Extensions of dualities and a new approach to the Fedorchuk duality

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Abstract
Applying a general categorical construction for the extension of dualities, we present a new proof of the Fedorchuk duality between the category of compact Hausdorff spaces with their quasi-open mappings and the category of complete normal contact algebras with suprema-preserving Boolean homomorphisms which reflect the contact relation.

1 Introduction

When $T : \mathcal{A} \rightarrow \mathcal{B}$ is a dual equivalence between categories $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{B}$ is a full subcategory of a category $\mathcal{C}$, it is not surprising that one can construct a category $\mathcal{D}$ containing $\mathcal{A}$ as a full subcategory, and a dual equivalence $\hat{T} : \mathcal{D} \rightarrow \mathcal{C}$ extending $T$: inside $\mathcal{C}$, one may just replace $\mathcal{B}$ by $\mathcal{A}$ and adjust the composition using the dual equivalence to obtain the category $\mathcal{D}$. This ad-hoc procedure, however, does not make for a naturally described category $\mathcal{D}$ since the definition of the hom-sets of $\mathcal{D}$ changes with the two types of objects involved. The principal goal of this paper is therefore to model the objects of a suitable extension category $\mathcal{D}$ of $\mathcal{A}$ dually equivalent to $\mathcal{C}$ in a natural way, as $\mathcal{A}$-objects provided with a structure that gives them a strong algebraic flavour. We apply simple general categorical constructions in order to achieve this.
goal, as summarized by Theorem 3.6 below. The key ingredient to the construction is our assumption that there exist a class $\mathcal{P}$ of $\mathcal{C}$-morphisms with certain properties, which one may interpret as presenting $\mathcal{B}$-objects as “projective covers” of $\mathcal{C}$-objects. Our role model is the Fedorchuk duality theorem, for which we give a new proof based on our general categorical construction (see Theorem 4.9). In this case the morphisms in $\mathcal{P}$ may be seen as generalized proximity structures on the objects of $\mathcal{A}$.

Recall that Fedorchuk [14] established a dual equivalence between the category $\text{CH}_\text{qop}$ of compact Hausdorff spaces with quasi-open continuous mappings and the category $\text{CNCA}_{\text{sup-ref}}$ of complete normal contact algebras and their suprema-preserving Boolean homomorphisms reflecting the contact relation. The notion of normal contact algebra had been introduced by de Vries [6] earlier under the name compingent algebra. He proved that there exists a dual equivalence between the category $\text{CH}$ of compact Hausdorff spaces and continuous mappings and the category $\text{CNCA}$ of complete normal contact algebras and their appropriate morphisms. Fedorchuk proved his duality result using de Vries’ duality theorem.

Here we give a direct proof of Fedorchuk’s duality theorem, by extending the restriction of the Stone duality between the category $\text{ECH}_\text{op}$ of extremally disconnected compact Hausdorff spaces with open continuous mappings and the category $\text{CBA}_{\text{sup}}$ of complete Boolean algebras and their suprema-preserving Boolean homomorphisms. (This restrictability of the Stone duality follows immediately from a result obtained in [9, Corollary 3.2(c)]; see also [7, Corollary 2.4(c)].) Now, considering $\mathcal{A} = \text{CBA}_{\text{sup}}$, $\mathcal{B} = \text{ECH}_\text{op}$, $\mathcal{C} = \text{CH}_\text{qop}$, and letting $\mathcal{P}$ be the class of all so-called irreducible $\mathcal{C}$-morphisms with domains in $\mathcal{B}$, our categorical extension theorem describes a category $\mathcal{D}$ dual to the category $\mathcal{C}$. We then show that the category $\mathcal{D}$ is equivalent to Fedorchuk’s category $\text{CNCA}_{\text{sup-ref}}$, thus completing a new proof of his duality theorem. Since all elements of the class $\mathcal{P}$ are projective covers, in this way we reveal the connection between Fedorchuk’s duality result and the theory of absolutes. (For the notions of projective cover and absolute, see Section 2.)

The paper is organized as follows. Section 2 contains all preliminary facts and definitions which are used in this paper. In Section 3, we present the categorical extension theorem for dualities (Theorem 3.6) which, in Section 4, is used to give our new proof of the Fedorchuk duality (Theorem 4.9).

If $\mathcal{C}$ denotes a category, we write $X \in |\mathcal{C}|$ if $X$ is an object of $\mathcal{C}$, and $f \in \mathcal{C}(X, Y)$ if $f$ is a morphism of $\mathcal{C}$ with domain $X$ and codomain $Y$.

For unexplained notation and notions we invite the reader to consult [1] for category theory, [13] for topology and [17] for Boolean algebras.

## 2 Preliminaries

Below we first recall the notions of contact algebra and normal contact algebra. They can be regarded as algebraic analogues of proximity spaces (see [12, 24, 5, 3, 20] for proximity spaces). Generally speaking, in this paper we work mainly with Boolean algebras with supplementary structures on them. In all cases, we will say that the structured Boolean algebra in question is complete if the underlying Boolean algebra
is complete. Our standard notation for the operations of a Boolean algebra $B$ is indicated by $B = (B, \land, \lor, *, 0, 1)$; note in particular that the complement in $B$ is denoted by $*$, and that 0 and 1 denote the least and largest element in $B$, not excluding the case $0 = 1$.

**Definition 2.1.** ([10]) A Boolean contact algebra, or, simply, contact algebra (abbreviated as CA), is a structure $(B, C)$, where $B$ is a Boolean algebra, and $C$ a binary relation on $B$, called a contact relation, which satisfies the following axioms:

(C1). If $a \neq 0$ then $aCa$.

(C2). If $aCb$ then $a \neq 0$ and $b \neq 0$.

(C3). $aCb$ implies $bCa$.

(C4). $aC(b \lor c)$ if and only if $aCb$ or $aCc$.

Two contact algebras $(B, C)$ and $(B', C')$ are said to be isomorphic if there exists a CA-isomorphism between them, i.e., a Boolean isomorphism $\varphi : B \rightarrow B'$ such that, for all $a, b \in B$, $aCb$ if and only if $\varphi(a)C'\varphi(b)$.

With $-C$ denoting the set complement of $C$ in $B \times B$, we shall consider two more properties of contact algebras:

(C5). If $a(-C)b$ then $a(-C)c$ and $b(-C)c^*$ for some $c \in B$.

(C6). If $a \neq 1$ then there exists $b \neq 0$ such that $b(-C)a$.

A contact algebra $(B, C)$ is called a Boolean normal contact algebra or, briefly, normal contact algebra (abbreviated as NCA) [6, 14] if it satisfies (C5) and (C6). (Note that if $0 \neq 1$, then (C2) follows from the axioms (C4), (C3), and (C6).)

The notion of normal contact algebra was introduced by Fedorchuk [14] under the name Boolean $\delta$-algebra, as an equivalent expression of the notion of compingent Boolean algebra by de Vries (see the definition below). We call such algebras “normal contact algebras” because they form a subclass of the class of contact algebras which naturally arise in the context of normal Hausdorff spaces (see [10]).

**Definition 2.2.** For a contact algebra $(B, C)$ we define a binary relation $\ll_C$ on $B$, called non-tangential inclusion, by

$$a \ll_C b \text{ if and only if } a(-C)b^*.$$ 

If $C$ is understood, we shall simply write $\ll$ instead of $\ll_C$.

The relations $C$ and $\ll$ are inter-definable. For example, normal contact algebras may be equivalently defined – and exactly in this way they were introduced under the name of compingent Boolean algebras by de Vries in [6] – as a pair consisting of a Boolean algebra $B$ and a binary relation $\ll$ on $B$ satisfying the following axioms:
(≪1). \( a \ll b \) implies \( a \leq b \).

(≪2). \( 0 \ll 0 \).

(≪3). \( a \leq b \ll c \leq t \) implies \( a \ll t \).

(≪4). \( a \ll c \) and \( b \ll c \) implies \( a \lor b \ll c \).

(≪5). If \( a \ll c \) then \( a \ll b \ll c \) for some \( b \in B \).

(≪6). If \( a \neq 0 \) then there exists \( b \neq 0 \) such that \( b \ll a \).

(≪7). \( a \ll b \) implies \( b^* \ll a^* \).

Indeed, if \( (B, C) \) is an NCA, then the relation \( \ll_C \) satisfies the axioms (≪1) – (≪7). Conversely, having a pair \( (B, \ll) \), where \( B \) is a Boolean algebra and \( \ll \) is a binary relation on \( B \) which satisfies (≪1) – (≪7), we define a relation \( C_{\ll} \) by \( aC_{\ll}b \) if and only if \( a \ll b^* \); then \( (B, C_{\ll}) \) is an NCA. Note that the axioms (C5) and (C6) correspond to (≪5) and (≪6), respectively. It is easy to see that contact algebras could be equivalently defined as a pair of a Boolean algebra \( B \) and a binary relation \( \ll \) on \( B \) subject to the axioms (≪1) – (≪4) and (≪7).

The most important example of a CA is given by the regular closed sets of an arbitrary topological space \( X \). Let us start with some standard notations and conventions that we use throughout the paper. For a subset \( M \) of \( X \), we denote by \( \text{cl}_X(M) \) (or simply \( \text{cl}(M) \)) the closure of \( M \) in \( X \), and by \( \text{int}(M) \) its interior. \( \text{CO}(X) \) denotes the set of all clopen (= closed and open) subsets of \( X \); trivially, \( (\text{CO}(X), \cup, \cap, \setminus, \emptyset, X) \) is a Boolean algebra. \( \text{RC}(X) \) (resp., \( \text{RO}(X) \)) denotes the set of all regular closed (resp., regular open) subsets of \( X \); recall that a subset \( F \) of \( X \) is said to be \textit{regular closed} (resp., \textit{regular open}) if \( F = \text{cl}(\text{int}(F)) \) (resp., \( F = \text{int}(\text{cl}(F)) \)).

Note that in this paper (unlike in [13]) compact spaces are not assumed to be Hausdorff.

**Example 2.3.** For a topological space \( X \), the collection \( \text{RC}(X) \) becomes a complete Boolean algebra under the operations

\[
F \lor G \overset{\text{df}}{=} F \cup G, \quad F \land G \overset{\text{df}}{=} \text{cl}(\text{int}(F \cap G)), \quad F^* \overset{\text{df}}{=} \text{cl}(X \setminus F), \quad 0 \overset{\text{df}}{=} \emptyset, \quad 1 \overset{\text{df}}{=} X.
\]

The infinite operations are given by the formulas

\[
\bigvee \{F_\gamma \mid \gamma \in \Gamma\} = \text{cl}(\bigcup_{\gamma \in \Gamma} F_\gamma) \overset{\text{df}}{=} \text{cl}(\bigcup_{\gamma \in \Gamma} \text{int}(F_\gamma)) = \text{cl}(\bigcap_{\gamma \in \Gamma} F_\gamma)),
\]

\[
\bigwedge \{F_\gamma \mid \gamma \in \Gamma\} = \text{cl}(\text{int}(\bigcap_{\gamma \in \Gamma} \{F_\gamma \mid \gamma \in \Gamma\})),
\]

One defines the relation \( \rho_X \) on \( \text{RC}(X) \) by setting, for each \( F, G \in \text{RC}(X) \),

\[F \rho_X G \text{ if and only if } F \cap G \neq \emptyset.\]
Clearly, $\rho_X$ is a contact relation on $\text{RC}(X)$, called the *standard contact relation of $X$*. The complete CA $(\text{RC}(X), \rho_X)$ is called a *standard contact algebra*. Note that, for $F, G \in \text{RC}(X)$,

$$F \ll_{\rho_X} G \text{ if and only if } F \subseteq \text{int}_X(G).$$

Thus, if $X$ is a normal Hausdorff space then the standard contact algebra

$$(\text{RC}(X), \rho_X)$$

is a complete NCA.

**Example 2.4.** Let $B$ be a Boolean algebra. Then there exist a largest and a smallest contact relation on $B$; the largest one, $\rho_l$, is defined by

$$a \rho_l b \iff (a \neq 0 \text{ and } b \neq 0),$$

and the smallest one, $\rho_s$, by

$$a \rho_s b \iff a \land b \neq 0.$$

Note that, for $a, b \in B$,

$$a \ll_{\rho_s} b \iff a \leq b,$$

hence $a \ll_{\rho_s} a$, for any $a \in B$. Thus $(B, \rho_s)$ is a normal contact algebra.

We will need the following definition and assertion from [10]:

**Definition 2.5.** ([10]) For a contact algebra $(B, C)$ one defines the relation $R_{(B, C)}$ on the set of all filters on $B$ by

(1) $f R_{(B, C)} g$ if and only if $f \times g \subseteq C,$

for all filters $f, g$ on $B$.

**Proposition 2.6.** (a) ([10, Lemma 3.5, p. 222]) Let $(B, C)$ be a contact algebra. Then, for all $a, b \in B$, $aCb$ if and only if there exist ultrafilters $u, v$ in $B$ such that $a \in u$, $b \in v$ and $u R_{(B, C)} v$.

(b) ([10, 11]) If $(B, C)$ is a normal contact algebra, then $R_{(B, C)}$ is an equivalence relation.

**Definition 2.7.** A non–empty subset $\sigma$ of a CA $(B, C)$ is called a *cluster* if for all $x, y \in B$,

(1) $(\text{CL}1)$ If $x, y \in \sigma$ then $xCy$;

(2) $(\text{CL}2)$ If $x \lor y \in \sigma$ then $x \in \sigma$ or $y \in \sigma$.

(3) $(\text{CL}3)$ If $xCy$ for every $y \in \sigma$, then $x \in \sigma$. 

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The next theorem is used later on and may be proved exactly as Theorem 5.8 of [20]:

**Theorem 2.8.** A subset \( \sigma \) of a normal contact algebra \((B, C)\) is a cluster if and only if there exists an ultrafilter \( u \) in \( B \) such that

\[
(2) \quad \sigma = \{a \in B : aCb \text{ for every } b \in u\}.
\]

Moreover, given \( \sigma \) and \( a_0 \in \sigma \), there exists an ultrafilter \( u \) in \( B \) satisfying (2) which contains \( a_0 \).

**Corollary 2.9.** Let \((B, C)\) be a normal contact algebra and \( u \) be an ultrafilter in \( B \). Then there exists a unique cluster \( \sigma_u \) in \((B, C)\) containing \( u \); it is defined by the formula

\[
(3) \quad \sigma_u = \{a \in B \mid aCb \text{ for every } b \in u\}.
\]

Let us fix the notation for the Stone Duality ([25, 17]). We will denote by \( \text{ZCH} \) the category of all zero-dimensional compact Hausdorff spaces (= Stone spaces) and their continuous mappings, and by \( \text{BA} \) the category of Boolean algebras and Boolean homomorphisms. The Stone contravariant functors which define the Stone duality will be denoted by

\[
S^a : \text{BA} \to \text{ZCH} \quad \text{and} \quad S^t : \text{ZCH} \to \text{BA}.
\]

For \( A \in |\text{BA}| \), \( S^a(A) \) is the set \( \text{Ult}(A) \) of all ultrafilters in \( A \) endowed with a topology having as an open base the family \( \{s_A(a) \mid a \in A\} \), where

\[
s_A(a) \overset{\text{df}}{=} \{u \in \text{Ult}(A) \mid a \in u\}
\]

for all \( a \in A \). For \( X \in |\text{ZCH}| \), one sets

\[
S^t(X) \overset{\text{df}}{=} \text{CO}(X).
\]

For morphisms \( f \in \text{ZCH}(X, Y) \) and \( \varphi \in \text{BA}(B_1, B_2) \) one puts

\[
S^t(f)(F) = f^{-1}(F) \quad \text{and} \quad S^a(\varphi)(u) = \varphi^{-1}(u)
\]

for all \( F \in \text{CO}(Y) \) and \( u \in \text{Ult}(B_2) \). Now, for every Boolean algebra \( A \), the map

\[
s_A : A \to S^t(S^a(A)), \quad a \mapsto s_A(a),
\]

is a Boolean isomorphism, and for every Stone space \( X \), the map

\[
t_X : X \to S^a(\text{CO}(X)), \quad x \mapsto u_x,
\]

is a homeomorphism; here, for every \( x \in X \),

\[
u_x \overset{\text{df}}{=} \{P \in \text{CO}(X) \mid x \in P\}.
\]

Moreover, \( s_A \) and \( t_X \) are natural in \( A \) and \( X \).
2.11. Let us recall some standard properties for a continuous map of topological spaces: \( f : X \to Y \) is

- **closed** if the image of each closed set is closed;
- **open** if the image of each open set is open;
- **perfect** if it is closed and has compact fibres;
- **quasi-open** ([18]) if \( \text{int}(f(U)) \neq \emptyset \) for every non-empty open subset \( U \) of \( X \);
- **skeletal** ([19]) if \( \text{int}(f^{-1}(\text{cl}(V))) \subseteq \text{cl}(f^{-1}(V)) \);
- **irreducible** if \( f(X) = Y \) and if, for every proper closed subset \( F \) of \( X \), \( f(F) \neq Y \).

Recall that, for a regular space \( X \), a space \( EX \) is called an **absolute of \( X \)** if there exists a perfect irreducible mapping \( \pi_X : EX \to X \) and every perfect irreducible preimage of \( EX \) is homeomorphic to \( EX \) (see, e.g., [22]). It is well-known that:

(a) the absolute is unique up to homeomorphism;

(b) a space \( Y \) is an absolute of a regular space \( X \) if and only if \( Y \) is an extremally disconnected Tychonoff space for which there exists a perfect irreducible mapping \( \pi : Y \to X \) (called the **projective cover of \( X \)**);

(c) if \( X \) is a compact Hausdorff space, then \( EX = S^a(\text{RC}(X)) \) and \( \pi_X(u) = \bigcap u \) for every \( u \in \text{Ult}(\text{RC}(X)) \) (= \( S^a(\text{RC}(X)) \), with \( S^a : \text{BA} \to \text{ZCH} \) the Stone contravariant functor).

2.12. Let \( \mathcal{C} \) be a subcategory of the category \( \text{Top} \) of all topological spaces and all continuous mappings between them. Recall that an object \( P \in |\mathcal{C}| \) is called a **projective object** in \( \mathcal{C} \) if for every \( g \in \mathcal{C}(P, Y) \) and every perfect surjection \( f \in \mathcal{C}(X, Y) \), there exists \( h \in \mathcal{C}(P, X) \) such that \( f \circ h = g \).

A. M. Gleason [15] proved:

_In the category \( \text{CH} \) of compact Hausdorff spaces and continuous mappings, the projective objects are precisely the extremally disconnected spaces._

3 Extensions of dualities

3.1. Every functor \( F : \mathcal{A} \to \mathcal{B} \) admits a factorization \( (F_1, \mathcal{B}_F, F_2) \) with

\[
F_1 : \mathcal{A} \to \mathcal{B}_F, \quad F_2 : \mathcal{B}_F \to \mathcal{B} \quad \text{and} \quad F_2 \circ F_1 = F,
\]

where \( F_1 \) is bijective on objects and \( F_2 \) full and faithful (i.e., the hom-maps of \( F_2 \) are bijective): simply define the class of objects of \( \mathcal{B}_F \) and its hom-sets by

\[
|\mathcal{B}_F| \overset{\text{df}}{=} |\mathcal{A}| \quad \text{and} \quad \mathcal{B}_F(A, A') \overset{\text{df}}{=} \mathcal{B}(F(A), F(A')).
\]
respectively, and let the composition in $\mathcal{B}_F$ be as in $\mathcal{B}$; $F_1$ maps objects identically and morphisms like $F$, while $F_2$ maps objects like $F$ and morphisms identically. (Note that this actually defines an orthogonal factorization system for categories and functors, known as the full-image factorization.)

3.2. Let $\mathcal{B}$ be a full subcategory of $\mathcal{C}$. For convenience we assume that $\mathcal{B}$ is replete in $\mathcal{C}$ (i.e., whenever $C \in |\mathcal{C}|$ is isomorphic to $B \in |\mathcal{B}|$, then $C \in |\mathcal{B}|$).

We form the *comma category* $\mathcal{B} \downarrow \mathcal{C}$.

Recall that its objects are all $\mathcal{C}$-morphisms with domain in $|\mathcal{B}|$, and a $\mathcal{B} \downarrow \mathcal{C}$-morphism $(u, v) : \pi \rightarrow \pi'$ is defined as a pair $(u, v)$, where $u \in \mathcal{C}(\text{dom}(\pi), \text{dom}(\pi'))$, $v \in \mathcal{C}(\text{cod}(\pi), \text{cod}(\pi'))$, and

$$v \circ \pi = \pi' \circ u;$$

composition is as in $\mathcal{C}$, i.e., for $(u, v) : \pi \rightarrow \pi'$ and $(u', v') : \pi' \rightarrow \pi''$,

$$(u', v') \circ (u, v) \doteq (u' \circ u, v' \circ v).$$

There are obvious functors

$$\text{Dom, Cod} : \mathcal{B} \downarrow \mathcal{C} \rightarrow \mathcal{C}$$

given by

$$\text{Dom}(\pi) \doteq \text{dom}(\pi), \quad \text{Dom}((u, v)) \doteq u, \quad \text{Cod}(\pi) \doteq \text{cod}(\pi), \quad \text{Cod}((u, v)) \doteq v.$$

Note that Dom actually takes values in $\mathcal{B}$.

3.3. In addition to $\mathcal{B}$ and $\mathcal{C}$ as in 3.2, we now consider a class $\mathcal{P}$ of $\mathcal{C}$-morphisms satisfying

(P1) $\text{dom}(\pi) \in |\mathcal{B}|$ for all $\pi \in \mathcal{P}$,

(P2) $\text{Iso}(\mathcal{B}) \subseteq \mathcal{P}$,

(P3) $\text{Iso}(\mathcal{C}) \circ \mathcal{P} \circ \text{Iso}(\mathcal{B}) \subseteq \mathcal{P}$.

By (P1) we can form the full subcategory $\mathcal{B} \downarrow_{\mathcal{P}} \mathcal{C}$

of $\mathcal{B} \downarrow \mathcal{C}$ with object class $|\mathcal{B} \downarrow_{\mathcal{P}} \mathcal{C}| = \mathcal{P}$. Then (P2), (P3) amount to asking $\mathcal{B} \downarrow_{\mathcal{P}} \mathcal{C}$ to contain as objects all identity morphisms of $\mathcal{B}$ and to be closed under isomorphisms in $\mathcal{B} \downarrow \mathcal{C}$. Considering the restriction

$$\text{Cod}^\mathcal{P} : \mathcal{B} \downarrow_{\mathcal{P}} \mathcal{C} \rightarrow \mathcal{C}$$

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of Cod, we form its full-image factorization

\[(\text{Cod}_1^\mathcal{P}, \mathcal{C}_\mathcal{P}, \text{Cod}_2^\mathcal{P})\],

(writing for brevity \(\mathcal{C}_\mathcal{P}\) instead of \(\mathcal{C}_{\text{Cod}^\mathcal{P}}\)). Explicitly then,

\[|\mathcal{C}_\mathcal{P}| = \mathcal{P} \text{ and } \mathcal{C}_\mathcal{P}(\pi, \pi') = \mathcal{C}(\text{cod}(\pi), \text{cod}(\pi'))\]

for all \(\pi, \pi' \in \mathcal{P}\), with functors

\[\text{Cod}_1^\mathcal{P} : \mathcal{B} \downarrow \mathcal{P} \mathcal{C} \rightarrow \mathcal{C}_\mathcal{P}, \text{ Cod}_2^\mathcal{P} : \mathcal{C}_\mathcal{P} \rightarrow \mathcal{C}\]

satisfying \(\text{Cod}^\mathcal{P} = \text{Cod}_2^\mathcal{P} \circ \text{Cod}_1^\mathcal{P}\).

Let \(\pi, \pi' \in \mathcal{P}\) and \(v \in \mathcal{C}(\text{cod}(\pi), \text{cod}(\pi'))\). Sometimes, for clarity, if we regard the \(\mathcal{C}\)-morphism \(v\) as a \(\mathcal{C}_\mathcal{P}\)-morphism between \(\pi\) and \(\pi'\), then we will denote it by \((\pi, v, \pi')\).

**3.4.** Given \(\mathcal{B}, \mathcal{C}, \mathcal{P}\) satisfying (P1)-(P3) of 3.3, we say that \(\mathcal{P}\) is a \((\mathcal{B}, \mathcal{C})\)-covering class if

(P4) \(\text{Cod}_2^\mathcal{P}\) has a right inverse.

Clearly, condition (P4) is equivalent to asking that:

(P4') for every \(C \in |\mathcal{C}|\), there exists \(\pi_C : B_C \rightarrow C\) in \(\mathcal{P}\),

with the understanding that there is a choice map \(C \mapsto \pi_C\). We will also need to consider the condition that

(P5) \(\text{Cod}_1^\mathcal{P}\) has a right inverse,

which we may express equivalently, as follows:

(P5') given \(\pi, \pi'\) in the class \(\mathcal{P}\), there is a functorial assignment

\[v \in \mathcal{C}(\text{cod}(\pi), \text{cod}(\pi')) \mapsto \hat{v} \in \mathcal{C}(\text{dom}(\pi), \text{dom}(\pi'))\],

where \(v \circ \pi = \pi' \circ \hat{v}\).

It is important to note that \(\hat{v}\) depends not only on \(v\), but also on its domain \(\pi\) in \(\mathcal{C}_\mathcal{P}\) and its codomain \(\pi'\) in \(\mathcal{C}_\mathcal{P}\); we write \((\pi, v, \pi')\) instead of \(\hat{v}\) whenever clarity demands doing so. Let us also mention that the stronger condition

(P5*) for all \(\pi, \pi'\) in the class \(\mathcal{P}\) and every \(v \in \mathcal{C}(\text{cod}(\pi), \text{cod}(\pi'))\) there is a unique \(\hat{v} \in \mathcal{C}(\text{dom}(\pi), \text{dom}(\pi'))\) such that \(v \circ \pi = \pi' \circ \hat{v}\),

obviously implies condition (P5') and, hence, condition (P5).

**3.5.** For the class \(\mathcal{P}\) satisfying conditions (P1)-(P5) as in 3.4, we let

\[H : \mathcal{C}_\mathcal{P} \rightarrow \mathcal{B} \downarrow \mathcal{P} \mathcal{C}\]

be a right inverse to \(\text{Cod}_1^\mathcal{P}\) as given by (P5). So, \(H\) is a functor with

\[H(\pi) = \pi\] and \(H(v) = (\hat{v}, v)\)
for all \( \pi, \pi' \in |C_P| = \mathcal{P} \) and every \( v \in \mathcal{C}_P(\pi, \pi') \), where \( \hat{v} \in \mathcal{C}(\text{dom}(\pi), \text{dom}(\pi')) \) satisfies \( v \circ \pi = \pi' \circ \hat{v} \). Now, let
\[
\widehat{C}_P
\]
denote the image of \( \mathcal{C}_P \) in \( \mathfrak{B} \downarrow_P \mathcal{C} \) under \( H \); this category has the same objects as the category \( \mathfrak{B} \downarrow_P \mathcal{C} \), but the morphisms in \( \widehat{C}_P \) are only those morphisms \((u, v)\) of \( \mathfrak{B} \downarrow_P \mathcal{C} \) for which \( u = \hat{v} \). We denote by \( H' : \mathcal{C}_P \to \widehat{C}_P \) the corestriction of \( H \), so that with the (non-full) inclusion functor \( E : \widehat{C}_P \to \mathfrak{B} \downarrow_P \mathcal{C} \) one has \( E \circ H' = H \).

**Proposition.** For the morphism class \( \mathcal{P} \) satisfying (P1)-(P5), the category \( \widehat{C}_P \) is equivalent to the category \( \mathcal{C} \).

**Proof.** With a right inverse \( G : \mathcal{C} \to \mathcal{C}_P, C \mapsto \pi_C \), to the functor \( \text{Cod}^p_2 \) as given by (P4), we have
\[
(\text{Cod}^p_2 \circ E) \circ (H' \circ G) = \text{Cod}^p_2 \circ \text{Cod}^p_1 \circ H \circ G = \text{Id}_{\mathcal{C}}.
\]
It therefore suffices to find a natural isomorphism
\[
\iota : \text{Id}_{\widehat{C}_P} \longrightarrow (H' \circ G) \circ (\text{Cod}^p \circ E)
\]
to confirm that \( H' \circ G : \mathcal{C} \to \widehat{C}_P \) is an equivalence of categories, with quasi-inverse \( \text{Cod}^p \circ E \). To this end, let \( \pi : B \to C \) be any element of the class \( \mathcal{P} \), considered as an object of \( \widehat{C}_P \); then, in our notation, \((H' \circ G)(\pi) = \pi_C \). Since \((\pi, 1_C, \pi_C) : \pi \to \pi_C \) is trivially an isomorphism in \( \mathcal{C}_P \), with inverse \((\pi_C, 1_C, \pi)\), by functoriality of \( H' \)
\[
\iota_\pi \overset{\text{df}}{=} H'((\pi, 1_C, \pi_C)) = ((\pi, 1_C, \pi_C), 1_C) : \pi \to \pi_C
\]
is an isomorphism in \( \widehat{C}_P \). Obviously, every \( \widehat{C}_P \)-morphism between two \( \widehat{C}_P \)-objects \( \pi \) and \( \pi' \) is of the form \( H'((\pi, v, \pi')) \), where \( v \in \mathcal{C}(\text{cod}(\pi), \text{cod}(\pi')) \). Having this in mind, it is easy to see that \( \iota_\pi \) is natural in \( \pi \), which completes the proof. \( \square \)

**3.6.** In addition to the data of 3.5, let us now consider a dual equivalence \( (S, T, \eta, \varepsilon) \) with contravariant functors
\[
T : \mathcal{A} \longrightarrow \mathcal{B} \quad \text{and} \quad S : \mathcal{B} \longrightarrow \mathcal{A}
\]
and natural isomorphisms
\[
\eta : \text{Id}_\mathcal{B} \longrightarrow T \circ S \quad \text{and} \quad \varepsilon : \text{Id}_\mathcal{A} \longrightarrow S \circ T.
\]
We define a category \( \mathcal{D} \) (which depends on the class \( \mathcal{P} \) and the above dual equivalence), as follows:
• objects in $\mathcal{D}$ are pairs $(A, \pi)$ with $A \in |\mathcal{A}|$ and $\pi : T(A) \to C$ in the class $\mathcal{P}$;

• morphisms $(\varphi, f) : (A, \pi) \to (A', \pi')$ in $\mathcal{D}$ are given by morphisms $\varphi \in \mathcal{A}(A, A')$, $f \in \mathcal{C}(\pi', \pi)$ such that $T(\varphi) = (\pi', f, \pi)$; in other words,

$$(T(\varphi), f) : \pi' \to \pi$$

is a morphism in $\widetilde{\mathcal{C}}_{\mathcal{P}}$;

• composition is as in $\mathcal{A}$ and $\mathcal{C}$; that is, $(\varphi, f)$ as above gets composed with $(\varphi', f') : (A', \pi') \to (A'', \pi'')$ by

$$(\varphi', f') \circ (\varphi, f) \overset{\text{def}}{=} (\varphi' \circ \varphi, f \circ f').$$

• the identity morphism of a $\mathcal{D}$-object $(A, \pi)$ is the $\mathcal{D}$-morphism $(1_A, 1_{\text{cod}(\pi)})$.

There is a full embedding

$$J : \mathcal{A} \to \mathcal{D}$$

defined by

$$J(A) = (A, 1_{T(A)}) \text{ and } J(\varphi) = (\varphi, T(\varphi))$$

for all $\varphi \in \mathcal{A}(A, A')$, which allows us to identify $\mathcal{A}$ with its image under $J$, i.e. treat $\mathcal{A}$ as a full subcategory of $\mathcal{D}$. Our goal is to establish a dual equivalence $(\tilde{S}, \tilde{T}, \tilde{\eta}, \tilde{\epsilon})$ with contravariant functors

$$\tilde{T} : \mathcal{D} \to \widetilde{\mathcal{C}}_{\mathcal{P}} \text{ and } \tilde{S} : \widetilde{\mathcal{C}}_{\mathcal{P}} \to \mathcal{D}$$

and natural isomorphisms

$$\tilde{\eta} : \text{Id}_{\widetilde{\mathcal{C}}_{\mathcal{P}}} \to \tilde{T} \circ \tilde{S} \text{ and } \tilde{\epsilon} : \text{Id}_{\mathcal{A}} \to \tilde{S} \circ \tilde{T},$$

in such a way that, after being composed with the equivalence of Proposition 3.5, its restriction to $\mathcal{A}$ and $\mathcal{B}$ returns $(S, T, \eta, \epsilon)$. It turns out to be easiest to just define $\tilde{T}$ and show that it is full and faithful and essentially surjective on objects, which determines the other components of the sought dual equivalence up to isomorphism. But even in order to achieve that we need an additional condition on the class $\mathcal{P}$, as follows:

(P6) If $\pi \in \mathcal{P}$ and $\alpha$ is a $\mathcal{B}$-isomorphism such that $\pi \circ \alpha = \pi$, then $\alpha$ is an identity morphism.

This means precisely that one asks the restriction $\text{Dom}^\mathcal{P} : \mathcal{B} \downarrow \mathcal{P} \to \mathcal{C}$ of the functor $\text{Dom}$ of 3.2 to be amnestic. We note that (P5*) implies (P6).

From now on, we will assume that the class $\mathcal{P}$ satisfies conditions (P1)-(P6) (but not (P5*)).

(P6) allows us to state the following simple lemma:
Lemma. Let \( \pi : B \to C \) be in the morphism class \( \mathcal{P} \) and \( \alpha : B' \to B \) a \( \mathcal{B} \)-isomorphism. Then \( \alpha = (\pi \circ \alpha, 1_C, \pi) \).

Proof. Set \( \beta \overset{\text{df}}{=} (\pi \circ \alpha, 1_C, \pi) \). Then, arguing as in the proof of Proposition 3.5, we obtain that \( \beta \) is a \( \mathcal{B} \)-isomorphism. Since \( \pi \circ \alpha = \pi \circ \beta \), we obtain that \( \pi = \pi \circ \alpha \circ \beta^{-1} \).

Thus, by (P6), \( \alpha \circ \beta^{-1} = 1_B \) and, therefore, \( \alpha = \beta \).

We define the contravariant functor \( \hat{T} \) by
\[
\hat{T}(A, \pi) \overset{\text{df}}{=} \pi \quad \text{and} \quad \hat{T}(\varphi, f) \overset{\text{df}}{=} (T\varphi, f)
\]
for all \( (\varphi, f) : (A, \pi) \to (A', \pi') \) in \( D \). Putting
\[
\hat{T} \overset{\text{df}}{=} \text{Cod}^\mathcal{P} \circ E \circ \hat{T} : D \to \mathcal{C}
\]
(see the Proof of Proposition 3.5), we can now prove the categorical extension theorem for dualities:

Theorem. For the morphism class \( \mathcal{P} \) satisfying (P1)-(P6), \( \hat{T} : D \to \mathcal{C} \) is a dual equivalence which extends the dual equivalence \( T : A \to \mathcal{B} \).

Proof. We first prove that the contravariant functor \( \hat{T} \) is full and faithful. So, let \( (A, \pi), (A', \pi') \in |D| \). Then \( \hat{T}(A, \pi) = \pi \) and \( \hat{T}(A', \pi') = \pi' \), and we have to prove that \( \hat{T} : D((A, \pi),(A', \pi')) \to \widehat{\mathcal{C}}(\pi', \pi) \) is a bijection. Let \( (\hat{v}, v) \in \widehat{\mathcal{C}}(\pi', \pi) \). Then \( \hat{v} : T(A') \to T(A) \) is a \( \mathcal{B} \)-morphism. Since \( T \) is a dual equivalence, there exists a unique \( \mathcal{A} \)-morphism \( \varphi : A \to A' \) such that \( \hat{v} = T(\varphi) \). Thus, \( (\varphi, v) \) is the unique \( D \)-morphism such that \( \hat{T}(\varphi, v) = (\hat{v}, v) \).

Next we show that \( \hat{T} \) is essentially surjective on objects. Let \( \pi \in |\widehat{\mathcal{C}}_{\mathcal{P}}| \), with \( \pi : B \to C \) as a morphism in \( \mathcal{C} \). Set \( A \overset{\text{df}}{=} S(B) \) and \( \pi \overset{\text{df}}{=} \pi \circ (\eta_B)^{-1} \); by (P3), \( \pi \in \mathcal{P} \). Thus \( (S(B), \pi) \in |D| \) and \( \hat{T}(S(B), \pi) = \pi \). Using the above Lemma we obtain that the \( \mathcal{B} \)-isomorphism \( (\eta_B)^{-1} \) satisfies \( (\eta_B)^{-1} = (\pi, 1_C, \pi) \). Thus, \( ((\eta_B)^{-1}, 1_C) : \pi \to \pi \) is a \( \widehat{\mathcal{C}}_{\mathcal{P}} \)-isomorphism.

This shows that \( \hat{T} \) is a dual equivalence. Since, by Proposition 3.5, \( \text{Cod}^\mathcal{P} \circ E \) is an equivalence, we have proved that \( \hat{T} \) is a dual equivalence.

Finally, identifying the category \( \mathcal{A} \) with its isomorphic copy \( J(\mathcal{A}) \), we obtain that \( \hat{T} \) is an extension of \( T \). Indeed, for every \( A \in |\mathcal{A}| \), one easily verifies that \( \hat{T}(J(A)) = \hat{T}(A, 1_{T(A)}) = \text{Cod}(\hat{T}(A, 1_{T(A)})) = \text{Cod}(1_{T(A)}) = T(A) \); also, for every \( \varphi \in \mathcal{A}(A, A') \), one has \( \hat{T}(J(\varphi)) = \hat{T}(\varphi, T(\varphi)) = \text{Cod}(\hat{T}(\varphi, T(\varphi))) = \text{Cod}(T(\varphi), T(\varphi)) = T(\varphi) \).

4 The Fedorchuk duality

In this section, we will use our Theorem 3.6 for obtaining a new proof of the Fedorchuk Duality Theorem. We first recall some statements about skeletal and quasi-open mappings.
It is well known that a function \( f : X \to Y \) is skeletal if and only if
\[
\int(\cl(f(U))) \neq \emptyset
\]
for every non-empty open subset \( U \) of \( X \) (see, e.g., [8]). Hence, every quasi-open mapping is skeletal. Also, if \( f : X \to Y \) is a continuous mapping, then \( f \) is a skeletal mapping if and only if for every \( F \in \RC(X) \), \( \cl(f(F)) \in \RC(Y) \) (see, e.g., [8]).

We also need the following result of A. Blaszczyk [4]:

**Lemma 4.1.** ([4]) A continuous mapping \( f : X \to Y \), where \( X \) and \( Y \) are topological spaces, is skeletal if and only if for every open dense subset \( V \) of \( Y \), \( \cl(f^{-1}(V)) = X \) holds.

Note that every closed irreducible mapping \( f : X \to Y \) is quasi-open (because, by a result of Ponomarev [21], for such mappings one has that for every non-empty open subset \( U \) of \( X \),
\[
(7) \quad f^\#(U) \overset{\text{df}}{=} \{ y \in Y \mid f^{-1}(y) \subseteq U \}
\]
is a non-empty open subset of \( Y \).

**Lemma 4.2.** Let \( f : X \to Y \) and \( g : Y \to Z \) be continuous mappings, \( g \circ f \) and \( f \) be quasi-open and \( \cl(f(X)) = Y \). Then the mapping \( g \) is quasi-open.

**Proof.** Let \( V \) be a non-empty open subset of \( Y \). Then \( V \cap f(X) \neq \emptyset \) and thus \( U \overset{\text{df}}{=} f^{-1}(V) \) is a non-empty open subset of \( X \). Hence \( W \overset{\text{df}}{=} \int((g \circ f)(U)) \) is a non-empty open subset of \( Z \). Since \( f(U) = f(f^{-1}(V)) \subseteq V \), we obtain that \( g(f(U)) \subseteq g(V) \). Hence \( W \subseteq \int(g(V)) \). Therefore, \( \int(g(V)) \neq \emptyset \). This shows that the mapping \( g \) is quasi-open. \( \square \)

4.3. It is well known that if \( X \) and \( Y \) are compact Hausdorff spaces, \( EX \) and \( EY \) are their respective absolutes, and \( \pi_X : EX \to X \), \( \pi_Y : EY \to Y \) are their respective projective covers, then, for every continuous mapping \( f : X \to Y \), there exists a continuous mapping \( \hat{f} : EX \to EY \) such that \( f \circ \pi_X = \pi_Y \circ \hat{f} \); also, the mapping \( f \) is surjective if and only if the mapping \( \hat{f} \) is surjective (see, e.g., [15], [27, 10M] and [16]). Indeed, the first assertion follows from the fact that \( \pi_Y \) is a perfect surjective mapping and \( EX \) is a projective object of the category \( \text{CH} \) (see the Gleason Theorem 2.12); the second one is an easy consequence of the irreducibility of the mapping \( \pi_Y \).

Further, Ju. Bereznitskij (cited in [22]) proved that if \( f \) is a continuous surjection, then, the mapping \( f \) is quasi-open if and only if the mapping \( \hat{f} \) is open. We will prove that this result is true even when \( f \) is not supposed to be a surjection.

**Proposition.** The mapping \( f \) is quasi-open if and only if the mapping \( \hat{f} \) is open.

**Proof.** (\( \Rightarrow \)) Since \( f \) and \( \pi_X \) are quasi-open mappings, their composition \( f \circ \pi_X \) is also a quasi-open mapping. Hence, the mapping \( \pi_Y \circ f \) is quasi-open. We will show now that the mapping \( \hat{f} \) is skeletal. For doing this, we will use the Blaszczyk Lemma 4.1.
So, let $U$ be an open dense subset of $EY$. We have to show that $\hat{f}^{-1}(U)$ is a dense subset of $EX$. Set $V \overset{df}{=} (\pi_Y)^\#(U)$. Then, by (7), $V$ is a non-empty open subset of $Y$ and $(\pi_Y)^{-1}(V) \subseteq U$. Using a result of Ponomarev [21] (or see [2, Proposition 2, page 345]), we obtain that $\pi_Y(\text{cl}(U)) = \text{cl}(V)$, i.e., $V$ is an open dense subset of $Y$. Since the mapping $\pi_Y \circ \hat{f}$ is quasi-open, the Blaszczyk Lemma 4.1 implies that the set $(\pi_Y \circ \hat{f})^{-1}(V)$ is dense in $EX$. Using the fact that $(\pi_Y)^{-1}(V) \subseteq U$, we obtain that $EX = \text{cl}(\hat{f}^{-1}((\pi_Y)^{-1}(V))) \subseteq \text{cl}(\hat{f}^{-1}(U)) \subseteq EX$. Thus, $\hat{f}^{-1}(U)$ is a dense subset of $EX$. So, $\hat{f}$ is a skeletal mapping. Since $\hat{f}$ is a closed mapping, we obtain that $\hat{f}$ is a quasi-open mapping (see, e.g., [8]). This implies that if $F \in RC(EX)$ then $\hat{f}(F) \in RC(EY)$. The spaces $EX$ and $EY$ are extremally disconnected and thus, $RC(EX) = \text{CO}(EX)$ and $RC(EY) = \text{CO}(EY)$. Since $\text{CO}(EX)$ is a base of $EX$, we obtain that $\hat{f}$ is an open mapping.

$(\Leftarrow)$ Since $\hat{f}$ and $\pi_Y$ are quasi-open mappings, their composition $\pi_Y \circ \hat{f}$ is also a quasi-open mapping. Hence, the mapping $f \circ \pi_X$ is quasi-open. Using the fact that $\pi_X$ is a surjection, we derive from Lemma 4.2 that $f$ is a quasi-open mapping. \hfill \Box

Recall the following theorem of M. Henriksen and M. Jerison [16]:

**Theorem 4.4.** Let $X$ and $Y$ be compact Hausdorff spaces, $\pi_X : EX \rightarrow X$ and $\pi_Y : EY \rightarrow Y$ be their projective covers, and $f : X \rightarrow Y$ be a continuous surjection. There exists a unique continuous mapping $\hat{f} : EX \rightarrow EY$ satisfying $f \circ \pi_X = \pi_Y \circ \hat{f}$ if and only if

$$(8) \quad \text{cl}(\text{int}(f^{-1}(F))) = \text{cl}(f^{-1}(\text{int}(F))) \quad \text{for every} \ F \in \text{RC}(Y).$$

**Remark 4.5.** In [16, Lemma 1 and Lemma 3], the expression “$((\pi_X)^{-1}(\alpha))$” has to be replaced by “$\text{cl}((\pi_X)^{-1}(\text{int}(\alpha)))$” (indeed, supposing that $(\pi_X)^{-1}(\alpha)$ is open for every $\alpha \in \text{RC}(X)$), we get, by the result of Ponomarev [21] cited above (see (7)), that $(\pi_X)^\#((\pi_X)^{-1}(\alpha))$ is open, i.e., that $\alpha$ is open for every $\alpha \in \text{RC}(X)$, which is true only when $X$ is extremally disconnected). Fortunately, all other statements in the paper [16] remain true, although their proofs have to be slightly repaired.

Clearly, every continuous skeletal mapping $f : X \rightarrow Y$, where $X$ and $Y$ are topological spaces, satisfies (8) ([19]). Hence, every quasi-open mapping $f : X \rightarrow Y$ satisfies (8) ([16, 22]). Note also that a continuous function $f : X \rightarrow Y$, where $X$ and $Y$ are topological spaces, satisfies (8) if and only if for every $U \in \text{RO}(Y)$, $f^{-1}(\text{Fr}(U))$ is a nowhere dense subset of $X$.

We will need the next assertion which is similar to Theorem 4.4 and which follows from a much more general theorem of Šapiro [23] (see also Uljanov [26]); for completeness of our exposition, we will supply it with a proof:

**Proposition 4.6.** Let $X$ and $Y$ be compact Hausdorff spaces, $\pi_X : EX \rightarrow X$ and $\pi_Y : EY \rightarrow Y$ be their projective covers, and $f : X \rightarrow Y$ be a quasi-open mapping. Then there exists a unique continuous mapping $\hat{f} : EX \rightarrow EY$ such that $f \circ \pi_X = \pi_Y \circ \hat{f}$. 

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Proof. The existence of such a mapping \( \hat{f} : EX \to EY \) was already established in 4.3. Suppose that \( g : EX \to EY \) is another continuous mapping such that \( f \circ \pi_X = \pi_Y \circ g \). Then, by Proposition 4.3, \( g(EX) \in \text{CO}(EY) \).

Set \( Z \overset{\text{def}}{=} f(X) \) and \( E \overset{\text{def}}{=} \text{cl}_{EY}((\pi_Y)^{-1}(\text{int}_Y(Z))) \). Since \( f \) is quasi-open, we obtain that \( Z \in \text{RC}(Y) \). Clearly \( E \in \text{CO}(EY) \) and hence, \( E \) is a extremally disconnected compact Hausdorff space. Since \( \pi_Y \) is a closed surjection, we obtain that \( \pi_Y(E) = Z \).

Set \( \pi_Z \overset{\text{def}}{=} \pi_Y \mid_E : E \to Z \). Then \( \pi_Z \) is an irreducible mapping. Indeed, suppose that there exists a closed proper subset \( F \) of \( E \) such that \( \pi_Z(F) = Z \). Then \( F \) and \( F \cup (EY \setminus E) \) are proper closed subsets of \( EY \), and \( \pi_Y(F \cup (EY \setminus E)) = Y \). Since \( \pi_Y \) is irreducible, we obtain a contradiction. Hence, \( E \) is the absolut of \( Z \) and \( \pi_Z \) is the projective cover of \( Z \). Let \( \alpha : \text{RC}(EY) \to \text{RC}(Y) \) be defined by the formula \( \alpha(G) \overset{\text{def}}{=} \pi_Y(G) \), for every \( G \in \text{RC}(EY)(= \text{CO}(EY)) \). Then \( \alpha \) is a Boolean isomorphism (see, e.g., [2, Corollary, p. 346]). Since \( \alpha(E) = \alpha(g(EX))(= Z) \), we obtain that \( g(EX) = E \). Now, applying Theorem 4.4 for \( \pi_X, \pi_Z \) and \( f \mid_X : X \to Z \), we conclude that \( g = \hat{f} \).

The next theorem was proved in [9, Corollary 3.2(c)] and [7, Corollary 2.4(c)]:

**Theorem 4.7.** ([9]) The category \( \text{ZCH}_{\text{qop}} \) of zero-dimensional compact Hausdorff spaces and quasi-open mappings is dually equivalent to the category \( \text{BA}_{\text{sup}} \) of Boolean algebras and suprema-preserving Boolean homomorphisms. The contravariant functors between these categories which realize the duality between them are the restrictions of the Stone contravariant functors \( S^t \) and \( S^a \) to the categories \( \text{ZCH}_{\text{qop}} \) and \( \text{BA}_{\text{sup}} \), respectively.

Using the well known fact that complete Boolean algebras correspond to extremally disconnected compact Hausdorff spaces under Stone duality, and arguing as in the end of the first part of proof of Proposition 4.3 (i.e., using the fact that a continuous mapping between two extremally disconnected compact Hausdorff spaces is quasi-open if and only if it is open), we obtain the following corollary:

**Corollary 4.8.** The category \( \text{ECH}_{\text{op}} \) of extremally disconnected compact Hausdorff spaces and open mappings is dually equivalent to the category \( \text{CBA}_{\text{sup}} \) of complete Boolean algebras and suprema-preserving Boolean homomorphisms. The contravariant functors between these categories which realize the duality between them are the restrictions of the Stone contravariant functors \( S^t \) and \( S^a \) to the categories \( \text{ECH}_{\text{op}} \) and \( \text{CBA}_{\text{sup}} \), respectively.

We will now formulate the Fedorchuk Duality Theorem.

Let \( \text{CH}_{\text{qop}} \) be the category of compact Hausdorff spaces and quasi-open mappings.

Let \( \text{CNCA}_{\text{sup-ref}} \) be the category whose objects are all complete normal contact algebras and whose morphisms \( \varphi : (A, C) \to (B, C') \) are all suprema-preserving Boolean homomorphisms \( \varphi : A \to B \) satisfying the following condition:

(F1) For all \( a, b \in A \), \( \varphi(a)C'\varphi(b) \) implies \( aCb \).
Theorem 4.9. (Fedorchuk [14]) The categories CH\textsuperscript{qop} and CNCA\textsubscript{sup-ref} are dually equivalent.

A new proof of Theorem 4.9. Set \(A \overset{df}{=} CBA\textsubscript{sup}, \ B \overset{df}{=} ECH\textsuperscript{op} \) and \(C \overset{df}{=} CH\textsuperscript{qop}\).

Then, clearly, \(B\) is a full subcategory of \(C\) and \(B\) is closed under \(C\)-isomorphisms. Let \(P\) be the class of all irreducible \(C\)-morphisms having as domain a \(B\)-object. It is obvious that conditions (P1)-(P3) are fulfilled. Using 2.11(c), we obtain that condition (P4) (and thus, condition (P4)) is also satisfied. Finally, Propositions 4.6 and 4.3 show that condition (P1)-(P3) are fulfilled. Using 2.11(c), we obtain that condition (P4) is fulfilled and thus, conditions (P5) and (P6) are satisfied. Set \(T\) be the class of all irreducible \(C\)-morphisms having as domain a \(B\)-object. It is obvious that conditions (P1)-(P3) are fulfilled. Using 2.11(c), we obtain that condition (P4) is fulfilled and thus, conditions (P5) and (P6) are satisfied. Set

\[
T \overset{df}{=} S^a \mid_A \text{ and } S \overset{df}{=} S^t \mid_B.
\]

Then, clearly, \(T : A \rightarrow B\) and \(S : B \rightarrow A\) together with \(\eta \overset{df}{=} t \mid_B\) and \(\epsilon \overset{df}{=} s \mid_A\) (i.e., \(\eta : Id_B \rightarrow T \circ S\) and \(\epsilon : Id_A \rightarrow S \circ T\)) realize the dual equivalence between the categories \(A\) and \(B\). Defining a category \(D\) as in 3.6, we obtain, according to Theorem 3.6, that there exist a full embedding \(J : A \rightarrow D\) and a dual equivalence \(\hat{T} : D \rightarrow C\) which extends the dual equivalence \(T : A \rightarrow B\). We will now show that the categories \(CNCA\textsubscript{sup-ref}\) and \(D\) are equivalent, completing in such a way the new proof of the Fedorchuk Duality Theorem.

Set \(\mathcal{E} \overset{df}{=} CNCA\textsubscript{sup-ref}\). We will define a functor \(F : D \rightarrow \mathcal{E}\).

Let \(D \in |D|\). Then, by 3.6, \(D = (A, \pi)\), where \(A \in |A|\), \(\pi \in P\), \(\pi : T(A) \rightarrow X\) and \(X \in |\mathcal{C}|\). Clearly, \(T(A) = S^a(A)\) and \(CO(T(A)) = RC(T(A))\). Set, for every \(a \in A\),

\[
\varphi_D(a) \overset{df}{=} \pi(s_A(a)).
\]

Then, by [2, Corollary, p. 346], \(\varphi_D : A \rightarrow RC(X)\) is a Boolean isomorphism. For every \(a, b \in A\), set

\[
aC_Db \quad \text{if} \quad \varphi_D(a) \cap \varphi_D(b) \neq \emptyset.
\]

Since \((RC(X), \rho_X)\) is an NCA, we obtain that \((A, C_D)\) is an NCA. (We have just transported the NCA-structure on RC(X) to A using the Boolean isomorphism \(\varphi_D\).)

Now we set

\[
F(D) \overset{df}{=} (A, C_D).
\]

Let us note that if \(D = (A, \pi) \in |D|\) then, for every \(a, b \in A\),

\[
aC_Db \quad \iff \quad \text{there exist } u, v \in \text{Ult}(A) \text{ such that } a \in u, b \in v \text{ and } \pi(u) = \pi(v).
\]

Let \((\alpha, f) \in \mathcal{D}(D, D')\), where \(D = (A, \pi)\) and \(D' = (A', \pi')\). Then \(\alpha \in A(A, A'), f \in C(\text{cod}(\pi'), \text{cod}(\pi))\) and \(\pi \circ T(\alpha) = f \circ \pi\'). We set

\[
F(\alpha, f) \overset{df}{=} \alpha.
\]

Then \(F(\alpha, f) \in \mathcal{E}(F(D), F(D'))\). Indeed, let \(a, b \in A\) and \(\alpha(a)C_{D'}\alpha(b)\). Then there exist \(u', v' \in \text{Ult}(A')\) such that \(\alpha(a) \in u', \alpha(b) \in v'\) and \(\pi'(u') = \pi'(v')\). Set \(u \overset{df}{=} \alpha^{-1}(u')\) and \(v \overset{df}{=} \alpha^{-1}(v')\). Then \(u, v \in \text{Ult}(A), a \in u \text{ and } b \in v\). Also, \(f(\pi'(u')) = f(\pi'(v'))\) and thus \(\pi(T(\alpha)(u')) = \pi(T(\alpha)(v'))\). Hence \(\pi(u) = \pi(v)\). Therefore, \(aC_Db\). This means that \(F(\alpha, f) (=\alpha)\) is an \(\mathcal{E}\)-morphism between \(F(D)\) and \(F(D')\). It is easy to see that \(F\) preserves identities and compositions. So, \(F : D \rightarrow \mathcal{E}\) is a functor.
We will now show that $F$ is a faithful functor. Let $D, D' \in |\mathcal{D}|$, where $D = (A, \pi)$ and $D' = (A', \pi')$. We have to prove that the restriction $F : \mathcal{D}(D, D') \rightarrow \mathcal{E}(F(D), F(D'))$ is an injection. Let $(\alpha, f), (\alpha_1, f_1) \in \mathcal{D}(D, D')$ and $(\alpha, f) \neq (\alpha_1, f_1)$. Then $F(\alpha, f) = \alpha$ and $F(\alpha_1, f_1) = \alpha_1$. We will show that $\alpha \neq \alpha_1$. Suppose that $\alpha = \alpha_1$. Then $T(\alpha) = T(\alpha_1)$. Thus $f \circ \pi' = T(\alpha) \circ \pi = T(\alpha_1) \circ \pi = f_1 \circ \pi'$. Now, having in mind that $\pi'$ is a surjection, we obtain that $f = f_1$, a contradiction. So, $F$ is a faithful functor.

Let us prove that $F$ is full. Let $D, D' \in |\mathcal{D}|$, where $D = (A, \pi)$ and $D' = (A', \pi')$. We have to prove that the restriction $F : \mathcal{D}(D, D') \rightarrow \mathcal{E}(F(D), F(D'))$ is a surjection. Let $\alpha \in \mathcal{E}(F(D), F(D'))$. Then $\alpha \in \mathcal{A}(A, A')$. Set $X \overset{df}{=} \text{cod}(\pi), X' \overset{df}{=} \text{cod}(\pi'), C \overset{df}{=} C_D$ and $C' \overset{df}{=} C_{D'}$. We have to find $f \in \mathcal{C}(X', X)$ such that $f \circ \pi' = \pi \circ T(\alpha)$.

Let us first show that if $u', v' \in T(A')$ and $\pi'(u') = \pi'(v')$, then

\begin{equation}
\pi(T(\alpha)(u')) = \pi(T(\alpha)(v')).
\end{equation}

Indeed, suppose that $\pi(T(\alpha)(u')) \neq \pi(T(\alpha)(v'))$. Set $u \overset{df}{=} T(\alpha)(u')$ and $v \overset{df}{=} T(\alpha)(v')$. Then $u = \alpha^{-1}(u'), v = \alpha^{-1}(v')$ and $\pi(u) \neq \pi(v)$. Since $X$ is a Hausdorff space, the points $\pi(u)$ and $\pi(v)$ have disjoint neighborhoods $U$ and $V$, where $\pi(u) \in U$ and $\pi(v) \in V$. Then there exist $a, b \in A$ such that $u \in s_A(a) \subseteq \pi^{-1}(U)$ and $v \in s_A(b) \subseteq \pi^{-1}(V)$. Thus $\pi(s_A(a)) \cap \pi(s_A(b)) = \emptyset$, i.e. $a(-C)b$. Since $\alpha$ is an $\mathcal{E}$-morphism, we obtain that $\alpha(a)(-C')\alpha(b)$, which means that $\pi'(s_{A'}(\alpha(a))) \cap \pi'(s_{A'}(\alpha(b))) = \emptyset$. This, however, is impossible, because $\varphi'(u') = \pi'(v')$, $u' \in s_{A'}(\alpha(a))$ (because $\alpha(a) \in u'$), and $v' \in s_{A'}(\alpha(b))$ (because $\alpha(b) \in v'$). Hence, $\pi(T(\alpha)(u')) = \pi(T(\alpha)(v'))$.

Let $x' \in X'$. Since $\pi'$ is a surjection, there exists a $u' \in T(A')$ such that $x' = \pi'(u')$. Set

\begin{equation}
f(x') \overset{df}{=} \pi(T(\alpha)(u')).
\end{equation}

Then (9) shows that in this way we have defined correctly a function $f : X' \rightarrow X$. Since $\pi'$ is a quotient mapping, we obtain that $f$ is a continuous function. We will show that $f$ is quasi-open. Indeed, let $U$ be a non-empty open subset of $X'$. Using the fact that $\pi'$ is a surjection and $T(\alpha)$ is an open mapping, we obtain that $V \overset{df}{=} T(\alpha)((\pi')^{-1}(U))$ is a non-empty open subset of $T(A)$. Thus, by (7), $W \overset{df}{=} \pi^\#(V)$ is a non-empty open subset of $X$. Since $W \subseteq \pi(V) = \pi(T(\alpha)((\pi')^{-1}(U))) = f(\pi'(((\pi')^{-1}(U)))) = f(U)$, we obtain that $f$ is a quasi-open mapping. Hence, $f \in \mathcal{C}(X', X)$ and $f \circ \pi' = \pi \circ T(\alpha)$. Therefore, $(\alpha, f) \in \mathcal{D}(D, D')$ and $F(\alpha, f) = \alpha$. So, the functor $F$ is full.

Now, we have to show that $F$ is essentially surjective on objects. Let $(A, C)$ be a complete NCA. Then $A$ is a complete Boolean algebra and $T(A) = S^\alpha(A)$. Thus $T(A)$ is the set $\text{Ult}(A)$ endowed with a topology $\mathcal{T}_A$ having as an open base the family $\{s_A(a) \mid a \in A\}$. Let $R_{(A,C)}$ be the relation on $\text{Ult}(A)$ defined in 2.5, i.e., for every $u, v \in T(A)$, $u R_{(A,C)}^c v$ if $u \times v \subseteq C$. For brevity, set $Y \overset{df}{=} T(A)$, $s \overset{df}{=} s_A$ and $R \overset{df}{=} R_{(A,C)}$. By Proposition 2.6(b), $R$ is an equivalence relation. Set $X \overset{df}{=} Y/R$ and let $\pi : Y \rightarrow X$ be the natural quotient mapping.
We will first prove that $\pi$ is irreducible. Suppose that there exists a proper closed subset $G$ of $Y$ such that $\pi(G) = X$. Let $u \in Y \setminus G$. Since $G$ is compact subset of $Y$ and $s : A \rightarrow \text{CO}(Y)$ is a Boolean homomorphism, there exists $a \in A$ such that $G \subseteq s(a) \subseteq Y \setminus \{u\}$. Hence $a \neq 1$; thus $a^* \neq 0$. Then, by (\langle \leq \rangle 6), there exists $b \neq 0$ such that $b \ll a^*$, i.e., $b(-C)a$. By [17, Corollary 2.17], there exists an ultrafilter $v$ in $A$ such that $b \in v$. Then $a^* \in v$, and hence $a \not\in v$. Thus, $v \not\subseteq s(a)$. Therefore, $v \in Y \setminus G$. Then there exists $w \in G$ such that $\pi(w) = \pi(v)$, i.e., $vRw$. Since $w \in G$, we obtain that $w \in s(a)$, i.e., $a \in w$. So, we have that $a \in w$, $b \in v$ and $v \times w \subseteq C$. Hence $aCb$, a contradiction. This shows that the mapping $\pi$ is irreducible.

Further, we will show that for every $u \in Y$, the equivalence class $[u]$ of $u$ is a closed subset of $Y$. Indeed, let $u \in Y$ and $v \in Y \setminus [u]$. Then $u(-R)v$, i.e., $u \times v \not\subseteq C$. Hence, there exist $a \in u$ and $b \in v$ such that $a(-C)b$. Then $v \in s(b)$. Also, $s(b) \cap [u] = \emptyset$. Indeed, if $w \in s(b)$ then $b \in w$, and since $a(-C)b$, we obtain that $u(-R)w$, i.e., $w \not\subseteq [u]$. Therefore, $[u]$ is a closed subset of $Y$.

We will now prove that $R$ is a closed equivalence relation on $Y$ (in the sense of [13, 2.4.9]). So, let $U$ be an open subset of $Y$; we have to prove that the union of all equivalence classes that are contained in $U$ is open in $Y$. Let $u \in Y$ be such that $[u] \subseteq U$. Since $[u]$ is compact, there exists $a \in A$ such that $[u] \subseteq s(a) \subseteq U$. Then $a \in u$. For every $M \subseteq Y$, we set

$$[M] \overset{df}{=} \bigcup \{[v] \mid v \in M\}.$$ 

Set $V \overset{df}{=} Y \setminus [s(a^*)]$. Then $[u] \subseteq V$. Indeed, we have that $[u] \cap s(a^*) = \emptyset$. Suppose that $[s(a^*)] \cap [u] \neq \emptyset$. Then there exists $v \in s(a^*)$ such that $[v] \cap [u] \neq \emptyset$. Thus $[v] = [u]$. Since $[u] \cap s(a^*) = \emptyset$, we obtain that $[v] \cap s(a^*) = \emptyset$, a contradiction. So, $[u] \subseteq V$.

Next, we will show that $V$ is open. Let $v \in V$. Suppose that for every $b \in v$, $s(b) \not\subseteq V$. Then, for every $b \in v$, $s(b) \cap [s(a^*)] \neq \emptyset$. Hence, for every $b \in v$ there exist $v_b \in s(b)$ and $w_b \in s(a^*)$ such that $v_b \in [w_b]$, i.e., for every $b \in v$ there exist $v_b, w_b \in Y$ such that $b \in v_b, a^* \in w_b$ and $v_bRw_b$. Using Proposition 2.6(a), we obtain that $a^*Cb$ for every $b \in v$. Thus, by Corollary 2.9, $a^* \in \sigma_v$, where $\sigma_v$ is the cluster generated by $v$.

Now, Theorem 2.8 gives us that there exists an ultrafilter $w$ in $A$ such that $a^* \in w$ and $\sigma_w = \sigma_v$. Hence $v \cup w \subseteq \sigma_v$ and thus, $v \times w \subseteq C$, i.e., $vRw$. Therefore, $[v] = [w]$. Since $w \in s(a^*)$, we obtain that $v \in [s(a^*)] = Y \setminus V$, a contradiction. Therefore, there exists $b \in v$ such that $s(b) \subseteq V$. Since $v \in s(b)$, we obtain that $V$ is an open subset of $Y$.

Finally, we will prove that $V$ is a subset of the union of all equivalence classes that are contained in $U$. Let $w \in Y$ and $[w] \cap V \neq \emptyset$. Suppose that $[w] \not\subseteq U$. Then $[w] \not\subseteq s(a)$. Hence, there exists $v \in (Y \setminus s(a)) \cap [w]$. Then $v \in s(a^*)$ and $[w] = [v] \subseteq [s(a^*)] = Y \setminus V$, a contradiction. Hence, $[w] \subseteq U$.

So, $R$ is a closed relation. Then, by the Alexandroff Theorem [13, Theorem 3.2.11], $X$ is a compact Hausdorff space, i.e., $X \in \mathcal{C}$. Since $\pi$ is a closed irreducible mapping, we obtain that $\pi$ is quasi-open (see the text after Lemma 4.1). Hence, $\pi \in \mathcal{C}(Y, X)$. Also, $Y = T(A) \in \mathcal{B}$. Therefore, $\pi \in \mathcal{P}$ and $D \overset{df}{=} (A, \pi)$ is a $D$-object.

By the definition of the relation $C_D$, we have that for every $a, b \in A$, $aCb$ if and
only if there exist \( u, v \in \text{Ult}(A) \) such that \( a \in u, b \in v \) and \( \pi(u) = \pi(v) \). Obviously, \( \pi(u) = \pi(v) \) if and only if \( uRv \). Hence, by Proposition 2.6(a), the relations \( C \) and \( C_D \) coincide. Thus, \( F(D) = (A, C) \). Therefore, the functor \( F \) is an equivalence.

Note that the category \( A \) is isomorphic to the full subcategory \( A' \) of \( \mathcal{E} \) having as objects all NCAs of the form \( (A, \rho^A_\alpha) \) (see Example 2.4 for \( \rho^A_\alpha \)). Indeed, if \( \alpha : A \rightarrow B \) is a (suprema-preserving) Boolean homomorphism, then \( \alpha \in \mathcal{E}((A, \rho^A_\alpha), (B, \rho^B_\beta)) \) because in this case condition (F1) (stated just before Theorem 4.9) is automatically fulfilled. Denoting by \( J' : A \rightarrow \mathcal{E} \) the functor defined by \( J'(A) = (A, \rho^A_\alpha) \) for every \( A \in |A| \), and by \( J'(\alpha) = \alpha \) for every \( \alpha \in A(A, B) \), we obtain that \( J' \) is an isomorphism and \( J' = F \circ J \).

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