THE SHARP BOUNDS OF THE SECOND AND THIRD HANKEL DETERMINANTS FOR THE CLASS $\mathcal{SL}^*$

SHAGUN BANGA AND S. SIVAPRASAD KUMAR

Abstract. The aim of the present paper is to obtain the sharp bounds of the Hankel determinants $H_2(3)$ and $H_3(1)$ for the well known class $\mathcal{SL}^*$ of starlike functions associated with the right lemniscate of Bernoulli. Further for $n = 3$, we find the sharp bound of the Zalcman functional for the class $\mathcal{SL}^*$. In addition, a couple of interesting results of $\mathcal{SL}^*$ is appended at the end.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, defined in the open unit disk $\Delta$. The subclass $\mathcal{S}$ of $\mathcal{A}$ consists of univalent functions. We say, $f$ is subordinate to $g$, denoted by $f \prec g$, if there exists a Schwartz function $\omega$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$, where $f$ and $g$ are analytic functions. For each $n \geq 2$, Zalcman conjectured the following coefficient inequality for the class $\mathcal{S}$:

$$|a_n^2 - a_{2n-1}| \leq (n - 1)^2.$$  

(1.1)

The above inequality also implies the Bieberbach conjecture $|a_n| \leq n$ (see [4]). Consider the class $\mathcal{SL}^*$ [24], given by

$$\mathcal{SL}^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \sqrt{1+z}, \quad z \in \Delta \right\}.$$

It is evident that if $\omega = zf'(z)/f(z)$, then the analytic characterization of the functions in $\mathcal{SL}^*$, is given by $|\omega^2 - 1| < 1$, which in fact is the interior of the right loop of the lemniscate of Bernoulli, with the boundary equation $\gamma_1 : (u^2 + v^2)^2 - 2(u^2 - v^2) = 0$. In 2009, Sokól [22] obtained the sharp bounds for $a_2$, $a_3$ and $a_4$ of functions in the class $\mathcal{SL}^*$, further it is conjectured that $|a_{n+1}| \leq 1/2n$ whenever $n \geq 1$, with the extremal function $f$ satisfying $zf'(z)/f(z) = \sqrt{1+z^2}$. Later, Shelly Verma [20] gave the proof for the sharp estimate of the fifth coefficient with the extremal function for $\mathcal{SL}^*$ using the characterization of positive real part functions in terms of certain positive semi-definite Hermitian form. Sokól [23] also dealt the radius problems for the class $\mathcal{SL}^*$. Recently, Ali et al. [2] have examined the radius of starlikeness associated with the lemniscate of Bernoulli. Some differential subordination results associated with lemniscate of Bernoulli is studied in [1][13].

The $q^{th}$ Hankel determinant for a function $f \in \mathcal{A}$, where $q, n \in \mathbb{N}$ is defined as follows:

$$H_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$  

(1.2)

This has been initially studied in [19]. This determinant has also been considered by several authors. It also plays an important role in the study of singularities (see [5]). Noor [18] studied the
rate of growth of $H_q(n)$ as $n \to \infty$ for functions in $\mathcal{S}$ with bounded boundary. The computation of the upper bound of $|H_q(n)|$ for several subclasses of $\mathcal{S}$ has always been a trendy problem in the field of geometric function theory. Hayami and Owa determined the second Hankel determinant $H_2(n)$ $(n = 1, 2, \ldots)$ for functions $f$ satisfying $\text{Re}(f(z)/z) > \alpha$ or $\text{Re} f'(z) > \alpha$ $(0 \leq \alpha < 1)$. Recently, Zaprawa obtained the upper bound of $|H_2(n)|$ for the class $T$ of typically real functions. Note that the Hankel determinant $H_2(1) := a_3 - a_2^2$ coincides with the famous Fekete-Szegő functional. In the year 1983, Bieberbach estimated the bound of the second Hankel determinant $H_2(2)$ $(n = 1, 2, \ldots)$ for functions in the Carathéodory class $\mathcal{P}$, defined by:

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \ (z \in \Delta),$$

with $\text{Re} p(z) > 0$ in $\Delta$. Recently, many authors have estimated the bound of $|H_2(2)|$ (see [3, 7, 9]). Recall that the second Hankel determinant is given by

$$H_2(3) = \begin{vmatrix} a_3 & a_4 & a_5 \\ a_4 & a_5 & \end{vmatrix} = a_3 a_5 - a_4^2. \quad (1.3)$$

Zaprawa investigated the Hankel determinant $H_2(3)$ for several classes of univalent functions. The estimate of the upper bound of the third order Hankel determinant, which is given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2), \quad (1.4)$$

requires the sharp bounds of the initial coefficients $(a_2, a_3, a_4$ and $a_5)$, Fekete-Szegő functional, second Hankel determinant $H_2(2)$ and the quantity $|a_4 - a_2a_3| =: L$. Using triangle inequality in $\text{(1.4)}$, the upper bound of $|H_3(1)|$ can be obtained as follows:

$$|H_3(1)| \leq |a_3||H_2(2)| + |a_4||L| + |a_5||H_2(1)|,$$

(see [12, 21, 25, 27]). Note that this computation does not yield a sharp bound for $H_3(1)$. It is pertinent to know that the computation of $|H_3(1)|$ and $|H_2(3)|$ is tedious if we desire to obtain the sharp bound. For the class $\mathcal{SL}^*$, Raza and Malik obtained the sharp bounds of $|H_2(1)|$ and $|H_2(2)|$ and the upper bound of $|H_3(1)|$. Thus, the sharp estimate of $|H_3(1)|$ for $\mathcal{SL}^*$ until now is an open problem. The study of the bound for third Hankel determinant has become an interesting problem only after the well known formula of expressing $p_4$ in terms of $p_1$ which was recently obtained in [14], which yields the sharp results in most of the cases. Kwon et al. improved the estimate of the third Hankel determinant for starlike functions. Recently, Kowalczyk et al. obtained the sharp bound of $|H_3(1)|$ for the class $T(\alpha) := \{f \in \mathcal{A} : \text{Re}(f(z)/z) > \alpha, \alpha \in [0, 1]\}$ and in [11] establish the sharp bound of the same for the class of convex functions. Zaprawa obtained the sharp bound of $|H_2(3)|$ for the class of typically real functions. Note that these are the only three (as per the knowledge of the authors) sharp bounds of $|H_2(3)|$ and $|H_3(1)|$ proved for any subclass of analytic functions till date.

For the class $\mathcal{SL}^*$, the known upper bound for $|H_3(1)|$ is $\frac{43}{576}$ (see [21]), whereas in this paper, we obtain a sharp estimate for the same which is equal to $\frac{1}{36}$. Further, we find the sharp bound of the second Hankel determinant $H_2(3)$ for the class $\mathcal{SL}^*$. Also, we estimate the sharp bound of the quantity $|a_3^2 - a_5|$ for the class $\mathcal{SL}^*$, which is the Zalcman functional, given in [14], when $n = 3$. In the last section, we establish few results pertaining to the sufficient condition for the functions in $\mathcal{S}$ to belong to the class $\mathcal{SL}^*$.
The following lemmas are required for the formulae of \( p_2, p_3 \) [17] and \( p_4 \) [14] in order to establish our main results.

**Lemma 1.1.** Let \( p \in \mathcal{P} \) and of the form \( 1 + \sum_{n=1}^{\infty} p_n z^n \). Then

\[
2p_2 = p_1^2 + \gamma(4 - p_1^2),
\]

and

\[
4p_3 = p_1^3 + 2p_1(4 - p_1^2)\gamma - p_1(4 - p_1^2)\gamma^2 + 2(4 - p_1^2)(1 - |\gamma|^2)\eta
\]

\[
8p_4 = p_1^4 + (4 - p_1^2)\gamma(p_1^2(\gamma^2 - 3\gamma + 3) + 4\gamma)
- 4(4 - p_1^2)(1 - |\gamma|^2)(p_1(\gamma - 1)\eta + \overline{\gamma}\eta^2 - (1 - |\eta|^2)\rho),
\]

for some \( \rho, \gamma \) and \( \eta \) such that \(|\rho| \leq 1, |\gamma| \leq 1 \) and \(|\eta| \leq 1 \).

**Lemma 1.2.** [20] Let \( a, b, c \) and \( d \) satisfy the inequalities \( 0 < c < 1, 0 < d < 1 \) and

\[
8d(1 - d)((cb - 2a)^2 + (c(d + c) - b)^2) + c(1 - c)(b - 2dc)^2 \leq 4e^2(1 - c)^2d(1 - d).
\]

If \( p \in \mathcal{P} \), then

\[
|ap_1^2 + dp_2^2 + 2cp_1p_3 - (3/2)bp_1p_2 - p_4| \leq 2.
\]

2. **Main Results**

We proceed with the following theorem.

**Theorem 2.1.** If \( f \in \mathcal{SL}^* \). Then we have

\[
|H_3(1)| \leq 1/36.
\]

The bound is sharp.

**Proof.** Let \( f \in \mathcal{SL}^* \) then from [20] p. 509, we have

\[
a_2 = \frac{p_1}{4}, \quad a_3 = \frac{1}{8}p_2 - \frac{3}{64}p_1^2, \quad a_4 = \frac{1}{12}p_3 - \frac{7}{96}p_1p_2 + \frac{13}{768}p_1^3
\]

and

\[
a_5 = -\frac{1}{16}\left(\frac{49}{384}p_1^4 - \frac{17}{24}p_1^2p_2 + \frac{1}{2}p_2^2 + \frac{11}{12}p_1p_3 - p_4\right).
\]

On simplifying the equation (1.1), we get

\[
H_3(1) = 2a_2a_3a_4 - a_3^2 - a_4^2 + a_3a_5 - a_2^2a_5.
\]

Since the class \( \mathcal{P} \) is invariant under the rotation, the value of \( p_1 \) lies in the interval \([0, 2]\). Let \( p := p_1 \) and substituting the above values of \( a_i \)'s in (2.4), we have

\[
H_3(1) = \frac{1}{2359296}\left(689p^6 - 3368p^4p_2 + 3520p^3p_3 + 24064pp_2p_3 + 3008p^2p_2^2
- 16128p^2p_4 - 13824p_2^3 - 16384p_3^2 + 18432p_2p_4\right).
\]

Using the equalities (1.5)-(1.7) and upon simplification, we arrive at

\[
H_3(1) = \frac{1}{2359296}\left(\nu_1(p, \gamma) + \nu_2(p, \gamma)\eta + \nu_3(p, \gamma)\eta^2 + \psi(p, \gamma, \eta)\rho\right).
\]
Where \( \rho, \eta, \gamma \in \overline{\Delta} \),
\[
\nu_1(p, \gamma) := 29p^6 + (4 - p^2)((4 - p^2)(944p^2\gamma^2 - 640p^2\gamma^3 - 2304\gamma^3 + 128p^2\gamma^4) - 116p^4\gamma + 752p^4\gamma^2 - 3456p^2\gamma^2 - 864p^4\gamma^3),
\nu_2(p, \gamma) := (4 - p^2)(1 - |\gamma|^2)(224p^3 + 3456p^3\gamma + (4 - p^2)(2432p\gamma - 512p\gamma^2)) ,
\nu_3(p, \gamma) := (4 - p^2)(1 - |\gamma|^2)((4 - p^2)(4096 - 512|\gamma|^2) + 3456p^2\gamma),
\psi(p, \gamma, \eta) := (4 - p^2)(1 - |\gamma|^2)(1 - |\eta|^2)(-3456p^2 + 4608\gamma(4 - p^2)).
\]
Further, by taking \( x := |\gamma|, y := |\eta| \) and using the fact \( |\rho| \leq 1 \), we have
\[
|H_3(1)| \leq \frac{1}{2359296} \left( |\nu_1(p, \gamma)| + |\nu_2(p, \gamma)|y + |\nu_3(p, \gamma)|y^2 + |\psi(p, \gamma, \eta)| \right)
\leq G(p, x, y),
\]
where
\[
G(p, x, y) := \frac{1}{2359296} \left( g_1(p, x) + g_2(p, x)y + g_3(p, x)y^2 + g_4(p, x)(1 - y^2) \right)
\]
with
\[
g_1(p, x) := 29p^6 + (4 - p^2)((4 - p^2)(944p^2x^2 + 640p^2x^3 + 2304x^3 + 128p^2x^4)
+ 116p^4x + 752p^4x^2 + 3456p^2x^2 + 864p^4x^3),
g_2(p, x) := (4 - p^2)(1 - x^2)(224p^3 + (4 - p^2)(2432px + 512px^2) + 3456p^3x),
g_3(p, x) := (4 - p^2)(1 - x^2)((4 - p^2)(4096 + 512x^2) + 3456p^2x),
g_4(p, x) := (4 - p^2)(1 - x^2)(3456p^2 + 4608x(4 - p^2)).
\]
Now we need to maximize \( G(p, x, y) \) in the closed cuboid \( S : [0, 2] \times [0, 1] \times [0, 1] \). We establish this by finding the maximum values in the interior of the six faces, on the twelve edges and in the interior of \( S \).

I. First we proceed with interior points of \( S \). Let \((p, x, y) \in (0, 2) \times (0, 1) \times (0, 1) \). In an attempt to find the points where the maximum value is attained in the interior of \( S \), we partially differentiate equation (2.5) with respect to \( y \) and on algebraic simplification, we get
\[
\frac{\partial G}{\partial y} = \frac{1}{73728}(4 - p^2)(1 - x^2)(8y(x - 1)(4(4 - p^2)(x - 8) + 27p^2)
+ p(4x(4 - p^2)(19 + 4x) + p^2(7 + 108x))).
\]
Now \( \frac{\partial G}{\partial y} = 0 \) yields
\[
y = \frac{p(4x(4 - p^2)(19 + 4x) + p^2(7 + 108x))}{4(x - 1)(4(4 - p^2)(x - 8) - 27p^2)} := y_0.
\]
For the existence of the critical points, \( y_0 \) should lie in the interval \((0, 1)\), which is possible only when
\[
p^3(7 + 108x) + 4px(4 - p^2)(19 + 4x) + 32(1 - x)(8 - x)(4 - p^2) < 216p^2(1 - x)
\]
and
\[
27p^2 > 4(4 - p^2)(8 - x).
\]
Now, we find the solutions satisfying both the inequalities (2.6) and (2.7) for the existence of critical points. Let \( g(x) := 16(8 - x)/(59 - 4x) \), which is decreasing function of \( x \) as \( g'(x) \) is negative for \( x \in (0, 1) \). Hence \( \min g(x)|_{x=1} = 112/55 \). Thus from equation (2.7), we can conclude that \( p > 1 \) for all \( x \in (0, 1) \). But for \( p \geq 1 \), the inequality (2.6) does not hold as it is not difficult to see
that $7p^3 \geq 216p^2(1-x)$ for all $x$. This shows that there does not exist any solution satisfying both the inequalities (2.6) and (2.7). Hence the function $G$ has no critical point in $(0, 2) \times (0, 1) \times (0, 1).

II. Here we consider the interior of all the six faces of the cuboid $S$.

On the face $p = 0$, $G(p, x, y)$ reduces to

$$h_1(x, y) := G(0, x, y) = \frac{2(1 - x^2)(y^2(x - 1)(x - 8) + 9x) + 9x^3}{576}, \quad x, y \in (0, 1). \quad (2.8)$$

We note that $h_1$ has no critical point in $(0, 1) \times (0, 1)$ since

$$\frac{\partial h_1}{\partial y} = \frac{y(1-x^2)(x-1)(x-8)}{144} \neq 0, \quad x, y \in (0, 1). \quad (2.9)$$

On the face $p = 2$, $G(p, x, y)$ reduces to $G(p, 0, y)$, given by

$$h_2(p, y) := \frac{128y^2(512 - 364p^2 + 59p^4) + 224p^3y(4 - p^2) + 13824p^2 - 3456p^4 + 29p^6}{2359296}, \quad (2.11)$$

where $p \in (0, 2)$ and $y \in (0, 1)$. We solve $\frac{\partial h_2}{\partial y} = 0$ and $\frac{\partial h_2}{\partial p} = 0$ to determine the points where the maxima occur. On solving $\frac{\partial h_2}{\partial y} = 0$, we get

$$y = -\frac{7p^3}{8(128 - 59p^2)} =: y_1. \quad (2.12)$$

For the given range of $y$, we should have $y_1 \in (0, 1)$, which is possible only if $p > p_0$, $p_0 \approx 1.47292$. A computation shows that $\frac{\partial h_2}{\partial p} = 0$ implies

$$256y^2(-182 + 59p^2) - 112y(-12p + 5p^3) + 87p^4 - 6912p^2 + 13824 = 0. \quad (2.13)$$

Substituting equation (2.12) in equation (2.13) and upon simplification, we get

$$75497472 - 107347968p^2 + 51265024p^4 - 8426096p^6 + 95167p^8 = 0. \quad (2.14)$$

A numerical computation shows that the solution of (2.14) in the interval $(0, 2)$ is $p \approx 1.39732$. Thus $h_2$ has no critical point in $(0, 2) \times (0, 1)$.

On the face $x = 1$, $G(p, x, y)$ reduces to

$$h_3(p, y) := G(p, 1, y) = \frac{36864 + 22784p^2 - 7920p^4 + 9p^6}{2359296}, \quad p \in (0, 2). \quad (2.15)$$

Solving $\frac{\partial h_3}{\partial p} = 0$, we get a critical point at $p =: p_0 \approx 1.2008$. A Simple calculation shows that $h_3$ attains its maximum value $\approx 0.0225817$ at $p_0$.

On the face $y = 0$, $G(p, x, y)$ reduces to

$$h_4(p, x) := G(p, x, 0) = \frac{1}{2359296} \left( 29p^6 + (4 - p^2)((4 - p^2)(944p^2x^2 + 640p^2x^3 - 2304x^3 + 128p^2x^4 + 4608x) + 116p^4x + 752p^4x^2 + 864p^4x^3 + 3456p^4x^2) \right). \quad (2.16)$$
Evaluating the equation (2.11) at

\[ A \text{ numerical computation shows that there does not exist any solution for the system of equations} \]

\[ \text{and} \]

\[ \text{and} \]

\[ p \]

In view of the equation (2.11) and by straightforward computation the maximum value of \( \lambda \) which is attained at \( y \) is attained at \( \lambda = 1 \).

Considering the equation (2.11), we have

\[ \frac{\partial h_4}{\partial x} = 0 \text{ and } \frac{\partial h_4}{\partial p} = 0 \text{ in } (0, 2) \times (0, 1). \]

On the face \( y = 1, G(p, x, y) \) reduces to

\[ G(p, x, 1) = \frac{1}{2359296} \left( 29 p^6 + (4 - p^2)(116 p^4 x + 752 p^4 x^2 + 3456 p^2 x^2 + 864 p^4 x^3 \right. \]

\[ + (1 - x^2)(224 p^3 + 3456 p^2 x + 3456 p^2 x) + (4 - p^2)((1 - x^2)(2432 p x \]

\[ + 512 p x^2 + 4096 + 512 x^2) + 944 p^2 x^2 + 640 p^2 x^3 + 2304 x^3 + 128 p^2 x^4) \] \)

\[ =: h_6(p, x). \]

Proceeding on the similar lines as in the previous case for face \( y = 0 \), again there is no solution for the system of equations \( \frac{\partial h_4}{\partial x} = 0 \) and \( \frac{\partial h_4}{\partial p} = 0 \) in \((0, 2) \times (0, 1)\).

III. Now we calculate the maximum values achieved by \( G(p, x, y) \) on the edges of the cuboid \( S \).

Considering the equation (2.11), we have \( G(p, 0, 0) =: s_1(p) = (29 p^6 - 3456 p^4 + 13824 p^2)/2359296 \).

It is easy to verify that the function \( s_1(p) = 0 \) for \( p =: \lambda_0 = 0 \) and \( p =: \lambda_1 \approx 1.43285 \) in the interval \([0, 2]\). We observe that \( \lambda_0 \) is the point of minima and the maximum value of \( s_1(p) \) is \( \approx 0.00596162 \), which is attained at \( \lambda_1 \).

Hence

\[ G(p, 0, 0) \leq 0.00596162, \quad p \in [0, 2]. \]

Evaluating the equation (2.11) at \( y = 1 \), we obtain \( G(p, 0, 1) = s_2(p) := (65536 - 32768 p^2 + 896 p^3 + 4096 p^4 - 224 p^5 + 29 p^6)/2359296 \). It is easy to verify that \( s_2(p) \) is decreasing function in \([0, 2]\) and hence attains its maximum value at \( p = 0 \).

Thus

\[ G(p, 0, 1) \leq \frac{1}{36}, \quad p \in [0, 2]. \]

In view of the equation (2.11) and by straightforward computation the maximum value of \( G(0, 0, y) \) is attained at \( y = 1 \). This implies

\[ G(0, 0, y) \leq \frac{1}{36}, \quad y \in [0, 1]. \]

As the equation (2.15) is independent of \( x \), we have \( G(p, 1, 1) = G(p, 1, 0) = s_3(p) := (9 p^6 - 7920 p^4 + 22784 p^2 + 36864)/2359296 \).

Now, \( s_3'(p) = 45568 p - 31680 p^3 + 54 p^5 = 0 \) for \( p =: \lambda_2 = 0 \) and \( p =: \lambda_3 \approx 1.2008 \) in the interval \([0, 2]\), where \( \lambda_2 \) is a point of minima and \( s_3(p) \) attains its maximum value at \( \lambda_3 \). We can conclude that

\[ G(p, 1, 1) = G(p, 1, 0) \leq 0.0225817, \quad p \in [0, 2]. \]
Substituting \( p = 0 \) in equation (2.15), we obtain \( G(0, 1, y) = 1/64 \). The Equation (2.10) is independent of all the variables \( p, x \) and \( y \). Thus the value of \( G(p, x, y) \) on the edges \( p = 2, x = 1; \) \( p = 2, x = 0; \) \( p = 2, y = 0 \) and \( p = 2, y = 1 \), respectively, is given by

\[
G(2, 1, y) = G(2, 0, y) = G(2, x, 0) = G(2, x, 1) = 29/36864, x, y \in [0, 1].
\]

Equation (2.11), yields \( G(0, 0, y) = y^2/36 \). A simple computation shows that

\[
G(0, 0, y) \leq 1/36, \quad y \in [0, 1].
\]

Using equation (2.8), we get \( G(0, x, 1) =: s_4(x) = (16 - 4x^2 + 9x^3 - 2x^4)/576 \). A simple computation shows that the function \( s_4 \) is decreasing in \([0, 1]\) and hence attains its maximum value at \( x = 0 \). Hence

\[
G(0, x, 1) \leq 1/36, \quad x \in [0, 1].
\]

Once again, by using the equation (2.8), we get \( G(0, x, 0) = s_5'(x) = -(x^2 - 2)/64 \). Performing a simple calculation, we get \( s_5'(x) = 0 \) for \( x =: x_0 = \sqrt{2}/\sqrt{3} \) and for \( 0 \leq x < x_0 \), \( s_5 \) is an increasing function and for \( x_0 < x \leq 1 \), it’s a decreasing function. Thus, it attains maximum value at \( x_0 \). Hence

\[
G(0, x, 0) \leq 0.0170103, \quad x \in [0, 1].
\]

In view of the cases I-III, the inequality (2.1) holds. Let the function \( f : \Delta \to \mathbb{C} \) be as follows

\[
f(z) = z \exp \left( \int_0^z \frac{\sqrt{1 + t^3} - 1}{t} dt \right) = z + \frac{z^4}{6} + \cdots. \tag{2.16}
\]

The sharpness of the bound \(|H_3(1)|\) is justified by the extremal function \( f \) given by (2.16), which belongs to the class \( \mathcal{S}\mathcal{L}^* \). For this function \( f \), we have \( a_2 = a_3 = a_5 = 0 \) and \( a_4 = 1/6 \), which clearly shows that \(|H_3(1)| = 1/36 \) using equation (2.4). This completes the proof. 

We now estimate the bound for the Hankel determinant \( H_2(3) \).

**Theorem 2.2.** Let \( f \in \mathcal{S}\mathcal{L}^* \). Then we have

\[
|H_2(3)| \leq \frac{1}{36}. \tag{2.17}
\]

The result is sharp.

**Proof.** We proceed here on the similar lines as in the proof of Theorem 2.1. Now, substituting the equalities (2.2)-(2.3) in (1.3) and with the assumption \( p_1 =: p \in [0, 2] \), we get

\[
H_2(3) = \frac{1}{1179648} \left( 103p^6 - 712p^4p_2 - 4608p_3^2 + 1984p^2p_2^2 + 5888pp_2p_3 
- 160p^3p_3 - 8192p_3^2 - 3456p^2p_4 + 9216p_2p_4 \right). \tag{2.18}
\]

Using the equalities (1.5)-(1.7) and simplifying the terms in the expression (2.18), we get

\[
H_2(3) = \frac{1}{1179648} \left( \zeta_1(p, \gamma) + \zeta_2(p, \gamma)\eta + \zeta_3(p, \gamma)\eta^2 + \xi(p, \gamma, \eta)\rho \right),
\]
where \( \rho, \eta \) and \( \gamma \in \Delta \),
\[
\zeta_1(p, \gamma) := -5p^6 + 4p^2\gamma(4-p^2)(-p^2 - 20(4-p^2)\gamma - 26p^2\gamma + 144\gamma + 36p^2\gamma^2 + 16\gamma^2(4-p^2) + 40\gamma^2(4-p^2)),
\]
\[
\zeta_2(p, \gamma) := 16p(4-p^2)(1-|\gamma|^2)(-5p^2 - 36p^2\gamma - 16\gamma^2(4-p^2) - 20\gamma(4-p^2)),
\]
\[
\zeta_3(p, \gamma) := 64(4-p^2)(1-|\gamma|^2)(4(4-p^2)(8+\gamma^2) - 9p^2\gamma),
\]
\[
\xi(p, \gamma, \eta) := 576(4-p^2)(1-|\gamma|^2)(1-|\eta|^2)(p^2 + 4\gamma(4-p^2)).
\]

By taking \( x := |\gamma|, y := |\eta| \) and using the fact \( |\rho| \leq 1 \), we get
\[
|H_2(3)| \leq \frac{1}{1179648} \left( |\zeta_1(p, \gamma)| + |\zeta_2(p, \gamma)|y + |\zeta_3(p, \gamma)|y^2 + |\xi(p, \gamma, \eta)| \right)
\leq F(p, x, y),
\]
where
\[
F(p, x, y) := \frac{1}{1179648} \left( q_1(p, x) + q_2(p, x)y + q_3(p, x)y^2 + q_4(p, x)(1-y^2) \right)
\]  \( (2.19) \)
with
\[
q_1(p, x) := 5p^6 + 4p^2x(4-p^2)(p^2 + 20(4-p^2)x + 26p^2x + 144x + 36p^2x^2 + 16x^3(4-p^2) + 40x^2(4-p^2)),
\]
\[
q_2(p, x) := 16p(4-p^2)(1-x^2)(5p^2 + 36p^2x + 16x^2(4-p^2) + 20x(4-p^2)),
\]
\[
q_3(p, x) := 64(4-p^2)(1-x^2)(4(4-p^2)(8+x^2) + 9p^2x),
\]
\[
q_4(p, x) := 576(4-p^2)(1-x^2)(p^2 + 4x(4-p^2)).
\]

In order to complete the proof, we need to maximize the function \( F(p, x, y) \) in the closed cuboid \( T : [0, 2] \times [0, 1] \times [0, 1] \). For this, we find the maximum values of \( F \) in \( T \) by considering all the twelve edges, interior of the six faces and in the interior of \( T \).

I. We proceed with interior points of \( T \). Let us assume \((p, x, y) \in (0, 2) \times (0, 1) \times (0, 1) \). To determine the points where the maximum value occur in the interior of \( T \), we partially differentiate equation \( (2.19) \) with respect to \( y \) and we get
\[
\frac{\partial F}{\partial y} = \frac{1}{73728} (4-p^2)(1-x^2)(8y(x-1)(4-p^2)(x-8) + 9p^2)

+ p(4x(4-p^2)(5+4x) + p^2(5+36x))).
\]

Now, \( \frac{\partial F}{\partial y} = 0 \) yields
\[
y = \frac{p(4x(4-p^2)(5+4x) + p^2(5+36x))}{8(x-1)(4-p^2)(8-x) - 9p^2} =: y_1.
\]

Now, \( y_1 \) should lie in the interval \( (0, 1) \) for the existence of the critical points. Thus, we have
\[
p^3(5+36x) + 4px(4-p^2)(5+4x) + 32(1-x)(8-x)(4-p^2) < 72p^2(1-x)
\]
and
\[
4(4-p^2)(8-x) < 9p^2.
\]

We try to find the solutions satisfying both the inequalities \( (2.20) \) and \( (2.21) \). Let us assume \( g(x) := 16(8-x)/(41-4x) \), which is decreasing function of \( x \) due to the fact that \( g'(x) \) is negative for \( x \in (0, 1) \). Therefore \( \min r(x)_{(x=1)} = 112/37 \). This implies \( p > 1 \) for all \( x \in (0, 1) \) using equation \( (2.21) \). But for \( p \geq 1 \), the inequality \( (2.20) \) does not hold as \( 5p^3 \geq 72p^2(1-x) \) for all \( x \). Thus we can conclude that there does not exist any solution satisfying \( (2.20) \) and \( (2.21) \). Thus
function $F$ has no critical point in $(0, 2) \times (0, 1) \times (0, 1)$.

II. Now, we consider the interior of all the six faces of the cuboid $T$. On the face $p = 0$,

$$k_1(x, y) := F(0, x, y) = \frac{1 - x^2}{288} \left( y^2(x - 1)(x - 8) + 9x \right), \quad x, y \in (0, 1). \quad (2.22)$$

A simple calculation shows that $\partial k_1 / \partial y = \partial h_1 / \partial y$. Thus equation (2.3) implies $k_1$ has no critical point in $(0, 1) \times (0, 1)$.

On the face $p = 2$,

$$F(2, x, y) = \frac{5}{18432}, \quad x, y \in (0, 1). \quad (2.23)$$

On the face $x = 0$,

$$k_2(p, y) := F(p, 0, y) = \frac{64y^2(512 - 292p^2 + 41p^4) + 80p^3y(4 - p^2) + 2304p^2 - 576p^4 + 5p^6}{1179648}, \quad p \in (0, 2) \text{ and } y \in (0, 1). \quad (2.24)$$

On solving $\frac{\partial k_2}{\partial y} = 0$, we get

$$y = \frac{5p^3}{8(41p^2 - 128)} =: y_1. \quad (2.25)$$

For the given range of $y$, $y_1$ should lie in the interval $(0, 1)$, which holds only if $p > p_0, p_0 \approx 1.7669$.

The computation shows that $\frac{\partial k_1}{\partial p} = 0$ implies

$$y^2(5248p^2 - 18688) + 40y(12p - 50p^3) + 2304 - 1152p^2 + 15p^4 = 0. \quad (2.26)$$

Let $p > p_0$ and substituting equation (2.25) in equation (2.26) and performing lengthy computation, we get

$$1048576 - 1196032p^2 + 449216p^4 - 57582p^6 + 615p^8 = 0. \quad (2.27)$$

The numerical computation shows that the solution of (2.27) for $p \in (0, 2)$ is $p =: p_0 \approx 1.39597$.

Thus $k_2$ has no critical point in $(0, 2) \times (0, 1)$.

On the face $x = 1$,

$$k_3(p) := F(p, 1, y) = \frac{7168p^2 - 2000p^4 + 57p^6}{1179648}, \quad p \in (0, 2). \quad (2.28)$$

To attain maximum value of $k_3$, we solve $\partial k_3 / \partial p = 0$ and get critical point at $p =: p_0 \approx 1.39838$.

Simple calculation shows that $k_3$ attains its maximum value $\approx 0.00576045$ at $p_0$.

On the face $y = 0$,

$$F(p, x, 0) = \frac{1}{1179648} \left( 5p^6 + (4 - p^2)(4 - p^2)(2304x(1 - x^2) + 80p^2x^2 \right. \right.$$

$$+ 160p^2x^3 + 64p^2x^4) + 4p^4x + 576p^2x^2 + 104p^4x^2 \right. \right.$$

$$+ 144p^4x^3 + 576p^2(1 - x^2)) =: k_4(p, x).$$

A complex computation shows that

$$\frac{\partial k_4}{\partial p} = \frac{1}{589824} \left( 2304p - 1152p^3 + 15p^5 + (-18432p + 4640p^3 - 12p^5)x \right. \right.$$

$$+ (1280p - 448p^3 - 72p^5)x^2 + (20992p - 6016p^3 + 48p^5)x^3 \right. \right.$$

$$+ (1024p - 1024p^3 + 192p^5)x^4 \right).$$
and
\[
\frac{\partial k_4}{\partial x} = \frac{1}{294912} \left( (p^2 - 4)((-256p^2 + 64p^4)x^3 + (6912 - 2208p^2 + 12p^4)x^2 \\
+ (-160p^2 - 12p^4)x - 2304 + 576p^2 - p^4) \right).
\]

The numerical computation shows that there does not exist any solution for the system of equations
\[
\frac{\partial k_5}{\partial p} = 0 \quad \text{and} \quad \frac{\partial k_5}{\partial x} = 0 \quad \text{in} \quad (0, 2) \times (0, 1).
\]

On the face \(y = 1\),
\[
F(p, x, 1) = \frac{1}{1179648} \left( 5p^6 + 4(4 - p^2)((4 - p^2)(80p^2x^2 + 64p^2x^4 + 160p^2x^3 \\
+ (1 - x^2)(526px^2 + 320px + 256(8 + x^2)) + 4p^4x + 104p^4x^2 \\
+ 576p^2x^2 + 144p^4x^3 + (1 - x^2)(80p^3 + 576p^3x + 576p^2x)) \right) =: k_5(p, x).
\]

Proceeding on the similar lines as in the previous case on the face \(y = 0\), again, the system of equations \(\partial k_5/\partial p = 0 \quad \text{and} \quad \partial k_5/\partial x = 0\) have no solution in \((0, 2) \times (0, 1)\).

III. We now consider the maximum values attained by \(F(p, x, y)\) on the edges of the cuboid \(T\): In view of the equation \(2.24\), we have
\[
F(p, 0, 0) = l_1(p) := 5p^6 - 576p^4 + 2304p^2 / 1179648.
\]

It is easy to compute that \(l_1'(p) = 0\) for \(p =: \lambda_0 = 0\) and \(p =: \lambda_1 \approx 1.43351\) in the interval \([0, 2]\), where \(\lambda_0\) is the point of minima and \(\lambda_1\) is the point of maxima. Hence
\[
F(p, 0, 0) \leq 0.00198843, \quad p \in [0, 2].
\]

Again, considering the equation \(2.24\), we obtain
\[
F(p, 0, 1) = l_2(p) := (32768 - 16384p^2 + 320p^3 + 2048p^4 - 80p^5 + 5p^6) / 1179648.
\]

Now, we note that \(l_2\) is a decreasing function in \([0, 2]\) and hence attains its maximum value at \(p = 0\). Thus,
\[
F(p, 0, 0) \leq \frac{1}{36}, \quad p \in [0, 2].
\]

Now, we observe that the equation \(2.28\) does not depend on the value of \(y\), hence we get
\[
F(p, 1, 1) = F(p, 1, 0) = l_3(p) := (7168p^3 - 2000p^4 + 57p^6) / 1179648.
\]

It is easy to verify that the function \(l_3\) has two critical points at \(p = 0\) and \(p =: \lambda_2 \approx 1.39838\) in the interval \([0, 2]\), where the maximum value is attained at \(\lambda_2\). Thus
\[
F(p, 0, 0) = F(p, 1, 0) \leq 0.0057645, \quad p \in [0, 2].
\]

On substituting \(p = 0\) in \(2.28\), we get \(F(0, 1, y) = 0\). In view of equation \(2.23\), which is independent of all the variables \(p, x, y\), the value of \(F(p, x, y)\) on the edges \(p = 2, x = 0; p = 2, x = 1; p = 2, y = 0\) and \(p = 2, y = 1\), respectively, is given by
\[
F(2, 0, y) = F(2, 1, y) = F(2, x, 0) = F(2, x, 1) = 5/18432, \quad x, y \in [0, 1].
\]

Evaluating equation \(2.24\) at \(p = 0\), we get
\[
F(0, 0, y) = l_4(y) := y^2 / 36. \quad \text{It is easy to verify that} \quad l_4 \quad \text{is an increasing function of} \quad y \quad \text{and hence attains maximum value at} \quad y = 1 \quad \text{in} \quad [0, 1]. \quad \text{Thus}
\]
\[
F(0, 0, y) \leq \frac{1}{36}, \quad y \in [0, 1].
\]

Using equation \(2.22\), we get \(F(0, 1, x) = l_5(x) := (8 - 7x^2 - x^4) / 288\). Since \(l_5\) is a decreasing function in \([0, 1]\), it attains maximum value at \(x = 0\). Thus
\[
F(0, x, 1) \leq \frac{1}{36}, \quad x \in [0, 1].
\]
Substituting $y = 0$ in equation (2.22), we obtain $F(0, x, 0) = l_6(x) := x(1 - x^2)/32$. A simple calculation shows that the function $l_6'(x) = 0$ at $x =: x_0 = \sqrt{3}/3$ and it is increasing in $(0, x_0)$ and decreasing in $(x_0, 1)$. Hence it attains the maximum value at $x = x_0$. Thus we conclude

$$F(0, x, 0) \leq \sqrt{3}/144, \quad x \in [0, 1].$$

Taking into account all the cases I-III, the inequality (2.17) holds. For the function given in (2.16), which belongs to the class $\mathcal{S}\mathcal{L}^*$, $a_3 = a_5 = 0$ and $a_4 = 1/6$. Thus $|H_2(3)| = 1/36$ for this function, which also proves the result is sharp. This completes the proof.

We note that for $n = 2$, the expression on the left of the inequality (1.1) reduces to the famous Fekete-Szegő functional. In the following theorem we obtain the Zalcman coefficient inequality for $n = 3$ for the class $\mathcal{S}\mathcal{L}^*$.

**Theorem 2.3.** Let $f \in \mathcal{S}\mathcal{L}^*$. Then

$$|a_3^2 - a_5| \leq \frac{1}{8}.$$

The estimate is sharp.

**Proof.** Using equation (2.23) and (2.23), we get

$$a_3^2 - a_5 = \frac{125}{1288}p_1^4 - \frac{43}{768}p_2^2p_2 + \frac{3}{64}p_2^2 + \frac{11}{192}p_1p_3 - \frac{1}{16}p_4.$$  (2.29)

Applying Lemma 1.2 with $a = 125/768$, $b = 43/72$, $c = 11/24$ and $d = 3/4$ in the equation (2.29), we get

$$|a_3^2 - a_5| \leq \frac{1}{8}.$$

Let the function $f : \Delta \rightarrow \mathbb{C}$, be defined as follows:

$$f(z) = z \exp \left( \int_0^z \frac{\sqrt{1 + t^2} - 1}{t} \, dt \right) = z + \frac{z^5}{8} + \cdots.$$  (2.30)

The equality holds for the function given in (2.30), which belong to $\mathcal{S}\mathcal{L}^*$ as $a_3 = 0$ and $a_5 = 1/8$, which contributes to the sharpness of the inequality. This completes the proof.

3. **Further Results**

Let $f$ and $g$ be analytic functions of the form, respectively

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

Then the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined by

$$(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Now, we derive the necessary and sufficient condition for a function $f \in \mathcal{S}$ to belong to the class $\mathcal{S}\mathcal{L}^*$ in the following theorem, involving the convolution concept.

**Theorem 3.1.** A function $f \in \mathcal{S}$ is in the class $\mathcal{S}\mathcal{L}^*$ if and only if

$$\frac{1}{z} (f \ast H_t(z)) \neq 0, \quad (z \in \Delta)$$  (3.1)

where

$$H_t(z) = \frac{z}{(1 - z)(1 - S(t))} \left( \frac{1}{1 - z} - S(t) \right)$$
and
\[ S(t) = \sqrt{t} + i \left( \pm \sqrt{1 + 4t - (t + 1)} \right), \quad (0 < t < 2). \]

**Proof.** Define \( p(z) = zf'(z)/f(z) \). As we know \( p(0) = 1 \), to prove the result, it suffices to show that \( f \in S L^* \) if and only if \( p(z) \notin \gamma_1 \), where
\[ \gamma_1 = \{(u^2 + v^2)^2 - 2(u^2 - v^2) = 0\}. \]
By taking \( u^2 = t \), we can give the parametric representation of the curve \( \gamma_1 \) as follows
\[ S(t) = \sqrt{t} + i \left( \pm \sqrt{1 + 4t - (t + 1)} \right), \quad (0 < t < 2). \]

For \( f \in S \), we have
\[ \frac{z}{(1 - z)^2} * f(z) = zf'(z) \quad \text{and} \quad \frac{z}{1 - z} * f(z) = f(z). \quad (3.2) \]
Using the above equations (3.1) and (3.2), we get
\[ \frac{1}{z} (f * H_t(z)) = \frac{f(z)}{z(1 - S(t))} \left( \frac{zf'(z)}{zf(z)} - S(t) \right) \neq 0, \]
which clearly shows that \( zf'(z)/f(z) \neq S(t) \). Hence \( 1/(z(f * H_t(z))) \neq 0 \) if and only if \( p(z) \notin \gamma_1 \) if and only if \( f \in S L^* \).

**Theorem 3.2.** The function
\[ \Theta(z) = \frac{z}{1 - \alpha z}, \quad (z \in \Delta) \]
belongs to the class \( S L^* \) if \( |\alpha| \leq 1/4 \).

**Proof.** By the definition of the class \( S L^* \), it suffices to show that the following inequality holds for the given range of \( \alpha \).
\[ \left| \left( \frac{1}{1 - \alpha z} \right)^2 - 1 \right| < 1. \quad (3.3) \]
The above inequality (3.3) holds whenever
\[ |2\alpha z - \alpha^2 z^2| < 1 + |\alpha z|^2 - 2Re(\alpha z), \]
which in turn holds if
\[ 2|\alpha z| \leq 1 - 2|\alpha z|, \]
which holds if
\[ |\alpha| \leq \frac{1}{4}. \]
Hence the function \( \Theta(z) \in S L^* \).

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