ZERO CYCLES ON GENERIC HYPERSURFACES OF LARGE DEGREE

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Abstract. We show that given a smooth projective variety $X$ over $\mathbb{C}$ with $\dim(X) \geq 3$, an ample line bundle $\mathcal{O}(1)$ on $X$ and an integer $n > 1$, any $n$ distinct points on a generic hypersurface of degree $d$ are linearly independent in $CH_0(X)$ if $d > 0$. This generalizes a result of C. Voisin.

Let $X$ be a smooth projective algebraic variety over $\mathbb{C}$ of dimension $r + 1 \geq 3$. Let $\mathcal{L}$ be an ample line bundle on $X$. Let $S^d = H^0(X, \mathcal{L}^d)$ and for a point $x \in X$, let $S^d_x = H^0(X, \mathcal{L}^d \otimes \mathcal{I}_x)$, where $\mathcal{I}_x$ is the ideal sheaf of $x$. Let $R = \oplus_{d \geq 0} S^d$.

Lemma 1. There exists an integer $m > 0$ such that, for all $d > 0$, the following holds:

(i) The natural map $S^d \otimes S^m \rightarrow S^{d+m}$ is surjective.
(ii) The natural map $S^d \otimes S^m_x \rightarrow S^{d+m}_x$ is surjective.

Proof. Let the ring $R$ be generated in degrees $\leq t$. Let $m_1 = t!$. It is easy to see that $a \cdot m_1$ satisfies condition (i) of the lemma for $d > a \cdot t! \cdot t(t + 1)/2$, for any positive integer $a$. Let $d'$ be a positive integer so that $\mathcal{L}^{d'}$ is generated by its global sections. There exists $m$, a positive multiple of $m_1$, such that the maps $S^{d'} \otimes S^m_x \rightarrow S^{d'+m}_x$ are surjective for all $x \in X$. For $d > 0$, both the maps $S^{d-d'} \otimes S^d \rightarrow S^d$ and $S^{d-d'} \otimes S^m_x \rightarrow S^{d+m}_x$ are surjective since $\mathcal{L}$ is ample, hence the map $S^{d-d'} \otimes S^d \otimes S^m_x \rightarrow S^{d+m}_x$ is also surjective. The following commutative diagram then shows that the map $S^d \otimes S^m_x \rightarrow S^{d+m}_x$ is surjective.

Let $\mathcal{X} \subset X \times S^d$ be the universal hyperplane section. For $s \in S^d$ we denote the fibre $p_2^*(s)$ by $X_s$, which we shall assume to be smooth. For a vector bundle $\mathcal{V}$ on $X$, by $\mathcal{V}(b)$ we shall mean $\mathcal{V} \otimes L^b$.

Proposition 1. For $d > 0$, the bundle $T\mathcal{X}(m)|_{X_s}$ is generated by its global sections.

Proof. The proof, given the previous lemma, is identical to Proposition 1.1 of [1] and is hence omitted.

Corollary 1. There exists a linear function $d(n)$ of $n$, such that for all $d \geq d(n)$, the vector bundle $\Omega^{d\dim S^d}_\mathcal{X}|_{X_s}$ separates any $n$ distinct points of $X_s$ i.e. the global
sections of the bundle surject onto the global sections of the bundle restricted to any subscheme consisting of \( n \) distinct reduced points.

**Proof.** \( \Omega^{\dim S_d}|_{X_s} \cong \Omega^r_{X_s} \otimes K_{X_s} \cong \wedge^r T X_s \otimes K_{X_s} \cong \wedge^r T X_s \otimes K_{X_s} \) is generated by global sections if \( d \gg 0 \). Since \( \mathcal{L} \) is ample, there exists a linear function \( d(n) \) such that \( K_X(d - r \cdot m)|_{X_s} \) separates \( n \) points if \( d \geq d(n) \). It follows that the tensor product also separates \( n \) distinct points.

**Theorem 1.** Let \( X \) be a smooth projective variety of dimension \( r + 1 \geq 3 \) and let \( \mathcal{L} \) be an ample line bundle on \( X \). Then there exists a linear function \( d(n) \) such that for all \( d \geq d(n) \), any \( n \) distinct points of a generic hypersurface \( X_s, s \in S^d \), are linearly independent in \( CH^r(X_s) \).

**Proof.** Suppose not. Then there exists an etale map \( S \to S^d \) and \( n \) distinct sections \( \sigma_1, ..., \sigma_n \) of \( X_S \) such that the classes of these sections in \( CH^r(X_S) \) are linearly dependent. We may assume that \( S \) is affine and that all the fibres are smooth. Consider the classes of these cycles, \( [\sigma_i] \), in the Hodge cohomology group \( H^r(X_S, \Omega^{\dim S_d}_{X_S}) \). By the Grothendieck-Serre duality, it is easy to see that as an element of \( Hom(H^0(X_S, \Omega^{\dim S_d}_{X_S}), H^0(S, \Omega^{\dim S_d}_{X_S})) \), \( [\sigma_i] \) is nothing but the restriction map \( \sigma^*_i \) on differential forms. By the previous corollary we see that all the \( \sigma^*_i \) are linearly independent, which is a contradiction.

**Corollary 2.** Let \( X, \mathcal{L} \) be as above. There exists a linear function \( d'(n) \) such that for all \( d \geq d'(n) \), the generic hypersurface \( X_s, s \in S^d \), does not contain any (possibly singular) \( n \)-gonal curves.

**Proof.** Follows easily from the theorem by considering two distinct elements in the linear system corresponding to a degree \( n \) map from the normalisation of the curve to \( \mathbb{P}^1 \).

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REFERENCES

[1] C. Voisin, *On a conjecture of Clemens on rational curves on hypersurfaces*, J. Differential Geom., 44 (1996), pp. 200–213.

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