Radiative corrections to the Casimir energy
and effective field theory

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February 21, 2001

Abstract

We discuss radiative corrections to the Casimir effect from an effective field theory point of view. It is an improvement and more complete version of a previous discussion by Kong and Ravndal. By writing down the most general effective Lagrangian respecting the symmetries and the boundary conditions, we are able to reproduce earlier results of Bordag, Robaschik and Wieczorek calculated in full QED. They obtained the correction $E_0^{(1)} = \frac{\pi^2 \alpha}{2560} m L^4$ to the Casimir energy. We find that this leading correction is due to surface terms in the effective theory, which we attribute to having dominant fluctuations localized on the plates.

PACS numbers: 12.20.Ds, 03.70.+k

Keywords: Effective field theory; Casimir effect; Quantum corrections

1 Introduction

Although the Casimir effect [1, 2] has been known for more than 50 years, the question of what are the leading quantum corrections to this effect is surprisingly still a subject of debate. The corrections we have in mind here are those that are caused by the coupling of the electromagnetic field to the electron field. These are first of all important for theoretical reasons, but recent improvements in experimental techniques [3] may lead to interesting confrontations between theory and experiment in the near future.

The first attempts to determine the quantum corrections to the Casimir energy $E_0$ were reported in a paper by Bordag, Robaschik, and Wieczorek [4] (BRW). These authors considered the quantum vacuum within the usual set-up with two perfectly conducting parallel plates using full QED. The electromagnetic field satisfies metallic boundary conditions, while the electron field does not feel the presence of the metallic plates. They found the correction $E_0^{(1)} = \frac{\pi^2 \alpha}{2560} m L^4$ to the well-known leading term $E_0^{(0)} = -\frac{\pi^2}{720} L^3$, where $L$ is the separation between the plates and $m$ is the electron mass. This correction emerges as an effect of vacuum polarization. Three years ago, one of us and Kong [5] (KR) studied the Casimir effect from an effective field theory [6] point of view. In that work it was argued that vacuum polarization does not have any effect on the Casimir energy. This would be just like for effective QED in free space where it is known that vacuum polarization has no physical consequence in the absence of external electrons. The leading corrections would then come from the Euler–Heisenberg effective Lagrangian and have...
the value \( E^{(1)}_0 = 11\pi^4 \alpha^2 / 27^3 5^3 m^4 L^7 \), in disagreement with the result of BRW. Subsequently, the use of effective QED in KR was criticized in Ref. \[7\]. The authors of \[7\] pointed out that even though the Casimir effect is a low energy phenomenon, a derivative expansion, as is typical of effective theories, could not be used in this case. The reason for this is that the evaluation of the corrections to the Casimir energy in the full theory involves an integral along the cut of the vacuum polarization tensor \( \Pi_{\mu\nu}(k^2) \), and this information is lost if \( \Pi_{\mu\nu} \) is expanded in powers of \( k^2 \). This might then “explain” why the results in KR were incorrect and indicate that the result of BRW was correct in the first place.

Thus, Ref. \[7\] gave the impression that effective field theory methods could not be used on the Casimir problem. However, effective field theory is not just a derivative expansion of the underlying theory but something more general. Indeed, effective field theory always works when the physical degrees of freedom are fields. This is because the effective theory is by definition constructed to give the same results as the underlying theory in a given situation, usually at low energies. There is therefore no reason to dismiss the results from KR on the grounds that they are based on effective field theory methods. We are in fact left with the following possibilities:

(a) The full QED result of BRW is correct and the effective QED discussion in KR is incomplete and for this reason gives the wrong answer. (b) The full QED calculation in BRW is incorrect and the results from effective QED in KR are correct. (c) Both the full and the effective QED results are wrong.

In this paper, we show that it is possibility (a) above that is correct by explicitly constructing the effective theory. We start in Sec. 2 by recalling how perturbation theory works in the presence of boundary conditions, as first discussed in BRW. We also sketch the calculation of the radiative corrections to the Casimir energy.

In Sec. 3 we turn to the construction of the effective theory. All possible terms of the Lagrangian that respect the symmetries must be written down. In this process we realize that there is a class of terms that were left out in the previous treatment of effective QED in KR. These are surface terms, where the contributions to the action live only on the two plates. We then establish the counting rules of the effective theory, and it is seen that the surface terms provide the leading correction to the Casimir energy.

In Sec. 4 we use this effective theory to calculate the Casimir energy corrections in terms of the low energy constant of the leading surface term in the effective Lagrangian. We are then able to determine this constant by matching to the BRW result of the full theory. This calculation demonstrates that effective field theory can be applied to the Casimir effect. However, more constants must be determined by matching before we can claim to have a workable effective theory with predictive power.

Sec. 5 contains a brief discussion of our results. In particular, we demonstrate one possible way of understanding the relevant surface terms as being “generated” from the full theory in the low energy approximation.

## 2 Functional derivation of the Casimir energy

In this section we make some general remarks and recall how the photon propagator is modified in the presence of the two plates, which allows perturbation theory to work the usual way. We also briefly sketch the calculation of the radiative corrections to the Casimir energy. There are no new results in this section.

The starting point is the path integral over the photon field \( A_\mu \),

\[
Z = \int \mathcal{D}A e^{iS},
\]  
(1)
with the Maxwell action \( S = \int d^4x (-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}) \). It is to be understood that only configurations respecting the boundary conditions are integrated over. The system we consider will always be the standard geometry of two perfectly conducting parallel plates positioned at \( z = 0 \) and \( z = L \). Hence the electromagnetic field respects the usual metallic boundary conditions \( n \times E = n \cdot B = 0 \) at the plates, where \( n = (0,0,1) \) is the normal vector to the plates in the positive \( z \)-direction. In four-vector notation the boundary condition reads \( n^\mu \tilde{F}_{\mu\nu}|_{z=0,L} = 0 \), where \( \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu
u\rho\sigma} F^{\rho\sigma} \) is the dual field strength tensor.

In order to make the boundary conditions explicit in the path integral, we may introduce a product of two delta functions enforcing these boundary conditions. At the same time we extend the integrations to all \( i \) configurations of \( A_\mu \):

\[
Z = \int D A \delta(n^\mu \tilde{F}_{\mu\nu}|_{z=0}) \delta(n^\mu \tilde{F}_{\mu\nu}|_{z=L}) e^{iS}.
\]

(2)

We then represent the delta functions by path integrals over two external fields \( B^i_\mu(x_\perp), i = 1, 2 \), \( x^\mu = (x^0, x^1, x^2) \), that lives only on the two plates at \( z = a_i \) with \( a_1 = 0, a_2 = L \). The complete path integral thus becomes

\[
Z = \int DA DB e^{iS'},
\]

(3)

with the modified action

\[
S' = -\frac{1}{4} \int d^4x F_{\mu\nu}(x) F^{\mu\nu}(x) - \int d^3x_\perp B^i_\mu(x_\perp)n^\mu \tilde{F}_{\mu\nu}(x_\perp,a_i).
\]

(4)

A summation over \( i \) is understood in this expression.

Apart from these constraints, the fields are assumed to exist both outside as well as in between the plates. This set-up is of course highly idealized, and in order to make the description more realistic we need, at least, to take into account the physical nature of the plates and the coupling of \( A_\mu \) to the electron field. In this paper we will consider corrections from the electron field, but where this electron field does not feel any boundary conditions at the plates. This is the same situation as in BRW and [8], and can be considered as a first step on the way to a more realistic treatment of corrections to the Casimir energy.

In BRW it was pointed out that standard Feynman perturbation theory can be used even in the presence of the metallic boundary conditions provided the correct modified photon propagator is used. An expression for this propagator may be obtained by first coupling external sources to the photon field in the action (3). Integrating out the photon field and the Lagrange multiplier fields \( B^i_\mu \) thus produces a functional of the external sources from which the propagator can be read off. The result is

\[
\langle 0|T A_\mu(x) A_\nu(x')|0 \rangle = iD_{\mu\nu}(x-x') - i\bar{D}_{\mu\nu}(x,x'),
\]

(5)

where \( D_{\mu\nu}(x-x') \) is the free propagator and \( \bar{D}_{\mu\nu}(x,x') \) is

\[
\bar{D}_{\mu\nu}(x,x') \equiv \int \frac{d^4k_\perp}{(2\pi)^3} \frac{-P^\perp_{\mu\nu}}{4\gamma \sin \gamma L} e^{-ik_\perp(x_\perp-x'_\perp)} e^{i\gamma|z-x'|} e^{i\gamma|z-x'|} e^{i\gamma|z-L|} e^{i\gamma|z'-L|} + e^{i\gamma|z-L|} e^{i\gamma|z'-L|}.
\]

(6)
The notation here is \( k_\perp = (k^0, k^1, k^2) \), \( \gamma = \sqrt{k_\perp^2} = \sqrt{k_0^2 - k_1^2 - k_2^2} \), and \( P_{\mu\nu}^\perp \) is the projection operator

\[
P_{\mu\nu}^\perp = \begin{cases} 
g_{\mu\nu} - \frac{k_\mu k_\nu}{k_\perp^2} & \text{for } \mu, \nu \neq 3, \\
0 & \text{for } \mu = 3 \text{ or } \nu = 3.
\end{cases} \tag{7}
\]

This modified propagator can then be used for calculating diagrams in perturbation theory when the interactions with the electrons are turned on. The correction to the Casimir energy can be found in this way by computing the relevant diagrams.

For later convenience we now briefly sketch how the corrections may be calculated. Our approach is essentially that of [8], which is an improved calculation compared to the original one in BRW. We are also inspired by the discussion in [9] which is another approach based on the functional formalism. The general idea is to make use of the field theory identity

\[
Z = e^{-iE_{\text{vac}}T}, \tag{8}
\]

where \( E_{\text{vac}} \) is the vacuum energy and \( T \) is the total time. The Casimir energy \( E_0 \) is defined to be \( E_0 = E_{\text{vac}}/A \) where \( A \) is the area of the plates. We may thus extract \( E_0 \) by evaluating a path integral.

We will first consider the simpler case of the plain Casimir effect without radiative corrections before we consider the full complexity of the problem. If we define a new field \( C_{\mu}^i \equiv -\eta^{\alpha} \epsilon_{\alpha\mu\nu\beta} \partial^\beta B_i^\nu \), we can write the second term in the action (11) as

\[
-\int d^3 x A_\mu P_{\mu\nu}^\perp C_i^\nu, \tag{9}
\]

where we also use the projection operator \( P_{\mu\nu}^\perp \). Now, since the photon field \( A_\mu \) couples directly to this field, we can without loss of generality change path integration variables from \( B_i^\mu \) to \( C_i^\mu \) as long as we make sure that \( C_i^\mu \) has the properties \( \partial_\mu C_i^\mu = 0 \) and \( C_i^3 = 0 \). The new path integral over \( C_i^\mu \) still imposes the same boundary conditions on \( A_\mu \). The two conditions on \( C_i^\mu \) means that this field has only two independent components, which implies that only two components of \( A_\mu \) couple to \( C_i^\mu \). The other two components thus decouples from the problem and has no physical consequence. On the other hand \( A_\mu \), or more generally any vector field, can be decomposed into \( A_\mu = P_{\mu\nu}^\perp A_\nu + A_{\mu}' \), where the two terms are orthogonal to each other. We can use this to rewrite the kinetic term of the photons:

\[
-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} A_\mu \Box P_{\perp}^\mu A_\nu + (\text{terms with } A_{\mu}') \tag{10}
\]

where we have used partial integration and the fact that \( P_{\mu\nu}^\perp (g^{\nu\rho} - \partial^\nu \partial^\rho/\Box) P_{\nu\rho}^\perp = P_{\mu\rho}^\perp \). The result of all this is that we are allowed to write the path integral over \( A_\mu \) and \( C_i^\mu \) with the action

\[
S'' = \frac{1}{2} \int d^4 x A_\mu(x) \Box P_{\perp}^\mu A_\nu(x) - \int d^3 x A_\mu(x, a_i) P_{\perp}^\mu C_i^\nu(x, a_i). \tag{11}
\]

The terms depending on \( A_{\mu}' \) have been omitted since they decouple from \( A_\mu \) and \( C_i^\mu \) and are therefore irrelevant to the Casimir energy. Note also that it has not been necessary to consider any specific gauge in order to arrive at this.

This path integral is Gaussian in \( A_\mu \) and may be evaluated to give a path integral over \( C_i^\mu \) alone with the new action

\[
S = -\frac{1}{2} \int d^3 x C_i^\mu M_{ij} P_{\perp}^\mu C_j^i, \tag{12}
\]

where \( M_{ij} \) is the operator whose form in momentum space is

\[
M_{ij} = \frac{i}{2\gamma} e^{i|a_i - a_j|} = \frac{i}{2\gamma} \begin{pmatrix} 1 & e^{i\gamma L} \\
-1 & e^{-i\gamma L} \end{pmatrix}. \tag{13}
\]
The path integral then becomes
\[ Z = \left( \text{Det} P_{\perp} M_{ij} \right)^{-1/2} = \left( \text{Det} M \right)^{-1} = e^{-\text{Tr} \ln \text{det} M}, \quad (12) \]
where 'Det M' means determinant with respect to both k-space and ij-indices, while 'det M' means determinant with respect to the ij-structure only. We have also used that \( P_{\perp} \) is a projection operator onto a two-dimensional space and thereby gives a multiplicity 2. Therefore,
\[ E_0 = -\frac{i}{AT} \text{Tr} \ln \text{det} M. \quad (13) \]
It is straightforward to evaluate this expression in momentum space, where \( \text{Tr} \rightarrow AT \int d^3k_{\perp}/(2\pi)^3 \), and
\[ \text{det} M = \frac{-1}{4\gamma^2} (1 - e^{2i\gamma L}). \quad (14) \]
When we take the logarithm of this, the term with \( \ln(-1/4\gamma^2) \) does not depend on \( L \) and can be ignored in this context. Thus we find that the Casimir energy is
\[ E_0 = -i \int \frac{d^3k_{\perp}}{(2\pi)^3} \ln \left( 1 - e^{2i\gamma L} \right) = -\frac{\pi^2}{720L^3}, \]
as it should be.

Let us now turn to the quantum corrections to this result. As is well-known, the full effect that the electrons have on the vacuum polarization to order \( \alpha \) can be summarized by replacing the kinetic term of free photons with a modified Lagrangian:
\[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \rightarrow -\frac{1}{4} F_{\mu\nu} [1 + \Pi(-\Box)] F^{\mu\nu}, \]
where \( \Pi(-\Box) \) is the usual renormalized vacuum polarization modulo the gauge invariant projection operator \( \Box g_{\mu\nu} - \partial_{\mu} \partial_{\nu} \).

This leads to a modification of the photon propagator and in turn this leads to a modification of \( M_{ij} \) and thereby its determinant, which now becomes
\[ \text{det} M = \frac{-1}{4\gamma^2} (1 - e^{2i\gamma L}) \left( 1 - 4i \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \frac{\Pi(k^2)}{k^2} \frac{\gamma (1 - e^{i\gamma L} \cos k_z L)}{1 - e^{2i\gamma L}} \right). \quad (15) \]
When the logarithm of this expression is inserted in formula (13) for the Casimir energy we recognize the first parenthesis as the contribution giving the leading term \( E_0^{(0)} = -\pi^2/720L^3 \) calculated above. The second parenthesis then gives the correction:
\[ E_0^{(1)} = -2i \int \frac{d^4k}{(2\pi)^4} \frac{\gamma}{\sin \gamma L} \left( e^{-i\gamma L} - \cos k_z L \right) \frac{\Pi(k^2)}{k^2}, \quad (16) \]
which is identical to Eq. (45) in [8], apart from a difference in sign due to different sign conventions for \( \Pi(k^2) \). It was evaluated there in the physically interesting limit where \( mL \gg 1 \):
\[ E_0^{(1)} = \frac{\pi^2 \alpha}{2560mL^4}. \quad (17) \]
This is the result that was first obtained by BRW.
3 Effective QED with two conducting plates

We now turn to effective field theory \[6\]. An effective field theory calculation starts with writing down the most general effective Lagrangian respecting the symmetries of the problem. Next, one assigns counting rules to each term in the effective Lagrangian, which allows us to calculate physical quantities from the effective theory in a systematic way. The free coefficients that multiplies each term in the effective Lagrangian – the “low energy constants” – are then determined by matching the results with the corresponding quantities calculated in the full theory. The result is a theory that in principle may be used to perform calculations to any desired order in the effective counting rules. This program is applied here to the Casimir energy which enables us to explicitly demonstrate that effective field theory works in this case.

Let us begin our investigations by writing down the effective Lagrangian, and then discuss it afterwards. It has the form

\[ L_{\text{eff}} = L_{\text{bulk}}^{\text{eff}} + L_{\text{surf}}^{\text{eff}}, \]

\[ L_{\text{bulk}}^{\text{eff}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{c_1}{m^2} F_{\mu\nu} n^\rho \partial_\rho n^\sigma \partial_\sigma F^{\mu\nu} + \frac{c_2}{m^2} F_{\mu\nu} n^\rho \partial^\sigma n^\rho \partial_\sigma F_{\rho\sigma} + \frac{c_3}{m^2} F_{\mu\nu} \Box F^{\mu\nu} + \text{higher orders}, \]

\[ L_{\text{surf}}^{\text{eff}} = -\frac{d_1}{4m} F_{\mu\nu} F^{\mu\nu} [\delta(z) + \delta(z - L)] + \text{higher orders}. \]

The terms fall into two categories, a “bulk” contribution where field values from the whole of space-time are used and a “surface” contribution where only field values at the two plates at \( z = 0 \) and \( z = L \) are used. It is to be understood that the metallic boundary conditions at the plates are accounted for by inserting delta functions in the path integral where \( \int d^4 x L_{\text{eff}} \) is the action.

But let us first comment on the symmetries of the problem. In free space, these would be gauge symmetry, Lorentz symmetry, translation symmetry and the discrete symmetries. In the presence of the two plates, however, parts of the Lorentz and translation symmetries are broken. Hence more terms are allowed in the effective Lagrangian. We can include some of these terms in a mock Lorentz invariant form by making use of the four-vector \( n^\mu \) from the last Section. Due to gauge invariance we are required to build the effective theory from \( F_{\mu\nu} \) rather than from \( A_\mu \). Thus, using \( F_{\mu\nu}'s, n's \) and derivatives we get an effective Lagrangian of the form \[13\]. Note that there can be no terms with an odd number of \( n \)'s. The reason for this is that \( n^\mu \) also appears in the delta function that enforce the proper boundary conditions in the full theory in the previous Section. It is then evident that there is an additional symmetry, \( n^\mu \rightarrow -n^\mu \), which rules out odd numbers of \( n \)'s.

The surface terms in \( L_{\text{surf}}^{\text{eff}} \) deserves special mention. They must \textit{a priori} be present in the action because they respect the symmetries. Let us also remember that surface terms are not unknown in the world of effective theories. The Wess–Zumino–Witten term in chiral perturbation theory has such an interpretation \[10\]. There is also a relation to the hyperfine splitting term, proportional to a delta function, in the effective Hamiltonian that is often used in the discussion of hydrogen-like atoms \[2\].

The effective Lagrangian in this problem is given in a natural way in terms of two derivative expansions – one in the bulk and one on the plates. The coefficient in front of each term has been scaled with the electron mass \( m \) so that the parameters \( c_i \) and \( d_i \) are dimensionless. The other scale in the theory, \( L \), is not expected to have any influence on a \textit{local} quantity like the effective Lagrangian. Only terms up to mass dimension six for the bulk part, and up to dimension five for the surface part, are displayed in Eq. \[13\]. It may appear that a third term with two \( n \)'s could be written down in \( L_{\text{eff}}^{\text{bulk}} \), proportional to \( F_{\mu\nu} n^\rho \partial^\sigma n^\rho \partial_\sigma F_{^\rho \nu} \). However, since...
the boundary conditions are taken care of by two delta functions inserted in the path integral, we are allowed to freely perform partial integrations. One may then check that this interaction is equivalent to the term with $c_1$ in (18). It may also appear that another term proportional to $n_\mu F^{\mu\rho} F_{\nu\rho} [\delta(z) + \delta(z - L)]$ could be written down in $\mathcal{L}_{\text{eff}}$, but due to the boundary conditions this term is equivalent to the one already given.

In order to assign counting rules for the effective theory we start with the “free” term $-\frac{1}{4} F_{\mu\nu}^2$. We may arbitrarily assign the order $p^2$ to this, where $p$ is a “small” momentum. Then $F_{\mu\nu}$ is of order $p$. Likewise we assign a factor $p$ for each derivative, so that the displayed terms with $c_i$ in $\mathcal{L}_{\text{eff}}$ are of order $p^4$. Furthermore, it is natural to assign a factor of $p$ to the $\delta$-functions in $\mathcal{L}_{\text{eff}}$ in agreement with dimensional analysis. Thus, the leading surface term, proportional to $d_1$, is of order $p^3$. This means that unless $d_1$ accidentally vanishes or is unusually small, the leading corrections to the Casimir energy in the effective theory comes from this term.

The order $p^4$ operators we have written in the effective Lagrangian (18) are all terms that vanish in free space. Indeed, it is well-known in free space effective field theory that terms in the action which vanishes due to the free field equations of motion can be removed by a field redefinition in the path integral [11]. This would remove the term proportional to $c_2$ in Eq. (18). On the other hand, the Uehling term proportional to $c_3$ may be removed by the transformation

$$A_\mu \rightarrow A_\mu - \frac{2c_3}{m^2} \Box A_\mu.$$  

However, this possibility is no longer open to us in the present case due to the nontrivial boundary conditions. The point is that even though one field configuration satisfies the required condition on the plates, this will in general not be true for the transformed configuration because of the presence of derivatives in the transformation law. This is not a problem in free space, since it is always implicitly understood that both the fields and their derivatives go to zero at infinity. Therefore, the Uehling term and other terms in the same situation must be kept and may a priori give rise to real physical effects.

The contribution to the Casimir energy from the various correction terms in (18) can be obtained from the field theory identity (8). From this we find the correction

$$E_{0}^{(1)} = -\frac{1}{AT} \langle S^{(1)} \rangle,$$  

where $S^{(1)}$ is a small perturbation of the leading order Maxwell action $S^{(0)}$. The expectation values in $\langle \cdots \rangle$ refer to the theory described by $S^{(0)}$ under the further restriction that the usual metallic boundary conditions $n^\mu F_{\mu\nu} = 0$ hold on the plates. This means that the contractions that appear in $\langle \cdots \rangle$ should be calculated using the modified photon propagator (5) of BRW, as discussed in Sec. 2.

4 Corrections to the Casimir energy in the effective theory

The leading order correction is expected to result from the lowest dimension operator in the Lagrangian (18), i.e.

$$\Delta \mathcal{L} = -\frac{d_1}{4m} F_{\mu\nu} F^{\mu\nu} [\delta(z) + \delta(z - L)].$$  

In order to make the contractions involved in calculating $\langle \Delta S \rangle$, it is appropriate to write $\Delta S$ so that it depends directly on $A_\mu$ instead of $F_{\mu\nu}$. For the plate at $z = a$, with $a$ either 0 or $L$, we have

$$\Delta S_a = -\frac{d_1}{4m} \int d^4 x F_{\mu\nu} F^{\mu\nu} \delta(z - a).$$  

7
We may perform partial integration which gives
\[ \Delta S_a = \frac{d_1}{2m} \int d^4x A_\mu \left[ \delta(z - a)\Box + \frac{i}{2}\delta''(z - a) \right] A^\mu, \quad (22) \]
where \( \delta''(z - a) = (d^2/dz^2)\delta(z - a) \). In this expression we have omitted terms that do not contribute to the result. Indeed, when (22) is contracted with the modified photon propagator, only the part with \( \bar{D}_{\mu
u}(x, x') \) carries \( L \)-dependence. This term involves the projection operator \( P_{\mu\nu} \), so that, without loss of generality, we may make the replacement
\[ \frac{1}{2} A_\mu[\cdots]A^\mu \rightarrow \frac{1}{2} P_{\mu\nu} A_\mu[\cdots]A_\nu. \quad (23) \]
The correction then becomes
\[ E^{(1)}_0 = -\frac{1}{ATm^2} \frac{d_1}{2m} \int d^4x \sum_{i=1}^2 P_{\mu\nu} \langle A_\mu(x)|\delta(z - a_i)\Box + \frac{i}{2}\delta''(z - a_i)|A_\nu(x)\rangle. \quad (24) \]
Let us first consider the term involving \( \delta'' \) and set \( i = 1 \). From Eq. (5) it then follows
\[ \langle A_\mu(x)|\delta''(z)|A_\nu(x)\rangle = \delta''(z)\epsilon(D_{\mu\nu}(x, x) - \bar{D}_{\mu\nu}(x, x)). \quad (25) \]
Since only \( \bar{D}_{\mu\nu} \) contains dependence on \( L \) we can disregard the part involving \( D_{\mu\nu} \). The right-hand side of (25) then becomes
\[ -i \int \frac{d^3k_\perp - P_{\mu\nu}\delta''(z)}{(2\pi)^3 4\gamma \sin \gamma L} \left[ e^{-i\gamma L(e^{2i\gamma |z|} + e^{2i\gamma |z-L|})} - 2e^{i\gamma (|z|+|z-L|)} \right], \quad (26) \]
which appears under a space-time integral. We may therefore partially integrate to remove the two \( z \)-derivatives on the \( \delta \)-function. Making use of identities such as
\[ \partial_z e^{i\gamma |z-a|} = 2i\gamma \delta(z-a) - \gamma^2 e^{i\gamma |z-a|}, \]
where \( \epsilon(z) \) is the sign function, this leads to
\[ \langle A_\mu(x)|\delta''(z)|A_\nu(x)\rangle = -iP_{\mu\nu} \delta(z) \int \frac{d^3k_\perp}{(2\pi)^3} \frac{\gamma e^{i\gamma L}}{\sin \gamma L}. \quad (28) \]
Similarly, the term with \( i = 2 \) gives
\[ \langle A_\mu(x)|\delta''(z-L)|A_\nu(x)\rangle = -iP_{\mu\nu} \delta(z-L) \int \frac{d^3k_\perp}{(2\pi)^3} \frac{\gamma e^{i\gamma L}}{\sin \gamma L}. \quad (29) \]
On the other hand, the terms in (24) involving \( \delta \Box \) are found to be equal to
\[ \langle A_\mu(x)|\delta(z - a_i)\Box A_\nu(x)\rangle = iP_{\mu\nu} \delta^4(0)\delta(z - a_i), \quad (30) \]
which do not give any \( L \)-dependent contribution to the correction \( E^{(1)}_0 \).
Collecting this information we find that the contribution to the Casimir energy is
\[ E^{(1)}_0 = -2d_1 \frac{m^2 P_{\mu\nu} 1}{2m} \frac{1}{2} (-iP_{\mu\nu}) \int \frac{d^3k_\perp}{(2\pi)^3} \frac{\gamma e^{i\gamma L}}{\sin \gamma L} \]
\[ = \frac{d_1}{m^2} i \int \frac{d^3k_\perp}{(2\pi)^3} \frac{\gamma e^{i\gamma L}}{\sin \gamma L} \]
\[ = \frac{d_1}{2\pi^2 m L^4} \int_{-\infty}^{\infty} dk \frac{k^3 e^{-k}}{\sinh k} \]
\[ = -\frac{\pi^2 d_1}{240 m L^4}, \quad (31) \]
where we have used that $P_{\mu\nu}P_{\mu\nu}^\perp = 2$. Thus, by choosing the numerical value

$$d_1 = \frac{-3\alpha}{32}$$

we are able to reproduce the full QED result $E_0^{(1)} = \pi^2\alpha / 2560mL^4$. This is a reasonable magnitude for a constant that represents radiative corrections from virtual electron-positron pairs.

## 5 Discussion

To summarize, we have constructed an effective field theory satisfying the appropriate boundary conditions, and have been able to reproduce the full QED result to next-to-leading order of the low energy expansion. This demonstrates that effective field theory works for the Casimir effect – as it does for any other field theoretical problem.

The Lagrangian of an effective field theory is constructed by writing down all possible terms that respect the symmetries and the coefficients that multiplies them are determined by a matching procedure. The question therefore never arises in an effective field theory calculation how the various terms in the effective Lagrangian “comes about”. Nevertheless, it may be interesting in the present case to see if we can gain some insight into the nature of the surface terms that are so important for our results. Referring back to Eq. (16) we have the following expression for the energy correction [4],

$$E_0^{(1)} = -2i \frac{d^4k}{(2\pi)^4} \frac{\gamma}{\sin \gamma L} \left( e^{-i\gamma L} - \cos k_z L \right) \frac{\Pi(k^2)}{k^2},$$

in the underlying theory. If now we are interested in the low energy content of this expression, we may try to simplify it by taking the low energy limit of $\Pi$:

$$\Pi(k^2) = c \frac{k^2}{m^2} + \mathcal{O}(k^4), \quad \text{for } k^2 \ll m^2,$$

with $c$ some unimportant constant. Inserting this in Eq. (33), we find that the term proportional to $\cos k_z L$ vanishes from the $k_z$-integration, and we get

$$E_0^{(1)} = -2i \frac{c}{m^2} \int \frac{d^4k}{(2\pi)^4} \frac{\gamma e^{i\gamma L}}{\sin \gamma L} + \mathcal{O}(k^4).$$

Here we have used that $e^{-i\gamma L} = e^{i\gamma L} - 2i \sin \gamma L$ and that the $\sin \gamma L$-part of this leads to an $L$-independent contribution. The $k_z$-integration here is divergent. However, we should remember that this form is valid only for $k^2 \ll m^2$. We can repair the situation by choosing a cutoff for the $k_z$-integration of the order $m$. We then get

$$E_0^{(1)} \sim \frac{i}{m} \int \frac{d^3k_\perp}{(2\pi)^3} \frac{\gamma e^{i\gamma L}}{\sin \gamma L} + \cdots$$

for the energy correction. On comparison, we see that this has the same form as the expression in the second line of Eq. (31), which is precisely the contribution from the surface term.

It is also possible to understand how the cut of the vacuum polarization tensor, with modes of relatively high momentum, $k^2 \geq 4m^2$, may contribute to the full QED energy correction. At first sight it would appear that the effects of virtual electron-positron pairs with high energies are unimportant to the Casimir problem. After all, such pairs are sharply localized while the
Casimir effect is controlled by the macroscopically large scale $L$, the distance between the plates. However, it is known from the literature on the Casimir effect that the fluctuations of the electromagnetic field are large near the plates and diverge as we approach them. In this region with violent fluctuations there will be an increase in the production of pairs. Thus, even sharply localized pairs may be expected to contribute significantly to the correction. This picture of what is going on means that part of the physics is localized on the plates and agrees very well with the presence of surface terms in the effective theory.

As mentioned in Sec. 3, the calculations we have done in this paper are only the first steps in a full-fledged effective field theory calculation. At higher orders of the counting rules more terms with their corresponding low energy constants enters the description and these constants must also be determined by matching. It will then be necessary to consider other quantities than the Casimir energy like, for example, the Green’s functions of the system. This would then allow us to perform more detailed calculations in the effective theory to any desired order of precision in the low energy expansion.

Finally, let us comment on the fact that we are using highly idealized plates which provides perfect metallic boundary conditions. More realistic plates would be associated with some cutoff $C < m$ representing the physical nature of the plates. More precisely, this means that modes of the electromagnetic field with momenta $p \gtrsim C$ are able to penetrate the plates and are not confined by them. This situation was investigated in [7]. The effective theory discussed in this paper can be viewed as a special case of this more general situation with the external momenta $p$ restricted by $1/L \leq p \ll C$. It is clear that effective field theory is powerful enough to provide a description of this more realistic case too. That, however, is beyond the scope of this paper.

Note: After this work was completed there appeared a paper by K. Melnikov on the archives [12] which also deals with radiative corrections to the Casimir energy and effective field theory. Melnikov also finds an effective Lagrangian that reproduces the BRW result. However, the term that is responsible for the corrections in his Lagrangian is a bulk term proportional to $n_\mu F^{\mu\rho} n_\nu F_{\nu\rho}$ (in our notation) with a coefficient involving the plate separation $L$. To our minds, it would be surprising if this term would be present in the correct effective Lagrangian. There are two reasons for this: First, the operator dimension of this term is 4 which is the same as the leading Maxwell term, and second, the parameter $L$ is a global property of the system and is not a priori expected to occur in a local quantity like the Lagrangian. To truly resolve the question of which effective Lagrangian is correct it is necessary to calculate some other quantity than the Casimir energy like for instance the propagator to order $\alpha$ in both the full and effective theories. One should then be able to determine which effective theory reproduces the result from the full theory.

Acknowledgments One of us (F.R.) wants to thank C.P. Burgess for many useful discussions.

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