Thresholds of Random Quasi-Abelian Codes

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Abstract

For a random quasi-abelian code of rate $r$, it is shown that the GV-bound is a threshold point: if $r$ is less than the GV-bound at $\delta$, then the probability of the relative distance of the random code being greater than $\delta$ is almost 1; whereas, if $r$ is bigger than the GV-bound at $\delta$, then the probability is almost 0. As a consequence, there exist many asymptotically good quasi-abelian codes with any parameters attaining the GV-bound.

Key words: Random quasi-abelian code, threshold, GV-bound, balanced code, cumulative weight enumerator.

1 Introduction

Random codes play an important role in Informatics, Statistical Physics and Coding Theory; for example, see [1], [17]. For a random linear code of rate $r$ over a finite field $F$ with $q$ elements, Varshamov [22] and Pierce [19] showed in fact that the GV-bound (see (1.1) below) is a threshold point: if $r$ is less than the GV-bound at $\delta$ where $0 < \delta < 1 - q^{-1}$, then the probability of the relative distance of the random linear code being greater than $\delta$ is almost 1; whereas, if $r$ is bigger than the GV-bound at $\delta$, then the probability is almost 0. Recently, in [8] the cumulative distance enumerators of random codes are introduced and their thresholds are investigated; as a consequence, the above threshold of random linear codes is redescribed explicitly with the parameters $r$ and $\delta$.

By means of random codes, [3] showed that, if 2 is primitive for infinitely many primes (this is a so-called Artin’s conjecture), then the asymptotically good binary quasi-cyclic codes exist. Later, [4] and [12] made big improvements from different points of view and proved that, without the Artin’s conjecture, the asymptotically good binary quasi-cyclic codes exist.

For a finite group $G$ of order $m$, any element $\sum_{z \in G} a_z z$ (with $a_z \in F$) of the group algebra $FG$ over the finite field $F$ can be viewed as a word $(a_z)_{z \in G}$...
of length $m$ over $F$. By extension, any element of the free module $(FG)^n$ of rank $n$ can be viewed as a word of length $mn$. Any $FG$-submodule $C$ of $(FG)^n$ is called a quasi-group code of index $n$. The code $C$ is just the so-called group code if $n = 1$; whereas it is just the usual quasi-cyclic code of index $n$ if $G$ is cyclic. And, $C$ is called a quasi-abelian code if $G$ is abelian; see [6], [23].

In 2006, Bazzi and Mitter [2] constructed a class of random binary quasi-abelian codes and a class of random binary dihedral group codes, and showed that the probability of the parameters of the random codes of any one of the two classes attaining GV-bound is large; as a consequence, within the two classes the asymptotically good codes exist. Soon after, with the similar random method Martínez-Pérez and Willems [14] proved that self-dual doubly-even binary dihedral group codes are asymptotically good.

We are interested in general random quasi-abelian codes and their thresholds. Modifying the random linear code ensemble in Shannon’s Information Theory (cf. [17], ch.6), in Section 2 we construct the general random quasi-abelian code ensemble, and state our main theorem, see Theorem 2.1 below, which asserts that the GV-bound is still a threshold point, i.e. the probability of the relative distance of the random code of the ensemble being greater than a given $\delta$ is almost 1 if the parameters are below the GV-bound; whereas, the probability is almost 0 if the parameters are beyond the GV-bound. The Varshamov-Pierce’s threshold for random linear codes mentioned above is the special case of our main theorem by taking the finite group to be trivial.

The proof of the main theorem consists of three parts. In Section 3, we extend a result on weights of so-called balanced codes; this result appeared in [15], [20] and [21] in a binary version, which played a key role in [2] and [14]. We generalize it to any $q$-ary version, see Theorem 3.3 below, so that we can treat any $q$-ary codes. Theorem 3.3 has independent significance; for example, from it quite a part of [2] can be extended to any $q$-ary case.

In Section 4, a threshold of the expectation of the cumulative weight enumerator of the random code of the ensemble is obtained in Theorem 4.1 below, from which the first part (“below the GV-bound”) of the main theorem follows immediately.

In Section 5, we prove the second part (“beyond the GV-bound”) of the main theorem by estimating the second moment of the cumulative weight enumerator of the random code of the ensemble.

From the random quasi-abelian code ensemble and the main theorem, in Section 6, we draw the random quasi-abelian codes of given rate $r$ and describe their thresholds; in particular, for any finite abelian group, for any $r$ and $\delta$ attaining the GV-bound, there is a series of quasi-abelian codes such that the limit of their rates and the limit of their relative distances are equal to $r$ and $\delta$ respectively.

In this paper, $h_q(x) = x \log_q(q - 1) - x \log_q x - (1 - x) \log_q(1 - x)$ with the convention that $0 \log_q 0 = 0$, the function $h_q(x)$ is called the $q$-ary entropy.
Each element \(a\) word \((\text{over } F)\) identified with each other; but note that for \(a, b\) in the algebra \(F G\), \(x = 1\) is a prime, and \(\prod G\) which is the \(q\)-ary asymptotic Gilbert-Varshamov bound, or GV-bound in short; note that \(g_q(x)\) for \(x \in [0, 1]\) is a convex function and has a unique zero point at \(x = 1 - q^{-1}\), hence \(g_q(x)\) is a strictly decreasing function for \(x \in [0, 1 - q^{-1}]\); see \([10]\ §2.10.6\]. About fundamentals on coding theory and group theory, please refer to \([10]\) and \([11]\) respectively.

2 Random quasi-abelian code ensembles

In this paper we always assume that \(F\) is a finite field with cardinality \(|F| = q = p^e\) where \(p\) is a prime, and \(G\) is a finite abelian group of order \(|G| = m\).

By \(FG = \{ \sum_{z \in G} a_z z \mid a_z \in F\}\) we denote the group algebra of \(G\) over \(F\). Each element \(a = \sum_{z \in G} a_z z\) of \(FG\) is viewed as a word \((a_z)_{z \in G}\) of length \(m\) over \(F\), and \(w(a) = w((a_z)_{z \in G})\) stands for the usual Hamming weight of the word \((a_z)_{z \in G}\). In this way, \(a = \sum_{z \in G} a_z z \in FG\) and word \((a_z)_{z \in G} \in F^m\) are identified with each other; but note that for \(a, b \in FG\) we have the product \(ab\) in the algebra \(FG\).

Let \(n\) be any positive integer. We consider the free \(FG\)-module of rank \(n\):

\[
(FG)^n = \{ \mathbf{a} = (a_1, \ldots, a_n) \mid a_i \in FG, \ i = 1, \ldots, n \}.
\]

Each element \(\mathbf{a} = (a_1, \ldots, a_n) \in (FG)^n\) is identified with a concatenated word \(((a_{i1})_{z \in G}, \ldots, (a_{in})_{z \in G})\) of length \(mn\) over \(F\), thus the Hamming weight \(w(\mathbf{a}) = w(a_1, \ldots, a_n) = w(a_1) + \cdots + w(a_n)\). As mentioned in Introduction, any submodule \(C\) of the \(FG\)-module \((FG)^n\) is said to be a quasi-abelian code of \(G\) over \(F\) (or quasi-\(FG\) code more precisely) with index \(n\). In particular, it is just the usual abelian code if \(n = 1\); whereas, it is just the usual quasi-cyclic code with index \(n\) if \(G\) is cyclic.

We always take the following parameters:

\[
r \in (0, 1), \quad \delta \in (0, \delta_0) \quad \text{where} \quad \delta_0 = 1 - q^{-1},
\]

and set \(k = \lfloor rn \rfloor\), the integer nearest to \(rn\). We consider the set of \(k \times n\) matrices over \(FG\):

\[
(FG)^{k \times n} = \left\{ A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} \mid a_{ij} \in FG \right\},
\]

which is viewed as a probability space with equiprobability. Following a notation in Shannon’s information theory, we call this probability space the random quasi-abelian code ensemble. In particular, if \(G = 1\) is trivial then \((FG)^{k \times n} = F^{k \times n}\) is just the usual random linear code ensemble; cf. \([17]\ ch.6\).
Take $A = (a_{ij})_{k \times n} \in (FG)^{k \times n}$, i.e. $A$ is a random $k \times n$ matrix over $FG$. We write $A = (A_1, \cdots, A_n)$ with $A_j = (a_{1j}, \cdots, a_{kj})^T$ being the $j$'th column of the matrix $A$, where the superscript "T" stands for the transpose. Then we have a random quasi-abelian code $C_A$ of index $n$ as follows:

$$C_A = \left\{ \mathbf{b}A = (\mathbf{b}A_1, \cdots, \mathbf{b}A_n) \mid \mathbf{b} = (b_1, \cdots, b_k) \in (FG)^k \right\},$$

(2.3)

where $\mathbf{b}A_j = b_1a_{1j} + \cdots + b_ka_{kj} \in FG$. Note that the rate $R(A) = \lim_{m \rightarrow \infty} \dim C_A$. It is obvious that $R(C_A) \leq \frac{k}{n} \approx r$, and $R(C_A) = \frac{k}{n}$ if and only if the FG-rank of $A$ is equal to $k$; so, we can get the random quasi-abelian codes of rate $r$ from the ensemble, see Section 6 below. About the rank of a matrix over a ring, please see [7, §2], or related refs such as [10].

By $\Delta(C_A)$ we denote the relative distance of the random quasi-abelian code $C_A$, i.e. $\Delta(C_A) = \frac{w(C_A)}{mn}$, where $w(C_A)$ denotes the minimum weight of $C_A$. Then $\Delta(C_A)$ is a random variable over the probability space $(FG)^{k \times n}$. We consider the asymptotic property (with $n \to \infty$) of $\Pr(\Delta(C_A) > \delta)$ which stands for the probability that $\Delta(C_A) > \delta$, and state our main theorem.

**Theorem 2.1.** Let notations be as in (2.1), (2.2) and (2.3). Then

$$\lim_{n \to \infty} \Pr(\Delta(C_A) > \delta) = \begin{cases} 1, & \text{if } r < g_\delta(
); \\ 0, & \text{if } r > g_\delta(\delta); \end{cases}$$

and both the limits converge exponentially.

If $G = 1$ is trivial, then $FG = F$ is just the finite field $F$ and the theorem exhibits just the threshold of random linear codes obtained by Vasharmov [22] and Pierce [19] (cf., [8, Corollary 3.2]), as mentioned in Introduction.

The key idea for the proof of the theorem is to estimate the first moment (i.e. the expectation) and the second moment of the cumulative weight enumerator of the random code $C_A$, so that we can bound $\Pr(\Delta(C_A) > \delta)$ suitably; for estimating the moments we need a result on weights of balanced codes which appeared in references, as we’ve seen so far, only in binary version, so we extend it to $q$-ary version first. Thus, as we mentioned in Introduction, the proof of the main theorem will be completed in Sections 3, 4 and 5.

3 The weights of balanced codes

Let $I = \{1, 2, \cdots, n\}$ be an index set; let $F^I = F^n$ be the set of all words over $F$ of length $n$. For any subset $I' = \{i_1, \cdots, i_d\}$ of $I$ with $1 \leq i_1 < \cdots < i_d \leq n$, we have a projection $\rho'$ from $F^I$ to $F^{I'}$ as follows: $\rho'(\mathbf{a}) = (a_{i_1}, \cdots, a_{i_d}) \in F^{I'}$ for any $\mathbf{a} = (a_1, \cdots, a_n) \in F^I$.

**Definition 3.1.** Let $C \subseteq F^n = F^I$. If there are subsets $I_1, \cdots, I_s$ (with repetition allowed) of the index set $I$ with every cardinality $|I_j| = d$ and an integer $t$ such that
then we say that $C$ is a \textit{balanced code} of $F^n$ with information length $d$, and $I_1, \cdots, I_s$ form a balanced system of information index sets of $C$.

\textbf{Remark 3.2.} For example, any group code $C$ (i.e. any ideal) of the group algebra $FG$ is a balanced code, see [2]; similarly, any coset $a + C$ for $a \in FG$ is a balanced code too.

For any word $a = (a_1, \cdots, a_n) \in F^n$, the fraction $w(a)/n$ is called the \textit{relative weight} of $a$. The following is a generalization of a result in [15], [20] and [21], where only the binary case is considered.

\textbf{Theorem 3.3.} Let $C$ be a balanced code of $F^n$ with information length $d$ and $B$ be a non-empty subset of $C$, and let $\omega = \frac{\sum_{b \in B} w(b)}{|B|}$ (the average relative weight of $B$). If $0 \leq \omega \leq 1 - q^{-1}$, then

$$|B| \leq q^{d h_q(\omega)}. \tag{3.1}$$

Before proving the theorem, we show two corollaries.

\textbf{Corollary 3.4.} Let $C$ be a balanced code of $F^n$ with information length $d$, let $C^{\leq \delta}$ be the set of the codewords of $C$ which relative weight are at most $\delta$. If $0 \leq \delta \leq 1 - q^{-1}$, then $|C^{\leq \delta}| \leq q^{d h_q(\delta)}$.

\textbf{Proof.} The average relative weight of $C^{\leq \delta}$ is at most $\delta$, and $h_q(x)$ is an increasing function in $[0, 1 - q^{-1}]$. \hfill $\square$

For $C \subseteq F^n$, the Cartesian product of $n'$ copies of $C$ in $(F^n)^{n'}$ is as follows:

$$C^{n'} = \{(c_1, \cdots, c_{n'}) \mid c_i \in C, \ i = 1, \cdots, n'\}. \tag{3.2}$$

\textbf{Corollary 3.5.} Let $C$ be a balanced code of $F^n$ with information length $d$. Then the product code $C^{n'}$ is a balanced code of $F^{n'n'}$ with information length $dn'$; in particular, if $0 \leq \delta \leq 1 - q^{-1}$ then $|(C^{n'})^{\leq \delta}| \leq q^{dn'h_q(\delta)}$.

\textbf{Proof.} Assume that the subsets $I_1, \cdots, I_s$ of the index set $I = \{1, \cdots, n\}$ form a balanced system of information index sets of $C$. We write the index set of the product code $C^{n'}$ as:

$$I^{n'} = \{1^{(1)}, \cdots, n^{(1)}_1, \cdots, 1^{(n')}, \cdots, n^{(n')}_1\}.$$

For each $I_j = \{j_1, \cdots, j_d\}$, we can form a subset $I^{n'}_j$ of $I^{n'}$ by concatenating $n'$ copies of $I_j$ as follows:

$$I^{n'}_j = \{j_1^{(1)}_1, \cdots, j_d^{(1)}_1, \cdots, j_1^{(n')}\}, \cdots, j_d^{(n')}\}.$$
Then it is easy to check that $I_{n}^{u} = I_{1}^{u} I_{2}^{u} \cdots I_{s}^{u}$ form a balanced system of information index sets of the product code $C^{n'}$.

The rest of this section contributes to the proof of the theorem.

**Proof of Theorem 3.3** First we assume that $d = n$, i.e. $C = F^{n}$ which is of course balanced (with $s = 1$, $I_{1} = I$ and $t = 1$), and prove the inequality (2): this is a key step of the proof.

Set $M = |B|$. Consider $B$ as a probability space with equiprobability. Each $\mathbf{b} \in B$ is an $n$-tuple: $\mathbf{b} = (b_{1}, \ldots, b_{n})$. For each index $i$, $1 \leq i \leq n$, we have a random variable $X_{i}$ defined over the probability space $B$ and taking values in $F$ as follows:

$$X_{i}(\mathbf{b}) = b_{i};$$

hence we have a discrete distribution function $p_{i}(a) = \Pr(X_{i} = a)$ for $a \in F$; we write the distribution as:

$$p_{i} = (p_{i}(a))_{a \in F}, \quad i = 1, \cdots, n.$$  

Set

$$p = \frac{p_{1} + \cdots + p_{n}}{n},$$

then $p$ is a distribution function. Denote $F^{*} = F \setminus \{0\}$ (which denotes the difference set). It is obvious that

$$\omega = \sum_{a \in F^{*}} p(a) = \sum_{a \in F^{*}} \sum_{i=1}^{n} \frac{p_{i}(a)}{n}; \quad (3.3)$$

hence we also have that

$$1 - \omega = p(0) = \sum_{i=1}^{n} \frac{p_{i}(0)}{n}. \quad (3.4)$$

Consider the random $n$-tuple $\mathbf{X} = (X_{1}, \ldots, X_{n})$ and its entropy with base $q$:

$$H_{q}(\mathbf{X}) = H_{q}(X_{1}, \cdots, X_{n}) = \sum_{\mathbf{a} \in F^{n}} -\Pr(\mathbf{X} = \mathbf{a}) \log_{q} \Pr(\mathbf{X} = \mathbf{a}).$$

For any $\mathbf{a} = (a_{1}, \cdots, a_{n}) \in F^{n}$, by the definition of the random variables $X_{i}$’s, we have

$$\Pr(\mathbf{X} = \mathbf{a}) = \begin{cases} \frac{1}{M}, & \mathbf{a} \in B; \\ 0, & \mathbf{a} \notin B. \end{cases}$$

So we get

$$H_{q}(\mathbf{X}) = H_{q}(X_{1}, \cdots, X_{n}) = \log_{q} M. \quad (3.5)$$

On the other hand, by an inequality for entropy of joint distribution (see [2, Theorem 2.6.6]), we have

$$H_{q}(X_{1}, \cdots, X_{n}) \leq H_{q}(X_{1}) + \cdots + H_{q}(X_{n}) = \sum_{i=1}^{n} \sum_{a \in F} -p_{i}(a) \log_{q} p_{i}(a);$$
so

\[ H_q(X) \leq \left( \sum_{i=1}^{n} -p_i(0) \log_q p_i(0) \right) + \left( \sum_{i=1}^{n} \sum_{a \in F^*} -p_i(a) \log_q p_i(a) \right). \]

Since \( -x \log_q x \) is a concave function, for the second bracket of the right hand side of the above inequality we get (with the help of Eqn (3.3))

\[
\sum_{i=1}^{n} \sum_{a \in F^*} -p_i(a) \log_q p_i(a) \leq \frac{n(q-1)}{(q-1)} \log_q \frac{\omega}{q-1}.
\]

that is

\[
\sum_{i=1}^{n} \sum_{a \in F^*} -p_i(a) \log_q p_i(a) \leq n \left( \omega \log_q (q-1) - \omega \log_q \omega \right).
\]

Similarly, with the help of Eqn (3.4) we can obtain

\[
\sum_{i=1}^{n} -p_i(0) \log_q p_i(0) \leq -n(1 - \omega) \log_q (1 - \omega).
\]

Thus we get

\[
H_q(X) \leq n \left( \omega \log_q (q-1) - \omega \log_q \omega - (1 - \omega) \log_q (1 - \omega) \right) = nh_q(\omega).
\]

Combining it with Eqn (3.5), we obtain that

\[
\log_q |B| = \log_q M \leq nh_q(\omega).
\]

which is just the inequality (3.1) since we have assumed that \( d = n \).

Next we turn to the general case. That is, there are subsets \( I_1, \ldots, I_s \) of the index set \( I = \{1, 2, \ldots, n\} \) with each \( |I_j| = d \) such that any index \( i \in I \) appears in exactly \( t \) members of the \( s \) subsets \( I_1, \ldots, I_s \); in particular, we have

\[
\sum_{i=1}^{n} \sum_{a \in F^*} -p_i(a) \log_q p_i(a) \leq \frac{n(q-1)}{(q-1)} \log_q \frac{\omega}{q-1}.
\]

\[
\sum_{i=1}^{n} -p_i(0) \log_q p_i(0) \leq -n(1 - \omega) \log_q (1 - \omega).
\]

Thus we get

\[
H_q(X) \leq n \left( \omega \log_q (q-1) - \omega \log_q \omega - (1 - \omega) \log_q (1 - \omega) \right) = nh_q(\omega).
\]

Combining it with Eqn (3.5), we obtain that

\[
\log_q |B| = \log_q M \leq nh_q(\omega).
\]

Set \( |B| = M \) again. For each \( I_j \), by \( \rho_j \) we denote the projection from \( F^n = F^I \) onto \( F^{I_j} \); then \( |\rho_j(B)| = M \).

Let \( \tilde{I} \) be the disjoint union of \( I_1, \ldots, I_s \) (though they may be not disjoint), so \( |\tilde{I}| = sd \), and \( F^\tilde{I} = F^{I_1} \times \cdots \times F^{I_s} \) is the product of \( F^{I_j} \) for \( j = 1, \ldots, s \), i.e. the words of \( F^\tilde{I} \) are the concatenations of the words of \( F^{I_j} \) for \( j = 1, \ldots, s \):

\[
F^\tilde{I} = \left\{(a_1, \ldots, a_s) \mid a_j \in F^{I_j}, \ j = 1, \ldots, s \right\}.
\]
Consider the following subset of $F^I$:

$$\hat{B} = \rho_1(B) \times \cdots \times \rho_s(B) = \left\{ (\rho_1(b_1), \cdots, \rho_s(b_s)) \ \bigg| \ b_1, \cdots, b_s \in B \right\}.$$  

Since $|\rho_j(B)| = M$ for $j = 1, \cdots, s$, we see that

$$|\hat{B}| = M^s. \quad (3.7)$$

Set \( \hat{w}(\hat{B}) = \sum_{\hat{b} \in \hat{B}} w(\hat{b}) \), which can be computed as follows:

\[
\hat{w}(\hat{B}) = \sum_{b_1, \cdots, b_s \in B} w(\rho_1(b_1), \cdots, \rho_s(b_s)) = \sum_{b_1, \cdots, b_s \in B} \sum_{j=1}^s w(\rho_j(b_j)) = \sum_{b_1, \cdots, b_s \in B} w(\rho_j(b_j)).
\]

For $j = 1$ we have that

\[
\sum_{b_1, \cdots, b_s \in B} w(\rho_1(b_1)) = \sum_{b_1 \in B} \sum_{b_2, \cdots, b_s \in B} w(\rho_1(b_1)) = \sum_{b_1 \in B} M^{s-1} w(\rho_1(b_1)) = M^{s-1} \sum_{b \in B} w(\rho_1(b)).
\]

Similarly, $\sum_{b_1, \cdots, b_s \in B} w(\rho_j(b_j)) = M^{s-1} \sum_{b \in B} w(\rho_j(b))$. So

$$\hat{w}(\hat{B}) = \sum_{j=1}^s M^{s-1} \sum_{b \in B} w(\rho_j(b)) = M^{s-1} \sum_{b \in B} \sum_{j=1}^s w(\rho_j(b)).$$

By (i) of Definition 3.1, we have

$$\sum_{j=1}^s w(\rho_j(b)) = w(\rho_1(b), \cdots, \rho_s(b)) = tw(b).$$

Recalling that $\omega = \sum_{b \in B} w(b) / nM$, we obtain that

$$\hat{w}(\hat{B}) = M^{s-1} t \sum_{b \in B} w(b) = M^{s-1} t \omega n M = \omega t M^s.$$  

By Eqns (3.6) and (3.7), we compute the average relative weight of $\hat{B}$ as follows:

$$\hat{w}(\hat{B})/sM^s = \omega t n M^s / sdM^s = \omega.$$  

Applying the conclusion proved in the first step (i.e. the case “\( d = n \)”) to the subset $\hat{B}$ of $F^I$, we obtain that $M^s = |\hat{B}| \leq q^{sdh_4(\omega)}$; in other words,

$$|B| = M \leq q^{dh_4(\omega)}.$$  

Theorem 3.3 is proved. \( \square \)
4 Cumulative weight enumerators of $C_A$

We keep the notations in (2.1), (2.2) and (2.3), and further set

$$\hat{N}_{C_A}(\delta) = \left| \left\{ b \in (FG)^k \ | \ 1 \leq w(bA) \leq mn\delta \right\} \right|, \quad (4.1)$$

which is a non-negative integral random variable defined over the probability space $(FG)^{k \times n}$. Obviously, $\hat{N}_{C_A}(\delta)$ stands for the number of such elements $b$ of $(FG)^k$ that $bA$ is a non-zero codewords of $C_A$ with relative weights at most $\delta$; so we call it the cumulative weight enumerator of the random code $C_A$; in particular (cf. [8, §3]),

$$\hat{N}_{C_A}(\delta) \geq 1 \iff \Delta(C_A) \leq \delta. \quad (4.2)$$

We are concerned with the asymptotic behavior of the expectation $E(\hat{N}_{C_A}(\delta))$. The following is the main result of this section.

**Theorem 4.1.** Let notation be as in (2.1), (2.2), (2.3) and (4.1). Then

$$\lim_{n \to \infty} E(\hat{N}_{C_A}(\delta)) = \begin{cases} 0, & r < g_q(\delta); \\ \infty, & r > g_q(\delta); \end{cases}$$

and both the limits converge exponentially.

Before proving the theorem, we show that the first part of Theorem 2.1 is an immediate consequence of the first part of the above theorem.

**Corollary 4.2.** If $r < g_q(\delta)$ then

$$\lim_{n \to \infty} \Pr(\Delta(C_A) > \delta) = 1$$

and the convergence speed is exponential.

**Proof.** By Eqn (4.1), Markov’s inequality (see [18, Theorem 3.1]) and the first part of Theorem 4.1, we have

$$\lim_{n \to \infty} \Pr(\Delta(C_A) \leq \delta) = \lim_{n \to \infty} \Pr(\hat{N}_{C_A}(\delta) \geq 1) \leq \lim_{n \to \infty} E(\hat{N}_{C_A}(\delta)) = 0. \quad \square$$

To prove Theorem 4.1 (and Theorem 2.1 also), a key step is to write $\hat{N}_{C_A}(\delta)$ as a sum of Bernoulli random variables.

For every $b \in (FG)^k$ we define a Bernoulli random variable over the probability space $(FG)^{k \times n}$:

$$X_b = \begin{cases} 1, & \text{if } 1 \leq w(bA) \leq mn\delta; \\ 0, & \text{otherwise}. \end{cases}$$

Set $X = \sum_{b \in (FG)^k} X_b$. It is obvious that $X_0 = 0$ and

$$\hat{N}_{C_A}(\delta) = \sum_{b \in (FG)^k} X_b = X. \quad (4.3)$$
Fixing any $b = (b_1, \ldots, b_k) \in (FG)^k$, we have an $FG$-homomorphism induced by $b$ as follows:

$$\beta_b : (FG)^{k \times n} \to (FG)^n, \quad A \mapsto bA = (bA_1, \ldots, bA_n). \quad (4.4)$$

For each $j$, $bA_j = b_1a_{1j} + \cdots + b_ka_{kj}$; so the set of $bA_j$ with $A_j$ running over $(FG)^k$ is an ideal of $FG$ generated by $b_1, \ldots, b_k$, we denote it by $I_b$:

$$I_b = FGb_1 + \cdots + FGb_k, \quad \text{for} \ b = (b_1, \ldots, b_k) \in (FG)^k;$$

and denote $d_b = \dim I_b$. Thus, the image of $\beta_b$ is the product code $I_b^n \subseteq (FG)^n$, and $\dim I_b^n = d_bn$.

Since $\beta_b$ is an $FG$-homomorphism, the number of the pre-images in $(FG)^{k \times n}$ of every $a \in I_b^n$ is equal to $\frac{q^{\delta n + k}}{q^{d_b n}}$, which is independent of the choice of $a$. And, by Remark 3.2 and Corollary 3.5, we have $|I_b| \leq q^{d_b n h_\delta}$. So

$$E(X_b) = \Pr \left( 1 \leq w(bA) \leq mn\delta \right) \leq \frac{q^{d_b n h_\delta} - 1}{q^{d_b n}};$$

that is

$$E(X_b) \leq q^{-d_b n g_\delta} - q^{-d_b n}, \quad \forall \ b \in (FG)^k. \quad (4.5)$$

For any ideal $I$ of $FG$, we denote $d_I = \dim I$ and set

$$I^{k*} = \left\{ b \in I^k \mid I_b = I \right\}; \quad (4.6)$$

in particular, $(FG)^{k*} = \{ b \in (FG)^k \mid I_b = FG \}$. Obviously, we have a disjoint union $(FG)^k = \bigcup_{I \leq FG} I^{k*}$, where the subscript “$I \leq FG$” means that $I$ runs over the ideals of $FG$. Thus, by the linearity of expectation, we get

$$E(X) = E \left( \sum_{b \in (FG)^k} X_b \right) = \sum_{0 \neq I \leq FG} \sum_{b \in I^{k*}} E(X_b). \quad (4.7)$$

To get a lower bound of $E(X_b)$ for $b \in (FG)^{k*}$, we recall an estimation of a partial sum of binomials:

$$q^{n h_\delta(k/n) - \frac{1}{2} \log_q n} \leq \sum_{i=1}^k \binom{n}{i} (q - 1)^i \leq q^{n h_\delta(k/n)}, \quad (4.8)$$

see [ Eqn.2.3]. One can also check the upper bound of (4.8) from Corollary 3.4 (by taking $C = F^n$, i.e. $k = n$ in the corollary), and check the lower bound by the argument in [ Lemma 9.2].

Let $b_1 \in (FG)^{k*}$, i.e. $I_{b_1} = FG$; then the image of $\beta_{b_1}$ in (4.4) is just the whole space $(FG)^n \cong F^{mn}$, so

$$E(X_{b_1}) = \Pr \left( 1 \leq w(b_1A) \leq mn\delta \right) = \frac{|(F^{mn})^{\leq \delta} | - 1}{|F^{mn}|}; \quad (4.9)$$
in particular,
\[ E(X_{b_1}) = E(X_{b_2}), \quad \forall \ b_1, b_2 \in (FG)^{k^*}. \quad (4.10) \]
Further, since \(|(FG)^{n\delta}| = \sum_{i=1}^{mn^\delta} \binom{mn}{i} (q - 1)^i\) and \(|F^{mn}| = q^{mn}\); by the inequality \((4.8)\) we get that
\[ E(X_{b_1}) \geq q^{-mng_\delta} - \frac{1}{2}\log_4(mn) - q^{-mn}, \quad \forall \ b_1 \in (FG)^{k^*}. \]
Moreover, since \(|(FG)^{k^*}| = \frac{|(FG)^{k^*}|}{|(FG)^k|} q^{nk}\), for \(b_1 \in (FG)^{k^*}\) we have
\[ |(FG)^{k^*}| \cdot E(X_{b_1}) \geq \frac{|(FG)^{k^*}|}{|(FG)^k|} (q^{mn} (\frac{k}{n} - g_\delta) - \frac{1}{2}\log_4(mn) - q^{-mn(1-\frac{k}{n})}). \quad (4.11) \]
Recalling from \((4.3)\) that \(N_{C_A}(\delta) = X\), we show a proof of Theorem 4.1

**Proof of Theorem 4.1**

Now we assume that \(r < g_\delta\). Since \(k = \lfloor rn \rfloor\), there is a positive number \(\gamma\) such that for large enough \(n\) we have \(\frac{k}{n} - g_\delta < -\gamma\). For any \(b \in I^{k^*}\) as above, since \(d_1 = d_1\), from Eqn \((4.3)\) we have \(E(X_b) \leq q^{-nd_1g_\delta}\). Further, because \(|I^{k^*}| \leq |I^k| = q^{dk}\), we get that
\[ \sum_{b \in I^{k^*}} E(X_b) \leq q^{dk}q^{-nd_1g_\delta} = q^{nd_1}\left(\frac{k}{n} - g_\delta\right) < q^{-\gamma d_1n}, \]
and the right hand side is exponentially convergent to 0 as \(n \to \infty\). Note that \(FG\) has only finitely many ideals, by Eqn \((4.7)\) we obtain that
\[ \lim_{n \to \infty} E(N_{C_A}(\delta)) = \lim_{n \to \infty} E(X) = \sum_{0 \neq I \leq FG} \lim_{n \to \infty} \sum_{b \in I^{k^*}} E(X_b) = 0. \]

In the following we assume that \(r > g_\delta\). Since \(k = \lfloor rn \rfloor\), there is a positive number \(\gamma\) such that for large enough \(n\) we have \(\frac{k}{n} - g_\delta > \gamma\) and \(1 - \frac{k}{n} > \gamma\). Fixing a \(b_1 \in (FG)^{k^*}\), from Eqn \((4.7)\) and the Eqn \((4.10)\) we have:
\[ E(X) \geq \sum_{b \in (FG)^{k^*}} E(X_b) = |(FG)^{k^*}| \cdot E(X_{b_1}). \]
Since \(k \to \infty\) as \(n \to \infty\), by Lemma \((4.3)\) below, we have \(\lim_{n \to \infty} \frac{|(FG)^{k^*}|}{|(FG)^k|} = 1\); so, by the inequality \((4.11)\), we obtain the following exponentially convergent limit:
\[ \lim_{n \to \infty} E(X) > \lim_{n \to \infty} \frac{|(FG)^{k^*}|}{|(FG)^k|} (q^{mn\gamma} - \frac{1}{2}\log_4(mn) - q^{-mn\gamma}) = \infty. \]
The proof of Theorem 4.1 is finished. \(\square\)
Lemma 4.3. Assume that \( m = |G| = p^m m' \) with \( m' \) coprime to \( p \), \( G \) has \( h \) irreducible characters over \( F \) with degree \( d_1, \ldots, d_h \) respectively, and \((FG)^{k*}\) is defined as in (4.6). Then the cardinality

\[
|(FG)^{k*}| = \prod_{j=1}^{h} q^{(p^m-1)d_j k} (q^{d_j k} - 1);
\]  

(4.12)

and

\[
\frac{|(FG)^{k*}|}{|(FG)^k|} = \prod_{j=1}^{h} (1 - q^{-d_j k}) \quad k \to \infty
\]

(4.13)

with exponential convergence speed.

Proof. By the assumptions, the abelian group \( G \) has a subgroup \( G' \) of order \( m' \) and a subgroup \( G'' \) of order \( p^m \) such that \( G = G'' \times G' \); hence we can assume that the group algebra \( FG' \) has \( h \) irreducible ideals \( E_j \) over \( F \) and denote \( d_j = \dim_F E_j \) for \( j = 1, \ldots, h \). Then each \( E_j \) is a field extension of \( F \) and

\[ FG' = E_1 \oplus E_2 \oplus \cdots \oplus E_h. \]

Since \( FG \cong FG'' \otimes_F FG' \) and \( FG'' \) is a local ring with head \( FG''/J(FG'') \cong F \) where \( J(FG'') \) denotes the Jacobson radical, we have

\[
FG \cong R_1 \oplus \cdots \oplus R_h,
\]

(4.14)

where \( R_j = FG'' \otimes_F E_j \) for \( j = 1, \ldots, h \) is a local algebra with

\[
R_j/J(R_j) \cong E_j, \quad \dim_F R_j = p^m d_j \quad \text{and} \quad \dim_F J(R_j) = p^m d_j - d_j.
\]

Thus we get

\[
(FG)^k \cong R_1^k \oplus \cdots \oplus R_h^k.
\]

A vector \( b_j = (b_{j1}, \ldots, b_{jk}) \) of \( R_j^k \) generates \( R_j \) (i.e. \( R_j b_{j1} + \cdots + R_j b_{jk} = R_j \)) if and only if the image of \( b_j \) in the residue \( R_j^k/J(R_j)^k \cong E_j^k \) is non-zero, i.e. \( R_j^k = R_j^k \setminus J(R_j)^k \) (the difference set). So we get

\[
|R_j^k| = q^{p^m d_j k} - q^{(p^m-1)d_j k} = q^{(p^m-1)d_j k} (q^{d_j k} - 1).
\]

It is easy to check that \((FG)^{k*} = R_1^{k*} \times \cdots \times R_h^{k*}\). We obtain that

\[
|(FG)^{k*}| = \prod_{j=1}^{h} q^{(p^m-1)d_j k} (q^{d_j k} - 1);
\]

hence

\[
\frac{|(FG)^{k*}|}{|(FG)^k|} = \prod_{j=1}^{h} (1 - q^{-d_j k}),
\]

which converges, as \( k \to \infty \), exponentially to 1. \( \square \)
5 Second moment method for the main theorem

In this section we keep the notations in Theorem 2.1 and Eqn (4.3).

In this section we always assume that \( r > g_\delta \) and prove that

\[
\lim_{n \to \infty} \Pr(X \geq 1) = 1 \quad \text{with exponential convergence speed;} \tag{5.1}
\]

which completes the proof of Theorem 2.1, since \( \Pr(\Delta(C_A) \leq \delta) = \Pr(X \geq 1) \), see (4.2), hence Eqn (5.1) implies that

\[
\lim_{n \to \infty} \Pr(\Delta(C_A) > \delta) = 0.
\]

By a known inequality, see [18, Theorem 6.10], we have that

\[
\Pr(X \geq 1) \geq \sum_{b \in (FG)^k} E(X_b) \frac{E(X|X_b = 1)}{E(X|X_b = 1)},
\]

where \( E(X|X_b = 1) \) denotes the conditional expectation, which is essentially involved in the second moment of \( X \). Such a way to investigate phase transitions (thresholds) by means of second moments is usually named the second moment method; e.g. see [9, Appendix].

Since \((FG)^{k*}\) is a part of \((FG)^k\), see Eqn (4.6), we have

\[
\Pr(X \geq 1) \geq \sum_{b \in (FG)^{k*}} E(X_b) \frac{E(X|X_b = 1)}{E(X|X_b = 1)}. \tag{5.2}
\]

By the linearity of expectations, for \( b_1 \in (FG)^k \) we have

\[
E(X|X_{b_1} = 1) = E\left( \sum_{b \in (FG)^k} X_b \middle| X_{b_1} = 1 \right) = \sum_{b \in (FG)^k} E(X_b|X_{b_1} = 1).
\]

By the conditional probability formula (and noting that \( X_b \)'s are 0-1 variables), we further have

\[
E(X_b|X_{b_1} = 1) = \frac{\Pr(X_b = 1 \& X_{b_1} = 1)}{\Pr(X_{b_1} = 1)} = \frac{E(X_b X_{b_1})}{E(X_{b_1})}.
\]

Set

\[
A(b, b_1) = \left\{ A \in (FG)^{k \times n} \mid 1 \leq w(bA), w(b_1A) \leq mn\delta \right\},
\]

then

\[
E(X_b X_{b_1}) = \Pr(X_b = 1 \& X_{b_1} = 1) = \frac{|A(b, b_1)|}{|(FG)^{k \times n}|} = \frac{|A(b, b_1)|}{q^{mnk}}.
\]

Thus we get that

\[
E(X_b|X_{b_1} = 1) = \frac{|A(b, b_1)|}{q^{mnk}E(X_{b_1})} \tag{5.3}
\]
For any invertible $k \times k$ matrix $Q$ over $FG$,

$$\mathcal{A}(bQ, b_1Q) = \{ A \in (FG)^{k \times n} \mid 1 \leq w(bQ A), w(b_1 Q A) \leq m n \delta \}$$

$$= \{ A \in (FG)^{k \times n} \mid QA \in \mathcal{A}(b, b_1) \} = \{ Q^{-1} A \mid A \in \mathcal{A}(b, b_1) \};$$

in particular, we have that $|\mathcal{A}(bQ, b_1Q)| = |\mathcal{A}(b, b_1)|$.

Now we can show that

$$E(X \mid X_{b_1} = 1) = E(X \mid X_{b_2} = 1), \quad \forall \ b_1, b_2 \in (FG)^{k*}.$$  \hspace{1cm} (5.4)

To see it, by [7, Proposition 2.11] we can take an invertible $k \times k$ matrix $Q$ over $FG$ such that $b_2 = b_1 Q$; then, by Eqns (5.3) and (4.10), we have

$$E(X_{b_1} \mid X_{b_1} = 1) = \frac{|\mathcal{A}(b, b_1)|}{q^{mnk} E(X_{b_1})} = \frac{|\mathcal{A}(bQ, b_1Q)|}{q^{mnk} E(X_{b_1})} = E(X_{bQ} \mid X_{b_2} = 1);$$

hence

$$E(X \mid X_{b_1} = 1) = \sum_{b \in (FG)^k} E(X_{b} \mid X_{b_1} = 1) = \sum_{b \in (FG)^k} E(X_{bQ} \mid X_{b_2} = 1);$$

noting that $bQ$ runs over $(FG)^k$ when $b$ runs over $(FG)^k$, we obtain that

$$E(X \mid X_{b_1} = 1) = \sum_{b \in (FG)^k} E(X_b \mid X_{b_2} = 1) = E(X \mid X_{b_2} = 1),$$

which is just Eqn (5.4).

From now on to the end of this section we fix $b_1 = (0, \ldots, 0, 1)$, which belongs obviously to $(FG)^{k*}$. By Eqns (5.2), (4.10) and (5.4), we have

$$\Pr(X \geq 1) \geq \frac{|FG^{k*} \cdot E(X_{b_1})}{E(X \mid X_{b_1} = 1)}.$$  \hspace{1cm} (5.5)

Thus, to prove Eqn (5.1), it is enough to prove that

$$\lim_{n \to \infty} \frac{E(X \mid X_{b_1} = 1)}{|FG^{k*} \cdot E(X_{b_1})} = \lim_{n \to \infty} \frac{\sum_{b \in (FG)^k} E(X_b \mid X_{b_1} = 1)}{|FG^{k*} \cdot E(X_{b_1})} = 1$$

and it converges exponentially.

For any $A \in (FG)^{k \times n}$, by $A_i$ we denote the $i$’th row of $A$. To compute $\mathcal{A}(b, b_1)$, we set

$$\mathcal{A}(b_1) = \{ A \in (FG)^{k \times n} \mid 1 \leq w(b_1 A) \leq m n \delta \};$$

since $b = (0, \cdots, 0, 1)$, it is clear that

$$\mathcal{A}(b_1) = \{ A \in (FG)^{k \times n} \mid 0 \neq A_k \in ((FG)^n)_{\leq \delta} \},$$
that is, \( \mathcal{A}(b_1) \) is the set of the \( k \times n \) matrices \( A \) over \( FG \) such that the \( k \)th row \( A_k \neq 0 \) and \( w(A_k) \leq mn \delta \); see the notation in Corollary 3.4.

Given any non-zero \( (a_{k1}, \ldots, a_{kn}) \in \left((FG)^n\right)^{\leq \delta} \), we denote

\[
\mathcal{A}(b_1)(a_{k1}, \ldots, a_{kn}) = \left\{ A \in (FG)^{k \times n} \mid A_k = (a_{k1}, \ldots, a_{kn}) \right\}.
\]

Then any \( b = (b_1, \ldots, b_{k-1}, b_k) \in (FG)^k \) induces a map:

\[
\beta_b : \quad \mathcal{A}(b_1)(a_{k1}, \ldots, a_{kn}) \rightarrow (FG)^n,
A \mapsto bA = b_1A_1 + \cdots + b_{k-1}A_{k-1} + b_kA_k;
\tag{5.6}
\]

Set \( 
\bar{b} = (b_1, \ldots, b_{k-1}), 
I_b = FGb_1 + \cdots + FGb_{k-1}
\) which is the ideal of \( FG \) generated by \( b_1, \ldots, b_{k-1} \), and set \( d_b = \dim I_b \). It is easy to see that the image of the map \( \beta_b \) is a coset of \( I_b^n \subseteq (FG)^n \) as follows

\[
I_b^n + b_kA_k,
\quad \text{with cardinality } |I_b^n + b_kA_k| = |I_b^n| = q^{d_b n};
\]

and the number of the pre-images in \( \mathcal{A}(b_1)(a_{k1}, \ldots, a_{kn}) \) of any \( a \in I_b^n + b_kA_k \) is equal to \( q^{mn(k-1)-d_bn} \), which is independent of the choices of \( a \) and \( (a_{k1}, \ldots, a_{kn}) \). Thus the cardinality of the pre-image in \( \mathcal{A}(b_1)(a_{k1}, \ldots, a_{kn}) \) of the set \( (I_b^n + b_kA_k)^{\leq \delta} \{0\} \) is

\[
\left( \left| (I_b^n + b_kA_k)^{\leq \delta} \right| - \lambda \right) q^{mn(k-1)-d_b n}
\]

with

\[
\lambda = \begin{cases} 
1, & \text{if } 0 \in I_b^n + b_kA_k; \\
0, & \text{otherwise};
\end{cases}
\tag{5.7}
\]

hence

\[
|\mathcal{A}(b, b_1)| = \sum_{0 \neq (a_{k1}, \ldots, a_{kn}) \in ((FG)^n)^{\leq \delta}} \left( \left| (I_b^n + b_kA_k)^{\leq \delta} \right| - \lambda \right) q^{mn(k-1)-d_b n};
\]

that is

\[
|\mathcal{A}(b, b_1)| = \left( \left| ((FG)^n)^{\leq \delta} \right| - 1 \right) \left( \left| (I_b^n + b_kA_k)^{\leq \delta} \right| - \lambda \right) q^{mn(k-1)-d_b n}.
\]

But \( \left| ((FG)^n)^{\leq \delta} \right| - 1 = q^{mn}E(X_{b_1}) \), see Eqn (4.10). By Eqn (5.8), we get that

\[
E(X_b | X_{b_1} = 1) = \left( \left| (I_b^n + b_kA_k)^{\leq \delta} \right| - \lambda \right) q^{-d_b n}.
\tag{5.8}
\]

By the disjoint union \( (FG)^{k-1} = \bigcup_{I \subseteq FG} I^{(k-1)^*} \) again, cf. Eqn (4.7) (but this time we consider \( \bar{b} \) which has length \( k-1 \), we have

\[
\sum_{b \in (FG)^k} E(X_b | X_{b_1} = 1) = \sum_{I \subseteq FG} \sum_{b \in I^{(k-1)^*}} \sum_{b \in FG} E(X_b | X_{b_1} = 1),
\]

15
Recalling that \( d \) where \( O \) large enough \( n \) and \( k \) since \( \bar{b} = (b_1, \ldots, b_{k-1}) \) and \( b = (b_1, \ldots, b_{k-1}, b_k) \). Thus
\[
\frac{\sum_{b \in (FG)^k} E(X_b | X_{b_1} = 1)}{|(FG)^k| \cdot E(X_{b_1})} = \sum_{0 \neq I \leq FG} S_I
\]
with
\[
S_I = \frac{\sum_{b \in I^{(k-1)}} \sum_{b_k \in FG} E(X_b | X_{b_1} = 1)}{|(FG)^k| \cdot E(X_{b_1})}.
\]
We compute the \( S_I \)'s for ideals \( I \) of \( FG \) into two cases.

**Case 1.** \( 0 \neq I \neq FG \); note that there are only finitely many such ideals of \( FG \). Let \( d_I = \dim I \); then \( d_I < m \). By Remark 3.2 and Corollary 3.5 we see that \( I^n_b + b_k A_k \) is a balanced code and
\[
\mu(I^n_b + b_k A_k)^{\delta \delta} - \lambda \leq \mu(I^n_b + b_k A_k)^{\delta \delta} \leq q^{d_k n h_k(\delta)}.
\]
By Eqn (5.8) and the above inequality, we obtain that
\[
E(X_b | X_{b_1} = 1) \leq q^{-d_k n g_k(\delta)}.
\]
Note that \( d_k = d_I \) for any \( b \in I^{(k-1)*} \), \( |I^{(k-1)*}| \leq |I^{(k-1)}| = q^{d_I(k-1)} \) and \( |FG| = q^m \). By the inequality (4.11) and the above inequality, we have
\[
S_I \leq \frac{\mu(I^{(k-1)*}) q^{d_I(k-1)} q^m q^{-d_k n g_k(\delta)}}{|(FG)^k|} \left( q^{mn(\frac{1}{n} - g_k(\delta)) - \frac{1}{2} \log_2 mn} - q^{-m n(1 - \frac{1}{n})} \right)
\]
\[
= \frac{|(FG)^k|}{\mu(I^{(k-1)*})} \frac{q^{d_I n (\frac{1}{n} - g_k(\delta)) - d_I + m}}{q^{mn(\frac{1}{n} - g_k(\delta)) - \frac{1}{2} \log_2 mn} - q^{-m n(1 - \frac{1}{n})}}.
\]
Since \( k = [\gamma n] \) and \( 1 > r > g_k(\delta) \), there is a real number \( \gamma > 0 \) such that for large enough \( n \) we have \( \frac{1}{n} - g_k(\delta) > \gamma \) and \( 1 - \frac{1}{n} > \gamma \); hence
\[
\frac{1}{S_I} \geq \frac{|(FG)^k|}{\mu(I^{(k-1)*})} \left( q^{\gamma (m - d_I)n + \frac{1}{2} \log_2 (mn) + d_I - m} - O(q^{-\gamma n}) \right),
\]
where \( O(q^{-\gamma n}) \) stands for a quantity bounded from above by a multiple of \( q^{-\gamma n} \). Recalling that \( d_I < m \) and \( \lim_{n \to \infty} \frac{|(FG)^k|}{\mu(I^{(k-1)*})} = 1 \) (see Eqn (4.13)), we get that
\[
\lim_{n \to \infty} S_I = 0, \quad \text{if} \quad I \neq FG,
\]
and the limit converges exponentially.

**Case 2.** \( I = FG \) and \( \bar{b} \in I^{(k-1)*} \). Then \( d_{\bar{b}} = m \), and \( I^n_{\bar{b}} = (FG)^n \), hence \( I^n_{\bar{b}} + b_k A_k = (FG)^n \); in particular, \( \lambda = 1 \) in Eqn (5.7). By Eqn (4.9), we get
\[
\mu(I^n_{\bar{b}} + b_k A_k)^{\delta \delta} - \lambda = \mu((FG)^n)^{\delta \delta} - 1 = q^{mn} E(X_{b_1}).
\]
By Eqn (5.8) we can compute
\[
S_I = \frac{|(FG)^{(k-1)*}| \cdot q^m \cdot q^{mn} E(X_{b_1}) \cdot q^{-mn}}{|(FG)^{k*}| \cdot E(X_{b_1})}
\]
\[
= \frac{|(FG)^{(k-1)*}| \cdot q^m}{|(FG)^{k*}|} = \frac{|(FG)^{(k-1)*}| \cdot E(X_{b_1})}{|(FG)^{k-1}| \cdot E(X_{b_1})}
\]
By the exponential convergence \( \lim_{n \to \infty} \frac{|(FG)^k|}{|(FG)^{k*}|} = 1 \) (see Eqn (4.13)) again, we get the following exponential convergent limit:
\[
\lim_{n \to \infty} S_I = 1, \quad \text{if } I = FG. \tag{5.11}
\]

Finally, by Eqn (5.9), Eqn (5.10) and Eqn (5.11), we obtain that
\[
\lim_{n \to \infty} \sum_{b \in (FG)^k} E(X_b | X_{b_1} = 1) = 1
\]
and it converges exponentially; this is just what Eqn (5.5) requires.

## 6 Random quasi-abelian codes

Keep notations in (2.1), (2.2) and (2.3).

Recall that for \( A \in (FG)^{k \times n} \) the rate \( R(C_A) = \frac{k}{n} \) if and only if the FG-rank of \( A \) is equal to \( k \); at that case we say that \( A \) is full-rank.

In order to get random quasi-abelian codes of rate \( \frac{k}{n} \approx r \), we consider the probability space \( \mathcal{F} \), which sample space is \( \{ A \in (FG)^{k \times n} \mid A \text{ is full-rank} \} \) and probability function is equiprobability. Take \( A \in \mathcal{F} \), construct \( C_A \) the same as in (2.3). By \( \Pr_{\mathcal{F}}(\Delta(C_A) > \delta) \) we emphasize that the probability is computed over the probability space \( \mathcal{F} \).

**Corollary 6.1.** Let notation be as above. Then
\[
\lim_{n \to \infty} \Pr_{\mathcal{F}}(\Delta(C_A) > \delta) = \begin{cases} 
1, & r < g_{\delta}(\delta); \\
0, & r > g_{\delta}(\delta);
\end{cases}
\]
and both the limits converge exponentially.

**Proof.** Let \( A \in (FG)^{k \times n} \). By the total probability formula we have
\[
\Pr(\Delta(C_A) > \delta) = \Pr(\Delta(C_A) > \delta \mid A \text{ is full-rank}) \cdot \Pr(A \text{ is full-rank}) + \Pr(\Delta(C_A) > \delta \mid A \text{ is not full-rank}) \cdot \Pr(A \text{ is not full-rank}).
\]
Noting that
\[
\Pr(\Delta(C_A) > \delta \mid A \text{ is full-rank}) = \Pr_{\mathcal{F}}(\Delta(C_A) > \delta);
\]
and by Lemma 6.3 below,

\[ \lim_{n \to \infty} \Pr(A \text{ is not full-rank}) = 0, \quad \lim_{n \to \infty} \Pr(A \text{ is full-rank}) = 1; \]

so we get

\[ \lim_{n \to \infty} \Pr(\Delta(C_A) > \delta) = \lim_{n \to \infty} \Pr_{\mathcal{F}}(\Delta(C_A) > \delta). \]

Then the corollary follows from Theorem 2.1 at once.

From the first part (the case \( r < g_q(\delta) \)) of the above corollary we obtain the following result immediately.

**Corollary 6.2.** For any \((r, \delta) \in (0, 1) \times (0, 1 - q^{-1})\) satisfying that \( r < g_q(\delta) \), there exists a series of quasi-FG codes \( C_1, C_2, \cdots \) such that:

(i) the length of \( C_i \) goes to infinity;

(ii) \( \lim_{i \to \infty} R(C_i) = r; \)

(iii) \( \lim_{i \to \infty} \Delta(C_i) \geq \delta. \)

**Lemma 6.3.** Let \( A \in (FG)^{k \times n} \) where \( k = [rn] \) and \( 0 < r < 1. \) Then \( \lim_{n \to \infty} \Pr (A \text{ is full-rank}) = 1, \) or equivalently, \( \lim_{n \to \infty} \Pr (A \text{ is not full-rank}) = 0; \) and both the limits converge exponentially.

**Proof.** Let \( A = (a_{\alpha \beta})_{k \times n} \) with \( a_{\alpha \beta} \in FG. \) We quote the decomposition of \( FG \) in [14] and adopt their notations. Each \( a_{\alpha \beta} \) can be written as

\[ a_{\alpha \beta} = (a_{\alpha \beta}^{(1)}, \cdots, a_{\alpha \beta}^{(h)}), \quad a_{\alpha \beta}^{(j)} \in R_j; \]

hence the matrix \( A \) can be rewritten as \( A = (A^{(1)}, \cdots, A^{(h)}) \) with \( A^{(j)} = (a_{\alpha \beta}^{(j)})_{k \times n} \) being \( k \times n \) matrix over the local algebra \( R_j \) (cf. [7, Eqn (2.2)]), and \( A \) is full-rank if and only if every \( A^{(j)} \) is full-rank over the field \( E_j \) for \( j = 1, \cdots, h, \) cf. [7] Lemma 2.2. For \( 1 \leq i \neq j \leq h, \) it is clear that \( A^{(i)} \in R_j^{k \times n} \) and \( A^{(j)} \in R_j^{k \times n} \) are randomly independent of each other. So we have

\[ \Pr (A \text{ is full-rank}) = \prod_{j=1}^{h} \Pr (A^{(j)} \text{ is full-rank}). \]  

Let \( j \) with \( 1 \leq j \leq h \) be given. First we claim that

\[ A^{(j)} \text{ is full-rank} \iff \bar{A}^{(j)} \text{ is full-rank}, \]

where \( \bar{A}^{(j)} = (\bar{a}_{\alpha \beta}^{(j)})_{k \times n} \) is the image of \( A^{(j)} \) in \( E_j^{k \times n}, \) i.e. each \( \bar{a}_{\alpha \beta}^{(j)} \) is the image of the element \( a_{\alpha \beta}^{(j)} \) in the residue filed \( E_j = R_j/J(R_j). \) To see it, we remark that \( A^{(j)} \) is full-rank if and only if it is right invertible, cf. [7] Lemma 2.6. Suppose that \( A^{(j)} \) is full-rank, then \( A^{(j)}B = I_{k \times k} \) for a \( B \in R_j^{n \times k}, \) where \( I_{k \times k} \)
stands for the identity $k \times k$ matrix; mapping them to matrices over $E_j$, we get
that $\bar{A}^{(j)}B = I_{k \times k}$, which implies that $A^{(j)}$ is full-rank over $E_j$. Conversely, if $A^{(j)}$ is full-rank over $E_j$, then $A^{(j)}B = I_{k \times k}$ for a $B \in R_j^{n \times k}$, hence
\[
A^{(j)}B = I_{k \times k} + C, \quad \text{with} \ C \in J(R_j)^{k \times k};
\]
since $C$ is a nilpotent matrix, $I_{k \times k} + C$ is an invertible matrix; hence $A^{(j)}$ is full-rank over $R_j$.

Next we claim that
\[
\Pr \left( A^{(j)} \text{ is not full-rank} \right) \leq q^{d_j(k-n)} \approx q^{d_j(r-1)n}. \tag{6.3}
\]
To see it, we note three points: $\bar{A}^{(j)}$ is not full-rank if and only if there a $(k-1)$-dimensional subspace of $E_j^k$ which contains all the columns of $\bar{A}^{(j)}$; the probability that a $(k-1)$-dimensional subspace of $E_j^k$ contains all the columns of $\bar{A}^{(j)}$ is $1/q^{d_jn}$ (recall that the cardinality $|E_j| = q^{d_j}$); the number of the $(k-1)$-dimensional subspaces of $E_j^k$ is $q^{d_jk} \leq q^{d_jk}$; thus
\[
\Pr \left( A^{(j)} \text{ is not full-rank} \right) \leq q^{d_jk} \cdot \frac{1}{q^{d_jn}} = q^{d_j(k-n)}.
\]
Each matrix in $E_j^{k \times n}$ has exactly $|J(R_j)|^{kn}$ inverse images in $R_j^{k \times n}$. So the claim \[6.3\] follows from the above inequality and the conclusion \[6.2\].

Finally, since $r-1 < 0$, from the inequality \[6.3\] we obtain
\[
\lim_{n \to \infty} \Pr \left( A^{(j)} \text{ is not full-rank} \right) \leq \lim_{n \to \infty} q^{d_j(r-1)n} = 0.
\]
By Eqn \[6.1\], we are done for the lemma. \hfill \Box

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