Dynamical quark loop light-by-light contribution to muon g-2 within the nonlocal chiral quark model

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Abstract

The hadronic corrections to the muon anomalous magnetic moment $a_{\mu}$, due to the gauge-invariant set of diagrams with dynamical quark loop light-by-light scattering insertions, are calculated in the framework of the nonlocal chiral quark model. These results complete calculations of all hadronic light-by-light scattering contributions to $a_{\mu}$ in the leading order in the $1/N_c$ expansion. The result for the quark loop contribution is $a_{\mu}^{\text{HLbL,Loop}} = (11.0 \pm 0.9) \cdot 10^{-10}$, and the total result is $a_{\mu}^{\text{HLbL,NAQM}} = (16.8 \pm 1.2) \cdot 10^{-10}$. This means, that the difference in values of $a_{\mu}$ between the standard model and the experiment is 2.4 $\sigma$. 

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I. INTRODUCTION

Experimental and theoretical research on lepton anomalous magnetic moments has a long and prominent history\(^1\). The most recent and precise measurements of the muon anomalous magnetic moment \(a_\mu\) were published in 2006 by the E821 collaboration at the Brookhaven National Laboratory \(^5\). The combined result, based on nearly equal samples of positive and negative muons, is

\[
a_\mu^{\text{BNL}} = 116.592\,080.0 \times 10^{-10} \quad [0.54 \text{ ppm}].
\]

Later on, this value was corrected \(^6, 7\) for a small shift in the ratio of the magnetic moments of the muon and the proton as

\[
a_\mu^{\text{BNL, CODATA}} = 116.592\,091.1 \times 10^{-10}.
\]

This exciting result is still limited by the statistical errors, and proposals to measure \(a_\mu\) with a fourfold improvement in accuracy were suggested at Fermilab (USA) \(^8\) and J-PARC (Japan) \(^9\). These plans are very important in view of a very accurate prediction of \(a_\mu\) within the standard model (SM). The dominant contribution in the SM comes from QED

\[
a_\mu^{\text{QED}} = 116.584\,718.951(80) \times 10^{-10} \quad [16].
\]

Other contributions are due to the electroweak corrections \(^10, 11\)

\[
a_\mu^{\text{EW}} = 15.36(0.1) \times 10^{-10} \quad [11],
\]

the hadron vacuum polarization (HVP) contributions in the leading, next-to-leading and next-next-to-leading order \(^12\) \(^14\),

\[
a_\mu^{\text{HVP, LO}} = 694.91(3.72)(2.10) \times 10^{-10} \quad [12],
\]

\[
a_\mu^{\text{HVP, NLO}} = -9.84(0.06)(0.04) \times 10^{-10} \quad [12],
\]

\[
a_\mu^{\text{HVP, NNLO}} = 1.24(0.01) \times 10^{-10} \quad [13],
\]

and the hadronic light-by-light (HLbL) scattering contribution (as it is estimated in \(^15\))

\[
a_\mu^{\text{HLbL}} = 10.5(2.6) \times 10^{-10}.
\]

\(^1\) For comprehensive reviews see \(^1 \sim 4\).
As a result, the total value for the SM contribution, if we take \( 8 \) for HLbL, is

\[
a^{\text{SM}}_{\mu} = 116 \, 591 \, 841 \times 10^{-10}. \tag{9}
\]

From the comparison of (2) with (9) it follows that there is a 3.11 standard deviation between theory and experiment. This might be an evidence for the existence of new interactions and stringently constrains the parametric space of hypothetical interactions extending the SM.

From above it is clear, that the main source of theoretical uncertainties comes from the hadronic contributions. The HVP contribution \( a^{\text{HVP,LO}}_{\mu} \), using analyticity and unitarity, can be expressed as a convolution integral over the invariant mass of a known kinematical factor and the total \( e^+e^- \rightarrow \gamma^* \rightarrow \text{hadrons} \) cross-section [17]. Then the corresponding error in \( a^{\text{HVP,LO}}_{\mu} \) essentially depends on the accuracy in the measurement of the cross-section [12, 14]. In near future it is expected, that new and precise measurements from CMD3 and SND at VEPP-2000 in Novosibirsk, BES III in Beijing and KLOE-2 at DAFNE in Frascati will allow to significantly increase the accuracy of the predictions for \( a^{\text{HVP,LO}}_{\mu} \).

On the other hand, the HLbL contribution \( a^{\text{HLbL}}_{\mu} \) cannot be calculated from first principles or (unlike to HVP) directly extracted from phenomenological considerations. Instead, it has to be evaluated using various QCD inspired hadronic models that correctly reproduce basic low- and high-energy properties of the strong interaction. Nevertheless, as will be discussed below, it is important for model calculations, that phenomenological information and well established theoretical principles should significantly reduce the number of model assumptions and the allowable space of model parameters.

Different approaches to the calculation of the contributions from the HLbL scattering process to \( a^{\text{HLbL}}_{\mu} \) have been suggested. These approaches can be classified into several types. The first one consists of various extended versions of the vector meson dominance model (VMD) supplemented by the ideas of the chiral effective theory, such as the hidden local symmetry model (HLS) [19], the lowest meson dominance (LMD) [20–22], and the resonance chiral theory (R\( _\chi \)T) [23, 24]. The second type of approaches is based on the consideration of effective models of QCD that use the dynamical quarks as effective degrees of freedom. The rest include different versions of the (extended) Nambu–Jona-Lasinio model (E)NJL [25], the constituent quark models with local interaction (CQM) [26–30], the models based on nonperturbative quark-gluon dynamics, like the nonlocal chiral quark model (N\( _\chi \)QM)
More recently, there have been attempts to estimate $a_{\mu}^{\text{HLbL}}$ within the dispersive approach (DA) \cite{41,42} and the so-called rational approximation (RA) approach \cite{43}.

The aim of this work is to complete calculations of the leading in $1/N_c$ HLbL contributions within the $N_{\chi}\text{QM}$ started in \cite{36,37} and compare the result with (8). Namely, in previous works we made detailed calculations of hadronic contributions due to the exchange diagrams in the channels of light pseudoscalar and scalar mesons. In the present work, the detailed calculation of the light quark loop contribution is given\footnote{Preliminary results of this work were announced in \cite{3}.}.

II. LIGHT-BY-LIGHT CONTRIBUTION TO $a_\mu$ IN THE GENERAL CASE

We start from some general consideration of the connection between the muon AMM and the light-by-light (LbL) scattering polarization tensor. The muon AMM for the LbL contribution can be extracted by using the projection \cite{44}

$$a_{\mu}^{\text{LbL}} = \frac{1}{48m_\mu} \Tr \left( (\hat{p} + m_\mu) [\gamma^\rho, \gamma^\sigma] (\hat{p} + m_\mu) \Pi_{\rho\sigma}(p, p) \right),$$

where

$$\Pi_{\rho\sigma}(p', p) = e^6 \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} \frac{1}{q_1^2(q_1 + q_2)^2(q_1 + k)^2} \times$$

$$\times \gamma^\rho \frac{\hat{p}^\prime - \hat{q}_2 + m_\mu}{(p^\prime - q_2)^2 - m_\mu^2} \gamma^\sigma \frac{\hat{p} + \hat{q}_1 + m_\mu}{(p + q_1)^2 - m_\mu^2} \gamma^\lambda \times$$

$$\times \frac{\partial}{\partial k_\rho} \Pi_{\mu\nu\lambda\sigma}(q_2, -(q_1 + q_2), k + q_1, -k),$$

where $m_\mu$ is the muon mass, $k_\mu = (p^\prime - p)_\mu$, and it is necessary to make the static limit $k_\mu \to 0$ after differentiation. Let us introduce the notation

$$\frac{\partial}{\partial k_\rho} \Pi_{\mu\nu\lambda\sigma}(q_2, -(q_1 + q_2), k + q_1, -k) = \Pi_{\rho\mu\nu\lambda\sigma}(q_2, -(q_1 + q_2), q_1) + O(k)$$

\footnote{First note, the tensor $\Pi_{\mu\nu\lambda\sigma}$ can be of any nature (QED, hadronic, etc.) Another note concerns the important result expressing the tensor $\Pi_{\rho\mu\nu\lambda\sigma}$ in the explicitly gauge-invariant form that was obtained in \cite{45}.}

\footnote{Preliminary results of this work were announced in \cite{3}.}

for the derivative of the four-rank polarization tensor, and rewrite Eqs. (10) and (11) in the form

$$a_{\mu}^{\text{LbL}} = \frac{e^6}{48m_\mu} \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} \frac{\Pi_{\rho\mu\nu\lambda\sigma}(q_2, -(q_3 + q_1), q_1, q_2, p) \Gamma_{\rho\mu\nu\lambda\sigma}(q_1, q_2, p)}{q_1^2q_2^2q_3^2(q_1 + q_2)^2 - m_\mu^2((p - q_2)^2 - m_\mu^2)},$$

$$2$$
where the tensor $T^{\rho\mu\nu\lambda\sigma}$ is the Dirac trace

$$T^{\rho\mu\nu\lambda\sigma}(q_1, q_2, p) = \text{Tr} \left( (\hat{p} + m_\mu)[\gamma^\rho, \gamma^\sigma](\hat{p} + m_\mu)\gamma^\mu(\hat{p} - \hat{q}_2 + m_\mu)\gamma^\nu(\hat{p} + \hat{q}_1 + m_\mu)\gamma^\lambda \right).$$

Taking the Dirac trace, the tensor $T^{\rho\mu\nu\lambda\sigma}$ becomes a polynomial in the momenta $p, q_1, q_2$. After that, it is convenient to convert all momenta into the Euclidean space, and we will use the capital letters $P, Q_1, Q_2$ for the corresponding counterparts of the Minkowskian vectors $p, q_1, q_2$, e.g. $P^2 = -p^2 = -m^2_\mu, Q_1^2 = -q_1^2, Q_2^2 = -q_2^2$. Then Eq. (13) becomes

$$a^{\text{lab}}_\mu = \frac{e^6}{48m_\mu} \int \frac{d^4E}{(2\pi)^4} \int \frac{d^4Q_1}{(2\pi)^4} \frac{d^4Q_2}{(2\pi)^4} \frac{T^{\rho\mu\nu\lambda\sigma} \Pi_{\rho\mu\nu\lambda\sigma}}{D_1 D_2},$$

$$D_1 = (P + Q_1)^2 + m^2_\mu = 2(P \cdot Q_1) + Q_1^2,$$

$$D_2 = (P - Q_2)^2 + m^2_\mu = -2(P \cdot Q_2) + Q_2^2.$$ Since the highest order of the power of the muon momentum $P$ in $T^{\rho\mu\nu\lambda\sigma}$ is two \(^4\) and $\Pi_{\rho\mu\nu\lambda\sigma}$ is independent of $P$, the factors in the integrand of (13) can be rewritten as

$$\frac{T^{\rho\mu\nu\lambda\sigma} \Pi_{\rho\mu\nu\lambda\sigma}}{D_1 D_2} = \sum_{a=1}^6 A_a \tilde{\Pi}_a,$$

with the coefficients

$$A_1 = \frac{1}{D_1}, \quad A_2 = \frac{1}{D_2}, \quad A_3 = \frac{(P \cdot Q_2)}{D_1}, \quad A_4 = \frac{(P \cdot Q_1)}{D_2}, \quad A_5 = \frac{1}{D_1 D_2}, \quad A_6 = 1,$$$$

where all $P$-dependence is included in the $A_a$ factors, while $\tilde{\Pi}_a$ are $P$-independent.

Then, one can average over the direction of the muon momentum $P$ (as was suggested in \cite{1} for the pion-exchange contribution)

$$\int \frac{d^4Q_1}{(2\pi)^4} \int \frac{d^4Q_2}{(2\pi)^4} \frac{A_a}{Q_1^2 Q_2^2 Q_3^2} \ldots = \frac{1}{2\pi^2} \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 dt \sqrt{1 - t^2} \frac{Q_1 Q_2}{Q_3^2} \langle A_a \rangle \ldots,$$

\(^4\) The possible combinations with momentum $P$ are

$$(P \cdot Q_1)^2 = (P \cdot Q_1)(D_1 - Q_1^2)/2, \quad (P \cdot Q_2)^2 = -(P \cdot Q_2)(D_2 - Q_2^2)/2,$$

$$(P \cdot Q_1)(P \cdot Q_2) = -(D_1 - Q_1^2)(D_2 - Q_2^2)/4,$$

$$(P \cdot Q_1) = (D_1 - Q_1^2)/2, \quad (P \cdot Q_2) = -(D_2 - Q_2^2)/2.$$
where the radial variables of integration \( Q_1 \equiv |Q_1| \) and \( Q_2 \equiv |Q_2| \) and the angular variable \( t = (Q_1 \cdot Q_2)/(|Q_1||Q_2|) \) are introduced. The averaged \( A_a \) factors are

\[
\langle A \rangle_1 = \langle \frac{1}{D_1} \rangle = \frac{R_1 - 1}{2m_\mu^2}, \quad \langle A \rangle_2 = \langle \frac{1}{D_2} \rangle = \frac{R_2 - 1}{2m_\mu^2},
\]

\[
\langle A \rangle_3 = \langle \frac{(P \cdot Q_2)}{D_1} \rangle = + (Q_1 \cdot Q_2) \frac{(1 - R_1)^2}{8m_\mu^2},
\]

\[
\langle A \rangle_4 = \langle \frac{(P \cdot Q_1)}{D_2} \rangle = - (Q_1 \cdot Q_2) \frac{(1 - R_2)^2}{8m_\mu^2},
\]

\[
\langle A \rangle_5 = \langle \frac{1}{D_1D_2} \rangle = \frac{1}{m_\mu^2} \frac{Q_1}{Q_2x} \arctan \left[ \frac{zx}{1 - zt} \right],
\]

\[
\langle A \rangle_6 = \langle 1 \rangle = 1,
\]

with

\[
x = \sqrt{1 - t^2}, \quad R_i = \sqrt{1 + \frac{4m_\mu^2}{Q_i^2}} \quad (i = 1, 2),
\]

\[
z = \frac{Q_1Q_2}{4m_\mu^2} (1 - R_1) (1 - R_2).
\]

After averaging the LbL contribution can be represented in the form

\[
a^\text{LbL}_\mu = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \rho^\text{LbL}(Q_1, Q_2),
\]

with the density \( \rho^\text{LbL}(Q_1, Q_2) \) being defined as

\[
\rho^\text{LbL}(Q_1, Q_2) = \frac{Q_1Q_2}{2\pi^2} \sum_{a=1}^6 \int dt \frac{\sqrt{1 - t^2}}{Q_3^2} \langle A_a \rangle \Pi_a.
\]

Thus, the number of momentum integrations in the original expression for (10) is reduced from eight to three. The transformations leading from (10) to (19), are of general nature, independent of the theoretical (model) assumptions on the form of the polarization tensors \( \Pi_a \). In particular, this 3D-representation is common for all hadronic LbL contributions: the pseudoscalar meson exchange contributions [1, 36], the scalar meson exchange contributions [37], and the quark loop contributions discussed in the present work. The next problem to be elaborated is the calculation of \( \rho^\text{HLbL}(Q_1, Q_2) \) in the framework of the model.
III. HADRONIC LIGHT-BY-LIGHT CONTRIBUTION TO $a_\mu$ WITHIN $N_\chi QM$

Let us briefly review the basic facts about the $N_\chi QM$\(^5\). The Lagrangian of the $SU(3)$ nonlocal chiral quark model with $SU(3) \times SU(3)$ symmetry has the form

$$\mathcal{L} = \bar{q}(x)(i\not\!\partial - m_c)q(x) + \frac{G}{2}[J_S^a(x)J_S^a(x) + J_{PS}^a(x)J_{PS}^a(x)]$$

$$- \frac{H}{4} T_{abc}[J_S^a(x)J_S^b(x)J_S^c(x) - 3J_S^a(x)J_{PS}^b(x)J_{PS}^c(x)],$$

where $q(x)$ are the quark fields, $m_c$ ($m_u = m_d \neq m_s$) is the diagonal matrix of the quark current masses, and $G$ and $H$ are the four- and six-quark coupling constants. The nonlocal structure of the model is introduced via the nonlocal quark currents

$$J_M^a(x) = \int d^4x_1 d^4x_2 F(x_1, x_2) \bar{q}(x - x_1) \Gamma_M \not\!\!q(x + x_2),$$

where $M = S$ for the scalar and $M = PS$ for the pseudoscalar channels, $\Gamma_S^a = \lambda^a$, $\Gamma_{PS}^a = i\gamma^5 \lambda^a$, and $F(x_1, x_2)$ is the form factor with the nonlocality parameter $\Lambda$ reflecting the nonlocal properties of the QCD vacuum. The $SU(2)$ version of the $N_\chi QM$ with $SU(2) \times SU(2)$ symmetry is obtained by setting $H$ to zero and taking only scalar-isoscalar and pseudoscalar-isovector currents. Within the $N_\chi QM$, spontaneous breaking of chiral symmetry occurs and the inverse propagator of the dynamical quark takes the form

$$S^{-1}(k) = \hat{k} - m(k^2)$$

where $m(k^2) = m_c + m_D F(k^2, k^2)$ is the dynamical quark propagator obtained by solving the Dyson-Schwinger equation. For numerical estimates two versions of the form factor (in momentum space) are used: the Gaussian form factor

$$F_G(k_E^2, k_E^2) = \exp \left(-k_E^2/\Lambda^2\right),$$

and the Lorentzian form factor

$$F_L(k_E^2, k_E^2) = \frac{1}{(1 + k_E^2/\Lambda^2)^2}.$$  

The second version is used in order to test the stability of the results to the nonlocality shape.

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\(^5\) More detailed information about the model is contained in our previous works [35, 37].
Next, it is necessary to introduce the gauge-invariant interaction with an external photon field $A_\mu$. The path-independent scheme, based on the rules for contour integrals, is used [46]. In our calculations we need to consider the quark-antiquark vertex with one-, two-, three- and four-photons (Fig. 1). The quark-photon vertex has the usual local part as well as the nonlocal piece

$$\Gamma^{(1)}_\mu(q_1) = \gamma_\mu + \Delta\Gamma^{(1)}_\mu(q_1), \quad \Delta\Gamma^{(1)}_\mu(q_1) = -(k + k')_\mu m^{(1)}(k, k').$$

The quark-antiquark vertices with more than one photon insertion are purely nonlocal. Their explicit form and the definition for the finite-difference derivatives $m^{(n)}$ are presented in the Appendix.

With the Feynman rules for the dynamical quark propagator (23) and the quark-photon vertices (26), (33), (34), and (35), the gauge invariant set of diagrams describing the polarization tensor $\Pi_{\mu\nu\lambda\rho}(q_2, -(q_1 + q_2), k + q_1, -k)$ is given in Fig. 2.

FIG. 1: The quark-photon vertex $\Gamma^{(1)}_\mu(q)$, the quark-two-photon vertex $\Gamma^{(2)}_{\mu\nu}(q_1, q_2)$, the quark-three-photon vertex $\Gamma^{(3)}_{\mu\nu\rho}(q_1, q_2, q_3)$, and the quark-four-photon vertex $\Gamma^{(4)}_{\mu\nu\rho\sigma}(q_1, q_2, q_3, q_4)$.

FIG. 2: The box diagram and the diagrams with nonlocal multiphoton interaction vertices that give the contributions to $\Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3)$. The numbers in front of the diagrams are the combinatoric factors.
FIG. 3: The 3D density $\rho(Q_1,Q_2)$ defined in Eq. (20). The density is given in units $10^{-10} \text{ GeV}^{-2}$.

IV. THE RESULTS

For the numerical estimates, the $SU(2)$- and $SU(3)$- versions of the $N_{\chi}QM$ model are used. In order to check the model dependence of the final results, we also perform calculations for different sets of model parameters.

In the $SU(2)$ model, the same scheme of fixing the model parameters as in [36, 37] is applied: fitting the parameters $\Lambda$ and $m_c$ by the physical values of the $\pi^0$ mass and the $\pi^0 \to \gamma\gamma$ decay width, and varying $m_D$ in the region $200 - 350 \text{ MeV}$.

For the $SU(3)$ version of the model, it is necessary to fix two more parameters: the current and dynamical masses of the strange quark. We suggest fixing them by fitting the $K^0$ mass and obtaining more or less reasonable values for the $\eta$ meson mass and the $\eta \to \gamma\gamma$ decay width. The main problem here is that the lowest value for the nonstrange dynamical mass $m_D$ is 240 MeV, because at lower $m_D$ the $\eta$ meson becomes unstable within the model approach. In addition, we use the parameters from [47] for the Gaussian ($G_I-G_{IV}$) and the Lorentzian ($L_I-L_{IV}$) form factors.

The important result, independent of the parameterizations, is the behavior of the density $\rho^{HLbl}(Q_1,Q_2)$, shown in Fig. 3. One can see, that $\rho^{HLbl}(Q_1,Q_2)$ is zero at the edges
FIG. 4: The 2D slice of the density $\rho(Q_1, Q_2)$ at $Q_2 = Q_1$. Different curves correspond to the contributions of topologically different sets of diagrams drawn in Fig. 2. The contribution of the box diagram with the local vertices, Fig. 2a, is the dot (olive) line (Loc); the box diagram, Fig. 2a, with the nonlocal parts of the vertices is the dash (red) line (NL_1); the triangle, Fig. 2b, and loop, Fig. 2c, diagrams with the two-photon vertices is the dash-dot (blue) line (NL_2); the loop with the three-photon vertex, Fig. 2d, is the dot-dot (magenta) line (NL_3); the loop with the four-photon vertex, Fig. 2e, is the dash-dot-dot (green) line (NL_4); the sum of all contributions (Total) is the solid (black) line. At zero all contributions are finite. The density is given in units $10^{-10}$ GeV$^{-2}$.

$(Q_1 = 0$ or $Q_2 = 0)$ and is concentrated in the low-energy region$^6$ ($Q_1 \approx Q_2 \approx 300$ MeV) providing the dominant contribution to $a_{\mu}^{HLLL}$. This behavior at the edges appears to be due to cancellations of contributions from different diagrams of Fig. 2.

In Fig. 4 the slice of $\rho^{HLLL}(Q_1, Q_2)$ in the diagonal direction $Q_2 = Q_1$ is presented together with the partial contributions from the diagrams of different topology. One can see, that the $\rho^{HLLL}(0, 0) = 0$ is due to a nontrivial cancellation of different diagrams of Fig. 2. This important result is a consequence of gauge invariance and the spontaneous violation of the chiral symmetry, and represents the low energy theorem analogous to the theorem for the Adler function at zero momentum. Another interesting feature is, that the large $Q_1$, $Q_2$ behavior is dominated by the box diagram with local vertices and local massive quark propagators in accordance with perturbative theory. All this is very important characteristics of the N\chi QM, interpolating the well-known results of the chiral perturbative theory at low momenta and the operator product expansion at large momenta. Earlier, similar results

$^6$ One should point out that the density for the mesonic exchanges has similar behavior.
were obtained for the two-point \cite{31,32} and three-point \cite{33} correlators.

The numerical results for the value of $a_{\mu}^{HLbL}$ are presented in Figs. 5, 6 for the $SU(2)$ and $SU(3)$ models. The estimates for the partial contributions to $a_{\mu}^{HLbL}$ (in $10^{-10}$) are the $\pi^0$ contribution $5.01(0.37)$ \cite{36}, the sum of the contributions from $\pi^0$, $\eta$ and $\eta'$ mesons $5.85(0.87)$ \cite{36}, the scalar $\sigma$, $a_0(980)$ and $f_0(980)$ mesons contribution $0.34(0.48)$ \cite{3,37}, and the quark loop contribution is $11.0(0.9)$ \cite{3}. The total contribution obtained in the leading order in the $1/N_c$ expansion is (see also \cite{3})

$$a_{\mu}^{HLbL,NQM} = 16.8(1.25) \cdot 10^{-10}. \quad (28)$$

The error bar accounts for the spread of the results depending on the model parameterizations. Comparing with other model calculations, we conclude that our results are quite close to the recent results obtained in \cite{30,38}.

V. CONCLUSIONS

In this paper, we have presented the results for the contribution of the dynamical quark loop mechanism for the light-by-light scattering to the muon anomalous magnetic moment within the nonlocal chiral quark model. In previous works \cite{3,36,37}, we calculated the corresponding contributions due to the exchange by pseudoscalar and scalar mesons. The basis of our model calculations is the spontaneous violation of the chiral symmetry in the
FIG. 6: The results for $a_{\mu}^{\text{HLbL}}$: the red dash line corresponds to the $SU(2)$-result, the black solid line is the $SU(3)$-result, the crosses correspond to the $G_{I}-G_{IV}$ set of model parameters, and the pluses to the $L_{I}-L_{IV}$ set of parameters for the $SU(3)$ model taken from [47]. In order to extrapolate the $SU(3)$-result to lower quark masses, we find that the maximal value of the difference between the total $SU(2)$- and $SU(3)$-results is 0.77. We add this number to the value of the $SU(2)$-result at 200 MeV. The horizontal dotted lines denote the lowest 15.57 (set $G_{III}$) and the highest 18.07 ($SU(2)$ model value 17.30 plus 0.77) values for the total $a_{\mu}^{\text{HLbL}}$ contribution. The black dash straight line is the extrapolation between the $SU(3)$-model endpoint result at 240 MeV and the highest value at 200 MeV.

model with the nonlocal four-fermion interaction and abelian gauge invariance. The first leads to the generation of the momentum-dependent dynamical quark mass, and the latter ensures the fulfillment of the Ward-Takahashi identities with respect to the quark-photon interaction.

In the present work, we derived the general expression for $a_{\mu}^{\text{HLbL}}$ as the three-dimensional integral in modulus of the two photon momenta and the angle between them. The integral is the convolution of the known kinematical factors and some projections of the four-photon polarization tensor. The latter is the subject of theoretical calculations.

Since our model calculations of the hadronic contributions are basically numerical, it is more convenient to present our results in terms of the density function $\rho^{\text{HLbL}}(Q_{1}, Q_{2})$. We observe some properties of this function that have model-independent character. Firstly, at zero momenta one has $\rho^{\text{HLbL}}(0, 0) = 0$ in spite of the fact that the partial contributions of
different diagrams are nonzero in this limit. This low-energy theorem is a direct consequence of the quark-photon gauge invariance and the spontaneous violation of the chiral symmetry. Secondly, at high momenta the density is saturated by the contribution from the box diagram with the local quark-photon vertices and local quark propagators in accordance with the perturbative theory. This is a consequence of the fact, that at small distances all non-perturbative nonlocal effects are washed out. Thirdly, with the model parameters chosen, the $\rho^{\text{HLbl}}(Q_1, Q_2)$ is concentrated in the region $Q_1 \approx Q_2 \approx 300$ MeV, which is a typical scale for light hadrons.

Summarizing the results of the present and previous works \cite{3, 36, 37}, we get the total hadronic contribution to $a^\text{HLbl}_\mu$ within the N$\chi$QM in the leading order in the $1/N_c$ expansion. The total result is given in Eq. (28). To estimate the uncertainty of this result, we vary some of the model parameters in physically reasonable interval and also study the sensitivity of the result with respect to different model parameterizations. If we add the result (28) to all other known contributions of the standard model to $a_\mu$, (3)-(7), we get that the difference between experiment (2) and theory is

$$a^\text{BNL,CODATA}_\mu - a^\text{SM}_\mu = 18.73 \times 10^{-10},$$

(29)

which corresponds to 2.43$\sigma$. If one uses the hadronic vacuum polarization contribution from the $\tau$ hadronic decays instead of $e^+e^-$ data

$$a^\text{HVP,LO-}\tau_\mu = 701.5(4.7) \times 10^{-10},$$

(30)

the difference decreases to $18.44 \times 10^{-10}$ (2.23$\sigma$) for the case of $a^\text{HLbl}_\mu$ from (8) \cite{15} and to $12.14 \times 10^{-10}$ (1.53$\sigma$) in our model (28).

Clearly, a further reduction of both the experimental and theoretical uncertainties is necessary. On the theoretical side, the calculation of the still badly known hadronic light-by-light contributions in the next-to-leading order in the $1/N_c$ expansion is the next goal. Work in this direction is now in progress, and we hope to report its results in the near future.

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VI. APPENDIX

Let us introduce the finite-difference derivatives

\[ f^{(1)}(a, b) = \frac{f(a + b) - f(b)}{(a + b)^2 - b^2}, \]

\[ f^{(n+1)}(a, \{b_i\}, b_1, b_2) = \frac{f^{(n)}(a, \{b_i\}, b_1) - f^{(n)}(a, \{b_i\}, b_2)}{(a + b_1)^2 - (a + b_2)^2}, \quad n = 1, 2, ... \]

Then, the quark-antiquark vertex with the two-photon insertions (Fig. 1b) is

\[ \Gamma^{(2)}_{\mu\nu}(q_1, q_2) = 2g_{\mu\nu}m^{(1)}(k, k') + \]

\[ (k + k_1)_\mu(k_1 + k')_\nu m^{(2)}(k, k_1, k') + \]

\[ (k + k_2)_\nu(k_2 + k')_\mu m^{(2)}(k, k_2, k'). \]

Here and below, \( k \) is the momentum of the incoming quark, \( k' \) is the momentum of the outgoing quark, \( q_i \) are the momenta of the incoming photons, and \( k_1 = k + q_1, k_{ij...k} = k + q_i + q_j + ... + q_k \).

The quark-three-photon vertex (Fig. 1c) is

\[ \Gamma^{(3)}_{\mu\nu\rho}(q_1, q_2, q_3) = -[2g_{\mu\nu}((k_{12} + k')_\rho m^{(2)}(k, k_{12}, k') + (k + k_3)_\rho m^{(2)}(k, k_3, k')) \]

\[ + (k + k_1)_\mu(k_{12} + k')_\nu(k_{12} + k')_\rho m^{(3)}(k, k_{12}, k') \]

\[ + (k + k_1)_\mu(k_{13} + k')_\nu(k_{13} + k')_\rho m^{(3)}(k, k_{13}, k') ] \]

\[ + [1 \rightleftharpoons 3, \mu \rightleftharpoons \rho] + [2 \rightleftharpoons 3, \nu \rightleftharpoons \rho]. \]
The quark-four-photon vertex (Fig. [1]) takes the form

\[
\Gamma^{(4)}_{\mu\nu\rho\tau}(q_1, q_2, q_3, q_4) = \left[ + 4g_{\mu\rho}g_{\nu\tau} \left( m^{(2)}(k, k_{12}, k') + m^{(2)}(k, k_{34}, k') \right) \\
+ 2g_{\mu\nu}(k + k_3)\rho(k_3 + k_{34})\tau m^{(3)}(k, k_{34}, k') \\
+ (k + k_3)\rho(k_{123} + k')\tau m^{(3)}(k, k_{123}, k') \\
+ (k_{12} + k_{123})\rho(k_{123} + k')\tau m^{(3)}(k, k_{12}, k_{123}, k') \\
+ (k_{124} + k')\rho(k_{124} + k_{124})\tau m^{(3)}(k, k_{12}, k_{124}, k') \\
+ (k_{124} + k')\rho(k_{124} + k_{124})\tau m^{(3)}(k, k_{12}, k_{124}, k') \\
+ (k_{124} + k')\rho(k_{124} + k_{124})\tau m^{(3)}(k, k_{12}, k_{124}, k') \\
+ (k_{124} + k')\rho(k_{124} + k_{124})\tau m^{(3)}(k, k_{12}, k_{124}, k') \\
\right] + \left[ 2 \equiv 4, \nu \equiv \tau \right] + \left[ 2 \equiv 3, \nu \equiv \rho \right].
\]

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