Tripartite and Sign Consensus for Clustering Balanced Social Networks

Giulia De Pasquale and Maria Elena Valcher

Abstract—In this paper we address two forms of consensus for multi-agent systems with undirected, signed, weighted, and connected communication graphs, under the assumption that the agents can be partitioned into three clusters, representing the decision classes on a given specific topic, for instance, the in favour, abstained and opponent agents.

We will show that under some assumptions on the cooperative/antagonistic relationships among the agents, simple modifications of DeGroot’s algorithm allow to achieve tripartite consensus (if the opinions of agents belonging to the same class all converge to the same decision) or sign consensus (if the opinions of the agents in the three clusters converge to positive, zero and negative values, respectively).

I. INTRODUCTION

Social networks consisting of a finite number of individuals that mutually interact in a cooperative or antagonistic way are frequently represented by signed graphs, whose positive edges represent friendly ties, while negative edges denote enmity relationships. Generally speaking, consensus for a network of agents is the problem of achieving a common objective, of converging to a common decision, by making use of information provided by neighbouring agents. During the past few decades, consensus problems have attracted the attention of scientists and researchers from various fields such as sociology, engineering and mathematics, as it can be seen from the large amount of scientific literature related to this topic [1], [5], [6], [7], [11]. However, these problems have been typically investigated under the assumption that the overall communication network is purely cooperative (pure “consensus”) [15], [16], [23] or “structurally balanced”, by this meaning that agents split into two groups of cooperative agents that compete with those of the other group (“bipartite consensus”) [1], [2], [8], [25].

On the other hand, when focusing on signed graphs that represent meaningful social relationships, cooperation and structural balance are only two of the possible sign configurations, and additional models have been considered [13]. If we restrict our attention to the case when the interpersonal appraisals between agents are reciprocal, namely the signed graph that models the interactions between the agents is undirected, in addition to structural balance also “clustering balance” may arise [5]. Specifically, when the sign attribution over the graph network is given according to the following rules: 1) the friend of my friend is my friend, 2) the enemy of my friend is my enemy, 3) the friend of my enemy is my enemy, clustering balance is obtained. If in addition to the previous three rules, the rule: 4) the enemy of my enemy is my friend is observed, then the network is structurally balanced. The aforementioned four rules are a cornerstone of the research on the opinion dynamics in social networks and are known in literature as “Heider’s rules” [11].

To the best of our knowledge, the literature on consensus problems over clustering balanced networks, by this meaning the problem of making all agents belonging to the same cluster in a clustering balanced network achieve a common decision, is quite limited [17], [18]. On the other hand, there have been some interesting research efforts aiming to explore the possibility of achieving group consensus for networks whose agents have been partitioned in disjoint groups. Such group partition, however, does not represent a balanced clusterization, according to Heider’s rules, and hence it is not suitable for formalising consensus problems in a sociological context [20], [21], [22], [26], [27]. Indeed, in all of these papers, the group partitioning is obtained according to the “indegree balanced condition”, that ensures that agents within the same group cooperate, while each agent has both cooperative and antagonistic relationships with the agents of every other group, but the weights of such relationships sum up to zero.

The ambition of this paper is to fill in a gap between the scientific literature regarding clustering balanced networks and the one related to consensus, by proposing two forms of consensus problems on clustering balanced networks and by providing conditions for their solvability. Specifically, given an undirected, signed, weighted, and connected network, with three disjoint and antagonistic clusters, we investigate under what conditions the opinions of (cooperative) agents belonging to the same cluster converge to the same value/decision or at least they converge to values/decisions having the same sign, and such a sign varies with the specific cluster (so, one cluster converges to a positive decision, one to a negative one and the members of the third cluster all converge to the zero value). These two targets correspond to two different notions of consensus that we will refer to as tripartite consensus and sign consensus, respectively. The sociological interpretation of these two problems is easily found in contexts such as elections, group decisions, bets, and every time agents are called to express their approval, disapproval, or abstention on a given topic or decision (see [12]) and hence split into three classes. Additional applications of these problems can be found in rendezvous problems for multi-robots systems or formation flights.
The results presented in this paper have been inspired by the work of C. Altafini [1], where the concept of bipartite consensus has been introduced for structurally balanced networks, by the tutorial paper of A.V. Proskurnikov and R. Tempo [19] on social networks, and by the works of J. Davis [6] and P. Cisneros-Velarde and F. Bullo [5], where the concept of clustering balance plays a major role. In our recent papers [17], [18] we have investigated consensus problems for clustering balanced networks with 3 or, in general, $k \geq 3$ clusters, under some “homogeneity” constraint on the (positive and negative) weights of the communication network, namely by assuming that the amount of trust/mistrust that each agent attributes to its friends/enemies is prefixed for all the agents in the same cluster. In this paper we will focus on networks that are partitioned into three clusters, and we will first show that, even without the homogeneity assumption, (tripartite) consensus can be obtained by means of a slightly revised version of De Groot’s distributed feedback control law. Subsequently, we will introduce sign consensus and show that also in that case, under some mild assumptions, a modified version of De Groot’s control law allows to successfully achieve the target.

The paper is organized as follows. Notation and preliminaries are first introduced. Section II formalises the tripartite consensus problem for a multi-agent network, whose agents are described as simple integrators and whose communication graph splits into 3 clusters. Section III provides a complete solution to this problem, under some mathematical assumptions formalising the existence of a strong relationship among the individuals of at least one cluster and, on the contrary, strong competition among the agents of adverse clusters. Section IV explores, under the same assumptions, the more general target of sign consensus, namely the case when the three clusters asymptotically converge to a positive, negative and neutral (namely zero) decision, respectively, but this decision within each cluster is not necessarily of the same modulus, just of the same sign.

Preliminaries. For $k, n \in \mathbb{Z}, k \leq n$, we denote by $[k,n]$ the integer set $\{k,k+1, \ldots, n\}$. The symbol $[A]_{i,j}$ denotes the $(i,j)$th entry of the matrix $A$, while $[v]_i$ is the $i$th entry of the vector $v$. A matrix (in particular, a vector) $A$ is nonnegative (denoted by $A \succeq 0$) [9] if all its entries are nonnegative. A matrix is strictly positive (denoted by $A \succ 0$) if all its entries are positive. A symmetric matrix $P \in \mathbb{R}^{n \times n}$ is positive (semi) definite if $x^T P x > 0$ ($x^T P x \geq 0$) for every $x \in \mathbb{R}^n, x \neq 0$, and when so we use the symbol $P > 0$ ($P \succeq 0$).

The notation $A = \text{diag}\{A_1, \ldots, A_n\}$ indicates a block diagonal matrix with diagonal blocks $A_1, \ldots, A_n$. $0_n$ and $1_n$ are the $n$-dimensional vectors with all entries equal to 0 and 1, respectively. A real square matrix $A$ is Hurwitz if every eigenvalue $\lambda$ in $\sigma(A)$, the spectrum of $A$, has negative real part, i.e., $\text{Re}(\lambda) < 0$.

A Metzler matrix is a real square matrix, whose off-diagonal entries are nonnegative. For $n \geq 2$, an $n \times n$ nonzero Metzler matrix $A$ is reducible [10], [14] if there exists a permutation matrix $P$ such that $P^TA P$ is block-triangular, otherwise it is irreducible.

Every Metzler matrix $A$ exhibits a real dominant (but not necessarily simple) eigenvalue [24], known as Frobenius eigenvalue and denoted by $\lambda_F(A)$. In other words, $\lambda_F(A) > \text{Re}(\lambda), \forall \lambda \in \sigma(A), \lambda \neq \lambda_F(A)$. If $A$ is also irreducible, then $\lambda_F(A)$ is necessarily simple.

The following technical result will be used extensively in this paper.

Lemma 1: [17] Let $D \in \mathbb{R}^{n \times n}$ be a diagonal matrix and let $A \in \mathbb{R}^{n \times n}$ be a symmetric nonnegative matrix, then:

i) $D - A$ is positive definite if and only if there exists a strictly positive vector $v \in \mathbb{R}^n$ such that $(D-A)v > 0$.

ii) If condition i) holds, then $(D - A)^{-1} \succeq 0$ and is symmetric.

II. TRIPARTITE CONSENSUS: PROBLEM STATEMENT

We consider a multi-agent system consisting of $N$ agents, each of them described as a continuous-time integrator (see [1], [15], [16], [23]). The overall system dynamics is described as

$$\dot{x}(t) = u(t),$$

where $x \in \mathbb{R}^N, u \in \mathbb{R}^N$, are the state and input variables, respectively.

Assumption 1 on the communication structure. [Connectedness and clustering] The communication among the $N$ agents is described by an undirected, signed and weighted communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is the set of vertices, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of arcs, and $A$ is the adjacency matrix of $\mathcal{G}$ that describes how agents interact. The $(i,j)$th entry of $A$, $[A]_{i,j}$, $i \neq j$, is nonzero if and only if the information about the status of the $j$th agent is available to the $i$th agent. We assume that the interactions between pairs of agents are symmetric and hence $A = A^T$. The interaction between the $i$th and the $j$th agents is cooperative if $[A]_{i,j} > 0$ and antagonistic if $[A]_{i,j} < 0$. Also, $[A]_{i,i} = 0$, for all $i \in [1,N]$. We also assume that the graph $\mathcal{G}$ is connected and clustering balanced, with three clusters, i.e., all the agents are grouped in 3 clusters, $\mathcal{V}_i, i \in [1,3]$, with $n_i = |\mathcal{V}_i|$, such that for every $i, j \in \mathcal{V}_i$, $p \in [1,k]$, $[A]_{i,j} \geq 0$, while for every $i \in \mathcal{V}_p, j \in \mathcal{V}_q$, $p, q \in [1,k], p \neq q$, $[A]_{i,j} \leq 0$. However, the agents cannot be grouped into a smaller number of clusters.

In this paper we want to extend some recent results obtained for tripartite consensus of a multi-agent system with undirected, signed, weighted, connected and clustering balanced communication graph, by relaxing the homogeneity constraint regarding mutual relationships between agents introduced in [17], [18]. Specifically, in [17], [18] we proved that if each agent in a cluster distributes the same amount of “trust” to the agents in its own group and “distrust” to
the agents belonging to adverse clusters, then it is possible to adopt a slightly modified version of DeGroot’s algorithm in such a way that agents belonging to the same cluster \( V_i, i \in [1, 3] \), asymptotically converge to the same decision, i.e.,

\[
\lim_{t \to +\infty} x_k(t) = c_i, \quad \forall k \in V_i.
\]

We now want to explore under what conditions tripartite consensus can still be achieved even if the aforementioned homogeneity constraint is removed.

For the sake of simplicity, in the following we will assume that the agents are ordered in such a way that the first \( n_1 \) agents belong to the cluster \( V_1 \), the subsequent \( n_2 \) to the cluster \( V_2 \), and the last \( n_3 \) to the cluster \( V_3 \). Clearly, \( n_1 + n_2 + n_3 = N \). This assumption entails no loss of generality, since it is always possible to reduce ourselves to this structure by means of a relabelling of the nodes/agents. Accordingly, the adjacency matrix of the graph \( G \) can be block-partitioned as follows

\[
A = \begin{bmatrix}
A_{1,1} & A_{1,2} & A_{1,3} \\
A_{2,1} & A_{2,2} & A_{2,3} \\
A_{3,1} & A_{3,2} & A_{3,3}
\end{bmatrix}
\]

with \( A_{i,j} \), \( i,j = 1, 2, 3 \). Let \( A_{i,j} = A_{j,i}^T \geq 0 \), and \( A_{i,j} = A_{j,i}^T = 0 \) for \( i \neq j \). Since the block-partitioned adjacency matrix is symmetric, the kernel of \( A \) includes a single vector in the kernel of \( A_{i,j} \) having the same “close friendship” assumption that we have adopted in Assumption 1.

We consider a distributed control law for the system (1) of the type

\[
u(t) = -Mx(t),
\]

where \( M \in \mathbb{R}^{N \times N} \) takes the form

\[
M = D - A,
\]

with \( D \in \mathbb{R}^{N \times N} \) a diagonal matrix partitioned according to the block-partition of \( A \), namely

\[
D = \text{diag}\{D_1, D_2, D_3\},
\]

and \( D_i \in \mathbb{R}^{n_i \times n_i} \), \( n_i \) being the cardinality of the \( i \)th cluster, \( i \in [1, 3] \). The diagonal entries of \( D_i \) represent the degree of stubbornness of each agent in \( V_i \). They quantify how much individuals in the \( i \)th cluster are convinced of their own opinions.

The overall multi-agent system is hence described as

\[
x(t) = -Mx(t),
\]

and the aim of this paper is to investigate if it is possible to choose the matrices \( D_i \) so that all the agents reach tripartite consensus, by this meaning that for almost every initial condition\(^2\) \( x(0) \in \mathbb{R}^N \) all the state variables associated to agents in the same cluster converge to the same value, namely

\[
\lim_{t \to +\infty} x(t) = [c_11_{n_1}^T, c_21_{n_2}^T, c_31_{n_3}^T]^T,
\]

for suitable \( c_i = c_i(x(0)) \in \mathbb{R}, i \in [1, 3] \), not all of them equal to zero.

It is worth noticing that while in the homogeneous case investigated in [17], [18], the desired goal was achieved by suitably choosing a stubbornness degree common to all the agents belonging to the same class, now the degree of stubbornness is individually tuned.

### III. Tripartite Consensus: Problem Solution

We first present necessary and sufficient conditions for tripartite consensus.

**Lemma 2:** [17] Given an undirected, signed, weighted and connected communication graph, \( G \), having 3 clusters, the multi-agent system (1), with communication graph \( G \) and distributed control law (3), and hence described as in (6), reaches tripartite consensus if and only if the following conditions hold:

(i) \( M \) is a singular positive semi-definite matrix,

(ii) The kernel of \( M \) is spanned by vectors of the type

\[
v = [v_1^T n_1, v_2^T n_2, v_3^T n_3]^T, v_i \in \mathbb{R}, i \in [1, 3].
\]

We now focus on the previous condition (ii), and provide the following lemma, whose easy proof is omitted.

**Lemma 3:** Given the matrix \( M \in \mathbb{R}^{N \times N} \) described as in (4), \( D \in \mathbb{R}^{N \times N} \) described as in (5) and \( D_i \in \mathbb{R}^{n_i \times n_i} \), for \( i \in [1, 3] \), diagonal matrices, the kernel of \( M \) includes a vector of the type \( v = [v_1^T n_1, v_2^T n_2, v_3^T n_3]^T \), \( v_i \in \mathbb{R}, i \in [1, 3] \), if and only if

\[
\text{rank} \left( \begin{bmatrix}
d_1 - a_{11} & -a_{12} & -a_{13} \\
-a_{21} & d_2 - a_{22} & -a_{23} \\
-a_{31} & -a_{32} & d_3 - a_{33}
\end{bmatrix} \right) < 3,
\]

where

\[
d_i := D_i1_{n_i}, \quad a_{ij} := A_{ij}1_{n_j}, \quad i, j \in [1, 3].
\]

Based on Lemmas 2 and 3, in the sequel we will provide conditions ensuring the existence of diagonal matrices \( D_i \) (equivalently, of vectors \( d_i = D_i1_{n_i} \)), for \( i \in [1, 3] \), such that the corresponding matrix \( M \) is a singular positive semi-definite matrix, with a simple eigenvalue in 0 and condition (8) holds. In fact, if 0 is a simple eigenvalue, in order to fulfill condition (ii) of Lemma 2 it is sufficient to prove that there exists a single vector in the kernel of \( M \) having the desired block structure.

It is worth noticing that \( a_{ij} \neq 0 \) for every pair \( i, j \in [1, 3], i \neq j \). In fact \( a_{ij} = 0 \) implies \( A_{ij} = 0 \) and hence also \( A_{ji} = 0 \) which means that \( V_i \) and \( V_j \) could be grouped together, thus contradicting the minimality of the partitioning into 3 clusters introduced in Assumption 1.

To solve the tripartite consensus problem, we introduce the same “close friendship” assumption that we have adopted in [17], [18] and that strengthens the relationships among agents in the same cluster.

**Assumption 2 on the communication structure.** [Close friendship] There exist two distinct indices \( i_1 \) and \( i_2 \) in \( [1, 3] \) such that the cluster \( V_{i_1} \) either consists of a single node/agent or for every pair of distinct agents \( (i, j) \in V_{i_1} \times V_{i_2} \) either one of the following cases applies (see Figure 1):

i) \((i,j)\) are friends (i.e., the edge \((i,j)\) belongs to \( E \) and it has positive weight);
ii) \((i, j)\) are enemies (i.e., the edge \((i, j)\) belongs to \(E\) and it has negative weight) of two (not necessarily distinct) vertices \(r\) and \(s\) in \(\mathcal{V}_i\), that belong to the same connected component in \(\mathcal{V}_i\).

To ensure that (11) holds, we iterate the same procedure, and impose condition:

\[
\mathbf{D}_2 - \mathbf{A}_{2,2} - \mathbf{A}_{2,1}(\mathbf{D}_1 - \mathbf{A}_{1,1})^{-1}\mathbf{A}_{1,2} \succ 0, \tag{13}
\]

and as well as condition (14).

To address condition (13), we first observe that by Lemma 1, part ii), \((\mathbf{D}_1 - \mathbf{A}_{1,1})^{-1}\) is symmetric and nonnegative, and hence so is \(\mathbf{A}_{2,2} + \mathbf{A}_{2,1}(\mathbf{D}_1 - \mathbf{A}_{1,1})^{-1}\mathbf{A}_{1,2}\). But then we can apply Lemma 1, part i), again, by assuming \(\mathbf{D} = \mathbf{D}_2\) and \(\mathbf{A} = \mathbf{A}_{2,2} + \mathbf{A}_{2,1}(\mathbf{D}_1 - \mathbf{A}_{1,1})^{-1}\mathbf{A}_{1,2}\). Indeed, if we impose the following constraint on \(\mathbf{d}_2\):

\[
\mathbf{d}_2 \gg \mathbf{a}_{22} + \mathbf{A}_{2,1}(\mathbf{D}_1 - \mathbf{A}_{1,1})^{-1}\mathbf{a}_{12} \succeq 0, \tag{15}
\]

then it is easy to verify that

\[
(\mathbf{D} - \mathbf{A})\mathbf{1}_{n_2} = \mathbf{d}_2 - \mathbf{a}_{22} - \mathbf{A}_{2,1}(\mathbf{D}_1 - \mathbf{A}_{1,1})^{-1}\mathbf{a}_{12} \gg 0.
\]

Therefore \(\mathbf{D} - \mathbf{A}\) is positive definite, namely (13) holds.

On the other hand, we can always choose (see [17]) the positive diagonal entries of the diagonal matrix \(\mathbf{D}_2\), namely the vector \(\mathbf{d}_2\), so that not only \(\mathbf{d}_2\) fulfills condition (15), but it is also sufficiently large to ensure that the entries of \([\mathbf{A}_{3,2} + \mathbf{A}_{3,1}(\mathbf{D}_1 - \mathbf{A}_{1,1})^{-1}\mathbf{A}_{1,2}][\mathbf{D}_2 - \mathbf{A}_{2,2} - \mathbf{A}_{2,1}(\mathbf{D}_1 - \mathbf{A}_{1,1})^{-1}\mathbf{A}_{1,2}]^{-1}[\mathbf{A}_{2,3} + \mathbf{A}_{2,1}(\mathbf{D}_1 - \mathbf{A}_{1,1})^{-1}\mathbf{A}_{1,3}]\) are small enough to guarantee that

\[
-(\mathbf{\Phi}_3 - \mathbf{D}_3) \approx \mathbf{A}_{3,3} + \mathbf{A}_{3,1}(\mathbf{D}_1 - \mathbf{A}_{1,1})^{-1}\mathbf{A}_{1,3}.
\]

By Assumption 2, for \(i_1 = 1\) and \(i_3 = 2\), the matrix \(\mathbf{A}_{3,3} + \mathbf{A}_{3,1}(\mathbf{D}_1 - \mathbf{A}_{1,1})^{-1}\mathbf{A}_{1,3}\) has positive off-diagonal entries, and hence the same is true for \(-(\mathbf{\Phi}_3 - \mathbf{D}_3)\). This ensures that \(-\mathbf{\Phi}_3\) is an irreducible Metzler matrix.

So, now, we are remained with proving that for a suitable choice of \(\mathbf{D}_3\) we can ensure that (14) holds. If we apply the vector \(\mathbf{1}_{n_3}\) on the right side of the matrix \(\mathbf{\Phi}_3\), by making use of reasonings similar to those just exploited to prove (13), we obtain

\[
\mathbf{\Phi}_3\mathbf{1}_{n_3} = \mathbf{d}_3 - \mathbf{a}_{33} - \mathbf{A}_{1,1}(\mathbf{D}_1 - \mathbf{A}_{1,1})^{-1}\mathbf{a}_{13}
\]

\[
\mathbf{a}_{33} + \mathbf{A}_{3,1}(\mathbf{D}_1 - \mathbf{A}_{1,1})^{-1}\mathbf{a}_{12} 
\]

\[
\mathbf{D}_2 - \mathbf{A}_{2,2} - \mathbf{A}_{2,1}(\mathbf{D}_1 - \mathbf{A}_{1,1})^{-1}\mathbf{A}_{1,2} \succ 0, \tag{16}
\]

and (11) hold.

Assume that

\[
\mathbf{d}_1 \gg \mathbf{a}_{11} \succeq 0. \tag{12}
\]

Then \((\mathbf{D}_1 - \mathbf{A}_{1,1})\mathbf{1}_{n_1} \gg 0\), and hence Lemma 1, part i), holds for \(\mathbf{v} = \mathbf{1}_{n_1}\), thus ensuring that \(\mathbf{D}_1 - \mathbf{A}_{1,1}\) is positive definite.
\[
\begin{bmatrix}
D_2 - A_{2,2} - A_{2,1}(D_1 - A_{1,1})^{-1}A_{1,2} & -A_{2,3} - A_{2,1}(D_1 - A_{1,1})^{-1}A_{1,3} \\
-A_{3,2} - A_{3,1}(D_1 - A_{1,1})^{-1}A_{1,2} & D_3 - A_{3,3} - A_{3,1}(D_1 - A_{1,1})^{-1}A_{1,3}
\end{bmatrix} \succeq 0.
\] (11)

\[
\Phi_3 := \begin{bmatrix}
D_3 - A_{3,3} - A_{3,1}(D_1 - A_{1,1})^{-1}A_{1,3} - [A_{3,2} + A_{3,1}(D_1 - A_{1,1})^{-1}A_{1,2}] \\
\cdot [D_2 - A_{2,2} - A_{2,1}(D_1 - A_{1,1})^{-1}A_{1,2}]^{-1}[A_{2,3} + A_{2,1}(D_1 - A_{1,1})^{-1}A_{1,3}] \geq 0 \text{ and singular.}
\end{bmatrix}
\] (14)

1) and 2) we can determine vectors \(d_i, i \in [1, 3]\), so that condition (B) holds.

Note that by assumption 1), \(a_{23} \ll 0\), and by assumption 2), either \(a_{12} \ll 0\) or \(a_{13} \ll 0\). In the sequel we will focus on the case \(a_{12} \ll 0\), the other case being completely equivalent.

We want to prove that we can always find vectors \(d_i, i \in [1, 3]\), consistent with the constraints (12), (15) and (16), so that (B) holds and hence there exist \(v_2, v_3\) such that

\[
\begin{bmatrix}
1 \\
v_2 \\
v_3
\end{bmatrix}
= \begin{bmatrix}
d_1 \\
v_2d_2 \\
v_3d_3
\end{bmatrix}.
\] (17)

This is equivalent to determining scalars \(v_2\) and \(v_3\) that make the vectors

\[
d_1 = a_{11} + v_2a_{12} + v_3a_{13} \quad (18)
\]

\[
d_2 = a_{22} + \frac{v_3}{v_2}a_{23} \quad (19)
\]

\[
d_3 = a_{33} + \frac{v_3}{v_2}a_{31} + \frac{v_3}{v_2}a_{32}, \quad (20)
\]

consistent with the constraints (12), (15) and (16).

We first note that since \(a_{12} \ll 0\), we can always choose \(v_2 < 0\) with large module, and \(v_3 > 0\) and small, so that \(v_2a_{12} + v_3a_{13} \gg 0\), which automatically implies that \(d_1\) satisfies condition (12). Also, we can choose the modules of \(v_2\) and \(v_3\) in such a way that the entries of \(d_1\) and hence of \(D_1\), are so large that \(a_{23} + A_{2,1}(D_1 - A_{1,1})^{-1}a_{13} \approx a_{23} < 0\) and hence \(a_{23} + A_{2,1}(D_1 - A_{1,1})^{-1}a_{13} \ll 0\), and also \(\frac{v_3}{v_2}a_{23} + A_{2,1}(D_1 - A_{1,1})^{-1}a_{13} \gg 0\). (21)

By making use of (21), we obtain that

\[
d_2 = a_{22} + \frac{v_3}{v_2}a_{23}
\]

\[
= a_{22} + \frac{1}{v_2}a_{21} - \frac{v_3}{v_2}A_{2,1}(D_1 - A_{1,1})^{-1}a_{13}
\]

\[
= a_{22} + \frac{1}{v_2}A_{2,1}[v_2(D_1 - A_{1,1})^{-1}a_{12}]
\]

\[
= a_{22} + \frac{1}{v_2}A_{2,1}(D_1 - A_{1,1})^{-1}a_{12},
\]

where we used the fact that condition (18) is equivalent to

\[
1_{n_1} = v_2(D_1 - A_{1,1})^{-1}a_{12} + v_3(D_1 - A_{1,1})^{-1}a_{13}.
\] (22)

So, this proves that also (15) holds. Finally, it is possible to prove (details are omitted due to page constraints) that if the identities (18) and (19) hold, then the constraints (20) and (16) are equivalent.

Hence we conclude that there exist suitable choices of \(d_i, i \in [1, 3]\), such that both conditions (A) and (B) are fulfilled and hence the overall multi-agent system reaches tripartite consensus.

Example 1: Consider the undirected, signed, weighted, connected and clustered communication graph, with three clusters of cardinals \(n_1 = 5, n_2 = 4, n_3 = 2\), respectively, and adjacency matrix composed of the sub-matrices

\[
A_{1,1} = \begin{bmatrix}
0 & 4 & 0 & 0 & 1 \\
4 & 0 & 3 & 10 & 2 \\
0 & 3 & 0 & 1 & 0 \\
10 & 1 & 0 & 1 & 0 \\
1 & 2 & 0 & 1 & 0
\end{bmatrix},
\]

\[
A_{1,2} = \begin{bmatrix}
1.5 & 1.5 & 0 & 1.5 \\
0.5 & 3 & 0 & 2.5 \\
3 & 0 & 5 & 2 \\
3 & 3 & 0 & 2 \\
0 & 0 & 5 & 2.5
\end{bmatrix},
\]

\[
A_{1,3} = \begin{bmatrix}
7 & 2 \\
0 & 3 \\
4 & 2 \\
8 & 0 \\
7 & 3
\end{bmatrix},
\]

\[
A_{2,1} = \begin{bmatrix}
0 & 6 & 0 & 2 \\
6 & 0 & 4 & 4 \\
0 & 4 & 0 & 0 \\
2 & 4 & 0 & 0
\end{bmatrix},
\]

\[
A_{2,2} = \begin{bmatrix}
3 & 4 \\
6 & 1 \\
1 & 0 \\
4 & 6
\end{bmatrix},
\]

\[
A_{2,3} = \begin{bmatrix}
0 & 6 \\
0 & 6 \\
0 & 6 \\
0 & 6
\end{bmatrix},
\]

all the others being deduced by symmetry. Assumption 2 holds for \(i_1 = 1\) and \(i_2 = 3\), and both assumptions 1) and 2) of Theorem 4 hold, since \(a_{23} \ll 0\), \(a_{12}\) and \(a_{13}\) are both strictly negative vectors. So, we can assume, for example, \((v_1, v_2, v_3) = (1.5, -8)\), and hence \(d_1 = [54.5 \ 13 \ 19.5 \ 36.74]^T\), \(d_2 = [17.6 \ 23.6 \ 5.5 \ 19.9]^T\) and \(d_3 = [18 \ 14.125]^T\). The dynamics of the state vector of the system, with random initial conditions \(x(0)\) taken as realizations of a gaussian vector with 0 mean and variance \(\sigma^2 = 4\), i.e. \(x(0) \sim \mathcal{N}(0, 4)\), is given in Fig. 2. The plot shows that tripartite consensus is reached after about 1.8 units of time with regime values \((c_1, c_2, c_3) = (0.22 \ 1.11 \ -1.76) = 0.22 \cdot (1.5 \ -8) = 0.22 \cdot (v_1 \ v_2 \ v_3)\).
IV. SIGN CONSENSUS

In this section we introduce the concept of sign consensus for which a formal definition is given in the following.

Definition 1 (Sign Consensus): The overall multi-agent system described as in (6), with $M \in \mathbb{R}^{N \times N}$ described as in (4), $D \in \mathbb{R}^{N \times N}$ described as in (5) and $D_i \in \mathbb{R}^{n_i \times n_i}$, for $i \in [1,3]$, diagonal matrices, whose interconnection topology is described by an undirected, signed and connected communication graph $G$, having 3 clusters, reaches sign consensus if there exists a relabelling of the three clusters such that, for every index $i \in V_2$, $\lim_{t \to \infty} x_i(t) = 0$, while for every $i, j \in V_1 \cup V_3$

$$\lim_{t \to \infty} \text{sgn}(x_i(t)) - \text{sgn}(x_j(t)) = 0, \quad \text{if } \exists m : i, j \in V_m,$$

$$\lim_{t \to \infty} \text{sgn}(x_i(t)) - \text{sgn}(x_j(t)) \neq 0, \quad \text{if } \nexists m : i, j \in V_m.$$

The following lemma provides necessary and sufficient conditions for sign consensus to be reached.

Lemma 5: Given an undirected, signed, weighted and connected communication graph $G$, having 3 clusters, the multi-agent system (1), with communication graph $G$ and distributed control law (3), and hence described as in (6), with $M \in \mathbb{R}^{N \times N}$ given in (4), $A$ in (2), $D \in \mathbb{R}^{N \times N}$ in (5) and $D_i \in \mathbb{R}^{n_i \times n_i}$, for $i \in [1,3]$, diagonal matrices reaches sign consensus if and only if the following conditions hold:

i) $M$ is a singular positive semi-definite matrix.

ii) There exists a reordering $\{i_1, i_2, i_3\}$ of the index set $\{1,2,3\}$ such that every nonzero vector in the kernel of $M$ can be expressed as $v = [v_{i_1}^T, v_{i_2}^T, v_{i_3}^T]^T$ with $v_{i_3} = 0$, and in the pair $(v_{i_1}, v_{i_2})$ one of the vectors is strictly positive and one is strictly negative.

Proof: Analogous to the proof of Lemma 2 (see [17]).

Theorem 6: Consider the multi-agent system (1), with undirected, signed, weighted and connected communication graph $G$ satisfying Assumption 1 and Assumption 2 for a suitable choice of $i_1, i_2 \in [1,3], i_1 \neq i_2$. Also, suppose that the following conditions hold:

a) every agent in $V_{i_1}$ has at least one enemy in $V_{i_2}$, which means that $A_{i_1,i_2} \cdot n_{i_2} \ll 0$, and

b) there exist vectors $v_{i_1} \in \mathbb{R}^{n_{i_1}}$ and $v_{i_2} \in \mathbb{R}^{n_{i_2}}$, one of them strictly positive and the other strictly negative, such that $A_{i_3,i_1}v_{i_1} + A_{i_3,i_2}v_{i_2} = 0$, where $i_3 = [1,3] \setminus \{i_1,i_2\}$.

Then there exist diagonal matrices $D_i \in \mathbb{R}^{n_i \times n_i}, i \in [1,3]$, such that the distributed control law (3), with $M \in \mathbb{R}^{N \times N}$ described as in (4), $D \in \mathbb{R}^{N \times N}$ described as in (5), makes the closed-loop multi-agent system (6) reach sign consensus.

Proof: We can always relabel the vertices in $V$ so that assumption a) and b) hold for $i_1 = 1, i_2 = 3, i_3 = 2$.

By Lemma 5, it will be sufficient to prove that under assumptions a)-b), it is always possible to choose the diagonal matrices $D_1, D_2$ and $D_3$ so that (A) the matrix $M$ is singular and positive semi-definite with a simple eigenvalue in 0, and (B) the kernel of $M$ includes the vector $v = [v_1^T, v_2^T, v_3^T]^T$, where $v_1 \in \mathbb{R}^{n_1}$ and $v_3 \in \mathbb{R}^{n_3}$ are two vectors satisfying assumptions b), and we assume w.l.o.g. that $v_1 \gg 0$ and $v_3 \ll 0$.

To prove (B) we note that solving the system of equations $Mv = 0v$ is equivalent to solve the system

$$
\begin{bmatrix}
D_1 - A_{1,1} & -A_{1,2} & -A_{1,3} \\
-A_{2,1} & D_2 - A_{2,2} & -A_{2,3} \\
-A_{3,1} & -A_{3,2} & D_3 - A_{3,3}
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= 0_N,
$$

and this in turn is equivalent to the three identities

$$D_1v_1 = A_{1,1}v_1 + A_{1,3}v_3 \quad (24)$$

$$0_{n_2} = A_{2,1}v_1 + A_{2,3}v_3 \quad (25)$$

$$D_3v_3 = A_{3,1}v_1 + A_{3,3}v_3 \quad (26)$$

Identity (25) holds by assumption b) and we note that the constraint (24) and (26) allow to uniquely determine the diagonal matrices $D_1$ and $D_3$, since they can be component-wise written as

$$[D_p]_{i,i} = \frac{1}{|V_p|} \left( \sum_{j \neq i} |A_{p,j}| v_p^j + \sum_{k=1}^{n_q} |A_{p,q}| \cdot k \cdot v_q^k \right) \quad (27)$$

for $p, q \in \{1,3\}, p \neq q$.

We are now remained with proving that, after having determined the matrices $D_1$ and $D_3$, it is always possible to choose $D_2$ so that (A) is satisfied. To do so we proceed as follows (see [4], page 651): we first verify that the upper diagonal block of $M$:

$$M = \begin{bmatrix}
D_1 - A_{1,1} & -A_{1,2} & -A_{1,3} \\
-A_{2,1} & D_2 - A_{2,2} & -A_{2,3} \\
-A_{3,1} & -A_{3,2} & D_3 - A_{3,3}
\end{bmatrix}, \quad (28)$$

is positive definite, namely condition

$$D_1 - A_{1,1} \succ 0 \quad (29)$$

holds, and then impose (by means of a suitable choice of $D_2$) that its Schur complement is positive semi-definite with a simple eigenvalue in 0, namely it verifies condition (30), and it has a simple eigenvalue in 0. Condition (24) ensures

3Note, however, that the values of $D_1$ and $D_3$ depend on the specific choice of the vectors $v_1$ and $v_3$ satisfying (25), which are not necessarily uniquely determined.
that

\[(D_1 - A_{1,1})v_1 = A_{1,3}v_3 \gg 0, \quad (31)\]

where we used the fact that \(v_3 \ll 0\) and \(A_{1,3}\) has no zero rows. Then Lemma 1, part i), holds with \(v = v_1\), thus ensuring that \(D_1 - A_{1,1}\) is positive definite.

To ensure that (30) holds for a suitable choice of \(D_2\), we iterate the same procedure, and impose condition (32):

\[D_{2,2} - A_{2,2} - A_{2,1}(D_1 - A_{1,1})^{-1}A_{1,2} > 0, \quad (32)\]
as well as condition (33).

To address condition (32), we first observe that by Lemma 1, part ii), \((D_1 - A_{1,1})^{-1}\) is symmetric and nonnegative, and hence so is \(A := A_{2,2} + A_{2,1}(D_1 - A_{1,1})^{-1}A_{1,2}\). Let us set \(a_{i2} := A_{i2}1_{n_2}, i \in [1, 2]\), and \(d_2 := D_21_{n_2}\), and impose the following constraint on \(d_2\):

\[d_2 \gg a_{22} + A_{2,1}(D_1 - A_{1,1})^{-1}a_{12}. \quad (34)\]

Then it is easy to verify that

\[(D - A)1_{n_2} = d_2 - a_{22} - A_{2,1}(D_1 - A_{1,1})^{-1}a_{12} \gg 0,\]

where \(D = D_2\). But then we can apply Lemma 1, part i), again, to claim that \(D - A\) is positive definite, namely (32) holds.

We now observe (see [17]) that we can always choose the positive diagonal entries of the diagonal matrix \(D_2\) sufficiently large to ensure that (not only (34) holds, but also) the entries of \([A_{i2} + A_{i1}(D_1 - A_{1,1})^{-1}A_{1,2}] [D_2 - A_{2,2} - A_{2,1}(D_1 - A_{1,1})^{-1}A_{1,2}]^{-1} [A_{2,3} + A_{2,1}(D_1 - A_{1,1})^{-1}A_{1,3}]\) are arbitrarily small and hence also the matrix \(A = -\Phi_3 + D_3\) has positive off-diagonal entries. This ensures that \(-\Phi_3\) is an irreducible Metzler matrix.

So, we now prove that (33) holds. We observe that from assumption a) for \(i_1 = 1\) and \(i_2 = 3\) it follows that \(A_{3,3} + A_{3,1}(D_1 - A_{1,1})^{-1}A_{1,3}\) is a nonnegative matrix whose off-diagonal entries are all positive. If we apply the vector \(-v_3 \gg 0\) on the right side of the matrix \(\Phi_3\), appearing in (33), we obtain

\[-\Phi_3v_3 = -D_3v_3 + A_{3,3}v_3 + A_{3,1}(D_1 - A_{1,1})^{-1}A_{1,3}v_3 + [A_{3,2} + A_{3,1}(D_1 - A_{1,1})^{-1}A_{1,2}] \cdot [D_2 - A_{2,2} - A_{2,1}(D_1 - A_{1,1})^{-1}A_{1,2}]^{-1} \cdot [A_{2,3}v_3 + A_{2,1}(D_1 - A_{1,1})^{-1}A_{1,3}]\]

We first note that by (26) we have \(-D_3v_3 + A_{3,3}v_3 = -A_{3,1}v_1\). On the other hand, from equation (24) one gets

\[A_{1,3}v_3 = (D_1 - A_{1,1})v_1, \quad (35)\]

from which it follows

\[A_{3,1}(D_1 - A_{1,1})^{-1}A_{1,3}v_3 = A_{3,1}v_1. \quad (36)\]

Therefore

\[\Phi_3v_3 = -D_3v_3 + A_{3,3}v_3 + A_{3,1}(D_1 - A_{1,1})^{-1}A_{1,3}v_3 = 0. \quad (37)\]

On the other hand, from (35) it also follows that \(A_{2,1}(D_1 - A_{1,1})^{-1}A_{1,3}v_3 = A_{2,1}v_1\), and making use of (25), this latter identity leads to

\[A_{2,1}(D_1 - A_{1,1})^{-1}A_{1,3}v_3 + A_{2,3}v_3 = 0. \quad (38)\]

So, by replacing (37) and (38) in the expression of \(-\Phi_3v_3\) we obtain the zero vector.

Since \(-\Phi_3\) is an irreducible Metzler matrix, this ensures [3] that 0 is the simple dominant eigenvalue of \(-\Phi_3\) and hence \(\Phi_3\) is positive semidefinite and singular with a simple eigenvalue in 0. Since the eigenvalues of \(M\) are the union of the eigenvalues of the matrices in (29) and (32), and the matrix \(\Phi_3\), that have been obtained from \(M\) by applying the Schur complement, then \(M\) is positive semidefinite with a simple eigenvalue in 0, and hence (A) holds.

To conclude, we have proved that by setting \(D_1\) and \(D_3\) as in (27), by choosing the diagonal entries of the diagonal matrix \(D_2\) sufficiently large, both conditions of Lemma 5 are fulfilled, and the overall multi-agent system reaches sign consensus.

\[\Phi_3 = \begin{bmatrix} 3 & 3 & 0 & 3 \\ 6 & 6 & 0 & 12 \\ 6 & 6 & 0 & 6 \\ 0 & 0 & 3 & 3 \end{bmatrix}, \quad \Phi_3 = \begin{bmatrix} 3.5 & 1 \\ 0 & 1.5 \\ 1.5 & 0 \\ 3.5 & 1.5 \end{bmatrix}, \quad \Phi_3 = \begin{bmatrix} 36 & 3 \\ 24 & 6 \\ 12 & 0 \\ 12 & 15 \end{bmatrix}. \]

Condition b) of Theorem 6 holds for \(v_3 = [2, 1, 1, 1, 2]^T \gg_0 0\), \(v_3 = -[0.5, 2]^T \ll 0\), while \(d_2 = a_{22} + A_{2,1}(D_1 - A_{1,1})^{-1}a_{12} + v\), where \(v\) is a vector whose entries are the absolute value of the entries of the realization of a gaussian vector with 0 mean and standard deviation \(\sigma = 200\).

The dynamics of the state vector of system (6), with random initial conditions \(x(0)\) taken as realizations of a gaussian vector with 0 mean and variance \(\sigma^2 = 4\), i.e. \(x(0) \sim N(0, 4)\), is given in Fig. 3.
the partition into clusters does not and for every agent $i$ feedback control law needs to be re-tuned. An exception is pose is not adaptive and in general if the weights change, the implementation is distributed. Also, the algorithm we pro-

The design of the proposed algorithm is centralised, but its 

V. CONCLUSIONS

In this paper we have investigated the tripartite and the sign consensus problems for a multi-agent system whose agents are described as simple integrators, and whose communications graph is undirected, signed and weighted, and 

Fig. 3. Sign consensus for Example 2.

REFERENCES

[1] C. Altafini. Consensus problems on networks with antagonistic interactions. IEEE Trans. Aut. Contr., 58, no. 4:935–946, 2013.
[2] D. Bauso, L. Giarré, and R. Pesenti. Quantized dissensus in networks of agents subject to death and duplication. IEEE Trans. Aut. Contr., 57:783–788, 2012.
[3] A. Berman and R.J. Plemmons. Nonnegative matrices in the mathematical sciences. Academic Press, New York, 1979.
[4] S. Boyd and L. Vandenberghe. Convex Optimization. Cambridge University Press, 2004.
[5] P. Cisneros-Velarde and F. Bullo. Signed network formation games and clustering balance. Dynamic Games and Applications, 2020.
[6] J.A. Davis. Clustering and structural balance in graphs. SAGE Social Science Collections, 20(2):181–187, 1957.
[7] M.H. DeGroot. Reaching a consensus. J. Amer. Statist. Assoc., 69, no. 345:118–121, 1974.
[8] D. Easley and J. Kleinberg. Networks, Crowds, and Markets. Reasoning About a Highly Connected World. Cambridge Univ. Press, Cambridge, U.K., 2010.
[9] L. Farina and S. Rinaldi. Positive linear systems: theory and applications. Wiley-Interscience, Series on Pure and Applied Mathematics, New York, 2000.
[10] G.F. Frobenius. Über Matrizen aus nicht Negativen Elementen. Sitzungsberichte der Königlich Preussischen Akademie der Wis-
senschaften, Berlin, Germany, 1912, pp.456–477, reprinted in Ges. Abh., Springer, Berlin, Germany, vol.3:546–567, 1968.
[11] F. Heider. Social perception and phenomenal causality. Psychological Review, 51(6):338–374, 1944.
[12] T. Jiang and J. Baras. Trust evaluation in anarchy: a case study on autonomous networks. pages 23–29, Barcelona, Spain, 2006.
[13] E.C. Johnsen. The micro-macro connection: exact structure and pro-
cess. In Roberts F., editor, Applications of Combinatorics and Graph Theory to the Biological and Social Sciences. The IMA Volumes in Mathematics and Its Appl., volume 17, pages 169–201. Springer,New York, NY, 1989.
[14] H. Minc. Nonnegative Matrices. J.Wiley & Sons, New York, 1988.
[15] R. Olfati-Saber, J.A. Fax, and R.M. Murray. Consensus and cooperation in networked multi-agent systems. Proc. of the IEEE, 95, no. 1:215–233, 2007.
[16] R. Olfati-Saber and R.M. Murray. Consensus problems in networks of agents with switching topology and time-delays. IEEE Trans. Aut. Contr., 49, no. 9:1520 –1533, 2004.
[17] G. De Pasquale and M.E. Valcher. Consensus for clusters of agents with cooperative and antagonistic relationships. submitted, available at http://arxiv.org/abs/2008.12398, 2020.
[18] G. De Pasquale and M.E. Valcher. Consensus problems on clustered networks. In Proc. of the 56th IEEE Conf. Decision and Control, pages 3675–3680, Jeju Island, Republic of Korea, 2020.
[19] A.V. Proskurnikov and R. Tempo. A tutorial on modeling and analysis of dynamic social networks. part i. Ann. Rev. Control., 43:65–79, 2017.
[20] J. Qin, Q. Ma, W.X. Zheng, and H. Gao. $h_{\infty}$ group consensus for clusters of agents with model uncertainty and external disturbance. In Prof. of the 54th IEEE Conference on Decision and Control, pages 2841–2846, Osaka, Japan, 2015.
[21] J. Qin and C. Yu. Cluster consensus control of generic linear multi-agent systems under directed topology with acyclic partition. Automatica, 49(9):2898–2905, 2013.
[22] J. Qin, C. Yu, and B.D.O. Anderson. On leaderless and leader-following consensus for interacting clusters of second-order multi-agent systems. Automatica, 74:214–221, 2016.
[23] W. Ren, R.W. Beard, and E.M. Atkins. Information consensus in multivehicle cooperative control. IEEE Control Sys. Magazine, 27 (2):71–82, 2007.
[24] N.K. Son and D. Hinrichsen. Robust stability of positive continuous time systems. Numerical Functional Analysis and Optimization, 17 (5–6):649–659, 1996.
[25] M.E. Valcher and P. Misra. On the consensus and bipartite consensus in high-order multi-agent dynamical systems with antagonistic interactions. Systems & Control Letters, 66(1):94–103, 2014.
[26] W. Xia and M. Cao. Clustering in diffusively coupled networks. Automatica, 47(11):2395–2405, 2011.
[27] J. Yu and L. Wang. Group consensus in multi-agent systems with switching topologies and communication delays. Systems and Control Letters, 59(6):340–348, 2010.