MOTIVIC SHEAVES REVISITED

DONU ARAPURA

Abstract. The purpose of this paper is to present a simplified construction of the author's category of motivic sheaves [A2], and to provide a simplified proof of a theorem of [A1] that the Leray spectral sequence can be lifted to this category.

Let us recall that given a subfield $k \subset C$, Nori defined an abelian category of mixed motives $\mathcal{M}(k)$, which received a universal cohomology theory for pairs of $k$-varieties. The book by Huber and Müller-Stach [HM] now gives a fairly detailed account of this story. In [A2], we generalized Nori’s construction to obtain an abelian category $\mathcal{M}(S)$, of motivic “sheaves” over a $k$-variety $S$. There is a faithful exact “Betti” realization functor $R_B$ from $\mathcal{M}(S)$ to the category of constructible sheaves on the analytic space $S_{an}$, and also an exact functor $R_\ell$ from $\mathcal{M}(S)$ to the category of constructible $\ell$-adic sheaves. There is also a Hodge realization from $\mathcal{M}(S)$ to the heart of a certain $t$-structure on the derived category of mixed Hodge modules. This is made explicit when $S$ is a curve in the fourth section of this paper.

For each projective morphism $f : X \to S$, there exists a motive $h^i_S(X) \in \mathcal{M}(S)$, such that $R_B(h^i_S(X)) = R^i f_*Z$ and $R_\ell(h^i_S(X)) = R^i f_*Z_\ell$.

The main result of this paper is that there exists a $\delta$-functor $h^* : \mathcal{M}(S) \to \mathcal{M}(k)$, such that $R_B(h^*(M)) = H^j(S, R_B(M))$. Furthermore, if $f : X \to S$ is projective, there is a spectral sequence converging to the Nori motive

$$M^{pq}E_2 = h^p(h^q_S(X)) \Rightarrow h^{p+q}(X)$$

whose image under $R_B$ is the Leray spectral sequence. This is an amalgam of the main theorem of [A1] and theorem 5.2.1 of [A2]. The proof here is simpler than either of the previous proofs. One of the goals of this paper is to give more transparent constructions and proofs of some results from the papers [A1, A2]. This is possible, in part, because of some developments over the intervening years. Nori’s Tannakian construction has been refined by various authors ([BLO], [BP], [HM] and [I]). In particular, using the set up by Barbieri Viale and Prest [BP], we are able to give the more direct and natural construction of $\mathcal{M}(S)$ used here. In addition, some of the complicated homological algebra from [A1] can be replaced by a technical result due to de Cataldo and Migliorini [CM], which gives a criterion for a map to be an isomorphism in the filtered derived category.

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1. The $\mathbb{N}^+$ construction

We will use the term quiver instead of (directed) graph used in [A2]. We will frequently apply category theoretic terminology to quivers. In particular, the words

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“vertex” and “object” (respectively “edge” and “morphism”) are used interchangeably. The set of objects of $\Delta$ is denoted by $\text{Ob} \, \Delta$. A subquiver $\Delta' \subseteq \Delta$ is full if for any edge in $\Delta$ joining $e, e' \in \text{Ob} \, \Delta'$, is in $\Delta'$. A morphism, functor, or representation between quivers $F : \Delta \to \Delta'$ is a pair of functions between vertices and edges which preserves incidence: the source/target of $F(e)$ is $F$ applied to the source/target of $e$. We say that a diagram of categories is 2-commutative, if any two paths between the same vertices are naturally isomorphic. We will recall the following generalization of Nori’s Tannakian construction due to Barbieri Viale and Prest [BP, pp 207, 214, 215], that we will refer to as the $N^+$ construction.

**Theorem 1.1.** Let $R$ be a commutative ring. Given a representation from a quiver to an $R$-linear abelian category $F : \Delta \to A$, there exists an $R$-linear abelian category $\mathcal{N}_R(F)$ and a 2-commutative diagram

$$
\begin{array}{c}
\Delta \\
\downarrow F \\
\mathcal{N}_R(F) \\
\downarrow \phi \\
A
\end{array}
$$

with $\phi$ $R$-linear faithful and exact. Furthermore, this is universal in the sense that given any other such factorization $\Delta \to B \to A$, we have a dotted arrow as drawn, which is unique up to natural isomorphism, and which makes the whole diagram 2-commutative

$$
\begin{array}{c}
\Delta \\
\downarrow F \\
\mathcal{N}_R(F) \\
\downarrow \phi \\
B \\
\downarrow G \\
\downarrow \phi \\
A
\end{array}
$$

It will be useful to briefly summarize the construction, since it will lead to somewhat more refined statements. One forms a preadditive category $R\Delta$ with the same objects as $\Delta$, and for morphisms take the free $R$-module generated by paths. (In the case where $R$ is not explicitly mentioned, we take $R = \mathbb{Z}$.) Given an $R$-linear preadditive category $C$, let $[C, R-\text{Mod}]$ denote the category of $R$-linear additive functors from $C$ to the category of $R$-modules, and let $[C, R-\text{Mod}]^{fp}$ be the full subcategory of finitely presented objects [BP, p 212]. Define

$$\text{Freyd}_R(\Delta) = [[R\Delta, R-\text{Mod}]^{fp}, R-\text{Mod}]^{fp}$$

This is an $R$-linear abelian category. Furthermore, there is a canonical representation $\Delta \to \text{Freyd}_R(\Delta)$, and $F$ has a canonical exact extension $\tilde{F} : \text{Freyd}_R(\Delta) \to A$. Then $\mathcal{N}_R(F)$, or $\mathcal{N}(F)$ when $R$ is understood, is the Serre quotient $\text{Freyd}_R(\Delta)/\ker \tilde{F}$, where $\ker \tilde{F} \subset \text{Freyd}_R(\Delta)$ is the full subcategory with objects $\{X \mid \tilde{F}(X) = 0\}$.

It should now be clear that the $N^+$ construction satisfies the following:

**Lemma 1.2.** Let $g : \Delta \to \Delta'$ be a morphism of quivers,

1. There is a 2-commutative diagram

$$
\begin{array}{c}
\Delta \\
\downarrow g \\
\Delta'
\end{array}
\quad
\begin{array}{c}
\text{Freyd}_R(\Delta) \\
\downarrow G \\
\text{Freyd}_R(\Delta')
\end{array}
$$

with $G$ exact.
(2) If there are representations \( F : \Delta \to A \) and \( F' : \Delta' \to A' \) to \( \mathbb{R} \)-linear abelian categories such that \( G \) sends objects of \( \ker \tilde{F} \) to \( \ker \tilde{F}' \), then we get an induced exact functor \( \mathcal{N}_R(F) \to \mathcal{N}_R(F') \) such that

```
\begin{array}{c}
\Delta \\
\downarrow^g \\
\Delta' \\
\mathcal{N}_R(F) \\
\downarrow \\
\mathcal{N}_R(F')
\end{array}
```

commutes.

(3) If there are representations \( F : \Delta \to A \) and \( F' : \Delta \to A' \) such that \( \ker \tilde{F} \subseteq \ker \tilde{F}' \), then \( \mathcal{N}_R(F') \) is a Serre quotient of \( \mathcal{N}_R(F) \).

**Corollary 1.3.** Suppose that \( F : \Delta \to A \) and \( F' : \Delta \to A' \) are two representations to abelian categories, that fit into a 2-commutative diagram

```
\begin{array}{c}
\Delta \\
\downarrow^g \\
\Delta' \\
A \\
\downarrow^G \\
A'
\end{array}
```

with \( G \) exact. Then there is an exact functor \( \mathcal{N}(F) \to \mathcal{N}(F') \) fitting into the obvious diagram.

The following will also be needed later.

**Lemma 1.4.** Suppose that \( \Delta = \bigcup \Delta_i \) is a directed union of quivers. If \( F : \Delta \to A \) is a representation into an \( \mathbb{R} \)-linear abelian category, then \( \mathcal{N}_R(F) \) is equivalent to the filtered 2-colimit

\[
\mathcal{N}_R(F|\Delta_i) \to \mathcal{N}_R(F)
\]

\( \mathcal{N}_R(F|\Delta_i) \) to \( \mathcal{N}_R(F) \) induces a functor

\[
\alpha : \mathcal{N}_R(F|\Delta_i) \to \mathcal{N}_R(F)
\]

The representations

\[
F|\Delta_i : \Delta_i \to \mathcal{N}_R(F|\Delta_i)
\]

patch to yield a representation of \( \Delta \). Hence, by theorem 1.1, we get

\[
\beta : \mathcal{N}_R(F) \to \mathcal{N}_R(F|\Delta_i)
\]

One checks \( \alpha \) and \( \beta \) are inverse up to natural equivalence. \[\square\]
2. Effective motivic sheaves

For the remainder of the paper, we fix a subfield $k \subset \mathbb{C}$ and a commutative noetherian ring $R$. By a $k$-variety, we mean a reduced separated scheme of finite type over $\text{Spec} \, k$. The symbols $S, X, Y$ should be assumed to be $k$-varieties, unless stated otherwise. Sheaves, and sheaf theoretic operations, should be understood to be with respect to the analytic topology $X_{an} = (X \times_{\text{Spec} \, k} \text{Spec} \, \mathbb{C})_{an}$, unless we explicitly indicate the étale topology by the decoration $\text{et}$. If $f : X \to S$ is a morphism of $k$-varieties and $Y \subset X$ is a closed subvariety, then the cohomology of the pair $(X, Y)$ relative to $S$ with coefficients in a sheaf $F$ on $X$ will be defined by

$$H^i_S(X, Y; F) = R^if_\ast j_{X,Y} ! F|_{X-Y}$$

and

$$\mathbb{H}^i_S(X, Y; F) = R^if_\ast j_{X,Y} ! F|_{X-Y},$$

where $j_{X,Y} : X - Y \to X$ is the inclusion. Note that $H^i_S$ is not cohomology with support in $S$. When $S$ is the point $\text{Spec} \, k$ and $F$ is constant, this agrees with what one usually means by cohomology of the pair. Let us say that a pair $(X \to S, Y)$ has the base change property if for any morphism $g : S' \to S$ of $k$-varieties, the canonical map gives an isomorphism

$$g^\ast H^i_S(X, Y; R) \cong H^i_{S'}(X_{S'}, Y_{S'}; R)$$

for all $i$, where $X_{S'} = (X \times S S')_{\text{red}}$. This property can certainly fail, e.g. for $(\mathbb{G}_m \hookrightarrow \mathbb{A}^1, \emptyset)$. We give some criteria for the property to hold.

**Lemma 2.1.** If $f$ is proper, then $(f : X \to S, Y)$ has the base change property. More generally, if $(f : X \to S, Y)$ has the base change property and $g : S \to T$ is proper, then $(g \circ f : X \to T, Y)$ has the base change property.

**Proof:** The first statement follows immediately from the proper base change theorem [Di, thm 2.3.26] or [KS, prop 2.6.7]. Let us prove the second. Consider the diagram

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow f' & & \downarrow f \\
S' & \underset{\pi}{\longrightarrow} & S \\
\downarrow g' & & \downarrow g \\
T' & \longrightarrow & T
\end{array}
$$

where both squares are Cartesian. Also let $Y' \subset X'$ denote the pullback of $Y$. Then by the proper base change theorem together with the hypothesis we have

$$p^\ast \mathbb{H}_T(X, Y; R) = p^\ast Rg_\ast \mathbb{H}_S(X, Y; R)$$

$$= Rg'_\ast \pi^\ast \mathbb{H}_S(X, Y; R)$$

$$= Rg'_\ast H_S(X', Y'; R)$$

$$= H_{T'}(X', Y'; R)$$

(Equality means that the canonical maps are isomorphisms.) \qed
Let us say that \((f : X \to S, Y)\) is controlled if there is a factorization of \(f\)
\[
\begin{array}{c}
X \\ \\
\overset{f_1}{\longrightarrow} \\
\overset{f_2}{\longrightarrow}
\end{array}
\begin{array}{c}
X' \\ \\
\overset{f_3}{\longrightarrow}
\end{array}
S
\]
such that \(f_2\) is proper, and \((f_1 : X_{an} \to X'_{an}, Y_{an})\) is a topological fibre bundle. To
be more explicit, the last condition means that there exists a topological space \(F\),
a closed subspace \(G \subseteq F\), an open covering \(\{U_i\}\) of \(X'_{an}\), and homeomorphisms of
pairs
\[
(f_i^{-1}U_i, f_i^{-1}U_i \cap Y) \cong (U_i \times F, U_i \times G)
\]
compatible with projection. The notion of being controlled is essentially the same
as the one defined in \([A2, 3.2.1]\).

**Lemma 2.2.** If \((f : X \to S, Y)\) is controlled, then it satisfies the base change
property.

**Proof.** Clearly, if \((f : X_{an} \to S_{an}, Y_{an})\) is topological fibre bundle, then it satisfies
the base change property. The general case of the lemma follows from this special
case and lemma 2.1. \(\square\)

**Lemma 2.3.** Suppose that \((X_i \to S, Y_i)\) satisfy the base change property for \(i = 1, 2,\) and suppose that \(H^*_S(X_i, Y_i; R)\) are flat modules. Then the fibre product
\[
(X_1 \times_S X_2 \to S, Y_1 \times_S X_2 \cup X_1 \times_S Y_2)
\]
has the base change property.

**Proof.** Let \((X_3, Y_3)\) denote the above fibre product. Given a morphism \(g : S' \to S,\)
the Künneth formula gives isomorphisms
\[
g^*H^*_S(X_3, Y_3; R) = \bigoplus_{j + \ell = i} g^*H^j_S(X_1, Y_1; R) \otimes g^*H^\ell_S(X_2, Y_2; R)
\]
\[
= \bigoplus_{j + \ell = i} H^j_{S'}(X_1S', Y_1S'; R) \otimes H^\ell_{S'}(X_2S', Y_2S'; R)
\]
\[
= H^i_{S'}(X_3S', Y_3S'; R)
\]
A proof of the Künneth formula can be found in \([Di, thm 4.3.14]\); although the
reference assumes \(R\) is a field, it suffices to assume that the cohomology of one of
the factors is flat. \(\square\)

Let \(S\) be a \(k\)-variety. Define a quiver \(\Delta(S)\) as follows. When \(S\) is connected, the
vertices are triples \((X \to S, Y, i)\) consisting of
- a quasi-projective morphism \(X \to S;\)
- a closed subvariety \(Y \subseteq X\) such that the pair \((X \to S, Y)\) has the base
  change property;
- a natural number \(i \in \mathbb{N}\).

One should think of \((X \to S, Y, i)\) as the symbol representing \(H^i_S(X, Y)\). Let us
refer to a pair \((X \to S, Y)\) satisfying the first two conditions as an admissible
pair. The set of edges, or morphisms, of \(\Delta(S)\) is the union of the following two sets:

**Type I:** Geometric morphisms
\[
(X \to S, Y, i) \rightarrow (X' \to S, Y', i)
\]
for every morphism of \(S\)-schemes \(X \to X'\) sending \(Y\) to \(Y'\).
Type II: Connecting or boundary morphisms

\((f : X \to S, Y, i + 1) \to (f|_Y : Y \to S, Z, i)\)

for every chain \(Z \subseteq Y \subseteq X\) of closed sets.

When \(S\) has several connected components \(S_i\), we take \(\Delta(S) = \prod \Delta(S_i)\).

Call a sheaf \(F\) of \(R\)-modules on \(S\) an \(k\)-constructible or simply constructible, if it has finitely generated stalks and if there exists a partition \(\Sigma = \{Z_i\}\) of \(S\) into Zariski locally closed sets such that \(F|_{Z_i}^{an}\) is locally constant. If \(\Sigma\) is given, then \(F\) is called constructible with respect to \(\Sigma\). The term “\(k\)-constructible” is meant to signify that even though the sheaf is on \(S\), the strata \(Z_i\) are defined over \(k\). Let Cons\((S^{an}, R)\) (or Cons\((S^{an}, \Sigma, R)\)) denote the full subcategory of the category of sheaves of \(R\)-modules consisting of \(k\)-constructible sheaves (with respect to \(\Sigma\)). It is abelian and \(R\)-linear.

Let \(\Delta(S)^{op}\) denote the opposite quiver, which means that the edges are reversed. We define a representation \(H : \Delta(S)^{op} \to \text{Cons}(S^{an}, R)\) which sends \((f : X \to S, Y, i)\) to

\(H(X \to S, Y, i ; R) := H^i_S(X^{an}, Y^{an}; R)\)

The action of \(H\) on edges is as follows. We start with the easier case of a morphism of type II. To \((f : X \to S, Y, i + 1) \to (f|_Y : Y \to S, Z, i)\)

we assign the connecting map

\(H^i_S(Y, Z; R) \to H^{i+1}_S(X, Y; R)\)

induced by the exact sequence

\[(2.1) 0 \to j_{XY!}R \to j_{XZ!}R \to j_{YZ!}R \to 0\]

For a morphism

\(g : (f : X \to S, Y, i) \to (f' : X' \to S, Y', i)\)

of type I, the map on cohomology

\[(2.2) H^i_S(X', Y'; R) = R^if_*j_{X'Y'!}R \to R^if_*j_{XY!}R = H^i_S(X, Y; R)\]

is constructed below. We have a commutative diagram of distinguished triangles

\[(2.3)\]

The dotted arrow above induces a map

\(\mathbb{R}f_*j_{XY!}R \to \mathbb{R}f_*\mathbb{R}g_*j_{XY!}R \cong \mathbb{R}f_*j_{XY!}R\)

which gives \((2.2)\)
Remark 2.4. It should be clear that one can define a representation of $\Delta(S)^{\text{op}}$ as above for any theory satisfying Grothendieck’s “six operations” formalism (as laid out in [BBD, pp 43-44] for example). In fact, one only needs a theory with direct images and extensions by zero, for which analogues of (2.1) and (2.3) exist.

Now we can apply the $N^+$ construction to obtain the category of effective motivic (constructible) sheaves $\mathcal{M}_\text{eff}(S, R) := N_R(H)$. If $R$ is understood, we write $\mathcal{M}_\text{eff}(S) = \mathcal{M}_\text{eff}(S, R)$ and $\mathcal{M}_\text{eff}(k) = \mathcal{M}_\text{eff}(\text{Spec } k)$. The category of motivic sheaves $\mathcal{M}(S)$ will be built from this in the next section by inverting a certain object.

Let $\Sigma$ be a finite partition of $S$ into locally closed sets, let $\Delta(S, \Sigma) \subset \Delta(S)$ denote the full subquiver of triples $(X \to S, Y, i)$ such that $H^i_S(X, Y) \in \text{Ob Cons}(S, \Sigma)$. Then we can consider the subcategory $\mathcal{M}_\text{eff}(S, \Sigma, R) = N_R(H|_{\Delta(S, \Sigma)}) \subset \mathcal{M}_\text{eff}(S, R)$ of motivic sheaves constructible with respect to $\Sigma$. We define the subcategory of motivic local systems as $\mathcal{M}_\text{eff}_\text{ls}(S, R) = \mathcal{M}_\text{eff}(S, \{S\}, R)$.

The image of $\mathcal{M}_\text{eff}_\text{ls}(S, R)$ under $R_B$ is contained in the category of local systems in the usual sense.

Remark 2.5. Let us compare the story so far with what was done in [A2].

1. In the earlier paper, $\mathcal{M}^{\text{eff}}(S)$ was not considered; $\mathcal{M}(S)$ was constructed in a single step. This required a more complicated definition of $\Delta(S)$, where objects had an extra parameter, and there was an additional set of morphisms.

2. Another change in the present definition of $\Delta(S)$ is to require that pairs have the base change property rather than the stronger condition that they be controlled. This condition is used later for the existence of inverse and direct images (Proposition 2.4 and Theorem 7.1).

3. In [A2], we only considered the case where $R$ was a field. There $\mathcal{M}(S)$ had coefficients in $\mathbb{Q}$.

4. The present construction corresponds to what were called premotivic sheaves in [A2]. There was an additional step of forcing $\mathcal{M}(\_)$ to be a stack in the Zariski topology. This could also be done here, but it is not necessary for the present purposes.

5. In [A2], the categories $\mathcal{M}(S, \Sigma)$ were defined first, and $\mathcal{M}(S)$ was taken to be the $2$-colimit.

Let us recapitulate the universal property of the $N^+$ construction in this context.

Theorem 2.6. There is a faithful exact $R$-linear functor to $R_B : \mathcal{M}^{\text{eff}}(S, R) \to \text{Cons}(S_{\text{an}}, R)$, and $H$ factors through it. This is universal in the sense that given any other such factorization $\Delta(S) \to B \to \text{Cons}(S_{\text{an}}, R)$ through an $R$-linear abelian category such that the last functor is exact and faithful, we have an essentially unique dotted arrow completing the diagram

$$
\begin{array}{ccc}
\Delta(S)^{\text{op}} & \xrightarrow{h} & \mathcal{M}^{\text{eff}}(S, R) \\
\downarrow & & \downarrow \\
B & \xleftarrow{R_B} & \text{Cons}(S_{\text{an}}, R)
\end{array}
$$
We call the upper functor $R_B$, the Betti realization. Given $(X \to S, Y) \in \text{Ob } \Delta(S)$, let $h^*_S(X, Y) = h(X \to S, Y, i)$.

Suppose that $f : T \to S$ is a morphism of $k$-varieties. We can define a morphism of quivers $f^* : \Delta(S)^{op} \to \Delta(T)^{op}$ which takes

$$(X \to S, Y, i) \mapsto (X_T \to T, Y_T, i)$$

Since $(X \to S, Y)$ has the base change property,

$$H(X_T \to T, Y_T, i) \cong f^* H(X \to S, Y, i)$$

Therefore corollary 1.3 can be applied to show that there is an exact functor

$$(2.4) \quad f^* : \mathcal{M}^{\text{eff}}(S, R) \to \mathcal{M}^{\text{eff}}(T, R)$$

which is compatible with $f^*$ for sheaves under Betti realization (compare [A2, 3.5.2]).

3. Étale realization

We want to discuss some other realizations of $\mathcal{M}^{\text{eff}}(S, R)$ starting with the étale realization with finite coefficients. We fix an embedding of the algebraic closure $\bar{k} \subset \mathbb{C}$, and let $X = X \times_{\text{Spec } k} \text{Spec } \bar{k}$ etc.

(R1) Let $R$ be a finite ring. We get a representation of $\Delta(S)^{op}$ to the category $\text{Cons}(S_{et}, R)$ of constructible $R$-modules on $S_{et}$, which sends

$$(X \to S, Y, i) \mapsto H^i_S(\bar{X}_{et}, \bar{Y}_{et}, R) := R^i f^\text{et}_* j^{\text{et}}_{\bar{X}, \bar{Y}} R_{\bar{X}, \bar{Y}}$$

This extends to a representation by remark 2.4. The comparison theorem [SGA4, exp XVI, thm 4.1; exp XVII, thm 5.3.3] plus theorem 2.6 implies that there is an exact faithful functor $R_{et} : \mathcal{M}^{\text{eff}}(S, R) \to \text{Cons}(S_{et}, R)$ (compare [A2, 3.4.6]), called the étale realization.

Before describing the $\ell$-adic realization, we need to recall some facts about $\ell$-adic sheaves. A constructible $\ell$-adic sheaf on a scheme $S$ is really an inverse system $\ldots \mathcal{F}_n \to \mathcal{F}_{n-1} \ldots$, where each $\mathcal{F}_n$ is a constructible sheaf of $\mathbb{Z}/\ell^n \mathbb{Z}$-modules on the étale topology $S_{et}$, such that each projection yields an isomorphism $\mathbb{Z}/\ell^{n-1} \mathbb{Z} \otimes \mathcal{F}_n \cong \mathcal{F}_{n-1}$. The collection of constructible $\ell$-adic sheaves can be made into a $\mathbb{Z}_\ell$-linear abelian category $\text{Cons}(S_{et}, \mathbb{Z}_\ell)$ with an appropriate definition [SGA5, exp V, VII]. Ekedahl [E] has constructed a triangulated category, that we denote by $D^b_{ch}(S, \mathbb{Z}_\ell)$, that behaves like the derived category of $\text{Cons}(S_{et}, \mathbb{Z}_\ell)$, and possesses Grothendieck’s six operations. More precisely, it has a $t$-structure with heart $\text{Cons}(S_{et}, \mathbb{Z}_\ell)$, and a conservative triangulated functor $\mathbb{Z}/\ell \otimes - : D^b_{et}(S, \mathbb{Z}_\ell) \to D^b(S_{et}, \mathbb{Z}/\ell \mathbb{Z})$. One has ordinary and extraordinary direct and inverse images in $D^b_{ch}(-, \mathbb{Z}_\ell)$ [E, thm 6.3], and these are compatible with the corresponding operations in $D^b(-, \mathbb{Z}/\ell \mathbb{Z})$. We can therefore define

$$H^i_S(\bar{X}_{et}, \bar{Y}_{et}, \mathbb{Z}_\ell) = R^i f^\text{et}_* j^{\text{et}}_{\bar{X}, \bar{Y}} R_{\bar{X}, \bar{Y}}$$

where the above operations and $t$-structure are used to define $R^i f^\text{et}_*$ etc.

In this paragraph, let us suppose that $k = \mathbb{C}$. Given a $\mathbb{C}$-variety $X$, define the site $X_{cl}$ with objects given by local homeomorphisms $U \to X_{an}$ and coverings are surjective families $\{U_i \to U\}$. There is an obvious map of sites $X_{cl} \to X_{an}$, which induces an equivalence of topoi, i.e. of categories of sheaves [SGA4, exp XI §4]. There is a canonical morphism of topoi $\epsilon : X_{et} \to X_{cl}$ which induces a map from étale to classical cohomology. Ekedahl’s construction, which is quite general, can
be applied to \(X_{ct}\), to yield a triangulated category \(D^b_{ek}(X_{ct}, \mathbb{Z}_\ell)\). Using the above equivalence of topoi and \([E, \text{thm 7.2}]\), we obtain equivalences
\[
D^b_{ek}(X_{ct}, \mathbb{Z}_\ell) \cong D^b_{ek}(X_{an}, \mathbb{Z}_\ell) \cong D^b_c(X_{an}, \mathbb{Z}_\ell)
\]
where the category on the right is the usual constructible derived category. One can see from construction that this equivalence is compatible with ordinary and proper direct images. We now come to the key comparison result.

**Proposition 3.1.** Suppose that \(f : X \to Y\) is a morphism of \(\mathbb{C}\)-varieties, and that \(\mathcal{F}\) is a constructible \(\ell\)-adic sheaf. Then there are canonical isomorphisms
\[
\epsilon^* R^i f^* \mathcal{F} \cong R^i f^* \epsilon^* \mathcal{F}
\]
where the direct images on the right are computed in the constructible derived categories.

**Proof.** Consider the distinguished triangle
\[
\epsilon^* R^i f^* \mathcal{F} \xrightarrow{\kappa} R^i f^* \epsilon^* \mathcal{F} \to C \xrightarrow{[1]}
\]
where \(\kappa\) is the canonical map. The usual comparison theorem \([\text{SGA4, exp XVI, thm 4.1; exp XVII, thm 5.3.3}]\) shows that \(\mathbb{Z}_\ell \otimes - = 0\). Since \(\mathbb{Z}_\ell \otimes -\) is conservative, \(C = 0\). The proof of the second isomorphism is similar. \(\square\)

(R2) We get a representation of \(\Delta(S)^{op}\) to the category \(\text{Cons}(S_{ct}, \mathbb{Z}_\ell)\) which sends
\[
(X \to S, Y, i) \mapsto H^i_{S}(\bar{X}_{et}, \bar{Y}_{et}, \mathbb{Z}_\ell)
\]
This extends to a representation by remark 2.4. The previous proposition plus theorem 2.6 implies that there is an exact faithful functor \(R_\ell : \mathcal{M}^{\text{eff}}(S, \mathbb{Z}_\ell) \to \text{Cons}(S_{ct}, \mathbb{Z}_\ell)\) (compare \([A2, 3.4.6]\)), called the \(\ell\)-adic realization.

(R3) We can take the product over all primes to get a representation
\[
(X \to S, Y, i) \mapsto \prod \ell H^i_{S}(\bar{X}_{et}, \bar{Y}_{et}, \mathbb{Z}_\ell) \in \text{Ob} \prod \ell \text{Cons}(S_{ct}, \mathbb{Z}_\ell)
\]
and this yields a realization
\[
\mathcal{R} : \mathcal{M}^{\text{eff}}(S, \widehat{\mathbb{Z}}_\ell) \to \prod \ell \text{Cons}(S_{ct}, \mathbb{Z}_\ell)
\]

(R4) If \(R'\) is a faithfully flat \(R\)-algebra, there is an \(R\)-linear exact change of coefficients functor \(R' \otimes_R - : \mathcal{M}^{\text{eff}}(S, R) \to \mathcal{M}^{\text{eff}}(S, R')\) fitting into a commutative diagram
\[
\begin{array}{ccc}
\mathcal{M}^{\text{eff}}(S, R) & \xrightarrow{R' \otimes R} & \mathcal{M}^{\text{eff}}(S, R') \\
\downarrow & & \downarrow \\
\text{Cons}(S_{an}, R) & \xrightarrow{R' \otimes R} & \text{Cons}(S_{an}, R')
\end{array}
\]
To see this, define \(\mathcal{M}^{\text{eff}}(S, R'/R)\) to be the category whose objects are triples \((M, L, \phi)\) with \((M, L) \in \mathcal{M}^{\text{eff}}(S, R') \times \text{Cons}(S_{an}, R)\) and \(\phi : R_B(M) \cong R' \otimes_R L\), and with the obvious notion of morphisms. Theorem 2.6 implies
the existence of an exact functor $\mathcal{M}^{\text{eff}}(S, R) \to \mathcal{M}^{\text{eff}}(S, R'/R)$. Compose this with the projection $\mathcal{M}^{\text{eff}}(S, R'/R) \to \mathcal{M}^{\text{eff}}(S, R')$.

(R5) By combining (R3) and (R4), and taking a projection, one obtains a realization

$$\mathcal{M}^{\text{eff}}(S, \mathbb{Z}) \xrightarrow{2\otimes_\mathbb{Z}} \mathcal{M}^{\text{eff}}(S, \hat{\mathbb{Z}}) \to \text{Cons}(S_{et}, \mathbb{Z}_\ell)$$

(The same sort of trick should be applicable to Ivorra’s category.)

4. HODGE REALIZATION OVER A CURVE

Nori constructed a Hodge realization $R_H$ from $\mathcal{M}^{\text{eff}}(k, \mathbb{Z})$ to the category of integral mixed Hodge structures using the representation that assigns to $(X, Y, i)$ the Deligne mixed Hodge structure on $H^i(X, Y; \mathbb{Z})$. Over a general base, things are more complicated. Saito [S] has defined his category of mixed Hodge modules $\text{MHM}(S)$ with the following properties:

1. Over a point $\text{MHM}(pt)$ is just the category of polarizable rational mixed Hodge structures. When $S$ is smooth, objects of $\text{MHM}(S)$ include polarizable variations of pure Hodge structures, and more generally admissible variations of mixed Hodge structures [S, thm 0.2].

2. The category $\text{MHM}(S)$ is abelian and $\mathbb{Q}$-linear. There is an exact faithful forgetful functor from $\text{MHM}(S)$ to the category of rational perverse sheaves $\text{Perv}(S)$ [BBD].

3. The previous functor extends to a triangulated functor $D^b \text{MHM}(S) \to D^b_c(S, \mathbb{Q})$ to the constructible derived category.

4. The standard operations $\mathbb{R}f_*, \mathbb{R}f^!, \ldots$ on $D^b_c(-)$ extend to operations $f^H_*, f^H!, \ldots$ on $D^b \text{MHM}(-)$ [S, thm 0.1].

There are two natural $t$-structures [BBD, §1.3] on $D^b \text{MHM}(S)$. The standard one has $\text{MHM}(S)$ as its heart. There is a second $t$-structure on $D^b \text{MHM}(S)$, that we call the classical $t$-structure, which corresponds to the usual one on $D^b_c(S)$ ([A2, appendix C], [S, rmk 4.6]). Let us call the heart of the classical $t$-structure, the category of constructible mixed Hodge modules, and denote it by $\text{CMHM}(S)$. It possesses a faithful exact functor to $\text{Cons}(S, \mathbb{Q})$. To each of the $t$-structures, there are associated cohomological functors $pH^* : D^b \text{MHM}(S) \to \text{MHM}(S)$ and $cH^* : D^b \text{MHM}(S) \to \text{CMHM}(S)$ respectively. In [A2, 3.4.7], we defined a Hodge realization functor

$$R_H : \mathcal{M}^{\text{eff}}(S, \mathbb{Q}) \to \text{CMHM}(S)$$

using the representation

$$(f : X \to S, Y, i) \mapsto cH_i f^*_i j^H_{X,Y} : \mathbb{Q}$$

Note that one can check that this is a representation with the help of remark 2.4.

The category $\text{CMHM}(S)$ is constructed abstractly, so the structure of its objects is not immediately obvious. We will give a more explicit alternative description of constructible mixed Hodge modules when $S$ is an irreducible smooth complex curve. A similar description is possible in general, but the notation becomes somewhat more cumbersome. We fix a partition $\Sigma = \{U, p_1, \ldots, p_n\}$ of $S$ into an open set $U$ and closed points $p_i$. Let $j : U \to S, i_1 : p_1 \to S, \ldots$ denote the inclusions. An admissible variation of mixed Hodge structures on $U$, consists of a $\mathbb{Q}$-local system $\mathcal{F}$, plus some other data which imply that all the stalks $\mathcal{F}_x$ are endowed with mixed Hodge structures. See [SZ, §3] for the precise definition. These form a $\mathbb{Q}$-linear
abelian category VMHS(U). Given an object \( \mathcal{F} \in \text{Ob VMHS}(U) \), we can view the perverse sheaf \( \mathcal{F}[1] \) as part of a mixed Hodge module by [S, thm 0.2]. So \( \mathcal{F} \) can be viewed as an object of \( D^b \text{MHM}(U) \). We define \( \text{CMHM}(S, \Sigma) \) to be the category with objects

\[
\{(\mathcal{F}, M_1, \ldots, \gamma_1, \ldots) \mid \mathcal{F} \in \text{Ob VMHS}(U), M_a \in \text{MHM}(pt), \gamma_a : M_a \to H^0(i_a^*j_a^*\mathcal{F})\}
\]

The object \( H^0(i_a^*j_a^*\mathcal{F}) \) is a mixed Hodge structure with underlying vector space \( i_a^*j_a^*\mathcal{F} \). We require that the gluing maps \( \gamma_a \) are morphisms of mixed Hodge structures.

A morphism \( (\mathcal{F}, M_1, \ldots) \to (\mathcal{F}', M_1', \ldots) \) is a collection of morphisms \( \mathcal{F} \to \mathcal{F}' \), \( M_a \to M_a' \) which are compatible with the gluing maps. It is not difficult to see that:

Lemma 4.1.

1. \( \text{CMHM}(S, \Sigma) \) is a \( \mathbb{Q} \)-linear abelian category.
2. The functor \( F : \text{CMHM}(S, \Sigma) \to \text{Cons}(S, \Sigma) \) which sends \( (\mathcal{F}, M_1, \ldots) \) to

\[
\ker \left[ j_*\mathcal{F} \oplus \bigoplus_a i_*M_a \to \bigoplus_a i_*i_a^*j_*\mathcal{F} \right],
\]

where the map is the difference of the adjunction map and \( \sum \gamma_a \), is exact and faithful.

Let us outline the construction of the Hodge realization

\[
H_H : \mathcal{M}_{\text{eff}}(S, \Sigma, \mathbb{Q}) \to \text{CMHM}(S, \Sigma)
\]

Given \((X, Y, i) \in \Delta(S, \Sigma)\), we need to assign an object \( H_H(X, Y, i) = (\mathcal{F}, M_1, \ldots) \) whose image under \( F \) is \( H^i_{\text{D}}(X, Y) \). Let \( X_{p_a} \) and \( Y_{p_a} \) denote fibres over \( p_a \), and let \( I_a : X_{p_a} \to X \) and \( J : X_U \to X \) denote the inclusions. We set \( M_a = H^i(X_{p_a}, Y_{p_a}) \) with the Deligne mixed Hodge structure. Since \((X, Y, i) \in \Delta(S, \Sigma), \mathcal{F} = H^i_U(X_U, Y_U)\) is a local system. It carries an admissible variation of mixed Hodge structure, namely

\[
(p^* H^i I_a^* j_a^* H^i_{XY}(Q)[-1])
\]

This can also be constructed by hand using methods of [SZ], but this is quite a long process. The adjunction maps

\[
I^*_a H^* j^*_a H^i_{XY}(Q) \to I^*_a H^* J^*_a H^i_{XY}(Q)
\]

induce maps on cohomology

\[
H^i(X_{p_a}, Y_{p_a}) \to i_{a*}j^* H^i_U(X_U, Y_U)
\]

These are gluing maps. For any sheaf \( \mathcal{G} \) on \( S \), one can check by examining stalks that

\[
0 \to \mathcal{G} \to j_*j^*\mathcal{G} \oplus \bigoplus_a i_{a*}i_a^*\mathcal{G} \to \bigoplus_a i_{a*}i_a^*j_*\mathcal{G}
\]

is exact. Applying this observation to \( \mathcal{G} = H^i_{\text{D}}(X, Y) \), shows that \( F(H_H(X, Y, i)) \cong \mathcal{G} \). It remains to check that \( H_H \) gives a representation, but this follows with the help of remark 2.4. We note that the image of \( \mathcal{M}_{\text{eff}}(S) \), under the Hodge realization, lies in \( \text{VMHS}(S) \).
5. Motivic sheaves

Given a category $A$ with an endofunctor $L : A \to A$, following [I, 7.6], we define a new category $A[L^{-1}]$ with objects $\text{Ob} A \times \mathbb{Z}$, and morphisms

$$\text{Hom}_{A[L^{-1}]}((a, n), (b, m)) = \lim_{\rightarrow} \text{Hom}_A(L^{i+n}a, L^{i+m}b)$$

The map $a \mapsto (a, 0)$ extends to a functor $A \to A[L^{-1}]$

**Lemma 5.1.**

1. If $A$ is $R$-linear abelian, and $L$ is $R$-linear and exact, then $A[L^{-1}]$ is $R$-linear abelian, and $A \to A[L^{-1}]$ is exact.
2. If $L$ is an equivalence, then $A$ is equivalent to $A[L^{-1}]$.
3. There exists a 2-commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{L} & A \\
\downarrow & & \downarrow \\
A[L^{-1}] & \xrightarrow{L'} & A[L^{-1}]
\end{array}$$

where $L'$ is an equivalence.

4. Given a functor $F : A \to B$ and a 2-commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{L} & A \\
\downarrow & & \downarrow \\
B & \xrightarrow{L''} & B
\end{array}$$

there exists a 2-commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{} & A[L^{-1}] \\
\downarrow & & \downarrow \\
B & \xrightarrow{} & B[L''^{-1}]
\end{array}$$

**Proof.** The first statement is [I, lemma 7.4]. If $L$ is an equivalence with quasi-inverse $L^{-1}$, one sees that $(a, n) \cong (L^{n}a, 0)$, and that

$$\text{Hom}_{A[L^{-1}]}((a, 0), (b, 0)) \cong \text{Hom}_A(a, b)$$

So the functor $A \to A[L^{-1}]$ is essentially surjective and fully faithful. One defines $L'(a, n) = (La, n+1)$, and checks this gives an auto-equivalence of $A[L^{-1}]$ extending $L$. The last statement is clear from the construction. \qed

**Corollary 5.2.** If the functor $L''$ in (4) is an equivalence, there is a 2-commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{} & A[L^{-1}] \\
\downarrow & & \downarrow \\
B & \xrightarrow{} & B[L''^{-1}]
\end{array}$$
Call \((X \to S, Y, i) \in \Delta(S)\) cellular if \(X/S\) is affine with equidimensional fibres, such that
\[H^j_S(X, Y; R) = 0\]
unless \(j = i\), and \(H^j_S(X, Y; R)\) is flat over \(R\). A basic example of cellular object is \((\mathbb{G}_m S \to S, \{1\}, 1)\). We refer to the corresponding motive \(h^1_S(\mathbb{G}_m, S, 1)\) as the Lefschetz motive. In the absolute case, a cellular object is what Nori calls a “good pair” [HM].

**Lemma 5.3.** Given a cellular object \((Z \to S, W, j)\), let \(M = H^j_S(Z, W; R)\). The map \(\zeta : \text{Ob} \Delta(S) \to \text{Ob} \Delta(S)\) given by
\[\zeta : (X \to S, Y, i) \mapsto (Z \times_S X \to S, W \times_S X \cup Z \times_S Y, j + i)\]
is a morphism of quivers. The Künneth isomorphism
\[H^{i+j}_S(Z \times S X, W \times_S X \cup Z \times_S Y; R) \oplus M \otimes R H^i(X, Y; R)\]
renders the diagram (5.2) \(\Delta(S)^{op} \xrightarrow{\zeta^{op}} \Delta(S)^{op}\)
\[\text{Cons}(S_{an}, R) \xrightarrow{M^{\otimes}} \text{Cons}(S_{an}, R)\]
2-commutative.

We omit the proof, but remark that the right side of (5.1) lies in \(\text{Ob} \Delta(S)\) by lemma 2.3. It follows from this lemma and corollary 1.3, that if \((Z \to S, W, j)\) is cellular, then we can construct an induced exact endofunctor
\[h^1_S(Z, W) \otimes - : \mathcal{M}^\text{eff}(S, R) \to \mathcal{M}^\text{eff}(S, R)\]
Define the exact endofunctor \(\mathbb{L}^\text{eff} : \mathcal{M}^\text{eff}(S, R) \to \mathcal{M}^\text{eff}(S, R)\) by \(\mathbb{L}^\text{eff} = h^1_S(\mathbb{G}_m, S, 1) \otimes -\). Set \(\mathcal{M}(S, R) := \mathcal{M}^\text{eff}(S, R)[[(\mathbb{L}^\text{eff})^{-1}]]\). Then there exists a 2-commutative diagram
\[\mathcal{M}(S, R) \xrightarrow{\mathbb{L}^\text{eff}} \mathcal{M}(S, R)\]
\[\mathcal{M}(S, R) \xrightarrow{\mathbb{L}} \mathcal{M}(S, R)\]
with \(\mathbb{L}\) invertible. Furthermore, \(\mathcal{M}(S, R)\) is the universal such category. We refer to this as the category of motivic sheaves with coefficients in \(R\), and \(\mathcal{M}(S) = \mathcal{M}(S, \mathbb{Q})\) simply as the category of motivic sheaves. The category \(\mathcal{M}^\text{eff}(S)\) is good enough for most purposes, but inverting \(\mathbb{L}\) becomes important in situations where one considers duals.

The 2-commutativity of (5.2) shows that there is a natural isomorphism \(R_B \circ \mathbb{L}^\text{eff} \cong R_B\). Therefore \(R_B\) extends to an exact functor \(\mathcal{M}(S, R) \to \text{Cons}(S_{an}, R)\) by the universal property. The construction of \(\mathcal{M}^\text{eff}(S, R)[[(\mathbb{L}^\text{eff})^{-1}]]\) shows that this is faithful. We note that the category \(\text{CMHM}(S)\) is stable under the operations \(M \mapsto M \otimes \mathbb{Q}(\pm 1)\).
Lemma 5.4. The functors $R\ell, R_H$ and $f^*$ extend to $\mathcal{M}(-)$. Tate twists in $\mathcal{M}(S)$ are compatible with these operations in the sense that

$$R\ell \circ L \cong \mathbb{Z}_\ell(-1) \otimes R\ell$$
$$R_H \circ L \cong \mathbb{Q}(-1) \otimes R_H$$
$$f^* \circ L \cong L \circ f^*$$

Proof. By Künneth, one gets an isomorphism of étale cohomology

$$H^i \left( \mathbb{G}_m \times \bar{S}, 1 \times \bar{X} \cup \mathbb{G}_m \times \bar{X} \cup \bar{S} \times \bar{Y} \right; \mathbb{Z}_\ell) \cong \mathbb{Z}_\ell(-1) \otimes H^i(\bar{X}, \bar{Y}; \mathbb{Z}_\ell)$$

Therefore

$$R\ell \circ L_{\text{eff}} \cong \mathbb{Z}_\ell(-1) \otimes R\ell$$

Similarly one checks that

$$R_H \circ L_{\text{eff}} \cong \mathbb{Q}(-1) \otimes R_H$$
$$f^* \circ L_{\text{eff}} \cong L_{\text{eff}} \circ f^*$$

on $\mathcal{M}_{\text{eff}}(-)$. The lemma is a formal consequence of these identities. \qed

The category $\mathcal{M}(k) = \mathcal{M}(\text{Spec } k)$ is precisely Nori's category of mixed motives. Let $\mathcal{M}_{\text{pure}}(k) \subset \mathcal{M}(k)$ denote the full subcategory generated by subquotients of $h^i(X)$, with $X$ smooth and projective. This subcategory can be related to other constructions. André defined a category of pure motives $\mathcal{A}$, by replacing algebraic cycles in Grothendieck’s construction by his motivated cycles. If one assumes the standard conjectures, then André’s and Grothendieck’s categories of motives would coincide.

Theorem 5.5. The category $\mathcal{M}_{\text{pure}}(k, \mathbb{Q})$ is equivalent to André’s category of pure motives.

Proof. See [A2, thm 6.4.1] or [HM, thm 10.2.7]. \qed

Corollary 5.6. The category $\mathcal{M}_{\text{pure}}(k, \mathbb{Q})$ is semisimple abelian.

Proof. [A, thm 0.4]. \qed

6. BACKGROUND ON CELLULAR DECOMPOSITIONS

We recall some background results from [A1, A2], along with some simplifications. First, we recall Jouanolou’s trick [J, lemma 1.5].

Lemma 6.1 (Jouanolou). If $S$ is a quasi-projective variety then there exists an affine variety $T$ and a smooth morphism $\pi : T \to S$ which is Zariski locally isomorphic to $\mathbb{A}^n_S$ for some $n$.

Fix $\pi : T \to S$ as above. Since $\mathbb{C}^n$ is contractible, then given $\mathcal{F} \in \text{Cons}(S_{an})$, we have an isomorphism $H^i(T, \pi^*\mathcal{F}) \cong H^i(S_{an}, \mathcal{F})$ for each $i$. So for our purposes, we lose nothing by working on $T$. The next result is key, and is a consequence of Beilinson’s basic lemma. First, we need some notation. Given a sheaf $\mathcal{F}$ on $T$, and closed sets $T_1 \subset T_2 \subset T$, let

$$H^i(T_2, T_1; \mathcal{F}) = H^i(T_2, j_{T_2T_1}!(\mathcal{F}|_{T_2-T_1}))$$

where

$$j_{T_2T_1} : T_2 - T_1 \hookrightarrow T_2$$

is the inclusion. When $\mathcal{F}$ is constant, this is just the cohomology of the pair in the usual sense.
Lemma 6.2. Given \( F \in \text{Cons}(T_{an}) \), there exist a chain
\[
\emptyset = T_{a-1} \subset T_0 \subset T_1 \subset \cdots = T
\]
of equidimensional closed sets with \( \dim T_i = i \), such that for all \( a \)
\[
H^i(T_a, T_{a-1}; F) = 0
\]
unless \( i = a \). Furthermore, \( T_\bullet \) can be chosen to refine a given chain.
Proof. [A1, lemma 3.7].

Let us say that an admissible pair \( (X \to S, Y) \) is \emph{cellular} with respect the chain \( T_\bullet \) if for all \( a \)
\begin{equation}
H^i(T_a, T_{a-1}; \pi^* H^*_S(X, Y)) = 0 \text{ if } i \neq a
\end{equation}
We say that an object \( (X \to S, Y, j) \) is cellular with respect to \( T_\bullet \), if \( (X \to S, Y) \) is. Let \( \Delta(S, T_\bullet) \subset \Delta(S) \) be the full sub quiver of triples \( (X \to S, Y, j) \) cellular with respect to \( T_\bullet \).

Corollary 6.3. \( \Delta(S) \) is a directed union of \( \Delta(S, T_\bullet) \), as \( T_\bullet \) runs over various chains (with \( T \) fixed as above).

Fix a commutative noetherian ring \( R \). Cohomology should be understood to take values in \( R \) if coefficients are not specified. Suppose that \( (X \to S, Y) \) is an admissible pair which is cellular with respect to \( T_\bullet \). Let \( F = \pi^* \mathbb{H}_S(X, Y; R) \in D^b(T_{an}, R) \), and \( G = G_{(X,Y)} = \mathbb{R} \Gamma F \in D^b(R\text{-mod}) \). We construct filtrations on \( G \) by
\[
P_\bullet(G) = \mathbb{R} \Gamma \tau_{\leq -} F, \quad \text{and} \quad F_\bullet G = \mathbb{R} \Gamma j_{\mathbb{T} T_\bullet}^* j_{\mathbb{T} T_\bullet}^* F
\]
Define a new filtration \( \text{Dec}(F) \) by \( \text{d\`ecalage} \) [D2, 1.3.3].

Lemma 6.4. The identity of \( G \) induces an isomorphism in the filtered derived category
\[
(G, P) \cong (G, \text{Dec}(F))
\]
Proof. This follows from de Cataldo-Migliorini [CM, prop 5.6.1]. The conditions of their proposition hold because of (6.1).
where the differentials are connecting maps. The whole spectral sequence can be constructed, in the usual manner \cite{W, §5.9}, from the exact couple

\[
E_1 = \bigoplus_p H^*(X_{T_p}, Y_{T_p} \cup X_{T_{p-1}})
\]

\[
D_1 = \bigoplus_p H^*(X, Y \cup X_{T_{p-1}})
\]

with maps

\[
D_1 \rightarrow E_1 \rightarrow D_1
\]

coming from the long exact sequence associated to the triples \((X, Y \cup X_{T_p}, Y \cup X_{T_{p-1}})\). For each \(i\), let us write \(K(i)^* = K(X, Y, i)^* = E_1^i\), i.e.

\[
K(i)^* = H^i(X_{T_0}, Y_{T_0} \cup X_{T_{0-1}}) \rightarrow H^{i+1}(X_{T_1}, Y_{T_1} \cup X_{T_{1-1}}) \rightarrow \ldots
\]

To summarize:

**Corollary 6.5.** With the same assumptions as above, there is an isomorphism of spectral sequences \(L^E_{pq} \cong F^E_{pq}\). In particular, there is an isomorphism of \(R\)-modules

\[
\phi : H^j(K(i)^*) \cong H^j(S, H^i_S(X, Y))
\]

where \(H^j\) stands for the \(j\)th cohomology module of a complex.

We need to understand the naturality properties of above the isomorphism. Given a morphism of \((X', Y' \rightarrow S, Y' \rightarrow S)\) of pairs, we have a morphism \(G_{(X,Y)} \rightarrow G_{(X',Y')}\) compatible with the filtrations \(P, F, Dec(F)\) and the isomorphism of lemma 6.4 (which is just the identity!). In particular, we can conclude that we have a morphism

\[
K(X, Y, i)^* \rightarrow K(X', Y', i)^*
\]

and a commutative diagram

\[
\begin{array}{ccc}
\mathcal{H}^j(K(X, Y, i)^*) & \rightarrow & H^j(S, H^i_S(X, Y)) \\
\downarrow & & \downarrow \\
\mathcal{H}^j(K(X', Y', i)^*) & \rightarrow & H^j(S, H^i_S(X', Y'))
\end{array}
\]

Let \(Z \subseteq Y \subseteq X\) be a chain of closed sets. Then the exact sequence

\[
0 \rightarrow j_{XY!}R \rightarrow j_{XZ!}R \rightarrow j_{YZ!}R \rightarrow 0
\]

gives rise to a distinguished triangle

\[
G_{(X,Y)} \rightarrow G_{(X,Z)} \rightarrow G_{(Y,Z)} \rightarrow G_{(X,Y)}[1]
\]
The last morphism is compatible with the filtrations leading to a commutative diagram

\[ \begin{array}{ccc} \mathcal{H}^i(K(Y, Z, i)^\bullet) & \xrightarrow{\phi} & H^i(S, H^i_S(Y, Z)) \\
\downarrow & & \downarrow \\
\mathcal{H}^i(K(X, Y, i + 1)^\bullet) & \xrightarrow{\phi} & H^i(S, H^{i+1}_S(X, Y)) \end{array} \]

7. Motivic Leray

Fix a subfield $k \subset \mathbb{C}$, and commutative noetherian ring $R$. Let $\mathcal{M}(S) = \mathcal{M}(S, R)$ for the rest of this section. Here is the main result of the paper. It refines theorem 3.1 of [A1], although the strategy of proof is closer to that of [A2, thm 5.2.1].

**Theorem 7.1.** Let $S$ be a quasiprojective $k$-variety. Then there exists a $\delta$-functor $\{ h^j : \mathcal{M}(S) \to \mathcal{M}(k) \}_{j=0,1,\ldots}$, such that for each $j$, the diagram

\[ \begin{array}{ccc} \mathcal{M}(S) & \xrightarrow{h^j} & \mathcal{M}(k) \\
\downarrow^{R_B} & & \downarrow^{R_B} \\
\text{Cons}(\text{S}_\text{an}) & \xrightarrow{H^j} & \text{R-mod} \end{array} \]

$2$-commutes. Given a controlled pair $(f : X \to S, Y)$, there exists a spectral sequence

\[ M^j_{p,q} = h^p(h^q_S(X, Y)) \Rightarrow h^{p+q}(X, Y) \]

in $\mathcal{M}(k)$ whose image under $R_B$ is isomorphic to the Leray spectral sequence

\[ L^j_{p,q} = H^p(S, H^q_S(X, Y)) \Rightarrow H^{p+q}(X, Y) \]

We will defer the proof until we have established some preliminary results. Define a new category $C$ whose objects are triples

\[(K^\bullet, F, \phi : H^*(S, F) \cong R_B \circ \mathcal{H}^*(K^\bullet))\]

where $K^\bullet \in \text{Ob } C^h(\mathcal{M}^\text{eff}(k))$, $F \in \text{Cons}(S, R)$, and $\phi$ an isomorphism of graded $R$-modules. A morphism $(K_1^\bullet, F_1, \phi_1) \to (K_2^\bullet, F_2, \phi_2)$ is pair of morphisms $K_1^\bullet \to K_2^\bullet$, $F_1 \to F_2$ which are compatible under $\phi_i$.

**Lemma 7.2.** The category $C$ is abelian, and the projections $p_1 : C \to C^h(\mathcal{M}^\text{eff}(k))$ and $p_2 : C \to \text{Cons}(S)$ are exact. The induced functor from $\overline{C} = C/\ker p_2$ to $\text{Cons}(S)$ is exact and faithful. The functor $\mathcal{H}^i \circ p_1 : C \to \mathcal{M}^\text{eff}(k)$ factors through the quotient $\overline{C}$.

**Proof.** The first two statements are straightforward and completely formal, so let us focus on the last. Let $\Sigma$ be the set of morphisms of $C$ whose kernel and cokernel lie in $\ker p_2$. Then by construction $\overline{C}$ is the localization $\Sigma^{-1}C$. So it suffices to prove that $\mathcal{H}^i \circ p_1$ takes $\Sigma$ to the set of isomorphisms. Let $f : (K_1^\bullet, F_1, \phi_1) \to (K_2^\bullet, F_2, \phi_2)$ be in $\Sigma$. Then $f$ induces an isomorphism $F_1 \cong F_2$, and therefore an isomorphism $H^i(F_1) \cong H^i(F_2)$. It follows that $f$ induces an isomorphism $\mathcal{H}^i(K_1) \cong \mathcal{H}^i(K_2).$ □
**Proposition 7.3.** There is a δ functor $h^* : \mathcal{M}^{\text{eff}}(S) \to \mathcal{M}^{\text{eff}}(\text{Spec } k)$ such that

$$
\begin{array}{ccc}
\mathcal{M}^{\text{eff}}(S) & \xrightarrow{h^*} & \mathcal{M}^{\text{eff}}(k) \\
\downarrow R_B & & \downarrow R_B \\
\text{Cons}(S) & \xrightarrow{h^*} & R\text{-mod}
\end{array}
$$

2-commutes.

**Proof.** By lemma 6.1, we can find an affine variety $T$ and an affine space bundle $p : T \to S$. We fix this choice. By corollary 6.3, $\Delta(S)$ is a directed union of $\Delta(S, T_\bullet)$. Therefore by lemma 1.4, $\mathcal{M}^{\text{eff}}(S)$ is the filtered 2-colimit of the categories

$$\mathcal{M}^{\text{eff}}(S, T_\bullet) = \mathcal{N}(H|_{\Delta(S, T_\bullet)})$$

Thus it suffices to define $h^*$ on these categories, and verify compatibility under refinement.

Given an object $(X, Y, i) \in \text{Ob } \Delta(S, T_\bullet)$, let $K_{T_\bullet}(X, Y, i)$ denote the sequence of motives in $\mathcal{M}^{\text{eff}}(k)$ given by

$$h^i(X_{T_0}, Y_{T_0} \cup X_{T_{-1}}) \xrightarrow{d} h^{i+1}(X_{T_1}, Y_{T_1} \cup X_{T_{-1}}) \xrightarrow{d} \ldots$$

where the maps $d$ are connecting maps. One can check immediately that $R_B(d^2) = 0$, so $d^2 = 0$ because $R_B$ is faithful. Therefore $K_{T_\bullet}(X, Y, i)$ is an object in the abelian category of bounded chain complexes $C^b(\mathcal{M}^{\text{eff}}(k))$. Its image $R_B(K(X, Y, i)) \in C^b(R\text{-mod})$ is the complex $K_i$ constructed in (6.3). We define

$$F_{T_\bullet}(X, Y, i) = (K_{T_\bullet}(X, Y, i), H^\bullet_S(X, Y, \phi) \in \text{Ob } C$$

where $\phi$ comes from corollary 6.5.

We claim that $F_{T_\bullet} : \Delta(S, T_\bullet)^{\text{op}} \to C$ is a representation. Given a morphism $(X', Y', i) \to (X, Y, i)$ of type I, one gets a diagram

$$
\begin{array}{ccc}
h^i(X_{T_0}, Y_{T_0} \cup X_{T_{-1}}) & \xrightarrow{d} & h^{i+1}(X_{T_1}, Y_{T_1} \cup X_{T_{-1}}) \\
\downarrow & & \downarrow \\
h^i(X'_{T_0}, Y'_{T_0} \cup X'_{T_{-1}}) & \xrightarrow{d} & h^{i+1}(X'_{T_1}, Y'_{T_1} \cup X'_{T_{-1}})
\end{array}
$$

It commutes because it does so after applying $R_B$. Therefore we have morphism $K_{T_\bullet}(X, Y, i) \to K_{T_\bullet}(X', Y', i)$. This can be completed to a morphism $F_{T_\bullet}(X, Y, i) \to F_{T_\bullet}(X', Y', i)$ by (6.4).

Similarly, given a morphism of type II associated to a triple $Z \subseteq Y \subseteq X$, one gets a commutative diagram

$$
\begin{array}{ccc}
h^i(X_{T_0}, Y_{T_0} \cup X_{T_{-1}}) & \xrightarrow{d} & h^{i+1}(X_{T_1}, Y_{T_1} \cup X_{T_{-1}}) \\
\downarrow & & \downarrow \\
h^{i+1}(Y_{T_0}, Z_{T_0} \cup Y_{T_{-1}}) & \xrightarrow{d} & h^{i+2}(Y_{T_1}, Z_{T_1} \cup Y_{T_{-1}})
\end{array}
$$

This can be extended to a morphism

$$(K_{T_\bullet}(X, Y, i), H^\bullet_S(X, Y, \phi) \to (K_{T_\bullet}(Y, Z, i + 1), H^{i+1}_S(Y, Z), \phi)$$

using (6.5). Thus $F_{T_\bullet}$ is a representation as claimed.
Let $\bar{F}_{T_\bullet} : \Delta(S,T_\bullet)^{op} \to \overline{C}$ be the composition of $F_{T_\bullet}$ with the quotient map. Since $H$ factors through $\bar{F}_{T_\bullet}$, by theorem 2.6, it extends to an exact functor

$$\bar{F}_{T_\bullet} : \mathcal{M}^{eff}(S,T_\bullet) \to \overline{C}$$

Let $h^j_{T_\bullet}$ denote the composite

$$(7.1) \quad \mathcal{M}^{eff}(S,T_\bullet) \xrightarrow{\bar{F}_{T_\bullet}} \overline{C} \xrightarrow{\mathcal{H}^{j,op}} \mathcal{M}^{eff}(k)$$

where the second arrow comes from the previous lemma. This forms a $\delta$-functor, since $\mathcal{H}^j \circ p_1$ does.

As noted already, by corollary 6.5

$$(7.2) \quad R_B(h^j_{T_\bullet}(h^i_S(X,Y))) \cong H^j(R_B(K_{T_\bullet}(X,Y,i)) \cong H^j(S,H^i_S(X,Y))$$

If $T'_\bullet \subset T_\bullet$, then one has a map of quivers $\Delta(S,T_\bullet) \to \Delta(S,T'_\bullet)$, and a corresponding map of complexes

$$(7.3) \quad K_{T_\bullet}(X,Y,i) \to K_{T'_\bullet}(X,Y,i)$$

This is a quasi-isomorphism by (7.2). Therefore $h^j_{T_\bullet}$ is compatible with refinement, so it extends to a functor $h^j$ on the 2-colimit $\mathcal{M}^{eff}(S)$.

\[\square\]

**Proof of theorem 7.1.** We first prove that the functor $h^j$ constructed in the last proposition has an extension $\mathcal{M}(S) \to \mathcal{M}(k)$ with the same properties. Given a complex $K^\bullet \in C^b(\mathcal{M}^{eff}(k))$, observe that

$$R_B(L^{eff}K^\bullet) \cong H^1(G_m, 1) \otimes_R R_B(K^\bullet) \cong R_B(K^\bullet)$$

Define $L^{eff} : C \to C$ by

$$L^{eff}(K^\bullet, F, \phi : H^*(S,F) \cong R_B \circ \mathcal{H}^*(K^\bullet)) = (L^{eff}K^\bullet, F, H^*(S,F) \cong R_B \circ \mathcal{H}^*(L^{eff}K^\bullet))$$

The composite $L^{eff} : C \to \overline{C}$ factors through $\overline{C}$.

Let $\lambda : \text{Ob} \Delta(S) \to \text{Ob} \Delta(S)$ be given by

$$(X \to S,Y,i) \mapsto (G_m \times X \to S, G_m \times Y \cup 1 \times X, i + 1)$$

Then

$$K(\lambda(X,Y,i)) \cong L^{eff}K(X,Y,i)$$

One can check that the diagram

$$\begin{array}{ccc}
\Delta(S,T_\bullet)^{op} & \xrightarrow{\bar{F}} & \overline{C} \\
\downarrow L^{eff} & & \downarrow L^{eff} \\
\mathcal{M}^{eff}(S,T_\bullet) & \xrightarrow{\bar{F}} & \overline{C} \\
\downarrow L^{eff} & & \downarrow L^{eff} \\
\mathcal{M}^{eff}(S,T_\bullet) & &
\end{array}$$

2-commutes. This implies that $\bar{F} : \mathcal{M}^{eff}(S,T_\bullet) \to \overline{C}$ extends to a functor $\mathcal{M}(S,T_\bullet) \to \overline{C}$. Composing with $\mathcal{H}^j \circ p_1$, and passing to the colimit, gives an extension $h^j : \mathcal{M}(S) \to \mathcal{M}(k)$ such that $L^{j} \circ h^j \cong h^j \circ L$. Any object of $M \in \text{Ob} \mathcal{M}(S)$ is isomorphic to $L^n M'$ with $M' \in \text{Ob} \mathcal{M}^{eff}(S)$. Therefore

$$R_B(h^j(M)) \cong R_B(L^n h^j(M')) \cong R_B(h^j(M')) \cong H^j(R_B(M))$$
Define an exact couple in $\mathcal{M}(k)$ by

\[ ME_1 = \bigoplus h^*(X_{T_p}, Y_{T_p} \cup X_{T_p-1}) \]

\[ MD_1 = \bigoplus h^*(X, Y \cup X_{T_p-1}) \]

with maps

\[ D_1 \rightarrow D_1 \rightarrow E_1 \]

induced by connecting maps as in (6.2). This generates a spectral sequence $ME_{pq}$.

Then the image of this exact couple under $R_B$ is (6.2). Therefore $R_B(ME_{pq}) \cong F\!E_{pq}^1$. Therefore by corollary $R_B(ME_{pq}) \cong L\!E_{pq}^2$. □

The proof actually gives a bit more than what was stated.

**Corollary 7.4** (of proof). With the same assumptions as in the theorem, there is a well defined triangulated functor $r\Gamma : D^b\mathcal{M}(S) \rightarrow D^b\mathcal{M}(k)$, such $h^iM = \mathcal{H}^i(r\Gamma M)$ for any $M \in \mathcal{M}(S)$.

**Proof.** The functor $p_2 \circ \bar{F}_T \ast$ extends to an exact functor

\[ C^b\mathcal{M}^{\text{eff}}(S, T_{\bullet}) \rightarrow C^b(C^b(\mathcal{M}^{\text{eff}}(k))) \]

from the category of single complexes to double complexes. Composing with the total complex, and projection, yields a functor

\[ C^b\mathcal{M}^{\text{eff}}(S, T_{\bullet}) \rightarrow D^b\mathcal{M}^{\text{eff}}(k) \]

The map (7.3) is a quasi-isomorphism by (7.2). Therefore the above map passes to the 2-colimit

\[ C^b\mathcal{M}^{\text{eff}}(S) \rightarrow D^b\mathcal{M}^{\text{eff}}(k) \]

This factors through $D^b\mathcal{M}^{\text{eff}}(S)$, and satisfies $h^iM = \mathcal{H}^i(r\Gamma M)$. One can check that this commutes with $L\!\mathcal{M}^{\text{eff}}$, therefore extends to $r\Gamma : D^b\mathcal{M}(S) \rightarrow D^b\mathcal{M}(k)$. □

**Theorem 7.5.** Let $X$ be a smooth and projective variety. Let $f : X \rightarrow S$ be a surjective projective morphism to another variety $S$, and assume that either $f$ is smooth and projective, or that $S$ is a smooth projective curve. Then there is a noncanonical decomposition

\[ h^i(X) \cong \bigoplus_{p+q=i} h^q(h^p_S(X)) \]

in $\mathcal{M}(k, \mathbb{Q})$.

**Proof.** The Leray spectral sequence

\[ LE_2 = H^p(S, H^q_S(X; \mathbb{Q})) \Rightarrow H^{p+q}(X; \mathbb{Q}) \]

degenerates at $E_2$, either by Deligne [D1, thm 1.5] when $f$ is smooth and projective, or by Zucker [Z, cor 15.15] when $S$ is a curve. This means that the differentials $d_2, d_3, \ldots$ are all zero. Therefore the same holds for the spectral sequence $ME_{pq}$, constructed in the previous theorem. It follows that there is a filtration $L^p$ on $h^i(X)$ such that

\[ Gr^p_L h^i(X) = h^q(h^p_S(X)) \]
Since $h^i(X)$ lies in $\mathcal{M}^{\text{pure}}(k)$, and this category is semisimple by corollary 5.6, it follows that the maps

$$h^i(X) \leftarrow L^p h^i(X) \rightarrow \text{Gr}_L^p h^i(X)$$

split. So we obtain an isomorphism

$$h^i(X) \cong \bigoplus_{p+q=i} h^q(h^p_S(X))$$

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Department of Mathematics, Purdue University, West Lafayette IN 47907, U.S.A.