors as those covariant functors which commute with direct limits and direct products, [14, Lemma 1]. This result was extended to locally finitely presented categories by H. Krause, [24, Chapter 9]. The influence of these functors is presented in [14] and [20].

Brown [12] and Strebel [31] used commuting properties of covariant Ext\(_{C}\)-functors with respect to direct limits in order to characterize groups of type (FP). In module theory an important ingredient used in the study of tilting classes (e.g. [19, Lemma 5.2.18 and Theorem 5.2.20]) is a homological characterization, [19, Theorem 4.5.6], of the closure \( \lim \rightarrow C \), where \( C \) is a class of \( FP_2 \)-modules. This is based on the fact that Ext\(_{C}\)(\( M, - \)) commutes with direct limits whenever \( M \) is an \( FP_2 \)-module, [19, Lemma 3.1.6]. In the case of Abelian groups, commuting properties of Ext\(_1\) functors with respect to particular direct limits were also studied in [4] and [30].

In this paper we will focus on commuting properties with respect to direct limits for the defect functor associated to a homomorphism in a locally finitely presented abelian category. Let us introduce basic notions which we will use in the sequel. Let \( M \) be an object in an additive category \( C \) with directed colimits, and \( G : C \rightarrow Ab \) a covariant functor. Furthermore suppose that \( \mathfrak{F} = (M_i, v_{ij})_{i, j \in I} \) is a directed system of objects in \( C \) such that there exists \( \lim \rightarrow M_i \) and let \( v_i : M_i \rightarrow \lim \rightarrow M_i \) be the canonical homomorphisms. Then \( (G(M_i), G(v_{ij})) \) is also a direct system, and we denote by \( \lim \rightarrow G(M_i) \) its direct limit. Moreover, we have a canonical homomorphism

\[
\Gamma_{\mathfrak{F}} : \lim \rightarrow G(M_i) \rightarrow G(\lim \rightarrow M_i)
\]

induced by the homomorphisms \( G(v_i) : G(M_i) \rightarrow G(\lim \rightarrow M_i), i \in I \).

We say that \( G \) commutes with \( \mathfrak{F} \) if \( \Gamma_{\mathfrak{F}} \) is an isomorphism. The functor \( G \) commutes with direct limits (direct unions, resp. direct sums) if the homomorphisms \( \Gamma_{\mathfrak{F}} \) are isomorphisms for all directed systems \( \mathfrak{F} \) (such that all \( v_{ij} \) are monomorphisms, resp. all direct sums).

Let \( C \) be an additive category with direct limits. We recall from [1] and [2] that an object \( M \) is finitely presented (finitely generated) respectively if and only if \( \text{Hom}_C(M, -) \) commutes with direct limits (of monomorphisms), i.e. the canonical homomorphisms

\[
\Psi^M_{\mathfrak{F}} : \lim \rightarrow \text{Hom}_C(M, M_i) \rightarrow \text{Hom}_C(M, \lim \rightarrow M_i)
\]

are isomorphisms for all direct systems \( \mathfrak{F} = (M_i, v_{ij}) \) (such that all \( v_{ij} \) are monomorphisms). The category \( C \) is finitely accessible if \( C \) has directed colimits and every object is a direct limit of finitely presented objects. A cocomplete finitely accessible category \( C \) is a locally finitely presented category.

The notion of defect functor associated to a homomorphism extends the defect functor of an exact sequence used in [7]. This functor represents generalizations for the following canonical functors: the Hom-covariant functor induced by an object, the Pext-covariant