LINKS WITH TRIVIAL ALEXANDER MODULE AND NONTRIVIAL MILNOR INVARIANTS

STAVROS GAROUFALIDIS

Abstract. Cochran constructed many links with Alexander module that of the unlink and some nonvanishing Milnor invariants, using as input commutators in a free group and as an invariant the longitudes of the links. We present a different and conjecturally complete construction, that uses elementary properties of clasper surgery, and a different invariant, the tree-part of the LMO invariant. Our method also constructs links with trivial higher Alexander modules and nontrivial Milnor invariants.

1. Introduction

1.1. History of the problem. Two of the best studied topological invariants a link $L$ in $S^3$ are its Alexander module $A(L)$ which measures the homology of the universal abelian cover of $S^3 - L$, and its collection of Milnor invariants $\bar{\mu}(L)$, which are concordance (and sometimes link homotopy) invariants, defined modulo a recursive indeterminacy. Let us say that $L$ has trivial Alexander module (resp. Milnor invariants) if $A(L) = A(O)$ (resp. $\bar{\mu}(L) = \bar{\mu}(O) = 0$) for an unlink $O$. Despite the indeterminacy of the Milnor invariants, note that the vanishing of all Milnor invariants is a well-defined statement.

Using the language of longitudes $\lambda_i$ of components of $L$, Milnor showed that a link $L$ has vanishing Milnor invariants iff $\lambda_i(L) \subset \pi_i$ for all $i$, where $\pi = \pi_1(S^3 - L)$ and $\pi_i = \cap_{n=1}^\infty \pi_n$ is the intersection of the lower central series $\pi_i$ of $\pi$, defined by $\pi_1 = \pi$ and $\pi_{n+1} = [\pi_n, \pi]$, see [Mi]. $L$ has trivial Alexander module iff there is a map $\pi \rightarrow F/[F, F], [F, F]$ which induces an isomorphism $\pi/[[\pi, \pi], [\pi, \pi]] \cong F/[F, F], [F, F]$.

It is natural to ask how independent are the conditions of trivial Alexander module and trivial Milnor invariants. In a sense, this question asks for a comparison between the lower central series and the commutator series of a link group.

In one direction, Levine showed that the vanishing of the Milnor invariants of a link $L$ implies that a localization $A(L)_S$ of its Alexander module (although not the Alexander module itself) vanishes, where $S \subset \mathbb{Z}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$ is the multiplicative set of polynomials that evaluate to $\pm 1$ at $t_1 = \cdots = t_r = 1$; see [Le]. A boundary link has vanishing Milnor invariants, and its Alexander module splits as a direct sum of a trivial module and a torsion module. It was shown in [GL2] that all torsion modules with the appropriate symmetry can be realized.

In the opposite direction, if $L$ has trivial Alexander module, then it is known that some low order Milnor invariants vanish, [Le, GL2]. For example, all nonrepeated (link homotopy) invariants with at most 5 indices vanish. On the other hand, Cochran constructed a class of links with trivial Alexander module and nontrivial Milnor invariants; such links are not even be concordant to homology boundary links.

Cochran’s construction used iteration, and used as a pattern certain elements in the lower central series of the free group. There is enough explicitness and control on the iteration that enabled Cochran to compute the longitudes directly and verify that these links have vanishing Alexander modules. Further, a geometric interpretation of Milnor invariants in terms of cycles on Seifert surfaces allowed Cochran to conclude that the constructed links have nontrivial Milnor invariants.

As an elementary application of the calculus of claspers, we will construct a plethora of links with vanishing Alexander module. For these links, we can compute the tree part of the LMO invariant (which can be identified with Milnor invariants, [HM]), using formal Gaussian integration. As a result, we will construct...
many (and conjecturally all) links with trivial Alexander module and nontrivial Milnor invariants. The next definition explains the patterns that we will use in our construction.

**Definition 1.1.** Let \( A^{tr}(r) \) (or simply, \( A^{tr} \), in case \( r \) is clear) denote the vector space over \( \mathbb{Q} \) generated by vertex-oriented unitrivalent trees, whose univalent vertices are labeled by \( r \) colors, modulo the AS and IHX relation. \( A^{tr}(r) \) is a graded vector space, where the degree of a graph is half the number of vertices. We will call a tree of degree 1 (with two univalent vertices and no trivalent ones) a **strut**.

A pattern \( \beta \) is an element of \( A^{tr}(r) \) which is represented by a tree which has a trivalent vertex \( v \) such that \( \beta - v \) has no strut components.

The next figure gives some examples of nonvanishing patterns:

![Graphs](https://via.placeholder.com/150)

**Theorem 1.** For every nonvanishing pattern \( \beta \in A^{tr}(m) \) there exists a link \( L(\beta) \) with \( r \) components such that \( A(L(\beta)) = A(\mathcal{O}) \), all Milnor invariants of degree less than \( m \) vanish and some Milnor invariant of degree \( m \) do not.

Our construction adapts without change to the case of links with trivial higher Alexander modules. Although classical, these modules appeared only recently in work of Cochran-Orr-Teichner [COT] and subsequent work of Cochran, [Co2]. Given a group \( \pi \), consider its commutator series defined by \( \pi(n+1) = [\pi(n), \pi(n)] \).

**Definition 1.2.** We will say that a link \( L \) in a homology sphere \( M \) has **trivial \( n \)-th Alexander module** if it has a map \( \pi \to F/F(n+1) \) which induces an isomorphism \( \pi/\pi(n+1) \cong F/F(n+1) \), where \( \pi = \pi_1(M - L) \).

The next definition explains the \( n \)-patterns which we will use.

**Definition 1.3.** Let \( c^{(n)} \) be a unitrivalent tree defined by

![Graphs](https://via.placeholder.com/150)

In other words, we are adding two univalent vertices in \( c^{(n+1)} \) to each of the univalent vertices of \( c^{(n)} \). An \( n \)-pattern \( \beta^{(n)} \) is an element of \( A^{tr}(r) \) which is represented by a tree \( \beta^{(n)} \) such that \( c^{(n)} \subset \beta^{(n)} \) and \( \beta^{(n)} - c^{(n)} \) has no strut components.

The proof of Theorem 1 generalizes without change to the following

**Theorem 2.** For every nonvanishing \( n \)-pattern \( \beta^{(n)} \in A^{tr}(m) \) there exists a link \( L(\beta^{(n)}) \) with \( r \) components with trivial \( n \)-th Alexander module, such that all Milnor invariants of degree less than \( m \) vanish and some Milnor invariant of degree \( m \) do not.

## 2. Constructing links by surgery on claspers

### 2.1. What is surgery on a clasper?

As we mentioned in the introduction, we will construct links of Theorem 2 using surgery on claspers. Since claspers play a key role in geometric constructions, as well as in the theory of finite type invariants, we include a brief discussion here. For a reference on claspers and their associated surgery, we refer the reader to [Gu2] and also to [GGP, Section 2] (where claspers were called clovers instead). It suffices to say that a clasper is a thickening of a trivalent graph, and it has a preferred set of loops, called the leaves. The degree of a clasper is the number of trivalent vertices (excluding those at the leaves). With our conventions, the smallest clasper is a Y-clasper (which has degree one and three leaves), so we explicitly exclude struts (which would be of degree zero with two leaves).

A clasper \( G \) of degree 1 is an embedding \( G : N \to M \) of a regular neighborhood of the graph \( \Gamma \).
in a 3-manifold $M$. Surgery on $G$ can be described by cutting $G(N)$ from $M$ (which is a genus 3 handlebody), twisting by a fixed diffeomorphism of its boundary (which acts trivially on the homology of the boundary) and gluing back. We will denote the result of surgery by $M_G$. Alternatively, we can describe surgery on $G$ by surgery on a framed six component link (the image of $L$) in $M$. The six component link consists of a 0-framed Borromean ring and an arbitrarily framed three component link, the so-called leaves of $G$. If one of the leaves bounds a 0-framed disk disjoint from the rest of $G$, then surgery on $G$ does not change the ambient 3-manifold $M$, although it can change an embedded link in $M$. In particular, surgery on a clasper of degree 1 is shown as follows:

In general, surgery on a clasper $G$ of degree $n$ can be described in terms of simultaneous surgery on $n$ claspers $G_1, \ldots, G_n$, which are obtained from $G$ after breaking its edges and inserting Hopf links as follows:

2.2. A basic principle. Surgery on a clasper is described by twisting by a surface diffeomorphism that acts trivially on homology, thus we have the basic principle:

\[
\text{Clasper surgery preserves the homology}
\]

Surgery on claspers with leaves of a restricted type has already been studied and used successfully in $\text{GR}$ (where the leaves were assumed null homologous in a knot complement), $\text{GL1}$ (and where the leaves where in the kernel of a map to a free group). It is important to study not only 3-manifolds but rather pairs of 3-manifolds together with a representation of their fundamental group into a fixed group. Claspers adapt well to this point of view, as we explain next.

Consider a pair $(N, \rho)$ of a 3-manifold $N$ (possibly noncompact) and a representation $\rho : \pi_1(N) \to \Gamma$ for some group $\Gamma$. Consider a clasper $G \subset N$ whose leaves are mapped to 1 under $\rho$. We will call such claspers $\rho$-null, or simply null, if $\rho$ is clear. Surgery on $G$ gives rise to a 4-manifold $W$ whose boundary consists of one copy of $N$ and one copy of $N_G$. We may think that $W$ is obtained by attaching $6n$ 2-handles on $N \times I$, where $n = \text{degree}(G)$. Since the cores of these handles lie in the kernel of $\rho$, it follows that $\rho$ extends over $W$, and in particular restricts to a representation $\rho_G$ on the end $N_G$ of $W$.

Lemma 2.1. We have $H_\ast(N, \rho) \cong H_\ast(N_G, \rho_G)$.

Proof. Let $\bar{N}$ (resp. $\bar{N}_G$) denote the cover of $N$ (resp. $N_G$) corresponding to $\rho$ (resp. $\rho_G$). Surgery on $G$ is equivalent to surgery on a collection $\{G_1, \ldots, G_k\}$ of degree 1 claspers, constructed by inserting Hopf links in the edges of $G$. Each $G_i$ lifts to a collection $\bar{G}_i$ of claspers in $\bar{N}$; let $\bar{G} = \bar{G}_1 \cup \ldots \bar{G}_k$. Then, $\bar{N}_G$ can be identified with $(\bar{N})_G$. Since clasper surgery preserves homology, the result follows.

We will adapt the above lemma in the following situation. Suppose that $G$ is a clasper in the complement of an unlink $X_0 = S^3 - \mathcal{O}$ of $r$ components whose leaves are null homologous in $X_0$, and let $(M, L)$ denote the result of surgery along $G$ on the pair $(S^3, \mathcal{O})$. It follows that $G$ lifts to a family $\bar{G}$ of claspers in $\bar{X}_0$ (the universal abelian cover of $X$) and that $\bar{X}$ is obtained from $\bar{X}_0$, by surgery on $\bar{G}$, where $X = M - L$. Since $A(L) = H_1(\bar{X}, \bar{x})$, and clasper surgery preserves homology, it follows that $A(M, L) = A(\mathcal{O})$. 

\[ \text{links with trivial Alexander module and nontrivial Milnor invariants} \]
The Aarhus integral in brief. This is the content of the next section. We describe surgery adequately, but also the invariants which we will use behave well with respect to \([G\text{u}2, H]\). From our point of view though, these pictures are complicated and unnecessary; since not only use various descriptions of surgery on a clasper that were discussed at length by Goussarov and Habiro at \([A]\), how this integration works. Consider an element

\[
\int_{S^3} Z(S^3, \mathcal{O})
\]

(where \(X\) is a set of variables in 1-1 correspondence with the components of \(C\)). Let us briefly recall from \([A]\) how this integration works. Consider an element

\[
s = \exp \left( \frac{1}{2} \sum_{x,y \in X} \left| Q_{xy} \right| \right) R,
\]
with $R$ a series of graphs that do not contain a strut whose legs are colored by $X$. Notice that $Q$ and $R$, the $X$-strutless part of $s$, are uniquely determined by $s$. Then, the integration $\int dX(s)$ glue all the $X$-colored legs of $R$ pairwise, using the negative inverse of the matrix $Q$. That is, when two legs $x, y$ of $R$ are glued, the resulting graph is multiplied by $-Q_{xy}$, the negative inverse of the matrix $Q_{xy}$.

It follows immediately that the tree-part $Z^{tr}(M, L)$ of $Z(M, L)$ depends only on the tree-part $Z^{tr}(S^3, \mathcal{O} \cup C)$ of $Z(S^3, \mathcal{O} \cup C)$.

3.2. **Claspers and the Aarhus integral.** Let us adapt the above discussion when the link $C$ is one that describes clasper surgery. Consider a null clasper $G \subset S^3 - \mathcal{O}$ of degree 1 constructed from a pattern $\beta$ and let $(M, L) = (S^3, \mathcal{O})_G$. Let $Z^{\min}(M, L)$ denotes the lowest degree nonvanishing tree part of $Z^{tr}(M, L)$. Assuming that the pattern is nonvanishing, and after we choose string-link representatives of $L \cup G$, we will show that

**Proposition 3.1.** We have

$$Z^{\min}(M, L) = \beta \in A^{tr}$$

It is clear that this concludes Theorem 1.

**Proof.** (of Proposition 3.1) Surgery on $G$ is equivalent to surgery on a 6 component link $C = C^e \cup C^l$; see Section 2.1. $C^e$ is a borromean link and $C^l$ consists of the leaves of $G$. In the obvious basis, the linking matrix of $C$ and its negative inverse are given as follows:

$$\begin{pmatrix} 0 & I \\ I & \text{lk}(C_i^l, C_j^l) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \text{lk}(C_i^e, C_j^e) & -I \\ -I & 0 \end{pmatrix}.$$

In particular, a univalent vertex labeled by a leaf has to be glued to a univalent vertex labeled by the corresponding edge. Let $A_i = \{C_i^e, C_i^l\}$ denote the arms of $G$ for $i = 1, 2, 3$. It is a key fact that surgery on any proper subcollection of the set $\{A_1, A_2, A_3\}$ of arms does not change the pair $(S^3, \mathcal{O})$. In other words, alternating with respect to the 8 subsets of the set of arms we have that

$$Z([\mathcal{O} \cup C], G) = Z([\mathcal{O} \cup C], \{A_1, A_2, A_3\})$$

The nontrivial contributions to the left hand side come from the $(\mathcal{O} \cup C)$-strutless part of $Z(S^3, \mathcal{O} \cup C)$ that consists of graphs with legs on $A_1$ and on $A_2$ and on $A_3$.

What kind of diagrams in $Z^{tr}(S^3, \mathcal{O} \cup C)$ contribute to the above sum? Consider a disjoint union $D$ of trees whose legs are labeled by $\mathcal{O} \cup C$. $D$ must have a leg (i.e., univalent vertex) labeled by $C_i^e$ or by $C_i^l$ for each $i = 1, 2, 3$. If $D$ has a leg labeled by $C_i^e$, then due to the shape of the gluing matrix, $D$ must have a $C_i^l$-labeled leg. Thus, in all cases, $D$ must have legs labeled by all three edges $C_i^e$ of $G$.

Consider a tree $T$ labeled by $\mathcal{O} \cup C$. If $T$ has a $C^l_i$-labeled leg, then it must either have legs labeled by all three edges of $G$, or else it must have a leg labeled by $C_i^l$. Indeed, $C_i^e$ is an unknot in a ball disjoint from $\mathcal{O} \cup C - \{C_i^e\}$, thus the rest of the trees of vanishing coefficient in $Z^{tr}(S^3, \mathcal{O} \cup C)$.

Consider further a vortex $Y$ (that is, a unitrivalent graph of the shape $Y$ with three univalent vertices and one trivalent one) whose legs are labeled by three leaves of $G$. Then, the coefficient of $Y$ in $Z(S^3, \mathcal{O} \cup C)$ is 1.

Consider further a tree $T$ with one univalent vertex labeled by a leaf $C_i^l$ of $G$ and all other vertices labeled by $\mathcal{O}$. Recall the corresponding rooted tree $T_i$ which is a component of $\beta - v$. Then the coefficient of $T$ in $Z^{tr}(S^3, \mathcal{O} \cup C)$ is zero if $\deg(T) < \deg(T_i)$ and equals to 1 if $T = T_i$. This, together with the above discussion and the gluing rules concludes the proof of Proposition 3.1. The argument is best illustrated by the following figure:
The above proposition and its proof generalize easily to the case of claspers \( G \) corresponding to nonvanishing \( n \)-patterns \( \beta^{(n)} \). In that case, if \((M, L)\) denote the corresponding link, we still have that
\[
Z^{\min}(M, L) = \beta^{(n)} \in A^{tr}
\]
which implies Theorem 2.

**Remark 3.2.** In the above discussion we have silently chosen dotted Morse link representatives (or equivalently, string-link representatives) and we ought to have normalized the Aarhus integral. But this does not affect the lowest degree nonvanishing tree part.

The links constructed by clasper surgery in Theorem 1 include the links that Cochran constructed via Seifert surfaces.

**Question 1.** Does Section 2 construct every link with trivial Alexander module?

**References**

[A] D. Bar-Natan, S. Garoufalidis, L. Rozansky and D. Thurston, *The Aarhus integral of rational homology 3-spheres I-III*, Selecta Math. in press.

[Co1] T. Cochran, *Links with trivial Alexander’s module but nonvanishing Massey products*, Topology 29 (1990) 189–204.

[Co2] _____, *Noncommutative knot theory*, preprint 2002 [math.GT/0206258]

[COT] _____, K. Orr and P. Teichner, *Knot concordance, Whitney towers and \( L^2 \) signatures*, [math.GT/9908117] to appear in the Annals of Math.

[CT] J. Conant and P. Teichner, *Grope cobordism of classical knots*, preprint 2000. [math.GT/0012118]

[GGP] S. Garoufalidis, M. Goussarov and M. Polyak, *Calculus of clovers and finite type invariants of 3-manifolds*, Geometry and Topology, 5 (2001) 75–108.

[GK] _____and A. Kricker, *A rational noncommutative invariant of boundary links*, preprint 2001. [math.GT/0105025]

[GL1] _____and J. Levine, *Homology surgery and invariants of 3-manifolds*, Geometry and Topology, 5 (2001) 551–578.

[GL2] _____and _____, *Analytic invariants of boundary links*, Journal of Knot Theory and its Rami. 11 (2002) 283–293.

[GL3] _____and _____, *Concordance and 1-loop clovers*, Algebraic and Geometric Topology, 1 (2001) 687–697.

[GR] _____and L. Rozansky, *The loop expansion of the Kontsevich integral, abelian invariants of knots and \( S \)-equivalence*, preprint [math.GT/0003167].

[Gu1] M. Goussarov, *Finite type invariants and \( n \)-equivalence of 3-manifolds*, C. R. Acad. Sci. Paris Ser. I. Math. 329 (1999) 517–522.

[Gu2] _____, *Knotted graphs and a geometrical technique of \( n \)-equivalence*, St. Petersburg Math. J. 12-4 (2001).

[HM] N. Habegger, G. Masbaum, *The Kontsevich Integral and Milnor’s Invariants*, Topology 39 (2000) 1253–1289.

[H] K. Habiro, *Clasper theory and finite type invariants of links*, Geometry and Topology 4 (2000), 1–83.

[LMO] T.T.Q. Le, J. Murakami, T. Ohtsuki, *A universal quantum invariant of 3-manifolds*, Topology 37 (1998) 539–574.

[Le] J. Levine, *Localization of link modules*, Low-dimensional topology (San Francisco, Calif., 1981) Contemp. Math., 20 213–229 AMS 1983.

[Mi] J.W. Milnor, *Link groups*, Ann. Math. 59 (1954) 177–195.

[Tr] L. Traldi, *Milnor’s invariants and the completion of link modules*, Transactions AMS 284 (1984) 401–424.

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA.

E-mail address: stavros@math.gatech.edu