ON THE RELATIVE TWIST FORMULA OF ℓ-ADIC SHEAVES

ENLIN YANG AND YIGENG ZHAO

ABSTRACT. We propose a conjecture on the relative twist formula of ℓ-adic sheaves, which can be viewed as a generalization of Kato-Saito’s conjecture. We verify this conjecture under some transversal assumptions.

We also define a relative cohomological characteristic class and prove that its formation is compatible with proper push-forward. A conjectural relation is also given between the relative twist formula and the relative cohomological characteristic class.

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1. INTRODUCTION

As an analogy of the theory of D-modules, Beilinson [Bei16] and T. Saito [Sai17a] define the singular support and the characteristic cycle of an ℓ-adic sheaf on a smooth variety respectively.

As an application of their theory, we prove a twist formula of epsilon factors in [UYZ], which is a modification of a conjecture due to Kato and T. Saito [KS08, Conjecture 4.3.11].

1.1. Kato-Saito’s conjecture.

1.1.1. Let X be a smooth projective scheme purely of dimension d over a finite field k of characteristic p. Let Λ be a finite field of characteristic ℓ ≠ p or Λ = ℂℓ. Let $F \in D_c^b(X, \Lambda)$ and $\chi(X, F)$ be the Euler-Poincaré characteristic of F. The Grothendieck L-function $\mathcal{L}(X, F, t)$ satisfies the following functional equation

\[ \mathcal{L}(X, F, t) = \varepsilon(X, F) \cdot t^{-\chi(X, F)} \cdot \mathcal{L}(X, D(F), t^{-1}), \]

where $D(F)$ is the Verdier dual $R\text{Hom}(F, Rf^!\Lambda)$ of F, $f : X \to \text{Spec} k$ is the structure morphism and

\[ \varepsilon(X, F) = \det(-\text{Frob}_k; Rf^!(X, F))^{-1} \]

is the epsilon factor (the constant term of the functional equation (1.1.1.1)) and Frob_k is the geometric Frobenius (the inverse of the Frobenius substitution).
1.1.1. In (1.1.1.1), both $\chi(X_k, F)$ and $\varepsilon(X, F)$ are related to ramification theory. Let $cc_{X/k}(F) = 0_X(CC(F, X/k)) \in CH_0(X)$ be the characteristic class of $F$ (cf. [Sai17a, Definition 5.7]), where $0_X : X \to T^*X$ is the zero section and $CC(F, X/k)$ is the characteristic cycle of $F$. Then $\chi(X_k, F) = \deg(\sigma_{cc_{X/k}}(F))$ by the index formula [Sai17a, Theorem 7.13]. The following theorem proved in [UYZ] gives a relation between $\varepsilon(X, F)$ and $cc_{X/k}(F)$, which is a modified version of the formula conjectured by Kato and T. Saito in [KS08, Conjecture 4.3.11].

**Theorem 1.1.3 (Twist formula, [UYZ, Theorem 1.5]).** We have

$$\varepsilon(X, F \otimes G) = \varepsilon(X, F)^{rank G} \cdot \det G(\rho_X(-cc_{X/k}(F)))$$

in $\Lambda^\times$,

where $\rho_X : CH_0(X) \to \pi_1^{ab}(X)$ is the reciprocity map defined by sending the class $[s]$ of a closed point $s \in X$ to the geometric Frobenius $\Frob_s$ and $\det G : \pi_1^{ab}(X) \to \Lambda^\times$ is the representation associated to the smooth sheaf $det G$ of rank $1$.

When $F$ is the constant sheaf $\Lambda$, this is proved by S. Saito [SS84]. If $F$ is a smooth sheaf on an open dense subscheme $U$ of $X$ such that $F$ is tamely ramified along $D = X \setminus U$, then Theorem 1.1.3 is a consequence of [Sai93, Theorem 1]. In [Vi09a, Vi09b], Vidal proves a similar result on a proper smooth surface over a finite field of characteristic $p > 2$ under certain technical assumptions. Our proof of Theorem 1.1.3 is based on the following theories: one is the theory of singular support [Bei16] and characteristic cycle [Sai17a], and another is Laumon’s product formula [Lau87].

1.2. $\varepsilon$-factorization.

1.2.1. Now we assume that $X$ is a smooth projective geometrically connected curve of genus $g$ over a finite field $k$ of characteristic $p$. Let $\omega$ be a non-zero rational 1-form on $X$ and $F$ an $\ell$-adic sheaf on $X$. The following formula is conjectured by Deligne and proved by Laumon [Lau87, 3.2.1.1]:

$$\varepsilon(X, F) = p^{[k:F_p][1-g]rank(F)} \prod_{v \in |X|} \varepsilon_v(F|_{X_v}, \omega).$$

For higher dimensional smooth scheme $X$ over $k$, it is still an open question whether there is an $\varepsilon$-factorization formula (respectively a geometric $\varepsilon$-factorization formula) for $\varepsilon(X, F)$ (respectively $\det Rf^*(F, X)$).

1.2.2. In [Bei07], Beilinson develops the theory of topological epsilon factors using $K$-theory spectrum and he asks whether his construction admits a motivic ($\ell$-adic or de Rham) counterpart. For de Rham cohomology, such a construction is given by Patel in [Pat12]. Based on [Pat12], Abe and Patel prove a similar twist formula in [AP17] for global de Rham epsilon factors in the classical setting of $D_X$-modules on smooth projective varieties over a field of characteristic zero. In the $\ell$-adic situation, such a geometric $\varepsilon$-factorization formula is still open even if $X$ is a curve. Since the classical local $\varepsilon$-factors depend on an additive character of the base field, a satisfied geometric $\varepsilon$-factorization theory will lie in an appropriate gerbe rather than be a super graded line (cf. [Bei07, Pat12]).

1.2.3. More generally, we could also ask similar questions in a relative situation. Now let $f : X \to S$ be a proper morphism between smooth schemes over $k$. Let $F$ be an $\ell$-adic sheaf on $X$ such that $f$ is universally locally acyclic relatively to $F$. Under these assumptions, we know that $Rf_*F$ is locally constant on $S$. Now we can ask if there is an analogue geometric $\varepsilon$-factorization for the determinant $\det Rf_*F$. This problem is far beyond the authors’ reach at this moment. But, similar to (1.1.3.1), we may consider twist formulas for $\det Rf_*F$. One of the purposes of this paper is to formulate such a twist formula and prove it under certain assumptions.
1.2.4. **Relative twist formula.** Let $S$ be a regular Noetherian scheme over $\mathbb{Z}[1/\ell]$ and $f: X \to S$ a proper smooth morphism purely of relative dimension $n$. Let $\mathcal{F} \in D_c^b(X, \Lambda)$ such that $f$ is universally locally acyclic relatively to $\mathcal{F}$. Then we conjecture that (see Conjecture 2.1.4) there exists a unique cycle class $cc_{X/S}(\mathcal{F}) \in CH^p(X)$ such that for any locally constant and constructible sheaf $\mathcal{G}$ of $\Lambda$-modules on $X$, we have an isomorphism of smooth sheaves of rank 1 on $S$

$$\det Rf_*(\mathcal{F} \otimes \mathcal{G}) \cong (\det Rf_*)^\otimes \det \mathcal{G}(cc_{X/S}(\mathcal{F}))$$

where $\det \mathcal{G}(cc_{X/S}(\mathcal{F}))$ is a smooth sheaf of rank 1 on $S$ (see 2.1.3 for the definition). We call (1.2.4.1) the relative twist formula. As an evidence, we prove a special case of the above conjecture in Theorem 2.4.4. It is also interesting to consider a similar relative twist formula for de Rham epsilon factors in the sense of [AP17]. We will pursue this question elsewhere.

1.2.5. If $S$ is moreover a smooth connected scheme of dimension $r$ over a perfect field $k$, we construct a candidate for $cc_{X/S}(\mathcal{F})$ in Definition 2.4.3. We also relate the relative characteristic class $cc_{X/S}(\mathcal{F})$ to the total characteristic class of $\mathcal{F}$. Let $K_0(X, \Lambda)$ be the Grothendieck group of $D_c^b(X, \Lambda)$. In [Sai17a, Definition 6.7.2], T. Saito defines the following morphism

$$cc_{X, \bullet}: K_0(X, \Lambda) \to CH_\bullet(X) = \bigoplus_{i=0}^{r+n} CH_i(X),$$

which sends $\mathcal{F} \in D_c^b(X, \Lambda)$ to the total characteristic class $cc_{X, \bullet}(\mathcal{F})$ of $\mathcal{F}$. Under the assumption that $f: X \to S$ is $SS(\mathcal{F}, X/k)$-transversal, we show that $(-1)^r \cdot cc_{X/S}(\mathcal{F}) = cc_{X, r}(\mathcal{F})$ in Proposition 2.5.2.

1.2.6. Following Grothendieck [SGA5], it’s natural to ask whether the following diagram

$$\begin{array}{ccc}
K_0(X, \Lambda) & \xrightarrow{cc_{X, \bullet}} & CH_\bullet(X) \\
\downarrow f_* & & \downarrow f_* \\
K_0(Y, \Lambda) & \xrightarrow{cc_{Y, \bullet}} & CH_\bullet(Y)
\end{array}$$

is commutative or not for any proper map $f: X \to Y$ between smooth schemes over $k$. If $k = \mathbb{C}$, the diagram (1.2.6.1) is commutative by [Gin86, Theorem A6]. By the philosophy of Grothendieck, the answer is no in general if char $(k) > 0$ (cf. [Sai17a, Example 6.10]). If $k$ is a finite field and if $f: X \to Y$ is moreover projective, as a corollary of Theorem 1.1.3, we prove in [UYZ, Corollary 5.26] that the degree zero part of (1.2.6.1) commutes. In general, motivated by the conjectural formula (1.2.4.1), we propose the following question. Let $f: X \to S$ and $g: Y \to S$ be smooth morphisms. Let $D_c^b(X/S, \Lambda)$ be the thick subcategory of $D_c^b(X, \Lambda)$ such that $f$ is $SS(\mathcal{F}, X/k)$-transversal. Let $K_0(X/S, \Lambda)$ be the Grothendieck group of $D_c^b(X/S, \Lambda)$. Then for any proper morphism $h: X \to Y$ over $S$, we conjecture that the following diagram commutes (see Conjecture 2.5.4)

$$\begin{array}{ccc}
K_0(X/S, \Lambda) & \xrightarrow{cc_{X, r}} & CH_r(X) \\
\downarrow h_* & & \downarrow h_* \\
K_0(Y/S, \Lambda) & \xrightarrow{cc_{Y, r}} & CH_r(Y)
\end{array}$$

1.2.7. As an evidence for (1.2.6.2), we construct a relative cohomological characteristic class

$$ccc_{X/S}(\mathcal{F}) \in H^{2n}(X, \Lambda(n))$$

in Definition 3.2.4 if $X \to S$ is smooth and $SS(\mathcal{F}, X/k)$-transversal. We prove that the formation of $ccc_{X/S}(\mathcal{F})$ is compatible with proper push-forward (see Corollary 3.3.4 for a precise
statement). Similar to [Sai17a, Conjecture 6.8.1], we conjecture that we have the following equality (see Conjecture 3.2.6)

\[(1.2.7.2) \quad \text{cl}(cc_{X/S}(\mathcal{F})) = ccc_{X/S}(\mathcal{F}) \quad \text{in} \quad H^{2n}(X, \Lambda(n))\]

where \(\text{cl}: \text{CH}^n(X) \to H^{2n}(X, \Lambda(n))\) is the cycle class map.

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**Notation and Conventions.**

1. Let \(p\) be a prime number and \(\Lambda\) be a finite field of characteristic \(\ell \neq p\) or \(\Lambda = \overline{\mathbb{Q}}_\ell\).
2. We say that a complex \(\mathcal{F}\) of étale sheaves of \(\Lambda\)-modules on a scheme \(X\) over \(\mathbb{Z}[1/\ell]\) is constructible (respectively smooth) if the cohomology sheaf \(\mathcal{H}^q(\mathcal{F})\) is constructible for every \(q\) and if \(\mathcal{H}^q(\mathcal{F}) = 0\) except finitely many \(q\) (respectively moreover \(\mathcal{H}^q(\mathcal{F})\) is locally constant for all \(q\)).
3. For a scheme \(S\) over \(\mathbb{Z}[1/\ell]\), let \(D^b_c(S, \Lambda)\) be the triangulated category of bounded complexes of \(\Lambda\)-modules with constructible cohomology groups on \(S\) and let \(K_0(S, \Lambda)\) be the Grothendieck group of \(D^b_c(S, \Lambda)\).
4. For a scheme \(X\), we denote by \(|X|\) the set of closed points of \(X\).
5. For any smooth morphism \(X \to S\), we denote by \(T^*_X(X/S) \subseteq T^*(X/S)\) the zero section of the relative cotangent bundle \(T^*(X/S)\) of \(X\) over \(S\). If \(S\) is the spectrum of a field, we simply denote \(T^*(X/S)\) by \(T^*X\).

2. **Relative twist formula**

2.1. **Reciprocity map.**

2.1.1. For a smooth proper variety \(X\) purely of dimension \(n\) over a finite field \(k\) of characteristic \(p\), the reciprocity map \(\rho_X: \text{CH}^n(X) \to \pi_1^{ab}(X)\) is given by sending the class \([s]\) of closed point \(s \in X\) to the geometric Frobenius \(\text{Frob}_s\) at \(s\). The map \(\rho_X\) is injective with dense image [KS83].

2.1.2. Let \(S\) be a regular Noetherian scheme over \(\mathbb{Z}[1/\ell]\) and \(X\) a smooth proper scheme purely of relative dimension \(n\) over \(S\). By [Sai94, Proposition 1], there exists a unique way to attach a pairing

\[(2.1.2.1) \quad \text{CH}^n(X) \times \pi_1^{ab}(S) \to \pi_1^{ab}(X)\]

satisfying the following two conditions:

1. When \(S = \text{Spec} k\) is a point, for a closed point \(x \in X\), the pairing with the class \([x]\) coincides with the inseparable degree times the Galois transfer \(\text{tran}_{k(x)/k}\) (cf. [Tat79, I]) followed by \(i_x^*\) for \(i_x: x \to X\)

\[
\text{Gal}(k^{ab}/k) \xrightarrow{\text{tran}_{k(x)/k}\times [k(x): k]} \text{Gal}(k(x)^{ab}/k(x)) \xrightarrow{i_x^*} \pi_1^{ab}(X).
\]

2. For any point \(s \in S\), the following diagram commutes

\[
\begin{array}{ccc}
\text{CH}^n(X) & \times & \pi_1^{ab}(S) \\
\downarrow & & \downarrow \\
\text{CH}^n(X_S) & \times & \pi_1^{ab}(s)
\end{array}
\]

\[
\begin{array}{ccc}
\pi_1^{ab}(X) & \to & \pi_1^{ab}(X) \\
\downarrow & & \downarrow \\
\pi_1^{ab}(X_S) & \to & \pi_1^{ab}(X_S)
\end{array}
\]
2.1.3. For any locally constant and constructible sheaf $G$ of $\Lambda$-modules on $X$ and any $z \in \text{CH}^n(X)$, we have a map

$$\pi_1^{ab}(S) \xrightarrow{(z, \bullet)} \pi_1^{ab}(X) \xrightarrow{\det G} \Lambda^\times$$

where $(z, \bullet)$ is the map determined by the paring (2.1.2.1) and $\det G$ is the representation associated to the locally constant sheaf $G$ of rank 1. The composition $\det G \circ (z, \bullet) : \pi_1^{ab}(S) \to \Lambda^\times$ determines a locally constant and constructible sheaf of rank 1 on $S$, which we simply denote by $\det G(z)$. Now we propose the following conjecture.

**Conjecture 2.1.4** (Relative twist formula). Let $S$ be a regular Noetherian scheme over $\mathbb{Z}[1/\ell]$ and $f : X \to S$ a smooth proper morphism purely of relative dimension $n$. Let $F \in D^b_c(X, \Lambda)$ such that $f$ is universally locally acyclic relatively to $F$. Then there exists a unique cycle class $cc_{X/S}(F) \in \text{CH}^n(X)$ such that for any locally constant and constructible sheaf $G$ of $\Lambda$-modules on $X$, we have an isomorphism

$$\det Rf_* (F \otimes G) \cong (\det Rf_* F)^{\otimes \text{rank} G} \otimes \det G(cc_{X/S}(F))$$

in $K_0(S, \Lambda)$, where $K_0(S, \Lambda)$ is the Grothendieck group of $D^b_c(S, \Lambda)$.

We call this cycle class $cc_{X/S}(F) \in \text{CH}^n(X)$ the relative characteristic class of $F$ if it exists. If $S$ is a smooth scheme over a perfect field $k$, we construct a candidate for $cc_{X/S}(F)$ in Definition 2.4.3.

As an evidence, we prove a special case of the above conjecture in Theorem 2.4.4. In order to construct a cycle class $cc_{X/S}(F)$ satisfying (2.1.4.1), we use the theory of singular support and characteristic cycle.

2.2. Transversal condition and singular support.

2.2.1. Let $f : X \to S$ be a smooth morphism of Noetherian schemes over $\mathbb{Z}[1/\ell]$. We denote by $T^*(X/S)$ the vector bundle $\text{Spec} \text{Sym}_{\mathcal{O}_X}(\Omega^1_{X/S})$ on $X$ and call it the relative cotangent bundle on $X$ with respect to $S$. We denote by $T_X^*(X/S) = X$ the zero-section of $T^*(X/S)$. A constructible subset $C$ of $T^*(X/S)$ is called conical if $C$ is invariant under the canonical $\mathbb{G}_m$-action on $T^*(X/S)$.

**Definition 2.2.2** ([Bei16, §1.2] and [HY17, §2]). Let $f : X \to S$ be a smooth morphism of Noetherian schemes over $\mathbb{Z}[1/\ell]$ and $C$ a closed conical subset of $T^*(X/S)$. Let $Y$ be a Noetherian scheme smooth over $S$ and $h : Y \to X$ an $S$-morphism.

(1) We say that $h : Y \to X$ is $C$-transversal relatively to $S$ at a geometric point $\bar{y} \to Y$ if for every non-zero vector $\mu \in C_{h(\bar{y})} = C \times_X \bar{y}$, the image $dh_{\bar{y}}(\mu) \in T^*_h(Y/S) := T^*(Y/S) \times_Y \bar{y}$ is not zero, where $dh_{\bar{y}} : T^*_{h(\bar{y})}(X/S) \to T^*_h(Y/S)$ is the canonical map. We say that $h : Y \to X$ is $C$-transversal relatively to $S$ if it is $C$-transversal relatively to $S$ at every geometric point of $Y$. If $h : Y \to X$ is $C$-transversal relatively to $S$, we put $h^C = dh(C \times_X Y)$ where $dh : T^*(X/S) \times_X Y \to T^*(Y/S)$ is the canonical map induced by $h$. By the same argument of [Bei16, Lemma 1.1], $h^C$ is a conical closed subset of $T^*(Y/S)$.

(2) Let $Z$ be a Noetherian scheme smooth over $S$ and $g : X \to Z$ an $S$-morphism. We say that $g : X \to Z$ is $C$-transversal relatively to $S$ at a geometric point $\bar{x} \to X$ if for every non-zero vector $\nu \in T^*_{g(\bar{x})}(Z/S)$, we have $dg_{\bar{x}}(\nu) \notin C_{\bar{x}}$, where $dg_{\bar{x}} : T^*_{g(\bar{x})}(Z/S) \to T^*_x(X/S)$ is the canonical map. We say that $g : X \to Z$ is $C$-transversal relatively to $S$ if it is $C$-transversal relatively to $S$ at all geometric points of $X$. If the base $B(C) := C \cap T^*_x(X/S)$ of $C$ is proper over $Z$, we put $g_* C := \text{pr}_1(dg^{-1}(C))$, where $\text{pr}_1 : T^*(Z/S) \times_Z X \to T^*(Z/S)$ denotes the first projection and $dg : T^*(Z/S) \times_Z X \to T^*(X/S)$ is the canonical map. It is a closed conical subset of $T^*(Z/S)$.

(3) A test pair of $X$ relative to $S$ is a pair of $S$-morphisms $(g, h) : Y \leftarrow U \to X$ such that $U$ and $Y$ are Noetherian schemes smooth over $S$. We say that $(g, h)$ is $C$-transversal relatively to
S if \( h : U \to X \) is \( C \)-transversal relatively to \( S \) and \( g : U \to Y \) is \( h^c C \)-transversal relatively to \( S \).

**Definition 2.2.3** ([Bei16, §1.3] and [HY17, §4]). Let \( f : X \to S \) be a smooth morphism of Noetherian schemes over \( \mathbb{Z}[1/\ell] \). Let \( \mathcal{F} \) be an object in \( D^b_p(X, \Lambda) \).

1. We say that a test pair \( (g, h) : Y \to U \to X \) relative to \( S \) is \( \mathcal{F} \)-acyclic if \( g : U \to Y \) is universally locally acyclic relatively to \( h^* \mathcal{F} \).
2. For a closed conical subset \( C \) of \( T^*(X/S) \), we say that \( \mathcal{F} \) is micro-supported on \( C \) relatively to \( S \) if every \( C \)-transversal test pair of \( X \) relative to \( S \) is \( \mathcal{F} \)-acyclic.
3. Let \( \mathcal{C}(\mathcal{F}, X/S) \) be the set of all closed conical subsets \( C' \subseteq T^*(X/S) \) such that \( \mathcal{F} \) is micro-supported on \( C' \) relatively to \( S \). Note that \( \mathcal{C}(\mathcal{F}, X/S) \) is non-empty if \( f : X \to S \) is universally locally acyclic relatively to \( \mathcal{F} \). If \( \mathcal{C}(\mathcal{F}, X/S) \) has a smallest element, we denote it by \( SS(\mathcal{F}, X/S) \) and call it the singular support of \( \mathcal{F} \) relative to \( S \).

**Theorem 2.2.4** (Beilinson). Let \( f : X \to S \) be a smooth morphism between Noetherian schemes over \( \mathbb{Z}[1/\ell] \) and \( \mathcal{F} \) an object of \( D^b_p(X, \Lambda) \).

1. ([HY17, Theorem 5.2]) If we further assume that \( f : X \to S \) is projective and universally locally acyclic relatively to \( \mathcal{F} \), the singular support \( SS(\mathcal{F}, X/S) \) exists.
2. ([HY17, Theorem 5.2 and Theorem 5.3]) In general, after replacing \( S \) by a Zariski open dense subscheme, the singular support \( SS(\mathcal{F}, X/S) \) exists, and for any \( s \in S \), we have

\[
SS(\mathcal{F}|_{X_s}, X_s/s) = SS(\mathcal{F}, X/S) \times_S s.
\]

3. ([Bei16, Theorem 1.3]) If \( S = \text{Spec} k \) and if \( X \) is purely of dimension \( d \), then \( SS(\mathcal{F}, X/S) \) is purely of dimension \( d \).

### 2.3. Characteristic cycle and index formula.

#### 2.3.1. Let \( k \) be a perfect field of characteristic \( p \). Let \( X \) be a smooth scheme purely of dimension \( n \) over \( k \), let \( C \) be a closed conical subset of \( T^*X \) and \( f : X \to \mathbb{A}^1_k \) a \( k \)-morphism. A closed point \( v \in X \) is called at most an isolated \( C \)-characteristic point of \( f : X \to \mathbb{A}^1_k \) if there is an open neighborhood \( V \subseteq X \) of \( v \) such that \( f : V - \{v\} \to \mathbb{A}^1_k \) is \( C \)-transversal. A closed point \( v \in X \) is called an isolated \( C \)-characteristic point if \( v \) is at most an isolated \( C \)-characteristic point of \( f : X \to \mathbb{A}^1_k \) but \( f : X \to \mathbb{A}^1_k \) is not \( C \)-transversal at \( v \).

**Theorem 2.3.2** (T. Saito, [Sai17a, Theorem 5.9]). Let \( X \) be a smooth scheme purely of dimension \( n \) over a perfect field \( k \) of characteristic \( p \). Let \( \mathcal{F} \) be an object of \( D^b_p(X, \Lambda) \) and \( \{C_\alpha\}_{\alpha \in I} \) the set of irreducible components of \( SS(\mathcal{F}, X/k) \). There exists a unique \( n \)-cycle \( CC(\mathcal{F}, X/k) = \sum_{\alpha \in I} m_\alpha [C_\alpha] \) (mod \( \mathbb{Z} \)) of \( T^*X \) supported on \( SS(\mathcal{F}, X/k) \), satisfying the following Milnor formula (2.3.2.1):

\[
\chi(X_\bar{k}, \mathcal{F}|_{X_\bar{k}}) = \deg(CC(\mathcal{F}, X/k), T^*_X X)_{T^*_X}. \tag{2.3.3.1}
\]

where \( \chi(X_\bar{k}, \mathcal{F}|_{X_\bar{k}}) \) denotes the Euler-Poincaré characteristic of \( \mathcal{F}|_{X_\bar{k}} \).
We give a generalization in Theorem 2.3.5. For a smooth scheme \( \pi: X \to \text{Spec}(k) \), and two objects \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) in \( D^b_c(X,\Lambda) \), we denote \( \mathcal{F}_1 \boxtimes_k^L \mathcal{F}_2 = \text{pr}_1^* \mathcal{F}_1 \otimes^L \text{pr}_2^* \mathcal{F}_2 \in D^b_c(X \times X,\Lambda) \), where \( \text{pr}_i: X \times X \to X \) is the ith projection, for \( i = 1, 2 \). We also denote \( D_X(\mathcal{F}_1) = R\text{Hom}(\mathcal{F}_1,\mathcal{K}_X) \), where \( \mathcal{K}_X = \mathcal{R}\pi^!\Lambda \).

**Lemma 2.3.4.** Let \( X \) be a smooth variety purely of dimension \( n \) over a perfect field \( k \) of characteristic \( p \). Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be two objects in \( D^b_c(X,\Lambda) \). Then the diagonal map \( \delta: \Delta = X \hookrightarrow X \times X \) is \( SS(\mathcal{F}_1,\mathcal{F}_2, X \times X/k) \)-transversal if and only if \( SS(\mathcal{F}_1,\mathcal{F}_2, X \times X/k) \subseteq T^\Delta_X(X \times X) \). If we are in this case, then the canonical map

\[ R\text{Hom}(\mathcal{F}_1,\Lambda) \otimes^L \mathcal{F}_2 \cong R\text{Hom}(\mathcal{F}_1,\mathcal{F}_2) \]

is an isomorphism.

**Proof.** The first assertion follows from the short exact sequence of vector bundles on \( X \) associated to \( \delta: \Delta = X \hookrightarrow X \times X \):

\[ 0 \to T^\Delta_X(X \times X) \to T^*(X \times X) \times_{X \times X} \Delta \xrightarrow{\delta_\ast} T^*X \to 0. \]

For the second claim, we have the following canonical isomorphisms

\[ R\text{Hom}(\mathcal{F}_1,\Lambda) \otimes^L \mathcal{F}_2 \cong R\text{Hom}(\mathcal{F}_1,\Lambda(n)[2n]) \otimes^L \Lambda(-n)[-2n] \otimes^L \mathcal{F}_2 \cong D_X \mathcal{F}_1 \otimes^L R\delta^! \Lambda \otimes^L \mathcal{F}_2 \]

(2.3.4.1)

\[ \cong \delta^!(\mathcal{F}_2 \boxtimes_k^L D_X \mathcal{F}_1) \otimes^L R\delta^! \Lambda \cong R\delta^!(\mathcal{F}_2 \boxtimes_k^L D_X \mathcal{F}_1) \]

(2.3.4.2)

\[ \cong R\delta^!(R\text{Hom}(\text{pr}_2^* \mathcal{F}_1, \text{pr}_1^! \mathcal{F}_2)) \cong R\text{Hom}(\delta^! \text{pr}_2^* \mathcal{F}_1, R\delta^! \text{pr}_1^! \mathcal{F}_2) \]

(2.3.4.3)

\[ \cong R\text{Hom}(\mathcal{F}_1,\mathcal{F}_2), \]

where

(1) follows from the purity for the closed immersion \( \delta \) [ILO14, XVI, Théorème 3.1.1];

(2) follows from the assumption that \( \delta \) is \( SS(\mathcal{F}_1,\mathcal{F}_2, D_X \mathcal{F}_1) \)-transversal by [Sai17a, Proposition 8.13 and Definition 8.5];

(3) follows from the Künneth formula [SGA5, Exposé III, (3.1.1)].

**Theorem 2.3.5.** Let \( X \) be a smooth projective variety purely of dimension \( n \) over an algebraically closed field \( k \) of characteristic \( p \). Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be two objects in \( D^b_c(X,\Lambda) \) such that the diagonal map \( \delta: \Delta = X \hookrightarrow X \times X \) is properly \( SS(\mathcal{F}_1,\mathcal{F}_2, X \times X/k) \)-transversal. Then we have

\[ (-1)^n \cdot \text{dim}_A \text{Ext}^i(\mathcal{F}_1,\mathcal{F}_2) = \deg(CC(\mathcal{F}_1,X/k),CC(\mathcal{F}_2,X/k))_{T^*X} \]

where \( \text{dim}_A \text{Ext}^i(\mathcal{F}_1,\mathcal{F}_2) = \sum_i (-1)^i \text{dim}_A \text{Ext}^i_{D^b_c(X,\Lambda)}(\mathcal{F}_1,\mathcal{F}_2) \).

**Proof.** By the isomorphisms (2.3.4.1), the left hand side of (2.3.5.1) equals to

\[ (-1)^n \cdot \chi(X,R\delta^!(\mathcal{F}_2 \boxtimes_k^L D_X \mathcal{F}_1)) = (-1)^n \cdot \chi(X,\delta^!(\mathcal{F}_2 \boxtimes_k^L D_X \mathcal{F}_1)) \]

(2.3.5.2)

\[ = (-1)^n \cdot \deg(CC(\delta^!(\mathcal{F}_2 \boxtimes_k^L D_X \mathcal{F}_1),X/k),T^*_X X)_{T^*X}. \]

Since \( \delta: X \to X \times X \) is properly \( SS(\mathcal{F}_1,\mathcal{F}_2, X \times X/k) \)-transversal, we have

\[ CC(\delta^!(\mathcal{F}_2 \boxtimes_k^L D_X \mathcal{F}_1),X/k) = (-1)^n \delta^! CC(D(\mathcal{F}_2 \boxtimes_k^L D_X \mathcal{F}_1),X \times X/k) \]

(2.3.5.3)

\[ = (-1)^n \delta^!(CC(\mathcal{F}_2,X/k) \times CC(\mathcal{F}_1,X/k)). \]
where the equality (2.3.5.3) follows from [Sai17a, Theroem 7.6], and (2.3.5.4) follows from [Sai17b, Theorem 2.2.2]. Consider the following commutative diagram

\[
\begin{array}{ccc}
T^*X \times T^*X & \xrightarrow{pr} & T^*(X \times X) \\
\downarrow & & \downarrow \\
T^*X & \cong & T^*_\Delta(X \times X) \\
\end{array}
\]

\[\xrightarrow{\delta} T^*X \rightarrow X.\]

We have \(\delta^*(CC(F_2, X/k) \times CC(F_1, X/k)) = d\delta^*pr^*(CC(F_2, X/k) \times CC(F_1, X/k))\) and
\[\deg(\delta^*(CC(F_2, X/k) \times CC(F_1, X/k)), T^*_X) = \deg(CC(F_1, X/k), CC(F_2, X/k))_{T^*_X}.\]

Then (2.3.5.1) follows from the above formula and (2.3.5.2).

**Remark 2.3.6.** If \(F_1\) is the constant sheaf \(\Lambda\), then Theorem 2.3.5 is the index formula (2.3.3.1). Theorem 2.3.5 can be viewed as the \(\ell\)-adic version of the global index formula in the setting of \(D_X\)-modules (cf. [Gin86, Theorem 11.4.1]).

### 2.4. Relative twist formula.

#### 2.4.1. Let \(S\) be a Noetherian scheme over \(\mathbb{Z}[1/\ell]\), \(f : X \rightarrow S\) a smooth morphism of finite type and \(F\) an object of \(D^b_X(X, \Lambda)\). Assume that the relative singular support \(SS(F, X/S)\) exists. A cycle \(B = \sum_{i \in I} m_i[B_i]\) in \(T^*(X/S)\) is called the **characteristic cycle of \(F\) relative to \(S\)** if each \(B_i\) is a subset of \(SS(F, X/S)\), each \(B_i \rightarrow S\) is open and equidimensional and if, for any algebraic geometric point \(\bar{s}\) of \(S\), we have

\[(2.4.1.1) \quad B_{\bar{s}} = \sum_{i \in I} m_i[(B_i)_{\bar{s}}] = CC(F|_{X_{\bar{s}}}, X_{\bar{s}}/\bar{s}).\]

We denote by \(CC(F, X/S)\) the characteristic cycle of \(F\) on \(X\) relative to \(S\). Notice that relative characteristic cycles may not exist in general.

**Proposition 2.4.2** (T. Saito, [HY17, Proposition 6.5]). Let \(k\) be a perfect field of characteristic \(p\). Let \(S\) be a smooth connected scheme of dimension \(r\) over \(k\), \(f : X \rightarrow S\) a smooth morphism of finite type and \(F\) an object of \(D^b_X(X, \Lambda)\). Assume that \(f : X \rightarrow S\) is \(SS(F, X/k)\)-transversal and that each irreducible component of \(SS(F, X/k)\) is open and equidimensional over \(S\). Then the relative singular support \(SS(F, X/S)\) and the relative characteristic cycle \(CC(F, X/S)\) exist, and we have

\[(2.4.2.1) \quad SS(F, X/S) = \theta(SS(F, X/k)),\]
\[(2.4.2.2) \quad CC(F, X/S) = (-1)^r\theta_*(CC(F, X/k)),\]

where \(\theta : T^*X \rightarrow T^*(X/S)\) denotes the projection induced by the canonical map \(\Omega^1_{X/k} \rightarrow \Omega^1_{X/S} \).

**Definition 2.4.3.** Let \(k\) be a perfect field of characteristic \(p\) and \(S\) a smooth connected scheme of dimension \(r\) over \(k\). Let \(f : X \rightarrow S\) be a smooth morphism purely of relative dimension \(n\) and \(F\) an object of \(D^b_X(X, \Lambda)\). Assume that \(f : X \rightarrow S\) is \(SS(F, X/k)\)-transversal. Consider the following cartesian diagram

\[(2.4.3.1) \quad \begin{array}{ccc}
T^*_S X & \xrightarrow{\theta} & T^*X \\
\downarrow & & \downarrow \\
X & \xrightarrow{0_{X/S}} & T^*(X/S)
\end{array}\]

where \(0_{X/S} : X \rightarrow T^*(X/S)\) is the zero section. Since \(f : X \rightarrow S\) is \(SS(F, X/k)\)-transversal, the refined Gysin pull-back \(0^\ell_{X/S}(CC(F, X/k))\) of \(CC(F, X/k)\) is a \(r\)-cycle class supported on \(X\). We define the **relative characteristic class** of \(F\) to be

\[(2.4.3.2) \quad cc_{X/S}(F) = (-1)^r \cdot 0^\ell_{X/S}(CC(F, X/k)) \quad \text{in} \quad CH^n(X).\]
Now we prove a special case of Conjecture 2.1.4.

**Theorem 2.4.4** (Relative twist formula). Let $S$ be a smooth connected scheme of dimension $r$ over a finite field $k$ of characteristic $p$. Let $f : X → S$ be a smooth projective morphism of relative dimension $n$. Let $F ∈ D^b_c(X,Λ)$ and $G$ a locally constant and constructible sheaf of $Λ$-modules on $X$. Assume that $f$ is properly $SS(F,X/k)$-transversal. Then there is an isomorphism

$$\det R_f^∗(F ⊗ G) \cong (\det R_f^∗F)^{⊗ \text{rank} G} \otimes \det G(\text{cc}_{X/S}(F)) \quad \text{in } K_0(S,Λ).$$

Note that we also have $\text{cc}_{X/S}(F) = (CC(F,X/S), T^*_X X)_{T^*(X/S)} \in \text{CH}^p(X)$.

**Proof.** We may assume $G \neq 0$. Since $G$ is a smooth sheaf, we have $SS(F,X/k) = SS(F⊗G,X/k)$. Since $f$ is proper and $SS(F,X/k)$-transversal, by [Sai17a, Lemma 4.3.4], $R_f^∗F$ and $R_f^∗(F⊗G)$ are smooth sheaves on $S$. For any closed point $s ∈ S$, we have the following commutative diagram

$$
\begin{array}{ccc}
T^*X \times_X X_s & \xrightarrow{θ_∗} & T^*(X/S) \times_X X_s \\
| pr & | & | pr \\
T^*X & \xrightarrow{θ} & T^*(X/S) & \xrightarrow{i} & X \\
\end{array}
$$

where $0_{X/S}$ and $0_{X_s}$ are the zero sections. Hence we have

$$\text{cc}_{X_s}(F|_{X_s}) = (CC(F|_{X_s},X_s/s), X_s)_{T^*X_s} = 0_{X_s}^! CC(F|_{X_s},X_s/s) \overset{(a)}{=} 0_{X_s}^! i^! CC(F,X/k) $$

$$= (-1)^r 0_{X_s}^! i^! CC(F,X/k) = (-1)^r 0_{X_s}^! \theta^* i^! CC(F,X/k)$$

$$= (-1)^r 0_{X_s}^! \theta^* i^! CC(F,X/k) = (-1)^r 0_{X_s}^! \theta^* i^! CC(F,X/S)$$

$$= 0_{X_s}^! \theta^* i^! CC(F,X/S) = i^! 0_{X_s}^! CC(F,X/S) = i^! \text{cc}_{X/S}(F),$$

where the equality (a) follows from [Sai17a, Theorem 7.6] since $f$ is properly $SS(F,X/k)$-transversal.

By Chebotarev density (cf. [Lau87, Théorème 1.1.2]), we may assume that $S$ is the spectrum of a finite field. Then it is sufficient to compare the Frobenius action. Then one use (2.4.4.2) and Theorem 1.1.3.

**Example 2.4.5.** Let $S$ be a smooth projective connected scheme over a finite field $k$ of characteristic $p > 2$. Let $f : X → S$ be a smooth projective morphism of relative dimension $n$, $χ = \text{rank} R_f^∗Ω^1_{X,k}$ the Euler-Poincaré number of the fibers and let $F$ be a constructible étale sheaf of $Λ$-modules on $S$. Then by the projection formula, we have $R_f^∗f^∗F \cong F ⊗ R_f^∗Ω^1_{X,k}$. Since $f$ is projective and smooth, $R_f^∗Ω^1_{X,k}$ is a smooth sheaf on $S$. Using Theorem 1.1.3, we get

$$\varepsilon(S,R_f^∗f^∗F) = \varepsilon(S,F)^χ \cdot \det R_f^∗Ω^1_{X,k}(−\text{cc}_{Y/k}(F)).$$

By [Sai94, Theorem 2], $\det R_f^∗Ω^1_{X,k} = κ_{X/S}(−\frac{1}{2}nχ)$, where $κ_{X/S}$ is a character of order at most 2 and is determined by the following way:

1. If $n$ is odd, then $κ_{X/S}$ is trivial.
2. If $n = 2m$ is even, then $κ_{X/S}$ is the quadratic character defined by the square root of

$$(-1)^{(χ−1)/2} \cdot δ_{\text{dr},X/S},$$

where $δ_{\text{dr},X/S} : (\det H_{\text{dr}}(X/S))^⊗2 \cong O_S$ is the de Rham discriminant defined by the non-degenerate symmetric bilinear form $H_{\text{dr}}(X/S)^⊗L.H_{\text{dr}}(X/S) → O_S[-2n]$, and $H_{\text{dr}}(X/S) = R_f^∗Ω^∗_{X/S}$ is the perfect complex of $O_S$-modules whose cohomology computes the relative de Rham cohomology of $X/S$.

Similarly, if $F$ is a locally constant and constructible étale sheaf of $Λ$-modules on $S$, then

$$\det R_f^∗f^∗F \cong \det(F ⊗ R_f^∗Ω^1_{X,k}) \cong (\det F)^⊗χ \otimes (\det R_f^∗Ω^1_{X,k})^⊗\text{rank} F$$

$$\cong (\det F)^⊗χ \otimes (κ_{X/S}(−\frac{1}{2}nχ))^{⊗\text{rank} F}. $$

2.4.5.2}
2.5. Total characteristic class.

2.5.1. In the rest of this section, we relate the relative characteristic class $cc_{X/S}(\mathcal{F})$ to the total characteristic class of $\mathcal{F}$. Let $X$ be a smooth scheme purely of dimension $d$ over a perfect field $k$ of characteristic $p$. In [Sai17a, Definition 6.7.2], T. Saito defines the following morphism

\[(2.5.1.1)\quad cc_{X, \cdot}: K_0(X, \Lambda) \to CH_\bullet(X) = \bigoplus_{i=0}^d CH_i(X),\]

which sends $\mathcal{F} \in D^b_c(X, \Lambda)$ to the total characteristic class $cc_{X, \cdot}(\mathcal{F})$ of $\mathcal{F}$. For our convenience, for any integer $n$ we put

\[(2.5.1.2)\quad cc_X^n(\mathcal{F}) := cc_{X, d-n}(\mathcal{F}) \quad \text{in} \quad CH^n(X).\]

By [Sai17a, Lemma 6.9], for any $\mathcal{F} \in D^b_c(X, \Lambda)$, we have

\[(2.5.1.3)\quad cc_X^d(\mathcal{F}) = cc_{X,0}(\mathcal{F}) = (CC(\mathcal{F}, X/k), T^*_X X)_{T^*_X} \quad \text{in} \quad CH_0(X),\]

\[(2.5.1.4)\quad cc_X^0(\mathcal{F}) = cc_{X,d}(\mathcal{F}) = (-1)^d \cdot \text{rank} \mathcal{F} \cdot [X] \quad \text{in} \quad CH_d(X) = \mathbb{Z}.\]

The following proposition gives a computation of $cc_X^n(\mathcal{F})$ for any $n$.

**Proposition 2.5.2.** Let $S$ be a smooth connected scheme of dimension $r$ over a perfect field $k$ of characteristic $p$. Let $f: X \to S$ be a smooth morphism purely of relative dimension $n$. Assume that $f$ is $SS(\mathcal{F}, X/k)$-transversal. Then we have

\[(2.5.2.1)\quad cc_X^n(\mathcal{F}) = (-1)^r \cdot cc_{X/S}(\mathcal{F}) \quad \text{in} \quad CH^n(X)\]

where $cc_{X/S}(\mathcal{F})$ is defined in Definition 2.4.3.

**Proof.** We use the notation of [Sai17a, Lemma 6.2]. We put $F = (T^*S \times_S X) \oplus \mathbb{A}^1_X$ and $E = T^*X \oplus \mathbb{A}^1_X$. We have a canonical injection $i: F \to E$ of vector bundles on $X$ induced by $df: T^*S \times_S X \to T^*X$. Let $i: \mathbb{P}(F) \to \mathbb{P}(E)$ be the canonical map induced by $i: F \to E$. By [Sai17a, Lemma 6.1.2 and Lemma 6.2.1], we have a commutative diagram:

\[
\begin{CD}
CH_r(\mathbb{P}(F)) @>{\tilde{i}^*}>> CH_{n+r}(\mathbb{P}(E)) \\
@. \cong @. \cong \\
\bigoplus_{q=0}^r CH_q(X) @>{\text{can}}>> \bigoplus_{q=0}^{n+r} CH_q(X) \\
@. \cong @. \cong \\
CH_r(X) @>{\text{can}}>> CH^n(X) @>{\text{can}}>> CH_r(X).
\end{CD}
\]

Since $f$ is smooth and $SS(\mathcal{F}, X/k)$-transversal, the intersection $SS(\mathcal{F}, X/k) \cap (T^*S \times_S X)$ is contained in the zero section of $T^*S \times_S X$. Thus the Gysin pull-back $i^*(CC(\mathcal{F}, X/k))$ is supported on the zero section of $T^*S \times_S X$. Let $CC(\mathcal{F}, X/k)$ be any extension of $CC(\mathcal{F}, X/k)$ to $\mathbb{P}(E)$ (cf. [Sai17a, Definition 6.7.2]). Then $i^*(CC(\mathcal{F}, X/k))$ is an extension of $i^*(CC(\mathcal{F}, X/k))$ to $\mathbb{P}(F)$. By [Sai17a, Definition 6.7.2], the image of $CC(\mathcal{F}, X/k)$ in $CH^n(X)$ by the right vertical map of (2.5.2.2) equals to $cc_X^n(\mathcal{F}) = cc_{X,r}(\mathcal{F})$. The image of $\tilde{i}^*(CC(\mathcal{F}, X/k))$ in $CH^n(X)$ by the left vertical map of (2.5.2.2) equals to $(-1)^r \cdot cc_{X/S}(\mathcal{F})$ (cf. (2.4.3.2)). Now the equality (2.5.2.1) follows from the commutativity of (2.5.2.2). \qed
2.5.3. Following Grothendieck [SGA5], it’s natural to ask the following question: is the diagram

$$
\begin{array}{ccc}
K_0(X, \Lambda) & \xrightarrow{cc_X} & CH_* (X) \\
\downarrow f_* & & \downarrow f_* \\
K_0(Y, \Lambda) & \xrightarrow{cc_Y} & CH_* (Y)
\end{array}
$$

(2.5.3.1)

commutative for any proper map $f: X \to Y$ between smooth schemes over a perfect field $k$? If $k = \mathbb{C}$, the diagram (2.5.3.1) is commutative by [Gin86, Theorem A.6]. By the philosophy of Grothendieck, the answer is no in general if char$(k) > 0$ (cf. [Sai17a, Example 6.10]). However, in [UYZ, Corollary 1.9], we prove that the degree zero part of the diagram (2.5.3.1) is commutative, i.e., if $f: X \to Y$ is a proper map between smooth projective schemes over a finite field $k$ of characteristic $p$, then we have the following commutative diagram

$$
\begin{array}{ccc}
K_0(X, \Lambda) & \xrightarrow{cc_{X,0}} & CH_0 (X) \\
\downarrow f_* & & \downarrow f_* \\
K_0(Y, \Lambda) & \xrightarrow{cc_{Y,0}} & CH_0 (Y).
\end{array}
$$

(2.5.3.2)

Now we propose the following:

**Conjecture 2.5.4.** Let $S$ be a smooth connected scheme over a perfect field $k$ of characteristic $p$. Let $f: X \to S$ be a smooth morphism purely of relative dimension $n$ and $g: Y \to S$ a smooth morphism purely of relative dimension $m$. Let $D^b_c(X/S, \Lambda)$ be the thick subcategory of $D^b_c(X, \Lambda)$ consists of $\mathcal{F} \in D^b_c(X, \Lambda)$ such that $f: X \to S$ is $SS(\mathcal{F}, X/k)$-transversal. Let $K_0(X/S, \Lambda)$ be the Grothendieck group of $D^b_c(X/S, \Lambda)$. Then for any proper morphism $h: X \to Y$ over $S$,

$$
\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
\downarrow f & & \downarrow g \\
S & & 
\end{array}
$$

(2.5.4.1)

the following diagram commutes

$$
\begin{array}{ccc}
K_0(X/S, \Lambda) & \xrightarrow{cc^X} & CH^n (X) \\
\downarrow h_* & & \downarrow h_* \\
K_0(Y/S, \Lambda) & \xrightarrow{cc^Y} & CH^m (Y).
\end{array}
$$

(2.5.4.2)

That is to say, for any $\mathcal{F} \in D^b_c(X, \Lambda)$, if $f$ is $SS(\mathcal{F}, X/k)$-transversal, then we have

$$h_*(cc^X_\mathcal{F}) = cc^Y_{Rh_* \mathcal{F}} \text{ in } CH^m (Y).$$

(2.5.4.3)

**Remark 2.5.5.** If $f$ is $SS(\mathcal{F}, X/k)$-transversal, by [Sai17a, Lemma 3.8 and Lemma 4.2.6], the morphism $g: Y \to S$ is $SS(Rh_* \mathcal{F}, Y/k)$-transversal. Thus we have a well-defined map $h_*: K_0(X/S, \Lambda) \to K_0(Y/S, \Lambda)$.

In next section, we formulate and prove a cohomological version of Conjecture 2.5.4 (cf. Corollary 3.3.4).

### 3. Relative cohomological characteristic class

In this section, we assume that $S$ is a smooth connected scheme over a perfect field $k$ of characteristic $p$ and $\Lambda$ is a finite field of characteristic $\ell$. To simplify our notations, we omit to write $R$ or $L$ to denote the derived functors unless otherwise stated explicitly or for $R \text{Hom}$. 
We briefly recall the content of this section. Let $X \to S$ be a smooth morphism purely of relative dimension $n$ and $\mathcal{F} \in D^b_c(X, \Lambda)$. If $X \to S$ is $SS(\mathcal{F}, X/k)$-transversal, we construct a relative cohomological characteristic class $\text{cccc}_{X/S}(\mathcal{F}) \in H^{2n}(X, \Lambda(n))$ following the method of [AS07, SGA5]. We conjecture that the image of the cycle class $\text{cc}_{X/S}(\mathcal{F})$ by the cycle class map $\text{cl}: \text{CH}^p(X) \to H^{2n}(X, \Lambda(n))$ is $\text{cccc}_{X/S}(\mathcal{F})$ (cf. Conjecture 2.1.4). In Corollary 3.3.4, we prove that the formation of $\text{cccc}_{X/S}\mathcal{F}$ is compatible with proper push-forward.

3.1. Relative cohomological correspondence.

3.1.1. Let $\pi_1: X_1 \to S$ and $\pi_2: X_2 \to S$ be smooth morphisms purely of relative dimension $n_1$ and $n_2$ respectively. We put $X := X_1 \times_S X_2$ and consider the following cartesian diagram

$\begin{array}{ccc}
X & \xrightarrow{pr_2} & X_2 \\
pr_1 \downarrow & & \downarrow \pi_2 \\
X_1 & \xrightarrow{\pi_1} & S
\end{array}$

(3.1.1.1)

Let $\mathcal{E}_i$ and $\mathcal{F}_i$ be objects of $D^b_c(X_i, \Lambda)$ for $i = 1, 2$. We put

$\mathcal{F} := \mathcal{F}_1 \boxtimes_S \mathcal{F}_2 := \text{pr}_1^* \mathcal{F}_1 \otimes^L \text{pr}_2^* \mathcal{F}_2,$

(3.1.1.2)

$\mathcal{E} := \mathcal{E}_1 \boxtimes_S \mathcal{E}_2 := \text{pr}_1^* \mathcal{E}_1 \otimes^L \text{pr}_2^* \mathcal{E}_2,$

(3.1.1.3)

which are objects of $D^b_c(X, \Lambda)$. Similarly, we can define $\mathcal{F}_1 \boxtimes_k^L \mathcal{F}_2$, which is an object of $D^b_c(X_1 \times_k X_2, \Lambda)$. We first compare $SS(\mathcal{F}_1 \boxtimes_k^L \mathcal{F}_2, X_1 \times_k X_1/k)$ and $SS(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2, X_1 \times_k X_1/k)$.

**Lemma 3.1.2.** Assume that $\pi_1: X_1 \to S$ is $SS(\mathcal{F}_1, X_1/k)$-transversal. Then we have

(3.1.2.1) $SS(\text{pr}_1^* \mathcal{F}_1, X/k) \cap SS(\text{pr}_2^* \mathcal{F}_2, X/k) \subseteq T^*_S X.$

Moreover, the closed immersion $i: X_1 \times_X X_2 \to X_1 \times_k X_2$ is $SS(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2, X_1 \times_k X_2/k)$-transversal and

(3.1.2.2) $SS(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2, X_1 \times_k X_2/k) \subseteq i^c(\text{SS}(\mathcal{F}_1 \boxtimes_k^L \mathcal{F}_2, X_1 \times_k X_2/k)).$

**Proof.** We first prove (3.1.2.1). Since $X_i \to S$ is smooth, we obtain an exact sequence of vector bundles on $X_i$ for $i = 1, 2$

(3.1.2.3) $0 \to T^* S \times_S X_i \xrightarrow{d\pi_i} T^* X_i \to T^* (X_i/S) \to 0.$

Since $\pi_1: X_1 \to S$ is $SS(\mathcal{F}_1, X_1/k)$-transversal, we have

(3.1.2.4) $SS(\mathcal{F}_1, X_1/k) \cap (T^* S \times_S X_1) \subseteq T^*_S S \times_S X_1.$

Consider the following diagram with exact rows and exact columns:

$\begin{array}{cccc}
0 & 0 & \ & \\
\downarrow & \downarrow & \ & \\
T^*(X_2/S) \times_X X_2 & \xrightarrow{\cong} T^*(X/X_1) & \ & \\
\downarrow & \downarrow & \ & \\
0 & T^* X_2 \times_X X_2 & T^* X & T^*(X_2/k) \to 0 & \ & \\
\downarrow & \downarrow & \downarrow & \ & \\
0 & T^* S \times_S X_2 & T^* X_1 \times_X X_1 & T^* (X_1/S) \times_X X_1 & 0 & \ & \\
\downarrow & \downarrow & \downarrow & \ & \\
0 & 0 & 0 & 0 & \\
\end{array}$

(3.1.2.5)

Since $pr_i$ is smooth, by [Sai17a, Corollary 8.15], we have

$SS(\text{pr}_1^* \mathcal{F}_i, X/k) = \text{pr}_1^* SS(\mathcal{F}_i, X_1/k) = SS(\mathcal{F}_i, X_1/k) \times_X X.$
It follows from (3.1.2.4) and (3.1.2.5) that $\text{pr}_1^* \mathcal{S} \mathcal{S}(\mathcal{F}_1, X_1/k) \cap \text{pr}_2^* \mathcal{S} \mathcal{S}(\mathcal{F}_2, X_2/k) \subseteq \mathcal{T}_X^* X$. Thus $\mathcal{S} \mathcal{S}(\text{pr}_1^* \mathcal{F}_1, X_1/k) \cap \mathcal{S} \mathcal{S}(\text{pr}_2^* \mathcal{F}_2, X_2/k) \subseteq \mathcal{T}_X^* X$. This proves (3.1.2.1).

Now we consider the cartesian diagram

$$
\begin{array}{c}
X = X_1 \times_\mathcal{S} X_2 \\
\downarrow i \\
\downarrow \\
\delta \\
S \\
\end{array}
\xrightarrow{\cong}
\begin{array}{c}
\delta \\
\downarrow \\
S \times_\mathcal{S} S \\
\end{array}

(3.1.2.6)

where $\delta: S \to S \times_\mathcal{S} S$ is the diagonal. We get the following commutative diagram of vector bundles on $X$ with exact rows:

$$
\begin{array}{ccccc}
0 & \longrightarrow & \mathcal{N}_{X/(X_1 \times_\mathcal{S} X_2)} & \longrightarrow & T^*(X_1 \times_\mathcal{S} X_2) \times_{X_1 \times_\mathcal{S} X_2} X & \longrightarrow & T^*X & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow & & \downarrow d & & \downarrow \\
0 & \longrightarrow & \mathcal{N}_{S/(S \times_\mathcal{S} S)} \times_\mathcal{S} X & \longrightarrow & T^*(S \times_\mathcal{S} S) \times_{S \times_\mathcal{S} S} X & \longrightarrow & T^*S \times_\mathcal{S} X & \longrightarrow & 0 \\
\end{array}

T^*S \times_\mathcal{S} X \longrightarrow (T^*S \times_\mathcal{S} X_1) \times_\mathcal{S} (T^*S \times_\mathcal{S} X_2)

where $\mathcal{N}_{S/(S \times_\mathcal{S} S)}$ is the conormal bundle associated to $\delta: S \to S \times_\mathcal{S} S$. By [Sai17b, Theorem 2.2.3], we have $\mathcal{S} \mathcal{S}(\mathcal{F}_1 \boxtimes_S^L \mathcal{F}_2, X_1 \times_\mathcal{S} X_2/k) = \mathcal{S} \mathcal{S}(\mathcal{F}_1, X_1/k) \times \mathcal{S} \mathcal{S}(\mathcal{F}_2, X_2/k)$. Therefore by (3.1.2.4), $\mathcal{N}_{X/(X_1 \times_\mathcal{S} X_2)} \cap \mathcal{S} \mathcal{S}(\mathcal{F}_1 \boxtimes_S^L \mathcal{F}_2, X_1 \times_\mathcal{S} X_2/k)$ is contained in the zero section of $\mathcal{N}_{X/(X_1 \times_\mathcal{S} X_2)}$. Hence $i: X \hookrightarrow X_1 \times_\mathcal{S} X_2$ is $\mathcal{S} \mathcal{S}(\mathcal{F}_1 \boxtimes_S^L \mathcal{F}_2, X_1 \times_\mathcal{S} X_2/k)$-transversal. Now the assertion (3.1.2.2) follows from [Sai17a, Lemma 4.2.4].

**Proposition 3.1.3.** Under the notation in 3.1.1, we assume that

1. $\mathcal{S} \mathcal{S}(\mathcal{E}_i, X_i/k) \cap \mathcal{S} \mathcal{S}(\mathcal{F}_i, X_i/k) \subseteq \mathcal{T}_X^* X_i$ for all $i = 1, 2$;
2. $\pi_1: X_1 \to S$ is $\mathcal{S} \mathcal{S}(\mathcal{E}_1, X_1/k)$-transversal or $\pi_2: X_2 \to S$ is $\mathcal{S} \mathcal{S}(\mathcal{F}_2, X_2/k)$-transversal;
3. $\pi_1: X_1 \to S$ is $\mathcal{S} \mathcal{S}(\mathcal{F}_1, X_1/k)$-transversal or $\pi_2: X_2 \to S$ is $\mathcal{S} \mathcal{S}(\mathcal{E}_2, X_2/k)$-transversal.

Then the following canonical map (cf. [Zh15, (7.2.2)] and [SGA5, Exposé III, (2.2.4)])

(3.1.3.1) $\underline{R \text{Hom}}(\mathcal{E}_1, \mathcal{F}_1) \boxtimes_S^L \underline{R \text{Hom}}(\mathcal{E}_2, \mathcal{F}_2) \to \underline{R \text{Hom}}(\mathcal{E}, \mathcal{F})$.

is an isomorphism.

If $S$ is the spectrum of a field, then the above result is proved in [SGA5, Exposé III, Proposition 2.3]. Our proof below is different from that of loc.cit. and is based on [Sai17a].

**Proof.** In the following, we put $\mathcal{E}_i^\vee := \underline{R \text{Hom}}(\mathcal{E}_i, \Lambda)$. Since $\mathcal{S} \mathcal{S}(\mathcal{E}_i, X_i/k) \cap \mathcal{S} \mathcal{S}(\mathcal{F}_i, X_i/k) \subseteq \mathcal{T}_X^* X_i$, Lemma 2.3.4 implies that

(3.1.3.2) $\mathcal{F}_i \otimes^L \mathcal{E}_i^\vee = \mathcal{F}_i \otimes^L \underline{R \text{Hom}}(\mathcal{E}_i, \Lambda) \overset{\cong}{\longrightarrow} \underline{R \text{Hom}}(\mathcal{E}_i, \mathcal{F}_i)$, for all $i = 1, 2$.

Hence we have

(3.1.3.3) $\underline{R \text{Hom}}(\mathcal{E}_1, \mathcal{F}_1) \boxtimes_S^L \underline{R \text{Hom}}(\mathcal{E}_2, \mathcal{F}_2) \cong (\mathcal{F}_1 \otimes^L \mathcal{E}_1^\vee) \boxtimes_S^L (\mathcal{F}_2 \otimes^L \mathcal{E}_2^\vee) \\
\cong (\mathcal{F}_1 \boxtimes_S^L \mathcal{F}_2) \otimes^L (\mathcal{E}_1^\vee \boxtimes_S^L \mathcal{E}_2^\vee)$.

Note that we also have

(3.1.3.4) $\mathcal{E}_1^\vee \boxtimes_S^L \mathcal{E}_2^\vee = \text{pr}_1^* \underline{R \text{Hom}}(\mathcal{E}_1, \Lambda) \otimes^L \text{pr}_2^* \underline{R \text{Hom}}(\mathcal{E}_2, \Lambda)$

(a) $\underline{R \text{Hom}}(\text{pr}_1^* \mathcal{E}_1, \underline{R \text{Hom}}(\text{pr}_2^* \mathcal{E}_2, \Lambda)) \\
\cong \underline{R \text{Hom}}(\mathcal{E}_1^\vee \boxtimes^L \mathcal{E}_2^\vee, \Lambda) = \mathcal{E}^\vee$, 

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where the isomorphism (a) follows from Lemma 2.3.4 by the fact that (cf. Lemma 3.1.2)
\[ SS(\text{pr}_1^* \mathcal{E}_1, X/k) \cap SS(\text{pr}_2^* \mathcal{E}_2, X/k) \subseteq T^*_X X. \]

By Lemma 3.1.2, we have
\[
SS(\mathcal{E}, X/k) \cap SS(\mathcal{F}, X/k) \\
\subseteq i^*(SS(\mathcal{E} \boxtimes_k \mathcal{F}, X_1 \times_k X_2/k)) \cap i^*(SS(\mathcal{F} \boxtimes_k \mathcal{E}, X_1 \times_k X_2/k)) \\
(b) \subseteq i^*(SS(\mathcal{E}_1, X_1) \times SS(\mathcal{E}_2, X_2)) \cap i^*(SS(\mathcal{F}_1, X_1) \times SS(\mathcal{F}_2, X_2)) \\
(c) \subseteq T^*_X X,
\]
where the equality (b) follows from [Sai17b, Theorem 2.2.3], and (c) follows from the assumptions (2) and (3) (cf. [Sai17b, Lemma 2.7.2]). Thus by Lemma 2.3.4, we have
\[ (3.1.3.5) \quad \mathcal{F} \otimes^L \mathcal{E}^\vee \cong R\mathcal{H}\text{om}(\mathcal{E}, \mathcal{F}). \]

Combining (3.1.3.3), (3.1.3.4) and (3.1.3.5), we get
\[ (3.1.3.6) \quad R\mathcal{H}\text{om}(\mathcal{E}_1, \mathcal{F}_1) \boxtimes_S L R\mathcal{H}\text{om}(\mathcal{E}_2, \mathcal{F}_2) \cong \mathcal{F} \otimes^L \mathcal{E}^\vee \cong R\mathcal{H}\text{om}(\mathcal{E}, \mathcal{F}). \]

This finishes the proof. \( \square \)

3.1.4. Künneth formula. We have the following canonical morphism
\[ (3.1.4.1) \quad \mathcal{F}_1 \boxtimes_S L R\mathcal{H}\text{om}(\mathcal{F}_2, \pi_2^! \mathcal{A}_S) \rightarrow R\mathcal{H}\text{om}(\text{pr}_2^* \mathcal{F}_2, \text{pr}_1^! \mathcal{F}_1), \]
by taking the adjunction of the following composition map
\[ \text{pr}_1^* \mathcal{F}_1 \otimes \text{pr}_2^* R\mathcal{H}\text{om}(\mathcal{F}_2, \pi_2^! \mathcal{A}_S) \otimes \text{pr}_2^* \mathcal{F}_2 \rightarrow \text{pr}_1^* \mathcal{F}_1 \otimes \text{pr}_2^* (\mathcal{F}_2 \otimes R\mathcal{H}\text{om}(\mathcal{F}_2, \pi_2^! \mathcal{A}_S)) \]
\[ \text{evaluation} \rightarrow \text{pr}_1^* \mathcal{F}_1 \otimes \text{pr}_2^* \pi_2^! \mathcal{A}_S \rightarrow \text{pr}_1^* \mathcal{F}_1 \otimes \text{pr}_1^! \mathcal{A}_X_1 \rightarrow \text{pr}_1^! \mathcal{F}_1. \]

Corollary 3.1.5. Assume that \( \pi_1: X_1 \rightarrow S \) is \( SS(\mathcal{F}_1, X_1/k) \)-transversal or \( \pi_2: X_2 \rightarrow S \) is \( SS(\mathcal{F}_2, X_2/k) \)-transversal. Then the canonical map (3.1.4.1) is an isomorphism.

If \( S \) is the spectrum of a field, then the above result is proved in [SGA5, Exposé III, (3.1.1)]. Our proof below is different from that of loc.cit.

Proof. By Proposition 3.1.3, we have the following isomorphisms
\[ \mathcal{F}_1 \boxtimes S L R\mathcal{H}\text{om}(\mathcal{F}_2, \pi_2^! \mathcal{A}_S) \overset{\text{Prop.} 3.1.3}{\cong} R\mathcal{H}\text{om}(\text{pr}_2^* \mathcal{F}_2, \text{pr}_1^* \mathcal{F}_1 \otimes \text{pr}_1^! \mathcal{A}_S) \]
\[ \overset{(a)}{=} R\mathcal{H}\text{om}(\text{pr}_2^* \mathcal{F}_2, \text{pr}_1^! \mathcal{F}_1), \]
where (a) follows from the fact that \( \text{pr}_1 \) is smooth (cf. [ILO14, XVI, Théorème 3.1.1] and [SGA4, XVIII, Théorème 3.2.5]). \( \square \)

Definition 3.1.6. Let \( X_i, \mathcal{F}_i \) be as in 3.1.1 for \( i = 1, 2 \). A relative correspondence between \( X_1 \) and \( X_2 \) is a scheme \( C \) over \( S \) with morphisms \( c_1: C \rightarrow X_1 \) and \( c_2: C \rightarrow X_2 \) over \( S \). We put \( c = (c_1, c_2): C \rightarrow X_1 \times_S X_2 \) the corresponding morphism. A morphism \( u: c_2^* \mathcal{F}_2 \rightarrow c_1^* \mathcal{F}_1 \) is called a relative cohomological correspondence from \( \mathcal{F}_2 \) to \( \mathcal{F}_1 \) on \( C \).

3.1.7. Given a correspondence \( C \) as above, we recall that there is a canonical isomorphism [SGA4, XVIII, 3.1.12.2]
\[ (3.1.7.1) \quad R\mathcal{H}\text{om}(c_2^* \mathcal{F}_2, c_1^* \mathcal{F}_1) \cong c^! R\mathcal{H}\text{om}(\text{pr}_2^* \mathcal{F}_2, \text{pr}_1^! \mathcal{F}_1). \]
3.1.8. For $i = 1, 2$, consider the following diagram of $S$-morphisms

$$
\begin{array}{ccc}
X_i & \xrightarrow{f_i} & Y_i \\
\pi_i \downarrow & & \downarrow q_i \\
S & & S
\end{array}
$$

where $\pi_i$ and $q_i$ are smooth morphisms. We put $X := X_1 \times_S X_2$, $Y := Y_1 \times_S Y_2$ and $f := f_1 \times_S f_2: X \to Y$. Let $\mathcal{M}_i \in D^b_c(Y_i, \Lambda)$ for $i = 1, 2$. We have a canonical map (cf. [Zh15, Construction 7.4] and [SGA5, Exposé III, (1.7.3)])

$$(3.1.8.1) \quad f^! \mathcal{M}_1 \boxtimes_S f^! \mathcal{M}_2 \to f^!(\mathcal{M}_1 \boxtimes_S \mathcal{M}_2)$$

which is adjoint to the composite

$$(3.1.8.2) \quad f_i(f^! \mathcal{M}_1 \boxtimes_S f^! \mathcal{M}_2) \xrightarrow{\sim} f_1 f^! \mathcal{M}_1 \boxtimes_S f_2 f^! \mathcal{M}_2 \xrightarrow{\text{adj}} \mathcal{M}_1 \boxtimes_S \mathcal{M}_2$$

where $(a)$ is the Künneth isomorphism [SGA4, XVII, Théorème 5.4.3].

**Proposition 3.1.9.** If $q_2: Y_2 \to S$ is $SS(\mathcal{M}_2, Y_2/k)$-transversal, then the map $(3.1.8.1)$ is an isomorphism.

If $S$ is the spectrum of a field, the above result is proved in [SGA5, Exposé III, Proposition 1.7.4].

**Proof.** Consider the following cartesian diagrams

$$
\begin{array}{ccc}
X_1 \times_S X_2 & \xrightarrow{f_1 \times \text{id}} & Y_1 \times_S X_2 & \xrightarrow{f_2} & X_2 \\
\text{id} \times f_2 & & f & & \text{id} \times f_2 \\
X_1 \times_S Y_2 & \xrightarrow{f_1 \times \text{id}} & Y_1 \times_S Y_2 & \xrightarrow{\text{pr}_2} & Y_2 \\
\text{pr}_1 & & f & & \text{pr}_1 \\
X_1 & \xrightarrow{f_1} & Y_1 & \xrightarrow{q_1} & S \\
\pi_1 & & \pi_1 & & \pi_1 \\
S & & S & & S.
\end{array}
$$

We may assume that $X_2 = Y_2$ and $f_2 = \text{id}$, i.e., it suffices to show that the canonical map

$$(3.1.9.1) \quad f^! \mathcal{M}_1 \boxtimes_S \mathcal{M}_2 \xrightarrow{\sim} (f_1 \times \text{id})^!(\mathcal{M}_1 \boxtimes_S \mathcal{M}_2).$$

is an isomorphism. Since we have

$$\mathcal{M}_2 \cong D_{Y_2} \mathcal{M}_2 \cong R\text{Hom}(D_{Y_2} \mathcal{M}_2, K_{Y_2})$$

$$\cong R\text{Hom}(D_{Y_2} \mathcal{M}_2, (-\dim S)[-2\dim S], q_2^! \Lambda_S),$$

we may assume $\mathcal{M}_2 = R\text{Hom}(\mathcal{L}_2, q_2^! \Lambda_S)$ for some $\mathcal{L}_2 \in D^b_c(Y_2, \Lambda)$. By [Sai17a, Corollary 4.9], we have $SS(\mathcal{M}_2, Y_2/k) = SS(\mathcal{L}_2, Y_2/k)$. Thus by assumption, the morphism $q_2: Y_2 \to S$ is $SS(\mathcal{L}_2, Y_2/k)$-transversal. By Corollary 3.1.5, we have an isomorphism

$$(3.1.9.2) \quad \mathcal{M}_1 \boxtimes_S R\text{Hom}(\mathcal{L}_2, q_2^! \Lambda_S) \cong R\text{Hom}(\text{pr}_2^! \mathcal{L}_2, \text{pr}_1^! \mathcal{M}_1) \quad \text{in} \quad D^b_c(Y_1 \times_S Y_2, \Lambda),$$

$$(3.1.9.3) \quad f^! \mathcal{M}_1 \boxtimes_S R\text{Hom}(\mathcal{L}_2, q_2^! \Lambda_S) \cong R\text{Hom}((f_1 \times \text{id})^* \text{pr}_2^! \mathcal{L}_2, \text{pr}_1^! f^! \mathcal{M}_1) \quad \text{in} \quad D^b_c(X_1 \times_S Y_2, \Lambda).$$
We have

\[
(f_1 \times \text{id})^! (\mathcal{M}_1 \boxtimes \mathcal{M}_2) = (f_1 \times \text{id})^! (\mathcal{M}_1 \boxtimes \mathcal{R}Hom(\mathcal{L}_2, q_2^! \Lambda_S))
\]

(3.1.9.2)

\[
\cong (f_1 \times \text{id})^! (\mathcal{R}Hom(\text{pr}_2^* \mathcal{L}_2, \text{pr}_1^* \mathcal{M}_1))
\]

(3.1.7.1)

\[
\cong \mathcal{R}Hom((f_1 \times \text{id})^* \text{pr}_2^* \mathcal{L}_2, (f_1 \times \text{id})^* \text{pr}_1^* \mathcal{M}_1)
\]

(3.1.9.4)

\[
\cong f_1^! \mathcal{M}_1 \boxtimes \mathcal{R}Hom(\mathcal{L}_2, q_2^! \Lambda_S) \cong f_1^! \mathcal{M}_1 \boxtimes \mathcal{L}_2.
\]

(3.1.5)

This finishes the proof. □

3.2. Relative cohomological characteristic class.

3.2.1. We introduce some notation for convenience. For any commutative diagram

\[
\begin{array}{ccc}
W & \xrightarrow{h} & V \\
\downarrow f & & \downarrow g \\
\text{Spec} & & \text{Spec}
\end{array}
\]

of schemes, we put

(3.2.1.1) \[ K_W := Rf^! \Lambda, \]

(3.2.1.2) \[ K_{W/V} := Rh^! \Lambda_V. \]

Under the notation in 3.1.1, by Proposition 3.1.9, we have an isomorphism

(3.2.1.3) \[ K_{X_1/S} \boxtimes \mathcal{L}_2 \cong K_{X/S}. \]

3.2.2. Consider a cartesian diagram

\[
\begin{array}{ccc}
E & \xrightarrow{e} & D \\
\downarrow e & & \downarrow d \\
C & \xrightarrow{c} & X
\end{array}
\]

of schemes over $k$. Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be objects of $D^b_c(X, \Lambda)$ and $\mathcal{F} \otimes \mathcal{G} \to \mathcal{H}$ any morphism. By the Künneth isomorphism [SGA4, XVII, Théorème 5.4.3] and adjunction, we have

\[
e_!(c^! \mathcal{F} \boxtimes \mathcal{L}_2 \otimes \mathcal{G}) \cong c_! c^! \mathcal{F} \otimes \mathcal{G} \to \mathcal{F} \otimes \mathcal{G} \to \mathcal{H}.
\]

By adjunction, we get a morphism

(3.2.2.2) \[ c^! \mathcal{F} \boxtimes \mathcal{L}_2 \otimes \mathcal{G} \to e^! \mathcal{H}. \]

Thus we get a pairing

(3.2.2.3) \[ \langle , \rangle : H^0(C, c^! \mathcal{F}) \otimes H^0(D, d^! \mathcal{G}) \to H^0(E, e^! \mathcal{H}). \]

3.2.3. Now we define the relative Verdier pairing by applying the map (3.2.2.3) to relative cohomological correspondences. Let $\pi_1 : X_1 \to S$ and $\pi_2 : X_2 \to S$ be smooth morphisms. Consider a cartesian diagram

\[
\begin{array}{ccc}
E & \xrightarrow{e} & D \\
\downarrow e & & \downarrow d=(d_1, d_2) \\
C & \xrightarrow{c=(c_1, c_2)} & X = X_1 \times_S X_2
\end{array}
\]
of schemes over \( S \). Let \( \mathcal{F}_1 \in D^b_c(X_1, \Lambda) \) and \( \mathcal{F}_2 \in D^b_c(X_2, \Lambda) \). Assume that one of the following conditions holds:

1. \( \pi_1: X_1 \to S \) is \( SS(\mathcal{F}_1, X_1/k) \)-transversal;
2. \( \pi_2: X_2 \to S \) is \( SS(\mathcal{F}_2, X_2/k) \)-transversal.

By Corollary 3.1.5, we have

\[
R\text{Hom}(pr_2^*\mathcal{F}_2, pr_1^*\mathcal{F}_1) \otimes^L R\text{Hom}(pr_1^*\mathcal{F}_1, pr_2^*\mathcal{F}_2) \xrightarrow{(3.2.3.2)} (\mathcal{F}_1 \boxtimes_S^L R\text{Hom}(\mathcal{F}_2, \pi_2^*\Lambda_S)) \otimes^L (R\text{Hom}(\mathcal{F}_1, \pi_1^*\Lambda_S) \boxtimes_S^L \mathcal{F}_2) \xrightarrow{\text{evaluation}} \pi_1^1\Lambda_S \boxtimes_S^L \pi_2^1\Lambda_S \xrightarrow{(3.2.1.3)} K_{X/S}.
\]

By (3.1.7.1), (3.2.2.2), (3.2.2.3) and (3.2.3.2), we get the following pairings

\[
(3.2.3.3) \quad c_1 R\text{Hom}(c_2^*\mathcal{F}_2, c_1^*\mathcal{F}_1) \otimes^L d_1 R\text{Hom}(d_1^*\mathcal{F}_1, d_2^*\mathcal{F}_2) \to c_1 K_{E/S},
\]

\[
(3.2.3.4) \quad \langle \cdot \rangle: \text{Hom}(c_2^*\mathcal{F}_2, c_1^*\mathcal{F}_1) \otimes \text{Hom}(d_1^*\mathcal{F}_1, d_2^*\mathcal{F}_2) \to H^0(E, e^1(K_{X/S})) = H^0(E, K_{E/S}).
\]

The pairing (3.2.3.4) is called the relative Verdier pairing (cf. [SGA5, Exposé III (4.2.5)]).

**Definition 3.2.4.** Let \( f: X \to S \) be a smooth morphism purely of relative dimension \( n \) and \( \mathcal{F} \in D^b_c(X, \Lambda) \). We assume that \( f \) is \( SS(\mathcal{F}, X/k) \)-transversal. Let \( c = (c_1, c_2): C \to X \times_S X \) be a closed immersion and \( u: c_2^*\mathcal{F} \to c_1^*\mathcal{F} \) be a relative cohomological correspondence on \( C \). We define the relative cohomological characteristic class \( ccc_{X/S}(u) \) of \( u \) to be the cohomology class \( \langle u, 1 \rangle \in H^n_{C \cap X}(X, K_{X/S}) \) defined by the pairing (3.2.3.4).

In particular, if \( C = X \) and \( c: C \to X \times_S X \) is the diagonal and if \( u: \mathcal{F} \to \mathcal{F} \) is the identity, we write

\[
ccc_{X/S}(\mathcal{F}) = \langle 1, 1 \rangle \quad \text{in} \quad H^{2n}(X, \Lambda(n))
\]

and call it the relative cohomological characteristic class of \( \mathcal{F} \).

If \( S \) is the spectrum of a perfect field, then the above definition is [AS07, Definition 2.1.1].

**Example 3.2.5.** If \( \mathcal{F} \) is a locally constant and constructible sheaf of \( \Lambda \)-modules on \( X \), then we have \( ccc_{X/S}(\mathcal{F}) = \text{rank} \mathcal{F} \cdot c_n(\Omega_X^{\vee}/S) \cap [X] \in CH^n(X) \).

**Conjecture 3.2.6.** Let \( S \) be a smooth connected scheme over a perfect field \( k \) of characteristic \( p \). Let \( f: X \to S \) be a smooth morphism purely of relatively dimension \( n \) and \( \mathcal{F} \in D^b_c(X, \Lambda) \). Assume that \( f \) is \( SS(\mathcal{F}, X/k) \)-transversal. Let \( \text{cl}: \text{CH}^n(X) \to H^{2n}(X, \Lambda(n)) \) be the cycle class map. Then we have

\[
(3.2.6.1) \quad \text{cl}(ccc_{X/S}(\mathcal{F})) = ccc_{X/S}(\mathcal{F}) \quad \text{in} \quad H^{2n}(X, \Lambda(n)),
\]

where \( ccc_{X/S}(\mathcal{F}) \) is the relative characteristic class defined in Definition 2.4.3.

If \( S \) is the spectrum of a perfect field, then the above conjecture is [Sai17a, Conjecture 6.8.1].

### 3.3. Proper push-forward of relative cohomological characteristic class.

**3.3.1.** For \( i = 1, 2 \), let \( f_i: X_i \to Y_i \) be a proper morphism between smooth schemes over \( S \). Let \( X := X_1 \times_S X_2 \), \( Y := Y_1 \times_S Y_2 \) and \( f := f_1 \times_S f_2 \). Let \( p_i: X \to X_i \) and \( q_i: Y \to Y_i \) be the canonical projections for \( i = 1, 2 \). Consider a commutative diagram

\[
(3.3.1.1) \quad \begin{array}{ccc}
X & \xrightarrow{c} & C \\
\downarrow{f} & & \downarrow{g} \\
Y & \xrightarrow{d} & D
\end{array}
\]
of schemes over $S$. Assume that $c$ is proper. Put $c_i = p_i c$ and $d_i = q_i d$. By [Zh15, Construction 7.17], we have the following push-forward maps for cohomological correspondence (see also [SGA5, Exposé III, (3.7.6)] if $S$ is the spectrum of a field):

\begin{align}
(3.3.1.2) & \quad f_*: \text{Hom}(c_*^L L_2, c_1^L L_1) \to \text{Hom}(d_*^L f_2 L_2, d_1^L (f_1^* L_1)), \\
(3.3.1.3) & \quad f_*: g_* \text{RHom}(c_*^L L_2, c_1^L L_1) \to \text{RHom}(d_*^L f_2 L_2, d_1^L (f_1^* L_1)).
\end{align}

**Theorem 3.3.2** ([SGA5, Théorème 4.4]). For $i = 1, 2$, let $f_i: X_i \to Y_i$ be a proper morphism between smooth schemes over $S$. Let $X := X_1 \times_S X_2$, $Y := Y_1 \times_S Y_2$ and $f := f_1 \times_S f_2$. Let $p_i: X_i \to X_1$ and $q_i: Y \to Y_i$ be the canonical projections for $i = 1, 2$. Consider the following commutative diagram with cartesian horizontal faces

\[
\begin{array}{cccc}
C' & \xleftarrow{c'} & C & \xrightarrow{c} & C'' \\
& \downarrow f' & & \downarrow g & \downarrow c'' \\
D' & \xleftarrow{d'} & D & \xrightarrow{d} & D'' \\
& \downarrow f'' & & \downarrow d'' & \downarrow d'' \\
Y & & & & Y
\end{array}
\]

where $c', c'', d'$ and $d''$ are proper morphisms between smooth schemes over $S$. Let $c_i' = p_i c', c_i'' = p_i c'', d_i' = q_i d', d_i'' = q_i d''$ for $i = 1, 2$. Let $L_i \in D^b(X_i, \Lambda)$ and we put $M_i = f_i^* L_i$ for $i = 1, 2$.

Assume that one of the following conditions holds:

1. $X_1 \to S$ is SS($L_1, X_1/k$)-transversal;
2. $X_2 \to S$ is SS($L_2, X_2/k$)-transversal.

Then we have the following commutative diagram

\[
\begin{array}{ccc}
& & f_*c_*\text{RHom}(c_2^L L_2, c_1^L L_1) \otimes^L & f_*c_*'\text{RHom}(c_1^L L_1, c_2^L L_2) \xrightarrow{(1)} f_*c_*K_{C/S} \\
(3.3.2.1) & & \downarrow{(2)} & \downarrow{(4)} & \downarrow{(4)} \\
& & d_*\text{RHom}(d_2^L M_2, d_1^L M_1) \otimes^L & d_*\text{RHom}(d_1^L M_1, d_2^L M_2) \xrightarrow{(3)} d_*K_{D/S}
\end{array}
\]

where (3) is given by (3.2.3.3), (1) is the composition of $f_*((3.2.3.3))$ with the canonical map $f_*c_*' \otimes^L f_*c_*'' \to f_*c_*' \otimes c_*''$ (2) is induced from (3.3.1.3), and (4) is defined by

\[
(3.3.2.2) \quad f_*c_*K_{C/S} \simeq d_*g_*K_{C/S} = d_*g_*K_{D/S} \xrightarrow{\text{adj}} d_*K_{D/S}.
\]

If $S$ is the spectrum of a field, this is proved in [SGA5, Théorème 4.4]. We use the same notation as loc.cit.

**Proof.** By [Sai17a, Lemma 3.8 and Lemma 4.2.6] and the assumption, one of the following conditions holds:

1. $Y_1 \to S$ is SS($M_1, Y_1/k$)-transversal;
2. $Y_2 \to S$ is SS($M_2, Y_2/k$)-transversal.

Now we can use the same proof of [SGA5, Théorème 4.4]. We only sketch the main step. Put

\begin{align}
(3.3.2.3) & \quad \mathcal{P} = L_1 \boxtimes^S \text{RHom}(L_2, K_{X_2/S}), \quad \mathcal{Q} = \text{RHom}(L_1, K_{X_1/S}) \boxtimes^S L_2 \\
(3.3.2.4) & \quad \mathcal{E} = M_1 \boxtimes^S \text{RHom}(M_2, K_{Y_2/S}), \quad \mathcal{F} = \text{RHom}(M_1, K_{Y_1/S}) \boxtimes^S M_2.
\end{align}
Then the theorem follows from the following commutative diagram

$$
\begin{array}{c}
H_{\mathcal{O}}(\mathcal{L}_2, \mathcal{L}_1) \otimes H_{\mathcal{O}}(\mathcal{L}_1, \mathcal{L}_2) \to H^0(C, K_{C/S}) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
H_{\mathcal{O}}(d^*_2 f_2 \mathcal{L}_2, d^*_1 f_1 \mathcal{L}_1) \otimes H_{\mathcal{O}}(d^*_1 f_1 \mathcal{L}_1, d^*_2 f_2 \mathcal{L}_2) \to H^0(D, K_{D/S}).
\end{array}
$$

**Corollary 3.3.3** ([SGA5, Corollaire 4.5]). Under the assumptions of Theorem 3.3.2, we have a commutative diagram

\[(3.3.3.1)\]  
\[\begin{array}{c}
\text{Hom}(c^*_2 \mathcal{L}_2, c^*_1 \mathcal{L}_1) \otimes \text{Hom}(c^*_1 \mathcal{L}_1, c^*_2 \mathcal{L}_2) \to H^0(C, K_{C/S}) \\
\downarrow (3.3.1.2) \otimes (3.3.1.2) \downarrow \quad \quad \downarrow g^* \\
\text{Hom}(d^*_2 f_2 \mathcal{L}_2, d^*_1 f_1 \mathcal{L}_1) \otimes \text{Hom}(d^*_1 f_1 \mathcal{L}_1, d^*_2 f_2 \mathcal{L}_2) \to H^0(D, K_{D/S}).
\end{array}\]

**Corollary 3.3.4.** Let \( S \) be a smooth connected scheme over a perfect field \( k \) of characteristic \( p \). Let \( f: X \to S \) be a smooth morphism purely of relative dimension \( n \) and \( g: Y \to S \) a smooth morphism purely of relative dimension \( m \). Assume that \( f \) is \( SS(F, X/k) \)-transversal. Then for any proper morphism \( h: X \to Y \) over \( S \),

\[(3.3.4.1)\]

\[f_\ast c_{X/S}(F) = c_{Y/S}(Rf_\ast F) \quad \text{in} \quad H^{2m}(Y, \Lambda(m)).\]

**Proof.** This follows from Corollary 3.3.3 and Definition 3.2.4. \(\square\)

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