Gaussian Time-Dependent Variational Principle for Bosons
Contact Interaction in one dimension

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Abstract

We investigate the Dirac time-dependent variational method using a Gaussian trial functional for an infinite one dimensional system of Bosons interacting through a repulsive contact interaction. The method produces a set of non-linear time dependent equations for the variational parameters. By solving the static equations we have calculated the ground state energy per particle. We have also considered small oscillations about equilibrium and obtain mode equations which lead us to a gapless dispersion relation. The existence of an exact numerical solution for the ground state energy and excitations obtained by Lieb allow us to compare with the Gaussian results. We can also, as the system becomes less dilute, see the improvement of the results as compared
with the Bogoliubov scheme.

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I. INTRODUCTION

Recently Bose-Einstein condensation in atomic traps was achieved \cite{1-3} in a dilute regime $(a^3 n << 1)$. This is in contrast with the Helium 4 regime where a much higher density leads to $(a^3 n \approx 1)$. For the high density regime there is a large ”depletion” $(\approx 90\%)$ \cite{4} where as for the dilute regime it is very small $(\approx 1\%)$ \cite{5}.

Theoretically these new experiments have been described very successfully with mean field theory without quantum fluctuations using the Gross Pitaevskii equation \cite{6} or the Bogoliubov scheme \cite{5}. The Gaussian self-consistent approximation presented here should be useful at in intermediate regime. Unfortunately due to three body recombinations there seems to be a limit \cite{7} for increasing the number of particles in the system. Recently it has been pointed out \cite{8} that using a strong magnetic field it would be possible to make the system effectively less dilute by drastically changing the scattering length. This makes the comparison between self-consistent results and dilute theories very important. In this context the one dimensional delta function case can produce some insight because a contact interaction can be used in the self-consistent theory in contrast with the 3 dimensional case \cite{9} - \cite{10}. The existence of an exact solution for the ground state energy and for particle and hole excitations makes the comparison very interesting, provide that we understand how the separated particle and hole excitations of the exact solution \cite{20} are connected to the particle-hole excitations given by the approximate methods.

The objective of this paper is to exhibit the most general way of obtaining time dependent equations of motion in the Gaussian approximation \cite{11}. This will lead to the so called generalized RPA, when one examines infinitesimal oscillations about equilibrium. The static solution in the uniform case can be obtained using several other methods \cite{12} - \cite{9} leading to a gap in the quasi-boson energy. We show here that the time dependent RPA equations lead to a gapless mode. In fact this must happen because particle number conservation symmetry is broken in the static solution, so the zero gap is exactly the associated Goldstone mode. This discussion can be seen as an alternative to the functional derivative \cite{13}, \cite{14} method.
in the Girardeau-Arnowitt \cite{15} approximation.

The Bogoliubov scheme, for a dilute or weak interacting system, can be obtained by a particular truncation of the Gaussian results. So that we can compare the Gaussian variational results, the dilute Bogoliubov scheme and the exact solution, for the particular case under discussion here.

The structure of this paper is as follows. Section 2 reviews the time-dependent variational principle and the canonical nature of the equations of motion arising from it. In section 3 we specialize to the one dimensional uniform case and examine the ground state energy and the excitations for both the approximate methods and the exact solution. Section 4 contains our numerical solutions and conclusions.

II. GENERAL FORMALISM

In this section we shall review some of the results of the Time-dependent Variational Principle \cite{11}, \cite{16} and show how it can be implemented in the non-relativistic case. First we define an effective action functional for the time-dependent quantum system

\[ S = \int L(t) dt = \int dt \langle \Psi, t | (i \partial_t - \hat{H}) | \Psi, t \rangle, \]  

(1)

where \(|\Psi, t\rangle\) is the quantum state of the system and \(\hat{H}\) is the Hamiltonian of the theory. For a system of non-relativistic interacting Bosons we have [we use the notation : \(\int_x = \int d^3 x\)]

\[ \hat{H} = \int_{x,y} \hat{\psi}(x)^\dagger h(x,y) \hat{\psi}(y) + \frac{1}{2} \int_{x,y} \hat{\psi}(y)^\dagger \hat{\psi}(y)^\dagger V(x-y) \hat{\psi}(x) \hat{\psi}(y), \]  

(2)

where the one body Hamiltonian \(h(x,y)\) may include a one body external potential. The creation and destruction operators \(\hat{\psi}^\dagger\) and \(\hat{\psi}\) can be written in the form

\[ \hat{\psi}(x) = \frac{1}{\sqrt{2}} \left[ \hat{\phi}(x) + i \hat{\pi}(x) \right] \]  

(3)

\[ \hat{\psi}(x)^\dagger = \frac{1}{\sqrt{2}} \left[ \hat{\phi}(x) - i \hat{\pi}(x) \right] \]
where $\hat{\phi}(x)$ is the field operator and $\hat{\pi}(x)$ the canonical field momentum.

We can obtain the time dependent Schrödinger equation by requiring that $S$ is stationary, supplemented by appropriate boundary conditions, under the most general variation of $|\Psi(t)|$. The variational scheme is implemented by choosing a trial wave functional describing the system. Working in the functional Shrödinger picture we replace the abstract state $|\Psi(t)|$ by a wave functional of the field $\phi'(x)$

$$|\Psi(t)| \rightarrow \Psi[\phi',t].$$  \hspace{1cm} (4)

The action of the operators $\hat{\phi}(x)$ and the canonical momentum $\hat{\pi}(x)$ are realized respectively by

$$\hat{\phi}(x)|\Psi(t)\rangle \rightarrow \phi'(x)\Psi[\phi',t]$$

$$\hat{\pi}(x)|\Psi(t)\rangle \rightarrow -i\frac{\delta}{\delta\phi'(x)}\Psi[\phi',t].$$ \hspace{1cm} (5)

The mean value of any operator is calculated by the functional integral

$$\langle \Psi, t | O | \Psi, t \rangle = \int (D\phi') \Psi^*[\phi',t]O\Psi[\phi',t],$$ \hspace{1cm} (6)

where $\Psi$ is normalized to unity. The Gaussian approximation consists in taking a Gaussian trial wave functional in its most general parametrization

$$\Psi[\phi',t] = N \exp \left\{ -\int_{x,y} \delta\phi'(x,t) \left[ \frac{G^{-1}(x,y,t)}{4} - i\Sigma(x,y,t) \right] \delta\phi'(y,t) \ight. \
+ \left. i \int_{x} \pi(x,t)\delta\phi'(x,t) \right\},$$ \hspace{1cm} (7)

with $\delta\phi'(x,t) = \phi'(x) - \phi(x,t)$. Due to the fact that the Hamiltonian commutes with the number of particles $\hat{N} = \int x \hat{\psi}^\dagger(x)\hat{\psi}(x)$ i.e.

$$[\hat{H}, \hat{N}] = 0$$ \hspace{1cm} (8)

we can actually define a more general trial functional.
\[ |\Psi', t\rangle = e^{-i\tilde{N}\theta(t)} |\Psi, t\rangle, \]  

where \( \theta(t) \) is another variational parameter introduced because of this continuous symmetry. Thus our variational parameters are \( \phi(x, t), \pi(x, t), \theta(t), G(x, y, t) \) and \( \Sigma(x, y, t) \), with \( G \) and \( \Sigma \) being real symmetric matrices. These quantities are related to the following mean-values:

\[
\langle \Psi', t|\hat{\phi}(x)|\Psi', t \rangle = \phi(x, t) = \langle \Psi', t|\hat{\pi}(x)|\Psi', t \rangle = \pi(x, t)
\]

\[
\langle \Psi', t|\hat{\phi}(x)\hat{\phi}(y)|\Psi', t \rangle = G(x, y, t) + \phi(x, t)\phi(y, t)
\]

\[
\langle \Psi', t|i\delta \delta_{t}^{}|\Psi', t \rangle = \int_{x,y} \Sigma(x, y, t)\hat{G}(y, x, t) + \int_{x} \pi(x, t)\hat{\phi}(x, t)
\]

\[
+\mathcal{N}\dot{\theta}(t) + \text{total time derivatives}. \tag{11}
\]

We may ignore the total time derivatives because they do not contribute to the equations of motion. If now we write the action we will get

\[
S = \int dt \left( \int_{x} \pi(x, t)\dot{\phi}(x, t) + \int_{x,y} \Sigma(x, y, t)\dot{G}(y, x, t) + \mathcal{N}\dot{\theta}(t) - \mathcal{H} \right), \tag{12}
\]

where

\[
\mathcal{H} = \langle \Psi', t|\hat{H}|\Psi', t \rangle \tag{13}
\]

and

\[
\mathcal{N} = \langle \Psi', t|\hat{N}|\Psi', t \rangle. \tag{14}
\]

From (12) we see that \((\mathcal{N}, \theta), (\pi, \phi)\) and \((\Sigma, G)\) are canonical pairs. Because of the symmetry \( \mathcal{H} \) has no dependence on \( \theta \) and it follows that \( \dot{\mathcal{N}} = 0 \) and \( \dot{\theta}(t) = \text{constant} \equiv \mu \). We can now write the remaining Hamilton equations,

\[
\dot{\phi}(x, t) = \frac{\delta(\mathcal{H} - \mu\mathcal{N})}{\delta\pi(x, t)},
\]

\[
\dot{\pi}(x, t) = -\frac{\delta(\mathcal{H} - \mu\mathcal{N})}{\delta\phi(x, t)}, \tag{15}
\]

\[
\dot{G}(x, y, t) = \frac{\delta(\mathcal{H} - \mu\mathcal{N})}{\delta\Sigma(x, y, t)},
\]

\[
\dot{\Sigma}(x, y, t) = -\frac{\delta(\mathcal{H} - \mu\mathcal{N})}{\delta G(x, y, t)}.\]
For convenience we introduce

$$
\psi(x, t) \equiv \langle \hat{\psi}(x) \rangle = \frac{\phi(x, t) + i\pi(x, t)}{\sqrt{2}},
$$

(16)

so that the equations for \( \phi \) and \( \pi \) become

$$
i\hat{\psi}(x, t) = \frac{\delta(H - \mu \mathcal{N})}{\delta \psi^*(x, t)}
$$

(17)

To obtain \( H - \mu \mathcal{N} \) we have to compute

$$
\mathcal{H} - \mu \mathcal{N} = \int (\mathcal{D}\phi')\Psi^*[\phi', t] \left[ \hat{H} - \mu \hat{\mathcal{N}} \right] \Psi[\phi', t]
$$

(18)

using (3) and (5) we have

\[
\begin{align*}
\mathcal{H} - \mu \mathcal{N} &= \int (\mathcal{D}\phi')\Psi^*[\phi', t] \left( \phi'(x) - \frac{\delta}{\delta \phi'(x)} \right) h(x, y) \left( \phi'(y) + \frac{\delta}{\delta \phi'(y)} \right) \Psi[\phi'] \\
&+ \int_{x,y} (\mathcal{D}\phi')\Psi^*[\phi'] \left( \phi'(x) - \frac{\delta}{\delta \phi'(x)} \right) \left( \phi'(y) - \frac{\delta}{\delta \phi'(y)} \right) V(x - y) \\
&\times \left( \phi'(x) + \frac{\delta}{\delta \phi'(x)} \right) \left( \phi'(y) + \frac{\delta}{\delta \phi'(y)} \right) \Psi[\phi']
\end{align*}
\]

(19)

All the functional integrals can be easily computed using an additional source term (Appendix A) leading to

\[
\begin{align*}
\mathcal{H} - \mu \mathcal{N} &= \int_{x,y} \left\{ \left[ h(x, y) - \mu \delta(x - y) \right] \rho(x, y, t) + \frac{1}{2} V(x - y) |\psi(x, t)|^2 |\psi(y, t)|^2 \right\} \\
&+ \frac{1}{2} \int_{x,y} V(x - y) \left[ R(y, x, t) R(x, y, t) + R(x, x, t) R(y, y, t) + D^*(x, y, t) D(x, y, t) \right] \\
&+ \int_{x,y} V(x - y) \left[ \frac{1}{2} \psi^*(x, t) \psi(y, t) R(x, y, t) + \frac{1}{2} \psi^*(y, t) \psi(x, t) R(y, x, t) + |\psi(x, t)|^2 R(y, y, t) \right] \\
&- \frac{1}{2} \int_{x,y} V(x - y) \left[ \psi(x, t) \psi(y, t) D^*(x, y, t) + \psi^*(x, t) \psi^*(y, t) D(x, y, t) \right]
\end{align*}
\]

(20)

and

$$
\rho(x, y, t) = \langle \psi^d(x) \psi(y) \rangle = \psi^*(x, t) \psi(y, t) + R(x, y, t)
$$

(21)

$$
\Delta(x, y, t) = -\langle \psi(x) \psi(y) \rangle = -\psi(x, t) \psi(y, t) + D(x, y, t)
$$

with
where the notation emphasises real and imaginary parts of the particles not in the condensate comes from the terms with no \(\psi\) the condensate is taken into account by the terms with two \(\psi\) contributions from the condensate field \(\psi\) as well as from the fluctuations \((G, \Sigma)\). The contribution from \(\psi^*\psi\) is the condensate density. So that the term with 4 \(\psi\)'s can be interpreted as the condensate self-interaction. The interaction of particles not in the condensate with the condensate is taken into account by the terms with two \(\psi\)'s. Finally the self-interaction of the particles not in the condensate comes from the terms with no \(\psi\) \((RR\) and \(DD)\).

We introduce the generalized potentials

\[
\begin{align*}
U_d(x, y, t) &= \delta(x - y) \int_z \rho(z, z, t)V(x - z) \\
U_e(x, y, t) &= \rho(x, y, t)V(x - y) \equiv U_e^r + iU_e^i \\
U_p(x, y, t) &= \Delta(x, y, t)V(x - y) \equiv U_p^r + iU_p^i
\end{align*}
\]

where the notation emphasises real and imaginary parts of \(U_p\). We also define the matrices

\[
\begin{align*}
A(x, y, t) &= h(x, y) - \mu + U_p^r(x, y, t) + U_e^r(x, y, t) + U_d(x, y, t) \\
B(x, y, t) &= h(x, y) - \mu - U_p^r(x, y, t) + U_e^r(x, y, t) + U_d(x, y, t) \\
C(x, y, t) &= h(x, y) - \mu + U_e(x, y, t) + U_d(x, y, t)
\end{align*}
\]

From Eqs. (25) and (27) we obtain an abstract matrix form of the equations of motion

\[
\begin{align*}
\dot{\Sigma} &= \frac{1}{8} G^{-1} A G^{-1} - 2\Sigma A\Sigma - \frac{B}{2} + \{U_p^r, \Sigma\} - [U_e^i, \Sigma] \\
\dot{G} &= \{A, \{G, \Sigma\}\} - \{U_p^r, G\} - [U_e^i, G] \\
\dot{\psi} &= C\psi - U_p\psi^*
\end{align*}
\]
where we have used the fact that $(\Sigma, G)$ are symmetric matrices. These equations (25) are the nonlinear field equations for an arbitrary interaction $V$ between the particles and contain any external potential through $h$. As an example the matrix product $G^{-1}A G^{-1}$ can be written in coordinate representation as

\[ \int_{z,w} G^{-1}(x, z, t) A(z, w) G^{-1}(w, y, t). \]  

The static equations can be obtained by setting the canonical momenta to zero, that is $\Sigma(x, t, t) = \pi(x, t) = 0, \dot{G}(x, y, t) = \dot{\phi}(x, t) = 0$. From (16) and (22) we then have

\[
\begin{align*}
R(x, y, 0) &\equiv R(x, y) = \frac{1}{2} \left[ \frac{G^{-1}(x, y)}{4} + G(x, y) - \delta(x - y) \right] \\
D(x, y, 0) &\equiv D(x, y) = \frac{1}{2} \left[ \frac{G^{-1}(x, y)}{4} - G(x, y) \right] \\
\psi(x, 0) &\equiv \psi(x) = \frac{\phi(x)}{\sqrt{2}}.
\end{align*}
\]

So that for the static case we have to self consistently solve

\[
\begin{align*}
\frac{1}{4} \int_{z,w} G^{-1}(x, z) A(z, w) G^{-1}(w, y) - B(x, y) &= 0 \\
&\int_{z} \left[ B(x, z)\psi(z) - 2\psi(x)\psi^2(z)V(x - z) \right] = 0.
\end{align*}
\]

using (21)-(24). We note that if we constraint $G = 1/2$ for the static solution this leads to $R = D = 0$ and $A = B$ so that equation (44) is to the usual nonlinear equation for the single quantity $\psi$ [6] obtained from a many body product wave function (permanent) for bosons. However the time dependent eqs. (25) are more general, because our trial Gaussian is actually a coherent state with an indefinite number of particles.

III. CONTACT INTERACTION IN THE ONE DIMENSIONAL UNIFORM CASE

A. Gaussian Approximation

We will specialize the results of the previous section for the one dimensional case and for a contact interaction so that
\( V(x - y) = \lambda \delta(x - y). \) \hspace{1cm} (29)

Because of the existence of exact numerical solutions we will treat the uniform case so that the momentum representation is the natural choice where the quantities \( A, B \) and \( G \) can simultaneously be diagonalized and \( \phi(k) = \phi \delta(k) \). The static equations (28) become [we use the notation : \( \int_{k} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \)]

\[
\frac{1}{4} G^{-2}(k) A(k) - B(k) = 0 \hspace{1cm} (30)
\]

\[
\phi \left[ B(0) - \lambda \phi^2 \right] = 0. \hspace{1cm} (31)
\]

The static version of \( A \) and \( B \) from (24) can be written as

\[
A(k) = e(k) - \mu + U_p + 2U + \lambda \frac{\phi^2}{2} \hspace{1cm} (32)
\]

\[
B(k) = e(k) - \mu - U_p + 2U + \frac{3}{2} \lambda \phi^2.
\]

where \( e(k) = \frac{\hbar^2 k^2}{2m} \) and the generalized potentials become

\[
U \equiv U_d = U_c = \lambda \int_{k'} R(k') \hspace{1cm} (33)
\]

\[
U_p = \lambda \int_{k'} D(k')
\]

So that we can write the solution for (30) as

\[
G(k) = \frac{1}{2} \sqrt{\frac{A(k)}{B(k)}} \hspace{1cm} (34)
\]

and for (31) we have

\[
\phi = 0 \hspace{1cm} (35)
\]

or

\[
B(0) = \lambda \phi^2 \hspace{1cm} (36)
\]

using (34) with (27) we can express \( D \) and \( R \) as functions of \( A, B \)
\[ D(k) = \frac{1}{2} \left[ \frac{G^{-1}(k)}{4} - G(k) \right] = \frac{1}{4} B(k) - A(k) \sqrt{A(k)B(k)} \]

\[ R(k) = \frac{1}{2} \left[ \frac{G^{-1}(k)}{4} + G(k) - 1 \right] = \frac{1}{2} \left\{ \frac{B(k) + A(k)}{2 \sqrt{A(k)B(k)}} - 1 \right\}. \]  

From (32), (33) and (34) we see that \( A(0) \) and \( B(0) \) must be positive so that if \( \lambda < 0 \) equation (36) demands that \( B(0) < 0 \) which is inconsistent with the previous statement. So the only possible solution in this case is \( \phi = 0 \). For \( \lambda > 0 \) the symmetry breaking solution \( \phi \neq 0 \), using (38), gives us

\[ \mu = \lambda \int_{k'} \left[ 2R(k') - D(k') \right] + \frac{\lambda}{2} \phi^2. \]  

Having solved for \( \mu \) we can rewrite \( A \) and \( B \) in (32) and (33) as

\[ A(k) = e(k) + 2\lambda \int_{k'} D_{k'} \equiv e(k) + 2\lambda a \]

\[ B(k) = e(k) + \lambda \phi^2 \equiv e(k) + 2\lambda b. \]

On the other hand using (37) and (39) we can write a pair of non-linear equations for \( a \) and \( b \)

\[ a = \frac{\lambda}{2} \int_{k'} \frac{[b - a]}{\sqrt{[e(k') + 2\lambda a][e(k') + 2\lambda b]}} \]

\[ b = \rho - \frac{1}{2} \int_{k'} \left\{ \frac{e(k) + \lambda [b + a]}{\sqrt{[e(k') + 2\lambda a][e(k') + 2\lambda b]}} - 1 \right\}, \]

where we have used the total density constraint

\[ \rho = \frac{\phi^2}{2} + \int_{k'} R(k'). \]  

which actually becomes our scale. This non-linear set of equations can be solved for \( \rho \), given \( a \) and \( b \). Once we have calculated \( a \) and \( b \) we can compute the chemical potential through
\[ \mu = \lambda [2\rho - a - b]. \] (42)

In the same fashion the ground state energy density \( (E/L = (\mathcal{H})/L) \) can be computed obtaining

\[ \frac{E}{L} = \frac{\lambda}{2} a^2 - \frac{\lambda}{2} b^2 - \lambda ab + \lambda \rho^2 + K, \] (43)

where \( K \), the contribution from the kinetic energy can also be computed in terms of \( a \) and \( b \) as

\[ K = \frac{1}{2} \int_{k'} e(k') \left\{ \frac{e(k') + \lambda [a + b]}{\sqrt{e(k') + 2\lambda a} [e(k') + 2\lambda b]} - 1 \right\}. \] (44)

As an aside we remark that for a dilute system we can approximate the self-consistent equations for \( a \) and \( b \) by truncating them at a second iteration. A first iteration on (40) takes \( a \approx 0, b \approx \rho \) which implies zero depletion and \( U_0 = 0 \). This leads us to a non-pairing theory (Gross Pitaevskii equation \( \mathbb{G} \)). Then the next iteration leads to

\[ a \approx \frac{\lambda \rho}{2} \int_{k'} \frac{1}{\sqrt{e(k')^2 + 2\lambda \rho e(k')}} \] (45)

\[ b \approx \rho - \frac{1}{2} \int_{k'} \left\{ \frac{e(k') + \lambda \rho}{\sqrt{e(k')^2 + 2\lambda \rho e(k')}} - 1 \right\}. \]

Then we can calculate \( R \) and \( D \) truncating the self-consistency and giving the same results as the Bogoliubov scheme. Physically this means neglecting the effect of the terms that take into account the self-intercation of the particles not in the condensate (\( DD \) and \( RR \)). This approximation is usually valid for dilute systems where these terms are not important. With this truncation the ground state energy can be easily computed giving

\[ \frac{E}{N} = \frac{\lambda \rho}{2} \left[ 1 - \frac{4}{3\pi} \sqrt{\gamma} \right], \] (46)

where the dimensionless parameter \( \gamma \) is

\[ \gamma = \frac{\lambda m}{\rho \hbar^2}. \] (47)
Returning to our discussion we determine the excitations through the RPA equations which can be found by expanding all quantities around their equilibrium value \[17\]. Thus we write

\[
G(k, k', t) = G(k)\delta(k - k') + \delta G(k, k', t)
\]

\[
\Sigma(k, k', t) \to \delta \Sigma(k, k', t)
\]

\[
\phi(k, t) = \phi\delta(k) + \delta\phi(k, t)
\]

\[
\pi(k, t) \to \delta\pi(k, t)
\]

Thus we have written \(G\) and \(\Sigma\) in the basis where the equilibrium \(G\) is diagonal and kept terms up to first order in small quantities, of course the diagonal basis is plane waves.

It will be useful to introduce new momentum coordinates so that

\[
P = k - k' \quad (49)
\]

\[
q = \frac{k + k'}{2} \quad (50)
\]

and

\[
\delta G(k, k') \to \delta G(P, q) \quad (51)
\]

We will see that \(P\) and \(q\) can be interpreted as total and relative momenta respectively of a pair of quasi-bosons. We can then write the RPA equations in a form where \(P\) is diagonal and can be considered as a dummy variable

\[
\delta\dot{G}(q, P, t) = s_K(q, P)\delta\Sigma(q, P, t) + c_K(q, P)\delta\pi(P, t) + \int_{q'} S_K(q, q', P)\delta\Sigma(q', P, t)
\]

\[
- \delta\dot{\Sigma}(q, P, t) = s_M(q, P)\delta G(q, P, t) + c_M(q, P)\delta\phi(P, t) + \int_{q'} S_M(q, q', P)\delta G(q', P, t) \quad (52)
\]

\[
\delta\dot{\phi}(P, t) = \delta\pi(P, t)A(P) + \int_{q'} c_K(q', P)\delta\Sigma(q', P, t)
\]

\[
- \delta\dot{\pi}(P, t) = \delta\phi(P, t)B(P) + \int_{q'} c_M(q', P)\delta G(q', P, t).
\]

We note that for a given value of \(P\) the \((\pi, \phi)\) degree of freedom is coupled to the much more numerous degrees of freedom \((\Sigma, G)\) which are labeled by \(q\). Different \(q\) values among
$(\Sigma, G)$ are also coupled. Introducing the notation $f(q' + P/2) = f_+^\prime$ and $f(q - P/2) = f_-$, we find non-diagonal matrices in $(q, q')$

\[ S_K(q, q', P) = \lambda \left[ G_+ + G_- \right] \left[ G_+^\prime + G_-^\prime \right] + \lambda \left[ G_+ - G_- \right] \left[ G_+^\prime - G_-^\prime \right] \]

\[ S_M(q, q', P) = \frac{\lambda}{2} + \frac{\lambda}{4} \left[ 1 - \frac{G_+^{-1} G_-^{-1}}{4} \right] \left[ 1 - \frac{G_+^{-1} G_-^{-1}}{4} \right] \]

\[ + \left[ \frac{G_+^{-1} G_-^{-1}}{4} \right] \left[ \frac{G_+^{-1} G_-^{-1}}{4} \right] \]

and diagonal elements

\[ s_K(q, P) = 2 \left[ A_+ G_- + A_- G_+ \right] \]

\[ s_M(q, P) = \frac{G_+^{-2} G_-^{-1} A_+ + G_-^{-2} G_+^{-1} A_-}{8}. \]

Finally we see the coupling elements between $(\pi, \phi)$ and $(\Sigma, G)$

\[ c_K(q, P) = \lambda \phi \left[ G_+ + G_- \right] \]

\[ c_M(q, P) = \frac{\lambda}{2} \phi \left[ 3 - \frac{G_+^{-1} G_-^{-1}}{4} \right], \]

which vanish when the symmetry in $\phi$ is conserved ($\phi = 0$). As pointed out above the equations are diagonal in $P$ so we can interpret it as the total momentum of a pair of quasi-bosons. Because $\delta G$, $\delta \Sigma$ and $\delta \phi$, $\delta \pi$ are canonical variables we may invert the definitions of momentum and coordinate. For convenience, we define column vectors

\[ \Theta(q, P, t) = \begin{pmatrix} \delta \Sigma(q, P, t) \\ \delta \pi(P, t) \end{pmatrix}, \quad \Pi(q, P, t) = - \begin{pmatrix} \delta G(q, P, t) \\ \delta \phi(P, t) \end{pmatrix}. \]

Then we can write a coupled oscillator Hamiltonian that corresponds to the RPA equations of motion in a suggestive matrix element form

\[ H_{RPA} = \frac{1}{2} \Pi M^{-1} \Pi + \frac{1}{2} \Theta K \Theta \]
where the matrixes \( M^{-1} \) and \( K \) are the generalizations of oscillator mass and spring constant

\[
K = \begin{pmatrix} S_K + s_K & c_K \\ c_K & A \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} S_M + s_M & c_M \\ c_M & B \end{pmatrix}.
\]

(58)

We may separate the diagonal part of \( H_{\text{RPA}} \) so that

\[
H_{\text{RPA}} = H_0 + H_{\text{int}},
\]

(59)

where

\[
H_0 = \frac{1}{2} \begin{pmatrix} \delta \Sigma & \delta \pi^* \end{pmatrix} \begin{pmatrix} s_K & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} \delta \Sigma \\ \delta \pi \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \delta G & \delta \phi^* \end{pmatrix} \begin{pmatrix} s_M & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \delta G \\ \delta \phi \end{pmatrix}
\]

(60)

\[
H_{\text{int}} = \frac{1}{2} \begin{pmatrix} \delta \Sigma & \delta \pi^* \end{pmatrix} \begin{pmatrix} S_K & c_K \\ c_M & 0 \end{pmatrix} \begin{pmatrix} \delta \Sigma \\ \delta \pi \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \delta G & \delta \phi^* \end{pmatrix} \begin{pmatrix} S_M & c_M \\ c_M & 0 \end{pmatrix} \begin{pmatrix} \delta G \\ \delta \phi \end{pmatrix}.
\]

Introducing the trivial canonical transformation

\[
\delta \Sigma \to \sqrt{s_M} \delta \Sigma \quad \delta G \to \frac{\delta G}{\sqrt{s_M}}
\]

(61)

\[
\delta \pi \to \sqrt{B} \delta \pi \quad \delta \phi \to \frac{\delta \phi}{\sqrt{B}}
\]

we obtain a simpler form for the diagonal part

\[
H_0 = \frac{1}{2} \begin{pmatrix} \delta \Sigma & \delta \pi^* \end{pmatrix} \begin{pmatrix} s_M s_K & 0 \\ 0 & AB \end{pmatrix} \begin{pmatrix} \delta \Sigma \\ \delta \pi \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \delta G & \delta \phi^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta G \\ \delta \phi \end{pmatrix}.
\]

(62)

If we define \( \Omega_1 \) and \( \Omega_2 \)

\[
\Omega_1(P) = \sqrt{A(P)B(P)} \\
\Omega_2(q, P) = \sqrt{s_K(q) s_M(q, P)}
\]

(63)

(64)

and use the definitions of \( s_K \) and \( s_M \) from (54) we get, after some algebra, the remarkable result

\[
\Omega_2(q, P) = \sqrt{A_+ B_+} + \sqrt{A_- B_-} = \Omega_1(k) + \Omega_1(k'),
\]

(65)
so that $\Omega_1(P)$ and $\Omega_2(q, P)$ can be interpreted as the one and two free quasi-boson energies. We note that $\Omega_2(0, P) = 2\Omega_1(P/2)$, which means that at zero relative momentum $\Omega_2(P, 0)$ corresponds to two quasi-bosons with momentum $P/2$. Thus the oscillations of the $\delta\phi$, $\delta\pi$ pair can be interpreted as a quasi-boson mode while the oscillations of $\delta G$, $\delta \Sigma$ can be interpreted as an interacting pair of these same quasi-bosons. When $\phi = 0$, we get $c_K = c_M = 0$ and the one and two quasi-bosons systems are calculated independently. When $\phi \neq 0$ we must rediagonalize so that our final modes will be mixtures of one and two quasi-bosons. The variable $q$ represents the internal motion of the quasi-boson pair with interaction given by the quantities $S$. In general this is a scattering problem and we must search for the scattering amplitude at a given energy and $P$, where the asymptotic conditions are determined by (64). In addition the coupling of one and two quasi-bosons will always lead to a bound state which is a particular mixture of the one quasi-boson mode with a bound component of the two quasi-bosons.

As we did for the static results it is straightforward to see that the truncation that gives the Bogoliubov results implies neglecting the coupling ($H_{\text{int}} = 0$) and will lead to the usual result

$$\Omega_1^b(P) = \sqrt{e^2(P) + 2\lambda \rho e(P)}.$$  \hspace{1cm} (66)

We can see that the Bogoliubov excitations consider no interaction between the quasi-bosons.

Note that because the Bogoliubov does not take into account the self-interaction of the particles not in the condensate we have $k' = 0$ which means $P = k$.

Returning to the dispersion relation for the bound mode one can finally eliminate the two quasi-boson components. To see this we try oscillatory solutions for (52) such as

$$\Theta(t) = \Theta e^{i\Omega t}$$

$$\Pi(t) = \Pi e^{i\Omega t},$$

and equations (52) can be written in a compact form
\[ \mathcal{M} \mathcal{X} = \mathcal{Y}, \quad (68) \]

that is

\[
\begin{pmatrix}
-\Omega & -s_K & 0 & -\lambda x \phi \\
s_M & \Omega & \lambda v \phi & 0 \\
0 & 0 & -\Omega & -A \\
0 & 0 & B & \Omega
\end{pmatrix}
\begin{pmatrix}
\delta G \\
\delta \Sigma \\
\delta \phi \\
\delta \pi
\end{pmatrix} =
\begin{pmatrix}
\lambda (xX + rR) \\
-\lambda yY - 2\lambda zZ \\
\lambda \phi X \\
-\lambda \phi (Y + 2Z)
\end{pmatrix}, \quad (69)
\]

where for simplicity we have used

\[
\begin{align*}
    r(q, P) &= G_+ - G_- \\
    x(q, P) &= G_+ + G_- \\
    y(q, P) &= \frac{1}{2} \left[ 1 + \frac{G_+^{-1} G_-^{-1}}{4} \right] \\
    z(q, P) &= \frac{1}{2} \left[ 1 - \frac{G_+^{-1} G_-^{-1}}{4} \right] \\
    v(q, P) &= y(q, P) + 2z(q, P).
\end{align*}
\]

and also

\[
\begin{align*}
    R(P) &= \int_{q'} r(q', P) \delta \Sigma(q', P) \\
    X(P) &= \int_{q'} x(q', P) \delta \Sigma(q', P) \\
    Y(P) &= \int_{q'} y(q', P) \delta G(q', P) \\
    Z(P) &= \int_{q'} z(q', P) \delta G(q', P)
\end{align*}
\]

Note that the condition \( \text{det} \mathcal{M} = 0 \) from the homogeneous equation \((\mathcal{Y} = 0)\) gives us back \( \Omega_1 \) and \( \Omega_2 \) discussed above.

In the discussion which follows we look for the bound state referred to above by holding \( \Omega < \Omega_1 < \Omega_2 \) so that it is not necessary here to include the usual scattering \( i\epsilon \) in the denominators. We can invert \( \mathcal{M} \) obtaining

\[
\delta G = -\frac{\lambda \Omega r R}{(\Omega^2 - \Omega_2^2)} - \left[ \frac{x}{(\Omega^2 - \Omega_2^2)} + \lambda \phi^2 \frac{s_K v + B x}{(\Omega^2 - \Omega_1^2)(\Omega^2 - \Omega_2^2)} \right] \lambda \Omega X
\]
\[ \delta \Sigma = \frac{\lambda s_K r R}{(\Omega^2 - \Omega_1^2)(\Omega^2 - \Omega_2^2)} + \left[ \frac{s_M x}{\Omega^2 - \Omega_1^2} + \lambda \phi^2 \frac{B s_M x + r \Omega^2}{(\Omega^2 - \Omega_1^2)(\Omega^2 - \Omega_2^2)} \right] \lambda X \]
\[ - \lambda^2 \phi^2 \Omega \frac{A v + s_M x}{(\Omega^2 - \Omega_1^2)(\Omega^2 - \Omega_2^2)} (Y + 2Z) - \lambda \Omega \frac{y Y + z Z}{\Omega^2 - \Omega_2^2} \]

Now we substitute (72) in the definitions of the quantities \( R, X, Y \) and \( Z \) (71). Because \( r(q, P) \) is an odd function i.e \( r(-q, P) = -r(q, P) \) it is easy to check we end up with a linear and homogeneous system for \( X, Y \) and \( Z \), that looks like

\[ W(P, \Omega).F = 0 \] (73)

where, omiting the \( P \) dependence we have

\[ W = \begin{pmatrix} W_{1,1} & W_{1,2} & W_{1,3} \\ W_{2,1} & W_{2,2} & W_{2,3} \\ W_{3,1} & W_{3,2} & W_{3,3} \end{pmatrix}, \quad F = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \] (74)

where the elements of \( W \) are given in the appendix B. The system (73) will have a non-trivial solution if

\[ \det W(P, \Omega(P)) = 0 \] (75)

so that we have for each \( P \) the corresponding energy \( \Omega(P) \). Numerically the problem reduces to calculating determinants of a \( 3 \times 3 \) matrix.

A very general property [18] of the dispersion relation \( \Omega(P) \) can be proven for the particular case where \( P = 0 \). In this case the first line of the matrix (74) using (40) and (64) is
\[
W_{1,1} = 1 - \lambda \frac{A(0) - B(0)}{A(0)} \int_q \frac{x^2(q,0)}{s_K(q,0)}, \quad W_{1,2} = 0, \quad W_{1,3} = 0 \tag{76}
\]

and now using (39)-(40) we have that
\[
\lambda \int_q \frac{x^2(q,0)}{s_K(q,0)} = \frac{\lambda}{2} \int_q \frac{1}{\sqrt{A(q)B(q)}} = \frac{A(0)}{A(0) - B(0)}. \tag{77}
\]

So that the first line of the matrix is zero, making the determinant vanish, showing that we will always have a gapless dispersion relation

independently of the value of \( \lambda \),

\[
\Omega(0) = 0. \tag{78}
\]

This zero mode of the RPA equations is the standard Goldstone mode as it’s structure is associated with the symmetry breaking (indefinite particle number) by the trial wave functional \[15\].

We note that the dispersion relation that comes from the RPA depends on the total momentum defined in (13) which means that \( \Omega \) is function of \((k - k')\), so that it takes into account that to obtain an excitation we remove a particle with momentum \( k' \) and create a particle with momentum \( k \). This result is very different from the Bogoliubov excitation where \( k' \) is taken to be zero. In other words the Gaussian approximation takes into account the effect of the depletion on the excitations.

\section*{B. The Exact Solution}

All the results summarized in this section were obtained by Lieb \[19\] who calculated the exact ground state energy and also the excitations in terms of independent particle and hole excitations. Our purpose is to make a connection with this work which showed that interacting bosons in one dimension can be analogous to a fermi gas. We will show how our modes correspond to Lieb’s particle-hole excitations.

First of all when \( \lambda \to \infty \) it is possible to recover the well-known result \[21\] that for an infinite coupling constant interacting bosons behave like a system of free fermions, so that the ground state energy can be easily computed by
\[
\frac{E}{N} = \frac{1}{\rho} \frac{1}{2\pi} \int_{-K_f}^{K_f} e(k) dk
\]

where \( K_f \) is the fermi- momentum. In this case it is trivial to calculate

\[
\rho = \frac{1}{2\pi} \int_{-K_f}^{K_f} dk
\]

which gives \( K_f = \pi \rho \). So that in this particular limit

\[
\frac{E}{N} = \frac{\hbar^2}{2m} \frac{\pi^2 \rho^2}{3}
\]

We can divide the particle hole excitations of the free fermi gas into two parts. The particular hole excitations that correspond to removing a particle from an occupied state to just above the fermi level \((K_f)\)

\[
\epsilon_h(k') = \frac{\hbar^2}{2m} \frac{\pi^2 \rho^2}{2} - \frac{\hbar^2 k'^2}{2m}
\]

where \( K_f < k' < K_f \) and the particle excitations where we remove a particle from the fermi level to an unoccupied state

\[
\epsilon_p(k) = \frac{\hbar^2}{2m} \frac{k^2}{2} - \frac{\hbar^2}{2m} \frac{\pi^2 \rho^2}{2}
\]

where \( k > K_f \) or \( k < -K_f \). To produce a particle hole excitation we must add these excitations giving us

\[
\epsilon_p(k, k') = \frac{\hbar^2}{2m} \frac{k^2}{2} - \frac{\hbar^2}{2m} \frac{\pi^2 \rho^2}{2}
\]

Using \( \epsilon_h \) and \( \epsilon_p \) one can look at \( E(k', k) \) as a function of the total and relative momentum \( P = k - k' \) and \( q = (k + k')/2 \). Because the hole excitations are limited to \((-K < k' < K)\) we need to separate two cases, for \( P > 0 \) we have

\[
\text{if } P < 2K_f \rightarrow \begin{cases} k = K_f \\ k' = K_f - P \end{cases}
\]

\[
\text{if } P > 2K_f \rightarrow \begin{cases} k = P - K_f \\ k' = -K_f \end{cases}
\]
The first case means fixing the momentum of the particle and moving the momentum of the hole while in the second one we fix the momentum of the hole at its lowest possible value and move the momentum of the particle. So that once we know $\epsilon_h$ and $\epsilon_p$ the threshold curve defined by the lowest value of $E(k',k)$ for a given $P$ will be given by

$$E(P) = \begin{cases} 
\epsilon_h(K_f - P) + \epsilon_p(K_f) & \text{for } P < 2K_f \\
\epsilon_h(-K_f) + \epsilon_p(P - K_f) & \text{for } P > 2K_f.
\end{cases} \quad (86)$$

Which gives us

$$E(P) = \begin{cases} 
2\pi \rho P - P^2 & \text{for } P < 2K_f \\
P^2 + 2\pi \rho P & \text{for } P > 2K_f
\end{cases} \quad (87)$$

Note that the $E(P)$ curve contains two parts. Finite range part $P < 2K_f$ where the contribution comes basically from the hole excitations. Infinite range part $P > 2K_f$ from the particle excitations. The more dilute the system the less finite range part in $E(P)$.

The generalization for bosons interacting with a finite $\lambda$ was carried out by Lieb and Lininger [19], [20]. The showed the ground-state can be calculated as

$$\frac{E}{N} = \frac{1}{\rho} \int_{-K}^{K} f(k)e(k)dk \quad (88)$$

where $f(k)$ is the solution of

$$2\gamma \rho \int_{-K}^{K} \frac{f(p)}{\rho^2 \gamma^2 + (p - k)^2} dp = 2\pi f(k) - 1 \quad (89)$$

with $\gamma$ given in (17). The condition

$$\int_{-K}^{K} f(k)dk = \rho \quad (90)$$

determines $K$. For the excitations Lieb defined two different basic interactions which he called ”particle” and ”hole” excitations. To determine these excitation Lieb showed that it was sufficient to solve new integral equations. For the particle energy

$$\epsilon_p(k) = \frac{\hbar^2 k^2}{2m} - \mu + \frac{\hbar^2}{m} \int_{-K}^{K} p J_p(p) dp \quad (91)$$
where $J_p(p)$ could be obtained by solving
\begin{equation}
2\pi J_p(p) = 2\gamma \rho \int_{-K}^{K} \frac{J_p(r)}{\rho^2 \gamma^2 + (p - r)^2} dr - \pi + 2 \tan^{-1} \left[ \frac{k - p}{\gamma \rho} \right] \tag{92}
\end{equation}
and for the hole energy $\epsilon_h(P, \lambda)$ he had
\begin{equation}
\epsilon_h(k') = \mu - \frac{\hbar^2 k'^2}{2m} + \frac{\hbar^2}{m} \int_{-K}^{K} p J_h(p) dp.
\end{equation}
where $J_h(p)$ was obtained from
\begin{equation}
2\pi J_h(p) = 2\gamma \rho \int_{-K}^{K} \frac{J_h(r)}{\rho^2 \gamma^2 + (p - r)^2} dr + \pi - 2 \tan^{-1} \left[ \frac{k' - p}{\gamma \rho} \right]. \tag{94}
\end{equation}
In the limit $\lambda \to \infty$ we can see that $J(p) \to 0$ and $\mu \to (\hbar^2 K^2)/(2m) = (\hbar^2 \pi^2 \rho)/(2m)$ recovering the free fermion results. Looking at the expressions for the ground state energy and the excitations for a given $\lambda$, Lieb interpreted them as those of a quasi-fermi gas where $K$ is an interaction dependent fermi momentum and the distributions factors $f(k)$ and $J(k)$ given a special weight for each $k$. Using this analogy is very reasonable, since it is correct in both the $\lambda = 0$ and $\lambda = \infty$. To obtain the threshold curve $E(P)$ we can use (86). The difference is that now $\epsilon_p$ and $\epsilon_h$ will have different curvatures depending on the interaction. Note that in his original work Lieb compared the Bogoliubov scheme with particle and holes excitations separately and got very good agreement with the particle excitations. This is interesting, as we pointed out earlier, and just tells us that the Bogoliubov scheme does not contain hole excitations. In general as $\gamma$ increases the contribution of the holes for the particle-hole excitation energy becomes more and more important and this effect is in part described by the Gaussian theory.

**IV. NUMERICAL RESULTS AND CONCLUSIONS**

For the numerical computations we follow Lieb and use the dimensionless coupling constant $\gamma$ and scale all lengths by $\rho$ and all energies by $(\hbar^2 \rho^3)/(2m)$. In these units we can write the ground state energy per particle as
\[
\frac{E}{N} = g(\gamma) \tag{95}
\]

For the Gaussian static results we solve the non-linear system (40)-(40) and determine \(a\) and \(b\) for \(0 < \gamma < 10\). After computing \(K\) defined in (44) one can get \(g(\gamma)\) using (43). In the Bogoliubov scheme (46) leads to

\[
g_B(\gamma) = \gamma \left[1 - \frac{4}{3\pi} \sqrt{\gamma}\right] \tag{96}
\]

Finally the exact result for \(g(\gamma)\) was obtained by Lieb solving (88)-(90). The results as a function of \(\gamma\) can be seen in Fig. 1. Where one can basically see the very good agreement of both Gaussian and Bogoliubov\) for low \(\gamma\) (\(\gamma \leq 1\)) with the difference that by construction the Gaussian result it is always an upper bound and the Bogoliubov energy is below the exact result. For higher values of \(\gamma\) the Bogoliubov results, as expected, collapses (\(\gamma \sim 5\)) while the Gaussian theory still gives a result. In the hard core limit (\(\gamma \to \infty\)) the Gaussian approximation fails to go to the finite free fermi energy of the exact solution.

To obtain the "exact" excitation energies we used Lieb's results for \(\epsilon_h\) and \(\epsilon_p\) obtained by solving (88)-(90). Using (86) we obtain the excitation curves \(E(P)\) as a function of \(P\) as shown in Fig. 2 for \(\gamma = 0, 0.787, 4.527\) and \(\infty\). Note that for \(P < 2K\) the contribution from the hole excitations produce quite interesting dispersion curves.

To illustrate the RPA results we plot in Fig.3, \(\Omega_1\), \(\Omega_2\) and \(\Omega\) as a function of \(P\) for \(\gamma = 7.551\). The relative momentum variable \(q\) will gives us a continuum that corresponds to a scattering region and one can see clearly how \(\Omega_1\) gets pushed down by the interaction producing the gapless \(\Omega\). We compare the bound state dispersion relation \(\Omega\) with the Bogoliubov one and with the exact results for \(\gamma = 0.787\) (Fig. 4) and \(\gamma = 4.527\) (Fig. 5). We note that the improvement of the RPA compared to the Bogoliubov result increases with \(\gamma\). When \(\gamma\) goes to infinity both the RPA and the Bogoliubov schemes fail.

The results of this paper can be summarized in three points

1) We have seen that the Gaussian variational method takes into account the self-interactions of the particles out of the condensate. This fact starts improving the ground
state energy results when compared with Bogoliubov theory once $\gamma$ increases ($\gamma > 2$). It is very difficult however for the Gaussian variational results to get close to the exact for high $\gamma$ values. In the one dimensional case when $\gamma \to \infty$ the Gaussian ground state energy diverges while the exact goes to the finite fermi energy.

2) For the particle hole excitations the Gaussian results take into account that when we remove a particle from an occupied state in order to make an excitation, this particle can be out of the condensate which does not happen in the Bogoliubov scheme. Again this seems an improved description for intermediate values of $\gamma$. As $\gamma$ increases the depletion starts to grow and in this particular case goes to a fermi sea when $\lambda \to \infty$ which can not be described by our methods since they do not deal with very short range correlations.

3) Another important point that is somehow related to item 2 is that to solve the RPA equation we had decoupled harmonic oscillators which, in the quasi-boson picture means that the RPA takes into account the interaction between the quasi-bosons generating new modes that are mixtures of one ($\delta \phi, \delta \pi$) and two quasi-bosons ($\delta G, \delta \Sigma$). In the Bogoliubov scheme we always have free quasi-bosons when their interaction starts to be relevant as $\gamma$ increases.

We can conclude that the Gaussian variational method can describe systems where the depletions can not be neglected (when dilute theories break down) but because of the absence of short range correlation will require corrections for highly depleted systems (like Helium 4).

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APPENDIX A:

If we carefully examine the expression (19) we see that the functional integral that must be computed involve moments of a Gaussian. Namely,

\[ \int (\mathcal{D}\phi') \Psi^*[\phi', t] \phi'(x_1) \ldots \phi'(x_4) \Psi[\phi', t] \]  

(A1)

where \( \Psi[\phi'] \) is our normalized trial wave functional given by (7). This functional integrals can be computed easily if we include a source term \([22]\) in the normalization integral i.e.

\[ \int (\mathcal{D}\phi') \Psi^*[\phi', t] e^{\int_x J(x) \delta \phi'(x) \Psi[\phi', t]} \]  

(A2)

Using the expression for \( \Psi[\phi', t] \) we complete the squares in the exponential getting

\[ \int (\mathcal{D}\phi') \Psi^*[\phi', t] e^{\int_x J(x) \delta \phi'(x) \Psi[\phi', t]} = e^{\frac{1}{2} \int_x J(x) G(x,y,t) J(y)} \]  

(A3)

The source term allows us to compute the functional integral of any moment of the Gaussian

\[ \int (\mathcal{D}\phi') \delta \phi'(x_1) \ldots \delta \phi'(x_n) e^{\int_x J(x) \delta \phi'(x) \frac{G^{-1}(x,y,t)}{2} \delta \phi'(y)} = \left. \frac{\delta}{\delta J(x_1)} \ldots \frac{\delta}{\delta J(x_n)} e^{\frac{1}{2} \int_x J(x) G(x,y,t) J(y)} \right|_{J=0} \]  

(A4)

For instance we can calculate

\[ \int (\mathcal{D}\phi') \Psi^*[\phi', t] \phi'(x) \phi'(y) V(x - y) \phi'(x) \phi'(y) \Psi[\phi', t] = 4 \phi(x, t) \phi(y, t) V(x - y) G((x, y, t) + \phi^2(x, t) V(x - y) G((x, y, t) + \phi^2(y, t) V(x - y) G((x, y, t) + 2G(x, y, t) V(x - y) G((x, y, t)) \]  

(A5)

Note that the last two terms of (A5) show,as expected, the mean field factorization.

APPENDIX B:

After some algebra one gets the elements of the matrix \( W \) which appears in equation (75). For calculating the dispersion relation, \( \Omega \) will be below the lowest pole appearing in (B1)-(B9).
\[ W_{1,1} = 1 - \lambda \int_q \frac{s_{Mx}^2}{(\Omega^2 - \Omega_2^2)} - \lambda^2 \phi^2 \int_q \frac{B s_{Mx}^2 + vx \Omega^2}{(\Omega^2 - \Omega_1^2)(\Omega^2 - \Omega_2^2)} \]  
(B1)

\[ W_{1,2} = \lambda^2 \Omega \phi^2 \int_q \frac{A_{vx} + s_{Mx}^2}{(\Omega^2 - \Omega_1^2)(\Omega^2 - \Omega_2^2)} + \lambda \Omega \int_q \frac{xy}{(\Omega^2 - \Omega_1^2)} \]  
(B2)

\[ W_{1,3} = 2 \lambda^2 \phi^2 \Omega \int_q \frac{A_{vx} + s_{Mx}^2}{(\Omega^2 - \Omega_1^2)(\Omega^2 - \Omega_2^2)} + 2 \lambda \Omega \int_q \frac{xz}{(\Omega^2 - \Omega_1^2)} \]  
(B3)

\[ W_{2,1} = \lambda \Omega \int_q \frac{xy}{(\Omega^2 - \Omega_2^2)} + \lambda^2 \phi^2 \Omega \int_q \frac{s_{Kvy} + B xy}{(\Omega^2 - \Omega_1^2)(\Omega^2 - \Omega_2^2)} \]  
(B4)

\[ W_{2,2} = 1 - \lambda^2 \phi^2 \int_q \frac{A_{s_{Kvy}} + xy \Omega^2}{(\Omega^2 - \Omega_1^2)(\Omega^2 - \Omega_2^2)} - \lambda \int_q \frac{s_{Kvy^2}}{(\Omega^2 - \Omega_1^2)} \]  
(B5)

\[ W_{2,3} = -2 \lambda^2 \phi^2 \int_q \frac{A_{s_{Kvy}} + xy \Omega^2}{(\Omega^2 - \Omega_1^2)(\Omega^2 - \Omega_2^2)} - 2 \lambda \int_q \frac{s_{Kvy^2}}{(\Omega^2 - \Omega_2^2)} \]  
(B6)

\[ W_{3,1} = \lambda \Omega \int_q \frac{xz}{(\Omega^2 - \Omega_2^2)} + \lambda^2 \phi^2 \Omega \int_q \frac{s_{Kvz} + B xz}{(\Omega^2 - \Omega_1^2)(\Omega^2 - \Omega_2^2)} \]  
(B7)

\[ W_{3,2} = -\lambda^2 \phi^2 \int_q \frac{A_{s_{Kvz}} + xz \Omega^2}{(\Omega^2 - \Omega_1^2)(\Omega^2 - \Omega_2^2)} - \lambda \int_q \frac{s_{Kvz^2}}{(\Omega^2 - \Omega_1^2)} \]  
(B8)

\[ W_{3,3} = 1 - 2 \lambda^2 \phi^2 \int_q \frac{A_{s_{Kvz}} + xz \Omega^2}{(\Omega^2 - \Omega_1^2)(\Omega^2 - \Omega_2^2)} - 2 \lambda \int_q \frac{s_{Kvz^2}}{(\Omega^2 - \Omega_2^2)} \]  
(B9)
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Figure Captions

FIG.1 The ground state energy per particle can be written as $E/N = \rho^2 g(\gamma)$. The curves give $g(\gamma)$ as a function of $\gamma$ for the Gaussian, the Bogoliubov and the exact result.

FIG.2 The exact threshold curves for the particle-hole excitation as a function of the total particle-hole momentum $P = k - k'$ as obtained using (86).

FIG.3 The free quasi-boson energy $\Omega_1(P)$ given by (63). The two free quasi-boson energy $\Omega_2(P, q)$ given by (64) and the new gapless $\Omega(P)$ mode energy obtained with the RPA calculation by solving (75). Note that, as expected, the gapless property of $\Omega(P)$ comes from the Goldstone mode associated with the symmetry breaking.

FIG.4 Comparison between the three different particle-hole excitations curves: Bogoliubov $\Omega^h_1(P)$ given by (66), the RPA $\Omega(P)$ by solving (75) and the exact using (86). These are given as a function of the total particle-hole momentum $P$ for $\gamma = 0.787$.

FIG.5 Comparison between the three particle-hole excitations curves: $\Omega^h_1(P)$ given by (66), $\Omega(P)$ by solving (75) and the exact using (86) as a function of the total particle-hole momentum $P$ for $\gamma = 5.527$. 
Fig. 1

\[ \gamma = \frac{4\pi^2 \lambda m}{\hbar^2 \rho} \]

\[ \frac{E}{N\rho^2} = g(\gamma) \]

- Gaussian Approx.
- Exact Result
- Bogoliubov

\[ \frac{\pi^2}{3} \]

\[ 0.0 \quad 2.0 \quad 4.0 \quad 6.0 \]

\[ 0.0 \quad 2.5 \quad 5.0 \quad 7.5 \quad 10.0 \]
Fig. 3

RPA results for $\gamma=7.551$

Scattering Region

Interaction

$\Omega_2(0,P)$

$\Omega_1(P)$

$\Omega(P)$
Fig. 5

\[ \gamma = 4.527 \]

![Graph showing the relationship between $E/\rho$ and $P/\rho$ for different models.](image)