Conformally Prescribed Scalar Curvature on Orbifolds

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Abstract: We study the prescribed scalar curvature problem in a conformal class on orbifolds with isolated singularities. We prove a compactness theorem in dimension 4, and an existence theorem which holds in dimensions $n \geq 4$. This problem is more subtle than the manifold case since the positive mass theorem does not hold for ALE metrics in general. We also determine the U(2)-invariant Leray–Schauder degree for a family of negative-mass orbifolds found by LeBrun.

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1. Introduction

To begin, we give the definition of a Riemannian orbifold.

**Definition 1.1.** We say that \((M, g)\) is a Riemannian orbifold of dimension \(n\) if it satisfies all of the following three conditions:

- \(M\) is a smooth manifold of dimension \(n\) with a smooth Riemannian metric away from a finite singular set \(\Sigma = \{(q_1, \ldots, q_l)\}\).
- Near each singular point \(q_j\), there exists a neighborhood \(U_j\) of \(q_j\), a nontrivial finite subgroup \(\Gamma_j \subset O(n)\) acting freely on \(\mathbb{R}^n\setminus \{0\}\) and a \(\Gamma_j\)-equivariant diffeomorphism \(\varphi_j : \overline{U}_j \to B_{\sigma_j}(0)\), where \(\overline{U}_j\) is the completion (by adding a point \(\tilde{q}_j\)) of the universal cover of \(U_j \setminus \{q_j\}\), and \(B_{\sigma_j}(0)\) is a ball of radius \(\sigma_j\) about the origin in \(\mathbb{R}^n\).
- The metric \((\varphi_j)^*g\) extends to a smooth Riemannian metric on \(B_{\sigma_j}(0)\), where \(\pi_j : \overline{U}_j \setminus \{\tilde{q}_j\} \to U_j \setminus \{q_j\}\) is a universal covering map. We extend \(\pi_j\) to \(\overline{U}_j\) by defining \(\pi_j(\tilde{q}_j) = q_j\).

Our convention is that if \(l = 0\), then \(\Sigma = \emptyset\), and \((M, g)\) is a smooth Riemannian manifold. But if \(l \geq 1\), then there must be nontrivial singular points. We next define the meaning of a \(C^k(M)\) function on a Riemannian orbifold \((M, g)\).

**Definition 1.2.** A function \(f : M \to \mathbb{R}\) is in \(C^k(M)\) if \(f : M \setminus \Sigma \to \mathbb{R}\) is in \(C^k(M \setminus \Sigma)\), and near each singularity, there exists a coordinate system \(\varphi_j\) such that the function \(f \circ \pi_j \circ \varphi_j^{-1} : B_{\sigma_j}(0) \to \mathbb{R}\) is in \(C^k(B_{\sigma_j}(0))\). We can also define the spaces \(C^\infty(M), C^{k,\alpha}(M), C^k_{loc}(M)\) in a similar fashion.

Note that since linear terms are never invariant under a nontrivial orbifold group, a \(C^1\) function necessarily has a critical point at any nontrivial orbifold singularity. In other words, \(\Sigma \subset \text{Crit}(f)\), where \(\text{Crit}(f) \equiv \{x \in M | \nabla_g f(x) = 0\}\).

**Remark 1.3.** In general, an orbifold can have higher-dimensional singular sets. So our definition is restrictive in that we only allow isolated quotient singularities.

Let \((M, g)\) be a compact Riemannian \(n\)-orbifold for \(n \geq 3\), and with positive scalar curvature \(R_g > 0\). Let \(K > 0\) be a positive \(C^2\) function on \(M\). We will study the following equation

\[-\Delta_g u + c(n) R_g u = K u^p, \tag{1.1}\]

where \(c(n) = \frac{n-2}{4(n-1)}\), \(1 < p \leq \frac{n+2}{n-2}\) and \(\Delta_g = \text{tr} \nabla^2 g\) is the Laplace-Beltrami operator associated with \(g\). Let \(L_g = \Delta_g - c(n) R_g\) denote the conformal Laplacian of the metric \(g\). When \(p = \frac{n+2}{n-2}\), the solution of Eq. (1.1) corresponds to the prescribed scalar curvature problem. That is, the metric given by \(\tilde{g} = u^{\frac{4}{n-2}} g\) has scalar curvature \(\tilde{R}_{\tilde{g}} = \frac{4(n-1)}{n-2} K\).

The Yamabe problem on manifolds is well-understood, and we refer to [Aub98a, LP87, Sch84, Yam60] and references therein. The prescribed scalar curvature problem on \(S^n\) and other manifolds has been extensively studied elsewhere; see for example [BAA08, CGY93, CL01, LI95, LI96, Mal02, SZ96]. Prescribed scalar curvature on manifolds is studied for example in [BACCH96, MM20, May17]. More recently, the Yamabe problem on singular spaces has been of interest; see for example [AB03, AB04, ACM14, ACM19, CLV21, Mon17, Mon18, Via10, Via13].

Analogous to the generalization from manifolds to orbifolds, there is the following generalization of asymptotically Euclidean metrics.
**Definition 1.4.** A complete Riemannian orbifold \((X^n, g)\) with finitely many singular points is called *asymptotically locally Euclidean* or ALE of order \(\tau\) if it has finitely many ends and for each end \(l\) there is a corresponding open subset \(E_l\), a finite subgroup \(\Gamma_l \subset O(n)\) acting freely on \(\mathbb{R}^n \setminus \{0\}\), and a diffeomorphism \(\psi_l : E_l \rightarrow (\mathbb{R}^n \setminus B_R(0))/\Gamma_l\), and such that under this identification,

\[
((\psi_l)_* g)_{ij} = \delta_{ij} + O(\rho^{-\tau}),
\]

\[
\hat{g}^k ((\psi_l)_* g)_{ij} = O(\rho^{-\tau - |k|}),
\]

for any partial derivative of order \(|k|\), as \(\rho \to \infty\), where \(\rho\) is the distance to some fixed basepoint. We call each end \(l\) an ALE end. If \(\Gamma_l = \{e\}\) for some \(l\), we call end \(l\) an asymptotically Euclidean (AE) end.

We will occasionally refer to an AE end as an ALE end, since AE is just the special case of ALE with \(\Gamma = \{e\}\). Next, we give the definition of the ADM mass on asymptotically locally Euclidean (ALE) orbifolds.

**Definition 1.5.** Given an \(n\)-dimensional ALE orbifold \((X, g)\) with asymptotic coordinates \(\{z^i\}\) and quotient group \(\Gamma_l\) near end \(l\), define the ADM mass as follows:

\[
m_l(g) = \frac{|\Gamma_l|}{\text{Vol}(S^{n-1})} \lim_{r \to \infty} \int_{S_r/\Gamma_l} \sum_{i,j=1}^n (\partial_i g_{ij} - \partial_j g_{ii})(\partial_j dV_z),
\]

where \(S_r/\Gamma_l\) is the hypersurface at \(|z| = r\), and \(dV_z\) is the Euclidean volume element. In the case \(X\) only has one end, the index \(l\) is omitted.

**Remark 1.6.** Bartnik proved that in the AE case, if \(\tau > (n-2)/2\) and \(R_g \in L^1\), then the mass is well-defined and independent of the choice of coordinates at infinity [Bar86]. Geometric invariance of the mass was also independently discovered in [Chr86]. A similar argument shows that the same result holds in the AE case.

**Remark 1.7.** Note that if \(|\Gamma_l| = \{e\}\) is the trivial group, the formula above defining the mass for the AE end \(l\) of the orbifold \((X, g)\) is consistent with [LP87, Definition 8.2]. Note also that if \(\Gamma_l \neq \{e\}\), our coefficient differs from [HL16] due to the factor of \(|\Gamma_l|\). Our convention has the advantage that it eliminates the need for writing extra factors of \(|\Gamma_l|\) in several formulas.

For any Riemannian orbifold \((M, g)\) with positive scalar curvature, we can construct a scalar-flat ALE orbifold \((X, \hat{g})\) by the following well-known procedure.

**Definition 1.8.** Take \(g\)-normal coordinates \(\{x^i\}\) centered at point \(\bar{x}\) and let \(r = |x|\) denote the distance function. Let \(\psi_{\bar{x}} > 0\) be the Green’s function of \(L_g\) with leading term \(r^{2-n}\) near \(\bar{x}\). Then \((X, \hat{g}_{\bar{x}}) = (M \setminus \bar{x}, \psi_{\bar{x}}^{4/(n-2)} g)\) is a scalar flat ALE orbifold. We will refer to \((\bar{x}, \hat{g}_{\bar{x}})\) as the conformal blow-up of \(g\) at the point \(\bar{x}\).

Of course, if \(\bar{x}\) is a smooth point of \(M\), then \((\bar{x}, \hat{g}_{\bar{x}})\) is an AE orbifold, but if \(\bar{x}\) is a singular point, then \((\bar{x}, \hat{g}_{\bar{x}})\) is an ALE orbifold.
1.1. Positive Mass Theorem on AE orbifolds. Our first result is that the positive mass theorem does hold for AE orbifolds.

**Theorem 1.9.** Let \((X, g_X)\) be an asymptotically locally Euclidean (ALE) \(n\)-dimensional Riemannian orbifold with finitely many isolated singular points, with \(R(g_X) \geq 0\), \(R(g_X) \in L^1\), and which is of order \(\tau_X > \frac{n-2}{2}\). Then for each asymptotically Euclidean (AE) end \(l, m_l(g_X) \geq 0\), and \(m_l(g_X) = 0\) for some AE end if and only if there are no nontrivial orbifold singularities, and \((X, g_X)\) is isometric to \((\mathbb{R}^n, g_{Euc})\).

This is proved in Sect. 2. The basic idea is to use a certain Green’s function for the Laplacian based at the orbifold points, which we use to reduce to the positive mass theorem for manifolds with concave boundary due to Hirsch-Miao [HM20] using the fundamental work of Schoen-Yau [SY79,SY81,SY17].

**Remark 1.10.** The positive mass theorem holds on ALE spaces only with some extra assumptions; see for example [Nak90]. In contrast, the positive mass theorem does not necessarily hold for arbitrary ALE metrics with nonnegative scalar curvature. The first examples were given in [LeB88], and many more in [HL16]. Theorem 1.9 shows that the study of the Yamabe equation on orbifolds can be substantially different from the manifold case. In dimension 4, we can define a mass function \(m : M \to \mathbb{R}\) by assigning the mass of the conformal blow-up at \(\bar{x} \in M\). In the manifold case, this is a smooth function [Hab00]. However, the above result shows that if \((M, g)\) is an orbifold and the conformal blow-up has negative mass at \(\bar{x} \in \Sigma\), then the mass function \(m\) is necessarily discontinuous at \(\bar{x}\); see Corollary 2.3 below.

1.2. Existence and compactness results. There is a long history of existence and compactness results for the Yamabe problem, which is the case when \(K = \text{constant}\). The fundamental idea for compactness was due to Schoen [Sch89b,Sch91,Sch89a]; compactness results for the Yamabe problem in low dimensions were then proved in [Dru03,KMS09,LZ05,LZ99,Mar05]. As mentioned above, existence and compactness results for variable \(K\) have been studied in great detail on \(S^n\) and on manifolds. Our next result generalizes many of these results in dimension four to the case of a Riemannian orbifold.

**Theorem 1.11.** Let \((M, g)\) be a compact Riemannian 4-dimensional orbifold with positive scalar curvature. Let \(K \in C^2(M)\) satisfy

\[
0 < \delta_1 \equiv \inf_M K, \quad (1.5)
\]

\[
0 < \delta_2 \equiv \inf_{\bar{x} \in \text{Crit}(K)} \left| m(\hat{g}_{\bar{x}}) + \frac{\Delta_g K(\bar{x})}{2K(\bar{x})} \right|, \quad (1.6)
\]

Then there exists some constant \(C\) depending only on \(M, g, \delta_1, \delta_2, \|K\|_{C^2(M)}\) such that

\[
1/C \leq u \leq C \quad \text{and} \quad \|u\|_{C^2,\alpha(M)} \leq C \quad (1.7)
\]

for all solutions \(u\) of (1.1) with \(p = 3\), where \(0 < \alpha < 1\). Moreover, if

\[
0 < \delta_3 \equiv \inf_{\bar{x} \in \text{Crit}(K)} \left( m(\hat{g}_{\bar{x}}) + \frac{\Delta_g K(\bar{x})}{2K(\bar{x})} \right), \quad (1.8)
\]

then (1.7) holds for all \(1 < 1 + \varepsilon < p \leq 3\) where \(C\) in addition depends upon \(\delta_3, \varepsilon\). Consequently, in this case there exists a solution \(u\) of (1.1) with \(p = 3\).
Theorem 1.11 will be proved in Sects. 3, 4, and 5. One of the main difficulties is due to the “discontinuity” of conformal normal coordinates at an orbifold point. In Sect. 5, we will show that if concentration of a sequence $u_k$ happens at an orbifold point, then the local maxima of $u_k$ must occur exactly at the orbifold point for $k$ sufficiently large; see Condition 3.7 which is a generalization of the isolated simple blow-up condition. Once we prove that blow-up points satisfy Condition 3.7; we can then fix the coordinates to be centered at the orbifold point, and then there is a contribution only from the mass function at the blow-up point. Note: our argument does not need continuity of the mass function which, as pointed out above, is not true in general anyway.

Remark 1.12. There should be analogous results for higher-dimensional orbifolds, but we have restricted to dimension 4 due to the considerable extra computations which would be required. We have also restricted to dimension 4 because this dimension involves a nice interplay between the mass and the Laplacian of $K$, and thus makes apparent that the orbifold case can exhibit quite different behavior than the manifold case (since the mass can be negative at orbifold points).

Remark 1.13. We emphasize that under the assumption (1.6), we are only claiming compactness; we do not have the existence in this general case. The main reason is that the subcritical method only works under the stronger assumption (1.8). However, the compactness result does allow one to define the Leray-Schauder degree. We expect that by imposing extra assumptions on $K$, one should be able to prove this degree is non-zero in some cases where (1.8) does not hold, but we do not pursue this here. Results along this line in the manifold case can be found in [BACCH96, Li96, MM20, May17].

1.3. Variational methods. Next, we have another existence result in dimensions $n \geq 4$ which is proved using variational methods. Roughly, this says that if $K$ is not too large away from the orbifold points, then we need assumption (1.8) to hold at only one orbifold point.

Theorem 1.14. Let $(M, g)$ be a compact Riemannian $n$-dimensional orbifold with singularities $\Sigma = \{(q_1, \Gamma_1), \cdots, (q_l, \Gamma_l)\}$ and positive scalar curvature. Let $K$ be a positive smooth function on $M$. Assume that

$$\max_{1 \leq i \leq l} |\Gamma_i|^2 K(q_i) \geq \sup K, \quad (1.9)$$

and

$$\begin{align*}
&\left\{ m(\hat{g}_{q_{i_0}}) + \frac{\Delta_g K(q_{i_0})}{2K(q_{i_0})} > 0, \quad n = 4, \\
&\Delta_g K(q_{i_0}) > 0, \quad n \geq 5, 
\right. \quad (1.10)
\end{align*}$$

for some $i_0$ such that $|\Gamma_{i_0}|^2 K(q_{i_0}) = \max_{1 \leq i \leq l} |\Gamma_i|^2 K(q_i)$, where $\hat{g}_{q_{i_0}}$ is as in Definition 1.8. Then there exists a positive smooth solution of Eq. (1.1) with $p = \frac{n+2}{n-2}$.

This will be proved in Sect. 6, which is closely related to [ES86, Theorem 2.1], [Aku12, Theorem 3.1] and [MM21, Proposition 5.1]. The analogue on manifolds is that (1.10) has to hold at one maximum point of $K$, which is only possible in dimension $n = 4$. Hence the theorem is quite special to the orbifold case in dimensions $n \geq 5$. We next apply this to some examples.

Consider $S^n \subset \mathbb{R}^{n+1}$, and let $\Gamma \subset \text{SO}(n)$ be a finite subgroup acting freely on $S^{n-1} \subset S^n \cap \{x_{n+1} = 0\}$. Then $S^n / \Gamma$ with the spherical metric $g_S$ is a Riemannian orbifold with 2 orbifold points $q_1$ and $q_2$. 
Corollary 1.15 (Spherical football orbifold). With $\Gamma$ as above, let $K : S^n / \Gamma \to \mathbb{R}_+$ be a smooth function and without loss a generality assume that $K(q_1) \geq K(q_2)$. If
\begin{align}
|\Gamma| \frac{2}{n-2} K(q_1) & \geq \sup K, \quad (1.11) \\
\Delta_{gS} K(q_1) & > 0, \quad (1.12)
\end{align}
then there exists a positive smooth solution of Eq. (1.1) with $g = gS$ and $p = \frac{n+2}{n-2}$.

We note that for $|\Gamma| = 2$, this result also follows from [LZ14, Theorem 1.12]. Another non-trivial example to which Theorem 1.14 applies is to the Calabi metric $g_{CAL(n)}$, which is a $U(n)$-invariant Ricci-flat Kähler ALE metric on the total space $X_n$ of the line bundle $O(-n) \to \mathbb{P}^{n-1}$. For details on the Calabi metric, see for example [MV20].

Corollary 1.16 (Calabi orbifold). For $n \geq 2$, take a $U(n)$-invariant conformal compactification $(\tilde{X}_n, \tilde{g}_{CAL(n)})$ of the Calabi metric $(X_n, g_{CAL(n)})$, such that the infinity of $g_{CAL(n)}$ is compactified to an orbifold point $\tilde{q}$, with quotient group $\mathbb{Z}/n\mathbb{Z}$. Let $K$ be a positive smooth function on $\tilde{X}_n$. Assume that
\begin{align}
n^{\frac{1}{n-1}} K(\tilde{q}) & \geq \sup K, \quad (1.13) \\
\Delta_{\tilde{g}_{CAL(n)}} K(\tilde{q}) & > 0, \quad (1.14)
\end{align}
then there exists a positive smooth solution of Eq. (1.1) with $g = \tilde{g}_{CAL(n)}$ and $p = \frac{n+1}{n-1}$.

Remark 1.17. In this case, from Theorem 1.3 and Remark 4.7 in [Via10], we know that there is no solution with $K = \text{constant} > 0$, which certainly does not contradict with the condition on the Laplacian at the orbifold point. However, Corollary 1.16 implies that any sufficiently small perturbation of a constant satisfying $\Delta_{\tilde{g}_{CAL(n)}} K(\tilde{q}) > 0$ does admit a solution. So the case of $K = \text{constant}$ is right on the boundary of existence. Furthermore, if we take a sequence of functions $K_k$ satisfying (1.14), but converging to a positive constant in $C^2$ norm, then this gives an example where bubbling must occur as $k \to \infty$.

For $n = 2$, $(X_2, g_{CAL(2)})$ is also known as the Eguchi-Hanson metric, which is included in the family of LeBrun metrics, which we discuss next.

1.4. LeBrun metrics. In this subsection, we will use the previous results to analyze the prescribed scalar curvature problem on the family of orbifolds mentioned above: the LeBrun negative mass metrics on $O_{\mathbb{P}^1}(-n)$. Note: these are 4-dimensional manifolds, and the integer $n = -c_1$, where $c_1$ is the first Chern class of this line bundle. These metrics possess an isometric $U(2)$-action. As we will see below, we have some results in the general case, but our most complete results are under a $U(2)$-symmetry assumption.

In [LeB88], LeBrun presented the first known example of a scalar-flat ALE metric of negative ADM mass. For any $n \in \mathbb{N}^*$, define
\begin{equation}
g_{LEB(n)} = \frac{1 + r^2}{n + r^2} dr^2 + (1 + r^2) \theta_1^2 + (1 + r^2) \theta_2^2 + \frac{r^2(n + r^2)}{1 + r^2} \theta_3^2, \quad (1.15)
\end{equation}
where $r \in (0, \infty)$ is the radial coordinate, and $\{\theta_1, \theta_2, \theta_3\}$ is a left-invariant coframe on $S^3 = SU(2)$. Attach a $\mathbb{P}^1$ at $r = 0$. After taking a quotient by $\Gamma_n = \mathbb{Z}/n\mathbb{Z}$, the metric
extends smoothly over this $\mathbb{P}^1$. Furthermore, the metric is ALE with group action $\Gamma_n$ near $r = \infty$ and the resulting manifold is diffeomorphic to $O_{\mathbb{P}^1}(-n)$.

The mass term defined in [LeB88] differs from our Definition 1.5 by a scaling factor. With a modification of the mass computed in [LeB88], we obtain

$$m(g_{LEB(n)}) = -2(n - 2). \quad (1.16)$$

For any $n \in \mathbb{N}^*$, $g_{LEB(n)}$ is scalar-flat. Note that $g_{LEB(1)}$ is conformal to the Fubini-Study metric on $\mathbb{P}^2$, and is known as the Burns metric, which is AE and has positive mass. As pointed out above, $g_{LEB(2)}$ is the Eguchi-Hanson metric, which is Ricci-flat ALE and has zero mass. For any $n \geq 3$, $g_{LEB(n)}$ has negative mass. More details and properties of LeBrun metrics can be found in [DL16, LeB88, Via10].

We next choose an orbifold compactification by defining

$$\tilde{g}_{LEB(n)} = \frac{1}{(n + r^2)^{\frac{1}{2}}} \cdot g_{LEB(n)}. \quad (1.17)$$

Then $(\tilde{O}_{\mathbb{P}^1}(-n), \tilde{g}_{LEB(n)})$ is a compact orbifold with singular point $\tilde{q}$ at $r = \infty$ with quotient group $\Gamma_n$. Its scalar curvature is computed to be

$$R_{\tilde{g}_{LEB(n)}} = \frac{24n(n + r^2)}{1 + r^2} > 0. \quad (1.18)$$

In this case, Theorem 1.14 specializes to the following.

**Theorem 1.18.** Consider $(\tilde{O}_{\mathbb{P}^1}(-n), \tilde{g}_{LEB(n)})$ for $n \in \mathbb{N}^*$. Let $K$ be a positive smooth function such that

$$\sup K \leq nK(\tilde{q}), \quad 4(n - 2)K(\tilde{q}) < \Delta_{\tilde{g}_{LEB(n)}} K(\tilde{q}). \quad (1.19)$$

Then there exists a positive smooth solution of Eq. (1.1) with $g = \tilde{g}_{LEB(n)}$ and $p = 3$.

Next, we turn to the U(2)-invariant problem. Define a function space $\mathcal{X}_n$

$$\mathcal{X}_n = \{ K : \tilde{O}_{\mathbb{P}^1}(-n) \to \mathbb{R} | K \text{ is smooth, U(2)-invariant, and } K > 0 \}. \quad (1.20)$$

Given any $K \in \mathcal{X}_n$, we are asking whether there exists a solution $u \in \mathcal{X}_n$ of the equation

$$-\Delta_{\tilde{g}_{LEB(n)}} u + \frac{1}{6} R_{\tilde{g}_{LEB(n)}} u = K u^3. \quad (1.21)$$

Decompose $\mathcal{X}_n$ into three disjoint subsets

$$\mathcal{X}_{n,+} = \{ K \in \mathcal{X}_n : \Delta_{\tilde{g}_{LEB(n)}} K(\tilde{q}) - 4(n - 2)K(\tilde{q}) > 0 \},$$

$$\mathcal{X}_{n,0} = \{ K \in \mathcal{X}_n : \Delta_{\tilde{g}_{LEB(n)}} K(\tilde{q}) - 4(n - 2)K(\tilde{q}) = 0 \}, \quad (1.22)$$

$$\mathcal{X}_{n,-} = \{ K \in \mathcal{X}_n : \Delta_{\tilde{g}_{LEB(n)}} K(\tilde{q}) - 4(n - 2)K(\tilde{q}) < 0 \}.$$
and a map
\[ F_{p,K,n} : \mathcal{\tilde{O}}_{\Lambda,n} \rightarrow C^2, \mathcal{\tilde{\varphi}1}(-n) \] by
\[ u + L^{-1}_{\tilde{\varphi}1}(Ku^p), \quad (1.24) \]
where \( L_{\tilde{\varphi}1}(n) = \Delta_{\tilde{\varphi}1}(n) - \frac{1}{6} R_{\tilde{\varphi}1}(n) \). Similar to [Sch84, LZ05, LZ99, KMS09, Mar05], we can define the \( U(2) \)-invariant Leray-Schauder degree of \( F_{p,K,n} \) in the region \( \mathcal{O}_\Lambda,n \) with respect to \( 0 \in C^2, \mathcal{\varphi1}(-n) \), which we denote by \( \text{deg}_{U(2)}(F_{p,K,n}, \mathcal{O}_\Lambda,n, 0) \).

Using Theorem 1.11 and some other special techniques, we present our last theorem, which gives a complete understanding of the \( U(2) \)-invariant Leray-Schauder degree.

**Theorem 1.19.** For any \( n \in \mathbb{N}^* \), there exists a constant \( C \), depending only on \( \mathcal{\varphi1}(-n) \), \( \tilde{\varphi}LEB(n) \) such that we have the following conclusions.

1. For all \( K \in X_{n,+} \) and \( \Lambda > C \), we have
\[ \text{deg}_{U(2)}(F_{3,K,n}, \mathcal{O}_\Lambda,n, 0) = -1. \quad (1.25) \]
Consequently, Eq. (1.21) admits at least one \( U(2) \)-invariant solution in \( X_n \).

2. For all \( K \in X_{n,-} \) and \( \Lambda > C \), we have
\[ \text{deg}_{U(2)}(F_{3,K,n}, \mathcal{O}_\Lambda,n, 0) = 0. \quad (1.26) \]

**Remark 1.20.** We note that vanishing of the Leray-Schauder degree does not give any information regarding the existence of a solution. However, we can moreover show that there is no solution at all for a large class of functions in \( X_{n,-} \); see Theorem 7.3. In particular, there is no solution for \( K = \text{constant} \) when \( n \geq 2 \). For \( n = 2 \) and \( K = \text{constant} \), nonexistence of any solution (symmetric or non-symmetric) was proved in [Via10]. However, it is still an open question whether the case \( K = \text{constant} \) possibly admits some non-symmetric solution when \( n > 2 \).

**Remark 1.21.** For any \( n \in \mathbb{N}^* \), the set \( X_{n,0} \) can be viewed as a “wall” in the space of positive radial functions \( X_n \), and the Leray-Schauder degree jumps by 1 upon crossing this wall, which is a phenomenon observed in many other geometric PDE problems.

All of the above results regarding the LeBrun metrics are proved in Sect. 7.

### 2. Properties of the Mass

In this section, we will prove Theorem 1.9. For simplicity, let us assume that there is exactly one orbifold point, which we denote as \( q \), with orbifold group \( \Gamma \subset O(n) \) and exactly one end which is AE (the argument below easily generalizes to the case of multiple orbifold singularities and multiple ends; see Remark 2.2 below). Let \( r \) denote a positive smooth function which is the Euclidean distance in the AE coordinate system, and near \( q \) is the distance to \( q \). We will first prove a lemma showing existence of a certain harmonic function on \( X \setminus \{q\} \).

**Lemma 2.1.** There exists a unique harmonic function \( H : X \setminus \{q\} \rightarrow \mathbb{R} \) which satisfies \( H > 1 \) and admits the expansion
\[ H = \begin{cases} r^{2-n} + O(r^{4-n-\epsilon}) & \text{as } r \to 0 \\ 1 + Ar^{2-n} + O(r^{2-n-\epsilon}) & \text{as } r \to \infty \end{cases}, \quad (2.1) \]
for \( \epsilon > 0 \) sufficiently small, for some constant \( A > 0 \).
Proof. Let \( \phi \) be the cutoff function

\[
\phi(t) = \begin{cases} 
1 & t \leq 1 \\
0 & t \geq 2
\end{cases},
\]

and consider

\[
h_0 = \phi(r/r_0)r^{2-n},
\]

for \( r_0 > 0 \) small. Since \( h_0 \) is harmonic with respect to the Euclidean metric near point \( q \), by expanding \( \Delta_g \) at the Euclidean metric, it is not hard to see that

\[
\Delta_g h_0 = \begin{cases} 
O(r^{2-n}) & r \to 0 \\
O(r) & r \geq 2r_0
\end{cases}.
\]

Denote \( X_q = X \setminus \{q\} \). The argument below uses weighted Hölder space theory; for background we refer to [Bar86,LP87]. Consider the doubly weighted Hölder space \( C^{k,\alpha}_{b_0,\delta_\infty}(X_q) \), which satisfies if \( u \in C^{k,\alpha}_{b_0,\delta_\infty}(X_q) \) then

\[
u = \begin{cases} 
O(r^{b_0}) & r \to 0 \\
O(r^{\delta_\infty}) & r \to \infty
\end{cases}.
\]

For any \( \epsilon > 0 \), from (2.4), we have

\[
\Delta_g h_0 \in C^{k-2,\alpha}_{2-n-\epsilon,-n+\epsilon}(X_q).
\]

Consider the operator

\[
\Delta_g : C^{k,\alpha}_{2-n-\epsilon,2-n+\epsilon}(X_q) \to C^{k-2,\alpha}_{2-n-\epsilon,-n+\epsilon}(X_q).
\]

The adjoint operator has domain \( (C^{k,\alpha}_{k-2,\epsilon,2-n+\epsilon}(X_q))^* \), and kernel elements lie in the doubly weighted space \( C^{k,\alpha}_{k-2,\epsilon,-n-\epsilon}(X_q) \). The removable singularity theorem then says that a kernel element \( u \) in this space extends to \( X \), and then \( u \in C^{k,\alpha}_{k-2,\epsilon}(X) \) (the weighted space on \( X \) with only a weight \( \delta_\infty = -\epsilon \) at infinity), so \( u \equiv 0 \). Thus for \( \epsilon > 0 \) sufficiently small, the operator in (2.7) is surjective, and we can solve for \( \Delta h_\epsilon = \Delta_g h_0 \), with \( h_\epsilon \in C^{k,\alpha}_{k-2,\epsilon,2-n+\epsilon}(X_q) \).

The function \( h \equiv h_0 - h_\epsilon \) satisfies \( \Delta_g h = 0 \), and by the existence of a harmonic expansion near \( \infty \), it admits the expansion.

\[
h = \begin{cases} 
r^{2-n} + O(r^{4-n-\epsilon}) & r \to 0 \\
A r^{2-n} + O(r^{2-n-\epsilon}) & r \to \infty
\end{cases},
\]

for some constant \( A \). We then define \( H = 1 + h \), which is harmonic. We have that \( \lim_{r \to \infty} H = 1 \), and \( \lim_{r \to 0} H = +\infty \). If \( H \) were not strictly larger than 1, then it would have an interior minimum. The strong maximum principle would then imply that \( H \) is constant, which is impossible. So \( H > 1 \), which clearly implies that \( A > 0 \). Obviously, \( H \) is unique. \( \Box \)
Proof of Theorem 1.9. For any constant $\delta > 0$, we define

$$H_\delta = \delta H + (1 - \delta),$$

(2.9)

which satisfies $\Delta_g H_\delta = 0$, $H_\delta > 1$, and admits the expansion

$$H_\delta = \begin{cases} \delta r^{2-n} + O(r^{4-n-\epsilon}) & r \to 0 \\ 1 + \delta Ar^{2-n} + O(r^{2-n-\epsilon}) & r \to \infty \end{cases},$$

(2.10)

for $\epsilon > 0$ sufficiently small, for the fixed constant $A$ from Lemma 2.1.

Next, we consider the metric $(X_q, g_\delta) = (X \setminus \{q\}, H_\delta^{-\frac{4}{n-2}} g_X)$. Near $r \sim \infty$, $g_\delta$ has a single AE end of order $\min\{\tau_X, n-2\}$. Since $q$ is an orbifold point, near $q$, $g_\delta$ has a single ALE end of order $\tau = 2 - \epsilon$. To see this, choose Riemannian normal coordinates $\{x^i\}$ for $g_X$ around $q$, then we have the expansions

$$g_X = dx^2 + O(|x|^2),$$

(2.11)

$$H_\delta = \delta |x|^{2-n} + O(|x|^{4-n-\epsilon}),$$

(2.12)

which yield the expansion

$$g_\delta = H_\delta^{-\frac{4}{n-2}} g_X = \delta^{-\frac{4}{n-2}} |x|^{-4} (1 + O(|x|^{2-\epsilon}))^{\frac{4}{n-2}} (dx^2 + O(|x|^2)),

(2.13)

as $|x| \to 0$. Next, define coordinates $y$ by

$$y = \delta^{\frac{n-2}{n-4}} x\left/ |x|^2 \right..$$

(2.14)

A computation then shows that

$$g_\delta = dy^2 + O(|y|^{-2+\epsilon})$$

(2.15)

as $|y| \to \infty$, so $g_\delta$ is indeed ALE of order $\tau = 2 - \epsilon$. Note also that the scalar curvature of $g_\delta$ is given by

$$R(g_\delta) = c(n)^{-1} H_\delta^{-\frac{n+2}{n-2}} (-\Delta_g H_\delta + c(n) R_g H_\delta) = H_\delta^{-\frac{4}{n-2}} R_g \geq 0.$$

(2.16)

Given $\delta > 0$, we can choose a very large distance sphere $\Sigma_\delta$ in the ALE end of $g_\delta$ which is strictly concave with respect to the normal pointing to the AE end. Let $X_{\Sigma_\delta}$ be the manifold with boundary obtained by removing the ALE end outside of $\Sigma$, which is a manifold with strictly concave boundary with a single AE end. From [HM20, Theorem 1.5 and Remark 1.7], we conclude that

$$m(X_{\Sigma_\delta}, g_\delta) \geq 0.$$

(2.17)

But an easy computation shows that

$$m(X_{\Sigma_\delta}, g_\delta) = m(X, g_X) + b(n) \delta A,$$

(2.18)

where $b(n) > 0$ is a dimensional constant. Since this is true for any constant $\delta > 0$, and $A$ is a fixed constant, we conclude that

$$m(X, g_X) \geq 0.$$

(2.19)
Note that if \( m(X, g_X) = 0 \), then we cannot conclude that \( m(X_{\Sigma}, g_{\Sigma}) = 0 \), since \( A > 0 \). Therefore we cannot directly use the equality case in [HM20, Theorem 1.5]. So to finish the proof, we instead argue as in [LP87, Lemma 10.7], to conclude that \( g_X \) is Ricci-flat if mass\( (X, g_X) = 0 \). Roughly, the argument is as follows. Take a variation \( g_t = g + th \), with \( g_0 = g_X \), where \( h \) is any compactly supported symmetric 2-tensor. There exists a solution \( \varphi_t \) of \( L_{g_t} \varphi_t = -c(n) R(g_X) \varphi_t \) for sufficiently small \( t \), which implies that \( g_t = \frac{\varphi_t^4}{\varphi_t^2} \) has \( R(g_t) \geq 0 \). Since \( m(g_t) \) is minimized at \( t = 0 \),

\[
0 = \frac{dm(g_t)}{dt} \bigg|_{t=0} = \int_X h^{jk} R_{jk} \, dV_g,
\]

which implies that \( g_X \) is Ricci-flat. Note that all steps in this argument are valid in the orbifold setting. Since \( g_X \) is asymptotically Euclidean, we have asymptotic equality in Bishop’s volume inequality (which holds for orbifolds; see [Bor92]). This implies that \( g_X \) is flat, which clearly implies that there can be no nontrivial orbifold singularities, and \((X, g_X)\) is isometric to \((\mathbb{R}^n, g_{\text{Euc}})\).

**Remark 2.2.** It is clear that the proof generalizes to the case of multiple ends and multiple orbifold points. For this, do the conformal blow-up at all orbifold points to get an ALE manifold with several ends. Then fix any AE end and take large distance spheres in all other ends of this blow-up space to get a manifold with exactly one AE end and concave boundary components, and apply the above argument to this space.

As mentioned above, in dimension 4, there are examples of ALE spaces which have negative mass [LeB88, HL16]. So we note the following corollary of Theorem 1.9.

**Corollary 2.3.** If \((X, g)\) is a 4-dimensional Riemannian orbifold with negative mass at an orbifold point \( q \), then the mass function \( m : X \to \mathbb{R} \) is necessarily discontinuous at \( q \).

2.1. Odd-dimensional cases. If the dimension \( n \) is odd, we have the following.

**Proposition 2.4.** Let \((X, g)\) is a compact Riemannian orbifold with isolated singularities and odd-dimensional. Then any nontrivial orbifold point must have \( \Gamma = \mathbb{Z}/2\mathbb{Z} \). Furthermore, \((X, g)\) is a good orbifold. That is, there is a \( \mathbb{Z}/2\mathbb{Z} \) action on a compact manifold \( \check{X} \) with finitely many fixed points such that \( X = \check{X} / (\mathbb{Z}/2\mathbb{Z}) \). Letting \( \pi : \check{X} \to X \) denote the quotient mapping, then \( \pi^* g \) is a smooth Riemannian metric on \( \check{X} \).

**Proof.** For \( n \) odd, any element \( A \in O(n) \) must have \( \pm 1 \) as an eigenvalue, so the only possibility for a nontrivial orbifold point is \( \Gamma = \mathbb{Z}/2\mathbb{Z} \). Near any singular point \( q \), a small distance sphere is homeomorphic to \( \mathbb{R}^{n-1} \), which is non-orientable if \( n \) is odd. So if there is any nontrivial orbifold point, then \( X \) contains a non-orientable 2-sided hypersurface, which implies that \( X \setminus \Sigma \) is non-orientable, where \( \Sigma \) is the finite set of singular points. Let \( \pi : X' \to X \setminus \Sigma \) denote the orientable double cover. Consider \( \tilde{X} = X' \cup \{\tilde{q}_1, \ldots, \tilde{q}_j\} \) where \( \tilde{q}_j \) are points, and extend \( \pi : \tilde{X} \to X \) by letting \( \pi(\tilde{q}_j) = q_j \). We extend to \( \mathbb{Z}/2\mathbb{Z} \) action to \( \tilde{X} \) with fixed points at \( \tilde{q}_j \), and endow \( \tilde{X} \) with the quotient topology. It is then straightforward to show that \( \tilde{X} \) is a smooth manifold and \( \pi^* g \) extends as a smooth Riemannian metric to \( \tilde{X} \). \( \square \)
Corollary 2.5. Let \((X, g)\) be a compact Riemannian orbifold with isolated singularities and odd-dimensional. Then the mass function \(m: X \to \mathbb{R}\) satisfies \(m > 0\) everywhere, unless \((X, g)\) is conformal to \((S^n, g_{\text{round}})\) or a “football” metric \(S^n/(\mathbb{Z}/2\mathbb{Z})\) with exactly 2 singular points.

Proof. At any smooth point of \(X\), the mass of the Green’s function metric is positive by Theorem 1.9. If the mass at a smooth point were zero, then the Green’s function metric would be Euclidean space, which would imply that \((X, g)\) is conformal to \((S^n, g_{\text{round}})\).

At a singular point \(q\), by uniqueness of the Green’s function metric, the Green’s function metric at \(q\) must be the \(\mathbb{Z}/2\mathbb{Z}\) quotient of the Green’s function of \((\tilde{X}, \pi^* g)\) at \(\tilde{q}\). Since \(\tilde{X}\) is a manifold, by the usual positive mass theorem, the Green’s function metric upstairs must have non-negative mass, so the Green’s function downstairs must also. If the mass at an orbifold point was 0, then the Green’s function metric upstairs would have to be Euclidean space. This implies that the Green’s function metric downstairs is a \(\mathbb{Z}/2\mathbb{Z}\)-quotient of Euclidean space, which implies that \((X, g)\) is conformal to a “football” metric \(S^n/(\mathbb{Z}/2\mathbb{Z})\).

The above remarks show that the odd-dimensional case of the orbifold Yamabe problem is equivalent to the \(\mathbb{Z}/2\mathbb{Z}\)-equivariant Yamabe problem on the manifold \(\tilde{X}\).

3. Compactness Preliminaries

Remarkable work in analyzing the blow-up points of Eq. (1.1) has been done in [Mar05], [LZ05], [LZ99], [KMS09]. In this section, we are going to quote some of their definitions and local results on manifolds, which also appear to hold on orbifolds by modification of proofs.

In the following context, we will write \(\mathbb{R}^n/\Gamma\) or \(B_r(\tilde{x})/\Gamma\). If \(\Gamma = \{e\}\) is the trivial group, it denotes the Euclidean space or a smooth ball; if \(\Gamma \neq \{e\}\) is a finite nontrivial group in \(O(n)\), it denotes the Euclidean cone or a quotient of a ball centered at a singular point \(\tilde{x}\).

3.1. Conformal scalar curvature equation. Instead of dealing with Eq. (1.1), we will study the following conformal scalar curvature equation. Let \(\Omega \subset \mathbb{R}^n/\Gamma\) be an open neighborhood of the origin, and suppose \(g\) is a Riemannian metric in \(\Omega\). Suppose also \(f\) is a positive \(C^1\) function defined in \(\Omega\). Consider a positive \(C^2\) function \(u\) satisfying

\[
L_g u + K f^{-\tau} u^p = 0,
\]

where \(K\) is a positive \(C^2\) function, \(1 < p \leq \frac{n+2}{n-2}\) and \(\tau = \frac{n+2}{n-2} - p\).

We note that (3.1) is scale invariant. To see this, let \(u\) be a solution to Eq. (3.1). For any \(s > 0\), define the rescaled solution \(v(y) = s^{\frac{2}{\tau-1}} u(sy)\). Then \(L_{\tilde{h}} v + \tilde{K} \tilde{f}^{-\tau} v^p = 0\), where \(\tilde{K}(y) = K(sy)\), \(\tilde{f}(y) = f(sy)\) and the components in metric \(h\) in normal coordinates are given by \(h_{ij}(y) = g_{ij}(sy)\). Note that \(v\) satisfies an equation of the same type as Eq. (3.1).

We also note that (3.1) is conformally invariant. To see this, suppose \(\tilde{g} = \phi^{\frac{4}{n-2}} g\) is a metric conformal to \(g\) and let \(u\) be a solution to Eq. (3.1). Then \(\phi^{-1} u\) satisfies

\[
L_{\tilde{g}} (\phi^{-1} u) + K (\phi f)^{-\tau} (\phi^{-1} u)^p = 0,
\]

which is again an equation of the same type.

These two properties, will be used later to take a rescaling of the coordinates and conformally map \(g\) to some conformal normal metric, without changing the type of Eq. (3.1).
3.2. Pohozaev Identity. Suppose \( u : B_\sigma(0)/\Gamma \rightarrow \mathbb{R} \) is a positive \( C^2 \) solution to the equation

\[
a^{ij}(x)\partial_i u + b^i(x)\partial_i u + c(x)u + K(x)u^p = 0, \tag{3.2}
\]

where \( p \neq -1, K \in C^1 \) and \( a^{ij}, b^i, c \) are continuous functions, \( 1 \leq i, j \leq n \). Define

\[
P(r, u) = \int_{|x|=r}/\Gamma \left( \frac{n-2}{2} u \frac{\partial u}{\partial \sigma} - r \frac{1}{2} |\nabla u|^2 + r \frac{1}{2} \frac{\partial u}{\partial r}^2 \right) d\sigma(r), \tag{3.3}
\]

for \( 0 < r < \sigma \). Then, we have the following lemma.

**Lemma 3.1** ([Mar05] Lemma 2.1). For all \( 0 < r < \sigma \),

\[
P(r, u) = -\int_{|x| \leq r}/\Gamma \left( x \cdot \nabla u + \frac{n-2}{2} u \right)((a^{ij} - \delta^{ij})\partial_i u + b^i \partial_i u)dx
\]

\[
+ \int_{|x| \leq r}/\Gamma \left( \frac{1}{2} x \cdot \nabla c + c \right)u^2 dx - \frac{r}{2} \int_{|x|=r}/\Gamma cu^2 d\sigma(r)
\]

\[
+ \frac{1}{p+1} \int_{|x| \leq r}/\Gamma (x \cdot \nabla K(x))u^{p+1}dx
\]

\[
+ \left( \frac{n}{p+1} - \frac{n-2}{2} \right) \int_{|x|=r}/\Gamma K(x)u^{p+1}dx
\]

\[- \int_{|x|=r}/\Gamma \frac{1}{p+1} K(x)r u^{p+1}d\sigma(r). \tag{3.4}
\]

3.3. Isolated and isolated simple blow-up points. Let \( \Omega = B_\sigma(\bar{x})/\Gamma \) be (a quotient of) an open ball centered at point \( \bar{x} \). Suppose \( \{g_k\} \) is a sequence of Riemannian metrics in \( \Omega \) converging, in the \( C^2_{loc} \) topology, to a metric \( g \). Let \( R_k \) denote the scalar curvature of \( g_k \) and \( R_g \) denote the scalar curvature of the limit metric \( g \). Suppose \( \{f_k\} \) is a sequence of positive \( C^1 \) functions converging in the \( C^1_{loc} \) topology to a positive function \( f \). Also suppose \( \{K_k\} \) is a sequence of positive \( C^2 \) functions converging in the \( C^2_{loc} \) topology to a positive function \( K_\infty \). Consider a sequence of positive \( C^2 \) functions \( u_k \) satisfying

\[
L_{g_k} u_k + K_k f_k^{-\tau_k} u_k^{p_k} = 0 \text{ in } \Omega, \tag{3.5}
\]

where \( 1 + \epsilon_0 < p_k \leq \frac{n+2}{n-2} \) for some \( \epsilon_0 > 0 \) and \( \tau_k = \frac{n+2}{n-2} - p_k \).

**Definition 3.2.** Suppose \( u_k \) is a sequence of positive functions satisfying Eq. (3.5). If \( \Gamma = \{e\} \), define \( \bar{x} \) to be an isolated blow-up point for \( u_k \) if there exists a sequence \( x_k \in \Omega \), converging to \( \bar{x} \), so that: 1) \( x_k \) is a local maximum point of \( u_k \); 2) \( M_k := u_k(x_k) \rightarrow \infty \) as \( k \rightarrow \infty \); 3) there exist \( r, C > 0 \) such that \( u_k(x) \leq C d_{g_k}(x, x_k)^{-\frac{2}{p_k-1}} \) for every \( x \in B_r(x_k) \subset \Omega \), where \( B_r(x_k) \) denotes the geodesic ball of radius \( r \), centered at \( x_k \), with respect to the metric \( g_k \).

If \( \Gamma \neq \{e\} \), let \( \pi : B_\sigma(\bar{x}) \rightarrow B_\sigma(\bar{x})/\Gamma \) be the projection map. We say that \( \bar{x} \) is an isolated blow-up point for \( u_k \) if \( \pi^*(\bar{x}) = \bar{x} \) is an isolated blow-up point for \( \pi^*(u_k) \) in the lifting-up space.
Define
\[ U_c(y) = \left( \frac{n(n-2)}{c} \right)^{\frac{n-2}{4}} (1 + |y|^2)^{\frac{2-n}{2}} \] (3.6)
on \mathbb{R}^n / \Gamma$, where $c$ is some positive constant. It is not hard to check that
\[ \Delta U_c(y) + c U_c^{\frac{n+2}{n-2}}(y) = 0. \] (3.7)

**Remark 3.3.** From now on, for each $k$, assume that we work in $g_k$-normal coordinates $\{x^i\}$ centered at point $x_k$. Then, we will simply write $u_k(x)$ instead of $u_k(\exp_{x_k}(x))$ and $|x|$ instead of $d_{g_k}(x, x_k)$.

Moreover, by [LP87, Theorem 5.1], there exists a conformal factor $\phi_k$, such that $\tilde{g}_k = \phi_k^{\frac{4}{n-2}} g_k$ is the conformal normal metric with conformal normal coordinates $\{\tilde{x}^i\}$ centered at $x_k$. Let $\tilde{u}_k = \phi_k^{-1} u_k$. By the property that Eq. (3.1) is conformally invariant as stated in Sect. 3.1, $u_k$ and $\tilde{u}_k$ satisfies the same type of conformal scalar curvature Eq. (3.1). Hence we may assume $g_k$ is already the conformal normal metric and $\{x^i\}$ is already the conformal normal coordinates centered at $x_k$.

Consider the change of variables
\[ y = M_k^{\frac{p_k-1}{2}} x, \] (3.8)
and define the rescaled metric and functions
\[ (h_k)_{ij}(y) = (g_k)_{ij}(M_k^{-\frac{p_k-1}{2}} y), \quad v_k(y) = M_k^{-1} u_k(M_k^{-\frac{p_k-1}{2}} y), \]
\[ \tilde{f}_k(y) = f_k(M_k^{-\frac{p_k-1}{2}} y), \quad \tilde{K}_k(y) = K_k(M_k^{-\frac{p_k-1}{2}} y) \quad \text{and} \quad \tilde{R}_k(y) = R_k(M_k^{-\frac{p_k-1}{2}} y), \] (3.9)
for $|y| < M_k^{\frac{p_k-1}{2}} r$, where $r$ is as in Definition 3.2. Using the property that Eq. (3.1) is rescale invariant as stated in Sect. 3.1, the rescaled functions satisfy
\[ L h_k v_k + \tilde{K}_k \tilde{f}_k^{-\nu_k} v_k^{p_k} = 0. \] (3.10)

The following property holds for an isolated blow-up point.

**Proposition 3.4** ([Mar05] Proposition 4.3). Assume $u_k$ is a sequence of positive functions satisfying Eq. (3.5) and $x_k \to \bar{x}$ is an isolated blow-up point. Moreover, if $\Gamma \neq \{e\}$, we require $x_k = \bar{x}$ for all large $k$’s. Assume $p_k \to \frac{n+2}{n-2}$, then there exist $R_k' \to \infty$ and $\epsilon_k \to 0$, such that after passing to a subsequence,
\[ \left\| v_k(y) - U_{K_{\infty}(\bar{x})}(y) \right\|_{C^2(B_{R_k'}(0))/\Gamma} \leq \epsilon_k, \] (3.11)
and
\[ \frac{R_k'}{\log(M_k)} \to 0, \] (3.12)
as $k \to \infty$. 
Proof. The case $K_k = \text{constant}$ is proved in [Mar05, Proposition 4.3]. For variable $K_k$, because $K_k \to K_\infty$ in the $C^0_{\text{loc}}$ norm, $\tilde{K}_k \to K_\infty(\tilde{x})$ in the $C^0_{\text{loc}}$ norm. By Eq. (3.10) and the proof of [Mar05, Proposition 4.3], after passing to a subsequence, $v_k \to v > 0$ in $C^2_{\text{loc}}$ norm, which satisfies

$$
\Delta v(y) + K_\infty(\tilde{x})v(y)^p = 0, \quad y \in \mathbb{R}^n / \Gamma
$$

$$
v(0) = 1, \quad \nabla v(0) = 0,
$$

(3.13)

where $p = \lim_{k \to \infty} p_k$ and $\Delta$ denotes the Euclidean Laplacian. Let $\tilde{v}(\tilde{y})$ be the lifting-up function of $v(y)$ in the universal covering space $\mathbb{R}^n$, where $\tilde{y}$ is the coordinate in $\mathbb{R}^n$. It turns out $\tilde{v}(\tilde{y})$ satisfies exactly the same equations as (3.13). By [CGS89], we must have $p = \frac{n+2}{n-2}$, and $\tilde{v}(\tilde{y}) = U_{K_\infty(\tilde{x})}(\tilde{y})$, which further implies $v(y) = U_{K_\infty(x)}(y)$ in $\mathbb{R}^n / \Gamma$. The proof of (3.12) is the same as in [Mar05].

**Remark 3.5.** Under the same assumptions as Proposition 3.4, we also have

$$
\|v_k(y) - U_{K_k(x_k)}(y)\|_{C^2(B_{g_k^J}(0)/\Gamma)} \leq \epsilon_k,
$$

(3.14)

which is simply because $U_{K_k(x_k)} \to U_{K_\infty(\tilde{x})}$ uniformly in $\mathbb{R}^n / \Gamma$.

**Definition 3.6.** Suppose $u_k$ is a sequence of positive functions satisfying Eq. (3.5) and $x_k \to \tilde{x}$ is an isolated blow-up point. If $\Gamma = \{e\}$, define

$$
\tilde{u}_k(r) = \frac{1}{Vol(S^{n-1})r^{n-1}} \int_{\partial B_r(x_k)} u_k d\sigma(r),
$$

(3.15)

where we are using $g_k$-normal coordinates and integrating with respect to the Euclidean volume form. We say $x_k \to \tilde{x}$ is an isolated simple blow-up point if there exists a real number $0 < \rho < r$ such that the functions

$$
\hat{u}_k(r) \equiv r^{\frac{2}{n-2}} \tilde{u}_k(r)
$$

(3.16)

have exactly one critical point in the interval $(0, \rho)$, for $k$ large.

If $\Gamma \neq \{e\}$, let $\pi : B_\sigma(\tilde{x}) \to B_\sigma(\tilde{x}) / \Gamma$ be the projection map. We say that $\tilde{x}$ is an isolated simple blow-up point for $u_k$ if $\pi^*(\tilde{x})$ is an isolated simple blow-up point for $\pi^*(u_k)$ in the lifting-up space.

Next, we give the special blow-up condition that we will work on.

**Condition 3.7.** Assume $u_k$ is a sequence of positive functions satisfying Eq. (3.5) and $x_k \to \tilde{x}$ is an isolated simple blow-up point. Moreover, if $\Gamma_{\tilde{x}} \neq \{e\}$, then we require that $x_k = \tilde{x}$ for all $k$ sufficiently large.

**Remark 3.8.** In Condition 3.7, the main reason why we require $x_k = \tilde{x}$ for all $k$ sufficiently large in the singular point case $\Gamma \neq \{e\}$ is the following. By assuming so, for each $k$, the geodesic ball centered at $x_k$ will be $B_\sigma(\tilde{x}) / \Gamma$. Then, when we later analyze some local integrals and let $k \to \infty$, we will not run into the case that integrals over smooth balls converge to an integral over a quotient of a smooth ball. We also note that eventually, we will prove in Corollary 5.4 that if blow-up occurs at a singular point, then Condition 3.7 must hold.
From now on, in this section, we will assume that the dimension \( n = 4 \); see Remark 1.12. In the following context, we will use \( C \) to denote various constants which only depend on the limit metric \( g, \inf K_\infty, \| K_\infty \|_{C^2} \) and possibly the chosen small radius \( \rho_1, \delta \) and \( \sigma \). The dependency is implied in the proof. Fix \( \delta > 0 \), and define

\[
\lambda_k = (2 - \delta) \frac{p_k - 1}{2} - 1. \tag{3.17}
\]

**Proposition 3.9 ([LZ99]).** Assuming Condition 3.7, for sufficiently small \( \delta > 0 \), there exists constants \( 0 < \rho_1 < \rho \) and \( C > 0 \) such that

\[
\begin{align*}
M_k^{\delta \cdot} u_k(x) &\leq C|x|^{-2+\delta}, \\
M_k^{\lambda_k} |\nabla u_k(x)| &\leq C|x|^{-3+\delta}, \\
M_k^{\lambda_k} |\nabla^2 u_k(x)| &\leq C|x|^{-4+\delta},
\end{align*}
\tag{3.18}
\]

for every \( x \) satisfying

\[
R_k^{-\frac{p_k-1}{2}} \leq |x| \leq \rho_1. \tag{3.19}
\]

As a consequence, it implies

\[
\begin{align*}
v_k(y) &\leq CM_k^{\delta \cdot} (1 + |y|)^{-2}, \\
|\nabla v_k(y)| &\leq CM_k^{\delta \cdot} (1 + |y|)^{-3}, \\
|\nabla^2 v_k(y)| &\leq CM_k^{\delta \cdot} (1 + |y|)^{-4},
\end{align*}
\tag{3.20}
\]

for every \( y \) satisfying

\[
|y| \leq \rho_1 M_k^{\frac{p_k-1}{2}}. \tag{3.21}
\]

**Proof.** The proof is the same as [LZ99, Lemma 7.3]. That proof was for \( n = 3 \), but directly generalizes to higher dimensions. \( \square \)

**Proposition 3.10.** Assuming Condition 3.7, then there exists a constant \( C \) such that

\[
|\nabla K_k(x_k)| \leq CM_k^{-2+2\delta}, \tag{3.22}
\]

as \( k \to \infty \). Consequently,

\[
|\nabla K_\infty(\bar{x})| = 0. \tag{3.23}
\]

**Proof.** The proof is very similar to [LZ99, Lemma 7.8], we provide here only a brief outline. Recall that \( \{x^i\} \) is the \( g_k \)-normal coordinates centered at point \( x_k \). For some fixed positive small \( \sigma \), let \( \eta \) be a smooth cutoff function such that \( \eta(x) = 1 \) for \( |x| \leq \sigma/2 \) and \( \eta(x) = 0 \) for \( |x| \geq \sigma \). Multiplying Eq. (3.5) by \( \eta(\partial u_k / \partial x^j) \), integrating by parts on \( \{|x| \leq \sigma\}/ \Gamma \), we get
\[
\int_{\{|x| \leq \sigma\}/\Gamma} \frac{\partial K_k}{\partial x^j} \eta f_k^{-\tau_k} u_k^{p_k+1} \, dx \\
\leq C \int_{\{|x| \leq \sigma\}/\Gamma} (|\nabla \eta| \cdot |\nabla u_k|^2 + \frac{\partial (\eta R_k)}{\partial x^j} |u_k^2 + \frac{\partial (\eta f_k^{-\tau_k})}{\partial x^j} |K_k u_k^{p_k+1}) \, dx. \quad (3.24)
\]

Then, using Proposition 3.4 in the ball \(|x| \leq R'_k M_k^{-2/p_k-1}\) and Proposition 3.9 in the annuli \(R'_k M_k^{-2/p_k-1} \leq |x| \leq \sigma\), together with the assumption that \(K_k\) converges to \(K_\infty\) in the \(C^0_{loc}\) norm, for large \(k\)'s we have

\[
\int_{\{|x| \leq \sigma\}/\Gamma} \frac{\partial K_k}{\partial x^j} u_k^{p_k+1} \, dx \leq C_1 M_k^{-2+2\delta}, \quad (3.25)
\]

where \(C_1\) depends on \(\inf K_\infty, \|K_\infty\|_{C^2}\) and \(\sigma\). Next, the power series expansion

\[
\frac{\partial K_k}{\partial x^j}(x) = \frac{\partial K_k}{\partial x^j}(0) + \frac{\partial^2 K_k}{\partial x^j \partial x^i}(0) \cdot x^i + O(|x|^2), \quad (3.26)
\]

implies that

\[
|\frac{\partial K_k}{\partial x^j}(0)| \leq |\frac{\partial K_k}{\partial x^j}(x)| + C_2 |x|, \quad (3.27)
\]

where \(C_2\) depends on \(\|K_\infty\|_{C^2}\). Multiplying by \(u_k^{p_k+1}\), integrating over \(|\{|x| \leq \sigma\}/\Gamma\) and using inequality (3.25), we get

\[
|\frac{\partial K_k}{\partial x^j}(0)| \int_{\{|x| \leq \sigma\}/\Gamma} u_k(x)^{p_k+1} \, dx \leq C \left( M_k^{-2+2\delta} + \int_{\{|x| \leq \sigma\}/\Gamma} |x| u_k(x)^{p_k+1} \, dx \right). \quad (3.28)
\]

The integral on the left limits to the volume of the bubble which is a finite constant, the integral on the right can be estimated similarly as above using Proposition 3.4 and Proposition 3.9. Therefore, we have proved

\[
|\nabla K_k(x_k)| \leq CM_k^{-2+2\delta}, \quad (3.29)
\]

where \(C\) is a constant depending on \(\inf K_\infty, \|K_\infty\|_{C^2}, g\) and \(\sigma\). \(\square\)

**Proposition 3.11** ([Mar05] Proposition 4.5). **Assuming Condition 3.7, then there exists a constant** \(C > 0\) **and** \(0 < \rho_1 < \rho\) **such that**

\[
M_k u_k(x) \leq Cd_{g_k}(x, x_k)^{-2} \quad (3.30)
\]

**for** \(x\) **satisfying** \(d_{g_k}(x, x_k) \leq \rho_1\).

**Proof.** The proof is very similar to the proof of [Mar05, Proposition 4.5]. That proof was assuming \(K_k = constant\). For variable \(K_k\), every step in the proof remains valid, except for Claim 2 of [Mar05, Proposition 4.5], which says that there exists \(C > 0\) such that

\[
\tau_k \leq CM_k^{-2+2\delta+o(1)} \log(M_k) \text{ ask } \to \infty. \quad (3.31)
\]
This estimate does not hold in our setting, however, a modification of his arguments shows that there exists a constant $C > 0$ such that

$$\tau_k \leq CM_k^{-2+6\delta+o(1)} \quad \text{as} \ k \to \infty. \quad (3.32)$$

To verify this, note that when $K_k$ is a variable function, there will be an extra term on the left hand side of [Mar05, inequality (4.18)]. That extra term is

$$\frac{1}{p_k + 1} \left| \int_{|x| \leq \frac{\rho_1}{2}} (\nabla K_k(x) \cdot x) u_k(x)^{p_k + 1} \, dx \right| \leq \frac{1}{p_k + 1} M_k^{3-p_k} \int_{|y| \leq \frac{\rho_1}{2} M_k^{\frac{1}{2}}} |\nabla_x K_k(M_k^{\frac{p_k-1}{2}} y)| \cdot |M_k^{\frac{p_k-1}{2}} y| v_k(y)^{p_k + 1} \, dy. \quad (3.33)$$

For small $\rho_1$ and large $k$, when $|x| \leq \frac{\rho_1}{2}$, by power series expansion and Proposition 3.10, we have

$$|\nabla K_k(x)| \leq |\nabla K_k(x_k)| + C|x| \leq C(M_k^{-2+2\delta} + |x|). \quad (3.34)$$

Then for $|y| \leq \frac{\rho_1}{2} M_k^{\frac{1}{2}}$ and large $k$,

$$|\nabla_x K_k(M_k^{\frac{p_k-1}{2}} y)| \cdot |M_k^{\frac{p_k-1}{2}} y| \leq C(M_k^{-2} + M_k^{-p_k + 1} |y|^2) \leq C(\rho_1 M_k^{2+2\delta} + M_k^{-p_k + 1}) |y|^2 \leq CM_k^{-2+2\delta} |y|^2. \quad (3.35)$$

On the other hand, by Proposition 3.9, we know

$$v_k(y)^{p_k + 1} \leq CM_k^{\frac{p_k^2}{2} - 1} (1 + |y|)^{-2(p_k + 1)}. \quad (3.36)$$

Therefore,

$$M_k^{3-p_k} \int_{|y| \leq \frac{\rho_1}{2} M_k^{\frac{1}{2}}} |\nabla_x K_k(M_k^{\frac{p_k-1}{2}} y)| \cdot |M_k^{\frac{p_k-1}{2}} y| v_k(y)^{p_k + 1} \, dy \leq CM_k^{-2+2\delta} M_k^{\frac{p_k^2}{2} - 1} \int_{|y| \leq \frac{\rho_1}{2} M_k^{\frac{1}{2}}} (1 + |y|)^{-2(p_k + 1)} |y|^2 \, dy \leq CM_k^{-2+6\delta + o(1)} \int_{\mathbb{R}^4} (1 + |y|)^{-8 + o(1)} |y|^2 \, dy \leq CM_k^{-2+6\delta + o(1)} \quad \text{as} \ k \to \infty. \quad (3.37)$$

Then (3.32) is proved following the same proof as in Claim 2 in [Mar05, Proposition 4.5].

**Corollary 3.12** ([Mar05] Corollary 4.6). Assuming Condition 3.7, after maybe passing to a subsequence, we have

$$M_k u_k \to h_{\bar{x}} \text{ in } C^2_{\text{loc}}((B_r(\bar{x}) \setminus \{\bar{x}\}) \setminus \Gamma), \quad (3.38)$$

where $M_k$ is as defined in Definition 3.2 and $h_{\bar{x}} = aG(\cdot, \bar{x})$ is a constant multiple of the standard Green’s function, i.e. $L_g(G(\cdot, \bar{x})) = \delta_{\bar{x}}$ is the Dirac delta function at point $\bar{x}$. (Here, $g$ stands for the limit metric.)
Proof. The proof of Marques remains valid for variable $K_k$. □

Then, we have the following:

**Proposition 3.13.** Assuming Condition 3.7, then

$$\tau_k \leq C M_k^{-2}, \quad (3.39)$$

and there exists $\delta > 0$ such that

$$|v_k(y) - U_{K_k(x_k)}(y)| \leq C M_k^{-2}, \quad (3.40)$$

$$|\nabla(v_k - U_{K_k(x_k)})(y)| \leq C M_k^{-2}(1 + |y|)^{-1}, \quad (3.41)$$

$$|\nabla^2(v_k - U_{K_k(x_k)})(y)| \leq C M_k^{-2}(1 + |y|)^{-2}, \quad (3.42)$$

for $|y| \leq \delta M_i^{p_i-1}$.

Proof. The proof for $K = constant$ is by [Mar05, Chapter 5]. We need to verify that Marques’ proof is valid for variable $K_k$. When $K_k$ is a variable function instead of a fixed constant, every $U_0$ in [Mar05, Chapter 5] has to be replaced by $U_{K_k(x_k)}$, then [Mar05, equation (5.1)] will become

$$|\tilde{K}_k(y) - K_k(x_k)| = |K_k(M_k^{-\frac{p_k-1}{2}}y) - K_k(0)| \leq |\nabla K_k(0)| M_k^{-\frac{p_k-1}{2}}|y| + C M_k^{-\frac{p_k-1}{2}} |y|^2. \quad (3.45)$$

By Proposition 3.9, $|\nabla K_k(0)| = |\nabla K_k(x_k)| \leq M_k^{-2+2\delta}$, hence

$$|\tilde{K}_k(y) - K_k(x_k)| \leq C \left( M_k^{-2+\delta+\sigma(1)} + M_k^{-2+\tau_k} |y| \right) |y|. \quad (3.46)$$

Then we have the estimate

$$\mathbb{B}_k \leq K_k(x_k) U_{K_k(x_k)}^3 (1 - \tilde{f}_k U_{K_k(x_k)}^{3-\tau_k}) + |\tilde{K}_k(y) - K_k(x_k)| \tilde{f}_k^{-\tau_k} U_{K_k(x_k)}^{p_k} \leq C \left( \tau_k \log(f_k U_{K_k(x_k)})(1 + |y|^2)^{-3} + (M_k^{-3+2\delta+\sigma(1)} + M_k^{-2+\tau_k} |y|)(1 + |y|^2)^{-2} \right).$$
Suppose
\[ \Lambda_k^{-1} \max\{M_k^{-2}, \tau_k\} \to 0 \quad \text{as} \ k \to \infty. \] (3.48)

Note that (3.32) implies \( \lim_{k \to \infty} M_k^{-1} = 1 \), so \( \lim_{k \to \infty} \Lambda_k^{-1} \to 0 \). Thus \( |Q_k| \to 0 \) as \( k \to 0 \) and [Mar05, Lemma 5.1] can be proved following the rest of his proof. Also, [Mar05, Lemma 5.3] remains valid with a similar modification to the term \( \tilde{Q}_k \) in his proof. Therefore, our proposition is proved for variable \( K_k \). \[ \square \]

4. Local Blow-Up Analysis

In this section, we will assume that the dimension \( n = 4 \); see Remark 1.12. We begin with the fundamental Pohozaev identity.

4.1. Application of Pohozaev Identity. Assuming Condition 3.7, recall that \( g_k \) is the conformal normal metric with conformal normal coordinates \( \{x^i\} \) centered at the point \( x_k \). By [LP87, Chapter 5] and the explicit computations in [LZ05, Chapter 2], we have the following for each \( i \). In \( x = 0 \) (the point \( x_k \)), the scalar curvature satisfies
\[ R_k(0) = 0, \quad (R_k)_{,i}(0) = 0. \] (4.1)

Locally in a neighborhood of \( x = 0 \), we have \( (g_k)_{ij} = \delta_{ij} + O(r^2) \) and \( \det(g_k) = 1 + O(r^N) \) for some \( N \geq 5 \). Write
\[ \Delta_{g_k} = \frac{1}{\sqrt{\det(g_k)}} \partial_i(\sqrt{\det(g_k)}(g_k)^{ij} \partial_j) = \Delta_0 + (b_k)_{,i} \partial_i + (d_k)_{ij} \partial_{ij}, \] (4.2)

where \( \partial_i = \partial/\partial x^i \), \( \partial_{ij} = \partial^2/\partial x^i \partial x^j \) and \( \Delta_0 = \sum_{i=1}^{n} \partial^2 / (\partial x^i)^2 \), then
\[ (b_k)_{,i} = O(r^2) \quad \text{and} \quad (d_k)_{ij} = g^{ij} - \delta_{ij} = O(r^2). \] (4.3)

By the change of variables and the rescaled metric defined in (3.8) and (3.9), we have
\[ \Delta_{h_k} = \tilde{\Delta}_0 + (\tilde{b}_k)_{,i} \tilde{\partial}_i + (\tilde{d}_k)_{ij} \tilde{\partial}_{ij}, \] (4.4)

where
\[ (\tilde{b}_k)_{,i}(y) = M_k^{-p_k - 1} (b_k)_{,i}(M_k^{-p_k - 1} y), \quad (\tilde{d}_k)_{ij}(y) = (d_k)_{ij}(M_k^{-p_k - 1} y), \] (4.5)

It follows that
\[ |(\tilde{b}_k)_{,i}(y)| = M_k^{3(p_k - 1)/2} O(|y|^2), \quad |(\tilde{d}_k)_{ij}(y)| = M_k^{-(p_k - 1)} O(|y|)^2. \] (4.6)
Using the above notation, we rewrite Eq. (3.5) in the \( \{x^i\} \) coordinates as following

\[
((d_k)_{ij} + \delta_{ij})\partial_{ij}u_k + (b_k)_i\partial_iu_k - \frac{1}{6}R_ku_k + K_k f_k^{-\tau} u_k^{p_k} = 0. \tag{4.7}
\]

For some small \( \sigma \), apply Lemma 3.1 to the above equation in \( \{|x| \leq \sigma\}/\Gamma \), to obtain

\[
P(\sigma,u_k) = -\int_{\{|x| \leq \sigma\}/\Gamma} \left( x \cdot \nabla_x u_k + u_k \right)((d_k)_{ij} \partial_{ij}u_k + (b_k)_i\partial_iu_k)
\]

\[
+ \frac{1}{12} \int_{\{|x| \leq \sigma\}/\Gamma} (x \cdot \nabla_x R_k + 2R_k)u_k^2
\]

\[
+ \frac{\tau_k}{p_k + 1} \int_{\{|x| = \sigma\}/\Gamma} K_k f_k^{-\tau} u_k^{p_k+1}
\]

\[
= \int_{\{|x| \leq \sigma\}/\Gamma} \left( u_k \frac{\partial u_k}{\partial v} - \frac{\sigma}{2} |\nabla u_k|^2 + \sigma \left| \frac{\partial u_k}{\partial v} \right|^2 \right).
\tag{4.8}
\]

Denote \( R_k^i = M_k^{\frac{p_k-1}{2}} \sigma \). Define the following

\[
I_{k,1} = -\frac{\sigma}{p_k + 1} \int_{\{|x| = \sigma\}/\Gamma} K_k f_k^{-\tau} u_k^{p_k+1} d\sigma(x)
\tag{4.9}
\]

\[
I_{k,2} = -\frac{\sigma}{p_k + 1} \int_{\{|y| \leq \sigma\}/\Gamma} \tilde{K}_k f_k^{-\tau} v_k^{p_k+1} d\sigma(y),
\tag{4.10}
\]

\[
I_{k,3} = \frac{\sigma}{12} \int_{\{|x| = \sigma\}/\Gamma} R_k u_k^2 d\sigma(x) = \frac{\sigma}{12} M_k^{\frac{7}{2} - \frac{3p_k}{2}} \int_{\{|y| = \sigma\}/\Gamma} \tilde{R}_k v_k^2 d\sigma(y),
\tag{4.11}
\]

\[
I_{k,4} = -\frac{1}{12} \int_{\{|x| \leq \sigma\}/\Gamma} (x \cdot \nabla_x R_k + 2R_k)u_k^2 dx
\tag{4.12}
\]

\[
I_{k,5} = \frac{1}{p_k + 1} \int_{\{|x| \leq \sigma\}/\Gamma} (x \cdot \nabla_x (K_k f_k^{-\tau}))u_k^{p_k+1} dx
\tag{4.13}
\]

\[
I_{k,6} = \frac{\tau_k}{p_k + 1} \int_{\{|x| \leq \sigma\}/\Gamma} K_k f_k^{-\tau} u_k^{p_k+1} dx = \frac{\tau_k}{p_k + 1} M_k^{\frac{\tau_k}{2}} \int_{\{|y| \leq \sigma\}/\Gamma} \tilde{K}_k f_k^{-\tau} v_k^{p_k+1} dy.
\tag{4.14}
\]
\[ I_{k,7} = \int_{|x|=\sigma/\Gamma} \left\{ \left( \left| \frac{\partial u_k}{\partial v_x} \right|^2 - \frac{1}{2} \left| \nabla_x u_k \right|^2 \right) \sigma + u_k \frac{\partial u_k}{\partial v_x} \right\} d\sigma(x) \]
\[ = M_k^\tau \int_{|y|=R_k'/\Gamma} \left\{ \left( \left| \frac{\partial v_k}{\partial v_x} \right|^2 - \frac{1}{2} \left| \nabla_y v_k \right|^2 \right) R_k' + v_k \frac{\partial v_k}{\partial v_y} \right\} d\sigma(y). \] (4.15)

Then (4.8) becomes
\[ I_{k,7} = I_{k,1} + I_{k,2} + I_{k,3} + I_{k,4} + I_{k,5} + I_{k,6}. \] (4.16)

4.2. Main Estimates. By \([CGS89]\), it is sufficient to consider the blow-up case as \( p_k \to 3 \), consequently \( \tau_k = 3 - p_k \to 0 \). Then we know \( \lim_{k \to \infty} M_k^\tau = 1 \) from (3.32).

We can estimate terms in the Pohozaev identity through the following lemmas. We will use \( C, C_1 \) to denote various positive constants independent of \( k \) and \( \sigma \). For notational simplicity, we will omit \( dx, dy, d\sigma(x) \) and \( d\sigma(y) \) terms in integrals.

**Lemma 4.1.** For small \( \sigma > 0 \),
\[ \lim_{k \to \infty} M_k^2 I_{k,1} = 0. \] (4.17)

**Proof.** Using (3.5),
\[ I_{k,1} = \frac{\sigma}{p_k + 1} \int_{|x|=\sigma/\Gamma} (L_g u_k) \cdot u_k. \] (4.18)

By Corollary 3.12, \( M_k u_k \to h \) in the \( C^2_{loc} \) norm, hence for small \( \sigma \),
\[ \lim_{k \to \infty} M_k^2 I_{k,1} = \lim_{k \to \infty} \frac{\sigma}{p_k + 1} \int_{|x|=\sigma/\Gamma} (L_g (M_k u_k)) \cdot (M_k u_k) \]
\[ = \frac{\sigma}{p_k + 1} \int_{|x|=\sigma/\Gamma} (L_g h) \cdot h = 0. \] (4.19)

\[ \square \]

**Lemma 4.2.** For \( \sigma \leq 1 \), there exists some constant \( C > 0 \) such that
\[ \limsup_{k \to \infty} M_k^2 |I_{k,2}| \leq C \sigma^2. \] (4.20)

**Proof.** Because \( U_{K_k(x_k)}(y) \) is a radial function and \( \{y^i\} \) is a rescale of the conformal normal coordinates, we know
\[ (\Delta_{h_k} - \tilde{\Delta}_0) U_{K_k(x_k)} = ((\tilde{b}_k)_i \tilde{\partial}_i + (\tilde{d}_k)_{ij} \tilde{\partial}_{ij}) U_{K_k(x_k)} = 0. \] (4.21)

Then, we have
\[ M_k^2 |I_{k,2}| \leq M_k^{2+\tau_k} \int_{|y|\leq \sigma M_k^{-1/2}} \left| ((\tilde{b}_k)_i \partial_i + (\tilde{d}_k)_{ij} \partial_{ij}) v_k \cdot |\nabla y v_k \cdot y + v_k| \right| \]
\[ = M_k^{2+\tau_k} \int_{|y|\leq \sigma M_k^{-1/2}} \left| ((\tilde{b}_k)_i \partial_i + (\tilde{d}_k)_{ij} \partial_{ij}) (v_k - U_{K_k(x_k)}) \right| \cdot |\nabla y v_k \cdot y + v_k|. \] (4.22)
For $\sigma \leq 1$, we have $|y| \leq \sigma M_k^{\frac{p_k-1}{2}} \leq M_k^{\frac{p_k-1}{2}}$, hence $M_k^{-1} \leq M_k^{-\frac{p_k-1}{2}} \leq C(1 + |y|)^{-1}$ for large $k$. By (4.6) and Proposition 3.13, for large $k$, we have

$$\left| (\tilde{b}_k)_i (y) \tilde{g}_l + (d_k)_{ij} (y) \tilde{g}_{ij} (u_k (y) - U_{K_k (x_k)} (y)) \right| \leq C M_k^{-2} M_k^{-(p_k-1)} |y|^2 (1 + |y|)^{-1} (M_k^{\frac{p_k-1}{2}} + (1 + |y|)^{-1})$$

$$\leq C M_k^{-1-p_k} |y|^2 (1 + |y|)^{-2} \leq C M_k^{-1-p_k}, \quad (4.23)$$

and

$$|y \cdot \nabla v_k (y) + u_k (y)| \leq |\nabla y U_{K_k (x_k)}| \cdot |y| + |\nabla y (v_k - U)| \cdot |y| + |v_k - U| + U \leq C \left( (|y|^2 (1 + |y|^2)^{-2} + M_k^{-2} (1 + |y|)^{-1}) |y| + M_k^{-2} (1 + |y|^2)^{-1} \right) \leq C \left( (1 + |y|^2)^{-1} + M_k^{-2} \right) \leq C (1 + |y|^2)^{-1}. \quad (4.24)$$

Thus, for large $k$,

$$M_k^2 |I_{k,2}| \leq C M_k^{2+p_k} M_k^{-1-p_k} \int_{\{|y| \leq \sigma M_k^{\frac{p_k-1}{2}} \}} (1 + |y|^2)^{-1} \leq C M_k^{1-p_k+p_k} \int_0^{\sigma M_k^{\frac{p_k-1}{2}}} r^3 (1 + r^2)^{-1} dr \leq C M_k^{1-p_k+p_k} ((\sigma M_k^{\frac{p_k-1}{2}})^2 + C_1) = C M_k^{p_k - \sigma^2 + C_1 M_k^{1-p_k+p_k}} \leq C \sigma^2 \quad \text{as } k \to \infty. \quad (4.25)$$

Lemma 4.3. There exist constants $C > 0$ and $0 < \delta < 1$ such that when $\sigma < \delta$,

$$\limsup_{k \to \infty} M_k^2 |I_{k,3}| \leq C \sigma^2. \quad (4.26)$$

Proof. Due to (4.1), there exist $C > 0$ and $0 < \delta < 1$ such that when $\sigma < \delta$, on $|x| = \sigma$, by power series expansion, $|R_k (x)| \leq C |x|^2 = C \sigma^2$. Hence $|\tilde{R}_k (y)| \leq C \sigma^2$ on $|y| = \sigma M_k^{\frac{p_k-1}{2}}$. On the other hand, on $|y| = \sigma M_k^{\frac{p_k-1}{2}} \leq M_k^{\frac{p_k-1}{2}}$, $M_k^{-1} \leq M_k^{-\frac{p_k-1}{2}} \leq |y|^{-1}$. By Proposition 3.13,

$$v_k^2 (y) \leq |v_k (y) - U_{K_k (x_k)} (y)| + U_{K_k (x_k)} (y) |v_k^2 (y) \leq C [M_k^{-2} + (1 + |y|^2)^{-1}]^2 \leq C |y|^{-4}. \quad (4.27)$$

Then, we have the estimate

$$M_k^2 |I_{k,3}| \leq \frac{\sigma}{12} M_k^{11-3p_k} \int_{\{|y| = \sigma M_k^{\frac{p_k-1}{2}} \}} |\tilde{R}_k v_k^2 | \leq C \sigma^3 M_k^{11-3p_k} \int_{\{|y| = \sigma M_k^{\frac{p_k-1}{2}} \}} |y|^{-4} \leq C \sigma^3 M_k^{11-3p_k} (\sigma M_k^{\frac{p_k-1}{2}})^{-1} = C \sigma^2 M_k^{2 \tau_k} \leq C \sigma^2 \quad \text{as } k \to \infty. \quad (4.28)$$

□
Lemma 4.4. There exist constants $C > 0$ and $0 < \delta < 1$ such that when $\sigma < \delta$, 
\[ \limsup_{k \to \infty} M_k^2 |I_{k,4}| \leq C \sigma^2. \]  
(4.29)

Proof. By (4.1), there exist $C > 0$ and $0 < \delta < 1$ such that when $\sigma < \delta$, in the ball $|x| \leq \sigma$, by power series expansion, we have 
\[ |R_k(x)| \leq C|x|^2, \quad |\nabla_x R_k(x)| \leq |\nabla_x R_k(0)| + C|x| \leq C|x|. \]  
(4.30)

The second inequality implies 
\[ |x \cdot \nabla_x R_k(x)| \leq |x| \cdot |\nabla_x R_k(x)| \leq C|x|^2. \]  
(4.31)

It follows that in the ball $|y| \leq \sigma M_k^{-\frac{p_k-1}{2}}$, $|\tilde{R}_k(y)| \leq C M_k^{-(p_k-1)}|y|^2$, and 
\[ |y \cdot \nabla_y \tilde{R}_k(y)| = |(M_k^{-\frac{p_k-1}{2}} y) \cdot \nabla_x R_k(M_k^{-\frac{p_k-1}{2}} y)| \leq C M_k^{-(p_k-1)}|y|^2. \]  
(4.32)

On the other hand, similar to (4.27), for $|y| \leq \sigma M_k^{-\frac{p_k-1}{2}}$ and large $k$, $v_k^2(y) \leq C(1 + |y|^2)^{-\delta}$. Then, for large $k$, we have the estimate 
\[
M_k^2 |I_{k,4}| \leq C M_k^{6-2p_k} \int_{|y| \leq \sigma M_k^{-\frac{p_k-1}{2}} / \Gamma} (|y \cdot \nabla_y \tilde{R}_k| + 2|\tilde{R}_k|) v_k^2 
\leq C M_k^{6-2p_k} M_k^{-(p_k-1)} \int_{|y| \leq \sigma M_k^{-\frac{p_k-1}{2}} / \Gamma} |y|^2 (1 + |y|^2)^{-\delta} 
\leq C M_k^{7-3p_k} \int_0^{\sigma M_k^{-\frac{p_k-1}{2}}} r^5 (1 + r^2)^{-\delta} dr 
\leq C M_k^{7-3p_k} (\sigma M_k^{-\frac{p_k-1}{2}})^2 + C_1 = C \sigma^2 M_k^{2\tau_k} + C C_1 M_k^{7-3p_k} \leq C \sigma^2 \text{ as } k \to \infty.
\]  
(4.33)

Next, we estimate the most important term $M_k^2 I_{k,5}$.

Lemma 4.5. There exists a constant $0 < \delta < 1$ such that when $\sigma < \delta$, 
\[ \lim_{k \to \infty} M_k^2 I_{k,5} = \frac{2 \Delta_x K_{\infty}(\bar{x}) \cdot Vol(S^2)}{3|\Gamma|K_{\infty}(\bar{x})^2}. \]  
(4.34)

Proof. First, note that $\lim_{k \to \infty} f_k^{-\tau_k}(x) = 1$, and 
\[ \lim_{k \to \infty} |\nabla_x f_k^{-\tau_k}(x)| = \lim_{k \to \infty} -\tau_k f_k^{-\tau_k-1}(x)|\nabla_x f_k(x)| = 0 \]  
(4.35)

uniformly for $|x| \leq \sigma$. It follows 
\[
\lim_{k \to \infty} M_k^2 I_{k,5} = \frac{1}{p_k + 1} M_k^2 \int_{|x| \leq \sigma / \Gamma}(x \cdot \nabla_x (K_k f_k^{-\tau_k})) u_k^{p_k+1} 
= \frac{1}{p_k + 1} M_k^2 \int_{|x| \leq \sigma / \Gamma}(x \cdot \nabla_x K_k) u_k^{p_k+1}
\]  

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Thus we have

\[ \delta < 0 \] such that when \( \sigma < \delta \), by power series expansion and Proposition 3.10, we have

\[
K_k(x) = K_k(0) + (K_k)_i(0)x^i + \frac{1}{2}(K_k)_{ij}(0)x^ix^j + O(|x|^3)
\]

for \( |x| \leq \sigma \), where \((K_k)_{ij}(0)\) denotes the second order partial derivatives of \( K_k \) in coordinates \( \{x^i\} \) at the point \( x_k \). It is not hard to verify that

\[
x \cdot \nabla_x[K_k(0) + O(M_k^{-2+2\delta} |x|) + O(|x|^3)] = O(M_k^{-2+2\delta} |x|) + O(|x|^3),
\]

\[
x \cdot \nabla_x \left( \frac{1}{2} (K_k)_{ij}(0)x^ix^j \right) = (K_k)_{ij}(0)x^ix^j.
\]

Thus

\[
x \cdot \nabla_x K_k(x) = (K_k)_{ij}(0)x^ix^j + O(M_k^{-2+2\delta} |x|) + O(|x|^3),
\]

which implies

\[
y \cdot \nabla_y \tilde{K}_k(y) = M_k^{-(p_k-1)}(K_k)_{ij}(0)y^iy^j + O(M_k^{-2-pk^{-1}+2\delta} |y|) + O(M_k^{-3-p_k^{-1}} |y|^3).
\]

Hence

\[
|y \cdot \nabla_y \tilde{K}_k(y)| \leq CM_k^{-(p_k-1)}(1 + |y|)^2
\]

and

\[
|y \cdot \nabla_y \tilde{K}_k(y) - M_k^{-(p_k-1)}(K_k)_{ij}(0)y^iy^j| \leq CM_k^{-3-p_k^{-1}/2}(1 + |y|)^3,
\]

for \( |y| \leq \sigma M_k^{-2} \) and large \( k \).

On the other hand, by power series expansion and Proposition 3.13,

\[
|v_k^{p_k+1}(y) - U_{K_k(x_k)}^{p_k+1}(y)| \leq C \cdot (p_k + 1)U_{K_k(x_k)}^{p_k}(y)|v_k(y) - U_{K_k(x_k)}(y)|
\]

\[
\leq C(1 + |y|^2)^{-p_k}M_k^{-2}.
\]

Together with estimate (4.41), for large \( k \), we have

\[
M_k^{2+\tau_k} \int_{\{|y| \leq \sigma M_k^{-2} \}/\Gamma} \langle y \cdot \nabla_y \tilde{K}_k \rangle |v_k^{p_k+1} - U_{K_k(x_k)}^{p_k+1}| \leq CM_k^{2+\tau_k}M_k^{-(p_k-1)}M_k^{-2} \int_{\{|y| \leq \sigma M_k^{-2} \}/\Gamma} (1 + |y|)^2(1 + |y|^2)^{-p_k}
\]

\[
\leq CM_k^{-(p_k-1)+\tau_k} \int_0^{\sigma M_k^{-2}} r^3(1 + r)^2(1 + r^2)^{-3+o(1)} dr
\]
\begin{align*}
&\leq CM_k^{-(p_k-1)+\tau_k}(\sigma M_k^{\frac{p_k-1}{2}} + C_1) \leq CM_k^{\frac{p_k-1}{2}+\tau_k} \to 0 \text{ as } k \to \infty. \quad (4.44)
\end{align*}

Therefore,
\begin{equation}
\lim_{k \to \infty} M_k^2 I_{k,5} = \lim_{k \to \infty} \frac{1}{p_k + 1} M_k^{2+\tau_k} \int_{|y| \leq \sigma M_k^{\frac{p_k-1}{2}} / \Gamma} (y \cdot \nabla \tilde{K}_k) U^{p_k + 1}_{K_k(x_k)}. \quad (4.45)
\end{equation}

Using estimate (4.42), for large \( k \), we have
\begin{align*}
M_k^{2+\tau_k} \int_{|y| \leq \sigma M_k^{\frac{p_k-1}{2}} / \Gamma} |y \cdot \nabla \tilde{K}_k(y) - M_k^{-(p_k-1)}(K_k)_{,ij}(0) y^i y^j| \cdot U^{p_k + 1}_{K_k(x_k)} &
\leq CM_k^{2+\tau_k} M_k^{3(p_k-1)} \int_{|y| \leq \sigma M_k^{\frac{p_k-1}{2}} / \Gamma} (1 + |y|)^3 (1 + |y|^2)^{-4+\sigma(1)} \\
&\leq CM_k^{2-3(p_k-1)+\tau_k} \int_0^{\sigma M_k^{\frac{p_k-1}{2}}} r^3 (1 + r)^3 (1 + r^2)^{-4+\sigma(1)} dr \\
&\leq CM_k^{2-3(p_k-1)+\tau_k} ((\sigma M_k^{\frac{p_k-1}{2}})^{-1+\sigma(1)} + C_1) \to 0 \text{ as } k \to 0. \quad (4.46)
\end{align*}

Therefore,
\begin{equation}
\lim_{k \to \infty} M_k^2 I_{k,5} = \lim_{k \to \infty} \frac{1}{p_k + 1} M_k^{2+\tau_k} \int_{|y| \leq \sigma M_k^{\frac{p_k-1}{2}} / \Gamma} (K_k)_{,ij}(0) y^i y^j U^{p_k + 1}_{K_k(x_k)}. \quad (4.47)
\end{equation}

Since \( U_{K_k(x_k)} \) is a radial function,
\begin{equation}
\int_{|y| \leq \sigma M_k^{\frac{p_k-1}{2}} / \Gamma} y^i y^j U_{K_k(x_k)} = \begin{cases} 0, & i \neq j, \\ \frac{1}{4} \int_{|y| \leq \sigma M_k^{\frac{p_k-1}{2}} / \Gamma} |y|^2 U_{K_k(x_k)}, & i = j. \end{cases} \quad (4.48)
\end{equation}

Thus we obtain
\begin{equation}
\lim_{k \to \infty} M_k^2 I_{k,5} = \lim_{k \to \infty} \frac{1}{p_k + 1} M_k^{2+\tau_k} \int_{|y| \leq \sigma M_k^{\frac{p_k-1}{2}} / \Gamma} \frac{1}{4} \Delta_x K_k(0) \cdot |y|^2 U^{p_k + 1}_{K_k(x_k)}, \quad (4.49)
\end{equation}

where \( \Delta_x K_k(0) = \Delta_x K_k(x_k) \) is the Laplacian of \( K_k \) in coordinates \( \{x^i\} \) at the point \( x_k \). Because \( \Delta_x K_k(x_k) \to \Delta_x K_{\infty}(\bar{x}) \) and \( U^{p_k + 1}_{K_k(x_k)} \to U^4_{K_{\infty}(\bar{x})} \) uniformly on \( \mathbb{R}^d / \Gamma \), we have
\begin{align*}
\lim_{k \to \infty} M_k^2 I_{k,5} &= \frac{1}{16} \int_{y \in \mathbb{R}^d / \Gamma} \Delta_x K_{\infty}(\bar{x}) \cdot |y|^2 \left( \frac{8}{K_{\infty}(\bar{x})} \right)^2 (1 + |y|^2)^{-4} \\
&= \frac{4\Delta_x K_{\infty}(\bar{x}) \cdot Vol(5^3)}{|\Gamma| K_{\infty}(\bar{x})^2} \int_0^\infty r^5 (1 + r^2)^{-4} dr \\
&= \frac{2\Delta_x K_{\infty}(\bar{x}) \cdot Vol(5^3)}{3|\Gamma| K_{\infty}(\bar{x})^2}. \quad (4.50)
\end{align*}

Finally, we have achieved the following proposition.
Proposition 4.6. Assuming Condition 3.7, we have the following inequality
\[
\lim_{\sigma \to 0} P(\sigma, h_{\bar{x}}) \geq \frac{2 \Delta_g K_\infty(\bar{x}) \cdot Vol(S^3)}{3|\Gamma|K_\infty(\bar{x})^2}. \tag{4.51}
\]
Moreover, if \(\tau_k = 0\) for all \(k\), we have the equality
\[
\lim_{\sigma \to 0} P(\sigma, h_{\bar{x}}) = \frac{2 \Delta_g K_\infty(\bar{x}) \cdot Vol(S^3)}{3|\Gamma|K_\infty(\bar{x})^2}. \tag{4.52}
\]
Here, \(\Delta_g\) is the Laplacian with respect to the limit conformal normal metric \(g\).

Proof. By Lemmas 4.1–4.5 and Eq. (4.16),
\[
\lim_{\sigma \to 0} \limsup_{k \to \infty} M_{k,7}^2 = \frac{2 \Delta_g K_\infty(\bar{x}) \cdot Vol(S^3)}{3|\Gamma|K_\infty(\bar{x})^2} + \lim_{\sigma \to 0} \limsup_{k \to \infty} M_{k,6}^2. \tag{4.53}
\]
By Corollary 3.12,
\[
\limsup_{k \to \infty} M_{k,7}^2 = \limsup_{k \to \infty} \int_{|x| = \sigma} \left\{ \left( \left| \frac{\partial (M_k u_k)}{\partial v_x} \right|^2 - \frac{1}{2} |\nabla_x (M_k u_k)|^2 \right) \sigma + (M_k u_k) \frac{\partial (M_k u_k)}{\partial v_x} \right\}
\]
\[
= P(\sigma, h_{\bar{x}}). \tag{4.54}
\]
On the other hand, recall that
\[
I_{k,6} = \frac{\tau_k}{p_k + 1} \int_{|x| < \sigma} K_k f_k^{-\tau_k} u_k^{p_k + 1}. \tag{4.55}
\]
It’s clear that \(I_{k,6} \geq 0\), and \(I_{k,6} = 0\) if \(\tau_k = 0\). Our proposition is proved. \(\Box\)

4.3. Green’s Function. Assuming Condition 3.7, Corollary 3.12 tells us that \(M_k u_k \to h_{\bar{x}} = aG(\cdot, \bar{x})\) in \(C^2_{loc}((B_r(\bar{x}) - \{\bar{x}\}) / \Gamma)\). We are going to determine the constant \(a\) and the regular part of \(G(\cdot, \bar{x})\) evaluated at \(\bar{x}\), where \(G(\cdot, \bar{x})\) is the standard Green’s function for \(L_g = \Delta_g - \frac{1}{6} R_g\) at the point \(\bar{x}\), and \(g\) is the limit metric, i.e. for any \(C^2\) function \(\phi\) and small \(\sigma > 0\),
\[
\int_{|x| \leq \sigma} G(L_g \phi) dV_g = \phi(\bar{x}). \tag{4.56}
\]
Hence Corollary 3.12 implies that
\[
\lim_{k \to \infty} \int_{|x| \leq \sigma} (M_k u_k)(L_g \phi) dV_{g_k} = a(\bar{x}). \tag{4.57}
\]

Lemma 4.7. The constant \(a\) in Corollary 3.12 is
\[
a = -\frac{4\sqrt{2}Vol(S^3)}{|\Gamma|\sqrt{K_\infty(\bar{x})}}. \tag{4.58}
\]
Proof. Recall Remark 3.3 and the change of variables in (3.8), (3.9). For any $C^2$ function $\phi$, define $\tilde{\phi}(y) = \phi(M_k^{\frac{p_k-1}{2}} y)$. Integrating by parts, we obtain

\[
\int_{\{ |x| \leq \sigma \}/\Gamma} (M_k u_k)(L_{g_k} \phi) = \int_{\{ |y| \leq \sigma M_k^{\frac{p_k-1}{2}} \}/\Gamma} \left( \Delta_x - \frac{1}{6} R_k \right) (M_k u_k) \cdot \phi = M_k^{-2(p_k-1)} \int_{\{ |y| \leq \sigma M_k^{\frac{p_k-1}{2}} \}/\Gamma} \left( M_k^{p_k-1} \Delta_y - \frac{1}{6} \tilde{R} \right) (M_k^2 v_k) \cdot \tilde{\phi} = (M_k^{\frac{p_k-1}{2}} \int_{\{ |y| \leq \sigma M_k^{\frac{p_k-1}{2}} \}/\Gamma} (\Delta_y v_k) \cdot \tilde{\phi}) - \frac{1}{6} M_k^{4-2p_k} \int_{\{ |y| \leq M_k^{\frac{p_k-1}{2}} \sigma \}/\Gamma} \tilde{R}_k v_k \tilde{\phi}.
\]

(4.59)

Since $\lim_{k \to \infty} M_k^{4-2p_k} = 0$, the second integral vanishes when $k \to \infty$. Thus

\[
\alpha(\tilde{x}) = \lim_{\sigma \to 0} \lim_{k \to \infty} \int_{\{ |x| \leq \sigma \}/\Gamma} (M_k u_k)(L_{g_k} \phi) = \lim_{\sigma \to 0} \lim_{k \to \infty} \int_{\{ |y| \leq M_k^{\frac{p_k-1}{2}} \sigma \}/\Gamma} (\Delta_y v_k) \cdot \phi(M_k^{\frac{p_k-1}{2}} y) = \left( \int_{\mathbb{R}^n} \Delta_y U_{K_{\infty}}(\tilde{x}) (y) dy \right) \cdot \phi(\tilde{x}),
\]

(4.60)

where the last equality is by Proposition 3.4. Therefore the constant $\alpha$ is

\[
\alpha = \int_{\mathbb{R}^n} \Delta_y U_{K_{\infty}}(\tilde{x}) (y) dy = -\int_{\mathbb{R}^n} K_{\infty}(\tilde{x}) U_{K_{\infty}}^3(\tilde{x}) (y) dy = -\frac{16\sqrt{2}}{\sqrt{K_{\infty}(\tilde{x})}} \int_{\mathbb{R}^n} (1 + |y|^2)^{-3} dy = -\frac{16\sqrt{2} Vol(S^3)}{|\Gamma| \sqrt{K_{\infty}(\tilde{x})}} \int_0^\infty r^3 (1 + r^2)^{-3} dr = -\frac{4\sqrt{2} Vol(S^3)}{|\Gamma| \sqrt{K_{\infty}(\tilde{x})}}.
\]

(4.61)

On the other hand, recall that we assume for each $k$, $g_k$ is the conformal normal metric with conformal normal coordinates centered at point $x_k$, hence the limit metric $g$ is the conformal normal metric with conformal normal coordinates centered at point $\tilde{x}$. Thus we have the following proposition.

**Proposition 4.8.** In $g$-conformal normal coordinates $\{x^i\}$ centered at $\tilde{x}$, $G(\cdot, \tilde{x})$ has the expansion

\[
G(\cdot, \tilde{x}) = b\psi_{\tilde{x}} = b[r^{-2} + A_{\tilde{x}} + O(r)],
\]

(4.62)

where $r = |x|$, $\psi_{\tilde{x}}$ is from Definition 1.8, $A_{\tilde{x}}$ is a constant, and

\[
b = -\frac{|\Gamma|}{2Vol(S^3)}.
\]

(4.63)
Proof. The proof is given by [LP87, Definition 6.2 and Lemma 6.4]. Note that our notation is different from [LP87]. Our point $\bar{x}$ is their point $P$; our operator $L_{\bar{g}}$ is equal to $-1/6$ multiplied with their box operator $\Box$; our $G(\cdot, \bar{x})$ is equal to $-6$ multiplied with their $\Gamma_P$, our $\psi_{\bar{x}}$ is their $G$. And the $\Gamma$ in our Eq. (4.63) is the quotient group near point $\bar{x}$. \[\blacksquare\]

Then we can relate the Pohozaev identity with the constant term $A_{\bar{x}}$.

**Lemma 4.9.** We have

$$\lim_{\sigma \to 0} P(\sigma, h_{\bar{x}}) = -\frac{16 \text{Vol}(S^3)}{|\Gamma| K_\infty(\bar{x})} \cdot A_{\bar{x}}. \tag{4.64}$$

**Proof.** We will write $G$ instead of $G(\cdot, \bar{x})$. Using $h_{\bar{x}} = aG$, we have

$$P(\sigma, h_{\bar{x}}) = a^2 \int_{|x| = \sigma} \left( \left( \frac{\partial G}{\partial \nu} \right)^2 - \frac{1}{2} |\nabla G|^2 \right) \sigma + G \frac{\partial G}{\partial \nu}. \tag{4.65}$$

By Proposition 4.8, on $|x| = \sigma$ for small $\sigma$,

$$G = b[\sigma^{-2} + A_{\bar{x}} + O(\sigma)], \quad \left| \frac{\partial G}{\partial \nu} \right| = b[-2\sigma^{-3} + O(1)], \quad |\nabla G| = b[-2\sigma^{-3} + O(1)]. \tag{4.66}$$

Then we have

$$\left( \left( \frac{\partial G}{\partial \nu} \right)^2 - \frac{1}{2} |\nabla G|^2 \right) \sigma + G \frac{\partial G}{\partial \nu} = b^2 \left( \frac{\sigma}{2} \cdot [-2\sigma^{-3} + O(1)]^2 + [\sigma^{-2} + A_{\bar{x}} + O(\sigma)] \cdot [-2\sigma^{-3} + O(1)] \right)$$

$$= b^2 \left( \frac{\sigma}{2} \cdot [4\sigma^{-6} + O(\sigma^{-3})] - 2\sigma^{-5} - 2A_{\bar{x}}\sigma^{-3} + O(\sigma^{-2}) \right)$$

$$= -2b^2 A_{\bar{x}} \sigma^{-3} + O(\sigma^{-2}). \tag{4.67}$$

Hence

$$P(\sigma, h_{\bar{x}}) = a^2 \int_{|x| = \sigma} \left( -2b^2 A_{\bar{x}} \sigma^{-3} + O(\sigma^{-2}) \right) = -\frac{2a^2 b^2 A_{\bar{x}} \cdot \text{Vol}(S^3)}{|\Gamma|} + O(\sigma). \tag{4.68}$$

Letting $\sigma \to 0$, using Lemma 4.7 and Proposition 4.8, we have

$$\lim_{\sigma \to 0} P(\sigma, h_{\bar{x}}) = -\frac{2a^2 b^2 A_{\bar{x}} \cdot \text{Vol}(S^3)}{|\Gamma|}$$

$$= (-2) \cdot \frac{\text{Vol}(S^3)}{|\Gamma|} \cdot \frac{32 \text{Vol}(S^3)^2}{|\Gamma|^2 K_\infty(\bar{x})} \cdot \frac{|\Gamma|^2}{4 \text{Vol}(S^3)^2} \cdot A_{\bar{x}} = -\frac{16 \text{Vol}(S^3)}{|\Gamma| K_\infty(\bar{x})} \cdot A_{\bar{x}}. \tag{4.69}$$

$\blacksquare$
Proposition 4.10. Assuming Condition 3.7, we have the following inequality

\[ A_{\tilde{x}} \leq -\frac{\Delta_{\tilde{g}} K_{\infty}(\tilde{x})}{24 K_{\infty}(\tilde{x})}. \]  \hspace{1cm} (4.70)

Moreover, if \( \tau_k = 0 \) for all \( k \), we have the equality

\[ A_{\tilde{x}} = -\frac{\Delta_{\tilde{g}} K_{\infty}(\tilde{x})}{24 K_{\infty}(\tilde{x})}. \]  \hspace{1cm} (4.71)

Here, \( \Delta_{\tilde{g}} \) is the Laplacian with respect to the limit conformal normal metric \( \tilde{g} \).

Proof. Assuming Condition 3.7, by Proposition 4.6, we have

\[ \lim_{\sigma \to 0} P(\sigma, h_{\tilde{x}}) \geq \frac{2 \Delta_{\tilde{g}} K_{\infty}(\tilde{x}) \cdot Vol(S^3)}{3|\Gamma| K_{\infty}(\tilde{x})^2}. \]  \hspace{1cm} (4.72)

By Lemma 4.10, we have

\[ \lim_{\sigma \to 0} P(\sigma, h_{\tilde{x}}) = -\frac{16 Vol(S^3)}{|\Gamma| K_{\infty}(\tilde{x})} \cdot A_{\tilde{x}}. \]  \hspace{1cm} (4.73)

Hence

\[ A_{\tilde{x}} \leq -\frac{2 \Delta_{\tilde{g}} K_{\infty}(\tilde{x}) \cdot Vol(S^3)}{3|\Gamma| K_{\infty}(\tilde{x})^2} \cdot \frac{|\Gamma| K_{\infty}(\tilde{x})}{16 Vol(S^3)} = -\frac{\Delta_{\tilde{g}} K_{\infty}(\tilde{x})}{24 K_{\infty}(\tilde{x})}. \]  \hspace{1cm} (4.74)

The case that \( \tau_k = 0 \) for all \( k \) follows similarly. \( \square \)

Moreover, by relating \( A_{\tilde{x}} \) with the mass, we can remove the assumption “conformal normal metric”, as follows.

Proposition 4.11. Assuming Condition 3.7, let \( g = \lim_{k \to \infty} g_k \) be the limit metric, but do not necessarily assume that \( g_k, g \) are conformal normal metrics. Let \( \tilde{g}_{\tilde{x}} = \psi^2_{\tilde{x}} g \) be the conformal blow-up of \( g \) at the point \( \tilde{x} \), as in Definition 1.8. We have the following inequality

\[ m(\tilde{g}_{\tilde{x}}) \leq -\frac{\Delta_{\tilde{g}} K_{\infty}(\tilde{x})}{2 K_{\infty}(\tilde{x})}. \]  \hspace{1cm} (4.75)

Moreover, if \( \tau_k = 0 \) for all \( k \), we have the equality

\[ m(\tilde{g}_{\tilde{x}}) = -\frac{\Delta_{\tilde{g}} K_{\infty}(\tilde{x})}{2 K_{\infty}(\tilde{x})}. \]  \hspace{1cm} (4.76)

Proof. For each \( k \), let \( r_k \) denote the \( g_k \)-distance function from the point \( x_k \). Assume \( \tilde{g}_k = \phi_k^2 g_k \) is the conformal normal metric with conformal normal coordinates \( \{\tilde{x}^i\} \) centered at point \( x_k \). By [LP87, Theorem 5.6], after applying a dilation and a translation to the coordinates \( \{\tilde{x}^i\} \), we may assume \( \phi_k(x_k) = 1 \) and \( \nabla \phi_k(x_k) = 0 \), in other words, for small \( r_k \), \( \phi_k \sim 1 + O(r_k^2) \). When \( k \to 0 \), we have that \( \tilde{g} \sim \phi^2 g \), where \( g, g_k, \phi \) are the limit of \( g_k, \tilde{g}_k, \phi_k \) as \( k \to \infty \). Moreover, \( \phi \sim 1 + O(r^2) \), where \( r \) is the \( g \)-distance function from the point \( \tilde{x} \). Next, recall that the conformal transformation law of the Laplacian for \( \tilde{g} = e^{2\phi} g \) is

\[ \Delta_{\tilde{g}} = e^{-2\phi} \Delta_g + (n - 2)e^{-2\phi} g^{ij} \frac{\partial \phi}{\partial x_j} \frac{\partial}{\partial x_i}. \]  \hspace{1cm} (4.77)
Thus, for any $f \in C^2$, we have $\Delta\tilde{g}f(\bar{x}) = \Delta_f f(\bar{x})$. Let $A_{\bar{x}}$ be the regular part corresponding to conformal blow-up of $\tilde{g}$ at point $\bar{x}$. Let $\hat{g}_{\bar{x}}$ be the conformal blow-up of $g$ at point $\bar{x}$. By [LP87, Lemma 9.7], we have

$$m(\hat{g}_{\bar{x}}) = 12A_{\bar{x}}. \quad (4.78)$$

Therefore, we know

$$m(\hat{g}_{\bar{x}}) \leq -\frac{\Delta \tilde{g} K_\infty(\bar{x})}{2K_\infty(\bar{x})} \iff A_{\bar{x}} \leq -\frac{\Delta \tilde{g} K_\infty(\bar{x})}{24K_\infty(\bar{x})},$$

$$m(\hat{g}_{\bar{x}}) = -\frac{\Delta \tilde{g} K_\infty(\bar{x})}{2K_\infty(\bar{x})} \iff A_{\bar{x}} = -\frac{\Delta \tilde{g} K_\infty(\bar{x})}{24K_\infty(\bar{x})}, \quad (4.79)$$

which implies that this proposition is equivalent to Proposition 4.10. \qed}

5. Blow-Up Points Must be Isolated and Simple

In [LZ99], they proved that on a 3-dimensional compact manifold, all blow-up points for (3.5) must be isolated simple blow-up points. The same result in higher dimensions is proved by [KMS09], [LZ05], but only for constant prescribed scalar curvature. Here, we will modify their proofs to show that the same result holds on 4-dimensional compact orbifolds, for a sequence of variable prescribed scalar curvatures.

Let $(M, g)$ be a compact Riemannian 4-dimensional orbifold with singularities

$$\Sigma = \{(q_1, \Gamma_1), \ldots, (q_l, \Gamma_l)\} \quad (5.1)$$

and positive scalar curvature $R_g$. Assume $u$ is a positive $C^2$ solution of Eq. (3.1) on $M$, where $K$ is a positive $C^2$ function and $f$ is a positive $C^1$ function. For any point $\bar{x} \in M$, define $\Omega_{\bar{x}, \sigma}$ in the following:

a) if $\bar{x}$ is a smooth point, define $\Omega_{\bar{x}, \sigma} = B_\sigma(\bar{x})$ for some $\sigma > 0$ such that its closure $\overline{\Omega_{\bar{x}, \sigma}}$ doesn’t include any singular point. In other words, $d_g(\bar{x}, \{q_1, \ldots, q_l\}) > \sigma$;

b) if $\bar{x} = q_j$ for some $1 \leq j \leq l$, then the neighborhood of $\bar{x}$ is locally a ball $B_\sigma(\bar{x})$ in the quotient space. Define $\Omega_{\bar{x}, \sigma} = \pi^{-1}(B_\sigma(\bar{x}))$ to be the lifting-up to a smooth ball.

Denote the lifting-up functions and metric still by $u$, $f$, $K$ and $g$.

First, let us recall a lemma from [LZ99].

**Lemma 5.1** ([LZ99] Lemma 5.1). Let $(M, g, u, K, f, \Omega_{\bar{x}, \sigma})$ be as defined above. Given any small $\varepsilon > 0$ and large $R' > 1$, there exists a large positive $C_0$, depending only on $(M, g, \|f\|_{C^1(M)}, \inf_M K, \|K\|_{C^2(M)}, \varepsilon$ and $R'$ such that for any compact $S \subset \Omega_{\bar{x}, \sigma}$, if $u$ satisfies

$$\max_{x \in \overline{\Omega_{\bar{x}, \sigma}} \setminus S} d_g(x, S)^{\frac{2}{p+1}} u(x) \geq C_0, \quad (5.2)$$

then we have $p > 3 - \varepsilon$ and for some local maximum point of $u$ in $\Omega_{\bar{x}, \sigma} \setminus S$, denoted as $x_0$,

$$\|u(x_0)^{-1} u\left(\exp_{x_0}\left(u(x_0)^{-\frac{p-1}{2}} x\right)\right) - U_{K(x_0)}(x)\|_{C^2(|x| \leq 2R')} < \varepsilon, \quad (5.3)$$

where $d_g(x, S)$ denotes the distance of $y$ to $S$, and $d_g(x, S) = 1$ if $S = \emptyset$. 

Proof. The case that $K$ is a positive constant and $\Omega_{\tilde{x}, \sigma}$ is replaced by a compact 3-dimensional manifold $M$ is proved in [LZ99, Lemma 5.1]. As mentioned in [LZ99, Section 7], it can be generalized to the case $K$ is a variable function. With a straightforward modification, it can also be generalized to higher dimensions. Moreover, note that the statement and proof in [LZ99, Lemma 5.1] is just a local argument in $M \setminus S$. Thus their proof remains valid in our case by choosing their $S$ to be any compact subset of $M$ satisfying $M \setminus S \subset B_{\sigma}(\tilde{x})/\Gamma$, and considering the lifting-up of $M \setminus S$ if $\tilde{x}$ is a singular point. \hfill \Box

Using Lemma 5.1, we can prove the following.

**Proposition 5.2** ([LZ99] Proposition 5.1). Let $(M, g, u, K, f, \Omega_{\tilde{x}, \sigma}$ be as defined in the beginning of this section. Given small $\varepsilon > 0$ and large $R'$, there exist some positive constants $C_0$ and $C_1$ depending on $M, g, \|f\|_{C^1(M)}, \inf_M K, \|K\|_{C^2(M)}$, $\varepsilon$ and $R'$ such that if

$$\max_{\Omega_{\tilde{x}, \sigma}} u > C_0,$$  \hfill (5.4)

then there exists some integer $N = N(\varepsilon)$ and $N$ local maximum points of $u$ denoted as $\{x_1, \cdots, x_N\} \subset \Omega_{\tilde{x}, \sigma}$, such that:

1) $3 - \varepsilon < p \leq 3$,

2) $B_{r_i}(x_i) \cap B_{r_j}(x_j) = \emptyset$ for $i \neq j$, where $r_j = R'u(x_j)^{-\frac{p-1}{2}}$,

and for each $j$,

$$\|u(x_j)^{-1}u\left(\exp_{x_j}\left((u(x_j)^{-\frac{p-1}{2}}x)\right) - U_{K(x_j)}(x)\right)\|_{C^2(\{x \mid r \leq 2R\})} < \varepsilon,$$  \hfill (5.5)

3) $d_{g}(x_i, x_j)^{\frac{2}{p-1}}u(x_j) \geq C_0$ for $j > i$, while $d_{g}(x_i, \{x_1, \cdots, x_N\})^{\frac{2}{p-1}}u(x) \leq C_1$ for all $x \in \Omega_{\tilde{x}, \sigma}$.

Proof. The case that $K$ is a positive constant and $\Omega_{\tilde{x}, \sigma}$ is replaced by a compact 3-dimensional manifold $M$ is proved in [LZ99, Proposition 5.1]. Briefly, that proof was completed by an induction process as following: first apply [LZ99, Lemma 5.1] with $\Omega_{\tilde{x}, \sigma}$ as mentioned above. Everything follows the same way and our proposition is proved. \hfill \Box

Next, we can rule out bubble accumulation.

**Proposition 5.3.** Let $(M, g, u, K, f, \Omega_{\tilde{x}, \sigma}$ be as defined in the beginning of this section. Let $\varepsilon, R', C_0, C_1$ and $\{x_1, \cdots, x_N\}$ be as defined in Proposition 5.2. If $\varepsilon$ is sufficiently small and $R'$ is sufficiently large, then there exists a positive constant $\tilde{C}$, which only depends on $M, g, \|f\|_{C^1(M)}, \inf_M K, \|K\|_{C^2(M)}$, $\varepsilon$ and $R'$, such that if $\max_{\Omega_{\tilde{x}, \sigma}} u \geq C_0$, then $d_{g}(x_j, x_l) \geq \tilde{C}$, for all $j \neq l$. 


Proof. The proof is similar to [KMS09, Proposition 8.3] and [LZ99, Proposition 5.2]. We will prove it by contradiction. Suppose that such a constant $C$ does not exist, then there exist sequences $p_k \to p \in (3 - \epsilon, 3]$ and $\{u_k\}$ with $\max_{\Omega_{k,\sigma}^c} u_k \geq C_0$ and

$$
\lim_{k \to \infty} \min_{j \neq l} d_g(x_j(u_k), x_l(u_k)) = 0.
$$

(5.6)

Without loss of generality, assume that

$$
\delta_k = d_g(x_1(u_k), x_2(u_k)) = \min_{j \neq l} d_g(x_j(u_k), x_l(u_k)) \to 0 \text{ as } k \to \infty.
$$

(5.7)

For each $k$, take normal coordinates $\{x^i\}$ centered at point $x_1(u_k)$ and consider change of variables $y = \delta_k^{-1} x$. Rescale $u_k$ by

$$
v_k(y) = \delta_k^{2p_k} u_k(\delta_k y), \quad \forall |y| < \delta_k^{-1}.
$$

(5.8)

Then $v_k$ satisfies

$$
L_{h_k} v_k + \tilde{K}_k \tilde{f}_k^{-\tau_k} v_k^{p_k} = 0,
$$

(5.9)

where $(h_k)_{ij}(y) = g_{ij}(\delta_k y), \tilde{K}_k(y) = K(\delta_k y)$ and $\tilde{f}_k(y) = f(\delta_k y)$. If $x_j(u_k) \in B_{\sqrt{\delta_k}}(x_1),$ denote by $y_j(u_k) = \delta_k^{-1} x_j(u_k)$ the $y$-coordinate of point $x_j(u_k)$. By the proof of [KMS09, Proposition 8.3], we have $y_1(u_k) = 0, y_2(u_k) \to \tilde{y}_2$ with $|\tilde{y}_2| = 1, \{0, \tilde{y}_2\}$ are isolated simple blow-up points for $\{v_k\}$, and

$$
v_k(0) v_k(y) \to a_1(|y|^{-2} + b_1 + O(y)) \text{ in } C^2_{\text{loc}}(\mathbb{R}^4 - S'),
$$

(5.10)

where $S'$ denotes the set of blow-up points for $\{v_k\}$ and $a_1, b_1 > 0$ are some positive constants. On the other hand,

$$
\tilde{K}_k(y) = K(\delta_k y) \to K(0) \text{ in } C^0_{\text{loc}} \text{ norm},
$$

$$
|\nabla y \tilde{K}_k(y)| = \delta_k |\nabla x K(\delta_k y)| \to 0 \text{ in } C^0 \text{ norm},
$$

$$
|\nabla^2 y \tilde{K}_k(y)| = \delta_k^2 |\nabla^2 x K(\delta_k y)| \to 0 \text{ in } C^0 \text{ norm}.
$$

(5.11)

Hence $\tilde{K}_k$ converges to the constant $K(0)$ in the $C^2_{\text{loc}}$ norm, where $K(0)$ by definition is the $K$ value at the limit point of $x_1(u_k)$ as $k \to \infty$, possibly by passing to a subsequence. Applying Proposition 4.10 to the blow-up sequence $\{v_k\}$, we get $b_1 \leq 0$, which contradicts $b_1 > 0$. Therefore our proposition is proved. \qed

Corollary 5.4. Let $(M, g)$ be a compact Riemannian 4-dimensional orbifold with positive scalar curvature. Suppose $\{f_k\}$ is a sequence of positive $C^1$ functions converging in the $C^1_{\text{loc}}$ topology to a positive function $f$. Also suppose $\{K_k\}$ is a sequence of positive $C^2$ functions converging in the $C^2_{\text{loc}}$ topology to a positive function $K_\infty$. Let $\{u_k\}$ be a sequence of positive solutions of Eq. (3.5) on $M$ with $g_k = g$ and $\max_M u_k \to \infty$. Then $p_k \to 3$ and the set of blow-up points is finite and consists only of isolated blow-up points. Moreover, if blow-up occurs at a singular point, i.e. $x_k \to \tilde{x}$ and $u_k(x_k) \to \infty$ where $\tilde{x}$ is a singular point, then there exists an integer $N \in \mathbb{N}^*$ such that for any $k > N$, $x_k = \tilde{x}$. 

Proof. By the assumption of \( f_k \) and \( K_k \), there exists a constant \( C_2 \) such that for large \( k \),
\[
\| f_k \|_{C^1(M)} \leq C_2 \| f \|_{C^1(M)}, \quad \inf_M K_k \geq C_2 \inf_M K, \quad \| K_k \|_{C^2(M)} \leq C_2 \| K \|_{C^2(M)}.
\]  
(5.12)

By Proposition 5.3, in each \( \Omega_\kappa, \sigma \), blow-up points must be isolated blow-up points and the number \( \mathcal{N}(u_k) \) as defined in Proposition 5.2 must have an uniformly upper bound, otherwise, there cannot exist a constant \( \tilde{C} \) such that \( d_k(x_i(u_k), x_j(u_k)) \geq \tilde{C} \) for all \( i \neq j \) and \( k \). A compact orbifold can be covered by finitely many sufficiently small open balls, which gives us finitely many \( \Omega_\kappa, \sigma \) as defined in the beginning of this section. Therefore the set of blow-up points on \( M \) is finite and consists only of isolated blow-up points.

Moreover, in the case that blow-up occurs at a singular point, i.e. \( x_k \to \tilde{x} \) and \( u_k(x_k) \to \infty \) where \( \tilde{x} \) is a singular point associated with a nontrivial quotient group \( \Gamma \), suppose there exists a subsequence, still denoted by \( (u_k, x_k) \), such that \( x_k \neq \tilde{x} \) for any \( k \). Consider the lifting-up space \( \Gamma_\kappa, \sigma \) as defined in the beginning of this section. Let \( \tilde{x}_k^{(1)}, \ldots, \tilde{x}_k^{(|\Gamma|)} \) denote the lifting-up points of \( x_k \). In the lifting-up space, it’s clear that
\[
d_k(\tilde{x}_k^{(1)}, \tilde{x}_k^{(2)}) \to 0 \quad \text{as} \quad k \to \infty,
\]
which contradicts against Proposition 5.3. Thus our corollary is proved. \( \square \)

Given Corollary 5.4, we are able to conclude the following.

**Proposition 5.5.** Let \( (M, g) \), \( f_k, K_k, g_k \) be as defined in Corollary 5.4. Assume \( u_k \) is a sequence of positive functions satisfying Eq. (3.5) and \( x_k \to \tilde{x} \) is an isolated blow-up point. Then \( \tilde{x} \) is an isolated simple blow-up point for \( \{u_k\} \).

Proof. The proof is similar to [KMS09, Lemma 8.2] and [LZ99, Proposition 4.1]. First, by Corollary 5.4, without loss of generality, we may assume \( x_k = \tilde{x} \) for all \( k \) if \( \tilde{x} \) is a singular point. Suppose \( \tilde{x} \) is an isolated blow-up point, but not an isolated simple blow-up point. Let \( x = \{x^i\} \) be normal coordinates centered at \( x_k \) and define the rescaled function
\[
v_k(y) = \frac{2}{\tau_k^{p_k-1}} u_k(\tau_k y), \quad \forall |y| < \tau_k^{-1}.
\]
(5.14)

Then \( v_k \) satisfies
\[
L_{h_k} v_k + \tilde{K}_k f_k^{\tau_k - p_k} v_k^{p_k} = 0,
\]
(5.15)
where \( (h_k)_{ij}(y) = (g_k)_{ij}(\tau_k y) \), \( \tilde{K}_k(y) = K_k(\tau_k y) \) and \( f_k(y) = f_k(\tau_k y) \). By the proof of [KMS09, Lemma 8.2], the origin \( y = 0 \) is an isolated simple blow-up point for \( \{v_k\} \) and
\[
v_k(0)v_k(y) \to h(y) = a_2(|y|^{-2} + b_2) \quad \text{in} \quad C^2_{loc}((\mathbb{R}^4 - \{0\})/\Gamma),
\]
where \( a_2 = b_2 = 1 \), where \( \Gamma = \{e\} \) if \( \tilde{x} \) is a smooth point, but \( \Gamma \) is the quotient group if \( \tilde{x} \) is a singular point.

On the other hand, because \( K_k \) converges to \( K_\infty \) in the \( C^2_{loc} \) norm, we know
\[
\tilde{K}_k(y) = K_k(\tau_k y) \to K_\infty(\tilde{x}) \quad \text{in} \quad C^0_{loc} \text{ norm}, \quad |\nabla_y \tilde{K}_k(y)| = \tau_k |\nabla \tilde{K}_k(\tau_k y)| \to 0 \quad \text{in} \quad C^0_{loc} \text{ norm},
\]
\[ |\nabla^2 \tilde{K}_k(y)| = \tau \frac{2}{k} |\nabla^2 K_k(\tau y)| \to 0 \text{ in } C^0_{loc} \text{ norm.} \quad (5.17) \]

Hence \( \tilde{K}_k \) converges to the constant \( K_\infty(\bar{x}) \) in the \( C^2_{loc} \) norm. Applying Proposition 4.10 to the blow-up sequence \( \{u_k\} \), we obtain \( b_2 \leq 0 \), which contradicts \( b_2 = 1 \). Therefore \( \bar{x} \) is an isolated simple blow-up points for \( \{u_k\} \). \( \square \)

**Corollary 5.6.** Let \((M, g), f_k, K_k, g_k\) be as defined in Corollary 5.4. Assume \( u_k \) is a sequence of positive functions satisfying Eq. (3.5), then Condition 3.7 is a necessary condition for any blow-up point.

**Proof.** This follows immediately from combining Corollary 5.4 and Proposition 5.5. \( \square \)

**Proof of Theorem 1.11.** To prove the upper bound \( u \leq C \) under assumption (1.8) in Theorem 1.11, suppose the contrary. Then there exist \( p_k \to 3 \) and \( \{u_k\} \) satisfying

\[ L_g u_k + K_k u_k^{p_k} = 0, \quad (5.18) \]

with \( \max_M u_k \to \infty \), where \( \{K_k\} \) is a sequence of positive \( C^2 \) functions converging in the \( C^2_{loc} \) topology to a positive function \( K_\infty \). Let \( x_k \) denote the point where \( u_k \) obtains a maximum, after possibly passing to a subsequence, we may assume \( x_k \to \bar{x} \) is a blow-up point. By Corollary 5.6, the sequence \((u_k, x_k)\) satisfies Condition 3.7, where \( g_k \) in Eq. (3.5) is the metric conformal to \( g \) with conformal normal coordinates centered at \( x_k \). By Proposition 4.11, we know

\[ m(\hat{g}, \bar{x}) \leq -\frac{\Delta_g K_\infty(\bar{x})}{2 K_\infty(\bar{x})}, \quad (5.19) \]

which contradicts against assumption (1.8) in Theorem 1.11. Therefore we know that \( u \leq C \) for \( u \) and \( C \) as stated in Theorem 1.11. Next, assume \( u \) obtains \( \sup_M u \) at a point \( P \), then \( \Delta u(P) \leq 0 \). By (1.1), \( u(P) \geq \sqrt{R_g(P)/(6K(P))} \geq c_0 \). By the Harnack inequality,

\[ \inf_M u \geq \frac{1}{1/c_1} u(P) \geq c_0/c_1 \geq 1/C, \quad (5.20) \]

for sufficiently large \( C \). By standard elliptic estimates, we conclude that \( \|u\|_{C^{2,\alpha}(M)} \leq C \). Furthermore, because for \( p < 3 \), there always exist subcritical solutions for any \( K \) (the proof on manifolds remains valid in the orbifold setting; see for example [ES86,LP87]). Take a sequence of subcritical solutions and let \( p \to 3 \), due to (1.7), they limit to a critical solution for \( p = 3 \). Thus the second part of Theorem 1.11 is proved.

The first part of Theorem 1.11 can be proved similarly, by fixing \( p_k = 3 \) in the above proof. \( \square \)

### 6. Variational Methods

Let \((M, g)\) be a compact Riemannian \( n \)-orbifold with singularities \( \Sigma = \{(q_1, \Gamma_1), \ldots, (q_l, \Gamma_l)\} \) and positive scalar curvature \( R_g \). Let \( K \) be a positive smooth function on \( M \). In this section, we will study Eq. (1.1) using a variational method. Consider the energy functional

\[ J_p(u, K, M) = \frac{\int_M (|\nabla u|^2 + c(n) R_g u^2 dV_{g}) \text{Vol}_g}{(\int_M K |u|^{p+1} dV_{g})^{\frac{2}{p+1}}}, \quad (6.1) \]
for $1 < p \leq \frac{n+2}{n-2}$ and $u \in W^{1,2}(M)$. Define the minimal energy to be
\[ E(p, K) = \inf_{u \in W^{1,2}(M)} J_p(u, K, M). \] (6.2)

Let $Q(S^n)$ denote the Sobolev quotient of $S^n$, which is also the minimal energy on $S^n$ for $K \equiv 1$, that is,
\[ Q(S^n) = \inf_{\phi \in W^{1,2}(S^n)} J_{\frac{n+2}{n-2}}(\phi, 1, S^n) = \frac{n(n-2)}{4} \text{Vol}(S^n)^{2/n}. \] (6.3)

The following theorem generalizes [ES86, Proposition 1.1] to the orbifold case, and also generalizes [Aku12, Theorem 3.1] from the case $K = constant$ to the case of variable $K$.

**Theorem 6.1.** Let $(M, g)$ be a compact Riemannian $n$-orbifold with positive scalar curvature and singularities $\Sigma_\Gamma = \{(q_1, \Gamma_1), \cdots, (q_l, \Gamma_l)\}$. Let $K$ be a positive smooth function on $M$. Define the modified maximum value of $K$
\[ B_K := \max \left\{ \sup_{x \in M} \{K(x)\}, \max_{1 \leq i \leq l} \{|\Gamma_i|^{\frac{2}{n-2}} K(q_i)\} \right\}. \] (6.4)

Then the following inequality always holds
\[ (B_K)^\frac{n-2}{2n} E\left(\frac{n+2}{n-2}, K\right) \leq Q(S^n). \] (6.5)

Furthermore, if
\[ (B_K)^\frac{n-2}{2n} E\left(\frac{n+2}{n-2}, K\right) < Q(S^n), \] (6.6)

then there exists a positive smooth solution $u$ of (1.1) with $p = \frac{n+2}{n-2}$, such that
\[ J_{\frac{n+2}{n-2}}(u, K, M) = E\left(\frac{n+2}{n-2}, K\right). \] (6.7)

**Proof.** We first prove the inequality (6.5). The quantity $B_K$ is attained at either a singular point or a smooth point. Consider any point $q$ with $|\Gamma|^{\frac{2}{n-2}} K(q) = B_K$, where $\Gamma$ is the orbifold group at $q$ if $q$ is a singular point, or $\Gamma = \{e\}$ if $q$ is a smooth point. Take a conformal mapping $g_q = u_q^{\frac{4}{n-2}} g$ such that $g_q$ is the conformal normal metric centered at the point $q$. Consider the Green’s function with the power series expansion
\[ G_q = \frac{1}{4n(n-1)\text{Vol}(S^{n-1})} (r_q^{2-n} + H_q), \] (6.8)
where $r_q$ is the geodesic distance from $q$ based on the metric $g_q$ and $H_q$ is the higher order term. Take a family of test functions
\[ \varphi_{q, \lambda} = u_q \cdot \left( \frac{\lambda}{1 + \lambda^2 (r_q^{2-n} + H_q)^{\frac{2}{n-2}}} \right)^{\frac{n-2}{2}}, \] (6.9)
where $\lambda > 0$. By [MM21, Proposition 5.1], we have the estimation

$$J_{n+2 \over n-2} (\varphi_{q, \lambda}, K, M) = \frac{c(n) \hat{c}_0}{|\Gamma|^{2 \over n} K(q)^{n-2 \over n}} + O(1/\lambda^2), \quad (6.10)$$

where

$$\hat{c}_0 = 4n(n-1) \left( \int_{\mathbb{R}^n} \frac{dx}{(1+r^2)^n} \right)^{2 \over n}. \quad (6.11)$$

Note that our functional $J_{n+2 \over n-2}$ is equal to $c(n)$ multiplied with the functional $J$ defined in [MM21, Chapter 1]. If $q$ is an orbifold point, we estimate integrals in $J_{n+2 \over n-2}$ by lifting everything up to the universal cover near the orbifold point $q$. Therefore, compared to [MM21, Proposition 5.1], our estimation (6.10) has an extra factor of $c(n)/|\Gamma|^{2 \over n}$. By direct computation, we know that

$$c(n) \hat{c}_0 = Q(S^n). \quad (6.12)$$

Clearly, we have

$$E \left( \frac{n+2}{n-2}, K \right) \leq \lim_{\lambda \to \infty} J_{n+2 \over n-2} (\varphi_{q, \lambda}, K, M) = \frac{Q(S^n)}{|\Gamma|^{2 \over n} K(q)^{n-2 \over n}}. \quad (6.13)$$

By our assumption, we have $|\Gamma|^{2 \over n} K(q)^{n-2 \over n} = B^{n-2 \over n}_K$, hence (6.5) is proved.

We next show that the strict inequality (6.6) implies the existence result. The proof is a modification of [Aku12, Theorem 3.1]. If we have proved the case that $M$ has only one singularity, the more general cases can be proved by an induction on the number of singularity points. Hence we may assume $M$ has only one singularity $(q, \Gamma)$. Let $X = M - \{q\}$. Note that

$$E \left( \frac{n+2}{n-2}, K \right) = \inf_{u \in C_\infty^\infty(X)} J_{n+2 \over n-2} (u, K, X). \quad (6.14)$$

Let $B_\rho$ denote the open geodesic ball centered at $q$ of radius $\rho$. Define

$$Y_k \equiv \inf_{u \in C_\infty^\infty(X \setminus B_{1/k})} J_{n+2 \over n-2} (u, K, X \setminus B_{1/k}), \quad (6.15)$$

for $k \in \mathbb{N}^*$. It follows that

$$Y_k \geq Y_{k+1} \geq Y_{k+2} \geq \cdots, \quad \text{and} \quad \lim_{k \to \infty} Y_k = E \left( \frac{n+2}{n-2}, K \right). \quad (6.16)$$

By (6.6), there exists a large integer $k_0$, such that

$$(B_K)^{n-2 \over n} Y_k < Q(S^n) \quad \text{for any } k \geq k_0. \quad (6.17)$$

Due to (6.4), it implies

$$\left( \sup_{x \in X \setminus B_{1/k}} \left[ K(x) \right] \right)^{n-2 \over n} Y_k < Q(S^n) \quad \text{for any } k \geq k_0. \quad (6.18)$$
Note that on the manifold with boundary \((N, \partial N) = (X - B_{1/k}, \partial B_{1/k})\), when we apply integration by parts to any function in \(C^\infty_c(X \setminus \overline{B_{1/k}})\), the boundary integral term always vanishes. As a result, the variational method used in [ES86, Proposition 1.1] remains valid here. Thus, (6.18) implies that for each \(k \geq k_0\), there exists a non-negative \(J_{\frac{2n}{n-2}}(\cdot, K, X \setminus \overline{B_{1/k}})\)-minimizer \(u_k \in C^\infty(X - B_{1/k})\), such that

\[
J_{\frac{2n}{n-2}}(u_k, K, X \setminus \overline{B_{1/k}}) = Y_k, \quad \int_{X \setminus \overline{B_{1/k}}} K u_k^{\frac{2n}{n-2}} dVol_g = 1, \quad (6.19)
\]

\[u_k = 0 \text{ on } \partial B_{1/k} \text{ and } u_k > 0 \text{ in } X \setminus \overline{B_{1/k}}. \quad (6.20)\]

Denote the zero extension of each \(u_k\) to \(M\) by also the same symbol \(u_k\). Suppose the sequence \(\{u_k\}\) has a uniform \(C^0\)-bound, i.e., there exists a constant \(C > 0\) such that \(\|u_k\|_{C^0(M)} \leq C\) for \(k \geq k_0\), then there exists a non-negative \(J_{\frac{2n}{n-2}}(\cdot, K, M)\)-minimizer \(u \in W^{1,2}(M)\) with \(\|u\|_{C^0(M)} \leq C\), such that

\[u_k \rightarrow u \text{ weakly in } W^{1,2}(M), \quad u_k \rightarrow u \text{ strongly in } L^2(M). \quad (6.21)\]

By (6.19), Lebesgue’s bounded convergence theorem, and the above uniform \(C^0\)-bound, we have

\[\int_M Ku_k^{\frac{2n}{n-2}} dVol_g = 1, \quad (6.22)\]

which further implies

\[u_k \rightarrow u \text{ strongly in } W^{1,2}(M). \quad (6.23)\]

By the standard elliptic estimate, we obtain that \(u \in C^\infty(M)\). The maximum principle implies that \(u > 0\) everywhere on \(M\); see for example [Aub98b, Proposition 3.75].

To complete the proof, it is sufficient to show a uniform \(C^0\)-bound for the sequence \(\{u_k\}\). For each \(k\), take the absolute maximum point \(x_k \in X\) of \(u_k\) and denote \(M_k \equiv u_k(x_k)\). Taking a subsequence if necessary, there exists a point \(\bar{x} \in M\) such that

\[\lim_{k \rightarrow \infty} x_k = \bar{x}. \quad (6.24)\]

Suppose that there is not a uniform \(C^0\)-bound for \(\{u_k\}\), that is \(\lim_{k \rightarrow \infty} M_k = \infty\). There will be two cases.

Case 1: \(\bar{x} = q\) (blow-up occurs at the singular point). In this case, we consider the universal cover of a small neighborhood around \(q\). Let \(\hat{x}_k\) be a sequence of lifting-up points of \(\{x_k\}\) in the same branch of the lifting-up space. Let \(\hat{u}_k, \hat{K}\) denote the lifting-up functions of \(u_k, K\), respectively. In the lifting-up space, for each \(k\), let \(\{\hat{x}_k^i\}\) be a normal coordinate system in a small ball \(B_\sigma(\hat{x}_k)\) centered at each \(\hat{x}_k\). Consider the change of variables \(\hat{y} = M_k^{\frac{2}{n-2}} \hat{x}\) and define the rescaled function \(\hat{v}_k(\hat{y}) = M_k^{-1} \cdot \hat{u}_k(M_k^{\frac{2}{n-2}} \hat{y})\). By Theorem 2.1 of Chapter 5 in [SY94],

\[\hat{v}_k \rightarrow \hat{v} \text{ in } C^2_{loc}(\mathbb{R}^n), \quad (6.25)\]

where \(\hat{v}\) satisfies the equation

\[-\Delta_0 \hat{v} = E\left(\frac{n+2}{n-2}, K(q)\right) \hat{v}^{\frac{n+2}{n-2}} \text{ on } \mathbb{R}^n. \quad (6.26)\]
Note that
\[ \int_{\mathbb{R}^n} K(q) \tilde{\nu}(y)^{2n \over n-2} dy = \lim_{r \to \infty} \int_{|\tilde{y}| \leq r} K(q) \tilde{\nu}(\tilde{y})^{2n \over n-2} d\tilde{y}. \] (6.27)

For each \( r > 0 \), by (6.25) and changing variables, we have
\[ \int_{|\tilde{y}| \leq r} K(q) \tilde{\nu}(\tilde{y})^{2n \over n-2} d\tilde{y} = \lim_{k \to \infty} \int_{|\tilde{x}| \leq r} \tilde{K}(M_k^{-2 \over n-2} \tilde{y}) \tilde{\nu}(\tilde{y})^{2n \over n-2} d\tilde{y} \]
\[ = \lim_{k \to \infty} \int_{|\tilde{x}| \leq M_k^{-2 \over n-2} r} \tilde{K}(\tilde{x}) \tilde{\nu}_k(\tilde{x})^{2n \over n-2} d\tilde{x} \]
\[ \leq |\Gamma| \cdot \lim_{k \to \infty} \int_M K u_k^{2n \over n-2} dVol_g = |\Gamma|, \] (6.28)

which implies
\[ \int_{\mathbb{R}^n} \tilde{\nu}^{2n \over n-2} \leq |\Gamma| K(q)^{-1}. \] (6.29)

Multiplying \( \tilde{\nu} \) with (6.26) and integrating by parts, we obtain
\[ \int_{\mathbb{R}^n} |\nabla \tilde{\nu}|^2 = E \left( \frac{n+2}{n-2}, K(q) \right) \int_{\mathbb{R}^n} \tilde{\nu}^{2n \over n-2}. \] (6.30)

Together with (6.29), we have
\[ Q(S^n) \leq J_{n+2 \over n-2} (\tilde{\nu}, 1, \mathbb{R}^n) = \frac{\int_{\mathbb{R}^n} |\nabla \tilde{\nu}|^2}{(\int_{\mathbb{R}^n} \tilde{\nu}^{2n \over n-2})^{n-2 \over n}} \]
\[ = E \left( \frac{n+2}{n-2}, K(q) \right) \left( \int_{\mathbb{R}^n} \tilde{\nu}^{2n \over n-2} \right)^{n-2 \over n} \leq E \left( \frac{n+2}{n-2}, K(q) \right) \left|\Gamma\right|^{2 \over n-2} \left( K(q) \right)^{n-2 \over n}. \] (6.31)

Note that \( q \) is a singular point on \( M \), so \( \left|\Gamma\right|^{2 \over n-2} K(q) \leq B_K \), which leads to a contradiction against (6.6).

Case 2: \( \bar{x} \neq q \) (blow-up occurs at a smooth point). The argument is identical to Case 1 with \( |\Gamma| = 1 \). Hence
\[ E \left( \frac{n+2}{n-2}, K(q) \right) \left( K(\bar{x}) \right)^{n-2 \over n} \geq Q(S^n). \] (6.32)

Since \( \bar{x} \) is a smooth point on \( M \), \( K(\bar{x}) \leq B_K \), which again leads to a contradiction against (6.6).

**Proof of Theorem 1.14.** The proof will be based on Theorem 6.1 and [MM21, Proposition 5.1]. Under the assumptions of Theorem 1.14, we simply denote the singular point \((q_{i_0}, \Gamma_{i_0})\) by \((q, \Gamma)\). Let \( \varphi_{q,\lambda} \) be as defined in (6.9). By [MM21, Proposition 5.1] and (6.12), we have the estimation
\[ J_{n+2 \over n-2} (\varphi_{q,\lambda}, K, M) = \frac{Q(S^n)}{|\Gamma|^2 K(q)^{n-2 \over n}} \left( 1 - \hat{c}_2 \frac{\Delta K(q)}{K(q) \lambda^2} - \hat{d}_1 \left( \begin{array}{c} H_q + O(\log \lambda) \\ \frac{\lambda}{2} \\ \frac{\lambda}{2} \\ \lambda \end{array} \right) \right) \]
\[ \text{for } n = 4 \]
\[ \text{for } n = 5 \]
\[ \text{for } n = 6 \]
\[ \text{for } n \geq 7 \] (6.33)
up to error $O(1/\lambda^4)$ for large $\lambda$, where
\[
\hat{c}_2 = \frac{\int_{\mathbb{R}^n} r^2 dx}{2n \int_{\mathbb{R}^n} (1+r^2)^n} \quad \text{and} \quad \hat{d}_1 = \frac{2n \int_{\mathbb{R}^n} r^n dx}{(n-2) \int_{\mathbb{R}^n} (1+r^2)^n}.
\tag{6.34}
\]
Moreover, in dimension 4, $\hat{c}_2 = 1/4$ and $\hat{d}_1 = 6$. Then, it is clear that under the following assumptions
\[
\begin{cases}
H_q + \frac{\Delta_g K(q)}{24 K(q)} > 0, & \text{for } n = 4, \\
\Delta_g K(q) > 0, & \text{for } n \geq 5,
\end{cases}
\tag{6.35}
\]
by choosing sufficiently large $\lambda$, we can get
\[
J_{\frac{n+2}{n-2}} (\varphi_q, \lambda, K, M) < \frac{Q(S^n)}{|\Gamma|^\frac{2}{n} K(q)^{\frac{n-2}{n}}}.
\tag{6.36}
\]
Note that we have $B_K = |\Gamma|^\frac{2}{n} K(q)$ under the assumption of Theorem 1.14, hence the above inequality implies
\[
(B_K)^{\frac{n-2}{n}} E\left(\frac{n+2}{n-2}, K \right) < Q(S^n).
\tag{6.37}
\]
By Theorem 6.1, there exists a positive smooth solution of Eq. (1.1). Moreover, by (4.78), we know that condition (1.10) is equivalent to condition (6.35) in dimension 4.

\section{LeBrun Metrics}
Recall the LeBrun metric $(O_{\mathbb{P}^1}(-n), \hat{g}_{LEB(n)})$ in $r$-coordinates and its conformal compactification $(\hat{O}_{\mathbb{P}^1}(-n), \hat{g}_{LEB(n)})$ from Sect. 1.4. On $\hat{O}_{\mathbb{P}^1}(-n)$, define $s = 1/r$ to be the inverted radial coordinate centered at the orbifold point $\hat{q}$. Then
\[
\hat{g}_{LEB(n)} = \frac{1 + s^2}{(1 + ns^2)^2} ds^2 + \frac{s^2 (s^2 + 1)}{(ns^2 + 1)^2} \left[ \theta_1^2 + \theta_2^2 + \theta_3^2 \right]
\tag{7.1}
\]
\[
= ds^2 + s^2 (\theta_1^2 + \theta_2^2 + \theta_3^2) + O(s^2) \quad \text{as } s \to 0.
\tag{7.2}
\]
Hence we may choose normal coordinates $\{x^i\}$ centered at $\hat{q}$ such that $s = |x| + O(|x|^3)$. Also note that $\hat{g}_{LEB(n)} = (n + r^2)^2 \cdot \hat{g}_{LEB(n)} = (s^{-2} + n)^2 \cdot \hat{g}_{LEB(n)}$ is scalar-flat, hence $\hat{g}_{LEB(n)}$ is the conformal blow-up of $\hat{g}_{LEB(n)}$ at the point $\hat{q}$, as in Definition 1.8. We will next study
\[
\Delta_{\hat{g}_{LEB(n)}} u - \frac{1}{6} R_{\hat{g}_{LEB(n)}} u = -K u^p, \quad 1 < p \leq 3.
\tag{7.3}
\]
In the $U(2)$-invariant case, Theorem 1.11 specializes to the following.

\footnote{The constant factor $\frac{2n}{n-2}$ was missing when the authors computed $\hat{d}_1$ in [MM21, Proposition 5.1]. We’ve corrected the mistake here.}
Proposition 7.1. Let $\mathcal{X}_n$, $\mathcal{X}_{n,+}$, $\mathcal{X}_{n,0}$, and $\mathcal{X}_{n,-}$ be as defined in (1.20) and (1.22). On the orbifold $(\widehat{\mathcal{O}}_{\mathbb{P}^1}(-n), \tilde{g}_{\text{LEB}(n)})$, for any $K \in \mathcal{X}_{n,+} \cup \mathcal{X}_{n,-}$, there exists some constant $C$, depending only on $\inf K$ and $\|K\|_{C^2}$, such that

$$1/C \leq u \leq C \text{ and } \|u\|_{C^{2,a}} \leq C,$$

(7.4)

for all $U(2)$-invariant solutions $u \in \mathcal{X}_n$ of (7.3) with $p = 3$, where $0 < \alpha < 1$. Moreover, if $K \in \mathcal{X}_{n,+}$, then (7.4) holds for all $1 < 1 + \varepsilon < p \leq 3$, where $C$ in addition depends on $\varepsilon$. Consequently, in this case there exists a $U(2)$-invariant solution $u$ of (7.3) with $p = 3$.

Proof. Assume $\{u_k\} \subset \mathcal{X}_n$ is a family of $U(2)$-invariant solutions of (7.3) with corresponding exponents $p_k \to 3$ as $k \to \infty$. Assume $u_k$ blows up at a point $\tilde{x}$ when $k \to \infty$. Suppose $\tilde{x}$ is in the set $\{s = s_0\}$ for some $s_0 > 0$. Because $u_k$ is $U(2)$-invariant, $u_k$ blows up on the entire set $\{s = s_0\}$, which is either a hypersurface if $s_0 < \infty$ or the $\mathbb{C}P^1$ component if $s_0 = \infty$. However, either case is impossible, because Proposition 5.3 implies that blow-up points are isolated. Therefore, $u_k$ must blow up at the orbifold point $\tilde{q}$. It is clear that if $K \in \mathcal{X}_{n,+}$ implies (1.8), and $K \in \mathcal{X}_{n,-}$ implies (1.6). By the proof and statement of Theorem 1.11, this proposition is proved.

If $u(s)$ satisfies Eq. (1.21), then $u^2 g_{\text{LEB}(n)}$ has scalar curvature $6K(s)$. This latter condition is equivalent to $v^2 g_{\text{LEB}(n)}$ having scalar curvature $6K(s)$, where

$$v(s) \equiv \frac{u(s)}{n + 1/s^2},$$

(7.5)

which is equivalent to

$$\Delta_{g_{\text{LEB}(n)}} v = -K v^3.$$  

(7.6)

Define another function space

$$\mathcal{Y}_n = \left\{ h(s) : (0, \infty) \to \mathbb{R}_+ \mid h(s) = \frac{f(s)}{n + 1/s^2}, f(s) \in \mathcal{X}_n \right\}.$$  

(7.7)

There is an obvious bijection between the solution set of Eq. (1.21) in $\mathcal{X}_n$ and the solution set of Eq. (7.6) in $\mathcal{Y}_n$.

Proposition 7.2. On $(\widehat{\mathcal{O}}_{\mathbb{P}^1}(-2), g_{\text{LEB}(2)})$, assume $K(s) \in \mathcal{X}_2$ satisfies

$$K'(s) \leq 0 \text{ for any } s \geq 0,$$

(7.8)

then there is no solution in $\mathcal{Y}_2$ that solves (7.6). Consequently, there is no solution in $\mathcal{X}_2$ that solves (1.21).

Proof. Suppose $v(s) \in \mathcal{Y}_2$ solves Eq. (7.6). It implies that $\tilde{g} = v^2 g_{\text{LEB}(2)}$ is a compact orbifold metric with $R_{\tilde{g}} = 6K(s)$, and it has an orbifold point $\tilde{q}$ at $s = 0$. Denote this orbifold by $\tilde{M}$. Define $\hat{v}(r) = v(1/r)$ and $\hat{K}(r) = K(1/r)$. Then in the $r$-coordinate, we have

$$\frac{2 + 3r^2}{r(1 + r^2)} \hat{v}'(r) + \frac{2 + r^2}{1 + r^2} \hat{v}''(r) = -\hat{K}(r)\hat{v}(r)^3.$$  

(7.9)
Denote the left hand side of the above equation by LHS. We observe that
\[ \text{LHS} \leq 0 \text{ and } \text{LHS} = \frac{[(2r + r^3)\dot{v}’(r)]’}{r(1 + r^2)}. \]  
(7.10)

It follows that
\[ [(2r + r^3)\dot{v}’(r)]’ \leq 0, \]
(7.11)
which implies that
\[ (2r + r^3)\dot{v}’(r) \leq 0. \]
(7.12)
Thus \( \dot{v}’(r) \leq 0 \) for any \( r \geq 0 \), which implies that
\[ v’(s) \geq 0, \text{ for any } s \geq 0. \]
(7.13)

Similar to the proof of [Via10, Theorem 1.3], letting \( E \) denote the traceless Ricci tensor, by the conformal transformation formula for \( E \) and the fact that \( g_{LEB(2)} \) is Ricci-flat, we have
\[ E_\hat{g} = v^{-1}(-2\nabla^2 v + (\Delta v/2)\hat{g}), \]
(7.14)
where \( \nabla \) and \( \Delta \) are respect to metric \( \hat{g} \). Using the argument of [Oba72], integrating \( v|E_\hat{g}|^2 \) on \( \hat{M} \), we obtain
\[ \int_{\hat{M}} v|E_\hat{g}|^2 dVol_\hat{g} = \int_{\hat{M}} vE^{ij}_{\hat{g}} \left\{ v^{-1}(-2\nabla^2 v + (\Delta v/2)\hat{g})_{ij} \right\} dVol_\hat{g} \]
\[ = -2 \int_{\hat{M}} E^{ij}_{\hat{g}} (\nabla^2 v)_{ij} dVol_\hat{g} \]
\[ = -2 \lim_{\epsilon \to 0} \int_{\hat{M} \setminus B_\epsilon(\hat{q})} E^{ij}_{\hat{g}} (\nabla^2 v)_{ij} dVol_\hat{g}, \]
(7.15)
where \( B_\epsilon(\hat{q}) \) denotes the geodesic ball centered at \( \hat{q} \) with radius \( \epsilon \), and the indices \( i = 0, 1, 2, 3 \) denote the directions \( ds, \theta_1, \theta_2, \theta_3 \) in the moving coframe. Integrating by parts, we have
\[ \int_{\hat{M}} v|E_\hat{g}|^2 dVol_\hat{g} = -2 \lim_{\epsilon \to 0} \left( \int_{\partial B_\epsilon(\hat{q})} E^{ij}_{\hat{g}} (\nabla v)_i v_j d\sigma_\hat{g} - \int_{\hat{M} \setminus B_\epsilon(\hat{q})} \nabla_j E^{ij}_{\hat{g}} (\nabla v)_i dVol_\hat{g} \right). \]
(7.16)
By (7.5), it is clear that \( v(s) = O(s^2) \) near \( s = 0 \). Also because the curvature and volume term for \( \hat{g} \) are bounded near \( s = 0 \), the first integral term on the right hand side goes to zero as \( \epsilon \to 0 \). For the second integral term on the right hand side, since \( v(s) \) is a radial function, the only nonvanishing component of \( (\nabla v)_i \) is \( (\nabla v)_0 = v’(s) \) along the \( \partial/\partial s \) direction. On the other hand, by the Bianchi identity,
\[ \nabla_j E^{ij}_{\hat{g}} = \nabla_j \left( R_{\hat{g}}^{ij} - \frac{1}{4} R_\hat{g} \cdot \hat{g}^{ij} \right) = \frac{1}{2} \nabla^i R_\hat{g} - \frac{1}{4} \nabla^i R_\hat{g} = \frac{3}{2} \nabla^i K. \]
(7.17)
Since \( K(s) \) is a radial function, the only nonvanishing component of \((\nabla K)^i\) is \((\nabla K)^0 = K'(s)\) along the \( ds \) direction. Hence
\[
\nabla_j E_{ij}^\hat{g} = \begin{cases} \frac{3}{2} K'(s), & i = 0, \\ 0, & i = 1, 2, 3. \end{cases}
\] (7.18)

It follows
\[
\int_M v|E_{ij}^\hat{g}|^2 dVol_{\hat{g}} = \lim_{\varepsilon \to 0} \int_\varepsilon^\infty C(s) K'(s)v'(s)ds,
\] (7.19)
where \( C(s) \) is a positive function in \( s \), depending on the volume term of \( \hat{g} \) at each \( s \). By (7.8) and (7.13), we know
\[
\lim_{\varepsilon \to 0} \int_\varepsilon^\infty C(s) K'(s)v'(s)ds \leq 0,
\] (7.20)
\[
\int_M v|E_{ij}^\hat{g}|^2 dVol_{\hat{g}} \leq 0,
\] (7.21)
which implies \( E_{ij}^\hat{g} = 0 \). By (7.18), we know \( K'(s) \equiv 0 \), so that \( K = \text{constant} \). By [Via10, Theorem 1.3], such a solution \( v \) does not exist. \( \Box \)

For each \( n \), we define another radial coordinate by
\[
t(r) = \log \left( \frac{n + r^2}{r^2} \right) \quad \text{and its inverse} \quad r(t) = \sqrt{\frac{n}{e^t - 1}}.
\] (7.22)

Hence
\[
t(s) = \log(n s^2 + 1) \quad \text{and} \quad s(t) = \sqrt{\frac{e^t - 1}{n}}.
\] (7.23)

We have defined three radial coordinates \( r, s, t \in [0, \infty) \). It is not hard to see that \( 1/r, s \) and \( t \) are monotonic to each other. By basic computations, on the ALE manifold \( \hat{O}_{\mathbb{P}^1}(-n) \), we have
\[
\Delta_{\text{SLEB}(n)} = \frac{n + 3r^2}{r(1 + r^2)} \frac{\partial}{\partial r} + \frac{n + r^2}{1 + r^2} \frac{\partial^2}{\partial r^2} = \frac{4(1 - e^{-t})^3}{e^{-t}(1 + (n - 1)e^{-t})} \frac{\partial^2}{\partial t^2}.
\] (7.24)

Next, we prove the following non-existence result.

**Theorem 7.3.** For any \( n \in \mathbb{N}^* \), on \((\hat{O}_{\mathbb{P}^1}(-n), \hat{g}_{\text{SLEB}(n)})\), there is no solution in \( X_n \) for Eq. (1.21) with
\[
K = K_{n, -}(s) = \frac{2 + ns^2}{n + ns^2} K_{2, -}(s),
\] (7.25)
where \( K_{2, -}(s) \in X_n \) is any monotonically decreasing function in \( s \). Moreover, we know
\[
K = K_{n, -}(s) \in X_{n, -} \text{ if } K_{2, -}''(0) < 0.
\] (7.26)

In particular, for \( n \geq 2 \), it implies that there is no \( U(2) \)-invariant solution for the Yamabe Problem \( K = \text{constant} \).
Proof. For such a $K = K_{n,-}(s)$, suppose there is a $u_n(s) \in \mathcal{X}_n$ that solves Eq. (1.21), then $v_n(s) = u_n(s)/(n + s^{-2}) \in \mathcal{Y}_n$ solves Eq. (7.6). Define

$$\tilde{v}_n(t) = v_n\left(\sqrt{\frac{e^t - 1}{n}}\right), \quad \tilde{K}_{n,-}(t) = K_{n,-}\left(\sqrt{\frac{e^t - 1}{n}}\right).$$

(7.27)

By (7.23) and (7.24), we know (7.6) is equivalent to the following equation

$$\tilde{v}_n''(t) = -\tilde{K}_{n,-}(t) \cdot \frac{e^{-t}(1 + (n - 1)e^{-t})}{4(1 - e^{-t})^3} \tilde{v}_n(t)^3.$$

(7.28)

It implies

$$\tilde{v}_n''(t) = - \frac{(1 + (n - 1)e^{-t}) \tilde{K}_{n,-}(t)}{1 + e^{-t}} \cdot \frac{e^{-t}(1 + e^{-t})}{4(1 - e^{-t})^3} \tilde{v}_n(t)^3.$$

(7.29)

Define

$$\tilde{K}_{2,-}(t) = \frac{(1 + (n - 1)e^{-t}) \tilde{K}_{n,-}(t)}{1 + e^{-t}},$$

(7.30)

then (7.29) becomes

$$\tilde{v}_n''(t) = -\tilde{K}_{2,-}(t) \cdot \frac{e^{-t}(1 + e^{-t})}{4(1 - e^{-t})^3} \tilde{v}_n(t)^3,$$

(7.31)

which can be viewed as (7.6) on $(\mathcal{O}_{p1}(-2), g_{LEB(2)})$ in the $t$-coordinate, with $K = \tilde{K}_{2,-}(t)$ and $v = \tilde{v}_n(t)$. To transform $\tilde{v}_n(t)$ and $\tilde{K}_{2,-}(t)$ back to $s$-coordinate on $\mathcal{O}_{p1}(-2)$, using (7.23) with $n = 2$, we define

$$v_2(s) = \tilde{v}_n(\log(2s^2 + 1)) \quad \text{and} \quad K_2(s) = \tilde{K}_{2,-}(\log(2s^2 + 1)).$$

(7.32)

Thus we know that on $\mathcal{O}_{p1}(-2)$, $v_2(s) \in \mathcal{Y}_2$ solves (7.6) with $K = K_2(s) \in \mathcal{X}_2$. By combining (7.25), (7.27) and (7.30), it is not hard to see that

$$K_2(s) = \frac{n + 2s^2}{2 + 2s^2} \tilde{K}_{n,-}(\log(2s^2 + 1)) = \frac{n + 2s^2}{2 + 2s^2} K_{n,-}\left(\sqrt{\frac{2}{n}s}\right) = K_{2,-}\left(\sqrt{\frac{2}{n}s}\right).$$

(7.33)

Our assumption that $K_{2,-}(s)$ is a monotonically decreasing function in $s$ implies $K_2'(s) \leq 0$ for all $s$, which leads to a contradiction against Proposition 7.2.

For the last part, assume $K(s) \in \mathcal{X}_n$. Near $s = 0$, $K$ has a power series expansion

$$K(s) = K(0) + \frac{1}{2} K''(0)s^2 + O(s^4).$$

(7.34)

By (4.77) and (7.2),

$$\Delta_{g_{LEB(n)}} K(0) = \Delta_{g_{LEB(n)}} K(0) = K''(0) + \frac{3}{s} K'(s) \bigg|_{s=0} = 4K''(0).$$

(7.35)

So, $K_{n,-}(s) \in \mathcal{X}_{n,-}$ is equivalent to

$$\frac{K_{n,-}''(0)}{K_{n,-}'(0)} < n - 2.$$

(7.36)
If \( K_{2,-}''(0) < 0 \), then we have

\[
\frac{d^2}{ds^2} \left( \frac{2 + n s^2}{n + n s^2} K_{2,-}(s) \right) \bigg|_{s=0} = n - 2 + \frac{K_{2,-}''(0)}{K_{2,-}'(0)} < n - 2.
\]

(7.37)

It follows \( K_{n,-}(s) \in X_{n,-} \). □

**Proof of Theorem 1.19.** By Proposition 7.1, and homotopy invariance of the Leray-Schauder degree, \( \text{deg}(F_{p,K,n}, \Omega_{\Lambda,n}, 0) \) is equal to a constant in either case (1) or (2) in Theorem 1.19. For any \( n \in \mathbb{N}^* \), Theorem 7.3 implies that

\[
\text{deg}(F_{3,K,n}, \Omega_{\Lambda,n}, 0) = 0,
\]

(7.38)

for \( K \in X_{n,-} \) satisfying the assumptions of Theorem 7.3. Homotopy invariance implies that the degree is zero for any \( K \in X_{n,-} \), which proves (2) of Theorem 1.19. Part (1) of Theorem 1.19 is proved by a standard subcritical degree counting argument; see [Sch91]. □

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