EQUIVARIANT MIRROR SYMMETRY FOR THE WEIGHTED PROJECTIVE LINE

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Abstract. In this paper, we establish equivariant mirror symmetry for the weighted projective line. This extends the results by B. Fang, C.C. Liu and Z. Zong, where the projective line was considered [Geometry & Topology 24:2049-2092, 2017]. More precisely, we prove the equivalence of the $R$-matrices for A-model and B-model at large radius limit, and establish isomorphism for $R$-matrices for general radius. We further demonstrate that the graph sum of higher genus cases are the same for both models, hence establish equivariant mirror symmetry for the weighted projective line.

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1. Introduction

In mirror symmetry for toric varieties, one aims at establishing equivalence between A-model invariants and the Ginzburg-Landau B-model invariants for a given toric variety.

On the A-model side, there are extensive studies for the equivariant Gromov-Witten theory. In [18], A.B. Givental computed all genus descendent invariants of equivariant Gromov-Witten theory of tori action with isolated fixed points for the smooth case. The process of recovering higher genus data is now known as Givental’s formula. In [23], Z. Zong gave all genus equivariant
Gromov-Witten invariants for GKM orbifolds by generalizing Givental’s formula to the orbifold case.

On the B-model side, B. Eynard and N. Orantin discovered a way to compute the topological expansion of matrix integrals in [12]. Eynard-Orantin’s topological recursion is related to Givental’s formula [11].

More recently, B. Fang, C.C. Liu and Z. Zong established the equivariant mirror symmetry for the projective line. They directly computed the $R$-matrices of both A and B-models, and applied Givental’s formula and Eynard-Orantin’s recursion, thereby proved the equivariant mirror symmetry for the projective line by calculating graph sums [14].

In this paper, we extend the equivariant mirror symmetry to the weighted projective line. First, we associate the equivariant Gromov-Witten invariants of the weighted projective line to the Eynard-Orantin invariants of the affine curve determined by the superpotential of its $T$-equivariant Landau-Ginzburg mirror. It is proved by calculating the graph sum and applying the main results in [12][17][18]. We use the equivalence of $R$-matrices in both models, which is established by computations with quantum Riemann-Roch, and integration on Lefschetz thimble at the large radius limit [22].

Secondly, we establish a precise correspondence between A-model genus $g, N$ point descendent equivariant Gromov-Witten invariants, and the Laplace transform of B-model Eynard-Orantin invariant along Lefschetz thimbles.

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2. THE FROBENIUS MANIFOLD

We assume $m$ and $n$ are coprime throughout this paper, i.e., $\mathbb{P}(m,n)$ has trivial generic stabilizer. We use bold form quantities $\mathbf{w}_\ell, \mathbf{w}_{-\ell}, \mathbf{p}, \mathbf{q}_\ell, \mathbf{q}_0, \mathbf{q}_{-\ell}$ to represent cohomology classes and normal form quantities $w_1, w_2, p, q_\ell, q_0, q_{-\ell}$ to represent complex numbers.

A weighted projective line $\mathbb{P}(m,n)$ is given by the fan below by standard toric construction:

\[ \begin{array}{ccc} \cdots & \cdots & \cdots \\
-m & n & \cdots \\
\end{array} \]

i.e., $\mathbb{P}(m,n) = (\mathbb{C}^2 \setminus \{(0,0)\})/\sim$, where $\sim$ is defined by $(z_1, z_2) \sim (\lambda^n z_1, \lambda^m z_2)$. A point in $\mathbb{P}(m,n)$ is given by homogeneous coordinates $[z_1 : z_2]$. Let $T = (\mathbb{C}^*)^2$ act on $\mathbb{P}(m,n)$ by $(t_1, t_2) \cdot [z_1, z_2] = [t_1 z_1, t_2 z_2]$.

2.1. Jacobian ring.

For $y \in \mathbb{C}$, let $Y = e^y \in \mathbb{C}^*$. Define the equivariant superpotential $W_T : \mathbb{C}^* \to \mathbb{C}$ by

\[ W_T(Y) = Y^m + \sum_{\ell=1}^{m-1} \tilde{q}_\ell Y^{\ell} + \sum_{\ell=1}^{n} \tilde{q}_{-\ell} Y^{-\ell} + \tilde{w}_m \log(Y^m) + \sum_{\ell=1}^{m-1} \tilde{w}_\ell \log(\tilde{q}_\ell Y^\ell) + \sum_{\ell=1}^{n} \tilde{w}_{-\ell} \log(\tilde{q}_{-\ell} Y^{-\ell}). \]

Let $x = W_T(e^y)$. The Jacobian ring of $W_T$ is

\[ \text{Jac}(W_T) = \mathbb{C}[w][Y]/\left< \frac{\partial W_T}{\partial y} \right> = \mathbb{C}[w][Y]/\left< mY^m + \sum_{\ell=1}^{m-1} \ell \tilde{q}_\ell Y^{\ell} - \sum_{\ell=1}^{n} \ell \tilde{q}_{-\ell} Y^{-\ell} + \tilde{p} \right>, \]

where $\tilde{p} = \sum_{\ell=1}^{m} \ell \tilde{w}_\ell - \sum_{\ell=1}^{n} \ell \tilde{w}_{-\ell}$.

Let $\{p_\alpha^\circ\}$ be the set of critical points of $W_T$. Define residual pairing $(f, g)$ on $\text{Jac}(W_T)$ by

\[ (f, g) = \sum_\alpha \text{Res}_{p_\alpha^\circ} \frac{f(Y)g(Y)}{(\partial W_T/\partial y)} \mathrm{d}Y. \]
2.2. Equivariant Gromov-Witten potentials.

The Chen-Ruan cohomology for \( WP(m,n) \) is \( H^*_{CR}(WP(m,n)) = \mathbb{C}\langle X, \tilde{X}\rangle / \langle X \tilde{X}, mX^n - n\tilde{X}^m \rangle \). For more details, please refer to [1] and the original papers [3, 4, 5].

Let \( \mathbb{C}[w] = \mathbb{C}[w_0, \ldots, w_{m-1}, w_1, \ldots, w_m] \) be the equivariant cohomology ring of a point with \( T = T^{m+n} \). Example 105 in [21] gives the equivariant Chen-Ruan cohomology \( H^*_{CR}(WP(m,n)) \equiv \mathbb{C}[w]/\langle X \tilde{X}, mX^n - n\tilde{X}^m + p \rangle \), with \( p = \sum_{\ell=1}^m (w_\ell - \sum_{\ell=1}^n (w_\ell - \ell) \).

Suppose that \( d > 0 \) or \( 2g - 2 + N > 0 \), then \( M_{g,n}(WP(m,n), d) \) is well defined. Let \( L_i \) be the i-th tautological bundle, whose restriction to a point \( [\Sigma, x_1, \ldots, x_n] \in M_{g,n}(WP(m,n), d) \) is isomorphic to \( T^*\Sigma \). Let \( \psi_i = c_1(L_i) \). Let \( ev : M_{g,n}(WP(m,n), d) \to WP(m,n) \) be the evaluation at the i-th point. For \( \gamma_1, \ldots, \gamma_N \in H^*(WP(m,n), \mathbb{C}), a_1, \ldots, a_N \in \mathbb{Z}_{\geq 0} \), define the orbifold descendant Gromov-Witten invariants as

\[
\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_N}(\gamma_N) \rangle_{WP(m,n), T} = \int_{[M_{g,n}(WP(m,n), d)]} \prod_{j=1}^N \psi_j^{a_j} \cdot ev_j^*(\gamma_j) \in \mathbb{C}[w].
\]

For \( 2g - 2 + M + N > 0 \) and given \( \gamma_1, \ldots, \gamma_M + N \in H^*(WP(m,n)) \), define

\[
\langle \gamma_1, \ldots, \gamma_N \rangle_{WP(m,n), T}^{g,M,N} = \sum_{a_1, \ldots, a_N \geq 0} \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_N}(\gamma_N) \rangle_{WP(m,n), T}^{g,M,N}. \]

This is actually the formal expansion of \( \sum_i \frac{x_i^{-1}}{z_i - 1} \) at \( z_i = 0 \). In the unstable case \( g = 0, d = 0, M + N = 1 \) or 2, we define

\[
\langle \gamma_1, \ldots, \gamma_N \rangle_{WP(m,n), T}^{g,M,N} = \frac{1}{z_1 + z_2} \int_{WP(m,n)} \gamma_1 \cup \gamma_2.
\]

Let \( t = \sum_i t_i T_i \), where \( \{T_i\} \) form a basis of \( H^*(WP(m,n), \mathbb{C}) \). Suppose that \( 2g - 2 + N + M > 0 \) or \( N > 0 \). Given \( \gamma_1, \ldots, \gamma_{M+N} \in H^*(WP(m,n)) \), we define

\[
\langle \gamma_1, \ldots, \gamma_N \rangle_{WP(m,n), T}^{g,M+N} = \sum_{a_1 \geq 0} \left( \frac{Q^d}{a!} \int_{WP(m,n)} \gamma_{N}^{a_1} \cdots \gamma_{N}^{a_{N}} \right). \]

Let

\[
u_j = \sum_{a \geq 0} (u_j)^a \langle a \rangle,
\]

\[
F_{g,N}^{WP(m,n), T}(u_1 \cdots u_N, t) = \sum_{a_1, \ldots, a_N \geq 0} \langle \tau_{a_1}(u_1) \cdots \tau_{a_N}(u_N) \rangle_{WP(m,n), T}^{g,M+N}. \]

For fixed \( M, N \in \mathbb{Z}_{\geq 0} \), consider \( \pi : \tilde{M}_{g,n+M}(WP(m,n), d) \to M_{g,n} \) which forgets the target and the last \( M \) marked points and stabilizes it. Let \( \tilde{L}_i \) be the pull-back of \( L_i \) along \( \pi \). Let \( \psi_i = \pi^*(\psi_i) = c_1(L_i) \) be the pull-back of \( \psi \)-classes in \( \tilde{M}_{g,n} \). Define

\[
F_{g,n}^{WP(m,n), T}(u_1 \cdots u_N, t) = \sum_{M,d,a_1,\ldots,a_N \geq 0} \frac{Q^d}{M!N!} \int_{[\tilde{M}_{g,n+M}(WP(m,n), d)]} \prod_{j=1}^N ev_j^*(u_j) \psi_j^{a_j} \prod_{i=1}^M ev_i^*(t).
\]
Let $F_{g,N}^{\WP(m,n),T}(u, t), \bar{F}_{g,N}^{\WP(m,n),T}(u, t)$ be such that all $u_j = u$.

We define the total descendent potential and the ancestor potential of $\WP(m, n)$ as follows, where $\hbar$ is an arbitrary parameter:

$$D_{\WP(m,n),T}(u) = \exp \left( \sum_{N,g} \hbar^{g-1} F_{g,N}(u, 0) \right),$$

$$A_{\WP(m,n),T}(u, t) = \exp \left( \sum_{N,g} \hbar^{g-1} \bar{F}_{g,N}(u, t) \right).$$

In fact, by Givental’s formula [17], we have

$$D_{\WP(m,n),T}(u) = \exp(F_{1,\WP(m,n),T}) \tilde{\delta}^{-1} A_{\WP(m,n),T}(u, t),$$

where $F_{1,\WP(m,n),T} = \sum_{N} F_{1,N}^{\WP(m,n),T}$. We shall give an equivalent graph sum formula in 3.2.

2.3. Quantum cohomology.

Following Iritani [19], we first compute the non-equivariant quantum cohomology ring $QH^*(\WP(m, n))$ with all twisted sectors added.

First we give a generalized definition of toric varieties. A toric variety is constructed from the following data:

- an $r$-dimensional algebraic torus $T \cong (\mathbb{C}^*)^r$. Set $N = \text{Hom}(\mathbb{C}^*, T)$;
- $M$ elements $D_i \in M = \text{Hom}(T, \mathbb{C}^*)$, such that $M \otimes \mathbb{R} = \sum \mathbb{R} \cdot D_i$;
- a vector $\eta \in M \otimes \mathbb{R}$ (that defines the stability condition).

The elements $\{D_i\}$ define a homomorphism $T \rightarrow (\mathbb{C}^*)^M$. Let $T$ act on $\mathbb{C}^M$ via this homomorphism. Set $A = \{I : \sum_{i \in I} \mathbb{R}_{>0} D_i \ni \eta\}, U_0 = \mathbb{C}^M \setminus \bigcup_{i \in A} C_i$. A toric orbifold is defined by the quotient stack $X = [U_0/T]$. Note that by setting $\eta = 0$ we obtain the original definition for toric varieties. Let $I_0 = \{i : I_0 \ni I \in A\}$. Let $\mathbb{D}_i$ be the image of $D_i$ in $M/\sum_{i \in I_0} \mathbb{Z} D_i \cong H^2_{CR}(X, \mathbb{Z})$.

Choose an integral basis $\{e_a\}$ of $M$. Assume that some elements in $\{e_a\}$ generates $\sum_{i \in I_0} \mathbb{R}_{>0} D_i$. Let $\bar{e}_a$ be the image of $e_a$ in $H^2(X, \mathbb{R})$. Let $m_{ia}$ be a matrix such that $D_i = \sum_a m_{ia} e_a$, where $m_{ia} \in \mathbb{Z}$. Note that $\mathbb{D}_i = \sum_a m_{ia} \bar{e}_a$, and that $\mathbb{D}_i = 0$ if $i \in I_0$. Let $k_{\text{eff}} = \{d \in \mathbb{N} \otimes \mathbb{Q} : i \cdot \langle D_i, d \rangle \in \mathbb{Z}_{\geq 0} \} \in A\}$. Let $q_a$ be coordinates on $\text{Hom}(\mathbb{N}, \mathbb{C}^*)$ with respect to the dual basis of $\{e_a\}$.

**Definition 2.1.** The $I$-function of $X$ is an $H^*_{CR}(X)$-valued power series defined by

$$I(q, z) = e^{\langle \sum_{a} e_a \log q_a \rangle / z} \sum_{d \in k_{\text{eff}}} q^d \cdot \prod_{\nu : i : \langle D_i, d \rangle \leq \nu < 0} (\mathbb{D}_i + \langle \langle D_i, d \rangle - \nu \rangle z)^{-1},$$

where $q^d = \prod_a q_a^{d_a}, d_a = \langle d, e_a \rangle$.

Let

$$\partial_a = q_a \frac{\partial}{\partial q_a}, D_i = \sum_a m_{ia} \cdot z \partial_a,$$

$$P_d = q^d \prod_{\nu : i : \langle D_i, d \rangle > \nu \geq 0} (D_i - \nu z) - \prod_{\nu : i : \langle D_i, d \rangle > \nu \geq 0} (D_i - \nu z).$$

By direct calculations (Lemma 4.6 in [19]) we have:

**Lemma 2.1.** It holds that $P_d(I(q, z)) = 0, \forall d \in \mathbb{N}$.

Let $\text{Eff}_X \subseteq H_2(X, \mathbb{Z})$ be the semigroup generated by effective stable maps. For an arbitrary $\tau \in H^*_C(X)$, let $\tau_{0,2}$ be the component of $\tau$ consisting of terms of degree 2, and $\tau' = \tau - \tau_{0,2}$. Choose an arbitrary basis $\{H_a\}$ of $H^*_C(X)$ as a $\mathbb{C}$-module, and $\{H^a\}$ its dual basis. Then $H^a$ could be regarded as in $H^*_C(X)$ by applying the Poincare duality.
Definition 2.2. The J-function

\[ J(τ, z) = e^{iτ, z/τ} (1 + \sum_{d} \frac{1}{d!} \sum_{α, β} \frac{1}{α, β} H^α \cdot J(τ, z) \cdot H^β). \]

Let \( I(q, z) = 1 + \frac{r(q)}{r} + o(q^{-1}) \) be the expansion of \( I \) with respect to \( z \) at \( z = \infty \). Here \( τ \) is called the mirror map.

Theorem 2.1 (Mirror Theorem). It holds that \( I(q, z) = J(τ(q), z) \).

Mirror theorem for weighted projective space was proved in [9]. General case was proved in [8].

On the other hand, consider the Frobenius algebra \( V = QH^*(\mathbb{P}(m, n)) \cong \text{Jac}(\mathbb{P}(m, n)) \).

Let \( H_a \) (which coincides with the \( H_a \) above in our case) be its basis as a \( \mathbb{C} \) vector space. We identify \( V \) and its tangent space \( T_p V \) for \( p \) a semisimple point on \( V \). Define quantum connection \( \nabla_a = z\partial_a - H_a \). Consider differential equations system: (Quantum Differential Equation, QDE)

\[ \nabla_a h = 0, \quad α = 1, \ldots, \dim V. \]

Let \( S_a = \sum_a \left( \left( H_a, \frac{H_a}{k} \right) \right)^{m} H^a \). This forms a set of fundamental solutions to the QDE.

Let \( (α, β) \) be the Poincaré paring \( \{α, β\} = \left( \left( H_a, \frac{H_a}{k} \right) \right)^{m} = (S_α, 1) \) by the duality of \( H_a \) and \( H^a \). Inducting on \( k \) with the Leibniz rule, we have \( \sum_i \left( z\partial_i - H_i \right) J(τ, z) = (H_{i_1}, \ldots, H_{i_s}, 1) \).

By \( P_d I = 0, I = J \), we know that for arbitrary \( α \),

\[ 0 = (P_d J, H_α) = ((q^d \prod_{i, v \cdot (d, d) > v > 0} \sum_{m} m_i - z\partial_a - \nu) - \sum_{i, v \cdot (d, d) > v > 0} \sum_{m} m_i - z\partial_a - \nu) J, H_α). \]

This implies for all \( d \),

\[ q^d \prod_{i, v \cdot (d, d) \leq 0} \sum_{m} m_i = \alpha (D, d) - \sum_{i, v \cdot (d, d) \geq 0} \sum_{m} m_i, H^a = 0. \]

Take \( d = e_α \), let \( X = (\sum_{j=0}^{n-1} (m-j)H_j)^{\frac{1}{n}}, X^m = (\sum_{j=0}^{n-1} (n-j)H_j)^{\frac{1}{m}}, \) then the above relations give \( H_τ = \frac{1}{m} \frac{1}{q^d X^m}, \ X = \frac{1}{m} \frac{1}{q^d X^m}, \ H_{-τ} = \frac{1}{m} \frac{1}{q^d X^m}, \ X^{m-1}. \)

So we obtain

\[ \mathbb{C}[H_{m}, \ldots, H_{m-1}]/\left( H_{m} - \frac{1}{m} q^{d} X^{m-1}, H_{m-1} - \frac{1}{m} q^{d} X^{m-1} \right) \cong \mathbb{C}[X_0, \bar{X}]/\left( X_0 - q^{d} X^{m-1}, q^{d} X^{m-1} - \sum_{m} (m-ℓ)q^{d} q^{\frac{1}{m}} X^{m-1} - \sum_{m} (n-ℓ)q^{d} q^{\frac{1}{m}} X^{m-1} \right). \]

By dimension argument, this is isomorphic to \( QH^*(\mathbb{P}(m, n)) \).
2.4. Equivariant quantum cohomology.

In our case, we may recover the equivariant quantum cohomology ring from the equivariant cohomology ring and the quantum cohomology ring. As a $\mathbb{C}[w]$-module, $QH^*_T(W\mathbb{P}(m,n), \mathbb{C}) \cong H_T^*(W\mathbb{P}(m,n), \mathbb{C})$. The product structure is given by $(\gamma_1, \gamma_2, \gamma_3) = \langle \gamma_1, \gamma_2, \gamma_3 \rangle_{0,3}(\mathbb{P}(m,n), T)$. With a slight abuse of notation, let $X, \tilde{X}$ be the image of $X$ and $\tilde{X}$ in $QH^*_T(W\mathbb{P}(m,n))$ in the $\mathbb{C}$-vector space isomorphism $QH^*(W\mathbb{P}(m,n)) \cong QH^*_T(W\mathbb{P}(m,n))$ (after regarding equivariant parameters as complex numbers).

Proposition 2.1. It holds that $QH^*_T(W\mathbb{P}(m,n)) \cong \mathbb{C}[X, \tilde{X}, p]/I(\cong \mathbb{C}[X, \tilde{X}]/I)$, where $I = \left\langle X \tilde{X} - q_0^{\frac{1}{m+n}}, mX^m-n\tilde{X}^n-\left(-p+\sum_{\ell=1}^{n-1}(n-\ell)q_0^{\frac{1}{m+n}}\tilde{X}^n-\sum_{\ell=1}^{m-1}(m-\ell)q_0^{\frac{1}{m+n}}X^{(m-\ell)}\right)\right\rangle$.

Proof. We prove by degree arguments.

First we calculate $X \times \tilde{X}$, where $*$ stands for the multiplication in $QH^*_T(W\mathbb{P}(m,n))$. We have $X \times \tilde{X} = \sum_{\alpha, \beta} \langle X, \tilde{X}, \gamma_\alpha \rangle \cdot \gamma_\alpha q^\beta$, where $\gamma_\alpha$ is the basis of $H^*_{CR}(W\mathbb{P}(m,n))$, and $\gamma_\alpha$ is its dual basis in $H^*_T(W\mathbb{P}(m,n))$. By the compactness of $W\mathbb{P}(m,n)$, we know that $\gamma_\alpha$ can be chosen in $H^*_T(W\mathbb{P}(m,n))$. Hence $X \times \tilde{X}$ does not contain equivariant parameters. So by taking non-equivariant limit, it holds that $X \times \tilde{X} = q_0^{\frac{1}{m+n}}$.

Next we calculate $mX^m-n\tilde{X}^n$, where $\alpha^k$ means the $k$-th power of $\alpha$ in $QH^*_T(W\mathbb{P}(m,n))$ for $k \in \mathbb{Z}_{\geq0}$ and $\alpha \in QH^*_T(W\mathbb{P}(m,n))$. Since $mX^m-n\tilde{X}^n \in QH^*_T(W\mathbb{P}(m,n))$, we can write it as

$mX^m-n\tilde{X}^n = \varphi(X, \tilde{X}, p = \sum_{\ell=1}^{m} \ell w_{\ell}, q = (q_{-n+1}, \ldots, q_{m-1}), q = (q_{n-1}, \ldots, q_{q_{m-1}}))\frac{1}{m}$.

By taking large radius limit and non-equivariant limit, we have

$\varphi(X, \tilde{X}, p, 0) = -p$.

Next we calculate $mX^m-n\tilde{X}^n = \sum_{\ell=1}^{n-1}(n-\ell)q_{-\ell}X^{(n-\ell)}-\sum_{\ell=1}^{m-1}(m-\ell)q_{\ell}X^{(m-\ell)}$.

By dimension argument, we drop the superscript $*$ and obtain the desired result.

Proposition 2.2. $QH^*_T(W\mathbb{P}(m,n)) \cong \text{Jac}(W_T)$ as Frobenius algebras.

Proof. Note that this could be proved by applying equivariant mirror theorem [8] or [10]. However we include a proof which recovers the pairing from its non-equivariant limit as usual.

We identify $\hat{q}_\ell$ with $q_\ell^{\frac{1}{m+n}}$ (for $\ell = 1, \ldots, m-1$), $\tilde{q}_{-\ell}$ with $q_{-n+\ell}^{\frac{1}{m+n}}$ (for $\ell = 1, \ldots, n$), $p$ with $\tilde{p}$, and $Y$ with $X$. 

Taking non-equivariant limit and applying mirror theorem, we have the isomorphism of non-equivariant limits $Q^*H(W\mathbb{P}(m,n)) \cong \text{Jac}(W)$ as Frobenius algebras. Here the non-equivariant limit is given by $w_i \to 0$ and $\tilde{w}_i \to 0$. We now prove that $w_i$ and $\tilde{w}_i$ affect neither the residue pairing nor the Poincaré pairing with 1, which directly leads to the result.

Let $(\cdot, \cdot)$ denote the non-equivariant pairing, $(\cdot, \cdot)_T$ denote the equivariant pairing. For an representative element $g(t) \in \oplus_{j=-n+1}^n C t^j$, we have the following results:

On the A-side,

$$ (g(X), 1) = \int_{\wedge X} X = \int_{\wedge X^2} g(X) = (g(X), 1)_T; $$

On the B-side, let $f(Y) = \partial W_T/\partial y$. By the residue formula,

$$ (g(Y), 1) = -Re s_{\infty} \frac{g(Y)}{f(Y)} \frac{dY}{Y} = \frac{1}{2\pi i} \lim_{R \to \infty} \int_0^{2\pi} \frac{g(Re^{i\theta})}{f(Re^{i\theta})} d\theta. $$

Hence we have

$$ (g(Y), 1) - (g(Y), 1)_T = \frac{1}{2\pi i} \lim_{R \to \infty} \int_0^{2\pi} \frac{g(Re^{i\theta}) \cdot p R^m e^{in\theta}}{f(Re^{i\theta})(f(Re^{i\theta}) + p R^m e^{in\theta})} d\theta = 0. $$

And as a result $(g_1(Y), g_2(Y))_T = (g_1(X), g_2(X))_T$. □

2.5. Basis.

Take $z_i (i = 0, \ldots, m + n - 1)$ to be the roots of $m z^m + \sum_{\ell = 1}^{m-1} \ell q_{-\ell} z^\ell - \sum_{\ell = 1}^n \ell q_{-\ell} z^{-\ell} + \tilde{p} = 0$ with respect to $z$. Assume that $z_i$ are distinct. Let $\phi_i = \prod_{j \neq i} \frac{z - z_j}{z_i - z_j}$. Then $\phi_i$ is a canonical basis, i.e., $\phi_i \cdot \phi_j = \delta_{ij} \phi_i$.

**Lemma 2.2.** Let $z_i$ be all roots of $f(z)$. Assume $z_i$’s are distinct. Then $\phi_j = \prod_{i \neq j} \frac{z - z_i}{z_j - z_i}$ is a representative of the canonical basis of $\mathbb{C}[z]/\langle f(z) \rangle$.

**Proof.** Observing that the constructed $\phi_i$ is characterized by $\phi_i(z_j) = \delta_{ij}$, we obtain the lemma directly from the Lagrange interpolation formula. □

By direct calculations, we get $\Delta^{\alpha} = \frac{1}{(\phi_{\alpha}, \phi_{\alpha})} = \frac{m \prod_{\ell \leq m} (z_n - z_{\ell})}{z_n^{m+1}}$. Now consider several different bases for $Q^*H(T(W\mathbb{P}(m,n), \mathbb{C})$:

- The natural basis $T_i = X^i$ and its dual basis $T^i$ with which $(T^i, T_j) = \delta^{ij}$.
- The canonical basis $\phi_i$ as defined above and its dual basis $\phi^i = \Delta^{\ell}(q) \phi_i$.
- The normalized canonical basis $\hat{\phi}_i = \sqrt{\Delta(q)} \cdot \phi_i$, and its dual basis $\hat{\phi}^i = \hat{\phi}_i$.

Regarding $\phi_i$ as a function of $q$, we often write it as $\phi_i(q)$, (same for $\hat{\phi}_i$, $\phi^i$ and $\hat{\phi}^i$). For an arbitrary point $pt \in Q^*H(T(W\mathbb{P}(m,n), \mathbb{C})$, let $t^i, u^i, \bar{u}^i$ be coordinates such that

$$ pt = \sum t^i T_i = \sum u^i \phi_i(q) = \sum \bar{u}^i \phi_i(0). $$

Regarding $t^i, u^i, \bar{u}^i$ as functions of $q$, we often write them as $t^i(q), u^i(q), \bar{u}^i(q)$. We often call $t^i(q)$ and $\bar{u}^i(q)$ as flat coordinates, and $u^i(q)$ as canonical coordinates.

3. Graph Sum Formula and the $R$-matrix

We first demonstrate graph sum formulas for A-model and B-model. These subsections follow [15].

3.1. Graph sum formula.

Given a connected graph $\Gamma$, we introduce the following notations:

- Let $V(\Gamma)$ denote the set of vertices in $\Gamma$.
- Let $E(\Gamma)$ denote the set of edges in $\Gamma$.
- Let $H(\Gamma)$ denote the set of half edges in $\Gamma$.
- Let $L^0(\Gamma)$ denote the set of ordinary leaves in $\Gamma$.
- Let $L^1(\Gamma)$ denote the set of dilation leaves in $\Gamma$. 

We now prove that $w_i$ and $\tilde{w}_i$ affect neither the residue pairing nor the Poincaré pairing with 1, which directly leads to the result.
By a half edge we mean either a leaf or an edge, together with a choice of one of the two vertices that it is attached to. Note that the order of two vertices attached to an edge does not affect the graph sum formula in this paper. With the above notations, we introduce the following labels:

- Genus \( g : V(\Gamma) \rightarrow \mathbb{Z}_{\geq 0} \);
- Marking \( \beta : V(\Gamma) \rightarrow \{1, \cdots, m+n\} \);
- Height \( k : H(\Gamma) \rightarrow \mathbb{Z}_{\geq 0} \).

Note that the marking on \( V(\Gamma) \) induces a marking on \( L(\Gamma) = L^0(\Gamma) \cup L^1(\Gamma) \) by \( \beta(\ell) = \beta(v) \) where \( \ell \) is attached to \( v \). Let \( H(v) \) be the set of all half edges attached to \( v \). Define the valence of \( v \in V(\Gamma) \) as \( \text{val}(v) = |H(v)| \). We say a labelled graph \( \Gamma = (\Gamma, g, \beta, k) \) is stable if \( 2g(\Gamma) - 2 + \text{val}(\Gamma) > 0, \forall v \in V(\Gamma) \). For a labelled graph \( \Gamma \), we define the genus by \( g(\Gamma) = \sum_{v \in V(\Gamma)} g(v) + (|E(\Gamma)| - |V(\Gamma)| + 1). \)

3.2. Givental’s formula and the A-model graph sum.

For a semisimple Frobenius algebra \( V \), let \( \{\phi_\alpha\} \) be its canonical basis. Under the identification of \( V \) and \( T_p V \), \( \phi_\alpha \) corresponds to a tangent vector in \( T_p V \). Let \( \{u_\alpha\} \) be Givental’s canonical coordinates corresponding to \( \phi_\alpha \), i.e., \( \phi_\alpha = \frac{\partial}{\partial u_\alpha} \). Let \( U = \text{diag}(u_1, \cdots, u_N) \). Take \( \Psi \) to be the base change of \( \hat{\phi}_\alpha \) to \( T_p \), i.e., \( \hat{\phi}_\alpha = \sum_{\beta} T_{\beta} \Psi_{\alpha}^\beta \). By Givental’s theorem\(^{[18]} \), there exists a unitary \( R(z) \) (i.e., \( R(z)R^T(-z) = \text{id} \)) such that \( S = \Psi R(z)e^{\frac{\psi}{\sqrt{z}}} \) is a fundamental solution of the QDE, with \( R(z) = \text{id} + R_1 z + \cdots \) a formal power series in \( z \). Furthermore, \( R(z) \) is unique up to a right multiplication of \( \exp(a_1 z + a_3 z^3 + a_5 z^5 + \cdots) \), where \( a_i \) are complex diagonal matrices.

The \( S \) operator is given by \( (a, S(b)) = \langle a, \frac{b}{\sqrt{z + \frac{1}{z}}} \rangle_{0,2}^{WP(m,n),T} \). The quantization of the \( S \) operator relates the ancestor potential and the descendant potential via Givental’s formula\(^{[17]} \), i.e.,

\[
D^{WP(m,n),T}(u) = \exp \left( F_1^{WP(m,n),T} \right) \hat{S}^{-1} A^{WP(m,n),T}(u, t).
\]

We now describe graph sum formulas for the ancestor potential \( A^{WP(m,n),T}(u, t) \) and the descendant potential with arbitrary primary insertions \( F_{g,N}^{WP(m,n),T}(u, t) \).

Let \( u = u^a T_a \). We assign weights to leaves, edges, and vertices of a labelled graph \( \Gamma \in \Gamma(W^P(m,n)) \) as follows.

1. Ordinary leaves. To each \( \ell \in L^0(\Gamma) \) we assign

\[
(E^u)^\beta_k(\ell) = \left[z^k \right] \sum_{\alpha=1}^{m+n} \frac{u^\alpha(z)}{\Delta^\alpha(q)} R^\alpha_\beta(-z).
\]

2. Dilaton leaves. To each \( \ell \in L^1(\Gamma) \) we assign

\[
(L^1)^\beta_k(\ell) = \left[z^{k-1} \right] \sum_{\alpha=1}^{m+n} \frac{1}{\Delta^\alpha(q)} R^\alpha_\beta(-z).
\]

3. Edges. To an edge connecting vertices marked by \( \alpha \) and \( \beta \), with heights \( k \) and \( \ell \) at the corresponding half-edges, we assign

\[
E^\alpha_{k,\ell}(e) = \left[z^k w^\ell \right] \left( \frac{1}{z + w} (\delta_{\alpha,\beta} - \sum_{\gamma=1}^{m+n} R^\alpha_\gamma(-z) R^\gamma_\beta(-w)) \right).
\]
Then it holds
\[ V^\beta_0(v) = \left( \sqrt{\Delta^\beta(q)} \right)^{2g - 2 + N} \left( \prod_{i=1}^{N} \tau_k \right)_g. \]

Hence the weight of \( \bar{\Gamma} \in \Gamma(W^\mathbb{P}(m, n)) \) is:
\[ w(\bar{\Gamma}) = \prod_{v \in \Gamma} V^\beta_0(v) \prod_{e \in E(\Gamma)} \mathcal{L}^\beta(v(e), \beta(e)) \cdot \prod_{\ell \in L^0(\Gamma)} (\mathcal{L}^\beta_k(\ell) \cdot \prod_{\ell \in L(\Gamma)} (\mathcal{L}^1_k(\ell)). \]

Then it holds
\[ \log(A_{\Gamma(2g-2+N, T)}(u, t)) = \sum_{g \geq 0} h^{g-1} \sum_{N \geq 0} \frac{w(\bar{\Gamma})}{\text{Aut}(\bar{\Gamma})}. \]

We define a new weight if we have \( N \) ordered variables \( (u_1, \ldots, u_N) \) and \( N \) ordered ordinary leaves \( \{\ell_1, \ldots, \ell_N\} \). Let
\[ S_{2g}^\beta(z) = (\phi_{\alpha}, S(\phi_{\alpha})), \quad u_j = \sum_{\alpha \geq 0} (u_j)_{\alpha} z^\alpha = \sum_{\alpha} u_j^\alpha \alpha, \]
\[ (\mathcal{L}^\alpha_u)_{k, \ell} \beta_j = [z^k] \left( \sum_{\alpha, \gamma} u^\alpha_j \sum_{\ell} \mathcal{L}^\beta_k \ell \right) R(-z)^{\beta_j}. \]

Let \( \bar{w}(\bar{\Gamma}) \) be the corresponding weight, then it holds similarly
\[ \sum_{g \geq 0} h^{g-1} \sum_{N \geq 0} F^\mathbb{P}(m, n), T(u_1, \ldots, u_N, t) = \sum_{g \geq 0} h^{g-1} \sum_{N \geq 0} \sum_{\Gamma \in \text{Aut}(\Gamma)} \frac{\bar{w}(\bar{\Gamma})}{\text{Aut}(\bar{\Gamma})}. \]

### 3.3. Eynard-Orantin recursion and the B-model graph sum.

Let \( \omega_{g, N} \) be defined recursively by the Eynard-Orantin topological recursion
\[ \omega_{0,1} = 0, \quad \omega_{0,2} = B(Y_1, Y_2) = \frac{dY_1 \otimes dY_2}{(Y_1 - Y_2)^2}. \]

When \( 2g - 2 + N > 0 \), we have
\[ \omega_{g, N}(Y_1, \ldots, Y_N) = \sum_{\alpha = 1}^{m+n} \text{Res}_{Y \to p^\alpha} \frac{-\int_{\xi \in Y} B(Y_N, \xi)}{2(\log(Y) - \log(\bar{Y}))} dW_T(\omega_{g-1, N-1}(Y, \bar{Y}, Y_1, \ldots, Y_{N-1}) \]
\[ + \sum_{g_1 + g_2 = g} \sum_{I \cup J = \{1, \ldots, N-1\}, I \cap J \neq \emptyset} \omega_{g_1, |I|+1}(Y, Y_I) : \omega_{g_2, |J|+1}(\bar{Y}, Y_J), \]

where \( Y \neq p^\alpha \) and \( \bar{Y} \neq Y \) are in a small neighborhood of \( p^\alpha \) such that \( W_T(Y) \neq W_T\bar{Y} \).

By definition, it holds that \( x = W_T(e^\alpha) \). Near any critical point \( v^\alpha(= \log p^\alpha) \), we define \( \zeta_\alpha, h^\alpha_k \) to satisfy \( x = u^\alpha - \zeta_\alpha^2, \ y = v^\alpha - \sum_{k=1}^{\infty} h^\alpha_k \zeta_\alpha^k \). Expand \( B(p^\alpha, p^\beta) \) in terms of \( \zeta_i \) as
\[ B(p^\alpha, p^\beta) = \frac{\delta_{\alpha, \beta}}{(\zeta_\alpha - \zeta_\beta)^2} + \sum_{k, \ell \geq 0} B^\alpha_{k, \ell} \zeta_\alpha^k \zeta_\beta^\ell d\zeta_\alpha \otimes d\zeta_\beta. \]

Let
\[ B^\alpha_{k, \ell} = \frac{(2k - 1)!!(2\ell - 1)!!}{2^{k + \ell + 1}} P^\alpha_{k, \ell}, \quad h^\alpha_k = \frac{(2k - 1)!!}{2^{k-1}} h^\alpha_{2k-1}, \]
\[ d\zeta^\alpha_k = (2k - 1)!!2^{-d} Re s_{p^\alpha \cdot p^\beta} B(p^\alpha, p^\beta)(\sqrt{-1} \zeta_\alpha)^{-2d-1}. \]
The B-model invariants $\omega_{g,N}$ can be expressed as graph sums. Given a labelled graph $\Gamma \in \tilde{G}_{g,N}(W\overline{P}(m,n))$ with $L^0(\Gamma) = \{\ell_1, \ldots, \ell_N\}$, we define its weight to be

$$\tilde{w}(\Gamma) = (-1)^g(\Gamma) - 1 + N \prod_{v \in V(\Gamma)} \left( \frac{h}{2} \right)^{2 - 2g - \text{val}(v)} \prod_{h \in H(v)} \tau_k(h) \prod_{e \in E(\Gamma)} \tilde{B}_{k(e),(\ell(e))}(\epsilon) \sum_{j=1}^{N} \frac{1}{\sqrt{2}} d_{k(e)}(Y_j) \prod_{\ell \in \Lambda^1(\Gamma)} (-1)^{\frac{1}{2}} \tilde{\chi}_{k(\ell)}(\epsilon).$$

We cite here the Theorem 3.7 in [11].

**Theorem 3.1.** For $2g - 2 + N > 0$, it holds

$$\omega_{g,N} = \sum_{\tilde{\Gamma} \in G_{g,N}(W\overline{P}(m,n))} \frac{\tilde{w}(\tilde{\Gamma})}{|\text{Aut}(\tilde{\Gamma})|}.$$

### 3.4. A-model large radius limit.

In 3.4 and 3.5, we assume $p$ and $z$ to be negative real numbers.

We denote $Q_1$ as the chart $W\overline{P}(m,n) \setminus \{[0 : 1]\}$, and $Q_2$ as the chart $W\overline{P}(m,n) \setminus \{[1 : 0]\}$. By Tseng [22] (see also Zong [23]), we have

$$\left(R^2_{\sigma}\right)_t |_{t=0,q=0} = \text{diag}((P_\sigma)_j^t), \text{ on } Q_\sigma, \text{ for } \sigma = 1, 2;$$

$$(P_\sigma)_j^t = \frac{1}{|G_\sigma|} \sum_{(h)} \chi_{\alpha_j}(h) \chi_{\alpha}(h^{-1}) \cdot \exp \left[ \sum_{t=1}^{\infty} \frac{(-1)^t}{t(t+1)} B_{t+1}(c_\sigma(h))(\frac{z}{w_\sigma})^t \right].$$

Let $\sigma = 1$. Then we have the following:

- $G = G_1 = \mathbb{Z}/m\mathbb{Z}$.
- $V_{\alpha_1+j} = \mathbb{C}$, with $\mathbb{Z}/m\mathbb{Z}$ action: $\bar{f} \circ z = e^{2\pi i \frac{f}{m}} \cdot z$. Then $\chi_{\alpha_1+j}(\bar{f}) = e^{2\pi i \frac{hf}{m}}$.
- $T = \{(\lambda_1, \lambda_2)\} \text{ acts on } Q_1$ by $(\lambda_1, \lambda_2) \circ z = (\lambda_2^m, \lambda_1)z$, i.e., $w_1 = \lambda_2^m, \lambda_1$.
- $c_\sigma(e^{2\pi i \frac{f}{m}}) = \frac{1}{m}$, where $0 \leq t \leq m$.

By [20] we have $\log \Gamma(z + s) = (z + s - \frac{1}{2}) \log z - z + \frac{1}{2} \log 2\pi + \sum_{t=1}^{\infty} \frac{(-1)^t}{t(t+1)} B_{t+1}(s) \frac{1}{2tan^2(s)}$. Let $\lambda = \frac{\lambda_1, \lambda_2}{m}$. Then we have

$$(P_1)^t_{\beta} = \frac{\sqrt{2\pi e^\lambda}}{m\sqrt{\lambda}} \sum_{h=0}^{m-1} \omega^{-h} \Gamma\left(\lambda + \frac{h}{m}\right) \lambda_{1-\lambda-\frac{h}{m}}.$$

### 3.5. B-model large radius limit.

Next we calculate the B-model $R$-matrix $\tilde{R}^\beta$ while $\tilde{q} \to 0$. Let

$$\tilde{R}^\beta_\alpha(\tilde{q}) = \sqrt{-2\pi z} \int_{\gamma_\beta} e^{\frac{W_T(Y)-W_T(p^\beta)}{z}} \theta_\alpha,$$

where:

- $p^\beta$ are critical points of $W_T$ for $\beta = 0, \ldots, m + n - 1$;
- for a fixed $\beta$, $\gamma_\beta = W_T^{-1}(W_T(p^\beta) + [0, +\infty))$;
- $\theta_\alpha = \frac{dz_\alpha}{z_\alpha}$, with $z_\alpha = c_\alpha \cdot (Y - p^\beta)$ and $c_\alpha, \epsilon \mathbb{C}$ such that $W_T(Y) - W_T(p^\beta) = \frac{1}{2} z_\alpha^2 + o(z^2)$. Noticing that $\tilde{R}^\beta_\alpha(\tilde{q})$ only involves terms of differences, it remains unchanged if we add a constant to $W_T$. More specifically, we replace $W_T$ by $W_T - \sum_{t=1}^{m-1} w_t \log \tilde{q}_t + \sum_{t=1}^{n} w_{-t} \log \tilde{q}_{-t}$. Further let $\tilde{R}_1^\beta_\alpha$ be the submatrix of the first $m$ columns and $m$ rows of $\lim_{\tilde{q} \to 0} \tilde{R}^\beta_\alpha(\tilde{q})$. It may be computed from $\int_{\gamma_\beta} \exp(\frac{W_T(Y) - W_T(p^\beta)}{z}) \theta_\alpha$ with $W_T = mYW + p\log Y$. In this new set-up, we have the following results:

- $p^\beta$ are roots of $0 = \frac{\partial W_T}{\partial \log Y} = mY^m + \frac{p}{m}$. This gives $p^\beta = m\sqrt{-\frac{1}{m}}e^{2\pi i \frac{\beta}{m}}$. 

• We claim that $\gamma_\beta = (0, +\infty) - p\beta$. In fact, direct calculation shows that $\hat{W}_T(t \cdot p^\beta) - \hat{W}_T(p^\beta) = (\frac{\gamma_\alpha}{m}) \cdot (t^m - 1 - \log t^m) \geq 0$.

• $\frac{\gamma_\alpha}{2} - \frac{\gamma_\beta}{2} = \frac{W_T(Y) - W_T(p^\alpha)}{(Y - p^\alpha)^2} = \frac{-mp}{2(Y - p^\alpha)^2}$. Taking $c_\alpha = -\frac{\gamma_\alpha}{2p^\alpha}$, we get $\theta_\alpha = \frac{\gamma_\alpha}{\gamma_\beta} \cdot \frac{dY}{(Y - p^\alpha)^2}$.

Let $\mu = \frac{\gamma_\alpha}{m^2}$, $\omega = e^{2\pi i \frac{m-\beta}{m}}$, $s = -\mu \gamma^m$. Integrating by parts,

$$\int_{s=0}^{\infty} e^{-s(\frac{\gamma_\alpha}{m} \frac{1}{\omega - (\frac{1}{m})})} = -\frac{m-1}{\omega} \sum_{h=0}^{m-1} \Gamma(\mu + \frac{h}{m}) \cdot \mu^{1-\mu - -} \cdot \omega^{-h}.$$ 

This equation differs from the main integral by $-\omega e^{\mu \frac{2\pi i}{m\sqrt{\mu}}}$. Hence for $1 \leq \alpha, \beta \leq m$,

$$(\hat{R}_1)^\beta_\alpha = e^{\mu \frac{2\pi i}{m\sqrt{\mu}}} \sum_{h=0}^{m-1} \Gamma(\mu + \frac{h}{m}) \cdot \mu^{1-\mu - -} \cdot \omega^{-h}.$$ 

**Proposition 3.1.** It holds that $(\hat{R}_1)^\beta_\alpha = (P_1)^\beta_\alpha$, for $0 \leq \alpha, \beta \leq m - 1$, if we identify $\lambda$ and $\mu$.

3.6. The general case.

It is obvious that $\hat{R}_\beta^\alpha(q)|_{q=0} = 0 = R_\beta^\alpha(q)|_{q=0}$ for $1 \leq \alpha \leq m < m + 1 \leq \beta \leq m + n$. Since both $\Psi \Re^{\alpha_\beta}$ and $\hat{\Psi} \Re^{\alpha_\beta}$ are solutions to the QDE on the Frobenius algebra $Q Hel(WP(m,n)) \cong \text{Jac}(WP(m,n))$, we have by Givental’s theorem $\hat{R}(\hat{q}) = R(q) \cdot A$, where $A = \exp(a_1 z + a_2 z^2 + a_3 z^3 + \cdots)$, with $a_i$’s diagonal matrices, and their diagonal entries are scalar. Considering the submatrix consisting of the first $m$ columns and the first $m$ rows of $R$ and $\hat{R}$, by the previous proposition we have $\lim_{q \to 0} P_1(q) = \hat{R} = \lim_{q \to 0} P_1(q)|_{m \times m}$. Comparing the diagonal elements, we find the submatrix of first $m$ columns and $m$ rows of $A$ as $A|_{m \times m} = I_{m \times m}$. Similarly, moving to the other chart we have $A|_{n \times n} = I_{n \times n}$. Hence $A = I$. This gives the following proposition.

**Proposition 3.2.** It holds that

$$R(q) = \hat{R}(\hat{q}).$$

4. All genus equivariant mirror symmetry

4.1. Calculations of the graph sum formula.

First observe that $\sqrt{\Delta^3} h_1^\alpha = \sqrt{2}$.

Let

$$\xi_{\alpha,0} = \frac{1}{\sqrt{-1}} \sqrt{\frac{2}{\Delta^3}} \frac{p^\alpha}{Y - p^\alpha}, \theta = \frac{d}{dW_T} W_k^\alpha = d((-1)^k \theta^k(\xi_{\alpha,0})), $$

$$(\tilde{u}_j)_k^\alpha = [z^j] \sum_{\alpha} S_\alpha^\beta(z) \frac{u_j^\beta(z)}{\sqrt{\Delta^3}(q)}.$$ 

Note that $d \xi_{\alpha,0} = d\xi_0^\alpha$.

**Theorem 1.** By identifying $W_k^\alpha(Y_j)$ and $\sqrt{-2}(\tilde{u}_j)_k^\alpha$, we have

$$\omega_{N,1} = (-1)^{g-1+N} \Delta^{W_{P(m,n)} T} \left( u_1, \cdots, u_N, t \right).$$

**Proof.** We prove by direct calculation as follows,

1. Vertices: This follows from $\sqrt{\frac{\Delta^3}{2}} h_1^\alpha(v) = 1$.
2. Edges: By $[\Pi], R_\beta^\alpha(z) = f_\beta^\alpha(\frac{1}{z})$ and the contribution of edges to weight in B-model is

$$\hat{B}_{k,\ell}^\alpha(\ell) = \left[ u^{-k} v^{-\ell} \right] \frac{uv}{u+v} \left( \delta_{\alpha,\beta} - \sum_{\gamma=1}^{m+n} f_{\gamma}^\alpha(u) f_{\gamma}^\beta(v) \right) = c_{k,\ell}^\alpha(\ell).$$
(3) Ordinary leaves: By $\frac{1}{z-m} = \sum_{s>0} (-\frac{m}{z})^s$ we know $-B_{k-1-i,0} = [z^{k-i}]R_\beta^\alpha(-z)$. From [14] we know $d\xi_k = W_\beta^\alpha - \sum_{i=0}^{k-1} \sum_\beta B_{k-1-i,0}^\alpha_\beta W_i^\beta$.

It holds after identifying $\frac{1}{\sqrt{-1}}W_k^\alpha(Y_j)$ and $(\tilde{u}_j)^\alpha_k$ that

$$
(L_d^{(\ell_j)})^{(\ell_j)}(\tilde{u}_j) = \sum_{i=0}^{m+n} \sum_\beta (\tilde{u}_j)^\beta_i [z^{k(\ell_j)}] R_\beta^\alpha(\ell_j)(-z)
$$

$$
= \sum_{i=0}^{m+n} \sum_\beta (\tilde{u}_j)^\beta_i ([z^{k(\ell_j)}] R_\beta^\alpha(-z))
$$

$$
= \frac{1}{\sqrt{-1}}d\xi_k(\ell_j)(Y_j).
$$

(4) Dilaton leaves: By [14] and the relation $R_\beta^\alpha(z) = f_\beta^{\alpha} \left( \frac{1}{z} \right)$, we have

$$
\tilde{h}_{k(\ell)}^\alpha = [u^{1-k(\ell)}] \sum_{\beta=1}^{m+n} \sqrt{-1} R_\beta^\alpha(\ell) \left( \frac{1}{u} \right) = [z^{k(\ell)-1}] \sum_{\beta=1}^{m+n} \sqrt{-1} R_\beta^\alpha(-z).
$$

By $h_1^\beta = \sqrt{\frac{2}{z^\alpha}}$, we know that $(L_{k(\ell)})^{(\ell)}(\tilde{z}) = \left( -\frac{1}{\sqrt{-1}} \right) \tilde{h}_{k(\ell)}^\alpha$.

\[ \square \]

4.2. The Laplace Transform.

Following Iritani [19] with slight modification, we define as follows [15].

**Definition 4.1** (equivariant Chern character). We define equivariant Chern character

$$
\tilde{c}_z : K_T(WP(m, n)) \to H^*_T(WP(m, n), \mathbb{Q}) \left[ \frac{1}{z} \right]
$$

by the following two properties which uniquely characterize it:

1. $\tilde{c}_z(\varepsilon_1 \oplus \varepsilon_2) = \tilde{c}_z(\varepsilon_1) + \tilde{c}_z(\varepsilon_2)$.
2. If $\mathcal{L}$ is a $T$-equivariant line bundle on $WP(m, n)$, then $\tilde{c}_z(\mathcal{L}) = \exp \left( -2\pi i \langle \varepsilon(z), \mathcal{L} \rangle \right)$.

**Definition 4.2** (equivariant K-theoretic framing). For $\forall \varepsilon \in K_T(WP^1(m, n))$, we define the K-theoretic framing of $\varepsilon$ by $\kappa(\varepsilon) = (-z)^{1-\langle \varepsilon(z), WP(m, n) \rangle} \Gamma \left( 1 - \langle \varepsilon(z), WP(m, n) \rangle \right) \tilde{c}_z(\varepsilon)$, where $(\mathcal{L})_{WP^1(m, n)}^T = \mathcal{L}$.

**Definition 4.3** (equivariant SYZ T-dual). Let $\mathcal{L} = \mathcal{O}_{WP(m, n)}(\ell_1 p_1 + \ell_2 p_2)$ be an equivariant ample line bundle on $WP(m, n)$, where $\ell_1, \ell_2 \in \mathbb{Z}$, such that $\ell_1 + \ell_2 > 0$. We define the equivariant SYZ T-dual $\text{SYZ}(\mathcal{L})$ of $\mathcal{L}$ be the figure below:

```
2\pi i \cdot \frac{\ell_1}{m} \quad \quad \quad \quad 2\pi i \cdot \frac{\ell_2}{m} + (+\infty)
```

```
\hline
\hline
-\infty \quad 2\pi i \cdot \frac{-\ell_1}{m} \quad \quad \quad \quad 2\pi i \cdot \frac{\ell_1}{m}
\hline
```

Extend the definition additively to the equivariant K-theory group $K_T(WP^1(m, n))$. By [15]:

**Theorem 4.1.**

$$
\left\langle \left\langle \kappa(\mathcal{L}) \right\rangle \right\rangle_{WP^1(m, n)}^T = \int_{\text{SYZ}(\mathcal{L})} e^{\frac{w_\mathcal{L}}{2}} dy.
$$
Corollary 4.1.

(1) By string equation:

$$\int_{\text{SYZ}(L)} \mathcal{W}_f e^\frac{w_T}{x} dy = \left\langle \frac{\kappa(L)}{z-\psi} \right\rangle_{0,1}^{\text{WP}(m,n),T} = \left\langle \frac{1}{z-\psi} \right\rangle_{0,2}^{\text{WP}(m,n),T};$$

(2) Integrating by parts,

$$-\left\langle \frac{\kappa(L)}{z-\psi} \right\rangle_{0,1}^{\text{WP}(m,n),T} = -z \int_{\text{SYZ}(L)} e^\frac{w_T}{x} dy = \int_{\text{SYZ}(L)} e^\frac{w_T}{x} y dx.$$

Define

$$S_{\beta}^\alpha(z) = \left\langle \frac{\phi_0(q), \hat{\phi}_0(q)}{z-\psi} \right\rangle_{0,2}^{\text{WP}(m,n),T}, S_{\beta}^\alpha(L) = \left\langle \frac{\phi_0(q), \kappa(L)}{z-\psi} \right\rangle_{0,2}^{\text{WP}(m,n),T}.$$

More generally, we have

**Proposition 4.1.**

$$S_{\beta}^\alpha(z) = -z \int_{y \in \gamma_{\beta}(L)} e^\frac{w_T}{x} d\xi_{\beta,0} \sqrt{-2},$$

$$S_{\beta}^\alpha(L) = -z \int_{y \in \text{SYZ}(L)} e^\frac{w_T}{x} d\xi_{\beta,0} \sqrt{-2}.$$

**Proof.** We prove the second equation as an example. The first one may be proved in a similar way.

Let $$f(Y) = \frac{\mathcal{W}_f}{Y}. $$ Then $$\Delta^\alpha = p^\alpha \cdot f'(p^\alpha). $$ By $$\hat{\phi}_0 = \sqrt{\Delta^\alpha} \phi_0 $$ and $$\xi_{\alpha,0} = \frac{1}{\sqrt{2}} \sqrt{\frac{2}{\Delta^\alpha \sqrt{-1}}}, $$ the desired proposition is equivalent to

$$\left\langle \frac{\phi_0(q), \kappa(L)}{z-\psi} \right\rangle_{0,2}^{\text{WP}(m,n),T} = z \int_{y \in \text{SYZ}(L)} e^\frac{w_T}{x} dY \cdot \frac{p^\beta}{(Y - p^\beta) \Delta^\alpha}.$$

By Corollary 4.1, we have

$$\left\langle \frac{1}{z-\psi} \right\rangle_{0,2}^{\text{WP}(m,n),T} = \int_{y \in \text{SYZ}(L)} e^\frac{w_T}{x} dY.$$

Applying $$z \frac{\partial}{\partial t_i} $$ to both sides, we have

$$\left\langle X_i, \frac{\kappa(L)}{z-\psi} \right\rangle_{0,2}^{\text{WP}(m,n),T} = \int_{y \in \text{SYZ}(L)} Y_i e^\frac{w_T}{x} dY.$$

Note that for the left hand side derivation, we have opened the double bracket and used the string equation.

This implies

$$\left\langle \frac{\phi_0(q), \kappa(L)}{z-\psi} \right\rangle_{0,2}^{\text{WP}(m,n),T}$$

$$= \int_{y \in \text{SYZ}(L)} e^\frac{w_T}{x} f(Y)Y^n dY \cdot \frac{1}{(Y - p^\beta) Y} \cdot (Y^n f(Y))'[p^\beta]$$

$$= -z \int_{y \in \text{SYZ}(L)} (Y - p^\beta) f'(p^\beta) \cdot e^\frac{w_T}{x} dY - \int_{y \in \text{SYZ}(L)} e^\frac{w_T}{x} f(Y)(Y^n - (p^\beta)^n) \cdot dY (Y - p^\beta)^n f'(p^\beta) Y$$

$$= z \int_{y \in \text{SYZ}(L)} e^\frac{w_T}{x} d(Y - p^\beta) \Delta^\alpha.$$


Theorem 2. By definition, 

\[ \int_{y \in SYZ(L)} \frac{w_Y}{Y} g(Y)f(Y) \frac{dY}{Y} = \left\langle \left\langle g(Y)f(Y), \frac{\kappa(L)}{z - \psi} \right\rangle \right\rangle_{0,2}^{WP(m,n),T} = \left\langle \left\langle 0, \frac{\kappa(L)}{z - \psi} \right\rangle \right\rangle_{0,2}^{WP(m,n),T} = 0, \]

where \( g(X) = \frac{1}{(p \beta^\alpha)^n (Y^n - (p \beta^\alpha)^n) (Y - p \beta^\alpha) \text{ is a polynomial of } Y}. \)

Integrating the second equation by parts, we have

\[ S^g_{\beta}(z) = -z^{k+1} \int_{y \in SYZ(L)} e^{\frac{w_Y}{g,Y}} W_{i \beta} \sqrt{-2} \]

Also notice that

\[ \sum_{\gamma=1}^{m+n} S^\gamma_{\alpha}(z) S^\gamma_{\beta}(-z) = (\phi_\alpha(0), \phi_\beta(0)) = \Delta^\alpha \delta_{\alpha \beta}, \]

\[ \sum_{\alpha=1}^{m+n} S^\alpha_{\beta}(z) S^\alpha_{\gamma}(z) = (\phi_\beta(0), \kappa(L)). \]

Theorem 2. It holds that

\[ \int_{y_1 \in SYZ(L_1)} \cdots \int_{y_N \in SYZ(L_N)} e^{\frac{w_{Y_1}}{g,Y_1} + \cdots + \frac{w_{Y_N}}{g,Y_N}} \omega_{g,N} = (-1)^{g-1} \left\langle \left\langle \kappa(L_1), \cdots, \kappa(L_N) \right\rangle \right\rangle_{g,N}. \]

Proof. By definition,

\[ \tilde{u}_j^\alpha(z) = \sum_{\beta_1}^{m+n} \sqrt{\Delta^\alpha(q)} \left\langle \left\langle \phi_\alpha(q), \phi_\beta(q) \right\rangle \right\rangle_{WP(m,n),T}^{0,2} u_j^\beta(z). \]

Taking the Laplace transform of \( \omega_{g,N} \), by Theorem 1 and definition of \( \tilde{u}_i \), we get

\[ \int_{y_1 \in SYZ(L_1)} \cdots \int_{y_N \in SYZ(L_N)} e^{\frac{w_{Y_1}}{g,Y_1} + \cdots + \frac{w_{Y_N}}{g,Y_N}} \omega_{g,N} = \int_{y_1 \in SYZ(L_1)} \cdots \int_{y_N \in SYZ(L_N)} e^{\frac{w_{Y_1}}{g,Y_1} + \cdots + \frac{w_{Y_N}}{g,Y_N}} (-1)^{g-1-N} \sum_{\beta_1, \alpha_i} \left\langle \left\langle \prod_{i=1}^{\tilde{N}} \tau_{\alpha_i}(\phi_{\beta_i}(0)) \right\rangle \right\rangle_{g,N} \cdot \prod_{i=1}^{\tilde{N}} \left( \frac{1}{\Delta^\alpha} \sum_{\alpha_1}^{m+n} \sum_{k \in \mathbb{Z}_{\geq 0}} \left[ z_{\alpha_i^1}^k \kappa_{\beta_i}(z_i) \right] \frac{W_{\alpha_i}(y_i)}{\sqrt{-2}} \right) \]

\[= (-1)^{g-1+N} \sum_{\beta_1, \alpha_i} \left\langle \left\langle \prod_{i=1}^{\tilde{N}} \tau_{\alpha_i}(\phi_{\beta_i}(0)) \right\rangle \right\rangle_{g,N} \cdot \prod_{i=1}^{\tilde{N}} \left( \frac{1}{\Delta^\alpha} \sum_{\alpha_1}^{m+n} \sum_{k \in \mathbb{Z}_{\geq 0}} \left[ z_{\alpha_i^1}^k \kappa_{\beta_i}(z_i) \right] \frac{W_{\alpha_i}(y_i)}{\sqrt{-2}} \right) \]

\[= (-1)^{g-1} \sum_{\beta_1, \alpha_i} \left\langle \left\langle \prod_{i=1}^{\tilde{N}} \tau_{\alpha_i}(\phi_{\beta_i}(0)) \right\rangle \right\rangle_{g,N} \cdot \prod_{i=1}^{\tilde{N}} \left( \frac{1}{\Delta^\alpha} \sum_{\alpha_1}^{m+n} \sum_{k \in \mathbb{Z}_{\geq 0}} \left[ z_{\alpha_i^1}^k \kappa_{\beta_i}(z_i) \right] \frac{W_{\alpha_i}(y_i)}{\sqrt{-2}} \right) \]

\[= (-1)^{g-1} \left\langle \left\langle \kappa(L_1), \cdots, \kappa(L_N) \right\rangle \right\rangle_{g,N}. \]

\[ \square \]
REFERENCES

[1] A. Adem, J. Leida, Y. Ruan. Orbifolds and Stringy Topology. Cambridge University Press, Cambridge, 2007.
[2] K. Behrend, B. Fantechi. The intrinsic normal cone. Inventiones Mathematicae 128(1):45-88, 1997.
[3] W. Chen, Y. Ruan. A new cohomology theory of orbifold, Communications in Mathematical Physics 248(1):1-31, 2004.
[4] W. Chen, Y. Ruan. Orbifold Gromov-Witten theory. arXiv:math/0103156 [math.AG], 2001.
[5] W. Chen, Y. Ruan. Orbifold quantum cohomology. arXiv:math/0005198 [math.AG], 2000.
[6] T. Coates. Wall-crossings in toric Gromov-Witten theory ii: local examples. arXiv:0804.2592 [math.AG], 2008.
[7] T. Coates, A. Corti, I. Hiroshi, H.H. Tseng, Computing genus-zero twisted Gromov-Witten invariants. Duke Mathematical Journal 147(3):377-438, 2009.
[8] T. Coates, A. Corti, H. Iritani, and H. Tseng. A mirror theorem for toric stacks. Compositio Mathematica 151(10):1878-1912, 2015.
[9] T. Coates, A. Corti, Y. Lee, and H. Tseng. The quantum orbifold cohomology of weighted projective space. Acta Mathematica 202:139-193, 2009.
[10] T. Coates, I. Hiroshi, H. Tseng. Wall-crossings in toric Gromov-Witten theory I: crepant examples. Geometry & Topology 13:2675-2744, 2009.
[11] P. Dunin-Barkowski, N. Orantin, S. Shadrin, L. Spitz. Identification of the Givental formula with the spectral curve topological recursion procedure. Math. Phys. 328(2): 669-700, 2014.
[12] B. Eynard, N. Orantin. Invariants of algebraic curves and topological expansion. Commun. Number Theory Phys. 1 (2):347-452, 2007.
[13] B. Fang. Central charges of T-dual branes for toric varieties. arXiv:1611.05153 [math.SG], 2016.
[14] B. Fang, C.M. Liu, Z. Zong. All genus open-closed mirror symmetry for affine toric Calabi-Yau 3-orbifolds, in Proceedings of Symposia in Pure Mathematics (V. Bouchard, C. Doran, S. MendezDiez, C Quigley eds.), Amer Mathematical Soc., 2016.
[15] B. Fang, C. Liu, Z. Zong. The Eynard-Orantin recursion and equivariant mirror symmetry for the projective line. Geometry & Topology 24:2049-2092, 2017.
[16] B. Fang, C. Liu, H. Tseng. Open-closed Gromov-Witten invariants of 3-dimensional Calabi-Yau smooth toric DM stacks. arXiv:1212.6073 [math.AG], 2012.
[17] A.B. Givental. Gromov-Witten invariants and quantization of quadratic Hamiltonians. Mosc. Math. J. 1(4): 551-568, 2001.
[18] A.B. Givental. Semisimple Frobenius structures at higher genus. Internat. Math. Res. Notices 23:1265-1286, 2001.
[19] H. Iritani. An integral structure in quantum cohomology and mirror symmetry for toric orbifolds. Advances in Mathematics 222:1016-1079, 2009.
[20] J. Kaczorowski, A. Perelli. A uniform version of Stirlings formula. Functiones et Approximatio 45: 89-96, 2011.
[21] C. Liu, Localization in Gromov-Witten theory and orbifold Gromov-Witten theory, Handbook of Moduli (Vol. II), 353-425, Adv. Lect. Math. (ALM) 25, International Press and Higher Education Press, Beijing, 2013.
[22] H. Tseng. Orbifold quantum Riemann-Roch, Lefschetz and Serre. Geometry & Topology 14 (1):1-81, 2010.
[23] Z. Zong. Equivariant Gromov-Witten theory of GKM orbifolds. arXiv:1604.07270 [math.SG], 2016.

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